WELL-POSEDNESS AND LONG-TIME BEHAVIOUR FOR A NONLINEAR PARABOLIC EQUATION WITH HYSTERESIS

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Abstract. The work deals with a study of a nonlinear parabolic equation with hysteresis, containing a nonlinear monotone operator in the diffusion term. The well-posedness of the model equation is addressed by using an implicit time discretization scheme in conjunction with the piecewise monotonicity of the hysteresis operator, and a fundamental inequality due to M. Hilpert. A characterization of the $\omega$-limit set of the solution is then given through the study of the long-time behaviour of the solution of the equation in which we investigate the convergence of trajectories to limit points.

1. Statement of the problem

We consider a nonlinear parabolic problem with hysteresis functionals whose the diffusion term is a monotone operator arising from a convex functional. The main purpose is to study the well-posedness and the long-time behaviour. The model problem is stated as follows

\begin{align}
\frac{\partial}{\partial t} (cu + w) - \text{div} a(\cdot, \nabla u) &= f \text{ in } Q = \Omega \times (0, T) \\
w(x,t) &= W[u(x, \cdot); x](t) \text{ in } Q \\
u &= 0 \text{ on } \partial \Omega \times (0, T) \text{ and } u(x, 0) = u_0(x) \text{ in } \Omega
\end{align}

(1.1)

where $u$ is the unknown function, $T > 0$ is the final time, and $\Omega$ is a sufficiently smooth open bounded set in $\mathbb{R}^d$ locally located on one side of its boundary. The known data in (1.1) are the functions $c, f, W, a$ and $u_0$, which are constrained as follows.

(A1) The function $a = (a_i)_{1 \leq i \leq d}$ is defined by $a_i(x, \lambda) = \frac{\partial J}{\partial \lambda_i}(x, \lambda)$ where the function $J: \Omega \times \mathbb{R}^d \to [0, +\infty)$ satisfies the following conditions:

(i) $J(\cdot, \lambda)$ is measurable and differentiable for all $\lambda \in \mathbb{R}^d$,
(ii) $J(x, \cdot)$ is strictly convex for almost all $x \in \Omega$,
(iii) There exist three constants $p \geq 2$, $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\alpha_1 |\lambda|^p \leq J(x, \lambda) \leq \alpha_2 (1 + |\lambda|^p)$$

for all $\lambda \in \mathbb{R}^d$ and for almost all $x \in \Omega$.

(A2) $c \in L^\infty(\Omega)$ with $c \geq \alpha > 0$ where $\alpha$ is a constant independent of $x \in \Omega$.

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(A3) \( f \in L^2(Q) \) and \( u_0 \in W_0^{1,p}(\Omega) \).

(A4) For every \( x \in \Omega \), the hysteresis operator \( W[; x] \) is continuous on \( C([0, T]) \) and piecewise increasing. Moreover \( W \) is affine bounded and there exist a function \( \kappa_0 \in L^2(\Omega) \) and a positive constant \( \gamma_0 \) such that, for all \( \ell \in \mathbb{N} \), the parameterized final value mapping

\[
(s, x) \mapsto W_f(s; x), \ s = (v_0, \ldots, v_\ell) \in S
\]

is measurable and satisfies

\[
|W_f(s; x)| \leq \kappa_0(x) + \gamma_0 \|s\|_\infty,
\]

where \( W_f \) denotes the generating functional of the hysteresis operator \( W \) and \( S \) is the set of all finite strings of real numbers, a string being as usual a vector having either finitely or countably infinitely many real components.

Further information concerning the construction of \( W_f \) can be found in [4].

Besides piecewise monotonicity and continuity, we need a further assumption on the hysteresis operator \( W[; x] \), which will ensure uniqueness of the solution:

(A5) The operator \( W[; x] \) maps \( W^{1,1}(0, T) \) for every \( x \in \Omega \) into itself, and there exist \( \gamma_1 > 0 \) and \( \kappa_1 \in L^2(\Omega) \) such that the condition

\[
|W[v; x]'(t)| \leq \kappa_1(x) + \gamma_1 |v'(t)| \quad \forall x \in \Omega, \text{ for a.e. } t \in (0, T)
\]

is satisfied for every \( v \in W^{1,1}(0, T) \).

Of special interest as far as applications are concerned in this work is an existence and uniqueness result and an asymptotic behaviour in time for an evolution parabolic equation modeling a diffusion process with hysteresis; these questions are addressed in this paper. The problem (1.1) is an equation that can be regarded as a model of heat conduction including phase transition.

Hysteresis is defined as rate independent memory effect. The basic feature of hysteresis behaviour is a memory effect and the irreversibility of the process. Hysteresis was mentioned for the first time in an article [9] on magnetism published in 1885. It is a nonlinear phenomenon that occurs in many natural and constructed systems, and because of the strong nonlinearity of this phenomenon which is usually non-smooth, it has not been easy to treat it mathematically for a very long time. Hence it was only in the early seventies that a group of Russian scientists led by M. A. Krasnoselskii initiated a systematic mathematical investigation of the phenomenon of hysteresis which resulted in the fundamental monograph of Krasnoselskii and Pokrovskii [13]. During that time, many mathematicians have contributed to the mathematical theory, and the important monographs of Mayergoyz [19] and Visintin [25] have emerged. It is very important to note that Visintin intensively investigated PDEs with hysteresis.

Hysteresis operators can be seen as nonlinear causal functional operators. One of the main characteristics of these operators is the fact that they have memory character, i.e. the value at some time \( t \) do not only depends on the value of \( t \) at this precise moment, but it also depends on the previous evolutions and on inputs up to the time \( t \). For further results and references concerning hysteresis operators, see e.g. [7, 8, 10, 11, 24] and the references therein.
The work is organized as follows. In Section 2 we gather necessary elementary tools together with some functional spaces, and we state our main result. Suitable properties of the function $a(x, \lambda)$ are detailed in Section 3. Section 4 is dedicated to the proof of the existence result. The technique we use for proving the existence result is based on approximation by implicit time discretization, a priori estimates and passage to the limit by compactness. This approximation procedure is often used and is quite convenient in the analysis of equations that include a memory operator, as in any time-step we solve a stationary problem in which this operator is reduced to a superposition with a nonlinear function. The details of the passage to the limit in the nonlinear diffusion term is worked out carefully. To obtain uniqueness result, a fundamental inequality due to Hilpert [12] is employed as well as the $L^1$-stability of the solution of the equation. The latter is done in Section 5. Finally, we prove in the last section a result related to the long-time behaviour of the solution to (1.1).

2. Functional setting and statement of the main results

2.1. Functional setting. In order to facilitate the reading of this paper, we collect here some mathematical tools, starting with some well-known inequalities which will be used in the work.

The set of non-negative integers is denoted by $\mathbb{N}$ and $\mathbb{N}^* \equiv \mathbb{N} - \{0\}$ while $\mathbb{R}$ stands for the set of real numbers. As usual we denote the non-negative real numbers by $\mathbb{R}_+$. For $1 \leq d \in \mathbb{N}$, $\mathbb{R}^d$ stands for the numerical space of variables $x = (x_1, ..., x_d)$. For any Banach space $X$, we shall denote by $X'$ its topological dual.

We will need some fundamental inequalities which are given below:

**Lemma 2.1** (Young inequality). Suppose that $1 < p, p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ then

$$|ab| \leq \frac{1}{p} |a|^p + \frac{1}{p'} |b|^{p'} \quad \forall a, b \in \mathbb{R}, \, \delta > 0.$$ 

**Lemma 2.2** (Poincaré inequality). Let $\Omega$ be an open bounded set in $\mathbb{R}^d$ and $p \geq 1$ a real number. Then there is a constant $C = C(\Omega, p) > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined as the space of all functions $u \in L^p(\Omega)$ having generalized partial derivatives $D^\alpha u \in L^p(\Omega)$ for every multi-index $\alpha$ satisfying $0 \leq |\alpha| \leq m$. Endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left( \sum_{0 \leq |\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^p \, dx \right)^{\frac{1}{p}},$$

$W^{m,p}(\Omega)$ becomes a Banach space which is separable for $1 \leq p < \infty$. In the case $p = 2$, one obtains a Hilbert space denoted by $H^m(\Omega)$ and with the inner product

$$\langle u, v \rangle = \sum_{0 \leq |\alpha| \leq m} \int_\Omega D^\alpha u(x) \overline{D^\alpha v(x)} \, dx.$$
Also, denoting by \( C_0^\infty(\Omega) \) the space of infinitely differentiable functions on \( \Omega \) with compact supports, we define the space \( W_0^{m,p}(\Omega) \) as the closure of \( C_0^\infty(\Omega) \) in \( W^{m,p}(\Omega) \), and we denote by \( W^{-m,p'}(\Omega) \) the topological dual of \( W_0^{m,p}(\Omega) \) (integers \( m \geq 1 \) and real number \( p > 1 \)). For \( p = 2 \), instead of \( H^1(\Omega) \), we may in general define the fractional Sobolev spaces \( H^\sigma(\Omega) \) with \( \sigma \in \mathbb{R} \) as in \([1, 2]\).

Next, we need to introduce another class of function spaces which will be employed for the variational treatment of our evolution problem, namely spaces of the type \( W^{m,p}(0, T; V) \), where \( T > 0 \) is some final time and \( V \) is a certain function space. If \( V = \mathbb{R}^d \), no additional difficulties occur, since all relevant properties of \( W^{m,p}(0, T) \) carry over to the finite product \( \prod_{i=1}^d W^{m,p}(0, T) \). In the infinite-dimensional case, the definition of \( W^{m,p}(0, T; V) \) uses the notion of Bochner integrals which attain values in \( V \). Let us give a very brief introduction to this notion. For our purposes, we may restrict ourselves to the case where \( V \) is a reflexive Banach space, since we will exclusively deal with the space \( V = W_0^{1,p}(\Omega) \).

**Definition 2.1.** A function \( u : [0, T] \to V \) is called Bochner measurable, if it is the pointwise limit of a sequence \( (u_n) \) of simple functions. The function \( u \) is called Bochner integrable if

\[
\lim_{n \to \infty} \int_0^T \|u(t) - u_n(t)\|_V dt = 0,
\]

in which case the integral of \( u \) is defined by

\[
\int_0^T u(t) dt = \lim_{n \to \infty} \int_0^T u_n(t) dt.
\]

We denote by \( L^p(0, T; V) \) \((1 \leq p < \infty)\) the space of all Bochner measurable functions \( u : [0, T] \to V \) for which

\[
\|u\|_{L^p(0, T; V)} = \left( \int_0^T \|u(t)\|^p_V dt \right)^{\frac{1}{p}} < \infty.
\]

Equipped with \( \|\cdot\|_{L^p(0, T; V)} \), \( L^p(0, T; V) \) is a Banach space. Similarly, we define the space \( L^\infty(0, T; V) \), using the norm

\[
\|u\|_{L^\infty(0, T; V)} = \text{ess sup}_{t \in [0, T]} \|u(t)\|_V.
\]

For \( 1 < p < \infty \), the space \( L^p(0, T; V) \) is separable. In addition, its topological dual is isomorphic to \( L^{p'}(0, T; V') \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). After the definition of the spaces \( L^p(0, T; V) \), the spaces \( W^{m,p}(0, T; V) \) are introduced using the concept of distributions with values in Banach spaces. For the details of this construction, we refer the reader to \([1]\).

We end this subsection with an important result related to the existence result for monotone operators. Let \( X \) be a real reflexive Banach space, let \( A : X \to X' \) \((X' \text{ the topological dual of } X)\) and let \( \langle \cdot, \cdot \rangle \) denote the duality pairing between \( X \) and \( X' \). We recall the following definitions. The operator

- \( A \) is monotone if \( \langle Au_1 - Au_2, u_1 - u_2 \rangle \geq 0 \) for all \( u_1, u_2 \in X \);
- \( A \) is strictly monotone if \( \langle Au_1 - Au_2, u_1 - u_2 \rangle > 0 \) whenever \( u_1 \neq u_2 \);
- \( A \) is hemicontinuous if \( \lim_{t \to 0} A(u + tv) = Au \) in \( X'\)-weak* for all \( u, v \in X \).
• $A$ is coercive if

$$\lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} = \infty.$$ 

Let us consider the operator equation of the form

Find $u \in X$ such that $Au = b$. (2.1)

The existence issue for (2.1) is given by the next result.

**Theorem 2.1** (Browder-Minty). Suppose that $A$ is strictly monotone, hemicontinuous and coercive. Then (2.1) has a unique solution $u \in X$ for every $b \in X'$.

The proof of the above theorem can be found in [17] or alternatively in [22] where several applications have been given for various properties of monotone operators.

2.2. **Statement of the main results.** We first define the notion of weak solution we will deal with in this work.

**Definition 2.2.** Let the assumptions (A1)-(A4) hold. We say that a function $u : Q \to \mathbb{R}$ is a weak solution of (1.1) if

$$\left\{ \begin{array}{l}
 u \in L^\infty(0, T; W_0^{1,p}(\Omega)) \text{ with } u' \in L^2(0, T; L^2(\Omega)), \\
 w = \mathcal{W}(u; \cdot) \in L^2(Q) \cap L^2(\Omega; C([0, T])) \text{ with } w' \in L^p(0, T; W^{-1,p'}(\Omega))
\end{array} \right.$$ 

and $u$ satisfies equation (2.2)

$$\begin{align*}
\int_Q cu'(x,t)\varphi(x,t)dxdt + \int_0^T &\langle w' \cdot t, \varphi \cdot t \rangle dt \\
+ \int_Q &a(x, \nabla u(x,t)) \cdot \nabla \varphi(x,t)dxdt = \int_Q f(x,t)\varphi(x,t)dxdt
\end{align*}$$

(2.2)

for all $\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$.

The first main purpose of the work is to prove the following result.

**Theorem 2.2.** Let the assumptions (A1)-(A4) hold. Then there exists at least a solution $u$ in the sense of Definition 2.2. Moreover if assumption (A5) is satisfied, then $u$ is unique and the following estimate holds:

$$\alpha \int_{t_1}^{t_2} \int_\Omega |u(x,t)|^2 dxdt + 2\sigma(u(t_2)) - 2\sigma(u(t_1)) \leq \frac{1}{\alpha} \int_{t_1}^{t_2} \int_\Omega |f(x,t)|^2 dxdt$$

(2.3)

for all $0 \leq t_1 \leq t_2 \leq T$. Here $\alpha > 0$ is the same as in assumption (A2) and $\sigma(\cdot)$ is defined by (3.9).

It is an urgent matter to make precise the comparison of our first main result in Theorem 2.2 with the existing ones in the literature. This kind of problem has already been considered in several work; see, e.g., [4, 5, 7, 14, 21, 25], just to cite a few. Most of these work deal with linear diffusion operators while very few treat nonlinear cases. Although assumptions (A1)-(A4) are the natural way to generalize the linear operators (like the Laplacian) or the nonlinear ones (like the $p$-Laplacian), to the best of our knowledge, there is no work in the literature dealing with nonlinear PDEs with hysteresis and exhibiting such kind of
nonlinearity in the diffusion term. One of the work with assumptions close to ours is [14] in which the authors considered a nonlinear diffusion operator of the form

$$- \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i \left( \frac{\partial u}{\partial x_i} \right) \text{ for } u \in W^{1,p}(\Omega)$$

where the $a_i$'s are linear or nonlinear monotone functions defined on $\mathbb{R}$; see [14, p. 42]. We believe that one of the main difficulties in obtaining the solutions of (1.1) is about obtaining an energy inequality like (3.10). But this is a mere consequence of Lemma 3.1 (see also Remark 3.1) that stems from some properties of the functional $J$. Hence the inequality (3.10) is in order, thanks to the monotonicity property of the hysteresis operator.

Theorem 2.2 will be proved in Section 4. To do this, we proceed in several steps. First of all we have to approximate our model problem by employing an implicit time discretization scheme of (2.2), which leads to the semilinear variational equation

$$b(u_m, \varphi) + \int_{\Omega} b_m(x, u_m(x)) \varphi(x) dx = 0 \text{ for } \varphi \in W^{1,p}_0(\Omega),$$

where $u_m = u(x, mh)$ and the functionals $b$ and $b_m$ are defined below in Section 4. We then derive the following uniform estimates

$$\sum_{m=1}^{\ell} h \left\| \frac{u_m - u_{m-1}}{h} \right\|_{L^2(\Omega)}^2 + \sup_{1 \leq k \leq \ell} \| \nabla u_k \|_{L^p(\Omega)}^p \leq C. \quad (2.4)$$

Next, defining the linear interpolates $u_\ell$ and $\tilde{u}_\ell$ (see Section 4), we prove that the estimate

$$\int_Q |u'_\ell|^2 dx \, dt + \sup_{0 \leq t \leq T} \left( \| \nabla u_\ell(t) \|_{L^p(\Omega)}^p + \| \nabla \tilde{u}_\ell(t) \|_{L^p(\Omega)}^p \right) \leq C,$$

holds uniformly in $\ell \in \mathbb{N}$, and further

$$\| u_\ell - \tilde{u}_\ell \|_{L^2(Q)} \leq \frac{C}{\ell}. \quad (2.5)$$

The linear interpolate $u_\ell$ is the approximate solution of the discretized problem

$$c \frac{\partial u_\ell}{\partial t} + \frac{\partial w_\ell}{\partial t} - \text{div} \, a(\cdot, \nabla \tilde{u}_\ell) = f_\ell \text{ in } W^{-1,p'}(\Omega) \text{ a.e. in } (0, T).$$

It is important to note that the estimate (2.5) above allows us to prove that the sequences $u_\ell$ and $\tilde{u}_\ell$ have the same strong limit in $L^2(Q)$. This estimate replaces its counterpart in the linear setting where the following one

$$\| u_\ell - \tilde{u}_\ell \|_{L^2(0,T;H^1_0(\Omega))} \leq \frac{C}{\ell} \quad (2.6)$$

is used, enabling to conclude that the sequence $u_\ell - \tilde{u}_\ell$ strongly converges to $0$ in $L^2(0,T;H^1_0(\Omega))$. Estimate (2.6) stems from the equality

$$\| u_\ell - \tilde{u}_\ell \|_{L^2(0,T;H^1_0(\Omega))}^2 = \frac{T}{3\ell} \sum_{m=1}^{\ell} \| \nabla u_m - \nabla u_{m-1} \|_{L^2(\Omega)}^2.$$
However, in the nonlinear framework, the above equality is out of reach, and we therefore replace it by the following one

\[ \|u_\ell - \tilde{u}_\ell\|_{L^2(Q)} = \frac{T}{3\ell} \sum_{m=1}^\ell \|u_m - u_{m-1}\|_{L^2(\Omega)}, \]

which, thanks to (2.4), ensures the equality of the weak limits of both sequences \((u_\ell)\) and \((\tilde{u}_\ell)\) in \(L^p(0,T; W_0^{1,p}(\Omega))\).

In order to state the next main result, we need a further notion. We define the \(\omega\)-limit set of a solution \(u\) of (1.1) by

\[ \omega(u) = \{ \varphi \in W_0^{1,p}(\Omega) : \exists t_n \to +\infty \text{ such that } \lim_{n \to \infty} \|u(\cdot, t_n) - \varphi\|_{L^p(\Omega)} = 0 \}. \]

It is known (see e.g. [6, p. 1019]) that if \(u : \mathbb{R}_+ \to L^2(\Omega)\) is a solution of (1.1) such that the range \(\{u(\cdot, t) : t \geq 1\}\) is relatively compact in \(L^2(\Omega)\), then the \(\omega\)-limit set \(\omega(u)\) is nonempty.

The next result is related to the existence of \(\omega\)-limit sets of trajectories of (1.1). Here, we deal with global solutions \(u \in L^2(\mathbb{R}_+; L^2(\Omega)) \cap L^\infty(\mathbb{R}_+; W_0^{1,p}(\Omega))\) of (1.1) given by Theorem 2.2 in which assumption (A3) is replaced by (A3)1 below:

(A3)1 \( f \in H^1(\mathbb{R}_+; L^2(\Omega)) \) and \( u_0 \in W_0^{1,p}(\Omega) \) where \( \mathbb{R}_+ = [0, \infty) \).

It is important to note that, in view of the estimate (3.10) where the constant \(C\) is independent of \(T\), such solutions exist by Theorem 2.2. At this level, we are not requiring uniqueness, but only the existence of solutions to (1.1).

**Theorem 2.3.** Assume that (A1), (A2), (A3)1 and (A4) hold. Then for any \(u\) given by Theorem 2.2 and any sequence of times \((t_n)_n\) such that \(t_n \to \infty\) with \(n\), there exist a subsequence of \((t_n)_n\) still denoted by \((t_n)_n\) and a function \(u_\infty \in W_0^{1,p}(\Omega)\) such that

\[ u(\cdot, t_n) \to u_\infty \text{ in } L^p(\Omega)\text{-strong}, \]

where \(u_\infty\) solves the stationary problem

\[ -\text{div } a(\cdot, \nabla u_\infty) = g \text{ in } \Omega, \]

the function \(g\) being equal either to 0 (if \(f\) depends on the time variable \(t\)) or to \(f\) (if \(f\) does not depend on \(t\)).

It is important to note that assumption (A3)1 on \(f\) entails the continuity of \(f\) with respect to \(t\), so that we could define \(f(\cdot, t)\) for any \(t \geq 0\). It is therefore made only for that purpose. It can thus be replaced by \(f \in L^2(\mathbb{R}_+; L^2(\Omega)) \cap C(\mathbb{R}_+; L^2(\Omega))\). However, our main purpose in proving the existence of the solution of (1.1) is to looking for the qualitative properties of the solutions \(u\) of (1.1) under a more general assumption on the behaviour of the source term \(f\) and on the coefficient functions \(a(\cdot, \lambda)\) with respect to both the time scale \(\tau = t/\varepsilon_n\) and the space scale \(y = x/\varepsilon_n\) when the coefficients depend on \(y = x/\varepsilon_n\) and \(\tau = t/\varepsilon_n\), where \(\varepsilon_n\) is a sequence of positive real numbers verifying \(0 < \varepsilon_n \leq 1\) with \(\varepsilon_n \to 0\) as \(n \to \infty\). This falls within the scope of homogenization theory, and depends carefully on properties of the coefficients of the operators in (1.1). This is another issue which will be addressed in a very subsequent work.
3. SOME USEFUL PROPERTIES OF THE FUNCTION $a(x, \lambda)$ AND A PRELIMINARY ESTIMATE

We need to derive some useful properties of the function $a$. Since the function $J(x, \cdot)$ is convex and has a growth of order $p$ (see in particular the right-hand side of the inequality in (1.2)) it emerges from [18, Proof of Theorem 2.1] that
\[
|\nabla_{\lambda} J(x, \lambda)| \leq C_3 (1 + |\lambda|^{p-1}) \text{ for all } \lambda \in \mathbb{R}^d, \text{ a.e. } x \in \Omega. \tag{3.1}
\]
Indeed, since $J(x, \cdot)$ is convex and differentiable, it holds that
\[
J(x, \lambda) - J(x, \mu) \geq \nabla_{\lambda} J(x, \mu) \cdot (\lambda - \mu) \text{ for all } \lambda, \mu \in \mathbb{R}^d \text{ and a.e. } x \in \Omega. \tag{3.2}
\]
Choosing $\mu = \lambda + he_i$ with $h \in \mathbb{R}$ and $e_i$ the $i$th vector of the canonical basis of $\mathbb{R}^d$, it follows that
\[
\frac{\partial J}{\partial \lambda_i}(x, \lambda) \leq \frac{1}{h} (J(x, \lambda) - J(x, \lambda + he_i)) \text{ if } h > 0
\]
\[
\frac{\partial J}{\partial \lambda_i}(x, \lambda) \geq \frac{1}{h} (J(x, \lambda) - J(x, \lambda + he_i)) \text{ if } h < 0.
\]
Hence, taking $|h| = |\lambda| + 1$ above and using the right-hand side of (1.2), we are led to
\[
\frac{\partial J}{\partial \lambda_i}(x, \lambda) \leq \frac{1}{|h|} (J(x, \lambda) + J(x, \lambda + he_i))
\]
\[
\leq \frac{C_2}{2} \left( 1 + (|\lambda| + 1)^p + |\lambda|^p \right) |\lambda| + 1 \leq C_3 (1 + |\lambda|^{p-1})
\]
since $p > 1$, where $C_3$ depends on $C_2$ and $p$. We also infer from (3.1) that
\[
|J(x, \lambda) - J(x, \mu)| \leq c_2 (1 + |\lambda|^{p-1} + |\mu|^{p-1}) |\lambda - \mu|, \text{ all } \lambda, \mu \in \mathbb{R}^d \text{ and a.e. } x \in \Omega.
\]
It also follows from (3.2) that
\[
\nabla_{\lambda} J(x, \lambda) \cdot (\lambda - t\mu) \geq J(x, \lambda) - J(x, t\mu)
\]
for all $\lambda, \mu \in \mathbb{R}^d$, $t \geq 0$ and a.e. $x \in \Omega$. Letting $t \to 0$ and using the left-hand side of (1.2), we get
\[
\nabla_{\lambda} J(x, \lambda) \cdot \lambda \geq c_1 |\lambda|^p - J(x, 0).
\]
Another consequence of (3.2) is the monotonicity of $\nabla_{\lambda} J(x, \cdot)$ expressed as follows:
\[
(\nabla_{\lambda} J(x, \lambda) - \nabla_{\lambda} J(x, \mu)) \cdot (\lambda - \mu) \geq 0, \text{ all } \lambda, \mu \in \mathbb{R}^d, \text{ a.e. } x \in \Omega. \tag{3.3}
\]
Indeed, (3.2) yields
\[
J(x, \lambda) - J(x, \mu) \geq \nabla_{\lambda} J(x, \mu) \cdot (\lambda - \mu)
\]
and
\[
J(x, \mu) - J(x, \lambda) \geq \nabla_{\lambda} J(x, \lambda) \cdot (\mu - \lambda).
\]
Adding these inequalities together, we obtain (3.3).

We summarize the above properties of $a$ here below.

(A6) The function $a : (x, \lambda) \mapsto a(x, \lambda)$ from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$ is therefore constrained as follows:

(H) For each given $\lambda \in \mathbb{R}^d$, the function $x \mapsto a(x, \lambda)$ is measurable from $\Omega$ into $\mathbb{R}^d$. 


There exists a positive constant $\alpha_3$ such that $a(x, \lambda) \cdot \lambda \geq \alpha_1 |\lambda|^p - \alpha_3$, where $\alpha_3 = \|J(\cdot, 0)\|_{L^\infty(\Omega)}$.

There is a constant $C_2 > 0$, such that, a.e. in $x \in \Omega$, for $\lambda_1, \lambda_2 \in \mathbb{R}^d$, 

$$|a(x, \lambda_1) - a(x, \lambda_2)| \leq C_2(1 + |\lambda_1| + |\lambda_2|)^{p-2} |\lambda_1 - \lambda_2|,$$

where the dot denotes the usual Euclidean inner product in $\mathbb{R}^d$, and $|\cdot|$ the associated norm.

The following result will be of interest in the sequel.

**Lemma 3.1.** Let $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy (A6) above. Assume that $u \in L^\infty(0, T; W_0^{1,p}(\Omega))$, $u' \in L^2(0, T; H_0^1(\Omega))$, and $\text{div}a(\cdot, \nabla u) \in L^{p'}(0, T; W^{-1,p'}(\Omega))$. Then the function $t \mapsto \sigma(u(t)) = \int_\Omega J(\cdot, \nabla u(t)) dx$ is absolutely continuous on $(0, T)$ and

$$\frac{d}{dt} \sigma(u(t)) = - \langle \text{div}a(\cdot, \nabla u(t)), u'(t) \rangle \text{ for a.e. } t \in [0, T] \quad (3.4)$$

where $u(t) = u(\cdot, t)$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $L^{p'}(0, T; W^{-1,p'}(\Omega))$ and $L^p(0, T; W_0^{1,p}(\Omega))$.

**Proof.** Let $h > 0$ be arbitrarily fixed. Then using inequality (3.2), we obtain, for a.e. $t \in (0, T)$,

$$a(\cdot, \nabla u(t)) \cdot (\nabla u(t + h) - \nabla u(t)) \leq J(\cdot, \nabla u(t + h)) - J(\cdot, \nabla u(t)) \leq a(\cdot, \nabla u(t + h)) \cdot (\nabla u(t + h) - \nabla u(t)).$$

Integrating the above inequalities with respect to $(x, t)$, we have, for a.e. $0 \leq t_1 \leq t_2 \leq T$,

$$- \int_{t_1}^{t_2} \left\langle \text{div}a(\cdot, \nabla u(\tau)), \frac{u(\tau + h) - u(\tau)}{h} \right\rangle d\tau$$

$$= \frac{1}{h} \int_{t_1}^{t_2} \int_\Omega a(\cdot, \nabla u(\tau)) \cdot (\nabla u(\tau + h) - \nabla u(\tau)) dx d\tau$$

$$\leq \frac{1}{h} \int_{t_1}^{t_2} \int_\Omega J(\cdot, \nabla u(\tau + h)) dxd\tau - \frac{1}{h} \int_{t_1}^{t_2} \int_\Omega J(\cdot, \nabla u(\tau)) dxd\tau$$

$$\leq \frac{1}{h} \int_{t_1}^{t_2} \int_\Omega a(\cdot, \nabla u(\tau + h)) \cdot (\nabla u(\tau + h) - \nabla u(\tau)) dx d\tau$$

$$= - \int_{t_1}^{t_2} \left\langle \text{div}a(\cdot, \nabla u(\tau + h)), \frac{u(\tau + h) - u(\tau)}{h} \right\rangle d\tau$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairings between $L^{p'}(0, T; W^{-1,p'}(\Omega))$ and $L^p(0, T; W_0^{1,p}(\Omega))$. But

$$\int_{t_1}^{t_2} \int_\Omega J(\cdot, \nabla u(\tau + h)) dxd\tau = \int_{t_1 + h}^{t_2 + h} \int_\Omega J(\cdot, \nabla u(\tau)) dxd\tau$$
and \( \int_{t_1+h}^{t_2+h} - \int_{t_1}^{t_2} = \int_{t_2}^{t_2+h} - \int_{t_1}^{t_1+h} \), so that (3.3) becomes

\[
- \int_{t_1}^{t_2} \left\langle \text{div} \mathbf{a} (\cdot, \nabla u(\tau)), \frac{u(\tau+h)-u(\tau)}{h} \right\rangle \, d\tau \\
\leq \frac{1}{h} \int_{t_2}^{t_2+h} \int_{\Omega} J(\cdot, \nabla u(\tau)) \, dx \, d\tau - \frac{1}{h} \int_{t_1}^{t_1+h} \int_{\Omega} J(\cdot, \nabla u(\tau)) \, dx \, d\tau \tag{3.6}
\]

\[
\leq - \int_{t_1}^{t_2} \left\langle \text{div} \mathbf{a} (\cdot, \nabla u(\tau+h)), \frac{u(\tau+h)-u(\tau)}{h} \right\rangle \, d\tau.
\]

We recall that all the integrals involved in (3.6) are well defined according to the assumption (A6) and the fact that \( u \in L^\infty(0,T; W^{1,p}_0(\Omega)) \). Letting \( h \to 0 \) in (3.6) (where we use the other assumptions in Lemma 3.1) yields, for a.e. \( t_1, t_2 \),

\[
- \int_{t_1}^{t_2} \left\langle \text{div} \mathbf{a} (\cdot, \nabla u(\tau)), u'(\tau) \right\rangle \, d\tau = \int_{\Omega} J(\cdot, \nabla u(t_2)) \, dx - \int_{\Omega} J(\cdot, \nabla u(t_1)) \, dx
\]

\[
= \sigma(u(t_2)) - \sigma(u(t_1)),
\]

thereby showing that the mapping \( t \mapsto \sigma(u(t)) \) is absolutely continuous on \((0,T)\) and that (3.4) is satisfied. \( \square \)

**Remark 3.1.** Note that Lemma 3.1 remains true if the assumptions

\[ u' \in L^2(0,T; H^1_0(\Omega)) \quad \text{and} \quad \text{div} \mathbf{a}(\cdot, \nabla u) \in L^p(0,T; W^{-1,p'}(\Omega)), \]

are replaced by the following ones therein:

\[ u' \in L^2(0,T; L^2(\Omega)) \quad \text{and} \quad \text{div} \mathbf{a}(\cdot, \nabla u) \in L^2(0,T; L^2(\Omega)), \tag{3.7} \]

the other ones remaining unchanged. In that case, the duality pairings \( \langle \cdot, \cdot \rangle \) will be replaced by the inner product in \( L^2(0,T; L^2(\Omega)) \) and we will proceed by approximation like in [3, Proposition 2.11] to obtain (3.4) for the approximating sequence and conclude like in [3, Lemma 3.3] after a limit passage.

This being so, to see what regularity can be expected for a solution to (1.1), we first present an informal argument. We test (1.1) by \( u' = \frac{\partial u}{\partial t} \) and integrate over \( \Omega \) to obtain, for \( t > 0 \),

\[
\int_{\Omega} c |u'(t)|^2 \, dx + \int_{\Omega} u'(t)w'(t) \, dx - \left\langle \text{div} \mathbf{a}(\cdot, \nabla u(t)), u'(t) \right\rangle = \int_{\Omega} u'(t)f(t) \, dx \tag{3.8}
\]

where we have used the abbreviation \( u(t) = u(\cdot, t) \). Assuming \( \mathcal{W}[\cdot; x] \) is piecewise monotone for every \( x \in \Omega \), then we have \( u'u' \geq 0 \), so that the second term of the left-hand side of (3.8) becomes non-negative, i.e. \( \int_{\Omega} u'(t)w'(t) \, dx \geq 0 \). Since (see (3.4))

\[
- \left\langle \text{div} \mathbf{a}(\cdot, \nabla u(t)), u'(t) \right\rangle = \frac{d}{dt} \sigma(u(t))
\]

where

\[
\sigma(u(t)) = \int_{\Omega} J(x, \nabla u(x,t)) \, dx, \tag{3.9}
\]
we integrate (3.8) with respect to \( t \) and apply appropriate Young’s inequality to its right-hand side to get

\[
\alpha \int_0^t \int_\Omega |u'(\tau)|^2 \, dx \, d\tau + \sigma(u(t)) - \sigma(u_0) \leq \frac{1}{2\alpha} \int_0^t \int_\Omega |f|^2 \, dx \, d\tau + \frac{\alpha}{2} \int_0^t \int_\Omega |u'(\tau)|^2 \, dx \, d\tau,
\]

where we have also used the inequality \( c \geq \alpha \) in assumption (A2). Using the left-hand side of inequality (1.2), we infer

\[
\int_0^t \int_\Omega |u'(t)|^2 \, dx \, dt + \|\nabla u(t)\|_{L^p(\Omega)}^p \leq C \int_0^T \int_\Omega |f|^2 \, dx \, dt + C \|\nabla u_0\|_{L^p(\Omega)}^p,
\]

where \( C \) depends only on \( \alpha, \alpha_1 \) and \( p \). Hence we find the a priori estimate

\[
\int_0^t \int_\Omega |u'(t)|^2 \, dx \, dt + \sup_{0 \leq t \leq T} \|\nabla u(t)\|_{L^p(\Omega)}^p \leq C \int_0^T \int_\Omega |f|^2 \, dx \, dt + C \|\nabla u_0\|_{L^p(\Omega)}^p. \tag{3.10}
\]

Since \( f \in L^2(0, T; L^2(\Omega)) \equiv L^2(\Omega) \) and \( u_0 \in W^{1,p}_0(\Omega) \), we are thus led to look for weak solutions \( u \) in the space

\[
Y = L^\infty(0, T; W^{1,p}_0(\Omega)) \cap \dot{H}^1(0, T; L^2(\Omega)),
\]

which implies \( u \in C([0, T], L^2(\Omega)) \) thanks to the continuous imbedding

\[
Y \hookrightarrow C([0, T], L^2(\Omega)).
\]

Since \( u(x, \cdot) \) is the input of the hysteresis operator \( W[\cdot; x] \) at each space point \( x \in \Omega \), the compactness of the imbedding

\[
Y \hookrightarrow L^2(\Omega; C([0, T])) \tag{3.11}
\]

will play a crucial role in the existence proof if the hysteresis operators \( W[\cdot; x] \) are continuous on \( C([0, T]) \). The proof of the compactness of the embedding (3.11) is given in [4] Corollary 3.2.3 for \( d = 1 \); for \( d > 1 \), we follow the argument of Visintin based on interpolation theory, and obtain the result from the chain of continuous imbeddings

\[
Y \hookrightarrow H^1(Q) \hookrightarrow H^\sigma(\Omega; H^{1-\sigma}(0, T)) \hookrightarrow L^2(\Omega; C([0, T])) \tag{3.12}
\]

for \( 0 < \sigma < \frac{1}{2} \), where the last imbedding is compact.

4. Proof of Theorem 2.2: Existence result

For the proof of the existence result, we proceed in three steps listed in the following subsections.
4.1. **Approximation and existence of approximate solutions.** We have to approximate our model problem by employing an implicit time discretization scheme of (2.2). To this end, let \( \ell \in \mathbb{N} \setminus \{0\} \) (\( \mathbb{N} \) the set of nonnegative integers) be given, and set \( h = T / \ell \) where \( T \) is the final time. In the sequel, we will denote by \( C \), some positive constants that may depend on \( \Omega \), \( T \), \( f \), and the initial data, but neither on \( \ell \) nor on \( m \in \{1, \ldots, \ell\} \).

For \( 1 \leq m \leq \ell \), we consider the semidiscrete problem on the time level \( t = mh \) for the unknown functions \( u_m, w_m : \Omega \to \mathbb{R} \) given by

\[
\frac{1}{h} \int_{\Omega} c(u_m - u_{m-1}) \varphi dx + \frac{1}{h} \int_{\Omega} (w_m - w_{m-1}) \varphi dx + \int_{\Omega} a(x, \nabla u_m) \cdot \nabla \varphi dx \\
= \int_{\Omega} f_m \varphi dx \text{ for all } \varphi \in W_0^{1,p}(\Omega)
\]  

\[ w_m(x) = W_f((u_0(x), \ldots, u_m(x)); x) \text{ for a.e } x \in \Omega \]  

where \( u_0(x) = u(x, 0) \) is given by Assumption (A3) and

\[ w_m(x) = u(x, mh) \text{ and } f_m(x) = \frac{1}{h} \int_{(m-1)h}^{mh} f(x, t) dt, \quad w_0(x) = W_f(u_0(x); x). \]  

We can rewrite (4.1) in the following form:

\[
cu_m - u_{m-1} \frac{h}{h} + w_m - w_{m-1} \frac{h}{h} - \text{div} a(x, \nabla u_m) = f_m \text{ in } W^{-1,p'}(\Omega). \]  

We rewrite (4.1) as a semilinear variational equation,

\[ b(u_m, \varphi) + \int_{\Omega} b_m(x, (u_m(x))) \varphi(x) dx = 0 \text{ for all } \varphi \in W_0^{1,p}(\Omega) \]  

where

\[ b(u, \varphi) = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx \text{ for } u, \varphi \in W_0^{1,p}(\Omega) \]

and the function \( b_m : \Omega \times \mathbb{R} \to \mathbb{R} \), which is defined by

\[ b_m(x, u) = \frac{1}{h} (cu + W_f((u_0(x), \ldots, u_m(x))); x) \]

\[ -\frac{1}{h} (cu_{m-1}(x) + w_{m-1}(x)) - f_m(x), \]

is measurable in \( x \) and continuous in \( u \). Moreover \( b_m(x, \cdot) \) is strictly increasing; indeed \( W[\cdot; x] \) is piecewise increasing for each fixed \( x \in \Omega \). Finally, we infer from (1.3) that, for all \( (x, u) \in \Omega \times \mathbb{R} \),

\[ |b_m(x, u)| \leq \kappa_2(x) + C \left( |w_{m-1}(x)| + \sum_{k=0}^{m-1} |u_k(x)| \right) + C |u|, \]

with a suitable positive constants \( C \) and some function \( \kappa_2 \in L^2(\Omega) \). Considering the induction over \( m \) (where we have used [1 Theorem 1.3.2] for the induction step) associated to Theorem 2.1, we derive, for each \( m \in \{1, \ldots, \ell\} \), the existence of a unique \( u_m \in W_0^{1,p}(\Omega) \) solution to (1.5). Moreover, the function \( w_m \), as defined by (4.2), belongs to \( L^2(\Omega) \).

With the sequence \( (u_m, w_m)_m \) in hands, the next step is to find appropriate uniform estimates which will be used in order to pass to the limit.
4.2. **Uniform estimates.** The goal here is first to derive the discrete version of (3.10), and next to apply it to obtain a continuous version similar to (3.10), but for a sequence of linear interpolates of $u_m$. The first result reads as follows.

**Lemma 4.1.** Let $u_m$ be defined by (4.5). Then one has

$$
\sum_{m=1}^{\ell} h \left\| \frac{u_m - u_{m-1}}{h} \right\|_{L^2(\Omega)}^2 + \sup_{1 \leq k \leq \ell} \| \nabla u_k \|_{L^p(\Omega)}^p \leq C
$$

(4.6)

for all positive integer $\ell$, where $C = C(p, u_0, \alpha, \alpha_1, \alpha_2)$.

**Proof.** Let the integer $m \geq 1$ be fixed. We insert $\varphi = u_m - u_{m-1}$ in (4.1) and we have

$$
\frac{1}{h} \int_{\Omega} c(u_m - u_{m-1})^2 dx + \frac{1}{h} \int_{\Omega} (w_m - w_{m-1})(u_m - u_{m-1}) dx
+ \int_{\Omega} a(x, \nabla u_m) \cdot (\nabla u_m - \nabla u_{m-1}) dx = \int_{\Omega} f_m (u_m - u_{m-1}) dx.
$$

(4.7)

Since $\mathcal{W}[\cdot; x]$ is piecewise increasing for every $x \in \Omega$, it holds $(u_m - u_{m-1})(w_m - w_{m-1}) \geq 0$ and so, using the fact that the second term of the left-hand side of (4.7) is non-negative and using Assumption (A2) we obtain

$$
\frac{1}{h} \alpha \left\| u_m - u_{m-1} \right\|_{L^2(\Omega)}^2 + \int_{\Omega} a(x, \nabla u_m) \cdot (\nabla u_m - \nabla u_{m-1}) dx \leq \int_{\Omega} f_m (u_m - u_{m-1}) dx.
$$

(4.8)

We sum both sides of (4.8) from $m = 1$ to $m = k$ (where $1 \leq k \leq \ell$) to get

$$
\sum_{m=1}^{k} \frac{1}{h} \alpha \left\| u_m - u_{m-1} \right\|_{L^2(\Omega)}^2 + \sum_{m=1}^{k} \int_{\Omega} a(x, \nabla u_m) \cdot (\nabla u_m - \nabla u_{m-1}) dx \leq \sum_{m=1}^{k} \int_{\Omega} f_m (u_m - u_{m-1}) dx.
$$

(4.9)

Using Cauchy-Schwarz’s inequality on the right-hand side of (4.9), we obtain

$$
\sum_{m=1}^{k} \int_{\Omega} f_m (u_m - u_{m-1}) dx \leq \left( \sum_{m=1}^{k} h \| f_m \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \left( \sum_{m=1}^{k} h \left\| \frac{u_m - u_{m-1}}{h} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
$$

(4.10)

Note that

$$
\sum_{m=1}^{k} h \| f_m \|_{L^2(\Omega)}^2 \leq \int_{0}^{T} \int_{\Omega} |f(x,t)|^2 dx dt \leq C.
$$

(4.11)

Hence we have the inequality

$$
\sum_{m=1}^{k} \int_{\Omega} f_m (u_m - u_{m-1}) dx \leq C \left( \sum_{m=1}^{k} h \left\| \frac{u_m - u_{m-1}}{h} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
$$

(4.12)

Now we have to deal with the second term of the left-hand side of (4.9). To this end, we first recall that, according to the definition of the function $a$ given in Assumption (A3), we have

$$
a(x, \nabla u_m) \cdot (\nabla u_m - \nabla u_{m-1}) = \nabla \chi J(x, \nabla u_m) \cdot (\nabla u_m - \nabla u_{m-1}).
$$

(4.13)
It follows from (3.2) that
\[ J(x, \nabla u_m) - J(x, \nabla u_{m-1}) \leq a(x, \nabla u_m) : (\nabla u_m - \nabla u_{m-1}). \] (4.14)
Hence
\[ \sum_{m=1}^{k} [J(x, \nabla u_m) - J(x, \nabla u_{m-1})] \leq \sum_{m=1}^{k} a(x, \nabla u_m) : (\nabla u_m - \nabla u_{m-1}), \] (4.15)
that is,
\[ J(x, \nabla u_k) - J(x, \nabla u_0) \leq \sum_{m=1}^{k} a(x, \nabla u_m) : (\nabla u_m - \nabla u_{m-1}). \] (4.16)
Integrating (4.16) over \( \Omega \) gives
\[ \int_{\Omega} (J(x, \nabla u_k) - J(x, \nabla u_0)) \, dx \leq \sum_{m=1}^{k} \int_{\Omega} a(x, \nabla u_m) : (\nabla u_m - \nabla u_{m-1}) \, dx. \] (4.17)
In view of (1.2) it holds that
\[ J(x, \nabla u_0) \leq \alpha_2 (1 + |\nabla u_0|^p) \] and \( \alpha_1 |\nabla u_k|^p \leq J(x, \nabla u_k), \)
so that
\[ \alpha_1 \|\nabla u_k\|_{L^p(\Omega)}^p \leq \int_{\Omega} \left( \sum_{m=1}^{k} a(x, \nabla u_m) : (\nabla u_m - \nabla u_{m-1}) \right) \, dx + \alpha_2 \left( |\Omega| + \|\nabla u_0\|_{L^p(\Omega)}^p \right) \] (4.18)
where \( |\Omega| = \text{meas}(\Omega) \) is the Lebesgue measure of \( \Omega \). Summarizing (4.9) to (4.18), we obtain
\[ \sum_{m=1}^{k} \frac{1}{h} \alpha \|u_m - u_{m-1}\|_{L^2(\Omega)}^2 + \alpha_1 \|\nabla u_k\|_{L^p(\Omega)}^p \leq C \left( \sum_{m=1}^{k} \frac{h}{h} \|u_m - u_{m-1}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} + \alpha_2 (|\Omega| + \|\nabla u_0\|_{L^p(\Omega)}^p). \] (4.19)
Applying Young’s inequality to the first term of the right-hand side of (4.19), we get
\[ C \left( \sum_{m=1}^{k} \frac{h}{h} \|u_m - u_{m-1}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \leq \frac{\alpha}{2} \sum_{m=1}^{\ell} \frac{h}{h} \|u_m - u_{m-1}\|_{L^2(\Omega)}^2 + C. \]
Coming back to (4.19), we end up with the following inequality
\[ \sum_{m=1}^{\ell} \frac{h}{h} \|u_m - u_{m-1}\|_{L^2(\Omega)}^2 + \sup_{1 \leq k \leq \ell} \|\nabla u_k\|_{L^p(\Omega)}^p \leq C \] (4.20)
where \( C = C(p, u_0, \Omega, \alpha, \alpha_1, \alpha_2) > 0. \)

Now, we have to define the linear interpolates. In order to emphasize the dependence on \( \ell \) of the sequence \( (u_m)_m \), we denote by \( u_{m,\ell} \) and \( w_{m,\ell} \) respectively the solutions of (4.11) and (4.2) for any \( \ell \in \mathbb{N} \), and by \( f_{m,\ell} \) the averages defined in (4.13) for \( 0 \leq m \leq \ell \). We define the piecewise linear interpolates as follows:
\[ u_{\ell}(x, (m + \tau)h) = \tau u_{m+1,\ell}(x) + (1 - \tau)u_{m,\ell}(x), \quad \tau \in [0, 1] \]
\[ w_{\ell}(x, (m + \tau)h) = \tau w_{m+1,\ell}(x) + (1 - \tau)w_{m,\ell}(x), \quad \tau \in [0, 1], \] (4.21)
as well as the constant interpolates
\[
\tilde{u}_{\ell}(x, (m + \tau)h) = u_{m+1,\ell}(x); \quad \tilde{f}_{\ell}(x, (m + \tau)h) = f_{m+1,\ell}(x), \quad \tau \in [0, 1]
\]
for \(0 \leq m \leq \ell - 1\). With the above notation, (4.1) reads as
\[
\int_{\Omega} cu_{\ell}(x, t)\varphi(x)dx + \int_{\Omega} w_{\ell}(x, t)\varphi(x)dx + \int_{\Omega} a(x, \nabla \tilde{u}_{\ell}) \cdot \nabla \varphi(x)dx = \int_{\Omega} \tilde{f}_{\ell}(x, t)\varphi(x)dx \quad \text{for all } \varphi \in W_{0}^{1,p}(\Omega) \text{ and a.e. } t \in (0, T).
\]
Thus (4.4) becomes
\[
cu + \partial w_{\ell} - \text{div}\ a(x, \nabla \tilde{u}_{\ell}) = \tilde{f}_{\ell} \text{ in } W^{-1,p'}(\Omega) \text{ a.e. in } (0, T).
\]

**Lemma 4.2.** Let \(u_{\ell}, w_{\ell}\) and \(\tilde{u}_{\ell}\) satisfying (4.22). Then there exists a positive constant \(C\) independent of \(\ell\) such that
\[
\|u_{\ell}'\|_{L^{2}(\Omega)}^{2} + \sup_{0 \leq t \leq T} \left( \|\nabla \tilde{u}_{\ell}(t)\|_{L^{p}(\Omega)}^{p} + \|\nabla u_{\ell}(t)\|_{L^{p}(\Omega)}^{p} \right) \leq C,
\]
\[
\|w_{\ell}\|_{L^{2}(\Omega)} \leq C, \quad \|a(\cdot, \nabla \tilde{u}_{\ell})\|_{L^{p'}(\Omega)} \leq C,
\]
and
\[
\left\| \frac{\partial}{\partial t} (cu_{\ell} + w_{\ell}) \right\|_{L^{p}(0,T;W^{-1,p'}(\Omega))} \leq C
\]
for all \(\ell \in \mathbb{N}\).

**Proof.** First of all, it follows from (4.22) that
\[
\int_{Q} (cu_{\ell}' + w_{\ell}')\varphi dxdt + \int_{Q} a(\cdot, \nabla \tilde{u}_{\ell}) \cdot \nabla \varphi dxdt = \int_{Q} \tilde{f}_{\ell}\varphi dxdt
\]
for all \(\varphi \in L^{p}(0,T;W_{0}^{1,p}(\Omega))\).

Proceeding exactly (multiply (4.23) by \(u_{\ell}'\) and integrate over \(\Omega \times (0, T)\)) as we did in Section 3 to obtain the estimate (5.10), we get mutatis mutandis:
\[
\int_{Q} |u_{\ell}'|^{2} dxdt + \sup_{0 \leq t \leq T} \|\nabla \tilde{u}_{\ell}(t)\|_{L^{p}(\Omega)}^{p} \leq C \int_{Q} |\tilde{f}_{\ell}|^{2} dxdt + C \|\nabla u_{0}\|_{L^{p}(\Omega)}^{p}.
\]
By virtue of the estimate (4.20) we obtain:
\[
\int_{Q} |u_{\ell}'|^{2} dxdt + \sup_{0 \leq t \leq T} \left( \|\nabla \tilde{u}_{\ell}(t)\|_{L^{p}(\Omega)}^{p} + \|\nabla u_{\ell}(t)\|_{L^{p}(\Omega)}^{p} \right) \leq C.
\]

Using the inequality (3.1), we get
\[
\|a(\cdot, \nabla u_{\ell})\|_{L^{p'}(\Omega)} \leq C (1 + \|\nabla u_{\ell}\|_{L^{p}(\Omega)}^{p}) \leq C
\]
where \(C\) is a positive constant depending on \(\text{meas}(\Omega)\) and \(T\), but not on \(\ell\).

Also
\[
\|u_{\ell}\|_{L^{p}(0,T;W_{0}^{1,p}(\Omega)) \cap H^{1}(0,T;L^{2}(\Omega))} \leq C
\]
and

\[ \| \tilde{u}_\ell \|_{L^\infty(0,T;W^{1,p}_0(\Omega))} \leq C \]  

(4.28)

The inequality (3.1) associated to (4.28) yield

\[ \| \mathbf{a}(\cdot, \nabla \tilde{u}_\ell) \|_{L^{p'}(Q)} \leq C. \]

Finally we find from (4.28) and (4.24) that for any \( \varphi \in L^p(0,T;W^{1,p}_0(\Omega)) \),

\[ \left| \int_Q u'_\ell \varphi dxdt \right| \leq C \| \varphi \|_{L^p(0,T;W^{1,p}_0(\Omega))}, \]

(4.29)

so that

\[ \| u'_\ell \|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C. \]

(4.30)

According to assumption (A4), \( \mathcal{W} \) is affine bounded, i.e. there exist \( L > 0 \) and \( \psi \in L^2(\Omega) \) such that for any measurable function \( u : \Omega \rightarrow C([0,T]) \) we have

\[ \| \mathcal{W}(u)(x,\cdot) \|_{C([0,T])} \leq L \| u(x,\cdot) \|_{C([0,T])} + \psi(x) \text{ a.e. in } \Omega, \]

(4.31)

and using (4.27) and (4.31), we get

\[ \| w_\ell \|_{L^2(Q)} \leq \sqrt{T} \| \tilde{u}_\ell \|_{L^2(\Omega;C([0,T]))} \leq \sqrt{T} L \| u_\ell \|_{L^2(\Omega;C([0,T]))} + \sqrt{T} \| \psi \|_{L^2(\Omega)} \leq C. \]

where \( C > 0 \) is independent of \( \ell \). So we obtain

\[ \| w_\ell \|_{L^2(Q)} \leq C. \]

The same reasoning as in (4.29) yields

\[ \left\| \frac{\partial}{\partial t} (cu_\ell + w_\ell) \right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C. \]

\[ \square \]

### 4.3. Passage to the limit.

Our goal here is to pass to the limit in each term of the variational formulation (4.25).

The a priori estimates we found in Lemma 4.2 allow us to conclude that, by a standard compactness result which can be found in [17], e.g., the sequence \( (u_\ell) \) stays in a compact subset of \( L^2(Q) \). Invoking some well-known results, we derive the existence of \( u, \tilde{u} \in L^\infty(0,T;W^{1,p}_0(\Omega)), v \in L^p(Q)^N, V \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \) and \( w \in L^2(Q) \) such that, up to a subsequence not relabeled, we have

\[ u_\ell \rightarrow u \text{ in } L^p(0,T;W^{1,p}_0(\Omega))-\text{weak and in } L^\infty(0,T;W^{1,p}_0(\Omega))-\text{weak*} \]

\[ \tilde{u}_\ell \rightarrow \tilde{u} \text{ in } L^p(0,T;W^{1,p}_0(\Omega))-\text{weak and in } L^\infty(0,T;W^{1,p}_0(\Omega))-\text{weak*} \]

\[ w_\ell \rightarrow w \text{ in } L^2(Q)-\text{weak} \]

\[ \frac{\partial u_\ell}{\partial t} \rightarrow \frac{\partial u}{\partial t} \text{ in } L^2(Q)-\text{weak} \]

\[ \mathbf{a}(\cdot, \nabla \tilde{u}_\ell) \rightarrow \mathbf{v} \text{ in } L^{p'}(Q)^N-\text{weak} \]

\[ \frac{\partial}{\partial t}(cu_\ell + w_\ell) \rightarrow V \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega))-\text{weak.} \]
It follows readily from (4.32) that \( cu_\ell \to cu \) in \( L^2(Q) \)-strong, so that, appealing to (4.33), we get at once
\[
V = \frac{\partial}{\partial t}(cu + w).
\]

We deduce from (4.30) that \( w' \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \) with \( w_\ell' \to w' \) in \( L^{p'}(0,T;W^{-1,p'}(\Omega)) \)-weak. Hence
\[
w \in L^2(Q) \text{ with } w' \in L^{p'}(0,T;W^{-1,p'}(\Omega))
\]

Let us next check that \( u = \tilde{u} \). To that end, we observe that
\[
\|u_\ell - \tilde{u}_\ell\|_{L^2(Q)}^2 = \sum_{m=0}^{\ell-1} \left( \int_{\frac{m+1}{h}}^{(m+1)h} (1 - \tau)^2 dt \right) \|u_{m+1} - u_m\|^2_{L^2(\Omega)}
= \sum_{m=0}^{\ell-1} \|u_{m+1} - u_m\|^2_{L^2(\Omega)} \int_{\frac{m+1}{h}}^{(m+1)h} \left( 1 + m - \frac{t}{h} \right)^2 dt
= \frac{h^3}{3} \sum_{m=0}^{\ell-1} \|u_{m+1} - u_m\|^2_{L^2(\Omega)}
\leq C \frac{h^3}{3}
\]

where for the last inequality above, we have used (4.6) (in Lemma 4.1). We thus obtain, as \( \ell \to \infty \), \( u_\ell - \tilde{u}_\ell \to 0 \) in \( L^2(Q) \)-strong. It follows from (4.32) that
\[
\tilde{u}_\ell = u_\ell + (\tilde{u}_\ell - u_\ell) \to u \text{ in } L^2(Q) \text{-strong,}
\]

so that \( u = \tilde{u} \). We therefore pass to the limit (as \( \ell \to \infty \)) in (4.25) and obtain
\[
\begin{cases}
\int_Q (cu' + w') \varphi dx dt + \int_Q v \cdot \nabla \varphi dx dt = \int_Q f \varphi dx dt \\
\text{for all } \varphi \in L^p(0,T;W^{1,p}_{0}(\Omega)).
\end{cases}
\]

The next step is to identify the functions \( v \) and \( w \) in terms of \( u \). Namely we must show that \( v = a(\cdot, \nabla u) \) and \( w = W(u) \). Let us first show that \( v = a(\cdot, \nabla u) \). We proceed as classically to get, using the monotonicity of \( a(x, \cdot) \) and the equality
\[
\limsup_{\ell \to \infty} \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla \tilde{u}_\ell dx dt = \int_Q v \cdot \nabla u dx dt
\]

that
\[
\limsup_{\ell \to \infty} \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla (\tilde{u}_\ell - u) dx dt = \limsup_{\ell \to \infty} \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla \tilde{u}_\ell dx dt - \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla u dx dt \leq 0.
\]

This implies that for all \( \varphi \in L^p(0,T;W^{1,p}_{0}(\Omega)) \),
\[
\liminf_{\ell \to \infty} \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla (\tilde{u}_\ell - \varphi) dx dt \geq \int_Q a(\cdot, \nabla u) \cdot \nabla (u - \varphi) dx dt
\]
which implies
\[ \liminf_{\ell \to \infty} \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla \tilde{u}_\ell \, dx \, dt - \int_Q \mathbf{v} \cdot \nabla \varphi \, dx \, dt \geq \int_Q a(\cdot, \nabla u) \cdot (u - \varphi) \, dx \, dt \tag{4.34} \]
Choosing \( \varphi = u \) in (4.34) yields
\[ \liminf_{\ell \to \infty} \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla \tilde{u}_\ell \, dx \, dt \geq \int_Q \mathbf{v} \cdot \nabla u \, dx \, dt. \]
It follows that
\[ \liminf_{\ell \to \infty} \int_Q a(\cdot, \nabla \tilde{u}_\ell) \cdot \nabla \tilde{u}_\ell \, dx \, dt = \int_Q \mathbf{v} \cdot \nabla u \, dx \, dt, \]
and thus
\[ \int_Q \mathbf{v} \cdot (u - \varphi) \, dx \, dt \geq \int_Q a(\cdot, \nabla u) \cdot (u - \varphi) \, dx \, dt \text{ for all } \varphi \in L^p(0, T; W^{1,p}_0(\Omega)). \]
We deduce that \( \mathbf{v} = a(\cdot, \nabla u) \). Recalling that \( \tilde{f}_\ell \to f \) in \( L^2(Q) \)-strong, we get that \((u, w)\) satisfies the equation
\[ c \frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} - \text{div} a(\cdot, \nabla u) = f. \]
It remains to check that the hysteresis equation in (1.1) holds, that is, \( w = \mathcal{W}(u) \). We already remarked that the a priori estimates we found yield
\[ u_\ell \to u \text{ in } L^p(0, T; W^{1,p}_0(\Omega)) \cap H^1(0, T; L^2(\Omega)) \)-weak. \tag{4.35} \]
On the other hand, by interpolation and after a suitable choice of representation in equivalence classes, we may deduce, from (3.12) (where the last inclusion is also compact) that, possibly extracting a subsequence, we have
\[ u_\ell \to u \text{ uniformly in } [0, T] \text{ and a.e. in } \Omega. \]
Using the strong continuity of the operator \( \mathcal{W} \), we get that
\[ \mathcal{W}(u_\ell; \cdot) \to \mathcal{W}(u; \cdot) \text{ uniformly in } [0, T] \text{ and a.e. in } \Omega. \]
Now, we define the functions
\[ z_\ell(x, t) = \mathcal{W}[u_\ell(x, \cdot); x](t) (\ell \in \mathbb{N}) \text{ and } z(x, t) = \mathcal{W}[u(x, \cdot); x](t). \]
The compactness of the imbedding (3.11) yields that \( u_\ell \to u \) in \( L^2(\Omega; C([0, T])) \)-strong; in particular, \( u_\ell(x, \cdot) \to u(x, \cdot) \) in \( C([0, T]) \), for a.e. \( x \in \Omega \). The fact that \( w = \mathcal{W}(u; \cdot) \) can be showed arguing as in [25, Section IV.1] in particular we have to use some interpolation results and exploit the continuity of the hysteresis operator \( \mathcal{W} \) uniformly in time, a.e. in space, which can be deduced from the locally Lipschitz continuity property of \( \mathcal{W} \). Thus, using the continuity of \( \mathcal{W} \) assumed in (A4), we have \( z_\ell(x, \cdot) \to z(x, \cdot) \) in \( C([0, T]) \), for a.e. \( x \in \Omega \). Next, note that, owing to the definition of the function \( h_{\nu, x} \) given in [15],
\[ \sup_{0 \leq t \leq T} |z_\ell(x, t)| \leq \kappa_0(x) + \gamma_0 \sup_{0 \leq t \leq T} |u_\ell(x, t)| \text{ for a.e. } x \in \Omega, \]
where the right-hand side converges in $L^2(\Omega)$. Hence $z_\ell \to z$ in $L^2(\Omega; C([0, T]))$-strong. Since $w_\ell$ is the linear interpolate of $z_\ell$, an analogous argument shows that $w_\ell - z_\ell \to 0$ in $L^2(\Omega; C([0, T]))$-strong.

In summary, as $w_\ell(x, \cdot)$ is the time interpolate given by \cite{1421}, we have

$$w_\ell \to \mathcal{W}(u; \cdot) \text{ uniformly in } [0, T] \text{ and a.e. in } \Omega.$$ 

Therefore, by \cite{4.33} we get $w = \mathcal{W}(u; \cdot)$ a.e. in $Q$. By \cite{4.31}, the sequence $(\|w_\ell(\cdot, t)\|_{C([0, T])})_t$ is uniformly integrable in $\Omega$ as the same holds for $u_\ell$. Hence we have shown that $w_\ell \to w = z$ in $L^2(\Omega; C([0, T]))$-strong. Finally we get that

$$w \in L^2(Q) \cap L^2(\Omega; C([0, T])) \text{ with } w' \in L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

This concludes the proof of the existence issue.

5. PROOF OF THEOREM 2.2: UNIQUENESS RESULT

The main purpose of this section is to prove uniqueness of the solution to \cite{11}, together with the estimate \cite{2.3}. It is important to remark that no information concerning the uniqueness of the solution is presented in the preceding subsection. This question has indeed remained unanswered for a number of years, and it is Hilpert \cite{12} who finally developed a technique to shown that in quite general situation, the solution of the initial-boundary value problem \cite{11} does in fact continuously depends on the right-hand side $f$ and on the initial data. His method will be presented in the sequel.

The next results can be found in \cite{12} or in \cite{4} in which some slightly modified results have been stated and proved, but we recall them here for the convenience of the reader. The following inequality will play a key role.

**Proposition 5.1 (Hilpert’s Inequality).** Consider the hysteresis operator $\mathcal{W}$ given by

$$\mathcal{W} [v; w_{-1}] (t) = q(\mathcal{F} [v; w_{-1}] (t)), \quad 0 \leq t \leq T, \quad (5.1)$$

with $w_{-1} \in \mathbb{R}$, where $q \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is an increasing function and where $\mathcal{F}$ is a hysteresis operator. Suppose that $v_1, v_2 \in W^{1,1}(0, T)$ and $w_{-1,1}, w_{-1,2} \in \mathbb{R}$ are given, and let $v = v_2 - v_1$, $w = w_2 - w_1$, where $w_i = \mathcal{W}[v_i, w_{-1,i}], i = 1, 2$. Then

$$\frac{d}{dt} w_+ (t) \leq w'(t) H (v(t)) \text{ a.e. in } (0, T), \quad (5.2)$$

where $w_+ = \max \{w, 0\}$ and where $H$ denotes the Heaviside function.

**Proof.** $\mathcal{F}$ maps $W^{1,1}(0, T)$ into itself. Hence, the chain rule can be applied to \cite{5.1}, and the time derivatives in \cite{5.2} are defined almost everywhere. If $q(x) = x$, i.e. if $\mathcal{W}[:]w_{-1}] = \mathcal{F}[:]w_{-1}$, that the crucial implication

$$w_2(t) < w_1(t), \quad v_2(t) \geq v_1(t) \implies w'_2(t) \geq 0, \quad w'_1(t) \leq 0, \quad (5.3)$$

holds almost everywhere in $(0, T)$. Since \cite{5.3} remains for almost every $t \in (0, T)$ if $w_i(t)$ is replace by $q(w_i(t))$, we see that \cite{5.3} is true for any increasing $q \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$.

Now, \cite{5.3} implies that

$$0 \leq w'(t) H (v(t)) \text{ if } w_2(t) < w_1(t).$$
Interchanging the indices 1 and 2, we also get
\[ w'(t) \leq w(t)H(u(t)) \text{ if } w_1(t) < w_2(t). \]

Finally, on the set \( \{ t : w_1(t) = w_2(t) \} \) both sides of (5.2) vanish, which concludes the proof of the assertion. \( \square \)

We now present the general stability result. In addition to (A4), we need further Assumption (A5) on the hysteresis operators \( \mathcal{W}[: ; x], x \in \Omega \).

**Remark 5.1.** Let (A5) be satisfied. Since any weak solution \((u, w)\) in the sense of Theorem 2.2 satisfies
\[ u \in H^1 \left( 0, T; L^2(\Omega) \right) \left( = L^2 \left( \Omega; H^1(0, T) \right) \subset L^2 \left( \Omega; W^{1,1}(0, T) \right) \right), \]
we can conclude that \( w \in L^2(\Omega; H^1(0, T)) \).

**Theorem 5.1 (L^1-Stability for the nonlinear heat equation with hysteresis).** Let (A4) hold, and let \( u_{0,1}, u_{0,2} \in W_0^{1,p}(\Omega) \) and \( f_1, f_2 \in L^2(\Omega) \) be given. Suppose that the parameterized hysteresis operator \( \mathcal{W}[:, x] \) satisfies (A5), and, at every space point \( x \in \Omega \), the inequality (5.2). Then any pair \((u_1, w_1)\) and \((u_2, w_2)\) of weak solutions to (1.1) in the sense of Theorem 2.2 satisfies, for almost every \( t \in (0, T) \),
\[
\int_\Omega c |u_2 - u_1| (x, t) dx + \int_\Omega |w_2 - w_1| (x, t) dx \leq \int_\Omega |u_{0,2} - u_{0,1}| (x) dx \\
+ \int_\Omega |w_2 - w_1| (x, 0) dx + \int_0^t \int_\Omega |f_2 - f_1| (x, \tau) dx d\tau. \tag{5.4}
\]

**Proof.** We proceed as in [4]. Let \( H_\varepsilon : \mathbb{R} \to \mathbb{R} \) denote the regularized Heaviside function defined by
\[
H_\varepsilon(x) = \begin{cases} 1 & \text{if } x \geq \varepsilon \\ \frac{x}{\varepsilon} & \text{if } 0 \leq x < \varepsilon \\ 0 & \text{if } x \leq 0. \end{cases}
\]
We set \( u = u_2 - u_1 \) and \( w = w_2 - w_1 \), where \( w_i(x, \cdot) = \mathcal{W}[u_i(x, \cdot); x], i = 1, 2 \). Clearly \( H_\varepsilon \circ u \in L^\infty(0, T; W_0^{1,p}(\Omega)) \), since \( H_\varepsilon \) is Lipschitz continuous. Hence, we may test the difference of the variational equations (2.2) for the pairs \((u_i, w_i)\) \((i = 1, 2)\) and the right-hand sides \( f_i \) by the function \( \varphi = (H_\varepsilon \circ u) 1_{(0, t)} \), for \( t \in (0, T) \), to obtain
\[
\begin{align*}
&\int_0^t \int_\Omega c u'H_\varepsilon(u) dx d\tau + \int_0^t \langle w', H_\varepsilon(u) \rangle d\tau \\
&\quad + \int_0^t \int_\Omega (a(\cdot, \nabla u_2) - a(\cdot, \nabla u_1)) \cdot \nabla (H_\varepsilon \circ u) dx d\tau = \int_0^t \int_\Omega (f_2 - f_1) H_\varepsilon(u) dx d\tau. \tag{5.5}
\end{align*}
\]
Applying the chain rule to the third integrand of the left-hand side of (5.5), we obtain
\[
(a(\cdot, \nabla u_2) - a(\cdot, \nabla u_1)) \cdot \nabla (H_\varepsilon \circ u) = H'_\varepsilon(u)(a(\cdot, \nabla u_2) - a(\cdot, \nabla u_1)) \cdot \nabla (u_2 - u_1) \geq 0,
\]
the last inequality being a consequence of the monotonicity of \( a(x, \cdot) \). Hence, in view of (5.5)
\[
\int_0^t \int_\Omega (cu' + w') (x, \tau) H_\varepsilon(u(x, \tau)) dx d\tau \leq \int_0^t \int_\Omega (f_2 - f_1) (x, \tau) H_\varepsilon(u(x, \tau)) dx d\tau.
\]
Since \( u', w' \in L^1(Q) \) and \( |H_\varepsilon \circ u| \leq 1 \), we may pass to the limit as \( \varepsilon \to 0 \) to arrive at
\[
\int_\Omega cu_+(x,t)dx + \int_\Omega \int_0^t w'(x,\tau)H(u(x,\tau))d\tau dx \leq \\
\int_\Omega cu_+(x,0)dx + \int_\Omega \int_0^t (f_2 - f_1)(x,\tau)H(u(x,\tau))dxd\tau.
\] (5.6)

We estimate the second integral on the left side of (5.6) from below using (5.2) to obtain
\[
\int_\Omega cu_+(x,t)dx + \int_\Omega w_+(x,t)dx \leq \int_\Omega cu_+(x,0)dx \\
+ \int_\Omega w_+(x,0)dx + \int_0^t (f_2 - f_1)(x,\tau)H(u(x,\tau))dxd\tau.
\] (5.7)

We now reverse the role of the indices 1 and 2 and add (5.7) to the corresponding inequality. Since \( 0 \leq H(v) + H(-v) \leq 1 \), the resulting inequality yields (5.4).

**Corollary 5.1 (Uniqueness).** Under the assumptions of Theorem 5.1, the weak solution in the sense of Theorem 2.2 is unique.

**Proof.** Assuming \( f_1 = f_2 \) and \( u_{0,1} = u_{0,2} \) in (5.4) yield at once \( u_1 = u_2 \). This ends the proof of Theorem 2.2.

**Proof of estimate (2.3).** We first note that thanks to Remark 3.1, the equality (3.4) (in Lemma 3.1) holds true. Indeed, we may rewrite the leading equation in (1.1) under the form
\[
\text{div}\mathbf{a}(\cdot, \nabla u) = -f + cu' + w'.
\] (5.8)

Using assumption (A5) we obtain that \( w' \in L^2(0,T;L^2(\Omega)) \). According to (A2), \( c \in L^\infty(\Omega) \) and since it is also known that \( u' \in L^2(0,T;L^2(\Omega)) \) (see e.g., (4.35)) then \( cu' \in L^2(0,T;L^2(\Omega)) \). So that, because of assumption (A3) on \( f \), the equality (5.8) yields \( \text{div}\mathbf{a}(\cdot, \nabla u) \in L^2(0,T;L^2(\Omega)) \). The assumptions given in Remark 3.1 are thus satisfied, in such a way that (3.4) holds, that is,
\[
\frac{d}{dt}\sigma(u(t)) = -(\text{div}\mathbf{a}(\cdot, \nabla u(t)), u'(t)) \quad \text{a.e. } t \in [0,T].
\]

Therefore we multiply (1.1) by \( u'(t) \) and integrate over \( \Omega \times [t_1,t_2] \) where \( 0 \leq t_1 < t_2 \leq T \), and proceed as we did in obtaining (3.10). This yields at once (2.3).

**6. Long time behaviour: Proof of Theorem 2.3**

We are concerned here with the proof of Theorem 2.3.

**Proof of Theorem 2.3** Let \( (t_n)_n \) be a sequence of times satisfying \( 0 \leq t_n \to \infty \) as \( n \to \infty \). Let \( u \) be determined by Theorem 2.2 (we do not need uniqueness at this level). Then owing to (3.10), we have that \( u \in C([0,\infty);L^2(\Omega)) \), so that \( u(\cdot, t_n) \equiv u(t_n) \) makes sense for all \( n \), and we have \( \sup_n \| \nabla u(t_n) \|_{L^p(\Omega)} \leq C \) where \( C > 0 \) is independent of \( n \). Therefore, up to a subsequence of \( (t_n)_n \) not relabeled, there exists a function \( u_\infty \in W^{1,p}_0(\Omega) \) such that \( u(t_n) \to u_\infty \) in \( W^{1,p}_0(\Omega) \)-weak and in \( L^p(\Omega) \)-strong. This shows (2.7).

The next step is to check that \( u_\infty \) solves (2.8). To proceed with, let us first observe that assuming \( f \) depending on the time variable \( t \), the hypothesis (A3) yields that \( f \in
$C([0, \infty); L^2(\Omega)) \cap L^2(0, \infty; L^2(\Omega))$, so that $f(t_n) \to 0$ in $L^2(\Omega)$-strong as $t \to +\infty$. Of course, if $f$ does not depend on $t$, then we do not need any further requirement on $f$ (like $(A3)_1$), but only $(A3)$. This being so, let $u_n(t) = u(t + t_n)$ for $t \in [0, 1]$, and $w_n = \mathcal{W}[u_n; \cdot]$. Then it is a fact that $u_n \in L^\infty(0, 1; W^{1,p}_0(\Omega)) \cap C([0, 1]; L^2(\Omega))$ solves the equation

$$
\begin{align*}
&\frac{\partial}{\partial t}(cu_n + w_n) - \text{div}\alpha(\cdot, \nabla u_n) = f_n \quad \text{in } Q_1 = \Omega \times (0, 1) \\
&w_n(x) = \mathcal{W}[u_n(x, \cdot); x] \quad \text{in } \Omega \\
&u_n = 0 \text{ on } \partial \Omega \times (0, 1) \text{ and } u_n(0) = u(t_n) \text{ in } \Omega
\end{align*}
$$

(6.1)

where $f_n(t) = f(t + t_n)$ for $t \in [0, 1]$. Proceeding as in Subsection 4.2 we obtain the estimate

$$
\|u'_n\|_{L^2(Q_1)} + \sup_{0 \leq t \leq 1} \|\nabla u_n\|_{L^p(\Omega)} \leq C
$$

(6.2)

where $C > 0$ does not depend neither on $t$, nor on $n$. Next, following the lines of Subsection 4.3 we infer the existence of $v \in L^\infty(0, 1; W^{1,p}_0(\Omega)) \cap C([0, 1]; L^2(\Omega))$ with $v' \in L^2(0, 1; L^2(\Omega))$ such that, up to a subsequence of $(u_n)_n$ keeping the same notation,

$$
u_n \to v \text{ in } L^p(0, 1; W^{1,p}_0(\Omega)) \text{-weak and in } L^\infty(0, 1; W^{1,p}_0(\Omega)) \text{-weak}^*$$

$$
u'_n \to v' \text{ in } L^2(Q_1) \text{-weak}
$$

and

$$
\alpha(\cdot, \nabla u_n) \rightharpoonup \alpha(\cdot, \nabla v) \text{ in } L^p(Q_1) \text{-weak.}
$$

Moreover using (6.2) (or (3.10)) we see that

$$
u'_n = u'_n(\cdot + t_n) \to 0 \text{ in } L^2(0, 1; L^2(\Omega)) \text{-strong}
$$

It follows that $v' = 0$, in such a way that $v$ is constant with respect to $t$, that is, $v(t) = v(0)$ in $L^2(\Omega)$ for all $t \in [0, 1]$. However it emerges from the equality $u_n(0) = u(t_n)$ (which yields $v(0) = u_\infty$) that $v(t) = u_\infty$ for all $t \in [0, 1]$. Therefore we obtain $w_n \rightharpoonup \mathcal{W}[u_\infty; \cdot]$ in $L^2(0, 1; L^2(\Omega))$-weak, and $\mathcal{W}[u_\infty; \cdot]$ does not depend on $t$. It readily follows that $v \equiv u_\infty \in W^{1,p}_0(\Omega)$ solves the equation (2.2), which amounts to (2.8) by suitable choice of test functions (namely choose $\varphi$ under the form $\varphi(x, t) = \chi(t)\phi(x)$ with $\chi \in C^\infty_0(0, 1)$ and $\phi \in W^{1,p}_0(\Omega)$). This concludes the proof of the theorem.

Conflict of interests.

The authors declare that there is no conflict of interest regarding the publication of this paper.

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