Weak extension theorem for measure-preserving homeomorphisms of noncompact manifolds

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(Received Apr. 14, 2008) (Revised June 28, 2008)

Abstract. In this paper we deduce weak type extension theorems for the groups of measure-preserving homeomorphisms of noncompact manifolds. As an application, we show that the group of measure-preserving homeomorphisms with compact support of a noncompact connected manifold, endowed with the Whitney topology, is locally contractible.

1. Introduction.

In this paper we study some topological properties of groups of measure-preserving homeomorphisms and spaces of measure-preserving embeddings in noncompact manifolds (cf. [4], [5], [8], [11], [12]). Suppose $M$ is a $\sigma$-compact topological $n$-manifold possibly with boundary and $U$ is an open subset of $M$. Let $\mathcal{E}(U, M)$ denote the space of proper embeddings of $U$ into $M$ endowed with the compact-open topology. The local deformation lemma for $\mathcal{E}(U, M)$ [6], [7] asserts that for any compact subset $C$ of $U$ and any compact neighborhood $K$ of $C$ in $U$ there exists a deformation $\varphi_t$ ($t \in [0, 1]$) of an open neighborhood $V$ of the inclusion map $i_U: U \subset M$ in $\mathcal{E}(U, M)$ such that $\varphi_0(f) = f$, $\varphi_1(f)|_C = i_C$ and $\varphi_t(f)|_{U \cap K} = f|_{U \cap K}$ ($t \in [0, 1]$) for each $f \in V$. For a subset $A$ of $M$ let $\mathcal{H}_A(M)$ denote the group of homeomorphisms $h$ of $M$ with $h|_A = id_A$ endowed with the compact-open topology. The local deformation lemma is equivalent to the following weak type extension theorem: for any compact neighborhood $L$ of $C$ in $U$ there exists a neighborhood $V$ of $i_U$ in $\mathcal{E}(U, M)$ and a homotopy $s_t: \mathcal{U} \to \mathcal{H}_{M \setminus L}(M)$ such that $s_0(f) = id_M$ and $s_1(f)|_C = f|_C$ ($f \in \mathcal{U}$).

This result motivates the following general formulation: Suppose $G$ is a topological group acting on $M$ with the unit element $e$. Consider the subspace $\mathcal{E}^G(U, M)$ defined by $\mathcal{E}^G(U, M) = \{g|_U \mid g \in G\}$, where $g$ denotes the homeo-

2000 Mathematics Subject Classification. Primary 28D05; Secondary 57S05.

Key Words and Phrases. group of measure-preserving homeomorphisms, extension, principal bundle, Whitney topology, sigma-compact manifolds.

This work is supported by Grant-in-Aid for Scientific Research (No. 19540078).
morphism on $M$ induced by $g \in G$. The weak extension theorem for the group action of $G$ on $M$ asserts that there exists a neighborhood $U$ of $i_U$ in $\mathcal{E}^G(U, M)$ and a homotopy $s_t : U \rightarrow G$ such that $s_0(f) = e$ and $s_t(f)|_C = f|_C$ ($f \in U$).

Suppose $\mu$ is a good Radon measure on $M$ with $\mu(\partial M) = 0$. Let $\mathcal{H}(M; \mu)$ and $\mathcal{H}(M; \mu\text{-reg})$ denote the subgroups of $\mathcal{H}(M)$ consisting of $\mu$-preserving homeomorphisms and $\mu$-biregular homeomorphisms of $M$ and let $\mathcal{E}^\mu(U, M; \mu\text{-reg})$ denote the subspace of $\mathcal{E}^\mu(U, M)$ consisting of $\mu$-biregular proper embeddings of $U$ into $M$. In [8] A. Fathi obtained a local deformation lemma for the space $\mathcal{E}^\mu(U, M; \mu\text{-reg})$ ([8, Theorem 4.1]). This is reformulated as the weak extension theorem for the group $\mathcal{H}(M; \mu\text{-reg})$ ([8, Corollary 4.2]). In the case $M$ is compact and connected, he also obtained a selection theorem for $\mu$-biregular measures on $M$ ([8, Theorem 3.3]) and used these results to deduce the weak extension theorem for the group $\mathcal{H}(M; \mu)$ ([8, Theorem 4.12]).

In this paper we are concerned with the case where $M$ is non-compact. In [4] R. Berlanga has already extended the selection theorem for $\mu$-biregular measures to the non-compact case ([4, Theorem 4.1]). We combine these results to obtain the weak extension theorem for the group $\mathcal{H}(M; \mu)$ (cf. Corollary 5.1).

**Theorem 1.1.** Suppose $M$ is an $n$-manifold, $\mu$ is a good Radon measure on $M$ with $\mu(\partial M) = 0$, $C$ is a compact subset of $M$, $U$ is an open neighborhood of $C$ in $M$. Then there exists a neighborhood $U$ of $i_U$ in $\mathcal{E}^{\mathcal{H}(M; \mu)}(U, M)$ and a homotopy $s : U \times [0, 1] \rightarrow \mathcal{H}(M; \mu)$ such that

1. for each $f \in U$
   a. $s_0(f) = \text{id}_M$, b. $s_1(f)|_C = f|_C$, c. if $f = \text{id}$ on $U \cap \partial M$, then $s_t(f) = \text{id}$ on $\partial M$ ($t \in [0, 1]$),
   d. $s_t(i_U) = \text{id}_M$ ($t \in [0, 1]$).

In comparison with topological or $\mu$-biregular homeomorphisms, “$\mu$-preserving homeomorphism” is a global property and we can not obtain a compactly supported weak extension theorem for the group $\mathcal{H}(M; \mu)$. This obstruction vanishes on the kernel of the end charge homomorphism $\psi^\mu$.

In [2] S. R. Alpern and V. S. Prasad introduced the end charge homomorphism $\psi^\mu$, which is a continuous homomorphism defined on the subgroup $\mathcal{H}_{E\mu}(M; \mu)$ of $\mu$-preserving homeomorphisms of $M$ which fix the ends of $M$. The kernel of $\psi^\mu$, $\ker \psi^\mu$, includes the subgroup $\mathcal{H}_{s\mu}(M; \mu)$ of $\mu$-preserving homeomorphisms of $M$ with compact support. If $h \in \mathcal{H}_{E\mu}(M, E; \mu)$ and $\psi^\mu(h) = 0$, then one can split moves of $\mu$-volume by $h$. Hence, we can obtain the compactly supported weak extension theorem for the subgroup $\ker \psi^\mu$ (cf. Theorem 5.2).

**Theorem 1.2.** Suppose $M$ is a connected $n$-manifold, $\mu$ is a good Radon measure on $M$ with $\mu(\partial M) = 0$, $C$ is a compact subset of $M$ and $U$ and $V$ are open
neighborhoods of \( C \) in \( M \) such that \( V \cap O \) is connected for each connected component \( O \) of \( M - C \). Then there exists a neighborhood \( U \) of \( i_U \) in \( \partial_{\ker c}(U, M) \) and a homotopy \( s : \mathcal{U} \times [0, 1] \to \mathcal{H}_{M - V, c}(M; \mu) \) such that

1. for each \( f \in \mathcal{U} \)
   - \( s_0(f) = \text{id}_M \),
   - \( s_1(f)|_C = f|_C \),
   - if \( f = \text{id} \) on \( U \cap \partial M \), then \( s_t(f) = \text{id} \) on \( \partial M \) (\( t \in [0, 1] \))
2. \( s_t(i_U) = \text{id}_M \) (\( t \in [0, 1] \)).

We also discuss a non-ambient deformation lemma for \( \mu \)-preserving embeddings (Theorem 5.3).

In the last section we study the group \( \mathcal{H}_c(M; \mu)_w \) endowed with the Whitney topology (cf. \([3]\)). It is known that the group \( \mathcal{H}(N) \) and the subgroup \( \mathcal{H}(N; \nu) \) are locally contractible for any compact \( n \)-manifold \( N \) and any good Radon measure \( \nu \) on \( N \) with \( \nu(\partial N) = 0 \) (\([7\), Corollary 1.1]\), \([8\), Theorem 4.4]\)). In \([3]\) it is shown that the group \( \mathcal{H}_c(M)_w \) consisting of homeomorphisms of \( M \) with compact support, endowed with the Whitney topology, is locally contractible. In this article, as an application of the weak extension theorem for \( \mathcal{H}_c(M; \mu) \), we show that the group \( \mathcal{H}_c(M; \mu)_w \) is also locally contractible for any connected \( n \)-manifold \( M \) (Theorem 6.1).

This paper is organized as follows. Section 2 is devoted to the general formulations and basic properties of local weak extension property and local weak section property for group actions. Section 3 contains fundamental facts related to Radon measures on manifolds (selection theorems for measures, end charge homomorphism, etc.). In Section 4 we recall the local deformation lemma for biregular embeddings and discuss some direct consequences of this lemma. In Section 5 we obtain the weak extension theorems for the groups \( \mathcal{H}(M; \mu) \), \( \ker c^\mu \) and \( \mathcal{H}_c(M; \mu) \) and a non-ambient deformation lemma for \( \mu \)-preserving embeddings. In Section 6 we recall basic facts on the Whitney topology and show that the group \( \mathcal{H}_c(M; \mu)_w \) is locally contractible for any connected \( n \)-manifold \( M \).

2. Fundamental facts on group actions.

2.1. Conventions.

For a topological space \( X \) and a subset \( A \) of \( X \), the symbols \( \text{Int}_XA \), \( c_XA \) and \( \text{Fr}_XA \) denote the topological interior, closure and frontier of \( A \) in \( X \). Let \( \mathcal{C}(X) \) denote the collection of all connected components of \( X \).

Suppose \( Y \) is a locally connected, locally compact Hausdorff space. Let \( \mathcal{H}(Y) \) denote the group of homeomorphisms of \( Y \) endowed with the compact-open topology. For a subset \( A \) of \( Y \), let \( \mathcal{H}(A) = \{ h \in \mathcal{H}(Y) \mid h|_A = \text{id}_A \} \) (with the subspace topology). The group \( \mathcal{H}(Y) \) and the subgroup \( \mathcal{H}(A) \) are topological
groups. In general, for any topological group \( G \), the symbols \( G_0 \) and \( G_1 \) denote the connected component and the path-component of the unit element \( e \) in \( G \).

For subspaces \( A \subset X \) of \( Y \) let \( \mathcal{E}(X, Y) \) denote the space of embeddings \( f : X \to Y \) endowed with the compact-open topology, and let \( \mathcal{E}_A(X, Y) = \{ f \in \mathcal{E}(X, Y) | f|_A = \text{id}_A \} \) (with the subspace topology). By \( i_X : X \subset Y \) we denote the inclusion map of \( X \) into \( Y \).

In this article, an \( n \)-manifold means a paracompact \( \sigma \)-compact (separable metrizable) topological \( n \)-manifold possibly with boundary. Suppose \( M \) is an \( n \)-manifold. The symbols \( \partial M \) and \( \text{Int} M \) denote the boundary and interior of \( M \) as a manifold. For a subspace \( X \) of \( M \), an embedding \( f : X \to M \) is said to be proper if \( f^{-1}(\partial M) = X \cap \partial M \). Let \( \mathcal{E}^*(X, M) \) denote the subspace of \( \mathcal{E}(X, M) \) consisting of proper embeddings \( f : X \to M \). For a subset \( A \) of \( X \) let \( \mathcal{E}^*_A(X, M) = \mathcal{E}^*(X, M) \cap \mathcal{E}_{A}(X, M) \).

By an \( n \)-submanifold of \( M \) we mean a closed subset \( N \) of \( M \) such that \( N \) is an \( n \)-manifold and \( \text{Fr}_M N \) is locally flat in \( M \) and transverse to \( \partial M \) so that (i) \( M - \text{Int}_M N \) is an \( n \)-manifold and (ii) \( \text{Fr}_M N \) and \( N \cap \partial M \) are \((n - 1)\)-manifolds with the common boundary \( (\text{Fr}_M N) \cap (N \cap \partial M) \). For simplicity, let \( \partial_1 N = \text{Fr}_M N \), \( \partial_2 N = N \cap \partial M \) and \( N^c = M - \text{Int}_M N \). More generally, for a subset \( U \) of \( M \) let \( \partial_1 U = U \cap \partial M \).

Suppose \( M \) is an \( n \)-manifold.

**Lemma 2.1** ([1, Theorem 0], cf. [9]). Suppose \( C \) is a compact subset of \( M \) and \( U \) is a neighborhood of \( C \) in \( M \). Then there exists a compact \( n \)-submanifold \( N \) of \( M \) such that \( C \subset \text{Int}_M N \) and \( N \subset U \).

**Lemma 2.2.**

1. If \( M \) is connected and \( L \) is an \( n \)-submanifold of \( M \) such that \( \partial_1 L \) is compact, then there exists a connected \( n \)-submanifold \( N \) of \( M \) such that \( L \subset \text{Int}_M N \) and \( N \cap L^c \) is compact.
2. Suppose \( C \) is a compact subset of \( M \).
   
   i. For any neighborhood \( U \) of \( C \) in \( M \) there exists a compact \( n \)-submanifold \( N \) of \( M \) such that \( C \subset \text{Int}_M N \), \( N \subset U \) and \( O - N \) is connected for each \( O \in \mathcal{C}(M - C) \).
   
   ii. If \( U \) is an open neighborhood of \( C \) in \( M \) such that \( U \cap O \) is connected for each \( O \in \mathcal{C}(M - C) \), then there exists a compact \( n \)-submanifold \( N \) of \( M \) such that \( C \subset \text{Int}_M N \), \( N \subset U \) and \( N \cap O \) is connected for each \( O \in \mathcal{C}(M - C) \).

**Proof.**

1. Since \( M \) is connected and \( \partial_1 L \) is compact, \( \mathcal{C}(L) \) is a finite collection. Since \( M \) is connected, there exists a finite collection of disjoint arcs \( \{ a_i \} \) in \( L^c \) such that
$L \cup (\bigcup \alpha_i)$ is connected. We apply Lemma 2.1 to $C = \partial_i L \cup (\bigcup \alpha_i)$ in the $n$-manifold $L'$ in order to find a compact $n$-submanifold $N_0$ of $L'$ such that $C \subset \text{Int}_L N_0$ and each $K \in \mathcal{C}(N_0)$ meets $C$. Then $N = L \cup N_0$ satisfies the required conditions.

(2) (i) We may assume that $M$ is connected (apply the connected case to each component of $M$). By Lemma 2.1 there exists a compact $n$-submanifold $N_1$ of $M$ such that $C \subset \text{Int}_M N_1$ and $N_1 \subset U$. Let $\mathcal{C} = \{O \in \mathcal{C}(M - C) \mid O \nsubseteq N_1\}$. Since $\mathcal{C}(N_1)$ is a finite collection, so is $\mathcal{C}$.

For each $O \in \mathcal{C}$, it is seen that $O$ is a connected $n$-manifold, $N_1 \cap O$ is an $n$-submanifold of $O$, $(N_1 \cap O) \cap = N_1 \cap O$ in $O$ and $\text{Fr}_O(N \cap O) = (\text{Fr}_M N_1) \cap O$ is compact (it is a union of components of $\text{Fr}_M N_1$). Thus, by (1) we can find a connected $n$-submanifold $L_O$ of $O$ such that $N_1 \cap O \subset \text{Int}_O L_O$ and $L_O \cap (N_1 \cap O)$ is compact. Note that $L_O$ is closed in $O$ so that it is also a connected $n$-submanifold of $M$. Let $L = \bigcup_{O \in \mathcal{C}} L_O$ Then, $N = L$ satisfies the required conditions. In fact, $C \subset M - L = \text{Int}_M N, N \subset N_1$, $\mathcal{C} = \{O \in \mathcal{C}(M - C) \mid O \nsubseteq N\}$ and $O - N = \text{Int}_M L_O$ for each $O \in \mathcal{C}$.

(ii) Since $\mathcal{C}(U - C) = \{O \cap U \mid O \in \mathcal{C}(M - C)\}$, by replacing $M$ by $U$, we may assume that $U = M$. Again we may assume that $M$ is connected. By Lemma 2.1 there exists a compact $n$-submanifold $N_1$ of $M$ such that $C \subset \text{Int}_M N_1$.

Consider the finite collection $\mathcal{C} = \{O \in \mathcal{C}(M - C) \mid O \nsubseteq N_1\}$. For each $O \in \mathcal{C}$, it is seen that $O$ is a connected $n$-manifold, $N_1 \cap O$ is an $n$-submanifold of $O$, $(N_1 \cap O) \cap = N_1 \cap O$ in $O$ and $\text{Fr}_O(N \cap O) = (\text{Fr}_M N_1) \cap O$ is compact. Thus, by (1) we can find a connected $n$-submanifold $K_O$ of $O$ such that $N_1 \cap O \subset \text{Int}_O K_O$ and $K_O \cap (N_1 \cap O)$ is compact. Then, $N = \bigcup_{O \in \mathcal{C}} K_O$ satisfies the required conditions. In fact, $\{O \in \mathcal{C}(M - C) \mid O \nsubseteq N\} \subset \mathcal{C}$ and $N \cap O = K_O$ for each $O \in \mathcal{C}$. \hfill \Box

2.2. Pull-backs.

For maps $B_1 \xrightarrow{\pi} B \xrightarrow{\pi'} E$, we obtain the pull-back diagram in the category of topological spaces and continuous maps:

$$
\begin{array}{ccc}
p^* E & \xrightarrow{p'} & E \\
\pi' \downarrow & & \downarrow \pi \\
B_1 & \xrightarrow{p} & B
\end{array}
$$

Explicitly, the space $p^* E$ and the maps $B_1 \xleftarrow{\pi^*} p^* E \xrightarrow{p^*} E$ are defined by

$$p^* E = \{(b_1, e) \in B_1 \times E \mid p(b_1) = \pi(e)\} \quad \text{and} \quad \pi^*(b_1, e) = b_1, \quad p^*(b_1, e) = e.$$
Suppose a topological group $G$ acts on spaces $B$ and $B_1$ transitively. Let $p : B_1 \to B$ be a $G$-equivariant map. Fix a point $b_1 \in B_1$ and let $b = p(b_1) \in B$ and let $G_b$ be the stabilizer of $b$ under the $G$-action on $B$. Consider the orbit map $\pi : G \to B$, $\pi(g) = gb$. Then the maps $B_1 \xrightarrow{p} B \xleftarrow{\pi} G$ induce the pull-back diagram:

$$
\begin{array}{ccc}
p^*G & \longrightarrow & G \\
\pi' \downarrow & & \downarrow \pi \\
B_1 & \longrightarrow & B
\end{array}
$$

The group $G_b$ acts freely on $p^*G$ on the right by $(x, g) \cdot h = (x, gh)$, $(x, g) \in p^*G$, $h \in G_b$. The induced map $p' : p^*G \to G$ admits a right inverse $r : G \to p^*G$, $r(g) = (gb_1, g)$ (i.e., $p'r = id_G$).

**Definition 2.1.** We say that the $G$-equivariant map $p : B_1 \to B$ has the local section property (LSP$_G$) at $b_1$ if there exists a neighborhood $U_1$ of $b_1$ in $B_1$ and a map $s_1 : U_1 \to G$ such that $\pi s_1 = p|_{U_1}$.

**Lemma 2.3.**

1. The map $p$ has LSP$_G$ at $b_1$ if and only if the induced map $\pi' : p^*G \to B_1$ is a principal $G_b$-bundle.

2. If the fiber $p^{-1}(b)$ is contractible, then the map $p' : p^*G \to G$ is a homotopy equivalence.

**Proof.**

1. Suppose the map $p$ has LSP$_G$ at $b_1$. Take any point $b_2 \in B_1$. Since $G$ acts on $B_1$ transitively, there exists a $g \in G$ with $b_2 = gb_1$. Then $U_2 = gU_1$ is a neighborhood of $b_2$ in $B_1$ and the map $s_2 : U_2 \to G$, $s_2(x) = gs_1(g^{-1}x)$ satisfies the condition $\pi s_2 = p|_{U_2}$ (i.e., $\pi s_2(x) = gs_1(g^{-1}x)b = g(p(g^{-1}x)) = p(x)$). The map $\pi' : p^*G \to B_1$ admits a local trivialization

$$
\phi : U_2 \times G_b \cong (\pi')^{-1}(U_2) = \bigcup_{x \in U_2} (\{x\} \times \pi^{-1}(p(x)))
$$

over $U_2$ defined by $\phi(x, h) = (x, s_2(x)h)$.

The converse is obvious.

2. It remains to show that $rp' \simeq id_{p^*G}$. There exists a contraction $\phi_t : p^{-1}(b) \to p^{-1}(b)$ ($t \in [0, 1]$) such that $\phi_t(p^{-1}(b)) = \{b_1\}$. If $(x, g) \in p^*G$, then $x \in
Thus, we can define a homotopy

$$\Phi_t : p'^*G \to p'^*G$$

from $\text{id}_{p'^*G}$ to $rp'$ by $\Phi_t(x, g) = (g\phi_t(g^{-1}x), g)$.

### 2.3. Group actions and spaces of embeddings.

Suppose a topological group $G$ acts continuously on a locally compact Hausdorff space $Y$. Each $g \in G$ induces $\hat{g} \in \mathcal{H}(Y)$ defined by $\hat{g}(y) = g y$ ($y \in Y$). Let $H$ be any subset of $G$. For subsets $A, B$ of $Y$ we have the following subsets of $H$:

$$H_A = \{ h \in H \mid \hat{h}|_A = \text{id}_A \}, \quad H(B) = H_{Y \setminus B}, \quad H_A(B) = H_A \cap H(B),$$

$$H_c = \{ h \in H \mid \text{supp} \hat{h} \text{ is compact} \}.$$ 

If $H$ is a subgroup of $G$, then these are subgroups of $H$.

For subsets $X \subset C \subset U$ of $Y$, the group $G_X(U)$ acts continuously on the space $\mathcal{E}_X(C, U)$ by the left composition $g \cdot f = \hat{g}f$ ($g \in G_X(U), f \in \mathcal{E}_X(C, U)$) and we have the following subspace of $\mathcal{E}_X(C, U)$:

$$\mathcal{E}^H_X(C, U) = H_X(U) i_C = \{ \hat{g}|_C \mid g \in H_X(U) \}$$

(with the compact-open topology).

Since $\mathcal{E}^H_X(C, U) = \mathcal{E}^{H_X}(C, U)$, by replacing $H$ by $H_X$ if necessary, we omit $X$ in the subsequent statements.

Consider the pull-back diagram:

$$
\begin{array}{ccc}
\mathcal{E}^G(U, Y) & \longrightarrow & \mathcal{E}^G(C, Y) \\
p' & \downarrow & \downarrow \pi \\
p & \pi' & \\
G & & G \\
\end{array}
$$

where $\pi(g) = \hat{g}|_C$ and $p(f) = f|_C$.

The group $G$ acts on the spaces $\mathcal{E}^G(U, Y)$ and $\mathcal{E}^G(C, Y)$ transitively. The restriction map $p$ is $G$-equivariant and has the fiber $p^{-1}(i_C) = \mathcal{E}^G_C(U, Y)$.

**DEFINITION 2.2.** We say that the pair $(U, C)$ has the local section property for $G$ (LSP$_G$) if the $G$-equivariant map $p : \mathcal{E}^G(U, Y) \twoheadrightarrow \mathcal{E}^G(C, Y)$ has LSP$_G$ at $i_U$.

**LEMMA 2.4.** The pair $(U, C)$ has LSP$_G$ if and only if the map $\pi' : p'^*G \to \mathcal{E}^G(U, Y)$ is a principal $G_C$-bundle.

This lemma follows directly from Lemma 2.3 (1).
LEMMA 2.5. Suppose there exists a path $h : [0, 1] \to G$ such that $h_0 = e$, $\widehat{h}_1(U) \subset C$ and $\widehat{h}_1(U) \subset U$, $\widehat{h}_1(C) \subset C$ ($t \in [0, 1]$). Then the following hold.

1. The map $p : \mathcal{E}^G(U, Y) \to \mathcal{E}^G(C, Y)$ is a homotopy equivalence.

2. There exists a strong deformation retraction $\chi_t$ ($t \in [0, 1]$) of $\mathcal{E}^G(U, Y)$ onto the singleton $\{i_U\}$.

3. The map $p^1 : p^1G \to G$ is a homotopy equivalence.

PROOF.

(1) We can define a map $p_1 : \mathcal{E}^G(C, Y) \to \mathcal{E}^G(U, Y)$ by $p_1(f) = f \widehat{h}_1|_U$. It follows that

(i) $p_1p(f) = f \widehat{h}_1|_U$ and a homotopy $\phi_t : \text{id} \simeq p_t p$ is defined by $\phi_t(f) = f \widehat{h}_1|_U$, and

(ii) $pp_1(f) = f \widehat{h}_1|_C$ and a homotopy $\psi_t : \text{id} \simeq pp_1$ is defined by $\psi_t(f) = f \widehat{h}_1|_C$.

(2) The contraction $\chi_t$ of $\mathcal{E}^G(U, Y)$ is defined by $\chi_t(f) = \widehat{h}_1^{-1}f \widehat{h}_1|_U$.

(3) The assertion follows from (2) and Lemma 2.3 (2).

Lemmas 2.4 and 2.5 yield the following consequence.

PROPOSITION 2.1. If a subset $C$ of $Y$ satisfies the condition $(*)$ below, then the map

$$G_C \subset G \xrightarrow{\pi} \mathcal{E}^G(C, Y) \text{ defined by } \pi(h) = \widehat{h}|_C$$

is a locally trivial bundle up to homotopy equivalences and hence has the exact sequence for homotopy groups.

$(*)$ There exists a subset $U$ of $Y$ such that (i) $C \subset U$, (ii) the pair $(U, C)$ has LSP$_G$, and (iii) there exists a path $h_t \in G$ ($t \in [0, 1]$) such that

$$h_0 = e, \quad \widehat{h}_1(U) = C, \quad \widehat{h}_1(U) \subset U, \quad \widehat{h}_1(C) \subset C \quad (t \in [0, 1]).$$

2.4. Weak extension property.

Suppose a topological group $G$ acts on an $n$-manifold $M$. Consider a pair $(H, F)$ of subsets of $G$ and a triple $(V, U, C)$ of subsets of $M$ such that $C \subset U \cap V$ (we do not assume that $F \subset H$ and $U \subset V$).

DEFINITION 2.3. We say that the triple $(V, U, C)$ has the weak extension property for $(H, F)$ (abbreviated as WEP$_{H,F}$ or WEP$(H, F)$) if there exists a neighborhood $\mathcal{W}$ of $i_U$ in $\mathcal{E}^H(U, M)$ and a homotopy $s : \mathcal{W} \times [0, 1] \to F(V)$ such that
(1) for each \( f \in \mathcal{U} \) (i) \( s_0(f) = e \), (ii) \( s_1(f)|_C = f|_C \), (iii) if \( f = \text{id} \) on \( \partial U \), then \( s_t(f) = \text{id} \) on \( \partial M \) (\( t \in [0, 1] \)),
(2) \( s_t(id_U) = e \) (\( t \in [0, 1] \)).

The map \( s_t : \mathcal{U} \to F(V) \) (\( t \in [0, 1] \)) is called the local weak extension map (LWE map). When \( H = F \), we simply say that \((V, U, C)\) has WEP. When \( V = U \), we say that the pair \((U, C)\) has WEP\(_{H,F}\). Note that WEP\(_G\) for \((U, C)\) implies LSP\(_G\) for \((U, C)\).

One of our interest is the following problem.

**Problem 2.1.** Given a class of triples \((V, U, C)\) in \( Y \) and a subset \( F \) of \( G \), determine the largest subset \( H \) of \( G \) for which each triple \((V, U, C)\) in this class has WEP\((H, F)\).

The next lemma easily follows from the definition.

**Lemma 2.6.** Suppose \((V, U, C)\) and \((V', U', C')\) are two triples of subsets in \( M \) such that \( C \subset U \cap V \) and \( C' \subset U' \cap V' \) and \((H, F)\) and \((H', F')\) are two pairs of subsets in \( G \). If (i) \((V, U, C)\) has WEP\((H, F)\), (ii) \( V \subset V' \subset U \subset U', C \supset C' \) and (iii) \( H \supset H', F \subset F' \), then \((V', U', C')\) has WEP\((H', F')\).

**Lemma 2.7.** Suppose \( F \) is a subgroup of \( G \). If two triples \((V_1, U_1, C_1)\) and \((V_2, U_2, C_2)\) have WEP\((H, F)\) and \( V_1 \cap V_2 = \emptyset \), then the triple \((V_1 \cup V_2, U_1 \cup U_2, C_1 \cup C_2)\) also has WEP\((H, F)\).

**Proof.** For \( i = 1, 2 \) let \( \mathcal{E}\_H(U_i, M) \supset \mathcal{U} \to F(V_i) \) be the associated LWE map for \((V_i, U_i, C_i)\). Take a neighborhood \( \mathcal{U} \) of \( i_{U_i \cup U_2} \) in \( \mathcal{E}\_H(U_1 \cup U_2, M) \) such that \( f|_{U_i} \in \mathcal{U} \) (\( i = 1, 2 \)) for each \( f \in \mathcal{U} \). Then the required LWE map \( s_t : \mathcal{U} \to F(V_1 \cup V_2) \) for \((V_1 \cup V_2, U_1 \cup U_2, C_1 \cup C_2)\) is defined by

\[
s_t(f) = s_1^1(f|_{U_1})s_1^2(f|_{U_2}) \quad \text{(the multiplication in } G)\.
\]

Note that \( s_t(f) = s_1(f|_{U_1}) \) on \( V_1 \) and \( s_t(f) = \text{id} \) on \( M - (V_1 \cup V_2) \). \( \square \)

**3. Spaces of Radon measures and groups of measure-preserving homeomorphisms.**

**3.1. Spaces of Radon measures.**

Suppose \( Y \) is a locally connected, locally compact, \( \sigma \)-compact (separable metrizable) space. Let \( \mathcal{B}(Y) \) denote the \( \sigma \)-algebra of Borel subsets of \( Y \). A **Radon measure** on \( Y \) is a measure \( \mu \) on the measurable space \((Y, \mathcal{B}(Y))\) such that \( \mu(K) < \)
for any compact subset $K$ of $Y$. Let $\mathcal{M}(Y)$ denote the set of Radon measures on $Y$. The weak topology $w$ on $\mathcal{M}(Y)$ is the weakest topology such that the function

$$\Phi_f : \mathcal{M}(Y) \to \mathbb{R} : \mu \mapsto \int_Y f \, d\mu$$

is continuous for any continuous function $f : Y \to \mathbb{R}$ with compact support. The set $\mathcal{M}(Y)$ is endowed with the weak topology $w$, otherwise specified.

For $\mu \in \mathcal{M}(Y)$ and $A \in \mathcal{B}(Y)$, the restriction $\mu|_A$ is the Radon measure on $A$ defined by $(\mu|_A)(B) = \mu(B)$ $(B \in \mathcal{B}(A))$.

**Lemma 3.1 ([4, Lemma 2.2]).** For any closed subset $A$ of $Y$, the map $\mathcal{M}(Y) \to \mathcal{M}(A) : \mu \mapsto \mu|_A$ is continuous at each $\mu \in \mathcal{M}(Y)$ with $\mu(\text{Fr}_M A) = 0$.

We say that $\mu \in \mathcal{M}(Y)$ is good if $\mu(p) = 0$ for any point $p \in Y$ and $\mu(U) > 0$ for any nonempty open subset $U$ of $Y$. For $A \in \mathcal{B}(Y)$ let $\mathcal{M}^A(Y)$ denote the subspace of $\mathcal{M}(Y)$ consisting of good Radon measures $\mu$ on $Y$ with $\mu(A) = 0$. For $\mu, \nu \in \mathcal{M}(Y)$, we say that $\nu$ is $\mu$-biregular if $\nu$ and $\mu$ have same null sets (i.e., $\nu(B) = 0$ if and only if $\mu(B) = 0$ for any $B \in \mathcal{B}(Y)$). For $\mu \in \mathcal{M}^A(Y)$ we set

$$\mathcal{M}^A_\mu(Y)_{\text{reg}} = \{ \nu \in \mathcal{M}^A(Y) \mid \nu \text{ is } \mu\text{-biregular} \} \quad \text{(with the weak topology)}.$$

For $h \in \mathcal{H}(Y)$ and $\mu \in \mathcal{M}(Y)$, the induced measures $h_\ast \mu, h \ast \mu \in \mathcal{M}(Y)$ are defined by

$$(h_\ast \mu)(B) = \mu(h^{-1}(B)) \text{ and } (h \ast \mu)(B) = \mu(h(B)) \quad (B \in \mathcal{B}(Y)).$$

The group $\mathcal{H}(Y)$ acts continuously on the space $\mathcal{M}(Y)$ by $h \cdot \mu = h_\ast \mu$. We say that $h \in \mathcal{H}(Y)$ is

(i) $\mu$-preserving if $h_\ast \mu = \mu$ (i.e., $\mu(h(B)) = \mu(B)$ for any $B \in \mathcal{B}(Y)$) and

(ii) $\mu$-biregular if $h_\ast \mu$ and $\mu$ have the same null sets (i.e., $\mu(h(B)) = 0$ if and only if $\mu(B) = 0$ for any $B \in \mathcal{B}(Y)$).

Let $\mathcal{H}(Y; \mu) \subset \mathcal{H}(Y; \mu\text{-reg})$ denote the subgroups of $\mathcal{H}(Y)$ consisting of $\mu$-preserving and $\mu$-biregular homeomorphisms of $Y$ respectively. For a subset $A$ of $Y$, the subgroups $\mathcal{H}_A(Y; \mu), \mathcal{H}_A(Y; \mu)_1, \mathcal{H}_A(Y; \mu), \mathcal{H}_A(Y; \mu\text{-reg})$, etc. are defined according to the conventions in Sections 2.1 and 2.3.

For spaces of embeddings, we use the following notations. Suppose $Y$ is a locally compact, $\sigma$-compact (separable metrizable) space and $\mu \in \mathcal{M}(Y)$. For any $X \in \mathcal{B}(Y)$, an embedding $f : X \to Y$ is said to be
respectively.

This result yields the following consequences on the product of Radon measures.

For a subset $A$ of $X$, let $E_A(X, Y; \mu)$ and $\mathcal{E}_A(X, Y; \mu)$ denote the subspaces of $E_A(X, Y)$ consisting of $\mu$-biregular embeddings and $\mu$-preserving embeddings respectively.

Suppose $M$ is a compact connected $n$-manifold and $\mu \in \mathcal{M}_g(M) = \mathcal{M}^{BM}(M)$.

**Theorem 3.1 ([10]).** If $\nu \in \mathcal{M}_g(M)$ and $\nu(M) = \mu(M)$, then there exists $h \in \mathcal{H}_g(M)$ such that $h_*\mu = \nu$.

Let $\mathcal{M}^{\mu}(M; \mu) = \{ \nu \in \mathcal{M}_g(M) \mid \nu(M) = \mu(M) \}$ (with the weak topology). (See Section 3.2 for the definition in the case where $M$ is noncompact.)

The group $\mathcal{H}(M; \mu)$ acts continuously on $\mathcal{M}^{\mu}(M; \mu)$ by $h \cdot \nu = h_*\nu$. This action induces the map

$$
\pi : \mathcal{H}(M; \mu) \rightarrow \mathcal{M}^{\mu}(M; \mu) : h \mapsto h_*\mu.
$$

**Theorem 3.2 ([8, Theorem 3.3]).** The map $\pi$ admits a section

$$
\sigma : \mathcal{M}^{\mu}(M; \mu) \rightarrow \mathcal{H}(M; \mu) : \nu \mapsto \sigma(\nu) = \nu.
$$

such that $(\pi \sigma = \text{id and } \sigma(\mu) = \text{id})$.

Next we recall basic facts on the product of measures. Suppose $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ are $\sigma$-finite measure spaces. Let $\mathcal{F} \times \mathcal{G}$ denote the $\sigma$-algebra on $X \times Y$ generated by the family $\{ A \times B \mid A \in \mathcal{F}, B \in \mathcal{G} \}$. For $G \in \mathcal{F} \times \mathcal{G}$ and $x \in X$, the slice $G_x \subset Y$ is defined by $G_x = \{ y \in Y \mid (x, y) \in G \}$. It is well known that

1. there exists a unique measure $\omega$ on the measurable space $(X \times Y, \mathcal{F} \times \mathcal{G})$ such that $\omega(A \times B) = \mu(A) \cdot \nu(B)$ ($A \in \mathcal{F}, B \in \mathcal{G}$) (we follow the convention $0 \cdot \infty = 0$),

2. for any $G \in \mathcal{F} \times \mathcal{G}$

   (i) $\nu(G_x) (x \in X)$ is an $\mathcal{F}$-measurable function on $X$ and

   (ii) $\omega(G) = \int_X \nu(G_x) \, d\mu(x)$.

This result yields the following consequences on the product of Radon measures.
PROPOSITION 3.1. Suppose \((X, \mu)\) and \((Y, \nu)\) are locally compact separable metrizable spaces with Radon measures. Then the following hold:

1. \(\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y)\).
2. There exists a unique \(\omega \in \mathcal{M}(X \times Y)\) such that \(\omega(A \times B) = \mu(A) \cdot \nu(B)\)
   \((A \in \mathcal{B}(X), B \in \mathcal{B}(Y))\).
3. For any \(G \in \mathcal{B}(X \times Y)\),
   \(\omega(G) = \int_X \nu(G_x) d\mu(x)\).

The measure \(\omega\) is called the product of \(\mu\) and \(\nu\) and denoted by \(\mu \times \nu\).

PROPOSITION 3.2. Suppose \(f : (X, \mu) \to (X_1, \mu_1)\) and \(g : (Y, \nu) \to (Y_1, \nu_1)\) are homeomorphisms between locally compact separable metrizable spaces with Radon measures. Then the product homeomorphism \(f \times g : (X \times Y, \mu \times \nu) \to (X_1 \times Y_1, \mu_1 \times \nu_1)\) has the following properties:

1. If \(f\) and \(g\) are biregular, then \(f \times g\) is biregular.
2. If \(f\) and \(g\) are measure-preserving, then \(f \times g\) is measure-preserving.

PROOF. For \(G \in \mathcal{B}(X \times Y)\), we have

(a) \((\mu \times \nu)(G) = \int_X \nu(G_x) d\mu(x)\) and

(b) \((\mu_1 \times \nu_1)((f \times g)(G)) = \int_{X_1} \nu_1(((f \times g)(G))_{x_1}) d\mu_1(x_1)\)

\[= \int_{X_1} \nu_1(g(G_{f^{-1}(x_1)})) d\mu_1(x_1).\]

(1) Note that

(i) \((\mu \times \nu)(G) = 0\) if and only if \(\nu(G_x) = 0\) \((\mu\text{-a.e. } x \in X)\)

\((\text{i.e., } \exists A \in \mathcal{B}(X)\text{ such that } \mu(A) = 0\text{ and } \nu(G_x) = 0\text{ } (x \in X - A))\),

(ii) \((\mu_1 \times \nu_1)((f \times g)(G)) = 0\) if and only if \(\nu_1(g(G_{f^{-1}(x_1)})) = 0\) \((\mu_1\text{-a.e. } x_1 \in X_1)\).

Since \(f\) and \(g\) are biregular, if (i) holds, then it follows that

\[f(A) \in \mathcal{B}(X_1), \mu_1(f(A)) = 0\text{ and } \nu_1(g(G_{f^{-1}(x_1)}) = 0 \text{ } (x_1 \in X_1 - f(A)).\]
This implies (ii). The same argument shows the opposite implication. This means that \( f \times g \) is biregular.

(2) Since \( f \) and \( g \) are measure-preserving, it follows that

\[
(\mu_1 \times \nu_1)((f \times g)(G)) = \int_{X_1} \nu_1(g(G_{f^{-1}(x_1)})) \, d\mu_1(x_1) = \int_{X_1} \nu(G_{f^{-1}(x_1)}) \, d\mu_1(x_1)
\]
\[
= \int_{X} \nu(G_x) \, d\mu(x) = (\mu \times \nu)(G).
\]

This means that \( f \times g \) is measure-preserving. We also note that \( (f \times g)^{\ast}(\mu_1 \times \nu_1) \in \mathcal{M}(X \times Y) \) satisfies the condition: for any \( A \in \mathcal{B}(X) \) and \( B \in \mathcal{B}(Y) \)

\[
((f \times g)^{\ast}(\mu_1 \times \nu_1))(A \times B) = (\mu_1 \times \nu_1)((f \times g)(A \times B)) = (\mu_1 \times \nu_1)(f(A) \times g(B))
\]
\[
= \mu_1(f(A)) \cdot \nu_1(g(B)) = \mu(A) \cdot \nu(B).
\]

By definition we have \( (f \times g)^{\ast}(\mu_1 \times \nu_1) = \mu \times \nu \). This also implies the conclusion. \( \square \)

We conclude this subsection with some remarks on collars of the boundary of a submanifold. Suppose \( M \) is an \( n \)-manifold and \( \mu \in \mathcal{M}_g(M) \).

**Remark 3.1.** Suppose \( N \) is an \( n \)-submanifold of \( M \) such that \( \partial_+ N \) is compact. Since \( \mu(\partial \mathcal{M}) = 0 \), we have \( \mu(\partial N) = \mu(\partial_+ N) \). Take a bicollar \( \partial_+ N \times [-1, 1] \) of \( \partial_+ N \) in \( M \). Since \( \partial_+ N \times [-1, 1] \) is compact, it follows that \( \mu(\partial_+ N \times [-1, 1]) < \infty \) and \( \{ t \in [-1, 1] : \mu(\partial_+ N \times \{t\}) \neq 0 \} \) is a countable subset of \([ -1, 1] \). Hence, we can modify \( N \) by adding or subtracting a thin collar of \( \partial_+ N \) so that \( \mu(\partial N) = \mu(\partial_+ N) = 0 \).

Let \( m \) denote the Lebesgue measure on the real line \( \mathbb{R} \).

**Lemma 3.2.** Suppose \( N \) is an \( n \)-submanifold of \( M \) such that \( \partial_+ N \) is compact and \( \mu(\partial_+ N) = 0 \) and suppose \( \nu \in \mathcal{M}_{g}^{\partial}(\partial_+ N) \). Then, there exists a bicollar \( E = \partial_+ N \times [a, b] \) \((a < 0 < b) \) of \( \partial_+ N \) in \( M \) such that \( \partial_+ N \neq \partial_+ N \times \{0\} \), \( N \cap E = \partial_+ N \times \{0\} \) and \( \mu|_E = \nu \times (m|_{[a,b]}) \).

**Proof.** Let \( \mathcal{G}(\partial_+ N) = \{ F_1, \cdots, F_m \} \). For each \( i = 1, \cdots, m \), choose a small bicollar \( E_i = F_i \times [a_i, b_i] \) \((a_i < 0 < b_i) \) such that \( F_i = F_i \times \{0\}, \ N \cap E_i = F_i \times [a_i, 0], \ \mu(\partial_+ E_i) = 0, \ \mu(F_i \times [a_i, 0]) = |a_i| \nu(F_i) \) and \( \mu(F_i \times [0, b_i]) = b_i \nu(F_i) \). We can apply Theorem 3.1 to
to replace the identification of the collar $E_i = F_i \times [a_i, b_i]$ so that $\mu|_{E_i} = \nu|_{F_i} \times (m|_{[0, a_i]} \times [0, b_i])$. Finally, take $a, b$ such that $\max a_i < a < b < \min b_i$ and set $E = \partial N \times [a, b] = \bigcup_i (F_i \times [a_i, b_i])$. 

3.2. End compactification and finite-end weak topology (cf. [2], [4]).

In order to extend the selection theorem 3.2 to the noncompact case, it is necessary to include the information of the ends. Suppose $Y$ is a noncompact, connected, locally connected, locally compact, separable metrizable space. Let $Y$ be the space $Y$ which assigns an end $e$ to replace the identification of the collar $E_i = F_i \times [a_i, b_i]$ so that $\mu|_{E_i} = \nu|_{F_i} \times (m|_{[0, a_i]} \times [0, b_i])$. Then, take $a, b$ such that $\max a_i < a < b < \min b_i$ and set $E = \partial N \times [a, b] = \bigcup_i (F_i \times [a_i, b_i])$.

$$N(e, K) = e(K) \cup \{ e' \in E_Y \mid e'(K) = e(K) \} \quad (K \in \mathcal{K}(Y)).$$

Then $\overline{Y}$ is a connected, locally connected, compact, metrizable space, $Y$ is a dense open subset of $\overline{Y}$ and $E_Y$ is a compact 0-dimensional subset of $\overline{Y}$.

For $h \in \mathcal{K}(Y)$ and $e \in E_Y$ we define $h(e) \in E_Y$ by $h(e)(K) = h(e(h^{-1}(K)))$ ($K \in \mathcal{K}(Y)$). Each $h \in \mathcal{K}(Y)$ has a unique extension $\overline{h} \in \mathcal{K}(\overline{Y})$ defined by $\overline{h}(e) = h(e)$ ($e \in E_Y$). The map $\mathcal{K}(Y) \to \mathcal{K}(\overline{Y}) : h \mapsto \overline{h}$ is a continuous group homomorphism. For $A \subset Y$ we set $\mathcal{K}_{A\cup E_Y}(Y) = \{ h \in \mathcal{K}_A(Y) \mid \overline{h}|_{E_Y} = \text{id}_{E_Y} \}$. Note that $\mathcal{K}_{A\cup E_Y}(Y)_0 = \mathcal{K}_A(Y)_0$.

Let $\mu \in \mathcal{M}(Y)$. An end $e \in E_Y$ is said to be $\mu$-finite if $\mu(e(K)) < \infty$ for some $K \in \mathcal{K}(Y)$. Let $E^\mu_Y = \{ e \in E_Y \mid e \text{ is } \mu \text{-finite} \}$. Then $\overline{Y} \cup E^\mu_Y$ is an open subset of $\overline{Y}$.

For $A \subset Y$ and $\mu \in \mathcal{M}^A(Y)$ we set

$$\mathcal{M}^A(Y, \mu; e\text{-reg}) = \{ \nu \in \mathcal{M}^A(Y) \mid \nu \text{ is } e\text{-biregular, } E^\nu_Y = E^\mu_Y \},$$

$$\mathcal{M}^A(Y, \mu) = \{ \nu \in \mathcal{M}^A(Y, \mu; e\text{-reg}) \mid \nu(Y) = \mu(Y) \}.$$

The finite-ends weak topology $ew$ on $\mathcal{M}^A(Y, \mu; e\text{-reg})$ is the weakest topology such that the function

$$\Phi : \mathcal{M}^A(Y, \mu; e\text{-reg}) \to R : \nu \mapsto \int_Y f|_Y d\nu$$

is continuous for each $f \in C_0(Y, [0, \infty])$. 

For $\mu \in \mathcal{M}(Y)$ and $A \subset Y$ we set

$$\mathcal{M}^A(Y, \mu; e\text{-reg}) = \{ \nu \in \mathcal{M}(Y) \mid \nu \text{ is } e\text{-biregular, } E^\nu_Y = E^\mu_Y \},$$

$$\mathcal{M}^A(Y, \mu) = \{ \nu \in \mathcal{M}^A(Y, \mu; e\text{-reg}) \mid \nu(Y) = \mu(Y) \}.$$
is continuous for any continuous function \( f : Y \cup E_Y^\mu \to \mathbb{R} \) with compact support.

There is an alternative description of this topology ([4, Section 3, p. 245]). Consider the space \( \mathcal{M}(Y \cup E_Y^\mu) \) (with the weak topology). Each \( \nu \in \mathcal{M}_0(Y; \mu\text{-e-reg}) \) has a natural extension \( \overrightarrow{\nu} \in \mathcal{M}_0(Y \cup E_Y^\mu) \) defined by \( \overrightarrow{\nu}(B) = \nu(B \cap Y) \) for \( B \in \mathcal{B}(Y \cup E_Y^\mu) \). The topology \( \text{ew} \) on \( \mathcal{M}_0^A(Y; \mu\text{-e-reg}) \) is the weakest topology for which the injection

\[
\iota : \mathcal{M}_0^A(Y; \mu\text{-e-reg}) \to \mathcal{M}(Y \cup E_Y^\mu) : \nu \mapsto \overrightarrow{\nu}
\]

is continuous. The symbol \( \mathcal{M}_0^A(Y; \mu\text{-e-reg})_{\text{ew}} \) denotes the space \( \mathcal{M}_0^A(Y; \mu\text{-e-reg}) \) endowed with the topology \( \text{ew} \).

We say that \( h \in \mathcal{H}(Y) \) is \( \mu\text{-end-biregular} \) if \( h \) is \( \mu\text{-biregular} \) and \( E_Y^{h,\mu} = E_Y^\mu \) (i.e., \( \overrightarrow{h}(E_Y^\mu) = E_Y^\mu \)). Let \( \mathcal{H}(Y; \mu\text{-e-reg}) \) denote the subgroup of \( \mathcal{H}(Y) \) consisting of \( \mu\text{-end-biregular} \) homeomorphisms of \( Y \).

Suppose \( M \) is a connected \( n \)-manifold and \( \mu \in \mathcal{M}_0^0(M) \). The group \( \mathcal{H}(M; \mu\text{-e-reg}) \) acts continuously on \( \mathcal{M}_0^0(M; \mu)_{\text{ew}} \) by \( h \cdot \nu = h_\ast \nu \). This action induces the map

\[
\pi : \mathcal{H}(M; \mu\text{-e-reg}) \to \mathcal{M}_0^0(M; \mu)_{\text{ew}} : h \mapsto h_\ast \mu.
\]

**Theorem 3.3 ([4, Theorem 4.1])**. The map \( \pi \) has a section

\[
\sigma : \mathcal{M}_0^0(M; \mu)_{\text{ew}} \to \mathcal{H}_0(M; \mu\text{-reg})_1 = \mathcal{M}_0(M; \mu\text{-reg})_1
\]

such that \( (\pi \sigma = \text{id} \text{ and } \sigma(\mu) = \text{id}_M) \).

### 3.3. End charge homomorphism.

We recall basic properties of the end charge homomorphisms defined in [2, Section 14]. Suppose \( Y \) is a connected, locally connected, locally compact separable, metrizable space. Let \( \mathcal{D}(E_Y) \) denote the algebra of clopen subsets of \( E_Y \) and let \( \mathcal{B}_c(Y) = \{ C \in \mathcal{B}(Y) \mid \text{Fr} Y \text{ is compact} \} \). For each \( C \in \mathcal{B}_c(Y) \) let

\[
E_C = \{ e \in E_Y \mid e(K) \subset C \text{ for some } K \in \mathcal{K}(Y) \} \quad \text{and} \quad \overline{C} = C \cup E_C \subset \overline{Y}.
\]

Note that (i) \( E_C \in \mathcal{D}(E_Y) \) and \( \overline{C} \) is a neighborhood of \( E_C \) in \( \overline{Y} \) with \( \overline{C} \cap E_Y = E_C \), (ii) for \( C, D \in \mathcal{B}_c(Y) \) it follows that \( E_C = E_D \) if and only if \( C \Delta D = (C - D) \cup (D - C) \) is relatively compact (i.e., has the compact closure) in \( Y \), (iii) if \( C \in \mathcal{B}_c(Y) \) and \( h \in \mathcal{H}_0(E_Y, Y) \), then \( h(C) \in \mathcal{B}_c(Y) \) and \( E_{h(C)} = E_C \).

An end charge of \( Y \) is a finitely additive signed measure \( c \) on \( \mathcal{D}(E_Y) \), that is, a
function $c : \mathcal{Q}(E_Y) \to \mathbb{R}$ which satisfies the following condition:

$$c(F \cup G) = c(F) + c(G) \quad \text{for} \quad F, G \in \mathcal{Q}(E_Y) \quad \text{with} \quad F \cap G = \emptyset.$$ 

Let $\mathcal{I}(Y)$ denote the space of end charges $c$ of $Y$ endowed with the weak topology (or the product topology). This topology is the weakest topology such that the function

$$\Psi_F : \mathcal{I}(Y) \to \mathbb{R} : c \mapsto c(F)$$

is continuous for any $F \in \mathcal{Q}(E_Y)$. For $\mu \in \mathcal{M}(Y)$ let

$$\mathcal{I}(Y, \mu) = \left\{ c \in \mathcal{I}(Y) \left| \begin{array}{l}
(i) \quad c(F) = 0 \text{ for } F \in \mathcal{Q}(E_Y) \text{ with } F \subseteq E_Y^0 \\
(ii) \quad c(E_Y) = 0
\end{array} \right. \right\}$$

(with the weak topology). Then $\mathcal{I}(Y)$ is a topological linear space and $\mathcal{I}(Y, \mu)$ is a linear subspace.

For $h \in \mathcal{H}_{E_Y}(Y; \mu)$ the end charge $c^h \in \mathcal{I}(Y, \mu)$ is defined as follows: For any $F \in \mathcal{Q}(E_Y)$ there exists $C \in \mathcal{B}_c(Y)$ with $E_C = F$. Since $\bar{\mathcal{I}}_{|E_Y} = \text{id}$, it follows that $E_C = E_{h(C)}$ and that $C \Delta h(C)$ is relatively compact in $Y$. Thus $\mu(C - h(C))$, $\mu(h(C) - C) < \infty$ and we can define

$$c^h(F) = \mu(C - h(C)) - \mu(h(C) - C) \in \mathbb{R}.$$ 

This quantity is independent of the choice of $C$.

**Proposition 3.3.** The end charge homomorphism $c^h : \mathcal{H}_{E_Y}(Y; \mu) \to \mathcal{I}(Y, \mu)$ is a continuous group homomorphism ([2, Section 14.9, Lemma 14.21 (iv)]).

In [12] we have shown that, for any connected $n$-manifold $M$ and $\mu \in \mathcal{M}_0^M(M)$, the end charge homomorphism $c^h : \mathcal{H}_{E_Y}(M; \mu) \to \mathcal{I}(M; \mu)$ has a (non-homomorphic) section $s : \mathcal{I}(M, \mu) \to \mathcal{H}_0(M; \mu)$.

For any subset $A$ of $Y$ we have the restriction of $c^h$

$$c^h_A : \mathcal{H}_{A \cap E_Y}(Y; \mu) \to S(Y, \mu).$$

The kernel of the homomorphism $c^h$ is denoted by $\ker c^h$. Note that $\mathcal{H}_c(M; \mu) \subset \ker c^h$ and $(\ker c^h)_A = \ker c^\mu_A$. By the definition, if $h \in \ker c^h$, then for any $C \in \mathcal{B}_c(Y)$ we have $\mu(C - h(C)) = \mu(h(C) - C)$.
LEMMA 3.3. Suppose $h \in \ker \psi^*$ and $C \in \mathcal{B}_c(Y)$. If $L \in \mathcal{B}(C \cap h(C))$ and $C - L$ is relatively compact in $Y$, then $h(C) - L$ is also relatively compact and $\mu(h(C) - L) = \mu(C - L)$.

PROOF. Since $\mu(C - h(C)) = \mu(h(C) - C)$, the assertion follows from the equalities:

$$h(C) - L = (h(C) - C) \cup ((C \cap h(C)) - L) \quad \text{and}$$

$$C - L = (C - h(C)) \cup ((C \cap h(C)) - L).$$

4. Weak extension theorem for biregular homeomorphisms.

Throughout this section $M$ is an $n$-manifold and $\mu \in \mathcal{H}_d(M)$. The weak extension theorem for the group $G = \mathcal{H}(M; \mu\text{-reg})$ is already obtained in [8]. In this section, we discuss some consequences of this extension theorem. In Section 5 we combine the weak extension theorem for $\mathcal{H}(M; \mu\text{-reg})$ and the selection theorem for $\mu$-biregular measures (Theorems 3.2 and 3.3) in order to obtain the weak extension theorems for the groups $\mathcal{H}(M; \mu)$ and $\ker \psi^*$.

First we recall the deformation theorem for $\mu$-biregular embeddings [8, Theorem 4.1]. For $X \in \mathcal{B}(M)$ and $A \subset X$, let $\mathcal{E}_A(X, M; \mu\text{-reg})$ denote the space of proper $\mu$-biregular embeddings $f : X \to M$ with $f|_A = \text{id}_A$, endowed with the compact-open topology (cf. Sections 2.1 and 3.1).

Suppose $C$ is a compact subset of $M$, $U \in \mathcal{B}(M)$ is a neighborhood of $C$ in $M$ and $D \subset E$ are two closed subsets of $M$ such that $D \subset \text{Int} M$.

THEOREM 4.1 ([8, Theorem 4.1]). For any compact neighborhood $K$ of $C$ in $U$, there exists a neighborhood $\mathcal{U}$ of $i_U$ in $\mathcal{E}_{E-U}(U, M; \mu\text{-reg})$ and a homotopy $\varphi : \mathcal{U} \times [0, 1] \to \mathcal{E}_{D-U}(U, M; \mu\text{-reg})$, such that

1. for each $f \in \mathcal{U}$,
   (i) $\varphi_0(f) = f$,   (ii) $\varphi_1(f)|_C = i_C$,   (iii) $\varphi_t(f)|_{U-K} = f|_{U-K}$ ($t \in [0, 1]$),
2. if $f = \text{id}$ on $\partial U$, then $\varphi_t(f) = \text{id}$ on $\partial U$ ($t \in [0, 1]$).

Theorem 4.1 is equivalent to the next weak extension theorem.

THEOREM 4.2 ([8, Corollary 4.2]). For any compact neighborhood $L$ of $C$ in $U$, there exists a neighborhood $\mathcal{W}$ of $i_U$ in $\mathcal{E}_{E-U}(U, M; \mu\text{-reg})$ and a homotopy $s : \mathcal{W} \times [0, 1] \to \mathcal{H}_{D\setminus(M-L)}(M; \mu\text{-reg})$, such that

1. for each $f \in \mathcal{W}$,
   (i) $s_0(f) = \text{id}_M$,   (ii) $s_1(f)|_C = f|_C$,   (iii) if $f = \text{id}$ on $\partial U$, then $s_t(f) = \text{id}$ on $\partial M$. 


(2) \( s_t(i_U) = \text{id}_M \) (\( t \in [0, 1] \)).

(In [8, Corollary 4.2] the map \( s_1 \) alone is mentioned.)

Now we discuss some consequences of Theorem 4.2 for the group \( G = \mathcal{H}(M; \mu\text{-reg}) \). Suppose \( X \) is a compact subset of \( M \). Note that \( G_X = \mathcal{H}_X(M; \mu\text{-reg}) \).

Suppose \( C \) is a compact subset of \( M \) with \( X \subset C \) and \( U \) is a neighborhood of \( C \) in \( M \). Consider the pull-back diagram:

\[
\begin{array}{ccc}
p'^*G_X & \to & G_X \\
p' \downarrow & & \downarrow \pi \\
\mathcal{E}^G_X(U, M) & \to & \mathcal{E}^G_X(C, M),
\end{array}
\]

where \( \pi(h) = h|_C \) and \( p(f) = f|_C \).

By Theorem 4.2 the pair \( (U, C) \) has WEP\( _G \). Hence it has LSP\( _G \) and also LSP\( _{G_X} \). Thus the next assertion follows from Lemma 2.4.

**Lemma 4.1.** The induced map \( \pi' : p'^*G_X \to \mathcal{E}^G_X(U, M) \) is a principal \( G_X \)-bundle.

Suppose \( N \) is a compact \( n \)-submanifold of \( M \) such that \( \mu(\partial N) = 0 \) and \( X \subset \text{Int}_M N \). Take any compact \( n \)-submanifold \( N_1 \) of \( M \) such that \( \mu(\partial N_1) = 0 \) and \( N_1 \) is obtained from \( N \) by adding an outer collar of \( \partial N \). We obtain the pull-back diagram:

\[
\begin{array}{ccc}
p'^*G_X & \to & G_X \\
p' \downarrow & & \downarrow \pi \\
\mathcal{E}^G_X(N_1, M) & \to & \mathcal{E}^G_X(N, M), \quad p^{-1}(i_N) = \mathcal{E}^G_X(N_1, M),
\end{array}
\]

**Lemma 4.2.** There exists a path \( h : [0, 1] \to G_X \) such that

\[
h_0 = \text{id}_M, \quad h_1(N_1) = N \quad \text{and} \quad h_t(N_1) \subset N_1, \quad h_t(N) \subset N \quad (t \in [0, 1]).
\]

**Proof.**

(1) Let \( m \) denote the Lebesgue measure on \( \mathbb{R} \). We can find a bicollar \( E = \partial_+ N \times [a, b] \) (\( a < 0, b > 1 \)) of \( \partial_+ N \) in \( M - X \) and \( \nu \in \mathcal{M}^2_\delta(\partial_+ N) \) such that

\[
(i) \partial_+ N = \partial_+ N \times \{0\}, \quad \partial_+ N_1 = \partial_+ N \times \{1\} \quad \text{and} \quad (ii) \mu_E = \nu \times (m|_{[a, b]}).
\]
This follows from the following observation. First take any bicollar $E' = \partial_i N \times [-1, 2]$ of $\partial_i N$ in $M - X$ which satisfies (i) and the weaker condition (ii') $\mu(\partial_i N \times \{-1\}) = \mu(\partial_i N \times \{2\}) = 0$. Let $G(\partial_i N) = \{F_1, \ldots, F_m\}$ and set $E_i' = F_i \times [-1, 2]$ ($i = 1, \ldots, m$). Choose any $\nu \in \mathcal{H}_g(\partial_i N)$ such that $\nu(F_i) = \mu(F_i \times [0, 1])$ ($i = 1, \ldots, m$). For each $i = 1, \ldots, m$, determine $a_i < 0$ and $b_i > 1$ by $\lfloor a_i, \lfloor \nu(F_i) = \mu(F_i \times [-1, 0]) \rfloor$ and $(b_i - 1)\nu(F_i) = \mu(F_i \times [1, 2])$, and reparametrize $F_i \times [-1, 0]$ to $F_i \times [a_i, 0]$ and $F_i \times [1, 2]$ to $F_i \times [1, b_i]$. We can apply Theorem 3.1 on $F_i \times [a_i, 0]$, $F_i \times [0, 1]$ and $F_i \times [1, b_i]$ to obtain a new identification $E_i' = F_i \times [a_i, b_i]$ so that $\mu_{E_i'} = \nu \times (m|_{[a_i, b_i]})$. Take $a, b$ such that $\max, a_i < a < 0$ and $1 < b < \min, b_i$, and set $E = \bigcup_i (F_i \times [a_i, b_i])$.

(2) Choose $\lambda \in \mathcal{H}_\partial([a, b])$ such that $\lambda$ is piecewise affine and $\lambda(0) = a/2$, $\lambda(1) = 0$. We obtain two isotopies

\[
\lambda_t \in \mathcal{H}_\partial([a, b]) \quad (t \in [0, 1]) \text{ defined by } \lambda_t(s) = (1 - t)s + t\lambda(s) \quad \text{and} \quad g_t \in \mathcal{H}_{\partial, N \times [a, b]}(\partial_i N \times [a, b]) \quad (t \in [0, 1]) \text{ defined by } g_t(y, s) = (y, \lambda_t(s)).
\]

Note that $\lambda_0 = \text{id}$, $\lambda([a, 1]) = [a, 0]$, $\lambda([a, 0]) \subset [a, 0]$ and $\lambda([a, 1]) \subset [a, 1]$. Since $\lambda_t$ is also piecewise affine, it is seen that $\lambda_t$ is $m|_{[a_i, b_i]}$-biregular. Then each $g_t$ is $\nu \times (m|_{[a_i, b_i]})$-biregular by Proposition 3.2. Finally, the required isotopy $h_t \in \mathcal{H}_E(M; \mu\text{-reg}) \subset G_X$ ($t \in [0, 1]$) is defined by $h_t|_E = g_t$. \hfill \Box

By Lemmas 4.1, 4.2 and 2.5 we have the following conclusions.

**Lemma 4.3.**

1. The induced map $\pi' : p^*G_X \to \mathcal{E}_X^G(N_1, M)$ is a principal $G_N$-bundle.
2. The map $p : \mathcal{E}_X^G(N_1, M) \to \mathcal{E}_X^G(N, M)$ is a homotopy equivalence.
3. There exists a strong deformation retraction $\chi_t$ ($t \in [0, 1]$) of $\mathcal{E}_X^G(N_1, M)$ onto the singleton $\{i_{N_1}\}$.
4. The map $p' : p^*G_X \to G_X$ is a homotopy equivalence.

**Corollary 4.1.** Suppose $X$ is a compact subset of $M$ and $N$ is a compact $n$-submanifold of $M$ such that $\mu(\partial N) = 0$ and $X \subset \text{Int}_M N$. Then the restriction map

\[
\mathcal{H}_N(M; \mu\text{-reg}) \subset \mathcal{H}_X(M; \mu\text{-reg}) \xrightarrow{\pi} \mathcal{E}_X^G(M; \mu\text{-reg})(N, M) \text{ defined by } \pi(h) = h|_N
\]

is a fibration up to homotopy equivalences and has the exact sequence for homotopy groups.
5. Weak extension theorem for measure-preserving homeomorphisms.

Throughout this section $M$ is an $n$-manifold and $\partial g(M)$. In this section we combine the weak extension theorem for $G = \mathcal{H}(M; \mu\text{-reg})$ (Theorem 4.2) and the selection theorem for $\mu$-biregular measures (Theorems 3.2 and 3.3) in order to obtain the weak extension theorems for the groups $H = \mathcal{H}(M; \mu)$ and $F = \ker c^\mu$. We also discuss a non-ambient weak deformation of measure-preserving embeddings (Theorem 5.3). Some application to the group $H_c = \mathcal{H}_c(M; \mu)$ endowed with the Whitney topology is provided in Section 6.

5.1. Weak extension theorem for $H(M; \mu)$.

We obtain the weak extension theorem for $H(M; \mu)$ in a general form (Theorem 5.1, cf. [8, Theorem 4.12]). This answers Problem 2.1 and also leads us to the weak extension theorem for $\ker c^\mu$ in Section 5.2. (Recall that $M$ is an $n$-manifold, $G = \mathcal{H}(M; \mu$-reg) and $H = \mathcal{H}(M; \mu$.)

For $A, B \in \mathcal{B}(M)$, consider the subset $G_{A, B}$ of $G$ defined by

$$G_{A, B} = \{ h \in G | h|_A \in \mathcal{E}(A, M; \mu) \text{ and } \mu(h(L)) = \mu(L) \ (L \in \mathcal{C}(M - B)) \}.$$ 

When $A = B$, we simply write $G_A$. For any $X \subset M$ we have the pair $(G_{A, B}^X, H_X)$ of subsets in $G_X$.

**Lemma 5.1.** Suppose $N$ is a compact $n$-submanifold of $M$ with $\mu(\partial N) = 0$, $U \in \mathcal{B}(M)$ is a neighborhood of $N$ in $M$ and $X$ is a closed subset of $\partial M$ with $X \cap N = \emptyset$. Then the triple $(M, U, N)$ has WEP($G^X_N, H_X$).

**Proof.**

Case 1: First we consider the case where $M$ is connected.

Since $\mathcal{E}_{G^X_{iU}}(U, M) \subset \mathcal{E}_{iU}^*(U, M; \mu\text{-reg})$, by Theorem 4.2 applied to $(U, C) = (M - X, N)$, there exists a neighborhood $\mathcal{W}$ of $i_U$ in $\mathcal{E}_{G^X_{iU}}(U, M)$ and a map $\sigma : \mathcal{W} \times [0, 1] \rightarrow (G_X)_t$ such that

(i) for each $f \in \mathcal{W}$ (a) $\sigma_0(f) = \text{id}_M$, (b) $\sigma_1(f)|_N = f|_N$, (c) if $f = \text{id}$ on $\partial - U$, then $\sigma_t(f) = \text{id}$ on $\partial M$,

(ii) $\sigma_t(i_U) = \text{id}_M$ ($t \in [0, 1]$).

(1) First we modify the map $\sigma$ to achieve the following additional condition:

(i) (b') $\sigma_1(f) \in H$.

Consider the induced map

$$\nu : \mathcal{W} \times [0, 1] \rightarrow \mathcal{M}_0^\mu(M; \mu)_{\text{reg}} \text{ defined by } \nu_t(f) = \sigma_t(f)$$

.
Since $M$ is connected, each $L \in \mathcal{U}(N^c)$ meets $\partial_{+}N$. Since $\partial_{+}N$ is compact, it follows that $\mathcal{U}(N^c)$ is a finite set. We note that $\nu_{1}(f)|_{L} \in \mathcal{M}_{2}^{\mu}(L;\mu|_{L})$ for any $f \in \mathcal{U}$ and $L \in \mathcal{U}(N^c)$. In fact, since $\nu_{1}(f) \in \mathcal{M}_{2}^{\mu}(M;\mu_{e}-\text{reg})$ and $\mu(\partial N) = 0$, we have $\nu_{1}(f)|_{L} \in \mathcal{M}_{2}^{\mu}(L;\mu|_{L}-\text{e-reg})$. It remains to show that $\nu_{1}(f)(L) = \mu(L)$. Since $f \in \mathcal{E}^{G,N}(U,M)$, there exists $h \in G^{N}_X$ such that $f = h|_{U}$. Then $k \equiv h^{-1}\sigma_{1}(f) \in \mathcal{H}_{N}(M)$. Since $M$ is connected, we see that $N \cap L \neq \emptyset$, and since $k = \text{id}$ on $N$, we have $k(L) = L$. Hence, $\sigma_{1}(f)(L) = h(L)$ and it follows that $\nu_{1}(f)(L) = \mu(\sigma_{1}(f)(L)) = \mu(h(L)) = \mu(L)$.

For each $L \in \mathcal{U}(N^c)$ we obtain the map $\mathcal{U} \to \mathcal{M}_{2}^{\mu}(L;\mu|_{L})_{\text{ev}} : f \mapsto \nu_{1}(f)|_{L}$. By the alternative description of the finite-ends weak topology and Lemma 3.1, this map is seen to be continuous (cf. [11, Lemma 3.2]). By Theorem 3.3 there exists a map

$$\eta_{L} : \mathcal{M}_{2}^{\mu}(L;\mu|_{L})_{\text{ev}} \longrightarrow \mathcal{H}_{\partial}(L;\mu|_{L}-\text{reg})_{1}$$

such that $\eta_{L}(\nu)|_{L} = \nu$ and $\eta_{L}(\mu|_{L}) = \text{id}_{L}$.

Define the map $\tau_{L} : \mathcal{U} \times [0,1] \to \mathcal{H}_{\partial}(L;\mu|_{L}-\text{reg})_{1}$ by

$$\tau_{L}(f,t) = \eta_{L}((1-t)\mu|_{L} + t\nu_{1}(f)|_{L}).$$

Combining $\tau_{L}$ ($L \in \mathcal{U}(N^c)$), we obtain the map

$$\tau : \mathcal{U} \times [0,1] \longrightarrow \mathcal{H}_{N,\partial M}(M;\mu-\text{reg})_{1}$$

defined by $\tau(f,t) = \tau_{L}(f,t)$ on $L \in \mathcal{U}(N^c)$

$$\text{id} \quad \text{on } N.$$  

Note that $\tau_{0}(f) = \text{id}_{M}$ and $\tau_{1}(f),\mu = \nu_{1}(f)$. Define a map

$$\sigma' : \mathcal{U} \times [0,1] \longrightarrow \mathcal{H}_{X}(M;\mu-\text{reg})_{1} \quad \text{by} \quad \sigma'_{t}(f) = \begin{cases} \sigma_{2t}(f) & (t \in [0,1/2]) \\ \sigma_{1}(f)\tau_{t-1}(f) & (t \in [1/2,1]). \end{cases}$$

Then the map $\sigma'$ satisfies the conditions (i) (a), (b), (c) and (ii). The condition (i) (b') is verified by

$$\sigma'_{t}(f),\mu = \sigma_{1}(f),\tau_{t}(f),\mu = \sigma_{1}(f),\nu_{1}(f) = \sigma_{1}(f),\sigma_{1}(f)^{+} \mu = \mu.$$  

(2) To see that the triple $(M,U,N)$ has WEP($G^{N}_X,H_X$), we construct a map
Consider the induced map
\[ \nu' : \mathcal{B} \times [0, 1] \to \mathcal{M}^\partial(M; \mu)_{\text{cw}} \text{ defined by } \nu'_t(f) = \sigma'_t(f)^* \mu. \]

It is seen that \( \nu'_0(f) = \nu'_1(f) = \mu \). By Theorem 3.3 there exists a map
\[ \eta : \mathcal{M}^\partial(M; \mu)_{\text{cw}} \to (\mathcal{B})_1 \] such that \( \eta(\nu_\ast \mu) = \nu \) and \( \eta(\mu) = \text{id}_M \).

The required map \( s \) is defined by \( s_t(f) = \sigma'_t(f) \eta(\nu'_t(f)) \) (\((f, t) \in \mathcal{B} \times [0, 1]\)). The conditions (iii) and (iv) are easily verified. For example, (iii) (b) is seen by
\[ s_1(f) = \sigma'_1(f) \eta(\nu'_1(f)) = \sigma'_1(f) \eta(\mu) = \sigma'_1(f) \quad \text{and} \quad s_t(f)|_N = \sigma'_t(f)|_N = f|_N. \]

Case 2: Next we treat the general case where \( M \) may not be connected.

By Lemma 2.6 we may assume that \( U \) is compact. Let \( M_1, \ldots, M_m \) denote the connected components of \( M \) which meet \( U \). For each \( i = 1, \ldots, m \), we set
\[ (U_i, N_i, X_i) = (U, N, X) \cap M_i \] and \( \mu_i = \mu|_{M_i} \). By Case 1, the triple \((M_i, U_i, N_i)\) in \( M_i \) has WEP for \((G_i, H_i) = (\mathcal{B}_X(M_i; \mu_i, \text{reg})^N, \mathcal{H}_X(M_i; \mu_i))\). Since the pair \((G_i, H_i)\) can be canonically identified with the subpair \((G^N_X(M_i), H_X(M_i))\) of \((G^N_X, H_X)\) and \( \mathcal{G}^{G_i}(U_i, M_i) = \mathcal{G}^{G_i(M_i)}(U_i, M) = \mathcal{G}^N_X(U_i, M) \cap \mathcal{E}(U_i, M_i) \) which is an open subset of \( \mathcal{E}^{G^N}(U, M) \), it is seen that the triple \((M_i, U_i, N_i)\) in \( M \) has WEP \( G^N_X, H_X \). Hence, by Lemma 2.7 \((\bigcup M_i, U, N)\) has WEP \( G^N_X, H_X \) and by Lemma 2.6 so is \((M, U, N)\).

**THEOREM 5.1.** Suppose \( C \) is a compact subset of \( M \), \( U \in \mathcal{B}(M) \) is a neighborhood of \( C \) in \( M \) and \( X \) is a closed subset of \( \partial M \) with \( X \cap C = \emptyset \). Then the triple \((M, U, C)\) has WEP \( G^U_C, H_X \).

**PROOF.** By Lemma 2.2 (2)(i) and Remark 3.1, there exists a compact \( n \)-submanifold \( N \) of \( M \) such that
\[ C \subset \text{Int}_M N, \quad N \subset \text{Int}_M U - X, \]
\[ O - N \text{ is connected for each } O \in \mathcal{G}(M - C) \quad \text{and} \quad \mu(\partial N) = 0. \]

We show that \( G^U_C \subset G^N \). Take any \( h \in G^U_C \). Since \( h|_U \in \mathcal{G}(U, M; \mu) \), we have
Then the triple \((M,U,N)\) has WEP\((G^X_X,H_X)\) and by Lemma 2.6 we conclude that the triple \((M,U,C)\) has WEP\((G^U_C,H_X)\).

Since \(H_X \subset G^{U,C}_X\), the next statement is an immediate consequence of Theorem 5.1 and Lemma 2.6.

**Corollary 5.1.** Suppose \(C\) is a compact subset of \(M\), \(U \in \mathcal{B}(M)\) is a neighborhood of \(C\) in \(M\) and \(X\) is a closed subset of \(\partial M\) with \(X \cap C = \emptyset\). Then the triple \((M,U,C)\) has WEP\((\mathcal{H}_X(M;\mu))\).

### 5.2. The weak extension theorem for \(\ker e^\mu\)

Suppose \(M\) is a connected \(n\)-manifold and \(\mu \in \mathcal{M}_e^c(M)\). In this section we deduce the weak extension theorem for the group \(F = \ker e^\mu\) (Theorem 5.2). (Recall that \(G = \mathcal{H}(M;\mu\text{-reg})\) and \(H = \mathcal{H}(M;\mu)\). Note that \(H_e = F_e\) and \(H(C) = F(C)\) for any compact subset \(C\) of \(M\).)

**Theorem 5.2.** Suppose \(C\) is a compact subset of \(M\), \(U\) and \(V\) are open neighborhoods of \(C\) in \(M\) such that \(V \cap O\) is connected for each \(O \in \mathcal{E}(M - C)\). Then, the triple \((V,U,C)\) has WEP\((\ker e^\mu,\mathcal{H}_e(M;\mu))\).

**Proof.**

(1) By Lemma 2.2 (2)(ii) and Remark 3.1, there exists a compact \(n\)-submanifold \(N\) of \(M\) such that

\[
C \subset \text{Int}_M N, \quad N \subset V,
\]

\(N \cap O\) is connected for each \(O \in \mathcal{E}(M - C)\) and \(\mu(\partial N) = 0\).

Note that \(\mathcal{E}(N - C) = \{N \cap O \mid O \in \mathcal{E}(M - C)\}\). Take compact subsets \(D\) and \(W\) of \(M\) such that \(C \subset \text{Int}_M D\), \(D \subset \text{Int}_M W\) and \(W \subset U \cap \text{Int}_M N\). Since \(N \subset V\) and \(W \subset U\), by Lemma 2.6 it suffices to show that the triple \((N,W,C)\) has WEP\((\ker e^\mu,\mathcal{H}_e(M;\mu))\).

Since \(\mathcal{E}^F(W,M) \subset \mathcal{E}^*(W,M;\mu\text{-reg})\), by Theorem 4.2 there exists a neighborhood \(\mathcal{V}\) of \(i_W\) in \(\mathcal{E}^F(W,M)\) and a map

\[
s : \mathcal{V} \to G(N) \quad \text{such that} \quad s(f)|_D = f|_D \quad \text{and} \quad s(i_W) = \text{id}_M.
\]
Replacing $\mathcal{U}$ by a smaller one, we may assume that $f(W) \subset N$ ($f \in \mathcal{U}$).

(2) Consider the $\pi$-manifold $N$ and $\mu|_N \in \mathcal{M}_{\pi}(N)$. By Theorem 5.1 the triple $(N, D, C)$ has WEP for

$$(G', H') = (\mathcal{H}_{\partial, N}(N; \mu|_N), \mathcal{H}_{\partial, N}(N; \mu|_N)).$$

Let $\delta^G(D, N) \ni \mathcal{U} \xrightarrow{H'} H'$ be the associated LWE map. Each $h' \in H'$ has a canonical extension $\psi(h') \in H(N)$ and this defines the canonical homeomorphism $\psi: H' \cong H(N)$.

(3) We show that $s(f)|_N \in G'$ for any $f \in \mathcal{U}$. Since $s(f) \in G(N)$, we have $s(f)|_N \in \mathcal{H}_{\partial, N}(N; \mu|_N)$. Since $f \in \delta^F(W, M)$, there exists $h \in F$ such that $f = h|_W$. Since $s(f)|_D = f|_D = h|_D \in \delta(D, M; \mu)$ and $s(f)(N) = N$, it follows that $s(f)|_D \in \delta(D, N; \mu|_N)$. Take any $L \in \mathcal{C}(N - C)$. Then there exists a unique $O \in \mathcal{C}(M - C)$ with $L = N \cap O$. Let $K = O - L = O - N$. Consider $g = h^{-1} s(f) \in \mathcal{H}_D(M)$. Since $M$ is connected, we have $O \cap D \neq \emptyset$ and since $g = \text{id}$ on $D$, we have $g(O) = O$ and so $s(f)(O) = h(O)$. Since $s(f) \in G(N)$, it follows that

$$s(f)(K) = K \quad \text{and} \quad s(f)(L) = s(f)(O - K) = s(f)(O) - K = h(O) - K.$$ 

Thus, we have $\mu(s(f)(L)) = \mu(h(O) - K)$. Since

$$Fr_M O \subset C, \quad O - K = L \subset N \quad \text{and} \quad K = s(f)(K) \subset s(f)(O) = h(O),$$

it follows that $O \in \mathcal{B}_{\pi}(M), K \subset O \cap h(O)$ and $O - K$ is relatively compact in $M$. Since $h \in F$, by Lemma 3.3 we have $\mu(h(O) - K) = \mu(O - K) = \mu(L)$. Therefore, we have $\mu(s(f)(L)) = \mu(L)$. This means that $s(f)|_N \in G'$.

(4) By (3), for any $f \in \mathcal{U}$, we have $s(f)|_N \in G'$ and $f|_D = s(f)|_D = (s(f)|_N)|_D \in \delta^G(D, N)$. Thus, we obtain the continuous map $\phi: \mathcal{U} \to \delta^G(D, N)$ defined by $\phi(f) = f|_D$. Replacing $\mathcal{U}$ by a smaller one, we may assume that $\phi(\mathcal{U}) \subset \mathcal{U}$. Finally, the associated LWE map $S_t: \mathcal{U} \to H(N)$ for WEP($F, H_\pi$) of the triple $(N, W, C)$ is defined by

$$S_t(f) = \psi\sigma_t^* \phi(f).$$

Since $H_\pi \subset F$, the next statement is an immediate consequence of Theorem 5.2 and Lemma 2.6.

**COROLLARY 5.2.** Suppose $C$ is a compact subset of $M$, $U$ and $V$ are open neighborhoods of $C$ in $M$ such that $V \cap O$ is connected for each $O \in \mathcal{C}(M - C)$. Then the triple $(V, U, C)$ has WEP($\mathcal{U}_\pi(M; \mu)$).
5.3. Non-ambient weak deformation of measure-preserving embeddings.

Suppose $M$ is an $n$-manifold and $\mu \in \mathcal{M}_\#^\partial(M)$. In this section we obtain a non-ambient weak deformation theorem for measure-preserving embeddings. For $X \in \mathcal{R}(M)$, let $\mathcal{E}^\#(X, M; \mu) = \mathcal{E}(X, M; \mu) \cap \mathcal{E}^\#(X, M)$ with the compact-open topology.

**Theorem 5.3.** Suppose $C$ is a compact subset of $M$ and $U \in \mathcal{R}(M)$ is a neighborhood of $C$ in $M$. Then there exists a neighborhood $U$ of $iU$ in $\mathcal{E}(U; M, \mu)$ with the compact-open topology.

**Lemma 5.2.** Suppose $N$ is a compact $n$-submanifold of $M$ with $\mu(\partial_+ N) = 0$ and $U \in \mathcal{R}(M)$ is a neighborhood of $N$ in $M$. Then the pair $(U, N)$ admits an LWD map in $M$.

**Proof.**

Case 1: First we treat the case where $N$ is connected.

1. By Lemma 3.2 there exists a bicollar $E = \partial_+ N \times [a, b]$ ($a < 0 < b$) of $\partial_+ N$ in $M$ such that

$$\partial_+ N = \partial_+ N \times \{0\}, \quad N \cap E = \partial_+ N \times [a, 0] \quad \text{and} \quad \mu|_E = \nu \times (m|_{[a, b]}),$$

where $\nu \in \mathcal{M}_\#^\partial(\partial_+ N)$ and $m$ is the Lebesgue measure on $\mathcal{R}$. Let $\mathcal{E}(\partial_+ N) = \{F_1, \ldots, F_m\}$ and $E_i = F_i \times [a, b]$ ($i = 1, \ldots, m$). For notational simplicity, we use the following notations:

$$E(I) = \partial_+ N \times I, \quad E_i(I) = F_i \times I \quad (I \subset [a, b]) \quad \text{and} \quad N_t = (N - E) \cup E[a, t] \quad (t \in [a, b]).$$

Take $\varepsilon > 0$ such that $a < -3\varepsilon$, $3\varepsilon < b$, and define $\alpha_t \in \mathcal{H}_{\partial}(a, b)$ ($t \in (-2\varepsilon, 2\varepsilon)$) by the conditions:

$$\alpha_t(s) = s + t \quad (s \in [-\varepsilon, \varepsilon]) \quad \text{and} \quad \alpha_t \text{ is affine on the intervals } [a, -\varepsilon] \text{ and } [\varepsilon, b] .$$

For each $i = 1, \ldots, m$, we obtain the isotopy

$$\phi_t^i = \text{id}_{F_i} \times \alpha_t \in \mathcal{H}_{\partial, E_i}(E_i; \mu|_{E_i}^\# \text{-reg}) \quad (t \in (-2\varepsilon, 2\varepsilon)).$$
Note that $\alpha_0 = \text{id}_{[a,b]}$ and $\phi_0^i = \text{id}_{E_i}$.

Take a small neighborhood $\mathcal{U}$ of $i_N$ in $\mathcal{E}^s(N, M; \mu \text{-reg})$ such that for any $g \in \mathcal{U}$ and $i = 1, \ldots, m$,

$$E_i[a, -\varepsilon] \subset g(N) \cap E_i \subset E_i[a, \varepsilon], \quad N - \varepsilon \subset g(N) \subset N + \varepsilon \quad \text{and} \quad g(F_i) \subset E_i(-\varepsilon, \varepsilon).$$

Then, for each $g \in \mathcal{U}$ and $i = 1, \ldots, m$, we have

(i) $(-\varepsilon - a)\nu(F_i) < \mu(g(N) \cap E_i) < (\varepsilon - a)\nu(F_i)$,

(ii) $\mu(\phi_i^i(g(N) \cap E_i)) = \mu(g(N) \cap E_i) + t\nu(F_i)$, since $\phi_i^i$ is $\mu$-preserving on $E_i(-\varepsilon, \varepsilon)$.

For each $i = 1, \ldots, m$, consider the map

$$c_i : \mathcal{U} \longrightarrow \mathbb{R} \text{ defined by } c_i(g) = \mu(g(N) \cap E_i).$$

Since $\mu(g(\partial N)) = 0$, the map $c_i$ is seen to be continuous. Note that $c_i(g) \in ((-\varepsilon - a)\nu(F_i), (\varepsilon - a)\nu(F_i))$.

(2) Next we construct a neighborhood $\mathcal{U}$ of $i_U$ in $\mathcal{E}^s(U, M; \mu)$ and a map $\eta : \mathcal{U} \times [0, 1] \longrightarrow \mathcal{E}^s(N, M; \mu \text{-reg})$ such that for any $f \in \mathcal{U}$ and $t \in [0, 1]$,

(iii) $\eta_0(f) = i_N$, $\eta_1(f) = f|_N$, $\eta_t(i_U) = i_N$, and (iv) $\mu(\eta_t(f)(N)) = \mu(N)$.

By Theorem 4.2 there exists a neighborhood $\mathcal{U}$ of $i_U$ in $\mathcal{E}^s(U, M; \mu)$ and a map

$$\sigma : \mathcal{U} \times [0, 1] \longrightarrow \mathcal{E}^s(M; \mu \text{-reg})$$

such that $\sigma_0(f) = \text{id}_M$, $\sigma_1(f)|_N = f|_N$ ($f \in \mathcal{U}$) and $\sigma_t(i_U) = \text{id}_M$ ($t \in [0, 1]$).

Replacing $\mathcal{U}$ by a smaller one, we may assume that $\sigma_t(f)|_N \in \mathcal{U}$ ($f \in \mathcal{U}$, $t \in [0, 1]$). Consider the map

$$\gamma : \mathcal{U} \times [0, 1] \longrightarrow \mathcal{U} \subset \mathcal{E}^s(N, M; \mu \text{-reg}) \text{ defined by } \gamma_t(f) = \sigma_t(f)|_N.$$ 

The map $\gamma$ satisfies the condition (iii). To achieve the condition (iv) we modify the map $\gamma$.

We define the maps $\lambda^i : \mathcal{U} \times [0, 1] \rightarrow \mathbb{R}$ and $\tau^i : \mathcal{U} \times [0, 1] \rightarrow (-2\varepsilon, 2\varepsilon)$ by

$$\lambda^i_t(f) = (1 - t)c_i(i_N) + tc_i(f|_N) \quad \text{and} \quad c_i(\gamma_t(f)) + \tau^i_t(f)\nu(F_i) = \lambda^i_t(f).$$

Since $\lambda^i_t(f), c_i(\gamma_t(f)) \in ((-\varepsilon - a)\nu(F_i), (\varepsilon - a)\nu(F_i))$, we have

$$|\tau^i_t(f)|\nu(F_i) = |\lambda^i_t(f) - c_i(\gamma_t(f))| < 2\varepsilon\nu(F_i).$$
The map $\tau^t$ has the following properties:

(v) $\tau^0_0(f) = \tau^0_0(f) = \tau^0_0(i_u) = 0$,
(vi) $\mu(\phi^{\tau^0_0}_{\tau^0_0}(\gamma^0_0(f)(N) \cap E_i)) = \mu(\gamma^0_0(f)(N) \cap E_i) + \tau^0_0(f)\nu(E_i) = \lambda^0_0(f)$.

The assertion (vi) follows from the property (1)(ii), while the assertion (v) follows from

\[
\tau^0_0(f)\nu(E_i) = \lambda^0_0(f) - c_i(\gamma_0(f)) = c_i(i_N) - c_i(x_N) = 0, \\
\tau^0_1(f)\nu(E_i) = \lambda^1_1(f) - c_i(\gamma_i(f)) = c_i(f|N) - c_i(f|N) = 0, \\
\tau^0_1(i_u)\nu(E_i) = \lambda^1_1(i_u) - c_i(\gamma_i(i_u)) = c_i(i_N) - c_i(x_N) = 0.
\]

The maps $\phi^M_i$ ($i = 1, \ldots, m$) are combined to induce the map

$\phi : \mathcal{U} \times [0, 1] \to \mathcal{M}_E(M; \mu\text{-reg})$ defined by $\phi_t(f)|_{E_i} = \phi^M_i(f)$ ($i = 1, \ldots, m$).

The desired map $\eta : \mathcal{U} \times [0, 1] \to \mathcal{M}(N, M; \mu\text{-reg})$ is defined by $\eta(f) = \phi_t(f)\gamma^0_0(f)$.

From (v) it follows that $\phi_0(f) = \phi_t(f) = \phi_t(i_u) = \text{id}_M$, since $\phi_t(f)|_{E_i} = \phi_t(f)|_{E_i} = \phi_0 = \text{id}_{E_i}$.

Thus, the map $\eta$ satisfies the condition (iii). To see the condition (iv), first note that

\[
\eta_t(f)(N) = \phi_t(f)\gamma^0_0(f)(N) = \phi_t(f)\left(N_a \cup \left(\bigcup_i (\gamma^0_0(f)(N) \cap E_i)\right)\right)
\]
\[
= N_a \cup \left(\bigcup_i \phi^M_i(\gamma^0_0(f)(N) \cap E_i)\right).
\]

Since $f$ is $\mu$-preserving, we have $\mu(f(N)) = \mu(N)$. Hence, from (vi) it follows that

\[
\mu(\eta_t(f)(N)) = \mu(N_a) + \sum_i \mu(\phi^M_i(\gamma^0_0(f)(N) \cap E_i)) = \mu(N_a) + \sum_i \lambda^0_0(f)
\]
\[
= \mu(N_a) + (1-t) \sum_i c_i(i_N) + t \sum_i c_i(f|N)
\]
\[
= (1-t) \left(\mu(N_a) + \sum_i c_i(i_N)\right) + t \left(\mu(N_a) + \sum_i c_i(f|N)\right)
\]
\[
= (1-t)\mu(N) + t\mu(f(N)) = \mu(N).
\]
(3) The required LWD map \( s \) is obtained as follows.

Theorem 3.2 yields a map \( \chi : \mathcal{M}_g^0(N; \mu|_X) \rightarrow \mathcal{M}_\theta(N; \mu|_X\text{-reg}) \), such that

\[
\chi(\omega), (\mu|_X) = \omega (\omega \in \mathcal{M}_g^0(N; \mu|_X)) \quad \text{and} \quad \chi(\mu|_X) = \text{id}_X.
\]

By the condition (2)(iv) we have the map \( \rho : U \times [0, 1] \rightarrow \mathcal{M}_g^0(N; \mu|_X) \) defined by \( \rho_t(f) = \eta_t(f)^* \mu \).

Since \( \rho_t(f) = \eta_t(f)^* \mu = \left((\phi_t(f)\sigma_t(f))^*\mu\right)|_N \), the map \( \rho \) is the composition of the following maps:

\[
U \times [0, 1] \xrightarrow{\rho_1} \mathcal{H}(\mu\text{-reg}) \xrightarrow{\rho_2} \mathcal{M}_g^0(N; \mu\text{-reg}) \xrightarrow{\rho_3} \mathcal{M}_g^0(N; \mu|_X\text{-reg}),
\]

where \( \rho_1(f, t) = \phi_t(f)\sigma_t(f), \rho_2(h) = h^*\mu \) and \( \rho_3(\omega) = \omega|_N \).

Since \( \mu(\partial_s N) = 0 \), by Lemma 3.1 the third map is continuous. Thus the continuity of the map \( \rho \) follows from the continuity of these maps. Finally, the map

\[
s : U \times [0, 1] \rightarrow \mathcal{E}^*(N, M; \mu)
\]

is defined by \( s_t(f) = \eta_t(f)\chi(\rho_t(f)) \).

Since \( s_t(f)^*\mu = \chi(\rho_t(f))^*(\eta_t(f)^*\mu) = \chi(\rho_t(f))^*\rho_t(f) = \mu|_N \), it follows that \( s_t(f) \) is \( \mu \)-preserving. If \( t = 0, 1 \) or \( f = i_U \), then by (2)(iii), \( \eta_t(f) \) is \( \mu \)-preserving, and so \( \rho_t(f) = \mu|_N \) and \( s_t(f) = \eta_t(f) \). Hence, by (2)(iii) the map \( s \) satisfies the required conditions: \( s_0(f) = i_N, s_1(f) = f|_N \) and \( s_t(i_U) = i_N \).

Case 2: Next we treat the general case where \( N \) may not be connected.

Let \( \mathcal{C}(N) = \{N_1, \ldots, N_m\} \). By Case 1, each pair \((U, N_i)\) \((i = 1, \ldots, m)\) admits an LWD map in \( M \)

\[
\mathcal{E}^*(U, M; \mu) \supset \mathcal{W}_i \xrightarrow{s_i} \mathcal{E}^*(N_i, M; \mu) \quad (t \in [0, 1]).
\]

For each \( i = 1, \ldots, m \), choose a neighborhood \( U_i \) of \( N_i \) in \( U \) such that \( U_i \cap U_j = \emptyset \) \((i \neq j)\).

We can find a small neighborhood \( \mathcal{W} \) of \( i_U \) in \( \mathcal{E}^*(U, M; \mu) \) such that \( \mathcal{W} \subset \mathcal{W}_i \) and \( s_i^t(N_i) \subset U_i \) \((f \in \mathcal{W})\) for each \( i = 1, \ldots, m \). An LWD map

\[
s : U \times [0, 1] \rightarrow \mathcal{E}^*(N, M; \mu)
\]

for \((U, N)\) is defined by \( s_t(f)|_{N_i} = s_i^t(f) \quad (i = 1, \ldots, m) \). \( \square \)
PROOF OF THEOREM 5.3. By Lemma 2.1 and Remark 3.1 there exists a compact \( n \)-submanifold \( N \) of \( M \) such that \( \mu(\partial N) = 0 \) and \( C \subset N \subset \text{Int}_MU \). By Lemma 5.2 the pair \((U, N)\) admits an LWD map

\[ \mathcal{E}^*(U, M; \mu) \supset \mathcal{W} \xrightarrow{\sigma_t} \mathcal{E}^*(N, M; \mu) \quad (t \in [0, 1]). \]

An LWD map \( s_t : \mathcal{W} \to \mathcal{E}^*(C, M; \mu) \) for \((U, C)\) is defined by \( s_t(f) = \sigma_t(f)|_C \). \( \square \)

6. Groups of measure-preserving homeomorphisms endowed with the Whitney topology.

Suppose \( M \) is a connected noncompact \( n \)-manifold and \( \mu \in \mathcal{M}_0^c(M) \). In [3, Proposition 5.3] we have shown that the group \( H^c_0(Y) \), endowed with the Whitney topology, is locally contractible. In this section we shall apply the weak extension theorem for \( H^c_0(M; \mu) \) (Corollary 5.2) to verify the local contractibility of the group \( H^c_0(M; \mu) \) endowed with the Whitney topology (Theorem 6.1).

6.1. Homeomorphism groups with the Whitney topology.

First we recall basic properties of the Whitney topology on homeomorphism groups (cf. [3, Section 4.3]). Suppose \( Y \) is a paracompact space and \( \text{cov}(Y) \) is the family of all open covers of \( Y \). For maps \( f, g : X \to Y \) and \( \mathcal{W} \in \text{cov}(Y) \), we say that \( f, g \) are \( \mathcal{W} \)-near and write \((f, g) \prec \mathcal{W}\) if every point \( x \in X \) admits \( U \in \mathcal{W} \) with \( f(x), g(x) \in U \). For each \( h \in \mathcal{H}(Y) \) and \( \mathcal{W} \in \text{cov}(Y) \), let

\[ \mathcal{W}(h) = \{ f \in \mathcal{H}(Y) \mid (f, h) \prec \mathcal{W} \}. \]

The Whitney topology on \( \mathcal{H}(Y) \) is generated by the base \( \mathcal{W}(h) \) (\( h \in \mathcal{H}(Y) \), \( \mathcal{W} \in \text{cov}(Y) \)). The symbol \( \mathcal{H}(Y)_w \) denotes the group \( \mathcal{H}(Y) \) endowed with the Whitney topology (while the symbol \( \mathcal{H}(Y) \) denotes the group \( \mathcal{H}(Y) \) with the compact-open topology). It is known that \( G = \mathcal{H}(Y)_w \) is a topological group. Recall the notations \( G_0 = \mathcal{H}_0(Y)_w \) (the identity component of \( G \)) and \( G_c = \mathcal{H}_c(Y)_w \) (the subgroup of \( G \) consisting of homeomorphisms with compact support). In [3, Sections 4.1, 4.3] it is shown that \( \mathcal{H}_0(Y)_w \subset \mathcal{H}_c(Y)_w \).

6.2. The box topology on topological groups.

The Whitney topology is closely related to box products (cf. [3]). Next we recall basic properties of (small) box products (cf. [3, Sections 1, 2]). The box product \( \bigsqcup_{n \geq 1} X_n \) of a sequence of topological spaces \((X_n)_{n \geq 1}\) is the product \( \prod_{n \geq 1} X_n \) endowed with the box topology generated by the base consisting of boxes \( \prod_{n \geq 1} U_n \) (\( U_n \) is an open subset of \( X_n \)). The small box product \( \bigsqcup_{n \geq 1} X_n \) of a sequence of pointed spaces \((X_n, *_n)_{n \geq 1}\) is the subspace of \( \bigsqcup_{n \geq 1} X_n \) defined by
\[ \square_{n \geq 1} X_n = \{ (x_n)_{n \geq 1} \in \square_{n \geq 1} X_n \mid \exists m \geq 1 \text{ such that } x_n = *_n \ (n \geq m) \}. \]

It has the canonical distinguished point \((*_n)_{n \geq 1}\). For a sequence of subsets \(A_n \subset X_n \ (n \geq 1)\), we set
\[ \square_{n \geq 1} A_n = \square_{n \geq 1} X_n \cap \square_{n \geq 1} A_n. \]

We say that a space \(X\) is (strongly) locally contractible at \(x \in X\) if every neighborhood \(V\) of \(x\) contains a neighborhood \(U\) of \(x\) which is contractible in \(V\) (rel. \(x\)). A pointed space \((X, *_0)\) is said to be locally contractible if \(X\) is locally contractible at any point of \(X\) and strongly locally contractible at \(*_0\). It is easily seen that if a topological group \(G\) is locally contractible at the identity element \(e\), then the pointed space \((G, e)\) is locally contractible ([3, Remark 1.9]). The next lemma follows from a straightforward argument.

**Lemma 6.1 ([3, Proposition 1.10])**. If pointed spaces \((X_i, *_i) \ (i \geq 1)\) are locally contractible, then the small box product \(\square_{i \geq 1} (X_i, *_i)\) is also locally contractible as a pointed space.

Suppose \(G\) is a topological group with the identity element \(e \in G\). A sequence of closed subgroups \((G_n)_{n \geq 1}\) of \(G\) is called a tower in \(G\) if it satisfies the following conditions:

\[ G_1 \subset G_2 \subset G_3 \subset \cdots \quad \text{and} \quad G = \bigcup_{n \geq 1} G_n. \]

Any tower \((G_n)_{n \geq 1}\) in \(G\) induces the small box product \(\square_{n \geq 1} (G_n, e)\) and the multiplication map
\[ p : \square_{n \geq 1} (G_n, e) \longrightarrow G \quad \text{defined by} \quad p(x_1, \ldots, x_m, e, e, \ldots) = x_1 \cdots x_m. \]

Note that \(\square_{n \geq 1} G_n\) is a topological group with the coordinatewise multiplication and the identity element \(e = (e, e, \ldots)\) and that the map \(p\) is well-defined and continuous ([3, Lemma 2.1]).

**Definition 6.1.** We say that \(G\) carries the box topology with respect to \((G_n)_{n \geq 1}\) if the map \(p : \square_{n \geq 1} G_n \to G\) is an open map.

Recall that \(G\) is the direct limit of \((G_n)_{n \geq 1}\) in the category of topological groups if any group homomorphism \(h : G \to H\) to an arbitrary topological group \(H\) is continuous provided the restriction \(h|G_n\) is continuous for each \(n \geq 1\). If \(G\)
carries the box topology with respect to \((G_n)_{n \geq 1}\), then \(G\) is the direct limit of \((G_n)_{n \geq 1}\) in the category of topological groups ([3, Proposition 2.7]). Note that the map \(p\) is an open map if it is open at \(e\) (i.e., for any neighborhood \(U\) of \(e\) in \(\square_{n \geq 1} G_n\), the image \(p(U)\) is a neighborhood of \(e\) in \(G\)). We say that a map \(f : X \to Y\) has a local section at \(y \in Y\) if there exists a neighborhood \(U\) of \(y\) in \(Y\) and a map \(s : U \to X\) such that \(fs = i_U\). If the map \(p\) has a local section \(s : U \to \square_{n \geq 1} G_n\) at \(e \in G\), then (i) we can adjust \(s\) so that \(s(e) = e\) and so (ii) the map \(p\) is open at \(e\).

Thus, the next lemma follows from Definition 6.1 and Lemma 6.1.

**Lemma 6.2.** Suppose the map \(p : \square_{n \geq 1} G_n \to G\) has a local section at \(e\). Then

1. \(G\) carries the box topology with respect to the tower \((G_n)_{n \geq 1}\).
2. if the subgroups \(G_n\) \((n \geq 1)\) are locally contractible, then \(G\) is also locally contractible.

**Proof.** Consider the maps \(\pi : \square_{n \geq 1} G_n \to \square_{i \geq 1} G_{n(i)}\) and \(\eta : \square_{i \geq 1} G_{n(i)} \to \square_{n \geq 1} G_n\) defined by

\[
\pi(\ldots, x_{n(i-1)+1}, \ldots, x_{n(i)}, \ldots) = (\ldots, (x_{n(i-1)+1}, \ldots, x_{n(i)}), \ldots) \quad \text{and} \\
\eta(\ldots, x_{i-1}, x_i, \ldots) = (\ldots, e, x_{i-1}, e, \ldots, e, x_i, \ldots), \quad \text{where } n(0) = 0.
\]

The maps \(p\) and \(p'\) have the factorizations \(p' = p\eta\) and \(p = p'\pi\), from which follows the assertion.

**6.3. Local contractibility of \(\mathcal{H}_c(M; \mu)_w\).**

Suppose \(M\) is a connected noncompact \(n\)-manifold and \(\mu \in \mathcal{M}_0^n(M)\). Let \(H = \mathcal{H}(M; \mu)\) and \(F = \ker c^\mu\). (Recall that the subscript \(w\) means the Whitney topology. For example, \(H_{c,w} = \mathcal{H}_c(M; \mu)_w\).

Consider any sequence \((K_i)_{i \geq 1}\) of compact subsets of \(M\) such that \(K_i \subset \text{Int}M\cap K_{i+1}\) \((i \geq 1)\) and \(M = \bigcup_{i \geq 1} K_i\). It induces a tower \(H(K_i) = \mathcal{H}_{M-K_i}(M; \mu)\) \((i \geq 1)\) of \(H_{c,w}\) and the multiplication map

\[
p : \square_{i \geq 1} H(K_i) \to H_{c,w}, \quad p(h_1, \ldots, h_m, \text{id}_M, \text{id}_M, \ldots) = h_1 \cdots h_m.
\]

**Theorem 6.1.**

1. The multiplication map \(p : \square_{i \geq 1} H(K_i) \to \mathcal{H}_c(M; \mu)_w\) has a local section at \(\text{id}_M\).
The group $\mathcal{H}_c(M;\mu)_w$ carries the box topology with respect to the tower $(H(K_i))_{i\geq 1}$.

The group $\mathcal{H}_c(M;\mu)_w$ is locally contractible.

We need some preliminary lemmas. Consider a sequence of compact connected $n$-submanifolds $(M_i)_{i\geq 1}$ of $M$ such that $M_i \subset \text{Int} M_{i+1}$ ($i \geq 1$) and $M = \bigcup_{i \geq 1} M_i$. Let $M_0 = \emptyset$ and $L_i = M_i - \text{Int} M_{i-1}$ ($i \geq 1$). There exists a sequence of compact $n$-submanifolds $(N_i)_{i\geq 1}$ of $M$ such that $L_i \subset \text{Int} M_i$, and $N_i \cap N_j \neq \emptyset$ if and only if $|i-j| \leq 1$. We call the sequence $(M_i, L_i, N_i)_{i\geq 1}$ an exhausting sequence for $M$.

**Lemma 6.4.** For any sequence $(K_i)_{i\geq 1}$ of compact subsets of $M$ there exists an exhausting sequence $(M_i, L_i, N_i)_{i\geq 1}$ for $M$ such that for each $i \geq 1$ (i) $K_i \subset M_i$, (ii) $\mu(\partial_i, M_i) = 0$ and (iii) the pair $(N_i, L_i)$ has WEP($F, H_c$).

**Proof.** By the repeated application of Lemma 2.1, we can find a sequence of compact connected $n$-submanifolds $(M_i)_{i\geq 1}$ of $M$ such that (i) $K_i \subset M_i \subset \text{Int} M_{i+1}$, $\mu(\partial_i, M_i) = 0$ ($i \geq 1$) and $M = \bigcup_{i \geq 1} M_i$, (ii) $L$ is noncompact and $M_{i+1} \cap L$ is connected for each $i \geq 1$ and each $L \in \mathcal{C}(M_i)$.

Let $M_i = \emptyset$ ($i \leq 0$) and $M_j = M_j - \text{Int} M_j$ ($j > i$).

(1) First we show that the pair $(N, K) = (M_{i-1}^{j+1}, M_i^j)$ has WEP($F, H_c$) for each $j > i \geq 0$. Let $\mathcal{C}(M_{i-1}^j) = \{C_1, \ldots, C_m\}$ and set $(N_k, K_k) = (N \cap C_k, K \cap C_k)$ ($k = 1, \ldots, m$). Since $(N_k, K_k)$ is a disjoint finite family, by Lemma 2.7 it suffices to show that each pair $(N_k, K_k)$ has WEP($F, H_c$).

Note that $\mathcal{C}(K_k) = \{E_0, E_1, \ldots, E_t\}$, where

$$E_0 = M_i \cup \bigcup_{k \neq k} C_k \quad \text{and} \quad \{E \in \mathcal{C}(M_j^t) \mid E \subset C_k\} = \{E_1, \ldots, E_t\}.$$ 

(If $i = 0$, we ignore $E_0$.) By the above condition (ii) it is seen that the intersections

$$N_k \cap E_0 = M_i \cap C_k \quad \text{and} \quad N_k \cap E_t = M_{i+1} \cap E_t \quad (t = 1, \ldots, \ell)$$

are connected. Hence, we can apply Theorem 5.2 to $(V, U, C) = (\text{Int} M_i, N_k, K_k)$ to conclude that this triple has WEP($F, H_c$). Thus, by Lemma 2.6 the pair $(N_k, K_k)$ also has WEP($F, H_c$).

(2) Now consider the subsequence $(M_{3i})_{i\geq 1}$. Let $L_i = M_{3i-4}^{3i-1}$ and $N_i = M_{3i-4}^{3i+1}$ ($i \geq 1$). Then, it is seen that $(M_{3i}, L_i, N_i)_{i\geq 1}$ is an exhausting sequence for $M$ and by (1) each pair $(N_i, L_i)$ has WEP($F, H_c$). $\square$
Suppose \((M_i, L, N_i)_{i \geq 1}\) is an exhausting sequence for \(M\). It induces a tower \((H(M_i))_{i \geq 1}\) of \(H_{c,w}\) and the multiplication map \(p : \square_{i \geq 1} H(M_i) \to H_{c,w}\).

**Lemma 6.5.** If each pair \((N_{2i}, L_{2i})\) \((i \geq 1)\) has WEP(\(H_c\)), then the map \(p : \square_{i \geq 1} H(M_i) \to H_{c,w}\) has a local section \(s : \mathcal{W} \to \square_{i \geq 1} H(M_i)\) at \(\text{id}_M\) such that \(s(\text{id}_M) = (\text{id}_M)_{i \geq 1}\).

**Proof.** We use the following notations: Let \(L_o = \bigcup_i L_{2i}, L_o = \bigcup_i L_{2i-1}\) and \(N_o = \bigcup_i N_{2i}\). Consider the continuous maps defined by

(a) \(r_c : H_{c,w} \to \square_i g^H(L_{2i}, M), r_c(h) = (h|_{L_{2i}}), \) and

\(r : H_{c,w} \to \square_i g^H(N_{2i}, M), r(h) = (h|_{N_{2i}}),\)

(b) \(\lambda : \square_i H(N_{2i}) \to H_i(N_{2i}), \lambda((g_i)_i)|_{N_{2i}} = g|_{N_{2i}}\) and

\(\lambda : \square_i H(L_{2i-1}) \to H_i(L_{2i}), \lambda((h_i)|_{L_{2i-1}}) = h|_{L_{2i-1}},\)

(c) \(\rho : \square_i H(N_{2i}) \times \square_i H(L_{2i-1}) \to H_{c,w}, \rho(g, h) = \lambda(g)\lambda_0(h).\)

Note that the map \(\lambda_0\) is a homeomorphism, since for any \(h \in H_i(L_o)\) we have \(h = \text{id}\) on \(\partial_i M_i\) and \(h(M_i) = M_i\), so that \(h(L_i) = L_i\) \((i \geq 1)\).

First we construct a local section of the map \(\rho\) at \(\text{id}_M\). By the assumption, for each \(i \geq 1\) there exists a neighborhood \(\mathcal{V}_i\) of the inclusion map \(i_{N_{2i}}\) in \(g^H(N_{2i}, M)\) and a map

\(\sigma : \mathcal{V}_i \to H(N_{2i})\) such that \(\sigma|_{L_{2i}} = f|_{L_{2i}}\) \((f \in \mathcal{V}_i)\) and \(\sigma(i_{N_{2i}}) = \text{id}_M.\)

Since \(\square_i \mathcal{V}_i\) is a neighborhood of \((i_{N_{2i}})_i\) in \(\square_i g^H(N_{2i}, M)\), the preimage \(\mathcal{W} = \rho^{-1}(\square_i \mathcal{V}_i)\) is a neighborhood of \(\text{id}_M\) in \(H_{c,w}\). The maps \((\sigma_i)_i\) determine the continuous maps

\(\sigma : \square_i \mathcal{V}_i \to \square_i H(N_{2i})\) defined by \(\sigma((f_i)_i) = (\sigma_i(f_i))_i\) and

\(\eta : \lambda \sigma r : \mathcal{W} \to H_i(N_{2i}).\)

For each \(g \in \mathcal{W}\) we have \(\eta(g) = g\) on \(L_0\) and \(\eta(g)^{-1} g \in H_{c,W} = H_{c,L_0}\). Thus we obtain the map

\(\phi : \mathcal{W} \to H_i(L_0)_{\omega}\) defined by \(\phi(g) = \eta(g)^{-1} g.\)

The required local section \(\zeta : \mathcal{W} \to \square_i H(N_{2i}) \times \square_i H(L_{2i-1})\) of the map \(\rho\) is defined by

\(\zeta(g) = (\sigma r(g), \lambda_0^{-1} \phi(g)).\)
In fact, we have

$$\rho \zeta(g) = \rho(\sigma r(g), \lambda^{-1}_0 \phi(g)) = \lambda(\sigma r(g)) \phi(g) = \eta(g)(\eta(g)^{-1}g) = g.$$  

Note that $$\zeta(\text{id}_M) = ((\text{id}_M), (\text{id}_M)).$$

For each $$h \in \mathcal{U}$$ the image $$\zeta(h) = ((f_i)_i, (g_i)_i)$$ satisfies the following conditions:

(i) $$h = \lambda((f_i)_i) \lambda((g_i)_i) = (f_1 f_2 \cdots)(g_1 g_2 \cdots) = f_1 f_2 g_2 f_3 g_3 \cdots.$$  
(ii) $$f_i \in H(N_{2i}) \subset H(M_{2i+1}), g_i \in H(L_{2i-1}) \subset H(M_{2i+2})$$ ($$i \geq 1$$).

(iii) $$(\text{id}_M, \text{id}_M, f_1, g_1, f_2, g_2, \ldots) \in \sqcup_{1 \leq i} H(M_i)$$ and

$$h = p(\text{id}_M, \text{id}_M, f_1, g_1, f_2, g_2, \ldots).$$  

Therefore, the required local section $$s : \mathcal{U} \rightarrow \sqcup_i H(M_i)$$ of the map $$p : \sqcup_i H(M_i) \rightarrow H_{c,w}$$ is defined by

$$s(h) = (\text{id}_M, \text{id}_M, f_1, g_1, f_2, g_2, \ldots).$$

This completes the proof. $\square$

**Lemma 6.6.** Suppose $$N$$ is a compact $$n$$-manifold, $$L$$ is a (locally flat) $$(n-1)$$-submanifold of $$\partial N$$ and $$\nu \in \mathcal{M}_{\partial}(N)$$. Then the group $$\mathcal{H}_L(N;\nu)$$ is locally contractible.

**Proof.** In [8, Theorem 4.4] the case where $$L = \emptyset$$ or $$\partial N$$ is verified. For the sake of completeness we include a proof. We may assume that $$N$$ is connected.

(1) First we see that the group $$G_L = \mathcal{H}_L(N;\nu\text{-reg})$$ is locally contractible. Since $$G_L$$ is a topological group, it suffices to show that it is semi-locally contractible at $$\text{id}_N$$, that is, a neighborhood of $$\text{id}_N$$ contracts in $$G_L$$. Using a collar $$L \times [0,2]$$ of $$L$$ in $$N$$ (cf. Lemma 3.2), we have a deformation of $$G_L$$ to $$G_{L \times [0,1]}$$ which fixes $$\text{id}_N$$. Applying Theorem 4.1 to $$(C, U, D, E) = (N, N, L, L \times [0,1])$$, we can find a neighborhood of $$\text{id}_N$$ in $$G_{L \times [0,1]}$$ which contracts in $$G_L$$. These deformations are combined to yield a desired contraction of a neighborhood of $$\text{id}_N$$ in $$G_L$$.

(2) Next we show that the group $$H_L = \mathcal{H}_L(N;\nu)$$ is a strong deformation retract (SDR) of $$G_L$$. By Theorem 3.2 the map $$\pi : G \rightarrow \mathcal{M}_{\partial}(N;\nu)$$ admits a section $$s : \mathcal{M}_{\partial}(N;\nu) \rightarrow G_{\partial} \subset G_L$$. This yields a homeomorphism of pairs

$$H_L \times (\mathcal{M}_{\partial}(N;\nu), \{\nu\}) \approx (G_L, H_L) : (h, \omega) \mapsto s(\omega)h.$$  

Since $$\mathcal{M}_{\partial}(N;\nu)$$ admits the “straight line contraction” to $$\{\nu\}$$, we obtain an SDR of $$G_L$$ onto $$H_L$$.

Finally, the conclusion follows from the observations (1) and (2). $\square$
PROOF OF THEOREM 6.1. (1), (3) By Lemma 6.4 there exists an exhausting sequence \((M_i, L_i, N_i)_{i \geq 1}\) for \(M\) such that \(\mu(\partial_i M_i) = 0\) \((i \geq 1)\) and each pair \((N_i, L_i)\) \((i \geq 1)\) has WEP\((H_{\partial_i})\). By Lemma 6.5 the multiplication map \(p': \square_{i \geq 1} H(M_i) \to H_{c,w}\) has a local section at \(\text{id}_M\). By Lemma 6.3 this implies the assertion (1) (consider a mixed sequence of \((K_i)\) and \((M_i)\)). By Lemma 6.6 the group \(H(M_i) \cong H_{\partial_i M_i}(M_i; \mu|_{M_i})\) is locally contractible for each \(i \geq 1\). Thus, by Lemma 6.2 (2) the group \(H_{c,w}\) is also locally contractible.

(2) The assertion follows from (1) and Lemma 6.2 (1).

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