MINIMUM RAINBOW $H$-DECOMPOSITIONS OF GRAPHS

LALE ÖZKAHYA AND YURY PERSON

Abstract. Given graphs $G$ and $H$, we consider the problem of decomposing a properly edge-colored graph $G$ into few parts consisting of rainbow copies of $H$ and single edges. We establish a close relation to the previously studied problem of minimum $H$-decompositions, where an edge coloring does not matter and one is merely interested in decomposing graphs into copies of $H$ and single edges.

1. Introduction and new results

For two graphs $G$ and $H$ and a proper coloring $\chi$ of $G$, a rainbow $H$-decomposition of $G$ is a partition of the edges of $G$ into $G_1, \ldots, G_t$ such that every $G_i$ is either a single edge or is rainbow-colored and isomorphic to $H$, where a rainbow coloring of $H$ assigns distinct colors to all edges of $H$. Rainbow $H$-decompositions present a variation on a well-studied topic of $H$-decompositions. Before we state our main results, we present the history of this general problem.

1.1. Previous work on $H$-decompositions. For two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edges of $G$ into $G_1, \ldots, G_t$ such that every $G_i$ is either a single edge or is isomorphic to $H$. An $H$-decomposition of $G$ with smallest possible $t$ is called minimum and $\phi(G, H) = t$ denotes its cardinality. Let $N(G, H)$ denote the maximum number of edge-disjoint copies of $H$ in $G$, then the following relation $\phi(G, H) = e(G) - (e(H) - 1)N(G, H)$ clearly holds.

We write $\phi(n, H) := \max_{G \in G_n} \phi(G, H)$ for a general function, where $G_n$ denotes the family of all graphs on $n$ vertices. This function was studied first by Erdős, Goodman and Pósa [5], who were motivated by the problem of representing graphs by set intersections. They showed that $\phi(n, K_3) = \text{ex}(n, K_3)$, where $\text{ex}(n, H)$ denotes the maximum size of a graph on $n$ vertices, that does not contain $H$ as a subgraph. Moreover, these authors proved in [5] that the only graph that maximizes this function is the complete balanced bipartite graph. Consequently, they conjectured that $\phi(n, K_r) = \text{ex}(n, K_r)$ and the only optimal graph is the Turán graph $T_{r-1}(n)$, the complete balanced $(r-1)$-partite graph on $n$ vertices, where the sizes of the partite sets differ from each other by at most one. Bollobás [3] verified this conjecture by showing that $\phi(n, K_r) = \text{ex}(n, K_r)$ for all $n \geq r \geq 3$.

Pikhurko and Sousa [17] proved that for any fixed graph $H$ with chromatic number $r \geq 3$, $\phi(n, H) = \text{ex}(n, H) + o(n^2)$ and they made the following conjecture.

Conjecture 1. For any graph $H$ with chromatic number at least 3, there is an $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$ for all $n \geq n_0$.  

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1
This conjecture has been verified by Sousa for clique extensions of order $r \geq 4$ \((n \geq r) \ [22]\), the cycles of length 5 \((n \geq 6)\) and 7 \((n \geq 10)\) \[21, 23\]. In a previous work \[16\] we verified Conjecture 1 for edge-critical graphs, where a graph $H$ is edge-critical if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$. Later Allen, Böttcher and Person \[1\] obtained general upper bounds for all graphs $H$ that improve the error term in the result of Pikhurko and Sousa \[17\] and extend the result on the edge-critical case.

Recently, Liu and Sousa \[15\] studied the following colored variant of the $H$-decomposition problem. Let $\phi_k(G, H)$ be the smallest number $t$ such that any graph $G$ of order $n$ and any coloring of its edges with $k$ colors admits a monochromatic $H$-decomposition with at most $t$ parts, where $H$ is a complete graph $K_r$. Later, Liu, Pikhurko and Sousa \[14\] generalized this by investigating $\phi_k(G, \mathcal{H})$, which is the smallest number $t$ such that any graph $G$ of order $n$ and any coloring of its edges with $k$ colors admits a monochromatic $\mathcal{H} = \{H_1, \ldots, H_k\}$-decomposition such that each part is either a single edge or forms a monochromatic copy of $H_i$ in color $i$, for some $1 \leq i \leq k$. Extending the results of Liu and Sousa \[15\], they solve this problem when each graph in $\mathcal{H}$ is a clique and $n \geq n_0(\mathcal{H})$ is sufficiently large.

1.2. New results. We call a rainbow $H$-decomposition of $G$ under the coloring $\chi$ with the smallest possible $t$ minimum and $\phi^R(G, H) = t$ denotes its cardinality. It is not difficult to see that $\phi^R(G, H) = e(G) - (e(H) - 1)N^R(G, H)$, where $N^R(G, H)$ denotes the maximum number of edge-disjoint rainbow copies of $H$ under this proper coloring $\chi$ of $G$.

In this paper, we study the function

$$\phi^R(n, H) := \max_{G \in \mathcal{G}_n} \max_{\chi} \phi^R(G, H),$$

where we maximize over all graphs $G$ from $\mathcal{G}_n$ and over all proper colorings $\chi$ of $G$. We will refer to decompositions that attain $\phi^R(n, H)$ as minimum rainbow $H$-decompositions of graphs.

Observe that $\phi^R(G, H) \geq \phi(G, H)$ for any graph $G$ and any proper edge coloring $\chi$ of $G$, otherwise it means that there are more than $N(G, H)$ edge-disjoint rainbow copies of $H$ in $G$ under $\chi$, which is a contradiction. Therefore, we have $\phi^R(n, H) \geq \phi(n, H)$. On the other hand, $\max_{\chi} \phi^R(G, H) \leq e(G)$, where equality is achieved when there is no copy of $H$ in $G$, otherwise one can always color $G$ properly so that a particular copy of $H$ is rainbow.

We prove the following result for any clique $K_r$, $r \geq 3$.

**Theorem 2.** For any $r \geq 3$ there is an $n_0$ such that any graph $G$ on $n \geq n_0$ vertices with some proper edge coloring $\chi$ that satisfies $\phi^R(G, K_r) \geq \text{ex}(n, K_r)$ must in fact be isomorphic to the Turán graph $T_{r-1}(n)$.

In particular, $\phi^R(n, K_r) = \text{ex}(n, K_r)$ for all $n \geq n_0$, and the only graph attaining $\phi^R(n, K_r)$ is the Turán graph $T_{r-1}(n)$.

We also obtain generalizations of the result of Pikhurko and Sousa \[17\] on $\phi(n, H) = \text{ex}(n, H) + o(n^2)$ (for non-bipartite $H$) and on our result from \[16\] about $\phi(n, H) = \text{ex}(n, H)$ for edge-critical graphs $H$. Since we provide only very rough sketches of these generalizations, we postpone their discussion to the concluding remarks section, Section 5.

Our proofs will combine stability approach with probabilistic techniques. The paper is organized as follows. In the section below, Section 2, we collect the various
results that we are going to use. In Section 3, we prove a new stability result about the function $\phi(G, K_r)$ and building on that, we show various stability results about the function $\phi^R(G, K_r)$. Our proof of the rainbow stability for $\phi^R(G, K_r)$, Theorem 12, is a main contribution of this paper. In Section 4 we provide the (sketch of the) proof of Theorem 2. Finally, we explain in Section 5 how the exact version for edge-critical $H$ and an approximate version for nonbipartite $H$ of the function $\phi^R(n, H)$ follow.

1.3. Notation. Throughout the sections, we omit floor and ceiling notations, since they do not affect our calculations. We use standard notations from graph theory. Thus, for $t \in \mathbb{N}$ we denote by $[t]$ the set $\{1, \ldots, t\}$. For a given graph $G = (V, E)$ and for a subset $U \subseteq V$ we denote by $E_G(U) = E \cap \binom{U}{2}$ and $G[U] = G(U, E_G(U))$. We set $e_G(U) = |E_G(U)|$, and for a vertex $v \in V$ we write $\deg_{G,U}(v) = |\{u \in U : \{v, u\} \in E(G)\}|$, i.e., we are only counting the neighbors of $v$ in $U$. Similarly, for two disjoint subsets $U, W \subseteq V$ we set $e_G(U, W) = \{|u, w\} \in E(G) : u \in U, w \in W\}$, $G[U, W] = G(U \cup W, E_G(U, W))$ and $e_G(U, W) = |E_G(U, W)|$. We will sometimes omit $G$ when there is no danger of confusion, and we write $\deg_{U}(v), e(U), E(U,W), e(U,W)$.

2. Tools

2.1. Probabilistic tools. We will make use of the following version of Chernoff’s inequality, see e.g. [10, Corollary 2.4 and Theorem 2.8].

**Theorem 3** (Chernoff’s inequality). Let $X$ be the sum of independent binomial random variables, then for any $\delta \in (0, 3/2]$ we have

$$
P[|X - E(X)| \geq \delta E(X)] \leq 2 \exp(-\delta^2 E(X)/3).
$$

Moreover, we have

$$
P[X \geq x] \leq \exp(-x) \text{ for } x \geq 7E(X).
$$

Another concentration result that we are going to employ is a theorem due to Kim and Vu [12]. We state it in a slightly less general version (without weights on the edges) suited for our purposes.

**Theorem 4** (Kim-Vu polynomial concentration result [12]). Let $H = (V, E)$ be a (not necessarily uniform) hypergraph on $n$ vertices whose edges have cardinality at most $k \in \mathbb{N}$, let $(X_v)_{v \in V}$ be a family of mutually independent binomial random variables and set $X := \sum_{e \in E} \prod_{v \in e} X_v$. Then, for any $\lambda > 1$, we have

$$
P \left[|X - E(X)| > a_k(EE')^{1/2}\lambda^k \right] < d_k e^{-\lambda n^{k-1}},
$$

where $a_k = 8^k k^{1/2}$, $d_k = 2e^2$, $n = |V|$, $E = \max(E(X), E')$ and $E' = \max_{i=1}^k E_i$, and the quantity $E_i$ is defined as follows

$$
E_i := \max_{A \subseteq V, |A| = i} E \left( \sum_{e \in E : A \subseteq e} \prod_{v \in e \setminus A} X_v \right).
$$
2.2. Extremal graph theoretic results. Let $N(n, m, K_r)$ denote the minimum of $N(G, K_r)$ over all graphs $G$ with $n$ vertices and $m$ edges and recall that $N(G, K_r)$ denotes the maximum number of edge-disjoint copies of $K_r$ in $G$. Below we summarize the lower bounds on the function $N(n, m, K_r)$ shown by Győri and Tuza [8] and by Hoi [9] which we are going to use.

Theorem 5. The following bounds hold:

(i) $N(n, ex(n, K_3) + m, K_3) \geq \left(\frac{5}{9} + o(1)\right) m$ (Győri and Tuza [8]), and

(ii) $N(n, ex(n, K_r) + m, K_r) \geq \frac{m}{(\frac{r}{2})^{r-2}}$ (Hoi [9, Theorem 1.1]).

We remark, that for particular $r$ better bounds are known ($N(n, ex(n, K_4) + m, K_4) \geq \frac{2m}{\gamma} - o(n^2)$ from [9]) and in the case $m = o(n^2)$ as well, see Győri [7]. We will combine the theorem above with the classical stability result due to Erdős [4] and Simonovits [20].

Theorem 6 (Stability theorem). For every $H$ with $\chi(H) = r \geq 3$, and every $\gamma > 0$ there exist a $\delta > 0$ and an $n_0$ such that the following holds. If $G$ is a graph on $n \geq n_0$ vertices with $\varepsilon(G) \geq ex(n, H) - \delta n^2$ and if it does not contain $H$ as a subgraph, then there exists a partition of $V(G) = V_1 \cup \ldots \cup V_{r-1}$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

Another versatile tool that we are going to apply is the regularity lemma of Szemerédi [24]. Before stating it, we introduce first the central concepts. We say that a bipartite graph $G = (V_1 \cup V_2, E)$, or simply $(V_1, V_2)$, is $\varepsilon$-regular if all pairs of subsets $U_i \subseteq V_i$, with $|U_i| \geq \varepsilon|V_i|$, $i = 1, 2$, satisfy

$$|d_G(U_1, V_2) - d_G(U_1, U_2)| \leq \varepsilon,$$

where $d_G(U_1, U_2) := \frac{e_G(U_1, V_2)}{|U_1||V_2|}$ is the density of the bipartite graph induced by the color classes $U_1$ and $U_2$. An $\varepsilon$-regular pair $(V_1, V_2)$ is called $(\varepsilon, d)$-regular if it has density at least $d$.

Now consider a partition $\{V_1, \ldots, V_t\}$ of $V$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_t| \leq |V_1| + 1$. We call such partition equitable. We refer to an equitable partition as $\varepsilon$-regular if it satisfies the condition that all but $\varepsilon \left(\frac{1}{2}\right)$ pairs $(V_i, V_j)$ are $\varepsilon$-regular, where $i < j \in [t]$. The vertex subsets $V_i$ are referred to as clusters or classes.

The regularity lemma states then the following.

Theorem 7 (Regularity lemma). For every integer $t_0 \geq 1$ and every $\varepsilon > 0$ there exist integers $T_0 = T_0(t_0, \varepsilon)$ and $n_0 = n_0(t_0, \varepsilon)$ such that every graph $G = (V, E)$ on at least $n_0$ vertices admits an $\varepsilon$-regular partition $V = V_1 \cup \ldots \cup V_t$ with $t_0 \leq t \leq T_0$.

The regularity lemma is accompanied by a very useful fact called the counting lemma, see e.g. a survey of Komlós and Simonovits [13]. We state here one version that will be enough for our needs.

Lemma 8 (Counting lemma). For every $r \geq 3$, every $\delta > 0$ and $\gamma > 0$ there exist an $\varepsilon > 0$ and $m_0$ such that the following holds. Let $m \geq m_0$, let $V_1, \ldots, V_r$ be vertex-disjoint subsets of size $m$ or $m + 1$ of some graph $G$, such that each pair $(V_i, V_j)$ (for $i \neq j \in [r]$) is $\varepsilon$-regular and has density $d_{ij} \geq \delta$. Then, for all $i \neq j$, all but at most $4r\varepsilon(m + 1)^2$ edges from $(V_i, V_j)$ lie in

$$\left(1 \pm \gamma\right) \prod_{\{k, \ell\} \in \binom{[r]}{2}, \{k, \ell\} \neq \{i, j\}} d_{k\ell} \prod_{s \in [r], s \neq i, j} |V_s|$$

(1)
copies $K$ of $K_r$ such that $|K \cap V_i| = 1$ for all $i$.

We will also need the following theorem of Pippenger and Spencer [18], see also Rödl [19] and Alon and Spencer [Theorem 4.7.1][2].

**Theorem 9.** For every integer $s \geq 2$, $k \geq 1$ and real $c_0 > 0$, there are $c_1 = c_1(s, c_0, k) > 0$ and $d_0 = d_0(s, c_0, k)$ such that for any $n$ and $D \geq d_0$ the following holds.

Every $s$-uniform hypergraph $\mathcal{H}$ on a set $V$ of $n$ vertices satisfying all of the following conditions
1. for all $x \in V$ but at most $c_1n$ of them, $\deg(x) = (1 \pm c_1)D$;
2. for all $x \in V$, $\deg(x) \leq kD$;
3. for any two distinct $x, y \in V$, $\text{codeg}(x, y) < c_1D$;
contains a matching consisting of at least $(1 - c_0)n/s$ hyperedges.

### 3. Stability results for rainbow $K_r$-decompositions

#### 3.1. Warm-up: stability for $\phi(G, K_r)$.

In [16] we proved the following approximate result about graphs $G \in \mathcal{G}_n$ with $\phi(G, H) \geq \text{ex}(n, H) - o(n^2)$.

**Theorem 10.** (Lemma 4 from [16]). For every $H$ with $\chi(H) = r \geq 3$, $H \neq K_r$, and for every $\gamma > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every graph $G$ on $n \geq n_0$ vertices the following is true. If

$$\phi(G, H) \geq \text{ex}(n, H) - \varepsilon n^2$$

then there exists a partition of $V(G) = V_1 \cup \ldots \cup V_{r-1}$ with $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

Our proof was built on a theorem of Pikhurko and Sousa [17] about weighted decompositions of graphs. Due to a technical calculation, the natural case $H = K_r$ remained uncovered. The following proposition provides a short proof of the stability for cliques for the function $\phi(G, K_r)$. It will be used to obtain the stability for cliques for the rainbow function $\phi^r(G, K_r)$.

**Proposition 11.** For every $r \geq 3$ and for every $\gamma > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every graph $G$ on $n \geq n_0$ vertices the following is true. If

$$\phi(G, K_r) \geq \text{ex}(n, K_r) - \varepsilon n^2$$

then there exists a partition of $V(G) = V_1 \cup \ldots \cup V_{r-1}$ with $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

**Proof.** Let $\gamma > 0$ and $r \geq 3$ be given. Take $\delta$ and $n_0$ as guaranteed by Theorem 6 on input $\gamma$ and $K_r$, and choose with foresight $\varepsilon := \frac{\min(n/2, \delta)}{m + \delta}$. Let $G$ be a graph on $n \geq n_0$ vertices with $\phi(G, K_r) \geq \text{ex}(n, K_r) - \varepsilon n^2$. Assume that $e(G) = \text{ex}(n, K_r) + m$, where $m$ might be negative. It follows from the lower bound on $\phi(G, K_r)$ and from the identity $\phi(G, K_r) = e(G) - \binom{r}{2} - 1)\text{ex}(G, K_r)$, that

$$N(G, K_r) \leq \frac{m + \varepsilon n^2}{\binom{r}{2} - 1}. \tag{2}$$

On the other hand we have for $m > 0$, by Theorem 5, that

$$N(G, K_3) \geq N(n, \text{ex}(n, K_3) + m, K_3) \geq \frac{(5 + o(1))m}{9}, \quad \text{and}$$

$$N(G, K_r) \geq N(n, \text{ex}(n, K_r) + m, K_r) \geq \frac{m}{\binom{r}{2} - (r - 2)},$$
and thus it follows with (2) that \( m \leq 10r\varepsilon n^2 \) (if \( m < 0 \) then we can use (2) directly). Therefore we obtain the following bound on \( N(G, K_r) \):

\[
N(G, K_r) \leq \frac{10r\varepsilon n^2 + \varepsilon n^2}{\binom{r}{2} - 1} \leq 10r\varepsilon n^2.
\]

We delete \( \binom{r}{2} N(G, K_r) \) edges from \( G \) making it \( K_r \)-free. Denote this new graph by \( G' \). Clearly, \( e(G') \geq \text{ex}(n, K_r) - \varepsilon n^2 - 10r\varepsilon n^2 \). Now we can apply Theorem 6 to \( G' \) to obtain the desired partition. \( \square \)

3.2. Rainbow stability. The aim of this section is to prove auxiliary tools that will imply the following stability result for minimum rainbow \( K_r \)-decompositions.

**Theorem 12.** For every \( r \geq 3 \) and for every \( \gamma > 0 \) there exist \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that for every graph \( G \) on \( n \geq n_0 \) vertices with a proper coloring \( \chi \) of its edges the following is true. If

\[
\phi_R^G(G, K_r) \geq \text{ex}(n, K_r) - \varepsilon n^2
\]

then there exists a partition of \( V(G) = V_1 \cup \ldots \cup V_{r-1} \) with \( \sum_{i=1}^{r-1} e(V_i) < \gamma n^2 \).

Notice that the case \( r = 3 \) is covered by Proposition 11, as any proper edge coloring colors a triangle with different colors.

### 3.2.1. Proof overview.

As already mentioned, Theorem 12 is a main contribution of this paper. We fix a collection of edge-disjoint (not necessarily rainbow) copies of \( K_r \) of maximum cardinality. The proof proceeds by an application of the regularity lemma of Szemerédi [24]. Then we identify various \( \varepsilon \)-regular pairs of not too small density that build \( r \)-partite graphs, where most edge-disjoint copies of \( K_r \) ‘live’. We need to argue then that we can find roughly that many rainbow copies instead. To do so, we would like to apply a Theorem of Pippenger and Spencer [18] (see also Alon and Spencer [Theorem 4.7.1][2]) to decompose ‘most’ of the edges into edge-disjoint rainbow copies. Before that we appropriately split \( \varepsilon \)-regular pairs, similarly as is done for example in [17]. This time however, the density of these ‘split’ \( \varepsilon \)-regular pairs may be much lower than \( \varepsilon \), which would make the usual approach to apply the counting lemma (and then to apply a Theorem of Pippenger and Spencer [18]) impossible. Our solution is to count the rainbow copies of \( K_r \) (via Lemma 13) before splitting the \( \varepsilon \)-regular pairs (of sufficiently high density) and only then to split them randomly. All we need to estimate is then the number of rainbow copies that will be in (random) subgraphs of \( \varepsilon \)-regular \( r \)-partite graphs, which we do by applying a concentration result of Kim and Vu [12] (more precisely, Lemma 14 above, which follows by an application of the result from Kim and Vu [12]).

### 3.2.2. Some auxiliary results.

The following lemma asserts that, in a proper edge coloring, for most of the edges from an “\( \varepsilon \)-regular environment” most of the cliques \( K_r \) that “sit” on them are in fact rainbow.

**Lemma 13.** For every \( r \geq 3 \), every \( \delta > 0 \) and \( \gamma > 0 \) there exist an \( \varepsilon > 0 \) and \( m_0 \) such that the following holds. Let \( m \geq m_0 \), let \( V_1, \ldots, V_r \) be vertex-disjoint subsets of size \( m \) or \( m + 1 \) of some graph \( G \), such that each pair \( (V_i, V_j) \) (for \( i \neq j \in [r] \)) is \( \varepsilon \)-regular and has density \( d_{ij} \geq \delta \). Further let \( \chi \) be any proper edge coloring of \( G \).
Thus, it follows that the total number of non-rainbow copies of $K_r$ between the sets $V_1, \ldots, V_r$. 

Proof. Let $\varepsilon > 0$ and $m_0$ be asserted by the counting lemma, Lemma 8, for supplied parameters $r$, $\delta$ and $\gamma/2$ (instead of $\gamma$). We clearly may assume that $r \geq 4$.

Let $G$ be a graph and let $V_1, \ldots, V_r$ be its vertex-disjoint subsets of size $m$ or $m + 1$. Then, clearly, it holds that all but at most $4r\varepsilon(m + 1)^2$ edges from $(V_i, V_j)$ lie in a copy of $K_r$. Assume that $K$ is not rainbow and let the vertex set of $K$ be $\{v_1, \ldots, v_r\}$. Then there exist four distinct indices $s_1, \ldots, s_4 \in [r]$ with $\chi(v_{s_1}, v_{s_2}) = \chi(v_{s_3}, v_{s_4})$. In the following we estimate the number of such non-rainbow copies of $K_r$, where we will make use of the fact that in a proper edge coloring every color class forms a matching and there are thus at most $r(m + 1)/2$ edges of the same color in $G$. We distinguish three cases:

1. one of the edges $v_{s_1}v_{s_2}, v_{s_3}v_{s_4}$ (say $v_{s_1}v_{s_2}$) equals to $v_iv_j$; there are at most $r(m + 1)/2$ choices for the edge $v_{s_3}v_{s_4}$ and, therefore, at most $\frac{r(m + 1)^2}{2} (m + 1)r^{-4} = o(m^{r-2})$ such non-rainbow copies $K$;

2. at least one of the edges $v_{s_1}v_{s_2}, v_{s_3}v_{s_4}$ (say $v_{s_1}v_{s_2}$) is incident to $v_i$ or $v_j$; then as $v_i$ and $v_j$ each has at most $(r - 2)(m + 1)$ neighbours among $V_\ell$ ($\ell \neq i, j$) this number is an upper bound on the number of possible colors with $\chi(v_{s_1}, v_{s_2}) = \chi(v_{s_3}, v_{s_4})$. Therefore, there are at most $(r - 2)(m + 1)^{r-4} = o(m^{r-2})$ non-rainbow copies $K$ where both edges $v_{s_1}v_{s_2}, v_{s_3}v_{s_4}$ are incident to $v_i, v_j$ respectively. Moreover, the number of non-rainbow copies $K$ with $\{v_{s_3}, v_{s_4}\} \cap \{v_i, v_j\} = \emptyset$ is at most $(r - 2)(m + 1)^{r-5} = o(m^{r-2})$. Summing up, this gives at most $o(m^{r-2})$ non-rainbow copies in this case.

3. both edges $v_{s_1}v_{s_2}, v_{s_3}v_{s_4}$ are disjoint from $\{v_i, v_j\}$. We write $m_2$ for the number of edges from $G[V_1 \ldots V_r]$ in color $i$ and observe that $m_1 \leq \frac{r(m + 1)}{2}$.

The number of non-rainbow copies $K$ is in this case at most

$$\sum_i m_1^2 m_2^6 \leq \frac{r(m + 1)}{2} m_2^6 \sum_i m_1 \leq \frac{r(m + 1)}{2} m_2^6 \left( \frac{m(r + 1)}{2} \right) = o(m^{r-2}).$$

Thus, it follows that the total number of non-rainbow copies of $K_r$ that contain $v_iv_j$ is $o(m^{r-2})$. Since the number of copies of $K_r$ is given by (4) and each $|V_i| \in \{m, m + 1\}$ we immediately infer (3) for $m_0$ sufficiently large. □
Later in our proof we will “sparsify” randomly our \( \varepsilon \)-regular pairs such that the densities of these pairs are much below \( \varepsilon \). As a consequence, we will not be able to count within these sparse pairs directly (as the density might be much less than the regularity parameter \( \varepsilon \)). Instead we count before sparsification and the following lemma asserts that as many rainbow copies remain as we would expect. Given an \( r \)-partite graph \( G \) with the classes \( V_1, \ldots, V_r \), and \( p_{ij} \in [0,1] \) for all \( \{i,j\} \in \binom{[r]}{2} \), we denote by \( G_{(p_{ij})} \) the random subgraph of \( G \) where each edge from \( (V_i, V_j) \) is included with probability \( p_{ij} \) independently of the other edges.

Let \( v_s v_t \) be an edge from \( G[V_s, V_t] \) and let \( K \) be a copy of \( K_r \) that contains \( v_s v_t \). We then say that the edge \( v_s v_t \) closes a copy of \( K \) in \( G_{(p_{ij})} \) if all edges of \( E(K) \setminus \{v_s v_t\} \) lie in \( G_{(p_{ij})} \).

**Lemma 14.** For \( r \geq 3 \), \( \gamma > 0 \) and \( \eta > 0 \) there exists a \( \beta > 0 \) and \( m_0 \) such that the following holds. Let \( G \) be an \( r \)-partite graph with classes \( V_1, \ldots, V_r \), each of size \( m \) or \( m+1 \), where \( m \geq m_0 \). Let \( v_1 v_2 \) be an edge from \( (V_1, V_2) \) and let \( \mathcal{K} \) be a family of some \( \delta m^{r-2} \) copies of \( K_r \) in \( G \) that contain \( v_1 v_2 \). Then for any sequence of \( p_{ij} > 0 \) (\( i \neq j \in [r] \)), such that

\[
\delta m^{r-2} \prod_{\{k, \ell\} \in \binom{[r]}{2}, \{k, \ell\} \neq \{1,2\}} p_{k\ell} \geq (m + 1)^{r-3+\gamma},
\]

it holds that the probability that the edge \( v_1 v_2 \) does not close in \( G_{(p_{ij})} \)

\[
(1 \pm \eta) \left( \prod_{\{k, \ell\} \in \binom{[r]}{2}, \{k, \ell\} \neq \{1,2\}} p_{k\ell} \right) \delta m^{r-2}
\]

many copies of \( K_r \) from \( \mathcal{K} \) is at most \( \exp(-m^\beta) \).

**Proof.** For every edge \( e \in E(G) \) let \( X_e \) be the indicator random variable whether the edge \( e \) is in \( G_{(p_{ij})} \). By the definition, we have \( \mathbb{P}(e \in G_{(p_{ij})}) = p_{ij} \) for \( e \in E(G(V_i, V_j)) \). Further we define the random variable \( X \) that counts the number of copies of \( K_r \) from \( \mathcal{K} \), that are closed by \( v_1 v_2 \), as follows:

\[
X := \sum_{K \in \mathcal{K}} \prod_{v_1 v_2 \in K, e \in E(K)} X_e.
\]

We clearly have

\[
\mathbb{E}(X) = |\mathcal{K}| \prod_{\{k, \ell\} \in \binom{[r]}{2}, \{k, \ell\} \neq \{1,2\}} p_{k\ell} \geq (m + 1)^{r-3+\gamma}, \tag{5}
\]

where the second inequality is an assumption of the lemma. We set \( k := \binom{r}{2} - 1 \) and we simply estimate the quantity

\[
E' := \max_{i \in [r]} \max_{|A| = i} \mathbb{E} \left( \sum_{K \in \mathcal{K}} \prod_{v_1 v_2 \in K, e \in E(K) \setminus (A \cup \{v_1 v_2\})} X_e \right)
\]
by \((m+1)^{r-3}\), since by choosing some edge \(f \neq v_1v_2\) there remain at most \((m+1)^{r-3}\) possible copies of \(K_r\) in \(G\) that contain \(v_1v_2\) and \(f\) as edges.

We may now apply Kim-Vu polynomial concentration (Theorem 4) with \(E = \max(\mathbb{E}(X), E') = \mathbb{E}(X)\) and \(E' = (m+1)^{r-3}\) as follows (with \(k = \binom{r}{2} - 1\), \(a_k = 8^k k!^{1/2}\), \(d_k = 2e^2\)):

\[
P\left[|X - \mathbb{E}(X)| \geq a_k(\mathbb{E}(E')^{1/2})^{k-1}\right] < d_k e^{-\lambda} \left(\frac{r}{2}\right)^{(m+1)^2}^{k-1}.
\]

By choosing \(\lambda = \left(\frac{\eta \mathbb{E}(X)}{a_k(\mathbb{E}(E')^{1/2})}\right)^{1/k}\) we have \(P[|X - \mathbb{E}(X)|] < d_k e^{-\lambda}(\binom{r}{2}(m+1))^{k-1}\). Using (5) we obtain the following lower bound on \(\lambda\), which will be sufficient for our purposes:

\[
\lambda \geq \left(\frac{\eta(m+1)^{(r-3+\gamma)/2}}{8^k k!^{1/2}(m+1)^{(r-3)/2}}\right)^{1/k} \geq m^{\gamma/((3k)}.
\]

Thus, we may estimate the probability \(P[|X - \mathbb{E}(X)|] < \eta \mathbb{E}(X)] < e^{-m^{\gamma/(3k)}}\), setting \(\beta = \gamma/(4k)\) and choosing \(m_0\) sufficiently large.

3.2.3. Main lemma. The following lemma shows that, for any graph \(G\) on \(n\) vertices and any proper edge coloring \(\chi\), the numbers (of edge-disjoint copies) \(N(G, K_r)\) and \(N^R(\chi, G, K_r)\) differ by at most \(o(n^2)\).

Lemma 15. For all \(r \geq 3\) and any \(c > 0\) there exists an \(n_0\) such that the following holds. In any proper edge coloring \(\chi\) of a graph \(G\) on \(\geq n_0\) vertices we have \(N(G, K_r) \leq N^R(\chi, G, K_r) + cn^2\).

Proof. Let \(G\) be given, let \(\chi\) be some fixed proper edge coloring of \(G\) and let \(K\) be a family of \(N(G, K_r)\) edge-disjoint copies (not necessarily rainbow) of \(K_r\) in \(G\).

We set \(\delta = c/3\) and \(\xi = \delta/4\). Then we choose \(c_0 = \xi, s = \binom{r}{2}\), and \(a_k = a(\xi, s, 0, k) > 0\) and \(d_0 = d_0(k, s, 0, k)\) be as asserted by Theorem 9 on input \(k, s\) and \(c_0\). We set \(\gamma = c_1/2\) and let \(\varepsilon'\) be as asserted by Lemma 13 on input \(r, \delta\) and \(\gamma\). Finally, we choose \(\varepsilon := \min\{\varepsilon', \delta/4, \delta c_1/(63r)\}\) and \(\varepsilon_0 = 2/\varepsilon\). Finally, let \(T\) be as asserted by the regularity lemma (Theorem 7) on input \(\varepsilon\) and \(\varepsilon_0\). We will also assume throughout the proof that \(n\) is sufficiently large, so that all asymptotic estimates hold.

An application of the regularity lemma. We apply regularity lemma to \(G\) with (carefully chosen) parameters \(\varepsilon > 0\) and \(t_0\) (lower bound on the number of clusters). We obtain an \(\varepsilon\)-regular partition of \(V(G)\) into \(V_1, \ldots, V_t\) where \(t \leq T = T(\varepsilon, t_0)\). We define a cluster graph \(R\) with the vertex set \(\{V_i: i \in [t]\}\), where \(ij \in E(R)\) whenever the density \(d(V_i, V_j) \geq \delta\). For convenience, we let \(d_{ij}\) denote \(d(V_i, V_j)\) for the remainder of the proof. First observe that the number of copies of \(K_r\) from \(\mathcal{K}\) with at least one edge not from the pairs \((V_i, V_j)\) with \(ij \in E(R)\) is at most

\[
\sum_{i=1}^{t} \binom{|V_i|}{2} + \sum_{ij \notin E(R), (V_i, V_j) \text{ not } \varepsilon\text{-regular}} |V_i||V_j| + \sum_{ij \notin E(R), d_{ij} < \delta} e(V_i, V_j) \leq n^2/t_0 + \varepsilon n^2 + \delta n^2/2 < \delta n^2.
\]
Thus, all but at most $\delta n^2$ copies from $K$ are completely contained within $\varepsilon$-regular pairs of density at least $\delta$, and we denote such set of copies by $K' \subseteq K$ and identify these copies with their vertex sets.

**Calculating $\alpha_S$.** Observe now that a copy of $K_r$ from $K'$ must lie between some $r$ clusters that form a copy of $K_r$ in $R$. We write $\text{supp}(T)$ for $\{i: V_i \cap T \neq \emptyset\}$, the *support* of $T$. For $S \in \binom{V(R)}{r}$, we denote by $|K'(S)|$ those copies $F$ of $K_r$ from $K'$ with $S = \text{supp}(V(K))$. Thus, $|K'| = \sum_{S \in \binom{V(R)}{r}} |K'(S)|$. Of course, if the graph $R[S]$ is not complete for some $S$ then $|K'(S)| = 0$.

For an $r$-element set $S \subseteq V(R)$ with at least one copy $F \in K'$ with $\text{supp}(F) = S$ we define the weight $\alpha_S$ as follows (for other $r$-element sets $S$ we set $\alpha_S = 0$):

$$
\alpha_S := \min_{(i,j) \in \binom{S}{2}} \frac{d_{ij}|K'(S)|}{\sum_{F \in K': (i,j) \in \text{supp}(V(F))}|F: F \in K'|}.
$$

(6)

Observe that for every $ij \in E(R)$ and $S \in \binom{V(R)}{r}$ with $S \supseteq \{i,j\}$ we have $\alpha_S \leq d_{ij}$. Consider now an arbitrary $ij \in E(R)$ with $|\{F: F \in K', (i,j) \in \text{supp}(V(F))\}| \neq 0$. From the equation

$$
|\{F: F \in K', (i,j) \in \text{supp}(V(F))\}| = \sum_{S \in \binom{V(R)}{r}: S \supseteq \{i,j\}} |K'(S)|
$$

we infer

$$
\sum_{S: S \supseteq \{i,j\}} \alpha_S \leq \sum_{S: S \supseteq \{i,j\}} \frac{d_{ij}|K'(S)|}{\sum_{F \in K': (i,j) \in \text{supp}(V(F))}|F: F \in K'|} = d_{ij}.
$$

Furthermore, every copy from $K'$ whose support contains $i$ and $j$ uses exactly one distinct edge from $E(V_i,V_j)$ and thus $|\{F: F \in K', (i,j) \in \text{supp}(V(F))\}| \leq d_{ij}|V_i||V_j|$. We let $i$ and $j$ be the elements of $S$ that yield the value of $\alpha_S$ in (6) and conclude with (7):

$$
|K'(S)| = \frac{\alpha_S}{d_{ij}} |\{F: F \in K', (i,j) \in \text{supp}(V(F))\}| \leq \alpha_S|V_i||V_j|.
$$

(8)

Roughly speaking, the values $\alpha_S$ will tell us below where we have to look for many edge-disjoint rainbow copies of $K_r$.

**Partitioning the edges of $\varepsilon$-regular pairs.** Now we partition randomly every pair $G[V_i,V_j]$ into bipartite subgraphs. We concentrate only on copies of $K'$ from graphs $G[\cup_{i \in S} V_i]$ with $\alpha_S \geq \delta /T^r$. Since there are $t \choose \delta t$ many $\alpha_S$ we will neglect (using (8)) less than

$$
\left(\frac{t}{\delta t}\right)^{\lfloor n/\delta t \rfloor^2} \leq \delta n^2
$$

(9)

edge-disjoint copies from $K'$. For each $ij \in E(R)$, we split the edges within each $\varepsilon$-regular pair $(V_i,V_j)$ randomly with probabilities $\alpha_S/d_{ij}$, where $i,j \in S \in \binom{V(R)}{r}$, as follows.

For every edge $e \in E(V_i,V_j)$ we consider the random variable $Y_{ij,e}$, which takes values in $\{S: S \in \binom{V(R)}{r}, i,j \in S\}$ with probability $P(Y_{ij,e} = S) = \alpha_S/d_{ij}$ and possibly some arbitrary other value with probability $1 - \sum_{S \in \binom{V(R)}{r}, i,j \in S} \alpha_S/d_{ij}$.

Thus, for a fixed $S \in \binom{V(R)}{r}$, the 2-set $\{i,j\} \subseteq S$ and an edge $e \in E(V_i,V_j)$, the indicator random variable $Z_{S,ij,e} := 1(Y_{ij,e} = S)$ is a Bernoulli variable with parameter $\alpha_S/d_{ij}$. In particular, for fixed $S$, the random variables $Z_{S,ij,e}$ are (mutually) independent.
In this way we obtain, for every $S$, a random $r$-partite subgraph $G_S \subseteq G[\bigcup_{i \in S} V_i]$ where $\mathbb{P}(e \in G_S) = \alpha_S/d_{ij}$ for every $e \in E(V_i, V_j)$ for some $i, j \in S$. For given $S \in \binom{V(G)}{r}$ with $\alpha_S \geq \delta/T^r$ and a 2-set $\{i, j\} \subseteq S$, we let $X_{ij}$ be the random variable which counts the number of edges from $E(V_i, V_j)$ chosen to be in $G_S$. Observe that the $Y_{ij}$’s make sure that the graphs $G_S$ are edge-disjoint for different sets $S$. Since $X_{ij}$ is the sum of $|E(V_i, V_j)| = d_{ij}|V_i||V_j|$ independent indicator random variables, which are distributed $\text{Be}(\alpha_S/d_{ij})$ we have $\mathbb{E}(X_{ij}) = \alpha_S|V_i||V_j| \geq (\delta/T^r)n/t^2$, and, by Chernoff’s inequality (Theorem 3):

$$\mathbb{P}(|X_{ij} - \mathbb{E}(X_{ij})| \geq \xi \mathbb{E}(X_{ij})) \leq 2 \exp\left(-\xi^2\delta n^2/(4t^2T^r)\right).$$

(10)

Thus, with probability at least $1 - 2\left(\frac{\xi^2}{8}\right)\left(\frac{T}{t}\right) \exp\left(-\xi^2\delta n^2/(4t^2T^r)\right) = 1 - o(1)$, for every $S$ with $\alpha_S \geq \delta/T^r$ and for all $i \neq j \in S$, we have that $|E_{G_S}(V_i, V_j)| = (1 \pm \xi)\alpha_S|V_i||V_j|$ and the density of every pair $(V_i, V_j)$ in $G_S$ is thus $(1 \pm \xi)\alpha_S$.

**Putting everything together.** To conclude the lemma we need to prove the following claim, whose proof we postpone first.

**Claim 16.** With probability $1 - o(1)$, we have that for every $S \in \binom{V(G)}{r}$, where $\alpha_S \geq \delta/T^r$, the graph $G_S$ contains at least $(1 - 3\xi)\alpha_S(n/t)^2$ edge-disjoint rainbow copies of $K_r$.

Since for every $S \in \binom{V(G)}{r}$ the graphs $G_S$ are edge-disjoint, we find at least

$$\sum_{S: \alpha_S \geq \delta/T^r} (1 - 3\xi)\alpha_S(n/t)^2 \geq (1 - 4\xi) \sum_{S: \alpha_S \geq \delta/T^r} |\mathcal{K}'(S)| \geq (1 - 4\xi)(|\mathcal{K}| - \delta n^2)$$

edge-disjoint rainbow copies of $K_r$. The last inequality follows since at most $\delta n^2$ edges lie in $G_S$ with small $\alpha_S < \delta/T^r$, cf. (9).

Thus, we have $N^R_{\chi}(G, K_r) \geq (1 - 4\xi)(|\mathcal{K}| - \delta n^2) - \delta n^2 \geq |\mathcal{K}| - 3\delta n^2$. It follows that

$$N(G, K_r) \leq N^R_{\chi}(G, K_r) + 3\delta n^2 \leq N^R_{\chi}(G, K_r) + cn^2.$$  

□

**Proof of Claim 16.** We fix some $S \in \binom{V(G)}{r}$ with $\alpha_S \geq \delta/T^r$. We define the auxiliary $\binom{r}{2}$-uniform hypergraph $\mathcal{H} := \mathcal{H}_S$ with the vertex set $V(\mathcal{H}_S) = E(G_S)$ as follows. Its hyperedges correspond to the edge-sets of all rainbow copies of $K_r$ in $G_S$. The number of edges of $G_S$ and thus the vertices of $\mathcal{H}$ is $(1 \pm 2\xi)\binom{r}{2}\alpha_S(n/t)^2$ with probability $1 - 2r \exp\left(-\xi^2\delta n^2/(4t^2T^r)\right) = 1 - o(1)$, see the application of Chernoff’s inequality (10).

Next we estimate the number of hyperedges of $\mathcal{H}$ and related quantities. By Lemma 13, for $i \neq j \in S$, all but at most $4r\varepsilon[n/t]^2$ edges from $(V_i, V_j)$ lie in

$$(1 \pm \gamma) \left( \prod_{\{k, \ell\} \in S \setminus \{i, j\}} d_{k\ell} \right) \prod_{a \in S, a \neq i, j} |V_a|$$

rainbow copies of $K_r$ between the sets $V_i$, $\ell \in S$. We refer to these edges as good. The remaining at most $4r\varepsilon[n/t]^2$ bad edges from $(V_i, V_j)$ lie in at most $[n/t]r^{-2}$ rainbow copies of $K_r$. 
We may now apply Lemma 14 (with \( p_{ij} = \alpha_S/d_{ij} \) for all \( i \neq j \in S \), \( \gamma_{L14} = 1/2 \) and \( \eta = \gamma \) obtaining the parameter \( \beta' \)) to these (rainbow) copies to conclude that, with probability \( 1 - n^2 e^{-[n/t]^3} \), each good edge closes in \( G_S \)

\[
(1 \pm \gamma) \prod_{\{k, \ell\} \in S \atop \{k, \ell\} \neq \{i, j\}} \frac{\alpha_S}{d_{k \ell}} (1 \pm \gamma) \left( \prod_{\{k, \ell\} \in S \atop \{k, \ell\} \neq \{i, j\}} d_{k \ell} \right) \prod_{a \in S, a \neq i, j} \left| V_a \right| = (1 \pm 2\gamma)\alpha_S^{(2)}(n/t)^{-2}
\]

rainbow copies (which are thus hyperedges in \( \mathcal{H} \)).

Recall that every bad edge from \( E(V_i, V_j) \) is included in \( G_S \) with probability \( \alpha_S/d_{ij} \) independently. We denote by \( B_{ij} \) the number of bad edges from \( E(V_i, V_j) \) that are included in \( G_S \). An application of Chernoff’s inequality guarantees that

\[
P[B_{ij} \geq 7(\alpha_S/d_{ij})4\epsilon\epsilon [n/t]^2] \leq \exp\left(-\left(\frac{\alpha_S}{d_{ij}}\right)4\epsilon\epsilon [n/t]^2\right),
\]

where we used \( E(B_{ij}) \leq (\alpha_S/d_{ij})4\epsilon\epsilon [n/t]^2 \). This implies that the above estimate holds for each \( \alpha_S \geq \delta/T^r \) and every \( \{i, j\} \in \binom{S}{2} \) with probability at most \( \binom{S}{2} (\delta/T^r)^{\exp\left(-\left(\frac{\alpha_S}{\delta}\right)4\epsilon\epsilon [n/t]^2\right)} \).

Further we need to bound the number of rainbow copies that a bad edge from \( G_S \) closes. Now we apply Lemma 14 (with \( p_{ij} = \alpha_S/d_{ij} \) for all \( i \neq j \in S \), \( \gamma_{L14} = 1/2 \) and \( \eta = 1/2 \) obtaining the parameter \( \beta' \)) to the at most \([n/t]^r \) rainbow copies (we can extend these to exactly this number \([n/t]^r \) by adding some arbitrary copies of \( K_r \)). We conclude that, with probability \( 1 - n^2 e^{-[n/t]^3} \), each bad edge closes in \( G_S \) at most

\[
\frac{3}{2} \left( \prod_{\{k, \ell\} \in S \atop \{k, \ell\} \neq \{i, j\}} \frac{\alpha_S}{d_{k \ell}} \right) [n/t]^r \leq 2(\alpha_S/\delta)^{\binom{S}{2}}(n/t)^{r-2}
\]

rainbow copies (which are thus hyperedges in \( \mathcal{H} \)).

To summarize: with probability at least

\[
1 - r^2 T^r \exp\left(-\xi^2 \delta n^2/(4t^2 T^r)\right) - r^2 T^r \exp\left(-\left(\frac{\alpha_S}{\delta}\right)4\epsilon\epsilon [n/t]^2\right)
\]

\[
- \frac{1}{2} n^2 T^r e^{-[n/t]^3} - n^2 T^r e^{-[n/t]^3} = 1 - o(1),
\]

we have the following for every \( S \) with \( \alpha_S \geq \delta/T^r \):

- the number of edges of \( G_S \) is \((1 \pm 2\xi)\binom{\binom{S}{2}}{2} \alpha_S (n/t)^2\), and
- each good edge (for some pair \((V_i, V_j)\) with \( d_{ij} \geq \delta \)) closes in \( G_S \)

\[
(1 \pm 2\gamma)\alpha_S^{(2)}(n/t)^{-2}
\]

rainbow copies of \( K_r \), and

- the number of bad edges in \( G_S \) is at most

\[
7(\binom{S}{2}) (\alpha_S/d_{ij})4\epsilon\epsilon [n/t]^2 \leq 7(\binom{S}{2}) (\alpha_S/\delta)4\epsilon\epsilon [n/t]^2,
\]

and

- each bad edge closes in \( G_S \) at most \( 2(\alpha_S/\delta)^{\binom{S}{2}}(n/t)^{r-2} \) rainbow copies of \( K_r \).
This means that all but at most \((\binom{n}{2})(\alpha_S/\delta)4r\varepsilon[n/t]^2 \leq c_1|V(H)|\) vertices \(e \in E(G_S) = V(H)\) have degree in the interval
\[
(1 \pm 2\gamma)\alpha_S^{(\binom{n}{2})^{-1}}(n/t)^{r-2}.
\]
Thus, if we set \(D := \alpha_S^{(\binom{n}{2})^{-1}}(n/t)^{r-2}\) then it is larger than \(d_0(\binom{n}{2}, c_0, k)\) since \(r \geq 3\). On the other hand, the degree of every vertex of \(H\) is at most \(2\delta^{-\binom{n}{2}+1}D\). Furthermore, any two edges in \(E(G_S)\) lie in at most \([n/t]^{r-3} = o(D)\) hyperedges. Thus, the assumptions of Theorem 9 are verified and hence \(H\) has a matching with at least
\[
(1 - c_0)\varepsilon(n/G \delta(r/2) \geq (1 - c_0)(1 - 2\xi)\alpha_S(n/t)^2 \geq (1 - 3\xi)\alpha_S(n/t)^2
\]
hyperedges.
We conclude that, with probability \(1 - o(1)\), every graph \(G_S\) (with \(\alpha_S \geq \delta/T^r\)) contains at least \((1 - 3\xi)\alpha_S(n/t)^2\) edge-disjoint rainbow copies of \(K_r\).

3.3. Proof of Theorem 12.

Proof of Theorem 12. For given \(r \geq 3\) and \(\gamma > 0\) let \(\varepsilon' := \frac{\varepsilon}{T^{11}}\) be as asserted by Proposition 11. We choose \(c := \varepsilon'/\varepsilon(K_r)\) and assume that \(n\) is large enough so that Lemma 15 is applicable. Finally we set \(\varepsilon := \varepsilon'/\varepsilon(K_r)\).

Let now \(G\) be a graph on \(n\) vertices with a proper coloring \(\chi\) of its edges such that \(\phi^R(\chi, G, K_r) \geq \varepsilon n/k^2\) holds. By Lemma 15, we have \(N(G, K_r) \leq N^R(\chi, G, K_r) + cn^2\) (for \(n\) large enough). Therefore, \(\phi^R(\chi, G, K_r) = \varepsilon(G) - (\varepsilon(K_r) - 1)N^R(\chi, G, K_r) \geq \varepsilon n/k^2\) implies
\[
\phi(G, K_r) = \varepsilon(G) - (\varepsilon(K_r) - 1)N(\chi, G, K_r) \geq \varepsilon(G) - (\varepsilon(K_r) - 1)(N^R(\chi, G, K_r) + cn^2) \geq \phi^R(\chi, G, K_r) - (\varepsilon(K_r) - 1)cn^2 \geq \varepsilon n/k^2 - (\varepsilon(K_r) - 1)cn^2 = \varepsilon n/k^2 - \varepsilon' n^2,
\]
and Proposition 11 yields the desired partition. \(\square\)

4. Proof of Theorem 2

In the following, we conclude with the proof of Theorem 2, which follows by using the same steps (Claims 7-9) of the proof of the main result from [16, Theorem 3] except a minor modification at the end which we are going to describe below. Although the main theorem in [16, Theorem 3] did not consider the case \(H = K_r\), the steps of the proof work verbatim for this case as well.

Sketch of the proof of Theorem 2. We will apply Theorem 12 in the form when \(\varepsilon = 0\), and we choose \(\gamma\) sufficiently small. We assume that there is a graph \(G\) on \(n\) vertices (\(n\) large enough) such that there exists a proper edge coloring of \(G\) with \(\phi^R(G, K_r) \geq \varepsilon(n, K_r)\) and \(G\) is not isomorphic to the Turán graph \(T_{r-1}(n)\). By following the steps of [16], we apply first Claim 7 from [16] and assume that \(\phi^R(G) = \varepsilon(n, K_r) + m\) for some \(m \geq 0\). We obtain a subgraph \(G'\) of \(G\) on \(n'\) vertices such that \(\delta(G') \geq \delta(T_{r-1}(n'))\) and \(\phi^R(G', K_r) \geq \varepsilon(n', K_r) + m\). Then, Theorem 12 asserts the existence of a partition of \(G'\) into \(r - 1\) parts that maximizes the number of edges between different parts so that the number \(m_2\) of edges within the partition classes is not zero but is at most \(\gamma n'^2\). It is observed that the order
of these \( r - 1 \) parts are almost balanced due to the maximality condition (Claim 8 from [16, Theorem 3]). This implies that \( e(G') \leq ex(n', K_r) + m_2 \).

In the final step, we find at least \( \left\lfloor \frac{m_2}{(r-1)} \right\rfloor + 1 \) many rainbow copies of \( K_r \). We find these copies iteratively (Claim 9 from [16, Theorem 3]). The only difference to the embedding in [16] is that we need to find at each iteration a copy of \( K_r \), which under the edge coloring \( \chi \) is rainbow. This is done by finding first a complete \((r-1)\)-partite graph \( F \) with parts of size at least \( 8(r-1)^3/3 \), so that one of the parts contains an edge (Claim 9 from [16, Theorem 3], this basically follows from an application of Theorem of Erdös and Stone [6]). A rainbow copy of \( K_r \) is guaranteed to exist in such a graph \( F \) by a result of Keevash, Mubayi, Sudakov, and Verstraëte [11, Lemma 2.2]. This allows us to find at least \( \left\lfloor \frac{m_2}{(r-1)} \right\rfloor + 1 \) many rainbow copies of \( K_r \) in \( G' \) under the edge coloring \( \chi \). Clearly, this implies that

\[
\phi^R_{\chi}(G', K_r) \leq ex(n', K_r) + m_2 - \left( \binom{r}{2} - 1 \right) \left( \left\lfloor \frac{m_2}{(r-1)} \right\rfloor + 1 \right) < ex(n, K_r),
\]

a contradiction. \( \square \)

5. CONCLUDING REMARKS

With essentially the same techniques, Theorem 2 can be generalized to any edge-critical graph of chromatic number at least 3 as follows.

**Theorem 17.** For any edge-critical graph \( H \) with chromatic number at least 3, there is an \( n_0 = n_0(H) \) such that \( \phi^R(n, H) = ex(n, H) \) for all \( n \geq n_0 \). Moreover, the only graph attaining \( \phi^R(n, H) \) is the Turán graph \( T_{\chi(H)-1}(n) \).

To do so one can prove the following stability result about \( \phi^R_{\chi}(n, H) \).

**Theorem 18.** For every \( H \) with \( \chi(H) = r \geq 3 \) and for every \( \gamma > 0 \) there exist \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that for every graph \( G \) on \( n \geq n_0 \) vertices with a proper coloring \( \chi \) of its edges the following is true. If

\[
\phi^R_{\chi}(G, H) \geq ex(n, H) - \varepsilon n^2
\]

then there exists a partition of \( V(G) = V_1 \cup \ldots \cup V_{r-1} \) with \( \sum_{i=1}^{r-1} e(V_i) < \gamma n^2 \).

Theorem 18 can be shown by generalizing the proof of Theorem 12. We provide here a very short sketch of a slightly different approach, which generalizes [16, Lemma 4]. We would like to stress however that this does not apply to cliques though.

*Proof sketch of Theorem 18.* The idea is again based on the regularity lemma, on the result of Pikhurko and Sousa [17, Theorem 1.1] and an application of Theorem 9.

We first fix an arbitrary proper edge coloring \( \chi \) of \( G \). The proof follows along the lines of the argument in [16, Lemma 4], up to the place when the auxiliary \( e(H) \)-uniform hypergraph is built (which is done in [16, Corollary 6]). The only difference is that our hypergraph consists now of rainbow copies of \( H \) (and these are the most copies of \( H \) by an argument similar to Lemma 13). The remainder of the proof stays the same. \( \square \)
As a direct consequence of Theorem 18 we obtain the following generalization of the result of Pikhurko and Sousa [17] on \(\phi(n, H)\) mentioned above.

**Theorem 19.** Let \(H\) be any graph of chromatic number at least three. Then we have
\[
\phi^R(n, H) = \text{ex}(n, H) + o(n^2).
\]

**Proof.** The lower bound follows by taking any proper coloring of any \(H\)-extremal graph on \(n\) vertices. The upper bound follows since any proper edge coloring \(\chi\) of any graph \(G\) on \(n\) vertices with \(\phi^R(G, H) \geq \text{ex}(n, H)\) admits a partition of \(V(G) = V_1 \cup \ldots \cup V_r - 1\) with \(\sum_{i=1}^{r-1} e(V_i) = o(n^2)\), implying \(e(G) \leq \text{ex}(n, H) + o(n^2)\) and thus \(\phi^R(n, H) = \text{ex}(n, H) + o(n^2)\). \(\Box\)

Finally, we would like to mention that Theorems 17 and 19 can be stated in a slightly general form, similar to the one of Theorem 2.

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_Hacettepe University, Department of Computer Engineering, Beytepe 06800 Ankara, Turkey_  
*E-mail address*: ozkahya@hacettepe.edu.tr

_Goethe-Universität, Institut für Mathematik, Robert-Mayer-Str. 10, 60325 Frankfurt am Main, Germany_  
*E-mail address*: person@math.uni-frankfurt.de