Aspects of differential calculus related to infinite-dimensional vector bundles and Poisson vector spaces

Helge Glöckner

Abstract
We prove various results in infinite-dimensional differential calculus which relate differentiability properties of functions and associated operator-valued functions (e.g., differentials). The results are applied in two areas:

(1) in the theory of infinite-dimensional vector bundles, to construct new bundles from given ones, like dual bundles, topological tensor products, infinite direct sums, and completions (under suitable hypotheses).

(2) in the theory of locally convex Poisson vector spaces, to prove continuity of the Poisson bracket and continuity of passage from a function to the associated Hamiltonian vector field.

Topological properties of topological vector spaces are essential for the studies, which allow hypocontinuity of bilinear mappings to be exploited. Notably, we encounter $k_{\mathbb{R}}$-spaces and locally convex spaces $E$ such that $E \times E$ is a $k_{\mathbb{R}}$-space.

Subject Classification. 26E15 (primary); 17B63, 22E65, 26E20, 46G20, 54B10, 54D50, 55R25, 58B10

Keywords and Phrases. Vector bundle, dual bundle, direct sum, completion, tensor product, cocycle, smoothness, analyticity, hypocontinuity, $k$-space, compactly generated space, infinite-dimensional Lie group, Poisson vector space, Poisson bracket, Hamiltonian vector field, group action, multilinear map

Contents

1 Preliminaries and notation 4
2 Differentiability properties of operator-valued maps 10
3 Compositions with hypocontinuous $k$-linear maps 14
4 Differentiability properties of $f^\land$ 16
5 Infinite-dimensional vector bundles 18
6 Completions of vector bundles 21
7 Tensor products of vector bundles 27
8 Locally convex direct sums of vector bundles 31
9 Dual bundles and cotangent bundles 34
10 Locally convex Poisson vector spaces 39
11 Continuity properties of the Poisson bracket 41
12 Continuity of the map taking $f$ to the Hamiltonian vector field $X_f$ 44
A Proofs for basic facts in Section II 45
B Smooth maps need not extend to the completion 49
Introduction

We study questions of infinite-dimensional differential calculus in the setting of Keller’s $C^{k}_c$-theory \cite{37} (going back to \cite{4}). Applications to infinite-dimensional vector bundles are given, and also applications in the theory of locally convex Poisson vector spaces.

Differentiability properties of operator-valued maps. Our results are centered around the following basic problem: Consider locally convex spaces $X$, $E$ and $F$, an open set $U \subseteq X$ and a map $f : U \to L(E, F)_b$ to the space of continuous linear maps, endowed with the topology of uniform convergence on bounded sets. How are differentiability properties of the operator-valued map $f$ related to those of $f \wedge : U \times E \to F$, $f \wedge(x, v) := f(x)(v)$?

We show that if $f \wedge$ is smooth, then also $f$ is smooth (Proposition 2.1). Conversely, exploiting the hypocontinuity of the bilinear evaluation map $L(E, F)_b \times E \to F$, $(\alpha, v) \mapsto \alpha(v)$, we find natural hypotheses on $E$ and $F$ ensuring that smoothness of $f$ entails smoothness of $f \wedge$ (Proposition 4.1; likewise for compact sets in place of bounded sets). Without extra hypotheses on $E$ and $F$, this conclusion becomes false, e.g. if $U = X$ is a non-normable real locally convex convex space with dual space $X' := L(X, \mathbb{R})$. Then $f := \text{id}_{X'} : X'_b \to X'_b$ is continuous linear and thus smooth, but $f \wedge : X'_b \times X \to \mathbb{R}$ is the bilinear evaluation map taking $(\lambda, x)$ to $\lambda(x)$, which is discontinuous for non-normable $X$ (see \cite{39}, p. 2) and hence not smooth in the sense of Keller’s $C^{\infty}_c$-theory. We also obtain results concerning finite order differentiability properties, as well as real and complex analyticity. Furthermore, $L(E, F)$ can be replaced with the space $L^k(E_1, \ldots, E_k, F)$ of continuous $k$-linear maps $E_1 \times \cdots \times E_k \to F$, if $E_1, \ldots, E_k$ are locally convex spaces.\footnote{Related questions also play a role in the comparative study of differential calculi \cite{37}.
}

As a very special case of our studies, the differential

$$f' : U \to L(E, F)_b$$

is $C^{r-2}$, for each $r \in \mathbb{N} \cup \{\infty\}$ with $r \geq 2$, locally convex spaces $E$ and $F$, and $C^r$-map $f : U \to F$ on an open set $U \subseteq E$ (see Proposition 2.2). This result is used, e.g., in \cite{18}, to study implicit functions from topological vector spaces to Banach spaces.

Applications to infinite-dimensional vector bundles. Apparently, mappings of the specific form just described play a vital role in the theory of vector bundles: If $F$ is a locally convex space, $M$ a (not necessarily finite-dimensional) smooth manifold, and $(U_i)_{i \in I}$ an open cover of $M$, then the smooth vector bundles $E \to M$, with fibre $F$, which are trivial over the sets $U_i$, can be described by cocycles $g_{ij} : U_i \cap U_j \to \text{GL}(F)$ such that $G_{ij} := g_{ij}^* : (U_i \cap U_j) \times F \to F$, $(x, v) \mapsto g_{ij}(x)(v)$ is smooth (Proposition 5.3 Remark 5.4. Then $g_{ij}$ is smooth as a mapping to the space $L(F)_b := L(F, F)_b$ (see Proposition 2.1).

In various contexts, for example when trying to construct dual bundles, we are in an
opposite situation: we know that each $g_{ij}$ is smooth, and would like to conclude that also the mappings $G_{ij}$ are smooth. Although this is not possible in general (as examples show), our results provide additional conditions ensuring that the conclusion is correct in the specific situation at hand. Notably, we obtain conditions ensuring the existence of a canonical dual bundle (Proposition 9.4). Without extra conditions, a canonical dual bundle need not exist (Example 9.5).

Besides dual bundles, we discuss a variety of construction principles of new vector bundles from given ones, including topological tensor products, completions, and finite or infinite direct sums. More generally, given a (finite- or infinite-dimensional) Lie group acting on the base manifold $M$, we discuss the construction of new equivariant vector bundles from given ones. Most of the constructions require specific hypotheses on the base manifold, the fibre of the bundle, and the Lie group.

As to completions, complementary topics were considered in the literature: Given an infinite-dimensional smooth manifold $M$, completions of the tangent bundle with respect to a weak Riemannian metric occur in [40, p. 549], in hypotheses for a so-called robust Riemannian manifold. Each $C^{r+1}$-map between open subsets of locally convex spaces locally factors over a $C^r$-map between open subsets of Banach spaces (see [28, Appendix A]).

We mention that multilinear algebra and vector bundle constructions can be performed much more easily in an inequivalent setting of infinite-dimensional calculus, the convenient differential calculus [39]. However, a weak notion of vector bundles is used there, which need not be topological vector bundles. Our discussion of vector bundles intends to pinpoint additional conditions ensuring that the natural construction principles lead to vector bundles in a stronger sense (which are, in particular, topological vector bundles).

The work [53] was particularly important for our studies. For an open subset $U$ of a Fréchet space $E$, smoothness of $f^\wedge: U \times E^k \to \mathbb{R}$ is deduced from smoothness of $f: U \to \Lambda^k(E')_b$ in the proof of [53, Proposition IV.6]. A typical hypocontinuity argument already appears in the proof of [53, Lemma IV.7]. In contrast to the local calculations in charts, the global structure on a dual bundle (and bundles of $k$-forms) asserted in the first remark of [53, p. 339] are problematic if Keller’s $C^\infty_c$-theory is used, without further hypotheses.

Applications related to locally convex Poisson vector spaces. In the wake of works by Odzijewicz and Ratiu on Banach-Poisson vector spaces and Banach-Poisson manifolds [47, 48], certain locally convex Poisson vector spaces were introduced [20], which generalize the Lie-Poisson structure on the dual space of a finite-dimensional Lie algebra going back to Kirillov, Kostant and Souriau. By now, the latter spaces can be embedded in a general theory of locally convex Poisson manifolds (see [44]; for generalizations of finite-dimensional Poisson geometry with a different thrust, cf. [6]). Recall that many important examples of bilinear mappings between locally convex topological vector spaces are not continuous, but at least hypocontinuous (cf. [9] for this classical concept). In Sections 11 and 12 we provide the proofs for two fundamental results in the theory of locally convex
Poisson vector spaces which are related to hypocontinuity\footnote{These proofs were stated in the preprint version of \cite{20}, but not included in the actual publication.} We show that the Poisson bracket associated with a continuous Lie bracket is always continuous (Theorem \ref{thm:1.1}) and that the linear map $C^\infty(E,\mathbb{R}) \rightarrow C^\infty(E,E)$ taking a smooth function to the associated Hamiltonian vector field is continuous (Theorem \ref{thm:12.1}). Ideas from \cite{20} and the current article were also taken further in \cite{25} Section 13.

Acknowledgements. A very limited first draft was written in 2001/02, supported by the research group FOR 363/1-1 of the German Research Foundation, DFG (working title: Bundles of locally convex spaces, group actions, and hypocontinuous bilinear mappings). The material was expanded in 2007, supported by DFG grant GL 357/5-1. Substantial extensions and a major rewriting were carried out in 2022.

\section{Preliminaries and notation}

We describe our setting of differential calculus and compile useful facts. Either references to the literature are given or a proof; the proofs can be looked up in Appendix A.

\textbf{Infinite-dimensional calculus.} We work in the framework of infinite-dimensional differential calculus known as Keller’s $C^k$-theory \cite{37}. Our main references are \cite{13} and \cite{30} (see also \cite{41}, \cite{32} \cite{42}, and \cite{43}). If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we let $\mathbb{D} := \{t \in \mathbb{K} : |t| \leq 1\}$ and $\mathbb{D}_\varepsilon := \{t \in \mathbb{K} : |t| \leq \varepsilon\}$ for $\varepsilon > 0$. We write $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All topological vector spaces considered in the article are assumed Hausdorff, unless the contrary is stated. For brevity, Hausdorff locally convex topological vector spaces will be called locally convex spaces. As usual, a subset $M$ of a $\mathbb{K}$-vector space is called balanced if $tx \in M$ for all $x \in M$ and $t \in \mathbb{D}$. The subset $M$ is called absolutely convex if it is both convex and balanced. If $q : E \rightarrow [0, \infty]$ is a seminorm on a $\mathbb{K}$-vector space $E$, we write $B^q_\varepsilon(x) := \{y \in E : q(y - x) < \varepsilon\}$ for $x \in E$ and $\varepsilon > 0$. We also write $\|x\|_q$ in place of $q(x)$. If $E$ is a locally convex $\mathbb{K}$-vector space, we let $E'$ be the dual space of continuous $\mathbb{K}$-linear functionals $\lambda : E \rightarrow \mathbb{K}$. We write $M^\circ := \{\lambda \in E' : \lambda(M) \subseteq \mathbb{D}\}$ for the polar of a subset $M \subseteq E$. If $\alpha : E \rightarrow F$ is a continuous $\mathbb{K}$-linear map between locally convex $\mathbb{K}$-vector spaces, we let $\alpha' : F' \rightarrow E'$, $\lambda \mapsto \lambda \circ \alpha$ be the dual linear map. We say that a mapping $f : X \rightarrow Y$ between topological spaces is a \textit{topological embedding} if it is a homeomorphism onto its image.

\subsection{1.1}

Let $E$ and $F$ be locally convex $\mathbb{K}$-vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $U \subseteq E$ be an open subset. A map $f : U \rightarrow F$ is called $C^0_{\mathbb{K}}$ if it is continuous, in which case we set $d^0 f := f$. Given $x \in U$ and $y \in E$, we define

$$df(x, y) := (D_y f)(x) := \lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}$$

if the limit exists (using $t \in \mathbb{K}^\times$ such that $x + ty \in U$). Let $r \in \mathbb{N} \cup \{\infty\}$. We say that a continuous map $f : U \rightarrow F$ is a $C^r_{\mathbb{K}}$-map if the iterated directional derivative

$$d^k f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1})(f)(x)$$

is continuous.
exists for all \( k \in \mathbb{N} \) such that \( k \leq r \) and all \((x, y_1, \ldots, y_k) \in U \times E^k\), and if the maps \( d^k f : U \times E^k \to F \) so obtained are continuous. Thus \( d^3 f = df \). If \( \mathbb{K} \) is understood, we write \( C^r \) instead of \( C^r_{\mathbb{K}} \). As usual, \( C^\infty \)-maps are also called smooth.

1.2 For \( k \in \mathbb{N} \), it is known that a map \( f : U \to F \) as before is \( C^k_{\mathbb{K}} \) if and only if \( f \) is \( C^1_{\mathbb{K}} \) and \( df : U \times E \to F \) is \( C^{k-1}_{\mathbb{K}} \) (cf. \cite{30} Proposition 1.3.10 or \cite{13} Lemma 1.14).

1.3 If \( \mathbb{K} = \mathbb{C} \), it is known that a map \( f : E \supseteq U \to F \) as before is \( C^\infty_{\mathbb{C}} \) if and only if it is complex analytic in the sense of \cite{3} Definition 5.6: \( f \) is continuous and for each \( x \in U \), there exists a 0-neighbourhood \( Y \subseteq E \) such that \( x + Y \subseteq U \) and \( f(x + y) = \sum_{n=0}^{\infty} \beta_n(y) \) for all \( y \in Y \) as a pointwise limit, where \( \beta_n : E \to F \) is a continuous homogeneous polynomial over \( \mathbb{C} \) of degree \( n \), for each \( n \in \mathbb{N} \). Furthermore, \( f \) is complex analytic if and only if \( f \) is \( C^\infty_{\mathbb{K}} \) and \( df(x, \cdot) : E \to F \) is complex linear for all \( x \in U \) (see \cite{13} Lemma 2.5). Complex analytic maps will also be called \( \mathbb{C} \)-analytic of \( C^{\infty}_{\mathbb{C}} \).

1.4 If \( \mathbb{K} = \mathbb{R} \), then a map \( f : U \to F \) as in \cite{14} is called real analytic (or \( \mathbb{R} \)-analytic, or \( C^\omega_{\mathbb{R}} \)) if it extends to a complex analytic mapping \( \tilde{U} \to F_{\mathbb{C}} \) on some open neighbourhood \( \tilde{U} \) of \( U \) in the complexification \( E_{\mathbb{C}} \) of \( E \).

In the following, \( r \in \mathbb{N}_0 \cup \{\infty, \omega\} \) (unless the contrary is stated). We use the conventions \( \infty + k := \infty - k := \infty \) and \( \omega + k := \omega - k := \omega \), for each \( k \in \mathbb{N} \). Furthermore, we extend the order on \( \mathbb{N}_0 \) to an order on \( \mathbb{N}_0 \cup \{\infty, \omega\} \) by declaring \( n < \infty < \omega \) for each \( n \in \mathbb{N}_0 \).

1.5 Compositions of composable \( C^r_{\mathbb{K}} \)-maps are \( C^r_{\mathbb{K}} \)-maps (see \cite{30} Proposition 1.3.4 and \cite{13} Propositions 2.7 and 2.8). Thus \( C^r_{\mathbb{K}} \)-manifolds modeled on locally convex \( \mathbb{K} \)-vector spaces can be defined in the usual way (see \cite{30} Chapter 3 for a detailed exposition). In this article, the word “manifold” (resp., “Lie group”) always refers to a manifold (resp., Lie group) modeled on a locally convex space.

The following basic fact will be used repeatedly.

**Lemma 1.6** For \( k \in \mathbb{N} \), let \( X, E_1, \ldots, E_k \), and \( F \) be locally convex \( \mathbb{K} \)-vector spaces, \( U \subseteq X \) be an open subset and

\[
f : U \times E_1 \times \cdots \times E_k \to F
\]

be a \( C^1_{\mathbb{K}} \)-map such that \( f^\vee (x) := f(x, \cdot) : E_1 \times \cdots \times E_k \to F \) is \( k \)-linear, for each \( x \in U \). Let \( x \in U \) and \( q \) be a continuous seminorm on \( F \). Then there exist a continuous seminorm \( p \) on \( X \) with \( B^1_1(x) \subseteq U \), and continuous seminorms \( p_j \) on \( E_j \) for \( j \in \{1, \ldots, k\} \) such that

\[
\|f(y, v_1, \ldots, v_k)\|_q \leq \|v_1\|_{p_1} \cdots \|v_k\|_{p_k} \quad \text{and} \quad \|f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k)\|_q \leq \|y - x\|_p \|v_1\|_{p_1} \cdots \|v_k\|_{p_k}
\]

for all \( y \in B^1_1(x) \) and \((v_1, \ldots, v_k) \in E_1 \times \cdots \times E_k\).
We shall also use the following fact:

**Lemma 1.7** Let $E$ and $F$ be locally convex $\mathbb{K}$-vector spaces, $k \geq 2$ be an integer and $f : U \times E^k \to F$ be a mapping such that $f(x, \cdot) : E^k \to F$ is $k$-linear and symmetric for each $x \in U$. Let $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. If

$$ h : U \times E \to F, \quad (x, y) \mapsto f(x, y, \ldots, y) $$

is $C^r_{\mathbb{K}}$, then also $f$ is $C^r_{\mathbb{K}}$. Notably, $f$ is continuous if $h$ is continuous.

**$k$-spaces, $k_\mathbb{R}$-spaces, $k_\infty$-spaces, and $k_\omega$-spaces.** We recall topological concepts.

1.8 A topological space $X$ is said to be **completely regular** if it is Hausdorff and its topology is initial with respect to the set $C(X, \mathbb{R})$ of all continuous real-valued functions on $X$.

Every locally convex space is completely regular, like every Hausdorff topological group (cf. [33, Theorem 8.2]). Compare [10] and [38] for the following.

1.9 A topological space $X$ is called a **$k$-space** if it is Hausdorff and a subset $A \subseteq X$ is closed if and only if $A \cap K$ is closed in $K$ for each compact subset $K \subseteq X$. Every metrizable topological space is a $k$-space, and every locally compact Hausdorff space. A Hausdorff space $X$ is a $k$-space if and only if, for each topological space, a map $f : X \to Y$ is continuous if and only if $f$ is $k$-continuous in the sense that $f|_K$ is continuous for each compact subset $K \subseteq X$. If $X$ is a $k$-space, then also every subset $M \subseteq X$ which is open or closed in $X$, when the induced topology is used on $M$.

1.10 A topological space $X$ is called a **$k_\mathbb{R}$-space** if it is Hausdorff and a function $f : X \to \mathbb{R}$ is continuous if and only if $f$ is $k$-continuous. Then also a map $f : X \to Y$ to a completely regular topological space $Y$ is continuous if and only if it is $k_\mathbb{R}$-continuous (as the latter condition implies continuity of $g \circ f$ for each $g \in C(Y, \mathbb{R})$). For more information, cf. [45].

Every $k$-space is a $k_\mathbb{R}$-space. The converse is not true: $\mathbb{R}^I$ is known to be a $k_\mathbb{R}$-space for each set $I$ (see [45], also [29]). If $I$ has cardinality $\geq 2^{\aleph_0}$, then $\mathbb{R}^I$ is not a $k$-space.\footnote{3If $\mathbb{R}^I$ was a $k$-space, then a certain non-discrete subgroup $G$ of $(\mathbb{R}^\mathbb{R}, +)$ constructed in [11] would be discrete, contradiction (see [30] Remark A.6.16(a) for more details). Compare also [45].}

The following facts are well known (cf. [45]):

**Lemma 1.11** (a) If a $k_\mathbb{R}$-space $X$ is a direct product $X_1 \times X_2$ of Hausdorff spaces and $X_1 \neq \emptyset$, then $X_2$ is a $k_\mathbb{R}$-space.

(b) Every open subset $U$ of a completely regular $k_\mathbb{R}$-space $X$ is a $k_\mathbb{R}$-space in the induced topology.

Notably, $U$ is a $k_\mathbb{R}$-space for each open subset $U$ of a locally convex space $E$ which is a $k_\mathbb{R}$-space. If $E \times E$ is a $k_\mathbb{R}$-space, then also $E$.\footnote{3If $\mathbb{R}^I$ was a $k$-space, then a certain non-discrete subgroup $G$ of $(\mathbb{R}^\mathbb{R}, +)$ constructed in [11] would be discrete, contradiction (see [30] Remark A.6.16(a) for more details). Compare also [45].}
Following [20], a topological space $X$ is called a $k^\infty$-space if the cartesian power $X^n$ is a $k$-space for each $n \in \mathbb{N}$, using the product topology. A Hausdorff space $X$ is called hemi-compact if $X = \bigcup_{n \in \mathbb{N}} K_n$ for a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets $K_n \subseteq X$ such that each compact subset of $X$ is a subset of some $K_n$. Hemi-compact $k$-spaces are also called $k_{\omega}$-spaces. If $X$ and $Y$ are $k_{\omega}$-spaces, then the product topology makes $X \times Y$ a $k_{\omega}$-space. Notably, every $k_{\omega}$-space is a $k^\infty$-space. See [2, 12, 27] for further information.

Finite products of metrizable spaces being metrizable, every metrizable topological space is a $k^\infty$-space. Recall that a locally convex space $E$ is said to be a Silva space or (DF)-space if it is the locally convex inductive limit of a sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces such that each inclusion map $E_n \to E_{n+1}$ is a compact operator. Every Silva space is a $k_{\omega}$-space (see [19] Example 9.4).

### Spaces of multilinear maps

Given $k \in \mathbb{N}$, locally convex $\mathbb{K}$-vector spaces $E_1, \ldots, E_k$ and $F$, and a set $\mathcal{S}$ of bounded subsets of $E_1 \times \cdots \times E_k$, we write $L^k(E_1, \ldots, E_k, F)_{\mathcal{S}}$ or $L^k_{\mathcal{S}}(E_1, \ldots, E_k, F)$ for the space of continuous $k$-linear maps $E_1 \times \cdots \times E_k \to F$, endowed with the topology $\mathcal{O}_{\mathcal{S}}$ of uniform convergence on the sets $B \in \mathcal{S}$. Recall that finite intersections of sets of the form

$$[B, U] := \{ \beta \in L^k(E_1, \ldots, E_k, F) : \beta(B) \subseteq U \}$$

yield a basis of $0$-neighbourhoods for this (not necessarily Hausdorff) locally convex vector topology, for $U$ ranging through the $0$-neighbourhoods in $F$ and $B$ through $\mathcal{S}$. If $\bigcup_{B \in \mathcal{S}} B = E_1 \times \cdots \times E_k$, then $\mathcal{O}_{\mathcal{S}}$ is Hausdorff. If $E_1 = \cdots = E_k$, we abbreviate $L^k(E, F)_{\mathcal{S}} := L^k_{\mathcal{S}}(E, E, E, \ldots, E, F)$. If $k = 1$ and $E := E_1$, we abbreviate $L(E, F)_{\mathcal{S}} := L^1(E, F)_{\mathcal{S}}$, $L_k(E, F)_{\mathcal{S}} := L^1_{\mathcal{S}}(E, E, F)$ and $L(E)_{\mathcal{S}} := L(E, E)_{\mathcal{S}}$. We write $\text{GL}(E) = L(E)^\times$ for the group of all automorphisms of the locally convex $\mathbb{K}$-vector space $E$. If $\mathcal{S}$ is the set of all bounded, compact, and finite subsets of $E_1 \times \cdots \times E_k$, respectively, we shall usually write “$b$,” “c,” and “p” in place of $\mathcal{S}$. For example, we shall write $L^k(E_1, \ldots, E_k, F)_b$, $L^k(E_1, \ldots, E_k, F)_c$, and $L^k(E_1, \ldots, E_k, F)_p$.

### Hipocontinuous multilinear maps

Beyond normed spaces, typical multilinear maps are not continuous, but merely hypocontinuous. Hypocontinuous bilinear maps are discussed in many textbooks. An analogous notion of hypocontinuity for multilinear maps (to be described presently) is useful us. It can be discussed like the bilinear case.
Lemma 1.15 For an integer \( k \geq 2 \), let \( \beta: E_1 \times \cdots \times E_k \to F \) be a separately continuous \( k \)-linear mapping and \( j \in \{2, \ldots, k\} \) such that, for each \( x \in E_1 \times \cdots \times E_{j-1} \), the map
\[
\beta^\vee(x) := \beta(x, \cdot): E_j \times \cdots \times E_k \to F
\]
is continuous. Let \( S \) be a set of bounded subsets of \( E_j \times \cdots \times E_k \). Consider the conditions:

(a) For each \( M \in S \) and each 0-neighbourhood \( W \subseteq F \), there exists a 0-neighbourhood \( V \subseteq E_1 \times \cdots \times E_{j-1} \) such that \( \beta(V \times M) \subseteq W \).

(b) The \((j-1)\)-linear map \( \beta^\vee: E_1 \times \cdots \times E_{j-1} \to L^{k-j+1}(E_j, \ldots, E_k, F)_S \) is continuous.

(c) \( \beta|_{E_1 \times \cdots \times E_{j-1} \times M}: E_1 \times \cdots \times E_{j-1} \times M \to F \) is continuous, for each \( M \in S \).

The (a) and (b) are equivalent, and (b) implies (c). If
\[
(\forall M \in S) \ (\exists N \in S) \ \exists M \subseteq N;
\]
then (a), (b), and (c) are equivalent.

Definition 1.16 A \( k \)-linear map \( \beta \) which satisfies the hypotheses and Condition (a) of Lemma 1.15 is called \( S \)-hypocontinuous in its arguments \((j, \ldots, k)\). If \( j = k \), we also say that \( \beta \) is \( S \)-hypocontinuous in the \( k \)-th argument. Analogously, we define \( S \)-hypocontinuity of \( \beta \) in the \( j \)-th argument, if \( j \in \{1, \ldots, k\} \) and a set \( S \) of bounded subsets of \( E_j \) are given.

We are mainly interested in \( b \)-, \( c \)-, and \( p \)-hypocontinuity, viz., in \( S \)-hypocontinuity with respect to the set \( S \) of all bounded subsets of \( E_j \times \cdots \times E_k \), the set \( S \) of all compact subsets, and the set \( S \) of all finite subsets, respectively. If \( S \) and \( T \) are sets of bounded subsets of \( E_j \times \cdots \times E_k \) such that \( S \subseteq T \) and \( \beta \) is \( T \)-hypocontinuous in its variables \((j, \ldots, k)\), then \( \beta \) is also \( S \)-hypocontinuous in the latter. The following is obvious from Lemma 1.15(c) (as the elements of a convergent sequence, together with its limit, form a compact set):

Lemma 1.17 If \( \beta: E_1 \times \cdots \times E_k \to F \) is \( c \)-hypocontinuous in some argument, or in its arguments \((j, \ldots, k)\) for some \( j \in \{2, \ldots, k\} \), then \( \beta \) is sequentially continuous. \( \square \)

In many cases, separately continuous bilinear maps are automatically hypocontinuous. Recall that a subset \( B \) of a locally convex space \( E \) is a barrel if it is closed, absolutely convex, and absorbing. The space \( E \) is called barrelled if every barrel is a 0-neighbourhood. See Proposition 6 in [9, Chapter III, §5, no. 3] for the following fact.

Lemma 1.18 If \( \beta: E_1 \times E_2 \to F \) is a separately continuous bilinear map and \( E_1 \) is barrelled, then \( \beta \) is \( S \)-hypocontinuous in its second argument, with respect to any set \( S \) of bounded subsets of \( E_2 \). \( \square \)

Evaluation maps are paradigmatic examples of hypocontinuous multilinear maps.
Lemma 1.19  Let $E_1,\ldots,E_k$ and $F$ be locally convex $\mathbb{K}$-vector spaces and $S$ be a set of bounded subsets of $E := E_1 \times \cdots \times E_k$ with $\bigcup_{M \in S} M = E$. Then the $(k+1)$-linear map
\[
\varepsilon: L^k(E_1,\ldots,E_k,F)_S \times E_1 \times \cdots \times E_k \to F, \quad (\beta,x) \mapsto \beta(x)
\]
is $S$-hypocontinuous in its arguments $(2,\ldots,k+1)$. If $k = 1$ and $E = E_1$ is barrelled, then $\varepsilon: L(E,F) \times E \to F$ is also hypocontinuous in the first argument, with respect to any locally convex topology $\mathcal{O}$ on $L(E,F)$ which is finer than the topology of pointwise convergence, and any set $T$ of bounded subsets of $(L(E,F),\mathcal{O})$.

Lemma 1.20  Let $k \geq 2$ be an integer, $E_1,\ldots,E_k$ and $F$ be locally convex spaces, and $\beta: E_1 \times \cdots \times E_k \to F$ be a $k$-linear map.

(a) If $\beta$ is sequentially continuous, then $\beta \circ f$ is continuous for each continuous function $f: X \to E_1 \times \cdots \times E_k$ on a topological space $X$ which is metrizable or satisfies the first axiom of countability.

(b) If $\beta$ is $c$-hypocontinuous in its arguments $(j,\ldots,k)$ for some $j \in \{2,\ldots,k\}$ and $X$ is a $k_{\mathbb{K}}$-space, then $\beta \circ f$ is continuous for each continuous function $f: X \to E_1 \times \cdots \times E_k$.

Lipschitz differentiable maps.  In Section 6 it will be useful to work with certain Lipschitz differentiable maps, instead of $C^r$-maps. We briefly recall concepts and facts.

Definition 1.21  Let $E$ and $F$ be locally convex $\mathbb{K}$-vector spaces, $U \subseteq E$ be open and $f: U \to F$ be a map. We say that $f$ is locally Lipschitz continuous or $LC_0^r$ if it has the following property: For each $x \in U$ and continuous seminorm $p$ on $E$, there exists a continuous seminorm $q$ on $E$ such that $B_p^r(x) \subseteq U$ and
\[
q(f(z) - f(y)) \leq p(z - y) \quad \text{for all } y,z \in B_p^r(x).
\]
Given $r \in \mathbb{N}_0 \cup \{\infty\}$, we say that $f$ is $LC_0^r$ if $f$ is $C^r_\mathbb{K}$ and $d^k f: U \times E^k \to F$ is $LC_0^r$ for each $k \in \mathbb{N}_0$ such that $k \leq r$.

Every $C^1$-map is $LC_0^0$ (see, for example, [24, Lemma 1.59]). As a consequence, for each $r \in \mathbb{N} \cup \{\infty\}$ every $C^r_\mathbb{K}$-map is $LC^{r-1}_\mathbb{K}$. Notably, every smooth map is $LC^\infty_\mathbb{K}$. Moreover, a $C^r_\mathbb{K}$-map with finite $r$ is $LC^r_\mathbb{K}$ if and only if $d^r f$ is $LC^r_\mathbb{K}$. The following facts are known, or part of the folklore.

Lemma 1.22  For locally convex spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and $r \in \mathbb{N}_0 \cup \{\infty\}$, we have:

(a) A map $f: E \supseteq U \to \prod_{j \in J} F_j$ to a direct product of locally convex spaces is $LC^r_\mathbb{K}$ if and only each component is $LC^r_\mathbb{K}$;

(b) Compositions of composable $LC^r_\mathbb{K}$-maps are $LC^r_\mathbb{K}$;

(c) Let $F$ be a locally convex space and $F_0 \subseteq F$ be a vector subspace which is closed in $F$, or sequentially closed. Then a map $f: E \supseteq U \to F_0$ is $FC^r_\mathbb{K}$ if and only if it is $FC^r_\mathbb{K}$ as a map to $F$.  

(d) A map $E \supseteq U \to P$ to a projective limit $P = \lim F_j$ of locally convex spaces is $\LC_r^c$ if and only if $p_j \circ f : U \to F_j$ is $\LC_r^c$ for all $j \in J$, where $p_j : P \to F_j$ is the limit map.

Our concept of local Lipschitz continuity is weaker than the one in [30] Definition 1.5.4.

The compact-open $C^r$-topology

1.23 If $E$ and $F$ are locally convex $\mathbb{K}$-vector spaces, $U \subseteq E$ is an open set and $r \in \mathbb{N}_0 \cup \{\infty\}$, then the vector space $C^r_{\mathbb{K}}(U, F)$ of all $C^r_{\mathbb{K}}$-maps $U \to F$ carries a natural topology (the “compact-open $C^r$-topology”), namely the initial topology with respect to the mappings

$$C^r_{\mathbb{K}}(U, F) \to C(U \times E^j, F)_{c.o.} \quad f \mapsto d^j f$$

for $j \in \mathbb{N}_0$ such that $j \leq r$, where the right hand side is endowed with the compact-open topology. Then $C^r_{\mathbb{K}}(U, F)$ is a locally convex $\mathbb{K}$-vector space. If $F$ is a complex locally convex space, then also $C^r_{\mathbb{K}}(U, F)$. See, e.g., [30] §1.7 for further information, or [22].

The following observation was useful in an earlier version of the manuscript, to enable exponential laws (as in [1]) to be applied. We retain it as it may be useful elsewhere.

Lemma 1.24 Let $L \in \{\mathbb{R}, \mathbb{C}\}$, $K \in \{\mathbb{R}, L\}$, $k \in \mathbb{N}$, $r \in \mathbb{N}_0 \cup \{\infty\}$, and $E_1, \ldots, E_k$ as well as $F$ be locally convex $L$-vector spaces. Then $L^k_L(E_1, \ldots, E_k, F)$ is a closed vector subspace of $C^r_{\mathbb{K}}(E_1 \times \cdots \times E_k, F)$ with respect to the compact-open $C^r$-topology. The latter induces on $L^k_L(E_1, \ldots, E_k, F)$ the compact-open topology.

2 Differentiability properties of operator-valued maps

Let $L \in \{\mathbb{R}, \mathbb{C}\}$, $K \in \{\mathbb{R}, L\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. In this section, we establish the following proposition.

Proposition 2.1 Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, $E_1, \ldots, E_k$ and $F$ be locally convex $L$-vector spaces, $X$ be a locally convex $K$-vector space, and $U \subseteq X$ be an open subset. Let $f : U \to L^k_L(E_1, \ldots, E_k, F)$ be a map such that

$$f^\wedge : U \times E_1 \times \cdots \times E_k \to F, \quad f^\wedge(x, v) := f(x)(v) \quad \text{for } x \in U, \; v \in E_1 \times \cdots \times E_k$$

is $C^r_{\mathbb{K}}$. Then the following holds:

(a) $f$ is $C^r_{\mathbb{K}}$ as a map to $L^k_L(E_1, \ldots, E_k, F)$.c.

(b) If $r \geq 1$, then $f$ is $C^{r-1}_{\mathbb{K}}$ as a map to $L^k_L(E_1, \ldots, E_k, F)$b.

Furthermore,

$$d^j f(x, y_1, \ldots, y_j)(v) = d^j(f^\wedge)((x, v), (y_1, 0), \ldots, (y_j, 0)) \quad (4)$$

for all $j \in \mathbb{N}$ with $j \leq r$ (resp., $j \leq r - 1$, in (b)), all $x \in U$, $v \in E_1 \times \cdots \times E_k$, and $y_1, \ldots, y_j \in X$. 

10
Corollary 2.2 Let $E$ and $F$ be locally convex $\mathbb{K}$-vector spaces and $f: U \to F$ be a $C^r_\mathbb{K}$-map on an open subset $U \subseteq E$, where $r \in \mathbb{N} \cup \{\infty, \omega\}$. Then the following holds:

(a) The map $f^{(k)}: U \to L^k_\mathbb{K}(E,F)_c$, $x \mapsto f^{(k)}(x) = d^k f(x,\cdot)$ is $C^{r-k}_\mathbb{K}$, for each $k \in \mathbb{N}$ such that $k \leq r$.

(b) The map $f^{(k)}: U \to L^k_\mathbb{K}(E,F)_b$ is $C^{r-k-1}_\mathbb{K}$, for each $k \in \mathbb{N}$ such that $k \leq r - 1$.

Furthermore, $d_j^{(f^{(k)})(x,y_1,\ldots,y_j)} = d^{j+k} f(x,\ldots,y_1,\ldots,y_j)$, for all $j \in \mathbb{N}$ with $j + k \leq r$ (resp., $j + k \leq r - 1$), all $x \in U$, and $y_1,\ldots,y_j \in E$.

Proof. For each $k \in \mathbb{N}$ such that $k \leq r$, the map $d^k f: U \times E^k \to F$ is $C^{r-k}_\mathbb{K}$ (see [30, Remark 1.3.13 and Exercise 2.2.7]) and $f^{(k)}(x) = d^k f(x,\cdot)$ is $k$-linear for each $x \in U$, by [30, Proposition 1.3.17] (or [7, Lemma 4.8]). Moreover, $(f^{(k)})^\wedge = d^k f$. Thus Proposition 2.1 applies with $f^{(k)}$ in place of $f$ and $r - k$ in place of $r$.

Given a topological space $X$ and locally convex space $F$, we endow the space $C(X,F)$ of continuous $F$-valued functions on $X$ with the compact-open topology. The next lemma will be useful when we discuss mappings to $L^k(E,F)_c$.

Lemma 2.3 Let $X$, $E$, and $F$ be locally convex $\mathbb{K}$-vector spaces, $U \subseteq X$ and $W \subseteq E$ be open subsets, and $f: U \times W \to F$ be a $C^r_\mathbb{K}$-map, with $r \in \mathbb{N}_0 \cup \{\infty\}$. Then also the map

$$f^\vee: U \to C(W,F), \quad x \mapsto f(x,\cdot)$$

is $C^{r}_\mathbb{K}$. If $\mathbb{K} = \mathbb{R}$ and $f$ admits a complex analytic extension $g: \tilde{U} \times \tilde{W} \to F_\mathbb{C}$ for suitable open neighbourhoods $\tilde{U}$ of $U$ in $X_\mathbb{C}$ and $\tilde{W}$ of $W$ in $E_\mathbb{C}$, then $f^\vee$ is real analytic.

Proof. We first assume that $r \in \mathbb{N}_0$, and proceed by induction. For $r = 0$, the assertion is well known (see, e.g., [30, Proposition A.6.17]). Now assume that $r \in \mathbb{N}$. Given $x \in U$ and $y \in X$, there exists $\epsilon > 0$ such that $x + \mathbb{D}^0_\epsilon y \subseteq U$, where $\mathbb{D}^0_\epsilon := \{t \in \mathbb{K}: |t| < \epsilon\}$. Consider

$$g: \mathbb{D}^0_\epsilon \times W \to F, \quad (t,w) \mapsto \begin{cases} \frac{f((x+ty,w)-f(x,w))}{t} & \text{if } t \neq 0; \\ df((x,w),(y,0)) & \text{if } t = 0. \end{cases}$$

Then $g(t,w) = f^1_0 df((x+stw),(y,0)) ds$, by the Mean Value Theorem. The integrand being continuous, also $g$ is continuous (by the Theorem on Parameter-Dependent Integrals, [30, Lemma 1.1.11]). Hence $g^\vee: V \to C(W,F)$ is continuous, by induction, and hence

$$\frac{f^\vee(x+ty) - f^\vee(x)}{t} = g^\vee(t) \to g^\vee(0)$$

---

4Alternatively, with a view towards Remark 2.4 we might use that $d^k f$ is a partial map of a certain $C^{r-k}_\mathbb{K}$-map $f^{(k)}$ if $k \neq \omega$ (cf. [7, Proposition 7.4]), which also settles the case $(r,\mathbb{K}) = (\omega,\mathbb{C})$ (see [7, Proposition 7.7]). The real analytic case follows via complex analytic extension.

5It is known that this topology coincides with the topology of uniform convergence on compact sets.
as \( t \to 0 \), where \( g^\vee(0) = df((x,\cdot),(y,0)) = h^\vee(x,y) \) with

\[
h: (U \times E) \times W \to F, \quad (x,y,w) \mapsto df((x,w),(y,0)).
\]

Since \( h \) is \( C^{r-1}_K \), the map \( df(h^\vee) = h^\vee \) is \( C^{r-1}_K \), by the inductive hypothesis. In particular, \( df \) is continuous and hence \( f \) is \( C^1_K \). Now \( f \) being \( C^1_K \) with \( df \) a \( C^{r-1}_K \)-map, \( f \) is \( C^r_K \).

The case \( r = \infty \). If \( f \) is \( C^{\infty}_K \), then \( f \) is \( C^k_K \) for each \( k \in \mathbb{N}_0 \). Hence \( f^\vee \) is \( C^k_K \) for each \( k \in \mathbb{N}_0 \) (by the case already treated), and thus \( f^\vee \) is \( C^{\infty}_K \).

Final assertion. By the \( C^\infty_C \)-case already treated, the map

\[
g^\vee: \tilde{U} \to C(\tilde{W},F_C)
\]

is \( C^\infty_C \). The restriction map

\[
\rho: C(\tilde{W},F_C) \to C(W,F_C), \quad \gamma \mapsto \gamma|_W
\]

being continuous \( C \)-linear and thus \( C^\infty_C \), it follows that the composition

\[
h := \rho \circ g^\vee: \tilde{U} \to C(W,F_C) = C(W,F)_C
\]

is \( C^\infty_C \) and hence complex analytic. Since \( h \) extends \( f^\vee \), we see that \( f^\vee \) is real analytic. \( \square \)

**Proof of Proposition 2.1** (a) Abbreviate \( E := E_1 \times \cdots \times E_k \). Because \( L^1_L(E_1,\ldots,E_k,F)_C \) is a closed \( K \)-vector subspace of \( C(E,F) \) and carries the induced topology, \( f \) will be \( C^r_K \) as a map to \( L^1_L(E_1,\ldots,E_k,F)_C \) if we can show that \( f \) is \( C^r_K \) as a map to \( C(E,F) \) (see [30, Lemma 1.3.19] and [13, Proposition 2.11]). Since \( f^\wedge \) is \( C^r_K \) and \( f = (f^\wedge)^\vee \), the latter follows from Lemma 2.3. This is obvious unless \( K = \mathbb{R} \) and \( r = \omega \). In this case, the map \( f^\wedge \) admits a \( C \)-analytic extension \( p: Q \to F_C \) to an open neighbourhood \( Q \) of \( U \times E \) in \( X_C \times E_C \). For each \( x \in U \), there exists an open, connected neighbourhood \( U_x \) of \( x \) in \( X_C \) and a balanced, open \( 0 \)-neighbourhood \( W_x \subseteq E_C \) such that \( U_x \times W_x \subseteq Q \) and \( U_x \cap X \subseteq U \). Let \( D := \{ z \in C: |z| < 1 \} \). Then

\[
q: U_x \times W_x \times D \to F_C, \quad (y,w,z) \mapsto p(y,zw) - z^kp(y,w)
\]

is a \( C \)-analytic map which vanishes on \( (U_x \times W_x \times D) \cap (X \times E \times \mathbb{R}) \). Hence \( q = 0 \), by the Identity Theorem (see [30, Theorem 2.1.16 (c)]). Then \( p(y,zw) = z^kp(y,w) \) for all \( z \in C \) such that \( |z| \leq 1 \), by continuity. This implies that the map

\[
g: U_x \times E_C \to F_C, \quad (y,w) \mapsto z^k p(y,z^{-1}w) \quad \text{for some } z \in C^x \text{ with } z^{-1}w \in W_x
\]

is well defined. Since \( g \) is \( C \)-analytic, the final statement of Lemma 2.3 applies.

(b) We prove the assertion for \( r \in \mathbb{N} \) first; then also the case \( r = \infty \) follows. If \( r = 1 \), let \( x \in U \). Given an open \( 0 \)-neighbourhood \( W \subseteq E \) such that \( B^2_1(0) \subseteq W \). By
Lemma [1.6] there exist continuous seminorms \( p \) on \( X \) and \( p_j \) on \( E_j \) for \( j \in \{1, \ldots, k\} \) such that \( B^p_r(x) \subseteq U \) and
\[
\|f^\vee(y,v) - f^\vee(x,v)\|_q \leq \|y - x\|_p \|v_1\|_{p_1} \cdots \|v_k\|_{p_k}
\]
for all \( y \in B^p_r(x) \) and all \( v = (v_1, \ldots, v_k) \in E_1 \times \cdots \times E_k \). Since \( B \) is bounded, we have
\[
C := \sup \{\|v_1\|_{p_1} \cdots \|v_k\|_{p_k} : v = (v_1, \ldots, v_k) \in B\} < \infty.
\]
Choose \( \delta \in [0,1] \) such that \( \delta C \leq 1 \). For each \( y \in B^p_r(x) \), we get \( \|f^\vee(y,v) - f^\vee(x,v)\|_q < \delta C \leq 1 \) for each \( v \in B \) and thus \( f^\vee(y,v) - f^\vee(x,v) \in B^p_1(0) \subseteq W \). Hence
\[
f(y) - f(x) \in [B,W] \quad \text{for each } y \in B^p_r(x).
\]
entailing that \( f \) is continuous.

Induction step: Now assume that \( r \geq 2 \). Given \( x \in U \) and \( y \in X \), there exists \( \varepsilon > 0 \) such that \( x + D^0_{\varepsilon} y \subseteq U \), where \( D^0_{\varepsilon} := \{ t \in \mathbb{K} : |t| < \varepsilon \} \). Consider
\[
g: D^0_{\varepsilon} \times E^k \to F, \quad (t,v) \mapsto \begin{cases} \frac{f(x+ty,v)-f^\vee(x,v)}{t} & \text{if } t \neq 0; \\ d(f^\vee)((x,v),(y,0)) & \text{if } t = 0. \end{cases}
\]
Then \( g \) is \( C^{r-1}_{\mathbb{K}} \) and hence \( C^1_{\mathbb{K}} \), as a consequence of [17] Propositions 7.4 and 7.7. Since \( g(t,v) \) is \( k \)-linear in \( v \), it follows that \( g^\vee : U \to L^k(E,F)_b \) is continuous, by induction. As a consequence,
\[
f(x + ty) - f(x) \overline{t} = g^\vee(t) \to g^\vee(0)
\]
as \( t \to 0 \), where \( g^\vee(0) = d(f^\vee)((x,\cdot),(y,0)) = h^\vee(x,y) \) with
\[
h: (U \times E^k) \times W \to F, \quad h((x,y),v) := d(f^\vee)((x,v),(y,0)).
\]
Since \( h \) is \( C^{r-1}_{\mathbb{K}} \) and \( h((x,y),v) \) is \( k \)-linear in \( v \), the map \( df = h^\vee \) is \( C^{r-1}_{\mathbb{K}} \), by induction. Hence \( df \) is continuous and thus \( f \) is \( C^r_{\mathbb{K}} \). Now \( f \) being \( C^1_{\mathbb{K}} \) with \( df \), \( C^{r-2}_{\mathbb{K}} \)-map, \( f \) is \( C^r_{\mathbb{K}} \).

The case \( \mathbb{K} = \mathbb{R}, r = \omega \). By [14] we may assume that \( L = \mathbb{R} \) (the case \( L = \mathbb{C} \) then follows). Given \( x \in U \), let \( g: U_x \times (E_c)^k \to F_c \) be as in the proof of (a). Then \( g \) is complex \( k \)-linear in the second variable and hence \( g^\vee \): \( U_x \to L^k_c(E_c,F_c)_b \) is \( \mathbb{C} \)-analytic, by the \( C^\infty \)-case already discussed. Because the map \( \rho: L^k_c(E_c,F_c)_b \to L^k_c(E,F)_b = (L^k(E,F)_b)_c \), \( \alpha \mapsto \alpha|_E \) is continuous \( \mathbb{C} \)-linear, the composition \( \rho \circ g^\vee \) is \( \mathbb{C} \)-analytic. But this mapping extends \( f|_{U_x} \cap E \). Hence \( f|_{U_x} \cap E \) is real analytic and hence so is \( f \), using that the open sets \( U_x \cap U \) form an open cover of \( U \).

**Formula for the differentials:** Let \( j \in \mathbb{N} \) with \( j \leq r, x \in U, v \in E_1 \times \cdots \times E_k \) and \( y_1, \ldots, y_j \in X \). Exploiting that \( ev_v: L^j_k(E_1,\ldots,E_k,F)_c \to F, \beta \mapsto \beta(v) \) is continuous and linear, we deduce that
\[
ev_v(d^j f(x,y_1,\ldots,y_j)) = d^j(ev_v \circ f)(x,y_1,\ldots,y_j) = d^j(f^\vee(\cdot,v))(x,y_1,\ldots,y_j)
\]
\[
= d^j(f^\vee((x,v),(y_1,0),\ldots,(y_j,0)))
\]
for $f$ as a map to $L_b^k(E_1, \ldots, E_k, F)_c$. If $j \leq r - 1$, the same calculation applies to $f$ as a mapping to $L_b^k(E_1, \ldots, E_k, F)_b$. \hfill \square$

For the special case of (a) when $r = 0$ and $X$ as well as $E_1 = \cdots = E_k$ are metrizable, see already [37, Lemma 0.1.2].

**Remark 2.4** We mention that the local convexity of $X$ and $E_1, \ldots, E_k$ (resp., $E$) in Lemma 1.6 Proposition 2.1 and Corollary 2.2 and their proofs is unnecessary (if one replaces continuous seminorms by continuous gauges in Lemma 1.6 and the proof of Proposition 2.1). Only the local convexity of $F$ is essential. In [18], we used Corollary 2.2 from the current article also in the case of non-locally convex domains.

## 3 Compositions with hypocontinuous $k$-linear maps

We study differentiability properties of compositions of the form $\beta \circ f$, where $\beta$ is $k$-linear map which need not be continuous.

**Lemma 3.1** Let $k \geq 2$ be an integer, $E_1, \ldots, E_k$, $X$, and $F$ be locally convex $\mathbb{K}$-vector spaces, $\beta: E_1 \times \cdots \times E_k \to F$ be a $k$-linear map, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ and $f: U \to E_1 \times \cdots \times E_k =: E$ be a $C_k^r$-map on an open subset $U \subseteq X$. Assume that

(a) $\beta$ is sequentially continuous and $X$ is metrizable; or

(b) For some $j \in \{2, \ldots, k\}$, the $k$-linear map $\beta$ is $c$-hypocontinuous in its variables $(j, \ldots, k)$. Moreover, $X \times X$ is a $k_\mathbb{R}$-space, or $r = 0$ and $X$ is a $k_\mathbb{R}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $X$ is a $k_\mathbb{R}$-space.

Then $\beta \circ f: U \to F$ is a $C_{k}^{r}$-map.

**Proof.** The case $r = 0$ was treated in Lemma 1.20. We first assume that $r \in \mathbb{N}$.

(a) Assuming (a), let $x \in U$, $y \in X$, and $(t_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{K} \setminus \{0\}$ such that $t_n \to 0$ as $n \to \infty$ and $x + t_n y \in U$ for all $n \in \mathbb{N}$. Using the components of $f = (f_1, \ldots, f_k)$, we can write the difference quotient $\frac{1}{t_n}(\beta(f(x + t_n y)) - \beta(f(x)))$ as the telescopic sum

$$
\sum_{\nu=1}^{k} \beta(f_1(x + t_n y), \ldots, f_{\nu-1}(x + t_n y), \frac{f_{\nu}(x + t_n y) - f_{\nu}(x)}{t_n}, f_{\nu+1}(x), \ldots, f_k(x)),
$$

which converges to

$$
\sum_{\nu=1}^{k} \beta(f_1(x), \ldots, f_{\nu-1}(x), df_{\nu}(x, y), f_{\nu+1}(x), \ldots, f_k(x)) = d(\beta \circ f)(x, y)
$$

as $n \to \infty$, using the sequential continuity of $\beta$. By Lemma 1.20, $d(\beta \circ f)$ is continuous, whence $\beta \circ f$ is $C_{k}^{1}$. If $r \geq 2$, then

$$
g_{\nu}: U \times X \to E, \quad (x, y) \mapsto (f_1(x), \ldots, f_{\nu-1}(x), df_{\nu}(x, y), f_{\nu+1}(x), \ldots, f_k(x))
$$
is a $C^{r-1}_{\mathbb{K}}$-map and $d(\beta \circ f) = \sum_{\nu=1}^{k} \beta \circ g_{\nu}$ is $C^{r-1}_{\mathbb{K}}$ by induction; thus $\beta \circ f$ is $C^{r}_{\mathbb{K}}$. If $r = \infty$, the preceding shows that $\beta \circ f$ is $C^{r}_{\mathbb{K}}$ for each $s \in \mathbb{N}_{0}$, whence $\beta \circ f$ is $C^{r}_{\mathbb{K}}$.

(b) If $X \times X$ is a $k_{\mathbb{K}}$-space, then $U \times X$ and $U$ are $k_{\mathbb{K}}$-spaces. By Lemma 1.17 $\beta$ is sequentially continuous. The argument from (a) shows that $d(\beta \circ f)(x, y)$ exists for all $(x, y) \in U \times X$ and is given by (4). Thus $d(\beta \circ f)$ is continuous, by Lemma 1.20, and thus $\beta \circ f$ is $C^{1}_{\mathbb{K}}$. Let $f$ be $C^{r+1}_{\mathbb{K}}$ now and assume $\beta \circ f$ is $C^{r}_{\mathbb{K}}$ with $r$th differential of the form

$$d^{r}(\beta \circ f)(x, y_{1}, \ldots, y_{r}) = \sum_{(I_{1}, \ldots, I_{r})} \beta(d^{I_{1}}f_{1}(x, y_{I_{1}}), \ldots, d^{I_{r}}f_{k}(x, y_{I_{r}}))$$

for $x \in U$ and $y_{1}, \ldots, y_{r} \in X$, where $(I_{1}, \ldots, I_{k})$ ranges through $k$-tuples of (possibly empty) disjoint sets $I_{1}, \ldots, I_{k}$ with $I_{1} \cup \cdots \cup I_{k} = \{1, \ldots, r\}$, and the following notation is used: For $\nu \in \{1, \ldots, k\}$, we let $|I_{\nu}| \in \mathbb{N}_{0}$ be the cardinality of $I_{\nu}$ and define $y_{I_{\nu}} := (y_{i_{1}}, \ldots, y_{i_{m}}) \in X^{m}$ if $i_{1} < i_{2} < \cdots < i_{m}$ are the elements of $I_{\nu}$, abbreviating $m := |I_{\nu}|$ (if $I_{\nu}$ is empty, the symbol $y_{I_{\nu}}$ is to be ignored). Holding $y_{1}, \ldots, y_{r}$ fixed, we can apply the case $r = 1$ to the function $d^{r}f(\cdot, y_{1}, \ldots, y_{r})$ and find that, for each $x \in U$ and $y_{r+1} \in X$, the directional derivative at $x$ in the direction $y_{r+1}$ exists and is given by

$$d^{r+1}(\beta \circ f)(x, y_{1}, \ldots, y_{r+1}) = \sum_{(I_{1}, \ldots, I_{r})} \sum_{\nu=1}^{k} \beta(d^{I_{1}}f_{1}(x, y_{I_{1}}), \ldots, d^{I_{r}}f_{k}(x, y_{I_{r}})),$$

$$d^{I_{\nu}}f_{\nu}(x, y_{I_{\nu}}, y_{r+1}), d^{I_{\nu+1}}f_{\nu+1}(x, y_{I_{\nu+1}}), \ldots, d^{I_{k}}f_{k}(x, y_{I_{k}})).$$

Thus also $d^{r+1}(\beta \circ f)$ is of the form (3), with $r + 1$ in place of $r$. Using Lemma 1.20, we deduce from the preceding formula that the map

$$U \times E \to F, \ (x, y) \mapsto d^{r+1}(\beta \circ f)(x, y, \ldots, y)$$

is continuous. Thus $d^{r+1}(\beta \circ f)$ is continuous, by Lemma 1.7, and thus $\beta \circ f$ is $C^{r+1}_{\mathbb{K}}$.

If $(r, \mathbb{K}) = (\infty, \mathbb{R})$, then $\beta \circ f$ is $C^{r}_{\mathbb{K}}$ for each $s \in \mathbb{N}_{0}$ and hence $C^{\infty}_{\mathbb{K}}$ (still assuming (b)). If $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $X$ is only assumed $k_{\mathbb{K}}$, then $\beta \circ f$ is continuous by the case $r = 0$. Moreover, the restriction $\beta \circ f|_{U \cap Y}$ is $C^{\infty}_{\mathbb{C}}$ for each finite-dimensional vector subspace $Y \subseteq X$, by case (a). Hence $f$ is $C^{\infty}_{\mathbb{C}}$ (and thus $C^{\infty}_{\mathbb{C}}$) as a mapping to a completion of $F$ (see [8. Theorem 6.2]). Then $f$ is also $C^{\infty}_{\mathbb{C}}$ as a map to $F$, as all of its iterated directional derivatives are in $F$.

Both in (a) and (b), it remains to consider the case $(r, \mathbb{K}) = (\omega, \mathbb{R})$. Then $f$ admits a $\mathbb{C}$-analytic extension $\tilde{f}: \tilde{U} \to (E_{1})_{\mathbb{C}} \times \cdots \times (E_{k})_{\mathbb{C}}$, defined on an open neighbourhood $\tilde{U}$ of $U$ in $X_{\mathbb{C}}$. The complex $k$-linear extension $\beta_{\mathbb{C}}: (E_{1})_{\mathbb{C}} \times \cdots \times (E_{k})_{\mathbb{C}} \to F_{\mathbb{C}}$ of $\beta$ is given by

$$z \mapsto \sum_{a_{1}, \ldots, a_{k}=0}^{1} i^{a_{1}+\cdots+a_{k}} \beta(x_{1,a_{1}}, \ldots, x_{k,a_{k}})$$

for $z = (x_{1,0} + ix_{1,1}, \ldots, x_{k,0} + ix_{k,1})$ with $x_{\nu,0} \in E_{\nu}$ and $x_{\nu,1} \in E_{\nu}$ for $\nu \in \{1, \ldots, k\}$. By the latter formula, $\beta_{\mathbb{C}}$ is sequentially continuous in the situation of (a), and $c$-hypocontinuous
in its arguments \((j,\ldots,k)\) in the situation of (b). The case \((\infty,\mathbb{C})\) shows that \(\beta_C \circ \tilde{f}\) is complex analytic. As this mapping extends \(\beta \circ f\), the latter map is real analytic. In case (b), we used here that \(X_C \cong X \times X\) is a \(k_\mathbb{R}\)-space. \(\square\)

Also the following variant will be useful.

**Lemma 3.2** Let \(X_1, X_2, E_1, E_2\) and \(F\) be locally convex \(K\)-vector spaces, and \(U_1 \subseteq X_1, U_2 \subseteq X_2\) be open subsets. Let \(r \in \mathbb{N}_0 \cup \{\infty, \omega\}\) and \(\beta: E_1 \times E_2 \to F\) be a \(K\)-bilinear map. Assume that \(X_1\) is finite-dimensional and \(\beta\) is \(c\)-hypocontinuous in its first variable. Then, for all \(C^r\)-maps \(f_1: U_1 \to E_1\) and \(f_2: U_1 \times U_2 \to E_2\), also the following map is \(C^r\):

\[
g: U_1 \times U_2 \to F, \quad (x_1, x_2) \mapsto \beta(f_1(x_1), f_2(x_1, x_2)).
\]

**Proof.** We first prove the assertion for \(r = 0\) (from which the case \(r = \infty\) follows). If \(r = 0\), we have to show that \(g\) is continuous. If \((x_1, x_2) \in U_1 \times U_2\), then \(x_1\) has a compact neighbourhood \(W = W_{x_1}\) in \(U_1\). Then \(f_1(W)\) is compact, and thus \(\beta|_{f_1(W) \times E_2}\) is continuous, by \(c\)-hypocontinuity. Hence \(g|_{W \times U_2} = \beta|_{f_1(W) \times E_2} \circ (f_1 \circ \pi_W, f_2)\) is continuous, where \(\pi_W: W \times U_2 \to W\) is the projection onto the first factor. Since \((W_{x_1} \times U_2)_{x_1 \in U_1}\) is an open cover of \(U_1 \times U_2\), the map \(g\) is continuous.

Since \(\beta\) is sequentially continuous by Lemma 1.17, we see as in the preceding proof that the directional derivative \(dg(x, y)\) exists for all \(x = (x_1, x_2) \in U_1 \times U_2\) and \(y = (y_1, y_2) \in X_1 \times X_2\), and is given by

\[
dg(x, y) = \beta(df_1(x_1, y_1), f_2(x)) + \beta(f_1(x_1), df_2(x, y)).
\] (7)

Note that \((x_1, y_1) \mapsto f_1(x_1)\) and \(df_1\) are \(C^{r-1}\)-mappings \(U_1 \times X_1 \to E_1\). Moreover, \(((x_1, y_1), (x_2, y_2)) \mapsto f_2(x_1, x_2)\) and \(((x_1, y_1), (x_2, y_2)) \mapsto df_2((x_1, x_2), (y_1, y_2))\) are \(C^{r-1}\)-maps \((U_1 \times X_1) \times (U_2 \times X_2) \to E_2\) (cf. 1.2). By induction, the right hand side of (7) is a \(C^{r-1}\)-map. Hence \(g\) is \(C^r\).

The case \((r, K) = (\omega, \mathbb{R})\) follows from the case \((\infty, \mathbb{C})\) as in the preceding proof. \(\square\)

**Remark 3.3** In a setting of differential calculus in which continuity on products is replaced with \(k\)-continuity (as championed by E.G.F. Thomas), every bilinear map \(\beta\) which is \(c\)-hypocontinuous in the second factor is smooth (see [51], Theorem 4.1); smoothness of \(\beta \circ f\) for a smooth map \(f\) then follows from the Chain Rule (cf. also [50]). Likewise, \(\beta\) is smooth in the sense of convenient differential calculus.

## 4 Differentiability properties of \(f^\wedge\)

For \(k = 1\), the following result is essential for our constructions of vector bundles.

**Proposition 4.1** Let \(L \in \{\mathbb{R}, \mathbb{C}\}, r \in \mathbb{N}_0 \cup \{\infty, \omega\}, K \in \{\mathbb{R}, L\}, k \in \mathbb{N}, E_1, \ldots, E_k\) and \(F\) be locally convex \(L\)-vector spaces, \(X\) be a locally convex \(K\)-vector space, and \(U \subseteq X\) be an open subset. Then the following holds.
(a) If \((X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)\) is a \(k_\mathbb{R}\)-space, or \(r = 0\) and \(X \times E_1 \times \cdots \times E_k\) is a \(k_\mathbb{R}\)-space, or \((r, \mathbb{K}) = (\infty, \mathbb{C})\) and \(X \times E_1 \times \cdots \times E_k\) is a \(k_\mathbb{R}\)-space, or all of the vector spaces \(E_1, \ldots, E_k\) are finite dimensional, then

\[
f^\wedge : U \times E_1 \times \cdots \times E_k \to F, \quad (x, y_1, \ldots, y_k) \mapsto f(x)(y_1, \ldots, y_k)
\]

is \(C^r_\mathbb{K}\) for each \(C^r_\mathbb{K}\)-map \(f : U \to L^k_\mathbb{K}(E_1, \ldots, E_k, F)_c\).

(b) If \(E := E_1 = E_2 = \cdots = E_k\) holds and, moreover, \((X \times E) \times (X \times E)\) is a \(k_\mathbb{R}\)-space or \(r = 0\) and \(X \times E\) is a \(k_\mathbb{R}\)-space, or \((r, \mathbb{K}) = (\infty, \mathbb{C})\) and \(X \times E\) is a \(k_\mathbb{R}\)-space, then

\[
f^\wedge : U \times E^k \to F, \quad (x, y_1, \ldots, y_k) \mapsto f(x)(y_1, \ldots, y_k)
\]

is \(C^r_\mathbb{K}\) for each \(C^r_\mathbb{K}\)-map \(f : U \to L^k_\mathbb{K}(E, F)_c\) such that \(f(x)\) is a symmetric \(k\)-linear map for each \(x \in U\).

(c) If \(X\) is finite dimensional, \(k = 1\), and \(E := E_1\) is barrelled, then \(f^\wedge : U \times E \to F, \ (x, y) \mapsto f(x)(y)\) is \(C^r_\mathbb{K}\) for each \(C^r_\mathbb{K}\)-map \(f : U \to L_\mathbb{K}(E, F)_c\).

(d) If all of the spaces \(E_1, \ldots, E_k\) are normable, then \(f^\wedge : U \times E_1 \times \cdots \times E_k \to F\) is \(C^r_\mathbb{K}\) for each \(C^r_\mathbb{K}\)-map \(f : U \to L^k_\mathbb{K}(E_1, \ldots, E_k, F)_b\).

**Proof.** Let \(ev : L^k_\mathbb{K}(E_1, \ldots, E_k, F)_c \times E_1 \times \cdots \times E_k \to F\) be the evaluation map, which is \(c\)-hypocontinuous in its arguments \((2, \ldots, k + 1)\) by Lemma 19

(a) Assuming the respective \(k_\mathbb{R}\)-property, the map \(f^\wedge = ev \circ (f \times \text{id}_{E_1 \times \cdots \times E_k})\) is \(C^r_\mathbb{K}\), by Lemma 3.1(b). If \(E_1, \ldots, E_k\) are finite dimensional, then \(L^k_\mathbb{K}(E_1, \ldots, E_k, F)_c\) equals \(L^k(E_1, \ldots, E_k, F)_b\), whence the conclusion of (a) is a special case of (d).

(b) By Lemma 3.1(b), the map

\[
g : U \times E \to F, \quad (x, y) \mapsto f^\wedge(x, y, \ldots, y)
\]

is \(C^r_\mathbb{K}\), as \(g = ev \circ (f \times \delta)\) with \(\delta : E \to E^k, y \mapsto (y, \ldots, y)\), which is continuous \(\mathbb{K}\)-linear. Then also \(f^\wedge\) is \(C^r_\mathbb{K}\), by Lemma 17.

(c) The bilinear map \(ev : L^k_\mathbb{K}(E, F)_c \times E \to F\) is \(c\)-hypocontinuous in its first argument, by Lemma 19 Hence \(f^\wedge = ev \circ (f \times \text{id}_E)\) is \(C^r_\mathbb{K}\), by Lemma 32.

(d) If \(E_1, \ldots, E_k\) are normable, then the evaluation map

\[
\varepsilon : L^k_\mathbb{K}(E_1, \ldots, E_k, F)_b \times E_1 \times \cdots \times E_k \to F
\]

is continuous \((k + 1)\)-linear and hence \(C^r_\mathbb{K}\), whence also \(f^\wedge = \varepsilon \circ (f \times \text{id}_{E_1 \times \cdots \times E_k})\) is \(C^r_\mathbb{K}\). □

**Remark 4.2** If \(X\) and all of \(E_1, \ldots, E_k\) are metrizable, then the topological space \((X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)\) is metrizable and hence a \(k\)-space. If \(X\) and all of \(E_1, \ldots, E_k\) are \(k_\mathbb{R}\)-spaces, then also \((X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)\) is a \(k_\mathbb{R}\)-space and hence a \(k\)-space. In either case, we are in the situation of (a).
5 Infinite-dimensional vector bundles

In this section, we provide foundational material concerning vector bundles modeled on locally convex spaces (cf. also [23] and [30, Chapter 3]). Notably, we discuss the description of vector bundles via cocycles, and define equivariant vector bundles.

Let \( L \in \{ \mathbb{R}, \mathbb{C} \} \), \( K \in \{ \mathbb{R}, L \} \), and \( r \in \mathbb{N}_0 \cup \{ \infty, \omega \} \). The word “manifold” always refers to a manifold modeled on a locally convex space. Likewise, the Lie groups we consider need not have finite dimension.

**Definition 5.1** Let \( M \) be a \( C^r_K \)-manifold and \( F \) a locally convex \( L \)-vector space. An \( L \)-vector bundle of class \( C^r_K \) over \( M \), with typical fibre \( F \), is a \( C^r_K \)-manifold \( E \), together with a surjective \( C^r_K \)-map \( \pi: E \to M \) and endowed with an \( L \)-vector space structure on each fibre \( E_x := \pi^{-1}(\{x\}) \), such that, for each \( x \in M \), there exists an open neighbourhood \( U \subseteq M \) of \( x \) and a \( C^r_K \)-diffeomorphism

\[
\psi: \pi^{-1}(U) \to U \times F
\]

(called a “local trivialization”) such that \( \psi(E_y) = \{ y \} \times F \) for each \( y \in U \) and the map \( \text{pr}_F \circ \psi|_{E_y}: E_y \to F \) is \( L \)-linear where \( \text{pr}_F: U \times F \to F \) is the projection.

**5.2** In the situation of Definition 5.1 let \( (\psi_i)_{i \in I} \) be an atlas of local trivializations for \( E \), \( i.e., \) a family of local trivializations

\[
\psi_i: \pi^{-1}(U_i) \to U_i \times F
\]

of \( E \) whose domains \( U_i \) cover \( M \). Then, given \( i, j \in I \), we have

\[
\psi_i(\psi_j^{-1}(x, v)) = (x, g_{ij}(x)(v))
\]

for \( x \in U_i \cap U_j \), \( v \in F \), for some function

\[
g_{ij}: U_i \cap U_j \to \text{GL}(F) \subseteq L(F).
\]

Here

\[
G_{ij}: (U_i \cap U_j) \times F \to F, \quad (x, v) \mapsto g_{ij}(x)(v)
\]

is \( C^r_K \), as \( \psi_i(\psi_j^{-1}(x, v)) = (x, G_{ij}(x, v)) \) is \( C^r_K \) in \( (x, v) \in (U_i \cap U_j) \times F \). By Proposition 2.1, \( g_{ij}: U_i \cap U_j \to L(F)_c \) is a \( C^r_K \)-map, and as a map to \( L(F)_b \), it is at least \( C^{r-1}_K \) (if \( r \geq 1 \)).

Note that the “transition maps” \( g_{ij} \) satisfy the “cocycle conditions”

\[
\left\{ \begin{array}{l}
(\forall i \in I) \ (\forall x \in U_i) \quad g_{ii}(x) = \text{id}_F \quad \text{and} \\
(\forall i, j, k \in I) \ (\forall x \in U_i \cap U_j \cap U_k) \quad g_{ij}(x) \circ g_{jk}(x) = g_{ik}(x).
\end{array} \right.
\]

**Proposition 5.3** Let \( L \in \{ \mathbb{R}, \mathbb{C} \}, K \in \{ \mathbb{R}, L \} \). Assume that

\[ ^6 \text{And hence an isomorphism of topological vector spaces, if we give } E_y \text{ the topology induced by } E. \]
(a) $M$ is a $C^r_\mathbb{K}$-manifold modeled on a locally convex $\mathbb{K}$-vector space $Z$;

(b) $E$ is a set and $\pi: E \to M$ a surjective map;

(c) $F$ is a locally convex $\mathbb{L}$-vector space;

(d) $(U_i)_{i \in I}$ is an open cover of $M$;

(e) $(\psi_i)_{i \in I}$ is a family of bijections $\pi^{-1}(U_i) \to U_i \times F$ such that $\psi_i(\pi^{-1}\{x\}) = \{x\} \times F$ for all $x \in U_i$;

(f) $g_{ij}(x)(v) := \text{pr}_F(\psi_i(\psi_j^{-1}(x,v)))$ depends $\mathbb{L}$-linearly on $v \in F$, for all $i,j \in I$, $x \in U_i \cap U_j$;

(g) $G_{ij}: (U_i \cap U_j) \times F \to F$, $G_{ij}(x,v) := g_{ij}(x)(v)$ is a $C^r_\mathbb{K}$-map.

Then there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_\mathbb{K}$ on $E$ making $\psi_i$ a local trivialization for each $i \in I$.

**Proof.** For $i,j \in I$, let $\text{pr}_{ij}: (U_i \cap U_j) \times F \to U_i \cap U_j$ be the projection onto the first component. As the maps

$$\psi_i \circ \psi_j^{-1}|_{(U_i \cap U_j) \times F} = (\text{pr}_{ij}, G_{ij})$$

are $C^r_\mathbb{K}$, there is a uniquely determined $C^r_\mathbb{K}$-manifold structure on $E$ making $\psi_i$ a $C^r_\mathbb{K}$-diffeomorphism for each $i \in I$. Given $x \in M$, we pick $i \in I$ with $x \in U_i$; we give $E_x := \pi^{-1}\{x\}$ the unique $\mathbb{L}$-vector space structure making the bijection $\text{pr}_F \circ \psi_i|_{E_x}: E_x \to F$ an isomorphism of vector spaces. It is easy to see that the vector space structure on $E_x$ is independent of the choice of $\psi_i$, and it is easily verified that we have turned $E$ into an $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ with the asserted properties. \hfill $\Box$

**Remark 5.4** Let $M$ be a $C^r_\mathbb{K}$-manifold, $F$ be a locally convex $\mathbb{L}$-vector space, $(U_i)_{i \in I}$ be an open cover of $M$, and $(g_{ij})_{i,j \in I}$ be a family of maps $g_{ij}: U_i \cap U_j \to \text{GL}(F)$ satisfying the cocycle conditions and such that

$$G_{ij}: (U_i \cap U_j) \times F \to F, \quad (x,v) \mapsto g_{ij}(x)(v)$$

is $C^r_\mathbb{K}$, for all $i,j \in I$. Using Proposition 5.3, the usual construction familiar from the finite-dimensional case provides an $\mathbb{L}$-vector bundle $\pi: E \to M$ of class $C^r_\mathbb{K}$, with typical fibre $F$, and a family $(\psi_i)_{i \in I}$ of local trivializations $\pi^{-1}(U_i) \to U_i \times F$, whose associated transition maps are the given $g_{ij}$'s. The bundle $E$ is unique up to canonical isomorphism.

Combining Proposition 5.3 and Proposition 4.4, we obtain:

**Corollary 5.5** Retaining the hypotheses (a)–(f) from Proposition 5.3 but omitting (g), consider the following conditions:

$$(g)' \quad g_{ij}(x) \in L(F) \text{ for all } i,j \in I, \quad x \in U_i \cap U_j, \quad \text{and } g_{ij}: U_i \cap U_j \to L(F)_c \text{ is } C^r_\mathbb{K};$$
In other words, \( L \) is an equivariant\(^6\) equivariant \( K \)-section. The mapping \( \beta \) is then equivariant in the sense that

\[
(\beta \circ \beta^{-1})(x) \in L(F) \text{ for all } i, j \in I, x \in U_i \cap U_j, \text{ and } g_{ij} : U_i \cap U_j \to L(F)_b \text{ is } C^r_K;
\]

(i) \((Z \times F) \times (Z \times F)\) is a \( k_{\mathbb{R}} \)-space, or \( r = 0 \) and \( Z \times F \) is a \( k_{\mathbb{R}} \)-space, or \((r, \mathbb{K}) = (\infty, \mathbb{C})\) and \( Z \times F \) is a \( k_{\mathbb{R}} \)-space;

(ii) \( \dim(M) < \infty \) and \( F \) is barrelled;

(iii) \( F \) is normable.

If \((g)''\) holds as well as (i) or (ii), then the conclusions of Proposition \[5.3\] remain valid. They also remain valid if \((g)''\) and (iii) hold. \(\square\)

Example 9.5 below shows that Conditions (a)–(f) and \((g)''\) alone are not sufficient for the conclusion of Proposition \[5.3\] without extra conditions on \( Z \) and \( F \). Note that (i) is satisfied if both \( Z \) and \( F \) are metrizable, or both \( Z \) and \( F \) are \( k_\omega \)-spaces.

**Equivariant vector bundles**

Beyond vector bundles, we shall discuss equivariant vector bundles in the following, i.e., vector bundles together with an action of a (finite- or infinite-dimensional) Lie group \( G \). Choosing \( G = \{ e \} \) as a trivial group, we obtain results about ordinary vector bundles (without a group action), as a special case.

For the remainder of this section, and also in Section \[6\], let \( \mathbb{L} \in \{ \mathbb{R}, \mathbb{C} \} \), \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{L} \} \), \( s \in \{ \infty, \omega \} \), and \( r \in \mathbb{N}_0 \cup \{ \infty, \omega \} \) with \( r \leq s \). Let \( G \) be a \( C^r_{\mathbb{K}} \)-Lie group (modeled on a locally convex \( \mathbb{K} \)-vector space \( Y \)) and \( M \) be a \( C^r_\mathbb{K} \)-manifold. We assume that a \( C^r_\mathbb{K} \)-action

\[
\alpha : G \times M \to M
\]

is given. Then \((M, \alpha)\) is called a \( G \)-manifold of class \( C^r_\mathbb{K} \).

**Definition 5.6** An equivariant \( \mathbb{L} \)-vector bundle of class \( C^r_\mathbb{K} \) over a \( G \)-manifold \((M, \alpha)\) of class \( C^r_\mathbb{K} \) is an \( \mathbb{L} \)-vector bundle \( \pi : E \to M \) of class \( C^r_\mathbb{K} \), together with a \( C^r_\mathbb{K} \)-action

\[
\beta : G \times E \to E
\]

such that \( \beta(g, E_x) \subseteq E_{\alpha(g, x)} \) for all \((g, x) \in G \times M\), and \( \beta(g, \cdot)|_{E_x} : E_x \to E_{\alpha(g, x)} \) is \( \mathbb{L} \)-linear.

In other words, \( \beta(g, \cdot) \) takes fibres linearly to fibres and coincides with \( \alpha(g, \cdot) \) on the zero section. The mapping \( \pi \) is then equivariant in the sense that \( \alpha \circ (\text{id}_G \times \pi) = \pi \circ \beta \).

**Example 5.7** If \( M \) is a \( G \)-manifold of class \( C^r_\mathbb{K} \), with \( r \geq 1 \), then the tangent bundle \( TM \) is an equivariant \( \mathbb{L} \)-vector bundle of class \( C^{r-1}_\mathbb{K} \) in a natural way, with \( \mathbb{L} := \mathbb{K} \). In fact, the action \( \alpha : G \times M \to M \) has a tangent map \( T\alpha : T(G \times M) \to TM \) which is \( C^{r-1}_\mathbb{K} \)-linear. Let \( 0_G : G \to TG \) be the 0-section. Identifying \( T(G \times M) \) with \( TG \times TM \) in the usual way, we obtain a \( C^{r-1}_\mathbb{K} \)-map \( \beta : G \times TM \to TM \) via

\[
\beta := (T\alpha) \circ (0_G \times \text{id}_{TM})\,.
\]

It is easy to see that \( \beta(g, v) = T_x(\alpha(g, \cdot))(v) \in T_{\alpha(g, x)}M \) for \( g \in G \) and \( v \in T_x M \), whence \( \beta(g, T_x M) \subseteq T_{\alpha(g, x)}M \) and \( \beta(g, \cdot)|_{T_x M} = T_x(\alpha(g, \cdot)) \). Clearly \( \beta \) is an action making \( TM \) an equivariant \( \mathbb{K} \)-vector bundle of class \( C^{r-1}_\mathbb{K} \) over the \( G \)-manifold \( M \).
Induced action on an invariant subbundle

Given an $\mathbb{L}$-vector bundle $\pi: E \to M$ of class $C^r_{K}$, with typical fibre $F$, we call a subset $E_0 \subseteq E$ a subbundle if there exists a sequentially closed $\mathbb{L}$-vector subspace $F_0 \subseteq F$ such that for each $x \in M$ there exists a local trivialization $\psi: \pi^{-1}(U) \to U \times F$ of $E$ such that $\psi(E_0 \cap \pi^{-1}(U)) = U \times F_0$. It readily follows from [30, Lemma 1.3.19] and [13, Proposition 2.11] that there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{K}$ on $\pi|_{E_0}: E_0 \to M$ making $\psi|_{\pi^{-1}(U) \cap E_0}: \pi^{-1}(U) \cap E_0 \to U \times F_0$ a local trivialization of $E_0$, for each local trivialization $\psi$ as before. Then the inclusion map $E_0 \to E$ is $C^r_{K}$, and a mapping $N \to E$ from a $C^r_{K}$-manifold $N$ to $E$ with image in $E_0$ is $C^r_{K}$ as a mapping to $E$ if and only if its co-restriction to $E_0$ is $C^r_{K}$, by the facts just cited. In the preceding situation, suppose that a $C^r_{K}$-Lie group $G$ acts $C^s_{K}$ on $M$ and $E$ is an equivariant vector bundle of class $C^r_{K}$ with respect to the action $\beta: G \times E \to E$. If $E_0$ is invariant under the $G$-action, i.e., if $\beta(G \times E_0) \subseteq E_0$, as a special case of the preceding observations we deduce from the $C^r_{K}$-property of $\beta$ that $\beta|_{G \times E_0}$ and thus also $\beta|_{G \times E_0}: G \times E_0 \to E_0$ is $C^r_{K}$. Summing up:

**Proposition 5.8** If $E$ is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{K}$ over a $G$-manifold $M$, then the action induced on any $G$-invariant subbundle $E_0$ is $C^r_{K}$ and thus makes the latter an equivariant $\mathbb{L}$-vector bundle of class $C^r_{K}$. \qed

6 Completions of vector bundles

Let $\pi: E \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{K}$, as in Definition 5.6 with typical fibre $F$ and $G$-actions $\alpha: G \times M \to M$ and $\beta: G \times E \to E$. Assume that $r \geq 1$. Our goal is to complete the fibre of the bundle, i.e., to find a $G$-equivariant vector bundle $\tilde{E}$ whose typical fibre is a completion of the locally convex space $F$, and which contains $E$ as a dense subset.

6.1 Let $\tilde{F}$ be a completion of $F$ such that $F \subseteq \tilde{F}$ and, for each $x \in M$, let $\tilde{E}_x$ be a completion of $E_x$ such that $E_x \subseteq \tilde{E}_x$. We may assume that the sets $\tilde{E}_x$ are pairwise disjoint for $x \in M$. Consider the (disjoint) union

$$\tilde{E} := \bigcup_{x \in M} \tilde{E}_x.$$

We shall turn $\tilde{E}$ into an equivariant vector bundle. Consider the map $\tilde{\beta}: G \times \tilde{E} \to \tilde{E}$, defined using the continuous extension $(\beta(g, \cdot)|_{E_x})^\gamma: \tilde{E}_x \to \tilde{E}_{\alpha(g,x)}$ of the linear mapping $\beta(g, \cdot)|_{E_x}: E_x \to E_{\alpha(g,x)}$ via

$$\tilde{\beta}(g, v) := (\beta(g, \cdot)|_{E_x})^\gamma(v)$$

for $g \in G$, $x \in M$, and $v \in \tilde{E}_x$. It is clear that $\tilde{\beta}$ makes $\tilde{E}$ a $G$-set. Let $\tilde{\pi}: \tilde{E} \to M$ be the map taking elements from $\tilde{E}_x$ to $x$. Then $\tilde{\pi}$ is $G$-equivariant. If $\psi: \pi^{-1}(U) \to U \times F$ is a local trivialization of $E$ and $\pr_F: U \times F \to F$, $(x, y) \mapsto y$, we define

$$\tilde{\psi}: \tilde{\pi}^{-1}(U) \to U \times \tilde{F}, \quad \tilde{E}_x \ni v \mapsto (x, (\pr_F \circ \psi|_{E_x})^\gamma(v)). \quad (8)$$
Then the following holds:

**Proposition 6.2** $(\tilde{E},\tilde{\beta})$ can be made an equivariant $\mathbb{L}$-vector bundle of class $C^{r-1}_\mathbb{K}$ over the $G$-manifold $M$, such that $\tilde{\psi}$ is a local trivialization of $\tilde{E}$ for each local trivialization $\psi$ of $E$.

**Remark 6.3** Omitting the hypothesis that $r \geq 1$, assume instead that $E$ is an equivariant $\mathbb{L}$-vector bundle of class $LC^{r}_\mathbb{K}$. That is, both $E$ and $M$ are $LC^{r}_\mathbb{K}$-manifolds (each admitting an atlas with transition maps of class $LC^{r}_\mathbb{K}$), a family of local trivializations can be chosen with $LC^{r}_\mathbb{K}$-transition maps, and the $G$-actions on $E$ and $M$ are $LC^{r}_\mathbb{K}$. Then also $\tilde{E}$ is an equivariant vector bundle of class $LC^{r}_\mathbb{K}$ (and hence of class $C^{r}_\mathbb{K}$).

**Extension of differentiable maps to subsets of the completions.** To enable the proof of Proposition 6.2 we need to discuss conditions ensuring that a $C^r$-map $f: E \supseteq U \to F$ (with locally convex spaces $E$ and $F$) can be extended to a $C^r$-map $\tilde{U} \to \tilde{F}$ on an open subset of the completion $\tilde{E}$ of $E$, or at least to a $C^{r-1}$-map. Although this is not possible in general, it is possible if $F$ is normed and $r$ is finite. This will be sufficient for our ends. The natural framework for the discussion of the problem are not $C^r$-maps, but Lipschitz differentiable maps, as in Definition 1.21.

**Proposition 6.4** Let $E$ be a locally convex $\mathbb{K}$-vector space, $(F, \| \cdot \|)$ be a Banach space over $\mathbb{K}$, $U \subseteq E$ be open and $f: U \to F$ be an $LC^{r}_\mathbb{K}$-map, where $r \in \mathbb{N}_{0}$. Let $\tilde{E}$ be a completion of $E$ such that $E \subseteq \tilde{E}$. Then $f$ extends to an $LC^{r}_\mathbb{K}$-map $\tilde{f}: \tilde{U} \to \tilde{F}$ on an open subset $\tilde{U} \subseteq \tilde{E}$ which contains $U$ as a dense subset.

The following lemma enables an inductive proof of Proposition 6.4.

**Lemma 6.5** Let $k \in \mathbb{N}$, $X$ be a locally convex $\mathbb{K}$-vector space, and $E_{1}, \ldots, E_{k}, F$ be locally convex $\mathbb{L}$-vector spaces, with completions $\tilde{X}, \tilde{E}_{1}, \ldots, \tilde{E}_{k}$ and $\tilde{F}$. Let $U \subseteq X$ be open and $f: U \times E_{1} \times \cdots \times E_{k} \to F$ be a map such that $f^{v}(x) := f(x, \cdot): E_{1} \times \cdots \times E_{k} \to F$ is $k$-linear over $\mathbb{L}$ for each $x \in U$. Assume that there exists an $LC^{r}_\mathbb{K}$-map $h: W \to \tilde{F}$ which extends $f$, defined on an open set $W \subseteq \tilde{X} \times \tilde{E}_{1} \times \cdots \times \tilde{E}_{k}$ in which $U \times E_{1} \times \cdots \times E_{k}$ is dense. Then there exists an $LC^{r}_\mathbb{K}$-map

$$\tilde{f}: \tilde{U} \times \tilde{E}_{1} \times \cdots \times \tilde{E}_{k} \to \tilde{F}$$

(9)

which extends $f$, for some open subset $\tilde{U} \subseteq \tilde{E}$ in which $U$ is dense. The maps $(\tilde{f})^{v}(x) := \tilde{f}(x, \cdot): \tilde{E}_{1} \times \cdots \times \tilde{E}_{k} \to F$ are $k$-linear over $\mathbb{L}$, for each $x \in \tilde{U}$.

**Proof.** For each $x \in U$, there exist an open neighbourhood $V_{x}$ of $x$ in $\tilde{X}$ and a balanced, open 0-neighbourhood $Q_{x} \subseteq \tilde{E}_{1} \times \cdots \times \tilde{E}_{k}$ such that $V_{x} \times Q_{x} \subseteq W$. After shrinking $V_{x}$, we may assume that $X \cap V_{x} = U$, whence $U \cap V_{x} = X \cap V_{x}$ is dense in $V_{x}$. Given $z \in \mathbb{L}$ such that $|z| \leq 1$, consider the map

$$V_{x} \times Q_{x} \to \tilde{F}, \quad (y, v) \mapsto h(y, zv) - z^{k}h(y, v).$$

22
This map vanishes, because it is continuous and vanishes on the dense subset \((V_x \cap X) \times (Q_x \cap (E_1 \times \cdots \times E_k))\). As a consequence, we obtain a well-defined map

\[
f_x : V_x \times \tilde{E}_1 \times \cdots \times \tilde{E}_k \rightarrow \tilde{F}, \quad (y,v) \mapsto z^{-k}h(y,zv)
\]

for \(y \in V_x, v \in \tilde{E}_1 \times \cdots \times \tilde{E}_k\) and \(z \in \mathbb{L} \setminus \{0\}\) with \(zv \in Q_x\). As \(f_x(y,v) = z^{-k}h(y,zv)\) is \(LC^r_x\) in \((y,v) \in V_x \times z^{-1}Q_x\) and these sets form an open cover of \(V_x \times \tilde{E}_1 \times \cdots \times \tilde{E}_k\), we see that \(f_x\) is \(LC^r_x\). Given \(x, y \in U\), the set \(U \cap V_x \cap V_y = X \cap V_x \cap V_y\) is dense in the open set \(V_x \cap V_y \subseteq \tilde{X}\). Since \(f_x, f_y\), and \(f\) coincide on the set \((U \cap V_x \cap V_y) \times E_1 \times \cdots \times E_k\), it follows that the continuous maps \(f_x\) and \(f_y\) coincide on the set \((V_x \cap V_y) \times \tilde{E}_1 \times \cdots \times \tilde{E}_k\) in which the former set is dense. Hence, setting \(\tilde{U} := \bigcup_{x \in U} V_x\), a well-defined map \(\tilde{f}\) as in (9) is obtained if we set

\[
\tilde{f}(y,v) := f_x(y,v) \quad \text{if} \quad x \in U, \ y \in V_x \quad \text{and} \quad v \in \tilde{E}_1 \times \cdots \times \tilde{E}_k.
\]

The final assertion follows by continuity from the linearity of the maps \(f^y(x)\) for \(x \in U\). \(\square\)

**Proof of Proposition 6.4.** We proceed by induction on \(r \in \mathbb{N}_0\).

The case \(r = 0\). Given \(x \in U\), there exists a continuous seminorm \(q\) on \(E\) such that \(B^q_1(x) \subseteq U\) and

\[
\|f(z) - f(y)\| \leq q(z - y) \quad \text{for all } y, z \in B^q_1(x).
\]

Then \(N_q := \{y \in E : q(y) = 0\}\) is a closed vector subspace of \(E\) and \(\|y + N_q\|_q := q(y)\) for \(y \in E\) defines a norm on \(E_q := E/N_q\) making the map \(\alpha_q : E \rightarrow E_q, y \mapsto y + N_q\) continuous linear. By (10), we have \(\|f(z) - f(y)\| = 0\) for all \(y, z \in B^q_1(x)\) such that \(y - z \in N_q\). Hence

\[
h : \alpha_q(B^q_1(x)) \rightarrow F, \quad y + N_q \mapsto f(y)
\]

is a well-defined map. Note that \(\alpha_q(B^q_1(x))\) is the open ball \(B := \{y \in E_q : \|y - \alpha_q(x)\|_q < 1\}\) in \(E_q\). Let \(\tilde{E}_q\) be the completion of the normed space \(E_q\); the extended norm will again be denoted by \(\|\|_q\). Applying (10) to representatives, we see that

\[
\|h(z) - h(y)\| \leq \|z - y\|_q \quad \text{for all } y, z \in B.
\]

Hence \(h\) satisfies a global Lipschitz condition (with Lipschitz constant 1), and hence \(h\) is uniformly continuous, entailing that \(h\) extends uniquely to a uniformly continuous map

\[
\tilde{h} : \tilde{B} \rightarrow F
\]

on the corresponding open ball \(\tilde{B} \subseteq \tilde{E}_q\). Then \(\|\tilde{h}(z) - \tilde{h}(y)\| \leq \|z - y\|_q\) for all \(y, z \in \tilde{B}\), by continuity. Let \(\tilde{\alpha}_q : \tilde{E} \rightarrow \tilde{E}_q\) be the continuous extension of the continuous linear map \(\alpha_q\). Then \(V_x := (\tilde{\alpha}_q)^{-1}(\tilde{B})\) is an open neighbourhood of \(x\) in \(\tilde{E}\) such that \(V_x \cap E = B^q_1(x) \subseteq U\). Moreover, \(f_x := h \circ \tilde{\alpha}_q|_{V_x}\) is a continuous map extending \(f|_{V_x \cap E}\), which furthermore satisfies

\[
\|f_x(z) - f_x(y)\| \leq \tilde{q}(z - y) \quad \text{for all } y, z \in V_x,
\]

(11)
where we use the continuous seminorm $\tilde{q} := \|\| \circ \tilde{\alpha}_q: \tilde{E} \to [0, \infty[$ extending $q$. Then 

$$
\tilde{U} := \bigcup_{x \in U} V_x
$$

is an open subset of $\tilde{E}$ and $E \cap \tilde{U} = U$ is dense in $\tilde{U}$. Given $x, y \in U$, the set $U \cap V_x \cap V_y = E \cap V_x \cap V_y$ is dense in the open set $V_x \cap V_y \subset \tilde{E}$. Since 

$$
f_x|_{U \cap V_x \cap V_y} = f|_{U \cap V_x \cap V_y} = f_y|_{U \cap V_x \cap V_y},
$$

it follows that $f_x|_{V_x \cap V_y} = f_y|_{V_x \cap V_y}$. Hence 

$$
\tilde{f}: \tilde{U} \to F, \quad z \mapsto f_x(z) \quad \text{for } x \in U \text{ such that } z \in V_x
$$

is a well-defined map. Since $\tilde{f}|_{V_x} = f_x$ is $LC_0^0$ for each $x \in U$ (by (11)), the map $\tilde{f}$ is $LC_0^0$. Furthermore, $\tilde{f}$ extends $f$ by construction.

Induction step. If $f$ is $LC_{r+1}^r$, then $f$ extends to an $LC_0^0$-map $\tilde{f}: \tilde{U} \to F$ on an open subset $\tilde{U} \subseteq \tilde{E}$ such that $\tilde{U} \cap E = U$, and $df: U \times E \to F$ extends to an $LC_{r+1}^r$-map $h: W \to F$ on an open subset $W$ of $\tilde{E} \times \tilde{E}$, by induction. Using Lemma [5.5] we find an open neighbourhood $V$ of $U$ in $\tilde{E}$ and an $LC_0^0$-map $g: V \times \tilde{E} \to F$ which extends $df$. After replacing $\tilde{U}$ and $V$ with their intersection, we may assume that $\tilde{U} = V$. If $x_0 \in \tilde{U}$ and $y_0 \in \tilde{E}$, there exist open neighbourhoods $Q$ of $x_0$ and $P$ of $y_0$ in $\tilde{E}$, and $\varepsilon > 0$ such that $Q + \mathbb{D}_\varepsilon P \subseteq \tilde{U}$. Then the map 

$$
\ell: Q \times P \times \mathbb{D}_\varepsilon \to F, \quad (x, y, t) \mapsto \int_0^1 g(x + sty, y) \, ds
$$

is continuous, being given by a parameter dependent weak integral with continuous integrand. For $(x, y, t)$ in the dense subset $(Q \cap E) \times (P \cap E) \times (\mathbb{D}_{\varepsilon} \setminus \{0\})$ of $Q \times P \times (\mathbb{D}_{\varepsilon} \setminus \{0\})$, the Mean Value Theorem implies that 

$$
\ell(x, y, t) = \frac{f(x + ty) - f(x)}{t} = \frac{\tilde{f}(x + ty) - \tilde{f}(x)}{t}.
$$

Then $\ell(x, y, t) = \frac{f(x + ty) - f(x)}{t}$ for all $(x, y, t) \in Q \times P \times (\mathbb{D}_{\varepsilon} \setminus \{0\})$, by continuity. Thus 

$$
\frac{f(x_0 + ty_0) - f(x_0)}{t} = \ell(x_0, y_0, t) \to \ell(x_0, y_0, 0) = g(x_0, y_0)
$$

as $t \to 0$. Hence $df(x_0, y_0) = g(x_0, y_0)$. Since $g$ is $LC_{r+1}$, it follows that $\tilde{f}$ is $LC_{r+1}^r$. □

The conclusion of Proposition 6.4 becomes false in general if the Banach space $F$ is replaced by a complete locally convex space. In fact, there exists a smooth map $E \to (\ell^1)\alpha$ from a proper, dense vector subspace $E$ of $\ell^1$ to a suitable power of $\ell^1$ which has no continuous extension to $E \cup \{x\}$ for any $x \in \ell^1 \setminus E$ (see Appendix [3]). Nonetheless, we have the following result.
Proposition 6.6 Let $k \in \mathbb{N}$, $X$ be a locally convex $\mathbb{K}$-vector space, and $E_1, \ldots, E_k, F$ be locally convex $\mathbb{L}$-vector spaces, with completions $\tilde{X}, \tilde{E}_1, \ldots, \tilde{E}_k$ and $\tilde{F}$, respectively. Let $U \subseteq X$ be open and $f : U \times E_1 \times \cdots \times E_k \to F$ be a mapping such that $f^\vee(x) := f(x, \cdot): E_1 \times \cdots \times E_k \to F$ is $k$-linear over $\mathbb{L}$ for each $x \in U$. If $f$ is $LC^r_{\mathbb{K}}$ for some $r \in \mathbb{N}_0 \cup \{\infty\}$ (resp., $C^r_{\mathbb{K}}$ for some $r \in \mathbb{N} \cup \{\infty, \omega\}$), then there exists a unique map

$$\tilde{f} : U \times \tilde{E}_1 \times \cdots \times \tilde{E}_k \to \tilde{F}$$

(12)

which is $LC^r_{\mathbb{K}}$ (resp., $C^r_{\mathbb{K}}$-1) and extends $f$. The maps $\tilde{f}^\vee(x) := \tilde{f}(x, \cdot) : \tilde{E}_1 \times \cdots \times \tilde{E}_k \to \tilde{F}$ are $k$-linear over $\mathbb{L}$, for each $x \in U$.

Proof. Abbreviate $E := E_1 \times \cdots \times E_k$ and $E := E_1 \times \cdots \times E_k$. Assume first that $r \neq \omega$. Since $LC^r_{\mathbb{K}}$-maps are continuous and $U \times E$ is dense in $U \times \tilde{E}$, there is at most one map $\tilde{f}$ with the asserted properties. We may therefore assume that $r \in \mathbb{N}_0$. We may also assume that $F$ is complete. Then $F = \lim_{\to} F_j$ for some projective system $((F_j)_{j \in J}, (p_{ij})_{i,j \in J})$ of Banach spaces $F_j$ and continuous linear maps $p_{ij} : F_j \to F_i$, with limit maps $p_j : F \to F_j$. We claim that $p_j \circ f : U \times E \to F_j$ has an $LC^r_{\mathbb{K}}$-extension $g_j := (p_j \circ f)^\vee : U \times \tilde{E} \to F_j$, for each $j \in J$. If this is true, then $p_{ij} \circ g_j = g_i$ for $i \leq j$, by uniqueness of continuous extensions. Hence, by the universal property of the projective limit, there exists a unique map $\tilde{f} : U \times \tilde{E} \to F$ such that $p_j \circ \tilde{f} = g_j$. Then $p_j \circ \tilde{f}|_{U \times E} = g_j|_{U \times E} = p_j \circ f$ and hence $\tilde{f}|_{U \times E} = f$. Furthermore, $f$ is $LC^r_{\mathbb{K}}$, by Lemma 11.22(d). To prove the claim, note that Proposition 6.4 yields an $LC^r_{\mathbb{K}}$-extension $h_j : W_j \to F_j$ of $p_j \circ f$ to an open subset $W_j \subseteq \tilde{X} \times \tilde{E}$ which contains $U \times E$ as a dense subset. Now Lemma 6.9 yields an open subset $U_j \subseteq \tilde{X}$ in which $U$ is dense, and an $LC^r_{\mathbb{K}}$-extension $e_j : U_j \times \tilde{E} \to F_j$ of $p_j \circ f$. Then $g_j := e_j|_{U \times \tilde{E}}$ is as desired.

We now consider the case $(r, \mathbb{K}) = (\omega, \mathbb{R})$. If $\mathbb{L} = \mathbb{C}$, by density of $U \times E$ in $U \times \tilde{E}$, for any real analytic extension $\tilde{f} : U \times \tilde{E} \to F \tilde{E}$ and $x \in U$, the map $\tilde{f}(x, \cdot)$ will be $k$-linear over $\mathbb{L}$. We may therefore assume that $\mathbb{L} = \mathbb{R}$. Let $h : W \to F_\mathbb{C}$ be a $\mathbb{C}$-analytic extension of $f$, defined on an open subset $W \subseteq X_\mathbb{C} \times E_\mathbb{C}$ such that $U \times E \subseteq W$. For each $x \in U$, there exist an open $x$-neighbourhood $U_x \subseteq U$ and balanced open 0-neighbourhoods $V_x \subseteq X$ and $W_x \subseteq E_\mathbb{C}$ such that $(U_x + iV_x) \times W_x \subseteq W$. We claim that there exists a $\mathbb{C}$-analytic map $g_x : (U_x + iV_x) \times E_\mathbb{C} \to F_\mathbb{C}$ such that $g_x|_{U_x \times E} = f|_{U_x \times E}$. For $x, y \in U$, the intersection $((U_x + iV_x) \times E_\mathbb{C}) \cap ((U_y + iV_y) \times E_\mathbb{C}) = ((U_x \cap U_y) + i(V_x \cap V_y)) \times E_\mathbb{C}$ is connected and meets $U \times E$ whenever it is non-empty. Hence, by the Identity Theorem, $g_x$ and $g_y$ coincide on the intersection of their domains. We therefore obtain a well-defined $\mathbb{C}$-analytic map $g : Q \times E_\mathbb{C} \to F_\mathbb{C}$ such that $g|_{(U_x + iV_x) \times E_\mathbb{C}} = g_x$ for each $x \in U$, using the open subset $Q := \bigcup_{x \in U} (U_x + iV_x)$ of $X_\mathbb{C}$. For each $x \in U$, the map $g(x, \cdot)|_E = g_x(x, \cdot)|_E = f(x, \cdot)$ is $k$-linear over $\mathbb{R}$. Using the Identity Theorem, we see that $g(x, \cdot)$ is $k$-linear over $\mathbb{C}$ for each $x \in U$, and hence for each $x \in Q$ by the Identity Theorem. By the case $(\infty, \mathbb{C})$, $g$ has a $\mathbb{C}$-analytic extension $\bar{g} : Q \times \tilde{E}_\mathbb{C} \to \tilde{F}_\mathbb{C}$. Since $g(U \times E) = f(U \times E) \subseteq F \subseteq \tilde{F}$ and $U \times E$ is dense in $U \times \tilde{E}$, we deduce that $\bar{g}(U \times \tilde{E}) \subseteq \tilde{F}$; we therefore obtain a map

$$\tilde{f} : U \times \tilde{E} \to \tilde{F}, \quad (x, y) \mapsto \tilde{g}(x, y)$$

25
for \( x \in U \), \( y \in \tilde{E} \). Since \( \tilde{g} \) is a \( \mathbb{C} \)-analytic extension for \( \tilde{f} \), the function \( \tilde{f} \) is \( \mathbb{R} \)-analytic. To prove the claim, consider for \( x \in U \) and \( n \in \mathbb{N} \) the \( \mathbb{C} \)-analytic map

\[
g_{x,n}: (U_x + iV_x) \times nW_x \to F_{\mathbb{C}}, \quad (z, y) \mapsto n^k h(z, (1/n)y).
\]

If \( n \leq m \) and \( y \in nW_x \cap E \), we have for all \( z \in U_x \)

\[
g_{x,m}(z, y) = m^k h(z, (1/m)y) = m^k f(z, (1/m)y) = f(z, y) = n^k f(t, (1/n)y) = g_{x,n}(z, y),
\]

whence \( g_{x,m}(z, y) = g_{x,n}(z, y) \) for all \( z \in U_x + iV_x \) and \( y \in nW_x \), by the Identity Theorem. Thus \( g_x: (U_x + iV_x) \times E_{\mathbb{C}} \to F_{\mathbb{C}}, (z, y) \mapsto g_{x,n}(z, y) \) if \( y \in nW_x \) is a well-defined \( \mathbb{C} \)-analytic extension of \( f|_{U_x \times E} \).

\[
\square
\]

**Proof of Proposition 6.2.** It suffices to prove the strengthening described in Remark 6.3. Let \( (\psi_i)_{i \in I} \) be a family of local trivializations \( \psi_i: \pi^{-1}(U_i) \to U_i \times F \) of an \( LC_{\mathbb{K}}^r \)-vector bundle \( E \) such that each local trivialization is some \( \psi_i \). Let \( (g_{ij})_{i,j \in I} \) be the corresponding cocycle and \( G_{ij} \) be the \( LC_{\mathbb{K}}^r \)-map \( g_{ij}^*: (U_i \cap U_j) \times F \to F \) which is \( \mathbb{L} \)-linear in the second argument. By Proposition 6.6 there is a unique \( LC_{\mathbb{K}}^r \)-map \( \tilde{G}_{ij}: U \times \tilde{F} \to \tilde{F} \) which extends \( G_{ij} \), and \( \tilde{G}_{ij} \) is \( \mathbb{L} \)-linear in the second argument. Thus, we obtain a map

\[
\tilde{g}_{ij}: U_i \cap U_j \to L_{\mathbb{L}}(\tilde{F}), \quad x \mapsto \tilde{G}_{ij}(x, \cdot).
\]

By continuity and density, for all \( i \in I \) we have \( \tilde{G}_{ii}(x, y) = y \) for all \( (x, y) \in U_i \times \tilde{F} \). Thus \( \tilde{g}_{ii}(x) = \text{id}_{\tilde{F}} \) for all \( x \in U_i \). For all \( i, j, k \in I \), we have

\[
\tilde{G}_{ij}(x, \tilde{G}_{jk}(x, y)) = \tilde{G}_{ik}(x, y) \quad \text{for all} \quad (x, y) \in (U_i \cap U_j \cap U_k) \times \tilde{F},
\]

as both sides are continuous in \((x, y)\) and equality holds for \( y \) in the dense subset \( F \) of \( \tilde{F} \); thus \( \tilde{g}_{ij}(x) \circ \tilde{g}_{jk}(x) = \tilde{g}_{ik}(x) \). Notably, \( \tilde{g}_{ij}(x) \circ \tilde{g}_{ji}(x) = \tilde{g}_{ii}(x) = \text{id}_{\tilde{F}} \) for all \( x \in U_i \) and \( i, j \in I \), entailing that \( \tilde{g}_{ij}(x) \in \text{GL}(\tilde{F}) \). By the preceding, the \( \tilde{g}_{ij} \) satisfy the cocycle conditions.

Let \( \tilde{E} \) and \( \tilde{\pi} \) be as in 6.1 define \( \tilde{\psi}_i: \tilde{\pi}^{-1}(U_i) \to U_i \times \tilde{F} \) as in (8), replacing \( \psi \) with \( \psi_i \). For all \( i, j \in I \) and \( x \in U_i \), we then have that

\[
\tilde{\psi}_i(\tilde{\psi}_j^{-1}(x, y)) = (x, \tilde{G}_{ij}(x, y))
\]

holds for all \( y \in \tilde{F} \), as equality holds for all \( y \in F \). As an analogue of Proposition 5.3 holds with \( LC_{\mathbb{K}}^r \)-maps in place of \( C_{\mathbb{K}}^r \)-maps, we get a unique \( \mathbb{L} \)-vector bundle structure of class \( LC_{\mathbb{K}}^r \) on \( \tilde{E} \) making \( \tilde{\psi}_i \) a local trivialization for each \( i \in I \).

It is apparent that \( \tilde{\beta}: G \times \tilde{E} \to \tilde{E} \) is an action, and \( E_x \) is taken \( \mathbb{L} \)-linearly to \( \tilde{E}_{\alpha(g,x)} \) by \( \tilde{\beta}(g, \cdot) \), for each \( g \in G \) and \( x \in M \). It only remains to show that \( \tilde{\beta} \) is \( LC_{\mathbb{K}}^r \). To this end, let \( g_0 \in G \) and \( x_0 \in M \); we show that \( \tilde{\beta} \) is \( LC_{\mathbb{K}}^r \) on \( U \times \tilde{\pi}^{-1}(V) \) for some open neighbourhood \( U \) of \( g_0 \) in \( G \) and an open neighbourhood \( V \) of \( x_0 \) in \( M \). Indeed, there exists a local trivialization \( \psi: \pi^{-1}(W) \to W \times F \) of \( E \) over an open neighbourhood \( W \) of \( \alpha(g_0, x_0) \).
in $M$. The action $\alpha$ being continuous, we find an open neighbourhood $U$ of $g_0$ in $G$ and an open neighbourhood $V$ of $x_0$ in $M$ over which $E$ is trivial, such that $\alpha(U \times V) \subseteq W$. Let $\phi: \pi^{-1}(V) \to V \times F$ be a local trivialization of $E$ over $V$. Then

$$\phi(\beta(g^{-1}, \psi^{-1}(\alpha(g, x), v))) = (x, A(g, x, v)) \quad \text{for all } g \in U, x \in V, \text{ and } v \in F,$$

for an $LC^r_-\text{map} A: U \times V \times F \to F$ which is $\mathbb{L}$-linear in the third argument. By Proposition 6.6, there is a unique extension of $A$ to an $LC^r_-\text{map}$

$$\tilde{A}: U \times V \times \tilde{F} \to \tilde{F},$$

and the latter is $\mathbb{L}$-linear in its third argument. For all $g \in U$ and $x \in V$, we then have

$$\tilde{\phi}(\tilde{\beta}(g^{-1}, \tilde{\psi}^{-1}(\alpha(g, x), v))) = (x, \tilde{A}(g, x, v))$$

for all $v \in \tilde{F}$, as equality holds for all $v \in F$. Thus $\tilde{\beta}$ is $LC^r_-$. \hfill \Box

7 Tensor products of vector bundles

Throughout this section, let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ such that $r \leq s$. Let $G$ be a $C^r_\mathbb{K}$-Lie group modeled on a locally convex $\mathbb{K}$-vector space $Y$, $M$ be a $C^r_\mathbb{K}$-manifold modeled on a locally convex $\mathbb{K}$-vector space $Z$, and $\alpha: G \times M \to M$ be a $C^r_\mathbb{K}$-action. For $k \in \{1, 2\}$, let $\pi_k: E_k \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ over $M$, whose typical fibre is a locally convex $\mathbb{L}$-vector space $F_k$. Let $\beta_k: G \times E_k \to E_k$ be the $G$-action of class $C^r_\mathbb{K}$. Consider the set $\mathcal{A}$ of all pairs of local trivializations of $E_1$ and $E_2$ trivializing these over the same open subset of $M$. Using an index set $I$, we have $\mathcal{A} = \{(\psi^i_1, \psi^i_2) : i \in I\}$, where $\psi^i_k: \pi_k^{-1}(U_i) \to U_i \times F_k$ is a local trivialization of $E_k$ for $k \in \{1, 2\}$, for each $i \in I$. Apparently, $(U_i)_{i \in I}$ is an open cover of $M$.

7.1 For our first result concerning tensor products, Proposition 7.6 we assume that $F_1$ is finite dimensional. Then, fixing a basis $e_1, \ldots, e_n$ for $F_1$, the map $\theta: (F_2)^n \to F_1 \otimes_{\mathbb{L}} F_2$, $(y_1, \ldots, y_n) \mapsto \sum_{\tau=1}^{n} e_{\tau} \otimes y_{\tau}$ is an isomorphism of $\mathbb{L}$-vector spaces. We give $F_1 \otimes_{\mathbb{L}} F_2$ the topology $\mathcal{T}$ making $\theta$ a homeomorphism. This topology makes $F_1 \otimes_{\mathbb{L}} F_2$ a locally convex $\mathbb{L}$-vector space and $\theta$ an isomorphism of topological $\mathbb{L}$-vector spaces. It is easy to check (and well known) that the topology $\mathcal{T}$ is independent of the chosen basis. Let $e^*_1, \ldots, e^*_n \in F_1'$ be the basis dual to $e_1, \ldots, e_n$. Our goal is to make the union

$$E_1 \otimes E_2 := \bigcup_{x \in M} (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$$

an equivariant $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ over $M$, with typical fibre $F_1 \otimes_{\mathbb{L}} F_2$; the tensor products $(E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ are chosen pairwise disjoint here for $x \in M$. Let $\pi: E_1 \otimes E_2 \to M$ be the mapping which takes $v \in (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ to $x$. 27
7.2 We define $\psi_i : \pi^{-1}(U_i) \rightarrow U_i \times (F_1 \otimes_L F_2)$ via

$$\psi_i(v) := (x, ((\text{pr}_{F_1} \circ \psi^1_1|(E_1)_x) \otimes (\text{pr}_{F_2} \circ \psi^2_1|(E_2)_x))(v))$$

for $x \in U_i$ and $v \in (E_1)_x \otimes_L (E_2)_x$, where $\text{pr}_{F_k} : M \times F_k \rightarrow F_k$ is the projection.

7.3 Given $i, j \in I$ and $x \in U_i \cap U_j$, we have $\psi_i^k((\psi_j^k)^{-1}(x, v)) = (x, G_{ij}^k(x, v))$ for all $k \in \{1, 2\}$ and $v \in F_k$, where $G_{ij}^k : (U_i \cap U_j) \times F_k \rightarrow F_k$ is $C^r_K$ and $g_{ij}^k(x) := G_{ij}^k(x, \cdot)$ an $L$-linear mapping. Then $e_{\sigma, \tau} : U_i \cap U_j \rightarrow \mathbb{K}$, $x \mapsto e_{\sigma}^i (G_{ij}^1(x, e_\tau))$ is $C^r_K$, and $\psi_i((\psi_j)^{-1}(x, v)) = (x, G_{ij}(x, v))$ for $x \in U_i \cap U_j$ and $v = \sum_{\tau=1}^n e_\tau \otimes v_\tau \in F_1 \otimes_L F_2$, where

$$G_{ij}(x, v) = (g_{ij}^1(x) \otimes g_{ij}^2(x))(v) = \sum_{\tau=1}^n (g_{ij}^1(x)e_\tau) \otimes (g_{ij}^2(x)v_\tau)$$

$$= \sum_{\sigma, \tau=1}^n e_\sigma \otimes (c_{\sigma, \tau}(x)g_{ij}^2(x)v_\tau) = \theta \left( \sum_{\tau=1}^n c_{\sigma, \tau}(x)G_{ij}^2(x, v_\tau) \right)_{\sigma=1}^n.$$

As $F_1 \otimes_L F_2 \rightarrow F_2$, $v \mapsto v_\tau = \text{pr}_\tau(\theta^{-1}(v))$ is a continuous linear map (where $\text{pr}_\tau : (F_2)^n \rightarrow F_2$ is the projection onto the $\tau$-component), in view of the preceding formula $G_{ij}$ is $C^r_K$. Thus, by Proposition 5.3, there is a unique $L$-vector bundle structure of class $C^r_K$ on $E_1 \otimes E_2$ making each $\psi_i$ a local trivialization.

7.4 Note that $\beta : G \times (E_1 \otimes E_2) \rightarrow E_1 \otimes E_2, (g, v) \mapsto (\beta_1(g, \cdot)|_{(E_1)_x} \otimes \beta_2(g, \cdot)|_{(E_2)_x})(v)$ for $g \in G$, $x \in M$, $v \in (E_1 \otimes E_2)_x$, defines an action of $G$ on $E_1 \otimes E_2$ by $L$-linear mappings, which makes $\pi : E_1 \otimes E_2 \rightarrow M$ an equivariant mapping and such that $\beta(g, \cdot)$ is $L$-linear on $(E_1)_x \otimes_L (E_2)_x$ for all $g \in G$ and $x \in M$.

7.5 To show that $\beta$ is $C^r_K$, let $g_0 \in G$ and $x_0 \in M$. We pick $i \in I$ such that $\alpha(g_0, x_0) \in U_i$. The mapping $\alpha$ being continuous, we find open neighbourhoods $U$ of $g_0$ in $G$ and $V$ of $x_0$ in $M$ such that $\alpha(U \times V) \subseteq U_i$. There is $j \in I$ such that $x_0 \in U_j \subseteq V$. For $k \in \{1, 2\}$, $g \in U$, $x \in U_j$ and $v \in F_k$, we have

$$\psi_i^k(\beta(g, (\psi_j^k)^{-1}(x, v))) = (\alpha(g, x), a_k(g, x, v))$$

for some $C^r_K$-map $a_k : U \times U_j \times F_k \rightarrow F_k$ which is $L$-linear in the final argument. Define $b_{\sigma, \tau} : U \times U_j \rightarrow \mathbb{L}$, $(g, x) \mapsto e_{\sigma}^1(a_1(g, x, e_\tau))$; then $b_{\sigma, \tau}$ is $C^r_K$. If $g \in U$, $x \in U_j$ and $v = \sum_{\tau=1}^n e_\tau \otimes v_\tau \in F_1 \otimes_L F_2$, then $\psi_i(\beta(g, (\psi_j)^{-1}(x, v)))$ equals

$$\left( \alpha(g, x), \sum_{\tau=1}^n a_1(g, x, e_\tau) \otimes a_2(g, x, v_\tau) \right) = \left( \alpha(g, x), \theta \left( \sum_{\tau=1}^n b_{\sigma, \tau}(g, x)a_2(g, x, v_\tau) \right)_{\sigma=1}^n \right),$$

which is a $C^r_K$-function of $(g, x, v)$. As a consequence, $\beta|_{U \times \pi^{-1}(U_j)}$ is $C^r_K$ and thus $\beta$, being $C^r_K$ locally, is $C^r_K$. Summing up:
Proposition 7.6 Let $G$ be a $C^r_\mathbb{K}$-Lie group and $M$ be a $G$-manifold of class $C^r_\mathbb{K}$. Let $E_1$ and $E_2$ be equivariant $\mathbb{L}$-vector bundles of class $C^r_\mathbb{K}$ over $M$. If the typical fibre of $E_1$ is finite dimensional, then $E_1 \otimes E_2$, as defined above, is an equivariant $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ over $M$. \hfill \Box

7.7 Instead of $\dim(F_1) < \infty$ (as in 7.1), assume that $F_1$ and $F_2$ are Fréchet spaces and the modeling spaces of $G$ and $M$ are metrizable. The completed projective tensor product

$$F := F_1 \hat{\otimes}_\pi F_2$$

over $\mathbb{L}$ then is a Fréchet space (cf. [52, p. 438, lines after Definitions 43.4]). We define

$$E := E_1 \hat{\otimes}_\pi E_2 := \bigcup_{x \in M} (E_1)_x \hat{\otimes}_\pi (E_2)_x,$$

where the $(E_1)_x \hat{\otimes}_\pi (E_2)_x$ for $x \in M$ are chosen pairwise disjoint. Let $\pi: E \to M$ be the map taking $v \in E_x := (E_1)_x \hat{\otimes}_\pi (E_2)_x$ to $x$. Define $\psi_i: \pi^{-1}(U_i) \to U_i \times (F_1 \hat{\otimes}_\pi F_2)$ via

$$\psi_i(v) := (x, ((\pr_{F_1} \circ \psi^1_i|_{(E_1)_x}) \hat{\otimes}_\pi (\pr_{F_2} \circ \psi^2_i|_{(E_2)_x}))(v))$$

for $x \in U_i$ and $v \in (E_1)_x \hat{\otimes}_\pi (E_2)_x$, where $\pr_{F_k}: M \times F_k \to F_k$ is the projection. Note that $\beta: G \times E \to E$, $(g, v) \mapsto ((\beta_1(g, \cdot)|_{(E_1)_x} \hat{\otimes}_\pi \beta_2(g, \cdot)|_{(E_2)_x}))(v)$ for $g \in G$, $x \in M$, $v \in E_x$ defines an action of $G$ on $E$ which makes $\pi: E \to M$ an equivariant mapping. We show:

Proposition 7.8 $\pi: E_1 \hat{\otimes}_\pi E_2 \to M$ admits a unique structure of equivariant $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ over $M$ such that $\psi_i$ is a local trivialization for each $i \in I$.

Proof. The uniqueness for prescribed local trivializations is clear. Let us show existence of the structure. Given $i, j \in I$ and $x \in U_i \cap U_j$, we have $\psi^k_j((\psi^k_j)^{-1}(x, v)) = (x, G^k_{ij}(x, v))$ for all $k \in \{1, 2\}$ and $v \in F_k$, where $G^k_{ij}: (U_i \cap U_j) \times F_k \to F_k$ is $C^r_\mathbb{K}$ and $g^k_{ij}(x) := G^k_{ij}(x, \cdot)$ an $\mathbb{L}$-linear mapping. By Proposition 2.1(a), the map $g^k_{ij}: U_i \cap U_j \to L(F_k)_c$ is $C^r_\mathbb{K}$. Now

$$L_{\mathbb{L}}(F_1)_c \times L_{\mathbb{L}}(F_2) \to L_{\mathbb{L}}(F_1 \hat{\otimes}_\pi F_2)_c, (S, T) \mapsto S \hat{\otimes}_\pi T$$

being continuous $\mathbb{L}$-bilinear (as recalled in Lemma 7.9), we deduce that

$$g_{ij}: U_i \cap U_j \to L_{\mathbb{L}}(F_1 \hat{\otimes}_\pi F_2)_c, x \mapsto g^1_{ij}(x) \hat{\otimes}_\pi g^2_{ij}(x)$$

is $C^r_\mathbb{K}$. Hence $G_{ij} := g^1_{ij}: (U_i \cap U_j) \times (F_1 \hat{\otimes}_\pi F_2) \to F_1 \hat{\otimes}_\pi F_2$, $(x, v) \mapsto g_{ij}(x)(v)$ is $C^r_\mathbb{K}$, by Proposition 2.1(a). We easily check that $\psi_i((\psi_j)^{-1}(x, v)) = (x, G_{ij}(x, v))$ holds for $G_{ij}$ as just defined, for all $x \in U_i \cap U_j$ and $v \in F_1 \hat{\otimes}_\pi F_2$. Hence $E_1 \hat{\otimes}_\pi E_2$ can be made an $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ in such a way that each $\psi_i$ is a local trivialization, by Proposition 5.3.

Note that $\beta(g, \cdot)$ is $\mathbb{L}$-linear on $E_x$ for all $g \in G$ and $x \in M$. To show that $\beta$ is $C^r_\mathbb{K}$, let $g_0$, $x_0$, $i$, $U$, $V$, $j$ and the $C^r_\mathbb{K}$-map $a_k$ be as in the proof of Proposition 7.6. By Proposition 2.1(a), $a_k: U \times U \to L(F_k)_c$, $(g, x) \mapsto a_k(g, x, \cdot)$ is $C^r_\mathbb{K}$. Hence

$$a: U \times U \to L(F_1 \hat{\otimes}_\pi F_2)_c, (g, x) \mapsto a_1^i(g, x) \hat{\otimes}_\pi a_2^i(g, x)$$

29
is $C^r_K$, by the Chain Rule and Lemma 7.9. Using Proposition 4.1(a), we find that the map $a^\lambda : U \times U_j \times (F_1 \otimes F_2) \to F_1 \otimes F_2$, $(g, x, v) \mapsto a(g, x)(v)$ is $C^r_K$. We easily verify that

$$\psi_1(\beta(g, (\psi_j)^{-1}(x, v))) = (\alpha(g, x), a^\lambda(g, x, v))$$

for all $(g, x, v) \in U \times U_j \times (F_1 \otimes F_2)$. Thus $\psi_1(\beta(g, (\psi_j)^{-1}(x, v)))$ is $C^r_K$ in $(g, x, v)$, which completes the proof. 

We used the following fact:

**Lemma 7.9** Let $E_1$, $E_2$, $F_1$, and $F_2$ be Fréchet spaces over $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$. Then the following bilinear map is continuous:

$$\Xi : L_\lambda(E_1, F_1)_c \times L_\lambda(E_2, F_2)_c \to L_\lambda((E_1 \otimes F_2), (F_1 \otimes F_2))_c, \ (S_1, S_2) \mapsto S_1 \otimes F_2 S_2.$$ 

**Proof.** Let $K \subseteq E_1 \otimes E_2$ be compact, $q$ be a continuous seminorm on $F_1 \otimes F_2$, and $\varepsilon > 0$. After increasing $q$, we may assume that $q = q_1 \otimes q_2$ for continuous seminorms $q_k$ on $F_k$ for $k \in \{1, 2\}$. By [52] p. 465, Corollary 2 to Theorem 45.2, $K$ is contained in the closed absolutely convex hull of $K_1 \otimes K_2$ for certain compact subsets $K_k \subseteq E_k$ for $k \in \{1, 2\}$. For all $S_k \in L(E_k, F_k)$ such that $\sup q_k(S_k(K_k)) \leq \sqrt{\varepsilon}$, we have

$$\sup q((S_1 \otimes S_2)(K)) \leq \sup q((S_1 \otimes S_2)(K_1 \otimes K_2)) = \sup q_1(S_1(K_1))q_2(S_2(K_2)) \leq \sqrt{\varepsilon^2} = \varepsilon,$$

using [52] Proposition 43.1. The assertion follows. 

**7.10** If $E_1$ and $E_2$ are Hilbert spaces over $\mathbb{L}$ with Hilbert space tensor product $E_1 \otimes E_2$, and also $F_1$ and $F_2$ are Hilbert space over $\mathbb{L}$, then the bilinear map

$$\Xi : L(E_1, F_1)_b \times L(E_2, F_2)_b \to L((E_1 \otimes F_2), (F_1 \otimes F_2))_b$$

is continuous, as $\|S_1 \otimes S_2\|_{\text{op}} \leq \|S_1\|_{\text{op}}\|S_2\|_{\text{op}}$. 

**7.11** Replace the hypotheses in 7.1 and 7.7 with the requirements that $G$ and $M$ are modeled on metrizable locally convex spaces, $r \geq 1$ and $F_1$, $F_2$ are Hilbert spaces. We now use 7.10 instead of Lemma 7.9 replace $F_1 \otimes F_2$ with the Hilbert space $F_1 \hat{\otimes} F_2$, Proposition 2.1(a) with Proposition 2.1(b) (so that operator-valued maps are only $C^{r-1}_K$) and use Proposition 4.1(b) with $r - 1$ in place of $r$. Repeating the proof of Proposition 7.8 we get:

**Proposition 7.12** On $E_1 \hat{\otimes} E_2 = \bigcup_{x \in M}(E_1)_x \hat{\otimes} (E_2)_x$, there is a unique equivariant $\mathbb{L}$-vector bundle structure of class $C^{r-1}_K$ over $M$ whose typical fibre is the Hilbert space $F_1 \hat{\otimes} F_2$, such that $\psi_i : \pi^{-1}(U_i) \to U_i \times (F_1 \hat{\otimes} F_2)$ is a local trivialization for each $i \in I$. 

**Remark 7.13** If $r \geq 1$, $G$ and $M$ are modeled on metrizable spaces and both $F_1$ and $F_2$ are pre-Hilbert spaces with Hilbert space completions $\bar{F}_1$ and $\bar{F}_2$, we can use the non-completed tensor product $F_1 \bar{\otimes} F_2 \subseteq \bar{F}_1 \otimes \bar{F}_2$ with the induced topology as the fibre and get an equivariant $\mathbb{L}$-vector bundle structure over $M$ of class $C^{r-1}_K$ over $M$ on $E_1 \otimes E_2 = \bigcup_{x \in M}(E_1)_x \otimes (E_2)_x$, exploiting that the $\mathbb{L}$-bilinear map $L_\lambda(F_1)_b \times L_\lambda(F_2)_b \to L_\lambda(F_1 \hat{\otimes} F_2)_b$, $(S_1, S_2) \mapsto S_1 \otimes S_2$ is continuous.
8 Locally convex direct sums of vector bundles

Let $L \in \{\mathbb{R}, \mathbb{C}\}$, $K \in \{\mathbb{R}, L\}$, $s \in \{\infty, \omega\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ such that $r \leq s$, $G$ be a $C^r_r$-Lie group modeled on a locally convex space $Y$, and $M$ be a $C^r_r$-manifold modeled on a locally convex $K$-vector space $Z$, together with a $C^r_r$-action $\alpha: G \times M \to M$.

8.1 Let $n \in \mathbb{N}$ and $\pi_k: E_k \to M$ be an equivariant $L$-vector bundle of class $C^r_r$ over $M$ for $k \in \{1, \ldots, n\}$, with typical fibre a locally convex $L$-vector space $F_k$; let $\beta_k: G \times E_k \to E_k$ be the $G$-action and $pr_{E_k}: M \times F_k \to F_k$ be the projection onto the 2nd component. We easily check: There is a unique $L$-vector bundle structure of class $C^r_r$ on the "Whitney sum"

$$E := E_1 \oplus \cdots \oplus E_n := \bigcup_{x \in M} (E_1)_x \times \cdots \times (E_n)_x,$$

with the apparent map $\pi: E \to M$, such that $\psi : \pi^{-1}(U) \to U \times F_1 \times \cdots \times F_n$, $v = (v_1, \ldots, v_n) \mapsto (\pi(v), pr_{F_1}(\psi_1(v_1)), \ldots, pr_{F_n}(\psi_n(v_n)))$ is a local trivialization of $E$, for all families $(\psi_k)_{k=1}^n$ of local trivializations $\psi_k: (\pi_k)^{-1}(U) \to U \times F_k$ which trivialize the $E_k$'s over a joint open subset $U$ of $M$. Then $\beta(g, v) := (\beta_1(g, v_1), \ldots, \beta_n(g, v_n))$ for $g \in G$, $v = (v_1, \ldots, v_n) \in E$ yields an action of $G$ on $E$. It is straightforward that $\beta$ is $C^r_r$. Thus:

**Proposition 8.2** If $E_1, \ldots, E_n$ are equivariant $L$-vector bundles of class $C^r_r$ over a $G$-manifold $M$ of class $C^r_r$, then also $E_1 \oplus \cdots \oplus E_n$ is an equivariant $L$-vector bundle of class $C^r_r$ over $M$. 

The following lemma allows infinite direct sums to be tackled.

**Lemma 8.3** Let $(E_i)_{i \in I}$ and $(F_i)_{i \in J}$ be families of locally convex spaces over $K \in \{\mathbb{R}, \mathbb{C}\}$, with locally convex direct sums $E := \bigoplus_{i \in I} E_i$ and $F := \bigoplus_{i \in J} F_i$, respectively. Let $V$ be an open subset of a locally convex $K$-vector space $Z$. Let $r \in \mathbb{N}_0 \cup \{\infty\}$, and assume that $f_i: V \times E_i \to F_i$ is a mapping which is linear in the second argument, for each $i \in I$. Moreover, assume that (a) or (b) holds:

(a) $Z$ is finite dimensional; or

(b) $Z$ and each $E_i$ is a $k_\omega$-space and $I$ is countable.

If $f_i$ is of class $C^r_r$ for each $i \in I$, then also the following map is $C^r_r$:

$$f: V \times E \to F, \quad (x, (v_i)_{i \in I}) \mapsto (f_i(x, v_i))_{i \in I}.$$

**Proof.** If (b) holds, we may assume that $I$ is countably infinite, excluding a trivial case. Thus, assume that $I = \mathbb{N}$. For each $n \in \mathbb{N}$, identify $E_1 \times \cdots \times E_n$ with a vector subspace of $E$, identifying $x \in E_1 \times \cdots \times E_n$ with $(x, 0)$. For each $n \in \mathbb{N}$, we then have

$$Z \times E = \bigcup_{n \in \mathbb{N}} (Z \times E_1 \times \cdots \times E_n) \quad \text{and} \quad V \times E = \bigcup_{n \in \mathbb{N}} (V \times E_1 \times \cdots \times E_n),$$

31
where \( Z \times E_1 \times \cdots \times E_n \) is a \( k_\omega \)-space in the product topology. The inclusion map
\[
\lambda_n: F_1 \times \cdots \times F_n \to \bigoplus_{i \in \mathbb{N}} F_i, \quad v \mapsto (v, 0)
\]
is continuous and \( \mathbb{K} \)-linear. Moreover,
\[
g_n: V \times E_1 \times \cdots \times E_n \to F_1 \times \cdots \times F_n, \quad (x, v_1, \ldots, v_n) \mapsto (f_1(x, v_1), \ldots, f_n(x, v_n))
\]
is a \( C_\mathbb{K}^r \)-map and so is \( f|_{V \times E_1 \times \cdots \times E_n} = \lambda_n \circ g_n \), for each \( n \in \mathbb{N} \). Hence \( f \) is \( C_\mathbb{K}^r \) on the open subset \( V \times E \) of \( Z \times E \), considered as the locally convex direct limit \( \lim \rightarrow (Z \times E_1 \times \cdots \times E_n) \), by [21] Proposition 4.5 (a)]. But this locally convex space equals \( Z \times \lim \rightarrow (E_1 \times \cdots \times E_n) = Z \times E \) with the product topology (see [34] Theorem 3.4]).

If (a) holds, it suffices to prove the assertion for \( r \in \mathbb{N}_0 \). We proceed by induction. The case \( r = 0 \). Let \((x, v) = (x, (v_i)_{i \in I}) \in V \times E \); we show that \( f \) is continuous at \((x, v) \). To this end, let \( Q \) be an absolutely convex, open \( 0 \)-neighbourhood in \( E \). There is a finite subset \( J \subseteq I \) such that \( v_i = 0 \) for all \( i \in I \setminus J \). Let \( N := |J| + 1 \). For each \( i \in I \), the intersection \( Q_i := (\frac{1}{N}Q) \cap F_i \) is an absolutely convex, open \( 0 \)-neighbourhood in \( F_i \). For the absolutely convex hull, we get \( \text{absconv}(\bigcup_{i \in I} Q_i) \subseteq \frac{1}{N}Q \). Since \( f_i \) is continuous for each \( i \in J \) and \( J \) is finite, we find a compact neighbourhood \( K \) of \( x \) in \( V \) such that \( f_i(y, v_i) - f_i(x, v_i) \in Q_i \) for all \( y \in K \) and \( i \in J \). Since \( f_i(K \times \{0\}) = \{0\} \), where \( K \) is compact and \( f_i \) is continuous, for each \( i \in I \) there is an absolutely convex, open \( 0 \)-neighbourhood \( P_i \) in \( E_i \) such that \( f_i(K \times P_i) \subseteq Q_i \). Then \( W := v + \text{absconv}(\bigcup_{i \in J} P_i) \) is an open neighbourhood of \( v \) in \( E \). Let \( y \in K \) and \( w \in W \) be given, say \( w = (w_i)_{i \in I} = v + (t_i p_i)_{i \in J} \) where \( p_i \in P_i \) and \( t_i \in I \in \bigoplus_{i \in J} \mathbb{R} \) such that \( t_i \in [0, 1] \) and \( \sum_{i \in J} t_i = 1 \). Then, for each \( i \in I \setminus J \), since \( v_i = 0 \) we obtain
\[
f_i(y, w_i) - f(x, v_i) = f_i(y, t_i p_i) = t_i f_i(y, p_i) \in t_i Q_i.
\]
For \( i \in J \) on the other hand, we have
\[
f_i(y, w_i) - f(x, v_i) = f_i(y, w_i - v_i) + (f_i(y, v_i) - f_i(x, v_i)) = t_i f_i(y, p_i) + (f_i(y, v_i) - f_i(x, v_i)) \in t_i Q_i + Q_i.
\]
As a consequence, \( f(y, w) - f(x, v) \in (\prod_{i \in I} t_i Q_i) + \sum_{i \in J} Q_i \subseteq \frac{1}{N}Q + \sum_{i \in J} \frac{1}{N}Q = Q \), using the convexity of \( Q \). We have shown that \( f \) is continuous at \((x, v) \).

Induction step. Let \( r \geq 1 \) and assume the assertion is true for all numbers \( < r \). Given \( u, v \in E \), \( x \in V \), and \( z \in Z \), we have \( u, v \in \bigoplus_{i \in J} E_i = \prod_{i \in J} E_i \) for some finite subset \( J \subseteq I \). The map \( f_J: V \times \prod_{i \in J} E_i \to \prod_{i \in J} F_i, (x, (v_i)_{i \in J}) \mapsto (f_i(x, v_i))_{i \in J} \) is \( C_\mathbb{K}^r \), whence
\[
\frac{d}{dt}(f_I((x, u) + t(z, v)) - f_I(x, u)) = \lim_{t \to 0} t^{-1}(f_I((x, u) + t(z, v)) - f_I(x, u)) = \lim_{t \to 0} t^{-1}(f((x, u) + t(z, v)) - f(x, u)) = df((x, u), (z, v))
\]
exists in \( \prod_{i \in J} F_i \) and thus in \( F \); its \( i \)th component is
\[
d f_i((x, u), (z, v)) = d_1 f_i(x, u, z) + d_2 f_i(x, u, v)
\]
32
in terms of partial differentials. Note that the mappings \( g_i : (V \times Z) \times (E_i \times E_i) \to F_i, \) \((x, z, u_i, v_i) \mapsto d_1f_i(x, u_i, z) \) and \( h_i : (V \times Z) \times (E_i \times E_i) \to F_i, \) \((x, z, u_i, v_i) \mapsto d_2f_i(x, u_i, v_i) = f_i(x, v_i) \) are \( C^r_{\mathbb{K}} \) and linear in \((u_i, v_i)\). By induction, the mappings

\[
g : (V \times Z) \times (E \times E) \to F, \quad (x, z, (u_i)_{i \in I}, (v_i)_{i \in I}) \mapsto (g_i(x, z, u_i, v_i))_{i \in I} \quad \text{and}
\]

\[
h : (V \times Z) \times (E \times E) \to F, \quad (x, z, (u_i)_{i \in I}, (v_i)_{i \in I}) \mapsto (h_i(x, z, u_i, v_i))_{i \in I}
\]

are \( C^r_{\mathbb{K}} \), using that \( E \times E \cong \bigoplus_{i \in I} (E_i \times E_i) \). Hence also \( df : (V \times E) \times (Z \times E) \to F \) is \( C^r_{\mathbb{K}} \), as \( df((x, u), (z, v)) = g(x, z, u, v) + h(x, z, u, v) \). Since \( df \) exists and is \( C^r_{\mathbb{K}} \), the continuous map \( f \) is \( C^r_{\mathbb{K}} \).

**Remark 8.4** The conclusion of Lemma 8.3 does not hold for \((r, \mathbb{K}) = (\omega, \mathbb{R})\) in the example \( I = \mathbb{N}, V = Z = \mathbb{R}, E_k = \mathbb{R}, f_k(r, t) := \frac{t}{1 + k^2t^2} \), using that the Taylor series of \( f_k(\cdot, t) \) around 0 has radius of convergence \( \frac{1}{\sqrt{k}} \) for all \( t \in \mathbb{R} \setminus \{0\} \) (cf. [14], Remark 3.14).

**8.5** Assuming now \( r \neq \omega \), let \((E_i)_{i \in I}\) be a family of equivariant \( \mathbb{L}\)-vector bundles \( \pi_i : E_i \to M \) of class \( C^r_{\mathbb{K}} \) with typical fibre \( F_i \) and \( G\)-action \( \beta_i : G \times E_i \to E_i \). We assume that (a) or (b) is satisfied:

(a) \( G \) and \( M \) are finite dimensional; or

(b) \( I \) is countable and each \( F_i \) as well as the modeling spaces of \( G \) and \( M \) are \( k_\omega \)-spaces.

Moreover, we assume:

(c) For each \( x \in M \), there exists an open neighbourhood \( U \) of \( x \) in \( M \), such that, for each \( i \in I \), the vector bundle \( E_i \) admits a local trivialization \( \psi_i : (\pi_i)^{-1}(U) \to U \times F_i \).

Thus, the \( C^r_{\mathbb{K}} \)-vector bundle \( E|_U \) is trivializable for each \( i \in I \). Define \( E := \bigcup_{x \in M} \bigoplus_{i \in I} (E_i)_x \) with pairwise disjoint direct sums and \( \pi : E \to M, \bigoplus_{i \in I} (E_i)_x \ni v \mapsto x \). Then

\[
\beta : G \times E \to E, \quad (g, (v_i)_{i \in I}) \mapsto (\beta_i(g, v_i))_{i \in I}
\]

is a \( G \)-action such that \( \beta(g, \cdot)|_{E_x} : E_x \to E_{\alpha(g, x)} \) is \( \mathbb{L}\)-linear for all \((g, x) \in G \times M\), where \( E_x := \pi^{-1}(\{x\}) \). We readily deduce from Proposition 5.3 and Proposition 8.3 that there is a unique \( \mathbb{L}\)-vector bundle structure of class \( C^r_{\mathbb{K}} \) on \( E \) such that

\[
\pi^{-1}(U) \to U \times \bigoplus_{i \in I} F_i, \quad E_x \ni (v_i)_{i \in I} \mapsto (x, (pr_{F_i}(\psi_i(v_i)))_{i \in I})
\]

is a local trivialization for \( E \), for each family \((\psi_i)_{i \in I}\) of local trivializations as above. The latter makes \( E \) an equivariant \( \mathbb{L}\)-vector bundle of class \( C^r_{\mathbb{K}} \). In fact, the \( C^r_{\mathbb{K}} \)-property of \( \beta \) can be checked using pairs of local trivializations, as in the proofs of Propositions 6.2, 7.6 and 7.8. Then apply Proposition 8.3 with \( F_i \) in place of \( E_i \) and \( Y \times Z \) in place of \( Z \). Thus:
Proposition 8.6 In the situation of 8.5, $\bigoplus_{i \in I} E_i$ is an equivariant $\mathbb{L}$-vector bundle of class $C^r_K$ over $M$. □

Remark 8.7 If $M$ is a $C^r_K$-manifold, then every $x \in M$ has an open neighbourhood $U$ which is $C^r_K$-diffeomorphic to a convex open subset $W$ in the modeling space $Z$ of $M$. If $W$ can be chosen $C^r_K$-paracompact, then every $C^r_K$-vector bundle over $U$ is trivializable (see [23, Corollary 15.10]). The latter condition is satisfied, for example, if $Z$ is finite dimensional, a Hilbert space, or a countable direct limit of finite-dimensional vector spaces (cf. [39, Corollary 16.16] and [17, Proposition 3.6]). If $(r, K) = (\infty, \mathbb{C})$ and $Z$ has finite dimension, then each finite-dimensional holomorphic vector bundle over a, say, polycylinder in $Z$ is $C^\infty_C$-trivializable (cf. [31]). Under suitable hypotheses, holomorphic Banach vector bundles over contractible bases are $C^\infty_C$-trivializable as well [49].

9 Dual bundles and cotangent bundles

In this section, we discuss conditions ensuring that a vector bundle has a canonical dual bundle. Let $L \in \{\mathbb{R}, \mathbb{C}\}$, $K \in \{\mathbb{R}, L\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, and $M$ be a $C^r_K$-manifold modeled on a locally convex space $Z$.

Definition 9.1 Let $\pi: E \to M$ be an $\mathbb{L}$-vector bundle of class $C^r_K$, with typical fibre $F$. Consider the disjoint union

$$E' := \bigcup_{x \in M} (E_x)' ;$$

let $p: E' \to M$ be the map taking $\lambda \in (E_x)'$ to $x$, for each $x \in M$. Given $t \in \mathbb{N}_0 \cup \{\infty, \omega\}$ such that $t \leq r$, we say that $E$ has a canonical dual bundle of class $C^t_K$ with respect to $S \in \{b, c\}$ if $E'$ can be made an $\mathbb{L}$-vector bundle of class $C^t_K$ over $M$, with typical fibre $F'_S$ and bundle projection $p$, such that

$$\tilde{\psi}: p^{-1}(U) \to U \times F'_S , \quad (E')_x = (E_x)' \ni \lambda \mapsto (x, ((\text{pr}_F \circ \psi|_{E_x})^{-1})(\lambda)) \quad (13)$$

is a local trivialization of $E'$, for each local trivialization $\psi: \pi^{-1}(U) \to U \times F$ of $E$.

To pinpoint situations where the dual bundle exists, we recall a fact concerning the formation of dual linear maps (see [20, Proposition 16.30]):

Lemma 9.2 Let $E$ and $F$ be locally convex spaces, and $S \in \{b, c\}$. If the evaluation homomorphism $\eta_{F,S}: F \to (F'_S)'_S$, $\eta_{F,S}(x)(\lambda) := \lambda(x)$ is continuous, then

$$\Theta: L(E,F)_S \to L(F'_S, E'_S)_S , \quad \alpha \mapsto \alpha'$$

is a continuous linear map. □

Remark 9.3 Let $F$ be a locally convex $K$-vector space over $K \in \{\mathbb{R}, \mathbb{C}\}$. It is known that $\eta_{F,b}$ is continuous if and only if $F$ is quasi-barrelled, i.e., every bornivorous barrel in $F$
is a 0-neighbourhood [36 Proposition 2 in Section 11.2]. In particular, \( \eta_{F,b} \) is continuous if \( F \) is bornological or barrelled. It is also known that \( \eta_{F,c} \) is continuous (and actually a topological embedding) if \( F \) is a \( k_\mathbb{R} \)-space. If \( \mathbb{K} = \mathbb{R} \), this follows from [10 Theorem 2.3] and [3 Lemma 14.3] (cf. also [3, Propositions 2.3 and 2.4]). If \( \mathbb{K} = \mathbb{C} \) and \( F \) is a \( k_\mathbb{R} \)-space, then \( \eta_{F,b,c} \) is a topological embedding for the real topological vector space \( \tilde{F} \mathbb{R} \) underlying \( F \).

Now \((\tilde{F} \mathbb{R})') = (F') \) as a real topological vector spaces, using that a continuous \( \mathbb{C} \)-linear functional \( \lambda: F \to \mathbb{C} \) is determined by its real part. Transporting the complex vector space structure from \( \tilde{F} \mathbb{C} \) to \( (F') \)′, the latter becomes a complex locally convex space. Thus \((\tilde{F} \mathbb{R})') \) can be identified with \((F')', \) and it is easy to verify that \( \eta_{F,b} \) corresponds to \( \eta_{F,b,c} \) if we make the latter identification.

**Proposition 9.4** Let \( \pi: E \to M \) be an \( \mathbb{L} \)-vector bundle of class \( C^r_\mathbb{K} \), with typical fibre \( F \).

Let \( S \in \{b, c\} \). If \( S = c \), let \( r_\pi := r \); if \( S = b \), assume \( r \geq 1 \) and set \( r_\pi := r - 1 \). Consider the following conditions:

1. **(α)** The modeling space \( Z \) of \( M \) is finite dimensional, \( \eta_{F,S} \) is continuous, and \( F^{\prime}_S \) is barreled\(^7\).
2. **(β)** \( \eta_{F,S} \) is continuous and, moreover, \((Z \times F^{\prime}_S) \times (Z \times F^{\prime}_S) \) is a \( k_\mathbb{R} \)-space, or \( r_\pi = 0 \) and \((Z \times F^{\prime}_S) \times (Z \times F^{\prime}_S) \) is a \( k_\mathbb{R} \)-space, or \((r, \mathbb{K}) = (\infty, \mathbb{C}) \) and \((Z \times F^{\prime}_S) \times (Z \times F^{\prime}_S) \) is a \( k_\mathbb{R} \)-space.

**Proof.** Let \( E' \) be the disjoint union \( \bigcup_{x \in M} (E_x)' \), and \( p: E' \to M \) be as in Definition 5.1. Let \((\psi_i)_{i \in I} \) be a family such that the \( \psi_i: \pi^{-1}(U_i) \to U_i \times F \) form the set of all local trivializations of \( E \). Let \((g_{ij})_{i,j \in I} \) be the associated cocycle (see 5.2). Then \( G_{ij} := g_{ij}^\psi \) is \( C^r_\mathbb{K} \) and hence \( g_{ij} = (G_{ij})' \) is \( C^r_\mathbb{K} \) by Proposition 2.1. Given \( i \in I \), we define \( \tilde{\psi}_i: p^{-1}(U_i) \to U_i \times F^{\prime}_S \) as in (13), using \( \psi_i \) instead of \( \psi \).

Then

\[
\tilde{\psi}_i(\tilde{\psi}_j^{-1}(x, \lambda)) = (x, ((\text{pr}_F \circ \psi_i|_{E_x})^{-1})' \circ (\text{pr}_F \circ \psi_j|_{E_x})'(\lambda)) = (x, (\text{pr}_F \circ \psi_j|_{E_x} \circ (\text{pr}_F \circ \psi_i|_{E_x})^{-1})'(\lambda)) = (x, g_{ij}(x)'(\lambda))
\]

for all \( x \in U_i \cap U_j \) and \( \lambda \in F' \) shows that

\[
(\tilde{\psi}_i \circ \tilde{\psi}_j^{-1})(x, \lambda) = (x, h_{ij}(x)(\lambda))
\]

where \( h_{ij}(x) := g_{ij}(x)' \in \text{GL}(F^{\prime}_S) \). If **(α)** or **(β)** holds, then \( \eta_{F,S}: F \to (F^{\prime}_S)' \) is continuous by hypothesis. If \( S = b \) and **(γ)** holds, then \( \eta_{F,b} \) is an isometric embedding (as is well

\(^7\)Example for \( S = b \): if \( F \) is a reflexive locally convex space, then \( \eta_{F,b} \) is continuous and \( F^{\prime}_b \) is barreled, being reflexive.
known) and hence continuous. Thus \( \Theta: L(F)_S \rightarrow L(F'_S)_S, \alpha \mapsto \alpha' \) is a continuous \( \mathbb{L} \)-linear map (Lemma 3.2). Since \( g_{ji}: U_i \cap U_j \rightarrow L(F)_S \) is \( C^r \), we deduce that \( h_{ij} = \Theta \circ g_{ji}: U_i \cap U_j \rightarrow L(F'_S)_S \) is \( C^r \). Thus Condition (g)’ of Corollary 5.5 is satisfied, with \( r_- \) in place of \( r \). Conditions (a)–(f) being apparent, the cited corollary provides an \( \mathbb{L} \)-vector bundle structure of class \( C^r \) on \( E' \).

Without specific hypotheses, a canonical dual bundle need not exist.

**Example 9.5** Let \( A \) be a unital, associative, locally convex topological \( \mathbb{K} \)-algebra whose group of units \( A^\times \) is open in \( A \), and such that the inversion map \( \iota: A^\times \rightarrow A^\times \) is continuous. Then \( \iota \) is smooth (and indeed \( \mathbb{K} \)-analytic), see [15]. We assume that the locally convex space underlying \( A \) is a non-normable Fréchet-Schwartz space and hence Montel, ensuring that \( L(A)_h = L(A)_c \). For example, we might take \( A := C^\infty(K, \mathbb{K}) \), where \( K \) is a connected, compact smooth manifold of positive dimension (cf. [15]). Let \( r, t \in \mathbb{N}_0 \cup \{\infty, \omega\} \) with \( t \leq r \) and \( S \in \{b, c\} \). We consider the trivial vector bundle

\[
pr_1: E := A^\times \times A \rightarrow A^\times.
\]

(Thus \( E \cong TA^\times \), the tangent bundle). Then \( E \) is a \( \mathbb{K} \)-vector bundle of class \( C^r_\mathbb{K} \) over the base \( A^\times \), with typical fibre \( A \). Both \( \psi_1 := id: A^\times \times A \rightarrow A^\times \times A \) and \( \psi_2: A^\times \times A \rightarrow A^\times \times A \), \( (a, v) \mapsto (a, av) \) are global trivializations of \( E \). Identifying \( E' := \bigcup_{a \in A^\times} (E_a)' \) with the set \( A^\times \times A' \), we consider the associated bijections \( \tilde{\psi}_i: E' = A^\times \times A' \rightarrow A^\times \times A' \) for \( i \in \{1, 2\} \) (cf. [13]). Thus \( \tilde{\psi}_1 = id \), and \( \tilde{\psi}_2(a, \lambda) = (a, \lambda(a^{-1})) \) for \( a \in A^\times, \lambda \in A' \). The map \( G_{ij}: A^\times \times A \rightarrow A, (a, v) \mapsto pr_2(\psi_i(\psi_j^{-1}(a, v))) \) is \( C^r_\mathbb{K} \) for \( i, j \in \{1, 2\} \), where \( pr_2: A^\times \times A \rightarrow A \) is the projection onto the second factor. Then also \( g_{ij}: A^\times \rightarrow L(A)_c = L(A)_h, a \mapsto G_{ij}(a, \cdot) \) is \( C^r_\mathbb{K} \), by Proposition 2.1(a). Now, \( A \) being Fréchet and thus barrelled, the evaluation homomorphism \( \eta_{A,h} \) is continuous; since \( A \) is metrizable and hence a \( k \)-space, also \( \eta_{A,c} \) is continuous (see Remark 9.3). Since \( g_{ij} \) is \( C^r_\mathbb{K} \), we deduce with Lemma 9.2 that also \( h_{ij}: A^\times \rightarrow L(A'_S)_S, a \mapsto (g_{ij}(a))' \) is \( C^r_\mathbb{K} \). Define

\[
h_{ij}: A^\times \times A'_S \rightarrow A'_S, \quad (a, \lambda) \mapsto h_{ij}(a)(\lambda)
\]

for \( i, j \in \{1, 2\} \). Then \( H_{12} \) is discontinuous. To see this, we compose \( H_{12} \) with the map \( ev_1: A'_b \rightarrow \mathbb{K}, \lambda \mapsto \lambda(1) \) which evaluates functionals at the identity element \( 1 \in A \), and recall that \( ev_1 \) is continuous. Then \( ev_1(H_{12}(a, \lambda)) = \lambda(g_{21}(a)(1)) = \lambda(a) \) for \( a \in A^\times \) and \( \lambda \in A' \). However, \( A \) being a non-normable locally convex space, the bilinear, separately continuous evaluation map \( \varepsilon: A \times A' \rightarrow \mathbb{K}, (a, \lambda) \mapsto \lambda(a) \) is discontinuous, and hence so is its restriction \( \varepsilon|_{A^\times \times A'_h} = ev_1 \circ H_{12} \) to the non-empty open subset \( A^\times \times A'_h \), as is readily verified. Now \( ev_1 \circ H_{12} \) being discontinuous, also \( H_{12} \) is discontinuous (and therefore not \( C^r_\mathbb{K} \)). As a consequence, also \( \tilde{\psi}_1 \circ \tilde{\psi}_2^{-1} = (pr_1, H_{12}) \) is discontinuous. Summing up:

There is no canonical vector bundle structure of class \( C^r_\mathbb{K} \) on \( E' \) because the two vector bundle structures on \( E' \) making \( \tilde{\psi}_1 \) (resp., \( \tilde{\psi}_2 \)) a global trivialization do not coincide.
Remark 9.6 In the preceding situation, set $M := A^x$, $F := A_b^x$, $I := \{1, 2\}$, $U_i := M$ for $i \in I$, and $\pi := \text{pr}_i : M \times F \to M$. If we let $M \times A_b^x$ play the role of $E$ in Proposition 5.3 and $\tilde{\psi}_i : \pi^{-1}(U_i) \to U_i \times F$ the role of $\psi_i$ in Proposition 5.3(e), then all of Conditions (a)–(f) of Proposition 5.3 and Condition (g)' of Corollary 5.5 are satisfied for $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ (with $\mathbb{L} := \mathbb{K}$). However, there is no $C^r_\mathbb{K}$-vector bundle structure on $M \times F$ making each $\tilde{\psi}_i$ a trivialization, as just observed, i.e., the conclusion of Corollary 5.5 becomes false.

Remark 9.7 Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, $t \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $t \leq r$ and $M$ be a $C^r_\mathbb{K}$-manifold modeled on a locally convex space $Z$. Then the tangent bundle $TM$ is a $\mathbb{K}$-vector bundle of class $C^{r-1}_\mathbb{K}$ over $M$, with typical fibre $Z$. Pick a locally convex topology $\mathcal{T}$ on $Z'$. Let $\mathcal{A}$ be the set of all maps $\tilde{\psi}$ as in (13), with $(Z', \mathcal{T})$ in place of $F_S'$, for $\tilde{\psi}$ ranging through the set of all local trivializations of $TM$ (alternatively, only those of the form $(\pi_{TU}, \phi)$ for charts $\phi : U \to V \subseteq Z$ of $M$, using the bundle projection $\pi_{TU} : TU \to U$). Let us say that $M$ has a canonical cotangent bundle of class $C^r_{\mathbb{K}}$ with respect to $\mathcal{T}$ if $T'M := \bigcup_{x \in M}(T_xM)'$ admits a $\mathbb{K}$-vector bundle structure of class $C^r_{\mathbb{K}}$ over $M$ with typical fibre $(Z', \mathcal{T})$, which makes each $\tilde{\psi} : p^{-1}(U) \to U \times (Z', \mathcal{T})$ a local trivialization (with $p : T'M \to M$, $(T_xM)' \ni \lambda \mapsto x$). Then the evaluation map

$$\varepsilon : (Z', \mathcal{T}) \times Z \to \mathbb{K}, \quad (\lambda, x) \mapsto \lambda(x)$$

must be continuous and hence $Z$ normable. For $\mathbb{K} = \mathbb{R}$, this is explained in [43, Remark 1.3.9] (written after Example 9.5 was found) if $r = \infty$. This implies the case $r \in \mathbb{N}$. As the diffeomorphism $\tilde{f}$ employed as a change of charts is real analytic, the case $(\omega, \mathbb{R})$ follows and also the complex case, using a $\mathbb{C}$-analytic extension of $\tilde{f}$. When $\mathcal{T}$ is the compact-open topology, existence of a canonical cotangent bundle for $M$ even implies that $Z$ is finite dimensional.\(^8\)

Cotangent bundles are not needed to define 1-forms on an infinite-dimensional manifold $M$. Following [3], these can be considered as smooth maps on $TM$ which are linear on the fibres (and a similar remark applies to differential forms of higher order).

**Differentiability properties of the $G$-action on the dual bundle**

Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $r \leq s$, and $G$ be a $C^s_{\mathbb{K}}$-Lie group modeled on a locally convex $\mathbb{K}$-vector space $Y$. Let $M$ be a $C^r_{\mathbb{K}}$-manifold modeled on a locally convex $\mathbb{K}$-vector space $Z$ and $\alpha : G \times M \to M$ be a $G$-action of class $C^r_{\mathbb{K}}$.

**Proposition 9.8** Let $\pi : E \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, with typical fibre $F$ and $G$-action $\beta : G \times E \to E$ of class $C^r_{\mathbb{K}}$. Let $S \in \{b, c\}$. If $S = c$, set $r_- := r$; if $S = b$, assume $r \geq 1$ and set $r_- := r - 1$. Consider the following conditions:

---

\(^8\)If $\varepsilon$ is continuous on $Z' \times Z$, then there exists a compact subset $K \subseteq Z$ and a 0-neighbourhood $W \subseteq Z$ such that $\varepsilon((K^\circ \times W) \subseteq D$. Hence $K^\circ \subseteq W^\circ$. Since $K^\circ$ is a 0-neighbourhood in $Z'^\circ$ and $W^\circ$ compact (by Ascoli’s Theorem), $Z'^\circ$ is locally compact and hence finite dimensional. As $Z'^\circ$ separates points on $Z$, also $Z$ must be finite dimensional.
(a) \( \eta_{F,S} \) is continuous, and, moreover, \((Y \times Z \times F'_S) \times (Y \times Z \times F'_S)\) is a \( k_\mathbb{R} \)-space, or \( r_- = 0 \) and \( Y \times Z \times F'_S \) is a \( k_\mathbb{R} \)-space, or \( (r, \mathbb{K}) = (\infty, \mathbb{C}) \) and \( Y \times Z \times F'_S \) is a \( k_\mathbb{R} \)-space;

(b) \( M \) and \( G \) are finite dimensional, \( \eta_{F,S} \) is continuous, and \( F'_S \) is barrelled; or

(c) \( F \) is normable.

If \( S = c \) and (a) or (b) holds, then \( E \) has a canonical dual bundle \( E' \) of class \( C^r_\mathbb{R} \) with respect to \( S \), and the map \( \beta^*: G \times E' \to E' \), defined using adjoint linear maps via

\[
\beta^*(g, \lambda) := (\beta(g^{-1}, \cdot)|_{E'_x})^t(\lambda)
\]

for \( g \in G, \lambda \in (E_x)' \), turns \( E' \) into an equivariant \( \mathbb{L} \)-vector bundle of class \( C^r_\mathbb{R} \) over the \( G \)-manifold \( M \). If \( S = b \) and (a), (b), or (c) is satisfied, then the same conclusion holds.

**Proof.** In view of Proposition 4.1, the hypotheses imply that \( E \) has a canonical dual bundle \( p: E' \to M \) of class \( C^r_\mathbb{R} \). It is apparent that \( \beta^*: G \times E' \to E' \) is an action, and \( E'_x \) is taken \( \mathbb{L} \)-linearly to \( E'_x \) by \( \beta^*(g, \cdot) \), for each \( g \in G \) and \( x \in M \). It therefore only remains to show that \( \beta^* \) is \( C^r_\mathbb{R} \). To this end, let \( g_0 \in G \) and \( x_0 \in M \); we show that \( \beta^* \) is \( C^r_\mathbb{R} \) on \( U \times p^{-1}(V) \), for some open neighbourhood \( U \) of \( g_0 \) in \( G \) and an open neighbourhood \( V \) of \( x_0 \) in \( M \). Indeed, there exists a local trivialization \( \psi: \pi^{-1}(W) \to W \times F \) of \( E \) over an open neighbourhood \( W \) of \( \alpha(g_0, x_0) \) in \( M \). The action \( \alpha \) being continuous, we find an open neighbourhood \( U \) of \( g_0 \) in \( G \) and an open neighbourhood \( V \) of \( x_0 \) in \( M \) over which \( E \) is trivial, such that \( \alpha(U \times V) \subseteq W \). Let \( \phi: \pi^{-1}(V) \to V \times F \) be a local trivialization of \( E \) over \( V \). Then

\[
\phi(\beta^{-1}(\psi^{-1}(\alpha(g, x), v))) = (x, A(g, x, v)) \quad \text{for all } g \in U, x \in V, \text{ and } v \in F,
\]

for a \( C^r_\mathbb{K} \)-map \( A: U \times V \times F \to F \) which is \( \mathbb{L} \)-linear in the third argument. By Corollary 2.1 the map \( a: U \times V \to L(F)_S, (g, x) \mapsto A(g, x, \cdot) \) is \( C^r_\mathbb{R} \). In view of the hypotheses, Lemmas 9.2 and 9.3 entail that also \( a^*: U \times V \to L(F'_S)_S, (g, x) \mapsto (a(g, x))^t \) is \( C^r_\mathbb{K} \)-map. Now, again using the specific hypotheses, Proposition 4.1 shows that also the mapping \( A^*: U \times V \times F'_S \to F'_S, (g, x, \lambda) \mapsto a^*(g, x)(\lambda) \) is \( C^r_\mathbb{R} \). However, for \( g \in U, x \in V, \) and \( \lambda \in F' \), we calculate

\[
\bar{\psi}(\beta^*(g, \bar{\phi}^{-1}(x, \lambda))) = \left( \alpha(g, x), \left( \left( \left. \left. \right| \right. \right)_{\alpha(g, x)}^\prime(\lambda) \right) \right)
\]

using notation as in (13). We conclude that \( \beta^*|_{U \times p^{-1}(V)} \) is \( C^r_\mathbb{R} \).

**Example 9.9** For elementary examples, recall that the group \( \text{Diff}(M) \) of all smooth diffeomorphisms of a \( \sigma \)-compact, finite-dimensional smooth manifold \( M \) can be made a smooth
Lie group, modeled on the \((\text{LF})\)-space \(\Gamma_c(TM)\) of compactly supported smooth vector fields on \(M\) (see [11, 26, 30]). The natural action \(\text{Diff}(M) \times M \to M\) is smooth [26]. In view of Example 5.7, Proposition 9.8(b), Proposition 7.6 and Proposition 5.8, we readily deduce that also the natural action of \(\text{Diff}(M)\) on \(TM\) is smooth, as well as the natural actions on \(T^*M := (TM)'\), \(TM^\otimes n \otimes (T^*M)^\otimes m\) for all \(n, m \in \mathbb{N}_0\), and the natural action on the subbundles \(S^n(T^*M)\) and \(\bigwedge^n T^*M\) of \((T^*M)^\otimes\) given by symmetric and exterior powers, respectively.

As a consequence, also the natural actions of \(\text{Diff}(M)\) on the associated spaces of smooth (or smooth compactly supported) sections are smooth (see [26]). For general results ensuring smoothness of the action on the space of smooth sections in a \(G\)-equivariant vector bundle, see [26, Proposition 7.4].

10 Locally convex Poisson vector spaces

We discuss a slight generalization of the concept of a locally convex Poisson vector space introduced in [20]. Fix \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\).

10.1 A bounded set-functor \(\mathcal{S}\) associates with each locally convex \(\mathbb{K}\)-vector space \(E\) a set \(\mathcal{S}(E)\) of bounded subsets of \(E\), such that \(\{\lambda(M) : M \in \mathcal{S}(E)\} \subseteq \mathcal{S}(F)\) for each continuous \(\mathbb{K}\)-linear map \(\lambda : E \to F\) between locally convex \(\mathbb{K}\)-vector spaces (cf. [20, Definition 16.15]).

Given locally convex \(\mathbb{K}\)-vector spaces \(E\) and \(F\), we shall write \(L(E, F)_\mathcal{S}\) as a shorthand for \(L_{\mathbb{K}}(E, F)_{\mathcal{S}(E)}\). We write \(E'_\mathcal{S} := L_{\mathbb{K}}(E, \mathbb{K})_{\mathcal{S}}\).

10.2 Throughout this section, we let \(\mathcal{S}\) be a bounded set-functor such that, for each locally convex space \(E\), we have \(\{K \subseteq E : K\text{ is compact}\} \subseteq \mathcal{S}(E)\).

Then \(\{x\} \in \mathcal{S}(E)\) for each \(x \in E\), entailing we get a continuous linear point evaluation

\[\eta_{E, \mathcal{S}}(x) : E'_\mathcal{S} \to \mathbb{K}, \quad \lambda \mapsto \lambda(x).\]

**Definition 10.3** A locally convex Poisson vector space with respect to \(\mathcal{S}\) is a locally convex \(\mathbb{K}\)-vector space \(E\) such that \(E \times E\) is a \(\mathbb{K}\)-space and

\[\eta_{E, \mathcal{S}} : E \to (E'_\mathcal{S})'_\mathcal{S}, \quad x \mapsto \eta_{E, \mathcal{S}}(x)\]

a topological embedding, together with a bilinear map \([, ,] : E'_\mathcal{S} \times E'_\mathcal{S} \to E'_\mathcal{S}, (\lambda, \eta) \mapsto [\lambda, \eta]\) which makes \(E'_\mathcal{S}\) a Lie algebra, is \(\mathcal{S}(E'_\mathcal{S})\)-hypocontinuous in its 2nd argument, and satisfies

\[\eta_{E, \mathcal{S}}(x) \circ \text{ad}_\lambda \in \eta_{E, \mathcal{S}}(E) \quad \text{for all } x \in E\text{ and } \lambda \in E', \tag{14}\]

writing \(\text{ad}_\lambda := \text{ad}(\lambda) := [\lambda, \cdot] : E' \to E'\).
Remark 10.4  (a) Definition 16.35 in [20] was more restrictive; $E$ was assumed to be a $k^\infty$-space there.

(b) In [20, 16.31 (b)], the following additional condition was imposed: For each $M \in \mathcal{S}(E'_S)$ and $N \in \mathcal{S}(E)$, the set $\varepsilon(M \times N)$ is bounded in $\mathbb{K}$, where $\varepsilon: E' \times E \to \mathbb{K}$ is the evaluation map. As we assume [10.2], the latter condition is automatically satisfied, by [20, Proposition 16.11 (a) and Proposition 16.14].

(c) Let us say that a locally convex space $E$ is $\mathcal{S}$-reflexive if $\eta_{E,\mathcal{S}}: E \to (E'_S)'_\mathcal{S}$ is an isomorphism of topological vector spaces.

(d) Of course, we are mostly interested in the case where $[,]$ is continuous, but only hypocontinuity is required for the basic theory.

Definition 10.5 Let $(E, [\,,\,])$ be a locally convex Poisson vector space with respect to $\mathcal{S}$, and $U \subset E$ be open. Given $f, g \in C^\infty_{\mathbb{K}}(U, \mathbb{K})$, we define a function $\{f, g\}: U \to \mathbb{K}$ via

$$\{f, g\}(x) := \langle [f'(x), g'(x)], x \rangle \quad \text{for } x \in U,$$

where $\langle \,,\, \rangle: E' \times E \to \mathbb{K}$, $\langle \lambda, x \rangle := \lambda(x)$ is the evaluation map and $f'(x) = df(x, \, \cdot \,)$. Condition (14) in Definition 10.3 enables us to define a map $X_f: U \to E$ via

$$X_f(x) := \eta_{E,S}^{-1}(\eta_{E,S}(x) \circ \text{ad}(f'(x))) \quad \text{for } x \in U.$$

10.6 Using Lemma 3.1 instead of [20, Theorem 16.26], we see as in the proof of [20, Theorem 16.40 (a)] that the function $\{f, g\}: U \to \mathbb{K}$ is $C^\infty_{\mathbb{K}}$. The $C^\infty_{\mathbb{K}}$-function $\{f, g\}$ is called the Poisson bracket of $f$ and $g$. Using Lemma 3.1 instead of [20, Theorem 16.26], we see as in the proof of [20, Theorem 16.40 (b)] that $X_f: U \to E$ is a $C^\infty_{\mathbb{K}}$-map; it is called the Hamiltonian vector field associated with $f$. As in [20, Remark 16.43], we see that the Poisson bracket just defined makes $C^\infty_{\mathbb{K}}(U, \mathbb{K})$ a Poisson algebra.

10.7 We shall write “$b$” and “$c$” in place of $\mathcal{S}$ if $\mathcal{S}$ is the bounded set functor taking a locally convex space $E$ to the set $\mathcal{S}(E)$ of all bounded subsets and compact subsets of $E$, respectively. Both of these satisfy the hypothesis of 10.2.

In the following, we describe new results for locally convex Poisson vector spaces over $\mathcal{S} = c$. We mention that the embedding property of $\eta_{E,c}$ is automatic in this case, as $E \times E$ is a $k_{\mathbb{R}}$-space in Definition 10.5; thus $E$ is a $k_{\mathbb{R}}$-space and Remark 9.3 applies.

Example 10.8 Let $(g_j)_{j \in J}$ be a family of finite-dimensional real Lie algebras $g_j$. Endow $g := \bigoplus_{j \in J} g_j$ with the locally convex direct sum topology, which coincides with the finest locally convex vector topology. Then $g$ is $c$-reflexive, like every vector space with its finest locally convex vector topology (see [33, Theorem 7.30 (a)]). As a consequence, also $g'_c$ is $c$-reflexive (cf. [33, Proposition 7.9 (iii)]). Using [16, Proposition 7.1], we see that the component-wise Lie bracket $g \times g \to g$ is continuous on the locally convex space $g \times g$, which
is naturally isomorphic to the locally convex direct sum $\bigoplus_{j \in J} (\mathfrak{g}_j \times \mathfrak{g}_j)$. We set $E := \mathfrak{g}_c'$ and give $E_c'$ the continuous Lie bracket $[,]$ making $\eta_{\mathfrak{g},c} : \mathfrak{g} \to (\mathfrak{g}_c')_c = E_c'$ an isomorphism of topological Lie algebras. Then

$$E = \mathfrak{g}_c' \cong \prod_{j \in J} (\mathfrak{g}_j)_c'$$

is a $k_\mathbb{K}$-space, being a cartesian product of locally compact spaces (see [45] or [29]). Thus $(E, [\cdot, \cdot])$ is a locally convex Poisson vector space over $\mathcal{S} = c$, in the sense of Definition 10.3. If $J$ has cardinality $\geq 2^{\aleph_0}$ and $\mathfrak{g}_j \neq \{0\}$ for all $j \in J$ (e.g., if we take an abelian 1-dimensional Lie algebra $\mathfrak{g}_j$ for each $j \in J$), then $E \cong \mathbb{R}^J$ is not a $k$-space. Hence $E$ is not a $k^\infty$-space, and hence it is not a Poisson vector space in the more restrictive sense of [20].

11 Continuity properties of the Poisson bracket

If $E$ and $F$ are locally convex $\mathbb{K}$-vector spaces and $U \subseteq E$ an open subset, we endow $C^\infty(U, F)$ with the compact-open $C^\infty$-topology. Our goal is the following result:

**Theorem 11.1** Let $(E, [\cdot, \cdot])$ be a locally convex Poisson vector space with respect to $\mathcal{S} = c$. Let $U \subseteq E$ be open. Then the Poisson bracket

$$\{\cdot, \cdot\} : C^\infty_{\mathbb{K}}(U, \mathbb{K}) \times C^\infty_{\mathbb{K}}(U, \mathbb{K}) \to C^\infty_{\mathbb{K}}(U, \mathbb{K})$$

is $c$-hypocontinuous in its second variable. If $[\cdot, \cdot] : E'_c \times E'_c \to E'_c$ is continuous, then also the Poisson bracket is continuous.

Various auxiliary results are needed to prove Theorem 11.1. With little risk of confusion with subsets of spaces of operators, given a 0-neighbourhood $W \subseteq F$ and a compact set $K \subseteq U$ we shall write $[K, W] := \{f \in C(U, F) : f(K) \subseteq W\}$.

**Lemma 11.2** Let $E, F$ be locally convex spaces and $U \subseteq E$ be open. Then the linear map

$$D : C^\infty_{\mathbb{K}}(U, F) \to C^\infty_{\mathbb{K}}(U, L(E, F)_c), \quad f \mapsto f'$$

is continuous.

**Proof.** By Corollary 2.2, $f' \in C^\infty_{\mathbb{K}}(U, L(E, F)_c)$ for each $f \in C^\infty_{\mathbb{K}}(U, F)$. As $D$ is linear and also $C^\infty(U, L(E, F)_c) \to C(U \times E^k, L(E, F)_c)$, $f \mapsto d^k f$ is linear for each $k \in \mathbb{N}_0$,

$$d^k \circ D : C^\infty(U, F) \to C(U \times E^k, L(E, F)_c)_{c.o.}$$

(17)

is linear, whence it will be continuous if it is continuous at 0. We pick a typical 0-neighbourhood in $C(U \times E^k, L(E, F)_c)_{c.o.}$, say $[K, V]$ with a compact subset $K \subseteq U \times E^k$ and a 0-neighbourhood $V \subseteq L(E, F)_c$. After shrinking $V$, we may assume that $V = [A, W]$ for some compact set $A \subseteq E$ and 0-neighbourhood $W \subseteq F$. 

41
We now recall that for \( f \in C^\infty_K(U, F) \), we have
\[
d^k(f')(x, y_1, \ldots, y_k) = d^{k+1} f(x, y_1, \ldots, y_k, \cdot): E \to F
\]
for all \( k \in \mathbb{N}_0, x \in U \) and \( y_1, \ldots, y_k \in E \) (cf. Corollary 2.2). Since \( [K \times A, W] \) is an open 0-neighbourhood in \( C(U \times E^{k+1}, F) \) and the map \( C^\infty(U, F) \to C(U \times E^{k+1}, F)_{c.o.}, f \mapsto d^{k+1} f \) is continuous, we see that the set \( \Omega \) of all \( f \in C^\infty(U, F) \) such that \( d^{k+1} f \in [K \times A, W] \) is a 0-neighbourhood in \( C^\infty(U, F) \). In view of (18), we have \( d^k(f') \in [K, [A, W]] \) for each \( f \in \Omega \). Hence \( d^k \circ D \) from (17) is continuous at 0, as required.

**Lemma 11.3** Let \( X \) be a Hausdorff topological space, \( F \) be a locally convex space, \( K \subseteq X \) be compact and \( M \subseteq C(X, F)_{c.o.} \) be compact. Let \( \text{ev}: C(X, F) \times X \to F, (f, x) \mapsto f(x) \) be the evaluation map. Then \( C(M \times K) \) is compact.

**Proof.** The map \( \rho: C(X, F)_{c.o.} \to C(K, F)_{c.o.}, f \mapsto f|_K \) is continuous by [10] §3.2 (2)]. Thus \( \rho(M) \) is compact in \( C(K, F)_{c.o.} \). The map \( \varepsilon: C(K, F) \times K \to F, (f, x) \mapsto f(x) \) is continuous by [10] Theorem 3.4.2]. Hence \( \text{ev}(M \times K) = \varepsilon(\rho(M) \times K) \) is compact.

**Lemma 11.4** Let \( E, F_1, F_2, \) and \( G \) be locally convex \( \mathbb{K} \)-vector spaces and \( \beta: F_1 \times F_2 \to G \) be a bilinear map which is c-hypocontinuous in its second argument. Let \( U \subseteq E \) be an open subset and \( r \in \mathbb{N}_0 \cup \{\infty\} \). Assume that \( E \times E \) is a \( k_\mathbb{K} \)-space, or \( r = 0 \) and \( E \) is a \( k_\mathbb{K} \)-space, or \( (r, \mathbb{K}) = (\infty, \mathbb{C}) \) and \( E \) is a \( k_\mathbb{K} \)-space. Then the following holds:

(a) We have \( \beta \circ (f, g) \in C^r_K(U, G) \) for all \( (f, g) \in C^r(U, F_1) \times C^r(U, F_2) \). The map
\[
C^r_K(U, \beta): C^r_K(U, F_1) \times C^r_K(U, F_2) \to C^r_K(U, G), \quad (f, g) \mapsto \beta \circ (f, g)
\]
is bilinear. For each compact subset \( M \subseteq C^r_K(U, F_2) \) and 0-neighbourhood \( W \subseteq C^r_K(U, G) \), there is a 0-neighbourhood \( V \subseteq C^r_K(U, F_1) \) such that \( C^r_K(U, \beta)(V \times M) \subseteq W \).

(b) For each \( g \in C^r_K(U, F_2) \), the map \( C^r_K(U, F_1) \to C^r_K(U, G), f \mapsto \beta \circ (f, g) \) is continuous and linear.

(c) If \( \beta \) is also c-hypocontinuous in its first argument, then \( C^r_K(U, \beta) \) is c-hypocontinuous in its second argument and c-hypocontinuous in its first argument.

(d) If \( \beta \) is continuous, then \( C^r_K(U, \beta) \) is continuous.

**Proof.** (a) By Lemma 3.1 \( \beta \circ (f, g) \in C^r_K(U, G) \). The bilinearity of \( C^r(U, \beta) \) is clear. It suffices to prove the remaining assertion for each \( r \in \mathbb{N}_0 \). To see this, let \( M \subseteq C^\infty_K(U, F_2) \) be a compact subset and \( W \subseteq C^\infty_K(U, G) \) be a 0-neighbourhood. Since the topology on \( C^\infty_K(U, G) \) is initial with respect to the maps inclusion map \( C^\infty(U, G) \to C^r_K(U, G) \) for \( r \in \mathbb{N}_0 \), there exists \( r \in \mathbb{N}_0 \) and a 0-neighbourhood \( Q \) in \( C^r_K(U, G) \) such that \( C^\infty_K(U, G) \cap Q \subseteq W \). If the assertion holds for \( r \), we find a 0-neighbourhood \( P \subseteq C^r_K(U, F_1) \) such that
\[ C^r_\mathbb{K}(U, \beta)(P \times M) \subseteq Q. \] Then \( V := C^\infty_\mathbb{K}(U, F_1) \cap P \) is a 0-neighbourhood in \( C^\infty_\mathbb{K}(U, F_1) \) and \( C^r_\mathbb{K}(U, \beta)(V \times M) \subseteq C^\infty_\mathbb{K}(U, G) \cap C^r_\mathbb{K}(U, \beta)(P \times M) \subseteq C^\infty_\mathbb{K}(U, G) \cap Q \subseteq W. \)

The case \( r = 0 \). Let \( M \subseteq C(U, F_2) \) be compact and \( W \subseteq C(U, G) \) be a 0-neighbourhood. Then \( [K, Q] \subseteq W \) for some compact subset \( K \subseteq U \) and some 0-neighbourhood \( Q \subseteq G \).

By Lemma 11.3, the set \( N := ev(M \times K) \subseteq F_2 \) is compact, where \( ev: C(U, F_2) \times U \to F_2 \) is the evaluation map. Since \( \beta \) is \( c \)-hypocontinuous in its second argument, there exists a 0-neighbourhood \( P \subseteq F_1 \) with \( \beta(P \times N) \subseteq Q \). Then \( \beta \circ ([K, P] \times M) \subseteq [K, Q] \subseteq W. \)

Induction step. Let \( M \subseteq C^r_\mathbb{K}(U, F_2) \) be a compact subset and \( W \subseteq C^r_\mathbb{K}(U, G) \) be a 0-neighbourhood. The topology on \( C^r(U, G) \) is initial with respect to the linear maps \( \lambda_1 : C^r_\mathbb{K}(U, G) \to C(U, G) \) \( f \mapsto f \) and \( \lambda_2 : C^r_\mathbb{K}(U, G) \to C^r(U \times E, G), f \mapsto df \) (by [22, Lemma A.1 (d)]). After shrinking \( W \), we may therefore assume that

\[ W = (\lambda_1)^{-1}(W_1) \cap (\lambda_2)^{-1}(W_2) \]

with absolutely convex 0-neighbourhoods \( W_1 \subseteq C(U, G) \) and \( W_2 \subseteq C^r_\mathbb{K}(U \times E, G) \). Applying the case \( r = 0 \) to \( C(U, \beta) \), we find a 0-neighbourhood \( V_1 \subseteq C(U, F_1) \) such that \( C(U, \beta)(V_1 \times M) \subseteq W_1 \). The map \( \delta_1 : C^r_\mathbb{K}(U, F_1) \to C^r(U \times E, F_1), f \mapsto df \) is continuous and \( \pi : U \times E \to U, (x, y) \mapsto x \) is smooth, whence \( \rho_1 : C^r_\mathbb{K}(U, F_1) \to C^r(U \times E, F_1), f \mapsto f \circ \pi \) is continuous (cf. [22, Lemma 4.4]). By (3), we have

\[ \lambda_2 \circ C^r_\mathbb{K}(U, \beta) = C^r_\mathbb{K}(U \times E, \beta) \circ (\delta_1 \circ \rho_2) + \lambda_1 \circ C^r_\mathbb{K}(U \times E, \beta) \circ (\delta_1 \circ \rho_2). \]

The subsets \( \rho_2(M) \subseteq C^r_\mathbb{K}(U \times E, F_2) \) and \( \delta_2(M) \subseteq C^r_\mathbb{K}(U \times E, F_2) \) are compact. Using the case \( r = 1 \) (with \( U \times E \) in place of \( U \)), which holds as the inductive hypothesis, we find 0-neighbourhoods \( V_2, V_3 \subseteq C^r_\mathbb{K}(U \times E, F_1) \) such that \( C^r_\mathbb{K}(U, \beta)(V_2 \times \rho_2(M)) \subseteq (1/2)W_2 \) and \( C^r_\mathbb{K}(U, \beta)(V_3 \times \delta_2(M)) \subseteq (1/2)W_2 \). Then \( Q := (\delta_1)^{-1}(V_2) \cap (\rho_1)^{-1}(V_3) \) is an open 0-neighbourhood in \( C^r_\mathbb{K}(U, F_1) \). Since \( (1/2)W_2 + (1/2)W_2 = W_2 \), we deduce from (19) that \( \lambda_2(C^r_\mathbb{K}(U, \beta)(Q \times M)) \subseteq C^r_\mathbb{K}(U \times E, \beta)(V_2 \times \rho_2(M)) + C^r_\mathbb{K}(U \times E, \beta)(V_3 \times \delta_2(M)) \subseteq W_2. \)

Thus \( C^r_\mathbb{K}(U, \beta)(Q \times M) \subseteq (\lambda_2)^{-1}(W_2) \). Now \( V := V_1 \cap Q \) is a 0-neighbourhood in \( C^r_\mathbb{K}(U, F_1) \) such that \( C^r_\mathbb{K}(U, \beta)(V \times M) \subseteq (\lambda_1)^{-1}(W_1) \cap (\lambda_2)^{-1}(W_2) = W. \)

(b) Since \( C^r_\mathbb{K}(U, \beta) \) is bilinear, the map \( f \mapsto \beta \circ (f, g) \) is linear. Its continuity follows from (a), applied with the singleton \( M := \{g\} \).

(c) By (a) just established, the condition in Lemma 11.15(a) is satisfied. By (b), the map \( C^r_\mathbb{K}(U, \beta) \) is continuous in its first argument. Interchanging the roles of \( F_1 \) and \( F_2 \), we see that \( C^r_\mathbb{K}(M, \beta) \) is also continuous in its second argument and hence \( c \)-hypocontinuous in its second argument. Likewise, \( C^r_\mathbb{K}(U, \beta) \) is \( c \)-hypocontinuous in its first argument.

(d) If \( \beta \) is continuous and hence smooth, then \( C^r(U, \beta) \) is smooth and hence continuous, as a very special case of [22, Proposition 4.16].

Proof of Theorem 11.1. By Lemma 11.2, the mapping \( D : C^\infty(U, \mathbb{K}) \to C^\infty(U, E'_2), f \mapsto f' \) is continuous and linear. By Lemma 11.3(c), the bilinear map

\[
C^\infty(U, [.,.]) : C^\infty(U, E') \times C^\infty(U, E') \to C^\infty(U, E'), (f, g) \mapsto (x \mapsto [f(x), g(x)])
\]

Note that the ordinary \( C^r \)-topology is used there, by [22, Proposition 4.19(d) and Lemma A2].
is $c$-hypocontinuous in its second argument; if $[\ldots]$ is continuous, then also $C^\infty(U,[\ldots])$, by Lemma 11.4(d). The evaluation map $\beta: E \times E'_c \rightarrow \mathbb{K}$, $(x, \lambda) \mapsto \lambda(x)$ is $c$-hypocontinuous in its first argument, by Proposition 11.19. As a consequence, $\beta_\lambda: C^\infty(U, E'_c) \rightarrow C^\infty(U, \mathbb{K})$, $f \mapsto \beta \circ (\text{id}_U, f)$ is continuous linear by Lemma 11.4(b). Since
\[
\{\ldots\} = \beta_* \circ C^\infty(U, [\ldots]) \circ (D \times D)
\]
by definition, we see that $\{\ldots\}$ is a composition of continuous maps if $[\ldots]$ is continuous, and hence continuous. In the general case, $\{\ldots\}$ is a composition of a bilinear map which is $c$-hypocontinuous in its second argument and continuous linear maps, whence $\{\ldots\}$ is $c$-hypocontinuous in its second argument.

\[\square\]

12 Continuity of the map taking $f$ to the Hamiltonian vector field $X_f$ 

In this section, we show continuity of the mapping which takes a smooth function to the corresponding Hamiltonian vector field, in the case $S = c$.

**Theorem 12.1** Let $(E, [\ldots])$ be a locally convex Poisson vector space with respect to $S = c$. Let $U \subseteq E$ be an open subset. Then the map
\[
\Psi: C^\infty_K(U, \mathbb{K}) \rightarrow C^\infty_K(U, E), \quad f \mapsto X_f
\]
(20)
is continuous and linear.

**Proof.** Let $\eta_E: E \rightarrow (E'_c)_c$ be the evaluation homomorphism and $V := \{A \in L(E'_c, E'_c): (\forall x \in E) \eta_E(x) \circ A \in \eta_E(E)\}$. Then $V$ is a vector subspace of $L(E'_c, E'_c)$ and $\text{ad}(E'_c) \subseteq V$. The composition map $\Gamma: (E'_c)_c \times L(E'_c, E'_c) \rightarrow (E'_c)_c$, $(\alpha, A) \mapsto \alpha \circ A$ is hypocontinuous with respect to equicontinuous subsets of $(E'_c)_c$, by Proposition 9 in [9, Chapter III, §5, no.5]. If $K \subseteq E$ is compact, then the polar $K^\circ$ is a 0-neighbourhood in $E'_c$, entailing that $(K^\circ)^\circ \subseteq (E'_c)^\circ$ is equicontinuous. Hence $\eta_E$ takes compact subsets of $E$ to equicontinuous subsets of $(E'_c)^\circ$, and hence
\[
\beta: E \times V \rightarrow E, \quad (x, A) \mapsto \eta_E^{-1}(\Gamma(\eta_E(x), A))
\]
is $c$-hypocontinuous in its first argument. By Lemma 11.4(c), $\beta_*: C^\infty(U, V) \rightarrow C^\infty(U, E)$, $f \mapsto \beta \circ (\text{id}_U, f)$ is continuous linear. Also the map $D: C^\infty(U, \mathbb{K}) \rightarrow C^\infty(U, E'_c)$, $f \mapsto f'$ is continuous linear by Lemma 11.2. Furthermore, $\text{ad} = [\ldots]' : E'_c \rightarrow L(E'_c, E'_c)$ is continuous linear since $[\ldots]$ is $c$-hypocontinuous in its second argument (see Lemma 11.15(b)), whence
\[
C^\infty(U, \text{ad}): C^\infty(U, E'_c) \rightarrow C^\infty(U, L(E'_c, E'_c)) , \quad f \mapsto \text{ad} \circ f
\]
is continuous linear (see, e.g., [22, Lemma 4.13]). Hence $\Psi = \beta_* \circ C^\infty(U, \text{ad}) \circ D$ is continuous and linear. \[\square\]
A Proofs for basic facts in Section 1

Proof of Lemma 1.6. Let $E := E_1 \times \cdots \times E_k$. Since $df: U \times E \times X \times E \to F$ is continuous and $df(x,0,0,0) = 0$, given $q$ there exist a continuous seminorm $p$ on $X$ such that $B^q_t(x) \subseteq U$, and continuous seminorms $p_j$ on $E_j$ for $j \in \{1, \ldots, k\}$ such that

$$\|df(y, v_1, \ldots, v_k, z, w_1, \ldots, w_k)\|_q \leq 1$$

for all $v_j, w_j \in B^p_j(0)$, $y \in B^p_t(x)$, and $z \in B^p_t(0)$. For $y \in B^p_t(x)$ and $(v_1, \ldots, v_k) \in B^p_t(0) \times \cdots \times B^p_t(0)$, the Mean Value Theorem (see [30, Proposition 1.2.6]) shows that

$$f(y, v_1, \ldots, v_k) = \int_0^1 df(y, tv_1, \ldots, tv_k, 0, v_1, \ldots, v_k) \, dt.$$ 

Since $\|df(y, tv_1, \ldots, tv_k, 0, v_1, \ldots, v_k)\|_q \leq 1$ for each $t$, it follows that $\|f(y, v_1, \ldots, v_k)\|_q \leq 1$ in the preceding situation. Because $f(y, \cdot)$ is $k$-linear, we deduce that (11) holds. To prove (2), we first note that (21) implies that

$$\|df(y, v_1, \ldots, v_k, z, 0, \ldots, 0)\|_q \leq \|z\|_p$$

for all $y \in B^p_t(x)$, $(v_1, \ldots, v_k) \in B^p_t(0) \times \cdots \times B^p_t(0)$ and $z \in X$, exploiting the linearity of $df(y, v_1, \ldots, v_k, z, 0, \ldots, 0)$ in $z$. We now use the Mean Value Theorem to write

$$f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k) = \int_0^1 df(x + t(y-x), v_1, \ldots, v_k, y-x, 0, \ldots, 0) \, dt$$

for $y \in B^p_t(x)$ and $(v_1, \ldots, v_k) \in B^p_t(0) \times \cdots \times B^p_t(0)$. By (22), we have

$$\|df(x + t(y-x), v_1, \ldots, v_k, y-x, 0, \ldots, 0)\|_q \leq \|y-x\|_p$$

and hence $\|f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k)\|_q \leq \|y-x\|_p$. Now (2) follows, using the $k$-linearity of the map $f(y, \cdot) - f(x, \cdot): E_1 \times \cdots \times E_k \to F$. \qed

Proof of Lemma 1.7. By the Polarization Formula for symmetric $k$-linear maps (see, e.g., [30, Proposition 1.6.19]), we have

$$f(x, y_1, \ldots, y_k) = \frac{1}{k!} \sum_{\varepsilon_1, \ldots, \varepsilon_k \in \{-1, 1\}} \varepsilon_1 \cdots \varepsilon_k h(x, \varepsilon_1 y_1 + \cdots + \varepsilon_k y_k)$$

for all $x \in U$ and $y_1, \ldots, y_k \in E$. Thus $f$ is $C^\infty_k$ if $h$ is so. \qed

Proof of Lemma 1.11. (a) Let $pr_2: X_1 \times X_2 \to X_2$, $(x, y) \mapsto y$ be the projection onto the second component and pick $x_0 \in X_1$. Since $pr_2$ is continuous, every $k$-continuous function $f: X_2 \to \mathbb{R}$ yields a $k$-continuous function $f \circ pr_2$ on $X$. Then $f \circ pr_2$ is continuous and hence also $f = (f \circ pr_2)(x_0, \cdot)$.

(b) Let $f: U \to \mathbb{R}$ be $k$-continuous and $x \in U$. As $X$ is completely regular, we find a continuous function $g: X \to \mathbb{R}$ with $g(x) \neq 0$ and support $\text{supp}(g) \subseteq U$. Define $h: X \to \mathbb{R}$
via \( h(y) := f(y)g(y) \) if \( y \in U \), \( h(y) := 0 \) if \( y \in X \setminus \text{supp}(g) \). If \( K \subseteq X \) is a compact subset, then each \( x \in K \) has a compact neighbourhood \( K_x \) in \( K \) which is contained in \( U \) or in \( X \setminus \text{supp}(g) \). In the first case, \( h|_{K_x} = f|_{K_x}g|_{K_x} \) is continuous by \( k \)-continuity of \( f \). In the second case, \( h|_{K_x} = 0 \) is continuous as well. Thus \( h|_K \) is continuous. Since \( X \) is a \( k_\mathbb{R} \)-space, continuity of \( h \) follows. Thus \( f \) is continuous on the open \( x \)-neighbourhood \( g^{-1}(\mathbb{R} \setminus \{0\}) \). □

A simple fact will be useful (see, e.g., [20, Lemma 1.13]).

**Lemma A.1** Let \( X \) be a topological space, \( F \) be a locally convex space, and \( BC(X, F) \) be the space of bounded \( F \)-valued continuous functions on \( X \), endowed with the topology of uniform convergence. Then \( \mu : BC(X, F) \times X \to F, (f, x) \mapsto f(x) \) is continuous. □

**Proof of Lemma 1.15**\(^{10} \) (a)\( \Leftrightarrow \)(b): \( \beta(V \times M) \subseteq W \) is equivalent to \( \beta^\vee(V) \in \lceil M, W \rceil \).

Hence (a) is equivalent to continuity of \( \beta^\vee \) in 0 and hence to its continuity (see Proposition 5 in [9, Chapter I, §1, no. 6]).

(b)\( \Rightarrow \)(c): If \( M \in \mathcal{S} \), then \( \varepsilon : L^{k-j+1}(E_j, \ldots, E_k, F)_S \times M \to F, \varepsilon(\alpha, x) := \alpha(x) \) is continuous as a consequence of Lemma A.1. Hence \( \beta|_{E_1 \times \cdots \times E_{j-1} \times M} = \varepsilon \circ (\beta^\vee \times \text{id}_M) \) is continuous.

(c)\( \Rightarrow \)(a) if (3) holds: Given \( M \in \mathcal{S} \) and a 0-neighbourhood \( W \subseteq F \), by hypothesis we can find \( N \in \mathcal{S} \) such that \( \mathbb{D}M \subseteq N \). By continuity of \( \beta|_{E_1 \times \cdots \times E_{j-1} \times N} \), there exist 0-neighbourhoods \( V_i \subseteq E_i \) for \( i \in \{1, \ldots, k\} \) such that \( \beta(V \times (N \cap U)) \subseteq W \), where \( V := V_1 \times \cdots \times V_{j-1} \) and \( U := V_j \times \cdots \times V_k \). Set \( a := \frac{j-1}{k-j+1} \). Since \( M \) is bounded, \( M \subseteq n^a U \) for some \( n \in \mathbb{N} \). Then \( \frac{1}{n^a} M \subseteq N \cap U \). Using that \( \beta \) is \( k \)-linear, we obtain \( \beta((\frac{1}{n}V) \times M) = \beta(V \times (\frac{1}{n^a} M)) \subseteq \beta(V \times (N \cap U)) \subseteq W \). □

**Proof of Lemma 1.19**. Given \( \alpha \in L^k(E_1, \ldots, E_k, F) \), we have \( \varepsilon^\vee(\alpha) = \varepsilon(\alpha, \cdot) = \alpha \), which is a continuous \( k \)-linear map. The map \( \varepsilon \) is also continuous in its first argument, as the topology on \( L^k(E_1, \ldots, E_k, F)_S \) is finer than the topology of pointwise convergence, by the hypothesis on \( \mathcal{S} \). The linear map \( \varepsilon^\vee : L^k(E_1, \ldots, E_k)_S \to L^k(E_1, \ldots, E_k)_S, \alpha \mapsto \alpha \) being continuous, condition (b) of Lemma 1.15 is satisfied by \( \varepsilon \) in place of \( \beta \) and hence also the equivalent condition (a), whence \( \varepsilon \) is \( \mathcal{S} \)-hypocontinuous in its arguments \( (2, \ldots, k+1) \).

Now assume that \( k = 1 \). Since \( \mathcal{O} \) is finer than the topology of pointwise convergence, the map \( \varepsilon \) remains separately continuous in the situation described at the end of the lemma. Hence, if \( E \) is barrelled, Lemma 1.18 ensures hypocontinuity with respect to \( \mathcal{T} \). □

**Proof of Lemma 1.20**. (a) The composition \( \beta \circ f \) is sequentially continuous and hence continuous, its domain \( X \) being first countable.

(b) Write \( f = (f_1, \ldots, f_k) \) with components \( f_j : X \to E_\nu \) for \( \nu \in \{1, \ldots, k\} \). If \( K \) is a compact subset of \( X \), then \( M := (f_j, \ldots, f_k)(K) \) is a compact subset of \( E_j \times \cdots \times E_k \). Since \( \beta|_{E_1 \times \cdots \times E_{j-1} \times M} \) is continuous by Lemma 1.15(c), the composition

\[
\beta \circ f|_K = \beta|_{E_1 \times \cdots \times E_{j-1} \times M} \circ f|_K
\]

\(^{10}\)If \( k = 2 \), see Proposition 3 and 4 in [9, Chapter III, §5, no. 3] for the equivalence (a)\( \Leftrightarrow \)(b) and the implication (b)\( \Rightarrow \)(c); (c)\( \Rightarrow \)(a) can be found in [20, Proposition 1.8].
is continuous. Thus $\beta \circ f$ is $k$-continuous and hence continuous, as $X$ is a $k_\mathbb{R}$-space and $F$ is completely regular. □

**Proof of Lemma 1.22.** (a) The case $r = 0$: Let $q$ be a continuous seminorm on $F := \prod_{j \in J} F_j$, and $x \in U$. After increasing $q$, we may assume that

$$q(y) = \max\{q_j(y_j) : j \in \Phi\} \quad \text{for all} \quad y = (y_j)_{j \in J} \in F,$$

for some non-empty, finite subset $\Phi \subseteq J$ and continuous seminorms $q_j$ on $F_j$ for $j \in \Phi$. If each $f_j$ is $LC^0_{\mathbb{K}}$, then we find a continuous seminorm $p_j$ on $E$ for each $j \in \Phi$ such that $B^p_j(x) \subseteq U$ and $q_j(f_j(z) - f_j(y)) \leq p_j(z - y)$ for all $z, y \in B^p_j(x)$. Then

$$p: E \to [0, \infty[, \quad y \mapsto \max\{p_j(y) : j \in \Phi\}$$

is a continuous seminorm on $E$ such that $B^p_j(x) \subseteq U$ and $q(f(z) - f(y)) \leq p(z - y)$ for all $z, y \in B^p_j(x)$. If $f$ is $LC^0_{\mathbb{K}}$, let us show that $f_j$ is $LC^0_{\mathbb{K}}$ for each $j \in J$. Let $q$ be a continuous seminorm on $F_j$ and $x \in U$. Let $pr_j: F \to F_j$, $(y_j)_{j \in J} \mapsto y_j$ be the continuous linear projection onto the $j$th component. Then $q \circ pr_j$ is a continuous seminorm on $F$, whence we find a continuous seminorm $p$ on $E$ such that $B^p_j(x) \subseteq U$ and $(q \circ pr_j)(f(z) - f(y)) \leq p(z - y)$ for all $z, y \in B^p_j(x)$. Since $(q \circ pr_j)(f(z) - f(y)) = q(f_j(z) - f_j(y))$, we see that $f_j$ is $LC^0_{\mathbb{K}}$.

If $r \in \mathbb{N} \cup \{\infty\}$, then $f$ is $C^r_{\mathbb{K}}$ if and only if each $f_j$ is $C^r_{\mathbb{K}}$, and $d^k f = (d^k f_j)_{j \in J}$ in this case for all $k \in \mathbb{N}_0$ such that $k \leq r$ (see [30, Lemma 1.3.3]). By the case $r = 0$, the map $d^k f$ is $LC^0_{\mathbb{K}}$ if and only if $d^k f_j$ is $LC^0_{\mathbb{K}}$ for all $j \in J$. The assertion follows.

(b) Let $E$, $F$, and $Y$ be locally convex $\mathbb{K}$-vector spaces and $f: U \to F$ as well as $g: V \to Y$ be $LC^0_{\mathbb{K}}$-maps on open subsets $U \subseteq E$ and $V \subseteq F$, such that $f(U) \subseteq V$.

If $r = 0$, let $x \in U$ and $q$ be a continuous seminorm on $Y$. There exists a continuous seminorm $p$ on $F$ such that $B^q_j(f(x)) \subseteq V$ and $q(g(b) - g(a)) \leq p(b - a)$ for all $a, b \in B^p_j(f(x))$. There exists a continuous seminorm $P$ on $E$ with $B^p_j(x) \subseteq U$ and $p(f(z) - f(y)) \leq P(z - y)$ for all $z, y \in B^p_j(x)$. Then $f(B^p_j(x)) \subseteq B^p_j(f(x))$ and hence

$$q(g(f(z)) - g(f(y))) \leq p(f(z) - f(y)) \leq P(z - y)$$

for all $z, y \in B^p_j(x)$. Thus $g \circ f: U \to Y$ is $LC^0_{\mathbb{K}}$.

If $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$ such that $k \leq r$, we can use Faà di Bruno’s Formula

$$d^k(g \circ f)(x, y) = \sum_{j=1}^{k} \sum_{P \in P_{k,j}} d^j g(f(x), d^{I_1}(x, y_{I_1}), \ldots, d^{I_j}(x, y_{I_j}))$$

for $x \in U$ and $y = (y_1, \ldots, y_k) \in E^k$, as in [30, Theorem 1.3.18]. Here $P_{k,j}$ is the set of all partitions $P = \{I_1, \ldots, I_j\}$ of $\{1, \ldots, k\}$ into $j$ disjoint, non-empty subsets $I_1, \ldots, I_j \subseteq \{1, \ldots, k\}$. For a non-empty subset $J \subseteq \{1, \ldots, k\}$ with elements $j_1 < \cdots < j_m$, let $y_J := (y_{j_1}, \ldots, y_{j_m})$. Using (a) and the case $r = 0$, we deduce from (24) that $d^k(g \circ f)$ is $LC^0_{\mathbb{K}}$.

(c) For each continuous seminorm $q$ on $F$, the restriction $q|_{F_0}$ is a continuous seminorm on $F_0$, and each continuous seminorm $Q$ on $F_0$ arises in this way. In fact, we find an open,
absolutely convex 0-neighbourhood $V \subseteq F$ such that $V \cap F_0 \subseteq B(0)$. Then the absolutely convex hull $W$ of $V \cup B(0)$ is a 0-neighbourhood in $F$ with $W \cap F_0 = B(0)$, whence $q|_{F_0} = Q$ holds for the Minkowski functional $q$ of $W$. The case $r = 0$ follows.

If $r \in \mathbb{N} \cup \{\infty\}$, let $\iota: F_0 \to F$ be the inclusion map and $f: U \to F_0$ be a map on an open subset $U \subseteq E$. Then $f$ is $C_r^\infty$ if and only if $\iota \circ f$ is $C_r^\infty$ and $d^k(\iota \circ f) = \iota \circ (d^k f)$ for all $k \in \mathbb{N}$ such that $k < r$ (see [30, Lemma 1.3.19]). By the case $r = 0$, each of the maps $d^k f$ is $LC^\infty$ if and only if $\iota \circ (d^k f)$ is so, from which the assertion follows.

(d) is immediate from (a) and (c).

**Proof of Lemma 1.24.** For each $x \in E_1 \times \cdots \times E_k =: E$, the point evaluation $ev_x: C^\infty_r(E, F) \to F$, $f \mapsto f(x)$ is linear and continuous, the compact-open $C^r$-topology on $C^\infty_r(E, F)$ being finer than the compact-open topology. Now $L^k(E_1, \ldots , E_k, F)$ is closed in $C^\infty_r(E, F)$, being the intersection of the closed sets

$$\{f \in C^\infty_r(E, F): ev_z(f) - a ev_x(f) - b ev_{x'}(f) = 0\}$$

for $j \in \{1, \ldots , k\}$, $x = (x_1, \ldots , x_k) \in E$, $x' := (x_1, \ldots , x_{j-1}, x'_j, x_{j+1}, \ldots , x_k)$ with $x'_j \in E_j$ and $z := (x_1, \ldots , x_{j-1}, ax_j + bx'_j, x_{j+1}, \ldots , x_k)$ with $a, b \in \mathbb{R}$.

Let $\iota: L^k_1(E_1, \ldots , E_k, F) \to C^\infty_r(E, F)$ be the inclusion map. Then $d^0 \circ \iota$ is the inclusion map $L^k_1(E_1, \ldots , E_k, F) \to C(E, F)_{c.o.}$, which is a topological embedding (and hence continuous). We claim: For all $j \in \mathbb{N}$ such that $j < r$, there are $m_j \in \mathbb{N}$ and $C^\infty_r$-maps $g_{j, \mu}: E^{j+1} \to E$ for $\mu \in \{1, \ldots , m_j\}$ such that

$$d^j \beta = \sum_{\mu=1}^{m_j} \beta \circ g_{j, \mu} \quad \text{for all } \beta \in L^k_1(E_1, \ldots , E_k, F). \quad (25)$$

If this is true, then $d^j \circ \iota: L^k_1(E_1, \ldots , E_k, F) \to C(E^{j+1}, F)_{c.o.}$ is a restriction of the mapping $\sum_{\mu=1}^{m_j} (C(g_{j, \mu}, F) \circ \iota)$ and hence continuous, by [30] Lemma A.6.9; thus $\iota$ is continuous. As $d^0 \circ \iota$ is a topological embedding, we deduce that also $\iota$ is a topological embedding.

To prove the claim, we proceed by induction. For all $\beta \in L^k_1(E_1, \ldots , E_k, F)$, $x = (x_1, \ldots , x_k) \in E$ and $y = (y_1, \ldots , y_k) \in E$, we have

$$d^1 \beta(x, y) = \sum_{\nu=1}^{k} \beta(x_1, \ldots , x_{\nu-1}, y_{\nu}, x_{\nu+1}, \ldots , x_k), \quad (26)$$

see [30] Example 1.2.3]. This establishes the claim for $j = 1$. Let $\pi: E^{j+2} \to E^{j+1}$ be the continuous linear projection $(x, y_1, \ldots , y_{j+1}) \mapsto (x, y_1, \ldots , y_j)$ and $\iota: E^{j+2} \to E^{2j+2}$ be the continuous linear map taking $(x, y_1, \ldots , y_{j+1})$ to $(x, y_1, \ldots , y_{j+1}, 0, \ldots , 0)$. If $1 \leq j < r$ and the claim holds for $j$, write $g_{j, \mu} = (g_{j, \mu, 1}, \ldots , g_{j, \mu, k})$ in components for $\mu \in \{1, \ldots , m_j\}$. Then $d^{j+1} \beta = \sum_{\mu=1}^{m_j} \sum_{\nu=1}^{k} \beta \circ h_{\mu, \nu}$ with $h_{\mu, \nu}: E^{j+2} \to E$,

$$h_{\mu, \nu} := (g_{j, \mu, 1} \circ \pi, \ldots , g_{j, \mu, \nu-1} \circ \pi, dg_{j, \mu, \nu} \circ \iota, g_{j, \mu, \nu+1} \circ \pi, \ldots , g_{j, \mu, k} \circ \pi).$$

Thus also $d^{j+1} \beta$ is of the asserted form and the claim holds for $j + 1$. □
B  Smooth maps need not extend to the completion

Let $E := \{(x_n)_{n \in \mathbb{N}} \in \ell^1 : (\exists N \in \mathbb{N})(\forall n \geq N) x_n = 0\}$ be the space of finite sequences, endowed with the topology induced by the real Banach space $\ell^1$ of absolutely summable real sequences. Then $E$ is a dense proper vector subspace of $\ell^1$, and $\ell^1$ is a completion of $E$. In this appendix, we provide a smooth map with the following pathological properties.

**Proposition B.1** There exists a smooth map $f : E \to F$ to a complete locally convex space $F$ which does not extend to a continuous extension to $E \cup \{z\}$ for any $z \in \ell^1 \setminus E$.

**Proof.** Given $z = (z_n)_{n \in \mathbb{N}} \in \ell^1 \setminus E$, the set $S := \{n \in \mathbb{N} : z_n \neq 0\}$ is infinite. For each $n \in \mathbb{N}$, we pick a smooth map $h_n : \mathbb{R} \to \mathbb{R}$ such that $h_n(z_n) = 1$; if $n \in S$, we also require that $h_n$ vanishes on some 0- neighbourhood. Endow $\mathbb{R}^N$ with the product topology. Then

$$g : \ell^1 \to \mathbb{R}^N, \quad x = (x_n)_{n \in \mathbb{N}} \mapsto (h_1(x_1) \cdots h_n(x_n))_{n \in \mathbb{N}}$$

is a smooth map, as its components $g_n : \ell^1 \to \mathbb{R}, x \mapsto h_1(x_1) \cdots h_n(x_n)$ are smooth. If $x = (x_n)_{n \in \mathbb{N}} \in E$, then there is $N \in S$ such that $x_n = 0$ for all $n \geq N$. Thus $g_n(x) = 0$ for all $n \geq N$ and hence $g(x) \in E$. Notably, $g(x) \in \ell^1$. It therefore makes sense to define

$$f_z : E \to \ell^1, \quad x \mapsto g(x).$$

We now show: $f_z : E \to \ell^1$ is a smooth map to $\ell^1$ which does not admit a continuous extension to $E \cup \{z\}$.

In fact, for $x$ and $N$ as above, there exists $\varepsilon > 0$ such that $h_N(t) = 0$ for each $t \in ]-\varepsilon, \varepsilon[$. Identify $\mathbb{R}^N$ with the closed vector subspace $\mathbb{R}^N \times \{0\}$ of $E$ and $\mathbb{R}^N$. Then

$$U := \{y = (y_n)_{n \in \mathbb{N}} \in E : |y_N| < \varepsilon\}$$

is an open neighbourhood of $x$ in $E$ such that $f_z(U) \subseteq \mathbb{R}^N$. Thus $f_z|U$ is smooth as a map to $\mathbb{R}^N$ and hence also as a map to $\ell^1$. As a consequence, $f_z : E \to \ell^1$ is smooth.

Now suppose that $p = (p_n)_{n \in \mathbb{N}} : E \cup \{z\} \to \ell^1$ was a continuous extension of $f_z$; we shall derive a contradiction. To this end, set $y_k := (z_1, \ldots, z_k, 0, 0, \ldots) \in E$ for $k \in \mathbb{N}$. Then $y_k \to z$ in $E$ as $k \to \infty$. The inclusion map $\ell^1 \to \mathbb{R}^N$ being continuous, we deduce that

$$p_n(y_k) \to p_n(z) \quad \text{as} \quad k \to \infty,$$

for each $n \in \mathbb{N}$. Since $p_n(y_k) = g_n(y_k) = h_1(z_1) \cdots h_n(z_n) = 1$ for all $k \geq n$, it follows that $p_n(z) = 1$ for all $n \in \mathbb{N}$ and thus $(1, 1, \ldots) = p(z) \in \ell^1$, which is absurd. Therefore $f_z$ has all of the asserted properties.

We now define $\Omega := \ell^1 \setminus E$ and endow $F := (\ell^1)^\Omega$ with the product topology. We let $f := (f_z)_{z \in \Omega} : E \to F$ be the map with components $f_z$ as defined before. By construction, $f$ has the properties described in Proposition B.1. $\square$
References

[1] Alzaareer, H. and A. Schmeding, *Differentiable mappings on products with different degrees of differentiability in the two factors*, Expo. Math. 33 (2015), 184–222.

[2] Ardanza-Trevijano, S. and M. J. Chasco, *The Pontryagin duality of sequential limits of topological Abelian groups*, J. Pure Appl. Algebra 202 (2005), 11–21.

[3] Banaszczyk, W., “Additive Subgroups of Topological Vector Spaces,” Springer, Berlin, 1991.

[4] Bastiani, A., *Applications différentiables et variétés différentiables de dimension infinie*, J. Anal. Math. 13 (1964), 1–114.

[5] Beggs, E., *De Rham's theorem for infinite-dimensional manifolds*, Quart. J. Math. 38 (1987), 131–154.

[6] Beltiţă D., T. Goliński, A.-B. Tumpach, *Queer Poisson brackets*, J. Geom. Phys. 132 (2018), 358–362.

[7] Bertram, W., H. Glöckner, and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. 22 (2004), 213–282.

[8] Bochnak, J. and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. 39 (1971), 77–112.

[9] Bourbaki, N., “Topological Vector Spaces, Chapters 1-5,” Springer, Berlin, 1987.

[10] Engelking, R., “General Topology,” Heldermann, Berlin, 1989.

[11] Ferrer, M. V., S. Hernández, and D. Shakhmatov, *A countable free closed non-reflexive subgroup of \( \mathbb{Z}^c \)*, Proc. Amer. Math. Soc. 145 (2017), 3599–3605.

[12] Franklin, S. P. and B. V. Smith Thomas, *A survey of \( k_\omega \)-spaces*, Topol. Proc. 2 (1978), 111–124.

[13] Glöckner, H., *Infinite-dimensional Lie groups without completeness restrictions*, pp. 43–59 in: Strasburger, A. et al. (eds.), “Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups” Banach Center Publications, Vol. 55, Warsaw, 2002.

[14] Glöckner, H., *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. 194 (2002), 347–409.

[15] Glöckner, H., *Algebras whose groups of units are Lie groups*, Studia Math. 153 (2002), 147–177.

[16] Glöckner, H., *Lie groups of measurable mappings*, Canadian J. Math. 55 (2003), 969–999.

[17] Glöckner, H., *Fundamentals of direct limit Lie theory*, Compos. Math. 141 (2005), 1551–1577.

[18] Glöckner, H., *Implicit functions from topological vector spaces to Banach spaces*, Israel J. Math. 155 (2006), 205–252.

[19] Glöckner, H., *Direct limits of infinite-dimensional Lie groups compared to direct limits in related categories*, J. Funct. Anal. 245 (2007), 19–61.

[20] Glöckner, H., *Applications of hypocontinuous bilinear maps in infinite-dimensional differential calculus*, pp. 171–186 in: S. Silvestrov, E. Paal, V. Abramov and A. Stolin (eds.), “Generalized Lie Theory in Mathematics, Physics and Beyond,” Springer, Berlin, 2008.
[21] Glöckner, H., *Direct limits of infinite-dimensional Lie groups*, pp. 243–280 in: K.-H. Neeb and A. Pianzola (eds.), “Developments and Trends in Infinite-Dimensional Lie Theory,” Birkhäuser, Basel, 2011.

[22] Glöckner, H., *Lie groups over non-discrete topological fields*, preprint, arXiv:math/0408008.

[23] Glöckner, H., *Differentiable mappings between spaces of sections*, preprint, arXiv:1308.1172.

[24] Glöckner, H., *Measurable regularity properties of infinite-dimensional Lie groups*, preprint, arXiv:1601.02568.

[25] Glöckner, H., *Smoothing operators for vector-valued functions and extension operators*, preprint, arXiv:2006.00254.

[26] Glöckner, H., *Patched locally convex spaces, almost local mappings, and diffeomorphism groups of non-compact manifolds*, manuscript, 2002.

[27] Glöckner, H., R. Gramlich, and T. Hartnick, *Final group topologies, Kac-Moody groups and Pontryagin duality*, Isr. J. Math. 177 (2010), 49–101.

[28] Glöckner, H. and J. Hilgert, *Aspects of control theory on infinite-dimensional Lie groups and G-manifolds*, preprint, arXiv:2007.11277.

[29] Glöckner, H. and N. Masbough, *Products of regular locally compact spaces are kR-spaces*, Topology Proc. 55 (2020), 35–38.

[30] Glöckner, H. and K.-H. Neeb, “Infinite Dimensional Lie Groups,” book in preparation.

[31] Grauert, H., *Analytische Faserungen über holomorph-vollständigen Räumen*, Math. Ann. 135 (1958), 263–273.

[32] Hamilton, R., *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7 (1982), 65–222.

[33] Hewitt, E. and K.A. Ross, “Abstract Harmonic Analysis I,” Springer, New York, 1979.

[34] Hirai, T., H. Shimomura, N. Tatsuuma, and E. Hirai, *Inductive limits of topologies, their direct products, and problems related to algebraic structures*, J. Math. Kyoto Univ. 41 (2001), 475–505.

[35] Hofmann, K. H. and S. A. Morris, “The Structure of Compact Groups,” de Gruyter, Berlin, 1998.

[36] Jarchow, H., “Locally Convex Spaces,” B. G. Teubner, Stuttgart, 1981.

[37] Kelley, J. L., “General Topology,” Springer, New York, 1975.

[38] Kriegl, A. and P. W. Michor, “The Convenient Setting of Global Analysis,” AMS, Providence, 1997.

[39] Micheli, M., P. W. Michor, and D. Mumford, *Sobolev metrics on diffeomorphism groups and the derived geometry of spaces of submanifolds*, Izv. Math. 77 (2013), 541–570.

[40] Michor, P. W., “Manifolds of Differentiable Mappings,” Shiva, Orpington, 1980.

[41] Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp. 1008–1057 in: B. DeWitt and R. Stora (eds.), “Relativity, Groups and Topology II,” North Holland, 1984.
[43] Neeb, K.-H., *Towards a Lie theory of locally convex groups*, Jpn. J. Math. 1 (2006), 291–468.

[44] Neeb, K.-H., H. Sahlmann, and T. Thiemann, *Weak Poisson structures on infinite dimensional manifolds and Hamiltonian actions*, pp. 105–135 in: V. Dobrev (ed.), “Lie Theory and its Applications in Physics,” Springer, Tokyo, 2014.

[45] Noble, N., *The continuity of functions on cartesian products*, Trans. Amer. Math. Soc. 149 (1970), 187–198.

[46] Noble, N., *k-groups and duality*, Trans. Amer. Math. Soc. 151 (1970), 551–561.

[47] Odzijewicz, A. and T. S. Ratiu, *Banach Lie-Poisson spaces and reduction*, Comm. Math. Phys. 243 (2003), 1–54.

[48] Odzijewicz, A. and T. S. Ratiu, *Extensions of Banach Lie-Poisson spaces*, J. Funct. Anal. 217 (2004), 103–125.

[49] Patyi, I., *On holomorphic Banach vector bundles over Banach spaces*, Math. Ann. 341 (2008), 455–482.

[50] Seip, U., “Kompakt erzeugte Vektorräume und Analysis,” Springer, Berlin, 1972.

[51] Thomas, E. G. F., “Calculus on Locally Convex Spaces,” preprint, University of Groningen, 1996.

[52] Treves, F., “Topological Vector Spaces, Distributions and Kernels,” Academic Press, New York, 1967.

[53] Wurzbacher, T., *Fermionic second quantization and the geometry of the restricted Grassmannian*, pp. 287–375 in: Huckleberry, A. T. and T. Wurzbacher (eds.), “Infinite-Dimensional Kähler Manifolds,” Birkhäuser, Basel, 2001.

**Helge Glöckner**, Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany. Email: glockner@math.upb.de