Computation of non-Gaussianity in loop quantum cosmology

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We summarize our investigations of the second-order perturbations in loop quantum cosmology (LQC). We shall discuss, primarily, two aspects. Firstly, whether the second-order contributions arising from the cosmic bounce, occurring at Planck scale, could be large enough to break the validity of perturbation theory. Secondly, the implications of the upper bounds on primordial non-Gaussianity, arrived at by the Planck collaboration, on the LQC phenomenology.

Keywords: Loop quantum cosmology; Primordial non-Gaussianity

1. Introduction

Loop quantum cosmology (LQC) provides an extension of the inflationary paradigm to the Planck era (see, for instance, Ref. 1). Over the past decade or so, there has been a research program aimed at investigating the viability of LQC as a theory of the pre-inflationary universe. Until now, investigations of primordial perturbations generated in LQC have focused mainly at the level of the power spectrum. In this work, we extend the analysis to the level of three-point functions, namely, the bispectrum of curvature perturbations. We will analyze primarily two aspects. Firstly, we check whether next-to-leading order corrections to the power spectrum are sub-leading. Secondly, we verify that the amount of non-Gaussianity as quantified by the dimensionless quantity $f_{NL}$ is compatible with the observations of cosmic microwave background (CMB) and investigate new predictions.

2. Computation of the Bispectrum in the Dressed Metric Approach

The system of interest is scalar perturbations $\delta \phi$ living on a Friedmann-Lemaître-Robertson-Walker (FLRW) metric sourced by a scalar field $\phi$. In LQC, such a system is described by a wavefunction $\Psi(v, \phi, \delta \phi)$, where $v \equiv a^3 V_0 4/\kappa$ with $a$ being the scale factor and $V_0$, the volume of the universe, introduced to regulate
infrared divergence. The dynamics is governed by the constraint equation, \( \hat{H} \Psi = 0 \), where the Hamiltonian operator can be split into the background and perturbed part as \( \hat{H} = \hat{H}_{\text{FLRW}} + \hat{H}_{\text{pert}} \). We are interested in solutions wherein \( \Psi(v, \phi, \delta \phi) = \Psi_0(v, \phi) \otimes \delta \Psi(v, \phi, \delta \phi) \), where \( \Psi_0 \) describes a quantum FLRW geometry and \( \delta \Psi \) describes the scalar perturbations.

The states \( \Psi_0 \) satisfies the equation \( \hat{H}_{\text{FLRW}} \Psi_0 = 0 \). It has been shown that, for states that are sharply peaked in the volume \( v \) during the entire evolution, the background geometry can be described by an effective classical Hamiltonian (see e.g. Ref. 1 and references therein). In the dressed metric approach, we are interested in quantum states \( \delta \Psi(v, \phi, \delta \phi) \) that are a small perturbation around such a quantum FLRW state \( \Psi_0(v, \phi) \). A detailed analysis shows that \( \delta \Psi(v, \phi, \delta \phi) \) are solutions to the Schrödinger equation, \( i\hbar \partial_t \delta \Psi = \langle \Psi_0 | \hat{H}_{\text{pert}} [N_{\phi}] | \Psi_0 \rangle \delta \Psi \), where \( \hat{H}_{\text{pert}} = \hat{H}^{(2)} + \hat{H}^{(3)} \), namely the Hamiltonian at second and third order in perturbations respectively and \( N_{\phi} \) is the lapse associated with relational time \( \phi \).

We are interested in computing the correlation functions of these scalar perturbations. The first step is to expand the perturbations in Fourier space and introduce creation and annihilation operators

\[
\delta \phi(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} \left( \hat{A}_{\vec{k}} \varphi_k(\eta) + \hat{A}^\dagger_{-\vec{k}} \varphi^*_k(\eta) \right) e^{i\vec{k} \cdot \vec{x}},
\]

where \( [\hat{A}_{\vec{k}}, \hat{A}^\dagger_{\vec{k}'}] = \hbar (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \) and \( [\hat{A}_{\vec{k}}, \hat{A}^\dagger_{\vec{k}'}] = 0 \). The dynamics of perturbations are governed by the second-order Hamiltonian with the background quantities determined using the effective background Hamiltonian. The scalar power spectrum of \( \delta \phi \) is defined as

\[
\langle 0 | \delta \phi(\eta) \delta \phi^*(\eta) | 0 \rangle \equiv \langle 2\pi \rangle^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{2\pi^2}{k^3} P_\delta(k, \eta),
\]

where \( |0\rangle \) is the vacuum annihilated by the operators \( \hat{A}_{\vec{k}} \) for all \( \vec{k} \). For the purpose of relating perturbations to the late time physics, it is convenient to express the power spectrum in terms of comoving curvature perturbations. The power spectrum of curvature perturbation, \( P_\kappa \), in terms of inflaton perturbation \( \delta \phi \), evaluated at the end of inflation is \( P_\kappa(k) \equiv \left( \frac{z_{e}(\eta_{0})}{z_{e}(\eta)} \right)^2 \frac{h k^3}{2\pi^2} |\varphi_k(\eta_{0})|^2 \), where \( z = -\frac{6 \pi a}{\kappa \pi a} \) with \( \kappa = 8\pi G \) and \( \rho_\phi \) and \( \pi a \) are momenta conjugate to \( \phi \) and \( \dot{\phi} \) respectively.

The self-interaction of scalar perturbation, at lowest order, is described by the third-order interaction Hamiltonian, \( \hat{H}_{\text{int}} \equiv \langle \Psi_0 | \hat{H}^{(3)} (N_{\phi}) | \Psi_0 \rangle \). The perturbations at this order are quantified using the scalar bispectrum, \( B_R(k_1, k_2, k_3) \), which is defined in terms of curvature perturbations by

\[
\langle 0 | \hat{R}_{\vec{k}_1} \hat{R}_{\vec{k}_2} \hat{R}_{\vec{k}_3} | 0 \rangle \equiv \langle 2\pi \rangle^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_R(k_1, k_2, k_3).
\]

It is often convenient to quantify the bispectrum using a dimensionless function, \( f_{NL} \), which can be defined as

\[
B_R(k_1, k_2, k_3) \equiv -\frac{6}{5} f_{NL}(k_1, k_2, k_3) \times (\Delta_{k_1} \Delta_{k_2} + \Delta_{k_1} \Delta_{k_3} + \Delta_{k_2} \Delta_{k_3}),
\]
where $\Delta k \equiv \frac{2\pi}{\ell}$ is the dimensionful power spectrum.

In order to compute bispectrum, we need to express it in terms of $\delta \phi$ as follows,

$$
\langle 0 | \hat{\mathcal{R}}_{\hat{k}_1} \hat{\mathcal{R}}_{\hat{k}_2} \hat{\mathcal{R}}_{\hat{k}_3} | 0 \rangle = \left( -\frac{a}{z} \right)^3 \langle 0 | \delta \phi_{\hat{k}_1} \delta \phi_{\hat{k}_2} \delta \phi_{\hat{k}_3} | 0 \rangle + \left( -\frac{3}{2} + \frac{3 V_\phi}{4 \kappa z^2} \right) \left( -\frac{a}{z} \right)^4 \times \left[ \int \frac{d^3p}{(2\pi)^3} \right] \langle 0 | \delta \phi_{\hat{k}_1} \delta \phi_{\hat{k}_2} \delta \phi_{\hat{k}_3} - p | 0 \rangle + \cdots ,
$$

where the symbols $(\hat{k}_i \leftrightarrow \hat{k}_j)$ indicate terms obtained by replacing $k_i$ with $k_j$ in

$$
\langle 0 | \delta \phi_{\hat{k}_1} (\eta) \delta \phi_{\hat{k}_2} (\eta) \delta \phi_{\hat{k}_3} (\eta) | 0 \rangle = \langle 0 | \delta \phi_{\hat{k}_1} (\eta) \delta \phi_{\hat{k}_2} (\eta) \delta \phi_{\hat{k}_3} (\eta) | 0 \rangle - i \hbar \int d\eta' \left[ \delta \phi_{\hat{k}_1} (\eta) \delta \phi_{\hat{k}_2} (\eta) \delta \phi_{\hat{k}_3} (\eta), \hat{H}^I_{\text{int}} (\eta') \right] | 0 \rangle ,
$$

where $I$ indicates fields in the interaction picture. Since, $\delta \phi^I$ is a Gaussian field, the first term vanishes and only the second term contributes. The second term in the RHS of Eq. (5) can be evaluated using Wick’s theorem and Eq. (2). Using Eqs. (5) and (6), one can compute the bispectrum and hence the function $f_{\text{NL}}$ using Eq. (1).

3. Numerical Method and Results

In this section, we will briefly describe our implementation of the formalism for computing $f_{\text{NL}}$ and the results we obtain. In order to compute $f_{\text{NL}}$, at the end of inflation, one needs to evolve the perturbations from an early time before the bounce until the perturbations leave the horizon during inflation at which point their amplitude freezes in time. We need to make three choices to do this computation.

Firstly, we need to specify the potential governing the field $\phi$. We choose the quadratic potential, $V(\phi) = m^2 \phi^2 / 2$, where $m = 6.4 \times 10^{-6} M_{\text{Pl}}$. Secondly, we need to choose a background geometry by specifying the value of $\phi$ and energy density, $\rho$, at the bounce. We work with $\phi_0 = 7.62 M_{\text{Pl}}$ and $\rho_B = 1 M^4_{\text{Pl}}$, where subscript $B$ denotes the bounce, so that the effects due to LQC appear at observable scales while respecting the Planck constraints on power spectrum. Finally, we need to choose an initial state for perturbations, which we choose to be a Minkowski initial state. More specifically, we choose $\varphi_k (\eta_0) = \frac{1}{\pi(\eta_0) \sqrt{2k}}$ and $\varphi_k' (\eta_0) = -i k + \frac{\dot{\varphi}_0 (\eta_0)}{\pi(\eta_0)} \varphi_k (\eta_0)$ as initial data for the modes, at conformal time $\eta_0 = -2.8 \times 10^3 T_{\text{Pl}}$ (the bounce takes place at $\eta = 0$). The initial time was chosen so that all the modes of interest, namely those between $k_{\text{min}} = k_s / 10$, and $k_{\text{max}} = 1000 k_s$, where $k_s / a(t_{\text{today}}) = 0.002 \text{ Mpc}^{-1}$ is the pivot scale, were in the adiabatic regime. We have investigated the effects of varying these choices in detail in Ref. 2.

To perform this computation, we use the platform provided by class. The computation was done in two stages. In the first stage we evolve the background
from very early times to the end of inflation. In the second step, we convert the time integral in Eq. (6) to a differential equation and evolve it together with the differential equation for the fourier modes. In the remaining part of this section, we will discuss the various results.

Fig. 1. The power spectrum and $f_{NL}(k_1, k_2, k_3)$ evaluated in the equilateral limit.

Fig. 2. The shape of the non-Gaussianity, $f_{NL}(k_1, k_2, k_3)$, evaluated with $k_1 = k_*/2$ and $k_*$. The figure shows the portions allowed by the condition $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$.

Fig. 1 depicts the scalar power spectrum and $f_{NL}$ in the equilateral limit. One can see that for $k \leq k_{LQC}$, where $k_{LQC}$ is the scale set by the spacetime curvature at the bounce, the spectra are strongly scale dependent while for $k >> k_{LQC}$, the spectra approach their slow roll values. At low wave numbers, the figure shows that $f_{NL}$ is oscillatory. We have depicted the $f_{NL}$ for all configurations in Fig. 2. In this figure, we have fixed the value of $k_1$ and varied $k_2$ and $k_3$ in such a way that they obey the triangle condition. This figure illustrates the shape of the non-Gaussianity. It can be seen that, in both the figures, the $f_{NL}$ peaks in the squeezed ($k_3 \ll k_1 \approx k_2$) - flattened ($k_1 \approx k_2 + k_3$) limit.

The primordial non-Gaussianity generated due to LQC has a characteristic enhancement of amplitude at scales comparable to $k_{LQC}$. By analyzing the integrals involved in the computation of $f_{NL}$, we can estimate the contribution to $f_{NL}$ from the epoch around the bounce. For modes, $k \geq k_{LQC}$, we can approximate the mode function as $\varphi_k \sim e^{-ik\eta}$. Then the contribution to the integral from time around
selects only the contribution from the time range 
expression with numerical result and we find a good match between the two.

dominant. The first perturbative correction to the two-point function of curvature
spectrum. For the perturbation theory to be valid, this contribution has to be sub-
be computed using Cauchy’s residue theorem and the spectral dependence of the

Finally let us compute the contribution to the power spectrum from the bis-

contribution to \( \Delta P \) from the bounce can be schematically written as,

\[
I(k_1, k_2, k_3) \sim \int_{-\Delta \eta}^{\Delta \eta} d\eta g(\eta) e^{i(k_1 + k_2 + k_3) \eta} \approx \int_{-\infty}^{\infty} d\eta g(\eta) e^{i\eta \eta} W(\eta, \Delta),
\]

where, \( g(\eta) \) is a combination of the functions depending on the background and the wavenumbers, \( k_i \equiv k_1 + k_2 + k_3 \) and \( W(\eta, \Delta \eta) \) is a window function which selects only the contribution from the time range \(-\Delta \eta < \eta < \Delta \eta\). This integral can be computed using Cauchy’s residue theorem and the spectral dependence of the integral can be written as \( e^{-\alpha k_1/k_{LQC}} \). In Fig. 3 we have compared the analytical expression with numerical result and we find a good match between the two.

Finally let us compute the contribution to the power spectrum from the bispectrum. For the perturbation theory to be valid, this contribution has to be sub-dominant. The first perturbative correction to the two-point function of curvature perturbation is given by

\[
\langle 0 | \mathcal{R}_{k_1} \mathcal{R}_{k_2} | 0 \rangle = (2\pi)^3 \delta(3)(\vec{k}_1 + \vec{k}_2) \frac{2\pi^2}{k_1^3} \hbar \left[ \mathcal{P}_R(k_1) + \Delta \mathcal{P}_R(k_1) \right],
\]

where

\[
\Delta \mathcal{P}_R(k_1) = \hbar \frac{k_1^4}{\pi^2} \left[ \left( \frac{\alpha}{z} \right)^3 \left[ -\frac{3}{2} + \frac{3 V_0 a^5}{4 \kappa p_\phi \pi a} + \frac{\kappa z^2}{4 a^2} \right] \int \frac{d^3p}{(2\pi)^3} B_{\phi}(k_1, \vec{p} - \vec{k}_1 - \vec{p}) \right. \\
+ \left. \left( \frac{\alpha}{z} \right)^4 \left[ \frac{3}{2} + \frac{3 V_0 a^5}{\kappa p_\phi \pi a} + \frac{\kappa z^2}{4 a^2} \right] \iota \int \frac{d^3p}{(2\pi)^3} |\phi_p|^2 |\phi_{\vec{k}_1 - \vec{p}}|^2 \right],
\]

where \( B_{\phi}(k_1, \vec{p} - \vec{k}_1 - \vec{p}) \) is the bispectrum of inflaton perturbations and all the quantities on the right are evaluated at the end of inflation. We have numerically plotted the relative amplitude of the first order correction, \( |\Delta \mathcal{P}_R / \mathcal{P}_R| \), in Fig. 3. We find that, as expected, the magnitude of first-order correction to the power spectrum is negligible. This result can be qualitatively understood as follows. The leading order contribution to \( \Delta \mathcal{P}_R(k_1) \) is given by the first term in Eq. (8) and it is
given by $\epsilon f_{NL} P_R^2$, where $\epsilon$ is the slow roll parameter of $O(10^{-2})$. Since, $f_{NL} \leq 10^4$ and $P_R \leq 10^{-7}$, we obtain $\Delta P_R/P_R \leq 10^{-4}$ as in Fig. 3.

4. Discussion

Let us conclude by making some remarks on the robustness of the results and its implication in the light of Planck data. We have verified the robustness of the results to a variation of the basic assumptions discussed in Sec. 2. For instance, we find that the effect of changing $\phi_b$ is only a shift in the scale which is sensitive to the effect of the bounce with respect to the scales observable today. An increase in $\rho_B$ also leads only to a similar shift in the scales sensitive to the curvature of the bounce, in addition, to an increase in amplitude of $f_{NL}$. The Planck mission has put strong constraints on certain models of scale invariant non-Gaussianity, but, it provides little information on the scale dependent non-Gaussianity as produced in LQC. Moreover, since the error bar on $f_{NL}$ goes as $1/\sqrt{\ell}$, at low multipoles, where the non-Gaussianity due to LQC is expected to be large, the error bar would be large. Considering the Planck error bars at low multipoles and demanding that the enhancement in $f_{NL}$ due to the LQC bounce appears at $\ell \lesssim 50$, one could try to arrive at constraints on the minimum value of scalar field at the bounce, for a given value of $\rho_B$. Furthermore, by demanding that the imprint of the bounce should be at observable scales, we can arrive at an upper bound on $\phi_B$. For instance, for $\rho_B = 1 M_{Pl}^4$, we obtain $7.46 M_{Pl} \leq \phi_B \leq 7.82 M_{Pl}$. It should be kept in mind that the constraint described above is a very conservative estimate. Most probably, the oscillations in $f_{NL}$ will relax the constraint on $\phi_B$ discussed above. A more detailed account of this work has been published in Ref. 2.

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