Heisenberg Groups in the Theory of the
Lattice Peierls Electron: the Irrational Flux
Case

P.P. Divakaran

Chennai Mathematical Institute*

92 G.N. Chetty Road, T. Nagar

Chennai-600 017, India.

E-mail: ppd@smi.ernet.in

*Work supported by the Department of Science Technology, Government of India.
Abstract

This paper establishes that the quantum mechanics of a charged particle moving in $C = \mathbb{R}^2$ (Landau) or $\mathbb{Z}^2 \subset \mathbb{R}^2$ (Peierls) in a uniform normal magnetic field $B$ is described in every detail by the (projective) representation theory of the appropriate euclidean group $E(2) = J \times A^2$ (where $J$ is the semidirect product; $J = SO(2, \mathbb{R})$, $A = \mathbb{R}$ and $J = \mathbb{Z}/4$, $A = \mathbb{Z}$ for $C = \mathbb{R}^2$ and $\mathbb{Z}^2$ respectively). The central extensions of $E(2)$ by the circle group $\mathbb{T}$ are of the form $\tilde{E}(2) = J \times \tilde{A}^2$ and hence the main object of study is the nilpotent group $\tilde{A}^2$. The unique representation property of the Heisenberg group $\tilde{R}^2$, $r \in H^2(\mathbb{R}^2, \mathbb{T}) = \mathbb{R}$ leads to a detailed description of the structure of the state space $H_r \cong L^2(\mathbb{R}^2)$ as a representation of $\tilde{R}^2_r$ and of $\tilde{E}_r(2, \mathbb{R})$. An essentially unique Hamiltonian $H_r$ for each $\tilde{E}_r(2, \mathbb{R})$ is also determined. The quantum theory that results is that of the Landau electron when $\{B\}$ is identified with $H^2(\mathbb{R}^2, \mathbb{T})$. For the Peierls case, the central extensions $\tilde{E}_\theta(2, \mathbb{Z}) = \mathbb{Z}/4 \times \tilde{\mathbb{Z}}^2_\theta$ are parametrised by $\theta \in [0, 2\pi) \cong \mathbb{T}$. When $\theta/2\pi$ is irrational, $\tilde{\mathbb{Z}}^2_\theta$ is an “almost Heisenberg” group in the sense that it has a distinguished irreducible representation on $L^2(\mathbb{Z})$. This is sufficient for a complete description of $\mathcal{H}_\theta \cong L^2(\mathbb{Z}^2)$ as a representation of $\tilde{\mathbb{Z}}^2_\theta$ and of $\tilde{E}_\theta(2, \mathbb{Z})$. When $\theta$ is identified with $\Phi$, the flux per plaquette modulo the flux quantum, the physics of the Peierls electron is fully determined by $\Phi$ and is periodic in $\Phi$ with one flux quantum as the period. The Hamiltonian $H_\theta$ in $\mathcal{H}_\theta$ is also determined by $\tilde{E}_\theta(2, \mathbb{Z})$ invariance; for only nearest neighbour hopping, $H_\theta$ is essentially the Harper Hamiltonian. Introduction of vector potentials and gauges is nowhere necessary.
1. General Introduction to the Problem and the Method

By the term Peierls electron [1] we mean a quantum system consisting of a particle of electric charge $e$ and mass $m$ in a constant uniform magnetic field $B$ and a periodic potential $V$. Since motion in the direction of $B$ is independent of $B$, the configuration space is taken to be the plane perpendicular to $B$ or any subset of it compatible with $V$, i.e. if $\Lambda \subset \mathbb{R}^2$ is the lattice defined by $V$, $\mathcal{C}$ should be invariant under automorphisms of $\Lambda$; in particular if $\mathcal{C} = \Lambda$, $V$ is constant on $\mathcal{C}$ and so can be dropped. The reason for this restriction (general discrete subsets of $\mathbb{R}^2$ have been considered in the literature [2]) is that our method of treating the problem is the representation theory of the group of automorphisms of $\mathcal{C}$. The specific case which is the main concern of this paper is the $\mathbb{Z}^2$ lattice Peierls electron, $\mathcal{C} = \Lambda = \mathbb{Z}^2$.

In his original study of the continuum problem, $\mathcal{C} = \mathbb{R}^2, \Lambda = \mathbb{Z}^2$, Peierls [1] made several perceptive simplifications of which the one most pertinent to us is the weak field or tight binding approximation. Later, Harper [3] in a careful study found a simple expression for the Hamiltonian of the electron within the Peierls scheme of approximations and in the extreme tight binding limit (in which wave functions are presumably supported in arbitrarily small neighbourhoods in $\mathbb{R}^2$ of the points of $\Lambda$):

$$H_\Phi = p_1 + p_2 + p_1^{-1} + p_2^{-1}$$

where $p_1$ and $p_2$ are unitary operators on the state space of the electron satisfying the commutator condition

$$p_1 p_2 p_1^{-1} p_2^{-1} = e^{i\Phi}$$
and $\Phi$, the only physical parameter left in the model, is the flux of $B$, in suitable units, through a unit cell or plaquette.

In the vast literature on the subject, it is sometimes claimed that eqn. (1) defines the exact Hamiltonian for the $Z^2$ Peierls electron, presumably because of the tight binding invoked in obtaining it. To establish rigorously the validity of such a claim, especially in view of the other poorly understood approximations made, one would first need a clear-cut prescription for the quantum mechanics of a charged particle living on a lattice and subject to a magnetic field. A magnetic field on a lattice is a physically ill-defined concept, a vector potential even more so. In practice, what is done is to associate certain unitary operators constructed from a continuum vector potential to links between the points of $\Lambda$ in such a way that in some “local” limit, the kinematics and dynamics of the continuum system are recovered. This procedure entails several arbitrary choices, e.g., the choice of a gauge, which obscure some central issues. One would like to know answers to questions such as: What is the exact characterisation of the state space $\mathcal{H}$? How do $p_1$ and $p_2$ operate on $\mathcal{H}$ and what is their physical significance? To what extent do eqns. (1) and (2) determine $H_{\Phi}$ as an operator on $\mathcal{H}$? How can we tell that $H_{\Phi}$ is the correct Hamiltonian for the problem and is it unique? etc.

It is the purpose of this paper and a sequel [4], working directly with the lattice problem, to show that all such questions can be posed precisely and answered in terms of the (projective) representations of the group $E$ of automorphisms of the configuration space ($Z^2$ in the present case). The reason why projective representations of Aut $\mathcal{C}$ are so effective in the quantum theory
of magnetic fields is best understood in the example of the familiar Landau electron, $C = \mathbb{R}^2$, $E = E(2, \mathbb{R}) = \text{SO}(2, \mathbb{R}) \rtimes \mathbb{R}^2$ ($\rtimes$ denotes the semidirect product). A uniform magnetic field normal to $\mathbb{R}^2$ is certainly invariant under $E(2, \mathbb{R})$ and the quantum theory should be capable of being described by the representations of $E(2, \mathbb{R})$. The linear (unitary) representations correspond of course to free particle motion. However, $E(2, \mathbb{R})$ has nontrivial projective representations and since, by Wigner’s theorem [5], they are as legitimate as linear representations, such representations are the only means of accommodating $E(2, \mathbb{R})$ invariant but non-free quantum mechanics. (Statements to the effect that a uniform magnetic field violates translation invariance are not rare in the literature). Because of the key role of this elementary fact in our considerations, section 3 is devoted to demonstrating that the quantum theory of the Landau electron is no more and no less than the theory of projective representations of $E(2, \mathbb{R})$.

The general setting for our approach is provided by the standard correspondence of the set of equivalence classes of projective representations (the qualifier “unitary” is implicit and will be dropped from now on) of a group $E$ with its 2nd cohomology group $H^2(E, \mathbb{T})$ ($\mathbb{T}$ is the circle group) and, equivalently, with the set of isomorphism classes of central extensions $\{\tilde{E}\}$ of $E$ by $\mathbb{T}$; in particular, every projective representation of $E$ in the class of $\alpha \in H^2(E, \mathbb{T})$ lifts to a linear representation of a central extension $\tilde{E}_\alpha$ having the property that its restriction to $\mathbb{T} \subset$ centre $\tilde{E}_\alpha$ is the natural character $t \mapsto t$ (see, for example, [6,7] for these well-known facts). The central extensions of $E(2, \mathbb{R})$ by $\mathbb{T}$ are all of the form $\tilde{E}_r(2, \mathbb{R}) \cong \text{SO}(2, \mathbb{R}) \rtimes \mathbb{R}^2_r$, $r \in \mathbb{R}$. The
real Heisenberg group $\bar{\mathbb{R}}^2_r$ thus assumes significance, especially its property of having, upto equivalence, just one irreducible representation restricting to a fixed character on $\mathbb{T}$ (the Stone-von Neumann theorem), conveniently realised on $L^2(\mathbb{R})$. The state space of a particle in $\mathcal{C} = \mathbb{R}^2$ is however isomorphic to $L^2(\mathbb{R}^2)$ (for the general theory of the state space of a system defined by $\mathcal{C}$ and $\mathcal{E}$, see [8,9]). We will study in section 3 the structure of $L^2(\mathbb{R}^2)$ as a $\widetilde{E}_r(2,\mathbb{R})$ module and see that $L^2(\mathbb{R}^2) \cong V_r \otimes V_{-r} = \mathcal{H}_r$ with $\bar{\mathbb{R}}^2_r$ acting irreducibly on $V_r$ and trivially on $V_{-r}$ ($V_{-r}$ will turn out to be an irreducible $\bar{\mathbb{R}}^2_{-r}$ module). We shall then use the above factorisation and a general characterisation of the Hamiltonian of a quantum system $(\mathcal{C}, \mathcal{E})$ [7,8] to determine the Hamiltonian $H_r$ in $\mathcal{H}_r$. It will be seen that $H_r$ is precisely the Landau Hamiltonian (when $r$ is put equal to $eB$) operating on $V_{-r}$ and having $V_r$ as degeneracy subspace. A bonus is that vector potentials and hence gauges are nowhere required to be invoked.

With the confidence thus acquired (a semi-heuristic account of the Landau electron from the symmetry point of view can be found in [10]), we turn to the $\mathbb{Z}^2$ Peirels electron in section 4. The relevant symmetry group is the integral euclidean group $E(2,\mathbb{Z}) = SO(2,\mathbb{Z}) \bar{x} \mathbb{Z}^2 = \mathbb{Z}/4\bar{x} \mathbb{Z}^2$. All central extensions of $E(2,\mathbb{Z})$ by $\mathbb{T}$ are of the form $\widetilde{E}_\theta(2,\mathbb{Z}) = \mathbb{Z}/4\bar{x} \mathbb{Z}_\theta^2$, parametrised by an angle $\theta \in \mathbb{T} = \{0 \leq \theta < 2\pi\}$. However, $\mathbb{Z}_\theta^2$ does not have the unique representation property for any value of $\theta$. The modifications in the theory of Heisenberg groups necessary to deal with the situation are described in section 2. The distinctive features are as follows.

A central extension $\tilde{G}$ of an abelian group $G$ by $\mathbb{T}$ is uniquely characterised
by the function \( c : G \times G \rightarrow T \) induced by the commutator map \((\tilde{g}, \tilde{h}) \mapsto \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}\) in \(\tilde{G}\) \([11]\). Denote the Pontryagin dual of \(G\) by \(\hat{G}\) and associate to every \(\tilde{G}\) a homomorphism \(\mu : G \rightarrow \hat{G}\) by \((\mu(g))(h) = c(g, h)\). Following Mumford \([12]\), we call a \(\tilde{G}\) for which \(\mu\) is an isomorphism a Heisenberg group as every such \(\tilde{G}\) has only one irreducible representation (upto equivalence) such that it restricts to a fixed character on \(T\).

Since \(\mathbb{Z}^2\) is not self-dual, \(\tilde{\mathbb{Z}}^2_\theta\) cannot be Heisenberg for any \(\theta\); depending on the value of \(\theta\), it belongs to one of two types of generalisations of Heisenberg groups:

- If \(\theta\) is an irrational multiple of \(2\pi\), \(\tilde{\mathbb{Z}}^2_\theta\) is a dense subgroup of a Heisenberg group; it has then a distinguished irreducible representation obtained by restriction.

- If \(\theta = 2\pi\nu/N\), \(\nu\) and \(N\) coprime, \(\tilde{\mathbb{Z}}^2_\theta\) is a central extension by \((\mathbb{N}\mathbb{Z})^2\) of the finite Heisenberg group \((\mathbb{Z}/N\mathbb{Z})^2\). Inequivalent irreducible representations of \(\tilde{\mathbb{Z}}^2_\theta\) are classified by the characters of \((\mathbb{N}\mathbb{Z})^2\).

In both cases, the relevant representations of \(\tilde{E}_\theta(2, \mathbb{Z})\) provide a full description of the \(\mathbb{Z}^2\) Peierls electron in flux per plaquette \(\Phi\) when \(\theta\) is identified with \(\Phi \mod \mathbb{Z}\) in units of the flux quantum \(2\pi/e\). The present paper is, however, confined to the irrational flux case (the rational case will be covered in a sequel \([4]\)). In many ways, the theory has parallels with the Landau electron; in particular, the state space has the decomposition \(\mathcal{H}_\Theta \cong L^2(\mathbb{Z}^2) \cong V_\Theta \otimes V_{-\Theta}\) with \(\tilde{\mathbb{Z}}^2_\theta\) acting irreducibly on \(V_\Theta\) and trivially on \(V_{-\Theta}\). The physics is controlled by the flux \(\Phi\) and is periodic in \(\Phi\) with period 1. We also determine
all possible Hamiltonians as self-adjoint operators on $H_\theta$ invariant under $\tilde{E}_\theta (2, \mathbb{Z})$ and find that the simplest of them has the Harper form, acting on $V_{-\theta}$ and having $V_\theta$ as degeneracy subspace.

Some of these results may come as a surprise to those who are familiar with the extensive literature on what are called magnetic translation groups (for irrational fluxes); see, for instance, [13] and the references cited there. While it is true that $\mathbb{Z}_\theta^2$ for $\theta$ irrational has a rich collection of inequivalent representations, known also from the theory of irrational rotation $C^*$-algebras [14,15], the one relevant for quantum mechanics is uniquely given as inherited from an embedding Heisenberg group.

The mathematical material, on Heisenberg groups and their appropriate generalisation, is gathered together in section 2. It is conceptually self-sufficient, though proofs are not always given in full detail. The omitted measure-theoretic and analytic elaborations are standard and can be supplied without difficulty by the reader.

2. Heisenberg Groups and Almost Heisenberg Groups

Let $G$ be an abelian group and $\mathbb{T}$ the circle group, both written multiplicatively ($\mathbb{T} \cong U(1)$) and $\tilde{G}$ a central extension of $G$ by $\mathbb{T}$. The commutator map in $\tilde{G}$, $(\tilde{g}, \tilde{h}) \mapsto \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$, associates to $\tilde{G}$ a function $c: G \times G \rightarrow \mathbb{T}$ which is homomorphic in each argument and is alternating: $c(g, g) = 1$ for all $g \in G$. If $\gamma$ is any 2-cocycle on $G$ associated to $\tilde{G}$, then $\gamma(g, h)\gamma(h, g)^{-1} = c(g, h)$ for all $g, h \in G$. So changing $\gamma$ by a coboundary does not affect $c$. Thus to every central extension of $G$ by $\mathbb{T}$ corresponds a unique element of the
abelian group $\mathcal{A}^2(G)$ of alternating bihomomorphic maps (or bicharacters) $G \times G \rightarrow \mathbb{T}$. These basic facts are easy to verify.

It is less trivial to establish that this correspondence is in fact bijective [11] (see also [16]):

2.1. $\mathcal{A}^2(G)$ is isomorphic to the 2nd cohomology group $H^2(G, \mathbb{T})$.

Let $\mathcal{B}^2(G)$ be the group of all bicharacters of $G$ and $\mathcal{S}^2(G)$ its subgroup of symmetric bicharacters. Then $\mathcal{A}^2(G) = \mathcal{B}^2(G)/\mathcal{S}^2(G)$. In general, however, $\mathcal{B}^2(G) \neq \mathcal{A}^2(G) \times \mathcal{S}^2(G)$; $c \in \mathcal{A}^2(G)$ is not necessarily skew symmetric, i.e., does not satisfy $c(g, h)c(h, g) = 1$. A sufficient condition for alternating to imply skewsymmetric is for $g \mapsto g^2$ to be an automorphism of $G$ so that we can define the square root $g^{1/2}$ of every $g \in G$ as the inverse of $g \mapsto g^2$. If this condition is met, given any $c \in \mathcal{A}^2(G)$, define $\gamma \in \mathcal{A}^2(G)$ by $\gamma(g, h) = c(g^{1/2}, h) = c(g, h)^{1/2}$ so that $c(g, h) = \gamma(g, h)^2 = \gamma(g, h) \gamma(h, g)^{-1}$. This gives us a canonical 2-cocycle which is itself skewsymmetric (and hence alternating) for every $\tilde{G}$. These points are explained in [11]. But, having raised them, for the reason that $\mathbb{Z}^2$ does not meet the sufficient condition of being divisible by 2 (in the usual terminology appropriate for additively written groups) we shall henceforth ignore them; for irrational central extensions $\mathbb{Z}^2$ which concern us here, it is always possible to choose a skewsymmetric 2-cocycle as will be seen below.

Given $\tilde{G}$ and the associated bicharacter $c \in \mathcal{A}^2(G)$, denote the Pontryagin dual of $G$ by $\hat{G}$ and define, following [12], a homomorphism $\mu : G \rightarrow \hat{G}$ by $(\mu(g))(h) = c(g, h)$. The map $\mu$ decides when $\tilde{G}$ has the unique representation
property [12]:

2.2. If $\tilde{G}$ is such that $\mu$ is an isomorphism of $G$ and $\tilde{G}$, then all irreducible representations of $\tilde{G}$ which restrict to $T \subset$ centre $\tilde{G}$ as the natural character $t \mapsto t$ are equivalent representations.

A central extension by $T$ of an abelian group $G$ for which $\mu$ is an isomorphism of $G$ and $\tilde{G}$ is a Heisenberg extension of $G$ or, simply, a Heisenberg group.

It is known [12] that equivalent realisations of the unique representation of a Heisenberg group are classified by the maximal isotropic subgroups, i.e., maximal subgroups $H$ of $G$, over which $\tilde{G}$ splits (so $H$ is a subgroup of $\tilde{G}$): for any such $H \subset G$, there is an action $U$ of $\tilde{G}$ on $L^2(G/H)$ which is linear (unitary) and irreducible. If $G$ is of the form $G = A \times A$ with $A$ self-dual, we may make the choice $H = A \times 1$. Writing $\tilde{g} \in \tilde{G}$ as $(a_1, a_2, t)$ with $(a_1, a_2) \in A \times A$, $t \in T$, the corresponding representation on $L^2(1 \times A)$ can be given as

\begin{align}
(U(1, 1, t)f)(x) &= tf(x), \\
(U(a_1, 1, 1)f)(x) &= c((a_1, 1), (1, x))f(x), \\
(U(1, a_2, 1)f)(x) &= f(a_2x).
\end{align}

We remark that the essential reason why this representation is irreducible is that $c$ is nondegenerate – i.e., there exists no $g \in G$, $g \neq Id$, such that $c(g, h) = 1$ for all $h \in G$ – which follows from $\mu$ being an isomorphism.

Suppose now that $G$ is a non-self-dual group of the form $G = A^2$ (so $A$ is also not self-dual) and $\tilde{G}$ a central extension of $G$ by $T$. $\tilde{G}$ still defines a
unique bicharacter of \( G, c \in \mathcal{A}^2(G) \), and a homomorphism \( \mu : G \rightarrow \hat{G} \) as earlier, but \( \mu \) cannot be an isomorphism. What we demand of \( \mu \) now is that it should be injective and that its image should be dense in \( \hat{G} \). We shall call a \( \tilde{G} \) for which \( \mu \) has the above property an \textit{almost Heisenberg} group. Again as before, \( A \times 1 \) is an isotropic subgroup of \( G : c(a, b) = (\mu(a))(b) = 1 \) for all \( a, b \in A \times 1 \). This means that, as maps from \( G \) into \( \mathbb{T} \), \( \{ \mu(a) \mid a \in A \times 1 \} \) have \( A \times 1 \) as kernel and hence define maps from \( G/(A \times 1) = 1 \times A \) into \( \mathbb{T} \). In other words, the restriction of \( \mu \) to \( A \times 1 \) maps it into \( 1 \times \hat{A} \) and has, by hypothesis, a dense image in \( 1 \times \hat{A} \). We have thus a dense inclusion of \( G = A \times A \) in the self-dual group \( G^* = A \times \hat{A} \) by \( (a_1, a_2) \mapsto (a_2, \mu(a_1)) \).

For any almost Heisenberg extension \( \tilde{G} \) of \( G = A^2 \), define a map \( c^* : G^* \times G^* \rightarrow \mathbb{T} \) by \( c^*((a_2, \mu(a_1)), (b_2, \mu(b_1))) = c((a_1, a_2), (b_1, b_2)) \). \( c^* \) is an alternating bicharacter defined, to begin with, on a dense subgroup of \( G^* \times G^* \) and, by continuity, on all of \( G^* \times G^* \). Correspondingly, we have a central extension \( \tilde{G}^* \) of \( G^* \) by \( \mathbb{T} \). Thus

**2.4.** Suppose \( G = A \times A \) is not self-dual and let \( \tilde{G} \) be an almost Heisenberg extension of \( G \). Then there exists a Heisenberg extension \( \tilde{G}^* \) of \( G^* = A \times \hat{A} \) of which \( \tilde{G} \) is a dense subgroup.

It follows that the irreducible representation of \( \tilde{G}^* \) restricts irreducibly to \( \tilde{G} \). Moreover, \( \hat{A} \) is evidently maximal isotropic for \( \tilde{G}^* \) and so this representation can be realised on \( L^2((\hat{A} \times A)/(\hat{A} \times 1)) = L^2(1 \times A) \):

**2.5.** Every almost Heisenberg extension \( \tilde{G} \) of \( G = A^2 \) has a distinguished irreducible representation with natural central character, obtained by restriction
from the unique irreducible representation (having natural central character) of the Heisenberg group $\tilde{G}^*$ associated to $\tilde{G}$. On $L^2(A)$, this representation is given by the formulae of eqns. (3), (4), (5).

The central extensions of $\mathbb{Z}^2$ relevant for the irrational flux Peierls electron (section 4) will turn out to be almost Heisenberg groups and the distinguished representation described in 2.5 is the only irreducible representation that comes into play in its quantum theory. However, the state space of a quantum system $(\mathcal{C}, G)$ corresponding to $\alpha \in H^2(G, \mathbb{T})$ is not an irreducible representation of $\tilde{G}_\alpha$, but rather the representation of $\tilde{G}_\alpha$ on, in general, $L^2$ sections of a certain line bundle over $\mathcal{C}$. The line bundle in question is that associated to $\alpha \in H^2(G, \mathbb{T})$ of a principal $\tilde{H}^2(G, \mathbb{T})$ bundle – by Pontryagin duality, $\alpha$ is a character of $\tilde{H}^2(G, \mathbb{T})$. (When $\mathcal{C}$ is not a manifold and $G$ is not a Lie group, the terminology is obviously meant in an algebraic sense). It is appropriate to name this representation as the wavefunction representation. Since in our applications $G$ is the translation group $\mathbb{R}^2$ or $\mathbb{Z}^2$, the state space $\mathcal{H}_\alpha$ is isomorphic to the space of $L^2$ functions on $G$ itself. Furthermore, the full symmetry groups of our systems are the euclidean groups $E = J \tilde{\times} G$ where $J$ is a subgroup of Aut $G$. Before studying how the central extensions of $E$ are related to those of $G$ and are represented on $L^2(G) = L^2(A^2)$, we exhibit a decomposition of $L^2(A^2)$ as a tensor product of irreducible representations of (almost) Heisenberg groups which is of great utility in all that follows.

To begin with, let $\tilde{G}_\alpha$, $\alpha \in \mathcal{A}^2(G)$, be a central extension of any abelian group $G$, not necessarily (almost) Heisenberg, but for which the canonical
choice of the associated 2-cocycle $\gamma_\alpha \in A^2(G)$ is possible. Define an action of $\tilde{G}_\alpha$ on $L^2(G) = \{\psi, \cdots\}$ by
\[(W(g, t)_\alpha \psi)(h) = t\gamma_\alpha(g, h)\psi(gh).\] (6)

The operators $W(g, t)_\alpha$ are clearly unitary on $L^2(G)$ and, by virtue of $\gamma_\alpha$ being bimultiplicative, furnish a representation of $\tilde{G}_\alpha$ for any $\alpha \in H^2(G, T)$. If, in addition, $\gamma_\alpha$ can be picked from $A^2(G)$, it is equally easy to verify that $L^2(G)$ is a representation of the direct product group $\tilde{G}_\alpha \times \tilde{G}_{\alpha^{-1}}$ for the action of each factor by eqn. (6), namely,
\[(W(g, t)_\alpha, (g', t')_{\alpha^{-1}})\psi)(h) = tt'\gamma_\alpha(g, h)\gamma_\alpha(g', h')^{-1}\psi(gg'h).\] (7)

We have used here the identity $\gamma_{\alpha^{-1}}(g, h) = \gamma_\alpha(g, h)^{-1}$ and also the skewsymmetry of $\gamma$. Note that $T \times T \subset$ centre $(\tilde{G}_\alpha \times \tilde{G}_{\alpha^{-1}})$ operates by $(t, t') \longrightarrow tt'$ so that, as a representation of either $\tilde{G}_\alpha$ or $\tilde{G}_{\alpha^{-1}}$, $L^2(G)$ is the lift of a projective representation of $G$.

Noting that if $\tilde{G}_\alpha$ is Heisenberg (almost Heisenberg), so is $\tilde{G}_{\alpha^{-1}}$, we have our key result:

**2.6.** Let $\tilde{G}_\alpha$ be a Heisenberg (respectively almost Heisenberg) extension of $G$. Then the representation of $\tilde{G}_\alpha \times \tilde{G}_{\alpha^{-1}}$ on $L^2(G)$ defined by eqn. (7) is irreducible. Thus $\mathcal{H}_\alpha \cong L^2(G)$ has the tensor product decomposition $\mathcal{H}_\alpha = V_\alpha \otimes V_{\alpha^{-1}}$ where $V_\alpha$ is the unique (respectively distinguished) irreducible representation of $\tilde{G}_\alpha$ having natural central character.

For the Heisenberg case, the proof is a simple extension of the proof of the irreducibility of the representation of $\tilde{G}_\alpha$ on $L^2(A)$ and will be found in
For the almost Heisenberg case, we do the obvious: embed \( \tilde{G}_\alpha \times \tilde{G}_{\alpha^{-1}} \) in the corresponding \( \tilde{G}^*_\alpha \times \tilde{G}^*_{\alpha^{-1}} \) and take the irreducible representation of the latter group on \( L^2(G^*) = L^2(A \times \hat{A}) \). This restricts to a representation of \( \tilde{G}_\alpha \times \tilde{G}_{\alpha^{-1}} \) irreducibly and, on taking Fourier transforms on \( 1 \times \hat{A} \), can be written as a representation on \( L^2(A \times A) \).

To conclude this account of the mathematical framework, we now consider the semidirect product groups \( E = J \tilde{\times} G \) where \( G \) as before is the (translation) group \( A^2 \) and \( J \) is a (rotation) subgroup of \( \text{Aut} \, G \). For the classification of central extensions of \( E \), we quote a general result (for a proof, see [7]):

2.7. For \( G \) an abelian group and \( J \) a subgroup of \( \text{Aut} \, G \), \( H^2(J \tilde{\times} G, \mathbb{T}) = H^2(J, \mathbb{T}) \times H^1(J, \hat{G}) \times H^2(G, \mathbb{T}) \).

Here \( H^2(G, \mathbb{T}) \) is the subgroup of \( H^2(G, \mathbb{T}) \) fixed pointwise by the action of \( J \) and \( H^1 \) is the 1st cohomology with coefficients in \( \hat{G} \) considered as a \( J \)-module; thus a \( \hat{G} \)-valued 1-cocycle on \( J \) is a map \( \varphi : J \rightarrow \hat{G} \) satisfying \( \varphi(\rho \sigma) = \varphi(\rho)(\rho \cdot \varphi(\sigma)) \) and it is a coboundary if there is a \( \chi \in \hat{G} \) such that \( \varphi(\rho) = (\rho \cdot \chi)\chi^{-1} \) for all \( \rho, \sigma \in J \). The following criterion for the vanishing of \( H^1 \) is useful.

2.8. If \( B \) is an abelian group divisible by 2 and \( J \) is an abelian subgroup of \( \text{Aut} \, B \), \( H^1(J, B) \) vanishes whenever there exists \( \rho_0 \in J \) such that \( \rho_0 \cdot \varphi(\rho) = \varphi(\rho)^{-1} \) for all 1-cocycles \( \varphi : J \rightarrow B \) and all \( \rho \in J \).

For proof, we have \( \varphi(\rho \sigma) = \varphi(\sigma \rho) \) implying the identity \( (\sigma \cdot \varphi(\rho))\varphi(\rho)^{-1} = (\rho \cdot \varphi(\sigma))\varphi(\sigma)^{-1} \) from the definition of a 1-cocycle. Choosing \( \sigma = \rho_0 \) and writing \( \varphi(\rho_0) = b_0 \in B \), this becomes \( \varphi(\rho)^{-2} = (\rho \cdot b_0)\overline{b_0}^{-1} \). Taking square
roots, we see that $\varphi$ is a coboundary.

In our applications, the conditions required for the vanishing of $H^1(J, \tilde{G})$ will be seen to be met. It will also turn out that $H^2(J, \mathbb{T}) = 0$ and $H^2(G, \mathbb{T})^J = H^2(G, \mathbb{T})$. Hence, in the rest of this section, we confine attention to central extensions of $E$ of the form $\tilde{E} = J \tilde{\times} \tilde{G}$, with $J$ acting trivially on $\mathbb{T} \subset$ centre $\tilde{G}$. Denoting by $\text{Aut}_0 \tilde{G}$ the subgroup of $\text{Aut} \tilde{G}$ fixing $\mathbb{T}$ pointwise, $J$ is thus a subgroup of $\text{Aut}_0 \tilde{G}$.

When $\tilde{G}$ is a Heisenberg group, it is a well-known fact that every Hilbert space $V$ on which $\tilde{G}$ has an irreducible representation $V$, unique up to equivalence, also carries a projective representation of $\text{Aut}_0 \tilde{G}$, the metaplectic representation: If $\rho \in \text{Aut}_0 \tilde{G}$, i.e. $\rho(g, t) = (\rho(g), t)$, and $U|_{\mathbb{T}}$ is the natural character, then $(g, t) \mapsto U_{\rho}(g, t) = U(\rho(g), t)$ is also an irreducible representation with $U_{\rho}|_{\mathbb{T}}$ also natural. By the unique representation theorem, there exist unitary operators $O(\rho)$ on $V$ such that $U_{\rho}(g, t) = O(\rho)U(\rho(g), t)O(\rho)^{-1}$ and $\rho \mapsto O(\rho)$ is clearly a representation, in general projective, of $\text{Aut}_0 \tilde{G}$ on $V$.

If $\tilde{G}$ is almost Heisenberg, then $\text{Aut}_0 \tilde{G}$ is a subgroup of $\text{Aut}_0$ of the Heisenberg group $\tilde{G}^*$. Hence, if $V$ is a Hilbert space on which the distinguished irreducible representation is realised, then, from the statement 2.5, there is a projective representation of $\text{Aut}_0 \tilde{G}$ on $V$. From this we draw the following conclusion relevant for our purpose.

2.9. Let $\tilde{G}$ be a Heisenberg (almost Heisenberg) group, $J$ a subgroup of $\text{Aut}_0 \tilde{G}$ such that $H^2(J, \mathbb{T}) = 0$ and $V$ a Hilbert space on which the unique
(distinguished) irreducible representation of $\tilde{G}$ is realised. Then there is an irreducible linear representation of $J \tilde{G}$ on $V$.

The above property carries over naturally to the wave function representation. When $\tilde{G}_\alpha$ and $\tilde{G}_{\alpha - 1}$ are Heisenberg, all representations of $\tilde{G}_\alpha \times \tilde{G}_{\alpha - 1}$ which restrict to either factor irreducibly and nontrivially and to the central subgroup $\mathbb{T} \times \mathbb{T}$ naturally are equivalent. And since $J$ fixes $\mathbb{T} \times \mathbb{T}$, it is a subgroup of $\text{Aut}_0(\tilde{G}_\alpha \times \tilde{G}_{\alpha - 1})$. Similar observations apply in the almost Heisenberg case and we have

2.10. $L^2(G) = V_\alpha \otimes V_{\alpha - 1}$ is an irreducible representation of $J \tilde{G}(\tilde{G}_\alpha \times \tilde{G}_{\alpha - 1})$ whenever $\tilde{G}_\alpha$ is Heisenberg or almost Heisenberg and $J \subset \text{Aut}_0\tilde{G}_\alpha$ has $H^2(J, \mathbb{T}) = 0$.

3. The Heisenberg Group of $\mathbb{R}^2$ and the Landau Electron

This section begins by studying the projective representations of the real euclidean group $E(2, \mathbb{R}) = SO(2, \mathbb{R}) \tilde{\times} \mathbb{R}^2$ on $L^2(\mathbb{R}^2)$ with a view to arrive at a description of the most general quantum system with configuration space $\mathbb{R}^2$ and symmetry group $E(2, \mathbb{R})$. The central extensions of the Lie algebra of $E(2, \mathbb{R})$ were first investigated by Bargmann [6]. The general theory needed to deal with the group is given in section 2 and is easy to apply.

First, we have $H^2(\mathbb{R}^2, \mathbb{T}) = \mathcal{A}^2(\mathbb{R}) \cong \mathbb{R}$, consisting of functions $c_r(x, y) = \exp(irx \wedge y)$ for $x, y \in \mathbb{R}^2$ and $r \in \mathbb{R}$, all written additively. $SO(2, \mathbb{R}) = \{\rho_\theta \mid 0 \leq \theta < 2\pi\}$ acts on these functions by $(\rho_\theta c_r)(x, y) = c_r(\rho_\theta x, \rho_\theta y)$, $\rho_\theta x = (x_1 \cos \theta + x_2 \sin \theta, -x_1 \sin \theta + x_2 \cos \theta)$. It is evident that $c_r(\rho_\theta x, \rho_\theta y) =$
so $H^2(\mathbb{R}^2, \mathbb{T})^{SO(2,\mathbb{R})} = H^2(\mathbb{R}^2, \mathbb{T})$. Also, $H^2(SO(2,\mathbb{R}), \mathbb{T}) = 0$ since $SO(2,\mathbb{R})$ is 1-dimensional. As for the $H^1$ contribution in the statement 2.7, the $SO(2,\mathbb{R})$ action on $\hat{\mathbb{R}}^2 \cong \mathbb{R}^2$ is $x \mapsto \rho_\theta x = \rho_\theta^{-1} x$ and it is immediately verified that $\rho_{\theta=\pi} x = -x$. Hence, by 2.8, $H^1(SO(2,\mathbb{R}), \mathbb{R}^2)$ also vanishes and we have

3.1. Inequivalent central extensions of $E(2,\mathbb{R})$ by $\mathbb{T}$ form a one real parameter family of groups $\tilde{E}_r(2,\mathbb{R}) = SO(2,\mathbb{R}) \times \mathbb{R}^2_r$.

The $SO(2,\mathbb{R})$ action on $\mathbb{R}^2_r$ is the one on $\mathbb{R}^2$ extended trivially to its centre $\mathbb{T}$. Therefore, by 2.9, an irreducible representation of the real Heisenberg group $\tilde{\mathbb{R}}^2_r$, $r \neq 0$, on $V_r$ say, is also an irreducible representation of $\tilde{E}_r(2,\mathbb{R})$. For classifying all actions of $\tilde{E}_r(2,\mathbb{R})$ on $V_r$, it is convenient to look at the corresponding Lie algebra actions. Choosing a basis $\{L, P_1, P_2, 1\}$ for Lie $\tilde{E}_\alpha(2,\mathbb{R})$ where $L$ is the angular momentum generating rotations, $P_1$ and $P_2$ are mutually perpendicular momenta generating translations and 1 generates the centre, we have the Lie brackets

\[ [P_1, P_2] = ir \]

\[ [L, P_1] = iP_2, \quad [L, P_2] = -iP_1. \]

One checks that $L + (P_1^2 + P_2^2)/2r$, $r \neq 0$, has vanishing brackets with $L, P_1$ and $P_2$ and, since $V_r$ is irreducible, is represented by a scalar $s$:

\[ L = s - \frac{1}{2r}(P_1^2 + P_2^2) \]

for some $s \in \mathbb{R}$. But we know the spectrum of $(P_1^2 + P_2^2)/2r$ in $V_r$ to be $\pm N \pm \frac{1}{2}$ depending on the sign of $r$ (the energy spectrum of the harmonic
oscillator) and the spectrum of \( L \) to be contained in \( \mathbb{Z} \), the characters of \( SO(2, \mathbb{R}) \). Hence \( s \in \mathbb{Z} + \frac{1}{2} \). (Note that working with the Lie algebra of \( \mathbb{R}^2_\tau \) is legitimate on account of \( \mathbb{R}^2 \) being simply connected [11,17]). Every \( s \) in this set defines a distinct set of characters of \( SO(2, \mathbb{R}) \) and a distinct irreducible representation of \( E_r(2, \mathbb{R}) \) on \( V_r \). More precisely, we have

**3.2.** Let \( V_r \) be an irreducible representation space of the real Heisenberg group \( \tilde{\mathbb{R}}^2_\tau \). Then, given any \( l_0 \in \mathbb{Z} \), there is an irreducible representation of \( E_r(2, \mathbb{R}) \) on \( V_r \) having the angular momentum decomposition

\[
V_r = \bigoplus_{l \leq l_0} V_{r,l} \quad \text{or} \quad V_r = \bigoplus_{l \geq l_0} V_{r,l}
\]

for \( r > 0 \) or \( r < 0 \) respectively, each (one dimensional) \( V_{r,l} \), with \( LV_{r,l} = lV_{r,l} \), occurring once in the sum.

In accordance with the general theory of section 2, a choice for \( V_r \) is \( V_r = L^2(\mathbb{R}) \), the space of functions of the momentum along a fixed direction. Turning to the wave function representation, statements **2.6** and **2.10** have the corollary

**3.3.** For \( r \neq 0 \), the state space \( \mathcal{H}_r \cong L^2(\mathbb{R}) \) is the unique irreducible representation \( V_r \otimes V_{-r} \) of \( \tilde{\mathbb{R}}^2_\tau \times \tilde{\mathbb{R}}^2_{-\tau} \). It is also an irreducible representation of \( SO(2, \mathbb{R}) \times (\tilde{\mathbb{R}}^2_\tau \times \tilde{\mathbb{R}}^2_{-\tau}) \).

The actions of \( \tilde{\mathbb{R}}^2_\tau \), \( \tilde{\mathbb{R}}^2_{-\tau} \) and \( SO(2, \mathbb{R}) \) on \( L^2(\mathbb{R}) \) are completely specified by the action of the corresponding Lie algebras. The interested reader will find them written down and their physical meaning discussed in [10]. We note that if \( Q_1 \) and \( Q_2 \) are noncentral basis vectors of Lie \( \tilde{\mathbb{R}}^2_{-\tau} \) satisfying the
Lie brackets

\[ [Q_1, Q_2] = -ir, \quad (11) \]

\[ [L, Q_1] = iQ_2, \quad [L, Q_2] = -iQ_1, \quad (12) \]

with \([P, Q] = 0\) then 3.3 has the consequence that \(L\) is a polynomial in \([P, Q]\). Its form is obtained by interpreting \(s\) in eqn. (10), which is a scalar in \(V_r\), as a polynomial in \(Q\). The brackets (11) and (12) then fix \(s\) and yield

\[ L = \frac{1}{2r}(Q_1^2 + Q_2^2) - \frac{1}{2r}(P_1^2 + P_2^2), \quad (13) \]

upto an additive integer scalar.

This completes the description of the kinematics, namely the structure of the state space, of the \(E(2, \mathbb{R})\)-symmetric quantum mechanics of a particle in \(\mathbb{R}^2\). One aspect of the general theory not invoked so far (because it has no significant physical role in magnetic field problems) is worthy of passing mention: for a system \((\mathcal{C}, E)\), every \(\mathcal{H}_\alpha, \alpha \in H^2(E, \mathbb{T})\), is a superselection sector \([7,8]\). In the following, we shall refer to \(\mathcal{H}_\alpha\) for each \(\alpha\) as a sector and to \(\mathcal{H}_{\alpha=0}\) as the trivial sector.

In approaching the question of dynamics, i.e., in looking for Hamiltonians to generate time evolution respecting the symmetries of the system, the central point to keep in mind is that there is no sense in which there is a unique Hamiltonian valid for all sectors \([7,8]\). To illustrate, let us assume that \(\mathcal{C}\) is a \(d\)-dimensional manifold which is a homogeneous space for the (connected) Lie group \(E, \mathcal{C} = E/R\), and let \(\{X_i\}\) be a basis for Lie \(E\) adapted to \(R\), i.e., \(\{X_i \mid \dim E - \dim R < i \leq \dim E\}\) is a basis for Lie \(R\), so that \(\{X_1, \ldots, X_d\}\)
is a vector space basis for Lie $E/\text{Lie } R$. \{${X}_1, \ldots, {X}_d$\} is thus a set of mutually perpendicular velocity vectors of the particle whose configuration space is $C$. Let $H_0$ be a nondegenerate symmetric quadratic polynomial in the velocity vectors, invariant under the adjoint action of $E$. In a representation of $E$, $H_0$ is represented by a selfadjoint operator and is a satisfactory free Hamiltonian in the trivial sector $\mathcal{H}_0$. To the extent that $E$-invariance fixes the symmetric coefficients occurring in $H_0$ upto an overall scale, $H_0$ is unique modulo an additive and a multiplicative scalar [7].

In a nontrivial sector $\mathcal{H}_\alpha$, $H_0$ is not the correct Hamiltonian because it cannot be invariant under $\tilde{E}_\alpha$ (as it should be) though it is still defined as an element of the symmetric algebra of Lie $\tilde{E}_\alpha$. (These aspects are examined in detail in [7]). The correct $\tilde{E}_\alpha$-invariant Hamiltonian $H_\alpha$ is found as follows [7]:

3.4. Suppose $E$ is a connected Lie group such that $H^2(E, \mathbb{T})$ is in bijective correspondence with $H^2(\text{Lie } E, \mathbb{R})$ and $R$ a subgroup of $E$ with $H^2(R, \mathbb{T}) = 0$. If $H_0$ is the (E-invariant) free Hamiltonian of the system $(C = E/R, E)$, then there is $X^{(\alpha)} \in \text{Lie } \tilde{E}_\alpha$ such that $H_\alpha = H_0 + X^{(\alpha)}$ is $\tilde{E}_\alpha$-invariant.

This $H_\alpha$ is a suitable kinetic energy in the sector $\mathcal{H}_\alpha$. It does not, indeed cannot, describe free (plane wave) motion – there is no free motion in a nontrivial sector.

The application of 3.4 to our system $E = E(2, \mathbb{R}), C = \mathbb{R}^2 = E(2, \mathbb{R})/\text{SO}(2, \mathbb{R})$ is immediate. The free Hamiltonian is of course $H_0 = (2m)^{-1}(P_1^2 + P_2^2)$. For $r \neq 0$, $\tilde{E}_r(2, \mathbb{R})$ does not leave $H_0$ invariant. But there is a unique
$X^{(r)} = rm^{-1}L \in \text{Lie } \tilde{E}_r(2, \mathbb{R})$ such that

$$H_r = H_0 + X^{(r)} = \frac{1}{2m}(P_1^2 + P_2^2) + \frac{r}{m}L$$

(14)

is fixed by $\tilde{E}_r(2, \mathbb{R})$. The expression (13) for $L$ simplifies this to

$$H_r = \frac{1}{2m}(Q_1^2 + Q_2^2),$$

(15)

i.e., on $\mathcal{H}_r = V_r \otimes V_{-r}$, $H_r$ is the operator $1 \otimes (Q_1^2 + Q_2^2)/2m$. The spectrum of $H_r$ is $(rm^{-1})(N - \frac{1}{2})$ with multiplicity one in $V_{-r}$; on the whole of $\mathcal{H}_r$, the eigenspaces of $H_r$ are $V_r$ for every eigenvalue and for all $r \neq 0$. The spectrum thus matches the energy eigenvalues and degeneracies of the Landau electron moving in a magnetic field $B$ on identifying the nonzero real number $r$ with $eB$.

We may now use $H_r$ to write down the Heisenberg equation of motion for any operator on $\mathcal{H}_r$. For $Q$ ($P$ and $J$ are automatically conserved) we find

$$\frac{dQ_1}{dt} = i[H_r, Q_1] = -\frac{r}{m}Q_2, \quad \frac{dQ_2}{dt} = \frac{r}{m}Q_1.$$  

(16)

These constitute the Lorentz force equation if $Q$ is taken to be proportional to the velocity $v$ (with $r = eB$); the velocity dependence of energy then fixes $Q = mv$.

The results of this section have established our claim that the projective representation theory of $E(2, \mathbb{R})$ on $L^2(\mathbb{R}^2)$ is the quantum theory of the Landau electron. The treatment may appear somewhat abstract, but has the great advantage of dispensing with all but the one essential physical parameter, namely the magnetic field. Many conceptual issues are thereby clarified.
especially the origin and (lack of) significance of gauges and gauge transformations, the origin of degeneracies, the fact that velocity is not proportional to momentum, ambiguities in the classical mechanics of (electro)magnetic problems, etc. A fuller account of these aspects will be found in [10].

4. The \(\mathbb{Z}^2\) Peierls Electron for Irrational Fluxes

In this section we take up a particle moving on the infinite planar square lattice, \(\mathcal{C} = \mathbb{Z}^2 \subset \mathbb{R}^2\), and having the discrete euclidean group \(E(2, \mathbb{Z}) = \mathbb{Z}/4 \times \mathbb{Z}^2\) as its group of symmetries. Denoting by \(\zeta\) the generator of \(\mathbb{Z}/4\) corresponding to an anticlockwise rotation by the angle \(\pi/2\), the action of \(\mathbb{Z}/4\) on \(\mathbb{Z}^2\) is \(\zeta \cdot (m_1, m_2) = (−m_2, m_1)\), \(m = (m_1, m_2) \in \mathbb{Z}^2\) (written additively).

A general bicharacter \(b\) on \(\mathbb{Z}^2\) is a function \(b(m, n) = \exp(i\theta_1 m_1 n_2 + i\theta_2 m_2 n_1 + i\theta_3 m_1 n_1 + i\theta_4 m_2 n_2)\) with \(\theta_i \in [0, 2\pi]\). If \(b\) is in addition alternating, then \(\theta_1 + \theta_2 = \theta_3 = \theta_4 = 0 \, (\text{mod } 2\pi)\). Hence

\[H^2(\mathbb{Z}^2, \mathcal{T}) = A^2(\mathbb{Z}^2) = \{c_\theta : \mathbb{Z}^2 \longrightarrow \mathcal{T} \mid c_\theta(m, n) = e^{i\theta m \cdot n}\} \cong \mathcal{T}.
\]

The action of \(\mathbb{Z}/4\) on \(A^2(\mathbb{Z}^2)\): \(\zeta \cdot c_\theta(m, n) = c_\theta(\zeta \cdot m, \zeta \cdot n)\) leaves every \(c_\theta\) fixed. Also, \(H^2(\mathbb{Z}/4, \mathcal{T})\) and \(H^1(\mathbb{Z}/4, \mathbb{Z}^2) = H^1(\mathbb{Z}/4, \mathcal{T}^2)\) both vanish, the latter because \(\rho_0 = \zeta^2\) meets the requirements of 2.8. Hence, as in the real case, central extensions of \(E(2, \mathbb{Z})\) and \(\mathbb{Z}^2\) are in 1-1 correspondence:

4.1. Every central extension of \(E(2, \mathbb{Z})\) by \(\mathcal{T}\) is of the form \(\tilde{E}(2, \mathbb{Z}) = \mathbb{Z}/4 \times \mathbb{Z}^2\), where \(\mathbb{Z}/4\) acts on \(\mathcal{T} \subset \mathbb{Z}^2\) trivially. Inequivalent central extensions of \(\mathbb{Z}^2\) and hence of \(E(2, \mathbb{Z})\) are parametrised by an angle \(\theta \in [0, 2\pi]\).

The relationship of the projective representations of \(E(2, \mathbb{Z})\) and of \(\mathbb{Z}^2\)
with the quantum mechanics of the $\mathbb{Z}^2$ Peierls electron becomes manifest already at this point. Reverting to the Landau electron briefly, we observe that the commutator of translations through $x$ and $y$ in $\bar{\mathbb{R}}^2$ has the physical meaning

$$c_r(x,y) = e^{i\pi x \wedge y} = e^{i\epsilon \Phi(x,y)}$$

where $\Phi$ the magnetic flux through a parallelogram having $x, y \in \mathbb{R}^2$ as adjacent sides. Embedding $\mathbb{Z}^2$ in $\mathbb{R}^2$ as the lattice generated by the vectors $(\xi_1,0), (0,\xi_2) \in \mathbb{R}^2$ and equating the commutator of $(1,0)$ and $(0,1)$ in $\bar{\mathbb{Z}}_\theta^2$ to that of $(\xi_1,0)$ and $(0,\xi_2)$ in $\bar{\mathbb{R}}^2_r$, we get

$$e^{i\theta} = e^{i\epsilon B \xi_1 \xi_2} = e^{i\epsilon \Phi} = e^{i2\pi \Phi/\Phi_0} \tag{17}$$

where the constant $\Phi$ is the flux through the plaquette bounded by the generators and $\Phi_0 = 2\pi/e$ is the flux quantum. Choosing units in which $\Phi_0 = 1$, we may identify $\theta$ with $2\pi \Phi$.

From the above we conclude:

**4.2.** The quantum mechanics of the $\mathbb{Z}^2$ Peierls electron is fully determined by, and is periodic in, one physical parameter, namely the flux per plaquette $\Phi$, with period equal to one flux quantum.

In particular, when the flux is integral, the motion of the particle is free hopping motion. It is also evident that the field $B$ itself is totally irrelevant.

The numerical work of Hofstadter [18] on the spectrum of the Harper Hamiltonian has demonstrated, very graphically, that it depends qualitatively on whether the flux is rational or irrational as a multiple of the flux
quantum. Such differences are reflections of the differences in the structure and representation theory of $\tilde{\mathbb{Z}}^2_{\theta}$ for rational and irrational values of $\theta/2\pi$. The commutator function $c_\theta$ defines the map $\mu_\theta : \mathbb{Z}^2 \rightarrow \mathbb{T}^2$ by $\mu_\theta(m_1, m_2) = (\exp(-i\theta m_2), \exp(i\theta m_1))$. If $\theta$ is rational, $\theta = 2\pi\nu/N$ with $\nu$ and $N$ coprime, then $\mu_\theta$ has kernel $(N\mathbb{Z})^2$. This rational flux case of the Peierls electron will be the subject of a sequel; in the rest of this paper, we deal with the case of irrational fluxes.

When $\theta$ is irrational, $\mu_\theta$ is an injective map and its image is dense in $\mathbb{T}^2$. Hence $\tilde{\mathbb{Z}}^2_{\theta}$ is an almost Heisenberg group. Moreover, $b_\theta^2 \neq 1$ for any irrational bicharacter $b_\theta$ and hence we can choose the 2-cocycle corresponding to $\tilde{\mathbb{Z}}^2_{\theta}$ canonically, as the skew symmetric square root of $c_\theta$ (see remarks following 2.1):

$$\gamma_\theta(m, n) = e^{\frac{1}{2}i\theta m \wedge n}$$ (18)

The distinguished irreducible representation and the wave function representation of $\tilde{\mathbb{Z}}^2_{\theta}$ can now be characterised. Denoting the noncentral generators of $\tilde{\mathbb{Z}}^2_{\theta}$ by $p_1 = (1, 0, 1)$, $p_2 = (0, 1, 1)$ ($\mathbb{Z}^2$ is written additively and $\mathbb{T}$ multiplicatively) and the corresponding generators of $\tilde{\mathbb{Z}}^2_{\theta}$ by $q_1, q_2$ satisfying

$$p_1 p_2 p_1^{-1} p_2^{-1} = e^{i\theta}, \quad q_1 q_2^{-1} q_1^{-1} q_2^{-1} = e^{-i\theta},$$ (19)

we have

4.3. For any irrational $\theta/2\pi, \tilde{\mathbb{Z}}^2_{\theta}$ has an irreducible representation on $L^2(\mathbb{Z})$ given by

$$(U(p_1)f)(m) = e^{i\theta m} f(m), \quad (U(p_2)f)(m) = f(m + 1).$$

24
By 2.9 this extends also to an irreducible representation of $\tilde{E}_\theta(2, \mathbb{Z})$.

4.4. There is an irreducible representation of $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_2^2$ and $\mathbb{Z} / 4 \mathbb{Z} \times (\mathbb{Z}_2^2 \times \mathbb{Z}_2^2)$ on $L^2(\mathbb{Z}^2)$ given by

\[
(W(p_1)\psi)(m_1, m_2) = e^{\frac{i}{2} m_2 \theta} \psi(m_1 + 1, m_2),
\]
\[
(W(p_2)\psi)(m_1, m_2) = e^{-\frac{i}{2} m_1 \theta} \psi(m_1, m_2 + 1),
\]
\[
(W(q_1)\psi)(m_1, m_2) = e^{-\frac{i}{2} m_2 \theta} \psi(m_1 + 1, m_2),
\]
\[
(W(q_2)\psi)(m_1, m_2) = e^{\frac{i}{2} m_1 \theta} \psi(m_1, m_2 + 1),
\]
\[
(W(\zeta)\psi)(m_1, m_2) = \psi(-m_2, m_1).
\]

If $V_\theta$ is the Hilbert space of the distinguished irreducible representation of $\mathbb{Z}_2^2$, $L^2(\mathbb{Z}^2)$ is isomorphic to $V_\theta \otimes V_{-\theta}$.

The second part of 4.4 is equivalent to the statement that $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$ has a unique irreducible representation up to equivalence with the property that its restriction to either factor is the distinguished irreducible representation.

This is the key representation-theoretic foundation of the quantum mechanics of the $\mathbb{Z}^2$ Peierls electron for irrational fluxes. The space $L^2(\mathbb{Z}^2)$ is the state space $\mathcal{H}_\theta$ for the action given above of $\tilde{E}_\theta(2, \mathbb{Z})$. Similar to the Landau case, only the subgroup $\tilde{E}_\theta(2, \mathbb{Z})$ of $\mathbb{Z} / 4 \mathbb{Z} \times (\mathbb{Z}_2^2 \times \mathbb{Z}_2^2)$ is directly related to the physical symmetries; the group $\mathbb{Z}_2^2$ just parametrises the possible inequivalent actions of $\tilde{E}_\theta(2, \mathbb{Z})$ (or the multiplicity of the distinguished representation of $\mathbb{Z}_2^2$) in $L^2(\mathbb{Z}^2)$.

The functions $\psi \in L^2(\mathbb{Z}^2)$ are the wave functions. The action of $p_1, p_2, q_1$ and $q_2$ given in 4.4 results from the canonical choice of the 2-cocycle $\gamma_\theta$ as an element of $\mathcal{A}(\mathbb{Z}^2)$, eqn. (18). We have the freedom to modify $\gamma_\theta$.
by a coboundary – the symmetric bimultiplicative function \( \exp(i\phi(m_1n_2 + m_2n_1)) \) – without changing the structure of the groups \( \tilde{\mathbb{Z}}^2_\theta \) and \( \tilde{E}_\theta(2,\mathbb{Z}) \). The representation \( W \) gets modified thereby to

\[
W'(p_1) = e^{iom_2}W(p_1), \quad W(p_2) = e^{iom_1}W(p_2),
\]

etc. But \( W \) and \( W' \) are both restrictions to a dense subgroup of irreducible representations \( W^* \) and \( W'^* \) of the group \( \tilde{\mathbb{Z}}^{2*}_\theta \times \tilde{\mathbb{Z}}^{2*}_\theta \) in which each factor is a Heisenberg extension of \( \mathbb{Z} \times \mathbb{T} \). Consequently \( W^* \) and \( W'^* \) are equivalent and hence so are \( W \) and \( W' \): \( W'(m,n) = SW(m,n)S^{-1} \) for some unitary operator \( S \) on \( \mathcal{H}_\theta \). The operators representing translations in \( \mathcal{H}_\theta \) for a given irrational flux form a one (angle) parameter family and are related among themselves by unitary operators \( S(\phi) \). Physically, \( S \) is a gauge transformation and the (unitary) equivalent representations of \( \mathbb{Z}/4\tilde{x}(\tilde{\mathbb{Z}}^2_\theta \times \tilde{\mathbb{Z}}^2_{-\theta}) \), parametrised by \( \phi \in [0,2\pi) \), are gauge-equivalent. The justification for this assertion is the corresponding phenomenon in the Landau case where, infinitesimally, \( S \) arises from an \( \mathbb{R}^2 \)-valued 2-coboundary added to a connection on a \( U(1) \) bundle, the vector potential, without changing the curvature [10]. In any case, our treatment bypasses all questions related to gauges, except in so far as the action of the symmetry group on wave functions is desired in an explicit form.

Our final task is the determination of the Hamiltonian(s) \( H_\theta \) governing time evolution in the sector \( \mathcal{H}_\theta \). As \( \mathcal{H}_\theta \) is irreducible under the action of \( \tilde{\mathbb{Z}}^2_\theta \times \tilde{\mathbb{Z}}^2_{-\theta} \), \( H_\theta \) as an operator on \( \mathcal{H}_\theta \) is a selfadjoint element of the algebra of operators representing this group. \( \tilde{E}_\theta(2,\mathbb{Z}) \) acts on the operator algebra by conjugation by unitary operators; \( H_\theta \) must be invariant under this action in
order to preserve the symmetries under time evolution.

Since we no longer have at our disposal infinitesimal operators representing momenta and velocities, the counterpart to the procedure followed for the Landau electron is to find the subalgebra of the group algebra \( \mathbb{C}\[\tilde{\mathbb{Z}}_2^2 \times \tilde{\mathbb{Z}}_2^2\] satisfying the two conditions of selfadjointness and pointwise invariance under \( \tilde{E}_\theta(2, \mathbb{Z}) \). Given the commutators of eqns. (19) and

\[
\zeta p_1 \zeta^{-1} = p_2, \quad \zeta p_2 \zeta^{-1} = p_1^{-1}, \quad (20)
\]

\[
\zeta q_1 \zeta^{-1} = q_2, \quad \zeta q_2 \zeta^{-1} = q_1^{-1}, \quad (21)
\]

this is a straightforward computation.

Confine attention first to \( \mathbb{C}[\tilde{\mathbb{Z}}_2^2] \). An element of this algebra has the general form

\[
\Omega_\theta = \sum w_{j_1 j_2} p_1^{j_1} p_2^{j_2} \quad (22)
\]

where the sum is over \( j_1, j_2 \in \mathbb{Z} \), using the fact that \( p_1 p_2 p_1^{-1} p_2^{-1} \) is in the centre of \( \tilde{\mathbb{Z}}_2^2 \), and hence is a scalar. Then

\[
p_1 \Omega_\theta p_1^{-1} = \sum w_{j_1 j_2} e^{ij_2 \theta} p_1^{j_1} p_2^{j_2} ,
\]

\[
p_2 \Omega_\theta p_2^{-1} = \sum w_{j_1 j_2} e^{-ij_1 \theta} p_1^{j_1} p_2^{j_2} .
\]

So for \( \tilde{\mathbb{Z}}_2^2 \) to fix \( \Omega_\theta \), we have the conditions

\[
\sum w_{j_1 j_2} (1 - e^{ij_2 \theta}) p_1^{j_1} p_2^{j_2} = 0,
\]

\[
\sum w_{j_1 j_2} (1 - e^{-ij_1 \theta}) p_1^{j_1} p_2^{j_2} = 0.
\]
Since the only relation among the generators of $\tilde{\mathbb{Z}}_2^2$ is eqn. (19), $p_1$ and $p_2$ generate $\tilde{\mathbb{Z}}_2^2$ freely modulo its centre. Hence the above equations hold only if the coefficients of $p_1^{j_1}p_2^{j_2}$ vanish for all nonzero $j_1, j_2 \in \mathbb{Z}$. But since $\theta$ is irrational, $\exp(i j \theta) \neq 1$ for any $j \neq 0$, leaving $w_{00}$ as the only nonzero coefficient: the centre of $C[\tilde{\mathbb{Z}}_2^2]$ is generated by the centre of $\tilde{\mathbb{Z}}_2^2$ and hence consists of scalars. It follows that the subalgebra of $C[\tilde{\mathbb{Z}}_2^2 \times \tilde{\mathbb{Z}}_2^2]$ fixed pointwise by $\tilde{E}_\theta(2, \mathbb{Z})$ is $C[\tilde{\mathbb{Z}}_2^2 \times \tilde{\mathbb{Z}}_2^2 - \theta]$ consisting of elements $\Omega_{-\theta}$ obtained from eqn. (22) by replacing $p$ by $q$. On this we have the $\mathbb{Z}/4$ action given by eqn. (21) implying that elements of $C[\tilde{\mathbb{Z}}_2^2 \times \tilde{\mathbb{Z}}_2^2 - \theta]$ invariant under $\tilde{E}_\theta(2, \mathbb{Z})$ are of the form

$$\Omega_{-\theta} = \sum w_{j_1, j_2}(q_1^{j_1}q_2^{j_2} + q_1^{-j_1}q_2^{-j_2} + q_1^{-j_1}q_2^{j_2} + q_1^{j_1}q_2^{-j_2})$$

for arbitrary complex coefficients $w_{j_1, j_2}$.

The requirement that $\Omega_{-\theta}$ be selfadjoint when $q_1$ and $q_2$ are represented by unitary operators imposes the final condition

$$\sum w_{j_1, j_2}(q_1^{j_1}q_2^{j_2} + e^{-ij_1,j_2 \theta}q_1^{-j_2}q_2^{j_1} + q_1^{j_1}q_2^{-j_2} + e^{ij_1,j_2 \theta}q_1^{-j_2}q_2^{j_1})$$

$$= \sum w_{j_1, j_2}(e^{-ij_1,j_2 \theta}q_1^{-j_2}q_2^{-j_1} + q_1^{j_2}q_2^{-j_1} + e^{ij_1,j_2 \theta}q_1^{j_1}q_2^{j_2} + q_1^{-j_2}q_2^{j_1}),$$

where we have ordered factors using the commutator of eqn. (19). But, again, since $q_1$ and $q_2$ are free generators of $\tilde{\mathbb{Z}}_2^2$ modulo its centre, the above relation can hold only if the coefficient of every monomial $q_1^{j_1}q_2^{j_2}, j_1 \neq 0, j_2 \neq 0$, is the same on both sides. Hence we must have

$$w_{j_1, j_2} = \overline{w}_{j_1, j_2}e^{ij_1,j_2 \theta}, \quad \overline{w}_{j_1, j_2} = \overline{w}_{j_1, j_2}e^{-ij_1,j_2 \theta}$$

simultaneously for all $j_1, j_2 \neq 0$. This is possible only if $\exp (2i j_1,j_2 \theta) = 1$ whenever $w_{j_1, j_2} \neq 0$ and, since $\theta$ is irrational, we have $w_{j_1, j_2} = 0$ unless $j_1 = 0$.
or $j_2 = 0$. The surviving terms in $\Omega_{-\theta}$ are therefore

$$\Omega_{-\theta} = \sum_{j \in \mathbb{Z}^+} (w_j + \overline{w}_j)(q_j^1 + q_{1-j}^1 + q_j^2 + q_{2-j}^2).$$  \hspace{1cm} (23)$$

We have thus demonstrated

**4.4.** The most general selfadjoint $\tilde{E}_\theta(2,\mathbb{Z})$-invariant element of $\mathcal{C}[\mathbb{Z}_\theta^2 \times \mathbb{Z}_-\theta^2]$ is given by eqn.(22) with arbitrary real coefficients.

According to the remarks earlier in this section, all possible Hamiltonians are obtained by restricting the sum in eqn.(23) to a finite number of terms. The Harper Hamiltonian results on keeping only the nearest neighbour terms, $j = 1$, upto an additive constant $w_0 + \overline{w}_0$ of no significance and a multiplicative constant $w_1 + \overline{w}_1$ which is just a scale. We summarise:

**4.5.** For the Peierls electron on $\mathbb{Z}^2$ in irrational flux per plaquette $\Phi = \theta/2\pi$,

i) the state space has the structure $\mathcal{H}_\theta = V_\theta \otimes V_{-\theta}$, where $V_\theta$ is the distinguished irreducible representation of the almost Heisenberg group $\mathbb{Z}_\theta^2$

ii) the $\tilde{E}_\theta(2,\mathbb{Z})$-invariant Hamiltonian restricted to the nearest neighbour term is, upto an additive and a multiplicative constant, of the form $1 \otimes H_\theta$, where $H_\theta$ acting on $V_{-\theta}$ is the Harper Hamiltonian;

iii) consequently, every energy eigenvalue has $V_\theta$ as degeneracy subspace.

As for the Landau electron, the infinite degeneracy of energy levels is a direct reflection of translation and euclidean invariance.

**Acknowledgements**

Discussions with M.S. Raghunathan, S. Ramanan and R.R. Simha of the Tata Institute of Fundamental Research, Mumbai and V.S. Sunder of
the Institute of Mathematical Sciences, Chennai have been invaluable in the
course of this work. The School of Mathematics of TIFR is thanked for its
hospitality and the IMSc for the use of its facilities.

References

1. Peierls, R.E.: Z. Phys 80, 763 (1933).

2. Lieb, E. and Loss, M.: Duke Math. J. 71, 335 (1993).

3. Harper, P.G.: Proc. Phys. Soc. A 68, 874 (1995).

4. Divakaran, P.P.: Heisenberg Groups in the Theory of the Lattice Peierls
Electron: The Rational Flux Case (Paper in preparation).

5. Wigner, E.P.: *Group Theory and its Applications to the Quantum Me-
chanics of Atomic Spectra*, New York: Academic Press, 1959.

6. Bargmann, V.: Ann Math. 59, 1 (1954).

7. Divakaran, P P.: Rev. Math. Phys.6, 167 (1994).

8. Divakaran, P P.:Phys. Rev. Lett. 79, 2159 (1997).

9. Divakaran, P.P.: in preparation.

10. Divakaran, P.P and Rajagopal, A.K.: Int. J. Mod. Phys. B9, 261(1995).

11. Raghunathan, M.S.: Rev. Math. Phys. 6, 207 (1994).

12. Mumford, D.,Nori, M and Norman, P.:*Tata Lectures on Theta III*,
Boston, Basel, Berlin: Birkhäuser, 1991.
13. Boon, M.H.: J.Math. Phys. 13, 1268 (1972).

14. Bellissard, J.: in Operator Algebras and Applications, Vol. 2, eds. Evans, D.E. and Takesaki, M., Cambridge: Cambridge University Press, 1988.

15. Davidson, K.R.: $C^*$-Algebra by Example, Delhi: Hindustan Book Agency, 1996.

16. Prasad, G. and Raghunathan, M.S.: Invent. Math. 92, 645(1998).

17. Varadarajan, V.S.: Geometry of Quantum Theory, Vol II, New York: Van Nostrand Reinhold, 1970.

18. Hofstadter, D.R.: Phys. Rev. B 14, 2239(1976).