ABSTRACT. For primes $p \equiv 4, 11 \pmod{15}$ explicit formulae are obtained for the cyclotomic numbers of order 15 over $\mathbb{GF}(p^2)$.

KEY WORDS AND PHRASES. Cyclotomic numbers, Gauss sums, Jacobisums, Eisenstein sums.

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1. INTRODUCTION.

Let $e \geq 2$ and $e \geq 1$ be integers and let $p$ be an odd prime such that $e$ divides $p^e - 1$. We set $q = p^e$ and denote the finite field with $q$ elements by $\mathbb{GF}(q)$. The positive integer $f$ is defined by $q = ef + 1$. We fix once and for all a generator $\gamma$ of the multiplicative group $\mathbb{GF}(q)^* = \mathbb{GF}(q) - \{0\}$, and set $g = \gamma^{1+p+\ldots+p^{e-1}}$ so that $g$ is a primitive root modulo $p$. For $\alpha \in \mathbb{GF}(q)^*$ the index of $\alpha$ with respect to $\gamma$, denoted by $\ind_\gamma \alpha$, is the unique integer $m$ such that $\alpha = \gamma^m$, $0 \leq m \leq q - 2$, and for $\alpha \in \mathbb{GF}(p)^*$ the index of $\alpha$ with respect to $g$, denoted by $\ind_g \alpha$, is the unique integer $n$ such that $\alpha \equiv g^n \pmod{p}$, $0 \leq n \leq p - 2$. These indices are related by the following congruence: for $\alpha \in \mathbb{GF}(p)^*$ we have

$$\ind_\gamma \alpha \equiv (1 + p + \ldots + p^{e-1}) \ind_g \alpha \pmod{p^e - 1}. \quad (1.1)$$

The congruence (1.1) will be used many times throughout the paper.

The number of solutions $\alpha \in \mathbb{GF}(q)^+ = \mathbb{GF}(q)^* - \{1\}$ of the pair of congruences

$$\ind_\gamma (\alpha - 1) \equiv h \pmod{e},$$

$$\ind_\gamma \alpha \equiv k \pmod{e}, \quad (1.2)$$

are counted by cyclotomic numbers of order 15 over $\mathbb{GF}(p^2)$.
is denoted by \((h, k)_e\), where \(h\) and \(k\) are integers such that \(0 \leq h \leq e - 1\), \(0 \leq k \leq e - 1\). The numbers \((h, k)_e\) are called the cyclotomic numbers of order \(e\) over \(GF(q)\), and they depend on \(p\), \(e\), \(\gamma\), and \(\eta\). It is a central problem in the theory of cyclotomy to obtain precise formulae for these numbers. This has been done for a number of values of \(e\) and \(\gamma\)—for small values of \(e\) these formulae are given in [11] and references to further results are given, for example, in [9].

The purpose of this paper is to give the first complete determination of the cyclotomic numbers \((h, k)_e\) in the case \(e = 15\), \(\gamma = 2\), \(p = 4, 11 \pmod{15}\) (so that \(q = p^2 \equiv 1 \pmod{15}\)). The two basic properties of cyclotomic numbers (6.1), (6.2) show that the \(e^2 = 225\) cyclotomic numbers take on at most 46 values. As we have assumed that \(p \not\equiv 1 \pmod{15}\) the additional property \((ph, pk)_e = (h, k)_e\) reduces the maximum number of possible different values to 28 in the case \(p = 4 \pmod{15}\) and 29 in the case \(p = 11 \pmod{15}\). It should be remarked that in the remaining case for which \(p^2 \equiv 1 \pmod{15}\), namely, \(p = 4 \pmod{15}\), the phenomenon of uniform cyclotomy occurs (see [2, Theorem I]) and there are just 3 different cyclotomic numbers.

In the case \(p = 4 \pmod{15}\), the 28 cyclotomic numbers are expressed (see Tables 13a, 13b, 13c) as linear combinations of \(p, 1, A^2, AB, B^2, AT, BT, AU, BU, T^2 - 15U^2, TU\), where the integers \(A, B, T, U\) are defined in Theorem 1 and satisfy
\[
p = A^2 - AB + B^2, \quad A \equiv -1 \pmod{3}, \quad B \equiv 0 \pmod{3}, \quad p = T^2 + 15U^2, \quad T \equiv -1 \pmod{3}.
\]

It should be noted that (1.3) alone does not distinguish between the two solutions \((A, B)\) and \((A - B, -B)\), while (1.4) determines \(T\) uniquely but determines \(U\) only up to sign.

For the case \(p = 11 \pmod{15}\), the 29 cyclotomic numbers are expressed (see Tables 14a - 14g) as linear combinations of \(p, 1, X, Y, U^2, V, W^2, VW, W^2\), where the integers \(X, U, V, W\) are defined in Theorem 2 and satisfy
\[
\begin{align*}
p &= X^2 + 5U^2 + 5V^2 + 5W^2, \quad X \equiv -1 \pmod{5}, \\
XW &= V^2 - UV - U^2.
\end{align*}
\]

Again, it should be noted that (1.5) alone does not distinguish between the four solutions \((X, U, V, W), (X, V, -U, -W), (X, -U, -V, W), (X, -V, U, -W)\) (see [10]).

As an application of the formulae for the cyclotomic numbers in the case \(p = 4 \pmod{15}\), we prove the following theorem in §7.

**THEOREM 8.** Let \(p = 4 \pmod{15}\) be a prime. Let \(\gamma\) be a generator of \(GF(p^2)\). Set \(g = \gamma^{p+1}\), so that \(g\) is a primitive root \(\pmod{p}\). The quantity
\[
\gamma^{(q-1)/5} - \gamma^{2(q-1)/5} - \gamma^{3(q-1)/5} + \gamma^{4(q-1)/5} \in GF(p)\] and a unique square root of \(5\) modulo \(p\) is given by
\[
\sqrt{5} \equiv \gamma^{(q-1)/5} - \gamma^{2(q-1)/5} - \gamma^{3(q-1)/5} + \gamma^{4(q-1)/5} \pmod{p}.
\]

Then we have
\[
\text{ind}_g (1 + \sqrt{5}) \equiv -U \pmod{3},
\]
where \(U\) is given by Theorem 1.

In particular, we see that \(\sqrt{5} \equiv 1 - \sqrt{5}\) (equivalently \(\sqrt{1 - \sqrt{5}}\)) is a cube modulo the prime \(p = 4 \pmod{15}\), if and only if, \(U \equiv 0 \pmod{3}\), where \(U\) is given by (1.4). A sketch of a proof of this, using classfield theory, has been given by Aigner [1]. This result complements that of Emma Lehmer (see [9: Theorem 2]) for the case \(p \equiv 1 \pmod{15}\).
2. BASIC PROPERTIES OF JACOBI, GAUSS AND EISENSTEIN SUMS.

With the notation of §1, we define the Jacobi sum $J_q(\beta^m, \beta^n)$ over $\text{GF}(q)$ of order $e$ for integers $m$ and $n$ by

$$J_q(\beta^m, \beta^n) = \sum_{\alpha \in \text{GF}(q)^*} \beta^m \text{ind} \gamma \alpha + n \text{ind} \gamma (1-\alpha), \quad (2.1)$$

where $\beta = \exp(2\pi i/e)$. These sums have the following well-known properties (see, for example, [11: Lemmas 13-16]):

$$J_q(\beta^m, \beta^n) = J_q(\beta^n, \beta^m) \equiv (-1)^{nf} J_q(\beta^{-m-n}, \beta^n). \quad (2.2)$$

$$J_q(\beta^m, \beta^n) = -1 \text{, if } e \nmid m, e \nmid n, \text{ or } e \mid m, e \nmid n, \quad (2.3)$$

$$J_q(\beta^m, \beta^n) = q - 2 \text{, if } e \mid m, e \mid n, \quad (2.4)$$

$$J_q(\beta^m, \beta^n) = (-1)^{nf+1}, \text{ if } e \mid m, e \nmid n, e \mid m + n, \quad (2.5)$$

$$J_q(\beta^{dm}, \beta^{dn}) = J_q(\beta^m, \beta^n). \quad (2.6)$$

If $(j, e) = 1$, the mapping which sends $\beta$ to $\beta^j$ is an automorphism of the cyclotomic field $\text{Q}(\beta)$, which we shall denote by $\sigma_j$. We have

$$\sigma_j(J_q(\beta^m, \beta^n)) = J_q(\beta^{jm}, \beta^{jn}). \quad (2.7)$$

Closely related to the Jacobi sum is the Gauss sum $G_q(\beta^m)$, defined for any integer $m$ by

$$G_q(\beta^m) = \sum_{\alpha \in \text{GF}(q)^*} \beta^m \text{ind} \gamma \alpha, \quad (2.8)$$

where $\text{tr} \alpha$ denotes the trace of $\alpha$ from $\text{GF}(q)$ to $\text{GF}(p)$, that is

$$\text{tr} \alpha = \alpha + \alpha^p + \alpha^{p^2} + \ldots + \alpha^{(p-1)},$$

and $\zeta = \exp(2\pi i/p)$. Gauss sums have the following properties (see for example [8: pp. 3-5] and [3: Theorem 2.12]):

$$G_q(\beta^m) = -1, e \mid m, \quad (2.9)$$

$$G_{p^2}(\beta^m) = p(-1)^{(p + 1)/e'} , \text{ if } e \mid m \text{ and } e' \mid p + 1, \quad (2.10)$$

where we have written $e'$ for $e/(e, m)$,

$$G_q(\beta^m) G_q(\beta^m) = (-1)^{mf} q, \text{ if } e \mid m, \quad (2.11)$$

$$G_q(\beta^{dm}) = G_q(\beta^m). \quad (2.12)$$

In addition, these sums satisfy the Davenport-Hasse relations [4: eqns. (0.8) and (0.9)] (see, for example, [8: p. 20, Theorem 5.1] and [3: §8]):

$$G_q(\beta^{m(q-1)/(p-1)}) = (-1)^{q-1} G(\beta^m), \quad (2.13)$$

where the Gauss sum $G(\beta^m)$ is defined in (2.18), and for $e = xy$ and an arbitrary integer $t$

$$G_q(\beta^{ty}) = \prod_{k=1}^{\text{v}-1} G_q(\beta^{kx}) = \beta^{ty} \prod_{k=1}^{\text{v}-1} G_q(\beta^{kx} + t). \quad (2.14)$$
The basic relationship between Gauss and Jacobi sums is (see, for example, [3: Theorem 2.2].)

\[ J_q(\beta^m, \beta^n) = \frac{G_q(\beta^m) G_q(\beta^n)}{G_q(\beta^{m+n})} \text{, if } e \not\mid m \text{, } e \not\mid n \text{, } e \not\mid m + n \] \hspace{1cm} (2.16)

We also define the Jacobi sum

\[ J(\beta^m, \beta^n) = \sum_{a \in \mathbb{GF}(p)^*} \zeta^m \text{ ind}_{\gamma} a + n \text{ ind}_{\gamma} (1 - a) \] \hspace{1cm} (2.17)

and the Gauss sum

\[ G(\beta^m) = \sum_{a \in \mathbb{GF}(p)^*} \zeta^m \text{ ind}_{\gamma} a \] \hspace{1cm} (2.18)

We note that if \( e = 1 \) (so that \( q = p \)), then \( J_q(\beta^m, \beta^n) = J(\beta^m, \beta^n) = G_q(\beta^m) = G(\beta^m) \). The sums \( J(\beta^m, \beta^n) \) and \( G(\beta^m) \) also have the properties (2.2), (2.3) with the right hand side replaced by \( p \), (2.4), (2.5) with the right hand side replaced by \( p - 2 \), (2.6), (2.7), (2.8), (2.10), (2.12) with the right hand side replaced by \((-1)^m p\), (2.13), and (2.16).

Next, we define the Eisenstein sums \( E(\beta^m) \) over \( \mathbb{GF}(p^2) \). We set

\[ E(\beta^m) = \sum_{b=0}^{p-1} \beta^m \text{ ind}_{\gamma} (1+bx) \] \hspace{1cm} (2.19)

where \( x \) is any element of \( \mathbb{GF}(p^2) \), such that

\[ \mathbb{GF}(p^2) = \{ a + bx | a, b = 0, 1, 2, \ldots, p - 1 \} \] \hspace{1cm} (2.20)

and \( x^2 \in \mathbb{GF}(p) \). (For example, we could take \( x = \gamma^{(p+1)/2} \).) Clearly, \( E(\beta^m) \) is independent of the choice of \( x \) as \( bx(1 \leq b \leq p - 1) \) runs twice through the square roots of the quadratic non-residues of \( \mathbb{GF}(p) \). Eisenstein sums have the properties (see, for example, [3: pp. 377-381])

\[ E(\beta^m) = p, \text{ if } e \not\mid m, \] \hspace{1cm} (2.21)

\[ E(\beta^m) = -(-1)^{\frac{p+1}{e'}} e', \text{ if } e \not\mid m \text{ and } e' \mid p + 1, \] \hspace{1cm} (2.22)

\[ E(\beta^m) E(\beta^{-m}) = p, \text{ if } e' \not\mid p + 1, \] \hspace{1cm} (2.23)

\[ E(\beta^m) = E(\beta^m), \] \hspace{1cm} (2.24)

\[ \sigma_j(E(\beta^m)) = E(\beta^{jm}), \text{ if } (j, e) = 1, \] \hspace{1cm} (2.25)

where, again, we have written \( e' \) for \( e/(e, m) \).

Finally, we remark that the Eisenstein sums are related to the Gauss sums as follows (see, for example, [3: Theorem 2.23]):

\[ E(\beta^m) = \beta^{m \text{ ind}_{\gamma} 2} G^2(\beta^m)/G(\beta^m), \text{ if } e' \not\mid p + 1. \] \hspace{1cm} (2.26)

3. DETERMINATION OF EISENSTEIN SUMS OVER \( \mathbb{GF}(p^2) \) FOR \( e = 15, p \equiv 4, 11 \pmod{15} \).

Throughout §3 and §4, we take \( e = 15, \gamma = 2, q = p^2 \) and \( p \equiv 4, 11 \pmod{15} \). We begin by determining the Eisenstein sums \( E(\beta^m) \) (\( m = 1, 2, \ldots, 14 \)) in the case \( p \equiv 4 \pmod{15} \).

**Theorem 1.** Let \( p \equiv 4 \pmod{15} \) be a prime. Let \( \gamma \) be a generator of \( \mathbb{GF}(p^2)^* \). Set \( g = \gamma^{p+1} \), \( \beta = \exp(2\pi i/15) \), and \( \omega = \beta^5 = \exp(2\pi i/3) = (-1 + \sqrt{-3}) \). Then

\[ \text{ind}_{10}^g \omega E(\beta) \] \hspace{1cm} (2.26)

\( E(\beta) \) is an integer of \( \mathbb{Q}(\sqrt{-15}) \) of the form \( T + U\sqrt{-15} \), where \( T \) and \( U \) are
rational integers, and we have

\[ E(\beta) = \frac{E(\beta^4)}{E(\beta^1)} = \frac{E(\beta^{14})}{E(\beta^1)} = \omega^{\text{ind}_g 10} (T + U\sqrt{-15}), \]  
\[ E(\beta^2) = \frac{E(\beta^4)}{E(\beta^2)} = \frac{E(\beta^{14})}{E(\beta^2)} = -\omega^{\text{ind}_g 10} (T + U\sqrt{-15}), \]  
\[ E(\beta^3) = \frac{E(\beta^6)}{E(\beta^3)} = \frac{E(\beta^9)}{E(\beta^3)} = E(\beta^{12}) = -1, \]  
\[ E(\beta^5) = \frac{E(\beta^{10})}{E(\beta^5)} = -\omega^{\text{ind}_g 2} (A + B\omega), \]  

where

\[ J(\beta, \beta) = A + B\omega. \]  

Moreover, A, B, T, U satisfy (1.3), (1.4), as well as the congruences given in the following chart.

| ind g 5 | A (mod 3) | B (mod 5) |
|---------|-----------|-----------|
| 0       | -T        | 0         |
| 1       | T         | 1         |
| 2       | 0         | -T        |

**PROOF.** From (1.1) we have for \( a \in \text{GF}(p)* \)

\[ \text{ind}_g a \equiv (1 + p) \text{ind}_g a \pmod{p^2 - 1}, \]  

so that

\[ \text{ind}_g a \equiv -\text{ind}_g a \pmod{3}, \]  

and

\[ \text{ind}_g a \equiv 0 \pmod{5}. \]  

From (2.18) and (3.8) we see that

\[ G(\beta^m) = G(\beta^n), \text{ if } m \equiv n \pmod{3}. \]  

We begin by proving (3.4). We have

\[ E(\beta^5) = \omega^{\text{ind}_g 2} G_q(\beta^5)/G(\beta^5) \]  
(by (2.26))

\[ = \omega^{\text{ind}_g 2} G_q(\beta^5)/G(\beta^2) \]  
(by (3.7) and (3.9))

\[ = -\omega^{\text{ind}_g 2} (G(\beta))^2/G(\beta^2) \]  
(by (2.14))

\[ = -\omega^{\text{ind}_g 2} J(\beta, \beta) \]  
(by (2.16))

\[ = -\omega^{\text{ind}_g 2} (A + B\omega), \]

where A and B satisfy (1.3) (see, for example, [7: Prop. 8.3.4]). Appealing to (2.25), we obtain \( E(\beta^{10}) = E(\beta^5) \), which completes the proof of (3.4).

We note that (3.3) follows from (2.22), and in view of (2.24) and (2.25), to complete the proof it suffices to prove

\[ E(\beta^2) = \omega^{\text{ind}_g 10} E(\beta) \]  
(3.10)

and

\[ E(\beta) = \omega^{\text{ind}_g 10} (T + U\sqrt{-15}). \]  
(3.11)

First, we prove (3.10). Taking \( t = 1, x = 3, y = 5 \) in (2.15) and appealing to
(2.12), we obtain
\[ q^2 G_q(\beta^5) = \omega \ \ \ \text{ind}_5 \ \ \ \ Y \ G_q(\beta) \ G_q(\beta^4) \ G_q(\beta^7) \ G_q(\beta^{10}) \ G_q(\beta^{13}) . \]

Applying (2.26) and (3.9), we have
\[ \text{ind}_5 \ \ \ \ G(\beta) \ E(\beta) = (G(\beta)) \ G(\beta^4) \ E(\beta^7) \ E(\beta^{10}) \ E(\beta^{13}) . \]

Multiplying both sides by \( G(\beta) \) and appealing to (2.12), (2.23), and (2.24), we obtain
\[ 2 \ \ \ \text{ind}_5 \ \ \ \ G(\beta) \ E(\beta)^2 = p^2 \ (E(\beta^2))^2 \ (E(\beta^5))^2 , \]
giving
\[ \text{ind}_5 \ \ \ \ G(\beta)^3 \ E(\beta) = \pm \ p \ E(\beta^2) \ E(\beta^5) . \]  

As
\[ G(\beta)^3 = J(\beta, \beta) \ G(\beta^2) \ G(\beta) = p(A + B\omega) , \]
we obtain from (3.4) and (3.12)
\[ \text{ind}_5 \ \ \ \ E(\beta^6) = \omega \ \ \ \text{ind}_5 \ \ \ \ E(\beta) . \]  

We now determine which sign holds in (3.13). Cubing both sides of (3.13), we obtain
\[ E(\beta^6)^3 = \pm \ E(\beta)^3 , \]
and so
\[ E(\beta^6) \equiv \pm \ E(\beta^3) \quad (\text{mod} \ 3) . \]  

Since \( E(\beta^3) = E(\beta^6) = -1 \), we must have the + sign holding in (3.14), giving (3.10).

Raising (3.10) to the 5th power and reducing modulo 5, we obtain
\[ E(\beta)^{10} \equiv (E(\beta^2))^5 \equiv \omega \ \ \ \text{ind}_5 \ \ \ \ E(\beta)^{10} \equiv \omega \ \ \ \text{ind}_5 \ \ \ \ E(\beta^5) \quad (\text{mod} \ 5) . \]

Then, appealing to (3.4), we obtain
\[ A + B\omega^2 \equiv \omega \ \ \ \text{ind}_5 \ \ \ \ (A + B\omega) \quad (\text{mod} \ 5) , \]
which gives
\[ B \equiv 0 \quad (\text{mod} \ 5) , \text{ if ind}_5 \equiv 0 \quad (\text{mod} \ 3) , \]
\[ A \equiv B \quad (\text{mod} \ 5) , \text{ if ind}_5 \equiv 1 \quad (\text{mod} \ 3) , \]
\[ A \equiv 0 \quad (\text{mod} \ 5) , \text{ if ind}_5 \equiv 2 \quad (\text{mod} \ 3) . \]  

Next, we prove (3.11). We have
\[ \text{ind}_5 \ \ \ \ E(\beta) = \omega \ \ \ \text{ind}_5 \ \ \ \ E(\beta^6) = \omega \ \ \ \text{ind}_5 \ \ \ \ E(\beta) . \]

From this equation, we see that \( \omega \ \ \ \text{ind}_5 \ \ \ \ E(\beta) \) is invariant under the automorphism \( \sigma_2 \) so that \( \omega \ \ \ \text{ind}_5 \ \ \ \ E(\beta) \) is an integer of \( Q(\sqrt{-15}) \). Hence, we have
\[ \text{ind}_5 \ \ \ \ E(\beta) = \frac{1}{2}(t + u\sqrt{-15}) , \ t \equiv u \quad (\text{mod} \ 2) , \]
where \( t \) and \( u \) are integers such that \( 4p = t^2 + 15u^2 \). Clearly, \( t \) and \( u \) cannot both be odd, so \( t = 2T, u = 2U \), and
\[ \text{ind}_5 \ \ \ \ E(\beta) = T + U\sqrt{-15} , \]
where \( p = T^2 + 15U^2 \). This proves (3.11).
We now show that $T \equiv -1 \pmod{3}$. Cubing (3.11), we obtain by (3.3)

$$T \equiv T^2 \equiv E(\beta)^3 \equiv E(\beta^3) \equiv -1 \pmod{3}.$$  

Finally, we prove (3.6). Raising (3.11) to the fifth power, we obtain by (3.4)

$$\text{ind}_{g^5}^2 (A + B\omega) \equiv E(\beta^5) \equiv E(\beta)^5 \equiv \omega^{-\text{ind}_{g^10}^5} T^5 \equiv \omega^{-\text{ind}_{g^10}^5} T \pmod{5},$$

that is

$$\text{ind}_{g^5}^T (A + B\omega) \pmod{5},$$

which together with (3.15) gives (3.6).

This completes the proof of Theorem 1.

Next we determine the Eisenstein sums $E(\beta^m)$ ($1 \leq m \leq 14$) in the case $p \equiv 11 \pmod{15}$.

**Theorem 2.** Let $p \equiv 11 \pmod{15}$ be a prime. Let $\gamma$ be a generator of $\text{GF}(p^2)^\times$.

Set $g = \gamma, \beta = \exp(2\pi i/15)$, and $\theta = \beta^3 = \exp(2\pi i/5) = (5^{\frac{1}{2}} - 1 + i(10+2(5)^{1/2}))/4$. Then

$$E(\beta) = \sum_{i=0}^{4} \text{ind}_{g^2}^T \text{ind}_{g^3}^T \sum_{i=0}^{4} t_i \theta^i,$$  

(3.16)

$$E(\beta^2) = \sum_{i=0}^{4} \text{ind}_{g^2}^T \sum_{i=0}^{4} t_i \theta^i,$$  

(3.17)

$$E(\beta^3) = -\sum_{i=0}^{4} t_i \theta^i,$$  

(3.18)

$$E(\beta^5) = -\sum_{i=0}^{4} t_i \theta^i,$$  

(3.19)

$$E(\beta^6) = -\sum_{i=0}^{4} t_i \theta^i,$$  

(3.20)

where the integers $t_i$ ($0 \leq i \leq 4$) satisfy

$$\sum_{i=0}^{4} t_i = -1$$

and are of the form

$$t_0 = \frac{1}{5} (4X - 1),$$  

(3.21)

$$t_1 = \frac{1}{5} (-1 - X) + U + V + W,$$  

(3.22)

$$t_2 = \frac{1}{5} (-1 - X) + U - V - W,$$  

(3.23)

$$t_3 = \frac{1}{5} (-1 - X) - U + V - W,$$  

(3.24)

$$t_4 = \frac{1}{5} (-1 - X) - U - V + W,$$  

(3.25)

where $(X, U, V, W)$ is a solution of

$$P = X^2 + 5U^2 + 5V^2 + 5W^2, X \equiv -1 \pmod{5},$$  

$$XW = V^2 - UV - U^2.$$  

(3.26)

Moreover, we have

$$\sum_{i=0}^{4} t_i \theta^i = X + U(i(5 + 2(5)^{1/2}) + V(i(5 - 2(5)^{1/2}) + W(5)^{1/2})$$
and
\[ 4 \sum_{i=0}^1 t_i \theta^{2i} = X + V'i(5 + 2(5)^{1/2}) - V'i(5 - 2(5)^{1/2}) - W(5)^{1/2} . \]

**Proof.** From (1.1), we have for \( a \in GF(p)^* \)
\[ \text{ind}_Y a = (1 + p) \text{ind}_g a \pmod{p^2 - 1} , \]
so that
\[ \text{ind}_Y a \equiv 0 \pmod{3} \] (3.27)
and
\[ \text{ind}_Y a \equiv 2 \text{ ind}_g a \pmod{5} . \] (3.28)
From (2.18) and (3.27) we see that
\[ G(\beta^m) = G(\beta^n) , \text{ if } m \equiv n \pmod{5} . \] (3.29)
We set
\[ A = G(\beta) , B = G(\beta^2) . \] (3.30)
From (3.29) and (3.30) we have
\[ G(\beta) = G(\beta^5) = G(\beta^{11}) = A . \] (3.31)
\[ G(\beta^2) = G(\beta^7) = G(\beta^{12}) = B . \]
Next appealing to (2.12), (3.29) and (3.31), we have
\[ G(\beta^3) = G(\beta^8) = G(\beta^{13}) = p/B , \] (3.32)
\[ G(\beta^4) = G(\beta^9) = G(\beta^{14}) = p/A . \] (3.33)
Also, from (2.18) and (3.27) we have
\[ G(1) = G(\beta^5) = G(\beta^{10}) = -1 . \] (3.34)
Next, we determine the values of the Gauss sums \( G_q(\beta^i) \) \((0 \leq i \leq 14) \) in terms of \( A, B, p \) and \( g \). Taking \( m = 4n \) in (2.14), we obtain
\[ G_q(\beta^{3n}) = - (G(\beta^{4n}))^2 . \] (3.35)
Thus, taking \( n = 1, 2, 3, 4 \) and appealing to (3.31), (3.32), (3.33), we obtain
\[ G_q(\beta^3) = - p^2/A^2 , \] (3.36)
\[ G_q(\beta^6) = - p^2/B^2 , \] (3.37)
\[ G_q(\beta^9) = - B^2 , \] (3.38)
\[ G_q(\beta^{12}) = - A^2 . \] (3.39)
Next, taking \( m = 5, 10 \) in (2.11), we obtain
\[ G_q(\beta^5) = G_q(\beta^{10}) = p . \] (3.40)
Also, taking \( m = 0 \) in (2.10), we have
\[ G_q(1) = - 1 . \] (3.41)
Further, from (2.13), we have
\[ G_q(\beta^{11m}) = G_q(\beta^m) . \] (3.42)
From the Davenport-Hasse relation (2.15) with \( x = 5, y = 3, t = 1 \), we obtain (appealing to (3.40))
\[ G_q(\beta^3) p^2 = 3 \text{ ind}_{\beta}^3 G_q(\beta) G_q(\beta^6) G_q(\beta^{11}) . \] (3.43)
and with \( x = 5, y = 3, t = 2 \), we have
\[ G_q(\beta^6) p^2 = 6 \text{ ind}_{\beta}^3 G_q(\beta^2) G_q(\beta^7) G_q(\beta^{12}) . \] (3.44)
Appealing to (3.36), (3.37), (3.42), with \( m = 1 \), the equation (3.43) becomes
\[ G_q(\beta)^2 = - 2 \text{ ind}_{\beta}^3 p^2 B^2/A^2 . \] (3.45)
and appealing to (3.37), (3.39), (3.42) with \( m = 2 \), the equation (3.44) becomes

\[ G_q(\beta^2)^2 = \theta^{-4} \text{ind}_{3}^{3} p/A^2 B^2 . \]  

(3.46)

Taking square roots in (3.45) and (3.46), we have

\[ G_q(\beta) = \pm \theta \text{ind}_{3}^{3} p B/A , \]  

(3.47)

\[ G_q(\beta^2) = \pm \theta^{-2} \text{ind}_{3}^{3} p^2 / A B . \]  

(3.48)

Raising both sides of (3.47) and (3.48) to the fifth power and reducing modulo 5, we see that the + signs must hold in both equations; that is,

\[ G_q(\beta) = \text{ind}_{3}^{3} p B/A , \]  

(3.49)

\[ G_q(\beta^2) = \theta^{-2} \text{ind}_{3}^{3} p^2 / A B . \]  

(3.50)

The values of the remaining \( G_q(\beta^i) \) follow immediately from (3.49) and (3.50) as

\[ G_q(\beta) = \overline{G_q(\beta^4)} = \overline{G_q(\beta^{11})} = \overline{G_q(\beta^{14})} \]  

(3.51)

\[ G_q(\beta^2) = \overline{G_q(\beta^7)} = \overline{G_q(\beta^{8})} = \overline{G_q(\beta^{14})} . \]  

(3.52)

We now appeal to (2.22), (2.26) and the above formulae for the \( G(\beta^i) \) and \( G_q(\beta^i) \) to obtain

\[ E(\beta) = \overline{E(\beta^4)} = \overline{E(\beta^{11})} = \overline{E(\beta^{14})} = - \theta \text{ind}_6^6 p B / A^2 , \]  

(3.53)

\[ E(\beta^2) = \overline{E(\beta^7)} = \overline{E(\beta^{8})} = \overline{E(\beta^{13})} = - \theta \text{ind}_6^6 p^2 / A B^2 , \]  

(3.54)

\[ E(\beta^3) = \overline{E(\beta^{12})} = - \theta \text{ind}_2^2 p B / A^2 , \]  

(3.55)

\[ E(\beta^5) = \overline{E(\beta^{10})} = -1 , \]  

(3.56)

\[ E(\beta^6) = \overline{E(\beta^9)} = - \theta \text{ind}_2^2 p^2 / A B^2 . \]  

(3.57)

We now examine \( E(\beta^3) \). We have

\[ 2 \text{ind}_2^2 \theta \text{ind}_6^6 p B / A^2 \]  

\[ = - \theta \text{ind}_2^2 G(\beta^4)^2 / G(\beta^3) \]  

\[ = - \theta \text{ind}_9^9 \Sigma_{\beta} g(\beta^4, \beta^4) \]  

(by (2.16) and (3.29))

\[ = - \theta \text{ind}_9^9 \Sigma_{\beta} g(p - 1) 4 \text{ind}_{a + 4 \text{ind}_y(1 - a)} \]  

\[ = - \theta \text{ind}_9^9 \Sigma_{\beta} g(p - 1) \text{ind}_{a + \text{ind}_y(1 - a)} \]  

\[ = - \theta \text{ind}_9^9 \text{ind}_{4 \Sigma_{\beta} g(\beta^4, \beta^4)} \]  

\[ = - \theta \text{ind}_9^9 \text{ind}_{4 \Sigma_{\beta} g(1, 1)} \]  

(in the notation of [5: eqn. (3.1)])

\[ = - \theta \text{ind}_9^9 \Sigma_{\beta} g(1, 1) \theta^{4} \]  

[5: p. 345]
\[ \sum_{j=0}^{4} B_5(j - 2 \text{ ind}_g 2, 1) \theta^j = \sum_{j=0}^{4} \left( B_5(j - 2 \text{ ind}_g 2, 1) - \frac{1}{5}(p - 1) \right) \theta^j = \sum_{j=0}^{4} t_j \theta^j, \]

where (see [5: eqn. (9.5)]) for \(0 \leq j \leq 4\)

\[ t_j = B_5(j - 2 \text{ ind}_g 2, 1) - \frac{1}{5}(p - 1). \tag{3.58} \]

From [5: eqn. (9.6)], we have

\[ \sum_{j=0}^{4} t_j = -1, \tag{3.59} \]

and [5: Theorem 8]

\[ 5t_0 + 1 = 4X, \quad t_1 + t_2 + t_3 + t_4 = 4U, \quad \]

\[ t_1 - t_2 + t_3 - t_4 = 4V, \quad t_1 - t_2 - t_3 + t_4 = 4W, \tag{3.60} \]

where \(X, U, V, W\) are integers satisfying (3.26). Solving (3.59) and (3.60) for \(t_0, t_1, t_2, t_3, t_4\), we obtain (3.21) - (3.25). This completes the proof of (3.18).

From (3.53) and (3.5b), we have

\[ E(\delta) = -\theta^{2 \text{ ind}_g 2} - \theta^{3 \text{ ind}_g 3} E(\theta^3), \]

which together with (3.18), gives (3.16).

Applying the automorphism \(\sigma_2\) to (3.16), we obtain (3.17), and applying the same automorphism to (3.18), we obtain (3.20).

Finally, we note that

\[ \sum_{i=0}^{4} t_i \theta^i = \sum_{i=1}^{4} (t_i - t_0) \theta^i = (-X + U + V + W)\theta^2 + (-X - U - V - W)\theta^3 + (-X - U - V + W)\theta^4 \]

\[ = X(-\theta^2 - \theta^3 - \theta^4) + U(\theta^2 - \theta^3 - \theta^4) + V(\theta^2 + \theta^3 - \theta^4) + W(\theta^2 - \theta^3 + \theta^4), \]

that is

\[ \sum_{i=0}^{4} t_i \theta^i = X + Ui(5 + 2(5)\frac{1}{2}) + Vi(5 - 2(5)\frac{1}{2}) + Wi(5 + 2(5)\frac{3}{2}) + W(5)\frac{1}{2}, \tag{3.61} \]

as required. Applying \(\sigma_2\) to (3.61), we obtain

\[ \sum_{i=0}^{4} t_i \theta^2i = X + Vi(5 + 2(5)\frac{1}{2}) - Ui(5 - 2(5)\frac{1}{2}) - W(5)\frac{1}{2}, \tag{3.62} \]

as

\[ \sigma_2(i(5 + 2(5)\frac{1}{2})) = \sigma_2(\theta + \theta^2 - \theta^3 - \theta^4) = \theta^2 + \theta^4 - \theta^6 - \theta^8 \]

\[ = -\theta^2 - \theta^3 + \theta^4 = -i(5 - 2(5)\frac{1}{2}), \]

\[ \sigma_2(i(5 - 2(5)\frac{1}{2})) = i(5 + 2(5)\frac{3}{2}), \]

and

\[ \sigma_2((5)\frac{1}{2}) = -(5)\frac{1}{2}. \]

This completes the proof of Theorem 2.
4. **DETERMINATION OF JACOBI SUMS FOR \( \epsilon = 15 \) AND \( p \equiv 4, 11 \pmod{15} \).**

The Jacobi sums of order 15 over \( \text{GF}(p^2) \) are conjugate to \( \text{J}_q(\beta^m, \beta) \) \((1 \leq m \leq 5)\), \( \text{J}_q(\beta^3, \beta^3) \) and \( \text{J}_q(\beta^5, \beta^5) \). We first evaluate these for \( p \equiv 4 \pmod{15} \). We prove

**THEOREM 3.** For \( p \equiv 4 \pmod{15} \), with the notation of Theorem 1, we have

\[
\begin{align*}
\text{J}_q(\beta, \beta) &= (A + Bw) (T + U\sqrt{-15}) , \\
\text{J}_q(\beta^2, \beta) &= (T + U\sqrt{-15})^2 , \\
\text{J}_q(\beta^3, \beta) &= p , \\
\text{J}_q(\beta^4, \beta) &= -\text{ind}_5 \left( T + U\sqrt{-15} \right)^2 , \\
\text{J}_q(\beta^5, \beta) &= -\omega^{-\text{ind}_5} \left( A + Bw \right) \left( T + U\sqrt{-15} \right) , \\
\text{J}_q(\beta^3, \beta^3) &= p , \\
\text{J}_q(\beta^5, \beta^5) &= -\left( A + Bw \right)^2 .
\end{align*}
\]

**PROOF.** From Theorem 1, we have

\[
\begin{align*}
\text{G}_q(\beta) &= \text{G}_q(\beta^4) = \text{G}_q(\beta^{11}) = \text{G}_q(\beta^{14}) = \omega^{\text{ind}_5} \left( T + U\sqrt{-15} \right) \text{G}(\beta) , \\
\text{G}_q(\beta^2) &= \text{G}_q(\beta^7) = \text{G}_q(\beta^{10}) = \omega^{-\text{ind}_5} \left( T + U\sqrt{-15} \right) \text{G}(\beta^2) , \\
\text{G}_q(\beta^3) &= \text{G}_q(\beta^6) = \text{G}_q(\beta^9) = \text{G}(\beta^{12}) = p , \\
\text{G}_q(\beta^5) &= \text{G}_q(\beta^{10}) = -\text{G}(\beta^2) ,
\end{align*}
\]

and

\[
\text{G}(\beta)^3 = p(A + Bw) , \quad \text{G}(\beta^2)^3 = p(A + Bw^2) , \quad \text{G}(\beta) \text{G}(\beta^2) = p .
\]

The values of the Jacobi sums (4.1) - (4.7) follow from (4.8) - (4.12) and (2.16). This completes the proof of Theorem 3.

Next, we evaluate the required Jacobi sums in the case \( p \equiv 11 \pmod{15} \). We prove

**THEOREM 4.** For \( p \equiv 11 \pmod{15} \), with the notation of Theorem 2, we have

\[
\begin{align*}
\text{J}_q(\beta, \beta) &= e^{2 \text{ind}_2 \frac{g^2}{4}} \left( \sum_{i=0}^{4} \frac{t_i^4}{j=0} \right) \left( \sum_{j=0}^{3j} \right) , \\
\text{J}_q(\beta^2, \beta) &= e^{-\text{ind}_3 \frac{g^3}{p}} , \\
\text{J}_q(\beta^3, \beta) &= -e^{-\text{ind}_3 \frac{g^3}{2}} \left( \sum_{i=0}^{4} \frac{t_i^4}{j=0} \right)^2 , \\
\text{J}_q(\beta^4, \beta) &= p , \\
\text{J}_q(\beta^5, \beta) &= -e^{-\text{ind}_3 \frac{g^3}{4}} \left( \sum_{i=0}^{4} \frac{t_i^4}{j=0} \right)^2 , \\
\text{J}_q(\beta^3, \beta^3) &= -e^{-\text{ind}_3 \frac{g^3}{2}} \left( \sum_{i=0}^{4} \frac{t_i^4}{j=0} \right)^2 , \\
\text{J}_q(\beta^5, \beta^5) &= p .
\end{align*}
\]
PROOF. Theorem 4 follows from (2.16), (3.36) - (3.40), (3.49) - (3.57) and Theorem 2.

5. DETERMINATION OF DICKSON-HURWITZ SUMS FOR e = 15 AND p \equiv 4, 11 (mod 15).

For any integers \( i \) and \( v \), the Dickson-Hurwitz sum \( B(e,i,v) \) is defined by

\[
B(e,i,v) = \sum_{h=0}^{e-1} (h, e - vh) e, \tag{5.1}
\]

where the cyclotomic number \( (h, k)_e \) was defined in (1.2).

Jacobi sums and Dickson-Hurwitz sums are related by the formula

\[
J_q(\beta^v, \beta^n) = (-1)^{n-1} \sum_{i=0}^{e-1} B(e,i,v) \beta^{ni}. \tag{5.2}
\]

The Dickson-Hurwitz sums have the well-known properties

\[
B(e,i,v) = B(e,i,e - v - 1) \tag{5.3}
\]

and

\[
B(e,i,0) = B(e,i,e - 1) = \begin{cases} f - 1, & \text{if } e \mid i \\ f, & \text{if } e \nmid i \end{cases} \tag{5.4}
\]

Taking \( n = 0 \) in (5.2), we obtain

\[
\sum_{i=0}^{e-1} B(e,i,v) = q - 2. \tag{5.5}
\]

In addition, Whiteman [12: Lemma 1] showed in the case \( e = 1 \) that if \( (v,e) = 1 \)

\[
B(e,i,v) = B(iv^{-1}, v^{-1}) \tag{5.6}
\]

It is very easily checked that, in fact, (5.6) holds for \( e \geq 1 \).

We next establish the relation

\[
(h, k)_e = (ph, pk)_e \tag{5.7}
\]

which will be needed in §6 as well as to establish property (5.8) below. Defining \( r \)

to be the least positive integer, such that \( p^r \equiv 1 \) (mod \( e \)), we have

\[
(h, k)_e = \begin{cases} \sum_{a \in GF(q)^+} 1 & \text{if } (h,k) \neq (1,1) \\ \sum_{a \in GF(q)^+} 1 & \text{if } (h,k) = (1,1) \end{cases} \tag{5.8}
\]

From (5.7) we see that

\[
B(e,pi,v) = B(e,i,v) \tag{5.9}
\]

as

\[
B(e,pi,v) = \sum_{h=0}^{e-1} (h, pi - vh) e
\]
From (5.3), (5.4), (5.6), (5.8), we see that for \( e = 15, q = p^2 \), the values of all the \( e^2 = 225 \) Dickson-Hurwitz sums \( B_{15}(i, v)(i, v = 0, 1, \ldots, 14) \) are known once the values of the 45 sums \( B_{15}(i, v)(i = 0, 1, 2, 3, 5, 6, 7, 10, 11; v = 1, 2, 3, 4, 5) \) have been determined in the case \( p \equiv 4 \pmod{15} \), and the values of the 50 sums \( B_{15}(i, v)(i = 0, 1, 2, 3, 4, 5, 6, 8, 9, 12; v = 1, 2, 3, 4, 5) \) have been determined in the case \( p \equiv 11 \pmod{15} \). This is clear in view of the following: for each \( i \)

\[
B_{15}(i, 0) = B_{15}(i, 14) = \begin{cases} f - 1, & \text{if } 15 \mid i \\ f, & \text{if } 15 \nmid i \end{cases}
\]

\[
B_{15}(i, v) = B_{15}(i, 14 - v) \quad (8 \leq v \leq 14), \tag{5.9}
\]

\[
B_{15}(i, 7) = B_{15}(13i, 13) = B_{15}(13i, 1),
\]

\[
B_{15}(i, 6) = B_{15}(i, 8) = B_{15}(2i, 2),
\]

and for each \( v \)

\[
B_{15}(i, v) = \begin{cases} B_{15}(4i, v), & \text{if } p \equiv 4 \pmod{15} \\ B_{15}(11i, v), & \text{if } p \equiv 11 \pmod{15} \end{cases}. \tag{5.10}
\]

We begin by determining the system of linear equations which have the nine Dickson-Hurwitz sums \( B_{15}(i, v)(i = 0, 1, 2, 3, 5, 6, 7, 10, 11) \) as solutions when \( p \equiv 4 \pmod{15} \).

**Theorem 5.** For \( p \equiv 4 \pmod{15} \) and for each integer \( v, 1 \leq v \leq 6 \), the nine Dickson-Hurwitz sums \( B_{15}(i, v)(i = 0, 1, 2, 3, 5, 6, 7, 10, 11) \) are the solutions of the matrix equation

\[
MB(v) = C(v), \tag{5.11}
\]

where the 9 \times 9 coefficient matrix of determinant \(-3375 = -3^3 \cdot 5^3\)

\[
M = \begin{bmatrix}
1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 \\
1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 \\
0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 \\
1 & 0 & -2 & 2 & -1 & 2 & 0 & -2 \\
0 & 2 & -2 & 0 & -1 & 0 & 2 & -1 \\
1 & 0 & -1 & 0 & 0 & -1 & 1 & -1 \\
0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\
0 & 0 & -1 & 0 & 1 & 0 & 1 & -1
\end{bmatrix}. \tag{5.12}
\]

is independent of \( v \),
The values of the $C_i(v)$ are given by

\begin{align*}
C_1(v) &= p^2 - 2, \\
C_2(v) &= \begin{cases} 
  p & \text{if } v = 1, 2, 3, 6 \\
  -1 & \text{if } v = 4, 5 
\end{cases}, \\
C_3(v) &= 0, \\
C_4(v) &= \begin{cases} 
  -A^2 + B^2 & \text{if } v = 1, 4, \\
  -2AB + B^2 & \text{if } v = 2, 3, 5, 6 
\end{cases}, \\
C_5(v) &= \begin{cases} 
  0 & \text{if } v = 2, 3, 5, 6 
\end{cases}, \\
C_6(1) &= AT - 3AU, C_7(1) = 4AU - 2BU, \\
C_8(1) &= 2AU + 2BU, C_9(1) = BT - 2AU + BU, \\
C_6(2) &= (T^2 - 15U^2) - 6TU, C_7(2) = 8TU, \\
C_8(2) &= 4TU, C_9(2) = -4TU, \\
C_6(3) &= p, C_7(3) = 0, C_8(3) = 0, C_9(3) = 0, \\
C_6(4) &= -(T^2 - 15U^2) + 6TU, C_7(4) = -8TU, \\
C_8(4) &= -4TU, C_9(4) = (T^2 - 15U^2) - 2TU, \text{ if } \text{ind}_5 \equiv 1 \pmod{3}, \\
C_6(4) &= (T^2 - 15U^2) - 6TU, C_7(4) = 4TU, \\
C_8(4) &= 8TU, C_9(4) = (T^2 - 15U^2) - 2TU, \text{ if } \text{ind}_5 \equiv 1 \pmod{3}, \\
C_6(5) &= 0, C_7(4) = 4TU, C_8(4) = -4TU, C_9(4) = -(T^2 - 15U^2) - 2TU, \text{ if } \text{ind}_5 \equiv 2 \pmod{3}, \\
C_6(5) &= -AT + 3AU, C_7(5) = -4AU + 2BU, \\
C_8(5) &= -2AU - 2BU, C_9(5) = -BT + 2AU - BU, \text{ if } \text{ind}_5 \equiv 0 \pmod{3}, \\
C_6(5) &= BT - 3BU, C_7(5) = 2AU + 2BU, C_8(5) = -2AU + 4BU, \\
C_9(5) &= -AT + BT - AU - BU, \text{ if } \text{ind}_5 \equiv 1 \pmod{3}, \\
C_6(5) &= AT - 3AU - BT + 3BU, C_7(5) = 2AU - 4BU, \\
C_8(5) &= 4AU - 2BU, C_9(5) = AT - AU + 2BU, \text{ if } \text{ind}_5 \equiv 2 \pmod{3}, \\
C_6(6) &= (T^2 - 15U^2) - 6TU, C_7(6) = 8TU, \\
C_8(6) &= 4TU, C_9(6) = -4TU. \\
\end{align*}

We remark that the value $v = 6$ has also been included in the statement of Theorem 5 as the solution of (5.11) yields an additional property of the Dickson-
Hurwitz sums of order 15, namely,

\[ B_{15}(i, 2) = B_{15}(i, 6) \quad (5.25) \]

so that by (5.3), (5.6) and (5.25), we obtain

\[ B_{15}(2i, 2) = B_{15}(i, 2) \quad (5.26) \]

**Proof.** The first row in the matrix equation (5.11) follows from (5.5) and (5.10).

The second and third rows are derived as follows. From (5.2) (with \( n = 3, e = 15 \)), we have

\[ J_q(\beta^{3v}, \beta) = \sum_{i=0}^{14} B_{15}(i, v) \beta^{3i} = 4 \sum_{j=1}^{4} S(j, v) \beta^{3j} \quad (5.27) \]

where

\[ S(j, v) = \sum_{i=0}^{14} B_{15}(j + 5i, v) \quad (5.28) \]

Appealing to (5.10) we deduce that

\[ S(4, v) = S(1, v), S(3, v) = S(2, v) \quad (5.29) \]

Thus expressing (5.27) in terms of the basis \( \{1, \beta, \beta^2, \beta^3\} \) for \( \text{GF}(\beta) \) we obtain

\[ J_q(\beta^{3v}, \beta) = (S(0, v) - S(1, v)) + (S(2, v) - S(1, v))(\beta^2 + \beta^3) \quad (5.30) \]

From (2.2), (2.4), (2.6) and Theorem 3 we have

\[ J_q(\beta^{3v}, \beta^3) = \left\{ \begin{array}{ll} p & \text{if } v = 1, 2, 3, 6 \\ -1 & \text{if } v = 4, 5 \end{array} \right. \quad (5.31) \]

and so

\[ S(0, v) - S(1, v) = C_2(v), S(2, v) - S(1, v) = C_3(v) \quad (5.32) \]

which are the second and third rows of (5.11). The fourth and fifth rows of (5.11) are obtained as follows. From (5.2) (with \( n = 5, e = 15 \)) we have

\[ J_q(\beta^{5v}, \beta^5) = \sum_{i=0}^{14} B_{15}(i, v) \beta^{5i} = 2 \sum_{j=0}^{2} T(j, v) \beta^{5j} \quad (5.33) \]

that is

\[ J_q(\beta^{5v}, \beta^5) = (T(0, v) - T(2, v)) + (T(1, v) - T(2, v)) \omega \]

where

\[ T(j, v) = \sum_{i=0}^{4} B_{15}(j + 3i, v) \quad (5.34) \]

From (2.4), (2.6) and Theorem 3 we have

\[ J_q(\beta^{5v}, \beta^5) = \left\{ \begin{array}{ll} -(A + B\omega)^2 & \text{if } v = 1, 4 \\ -1 & \text{if } v = 2, 3, 5, 6 \end{array} \right. \quad (5.35) \]

From (5.32) and (5.34) we derive

\[ T(0, v) - T(2, v) = C_4(v), T(1, v) - T(2, v) = C_5(v) \quad (5.36) \]

which are the required fourth and fifth equations.

The sixth, seventh, eighth, ninth rows of (5.11) are obtained by equating the coefficients of \( 1, \beta, \beta^2 \) and \( \beta^5 \) respectively in

\[ J_q(\beta^v, \beta) = \sum_{i=0}^{14} B_{15}(i, v) \beta^i \quad (5.37) \]
after representing both sides in terms of a basis \( \{1, \beta, \ldots, \beta^7\} \) of \( Q(\beta) \) with the values of the Jacobi sums \( J_q(\beta^v, \beta)(v = 1, 2, \ldots, 6) \) given by Theorem 3. The right-hand side of (5.36) in terms of the basis \( \{1, \beta, \ldots, \beta^7\} \) is
\[
\sum_{i=0}^{7} W(i, v) \beta^i
\]
where
\[
\begin{align*}
W(0,v) &= B_{15}(0,v) - B_{15}(2,v) - B_{15}(6,v) + B_{15}(7,v) - B_{15}(10,v) + B_{15}(11,v), \\
W(1,v) &= W(4,v) = B_{15}(1,v) + B_{15}(2,v) - B_{15}(7,v) - B_{15}(11,v), \\
W(2,v) &= -W(3,v) = W(7,v) = B_{15}(2,v) - B_{15}(3,v) + B_{15}(6,v) - B_{15}(11,v), \\
W(5,v) &= -B_{15}(2,v) + B_{15}(5,v) + B_{15}(7,v) - B_{15}(10,v), \\
W(6,v) &= 0.
\end{align*}
\]
When \( v = 1 \) we have by Theorem 3
\[
J_q(\beta, \beta) = (A + B \beta^5)(T + U(-3 + 4 \beta + 2 \beta^2 - 2 \beta^3 + 4 \beta^4 - 2 \beta^5 + 2 \beta^7))
\]
\[
= (AT - 3AU) + (4AU - 2BU)\beta + (2AU + 2BU)\beta^2 + (-2AU - 2BU)\beta^3 + (4AU - 2BU)\beta^4 + (BT - 2AU + BU)\beta^5 + (2AU - 2BU)\beta^7,
\]
from which (5.19) follows. When \( v = 2 \) we have
\[
J_q(\beta^2, \beta) = (T + U(-15)^4)^2
\]
\[
= (((T^2 - 15U^2) - 6TU) + 8TU_B + 4TU_B^2 - 4TU_B^3 + 8TU_B^4 - 4TU_B^5 + 4TU_B^7),
\]
from which (5.20) follows. The remaining \( v \) can be treated similarly. It should be noted that for \( v = 4 \) and 5 there are three cases to be considered depending upon \( \text{ind}_5 \) (mod 3).

This completes the proof of Theorem 5.

We next determine the system of linear equations which have the ten Dickson-Hurwitz sums \( B_{15}(i, v)(i = 0, 1, 2, 3, 4, 5, 6, 8, 9, 12) \) as solutions of the matrix equation
\[
MB(v) = C(v),
\]
where the 10x10 coefficient matrix of determinant \(-6075 = -3^5 \cdot 5^2\)
\[
M = \begin{bmatrix}
1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & -1 & 0 \\
0 & 2 & 0 & 0 & -2 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & -2 & 0 & 2 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\]
is independent of \( v \),
The values of the $C_i(v)$ are given by

\begin{align*}
C_1(v) &= p^2 - 2, \\
C_2(v) &= \begin{cases} 
-1 & \text{if } v = 2, 3, 5, \\
p & \text{otherwise,}
\end{cases} \\
C_i(1) &= C_i(3) = Q(1 + \text{ind}_g 2) - Q(3 - i + \text{ind}_g 2), i = 3, 4, 5, 6, \\
C_i(2) &= Q(3 + \text{ind}_g 2) - Q(4 + 2i + \text{ind}_g 2), i = 3, 4, 5, 6, \\
C_i(4) &= C_i(5) = -1, C_j(4) = C_j(5) = 0, j = 4, 5, 6, \\
C_i(1) &= R(2i + 1 + \text{ind}_g 2) - R(3 + \text{ind}_g 2), i = 7, 8, 9, 10, \\
C_i(2) &= \begin{cases} 
p, & \text{if } \text{ind}_g 3 \equiv 2 \pmod{5}, \\
-p, & \text{if } \text{ind}_g 3 \equiv 3i \pmod{5}, i = 7, 8, 9, 10, \\
0, & \text{otherwise,}
\end{cases} \\
C_i(3) &= Q(1 + \text{ind}_g 2 + 3\text{ind}_g 3) - Q(2 - i + \text{ind}_g 2 + 3\text{ind}_g 3), i = 7, 8, 9, 10, \\
C_i(4) &= p, C_j(4) = 0, j = 8, 9, 10, \\
C_i(5) &= R(3 + \text{ind}_g 2 + 2\text{ind}_g 3) - R(2i + 1 + \text{ind}_g 2 + 2\text{ind}_g 3), i = 7, 8, 9, 10,
\end{align*}

where (the $t_i$ are specified in Theorem 2)

\begin{align*}
Q(0) &= t_0^2 + 2t_1t_4 + 2t_2t_3, \\
Q(1) &= t_2^2 + 2t_0t_4 + 2t_1t_3, \\
Q(2) &= t_4^2 + 2t_0t_3 + 2t_1t_2, \\
Q(3) &= t_1^2 + 2t_0t_2 + 2t_3t_4, \\
Q(4) &= t_3^2 + 2t_0t_1 + 2t_2t_4, \\
R(0) &= t_0^2 + t_1t_2 + t_1t_3 + t_2t_4 + t_3t_4, \\
R(1) &= t_2^2 + t_0t_1 + t_0t_3 + t_1t_4 + t_3t_4, \\
R(2) &= t_4^2 + t_0t_1 + t_0t_2 + t_1t_3 + t_2t_3, \\
R(3) &= t_1^2 + t_0t_3 + t_0t_4 + t_2t_3 + t_2t_4, \\
R(4) &= t_3^2 + t_0t_2 + t_0t_4 + t_1t_2 + t_1t_4,
\end{align*}
and

$$Q(n) = Q(n + 5), \quad R(n) = R(n + 5)$$

for all integers \( n \).

**Proof.** The first row in the matrix equation (5.38) follows from (5.5) and (5.10).

The second row is derived as follows. From (5.2) (with \( n = 5, e = 15 \)) we have

$$J_0(\beta^{3v}, \beta^5) = \sum_{i=0}^{14} B_{15}(i, v) \beta^{5i}$$

$$\begin{align*}
&= \sum_{j=0}^{2} \sum_{i=0}^{4} (B_{15}(3i + j, v)) \omega^j \times \\
&= \sum_{j=0}^{1} \sum_{i=0}^{4} (B_{15}(3i + j, v)) - B_{15}(3i + 2, v)) \omega^j \times \\
&= B_{15}(0, v) - B_{15}(1, v) - B_{15}(2, v) + B_{15}(3, v) - B_{15}(4, v) \\
&\quad - B_{15}(5, v) + B_{15}(6, v) - B_{15}(8, v) + B_{15}(9, v) + B_{15}(12, v). 
\end{align*}$$

The values of \( C_2(v) \) now follow from (4.19), (2.4) and (2.6).

The third, fourth, fifth, and sixth rows are derived as follows. From (5.2) (with \( n = 3, e = 15 \)), we have

$$J_0(\beta^{3v}, \beta^3) = \sum_{i=0}^{14} B_{15}(i, v) \beta^{3i} = \sum_{j=0}^{14} S(j, v) \omega^j,$$

where

$$S(j, v) = \sum_{i=0}^{2} B_{15}(j + 5i, v).$$

Hence, as \( \omega^4 = -1 - \omega - \omega^2 - \omega^3 \), we have

$$J_0(\beta^{3v}, \beta^3) = \sum_{j=0}^{3} T(j, v) \omega^j,$$

where

$$T(j, v) = S(j, v) - S(4, v).$$

Appealing to (5.10) we obtain

$$T(0, v) = B_{15}(0, v) - 2B_{15}(4, v) + 2B_{15}(5, v) - B_{15}(9, v),$$

$$T(1, v) = 2B_{15}(1, v) - 2B_{15}(4, v) + B_{15}(6, v) - B_{15}(9, v),$$

$$T(2, v) = 2B_{15}(2, v) - 2B_{15}(4, v) - B_{15}(9, v) + B_{15}(12, v),$$

$$T(3, v) = B_{15}(3, v) - 2B_{15}(4, v) + 2B_{15}(8, v) - B_{15}(9, v).$$

The values of \( C_3(v), C_4(v), C_5(v), \) and \( C_6(v) \) follow from (4.18), (2.2), (2.4), (2.6), and (2.7), by equating coefficients of \( 1, \omega, \omega^2 \) and \( \omega^3 \) in (5.53).

The seventh, eighth, ninth, tenth rows of (5.38) are obtained by equating the coefficients of \( 1, \omega^2 \) and \( \omega^3 \) respectively in

$$J_0(\beta^{3v}, \beta^3) = (B_{15}(0, v) + B_{15}(2, v) - B_{15}(5, v) - B_{15}(12, v))$$

$$\quad + (B_{15}(2, v) + B_{15}(3, v) - B_{15}(8, v) - B_{15}(12, v)) \omega$$

$$\quad + (-B_{15}(1, v) + B_{15}(2, v) + B_{15}(6, v) - B_{15}(12, v)) \omega^2$$

$$\quad + (B_{15}(2, v) - B_{15}(4, v) + B_{15}(9, v) - B_{15}(12, v)) \omega^3,$$

which was obtained from (5.2) (with \( e = 5, n = 1 \)) by expressing the right-hand side in terms of a basis \( (1, \beta, \ldots, \beta^5) \) of \( Q(\beta) \) and then using (5.10) to express it in
The values of the Jacobis sums $J_q(\theta^v, \theta^n)(v = 1, 2, \ldots, 6)$ are given by Theorem 4.

This completes the proof of Theorem 6.

Next we use Theorems 5 and 6 to determine the values of the Dickson-Hurwitz sums. We treat the case $p \equiv 4 \pmod{15}$ first. Inverting the matrix $M$ given in (5.12), we obtain from (5.11)

$$B(v) = \frac{1}{15}NC(v),$$  \hspace{1cm} (5.54)

where

$$N = \begin{bmatrix}
1 & 4 & 2 & 2 & -1 & 8 & 2 & 4 & -4 \\
1 & -1 & 2 & -1 & 2 & 1 & 6 & -2 & 1 \\
1 & -1 & -3 & -1 & -1 & 1 & 2 & 3 & -2 \\
1 & -1 & -3 & 2 & -1 & -2 & 2 & -6 & 1 \\
1 & 4 & 2 & -1 & -1 & -4 & 2 & -2 & 8 \\
1 & -1 & 2 & 2 & -1 & -2 & 3 & 4 & 1 \\
1 & -1 & -3 & -1 & 2 & 1 & -4 & 3 & 1 \\
1 & 4 & 2 & -1 & 2 & -4 & -4 & -2 & -4 \\
1 & -1 & 2 & -1 & -1 & 1 & -3 & -2 & -2
\end{bmatrix}.  \hspace{1cm} (5.55)$$

Substituting the values for $C_1(v), \ldots, C_9(v)$ specified in Theorem 5 into (5.54) we obtain the following tables of values of $B_{15}(i, v)$ ($i = 0, 1, 2, 3, 5, 6, 7, 10, 11; v = 1, 2, 3, 4, 5$).

| Table 1: Dickson-Hurwitz sums $B_{15}(i, 1)$, $p \equiv 4 \pmod{15}$ |
|--------------------------------|
| $p^2$ | $p$ | $1$ | $A^2$ | $AB$ | $B^2$ | $AT$ | $BT$ | $AU$ | $BU$ |
|---|---|---|---|---|---|---|---|---|---|
| 15 $B_{15}(0,1)$: | 1 | 4 | -2 | -2 | 2 | 1 | 8 | -4 | 
| 15 $B_{15}(1,1)$: | 1 | -1 | -2 | 1 | -4 | 1 | 1 | 15 | -15 |
| 15 $B_{15}(2,1)$: | 1 | -1 | -2 | 1 | 2 | -2 | 1 | -2 | 15 |
| 15 $B_{15}(3,1)$: | 1 | -1 | -2 | -2 | 2 | 1 | -2 | 1 | -15 |
| 15 $B_{15}(5,1)$: | 1 | 4 | -2 | 1 | 2 | -2 | -4 | 8 | 
| 15 $B_{15}(6,1)$: | 1 | -1 | -2 | -2 | 2 | 1 | -2 | 1 | 15 |
| 15 $B_{15}(7,1)$: | 1 | -1 | -2 | 1 | -4 | 1 | 1 | 1 | -15 |
| 15 $B_{15}(10,1)$: | 1 | 4 | -2 | 1 | -4 | 1 | -4 | -4 | 
| 15 $B_{15}(11,1)$: | 1 | -1 | -2 | 1 | 2 | -2 | 1 | -2 | -15 |

| Table 2: Dickson-Hurwitz sums $B_{15}(i, 2)$, $p \equiv 4 \pmod{15}$ |
|--------------------------------|
| $p^2$ | $p$ | $1$ | $T^2$ | $U^2$ | $TU$ |
|---|---|---|---|---|---|
| 15 $B_{15}(0,2)$: | 1 | 4 | -4 | 8 | -120 |
| 15 $B_{15}(1,2)$: | 1 | -1 | -1 | 1 | -15 | 30 |
| 15 $B_{15}(2,2)$: | 1 | -1 | -1 | 1 | -15 | 30 |
| 15 $B_{15}(3,2)$: | 1 | -1 | -4 | -1 | 4 | -4 |
| 15 $B_{15}(10,1)$: | 1 | 4 | -1 | -1 | -15 | -30 |
| 15 $B_{15}(11,1)$: | 1 | -1 | -1 | 1 | -15 | -30 |
Table 3: Dickson-Hurwitz sums $B_{15}(i, 3)$, $p \equiv 4 (\mod 15)$

| $p^2$ | $p$ | $1$ |
|-------|-----|-----|
| $15 B_{15}(0,3)$: | 1 | 12 | -4 |
| $15 B_{15}(1,3)$: | 1 | -1 |
| $15 B_{15}(2,3)$: | 1 | -1 |
| $15 B_{15}(3,3)$: | 1 | -3 | -4 |
| $15 B_{15}(5,3)$: | 1 | -1 |
| $15 B_{15}(6,3)$: | 1 | -3 | -4 |
| $15 B_{15}(7,3)$: | 1 | -1 |
| $15 B_{15}(10,3)$: | 1 | -1 |
| $15 B_{15}(11,3)$: | 1 | -1 |

Table 4a: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 4 (\mod 15)$, $\text{ind}_g 5 \equiv 0 (\mod 3)$

| $p^2$ | $1$ | $A^2$ | $A B$ | $B^2$ | $T^2$ | $U^2$ |
|-------|-----|-----|-------|-----|-----|-----|
| $15 B_{15}(0,4)$: | 1 | -6 | -2 | 2 | 1 | -8 | 120 |
| $15 B_{15}(1,4)$: | 1 | -1 | 1 | -4 | 1 | -1 | -30 | 15 |
| $15 B_{15}(2,4)$: | 1 | -1 | 1 | 2 | -2 | 1 | -30 | 15 |
| $15 B_{15}(3,4)$: | 1 | -1 | -2 | 2 | 1 | 2 | -30 |
| $15 B_{15}(5,4)$: | 1 | -6 | 1 | 2 | -2 | 4 | -60 |
| $15 B_{15}(6,4)$: | 1 | -1 | -2 | 2 | 1 | 2 | -30 |
| $15 B_{15}(7,4)$: | 1 | -1 | 1 | -4 | 1 | -1 | 30 | 15 |
| $15 B_{15}(10,4)$: | 1 | -6 | 1 | -4 | 1 | 4 | -60 |
| $15 B_{15}(11,4)$: | 1 | -1 | 1 | 2 | -2 | -1 | 30 | 15 |

Table 4b: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 4 (\mod 15)$, $\text{ind}_g 5 \equiv 1 (\mod 3)$

| $p^2$ | $1$ | $A^2$ | $A B$ | $B^2$ | $T^2$ | $U^2$ |
|-------|-----|-----|-------|-----|-----|-----|
| $15 B_{15}(0,4)$: | 1 | -6 | -2 | 2 | 1 | 4 | -60 |
| $15 B_{15}(1,4)$: | 1 | -1 | 1 | -4 | 1 | 2 | -30 |
| $15 B_{15}(2,4)$: | 1 | -1 | 1 | 2 | -2 | 1 | -30 | 15 |
| $15 B_{15}(3,4)$: | 1 | -1 | -2 | 2 | -1 | 4 | -60 |
| $15 B_{15}(5,4)$: | 1 | -6 | 1 | 2 | -2 | 1 | -30 | 15 |
| $15 B_{15}(6,4)$: | 1 | -1 | -2 | 2 | 1 | 1 | -8 | 120 |
| $15 B_{15}(7,4)$: | 1 | -1 | 1 | -4 | 1 | 2 | -30 |
| $15 B_{15}(10,4)$: | 1 | -6 | 1 | -4 | 1 | -8 | 120 |
| $15 B_{15}(11,4)$: | 1 | -1 | 1 | 2 | -2 | -1 | -30 | 15 |
Table 4c: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 4 \pmod{15}$, $\text{ind}_5 = 2 \pmod{3}$

| $p^2$ | 1 | $A^2$ | $AB$ | $B^2$ | $T^2$ | $TU$ | $U^2$ |
|-------|---|-------|------|-------|-------|------|------|
| $15 B_{15}(0,4)$: | 1 | -6 | -2 | 2 | 1 | 4 | -60 |
| $15 B_{15}(1,4)$: | 1 | -1 | 1 | -4 | 1 | -1 | 30 | 15 |
| $15 B_{15}(2,4)$: | 1 | -1 | 1 | 2 | -2 | 2 | -30 |
| $15 B_{15}(3,4)$: | 1 | -1 | -2 | 2 | 1 | -1 | 30 | 15 |
| $15 B_{15}(5,4)$: | 1 | -6 | 1 | 2 | -2 | -8 | 120 |
| $15 B_{15}(6,4)$: | 1 | -1 | -2 | 2 | 1 | -1 | -30 | 15 |
| $15 B_{15}(10,4)$: | 1 | -6 | 1 | -4 | 1 | 4 | -60 |
| $15 B_{15}(11,4)$: | 1 | -1 | 1 | 2 | -2 | 2 | -30 |

Table 5a: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 4 \pmod{15}$, $\text{ind}_5 = 0 \pmod{3}$

| $p^2$ | 1 | AT | BT | AU | BU |
|-------|---|----|----|----|----|
| $15 B_{15}(0,5)$: | 1 | -8 | -8 | 4 |
| $15 B_{15}(1,5)$: | 1 | -1 | -1 | -15 | 15 |
| $15 B_{15}(2,5)$: | 1 | -1 | 2 | -15 |
| $15 B_{15}(3,5)$: | 1 | -3 | 2 | -1 | 15 |
| $15 B_{15}(5,5)$: | 1 | -5 | 4 | -8 |
| $15 B_{15}(6,5)$: | 1 | -3 | 2 | -1 | -15 |
| $15 B_{15}(7,5)$: | 1 | -1 | -1 | 15 | -15 |
| $15 B_{15}(10,5)$: | 1 | -5 | 4 | 4 |
| $15 B_{15}(11,5)$: | 1 | -1 | 2 | 15 |

Table 5b: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 4 \pmod{15}$, $\text{ind}_5 = 1 \pmod{3}$

| $p^2$ | 1 | AT | BT | AU | BU |
|-------|---|----|----|----|----|
| $15 B_{15}(0,5)$: | 1 | -8 | 4 | 4 |
| $15 B_{15}(1,5)$: | 1 | -1 | 2 | 15 |
| $15 B_{15}(2,5)$: | 1 | 2 | -1 | 15 |
| $15 B_{15}(3,5)$: | 1 | -3 | -1 | -1 | 15 | -15 |
| $15 B_{15}(5,5)$: | 1 | -5 | -8 | 4 |
| $15 B_{15}(6,5)$: | 1 | -3 | -1 | -1 | -15 | 15 |
| $15 B_{15}(7,5)$: | 1 | -1 | 2 | -15 |
| $15 B_{15}(10,5)$: | 1 | -5 | 4 | -8 |
| $15 B_{15}(11,5)$: | 1 | 2 | -1 | -15 |
Table 5c: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 4 (\text{mod } 15)$, $\text{ind}_5 g \equiv 2 (\text{mod } 3)$

| $p^2$ | 1   | AT | BT | AU | BU |
|-------|-----|----|----|----|----|
| 15    | B_{15}(0,5) | 1  | -8 | 4  | -8 |
| 15    | B_{15}(1,5) | 1  | 2  | -1 |    |
| 15    | B_{15}(2,5) | 1  | -1 | -1 | 15 |
| 15    | B_{15}(3,5) | 1  | -3 | -1 | 2  |
| 15    | B_{15}(4,5) | 1  | -5 | 4  |    |
| 15    | B_{15}(5,5) | 1  | -3 | -1 | 2  |
| 15    | B_{15}(6,5) | 1  | 2  | -1 | 15 |
| 15    | B_{15}(7,5) | 1  | -5 | -8 | 4  |
| 15    | B_{15}(8,5) | 1  | -1 | -1 | -15|
| 15    | B_{15}(9,5) | 1  | -1 | -1 | -15|

Next, we treat the case $p \equiv 11 (\text{mod } 15)$. Inverting the matrix $M$ given in (5.39), we obtain from (5.38)

$$B(v) = \frac{1}{15} N C(v),$$

where

$$N = \begin{bmatrix}
1 & 2 & 4 & -1 & -1 & 8 & -2 & -2 & -2 \\
1 & -1 & -1 & 4 & -1 & -1 & 1 & 1 & -4 \\
1 & -1 & -1 & -1 & 4 & -1 & 1 & 1 & 1 \\
1 & 2 & -1 & -1 & -1 & 4 & -2 & 8 & -2 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -4 \\
1 & -1 & 4 & -1 & -1 & -4 & 1 & 1 & 1 \\
1 & 2 & -1 & 4 & -1 & -1 & -2 & -2 & 8 \\
1 & -1 & -1 & -1 & -1 & 4 & -4 & 1 & 1 \\
1 & 2 & -1 & -1 & -1 & -2 & -2 & -2 & 8 \\
1 & 2 & -1 & -1 & 4 & -1 & -2 & -2 & -2 \\
\end{bmatrix}.$$ (5.57)

Substituting the values of the $C_i(v)$ given in Theorem 6 into (5.56), we obtain the Dickson-Hurwitz sums in terms of the $Q(i)$ and $R(i)$ ($0 \leq i \leq 4$). There will be 61 tables as follows:

| $v$ | no. of tables |
|-----|---------------|
| 1   | 5 (depending upon $\text{ind}_g 2 \ (\text{mod } 5)$) |
| 2   | 25 (depending upon $\text{ind}_g 2$, $\text{ind}_g 3 \ (\text{mod } 5)$) |
| 3   | 25 (depending upon $\text{ind}_g 2$, $\text{ind}_g 3 \ (\text{mod } 5)$) |
| 4   | 1 |
| 5   | 5 (depending upon ($\text{ind}_g 2 + 2 \text{ind}_g 3$)(mod 5)) |

As we cannot display them all here, we just give one table for each value of $v$. 
Table 6: Dickson-Hurwitz sums $B_{15}(i,1)$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 0 \pmod{5}$

| $p^2$ | $p$ | $Q(0)$ | $Q(1)$ | $Q(2)$ | $Q(3)$ | $Q(4)$ | $R(0)$ | $R(1)$ | $R(2)$ | $R(3)$ | $R(4)$ |
|-------|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 15 $B_{15}(0,1)$: | 1 | 2 | -2 | -4 | 1 | 1 | 1 | 1 | 8 | -2 | -2 | -2 |
| 15 $B_{15}(1,1)$: | 1 | -1 | -2 | 1 | 1 | 1 | 1 | -4 | 1 | 1 | 1 | 1 |
| 15 $B_{15}(2,1)$: | 1 | -1 | -2 | 1 | 1 | 1 | -4 | 1 | 1 | 1 | 1 | -4 |
| 15 $B_{15}(3,1)$: | 1 | 2 | -2 | 1 | 1 | -4 | 1 | 1 | -2 | -2 | 8 | -2 |
| 15 $B_{15}(4,1)$: | 1 | -1 | -2 | 1 | -4 | 1 | 1 | 1 | -4 | 1 | 1 | 1 |
| 15 $B_{15}(5,1)$: | 1 | -1 | -2 | -4 | 1 | 1 | 1 | 1 | -4 | 1 | 1 | 1 |
| 15 $B_{15}(6,1)$: | 1 | 2 | -2 | 1 | 1 | 1 | 1 | -4 | -2 | -2 | -2 | 8 |
| 15 $B_{15}(7,1)$: | 1 | -1 | -2 | 1 | -4 | 1 | 1 | 1 | -4 | 1 | 1 | 1 |
| 15 $B_{15}(8,1)$: | 1 | 2 | -2 | 1 | -4 | 1 | 1 | 1 | -2 | 8 | -2 | -2 |
| 15 $B_{15}(9,1)$: | 1 | 2 | -2 | 1 | -4 | 1 | 1 | 1 | -2 | 8 | -2 | -2 |
| 15 $B_{15}(10,1)$: | 1 | 2 | -2 | 1 | -4 | 1 | -2 | -2 | -2 | 8 | -2 | -2 |

Table 7: Dickson-Hurwitz sums $B_{15}(i,2)$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 1 \pmod{5}$, $\text{ind}_g 3 \equiv 2 \pmod{5}$.

| $p^2$ | $p$ | $Q(0)$ | $Q(1)$ | $Q(2)$ | $Q(3)$ | $Q(4)$ |
|-------|-----|--------|--------|--------|--------|--------|
| 15 $B_{15}(0,2)$: | 1 | 2 | -4 | 1 | -4 | 1 | 1 | 1 |
| 15 $B_{15}(1,2)$: | 1 | -1 | -1 | 1 | 1 | 1 | -4 | 1 |
| 15 $B_{15}(2,2)$: | 1 | 4 | -1 | -4 | 1 | 1 | 1 | 1 |
| 15 $B_{15}(3,2)$: | 1 | 2 | -4 | 1 | 1 | -4 | 1 | 1 |
| 15 $B_{15}(4,2)$: | 1 | -1 | -1 | 1 | 1 | 1 | 1 | -4 |
| 15 $B_{15}(5,2)$: | 1 | -1 | -1 | 1 | -4 | 1 | 1 | 1 |
| 15 $B_{15}(6,2)$: | 1 | 2 | -4 | 1 | 1 | 1 | -4 | 1 |
| 15 $B_{15}(7,2)$: | 1 | -1 | -1 | 1 | 1 | -4 | 1 | 1 |
| 15 $B_{15}(8,2)$: | 1 | 2 | -4 | 1 | 1 | 1 | -4 | 1 |
| 15 $B_{15}(9,2)$: | 1 | -8 | -4 | -4 | 1 | 1 | 1 | 1 |
| 15 $B_{15}(10,2)$: | 1 | -8 | -4 | -4 | 1 | 1 | 1 | 1 |
Table 8: Dickson-Hurwitz sums $B_{15}(i,3)$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 0 \pmod{5}$, $\text{ind}_g 3 \equiv 2 \pmod{5}$.

| $p^2$ | $p$ | 1 | Q(0) | Q(1) | Q(2) | Q(3) | Q(4) |
|-------|-----|---|------|------|------|------|------|
| 15 $B_{15}(0,3)$: | 1 | 0 | -4 | -2 | 3 | 3 | -7 | 3 |
| 15 $B_{15}(1,3)$: | 1 | 0 | -1 | 0 | 5 | 0 | 0 | -5 |
| 15 $B_{15}(2,3)$: | 1 | 0 | -1 | 0 | 0 | 0 | -5 | 5 |
| 15 $B_{15}(3,3)$: | 1 | 0 | -4 | 3 | 3 | -12 | 3 | 3 |
| 15 $B_{15}(4,3)$: | 1 | 0 | -1 | 5 | -5 | 0 | 0 | 0 |
| 15 $B_{15}(5,3)$: | 1 | 0 | -1 | -5 | 0 | 0 | 5 | 0 |
| 15 $B_{15}(6,3)$: | 1 | 0 | -4 | 3 | -7 | 3 | 3 | -2 |
| 15 $B_{15}(8,3)$: | 1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 |
| 15 $B_{15}(9,3)$: | 1 | 0 | -4 | -7 | -2 | 3 | 3 | 3 |
| 15 $B_{15}(12,3)$: | 1 | 0 | -4 | 3 | 3 | 3 | -2 | -7 |

Table 9: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 11 \pmod{15}$

| $p^2$ | $p$ | 1 |
|-------|-----|---|
| 15 $B_{15}(0,4)$: | 1 | 10 | -6 |
| 15 $B_{15}(1,4)$: | 1 | 0 | -1 |
| 15 $B_{15}(2,4)$: | 1 | 0 | -1 |
| 15 $B_{15}(3,4)$: | 1 | 0 | -1 |
| 15 $B_{15}(4,4)$: | 1 | 0 | -1 |
| 15 $B_{15}(5,4)$: | 1 | -5 | -6 |
| 15 $B_{15}(6,4)$: | 1 | 0 | -1 |
| 15 $B_{15}(8,4)$: | 1 | 0 | -1 |
| 15 $B_{15}(9,4)$: | 1 | 0 | -1 |
| 15 $B_{15}(12,4)$: | 1 | 0 | -1 |
Table 10: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 11(\text{mod } 15)$, $\text{ind}_9^2 + 2 \text{ind}_9^3 \equiv 4(\text{mod } 5)$

| $p^2$ | $p$ | 1 | $R(0)$ | $R(1)$ | $R(2)$ | $R(3)$ | $R(4)$ |
|-------|-----|---|---------|---------|---------|---------|---------|
| 15 $B_{15}(0,5)$: | 1 | 0 | -8 | 2 | 2 | 2 | 2 | -8 |
| 15 $B_{15}(1,5)$: | 1 | 0 | 0 | -1 | -1 | 4 | -1 | -1 |
| 15 $B_{15}(2,5)$: | 1 | 0 | 0 | -1 | -1 | 4 | -1 | -1 |
| 15 $B_{15}(3,5)$: | 1 | 0 | -3 | 2 | -8 | 2 | 2 | 2 |
| 15 $B_{15}(4,5)$: | 1 | 0 | 0 | 4 | -1 | -1 | -1 | -1 |
| 15 $B_{15}(5,5)$: | 1 | 0 | -5 | -1 | -1 | -1 | -1 | 4 |
| 15 $B_{15}(6,5)$: | 1 | 0 | -3 | 2 | 2 | 2 | -8 | 2 |
| 15 $B_{15}(8,5)$: | 1 | 0 | 0 | -1 | 4 | -1 | -1 | -1 |
| 15 $B_{15}(9,5)$: | 1 | 0 | -3 | -8 | 2 | 2 | 2 | 2 |
| 15 $B_{15}(12,5)$: | 1 | 0 | -3 | 2 | 2 | -8 | 2 | 2 |

6. EVALUATION OF CYCLOTOIC NUMBERS $(h,k)_{15}$ OVER $GF(p^2)$.

The cyclotomic numbers have the following well-known properties (see, for example, [11: p. 25])

\[
(h,k)_e = (e-h,k-h)_e,
\]

\[
(k,h)_e = (h,k)_e, \quad \text{if } f \text{ is even},
\]

\[
(h,k)_e = (k + \frac{1}{2}e, h + \frac{1}{2}e)_e, \quad \text{if } f \text{ is odd},
\]

as well as the property

\[
(h, k)_e = (ph, pk)_e
\]

noted in (5.7). Taking $e = 15$, $x = 2$, $q = p^2$, $p \equiv 4, 11(\text{mod } 15)$, and appealing to (5.7), (6.1) and (6.2), we see that each of the $e^2 = 225$ cyclotomic numbers $(h, k)_{15}$ is equal to one of the 28 cyclotomic numbers

\[
\begin{align*}
(0, k)_{15}, & \quad k = 0, 1, 2, 3, 5, 6, 7, 10, 11, \\
(1, k)_{15}, & \quad k = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, \\
(2, k)_{15}, & \quad k = 5, 7, 8, 9, 12, \\
(3, k)_{15}, & \quad k = 6, 9, \\
(5, 10)_{15}, & \quad
\end{align*}
\]

when $p \equiv 4 (\text{mod } 15)$, and to one of the 29 cyclotomic numbers

\[
\begin{align*}
(0, k)_{15}, & \quad k = 0, 1, 2, 3, 4, 5, 6, 8, 9, 12, \\
(1, k)_{15}, & \quad k = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, \\
(2, k)_{15}, & \quad k = 5, 7, 8, 10, 12, \\
(3, k)_{15}, & \quad k = 6, 9, \\
(5, 10)_{15}, & \quad
\end{align*}
\]

when $p \equiv 11 (\text{mod } 15)$. 

When $p \equiv 4 (\text{mod } 15)$, and to one of the 29 cyclotomic numbers

\[
\begin{align*}
(0, k)_{15}, & \quad k = 0, 1, 2, 3, 4, 5, 6, 8, 9, 12, \\
(1, k)_{15}, & \quad k = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, \\
(2, k)_{15}, & \quad k = 5, 7, 8, 10, 12, \\
(3, k)_{15}, & \quad k = 6, 9, \\
(5, 10)_{15}, & \quad
\end{align*}
\]
These relationships are exhibited in Tables 11 and 12, in which \((h, k)_{15}\) is in row \(h (0 \leq h \leq 14)\) and column \(k (0 \leq k \leq 14)\), and we have written \(A\) for 10, \(B\) for 11, \(C\) for 12, and \(D\) for 13.

Table 11: Relationships between cyclotomic numbers \((h, k)_{15}\) over \(GF(p^2), p \equiv 4 (\text{mod} \ 15)\)

|   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1 | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2 | 2  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 3 | 3  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
| 4 | 4  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
| 5 | 5  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  |
| 6 | 6  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
| 7 | 7  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  |

Table 12: Relationships between cyclotomic numbers \((h, k)_{15}\) over \(GF(p^2), p \equiv 11 (\text{mod} \ 15)\)

|   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1 | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2 | 2  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 3 | 3  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
| 4 | 4  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
| 5 | 5  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  |
| 6 | 6  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
| 7 | 7  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  |

In order to determine explicit formulae for the cyclotomic numbers in (6.3) and (6.4), we express each cyclotomic number \((h, k)_{15}\) in terms of the Dickson-Hurwitz sums \(B_{15}(i, v)(i, v = 0, 1, \ldots, 14)\) and then appeal to the formulae for the \(B_{15}(i, v)\) derived in §5.
Whiteman gave the following formula, which expresses each cyclotomic number of order $e$, where $e$ is an odd prime, as a linear combination of certain Dickson-Hurwitz sums of order $e$ [12: Theorem 1]. We give the formula in the context of $\text{GF}(q)$, as Whiteman's proof for $\text{GF}(p)$ goes over without difficulty to this case. We have, for $e$ an odd prime and $r, s = 0, 1, \ldots, e-1$,

$$e^2(r, s)_e = (q - 1)(1 - e) + e(1 - \delta_r) + e \sum_{v=0}^{e-1} B_e(vr + s, v)$$  \hfill (6.5)

where

$$\delta_r = \begin{cases} 1, & \text{if } r \equiv 0 \pmod{e}, \\ 0, & \text{if } r \not\equiv 0 \pmod{e}. \end{cases}$$  \hfill (6.6)

In Theorem 7, we give the corresponding formula to (6.5), where $e$ is the product of two distinct odd primes $x$ and $y$. The formula will be used with $e = 15$, $x = 3$, $y = 5$, $q = p^2$, $p \equiv 4, 11 \pmod{15}$.

We use the well-known identity

$$B_x(i, v) = \sum_{j=0}^{y-1} B_e(i + jx, v), \quad e = xy,$$  \hfill (6.7)

to simplify formulae appearing in the proof of Theorem 7, often in the form

$$\sum_{j=1}^{y-1} B_e(i + jx, v) = B_x(i, v) - B_e(i, v).$$  \hfill (6.8)

We also use a special case of the Ramanujan sum [6: pp. 237-238]

$$\sum_{n=1}^{e-1} \beta_{nk} = \begin{cases} 1, & \text{if } x \mid k, y \mid k \\ -(x-1), & \text{if } x \mid k, y \not\mid k \\ -(y-1), & \text{if } x \not\mid k, y \mid k \\ (x-1)(y-1), & \text{if } x \not\mid k, y \not\mid k \end{cases}$$  \hfill (6.9)

where the prime (') indicates that the summation variable is restricted to values coprime with $e$. We prove

**THEOREM 7.** Let $e = xy$ be the product of two distinct odd primes $x$ and $y$. Then for $r, s = 0, 1, 2, \ldots, e-1$ we have

$$e^2(r, s)_e = q + 1 - e(\delta_r + \delta_s + \delta_{r-s})$$

\[+ \sum_{v=1}^{e-1} [(e-1)B_e(vr+s,v)-(x-1)B_x(vr+s,v)-(y-1)B_y(vr+s,v)+\sum_{u=1}^{e-1} B_e(vr+s+u,v)]\]

\[+ \sum_{v=1}^{e-2} [(e-1)B_e(vs+r,v)-(x-1)B_x(vs+r,v)-(y-1)B_y(vs+r,v)+\sum_{u=1}^{e-1} B_e(vs+r+u,v)]\]

\[+ y \sum_{w=1}^{y-2} B_y(ws+r,w)-(y-2)(q-2)+ x \sum_{w=1}^{x-2} B_x(ws+r,w)-(x-2)(q-2)\]

\[+ (e-1)(B_e(rax+sby,ax)+B_e(sax+rby,ax))\]

\[+ \sum_{j=1}^{e-1} (B_e(rax+sby+j,ax)+B_e(sax+rby+j,ax))\]

\[- 4(q-1) + 2f(x+y) + (x-1)(\delta_{sy} + \delta_{ry}) + (y-1)(\delta_{rx} + \delta_{sx}),\] where $a$ and $b$ are integers such that
PROOF. It is well-known that the cyclotomic numbers \((r,s)\) are related to the Jacobis sums \(J(q^m, q^n)\) by the formula

\[
e^2(r, s)_e = \sum_{m=0}^{e-1} \sum_{n=0}^{e-1} J(q^m, q^n)q^{mr-ns}.
\]  

(6.11)

Splitting the double sum on the right-hand side into five sums according as

\[m = n = 0; m = 0, n \neq 0; m \neq 0, n = 0; m \neq 0, n \neq 0, m + n \neq e; m \neq 0, n \neq 0, m + n \neq e;\]

we obtain, appealing to (2.4),

\[
e^2(r, s)_e \approx q+1 - e(\delta_r + \delta_s + \delta_{r-s}) + \sum_{m,n=1}^{e-1} J(q^m, q^n)q^{mr-ns}.
\]  

(6.12)

The double sum on the right-hand side of (6.12) will be broken up into six sums \(S_i\) as follows:

\[
S_1 : m + n \neq e, (m, e) > 1, (n, e) = 1,
S_2 : m + n \neq e, (m, e) = 1,
S_3 : m + n \neq e, (m, e) = (n, e) = x,
S_4 : m + n \neq e, (m, e) = (n, e) = y,
S_5 : (m, e) = x, (n, e) = y,
S_6 : (m, e) = y, (n, e) = x.
\]

First, we evaluate \(S_1\). We have

\[
S_1 = \sum_{(m, e) > 1} \sum_{n=1}^{(m, e) > 1} J(q^m, q^n)q^{mr-ns}.
\]

(6.13)

Appealing to (6.9), we obtain

\[
S_1 = \sum_{(v, e) > 1} \left[ (x-1)(y-1)B_e(vr+s, v) - (x-1) \sum_{u=1}^{y-1} B_e(vr+sxu, v) \right]
\]

(6.13)

Similarly, we obtain
Next, we evaluate $S_3$. We have

$$S_3 = \sum_{t=1}^{y-1} \sum_{u=1}^{y-1} q(\beta^t, \beta^u) \cdot \sum_{v=1}^{x-1} \sum_{w=1}^{x-1} B(x, w) \cdot \sum_{i=1}^{y-1} B(i, w) \cdot \sum_{j=1}^{y-1} B(j, w).$$

That is

$$S_3 = \sum_{t=1}^{y-1} \sum_{u=1}^{y-1} q(\beta^t, \beta^u) \cdot \sum_{v=1}^{x-1} \sum_{w=1}^{x-1} B(x, w) \cdot \sum_{i=1}^{y-1} B(i, w).$$

Similarly we have

$$S_4 = x \sum_{t=1}^{y-1} \sum_{u=1}^{x-1} B(x, w) \cdot \sum_{i=1}^{y-1} B(i, w).$$

Next we evaluate $S_5$. We have

$$S_5 = \sum_{m=1}^{e-1} \sum_{n=1}^{e-1} q(\beta^m, \beta^n) \cdot \sum_{t=1}^{x-1} \sum_{u=1}^{y-1} q(\beta^t, \beta^u) \cdot \sum_{v=1}^{x-1} \sum_{w=1}^{x-1} B(x, w) \cdot \sum_{i=1}^{y-1} B(i, w).$$

Now we replace $t$ by $iax$ (mod $y$) and $u$ by $jby$ (mod $x$) to obtain

$$S_5 = \sum_{i=1}^{y-1} \sum_{j=1}^{y-1} q(\beta^{iax}, \beta^{jby}) \cdot \sum_{t=1}^{x-1} \sum_{u=1}^{y-1} q(\beta^t, \beta^u) \cdot \sum_{v=1}^{x-1} \sum_{w=1}^{x-1} B(x, w) \cdot \sum_{i=1}^{y-1} B(i, w).$$

This gives

$$S_5 = \text{tr}(J_q(\beta^{ax}, \beta^{by}) \cdot B(x, w)).$$

(by (2.2) and (6.10))
\[
\begin{align*}
&= \sum_{i=0}^{e-1} \beta_{e}(i, ax) B_{e}(i, ax) e^{-i \pi a x - s y} \\
&= \sum_{j=0}^{e-1} \beta_{e}(r a x + s b y, j, ax) e^{j} \\
&= \sum_{j=0}^{e-1} \beta_{e}(r a x + s b y, j, ax) \sum_{i=0}^{e-1} \beta_{i} j \\
&= (x-1)(y-1)B_{e}(r a x + s b y, ax) - (x-1) \sum_{t=1}^{e-1} B_{e}(r a x + s b y + x t, ax) \\
&\quad - (y-1) \sum_{t=1}^{e-1} B_{e}(r a x + s b y + y t, ax) + \sum_{j=1}^{e-1} B_{e}(r a x + s b y + j, ax)
\end{align*}
\]

that is

\[
S_{5} = (e-1)B_{e}(r a x + s b y, ax) - (x-1)B_{x}(s b y, 0) - (y-1)B_{y}(r a x, ax) \\
\quad + \sum_{j=1}^{e-1} B_{e}(r a x + s b y + j, ax).
\]

Appealing to (5.4) we have

\[
B_{x}(s b y, 0) = f y - \delta_{s y}
\]

and

\[
B_{y}(r a x, ax) = B_{y}(r a x, e-1 - b y) = B_{y}(r a x, e-1) = f x - \delta_{r x}.
\]

Similarly we obtain

\[
S_{6} = (e-1)B_{e}(s a x + r b y, ax) - (x-1)B_{x}(r b y, 0) - (y-1)B_{y}(s a x, ax) \\
\quad + \sum_{j=1}^{e-1} B_{e}(s a x + r b y + j, ax),
\]

where as above we have

\[
B_{x}(r b y, 0) = f y - \delta_{r y}
\]

and

\[
B_{y}(s a x, ax) = f x - \delta_{s x}.
\]

This completes the proof of Theorem 7.

Applying Theorem 7 and appealing to the tables of the Dickson-Hurwitz sums obtained in §5, we obtain formulae for the cyclotomic numbers of order 15 over GF(p^2), p \equiv 4, 11(\text{mod } 15).

In the case p \equiv 4(\text{mod } 15) there are 3 tables depending upon the value of \text{ind}_g(\text{mod } 3). All 3 tables are given below (see Tables 13a, 13b, 13c). The values of the cyclotomic numbers are given as linear combinations of

\[
p^{2}, p, 1, A^{2}, AB, B^{2}, AT, BT, AU, BU, T^{2}-15U^{2}, TU,
\]

using the following abbreviations

\[
\begin{align*}
x &= -2A^{2} + 2AB + B^{2}, \\
y &= A^{2} - 4AB + B^{2}, \\
z &= A^{2} + 2AB - B^{2}.
\end{align*}
\]
In the case \( p \equiv 11 \pmod{15} \) there are 25 tables depending upon the values of \( \text{ind}_g 2 \pmod{5} \) and \( \text{ind}_g 3 \pmod{5} \). The values of the cyclotomic numbers are given as linear combinations of

\[
p^2, p, 1, XU, XV, U^2, UV, UW, v^2, VW, W^2
\]

after eliminating the \( Q(i) \) and \( R(i) \) \( (0 \leq i \leq 4) \) by means of the following formulae (see (3.21) - (3.26), (5.51), 5.52):

\[
\begin{align*}
Q(0) &= \frac{4}{5}p + \frac{1}{5} - 8u^2 - 8v^2, \\
Q(1) &= -\frac{1}{5}p + \frac{1}{5} - 2XU - 2XV - 2U^2 - 4UV - 6UW + 6V^2 + 2VW, \\
Q(2) &= -\frac{1}{5}p + \frac{1}{5} - 2XU + 2XV + 6U^2 + 4UV - 2UW - 2V^2 - 6VW, \quad (6.20) \\
Q(3) &= -\frac{1}{5}p + \frac{1}{5} + 2XU - 2XV + 6U^2 + 4UV + 2UW - 2V^2 + 6VW, \\
Q(4) &= -\frac{1}{5}p + \frac{1}{5} + 2XU + 2XV - 2U^2 - 4UV + 6UW + 6V^2 - 2VW,
\end{align*}
\]

and

\[
\begin{align*}
R(0) &= \frac{4}{5}p + \frac{1}{5} - 4U^2 - 4V^2 - 8W^2, \\
R(1) &= -\frac{1}{5}p + \frac{1}{5} + 2XU + 2U^2 - 4UV - 2UW + 4VW + 2W^2, \\
R(2) &= -\frac{1}{5}p + \frac{1}{5} + 2XU + 4UV - 4UW + 2V^2 - 2VW + 2W^2, \quad (6.21) \\
R(3) &= -\frac{1}{5}p + \frac{1}{5} - 2XU + 4UV + 4UW + 2V^2 + 2VW + 2W^2, \\
R(4) &= -\frac{1}{5}p + \frac{1}{5} - 2XU + 2U^2 - 4UV + 2UW - 4VW + 2W^2.
\end{align*}
\]

Only 7 of these tables are presented in this paper as the remainder can be deduced from them as indicated below.

For \( n = 2, 3, 4 \) it is convenient to define \( n\* = 8, 2, 4 \) respectively. The 18 cases not covered in the given tables can be divided into 3 classes as follows:

(a) \( \text{ind}_g 2 \equiv 0 \pmod{5}, \text{ind}_g 3 \equiv n \pmod{5}, n = 2, 3, 4, \)

(b) \( \text{ind}_g 2 \equiv n \pmod{5}, \text{ind}_g 3 \equiv 0 \pmod{5}, n = 2, 3, 4, \)

(c) \( \text{ind}_g 2 \equiv m \pmod{5}, \text{ind}_g 3 \equiv n \pmod{5}, m = 1, 2, 3, 4; n = 2, 3, 4, \)

For cases in class (a) (resp. (b), resp. (c)) the value of \( (h,k)_{15} \) may be deduced from Table 14b (resp. Table 14c, resp. Table 14d (if \( m = n \pmod{5} \)), Table 14e if \( m = 2n \pmod{5} \)), Table 14f (if \( m = 3n \pmod{5} \)), Table 14g (if \( m = 4n \pmod{5} \)) by looking up the value of \( (n\*, n\*)_{15} \) and replacing \( X, U, V, W \) by

\[
\begin{align*}
X, -V, U, -W, \text{ if } n = 2, \\
X, V, -U, -W, \text{ if } n = 3, \\
X, -U, -V, W, \text{ if } n = 4.
\end{align*}
\]

These transformations are clear in view of the basic properties

\[
(h, k)_\gamma = (m^{-1}h, m^{-1}k)_{\gamma m},
\]

\[
\text{ind}_{\gamma m} a = m^{-1} \text{ind}_{\gamma} a \pmod{q-1},
\]

\[
E_{\gamma m}(\delta^{n}) = E_{\gamma}(\delta^{m^{-1}n}),
\]
where \((m, q-1) = 1\) and the dependence upon the generator \(\gamma\) or \(\gamma^m\) of \(GF(q)^*\) is shown by writing it as a subscript.

Table 13a: Cyclotomic numbers \((h,k)_{15}, p \equiv 4(\text{mod } 15), \text{ind}_5^g \equiv 0(\text{mod } 3)\)

|\(h,k\) | \(p^2\) | \(p\) | \(1\) | \(A,B\) | \(AT\) | \(BT\) | \(AU\) | \(BU\) | \(T^2-15U^2\) | \(TU\) |
|---|---|---|---|---|---|---|---|---|---|---|
|\((0,0)\) | 1 | 36 | -44 | x | 48 | -60 | 0 | 0 | 60 | 0 |
|\((0,1)\) | 1 | -3 | -14 | y | 13 | 13 | 45 | -45 | 2 | 180 |
|\((0,2)\) | 1 | -3 | -14 | z | -17 | 4 | 15 | -30 | -4 | 120 |
|\((0,3)\) | 1 | -9 | -14 | x | -2 | -5 | -60 | -15 | -10 | 60 |
|\((0,5)\) | 1 | 12 | -14 | z | -12 | 24 | 0 | 0 | -24 | 0 |
|\((0,6)\) | 1 | -9 | -14 | x | -2 | -5 | 60 | 15 | -10 | -60 |
|\((0,10)\) | 1 | 12 | -14 | y | -12 | 0 | 0 | 12 | 0 |
|\((0,11)\) | 1 | -3 | -14 | z | -17 | 4 | -15 | 30 | -4 | -120 |
|\((1,2)\) | 1 | 0 | 1 | x | 7 | -8 | -15 | -30 | -7 | -30 |
|\((1,3)\) | 1 | -3 | 1 | y | -2 | -2 | 60 | -60 | 2 | -60 |
|\((1,4)\) | 1 | 12 | 1 | z | -2 | 4 | 30 | 60 | -4 | 60 |
|\((1,5)\) | 1 | 0 | 1 | x | -8 | 7 | 30 | -15 | 5 | -60 |
|\((1,6)\) | 1 | -3 | 1 | y | -2 | -2 | 0 | 0 | 2 | 0 |
|\((1,7)\) | 1 | -3 | 1 | z | -2 | 4 | 30 | -30 | 11 | -30 |
|\((1,8)\) | 1 | 0 | 1 | x | 7 | 8 | 15 | 30 | -7 | 30 |
|\((1,9)\) | 1 | -3 | 1 | y | -2 | -2 | -60 | 60 | 2 | 60 |
|\((1,10)\) | 1 | -3 | 1 | z | -13 | -11 | -15 | -15 | -4 | 60 |
|\((1,12)\) | 1 | 12 | 1 | y | -2 | -2 | -60 | 60 | 2 | 60 |
|\((1,13)\) | 1 | -3 | 1 | z | -8 | 7 | -30 | 30 | -13 | 30 |
|\((2,5)\) | 1 | -2 | 1 | y | -2 | -2 | 0 | 0 | 2 | 0 |
|\((2,7)\) | 1 | 0 | 1 | x | -8 | 7 | 30 | -15 | 8 | 60 |
|\((2,8)\) | 1 | 12 | 1 | y | -2 | -2 | -60 | 60 | 2 | 60 |
|\((2,9)\) | 1 | 12 | 1 | z | -2 | 4 | -30 | -60 | -4 | -60 |
|\((2,12)\) | 1 | -3 | 1 | z | -13 | -11 | 15 | 15 | -4 | -60 |
|\((3,6)\) | 1 | 6 | 1 | y | -2 | 10 | -30 | 30 | 5 | 30 |
|\((3,9)\) | 1 | 6 | 1 | x | -2 | 10 | 30 | -30 | 5 | -30 |
|\((5,10)\) | 1 | 0 | 1 | x | 12 | 12 | 0 | 0 | -12 | 0 |

Table 13b: Cyclotomic numbers \((h,k)_{15}, p \equiv 4(\text{mod } 15), \text{ind}_5^g \equiv 1(\text{mod } 3)\)

|\(h,k\) | \(p^2\) | \(p\) | \(1\) | \(A,B\) | \(AT\) | \(BT\) | \(AU\) | \(BU\) | \(T^2-15U^2\) | \(TU\) |
|---|---|---|---|---|---|---|---|---|---|---|
|\((0,0)\) | 1 | 36 | -44 | x | 48 | 12 | 0 | 0 | 60 | 0 |
|\((0,1)\) | 1 | -3 | -14 | y | -17 | 13 | 15 | 15 | -4 | 120 |
|\((0,2)\) | 1 | -3 | -14 | z | -17 | 26 | 45 | 0 | 2 | 180 |
|\((0,3)\) | 1 | -9 | -14 | x | -2 | 7 | 60 | -75 | -10 | -60 |
|\((0,5)\) | 1 | 12 | -14 | z | -12 | 24 | 0 | 0 | 12 | 0 |
|\((0,6)\) | 1 | -9 | -14 | x | -2 | 7 | -60 | 75 | -10 | 60 |
|\((0,10)\) | 1 | 12 | -14 | z | -2 | 13 | -15 | -15 | -4 | -120 |
|\((0,11)\) | 1 | -3 | -14 | z | -13 | -26 | -45 | 0 | 2 | -180 |
|\((1,2)\) | 1 | 0 | 1 | x | 7 | 1 | 15 | -45 | -7 | 30 |
|\((1,3)\) | 1 | -3 | 1 | y | -2 | -2 | 30 | 0 | 11 | -30 |
|\((1,4)\) | 1 | 12 | 1 | z | -2 | 4 | -30 | 0 | -13 | 30 |
|\((1,5)\) | 1 | 0 | 1 | x | -8 | 1 | 30 | -15 | 8 | 60 |
|\((1,6)\) | 1 | -3 | 1 | y | -2 | -2 | -15 | -30 | -4 | -60 |
|\((1,7)\) | 1 | -3 | 1 | z | -2 | 4 | 60 | 0 | 2 | -60 |
|\((1,8)\) | 1 | 0 | 1 | x | 7 | 1 | -15 | 45 | -7 | -30 |
|\((1,9)\) | 1 | -3 | 1 | y | -2 | -2 | -30 | 0 | 11 | 30 |
|\((1,10)\) | 1 | -3 | 1 | z | -2 | 4 | 0 | 0 | 2 | 0 |
|\((1,12)\) | 1 | 12 | 1 | y | -2 | -2 | -30 | 90 | -4 | -60 |
|\((1,13)\) | 1 | -3 | 1 | z | -2 | 4 | -60 | 0 | 2 | 60 |
|\((2,5)\) | 1 | -2 | 1 | y | -2 | -2 | 0 | 0 | 2 | 0 |
|\((2,7)\) | 1 | 0 | 1 | x | -8 | 1 | -30 | 15 | 8 | -60 |
|\((2,8)\) | 1 | 12 | 1 | y | -2 | -2 | 30 | -90 | -4 | -60 |
|\((2,9)\) | 1 | 12 | 1 | z | -2 | 4 | 30 | 0 | -13 | -30 |
|\((2,12)\) | 1 | -3 | 1 | z | -2 | 4 | 0 | 0 | 2 | 0 |
|\((3,6)\) | 1 | 6 | 1 | x | -2 | -8 | 30 | 0 | 5 | -30 |
|\((3,9)\) | 1 | 6 | 1 | x | -2 | -8 | -30 | 0 | 5 | 30 |
|\((5,10)\) | 1 | 0 | 1 | x | 12 | -24 | 0 | 0 | -12 | 0 |
CYCLOTOMY OF ORDER 15 OVER GF(p^{2})

Table 13c: Cyclotomic numbers (h,k)_{15}, p \equiv 4 \pmod{15}, \text{ind}_5 g \equiv 2 \pmod{3}

| p^2 | p   | 1   | A,B | AT  | BT   | AU | BU | T^2-15U^2 | TU |
|-----|-----|-----|-----|-----|------|----|----|-----------|----|
| 225 | 0, 0 | 1   | 36  | -44 | x    | -24| 12 | 0         | 24 |
| 225 | 0, 1 | 1   | -3  | -14 | y    | 13 | -17| -15       | -15| 2         | 60 |
| 225 | 0, 2 | 1   | -3  | -14 | z    | 13 | 4  | -15       | 30 | 2         | 60 |
| 225 | 0, 3 | 1   | -9  | -14 | x    | -14| 7  | 0         | 45 | -16       | 0  |
| 225 | 0, 4 | 1   | 12  | -14 | z    | -12| 24 | 0         | 0  | 12        | 0  |
| 225 | 0, 5 | 1   | -9  | -14 | x    | -14| 7  | 0         | 45 | -16       | 0  |
| 225 | 0, 6 | 1   | -9  | -14 | y    | 13 | -17| 15       | 15 | 2         | -60|
| 225 | 0, 7 | 1   | -9  | -14 | y    | -12| -12| 0         | 0  | 12        | 0  |
| 225 | 0, 8 | 1   | -9  | -14 | z    | 13 | 4  | 15       | -30| 2         | -60|
| 225 | 0, 9 | 1   | 0   | 1   | x    | -2 | 1  | 0         | -15| 2         | 0  |
| 225 | 1, 0 | 1   | 0   | 1   | y    | -2 | 1  | 0         | -15| 2         | 0  |
| 225 | 1, 1 | 1   | 0   | 1   | z    | -2 | 2  | 0         | 30 | 2         | 120|
| 225 | 1, 2 | 1   | 0   | 1   | y    | -2 | 2  | 0         | 30 | 2         | 120|
| 225 | 1, 3 | 1   | 0   | 1   | z    | -2 | 2  | 0         | 30 | 2         | 120|
| 225 | 1, 4 | 1   | 0   | 1   | y    | -2 | 2  | 0         | 30 | 2         | 120|
| 225 | 1, 5 | 1   | 0   | 1   | z    | -2 | 2  | 0         | 30 | 2         | 120|
| 225 | 1, 6 | 1   | 0   | 1   | y    | -2 | 2  | 0         | 30 | 2         | 120|

Table 14a: Cyclotomic numbers (h,k)_{15}, p \equiv 11 \pmod{15}, \text{ind}_5 g \equiv 0 \pmod{5}, \text{ind}_3 g \equiv 0 \pmod{5}

| p^2 | p   | 1   | XU  | XV  | U^2 | UV | UW | V^2 | VW | W^2 |
|-----|-----|-----|-----|-----|-----|----|----|-----|----|-----|
| 225 | 0, 0 | 1   | -82 | -44 | 0   | 0  | 480| 0   | 480| 0   | 240|
| 225 | 0, 1 | 1   | 13  | -14 | -20| -110| -10| 160| -30| -180| 60 | -130|
| 225 | 0, 2 | 1   | 13  | -14 | -110| 20 | -180| -160| -60| -10 | -30 | -130|
| 225 | 0, 3 | 1   | 13  | -14 | 50 | -50 | 30 | 100| 150| -70 | 150 | 140 |
| 225 | 0, 4 | 1   | 13  | -14 | 20 | 110 | -10| 160| 30 | -180| 60 | -130|
| 225 | 0, 5 | 1   | -37 | -14 | 0  | 0  | 180| 0  | 180| 0   | 120|
| 225 | 0, 6 | 1   | 13  | -14 | -50| -50 | 70 | 100| 150| 30 | -150| 140 |
| 225 | 0, 7 | 1   | 13  | -14 | 110| -20 | -160| -160| 60 | 10  | -30 | -130|
| 225 | 0, 8 | 1   | 13  | -14 | 50 | 50  | 70 | 100| 150| 30 | 150 | 140 |
| 225 | 0, 9 | 1   | 13  | -14 | -50| 50  | 30 | -100| 150| -70 | -150| 140 |
| 225 | 0,12 | 1   | 13  | -14 | -50| 50  | 30 | -100| 150| -70 | -150| 140 |
| 225 | 1, 2 | 1   | -2  | 1   | 0  | 0  | -80| -40| 0  | 100 | 0   | 20  |
| 225 | 1, 3 | 1   | -2  | 1   | 0  | 0  | -20| 100| -60| -20 | -180| 20  |
| 225 | 1, 4 | 1   | -2  | 1   | 0  | 0  | -40| 220| 0  | 220 | 0   | 20  |
| 225 | 1, 5 | 1   | -2  | 1   | 40 | -80| 80 | 80 | 0  | 60  | 0   | 20  |
| 225 | 1, 6 | 1   | -2  | 1   | 20 | -40| 40 | -40| 20 | 60  | 0   | -120|
| 225 | 1, 7 | 1   | -2  | 1   | 0  | 0  | -20| -100| -180| -20 | 60  | 20  |
| 225 | 1, 8 | 1   | -2  | 1   | 0  | 0  | 100| 40 | 0  | -80 | 0   | 20  |
| 225 | 1, 9 | 1   | -2  | 1   | 0  | 0  | -20| -100| 180| -20 | 60  | 20  |
| 225 | 1,10 | 1   | -2  | 1   | -20| 40  | -40| -20| 60  | 0   | 120 | 20  |
| 225 | 1,11 | 1   | -2  | 1   | -40| 80  | 80 | -80| 0  | -60 | 0   | 20  |
| 225 | 1,13 | 1   | -2  | 1   | 0  | 0  | -20| 100| 60  | -20 | 180 | 20  |
| 225 | 2, 5 | 1   | -2  | 1   | 40 | 20  | 0  | 20 | -120| -40 | -60 | 20  |
| 225 | 2, 7 | 1   | -2  | 1   | 80 | 40  | -60| 80 | 0   | 80  | 0   | 20  |
| 225 | 2, 8 | 1   | -2  | 1   | 0  | 0  | 220| 40 | 0  | 40  | 0   | 20  |
| 225 | 2,10 | 1   | -2  | 1   | -80| -40| -60| 80 | 0   | 80  | 0   | 20  |
| 225 | 2,12 | 1   | -2  | 1   | -40| -20| 0  | 20 | 120 | -40 | 60  | 20  |
| 225 | 3, 6 | 1   | -2  | 1   | 0  | 0  | -20| -200| 0  | -20 | 0   | -160|
| 225 | 3, 9 | 1   | -2  | 1   | 0  | 0  | -20| 200| 0  | -20 | 0   | -160|
| 225 | 5,10 | 1   | 98  | 0   | 0  | -240| 0  | 0  | -240| 0   | -480|
| p^2 | p  | XU | XV | U^2 | UV | UW | V^2 | VW | W^2 |
|-----|----|----|----|-----|----|----|-----|----|-----|
| 225(0,0) | 1  | 68 | -44 | -180 | 60 | -180 | -360 | 180 | -60 |
| 225(0,1) | 1  | 13 | -14 | 40 | -30 | -90 | -120 | -230 | 0 |
| 225(0,2) | 1  | 13 | -14 | 50 | 100 | -140 | 80 | -60 | -30 |
| 225(0,3) | 1  | -37 | -14 | 70 | 10 | -30 | -60 | 30 | 190 |
| 225(0,4) | 1  | 13 | -14 | 0 | -50 | -130 | 240 | 30 | -160 |
| 225(0,5) | 1  | -37 | -14 | 0 | 0 | 180 | 0 | 0 | 180 |
| 225(0,6) | 1  | 13 | -14 | 70 | -90 | -30 | -60 | 130 | -210 |
| 225(0,7) | 1  | -37 | -14 | 10 | -20 | 180 | 0 | 60 | 130 |
| 225(0,8) | 1  | 13 | -14 | -30 | 10 | 170 | -60 | 30 | -10 |
| 225(0,9) | 1  | 13 | -14 | -130 | 10 | 70 | 140 | 30 | -110 |
| 225(1,2) | 1  | -2 | 1 | -10 | 20 | 0 | 40 | 60 | 70 |
| 225(1,3) | 1  | -2 | 1 | 50 | 0 | 120 | 40 | 40 | -110 |
| 225(1,4) | 1  | -2 | 1 | -10 | 20 | -60 | -140 | 0 | 70 |
| 225(1,5) | 1  | -2 | 1 | -20 | -60 | 60 | 0 | -140 | 60 |
| 225(1,6) | 1  | -2 | 1 | 30 | 40 | 20 | -60 | 60 | -10 |
| 225(1,7) | 1  | -2 | 1 | 20 | -40 | 0 | -80 | 120 | 160 |
| 225(1,8) | 1  | -2 | 1 | -10 | -30 | 30 | -80 | 10 | -110 |
| 225(1,9) | 1  | -2 | 1 | -10 | 20 | 60 | -20 | -120 | -50 |
| 225(1,10) | 1  | -2 | 1 | -50 | 0 | 0 | 120 | 40 | -90 |
| 225(1,11) | 1  | -2 | 1 | 0 | 100 | -40 | 0 | 60 | -40 |
| 225(1,12) | 1  | -2 | 1 | -10 | -30 | 90 | 160 | 10 | 70 |
| 225(2,5) | 1  | -2 | 1 | -40 | -20 | -20 | -100 | -120 | -20 |
| 225(2,7) | 1  | 23 | 1 | 70 | 10 | -30 | 60 | -90 | -50 |
| 225(2,8) | 1  | -2 | 1 | -40 | 30 | -30 | 40 | -50 | 40 |
| 225(2,9) | 1  | -2 | 1 | -40 | -20 | -80 | 40 | 60 | 100 |
| 225(2,10) | 1  | 23 | 1 | 10 | -20 | -180 | -60 | 120 | -110 |
| 225(3,6) | 1  | -2 | 1 | 20 | 60 | 120 | 40 | -20 | 40 |
| 225(3,9) | 1  | -2 | 1 | 20 | -40 | -180 | 40 | -120 | 40 |
| 225(5,10) | 1  | 23 | 1 | 90 | -30 | 90 | 180 | -90 | 30 |
Table 14c: Cyclotomic numbers \((h,k)_{15}\), \(p \equiv 11 (\text{mod } 15)\), \(\text{ind}_g^2 \equiv 1 (\text{mod } 5)\), \(\text{ind}_3 \equiv 0 (\text{mod } 5)\)

| \(p^2\) | \(p\) | \(l\) | \(XU\) | \(XV\) | \(U^2\) | \(UV\) | \(UW\) | \(V^2\) | \(VW\) | \(W^2\) |
|--------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 225(0, 0) | 1    | -7   | -44   | 90    | 30    | 30    | 300   | 330   | -270  | -210  | -60   |
| 225(0, 1) | 1    | -17  | -14   | -20   | 50    | 130   | -200  | 90    | 180   | 20    | 30    |
| 225(0, 2) | 1    | -7   | -14   | -20   | 0     | 180   | 0     | -240  | -120  | -20   | 20    |
| 225(0, 3) | 1    | 13   | -14   | -10   | 30    | -70   | 100   | -70   | 230   | -110  | -60   |
| 225(0, 4) | 1    | -17  | -14   | 50    | -40   | 40    | -40   | 0     | 70    | 250   | 170   |
| 225(0, 5) | 1    | -7   | -14   | 30    | 0     | 0     | 120   | 120   | -90   | -90   | -30   |
| 225(0, 6) | 1    | 13   | -14   | -110  | -70   | 130   | 100   | 30    | 30    | -10   | -60   |
| 225(0, 7) | 1    | 13   | -14   | -40   | 90    | -250  | -80   | -70   | 40    | -30   | -10   |
| 225(0, 8) | 1    | 13   | -14   | -10   | -70   | -170  | -100  | 30    | 130   | -110  | 140   |
| 225(0, 9) | 1    | 13   | -14   | -10   | -70   | -170  | -100  | 30    | 130   | -110  | 140   |
| 225(0,10) | 1    | 38   | -14   | 40    | -120  | -120  | 0     | -120  | -120  | 40    | -160  |
| 225(1, 0) | 1    | 3    | 1    | -40   | 0     | -20   | 20    | -20   | -60   | -60   | 20    |
| 225(1, 1) | 1    | -2   | 1    | -50   | -40   | 40    | 60    | 60    | -30   | 130   | -30   |
| 225(1, 2) | 1    | 18   | 1    | -10   | 30    | 10    | 80    | 70    | -150  | -90   | 20    |
| 225(1, 3) | 1    | 13   | 1    | 10    | 50    | -110  | -20   | 30    | 30    | -10   | -180  |
| 225(1, 4) | 1    | 13   | 1    | -40   | -10   | -110  | -40   | 30    | -80   | 40    | -130  |
| 225(1, 5) | 1    | -12  | 1    | -10   | 30    | 70    | 140   | -110  | 90    | -30   | 20    |
| 225(1, 6) | 1    | -2   | 1    | 40    | -10   | 10    | 0     | 30    | 60    | 40    | -30   |
| 225(1, 7) | 1    | 18   | 1    | 20    | -30   | -170  | -100  | -50   | -120  | 0     | -130  |
| 225(1, 8) | 1    | 17   | 1    | 10    | -10   | 70    | 40    | -90   | 90    | -70   | 120   |
| 225(1, 9) | 1    | -2   | 1    | 80    | -40   | 40    | 60    | -120  | -20   | 40    | 20    |
| 225(1,10) | 1    | -17  | 1    | 10    | -10   | 10    | -120  | 30    | 150   | -110  | 120   |
| 225(1,11) | 1    | -7   | 1    | -20   | 0     | 0     | 120   | 0     | 60    | 100   | 20    |
| 225(1,12) | 1    | -2   | 1    | -40   | 0     | -100  | 40    | 80    | 20    | 40    | 120   |
| 225(2, 0) | 1    | 13   | 1    | 70    | 20    | 40    | 60    | 60    | -30   | 130   | -30   |
| 225(2, 1) | 1    | 13   | 1    | -50   | -30   | 150   | -60   | 30    | -30   | -110  | 20    |
| 225(2, 2) | 1    | -7   | 1    | 20    | -30   | -140  | -10   | -20   | -80   | -30   | -30   |
| 225(2, 3) | 1    | -2   | 1    | 40    | 80    | -20   | 0     | -120  | -120  | 40    | -60   |
| 225(2, 4) | 1    | -7   | 1    | 10    | 30    | 130   | -100  | 130   | 30    | 90    | 140   |
| 225(2, 5) | 1    | 38   | 1    | 0     | 120   | 120   | -240  | -120  | 0     | 240   | 120   |
| \( p^2 \) | \( p \) | \( l \) | \( XU \) | \( XV \) | \( U^2 \) | \( UV \) | \( UW \) | \( V^2 \) | \( VW \) | \( W^2 \) |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 225(0, 0) | 1   | 53  | -44 | 90  | 30  | 150 | -100| -390| -390| 30  | -60 |
| 225(0, 1) | 1   | -7  | -14 | -40 | 90  | 50  | 120 | -110| 160 | -80 | 70  |
| 225(0, 2) | 1   | -17 | -14 | -20 | 0   | 20  | 40  | -40 | 40  | 180 | 220 |
| 225(0, 3) | 1   | -27 | -14 | -10 | 30  | -150| 20  | 10  | 110 | -270| 140 |
| 225(0, 4) | 1   | -7  | -14 | -90 | -60 | 20  | -160| 100 | 90  | 150 | -30 |
| 225(0, 5) | 1   | -7  | -14 | 30  | 0   | 0   | 120 | 120 | -90 | -90 | 30  |
| 225(0, 6) | 1   | 23  | -14 | -10 | 30  | -250| 10  | 10  | 130 | -260| -130|
| 225(0, 7) | 1   | 3   | -14 | 20  | -30 | -90 | 80  | 130 | -100| -60 | 40  |
| 225(0, 8) | 1   | 23  | -14 | 90  | 30  | 50  | 20  | 10  | -90 | 30  | -60 |
| 225(0, 9) | 1   | -2  | -14 | 40  | -120| 200 | -80 | -40 | 160 | -120| 40  |
| 225(0, 10)| 1   | -2  | 1   | 10  | 0   | 20  | -40 | 40  | -50 | -30 | 70  |
| 225(0, 11)| 1   | 8   | 1   | 40  | -20 | -120| 0   | 60  | 80  | 80  | -80 |
| 225(0, 12)| 1   | 13  | 1   | -20 | 0   | -40 | 80  | -80 | -200| 60  | -80 |
| 225(0, 13)| 1   | 8   | 1   | -40 | 0   | 80  | 120 | 40  | 40  | -80 | -80 |
| 225(0, 14)| 1   | 8   | 1   | 30  | 0   | -100| 20  | -20 | -90 | 90  | -30 |
| 225(0, 15)| 1   | 28  | 1   | -20 | 0   | -160| 40  | -80 | -180| -80 |
| 225(0, 16)| 1   | -7  | 1   | 40  | 40  | 0   | -60 | 60  | 20  | -80 |
| 225(0, 17)| 1   | -17 | 1   | 40  | 30  | 110 | 80  | -50 | 100 | 0   | 70  |
| 225(0, 18)| 1   | -22 | 1   | 20  | -30 | 110 | -120| 10  | 100 | -20 | 70  |
| 225(0, 19)| 1   | -7  | 1   | 30  | -90 | -70 | 20  | -110| 90  | -30 | 120 |
| 225(0, 20)| 1   | -7  | 1   | 10  | 40  | 120 | -60 | 180 | -10 | -10 | 70  |
| 225(1, 5) | 1   | -2  | 1   | -20 | 0   | 80  | 100 | -100| -20 | 0   | -80 |
| 225(1, 6) | 1   | 18  | 1   | -40 | 0   | -60 | 80  | 40  | 80  | 120 | 20  |
| 225(1, 7) | 1   | -7  | 1   | -20 | -50 | 90  | 60  | -30 | 20  | -40 | 70  |
| 225(1, 8) | 1   | 13  | 1   | -50 | -30 | -10 | -20 | -10 | -170| -30 | -80 |
| 225(1, 9) | 1   | 3   | 1   | -10 | 30  | -90 | -220| -110| 50  | -30 | 20  |
| 225(2, 5) | 1   | 8   | 1   | 40  | -20 | 0   | 120 | 60  | -40 | 80  | 40  |
| 225(2, 6) | 1   | -17 | 1   | -110| 30  | 50  | 20  | 10  | 10  | 30  | 40  |
| 225(5, 10)| 1   | 8   | 1   | 0   | 120 | 60  | 0   | 240 | 60  | 120 | 120 |
Table 14e: Cyclotomic numbers \((h, k)\) over \(\mathbb{F}(\sqrt{\rho})\)

\[
\begin{array}{cccccccccc}
\hline
p^2 & p & l & XU & XV & U^2 & UV & UW & V^2 & VW & W^2 \\
\hline
225(0,0) & 1 & 23 & -44 & 90 & 90 & 30 & 300 & -210 & 330 & 270 & -60 \\
225(0,1) & 1 & -7 & -14 & -10 & -60 & 180 & 200 & 60 & -70 & 30 & 170 \\
225(0,2) & 1 & -17 & -14 & -10 & -60 & 20 & 40 & -20 & 190 & -210 & 70 \\
225(0,3) & 1 & -7 & -14 & -110 & -110 & -70 & 100 & -10 & -170 & 70 & -60 \\
225(0,4) & 1 & 3 & -14 & 40 & 40 & 60 & -120 & -60 & -100 & -240 & -80 \\
225(0,5) & 1 & -7 & -14 & 0 & -30 & -90 & -120 & 90 & 0 & 120 & -30 \\
225(0,6) & 1 & -7 & -14 & -10 & 90 & 30 & -100 & -210 & -70 & -30 & 140 \\
225(0,8) & 1 & -7 & -14 & -20 & 10 & -70 & -200 & 50 & -20 & 100 & -30 \\
225(0,9) & 1 & 18 & -14 & 40 & 40 & -120 & 0 & 40 & 80 & 120 & 40 \\
225(0,12) & 1 & 43 & -14 & -10 & 90 & -70 & 100 & 190 & -170 & -30 & -260 \\
225(1,2) & 1 & 8 & 1 & 10 & -30 & -90 & 0 & -30 & 10 & 10 & -80 \\
225(1,3) & 1 & 3 & 1 & -40 & 50 & -70 & -100 & 30 & -20 & -60 & 70 \\
225(1,4) & 1 & -22 & 1 & 10 & 0 & 120 & -60 & -60 & 190 & 70 & 70 \\
225(1,5) & 1 & 8 & 1 & 20 & 0 & 120 & -40 & 120 & -100 & 60 & 20 \\
225(1,6) & 1 & -12 & 1 & -20 & 10 & -30 & -60 & -50 & 140 & 0 & 70 \\
225(1,7) & 1 & 23 & 1 & -50 & -30 & -30 & -60 & -90 & -110 & 130 & -80 \\
225(1,8) & 1 & -12 & 1 & 20 & 50 & -10 & 80 & -30 & -20 & 60 & 70 \\
225(1,9) & 1 & -7 & 1 & 10 & 0 & 0 & 120 & 120 & 10 & 10 & 70 \\
225(1,10) & 1 & 8 & 1 & 20 & 30 & -90 & 80 & 30 & -100 & -60 & -130 \\
225(1,11) & 1 & 3 & 1 & -110 & 10 & -30 & 60 & 70 & 50 & -90 & -80 \\
225(1,13) & 1 & 18 & 1 & 80 & -40 & -160 & -40 & 0 & -80 & 0 & -80 \\
225(2,5) & 1 & -17 & 1 & 20 & 30 & 170 & 40 & 70 & 40 & 60 & 70 \\
225(2,7) & 1 & 8 & 1 & -20 & -20 & 80 & -80 & 20 & 40 & -140 & 120 \\
225(2,8) & 1 & 3 & 1 & -10 & -10 & 50 & 20 & 90 & 10 & -30 & -80 \\
225(2,10) & 1 & -2 & 1 & 20 & 0 & 80 & -80 & -80 & 40 & 60 & -80 \\
225(2,12) & 1 & 8 & 1 & -20 & 10 & -10 & 160 & -190 & -80 & -20 & -30 \\
225(3,6) & 1 & 3 & 1 & -10 & -10 & -70 & -100 & 90 & 130 & -30 & 40 \\
225(3,9) & 1 & -22 & 1 & 40 & -60 & 180 & 0 & -60 & -20 & -60 & 40 \\
225(5,10) & 1 & 23 & 1 & 90 & -90 & -150 & -60 & -30 & -30 & -90 & 120 \\
\end{array}
\]
Table 14f: Cyclotomic numbers \((h,k)_{15}\), \(p = 11 \text{(mod 15)}\), \(\text{ind}_2 = 3 \text{(mod 5)}\), \(\text{ind}_3 = 1 \text{(mod 5)}\)

| \(p^2\) | \(p\) | \(1\) | \(XU\) | \(XV\) | \(U^2\) | \(UV\) | \(UW\) | \(V^2\) | \(VW\) | \(W^2\) |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 225(0,0) | 1   | -7  | -44 | -30 | 90  | 210 | 180 | 270 | 150 | -90 | 540  |
| 225(0,1) | 1   | 3   | -14 | 20  | -60 | 0   | -80 | 180 | 160 | -60 | -80  |
| 225(0,2) | 1   | 13  | -14 | -80 | -10 | 10  | 80  | 150 | -200| 0   | -130 |
| 225(0,3) | 1   | -37 | -14 | -30 | -10 | 210 | -220| -30 | 150 | -90 | -60  |
| 225(0,4) | 1   | 13  | -14 | 30  | 20  | -60 | 160 | -60 | 130 | 270 | -130 |
| 225(0,5) | 1   | -7  | -14 | 0   | 30  | -90 | -120| -90 | 0   | -120| -30  |
| 225(0,6) | 1   | -12 | -14 | 20  | -60 | 60  | 280 | -180| 100 | 60  | 40   |
| 225(0,7) | 1   | -7  | -14 | 30  | 20  | -60 | -40 | -180| -30 | -90 | -30  |
| 225(0,8) | 1   | 13  | -14 | -30 | -10 | -90 | -20 | -30 | 50  | -90 | 140  |
| 225(0,9) | 1   | 13  | -14 | 70  | -10 | 10  | 220 | -30 | -50 | 210 | 140  |
| 225(1,2) | 1   | -2  | 1   | -40 | 40  | 20  | -40 | 0   | 0   | 0   | 20   |
| 225(1,3) | 1   | 8   | 1   | 0   | -70 | -30 | 40  | -150| -60 | 0   | -30  |
| 225(1,4) | 1   | -2  | 1   | 20  | 40  | 20  | 140 | 60  | -60 | -120| 20   |
| 225(1,5) | 1   | 18  | 1   | 60  | 0   | -120| -80 | 0   | 40  | 0   | -80  |
| 225(1,6) | 1   | -2  | 1   | -30 | -10 | 30  | 100 | -30 | -70 | -90 | 20   |
| 225(1,7) | 1   | -2  | 1   | 20  | 40  | 140 | -40 | -120| 120 | -60 | 20   |
| 225(1,8) | 1   | -22 | 1   | 0   | -40 | 40  | 0   | 40  | 0   | 120 | 0    |
| 225(1,9) | 1   | -2  | 1   | -10 | -50 | -70 | -100| 90  | 30  | -30 | 20   |
| 225(1,10)| 1   | -12 | 1   | -10 | 0   | -80 | 0   | 10  | 150 | 70  |      |
| 225(1,11)| 1   | -2  | 1   | 0   | -40 | 180 | -80 | 0   | -160| 0   | 20   |
| 225(1,13)| 1   | 8   | 1   | -60 | 50  | -150| -80 | -30 | 60  | 60  | -30  |
| 225(2,5) | 1   | -2  | 1   | 70  | -10 | -50 | 20  | 90  | 10  | -30 | 20   |
| 225(2,7) | 1   | 38  | 1   | 0   | -40 | -60 | 80  | 0   | -120| 0   | -180 |
| 225(2,8) | 1   | 8   | 1   | 30  | 20  | 0   | -80 | 60  | -90 | 30  | -30  |
| 225(2,10)| 1   | -2  | 1   | 40  | 80  | 40  | 80  | 0   | -20 | 0   | 20   |
| 225(2,12)| 1   | -22 | 1   | -30 | -10 | 210 | 80  | 30  | 150 | 90  | 120  |
| 225(3,6) | 1   | 23  | 1   | -30 | -10 | -90 | -20 | -30 | -150| -90 | -60  |
| 225(3,9) | 1   | -2  | 1   | 20  | 40  | -40 | 80  | 120 | 0   | 60  | -160 |
| 225(5,10)| 1   | 38  | 1   | -120| 0   | -240| 0   | 60  | 0   | -180|      |
7. AN APPLICATION: PROOF OF THEOREM 8.

Appealing to the well-known congruence (stated here for GF(q))
\[ \text{ind}_Y (1 - \gamma^v) \equiv (q - 1)/2 + \sum_{u=1}^{e-1} u(u,v)_e \pmod{e}, \] (7.1)

see for example [9: p. 499], and taking \( e = 15, q = p^2, p \equiv 4(\text{mod} 15), \) we obtain, as (see (1.6))
\[ \frac{1}{2}(1 + \sqrt{5}) \equiv -\gamma^6(1 - \gamma^6)(1 - \gamma^3)^{-1} \pmod{p}, \] (7.2)

the congruence
\[ \text{ind}_Y \frac{1}{2}(1 + \sqrt{5}) \equiv \sum_{u=1}^{14} u((u,6)_{15} - (u,3)_{15}) \pmod{3}. \] (7.3)

Then, making use of Table 11, we obtain
\[ \text{ind}_Y \frac{1}{2}(1 + \sqrt{5}) \equiv (1,3)_{15} + (1,4)_{15} + (1,7)_{15}^* + (1,10)_{15} + (2,5)_{15} + (2,8)_{15}^* \]
\[ - (1,9)_{15} - (1,12)_{15} - (1,13)_{15} - (2,9)_{15} - (2,12)_{15} \pmod{3}. \] (7.4)

Substituting the values given in Tables 13a, 13b, 13c for \((1,3)_{15}, \ldots, (2,12)_{15}\) into (7.4) we obtain
\[ \text{ind}_Y \frac{1}{2}(1 + \sqrt{5}) \equiv \frac{1}{3}(T - A - B) \pmod{3}, \] (7.5)

independently of the value of \( \text{ind}_3 5(\text{mod} 3). \)

Reducing the expressions for \(225(0,0)_{15} - (0,1)_{15}\) given by Tables 13a, 13b, 13c modulo 9, and appealing to (1.3) and (1.4), we obtain

| \( p^2 \) | \( p \) | \( 1 \) | \( XU \) | \( XV \) | \( U^2 \) | \( UV \) | \( UW \) | \( V^2 \) | \( VW \) | \( W^2 \) |
|------|------|-----|------|------|------|------|------|------|------|------|
| 225 | 0 | 0 | 53 | -44 | 30 | -270 | -210 | 60 | 150 | -30 | -150 | -60 |
| 225 | 0 | 1 | 33 | -14 | 60 | 60 | -140 | -120 | 100 | -250 | -50 | -130 |
| 225 | 0 | 2 | 1 | 7 | -14 | 60 | -30 | 90 | -240 | -30 | 20 | 60 | -30 |
| 225 | 0 | 3 | 1 | 2 | -14 | 80 | 80 | 40 | 60 | 0 | -280 | 0 | -160 |
| 225 | 0 | 4 | 1 | -37 | -14 | 20 | 50 | 110 | -120 | 10 | 300 | -20 | 270 |
| 225 | 0 | 5 | 1 | 7 | -14 | -30 | 0 | 120 | -120 | -90 | 90 | -30 |
| 225 | 0 | 6 | 1 | -27 | -14 | -70 | 30 | 190 | 60 | 250 | 170 | -50 | 140 |
| 225 | 0 | 8 | 1 | 17 | -14 | -20 | 40 | 160 | -60 | 20 | 120 | 20 |
| 225 | 0 | 9 | 1 | 23 | -14 | -70 | -70 | -10 | 60 | -50 | -30 | -50 | -60 |
| 225 | 0,12 | 1 | 23 | -14 | 30 | 30 | -210 | 60 | -150 | 170 | -150 | -60 |
| 225 | 1 | 2 | 1 | -2 | 30 | -30 | 50 | 40 | -70 | -30 | -10 | 20 |
| 225 | 1,3 | 1 | -7 | 1 | -50 | 40 | 100 | 20 | 20 | 110 | 10 | -30 |
| 225 | 1,4 | 1 | -2 | 0 | -60 | -40 | -20 | 80 | -140 | -20 | -80 | 140 |
| 225 | 1,5 | 1 | 18 | 1 | -40 | 60 | 140 | 0 | -20 | -40 | 40 | 20 |
| 225 | 1,6 | 1 | -7 | 1 | -10 | -40 | 80 | 0 | 40 | -80 | -50 | 70 |
| 225 | 1,7 | 1 | 13 | 1 | 30 | 30 | 50 | 220 | 50 | -90 | 50 | 20 |
| 225 | 1,8 | 1 | -7 | 1 | -50 | -20 | -20 | 20 | 80 | -10 | -110 | -30 |
| 225 | 1,9 | 1 | 13 | 1 | 30 | 0 | -100 | -80 | -40 | -90 | -70 | -130 |
| 225 | 1,10 | 1 | -12 | 1 | 20 | 0 | -20 | 0 | -80 | 80 | -140 | 20 |
| 225 | 1,11 | 1 | 8 | 1 | 80 | 20 | 40 | 0 | -20 | 60 | 40 | -180 |
| 225 | 1,13 | 1 | -7 | 1 | -20 | -20 | 100 | 20 | -160 | -40 | -80 | 120 |
| 225 | 2,5 | 1 | 8 | 1 | -30 | 0 | -180 | -60 | 60 | -10 | 30 | -30 |
| 225 | 2,7 | 1 | 13 | 1 | -10 | 50 | 70 | -140 | 30 | 50 | 90 | 20 |
| 225 | 2,8 | 1 | 8 | 1 | 40 | 10 | -110 | -40 | -70 | 20 | 40 | -30 |
| 225 | 2,10 | 1 | 2 | 1 | 50 | -30 | -30 | 60 | 30 | 50 | 90 | 120 |
| 225 | 2,12 | 1 | -2 | 1 | 50 | -10 | 70 | 40 | 150 | -10 | -150 | -20 |
| 225 | 3,5 | 1 | 6 | 1 | -20 | -20 | 40 | 40 | -100 | 20 | 100 | -60 |
| 225 | 3,9 | 1 | 17 | 1 | 30 | 30 | -10 | -140 | 50 | -30 | 50 | 140 |
| 225 | 5,10 | 1 | 8 | 1 | 60 | 0 | 240 | -120 | -120 | -120 | -60 | 120 |
Combining (7.5) and (7.6) we get
\[ \text{ind}_g \frac{1}{2} (1 + \sqrt{5}) \equiv U^3 \equiv U \pmod{3}, \]
that is
\[ \text{ind}_g \frac{1}{2} (1 + \sqrt{5}) \equiv -U \pmod{3}, \]
which completes the proof of Theorem 8.

We remark that Theorem 8 can also be proved similarly to the proof of the case \( p \equiv 1 \pmod{15} \) [9: pp. 499-501].

8. CONCLUDING REMARKS.

During the preparation of this paper, a number of Eisenstein sums, Jacobi sums, Dickson-Hurwitz sums, cyclotomic numbers, were computed for several values of \( p \). These computations were performed using the facilities of the Computing Laboratory of the Department of Mathematics and Statistics at Carleton University and the numerical values obtained were of considerable use in formulating and checking the results. The help of Dr. B. C. Mortimer is acknowledged.

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