Critical Phenomena and Thermodynamic Geometry of RN-AdS Black Holes

Chao Niu and Yu Tian

College of Physical Sciences, Graduate University of Chinese Academy of Sciences, Beijing 100049, China

Xiao-Ning Wu

Institute of Mathematics, Academy of Mathematics and System Science, CAS, Beijing 100190, China and Hua Loo-Keng Key Laboratory of Mathematics, CAS, Beijing 100190, China

(Dated: January 19, 2013)

Abstract

The phase transition of Reissner-Nordström black holes in \((n+1)\)-dimensional anti-de Sitter spacetime is studied in details using the thermodynamic analogy between a RN-AdS black hole and a van der Waals liquid gas system. We first investigate critical phenomena of the RN-AdS black hole. The critical exponents of relevant thermodynamical quantities are evaluated. We find identical exponents for a RN-AdS black hole and a Van der Waals liquid gas system. This suggests a possible universality in the phase transitions of these systems. We finally study the thermodynamic behavior using the equilibrium thermodynamic state space geometry and find that the scalar curvature diverges exactly at the van der Waals-like critical point where the heat capacity at constant charge of the black hole diverges.

PACS numbers: 04.70.Dy, 04.50.-h, 04.20.Cv, 04.40.Nr
I. INTRODUCTION

Black hole is one of the most interesting objects in physics. The study of black hole thermodynamics [1, 2] is therefore quite important. Black holes are indeed thermodynamical objects with a physical temperature and an entropy. It has been known over the past few decades that thermodynamics of black holes provides an important tool for understanding several issues involving quantum theories of gravity. These have been intensely discussed for the recent past. However, there is no microscopic or statistical description behind their thermodynamical behavior, although thermodynamic studies of black holes do indicate extremely rich phase structures and critical phenomena in these systems. We can consider black holes as states in a thermodynamical ensemble and to study phase transition in black holes. A well-known example is due to Hawking and Page [3]. Motivated by these ideas, much work has been done on phase structure of black holes, quite rich phase structure and critical phenomena has been found [4].

Recently, the study of phase transitions of black holes in asymptotically anti de-Sitter spacetime [5–10] has focused much interest since these transitions have been related with holographic superconductivity [12–14] in the context of the AdS/CFT correspondence (see relevant reviews in [15]). In this paper, we first review a thermodynamic analogy between a (n+1)-dimensional RN-AdS black hole and a van der Waals liquid gas system first discovered in [5, 11]. From this analogy we calculate the critical exponents of relevant thermodynamical quantities and discuss the scaling symmetry of the free energy. Then we compare the critical exponents with the known case of a four dimensional RN-AdS black hole [5] and a van der Waals liquid gas system [16] and also check whether these exponents satisfy the scaling law for the singular part of the free energy near criticality. Among the results we get, we find that the critical exponents of the four dimensional RN-AdS black hole in [5] have some errors. But we find identical exponents between the (n+1)-dimensional RN-AdS black hole and the van der Waals liquid gas system, and possible universality in the phase transitions of these systems. Furthermore we study the phase transition using a geometrical perspective of equilibrium thermodynamics. This approach has been developed over the last few decades [9, 10, 17–19]. We find the scalar curvature diverges precisely at the van der Waals-like critical point where the heat capacity at constant charge of the black hole diverges.

This paper is organized as follows. In section II we briefly discuss the thermodynamics of
the \((n+1)\)-dimensional RN-AdS black hole, mainly using it to establish our notations and obtain formulae of thermodynamic functions for later use. In section III we study the critical behavior of the \((n+1)\)-dimensional RN-AdS black hole at the van der Waals-like critical point. Further, in section IV, we study the state space scalar curvature of the \((n+1)\)-dimensional RN-AdS black hole in detail. Finally, section V contain a discussion of our results.

II. PHASE STRUCTURE OF A \((n+1)\)-DIMENSIONAL RN-ADS BLACK HOLE

The theme of the present Sec. is to give an overview of the singular behavior of the heat capacity at constant charge of a \((n+1)\)-dimensional RN-AdS black hole which forms the background of this work. For more details of the spherical case, see [11]. Recently, motivated by the study of holographic superconductivity [12, 13], plane symmetric and hyperbola symmetric cases in special dimensions are also discussed [20].

Now we consider general \((n+1)\)-dimensional RN-AdS black holes. The form of the spacetime metric is

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}^2,
\]

where

\[
f(r) = k - \frac{8\Gamma(\frac{n}{2})M}{(n-1)\pi^{\frac{n}{2}-1}r^{n-2}} + \frac{Q^2}{r^{2n-4}} + \frac{r^2}{l^2}
\]

and \(\Lambda = -\frac{n(n-1)}{2l^2}\) is the cosmological constant. Here \(k = 1, 0, -1\) corresponds to the sphere, plane and hyperbola symmetric cases respectively and \(d\Omega_{n-1}\) is the metric of the associated \((n-1)\)-dimensional base manifold.

The mass of the black hole is given by

\[
M = \frac{(n-1)\pi^{\frac{n}{2}-1}}{8\Gamma(\frac{n}{2})}(kr^{n-2}_+ + Q^2r^{n-2}_+ + \frac{r^n}{l^2}),
\]

where \(r_+\) is the value of \(r\) at the horizon.

Using the Bekenstein-Hawking formula, we have

\[
S = \frac{A_\mu}{4} = \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})}r^{n-1}_+.
\]

---

1 Throughout we shall adopt Planck units in which \(G = \hbar = c = k_B = 1\), where all symbols have their usual meanings.
It is now possible to determine the other thermodynamic entities using the basic thermo-
dynamic relations

\[ \delta M = T \delta S + \Phi \delta Q. \]  

These are defined as

\[
T = \left( \frac{\partial M}{\partial S} \right)_Q = \frac{1}{4\pi} \frac{-\frac{2\Lambda}{n-1}r_+^{2n-2} + (n-2)kr_+^{2n-4} - (n-2)Q^2}{r_+^{2n-3}} 
\]

(5)

\[
\Phi = \left( \frac{\partial M}{\partial Q} \right)_S = \frac{(n-1)\pi \frac{n}{2} - 1}{4\Gamma(n/2)} \frac{Q}{r_+^{n-2}} 
\]

(6)

\[
C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = \frac{2(n-1)\pi \frac{n}{2} + 1}{\Gamma(n/2)} \frac{-\frac{2\Lambda}{n-1}r_+^{2n-2} - (n-2)kr_+^{2n-4} + (n-2)(2n-3)Q^2}{r_+^{3n-4}T} 
\]

(7)

where \( \Phi \) is the potential difference between the horizon and infinity, \( T \) is the Hawking
temperature, \( S \) is the entropy and \( C_Q \) is the heat capacity at constant charge of the black
hole.

For a non-extreme black hole, it can be seen from (7) that \( C_Q \) is always positive and
regular for the \( k = 0, -1 \) cases, which tells us that there is no phase transition happening. However, for the spherically symmetry case \( (k = 1) \), \( C_Q \) will become singular for a certain
set of black hole parameters \( (M, Q) \) at which

\[-\frac{2\Lambda}{n-1}r_+^{2n-2} - (n-2)kr_+^{2n-4} + (n-2)(2n-3)Q^2 = 0. \]  

(8)

Considering the equation (8), we then find that the critical points are given in terms of the
radius of the event horizon as \( r_1, r_2 (r_1 < r_2) \) when \( Q^2 < -\frac{(n-2)^2}{2\Lambda}r_+^{-2} + \frac{1}{(n-1)(2n-3)} =: Q_c^2 \). For
the special value \( Q^2 = Q_c^2 \), the two horizons degenerate, so we denote \( r_c := r_1 = r_2 = \frac{n-2}{\sqrt{2\Lambda}}. \)

For fixed \( Q \) so that \( Q^2 < Q_c^2 \), \( C_Q < 0 \) when \( r_1 < r_+ < r_2 \) and \( C_Q > 0 \) when \( r_+ < r_1 \)
and \( r_+ > r_2 \), so across the critical points at \( r_1 \) and \( r_2 \), there is a change of thermodynamic
stability of a black hole (see Fig.1).

In the limit \( Q^2 \) approaches the critical value \( Q_c^2 \), both \( r_1 \) and \( r_2 \) degenerate into \( r_c \). In this
case, \( C_Q \) remains positive and the unstable phase of a black hole disappears (see Fig.2). When \( Q^2 \) is greater than \( Q_c^2 \), the heat capacity \( C_Q \) of the RN-AdS black hole is always
regular.
III. CRITICAL BEHAVIOR AT THE VAN DER WAALS-LIKE CRITICAL POINT

As described in II, when the charge of a RN-AdS black hole reaches the critical value $Q_c$, the critical points at $r_1$ and $r_2$ degenerate into a single critical point located at $r_c$. The thermally unstable phase of a RN-AdS black hole disappears (see Fig. (2)). The theme of this section is to study the critical thermodynamic behavior of a RN-AdS black hole near $r_c$. To this end, we shall first review a thermodynamic analogy between a RN-AdS black hole and a van der Waals liquid gas system. The analogy, though incomplete, will still serve
as a very useful guide in the study of the critical behavior of a RN-AdS black hole in the vicinity of \( r_c \).

A. Thermodynamic analogy with a van der Waals liquid gas system

Given the potential at the event horizon \( \Phi = \frac{(n-1)\pi \frac{2}{n} - 1}{4\Gamma(\frac{n}{2}) r_+} Q \), the equation of state \( [3] \) can be rewritten as

\[
T = \frac{1}{4\pi} \frac{(n-2)}{(n-1)\pi \frac{2}{n} - 1} \frac{2}{n-2} - \frac{(n-2)}{(n-1)\pi \frac{2}{n} - 1} \frac{2}{n-2} \frac{2n-2}{n-2} - \frac{2\Lambda}{n-1} Q^{\frac{n+1}{n-2}}.
\]

(9)

In term of the thermodynamical variables \((Q, \Phi)\), we have

\[
C_Q = \frac{(n-1)\pi \frac{2}{n} - 1}{2\Gamma(\frac{n}{2})} \times \\
\frac{(n-2)\Phi^{n-2}}{(n-1)\pi \frac{2}{n} - 1} \frac{2}{n-2} - \frac{(n-2)\Phi^{n-2}}{(n-1)\pi \frac{2}{n} - 1} \frac{2}{n-2} \frac{2n-2}{n-2} - \frac{2\Lambda}{n-1} Q^{\frac{n+1}{n-2}}
\]

(10)

and

\[
\left( \frac{\partial Q}{\partial \Phi} \right)_T = \frac{Q}{\Phi} \frac{(n-2)(2n-3)}{(n-2)(2n-3)} \frac{2}{n-2} \frac{2n-2}{n-2} - \frac{(n-2)}{(n-1)\pi \frac{2}{n} - 1} \frac{2}{n-2} \frac{2n-2}{n-2} - \frac{2\Lambda}{n-1} Q^{\frac{n+1}{n-2}}.
\]

(11)

It may be inferred from \( [11] \) that, like a subcritical isotherm of a van der Waals liquid gas system in the \((P, V)\) phase plane, an isotherm of a RN-AdS black hole with \( T > T_c \) also has a local maxima and minima located respectively at \( \Phi_1 \) and \( \Phi_2 \). Along the segment of the isotherm between \( \Phi_1 \) and \( \Phi_2 \), a RN-AdS black hole is in a thermally unstable phase with \( \left( \frac{\partial Q}{\partial \Phi} \right)_T > 0 \) (see Fig.\( [3] \)).

In the limit when \( T_c \) is reached, the shape of the isotherm undergo noticeable change (see Fig.\( [11] \)) and the critical points located at \( \Phi_1 \) and \( \Phi_2 \) on a subcritical isotherm coalesce into a single critical point located at \( \Phi_c := \frac{(n-1)\pi \frac{2}{n} - 1}{4\Gamma(\frac{n}{2}) r_c} Q_c = \frac{(n-1)\pi \frac{2}{n} - 1}{4\Gamma(\frac{n}{2}) \sqrt{(n-1)(2n-3)}} \) at the critical isotherm. The critical point at \( \Phi_c \) coincides with that located at \( r_c \) on the critical isocharge curve with \( Q = Q_c \).
FIG. 3: The isotherm of a RN-AdS black hole along which $T > T_c$. The local maxima and minima located respectively at $\Phi_1$ and $\Phi_2$ are critical points of $C_Q$. For $\Phi \in (\Phi_1, \Phi_2)$, the black hole is unstable with $(\frac{\partial Q}{\partial \Phi})_T > 0$.

Like the case of the van der Waals liquid gas system, the critical point at the critical isotherm (along which $T = T_c$) of a RN-AdS black hole is also a point of inflection of the critical isotherm and may be characterized by

\[
\left. \frac{\partial Q}{\partial \Phi} \right|_c = 0
\]
\[
\left. \frac{\partial^2 Q}{\partial \Phi^2} \right|_c = 0
\]

where the subscript $c$ denotes the corresponding quantity evaluated at the critical point at $r_c$ from now on. In view of the above similarities, if we formally identify the variables $(Q, \Phi)$ of a RN-AdS black hole with $(P, V)$ of a van der Waals liquid gas system, then we see that, at least at a qualitative level, the phase structure of a RN-AdS black hole does bear certain remarkable resemblances to that of a van der Waals liquid gas system.

**B. The introduction of an order parameter**

In analogy to a van der Waals liquid gas system, an order parameter in the RN-AdS context which measures the phase change across the critical at $r_c$ may also be defined in
FIG. 4: The critical isotherm along which $T = T_c$. The point of inflection located at $\Phi_c$ is a critical point of $C_Q$, $C_Q > 0$ along the critical isotherm.

terms of the Maxwell equal-area law. To do so, in the $(Q, \Phi)$ phase plane, fix a subcritical isotherm and draw a horizontal line which interests the subcritical isotherm at points $a, d, b$ (see Fig.5) such that the area bounded by the horizontal line segment $ad$ and the isotherm is equal to that bounded by the line segment $db$ and the isotherm.

As in the case of a van der Waals liquid gas system, define

$$\eta = \Phi_b - \Phi_a$$

as the order parameter to describe the phase change of a RNAdS black hole near $r_c$.

C. Critical exponents

Near the critical point at the critical isotherm, the critical behavior of a van der Waals liquid gas system may be described in terms of

$$(1) \quad P - P_c \sim (V - V_c)^\delta$$

$$(2) \quad \frac{V_g - V_l}{V_c} \sim (-\epsilon)^\beta$$
FIG. 5: A horizontal line is drawn which connects points \( a \) and \( b \) of the subcritical isotherm. The area bounded by the line segment \( ad \) and the isotherm is equal to that bounded by the line segment \( db \) and the isotherm.

\[
\begin{align*}
(3) \quad C_P & \sim (-\epsilon)^{-\alpha'} \quad (T < T_c) \\
& \sim \epsilon^{-\alpha} \quad (T > T_c)
\end{align*}
\]

\[
\begin{align*}
(4) \quad \kappa_T & \sim (-\epsilon)^{-\gamma'} \quad (T < T_c) \\
& \sim \epsilon^{-\gamma} \quad (T > T_c).
\end{align*}
\]

With the formal correspondence \( (Q, \Phi) \leftrightarrow (P, V) \) as described in the preceding subsection, analogous quantities may also be defined for a RN-AdS black hole. The concrete values of the corresponding critical exponents in the case of a RN-AdS black hole can also be worked out as follows.

1. **The degree of the critical isotherm \( \delta \)**

Using the equation of state \([9]\), we have

\[
Q^{\frac{1}{n-2}} = -\frac{n-1}{4\Lambda} \left( 4\pi \left( \frac{4\Gamma\left(\frac{n}{2}\right)}{(n-1)\pi^{\frac{n}{2}-1}} \Phi \right)^{\frac{1}{n-2}} T - \right.
\]

\[
\sqrt{16\pi^2 \left( \frac{4\Gamma\left(\frac{n}{2}\right)}{(n-1)\pi^{\frac{n}{2}-1}} \Phi \right)^{\frac{2}{n-2}} T^2 - \frac{8\Lambda(n - 2)}{n - 1} \left( \frac{4\Gamma\left(\frac{n}{2}\right)}{(n-1)\pi^{\frac{n}{2}-1}} \Phi \right)^{\frac{2n-2}{n-2}} - \left( \frac{4\Gamma\left(\frac{n}{2}\right)}{(n-1)\pi^{\frac{n}{2}-1}} \Phi \right)^{\frac{2}{n-2}} \right) \left( \frac{4\Gamma\left(\frac{n}{2}\right)}{(n-1)\pi^{\frac{n}{2}-1}} \Phi \right)^{\frac{2n-2}{n-2}} - \left( \frac{4\Gamma\left(\frac{n}{2}\right)}{(n-1)\pi^{\frac{n}{2}-1}} \Phi \right)^{\frac{2}{n-2}} \right).
\]

\[(13)\]
In order to examine the neighborhood of the critical point, we introduce expansion parameter $\epsilon = (T/T_c) - 1$ and $\omega = (\Phi/\Phi_c) - 1$. In the neighborhood of the critical point, (13) can be written

$$Q = a_{00} + a_{10}\epsilon + a_{01}\omega + a_{11}\epsilon\omega + a_{20}\epsilon^2 + a_{02}\omega^2 + a_{21}\epsilon^2\omega + a_{12}\epsilon\omega^2 + a_{30}\epsilon^3 + a_{03}\omega^3 + \cdots \quad (14)$$

where $a_{\mu\nu}$ the coefficient of $\epsilon^\mu\omega^\nu$ in the expansion.

Let $\epsilon = 0$ in (14). This gives

$$Q = a_{00} + a_{01}\omega + a_{02}\omega^2 + a_{03}\omega^3 + \cdots \quad (15)$$

where

$$a_{00} = Q_c$$
$$a_{01} = a_{02} = 0$$
$$a_{03} \neq 0.$$  

This means

$$\delta = 3.$$  

2. The degree of the coexistence curve $\beta$

In the neighbourhood of the critical point, we have (14). The values of $\omega$ on either side of the coexistence curve can be found from the conditions that along the isotherm,

$$\int_{\Phi_a}^{\Phi_b} \Phi dQ = 0 \quad (16)$$

and

$$Q(\Phi_a) = Q(\Phi_b). \quad (17)$$

Let $\Phi_a = \Phi_c(1 - \omega_a)$ and $\Phi_b = \Phi_c(1 + \omega_b)$. Substitute (14) into (16) and (17), we have

$$a_{11}\epsilon(\omega_b + \omega_a) + a_{21}\epsilon^2(\omega_b + \omega_a) + \frac{1}{2}(a_{11} + 2a_{12})\epsilon(\omega_b^2 - \omega_a^2) + a_{03}(\omega_b^3 + \omega_a^3) = 0 \quad (18)$$

and

$$a_{11}\epsilon(\omega_b + \omega_a) + a_{21}\epsilon^2(\omega_b + \omega_a) + a_{12}\epsilon(\omega_b^2 - \omega_a^2) + a_{03}(\omega_b^3 + \omega_a^3) = 0. \quad (19)$$
In order for \((18)\) and \((19)\) to be consistent, we must have \(\omega_a = \omega_b\). If we plug this into \((18)\) or \((19)\), we get \(\omega_a = \omega_b = \omega\). This gives

\[
\omega^2 = -\frac{1}{a_{03}}(a_{11}\epsilon + a_{21}\epsilon^2).
\]

Thus,

\[
\omega_b \approx \omega_a = \sqrt{-\frac{a_{11}}{a_{03}}\epsilon} = (n - 2)\sqrt{6}\epsilon.
\]

This means

\[
\beta = \frac{1}{2}.
\]

3. The heat capacity exponent \(\alpha\)

From \((3)\), \((5)\) and \((6)\), we have

\[
C_\Phi = \frac{(n - 1)\pi^{\frac{n}{2}}}{2\Gamma\left(\frac{n}{2}\right)} \times \frac{(n - 2)Q^{\frac{n-1}{2}} (\frac{4\Gamma(\frac{n}{2})}{(n-1)^{\frac{n}{2}-1}} \Phi)^{\frac{1}{2}} - (n - 2)Q^{\frac{n-1}{2}} (\frac{4\Gamma(\frac{n}{2})}{(n-1)^{\frac{n}{2}-1}} \Phi)^{\frac{1}{2}} - 2\Lambda^{\frac{n+1}{2}} - 2\Lambda^{\frac{n+1}{2}}}{(n - 2)\left(\frac{4\Gamma(\frac{n}{2})}{(n-1)^{\frac{n}{2}-1}} \Phi\right)^{\frac{n+1}{2}} - (n - 2)\left(\frac{4\Gamma(\frac{n}{2})}{(n-1)^{\frac{n}{2}-1}} \Phi\right)^{\frac{n+1}{2}} - 2\Lambda^{\frac{n+1}{2}} \left(\frac{4\Gamma(\frac{n}{2})}{(n-1)^{\frac{n}{2}-1}} \Phi\right)^{\frac{n+1}{2}}}. \tag{20}
\]

From \((20)\), we see that \(C_\Phi\) display no singular behavior at the critical point. Therefore

\[
\alpha = \alpha' = 0.
\]

4. The isothermal compressibility exponent \(\gamma\)

Let us compute \((\partial Q / \partial \omega)_{\epsilon}\). We obtain

\[
\left(\frac{\partial Q}{\partial \omega}\right)_{\epsilon} = a_{11}\epsilon + a_{21}\epsilon^2 + 2a_{12}\epsilon\omega + 3a_{03}\omega^2.
\]

For \(T < T_c\) one approaches the critical point along the critical isochore. Then set \(\omega = 0\), we obtain

\[
\left(\frac{\partial Q}{\partial \omega}\right)_{\epsilon} = a_{11}\epsilon = 2^{2-\frac{n}{2}}(n-2)^{n-2}\sqrt{(n-1)(2n-3)(-\Lambda)^{1-\frac{n}{2}}} \epsilon
\]

for \(\omega = 0\). Therefore

\[
\gamma' = 1.
\]
For $T > T_c$ one approaches the critical point along the coexistence curve. Then set $\omega = \sqrt{-\frac{a_{11}}{a_{03}}} \epsilon$, we obtain

$$\left(\frac{\partial Q}{\partial \omega}\right)_\epsilon = -2a_{11}\epsilon = -2^{\frac{n-2}{n}}(n-2)^{n-2} \sqrt{(n-1)(2n-3)}(-\Lambda)^{1-\frac{n}{2}}\epsilon$$

for $\omega = \sqrt{-\frac{a_{11}}{a_{03}}} \epsilon$. Therefore

$$\gamma = 1.$$  

D. Scaling symmetry for the Gibbs free energy near criticality

In the case of a van der Waals liquid gas system, scaling symmetry exists for the singular part of the Gibbs free energy near the critical point located at the critical isotherm and the critical exponents may all be expressed in terms of the two independent homogeneity degrees of the Gibbs energy \[21\]. In this subsection, we shall show that similar scaling symmetry also exists for a RN-AdS black hole. The behavior of Gibbs free energy near the critical point is similar to van der Waals system, from which scaling laws for the critical exponents can be derived. Scaling symmetry in the black hole critical phenomena was first discussed in \[22\] in the context of Kerr-Newman black holes.

Sufficiently close to $r_c$, the Gibbs free energy for a RN-AdS black hole may be written as $G = G_r + G_s$. Here $G_r$ is the regular part of the Gibbs free energy whose second order partial derivatives are well behaved at the critical point at $r_c$, and $G_s$ is the part of the Gibbs free energy responsible for the singular thermodynamic behavior of a RN-AdS black hole near $r_c$. $G_s$ can be further worked out to be

$$G_s = a\epsilon^2 + b\omega^{4/3} \hspace{1cm} (21)$$

for some constant $a, b$ dependent on $\Lambda$. From \[21\], we find

$$G(\lambda^p\epsilon, \lambda^q\omega) = \lambda G(\epsilon, \omega) \hspace{1cm} (22)$$

with $p = \frac{1}{2}, q = \frac{3}{4}$ and $\lambda$ a real constant. As in the case of a van der Waals liquid-gas system,
the critical exponents derived in the previous section can be expressed in terms of $p, q$ as

$$\alpha = 1 - \frac{1}{p}$$
$$\beta = \frac{1-q}{p}$$
$$\gamma = \frac{2q-1}{p}$$
$$\delta = \frac{q}{1-q}.$$  \tag{23}

From (23), it may also be seen that the critical exponents in the critical regime of $r_c$ are not independent. They are related by following equations [16, 21]:

$$\alpha + 2\beta + \gamma = 2$$
$$\alpha + \beta(\delta - 1) = 2$$
$$\gamma(\delta - 1) = (2 - \alpha)(\delta - 1)$$
$$\gamma = \beta(\delta - 1).$$  \tag{24}

Apart from obtaining the algebraic relations among the critical exponents, (23) or (24) also enable us to give a consistency check of the validity of the critical exponents obtained in Sec. III C.

**IV. STATE SPACE SCALAR CURVATURE FOR RN-ADS BLACK HOLE**

In this section, we will study the critical phenomena of RN-AdS black holes using thermodynamic geometry. The Hessian of the entropy function (or other thermodynamical potentials) can be thought of as a metric tensor on the state space [9, 10, 17–19]. In the context of thermodynamical fluctuation theory Ruppeiner has argued that the Riemannian geometry of this metric gives insight into the underlying statistical mechanical system. In this picture the occurrence of a van der Waals critical point is connected with the divergence of the state space scalar curvature. The metric as defined by Ruppeiner [17] is given by,

$$g_{ij} = -\frac{\partial^2 S(x)}{\partial x^i \partial x^j}$$  \tag{25}

where the coordinates $x^i$ are chosen to be the extensive variables of the system. In fact, it is convenient to use the Weinhold metric which is defined in the following way [19],

$$g^W_{ij} = -\frac{\partial^2 U(x)}{\partial x^i \partial x^j}$$  \tag{26}
where we use $U$ to denote the internal energy. It is well known that the line elements in Ruppeiner geometry and the Weinhold geometry are conformally related by \[23, 24\]

$$dS_R^2 = \frac{1}{T}dS_W^2$$

(27)

where $T$ is the temperature of the RN-AdS black hole. In this picture we will consider $U = M - Q\Phi$, $x^1 = S$ and $x^2 = \Phi$.

From (2), (3), (5), (6) and (7), we have

$$M = \frac{(n - 1)S^{2/3}}{4\pi} \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \times$$

$$\left( 1 + \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \Phi^2 - \frac{2\Lambda}{n(n - 1)} \left( \frac{2\Gamma(\frac{n}{2})S}{\pi^{\frac{n}{2}}} \right) \Phi \right)$$

(28)

$$T = \frac{1}{4\pi} \left( \frac{n - 2}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi^2 + (n - 2) \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n - 1}}$$

(29)

$$C_Q = (n - 1)S \times$$

$$\frac{- (n - 2) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi^2 + (n - 2) \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n - 1}}}{(n - 2)(2n - 3) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi^2 - (n - 2) \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n - 1}}}$$

(30)

Now using \[23, 24\] we can easily calculate the Ruppeiner metric

$$g_{ss} = -\frac{1}{(n - 1)S} \times$$

$$\frac{- (n - 2) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi^2 + (n - 2) \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n - 1}}}{- (n - 2) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi^2 - (n - 2) \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n - 1}}}$$

$$g_{s\Phi} = \frac{-2(n - 1) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} S}{- (n - 2) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi^2 + (n - 2) \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n - 1}}}$$

$$= g_{\Phi S}$$

$$g_{\Phi \Phi} = \frac{-2(n - 2) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi}{- (n - 2) \left( \frac{4\Gamma(\frac{n}{2})}{(n - 1)\pi^{\frac{n}{2} - 1}} \right)^2 \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} \Phi^2 + (n - 2) \left( \frac{\pi^{\frac{n}{2}}}{2\Gamma(\frac{n}{2})} \right) \frac{\pi}{n} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n - 1}}}.$$
Observe that all the metric components have an identical denominator which appears in the expression of the temperature.

Our concern is the scalar curvature of the Ruppeiner metric, which is

$$ R = \frac{C(S, \Phi)}{A(S, \Phi)B^2(S, \Phi)}, $$

where

$$ A(S, \Phi) = -(n - 2)\left(\frac{4\Gamma(n)}{(n - 1)\pi^{n-1}}\right)^2 \left(\frac{\pi^{\frac{n}{2}}}{2\Gamma(n)}\right)^{\frac{2}{n+2}} \Phi^2 $$

$$ + (n - 2)\left(\frac{\pi^{\frac{n}{2}}}{2\Gamma(n)}\right)^{\frac{2}{n-1}} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n-1}}, \quad (31) $$

$$ B(S, \Phi) = (n - 2)(2n - 3)\left(\frac{4\Gamma(n)}{(n - 1)\pi^{n-1}}\right)^2 \left(\frac{\pi^{\frac{n}{2}}}{2\Gamma(n)}\right)^{\frac{2}{n+2}} \Phi^2 $$

$$ - (n - 2)\left(\frac{\pi^{\frac{n}{2}}}{2\Gamma(n)}\right)^{\frac{2}{n-1}} - \frac{2\Lambda}{n - 1} S^{\frac{2}{n-1}}, \quad (32) $$

and $C(S, \Phi)$ is a regular function whose explicit form is irrelevant to the singular behavior of $R$. The function $A(S, \Phi)$ is always positive due to the nonextremal condition, as can be seen from (29). The function $B(S, \Phi)$ is identical with the denominator of the heat capacity at constant charge (30). Hence the scalar curvature will diverge exactly at those points at which the heat capacity diverges. It is easy to see that there are two singular points when temperature above $T_c$ and these two points coincide as $T = T_c$, so the thermal geometry method gives the same result which we have found in section III.

V. DISCUSSIONS

In the present work, We have obtained different thermodynamic entities like temperature, potential and heat capacity at constant charge for a $(n + 1)$-dimensional RN-AdS black hole from the first law of black hole thermodynamics. The heat capacity shows a divergence at the van der Waals-like critical point. Moreover, we have investigated the critical behavior of the $(n + 1)$-dimensional RN-AdS black hole at the van der Waals-like critical point. One of the striking characteristics of the phase transition is the fact that many measures of a system’s behavior near a critical point are independent of the details of the interactions between the particles making up the system. The universal features are not only independent of
the numerical details of the interparticle interactions, but are also independent of the most fundamental aspects of the structure of the system. The critical exponents of a \((n + 1)\)-dimensional RN-AdS black hole and a van der Waals liquid gas system are exactly the same. This result is quite interesting because of the differences in the physical property of the two systems.

We have also studied the phase transition using the geometrical perspective of equilibrium thermodynamics. We find that both the scalar curvature and the heat capacity at constant charge have a common denominator and hence diverge at identical points. This shows that a divergence in the scalar curvature corresponds to a divergence in the heat capacity at constant charge thereby suggesting the occurrence of a phase transition. Moreover, there is another factor in the denominator of the scalar curvature which is the same expression arising in the expression for the temperature \((29)\). So we cannot put it equal to zero due to the nonextremal condition. Therefore we can easily get information about the occurrence of phase transition from the scalar curvature.

We have found that the van der Waals-like critical behavior does not present in the planar or hyperbolic RN-AdS black holes. But we know that there is another type of phase transition associated with a scalar hair in the planar case \([25]\). Based on the AdS/CFT correspondence, this type of phase transition describes the superconductivity phase transition in the dual boundary system (see \([15]\) and references therein), which has been proved a powerful tool to study superconductivity phenomena. In our case, the boundary system dual to the spherical RN-AdS black hole should also have critical behavior dual to the van der Waals-like critical behavior in the bulk. So, it is quite natural to ask whether this dual boundary theory also describes some important, realistic phenomena of phase transition in physics.

We have also observed that RN-dS black holes do not possess the van der Waals-like phase transition, while the Hawking-Page phase transition can occur in this kind of background \([26]\). So, we see that the asymptotic AdS background is crucial for the van der Waals-like phase transition. It is interesting to investigate the underlying mechanism of this phenomenon.
Acknowledgments

We thank Prof. Y. Ling for helpful discussions. This work is partly supported by the National Natural Science Foundation of China (Grant Nos. 10705048, 10731080 and 11075206) and the President Fund of GUCAS.

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