Analysis of Sinc Projection Methods for Weakly Singular Nonlinear Integral Equations

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Abstract

In this paper, two numerical schemes for nonlinear integral equation of Fredholm type with weakly singular kernel are proposed. These numerical methods combine sinc-collocation and sinc-convolution approximations with Newton and steepest descent iterative methods that involve solving a nonlinear system of equations. The convergence rate of the approximation schemes are also analyzed. Numerical experiments have been performed to illustrate the sharpness of the theoretical estimates and the sensitivity of the solution with respect to some parameters in the equation.

Keywords: nonlinear integral operator, sinc-collocation method, sinc-convolution method.

Mathematics Subject Classification (2010): 65J15, 65L60, 45E10.

1 Introduction

The aim of this paper is to study of the numerical solution of nonlinear Fredholm integral equation

\[ u(t) = g(t) + \int_a^b f(|t - s|)k(t, s)\psi(s, u(s))ds, \quad -\infty < a \leq t \leq b < \infty, \] (1)

where \( u(t) \) is an unknown function to be determined and \( k(t, s), \psi(s, u) \) and \( g(t) \) are given functions. Eq. (1) is an algebraic weakly singular integral equation, whenever \( f(t) \) is defined by \( t^{-\lambda}, 0 < \lambda < 1 \). A more general type of this equation, so-called Urysohn weakly singular integral equation [28], is defined as

\[ u(t) = g(t) + \int_a^b f(|t - s|)k(t, s, u(s))ds, \quad -\infty < a \leq t \leq b < \infty. \] (2)

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Linear and nonlinear integral equations with weakly singular kernels arise in various applications such as astrophysics. In the potential theory, the boundary integral equations of the Laplace and Helmholtz operators are described as a linear combination of weakly singular operators.

It is well-known that the solution of weakly singular integral equation has some singularities near the boundaries. This is the main point in designing the numerical schemes for them. There is a considerable interest in the numerical analysis of linear and nonlinear integral equations with weakly singular kernels. This interest has been followed by some projection schemes such as Galerkin, collocation and product integration methods with singularity preserving approaches which find an approximation with optimal error bound. Some of these numerical schemes are classified in Section 2. It is worth mentioning that the numerical solution of Eq. (1) with the smooth kernel is comprehensively studied, for more information see [3, 15].

The first objective of this study is to investigate the analysis of sinc-collocation method for nonlinear weakly singular Fredholm integral equation. In [14], the authors have studied this subject and obtained the rate of convergence $O(\|A^{-1}\|_\infty(3 + \log(N))\sqrt{N}\exp(-\sqrt{\pi d\lambda N}))$. We propose an alternative error analysis for the approximate solution chosen from an appropriate finite dimensional space built by the shifted sinc functions and considering the singularity properties of the exact solution. However, we encounter a term $\xi_N$ in the upper bound which depends on $N$ and is unavoidable due to the nature of the projection methods.

For the second objective, we present and analyze a numerical scheme using the key idea of how to approximate the following nonlinear convolution

$$r(x) = \int_a^x k(x, x - t, t)dt,$$

which is called sinc-convolution method. Here, we replace the variables with the single exponential transformation introduced in Section 3.

The paper is structured as follows. In Section 2 we review some popular numerical schemes for the solution of weakly singular integral equations. In order to make the paper self-contained, the basic properties of the sinc approximation method are introduced in Section 3. Two numerical schemes based on sinc-collocation and convolution methods will be studied in Section 4. Furthermore, this section contains a complete convergence analysis for the proposed methods. Finally, some numerical experiments are illustrated to show the consistency with the theoretical estimates on order of convergence in Section 5.

2 Numerical methods for weakly singular integral equations

There are a large amount of literature devoted to the existence and (locally) uniqueness issue of the (nonlinear) linear Fredholm integral equations (see [11].

2
and references therein). Study on the regularity of solution is crucial for finding an appropriate numerical method to approximate the solution and it is discussed for the linear case in [22] and Kaneko et al. have extended the results for the nonlinear integral equation [11] as the following theorem states:

**Theorem 1** Let for an positive integer $m$, $\psi \in C^{m+1}([a,b] \times [a,b])$ and $g \in C^{(0,\lambda)}[a,b] \cap C^m(a,b)$, where $C^{(0,\lambda)}[a,b]$ is an $\lambda$-Hölder continuous space. Furthermore, assume that $\psi \in C^{(0,1)}([a,b] \times (\infty, \infty))$ and for $m \geq 2$, $\psi \in C^{m-1}([a,b] \times (\infty, \infty))$, then any solution of (1) belongs to $C^{m}(a,b) \cap C^{0,\lambda}(a,b)$.

Several researches have been done to establish numerical techniques for finding an approximation of the linear or nonlinear weakly singular integral equations. Most of them have been utilized to match the effect of a weakly singular kernel to the solution of integral equation. In the following some of numerical schemes have been categorized:

**Galerkin method:** This method finds a numerical solution via projecting the space of the problem to an appropriate finite dimensional space by an orthogonal projection. Choosing a suitable basis functions leads to the Legendre-Galerkin method [10] or singularity preserving Galerkin method [7].

**Collocation method:** It uses interpolation projection to find an approximate solution. For example using Legendre basis functions leads to Legendre-collocation method [10], or combination of different types of finite dimensional spaces could be used to define a hybrid collocation method [6].

**Nyström method:** To utilize the Nyström method for the weakly singular integral equation, it is combined with interpolation of the kernel to construct a new method so-called product integration scheme introduced by Schneider in [22]. Furthermore, Bremer and Gimbutas applied the modified Nyström method benefits from scaled functions values [4] for weakly singular integral equations in $\mathbb{R}^3$.

**Petrov-Galerkin method:** It is an orthogonal projection method with different ansatz and test spaces. In [9, 8], a general framework based on the collectively compact operator theory is established to analyze the discrete Petrov-Galerkin method for weakly singular Fredholm integral equations.

The Galerkin and Petrov-Galerkin methods could be studied for Eq. (1), but they are not the subject of this paper. In this work, we present numerical schemes based on sinc approximation; sinc-convolution and sinc-collocation methods for the nonlinear Fredholm weakly singular integral equations. Sinc-convolution is introduced in [23] to collocate indefinite integrals of convolution type and it could be interpreted as a special type of Nyström method. It will be shown that this method has the exponential rate of convergence. For a comprehensive study of sinc-convolution method and its application to different kinds of equations, we refer to [24, 25]. Furthermore, sinc-collocation method and its
features are studied in this paper. Equations (1) and (2) can be expressed in
the operator form as
\[(I - K_i)u = g, \quad i = 1, 2, \quad (3)\]
where \((K_1u)(t) = \int_a^b f(|t - s|)k(t,s)\psi(s,u(s))ds\) and \((K_2u)(t) = \int_a^b f(|t - s|)k(t,s,u(s))ds\). These operators are defined on an appropriate Banach space
\(X\) discussed in more detail in Section 3. We suppose that the unknown solution
\(u(t)\) to be determined is geometrically isolated \([12]\), in other words, there is
some ball
\[B(u, r) = \{ x \in X : \| x - u \| \leq r \}, \]
with \(r > 0\), that contains no solution of (1) other than \(u\).

3 Preliminaries

Some basic definitions and theorems on sinc function, sinc interpolation and
quadrature are stated.

3.1 Sinc interpolation

The sinc function is defined on the whole real line by
\[
sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}
\]

It is well-known that the sinc approximation for a function \(f\) is expressed as
\[
f(t) \approx \sum_{j=-N}^{N} f(jh)S(j,h)(t), \quad t \in \mathbb{R}, \quad (4)
\]
where the basis function \(S(j,h)(t)\) is defined by
\[
S(j,h)(t) = \text{sinc}\left(\frac{t}{h} - j\right), \quad j \in \mathbb{Z}, \quad (5)
\]
and \(h\) is a step size appropriately chosen depending on a given positive integer
\(N\), and \((5)\) is called \(j\)th sinc function. Eq. (4) can be adapted to approximate on
general intervals with the aid of appropriate variable transformations \(t = \varphi(x)\).
As the transformation function \(\varphi(x)\) appropriate single exponential and
double exponential transformations are applied [11, 21, 27]. The single exponential
transformation and its inverse could be introduced respectively as below
\[
\varphi_{a,b}(x) = \frac{b - a}{2} \tanh\left(\frac{x}{2}\right) + \frac{b + a}{2},
\]
\[
\phi_{a,b}(t) = \log\left(\frac{t - a}{b - t}\right).
\]
The superscripts $a$ and $b$ in notation of the transformations play an important role in the application of sinc-collocation method for weakly singular integral equations [17]. In order to define a convenient function space, the strip domain

$$
\mathcal{D}_d = \{z \in \mathbb{C} : |\Im z| < d\},
$$

for some $d > 0$ is introduced. When it is incorporated with this transformation, the condition should be considered on the translated domain

$$
\varphi(\mathcal{D}_d) = \left\{ z \in \mathbb{C} \mid \arg\left(\frac{z-a}{b-z}\right) < d \right\}.
$$

The following definitions and theorems are considered for further details of the procedure.

**Definition 2** ([14]) Let $\mathcal{D}$ be a bounded and simply connected domain. Then $H^\infty(\mathcal{D})$ denotes the family of analytic functions $f$ on $\mathcal{D}$ such that the norm $\|f\|_{H^\infty(\mathcal{D})}$ is finite, where

$$
\|f\|_{H^\infty(\mathcal{D})} = \sup_{z \in \mathcal{D}} |f(z)|.
$$

**Definition 3** ([14]) Let $\alpha$ and $C$ be positive constants, and let $\mathcal{D}$ be a bounded and simply connected domain which satisfies $(a, b) \subset \mathcal{D}$. Then $\mathcal{L}_\alpha(\mathcal{D})$ denotes the family of all functions $f \in H^\infty(\mathcal{D})$ which satisfy

$$
|f(z)| \leq C|Q(z)|^\alpha,
$$

for all $z$ in $\mathcal{D}$ where $Q(z) = (z - a)(b - z)$.

The next theorem shows the exponential convergence of the sinc approximation.

**Theorem 4** ([18]) Let $f \in \mathcal{L}_\alpha(\varphi_{a, b}(\mathcal{D}_d))$ for $d$ with $0 < d < \pi$. Suppose that $N$ be a positive integer, and $h$ be given by the formula $h = \sqrt{\frac{\pi d}{\alpha N}}$. Then there exists a constant $C$ independent of $N$, such that

$$
\left\|f(t) - \sum_{j=-N}^{N} f(\varphi_{a, b}(jh))S(j, h)(\phi(t))\right\| \leq C\sqrt{N}\exp(-\sqrt{\pi d\alpha N}),
$$

where

$$
C = \frac{2K(b-a)^{2\alpha}}{\alpha} \left[ \frac{2}{\pi d(1-e^{-2\sqrt{\pi d\alpha}})(\cos(\frac{\pi}{4}))^{2\alpha} + \sqrt{\frac{\alpha}{\pi d}}} \right].
$$

According to Theorems 4, in order to achieve exponential convergence, a function to be approximated by the sinc approximation should belong to $\mathcal{L}_\alpha(\mathcal{D})$. By the condition (6), such a function is required to be zero at the endpoints, which is restrictive in practice. However, it can be relaxed to the following function space $\mathcal{M}_\alpha(\mathcal{D})$ with $0 < \alpha \leq 1$ and $0 < d < \pi$.  

5
Definition 5 ([24]) Let $\mathcal{D}$ be a simply connected and bounded domain which contains $(a, b)$. The family $\mathcal{M}_\alpha(\mathcal{D})$ contains all analytical functions which are continuous on $\overline{\mathcal{D}}$ such that the transformation

$$G[f](t) = f(t) - [(\frac{b-t}{b-a})f(a) + (\frac{t-a}{b-a})f(b)],$$

belongs to $\mathcal{L}_\alpha(\mathcal{D})$.

3.2 Sinc quadrature

Sinc approximation by incorporating with single exponential transformation could be applied to definite integration based on the function approximation to designate the sinc quadrature. The following theorem includes error bound for the sinc quadrature of $f$ on $(a, b)$.

Theorem 6 ([18]) Let $(fQ) \in \mathcal{L}_\alpha(\varphi_{a,b}(D_d))$ for $d$ with $0 < d < \pi$. Suppose that $N$ be a positive integer and $h$ is selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}.$$

Then

$$\left| \int_a^b f(s) \, ds - h \sum_{j=-N}^{N} f(\varphi_{a,b}(jh))(\varphi_{a,b})'(jh) \right| \leq C(b-a)^{2\alpha-1} \exp(-\sqrt{\pi d \alpha N}),$$

where $C$ is a constant independent of $N$.

4 Two numerical schemes

4.1 Sinc-collocation

In this section the sinc-collocation and its feature for the nonlinear Fredholm integral equation with weakly singular kernel is discussed. A sinc approximation $u_N$ to the solution $u \in \mathcal{M}_\lambda(\varphi_{a,b}(D_d))$ of Eq. (1) is described in this part. For this aim the interpolation operator $\mathcal{P}_N : \mathcal{M}_\lambda \to X$ is defined as follows

$$\mathcal{P}_N[u](t) = \mathcal{L}u(t) + \sum_{j=-N}^{N} [u(t_j) - (\mathcal{L}u)(t_j)]S(j, h)(\varphi_{a,b}(t)),$$

where

$$\mathcal{L}[u](t) = \left( \frac{b-t}{b-a} \right)\lambda u(a) + \left( \frac{t-a}{b-a} \right)^{1-\lambda} u(b).$$

In this formula, the sinc points $t_j$ are defined by the formula

$$t_j = \begin{cases} 
    a, & j = -N - 1, \\
    \varphi_{a,b}(jh), & j = -N, \ldots, N, \\
    b, & j = N + 1.
\end{cases} \quad (8)$$

6
The approximate solution could be in form of
\[ u_N(t) = c_{-N-1}(\frac{b-t}{b-a})^\lambda + \sum_{j=-N}^{N} c_j S(j, h)(\phi_{a,b}(t)) + c_{N+1}(\frac{t-a}{b-a})^{1-\lambda}, \] (9)

where the singularity exponent parameter \(\lambda\) introduced in Eq. (1). We notice that choice of the basis functions incorporate with sinc function reflects the singularity of the exact solution. Applying the operator \(P_N\) to both sides of Eq. (1) gives us the following approximate equation in the operator form
\[ u_N = P_N g + P_N K u_N. \] (10)

This equation could be rewritten as
\[ u_N(t_i) = g(t_i) + \int_{a}^{b} f(|t_i - s|)k(t_i, s)\psi(s, u_N(s))ds, \quad i = -N - 1, \ldots, N + 1, \] (11)

so the collocation method for solving Eq. (11) amounts to solve Eq. (11) for \(N\) sufficiently large. Using the theory of holomorphic function space along with the singularity preserving representation of the approximate solution incorporate a mechanism for approximating the singular integrals which arise from the discretization of weakly singular integral operators. Let us have the following presentation for Eq. (11):
\[ u_N(t_i) = \int_{a}^{t_i} f(|t_i - s|)k(t_i, s)\psi(s, u_N(s))ds + g(t_i), \quad i = -N - 1, \ldots, N + 1. \] (12)

Due to the complexity of the integral kernel, we are interested in approximating the integral operator in (12) by the quadrature formula presented in (7). We notice that in order to use the sinc quadrature method properly, the intervals \([a, t_i]\) and \([t_i, b]\) should be transformed to the whole real line. So, Eq. (12) could be written as
\[ u_N(t_i) = h|t_i - a|^\lambda \sum_{j=-N}^{N} \frac{1}{(1 + e^{jh})^\lambda(1 + e^{-jh})} k(t_i, \varphi_{a,t_i}(jh)) \psi(\varphi_{a,t_i}(jh), u_N(\varphi_{a,t_i}(jh))) + \\
h|b - t_i|^\lambda \sum_{j=-N}^{N} \frac{1}{(1 + e^{jh})^\lambda(1 + e^{-jh})} k(t_i, \varphi_{t_i,b}(jh)) \psi(\varphi_{t_i,b}(jh), u_N(\varphi_{t_i,b}(jh))) + \\
g(t_i), \quad i = -N - 1, \ldots, N + 1. \] (13)

This numerical procedure leads us to replace Eq. (13) with
\[ u_N - P_N K u_N = P_N g, \] (14)
where the discrete operator $K_Nu$ is defined as
\[
(K_Nu)(t) := h|t - a|^{\lambda} \sum_{j=-N}^{N} \frac{1}{(1 + e^{jh})\lambda(1 + e^{-jh})} k(t, \varphi_{a,t_i}(jh)) \psi(\varphi_{a,t_i}(jh), u(\varphi_{a,t_i}(jh))) \\
+ h|b - t|^{\lambda} \sum_{j=-N}^{N} \frac{1}{(1 + e^{jh})\lambda(1 + e^{-jh})} k(t, \varphi_{a,b}(jh)) \psi(\varphi_{a,b}(jh), u(\varphi_{a,b}(jh))).
\]
(15)

The Eq. (14) is the operator form of discrete collocation method based on the sinc basis function. By solving the nonlinear system of equations (14), the unknown coefficients in $u_N$ are determined.

4.1.1 Convergence analysis

In this subsection we give an error analysis for the sinc-collocation method. We state the following lemmas which are used subsequently.

Lemma 7 ([24]) Let $h > 0$. Then it holds that
\[
\sup_{x \in \mathbb{R}} \sum_{j=-N}^{N} |S(j, h)(x)| \leq \frac{2}{\pi}(3 + \log(N)).
\] (16)

This lemma concludes that $\|P_N\| \leq C \log(N)$ where $C$ is a constant independent of $N$ and $P_N$ is the interpolation operator constructed on the sinc points.

Lemma 8 ([17]) Let $d$ be a constant with $0 < d < \pi$. Define a function $\varphi_1$ as
\[
\varphi_1(x) = \frac{1}{2} \tanh\left(\frac{x}{2}\right) + \frac{1}{2}.
\]

Then there exists a constant $c_d$ such that for all $x \in \mathbb{R}$ and $y \in [-d, d]$,
\[
|\{\varphi_{a,b}\}'(x + iy)| \leq (b - a)c_d\varphi_1'(x),
\] (17)
\[
|\varphi_{0,1}(x + iy)| \geq \varphi_1(x).\] (18)

In addition, if $t \leq x$,
\[
|\varphi_{a,b}(x + iy) - \varphi_{a,b}(t + iy)| \geq (b - a)\{\varphi_1(x) - \varphi_1(t)\}.\] (19)

With the aid of Lemma 8, the analytical behavior of the solution is investigated for a general kernel function. It is convenient to define the following nonlinear operators which will be used in the next theorem
\[
(K^1u)(t) = \int_a^t |t - s|^{-\lambda}k(t, s, u(s))ds,
\]
\[
(K^2u)(t) = \int_t^b |t - s|^{-\lambda}k(t, s, u(s))ds.
\] (20)
Theorem 9 Let $d$ be a constant with $0 < d < \pi$ and $D = (\varphi_{a,b})^{-1}(D_d)$. Suppose that $k(z,.,v) \in H^\infty(D)$ for all $z$ and $v$ belong to $D$ and $k(z,.,v) \in H^\infty(D)$ for all $z$ and $w$ belong to $D$. Moreover, $k(.,v,w) \in M_{1-\lambda}(D)$ for all $v$, $w \in D$, $k(z,v,w)$ is bounded for $z$, $v$ and $w$ in $D$ and $y \in M_\beta(D)$. Then $u$, the solution to (1) belongs to $M_\gamma(D)$, where $\gamma = \min(1-\lambda, \beta)$.

Proof. In [13, p. 83], sufficient conditions have been mentioned to have a nonlinear analytic operator and then analytic solution. So it is adequate to show that $Ku$ is $(1-\lambda)$-Hölder continuous. For this aim, we show the operators defined in (20) have this property. To proof the $(1-\lambda)$-Hölder continuity of $K^1u$ and $K^2u$, the idea of Lemma A.2. in [17] is extended to the nonlinear case. Set $x = \text{Re}[(\varphi_{a,b})^{-1}(z)]$, $y = \text{Im}[(\varphi_{a,b})^{-1}(z)]$ and $v = \varphi_{a,b}(t+iy)$ as a variable transformation,

$$(K^1u)(z) - (K^1u)(a) = \int_a^z |z-v|^{-\lambda}k(z,v,u(v))dv - 0$$

$$= \int_a^z |\varphi_{a,b}(x+iy) - \varphi_{a,b}(t+iy)|^{-\lambda}k(x+iy,t+iy,u(t+iy))(\varphi_{a,b})'(t+iy)dt.$$ 

Applying the absolute value on both sides of the above equation and using Eqs. (17) and (19), we have

$$|(K^1u)(z) - (K^1u)(a)| \leq \int_a^z (|b-a|)^{-\lambda}(|\varphi_1(x) - \varphi_1(t)|)^{-\lambda}M_k(b-a)c_d\varphi_1'(t)dt$$

$$\leq \frac{M_kc_d}{1-\lambda}(|b-a|\varphi_1(x))^{1-\lambda},$$

where $M_k = \max_{D} |k(z,w,v)|$. In addition, by using (18), the inequality $(b-a)|\varphi_1(x)| \leq |z-a|$ is concluded. So

$$|(K^1u)(z) - (K^1u)(a)| \leq \frac{M_kc_d}{1-\lambda}|z-a|^{(1-\lambda)}.$$

(21)

Now, the $(1-\lambda)$-Hölder continuity at the point $b$ is considered

$$(K^1u)(b) - (K^1u)(z) = \int_a^b |b-v|^{-\lambda}\left\{k(b,v,u(v)) - k(z,v,u(v))\right\}dv$$

$$+ \int_a^b \left\{|b-v|^{-\lambda} - |z-v|^{-\lambda}\right\}k(z,v,u(v))dv$$

$$- \int_b^z |z-v|^{-\lambda}k(z,v,u(v))dv.$$ 

Since $k(.,v,w) \in M_{1-\lambda}(D)$, there exists $M_1$ such that

$$\left| \int_a^b |b-v|^{-\lambda}\left\{k(b,v,u(v)) - k(z,v,u(v))\right\}dv \right| \leq M_1|b-z|^{(1-\lambda)} \int_a^b |b-v|^{-\lambda}dv$$

$$\leq \frac{M_1|b-a|^{1-\lambda}}{1-\lambda}|b-z|^{1-\lambda}.$$
The third term is bounded by

\[ \left| \int_b^b \left| z - v \right|^{-\lambda} k(z, v, u(v)) dv \right| \leq \frac{M_b}{1 - \lambda} |b - z|^{1 - \lambda}. \] (22)

Integration by part, the Hölder continuity of the function \( F(z) = z^{1-\lambda} \) and the assumptions on \( k(z, w, .) \) and \( k(z, ., v) \in H^\infty(D) \) give

\[ |(K_1 u)(b) - (K_1 u)(z)| \leq \frac{M_2}{1 - \lambda} |b - z|^{(1-\lambda)}. \]

The \((1-\lambda)\)-Hölder continuity of the operator \( K^2(u) \) can be proved in a similar manner. Therefore, the result is achieved. \( \blacksquare \)

The Fréchet derivative of the nonlinear operators \( K \) and \( K_N \) for all \( u \) is stated by

\[ (K'u)(t) = \int_a^b f(|t - s|)k(t, s)\frac{\partial \psi}{\partial u}(s, u(s))x(s)ds, \quad t \in [a, b], \quad x \in X, \] (23)

and

\[ (K'Nu)(t) = h[t - a]^\lambda \sum_{j=-N}^N \frac{1}{(1 + e^{jh})^\lambda(1 + e^{-jh})} k(t, \varphi_{a,t}(jh)) \frac{\partial \psi}{\partial u}(\varphi_{a,t}(jh), u(\varphi_{a,t}(jh)))x(jh) \]

\[ + h|b - t|^\lambda \sum_{j=-N}^N \frac{1}{(1 + e^{jh})^\lambda(1 + e^{-jh})} k(t, \varphi_{t,b}(jh)) \frac{\partial \psi}{\partial u}(\varphi_{t,b}(jh), u(\varphi_{t,b}(jh)))x(jh) \] (24)

**Theorem 10** Suppose that \( u(t) \) is an exact solution of Eq. (1) and \( I - K'u \) is a non-singular operator. Also, the term \( \frac{\partial^2 \psi}{\partial u^2}(t, s, u) \) is well-defined and continuous on its domain. Furthermore, assume that \( g \in M_\lambda(\varphi_{a,b}(D_d)) \) and \( Ku \in M_\lambda(\varphi_{a,b}(D_d)) \) for all \( u \in \mathcal{B}(u, r) \) with \( r > 0 \). Then there exists a constant \( C \) independent of \( N \) such that

\[ ||u - u_N|| \leq C\xi_N \sqrt{N} \log(N + 1) \exp(-\sqrt{\pi d\lambda N}), \] (25)

where \( \xi_N = ||(I - \mathcal{P}_N(K_N)'(u))^{-1}||. \)

**Proof.** To find an upper bound for the error, we subtract (3) from (11) and obtain

\[ u - u_N = Ku - \mathcal{P}_N K_N u_N + g - \mathcal{P}_Ng. \]

The above relation could be written as

\[ u - u_N = (I - \mathcal{P}_N(K'_N)(u))^{-1} \{(g - \mathcal{P}_N g) \]

\[ + (Ku - \mathcal{P}_N Ku) + \mathcal{P}_N(Ku - K_N u) \]

\[ + \mathcal{P}_N(K_N u - K_N u_N - (K'_N)(u)(u - u_N)) \}. \] (26)
Finally, the following relation is obtained
\[ \|u - u_N\| \leq \|(I - P_N(K_N'))(u)\|^{-1}\{\|g - P_Ng\| \]
\[ + \|Ku - P_NKu\| + \|P_N\|\|Ku - KNu\|\} + \|P_N\|O(\|u - u_N\|^2). \tag{27} \]

Because of \(g, Ku \in \mathcal{M}_\lambda(\varphi_{a,b}(D_d))\), we can apply Theorem 4 and get
\[ \|g - P_Ng\| \leq C_1\sqrt{N}\exp(-\sqrt{\pi d\lambda N}), \]
\[ \|Ku - P_NKu\| \leq C_2\sqrt{N}\exp(-\sqrt{\pi d\lambda N}). \]
By using Theorem 7, the following result is concluded
\[ \|Ku - KNu\| \leq C_3\exp(-\sqrt{\pi d\lambda N}), \]
and finally \(\|P_N\|\) is estimated by conclusion of Lemma 7. So
\[ \|u - u_N\| \leq C\xi N \log(N + 1)\sqrt{N}\exp(-\sqrt{\pi d\lambda N}). \]

4.2 Sinc-convolution

Let \(f(t)\) be a function with singularity at the origin and \(g(t)\) be a function with singularities at both endpoints. The method of sinc-convolution is based on an accurate approximation of following integrals
\[ p(s) = \int_a^s f(s - t)g(t)dt, \quad s \in (a, b), \]
\[ q(s) = \int_s^b f(t - s)g(t)dt, \quad s \in (a, b), \tag{28} \]
and then it could be used to approximate the definite convolution integral
\[ \int_a^b f(|s - t|)g(t)dt. \tag{29} \]

In order to make such appropriate approximation, the following notations are defined.

**Definition 11** For a given positive integers \(N\), let \(D_N\) and \(V_N\) denote linear operators acting on function \(u\) by
\[ D_Nu = \text{diag}[u(t_{-N}), \ldots, u(t_N)], \]
\[ V_Nu = (u(t_{-N}), \ldots, u(t_N))^T. \tag{30} \]
where the superscript $T$ specifies the transpose, diag symbolizes the diagonal matrix. Set the basis functions as follow

$$
\gamma_j(t) = S(j, h)(\varphi_{a,b}(t)), \quad j = -N, \ldots, N,
$$

$$
\omega_j(t) = \gamma_j(t), \quad j = -N, \ldots, N,
$$

$$
\omega_{-N}(t) = \frac{b - t}{b - a} - \sum_{j=-N+1}^{N} \frac{1}{1 + e^{jh}} \gamma_j(t),
$$

$$
\omega_N(t) = \frac{t - a}{b - a} - \sum_{j=-N}^{-1} \frac{e^{jh}}{1 + e^{jh}} \gamma_j(t).
$$

With the aid of these basis functions for a given vector $c = (c_{-N}, \ldots, c_{N})^T$, we consider a linear combination, symbolized by $\Pi_N$ as follows

$$
(\Pi_N c)(t) = \sum_{j=-N}^{N} c_j \omega_j(t). \quad (32)
$$

Let us define the interpolation operator $\mathcal{P}^c_N : \mathcal{M}_A(\mathcal{D}) \to X_N = \text{span}\{\omega_j(t)\}_{j=-N}^{N}$ as follows

$$
\mathcal{P}^c_N f(t) = \sum_{j=-N}^{N} f(t_j) \omega_j(t),
$$

where $t_j$ are are defined in (35). The numbers $\sigma_k$ and $e_k$ are determined by

$$
\sigma_k = \int_0^k \text{sinc}(t) dt, \quad k \in \mathbb{Z},
$$

$$
e_k = \frac{1}{2} + \sigma_k. \quad (33)
$$

An $(2N + 1) \times (2N + 1)$ (Toeplitz) matrix $I^{(-1)}$ is defined by $I^{(-1)} = [e_{i-j}]$, with $e_{i-j}$ representing the $(i, j)^{th}$ element of $I^{(-1)}$. In addition, the operators $I^+$ and $I^-$ are specified as follows

$$
(I^+ g)(t) = \int_a^t g(s) ds,
$$

$$
(I^- g)(t) = \int_t^b g(s) ds. \quad (34)
$$

The following discrete operators $I^+_N$ and $I^-_N$ approximate the operators $I^+$ and $I^-$ as follows

$$
(I^+_N g)(t) = \Pi_N A^{(1)} V_N g(t), \quad A^{(1)} = h f^{(-1)} D_N \left( \frac{1}{\varphi_{a,b}} \right),
$$

$$
(I^-_N g)(t) = \Pi_N A^{(2)} V_N g(t), \quad A^{(2)} = h (f^{(-1)})^T D_N \left( \frac{1}{\varphi_{a,b}} \right). \quad (35)
$$
For a function $f$, the operator $\mathcal{F}[f](s)$ is defined by

\[
\mathcal{F}[f](s) = \int_0^c e^{-\frac{t}{s}} f(t) dt,
\]

(36)

and it is assumed that Eq. (36) is well-defined for some $c \in [b - a, \infty]$ and for all $s$ on the right half of the complex plane, $\Omega^+ = \{ z \in \mathbb{C} : \Re(z) > 0 \}$.

Sinc-convolution method provides formulae of high accuracy and allows $f(s)$ to have an integrable singularity at $s = b - a$ and for $g$ to have singularities at both endpoints of $(a, b)$ [25]. This property of the sinc-convolution makes it suitable to approximate the weakly singular integral equations.

Now for convenience, some useful theorems related to sinc-convolution method are introduced. The following theorem predicts the convergence rate of the sinc-convolution method.

**Theorem 12** ([25]) (a) Suppose that the integrals $p(t)$ and $q(t)$ in (28) exist and are uniformly bounded on $(a, b)$, and let $\mathcal{F}$ be defined as (36). Then the following operator identities hold

\[
p = \mathcal{F}(I^+)g, \quad q = \mathcal{F}(I^-)g.
\]

(37)

(b) Assume that $\frac{g}{\varphi_{a,b}} \in \mathcal{L}_x(D)$. If for some positive $C'$ independent of $N$, we have $|\mathcal{F}'(s)| \leq C'$ for all $\Re(s) \geq 0$, then there exists a constant $C$, independent of $N$ such that

\[
\| p - \mathcal{F}(I^+_N)g \| \leq C\sqrt{N} \exp(-\sqrt{\pi\lambda dN}),
\]

\[
\| q - \mathcal{F}(I^-_N)g \| \leq C\sqrt{N} \exp(-\sqrt{\pi\lambda dN}).
\]

(38)

4.2.1 Sinc-convolution scheme

In order to practically use of the convolution method, it is assumed that the dimension of matrices, $2N + 1$, is such that the matrices $A^{(1)}$ and $A^{(2)}$ are diagonalizable [25] as follows

\[
A^{(j)} = X^{(j)}S(X^{(j)})^{-1}, \quad j = 1, 2,
\]

(39)

where

\[
S = \text{diag}(s_{-N}, \ldots, s_N),
\]

\[
X^{(1)} = [x_{k,l}], \quad (X^{(1)})^{-1} = [x^{k,l}],
\]

\[
X^{(2)} = [\xi_{k,l}], \quad (X^{(2)})^{-1} = [\xi^{k,l}].
\]

(40)

The integral at Eq. (1) has been split into the following integrals

\[
\int_a^b |t-s|^{-\lambda}k(t, s, u(s))ds = \int_a^t |t-s|^{-\lambda}k(t, s, u(s))ds + \int_t^b |t-s|^{-\lambda}k(t, s, u(s))ds.
\]

(41)
Based on formulae \(35\), the two discrete nonlinear operators are defined

\[
\begin{align*}
(\mathcal{K}_N^1 u)(t) &= \Pi_N A^{(1)} V_N k(t, s, u(s)), \\
(\mathcal{K}_N^2 u)(t) &= \Pi_N A^{(2)} V_N k(t, s, u(s)).
\end{align*}
\]  

The approximate solution takes the form

\[
u_N^c(t) = \sum_{j=-N}^{N} c_j \omega_j(t),
\]

where \(c_j\)s are unknown coefficients to be determined. The integrals in right hand side of \(41\) are approximated by the formulae \(35\), \(37\) and \(39\). We substitute these approximations in \(1\) and then the approximated equation is collocated at the sinc points. This process reduces solving \(1\) to solving the following finite dimensional system of equations

\[
c_j - \sum_{k=-N}^{N} x_{j,k} \sum_{l=-N}^{N} x_{k,l} F(s_k) k(z_j, z_l, c_l) - \sum_{k=-N}^{N} \xi_{j,k} \sum_{l=-N}^{N} \xi_{k,l} F(s_k) k(z_j, z_l, c_l) = y(z_j),
\]

for \(j = -N, \ldots, N\).

Equation \(43\) can be expressed in the operator notation as follows

\[
u_N^c - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c - \mathcal{P}_N^c \mathcal{K}_N^2 u_N^c = \mathcal{P}_N^c y
\]

### 4.2.2 Convergence analysis

The convergence analysis of sinc-convolution method is discussed in this section. The main result is formulated in the following theorem.

**Theorem 13**: Suppose that \(u(t)\) is an exact solution of Eq. \(1\) and the kernel \(k\) satisfies the Lipschitz condition with respect to the third variable by \(L\). Also, let the assumptions of Theorem 9 be fulfilled. Then there exists a constant \(C\) independent of \(N\) such that

\[
\|u - u_N^c\| \leq C \sqrt{N} \log(N) \exp(-\sqrt{\pi d \lambda N}).
\]

**Proof.** By subtracting Eq. \(1\) from Eq. \(44\), the following term has been obtained

\[
\|u - u_N^c\| \leq \|\mathcal{K}_N^1 u - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c\| + \|\mathcal{K}_N^2 u - \mathcal{P}_N^c \mathcal{K}_N^2 u_N^c\| + \|y - \mathcal{P}_N^c y\|.
\]

Finding an upper bound for the first and second terms is almost the same. For this aim the first term is rewritten as follows

\[
\mathcal{K}_N^1 u - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c = \mathcal{K}_N u - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c + \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c - \mathcal{P}_N^c \mathcal{K}_N^1 u_N^c,
\]
so we have

\[ \| K^1 u - P_N K_N^1 u_N \| \leq \| K^1 u - K_N^1 u_N \| \]

\[ + \| K_N^1 u_N - P_N K^1 u_N \| + \| P_N \| \| K^1 u_N - K_N^1 u_N \|, \]  

(45)

the second term is bounded by Theorem 4. Due to the Lipschitz condition, the
first term is bounded by

\[ \| K^1 u - K_N^1 u_N \| \leq C_1 \| u - u_N \|, \]

where \( C_1 \) is a suitable constant. In addition, Theorem 12 and conclusion of
Lemma 7 help us to find the following upper bound

\[ \| P_N \| \| K_N^1 u_N - K_N^1 u_N \| \leq C_2 \sqrt{N} \log(N) \exp(-\sqrt{\pi d} N). \]  

(46)

Finally, we get

\[ \| u - u_N \| \leq C \sqrt{N} \log(N) \exp(-\sqrt{\pi d} N). \]  

(47)

5 Numerical experiments

This section is devoted to the numerical experiments concerning the accuracy
and rate of convergence of the presented methods in this paper. The proposed
algorithms are executed in Mathematica® on a PC with 4.00 GHz Intel® Core™
i5 dual processor with 16 GB RAM. For the solution of nonlinear systems which
arise in the formulation of proposed methods, we have used Mathematica routine
FindRoot which works based on the steepest descent and Newton iteration
algorithms. In these examples, an initial guess for the Newton iteration is se-
lected by the steepest descent method [5]. The convergence rate of the sinc
convolution and sinc-convolution methods depends on two parameters \( \alpha \) and \( d \).
In fact the parameter \( d \) indicates the size of the holomorphic domain of \( u \). So
the parameter \( \alpha \) is determined by Theorem 9 and the important parameter \( d \)
value is 3.14. For all examples \( c \) in formula (36) is chosen the infinity.

Example 14 ([19]) Consider the integral equation

\[ u(t) - \int_0^1 |t - s|^{0.5} u^2(s) ds = g(t), \quad t \in (0, 1), \]  

(48)

where

\[
g(t) = |t(1-t)|^{0.5} + \frac{16}{15} t^{0.5} + 2t^2(1-t)^{0.5} \\
+ 4\frac{3}{5}(1-t)^{0.5} + 2\frac{2}{5}(1-t)^{0.5} \\
- 4\frac{3}{5}t^{0.5} - 2t(1-t)^{0.5} - \frac{2}{3}(1-t)^{0.5},
\]

with the exact solution \( u(t) = \sqrt{t(1-t)} \). The exact solution has singularity
near the zero. The numerical results have been shown by Figure 1.
Example 15 \cite{26} In this example, we consider the following integral equation
\[ u(t) - \int_0^1 |t-s|^{-\frac{1}{3}} u^2(s)ds = g(t), \quad t \in (0,1). \]  
(49)

The function $g(t)$ is chosen such that $u(t) = t^{\frac{3}{2}}$ be the exact solution. The first derivative of the exact solution has singularity near the zero. Figure 2 shows the error results achieved for the sinc-convolution and sinc-collocation methods.

Example 16 Consider the integral equation
\[ u(t) - \int_0^1 |t-s|^{-\frac{1}{2}} \cos(s + u(s))ds = g(t), \]
where \( g(t) \) is selected so that \( u(t) = \cos(t) \). This example with an infinitely smooth solution is discussed in [14], here we compare sinc-collocation and sinc-convolution solutions of it.

**Example 17** In this experiment, we explore the sensitivity of the methods to the parameter \( \lambda \in (0, 1) \) in weakly singular integral equation. We consider the equation

\[
u(t) - \int_0^1 \frac{1}{|t-s|^\lambda} u^2(s) ds = g(t),
\]

with the exact solution \( u_\lambda(t) = t^{2-\lambda} \). We choose \( \lambda = \frac{k}{10} \), for \( k \in \{1, \cdots, 9\} \) and the errors for the sinc-collocation method are reported in Figure 4.
Conclusion

In this paper, the sinc collocation and sinc convolution methods were considered, and rigorous proofs of the exponential convergence of the schemes are obtained. As the theoretical arguments show directly applying the collocation method with sinc basis functions leads to the parameter $\xi_N$ in the error upper bound. This parameter is unavoidable due to non-uniformly boundedness of the sinc interpolation operator. So, the numerical method based on sinc convolution has been suggested. It was shown theoretically and numerically that convolution scheme is more accurate and achieve exponential convergence with respect to $N$. The main advantage of the sinc methods for the weakly singular kernels is the fact that they disregard the singularity in boundaries. The matter is worthy of the handling discrete sinc-convolution operators and extending to the case of full implicit integral equations by utilizing double exponential sinc method.

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