DEFORMATIONS AND INVERSION FORMULAS FOR
FORMAL AUTOMORPHISMS IN NONCOMMUTATIVE
VARIABLES

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Abstract. Let \( z = (z_1, z_2, ..., z_n) \) be noncommutative free variables and \( t \) a formal parameter which commutes with \( z \). Let \( k \) be any unital integral domain of any characteristic and \( F_t(z) = z - H_t(z) \) with \( H_t(z) \in k[[t]]\langle\langle z\rangle\rangle^n \) and the order \( o(H_t(z)) \geq 2 \). Note that \( F_t(z) \) can be viewed as a deformation of the formal map \( F(z) := z - H_{t=1}(z) \) when it makes sense (for example, when \( H_t(z) \in k[[t]]\langle\langle z\rangle\rangle^n \)). The inverse map \( G_t(z) \) of \( F_t(z) \) can always be written as \( G_t(z) = z + M_t(z) \) with \( M_t(z) \in k[[t]]\langle\langle z\rangle\rangle^n \) and \( o(M_t(z)) \geq 2 \). In this paper, we first derive the PDE's satisfied by \( M_t(z) \) and \( u(F_t), u(G_t) \in k[[t]]\langle\langle z\rangle\rangle^n \) with \( u(z) \in k\langle\langle z\rangle\rangle \) in the general case as well as in the special case when \( H_t(z) = tH(z) \) for some \( H(z) \in k\langle\langle z\rangle\rangle^n \). We also show that the formal power series above are actually characterized by certain Cauchy problems of these PDE's. Secondly, we apply the derived PDE's to prove a recurrent inversion formula for formal maps in noncommutative variables. Finally, for the case \( \text{char. } k = 0 \), we derive an expansion inversion formula by the planar binary rooted trees.

1. Introduction

Let \( z = (z_1, z_2, ..., z_n) \) be \( n \) noncommutative free variables and \( t \) a formal parameter which commutes with \( z \). We fix a unital integral domain \( k \) of any characteristic and denote by \( k\langle\langle z\rangle\rangle \) and \( k[[t]]\langle\langle z\rangle\rangle \) the algebras of formal power series in \( z \) over \( k \) and \( k[[t]] \), respectively. In this paper, we first study the deformations of automorphisms of \( k\langle\langle z\rangle\rangle \) parameterized by the formal parameter \( t \) and then derive some inversion formulas for the automorphisms of \( k\langle\langle z\rangle\rangle \). More precisely, we consider the automorphisms \( F_t(z) \) of \( k[[t]]\langle\langle z\rangle\rangle \) over \( k[[t]] \) of the form

\[ F_t(z) = z - H_t(z) \]

where \( H_t(z) \in k[[t]]\langle\langle z\rangle\rangle^n \) and the order \( o(H_t(z)) \geq 2 \). We also show that the formal power series above are actually characterized by certain Cauchy problems of these PDE's. Secondly, we apply the derived PDE's to prove a recurrent inversion formula for formal maps in noncommutative variables. Finally, for the case \( \text{char. } k = 0 \), we derive an expansion inversion formula by the planar binary rooted trees.
\[ F_t(z) = z - H_t(z) \] with \[ H_t(z) \in k[[t]]\langle\langle z\rangle\rangle^{\times n} \] and the order \( o(H_t(z)) \geq 2 \). Note that \( F_t(z) \) can be viewed as a general deformation parameterized by \( t \) of the formal map \( F(z) := z - H_{t=1}(z) \) when it exists (for example, when \( H_t(z) \in k[t]\langle\langle z\rangle\rangle^{\times n} \), or when \( k = \mathbb{C} \) and all coefficients of \( H_t(z) \) are holomorphic functions of \( t \) which are convergent over an open subset of \( \mathbb{C} \) containing the closed unit disk). In particular, this is indeed the case for the special deformation \( F_t(z) = z - tH(z) \) with \( H(z) \in k\langle\langle z\rangle\rangle^{\times n} \), i.e. \( H_t(z) = tH(z) \). We will always denote by \( G_t(z) \) the formal inverse map of \( F_t(z) \) and write it as \( G_t(z) = z + M_t(z) \) with \( M_t(z) \in k[[t]]\langle\langle z\rangle\rangle^{\times n} \) and \( o(M_t(z)) \geq 2 \). When \( F_t(z) \) is the special deformation \( F_t(z) = z - tH(z) \) above, we also write its inverse as \( G_t(z) = z + tN_t(z) \) with \( N_t(z) \in k[[t]]\langle\langle z\rangle\rangle^{\times n} \). In the first part of this paper, we derive the PDE’s in \( z \) and \( t \) satisfied by \( M_t(z) \), \( N_t(z) \) \( u(F_t) \) and \( u(G_t) \) \( u(z) \in k\langle\langle z\rangle\rangle \). In particular, we show that \( N_t(z) \) is a formal power series of the Cauchy problem of a Burgers-like PDE (see Theorem \[4.3\] and Remark \[4.4\]). When \( \text{char.} \ k = 0 \), \( N_t(z) \) is actually the unique power series solution of a Cauchy problem of the PDE; while when \( \text{char.} \ k = p > 0 \), \( N_t(z) \) is completely determined by this property together with its coefficients of \( t^m \) \( (m \geq 1) \), which can be calculated by some other methods (see Corollary \[5.2\] and Theorem \[5.5\]). In addition, we also discuss some other characterizing properties of \( N_t(z) \). In the second part of this paper, we apply the PDE satisfied by \( N_t(z) \) to derive a recurrent inversion formula and, when \( \text{char.} \ k = 0 \), an expansion inversion formula by the planar binary rooted trees for formal maps in noncommutative free variables. Note that the special deformation \( F_t(z) = z - tH(z) \) for commutative variables \( z \) over any unital commutative ring \( k \) of characteristic zero has been studied in \[Z2\]. Here we not only generalize the results in \[Z2\] to formal maps in noncommutative variables, but also give some inversion algorithms for the case when the base ring \( k \) has \( \text{char.} \ k = p > 0 \). When \( \text{char.} \ k = 0 \), the expansion inversion formula by the planar binary rooted trees for the symmetric maps in \[Z2\] is also generalized to general automorphisms.

The problem seeking various inversion formulas of formal maps in commutative variables has a long history in mathematics. It started with the Lagrange’s inversion formula in one variable by L. Lagrange \[L\] in 1770, then the Jacobi’s inversion formula by C. G. J. Jacobi \[J1\] in 1830 and \[J2\] in 1844. Later, motivated by the well-known Jacobian conjecture proposed by O. H. Keller \[Ke\] in 1939, more inversion formulas have been proved (see \[BCW\], \[E\], \[Sm\] and references there for more history and known results on the Jacobian conjecture). In 1965, I.
G. Good \cite{Go} generalized the Lagrange’s inversion formula to the multiple variable case. In 1974, Gurjar (unpublished) and later Abhyankar \cite{Ab} proved so-called Abhyankar-Gurjar inversion formula. In 1981, H. Bass, E. Connell and D. Wright \cite{BCW} and D. Wright \cite{Wr} proved the so-called Bass-Connell-Wright’s tree expansion formula. Very recently, D. Wright and the author \cite{WZ} generalized this formula to tree expansion formulas for the D-log and the formal flow of formal maps. In \cite{Z2} and \cite{Z3}, the author proved a recurrent inversion formula in general and a non-recurrent formula for the symmetric maps which satisfy the Jacobian condition. The later was mainly motivated by the remarkable symmetric reduction on the Jacobian conjecture achieved recently by M. de Bonlt and A. van den Essen in \cite{BE} and G. Meng in \cite{Me}.

On the other hand, comparing with the commutative case, it seems not many inversion formulas for formal automorphisms in noncommutative variables are known in the literature. But, for an interesting approach to this problem, see \cite{Ge}; for several $q$-analogue inversion formulas see \cite{An, Ga, GH}.

One remark is that, based on some results obtained in this paper, later, in the followed papers \cite{Z4}, \cite{Z5} and \cite{Z6}, some connections of the commutative or noncommutative inversion problem with the Hopf algebra $NSym$ of noncommutative symmetric functions, which were first introduced and studied in \cite{GKLLRT}, and the Grossman-Larson Hopf algebra \cite{GL, F} of labeled rooted trees will be studied. In particular, more inversion formulas in both commutative and noncommutative cases will be derived in \cite{Z5}. The tree expansion formulas obtained in \cite{BCW}, \cite{Wr} and \cite{WZ} for the inverse map, the D-Log’s and the formal flows in the commutative case will also be generalized in \cite{Z6} to the noncommutative case.

The arrangement of this paper is as follows. In Section 2, we first fix some notation which will be used throughout the paper. We then consider certain properties of derivations and differential operators in noncommutative variables. In particular, we prove two chain rules for the derivations of $k\langle\langle z\rangle\rangle$ and $k[[t]]\langle\langle z\rangle\rangle$, respectively (see Lemma 2.1 and 2.4). In Section 3 and 4, we study the general deformation $F_t(z) = z - H_t(z)$ with $H_t(z) \in k[[t]]\langle\langle z\rangle\rangle^{\times n}$ and the special deformation $F_t(z) = z - tH(z)$ with $H(z) \in k\langle\langle z\rangle\rangle^{\times n}$, respectively. We not only derive the PDE’s satisfied by $M_t(z)$, $N_t(z)$ as well as formal power series of the forms $u(F_t)$ and $u(G_t)$ with $u(z) \in k\langle\langle z\rangle\rangle$, but also show that the elements above are also characterized by certain Cauchy problems of these PDE’s. Note that, not all these results are needed later for the derivations of the inversion formulas in the second part of this paper,
but will be crucial for the followed papers [Z4], [Z5] and [Z6]. In Section 5 we apply some results obtained in Section 4 to derive a recurrent inversion formula for formal maps in noncommutative variables over a base ring $k$ of any characteristic. In Section 6 we assume our base ring $k$ has characteristic zero and prove an expansion inversion formula by the planar binary rooted trees.

One final remark is as follows. For simplicity, we mainly focus on formal maps in noncommutative variables $z$. But most of the results obtained in this paper have their analogs for commutative variables, which can be derived either by taking the quotient over the ideal generated by the commutators of $z_i$’s or by applying parallel arguments.

Acknowledgment: The author would like to thank the referee for pointing out misprints and providing valuable suggestions.

2. Chain Rules in the Noncommutative Case

In this section, we consider certain properties of derivations and differential operators in noncommutative free variables. In particular, we prove two variations of the usual chain rule in the commutative case for the derivations in the noncommutative case (see Lemma 2.1 and 2.4). These chain rules will be crucial for our later arguments.

First, let us fix the following notation that will be used throughout this paper.

Notation:

(1) The base rings $k$ throughout this paper will always be assumed to be unital integral domains.

(2) We fix $n \geq 1$ and let $z = (z_1, z_2, ..., z_n)$ be $n$ noncommutative variables. For any unital integral domain $k$, we denote by $k\langle z \rangle$ and $k\langle \langle z \rangle \rangle$ the algebras of (noncommutative) polynomials and formal power series in $z_i$ ($1 \leq i \leq n$) over $k$, respectively.

(3) For any unital integral domain $k$, note that the set of endomorphisms $\phi$ of $k\langle \langle z \rangle \rangle$ as a $k$-algebra is in 1-1 correspondence with the set of $n$-vectors $(F_1(z), F_2(z), \ldots, F_n(z)) \in k\langle \langle z \rangle \rangle^{\times n}$ via $F_i(z) = \phi(z_i)$ ($1 \leq i \leq n$). So, in this paper, by a formal endomorphism of $k\langle \langle z \rangle \rangle$ or a formal map in $z$, we simply mean a $n$-vector $F(z) = (F_1(z), F_2(z), \ldots, F_n(z))$ with $F_i(z) \in k\langle \langle z \rangle \rangle$ ($1 \leq i \leq n$). When each $F_i(z)$ is a polynomial in $z$, we say $F(z)$ is a polynomial endomorphism of $k\langle \langle z \rangle \rangle$ or simply a polynomial map in $z$. 
(4) For any $m \geq 1$ and $U(z) = (U_1(z), \ldots, U_m(z)) \in k\langle \langle z \rangle \rangle^m$, we define the order $o(U(z))$ of $U(z)$ to be

$$o(U(z)) := \min_{1 \leq i \leq k} o(U_i(z))$$

and, when $U(z) \in k\langle \langle z \rangle \rangle^m$, the degree $\deg(U(z))$ of $U(z)$ to be

$$\deg U(z) := \max_{1 \leq i \leq k} \deg U_i(z).$$

When the base ring is (as it frequently will be in this paper) the polynomial algebra $k[t]$ or the formal power series algebra $k[[t]]$ in a central parameter $t$ over a unital integral domain $k$, the notation $o(U_t(z))$ and $\deg U_t(z)$ above always stand for the order and the degree of $U_t(z)$ with respect to $z$, respectively. In other words, $t$ will not be treated as a variable as $z_i$'s but a scalar parameter which commutes with $z_i$'s.

(5) All $n$-vectors in this paper are supposed to be column vectors unless stated otherwise. For any vector or matrix $U$, we denote by $U^\tau$ its transpose.

Now let $k$ be a unital integral domain of any characteristic and $k\langle \langle z \rangle \rangle$ fixed as above. By a derivation of $k\langle \langle z \rangle \rangle$, we mean a homomorphism of abelian groups $\delta : k\langle \langle z \rangle \rangle \to k\langle \langle z \rangle \rangle$ that satisfies the Leibniz rule, i.e. for any $f, g \in k\langle \langle z \rangle \rangle$, we have

$$\delta(fg) = (\delta f)g + f(\delta g).$$

(2.1) A derivation $\delta$ of $k\langle \langle z \rangle \rangle$ is said to be a $k$-derivation if it annihilates all elements of $k \subset k\langle \langle z \rangle \rangle$. In other words, it is also a $k$-linear map from $k\langle \langle z \rangle \rangle$ to $k\langle \langle z \rangle \rangle$. We will denote by $\text{Der}_k\langle \langle z \rangle \rangle$ or $\text{Der}_{\langle \langle z \rangle \rangle}$, when the base ring $k$ is clear in the context, the set of all $k$-derivations of $k\langle \langle z \rangle \rangle$.

The unital subalgebra of $\text{End}_k(k\langle \langle z \rangle \rangle)$ generated by all $k$-derivations of $k\langle \langle z \rangle \rangle$ will be denoted by $\mathcal{D}\langle \langle z \rangle \rangle$ or $\mathcal{D}_k\langle \langle z \rangle \rangle$. Elements of $\mathcal{D}\langle \langle z \rangle \rangle$ will be called (formal) differential operators in the noncommutative variables $z_i$ ($1 \leq i \leq n$).

For any $1 \leq i \leq n$ and $u(z) \in k\langle \langle z \rangle \rangle$, we denote by $\left[u(z)\frac{\partial}{\partial z_i}\right]$ the $k$-derivation which maps $z_i$ to $u(z)$ and $z_j$ to 0 for any $j \neq i$. For any $\vec{u} = (u_1, u_2, \cdots, u_n) \in k\langle \langle z \rangle \rangle^n$, we set

$$[\vec{u} \frac{\partial}{\partial z}] := \sum_{i=1}^{n} [u_i \frac{\partial}{\partial z_i}].$$

\(^1\)The reason we put a bracket $[\cdot]$ in the notation for derivations of $k\langle \langle z \rangle \rangle$ is to avoid any possible confusion caused by a subtle point described in the Warning below.
Furthermore, for any matrix $M_{m \times n}$ with row vectors $M_j(z) \in k\langle\langle z\rangle\rangle^{\times n}$ ($1 \leq j \leq m$), we set

$$M \frac{\partial}{\partial z} := ([M_1 \frac{\partial}{\partial z}], [M_2 \frac{\partial}{\partial z}], \ldots, [M_m \frac{\partial}{\partial z}]) \in \text{Der}(\langle\langle z\rangle\rangle)^{\times n}. \tag{2.3}$$

**Warning:** Unlike in the commutative case, in general, we do not have $u(z) \frac{\partial}{\partial z} g(z) = u(z) \frac{\partial g}{\partial z}$ for all $u(z), g(z) \in k\langle\langle z\rangle\rangle$. For example, let $g = z_jz_i$ with $j \neq i$, we have

$$[u \frac{\partial}{\partial z_j}] (z_jz_i) = ([u \frac{\partial}{\partial z_j}] z_j)z_i + z_j([u \frac{\partial}{\partial z_j}] z_i) = z_ju(z),$$

$$u(z) \frac{\partial g}{\partial z_j} = u(z)z_j,$$

which are not equal unless $u(z)$ commutes with $z_j$.

With the notation above, it is easy to see that any $k$-derivations $\delta$ of $k\langle\langle z\rangle\rangle$ can be written uniquely as $\sum_{i=1}^n \left[f_i(z) \frac{\partial}{\partial z_i}\right]$ with $f_i(z) = \delta z_i \in k\langle\langle z\rangle\rangle$ ($1 \leq i \leq n$).

Finally, for any automorphism $F(z)$ of $k\langle\langle z\rangle\rangle$ and any $\delta \in \text{Der}(\langle\langle z\rangle\rangle)$, we define $F_*(\delta) \in \text{Der}(\langle\langle z\rangle\rangle)$ by setting, for any $u(z) \in k\langle\langle z\rangle\rangle$,

$$F_*(\delta) u(z) := \left(\delta(u(F^{-1}))\right) (F). \tag{2.4}$$

We call $F_*(\delta)$ the induced action of $F(z)$ on $\delta$.

Next, let us consider the chain rules for derivations of $k\langle\langle z\rangle\rangle$ and $k[[t]]\langle\langle z\rangle\rangle$. The usual chain rule for derivations in the commutative case certainly does not hold anymore in the noncommutative case. But it has the following two variations in certain special cases, see Lemma 2.1 and 2.4 below.

First, let us consider the following chain rule for $k$-derivations of $k\langle\langle z\rangle\rangle$.

**Lemma 2.1. (Chain Rule for $k$-Derivations)**

Let $\delta$ be a $k$-derivation of $k\langle\langle z\rangle\rangle$ and $F(z) = (F_1(z), \cdots, F_n(z))$ an automorphism of $k\langle\langle z\rangle\rangle$. Then, for any $u(z) \in k\langle\langle z\rangle\rangle$, we have

$$\delta(u(F)) = \left[ (\delta F)(F^{-1}) \frac{\partial}{\partial z} \right] u \circ F, \tag{2.5}$$

or equivalently,

$$(F^{-1})_* (\delta) = \left[ (\delta F)(F^{-1}) \frac{\partial}{\partial z} \right], \tag{2.6}$$

where $\delta F := (\delta F_1(z), \delta F_2(z), \cdots, \delta F_n(z))$. 

Proof: It is easy to see that Eqs. (2.5) and (2.6) are equivalent to each other via composing with $F$ or $F^{-1}$ from right. So it will be enough to show Eq. (2.6).

First, note that both sides of Eq. (2.6) are $k$-derivations of $k\langle\langle z \rangle\rangle$. Secondly, it is easy to check directly that, for any $1 \leq i \leq n$, both derivations send $z_i$ to $(\delta F_i)(F^{-1})$. Hence they must be same as $k$-derivations of $k\langle\langle z \rangle\rangle$ and Eq. (2.6) holds. 

Note that, when $z_i$’s are commutative variables, Eq. (2.5) becomes the usual chain rule. It is worth to mention that, the chain rule Eq. (2.5) or (2.6) also has a very simple form for endomorphisms of $k\langle\langle z \rangle\rangle$ in terms of the Jacobian matrices. Here, for any sequence $U(z) = (U_1(z), \cdots, U_m(z))$ of $k\langle\langle z \rangle\rangle^m$, we define the Jacobian matrix to be $JU(z) = \left(\frac{\partial}{\partial z_j} U_i\right)$ as in the commutative case and set $\tilde{J}U(z) = (JU)^\tau(z) = \left(\frac{\partial}{\partial z_i} U_j\right)$.

Corollary 2.2. Let $U(z) = (U_1, \cdots, U_m) \in k\langle\langle z \rangle\rangle^m$ and $F(z)$ an automorphism of $k\langle\langle z \rangle\rangle$. Then, we have

\begin{equation}
(2.7) \quad \tilde{J}(U(F))(z) = \left(\left[\tilde{J}F(F^{-1})\frac{\partial}{\partial z}\right]^\tau U\right)(F),
\end{equation}

where the matrix $\left(\left[\tilde{J}F(F^{-1})\frac{\partial}{\partial z}\right]^\tau U\right)$ in the equation above is the formal “product” of the column vector $\left[\tilde{J}F(F^{-1})\frac{\partial}{\partial z}\right]^\tau \in \text{Der}(\langle\langle z \rangle\rangle)^n$ with the row vector $U(z) = (U_1, \cdots, U_m)$.

In particular, when $m = n$ and $U(z) = G(z) := F^{-1}(z)$, we have

\begin{equation}
(2.8) \quad \left[\tilde{J}F(G)\frac{\partial}{\partial z}\right] G = I_{n \times n} = \left[\tilde{J}G(F)\frac{\partial}{\partial z}\right] F(z).
\end{equation}

The proof of Eq. (2.7) is straightforward, just to apply Eq. (2.5) or (2.6) to the entries of the matrix $\tilde{J}(U(F))(z)$; while Eq. (2.8) is an immediate consequence of Eq. (2.7) and the fact $G(F(z)) = z = F(G(z))$.

Note that, when $z$ are commutative variables, Eq. (2.8) is same as $JF(G)JG = I_{n \times n} = JG(F)JF$. But, unlike in the commutative case, $JF(G)$ in general is not the multiplication inverse matrix of $JF$. This can be seen from the following example.

Example 2.3. Let $F(x, y) = (F_1, F_2)$ be the automorphism of $k\langle\langle x, y \rangle\rangle$ with

\begin{align*}
F_1(x, y) &= e^x - 1, \\
F_2(x, y) &= ye^{-x}.
\end{align*}
Its inverse map \( G(x, y) = (G_1, G_2) \) is given by
\[
G_1(x, y) = \ln(1 + x), \\
G_2(x, y) = y(1 + x).
\]

Now consider the Jacobian matrices
\[
J_F(x, y) = \begin{pmatrix} e^x & 0 \\ -ye^{-x} & e^{-x} \end{pmatrix}, \\
J_G(x, y) = \begin{pmatrix} \frac{1}{1+x} & 0 \\ y & 1 + x \end{pmatrix}
\]

But, on the other hand,
\[
J_G(F_{1}, F_{2}) = \begin{pmatrix} e^{-x} & 0 \\ ye^{-x} & e^{x} \end{pmatrix}, \\
(J_F)^{-1}(x, y) = \begin{pmatrix} e^{-x} & 0 \\ ye^{-2x} & e^{x} \end{pmatrix}
\]

Hence \( J_G(F) \neq (JF)^{-1} \) unless \( x \) and \( y \) commute with each other.

The second chain rule we will need later is the following. Let \( t \) be a formal parameter which commutes with \( z \) and \( k[[t]] \) the formal power series in \( t \) over \( k \). Note that the derivation \( \frac{\partial}{\partial t} \) of \( k[[t]] \) can be extended naturally to a derivation of \( k[[t]]\langle\langle z \rangle\rangle \), which we still denote by \( \frac{\partial}{\partial t} \), by setting \( \frac{\partial z_i}{\partial t} = 0 \) for any \( 1 \leq i \leq n \).

**Lemma 2.4.** Let \( F_t = (F_{t,1}, F_{t,2}, \ldots, F_{t,n}) \) be an automorphism of \( k[[t]]\langle\langle z \rangle\rangle \) (as an algebra over \( k[[t]] \)) with inverse map \( F_t^{-1}(z) \). Then, for any \( u_t(z) \in k[[t]]\langle\langle z \rangle\rangle \), we have

\[
\frac{\partial u_t(F_{t})}{\partial t} = \frac{\partial u_t}{\partial t}(F_{t}) + \left[ \frac{\partial F_{t}}{\partial t}(F_{t}^{-1}) \frac{\partial}{\partial z} \right] u_t(F_{t}). \tag{2.9}
\]

**Proof:** The proof is similar as the one for Lemma 2.1 which goes as follows.

First, composing \( F_t^{-1} \) to Eq. (2.9) from right, we get

\[
\frac{\partial u_t(F_{t})}{\partial t} \circ F_t^{-1} = \frac{\partial u_t}{\partial t}(z) + \left[ \frac{\partial F_{t}}{\partial t}(F_{t}^{-1}) \frac{\partial}{\partial z} \right] u_t, \tag{2.10}
\]

which is equivalent to Eq. (2.9).

Secondly, we define the maps \( \delta_1, \delta_2 : k[[t]]\langle\langle z \rangle\rangle \rightarrow k[[t]]\langle\langle z \rangle\rangle \) by setting

\[
\delta_1(u_t) = \frac{\partial u_t(F_{t})}{\partial t} \circ F_t^{-1}, \tag{2.11}
\]
\[
\delta_2(u_t) = \frac{\partial u_t}{\partial t} + \left[ \frac{\partial F_{t}}{\partial t}(F_{t}^{-1}) \frac{\partial}{\partial z} \right] u_t \tag{2.12}
\]

for any \( u_t(z) \in k[[t]]\langle\langle z \rangle\rangle \).

Hence, it will be enough to show \( \delta_1 = \delta_2 \). But, again, it is easy to see that \( \delta_i \ (i = 1, 2) \) both are derivations of \( k[[t]]\langle\langle z \rangle\rangle \). (Actually,
\[\delta_1 = (F_t^{-1})_* \left( \frac{\partial}{\partial t} \right)\]. Therefore, it will be enough to show they have same values when \(u_t(z) = t\) and \(u_t(z) = z_i\) for any \(1 \leq i \leq n\). But, for these cases, we have

\[\delta_1(t) = 1 = \delta_2(t),\]
\[\delta_1(z_i) = \frac{\partial F_t^{-1}}{\partial t}(F_t^{-1}) = \delta_2(z_i)\]

for any \(1 \leq i \leq n\). \(\Box\)

3. General Deformations

Let \(k\) be a unital integral domain of any characteristic and \(z = (z_1, z_2, \ldots, z_n)\) and \(t\) as in the previous section, i.e. \(z_i\) (\(1 \leq i \leq n\)) are \(n\) free noncommutative variables and \(t\) is a formal parameter which commutes with \(z_i\)’s. In this section, we study the general deformation of automorphisms of \(k\langle\langle z\rangle\rangle\) parameterized by \(t\). More precisely, we study automorphisms \(F_t(z)\) of \(k[[t]]\langle\langle z\rangle\rangle\) over \(k[[t]]\) of the form \(F_t(z) = z - H_t(z)\) with \(H_t(z) \in k[[t]]\langle\langle z\rangle\rangle^{\times n}\) and \(o(H_t(z)) \geq 2\). Note that, when \(F(z) := F_{t=1}(z)\) makes sense (for example, when \(H_t(z) \in k[t]\langle\langle z\rangle\rangle^{\times n}\), or when \(k = \mathbb{C}\) and all coefficients of \(H_t(z)\) are holomorphic functions of \(t\) which are convergent over an open subset of \(\mathbb{C}\) containing the closed unit disk), \(F_t(z)\) can be viewed as a deformation of the automorphism \(F(z) := F_{t=1}(z)\) of \(k\langle\langle z\rangle\rangle\). We will denote by \(G_t(z)\) and \(G(z)\) the formal inverse maps of \(F_t(z)\) and \(F(z) = F_{t=1}(z)\) (again, when it exists), respectively. We will always write \(G_t(z)\) as \(G_t(z) = z + M_t(z)\) for some \(M_t(z) \in k[t]\langle\langle z\rangle\rangle^{\times n}\) with \(o(M_t(z)) \geq 2\). Note that, when \(F(z) = F_{t=1}(z)\) and \(G_{t=1}(z)\) both make sense, by the uniqueness of inverse maps, we have \(G_{t=1}(z) = G(z)\). In this section, we first derive the PDE’s satisfied by \(M_t(z)\), \(u(F_t)\) and \(u(G_t)\) with \(u(z) \in k\langle\langle z\rangle\rangle\) (see Eqs. \((3.1)\), \((3.10)\) and \((3.11)\)). We then in Theorem \(3.4\) show that, when \(\text{char. } k = 0\), the power series \(u(F_t)\) and \(u(G_t)\) \((u(z) \in k\langle\langle z\rangle\rangle)\) are actually characterized by the PDE’s \((3.10)\) and \((3.11)\), respectively. When \(\text{char. } k = p > 0\), \(u(F_t)\) and \(u(G_t)\) still satisfy the PDE’s \((3.10)\) and \((3.11)\), respectively but they are only uniquely determined by these PDE’s together with their coefficients of \(\ell^m\) \((m \geq 0)\) (see Remark \(3.3\)).

Let us start with the following simple lemma.

**Lemma 3.1.** Let \(F_t(z)\), \(H_t(z)\), \(G_t(z)\), \(M_t(z)\) as fixed above. Then we have

\[(3.1) \quad M_t = H_t(G_t),\]
\[(3.2) \quad H_t = M_t(F_t),\]
\begin{equation}
\frac{\partial H_t}{\partial t}(z) = \left[ \frac{\partial M_t(F_t)}{\partial t} \frac{\partial}{\partial z} \right] F_t(z),
\end{equation}

(3.3)

\begin{equation}
\frac{\partial M_t}{\partial t}(z) = \left[ \frac{\partial H_t(G_t)}{\partial t} \frac{\partial}{\partial z} \right] G_t(z).
\end{equation}

(3.4)

**Proof:** Since \( F_t(G_t(z)) = z \), we have

\begin{equation}
z + M_t(z) - H_t(G_t(z)) = z.
\end{equation}

(3.5)

Hence Eq. (3.1) holds. Similarly, Eq. (3.2) follows from \( G_t(F_t(z)) = z \).

To show Eq. (3.3), applying \( \frac{\partial}{\partial t} \) to Eq. (3.1) and using the chain rule Eq. (2.9), we have

\begin{align*}
\frac{\partial M_t}{\partial t} = \frac{\partial H_t(G_t)}{\partial t} + \left( \left[ \frac{\partial M_t(F_t)}{\partial t} \frac{\partial}{\partial z} \right] H_t \right) (G_t),
\end{align*}

\begin{align*}
&= \frac{\partial H_t}{\partial t}(G_t) + \left( \left[ \frac{\partial M_t(F_t)}{\partial t} \frac{\partial}{\partial z} \right] H_t \right) (G_t).
\end{align*}

Therefore, we have

\begin{align*}
\frac{\partial H_t}{\partial t}(G_t) &= \frac{\partial M_t}{\partial t} - \left( \left[ \frac{\partial M_t(F_t)}{\partial t} \frac{\partial}{\partial z} \right] H_t \right) (G_t) \\
&= \left( \left[ \frac{\partial M_t(F_t)}{\partial t} \frac{\partial}{\partial z} \right] (z - H_t) \right) (G_t) \\
&= \left( \left[ \frac{\partial M_t(F_t)}{\partial t} \frac{\partial}{\partial z} \right] F_t \right) (G_t).
\end{align*}

Composing with \( F_t \) from right to the equation above, we get Eq. (3.3). Eq. (3.4) can be proved similarly by applying \( \frac{\partial}{\partial t} \) to Eq. (3.2). \( \square \)

Now, we set

\begin{equation}
h(t) := \left[ \frac{\partial M_t(F_t)}{\partial t} \frac{\partial}{\partial z} \right],
\end{equation}

(3.6)

\begin{equation}
m(t) := \left[ \frac{\partial H_t(G_t)}{\partial t} \frac{\partial}{\partial z} \right].
\end{equation}

(3.7)

**Lemma 3.2.**

\begin{equation}
(G_t)_*(h(t)) = m(t),
\end{equation}

(3.8)

\begin{equation}
(F_t)_*(m(t)) = h(t).
\end{equation}

(3.9)

**Proof:** Note that Eq. (3.9) follows immediately when we apply \( (F_t)_* \) to Eq. (3.8). So we only need show Eq. (3.8).
First, applying the chain rule Eq. (2.6) with \( \delta = h(t) \) and Eq. (3.6), we have

\[
(G_t)_*(h(t)) = \left( (h(t)F_t) \left( G_t \frac{\partial}{\partial z} \right) \right) = \left( \left( \frac{\partial M_t}{\partial t} \left( G_t \frac{\partial}{\partial z} \right) \right) F_t \right) \left( G_t \frac{\partial}{\partial z} \right)
\]

Applying Eqs. (3.3) and (3.7):

\[
= \left[ \frac{\partial H_t}{\partial t} \left( G_t \frac{\partial}{\partial z} \right) \right]
\]

\[= m(t).\]

\( \square \)

**Proposition 3.3.** For any \( u(z) \in k\langle z \rangle \), we have

\[
\frac{\partial u(F_t)}{\partial t} = -(m(t)u)(F_t) = -h(t) u(F_t), \tag{3.10}
\]

\[
\frac{\partial u(G_t)}{\partial t} = (h(t)u)(G_t) = m(t) u(G_t). \tag{3.11}
\]

**Proof:** Here we only give a proof for Eq. (3.10). Eq. (3.11) can be proved by a similar argument.

By the chain rule Eq. (2.9), we have

\[
\frac{\partial u(F_t)}{\partial t} = \frac{\partial u}{\partial t}(F_t) + \left( \left[ \frac{\partial F_t}{\partial t} \left( G_t \frac{\partial}{\partial z} \right) \right] u \right) (F_t)
\]

\[= \left[ \frac{\partial H_t}{\partial t} \left( G_t \frac{\partial}{\partial z} \right) \right] u (F_t)
\]

\[= -(m(t)u)(F_t).
\]

Hence, we get the first part of Eq. (3.10). To show the second part, first, by Eqs. (3.8) and (2.4), we have

\[
m(t)u(z) = ((G_t),h(t)) u(z)
\]

\[= (h(t)(u(F_t))(G_t).
\]

Composing with \( F_t \) from right to the equation above, we get

\[
(m(t)u)(F_t) = h(t) u(F_t). \tag{3.13}
\]

Combining Eqs. (3.12) and (3.13), we have

\[
\frac{\partial u(F_t)}{\partial t} = -(m(t)u)(F_t)
\]
\[ = -h(t) u(F_t), \]

which is the second part of Eq. (3.10). \[\square\]

Actually, when \(\text{char. } k = 0\), elements of \(k[[t]]\langle\langle z\rangle\rangle\) of the forms \(u(F_t)\) and \(u(G_t)\) for some \(u(z) \in k\langle\langle z\rangle\rangle\) are characterized by Eqs. (3.10) and (3.11), respectively. This can be seen from the following theorem.

**Theorem 3.4.** Assume that the base ring \(k\) has char. \(k = 0\), then

(a) For any \(U_t(z) \in k[[t]]\langle\langle z\rangle\rangle\), \(U_t(z) = u(F_t(z))\) for some \(u(z) \in k\langle\langle z\rangle\rangle\) iff \(U_t(z)\) satisfies the PDE

(3.14) \[ \frac{\partial U_t(z)}{\partial t} = -h(t)U_t(z). \]

(b) For any \(V_t(z) \in k[[t]]\langle\langle z\rangle\rangle\), \(V_t(z) = u(G_t(z))\) for some \(u(z) \in k\langle\langle z\rangle\rangle\) iff \(V_t(z)\) satisfies the PDE

(3.15) \[ \frac{\partial V_t(z)}{\partial t} = m(t)V_t(z). \]

**Proof:** (a) The \((\Rightarrow)\) part is just Proposition 3.3. Conversely, suppose \(U_t(z) \in k[[t]]\langle\langle z\rangle\rangle\) satisfies Eq. (3.14). Set \(\tilde{U}_t(z) = U_t(G_t(z))\). By the chain rule Eq. (2.9), we have

\[
\frac{\partial \tilde{U}_t(z)}{\partial t} = \frac{\partial U_t}{\partial t}(G_t) + \left( \frac{\partial G_t}{\partial t}(F_t) \frac{\partial}{\partial z} U_t \right)(G_t)
\]

\[
= \left( \frac{\partial U_t}{\partial t} + \frac{\partial G_t}{\partial t}(F_t) \frac{\partial}{\partial z} U_t \right)(G_t)
\]

\[
= \left( \frac{\partial U_t}{\partial t} + h(t)U_t \right)(G_t)
\]

\[
= 0.
\]

Therefore, if we set \(u(z) := \tilde{U}_t(z) = U_t(G_t(z))\), then \(u(z) \in k\langle\langle z\rangle\rangle\) and \(U_t(z) = u(F_t)\). Hence we have proved (a).

(b) can be proved similarly. \[\square\]

**Remark 3.5.** From the proof of Theorem 3.4 above, one can see that, when the base ring \(k\) has char. \(k = p > 0\), the \((\Rightarrow)\) part of the theorem still holds; while the \((\Leftarrow)\) part is not true in general. But, if the coefficients of \(t^m\) \((m \geq 0)\) of \(U_t(z)\) and \(V_t(z)\) are given or fixed, \(U_t(z)\) and \(V_t(z)\) are still uniquely determined by Eqs. (3.14) and (3.15), respectively. This can be easily seen by viewing \(U_t(z)\) and \(V_t(z)\) as formal power series in \(t\) over the ring \(k\langle\langle z\rangle\rangle\) and solving Eqs. (3.14) and (3.15) recursively. For a more detailed discussion on a similarly situation, see Section 5.
4. A Special Deformation

In this section, we will focus on a special family of deformations of automorphisms of $k\langle\langle z \rangle\rangle$. We start with a fixed automorphism $F(z)$ of $k\langle\langle z \rangle\rangle$ and always assume that $F(z)$ has the form $F(z) = z - H(z)$ with $o(H(z)) \geq 2$. We set $F_t(z) = z - tH(z)$ and write its inverse map as $G_t(z) = z + tN_t(z)$ with $N_t(z) \in k[[t]][\langle\langle z \rangle\rangle]^{\times n}$ and $o(N_t(z)) \geq 2$. In terms of the notation in Section 2, we have

$$H_t(z) = tH(z),$$

(4.1)

$$M_t(z) = tN_t(z).$$

(4.2)

We first apply the results obtained in the previous section for the general deformations to the special deformation above. In particular, we show in Theorem 4.3 that $N_t(z)$ is a power series solution of a Cauchy problem of the PDE involved (see Eqs. (4.10) and (4.11)). One interesting aspect of this fact is that, when passing to the commutative case, the PDE (4.10) is almost the Burgers’ equation in Diffusion theory, see Remark [4.3]. When $F_t(z) = z - tH(z)$ is a symmetric map, i.e. $H(z)$ is the gradient vector $\nabla P(z)$ for some $P(z) \in k[[z]]$, it can be further linked to the Heat equation. For more discussion in this direction, see [Z2] and [Z3]. The PDE (4.10) in Theorem 4.3 is also the starting point for the inversion formulas that will be derived in next two sections. Besides the property of $N_t(z)$ given in Theorem 4.3, other characterizing properties of $N_t(z)$ are also derived (see Lemma 4.7 and Proposition 4.8).

First, let us work out the special forms for the differential operators $h(t)$ and $m(t)$ defined in Eqs. (3.6) and (3.7), respectively, for the special deformation $F_t(z) = z - tH(z)$ with $H(z) \in k\langle\langle z \rangle\rangle^{\times n}$ and $o(H(z)) \geq 2$.

**Lemma 4.1.** With the notation above, we have

(4.3) $m(t) = \left[ N_t(z) \frac{\partial}{\partial z} \right],$

(4.4) $h(t) = \sum_{m \geq 1} t^{m-1} \left[ C_m(z) \frac{\partial}{\partial z} \right],$

where $C_m(z) \in k\langle\langle z \rangle\rangle^{\times n}$ ($m \geq 1$) are defined recursively by

(4.5) $C_1(z) = H(z),$ 

(4.6) $C_m(z) = \left[ C_{m-1}(z) \frac{\partial}{\partial z} \right] H,$

for any $m \geq 2.$
**Proof:** First, by Lemma 3.1 and Eqs. (4.1) and (4.2), it is easy to see that, we have

\[(4.7) \quad N_t(F_t(z)) = H(z),\]
\[(4.8) \quad H(G_t) = N_t(z).\]

By Eqs. (3.7), (4.1) and also the equations above, we have

\[
m(t) = \left[ \frac{\partial H_t}{\partial t}(G_t) \frac{\partial}{\partial z} \right]
= \left[ H(G_t) \frac{\partial}{\partial z} \right]
= \left[ N_t(z) \frac{\partial}{\partial z} \right].
\]

Hence, we get Eq. (4.3).

To show Eq. (4.4), we first write \(h(t)\) as in Eq. (4.4) for some \(C_m(z) \in k\langle \langle z \rangle \rangle^n (m \geq 1)\), and then show that \(C_m(z)\)'s also satisfy Eqs. (4.5) and (4.6). Consequently, \(C_m(z) (m \geq 1)\) will be uniquely determined by Eqs. (4.5) and (4.6).

First, by Eqs. (3.3) and (3.6), we have

\[
H(z) = \frac{\partial H_t}{\partial t}(z)
= h(t) F_t(z)
= \sum_{m \geq 1} t^{m-1} \left[ C_m(z) \frac{\partial}{\partial z} \right] (z - tH(z))
= \sum_{m \geq 1} t^{m-1} C_m(z) - t \sum_{m \geq 1} t^{m-1} \left[ C_m(z) \frac{\partial}{\partial z} \right] H(z)
= C_1(z) + \sum_{m \geq 2} t^{m-1} \left( C_m(z) - t \left[ C_{m-1}(z) \frac{\partial}{\partial z} \right] H(z) \right).
\]

Then, by comparing the coefficients of \(t^{m-1} (m \geq 1)\) in the equation above, we see that \(C_m(z) (m \geq 1)\) indeed satisfy Eqs. (4.5) and (4.6).

By using the mathematical induction on \(m \geq 1\), it is easy to check that, when \(z_i\)'s are commutative variables, \(C_m(z)\) further has the following simple form.

**Corollary 4.2.** For commutative variables \(z_i (1 \leq i \leq n)\), we have

\[(4.9) \quad C_m(z) = (JH)^{m-1} H,\]
for any \( m \geq 1 \).

By Eqs. (4.3), (4.8) and Theorem 3.4, (b) with \( u(z) = H_i(z) \) \((1 \leq i \leq n)\) for the special deformation \( F_t \), it is easy to see that we have the following theorem on \( N_t(z) \), which later will imply an effective recurrent inversion formula for \( G_t(z) \) (see Theorem 5.5).

**Theorem 4.3.** Let \( k \) be a unital integral domain of any characteristic and \( H(z) \in k\langle\langle z\rangle\rangle^{\times n} \), \( N_t(z) \in k[[t]]\langle\langle z\rangle\rangle^{\times n} \) as above, then, \( N_t(z) \) is a power series solution in \( k[[t]]\langle\langle z\rangle\rangle \times n \) of the following Cauchy problem of PDE’s in noncommutative variables.

\[
\frac{\partial N_t}{\partial t} = \left[ N_t \frac{\partial}{\partial z} \right] N_t \tag{4.10}
\]

\[
N_{t=0}(z) = H(z). \tag{4.11}
\]

**Remark 4.4.** Note that, in the commutative case, the PDE (4.10) becomes

\[
\frac{\partial N_t}{\partial t} = JN_t \cdot N_t. \tag{4.12}
\]

which was first proved in [Z1] (unpublished) and later in [Z2]. Interestingly, the PDE above is almost the classical Burgers’ equation in Diffusion theory, which has the form

\[
\frac{\partial N_t}{\partial t} = (JN_t)^T \cdot N_t. \tag{4.13}
\]

In particular, when \( N_t \) is the gradient vector of \( Q_t \) for some \( Q_t \in k[[t]][[z]] \), Eqs. (4.12) and (4.13) coincide. Furthermore, in this case, Eq. (4.12) is also closely related with the Heat equation. For more detailed discussions on the connections among these three PDE’s in the commutative case, see [Z2] and [Z3].

Next, we derive more properties of \( N_t(z) \). The first interesting property of \( N_t(z) \) is the following proposition. It essentially says that \( \{N_t(z) | t \in k\} \) gives a family of automorphisms of \( k[[t]]\langle\langle z\rangle\rangle \) which are “closed” under the inverse operation.

**Proposition 4.5.** For any \( s \in k \), the formal inverse of \( U_{s,t}(z) := z - sN_t(z) \) is given by \( V_{s,t}(z) := z + sN_{t+s}(z) \). Actually, \( U_{s,t}(z) = F_{t+s} \circ G_t(z) \) and \( V_{s,t}(z) = F_t \circ G_{s+t}(z) \).

**Proof:**

\[
F_{t+s} \circ G_t(z) = G_t(z) - (t + s)H(G_t(z))
= z + tN_t(z) - (t + s)N_t(z)
= z - sN_t(z)
\]
Similarly, we can prove \( V_{s,t}(z) = F_t \circ G_{s+t}(z) \). Hence we have \( U_{s,t}^{-1}(z) = V_{s,t}(z) \). \( \square \)

In the rest of this section, we will assume the base ring \( k \) has char. \( k = 0 \). Below we show that \( N_t(z) \) in this case is actually characterized by the Cauchy problem Eqs. (4.10) and (4.11) in Theorem 4.3.

**Proposition 4.6.** For any \( H(z) \in k\langle\langle z\rangle\rangle \times n \) and \( N_t(z) \in k[[t]]\langle\langle z\rangle\rangle \times n \) with \( o(H(z)) \geq 2 \) and \( o(N_t(z)) \geq 2 \), respectively. The following statements are equivalent.

1. The formal map \( G_t(z) = z + tN_t(z) \) is the inverse of \( F_t(z) = z - tH(z) \).
2. \( N_t(z) \in k[[t]]\langle\langle z\rangle\rangle \) is the unique power series solution of the Cauchy problem Eqs. (4.10) and (4.11).

**Proof:** First, (1) \( \Rightarrow \) (2) is exactly Theorem 4.3. To show (2) \( \Rightarrow \) (1), we assume that the formal inverse of \( F_t(z) = z - tH(z) \) is given by \( G_t(z) = z + t\tilde{N}_t(z) \). By Theorem 4.3, we know that \( \tilde{N}_t(z) \) also satisfies Eqs. (4.10) and (4.11). But, by Corollary 5.2 (a) in next section, the power series solution in \( k[[t]]\langle\langle z\rangle\rangle \) of Eqs. (4.10) and (4.11) is actually unique. Hence we have \( \tilde{N}_t(z) = N_t(z) \) and (2) \( \Rightarrow \) (1) follows. \( \square \)

Another characterizing property of \( N_t(z) \) (see Proposition 4.8 below) can be derived as follows. First, we need the following lemma.

**Lemma 4.7.** For any \( u(z) \in k\langle\langle z\rangle\rangle \), the unique power series solution \( U_t(z) \) in \( z \) and \( t \) of the following Cauchy problem

\[
\begin{align*}
\frac{\partial U_t}{\partial t} &= \left[ N_t \frac{\partial}{\partial z} \right] U_t, \\
U_{t=0}(z) &= u(z).
\end{align*}
\]

(4.14)

is given by \( U_t(z) = u(z + tN_t(z)) \).

**Proof:** By a similar argument as in the proof of Lemma 5.1 in next section, it is easy to check that the power series solution in \( z \) and \( t \) of the Cauchy problem Eq. (4.14) is unique. So it will be enough to show that \( U_t(z) = u(z + tN_t(z)) \) satisfies Eq. (4.14). First, the boundary condition in Eq. (4.14) is obviously satisfied by \( U_t(z) \). Secondly, by Theorem 3.4, (b) and Eq. (4.3), \( U_t(z) \) also satisfies the PDE in Eq. (4.14). \( \square \)

**Proposition 4.8.** For any \( N_t(z) \in k[[t]]\langle\langle z\rangle\rangle \times n \) with \( o(N_t(z)) \geq 2 \), the following two statements are equivalent.
(a) $z + tN_t(z)$ is the formal inverse map of $F_t(z) = z - tH(z)$ for some $H(z) \in k\langle\langle z \rangle\rangle^{\times n}$.

(b) Lemma 4.7 holds for $N_t(z)$.

Proof: First, $(a) \Rightarrow (b)$ follows from Lemma 4.7. To show $(b) \Rightarrow (a)$, let $U_{t,i}(z) (1 \leq i \leq n)$ be the unique power series solution of the Cauchy problem (4.14) with $u(z) = z_i$. Set $\tilde{U}_t(z) = (U_{t,1}(z), \ldots, U_{t,n}(z))$. Note that Eq. (4.14) for $U_{t,i}(z) (1 \leq i \leq n)$ can be written as

\begin{equation}
\frac{\partial \tilde{U}_t}{\partial t} = \left[ N_t \frac{\partial}{\partial z} \right] \tilde{U}_t.
\end{equation}

(4.15)

Since, by our condition on $N_t(z)$, Lemma 4.7 holds for $N_t(z)$, so we have

\begin{equation}
\tilde{U}_t(z) = z + tN_t(z).
\end{equation}

(4.16)

Applying $\frac{\partial}{\partial t}$ to the equation above, we get

\begin{equation}
\frac{\partial \tilde{U}_t}{\partial t} = N_t + t\frac{\partial N_t}{\partial t}.
\end{equation}

(4.17)

Combining the equation above with Eqs. (4.15) and (4.16), we have

\begin{equation}
N_t + t\frac{\partial N_t}{\partial t} = \left[ N_t \frac{\partial}{\partial z} \right] (z + tN_t) = N_t + t \left[ N_t \frac{\partial}{\partial z} \right] N_t.
\end{equation}

Therefore, we have

\begin{equation}
\frac{\partial N_t}{\partial t} = \left[ N_t \frac{\partial}{\partial z} \right] N_t.
\end{equation}

(4.18)

Set $H(z) := N_{t=0}(z)$. Therefore, $N_t(z)$ is a formal power series solution of the Cauchy problem Eqs. (4.10) and (4.11). Then, by Proposition 4.6 we see that $(a)$ holds.

\[ \square \]

5. A Recurrent Inversion Formula for automorphisms in Noncommutative Variables

In this section, we apply some results obtained in Section 4 to derive a recurrent inversion formula for formal maps in noncommutative variables (see Theorem 5.3). This will generalize the recurrent inversion formula in [Z2] for the commutative case with $\text{char. } k = 0$ to the noncommutative case over a base ring $k$ of any characteristic.
Lemma 5.1. Let $W_t(z) \in k[[t]]\langle\langle z\rangle\rangle$ be a solution of Eqs. (4.10) and (4.11). We write $W_t(z)$ as
\begin{equation}
W_t(z) = \sum_{m=1}^{\infty} W_m(z)t^{m-1}.
\end{equation}
with $W_m(z) \in k\langle\langle z\rangle\rangle$ $(m \geq 1)$. Then, the sequence $\{W_m(z)|m \geq 1\}$ satisfies the following recurrent relations:
\begin{equation}
W_1(z) = H(z),
\end{equation}
\begin{equation}
(m-1)W_m(z) = \sum_{k+l=m, \ k,l \geq 1} \left[ W_k \frac{\partial}{\partial z} \right] W_l
\end{equation}
for any $m \geq 2$.

Proof: First, Eq. (5.2) follows directly from Eq. (4.11). Secondly, by Eq. (4.10), we have
\begin{equation}
\sum_{m=1}^{\infty} (m-1)W_m(z)t^{m-2} = \left( \sum_{k=1}^{\infty} t^{k-1} \left[ W_k \frac{\partial}{\partial z} \right] \right) \left( \sum_{l=1}^{\infty} W_l(z)t^{l-1} \right).
\end{equation}
For any $m \geq 2$, by comparing the coefficients of $t^{m-2}$ of the both sides of the equation above, we get Eq. (5.3). \qed

Some direct consequences of the lemma above are given by the following three corollaries.

Corollary 5.2. (a) When char. $k = 0$, the power series solutions in $k[[t]]\langle\langle z\rangle\rangle$ of the Cauchy problem Eqs. (4.10) and (4.11) is unique.
(b) When char. $k = p > 0$, there are infinitely many solutions $W_t(z)$ in $k[[t]]\langle\langle z\rangle\rangle$ of the Cauchy problem Eqs. (4.10) and (4.11). Actually, for any fixed $W_{mp+1} \in k\langle\langle z\rangle\rangle$ $(m \geq 1)$, there exists one and only one solution of Eqs. (4.10) and (4.11).

Let $H(z)$ and $N_t(z)$ be fixed as in Section 4. We define the sequence $\{N_m(z) \in k\langle\langle z\rangle\rangle|m \geq 1\}$ by writing
\begin{equation}
N_t(z) = \sum_{m \geq 1} t^{m-1}N_m(z).
\end{equation}

Corollary 5.3. Suppose that the base ring $k$ has char. $k = p > 0$. Then, for any $m \geq 1$ and $m \equiv 1 \mod p$, we have
\begin{equation}
\sum_{k+l=m, \ k,l \geq 1} \left[ N_k \frac{\partial}{\partial z} \right] N_l(z) = 0.
\end{equation}
Proof: By Theorem 4.3 and Lemma 5.1, we know the sequence \( \{N_{[m]}(z) \in k\langle z \rangle | m \geq 1 \} \) satisfies the recurrent relations Eqs. (5.2) and (5.3). Hence the corollary follows immediately from Eq. (5.3). \( \square \)

Corollary 5.4. For any unital integral domain \( k \) of any characteristic, we have
(a) \( o(N_{[m]}(z)) \geq m + 1 \) for any \( m \geq 1 \).
(b) Suppose \( H(z) \in k\langle z \rangle^{\times n} \), then, for any \( m \geq 1 \), \( N_{[m]}(z) \in k\langle z \rangle^{\times n} \) with \( \deg N_{[m]}(z) \leq m(\deg H - 1) + 1 \).
(c) If \( H(z) \) is homogeneous of degree \( d \geq 2 \), then, \( N_{[m]}(z) \) is homogeneous of degree \( (d - 1)m + 1 \) for any \( m \geq 1 \).

Proof: Again, by Theorem 4.3 and Lemma 5.1, we know that the sequence \( \{N_{[m]}(z) \in k\langle z \rangle | m \geq 1 \} \) satisfies the recurrent relations Eqs. (5.2) and (5.3). If \( \text{char. } k = 0 \), the corollary can be easily proved by the mathematical induction on \( m \geq 1 \) via the recurrent relation Eq. (5.3). But, if \( \text{char. } k = p > 0 \), the induction breaks down when \( m \equiv 1 \pmod{p} \). However, we can fix this problem as follows. Suppose the corollary holds for all \( 1 \leq l \leq kp \) for some \( k \geq 1 \). We consider \( N_{[m]}(z) \) with \( m = kp + 1 \). By Eq. (5.2), we have
\[
H = N_t(z - tH) = \sum_{l \geq 1} t^{l-1} N_{[l]}(z - tH).
\]
Comparing the coefficients of \( t^{m-1} \) in the equation above, we have
\[
N_{[m]}(z) = -\text{Res}_t \sum_{l=1}^{m-1} t^{l-m-1} N_{[l]}(z - tH).
\]
(5.6)

Note that, for any \( 1 \leq l \leq m \), \( \text{Res}_t t^{l-m-1} N_{[l]}(z - tH) \) as the coefficient of \( t^{m-l} \) of \( N_{[l]}(z - tH) \) is obtained by replacing \( (m - l) \) copies \( z_i \)'s by \((-H_i)\)'s in all possible ways for each monomial of \( N_{[l]}(z - tH) \). With this observation, it is easy to see that our mathematical induction arguments still can go through at \( m = kp + 1 \). \( \square \)

Note that, by Corollary 5.4 (a), the infinite sum \( \sum_{m=1}^{\infty} N_{[m]}(z)t_0^{m-1} \) makes sense for any \( t = t_0 \in k \). In particular, when \( t = 1 \), \( G_{t=1}(z) \) gives us the formal inverse \( G(z) \) of \( F(z) \). Now we can summarize the results above to formulate the following recurrent inversion formula.

Theorem 5.5. (Recurrent Inversion Formula)
Let \( k \) be any integral domain of any characteristic. Let \( H(z), N_t(z) \) and \( \{N_{[m]}(z) | m \geq 1 \} \) fixed as before. Then
(a) If char. \( k = 0 \), \( \{ N_{[m]}(z) | m \geq 1 \} \) are completely determined by
\begin{equation}
N_{[1]}(z) = H(z),
\end{equation}
\begin{equation}
N_{[m]}(z) = \frac{1}{m-1} \sum_{k+l=m} \left[ N_{[k]} \frac{\partial}{\partial z} \right] N_{[l]}(z)
\end{equation}
for any \( m \geq 2 \).

(b) If char. \( k = p > 0 \), the recurrent relations above still hold for any \( m \geq 2 \) and \( m \not\equiv 1 \pmod{p} \). When \( m = kp + 1 \) for some \( k \geq 1 \), \( N_{[m]}(z) \) can be obtained by Eq. \((5.6)\).

When char. \( k = p > 0 \), the inverse maps \( G(z) \) can also be obtained by the following symbolic calculation.

**Algorithm 5.6. (An Inversion Algorithm when char. \( k = p > 0 \))**

**Step 1:** Let \( S \) be the set of the ordered triples \((i; I, J)\) with \( 1 \leq i \leq n \) and \( I, J \in (\mathbb{N}^+)^m \) for some \( m \geq 1 \) such that the monomial \( z_{i_1}^{j_1}z_{i_2}^{j_2} \cdots z_{i_m}^{j_m} \) appears in \( H_i(z) \) with a nonzero coefficient, say, \( a_i^j(i) \in k \). Now let \( A := \{ A_i^j(i) | (i; I, J) \in S \} \) be a set of free commutative variables and define \( \bar{F}(z) \in \mathbb{Z}[A]\langle\langle z\rangle\rangle^\times n \) by replacing \( a_i^j(i) \) by \( A_i^j(i) \) in \( F(z) \) for each triple \((i; I, J) \in S\).

**Step 2:** We view \( \bar{F}(z) \) as an automorphism of \( \mathbb{Z}[A]\langle\langle z\rangle\rangle \) over the base ring \( \mathbb{Z}[A] \) which is of characteristic zero. Now we can apply the recurrent formulas Eqs. \((5.7)\) and \((5.8)\) to calculate the inverse map \( \bar{G}(z) \) of \( \bar{F}(z) \). Note that coefficients of all monomials of \( \bar{G}(z) \) are also in the base ring \( \mathbb{Z}[A] \).

**Step 3:** To recover the inverse map \( G(z) \) from \( \bar{G}(z) \), we simply change all coefficients of \( \bar{G}(z) \) by replacing each \( A_i^j(i) \) by \( a_i^j(i) \) and each integer by its congruence class modulo \( p \).

**Remark 5.7.** In Step 1 of the algorithm above, we may lift those co-efficients \( a_i^j(i) \in k \) which lie in \( \mathbb{Z}_p \subseteq k \) to any their pre-images in \( \mathbb{Z} \) instead of the corresponding formal variables \( A_i^j(i) \). This may reduce the number of formal variables \( A_i^j(i) \) involved and simplifies the algorithm substantially under certain circumstances.

Next let us consider the following example for the recurrent formula Eq. \((5.8)\) in Theorem 5.5

**Example 5.8.** Let \( k \) be any integral domain with char. \( k = 0 \) and \( x, y \) two noncommutative free variables. Let \( ad_y : k\langle\langle x, y\rangle\rangle \to k\langle\langle x, y\rangle\rangle \) be the \( k \)-linear map with \( ad_y(u) = yu - uy \) for any \( u \in k\langle\langle x, y\rangle\rangle \).
Let $F(x, y) = (F_1(x, y), F_2(x, y))$ be the automorphism of the formal power series algebra $k\langle\langle x, y \rangle\rangle$ with
\[
\begin{cases}
F_1(x, y) = x - (yx - xy) = (1 - ad_y)(x), \\
F_2(x, y) = y,
\end{cases}
\]
where 1 in the expression denotes the identity map of $k\langle\langle x, y \rangle\rangle$.

Now let us apply the recurrent formula in Theorem 5.5 to determine the inverse map $G(x, y)$ of $F(x, y)$.

Let $t$ be a central parameter and $F_t(x, y)$ be the special deformation discussed in Section 4, i.e.
\[
\begin{cases}
F_{t,1}(x, y) = x - t(yx - xy) = (1 - t ad_y)(x), \\
F_{t,2}(x, y) = y.
\end{cases}
\]

Let $G_t(x, y) = z + t N_t(x, y)$ be the inverse map of $F_t(x, y)$. From the equation $F_{t,2}(G_{t,1}, G_{t,2}) = y$, we see that $G_{t,2}(x, y) = y$ and the second component of $N_t(x, y)$ must be zero. Therefore, there exists $u_t(x, y) = \sum_{m \geq 1} t^{m-1} u_m(x, y) \in k\langle\langle x, y \rangle\rangle$ such that
\[
N_t(x, y) = (u_t(x, y), 0),
\]
\[
N_{[m]}(x, y) = (u_m(x, y), 0)
\]
for any $m \geq 1$.

With $N_{[m]}(x, y)$ having the form above, one can easily check that Eqs. (5.7) and (5.8) in Theorem 5.5 become
\[
u_1(x, y) = xy - yx = ad_y(x),
\]
\[
u_m(x, y) = \frac{1}{m - 1} \sum_{k,l \geq 1} \left[u_k(x, y) \frac{\partial}{\partial x} \right] u_l(x, y).
\]
for any $m \geq 2$.

We claim that, for any $m \geq 1$, $u_m(x, y) = ad_y^m(x)$. This can be easily checked inductively by using Eqs. (5.11) and (5.12) above along with the following simple observation: for any $k, l \geq 0$, we have
\[
\left[ ad_y^k(x) \frac{\partial}{\partial x} \right] ad_y(l) = \left[ ad_y^k(x) \frac{\partial}{\partial x} \right] x \\
= ad_y^l ad_y(x) \\
= ad_y^{k+l}(x).
\]

Therefore, we have
\[
G_{t,1}(x, y) = x + t N_{t,1}(x, y)
\]
In particular, the inverse map \(G(x, y)\) of \(F(x, y)\) is given by

\[
\begin{align*}
G_1(x, y) &= (1 - t ad_y)^{-1}(x) = \sum_{m \geq 0} t^m ad_y^m(x), \\
G_2(x, y) &= y.
\end{align*}
\]

Note that the formula of \(G_t(x, y)\) derived above can also be checked directly as follows.

Second Proof: It is enough to check directly that

\[
\begin{align*}
F_{t,1}(G_{t,1}, G_{t,2}) &= x, \\
F_{t,2}(G_{t,1}, G_{t,2}) &= y.
\end{align*}
\]

The second equation is obvious. Now consider the first one:

\[
\begin{align*}
F_{t,1}(G_{t,1}, G_{t,2}) &= (1 - t ad_y)(G_{t,1}(x, y)), \\
&= (1 - t ad_y)((1 - t ad_y)^{-1}(x)), \\
&= x.
\end{align*}
\]

\[\square\]

Remark 5.9. Considering the (commutative) Jacobian conjecture (see [BCW] and [E]), a naive noncommutative generalization of the Jacobian conjecture would be: for any polynomial map \(F\) of \(k\langle\langle z\rangle\rangle\) with the Jacobian matrix \(JF\) (defined before Corollary 2.2) multiplicatively invertible, i.e. \(JF \in GL_n(k\langle\langle z\rangle\rangle)\), it must be a polynomial automorphism of \(k\langle\langle z\rangle\rangle\). The simple example above shows that this noncommutative generalization of the Jacobian conjecture is simply false, for the Jacobian matrix \(JF_t(x, y)\) is the \(2 \times 2\) identity matrix; while the inverse map \(G_t(x, y)\) is not a polynomial map. For a correct noncommutative generalization of the Jacobian conjecture, see [Se] and [MSY].

Finally, let us consider the following example for the symbolic Algorithm 5.6 when the base ring \(k\) has char. \(k = p > 0\).

Example 5.10. Let \(k\) be an integral domain of char. \(k = p > 0\) and \(F(x, y) = (F_1(x, y), F_2(x, y))\) the polynomial map of \(k\langle\langle x, y\rangle\rangle\) given by

\[
\begin{align*}
F_1(x, y) &= x - s_0(yx - xy) = (1 - s_0 ad_y)(x), \\
F_2(x, y) &= y,
\end{align*}
\]
where $s_0$ is any fixed element of $k$.

Let $s$ be a formal variable. Applying Step 1 of Algorithm 5.6, we get the polynomial map $\tilde{F}_s(x, y)$ of $\mathbb{Z}[s][\langle x, y \rangle]$ with
\[
\begin{align*}
\tilde{F}_{s,1}(x, y) &= x - s(yx - xy) = (1 - s \text{ad}_y)(x), \\
\tilde{F}_{s,2}(x, y) &= y.
\end{align*}
\]

Now we consider Step 2 of Algorithm 5.6. By the arguments in Example 5.8 above (with the central parameter $t$ replaced by $s$), we see that the inverse map $\tilde{G}_s(x, y)$ of $\tilde{F}_s(x, y)$ is given by
\[
\begin{align*}
\tilde{G}_{s,1}(x, y) &= \sum_{m \geq 0} s^m \text{ad}_y^m(x), \\
\tilde{G}_{s,2}(x, y) &= y.
\end{align*}
\]

Finally apply Step 3, we get the inverse map $G(x, y)$ of $F(x, y)$ with
\[
\begin{align*}
G_1(x, y) &= \sum_{m \geq 0} s_0^m \text{ad}_y^m(x), \\
G_2(x, y) &= y.
\end{align*}
\]

For example, when $k = \mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$ and $s_0 = 4 = -1$, the inverse map $G(x, y)$ is given by
\[
\begin{align*}
G_1(x, y) &= \sum_{m \geq 0} 4^m \text{ad}_y^m(x) = \sum_{m \geq 0} (-1)^m \text{ad}_y^m(x), \\
G_2(x, y) &= y.
\end{align*}
\]

6. An Expansion Inversion Formula by the Planar Binary Rooted Trees

In this section, we always assume the base ring $k$ has char. $k = 0$. We derive an expansion inversion formula by the planar binary rooted trees for the inverse map $G(z)$ of automorphisms $F(z)$ of $k[\langle z \rangle]$ (see Theorem 6.2). Note that, unlike the tree expansion formula in [Z2], which only holds for the symmetric maps in commutative variables, the tree expansion inversion formula derived here works for all formal automorphisms in commutative or noncommutative variables.

First let us fix the following notations and conventions. By a rooted tree we mean a finite 1-connected graph with one vertex designated as its root. In a rooted tree there are natural ancestral relations between vertices. We say a vertex $w$ is a child of vertex $v$ if the two are connected by an edge and $w$ lies further from the root than $v$. We define the degree of a vertex $v$ of $T$ to be the number of its children. A vertex is called a leaf if it has no children. A rooted tree $T$ is said to be a binary if every non-leaf vertex of $T$ has exactly two children. A rooted tree $T$ is said to be a planar if the set of all
children of each non-leaf vertex of $T$ is given a fixed linear order. A planar rooted forest is an ordered disjoint union of finitely many planar rooted trees. A planar binary rooted tree is a rooted tree which is both planar and binary. When we speak of isomorphisms between rooted trees, we will always mean root-preserving isomorphisms.

**Notation:**

Once and for all, we fix the following notation for the rest of this paper.

1. We let $T$ (resp. $B$) be the set isomorphism classes of all rooted trees (resp. binary rooted trees). We denote by $\mathbb{T}^p$ (resp. $\mathbb{B}^p$) the set of all planar rooted trees (resp. planar binary rooted trees). For any $m \geq 1$, we let $\mathbb{T}_m$, $\mathbb{B}_m$, $\mathbb{T}_m^p$ and $\mathbb{B}_m^p$ be the set of elements of $\mathbb{T}$, $\mathbb{B}$, $\mathbb{T}^p$ and $\mathbb{B}^p$, respectively, with $m$ vertices.

2. We call the rooted tree with one vertex the singleton, denoted by $\circ$. For convenience, we also view the empty set as a rooted tree, denoted by $\emptyset$.

3. For any rooted tree $T$, we set the following notation:
   - $\text{rt}_T$ denotes the root vertex of $T$.
   - $|T|$ denotes the number of the vertices of $T$ and $l(T)$ the number of leaves.
   - $\hat{T}$ denotes the rooted tree obtained by deleting all the leaves of $T$.

For any set of rooted trees $T_1, T_2, \ldots, T_d$, we define $B_+(T_1, T_2, \ldots, T_d)$ to be the rooted tree obtained by connecting all roots of $T_i$ ($i = 1, 2, \ldots, d$) to a single new vertex, which is set to the root of the new rooted tree $B_+(T_1, T_2, \ldots, T_d)$. For any rooted forest, say $T_1, T_2, \ldots, T_d$ ordered by their indices, we define $B_+(T_1, T_2, \ldots, T_d)$ similarly, except we also order the set of children of the new root, which is set of roots of $T_i$'s, as the same order of $T_i$'s. Note that, for any $T_1, T_2 \in \mathbb{B}$, we have $B_+(T_1, T_2) \in \mathbb{B}$.

Next let us recall $T$-factorial $T!$ of rooted trees $T$, which was first introduced by D. Kreimer [Kr]. It is defined inductively as follows.

1. For the empty rooted tree $\emptyset$ and the singleton $\circ$, we set $\emptyset! = 1$ and $\circ! = 1$.

2. For any rooted tree $T = B_+(T_1, T_2, \ldots, T_d)$, we set

$$ T! = |T| T_1! T_2! \cdots T_d!.$$

Note that, for the chains $C_m$ ($m \in \mathbb{N}$), i.e. the rooted trees with $m$ vertices and height $m - 1$, we have $C_m! = m!$. Therefore the $T$-factorial of rooted trees can be viewed as a generalization of the usual factorial of natural numbers.
Lemma 6.1. (a) For any non-empty binary rooted tree $T$, we have

\begin{align}
|T| &= 2l(T) - 1, \quad (6.2) \\
|\hat{T}| &= l(T) - 1. \quad (6.3)
\end{align}

(b) For any $T \in \mathcal{B}$ with $T = B_+(T_1, T_2)$, we have

\begin{equation}
\hat{T}! = (l(T) - 1)\hat{T}_1!\hat{T}_2! \quad (6.4)
\end{equation}

Proof: (a) can be proved easily by induction on the number of vertices. See Lemma 5.1 in [22], for example.

(b) Note that, by the definition of the operation $B_+ \hat{T}$, we have $\hat{T} = B_+(\hat{T}_1, \hat{T}_2)$. By Eqs. (6.1) and (6.3), we also have

\begin{equation}
\hat{T}! = |\hat{T}| \hat{T}_1!\hat{T}_2! = (l(T) - 1)\hat{T}_1!\hat{T}_2! \quad (6.5)
\end{equation}

Hence we have Eq. (6.4). \(\square\)

Now we fix an automorphism $F(z) = z - H(z)$ of $k\langle\langle z\rangle\rangle$ with $o(H(z)) \geq 2$. Let $F_t(z) = z - tH(z)$ and $G_t(z) = z + tN(z)$ as in Section 4.

We assign a $n$-sequence $N_T(z) \in k\langle\langle z\rangle\rangle \times n$ for each non-empty planar binary rooted tree $T$ as follows.

(1) For $T = \emptyset$, we set $N_T(z) = z$.

(2) For $T = o$, we set $N_T(z) = H(z)$.

(3) For any planar binary rooted tree $T = B_+(T_1, T_2)$, we set

\begin{equation}
N_T(z) = \left[N_{T_1}(z) \frac{\partial}{\partial z}\right] N_{T_2}(z). \quad (6.6)
\end{equation}

Now we are ready to state and prove the main theorem of this section.

Theorem 6.2. For any $m \geq 1$, we have

\begin{equation}
N_{[m]}(z) = \sum_{T \in \mathcal{B}_2} \frac{1}{T!} N_T(z) = \sum_{T \in \mathcal{B}_2 \atop l(T) = m} \frac{1}{T!} N_T(z). \quad (6.7)
\end{equation}

Therefore, by Eq. (5.4) we have

\begin{align}
N_t(z) &= \sum_{T \in \mathcal{B}_2} \frac{t^{l(T)-1}}{T!} N_T(z), \quad (6.8) \\
G_t(z) &= \sum_{T \in \mathcal{B}_2} \frac{t^{l(T)}}{T!} N_T(z). \quad (6.9)
\end{align}
Proof: Note that, by Eq. (6.2) in Lemma 6.1, we have
\[ B_{2m-1}^p = \{ T \in B^p \mid l(T) = m \} \]
\[ B_{2m}^p = \emptyset, \]
for any \( m \geq 1 \). Hence the two sums in Eq. (6.7) are equal to each other.

To prove Eq. (6.7), we first set, for any \( m \geq 1 \),
\[ V_{[m]}(z) = \sum_{T \in B_{2m-1}^p} \frac{1}{T!} N_T(z). \]
Then, by Theorem 5.5, to show that \( V_{[m]}(z) = N_{[m]}(z) \) for any \( m \geq 1 \), it will be enough to show that the sequence \( \{ V_{[m]}(z) \in k\langle\langle z\rangle\rangle \mid m \geq 1 \} \) also satisfies Eqs. (5.7) and (5.8).

For the case \( m = 1 \), since there is only one planar binary rooted tree \( T \) with \( l(T) = 1 \), namely, \( T = \circ \), we have \( V_{[1]}(z) = N_{T=\circ}(z) = H(z) \). Hence Eq. (5.7) is satisfied.

For any \( m \geq 2 \), we consider
\[ \sum_{k,l \geq 1} \frac{1}{k!l!} \left( \sum_{T \in B_{2m}^p, l(T) = k, l(T_2) = l, k,l \geq 1, k+l = m} N_{B_k(T_1, T_2)}(z) \right) \]
Applying Eq. (6.4) in Lemma 6.1
\[ = \sum_{T \in B_{2m}^p, l(T) = m} \frac{1}{T!} N_T(z) \]
\[ = V_{[m]}(z). \]
Hence we have Eq. (5.8) for \( V_{[m]}(z) \)'s. □

Next let us consider the tree expansion formula Eq. (6.7) for the polynomial map in Example 5.8 (see Example 6.4 below). As a bi-product, we will get a proof for the following identities of the \( T \)-factorials \( \hat{T}! \) (\( T \in B^p \)).
Proposition 6.3. For any \( m \geq 1 \), we have
\[
\sum_{T \in B_{2m-1}^p} \frac{1}{T!} = \sum_{T \in B^p_{(T)=m}} \frac{1}{T!} = 1.
\]

Note that the first equation in Eq. (6.10) simply follows from the identity Eq. (6.2).

Example 6.4. Let \( F(x, y) \) and all the related notation as in Example 5.8. Note that in this case \( H(x, y) = (ad_y(x), 0) \). From Eq. (6.6), it is easy to see inductively that, for any non-empty planar binary rooted tree \( T \), the second component of \( N_T(x, y) \) is also 0. So we may write \( N_T(x, y) = (u_T(x, y), 0) \) for some \( u_T(x, y) \in k\langle\langle x, y \rangle\rangle \). With this notation fixed, it is easy to check that, for any \( T \in B^p \) with \( T = B(T_1, T_2) \), Eqs. (6.6) and (6.7) become respectively
\[
\begin{align*}
(6.11) \quad u_T(x, y) &= \left[ u_{T_1}(x, y) \frac{\partial}{\partial x} \right] u_{T_2}(x, y), \\
(6.12) \quad u_m(x, y) &= \sum_{T \in B_{2m-1}^p} \frac{1}{T!} u_T(x, y) = \sum_{T \in B^p_{(T)=m}} \frac{1}{T!} u_T(x, y).
\end{align*}
\]

Claim: for any \( T \in B^p \) with \( T \neq \emptyset \), we have
\[
(6.13) \quad u_T(x, y) = ad_y^{l(T)}(x).
\]

Proof of Claim: We use the mathematical induction on the number \( l(T) \) of leaves of \( T \in B^p \).

First, when \( T = \circ \), by the definition of \( N_T(x, y) \), we know that \( u_T(x, y) \) is the first component of \( H(x, y) \), which is \( ad_y(x) \). Hence Eq. (6.13) holds in this case.

Now assume Eq. (6.13) holds for any \( T \in B^p \) with \( l(T) \leq m - 1 \) for some \( m \geq 2 \), and consider the case for \( T \in B^p \) with \( l(T) = m \). Write \( T = B(T_1, T_2) \) with \( T_i \in B^p \) (\( i = 1, 2 \)) and \( l(T_i) < m \). By Eq. (6.6) and the induction assumption, we have
\[
\begin{align*}
u_T(x, y) &= \left[ u_{T_1}(x, y) \frac{\partial}{\partial x} \right] u_{T_2}(x, y) \\
&= \left[ ad_y^{l(T_1)}(x) \frac{\partial}{\partial x} \right] ad_y^{l(T_2)}(x) \\
&= ad_y^{l(T_2)}(x) \left[ ad_y^{l(T_1)}(x) \right] \\
&= ad_y^{l(T_1)+l(T_2)}(x)
\end{align*}
\]
Now, from Eqs. (6.12) and (6.13), we see that, for any $m \geq 1$, the first component $u_m(x, y)$ of $N_m(x, y)$ is given by
\[
(6.14) \quad u_m(x, y) = \sum_{T \in B_{p2m-1}} \frac{1}{T!} ad_{l(T)}(x) = \left( \sum_{T \in B_{p2m-1}} \frac{1}{T!} \right) ad^m_y(x).
\]

But from Example 5.8, we also know that
\[
(6.15) \quad u_m(x, y) = ad^m_y(x),
\]
for any $m \geq 1$.

Comparing Eqs. (6.14) and (6.15), we get the identity Eq. (6.10).

Finally, to make our arguments more complete and also the tree expansion formula Eq. (6.7) more convincing, let us end this paper with the following direct proof of Proposition 6.3.

2nd Proof of Proposition 6.3: Let $s$ be a formal variable and $a(s)$ the following generating function
\[
(6.16) \quad a(s) = \sum_{T \in B^p} \frac{1}{T!} s^{l(T)-1} = \sum_{m \geq 1} \left( \sum_{T \in B^p} \frac{1}{T!} \right) s^{m-1}.
\]

Consider
\[
a^2(s) = \left( \sum_{T_1 \in B^p} \frac{1}{T_1!} s^{l(T_1)-1} \right) \left( \sum_{T_2 \in B^p} \frac{1}{T_2!} s^{l(T_2)-1} \right)
= \sum_{(T_1, T_2) \in B^p \times B^p} \frac{1}{T_1! T_2!} s^{l(T_1)+l(T_2)-2}
\]
Re-indexing the terms in the sum above by $T := B_+(T_1, T_2)$ and noting that any $T \in B^p$ with $|T| \geq 2$ can appear once and only once as $B_+(T_1, T_2)$ for some $T_i \in B^p$:}
\[
= \sum_{T \in B^p \atop T = B_+(T_1, T_2)} \frac{1}{T_1! T_2!} s^{l(T_1)+l(T_2)-2}
\]
Applying Eq. (6.4):}
\[
= \sum_{T \in B^p \atop |T| \geq 2} \frac{l(T) - 1}{T!} s^{l(T)-2}
\]
Noting that when $|T| = 1$, we have $T = e$ and $l(T) = 1$:

$$
\sum_{T \in B^p} \frac{l(T) - 1}{T!} s^{l(T) - 2}
$$

$$
= \frac{d}{ds} a(s).
$$

Therefore, we see that $a(s)$ satisfies the equations

$$
\begin{cases}
\frac{da(s)}{ds} = a^2(s), \\
a(0) = 1.
\end{cases}
$$

But it is easy to check that the only formal power series solution of the equations above is $(1 - s)^{-1}$. Therefore, we have $a(s) = (1 - s)^{-1}$. By comparing the coefficients of $s^{m-1}$ ($m \geq 1$) of $a(s)$ and $(1 - s)^{-1}$, we get Eq. (6.10). □

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