Cohomology of coherent sheaves on a proper scheme

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Let $f : X \to S$ be a proper morphism where $S = \text{Spec } A$ where $A$ is a noetherian ring. Let $\mathcal{F}$ be a coherent sheaf on $X$. Then a theorem of Grothendieck says that the cohomology groups $H^i(X, \mathcal{F})$ are $A$-modules of finite type. We set out to give an algorithm to compute these $A$-modules when $A$ is sufficiently algorithmic. We found general result about a $\Gamma(X, -)$-acylic resolution of $\mathcal{F}$.

We will first define a coherent sheaf $\mathcal{G}$ on $X$ to be primitive if $\text{support } (\mathcal{G}) = \bar{x}$ where $x$ is a point of $X$ and the natural mapping $\mathcal{G} \to i_x^*(\tilde{\mathcal{G}}_x)$ is injective where $i_x : \text{Spec } \mathcal{O}_{X,x} \to X$ is the natural morphism. The associated points of $\mathcal{G}$ is denoted by $\text{Ass}(\mathcal{G})$, which is a finite set.

The theorem we prove is the following.

**Theorem 1.** There exists an inclusion $\mathcal{G} \hookrightarrow \bigoplus_{x \in \text{Ass}(\mathcal{G})} \mathcal{H}_x$ where a) each $\mathcal{H}_x$ is a primitive coherent sheaf with support $\bar{x}$, b) each $\mathcal{H}_x$ is $\Gamma(X, -)$-acylic and its group of sections $\Gamma(X, \mathcal{H}_x)$ is an $A$-module of finite type, and c) $\alpha$ is an isomorphism on an open dense subset of the support of $\mathcal{G}$.

This theorem gives a method to compute the cohomology of $\mathcal{F}$. One applies the theorem inductively to construct a $\Gamma(X, -)$-acylic coherent resolution $0 \to \mathcal{F} \to \mathcal{F}^*$ and simply computes $H^i(X, \mathcal{F})$ as the $i$-homology group of the complex $\Gamma(X, \mathcal{F}^*)$.

Thus Theorem 1 implies

**Corollary (Grothendieck).** The cohomology groups $H^i(X, \mathcal{F})$ are $A$-modules of finite type.

§1. A lemma.

We will use

**Lemma 2.** If $\mathcal{F}$ is a coherent sheaf on $X$, there exists an embedding $\mathcal{F} \hookrightarrow \bigoplus_{x \in \text{Ass}(\mathcal{F})} \mathcal{G}_x$ where $\mathcal{G}_x$ is a primitive coherent sheaf with support $\bar{x}$ and $\beta$ is an isomorphism in an open dense subset of support $(\mathcal{F})$.

**Proof.** We have an injection $\mathcal{F} \hookrightarrow \bigoplus_{x \in \text{Ass}(\mathcal{F})} i_x^*(\widetilde{\mathcal{F}}_x)$ because the kernel would have no associated points. Similarly, we have the inclusion $\mathcal{F} \hookrightarrow \lim_{n} F_x/m_x^n F$ where $m_x$ is the maximal ideal of $\mathcal{O}_{x,x}$. As $F$ is coherent it follows that $\mathcal{F} \hookrightarrow \bigoplus_{x \in \text{Ass}(\mathcal{F})} i_x^*(\mathcal{F}_x/\widetilde{m}_x^n \mathcal{F}_x)$ where the $n_x$ are sufficiently big. This is our embedding if $\mathcal{G}_x$ is the image of $\mathcal{F}$ in $i_x^*(\mathcal{F}_x/\widetilde{m}_x^{n_x})$. QED

By the lemma to prove the theorem we may assume that $\mathcal{G}$ is primitive with support $\bar{x}$ for some $x$ in $X$. 

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§2. The proof.

To prove the theorem we will use basically Grothendieck’s idea of using Chow coverings. Let \( Z \) be a closed subscheme of \( X \) such that the support \((Z) = \bar{x}\) and \( G \) admits the structure of a \( \mathcal{O}_{Z} \)-module. We will modify Chow lemma to the non-reduce case.

Let \( Z = \bigcup U_i \) be a finite affine covering of \( Z \) by dense open subsets. Let \( \bar{U}_i \supset U_i \) be a projective imbedding for each \( i \). Take \( W \) to be the schematic closure of \( \cap U_i \) embedded in \( X \times_S \prod \bar{U}_i \). Then by the usual reasoning the projection \( W \to \prod \bar{U}_i \) is set-theoretically injective proper morphism. Thus \( W \) is a projective scheme over \( S \).

Let \( M \) be a very ample sheaf on \( \prod \bar{U}_i = K \). Then \( N = \pi_Z^* M \) is very ample on \( W \) and relatively ample for the projective morphism \( \pi_Z^*: W \to Z \), which is an isomorphism between open dense subsets.

Now let \( L_x \) be the maximal primitive quotient of \( \pi_Z^* G \) with support \( W \). By projective theory the \( K_x \equiv L_x \otimes M^\otimes n \) is \( \Gamma(W, -) \)-acylic and \( \pi_{Z*} \)-acylic for \( n \gg 0 \). Thus \( \pi_{Z*}(K_x) = H_x \) is a \( \Gamma(X, -) \)-acylic coherent primitive sheaf, by the degenerate Leray spectral sequence. It has support \( \bar{x} \) and its sections are isomorphic to \( \Gamma(W, K_x) \), which is an \( A \)-module of finite type.

To get the inclusion \( F \to H_x \) just multiply the natural map \( F \to \pi_{Z*} L_x \) by a section of \( M^\otimes n \) which is general. This proves the theorem.

References

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