The Coulomb-Oscillator Relation on n-Dimensional Spheres and Hyperboloids

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Abstract

In this paper we establish a relation between Coulomb and oscillator systems on n-dimensional spheres and hyperboloids for \( n \geq 2 \). We show that, as in Euclidean space, the quasiradial equation for the \( n+1 \) dimensional Coulomb problem coincides with the \( 2n \)-dimensional quasiradial oscillator equation on spheres and hyperboloids. Using the solution of the Schrödinger equation for the oscillator system, we construct the energy spectrum and wave functions for the Coulomb problem.

1 Introduction

It has long been known that the Coulomb and oscillator potentials are two paradigms in quantum mechanics that possess dynamical or hidden symmetries: \( O(n+1) \) for motion in a Coulomb field and \( SU(n) \) for the oscillator. On the other hand the connections with these two Lie groups of dynamical symmetries provide relations between the Coulomb and oscillator systems. In particular the \( (n+1) \) radial Schrödinger equation for the Coulomb system is identical to the oscillator equation for \( 2n \)-dimensions by the duality transformation. It is also known that the complete relation (not only for the radial part) is possible for only special dimensions \( (2,2) \), \( (3,4) \) and \( (5,8) \) respectively. The dual mappings in these cases are so-called Levi-Civita, Kustaanheimo-Stiefel and Hurwitz transformations.

The generalization of the Coulomb problem to the three-sphere has been done in the famous article of Schrödinger and for the \( n \)-dimensional hyperboloid. Later the Coulomb and

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oscillator problem on spheres and pseudospheres was discussed from many point of view in 
[8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In a previous article [20] we have constructed a series of complex mappings $S_{2C} \rightarrow S_2$, $S_{4C} \rightarrow S_3$ and $S_{8C} \rightarrow S_5$, which extend to spherical geometry the Levi-Civita, Kustaanheimo-Stiefel and Hurwitz transformations, well known for Euclidean space. We have shown that these transformations establish a correspondence between Coulomb and oscillator problems in classical and quantum mechanics for dimensions (2,2), (3,4) and (5,8) on the spheres. A detailed analysis of the real mapping on the curved space has been done in [21]. It was shown that in the stereographic projection (see also the paper [22]) the relation between Coulomb and oscillator problems functionally coincide with the flat space Levi-Civita and Kustaanheimo-Stiefel transformations.

In the present paper we find the relation between the quasiradial Schrödinger equations for Coulomb and oscillator problems on the $n$-dimensional sphere and two-sheeted hyperboloids for $n \geq 2$.

## 2 Coulomb-oscillator relation on n-sphere

The Schrödinger equation describing the nonrelativistic quantum motion on the $n$-dimensional sphere: $s_0^2 + s_1^2 + \cdots + s_n^2 = R^2$, where $s_i$ are Cartesian coordinates in ambient Euclidean $(n+1)$-space, has the following form ($\hbar = \mu = 1$)

$$ \mathcal{H}\Psi = \left( -\frac{1}{2} \Delta_{LB} + V(s) \right) \Psi = E\Psi \quad (1) $$

where the Laplace-Beltrami operator in arbitrary curvilinear coordinates $\xi_\mu$ is

$$ \Delta_{LB} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_\mu} g^{\mu\nu} \sqrt{g} \frac{\partial}{\partial \xi_\nu}, \quad g = \det \|g_{\mu\nu}\|, \quad g_{\alpha\mu} g^{\mu\nu} = \delta^\nu_\alpha. \quad (2) $$

For any central potential $V(\chi)$ the Schrödinger equation admits separation of variables in hyperspherical coordinates

\[
\begin{align*}
    s_0 &= R \cos \chi \\
    s_1 &= R \sin \chi \cos \vartheta_1 \\
    s_2 &= R \sin \chi \sin \vartheta_1 \cos \vartheta_2 \\
    &\vdots \\
    s_{n-1} &= R \sin \chi \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{n-2} \cos \varphi \\
    s_n &= R \sin \chi \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{n-2} \sin \varphi.
\end{align*}
\]

where $\chi, \vartheta_1, \ldots \vartheta_{n-2} \in [0, \pi], \varphi \in [0, 2\pi)$. We can separate the angular part of the wave function using the ansatz

$$ \Psi(\chi, \vartheta_1, \ldots, \vartheta_{n-2}, \varphi) = \mathcal{R}(\chi) Y_{l_1,l_2,l_{n-2}}(\vartheta_1, \ldots, \vartheta_{n-2}, \varphi) \quad (3) $$

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where \( l_i \) are the angular hypermomenta and \( L \) is total angular momentum, and the hyperspherical function \( Y_{L,l_1,l_2,l_3} (\vartheta_1, \cdots \vartheta_{n-2}, \varphi) \) is the solution of the Laplace-Beltrami eigenvalue equation on the \( n-1 \) dimensional sphere. After separation of variables in (1) we obtain the quasiradial equation

\[
\frac{1}{\sin^{n-1} \chi} \frac{d}{d\chi} \sin^{n-1} \chi \frac{d\mathcal{R}(\chi)}{d\chi} + \left[ 2R^2 E - \frac{L(L + n - 2)}{\sin^2 \chi} - 2R^2 V(\chi) \right] \mathcal{R}(\chi) = 0. \tag{4}
\]

Using the substitution

\[
Z(\chi) = (\sin \chi)^{n-1} \mathcal{R}(\chi) \tag{5}
\]

we find

\[
\frac{d^2 Z}{d\chi^2} + \left[ \tilde{E} - \frac{(2L + n - 1)(2L + n - 3)}{4 \sin^2 \chi} - 2R^2 V(\chi) \right] Z = 0 \tag{6}
\]

where \( \tilde{E} = 2R^2 E + (n-1)^2 / 4 \) and the quasiradial wave function \( Z(\chi) \) satisfies the normalization condition

\[
\int_0^\pi Z(\chi)Z^*(\chi) R^n \, d\chi = 1. \tag{7}
\]

2.1. Let us now consider the \( n \)-dimensional oscillator potential \([8, 9]\)

\[
V(\chi) = \frac{\omega^2 R^2}{2} \left( s_1^2 + s_2^2 + \ldots + s_n^2 \right) = \frac{\omega^2 R^2}{2} \tan^2 \chi. \tag{8}
\]

Substituting the oscillator potential in equation (6) we obtain the Pöschl-Teller type equation

\[
\frac{d^2 Z}{d\chi^2} + \left[ \epsilon - \frac{\nu^2 - \frac{1}{4}}{\cos^2 \chi} - \frac{(L + \frac{n-2}{2})^2 - \frac{1}{4}}{\sin^2 \chi} \right] Z = 0 \tag{9}
\]

where \( \nu = \sqrt{\omega^2 R^4 + 1/4} \) and \( \epsilon = \tilde{E} + \omega^2 R^4 \). The solution of the above equation, regular for \( \chi \in [0, \pi/2] \) and expressed in terms of the hypergeometric function, is \([23]\)

\[
Z(\chi) \equiv Z_{n_r \nu}^n(\chi) = \sqrt{\frac{2(2n_r + L + \nu + \frac{n}{2})\Gamma(n_r + L + \nu + \frac{n}{2})\Gamma(n_r + L + \frac{n}{2})}{R^n \Gamma(L + \frac{n}{2})^2 \Gamma(n_r + \nu + 1)(n_r)!}} (\sin \chi)^{L + \frac{n-1}{2}} (\cos \chi)^{\nu + \frac{1}{2}} F_1(-n_r, n_r + L + \nu + \frac{n}{2}; L + \frac{n}{2}; \sin^2 \chi), \tag{10}
\]

and the \( \epsilon \) is quantized as

\[
\epsilon = (2n_r + L + \nu + \frac{n}{2})^2 \tag{11}
\]

where \( n_r + L = 0, 1, 2, \ldots \) is a “quasiradial” quantum number. The energy spectrum of the \( n \)-dimensional oscillator is given by

\[
E_N^n(R) = \frac{1}{2R^2} \left[ (N + 1)(N + n) + (2\nu - 1)(N + \frac{n}{2}) \right] \tag{12}
\]
where $N = 2n_r + L = 0, 1, ...$ is principal quantum number. In the contraction limit $R \to \infty$, $\chi \to 0$ and $R\chi \sim r$–fixed and $\nu \sim \omega R^2$, we see that

$$\lim_{R \to \infty} E_n^N(R) = \omega(N + \frac{n}{2})$$

(13)

and

$$\lim_{R \to \infty} (R)^{\frac{n-1}{2}} Z_{N,L\nu}^n(\chi) = \frac{(\omega)^{\frac{L+n}{2}} \sqrt{2\Gamma\left(\frac{N+L+n}{2}\right)}}{\Gamma(L + \frac{n}{2})} \left\{ \frac{\omega R^2}{2} \right\}_{1F1}(-\frac{N-L}{2}, L + \frac{n}{2}, \omega R^2).$$

(14)

Formula (14) coincides with the known formula for $n$-dimensional flat radial wave functions [24].

2.2. The potential, which is the analogue of the the Coulomb potential on the n-dimensional sphere, has the following form [6, 8, 9]:

$$V(\chi) = -\frac{\alpha}{R} s_0 \frac{s_1}{s_1^2 + s_2^2 + ... + s_n^2} = -\frac{\alpha}{R} \cot \chi.$$  

(15)

The Schrödinger equation (6) for this potential is

$$\frac{d^2 Z}{d\chi^2} + \left[ \tilde{E} - \frac{(2L + n - 1)(2L + n - 3)}{4\sin^2 \chi} + 2\alpha R \cot \chi \right] Z = 0.$$  

(16)

We make now a transformation to the new variable $\theta \in [0, \frac{\pi}{2}]$

$$e^{i\chi} = \cos \theta,$$

(17)

which is possible if we continue the variable $\chi$ in the complex domain $G$: Re $\chi = 0$, 0 $\leq$ Im $\chi < \infty$ (see Fig.1). We complexify also the coupling constant $\alpha$ by putting $k = i\alpha$ such that

$$\alpha \cot \chi = k(1 - 2 \sin^{-2} \theta).$$  

(18)
As result we obtain the equation

\[ \frac{d^2W}{d\theta^2} + \left[ \epsilon - \frac{\nu^2 - \frac{1}{4}}{\cos^2 \theta} - \frac{(2L + n - 2)^2 - \frac{1}{4}}{\sin^2 \theta} \right] W = 0 \] (19)

where \( W(\theta) = (\cot \theta)^{\frac{1}{2}} Z(\theta) \) and

\[ \epsilon = \tilde{E} + 2kR, \quad \nu^2 = \tilde{E} - 2kR. \] (20)

From above equation we see that, up to the substitution (20) and transformation \( L \to 2L \), the quasiradial equation (19) for the \( n^{\text{coul}} = (d + 1) \)-dimensional Coulomb problem coincides with the \( n^{\text{osc}} = 2d \)-dimensional quasiradial oscillator equation (9). This means that relations between these two systems are possible only for oscillators in even dimensions: \( n^{\text{osc}} = 2, 4, 6, 8,... \).

Thus equation (19) describes the \( 2(n - 1) \)-dimensional oscillator quasiradial functions with even angular momentum \( 2L \). The regular, for \( \theta \in [0, \pi/2] \) and \( \nu \leq 1/4 \), solution of this equation according to (10) has the form

\[
Z(\theta) = \frac{W(\theta)}{\sqrt{\cot \theta}} = Z_{n_r, L}(\theta) = C_{n_r, L}(\nu) (\sin \theta)^{2L + n - 1} (\cos \theta)^{\nu} \\
\times {}_2F_1(-n_r, n_r + 2L + \nu + n - 1; 2L + n - 1; \sin^2 \theta) \] (21)
where $C_{n_r L}^n (\nu)$ is the normalization constant. To compute the constant $C_{n_r L}^n (\nu)$ for the corresponding Coulomb quasiradial function we require that the wave function (21) satisfy the normalized condition
\[
R^n \int_0^\pi Z_{n_r L} Z_{n_r L}^* d\chi = 1,
\]
(22)

where the symbol “$\circ$” means the complex conjugate together with the inversion $\chi \rightarrow -\chi$, i.e. $Z^\circ (\chi) = Z^* (-\chi)$. [We choose the scalar product as $Z^\circ$ because for $\chi \in \mathcal{G}$ and real $\alpha$, and $\tilde{E}$ the function $Z^\circ (\chi)$ also belongs to the solution space of \{(16)\}.] By analogy to the work \cite{24} we consider the integral over contour $G$ in the complex plane of variable $\chi$ (see Fig.1)
\[
\oint Z_{n_r L} (\chi) Z_{n_r L}^* (\chi) d\chi = \int_0^\pi Z_{n_r L} (\chi) Z_{n_r L}^* (\chi) d\chi + \int_{\pi + i\infty}^{\pi + i\infty} Z_{n_r L} (\chi) Z_{n_r L}^* (\chi) d\chi
\]
(23)

Using the facts that the integrand vanishes as $e^{2i\nu \chi}$ and that $Z_{n_r L} (\chi)$ is regular in the domain $\mathcal{G}$ (see Fig.1), then according to the Cauchy theorem we have
\[
\int_0^\pi Z_{n_r L} (\chi) Z_{n_r L}^* (\chi) d\chi = (1 - e^{2i\nu}) \int_0^\infty Z_{n_r L} (\chi) Z_{n_r L}^* (\chi) d\chi.
\]
(24)

Making the substitution \cite{17} in the right integral of eq. \cite{(24)}, we find
\[
\int_0^\pi Z_{n_r L} (\chi) Z_{n_r L}^* (\chi) d\chi = i (1 - e^{2i\nu}) \int_0^{2\pi} [Z_{n_r L}] \tan \theta d\theta,
\]
(25)

and after integration over the angle $\theta$ we finally get \cite{25}
\[
C_{n_r m}^n (\nu) = \frac{(-2i\nu)(\nu + 2n_r + 2L + n - 1)}{R^n 1 - e^{2i\nu}} (2n_r + 2L + n - 1) (n_r + 2L + n - 2)! \Gamma(n_r + \nu + 1).
\]
(26)

Comparing now the eqs. \cite{(11)} with \cite{(20)} and putting $k = i\alpha$, we get
\[
\nu = - (n_r + L + \frac{n - 1}{2}) + i\sigma, \quad \sigma = \frac{\alpha R}{n_r + L + \frac{n - 1}{2}}
\]
(27)

and obtain the energy spectrum for the Coulomb problem
\[
E_n = \frac{N(N + n - 1)}{2R^2} - \frac{\alpha^2}{2(N + \frac{n - 1}{2})^2}, \quad N = n_r + L = 0, 1, 2, ...
\]
(28)

Returning to the variable $\chi$, we see that the Coulomb quasiradial wave function has the form
\[
Z_{NL} (\chi) = C_{NL} (\sigma) (\sin \chi)^{L + \frac{n - 1}{2}} \exp [-i\chi (N - L - i\sigma)]
\]
\[
\times _2 F_1 (-N + L, L + \frac{n - 1}{2} + i\sigma; 2L + n - 1; 1 - e^{2i\chi}),
\]
(29)
where the normalization constant $C_{NL}(\sigma)$ is

$$C_{NL}^n(\sigma) = 2^{L+n-1} e^{\frac{\sigma^2}{2}} \frac{\Gamma(L + \frac{n-1}{2} - i\sigma)}{\Gamma(2L + n - 1)} \sqrt{\frac{(N + \frac{n-1}{2})^2 + \sigma^2}{2R^n\pi (N + \frac{n-1}{2})(N - L)!}}. \quad (30)$$

Thus by using the relation between Coulomb and oscillator systems we have constructed the quasiradial wave functions and energy spectrum for a Coulomb system on the n-dimensional sphere.

Finally, note that in the contraction limit $R \to \infty$ (see for details [14]) it is easy to recover the well known formulas for the flat space n-dimensional Coulomb problem both for discrete and continuous spectrum [1].

3 Coulomb-oscillator relation on the n-dimensional two-sheeted hyperboloid

The pseudospherical coordinates on n-dimensional two-sheeted hyperboloid: $s_0^2 - s_1^2 - s_2^2 - \cdots - s_n^2 = R^2$, $s_0 \geq R$, are

$$

to = R \cosh \tau \\
s_1 = R \sinh \tau \cos \vartheta_1 \\
s_2 = R \sinh \tau \sin \vartheta_1 \cos \vartheta_2 \\
\vdots \\
s_{n-1} = R \sinh \tau \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{n-2} \cos \varphi \\
s_n = R \sinh \tau \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{n-2} \sin \varphi
$$

where $\tau \in [0, \infty)$. Variables in the Schrödinger equation (1) may be separated for any central potential $V(\tau)$ by the ansatz

$$
\Psi(\tau, \vartheta_1, \cdots \vartheta_{n-2}, \varphi) = \mathcal{R}(\tau) Y_{Ll_1,l_2,l_{n-2}}(\vartheta_1, \cdots \vartheta_{n-2}, \varphi) \quad (31)
$$

where, as in previous case $l_i$ are the angular hypermomenta and $L$ is total angular momentum, and the hyperspherical function $Y_{Ll_1,l_2,l_{n-2}}(\vartheta_1, \cdots \vartheta_{n-2}, \varphi)$ is the solution of Laplace-Beltrami equation on the $n-1$ dimensional sphere. After the separation of variables we find the quasiradial equation

$$
\frac{1}{\sinh^{n-1} \tau} \frac{d}{d\tau} \sinh^{n-1} \tau \frac{d\mathcal{R}}{d\tau} + \left[ \frac{2R^2E - L(L + n - 2)}{\sinh^2 \tau} \right] \mathcal{R} = 0. \quad (32)
$$

Using now the substitution

$$
Z(\tau) = (\sinh \tau)^{\frac{n-1}{2}} \mathcal{R}(\tau) \quad (33)
$$

we come to the equation

$$
\frac{d^2Z}{d\tau^2} + \left[ \tilde{E} - \frac{(2L + n - 1)(2L + n - 3)}{4 \sinh^2 \tau} \right] Z = 0 \quad (34)
$$
where $\tilde{E} = 2R^2E - \frac{(n-1)^2}{4}$ and the quasiradial wave function $Z(\tau)$ satisfy the normalization condition

$$\int_0^\infty Z(\tau)Z^*(\tau)R^n\,d\tau = 1$$

(35)

### 3.1. The oscillator potential on the two-sheeted n-dimensional hyperboloid is given by the potential

$$V(\tau) = \frac{\omega^2R^2}{2} \frac{s_1^2 + s_2^2 + \ldots + s_n^2}{s_0^2} = \frac{\omega^2R^2}{2} \tanh^2 \tau.$$  

(36)

From equation (34) we obtain

$$\frac{d^2Z}{d\tau^2} + \left[ \epsilon + \frac{\nu^2 - \frac{1}{4}}{\cosh^2 \tau} - \frac{(L + \frac{n-2}{2})^2 - \frac{1}{4}}{\sinh^2 \tau} \right] Z = 0$$

(37)

where $\nu = \sqrt{\omega^2R^4 + \frac{1}{4}}$ and $\epsilon = \tilde{E} - \omega^2R^4$. Thus the oscillator problem on the hyperboloid is described by the modified Pöschl-Teller equation and, unlike the oscillator equation on the sphere which has only bound spectrum, the equation (37) possesses both bound and unbound states.

The discrete wave-functions regular on the line $\tau \in [0, \infty)$, have the form \[\text{[16, 19, 26]}\]

$$Z(\tau) \equiv Z_{n_rL}(\tau) = \frac{1}{\Gamma(L + \frac{n}{2})} \left[ \frac{2(n - L - 2n_r - \frac{n}{2})\Gamma(\nu - n_r)\Gamma(n_r + L + \frac{n}{2})}{R^n(n_r)\Gamma(\nu - L - n_r - \frac{n}{2} + 1)} \right] \times \left( \sinh \tau \right)^{L + \frac{n-1}{2}} \left( \cosh \tau \right)^{2n_r - \nu + \frac{1}{4}} \mathcal{F}_1(-n_r, -n_r + \nu; L + \frac{n}{2}; \tanh^2 \tau),$$

(38)

with $n_r = 0, 1, \ldots, \left[\frac{1}{2}(\nu - L - \frac{n}{2})\right]$. The $\epsilon$ is quantized by

$$\epsilon = -(2n_r + L - \nu + n/2)^2$$

(39)

and the energy spectrum for the quantum oscillator on the $n$-dimensional two-sheeted hyperboloid takes the value

$$E_N^n(R) = \frac{1}{2R^2} \left[ -N(N + n - 1) + (2\nu - 1)(N + \frac{n}{2}) \right].$$

(40)

Here $N = 2n_r + L$ is a principal quantum number and the bound state solution is possible only for

$$0 \leq N \leq \left\lfloor \nu - \frac{n}{2} \right\rfloor$$

(41)

In the contraction limit $R \to \infty$, $\tau \sim r/R$ and $\nu \sim \omega R^2$ we see that the continuous spectrum is vanishing while the discrete spectrum is infinite, and it is easy to reproduce the oscillator energy spectrum \[\text{[13]}\] and wave function \[\text{[14]}\].
3.2. The Coulomb potential on the two-sheeted n-dimensional hyperboloid has the form \[ V(\tau) = -\frac{\alpha}{R} \left( \frac{s_0}{\sqrt{s_1^2 + s_2^2 + \ldots + s_N^2}} - 1 \right) = -\frac{\alpha}{R}(\coth \tau - 1). \] (42)

Substituting potential (42) in Schrödinger equation (34) we arrive at
\[
\frac{d^2 Z}{d\tau^2} + \left[ (\tilde{E} - 2\alpha R) - \frac{(2L + n - 1)(2L + n - 3)}{4 \sinh^2 \tau} + 2\alpha R \coth \tau \right] Z = 0,
\] (43)
which is known as the Manning-Rosen potential problem [27].

Making the transformation from variable \( \tau \) \((0 \leq \tau < \infty)\), to the new variable \( \mu \) \(\in [0, \infty)\)
\[ e^\tau = \cosh \mu, \] (44)
and setting \( Z(\mu) = W(\mu)/\sqrt{\coth \mu} \), we go to the modified Pöschl-Teller equation
\[
\frac{d^2 W}{d\mu^2} + \left[ \tilde{E} + \frac{(-\tilde{E} + 4\alpha R) - \frac{1}{4} - (2L + n - 2)^2 - \frac{1}{4}}{\cosh^2 \mu - \sinh^2 \mu} \right] W = 0.
\] (45)

It can be see from the eq.(45) with the substution \( \epsilon = \tilde{E}, \quad \nu^2 = -\tilde{E} + 4\alpha R \). (46)
and the transformation \( L \to 2L \), the quasiradial equation (19) for \( n_{\text{coul}} = 2d + 1 \)-dimensional Coulomb problem coincides with the \( n_{\text{osc}} = 2d \) - dimensional quasiradial oscillator equation (37).

Thus the regular for \( \mu \in [0, \infty) \) solution of equation (43) or (45) has the form
\[
Z(\mu) = \frac{W(\mu)}{\sqrt{\coth \mu}} \equiv Z_{n_r,L}(\mu) = A_{n_r,L}(\nu) (\sinh \mu)^{L + \frac{\sigma}{2}} (\cosh \mu)^{2n_r - \nu} \times \binom{2F1}{-n_r, -n_r + \nu; L + \frac{n_r}{2}; \tanh^2 \mu}
\] (47)
where \( A_{n_r,L}(\nu) \) is the normalization constant. The constant \( A_{n_r,L}(\nu) \) is computed from the requirement that the wave function (47) satisfies the normalized condition
\[
R^n \int_0^\infty |Z_{n_r,L}(\tau)|^2 d\tau = R^n \int_0^\infty |Z_{n_r,L}(\mu)|^2 \tanh \mu d\mu = 1.
\] (48)
and has the following form
\[
A_{n_r,L}(\nu) = \frac{1}{\Gamma(L + \frac{n_r}{2})} \left[ \frac{2\nu(\nu - L - 2n_r - \frac{n}{2})\Gamma(\nu - n_r)\Gamma(n_r + L + \frac{n}{2})}{R^n(L + 2n_r + \frac{n}{2})(n_r)!\Gamma(\nu - L - n_r - \frac{n}{2} + 1)} \right].
\] (49)
Comparing now eq.(46) with (38) and passing from the oscillator to the Coulomb angular quantum number \( L \to 2L \) and dimension \( n \to 2(n - 1) \), we get
\[
\nu = (n_r + L + \sigma + \frac{n - 1}{2}), \quad \sigma = \frac{\alpha R}{n_r + L + \frac{n - 1}{2}}.
\] (50)
Thus the discrete energy spectrum of the Coulomb problem on the \( n \)-dimensional two-sheeted hyperboloid is described by the formula

\[
E^n_N(R) = -\frac{N(N+n-1)}{2R^2} - \frac{\alpha^2}{2(N+n-1/2)^2} + \frac{\alpha}{R},
\]

(51)

where \( N = n_r + L \) is the principal quantum number and the bound states occur for

\[
0 \leq N \leq \left\lfloor \sigma - \frac{n-1}{2} \right\rfloor.
\]

(52)

The discrete wave function has the form

\[
Z^n_{NL}(\tau) = A^n_{NL}(\sigma) (\sinh \tau)^{L+n-1/2} e^{\tau(N-L-\sigma)}
\times \ _2F_1 \left( -N + L, L + \frac{n-1}{2} + \sigma; 2L + n - 1; 1 - e^{-2\tau} \right),
\]

(53)

where the normalization constant \( A^n_{NL}(\sigma) \) is

\[
A^n_{NL}(\sigma) = \frac{2^{L+n-1}}{\Gamma(2L+n-1)} \sqrt{\frac{[\sigma^2 - (N+n-1/2)^2] \Gamma(N+L+n-1) \Gamma(\sigma + L + n-1/2)}{R^{n(L+n-1/2)}(N-L)! \Gamma(\sigma - L - n-1/2 + 1)}}.
\]

(54)

The solution for the Coulomb quasiradial equation, both for energy spectrum and wave functions, is identical to that given in the paper [12] by a path integral approach. We not consider here the contraction limit \( R \to \infty \) to flat Euclidean space for the Coloumb problem because it has been done already in the same article [12].

It should be noted that instead of substitution (44) it is possible to use the trigonometric transformation

\[
e^{-\tau} = \cos \varphi, \quad \varphi \in [0, \pi/2].
\]

(55)

It is easy to see that in this case, up to the permutation

\[
\epsilon = -\tilde{E} + 4\alpha R, \quad \nu^2 = -\tilde{E},
\]

(56)

and transformation \( L \to 2L \), the quasiradial equation (43) for the \( n_{\text{coul}} = (d+1) \) - dimensional Coulomb problem passes to the \( n_{\text{osc}} = 2d \) - dimensional quasiradial oscillator equation (9). Thus the Coulomb problem on the two-sheeted hyperboloid is related to the oscillator problem on the sphere or two-sheeted hyperboloid.

### 4 Coulomb-oscillator relation on the n-dimensional one-sheeted hyperboloid

Pseudospherical coordinates on the \( n \)-dimensional one-sheeted hyperboloid: \( s_0^2 - s_1^2 - s_2^2 - \cdots - s_n^2 = -R^2 \) are

\[
s_0 = R \sinh \tau
\]
\[
s_1 = R \cosh \tau \cos \vartheta_1 \\
s_2 = R \cosh \tau \sin \vartheta_1 \cos \vartheta_2 \\
... \\
s_{n-1} = R \cosh \tau \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{n-2} \cos \varphi \\
s_n = R \cosh \tau \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{n-2} \sin \varphi
\]

where \( \tau \in (-\infty, \infty) \). Variables in Schrödinger equation (1) may be separated using the ansatz

\[
\Psi(\tau, \vartheta_1, \cdots \vartheta_{n-2}, \varphi) = R(\tau) Y_{l_1, l_2, l_{n-2}}(\vartheta_1, \cdots \vartheta_{n-2}, \varphi)
\]

where as in the previous case the \( l_i \) are the angular hypermomenta, \( L \) is total angular momentum, and the hyperspherical function \( Y_{l_1, l_2, l_{n-2}}(\vartheta_1, \cdots \vartheta_{n-2}, \varphi) \) is the solution of the Laplace-Beltrami equation on the \( n - 1 \) dimensional sphere. After separation of variables we find the quasiradial equation

\[
\frac{1}{\cosh^{n-1} \tau} \frac{d}{d \tau} \cosh^{n-1} \tau \frac{dR}{d \tau} + \left[ 2R^2 E + \frac{L(L + n - 2)}{\cosh^2 \tau} - 2R^2 V(\tau) \right] R = 0.
\]

Using now the substitution

\[
Z(\tau) = (\cosh \tau)^{\frac{n-1}{2}} R(\tau)
\]

we come to the equation

\[
\frac{d^2 Z}{d \tau^2} + \left[ \bar{E} + \frac{(2L + n - 1)(2L + n - 3)}{4 \cosh^2 \tau} - 2R^2 V(\tau) \right] Z = 0
\]

where \( \bar{E} = 2R^2 E - \frac{(n-1)^2}{4} \) and the quasiradial wave function \( Z(\tau) \) satisfies the normalization condition

\[
\int_{-\infty}^{\infty} Z(\tau) Z^*(\tau) R^n d\tau = 1.
\]

4.1 The oscillator potential on the \( n \)-dimensional one-sheeted hyperboloid is given by

\[
V(\tau) = \frac{\omega^2 R^2}{2} \frac{s_1^2 + s_2^2 + \cdots + s_n^2}{s_0^2} = \frac{\omega^2 R^2}{2} \coth^2 \tau,
\]

so for equation (60) we have

\[
\frac{d^2 Z}{d \tau^2} + \left[ \epsilon + \frac{(L + \frac{n-2}{2})^2 - \frac{1}{4}}{\cosh^2 \tau} - \frac{\nu^2 - \frac{1}{4}}{\sinh^2 \tau} \right] Z = 0
\]

where \( \nu = \sqrt{\omega^2 R^4 + \frac{1}{4}} \), \( \epsilon = \bar{E} - \omega^2 R^4 \). As in the previous case the oscillator system is described by the modified Pöschl-Teller equation and possesses discrete and continuous spectrum. However, differing from the motion on the two-sheeted hyperboloid, the number of bound states
depends on the total angular momentum. The discrete state wave functions regular on the line \( \tau \in (-\infty, \infty) \) are

\[
Z(\tau) \equiv Z_{n,L}(\tau) = \sqrt{\frac{(L - \nu - 2n_r + \frac{n_r}{2} - 2)\Gamma(L - n_r + \frac{n_r}{2} - 1)\Gamma(n_r + \nu + 1)}{R^n(n_r)!\Gamma(\nu + 1)^2\Gamma(L - \nu - n_r + \frac{n_r}{2} - 1)}} \times (\sinh \tau)^{\nu + \frac{1}{2}} (\cosh \tau)^{2n_r - L - \frac{\nu}{2} + \frac{3}{2}} F_1(-n_r, -n_r + L + \frac{n_r}{2} - 1; \nu + 1; \tanh^2 \tau),
\]

and

\[
\epsilon = -(2n_r - L + \nu - \frac{n_r}{2} + 2)^2
\]

where the bound states occur for \( n_r = 0, 1, ..., n_r^{\text{max}} = [\frac{1}{2}(L - \nu + \frac{n_r}{2} - 2)] \). The last formula means that the discrete spectrum depends on quantum number \( L \) and the energy spectrum of the oscillator system takes the form

\[
E_{n,L}(R) = -\frac{1}{2R^2}[(2n_r - L + 2)(2n_r - L - n + 3) + (2\nu - 1)(2n_r - L - \frac{n_r}{2} + 2)].
\]

4.2 The Coulomb potential on the \( n \)-dimensional hyperboloid has the form

\[
V(\tau) = -\frac{\alpha}{R} \left( \frac{s_0}{\sqrt{s_1^2 + s_2^2 + ... + s_n^2}} + 1 \right) = -\frac{\alpha}{R} (\tanh \tau + 1).
\]

The Schrödinger equation for this potential is

\[
\frac{d^2 Z}{d\tau^2} + \left[ (\tilde{E} + 2\alpha R) + \frac{(2L + n - 1)(2L + n - 3)}{4 \cosh^2 \tau} + 2\alpha R \tanh \tau \right] Z = 0,
\]

which coincides with the Rosen-Morse equation.

Making the transformation from variable \( \tau (-\infty < \tau < \infty) \), to the new variable \( \mu \in [0, \infty) \)

\[
e^\tau = \sinh \mu,
\]

we go to the equation

\[
\frac{d^2 W}{d\mu^2} + \left[ (\tilde{E} + 4\alpha R) + \frac{(2L + n - 2)^2 - \frac{1}{4}}{\cosh^2 \mu} - \frac{(-\tilde{E}) - \frac{1}{4}}{\sinh^2 \mu} \right] W = 0
\]

where \( W(\mu) = (\tanh \mu)^\frac{1}{2} Z(\mu) \). From this equation we see that, up to the substitution

\[
\tilde{E} \rightarrow \tilde{E} + 4\alpha R, \quad \nu^2 = -\tilde{E},
\]

and the simultaneous transformation for total angular momentum \( L \rightarrow 2L \), the quasiradial equation for the Coulomb problem on the \( n_{\text{coul}} = d+1 \)-dimensional one-sheeted hyperboloid coincides with the \( n_{\text{osc}} = 2d \)-dimensional quasiradial oscillator equation.
Comparing now eq. (70) with (63) and taking into account the eqs. (65) and (71), we see that the discrete wave function satisfying the normalization condition
\[
R^n \int_{-\infty}^{\infty} |Z_{n_rL}(\tau)|^2 d\tau = R^n \int_{-\infty}^{\infty} |Z_{n_rL}(\mu)|^2 \coth \mu d\mu = 1.
\] (72)
has the form
\[
Z_{n_rL}(\tau) = \frac{2^{n_r-L-\frac{3}{2}}}{\Gamma(L-n_r+n-\frac{3}{2})} \sqrt{\frac{\Gamma(2L-n_r+n-2)\Gamma(L+n-\frac{3}{2})}{R^n(L-n_r+n-\frac{3}{2})(n_r)!\Gamma(L-\sigma+\frac{1}{2})}} \times \left( \cosh \tau \right)^{n_r-L-\frac{n-1}{2}} (e^\tau)^{(\sigma-1)} \times _2F_1 \left(-n_r,-n_r+L+n-2; L-n_r+n-\frac{3}{2}+\sigma; \frac{1}{1+e^{-2\tau}} \right),
\] (73)
with the discrete energy spectrum of the Coulomb problem described by the formula
\[
E_n = -\frac{(L-n_r-1)(L-n_r+n-2)}{2R^2} - \frac{\alpha^2}{2(L-n_r+n-\frac{3}{2})^2} - \frac{\alpha}{R}.
\] (74)
Bound states occur for \(n_r = 0, 1, \ldots, n_r^{\text{max}} = [(L+n-3L\sigma)]\).

Finally note that in distinction to the sphere and two-sheeted hyperboloid, the contraction limit \(R \to \infty\) on one-sheeted hyperboloids for the oscillator and Coulomb problems makes no sense.

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