ON SOME PROPERTIES OF BIRATIONAL DERIVED SPLINTERS

SHIJI LYU

Abstract. A Noetherian reduced ring $A$ is called a birational derived splinter if for all proper birational maps $X \to \text{Spec}(A)$, the canonical map $A \to Rf_*O_X$ splits. In equal characteristic zero this property characterizes rational singularities, but much less can be said in positive or mixed characteristics. In this paper, we prove some fundamental properties of this notion, including the behavior under localization, taking a pure subring, taking direct limit, and along an étale extension. In particular, direct limit of rational singularities in characteristic zero has rational singularities. Then, we study residue extensions (in arbitrary characteristic), and openness and regular extensions in positive characteristic, parallel to Datta-Tucker and the author’s previous works on splinters.

1. Introduction

In algebraic geometry, it is extremely useful to consider proper birational models $Y \to X$ of a given algebraic variety $X$. One question that naturally arises is how to compare the (coherent or local) cohomology of $X$ and that of $Y$. Works in this direction include [CR15].

In this paper, we shall specialize to the problem of local splitting. More explicitly, we will study the (Noetherian reduced) rings $A$ such that for all proper birational maps $f : X \to \text{Spec}(A)$, the canonical map $A \to Rf_*O_X$ splits in the derived category of $A$. We say $A$ is a birational derived splinter.

There are two closely related notions, splinters and derived splinters, defined by considering finite surjective and proper surjective $f$’s respectively instead of proper birational $f$’s. Regular rings are derived splinters [Bha18], and splinters and derived splinters are the same in positive and mixed characteristic ([Bha12] and the forthcoming work [BL]), so in particular, splinters are birational derived splinters. In characteristic zero, a result of Kovács [Kov00] shows that for affine varieties over the complex numbers, the notion of birational derived splinter is the same as the notion of rational singularity. Recently, Murayama [Mur21b] has extended the Kodaira-type vanishing theorems to general schemes of equal characteristic zero, from which it follows that the notion of birational derived splinter is the same as the notion of rational singularity for all quasi-excellent rings of equal characteristic zero, as noted below in Proposition 3.5.
However, the notion of birational derived splinters does not seem to be very well studied itself, especially in positive and mixed characteristics. This is the aim of this paper. Our first main result is a collection of fundamental properties of birational derived splinters (Corollary 3.11, Corollary 3.12, Theorem 4.4, and Theorem 5.1):

**Theorem 1.1.** Let $A$ be a Noetherian ring. Then the followings hold.

1. $A$ is a birational derived splinter if and only if all localizations of $A$ at prime (resp. maximal) ideals of $A$ are birational derived splinters.
2. If $A$ is a birational derived splinter, then for all cyclically pure ring maps $A' \rightarrow A$, $A'$ is a Noetherian birational derived splinter.
3. If $A$ is the direct limit of a system of Noetherian birational derived splinters, then $A$ is a birational derived splinter.
4. If $A$ is a birational derived splinter and $B$ is an étale $A$-algebra, then $B$ is a birational derived splinter.

We note that of these results, the most difficult one is (3). This is because of the lack of flatness, as mentioned in [AD21, Remark 5.3.3(2)]. Specializing to characteristic zero, we have the following consequence, which the author believes to be new:

**Corollary 1.2** (=Corollary 4.5). Let $A$ be a Noetherian quasi-excellent $\mathbb{Q}$-algebra. If $A$ is the direct limit of a system of Noetherian quasi-excellent $\mathbb{Q}$-algebras that have rational singularities, then $A$ has rational singularities.

Next, we can employ the methods in the author’s previous work [Lyu22] and part of [DT21] on splinters to the study of birational derived splinters. As in [Lyu22], we use the notion of a G-ring to state our results in maximal generality. Unfamiliar readers can refer to [Stacks, Tag 0708] and [Mat80, (34.A)].

The following main results are parallel to [Lyu22, Theorem 1.3], [Lyu22, Theorem 2.14], [DT21, Theorem 1.0.1], and [Lyu22, Theorem 1.1]. Note that for the results in characteristic $p$, we involve $F$-purity, since it is required by our method. See §8 for questions about possible strengthening of these results.

**Theorem 1.3** (=Theorem 6.4). Let $(S, \mathfrak{m}) \rightarrow (S', \mathfrak{m}')$ be a regular homomorphism of Noetherian local rings with $\mathfrak{m}' = \mathfrak{m}S'$. If $S$ is a birational derived splinter, so is $S'$. In particular, the completion of a local G-ring that is a birational derived splinter is a birational derived splinter.

**Theorem 1.4** (see Corollary 7.5 and Theorem 7.8(1)). Let $R$ be a Noetherian local $F_p$-algebra, and $A$ an $R$-algebra essentially of finite type. Assume $R$ is $F$-pure, and that $R \rightarrow A$ is $F$-pure [Has10, (2.1)].

Then the locus of prime ideals $\mathfrak{p}$ of $A$ such that $A_\mathfrak{p}$ is a birational derived splinter (resp., a splinter, a normal local ring) is open in Spec$(A)$.
Theorem 1.5 (=Theorem 7.8(2)). Let $A$ be a Noetherian $\mathbb{F}_p$-algebra. Assume either that $A$ is $F$-finite, or that $A$ is essentially of finite type over a Noetherian local $G$-ring.

Then the locus of prime ideals $p$ of $A$ such that $A_p$ is an $F$-pure birational derived splinter is open in $\text{Spec}(A)$.

Here the corresponding result for the normal locus is well-known, see Lemma 7.1 below, and for the splinter locus it is [DT21, Theorem 1.0.1].

Theorem 1.6 (=Theorem 7.9). Let $S \rightarrow A$ be a regular homomorphism of Noetherian $\mathbb{F}_p$-algebras. If $S$ is an $F$-pure birational derived splinter, so is $A$.

Finally, we mention a curious result, which, as does Theorem 1.4, fails for a general $F$-pure Noetherian ring $A$, even if we assume $A$ is a $G$-ring. See Remark 7.7.

Theorem 1.7 (=Corollary 7.6). Let $R$ be a Noetherian local $\mathbb{F}_p$-algebra, $A$ an $R$-algebra essentially of finite type. Assume $R$ is $F$-pure, and that $R \rightarrow A$ is $F$-pure [Has10, (2.1)]. Then the normalization of $A$ is finite over $A$.

The layout of this paper is as follows. §2 contains necessary general results of rings, modules, schemes, and morphisms. Since our method requires the use of non-Noetherian rings, we need to introduce and study a variant of proper birational morphisms that is more robust. This is done in §§2.3. Another class of morphisms may work as well. In §3 we study birational derived splinters and a non-Noetherian variant of which defined using the class of morphisms defined in §§2.3. In §4 we treat direct limit, and in §5 we treat étale extensions. The rest of the main results are proved in §6 and §7. We ask some questions in §8.

Acknowledgements. We thank Rankeya Datta, Linquan Ma, Takumi Murayama, and Kevin Tucker, discussions with whom for the previous work [Lyu22] persist to be helpful; we also thank János Kollár, Longke Tang, and Chenyang Xu for helpful discussions.

2. Preparations

For a ring $A$, the expression “$\dim A = 0$” means “every prime ideal of $A$ is maximal.” The reduction of $A$ is denoted by $A_{\text{red}}$. We do not distinguish between $A$-modules and quasi-coherent $\mathcal{O}_{\text{Spec}(A)}$-modules, and similarly $D(A) = D(\text{QCoh}(\mathcal{O}_{\text{Spec}(A)})) = D_{\text{QCoh}}(\mathcal{O}_{\text{Spec}(A)})$, cf. [Stacks, Tag 0620].

2.1. Residue extensions.

Definition 2.1. A residue extension is a flat local map $\varphi : (A, m, k) \rightarrow (B, n, l)$ of Noetherian local rings such that $n = mB$. A separable residue extension is a residue extension where $l/k$ is separable [Stacks, Tag 0300]; a
regular residue extension is a residue extension where \( \varphi \) is regular [Stacks, Tag 07BZ].

A regular residue extension is always separable (cf. [Stacks, Tag 0322]). The converse holds in the case \( A \) is a G-ring:

**Lemma 2.2** ([And74]). Let \( \varphi : A \to B \) be a separable residue extension of Noetherian local rings. If \( A \) is a G-ring then \( \varphi \) is a regular residue extension.

**Proof.** We have a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\iota_A} & & \downarrow{\iota_B} \\
A^\wedge & \xrightarrow{\varphi^\wedge} & B^\wedge
\end{array}
\]

of flat local maps of Noetherian local rings. Since \( \varphi \) is a separable residue extension, so is \( \varphi^\wedge \), thus \( \varphi^\wedge \) is regular [Stacks, Tag 07PM], and so is \( \iota_A \) by the definition of a G-ring. Since \( \iota_B \) is faithfully flat we see \( \varphi \) regular by [Stacks, Tags 07QI and 07NT]. \( \square \)

### 2.2. Total fraction rings.

**Lemma 2.3.** Let \( A \) be a ring. Let \( S \) be a multiplicative subset of \( A \) that consists of nonzerodivisors. If \( \dim(S^{-1}A) = 0 \), then the total fraction ring of \( A \) is \( S^{-1}A \).

**Proof.** It suffices to show that every nonzerodivisor \( x \in S^{-1}A \) is invertible. If not, take a maximal ideal \( m \) of \( S^{-1}A \) that contains \( x \). Since every prime ideal of \( S^{-1}A \) is maximal, \( m \) is a minimal prime and \( x \) is nilpotent in \( (S^{-1}A)_m \), contradiction. \( \square \)

**Lemma 2.4.** Let \( X \) be a set, \( A_x \ (x \in X) \) be a family of reduced rings with total fraction rings \( K_x \). If every \( A_x \) has finitely many minimal primes, then \( K := \prod_x K_x \) is the total fraction ring of \( A := \prod_x A_x \), and \( \dim K = 0 \).

**Proof.** Let \( S_x \) be the set of nonzerodivisors of \( A_x \). Then the set of nonzerodivisors of \( A \) is \( S = \prod_x S_x \), and \( S^{-1}A = K \), so \( K \) is the total fraction ring of \( A \). By [Stacks, Tags 02LX and 00EW], \( K_x \) is a product of fields, hence so is \( K \). Thus \( \dim K = 0 \), see [Stacks, Tags 092G and 092F]. \( \square \)

### 2.3. M-morphisms.

To define birational derived splinters, we need to make sense of what is “birational.” We use the following class of morphisms that behaves better with respect to limit and base change. The name “M-morphism” indicates its close relation to modifications [Stacks, Tag 0AAZ]. See Lemma 2.11 below.

**Definition 2.5.** Let \( A \) be a ring. Let \( f : X \to S = \Spec(A) \) be a morphism of schemes. We say \( f \) is an M-morphism if \( f \) is proper and of finite presentation, and for the total fraction ring \( K \) of \( A_{\text{red}} \), the base change of \( f \) to \( \Spec(K) \) is an isomorphism.
Note that we do not assume the irreducible components of $X$ lying above those of $S$.

**Lemma 2.6.** Let $A \to B$ be a ring map and $f : X \to \Spec(A)$ an $M$-morphism.

Assume that every nonzerodivisor of $A_{\text{red}}$ is mapped to a nonzerodivisor of $B_{\text{red}}$. Then the base change of $f$ to $B$ is an $M$-morphism.

**Proof.** We may assume both $A$ and $B$ are reduced. Now every nonzerodivisor of $A$ is mapped to a nonzerodivisor of $B$, so the total fraction ring $L$ of $B$ is naturally a $K$-algebra where $K$ is the total fraction ring of $A$. This proves the lemma. \hfill \square

**Lemma 2.7.** Let $A$ be a ring, $U$ a multiplicative subset of $A$, $B = U^{-1}A$.

Let $S = \Spec(A)$ and let $T$ be either a quasi-compact open subscheme of $S$ or $\Spec(B)$. Let $g : Y \to T$ be a proper morphism of finite presentation. Then there exists a proper morphism $f : X \to S$ of finite presentation such that $X \times_S T \cong Y$.

**Proof.** Assume first that $T = \Spec(B)$. We know that $B = \colim_{u \in U} A_u$, so there exists an element $u \in U$ and a proper morphism $f_1 : X_1 \to \Spec(A_u)$ of finite presentation such that $X_1 \times_{\Spec(A_u)} T \cong Y$, see [Stacks, Tags 012M and 081F]. So we may replace $T$ by $\Spec(A_u)$, and we may always assume $T$ a quasi-compact open subscheme of $S$. By Nagata compactification [Stacks, Tag 0F41], there exists an open immersion $j : Y \to X_2$ where $X_2$ is proper over $S$. Note that $j$ is quasi-compact as $T$ is. We may replace $X_2$ by the scheme-theoretic image of $j$ to assume $j$ scheme-theoretically dominant, in particular $X_2 \times_S T = Y$. Note that $X_2$ is not necessarily of finite presentation over $S$. However, by [Stacks, Tag 092R], $X_2 = \lim X_i$ where the transition maps are closed immersions and each $X_i$ is proper of finite presentation over $S$. The system $Y = X_2 \times_S T \to X_i \times_S T$ of morphisms of $X_i$-schemes satisfies [Stacks, Tag 081D] (as $X_2 \to X_i$ is a closed immersion), so by [Stacks, Tag 081E], we see that there exists an $i_0$ such that $Y = X_{i_0} \times_S T$. We take $X = X_{i_0}$. \hfill \square

**Lemma 2.8.** Let $A$ be a ring, $U$ a multiplicative subset of $A$, $B = U^{-1}A$. Let $K$ be the total fraction ring of $A$ and assume $\dim K = 0$; also assume $A$ is reduced and integrally closed in $K$.

Let $S = \Spec(A)$, and $T = \Spec(B)$. Let $g : Y \to T$ be an $M$-morphism. Then there exists an $M$-morphism $f : X \to S$ such that $X \times_S T \cong Y$.

**Proof.** By Lemma 2.7, there exists an $X_0$ proper of finite presentation over $S$ such that $X_0 \times_S T = Y$. In particular $h : X_0 \times_S \Spec(K) \to \Spec(K)$ is proper of finite presentation. We note that $h$ is flat since $K$ is reduced and $\dim K = 0$, see [Stacks, Tag 092F]. By [Stacks, Tags 0D4J and 02LS], there exists a unique clopen subset $E_1$ of $\Spec(K)$ such that $h$ is finite over $E_1$ and that $h$ does not have any finite fiber outside $E_1$. Then $h_* O_{X_0 \times_S \Spec(K)}$ is finite locally free over $E_1$. We now see that there is a unique clopen
subset $E_0$ of $E_1$ such that $h$ is an isomorphism over $E_0$ and the fibers of $h$ outside $E_0$ are either empty or non-trivial. Since $A$ is integrally closed in $K$, $E_0$ comes from a clopen subset $E$ of $S$, and it is now clear that

$$X := (X_0 \times_S E) \sqcup (S \setminus E) \to S$$

is a well-defined $M$-morphism.

Finally, we show $X \times_S T \cong Y$. Since $X_0 \times_S T \cong Y$, it suffices to show that the image of the morphism $T \to S$ is contained in $E$. As $\dim K = 0$, $\dim(U^{-1}K) = 0$, so $U^{-1}K$ is the total fraction ring of $B$, see Lemma 2.3. Therefore by construction and the fact that $g$ is an $M$-morphism, the image of $\Spec(U^{-1}K)$ in $\Spec(K)$ is contained in $E_0$. Since $\Spec(U^{-1}K) \to T$ is dominant (cf. [Stacks, Tag 00EW]) the image of $T$ is therefore contained in $E$ as desired.

**Lemma 2.9.** Let $A$ be an integral domain and $f : X \to \Spec(A)$ a proper surjective morphism of finite presentation. If there exists a (nonzero) $A$-algebra $B$ such that $f_B : X \times_{\Spec(A)} \Spec(B) \to \Spec(B)$ is an $M$-morphism, then $f$ is an $M$-morphism.

**Proof.** Replace $B$ by its reduction, and then a residue field of its total fraction ring, we may assume $B$ a field. Let $p = \ker(A \to B)$. Since $A$ is an integral domain it suffices to show $f_A$ an $M$-morphism. Thus we may assume $(A, p, k)$ local. $B$ is now a faithfully flat $k$-algebra, so $f_k$ is an isomorphism. By Zariski’s Main Theorem (cf. [Stacks, Tag 02UP]) $f$ is finite. By Nakayama’s Lemma $f$ is a closed immersion. Since $f$ is surjective and $A$ is reduced, $f$ is an isomorphism, as desired.

**Corollary 2.10.** Let $(A_i)_i$ be a direct system of integral domains and let $A = \colim A_i$. Let $f : X \to S := \Spec(A)$ be an $M$-morphism. Then there exists an index $i$ and an $M$-morphism $f_i : X_i \to S_i := \Spec(A_i)$ whose base change to $A$ is $f$.

**Proof.** There exists an index 0 and a morphism $f_0 : X_0 \to S_0$ of finite presentation whose base change to $A$ is $f$, see [Stacks, Tag 012M]. We may assume $f_0$ proper surjective by [Stacks, Tags 081F and 07RR]. Then $f_0$ is an $M$-morphism by Lemma 2.9.

Birational morphisms are defined as in [Stacks, Tag 01RN].

**Lemma 2.11.** Let $A$ be a Noetherian reduced ring, $f : X \to S = \Spec(A)$ be an $M$-morphism. Then there exists a reduced closed subscheme $Z$ of $X$ such that $Z \to S$ is birational.

Conversely, a proper birational morphism $g : Y \to S$ is an $M$-morphism.

**Proof.** For the first part we can just take $Z$ to be the scheme-theoretic image of $X \times_S \Spec(K)$ in $X$ where $K$ is the total fraction ring of $A$.

Every proper $A$-scheme is of finite presentation since $A$ is Noetherian. Therefore the second part follows immediately from [Stacks, Tag 08BAB].

2.4. Splitting in the derived category: trace ideals. Let $A$ be a ring, $f : X \to S = \Spec(A)$ be a qcqs morphism of schemes. Then $Rf_* \mathcal{O}_X$ lies in
ON SOME PROPERTIES OF BIRATIONAL DERIVED SPLINTERS

$D^b(A)$, see [Stacks, Tag 0805]. The formation of $Rf_*\mathcal{O}_X$ commutes with flat base change, see [Stacks, Tag 081B]. Therefore if $A \to Rf_*\mathcal{O}_X$ splits, then the same is true after any flat base change $A \to B$.

The canonical map $A \to Rf_*\mathcal{O}_X$ splits if and only if the induced map $\text{Hom}_{D(A)}(Rf_*\mathcal{O}_X, A) \to \text{Hom}_{D(A)}(A, A)$ of $A$-modules is surjective, by general category theory. The formation of this map commutes with flat base change if $Rf_*\mathcal{O}_X$ is pseudo-coherent [Stacks, Tag 08CB], see [Stacks, Tags 0A6A and 0A64]. This holds when $A$ is Noetherian and $f$ is proper (cf. [Stacks, Tag 08EB]).

Using the identification $A = \text{Hom}_{D(A)}(A, A)$, the morphism $f$ determines an ideal of $A$, namely the image of $\text{Hom}_{D(A)}(Rf_*\mathcal{O}_X, A) \to \text{Hom}_{D(A)}(A, A) = A$. We denote this ideal by $t(f)$ or $t(X/A)$. As noted in the previous paragraph, $A \to Rf_*\mathcal{O}_X$ splits if and only if $t(f) = A$ and the formation of $t(f)$ commutes with flat base change if $A$ is Noetherian and $f$ is proper. Note that in general, for any flat ring map $A \to B$, we have at least $t(X/A)B \subseteq t(X_B/B)$.

3. BIRATIONAL DERIVED SPLINTERS

In this section we study some fundamental properties of birational derived splinters, and the non-Noetherian variant MDSs as defined below. Many results are parallel to those in [Lyu22, §3], and we do refrain from writing too many “cf.”s. In this section and what follows, we shall use the term “birational derived splinter” for Noetherian rings and “MDS” for general rings to state our results, and these notions are the same for Noetherian rings, Lemma 3.4.

**Definition 3.1.** A ring $A$ is an M-derived splinter, shorthand MDS, if for every M-morphism (Definition 2.5) $f : X \to \text{Spec}(A)$ the map $A \to Rf_*\mathcal{O}_X$ splits in $D(A)$.

**Lemma 3.2.** An MDS $A$ is reduced and integrally closed in its total fraction ring.

**Proof.** Let $a \in A$ be nilpotent. Then $\text{Spec}(A/aA) \to \text{Spec}(A)$ is an M-morphism (Definition 2.5), so $A \to A/aA$ is a split map of $A$-modules. Thus $a = 0$ and $A$ is reduced.

Let $K$ be the total fraction ring of $A$ and $B \subseteq K$ be a finite $A$-algebra. We have $B = \text{colim}_i B_i$ where $B_i$ are finite of finite presentation over $A$ and the transition maps $B_i \to B_j$ are surjective, see [Stacks, Tag 09YY]. Using either [Stacks, Tag 081E] or [Stacks, Tag 00Q0] we see that $(B_i)_K = K$ for large $i$, so $\text{Spec}(B_i) \to \text{Spec}(A)$ is an M-morphism. Thus $A \to B_i$ is split, and we see $aB_i \cap A = aA$. Taking colimit, we see that $aB \cap A = aA$ for all $a \in A$. Since $B \subseteq K$, $A = B$, as desired. \qed

**Definition 3.3.** A Noetherian ring $A$ is a birational derived splinter, if $A$ is reduced and for every proper birational morphism $f : X \to \text{Spec}(A)$ the map $A \to Rf_*\mathcal{O}_X$ splits in $D(A)$. 
Here, the notion of a birational morphism to a not necessarily integral scheme is defined in \[ \text{Stacks, Tag 01RN}. \]

This is the same notion as Definition 3.1 for Noetherian rings:

**Lemma 3.4.** Let \( A \) be a Noetherian ring. Then \( A \) is an MDS if and only if \( A \) is a birational derived splinter, in which case \( A \) is normal.

**Proof.** A Noetherian MDS is normal by Lemma 3.2, so we always have \( A \) reduced. The equivalence of MDS and birational derived splinter follows immediately from Lemma 2.11. \( \square \)

In equal characteristic zero, we have the following characterization of birational derived splinters.

**Proposition 3.5** (cf. \[ \text{Kov00} \]). Let \( A \) be a quasi-excellent \( \mathbb{Q} \)-algebra. Then the followings are equivalent.

1. \( A \) is a birational derived splinter.
2. There exists a resolution of singularities \( f : X \to \text{Spec}(A) \) such that \( A \to Rf_*\mathcal{O}_X \) splits.
3. For any resolution of singularities \( f : X \to \text{Spec}(A) \), we have \( A = Rf_*\mathcal{O}_X \).

If the equivalent conditions above hold, we say \( A \) has rational singularities, cf. \[ \text{Kol13, Definition 2.76}. \]

**Proof.** In all cases \( A \) is normal. The existence of resolutions \[ \text{Tem12, Theorem 1.2.1} \] shows that (3) implies (1) and that (1) implies (2). It suffices to show (2) implies (3).

We make the following observation. Let \( p \in \text{Spec}(A) \) and let \( A' \) be the completion of \( A_p \). Then \( A_p \to A' \) is regular by definition, so \( A' \) is normal and if \( f : X \to \text{Spec}(A) \) is a resolution, then the base change \( f' : X' := \text{Spec}(A) \times \text{Spec}(A') \to \text{Spec}(A') \) is also a resolution, see for example \[ \text{Stacks, Tags 0C22 and 033A}. \] Since \( p \) was arbitrary, by flat base change (cf. §§2.4) and descent, we may always assume \( A \) complete local. In particular, \( A \) has a dualizing complex \[ \text{Stacks, Tag 0BFR} \] and is excellent.

Assume (2). \[ \text{CR15} \] and the existence of resolutions \[ \text{Tem12, Theorem 1.2.1} \] show that for all resolutions \( g : Y \to \text{Spec}(A) \), \( A \to Rg_*\mathcal{O}_Y \) splits. Since we have the Grauert-Riemenschneider vanishing theorem \[ \text{Mur21b, Theorem A} \] in this case, \[ \text{Kol13, Corollary 2.75} \] shows that \( Rg_*\mathcal{O}_Y = A \), as desired. \( \square \)

**Remark 3.6.** Note that a quasi-excellent \( \mathbb{Q} \)-algebra that has rational singularities is Cohen-Macaulay, see \[ \text{Kol13, Proposition 2.77}. \] Thus by Lemma 3.9 below and \[ \text{Mat80, (34.A)} \] we see a G-ring that contains \( \mathbb{Q} \) and is a birational derived splinter is Cohen-Macaulay.

If we drop the excellence assumption completely, then the definition of rational singularities above may be vacuous, cf. \[ \text{EGA IV}_2, \text{Proposition 7.9.5}. \] The author does not know, for example, if all Noetherian birational derived splinter containing \( \mathbb{Q} \) are Cohen-Macaulay.
Lemma 3.7. Let $X$ be a set and $A_x (x \in X)$ a family of MDSs. Then $A := \prod_{x \in X} A_x$ is an MDS.

Proof. We note first that a product of reduced rings is reduced, so by Lemma 3.2 we see $A$ and all $A_x$ reduced.

Let $g : Y \to \text{Spec}(A)$ be an $\mathcal{M}$-morphism and $g_x : Y_x \to \text{Spec}(A_x)$ be the base change of $g$ along the projection map. Note that a nonzerodivisor of $A$ is just an element all of whose coordinates are nonzerodivisors. Thus by Lemma 2.6, $g_x$ is an $\mathcal{M}$-morphism and thus $A_x \to Rg_{x*}\mathcal{O}_{Y_x}$ splits in $D(A_x)$. Taking product, we see that $A \to \prod_x Rg_{x*}\mathcal{O}_{Y_x}$ splits in $D(A)$. Since $A \to A_x$ is flat, $Rg_{x*}\mathcal{O}_{Y_x} = Rg_x\mathcal{O}_Y \otimes_A A_x$, so $A \to \prod_x Rg_{x*}\mathcal{O}_{Y_x}$ factors through $A \to Rg_x\mathcal{O}_Y$ and thus the latter map splits. □

Remark 3.8. One can do the proof in terms of explicit complexes if not comfortable with infinite products in the derived category. Cover $Y$ by affine opens $U^{(j)} (1 \leq j \leq n)$. Then we have the Čech complex $\check{C}^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{O})$ that computes $Rg_x\mathcal{O}_{Y_x}$, and we see that the complex $\check{C}^\bullet_x := \check{C}^\bullet \otimes_A A_x$ is a Čech complex that computes $Rg_{x*}\mathcal{O}_{Y_x}$. Denote by $\varphi_x$ the canonical map $A_x \to \check{C}^\bullet_x$, so $\varphi_x$ splits in $D(A_x)$, which means that there exists an $A_x$-complex $F_x^\bullet$ together with a map $\psi_x : F_x^\bullet \to A_x$ and a quasi-isomorphism $s_x : F_x^\bullet \to \check{C}^\bullet_x$ such that $H^0(\psi_x) \circ H^0(s_x)^{-1} \circ H^0(\varphi_x) = \text{id}_{A_x}$. Taking product of these data we get an $A$-complex $F^\bullet = \prod_x F_x^\bullet$, a map $\psi = \prod_x \psi_x$ and a quasi-isomorphism $s = \prod_x s_x$ satisfying the same conditions and we see $\prod_x A_x \to \prod_x \check{C}^\bullet_x$ splits in $D(A)$. Finally, we have the canonical factorization $A = \prod_x A_x \to \check{C}^\bullet \to \prod_x \check{C}^\bullet_x$ in the category of cochain complexes of $A$-modules. Thus $A \to \check{C}^\bullet$ splits in $D(A)$ as desired.

Lemma 3.9. Let $A$ be an MDS with total fraction ring $K$. Assume that $\dim K = 0$. Then any localization $S^{-1}A$ is an MDS.

Proof. We need to find a splitting $S^{-1}A \to Rg_x\mathcal{O}_Y$ for every $\mathcal{M}$-morphism $g : Y \to \text{Spec}(S^{-1}A)$. By Lemma 3.2 $A$ is reduced and integrally closed in $K$, so Lemma 2.8 applies and shows that $g$ extends to an $\mathcal{M}$-morphism $f : X \to \text{Spec}(A)$, and $A \to Rf_{x*}\mathcal{O}_X$ splits. By flat base change, see §§2.4, $S^{-1}A \to Rg_x\mathcal{O}_Y$ splits as desired. □

Recall that a ring map $A \to B$ is pure if it is universally injective as a map of $A$-modules. See [Stacks, Tag 058H]. The fact that $B$ is not necessarily assumed Noetherian in the following result is essential later on.

Proposition 3.10. Let $A \to B$ be a pure ring map that sends nonzerodivisors to nonzerodivisors (for example, a faithfully flat ring map). Assume that $A$ is Noetherian and that $B$ is an MDS. Then $A$ is a birational derived splinter.

Proof. Since $B$ is an MDS, it is reduced by Lemma 3.2, so $A$ is reduced as a pure ring map is injective.
Let \( f : X \to S = \text{Spec}(A) \) be an M-morphism, and \( T = \text{Spec}(B) \). The base change \( f_T : X_T \to T \) is also an M-morphism, see Lemma 2.6. Since \( B \) is an MDS, \( B \to Rf_{T*}O_{X_T} \) splits in \( D(B) \).

Let \( m \) be a maximal ideal of \( A \) and let \( A' \) be the completion of \( A \) at \( m \). By flat base change, see §§ 2.4, \( B' \to Rf'_{T*}O_{X'_T} \) splits in \( D(B') \), where \((-)'\) is the base change along \( A \to A' \). We also note that \( A' \to B' \) is pure, and that \( A' \) is a Noetherian complete local ring, so \( A' \to B' \) splits as a map of \( A' \)-modules by [Fed83, Lemma 1.2]. Therefore \( A' \to Rf'_{T*}O_{X'_T} \) splits in \( D(A') \), and since this map factors through \( A' \to Rf'_sO_{X'} \), we see that \( A' \to Rf'_sO_{X'} \) splits in \( D(A') \) as well.

Using the observations made in §§ 2.4, we see that the ideal \( t(f) \) satisfies \( t(f)A' = A' \). Since \( m \) was arbitrary, we see \( t(f) = A \), as desired. □

**Corollary 3.11.** Let \( A \) be a Noetherian ring. The followings are equivalent.

1. \( A \) is a birational derived splinter.
2. \( A_p \) is a birational derived splinter for all prime ideals \( p \) of \( A \).
3. \( A_m \) is a birational derived splinter for all maximal ideals \( m \) of \( A \).

**Proof.** We have \( 1 \Rightarrow 2 \) by Lemma 3.9, and \( 2 \Rightarrow 3 \) is trivial.

The implication \( 3 \Rightarrow 1 \) follows from a similar but much simpler argument as in the proof of Proposition 3.10, but we can also obtain the result directly as follows. For a Noetherian ring \( A \), the map \( A \to \prod_mA_m \), where \( m \) runs over all maximal ideals of \( A \), is faithfully flat, cf. [Stacks, Tags 05CZ and 05CY].

Our result now follows from Lemma 3.7 and Proposition 3.10. □

In the Noetherian case, we also have the following strengthening of Proposition 3.10. Together with Proposition 3.5 and [HH95, Lemma 2.2], we recover [Mur21b, Theorem C] (note that our definition is slightly different from [Mur21b, Definition 7.2]).

**Corollary 3.12.** Let \( A \to B \) be a cyclically pure ring map, that is, for all ideals \( I \subseteq A \) we have \( IB \cap A = I \). If \( B \) is a Noetherian birational derived splinter, so is \( A \), and \( A \to B \) is pure.

**Proof.** From Lemma 3.4 we see \( B \) normal. By [Has10, Corollary 3.12] (which ultimately comes from [Hoc77]), \( A \) is a Noetherian normal ring and \( A \to B \) is pure.

For every \( p \in \text{Spec}(A) \), \( A_p \to B_p \) is pure, so [HH95, Lemma 2.2] shows that there exists \( q \in \text{Spec}(B) \) such that \( q \cap A \subseteq p \) and \( A_p \to B_q \) is pure. Now \( A_p \) and \( B_q \) are integral domains, so \( A_p \to B_q \) sends nonzerodivisors to nonzerodivisors. By Proposition 3.10 (and Lemma 3.9), \( A_p \) is a birational derived splinter. Since this holds for all \( p \in \text{Spec}(A) \) we see \( A \) is a birational derived splinter from Corollary 3.11. □

4. Direct limit

In this section we consider the direct limit of a system of birational derived splinters. We need to do some work on derived categories and direct limit.
CONVENTION: by a complex of modules over a ring $A$ we mean a cochain complex in the abelian category of $A$-modules. We usually denote it by $\square^\bullet$ where $\square$ is a capital Latin letter, in which case $\square$ denotes the corresponding object in the derived category $D(A)$.

4.1. Splitting in the derived category: from perfect to pseudo-coherent. In this subsection we establish the following result. This is inspired by a notion of purity in general categories, cf. [AR94, §2.D].

**Lemma 4.1.** Let $R$ be a ring, $E$ a bounded object in $D(R)$, $F$ a pseudo-coherent object in $D(R)$ [Stacks, Tag 064Q].

Let $\varphi : E \to F$ be a map in $D(R)$. If for all perfect objects [Stacks, Tag 0657] $K \in D(R)$ and all factorizations $E \to K \to F$ of $\varphi$ the map $E \to K$ is a split mono in $D(R)$, then $\varphi$ is a split mono in $D(R)$.

**Proof.** By definition, we may represent the object $F$ by a complex $F^\bullet$ of finite free $R$-modules such that $F^i = 0$ for $i$ outside an interval $(-\infty, b]$ ($b \in \mathbf{Z}$). Choose any complex $E^\bullet$ that represents $E$ together with a map of complexes $\varphi^\bullet : E^\bullet \to F^\bullet$ that represent $\varphi$. Since $E$ is bounded, we may assume that $E^i$ are zero for $i$ outside an interval $[a, b']$ ($a, b' \in \mathbf{Z}, a \leq b'$).

Next, let $K^\bullet$ be the complex defined by $K^i = F^i$ ($i \geq a - 1$) and $K^i = 0$ ($i < a - 1$). (This is the stupid truncation $\sigma_{\geq a - 1}F^\bullet$, see [Stacks, Tag 0118].) Then $K^\bullet$ is a perfect complex of $\text{Tor}$-amplitude $[a - 1, b]$ that canonically maps to $F^\bullet$, and the map $\varphi^\bullet$ factors as $E^\bullet \xrightarrow{\psi^\bullet} K^\bullet \to F^\bullet$.

Now, by assumption, the map $\psi : E \to K$ is a split mono in $D(R)$. This means that there exists a quasi-isomorphism of complexes $s^\bullet : K^\bullet \to E^\bullet$ and a map of complexes $\psi^\bullet : K^\bullet \to E^\bullet$ such that $\psi^\bullet \circ s^{-1} \circ \psi = \text{id}_E$ in $D(R)$. If $s^\bullet : K^\bullet \to E^\bullet$ is a quasi-isomorphism of complexes, then we may replace $K^\bullet, s, \psi^\bullet$ by $K'^\bullet, s \circ s', \psi^\bullet \circ s'$. Therefore we may assume $K'^\bullet$ are zero for $i \notin [a - 1, b']$.

Now, the composition $\theta : K'^\bullet \xrightarrow{s} K^\bullet \to F^\bullet$ is bijective on $H^i$ ($i \geq a$) and surjective on $H'^i$ ($i \geq a - 1$). Thus one can find a complex $F'^\bullet$ with $F'^i = K'^i$ when $i \geq a - 1$ and a quasi-isomorphism $s'^\bullet : F'^\bullet \to F^\bullet$ that agrees with $\theta$ on degrees $\geq a - 1$. This is by the same process as that to find a projective resolution of a complex, see for example the proof of [Stacks, Tag 0577]. Since $E^i = 0$ for $i < a$ and $F'^i = K'^i$ for $i \geq a - 1$, $\psi'^\bullet : K'^\bullet \to E^\bullet$ extends canonically to a map $\psi'^\bullet : F'^\bullet \to E^\bullet$ of complexes by putting $\psi'^i = 0$ when $i < a - 1$. Since $F$ and $K$ also only differ in degrees $< a - 1$, $\psi'^\bullet \circ s'^{-1} \circ \varphi = \psi^\prime \circ s^{-1} \circ \psi = \text{id}_E$, as desired.  

4.2. Splitting in the derived category: limits. In this subsection fix a direct system $(R_i)_{i \in A}$ of rings with colimit $R$.

Let $C^\bullet_i$ be complexes of $R_i$-modules equipped with compatible maps of $R_i$-complexes $C^\bullet_i \to C^\bullet_j$ for $i \leq j$. Then the termwise colimit $C^\bullet = \text{colim}_i C^\bullet_i$ is canonically a complex of $R$-modules and $H^n(C^\bullet) = \text{colim}_i H^n(C^\bullet_i)$ for all $n \in \mathbf{Z}$, since filtered colimit is exact.
Let $K^\bullet$ be a complex of finitely presented $R$-modules with finitely many nonzero terms. Then there exists an index $i_0$ and a complex $K_{i_0}^\bullet$ of finitely presented $R_{i_0}$-modules with finitely many nonzero terms such that $K_{i_0}^\bullet \otimes_{R_{i_0}} R = K^\bullet$ (cf. [Stacks, Tag 05N7]). Note that in general $K_{i_0}^\bullet \otimes_{R_{i_0}} R \neq K \in D(R)$. Fix such a choice of $i_0$ and $K_{i_0}^\bullet$ and denote by $K^\bullet$ the tensor product $K_{i_0}^\bullet \otimes_{R_{i_0}} R_i$ for $i \geq i_0$, so $K^\bullet = \operatorname{colim}_{i \geq i_0} K_i^\bullet$. Then we have (see [Stacks, Tags 05DQ and 0G8P])

$$\operatorname{Hom}_R(K^\bullet , C^\bullet) = \operatorname{Hom}_{R_{i_0}}(K_{i_0}^\bullet , C^\bullet)$$
$$= \operatorname{Hom}_{R_{i_0}}(K_{i_0}^\bullet , \operatorname{colim}_{i \geq i_0} C_i^\bullet)$$
$$= \operatorname{colim}_{i \geq i_0} \operatorname{Hom}_{R_{i_0}}(K_{i_0}^\bullet , C_i^\bullet)$$
$$= \operatorname{colim}_{i \geq i_0} \operatorname{Hom}_{R_i}(K_i^\bullet , C_i^\bullet).$$

Now, let $K \in D(R)$ be perfect [Stacks, Tag 0657] and let $\psi : K \to C$ be a map in $D(R)$. Then there exists a complex $K^\bullet$ of finite projective $R$-modules with finitely many nonzero terms and a map of complexes $\psi^\bullet : K^\bullet \to C^\bullet$ (see the proof of [Stacks, Tag 0658]) that represents $\psi$. Fix a choice of $i_0$ and $K_{i_0}^\bullet$ as above, and enlarge $i_0$ if necessary, we may assume that there exists a map of $R_{i_0}$-complexes $\psi_{i_0} : K_{i_0}^\bullet \to C_{i_0}^\bullet$ compatible with $\psi$. Since being finite projective is the same as being a direct summand of a finite free module we see from [Stacks, Tag 05N7] that enlarging $i_0$ if necessary we may assume each term of $K_{i_0}^\bullet$ finite projective, in particular flat. Then $K_{i_0} \otimes^{L}_{R_{i_0}} R = K \in D(R)$, and we have the following commutative diagram in $D(R)$:

$$\begin{array}{ccc}
K_{i_0} \otimes^{L}_{R_{i_0}} R & \xrightarrow{\psi_{i_0} \otimes 1} & C_{i_0} \otimes^{L}_{R_{i_0}} R \\
\downarrow & & \downarrow \\
K & \xrightarrow{\psi} & C
\end{array}$$

If, furthermore, we are given an integer $n$ and a cohomology class $\alpha_{i_0} \in H^n(C_{i_0})$ such that the image $\alpha \in H^n(C)$ comes from a $\beta \in H^n(K^\bullet)$, then $\beta$ comes from some $\beta_{i_0} \in H^n(K_{i_0}^\bullet)$ after possibly enlarging $i_0$. The class $\alpha_{i_0}$ does not necessarily agree with $\psi_{i_0}(\beta_{i_0})$. However, since $H^n(C^\bullet) = \operatorname{colim}_{i} H^n(C_i^\bullet)$, after possibly enlarging $i_0$ we will have $\alpha_{i_0} = \psi_{i_0}(\beta_{i_0})$. The discussions above lead us to

**Lemma 4.2.** Let $R = \operatorname{colim}_{i} R_i$ be a direct limit of rings. Let $i_0$ be an index and $f_{i_0} : X_{i_0} \to \operatorname{Spec}(R_{i_0})$ be a quasi-compact and separated morphism of schemes. Let $f_i : X_i \to \operatorname{Spec}(R_i)$ and $f : X \to \operatorname{Spec}(R)$ be the base change of $f_{i_0}$ to $R_i$ and $R$ respectively.

Let $K \in D(R)$ be a perfect object and $\psi : K \to Rf_\ast \mathcal{O}_X \in D(R)$ a map in $D(R)$. Assume that there exists a class $\beta \in H^0(K)$ that maps to $1 \in f_\ast \mathcal{O}_X$.

Then for large enough $i$, $\psi$ factors through the canonical map $Rf_\ast \mathcal{O}_X \otimes^{L}_{R_i} R \to Rf_\ast \mathcal{O}_X$, in a way that $\beta \in H^0(K)$ is mapped to $1 \in H^0(Rf_\ast \mathcal{O}_X \otimes^{L}_{R_i} R)$. 
Proof. Fix a finite affine cover \( X_{i_0} = \bigcup_{j=1}^m U_{i_0}^{(j)} \). Then the Čech complex \( \check{C}_{i_0}^* \) associated with this open cover and the structure sheaf \( \mathcal{O}_{X_{i_0}} \) computes \( Rf_{i_0, *} \mathcal{O}_{X_{i_0}} \), see [Stacks, Tag 01XD]. We can use the base change of this open cover to compute \( Rf_{i_*} \mathcal{O}_{X_i} (i \geq i_0) \) and \( Rf_* \mathcal{O}_X \) with corresponding Čech complexes \( \check{C}_i^* = \check{C}_{i_0}^* \otimes_{R_{i_0}} R_i \) and \( \check{C}^* = \check{C}_{i_0}^* \otimes_{R_{i_0}} R = \colim_{i \geq i_0} \check{C}_i^* \).

The canonical class \( 1 \in H^0(\check{C}^*_i) \) is defined by the unit element of the ring \( \check{C}^0_i = \prod_{j=1}^m \mathcal{O}_{X_i}(U_i^{(j)}) \). Therefore we can apply the discussions above to the system \( \check{C}_i^* \) and for some \( i \geq i_0 \) find a commutative diagram

\[
\begin{array}{ccc}
K_i \otimes_{R_i} R & \psi_i \otimes 1 \longrightarrow & \check{C}_i \otimes_{R_i} R \\
\downarrow & & \downarrow \\
K & \psi \longrightarrow & \check{C}
\end{array}
\]

in \( D(R) \) and a class \( \beta_i \in H^0(K_i^*) \) that maps to \( \beta \in H^0(K) \) and to \( 1 \in H^0(\check{C}_i) \) by \( \psi_i \). We win. \( \square \)

Remark 4.3. Using the language of derived \( \infty \)-categories [HA, 1.3.2], we can prove the lemma above as follows. We may assume that \( i_0 \) is the smallest element of the index set. The Čech complex consideration tells us \( Rf_* \mathcal{O}_X = \colim_i Rf_{i_*} \mathcal{O}_{X_i} \) in the (bounded-above) derived \( \infty \)-category of abelian groups, and this identification preserves \( R_i \)-structure. (This colimit is the “homotopy colimit” (cf. [Stacks, Tag 0902]).) Therefore

\[
Rf_* \mathcal{O}_X = \colim_{(i,j); i \leq j} \left( Rf_{i_*} \mathcal{O}_{X_i} \otimes_{R_i} R_j \right)
\]

since \( \{(i, i) \mid i \in \Lambda \} \) is a cofinal subset of \( \{(i, j) \mid i \leq j \} \). Taking colimit with respect to \( j \) we get

\[
Rf_* \mathcal{O}_X = \colim_i \left( Rf_{i_*} \mathcal{O}_{X_i} \otimes_{R_i} R \right)
\]

in the (bounded-above) derived \( \infty \)-category of \( R \)-modules. The perfect object \( K \) is a compact object in this \( \infty \)-category (cf. [HA, Propositions 7.2.4.2 and 7.1.1.15]), so we obtain a factorization \( K \to Rf_{i_*} \mathcal{O}_{X_i} \otimes_{R_i} R \) for some \( i \). Enlarging \( i \) we may assume \( \beta \) is mapped to \( 1 \) and we win.

4.3. Direct limit of Noetherian birational derived splinters. We turn to the main result of this section.

Theorem 4.4. Let \((S_i)_{i} \) be a direct system of Noetherian rings with colimit \( S \). Assume that \( S \) is Noetherian, and that each \( S_i \) is a birational derived splinter. Then \( S \) is a birational derived splinter.

Proof. Since \( S \) and all \( S_i \) are Noetherian, the problem is local by Corollary 3.11. We may therefore assume that \( S \) is local, that each \( S_i \) is local, and that the transition maps are local. By Lemma 3.4 each \( S_i \) is normal, hence an integral domain.

Let \( f : X \to \text{Spec}(S) \) be an \( M \)-morphism. By Corollary 2.10, there exists an index \( i_0 \) and an \( M \)-morphism \( f_{i_0} : X_{i_0} \to \text{Spec}(S_{i_0}) \) whose base
change to $S$ gives $f$. As usual, denote by $f_i : X_i \to \text{Spec}(S_i)$ the base change of $f_0$ to $\text{Spec}(S_i)$ where $i \geq i_0$. By Lemma 2.9 each $f_i$ is an $\mathbb{M}$-morphism and thus $S_i \to Rf_{i*}\mathcal{O}_{X_i}$ splits in $D(S_i)$. Therefore we obtain a map $\epsilon_i : Rf_{i*}\mathcal{O}_{X_i} \otimes_{S_i}^L S \to S$ in $D(S)$ that sends 1 to 1.

Now, let $K \in D(S)$ be perfect and let $S \xrightarrow{\psi} K \xrightarrow{\delta_i} Rf_{i*}\mathcal{O}_{X_i}$ be a factorization of the canonical map $S \to Rf_*\mathcal{O}_X$. Since $S$ is Noetherian, $Rf_*\mathcal{O}_X \in D(S)$ is pseudo-coherent (cf. [Stacks, Tag 08E8]). Thus by Lemma 4.1 it suffices to show that $\psi$ is a split mono in $D(S)$.

Denote by $\beta$ the image of $1 \in S$ in $H^0(K)$, so $\psi$ maps $\beta$ to 1. By Lemma 4.2, for large $i$ there exists a map $\delta_i : K \to Rf_{i*}\mathcal{O}_{X_i} \otimes_{S_i}^L S$ in $D(S)$ that sends $\beta$ to 1. Composing with $\epsilon_i$, we get a map $K \to S$ in $D(S)$ that sends $\beta$ to 1, in other words, a splitting of $\varphi$, as desired.

Combine with Proposition 3.5, in equal characteristic zero we have the following result.

**Corollary 4.5.** Let $A$ be a Noetherian quasi-excellent $\mathbb{Q}$-algebra. If $A$ is the direct limit of a system of Noetherian quasi-excellent $\mathbb{Q}$-algebras that have rational singularities, then $A$ has rational singularities.

In general, in view of Popescu’s theorem [Stacks, Tag 07GC], we have the following.

**Corollary 4.6.** Let $S_0 \to S$ be a regular map of Noetherian rings. If every smooth $S_0$-algebra is a birational derived splinter, then $S$ is a birational derived splinter.

**Remark 4.7.** Assume both $S_0$ and $S$ are complete local and the map $S_0 \to S$ is local. Let $k_0$ (resp. $k$) be the residue field of $S_0$ (resp. $S$). If $k/k_0$ is separable [Stacks, Tag 0300], we can argue as follows to prove Corollary 4.6 avoiding previous derived category techniques. By [Stacks, Tags 03C3 and 050i] we can find a residue extension (Definition 2.1) $S_0 \to T$ that realizes $k_0 \to k$. $T$ is then formally smooth over $S_0$ by [Stacks, Tag 07PM] and thus maps to $S$. Now it is clear that $S$ is isomorphic to a power series algebra over $T$, thus a residue extension of a localization of a polynomial algebra over $S_0$. By Lemma 2.2 this residue extension is regular. Then we conclude by Theorem 1.3, which is proved independent of the methods in this section (modulo the proof of Corollary 5.2).

However, in general $k/k_0$ is not necessarily separable. A counterexample is given by $S_0 = k_0 = F_p(t)$, $S = S_0[X]_{(X^p - t)}$, so $k = F_p(t^{1/p})$.

5. \textit{\'{E}tale ascent}

We remark the following result on birational maps, which can be understood as a version of Chow’s lemma for algebraic spaces [RG71, Corollaire 5.7.13]. Nevertheless, for the reader’s convenience, we (re-)do the proof using just schemes. The arguments here are similar, but not identical, to those in [RG71, §5.7].
Theorem 5.1. Let $A$ be a Noetherian reduced ring, $A \to B$ an étale ring map. Let $g : Y \to T := \text{Spec}(B)$ be an $M$-morphism. Then there exists an $M$-morphism $f : X \to S := \text{Spec}(A)$ that admits a morphism $X \times_S T \to Y$ of $T$-schemes.

In particular, if $A$ is a birational derived splinter, so is $B$ (cf. [DT19, Theorem A]).

Proof. The “in particular” statement follows from flat base change, see §§2.4.

By Zariski’s Main Theorem there exists a factorization $T \rightarrow T \rightarrow S$ where $\pi$ is finite and $j$ is an open immersion. Write $T = \text{Spec}(B)$. We can replace $T$ by the scheme-theoretic image of $j$ to assume $j$ scheme-theoretically dominant, in particular, for the total fraction ring $K$ of $A$, $B \otimes_A K = B \otimes_A K$, because a dense subset of the discrete space $\text{Spec}(B \otimes_A K)$ must be the whole space.

By Lemma 2.7 (which has a much shorter proof in the Noetherian case), there exists a proper morphism $\overline{g} : Y \to \overline{T}$ such that $Y \times_{\overline{T}} T = Y$. We know $B \otimes_A K$ is the total fraction ring of $B = B_{\text{red}}$ (cf. Lemma 2.3). Since $g$ is an $M$-morphism, $g$ is an isomorphism after the base change $A \to K$. Since $B \otimes_A K = B \otimes_A K$ the same is true for $\overline{g}$. Picture:

$$
\begin{array}{ccc}
Y \times_S \text{Spec}(K) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
T \times_S \text{Spec}(K) & \longrightarrow & T \\
\text{finite flat} & & \\
\text{Spec}(K) & \longrightarrow & S \\
\end{array}
$$

By [Stacks, Tag 02KB] we know that a finite flat morphism of finite presentation is finite locally free (see [Stacks, Tag 02KA] for definition). By [Stacks, Tag 06AC], the composition $Y \to T \to S$ is finite locally free after the base change $\text{Spec}(A_a) \to S$ for some nonzerodivisor $a \in A$. By a consequence [Stacks, Tag 0B49] of “flattening” [RG71, Théorème 5.2.2], there exists a blow-up $f : X \to S$ of an ideal supported on $\text{Spec}(A/aA)$ set-theoretically and a closed subscheme $Y' \subseteq Y \times_S X$ finite locally free over $X$ that agrees with $Y \times_S X$ over $\text{Spec}(A_a) \subseteq S$. The last bit follows from the definition of strict transform [Stacks, Tag 080D]. We see that $f : X \to S$ is proper and that the base change of $f$ to $\text{Spec}(A_a)$ is an isomorphism, so $f$ is an
M-morphism. Picture:

\[
\begin{array}{cccc}
Y' & \longrightarrow & Y \times_S X & \longrightarrow & Y \\
\downarrow & & \downarrow \pi & & \\
T \times_S X & \longrightarrow & T & \longrightarrow & \\
\downarrow & & \downarrow \pi & & \\
X & \longrightarrow & S & & \\
\end{array}
\]

Let \( Y' = Y \times_T T \) and \( h : Y' \to T \times_S X \) be the morphism induced by \( g \). Then the following diagram of schemes is commutative:

\[
\begin{array}{cccc}
Y' & \longrightarrow & Y \\
\downarrow h & & \downarrow g \\
T \times_S X & \longrightarrow & T \\
\downarrow & & \downarrow \\
X & \longrightarrow & S \\
\end{array}
\]

It now suffice to show that \( h \) is an isomorphism, so \( T \times_S X \) dominates \( Y \).

By our choice \( Y' \to X \) is quasi-finite and flat. Since \( T \times_S X \to X \) is étale, \( h \) is quasi-finite and flat (cf. [Stacks, Tag 04R3]). However, by construction, \( h \) is proper, so \( h \) is finite flat, and we need to show that the degree of \( h \) is 1. We have \( Y' = Y \times_S X \) over \( \operatorname{Spec}(A_a) \), so \( Y' = Y \times_S X \) over \( \operatorname{Spec}(A_a) \); also \( Y = T \) after the base change \( A \to K \) as noted before. Thus \( h \) is an isomorphism after the base change \( A \to K \), so the locus where \( \deg h = 1 \) contains the closure of the image of \( b : T \times_S X \times_S \operatorname{Spec}(K) \to T \times_S X \). However, \( X \to S \) is a blowup, so a nonzerodivisor in \( A \) is always a nonzerodivisor on \( X \). Thus \( X \times_S \operatorname{Spec}(K) \to X \) is (scheme-theoretically) dominant. Then the same is true after the flat base change \( T \to S \), so \( b \) is dominant and \( h \) is an isomorphism. \( \square \)

**Corollary 5.2.** Let \( A \) be a Noetherian birational derived splinter and let \( B \) be a direct limit of étale \( A \)-algebras. If \( B \) is Noetherian, then \( B \) is a birational derived splinter.

**Proof.** Immediate from Theorem 4.4. Alternatively, one can apply Corollaries 3.11 and 2.10 and flat base change (§§2.4); details of this route are left to the reader. \( \square \)

**Remark 5.3.** We can also deduce from Theorem 5.1 that if \( A \to B \) is a local map of Noetherian local rings that is essentially étale, and \( A \) is pseudo-rational in the sense of [LT81], then so is \( B \).
6. Regular residue extensions

We continue to use our ad-hoc definition of ultraproduct of local rings [Lyu22, Definition 3.1].

**Definition 6.1.** Let $X$ be an index set and $A_x$ $(x \in X)$ be a family of local rings. An *ultraproduct* of the rings $A_x$ $(x \in X)$ is the localization of the ring $\prod_x A_x$ at a maximal ideal.

An *ultrapower* of a local ring $A$ is an ultraproduct of a family $A_x$ $(x \in X)$ where each $A_x = A$.

For convenience, we state the next two lemmas for ultrapowers. One can easily extend them to general ultraproducts.

**Lemma 6.2.** Let $A$ be a local domain. Then any ultrapower of $A$ is an integral domain.

*Proof.* Let $A^\natural = (A^X)^M$ be an ultrapower of $A$ where $X$ is an index set. To show $A^\natural$ an integral domain, it suffices to show that if $a = (a_x)_{x \in X}$ and $b = (b_x)_{x \in X}$ are two elements in $A^X$ such that $ab = 0$, then there exists an $s \notin M$ such that $sa = 0$ or $sb = 0$.

Let $U = \{x \in X \mid a_x = 0\}$ and $V = \{x \in X \mid b_x = 0\}$, so $U \cup V = X$ as $A$ is an integral domain. Let $e_U \in A^X$ be the idempotent defined by $(e_U)_x = 1$ when $x \in U$ and $(e_U)_x = 0$ when $x \notin U$ and define $e_V$ similarly. Then $(1 - e_U)(1 - e_V) = 0$, so either $e_U \notin M$ or $e_V \notin M$. But $e_U a = 0 = e_V b$. We win. □

**Lemma 6.3.** Let $S$ be a local MDS, $S^\natural$ an ultrapower of $S$. If $S$ has finitely many minimal primes, then $S^\natural$ is an MDS.

*Proof.* $S^\natural$ is a localization of some $S^X$ which is an MDS by Lemma 3.7.

Let $K$ be the total fraction ring of $S$. Note that $S$ is reduced by Lemma 3.2. By Lemma 2.4, $K^X$ is the total fraction ring of $S^X$ and $\dim(K^X) = 0$. The result now follows from Lemma 3.9. □

**Theorem 6.4.** Let $(S, m) \to (S', m')$ be a regular homomorphism of Noetherian local rings with $m' = mS'$. If $S$ is a birational derived splinter, so is $S'$. In particular, the completion of a local G-ring that is a birational derived splinter is a birational derived splinter.

*Proof.* The proof is mostly verbatim to the proof of [Lyu22, Theorem 1.4], replacing the ingredients Lemma 2.2, Theorem 2.8, and Lemma 3.2 there by our Proposition 3.10, Corollary 5.2, and Lemma 6.3 respectively. The only difference occurs at the last step, where for Proposition 3.10 to apply, we need to know that the pure map $S' \to S^\natural$ sends nonzerodivisors to nonzerodivisors. However, $S$ is an integral domain (Lemma 3.4), hence so is $S^\natural$ (Lemma 6.2). We win.

**Corollary 6.5.** Let $S$ be a Noetherian local G-ring that is a birational derived splinter. Then any separable residue extension (Definition 2.1) of $S$ is a birational derived splinter.
Proof. In view of Lemma 2.2, this is the special case of Theorem 6.4 where $S$ is a G-ring.

7. OPENNESS IN PRIME CHARACTERISTIC AND REGULAR ASCENT

We use the notations in [DT21, §§2.3] (Frobenius pushforward), and in our §§2.4 (trace ideal).

We need one preparation before we go into the argument.

Lemma 7.1. Let $A$ be as in Theorem 1.5. Then $A$ is quasi-excellent. Moreover, the normal, Cohen-Macaulay, and $F$-pure loci of $\text{Spec}(A)$ are open.

Proof. By a theorem of Kunz, $F$-finite Noetherian rings are excellent, see [Mat80, Theorem 108]. Noetherian local G-rings are quasi-excellent, and the property of being quasi-excellent is preserved by extensions essentially of finite type, see [Mat80, (34.A)]. We got quasi-excellence.

Since $A$ is quasi-excellent, [EGA IV, Prop. 6.11.8] shows that the function $p \mapsto \dim A_p - \text{depth} A_p$ is constructible and upper semi-continuous on $\text{Spec}(A)$. Thus the Cohen-Macaulay locus of $A$ is open. The normal locus is open as well by, for example, ibid., Corollaire 6.13.5.

Finally, openness of $F$-pure locus is trivial in the $F$-finite case (as pure is equivalent to split, see [Stacks, Tag 058L]) and is [Mur21a, Corollary 3.5] in the other case. □

Definition 7.2 ([DT21, Definition 3.1.1]). An ideal $a$ of an $\mathbf{F}_p$-algebra $A$ is uniformly $F$-compatible if for all $e \in \mathbb{Z}_{>0}$ and all $A$-linear maps $\varphi : F_e^* A \to A$, we have $\varphi(F_e^* a) \subseteq a$.

Lemma 7.3 (cf. [DT21, Lemma 3.2.3]). Let $A$ be an $\mathbf{F}_p$-algebra and $X$ a qcqs $A$-scheme. Then the ideal $t(X/A)$ is uniformly $F$-compatible.

Proof. Let $f : X \to \text{Spec}(A)$ be the structural map and consider the following commutative diagram of schemes

\[
\begin{array}{ccc}
X & \xrightarrow{F_e} & X \\
\downarrow f & & \downarrow f \\
\text{Spec}(A) & \xrightarrow{F_e} & \text{Spec}(A)
\end{array}
\]

where $F_e$ is the $e$-th iterated absolute Frobenius of $X$ or $\text{Spec}(A)$. The map $\mathcal{O}_X \to F_e^* \mathcal{O}_X$ induces a map $Rf_* \mathcal{O}_X \to Rf_* F_e^* \mathcal{O}_X = F_e^* Rf_* \mathcal{O}_X$, the identity because $F_e$ is affine. If $\eta : Rf_* \mathcal{O}_X \to A$ is a map in $D(A)$, and $\varphi : F_e^* A \to A$ is an $A$-linear map, then the composition

\[
\eta' : Rf_* \mathcal{O}_X \to Rf_* F_e^* \mathcal{O}_X = F_e^* Rf_* \mathcal{O}_X \xrightarrow{F_e \eta} F_e^* A \xrightarrow{\varphi} A
\]

is also a map in $D(A)$. If the image of $\eta$ in $A$ is $a$ then it is clear that the image of $\eta'$ in $A$ is $\varphi(F_e^* a)$, so $\varphi(F_e^* t(X/A)) \subseteq t(X/A)$ as desired. □
Theorem 7.4 (cf. [DT21, Theorem 4.3.1]). Let $A$ be as in Theorem 1.4, or be as in Theorem 1.5 and $F$-pure.

Then the set of ideals $\Sigma_A := \{ t(X/A) \mid X \text{ proper over } A \}$ is finite.

Proof. If $A$ is $F$-finite, then $A$ is $F$-split since $A$ is Noetherian and $F$-pure. Therefore the number of uniformly $F$-compatible ideals of $A$ is finite, see [DT21, Proposition 3.4.1], so we conclude by Lemma 7.3 that $\Sigma_A$ is finite.

If $A$ is essentially of finite type over a Noetherian local $G$-ring, then since $A$ is $F$-pure, there exists a faithfully flat ring map $A \to B$ where $B$ is Noetherian, $F$-finite, and $F$-pure by [Mur21a, Theorem 3.4] (cf. the proof of [Mur21a, Corollary 3.5]) and $\Sigma_B$ is finite by what we just proved. For $t(X/A) \in \Sigma_A$, we have $t(X/A)B = t(X_B/B) \in \Sigma_B$ (see §§2.4; this is why we need properness), so the finiteness of $\Sigma_B$ implies that of $\Sigma_A$.

Finally, when $A$ is as in Theorem 1.4, we can find a faithfully flat ring map $R \to R'$ where $R'$ is Noetherian, $F$-finite, and $F$-pure, see [Lyu22, Lemma 2.13]. By [Has10, Proposition 2.4] $R' \otimes_R A$ is $F$-pure, and is Noetherian and $F$-finite since it is essentially of finite type over $R'$. The rest of the argument above carries verbatim. □

Corollary 7.5. Let $A$ be as in Theorem 1.4. Then the normal (resp. splinter) locus of $A$ is open.

Proof. By [Lyu22, Proposition 2.12], openness of the splinter locus follows from Theorem 7.4. The proof of the openness of the normal locus follows the same lines, but let us do it below for the reader’s convenience.

For every $p \in \text{Spec}(A)$ such that $A_p$ is normal, since $A$ is Noetherian, there exists $f \in A, f \notin p$ such that $A_f$ is an integral domain. We may thus assume from the outset that $A$ is an integral domain.

For a Noetherian integral domain $C$ put $\Sigma_{\nu}(C) = \{ t(D/C) \mid D \text{ is a finite extension of } C \text{ in the fraction field of } C \}$. It is elementary to see that $\Sigma_{\nu}(C_P) = \{ IC_P \mid I \in \Sigma_{\nu}(C) \}$ for all $P \in \text{Spec}(C)$. Also, if $C \subseteq D$ is an extension of integral domains of the same fraction field, then $C \to D$ splits as a map of $C$-modules if and only if $C = D$. One sees that the normal locus of a Noetherian integral domain $C$ is always $\text{Spec}(C) \setminus \bigcup_{a \in \Sigma_{\nu}(C)} V(a)$.

Now, by Theorem 7.4, $\Sigma_{\nu}(A) \subseteq \Sigma_A$ is finite. This concludes the proof. □

Corollary 7.6. Let $A$ be as in Theorem 1.4. Then the normalization of $A$ is finite over $A$.

Proof. By [Has10, Proposition 2.4] $A$ is $F$-pure, in particular reduced. Thus by Corollary 7.5 $A_f$ is normal for some nonzerodivisor $f \in A$. Moreover, for all $p \in \text{Spec}(A)$, the completion $A_p^\wedge$ is $F$-pure (see for example [Has10, Lemma 3.26]), thus reduced, so the normalization of $A_p$ is finite [Stacks, Tag 032Y]. Our result now follows from [EGA IV2, Proposition 6.13.6], which is stated for domains but works for reduced rings as well. □
Remark 7.7 (cf. [DT19, Example 4.0.3]). The previous corollaries (and thus Theorem 7.4) fail for a general $F$-pure Noetherian ring, and in fact, fails even for a general $F$-pure Noetherian G-ring.

For a counterexample, we let $k$ be an algebraically closed field of characteristic $p$, and let $(A_0, m_0)$ be a $k$-algebra essentially of finite type over $k$ that is a non-normal $F$-pure integral domain. For instance we can take $A_0 = k[x, y]/(xy + x^3 + y^3)$, so $A_0^\wedge = k[[x, y]]/(xy)$. Then by [Hoc73], there is a Noetherian integral domain $A$ which has local rings at maximal ideals of the form $A_0^\wedge K$ where $K/k$ is a field extension (thus is $F$-pure and a G-ring), and has non-open normal locus, and thus the normalization of $A$ is not finite [EGA IV$_2$, Proposition 6.13.2]. Since a non-normal ring is neither a splinter nor a birational derived splinter, [Hoc73] also shows that such $A$ constructed has non-open splinter and birational derived splinter locus.

Now we prove our main theorem about openness of the birational derived splinter locus. It is similar to the proof of Corollary 7.5 and [Lyu22, Theorem 2.14], except for extra care of normality due to the assumptions in Lemma 2.8.

Theorem 7.8. The followings hold.

(1) Let $A$ be as in Theorem 1.4. Then the locus of prime ideals $p$ of $A$ such that $A_p$ is a birational derived splinter is open.

(2) Let $A$ be as in Theorem 1.5. Then the locus of prime ideals $p$ of $A$ such that $A_p$ is an $F$-pure birational derived splinter is open.

Proof. In case (2), we may assume $A$ $F$-pure by Lemma 7.1. Thus in both cases we need to show the locus

$$\{ p \in \text{Spec}(A) \mid A_p \text{ is a birational derived splinter} \}$$

is open. A Noetherian birational derived splinter is normal by Lemma 3.4, so we may assume $A$ normal by either Lemma 7.1 or Corollary 7.5.

We have shown that $\Sigma_A$ is finite (Theorem 7.4), in particular,

$$\Sigma' := \{ t(f) \mid f \text{ is an M-morphism to } \text{Spec}(A) \}$$

is finite. For every $p \in \text{Spec}(A)$ and every M-morphism $g : Y \rightarrow \text{Spec}(A_p)$, $g$ extends to an M-morphism $f : X \rightarrow \text{Spec}(A)$, see Lemma 2.8, and $t(f)A_p = t(g)$, see §§2.4. On the other hand, a localization of an M-morphism is an M-morphism (Lemma 2.6; note that our ring $A$ is reduced). We conclude that the MDS locus is

$$\{ p \in \text{Spec}(A) \mid aA_p = A_p, \forall a \in \Sigma' \}$$

$$= \bigcap_{a \in \Sigma'} (\text{Spec}(A) \setminus V(a))$$

which is open since $\Sigma'$ is finite. □

Theorem 7.9. Let $S \rightarrow A$ be a regular homomorphism of Noetherian $F_p$-algebras. If $S$ is an $F$-pure birational derived splinter, so is $A$.
Proof. Note that $F$-purity localizes and ascends along regular maps, see [Has10, Proposition 2.4], so in the (mostly implicit) dévissage process below $S$ is always $F$-pure and so is $A$.

We only need to consider the special case where $A$ a smooth $S$-algebra, see Corollary 4.6. Now, the proof of [Lyu22, Theorem 1.1] applies here, once we replace the ingredients Theorem 2.9, Corollary 2.6, Lemma 2.2, Theorem 2.14, and Theorem 1.3 there by our Corollary 5.2, Corollary 3.11, Proposition 3.10, Theorem 7.8(1), and Theorem 6.4 respectively.

□

Remark 7.10. In fact, we only used Theorem 6.4 in the case $S' = S[Y]_{mS[Y]}$. In this case, the map $S \to S'$ actually satisfies the statement of Theorem 5.1. This is because an M-morphism $Z \to \text{Spec}(S')$ is dominated by a blowup of an ideal $J \subseteq S'$; and if $J = IS'$ for an ideal $I \subseteq S[Y]$, then the base change of the blowup of $\text{Spec}(S)$ of the ideal generated by the coefficients of elements of $I$ to $S'$ dominates $Z$. This construction appears in [LT81] immediately before its Theorem 2.1; details omitted.

8. Questions

Question 8.1. Let $A$ be essentially of finite type over a local G-ring. Is the birational derived splinter locus of $A$ always open? What about a general quasi-excellent ring, or an excellent ring?

In equal characteristic zero, the birational derived splinter locus of a quasi-excellent ring is always open, as easily follows from Proposition 3.5.

Question 8.2. Let $S \to A$ be a regular homomorphism of Noetherian rings. If $S$ is a birational derived splinter, is $A$ necessarily a birational derived splinter?

In equal characteristic zero, if both $S$ and $A$ are quasi-excellent, this is an easy consequence of Proposition 3.5 again.

We can ask if a strengthening holds, see Question 8.4 below.

Question 8.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring, $t \in \mathfrak{m}$ a nonzerodivisor. If $A/tA$ is a birational derived splinter, is $A$ necessarily a birational derived splinter?

Question 8.4. Let $S \to A$ be a flat homomorphism of Noetherian rings. If $S$ is a birational derived splinter, and the fibers of $S \to A$ are geometrically birational derived splinters, is $A$ necessarily a birational derived splinter?

These two questions are closely related. If we replace “birational derived splinter” by “splinter” or “derived splinter,” then negative answers are given in [Sin99], since $F$-regularity implies splinter (see [MP, Theorem 3.5] and [LS99]) and thus derived splinter by [Bha12]. However, the statements for birational derived splinter may be true, and they hold for related singularity types: rational singularity in characteristic zero [Elk78], $F$-rationality [MP, Theorems 5.1 and 7.12], and BCM-rationality ([MS18, Proposition 3.4], for Question 8.3).
We shall mention that if both Questions 8.3 and 8.4 have affirmative answers, then we will have further results about fibers of a map of Noetherian rings geometrically being birational derived splinters. See [Mur22].

**REFERENCES**

[AD21] B. Antieau and R. Datta, *Valuation rings are derived splinters*. Math. Zeitschrift 299 (2021), 827-851.

[And74] M. André, *Localisation de la lissité formelle*. Manuscripta Math. 13 (1974), 297-307.

[AR94] J. Adámek and J. Rosisky, *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series 189, Cambridge University Press (1994).

[Bha12] B. Bhatt, *Derived splinters in positive characteristic*. Compos. Math. 148 (2012), 1757–1786.

[Bha18] B. Bhatt, *On the direct summand conjecture and its derived variant*. Invent. Math. 212(2) (2018), 297–317.

[BL] B. Bhatt and J. Lurie, *A p-adic Riemann-Hilbert functor: Z/p^n-coefficients*. In preparation.

[CR15] A. Chatzistamatiou and K. Rülling, *Vanishing of the higher direct images of the structure sheaf*. Compos. Math. 151.11 (2015), 2131–2144.

[DT19] R. Datta and K. Tucker, *On some permanence properties of (derived) splinters*. (2019). https://arxiv.org/abs/1909.06891

[DT21] R. Datta and K. Tucker, *Openness of splinter loci in prime characteristic*. (2021). https://arxiv.org/abs/2103.10525

[EGA IV] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie. Inst. Hautes Études Sci. Publ. Math. 24 (1965).

[Elk78] R. Elkik, *Singularités rationelles et déformations*. Invent. Math. 47 (1978), 139-148.

[Fed83] R. Fedder, *F-purity and rational singularity*. Trans. Amer. Math. Soc. 278.2 (1983), 461–480.

[HA] J. Lurie, *Higher Algebra*. (2017). https://www.math.ias.edu/~lurie/papers/HA.pdf

[Has10] M. Hashimoto, *F-pure homomorphisms, strong F-regularity, and F-injectivity*. Comm. Algebra 38.12 (2010), 4569–4596.

[HH95] M. Hochster and C. Huneke, *Applications of the existence of big Cohen-Macaulay algebras*. Adv. Math. 113.1 (1995), 45–117.

[Hoc73] M. Hochster, *Non-openness of loci in Noetherian rings*. Duke Math. J. 40(1) (1973), 215–219.

[Hoc77] M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*. Trans. Amer. Math. Soc. 231.2 (1977), 463–488.

[Kol13] J. Kollár, *Singularities of the minimal model program*. With a collaboration of S. J. Kovács. Cambridge Tracts in Math., Vol. 200. Cambridge: Cambridge Univ. Press (2013).

[Kov00] S. J. Kovács, *A characterization of rational singularities*, Duke Math. J. 102.2 (2000), 187–191.

[LS99] G. Lyubeznik and K. E. Smith, *Strong and weak F-regularity are equivalent for graded rings*. Amer. J. Math., 121.6 (1999), 1279–1290.

[Lyu22] S. Lyu, *Permanence properties of splinters via ultrapower*. (2022). https://arxiv.org/abs/2203.00019

[LT81] J. Lipman and B. Teissier, *Pseudorational local rings and a theorem of Briançon-Skoda about integral closures of ideals*. Michigan Math. J. 28.1 (1981), 97–116.
[Mat80] H. Matsumura, Commutative algebra. Second ed. Mathematics Lecture Note Series Vol. 56, Reading, MA: Benjamin/Cummings Publishing Co., Inc. (1980).

[MP] L. Ma and T. Polstra, $F$-singularities: a commutative algebra approach. https://www.math.purdue.edu/~ma326/F-singularitiesBook.pdf

[MS18] L. Ma and K. Schewede, Singularities in mixed characteristic via perfectoid big Cohen-Macaulay algebras. Duke Math. J. 170, no. 13 (2021), 2815-2890.

[Mur21a] T. Murayama, The gamma construction and asymptotic invariants of line bundles over arbitrary fields. Nagoya Math. J. 242 (2021), 165–207.

[Mur21b] T. Murayama, Relative vanishing theorems for $\mathbb{Q}$-schemes. (2021). https://arxiv.org/abs/2101.10397

[Mur22] T. Murayama, A uniform treatment of Grothendieck’s localization problem. Compositio Mathematica 158.1 (2022), 57–88.

[RG71] M. Raynaud and L. Gruson, Critères de platitude et de projectivité. Techniques de “platification” d’un module. Invent. Math. 13 (1971), 1–89.

[Sin99] A. K. Singh, $F$-regularity does not deform. Amer. J. Math., 121.4 (1999), 919–929.

[Stacks] The Stacks Project authors, The Stacks Project. https://stacks.math.columbia.edu/

[Tem12] M. Temkin, Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case. Duke Math. J. 161.11 (2012), 2207-2254.