On Generalized Integrals and Ramanujan-Jacobi Special Functions

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Abstract

In this article we consider new generalized functions for evaluating integrals and roots of functions. The construction of these generalized functions is based on Rogers-Ramanujan continued fraction, the Ramanujan-Dedekind eta, the elliptic singular modulus and other similar functions. We also provide modular equations of these new generalized functions and remark some interesting properties.

1 Introduction

Let

\[ \eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \]  

(1)

denotes the Dedekind eta function which is defined in the upper half complex plane. It not defined for real \( \tau \).

Let for \(|q| < 1\) the Ramanujan eta function be

\[ f(-q) = \prod_{n=1}^{\infty} (1 - q^n). \]  

(2)

The following evaluation holds (see [12]):

\[ f(-q) = 2^{1/3} \pi^{-1/2} q^{-1/24} k^{1/12} k^*^{1/3} K(k)^{1/2}, \]  

(3)

where \( k = k_r \) is the elliptic singular modulus, \( k^* = \sqrt{1 - k^2} \) and \( K(x) \) is the complete elliptic integral of the first kind.

The Rogers-Ramanujan continued fraction is (see [8],[9],[14]):

\[ R(q) := \frac{q^{1/5}}{1 + \frac{q^1}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}, \]  

(4)
which have first derivative (see [5]):
\[ R'(q) = 5^{-1} q^{-5/6} f(-q)^4 R(q) \sqrt[6]{R(q)^5 - 11 - R(q)^5} \] (5)

and also we can write
\[ \frac{dR(q)}{dk} = 5^{-1} \cdot 2^{1/3} (k^*)^{-2/3} R(q) \sqrt[6]{R(q)^5 - 11 - R(q)^5}. \] (6)

Ramanujan have proven that
\[ R(q)^{-5} - 11 - R(q)^5 = \frac{f(-q)^6}{q f(-q^5)^6}. \] (7)

Also holds the following interesting identity
\[ \frac{dk}{dq} = -\frac{2k(k^*)^2 K(k)^2}{q^2 \pi^2}, \] (8)

which is a result of Ramanujan (see [7] and [8]) for the first derivative of \( k = k_r \) with respect to \( q = e^{-\pi \sqrt{r}} \), \( r > 0 \).

2 Propositions

Definition 1.
For any smooth function \( G \), we define \( m_G(A) \) to be such that
\[ A = \pi \int_{\sqrt{m_G(A)}}^{+\infty} \eta \left( \frac{it}{2} \right)^4 G\left( R(e^{-\pi t}) \right) dt. \] (9)

Theorem 1.
If
\[ 5 \int_{0}^{y} \frac{G(t)}{t^{1/5} - 11 - t^5} dt = A, \] (10)
then
\[ y = R\left( e^{-\pi \sqrt{m_G(A)}} \right). \] (11)

Proof.
From [5] it is known that if \( a, b \in (0, 1) \) then
\[ \int_{b}^{a} f(-q)^4 q^{-5/6} G(R(q)) dq = 5 \int_{R(b)}^{R(a)} \frac{G(x)}{x^{1/5} - 11 - x^5} dx, \] (12)

which is equivalent to write
\[ -\pi \int_{b_1}^{a_1} f(-e^{-t})^4 e^{-t \pi / 6} G\left( R(e^{-\pi t}) \right) dt = 5 \int_{R(e^{-\pi t})}^{R(e^{-\pi t_1})} \frac{G(x)}{x^{1/5} - 11 - x^5} dx. \]
Setting $b_1 = +\infty$, $a_1 = \sqrt{m_G(A)}$ and then using Definition 1 and the Dedekind eta expansion (1) we get the result.

Also differentiating (8) one can get
\[
\frac{dm_G^{(-1)}(r)}{dr} = -\frac{\pi\eta((\sqrt{r}/2)^4 G(R(q)))}{2\sqrt{r}}, q = e^{-\pi\sqrt{r}}, r > 0
\]  
(13)

and if
\[
\phi(r) := -\frac{2\sqrt{r}}{\pi\eta((\sqrt{r}/2)^4 G(R(q)))},
\]  
(14)

then

**Proposition 1.**

If $A > 0$, then
\[
\frac{d}{dA} m_G(A) = \phi(m_G(A)).
\]  
(15)

Derivating (10) with respect to $q_{m_G} = e^{-\pi\sqrt{m_G}}$ we get
\[
\frac{5G(R(q_{m_G}))}{R(q_{m_G})\sqrt{R(q_{m_G})}^{5/2} - 11 - R(q_{m_G})^{5/2}} \frac{dR(q_{m_G})}{dq_{m_G}} = \frac{dA}{dk_{m_G}} \frac{1}{dq_{m_G}},
\]

or the equivalent, using (8), (5) and (3):

**Theorem 2.**

\[
\frac{dA}{dk_{m_G}} = 3\sqrt{2} G(y(A)) \left(\frac{k_{m_G}}{k_{m_G}^*}\right)^{2/3}.
\]  
(16)

Also with integration of (16) we have
\[
3\sqrt{2} \kappa_{m_G} \cdot 2F_1\left[ \begin{array}{c} 1 \\dfrac{1}{3} \dfrac{7}{6} \dfrac{k_{m_G}^2}{6} \end{array} \right] = \int_0^A \frac{1}{G\left( R\left( e^{-\pi\sqrt{m_G(t)}} \right) \right)} dt.
\]  
(17)

Hence

**Definition 2.**

We define the function $h$ as
\[
x = \int_0^{h(x)} \frac{dt}{G\left( R\left( e^{-\pi\sqrt{m_G(t)}} \right) \right)} = \int_0^{h(x)} \frac{dt}{G\left( y(t) \right)}.
\]  
(18)
Theorem 3.
Set $m_G(A) = r$, then

$$A = h \left( 3 \sqrt{2k_r \cdot 2F_1}\left[ \frac{1}{6}, \frac{1}{3}; \frac{7}{6}; k_r^2 \right] \right), \quad (19)$$

or better

$$m_G^{(-1)}(r) = h \left( 3 \sqrt{2k_r \cdot 2F_1}\left[ \frac{1}{6}, \frac{1}{3}; \frac{7}{6}; k_r^2 \right] \right). \quad (20)$$

From relation (10) differentiating we have

$$5 \frac{G(y(A))y'(A)}{y(A) \sqrt{y(A)^{-5} - 11 - y(A)^6}} = 1.$$  

Hence

$$5 \int_0^y(A) \frac{dt}{t \sqrt{t^{-5} - 11 - t^5}} = \int_0^A \frac{dt}{G(y(t))} = b_{m_G}, \quad (21)$$

where

$$b_r = 3 \sqrt{2k_r \cdot 2F_1}\left[ \frac{1}{6}, \frac{1}{3}; \frac{7}{6}; k_r^2 \right]. \quad (22)$$

Also if $m(x)$ is the function defined as (see [13])

$$\pi \int_{\sqrt{m(r)}}^{+\infty} \eta(it/2)\eta dt = r. \quad (23)$$

Then we have also

$$b^{(-1)}(r) = m(r). \quad (24)$$

Hence

$$y(A) = R \left( e^{-\pi \sqrt{m_G}} \right) = F_1(b_{m_G}), \quad (25)$$

where we have define $F_1$ by

$$x = 5 \int_0^{F_1(x)} \frac{dt}{t \sqrt{t^{-5} - 11 - t^5}}. \quad (26)$$

Theorem 4.

$$5 \int_0^A \frac{G(t)dt}{t \sqrt{t^{-5} - 11 - t^5}} = h \left( 5 \int_0^A \frac{dt}{t \sqrt{t^{-5} - 11 - t^5}} \right). \quad (27)$$

Proof.
From relations (21) and (19) and (10) we have

$$h \left( 5 \int_0^y(A) \frac{dt}{t \sqrt{t^{-5} - 11 - t^5}} \right) = h(b_{m_G}) = A = 5 \int_0^y(A) \frac{G(t)dt}{t \sqrt{t^{-5} - 11 - t^5}}. \quad (28)$$
which by inversion of \( y(A) \) we get the desired result.

Also from relation (20) is

\[
m_G^{(-1)}(A) = h(b_A). \tag{29}
\]

Hence knowing the function \( h \) we know almost everything. For this, for a given function \( P(x) \) we can set

\[
G(x) = 5^{-1}xP'(x)\sqrt[5]{x^{-5} - 11 - x^5}, \tag{30}
\]

then

\[
m_G^{(-1)}(A) = P\left(R\left(e^{-\pi\sqrt{A}}\right)\right) = h(b_A). \tag{31}
\]

Hence inverting \( b_A \) we get the function \( h \). Note also that \( P(A) = y^{(-1)}(A) \) and

\[
m_G(A) = v_1(y(A)), \tag{32}
\]

where \( v_1(x) \) is the inverse function of \( R(e^{-\pi\sqrt{x}}) \).

**Remarks.**

1) Note by definition that when we know \( G \) the \( m_G^{(-1)}(A) \) is a closed form formula and \( m_G(A) \) is not (it needs inversion).

2) Continuing from relation (30) and \( y(x) = P^{(-1)}(x) \) we have

\[
G(y(x)) = 5^{-1}P^{(-1)}(x)P'(P^{(-1)}(x))\sqrt[5]{(P^{(-1)}(x))^{-5} - 11 - (P^{(-1)}(x))^5}. \tag{33}
\]

From equation (28) we get also

**Corollary 1.**

\[
h^{(-1)}(A) = 5 \int_0^{y(A)} \frac{dt}{t\sqrt[5]{t^{-5} - 11 - t^5}}. \tag{33}
\]

**Corollary 2.**

\[
y'(A) = 5^{-1}h'_1(A)y(A)\sqrt[5]{y(A)^{-5} - 11 - y(A)^6}, \tag{34}
\]

with

\[
h_1(A) = h^{(-1)}(A) = b_{m_G}(A) = (b \circ m_G)(A) \tag{35}
\]

and also

\[
\frac{dy(A)}{dk_{m_G}} = 5^{-1} \cdot 2^{1/3} \left(k_{m_G}\right)^{-2/3} \left(5^{-1}2^{-1/3}y(A)\sqrt[5]{y(A)^{-5} - 11 - y(A)^6} \right)^{2/3}. \tag{36}
\]

Using the above one can show with differentiation the following
Theorem 5.

\[ h_1(A) = \pi \int_{\sqrt{mG(A)}}^{\infty} \frac{\eta(it/2)^4}{t^5 - 11 - t^5} \, dt = b_{mG(A)} = 5 \int_{0}^{\eta(A)} \frac{dt}{t^5 - 11 - t^5}. \] (37)

Also another interesting theorem arises from the definition of \( h(x) \). By setting

\[ h_1(t) := \left( \frac{1}{h_1'(x)} \right)^{(1)}(t), \] (38)

then

Theorem 6.

\[ 5 \int_{0}^{G_1(x)} \frac{dt}{t^5 - 11 - t^5} = \int_{c}^{x} \frac{h_1'(t)}{t} \, dt. \] (39)

Proof.

From relation (18) we have

\[ G(y(t)) = \frac{1}{h_1'(t)}, \]

hence inverting

\[ y_i(G_i(t)) = h_1(t). \]

Taking the derivatives in both parts we get

\[ 5 \frac{G_1'(t)}{G_i(t)^5/5 - 11 - G_i(t)^5} = \frac{h_1'(t)}{t}. \]

Lastly integrating the above relation in both parts we get the result.

Equation (39) can also written in the next form using (26)

\[ G_i(x) = F_1 \left( \int_{c}^{x} \frac{h_1'(t)}{t} \, dt \right). \] (40)

Suppose the \( n \)-th order modular equation of \( y(A) \) is

\[ P_n(A) = y \left( n \cdot y^{-1}(A) \right). \] (41)

Then we can find \( P_n(x) \) by solving

\[ \int_{0}^{P_n(A)} \frac{G(t)}{t^5 - 11 - t^5} \, dt = n \int_{0}^{A} \frac{G(t)}{t^5 - 11 - t^5} \, dt. \] (42)
with respect to $P_n(A)$. Setting $Q_n(A)$ to be

$$Q_n(A) := m_G \left( n^2 \cdot m_G^{(-1)}(A) \right),$$

the $n$-th degree modular equation of $m_G(A)$ and using (15) we have

$$m_G^{(-1)}(A) = \int_0^A \frac{dt}{\phi(t)}$$

and

$$\int_0^{Q_n(A)} \frac{dt}{\phi(t)} = n \int_0^A \frac{dt}{\phi(t)}.$$ 

Also

$$Q_n^{(-1)}(A) = Q_{1/n}(A)$$

and

$$\int_0^{Q_{nm}(A)} \frac{dt}{\phi(t)} = n \int_0^{Q_m(A)} \frac{dt}{\phi(t)} = nm \int_0^A \frac{dt}{\phi(t)}$$

and

$$\int_0^{Q_n(Q_m(A))} \frac{dt}{\phi(t)} = n \int_0^{Q_m(A)} \frac{dt}{\phi(t)} = nm \int_0^A \frac{dt}{\phi(t)}.$$

Hence

$$\int_0^{Q_n(Q_m(A))} \frac{dt}{\phi(t)} = \int_0^{Q_{nm}(A)} \frac{dt}{\phi(t)}$$

and consequently

$$Q_n(Q_m(A)) = Q_m(Q_n(A)) = Q_{nm}(A),$$  

$$Q_1(A) = A$$ and if $n = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$

$$Q_n(x) = (Q_{p_1} \circ \ldots \circ Q_{p_1}) \circ (Q_{p_2} \circ \ldots \circ Q_{p_2}) \circ \ldots \circ (Q_{p_s} \circ \ldots \circ Q_{p_s}),$$

respectively iterated $a_1, a_2, \ldots, a_s$ times.

Hence

$$y \left( m_G^{(-1)} \circ m_G^{(-1)} \circ Q_n \circ m_G \circ m_G(A) \right) = R \left( e^{-\pi \sqrt{m_G^{(-1)} \circ Q_n \circ m_G \circ m_G(A)}} \right) = R \left( e^{-\pi \sqrt{m_G(A)}} \right) = \Omega_n \left( y(A) \right).$$

But

$$m_G^{(-1)} \circ m_G^{(-1)} \circ Q_n \circ m_G \circ m_G(A) = m_G^{(-1)} \left( n^2 \cdot m_G(A) \right) \cdot m_G(A).$$

By this we lead to the following
Theorem 7.

\[ y (Q^*_n (A)) = \Omega_n (y(A)), \]

where \( \Omega_n (A) \) is the \( n \)-th modular equation of the Rogers-Ramanujan continued fraction and \( Q^*_n (A) \) is the \( n \)-th order modular equation of \( m^{(-1)}_G(A) \) i.e \( Q^*_n (A) = m^{(-1)}_G (n^2 \cdot m_G (A)). \)

Theorem 8.

If we know \( f_0 = y \) and \( y_i = f_0^{(-1)} \), then

\[ m_G (A) = b^{(-1)} \circ F^{(-1)}_1 \circ f_0 (A) \quad : (\nu 1) \]

and for all \( n \) positive real:

\[ m^{(-1)}_G (n^2) = f_0^{(-1)} \circ \Omega_n \circ f_0 \circ m^{(-1)}_G (1) = f_0^{(-1)} \left( R \left( e^{-\pi n} \right) \right), \quad : (\nu 2) \]

where \( n \) can take and positive real values as long as \( \Omega_n (x) = v \left( n^2 \cdot v^{(-1)} (x) \right) \)

and \( v(x) = R \left( e^{-\pi v^{(-1)} (x)} \right), \) i.e. \( v(x) \) is the Rogers-Ramanujan continued fraction.

The constant function \( \Omega_n (x) \) is the \( n \)-th degree modular equation of the \( v(x) \).

Hence knowing \( y(x) \) and \( y^{(-1)} (x) = y_i(x) \) we know respectively \( m_G (x) \) and \( m^{(-1)}_G (x) \).

Examples.

1) If \( f(x) = x^2 + 2x \) then \( f^{(-1)} (x) = -1 + \sqrt{1 + x} \) and

\[ m_G (x) = b^{(-1)} \circ F^{(-1)}_1 (-1 + \sqrt{1 + x}). \]

2) If

\[ G(t) = \frac{t \sqrt{t^5 - 11 - t^5}}{t + 1}, \]

then \( y(x) = -1 + e^{x/5} \) and \( y_i(x) = 5 \log (x + 1) \). Hence exist constant \( c_1 \) such that

\[ m^{(-1)}_G (n^2) = 5 \log (1 + \Omega_n (c_1)), \forall n > 0 \]

3) If

\[ G(x) = \frac{x \sqrt{x^5 - 11 - x^5}}{5 \sqrt{1 - k \sin(x)^2}}, \]

then

\[ y(x) = E(x, k) \text{ and } y_i(x) = \text{am}(x, k), \]
where am is the Jacobi amplitude i.e the inverse of the incomplete elliptic integral of the first kind $E[x, k]$, with $k$ a parameter. Hence

$$m_G^{(-1)}(4x) = Q_4^* \left( m_G^{(-1)}(x) \right), \quad Q_4^*(x) = E \left[ \omega_2(\text{am}(x, k)), k \right].$$

If $k = 1$

$$m_G^{(-1)}(n^2) = \log \left[ \sec \left( \Omega_n(t) \right) + \tan \left( \Omega_n(t) \right) \right], \quad (48)$$

where

$$t = -\frac{\pi}{2} + 2 \arctan \left( e^{m_G^{(-1)}} \right), \quad (49)$$

for every $n \in \mathbb{R}^*_+$, ($t$ is a constant). Also for $k = 1/2$

$$m_G^{(-1)}(4) = E \left[ \frac{1}{2} \left( -1 - \sqrt{5} + \sqrt{2 \left( 5 + \sqrt{5} \right)} \right), \frac{1}{2} \right]. \quad (50)$$

4) If $n \in \mathbb{N}$ and $y(x) = BR(x)$ (the Bring Radical function), then

$$m_G^{(-1)}(n^2) = (\Omega_n \circ F_1 \circ b_1)^5 + \Omega_n \circ F_1 \circ b_1 \quad (51)$$

and $F_1$ defined from (26).

3 Further transformations

The function $k_i(A)$ is the inverse function of the elliptic singular modulus $k_A = k(A)$. We have

$$k_i(x) = \left( \frac{K \left( \sqrt{1 - x^2} \right)}{K(x)} \right)^2, \quad 0 < x < 1. \quad (51)$$

Also we define

$$Q_G(x) := m_G^{(-1)}(k_i(x)), \quad (52)$$

then

$$m_G^{(-1)}(r) = Q_G(k_r). \quad (53)$$

From (11) we have

$$y \left( Q_G(k_r) \right) = R \left( e^{-\pi \sqrt{r}} \right). \quad (54)$$

We are interested to find an expression for $G$. From (10) we get

$$5 \int_0^{y(Q_G(k_r))} \frac{G(t)}{t \sqrt{t-5} - t^2} dt = Q_G(k_r),$$

or equivalently

$$5 \int_0^{R(q)} \frac{G(t)dt}{t \sqrt{t-5} - t^2} dt = Q_G(k_r) \quad (54)$$
and differentiating the last relation we get
\[ \frac{5 G(R(q)) R'(q)}{R(q) \sqrt{R(q)^{-5} - 11 - R(q)^5}} = Q'_G(k_r) \frac{dk}{dq}. \]

Or, using (3),(5),(8) we arrive to
\[ Q'_G(k_r) = \frac{2^{1/3}}{(k_r k_r^*)^{2/3}} G(R(q)). \quad (55) \]

But from the fact that Rogers-Ramanujan’s continued fraction is algebraic function of elliptic singular moduli we have \( R(q) = F(k_r) \). Hence
\[ Q'_G(A) = \frac{2^{1/3}}{(A \sqrt{1 - A^2})^{2/3}} G(F(A)). \quad (56) \]

Inverting \( F \) we get
\[ G(A) = 2^{-1/3} \left( F_i(A) \sqrt{1 - F_i(A)^2} \right)^{2/3} Q'_G(F_i(A)). \quad (57) \]

The study of \( F(A) \) has to reveal some interesting properties of the general function
\[ y_s(A) = R \left( e^{-\pi \sqrt{k_i(s(A))}} \right), \quad (58) \]
where \( s(A) \) is “arbitrary” function. Along with \( s(A) \), we attach the function \( \sigma(A) \), which satisfies the condition \( s'(A) = \sigma(s(A)) \) and
\[ m_G(A) = k_i(s(A)). \quad (59) \]

It holds from the definition of \( y_s \) and Theorem 5 and (22):
\[ \frac{1}{G_s(y_s(A))} = \frac{1}{G(y(A))} = h'_i(A) = \frac{d}{dA} \left( 3 \sqrt{2s(A)} \cdot 2F_1 \left[ \frac{1}{3}, \frac{1}{6}, \frac{7}{6}; s(A)^2 \right] \right) = \frac{2^{1/3}}{s(A)^{2/3} (1 - s(A)^2)^{1/3}} s'(A). \quad (60) \]

Hence
\[ \frac{1}{G \left( R \left( e^{-\pi \sqrt{k_i(A)}} \right) \right)} = \frac{2^{1/3}}{A^{2/3} (1 - A^2)^{1/3}} \sigma(A). \quad (61) \]

Inverting the \( F(x) \) function (see and Appendix) we get

**Theorem 9.**
\[ Q(A) = s^{(-1)}(A) \quad (62) \]

and
\[ G(A) = G_s(A) = \frac{2^{-1/3} \left( F_i(A) \sqrt{1 - F_i(A)^2} \right)^{2/3}}{\sigma(F_i(A))}. \quad (63) \]
Theorem 10.
Suppose $G_0(x)$ is that of (155),(65) below and $F(x) = R \left( e^{-\pi \sqrt{k_i(x)}} \right)$. Both functions are "constant" and algebraic. If
\[
5 \int_0^{y_s(A)} \frac{G_0(t)}{\sigma(F_i(t))} \frac{dt}{t^{3/5} - 11 - t^3} = A,
\]
then $y_s(A) = y(A)$ is that of (58), with $s'(A) = \sigma(s(A))$ and
\[
G(t) = \frac{G_0(t)}{\sigma(F_i(t))}.
\]

Proof.
Inverting $s(A)$ we get
\[
5 \int_0^{R \left( e^{-\pi \sqrt{k_i(A)}} \right)} \frac{G_0(t)}{\sigma(F_i(t))} \frac{dt}{t^{3/5} - 11 - t^3} = s_i(A).
\]
Then differentiating (the $h$ is refering to $R \left( e^{-\pi \sqrt{k_i(A)}} \right)$ function)
\[
\frac{G_0 \left( R \left( e^{-\pi \sqrt{k_i(A)}} \right) \right)}{\sigma(A)} h_i'(A) = s'_i(A).
\]
Using now Corollary 2 and (60),(61), we get
\[
\frac{G_0 \left( R \left( e^{-\pi \sqrt{k_i(A)}} \right) \right)}{\sigma(A)} \frac{2^{1/3}}{A^{2/3}(1 - A^2)^{1/3}} = s'_i(A).
\]
Equivalently
\[
\frac{2^{-1/3} A^{2/3}(1 - A^2)^{1/3}}{\sigma(A)} \frac{2^{1/3}}{A^{2/3}(1 - A^2)^{1/3}} = s'_i(A)
\]
and finally
\[
\sigma(A) s'_i(A) = 1,
\]
which is true hence we get (64).

Note.
For the function $G_0(x)$ holds
\[
G_0 \left( F(x) \right) = \frac{\left( x \sqrt{1 - x^2} \right)^{2/3}}{\sqrt{2}}
\]
and hence $G_0$ is algebraic (see also Appendix for $G_0$).
Theorem 11.
We set \( c(A) = 3 \sqrt[3]{2A \cdot 2} \cdot _2F_1 \left[ \frac{1}{6}, \frac{1}{3}; \frac{7}{6}; A^2 \right] \), then
\[
Q_G^{-1} (h(A)) = c_i(A). \tag{66}
\]
Also
\[
\phi(A) = \frac{\sigma (k(A))}{k'(A)}, \tag{67}
\]
where \( \phi(x) \) is that of (14). Also
\[
m_G^{-1}(A) = \int_0^{k(A)} \frac{dt}{\sigma(t)}. \tag{67.1}
\]

Proof.
From Theorem 5 and (35) we have \( h_i \left( m_G^{-1}(A) \right) = b_A \) or \( h_i (Q(A)) = c(A) \).
Inverting we get the first result. For the second result we have
\[ m_G'(A) = \phi(m_G(A)), \]
or from (59)
\[ k_i'(s(A))s'(A) = \phi(m_G(A)), \]
or
\[ k_i'(s(A))s'(A) = \phi(k_i(s(A))), \]
or
\[ \frac{k_i'(A)}{s_i'(A)} = \phi(k_i(A)), \]
or
\[ s_i'(A) = \frac{k_i'(A)}{\phi(k_i(A))}, \]
or
\[ s_i(A) = \int_0^{k_i(A)} \frac{dt}{\phi(t)} = \int_0^A \frac{dt}{\sigma(t)}. \tag{68}
\]
Differentiating the last relation and inverting \( k_A = k(A) \) we get the result.

Note.
One can see immediately that
\[
h \left( 3 \sqrt[3]{2A \cdot 2} \cdot _2F_1 \left[ \frac{1}{3}, \frac{1}{6}; \frac{7}{6}; A^2 \right] \right) = s_i(A), \tag{69}
\]
which means that \( h \) and \( s_i \) "generalized" functions are essentially the same. We are going to describe this kind of relation between generalized functions.

Definition 3.
We say that a function \( f \) is generalized, if it is not "constant" function.
Definition 4.
We say that two invertible generalized functions $f, g$ are equivalent $f \equiv g$, if exist constant functions $\alpha_1(x), \beta_1(x), \gamma_1(x)$ such that
\[
f(x) = \frac{\alpha_1 (g(\beta_1(x)))}{\gamma_1(x)}.
\] (70)

Proposition 2.
The notation $\equiv$ is an equivalence relation i.e. it has the following properties
i. **Reflection**: $f \equiv f$
ii. **Symmetry**: If $f \equiv g$ then $g \equiv f$
iii. **Transition**: If $f \equiv g$ and $g \equiv h$, then $f \equiv h$.

Proposition 3.
We have the following equivalences of functions
\[
y \equiv m_G \equiv h_i \equiv Q_G^{-1} \equiv s
\] (71)
\[
y_i \equiv m_G^{-1} \equiv h \equiv Q_G \equiv s_i
\] (72)
\[
y_i' \equiv G \equiv h' \equiv \phi \equiv m_G^{-1}, \equiv Q'_G \equiv s \equiv s_i'
\] (73)

Theorem 12.
\[
y'(A) = 5^{-1} \sqrt[3]{2} \left(s(A) \sqrt{1 - s(A)^2}\right)^{-2/3} \cdot s'(A) y(A)^{1/3} y(A)^{-5} - 11 - y(A)^5.
\] (74)

Set now
\[
U(x) := 256 \frac{(1 - x^2 + x^4)^3}{x^4 (1 - x^2)^2}
\] (75)
and
\[
U_2(x) = \frac{1 - \sqrt{1 - U(x)^2}}{1 + \sqrt{1 - U(x)^2}} = 16 \frac{(1 + 14x^2 + x^4)^3}{x^2 (1 - x^2)^3},
\] (76)
then we have the next

Theorem 13.
For an arbitrary $G(x)$ the value of $\sigma(x)$ is given from
\[
\sigma(x) = \frac{(x \sqrt{1 - x^2})^{2/3}}{\sqrt{2 \cdot G(F(x))}}
\] (77)
and the value of $s(x)$ from
\[
\int_0^{s(x)} \frac{dt}{\sigma(t)} = x.
\] (78)
Then the value of \( y(x) \) at \( x = A \) can evaluated from
\[
U_2(s(A))^{1/3} Y^{5/3} = Y^2 + 250Y + 3125, \tag{79}
\]
where
\[
Y = y(A)^{-5} - 11 - y(A)^5. \tag{80}
\]

**Notes.**

I. Assume that \( G(A) = G^*(F_i(A)) \), where \( G^*(A) \) known. Then from (77) we have
\[
\sigma(A) = 2^{-1/3} \left( \frac{A \sqrt{1 - A^2}}{G^*(A)} \right)^{2/3}. \tag{81}
\]
Hence
\[
\int_0^A \frac{dt}{\sigma(t)} = \sqrt[3]{2} \int_0^A \frac{G^*(t)dt}{(t \sqrt{1 - t^2})^{2/3}} = s_i(A). \tag{82}
\]
Let \( G(t) = \sqrt[3]{1 - F_i(t)^2} \), then \( s(A) = \frac{A^3}{4} \).

i. If \( A_0 = 3 \sqrt[3]{6 - 4\sqrt{2}} \) we have
\[
j^{1/3} = U_2(s(A_0))^{1/3} = 12.
\]
Then equation (79) becomes
\[
12Y^{5/3} = Y^2 + 250Y + 3125
\]
and
\[
Y = 125(2 + \sqrt[5]{5}).
\]
Hence
\[
y(A_0) = y\left( \sqrt[3]{6 - 4\sqrt{2}} \right) = \frac{2}{\sqrt[3]{2}(5 + \sqrt{5}) + \sqrt{5} + 1}.
\]
Finally the function \( y(A) \) which is a solution of
\[
5 \int_0^{y(A)} \frac{\sqrt[3]{1 - F_i(t)^2}}{t \sqrt{t^5 - 11 - t^5} }dt = A,
\]
is
\[
y(A) = R\left( e^{-\pi \sqrt{s_i(A^3/54)}} \right)
\]
and can be determined in closed form up to a 6th degree polynomial equation (that of (79)).

ii. If
\[
A_0 = 3 \sqrt[3]{66 + 48\sqrt{2} - 8\sqrt{940 + 99\sqrt{2}}},
\]

then 
\[ j^{1/3} = U_2 (s(A_0))^{1/3} = 66 \]
and equation (79) becomes
\[ 66Y^{5/3} = Y^2 + 250Y + 3125, \]
with solution
\[ Y = \frac{125}{2} \left( 1147 + 513\sqrt{5} + \sqrt{2630810 + 1176534\sqrt{5}} \right). \]
Hence
\[ y \left( 3\sqrt[3]{66 + 48\sqrt{2} - 8\sqrt{140 + 99\sqrt{2}}} \right) = \sqrt[3]{\frac{-11 - \sqrt{125 + 22Y + Y^2}}{2}}. \]

II. If for a function \( y(A) \) we know \( G \), then from (77) we have
\[ \int_0^A \frac{dt}{\sigma(t)} = s_i(A). \]: (eq1)
Knowing \( s(A) \) we solve
\[ \sqrt[3]{\frac{1 + 14s(A)^2 + s(A)^4}{s(A)^{2/3} (1 - s(A))^{4/3}}} \cdot Y^{5/3} = Y^2 + 250Y + 3125 \]: (eq2)
and we get that
\[ y(A) = y_s(A) = \sqrt[3]{\frac{-11 - \sqrt{125 + 22Y + Y^2}}{2}}. \]: (eq3)
Hence we find the closed form of \( y(A) \) in (64) (and hence to the problem (10)) up to the inverting of the integral of (eq1) and solving the sextic equation (eq2).

i) Suppose that \( \sigma(A) = A + 1 \), then \( s(A) = e^A - 1 \). This case corresponds to
\[ 5 \int_0^{y(x)} \frac{G_0(t)}{F_1(t) + 1} \frac{dt}{t \sqrt{t - 5} - 11 - t^3} = x \]
and is solvable up to the sextic equation (eq2). Also
\[ y(A) = R \left( e^{-\pi \sqrt{k_i(x^{a-1})}} \right). \]

ii) Another example is with \( \sigma(A) = 1 \). This leads to \( s(A) = A \) and corresponds to
\[ 5 \int_0^{y(x)} \frac{G_0(t)}{t \sqrt{t - 5} - 11 - t^3} dt = x. \]
The first derivative according to Theorem 12 is
\[
y'(A) = 5^{-1} \sqrt[3]{2} \left( A \sqrt{1 - A^2} \right)^{-2/3} y(A) \sqrt[3]{y(A)^{-5} - 11 - y(A)^5}
\]
and
\[
y(A) = R \left( e^{-\pi \sqrt{k_r(A)}} \right).
\]
For more details see section Applications.

iii) If \( \sigma(A) = 1/A \) we get \( s(A) = \sqrt{2A} \), hence
\[
y'(A) = \frac{5^{-1}}{\sqrt{2} (1 - 2A)^{1/3} A^{5/6}} y(A) \sqrt[3]{y(A)^{-5} - 11 - y(A)^5},
\]
with
\[
y(A) = R \left( e^{-\pi \sqrt{k_r(\sqrt{2A})}} \right).
\]

iv) If \( \sigma(A) = \sqrt{1 - A^2} \sqrt{1 - kA^2} \), then \( s(A) = \text{sn}(A, k) \) and the solution \( y(x) \) of
\[
5 \int_0^{y(x)} \frac{G_0(t) F_i(t) dt}{t \sqrt{t^2 - 11 - t^5}} = x,
\]
is given from (eq2) and (eq3). Essentialy the function \( y(x) = y_k(x) = y(x, k) \) is algebraic function of \( s(A) = \text{sn}(A, k) \) and hence double periodic elliptic function. Also
\[
\frac{y_k'(A)^2}{1 - k F_i(y_k(A))^2} = \frac{y_i'(A)^2}{1 - l F_i(y_i(A))^2} = C(A)
\]
and
\[
y_k'(A) = 5^{-1} \sqrt[3]{2} \frac{\text{cn}(A, k)^{1/3} \text{dn}(A, k)}{\text{sn}(A, k)^{2/3}} y_k(A) \sqrt[3]{y_k(A)^{-5} - 11 - y_k(A)^5},
\]
with
\[
y(A) = y_k(A) = R \left( e^{-\pi \sqrt{k_r(\text{sn}(A, k))}} \right).
\]

III. Another notation but not so detailed can found using (58),(77),(78) and relation \( R(q) = F(k_r) \). We have
\[
y_i(A) = 3 \sqrt[3]{2} \int_0^{F_i(A)} \frac{G(F(t))}{(t \sqrt{1 - t^2})^{2/3}} dt. \tag{83}
\]
As application we set $B(x, a, b) = \int_{0}^{x} t^{a-1} (1 - t)^{b-1} dt$ to be the incomplete beta function. Then if

$$G (F(A)) = \frac{1}{\sqrt{2}} \left( A \sqrt{1 - A^2} \right)^{2/3} (A - A^2)^{a-1},$$

we have

$$y(B(x, a, a)) = F(x) \quad (84)$$

and (see [13]) the solution of

$$\frac{B (1 - \beta_r, a, a)}{B(\beta_r, a, a)} = r, \quad r > 0, \quad 0 < \beta_r < 1, \quad (85)$$

is equivalent to

$$B(\beta_r, a, a) = \frac{\Gamma(a)^2}{\Gamma(2a)(r + 1)}. \quad (86)$$

Hence we get

$$y \left( \frac{\Gamma(a)^2}{\Gamma(2a)(r + 1)} \right) = F(\beta_r). \quad (87)$$

IV. Also from

$$\int_{0}^{A} \frac{dt}{\sigma(t)} = s_i(A),$$

we have

$$\int_{0}^{k_r} \frac{dt}{\sigma(t)} = s_i(k_r).$$

Hence from (59): $s_i(A) = m_{G}^{(-1)}(k_i(A))$ or equivalently $s_i(k_r) = m_{G}^{(-1)}(r)$ and we get

$$\int_{0}^{k_r} \frac{dt}{\sigma(t)} = m_{G}^{(-1)}(r). \quad (88)$$

V. If $G(F(A))$ is polynomial

$$G (F(x)) = \sum_{n=0}^{M} c_n x^n, \quad (89)$$

then using

$$\int_{0}^{A} \frac{t^n}{(t^\sqrt{1 - t^2})^{2/3}} dt = 3^{2/3} \frac{A^{n+1/3}}{3n + 1} {2F_1 \left[ \frac{1}{3}, \frac{3n + 1}{6}; \frac{3n + 7}{6}; A^2 \right]}, \quad (90)$$

we get

$$y \left( 3^{2/3} \sum_{n=0}^{M} c_n \frac{A^{n+1/3}}{3n + 1} \cdot {2F_1 \left[ \frac{1}{3}, \frac{3n + 1}{6}; \frac{3n + 7}{6}; A^2 \right]} \right) = F(A), \quad (91)$$

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or if someone prefers
\[
3 \sqrt[3]{2} \sum_{n=0}^{M} c_n \frac{A^{n+1/3}}{3n+1} \cdot 2F_1 \left[ \frac{1}{3}, \frac{3n+1}{6}, \frac{3n+7}{6} : A^2 \right] + c = y_i (F(A)). \tag{92}
\]

and consequently:

**Theorem 14.**

If
\[
\sigma(A) = \frac{\sqrt{1 - A^2}^{2/3}}{\sqrt{2} \sum_{m=0}^{M} c_m A^m},
\]

then
\[
y_i(A) = 3 \sqrt[3]{2} \sum_{n=0}^{M} c_n \frac{F_i(A)^{n+1/3}}{3n+1} \cdot 2F_1 \left[ \frac{1}{3}, \frac{3n+1}{6}, \frac{3n+7}{6} : F_i(A)^2 \right] + c. \tag{94}
\]

Assume function $G$ as in (89), then from the fact that $F$ is algebraic, there exists coefficients $a_{n,l}$ such that
\[
\sum_{n,l=0}^{N} a_{kl} G(x)^n x^l = 0. \tag{95}
\]

But then also
\[
\sum_{n,l=0}^{N} a_{nl} (F(x))^n F(x)^l = 0
\]

and
\[
\sum_{n,l=0}^{N} a_{nl} \left( \sum_{m=0}^{M} c_m x^m \right)^n F(x)^l = 0. \tag{96}
\]

Finally
\[
\sum_{n,l=0}^{N} a_{nl} \left( \sum_{m=0}^{M} c_m (k_r)^m \right)^n R(q)^l = 0. \tag{97}
\]

Hence knowing $c_m$ we find $a_{nl}$ from relation (97) and equation
\[
x^2 (1 - x^2)^4 \left( y^{20} - 228y^{15} + 494y^{10} + 228y^5 + 1 \right)^3 +
+ 16 \left( x^4 + 14x^2 + 1 \right)^3 y^3 \left( y^{10} + 11y^5 - 1 \right)^5 = 0,
\]

(which is Klein formula for the icosahedron $y = F(x)$). That is the $a_{nl}$ of (95) can be found from that of $c_m$, by equating coefficients of the identity:
\[
\sum_{n,l=0}^{N} a_{nl} \left( \sum_{m=0}^{M} c_m x^m \right)^n y^l = x^2 (1 - x^2)^4 \left( y^{20} - 228y^{15} + 494y^{10} + 228y^5 + 1 \right)^3 +
\]
If hapens $G(F(x)) = \psi(x)$ be more complicated, for example algebraic, then we solve the equation $G = \psi(x)$ with respect to $x$, $x = \psi^{-1}(G) = \psi(G)$ and

$$\sum_{n,l=0}^{60} A_{nl} (\psi(G))^{n} x^{l} = 0,$$  \hspace{1cm} (100)

where the $A_{nl}$ are that of Klein’s equation

$$\sum_{n,l=0}^{60} A_{nl} x^{n} y^{l} = x^{2} (1 - x^{2})^{4} (y^{20} - 228y^{15} + 494y^{10} + 228y^{5} + 1)^{3} +$$

$$+ 16 (x^{4} + 14x^{2} + 1)^{3} y^{5} (y^{10} + 11y^{5} - 1)^{5}. \hspace{1cm} (101)$$

Hence in general:

**Theorem 15.**

If $G(F(x)) = \psi(x)$ is known resonable function (polynomial algebraic etc...), then the minimal equation for $G$ is (100), with coefficients $A_{nl}$ that of (101).

**Example.**

For $G(F(x)) = 2^{-1/3}(x^{2} - x^{4})^{1/3}$, we get $Q_{G}(x) = 1$, hence $s_{i}(x) = x$ and $y(x) = F(x)$, where $G(x) = G_{0}(x)$ is solution of

$$\sum_{n,l=0}^{60} A_{nl} \left( \sqrt{1 - \sqrt{1 - 8G(x)^{2}}} \right)^{n} x^{l} = 0. \hspace{1cm} (102)$$

The coefficients $A_{nl}$ are that of (101).

## 4 Solution of General Equations and Inversion

Consider a polynomial $w(x)$ and the equation

$$w(x) = \lambda \hspace{1cm} (103)$$

Set $G(x) = 5^{-1}xw'(x)\sqrt{x^{5} - 11 - x^{2}}$, then

$$m_{G}^{(-1)}(r) = w\left( R\left( e^{-\pi\sqrt{r}} \right) \right) \hspace{1cm} (104)$$

and the solution of (103) is $x = R\left( e^{-\pi\sqrt{r}} \right)$, where $r = m_{G}(\lambda)$.

Taking the derivatives with respect to $A$ in $w(f^{(-1)}(A)) = A$, we lead to

$$w'\left( f^{(-1)}(A) \right) f^{(-1)'(A)} = 1, \hspace{1cm} (105)$$
or
\[ f^{(-1)'}(A) = \frac{1}{w'(f^{(-1)}(A))}. \]  
(106)

Hence if we set \( W(A) = \frac{1}{w'(A)} \), then clearly
\[ f^{(-1)'}(A) = W\left(f^{(-1)}(A)\right). \]  
(107)

Setting where \( f^{(-1)}(A) = y(A) \), with \( y(A) \) that of (9) and (10) and taking the derivatives with respect to \( A \) we have
\[
\frac{G\left(f^{(-1)}(A)\right) f^{(-1)'(A)}}{f^{(-1)}(A) \sqrt[6]{f^{(-1)}(A)^{-5} - 11 - (f^{(-1)}(A))^{5}}} = 5 \left(\frac{G\left(f^{(-1)}(A)\right) W\left(f^{(-1)}(A)\right)}{f^{(-1)}(A) \sqrt[6]{f^{(-1)}(A)^{-5} - 11 - (f^{(-1)}(A))^{5}}}\right) = 1. \]  
(108)

After inverting \( f^{(-1)}(x) \) we have
\[ G(x)W(x) = 5^{-1}x \sqrt[6]{x^{-5} - 11 - x^5}, \]
or equivalently
\[ G(x) = 5^{-1}xw'(x) \sqrt[6]{x^{-5} - 11 - x^5} \]  
(109)

and it will be
\[ f^{(-1)}(A) = R\left(e^{-\pi\sqrt{m_G(A)}}\right). \]  
(110)

From the above we can state the following theorem

**Theorem 16.**

The equation \( w(y(x)) = x \) have solution
\[ y(x) = R\left(e^{-\pi\sqrt{m_G(x)}}\right), \]  
(111)

with \( G \) that of relation (109) and \( m_G(A) \) as defined in (9).

**Example.**

Let \( \rho_1 = \frac{1}{2} \left(11 - 5\sqrt{5}\right) \), \( \rho_2 = \frac{1}{2} \left(11 + 5\sqrt{5}\right) \) and consider the equation
\[
\frac{6a + 14}{6a + 11} F_{Ap}\left(\frac{6a + 11}{30}, \frac{1}{6}, \frac{1}{6}, \frac{6a + 41}{30}, \rho_1 x^5, \rho_2 x^5\right) = A, \]  
(112)

where \( a \) is parameter. The function \( G \) is \( G(x) = \frac{e^{a+1}}{8}, \) and
\[ x = R\left(e^{-\pi\sqrt{m_G(A)}}\right), \]  
(113)
where the $m_G(x)$ is given from
\[ x = \frac{\pi}{5} \int_{\sqrt{m_G(x)}}^{\infty} \eta(it/2)^4 R(e^{-\pi t})^{a+1} dt. \] (114)

Hence if
\[ g_a(x) = 6 \frac{x^{a+\frac{41}{6a+11}}}{6a+11} F_{Ap} \left( \frac{6a+11}{30}, \frac{1}{6}, \frac{1}{6}, \frac{6a+41}{30}, \rho_1 x^5, \rho_2 x^5 \right), \]
then
\[ g_a \left( R(e^{-\pi A}) \right) = \frac{\pi}{5} \int_{A}^{\infty} \eta(it/2)^4 R(e^{-\pi t})^{a+1} dt. \] (115)

**Theorem 17.**
Given the equation $P(x) = a : (\epsilon)$, the inverse of $P(x)$ is $y(x)$, then $\epsilon$ is equivalent to
\[ \int_{0}^{F_i(x)} \frac{dt}{\sigma(t)} = a. \] (116)

**Proof.**
Easy

**Example.**
If $\sigma(x) = x + 1$, then
\[ P(x) = 5 \int_{0}^{x} \frac{G_0(t)}{F_i(t) + 1} \frac{dt}{t \sqrt{t^5 - 11 - t^5}} \]
and the equation $P(x) = a$ have solution $x$ such that
\[ \int_{0}^{F_i(x)} \frac{dt}{t + 1} = a, \]
or equivalently $x = F(e^a - 1)$. In general we have the next formula

If $|x| < 1$ then
\[ \int_{0}^{x} \frac{f(-q)^5 R(q) g'(R(q))}{f(-q^5)} dq = 5g(R(x)), \] (117)
which is consequence of the next identity
\[ \frac{R'(q)}{R(q)} = \frac{f(-q)^5}{5qf(-q^5)} \] (118)
Relation (118) was given by Ramanujan (see [3]). We know that
\[
\int_{q_1}^{q_2} f(-q)^4 q^{-5/6} R(q)^{5\nu} dq = -\pi \int_{\sqrt{r_1}}^{\sqrt{r_2}} \eta(it/2)^4 R(e^{-\pi t})^{5\nu} dt
\]
and (see [13]):
\[
C(\nu) := \int_0^1 f(-q)^4 q^{-5/6} R(q)^{5\nu} dq =
\]
\[
= \Gamma \left( \frac{5}{6} \right) \left( \frac{11 + 5\sqrt{5}}{2} \right)^{-\frac{5}{6}-\nu} \frac{\Gamma \left( \frac{1}{6} + \nu \right)}{\Gamma(1 + \nu)} {}_2F_1 \left( \frac{1}{6}, \frac{1}{6} + \nu; 1 + \nu; \frac{11 - 5\sqrt{5}}{11 + 5\sqrt{5}} \right),
\]
where \( \nu \geq 0 \). Hence

**Theorem 18.**

If \( G(x) \) is a polynomial (or analytic function when \( n \to +\infty \) of the form
\[
G(x) = \sum_{m=0}^{n} a_n x^{p_m},
\]
with \( p_m \) — positive reals, (lim \( p_m = +\infty \)) and \( R(1) = \frac{\sqrt{5} - 1}{2} \), then
i) \( m_G^{-1}(0) = \pi \int_0^\infty \eta(it/2)^4 G(R(e^{-\pi t})) dt = \sum_{m=0}^{n} a_m C \left( 5^{-1} p_m \right) \).

ii) If \( y \) is a smooth function and \( G \) is of the form (120) then the equation
\[
y(x) = \sqrt{5} - 1 \]
have a solution
\[
x = x_0 = \sum_{m=0}^{n} a_m C(5^{-1} p_m).
\]

**Example.**

Let \( G(x) = e^{-x} - 1 \). Then \( a_m = \frac{(-1)^m}{m!} \), \( m = 1, 2, \ldots \) and
\[
5 \int_0^{g(x)} \frac{e^{-t} - 1}{t^{1/5} - 11 - t^5} dt = x
\]
and the solution of \( y(x) = \frac{\sqrt{5} - 1}{2} \) is
\[
x = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} C \left( \frac{m}{5} \right).
\]
Note here that if $G$ has finite expansion (120) the result (123) becomes more meaningful since the evaluation of the root of (122) by hypergeometric functions is better than the integral:

$$
5 \int_0^{\sqrt{5}-1} \frac{G(t)}{t^5 - 5 - t^5} dt. \quad (126)
$$

Assume now

$$
G^*(t) := G(t^\alpha) \sqrt{\frac{t^{-5} - 11 - t^5}{(at)^{-5} - 11 - (at)^5}}. \quad (126.1)
$$

If $y^*(A)$ corresponds to $G^*(t)$, one can easily see that

$$
y^*(A) = \frac{y(A)}{\alpha}. \quad (126.2)
$$

Hence we have the next theorem which is generalization of Theorem 18:

**Theorem 18.1**

Assume the function $G(t)$ is given near the origin by (under certain converging conditions):

$$
G(t) = \sum_{m=0}^{\infty} a_m t^{p_m}, \quad (126.3)
$$

where $p_m$ is any increasing sequence of positive real numbers with $\lim p_m = +\infty$. Then the function $y(A)$ defined as

$$
5 \int_0^{y(A)} \frac{G(t)}{t^5 - 5 - t^5} dt = A, \quad (126.4)
$$

have the following property: Every equation of the form

$$
y(x) = \alpha \sqrt{\frac{5 - 1}{2}}, \quad (126.5)
$$

have solution $x$ such that

$$
x = \sum_{m=0}^{\infty} a^*_m(\alpha) C \left(5^{-1} p^*_m\right), \quad (126.6)
$$

where $a^*_m(\alpha)$ is such that

$$
G(x\alpha) \sqrt{\frac{x^{-5} - 11 - x^5}{(ax)^{-5} - 11 - (ax)^5}} = \sum_{m=0}^{\infty} a^*_m(\alpha)x^{p^*_m}. \quad (126.7)
$$
Hence if we set $\xi^{-1} = \frac{\sqrt{5} - 1}{2}$, then the inverse of $y(A)$ is

$$y_i(A) = \sum_{m=0}^{\infty} a_m^*(\xi A) C \left( 5^{-1} p_m^* \right),$$  \hspace{1cm} (126.8)

provided the convergence of (126.1),(126.3),(126.4),(126.6),(126.7),(126.8).

Set

$$G = G_1(t) = \frac{t \sqrt{t^5 - 11 - t^5}}{5 \sqrt{1 - t^2} \sqrt{1 - k^2 t^2}}.$$  \indent Then

$$\int_0^y G_1(t) \frac{dt}{t \sqrt{t^5 - 11 - t^5}} = \int_0^y \frac{1}{\sqrt{1 - t^2} \sqrt{1 - k^2 t^2}} dt = A. \hspace{1cm} (127)$$

Hence (see [4]):

$$y = \text{sn}(A, k) = R \left( e^{-\pi \sqrt{m_G}} \right) \quad \text{and} \quad A = h(b_{m_G}),$$

where

$$b_{m_G} = 3 \sqrt{2k_{m_G}} \cdot 2F_1 \left[ \frac{1}{6}, \frac{1}{3}, \frac{7}{6}; k_{m_G}^2 \right]$$

and

$$h^{-1}(x) = \int_0^x \frac{du}{G_1(\text{sn}(u, k_r))} = 5 \int_0^x \frac{\text{dn}(u) \text{cn}(u)}{\text{sn}(u) \sqrt{\text{sn}(u)^{-5} - 11 - \text{sn}(u)^5}} du.$$  \indent Hence if $y(A)$ is defined by (9),(10) and $\text{sn}(u) = \text{sn}(u, k)$, $\text{dn}(u) = \text{dn}(u, k)$, then

**Theorem 19.**

$$\int_0^{E[\arcsin(y(A)), k]} \frac{\text{dn}(u) \text{cn}(u)}{\text{sn}(u) \sqrt{\text{sn}(u)^{-5} - 11 - \text{sn}(u)^5}} du =$$

$$= 3 \sqrt{2k_{m_G}} \cdot 2F_1 \left[ \frac{1}{6}, \frac{1}{3}, \frac{7}{6}; k_{m_G}^2 \right], \hspace{1cm} (128)$$

where $k$ is independent parameter, $m_G = m_G(A)$ and $E$ denotes the incomplete elliptic integral of the first kind, $y$ is the function defined in (9),(10),(11).

Inverting the above integral we get a formula for the Rogers-Ramanujan continued fraction:

Set

$$\int_0^{H_a(x)} \frac{\text{dn}(u) \text{cn}(u)}{\text{sn}(u) \sqrt{\text{sn}(u)^{-5} - 11 - \text{sn}(u)^5}} du = x, \hspace{1cm} (129)$$

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then
\[ R(q) = \text{sn}(H_o(b_r), k). \] (130)

For the function sn we have \( G(t) = G_1(t) \) and if the equation
\[ \text{sn}(x, k) = a, \]
have \( m_1 \) such that \( R\left(e^{-\pi \sqrt{m_1}}\right) = a \). The solution is \( x = x_1 \):

\[ x_1 = E\left[\arcsin(a), k\right] = \pi \int_{\sqrt{m_1}}^{+\infty} \eta(it/2)^4 G_1\left(R\left(e^{-\pi t}\right)\right) dt. \]

Taking derivatives in (129) we get
\[ H_o'(H_o^{(-1)}(x)) = \frac{1}{H_o^{(-1)}(x)} = \frac{\text{sn}(x) \sqrt{\text{sn}(x)^{-5} - 11 - \text{sn}(x)^5}}{5\text{cn}(x)\text{dn}(x)}. \] (131)

Hence
\[ H_o^{(-1)}'(E\left[\arcsin(x), k^2\right]) = \frac{5\sqrt{1 - k^2x^2} \sqrt{1 - x^2}}{x \sqrt{x^2 - 11 - x^5}}. \] (132)

According to (129) and (130), the function \( H_o = H_o(x, m) \), takes special values
\[ H_o(b_r) = H_o(b_r, m) = E(\arcsin(R(q)), m), \]
where \( q = e^{-\pi \sqrt{r}}, r > 0 \) for all \( 0 < m < 1 \).

By this way \( H_o \) can evaluated with known functions
\[ H_o(A, k) = E\left(\arcsin\left(R\left(e^{-\pi \sqrt{m(A)}}\right)\right), k\right), \] (132.1)
with \( m(A) \) that of (23),(24).

5 More integrals

Let \( F_1 \) be the function introduced in the above sections i.e. \( F_1(x) = R\left(e^{-\pi \sqrt{m(x)}}\right) \),
then \( F_1 \) is such that
\[ F_1'(x) = 5^{-1}F_1(x) \sqrt{(F_1(x))^{-5} - 11 - (F_1(x))^5}. \] (133)

Also \( F_1^{(-1)} \) is a specific Appell function
\[ F_{Ap}[a, b_1, b_2, c, x, y] := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_{m} (b_2)_{n}}{m! n!} x^m y^n. \] (134)

More precisely
\[ F_1^{(-1)}(x) = 6x^{5/6}F_{Ap}\begin{bmatrix} 1 & 1 & 1 & 7 & -2x^{5} & -2x^{5} \\ 6 & 0 & 6 & 6 & 11 + 5\sqrt{3} & 11 - 5\sqrt{3} \end{bmatrix}. \] (135)
Set also \( t = t(w) \) such that

\[
 t(w) = F_1 \left[ (-1)^{m+1} a^{m-1} D^{-m+1/2} B \left( \frac{-b + \sqrt{D} - 2aw}{2\sqrt{D}}, 1 - m, 1 - m \right) \right],
\]

where \( B(x, a, b) = \int_0^x t^{a-1} (1 - t)^{b-1} dt \), \( D = b^2 - 4ac \).

Also let

\[
 U(a, b, c; m; x) := (-1)^{m+1} a^{m-1} D^{-m+1/2} B \left( \frac{-b + \sqrt{D} - 2ax}{2\sqrt{D}}, 1 - m, 1 - m \right).
\]

Then

\[
 \frac{d}{dx} U(a, b, c; m; x) = \frac{1}{(ax^2 + bx + c)^m} \Leftrightarrow \int_A^B \frac{dt}{(at^2 + bt + c)^m} = U(a, b, c; m; B) - U(a, b, c; m; A).
\]

Hence we get

\[
 5 \int_{t_0}^{t_1} \frac{G(t)dt}{t \sqrt{t^5 - 11 - t^5}} = \int_{w_0}^{w_1} \frac{G \left( F_1 \left[ (-1)^{m+1} a^{m-1} D^{-m+1/2} B \left( \frac{-b + \sqrt{D} - 2aw}{2\sqrt{D}}, 1 - m, 1 - m \right) \right] \right)}{(aw^2 + bw + c)^m} dw,
\]

where \( w_1, w_0, t_0, t_1 \) are such that

\[
 U_i \left( a, b, c; m; F_1^{-1} (t_1) \right) = w_1 \text{ and } U_i \left( a, b, c; m; F_1^{-1} (t_0) \right) = w_0.
\]

Suppose that we wish to evaluate the integral

\[
 I := \int_{w_0}^{w_1} \frac{f(w)}{\sqrt{aw^2 + bw + c}} dw.
\]

Then easily from (139) with \( m = 1/2 \)

\[
 I = 5 \int_{F_1(U(w_1))}^{F_1(U(w_0))} \frac{f \left( U_i \left( F_1^{-1} (t) \right) \right)}{t \sqrt{t^5 - 11 - t^5}} dt = y_i (F_1(U(w_1))) - y_i (F_1(U(w_0))) = h(U(w_1)) - h(U(w_0)),
\]

where \( y(x) = R \left( e^{-\pi \sqrt{mc(x)}} \right) \) and

\[
 G(x) = f \left( U_i \left( F_1^{-1} (x) \right) \right)
\]

and

\[
 U_i(x) := \frac{1}{2a} \left( \sqrt{b^2 - 4ac} \cdot \sinh \left( \sqrt{ax} \right) - b \right).
\]
Until now we have evaluated $G(x)$. From Theorem 4 we have
\[ h'(5 \int_0^x \frac{dt}{t\sqrt{5-11-t^5}}) \frac{1}{\sqrt{x^5-11-x^5}} = \frac{5}{x\sqrt{x^5-11-x^5}} G(x). \]
Hence
\[ h'(x) = G(F_1(x)) = f(U_i(x)), \quad (145) \]
which is a reasonable equation to find $h$. In general we can state the following

**Theorem 20.**
We have
\[ \int_{w_0}^{w_1} \frac{f(w)}{(aw^2 + bw + c)^m} dw = h(U(w_1)) - h(U(w_0)), \quad (146) \]
where
\[ G(x) = f(U_i(F_1^{(-1)}(x))). \quad (147) \]

**Theorem 20.1**
Assume that
\[ G(x) = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} x^n. \quad (147.1) \]
Then
\[ h(A) = Q_G(c_i(A)) = 5F_1(A)^{5/6} \times \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} \frac{F_1(A)^n}{5/6 + n} F_{Ap} \left[ \frac{1}{6} + \frac{n}{5}, \frac{1}{6}, \frac{1}{6}, \frac{7}{6}, \frac{n}{5}, \rho_1 F_1(A) \right]. \quad (147.2) \]

**Proof.**
See section Applications paragraph 6.2.

**Notes.**
More general if
\[ G(x) = \sum_{n=0}^{\infty} G_n x^{p_n}, \quad (147.3) \]
where $p_n$ is increasing sequence of positive real numbers, with $\lim p_n = +\infty$, then
\[ h(A) = Q_G(c_i(A)) = 5F_1(A)^{5/6} \times \sum_{n=0}^{\infty} G_n \frac{F_1(A)^{p_n}}{5/6 + p_n} F_{Ap} \left[ \frac{1}{6} + \frac{p_n}{5}, \frac{1}{6}, \frac{1}{6}, \frac{7}{6}, \frac{p_n}{5}, \rho_1 F_1(A) \right]. \quad (147.4) \]

**Theorem 21.**
It is $G(F_1(x)) = f(U_i(x))$ and
\[ Q_G(A) = h(c_A) = \int_0^A f(U_i(t))dt = \int_0^A G(F_1(t))dt. \quad (148) \]
Proof.
From (145) we have \( h'(b_A) = G(R(q)) = f(U_i(b_A)) \) inverting \( k(A) \) we get \( h'(c_A) = G(F(A)) = f(U_i(c_A)) \). Hence \( h'(c_A)c_A' = f(U_i(c_A)c_A') \) and integrating, \( h(c_A) = Q_G(A) = \int_0^{c_A} f(U_i(t))dt = \int_0^{c_A} G(F_1(t))dt \). Hence we get the result.

Example.
If \( G(t) = \sqrt[4]{t} \), then from (167) below
\[
Q_G(A) = \int_0^{c_A} \sqrt[F_1(t)]{t}dt = 5F_1(c_A)F_{Ap}\left[\frac{1}{5}, \frac{1}{6}, \frac{1}{6}, \frac{11 - 5\sqrt{5}}{2}F_1(c_A)^5, \frac{11 + 5\sqrt{5}}{2}F_1(c_A)^5\right].
\]
Hence \( y(A) = R\left(e^{-\pi\sqrt[A]{(Q_G^{(-1)}(A))}}\right) \) and holds the following semi-algebraic relation for the function \( y(x) \):
\[
y\left(5F_1(c_A)F_{Ap}\left[\frac{1}{5}, \frac{1}{6}, \frac{1}{6}, \frac{11 - 5\sqrt{5}}{2}F_1(c_A)^5, \frac{11 + 5\sqrt{5}}{2}F_1(c_A)^5\right]\right) = F(A).
\]

Theorem 22.
Given \( G \) there holds the relation
\[
y_i(G_i(x)) = xF_1^{(-1)}(G_i(x)) - \int_0^x F_1^{(-1)}(G_i(t))dt, \quad (149)
\]
where \( F_1^{(-1)} \) is the Appell function of (135).

Proof.
Integration by parts.

Example.
Suppose \( G(x) = \log\left(F_1^{(-1)}(x) + 1\right) \), then \( F_1^{(-1)}(G_i(x)) = e^x - 1 \) hence
\[
y_i(F_1(e^x - 1)) = e^x(x - 1)
\]
and \( h(x) = \int_0^x \log(t + 1)dt = (x + 1)\log(x + 1) - x \).

Also from Theorems 17 and 19 we get

Theorem 23.
Let \( \xi^{-1} = \sqrt[2]{\frac{-11 + 5\sqrt{5}}{2}} \) and
\[
U_i\left(a, b, c; m; F_1^{(-1)}\left(\xi^{-1}\right)\right) = p_1 \text{ and } U_i\left(a, b, c; m; F_1^{(-1)}(0)\right) = p_0, \quad (150)
\]

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then
\[ \int_{p_0}^{p_1} G(F_1[U(a, b, c; m; w)]) \, dw = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} C \left( \frac{n}{5} \right). \]  

(151)

**Corollary 1.**
\[ 5 \int_{p_0}^{p_1} F_1[U(a, b, c; m; w)] \, dw = C \left( \frac{1}{5} \right) = \left( \frac{2}{11 + 5\sqrt{5}} \right)^{11/30} \frac{\Gamma \left( \frac{11}{30} \right) \Gamma \left( \frac{5}{3} \right) \Gamma \left( \frac{2}{3} \right)}{\Gamma \left( \frac{4}{3} \right)} 2F_1 \left( \frac{1}{6}, \frac{11}{30}, \frac{6}{5}; -\frac{123}{2} + \frac{55\sqrt{5}}{2} \right). \]

(152)

### 6 Applications

**6.1 The study of** \( y(x) = F(x) = R \left( e^{-\pi \sqrt{k_i(x)}} \right) \) **function**

Assume that \( y(x) = F(x) = R \left( e^{-\pi \sqrt{k_i(x)}} \right) \), then from Theorem’s 1,5 we have \( m_{G_0}(x) = k_i(x) \) and

\[ 5 \int_{0}^{y(x)} \frac{dt}{t^{1/2} - 11 - t^3} = 3 \sqrt{2} x \cdot 2F_1 \left[ \frac{1}{6}, \frac{1}{3}; \frac{7}{6}; x^2 \right] = \int_{0}^{x} \frac{dt}{G_0(y(t))}. \]

Differentiating the above equation we get

\[ h_i'(x) = \frac{1}{G_0(y(x))} = \frac{2^{1/3}}{x^{2/3}(1 - x^2)^{1/3}}. \]

(153)

Hence

\[ y_i(x) = \sqrt{\frac{1 - \sqrt{1 - 8G_0(x)^3}}{2}} \]

(154)

and

\[ 5 \int_{0}^{x} \frac{G_0(t)}{t^{1/2} - 11 - t^3} dt = y_i(x). \]

Hence the final equation for evaluating the \( G \)-function is

\[ \sqrt{\frac{1 - \sqrt{1 - 8G_0(x)^3}}{2}} = 5 \int_{0}^{x} \frac{G_0(t)}{t^{1/2} - 11 - t^3} dt, \]

which under differentiation becomes

\[ \frac{5}{x^{6/3} - x^5 + \frac{1}{x^6} - 11} = \sqrt{\frac{3G_0(x)G_0'(x)}{\frac{1}{2} - 4G_0(x)^3 \sqrt{1 - \sqrt{1 - 8G_0(x)^3}}}} \]

Solving the last differential equation, we get the following relation for \( G_0(x) \):

\[ \sqrt{3} \cdot 2^{2/3} \cdot \sqrt{G_0(x)} \cdot 2F_1 \left[ \frac{1}{6}, \frac{11}{5}, \frac{1}{2}; \frac{1 - \sqrt{1 - 8G_0(x)^3}}{2} \right] = 29 \]
\[
F = 6x^{5/6} F_{Ap} \left[ \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 7; \frac{11 - 5\sqrt{5}}{2} x^5, \frac{11 + 5\sqrt{5}}{2} x^5 \right] = F_1^{(-1)}(x). \tag{155}
\]

From (155) the partial evaluation of \(G_0(x)\) follows.

### 6.2 The case of \(G(x) = 1\) function

In case \(G(x) = 1\), then from relation (26)

\[
y(x) = F_1(x). \tag{156}
\]

From \(h(x) = x\), we have

\[
F_1'(x) = 5^{-1} F_1(x) \sqrt{F_1(x)-5} - 11 - F_1(x)^5 \tag{157}
\]

and from (56)

\[
Q'_G(x) = s'_1(x) = \frac{1}{\sigma(x)} = \frac{2^{1/3}}{(x\sqrt{1-x^2})^{2/3}}. \tag{158}
\]

Hence

\[
Q_G(x) = 2^{1/3} \int \frac{dx}{(x\sqrt{1-x^2})^{2/3}} = 3\sqrt{2x} \cdot 2F_1 \left[ \frac{1}{6}, \frac{1}{3}; \frac{7}{6}; x^2 \right] + c. \tag{159}
\]

The modular equation for \(y_i(x)\) is \(P_n(x)\) and

\[
P_n(x) = 5 \int_0^{\pi/2} \frac{dF_1(x)}{t^{\sqrt{t^2 - 5} - 11 - t^5}}. \tag{160}
\]

Also

\[
m_G(x) = b_x^{(-1)} = b^{(-1)}(x). \tag{161}
\]

Avoiding the inverse of \(b_r\), which is a hypergeometric function (more precisely, Beta function) we can use the function \(m_G(x) = m(x)\) defined by (see relation (23))

\[
\pi \int_0^{\infty} \frac{\eta(it/2)^4}{\sqrt{n^{m(x)}}} dt = x. \tag{162}
\]

Hence

\[
y(x) = F_1(x) = R \left( e^{-\pi\sqrt{n^{m(x)}}} \right) \tag{163}
\]

and

\[
m(x) = k_i(s(x)).
\]

Hence

\[
Q(k_r) = \pi \int_0^{\infty} \eta(it/2)^4 dt = b^{(-1)}(x) = m(r). \tag{164}
\]

\[
F_1^{(-1)}(x) = \frac{5}{t^{\sqrt{t^2 - 5} - 11 - t^5}}. \tag{165}
\]
Hence
\[
\int F_1(t)^{\nu} \, dt = \int \frac{x^{\nu}}{x^{\sqrt{x^5}} - 11 - x^5} \, dx =
\]
\[
= \frac{30x^{5/6 + \nu}}{5 + 6\nu} F_{Ap} \left[ \frac{1}{6} + \frac{\nu}{5} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{7}{6} + \frac{\nu}{5} \cdot \frac{11 - 5\sqrt{5}}{2} x^5, \frac{11 + 5\sqrt{5}}{2} x^5 \right],
\]
where we have made the change of variable \( x = F_1(t) \). Hence setting \( \rho_1 = \frac{11-5\sqrt{5}}{2} \), \( \rho_2 = \frac{11+5\sqrt{5}}{2} \) we get

\[
\int F_1(t)^{\nu} \, dt = \frac{5F_1(t)^{5/6 + \nu}}{5/6 + \nu} F_{Ap} \left[ \frac{1}{6} + \frac{\nu}{5} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{7}{6} + \frac{\nu}{5} \cdot \rho_1 F_1(t)^{5} , \rho_2 F_1(t)^{5} \right] + c.
\]

Hence if \( G(t) = t^\nu \), then

\[
y \left( \frac{5F_1(c_A)^{5/6 + \nu}}{5/6 + \nu} F_{Ap} \left[ \frac{1}{6} + \frac{\nu}{5} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{7}{6} + \frac{\nu}{5} \cdot \rho_1 F_1(c_A)^{5} , \rho_2 F_1(c_A)^{5} \right] \right) = F(A).
\]

More general we have: If

\[
G(t) = \sum_{n=1}^{\infty} \frac{G^{(n)}(0)}{n!} t^n,
\]
then

\[
\int G(F_1(t)) \, dt = 5F_1(c_A)^{5/6} \times 
\]
\[
\sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} F_1(c_A)^n \frac{F_{Ap} \left[ \frac{1}{6} + \frac{n}{5} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{7}{6} + \frac{n}{5} \cdot \rho_1 F_1(c_A)^{5} , \rho_2 F_1(c_A)^{5} \right]}{5/6 + n} 
\]
and

\[
y_i(F(A)) = 5F_1(c_A)^{5/6} \times 
\]
\[
\sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} F_1(c_A)^n \frac{F_{Ap} \left[ \frac{1}{6} + \frac{n}{5} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{7}{6} + \frac{n}{5} \cdot \rho_1 F_1(c_A)^{5} , \rho_2 F_1(c_A)^{5} \right]}{5/6 + n}.
\]

Also

\[
Q_G(A) = 5F_1(c_A)^{5/6} \times 
\]
\[
\sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} F_1(c_A)^n \frac{F_{Ap} \left[ \frac{1}{6} + \frac{n}{5} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{7}{6} + \frac{n}{5} \cdot \rho_1 F_1(c_A)^{5} , \rho_2 F_1(c_A)^{5} \right]}{5/6 + n}.
\]

Hence

\[
h(A) = Q_G(c_i(A)) = 5F_1(A)^{5/6} \times 
\]
\[
\sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} F_1(A)^n \frac{F_{Ap} \left[ \frac{1}{6} + \frac{n}{5} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{7}{6} + \frac{n}{5} \cdot \rho_1 F_1(A)^{5} , \rho_2 F_1(A)^{5} \right]}{5/6 + n}.
\]

This can be seen as the evaluation of Theorem 4 (evaluation with inverse integrals). However now we have infinite series expansion. The above equations can also help find \( y(A) \) in case \( G(A) \) is a polynomial.
6.3 The Case of $G(F(x)) = x$ function

From (77) we have

$$\sigma(x) = \frac{(x\sqrt{1-x^2})^{2/3}}{\sqrt{2} \cdot x}$$  \hspace{1cm} (168)

and

$$Q_G(x) = s_i(x) = \frac{3\sqrt{3\pi t}}{2\sqrt{4}} \cdot \phi \left[ \frac{1}{3} \frac{1}{3} \frac{5}{3} \cdot \frac{x^2}{3} \right] + c.$$  \hspace{1cm} (169)

Hence

$$g \left( \frac{3\sqrt{3\pi t}}{2\sqrt{4}} \cdot \phi \left[ \frac{1}{3} \frac{1}{3} \frac{5}{3} \cdot \frac{x^2}{3} \right] + c \right) = F(x).$$  \hspace{1cm} (170)

$$h_1^{(-1)}(x) = G(y(x)) = G(F(k_m G)) = k(m G(x)) = Q_G^{(-1)}(x).$$

Hence $h_1(x) = Q_G(x)$

$$h_1'(x) = s'_i(x) = \frac{1}{\sigma(x)} = \frac{\sqrt{3\pi t}}{x\sqrt{1-x^2}^{2/3}}.$$

6.4 The Case of Jacobi Theta Functions

For an extended version of this section one can see [15].

For $a, p$ positive rationals with $a < p$ and $|q| < 1$ we define the forms

$$[a, p; q] := \prod_{n=0}^{\infty} (1 - q^{np+a})(1 - q^{np+p-a})$$  \hspace{1cm} (171)

and

$$\theta(a, p; q) := [a, p; q]^* := q^{C_0} [a, p; q],$$  \hspace{1cm} (172)

where $C_0 := p/12 - a/2 + a^2/(2p)$.

With the help of Jacobi triple product identity it can be shown that

$$\vartheta \left( \frac{p}{2}, \frac{p}{2} - a; q \right) = q^{C_0} \vartheta_1(p \tau) [a, p; q],$$  \hspace{1cm} (173)

where

$$\vartheta(a, b; q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{an^2+bn}, \quad |q| < 1,$$  \hspace{1cm} (174)

is a theta function and

$$\vartheta_1(\tau) := \prod_{n=1}^{\infty} (1 - q^n), \quad q^{i\pi}, \quad \tau = i\sqrt{r}, \quad r > 0,$$  \hspace{1cm} (175)

is the Ramanujan-Dedekind eta function.
We also define
\[ Q_{\{a,p\}}(x) := \left[ a, p; e^{-\pi \sqrt{k_i(x)}} \right]^*. \] (176)

Here \( Q_{\{a,p\}}(x) \) is the \( Q_G(x) \) function defined in Section 3 relation (52) above. In the case of Jacobi theta functions describes their algebraic part (conjecture).

We will try to characterize these functions \( Q_{\{a,p\}}(x) \). For this, assume that \( P_n \) is the \( n \)-th modular equation of \( \theta(a, p; q) \), then
\[ \theta(a, p; q^n) = P_n(\theta(a, p; q)). \] (177)

Also assume that our conjecture holds, then
\[ Q_{\{a,p\}}(k_n^2r) = P_n(Q_{\{a,p\}}(k_r)). \]

By inverting \( k_r \), we get
\[ Q_{\{a,p\}}(k_n^2k_i(x)) = P_n(Q_{\{a,p\}}(x)). \]

Setting
\[ S_n(x) := k_n^2k_i(x), \] (178)
we lead to the next

**Theorem 24.**
If the \( n \)-th modular equation of \( \theta(a, p; q) \) is that of (177), then
\[ k_n^2k_i(x) = S_n(x) = Q_{\{a,p\}}^{-1}(P_n(Q_{\{a,p\}}(x))), \ n = 2, 3, 4, \ldots \] (179)

If one manages to solve equation (179) with respect to \( Q_{\{a,p\}}(x) \) for given \( a, p \), then
\[ \sum_{n=-\infty}^{\infty} (-1)^n q^{pn^2/2+(p-2a)n/2} = q^{-\frac{\pi^2}{24}+\frac{\pi^2}{24}\eta(q)p}Q_{\{a,p\}}(k_r), \forall r > 0 \] (180)

and \( Q_{\{a,p\}}(x) \) will be a root of a minimal polynomial of degree \( \nu = \nu(a, p, x) \).

Note that in case of rational \( x \in (0, 1) \) and \( a, p \) rational with \( 0 < a, p \), then the degree \( \nu \) is independent of \( x \) and the minimal polynomial of \( Q_{\{a,p\}}(x) \) will have integer coefficients.

**Example.**
The 2nd degree modular equation of \( A(1, 4; q) \) is
\[ 16u^8 + u^{16}v^8 - v^{16} = 0. \] (181)
If we solve with respect to \( v \), we get \( v = P_2(u) \), where \( v = A(1,4;q^2) \) and \( u = A(1,4;q) \). Moreover

\[
P_2(w) = \left( \frac{w^{16} + w^4 \sqrt{64 + w^{24}}}{2^{1/8}} \right)^{1/8}.
\] (182)

It is \( n = 2 \) then hold (see [9])

\[
k_{4r} = \frac{1 - \sqrt{1 - k_r^2}}{1 + \sqrt{1 - k_r^2}}.
\] (183)

Hence

\[
S_2(x) = k_{4k_i(x)} = \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}}.
\] (184)

Finally we get from the relation (179) of Theorem 24:

\[
\sqrt[8]{Q_{\{1,4\}}(x)^{16} + Q_{\{1,4\}}(x)^4 \sqrt{Q_{\{1,4\}}(x)^{24} + 64}} = Q_{\{1,4\}} \left( \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \right),
\] (185)

which has indeed a solution

\[
Q_{\{1,4\}}(x) = \left( \frac{4(1 - x^2)}{\sqrt{x}} \right)^{1/2}.
\]

Note.
We note that function \( m(q) = k_r^2 \) is implemented in program Mathematica. However a useful expansion is

\[
k_r = \sqrt{m(q)} = 4q^{1/2} \exp \left(-4 \sum_{n=1}^{\infty} q^n \sum_{d|n} \frac{(-1)^{d+n/d}}{d} \right),
\] (186)

where \( q = e^{-\pi \sqrt{r}} \), \( r > 0 \).

Continuing we denote

\[
\theta(q) := \theta_{\{a,p\}}(q) = q^{p/12 - a/2 + a^2/(2p)} \frac{\vartheta \left( \frac{p}{2}, \frac{p-2a}{2}; q \right)}{f(-q^p)}, \quad q = e^{-\pi \sqrt{r}}.
\] (187)

In the case of Jacobi theta functions we set

\[
Q(x) := Q_{\{a,p\}}(x) = e^{-\pi \sqrt{k_i(x)}}.
\] (188)

But it holds \( y(x) = F(k (m_G(x))) \) and \( m_G(x) = k_1 (Q_i(x)) \), hence \( y(x) = F(Q_i(x)) \), inverting

\[
y_i(x) = \theta_{\{a,p\}} \circ k_i \circ F_i(x).
\] (189)
Hence we get the following

**Theorem 25.**
If \( q = e^{-\pi \sqrt{r}} \), \( r > 0 \), then
\[
y(q) = R(q) \quad \text{and} \quad m_{G}^{(-1)}(r) = \theta(q).
\] (190)

**Theorem 26.**
The \( n \)-th modular equation of \( \theta(q) = \theta_{\{a,p\}}(q) \) is
\[
P_n(x) = Q_{\{a,p\}} \left( k \left( n^2 k \left( Q_{\{a,p\}}^{(-1)}(x) \right) \right) \right).
\] (191)

Also \( m_{G}(x) = k_{i} \left( Q_{\{a,p\}}^{(-1)}(x) \right) \), \( m_{G}^{(-1)}(n) = P_{\sqrt{n}} \left( Q_{\{a,p\}}(1) \right) \), \( n > 0 \) and
\[
y(x) = R \left( e^{\pi \sqrt{k_{i} \left( Q_{\{a,p\}}^{(-1)}(x) \right)}} \right).
\] (192)

**Theorem 27.**
If \( q = e^{-\pi \sqrt{r}} \), \( r > 0 \), then
\[
\frac{d\theta(q)}{dr} = \frac{1}{\phi(r)}.
\] (193)

**Proof.**
From \( m_{G}^{(-1)}(A) = \theta(q) \), \( q = e^{-\pi \sqrt{A}} \) we get
\[
5 \int_{0}^{R(q)} \frac{G(t)dt}{t^{5/6} - 11 - t^{5}} = \theta(q).
\] (194)

After derivating the above relation and using (5), we get
\[
G \left( R(q) \right) q^{-5/6}f(-q)^{4} = \theta'(q).
\] (195)

Using (14), we get the result.

**Notes.**
Assuming the above we have
\[
\theta'(q) = q^{-1} \eta(z)^{4}G \left( R(q) \right),
\]
where
\[
\eta(z) := q^{1/24}f(-q), \quad q = e^{2\pi iz}, \quad \text{Im}(z) > 0,
\]
is the Dedekind eta function (see also relation (1)).

**Conjecture 1.**
In the case $Q_G(x) = Q_{\{a,p\}}(x)$ we have that $G(R(q))$ is root of polynomial with integer coefficients.

**Definition 4.**
We call theta function of the $G$–transformation of Theorem 1 the function $m_G^{-1}(A)$, where $q = e^{-\pi\sqrt{A}}$, $A > 0$. By this way the definition of the theta functions is generalized and related with the $G$–transform.

Therefore for the function $m_G(A)$ holds

$$\theta \left( e^{-\pi\sqrt{m_G(A)}} \right) = A$$

(196)

and

$$R \left( e^{-\pi\sqrt{m_G(A)}} \right) = y(A).$$

(197)

**Theorem 28.**
For the modularity of $m_G^{-1}(A)$ we have the next relation

$$m_G^{-1} \left( \frac{1}{A} \right) = Q_G \left( \sqrt{1 - Q_G^{-1} \left( m_G^{-1}(A) \right)^2} \right).$$

(198)

**Proof.**
From relation (53) and the identity $k_1/r = k'_r$, we get the result.

**Example.**
Suppose $a = 1$, $p = 4$, then

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n} = q^{1/24} q(4\tau)Q_{\{1,4\}}(k_r).$$

(199)

Then $Q_{\{1,4\}}(x)$ will be

$$Q(x) = \sqrt[12]{2^{1/12}} \sqrt{1 - x^2 / x}.$$

For a certain $G$ we have from (62) and (64.1):

$$\sigma(x) = -6 \cdot 2^{5/6} x^{13/12} (1 - x^2)^{11/12} (1 + x^2)^{-1}.$$

Hence

$$y(x) = R \left( e^{-\pi\sqrt{k_i \left( \frac{1}{4} (-x^{12} + \sqrt{64 + x^{24}}) \right)} \right).$$

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