GRB Light Curves in the Relativistic Turbulence Model

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ABSTRACT

Relativistic Turbulence provides an alternative to internal shocks as a mechanism for producing GRBs’ variable light curves with efficient conversion of energy to radiation. In this model the relativistic outflow is broken into small eddies moving relativistically in the outflow’s rest frame. Variability arises because an observer sees an eddy only when its velocity points towards him and only a small fraction of the eddies are observed. Relativistic turbulence with a significant relativistic velocity requires converting and maintaining a large fraction of the overall energy into turbulent motion. While it is not clear how this is achieved, we explore here, using a toy model, the constraints the model parameters results in light curves comparable to the observations. We find that a tight relation between the size of the eddies and the bulk and turbulent Lorentz factors is needed and that the variability level determines the turbulent Lorentz factor. While the model successfully produces the observed variability there are several inconsistencies with other properties of the light curves. Most of which, but not all, might be resolved if the central engine is active for a long time producing a number of shells, resembling to some extent the internal shocks model.

1. Introduction

The temporal variability seen in GRB light curves played a major role in the understanding how GRBs operate. Standard external shocks, in which the relativistic ejecta is slowed down by external medium, cannot produce sufficiently the highly variable light curves (Sari & Piran 1997). The alternative was internal dissipation of the bulk energy within a relativistic outflow (e.g., internal shocks). In internal shocks the observed time scales are determined by the inner engine and they reflect its activity. While internal shocks resolve the variability (by imposing it on the central engine) and agree with other properties of GRB light curves (e.g., Nakar & Piran 2002b, Ramirez-Ruiz & Fenimore 2000)) they suffers from several drawbacks. First and foremost is their low efficiency (Kobayashi et al. 1997, Daigne & Mochkovitch 1998) unless the relative Lorentz factor between the shells is very large (Kobayashi et al. 1997, Kobayashi & Sari 2001, Beloborodov 2000). This is particularly troublesome in view of the high efficiency implied from comparison of the prompt \(\gamma\)-rays luminosity and the kinetic energy that remains in the outflow. Other problems arise when one considers detailed models for the emission mechanisms of the prompt \(\gamma\)-rays (Kumar & McMahon 2008).

External shocks can produce a highly variable light curve if the outflow is slowed down by external clumps. Each clump produces a short pulse. However, this process will inevitably be extremely inefficient (Sari & Piran 1997). The overall covering factor of the emitting regions must be as small as \(\delta t/T\) (where \(\delta t\) is the duration of an individual pulse and \(T\) is the burst’ duration). Observed values of \(\delta t/T\) are typically \(\sim 0.01\) and in extreme cases can be as low as \(10^{-4}\) (Nakar & Piran 2002a).

Lyutikov & Blandford (2002, 2003) (see also Lazar 2005, Lyutikov 2006, Narayan & Kumar 2008) proposed relativistic turbulence as an alternative. In this model a fluid shell that moves with a bulk Lorentz factor \(\Gamma\) contains eddies with random macroscopic relativistic velocities (with a Lorentz factor \(\gamma\')). An eddy is observed only when its radiation cone (with an opening angle of the order of \(1/\Gamma\gamma\)' in the lab frame) points towards the observer, producing a pulse that is much
shorter than the burst duration. The filling factor of these eddies may be as high as unity, recovering high efficiency. However, as only a small fraction of the eddies are observed at any given time, this model can potentially produce rapid variability while maintaining high efficiency (Lyutikov 2006). Unlike internal shocks the overall duration of the burst, is given in this model by the larger between the angular time and the shell light crossing time \( \max \{ R/c \Gamma^2, \Delta/c \} \), where \( R \) and \( \Delta \) are the shell radius and width respectively and \( c \) is the light speed. The temporal variability is then dictated by the turbulent Lorentz factor, \( \gamma' \). In this model the observed variability does not reflect the engine variability and if \( R/\Gamma^2 > \Delta \) the burst duration does not reflect the total engine activity time.

Even though it is unclear if large scale relativistic turbulence can be generated, we assume in this paper that it does and examine the conditions under which this model can produce the main temporal features of the observed light curve (see L05). Lacking a model for relativistic turbulence we consider a simplified toy model that includes the essential ingredients. This toy model is sufficient to derive constraints on the parameters and to explore its basic features. We describe our toy model and derive analytic constraints in \( \S 2 \). We present numerical simulations in \( \S 3 \) and we summarize the results and compare with observations in \( \S 4 \).

2. Relativistic Turbulence

We construct a simple kinematic toy model (figure 1) that mimics the essential features of relativistic turbulence (L05). The shell is divided into discrete randomly distributed emitters that have randomly oriented relativistic velocities, with a Lorentz factor \( \gamma' \) in the shell’s comoving frame. In its own rest frame, each emitter has a size \( \psi R \) and it emits isotropically in this frame. Note that there are three frames: The lab frame; the shell’s frame, denoted by a prime, which is boosted radially with a Lorentz factor \( \Gamma \) relative to the lab; and the frame of each emitter, denoted by two primes, which is boosted by (randomly oriented) \( \gamma' \) relative to the shell frame. The observer is, of course, at rest relative to the lab frame. However, the observer time, namely the arrival time of photons (denoted \( t \)) differs from the lab time by the usual time of flight argument \( \psi \).

1Note the emitters have a fixed angular size. If instead the physical size is fixed the angular size would decrease by a factor of 2 when the shells expands from \( R_0 \) to \( 2R_0 \). This will introduce numerical factors of order unity in our analysis.

\[
\Lambda = \left[ \gamma \left( 1 - \beta \cdot \cos \alpha \right) \right]^{-1},
\]

(1)

where \( \gamma \), \( \beta \) and \( \alpha \) are the Lorentz factor, velocity and the angle between the velocity and the line to the observer (both in the lab frame). The flux at a given frequency that reaches the observer from this emitter is:

\[
F_\nu = \int I_\nu' \Lambda^3 d\Omega_i \approx I_\nu' \Lambda^3 \psi^2 R^2 \frac{D^2}{D},
\]

(2)

where \( I_\nu' \), is the specific intensity in the emitters’ frame and the second relation holds for a small enough emitter (\( D \) is the distance to the observer). An implicit K correction arises from the difference between the observed \( \nu \) and \( \nu' \).

The maximal Doppler boost, \( \Lambda_{max} = 4\gamma' \Gamma \), is obtained when an eddy moves on the line of sight directly towards the observer. The flux decreases like the third
power of $\Lambda$ plus a K correction. Therefore, when estimating the probability that an emitter is seen by the observer we can safely ignore emission from emitters with $\Lambda < \Lambda_{\text{max}}/2$ (this is checked later numerically). Given an emitter at an angle $\theta$ we calculate the probability that it will be seen by the observer $S(\theta, \Gamma, \gamma')$. The probability to see an emitter located at $\theta = 0$ is $S(0, \Gamma, \gamma') \approx 1/4\gamma^2$. Similarly, $S(1/\Gamma, \Gamma, \gamma') = 0$. This suggests (as verified numerically) that $S$ scales with the Lorentz factors as $S(\theta, \Gamma, \gamma') = \gamma'^{-2} S(\theta\Gamma)$.

(L05). The average probability, $P$, that an emitter will be visible from an arbitrary position on the shell is:

$$P(\Gamma, \gamma') \approx \frac{1}{2(\Gamma\gamma')^2} \int_0^1 S(\theta\Gamma)(\theta\Gamma)d(\theta\Gamma) \approx \frac{0.3}{4\pi(\Gamma\gamma')^2}. \quad (3)$$

The factor 0.3 was evaluated numerically (L05) and can be ignored at the accuracy level of our discussion (thus confirming the order of magnitude estimate of Lyutikov [2006]).

The arrival time from an emitter at $R, \theta$ is:

$$T = \frac{R - R_0}{2c\Gamma^2} + \frac{R\theta^2}{2c} + \frac{x}{c}, \quad (4)$$

where $x(\ll \Delta)$ is the distance of the emitter from the front of the shell. As the last photons will arrive from $2R_0$, an angle of $1/\Gamma$ and $x = \Delta$, the overall duration of the burst will be a function only of $\Delta$ and $\Gamma$ (and not of the turbulent Lorentz factor, $\gamma'$):

$$T \approx \frac{R_0 d}{c\Gamma^2}. \quad \quad (5)$$

where we define $d \equiv \Delta \Gamma^2/R$. As the shell is expected to expand relativistically in its own frame, $d \gg 1$. For $d > 1$ the shell’s width, as well as $T$, are determined by the engine activity time while for $d = 1$ they don’t.

The duration of a pulse arriving from a single emitter is the longest of the three following time scales:

(i) The duration over which the emitter pointing towards the observer, namely the duration over which the direction of motion varies by an angle $1/\Gamma\gamma'$ in the lab frame ($1/\gamma'$ in the shell’s frame). As the emitter is confined to the shell it should make at least a $\pi/2$ turn during $\Delta' / c$ (shell’s frame). This implies that a turn by $1/\gamma'$ is faster than $\Delta'/c\gamma'$. Causality puts a lower limit on the turning time of $R\psi/c$ (in the shell frame).

Allowing for the uncertainty in this value we denote it as $\tau'$:

$$R\psi/c \leq \tau' \leq \Delta'/c\gamma'. \quad (6)$$

In the observer’s frame, this translates to:

$$R\psi/\Gamma\gamma'^2c \leq \tau \leq \Delta'/c\gamma'^3. \quad (7)$$

(ii) The emitter’s light crossing time in the lab frame (in the direction along the line of sight). For an emitter moving towards the observer this time is $R\psi/\gamma\Gamma$.

(iii) The angular time scale – At the largest possible angle, where the emitter is still visible by the observer, $1/\gamma\Gamma\gamma$, the time difference between the first and the last photon would be $\frac{1}{2}R\psi \sin(1/\gamma\Gamma) \approx R\psi/c\gamma$. Overall (ii) and (iii) are of the same order and much larger than (i). Thus (c.f. Lyutikov [2006]):

$$\delta t \approx R\psi/c\gamma\Gamma. \quad (8)$$

Using Eqs. 5 and 8 we express, $N_p$, the number of (possibly overlapping) pulses expected in a burst:

$$N_p \equiv n_p \frac{T}{\delta t} = n_p \frac{d\gamma'}{\psi\Gamma}, \quad \quad (9)$$

where $n_p$ is the occupation number of pulses at any given observer time (i.e., $n_p \gg 1$ implies many overlapping pulses while $n_p \ll 1$ implies long quiescent periods between isolated pulses).

The number of emitters in a shell is $4\pi R^2 \Delta'/((R\psi)^3) = 4\pi \Delta\Gamma/R\psi^3$. As the shell expands the emitters obtain new random directions (which differ by more than $1/\gamma'$, in the shell’s frame, than the previous ones) after a time $\tau'$. Thus the total number of independent emitters, $N_{\text{tot}}$, is larger by a factor $R/((c\Gamma\gamma')^2)$, the ratio of the total duration over which the radius doubles and $\tau'$. Finally we introduce a filling factor $f \leq 1$ allowing for the possibility that not all emitters are active all the time or that space is not fully covered by emitters. Overall we find:

$$N_{\text{tot}} = \frac{4\pi f d}{\psi^3} \frac{R}{\Gamma^2 c\gamma'}, \quad \quad (10)$$

The condition $N_p = PN_{\text{tot}}$ yields:

$$n_p = \frac{f d}{\gamma^3\Delta'\psi^2} \frac{R}{c\gamma'}, \quad \quad (11)$$

and using $\frac{f}{d(\gamma\Gamma\psi)^2} \leq n_p \leq \frac{f}{(\gamma\Gamma\psi)^3}. \quad \quad (12)$

Note that for a hydrodynamic external shock $d \lesssim 1$ (Sari & Piran [1997]) but this might not be relevant here.
Observationally, we have \( n_p \approx 1 \) since many overlapping pulses reduce the observed variability, whereas very frequent long quiescent times between individual pulses are not observed.

If the shell is in the freely expanding phase (i.e., \( d \approx 1 \) ) \( n_p \) will be of order unity if:

\[
\psi \approx f^{1/k} \frac{1}{\gamma/T}.
\]

where \( k \) is between 2 and 3. Narayan & Kumar (2008) have pointed out that \( \psi = 1/\gamma T \) if one requires that the emitters are of the maximal causally allowed size. Note that \( n_p \) depends quite sensitively on \( \gamma/T \psi \) and it increases rapidly if \( \psi \) is smaller than \( 1/\gamma T \). This implies, for example, that a significant number of small eddies, that may arise in a turbulent cascade may be problematic. On the other hand as \( \psi \leq 1/\gamma T \) one can get \( n_p < 1 \) only by reducing the filling factor \( f \).

Using the relations [13] and [9] and assuming the causal limit for \( \tau' \) we obtain:

\[
\gamma' \approx \left(\frac{f}{n_p}\right)^{1/6} \sqrt{\frac{T}{d\delta t}}.
\]

For typical observed values this leads to \( \gamma' \approx 10/\sqrt{d} \). Note that while the model determines \( \gamma' \) it does not constrain \( \Gamma \) and \( R \).

3. Numerical Simulations

We have simulated numerically light curves produced by this model (L05). We populate the volume between \( R_0 \) and \( 2R_0 \) with \( N_{\text{tot}} \) emitters of identical angular size \( \psi \) and with randomly oriented (constant in size) velocities. For each emitter the arrival time, \( t \), duration, \( \delta t \), and flux, \( F_\nu \), are calculated according to Eqs. [3], [8] and [2] respectively. The flux is calculated assuming that the radiation efficiency is constant per unit mass for all emitters in their rest frame, namely that \( F_\nu \propto (\psi R)^{-3} \), therefore \( F_\nu \propto A^3/(\psi R) \). We approximate each pulse as a Gaussian with the above parameters and we then construct a light curve summing over all contributions. We do not use the aforementioned cut-off at \( \Lambda/A_{\text{max}} = 1/2 \). Instead we sum the contributions from all emitters, as weak as they may be. The model is parametrized by \( \gamma'/\Gamma, \psi, R, d \) and \( f \). From those, \( \Gamma, R \) and \( d \) set the burst duration (as defined in Eq. [3]), and time is normalized by this scale so that the value of \( R \) and \( \Gamma \) are irrelevant as long as \( \Gamma \gg 1 \). The other parameters satisfy Eq. [14] with \( T/\delta t \approx 100; \psi \leq 1/\gamma T \) and the right equality in relation [12] (corresponding to \( \tau' = \tau_{\text{min}} \)).

Figure 2 depicts the resulting light curves for four choices of parameters. The two upper panels have \( n_p = 1 \) with different emitter sizes. Naturally, as the emitters are smaller on the right panel it has more pulses than the left one. Still, both light curves are highly variable and densely filled with non-overlapping pulses. However, with \( d = 1 \), one can see clearly an underlying overall envelope of the pulses. The bottom panels are with \( d = 10 \), which stretches this envelope. The lower left panel depicts a very low \( n_p \) with a rather sparse light curve. The lower right panel depicts a light curve of a wide shell and \( n_p = 0.7 \), which is rather similar to observed bursts. For \( n_p > 1 \) the pulses are overlapping and all variability is erased, leaving only the envelope.

![Numerical Simulations](image-url)
normal distribution of $\gamma'$ and corresponding emitter sizes according to the required values of $n_p$. While this increased the spread in the pulse heights, in particular individual very bright pulses appear, the over all features of the light curves remained unchanged. It is interesting to note that in a case of distribution of parameters (like $\psi$ and $\gamma'$) the constraint on $n_p$ becomes a constraint on the moments of the distributions. For $d \approx 1$ and $\psi = 1/\gamma' \Gamma$ we have:

$$n_p = f \frac{\langle (\gamma'-2) \psi^{-1} \rangle}{\Gamma \langle \psi^3 \rangle \langle \gamma' / \psi \rangle} = f \frac{\langle \gamma^{-2} \rangle}{\langle \gamma^{-3} \rangle \langle \gamma'^2 \rangle}.$$  \ 

(15)

For reasonable distributions this expression yields correcting factors of order unity to Eq. 14

4. Discussion and Conclusions

We have derived a condition on the parameters of relativistic turbulence needed for producing a variable GRB light curve. This is characterized by the relation $n_p \approx 1$ which arises from the condition that typical pulses don’t overlap and are not too sparse either. It is remarkable that causality arguments suggest that the implied relation $\psi = 1/\gamma' \Gamma$ between the angular size of the emitters, $\psi$, and the turbulent and the bulk Lorentz factors holds naturally in a model of relativistic eddies (Narayan & Kumar 2008). However, this condition holds when the turbulent eddies are of the maximal possible size and it might break by a cascade that produces a significant fraction of much smaller eddies.

Our numerical simulations of the resulting light curve show that for $0.03 < n_p < 3$ one obtains light curves that resemble observed GRBs (see figure 2). The resulting light curves do not change qualitatively when we introduce a distribution of turbulent Lorentz factors and corresponding sizes.

The numerical simulations reveal an underlying envelope of a rising and falling light curve in case of a single expanding shell with $d \approx 1$. Only a small fraction of the volume and hence hence fewer pulses are seen early on. Similarly at $t > (d + 1) R_0 / 2cT^2$ pulses from small $\theta$ values are not seen, implying that for $d \approx 1$ only lower amplitude pulses (on average) are observed at the second half of the burst. Relaxing the assumption of a single homogenous emitting shell moving with a constant Lorentz factor $\Gamma$ and radiating between $R_0$ and $2R_0$, will change the shape of the envelope, but in general a temporal evolution is expected. On the other hand no strong evolution is expected for $d \gg 1$ or if there are several shells. Both cases are similar to the internal shock model in the sense that the burst duration reflects the engine’s activity. This solution becomes marginal if $\tau'$ is determined by causality, since $\gamma' \approx 1$ requires $d \lesssim 10$ (see Eq. 14).

Bursts which depict several periods of intense activity separated by quiescent periods cannot be produced by a single shell, regardless of the parameters chosen. However, a combination of several such shells could produce such light curves. This case is similar to $d > 1$ (with $d$ measuring the distance between the first and last shells) and therefore has the same limit of $d \lesssim 10$.

It seems that with proper, and rather reasonable conditions, relativistic turbulence can produce (efficiently) the observed highly variable GRB light curves. We turn now to several shortcomings that have to be resolved. First and foremost is the question how such macroscopic relativistic motions can be generated and sustained over the period of time needed. Note that before any photons are emitted, one needs to convert $\sim 1 - 1/\gamma'$ of the initial total energy to the kinetic energy of the eddies. Further dissipation in the eddies’ frame is needed to generate the radiation.

Additional questions involve the shape and other properties of individual pulses with those seen in observed pulses:

• GRBs show a clear difference between the fast rise and the slow decline of individual pulses (Norris et al. 1996). Here, the light curve of an individual pulse results from a combination of the motion of the emitter, its orientation relative to the observer, its width as well as intrinsic inhomogeneities within the emitter. As the emitter was radiating long before its velocity points towards the observer and it continues to emit long after it moves away from the observer there is no reason (on average) for a difference between the rising and falling phases of an individual pulse.

• The temporal structure of the first and second halves of GRB light curves are similar (Ramirez-Ruiz & Fenimore 2000). The light curves produced in the model have an overall envelope of single pulse which favors stronger pulses in the first half and weaker ones in the second. As mentioned earlier this might be resolved

\[\text{Note that systematic variation of the emitter properties on a time scale of } \tau' \text{ will result in a strong signature differentiating between early and late phases of the overall light curve, which is not observed. On the other hand non-systematic variations (e.g., deceleration and acceleration) are expected to result in similar affects on the temporal structure of rising and decaying parts of pulses.}\]
by a combination of several emitting shells or with a very wide shells, but here some fine tuning is required in order to keep $\gamma' \gg 1$.

- Weaker and denser pulses (arriving from eddies not moving directly towards the observer) continues after $R/c\Gamma^2$ producing the typical envelope of high latitude emission (Kumar & Panaitescu 2000). This is consistent with some rapid declines seen in the early afterglow. However in many cases the decline is faster. In the standard internal shocks model this is attributed to the dominant contribution of the last pulse, that shifts the zero point of the time from which the slope is calculated. Such an option does not arise here unless once more we allow for several shells or a single wide shell.

- The duration of an observed pulse is correlated with the interval between this pulse and the preceding one (Nakar & Piran 2002, Quilligan et al. 2002). While relativistic turbulence with $n_p \approx 1$ produces pulses and separations of comparable width there is no reason that individual pulses will be correlated with the intervals.

- Relativistic turbulence predicts the standard Doppler induced correlation between the intensity and $E_{\text{peak}}$. While stronger peaks are typically harder, it is not clear that this specific relationship is satisfied.

We could not find obvious modifications that will address all these issues. This does not mean that those won’t be found in the future, but it suggests that the simple version of the model might not be enough. We have pointed out that a simple extension of a wide shell $d \gg 1$ or several separated shells might resolve some of the issues. However this makes the model, as least as far as central engine properties more similar to those inferred in the internal shocks model.

This research is supported by the ISF center of excellence in High energy Astrophysics (TP & AL), a Marie Curie IRG grant (EN) and by the Schwartzmann chair (TP).

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