Bi-seasonal discrete time risk model with income rate two

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ABSTRACT
This article proceeds calculation of ultimate time survival probability for bi-seasonal discrete time risk model when premium rate equals two. The same model with income rate equal to one was investigated in 2014 by Damarackas and Siaulys. In general, discrete time and related risk models deal with possibility for a certain version of random walk to hit a certain threshold at least once in time. In this research, the mentioned threshold is the line $u + 2t$ and random walk consists from two interchangeably occurring independent but not necessarily identically distributed random variables. Most of proved theoretical statements are illustrated via numerical calculations. Also, there are raised a couple of conjectures that a certain recurrent determinants are non-vanishing.

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1. Introduction

Modeling of large values has many interests across various nature sciences. Insurers may be concerned on large pay offs, biologists on vanishing of some population and so on. Models, estimating a likelihood of such events, are often random walk based. In this research, we define the random walk (r.w.) by a sum of random variables (r.vs.) $\sum_{i=1}^{t} Z_i$, $t \in \mathbb{N}$. Then, the bi-seasonal discrete time risk model with a generalized premium rate $W(t)$ is defined as follows

$$W(t) = u + \kappa t - \sum_{i=1}^{t} Z_i,$$

where

- $t, \kappa \in \mathbb{N}$ and $u \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,
- $Z_{2i-1} \overset{d}{=} X$, $Z_{2i} \overset{d}{=} Y$ for all $i \in \mathbb{N}$ and $X$, $Y$ are independent integer valued non-negative random variables which may be distributed differently.

Modeling insurers or other individuals wealth by (1), the parameter $u \in \mathbb{N}_0$ is deemed as initial surplus, $\kappa \in \mathbb{N}$ as income or premium rate per unit of time and r.vs. $Z_i$, $i \in \mathbb{N}$ are considered as occurring random claim amounts. Such type of models are discrete
versions of more general Andersen model (see Andersen (1957)) and are widely studied all across the world. In fact, the study is heavily related to the maximum distribution of partial sums of random variables and a great initial sources on that are Spitzer (1956), Spitzer (1988), and Feller (1968). Scrolling across the timeline, an observable works of Gerber and Shiu on the risk collective models could be highlighted: Gerber (1988a), Gerber (1988b), Shiu (1988) and Shiu (1989). Recently, many research articles on the related risk models as in (1) are occurring per year, see for example Răducan, Vernic, and Zbaganu (2015), Asmussen and Albrecher (2010), Dickson (1999), Shimizu and Zhang (2019), Kievinaitė and Šiaulys (2018), Castañer et al. (2013), Kizinevič and Šiaulys (2018) and references therein.

The main concern of some processes, modeled by (1), is whether its deterministic part \( u + \kappa t \) is greater than random part \( \sum_{i=1}^{t} Z_i \) for all natural \( t \) up to some \( T \in \mathbb{N} \) or even when \( T \to \infty \). In some financial context, that is to know the likelihood whether initial savings and earnings are always sufficient to cover incurred expenses. More precisely, we are interested to calculate the probabilities

\[
\varphi(u, T) := \mathbb{P}\left( \bigcap_{t=1}^{T} \{ W(t) > 0 \} \right) = \mathbb{P}\left( \max_{1 \leq t \leq T} \left\{ \sum_{i=1}^{t} (Z_i - \kappa) \right\} < u \right),
\]

\[
\varphi(u) := \mathbb{P}\left( \bigcap_{t=1}^{\infty} \{ W(t) > 0 \} \right) = \mathbb{P}\left( \sup_{t \geq 1} \left\{ \sum_{i=1}^{t} (Z_i - \kappa) \right\} < u \right),
\]

where the first one \( \varphi(u, T) \) is called the finite time survival probability and the later \( \varphi(u) \) – ultimate time survival probability. Algorithms for \( \varphi(u, T) \) are a lot more simple than for \( \varphi(u) \). Due to complexity of \( \varphi(u) \), \( u \in \mathbb{N}_0 \) with arbitrary natural \( \kappa \), we restrict the model (1) to \( \kappa = 2 \), where \( \kappa = 1 \) case was studied in Damarackas and Šiaulys (2014). Even such a little change from \( \kappa = 1 \) to \( \kappa = 2 \) has a significant impact on expressions of the ultimate time survival probability \( \varphi \).

Let us demonstrate where the expressions of \( \varphi(u) \), \( u \in \mathbb{N}_0 \) are stemming from. First we need to introduce some notations. For \( u \in \mathbb{N}_0 \) we denote

\[
x_u := \mathbb{P}(X = u), \quad y_u := \mathbb{P}(Y = u), \quad s_u := \mathbb{P}(X + Y = u),
\]

\[
X(u) := \sum_{i=0}^{u} x_i, \quad Y(u) := \sum_{i=0}^{u} y_i, \quad S(u) := \sum_{i=0}^{u} s_i,
\]

\[
\bar{X}(u) := 1 - X(u), \quad \bar{Y}(u) := 1 - Y(u), \quad \bar{S}(u) := 1 - S(u).
\]

By the formula \( \mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c) \), rearrangements and other techniques from elementary probability theory, for the bi-seasonal discrete time risk model with income rate two, we have

\[
\varphi(u) = \mathbb{P}\left( \bigcap_{t=1}^{\infty} \left\{ u + 2t - \sum_{i=1}^{t} Z_i > 0 \right\} \right) = \mathbb{P}\left( \bigcap_{t=2}^{\infty} \left\{ u + 2t - \sum_{i=1}^{t} Z_i > 0 \right\}, \ Z_1 < u + 2 \right)
\]

\[
= \mathbb{P}\left( \bigcap_{t=2}^{\infty} \left\{ u + 2t - \sum_{i=1}^{t} Z_i > 0 \right\} \right) - \mathbb{P}\left( \bigcap_{t=2}^{\infty} \left\{ u + 2t - \sum_{i=1}^{t} Z_i > 0 \right\}, \ Z_1 \geq u + 2 \right)
\]
\[
= \mathbb{P}\left( \bigcap_{t=3}^{\infty} \left\{ u + 2t - Z_1 - Z_2 - \sum_{i=3}^{t} Z_i > 0 \right\}, Z_1 + Z_2 < u + 4 \right)
\]
\[
- \mathbb{P}\left( \bigcap_{t=3}^{\infty} \left\{ u + 2t - Z_1 - Z_2 - \sum_{i=3}^{t} Z_i > 0 \right\}, Z_1 \geq u + 2, Z_1 + Z_2 < u + 4 \right)
\]
\[
= \sum_{k=0}^{u+3} \mathbb{P}\left( \bigcap_{t=1}^{\infty} \left\{ u + 2(t-2) + 4 - Z_1 - Z_2 - \sum_{i=1}^{t-2} Z_i > 0 \right\}, Z_1 + Z_2 = k \right)
\]
\[
- (x_{u+3}y_0 + x_{u+2}y_1)\mathbb{P}\left( \bigcap_{t=1}^{\infty} \left\{ 1 + 2t - \sum_{i=1}^{t} Z_i > 0 \right\} \right)
\]
\[
- x_{u+2}y_0\mathbb{P}\left( \bigcap_{t=1}^{\infty} \left\{ 2 + 2t - \sum_{i=1}^{t} Z_i > 0 \right\} \right)
\]
\[
= \sum_{k=0}^{u+3} \phi(u + 4 - k)s_k - (x_{u+3}y_0 + x_{u+2}y_1)\phi(1) - x_{u+2}y_0\phi(2)
\]
\[
= \sum_{k=1}^{u+4} \phi(k)s_{u+4-k} - (x_{u+3}y_0 + x_{u+2}y_1)\phi(1) - x_{u+2}y_0\phi(2).
\]
It is curious that the number of initial values, needed for (2), might be reduced by one, finding the relation between $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, $\varphi(3)$. Before demonstrating that, we introduce the net profit condition. We say that the net profit condition holds for the model (1) with $\kappa = 2$ if $\mathbb{E}S < 4$. This condition is crucial trying to avoid guaranteed ruin (survival with probability 0) as time grows ultimately, see Theorem 2.6 in Section 2. An intuitive understanding of the net profit condition is simple. The expectation of $W(t)$ in (1) for even or odd $t \in \mathbb{N}$ is:

$$\mathbb{E}W(2t) = u + t(2\kappa - \mathbb{E}S),$$
$$\mathbb{E}W(2t - 1) = u + \kappa + \mathbb{E}X + t(2\kappa - \mathbb{E}S),$$

where the sign of $2\kappa - \mathbb{E}S$ influences the sign of $\mathbb{E}W(t)$ and possibility that $W(t) > 0$ for some natural $t$'s.

We now turn back to the relation between $\varphi(0), \ldots, \varphi(3)$. Summing up the both sides of (2), we obtain

$$\sum_{u=0}^{v} \varphi(u) = \sum_{u=0}^{v} \sum_{k=1}^{u+4} \varphi(k) s_{u+4-k} - \left( (X(v + 3) - X(2))y_0 + (X(v + 2) - X(1))y_1 \right) \varphi(1)$$
$$- \left( (X(v + 2) - X(1))y_0 \varphi(2), \right. \tag{4}$$

where

$$\sum_{u=0}^{v} \sum_{k=1}^{u+4} \varphi(k) s_{u+4-k} = \sum_{k=1}^{3} \varphi(k) \sum_{u=0}^{v} s_{u+4-k} + \sum_{k=4}^{v+4} \varphi(k) \sum_{u=k-4}^{v} s_{u+4-k}$$
$$= \sum_{k=1}^{3} \varphi(k) (S(v + 4 - k) - S(3 - k)) + \sum_{k=4}^{v+4} \varphi(k) S(v + 4 - k). \tag{5}$$

Inserting (5) into (4) and rearranging we obtain

$$\sum_{k=0}^{v+4} \varphi(k) S(v + 4 - k) - \sum_{u=v+1}^{v+4} \varphi(u)$$
$$= \sum_{k=1}^{3} \varphi(k) (S(v + 4 - k) - S(3 - k)) - \sum_{k=0}^{3} \varphi(k) S(v + 4 - k)$$
$$- \left( (X(v + 3) - X(2))y_0 + (X(v + 2) - X(1))y_1 \right) \varphi(1)$$
$$- \left( (X(v + 2) - X(1))y_0 \varphi(2). \right. \tag{6}$$

If $v \to \infty$ and $\mathbb{E}S < 4$, from the last equation, by Lemma 1, we get

$$\varphi(0) + (\bar{X}(2)y_0 + \bar{X}(1)y_1) \varphi(1) + \bar{X}(1)y_0 \varphi(2) + \sum_{k=1}^{3} \varphi(k) S(3 - k) = 4 - \mathbb{E}S. \tag{7}$$

Adding equality (7) to the list of equations in (3), we are able to express $\varphi(n)$ via $\varphi(0)$, $\varphi(1)$ and $\varphi(2)$ for all $n \in \mathbb{N}_0$. Namely that is the main idea for finding a needed initial values for the recurrence relation in (2).

The rest of the article is structured as follows. In Section 2, Theorem 2.1 serves purpose for the expression of finite time survival probability $\varphi(u, T)$ while Theorems 2.2–2.5 deal
with expressions of the ultimate time survival probability under the net profit condition. Expressions of \( \varphi \) under \( ES < 4 \) in Theorems 2.2–2.5 are dependent on the lowest value of the distribution of \( S = X + Y \). The last Theorem 2.6 demonstrates that survival is impossible in all except few trivial cases if the net profit condition is violated \( ES \geq 4 \).

2. Statements

In this section we formulate all of the statements which are necessary to express \( \varphi(u, T) \) and \( \varphi(u) \) – the finite and ultimate time survival probabilities accordingly, for the model (1) with \( \kappa = 2 \). In fact, the distribution function \( \varphi(u, T) \) can be calculated by following the ideas in Blaževièius, Bieliauskièienè, and Šiaulys (2010, Theorems 1–4), however Theorem 2.1 below is specially adopted for the bi-seasonal discrete time risk model with income rate two. Our reason to have it, is an interest to the broader view as \( T \) grows and \( \varphi(u, T) \) approximates \( \varphi(u) \), see Section 4. All of the statements formulated in this section are proved in the later Section 3.

**Theorem 2.1.** For the finite time survival probability \( \varphi(u, T) \) of the bi-seasonal discrete time risk model with income rate two, the following expressions are true:

\[
\varphi(u, 1) = X(u + 1), \quad \varphi(u, 2) = \sum_{k=0}^{u+1} x_k Y(u + 3 - k),
\]

\[
\varphi(u, T) = \sum_{k=0}^{u+3} \varphi(u + 4 - k, T - 2)s_k - x_{u+2}y_0\varphi(2, T - 2) - (x_{u+2}y_1 + x_{u+3}y_0)\varphi(1, T - 2), \quad T \geq 3.
\]

We now turn to the ultimate time. As mentioned, expressions of \( \varphi \) are heavily dependent on the lowest value of convolution \( S = X + Y \). For \( s_0 > 0 \) let us define four recurrent sequences \( \alpha_n, \beta_n, \gamma_n, \delta_n \). For \( n = 0, 1, 2, 3 \) and for \( n = 4, 5, \ldots \)

| \( n \) | \( \alpha_n \) | \( \beta_n \) | \( \gamma_n \) | \( \delta_n \) |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | \(-\frac{1}{s_0}\)| \(-\frac{(5/2-\kappa)(2y_0+\kappa)(2)(y_1+y_0)}{s_0}\)| \(-\frac{5/2+\kappa(2y_0+\kappa)(y_1)}{s_0}\)| \(\frac{1}{s_0}\)|

\[
\alpha_n = \frac{1}{s_0} \left( \alpha_{n-4} - \sum_{k=1}^{n-1} s_{n-k} \alpha_k \right), \quad \beta_n = \frac{1}{s_0} \left( \beta_{n-4} - \sum_{k=1}^{n-1} s_{n-k} \beta_k + x_{n-1}y_0 + x_{n-2}y_1 \right),
\]

\[
\gamma_n = \frac{1}{s_0} \left( \gamma_{n-4} - \sum_{k=1}^{n-1} s_{n-k} \gamma_k + x_{n-2}y_0 \right), \quad \delta_n = \frac{1}{s_0} \left( \delta_{n-4} - \sum_{k=1}^{n-1} s_{n-k} \delta_k \right).
\]

**Theorem 2.2.** If \( s_0 > 0 \) and \( ES < 4 \), for the ultimate time survival probability of the bi-seasonal discrete time risk model with income rate two, the following relations hold:
\[
\begin{pmatrix}
\alpha_{n+1} - \alpha_n & \beta_{n+1} - \beta_n & \gamma_{n+1} - \gamma_n \\
\alpha_{n+2} - \alpha_n & \beta_{n+2} - \beta_n & \gamma_{n+2} - \gamma_n \\
\alpha_{n+3} - \alpha_n & \beta_{n+3} - \beta_n & \gamma_{n+3} - \gamma_n
\end{pmatrix}
\times
\begin{pmatrix}
\varphi(0) \\
\varphi(1) \\
\varphi(2)
\end{pmatrix}
\times
(4 - \text{ES})
\]

Remark 1. It holds that \( \varphi(n) \leq \varphi(n + 1) \leq 1 \) for all \( n \in \mathbb{N}_0 \). Also, Lemma 1 implies \( \varphi(n) \approx 1 \) if \( n \) is large enough. Therefore, in practical applications the right hand side of (8) is assumed \((0, 0, 0)^T\) and three needed initial values \( \varphi(0), \varphi(1), \varphi(2) \) are solved out from (8).

Remark 2. We can not prove the system matrix in (8) being nonsingular for all \( n \in \mathbb{N}_0 \). On the other hand, we never find such matrix being singular with any chosen underlying distributions. Attempts to prove and numerical calculations raise the following conjecture.

Conjecture 1. Let \( D_n \) denote the principal determinant of the system matrix in (8). Then, \( 1 \leq D_{2n} \leq D_{2n+2} \) and \( -1 \geq D_{2n+1} \geq D_{2n+3} \) for all \( n \in \mathbb{N}_0 \).

We now assume \( s_0 = 0 \) and \( s_1 > 0 \). For \( s_1 > 0 \) let us define three recurrent sequences \( \bar{\alpha}_n, \bar{\beta}_n, \bar{\delta}_n \). For \( n = 0, 1, 2 \)

| \( n \) | \( \bar{\alpha}_n \) | \( \bar{\beta}_n \) | \( \bar{\delta}_n \) |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 2 | \(-\frac{1}{s_1 + (2)\gamma_0 + \bar{\alpha}_1}\) | \(-\frac{(2)\gamma_0 + (1)\gamma_1}{s_1 + (1)\gamma_0}\) | \(-\frac{1}{s_1 + (1)\gamma_0}\) |

and for \( n = 3, 4, \ldots \)

\[ \bar{\alpha}_n = \frac{1}{s_1} \left( \alpha_{n-3} - \sum_{k=1}^{n-1} s_{n+1-k} \alpha_k + \alpha_{n-1} \gamma_0 \bar{\alpha}_2 \right), \]

\[ \bar{\beta}_n = \frac{1}{s_1} \left( \beta_{n-3} - \sum_{k=1}^{n-1} s_{n+1-k} \beta_k + \beta_{n-1} \gamma_0 \beta_2 \right), \]

\[ \bar{\delta}_n = \frac{1}{s_0} \left( \delta_{n-3} - \sum_{k=1}^{n-1} s_{n+1-k} \delta_k + \delta_{n-1} \gamma_0 \delta_2 \right). \]

Theorem 2.3. If \( s_0 = 0, s_1 > 0 \) and \( \text{ES} < 4 \), for the ultimate time survival probability of the bi-seasonal discrete time risk model with income rate two, the following relations are true:
(\bar{x}_{n+1} - \bar{x}_n, \bar{\beta}_{n+1} - \bar{\beta}_n) \times \left( \begin{array}{c} \varphi(0) \\ \varphi(1) \end{array} \right) + \left( \begin{array}{c} \bar{\delta}_{n+1} - \bar{\delta}_n \\ \bar{\delta}_{n+2} - \bar{\delta}_n \end{array} \right) \times (4 - \mathbb{E}S) \\
= \left( \begin{array}{c} \varphi(n+1) - \varphi(n) \\ \varphi(n+2) - \varphi(n) \end{array} \right), \ n \in \mathbb{N}_0, 
\tag{9}
\]

\[
\varphi(2) = \frac{-\varphi(0) - (S(2) + \bar{X}(2)y_0 + \bar{X}(1)y_1)\varphi(1) + 4 - \mathbb{E}S}{s_1 + \bar{X}(1)y_0},
\]

\[\varphi(u) = \frac{1}{s_1} \left( \varphi(u-3) + (x_u y_0 + x_{u-1} y_1)\varphi(1) + x_{u-1} y_0 \varphi(2) - \sum_{k=1}^{u-1} s_{u+1-k} \varphi(k) \right),
\]

\[u = 3, 4, \ldots\]

**Remark 3.** We note that conditions \(s_0 = 0\) and \(s_1 > 0\) imply that denominator of \(\varphi(2)\) in Theorem 2.3 and in expressions \(\bar{x}_2, \bar{\beta}_2, \bar{\gamma}_2\) above, is \(s_1 + \bar{X}(1)y_0 = y_0 + x_0 y_1 > 0\).

**Remark 4.** Practical applications of Theorem 2.3 are described the same way as for Theorem 2.2 in Remark 1. The non-vanishing of system matrix in (9) is unknown for all \(n \in \mathbb{N}_0\). Arguing the same as in Remark 2, the following conjecture is raised.

**Conjecture 2.** Let \(\bar{D}_n\) denote the principal determinant of the system matrix in (9). Then, \(1 \leq \bar{D}_n \leq \bar{D}_{n+1}\) for all \(n \in \mathbb{N}_0\).

We next turn to the case \(s_0 = s_1 = 0, s_2 > 0\). This, in turn, has three underlying scenarios which impact the expression of survival probability \(\varphi\):

\(S_1: x_0 = 0, \ y_0 = 0, \ x_1 > 0, \ y_1 > 0,\)

\(S_2: x_0 > 0, \ y_0 = 0, \ y_1 = 1, \ y_2 > 0,\)

\(S_3: x_0 = 0, \ y_0 > 0, \ x_1 = 0, \ x_2 > 0.\)

Scenarios \(S_1\) and \(S_2\) imply that \(s_2 - x_2 y_0 = x_1 y_1 + x_0 y_2 > 0\), while \(S_3\) implies \(s_2 - x_2 y_0 = 0\). Under \(S_1\) or \(S_2\) let us define two recurrent sequences for \(n \in \mathbb{N}\):

\[
\hat{x}_0 = 1, \ \hat{x}_1 = -\frac{1}{\bar{X}(1)y_1 + s_2}, \ \hat{x}_n = \frac{1}{s_2} \left( \hat{x}_{n-2} - \sum_{k=1}^{n-1} \hat{x}_{k+1} \hat{s}_{n+2-k} + x_n y_1 \hat{x}_1 \right), \ n \geq 2,
\]

\[
\hat{\delta}_0 = 0, \ \hat{\delta}_1 = \frac{1}{\bar{X}(1)y_1 + s_2}, \ \hat{\delta}_n = \frac{1}{s_2} \left( \hat{\delta}_{n-2} - \sum_{k=1}^{n-1} \hat{\delta}_{k+1} \hat{s}_{n+2-k} + x_n y_1 \hat{\delta}_1 \right), \ n \geq 2.
\]

**Remark 5.** We note that scenarios \(S_1\) and \(S_2\) accordingly imply \(\hat{x}_1 = -1/y_1 = -\hat{\delta}_1\) and \(\hat{x}_1 = -1/(x_0 y_2) = -\hat{\delta}_1\).
Under condition $S_3$ let us also define two recurrent sequences for $n \in \mathbb{N}$:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = -\frac{y_0 + y_1}{y_0}, \quad \tilde{x}_n = \frac{1}{s_2} \left( \tilde{x}_{n-2} - \sum_{k=1}^{n-1} \tilde{x}_k s_{n+2-k} + (x_{n+1} - x_n)y_0 \right), \quad n \geq 3,$$

$$\tilde{\delta}_1 = 0, \quad \tilde{\delta}_2 = \frac{1}{y_0}, \quad \tilde{\delta}_n = \frac{1}{s_2} \left( \tilde{\delta}_{n-2} - \sum_{k=1}^{n-1} \tilde{\delta}_k s_{n+2-k} + x_n \right), \quad n \geq 3,$$

**Theorem 2.4.** If $s_0 = s_1 = 0, s_2 > 0$ and $\mathbb{E}S < 4$, the ultimate time survival probability of the bi-seasonal discrete time risk model with income rate two is expressed as follows.

Under scenarios $S_1$ or $S_2$:

$$(\tilde{x}_{n+1} - \tilde{x}_n)\varphi(0) + (\tilde{\delta}_{n+1} - \tilde{\delta}_n)(4 - \mathbb{E}S) = \varphi(n + 1) - \varphi(n), \quad n \in \mathbb{N}_0,$$

$$\varphi(1) = \tilde{x}_1 \varphi(0) + \tilde{\delta}_1(4 - \mathbb{E}S).$$

Under condition $S_3$:

$$\varphi(0) = 0, \quad (\tilde{x}_{n+1} - \tilde{x}_n)\varphi(1) + (\tilde{\delta}_{n+1} - \tilde{\delta}_n)(4 - \mathbb{E}S) = \varphi(n + 1) - \varphi(n), \quad n \in \mathbb{N},$$

$$\varphi(2) = -\frac{y_0 + y_1}{y_0} \varphi(1) + \frac{4 - \mathbb{E}S}{y_0}.$$ (10)

The remaining values of the survival probability are calculated by

$$\varphi(u) = \frac{1}{s_2} \left( \varphi(u - 2) - \sum_{k=1}^{u-1} \varphi(k)s_{u+2-k} + (x_{u+1}y_0 + x_u y_1)\varphi(1) + x_u y_0 \varphi(2) \right)$$

for $u = 2, 3, \ldots$ under $S_1$ or $S_2$ and for $u = 3, 4, \ldots$ under $S_3$.

In addition, $\tilde{x}_{n+1} - \tilde{x}_n \neq 0$ for all $n \in \mathbb{N}_0$ and $\tilde{x}_{n+1} - \tilde{x}_n \neq 0$ for all $n \in \mathbb{N}$.

**Remark 6.** As commented in Remark 1, the practical application of Theorem 2.4 is based on $\varphi(n + 1) - \varphi(n) \approx 0$ when $n$ is large enough.

We now turn to the last case of $\varphi$ dependencies on r.v. $S$, which is $s_0 = s_1 = s_2 = 0$ and $s_3 > 0$. This, in turn, has four underlying scenarios, which impact an expressions of $\varphi$:

$\mathcal{V}_1 : x_0 = 0, \quad y_0 = 0, \quad x_1 = 0, \quad y_1 > 0, \quad x_2 > 0,$

$\mathcal{V}_2 : x_0 = 0, \quad y_0 = 0, \quad y_1 = 0, \quad x_1 > 0, \quad y_2 > 0,$

$\mathcal{V}_3 : x_0 = 0, \quad y_0 > 0, \quad x_1 = 0, \quad x_2 = 0, \quad x_3 > 0,$

$\mathcal{V}_4 : x_0 > 0, \quad y_0 = 0, \quad y_1 = 0, \quad y_2 = 0, \quad y_3 > 0.$

Formulas of $\varphi$ under scenarios form $\mathcal{V}_1$ to $\mathcal{V}_4$ dictate a need to evaluate $s_3 - x_3 y_0$ and $s_3 - x_3 y_0 - x_2 y_1$. We do that in the following table

| Scenario | $s_3 - x_3 y_0$ | $s_3 - x_3 y_0 - x_2 y_1$ |
|----------|-----------------|-----------------------------|
| $\mathcal{V}_1$ | $> 0$ | $= 0$ |
| $\mathcal{V}_2$ | $> 0$ | $> 0$ |
| $\mathcal{V}_3$ | $= 0$ | $= 0$ |
| $\mathcal{V}_4$ | $> 0$ | $> 0$ |

**Theorem 2.5.** If $s_0 = s_1 = s_2 = 0, s_3 > 0$ and $\mathbb{E}S < 4$, the ultimate time survival probability of the bi-seasonal discrete time risk model with income rate two, satisfies:
Under $V_2$ or $V_4$ : $\varphi(0) = 4 - \mathbb{E}S$, $\varphi(1) = \frac{\varphi(0)}{x_1y_2 + x_0y_3}$,

Under $V_1$ : $\varphi(0) = 0$, $\varphi(1) = \frac{4 - \mathbb{E}S}{y_1}$,

Under $V_3$ : $\varphi(0) = \varphi(1) = 0$, $\varphi(2) = \frac{4 - \mathbb{E}S}{y_0}$.

The remaining values of the survival probability for $u = 2, 3, \ldots$ are calculated by

$$\varphi(u) = \frac{1}{s_3}\left(\varphi(u - 1) + (x_{u+2}y_0 + x_{u+1}y_1)\varphi(1) + x_{u+1}y_0\varphi(2) - \sum_{k=1}^{u-1} s_{u+3-k}\varphi(k)\right).$$

It is easy to see that $s_0 = \ldots = s_3 = 0$ leads to the unsatisfied net profit condition $\mathbb{E}S \geq 4$. If that happens, the following statement is true.

**Theorem 2.6.** If the net profit condition is not satisfied, then the ultimate time survival probability of the bi-seasonal discrete time risk model with income rate two is expressed as follows:

- $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$ if $\mathbb{E}S > 4$,
- $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$ if $\mathbb{E}S = 4$ and $s_4 < 1$,
- If $\mathbb{E}S = 4$ and $s_4 = 1$, then the following sub-cases arise:
  - $\varphi(0) = \varphi(1) = \varphi(2) = 0$ and $\varphi(u) = 1$ for $u \geq 3$ if $x_4 = y_0 = 1$,
  - $\varphi(0) = \varphi(1) = 0$ and $\varphi(u) = 1$ for $u \geq 2$ if $x_3 = y_1 = 1$,
  - $\varphi(0) = 0$ and $\varphi(u) = 1$ for $u \geq 1$ if $x_2 = y_2 = 1$ or $x_1 = y_3 = 1$ or $x_0 = y_4 = 1$.

It is worth mentioning that a similar model to (1), with $\kappa = 2$ and $X \overset{d}{=} Y$, was studied in (Grigutis and Šiaulys 2020, Section 2) where similar recurrent matrices $2 \times 2$ as in (9) were obtained. These matrices recently have been studied in Grigutis and Jankauskas (2021), where some results on its nonsingularity were obtained. Moreover, it was shown in Grigutis and Jankauskas (2021) that a required initial values of survival probability for a homogeneous model ($X \overset{d}{=} Y$) with income rate two, have expressions via a certain roots of probability generating function. It is very likely that the same ideas as in Grigutis and Jankauskas (2021) are applicable to Conjectures 1 and 2, and also for finding the exact initial values of $\varphi$, which are approximately expressed in Theorems 2.2–2.4. On the other hand, approximate expressions of $\varphi$ as given in Theorems 2.2–2.4 do not require nor probability generating functions, nor any knowledge on the roots of a certain type of power series.

**3. Proofs**

In this section we prove all of the statements formulated in the previous Section 2. We start with an auxiliary lemma on survival probability $\varphi$. Let us denote $\varphi(\infty) := \lim_{u \to \infty} \varphi(u)$.

**Lemma 1.** For the ultimate time survival probability of the bi-seasonal discrete time risk model with income rate two, the following relations hold:

$$\varphi(\infty) = 1, \text{ if } \mathbb{E}S < 4,$$

$$\lim_{v \to \infty} \sum_{k=0}^{v+4} \varphi(k)\tilde{S}(v+4-k) = \varphi(\infty) \cdot \mathbb{E}S.$$
Proof. The first equality (12) is implied by replicating the proof line by line of the bi-seasonal model with income rate one in Damarackas and Šiaulys (2014, p. 935–936).

The remaining equality (13) follows due to the same lower and upper estimates of

\[
\lim_{k \to \infty} \sum_{k=0}^{\nu+1} \phi(k) \tilde{S}(\nu + 4 - k),
\]

see (Grigutis and Šiaulys 2020, p. 4) or (Grigutis and Šiaulys 2020, p. 13).

\[\Box\]

Proof of Theorem 2.1. The definition of the finite-time ruin probability implies

\[\phi(u, 1) \equiv \mathbb{P}(u + 2 - Z_1 > 0) = X(u + 1)\]

In the same manner for \( T = 2 \)

\[\phi(u, 2) = \mathbb{P}(\{W(1) > 0\} \cap \{W(2) > 0\}) = \mathbb{P}(\{Z_1 < u + 2\} \cap \{Z_1 + Z_2 < u + 4\}) = \mathbb{P}(\{X \leq u + 1\} \cap \{X + Y \leq u + 3\}) = \sum_{k=0}^{\nu+1} \mathbb{P}(X = k) \mathbb{P}(Y \leq u + 3 - k)
\]

\[= \sum_{k=0}^{\nu+1} x_k Y(u + 3 - k).\]

For \( T \geq 3 \), by the similar arguments as obtaining (2), we get

\[\phi(u, T) = \mathbb{P}\left(\bigcap_{t=1}^{T} \left\{u + 2t - \sum_{i=1}^{t} Z_i > 0\right\}\right)
\]

\[= \mathbb{P}\left(\bigcap_{t=2}^{T} \left\{u + 2t - \sum_{i=1}^{t} Z_i > 0\right\} \cap \{Z_1 < u + 2\}\right)
\]

\[= \mathbb{P}\left(\bigcap_{t=2}^{T} \left\{u + 2t - \sum_{i=1}^{t} Z_i > 0\right\}\right) - \mathbb{P}\left(\bigcap_{t=2}^{T} \left\{u + 2t - \sum_{i=1}^{t} Z_i > 0\right\} \cap \{Z_1 \geq u + 2\}\right)
\]

\[= \mathbb{P}\left(\bigcap_{t=3}^{T} \left\{u + 2t - Z_1 - Z_2 - \sum_{i=3}^{t} Z_i > 0\right\} \cap \{Z_1 + Z_2 < u + 4\}\right)
\]

\[= \sum_{k=0}^{\nu+3} \mathbb{P}(Z_1 + Z_2 = k) \mathbb{P}\left(\bigcap_{t=2}^{T-2} \left\{u + 2(t - 2) + 4 - k - \sum_{i=3}^{t-2} Z_i > 0\right\} \cap \{Z_1 + Z_2 = k\}\right)
\]

\[= (x_{u+2} y_1 + x_{u+3} y_0) \mathbb{P}\left(\bigcap_{t=1}^{T-2} \left\{1 + 2t - \sum_{i=1}^{t} Z_i > 0\right\}\right)
\]

\[= x_{u+2} y_0 \mathbb{P}\left(\bigcap_{t=1}^{T-2} \left\{2 + 2t - \sum_{i=1}^{t} Z_i > 0\right\}\right)
\]

\[= \sum_{k=0}^{\nu+3} \phi(u + 4 - k, T - 2) s_k - (x_{u+2} y_1 + x_{u+3} y_0) \phi(1, T - 2) - x_{u+2} y_0 \phi(2, T - 2). \]
Proof of Theorem 2.2. Let \( z_n, \beta_n, \gamma_n \) and \( \delta_n \) be the recurrent sequences defined prior to Theorem 2.2. We aim to show that for all \( n \in \mathbb{N}_0 \)

\[
\phi(n) = z_n\phi(0) + \beta_n\phi(1) + \gamma_n\phi(2) + \delta_n(4 - ES). \tag{14}
\]

If \( n = 0, 1 \) or \( n = 2 \), the statement is evident. If \( n = 3 \), the relation results from (7)

\[
\phi(3) = -\frac{1}{s_0}\phi(0) - \frac{X(2)y_0 + X(1)y_1 + S(2)}{s_0}\phi(1) - \frac{X(1)y_0 + S(1)}{s_0}\phi(2) + \frac{1}{s_0}(4 - ES) = z_3\phi(0) + \beta_3\phi(1) + \gamma_3\phi(2) + \delta_3(4 - ES).
\]

For \( n \geq 4 \) we use induction. Then, by (2) and induction hypothesis,

\[
\phi(n) = \frac{1}{s_0}\left( \phi(n - 4) + (x_{n-1}y_0 + x_{n-2}y_1)\phi(1) + x_{n-2}y_0\phi(2) - \sum_{k=1}^{n-1} \phi(k)s_{n-k} \right)
\]

\[
= \frac{1}{s_0}(z_{n-4}\phi(0) + \beta_{n-4}\phi(1) + \gamma_{n-4}\phi(2) + \delta_{n-4}(4 - ES) + (x_{n-1}y_0 + x_{n-2}y_1)(z_1\phi(0) + \beta_1\phi(1) + \gamma_1\phi(2) + \delta_1(4 - ES)) + x_{n-2}y_0(z_2\phi(0) + \beta_2\phi(1) + \gamma_2\phi(2) + \delta_2(4 - ES)) - \sum_{k=1}^{n-1} s_{n-k}(z_k\phi(0) + \beta_k\phi(1) + \gamma_k\phi(2) + \delta_k(4 - ES)))
\]

\[
= \phi(0)\frac{1}{s_0}\left( z_{n-4} - \sum_{k=1}^{n-1} s_{n-k}z_k \right) + \phi(1)\frac{1}{s_0}\left( \beta_{n-4} - \sum_{k=1}^{n-1} s_{n-k}\beta_k + x_{n-1}y_0 + x_{n-2}y_1 \right) + \phi(2)\frac{1}{s_0}\left( \gamma_{n-4} - \sum_{k=1}^{n-1} s_{n-k}\gamma_k + x_{n-2}y_0 \right) + (4 - ES)\frac{1}{s_0}\left( \delta_{n-4} - \sum_{k=1}^{n-1} s_{n-k}\delta_k \right)
\]

\[
= z_n\phi(0) + \beta_n\phi(1) + \gamma_n\phi(2) + \delta_n(4 - ES).
\]

Consequently, Equation (14) holds for all \( n \in \mathbb{N}_0 \). Subtractions of \( \phi(n+1) - \phi(n) \), \( \phi(n+2) - \phi(n) \) and \( \phi(n+3) - \phi(n) \) obtain the system (8) where the remaining equalities in Theorem 2.2 are implied by (7) and (2). \( \square \)

Proof of Theorem 2.3. Let \( \bar{z}_n, \bar{\beta}_n \) and \( \bar{\delta}_n \) be the recurrent sequences defined prior to Theorem 2.3. Then, arguing the same as in proof of Theorem 2.2, we show that

\[
\phi(n) = \bar{z}_n\phi(0) + \bar{\beta}_n\phi(1) + \bar{\delta}_n(4 - ES), \quad n \in \mathbb{N}_0.
\]

If \( n = 0, 1 \) or \( n = 2 \), the equation is obvious. For \( n = 2 \), it follows by (7)

\[
\phi(2) = -\frac{1}{s_1 + X(1)y_0}\phi(0) - \frac{X(2)y_0 + X(1)y_1 + S(2)}{s_1 + X(1)y_0}\phi(1) + \frac{1}{s_1 + X(1)y_0}(4 - ES) = \bar{z}_2\phi(0) + \bar{\beta}_2\phi(1) + \bar{\delta}_2(4 - ES)
\]

and, for any \( n \in \mathbb{N}_0 \), induction does its work as in the previous proof and the rest is implied by (7) and (2). \( \square \)

Proof of Theorem 2.4. Let us consider the cases \( S_1 \) and \( S_2 \). Recall that \( \bar{z}_n \) and \( \bar{\delta}_n \) are the recurrent sequences defined prior to Theorem 2.4. Then, the following equality is true
\[ \varphi(n) = \tilde{\varphi}_n \varphi(0) + \tilde{\delta}_n (4 - ES), \quad n \in \mathbb{N}. \]  
(16)

Indeed, if \( n = 0 \) the equation is evident, if \( n = 1 \), from (7) we get
\[ \varphi(2) = \frac{1}{X(1)y_0} (-\varphi(0) - (\tilde{X}(2)y_0 + \tilde{X}(1)y_1 + s_2)\varphi(1) + 4 - ES). \]  
(17)

By setting \( n = 0 \) into (2) we get
\[ \varphi(2) = \frac{1}{s_2 - x_2y_0} (\varphi(0) - (s_3 - x_3y_0 - x_2y_1)\varphi(1)). \]  
(18)

Equating (17) to (18) and rearranging we obtain
\[ \varphi(1) = -\varphi(0) + \frac{1 + \frac{\tilde{X}(1)y_0}{s_2 - x_2y_0}}{\tilde{X}(2)y_0 + \tilde{X}(1)y_1 + s_2 - \frac{\tilde{X}(1)y_0(s_3 - x_3y_0 - x_2y_1)}{s_2 - x_2y_0}} \frac{4 - ES}{\tilde{X}(2)y_0 + \tilde{X}(1)y_1 + s_2 - \frac{\tilde{X}(1)y_0(s_3 - x_3y_0 - x_2y_1)}{s_2 - x_2y_0}} = \tilde{\varphi}_1 \varphi(0) + \tilde{\delta}_1 (4 - ES). \]

Observing that \( y_0 = 0 \) under scenarios \( S_1 \) and \( S_2 \), we confirm the Equation (16) for \( n = 1 \).

For \( n \geq 2 \), Equation (16) follows by mathematical induction the same way as proving Theorems 2.2 and 2.3. Subtracting \( \varphi(n + 1) - \varphi(n) \) we obtain the equation defined in Theorem 2.4
\[ (\tilde{\varphi}_{n+1} - \tilde{\varphi}_n) \varphi(0) + (\tilde{\delta}_{n+1} - \tilde{\delta}_n) (4 - ES) = \varphi(n + 1) - \varphi(n). \]

Considering the last case \( S_3 \) we note that \( \varphi(0) = 0 \). This is due to the first possible claim (r.v. \( X \)) at \( t = 1 \) is greater or equal to 2. For the following survival probabilities \( \varphi(u), u \geq 1 \), let \( \tilde{\varphi}_n \) and \( \tilde{\delta}_n \) be the recurrent sequences defined prior to Theorem 2.4. Then, the following equality holds for all \( n \in \mathbb{N} \)
\[ \varphi(n) = \tilde{\varphi}_n \varphi(1) + \tilde{\delta}_n (4 - ES). \]  
(19)

Indeed, it is evident for \( n = 1 \) and for \( n = 2 \) is implied by (7)
\[ \varphi(2) = -\frac{1}{y_0} \varphi(1)(y_0 + y_1) + \frac{1}{y_0} (4 - ES) = \tilde{\varphi}_2 \varphi(1) + \tilde{\delta}_2 (4 - ES). \]

For \( n \geq 3 \), it is confirmed by induction and the rest is evident by setting the difference \( \varphi(n + 1) - \varphi(n) \) and etc.

It remains to show that we do not divide by zero obtaining an expression of \( \varphi(0) \) from Equation (10) or \( \varphi(1) \) from Equation (11). As Theorem 2.4 deals with three underlying scenarios related to where the distribution of \( X + Y \) can start not to violate the net profit condition and \( s_0 = s_1 = 0, s_2 > 0 \), we have three slightly different types of recurrent sequences to go through. Under scenario \( S_1 \) we have
\[ \tilde{\varphi}_0 = 1, \quad \tilde{\varphi}_1 = -\frac{1}{y_1}, \quad \tilde{\varphi}_n = \frac{1}{s_2} \left( \tilde{\varphi}_{n-2} - \sum_{k=1}^{n-1} \tilde{\varphi}_k s_{n+2-k} - x_n \right), \quad n = 2, 3, \ldots, \]

while \( S_2 \) implies
\[ \hat{x}_0 = 1, \quad \hat{x}_1 = -\frac{1}{x_0 y_2}, \quad \hat{x}_n = \frac{1}{s_2} \left( \hat{x}_{n-2} - \sum_{k=1}^{n-1} \hat{x}_k s_{n-2-k} \right), \quad n = 2, 3, \ldots \]

The proof of \( \hat{x}_{n+1} - \hat{x}_n \neq 0 \) for all \( n \in \mathbb{N}_0 \) under scenarios \( S_1 \) or \( S_2 \) is almost identical to the one given in Damarackas and Šiaulys (2014, p. 937) or Grigutis and Šiaulys (2020, p. 16) accordingly.

Under the case \( S_3 \) we have

\[ \hat{x}_1 = 1, \quad \hat{x}_2 = -\frac{y_0 + y_1}{y_0}, \quad \hat{x}_n = \frac{1}{s_2} \left( \hat{x}_{n-2} - \sum_{k=1}^{n-1} \hat{x}_k s_{n-2-k} + (x_{n+1} - x_n)y_0 \right), \quad n = 3, 4, \ldots \]

The property \( \hat{x}_{n+1} - \hat{x}_n \neq 0 \) for \( n \in \mathbb{N} \) is implied by \( 1 \leq \hat{x}_{2n+1} \leq \hat{x}_{2n+3} \) and \( -1 \geq \hat{x}_{2n} \geq \hat{x}_{2n+2} \). To verify the last inequalities we utilize induction. If \( n = 1 \), then

\[ \hat{x}_4 = \frac{\hat{x}_2 - \sum_{k=1}^{3} \hat{x}_k s_{n-2-k} + (x_5 - x_4)y_0}{s_2} \leq \frac{\hat{x}_2 - \hat{x}_2 s_4 + x_5y_0}{s_2} \leq \hat{x}_2 \leq -1 \]

and

\[ \hat{x}_3 = \frac{1 - \hat{x}_1 s_4 - \hat{x}_2 s_3 + (x_4 - x_3)y_0}{s_2} = \frac{1}{y_0} \left( \frac{1}{\hat{x}_2} - y_2 + y_1 + \frac{y_1^2}{y_0} \right) \geq 1 = \hat{x}_1. \]

For arbitrary \( n \in \mathbb{N} \), under induction hypothesis, it follows

\[ \hat{x}_{2n+2} = \frac{1}{s_2} \left( \hat{x}_{2n} - \sum_{k=1}^{n+1} s_{2n+4-k} \hat{x}_k + (x_{2n+3} - x_{2n+2})y_0 \right) \]

\[ = \frac{1}{s_2} \left( \hat{x}_{2n} - (s_3 \hat{x}_{2n+1} + \ldots + s_{2n+3} \hat{x}_1) - (s_4 \hat{x}_{2n} + \ldots + s_{2n+2} \hat{x}_2) + (x_{2n+3} - x_{2n+2})y_0 \right) \]

\[ \leq \frac{1}{s_2} \left( \hat{x}_{2n} - \hat{x}_1 (s_3 + \ldots + s_{2n+3}) - \hat{x}_{2n} (s_4 + \ldots + s_{2n+2}) + x_{2n+3}y_0 \right) \]

\[ \leq \frac{1}{s_2} \left( \hat{x}_{2n} (1 - s_4 - \ldots - s_{2n+2}) - \hat{x}_1 (s_3 + \ldots + s_{2n+1}) \right) \leq \hat{x}_{2n} \]

and

\[ \hat{x}_{2n+3} = \frac{1}{s_2} \left( \hat{x}_{2n+1} - \sum_{k=1}^{n+2} s_{2n+5-k} \hat{x}_k + (x_{2n+4} - x_{2n+3})y_0 \right) \]

\[ = \frac{1}{s_2} \left( \hat{x}_{2n+1} - (s_4 \hat{x}_{2n+1} + \ldots + s_{2n+4} \hat{x}_1) - (s_3 \hat{x}_{2n+2} + \ldots + s_{2n+3} \hat{x}_2) \right. \]

\[ + (x_{2n+4} - x_{2n+3})y_0 \)

\[ \geq \frac{1}{s_2} \left( \hat{x}_{2n+1} - \hat{x}_{2n+1} (s_4 + \ldots + s_{2n+4}) - x_{2n+3}y_0 \right) \geq \hat{x}_{2n+1}. \]

It is true that we can avoid differences of \( \varphi(n+1) - \varphi(n) \) and etc. in Theorems 2.2–2.4. Instead of that, we can utilize expressions (14–16) accordingly and obtain the needed initial values \( \varphi(0), \varphi(1), \varphi(2) \) and so on based on \( \varphi(n) \approx 1 \) if \( n \) is sufficiently large. However, for
some slowly growing $\varphi(n)$ the approximation $\varphi(n + 1) - \varphi(n) \approx 0$ seems to be more precise than $\varphi(n) \approx 1$. Some thoughts on that are given in Grigutis and Šiaulys (2020, Sec. 5).

**Proof of Theorem 2.5.** We start with the cases $V_2$ and $V_4$. For $u = 0$ or $u = 1$, the Equation (2) accordingly implies

$$\varphi(1) = \frac{\varphi(0)}{s_3 - x_3y_0 - x_2y_1}, \quad \varphi(2) = \frac{\varphi(1)(1 - s_4 + x_4y_0 + x_3y_1)}{s_3 - x_3y_0}.$$  

From this

$$\varphi(2) = \frac{\varphi(0)(1 - s_4 + x_4y_0 + x_3y_1)}{(s_3 - x_3y_0)(s_3 - x_3y_0 - x_2y_1)}.$$  

Finally, equality (7) can be rewritten as

$$\varphi(0) + \varphi(0) \frac{X(2)y_0 + X(1)y_1}{s_3 - x_3y_0 - x_2y_1} + \varphi(0) \frac{X(1)y_0(1 - s_4 + x_4y_0 + x_3y_1)}{(s_3 - x_3y_0)(s_3 - x_3y_0 - x_2y_1)} = 4 - \mathbb{E}S.$$  

Therefore

$$\varphi(0) = \frac{4 - \mathbb{E}S}{1 + \frac{X(2)y_0 + X(1)y_1}{s_3 - x_3y_0 - x_2y_1} + \frac{X(1)y_0(1 - s_4 + x_4y_0 + x_3y_1)}{(s_3 - x_3y_0)(s_3 - x_3y_0 - x_2y_1)}} = 4 - \mathbb{E}S$$  

as $y_0 = y_1 = 0$ under scenarios $V_2$ and $V_4$.

For the next case $V_1$, if $u = 0$, from Equation (2) we have

$$\varphi(0) = \varphi(1)(s_3 - x_2y_1) = 0$$  

so Equation (7) gives

$$\varphi(1) = \frac{4 - \mathbb{E}S}{y_1}.$$  

The result $\varphi(0) = 0$ is natural due to min$X = 2$.

For the last case $V_3$, if $u = 0$, from (2) we get

$$\varphi(0) = \varphi(1)(s_3 - x_3y_0) = 0$$  

and, if $u = 1$,

$$\varphi(1) = \varphi(1)s_4 + \varphi(2)s_3 - \varphi(1)(x_4y_0 + x_3y_1) - \varphi(2)x_3y_0$$  

therefore

$$\varphi(1)(1 - s_4 + x_4y_0 + x_3y_1) = \varphi(2)(s_3 - x_3y_0) = 0.$$  

Consequently, Equation (7) implies

$$\varphi(2) = \frac{4 - \mathbb{E}S}{y_0}.$$  

Also, $\varphi(0) = \varphi(1) = 0$ is because of min$X = 3$.

The remaining survival probabilities $\varphi(u)$, $u \geq 2$, are implied by (2).

**Proof of Theorem 2.6.** The first case, $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$ if $\mathbb{E}S > 4$, is implied by combining Equation (6) with (13), which gives
\[
\varphi(0) + (\bar{X}(2)y_0 + \bar{X}(1)y_1)\varphi(1) + \bar{X}(1)y_0\varphi(2) + \sum_{k=1}^{3} \varphi(k)S(3 - k) = (4 - \mathbb{E}S) \cdot \varphi(\infty).
\]
\hspace{1cm} (20)

As the left hand side of (20) is non-negative and \(4 - \mathbb{E}S < 0\), then \(\varphi(\infty) = 0\), and consequently \(\varphi(u) = 0\) for all \(u \in \mathbb{N}_0\) as \(\varphi\) is non-decreasing.

For the second case, \(\varphi(u) = 0\) for all \(u \in \mathbb{N}_0\) if \(\mathbb{E}S = 4\) and \(s_4 < 1\), we observe that (20) becomes
\[
\varphi(0) + (x_0y_2 + y_0 + y_1)\varphi(1) + (x_0y_1 + y_0)\varphi(2) + x_0y_0\varphi(3) = 0.
\]
\hspace{1cm} (21)

The rest for this second case is concluded in the same manner as Theorems 2.2–2.5 are structured. Indeed, if \(s_0 > 0\) as in Theorem 2.2, then Equation (21) implies \(\varphi(0) = \varphi(1) = \varphi(2) = \varphi(3) = 0\) and \(\varphi(u) = 0\) for \(u \geq 4\) follows by Equation (2). If \(s_0 = 0\) and \(s_1 > 0\) (as in Theorem 2.3), which consists from two underlying cases \(x_0 = 0\), \(y_0 > 0\), \(x_1 > 0\) or \(x_0 > 0\), \(y_0 = 0\), \(y_1 > 0\), then (21) implies \(\varphi(0) = \varphi(1) = \varphi(2) = 0\) and Equation (2) does the rest. Two remaining dependencies \(s_0 = s_1 = 0\), \(s_2 > 0\) and \(s_0 = s_1 = s_2 = 0\), \(s_3 > 0\) follow by the same arguments.

The third and last case on \(\varphi\) expressions when \(\mathbb{E}S = 4\) and \(s_4 = x_4y_0 + x_3y_1 + x_2y_2 + x_1y_3 + x_0y_4 = 1\) is implied by the definition of the bi-seasonal discrete time risk model with income rate two (see (1)). Then the modeled function \(W(t)\) equals:

- \(W(t) = u - 2 \cdot 1_{\{t \text{ is odd}\}}\) if \(x_4 = y_0 = 1\),
- \(W(t) = u - 1 \cdot 1_{\{t \text{ is odd}\}}\) if \(x_3 = y_1 = 1\),
- \(W(t) = u\) if \(x_2 = y_2 = 1\),
- \(W(t) = u + 1 \cdot 1_{\{t \text{ is odd}\}}\) if \(x_1 = y_3 = 1\),
- \(W(t) = u + 2 \cdot 1_{\{t \text{ is odd}\}}\) if \(x_0 = y_4 = 1\).

The proof follows identifying when \(W(t) > 0\) for all \(t = 1, 2, \ldots\)

Similar thoughts as given in this proof appear in Damarackas and Šiaulys (2014, 935). \(\square\)

4. Numerical examples

In this section, using program Mathematica (2012), we demonstrate numerical outputs of Theorems from Section 2 assuming that r.v.s., generating the bi-seasonal discrete time risk model with income rate two, follow the displaced Poisson distribution \(\mathcal{P}(\lambda, \xi)\) with parameters \(\lambda > 0\) and \(\xi \in \mathbb{N}_0\), which PMF is
\[
\mathbb{P}(X = m) = e^{-\lambda} \frac{\lambda^{m-\xi}}{(m-\xi)!}, \quad m = \xi, \xi + 1, \ldots
\]

It can be verified that \(\mathbb{E}X = \lambda + \xi\) and \(X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2, \xi_1 + \xi_2)\) if \(X \sim \mathcal{P}(\lambda_1, \xi_1)\), \(Y \sim \mathcal{P}(\lambda_2, \xi_2)\) and \(X, Y\) are independent. See Staff (1967) for more information on displaced Poisson distribution.

All of the output tables of \(\varphi(u, T)\) and \(\varphi(u)\) below are structured as follows: the present survival probabilities are rounded up to three decimal places except when the
numbers are 0 or 1; parameters $T$ and $u$ are chosen to reflect changes of survival probabilities; the size of $n = 150$ is considered as high enough to reach a sufficient accuracy when Theorems 2.2–2.4 are employed to find a needed initial values of $\varphi$.

**Example 4.1.** Let $X \sim \mathcal{P}(1,0)$ and $Y \sim \mathcal{P}(2,0)$. Using Theorem 2.1 and Theorem 2.2 we obtain Table 1.

**Table 1.** Survival probabilities for $X \sim \mathcal{P}(1,0)$ and $Y \sim \mathcal{P}(2,0)$.

| $T$ | $u$ | 0   | 1   | 2   | 3   | 4   | 5   | 10  | 20  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     |     |     |     |     |
| 1   | 0.736| 0.920| 0.981| 0.996| 0.999| 1   | 1   | 1   | 1   |
| 2   | 0.564| 0.788| 0.909| 0.965| 0.988| 0.996| 1   | 1   | 1   |
| 3   | 0.517| 0.771| 0.898| 0.959| 0.985| 0.995| 1   | 1   | 1   |
| 4   | 0.505| 0.727| 0.863| 0.936| 0.972| 0.989| 1   | 1   | 1   |
| 5   | 0.499| 0.720| 0.857| 0.932| 0.969| 0.987| 1   | 1   | 1   |
| 10  | 0.460| 0.673| 0.813| 0.898| 0.946| 0.972| 0.999| 1   | 1   |
| 20  | 0.446| 0.656| 0.795| 0.882| 0.933| 0.962| 0.998| 1   | 1   |
| 30  | 0.443| 0.652| 0.791| 0.878| 0.930| 0.960| 0.998| 1   | 1   |
| 50  | 0.442| 0.650| 0.790| 0.876| 0.928| 0.959| 0.997| 1   | 1   |

**Table 2.** Survival probabilities for $X \sim \mathcal{P}(1,1)$ and $Y \sim \mathcal{P}(19/10,0)$. Using Theorem 2.1 and Theorem 2.3 we obtain Table 2.

| $T$ | $u$ | 0   | 1   | 2   | 3   | 4   | 5   | 10  | 20  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     |     |     |     |     |
| 1   | 0.368| 0.736| 0.920| 0.981| 0.996| 0.999| 1   | 1   | 1   |
| 2   | 0.259| 0.581| 0.803| 0.919| 0.970| 0.990| 1   | 1   | 1   |
| 3   | 0.223| 0.518| 0.743| 0.877| 0.947| 0.979| 1   | 1   | 1   |
| 4   | 0.192| 0.458| 0.677| 0.823| 0.910| 0.957| 1   | 1   | 1   |
| 5   | 0.177| 0.428| 0.641| 0.791| 0.886| 0.942| 0.999| 1   | 1   |
| 10  | 0.130| 0.324| 0.505| 0.652| 0.765| 0.847| 0.990| 1   | 1   |
| 20  | 0.098| 0.248| 0.396| 0.525| 0.634| 0.724| 0.951| 1   | 1   |
| 30  | 0.084| 0.214| 0.343| 0.460| 0.562| 0.649| 0.908| 0.998| 1   |
| 50  | 0.076| 0.193| 0.311| 0.419| 0.515| 0.599| 0.869| 0.994| 1   |
| 100 | 0.057| 0.144| 0.234| 0.319| 0.395| 0.465| 0.731| 0.952| 0.995|
| $\infty$ | 0.037| 0.094| 0.152| 0.208| 0.259| 0.307| 0.506| 0.748| 0.872|

**Table 3.** Survival probabilities for $X \sim \mathcal{P}(1,1)$ and $Y \sim \mathcal{P}(9/10,1)$. Using Theorem 2.1 and Theorem 2.4 we obtain Table 3.

| $T$ | $u$ | 0   | 1   | 2   | 3   | 4   | 5   | 10  | 20  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     |     |     |     |     |
| 1   | 0.368| 0.736| 0.920| 0.981| 0.996| 0.999| 1   | 1   | 1   |
| 2   | 0.284| 0.629| 0.850| 0.950| 0.986| 0.996| 1   | 1   | 1   |
| 3   | 0.237| 0.552| 0.784| 0.910| 0.967| 0.989| 1   | 1   | 1   |
| 4   | 0.212| 0.506| 0.739| 0.878| 0.949| 0.980| 1   | 1   | 1   |
| 5   | 0.191| 0.466| 0.695| 0.844| 0.927| 0.968| 1   | 1   | 1   |
| 10  | 0.145| 0.366| 0.572| 0.729| 0.837| 0.908| 0.997| 1   | 1   |
| 20  | 0.111| 0.286| 0.459| 0.605| 0.720| 0.807| 0.981| 1   | 1   |
| 30  | 0.096| 0.249| 0.404| 0.538| 0.650| 0.740| 0.957| 1   | 1   |
| 50  | 0.087| 0.227| 0.370| 0.496| 0.604| 0.693| 0.933| 0.999| 1   |
| 100 | 0.067| 0.175| 0.288| 0.390| 0.482| 0.562| 0.828| 0.984| 0.999|
| $\infty$ | 0.048| 0.127| 0.209| 0.286| 0.355| 0.417| 0.649| 0.873| 0.954|

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Example 4.4. Let $X \sim \mathcal{P}(1/2, 2)$ and $Y \sim \mathcal{P}(1/3, 1)$. Using Theorem 2.1 and Theorem 2.5 we obtain Table 4.

Table 4. Survival probabilities for $X \sim \mathcal{P}(1/2, 2)$ and $Y \sim \mathcal{P}(1/3, 1)$.

| $\tau_u$ | 0   | 1   | 2   | 3   | 4   | 5   | 10  | 15  | 20  | 25  |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1        | 0.607 | 0.910 | 0.986 | 0.998 | 1   | 1   | 1   | 1   | 1   | 1   |
| 2        | 0.435 | 0.797 | 0.948 | 0.990 | 0.998 | 1   | 1   | 1   | 1   | 1   |
| 3        | 0.395 | 0.758 | 0.928 | 0.983 | 0.997 | 0.999 | 1   | 1   | 1   | 1   |
| 4        | 0.346 | 0.760 | 0.894 | 0.969 | 0.992 | 0.998 | 1   | 1   | 1   | 1   |
| 5        | 0.329 | 0.678 | 0.878 | 0.961 | 0.989 | 0.997 | 1   | 1   | 1   | 1   |
| 10       | 0.262 | 0.573 | 0.788 | 0.906 | 0.962 | 0.986 | 1   | 1   | 1   | 1   |
| 20       | 0.219 | 0.494 | 0.705 | 0.838 | 0.916 | 0.959 | 0.999 | 1   | 1   | 1   |
| 30       | 0.202 | 0.459 | 0.662 | 0.799 | 0.884 | 0.936 | 0.998 | 1   | 1   | 1   |
| 40       | 0.192 | 0.429 | 0.567 | 0.773 | 0.862 | 0.918 | 0.996 | 1   | 1   | 1   |
| 50       | 0.186 | 0.425 | 0.620 | 0.756 | 0.846 | 0.905 | 0.994 | 1   | 1   | 1   |
| 100      | 0.173 | 0.398 | 0.583 | 0.716 | 0.808 | 0.872 | 0.985 | 0.999 | 1   | 1   |
| $\infty$ | 0.167 | 0.383 | 0.563 | 0.693 | 0.784 | 0.849 | 0.974 | 0.996 | 0.999 | 1   |

Example 4.5. Let $X \sim \mathcal{P}(2, 1)$ and $Y \sim \mathcal{P}(1, 1)$. Note that $\mathbb{E}S = 5 > 4$ in the case under consideration. Using Theorem 2.1 and Theorem 2.6 we obtain Table 5.

Table 5. Survival probabilities for $X \sim \mathcal{P}(2, 1)$ and $Y \sim \mathcal{P}(1, 1)$.

| $\tau_u$ | 0   | 1   | 2   | 3   | 4   | 5   | 10  | 20  | 30  | 50  |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1        | 0.135 | 0.406 | 0.677 | 0.857 | 0.947 | 0.983 | 1   | 1   | 1   | 1   |
| 2        | 0.100 | 0.324 | 0.581 | 0.782 | 0.903 | 0.962 | 1   | 1   | 1   | 1   |
| 3        | 0.054 | 0.194 | 0.391 | 0.589 | 0.750 | 0.862 | 0.998 | 1   | 1   | 1   |
| 4        | 0.045 | 0.166 | 0.343 | 0.532 | 0.696 | 0.820 | 0.996 | 1   | 1   | 1   |
| 5        | 0.029 | 0.112 | 0.243 | 0.401 | 0.558 | 0.696 | 0.982 | 1   | 1   | 1   |
| 10       | 0.010 | 0.042 | 0.099 | 0.179 | 0.278 | 0.388 | 0.854 | 1   | 1   | 1   |
| 20       | 0.002 | 0.008 | 0.020 | 0.038 | 0.065 | 0.102 | 0.417 | 0.946 | 0.999 | 1   |
| 30       | 0    | 0.002 | 0.005 | 0.010 | 0.017 | 0.029 | 0.161 | 0.723 | 0.978 | 1   |
| 40       | 0    | 0.001 | 0.001 | 0.003 | 0.005 | 0.008 | 0.058 | 0.439 | 0.874 | 1   |
| 50       | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0   |
| 100      | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0   |
| $\infty$ | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0   |

As a general overview on the results of survival probabilities present in the above Examples 4.1–4.5, it can be commented that the difference of $\mathbb{E}S$ to 4 makes a high impact on the likelihood of survival. Example 4.1 shows that expenses, represented as random claims, which are not too harsh on average ($\mathbb{E}S = 3$), may be well covered by the initial surplus $u$ and guaranteed survival is reached when $u = 15$. On the other hand, Table 5 illustrates that a quite confident short term survival possibility can be achieved with a sufficient level of initial savings even when an occurring random expenses are more “aggressive”.

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