Critical Angle at a Moving Interface Formed by a Space-Time Modulation Step

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\textit{Abstract} – This paper addresses the problem of total reflection at a moving interface formed by a space-time modulation step. Specifically, it derives, using the technique of frame hopping with Lorentz transformation, the formula for the corresponding critical angle beyond which the transmitted field is evanescent. It shows that the critical angle versus the modulation velocity monotonically decreases and reaches zero at the velocity where the incident wave cannot catch up with the interface any more. It further shows that the maximum transmission angle is smaller (resp. larger) than \( \pi/2 \) (position of the interface) for a modulation that is codirectional (resp. contradirectional) to the direction of wave propagation. The theory is illustrated and validated by full-wave FDTD simulation.

I. INTRODUCTION

Total internal reflection is an ubiquitous phenomenon in optics [1]. First suggested in connection with rainbows by Theodoric of Freiberg in the early XIV\textsuperscript{th} century, it was studied by several scientists in the subsequent centuries until it became fully understood and characterized, by Augustin-Jean Fresnel, in terms of what is nowadays called the \textit{critical angle}. The critical angle is the angle of light incident at an interface between two different media at which the transmitted wave propagates along the interface and beyond which total internal reflection occurs. Given the fundamental nature of total internal reflection, the critical angle is of paramount importance in many applications, including optical fibers, prisms, spatial filters, etc.

The emergence of \textit{space-time metamaterials}, formed by the traveling-wave modulation of a host medium [2, 3], prompts for generalizing the concepts of total internal reflection and critical angle from stationary interfaces to \textit{moving interfaces}. In such interfaces, the scattering phenomenology also includes space-time transitions, i.e., Doppler shifting in terms of temporal frequencies and rotated diffraction in terms of spatial frequencies. These transitions should bring about new physics and require therefore new analysis. The problem of a \textit{moving-matter} interface has been solved [4], but the problem of a \textit{moving-perturbation} interface, which does not involve any net motion of matter and which is relevant to space-time metamaterials of interest [6, 7], is still an open problem\textsuperscript{1}.

This paper resolves this problem of total internal reflection at a moving interface formed by a space-time modulation step. Specifically, it investigates the condition of total reflection, provides the exact formula for the critical angle as a function of the modulation velocity, and validates these theoretical results by numerical full-wave simulation.

II. DESCRIPTION OF THE PROBLEM

Figure 1 depicts the problem of interest, namely the determination of the general critical angle at an interface between two (homogeneous, isotropic, nondispersive and linear) media with refractive indices \( n_1 \) and \( n_2 \), respectively. Figure 1a shows, for the sake of comparison, the case of a stationary interface, where the transmitted wave at the critical angle propagates exactly along the interface (\( x \)-direction, \( \theta_i = \pi/2 \)), so that the wave spectrum in the direction perpendicular to the interface (\( z \)-direction) is purely evanescent. Figure 1b represents the actual space-time modulation step, where the motion of the interface corresponds to the moving perturbation produced by an external step source, whereby the atoms and molecules (represented by dots in the figure) transversely oscillate in the polarization process without moving in the direction of the perturbation. In general, the space-time modulation step interface is given by the refractive index function

\[
n(z, t) = n(z - v_m t) = n_1 + (n_2 - n_1) H(z - v_m t),
\]

\textsuperscript{1}We found that the result in [5] is incorrect.
where $H(\cdot)$ is the Heaviside step function and $v_m$ is the modulation (perturbation) velocity of the interface.

Fig. 1: Critical angle at a moving interface formed by a space-time modulation step. (a) $v_m = 0$ (stationary interface). (b) $v_m > 0$ (moving-perturbation interface).

III. ANALYTICAL RESOLUTION

We shall solve the problem using the frame hopping technique [8]. This technique involves the lab frame, $K$, where the experiment is to be performed, and the comoving frame, $K'$, which moves with the perturbation, i.e., at the velocity $v_m$. We also assume a monochromatic plane wave, which can easily be later generalized to more complex waves using superposition, given the assumed linearity of the problem.

In the $K'$ frame, the interface is purely spatial and therefore no temporal frequency transition occurs, i.e., $\Delta \omega' = 0$ or $\omega'_0 = \omega'_1$. This relation can be transposed to the $K$ frame upon substitution of the spectral Lorentz transformation $\omega' = \gamma(\omega - v_m k_z)$, which, using also $\omega = ck/n$ to eliminate $\omega$ and $k_z = k \cos \theta$ to involve the angles, leads to the relation $k_z(1/n_1 - \beta \cos \theta_1) = k_z(1/n_2 - \beta \cos \theta_1)$ where $\beta = v_m/c$. Moreover, the $K'$ phase matching condition $k'_m = k'_1$ remains unchanged under inverse Lorentz transformation, i.e., $k_m = k_{x1}$, or $k_m \sin \theta_1 = k_x \sin \theta_1$, since motion is restricted to the $z$-direction. Taking the ratio of the two so-obtained expressions and solving for $\cos \theta_1$ yields

$$\cos \theta_1 = \frac{\beta n_1^2 \sin^2 \theta_1 + (1 - \beta n_1 \cos \theta_1) \sqrt{n_2^2 (\beta^2 n_1^2 - 2 \beta n_1 \cos \theta_1 + 1) - n_1^2 \sin^2 \theta_1}}{n_2 (\beta^2 n_1^2 - 2 \beta n_1 \cos \theta_1 + 1)}. \tag{2}$$

The critical angle corresponds to the limit where $\cos \theta_1 = k_x/k_z$ becomes complex, which occurs when the radicand in the numerator of this expression becomes negative. The corresponding incident angle is the sought-after critical angle, which reads

$$\theta_{ic} = \arcsin (n_2/n_1) - \arcsin (\beta n_2). \tag{3}$$

Equation (3) reveals that $\theta_{ic}$ varies with the modulation velocity, $v_m$ (through $\beta = v_m/c$). When $v_m > 0$ (codirectional modulation), $\theta_{ic}$ decreases with increasing $v_m$ until it reaches zero, which corresponds to the situation where the incident wave cannot catch up with the moving interface. Conversely, when $v_m < 0$ (contradirectional modulation), $\theta_{ic}$ increases with increasing $|v_m|$. For $v_m = 0$ (no modulation), the formula reduces, as expected, to the conventional result $\theta_{ic} = \arcsin (n_2/n_1)$. Interestingly, the transmitted angle at the critical angle incidence is no longer $\pi/2$ in the presence of the modulation: we have then $\theta_{tc} \leq \pi/2$ for $v_m \geq 0$.

IV. RESULTS AND DISCUSSION

Figure 2 presents a numerical full-wave illustration of the theory presented in Sec. III for Gaussian incident wave, using the Finite-Difference Time-Domain (FDTD) approach described in [9]. Figure 2a shows a snapshot of the scattered field distribution for an incident angle that is slightly larger than the critical angle, given by Eq. (3). The fact that transmitted wave is evanescent despite the (visible) phase propagation $\theta_{tc}$ evidences that $\theta_{tc} < \pi/2$.\footnote{At $\theta_1 = \theta_{ic}$, the transmitted wave really propagates under the angle $\theta_{tc}$, which might a priori give the wrong impression that it undergoes regular refraction transmission! Taking a slightly larger incident angle avoids this potential confusion.}
Fig. 2: FDTD simulation illustrating the space-time modulation critical angle in Fig. 1b, with the parameters $n_1 = 1.5, n_2 = 1.3$ and $v_m = 0.2c$, corresponding to $\theta_{tc} = 45^\circ$ [Eq. (3)] and $\theta_{ic} = 75^\circ$ (equation not given here).
(a) Snapshot taken shortly after beam front hit the interface. (b) Normalized $x$-Fourier transform of the incident wave and transmitted wave just before and beyond the interface, respectively. Here, the incident angle is chosen as slightly larger than $\theta_{ic}$, namely $\theta_i = 48^\circ$, to clear proves the evanescence associated with total reflection.

The spatial field representation in Fig. 2a provides, as we just saw, useful qualitative information on the space-time interface scattering phenomenology, but no precise information on the scattering directions. To access such information, Fig. 2b plots the spatial Fourier transform of the fields along the $x$-direction at either side of the interface ($z = v_m t \pm \delta z$), as in [10]. The result of $\theta_t = 73.4^\circ$ closely corresponds to the theoretically predicted angle of $\theta_{tc} = 75^\circ$, which validates the theory of the paper.

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