ON THE NON-VANISHING OF $L$-FUNCTIONS ASSOCIATED TO CUSP FORMS OF HALF-INTEGRAL WEIGHT

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ABSTRACT. We prove a strengthening of Muić’s integral non-vanishing criterion for Poincaré series on unimodular locally compact Hausdorff groups and use it to prove a result on non-vanishing of $L$-functions associated to cusp forms of half-integral weight.

1. Introduction

In [6, Theorem 4-1], G. Muić proved an integral criterion for non-vanishing of Poincaré series on unimodular locally compact Hausdorff groups. In [8], this criterion was refined (see [8, Lemmas 2-1 and 3-1]) and used to study the non-vanishing of classical Poincaré series. It found further applications in [7] (resp., in [16] and [17]), where it was used to study the non-vanishing of Poincaré series associated to certain $K$-finite matrix coefficients of integrable representations of $\text{SL}_2(\mathbb{R})$ (resp., the metaplectic cover of $\text{SL}_2(\mathbb{R})$) and to prove results on non-vanishing of the corresponding cusp forms of integral (resp., of half-integral) weight. In [9], the criterion provided results on non-vanishing of $L$-functions associated to cusp forms of integral weight. In the first part of this paper, we prove a strengthening of this criterion (Theorem 3-2) and derive from it a non-vanishing criterion for Poincaré series of half-integral weight on the upper half-plane $\mathcal{H}$ (Theorem 4-5). Our proofs are considerably shorter than the proofs of criteria in [6] and [8] because we eliminate the need to approximate $L^1$-functions by continuous functions with compact support.

In the second part of the paper, we use techniques of [9] and Theorem 4-5 to prove results on analytic continuation and non-vanishing of $L$-functions associated to cusp forms of half-integral weight. We work in the following setting. The group $\text{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ by linear fractional transformations:

$$g.z := \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}.$$
The metaplectic cover of $SL_2(\mathbb{R})$ is defined as the group
\[
SL_2(\mathbb{R})^\sim := \left\{ \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \eta_\sigma \right\} \in SL_2(\mathbb{R}) \times \mathbb{C}^H : \eta_\sigma \text{ is holomorphic and } \eta_\sigma^2(z) = cz + d \text{ for all } z \in \mathcal{H} \right\}
\] (1-1)
with multiplication rule
\[
\sigma_1 \sigma_2 := \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \eta_{\sigma_1} \left( \begin{array}{cc} g_{\sigma_2} & \eta_{\sigma_2} \\ \eta_{\sigma_2} & c_{\sigma_2} \end{array} \right), \eta_\sigma \right) \in SL_2(\mathbb{R})^\sim.
\] (1-2)

Let $\Gamma$ be a discrete subgroup of finite covolume in $SL_2(\mathbb{R})^\sim$ such that $\infty$ is a cusp of $P(\Gamma)$, where $P : SL_2(\mathbb{R})^\sim \rightarrow SL_2(\mathbb{R})$ is the projection onto the first coordinate. Let $m \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$, and let $\chi : \Gamma \rightarrow \mathbb{C}^\times$ be a character of finite order such that $\eta_\chi^{-2m} = \chi(\gamma)$ for all $\gamma \in \Gamma_\infty$. The standardly defined space $S_m(\Gamma, \chi)$ of cusp forms (see Section 2) is a finite-dimensional Hilbert space under the Petersson inner product $\langle \cdot, \cdot \rangle_{S_m(\Gamma, \chi)}$. Let us mention that the spaces $S_m(N, \psi)$ from [14], where $N \in 4\mathbb{Z}_{>0}$ and $\psi$ is a Dirichlet character modulo $N$, are of this form (see [17, §9]).

For $f \in S_m(\Gamma, \chi)$ with Fourier expansion $f(z) = \sum_{n=1}^{\infty} a_n(f)e^{2\pi i n z}$, the $L$-function of $f$ is defined by the formula
\[
L(s, f) := \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}, \quad \Re(s) > \frac{m}{2} + 1.
\]
In Section 5, we determine Poincaré series $\Psi_{\Gamma, m, \chi, s} \in S_m(\Gamma, \chi)$ and a constant $c_{\Gamma, m} \in \mathbb{R}_{>0}$ that depends only on $\Gamma$ and $m$, with the following properties: for $\Re(s) > \frac{m}{2} + 1$, we have
\[
L(s, f) = c_{\Gamma, m} \langle f, \Psi_{\Gamma, m, \chi, m-s} \rangle_{S_m(\Gamma, \chi)}, \quad f \in S_m(\Gamma, \chi),
\] (1-3)
and if $m \in \frac{9}{2} + \mathbb{Z}_{\geq 0}$, then the formula (1-3) defines an analytic continuation of $L(\cdot, f)$ to the half-plane $\Re(s) > \frac{m}{2}$. This result is a half-integral weight version of [9, Theorem 3-9 and (3-10)].

Next, we apply our non-vanishing criterion, Theorem 4-5, to prove a result on non-vanishing of Poincaré series $\Psi_{\Gamma, m, \chi, m-s}$ for $\frac{m}{2} < \Re(s) < m - 1$. The main idea of the proof is the same as that of the proof of [9, Lemma 4-2], but our proof uses stronger estimates which relax the final sufficient condition for the non-vanishing of cusp forms $\Psi_{\Gamma, m, \chi, m-s}$. The final result is given in Theorem 6-3.

Theorem 6-3 translates via (1-3) into a sufficient condition for the inequality
\[
L(s, \Psi_{\Gamma, m, \chi, m-s}) > 0
\]
to hold (see Corollaries 6-9 and 6-10). This result adds to a series of results on non-vanishing of $L$-functions associated to certain cusp forms that was started by Kohnen’s paper [3]. In that paper, W. Kohnen studied the non-vanishing in the critical strip of the sum of appropriately normalized Hecke $L$-functions for $S_m(SL_2(\mathbb{Z}))$, where $m \in \mathbb{Z}_{\geq 4}$. His method is based on estimating the first Fourier coefficient of the cusp form that represents, in the sense of the Riesz representation theorem, the linear functional $S_m(SL_2(\mathbb{Z})) \rightarrow \mathbb{C}$, $f \mapsto c_{m,s}L(s, f)$, where $c_{m,s} \in \mathbb{C}^\times$ depends only on $m$ and $s$. Kohnen’s method inspired Muić’s work in [9] and was adapted in [11] to the case of cusp forms with non-trivial level and character, and
in [13] and [4] to the case of cusp forms of half-integral weight.

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2. Preliminaries

Let \( \sqrt{\cdot} : \mathbb{C} \to \mathbb{C} \) be the branch of the complex square root such that \( \arg(\sqrt{z}) \in ]-\frac{\pi}{2}, \frac{\pi}{2}] \) for all \( z \in \mathbb{C}^\times \). We write \( i := \sqrt{-1} \).

The group \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathbb{C} \cup \{\infty\} \) by linear fractional transformations:

\[
g. z := \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathbb{C} \cup \{\infty\}.
\]

The half-plane \( \mathcal{H} := \mathbb{C}_{\Im(z) > 0} \) is an orbit of this action. We note that

\[
\Im(g.z) = \frac{\Im(z)}{|cz + d|^2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \mathcal{H}.
\]

We define the group \( \text{SL}_2(\mathbb{R})^\sim \) by formulae (1-1) and (1-2). We use shorthand notation \((g_\sigma, \eta_\sigma(i))\) for elements \( \sigma = (g_\sigma, \eta_\sigma) \) of \( \text{SL}_2(\mathbb{R})^\sim \) and define the smooth structure of \( \text{SL}_2(\mathbb{R})^\sim \) by requiring that the Iwasawa parametrization \( \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} \to \text{SL}_2(\mathbb{R})^\sim \),

\[
(x, y, t) \mapsto \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1 \right) \left( \begin{pmatrix} y^\frac{i}{2} & 0 \\ 0 & y^{-\frac{i}{2}} \end{pmatrix}, y^{-\frac{1}{4}} \right) \left( \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, e^{i\frac{t}{2}} \right),
\]

be a local diffeomorphism. With this smooth structure, \( \text{SL}_2(\mathbb{R})^\sim \) is a connected Lie group, and the projection \( P : \text{SL}_2(\mathbb{R})^\sim \to \text{SL}_2(\mathbb{R}) \), \( P(\sigma) := g_\sigma \), is a smooth covering homomorphism of degree 2. Next, let us denote the center of \( \text{SL}_2(\mathbb{R})^\sim \) by \( Z(\text{SL}_2(\mathbb{R})^\sim) \). For a discrete subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{R})^\sim \), we define

\[
Z(\Gamma) := \Gamma \cap Z(\text{SL}_2(\mathbb{R})^\sim) \quad \text{and} \quad \varepsilon_\Gamma := |Z(\Gamma)|.
\]

The group \( \text{SL}_2(\mathbb{R})^\sim \) acts on \( \mathcal{H} \) by the formula

\[
\sigma.z := g_\sigma.z, \quad \sigma \in \text{SL}_2(\mathbb{R})^\sim, \quad z \in \mathcal{H}.
\]

Let us denote the three factors on the right-hand side of (2-2), from left to right, by \( n_x, a_y, \) and \( \kappa_t \). We have

\[
n_x a_y \kappa_t i = x + iy, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}_{>0}, \quad t \in \mathbb{R}.
\]

The group \( K := \{ \sigma \in \text{SL}_2(\mathbb{R})^\sim : \sigma i = i \} = \{ \kappa_t : t \in \mathbb{R} \} \) is a maximal compact subgroup of \( \text{SL}_2(\mathbb{R})^\sim \). Its unitary dual consists of the characters \( \chi_n : K \to \mathbb{C}^\times, \) \( n \in \frac{1}{2} \mathbb{Z} \),

\[
\chi_n(\kappa_t) := e^{-int}, \quad t \in \mathbb{R}.
\]

Let \( \nu \) be the \( \text{SL}_2(\mathbb{R}) \)-invariant Radon measure on \( \mathcal{H} \) defined by the formula

\[
\int_{\mathcal{H}} f \, d\nu = \int_{\mathbb{R}} \int_{0}^{\infty} f(x + iy)y^{-2} \, dy \, dx, \quad f \in C_c(\mathcal{H}).
\]
We fix the following Haar measure on $\text{SL}_2(\mathbb{R})^\sim$:

$$
\int_{\text{SL}_2(\mathbb{R})^\sim} \varphi \, d\mu_{\text{SL}_2(\mathbb{R})^\sim} := \frac{1}{4\pi} \int_0^{4\pi} \varphi(n_xa_y\kappa_t) \, dv(x + iy) \, dt
$$

for all $\varphi \in C_c(\text{SL}_2(\mathbb{R})^\sim)$.

Next, let $G$ be a locally compact Hausdorff group that is second-countable and unimodular, with Haar measure $\mu_G$. For a discrete subgroup $\Gamma$ of $G$, we denote by $\mu_{\Gamma \backslash G}$ the unique Radon measure on $\Gamma \backslash G$ such that

$$
\int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \varphi(\gamma g) \, d\mu_{\Gamma \backslash G}(g) = \int_{\Lambda \backslash G} \varphi \, d\mu_{\Lambda \backslash G}, \quad \varphi \in L^1(\Lambda \backslash G)
$$

(see [8, (2-2)]). In the case when $G = \text{SL}_2(\mathbb{R})^\sim$ and $\mu_G = \mu_{\text{SL}_2(\mathbb{R})^\sim}$, we have, for all $\varphi \in C_c(\text{SL}_2(\mathbb{R})^\sim)$,

$$
\int_{\text{SL}_2(\mathbb{R})^\sim} \varphi \, d\mu_{\text{SL}_2(\mathbb{R})^\sim} = \frac{1}{4\pi \varepsilon} \int_0^{4\pi} \int_{\Gamma \backslash \mathcal{H}} \varphi(n_xa_y\kappa_t) \, dv(x + iy) \, dt.
$$

For every $m \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, we have the following right action of $\text{SL}_2(\mathbb{R})^\sim$ on $\mathbb{C}^H$:

$$
(f \mid_m \sigma)(z) := f(g_\sigma z)\eta_\sigma(z)^{-2m}, \quad f \in \mathbb{C}^H, \quad \sigma \in \text{SL}_2(\mathbb{R})^\sim, \quad z \in \mathcal{H}.
$$

Let us note that for $\delta \in Z(\text{SL}_2(\mathbb{R})^\sim) = P^{-1}(\{\pm I_2\}) = \{\kappa_n : n \in \mathbb{Z}\}$, we have

$$
\eta_{\delta}^{-2m}(z) = \chi_m(\delta), \quad z \in \mathcal{H},
$$

and

$$
(f \mid_m \delta = \chi_m(\delta)f, \quad f \in \mathbb{C}^H.
$$

Next, let $\Gamma$ be a discrete subgroup of finite covolume in $\text{SL}_2(\mathbb{R})^\sim$, $\chi : \Gamma \rightarrow \mathbb{C}^\times$ a character of finite order, and $m \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. The space $S_m(\Gamma, \chi)$ of cusp forms for $\Gamma$ of weight $m$ with character $\chi$ by definition consists of holomorphic functions $f : \mathcal{H} \rightarrow \mathbb{C}$ that satisfy

$$
f \mid_{m} \gamma = \chi(\gamma)f, \quad \gamma \in \Gamma,
$$

and vanish at all cusps of $P(\Gamma)$ (see the beginning of [17, §3] for a detailed explanation of the last condition). $S_m(\Gamma, \chi)$ is a finite-dimensional Hilbert space under the Petersson inner product

$$
\langle f_1, f_2 \rangle_{S_m(\Gamma, \chi)} := \varepsilon^{-1}_m \int_{\Gamma \backslash \mathcal{H}} f_1(z)\overline{f_2(z)}\Im(z)^m \, dv(z), \quad f_1, f_2 \in S_m(\Gamma, \chi).
$$

For $m \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, we introduce the classical lift of a function $f : \mathcal{H} \rightarrow \mathbb{C}$ to the function $F_f : \text{SL}_2(\mathbb{R})^\sim \rightarrow \mathbb{C}$,

$$
F_f(\sigma) := (f \mid_m \sigma)(i).
$$
Equivalently,
\[(2.9) \quad F_f(n_x a_y \kappa_t) = f(x + iy) g^{\frac{\kappa}{2} e^{-i\kappa t}}, \quad x \in \mathbb{R}, \ y \in \mathbb{R}_{>0}, \ t \in \mathbb{R}.\]

Let \( \Gamma \) be a discrete subgroup of \( \text{SL}_2(\mathbb{R})^\sim \) and \( \chi : \Gamma \to \mathbb{C}^\times \) a unitary character. One checks easily that for every \( f : \mathcal{H} \to \mathbb{C} \), the following equivalence holds:
\[(2.10) \quad f|_m \gamma = \chi(\gamma)f, \quad \gamma \in \Gamma \quad \iff \quad F_f(\gamma \cdot) = \chi(\gamma)F_f, \quad \gamma \in \Gamma.\]

Moreover, if these equivalent conditions are satisfied and \( f \) is measurable, then we have
\[(2.11) \quad \int_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} |F_f| \, d\mu_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} = \int_{\Gamma \setminus \mathcal{H}} |f(z) \Im(z)^{\frac{\kappa}{2}}| \, dv(z).\]

3. A non-vanishing criterion for Poincaré series on locally compact Hausdorff groups

The proof of the following lemma is given in the first part of [8, proof of Lemma 2-1].

**Lemma 3.1.** Let \( G \) be a locally compact Hausdorff group that is second-countable and unimodular, with Haar measure \( \mu_G \). Let \( \Gamma \) be a discrete subgroup of \( G \), \( \Lambda \) a subgroup of \( \Gamma \), and \( \chi : \Gamma \to \mathbb{C}^\times \) a unitary character. Let \( \varphi : G \to \mathbb{C} \) be a measurable function with the following properties:

(F1) \( \varphi(\lambda g) = \chi(\lambda)\varphi(g), \quad \lambda \in \Lambda, \ g \in G \)

(F2) \( |\varphi| \in L^1(\Lambda \setminus G) \).

Then, we have the following:

1. The Poincaré series

\[
(P_{\Lambda \setminus \Gamma, \chi} \varphi)(g) := \sum_{\gamma \in \Lambda \setminus \Gamma} \overline{\chi(\gamma)} \varphi(\gamma g), \quad g \in G,
\]

converges absolutely almost everywhere on \( G \).

2. \( (P_{\Lambda \setminus \Gamma, \chi} \varphi)(\gamma \cdot) = \chi(\gamma)P_{\Lambda \setminus \Gamma, \chi} \varphi, \quad \gamma \in \Gamma \).

3. \( |P_{\Lambda \setminus \Gamma, \chi} \varphi| \in L^1(\Gamma \setminus G) \).

The following theorem is a strengthening of the integral non-vanishing criterion [8, Lemma 2-1].

**Theorem 3.2.** Let \( G, \Gamma, \Lambda, \chi, \) and \( \varphi \) satisfy the assumptions of Lemma 3.1. Then,
\[
\int_{\Gamma \setminus G} |P_{\Lambda \setminus \Gamma, \chi} \varphi| \, d\mu_{\Gamma \setminus G} > 0
\]

if there exists a Borel measurable set \( C \subseteq G \) with the following properties:

(C1) \( CC^{-1} \cap \Gamma \subseteq \Lambda \).

(C2) Writing \( (\Lambda C)^c := G \setminus \Lambda C \), we have

\[
(3.3) \quad \int_{\Lambda \setminus \Lambda C} |\varphi| \, d\mu_{\Lambda \setminus G} > \int_{\Lambda \setminus (\Lambda C)^c} |\varphi| \, d\mu_{\Lambda \setminus G}
\]

for some measurable function \( |\cdot| : \mathbb{C} \to \mathbb{R}_{>0} \) with the following properties:

(M1) \( |0| = 0 \).
Proof. Suppose that a Borel measurable set $C \subseteq G$ satisfies (C1) and (C2). Let us denote by $1_A$ the characteristic function of $A \subseteq G$. The property (C1) implies the following:

$$\text{(M2) } |z| = |\bar{z}|, \quad z \in \mathbb{C}. $$

$$\text{(M3) } \sum_{n=1}^{\infty} |z_n| \leq \sum_{n=1}^{\infty} |z_n| \text{ for every } (z_n)_{n \in \mathbb{Z}^+} \subseteq \mathbb{C} \text{ such that } \sum_{n=1}^{\infty} |z_n| < \infty. $$

Namely, if $1_A(g) \neq 0$ and $1_A(\gamma') \neq 0$ for some $\gamma, \gamma' \in \Gamma$, i.e., if $\gamma,g,\gamma' \in \Lambda_C$, then $\gamma \gamma'^{-1} = (\gamma g)(\gamma')^{-1} \in \Lambda C^{-1}\Lambda \cap \Gamma = \Lambda (\Lambda C^{-1}\Lambda \cap \Gamma) \Lambda \overset{(C1)}{=} \Lambda$, hence $\Lambda g = \Lambda g'$. Now we have

$$\text{(3-4) } \quad \int_{\Gamma \setminus G} |P_{\Lambda \setminus \Lambda, \chi} (\varphi \cdot 1_{\Lambda C})| \, d\mu_{\Gamma \setminus G}$$

$$\quad = \int_{\Gamma \setminus G} \left| \sum_{\gamma \in \Lambda \setminus \Gamma} \bar{\chi}(\gamma) \varphi(\gamma g) 1_{\Lambda C}(\gamma g) \right| \, d\mu_{\Gamma \setminus G}(g)$$

$$\quad \overset{(3-4), (M1),(M2)}{=} \int_{\Gamma \setminus G} \sum_{\gamma \in \Lambda \setminus \Gamma} |\varphi(\gamma g) 1_{\Lambda C}(\gamma g)| \, d\mu_{\Gamma \setminus G}(g)$$

$$\quad \overset{(2-4)}{=} \int_{\Lambda \setminus \Gamma} |\varphi \cdot 1_{\Lambda C}| \, d\mu_{\Lambda \setminus \Gamma}$$

$$\text{(3-5) } \quad \overset{(M1)}{=} \int_{\Lambda \setminus \Lambda C} |\varphi| \, d\mu_{\Lambda \setminus \Gamma}. $$

On the other hand,

$$\text{(3-6) } \quad \int_{\Gamma \setminus G} |P_{\Lambda \setminus \Lambda, \chi} (\varphi \cdot 1_{(\Lambda C)^{\circ}})| \, d\mu_{\Gamma \setminus G}$$

$$\quad = \int_{\Gamma \setminus G} \left| \sum_{\gamma \in \Lambda \setminus \Gamma} \bar{\chi}(\gamma) \varphi(\gamma g) 1_{(\Lambda C)^{\circ}}(\gamma g) \right| \, d\mu_{\Gamma \setminus G}(g)$$

$$\quad \overset{(M3)}{\leq} \int_{\Gamma \setminus G} \sum_{\gamma \in \Lambda \setminus \Gamma} |\varphi(\gamma g) 1_{(\Lambda C)^{\circ}}(\gamma g)| \, d\mu_{\Gamma \setminus G}(g)$$

$$\quad \overset{(2-4)}{=} \int_{\Lambda \setminus G} |\varphi \cdot 1_{(\Lambda C)^{\circ}}| \, d\mu_{\Lambda \setminus G}$$

$$\quad \overset{(M1)}{=} \int_{\Lambda \setminus (\Lambda C)^{\circ}} |\varphi| \, d\mu_{\Lambda \setminus G}.$$
Now we have
\[
\int_{\Gamma \setminus G} |P_{\Lambda \setminus \Gamma, \chi} \varphi| \, d\mu_{\Gamma \setminus G}
\]
\[
\ge \int_{\Gamma \setminus G} |P_{\Lambda \setminus \Gamma, \chi} (\varphi \cdot 1_{AC})| \, d\mu_{\Gamma \setminus G} - \int_{\Lambda \setminus G} |\varphi| \, d\mu_{\Lambda \setminus G}
\]
\[
\ge \int_{\Lambda \setminus AC} |\varphi| \, d\mu_{\Lambda \setminus G} - \int_{\Lambda \setminus (AC)^c} |\varphi| \, d\mu_{\Lambda \setminus G}
\]
\[
> 0,
\]
from which it follows by (M1) that $|P_{\Lambda \setminus \Gamma, \chi} \varphi| \not\equiv 0$ in $L^1(\Gamma \setminus G)$, hence
\[
\int_{\Gamma \setminus G} |P_{\Lambda \setminus \Gamma, \chi} \varphi| \, d\mu_{\Gamma \setminus G} > 0.
\]

In the case when $|\cdot| = |\cdot| : \mathbb{C} \to \mathbb{R}_{\ge 0}$, Theorem 3-2 is a variant of [8, Lemma 2-1]. More generally, we have the following lemma.

**Lemma 3-8.** Let $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that $f(0) = 0$. Suppose that $f$ is concave, i.e., that
\[
f((1-t)x + ty) \ge (1-t)f(x) + tf(y), \quad t \in [0,1], \ x, y \in \mathbb{R}_{\ge 0}.
\]
Then, the function $|\cdot| : \mathbb{C} \to \mathbb{R}_{\ge 0}$,
\[
|z| := f(|z|),
\]
is measurable and has the properties (M1)–(M3) of Theorem 3-2.

**Proof.** We note that $f$ is subadditive and non-decreasing by [15, Theorem 2]. Next, we prove that $|\cdot|$ satisfies (M3), the other claims being trivial.

First, let us show that for every $(x_n)_{n\in\mathbb{Z}_{>0}} \subseteq \mathbb{R}_{\ge 0}$ such that $\sum_{n=1}^{\infty} x_n < \infty$, we have
\[
f \left( \sum_{n=1}^{\infty} x_n \right) \le \sum_{n=1}^{\infty} f(x_n).
\]
This holds trivially if $\sum_{n=1}^{\infty} x_n = 0$, so suppose that $\sum_{n=1}^{\infty} x_n > 0$. By subadditivity, we have $f \left( \sum_{n=1}^{N} x_n \right) \le \sum_{n=1}^{N} f(x_n)$ for all $N \in \mathbb{Z}_{>0}$, from which (3-9) follows by applying $\lim_{N \to \infty}$ since by concavity $f$ is continuous on $\mathbb{R}_{>0}$ [10, Theorem 1.3.3].

Now, for every $(z_n)_{n\in\mathbb{Z}_{>0}} \subseteq \mathbb{C}$ such that $\sum_{n=1}^{\infty} |z_n| < \infty$, we have
\[
|\sum_{n=1}^{\infty} z_n| = f \left( \sum_{n=1}^{\infty} |z_n| \right) \le f \left( \sum_{n=1}^{\infty} |z_n| \right) \overset{(3-9)}{=} \sum_{n=1}^{\infty} f(|z_n|) = \sum_{n=1}^{\infty} |z_n|,
\]
where the first inequality holds because $f$ is non-decreasing. \qed
4. A non-vanishing criterion for Poincaré series of half-integral weight

Lemma 4-1. Let $\Gamma$ be a discrete subgroup of $\text{SL}_2(\mathbb{R})^\sim$, $\Lambda$ a subgroup of $\Gamma$, $\chi : \Gamma \to \mathbb{C}^\times$ a unitary character, and $m \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Let $f : \mathcal{H} \to \mathbb{C}$ be a measurable function with the following properties:

1. $f|^m_\chi \lambda = \chi(\lambda)f, \lambda \in \Lambda$.
2. $\int_{\Lambda \setminus \mathcal{H}} \left| f(z) \Im(z) \right|^2 |dv(z)| < \infty$.

Then, we have the following:

1. The series

$$P_{\Lambda \setminus \Gamma, \chi} f := \sum_{\gamma \in \Lambda \setminus \Gamma} \overline{\chi(\gamma)} f|^m_\gamma$$

converges absolutely almost everywhere on $\mathcal{H}$.

2. $(P_{\Lambda \setminus \Gamma, \chi} f)|^m_\gamma = \chi(\gamma)P_{\Lambda \setminus \Gamma, \chi} f, \gamma \in \Gamma$.

3. $\int_{\Gamma \setminus \mathcal{H}} \left| (P_{\Lambda \setminus \Gamma, \chi} f)(z) \Im(z) \right|^2 |dv(z)| = \varepsilon \int_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} \left| P_{\Lambda \setminus \Gamma, \chi} Ff \right| d\mu|_{\text{SL}_2(\mathbb{R})^\sim} < \infty$.

Proof. By (f1), the terms of the series $P_{\Lambda \setminus \Gamma, \chi} f$ are well-defined. Next, the function $Ff$ satisfies the assumptions of Lemma 3-1: it satisfies (F1) by (2-10) and (f1), and it satisfies (F2) by (2-11) and (f2). By Lemma 3-1, the series $P_{\Lambda \setminus \Gamma, \chi} Ff$ converges absolutely almost everywhere on $\text{SL}_2(\mathbb{R})^\sim$, and this implies the claim (1) and the equality

$$F_{P_{\Lambda \setminus \Gamma, \chi} f} = P_{\Lambda \setminus \Gamma, \chi} Ff,$$

as is easily checked by following the definitions. The claim (2) is easy to check directly; alternatively, it follows by (2-10) and (4-2) from Lemma 3-1. Finally, the equality in the claim (3) holds by (2-11) and (4-2), and its right-hand side is finite by Lemma 3-1.(3). \qed

Lemma 4-3. Let $\Gamma$, $\Lambda$, $\chi$, $m$, and $f$ satisfy the assumptions of Lemma 4-1. If

$$\chi|^m_{Z(\Gamma)} \neq \chi|^m_{Z(\Gamma)};$$

then $P_{\Lambda \setminus \Gamma, \chi} f \equiv 0$.

Proof. The claim is clear from the equality

$$\chi(m(\delta)P_{\Lambda \setminus \Gamma, \chi} f)|^m_\gamma = \chi(\delta)P_{\Lambda \setminus \Gamma, \chi} f, \quad \delta \in Z(\Gamma),$$

where the second equality holds by Lemma 4-1.(2). \qed

Theorem 4-5. Let $\Gamma$, $\Lambda$, $\chi$, $m$, and $f$ satisfy the assumptions of Lemma 4-1. Suppose that

$$\chi|^m_{Z(\Gamma)} = \chi|^m_{Z(\Gamma)};$$

Then,

$$\int_{\Gamma \setminus \mathcal{H}} \left| (P_{\Lambda \setminus \Gamma, \chi} f)(z) \Im(z) \right|^2 |dv(z)| > 0$$

if there exists a Borel measurable set $S \subseteq \mathcal{H}$ with the following properties:

1. If $z_1, z_2 \in S$ and $z_1 \neq z_2$, then $\Gamma z_1 \neq \Gamma z_2$. 

(S2) Writing \((\Lambda.S)^c := \mathcal{H} \setminus \Lambda.S\), we have
\[
\int_{\Lambda \setminus \Lambda.S} |f(z)\bar{\Omega}(z)\frac{1}{m} \sum_{y} \sum_{\delta \in \Lambda \setminus \Lambda.S} \eta(z)\Gamma_{\delta} \Gamma_{\gamma} f | \, dv(z) > \int_{\Lambda \setminus (\Lambda.S)^c} |f(z)\bar{\Omega}(z)\frac{1}{m} \sum_{y} \sum_{\delta \in \Lambda \setminus \Lambda.S} \eta(z)\Gamma_{\delta} \Gamma_{\gamma} f | \, dv(z)
\]
for some measurable function \(|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}\) that satisfies (M1)–(M3) of Theorem 3.2.

Proof. Suppose that a Borel measurable set \(S \subseteq \mathcal{H}\) satisfies (S1) and (S2). We have
\[
P_{\Lambda \setminus \Gamma, \chi f} = \sum_{\gamma \in \Lambda \setminus \Lambda.\Gamma} \sum_{\delta \in \Lambda \setminus \Lambda.\Gamma} \chi(\delta) \chi_{m}(\delta) \frac{1}{m} \sum_{y} \sum_{\delta \in \Lambda \setminus \Lambda.\Gamma} \eta(z)\Gamma_{\delta} \Gamma_{\gamma} f | \, dv(z)
\]

hence it suffices to prove the non-vanishing of the series \(P_{\Lambda \setminus \Gamma, \chi f}\). In other words, we may assume without loss of generality that \(Z(\Gamma) \subseteq \Lambda\). We may also assume that \(S\) contains no elliptic points for \(\Gamma\). Under these assumptions, by (S1) and (2-3) the set
\[
C := \{n_{x}a_{y} : x + iy \in S\} \subseteq SL_{2}(\mathbb{R})^{-}
\]
satisfies \(CC^{-1} \cap \Gamma \subseteq Z(\Gamma) \subseteq \Lambda\), i.e., \(C\) has the property \((C1)\) of Theorem 3.2.

Our goal is to apply Theorem 3.2 to the series \(P_{\Lambda \setminus \Gamma, \chi f}\). Since we have seen in the proof of Lemma 4.1 that the function \(F_{f}\) satisfies the conditions \((F1)\) and \((F2)\) of Lemma 3.1, it remains to prove the inequality
\[
(4-8) \quad \int_{\Lambda \setminus (\Lambda.S)^c} |F_{f}| \, d\mu_{\Lambda \setminus SL_{2}(\mathbb{R})^{-}} > \int_{\Lambda \setminus (\Lambda.S)^c} |F_{f}| \, d\mu_{\Lambda \setminus SL_{2}(\mathbb{R})^{-}},
\]
where \((\Lambda.C)^c := SL_{2}(\mathbb{R})^{-} \setminus \Lambda.C\). To this end, we note that
\[
(4-9) \quad \Lambda.C = \{n_{x}a_{y} : x + iy \in \Lambda.S\} \subseteq SL_{2}(\mathbb{R})^{-}
\]
hence we have
\[
(4-10) \quad \int_{\Lambda \setminus (\Lambda.C)^c} |F_{f}| \, d\mu_{\Lambda \setminus SL_{2}(\mathbb{R})^{-}} = \int_{\Lambda \setminus (\Lambda.C)^c} |F_{f}| \, d\mu_{\Lambda \setminus SL_{2}(\mathbb{R})^{-}}
\]
We see analogously that
\[
(4-11) \quad \int_{\Lambda \setminus (\Lambda.C)^c} |F_{f}| \, d\mu_{\Lambda \setminus SL_{2}(\mathbb{R})^{-}} = \varepsilon_{\Lambda}^{-1} \int_{\Lambda \setminus (\Lambda.S)^c} |F_{f}| \, d\mu_{\Lambda \setminus SL_{2}(\mathbb{R})^{-}}
\]
By (4-10) and (4-11), (S2) implies (4-8).

Thus, by Theorem 3-2 \( \int_{\Gamma \backslash SL_2(\mathbb{R})} |P_{\Lambda \backslash \Gamma, \chi} F_f| \, d\mu_{\Gamma \backslash SL_2(\mathbb{R})} > 0 \). By Lemma 4-1.(3), this implies (4-7).

We conclude this section by one more lemma on convergence of Poincaré series of half-integral weight (cf. [9, Lemma 2-4]).

**Lemma 4.12.** Let \( \Gamma, \Lambda, \chi, m, \) and \( f \) satisfy the assumptions of Lemma 4-1. Suppose in addition that \( f \) is holomorphic. Then, the series \( P_{\Lambda \backslash \Gamma, \chi} f \) converges absolutely and locally uniformly on \( \mathcal{H} \) and defines an element of \( S_m(\Gamma, \chi) \).

**Proof.** The series \( P_{\Lambda \backslash \Gamma, \chi} f \) converges absolutely and locally uniformly on \( \mathcal{H} \) by the half-integral weight version of [5, Theorem 2.6.6.(1)]. Since Lemma 4-1.(2) also holds, it remains to prove that \( P_{\Lambda \backslash \Gamma, \chi} f \) vanishes at every cusp of \( P(\Gamma) \).

Let \( x \) be a cusp of \( P(\Gamma) \), and let \( \sigma \in SL_2(\mathbb{R}) \) such that \( \sigma \cdot \infty = x \). Let

\[
((P_{\Lambda \backslash \Gamma, \chi} f)_{\sigma m}) (z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \sigma z}, \quad z \in \mathcal{H},
\]

be the Fourier expansion of \( (P_{\Lambda \backslash \Gamma, \chi} f)_{\sigma m} \). We need to prove that \( a_n = 0 \) for all \( n \in \mathbb{Z}_{\leq 0} \).

Let \( h \in \mathbb{R}_{>0} \) such that \( Z(SL_2(\mathbb{R})^{-}) \sigma^{-1}\Gamma \sigma = Z(SL_2(\mathbb{R})^{-}) \langle m_h \rangle \). Since \( \left| (P_{\Lambda \backslash \Gamma, \chi} f)_{\sigma m} \right| \) is \( h \)-periodic, by the half-integral weight version of [9, Lemma 2-1] it suffices to prove that

\[
I := \int_{[0, h[ \times [h, \infty[} \left| ((P_{\Lambda \backslash \Gamma, \chi} f)_{\sigma m}) (z) \mathfrak{F}(z)_{\sigma m} \right| dv(z)
\]

is finite. We have

\[
I \overset{(2-1)}{=} \int_{[0, h[ \times [h, \infty[} \left| (P_{\Lambda \backslash \Gamma, \chi} f) (\sigma z) \mathfrak{F}(\sigma z)_{\sigma m} \right| dv(z)
\]

\[
= \int_{\sigma \cdot [0, h[ \times [h, \infty[} \left| (P_{\Lambda \backslash \Gamma, \chi} f) (z) \mathfrak{F}(z)_{\sigma m} \right| dv(z)
\]

\[
\leq \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Lambda \backslash \Gamma} \left| (f_{\sigma m} \gamma) (z) \mathfrak{F}(z)_{\sigma m} \right| dv(z)
\]

\[
= \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Lambda \backslash \Gamma} \left| f(\gamma z) \mathfrak{F}(z)_{\sigma m} \right| dv(z)
\]

\[
\overset{(2-1)}{=} \int_{\varepsilon_\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Lambda \backslash \Gamma} \mathfrak{F}(\gamma z)_{\sigma m} \right| dv(z)
\]

\[
\overset{(2-2)}{=} \int_{\varepsilon_\Lambda \backslash \mathcal{H}} \mathfrak{F}(z)_{\sigma m} \right| dv(z) \overset{(2-2)}{<} \infty,
\]

where the first inequality holds because by [5, Corollary 1.7.5] any two distinct points of \( \sigma \cdot ([0, h[ \times [h, \infty[] \) are mutually \( \Gamma \)-inequivalent. \( \square \)

### 5. Analytic Continuation of \( L \)-functions

Throughout this section, let \( m \in \frac{5}{2} + \mathbb{Z}_{\geq 0} \), let \( \Gamma \) be a discrete subgroup of finite covolume in \( SL_2(\mathbb{R})^{-} \) such that \( \infty \) is a cusp of \( P(\Gamma) \), and let \( \chi : \Gamma \to \mathbb{C}^\times \) be a character of finite order...
such that
\[ n^{-2m} = \chi(\gamma), \quad \gamma \in \Gamma_\infty. \]
Let \( h \in \mathbb{R}_{>0} \) such that
\[ Z(\text{SL}_2(\mathbb{R})^\sim) \Gamma_\infty = Z(\text{SL}_2(\mathbb{R})^\sim) \langle n_h \rangle. \]

It is well-known that the classical Poincaré series
\[ \psi_{\Gamma, n, m, \chi} := \sum_{\Gamma \backslash \Gamma_\infty} \chi(\gamma) e^{2\pi i n \gamma} \] converges absolutely and locally uniformly on \( H \) and define elements of \( S_m(\Gamma, \chi) \). The following lemma is a half-integral weight variant of [9, Theorem 2-10] that generalizes their construction.

\textbf{Lemma 5-3.} Let \( (a_n)_{n \in \mathbb{Z}_{>0}} \subseteq \mathbb{C} \) such that \( \sum_{n=1}^{\infty} |a_n| n^{1-m} < \infty \). Then, the double series
\[ S := \sum_{\gamma \in \Gamma \backslash \Gamma_\infty} \sum_{n=1}^{\infty} \chi(\gamma) a_n e^{2\pi i n \gamma} |m \gamma \]
converges absolutely and uniformly on compact sets in \( H \) and defines an element of \( S_m(\Gamma, \chi) \) that coincides with
\[ P_{\Gamma \backslash \Gamma_\infty} \left( \sum_{n=1}^{\infty} a_n e^{2\pi i n \gamma} \right) = \sum_{n=1}^{\infty} a_n \psi_{\Gamma, n, m, \chi}. \]

All the series in (5-5) converge absolutely and uniformly on compact sets in \( H \). Moreover, the series \( \sum_{n=1}^{\infty} a_n \psi_{\Gamma, n, m, \chi} \) converges to \( S \) in the topology of \( S_m(\Gamma, \chi) \).

\textbf{Proof.} The terms of the series \( S \) are well-defined, i.e., we have
\[ e^{2\pi i \gamma} |m \gamma \gamma = \chi(\gamma) e^{2\pi i \gamma}, \quad \gamma \in \Gamma_\infty, \]
since by (5-1) the equality (5-6) is equivalent to \( h \)-periodicity of the function \( e^{2\pi i \gamma} \).

The double series \( S \) and all the series in (5-5) converge absolutely and uniformly on compact sets in \( H \) by [5, Corollary 2.6.4] and the following estimate:
\[
\int_{\Gamma \backslash H} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \sum_{n=1}^{\infty} |a_n| \left( e^{2\pi i \gamma} |m \gamma \right) (z) \Im(z)^{\frac{m}{2}} dv(z)
\]
\[(2-1) = \sum_{n=1}^{\infty} |a_n| \int_{\Gamma \backslash H} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left| e^{2\pi i \gamma} \Im(z)^{\frac{m}{2}} \right| dv(z)
\]
\[ = \sum_{n=1}^{\infty} |a_n| \int_{\Gamma \backslash H} \left| e^{2\pi i \gamma} \Im(z)^{\frac{m}{2}} \right| dv(z)
\]
\[ = \sum_{n=1}^{\infty} |a_n| \int_{0}^{h} \int_{0}^{\infty} e^{-2\pi i n y} y^{\frac{m}{2}-2} dy dx
\]
\[ = \sum_{n=1}^{\infty} |a_n| n^{1-\frac{m}{2}} \frac{h^{\frac{m}{2}}}{(2\pi)^{\frac{m}{2}-1}} \Gamma \left( \frac{m}{2} - 1 \right) < \infty. \]
In the last equality we use the standard notation for the gamma function: $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t} \, dt$, $\Re(s) > 0$. Now it is clear that $S = (5\text{-}5)$.

Since $S_m(\Gamma, \chi)$ is a finite-dimensional Hausdorff topological vector space, it is a closed subspace of $C(\mathcal{H})$ with the topology of uniform convergence on compact sets. Thus, the fact that the series $\sum_{n=1}^\infty a_n \psi_{\Gamma,n,m,\chi}$ converges to $S$ uniformly on compact sets in $\mathcal{H}$ implies that $S$ belongs to $S_m(\Gamma, \chi)$ and that the series $\sum_{n=1}^\infty a_n \psi_{\Gamma,n,m,\chi}$ converges to $S$ in the topology of $S_m(\Gamma, \chi)$.  

The following lemma (cf. [9, Lemma 5\text{-}1]) will be applied in the proof of Lemma 5\text{-}13 to prove the estimate (5\text{-}18) that will be used in the proof of Theorem 6\text{-}3.

**Lemma 5\text{-}7.** Let $a \in \mathbb{R}_{>\frac{1}{2}}$. Then,

\[(5\text{-}8)\quad \Gamma(a) \int_\mathbb{R} \frac{dx}{(x^2 + 1)^a} = \sqrt{\pi} \Gamma\left(a - \frac{1}{2}\right).\]

**Proof.** We have

\[
\Gamma(a) \int_\mathbb{R} \frac{dx}{(x^2 + 1)^a} = \int_\mathbb{R} \int_0^\infty \left(\frac{y}{x^2 + 1}\right)^{a-1} e^{-y} \frac{dy}{x^2 + 1} \, dx \\
= \int_\mathbb{R} \int_0^\infty t^{a-1} e^{-(x^2+1)t} \, dt \, dx = \int_\mathbb{R} t^{a-1} e^{-t} \left(\int_\mathbb{R} e^{-x^2 t} \, dx\right) \, dt \\
= \int_0^\infty t^{a-\frac{3}{2}} e^{-t} \left(\int_\mathbb{R} e^{-u^2} \, du\right) \, dt = \sqrt{\pi} \Gamma\left(a - \frac{1}{2}\right). \quad \square
\]

By the half-integral weight version of [5, Theorem 2.6.10], every $f \in S_m(\Gamma, \chi)$ has the Fourier expansion

\[(5\text{-}9)\quad f(z) = \sum_{n=1}^\infty a_n(f) e^{2\pi inz^2}, \quad z \in \mathcal{H},\]

where

\[
a_n(f) = \frac{\varepsilon \Gamma(4\pi n)^{m-1}}{h^m \Gamma(m-1)} \langle f, \psi_{\Gamma,n,m,\chi} \rangle_{S_m(\Gamma, \chi)}, \quad n \in \mathbb{Z}_{>0}.\]

The $L$-function of $f$ is the function $L(\cdot, f) : \mathbb{C}_{\Re(s) > \frac{m}{2} + 1} \to \mathbb{C}$,

\[(5\text{-}10)\quad L(s, f) := \sum_{n=1}^\infty \frac{a_n(f)}{n^s}, \quad \Re(s) > \frac{m}{2} + 1.\]

Since by a standard argument $a_n(f) = O\left(n^{\frac{m}{2}}\right)$ (e.g., see [5, Corollary 2.1.6]), the series (5\text{-}10) converges absolutely and uniformly on compact sets in $\mathbb{C}_{\Re(s) > \frac{m}{2} + 1}$, hence $L(\cdot, f)$ is holomorphic on $\mathbb{C}_{\Re(s) > \frac{m}{2} + 1}$.

We define $F : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$,

\[
F(z, s) := \sum_{n=1}^\infty n^{s-1} e^{2\pi inz^2}.
\]
The series on the right-hand side obviously converges absolutely and locally uniformly on $\mathcal{H} \times \mathbb{C}$, hence $F$ is holomorphic on $\mathcal{H} \times \mathbb{C}$.

Next, let $\Re(s) > \frac{m}{2} + 1$. We have

\begin{equation}
L(s, f) = \sum_{n=1}^{\infty} \langle f, n^{m-1-s} \psi_{\Gamma, n, m, \chi} \rangle_{S_m(\Gamma, \chi)}.
\end{equation}

By Lemma 5-3, the series $\sum_{n=1}^{\infty} n^{m-1-s} \psi_{\Gamma, n, m, \chi}$ converges in $S_m(\Gamma, \chi)$ and we have

\begin{equation}
\sum_{n=1}^{\infty} n^{m-1-s} \psi_{\Gamma, n, m, \chi} = P_{\Gamma, \chi}(F(\cdot, m - \overline{s})),
\end{equation}

where the latter series converges absolutely and uniformly on compact sets in $\mathcal{H}$. Thus, it follows from (5-11) that

\begin{equation}
L(s, f) = \sum_{n=1}^{\infty} \langle f, n^{m-1-s} \psi_{\Gamma, n, m, \chi} \rangle_{S_m(\Gamma, \chi)} = P_{\Gamma, \chi}(F(\cdot, m - \overline{s})),
\end{equation}

This motivates the following lemma.

**Lemma 5-13.** Let $\Re(s) < \frac{m}{2} - 1$ or $1 < \Re(s) < \frac{m}{2}$. Then, the series

\[ \Psi_{\Gamma, m, \chi, s} := P_{\Gamma, \chi}(F(\cdot, s)) \]

converges absolutely and uniformly on compact sets in $\mathcal{H}$ and defines an element of $S_m(\Gamma, \chi)$.

**Proof.** In the case when $\Re(s) < \frac{m}{2} - 1$, the claim follows from the above discussion.

Let us prove the claim in the case when $1 < \Re(s) < \frac{m}{2}$ by applying Lemma 4-12 to the Poincaré series $P_{\Gamma, \chi}(F(\cdot, s))$. We prove the inequality

\begin{equation}
\int_{\Gamma, \chi} |F(z, s)|^{\frac{m}{2}} dv(z) < \infty, \quad 1 < \Re(s) < \frac{m}{2},
\end{equation}

the other conditions of Lemma 4-12 being satisfied trivially. For every $\varepsilon \in \mathbb{R}_{>0}$, we can write the left-hand side of (5-14) as the sum

\[
\int_{[0, h] \times [0, \varepsilon]} |F(z, s)|^{\frac{m}{2}} dv(z) + \int_{[0, h] \times [\varepsilon, \infty]} |F(z, s)|^{\frac{m}{2}} dv(z).
\]

\[= I_1 + I_2 \]
First, we estimate $I_2$: we have

$$I_2 = \int_{[0,h] \times [0,\infty]} \left| \sum_{n=1}^{\infty} n^{s-1} e^{2\pi inz} \mathfrak{K}(z)^{\frac{m}{2}} \right| \, dv(z)$$

$$\leq \sum_{n=1}^{\infty} n^{\Re(s)-1} \int_{0}^{h} \int_{0}^{\infty} e^{-2\pi yz n^{\frac{m}{2}}} y^{\frac{m}{2}-2} \, dy \, dx$$

$$\leq \left( \sum_{n=1}^{\infty} n^{\Re(s)-1} e^{-2\pi (n-1) \xi n^{\frac{m}{2}}} \right) \frac{h^{\frac{m}{2}}}{(2\pi)^{\frac{m}{2}}} \Gamma \left( \frac{m}{2} - 1 \right),$$

(5-15)

and the right-hand side is finite by d’Alembert’s ratio test. To estimate $I_1$, we will use the Lipschitz summation formula (see [2, Theorem 1]):

$$\sum_{n=1}^{\infty} n^{s-1} e^{2\pi inz} = \Gamma(s)(2\pi)^{-s} e^{\frac{i\pi s}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^s}, \quad z \in \mathcal{H}, \; \Re(s) > 1,$$

(5-16)

where

$$z^s := e^{s(\log|z| + i\arg(z))}, \quad z \in \mathcal{H}, \; s \in \mathbb{C}, \; \arg(z) \in ]0, \pi[.$$

We note that for all $z \in \mathcal{H}$ and $s \in \mathbb{C},$

$$|z^s| = |z|^{\Re(s)} e^{-\Im(s) \arg(z)} \in \left[ |z|^{\Re(s)} \min \{ e^{-\pi \Im(s)}, 1 \}, |z|^{\Re(s)} \max \{ e^{-\pi \Im(s)}, 1 \} \right].$$

(5-17)

Now, for $1 < \Re(s) < \frac{m}{2}$ we have

$$I_1 = \int_{[0,h] \times [0,\epsilon]} \left| \frac{\Gamma(s)(2\pi)^{-s} e^{\frac{i\pi s}{2}} h^s}{\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^s}} \mathfrak{K}(z)^{\frac{m}{2}} \right| \, dv(z)$$

$$\leq \Gamma(\Re(s)) \left( \frac{h}{2\pi} \right)^{\Re(s)} e^{\frac{\pi \Im(s)}{2}} \int_{[0,h] \times [0,\epsilon]} \mathfrak{K}(z)^{\frac{m}{2}} \frac{\mathfrak{K}(z)^{\frac{m}{2}}}{|z|^\Re(s)} \, dv(z) \max \{ e^{\pi \Im(s)}, 1 \}$$

$$= \Gamma(\Re(s)) \left( \frac{h}{2\pi} \right)^{\Re(s)} e^{\frac{\pi \Im(s)}{2}} \int_{[0,h] \times [0,\epsilon]} \mathfrak{K}(z)^{\frac{m}{2}} \frac{\mathfrak{K}(z)^{\frac{m}{2}}}{|z|^\Re(s)} \, dv(z)$$

$$= \Gamma(\Re(s)) \left( \frac{h}{2\pi} \right)^{\Re(s)} e^{\frac{\pi \Im(s)}{2}} \int_{[0,\epsilon]} \int_{[0,\epsilon]} \frac{dy \, dx}{(\frac{z}{\sqrt{x^2 + y^2}} + 1)^{\frac{\Re(s)}{2}}}$$

$$= \Gamma(\Re(s)) \left( \frac{h}{2\pi} \right)^{\Re(s)} e^{\frac{\pi \Im(s)}{2}} \frac{\sqrt{\pi} \Gamma \left( \frac{\Re(s)-1}{2} \right)}{\Gamma \left( \frac{\Re(s)}{2} \right)} \cdot \frac{e^{\frac{m}{2} - \Re(s)}}{e^{\frac{m}{2} - \Re(s)} - \infty}. $$

(5-18)
This concludes the proof of (5-14).

The main result of this section is the following theorem.

**Theorem 5-19.** Let \( f \in S_m(\Gamma, \chi) \). Then, the function \( \ell(\cdot, f) : \mathbb{C}_{\Re(s)>\frac{m}{2}} \cup \mathbb{C}_{\Re(s)<m-1} \rightarrow \mathbb{C} \),

\[
(5-20) \quad \ell(s, f) := \frac{\varepsilon \Gamma(4\pi)^{m-1}}{h^m \Gamma(m-1)} \langle f, \Psi_{\Gamma, m, \chi, m-\pi} \rangle_{S_m(\Gamma, \chi)},
\]

is well-defined and holomorphic. If \( m > 4 \), then \( \ell(\cdot, f) \) is an analytic continuation of \( L(\cdot, f) \) to the half-plane \( \mathbb{C}_{\Re(s)>\frac{m}{2}} \).

**Proof.** The function \( \ell(\cdot, f) \) is well-defined by Lemma 5-13. Its restriction to the half-plane \( \mathbb{C}_{\Re(s)>\frac{m}{2}+1} \) is holomorphic since it coincides with \( L(\cdot, f) \) by (5-12).

Let us prove that \( \ell(\cdot, f) \) is holomorphic on \( \mathbb{C}_{\Re(s)<m-1} \). We have

\[
(5-21) \quad \ell(s, f) = \frac{(4\pi)^{m-1}}{h^m \Gamma(m-1)} \int_{\Gamma \backslash \mathcal{H}} f(z) \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \chi(\gamma) F(\gamma, m-\pi) \mathcal{F}(z)^m dv(z)
\]

Since the integrand in (5-21) is holomorphic in \( s \) for every fixed \( z \in \mathcal{H} \), by [5, Lemma 6.1.5] it suffices to prove that

\[
(5-22) \quad \int_{\Gamma_{\infty} \setminus \mathcal{H}} \left| f(z) \mathcal{F}(z, m-\pi) \mathfrak{J}(z)^m \right| dv(z)
\]

is bounded, as a function of \( s \), on every compact subset of \( \mathbb{C}_{\Re(s)<m-1} \). This is true because (5-22) is dominated by

\[
(5-23) \quad \left( \sup_{z \in \mathcal{H}} \left| f(z) \mathfrak{J}(z)^{\frac{m}{2}} \right| \right) \cdot \int_{\Gamma_{\infty} \setminus \mathcal{H}} \left| \mathcal{F}(z, m-\pi) \mathfrak{J}(z)^{\frac{m}{2}} \right| dv(z),
\]

which is bounded, as a function of \( s \), on every compact subset of \( \mathbb{C}_{\Re(s)<m-1} \) since \( \sup_{z \in \mathcal{H}} \left| f(z) \mathfrak{J}(z)^{\frac{m}{2}} \right| \) is finite by a well-known argument (see [5, Theorem 2.1.5]), while the integral in (5-23) is dominated by the sum of (5-15) and (5-18) by the proof of Lemma 5-13.

Let us prove the last claim of the theorem. If \( m > 4 \), then \( m-1 > \frac{m}{2} + 1 \), hence the domain of \( \ell(\cdot, f) \) is the half-plane \( \mathbb{C}_{\Re(s)>\frac{m}{2}} \). Since by (5-12) \( \ell(\cdot, f) \) coincides with \( L(\cdot, f) \) on the half-plane \( \mathbb{C}_{\Re(s)>\frac{m}{2}+1} \), the claim follows.

\[ \square \]

6. Non-vanishing of \( L \)-functions

Let \( m, \Gamma, \chi, \) and \( h \) satisfy the assumptions of the first paragraph of Section 5. Let

\[ N := \inf \left\{ |c| : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P(\Gamma \setminus \Gamma_{\infty}) \right\}. \]
It follows from [5, Lemma 1.7.3] that
\begin{equation}
\tag{6-1}
hN \geq 1.
\end{equation}
In particular, \(N > 0\).

In the proof of the following theorem, we will use the notion of the median, \(M_{\Gamma(a,b)}\), of the gamma distribution with parameters \(a, b \in \mathbb{R}_{>0}\). It is defined as the unique \(M_{\Gamma(a,b)} \in \mathbb{R}_{>0}\) such that
\[
\int_{0}^{M_{\Gamma(a,b)}} x^{a-1} e^{-\frac{x}{b}} \, dx = \int_{M_{\Gamma(a,b)}}^{\infty} x^{a-1} e^{-\frac{x}{b}} \, dx.
\]

By [1, Theorem 1], we have
\begin{equation}
\tag{6-2}
a - \frac{1}{3} < M_{\Gamma(a,1)} < a, \quad a \in \mathbb{R}_{>0}.
\end{equation}

**Theorem 6-3.** Suppose that
\begin{equation}
\tag{6-4}
m \geq \frac{4\pi}{Nh} + \frac{8}{3}.
\end{equation}
Let \(1 < \Re(s) < \frac{m}{2}\). If
\begin{equation}
\tag{6-5}
e^{\frac{\pi}{2}|\Im(s)|} \Gamma \left( \frac{\Re(s) + 1}{2} \right) \Gamma \left( \frac{\Re(s) - 1}{2} \right) \frac{2^{\frac{m}{2} - 1}}{\Gamma \left( \frac{m}{2} - 1 \right)} \frac{\pi}{\frac{m}{2} - \Re(s)} \leq \pi,
\end{equation}
then \(\Psi_{\Gamma,m,\chi,s} \neq 0\).

**Proof.** We will prove the theorem by applying Theorem 4-5 to the Poincaré series \(\Psi_{\Gamma,m,\chi,s} = P_{\Gamma_{\infty} \setminus \Gamma, \chi} (F(\cdot, s))\) with \(|\cdot| = |\cdot|\). Let us prove that the assumptions of Theorem 4-5 are satisfied. The function \(F(\cdot, s)\) satisfies the condition (f1) for \(\Lambda = \Gamma_{\infty}\) since it is \(h\)-periodic and (5-1) holds, and by (5-14) it satisfies (f2). The condition (4-6) is satisfied by (5-1) and (2-6).

Next, we define
\begin{equation*}
S := [0, h[ \times \left[ \frac{1}{N}, \infty \right] .
\end{equation*}

Let us prove that \(S\) has the property (S1) of Theorem 4-5. Suppose that \(z \in S\) and \(\gamma = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \eta \right) \in \Gamma\) are such that \(\gamma.z \in S\). We need to prove that \(\gamma.z = z\). First we note that \(c = 0\), otherwise we would have
\[
\frac{1}{N} < \Im(\gamma.z) = \frac{2}{(c \Re(z) + d)^2 + (c \Im(z))^2} \leq \frac{\Im(z)}{c^2 \Im(z)} = \frac{1}{c^2 N^2} < \frac{1}{N^2} = \frac{1}{N}.
\]
Thus, \(\gamma \in \Gamma_{\infty}\), hence by (5-2) \(\gamma.z = z + kh\) for some \(k \in \mathbb{Z}\). Since \(\Re(z), \Re(\gamma.z) \in [0, h[\), it follows that \(k = 0\), hence \(\gamma.z = z\).

It remains to prove that \(S\) has the property (S2), i.e., writing \((\Gamma_{\infty} \setminus S)^c := \mathcal{H} \setminus \Gamma_{\infty} \setminus S\), to prove the inequality
\begin{equation}
\tag{6-6}
\int_{\Gamma_{\infty} \setminus \Gamma_{\infty} \setminus S} |F(z, s) \Im(z)^{\frac{\pi}{N}}| \, d\nu(z) > \int_{\Gamma_{\infty} \setminus (\Gamma_{\infty} \setminus S)^c} |F(z, s) \Im(z)^{\frac{\pi}{N}}| \, d\nu(z).
\end{equation}
We have

\[
\int_{\Gamma_{\infty} \setminus \Gamma_{\infty}.S} \left| F(z, s) \mathcal{Z}(z) \sqrt{2} \right| \, dv(z)
\]

\[
\overset{(6-1)}{=} \int_{S} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n \frac{z}{2}} \mathcal{Z}(z) \sqrt{2} \, dv(z)
\]

\[
= \int_{\frac{1}{2}}^{1} \int_{0}^{h} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i (n-1) \frac{z}{2} i} \, dx \, e^{-2\pi \frac{m}{2} \sqrt{y} y^2 - 2} \, dy
\]

\[
\geq \int_{\frac{1}{2}}^{1} \int_{0}^{h} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i (n-1) \frac{z}{2} i} \, dx \, e^{-2\pi \frac{m}{2} \sqrt{y} y^2 - 2} \, dy
\]

\[
= \int_{\frac{1}{2}}^{1} \sum_{n=1}^{\infty} n^{s-1} \int_{0}^{h} e^{2\pi i (n-1) \frac{z}{2} i} \, dx \, e^{-2\pi \frac{m}{2} \sqrt{y} y^2 - 2} \, dy
\]

\[
= h \left( \frac{h}{2\pi} \right) \int_{\frac{m}{2}}^{\infty} e^{-y \frac{m}{2} y^2 - 2} \, dy.
\]

Since

\[
\frac{2\pi}{Nh} \overset{(6-4)}{\leq} \frac{m}{2} - \frac{4}{3} < M_{\Gamma(\frac{m}{2} - 1, 1)},
\]

it follows that

\[
\int_{\Gamma_{\infty} \setminus \Gamma_{\infty}.S} \left| F(z, s) \mathcal{Z}(z) \sqrt{2} \right| \, dv(z) > h \left( \frac{h}{2\pi} \right)^{\frac{m}{2} - 1} \frac{1}{2} \int_{0}^{\infty} e^{-y \frac{m}{2} y^2 - 2} \, dy
\]

\[
= \left( \frac{h}{2\pi} \right)^{\frac{m}{2}} \pi \Gamma \left( \frac{m}{2} - 1 \right).
\]

On the other hand, since \([0, h[ \times [0, \infty[\) is a fundamental domain for the action of \(\Gamma_{\infty}\) on \(\mathcal{H}\), we have

\[
\int_{\Gamma_{\infty} \setminus \Gamma_{\infty}.S^c} \left| F(z, s) \mathcal{Z}(z) \sqrt{2} \right| \, dv(z) = \int_{[0, h[ \times [0, \infty[} \left| F(z, s) \mathcal{Z}(z) \sqrt{2} \right| \, dv(z)
\]

\[
\overset{(5-18)}{\leq} e^{\frac{\pi}{2} |\mathcal{Z}(s)|} \left( \frac{h}{\pi} \right)^{\Re(s)} \Gamma \left( \frac{|\mathcal{Z}(s)|}{2} \right) \Gamma \left( \frac{\Re(s) - 1}{2} \right) \frac{1}{\sqrt{\pi}} \sqrt{\Gamma(\Re(s))} \left( \frac{1}{\pi} \right)^{\frac{m}{2} - \Re(s)}
\]

\[
= \frac{e^{\frac{\pi}{2} |\mathcal{Z}(s)|}}{2} \left( \frac{h}{\pi} \right)^{\Re(s)} \Gamma \left( \frac{\Re(s) + 1}{2} \right) \Gamma \left( \frac{\Re(s) - 1}{2} \right) \left( \frac{1}{\pi} \right)^{\frac{m}{2} - \Re(s)}
\]

(6-8)

where the last equality is obtained by applying the Legendre duplication formula [12, 19.(2)]:

\[
\frac{\sqrt{\pi} \Gamma(2z)}{2^{2z} \Gamma(z)} = \frac{1}{2} \Gamma \left( z + \frac{1}{2} \right), \quad z \in \mathbb{C} \setminus \frac{1}{2} \mathbb{Z}_{\leq 0}.
\]
Inequalities (6-7) and (6-8) imply that (6-6) holds if we have
\[
\left(\frac{h}{2\pi}\right)^{\frac{m}{\pi}} \pi^\Gamma \left(\frac{m}{\pi} - 1\right) \geq \frac{e^{\frac{\pi}{2}}}{} \pi^\Gamma \left(\frac{m}{\pi} - 1\right) \pi^\Gamma \left(\frac{m}{\pi} - \frac{\Re(s)}{2}\right),
\]
and this inequality is obviously equivalent to (6-5).

\[\blacksquare\]

**Corollary 6-9.** Suppose that \( m \in \frac{9}{2} + \mathbb{Z}_{\geq 0} \). Let \( \frac{m}{2} < \Re(s) < m - 1 \). Suppose that
\[
\frac{N\pi}{h} \geq \max \left\{ \frac{4}{m - \frac{s}{3}}, \left(\frac{e^{\frac{\pi}{2}}}{} \pi^\Gamma \left(\frac{m-\Re(s)}{2}\right) \pi^\Gamma \left(\frac{m-\Re(s)-1}{2}\right) 2^{\frac{m-1}{2}} \right) \right\}.
\]
Then,
\[
L(s, \Psi_{\Gamma,m,\chi,m-\pi}) > 0.
\]

**Proof.** By Theorem 6-3 \( \Psi_{\Gamma,m,\chi,m-\pi} \neq 0 \), hence
\[
0 < \langle \Psi_{\Gamma,m,\chi,m-\pi}, \Psi_{\Gamma,m,\chi,m-\pi} \rangle_{S_m(\Gamma,\chi)} \leq \frac{h^m \Gamma(m-1)}{\varepsilon \Gamma(4\pi)^{m-1}} \ell(s, \Psi_{\Gamma,m,\chi,m-\pi}).
\]
Thus, \( \ell(s, \Psi_{\Gamma,m,\chi,m-\pi}) > 0 \), i.e., by Theorem 5-19, \( L(s, \Psi_{\Gamma,m,\chi,m-\pi}) > 0 \).

\[\blacksquare\]

**Corollary 6-10.** Let \( \varepsilon \in \mathbb{R}_{\geq 1}, \nu \in \mathbb{R}_\geq, \) and \( \eta \in \mathbb{R}_{>0} \). For \( m \in \frac{9}{2} + \mathbb{Z}_{\geq 0} \), we define
\[
C_m := \left[ \frac{m}{2} + \varepsilon, \frac{m}{2} + \nu \right] \times [-\eta, \eta] \subseteq \mathbb{C}.
\]
There exists \( m_0 \in \frac{9}{2} + \mathbb{Z}_{\geq 0} \) such that for all \( m \in m_0 + \mathbb{Z}_{\geq 0} \) and \( s \in C_m \), for every discrete subgroup \( \Gamma \) of finite covolume in \( \text{SL}_2(\mathbb{R})^\sim \) such that \( \infty \) is a cusp of \( P(\Gamma) \) and for every character \( \chi: \Gamma \to \mathbb{C}^\times \) of finite order that satisfies \( \eta^{-2m} = \chi(\gamma) \) for all \( \gamma \in \Gamma_\infty \), we have
\[
L(s, \Psi_{\Gamma,m,\chi,m-\pi}) > 0.
\]

**Proof.** If \( m > 2\nu + 2 \), then \( \frac{m}{2} + \nu < m - 1 \), hence \( C_m \subseteq \mathbb{C}_{\frac{m}{2} < \Re(s) < m-1} \). Thus, by Corollary 6-9 and by (6-1) it suffices to prove that there exists \( m_0 \in \frac{9}{2} + \mathbb{Z}_{\geq 0} \) such that for all \( m \in m_0 + \mathbb{Z}_{\geq 0} \) and \( s \in C_m \), we have
\[
\frac{1}{\pi} \geq \frac{4}{m - \frac{s}{3}}
\]
and
\[
\left(\frac{e^{\frac{\pi}{2}}}{} \pi^\Gamma \left(\frac{m-\Re(s)}{2}\right) \pi^\Gamma \left(\frac{m-\Re(s)-1}{2}\right) 2^{\frac{m-1}{2}} \right) \leq \frac{e^{\frac{\pi}{2}}}{} \pi^\Gamma \left(\frac{m-\Re(s)}{2}\right) \pi^\Gamma \left(\frac{m-\Re(s)-1}{2}\right) 2^{\frac{m-1}{2}}.
\]
The inequality (6-11) obviously holds if \( m \geq \frac{8}{3} + 4\pi \). Next, if \( m \geq 2\nu + 10 \), then for every \( s \in C_m \), \( \frac{m-\Re(s)+1}{2} \) and \( \frac{m-\Re(s)-1}{2} \) belong to the interval \( [2, \infty[ \), on which the gamma function is non-decreasing, hence the right-hand side of (6-12) is bounded from above for all \( s \in C_m \) by
\[
\frac{e^{\frac{\pi}{2}}}{} \pi^\Gamma \left(\frac{m}{4} + \frac{1-\varepsilon}{2}\right) \pi^\Gamma \left(\frac{m}{4} - \frac{1+\varepsilon}{2}\right) 2^{\frac{m-1}{2}}.
\]
On the other hand, the left-hand side of (6-12) is, for all \( s \in C_m \), greater than or equal to \( \left( \frac{1}{\pi} \right)^{\nu} \), hence (6-12) holds for all \( s \in C_m \) if

\[
\left( \frac{1}{\pi} \right)^{\nu} \geq \frac{\varepsilon e^{\pi} \Gamma \left( \frac{m}{4} + \frac{1-\varepsilon}{2} \right) \Gamma \left( \frac{m}{4} - \frac{1+\varepsilon}{2} \right) 2^{\frac{m}{2} - 1}}{\pi \Gamma \left( \frac{m}{2} - 1 \right) \varepsilon},
\]

i.e., if we have

\[
\frac{\varepsilon}{2} e^{\pi} \Gamma \left( \frac{1}{2} - \nu \right) \geq \Gamma \left( \frac{m}{4} + \frac{1-\varepsilon}{2} \right) \Gamma \left( \frac{m}{4} - \frac{1+\varepsilon}{2} \right) \frac{2^{\frac{m}{2} - 2}}{\sqrt{\pi} \Gamma \left( \frac{m}{2} - 1 \right)} =: R(m).
\]

Thus, to finish the proof of the corollary, it suffices to prove that \( \lim_{m \to \infty} R(m) = 0 \). By applying the Legendre duplication formula \([12, 19.(2)]\)

\[
\frac{2^{2z-1}}{\sqrt{\pi} \Gamma(2z)} = \frac{1}{\Gamma(z) \Gamma(z + \frac{1}{2})}, \quad z \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}_{\leq 0},
\]

we see that

\[
R(m) = \frac{\Gamma \left( \frac{m}{4} + \frac{1-\varepsilon}{2} \right) \Gamma \left( \frac{m}{4} - \frac{1+\varepsilon}{2} \right)}{\Gamma \left( \frac{m}{4} - \frac{1}{2} \right) \Gamma \left( \frac{m}{4} \right)}.
\]

Let us write \( m = 4n + l \) with \( n \in \mathbb{Z}_{>0} \) and \( l \in \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \} \). We note that \( R(4n + l) \) equals

\[
(6-13) \quad \frac{\Gamma \left( n + \frac{l}{4} + \frac{1-\varepsilon}{2} \right) \Gamma \left( n + \frac{l}{4} - \frac{1+\varepsilon}{2} \right)}{\Gamma(n) n^{\frac{l+1-\varepsilon}{2}} \Gamma \left( \frac{n+l}{4} - \frac{1}{4} \right) \Gamma \left( \frac{n+l}{4} \right)} \frac{1}{n^{\frac{1}{2} - \varepsilon}}.
\]

For every \( l \in \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \} \), the right-hand side of (6-13) converges to 0 as \( n \to \infty \) since by \([12, 18. \text{Lemma } 7}\) we have

\[
\lim_{n \to \infty} \frac{\Gamma(n+s)}{(n-1)! n^s} = 1, \quad s \in \mathbb{C}.
\]

Thus, \( \lim_{m \to \infty} R(m) = 0 \). This proves the corollary. \( \square \)

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