The Membrane Paradigm for Gauss-Bonnet gravity

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Abstract

We construct the membrane paradigm for black objects in Einstein-Gauss-Bonnet gravity in spacetime dimensions $\geq 5$. As in the case of general relativity, for the observers outside the horizon the dynamics of the perturbations of the horizon can be modelled as a membrane endowed with fluid-like properties. We derive the stress-tensor for this membrane fluid and study the perturbation around static backgrounds with constant curvature horizon cross-section to express the stress tensor in the form of a Newtonian viscous fluid with pressure, shear viscosity and bulk viscosity. The ratio of the shear viscosity and the entropy density is shown to generically violate the bound suggested by Policastro, Son and Starinets. We evaluate the transport coefficients for some static geometries. For the black brane geometry our results match with those available in the literature. For the spherically symmetric AdS black hole our results can be interpreted as a holographic prediction for the transport coefficients and viscosity to the entropy density ratio for the dual conformal field theory living on the boundary, $S^3 \times R$. 

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1 Introduction

Gravitation being the manifestation of the curvature of spacetime, affects the causal structure of the spacetime. This can lead to the existence of regions which are causally inaccessible to a class of observers. An example of such a region is the portion of spacetime inside the event horizon of a black hole, which is causally disconnected from any outside observer. Hence, the relevant physics for the observers outside the hole must be independent of what is happening inside the hole. This observation forms the basis of the membrane paradigm for black holes.

The membrane paradigm [1, 2] is an approach in which the interaction of the black hole with the outside world is modelled by replacing the black hole by a membrane of fictitious fluid “living” on the horizon. The mechanical interactions of the black hole with the outside world are then captured by the (theory dependent) transport coefficients of the fluid. The electromagnetic interaction of the black hole is described by endowing the horizon with conductivity and so forth. This formalism provides an intuitive and elegant understanding of the physics of the event horizon in terms of a simple non-relativistic language and also serves as an efficient computational tool useful in dealing with some astrophysical problems. After the advent of holography, the membrane paradigm took a new life in which the membrane fluid is conjectured to provide the long wavelength description of the strongly coupled quantum field theory at a finite temperature [3].

The original membrane paradigm [1, 2] was constructed for black holes in general relativity. The membrane fluid has the shear viscosity, \( \eta = 1/16\pi G \). Dividing this by the Bekenstein-Hawking entropy density, \( s = 1/4G \), gives a dimensionless number, \( \eta/s = 1/4\pi \). The calculation which leads to this ratio relies only on the dynamics and the thermodynamics of the horizon in classical general relativity. But, interestingly enough, it was found in [3] that the same ratio is obtained in the holographic description of the hydrodynamic limit of the strongly coupled \( N = 4 \), \( U(N) \) gauge theory at finite temperature which is dual to general relativity in the limit \( N \to \infty \) and \( \lambda_t \to \infty \), where \( N \) is the number of colors and \( \lambda_t \) is the ’t Hooft coupling. The authors of [4] conjectured that this ratio is a universal lower bound for all the materials, this is called the KSS bound. The relationship between the membrane paradigm calculations and the holographically derived KSS bound was explained as a consequence of the trivial renormalization group (RG) flow from IR to UV in the boundary gauge theory as one moves the outer cutoff surface from the horizon to the boundary of spacetime [5]. The universality of this bound, how it might be violated, and the triviality of the RG flow in the long wavelength limit at the level of the linear response were also clarified in [6].

The general theory of relativity, which is based upon the Einstein-Hilbert action functional, is the simplest theory of gravity one can write guided by the principle of diffeomorphism invariance while containing only the time derivatives of second order in the equation of motion. Although such a simple choice of the action functional has so far been adequate to explain all the experimental and observational results, there is no reason to believe that this choice is fundamental. Indeed, it is expected on various general grounds that the low energy limit of any quantum theory of gravity will contain higher derivative correction terms. In fact, in string theory the low energy effective action generically contains terms which are higher order in curvature due to the stringy (\( \alpha' \)) corrections. In the context of holographic duality, such \( \alpha' \) modifications correspond to \( 1/\lambda_t \) corrections. The specific form of these terms depends ultimately on the detailed features of the quantum theory. From the classical point of view, a simple modification of the Einstein-Hilbert action is to include the higher order curvature terms preserving the diffeomorphism invariance and still leading to an equation of motion containing no more than second order time derivatives. In fact this generalization is unique [7] and goes by the name of Lanczos-Lovelock gravity, of which the lowest order correction (second order in curvature) appears as the Gauss-Bonnet (GB) term in spacetime dimensions \( D > 4 \). Einstein-Gauss-Bonnet (EGB) gravity is free from ghosts [8, 9] and leads to a well-defined initial value problem. Black hole solutions in EGB gravity have been studied extensively and are found to have various interesting features [10, 11]. The entropy of these black holes is no longer proportional to the area of the horizon but contains a curvature dependent term [12, 13, 14, 15]. Hence, unlike in general relativity, the entropy density of the horizon in EGB gravity is not a constant but depends on the horizon curvature. Now, the form of the membrane stress tensor in the fluid model of the horizon is also theory dependent and therefore the transport coefficients of the membrane fluid will change due to the presence of the GB term in the action. Hence, it is of interest
to investigate the membrane paradigm and calculate the transport coefficients for the membrane fluid in the EGB gravity. The violation of the KSS bound due to the GB term in the action has already been shown in [16] using other methods.

In this paper, we use the action principle formalism of the membrane paradigm as constructed in [17] and we extend it to EGB gravity. We first derive the membrane stress-energy tensor on the stretched horizon for EGB theory. After regularization and restriction to the linearized perturbations of static black backgrounds with horizon cross-section of constant curvature, we express the membrane stress tensor in the form of a Newtonian viscous fluid described by certain transport coefficients.

We find the ratio of the shear viscosity to the entropy density for the membrane fluid corresponding to the black brane solution in EGB theory with cosmological constant $\Lambda = -(D-1)(D-2)/2l^2$ to be
\[
\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 - \frac{2(D-1)\lambda}{(D-3)l^2} \right],
\]
where $\lambda = (D-3)(D-4)\alpha'$ and $\alpha'$ is the GB coupling constant. This matches with the result found in [16]. Notice that the calculation in [16] is done at the boundary of the spacetime at infinity while our calculation refers to the horizon. This naturally leads to the conclusion that in the EGB theory, as in general relativity, the ratio $\eta/s$ is universal i.e., scale independent in the sense of [5]. That this result is true is argued in [6] and our calculation combined with the result of [16] provides a support for their conclusion. The same result is reported in a recent study in [18] following the proposal of [5]. Also, the presence of the GB term violates the KSS bound, $\eta/s \geq 1/4\pi$, for any $\alpha' > 0$.

Besides the black brane, our method also provides the value of this ratio for the five dimensional spherically symmetric AdS black hole in EGB gravity,
\[
\frac{\eta}{s} = \frac{r_h^4}{4\pi (r_h^2 + \alpha') (r_h^2 + 3\alpha')} \left( 1 - \frac{2\alpha'}{l^2} \right),
\]
where $r_h$ is the radius of the horizon and $l$ is related to the cosmological constant by $\Lambda = -6/l^2$. As far as we are aware this is a new result. Our method thus predicts that the hydrodynamic limit of the conformal field theory (CFT) state dual to the 5D-spherically symmetric AdS black hole in EGB theory with AdS boundary conditions has the viscosity to entropy ratio given by (2), where $r_h$ is understood to be a function of the Hawking temperature of the black hole. As in the case of the black brane, the KSS viscosity bound is violated for any $\alpha' > 0$. The difference between the black brane and black hole results are further discussed in section 6.

This paper is organized as follows: we begin in section 2 where we present the geometric setup of the membrane paradigm. In section 3 we review the action based membrane paradigm approach for the black holes in general relativity. This construction is then generalized in section 4 where we construct the membrane paradigm for the black objects in the EGB gravity. In this section we obtain the stress tensor for the membrane fluid and we derive the expressions for the transport coefficients of the fluid. In section 5 we evaluate these transport coefficients for some specific black geometries. Finally, we conclude with the summary and discussion in section 6.

We adopt the metric signature ($- ,+ ,+ ,+ ,...$) and our sign conventions are same as those of MTW [19]. All the symbols that we will be using in the main body of the paper are defined when introduced for the first time. For the convenience of the reader we have also included a table in the appendix summarizing these symbols and their meanings.

## 2 Geometric set-up

In this section we elaborate on the geometric set-up necessary to construct the membrane paradigm. We include it in this paper to keep it self-contained but the interested reader can find a detailed discussion in the monograph [11].
The event horizon, \( H \), of the black hole in \( D \) spacetime dimensions, is a \((D-1)\)-dimensional null hypersurface generated by the null geodesics \( l^a \). We choose a non-affine parameterization such that the null generators satisfy the geodesic equation \( l^a \nabla_a l^b = \kappa l^b \), where \( \kappa \) is a constant non-affine coefficient. For a stationary spacetime, \( l^a \) coincides with the null limit of the time-like Killing vector and \( \kappa \) can then be interpreted as the surface gravity of the horizon.

Next we introduce a time-like surface positioned just outside \( H \) which is called the stretched horizon and denoted by \( \mathcal{H}_s \). One can think of \( \mathcal{H}_s \) as the world-tube of a family of fiducial observers just outside the black hole horizon. The four velocity of these fiducial observers is denoted by \( u^a \). Just as \( H \) is generated by the null congruence \( l^a \), \( \mathcal{H}_s \) is generated by the time-like congruence \( u^a \). The unit normal to \( \mathcal{H}_s \) is denoted by \( n^a \) and is taken to point away from the horizon into the bulk. We relate the points on \( \mathcal{H}_s \) and \( H \) by ingoing light rays parametrized by an affine parameter \( \gamma \), such that \( \gamma = 0 \) is the position of the horizon and \((\partial/\partial \gamma)^a l_a = -1 \) on the horizon. Then, in the limit \( \gamma \to 0 \), when the stretched horizon approaches the true one, \( u^a \to \delta^{-1} l^a \) and \( n^a \to \delta^{-1} l^a \) where \( \delta = \sqrt{2\kappa \gamma} \).

The induced metric \( h_{ab} \) on \( \mathcal{H}_s \) can be expressed in terms of the spacetime metric \( g_{ab} \) and the covariant normal \( n_a \) as \( h_{ab} = g_{ab} - n_a n_b \). Similarly, the induced metric \( \gamma_{ab} \) on the \((D-2)\)-dimensional space-like cross-section of \( \mathcal{H}_s \) orthogonal to \( u_a \) is given by \( \gamma_{ab} = h_{ab} + u_a u_b \). The extrinsic curvature of \( \mathcal{H}_s \) is defined as \( K^a_{b} = h_{a}^{c} \nabla_c n^a \). Using the limiting behavior of \( n^a \) and \( u^a \) it is easy to verify that in the limit that \( \delta \to 0 \) various components of the extrinsic curvature behave as

\[
\text{As } \delta \to 0 : \quad K^a_{b} = K^a_{b} u_a u^b = g \sim \frac{\kappa}{\delta},
\]

\[
K^a_{b} = K^a_{b} u_a \gamma^b_B = 0,
\]

\[
K^a_{b} = K^a_{b} u_A \gamma^b_B \sim \frac{k^a_B}{\delta}.
\]

\[
K = K_{ab} g^{ab} \sim \frac{(\theta + \kappa)}{\delta},
\]

(3)

where \( \theta \) is the expansion scalar of \( l^a \) and \( k^a_B \) is the extrinsic curvature of the \((D-2)\)-dimensional space-like cross-section of the true horizon \( H \). Note that apriori the projection of the extrinsic curvature of \( \mathcal{H}_s \) on the cross-section of \( \mathcal{H}_s \) has nothing to do with the extrinsic curvature of the cross-section (orthogonal to \( u^a \)) as embedded in \( \mathcal{H}_s \), i.e., there is, in general, no relationship between the pull-back of \( \nabla_a n_b \) and \( \nabla_a u_b \) to the cross-section of the stretched horizon. However, in the null limit (\( \delta \to 0 \)) both \( u^a \) and \( n^a \) map to the same null vector \( l^a \) and we have \( K^a_{b} \to \delta^{-1} k^a_B \). Finally, we decompose \( k_{AB} \) into its trace-free and trace-full part as

\[
k_{AB} = \sigma_{AB} + \frac{1}{(D-2)} \theta \gamma_{AB},
\]

(4)

where \( \sigma_{AB} \) is the shear of \( l^a \). It is clear from equation (3) that in the null limit, various components of the extrinsic curvature diverge and we need to regularize them by multiplying by a factor of \( \delta \). The physical reason behind such infinities is that, as the stretched horizon approaches the true one, the fiducial observers experience more and more gravitational blue shift; on the true horizon, the amount of blue shift is infinite.

This completes the description of our geometric set-up. Next, we review the derivation of the black hole membrane paradigm in standard Einstein gravity.

3 The membrane paradigm in Einstein gravity

In this section we construct the membrane paradigm in Einstein gravity in four spacetime dimensions. Our construction will closely follow the action approach of [17]. Our purpose is to fix the notation and emphasize

\footnote{A, B denote the indices on the cross-section of the horizon.}
the points in the construction which will be of importance for the corresponding construction in the EGB gravity. We will highlight the steps which will be different in the EGB case and where one has to make assumptions. For the construction of the membrane stress tensor we will work exclusively with differential forms and only in the end do we go back to the metric formalism.

In the rest of this paper, unless otherwise explicitly written, we will work with the units such that $16\pi G = 1$. The small Roman letters on the differential forms are the Lorentz indices while in the spacetime tensors we will not differentiate between the Lorentz and the world indices, this being understood that one can always use the vierbeins to convert the indices from the Lorentz to the world and vice-versa.

In the Cartan formalism, the Einstein-Hilbert lagrangian is written in terms of the vector valued vierbein one-form $e^a$, related to the metric by $g = \eta_{ab}e^a \otimes e^b$, and the Lorentz Lie-algebra valued torsion-free connection one-form $\omega^{ab}(e)$ defined by the equation $De^a = de^a + \omega^a_b \wedge e^b = 0$. The action is then

$$ S_{EH} = \frac{1}{2\pi} \int_M \Omega^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} + \frac{1}{2\pi} \int_{\partial M} \theta^{ab} \wedge e^c \wedge e^d \epsilon_{abcd}, \quad (5) $$

where, $\Omega^{ab}$ is the curvature of the torsion-free connection given by $\Omega^{ab} = dw^{ab} + \omega^a_c \wedge e^b$, and $\theta^{ab}$ is the second fundamental form on the boundary $\partial M$ of $M$, \cite{20}. It is related to the extrinsic curvature by

$$ \theta^{ab} = (n.a)(n^a K^b - n^b K^a)e^a. \quad (6) $$

In our case, the boundary $\partial M$ of $M$ consists of the outer boundary at spatial infinity and the inner boundary at the stretched horizon, $\mathcal{H}_s$. Variation of the action with respect to $e^a$ can be separated into the contribution from $\omega(e)$ and the rest. Variation with respect to $\omega$ of the bulk part of the action yields a total derivative which, after the integration by parts, gives a contribution identical to the negative of the variation of the boundary part. Thus the variation of the action with respect to $\omega$ vanishes identically. In the absence of the inner boundary, variation of $S_{EH}$ with respect to the vierbein, $e^a$, under the Dirichlet boundary condition (holding the vierbein fixed on the outer boundary) yields the equation of motion for the vierbein. But when the inner boundary is present, there is no natural way to fix the vierbein there. The physical reason for this is simply that the horizon acts as a boundary only for a class of observers, and surely the metric is not fixed there. However, because it is a causal boundary the dynamics outside the horizon is not affected by what happens inside. Hence, for the consideration of the outside dynamics one can imagine some fictitious matter living on the stretched horizon whose contribution to the variation of the action cancels that of the inner boundary. This is the basic idea of the construction of stress-tensor `a la Brown and York, \cite{21}. Since we will be interested in the boundary term on the stretched horizon which is a time-like hypersurface, from now on we will put $n.n = 1$.

Hence, variation of the action with respect to the vierbein gives

$$ \delta S_{EH} = \int_M \Omega^{ab} \wedge e^c \wedge \delta e^d \epsilon_{abcd} + \int_{\partial M} \theta^{ab} \wedge e^c \wedge \delta e^d \epsilon_{abcd}. \quad (7) $$

The bulk term just gives the equation of motion. Using the Dirichlet boundary condition on the outer boundary and the expression of $\theta$ given in equation (6), the surviving contribution of the variation of the total action $S_{total} = S_{EH} + S_{matter}$ comes solely from the inner boundary and is given by

$$ \delta S_{total} = \int_{\mathcal{H}_s} K^b_m e^m \wedge e^c \wedge \delta e^d \epsilon_{bcd}, \quad (8) $$

where $\epsilon_{abcd} = -n^a \epsilon_{abcd}$. This surviving contribution can be interpreted as due to a fictitious matter source residing on $\mathcal{H}_s$ whose stress tensor is given by

$$ t^{ab} = 2(Kh^{ab} - K^{ab}). \quad (9) $$

\footnote{Our sign convention is such that the plus (minus) signs apply to a time-like (space-like) boundary. It should be noted that this is different from \cite{20} because their definition of extrinsic curvature differs from ours by a negative sign.}
In terms of \( t_{ab} \) the on-shell variation of \( S_{EH} \) becomes

\[
\delta S_{EH} = -\frac{1}{2} \int_{H_s} t_{ab} \delta h^{ab} + \frac{1}{2} \int_{M} T_{ab} \delta g^{ab},
\]

(10)

where, \( T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{ab}} \) is the external matter’s stress energy tensor. Now, restricting the variation \( \delta \xi \) corresponding to the diffeomorphism generated by a vector field \( \xi^a \) which is arbitrary in the bulk, tangential to the inner boundary and vanishes on the outer boundary and the boundary of \( H_s \), the diffeomorphism invariance of the theory ensures that

\[
\delta \xi S_{EH} = -\frac{1}{2} \int_{H_s} t_{ab} \delta \xi^{ab} + \frac{1}{2} \int_{M} T_{ab} \delta \xi^{ab} = 0
\]

(11)

\[
\Rightarrow -\int_{H_s} t_{ab} D^a \xi^b + \int_{M} T_{ab} \nabla^a \xi^b = 0,
\]

where, \( D^a \) is the covariant derivative of the induced metric on the inner boundary \( H_s \), and \( \nabla^a \) is the covariant derivative of the spacetime metric. Using integration by parts and the afore-mentioned conditions on \( \xi^a \), equation (11) gives

\[
D^a t_{ab} = -T_{ac} n^a h^{cb},
\]

(12)

where the negative sign on the right hand side arises because we have chosen \( n^a \) as pointing away from the stretched horizon into the bulk. The right hand side of this equation can be interpreted as the flux of external matter crossing the horizon from the bulk. Then equation (12) has the interpretation of the continuity equation satisfied by the fictitious matter living on the stretched horizon.

At this stage, we would like to point out a difference between our approach and that of [17]. In [17], the Gibbons-Hawking boundary term is considered to be only on the outer boundary. Therefore, one needs to show that a certain contribution containing the derivatives of the variation of the metric on the stretched horizon vanishes in the limit as the stretched horizon reaches the true horizon. In our approach there is no such requirement because we have the boundary term on the entire boundary which includes the stretched horizon. We believe that our approach is conceptually transparent and computationally simpler than the one in [17].

We have derived the form of the membrane stress tensor for the particular case of \( D = 4 \) spacetime dimensions, but it is easy to check that the form of the stress tensor in equation (9) remains unchanged for a general \( D \)-dimensional spacetime. Then the components of the membrane stress tensor \( t_{ab} \), evaluated on the stretched horizon in the basis \( (u^a, x^A) \) are given by

\[
t_{uu} = \rho = -2\theta_s,
\]

\[
t^{AB} = 2 \left( -\sigma_s^{AB} + \frac{(D-3)}{(D-2)} \theta_s \gamma^{AB} + g \gamma^{AB} \right),
\]

(13)

where \( g = \kappa/\delta \). In deriving this expression, we have replaced \( K^{AB} \) by the expression: \( \sigma_s^{AB} + \frac{\theta_s}{(D-2)} \gamma^{AB} \), where \( \theta_s \) is the expansion and \( \sigma_s^{AB} \) is the shear of the congruence generated by the time-like vector field \( u^a \) on \( H_s \). As pointed out in [11, 2], this replacement is valid only up to \( O(\delta) \). Since we are ultimately interested in the limit \( \delta \to 0 \), any \( O(\delta) \) error does not contribute. Although this is certainly true for general relativity, for the EGB gravity such \( O(\delta) \) terms will play an important role and they actually contribute in the limit that the stretched horizon becomes the true horizon.

The particular form of the components of the membrane stress tensor in equation (13) has an interpretation: the fictitious matter on the stretched horizon can be regarded as a \((D-2)\)-dimensional viscous fluid.
with the energy density and transport coefficients given by

\[
\begin{align*}
\text{Energy density} & : \quad \rho_s = -2\theta_s, \\
\text{Pressure} & : \quad p_s = 2\kappa, \\
\text{Shear Viscosity} & : \quad \eta_s = 1, \\
\text{Bulk Viscosity} & : \quad \zeta_s = -2 \left( \frac{D-3}{D-2} \right).
\end{align*}
\]

Hence the entire $t_{ab}$ on the stretched horizon can be expressed as,

\[
t_{ab} = \rho_s u_a u_b + \gamma^A_a \gamma^B_b \left( p_s \gamma_{AB} - 2\eta_s \sigma_s AB - \zeta_s \theta_s \gamma_{AB} \right).
\]

(14)

Substituting these quantities in the conservation equation (12) then gives the evolution equation for the energy density,

\[
\mathcal{L}_a \rho_s + \rho_s \theta_s = -p_s \theta_s + \zeta_s \theta_s^2 + 2\eta_s \sigma_s^2 + T_{ab} n^a u^b.
\]

(15)

The evolution equation matches exactly with energy conservation equation of a viscous fluid. We stress the fact that equation (15) is a direct consequence of the conservation equation (12) and the form of the stress tensor in equation (14). In fact, in any theory of gravity once we can express the stress tensor of the fictitious matter obeying the conservation equation on the stretched horizon in a form analogous to the one in equation (14), the conservation equation will automatically imply an evolution equation of the form (15). Notice that the conservation equation is only valid on-shell, which means that the equations of motion of the theory have to be satisfied for it to hold.

From the analysis of section 2 it is evident that as the stretched horizon approaches the true horizon the membrane stress tensor in equation (13) diverges as $\delta^{-1}$ due to the large blue shift near the horizon. This divergence is regulated by simply multiplying it by $\delta$. This limiting and regularization procedure then yields a stress tensor of a fluid living on the cross sections of the horizon itself in terms of the quantities intrinsic to the horizon. This stress tensor is

\[
t^{(H)}_{AB} = (p \gamma_{AB} - 2\eta \sigma_{AB} - \zeta \theta \gamma_{AB}),
\]

(16)

and the energy density and the transport coefficients become,

\[
\begin{align*}
\text{Energy density} & : \quad \rho = -2\theta, \\
\text{Pressure} & : \quad p = 2\kappa, \\
\text{Shear Viscosity} & : \quad \eta = 1, \\
\text{Bulk Viscosity} & : \quad \zeta = -2 \left( \frac{D-3}{D-2} \right),
\end{align*}
\]

where $\theta$ is the expansion of the null generator of the true horizon as discussed in section 2. Similarly, the regularization of the evolution equation (15) gives

\[
\mathcal{L}_a \rho + \rho \theta = -p \theta + \zeta \theta^2 + 2\eta \sigma^2 + T_{ab} n^a u^b,
\]

(17)

which is just the Raychaudhuri equation of the null congruence generating the horizon. It should be noted that our approach for deriving the evolution equation is different from the one used in [17]. In principle, one can just take the Lie derivative of the energy density with respect to the generator of the horizon to obtain the Raychaudhuri equation as in [17] and then use the Einstein equation to replace the curvature dependence in terms of the matter energy-momentum tensor. We have followed an indirect approach in which we derive...
the evolution equation via the continuity equation \[ \frac{\eta}{s} = \frac{1}{4\pi}. \] (18)

Note that this is a dimensionless constant independent of the parameters of the horizon. Comparing this with the KSS bound, \( \eta/s \geq 1/4\pi \), we see that the bound is saturated in general relativity. Interestingly, for any gravity theory with a Lagrangian depending on the Ricci scalar only, the value of this ratio is same as that in general relativity and therefore the KSS bound is saturated in these theories \[23\].

Another important fact is that the bulk viscosity associated with the horizon is negative. Clearly, the fluid corresponding to the horizon is not an ordinary fluid and as explained in \[1\], the negative bulk viscosity is related to the teleological nature of the event horizon.

4 The membrane paradigm in Einstein-Gauss-Bonnet gravity

EGB gravity is a natural generalization of general relativity which includes terms higher order in curvature but in just such a way that the equation of motion remains second order in time.

The action of the theory is given by

\[ S_{total} = S_{EH} + \alpha' S_{GB} + S_{matter} \] (19)

where, \( S_{EH} \) and \( S_{matter} \) are the contribution of the Einstein-Hilbert and the matter, respectively, while \( S_{GB} \) is the Gauss-Bonnet addition to the action. From the analysis in the section 3 we know how to take care of the \( S_{EH} \). So, we can exclusively work with the GB term now. The GB contribution to the total action, in \( D = 5 \), is given by \[20\]

\[ S_{GB} = \int_M \Omega^{ab} \wedge \Omega^{cd} \wedge \epsilon^{abcdef} + 2 \int_{\partial M} \theta^{ab} \wedge (\Omega - \frac{2}{3} \theta \wedge \theta)^{cd} \wedge \epsilon^{abcdef}. \] (20)

As in the case of general relativity discussed in section 3 the variation of \( S_{GB} \) with respect to the connection \( \omega \) vanishes identically. Variation with respect to the vierbein \( e^a \) under the Dirichlet boundary condition yields the equation of motion. Using the torsion-free condition \( \mathcal{D} e^a = 0 \) on the connection, this equation can be shown to be the same as that obtained in the metric formalism. In the presence of the inner boundary at the stretched horizon, \( \mathcal{H}_s \), we need the variation of the boundary part of the \( S_{GB} \) due to the variation of the vierbein on the inner boundary. This is obtained from equation (21), which after variation with respect to \( e^a \) and then using the relations \( \theta^{ab} = (\eta^{a} K_b^c - n^b K_c^d) e^c \) and \( \Omega^{ab} = \frac{1}{2} R_{ab}^{cd} e^m \wedge \epsilon^n \), gives

\[ \delta S_{GB} \big|_{\text{bndy}} = 4 \int_{\partial M} K_a \left( \frac{1}{2} h^r h^q h^s R_{pq}^{rs} + \frac{2}{3} K^c_{mn} K^d \right) 4! \delta^{[amnb]} \delta e^f, \] (21)

The projections of the spacetime Riemann tensor can be written in terms of the Riemann tensor intrinsic to \( \mathcal{H}_s \) and the extrinsic curvature of \( \mathcal{H}_s \) using the Gauss-Codazzi equation

\[ h^{cd}_{pq} = \hat{R}^{cd}_{ab} - K^c_{mk} K^d_{bn} + K^c_{mb} K^d_{an}, \] (22)

where \( \hat{R}^{cd}_{ab} \) is the Riemann tensor intrinsic to \( \mathcal{H}_s \). Thus the variation of the boundary term can be written in terms of the quantities intrinsic to the boundary,

\[ \delta S_{GB} \big|_{\text{bndy}} = 4 \int_{\partial M} K_a \left( \frac{1}{2} \hat{R}^{cd}_{mn} + \frac{1}{3} K^c_{mn} R^d \right) 4! \delta^{[amnb]} \delta e^f, \] (23)
As in the case of general relativity, we can interpret the variation \( \delta S \) of the true horizon, the extrinsic curvature of the stretched horizon diverges as we take the limit to the horizon. Since, as we take the limit to the horizon, the stress tensor is linear in the extrinsic curvature while in the latter the stress tensor contains terms cubic in the extrinsic curvature of the stretched horizon. This can be evaluated to be

\[
\delta S_{GB}^{\text{bdy}} = 4 \int_{\partial M} \left( 2K_{mn}\hat{P}^{mn} - J^a_b + \frac{1}{3}J^a_b \right) \delta\epsilon_a^b.
\]

(24)

where we have defined,

\[
\hat{P}^{mn} = \tilde{R}^{mn} + 2\tilde{R}[m\tilde{h}^n] + 2\tilde{R}_{\tilde{a}}[\tilde{h}^m\tilde{h}^n] + \tilde{R}h_{[m}\tilde{h}_{n]},
\]

(25)

\[
J^a_b = K^2K^a_b - KK^{cd}K^{cd}_b + 2K^a_bK^c_d - 2K^a_bK^c_e,
\]

(26)

\[
J = K^3 - 3KK^{cd}K^{cd} + 2K^a_bK^b_cK^c_e.
\]

(27)

As in the case of general relativity, we can interpret the variation \( \delta S_{\text{total}} \) as due to a fictitious matter living on the membrane whose stress energy tensor is given by the coefficient of \( \delta\epsilon_a^b \). Thus from the equation \([24]\) we can read off the membrane stress tensor due to the GB term (now including the GB coupling \( \alpha' \) and a negative sign arising from the fact that we are defining the stress tensor with covariant indices) and we get,

\[
t_{ab}|_{(GB)} = -4\alpha' \left( 2K_{mn}\hat{P}^{mn}_a - J_{ab} + \frac{1}{3}Jh_{ab} \right).
\]

(28)

Adding to this the contribution coming from the Einstein-Hilbert action we get the total stress tensor for the membrane,

\[
t_{ab} = 2(Kh_{ab} - K_{ab}) - 4\alpha' \left( 2K_{mn}\hat{P}^{mn}_a - J_{ab} + \frac{1}{3}Jh_{ab} \right).
\]

(29)

Although, we have derived the membrane stress tensor for the particular case of \( D = 5 \), the result can be easily generalized to arbitrary dimensions and the form in equation \([29]\) remains unchanged. By the same arguments as discussed in the case of general relativity, this \( t_{ab} \) also satisfies the continuity equation \([12]\). This can also be verified explicitly by taking the divergence of \( t_{ab} \) and using the appropriate projections of the equation of motion \([24]\).

Note that a crucial difference between the membrane stress tensors for general relativity and EGB gravity is that in the former the stress tensor is linear in the extrinsic curvature while in the latter the stress tensor contains terms cubic in the extrinsic curvature of the stretched horizon. Since, as we take the limit to the true horizon, the extrinsic curvature of the stretched horizon diverges as \( \delta^{-1} \), one would expect higher order divergences in the case of EGB gravity coming from the contribution of the GB term to the stress tensor. The regularization procedure used to tame the divergence coming from the Einstein-Hilbert term involves multiplication with \( \delta \), which does not tame the cubic order divergence coming from the GB term. Clearly one needs either a new regularization procedure or some well-motivated prescription which justifies neglect of the terms that lead to higher order divergences in the limit when the stretched horizon approaches the true one. In this paper, we will adopt the latter approach.

We will restrict attention to background geometries which are static so that the expansion and the shear of the null generators of the true horizon are zero. Next, we will consider some arbitrary perturbation of this background which may arise due to the flux of matter flowing into the horizon. As a result, the horizon becomes time dependent and acquires expansion and shear. We will assume this perturbation of the background geometry to be small so that we can work in the linear order of perturbation and ignore all the higher order terms. Essentially, our approximation mimics a slow physical-process version of the dynamics of the horizon \([25]\). We also restrict ourselves to the terms proportional to the first derivative of the observer’s four velocity \( u^a \) which plays the role of the velocity field for the fluid. We will discard all higher order derivatives of the velocity except the linear one so that we can write the membrane stress tensor as a Newtonian viscous fluid. In this limited setting we will see that the only divergence that survives is of \( O(\delta^{-1}) \) which can be regularized in the same fashion as in the case of general relativity. Therefore, when we encounter a product of two quantities \( X \) and \( Y \), we will always express such a product as,

\[
XY \approx \dot{X}\dot{Y} + \dot{X}\delta Y + \dot{Y}\delta X,
\]

(30)
where $\delta X$ is the value of the quantity $X$ evaluated on the static background and $\delta X$ is the perturbed value of $X$ linear in perturbation.

In order to implement this scheme, we first define a quantity $Q_{ab}$, whose importance will be apparent later, as

$$Q_{ab} = KK_{ab} - K_{ac}K_{bc}^{c},$$  \hspace{1cm} (31)$$

in terms of which, we write

$$J_{ab} = K_{ab}Q - 2K_{ac}Q_{bc},$$  \hspace{1cm} (32)$$

where $Q$ is the trace of $Q_{ab}$.

Now we observe the following facts. First, the components of the extrinsic curvature of $H_s$ in the backgrounds that we are interested in is $O(\delta)$. In particular, for the static spacetimes one can choose the cross-sections of the $H_s$ such that the pull-back of the extrinsic curvature to these cross-sections is $K_{AB} = \frac{2}{r} \gamma_{AB}$. Secondly, for these backgrounds $Q_{ab}$ and $Q$ defined as above are finite on $H_s$ and remain finite in the limit as $H_s$ reaches the true horizon. In fact, $Q_{AB}$ for the background, in the limit that $H_s$ approaches the true horizon, is simply $Q_{AB} = \frac{2}{r} \gamma_{AB}$. Finally, the linearized $Q_{AB}$ and $Q$ have the most singular terms given by

$$\delta Q_{AB} \sim \frac{1}{\delta^2} \kappa \left( \sigma_{AB} + \frac{\theta}{(D-2)} \gamma_{AB} \right),$$  \hspace{1cm} (33)$$

$$\delta Q \sim \frac{1}{\delta^2} 2k\theta.$$

Notice that the perturbation of the non affine coefficient $\kappa$ can always be gauged away by choosing a suitable parametrization of the horizon. This remark means that we can put $\delta \kappa = 0$. So, without any loss of generality, we set $\kappa$ equal to the surface gravity of the background black-geometry. Now, as an illustration of the perturbation scheme mentioned in the equation (31) and the reason for the definition of the $Q_{ab}$, we use the facts mentioned in the previous paragraph to evaluate a term $K_{AC}Q_{D}^{C}$ contributed by $J_{AD}$ which we encounter in the projection of $t_{ab}$ on the cross-section of $H_s$. We evaluate this term as follows:

$$K_{AC}Q_{D}^{C} \approx \dot{K}_{AC}\dot{Q}_{D}^{C} + \delta K_{AC}\dot{Q}_{D}^{C} + \dot{K}_{AC}\delta Q_{D}^{C}$$

$$= \frac{\delta}{r} \gamma_{AC}\dot{Q}_{D}^{C} + \frac{1}{\delta} \left( \sigma_{AC} + \frac{\theta}{(D-2)} \gamma_{AC} \right) \dot{Q}_{D}^{C} + \frac{\delta}{r} \gamma_{AC}\frac{1}{\delta^2} \kappa \left( \sigma_{D}^{C} + \frac{\theta}{(D-2)} \gamma_{D}^{C} \right)$$

$$\sim \frac{1}{\delta} \left( \sigma_{AC} \dot{Q}_{D}^{C} + \frac{\theta}{(D-2)} \dot{Q}_{AD} + \frac{\kappa}{r} \sigma_{AD} + \frac{\kappa}{r} \frac{\theta}{(D-2)} \gamma_{AD} \right)$$

$$\sim \frac{1}{\delta} \left( \frac{\kappa}{r} \sigma_{AD} + \frac{\theta}{(D-2)} \frac{\kappa}{r} \gamma_{AD} + \frac{\kappa}{r} \frac{\theta}{(D-2)} \gamma_{AD} \right).$$  \hspace{1cm} (35)$$

In the steps above we have dropped the terms which are of $O(1)$ or $O(\delta)$ because after regularization (i.e., multiplying by $\delta$) those terms will make no contribution. In this way one sees that our method of approximation is consistent. At the linear order in perturbation the only divergence that comes up in the membrane stress energy tensor is of $O(\delta^{-1})$ and therefore the whole stress tensor can be regularized simply by multiplying with $\delta$ exactly as in general relativity. The sample calculation above elaborates on how it is done for one particular term. Using the Gauss-Codacci equation and the Raychaudhari equation it can be shown that the projections of the curvature of the $H_s$ on the cross-section of $H_s$ is equal to the curvature of the cross-section in the limit $\delta \to 0$.

Before writing down the stress tensor there is one more restriction that we are going to put on the background geometry. We require that the cross-section of the horizon of the background geometry be a
space of constant curvature, i.e., 
\[ (D-2) \hat{R}_{ABCD} = c (\gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC}), \]
where \( c \) is a constant of dimension \( \text{length}^{-2} \) which is related to the intrinsic Ricci scalar of the horizon cross-section as
\[ c = \frac{(D-2) \hat{R}}{(D-3)(D-2)}. \]  
(36)

This assumption regarding the intrinsic geometry of the horizon cross-section is necessary for the stress tensor to be of the form of an isotropic viscous fluid. Using these observations and approximations the contribution of different terms in the membrane stress energy tensor due to the GB term in the action are obtained as:
\[ K_{mn} \hat{\rho}^{mn} \gamma^{a} = \frac{1}{\delta} \sigma_{AB} \frac{(D-3)(D-4)}{2} \eta + \frac{1}{\delta} \gamma_{AB} \frac{(D-3)(D-4)}{2} \zeta \theta_{AB}, \]
(37)

All the steps required to obtain the regularized membrane stress tensor are laid out now. Our perturbative strategy and the restriction that the horizon’s cross-section be the space of constant curvature then yields the membrane stress tensor as
\[ \tilde{t}^{(H)}_{\gamma AB} = p \gamma_{AB} - 2\eta \sigma_{AB} - \zeta \theta_{AB}, \]  
(38)

where,
\[ p = 2\kappa + 4\alpha' \kappa (D-4)(D-3), \]  
(39a)
\[ \eta = 1 - 4\alpha' \left[ \frac{(D-5)(D-4)}{2} + (D-4) \frac{\kappa}{r} \right], \]  
(39b)
\[ \zeta = -2 \left[ \frac{(D-3)(D-4)}{(D-2)} + 4\alpha' \left( \frac{(D-5)(D-4)(D-3)}{(D-2)} + \frac{(D-4)(D-3)}{(D-2)} \right) \frac{2\kappa}{r} \right]. \]  
(39c)

We stress that while we are working only at the linear order in the perturbations, the \( \alpha' \) corrections in the transport coefficients given in equations (39) are non-perturbative. This simply means that the theory is exactly the EGB theory and the background spacetimes of interest are the exact static solutions of this theory. We are perturbing these backgrounds slightly and therefore we care about only the linear order terms in the perturbations. This approach differs from some of the other work in the literature, see for example [26], where one considers the effect of the GB term in the action as a small perturbation of general relativity. Also, note that the bulk viscosity \( \zeta \) and shear viscosity \( \eta \) of the membrane fluid are related as,
\[ \zeta = -2 \frac{(D-3)}{(D-2)} \eta. \]  
(40)

This relationship holds in general relativity and interestingly enough it survives in the EGB gravity. In fact, this relationship is also true for any gravity theory with a Lagrangian which is an arbitrary function of the Ricci scalar [23] and it shows that the bulk viscosity of the membrane fluid is always negative as long as the shear viscosity is positive.
The energy density of the fluid is given by regularizing (i.e., multiplying by $\delta$) the component of $t_{ab}$ along fiducial observers,

$$\rho = -2\theta - 4\alpha'\theta [(D - 4)(D - 3)c]. \quad (41)$$

The continuity equation (12) yields the equation describing the evolution of the energy density along the null generators of the horizon. Note that we are actually applying the linearized continuity equation and in order to derive the evolution equation we have to keep the induced metric on the stretched horizon fixed. As in the case of general relativity, we first write down the equation on the stretched horizon keeping the terms which are linear in perturbations and have $O(\delta^{-1})$ divergence in $t_{ab}$ which gives $O(\delta^{-2})$ divergence in the continuity equation. This is regularized by multiplying the whole equation by $\delta^2$. This again has the form as that in the equation (15) with $p$, $\eta$ and $\zeta$ now given by the equations (39).

## 5 Specific background geometries

In this section we will study some specific static background geometries. We will calculate the transport coefficients and the ratio $\eta/s$ for the membrane fluid.

### 5.1 Black brane background

The D-dimensional black brane solution of EGB gravity is given by [16]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 \left(\sum_{i=1}^{D-2} dx_i^2\right), \quad (42)$$

where the function $f(r)$ is given by

$$f(r) = \frac{r^2}{2\lambda} \left[1 - \sqrt{1 - \frac{4\lambda}{r^2} \left(1 - \left(\frac{r_h}{r}\right)^{D-1}\right)}\right]. \quad (43)$$

with $\lambda$ defined in terms of the Gauss-Bonnet coupling $\alpha'$ as $\lambda = (D - 3)(D - 4)\alpha'$. The location of the horizon is denoted by $r_h$ in terms of which, the Hawking temperature and the entropy density (in the units $16\pi G = 1$) is given by

$$T = \frac{(D - 1)r_h}{4\pi l^2}, \quad (44)$$

$$s = 4\pi. \quad (45)$$

Note that the intrinsic curvature of the cross-section of the planar horizon is zero, i.e., $c = 0$. Also, the entropy density of the planar horizon in EGB theory is same as that in general relativity [12]. From equation (11), the energy density for the fluid is just $-2\theta$ which is same as that in general relativity, i.e., there is no correction in the energy density due to the Gauss-Bonnet coupling. The transport coefficients for the fluid description of the horizon can now be obtained from the equations (39) as,

$$p = (D - 1)\frac{r_h}{l^2}, \quad (46)$$

$$\eta = 1 - 2\frac{(D - 1)}{(D - 3)} \frac{\lambda}{l^2}, \quad (47)$$

$$\zeta = -2\frac{(D - 3)}{(D - 2)} + 4\frac{(D - 1)}{(D - 2)} \frac{\lambda}{l^2}. \quad (48)$$

---

Note that our notation is different from the one in [16]. We define the entropy density as the entropy per unit cross-section area of the horizon. For planar black holes the total entropy is infinite because the horizon cross-section has an infinite area, but the entropy density is well defined.
The dimensionless ratio \( \eta/s \) is
\[
\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 - \frac{2(D-1)}{(D-3)} \frac{\lambda}{l^2} \right].
\] (49)

We see that the value of \( \eta/s \) calculated from the membrane paradigm matches the one found by other methods in the literature, see for example [16] where one of the calculations utilizes the Kaluza-Klein reduction to express the transverse metric perturbation in \( D \)-dimensions as the vector potential in \( (D-1) \)-dimensions and then uses the membrane paradigm results for the electromagnetic interaction of the black hole in \( (D-1) \)-dimensions. It is evident that the KSS bound is violated for any \( \lambda > 0 \). Also, the bulk viscosity becomes positive when \( \lambda/l^2 > (D-3)/(2(D-1)) \) and for the same range of \( \lambda \) the shear viscosity is negative. In fact, as pointed out in [16] and [6], the gravitons in the theory become strongly coupled as \( \lambda/l^2 \to (D-3)/(2(D-1)) \). For \( \lambda/l^2 > (D-3)/(2(D-1)) \) the theory becomes unstable.

5.2 Boulware-Deser black hole background

The Boulware-Deser black hole [10, 11] is a spherically symmetric, asymptotically flat solution of the vacuum EGB. It is a natural generalization of the Schwarzschild spacetime in general relativity. The metric is given by
\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2,
\] (50)
where \( d\Omega^2 \) is the metric on a \( (D-2) \)-dimensional sphere and
\[
f(r) = 1 + \frac{r^2}{2\lambda} \left[ 1 - \sqrt{1 + \frac{4\lambda M}{r^{D-1}}} \right].
\] (51)

The constant \( M \) is proportional to the ADM mass of the spacetime. This spacetime is asymptotically flat and reduces to the \( D \)-dimensional Schwarzschild solution when \( \alpha' \to 0 \). The location of the horizon is obtained from the solution of the equation,
\[
r_h^{D-3} + \lambda r_h^{D-5} = M.
\] (52)

The surface gravity of the horizon is obtained as \( \kappa = f'(r_h)/2 \) which gives the Hawking temperature of the horizon as [11]
\[
T = \frac{(D-3)}{4\pi r_h} \left[ \frac{r_h^2 + (D-5)}{(D-3)} \frac{\lambda}{(r_h^2 + 2\lambda)} \right].
\] (53)

For Boulware-Deser black hole, the entropy associated with the horizon is no longer proportional to the area \( A \) but is given by (in the units \( 16\pi G = 1 \)) [11],
\[
S = 4\pi A \left[ 1 + \frac{(D-2)}{(D-4)} \frac{\lambda}{r_h^2} \right].
\] (54)

We can now evaluate the transport coefficients of the membrane fluid when the background is chosen as the \( D \)-dimensional Boulware-Deser black hole. Note that, unlike the black brane case, here the curvature of the

\footnote{The spherically symmetric solution of Einstein-Gauss Bonnet gravity in vacuum admits two different branches. For our purpose, we are only interested in the branch which has a general relativity limit when \( \lambda \to 0 \).}
The Hawking temperature associated with this horizon is non-zero and will contribute to the transport coefficients. Using equation (39), these coefficients are calculated to be

\[ p = \frac{(D-3) + (D-5)}{r_h^2} \] (55)

\[ \eta = 1 - \frac{4}{(D-3)(r_h^2 + 2\lambda)} + \frac{2(D-5)}{(D-3)(r_h^2 + 2\lambda)} \lambda^2 \] (56)

\[ \zeta = -2\frac{(D-3)r_h^2}{(D-2)(r_h^2 + 2\lambda)} - \frac{4(D-5)}{(D-2)(r_h^2 + 2\lambda)} - \frac{4(D-5)}{(D-2)(r_h^2 + 2\lambda)} \lambda^2 \] (57)

The position of the horizon is determined by the equation,

\[ \lambda > \frac{2}{D-3} \] (58)

The ratio of the shear viscosity to the entropy density is

\[ \frac{\eta}{s} = \frac{(D-4)}{4\pi(D-3)(r_h^2 + 2\lambda)} \left[ \frac{(D-3)r_h^2 + 2\lambda(D-5)r_h^2 + 2\lambda^2(D-5)}{(D-4)r_h^2 + 2(D-2)\lambda} \right]. \] (59)

For \( \lambda = 0 \) this reduces to the familiar result found in general relativity in equation (18). The ratio violates the KSS bound for any \( \lambda > 0 \).

In the particular case of \( D = 5 \), the transport coefficients and the ratio \( \eta/s \) are

\[ p = \frac{2}{r_h}, \quad \eta = \frac{r_h^2}{(r_h^2 + 2\lambda)}, \quad \text{and} \quad \zeta = -\frac{4r_h^2}{3(r_h^2 + 2\lambda)}, \] (60)

\[ \frac{\eta}{s} = \frac{r_h^4}{4\pi(r_h^2 + 2\lambda)(r_h^2 + 6\lambda)}. \] (61)

5.3 Boulware-Deser-AdS black hole background

The spherically symmetric solution of EGB gravity with a negative cosmological constant \( \Lambda = -(D-1)(D-2)/2l^2 \) is given by [27],

\[ f(r) = 1 + \frac{r^2}{2\lambda} \left[ 1 - \sqrt{1 + \frac{4\lambda M}{r^D-1} - \frac{4\lambda l^2}{r^D}} \right]. \] (62)

The position of the horizon is determined by the equation,

\[ r_h^{D-3} + \lambda r_h^{D-5} + \frac{r_h^{D-1}}{l^2} = M. \] (63)

The Hawking temperature associated with this horizon is

\[ T = \frac{1}{4\pi r_h l^2 (r_h^2 + 2\lambda)} \left[ (D-1)r_h^4 + (D-3)r_h^2 l^2 + (D-5)\lambda l^2 \right]. \] (64)

Using equations (59), the various transport coefficients are obtained as

\[ p = \frac{(D-3)}{r_h} + \frac{(D-1)}{l^2} + \frac{(D-5)}{r_h^2} \lambda, \] (65)

\[ \eta = 1 - \frac{4}{(D-3)(r_h^2 + 2\lambda)} - \frac{2(D-1)}{(D-3)(r_h^2 + 2\lambda)} \lambda r_h^2 + \frac{2(D-5)}{(D-3)(r_h^2 + 2\lambda)} \lambda^2 \] (66)

\[ \zeta = -\frac{4(D-3)r_h^2}{(D-2)(r_h^2 + 2\lambda)} - \frac{4(D-5)}{(D-2)(r_h^2 + 2\lambda)} - \frac{4(D-1)}{(D-2)(r_h^2 + 2\lambda)} \] (67)
Note that in the limit $\Lambda \to 0$, i.e., $l \to \infty$, these coefficients reduce to those for the vacuum case discussed in the section 5.2.

The ratio of shear viscosity and entropy density is,

$$\frac{\eta}{s} = \frac{(D-4)}{4\pi(D-3)l^2(r_h^2 + 2\lambda)} \left[ \frac{(D-3)l^2r_h^4 + 2\lambda(D-5)l^2r_h^2 + 2\lambda^2l^2(D-5) - 2\lambda(D-1)r_h^2}{(D-4)r_h^2 + 2(D-2)\lambda} \right].$$

(67)

In particular, for $D = 5$ the ratio is

$$\frac{\eta}{s} = \frac{r_h^4}{4\pi (r_h^2 + 2\lambda)(r_h^2 + 6\lambda)} \left( 1 - \frac{4\lambda}{l^2} \right).$$

(68)

The ratio is positive and violates the KSS bound as long as $0 < \lambda < l^2/4$.

The result in equation (68) can be viewed as a prediction that the long wavelength hydrodynamic limit of the dual CFT living on the boundary $S^3 \times R$ has the ratio $\eta/s$ given by (68). Also, in the limit of high black hole temperature (which corresponds to the limit $r_h \to \infty$), the ratio $\eta/s$ for Boulware-Deser-AdS black hole reduces to the ratio for the black brane (49).

6 Summary and Discussion

In this paper we have proposed a perturbative scheme to derive the membrane stress tensor and the transport coefficients for the membrane fluid in Einstein-Gauss-Bonnet gravity. We used the action principle formalism to determine the membrane stress tensor on the stretched horizon. Our derivation is slightly different from the one given in [17], since we include the Gibbons-Hawking boundary term at infinity as well as on the stretched horizon. As a result, all the contribution from the bulk part under the variation with respect to $\omega$ vanishes automatically. (In [17] it was argued that without the Gibbons-Hawking term, these contributions vanish in the limit when the stretched horizon approaches the true horizon.) Our method can be easily generalized to the higher order Lovelock theories.

The membrane stress tensor in EGB gravity has terms cubic in the extrinsic curvature and in the limit that the stretched horizon approaches the true horizon these terms are cubically divergent in $\delta^{-1}$. In order to tame these divergences we studied only the perturbations about the static black geometries. We found that restricting to the linear order in perturbations on the stretched horizon, the membrane stress tensor is in fact only linearly divergent. Therefore, at the linear order the divergence structure of the GB contribution to the stress tensor is identical to that of the Einstein contribution. Hence the whole membrane stress tensor could be regularized in the same way as in general relativity: simply absorb one power of the divergent factor in the definition of the stress tensor. We expect that this method can be generalized to the higher order Lovelock theories. We also believe that our method to obtain the evolution equation for the energy density is particularly easy to adopt to the Lovelock theories because it avoids manipulating the equation of motion in order to write the curvature term in the Raychaudhuri equation in terms of the energy-momentum tensor of the matter.

In order to write the membrane stress tensor in the form of an isotropic viscous fluid we had to restrict the background geometries to those which have a constant curvature horizon cross-section. This looks like a severe restriction, and it should be possible to bypass this requirement by modelling the horizon as an anisotropic fluid with tensorial transport coefficients. But even with this restriction we are able to study interesting cases discussed in section 5. Nevertheless, it would be interesting to study the more general case by lifting this restriction and study backgrounds with non-constant curvature horizon cross-section.

The transport coefficients of the membrane fluid for the horizon in EGB gravity are given in equations (39). We notice a relation between the shear viscosity and the bulk viscosity,

$$\zeta = -\frac{2(D-3)}{(D-2)}\eta.$$
which says that if the shear viscosity is positive then the bulk viscosity is negative and vice-versa. The negative value of $\zeta$ of the membrane fluid is related to the teleological nature of the horizon. This relation also gives a consistency check that the allowed range of the GB coupling for the fluid description to make sense is the same whether calculated from the positivity of $\eta$ or the negativity of $\zeta$. It is quite possible that the same relationship between these two transport coefficients is true for the higher order Lovelock theories and is worth investigating.

Some particular static spacetimes are analyzed in section 5. For the black brane solution the EGB gravity, the membrane paradigm calculation gives a value of $\eta/s$ which agrees with that found in the literature, see for example [16] which calculates the ratio from both linear response theory and a Kaluza-Klein compactified version of the membrane paradigm. The KSS bound is violated for any value of positive GB coupling. From the other cases we have studied it appears that the violation of the KSS bound is a generic feature of the EGB gravity except for the case of an extremal ($\kappa = 0$) black brane solution, where all the GB corrections vanish.

From the purely classical point of view via the membrane paradigm and an input from black hole thermodynamics, we have seen that in general relativity the values of viscosity and entropy density are constants independent of the solution. In particular, they are the same for the black brane and the Boulware-Deser-AdS black hole. When the $\alpha'$ corrections are taken into account, the entropy density for the black brane does not change (because the horizon is flat), but the entropy density for the black hole increases (see [34]). Also, the viscosity of both the black brane ([17] and the black hole ([55] decreases as $\alpha'$ is turned on. Consequently, in both cases $\alpha'$ affects the ratio $\eta/s$ and causes it to fall below $1/4\pi$, the value in general relativity.

Let us try to understand the reason for the difference between the black brane and black hole entropy densities and shear viscosity from a dual CFT viewpoint. If a dual CFT exists, it is not the same one that is dual to Type IIB string theory, since in that case the leading correction to the gravitational action is an $O(\alpha'^3 R^4)$ term [28], not an $O(\alpha')$ Gauss-Bonnet term. But let us suppose that some dual CFT does exist. The black hole in the bulk is then dual to a finite temperature CFT on the conformal boundary $S^3 \times R$ of the bulk, and the black brane in the bulk is dual to the same CFT on $R^4 \subset S^3 \times R$ and is thermal with respect to a different conformal Killing generator.

The $\alpha'$ corrections to entropy density and shear viscosity presumably correspond to the corrections due to finite 't Hooft coupling (or something analogous) in the dual CFT. In the limit of infinite 't Hooft coupling $\lambda_t$ (i.e., when the holographic dual to the boundary theory is just classical general relativity), the hydrodynamic limit of the boundary theory has the same entropy density and viscosity for both the black hole and the black brane duals. When the effects due to finite $\lambda_t$ are taken into account they become different. Evidently the corrections due to finite 't Hooft coupling must be sensitive to whether the dual CFT is on $S^3 \times R$ or $R^4$. Why might that be?

First note that even with infinite 't Hooft coupling, there is a difference, in that on $S^3 \times R$ there is a critical temperature, set by the radius of the $S^3$, below which the CFT is confining and is dual to AdS without a black hole [29]. However, once above the critical temperature, the leading order entropy density and viscosity are strictly identical to those on $R^4$. Perhaps we may interpret this as being due to an infinitely sharp phase transition in the presence of an infinite 't Hooft coupling.

Now consider the situation at finite 't Hooft coupling. When $r_h \gg l$, the black hole temperature [63] is $T \sim r_h/l^2 \gg 1/l$. (Here we use the fact that for a stable theory we must have $\lambda < l^2/4$, so also $r_h \gg \sqrt{\lambda}$.) In this high temperature regime, $s$ and $\eta$ become indistinguishable from their values for the black brane. In the dual CFT, this is to be expected since the thermal wavelength is much smaller than the size $\sim l$ of the $S^3$. Away from this limit, the thermal state on $S^3 \times R$ is different, in its local properties, from the thermal state dual to the black brane.

It remains somewhat puzzling however, from the dual CFT viewpoint, why there would be no $\alpha'$ correction to the entropy density of the black brane, given that there is such a correction to its viscosity. In fact, in the case of the super Yang-Mills theory dual to Type IIB string theory, there is indeed an $\alpha'^3$ correction to the black brane entropy, seen from an $O(\alpha'^3 R^3)$ correction to the gravitational action [28]. While it has not been computed in the dual CFT, the duality asserts that this correction arises from a finite 't Hooft coupling effect in the CFT. By contrast, the absence of an $\alpha'$ correction to the black brane entropy in EGB theory
suggests that, if there indeed is a field theory dual to EGB gravity, for some reason it has no $O(\alpha')$, finite ‘t Hooft coupling correction to the thermal entropy on $R^4$.

Let us spell this out a little more explicitly. To obtain the CFT entropy density from the horizon entropy density one must multiply by $(r_h/l)^{D-2}$, the ratio of horizon ‘area’ to spatial volume ($\int \Pi_{i=1}^{D-2} dx_i$) of the CFT. Although the black brane metric (43) receives corrections from $\alpha'$, the ratio $r_h/l = 4\pi TL/(D-1)$ in equation (44) is independent of $\alpha'$ when expressed in terms of $T$. Thus, for $D = 5$, the entropy density of the CFT is $\propto (TL)^3(l_P)^{-3} = T^3(l/l_P)^3$, where $l_P$ is the 5-dimensional Planck length. For the black hole, by contrast, equation (63) shows that $r_h/l$ depends on $\alpha'$ at fixed temperature, and also the horizon entropy receives an $\alpha'$ correction. Hence, the entropy density in the CFT must also receive an $\alpha'$ correction. This discrepancy is puzzling, since although the finite size of the $S^3$ could naturally affect the temperature dependence of the entropy at low temperature, it would seem that the mere presence or absence of finite coupling effects in the CFT entropy density should be independent of the finite size of the $S^3$.

Finally, we remind the reader that our results are derived for the linear-order perturbations of the static background geometries where the cross-section of the horizon is a space of constant curvature, and we have evaluated only the first-order transport coefficients keeping only the terms of first-order in derivatives of velocity. It will be interesting to generalize this procedure to calculate the higher-order transport coefficients.

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# Appendix: Table of important symbols and their meanings

$D$ denotes the spacetime dimension. The metric signature is $(-,+,+,...)$.

| Symbol | Meaning |
|--------|---------|
| $H$   | True horizon, a $(D-1)$-dimensional null-hypersurface |
| $\mathcal{H}_s$ | Stretched horizon, a $(D-1)$-dimensional time-like hypersurface with tangent $u^a$ and normal $n^a$ |
| $(a, b, c, ...)$ | Spacetime indices |
| $(A, B, C, ...)$ | Indices on the $(D-2)$-dimensional cross-section of the true/stretched horizon |
| $l^a$ | Null generator of the true horizon parametrized by a non-affine parameter. Obeys the geodesic equation: $l^a \nabla_a l^b = \kappa l^b$ |
| $h_{ab}$ | Induced metric on the stretched horizon |
| $\gamma_{ab}$ | Induced metric on the cross-section of the stretched horizon, which in the null limit is identified with the metric on the cross-section of the true horizon |
| $K_{ab}$ | Extrinsic curvature of the stretched horizon defined as, $K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab}$ |
| $k_{AB}$ | Extrinsic curvature of the cross-section of the true horizon defined as $k_{AB} = \frac{1}{2} \mathcal{L}^a l^b \gamma_{AB}$ |
| $\theta, \theta_s$ | Expansion of the true/stretched horizon |
| $\sigma^{ab}, \sigma^a_{ab}$ | Shear of the true/stretched horizon |
| $\tilde{R}_{abcd}$ | Riemann tensor intrinsic to the stretched horizon |
| $(D-2)\tilde{R}_{ABCD}$ | Intrinsic Riemann tensor of the $(D-2)$-dimensional cross-section of the stretched horizon in the background geometry, which in the null limit is identified with the intrinsic Riemann tensor of the cross-section of the true horizon |
| $(D-2)\tilde{R}$ | Intrinsic Ricci scalar of the $(D-2)$-dimensional cross-section of the stretched horizon in the background geometry, which in the null limit is identified with the intrinsic Ricci scalar of the cross-section of the true horizon |
| $\delta$ | The parameter which measures the deviation of the stretched horizon from the true horizon |
| $c$ | Defined as $c = \frac{(D-2)\tilde{R}}{(D-3)(D-2)}$ (For the planar horizon, $c = 0$) |
| $\alpha'$ | Gauss-Bonnet coupling constant |
| $\lambda$ | Constant of dimension $\text{length}^2$ related to the Gauss-Bonnet coupling $\alpha'$ as $\lambda = (D-3)(D-4)\alpha'$ |
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