Timelike Geodesic Motion in Hořava-Lifshitz Spacetime

Juhua Chen\textsuperscript{1,2} and Yongjiu Wang\textsuperscript{1}

College of Physics and Information Science,
Hunan Normal University, Changsha, Hunan 410081, P. R. China

Department of Physics & Astronomy,
University of Missouri, Columbia, MO 65211, USA

Abstract

Recently Hořava proposed a non-relativistic renormalisable theory of gravitation. When restricted to satisfy the condition of detailed balance, this theory is intimately related to topologically massive gravity in three dimensions, and the geometry of the Cotton tensor. At long distances, this theory is expected to flow to the relativistic value $\lambda = 1$, and could therefore serve as a possible candidate for a UV completion of Einstein general relativity or an infrared modification thereof.

In this paper under allowing the lapse function to depend on the spatial coordinates $x^i$ as well as $t$, we obtain the spherically symmetric solutions. And then by analyzing the behavior of the effective potential for the particle, we investigate the timelike geodesic motion of particle in the Hořava-Lifshitz spacetime. We find that the nonradial particle falls from a finite distance to the center along the timelike geodesics when its energy is in an appropriate range. However, we find that it is complexity for radial particle along the timelike geodesics. There are follow different cases due to the energy of radial particle: 1) When the energy of radial particle is higher than a critical value $E_C$, the particle will fall from infinity to the singularity directly; 2) When the energy of radial particle equals to the critical value $E_C$, the particle orbit is unstable at $r = r_C$, i.e. the particle will escape from $r = r_C$ to the infinity or to the singularity, which is determined by the initial conditions of the particle; 3) When the energy of radial particle is in a proper range, the particle will rebound to the infinity or plunge to the singularity from a infinite distance, which is also determined by the initial conditions of the particle.

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\textsuperscript{*}Electronic address: jhchen@hunnu.edu.cn
I. INTRODUCTION

P. Hořava suggested a renormalizable four-dimensional theory of gravity, which admits the Lifshitz scale-invariance in time and space \([1, 2]\). The theory is with a broken Lorentz symmetry at short distances, while at large distances, higher derivative terms do not contribute and the theory runs to the standard GR, if a coupling, which controls the contribution of trace of the extrinsic curvature has a particular value \((\lambda = 1)\). In this specific limit, post-Newtonian corrections of the Hořava Gravity coincide with those of the standard GR.

Up to now there are several versions of Hořava gravity, which are classified according to whether or not the detailed balance and the projectability conditions are imposed. Since Hořava theory is modelled after a scalar field model studied by Lifshitz \([3]\), the theory is called Hořava-Lifshitz (HL) gravity. So far most of the work on the HL theory has abandoned the projectability condition but maintained detailed balance \([4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]\). One of the main reasons is that the resulting theory is much simpler to deal with, giving local rather than global energy constraints.

Specific solutions of this simplest version of Hořava-Lifshitz gravity have recently been analyzed \([22, 23, 24, 25, 26, 27, 28]\). Homogeneous vacuum solutions with gravitational waves were studied in Ref. \([29]\), and black hole solutions were analyzed in Ref. \([30, 31, 32, 33, 34]\). G. Calcagni et al \([35, 36, 37, 38, 39]\) gave cosmological solutions with a Lifshitz scalar matter which is the analogs of the Friedmann equations in Hořava-Lifshitz gravity include a term which scales as dark radiation and contributes a negative term to the energy density. Thus, it is possible in principle to obtain a nonsingular cosmological evolution with the Big Bang of Standard and Inflationary Cosmology replaced by a bounce \([40, 41]\). In last months some author focused on investigating the various properties and consequences of the Hořava-Lifshitz gravity. Konoplya considered the potentially observable properties of black holes in the Hořava-Lifshitz gravity with Minkowski vacuum: the gravitational lensing and quasinormal modes \([42, 43]\). Nishioka \([44]\) derived the detailed balance condition as a solution to the Hamilton-Jacobi equation in the Hořava-Lifshitz gravity, which proposes the existence of the d-dimensional quantum field theory with the effective action on the future boundary of the \((d + 1)\)-dimensional Hořava-Lifshitz gravity from the viewpoint of the holographic renormalization group. Myung \([45, 46, 47]\) studied the thermodynamics of black holes in the deformed Hořava-Lifshitz gravity. By using the canonical Hamiltonian
method, Rong-Gen Cai \cite{48, 49, 50, 51, 52} obtained the mass and entropy of the black holes with general dynamical coupling constant $\lambda$ in Hořava-Lifshitz Gravity.

Timelike geodesic of particles in a given space-time is very interesting topics in general relativity. By using the method of an effective potential, Jaklitsch et al \cite{53} investigated the time-like geodesic structure with a positive cosmological constant and Cruz et al \cite{54} studied the geodesic structure of the Schwarzschild Anti-de Sitter black hole. The analysis of the effective potential for null geodesics in Reissner-Nordström-de Sitter and Kerr-de Sitter space-time was carried out in Ref. \cite{55}. Podolsky \cite{56} investigated all possible geodesic motions in the extreme Schwarzschild-de Sitter space-time. Chen and Wang \cite{57, 58, 59, 60, 61, 62, 63} have investigated the orbital dynamics of the test particle in several gravitational fields with an electric dipole and a mass quadrupole, and in the extreme Reissner-Nordström black hole space-time. The exact solutions in the closed analytic form for the geodesic motion in the Kottler space-time were considered by Kraniotis \cite{64}. The exact solutions of the time-like geodesics were used to investigate the perihelion precession of the planet Mercury. By solving the Hamilton-Jacobi partial differential equation, Kraniotis and Whitehouse \cite{65} investigated the geodesic motion of the massive particle in the Kerr and Kerr-(anti)de Sitter gravitational field. In this paper we plan to extend Jaklitsch’s effective potential method to concentrate on the time-like geodesic motion in Hôrava-Lifshitz Spacetime.

II. TIMELIKE GEODESIC EQUATION AND EFFECTIVE POTENTIAL

Recently a new four-dimensional non relativistic renormalizable theory of gravity was proposed by Hořava \cite{2}. It is believed that this theory is a UV completion for the Einstein theory of gravitation. We use the ADM decomposition method \cite{30, 31, 32, 33, 34} to find spherically symmetric solutions. We start from the four-dimensional metric written in the ADM formalism, Let us consider the ADM decomposition of the metric in standard GR,

$$ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt),$$

where the lapse, shift and 3-metric $N, N^i$ and $g_{ij}$ are all functions of $t$ and $x^i$. The lapse function $N$ is viewed as a gauge field for time reparameterisations, and it is effectively restricted to depend only on $t$, but not the spatial coordinates $x^i$. A closer parallel with general relativity is achieved if this projectability restriction is relaxed. The action for the
fields of the theory is

\[
S = \int dt d^3x \sqrt{gN} \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu^2}{2w^2} \epsilon^{ijk} R^{(3)}_{lk} \nabla_j R^{(3)}_{ij} \right. \\
\left. - \frac{\kappa^2 \mu^2}{8} R^{(3)}_{ij} R^{(3)}_{ij} + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( 1 - \frac{4\lambda}{4} (R^{(3)})^2 + \Lambda_W R^{(3)} - 3\Lambda_W^2 \right) + \mu^4 R^{(3)} \right\}. 
\]

(2)

We should note that

\[
K_{ij} = \frac{1}{2N} (g_{ij} - \nabla_i N_j - \nabla_j N_i) ,
\]

is the second fundamental form, and

\[
C^{ij} = \epsilon^{ik\ell} \nabla_k \left( R^{(3)}_{j \ell} - \frac{1}{4} R^{(3)} \delta^j_\ell \right),
\]

is the Cotton tensor. \(\kappa, \lambda, w\) are dimensionless coupling constants, whereas \(\mu, \Lambda_W\) are dimensionful of mass dimensions \([\mu] = 1, [\Lambda_W] = 2\). The action (2) is the action in Ref.\cite{2} where we have added the last term, which represents a soft violation of the detailed balance condition.

We will now consider the limit of this theory, i.e. \(\Lambda_W \to 0\). In this particular limit, the theory turns out to be

\[
S = \int dt d^3x \sqrt{gN} \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu^2}{2w^2} \epsilon^{ijk} R^{(3)}_{lk} \nabla_j R^{(3)}_{ij} \right. \\
\left. - \frac{\kappa^2 \mu^2}{8} R^{(3)}_{ij} R^{(3)}_{ij} + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( 1 - \frac{4\lambda}{4} (R^{(3)})^2 + \mu^4 R^{(3)} \right) \right\}. 
\]

(5)

For the \(\lambda = 1(\omega = 16\mu^2/\kappa^2)\) case, we get the static, spherically symmetric metric for asymptotically flat space-time

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

(6)

where the lapse function is given by

\[
f(r) = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)},
\]

(7)

where \(M\) is an integration constant, with dimension \([M]=-1\). For \(r \gg (M/\omega)^{1/3}\), we can get the usual behavior of Schwarzschild black hole

\[
f(r) \approx 1 + \frac{2M}{r} + \mathcal{O}(r^{-4}),
\]

(8)
The lapse function vanished at the zeros of the equation

\[ 1 + \omega^2 r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)} = 0, \]  

there are two event horizons at

\[ r_{\pm} = M \left( 1 \pm \sqrt{1 - \frac{1}{2\omega M^2}} \right). \]  

Avoiding naked singularity at the origin implies \( \omega M^2 \geq \frac{1}{2} \). When \( \omega M^2 \gg 1 \) means the regime of the convetional General Relativity, the outer horizon approaches

\[ r_+ \approx 2M - \frac{1}{4\omega M^2} + \ldots \]  

The outer event horizon is lower than the usual Schwarzschild event horizon, \( r_{+Sch} = 2M \), and the inner one will approach the singularity \( r_- \to 0 \).

In order to investigate the time-like geodesics of particles in the space-time which is described by (6), we solve the Euler-Lagrange equation for the variation problem associated with this metric. The corresponding Lagrangian is

\[ L = -f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2), \]  

where the dots represent the derivative with respect to the affine parameter \( \tau \), along the geodesics. The equations of motion are

\[ \Pi_q - \frac{\partial L}{\partial \dot{q}} = 0, \]  

where \( \Pi_q = \frac{\partial L}{\partial \dot{q}} \) is the momentum conjugate to coordinate \( q \). For the Lagrangian is independent of \( (t, \varphi) \), the corresponding conjugate momentum is conserved, thus

\[ \Pi_t = -(1 + \omega^2 r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)})\dot{t} = -E, \]  

\[ \Pi_\varphi = r^2 \sin^2\theta \dot{\varphi} = L. \]  

Choosing the conditions \( \theta = \pi/2 \) and \( \dot{\theta} = 0 \), then the above equation will be simplified into

\[ \Pi_\varphi = r^2 \dot{\varphi} = L. \]
Substituting Eqs (14, 16) into Eq (12), we obtain

\[ 2L = h = \frac{E^2}{1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}} - \frac{\dot{r}^2}{1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}} - \frac{L^2}{r^2}. \]  

(17)

Now we solve the above equation for \( \dot{r}^2 \) in order to obtain the radial equation, which allows us to characterize possible moments of test particles without an explicit solution of the equation of motion in an invariant plane

\[ \dot{r}^2 = E^2 - \left(1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}\right) \left(h + \frac{L^2}{r^2}\right). \]  

(18)

We can rewrite above equation as a one-dimensional one

\[ \dot{r}^2 = E^2 - V_{eff}^2, \]  

(19)

### III. TIMELIKE GEODESICS

For the time-like geodesics, \( h = 1 \), and the effective potential turns into

\[ V_{eff}^2 = \left(1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}\right) \left(1 + \frac{L^2}{r^2}\right). \]  

(20)

From effective potential for particle along the time-like geodesic, we investigate the radial motion equation (18) for two cases: non-radial and radial particles in the following subsections, respectively.

#### A. Timelike geodesics of radial particles

The radial geodesic corresponds to the motion of the particle without angular momentum \( L = 0 \), which means the particles fall from, in a finite distance, to the center. So the effective potential \( V_{eff}^2 \) has the following form

\[ V_{eff}^2 = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}, \]  

(21)

and the corresponding radial motion equation takes the following form:

\[ \dot{r}^2 = E^2 - \left(1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}\right). \]  

(22)
FIG. 1: The effective potentials $V^2_{eff}$ for radial particles vs $r$ with the parameters $\omega = 1, L = 0$ and $M = 1$. The dashed line corresponds to the effective potential in the Schwarzschild spacetime with the same parameter values.

FIG. 2: The radial motion of radial particles in $(v$ vs $r$) phase space with the parameters $E = 0.5, \omega = 1$ and $M = 1$.

In order to investigate the properties of this effective potential, we simulate it in Fig.1. From this figure we can find that particles always plunge into the horizon from an upper distance which is determined by the motion energy $E$ of particles. At the same time we numerically perform the evolution of the radial particles in $(v$-$r$) phase space in Fig.2, we see that the velocity of particles, with a constant energy $E$, will tend to zero when the particles reach a finite distance, then radial particles will turn back and fall to the center.
FIG. 3: The effective potentials \( V_{\text{eff}}^2 \) for non-radial particles vs \( r \) at different values of \( L^2 \): \( 80M^2, 40M^2 \) and \( 10M^2 \) (from top to bottom) with \( \omega = 1 \) and \( h = M = 1 \). The dashed lines correspond to the effective potentials in the Schwarzschild spacetime with the same parameter values.

FIG. 4: The effective potentials \( V_{\text{eff}}^2 \) for non-radial particles vs \( r \) at different values of \( L^2 = 80M^2, \omega = 1 \) and \( h = M = 1 \). The dashed line corresponds to the effective potential in the Schwarzschild spacetime with the same parameter values.

B. Timelike geodesics non-radial particles

\[
\dot{r}^2 = E^2 - \left( 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)} \right) \left( 1 + \frac{L^2}{r^2} \right), \tag{23}
\]

where \( V_{\text{eff}}^2 \) is defined as an effective potential and expressed as

\[
V_{\text{eff}}^2 = \left( 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)} \right) \left( 1 + \frac{L^2}{r^2} \right). \tag{24}
\]
We have plotted the effective potential for non-radial particles with different values of $L^2 : 80M^2, 40M^2$ and $10M^2$ (from top to bottom) with $\omega = 1$ and $h = M = 1$ in Fig. 3 as an example. From the motion equation [23] and Fig. 4, we can divided the time-like geodesics for non-radial particles into several cases according to different values of $E$ for the non-radial particles as follows:

(I) when $E^2 > E_C^2$, the non-radial particle with enough energy will directly fall from infinity to the singularity.

(II) when $E^2 = E_C^2$, the orbit of the non-radial particle is unstable at $r = r_C$. The non-radial particle in this kind of orbit will escape from $r = r_C$ to the infinity, or to the singularity, which is determined by the initial conditions of the particle.

(III) when $E_1^2 < E^2 < E_C^2$, the non-radial particle in this kind of orbit will rebound from $r = r_D$ to the infinity, or to the singularity from $r = r_B$, which is also determined by the initial condition of the particle.

(IV) when $E^2 < E_1^2$, the non-radial particle will plunge into the singularity directly.

IV. CONCLUSIONS

In this paper we have investigated timelike geodesic motion for radial and non-radial particles in Hořava-Lifshitz spacetime. By analyzing the behavior of the effective potential for the particle, we investigate the timelike geodesic motion of particle in the Hořava-Lifshitz spacetime. We find that the nonradial particle falls from a finite distance to the center along the timelike geodesics when its energy is in an appropriate range. However, we find that it is complexity for radial particle along the timelike geodesics. There are follow different cases due to the energy of radial particle: When the energy of radial particle is higher than a critical value $E_C$, the particle will fall from infinity to the singularity directly; When the energy of radial particle equals to the critical value $E_C$, the particle orbit is unstable at $r = r_C$, i.e. the particle will escape from $r = r_C$ to the infinity or to the singularity, which is determined by the initial conditions of the particle; When the energy of radial particle is in a proper range, the particle will rebound to the infinity or plunge to the singularity from a infinite distance, which is also determined by the initial conditions of the particle. At the same time we have compared all cases of timelike geodesic motion for radial and non-radial particles in the Hořava-Lifshitz spacetime with those in the Schwarzschild spacetime. We
found that properties of Hořava-Lifshitz spacetime in large scale will run to the standard GR spacetime which is suggested by Hořava.

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