TOPOLOGICAL STRUCTURES OF LARGE SCALE INTERACTING SYSTEMS VIA UNIFORM FUNCTIONS AND FORMS

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ABSTRACT. In this article, we investigate the topological structure of large scale interacting systems on infinite graphs, by constructing a suitable cohomology which we call the uniform cohomology. This provides a new perspective for the identification of macroscopic observables from the microscopic system. As a straightforward application of our theory when the underlying graph has a free action of a group, we prove a certain decomposition theorem for shift-invariant closed uniform forms. This result is a uniform version in a very general setting of the decomposition result for shift-invariant closed $L^2$-forms originally proposed by Varadhan, which has repeatedly played a key role in the proof of the hydrodynamic limits of nongradient large scale interacting systems. In a subsequent article [3], we use this result as a key to prove Varadhan’s decomposition theorem for a general class of large scale interacting systems.

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1. INTRODUCTION

1.1. Introduction. One of the fundamental problems in the natural and social sciences is to explain macroscopic phenomena that we can observe from the rules governing the microscopic system giving rise to the phenomena. Hydrodynamic limit provides a rigorous mathematical method to derive the deterministic partial differential equations describing the time evolution of macroscopic parameters, from the stochastic dynamics of a microscopic large scale interacting system. The heart of this method is to take the limit with respect to proper space-time scaling, so that the law of large numbers absorbs the large degree of freedom of the microscopic system, allowing to extract the behavior of the macroscopic parameters which characterize
the equilibrium states of the microscopic system. Hence, techniques from probability theory including various estimates on Markov processes and their stationary distributions have played a central role. In this article, we introduce a novel, geometric perspective to the theory of hydrodynamic limits. Instead of using the law of large numbers, we construct a new cohomology theory for microscopic models to identify the macroscopic observables and give interpretations to the mechanism giving the macroscopic partial differential equations. Our main theorem gives an analogue of Varadhan’s decomposition of closed $L^2$-forms, which has played a key role in the proofs of the hydrodynamic limits of nongradient systems.

Initially, many of the techniques developed in the theory of hydrodynamic limit were specific to the interacting system under consideration. In the seminal article [7], Guo, Papanicolaou and Varadhan introduced a widely applicable strategy known as the entropy method for proving the hydrodynamic limit when the interacting system satisfies a certain condition known as the gradient condition. Furthermore, Varadhan in [18] introduced a novel, refined strategy for systems which do not necessarily satisfy the gradient condition, relying on proving the so-called decomposition of closed $L^2$-forms. Although this strategy has been successful in proving the hydrodynamic limit for a number of nongradient systems [6, 9, 12, 13, 15, 16, 19], the implementation in practice has proven notoriously difficult, requiring arguments with sharp spectral gap estimates specific to the system under consideration (see for example [8, Section 7]). Due to the restrictiveness of the gradient condition, many interesting microscopic systems are known or expected to be nongradient. Thus it is vital to understand the mechanism of Varadhan’s strategy and construct model independent criteria for implementation applicable to a wide variety of models.

The motivation of this article is to systematically investigate various large scale interacting systems in a unified fashion, especially to understand the mechanism in which similar decompositions seemingly independent of the stochastic data appear in the proofs of the hydrodynamic limits. For this goal, we introduce a general framework encompassing a wide variety of interacting systems, including the different variants of the exclusion processes and the lattice-gas with energy (see Examples 1.3 and 2.18). We let $\mathbb{B}$ be a certain infinite graph which we call a locale, generalizing the typical Euclidean lattice modeling the space where the microscopic dynamics takes place. We let $S$ be a set expressing the possible states at each vertex, such as the number of particles or amount of energy, and let $S^X := \prod_{x \in X} S$ be the configuration space expressing all of the possible configurations of states on $X$. The dynamics of a microscopic stochastic system is usually expressed by a generator. However, in our framework, we focus on the interaction $\phi$ – a certain map $\phi: S \times S \to S \times S$ encoding the permitted change in states on adjacent vertices (see Definition 2.4). The interaction gives $S^X$ a geometric structure, that of a graph whose edges correspond to the transitions, i.e., all possible change of the configuration at a single instant (see §1.2). This structure is independent of the transition rate – stochastic data which encodes the expected frequency of the transitions.

In this article, we construct the uniform cohomology reflecting the topological property of the geometric structure of $S^X$, by replacing the space of functions on $S^X$ with a new class of functions called the uniform functions, which considers the distances between the vertices of the locale. Our key result, Theorem 6 states that under general assumptions, the zeroth uniform cohomology is isomorphic to the space of conserved quantities – functions on $S$ whose sums
are conserved by $\phi$. This cohomology is finite dimensional even though $S^X$ in general has an infinite number of connected components. For the cases where the hydrodynamic limit is proven, conserved quantities are known to correspond to the macroscopic parameters which characterize the equilibrium (or stationary) measures of the microscopic system. Thus, we believe uniform cohomology gives an alternative justification for the origin of the macroscopic observables. In addition, Theorem 6 also states that the uniform cohomology of $S^X$ for degrees other than zero vanish. The essential case is for degree one, where we prove that any closed uniform form is the differential of a uniform function.

Our main theorem, Theorem 1, gives a certain structure theorem for closed uniform forms that are shift-invariant, i.e., invariant by the action of a group. Here, we assume the existence of a free action of a group on the locale, which ensures a certain homogeneity. The theorem is obtained as a straightforward application of group cohomology to Theorem 6. If we choose a fundamental domain of $X$ for the action of the group, then we obtain a decomposition theorem in the spirit of the decomposition of Varadhan (see Theorems 1, 3 and 5 of §1.3). The closed forms of Varadhan are $L^2$-forms for the equilibrium measure arising from the choice of the transition rate. Although uniform functions and forms are defined algebraically without the need for any stochastic data, our shift-invariant forms in fact form a common core of the various spaces of shift-invariant $L^2$-forms constructed for each choice of the transition rate, and will in subsequent research play a crucial role in proving Varadhan’s decomposition for $L^2$-forms (see for example [3]). Our main theorem indicates that the specification of Varadhan’s decomposition is determined by the underlying geometric structure of the model. Moreover, our theory gives a cohomological interpretation of the dimension of the space of shift-invariant closed forms modulo the exact forms – whose origin up until now had been a mystery. The proof of our main theorem does not require any spectral gap estimates and can be applied universally to a wide variety of systems.

Currently, all existing research concerning hydrodynamic limits for non-gradient systems deal exclusively with the case when the locale is the Euclidean lattice $\mathbb{Z}^d$, with an action of $G = \mathbb{Z}^d$ given by the translation. Our decomposition theorem is valid for far general infinite locales and groups, including various crystal lattices with their group of translations and Cayley graphs associated to finitely generated infinite groups with natural action of the group. The theorem is also true for systems with multiple linearly independent conserved quantities. Our result provides crucial insight into the formulation of Varadhan’s decomposition in these general settings. One of our goals is to find a more intuitive and universal proof of the hydrodynamic limits for non-gradient models. In a subsequent article [3], we prove Varadhan’s decomposition theorem for a general class of models, where the main result of the present paper is the key for the generalization. Furthermore, we will use this decomposition theorem to perform scaling limits for the general class of models.

Our theory is constructed from scratch, using only algebraic and combinatorial methods. In particular, no probability theory, measure theory, or analytic methods are used. Most importantly, we have taken care to make this article including the proof of our main result self-contained, except for the proof of the well established long exact sequence arising in group cohomology (see §5.2). Thus, we believe our article should be accessible to mathematicians in a wide range of disciplines. We hope this article would introduce to a broad audience interesting mathematical
concepts related to typical large scale interacting systems, and to researchers in probability theory potentially powerful cohomological techniques that may be relevant in indentifying important structures of stochastic models.

The remainder of this section is as follows. In §1.2, we describe our framework and present some examples. Then in §1.3 we state Theorem 1, the main theorem of our article, asserting the decomposition for shift-invariant uniform closed forms. We then explain its relation to the decomposition by Varadhan. Finally, in §1.4 we provide an overview of our article and the outline of the proof of our main theorem.

1.2. The Large Scale Interacting System. In this subsection, we introduce the various objects in our framework describing large scale interacting systems and give natural assumptions which ensure our main theorem. The precise mathematical definitions of the objects in the triple \((X, S, \phi)\) given in §1.1 are as follows. We define a locale \((X, E)\) to be any locally finite simple symmetric directed graph which is connected (see §2.1 for details). Here, \(X\) denotes the set of vertices and \(E \subset X \times X\) denotes the set of directed edges of the locale. By abuse of notation (see Remark 2.8), we will often denote the locale \((X, E)\) with the same symbol as its set of vertices \(X\). The condition that \(X\) is connected and locally finite implies that the set of vertices of \(X\) is countable. If the set of vertices is an infinite set, then we say that \(X\) is an infinite locale. We define the set of states \(S\) as a nonempty set with a designated element \(\star \in S\) which we call the base state, and we define the symmetric binary interaction, or simply an interaction \(\phi\) on \(S\) to be a map \(\phi: S \times S \to S \times S\) such that for any pair of states \((s_1, s_2) \in S \times S\) satisfying \(\phi(s_1, s_2) \neq (s_1, s_2)\), we have \(\hat{\iota} \circ \phi \circ \hat{\iota} \circ \phi(s_1, s_2) = (s_1, s_2)\), where \(\hat{\iota}: S \times S \to S \times S\) is the bijection obtained by exchanging the components of \(S \times S\). The ordering of \(S \times S\) determines the direction of the interaction, and the condition intuitively means that if we execute the interaction and if it is nontrivial, then further executing the interaction in the reverse direction takes us back to where we started. To realize the full large scale interacting system, we also need to choose a transition rate. However, this is outside the scope of the current article.

The most typical example of an infinite locale is given by the Euclidean lattice \(\mathbb{Z}^d = (\mathbb{Z}^d, \mathbb{B}^d)\) for integers \(d \geq 1\), where \(\mathbb{Z}^d\) is the \(d\)-fold product of the set of integers \(\mathbb{Z}\), and

\[\mathbb{B}^d := \{ (x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \mid |x - y| = 1 \} .\]

Here, we let \(|x - y| := \sum_{j=1}^{d} |x_j - y_j|\) for any \(x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d)\) in \(\mathbb{Z}^d\). Crystal lattices such as the triangular and hexagonal lattices as well as Cayley graphs associated to finitely generated infinite groups (see Figure 1) are other examples of infinite locales.

We say that a locale is weakly transferable, if for any ball \(B \subset X\), the complement \(X \setminus B\) is a nonempty finite disjoint union of connected infinite graphs. By definition, a weakly transferable locale is an infinite locale. We will also consider a stronger condition on the locale which we call transferable (see Definition 4.16 for the precise definition). Immediately from the definition, we see that if for any ball \(B \subset X\), the complement \(X \setminus B\) is a connected infinite graph, then \(X\) is transferable. The Euclidean lattice \(\mathbb{Z}^d = (\mathbb{Z}^d, \mathbb{B}^d)\) for \(d > 1\), crystal lattices such as the triangular and hexagonal lattices, as well as the Cayley graph for a finitely generated free group generated by \(d > 1\) elements give examples of transferable locales (see Remark 4.19). The Euclidean lattice \(\mathbb{Z} = (\mathbb{Z}, \mathbb{F})\) for \(d = 1\), which is also the Cayley graph for a free group generated...
Note that the sum is a finite sum since \( \sum_i \) the configuration \( \sigma \) function \( \Phi \) finite quantity \( (2) \) quantities for the interaction \( \phi \) are conserved by the interaction. We denote by \( \text{Consv} \) state, such as the number of particles or energy depending on the physical context, whose sums \( (2) \) for any instant, is defined as \( \Phi \) graph \( (\text{directed graph (see Lemma 2.5 for a proof)). Again by abuse of notation, we will often denote the \( 1 \) (1) is not connected, simple nor locally finite as a graph, and the set of vertices is not countable. We call any element \( \sigma \) to be the configuration whose components are all at base state \( \sigma \). Next, for any \( \sigma = (1,2,3) \in \mathbb{R}^3 \), we let \( (\eta_{x_1}', \eta_{x_2}') = \phi(\eta_{x_1}, \eta_{x_2}) \). We define \( \eta^e = (\eta^e) \in S^X \) by

\[
\eta^e_x = \begin{cases} 
\eta_x & x \neq x_1, x_2 \\
\eta'_x & x = x_1, x_2.
\end{cases}
\]

The transition structure on \( S^X \), expressing all possible change of the configuration at a single instant, is defined as \( \Phi := \{ (\eta, \eta^e) \mid \eta \in S^X, e \in E \} \subset S^X \times S^X \). Then \( (S^X, \Phi) \) is a symmetric directed graph (see Lemma 2.5 for a proof). Again by abuse of notation, we will often denote the graph \( (S^X, \Phi) \) by \( S^X \) (see also Remark 2.8). We remark that generally, \( S^X \) on an infinite locale is not connected, simple nor locally finite as a graph, and the set of vertices is not countable.

Next, we introduce the conserved quantity for the interaction \( \phi \), which we define to be any function \( \xi : S \to \mathbb{R} \) satisfying \( \xi(*) = 0 \) and

\[
\xi(s_1) + \xi(s_2) = \xi(s'_1) + \xi(s'_2)
\]

for any \( (s_1, s_2) \in S \times S \), where \( (s'_1, s'_2) := \phi(s_1, s_2) \). This function gives values reflecting the state, such as the number of particles or energy depending on the physical context, whose sums are conserved by the interaction. We denote by \( \text{Consv}^\phi(S) \) the \( \mathbb{R} \)-linear space of all conserved quantities for the interaction \( \phi \).

Denote by \( S^X_\ast \) the subset of \( S^X \) consisting of elements \( \eta = (\eta_x) \) such that \( \eta_x = * \) for all but finite \( x \in X \). Then \( S^X_\ast \) also has a structure of a graph induced from that of \( S^X \). Any conserved quantity \( \xi \in \text{Consv}^\phi(S) \) defines a map \( \xi_X : S^X_\ast \to \mathbb{R} \) by \( \xi_X(\eta) := \sum_{x \in X} \xi(\eta_x) \) for any \( \eta \in S^X_\ast \). Note that the sum is a finite sum since \( \eta \in S^X_\ast \). We call the value \( \xi_X(\eta) \) a conserved quantity of the configuration \( \eta \).
Throughout this article, we let $\mathbb{N} = \{0, 1, \ldots, \}$ denote the set of natural numbers. We consider the following properties of an interaction, which will play an important role in our main theorem.

**Definition 1.1.** For an interaction $\phi$ on $S$, let $c_\phi := \dim_\mathbb{R} \text{Consv}^\phi(S)$.

1. We say that the interaction $\phi$ is irreducibly quantified, if for any locale $X$, if the configurations $\eta, \eta' \in S^X_\phi$ have the same conserved quantities, i.e. if $\xi_X(\eta) = \xi_X(\eta')$ for any $\xi \in \text{Consv}^\phi(S)$, then there exists a finite path (see §2.1) from $\eta$ to $\eta'$ in $S^X_\phi$.

2. We say that the interaction $\phi$ is simple, if $c_\phi = 1$, and for any nonzero conserved quantity $\xi \in \text{Consv}^\phi(S)$, the monoid generated by $\xi(S)$ via addition in $\mathbb{R}$ is isomorphic to $\mathbb{N}$ or $\mathbb{Z}$.

A monoid is defined to be a set with a binary operation that is associative and has an identity element, the first examples being $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$ with the operation being the usual addition and with identity element 0. Provided $c_\phi = 1$, the second condition in Definition 1.1 (2) is satisfied for example if there exists $\xi \in \text{Consv}^\phi(S)$ such that $\xi(S) \subset \mathbb{N}$ and $1 \in \xi(S)$, or $\xi(S) \subset \mathbb{Z}$ and $\pm 1 \in \xi(S)$.

**Remark 1.2.** The second condition of Definition 1.1 (1) implies that any configurations with the same conserved quantities are in the same connected component of the configuration space $S^X_\phi$. The configuration space on an infinite locale $X$ usually has an infinite number of connected components. If the above condition is satisfied, then we may prove that the connected components of $S^X_\phi$ are characterized by its conserved quantities (see Remark 2.31). This condition is equivalent to the condition that the associated stochastic process on the configurations with fixed conserved quantities are irreducible.

The following are examples of interactions and corresponding conserved quantities.

**Example 1.3.** (1) The most basic situation is when $S = \{0, 1\}$ with base state $\ast = 0$. The map $\phi: S \times S \to S \times S$ defined by exchanging the components of $S \times S$ is an interaction (see Figure 2). The conserved quantity $\xi: S \to \mathbb{N}$ given by $\xi(s) = s$ gives a basis of the one-dimensional $\mathbb{R}$-linear space $\text{Consv}^\phi(S)$. This interaction is simple. The stochastic process induced from this interaction via a choice of a transition rate is called the exclusion process.

![Figure 2. Exclusion Process](image)

(2) Consider the case $S = \{0, 1, \ldots, \kappa\}$ with base state $\ast = 0$ for some integer $\kappa > 1$. The map $\phi: S \times S \to S \times S$ defined by exchanging the components of $S \times S$ is an interaction (see Figure 3). For $i = 1, \ldots, \kappa$, let $\xi^{(i)}$ be the conserved quantity given by $\xi^{(i)}(s) = 1$ if $s = i$ and $\xi^{(i)}(s) = 0$ otherwise. Then $\xi^{(1)}, \ldots, \xi^{(\kappa)}$ gives a basis of the $\mathbb{R}$-linear
space $\text{Cons}_\phi(S)$. The stochastic process induced from this interaction via a choice of a transition rate is called the 
\textit{multi-color exclusion process}, or more generally, the 
\textit{multi-species exclusion process}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{multi-species-exclusion-process.png}
\caption{Multi-species Exclusion Process}
\end{figure}

See Example [2.18] in §2.3 for other examples of interactions covered by our theory. We will prove in Proposition [2.27] that all of the interactions in Examples [1.3] and [2.18] are irreducibly quantified.

\textbf{Remark 1.4.} For $s \in S$ in the interactions in Example [1.3], $s = 0$ describes the state where there are no particles at the vertex, and $s = i$ for an integer $i > 0$ the state where there exists a particle of type labeled as $i$ (referred to as color $i$ or species $i$) at the vertex. The exclusion in the exclusion and the multi-species exclusion processes signify that at most one particle is allowed to occupy each vertex. The conserved quantity $\xi^{(i)}$ returns 1 if a particle of species $i$ occupies the vertex and 0 otherwise. Then $\xi_X^{(i)}(\eta) := \sum_{x \in X} \xi^{(i)}(\eta_x)$ for a configuration $\eta = (\eta_x) \in S_X^*$ on a locale $X$ expresses the total number of particles of species $k$ in the configuration.

The hydrodynamic limits of the exclusion process for certain choices of transition rates of nongradient type have been studied by Funaki, Uchiyama and Yau [6] and Varadhan and Yau [19] (see also Theorem [1.3] of §1.3). For the multi-species exclusion process, a variant has been studied by Quastel [13] and Erignoux [5]. See Remark [2.20] for other known cases corresponding to the interactions given in Example [2.18]. Up until now, all of the interacting systems of nongradient type whose hydrodynamic limits have been proved are models over the Euclidean lattice.

Typical research in hydrodynamic limit investigates the stochastic process of large scale interacting systems obtained from a specific interaction with a specific transition rate on a specific locale. One important purpose of this article is to construct a mathematical framework to study many types of models at once and to find specific conditions on the locale and interactions to allow for a suitable theory. The notion of transferable locales and irreducibly quantified interactions, which we believe are new and have not previously appeared in literature, are steps in this direction. The distinctive feature of our framework is the separation of the stochastic data from the geometric data, as well as the separation of the set of states and the interaction from the underlying locale. The theory works best when $S$ is discrete, a case which already covers a wide variety of models. In future research, we hope to generalize our framework to include known models with more general $S$, such as $S = \mathbb{R}$ and $S = \mathbb{R}_{\geq 0}$, where a more subtle notion of uniform functions incorporating smoothness should be necessary for compatibility with existing models.
1.3. Main Theorem and Relation to Varadhan’s Decomposition. The goal of our article is to study the topological properties of the configuration space $S^X$ with transition structure via a newly defined class of uniform functions and forms. In this subsection, we introduce Theorem [4] which is the main theorem of this article, giving a decomposition of shift-invariant closed uniform forms. We will then discuss its relation to Varadhan’s decomposition of shift-invariant closed $L^2$-forms.

We first introduce notations concerning functions and forms on the configuration space with transition structure. Consider the triple $(X, S, \phi)$, and let $S^X$ be the corresponding configuration space with transition structure. For any set $A$, we let $C(A) := \text{Map}(A, \mathbb{R})$ be the $\mathbb{R}$-linear space of maps from $A$ to $\mathbb{R}$. We say that a function $f \in C(S^X)$ is local, if there exists a finite $\Lambda \subset X$ such that $f$ is in the image of $C(S^\Lambda)$ with respect to the inclusion $C(S^\Lambda) \hookrightarrow C(S^X)$ induced from the projection $S^X \to S^\Lambda$. Any local function may be regarded as a function in $C(S^X)$ via the map induced from the projection $S^X \to S^\Lambda$. We denote by $C_{\text{loc}}(S^X)$ the space of local functions on $S^X$, which is a subspace of both $C(S^X)$ and $C(S^X)^\Lambda$. We define the space of uniform functions to be a certain $\mathbb{R}$-linear subspace $C_{\text{unif}}(S^X)$ of $C(S^X)$ containing the space of local functions $C_{\text{loc}}(S^X)$, and we let $C_0^{\text{unif}}(S^X)$ be the subspace of $C_{\text{unif}}(S^X)$ consisting of function $f$ satisfying $f(*) = 0$ (see Definition [8.5]). We define the space of closed uniform forms $Z_1^{\text{unif}}(S^X)$ to be a certain $\mathbb{R}$-linear subspace of $\prod_{e \in E} C_{\text{loc}}(S^X)$ (see Definition [3.10]), and we define the differential $\partial : C_0^{\text{unif}}(S^X) \to Z_1^{\text{unif}}(S^X)$ by $\partial f := (\nabla_e f)$, where $\nabla_e f$ for any $e \in E$ is the function defined by

$$\nabla_e f(\eta) := f(\eta^e) - f(\eta)$$

for any $f \in C_0^{\text{unif}}(S^X)$ and $\eta \in S^X_\Lambda$. The differential $\partial$ is induced from the differential of the standard cochain complex associated with the graph $(S^X, \Phi)$ (see §2.2).

For our main theorem, we consider a locale with a free action of a group. Let $G$ be a group, and we assume that the locale $X$ has a free action of a group $G$. This induces actions of $G$ on various functions and forms. Both the Euclidean lattice and crystal lattices such as the triangular and hexagonal lattices of dimension $d$ have natural free actions of $G = \mathbb{Z}^d$. For any $\mathbb{R}$-linear space with an action of $G$, we denote by $U^G$ the $G$-invariant subspace of $U$. We will often say shift-invariant to mean $G$-invariant if the group $G$ is understood. We denote by $C := (Z_1^{\text{unif}}(S^X))^G$ the space of shift-invariant closed uniform forms, and by $E := \partial(C_0^{\text{unif}}(S^X))^G$ the image by $\partial$ of the space of shift-invariant uniform functions.

**Remark 1.5.** The existence of a free action of $G$ ensures that the locale $X$ is homogenous to a certain degree. We understand the quotient $C/E$ to philosophically represent the first uniform cohomology of the quotient space $S^X/G$, a topological space which can be interpreted as a model of an infinitesimal neighborhood of a macroscopic point.

We denote by $H^1(G, \text{Consv}^\phi(S))$ the first group cohomology of $G$ with coefficients in $\text{Consv}^\phi(S)$ (see Definition [5.11]). Our decomposition theorem giving the uniform analogue of Varadhan’s decomposition is as follows.

**Theorem 1** (=Theorem [5.17]). For the triple $(X, S, \phi)$, assume that the interaction $\phi$ is irreducibly quantified, and that $X$ has a free action of a group $G$. If $X$ is transferable, or if the interaction
\(\phi\) is simple and \(X\) is weakly transferable, then we have a canonical isomorphism

\[(4)\quad C/\mathcal{E} \cong H^1(G, \text{Consv}^\phi(S)).\]

Moreover, a choice of a fundamental domain for the action of \(G\) on \(X\) gives a natural decomposition

\[C \cong \mathcal{E} \oplus H^1(G, \text{Consv}^\phi(S))\]

of \(\mathbb{R}\)-linear spaces.

If the rank of the maximal abelian quotient \(G^{ab}\) of \(G\) is finite, then we have the following.

**Corollary 2** (=Corollary 5.19). *Let the assumptions be as in Theorem 7. Moreover, suppose that \(G^{ab}\) is of finite rank \(d\). If we fix a generator of the free part of \(G^{ab}\), then we have an isomorphism \(H^1(G, \text{Consv}^\phi(S)) \cong \bigoplus_{j=1}^{d} \text{Consv}^\phi(S)\). A choice of a fundamental domain of \(X\) for the action of \(G\) gives a decomposition*

\[(5)\quad C \cong \mathcal{E} \oplus \bigoplus_{j=1}^{d} \text{Consv}^\phi(S).\]

The decomposition (5) decomposes any shift-invariant closed uniform form in \(C\) as a unique sum of a form in \(\mathcal{E}\), closed forms whose potential are shift-invariant uniform functions, and a form obtained as the image with respect to the isomorphism (5) of elements in \(\bigoplus_{j=1}^{d} \text{Consv}^\phi(S)\). The space \(\mathcal{E}\) corresponds to the part which averages out to zero when taking a proper space-time scaling limit. Hence the decomposition theorem implies that the macroscopic property of our model may be completely expressed in terms of forms arising from the space \(\bigoplus_{j=1}^{d} \text{Consv}^\phi(S)\), which are related to the flow of conserved quantities in each of the directions induced by the action of the group \(G\).

In addition to the geometric data \((X, S, \phi)\), if we fix a transition rate, then this gives a shift-invariant equilibrium measure on the configuration space and a compatible inner product on the space of forms. If we consider the case when the locale is the Euclidean lattice \(X = (\mathbb{Z}^d, \mathbb{E}^d)\) with standard translation by the group \(G = \mathbb{Z}^d\), and if \((S, \phi)\) is the exclusion process of Example 1.3 (1), a typical choice of a transition rate gives rise to the product measure \(\nu = \mu_p \otimes \mathbb{Z}^d\) on \(S^X = \{0, 1\}^{\mathbb{Z}^d}\), where \(\mu_p\) is the probability measure on \(S = \{0, 1\}\) given by

\[\mu_p(s = 1) = p, \quad \mu_p(s = 0) = 1 - p\]

for some real number \(0 < p < 1\). We denote by \(L^2(\nu)\) the usual \(L^2\)-space of square integrable functions on \(\{0, 1\}^{\mathbb{Z}^d}\) with respect to the measure \(\nu\). The space of local functions \(C_{\text{loc}}(\{0, 1\}^{\mathbb{Z}^d})\) is known to be a dense subspace of \(L^2(\nu)\). We let \(\xi: \{0, 1\} \to \mathbb{N}\) be the conserved quantity given by \(\xi(s) = s\), which gives a basis of \(\text{Consv}^\phi(\{0, 1\})\). For any \(x \in X\), we let \(\xi_x: \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}\) be the function defined by \(\xi_x(\eta) := \eta_s\) for any \(\eta = (\eta_s) \in \{0, 1\}^{\mathbb{Z}^d}\). For any \(x = (x_j) \in \mathbb{Z}^d\), denote by \(\tau_x\) the translation of \((\mathbb{Z}^d, \mathbb{E}^d)\) by \(x\). In this case, Varadhan’s decomposition of shift-invariant closed \(L^2\)-forms proved by Funaki, Uchiyama and Yau is the following.
Theorem 3 ([6, Theorem 4.1]). Let $\omega = (\omega_e) \in \prod_{e \in E} L^2(\nu)$ be a shift-invariant closed $L^2$-form. Then there exists a set of constants $a_1, \ldots, a_d \in \mathbb{R}$ and a series of local functions $(f_n)_{n \in \mathbb{N}}$ in $C_{\text{loc}}(\{0, 1\}^{\mathbb{Z}^d})$ such that

$$
\omega_e = \lim_{n \to \infty} \nabla_e \left( \sum_{x \in \mathbb{Z}^d} \tau_x(f_n) + \sum_{j=1}^d a_j \sum_{x \in \mathbb{Z}^d} x_j \xi_x \right)
$$

in $L^2(\nu)$ for any $e \in \mathbb{E}$.

The same statement for certain transition rates giving non-product measures on $\{0, 1\}^{\mathbb{Z}^d}$ was proved by Varadhan and Yau [19], requiring different spectral gap estimates. The uniform version of Theorem 3 obtained by applying Corollary 2 to the above model for the fundamental domain $\Lambda_0 = \{(0, \ldots, 0)\}$ of $X = \mathbb{Z}^d$ for the action of $G = \mathbb{Z}^d$, is given as follows.

Theorem 4. Let $\omega = (\omega_e) \in \prod_{e \in E} C_{\text{loc}}(\{0, 1\}^{\mathbb{Z}^d})$ be a shift-invariant closed form. Then there exists a set of constants $a_1, \ldots, a_d \in \mathbb{R}$ and a local function $f$ in $C_{\text{loc}}(\{0, 1\}^{\mathbb{Z}^d})$ satisfying $f(\star) = 0$ such that

$$
\omega_e = \nabla_e \left( \sum_{x \in \mathbb{Z}^d} \tau_x(f) + \sum_{j=1}^d a_j \sum_{x \in \mathbb{Z}^d} x_j \xi_x \right)
$$

in $C_{\text{loc}}(\{0, 1\}^{\mathbb{Z}^d})$ for any $e \in \mathbb{E}$.

Our proof does not require a choice of the transition rate, thus completely independent of the measure. We remark that the sum in the brackets on the right hand side of Theorem 4 is a formal sum which has meaning only after taking $\nabla_e$, whereas the sum in the brackets on the right hand side of Theorem 4 has meaning as a uniform function in $C^{0}_{\text{unif}}(\{0, 1\}^{\mathbb{Z}^d})$.

Next, consider a general $(X, S, \phi)$ satisfying the assumptions of Theorem 4 and suppose that $X$ has a free action of $G = \mathbb{Z}^d$. We fix the generator of $G$ to be the standard basis of $\mathbb{Z}^d$, and we denote an element of $G = \mathbb{Z}^d$ by $\tau = (\tau_j) \in \mathbb{Z}^d$ instead of $x = (x_j)$ since the locale $X$ in general does not coincide with $G$. The disassociation of the group $G$ from the locale $X$ is another distinctive feature of our framework. If we fix a fundamental domain $\Lambda_0$ of $X$ for the action of $G$, then Corollary 2 in this case gives the following.

Theorem 5. Assume for simplicity that $c_\phi$ is finite, and fix a basis $\xi^{(1)}, \ldots, \xi^{(c_\phi)}$ of $\text{Consv}^\phi(S)$. Let $\omega = (\omega_e) \in \prod_{e \in E} C_{\text{loc}}(S^X)$ be a shift-invariant closed uniform form. In other words, let $\omega \in C$. Then there exists $a_{ij} \in \mathbb{R}$ for $i = 1, \ldots, c_\phi$ and $j = 1, \ldots, d$ and a shift-invariant uniform function $F$ in $C^{0}_{\text{unif}}(S^X)$ such that

$$
\omega = \partial \left( F + \sum_{i=1}^{c_\phi} \sum_{j=1}^d a_{ij} \sum_{\tau \in G} \tau_j \xi^{(i)} (\tau_{\Lambda_0}) \right),
$$

(6)

where we let $\xi_W$ be the function in $C^{0}_{\text{unif}}(S^X)$ defined as $\xi_W := \sum_{x \in W} \xi_x$ for any conserved quantity $\xi \in \text{Consv}^\phi(S)$ and $W \subset X$. 
In Theorem 5 we remark that $\partial F \in \mathcal{E}$, and by Remark 5.20

\begin{equation}
\omega_{\rho} := \partial \left( \sum_{i=1}^{c_{\Phi}} \sum_{j=1}^{d_{\Phi}} a_{ij} \left( \sum_{\tau \in G} \tau \xi_{\tau(\Lambda_{0})}^{(j)} \right) \right) \in C
\end{equation}

is the image of $\rho = (\sum_{i=1}^{c_{\Phi}} a_{i} \xi^{(i)}, \ldots, \sum_{i=1}^{c_{\Phi}} a_{id} \xi^{(i)}) \in \bigoplus_{i=1}^{d_{\Phi}} \text{Consv}^{\Phi}(S)$ through the isomorphism 5 for our choice of $\Lambda_{0}$. The equality $\omega = \partial F + \omega_{\rho}$ of (6) is precisely the decomposition given by 5.

In fact, Theorem 4 is a special case of Theorem 5 as follows. If $\Lambda_{0}$ is finite, then we may see from the definition that any $\omega \in \prod_{e \in E} C_{\text{loc}}(S^{X})$ which is closed and shift-invariant is uniform. In addition, again if $\Lambda_{0}$ is finite, any shift-invariant uniform function $F \in C_{\text{unif}}^{0}(S^{X})$ is of the form $F = \sum_{e \in E} \tau(f)$ for some local function $f \in C_{\text{loc}}(S^{X})$ satisfying $f(*) = 0$ (see Lemma 5.15). Here, $\tau(f)$ denotes the image of $f$ with respect to the action of $\tau \in \mathbb{Z}^{d}$. For the case $X = \mathbb{Z}^{d}$ with the action of $G = \mathbb{Z}^{d}$ given by the standard translation, if we let $\Lambda_{0} = \{(0, \ldots, 0)\}$, then we have $\tau_{x}(\Lambda_{0}) = \{x\}$ for any $x \in \mathbb{Z}^{d}$. From these observations and the definition of the differential $\partial$, we see that Theorem 4 follows from Theorem 5.

In the general setting of Theorem 1, the choice of a transition rate satisfying certain conditions gives an inner product compatible with the norm on the space of $L^{2}$-forms. The existence of Varadhan’s decomposition amounts to the following question. See [2, Conjecture 5.5] for a precise formulation of this question as a conjecture.

**Question.** Assume that the fundamental domain of the action of $G$ on the vertices of $X$ is finite. For a suitable definition of closed $L^{2}$-forms, if $\omega$ is a shift-invariant closed $L^{2}$-form, then does there exist $\omega_{n} \in \mathcal{E}$ for $n \in \mathbb{N}$ and $\rho \in H^{1}(G, \text{Consv}^{\Phi}(S))$ such that

$$
\lim_{n \to \infty} (\omega_{n} + \omega_{\rho}) = \omega
$$

Here, we let $\omega_{\rho}$ be the element in $C$ corresponding to $\rho$ in the decomposition 5 of Corollary 2 given for a choice of a fundamental domain of $X$ for the action of $G$.

The question is answered affirmatively for the cases that Varadhan’s decomposition are shown. Although our local forms construct the core of the $L^{2}$-space, and local closed forms in our sense are closed forms in the sense of the $L^{2}$-space, it is currently not generally known whether our local closed forms form a core of closed forms in the sense of $L^{2}$-spaces. Nevertheless, in subsequent research, we prove Varadhan’s decomposition for certain locales using Theorem 1 of this article, when $S$ is finite and $\mu$ is a product measure, assuming a certain spectral gap estimate pertaining to the interaction (cf. (3)). Through this process, we hope to understand the role played by the sharp spectral gap estimates in the proof of hydrodynamic limits for nongradient systems, a question which has been an important open question for the past thirty years (see for example [8, Preface]).

Let $C_{L^{2}}$ and $\mathcal{E}_{L^{2}}$ be the shift-invariant closed and exact forms for the $L^{2}$-space. The inner product on the $L^{2}$-space defines an orthogonal decomposition

\begin{equation}
C_{L^{2}} \cong \mathcal{E}_{L^{2}} \oplus H^{1}(G, \text{Consv}^{\Phi}(S))
\end{equation}
which is different from \( \mathcal{S} \). By reinterpreting the method in hydrodynamic limits for obtaining the macroscopic deterministic partial differential equation from the microscopic system, we have come to understand that the diffusion matrix associated with the macroscopic partial differential equation is given precisely by the matrix relating the two decompositions \( \mathcal{S} \) and \( \mathcal{S} \). One critical observation from this fact is that the size of the diffusion matrix of our system should be \( c_\phi d \), the dimension of \( \bigoplus_{j=1}^d \text{Consv}^\phi(S) \).

Through our investigation, we have come to see the stochastic data consisting of the measure and compatible inner product as a certain analogy of differential geometric data on Riemannian manifolds – the volume form and the metric. Through this analogy, the orthogonal decomposition \( \mathcal{S} \) may be regarded as a differential geometric decomposition given as a certain analogue of the Hodge-Kodaira decomposition in Riemannian geometry, whereas the decomposition \( \mathcal{S} \) is viewed as a more topological decomposition. In light of this analogy, it would be interesting to interpret the diffusion matrix relating the topological and measure theoretic structures of \( C_{L^2} \) as an analogy of the period matrix in Hodge theory comparing the topological and differential geometric structures of the manifold. Such ideas will be explored in future research.

1.4. Overview. In this subsection, we give an overview of the proof of Theorem 1. The key result for the proof is Theorem 6 below concerning the property of uniform cohomology. Consider the triple \( (X, S, \phi) \). The uniform cohomology is defined for a configuration space with transition structure as follows.

**Definition 1.6.** We define the *uniform cohomology* \( H^m_{\text{unif}}(S^X) \) for \( m \in \mathbb{Z} \) of the configuration space \( S^X \) with transition structure to be the cohomology of the cochain complex

\[
C^0_{\text{unif}}(S^X) \xrightarrow{\partial} Z^1_{\text{unif}}(S^X)
\]

which is zero in degrees \( m \neq 0, 1 \). Concretely, we have \( H^0_{\text{unif}}(S^X) := \text{Ker} \partial, H^1_{\text{unif}}(S^X) := Z^1_{\text{unif}}(S^X)/\text{Im} \partial, \) and \( H^m_{\text{unif}}(S^X) = \{0\} \) in degrees \( m \neq 0, 1 \). The uniform cohomology is philosophically the *reduced cohomology* in the sense of topology of the pointed space consisting of the configuration space \( S^X \) and base configuration \( \star \in S^X \).

**Theorem 6** (=Theorem 5.8). *For the triple \( (X, S, \phi) \), assume that the interaction \( \phi \) is irreducibly quantified. If \( X \) is transferable, or if the interaction \( \phi \) is simple and \( X \) is weakly transferable, then we have

\[
H^m_{\text{unif}}(S^X) \cong \begin{cases} 
\text{Consv}^\phi(S) & m = 0 \\
\{0\} & m \neq 0.
\end{cases}
\]

The configuration space \( S^X \) with transition structure viewed geometrically as a graph generally has an infinite number of connected components. Hence it may be surprising that \( H^0_{\text{unif}}(S^X) \) is finite dimensional. This calculation very beautifully reflects the fact that assuming the conditions of the theorem, the connected components of the graph \( S^X \) can be determined from the values of its conserved quantities (see Remark 2.33). By the definition of uniform cohomology \( H^m_{\text{unif}}(S^X) \), Theorem 6 is equivalent to the existence of an exact sequence

\[
0 \longrightarrow \text{Consv}^\phi(S) \longrightarrow C^0_{\text{unif}}(S^X) \overset{\partial}{\longrightarrow} Z^1_{\text{unif}}(S^X) \longrightarrow 0.
\]
A large portion of our article is dedicated to the construction of \( (9) \).

First, in §2.1 we introduce our model and the associated configuration space. Then in §2.2 we define naïve cohomology, which is the usual simplicial cohomology for the configuration space with transition structure, and the notion of closed forms. In §2.3 and §2.4 we introduce the notion of a conserved quantity and give its relation to the naïve \( H^0 \) of the configuration space.

In §3.1 we introduce the notion of local functions with exact support and prove that any function \( f \in C(S^X) \) may be expanded uniquely as a possibly infinite sum of local functions with exact support. We say that \( f \in C(S^X) \) is uniform if the diameter of support in the expansion of \( f \) is uniformly bounded. In §3.2 we first note that the function \( \xi_X = \sum_{x \in X} \xi_x \) for any conserved quantity \( \xi \in \text{Consv}^\phi(S) \) is uniform. Since \( \xi_X(\star) = 0 \), the correspondence \( \xi \mapsto \xi_X \) gives a natural inclusion \( \text{Consv}^\phi(S) \hookrightarrow C^0_{\text{unif}}(S^X) \). Assume now that the interaction is irreducibly quantified. We prove in Theorem 3.7 that this inclusion gives an isomorphism

\[
\text{Consv}^\phi(S) \cong \text{Ker} \partial.
\]

It remains to prove that \( \partial \) is surjective. In Definition 3.10 we define the notion of uniform forms. For the remainder of §3.4 we assume in addition that \( X \) is strongly transferable, that is \( X \setminus B \) is an infinite connected graph for any ball \( B \) in \( X \). We consider a function \( f \in C(S^X) \) such that \( \partial f \) is uniform, and we construct in Proposition 3.18 of §3.3 a symmetric pairing \( h_f : \mathcal{M} \times \mathcal{M} \to \mathbb{R} \) on a certain additive submonoid \( \mathcal{M} \subset \mathbb{R}^+ \) satisfying the cocycle condition

\[
h_f(\alpha, \beta) + h_f(\alpha + \beta, \gamma) = h_f(\beta, \gamma) + h_f(\alpha, \beta + \gamma)
\]

for any \( \alpha, \beta, \gamma \in \mathcal{M} \). We then prove in Proposition 3.19 of §3.4 that if \( h_f \equiv 0 \), then \( f \in C^0_{\text{unif}}(S^X) \).

In §4 we consider the case when \( X \) is weakly transferable. This section is technical and can be skipped if the reader is only interested in the strongly transferable case. In §4.2 we again construct a pairing \( h_f \) for any \( f \in C(S^X) \) such that \( \partial f \) is uniform, and prove in Proposition 4.13 the cocycle condition for \( h_f \). We note that in general, the pairing \( h_f \) may not be symmetric. We prove in Proposition 4.14 of §4.3 a weakly transferable version of Proposition 3.19. Finally, we prove in §4.4 that the pairing \( h_f \) is symmetric if the locale \( X \) is transferable.

In §5 we complete the proof that \( \partial \) is surjective (see Theorem 5.2 for details). The method of proof is as follows. For any closed uniform form \( \omega \in Z^1_{\text{unif}}(S^X) \), since \( \omega \) is closed, by Lemma 2.16 there exists \( f \in C(S^X) \) such that \( \partial f = \omega \). Note that \( f \) may not necessarily be a uniform function. By the previous argument, there exists a pairing \( h_f : \mathcal{M} \times \mathcal{M} \to \mathbb{R} \) satisfying the cocycle condition. We prove in Lemma 5.3 that if the pairing \( h_f \) is symmetric, or in Lemma 5.5 if the interaction is simple, there exists a map \( h : \mathcal{M} \to \mathbb{R} \) such that

\[
h_f(\alpha, \beta) = h(\alpha) + h(\beta) - h(\alpha + \beta)
\]

for any \( \alpha, \beta \in \mathcal{M} \). We modify the function \( f \) using the map \( h \) to obtain a function with the same \( \partial f \) but satisfies \( h_f \equiv 0 \). By Proposition 3.19 or Proposition 4.14 we see that \( f \in C^0_{\text{unif}}(S^X) \). This proves that the differential \( \partial \) in (9) is surjective, completing the proof of Theorem 6.

Now for the proof of Theorem 6, suppose that the locale \( X \) has a free action of a group \( G \). This gives a natural action of \( G \) on various spaces of functions and forms, and the boundary homomorphism of the long exact sequence for group cohomology (4.17) associated with the short
exact sequence (9) immediately gives an inclusion
\[ C/E\overset{\partial}{\longrightarrow} H^1(G, \text{Consv}^\phi(S)) \]
Note that we have \( H^1(G, \text{Consv}^\phi(S)) = \text{Hom}(G, \text{Consv}^\phi(S)) \) (see (45)). For a fixed fundamental domain \( \Lambda_0 \) of \( X \) for the action of \( G \), we let
\[ \omega_\rho := \partial \left( \sum_{\tau \in G} \rho(\tau)_{\tau(\Lambda_0)} \right) \in C \]
for any \( \rho \in \text{Hom}(G, \text{Consv}^\phi(S)) \), noting that \( \rho(\tau) \in \text{Consv}^\phi(S) \) for any \( \tau \in G \). The relation of \( \omega_\rho \) to the form in (7) is explained in Remark 2.8. By explicit calculation, we see in Proposition 5.18 that \( \delta(\omega_\rho) = \rho \), hence \( \delta \) is surjective. The \( \mathbb{R} \)-linear map \( \omega \mapsto (\omega - \omega_\rho, \rho) \) for \( \rho := \delta(\omega) \) gives a decomposition of \( \mathbb{R} \)-linear spaces
\[ C \cong E \oplus H^1(G, \text{Consv}^\phi(S)) \]
completing the proof of Theorem 1. Finally, in §5.4 we let \( X = \mathbb{Z} \) and consider the multi-color exclusion process for \( S = \{0, 1, 2\} \). Then \( X \) is not transferable and \( c_\phi = 2 \). We prove that \( \delta \) is not surjective in this case.

2. Configuration Space and Conserved Quantities

In this section, we will introduce the configuration space of a large scale interacting system and investigate its naive cohomology. We will then introduce the notion of a conserved quantity.

2.1. Configuration Space and Transition Structure. In this subsection, we will give a graph structure which we call the transition structure on the configuration space of states on a locale. We first review some terminology related to graphs.

A directed graph \((X, E)\), or simply a graph, is a pair consisting of a set \( X \), which we call the set of vertices, and a subset \( E \subset X \times X \), which we call the set of directed edges, or simply edges. For any \( e \in E = X \times X \), we denote by \( o(e) \) and \( t(e) \) the first and second components of \( e \), which we call the origin and target of \( e \), so that \( e = (o(e), t(e)) \in X \times X \). For any \( e \in E \), we let \( \bar{e} := (t(e), o(e)) \), which we call the opposite of \( e \). We say that a directed graph \((X, E)\) is symmetric if \( \bar{e} \in E \) for any \( e \in E \), and simple if \( (x, x) \notin E \) for any \( x \in X \). For any \( e = (o(e), t(e)) \in E \), we will often use \( e \) to denote the set \( e = \{o(e), t(e)\} \). We say that \((X, E)\) is locally finite, if for any \( x \in X \), the set \( \{e \in E \mid x \in e\} \) is finite. In this article, by abuse of notation (see Remark 2.8), we will often simply denote the graph \((X, E)\) by its set of vertices \( X \).

We define a finite path on the graph \( X \) to be a finite sequence \( \tilde{p} := (e_1, e_2, \ldots, e_N) \) of edges in \( E \) such that \( t(e_i) = o(e_{i+1}) \) for any integer \( 0 < i < N \). We denote by \( \text{len}(\tilde{p}) \) the number of elements \( N \) in \( \tilde{p} \), which we call the length of \( \tilde{p} \). We let \( o(\tilde{p}) := o(e_1) \) and \( t(\tilde{p}) := t(e_N) \), and we say that \( \tilde{p} \) is a path from \( o(\tilde{p}) \) to \( t(\tilde{p}) \). For paths \( \tilde{p}_1, \tilde{p}_2 \) such that \( t(\tilde{p}_1) = o(\tilde{p}_2) \), we denote by \( \tilde{p}_1 \tilde{p}_2 \) the path from \( o(\tilde{p}_1) \) to \( t(\tilde{p}_2) \) obtained as the composition of the two paths. If a path \( \tilde{p} \) satisfies \( o(\tilde{p}) = t(\tilde{p}) \), then we say that \( \tilde{p} \) is a closed path. For any \( x, x' \in X \), we denote by \( P(x, x') \) the set of paths from \( x \) to \( x' \). We define the graph distance \( d_X(x, x') \) between \( x \) and \( x' \) by
\[ d_X(x, x') := \inf_{\tilde{p} \in P(x, x')} \text{len}(\tilde{p}) \]
if \( P(x, x') \neq \emptyset \), and \( d_X(x, x') := \infty \) otherwise. We say that any subset \( Y \subseteq X \) is connected, if \( d_Y(x, x') < \infty \) for any \( x, x' \in Y \), where \( d_Y \) is the graph distance on the graph \((Y, E_Y)\) given by \( E_Y := E \cap (Y \times Y) \).

**Definition 2.1.** We define the locale to be a locally finite simple symmetric directed graph \( X = (X, E) \) which is connected. If the set of vertices of \( X \) is an infinite set, then we say that \( X \) is an infinite locale.

We will use the terminology locale to express the discrete object that models the space where the dynamics under question takes place. We understand the connectedness to be an important feature of the locale.

**Example 2.2.** (1) The most typical example of an infinite locale is the Euclidean lattice \( \mathbb{Z}^d = (\mathbb{Z}^d, \mathbb{B}^d) \) for integers \( d \geq 1 \), where \( \mathbb{Z}^d \) is the \( d \)-fold product of \( \mathbb{Z} \), and

\[
\mathbb{B}^d := \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \mid |x - y| = 1\}.
\]

Here, we let \( |x - y| := \sum_{j=1}^{d} |x_j - y_j| \) for any \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{Z}^d \).

(2) For integers \( d \geq 1 \) and \( n > 0 \), a variant of the Euclidean lattice is given by the Euclidean lattice with nearest \( n \)-neighbor \( \mathbb{Z}_n^d = (\mathbb{Z}^d, \mathbb{B}_n^d) \), where

\[
\mathbb{B}_n^d := \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \mid 0 < |x - y| \leq n\}.
\]

Figure 4. The Euclidean Lattices \( \mathbb{Z}^2 \) and with nearest 2-neighbor \( \mathbb{Z}_2^2 \)

(3) Many types of crystal lattices such as the triangular, hexagonal, and diamond lattices are infinite locales (see for example [17, Example 3.4, Example 8.3] and [10, §5]).

(4) Let \( G \) be a finitely generated group and let \( S \subseteq G \) be a minimal set of generators. Then the associated Cayley graph \((G, E_S)\) given by \( E_S := \{ (\tau, \tau \sigma), (\tau, \tau \sigma^{-1}) \mid \tau \in G, \sigma \in S \} \) is a locale. If \( G \) is infinite, then the associated Cayley graph is an infinite locale.

(5) Let \( X_1 = (X_1, E_1) \) and \( X_2 = (X_2, E_2) \) be locales. If \( X := X_1 \times X_2 \) and

\[
E := \{(o(e_1), o(e_2)), (t(e_1), o(e_2)) \mid e_1 \in E_1, e_2 \in E_2 \}
\]

\[
\cup \{(o(e_1), o(e_2)), (o(e_1), t(e_2)) \mid e_1 \in E_1, e_2 \in E_2 \} \subset (X_1 \times X_2) \times (X_1 \times X_2),
\]

then \((X, E)\) is a locale, which we denote \( X_1 \times X_2 \). We say that \( X \) is a product of \( X_1 \) and \( X_2 \). Note that \((\mathbb{Z}^d, \mathbb{B}^d)\) coincides with the \( d \)-fold product \((\mathbb{Z}, \mathbb{B}) \times \cdots \times (\mathbb{Z}, \mathbb{B})\).

(6) Suppose \( X = (X, E) \) is a locale, and let \( Y \subseteq X \) be a connected subset. If we let \( E_Y := E \cap (Y \times Y) \subseteq X \times X \), then \( Y = (Y, E_Y) \) gives a graph which is a locale. We call \( Y \) a sublocale of \( X \).
Next, we introduce the set of states, which is a nonempty set \( S \) expressing the possible states the model may take at each vertex, and the configuration space \( S^X \) for \( S \) on \( X \).

**Definition 2.3.** We define the set of *states* to be a nonempty set \( S \). We call any element of \( S \) a *state*. We will designate an element \(* \in S\), which we call the *base state*. If \( S \subset \mathbb{R} \) and \( 0 \in S \), then we will often take the base state \(* \) to be 0. For any locale \( X = (X, E) \), we define the configuration space for \( S \) on \( X \) by

\[
S^X := \prod_{x \in X} S.
\]

We call an element \( \eta = (\eta_x) \) of \( S^X \) a configuration. We denote by \( * \) the base configuration, which is the configuration in \( S^X \) whose components are all at base state.

Now, we introduce the symmetric binary interaction, which expresses the interaction between states on adjoining vertices.

**Definition 2.4.** A symmetric binary interaction, which we simply call an interaction on \( S \), is a map \( \phi : S \times S \to S \times S \) such that

\[
\hat{\iota} \circ \phi \circ \hat{\iota} \circ \phi(s_1, s_2) = (s_1, s_2)
\]

for any \((s_1, s_2) \in S \times S\) satisfying \( \phi(s_1, s_2) \neq (s_1, s_2) \), where \( \hat{\iota} : S \times S \to S \times S \) is the bijection obtained by exchanging the components of \( S \times S \).

Examples of the set of states \( S \) and interactions \( \phi \) are given in Example 2.18 of §2.3. Throughout this article, a triple \((X, S, \phi)\) indicates that \( X \) is a locale, \( S \) is a set of states, and \( \phi \) is an interaction. We will next define the configuration space with transition structure associated with such a triple. We first prepare a lemma.

**Lemma 2.5.** For a locale \( X = (X, E) \) and the set of states \( S \), let \( S^X \) be the configuration space for \( S \) on \( X \). Let \( \phi : S \times S \to S \times S \) be an interaction on \( S \). For any \( e = (o(e), t(e)) \in E \subset X \times X \), we define the map \( \phi_e : S^X \to S^X \) by \( \phi_e(\eta) := \eta^e \), where \( \eta^e = (\eta^e_x) \in S^X \) is defined as in (1). In other words, \( \phi_e(\eta) \) is obtained by applying \( \phi \) to the \( o(e) \) and \( t(e) \) components of \( \eta \). If we denote by \( \Phi \) the image of the map

\[
E \times S^X \to S^X \times S^X, \quad (e, \eta) \mapsto (\eta, \phi_e(\eta)),
\]

then the pair \((S^X, \Phi)\) is a symmetric directed graph.

**Proof.** It is sufficient to prove that the directed graph \((S^X, \Phi)\) is symmetric. By the definition of an interaction, we have \( \hat{\iota} \circ \phi \circ \hat{\iota} \circ \phi(s_1, s_2) = (s_1, s_2) \) for any \((s_1, s_2) \in S \times S\) such that \( \phi(s_1, s_2) \neq (s_1, s_2) \). This shows that for any \( \eta \in S^X \), if \( \phi_e(\eta) \neq \eta \), then we have \( \phi_e \circ \phi_e(\eta) = \eta \).

Consider the element \((\eta, \phi_e(\eta)) \in \Phi\). If \( \phi_e(\eta) = \eta \), then \((\phi_e(\eta), \eta) = (\eta, \eta) \in \Phi\). If \( \phi_e(\eta) \neq \eta \), then \((\phi_e(\eta), \eta) = (\phi_e(\eta), \hat{\iota} \circ \phi_e(\eta)) \), which is an element in \( \Phi \) since it is the image of \((\hat{\iota}, \phi_e(\eta)) \in E \times S^X \) by the map (1).

**Remark 2.6.** In fact, the condition (11) that we impose on the interaction is simply a sufficient condition and not a necessary condition for our theory. The property that we actually use is that \((S^X, \Phi)\) is a symmetric directed graph.
Definition 2.7. For a locale $X$ and a set of states $S$, if we fix an interaction $\phi : S \times S \to S \times S$, then Lemma 2.5 implies that $\Phi$ gives a structure of a symmetric directed graph

$$S^X = (S^X, \Phi)$$

on the configuration space $S^X$. We call this structure the transition structure, and we call any element $\varphi \in \Phi$ a transition. In particular, we say that $\varphi = (\eta, \phi_\varepsilon(\eta)) \in \Phi$ is a transition of $\eta$ by $\varepsilon$. Following the convention in literature, we will often denote $\phi_\varepsilon(\eta)$ by $\eta^\varepsilon$.

Remark 2.8. Our convention of denoting the locale $(X, E)$ by $X$ and the configuration space with transition structure $(S^X, \Phi)$ by $S^X$ follows similar convention as that of topological spaces, where the topological space and its underlying set is denoted by the same symbol. We are interpreting the set of edges of a graph as giving a geometric structure to the set of vertices.

For the configuration space $S^X$ with transition structure, the edges $\Phi$ expresses all the possible transitions on the configuration space with respect to the interaction $\phi$. For any element $\eta = (\eta_x) \in S^X$, we define the support of $\eta$ to be the set $\text{Supp}(\eta) := \{x \in X \mid \eta_x \neq *\} \subset X$. The subset $S^X_\ast := \{\eta = (\eta_x) \in S^X \mid |\text{Supp}(\eta)| < \infty\} \subset S^X$

of the configuration space will play an important role in our theory. If we let $\Phi_\ast := \Phi \cap (S^X_\ast \times S^X_\ast)$, then $S^X_\ast = (S^X_\ast, \Phi_\ast)$ is again a symmetric directed graph, which we refer to again as a configuration space with transition structure.

For any set $A$, we let $C(A) := \text{Map}(A, \mathbb{R})$ be the $\mathbb{R}$-linear space of maps from $A$ to $\mathbb{R}$. As in §1.2 for any finite $\Lambda \subset X$, we let $S^\Lambda := \prod_{x \in \Lambda} S$. Following standard convention, we let $S^0 = \{\ast\}$ if $\Lambda = \emptyset$. The natural projection $S^X \subset S^X$ given by mapping $\eta = (\eta_x)_{x \in X} \in S^X$ to $\eta|_{\Lambda} := (\eta_x)_{x \in \Lambda}$ induces a natural injection $C(S^\Lambda) \hookrightarrow C(S^X)$. From now on, we will identify $C(S^\Lambda)$ with its image in $C(S^X)$. Note that any $f \in C(S^X_\ast)$ is a function in $C(S^\Lambda)$ if and only if $f$ as a function for $\eta \in S^X$ depends only on $\eta|_{\Lambda} \in S^\Lambda$. We call any such function a local function. We denote by $C_{\text{loc}}(S^X)$ the space of local functions, which is a subspace of both $C(S^X)$ and $C(S^X_\ast)$. If the set of vertices of $X$ is finite, then we simply have $C_{\text{loc}}(S^X) = C(S^X) = C(S^X_\ast)$. Our methods are of interest predominantly for the case when $X$ is an infinite locale.

Since we will later define the space of uniform functions as a subspace of the space of functions $C(S^X_\ast)$, we will first investigate the cohomology of the graph $(S^X_\ast, \Phi_\ast)$.

2.2. Naïve Cohomology of the Configuration Space. In this subsection, we will define the cohomology of a configuration space with transition structure. For any set $A$, we let $\mathbb{Z}[A] := \bigoplus_{a \in A} \mathbb{Z}a$ the free $\mathbb{Z}$-module generated by elements of $A$, which is an abelian group consisting of elements of the form $\sum_{a \in A} n_a a$, where $n_a \in \mathbb{Z}$ and $a \in I$ for some finite index set $I \subset A$. We let $S^X_\ast = (S^X_\ast, \Phi_\ast)$ be the configuration space for a state $S$ on a locale $X$ with transition structure $\Phi_\ast$ induced from an interaction $\phi : S \times S \to S \times S$.

For any $\varphi = (o(\varphi), t(\varphi)) \in \Phi_\ast$, we let $\bar{\varphi} := (t(\varphi), o(\varphi)) \in \Phi_\ast$, which we call the opposite transition. By taking the opposite transition, we have an automorphism $\mathbb{Z}[\Phi_\ast] \to \mathbb{Z}[\Phi_\ast]$ of
We define the complex $C_\ast(S^X, \mathbb{Z})$ to be

$$C_m(S^X, \mathbb{Z}) := \begin{cases} \mathbb{Z}[S^X] & m = 0, \\ \mathbb{Z}[^{alt}\Phi_\ast] & m = 1 \\ \{0\} & \text{otherwise}, \end{cases}$$

where $\mathbb{Z}[\Phi_\ast]^{alt}$ is the quotient of $\mathbb{Z}[\Phi_\ast]$ by the $\mathbb{Z}$-submodule generated by elements $\varrho \in \mathbb{Z}[\Phi_\ast]$ satisfying $\bar{\varrho} = \varrho$. Note that for any transition of the form $\varphi = (\eta, \eta')$ for some $\eta \in S^X$, the class of $\varphi$ in $\mathbb{Z}[\Phi_\ast]^{alt}$ is zero, since $\varphi = \bar{\varphi}$. We define the differentials $\partial_m : C_m(S^X, \mathbb{Z}) \to C_{m-1}(S^X, \mathbb{Z})$ for $m \in \mathbb{Z}$ by $\partial_1 \varphi := t(\varphi) - o(\varphi)$ for any $\varphi = (o(\varphi), t(\varphi)) \in \Phi_\ast$, and $\partial_m := 0$ for $m \neq 1$.

**Definition 2.9.** We define the homology group of the configuration space $S^X$ by

$$H_m(S^X) := H_m(C_\ast(S^X, \mathbb{Z})),$$

where $H_m(C_\ast(S^X, \mathbb{Z})) := \text{Ker} \partial_m/\text{Im} \partial_{m+1}$ for any integer $m \in \mathbb{Z}$.

We define the cochain complex $C^\ast(S^X)$ associated with $S^X$ to be the complex given by

$$C^m(S^X) := \text{Hom}_\mathbb{Z}(C_m(S^X, \mathbb{Z}), \mathbb{R}),$$

where $\text{Hom}_\mathbb{Z}$ denotes the set of homomorphisms of $\mathbb{Z}$-modules. For any $m \in \mathbb{Z}$, we have a differential $\partial^m : C^m(S^X) \to C^{m+1}(S^X)$ induced from the differential $\partial_{m+1}$ on $C_\ast(S^X, \mathbb{Z})$. In other words, for any $f \in C^m(S^X)$, we let $\partial^m f := f \circ \partial_{m+1} \in C^{m+1}(S^X)$.

**Remark 2.10.** For $S^X \ast (S^X, \Phi_\ast)$, we have by definition

$$C^0(S^X) = C(S^X) := \text{Map}(S^X, \mathbb{R}), \quad C^1(S^X) = \text{Map}^{alt}(\Phi_\ast, \mathbb{R}),$$

where $\text{Map}^{alt}(\Phi_\ast, \mathbb{R}) := \{\omega : \Phi_\ast \to \mathbb{R} \mid \forall \varphi \in \Phi_\ast \omega(\bar{\varphi}) = -\omega(\varphi)\}$. For any $f \in C(S^X)$, the element $\partial f := \partial^0 f \in C^1(S^X)$ corresponds to the map $\partial f : \Phi_\ast \to \mathbb{R}$ given by $\partial f(\varphi) = f(\eta^\circ) - f(\eta)$ for any $\varphi = (\eta, \eta^\circ) \in \Phi_\ast$. We will often call an element in $C^1(S^X)$ a form. We say that a form $\omega \in C^1(S^X)$ is exact, if there exists a function $f \in C(S^X)$ such that $\partial f = \omega$.

In what follows, we will give explicit representations of the modules $C^1(S^X)$ and the differential $\partial$ in terms of functions on $S^X$. By the definition of $\Phi_\ast$ in ( ), we have a surjection $E \times S^X \to \Phi_\ast$. This shows that we have natural injections

$$C^1(S^X) \subseteq \text{Hom}_\mathbb{Z}(\mathbb{Z}[\Phi_\ast], \mathbb{R}) \subseteq \text{Hom}_\mathbb{Z}(\mathbb{Z}[E \times S^X], \mathbb{R})$$

$$\text{Map}^{alt}(\Phi_\ast, \mathbb{R}) \subseteq \text{Map}(\Phi_\ast, \mathbb{R}) \subseteq \text{Map}(E \times S^X, \mathbb{R}).$$

Hence we may view a form $\omega \in C^1(S^X)$ as a family of functions $\omega = (\omega_\epsilon)_{\epsilon \in E}$ through the identification

$$\text{Map}(E \times S^X, \mathbb{R}) = \prod_{\epsilon \in E} C(S^X),$$

where $\omega_\epsilon : S^X \to \mathbb{R}$ is the map defined as

$$\omega_\epsilon(\eta) := \omega(\varphi), \quad \varphi = (\eta, \eta^\circ) \in \Phi_\ast \subseteq S^X \times S^X.$$
Conversely, any family of functions \((\omega_e) \in \prod_{e \in E} C(S^X_e)\) comes from an element \(\omega \in C^1(S^X_e)\) if and only if \(\omega_e(\eta) = \omega_e'(\eta)\) if \(\eta^e = \eta'^e\), \(\omega_e(\eta) = 0\) if \(\eta^e = \eta'\), and \(\omega_e(\eta) = -\omega_e(\eta^e)\) for any \((e, \eta) \in E \times S^X_e\). For any \(f \in C(S^X_e)\), if we view \(\partial f\) as an element \(\partial f = ((\partial f)_e)_{e \in E} \in \prod_{e \in E} C(S^X_e)\), then we see that \((\partial f)_e\) is a function satisfying \((\partial f)_e(\eta) = f(\eta^e) - f(\eta)\) for any \(\eta \in S^X_e\). If we define the function \(\nabla f \in \text{Map}(S^X_e, \mathbb{R})\) for any \(e \in E\)
\[
\nabla f(\eta) := f(\eta^e) - f(\eta)
\]
for any \(\eta \in S^X_e\), then we have \(\partial f = (\nabla f)_e\) by construction. Hence our differential coincides with the differential \(\partial\) of \([1, 2]\).

**Definition 2.11.** The form \(\omega = (\omega_e)\) in \(C^1(S^X_e)\) is exact if and only if there exists a function \(f : S^X_e \to \mathbb{R}\) in \(C(S^X_e)\) such that \(\omega_e = \nabla f(\eta)\) for any \(e \in E\).

In what follows, we will often identify a form \(\omega \in C^1(S^X_e)\) with its representation \(\omega = (\omega_e)_{e \in E}\) in \(\prod_{e \in E} C(S^X_e)\). We now define the naïve cohomology of the configuration space \(S^X_e\) as follows.

**Definition 2.12.** We define the *naïve cohomology group* of the configuration space \(S^X_e\) with transition structure by
\[
H^m(S^X_e) := H^m(C^*(S^X_e)),
\]
where \(H^m(C^*(S^X_e)) := \text{Ker} \partial^m / \text{Im} \partial^{m-1}\) for any \(m \in \mathbb{Z}\).

Although \(H^m(S^X_e)\) is the standard simplicial cohomology of \(S^X_e\) and reflects the topological structure of \(S^X_e\), it is naïve in the sense that it does not reflect the situation that we would like for our model to represent. For example, the naïve \(H^1\) may be infinite dimensional as follows.

**Remark 2.13.** Let \(X = (\mathbb{R}^d, \mathbb{R}^d)\) and \(S = \{0, 1, \ldots, \kappa\}\) for an integer \(d > 1\) and some natural number \(\kappa > 0\). If we let \(S^X_e\) be the configuration space for \(S\) on \(X\) with transition structure for the multi-species exclusion process of Example[1, 3] then we have \(\dim_{\mathbb{R}} H^1(S^X_e) = \infty\).

This may be seen as follows. For any \(x \in X\), let \(1_x \in S^X_e\) be the configuration with 1 in the \(x\) component and 0 in the other components. For any edge \(e = (x, x') \in E\), the configuration \(1_x^e\) is the configuration \(1_{x'}\) with 1 in the \(x'\) component and 0 in the other components, hence \(1_e := (1_x, 1_{x'})\) is a transition of \(S^X_e\). If we let \(\omega \in C^1(S^X_e) = \text{Map}_{\text{alt}}(\Phi_\ast, \mathbb{R})\) be the form given by \(\omega(1_e) = -\omega(1_\varepsilon) = 1\) and \(\omega(\varphi) = 0\) for \(\varphi \neq 1_e, 1_\varepsilon\), then we may prove that \(\omega\) gives a nonzero element in \(H^1(S^X_e)\). In addition, we may prove that such \(\omega\) for a finite set of edges in \(E\) which do not share common vertices give linearly independent elements of \(H^1(S^X_e)\), giving our assertion.

The group \(H^1(S^X_e)\) is in general large, since there are many linearly independent forms which are not exact. We will consider a class of forms called *closed forms* in \(C^1(S^X_e)\) which are in fact always exact. We recall that a finite path on a graph \(S^X_e\) is a finite sequence \((\varphi^1, \ldots, \varphi^N)\) of transitions of \(S^X_e\) such that \(t(\varphi^i) = o(\varphi^{i+1})\) for any integer \(0 < i < N\). For a form \(\omega \in C^1(S^X_e)\) on \(S^X_e\), we define the integral of \(\omega\) with respect to the path \(\tilde{\gamma} = (\varphi^1, \ldots, \varphi^N)\) by
\[
\int_{\tilde{\gamma}} \omega := \sum_{j=1}^N \omega(\varphi^j).
\]
Definition 2.14. We say that a form \( \omega \in C^1(S^X_\ast) = \text{Map}^\text{alt}(\Phi_\ast, \mathbb{R}) \) is closed, if for any closed path \( \vec{y} \) in \( S^X_\ast \), we have

\[
\int_{\vec{y}} \omega = 0.
\]

We will denote by \( Z^1(S^X_\ast) \) the subset of closed forms in \( C^1(S^X_\ast) \).

We may prove that any exact form is a closed form.

Lemma 2.15. If \( \omega \in C^1(S^X_\ast) \) is exact, of the form \( \omega = \partial f \) for some \( f \in C(S^X_\ast) \), then \( \omega \) is closed.

Proof. If \( \omega = \partial f \) for some \( f \in C(S^X_\ast) \), then we have

\[
\int_{\vec{y}} \omega = f(t(\vec{y})) - f(o(\vec{y}))
\]

for any finite path \( \vec{y} \) in \( S^X_\ast \). In particular, for any closed path \( \vec{y} \), we have \( \int_{\vec{y}} \omega = 0 \), which implies that \( \omega \) is a closed form. \( \square \)

The converse is also true, as follows.

Lemma 2.16. Let \( \omega \) be a closed form in \( C^1(S^X_\ast) \). Then \( \omega \) is exact. In other words, there exists \( f \in C(S^X_\ast) \) such that \( \omega = \partial f \).

Proof. Consider the equivalence relation on \( S^X_\ast \) given by \( \eta \sim \eta' \) if there exists a finite path \( \vec{y} \) from \( \eta \) to \( \eta' \). We let \( S^X_0 \subset S^X_\ast \) be a representative of the set \( S^X_\ast / \sim \). We define a function \( f : S^X_\ast \rightarrow \mathbb{R} \) in \( C(S^X_\ast) \) as follows. For any \( \eta \in S^X_\ast \), there exists some \( \eta_0 \in S^X_0 \) and a path \( \vec{y} \) from \( \eta_0 \) to \( \eta \). We let

\[
f(\eta) := \int_{\vec{y}} \omega.
\]

The value is independent of the choice of the path \( \vec{y} \) since \( \omega \) is closed. For a transition \( \varphi = (\eta, \eta') \) in \( \Phi_\ast \), let \( \vec{y} \) and \( \vec{y}' \) be paths from some \( \eta_0 \in S^X_0 \) to \( \eta \) and \( \eta' \). Then we have

\[
\partial f(\varphi) = f(\eta') - f(\eta) = \int_{\vec{y}'} \omega - \int_{\vec{y}} \omega = \int_{\vec{q}} \omega = \omega(\varphi),
\]

where \( \vec{q} \) is the path \( \vec{q} = (\varphi) \) of length 1 from \( \eta \) to \( \eta' \). This proves that \( \partial f = \omega \), hence the form \( \omega \) is exact as desired. \( \square \)

In §2.4, we will study the naïve cohomology group \( H^0(S^X_\ast) \).

2.3. Conserved Quantities and Irreducibly Quantified Interactions. In this subsection, we introduce the notion of a conserved quantity, which are certain invariants of states preserved by the interaction. Using the conserved quantities, we will introduce the important notion for an interaction to be irreducibly quantified. In what follows, let \( S \) be a set of states with base state \( \ast \in S \), and we fix an interaction \( \phi : S \times S \rightarrow S \times S \) on \( S \).
**Definition 2.17.** A *conserved quantity* for the interaction \( \phi \) is a map \( \xi : S \to \mathbb{R} \) satisfying \( \xi(*) = 0 \) and
\[
\xi(s_1) + \xi(s_2) = \xi(s'_1) + \xi(s'_2)
\]
for any \( (s_1, s_2) \in S \times S \) and \( (s'_1, s'_2) := \phi(s_1, s_2) \). We denote by \( \text{Consv}^\phi(S) \) the \( \mathbb{R} \)-linear subspace of \( \text{Map}(S, \mathbb{R}) \) consisting of the conserved quantities for the interaction \( \phi \).

Examples of interactions and corresponding conserved quantities are given as follows.

**Example 2.18.**

1. Let \( S = \{0, 1, \ldots, \kappa\} \) with base state \( * = 0 \) for some integer \( \kappa > 0 \). For the multi-species exclusion process of Example 1.3(2), the conserved quantities given by \( \xi(i)(s) = 1 \) if \( s = i \) and \( \xi(i)(s) = 0 \) otherwise for \( i = 1, \ldots, \kappa \) give a basis of the \( \mathbb{R} \)-linear space \( \text{Consv}^\phi(S) \).

2. Let \( S = \mathbb{N} \) or \( S = \{0, \ldots, \kappa\} \subset \mathbb{N} \) for some integer \( \kappa > 0 \). We let \( * = 0 \) be the base state. The map \( \phi : S \times S \to S \times S \) defined by
\[
\phi(s_1, s_2) := \begin{cases} 
(s_1 - 1, s_2 + 1) & \text{if } s_1 - 1, s_2 + 1 \in S \\
(s_1, s_2) & \text{otherwise}
\end{cases}
\]
is an interaction. The stochastic process induced from this interaction when \( S \) is finite is the generalized exclusion process. Note that for any \( n > 1 \) in \( S \), we have \( \phi(n, 0) = (n - 1, 1) \), hence for any conserved quantity \( \xi \), equation (2) inductively gives
\[
\xi(n) = \xi(n) + \xi(0) = \xi(n - 1) + \xi(1) = \xi(n - 2) + 2\xi(1) = \cdots = n\xi(1),
\]
hence the conserved quantity \( \xi : S \to \mathbb{N} \) given by \( \xi(s) = s \) gives a basis of the one-dimensional \( \mathbb{R} \)-linear space \( \text{Consv}^\phi(S) \). This interaction is simple in the sense of Definition 1.1(2).

![Figure 5. Generalized Exclusion Process](image)

3. Let \( S = \mathbb{N} \), or let \( S = \{0, \ldots, \kappa\} \subset \mathbb{N} \) for some integer \( \kappa > 1 \), with base state \( * = 0 \). The map \( \phi : S \times S \to S \times S \) defined by
\[
\phi(s_1, s_2) := \begin{cases} 
(s_2, s_1) & s_1 > 0, s_2 = 0 \\
(s_1 - 1, s_2 + 1) & s_1 > 1, s_2 > 0, s_2 + 1 \in S \\
(s_1, s_2) & \text{otherwise}
\end{cases}
\]
is an interaction. The functions \( \xi^{(1)}(s) = s \) and
\[
\xi^{(2)}(s) = \begin{cases} 
1 & s > 0 \\
0 & s = 0
\end{cases}
\]
gives a basis of the two dimensional \( \mathbb{R} \)-linear space \( \text{Consv}^\phi(S) \). This interaction is the \textit{lattice gas with energy}. Intuitively, \( S \) represents the amount of energy on the vertex, with \( s = 0 \) representing the fact that there are no particles.

(4) For \( S = \{-1, 0, 1\} \) with base state \( * = 0 \), the map \( \phi: S \times S \to S \times S \) defined by
\[
\phi((0, 0)) = (-1, 1), \quad \phi((-1, 1)) = (1, -1), \quad \phi((1, -1)) = (0, 0),
\]
and \( \phi((s_1, s_2)) = (s_2, s_1) \) for \( s_1, s_2 \in S \) such that \( s_1 + s_2 \neq 0 \) is an interaction. The conserved quantity \( \xi: S \to \mathbb{Z} \) given by \( \xi(s) = s \) gives a basis of the one-dimensional \( \mathbb{R} \)-linear space \( \text{Consv}^\phi(S) \). This interaction is simple in the sense of Definition 1.1 (2).

(5) For \( S = \{0, 1\} \) with base state \( * = 0 \), the map \( \phi: S \times S \to S \times S \) defined by
\[
\phi((0, s)) = (1, s), \quad \phi((1, s)) = (0, s)
\]
for any \( s \in S \) does not satisfy (10). However, \( (S^X, \Phi) \) is a symmetric directed graph for any locale \( X \) and our theory applies also to this case (see also Remark 2.6). This case is known as the \textit{Glauber model}. Since \( \phi(0, 0) = (1, 0) \), equation (12) gives \( \xi(1) = \xi(0) = 0 \) for any conserved quantity \( \xi \), hence we have \( c^\phi = \text{dim}_\mathbb{R} \text{Consv}^\phi(S) = 0 \).

The exclusion process of Example 1.3 (1) is a special case of both the multi-species exclusion process and the generalized exclusion process, with the set of states given by \( S = \{0, 1\} \).

Let \( X \) be a locale, and let \( S^X_\eta \) be the configuration space with transition structure associated with our interaction. Note that if \( \xi \) is a conserved quantity, then \( \xi \) defines a function \( \xi_X: S^X_\eta \to \mathbb{R} \) given by
\[
\xi_X(\eta) := \sum_{x \in X} \xi(\eta_x)
\]
for any \( \eta = (\eta_x) \in S^X_\eta \). The sum is well-defined since by definition, \( \eta_x = * \) outside a finite number of \( x \in X \).

**Remark 2.19.** The interactions in Example 2.18 have the following interpretations. See Remark 1.4 for the case of the multi-species exclusion process.

(2) For \( s \in S \) in the process in Example 2.18 (2), \( s = 0 \) describes the state where there are no particles at the vertex, and \( s = k \) for an integer \( k > 0 \) the state where there are \( k \) indistinguishable particles at the vertex. Then for the conserved quantity given by \( \xi(s) = s \) for any \( s \in S \), the value \( \xi_X(\eta) \) for a configuration \( \eta \in S^X_\eta \) expresses the total number of particles in the configuration. If \( S = \{0, \ldots, \kappa\} \) for some integer \( \kappa > 0 \), then this process is called the generalized \textit{exclusion} process, since at most \( \kappa \) particles are allowed to occupy the vertex. If \( S = \mathbb{N} \), then the process includes the trivial case where the particles evolve as independent random walks with no interactions between the particles, as described in [8, Chapter 1].

(3) For \( s \in S \) the process in Example 2.18 (3), \( s = 0 \) describes the state where there are no particles at the vertex, and \( s = k \) for an integer \( k > 0 \) the state where there is a particle with energy \( k \) at the vertex. Then \( \xi^{(1)}_X(\eta) \) for a configuration \( \eta \in S^X_\eta \) expresses the total energy of the particles in the configuration, and \( \xi^{(2)}_X(\eta) \) expresses the total number of particles in the configuration.
(4) For the process in Example 2.18 (4), $s$ describes the spin of the particle at the vertex.
Then for the conserved quantity given by $\xi(s) = s$ for any $s \in S$, the value $\xi_X(\eta)$ for a
configuration $\eta \in S^X$ expresses the total spin of the configuration.
(5) For $s \in S$ in the Glauber model in Example 2.18 (5), one interpretation is that $s = 0$
describes the state where there are no particles at the vertex, and $s = 1$ describes the
state where there is a single particle at the vertex. Since the interaction allows for
the creation and annihilation of particles, there are no nontrivial conserved quantities.
Another interpretation is that $s$ describes the spin of the particle at each vertex.

Remark 2.20. The hydrodynamic limits for the generalized exclusion process has been studied
by Kipnis-Landim-Olla [9]. See also [8, Chapter 7]. The case of lattice gas with energy has
been studied by Nagahata [11], and the stochastic process induced from this interaction in (4)
was studied by Sasada [15]. In all of the known cases, the underlying locale is taken to be the
Euclidean lattice.

Remark 2.21. If we consider $S = \{0, 1\}$ with base state $\ast = 0$ and the interaction $\phi$ given by
$$\phi((0, 0)) = (1, 1), \quad \phi((1, 0)) = (1, 0), \quad \phi((0, 1)) = (0, 1), \quad \phi((1, 1)) = (0, 0),$$
then a map $\xi : S \to \mathbb{R}$ satisfying $\xi(0) = 0$ and $\xi(1)$ would imply that $\xi(1) = 0$, hence $\xi \equiv 0$. In
fact, this interaction is not irreducibly quantified. In order to deal with such a model, it may be
necessary to consider conserved quantities with values in $\mathbb{Z}/2\mathbb{Z}$.

The important notion of irreducibly quantified is defined as follows.

Definition 2.22. We say that an interaction $\phi : S \times S \to S \times S$ is irreducibly quantified, if for
any locale $X$ and configurations $\eta, \eta' \in S^X$, if
$$\xi_X(\eta) = \xi_X(\eta')$$
for any conserved quantity $\xi \in \text{Consv}^\phi(S)$, then there exists a finite path $\gamma$ from $\eta$ to $\eta'$ in $S^X$.

The remainder of this section is devoted to the proof of Proposition 2.27 which asserts that
the interactions given in Example 2.18 are all irreducibly quantified. For the proof, we first
introduce the notion of exchangeability.

Definition 2.23. We say that an interaction $\phi : S \times S \to S \times S$ is exchangeable, if for any $(s_1, s_2) \in
S \times S$, there exists a map $\psi : S \times S \to S \times S$ obtained as the combination of maps $\phi$ and $\psi$ such that
$\psi(s_1, s_2) = (s_2, s_1) \in S \times S$. Here, if we write $\phi(s_1, s_2) = (\phi_1(s_1, s_2), \phi_2(s_1, s_2)) \in S \times S$, then
we let $\phi(s_1, s_2) := (\phi_2(s_2, s_1), \phi_1(s_2, s_1))$ for any $(s_1, s_2) \in S \times S$. In other words, $\phi := \tilde{\iota} \circ \phi \circ \tilde{\iota}$,
where $\tilde{\iota} : S \times S \to S \times S$ is the bijection obtained by exchanging the components of $S \times S$.

It is straightforward to check that the interactions in Example 2.18 are all exchangeable. Since
$\phi \circ \phi(s_1, s_2) = (s_1, s_2)$ if $\phi(s_1, s_2) = (s_1, s_2)$, the map $\psi$ is necessarily of the form $\phi^n$ or $\tilde{\phi}^n$
for some integer $n \geq 0$. In what follows, consider a triple $(X, S, \phi)$ and the associated configuration
space $S^X$ with transition structure. We first prepare some notations.

Definition 2.24. Let $\eta$ be a configuration in $S^X$, and let $x, y$ be vertices in $X$.

1. We define $\eta^{x,y} \in S^X$ to be the configuration whose component outside $x, y \in X$ coincides
with that of $\eta$, and the $x$ and $y$ components are given by $\eta^{x,y}_x := \eta_y$ and $\eta^{x,y}_y := \eta_x$. 
(2) We define \( \eta^{x \rightarrow y} \) to be the configuration whose component outside \( x, y \in X \) coincides with that of \( \eta \), and the \( x \) and \( y \) components are given by \( \eta^{x \rightarrow y}_x := \phi_1(\eta_x, \eta_y) \) and \( \eta^{x \rightarrow y}_y := \phi_2(\eta_x, \eta_y) \), where \( \phi(\eta_x, \eta_y) = (\phi_1(\eta_x, \eta_y), \phi_2(\eta_x, \eta_y)) \in S \times S \) for the interaction \( \phi \) on \( S \).

A path in \( S^X \) is a sequence of transitions, and a transition is induced by an interaction on some edge of the locale. Hence for a sequence of edges \( e = (e^1, \ldots, e^N) \) in \( E \) and \( \eta \in S^X \), if we let \( \eta^0 := \eta \) and \( \eta^i := (\eta^{i-1})^{e_i} \) for \( i = 1, \ldots, N \), then \( \varphi^i = (\eta^i, \eta^{i+1}) \) is a transition and \( \gamma_\eta^i := (\varphi^1, \ldots, \varphi^N) \) gives a path from \( \eta \) to \( \eta^N := \eta^N \) in \( S^X \). We call this path \( \gamma_\eta^i \) the path with origin \( \eta \) induced by \( e \), or a path obtained by applying the edges \( e \) to \( \eta \).

We prove some existences of paths between certain configurations, when the interaction is exchangeable.

**Lemma 2.25.** If an interaction \( \phi : S \times S \rightarrow S \times S \) is exchangeable, then for any configuration \( \eta \in S^X \) and vertices \( x, y \in X \), there exists a path \( \tilde{\gamma} \) from \( \eta \) to \( \eta^{x \rightarrow y} \) in \( S^X \).

**Proof.** Since \( X \) is connected, there exists a path \( \tilde{\eta} = (e^1, \ldots, e^N) \) from \( x \) to \( y \). Note that by definition, \( x = o(e^1) \) and \( y = t(e^N) \). Since \( \phi \) is exchangeable, applying \( e^1 \) or \( \tilde{e}^1 \) sufficiently many times to \( \eta \), we obtain a path from \( \eta \) to \( \eta^{o(e^1), t(e^1)} \). Repeating this process for \( e^2, \ldots, e^N \), we obtain a path \( \tilde{\gamma}_1 \) from \( \eta \) to \( \eta' \), where \( \eta' \) is such that the components of \( \eta' \) coincides with that of \( \eta \) outside the vertices appearing in the edges \( e^1, \ldots, e^N \), \( \eta^{i, t(e^i)} = \eta^{o(e^i), t(e^i)} \) for \( i = 1, \ldots, N-1 \), and \( \eta'_y = \eta_x \). Then, reversing the above process, first by applying \( e^{N-1} \) or \( \tilde{e}^{N-1} \) sufficiently many times to \( \eta' \), we obtain a path from \( \eta' \) to \( \eta'^{o(e^{N-1}), t(e^{N-1})} \). Repeating this process for \( e^{N-2}, \ldots, e^1 \), we see that we obtain a path \( \tilde{\gamma}_2 \) from \( \eta' \) to \( \eta^{x \rightarrow y} \). Then the composition \( \tilde{\gamma} := \tilde{\gamma}_1 \tilde{\gamma}_2 \) gives a path from \( \eta \) to \( \eta^{x \rightarrow y} \) in \( S^X \) as desired. \( \square \)

**Lemma 2.26.** If an interaction \( \phi : S \times S \rightarrow S \times S \) is exchangeable, then for any configuration \( \eta \in S^X \) and vertices \( x, y \in X \), there exists a path \( \tilde{\gamma} \) from \( \eta \) to \( \eta^{x \rightarrow y} \) in \( S^X \).

**Proof.** Again, since \( X \) is connected, there exists a path \( \tilde{\eta} = (e^1, \ldots, e^N) \) from \( x = o(e^1) \) to \( y = t(e^N) \). If we let \( x' = o(e^N) \), then \( x' \) is a vertex connected to \( y \) by the edge \( e^N \). Then by construction, if we let \( \eta' := (\eta^{x, x'})^{e^N} \), then \( \eta' \) is a configuration whose component outside \( x, x', y \in X \) coincides with that of \( \eta \), and the \( x, x' \) and \( y \) components are given by \( \eta'_x = \eta'_x \), \( \eta'_{x'} = \phi_1(\eta_x, \eta_{x'}) \), and \( \eta'_y = \phi_2(\eta_x, \eta_y) \). Thus we have \( (\eta'^{x, x'}) = (\eta^{x \rightarrow y}) \). By Lemma 2.25, there exists a path \( \tilde{\gamma}_1 \) from \( \eta \) to \( \eta^{x, x'} \) and a path \( \tilde{\gamma}_2 \) from \( \eta' \) to \( \eta^{x \rightarrow y} \) in \( S^X \). If we denote by \( \tilde{\varphi} \) the path given by a single transition \( \varphi = (\eta^{x, x'}, \eta') \), then the composition \( \tilde{\gamma} := \tilde{\gamma}_1 \tilde{\varphi} \tilde{\gamma}_2 \) gives a path from \( \eta \) to \( \eta^{x \rightarrow y} \) in \( S^X \) as desired. \( \square \)

Using the above results, we may prove that various interactions are irreducibly quantified.

**Proposition 2.27.** The interactions of Example 2.18 are all irreducibly quantified.

**Proof.** By Example 2.18 we see that \( \text{Cons}^\phi(S) \) are finite dimensional. Let \( X \) be a locale and let \( \eta, \eta' \in S^X \) such that \( \xi_x(\eta) = \xi_x(\eta') \) for any conserved quantity \( \xi \in \text{Cons}^\phi(S) \). We prove the existence of a path \( \tilde{\gamma} \) from \( \eta \) to \( \eta' \) by induction on the order of the set \( \Delta(\eta, \eta') := \{ x \in X \mid \eta_x \neq \eta'_x \} \). Note that \( \Delta(\eta, \eta') \) is finite since the supports of \( \eta \) and \( \eta' \) are finite. If \(|\Delta(\eta, \eta')| = 0,\)
then $\eta = \eta'$ and there is nothing to prove. Suppose $|\Delta(\eta, \eta')| > 0$ and that the assertion is proved for $\eta'', \eta'$, where $\eta''$ is any configuration in $S^X$ such that $|\Delta(\eta'', \eta')| < |\Delta(\eta, \eta')|$ and $\xi_x(\eta'') = \xi_x(\eta')$ for any conserved quantity $\xi \in \text{Cons}^d(S)$.

(1) Consider the case of the multi-species exclusion process of Example 2.18(1) with conserved quantity $\xi^{(1)}, \ldots, \xi^{(c,s)}$. Let $x \in \Delta(\eta, \eta')$. If we let $\eta_s = i \in S$, then since $\xi^{(i)}(\eta) = \xi^{(i)}(\eta')$, there exists $y \in \Delta(\eta, \eta')$ such that $\eta_y = i$. By Lemma 2.25, there exists a path $\tilde{y}_1$ from $\eta$ to $\eta^{x,y}$. Note that $\eta^{x,y}$ coincides with $\eta$ outside $x, y$, and we have $\eta^{x,y}_x = \eta_s = i = \eta'_y$, which implies that $\xi_x(\eta) = \xi_x(\eta^{x,y}) = \xi_x(\eta')$ for any conserved quantity and $|\Delta(\eta^{x,y}, \eta')| < |\Delta(\eta, \eta')|$. Hence by the induction hypothesis, there exists a path $\tilde{y}_2$ from $\eta^{x,y}$ to $\eta'$ in $S^X$. Our assertion is proved by taking $\tilde{y} := \tilde{y}_1\tilde{y}_2$.

(2) Consider the case of the generalized exclusion process of Example 2.18(2) with conserved quantity $\xi$ given by $\xi(s) = s$. Again let $x \in \Delta(\eta, \eta')$ such that $\eta_x > \eta'_x$. Since $\xi_x(\eta) = \xi_x(\eta')$, there exists $y \in \Delta(\eta, \eta')$ such that $\eta_y < \eta'_y$. Let $M := \min(\eta_x - \eta'_x, \eta'_y - \eta_y)$, and let $\eta^0 = \eta$ and $\eta^i = (\eta^{i-1})^{s+y}$ for $i = 1, \ldots, M$. By Lemma 2.26, there exists a path $\tilde{y}_i$ from $\eta^{i-1}$ to $\eta^i$ in $S^X$ for $1 < i < M$. We have $\xi_x(\eta^M) = \xi_x(\eta) = \xi_x(\eta')$, and since $\eta^M_x = \eta_x - M$ and $\eta^M_y = \eta'_y + M$, we have either $\eta^M_x = \eta'_x$ or $\eta^M_y = \eta'_y$. This shows that $|\Delta(\eta^M, \eta')| < |\Delta(\eta, \eta')|$. Hence by the induction hypothesis, there exists a path $\tilde{y}'$ from $\eta^M$ to $\eta'$ in $S^X$. Our assertion is proved by taking $\tilde{y} := \tilde{y}_1 \cdots \tilde{y}_M \tilde{y}'$.

(3) Consider the case of the lattice gas with energy of Example 2.18(3) with conserved quantity $\xi^{(1)}$ and $\xi^{(2)}$. Suppose there exists $x \in \Delta(\eta, \eta')$ such that $\eta'_x = 0$. Then, since $\xi^{(2)}(\eta) = \xi^{(2)}(\eta')$, there exists $y \in \Delta(\eta, \eta')$ such that $\eta_y = 0$. Then $\eta^{x,y}$ satisfies $\xi_x(\eta^{x,y}) = \xi_x(\eta')$ for any conserved quantity $\xi$, and $|\Delta(\eta^{x,y}, \eta')| < |\Delta(\eta, \eta')|$. Hence by the induction hypothesis, there exists a path $\tilde{y}_2$ from $\eta^{x,y}$ to $\eta'$ in $S^X$. If we let $\tilde{y}_1$ be the path from $\eta$ to $\eta^{x,y}$ given in Lemma 2.25, then $\tilde{y} := \tilde{y}_1 \tilde{y}_2$ satisfies the desired property. Otherwise, we have $\eta_x \neq 0$ and $\eta'_x \neq 0$ for any $x \in \Delta(\eta, \eta')$. Since $\xi^{(1)}(\eta) = \xi^{(1)}(\eta')$, there exists $x, y \in \Delta(\eta, \eta')$ such that $\eta_x > \eta'_x$ and $\eta_y < \eta'_y$. As in (2), let $M := \min(\eta_x - \eta'_x, \eta'_y - \eta_y)$, and let $\eta^0 = \eta$ and $\eta^i = (\eta^{i-1})^{s+y}$ for $i = 1, \ldots, M$. By Lemma 2.26, there exists a path $\tilde{y}_1$ from $\eta^{i-1}$ to $\eta^i$ in $S^X$ for $1 < i < M$. We have $\xi_x(\eta^M) = \xi_x(\eta) = \xi_x(\eta')$ and either $s^M = \eta'_x$ or $s^M = \eta'_y$, which shows that $|\Delta(\eta^M, \eta')| < |\Delta(\eta, \eta')|$. Hence by the induction hypothesis, there exists a path $\tilde{y}'$ from $\eta^M$ to $\eta'$ in $S^X$. Our assertion is proved by taking $\tilde{y} := \tilde{y}_1 \cdots \tilde{y}_M \tilde{y}'$.

(4) The case of Example 2.18(4) is proved in a similar fashion as that of (1), but by first using the interaction $\phi((0,0)) = (1, -1)$ and $\phi((-1,1)) = (0,0)$ to equalize the number of vertices whose states are at $+1$ and $-1$.

(5) Consider the Glauber model of Example 2.18(5). Since in this case, the only conserved quantity is the zero map, the condition for the conserved quantity is always satisfied. Let $x \in \Delta(\eta, \eta')$. If we let $e \in E$ be any edge such that $o(e) = x$, then $\eta^e$ coincides with $\eta$ outside $x$ and we have $\eta^e_x = \eta'_x$. This shows we have $|\Delta(\eta^e, \eta')| < |\Delta(\eta, \eta')|$. Hence by the induction hypothesis, there exists a path $\tilde{y}'$ from $\eta^e$ to $\eta'$ in $S^X$. Our assertion is proved by taking $\tilde{y} := \tilde{y} \tilde{y}'$, where $\tilde{y}$ is the path given by the transition $\varphi = (\eta, \eta^e)$. □
2.4. $H^0$ of the Configuration Space. For the triple $(X, S, \phi)$, let $S^X_* = (S^X_*, \Phi_*)$ be the associated configuration space with transition structure. In this subsection, we will use the conserved quantities to investigate the naïve cohomology $H^0(S^X_*)$.

We say that a function $f \in C(S^X_*)$ is horizontal, if $\delta f = 0$. We first prove the following result concerning horizontal functions.

**Lemma 2.28.** Suppose $f \in C(S^X_*)$ is horizontal. Then for any configuration $\eta, \eta' \in S^X_*$, if there exists a finite path $\vec{\gamma}$ from $\eta$ to $\eta'$, then we have $f(\eta) = f(\eta')$.

**Proof.** Let $\vec{\gamma} = (\phi^1, \ldots, \phi^N)$ be a path with $o(\phi^1) = \eta$ and $t(\phi^N) = \eta'$. Since $\delta f = 0$, we have

$$\partial f(\phi^i) = f(o(\phi^i)) - f(t(\phi^i)) = 0$$

for any integer $i = 1, \ldots, N$. Our assertion follows from the fact that since $\vec{\gamma}$ is a path, we have $t(\phi^i) = o(\phi^{i+1})$ for any integer $i$ such that $1 < i < N$. \qed

**Remark 2.29.** Lemma 2.28 implies that any horizontal function in $C(S^X_*)$ is constant on each connected component of $S^X_*$. Let $\xi$ be a conserved quantity in $\text{Consv}^\phi(S)$. If we associate to $\xi$ the function $\xi_X : S^X_* \to \mathbb{R}$ of (13), then this induces a homomorphism of $\mathbb{R}$-linear spaces $\text{Consv}^\phi(S) \to C(S^X_*)$. This homomorphism is injective since for a fixed $x \in X$, if we let $\eta \in S^X_*$ be the configuration with $s$ in the $x$-component and at base state in the other components, then we have $\xi(s) = \xi_X(\eta)$, hence $\xi_X = \xi^X$ implies that $\xi = \xi'$ for any $\xi, \xi' \in \text{Consv}^\phi(S)$.

**Lemma 2.30.** Let $\xi$ be a conserved quantity for the interaction $\phi$. Then the function $\xi_X \in C(S^X_*)$ defined by (13) is horizontal. In particular, $\xi_X$ defines a class in $H^0(S^X_*)$.

**Proof.** For any $\varphi = (\eta, \eta^e) \in \Phi_*$, we have

$$\partial \xi_X(\varphi) := \xi_X(\eta^e) - \xi_X(\eta) = \sum_{x \in X} \xi(\eta^e_x) - \sum_{x \in X} \xi(\eta_x).$$

If we let $e = (x_1, x_2) \in E \subset X \times X$, then by definition of $\phi_e$ given in Lemma 2.5, we have $(\eta^e_{x_1}, \eta^e_{x_2}) = \phi(\eta_{x_1}, \eta_{x_2})$ and $\eta^e_x = \eta_x$ for $x \neq x_1, x_2$. This shows that

$$\partial \xi_X(\varphi) = \sum_{x \in X} \xi(\eta^e_x) - \sum_{x \in X} \xi(\eta_x) = (\xi(\eta^e_{x_1}) + \xi(\eta^e_{x_2})) - (\xi(\eta_{x_1}) + \xi(\eta_{x_2})) = 0$$

as desired, where the last equality follows from (2). \qed

Lemma 2.30 shows that (13) induces an injective homomorphism of $\mathbb{R}$-linear spaces

$$(14) \quad \text{Consv}^\phi(S) \hookrightarrow H^0(S^X_*).$$

**Remark 2.31.** We remark that $H^0(S^X_*)$ is in general very large, containing the $\mathbb{R}$-algebra generated by the image of $\text{Consv}^\phi(S)$. For example, the function $(\xi_X)^2$ on $S^X_*$ for any conserved quantity $\xi \in \text{Consv}^\phi(S)$ also defines an element of $H^0(S^X_*)$. The property (12) ensures that $\xi_X$ is an extensive quantity, i.e. additive with respect to the size of the system. We will prove in Theorem 5.8 that under suitable conditions, (14) gives an isomorphism between $\text{Consv}^\phi(S)$ and the 0-th uniform cohomology of the configuration space with transition structure.
Let $\eta = (\eta_x)$ and $\eta' = (\eta'_x)$ be configurations in $S^X_*$, and let $\gamma$ be a finite path from $\eta$ to $\eta'$. If $\xi$ is a conserved quantity for the interaction $\phi$, then Lemma 2.30 and Lemma 2.28 give the equality

$$\xi_X(\eta) = \xi_X(\eta').$$

This shows that $\xi_X$ is constant on each of the connected components of $S^X_*$. From now until the end of this subsection, we assume that $X$ is an infinite locale. We define a monoid to be any set with a binary operation that is associative and has an identity element. We say that a monoid is commutative if the operation is commutative.

**Definition 2.32.** Assume for simplicity that $\text{Consv}^\phi(S)$ is finite dimensional, and fix an $\mathbb{R}$-linear basis $\xi^{(1)}, \ldots, \xi^{(c_\phi)}$ of $\text{Consv}^\phi(S)$. We define the map

$$\xi_X: S^X_* \to \mathbb{R}^{c_\phi}$$

by $\xi_X(\eta) := (\xi^{(1)}(\eta), \ldots, \xi^{(c_\phi)}(\eta))$ for any $\eta = (\eta_x) \in S^X_*$, where $\xi^{(i)}(\eta) := \sum_{x \in X} \xi^{(i)}(\eta_x)$ for any $i = 1, \ldots, c_\phi$. Assuming that $X$ is an infinite locale, we let $M := \xi_X(S^X_*)$, which we view as a commutative monoid with operation induced from the addition on $\mathbb{R}^{c_\phi}$. Then the monoid $M$ is determined independently up to a natural isomorphism of the choice of the basis $(\xi^{(1)}, \ldots, \xi^{(c_\phi)})$.

**Remark 2.33.** We may define the commutative monoid $M$ of Definition 2.32 intrinsically as an additive submonoid of $\text{Hom}_\mathbb{R}(\text{Consv}^\phi(S), \mathbb{R})$ as follows. We define a map

$$\xi_X^{\text{univ}}: S^X_* \to \text{Hom}_\mathbb{R}(\text{Consv}^\phi(S), \mathbb{R})$$

by $\eta \mapsto (\xi \mapsto \xi_X(\eta))$, and we let $M := \xi_X^{\text{univ}}(S^X_*).$ In fact, the monoid $M$ is defined independently of the choice of the infinite locale $X$. In what follows, we will denote the map $\xi_X^{\text{univ}}: S^X_* \to M$ simply by $\xi_X$. If $\text{Consv}^\phi(S)$ is finite dimensional, a choice of a basis $\xi^{(1)}, \ldots, \xi^{(c_\phi)}$ of $\text{Consv}^\phi(S)$ gives an isomorphism $\text{Hom}_\mathbb{R}(\text{Consv}^\phi(S), \mathbb{R}) \cong \mathbb{R}^{c_\phi}$, and the monoid $M$ maps to the $M$ of Definition 2.32 through this isomorphism.

By Lemma 2.30 and Lemma 2.28 if the configurations $\eta, \eta' \in S^X_*$ are connected by a path, then it is in the same fiber of the map $\xi_X$. On the other hand, the condition that the interaction is irreducibly quantified implies that the fibers of the map $\xi_X$ are connected. Thus in this case, the connected components of $S^X_*$ correspond bijectively with the elements of $M$. The authors appreciate Hiroyuki Ochiai for suggesting this formulation.

By Remark 2.33 if the interaction is irreducibly quantified, then the connected components of $S^X_*$ correspond bijectively with the elements of the commutative monoid $M$. In particular, if $c_\phi = 0$, then $S^X_*$ is connected. Moreover, if $c_\phi > 0$, then we have $\dim \mathbb{R} H^0(S^X_*) = \infty$.

In the next section, we will define the notion of uniform functions and forms which are better suited to extract the topological properties of the configuration space with transition structure when the locale $X$ is infinite. We will define and calculate the uniform cohomology of $S^X$ in §5.
3. Uniform Functions and Uniform Forms

In this section, we will define the notion of uniform functions and uniform forms, which are functions and forms which reflect the geometry of the underlying locale. We will then investigate its properties, including a criterion for a function to be uniform.

3.1. Uniform Functions on the Configuration Space. For the triple \((X, S, \phi)\), let \(S^X = \prod_{x \in X} S\) be the configuration space for \(S\) on \(X\). We let \(S^X_\ast \subset S^X\) be the subset consisting of configurations with finite support. In this subsection, we will prove the existence of a canonical expansion of uniform forms in \(C(S^X_\ast)\) in terms of local functions with exact support (see Definition 3.1), and we will introduce the notion of a uniform function, which are functions which reflect the geometry of the underlying locale.

For a finite \(\Lambda \subset X\), there exists a natural inclusion \(t^\Lambda : S^\Lambda \hookrightarrow S^\Lambda \times S^X_\ast = S^X_\ast\) given by \(\eta_\Lambda \mapsto (\eta_\Lambda, \ast)\) for any \(\eta_\Lambda \in S^\Lambda\), where \(\ast \in S^X_\ast\) is the element whose components are all at base state. By abuse of notation, we will often denote \(t^\Lambda(\eta|\Lambda)\) by \(\eta|\Lambda\). This inclusion induces a homomorphism

\[
\tau^\Lambda : C(S^X) \to C(S^\Lambda) \subset C(S^X_\ast),
\]

which may be regarded as an \(\mathbb{R}\)-linear operator on the set of functions \(C(S^X_\ast)\). Note that we have \(\tau^\Lambda f(\eta) = f(\eta|\Lambda)\) for any \(f \in C(S^X_\ast)\) and \(\eta \in S^X\). By definition, if \(f \in C(S^\Lambda)\), then we have \(\tau^\Lambda f = f\). For any \(\Lambda, \Lambda' \subset X\), we have \(\tau^\Lambda \tau^\Lambda' = \tau^\Lambda' \tau^\Lambda = \tau^{\Lambda \Lambda'}\).

**Definition 3.1.** For any finite \(\Lambda \subset X\), we let

\[
C_\Lambda(S^X) := \{ f \in C(S^\Lambda) \mid f(\eta) = 0 \text{ if } \exists x \in \Lambda \text{ such that } \eta_x = \ast \}.
\]

We call any function \(f \in C_\Lambda(S^X)\) a local function with exact support \(\Lambda\).

Any function in \(C(S^X_\ast)\) has a unique expansion in terms of local functions with exact support, as will be shown in Proposition 3.3. We first start with the following lemma.

**Lemma 3.2.** Let \((f_\Lambda)\) be a set of functions such that \(f_\Lambda \in C_\Lambda(S^X)\) for any finite \(\Lambda \subset X\). Then the sum

\[
f := \sum_{\Lambda \subset X, |\Lambda| < \infty} f_\Lambda
\]

defines a function in \(C(S^X_\ast)\).

**Proof.** By definition, for any \(\eta = (\eta_x) \in S^X_\ast\), the support \(\text{Supp}(\eta) \subset X\) satisfies \(|\text{Supp}(\eta)| < \infty\). Then we have

\[
f(\eta) := \sum_{\Lambda \subset X, |\Lambda| < \infty} f_\Lambda(\eta) = \sum_{\Lambda \subset \text{Supp}(\eta)} f_\Lambda(\eta),
\]

where the last sum is defined since it is a finite sum. We see that the sum defines a function \(f : S^X \to \mathbb{R}\) as desired. \(\square\)

The expansion of functions in \(C(S^X_\ast)\) in terms of local functions with exact support is given as follows.
Proposition 3.3. For any \( f \in C(S^X) \), there exists a unique expansion

\[
f = \sum_{\Lambda \subset X, |\Lambda| < \infty} f_\Lambda,
\]

in terms of local functions with exact support \( f_\Lambda \in C_\Lambda(S^X) \) for finite \( \Lambda \subset X \).

Proof. We construct \( f_\Lambda \) by induction on the order of \( \Lambda \). Suppose an expansion of the form (16) exists. Note that for any \( \Lambda, \Lambda' \subset X \) such that \( \Lambda' \not\subset \Lambda \), we have \( \iota^\Lambda f_{\Lambda'} = 0 \) since \( f_{\Lambda'} \in C_{\Lambda'}(S^X) \), hence if we apply the \( \mathbb{R} \)-linear operator \( \iota^\Lambda \) of (15) on (16), then we obtain the equality

\[
\iota^\Lambda f = \sum_{\Lambda' \subset \Lambda} f_{\Lambda'}.
\]

Hence this shows that assuming the existence of the expansion, \( f_\Lambda \) is uniquely given inductively for the set \( \Lambda \) by

\[
f_\Lambda = \iota^\Lambda f - \sum_{\Lambda' \subset \Lambda} f_{\Lambda'}.
\]

We will prove by induction on the order of \( \Lambda \) that the function \( f_\Lambda \) inductively given by (17) is a function in \( C_\Lambda(S^X) \). We let \( \ast \in S^X \) be the base state. For \( \Lambda = \emptyset \), equation (17) gives \( f_\emptyset = \iota^\emptyset f \), which shows that \( f_\emptyset \) is an element in \( C_\emptyset(S^X) \). Note that \( f_\emptyset \) is the constant function given by \( f_\emptyset(\eta) = f(\ast) \) for any \( \eta \in S^X \). For \( |\Lambda| > 0 \), suppose \( f_{\Lambda'} \in C_{\Lambda'}(S^X) \) for any \( \Lambda' \subsetneq \Lambda \). Then by (17), the function \( f_\Lambda \) is a function in \( C(S^\Lambda) \). Next, we prove that \( f \in C_\Lambda(S^X) \). Suppose \( \eta = (\eta_x) \in S^X \) satisfies \( \eta_x = \ast \) for some \( x \in \Lambda \). If \( \Lambda' \subsetneq \Lambda \) and \( x \in \Lambda' \), then we have \( f_{\Lambda'}(\eta) = 0 \) since \( f_{\Lambda'} \in C_{\Lambda'}(S^X) \). Hence by (17), we have

\[
f_\Lambda(\eta) = \iota^\Lambda f(\eta) - \sum_{\Lambda' \subset \Lambda} f_{\Lambda'}(\eta).
\]

Note that since \( \eta_x = \ast \), we have \( \iota^\Lambda f(\eta) = \iota^{\Lambda \setminus \{x\}} f(\eta) \) by definition of \( \iota^\Lambda \), hence we have

\[
f_\Lambda(\eta) = \iota^{\Lambda \setminus \{x\}} f(\eta) - \sum_{\Lambda' \subset \Lambda \setminus \{x\}} f_{\Lambda'}(\eta) = \left( \iota^{\Lambda \setminus \{x\}} f(\eta) - \sum_{\Lambda' \subset \Lambda \setminus \{x\}} f_{\Lambda'}(\eta) \right) - f_{\Lambda \setminus \{x\}}(\eta) = f_{\Lambda \setminus \{x\}}(\eta) - f_{\Lambda \setminus \{x\}}(\eta) = 0.
\]

This proves that \( f_\Lambda \in C_\Lambda(S^X) \) as desired. The sum (16) gives the function \( f \) in \( C(S^X) \), since for any \( \eta \in S^X \), we have \( f(\eta) = \sum_{\Lambda \subset \Lambda'} f_{\Lambda'}(\eta) \) for \( \Lambda' \subset \Lambda \).

\[
\Box
\]

Corollary 3.4. If \( f \in C(S^\Lambda) \) for some finite \( \Lambda \subset X \), then we have a unique expansion

\[
f = \sum_{\Lambda' \subset \Lambda} f_{\Lambda'},
\]

in terms of local functions with exact support \( f_{\Lambda'} \in C_{\Lambda'}(S^X) \) for \( \Lambda' \subset \Lambda \).

Proof. Our statement follows by applying the \( \mathbb{R} \)-linear operator \( \iota^\Lambda \) to the expansion (16) of Proposition 3.3 noting that \( \iota^\Lambda f = f \) and \( \iota^\Lambda f_{\Lambda'} = 0 \) if \( \Lambda' \not\subset \Lambda \).

\[
\Box
\]
For any \( \Lambda \subset X \), we define the diameter \( \text{diam}(\Lambda) \) of \( \Lambda \) by

\[
\text{diam}(\Lambda) := \sup_{x, x' \in \Lambda} d_X(x, x').
\]

Since \( X \) is connected, if \( \Lambda \subset X \) is finite, then we have \( \text{diam}(\Lambda) < \infty \). Uniform functions are defined as follows.

**Definition 3.5.** We say that a function \( f \in C(S^X_*) \) is uniform, if there exists \( R > 0 \) such that the canonical expansion of \( f \) is given by

\[
f = \sum_{\Lambda \subset X, \text{diam}(\Lambda) \leq R} f_\Lambda.
\]

In other words, \( f_\Lambda = 0 \) in the expansion \( f \) if \( \text{diam}(\Lambda) > R \). We denote by \( C_{\text{unif}}(S^X) \) the \( \mathbb{R} \)-linear subspace of \( C(S^X) \) consisting of uniform functions, and by \( C^0_{\text{unif}}(S^X) \) the subspace of \( C_{\text{unif}}(S^X) \) consisting of functions satisfying \( f(\ast) = 0 \).

For any \( x \in X \) and \( \Lambda \subset X \), we let \( d_X(x, \Lambda) := \inf_{x' \in \Lambda} d_X(x, x') \), and for \( \Lambda, \Lambda' \subset X \), we let \( d_X(\Lambda, \Lambda') := \inf_{(x, x') \in \Lambda \times \Lambda'} d_X(x, x') \).

**Remark 3.6.** Suppose \( f \in C^0_{\text{unif}}(S^X) \) so that there exists some \( R > 0 \) such that \( f_\Lambda \equiv 0 \) in the expansion \( f \) if \( \text{diam}(\Lambda) > R \). Let \( \Lambda \) and \( \Lambda' \) be subsets of \( X \) such that \( d_X(\Lambda, \Lambda') > R \). By Corollary 3.4, we have

\[
\tilde{t}^{\Lambda \cup \Lambda'} f = \sum_{\Lambda'' \subset \Lambda \cup \Lambda'} f_{\Lambda''}, \quad \tilde{t}^\Lambda f = \sum_{\Lambda'' \subset \Lambda} f_{\Lambda''} \quad \text{and} \quad \tilde{t}^{\Lambda'} f = \sum_{\Lambda'' \subset \Lambda'} f_{\Lambda''}.
\]

Note that if \( \Lambda'' \subset \Lambda \cup \Lambda' \) satisfies \( \Lambda'' \cap \Lambda \neq \emptyset \) and \( \Lambda'' \cap \Lambda' \neq \emptyset \), then we have \( \text{diam}(\Lambda'') > R \), hence \( f_{\Lambda''} = 0 \) from our choice of \( R \). This shows that we have \( \tilde{t}^{\Lambda \cup \Lambda'} f = \tilde{t}^\Lambda f + \tilde{t}^{\Lambda'} f \), where we have used the fact that \( f(\ast) = \tilde{t}^0 f = 0 \).

## 3.2. Horizontal Uniform Functions

Consider the triple \( (X, S, \phi) \). From now until the end of §3, we assume that the interaction \( \phi \) is irreducibly quantified in the sense of Definition 2.22. The purpose of this subsection is to prove the following theorem.

**Theorem 3.7.** For the triple \( (X, S, \phi) \), assume that \( X \) is an infinite locale and the interaction \( \phi \) is irreducibly quantified. Let \( f \) be a uniform function in \( C^0_{\text{unif}}(S^X) \). If \( f \) is horizontal, i.e. if \( \partial f = 0 \), then there exists a conserved quantity \( \xi : S \to \mathbb{R} \) such that

\[
f(\eta) = \sum_{x \in X} \xi(\eta_x)
\]

for any \( \eta = (\eta_x) \in S^X \).

Theorem 3.7 implies that assuming that \( X \) is infinite and \( \phi \) is irreducibly quantified, any uniform function which is constant on the connected components of \( S^X \) coincides with \( \xi_X \) for some conserved quantity \( \xi \in \text{Consv}^\phi(S) \). We will give the proof of Theorem 3.7 at the end of this subsection. We first prove the following lemma.

**Lemma 3.8.** Suppose \( f \in C(S^X_*) \), and for any \( x \in X \), let \( f_{\{x\}} \) be the function with exact support \( \Lambda = \{x\} \) in the canonical expansion \( f \). If \( f \) is horizontal, then the functions

\[
\tilde{t}^{\{x\}} f_{\{x\}} : S \to \mathbb{R}
\]

are all equal as \( x \) varies over \( X \).
Proof. Consider the expansion
\[ f = \sum_{\Lambda \subset X, |\Lambda| < \infty} f_\Lambda \]
of (16). We let \( s \) be any element in \( S \). For \( x, x' \in X \), we let \( \eta := \iota^{(x)}(s) \) and \( \eta' := \iota^{(x')}(s) \) be the configuration in \( S^X \) with \( s \) respectively in the \( x \) and \( x' \) components, and base state \( * \) in the other components. This implies that
\[ \sum_{x \in X} \xi(\eta_x) = \sum_{x \in X} \xi(\eta'_x) = \xi(s), \]
in other words that \( \xi_X(\eta) = \xi_X(\eta') \) for any conserved quantity \( \xi \in \text{Cons}_X^\Phi(S) \). Hence from the fact that the interaction is irreducibly quantified, there exists a finite path \( \gamma \) from \( \eta \) to \( \eta' \) in \( S^X \).

By Lemma 2.28 noting that \( \partial f = 0 \), we have
\[ \iota^{(x)} f_{(x)}(s) = f(\eta) = f(\eta') = \iota^{(x')} f_{(x')}(s), \]
which shows that \( \iota^{(x)} f_{(x)} : S \to \mathbb{R} \) is independent of the choice of \( x \in X \) as desired. \( \square \)

Next, we prove the following lemma.

Lemma 3.9. Assume that \( X \) is an infinite locale, and suppose \( f \) is a uniform function in \( C^0_{\text{unif}}(S^X) \). If \( f \) is horizontal, then we have
\[ f = \sum_{x \in X} f_{(x)}, \]
where \( f_{(x)} \) is the function with exact support \( \Lambda = \{x\} \) in the canonical expansion (16).

Proof. Since \( f \) is uniform, there exists an \( R > 0 \) such that \( f_\Lambda \equiv 0 \) if \( \text{diam}(\Lambda) > R \). By Proposition 3.3, it is sufficient to prove that for any finite \( \Lambda \subset X \) satisfying \( |\Lambda| > 1 \), we have \( f_\Lambda \equiv 0 \) in the expansion of (16). We will prove this by induction on the order of \( \Lambda \subset X \). We consider a finite \( \Lambda \subset X \) such that \( n := |\Lambda| > 1 \), and assume that \( f_{\Lambda'} \equiv 0 \) for any \( \Lambda' \subset X \) such that \( 1 < |\Lambda'| < n \). Note that this condition is trivially true for \( n = 2 \). We let \( \Lambda_n \subset X \) be a subset of \( X \) with \( n \) elements such that \( \text{diam}(\Lambda_n) > R \). Such \( \Lambda_n \) exists since \( X \) is a locally finite infinite graph that is connected. Then by construction, we have \( f_{\Lambda_n} \equiv 0 \). We fix a bijection between the set \( \{1, \cdots, n\} \) and the sets \( \Lambda \) and \( \Lambda_n \), which induces bijections \( S^n \equiv S^\Lambda \) and \( S^n \equiv S^{\Lambda_n} \). For any \( (\eta_i) \in S^n \), which we view as an element in \( S^\Lambda \) and \( S^{\Lambda_n} \), we let \( \eta := \iota^\Lambda((\eta_i)) \) and \( \eta' := \iota^{\Lambda_n}((\eta_i)) \).

Then for any conserved quantity \( \xi : S \to \mathbb{R} \), we have
\[ \sum_{x \in X} \xi(\eta_x) = \sum_{x \in X} \xi(\eta'_x) = \sum_{i=1}^n \xi(\eta_i), \]
in other words, \( \xi_X(\eta) = \xi_X(\eta') \). Since the interaction is irreducibly quantified, there exists a finite path \( \gamma \) from \( \eta \) to \( \eta' \) in \( S^X \). By Lemma 2.28 and our condition that \( \partial f = 0 \), we have \( f(\eta) = f(\eta') \). Note that by Lemma 3.8 if we let \( \zeta := \iota^{(x)} f_{(x)} : S \to \mathbb{R} \), then \( \zeta \) is independent of
the choice of $x \in X$. Corollary [3.4] implies that we have

$$f(\eta) = \iota^\Lambda f(\eta) = f_\Lambda(\eta) + \sum_{\Lambda' \subseteq \Lambda} \sum_{x \in \Lambda'} f_{\Lambda'}(\eta) = f_\Lambda(\eta) + \sum_{x \in \Lambda} f_{\{x\}}(\eta) = f_\Lambda(\eta) + \sum_{\Lambda' \subseteq \Lambda} \xi(\eta_i).$$

$$f(\eta') = \iota^\Lambda f(\eta') = f_\Lambda(\eta') + \sum_{\Lambda' \subseteq \Lambda} \sum_{x \in \Lambda'} f_{\Lambda'}(\eta') = \sum_{x \in \Lambda} f_{\{x\}}(\eta') = \sum_{i=1}^n \xi(\eta_i),$$

hence we have $f_\Lambda(\eta) = 0$. Since this was true for any $(\eta_i) \in S^n$, we have $f_\Lambda \equiv 0$ as desired. Our assertion now follows by induction on $n$. \qed

We may now prove Theorem 3.7.

**Proof of Theorem 3.7** Let $\xi := \iota^{\{x\}} f_{\{x\}} : S \rightarrow \mathbb{R}$, which by Lemma [3.8] is independent of the choice of $x \in X$. Then by Lemma [3.9] and the definition of $\iota^{\{x\}} f_{\{x\}}$, we have

$$f(\eta) = \sum_{x \in X} \xi(\eta_x)$$

for any $\eta = (\eta_x) \in S^X_x$. It is sufficient to show that $\xi$ is a conserved quantity. First, note that we have $\xi(\ast) = f_{\{x\}}(\ast) = 0$ for any $x \in X$ since $f_{\{x\}}$ has exact support $\{x\}$. Consider an edge $e = (o(e), t(e)) \in E \subseteq X \times X$, and for $(s_1, s_2) \in S \times S$, let $\eta \in S^X_x$ be the configuration with $s_1$ in the $o(e)$ component, $s_2$ in the $t(e)$ component, and base states $\ast$ at the other components. Then $\eta^e$ is the configuration with $s_1'$ in the $o(e)$ component, $s_2'$ in the $t(e)$ component, and base states at the other components, where $(s_1', s_2') = \phi(s_1, s_2)$. If we let $\varphi = (\eta, \eta^e)$ be the transition from $\eta$ to $\eta^e$, then $\partial f = 0$ implies that

$$\partial f(\varphi) = f(\eta^e) - f(\eta) = (\xi(s_1') + \xi(s_2')) - (\xi(s_1) + \xi(s_2)) = 0.$$ 

This shows that $\xi$ satisfies [12], hence it is a conserved quantity as desired. \qed

### 3.3. Pairings for Functions with Uniform Differentials

Let $X$ be an infinite locale and assume that the interaction is irreducibly quantified. In this subsection, we further define the notion of *uniform forms*. Next, we will prove Proposition 3.18 which associates a certain pairing $h_f : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ to any function $f \in C(S^X_x)$ whose differential $\partial f \in C^1(S^X_x)$ is a uniform form. Here $\mathcal{M}$ is the commutative monoid given in Definition 2.32 (see also Remark 2.33).

A **ball** in $X$ is a set of the form $B(x, R) := \{x' \in X \mid d_X(x, x') \leq R\}$ for some $x \in X$ and constant $R > 0$. We say that $x$ is the *center* and $R > 0$ is the *radius* of $B(x, R)$. If $B$ is a ball in $X$, then we denote by $r(B)$ the radius of $B$. For any $\Lambda \subseteq X$, we let $B(\Lambda, R) := \bigcup_{x \in \Lambda} B(x, R)$, which we call the $R$-**thickening** of $\Lambda$. In particular, for any edge $e = (o(e), t(e)) \in E \subseteq X \times X$, we let $B(e, R) := B(o(e), R) \cup B(t(e), R)$. For any $R > 0$, we define the set of **$R$-uniform forms** $C^1_R(S^X_x) \subset C^1(S^X_x) \subset \prod_{e \in E} \text{Map}(S^X_x, \mathbb{R})$ by

$$C^1_R(S^X_x) := C^1(S^X_x) \cap \prod_{e \in E} C(S^{B(e, R)}).$$
Definition 3.10. We define the space of uniform forms on $S^X$ to be the $\mathbb{R}$-linear space
\[ C^1_{\text{unif}}(S^X) := \bigcup_{R > 0} C^1_R(S^X). \]
We define a closed uniform form to be a uniform form which is closed in the sense of Definition 2.14. We will denote by $Z^1_{\text{unif}}(S^X)$ the space of closed uniform forms.

For any subset $Y \subset X$, we denote by $\text{pr}_Y$ the map of sets $\text{pr}_Y : S^X \rightarrow S^Y$ induced from the natural projection $\text{pr}_Y : S^X \rightarrow S^Y$. For any $\eta \in S^X$, we will often denote $\text{pr}_Y(\eta)$ by $\eta|_Y$. By abuse of notation, we often write $\eta|_Y$ for the configuration $\eta_Y(\eta|_Y)$ in $S^X$. We say that the configurations $\eta, \eta' \in S^X$ coincide outside $Y$, if $\eta|_{Y^c} = \eta'|_{Y^c}$ for $Y^c := X \setminus Y$.

For any conserved quantity $\xi : S \rightarrow \mathbb{R}$ and $W \subset X$, we define the function $\xi_W : S^X \rightarrow \mathbb{R}$ by $\xi_W(\eta) := \sum_{\eta \in \mathcal{E}_W} \xi(\eta_x)$ for any $\eta \in S^X$. The following result concerns the values of functions whose differential are uniform.

Lemma 3.11. Let $f \in C(S^X)$ and assume that $\partial f \in C^1_R(S^X)$ for some $R > 0$. Let $Y \subset X$ be a sublocale, and suppose $\eta, \eta' \in S^X$ are configurations which coincide outside $Y$ and satisfy $\xi_Y(\eta) = \xi_Y(\eta')$ for any conserved quantity $\xi \in \text{Consv}^\phi(S)$. Suppose that the interaction is irreducibly quantified. If $\tilde{Y}$ is any subset of $X$ such that $B(Y, R) \subset \tilde{Y}$, then we have
\[ f(\eta') - f(\eta) = f(\eta'|_{\tilde{Y}}) - f(\eta|_{\tilde{Y}}). \]

Proof. Since $\partial f \in C^1_R(S^X)$, we have $\nabla_e f \in C(S^{R(e),R})$. Hence since $B(Y, R) \subset \tilde{Y}$, we have
\[ \nabla_e f(\eta) = \nabla_e f(\eta|_{\tilde{Y}}) \]
for any $e \subset Y$. The condition $\xi_Y(\eta) = \xi_Y(\eta')$ implies that $\xi_Y(\eta|_Y) = \xi_Y(\eta'|_Y)$ for any conserved quantity $\xi$. Since the interaction is irreducibly quantified, there exists a path $\tilde{y}|_Y$ from $\eta|_Y$ to $\eta'|_Y$ in $S^Y$. If we let $e = (e^1, \ldots, e^N)$ be the sequence of edges in $Y$ such that $\tilde{y}|_Y = \tilde{y}^e|_Y$, then since $\eta$ and $\eta'$ coincide outside $Y$, the path $\tilde{y} := \tilde{y}^e|_{\tilde{Y}}$ gives a path from $\eta$ to $\eta'$ and the path $\tilde{y}|_{\tilde{Y}} := \tilde{y}^e|_{\tilde{Y}}$ gives a path from $\eta|_{\tilde{Y}}$ to $\eta'|_{\tilde{Y}}$. Note that for any $e \in E$, by definition, $\nabla_e f(\eta) = f(\eta^e) - f(\eta)$. Hence
\[ f(\eta') - f(\eta) = \sum_{i=1}^N \nabla_{e^i} f(\eta^{i-1}), \quad f(\eta'|_{\tilde{Y}}) - f(\eta|_{\tilde{Y}}) = \sum_{i=1}^N \nabla_{e^i} f(\eta^{i-1}|_{\tilde{Y}}), \]
where we let $\eta^0 := \eta$ and $\eta^i := (\eta^{i-1})^{e^i}$ for $i = 1, \ldots, N$. Our assertion now follows from (18).

For the remainder of this article, we consider the map
\[ \xi_X : C(S^X) \rightarrow \mathcal{M} \]
given in Remark 2.33. For finite $\Lambda$, we denote by $\mathcal{M}_{|\Lambda|} := \xi_\Lambda(\Lambda^X) \subset \mathcal{M}$ the image of $S^X$ with respect to the map $\xi_\Lambda$. As the notation suggests, the set $\mathcal{M}_{|\Lambda|}$ as a subset of $\mathcal{M}$ depends only on the cardinality of $\Lambda$. We have the following.
**Proposition 3.12.** Let \( f \in C(S^X) \) and suppose \( \partial f \in C^1_R(S^X) \) for some \( R > 0 \). Then for any finite connected \( \Lambda, \Lambda' \subset X \) such that \( d_X(\Lambda, \Lambda') > R \), there exists a function \( h_f^{\Lambda, \Lambda'} : \mathcal{M}_{|\Lambda|} \times \mathcal{M}_{|\Lambda'|} \to \mathbb{R} \) such that

\[
\iota^{\Lambda \cup \Lambda'} f(\eta) - \iota^\Lambda f(\eta) - \iota^\Lambda' f(\eta) = h_f^{\Lambda, \Lambda'}(\xi_\Lambda(\eta), \xi_{\Lambda'}(\eta))
\]

for all \( \eta \in S^X \).

**Proof.** Let \( \Lambda, \Lambda' \) be finite connected subsets of \( X \) such that \( d_X(\Lambda, \Lambda') > R \). Noting that \( \iota^W f(\eta) = f(\eta|_W) \) for any \( W \subset X \), it is sufficient to prove the statement for \( \eta \in S^X_{\ast} \) with support in \( \Lambda \cup \Lambda' \). Consider configurations \( \eta, \eta' \in S_X^\ast \) such that Supp(\( \eta \)), Supp(\( \eta' \)) \subset \Lambda \cup \Lambda' \), and satisfying \( \xi_\Lambda(\eta) = \xi_\Lambda(\eta') \) and \( \eta|_{\Lambda'} = \eta'|_{\Lambda'} \). Then by construction, we have \( \xi_X(\eta) = \xi_X(\eta') \).

Since the interaction is irreducibly quantified and the configurations \( \eta \) and \( \eta' \) coincide outside \( \Lambda \), by Lemma 3.11 applied to \( \Lambda \), which is a locale, and \( \tilde{Y} := B(\Lambda, R) \), we have

\[
f(\eta') - f(\eta) = f(\eta'|_{B(\Lambda, R)}) - f(\eta|_{B(\Lambda, R)}).
\]

Since \( \eta' = \eta'|_{\Lambda \cup \Lambda'} \) and \( \eta = \eta|_{\Lambda \cup \Lambda'} \), noting that \( (\Lambda \cup \Lambda') \cap B(\Lambda, R) = \Lambda \), we have

\[
f(\eta'|_{\Lambda \cup \Lambda'}) - f(\eta'|_{\Lambda}) = f(\eta|_{\Lambda}) - f(\eta|_{\Lambda}).
\]

This shows that the function \( \iota^{\Lambda \cup \Lambda'} f(\eta) - \iota^\Lambda f(\eta) \) depends only on \( \xi_\Lambda(\eta) \) and \( \eta|_{\Lambda} \) if \( d_X(\Lambda, \Lambda') > R \). Since \( \iota^\Lambda f(\eta) \) also depends only on \( \eta|_{\Lambda} \), we see that

\[
(19) \quad \iota^{\Lambda \cup \Lambda'} f(\eta) - \iota^\Lambda f(\eta) - \iota^\Lambda' f(\eta)
\]

is a function which depends only on \( \xi_\Lambda(\eta) \) and \( \eta|_{\Lambda} \). Due to symmetry, we can also see that \( \iota^{\Lambda \cup \Lambda'} \) depends only on \( \xi_{\Lambda'}(\eta) \) and \( \eta|_{\Lambda'} \). This implies that \( (19) \) only depends on \( \xi_\Lambda(\eta) \) and \( \xi_{\Lambda'}(\eta) \), hence there exists a function \( h_f^{\Lambda, \Lambda'} : \mathcal{M}_{|\Lambda|} \times \mathcal{M}_{|\Lambda'|} \to \mathbb{R} \) such that

\[
\iota^{\Lambda \cup \Lambda'} f(\eta) - \iota^\Lambda f(\eta) - \iota^\Lambda' f(\eta) = h_f^{\Lambda, \Lambda'}(\xi_\Lambda(\eta), \xi_{\Lambda'}(\eta))
\]

for any \( \eta \in S^X \) as desired. \( \square \)

**Lemma 3.13.** Let \( f \in C(S^X) \) and \( \partial f \in C^1_R(S^X) \) for some \( R > 0 \). For finite connected subsets \( \Lambda, \Lambda', \Lambda'' \) in \( X \), suppose \( \Lambda' \) and \( \Lambda'' \) are in the same connected component of \( X \setminus B(\Lambda, R) \). Then the functions \( h_f^{\Lambda, \Lambda'} \) and \( h_f^{\Lambda, \Lambda''} \) of Proposition 3.12 satisfy

\[
h_f^{\Lambda, \Lambda'}(\alpha, \beta) = h_f^{\Lambda, \Lambda''}(\alpha, \beta)
\]

for any \( \alpha \in \mathcal{M}_{|\Lambda|} \) and \( \beta \in \mathcal{M}_{|\Lambda'| \cap \mathcal{M}_{|\Lambda''|}} \).

**Proof.** Let \( \alpha \in \mathcal{M}_{|\Lambda|} \) and \( \beta \in \mathcal{M}_{|\Lambda'| \cap \mathcal{M}_{|\Lambda''|}} \). Then since \( \Lambda \cap \Lambda' = \Lambda \cap \Lambda'' = \emptyset \), there exists states \( \eta' \) and \( \eta'' \in S^X_{\ast} \) at base state outside \( \Lambda \cup \Lambda' \) and \( \Lambda \cup \Lambda'' \) respectively satisfying \( \xi_\Lambda(\eta') = \xi_\Lambda(\eta'') = \alpha \) and \( \xi_{\Lambda'}(\eta') = \xi_{\Lambda''}(\eta'') = \beta \). Moreover, we may choose \( \eta' \) and \( \eta'' \) so that they coincide on \( \Lambda \). Let \( Y \) be a sublocale of \( X \) containing \( \Lambda' \cup \Lambda'' \) such that \( d_X(Y, \Lambda) > R \). By our choice of \( \eta' \) and \( \eta'' \), we have \( \xi_X(\eta') = \xi_X(\eta'') = \alpha + \beta \). Since the interaction is irreducibly quantified, by Lemma 3.11 applied to \( Y \) and \( \tilde{Y} = B(Y, R) \), we have

\[
(20) \quad f(\eta'') - f(\eta') = f(\eta''|_{B(Y, R)}) - f(\eta'|_{B(Y, R)}).
\]
Since \( \eta' \) and \( \eta'' \) are at base state outside \( \Lambda \cup \Lambda' \) and \( \Lambda \cup \Lambda'' \) and \( \Lambda \cap B(Y, R) = \emptyset \), we have \( \eta' = \eta'|_{\Lambda \cup \Lambda'}, \eta'' = \eta''|_{\Lambda \cup \Lambda''}, \eta'|_{B(Y, R)} = \eta'|_{\Lambda'}, \) and \( \eta''|_{B(Y, R)} = \eta''|_{\Lambda''} \). Noting that \( \iota^W f(\eta) = f(\eta|_W) \) for any \( W \subset X \), equation (20) gives
\[
\iota^{\Lambda \cup \Lambda'} f(\eta'') - \iota^{\Lambda'} f(\eta') = \iota^{\Lambda \cup \Lambda'} f(\eta') - \iota^{\Lambda} f(\eta').
\]
Noting also that \( \iota^\Lambda f(\eta') = \iota^\Lambda f(\eta'') \) since \( \eta' \) and \( \eta'' \) coincide on \( \Lambda \), by the definition of \( h^{\Lambda, \Lambda'}_f \) and \( h^{\Lambda, \Lambda''}_f \), we have
\[
\begin{align*}
 h^{\Lambda, \Lambda'}_f(\alpha, \beta) &:= \iota^{\Lambda \cup \Lambda'} f(\eta') - \iota^\Lambda f(\eta') - \iota^{\Lambda'} f(\eta') \\
 &= \iota^{\Lambda \cup \Lambda'} f(\eta'') - \iota^\Lambda f(\eta'') - \iota^{\Lambda'} f(\eta'') = h^{\Lambda, \Lambda''}_f(\alpha, \beta)
\end{align*}
\]
as desired. \( \square \)

Let \( \mathcal{M} \) be the commutative monoid defined in Definition 2.32. We will next construct a well-defined pairing \( h_f: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R} \). We first consider some conditions on the locale.

**Definition 3.14.** We say that a locale \( X \) is weakly transferable, if for any ball \( B \subset X \), the complement \( X \setminus B \) is a nonempty finite disjoint union of connected infinite graphs. In particular, if \( X \setminus B \) is a connected infinite graph for any ball \( B \subset X \), then we say that \( X \) is strongly transferable.

Immediate from the definition, weakly transferable locales are infinite locales.

**Remark 3.15.** Consider the Euclidean lattice given in Example 2.2(1). Then \( \mathbb{Z}^d = (\mathbb{Z}^d, \mathbb{E}^d) \) for \( d > 1 \) is strongly transferable. The Euclidean lattice \( \mathbb{Z} = (\mathbb{Z}, \mathbb{E}) \) is weakly transferable, but not strongly transferable.

For any \( R > 0 \), consider the set
\[
\mathcal{F}_C R := \{ (\Lambda, \Lambda') \mid \Lambda, \Lambda': \text{ finite nonempty connected } \subset X, d_X(\Lambda, \Lambda') > R \}.
\]
For any \((\Lambda_1, \Lambda'_1), (\Lambda_2, \Lambda'_2) \in \mathcal{F}_C R\), we denote \((\Lambda_1, \Lambda'_1) \leftrightarrow (\Lambda_2, \Lambda'_2)\) if \( \Lambda_1 = \Lambda_2 \) and \( \Lambda'_1 \) and \( \Lambda'_2 \) are in the same connected component of \( X \setminus B(\Lambda_1, R) = X \setminus B(\Lambda_2, R) \), or \( \Lambda'_1 = \Lambda'_2 \) and \( \Lambda_1 \) and \( \Lambda_2 \) are in the same connected component of \( X \setminus B(\Lambda'_1, R) = X \setminus B(\Lambda'_2, R) \). Note that we have \((\Lambda_1, \Lambda'_1) \leftrightarrow (\Lambda_2, \Lambda'_2)\) if and only if \((\Lambda'_1, \Lambda_1) \leftrightarrow (\Lambda'_2, \Lambda_2)\).

**Definition 3.16.** We denote by \( \mathcal{A}_R \) the subset of \( \mathcal{F}_C R \) consisting of pairs \((\Lambda, \Lambda')\) such that at least one of \( \Lambda \) and \( \Lambda' \) are balls.

From now until the end of this subsection, we assume that \( X \) is strongly transferable. In this case, we will prove that the pairing \( h_f^{\Lambda, \Lambda'}: \mathcal{M}_{|\Lambda|} \times \mathcal{M}_{|\Lambda'|} \rightarrow \mathbb{R} \) of Proposition 3.12 associated with \( f \) is independent of the choice of \((\Lambda, \Lambda') \in \mathcal{A}_R\), and defines a well-defined pairing \( h_f: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R} \) satisfying a certain cocycle condition. We will address the weakly transferable case in \( \S 4 \).

**Lemma 3.17.** Suppose \( X \) is strongly transferable. Then for any \((\Lambda_1, \Lambda'_1), (\Lambda_2, \Lambda'_2) \in \mathcal{A}_R\), we have
\[
h_f^{\Lambda_1, \Lambda'_1}(\alpha, \beta) = h_f^{\Lambda_2, \Lambda'_2}(\alpha, \beta)
\]
for any \( \alpha \in \mathcal{M}_{|\Lambda_1|} \cap \mathcal{M}_{|\Lambda_2|} \) and \( \beta \in \mathcal{M}_{|\Lambda'_1|} \cap \mathcal{M}_{|\Lambda'_2|} \).
Proof. We first consider the case when \( \Lambda_1, \Lambda'_1, \Lambda_2, \Lambda'_2 \) are all balls. Note that for a ball \( B \), the \( R \)-thickening \( B(B, R) \) is also a ball. Let \( B \) be a ball such that \(|B| \geq |\Lambda_i|\) and \( d_X(\Lambda_i, B) > R \) for \( i = 1, 2 \). Then since \( X \) is strongly translatable, the sets \( X \setminus B(\Lambda_1, R), X \setminus B(\Lambda_2, R) \) and \( X \setminus B(B, R) \) are all sublocales, hence we have

\[
(\Lambda_1, \Lambda'_1) \leftrightarrow (\Lambda_1, B) \leftrightarrow (\Lambda_2, B) \leftrightarrow (\Lambda_2, \Lambda'_2).
\]

Our assertion follows from Lemma 3.13. Now, consider the case for general \((\Lambda_i, \Lambda'_i) \in \mathcal{A}_R\). By replacing the component which is not a ball with a ball of sufficiently large order in the complement, we see that there exists a pair of balls \((B_i, B'_i) \in \mathcal{A}_R\) such that \(|B_i| \geq |\Lambda_i|, |B'_i| \geq |\Lambda'_i|\) for \( i = 1, 2 \) satisfying \((\Lambda_i, \Lambda'_i) \leftrightarrow (B_i, B'_i)\). Again by Lemma 3.13 we see that \( h_{\Lambda, \Lambda'}^B(\alpha, \beta) = h_{B, B'}^B(\alpha, \beta) \) for any \( \alpha \in \mathcal{M}_{|\Lambda|} \) and \( \beta \in \mathcal{M}_{|\Lambda'|} \). Our assertion now follows from our assertion for balls. \( \square \)

**Proposition 3.18.** For the triple \((X, S, \phi)\), assume that \( X \) is strongly translatable and that the interaction \( \phi \) is irreducibly quantified. Let \( f \in C(S^X) \) be a function such that \( \partial f \in C^1_R(S^X) \) for some constant \( R > 0 \). Then there exists a pairing

\[
h_f: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}
\]

such that for any \((\Lambda, \Lambda') \in \mathcal{A}_R\), we have

\[
\iota^{\Lambda \cup \Lambda'} f(\eta) - \iota^\Lambda f(\eta) - \iota^{\Lambda'} f(\eta) = h_f(\xi_\Lambda(\eta), \xi_{\Lambda'}(\eta))
\]

for any \( \eta \in S^X \). Moreover, the pairing \( h_f \) is symmetric, in other words \( h_f(\alpha, \beta) = h_f(\beta, \alpha) \) for any \( \alpha, \beta \in \mathcal{M} \), and satisfies the cocycle condition

\[
h_f(\alpha, \beta) + h_f(\alpha + \beta, \gamma) = h_f(\beta, \gamma) + h_f(\alpha, \beta + \gamma)
\]

for any \( \alpha, \beta, \gamma \in \mathcal{M} \).

**Proof.** Note that for any \( \alpha, \beta \in \mathcal{M} \), there exists \( k \in \mathbb{N} \) such that \( \alpha, \beta \in \mathcal{M}_k \). By Lemma 3.17 the pairing \( h_{\Lambda, \Lambda'}^f: \mathcal{M}_{|\Lambda|} \times \mathcal{M}_{|\Lambda'|} \rightarrow \mathbb{R} \) of Proposition 3.12 associated with \( f \) is independent of the choice of \((\Lambda, \Lambda') \in \mathcal{A}_R\), hence we have a pairing \( h_f: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R} \). The equality of \((21)\) follows from Proposition 3.12. In addition, for \((\Lambda, \Lambda') \in \mathcal{A}_R\), we have \((\Lambda', \Lambda) \in \mathcal{A}_R\), which by \((21)\) implies that \( h_f \) is symmetric. In order to prove the cocycle condition, fix an arbitrary \( \alpha, \beta, \gamma \in \mathcal{M} \), and let \( k \in \mathbb{N} \) be such that \( \alpha, \beta, \gamma \in \mathcal{M}_k \). Let \( B_1 \) be any ball with \(|B_1| > k \). Since \( X \) is strongly translatable, \( X \setminus B(B_1, R) \) is an locale. Take any ball \( B_2 \) with \(|B_2| > k \) in \( X \setminus B(B_1, R) \). Let \( B \) be a ball sufficiently large containing both \( B_1 \) and \( B_2 \). Then again since \( X \) is strongly translatable, \( X \setminus B(B, R) \) is an locale. Take any ball \( B_3 \) with \(|B_3| > k \) in \( X \setminus B(B, R) \). Then the balls \( B_1, B_2, B_3 \subset X \) satisfy \(|B_i| \geq k \) and \( d_X(B_i, B_j) > R \) for \( i \neq j \). Since \( \alpha, \beta, \gamma \in \mathcal{M}_k \), there exists \( \eta \in S^X \) such that \( \xi^{B_1}(\eta) = \alpha, \xi^{B_2}(\eta) = \beta, \xi^{B_3}(\eta) = \gamma \) and is at base state outside \( B_1 \cup B_2 \cup B_3 \). We let \( \Lambda \subset X \setminus B(B_3, R) \) be a finite connected subset of \( X \) such that \( B_1 \cup B_2 \subset \Lambda \), and let \( \Lambda' \subset X \setminus B(B_1, R) \) be a finite connected subset of \( X \) such that \( B_2 \cup B_3 \subset \Lambda' \). For example, we may take \( \Lambda \) to be the union of \( B_1, B_2 \) and points on a path in \( X \setminus B(B_3, R) \) from the center of \( B_1 \) to the center of \( B_2 \), and similarly for \( \Lambda' \). Note by construction, we have \((B_1, \Lambda'), \)}
\((\Lambda, B_3) \in \mathcal{A}_R\). Then by Lemma 3.13 and the definition of \(h\), we have

\[
\begin{align*}
\quad h_f(\alpha, \beta) &= h_f^{B_1, B_2}(\alpha, \beta) = \iota^{B_1 \cup B_2} f(\eta) - \iota^{B_1} f(\eta) - \iota^{B_2} f(\eta) \\
\quad h_f(\alpha + \beta, \gamma) &= h_f^{\Lambda, B_3}(\alpha, \beta, \gamma) = \iota^{\Lambda \cup B_3} f(\eta) - \iota^\Lambda f(\eta) - \iota^{B_3} f(\eta) \\
\quad h_f(\beta, \gamma) &= h_f^{B_2, B_3}(\alpha, \beta) = \iota^{B_2 \cup B_3} f(\eta) - \iota^{B_2} f(\eta) - \iota^{B_3} f(\eta) \\
\quad h_f(\alpha, \beta + \gamma) &= h_f^{B_1, \Lambda'}(\alpha, \beta, \gamma) = \iota^{B_1 \cup \Lambda'} f(\eta) - \iota^{B_1} f(\eta) - \iota^{\Lambda'} f(\eta).
\end{align*}
\]

Since \(\eta\) is at base state outside \(B_1 \cup B_2 \cup B_3\), we have

\[
\iota^{\Lambda \cup B_3} f(\eta) = \iota^{B_1 \cup B_2 \cup B_3} f(\eta) = \iota^{B_1 \cup B_2} f(\eta) = \iota^{B_1} f(\eta), \quad \iota^{\Lambda} f(\eta) = \iota^{B_2 \cup B_3} f(\eta) = \iota^{B_2} f(\eta), \quad \iota^{\Lambda'} f(\eta) = \iota^{B_1 \cup \Lambda'} f(\eta) = \iota^{B_1} f(\eta),
\]

which proves our assertion. \(\square\)

### 3.4. Criterion for Uniformity

In this subsection, we prove Proposition 3.19, which gives a criterion for a function \(f \in C(S_X^X)\) to be uniform when \(X\) is strongly transferable. The weakly transferable case will be addressed in Proposition 4.14. This result will play an essential role in the proof of Theorem 5.2. As in §3.3, we assume here that the interaction is irreducibly quantified.

**Proposition 3.19.** For the triple \((X, S, \phi)\), assume that \(X\) is strongly transferable and that the interaction \(\phi\) is irreducibly quantified. Let \(f \in C(S_X^X)\) be a function such that \(\partial f \in C^1_R(S^X)\) for some \(R > 0\), and let \(h_f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}\) be the pairing given in Proposition 3.18. If \(h_f \equiv 0\), then we have \(f \in C^0_{\text{unif}}(S^X)\).

We first prove a lemma to characterize the functions in \(C^0_{\text{unif}}(S^X)\), which does not require \(X\) to be strongly transferable. For any ball \(B(x, R)\), we denote by \(B^*(x, R) := B(x, R) \setminus \{x\}\) the punctured ball.

**Lemma 3.20.** Suppose \(f \in C(S_X^X)\). Then \(f \in C^0_{\text{unif}}(S^X)\) if and only if there exists \(R > 0\) such that for any finite \(\Lambda \subset X\) and \(x \in \Lambda\), we have

\[
\iota^\Lambda f - \iota^{\Lambda \setminus \{x\}} f = \iota^{\Lambda \cap B(x, R)} f - \iota^{\Lambda \cap B^*(x, R)} f.
\]

**Proof.** First observe that for any \(f \in C(S_X^X)\), we have

\[
\iota^\Lambda f = \sum_{\Lambda' \subset \Lambda} f_{\Lambda'} = \sum_{\Lambda' \subset \Lambda \setminus \{x\}} f_{\Lambda'} + \sum_{\Lambda' \subset \Lambda, x \in \Lambda'} f_{\Lambda'} = \iota^{\Lambda \setminus \{x\}} f + \sum_{\Lambda' \subset \Lambda, x \in \Lambda'} f_{\Lambda'}
\]

where \(f_{\Lambda}\) is the local function with exact support in the canonical expansion (16).

Suppose \(f \in C^0_{\text{unif}}(S^X)\). Then there exists \(R > 0\) such that \(f_{\Lambda'} \equiv 0\) for any finite \(\Lambda' \subset X\) satisfying \(\text{diam}(\Lambda') > R\). Then for any finite \(\Lambda \subset X\) and \(x \in \Lambda\), we have

\[
\sum_{\Lambda' \subset \Lambda, x \in \Lambda'} f_{\Lambda'} = \sum_{\Lambda' \subset \Lambda \cap B(x, R), x \in \Lambda'} f_{\Lambda'} = \sum_{\Lambda' \subset \Lambda \cap B^*(x, R)} f_{\Lambda'} = \iota^{\Lambda \cap B(x, R)} f - \iota^{\Lambda \cap B^*(x, R)} f.
\]

This proves the necessity of \((\Lambda, B_3) \in \mathcal{A}_R\).
This gives (23). Next, we prove the converse. By the same argument, if (23) holds, then we have
\[ \sum_{\Lambda' \subset \Lambda \cap B(x,R), x \in \Lambda'} f_{\Lambda'} = \sum_{\Lambda' \subset \Lambda \cap B(x,R), x \in \Lambda'} f_{\Lambda'} \]
for any finite \( \Lambda \subset X \) and \( x \in \Lambda \). Suppose there exists finite \( \Lambda \subset X \) such that \( \text{diam}(\Lambda) > R \) and \( f_{\Lambda} \neq 0 \). By iteratively replacing \( \Lambda' \subset \Lambda \) by \( \Lambda \) if necessary, we may assume that for any \( \Lambda' \subset \Lambda \), we have \( f_{\Lambda'} \equiv 0 \) or \( \text{diam}(\Lambda') \leq R \). However, for this \( \Lambda \) and an arbitrary \( x \in \Lambda \), we have
\[ \sum_{\Lambda' \subset \Lambda, x \in \Lambda'} f_{\Lambda'} = f_{\Lambda} + \sum_{\Lambda' \subset \Lambda \cap B(x,R), x \in \Lambda'} f_{\Lambda'} . \]
This contradicts our hypothesis that \( f_{\Lambda} \equiv 0 \), hence our assertion is proved.

Before the proof of Proposition 3.19, we prepare an additional lemma. Here, we will assume that \( X \) is strongly transferable. The weakly transferable case will be treated in §4.

Lemma 3.21. For the triple \((X, S, \phi)\), assume that \( X \) is strongly transferable and that the interaction \( \phi \) is irreducibly quantified. Suppose \( f \in C(S^X) \) and \( \delta f \in C^1_{\mathbb{R}}(S^X) \) for some \( R > 0 \), and let \( h_f : M \times M \rightarrow \mathbb{R} \) be the pairing given in Proposition 3.18. If \( h_f \equiv 0 \), then for any \( x \in X \) and finite \( \Lambda \subset X \) such that \( d_X(x, \Lambda) > R \), we have
\[ t^{\Lambda \cup B(x,R)} f - t^{\Lambda \cup B^+(x,R)} f = t^{B(x,R)} f - t^{B^+(x,R)} f . \]

Proof. Note that by the definition of the pairing \( h_f \) given in (21), we have
\[ t^{\Lambda \cup \Lambda'} f = t^{\Lambda} f + t^{\Lambda'} f \]
for any \((\Lambda, \Lambda') \in \mathcal{A}_R\). If \( \Lambda \) is a ball, then \((\Lambda, B(x,R))\) and \((\Lambda, B^+(x,R))\) are both pairs in \( \mathcal{A}_R \), hence our assertion immediately follows by applying (24) to the left hand side. Consider a general finite \( \Lambda \subset X \) such that \( d_X(x, \Lambda) > R \). It is sufficient to prove that
\[ f(\eta|_{\Lambda \cup B(x,R)}) - f(\eta|_{\Lambda \cup B^+(x,R)}) = f(\eta|_{B(x,R)}) - f(\eta|_{B^+(x,R)}) \]
for any \( \eta \in S^X \) such that \( \text{Supp}(\eta) \subset \Lambda \cup B(x,R) \). Since \( X \) is strongly transferable, \( X \setminus B(x, 2R) \) is a locale. Let \( B \subset X \setminus B(x, 2R) \) be a ball of order greater than that of \( \Lambda \) and satisfying \( d_X(\Lambda, B) > R \). Such a set \( B \) exists since \( X \setminus B(x, 2R) \) is infinite. Now choose a \( \eta' \in S^X \) such that \( \eta' \) coincides with \( \eta \) outside \( \Lambda \cup B \), as is base state on \( \Lambda \), and \( \xi_\Lambda(\eta') = \xi_\Lambda(\eta) \). This implies that \( \xi_X(\eta') = \xi_X(\eta) \). Since \( X \) is strongly transferable, the complement \( Y := X \setminus B(x, R) \) is a locale. From our condition that the interaction is irreducibly quantified, by Lemma 3.11 applied to \( Y = X \setminus B(x, R) \) and \( Y' = X \setminus \{x\} \), we have
\[ f(\eta') - f(\eta) = f(\eta'|_{X \setminus \{x\}}) - f(\eta|_{X \setminus \{x\}}) . \]
Noting that \( \eta' = \eta'|_{B \cup B(x,R)} \) and \( \eta = \eta|_{\Lambda \cup B(x,R)} \), we see that
\[ f(\eta'|_{B \cup B(x,R)}) - f(\eta|_{\Lambda \cup B(x,R)}) = f(\eta'|_{B \cup B^+(x,R)}) - f(\eta|_{\Lambda \cup B^+(x,R)}) . \]
Since \( B \subset X \setminus B(x, 2R) \), we have \( d_X(B, B(x,R)) > R \), hence \((B, B(x,R))\) and \((B, B^+(x,R))\) are both pairs in \( \mathcal{A}_R \). Our condition (24) on \( f \) applied to this pair implies that
\[ f(\eta'|_{B \cup B(x,R)}) = f(\eta'|_B) + f(\eta'|_{B(x,R)}), \quad f(\eta'|_{B \cup B^+(x,R)}) = f(\eta'|_B) + f(\eta'|_{B^+(x,R)}) . \]
Hence (26) gives
\[ f(\eta'|_{B(x,R)}) - f(\eta|_{A\cup B(x,R)}) = f(\eta'|_{B^*(x,R)}) - f(\eta|_{A\cup B^*(x,R)}). \]
Our assertion (25) follows from the fact that \( \eta \) and \( \eta' \) coincide on \( B(x,R) \).
\[ \square \]

We may now prove Proposition 3.19.

**Proof of Proposition 3.19.** Suppose \( f \in C(S^X) \) and \( \partial f \in C^1_R(S^X) \), and that \( h_f \equiv 0 \) for the pairing \( h_f \) of Proposition 3.18. By Lemma 3.20, it is sufficient to check that for any fixed finite subset \( \Lambda \subset X \) and \( x \in \Lambda \), we have
\[ t^\Lambda f - t^\Lambda \{x\} f = t^{\Lambda \cap B(x,R)} f - t^{\Lambda \cap B^*(x,R)} f. \]
We let \( \Lambda' := \Lambda \setminus (\Lambda \cap B(x,R)) \). Then the above equation may be written as
\[ (27) \quad t^{\Lambda' \cup (\Lambda \cap B(x,R))} f - t^{\Lambda' \cup (\Lambda \cap B^*(x,R))} f = t^{\Lambda \cap B(x,R)} f - t^{\Lambda \cap B^*(x,R)} f. \]
In order to prove this, it is sufficient to prove that
\[ (28) \quad t^{\Lambda' \cup B(x,R)} f - t^{\Lambda' \cup B^*(x,R)} f = t^B(x,R) f - t^B^*(x,R) f, \]
since (27) may be obtained by applying \( t^{\Lambda \cup (\Lambda \cap B(x,R))} \) to both sides of (28). Hence it is sufficient to prove that for any finite \( \Lambda \subset X \) such that \( \Lambda \cap B(x,R) = \emptyset \), we have
\[ t^{\Lambda \cup B(x,R)} f - t^{\Lambda \cup B^*(x,R)} f = t^B(x,R) f - t^B^*(x,R) f, \]
which is precisely Lemma 3.21. \[ \square \]

4. Pairing and Criterion in the Weakly Transferable Case

In this section, we will extend Proposition 3.18 concerning the construction of the pairing \( h_f \) and Proposition 3.19 concerning the criterion for uniformity to the case when the locale \( X \) is weakly transferable. The reader interested only in the strongly transferable case may skip to §5.

We first start with the classification of objects in \( \mathcal{B}_R \) which generalizes the set of pairs \( \mathcal{A}_R \) given in Definition 3.16 as follows.

**Definition 4.1.** For any \( r \in \mathbb{N} \), we denote by \( \mathcal{B}_R^r \) the subset of \( \mathcal{F}_R \) consisting of pairs of balls \( (B, B') \) such that the radii of \( B \) and \( B' \) are at least \( r \). Furthermore, we let \( \mathcal{B}_R := \mathcal{B}_R^0 \).

We define the relation \( \sim_r \) to be the equivalence relation in \( \mathcal{B}_R^r \) generated by the relations \( (B, B') \leftrightarrow (B, B'') \) and \( (B', B) \leftrightarrow (B'', B) \). In this subsection, we first study the equivalence relation \( \sim_r \) on \( \mathcal{B}_R^r \). In particular, we have the following.

**Proposition 4.2.** For any \( r \in \mathbb{N} \), there are at most two equivalence classes with respect to the equivalence relation \( \sim_r \) in \( \mathcal{B}_R^r \). Moreover, there is only one equivalence class if \( (B, B') \sim_r (B', B) \) for some \( (B, B') \in \mathcal{B}_R^r \).

The following observation will be used throughout this section.
**Remark 4.3.** Let $X$ be a weakly transferable locale, and let $B \subseteq X$ be a ball.

1. For any finite set $\Lambda \subseteq X$ and $r, R > 0$, there exists a ball $B'$ of radius at least $r$ such that $d_X(\Lambda, B') > R$ and $B' \subseteq (X \setminus B)$.
2. For any finite connected set $\Lambda \subseteq X \setminus B$ and $r, R > 0$, there exists a ball $B'$ of radius at least $r$ such that $d_X(B, B') > R$ and $B'$ and $\Lambda$ are in the same connected component of $X \setminus B$.

In order to prove Proposition 4.2, we first prove Lemma 4.4 and Lemma 4.5.

**Lemma 4.4.** For any $r \in \mathbb{N}$, suppose there exists three balls $B_1, B_2, B_3$ of radii at least $r$ in $X$ such that $d_X(B_i, B_j) > 2R$ for $i \neq j$. Then either $(B_1, B_2) \sim_r (B_1, B_3)$ or $(B_1, B_2) \sim_r (B_3, B_2)$ holds.

**Proof.** Suppose $B_2$ and $B_3$ are not in the same connected component of $X \setminus B(B_1, R)$. Since $X$ is connected, there exists a path $\vec{p} = (e^1, \ldots, e^N)$ such that $o(\vec{p}) \in B_2$ and $t(\vec{p}) \in B_3$. We take the shortest among such paths, and let $p^i = t(e^i)$ for $i = 1, \ldots, N$. Since $B_2$ and $B_3$ are not in the same connected component of $X \setminus B(B_1, R)$, there exists $j$ such that $p^j \in B(B_1, R)$. Since $d_X(B_1, B_2) > 2R$, we have $p^j \not\in B(B_2, R)$, hence $j > R$. Then for any $i \geq j$, we have $p^i \in X \setminus B(B_2, R)$, since we have taken $\vec{p}$ to be the shortest path from a vertex in $B_2$ to a vertex in $B_3$. Then $\vec{p}' = (p^j, p^{j+1}, \ldots, p^N)$ gives a path in $X \setminus B(B_2, R)$ from $p^j$ to a vertex in $B_3$. Since $p^j \in B(B_1, R)$, there exists a path in $X \setminus B(B_2, R)$ from an element in $B_1$ to the vertex $p^j$. The combination of this path with $\vec{p}'$ gives a path in $X \setminus B(B_2, R)$ from a vertex in $B_1$ to a vertex in $B_3$, hence $B_1$ and $B_3$ are in the same connected component of $X \setminus B(B_2, R)$ as desired. \hfill \Box

**Lemma 4.5.** For any $r \in \mathbb{N}$, suppose there exists three balls $B_1, B_2, B_3$ of radii at least $r$ in $X$ such that $d_X(B_i, B_j) > 2R$ for $i \neq j$. Then $(B_1, B_2) \sim_r (B_1, B_3)$ or $(B_1, B_2) \sim_r (B_3, B_1)$ holds.

**Proof.** By Lemma 4.4, at least one of $(B_1, B_2) \sim_r (B_1, B_3)$ or $(B_1, B_2) \sim_r (B_3, B_2)$ holds. Also, by reversing the roles of $B_2$ and $B_3$, we see that at least one of $(B_1, B_3) \sim_r (B_1, B_2)$ or $(B_1, B_3) \sim_r (B_2, B_3)$ holds. Hence if $(B_1, B_2) \sim_r (B_1, B_3)$, then we have $(B_1, B_2) \sim_r (B_3, B_2)$ and $(B_1, B_3) \sim_r (B_2, B_3)$, where the last equivalence implies that $(B_3, B_2) \sim_r (B_3, B_1)$ by symmetry. This implies that $(B_1, B_2) \sim_r (B_3, B_2) \sim_r (B_3, B_1)$ as desired. \hfill \Box

We may now prove Proposition 4.2.

**Proof of Proposition 4.2.** We fix a pair $(B_1, B_1') \in \mathcal{B}^r_R$. We let $B'' \subset X \setminus B(B_1, 2R)$ be a ball of radius at least $r$ which is in the same connected component as $B_1'$ in $X \setminus B(B_1, R)$. Then we have $(B_1, B_1') \sim_r (B_1, B'')$. Consider any $(B_2, B_2') \in \mathcal{B}^r_R$. We let $B$ be a ball sufficiently large containing $B_1 \cup B''$. Then $B(B, 2R)$ is also a ball, and $X \setminus B(B, 2R)$ decomposes into a finite sum of locales. We let $B_3 \subset X \setminus B(B, 2R)$ be a close ball of radius at least $r$ which is in the same connected component as $B_2$ in $X \setminus B(B_3, R)$. Then by definition, we have $(B_2, B_2') \sim_r (B_3, B_3')$. Finally, we let $B'$ be the ball sufficiently large containing $B_1 \cup B'' \cup B_3$, and we let $B_3' \subset X \setminus B(B', 2R)$ be a close ball of radius at least $r$ which is in the same connected component of $X \setminus B(B_3, R)$ as $B_3'$. Then by construction, we have $(B_2, B_3') \sim_r (B_3, B_3') \sim_r (B_3, B_3'')$. By our construction of $B''$ and $B_3'$ and Lemma 4.5, either $(B_2, B'') \sim_r (B_1, B'')$ or $(B_1, B'') \sim_r (B_3', B_1)$ holds. Furthermore, either $(B_2, B_3') \sim_r (B_3, B_3')$ or $(B_3', B_1) \sim_r (B_3', B_3)$. Combining the two, noting that $(B_2, B_3') \sim_r (B_3, B_3')$, we obtain our assertion. \hfill \Box
As a consequence of Proposition 4.2, we have the following.

**Proposition 4.6.** Let \( X = (X, E) \) be a weakly transferable locale, and let \( R > 0 \). For any integer \( r \in \mathbb{N} \), we choose an equivalence \( \mathcal{C}_R^r \) in \( \mathcal{B}_R \) with respect to the equivalence relation \( \sim_r \). Then one of the following holds.

1. For any \( r \in \mathbb{N} \), we have \( \mathcal{B}_R^r = \mathcal{C}_R^r \).
2. There exists \( r_0 \in \mathbb{N} \) such that for any \( r < r_0 \), we have \( \mathcal{B}_R^r = \mathcal{C}_R^r \), and for \( r \geq r_0 \), we have

\[
\mathcal{B}_R^r = \mathcal{C}_R^r \cup \overline{\mathcal{C}_R^r},
\]

where \( \overline{\mathcal{C}_R^r} := \{(\Lambda', \Lambda) \mid (\Lambda, \Lambda') \in \mathcal{C}_R^r\} \).

Moreover, we may choose \( \mathcal{C}_R^r \) so that \( \mathcal{C}_R^r = \mathcal{C}_R^{r_0} \cap \mathcal{B}_R^r \) for any \( r \geq r_0 \), where we let \( r_0 = 0 \) when (1) holds.

**Proof.** Suppose (1) does not hold. Then by Proposition 4.2, there exists \( r \in \mathbb{N} \) such that for any \((B, B') \in \mathcal{B}_R^r\), we have \((B, B') \not\sim_r (B', B)\). Then for any \( r' \geq r \) and \((B, B') \in \mathcal{B}_R^{r'}\), if \((B, B') \sim_{r'} (B', B)\) then this would imply that \((B, B') \sim_r (B', B)\). Hence our assumption implies that \((B, B') \not\sim_r (B', B)\). We take \( r_0 \) to be the minimum of such \( r \). Then (29) follows from Proposition 4.2.

Note that for any \( r \geq r_0 \) and \((B_1, B'_1)\) and \((B_2, B'_2)\) in \( \mathcal{B}_R^r \), we have \((B_1, B'_1) \sim_{r_0} (B_2, B'_2)\) if and only if \((B_1, B'_1) \sim_r (B_2, B'_2)\). This may be proved as follows. It is immediate from the definition that if \( \sim_r \) holds, then \( \sim_{r_0} \) holds. On the other hand, if \((B_1, B'_1) \sim_{r_0} (B_2, B'_2)\) and if \((B_1, B'_1) \not\sim_r (B_2, B'_2)\), then we would have \((B_1, B'_1) \sim_r (B'_2, B_2)\), which would imply that \((B_1, B'_1) \sim_{r_0} (B'_2, B_2)\). Then we would have \((B_2, B'_2) \sim_{r_0} (B'_2, B_2)\), contradicting our choice of \( r_0 \).

If we choose an equivalence class \( \mathcal{C}_R^{r_0} \) of \( \mathcal{B}_R^0 \) with respect to the equivalence \( \sim_{r_0} \), then we may take \( \mathcal{C}_R := \mathcal{C}_R^{r_0} \cap \mathcal{B}_R^r \) for any \( r \geq r_0 \), which gives an equivalence class of \( \mathcal{B}_R^r \) with respect to the equivalence \( \sim_r \), satisfying the condition of our assertion. \( \square \)

**Definition 4.7.** Let \( X = (X, E) \) be a weakly transferable locale, and let \( R > 0 \). If (1) of Proposition 4.6 holds, then we say that \( \mathcal{B}_R \) has a unique class, and we define the integer \( r_0 \) to be one. If (2) of Proposition 4.6 holds, then we say that \( \mathcal{B}_R \) is split, and we define \( r_0 \) to be the minimum integer satisfying (29). Moreover, we will fix an equivalence class \( \mathcal{C}_R^r \) of \( \mathcal{B}_R^r \) with respect to the equivalence relation \( \sim_r \) so that \( \mathcal{C}_R^r = \mathcal{C}_R^{r_0} \cap \mathcal{B}_R^r \) for \( r \geq r_0 \).

4.2. **Pairing for Functions in the Weakly Transferable Case.** In this subsection, we will construct and prove the cocycle condition for the pairing \( h_f : M \times M \to \mathbb{R} \). We let \( X \) be a weakly transferable locale. We let \( \mathcal{B}_R \) as in Definition 4.1 and we fix an equivalence class \( \mathcal{C}_R^r \) of \( \mathcal{B}_R^r \) with respect to the equivalence relation \( \sim_r \) as in Definition 4.7. We define the pairing \( h_f \) as follows.

**Definition 4.8.** Let \( f \in C(S_X^k) \) such that \( \partial f \in C^l(S_X^k) \) for some \( R > 0 \). We define the pairing \( h_f : M \times M \to \mathbb{R} \) as follows. For any \( \alpha, \beta \in M \), we let \( k \geq r_0 \) such that \( \alpha, \beta \in M_k \). By taking an arbitrary \((B, B') \in \mathcal{C}_R^k\), we let

\[
h_f(\alpha, \beta) := h_f^{B, B'}(\alpha, \beta),
\]
where \( h_f^{B, B'} \) is the pairing \( h_f^{\Lambda, \Lambda'} \) defined in Proposition 3.12 for \((\Lambda, \Lambda') = (B, B')\). Note that we have \(|B| \geq r(B)\) and \(|B'| \geq r(B')\), hence \(|B|, |B'| \geq k\).

For the remainder of this subsection, we let \( R > 0 \), and we fix a \( f \in C(S^X)\) such that \( \partial f \in C^1(S^X) \). If \((B_1, B_1'), (B_2, B_2') \in \mathcal{C}_R^k\), then we have \((B_1, B_1') \sim_k (B_2, B_2')\), hence by Lemma 3.13 we see that

\[
h_f^{B_1, B_1'}(\alpha, \beta) = h_f^{B_2, B_2'}(\alpha, \beta)
\]

for any \( \alpha, \beta \in \mathcal{M}_k \). Since \( \mathcal{C}_R^k = \mathcal{C}_R^{k_0} \cap \mathcal{B}_R^k \), this shows that \( h_f \) is independent of the choices of \( k \geq r_0 \) satisfying \( \alpha, \beta \in \mathcal{M}_k \) and the pair \((B, B') \in \mathcal{C}_R^k\).

When \( X \) is weakly transferable, the pairing \( h_f \) may not be symmetric.

**Lemma 4.9.** If \( \mathcal{B}_R \) has a unique class, then the pairing \( h_f \) is symmetric.

**Proof.** If \( \mathcal{B}_R \) has a unique class, then \((B, B') \in \mathcal{C}_R^k\) implies that \((B', B) \in \mathcal{C}_R^k\). This shows that for any \( k \geq r_0 \) and \( \alpha, \beta \in \mathcal{M}_k \), we have

\[
h_f(\alpha, \beta) = h_f^{B, B'}(\alpha, \beta) = h_f^{B', B}(\beta, \alpha) = h_f(\beta, \alpha)
\]

as desired. \( \square \)

**Remark 4.10.** If \( \mathcal{B}_R \) is split, then \( h_f \) may not necessarily be symmetric.

The pairing \( h_f \) may be calculated as follows.

**Proposition 4.11.** For any \((\Lambda, \Lambda') \in \mathcal{A}_R\) and for any \( \alpha \in \mathcal{M}_{|\Lambda|} \) and \( \beta \in \mathcal{M}_{|\Lambda'|} \), we have

\[
h_f(\alpha, \beta) = h_f^{\Lambda, \Lambda'}(\alpha, \beta) \quad \text{or} \quad h_f(\beta, \alpha) = h_f^{\Lambda, \Lambda'}(\alpha, \beta).
\]

In particular, if \( h_f \) is symmetric, then we have

\[
h_f(\alpha, \beta) = h_f^{\Lambda, \Lambda'}(\alpha, \beta).
\]

**Proof.** Fix an arbitrary \((\Lambda, \Lambda') \in \mathcal{A}_R\) and \( \alpha \in \mathcal{M}_{|\Lambda|}, \beta \in \mathcal{M}_{|\Lambda'|} \). Let \( k \geq r_0 \) such that \( k \geq \max\{|\Lambda|, |\Lambda'|\} \). Assume without loss of generality that \( \Lambda \) is a ball. Since \( X \) is weakly transferable, the connected component of \( X \setminus B(\Lambda, R) \) containing \( \Lambda' \) is infinite, hence contains a closed ball \( B' \) such that \( r(B') \geq k \). Then \((\Lambda, \Lambda') \leftrightarrow (\Lambda, B')\). Again, the connected component of \( X \setminus B(B', R) \) containing \( \Lambda \) is infinite, hence contains a ball \( B \) such that \( r(B) \geq k \). Then \((\Lambda, B') \leftrightarrow (B, B')\). By construction, \((B, B') \in \mathcal{B}_R^k\). If \((B, B') \in \mathcal{B}_R^k\), then by Lemma 3.13 noting that \(|B|, |B'| \geq k\), we have

\[
h_f^{\Lambda, \Lambda'}(\alpha, \beta) = h_f^{B, B'}(\alpha, \beta) = h_f(\alpha, \beta).
\]

If \((B, B') \notin \mathcal{B}_R^k\), then \((B', B) \in \mathcal{C}_R^k\), hence we have

\[
h_f^{\Lambda, \Lambda'}(\alpha, \beta) = h_f^{B, B'}(\alpha, \beta) = h_f^{B', B}(\beta, \alpha) = h_f(\beta, \alpha)
\]

as desired. \( \square \)
In order to prove the cocycle condition of $h_f$, we will prepare the following lemma concerning the existence of a triple of balls $B_1, B_2, B_3$ sufficiently large and sufficiently apart satisfying the condition $(B_1, B_2) \leftrightarrow (B_1, B_3) \leftrightarrow (B_2, B_3)$. Note that if $X$ is strongly transferable, then this condition is automatically satisfied if $d_X(B_i, B_j) > R$ for $i \neq j$.

**Lemma 4.12.** For any $r \in \mathbb{N}$, there exists balls $B_1, B_2, B_3$ of radii at least $r$ in $X$ such that $d_X(B_i, B_j) > R$ for $i \neq j$ and

$$ (30) \quad (B_1, B_2) \leftrightarrow (B_1, B_3) \leftrightarrow (B_2, B_3). $$

Moreover, we may take $B_1, B_2, B_3$ so that $(B_1, B_2), (B_1, B_3), (B_2, B_3) \in \mathcal{C}_R^r$.

**Proof.** Take $B_1$ to be a ball in $X$ such that $r(B_1) \geq r$. We then take $B_2$ to be an arbitrary ball in $X \setminus B(B_1, 2R)$ such that $r(B_2) \geq r$. Finally, we take a ball $\Lambda$ containing $B_1$ and $B_2$, and let $B_3$ be a ball in $X \setminus B(\Lambda, 2R)$ with $r(B_3) \geq r$. By construction, we have $d_X(B_i, B_j) > 2R$ for $i \neq j$. By Lemma 4.5, the following holds.

1. Either $(B_1, B_2) \leftrightarrow (B_1, B_3)$ or $(B_1, B_2) \leftrightarrow (B_3, B_2)$.
2. Either $(B_1, B_3) \leftrightarrow (B_1, B_2)$ or $(B_1, B_3) \leftrightarrow (B_2, B_3)$.
3. Either $(B_2, B_3) \leftrightarrow (B_2, B_1)$ or $(B_2, B_3) \leftrightarrow (B_1, B_3)$.

From this observation, we see that at least one of the following holds.

(a) $(B_1, B_2) \leftrightarrow (B_1, B_3)$ and $(B_1, B_3) \leftrightarrow (B_2, B_3)$.
(b) $(B_1, B_2) \leftrightarrow (B_3, B_2)$ and $(B_1, B_3) \leftrightarrow (B_2, B_3)$.
(c) $(B_1, B_3) \leftrightarrow (B_1, B_2)$ and $(B_2, B_3) \leftrightarrow (B_2, B_1)$.

In fact, if (a) does not hold, then either $(B_1, B_2) \leftrightarrow (B_1, B_3)$ or $(B_1, B_3) \leftrightarrow (B_2, B_3)$ holds. If $(B_1, B_2) \leftrightarrow (B_1, B_3)$, then by (1), we have $(B_1, B_2) \leftrightarrow (B_3, B_2)$ and by (2), we have $(B_1, B_3) \leftrightarrow (B_2, B_3)$, hence (b) holds. If $(B_1, B_3) \leftrightarrow (B_2, B_3)$, then by (2), we have $(B_1, B_2) \leftrightarrow (B_3, B_2)$ and by (3), we have $(B_2, B_3) \leftrightarrow (B_2, B_1)$, hence (c) holds.

If (a) holds, then we have (30). If (b) holds, then we obtain (30) by taking $B_1$ to be $B_2$ and $B_2$ to be $B_1$. If (c) holds, then we obtain (30) by taking $B_2$ to be $B_3$ and $B_3$ to be $B_2$. Our last assertion is proved if $(B_1, B_2) \in \mathcal{C}_R^r$. If $(B_1, B_2) \notin \mathcal{C}_R^r$, our assertion is proved by taking $B_1$ to be $B_3$ and $B_3$ to be $B_1$. \qed

We are now ready to prove the cocycle condition for our $h_f$.

**Proposition 4.13.** Assume that $X$ is a weakly transferable locale. Let $R > 0$ and let $f \in C(S^X_\ast)$ such that $\partial f \in C^1_R(S^X)$. Then the pairing $h_f : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ satisfies

$$ h_f(\alpha, \beta) + h_f(\alpha + \beta, \gamma) = h_f(\beta, \gamma) + h_f(\alpha, \beta + \gamma). $$

for any $\alpha, \beta, \gamma \in \mathcal{M}$.

**Proof.** For an arbitrary $\alpha, \beta, \gamma \in \mathcal{M}$, there exists $k \geq r_0$ such that $\alpha, \beta, \gamma, \in \mathcal{M}_k$. Let $B_1, B_2, B_3$ be balls in $X$ satisfying the condition of Lemma 4.12 for $r = k$. Then since $\alpha, \beta, \gamma \in \mathcal{M}_k$, there exists $\eta \in S^X$ such that $\xi^{B_1}(\eta) = \alpha, \xi^{B_2}(\eta) = \beta, \xi^{B_3}(\eta) = \gamma$ and is at base state outside $B_1 \cup B_2 \cup B_3$. Then since $(B_1, B_3) \leftrightarrow (B_2, B_3)$, there exists a finite connected subset $\Lambda \subset X \setminus B(B_3, R)$ such that $B_1 \cup B_2 \subset \Lambda$, and since $(B_1, B_2) \leftrightarrow (B_1, B_3)$, there exists a finite connected subset $\Lambda' \subset X \setminus B(B_1, R)$ such that $B_2 \cup B_3 \subset \Lambda'$. Note by construction, we have
(B_1, \Lambda'), (\Lambda, B_3) \in \mathcal{A}_R$, and $(B_1, B_2), (B_2, B_3) \in \mathcal{C}_R^k$. Then by Lemma 3.13 and the definition of $h$, we have
\[ h_f(\alpha, \beta) = h_f^{B_1, B_2}(\alpha, \beta) = t^{B_1 \cup B_2} f(\eta) - t^{B_1} f(\eta) - t^{B_2} f(\eta) \]
\[ h_f(\alpha + \beta, \gamma) = h_f^{\Lambda, B_3}(\alpha + \beta, \gamma) = t^{\Lambda \cup B_3} f(\eta) - t^{\Lambda} f(\eta) - t^{B_3} f(\eta) \]
\[ h_f(\beta, \gamma) = h_f^{B_2, B_3}(\alpha, \beta) = t^{B_2 \cup B_3} f(\eta) - t^{B_2} f(\eta) - t^{B_3} f(\eta) \]
\[ h_f(\alpha, \beta + \gamma) = h_f^{B_1, B_3}(\alpha, \beta + \gamma) = t^{B_1 \cup \Lambda'} f(\eta) - t^{B_1} f(\eta) - t^{\Lambda'} f(\eta). \]

Since $\eta$ is at base state outside $B_1 \cup B_2 \cup B_3$, we have
\[ t^{\Lambda \cup B_3} f(\eta) = t^{B_1 \cup \Lambda'} f(\eta) = t^{B_1 \cup B_2 \cup B_3} f(\eta), \quad t^{\Lambda} f(\eta) = t^{B_1 \cup B_2} f(\eta), \quad t^{\Lambda'} f(\eta) = t^{B_2 \cup B_3} f(\eta), \]
which proves our assertion. \hfill \Box

4.3. **Criterion for Uniformity in the Weakly Transferable Case.** In this subsection, we will prove Proposition 4.14, which is a weakly transferable version of Proposition 3.19.

**Proposition 4.14.** Assume that the locale $X$ is weakly transferable. Suppose $f \in C(X)$ satisfies $\partial f \in C^1_R(S_X)$ for $R > 0$, and let $h_f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the pairing defined in Definition 4.8. If $h_f \equiv 0$, then we have $f \in C^0_{\text{unif}}(S_X)$.

The proof of Proposition 4.14 is the same as that of Proposition 3.19 but by using Lemma 4.15 below which is valid even when $X$ is weakly transferable, instead of Lemma 3.21, which assumed that $X$ is strongly transferable.

**Lemma 4.15.** Assume that $X$ is weakly transferable. Suppose $f \in C(S_X)$ and $\partial f \in C^1_R(S_X)$, and let $h_f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the pairing defined in Definition 4.8. If $h_f \equiv 0$, then for any $x \in X$ and finite $\Lambda \subset X$ such that $d_X(x, \Lambda) > R$, we have
\[ t^{\Lambda \cup B(x, R)} f - t^{\Lambda \cup B^*(x, R)} f = t^{B(x, R)} f - t^{B^*(x, R)} f, \]
where $B^*(x, R) := B(x, R) \setminus \{x\}$ is the punctured ball.

**Proof.** Since $h_f \equiv 0$, by the definition of the pairing $h_f$ given in Definition 4.8 and Proposition 3.12, we have
\[ t^{\Lambda \cup \Lambda'} f = t^{\Lambda} f + t^{\Lambda'} f \]
for any $(\Lambda, \Lambda') \in \mathcal{A}_R$. Again, if $B$ is a ball, then $(B, B(x, R))$ and $(B, B^*(x, R))$ are both pairs in $\mathcal{A}_R$, hence our assertion immediately follows by applying (31) to the left hand side. Consider a general finite $\Lambda \subset X$ such that $d_X(x, \Lambda) > R$. It is sufficient to prove that
\[ f(\eta|_{\Lambda \cup B(x, R)}) - f(\eta|_{\Lambda \cup B^*(x, R)}) = f(\eta|_{B(x, R)}) - f(\eta|_{B^*(x, R)}) \]
for any $\eta \in S_X$ such that $\text{Supp}(\eta) \subset \Lambda \cup B(x, R)$. Since $X$ is weakly transferable, $Y := X \setminus B(x, R)$ decomposes into a finite number of locales $Y_i = Y_1 \cup \cdots \cup Y_K$. We may reorder the $Y_i$ so that there exists integer $k$ such that $Y_i \cap \Lambda \neq \emptyset$ for $i \leq k$ and $Y_i \cap \Lambda = \emptyset$ for $i > k$. Let $\Lambda_i := \Lambda \cap Y_i$. Then we have $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$.

Let $B'_i$ be a ball such that $B(\Lambda_i, R) \cup B(x, R) \subset B'_i$. Let $B_1 \subset (X \setminus B(B'_1, R)) \cap Y_1$, $d_X(\Lambda, B_1) > R$ be a ball of order greater than that of $\Lambda_1$. Such a set $B_1$ exists since $Y_1$ is an infinite set. Now
choose a $\eta' \in S^X$ such that $\eta'$ coincides with $\eta$ outside $\Lambda_1 \cup B_1$, is as base state on $\Lambda_1$, and 
$\xi_{B_1}(\eta') = \xi_{\Lambda_1}(\eta)$. This implies that $\xi_X(\eta') = \xi_X(\eta)$. Since the transition is irreducibly quantified, by Lemma 3.11 applied to $Y_1$ and $Y_1 := X \setminus \{x\}$, we have

$$f(\eta') - f(\eta) = f(\eta'|X\setminus\{x\}) - f(\eta|X\setminus\{x\}).$$

Noting that $\eta' = \eta'|_{B_1 \cup B_1'}$ and $\eta = \eta|_{B_1'}$, we see that

$$(33) \quad f(\eta'|_{B_1 \cup B_1'}) - f(\eta|_{B_1'}) = f(\eta'|_{B_1 \cup (B_1')\setminus\{x\}}) - f(\eta|_{B_1\setminus\{x\}}).$$

By our choice of $B_1$, we have $d_X(B_1, B_1') > R$, hence $(B_1, B_1')$ and $(B_1, B_1' \setminus \{x\})$ are both pairs in $\not A. R$. Our condition (31) on $f$ applied to this pair implies that

$$f(\eta'|_{B_1 \cup B_1'}) = f(\eta'|_{B_1}) + f(\eta'|_{B_1'}), \quad f(\eta'|_{B_1 \cup (B_1')\setminus\{x\}}) = f(\eta|_{B_1}) + f(\eta'|_{B_1\setminus\{x\}}).$$

Hence (33) gives $f(\eta'|_{B_1}) = f(\eta|_{B_1})$. In particular, since $\eta'$ is at base state outside $B_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)$ and $\eta$ is at base state outside $\Lambda \cup B(x, R)$, we have

$$f(\eta'|_{\Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)}) = f(\eta|_{\Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)}).$$

Note that $\eta'|_{\Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)} = \eta|_{\Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)}$ since $\eta'$ and $\eta$ coincides outside $\Lambda_1 \cup B_1''$. Hence we see that

$$f(\eta|_{\Lambda \cup B(x, R)}) = f(\eta|_{\Lambda \cup B(x, R)}.R) = f(\eta|_{\Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)}).$$

By applying the same argument with $\Lambda$ replaced by $\Lambda_2 \cup \cdots \cup \Lambda_k$ and $\Lambda_1$ replaced by $\Lambda_2$, we have

$$f(\eta|_{\Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)} = f(\eta|_{\Lambda_2 \cup \cdots \cup \Lambda_k \cup B(x, R)}).$$

By repeating this process, we see that

$$f(\eta|_{\Lambda \cup B(x, R)}) = f(\eta|_{\Lambda \cup B(x, R)}).$$

Our assertion (32) follows from the fact that $\eta$ is supported on $\Lambda \cup B(x, R)$ and the fact that $\iota^W f(\eta) = f(\eta|_W)$ for any $W \subset X$. 

4.4. Transferability. In this subsection, we will introduce the notion of transferability for a locale $X$, which ensures that the pairing $h_f$ defined in Definition 4.8 is symmetric.

**Definition 4.16.** Let $X$ be a locale. We say that $X$ is *transferable*, if $X$ is weakly transferable and satisfies either one of the following conditions.

(a) There exists a ball $B$ such that $X \setminus B$ has three or more disjoint connected components (which are all infinite since $X$ is weakly transferable).

(b) For any $r \in \mathbb{N}$, there exists a ball $B$ of radius at least $r$ such that $X \setminus B$ is connected.

Note that by (b), if $X$ is strongly transferable, then it is transferable.

**Example 4.17.** The following subgraphs $X = (X, E)$ of the Euclidean lattice $\mathbb{Z}^2 = (\mathbb{Z}^2, \mathbb{E})$ give examples of locales which are transferable but not strongly transferable.

1. Let $X := \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 x_2 = 0\}$ and $E := (X \times X) \cap \mathbb{E}$. Then $(X, E)$ is transferable by Definition 4.16(a).
(2) Let \( X := \{ (x_1, x_2) \in \mathbb{Z}^2 \mid x_1 \geq 0 \} \cup \{ (x_1, x_2) \in \mathbb{Z}^2 \mid x_2 = 0 \} \) and \( E := (X \times X) \cap \mathbb{E} \). Then \((X, E)\) is transferable by Definition 4.16(b).

![Figure 6. Examples of Transferable but not Strongly Transferable Locales](image)

**Remark 4.18.** The locale \( \mathbb{Z} = (\mathbb{Z}, \mathbb{E}) \) is weakly transferable but not transferable, since the complements of balls always have exactly two connected components. The locale \( \mathbb{Z}^d = (\mathbb{Z}^d, \mathbb{E}^d) \) for \( d > 1 \) is transferable, since it is strongly transferable.

**Remark 4.19.** Let \( G \) be a finitely generated free group \( G \) with set of generators \( S \), and let \((G, E_S)\) be the Cayley graph given in Example 2.2(4). If the number of free generators of \( G \) is equal to one, then \( G = \mathbb{Z} \), hence \((G, E_S)\) is weakly transferable but not transferable. If the number of free generators of \( G \) is \( d > 1 \), then \((G, E_S)\) is transferable. This follows from Definition 4.16(a) and the fact that \( G \setminus \{ \text{id}_G \} \) has exactly \( 2d \) connected components (see Figure 1 for the case \( d = 2 \)).

More generally, for a finitely generated infinite group \( G \), it is known that the number of ends of a Cayley graph \((G, E_S)\) is either 1, 2 or infinity, and the number of ends is 2 if and only if it has an infinite cyclic subgroup of finite index (see [4] (1.6 Theorem)). In the case that \((G, E_S)\) is weakly transferable, then it is strongly transferable, not transferable, or transferable if and only if the number of ends is respectively 1, 2, or infinity.

**Proposition 4.20.** Suppose \( X \) is weakly transferable. Then \( X \) is transferable if and only if \( \mathcal{B}_R \) has a unique equivalence class for any \( R > 0 \) in the sense of Definition 4.7.

**Proof.** Suppose (a) of Definition 4.16 holds for the ball \( B \). Let \( Y_1, Y_2, Y_3 \) be three distinct connected components (there may be more) of \( X \setminus B \). For any \( r \in \mathbb{N} \), there exists balls \( B_1, B_2, B_3 \) of radii at least \( r \) such that \( B(B_i, R) \subset Y_i \) for \( i = 1, 2, 3 \) and \( d_X(B_i, B_j) > R \) for any \( i \neq j \). In particular, by exchanging the numberings \( i = 1, 2 \) if necessary, we may assume that \((B_1, B_2) \in \mathcal{C}_R \). By construction, there exists paths from \( B \) to \( Y_1 \) and \( B \) to \( Y_3 \) which do not pass through \( Y_2 \), hence \( Y_1 \) and \( Y_3 \) are in the same connected component of \( X \setminus Y_2 \). Hence in particular, \( B_1 \) and \( B_3 \) are in the same connected component of \( X \setminus B(B_2, R) \), and we have \((B_1, B_2) \leftrightarrow (B_3, B_2)\). In the same manner, we may prove that \((B_1, B_3) \leftrightarrow (B_1, B_2)\), and \((B_1, B_3) \leftrightarrow (B_2, B_3)\). This shows that we have \((B_1, B_2) \sim_r (B_3, B_2) \sim_r (B_3, B_1) \sim_r (B_2, B_1)\). By Proposition 4.2, we see that \( \mathcal{B}_R \) has a unique equivalence class with respect to the equivalence relation \( \sim_r \).

Next, suppose (b) of Definition 4.16 holds. Fix a \( r \in \mathbb{N} \) and \( R > 0 \), and let \( B(x, R') \) be the ball given by (2) of radius \( R' \geq k + R \) such that \( X \setminus B(x, R') \) is connected. If we let \( B_1 := B(x, R' - R)\),
then $B_1$ is a ball of radius at least $r$ such that $X \setminus B(B_1, R)$ is connected. Take a ball $B_2$ of radius at least $r$ and $d_X(B_1, B_2) > R$, and a finite connected $\Lambda$ such that $B_1 \cup B_2 \subset \Lambda$. Since $X$ is weakly transferable, $X \setminus B(B_2, R)$ decomposes into a finite number of locales. We let $Y$ be the connected component of $X \setminus B(B_2, R)$ containing $B_1$, and we take $B_3$ to be a ball of radius at least $r$ satisfying $B_3 \subset Y \setminus B(B_1, R)$ and $d_X(\Lambda, B_3) > R$. Since $X \setminus B(B_1, R)$ is connected by our construction of $B_1$, we have $(B_1, B_2) \leftrightarrow (B_1, B_3)$. Also since $B_1, B_3 \subset Y$, we have $(B_2, B_1) \leftrightarrow (B_2, B_3)$. Moreover, since $B(B_3, R) \cap \Lambda = \emptyset$ and $\Lambda$ is connected, we have $(B_1, B_3) \leftrightarrow (B_2, B_3)$. This shows that $(B_1, B_2) \sim_r (B_1, B_3) \sim_r (B_2, B_3) \sim_r (B_2, B_1)$, which by Proposition 4.2 shows that $\mathcal{B}_R$ has a unique equivalence class with respect to the equivalence relation $\sim_r$.

Finally, we prove that if there exists $r \in \mathbb{N}$ such that for any ball $B$ of radius at least $r$, the set $X \setminus B$ has exactly two connected components, then for any $R > 0$, there are two equivalence classes in $\mathcal{B}_R$ with respect to $\sim_r$. We first prove that in this situation, for any ball $B$ of radius at least $r$, there exists a decomposition $X \setminus B = Y_B^+ \cup Y_B^-$ into infinite locales, where $Y_B^+ \cap Y_B^-$, are infinite sets and $Y_B^+ \cap Y_B^-$ are finite sets for any balls $B, B'$ of radius at least $r$. This follows from the fact that one of $Y_B^+ \cap Y_B^-$ or $Y_B^+ \cap Y_B^-$ must be infinite since $Y_B^+$ is infinite. If both are infinite, then there exists a ball $B$ satisfying $B \cup B' \subset B$ and $X \setminus B$ has at least three disjoint connected components in $Y_B^+ \cap Y_B^-$, $Y_B^+ \cap Y_B^-$ and $Y_B^-$, contradicting our assertion. Hence exactly one of $Y_B^+ \cap Y_B^-$ and $Y_B^+ \cap Y_B^-$ is infinite and the other is finite. So we choose the $\pm$ label of $B'$ so that $Y_B^+ \cap Y_B^-$ is infinite and $Y_B^+ \cap Y_B^-$ is finite.

Consider any $R > 0$. For each $(B, B') \in \mathcal{B}_R$, we denote $B < B'$ if $B' \subset Y_B^+$. This is equivalent to the condition that $B \subset Y_B^-$, for the following reason. Consider $x \in Y_B^- \cap Y_B^-$ and $y \in B' \subset Y_B^+$.

Since $X$ is connected, there exists a path $p$ from $x$ to $y$, which must go through $B$ since $Y_B^-$ and $Y_B^+$ are disjoint and connected by $B$. This shows that $x$ hence $Y_B^-$ is connected to $B$ in $X \setminus B'$. Hence for any $(B, B') \in \mathcal{B}_R$, either one of $B < B'$ or $B > B'$ hold. Moreover, if $B < B'$, then $B' \subset Y_B^+$ and $B \subset Y_B^-$.

This shows that for any $(B_1, B_1'), (B_2, B_2') \in \mathcal{B}_R$, we have $(B_1, B_1') \leftrightarrow (B_2, B_2')$ if and only if we have $B_1 < B_1'$ and $B_2 < B_2'$, or $B_1 > B_1'$ and $B_2 > B_2'$. In other words, pairs are in the same equivalence class if and only if their order is preserved. This proves that $\mathcal{B}_R$ has exactly two equivalence classes as desired.

Corollary 4.21. If $X$ is transferable, then for any $f \in C(S^X)$ such that $\partial f \in C^1_{\text{unif}}(S^X)$, the pairing $h_f$ defined in Definition 4.8 is symmetric.

Proof. By Proposition 4.20, if $X$ is transferable, then $\mathcal{B}_R$ has a unique class for any $R > 0$ in the sense of Definition 4.7. Our assertion now follows from Lemma 4.9.

5. Uniform Cohomology and the Main Theorem

In this section, we will introduce and calculate the uniform cohomology of a configuration space with transition structure satisfying certain conditions. We will then consider a free action of a group $G$ on the locale, and prove our main theorem, Theorem 5.17, which coincides with Theorem 1 of §1.3.

5.1. Uniform Cohomology of the Configuration Space. In this subsection, we will define the uniform cohomology of a configuration space with transition structure. We will then prove our
key theorem, Theorem 5.2, which states that under certain conditions, any closed uniform form is integrable by a uniform function. This result is central to the proof of our main theorem.

Consider the triple \((S, X, \phi)\) and the associated configuration space with transition structure. In what follows, let \(Z^1(S^X)\) be the set of closed forms of \(S^X\), and \(Z^1_{\text{unif}}(S^X) := C^1_{\text{unif}}(S^X) \cap Z^1(S^X)\) the set of uniform closed forms. We first prove the following lemma concerning monoids.

**Lemma 5.1.** Let \(f \in C^0_{\text{unif}}(S^X)\). Then we have \(\partial f \in Z^1_{\text{unif}}(S^X)\).

**Proof.** By the definition of \(f \in C^0_{\text{unif}}(S^X)\), there exists \(R > 0\) such that

\[
f = \sum_{\Lambda \in X, \text{diam}(\Lambda) \leq R} f_\Lambda
\]

for \(f_\Lambda \in C_{\Lambda}(S^X) \subset C(S^\Lambda)\). Then for any \(e \in E\), we have \(\nabla_e f_\Lambda \in C(S^{\Lambda \cup e})\) and \(\nabla_e f_\Lambda \equiv 0\) if \(\Lambda \cap e = \emptyset\), hence \(\partial f \in C^1_{R+1}(S^X) \subset C^1_{\text{unif}}(S^X)\). The fact that \(\partial f \in Z^1(S^X)\) follows from Lemma 2.15, which states that exact forms are closed.

Lemma 5.1 shows that the differential \(\partial\) induces an homomorphism

\[
\partial : C^0_{\text{unif}}(S^X) \rightarrow Z^1_{\text{unif}}(S^X).
\]

As in Definition 1.6, we define the uniform cohomology \(H^m_{\text{unif}}(S^X)\) of the configuration space \(S^X\) with transition structure to be the cohomology of the complex \((34)\). Our choice for \(C^0_{\text{unif}}(S^X)\) of restricting to functions satisfying \(f(\ast) = 0\) implies that uniform cohomology is philosophically the reduced cohomology in the sense of topology of the pointed space consisting of the configuration space \(S^X\) and base configuration \(\ast \in S^X\).

Now, let \(X\) be a locale which is weakly transferable in the sense of Definition 4.16 and assume that the interaction is irreducibly quantified. Let \(c_\phi = \dim_\mathbb{R} \text{Consv}^\phi(S)\). Our theorem concerning the integration of uniform forms is as follows.

**Theorem 5.2.** Let \((X, S, \phi)\) be a triple such that the interaction \(\phi\) is irreducibly quantified. If the locale \(X\) is transferable, or the interaction \(\phi\) is simple and \(X\) is weakly transferable, then for any \(\omega \in Z^1_{\text{unif}}(S^X)\), there exists \(f \in C^0_{\text{unif}}(S^X)\) such that \(\partial f = \omega\).

The proof of Theorem 5.2 is based on Lemma 5.3 and Lemma 5.6 below. A homomorphism of monoids is a map of sets preserving the binary operation and the identity element. We say that a monoid \(\mathcal{E}\) with operation \(+_{\mathcal{E}}\) satisfies the (right)-cancellation property, if \(a +_{\mathcal{E}} b = a' +_{\mathcal{E}} b\) implies that \(a = a'\) for any \(a, a', b \in \mathcal{E}\). Note that any monoid obtained as a submonoid of a group satisfies the cancellation property. We will first prove the following lemma concerning commutative monoids satisfying the cancellation property.

**Lemma 5.3.** Assume that \(\mathcal{E}\) is a commutative monoid with operation \(+_{\mathcal{E}}\) satisfying the cancellation property. If there exists an injective homomorphism of monoids

\[
i: \mathbb{R} \hookrightarrow \mathcal{E},
\]

where we view \(\mathbb{R}\) as an abelian group via the usual addition, then there exists a homomorphism of monoids \(\pi: \mathcal{E} \rightarrow \mathbb{R}\) such that \(\pi \circ i = \text{id}_\mathbb{R}\).
Proof. The statement follows from the fact that $\mathbb{R}$ is a divisible abelian group (see Remark 5.4). We will give a proof for the sake of completeness. We consider the set of pairs $(\mathcal{N}, \pi_\mathcal{N})$, where $\mathcal{N}$ is a submonoid of $\mathcal{E}$ containing $i(\mathbb{R})$ and $\pi_\mathcal{N} : \mathcal{N} \to \mathbb{R}$ is an homomorphism of monoids such that $\pi_\mathcal{N} \circ i = \text{id}_\mathbb{R}$.

We consider an order on the set of such pairs such that $(\mathcal{N}, \pi_\mathcal{N}) \leq (\mathcal{N}', \pi_\mathcal{N}')$ if and only if $\mathcal{N} \subseteq \mathcal{N}'$ and $\pi_\mathcal{N}' |_{\mathcal{N}} = \pi_\mathcal{N}$. Let $((\mathcal{N}_i, \pi_{\mathcal{N}_i}))_{i \in I}$ be a totally ordered set of such pairs. If we let $\mathcal{N} := \bigcup_{i \in I} \mathcal{N}_i$, and if we let $\pi_{\mathcal{N}} : \mathcal{N} \to \mathbb{R}$ be a homomorphism of monoids obtained as the collection of $\pi_{\mathcal{N}_i}$, then we have $\pi_{\mathcal{N}} \circ i = \text{id}_\mathbb{R}$. Hence $(\mathcal{N}, \pi_\mathcal{N})$ is a maximal element of the totally ordered set $((\mathcal{N}_i, \pi_{\mathcal{N}_i}))_{i \in I}$. By Zorn’s lemma, there exists a maximal pair $(\mathcal{M}, \pi_\mathcal{M})$ in the set of all such pairs.

We will prove by contradiction that $\mathcal{M} = \mathcal{E}$. Suppose $\mathcal{M} \subsetneq \mathcal{E}$. Let $w \in \mathcal{E}$ such that $w \notin \mathcal{M}$, and let $\mathcal{M}' := \mathcal{M} +_\mathcal{E} \mathbb{N} w$. Note that since $\mathcal{E}$ is commutative, if there exists a nontrivial algebraic relation between elements of $\mathcal{M}$ and $\mathbb{N} w$, then there would exist $a, b \in \mathcal{M}$ such that $a +_\mathcal{E} m w = b +_\mathcal{E} n w$ for some $m, n \in \mathbb{N}$. By the cancellation property of $\mathcal{E}$, this implies that $a +_\mathcal{E} n w = b$ for some $a, b \in \mathcal{M}$ and integer $n > 0$. In the case that there exists an integer $n > 0$ and $a, b \in \mathcal{M}$ such that $a +_\mathcal{E} n w = b$, we define $u$ be an element in $\mathbb{R}$ such that $nu = (\pi_\mathcal{M}(b) - \pi_\mathcal{M}(a)) \in \mathbb{R}$.

If $n', a', b'$ satisfies the same condition, then we have $n'a +_\mathcal{E} n'n w = n'b$ and $na' +_\mathcal{E} nn'w = nb'$. Combining this equality, we have

$$n'a +_\mathcal{E} n'n w +_\mathcal{E} nb' = n'b +_\mathcal{E} na' +_\mathcal{E} nn'w,$$

hence $n'a +_\mathcal{E} nb' = n'a +_\mathcal{E} nb'$ since $\mathcal{E}$ is commutative and satisfies the cancellation property. This shows that we have

$$n'(\pi_\mathcal{M}(b) - \pi_\mathcal{M}(a)) = n(\pi_\mathcal{M}(b') - \pi_\mathcal{M}(a')),$$

which implies that $u$ is independent of the choice of $a, b \in \mathcal{M}$ and $n > 0$. On the other hand, if for any integer $n > 0$ and $a, b \in \mathcal{M}$, we have $a +_\mathcal{E} n w \neq b$, then we let $u = 0$. In both cases, we define the map $\pi_\mathcal{M}' : \mathcal{M} \to \mathbb{R}$ by $\pi_\mathcal{M}'(a + nw) = \pi_\mathcal{M}(a) + nu$ for any $a \in \mathcal{M}$ and $n \in \mathbb{N}$. This gives a homomorphism of monoids satisfying $\pi_\mathcal{M}' \circ i = \text{id}_\mathbb{R}$, hence we have $(\mathcal{M}, \pi_\mathcal{M}) < (\mathcal{M}', \pi_\mathcal{M}')$, which contradicts the fact that $(\mathcal{M}, \pi_\mathcal{M})$ is maximal for such pairs.

This shows that we have $\mathcal{M} = \mathcal{E}$. Then $\pi := \pi_\mathcal{M} : \mathcal{E} \to \mathbb{R}$ is a homomorphism of monoids satisfying $\pi \circ i = \text{id}_\mathbb{R}$ as desired. 

Remark 5.4. Assume that $D$ is a divisible abelian group, that is, for any $v \in D$ and integer $n > 0$, there exists $u \in D$ such that $un = v$. Then we may prove that if $i : D \to \mathcal{E}$ is an injective homomorphism into a commutative monoid $\mathcal{E}$ satisfying the cancellation property, then there exists a homomorphism of monoids $\pi : \mathcal{E} \to D$ such that $\pi \circ i = \text{id}_D$. Indeed, by the Grothendieck construction, the monoid $A := (\mathcal{E} \times \mathcal{E}) / \sim$ defined via the equivalence relation $(a, b) \sim (a', b')$ if $a +_\mathcal{E} b' = a' +_\mathcal{E} b$ for any $(a, b), (a', b') \in \mathcal{E} \times \mathcal{E}$ is an abelian group, since $\mathcal{E}$ satisfies the cancellation property. In addition, we have an injective homomorphism $\mathcal{E} \to A$ given by mapping $a \in \mathcal{E}$ to the class of $(a, 0)$ in $A$. Since the composition $i_A : D \to \mathcal{E} \to A$ is an injective homomorphism of abelian groups, from the fact that $D$ is divisible hence an injective object in the category of abelian groups (see [14, 4.1.2]), there exists a homomorphism
\( \pi: A \to D \) such that \( \pi_A \circ t_A = \text{id}_D \). Then \( \pi := \pi_A\vert_\mathcal{E} \) satisfies \( \pi \circ t = \text{id}_D \) as desired. The authors thank Kei Hagihara and Shuji Yamamoto for discussion concerning this remark as well as the proof of Lemma 5.3.

Next, let \( \mathcal{M} \) be a commutative monoid with operation \( + \) and identity element \( 0 \). We first give a construction of extensions of monoids arising from a pairing \( H: \mathcal{M} \times \mathcal{M} \to \mathbb{R} \) satisfying \( H(0, 0) = 0 \) and

\[
H(\alpha, \beta) + H(\alpha + \beta, \gamma) = H(\beta, \gamma) + H(\alpha, \beta + \gamma)
\]

for any \( \alpha, \beta, \gamma \in \mathcal{M} \). We let \( \mathcal{E} := \mathcal{M} \times \mathbb{R} \), and we denote an element of \( \mathcal{E} \) by \( [\alpha, u] \) for \( \alpha \in \mathcal{M} \) and \( u \in \mathbb{R} \). Define the binary operation \( +_{\mathcal{E}} \) on \( \mathcal{E} \) by

\[
[\alpha, u] +_{\mathcal{E}} [\beta, v] := [\alpha + \beta, u + v - H(\alpha, \beta)]
\]

for any \( \alpha, \beta \in \mathcal{M} \) and \( u, v \in \mathbb{R} \). The element \( [0, 0] \) is an identity element with respect to \( +_{\mathcal{E}} \), and the cocycle condition (35) ensures that \( +_{\mathcal{E}} \) is associative. Hence \( \mathcal{E} \) is a monoid with respect to the operation \( +_{\mathcal{E}} \), which fits into the exact sequence of monoids

\[
0 \to \mathbb{R} \to \mathcal{E} \to \mathcal{M} \to 0.
\]

Here, the arrows are homomorphisms of monoids with the injection \( \mathbb{R} \to \mathcal{E} \) given by \( u \mapsto [0, u] \), and the surjection \( \mathcal{E} \to \mathcal{M} \) given by \( [\alpha, u] \mapsto \alpha \). If the pairing \( H \) is symmetric, then we see that \( \mathcal{E} \) is a commutative monoid. Note that if \( \mathcal{M} \) satisfies the right cancellation property, then \( \mathcal{E} \) also satisfies the right cancellation property. In fact, if \( [\alpha, u] +_{\mathcal{E}} [\beta, v] = [\alpha', u'] +_{\mathcal{E}} [\beta, w] \) for some \( \alpha, \alpha', \beta \in \mathcal{M} \) and \( u, u', v, w \in \mathbb{R} \), then we have

\[
[\alpha + \beta, u + v - H(\alpha, \beta)] = [\alpha' + \beta, u' + v - H(\alpha', \beta)]
\]

This shows that \( \alpha + \beta = \alpha' + \beta \), hence \( \alpha = \alpha' \) since \( \mathcal{M} \) satisfies the right cancellation property. Then we have \( u + v - H(\alpha, \beta) = u' + v - H(\alpha', \beta) = u' + v - H(\alpha, \beta) \) in \( \mathbb{R} \), hence \( u = u' \). This shows that \( [\alpha, u] = [\alpha', u'] \), proving that the monoid \( \mathcal{E} \) satisfies the right cancellation property.

**Lemma 5.5.** Let \( \mathcal{M} \) be a commutative monoid satisfying the cancellation property, and let \( H: \mathcal{M} \times \mathcal{M} \to \mathbb{R} \) be a symmetric pairing satisfying the cocycle condition (35). Then there exists a map

\[
h: \mathcal{M} \to \mathbb{R}
\]

such that \( H(\alpha, \beta) = h(\alpha) + h(\beta) - h(\alpha + \beta) \) for any \( \alpha, \beta \in \mathcal{M} \).

**Proof.** If we prove our assertion for the pairing \( \widetilde{H}(\alpha, \beta) := H(\alpha, \beta) - H(0, 0) \) and show that there exists a map \( \tilde{h}: \mathcal{M} \to \mathbb{R} \) satisfying our condition, then our assertion is proved also for \( H(\alpha, \beta) \) by taking \( h(\alpha) := \tilde{h}(\alpha) + H(0, 0) \). Hence by replacing \( H \) by \( \tilde{H} \), we may assume that \( H(0, 0) = 0 \). Consider the extension \( \mathcal{E} \) given in (36) corresponding to the pairing \( H \). Then \( \mathcal{E} \) is a commutative monoid since \( \tilde{H} \) is symmetric. Hence by Lemma 5.3 there exists a homomorphism of monoids \( \pi: \mathcal{E} \to \mathbb{R} \) such that \( \pi([0, u]) = u \) for any \( u \in \mathbb{R} \). For any \( \alpha \in \mathcal{M} \), choose an arbitrary \( u \in \mathbb{R} \) and let

\[
\overline{\alpha} := [\alpha, u] +_{\mathcal{E}} [0, -\pi([\alpha, u])] \in \mathcal{E}
\]
If $v \in \mathbb{R}$, then we have $[\alpha, v] = [\alpha, u] + \mathcal{E} [0, w]$ for some $w \in \mathbb{R}$ since the classes of $[\alpha, v]$ and $[\alpha, u]$ coincide on $\mathcal{M} \cong \mathcal{E}/\mathbb{R}$, hence

$$[\alpha, v] +_\mathcal{E} [0, -\pi([\alpha, v])] = [\alpha, u] +_\mathcal{E} [0, w] +_\mathcal{E} [0, -\pi([\alpha, u])] +_\mathcal{E} [0, -\pi([0, w])] = [\alpha, u] +_\mathcal{E} [0, w] +_\mathcal{E} [0, -\pi([\alpha, u])] +_\mathcal{E} [0, -w] = [\alpha, u] +_\mathcal{E} [0, -\pi([\alpha, u])] = \tilde{\alpha},$$

hence $\tilde{\alpha}$ is independent of the choice of $u \in \mathbb{R}$. We define $h: \mathcal{M} \to \mathbb{R}$ to be the map defined by $\tilde{\alpha} = [\alpha, h(\alpha)]$.

Note that by (37), for any $\alpha, \beta \in \mathcal{M}$, we have

$$\tilde{\alpha} +_\mathcal{E} \tilde{\beta} = [\alpha, u] +_\mathcal{E} [0, -\pi([\alpha, u])] +_\mathcal{E} [\beta, H(\alpha, \beta)] +_\mathcal{E} [0, -\pi(\beta, H(\alpha, \beta))] = [\alpha + \beta, u] +_\mathcal{E} [0, -\pi([\alpha, u]) - \pi(\beta, H(\alpha, \beta))] = [\alpha + \beta, u] +_\mathcal{E} [0, -\pi([\alpha + _\mathcal{E} \beta, u])] = \tilde{\alpha} + \tilde{\beta}.$$ 

This shows that

$$[\alpha + \beta, h(\alpha + \beta)] = \tilde{\alpha} + \tilde{\beta} = \tilde{\alpha} +_\mathcal{E} \tilde{\beta} = [\alpha, h(\alpha)] +_\mathcal{E} \beta, h(\beta)] = [\alpha + \beta, h(\alpha) + h(\beta) - H(\alpha, \beta)].$$

Hence we have $h(\alpha + \beta) = h(\alpha) + h(\beta) - H(\alpha, \beta)$ for any $\alpha, \beta \in \mathcal{M}$ as desired.

Next, we consider the case when $\mathcal{M} \cong \mathbb{N}$ or $\mathcal{M} \cong \mathbb{Z}$.

**Lemma 5.6.** Assume that $\mathcal{M} \cong \mathbb{N}$ or $\mathcal{M} \cong \mathbb{Z}$, viewed as a commutative monoid with respect to addition, and let $H: \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a pairing satisfying the cocycle condition (35). Then there exists a map $h: \mathcal{M} \to \mathbb{R}$ such that $H(\alpha, \beta) = h(\alpha) + h(\beta) - h(\alpha + \beta)$ for any $\alpha, \beta \in \mathcal{M}$.

**Proof.** Again, as in the proof of Lemma 5.6, by replacing the pairing $H$ by $H - H(0, 0)$, we may assume that $H(0, 0) = 0$. We first treat the case $\mathcal{M} \cong \mathbb{N}$, which we regard as a commutative monoid with respect to the usual addition. Consider the extension $\mathcal{E}$ given in (36) corresponding to the pairing $H$. Note that $[1, 0]$ gives an element in $\mathcal{E}$. For any $\alpha \in \mathcal{M}$ corresponding to $n \in \mathbb{N}$, we define $h(\alpha)$ to be the element in $\mathcal{E}$ given by the formula

$$n[1, 0] = [1, 0] +_\mathcal{E} \cdots +_\mathcal{E} [1, 0] = [\alpha, h(\alpha)],$$

where $n[1, 0]$ is the $n$-fold sum of $[1, 0]$ with respect to the operator $+_\mathcal{E}$. Then for $\beta \in \mathcal{M}$ corresponding to $n' \in \mathbb{N}$, we have $n'[1, 0] = [\beta, h(\beta)]$ and

$$[\alpha, h(\alpha)] +_\mathcal{E} [\beta, h(\beta)] = (n + n')[1, 0] = [\alpha + \beta, h(\alpha + \beta)].$$

By the definition of the operation $+_\mathcal{E}$, we have

$$[\alpha, h(\alpha)] +_\mathcal{E} [\beta, h(\beta)] = [\alpha + \beta, h(\alpha) + h(\beta) - H(\alpha, \beta)].$$
Since the operation $+$ is associative, comparing this equality with (38), we have $h(\alpha + \beta) = h(\alpha) + h(\beta) - H(\alpha, \beta)$ for any $\alpha, \beta \in M$. Hence we see that $h : M \to \mathbb{R}$ satisfies the requirement of our assertion. The case for $M \cong \mathbb{Z}$ may be proved precisely in the same manner. In this case, we regard $M$ and $\mathcal{E}$ as abelian groups instead of commutative monoids.

\[\text{□}\]

**Remark 5.7.** For the case that $M$ is an abelian group, the extension $\mathcal{E}$ in Lemma 5.5 corresponding to the symmetric cocycle $H : M \times M \to \mathbb{R}$ is a commutative group. The existence of $h : M \to \mathbb{R}$ for this case corresponds to the well-known fact that such an extension $\mathcal{E}$ splits, since the additive group of $\mathbb{R}$ is divisible hence an injective object in the category of abelian groups (see [14, 4.1.2]). The statement for Lemma 5.6 corresponds to the fact that the group cohomology $H^2(\mathbb{Z}, \mathbb{R}) = \{0\}$. The existence of $h : M \to \mathbb{R}$ ensures that the cocycle $H : M \times M \to \mathbb{R}$ is in fact symmetric in this case.

We are now ready to prove Theorem 5.2.

**Proof of Theorem 5.2** Suppose $\omega \in Z_{\text{unif}}(S^X) = C^1_{\text{unif}}(S^X) \cap Z^1(S^X)$. Since $\omega \in Z^1(S^X)$, by Lemma 2.16 there exists $f \in C(S^X)$ such that $\partial f = \omega$. Furthermore, since $\omega \in C^1_{\text{unif}}(S^X)$, there exists $R > 0$ such that $\partial f = \omega \in C^1(S^X)$. Hence by Definition 4.8 and Proposition 4.13 (see also Proposition 3.18), there exists a pairing $h_f : M \times M \to \mathbb{R}$ satisfying the cocycle condition (22). By Lemma 4.9 if $X$ is transferable (see also Proposition 3.18) and by Remark 5.7 if the interaction is simple, we see that $h_f$ is symmetric. Note that here we have used the fact that $M \cong \mathbb{Z}$ or $M \cong \mathbb{N}$ if the interaction is simple. Hence by Proposition 4.11 (see also Proposition 3.18), we have

$$t^{\Lambda \Lambda'} f(\eta) - t^\Lambda f(\eta) = h_f(\xi_\Lambda(\eta), \xi_{\Lambda'}(\eta))$$

for any $(\Lambda, \Lambda') \in \mathcal{A}_R$ and $\eta \in S^X$. Note that $M$ satisfies the cancellation property since it is a submonoid of an abelian group. Hence by Lemma 5.5 there exists $h : M \to \mathbb{R}$ such that

$$h_f(\alpha, \beta) = h(\alpha) + h(\beta) - h(\alpha + \beta)$$

to any $\alpha, \beta \in M$. We define the function $g \in C(S^X)$ by

$$g := f + h \circ \xi_X.$$  

We will prove that $g$ is uniform. By Lemma 2.30, for any conserved quantity $\xi : S \to \mathbb{R}$, the function $\xi_X \in C(S^X)$ is horizontal. This implies that $\nabla_e \xi_X = 0$, hence $\xi_X(\eta^e) = \xi_X(\eta)$ for any $e \in E$. This shows that $h \circ \xi_X(\eta^e) = h \circ \xi_X(\eta)$ for any $e \in E$, hence $\nabla_e (h \circ \xi_X) = 0$, which implies that $\partial (h \circ \xi_X) = 0$. This gives the formula $\partial g = \partial f = \omega$. Furthermore, noting that $t^\Lambda (h \circ \xi_X(\eta)) = h \circ \xi_\Lambda(\eta)$ for any $\Lambda \subset X$, we have

$$t^{\Lambda \Lambda'} g(\eta) - t^\Lambda g(\eta) = t^{\Lambda \Lambda'} f(\eta) - t^\Lambda f(\eta) + h \circ \xi_{\Lambda \Lambda'}(\eta) - h \circ \xi_\Lambda(\eta) - h \circ \xi_{\Lambda'}(\eta) = 0$$

for any $(\Lambda, \Lambda') \in \mathcal{A}_R$ and $\eta \in S^X$, where we have used the coboundary condition (39) and the fact that $\xi_{\Lambda \Lambda'}(\eta) = \xi_\Lambda(\eta) + \xi_{\Lambda'}(\eta)$. From the definition of the pairing given in Definition 4.8, we see that $h_g \equiv 0$. Hence by Proposition 4.14, we see that $g \in C^0_{\text{unif}}(S^X)$. Our assertion is now proved by replacing $f$ by $g$.

By using Theorem 5.2, we may calculate the uniform cohomology of $S^X$ as follows.
**Theorem 5.8.** Let $X$ be a locale, and assume that the interaction is irreducibly quantified. If either $X$ is transferable, or the interaction is simple and $X$ is weakly transferable, then we have

$$H^m_{\text{unif}}(S^X) \cong \begin{cases} \text{Consv}^\phi(S) & m = 0, \\ \{0\} & m \neq 0. \end{cases}$$

In particular, we have $\dim_{\mathbb{R}} H^0_{\text{unif}}(S^X) = \dim_{\mathbb{R}} \text{Consv}^\phi(S)$.

**Proof.** As stated in §1.4 Theorem 5.8 is equivalent to the fact that the sequence

$$0 \longrightarrow \text{Consv}^\phi(S) \longrightarrow C^0_{\text{unif}}(S^X) \xrightarrow{\partial} Z^1_{\text{unif}}(S^X) \longrightarrow 0$$

is exact. A conserved quantity $\xi: S \rightarrow \mathbb{R}$ defines a uniform function $\xi_X: S^X \rightarrow \mathbb{R}$ whose definition $\xi_X := \sum_{x \in X} \xi_x$ is the canonical expansion (16) with each $\xi_x \in C_{(x)}(S^X)$. This shows that $\xi$ is uniform satisfying $\xi(X(\bullet)) = 0$, hence (14) induces an inclusion

$$(41) \quad \text{Consv}^\phi(S) \hookrightarrow C^0_{\text{unif}}(S^X).$$

On the other hand, if $f \in C^0_{\text{unif}}(S^X)$ satisfies $\partial f = 0$, then by Theorem 3.7, there exists a conserved quantity $\xi: S \rightarrow \mathbb{R}$ such that $f(\eta) = \xi_X(\eta)$, which shows that $f$ is in the image of (41). This proves that we have an isomorphism

$$(42) \quad \text{Consv}^\phi(S) \cong \text{Ker} \partial = H^0_{\text{unif}}(S^X).$$

From Theorem 5.2 we see that the differential $\partial$ is surjective, hence the short exact sequence (40) is exact. Our assertion now follows from the definition of uniform cohomology given in Definition 1.6. \qed

**Remark 5.9.** The fact that $H^m_{\text{unif}} = \{0\}$ for $m \neq 0$ reflects the fact that we are viewing the configuration space as modeling a space with a simple topological structure whose only topological feature is its connected components. The $H^0_{\text{unif}}$ is expressed in terms of the conserved quantities and is finite dimensional if $\text{Consv}^\phi(S)$ is finite dimensional.

**Example 5.10.** In each of the examples of Example 2.18 we have the following.

1. In the case of the multi-species exclusion process, we have $H^0_{\text{unif}}(S^X) \cong \mathbb{R}^\kappa$, where $\kappa > 0$ is such that $S = \{0, \ldots, \kappa\}$.
2. In the case of the generalized exclusion process, we have $H^0_{\text{unif}}(S^X) \cong \mathbb{R}$.
3. In the case of the lattice gas with energy process, we have $H^0_{\text{unif}}(S^X) \cong \mathbb{R}^2$.
4. For the interaction of Example 2.18(4), we have $H^0_{\text{unif}}(S^X) \cong \mathbb{R}$.
5. For the Glauber Model of Example 2.18(5), we have $H^0_{\text{unif}}(S^X) \cong \{0\}$.

**5.2. Group Action on the Configuration Space.** In this subsection, we first review basic definitions and results concerning group cohomology of a group $G$ acting on an $\mathbb{R}$-linear space $V$. We will then consider the action of a group $G$ on the locale $X$.

We say that an $\mathbb{R}$-linear space $V$ is a $G$-module, if any $\sigma \in G$ gives an $\mathbb{R}$-linear homomorphism $\sigma: V \rightarrow V$ such that $(\sigma \tau)(v) = \tau(\sigma(v)) = \tau \circ \sigma(v)$ for any $\sigma, \tau \in G$ and $v \in V$. In what follows, we will often simply denote $\sigma(v)$ by $\sigma v$. For any $G$-module $V$, we denote by $V^G$ the $G$-invariant subspace of $V$, defined by $V^G := \{v \in V | \sigma v = v \forall \sigma \in G\}$.
**Definition 5.11.** Let $V$ be a $G$-module. The zeroth group cohomology $H^0(G, V)$ of $G$ with coefficients in $V$ is given as

$$H^0(G, V) := V^G.$$  

Furthermore, we let

$$Z^1(G, V) := \{\rho : G \to V \mid \rho(\sigma \tau) = \sigma \rho(\tau) + \rho(\sigma) \forall \sigma, \tau \in G\},$$

$$B^1(G, V) := \{\rho \in Z^1(G, V) \mid \exists v \in V, \rho(\sigma) = (\sigma - 1)v \forall \sigma \in G\},$$

where 1 is the identity element of $G$. Then the first group cohomology $H^1(G, V)$ of $G$ with coefficients in $V$ is given as

$$H^1(G, V) := Z^1(G, V)/B^1(G, V).$$

In particular, if the action of $G$ on $V$ is trivial, in other words, if $\sigma v = v$ for any $v \in V$ and $\sigma \in G$, then we have $H^0(G, V) = V$ and

$$H^1(G, V) = \text{Hom}(G, V),$$

where $\text{Hom}(G, V)$ denotes the set of homomorphisms of groups from $G$ to $V$.

**Remark 5.12.** The group cohomology of $G$ with coefficients in $V$ is usually defined using the right derived functor of the functor $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ applied to $V$. In other words, $H^m(G, V) := \text{Ext}^m_{\mathbb{Z}[G]}(\mathbb{Z}, V)$ for any integer $m \in \mathbb{Z}$ (see for example [1, §2]). Definition 5.11 is the well-known description of this cohomology group in terms of explicit cocycles (see [1, §2]).

Let $V$ and $V'$ be $G$-modules. We say that an $\mathbb{R}$-linear homomorphism

$$\pi : V \to V'$$

is a $G$-homomorphism, if $\pi(\sigma v) = \sigma \pi(v)$ for any $\sigma \in G$ and $v \in V$. By definition, we have $\pi(V^G) \subset \pi(V)^G$, where $\pi(V)$ is the $G$-submodule of $V$ defined to be the image of $V$ with respect to $\pi$. Note that $\pi$ gives an exact sequence

$$0 \longrightarrow \text{Ker } \pi \longrightarrow V \longrightarrow \pi(V) \longrightarrow 0$$

of $G$-modules, which by the standard theory of cohomology of groups (see for example [1, (1.3)]) gives rise to the long exact sequence

$$0 \longrightarrow (\text{Ker } \pi)^G \longrightarrow V^G \longrightarrow \pi(V)^G \longrightarrow H^1(G, V) \longrightarrow H^1(G, \text{Ker } \pi) \longrightarrow \cdots.$$  

The homomorphism $\delta$ is given explicitly as follows. For any $\omega \in \pi(V)^G$, choose a $v \in V$ such that $\pi(v) = \omega$. Then $\delta(\omega) \in H^1(G, \text{Ker } \pi)$ is the class given by the cocycle satisfying

$$\delta(\omega)(\sigma) = (1 - \sigma)v$$

for any $\sigma \in G$ (see [1, §2 p.97]). Note that since $\omega = \pi(v)$ is invariant under the action of $G$, we have $\pi(\delta(\omega)(\sigma)) = (1 - \sigma)\pi(v) = 0$, hence $\delta(\omega)(\sigma) \in \text{Ker } \pi$ for any $\sigma \in G$. Our choice of the sign of the homomorphism $\delta$ in (45) is to ensure compatibility with standard sign conventions used in probability theory.

In what follows, let $X$ be a locale, and let $G$ be a group.
Definition 5.13. An automorphism of a locale $X$ is a bijective map of sets $\sigma : X \to X$ such that $\sigma(E) = E \subseteq X \times X$. The set Aut$(X)$ of all automorphisms of $X$ forms a group with respect to the operation given by composition of automorphisms. We say that $X$ has an action of $G$, if there exists a homomorphism of groups $G \to \text{Aut}(X)$ so that any $\sigma \in G$ induces an automorphism $\sigma : X \to X$.

Example 5.14. (1) Let $d$ be an integer $> 0$. Consider the Euclidean lattice $X = (\mathbb{Z}^d, \mathbb{R}^d)$ and let $G = \mathbb{Z}^d$. For any $\tau \in G$, if we define the automorphism $\tau : X \to X$ by $\tau(x) := x + \tau$ for any $x \in X$, then this gives an action of $G$ on $X$.

(2) The group $G = \mathbb{Z}^2$ acts on the triangular and hexagonal lattices via translation. The group $G = \mathbb{Z}^3$ acts on the diamond lattice also via translation.

(3) If $G$ is a finitely generated group with set of minimal generators $S$, then left multiplication by elements of $G$ gives an action of the group $G$ on the Cayley graph $(G, E_S)$.

From now until the end of §5.3 we assume that $X$ has an action of a group $G$. If we denote the group action from the left, then $S^X_e = (S^X, \Phi_e)$ has a natural $G$-action given by $\eta^\sigma := (s_{\sigma(x)})$ for any $\eta = (\eta_x)$ and $\sigma \in G$. Then $C^0(S^X_e) = C(S^X_e)$ and $C^1(S^X_e)$ have natural $G$-actions given for any $\sigma \in G$ by $\sigma(f) = f \circ \sigma$ for any function $f \in C(S^X_e)$, and $\sigma(\omega) = \omega \circ \sigma$ for any form $\omega \in C^1(S^X_e)$. Since the action of $G$ preserves the distance on the locale $X$ and preserves closed forms on $C^1(S^X_e)$, the groups $C_{\text{unif}}(S^X_e)$ and $Z_{\text{unif}}^1(S^X_e)$ have induced $G$-module structures.

In case of functions and forms, we will use the term shift-invariant interchangeably with the term $G$-invariant, when the group $G$ is understood. We say that a subset $\Lambda_0 \subseteq X$ is a fundamental domain of $X$ for the action of $G$, if it represents the set of orbits of the vertices of $X$ for the action of $G$.

Lemma 5.15. Suppose $X$ has an action of a group $G$, and assume that the set of orbits of the vertices of $X$ for the action of $G$ is finite. Then for any shift-invariant uniform function $F \in C^0_{\text{unif}}(S^X)$, there exists a local function $f \in C^1_{\text{loc}}(S^X)$ satisfying $f(\ast) = 0$ such that

$$F = \sum_{\tau \in G} \tau(f)$$

in $C_{\text{unif}}^0(S^X)$.

Proof. By definition, $F(\ast) = 0$. Since $F$ is uniform, there exists $R > 0$ such that the expansion (16) of $F$ in terms of local functions with exact support is given by

$$F = \sum_{\Lambda \subseteq X, \text{diam}(\Lambda) \leq R} F_\Lambda.$$

Since $F$ is shift-invariant, we have $\tau(F_\Lambda) = \tau(F)_{\tau(\Lambda)} = F_{\tau(\Lambda)}$ for any $\tau \in G$. Let $J_R^*$ be the set of nonempty finite $\Lambda \subseteq X$ such that $\text{diam}(\Lambda) \leq R$. Then $J_R^*$ has a natural action of $G$. We denote by $\sim$ the equivalence relation on $J_R^*$ given by $\Lambda \sim \Lambda'$ if $\Lambda' = \tau(\Lambda)$ for some $\tau \in G$. Let $\Lambda_0 \subseteq X$ be a fundamental domain of $X$ for the action of $G$. Then any equivalence class of $J_R^*$ with respect to the relation $\sim$ contains a representative that intersects with $\Lambda_0$. Since $\Lambda_0$ is finite, and the diameters of the sets in $J_R^*$ are bounded, this implies that $J_R^*/\sim$ is finite. For each
equivalence class $C$ of $T^*_R/\sim$, let $C_0 := \{\Lambda \in C \mid \Lambda \cap \Lambda_0 \neq \emptyset\}$, which is again finite. If we let

$$f_C := \frac{1}{|C_0|} \sum_{\Lambda \in C_0} F_{\Lambda},$$

then it is a finite sum hence gives a local function in $C_{\text{loc}}(S^X)$. Then since $T^*_R/\sim$ is finite,

$$f := \sum_{C \in T^*_R/\sim} f_C$$

again defines a local function in $C_{\text{loc}}(S^X)$, which by construction satisfies (46) as desired. □

The action of $G$ on $\text{Consv}^\phi(S)$ viewed as a subspace of $C^0_{\text{unit}}(S^X)$ is given by the trivial action. Applying (44) to the short exact sequence (40), we obtain the long exact sequence

$$0 \longrightarrow \text{Consv}^\phi(S) \longrightarrow (C^0_{\text{unit}}(S^X))^G \xrightarrow{\partial} (Z^1_{\text{unit}}(S^X))^G$$

$$\xrightarrow{\delta} H^1(G, \text{Consv}^\phi(S)) \longrightarrow H^1(G, C^0_{\text{unit}}(S^X)) \longrightarrow \cdots.$$

Remark 5.16. We may view the cohomology group $H^1(G, \text{Consv}^\phi(S))$ as a group which philosophically reflects the first reduced cohomology group of the quotient space $S^X/G$ with fixed base point $\star/G$. Intuitively, we are viewing $S^X$ as a model of the configuration space on $X$ which we view as an infinitely magnified version of a point in a macroscopic space. In this context, the cohomology $H^1(G, \text{Consv}^\phi(S))$ is regarded as representing the flow of the conserved quantities at this point induced from the action of $G$. More generally, for $m \in \mathbb{Z}$, we may view the $m$-th group cohomology $H^m(G, \text{Consv}^\phi(S))$ as a group philosophically reflecting the $m$-th reduced cohomology group of the quotient space $S^X/G$ with fixed base point $\star/G$.

5.3. Group Cohomology of the Configuration Space. In this subsection, we will prove Theorem [5.17] which is the main theorem of this article. We say that an action of a group $G$ on a locale $X$ is free, if $\sigma(x) = \tau(x)$ implies that $\sigma = \tau$ for any $x \in X$ and $\sigma, \tau \in G$. Throughout this subsection, we assume that $X$ has a free action of a group $G$.

As in §1.2, denote by $C := (Z^1_{\text{unit}}(S^X))^G$ the space of shift-invariant closed uniform forms, and by $\mathcal{E} := \partial(C^0_{\text{unit}}(S^X)^G)$ the image by $\partial$ of the space of shift-invariant uniform functions. The main theorem of this article is the following, given as Theorem [11] in §1.3

Theorem 5.17. Let the triple $(X, S, \phi)$ be as in Theorem [5.3] and assume that the action of $G$ on the locale $X$ is free. Then the boundary morphism $\delta$ of (47) gives a canonical isomorphism

$$H^1(G, \text{Consv}^\phi(S)) \cong C/\mathcal{E}.$$

Moreover, a choice of a fundamental domain for the action of $G$ on $X$ gives an $\mathbb{R}$-linear homomorphism $\lambda : H^1(G, \text{Consv}^\phi(S)) \to C$ such that $\delta \circ \lambda = \text{id}$, which gives a decomposition

$$C \cong \mathcal{E} \oplus H^1(G, \text{Consv}^\phi(S)).$$
In order to prove Theorem 5.17, we first prove Proposition 5.18 concerning the existence of a section of $\delta$. Let $\Lambda_0$ be a fundamental domain of $X$ for the action of $G$. Since the action of $G$ on $X$ is free, any $x \in X$ may be uniquely written as $\sigma(x_0)$ for some $x_0 \in \Lambda_0$ and $\sigma \in G$. Then for $\xi \in \text{Consv}^\phi(S)$, we have

$$\xi x = \sum_{\sigma \in G} \xi_{\sigma(\Lambda_0)},$$

where $\xi_W := \sum_{x \in W} \xi_x$ for any $W \subset X$.

**Proposition 5.18.** Let $X$ be a locale with a free action of a group $G$, and let $\Lambda_0 \subset X$ be a fundamental domain of $X$ for the action of $G$. For any $\rho \in Z^1(G, \text{Consv}^\phi(S))$, we let $\omega_\rho := \delta(\theta_\rho)$, where

$$\theta_\rho := \sum_{\tau \in G} \rho(\tau)_{\tau(\Lambda_0)} \in C^0_{\text{unif}}(S^X).$$

Then we have $\delta(\omega_\rho) = \rho$.

**Proof.** By definition, the map is $\mathbb{R}$-linear. Since $\rho$ is a cocycle with values in $\text{Consv}^\phi(S)$ and the group $G$ acts trivially on $\text{Consv}^\phi(S)$, we have $\rho(\sigma \tau) = \rho(\tau) + \rho(\sigma)$ for any $\sigma, \tau \in G$. Note that

$$\sigma(\theta_\rho) = \sum_{\tau \in G} \sigma(\rho(\tau)_{\tau(\Lambda_0)}) = \sum_{\tau \in G} \rho(\tau)_{\sigma \tau(\Lambda_0)} = \sum_{\tau \in G} \left(\rho(\sigma \tau)_{\sigma \tau(\Lambda_0)} - \rho(\sigma)_{\sigma \tau(\Lambda_0)}\right) = \sum_{\tau \in G} \left(\rho(\tau)_{\tau(\Lambda_0)} - \rho(\sigma)_{\tau(\Lambda_0)}\right) = \theta_\rho - \rho(\sigma)_X,$$

hence we have $(1 - \sigma)\theta_\rho = \rho(\sigma)_X$ for any $\sigma \in G$. Since $\rho(\sigma)$ is a conserved quantity, $\rho(\sigma)_X$ is horizontal by Lemma 2.30, hence we have $(1 - \sigma)\omega_\rho = (1 - \sigma)\delta\theta_\rho = \delta(\theta_\rho) = 0$ for any $\sigma \in G$. This implies that we have $\omega_\rho \in C$. By the explicit description of the homomorphism $\delta$ in (45), we see that $\delta(\omega_\rho) = \rho$ as desired. $\square$

We may now prove Theorem 5.17.

**Proof of Theorem 5.17.** By the definition of $C$ and $E$, the long exact sequence (47) gives the exact sequence

$$0 \longrightarrow E \longrightarrow C \overset{\delta}{\longrightarrow} H^1(G, \text{Consv}^\phi(S)).$$

By Proposition 5.18, for any $\rho \in H^1(G, \text{Consv}^\phi(S))$, there exists $\omega_\rho \in C$ such that $\delta(\omega_\rho) = \rho$. This implies that $\delta$ is surjective. By construction, the map $\rho \mapsto \omega_\rho$ is $\mathbb{R}$-linear, hence we have a decomposition $C \cong E \oplus H^1(G, \text{Consv}^\phi(S))$, given explicitly by mapping any $\omega \in C$ to the element $(\omega - \omega_\rho, \rho) \in E \oplus H^1(G, \text{Consv}^\phi(S))$, where $\rho := \delta(\omega)$. $\square$

As a corollary of Theorem 5.17, we have the following result, which coincides with Corollary 2 of the introduction.

**Corollary 5.19.** Let the triple $(X, S, \phi)$ and the $G$-action be as in Theorem 5.17. Assume in addition that the abelian quotient $G^{ab}$ of $G$ is of finite rank $d$. If we choose a generator of the free
part of $G^{ab}$, then we have an isomorphism $H^1(G, \text{Consv}^\phi(S)) \cong \bigoplus_{j=1}^d \text{Consv}^\phi(S)$. A choice of a fundamental domain of $X$ for the action of $G$ gives a decomposition

\begin{equation}
C \cong E \oplus \bigoplus_{j=1}^d \text{Consv}^\phi(S).
\end{equation}

**Proof.** The group $G$ acts trivially on $\text{Consv}^\phi(S)$, hence

\[ H^1(G, \text{Consv}^\phi(S)) = \text{Hom}(G, \text{Consv}^\phi(S)) = \text{Hom}_{\mathbb{Z}}(G^{ab}/G^{\text{tors}}, \text{Consv}^\phi(S)), \]

where $G^{\text{tors}}$ is the torsion subgroup of $G^{ab}$ and the last equality follows from the fact that $\text{Consv}^\phi(S)$ is abelian and torsion free. This implies that any element in $H^1(G, \text{Consv}^\phi(S))$ is determined by the image of the generators of the free part of $G^{ab}$, hence if we fix such a set of generators, then we have an isomorphism

\[ H^1(G, \text{Consv}^\phi(S)) \cong \bigoplus_{j=1}^d \text{Consv}^\phi(S). \]

Our assertion now follows from Theorem 5.17. $\square$

**Remark 5.20.** Let the triple $(X, S, \phi)$ be as in Theorem 5.8, and assume that $G = \mathbb{Z}^d$ and that the action of $G$ on $X$ is free. The standard basis of $G = \mathbb{Z}^d$ gives an isomorphism $H^1(G, \text{Consv}^\phi(S)) \cong \bigoplus_{j=1}^d \text{Consv}^\phi(S)$, by associating to the element

\[ \rho = (\zeta^{(1)}, \ldots, \zeta^{(d)}) \in \bigoplus_{j=1}^d \text{Consv}^\phi(S) \]

the cocycle $\rho : G \to \text{Consv}^\phi(S)$ given by $\rho(\tau) := \sum_{j=1}^d \tau_j \zeta^{(j)}$ for any $\tau = (\tau_j) \in \mathbb{Z}^d$. If we fix a fundamental domain $\Lambda_0$ of $X$ for the action of $G$, then the function $\theta_\rho$ in Proposition 5.18 is given by

\[ \theta_\rho = \sum_{j=1}^d \left( \sum_{\tau \in G} \tau_j \zeta^{(j)}(\tau_{\Lambda_0}) \right). \]

Hence the form in $C$ corresponding to $\rho = (\zeta^{(1)}, \ldots, \zeta^{(d)})$ is given by $\omega_\rho = \partial \left( \sum_{j=1}^d \sum_{\tau \in G} \tau_j \zeta^{(j)}(\tau_{\Lambda_0}) \right)$ as stated in (7) of §1.4. Then Corollary 5.19 implies that any shift-invariant closed local form $\omega$ decomposes as

\begin{equation}
\omega = \partial (F + \theta_\rho) = \partial F + \omega_\rho,
\end{equation}

for some shift-invariant uniform function $F$ in $C^0_{\text{unif}}(S^X)$ and $\rho \in \bigoplus_{j=1}^d \text{Consv}^\phi(S)$. For the case when $c_\phi = \dim_{\mathbb{R}} \text{Consv}^\phi(S)$ is finite, if we fix a basis $\zeta^{(1)}, \ldots, \zeta^{(c_\phi)}$ of $\text{Consv}^\phi(S)$, then we have $\zeta^{(i)} = \sum_{i=1}^{c_\phi} a_{ij} \zeta^{(i)}$ for some $a_{ij} \in \mathbb{R}$, $i = 1, \ldots, c_\phi$, $j = 1, \ldots, d$. This shows that

\[ \theta_\rho = \sum_{i=1}^{c_\phi} \sum_{j=1}^d a_{ij} \left( \sum_{\tau \in G} \tau_j \zeta^{(i)}(\tau_{\Lambda_0}) \right), \]
which with (50) gives the representation of Theorem 5 of the introduction.

5.4. A Counterexample: Case when $X = (\mathbb{Z}, \mathbb{B})$ and $c_\phi = 2$. In Theorem 5.2, we assumed that the interaction is simple in the weakly transferable case. In this subsection, we give an example of a weakly transferable locale $X$ and an interaction $\phi$ that is irreducibly quantified and satisfies $c_\phi > 1$, but the uniform cohomology $H^1_{\text{unif}}(S^X)$ does not satisfy the conclusion of Theorem 5.2. More precisely, we prove the following.

**Proposition 5.21.** Suppose $\kappa = 2$ so that $S = \{0, 1, 2\}$, and let $X = (\mathbb{Z}, \mathbb{B})$, which is weakly transferable. If we consider the configuration with transition structure $S^X$ for the interaction $\phi : S \times S \to S \times S$ given in Example 7.3 (2), then we have $H^1_{\text{unif}}(S^X) \neq 0$.

**Proof.** We will explicitly construct a form $\omega \in Z^1_{\text{unif}}(S^X)$ which is not integrable by a function $F \in C^0_{\text{unif}}(S^X)$. For any $W \subset S^X$, let $1_W$ be the characteristic function of $W$ on $S^X$. Let $\omega = (\omega_e)_{e \in E} \in \text{Map}(\Phi, \mathbb{R})$ such that

$$\omega_e := 1_{\{\eta_{\ell(e)} = 2, \eta_{r(e)} = 1\}} - 1_{\{\eta_{\ell(e)} = 1, \eta_{r(e)} = 2\}}$$

if $t(e) \geq o(e)$ and $\omega_e := \omega_{e^{-}}$ otherwise. We let $f \in C(S^X)$ be the function

$$f = \sum_{y > x} 1_{\{\eta_x = 1, \eta_y = 2\}}.$$

Then for any $\eta \in S^X$ and $e \in \mathbb{B}$, we have

$$\nabla_e f(\eta) = f(\eta^e) - f(\eta) = \sum_{y > x} 1_{\{\eta_x = 1, \eta_y = 2\}} (\eta^e) - \sum_{y > x} 1_{\{\eta_x = 1, \eta_y = 2\}} (\eta).$$

If we suppose $t(e) \geq o(e)$, then the sum of $\nabla_e f(\eta)$ cancel outside $e = (x, y)$, and we have

$$\nabla_e f(\eta) = 1_{\{\eta_{\ell(e)} = 2, \eta_{r(e)} = 1\}} (\eta) - 1_{\{\eta_{\ell(e)} = 1, \eta_{r(e)} = 2\}} (\eta) = \omega_e.$$

Furthermore, we have $\nabla_{e^{-}} f(\eta) = \omega_e = \omega_{e^{-}}$, hence $\partial f = \omega$, which by Lemma 2.15 shows that $\omega \in Z^1(S^X)$. By definition, we have $\omega \in C^1(S^X) \subset C^0_{\text{unif}}(S^X)$, hence $\omega \in Z^1_{\text{unif}}(S^X)$.

We prove our assertion by contradiction. Suppose there exists $F \in C^0_{\text{unif}}(S^X)$ such that $\partial F = \omega$. By taking $F - F(\star)$ if necessary, we may assume that $F(\star) = 0$. For any $\eta, \eta' \in S^X$, if $\xi_X(\eta) = \xi_X(\eta')$, then since the interaction is irreducibly quantified by Proposition 2.27 there exists a path $\overline{\gamma}$ from $\eta$ to $\eta'$. Since $\partial(f - F) = 0$, by Lemma 2.28 we have $(f - F)(\eta) = (f - F)(\eta')$. This shows that there exists $h : \mathcal{M} \to \mathbb{R}$ such that $f(\eta) - F(\eta) = h \circ \xi_X(\eta)$ for any $\eta \in S^X$. By Remark 3.6, we have $t^{\Lambda \cup \Lambda'} F = t^\Lambda F + t^{\Lambda'} F$ for any pair $(\Lambda, \Lambda') \in \mathcal{S}_R$. Hence

$$t^{\Lambda \cup \Lambda'} f(\eta) - t^\Lambda f(\eta) - t^{\Lambda'} f(\eta) = t^{\Lambda \cup \Lambda'} (f - F)(\eta) - t^\Lambda (f - F)(\eta) - t^{\Lambda'} (f - F)(\eta)$$

$$= h \circ \xi_{\Lambda \cup \Lambda'}(\eta) - h \circ \xi_\Lambda(\eta) - h \circ \xi_{\Lambda'}(\eta)$$

$$= h \circ (\xi_\Lambda(\eta) + \xi_{\Lambda'}(\eta)) - h \circ \xi_\Lambda(\eta) - h \circ \xi_{\Lambda'}(\eta).$$

In particular, for $\eta, \eta' \in S^X$, if $\xi_\Lambda(\eta) = \xi_{\Lambda'}(\eta')$ and $\xi_\Lambda(\eta') = \xi_{\Lambda'}(\eta)$, then we have

$$t^{\Lambda \cup \Lambda'} f(\eta) - t^\Lambda f(\eta) - t^{\Lambda'} f(\eta) = t^{\Lambda \cup \Lambda'} f(\eta') - t^\Lambda f(\eta') - t^{\Lambda'} f(\eta').$$
In what follows, we prove that this it not the case. For any \((\Lambda, \Lambda') \in \mathcal{A}_R\), we have

\[
\begin{align*}
t^{\Lambda \cup \Lambda'} f - t^{\Lambda} f - t^{\Lambda'} f & = \sum_{y > x, x, y \in \Lambda \cup \Lambda'} 1_{\{\eta_x = 1, \eta_y = 2\}} - \sum_{y > x, x, y \in \Lambda} 1_{\{\eta_x = 1, \eta_y = 2\}} - \sum_{y > x, x, y \in \Lambda'} 1_{\{\eta_x = 1, \eta_y = 2\}} \\
& = \sum_{y > x, y \in \Lambda, x \in \Lambda'} 1_{\{\eta_x = 1, \eta_y = 2\}} + \sum_{y > x, x \in \Lambda, y \in \Lambda'} 1_{\{\eta_x = 1, \eta_y = 2\}}.
\end{align*}
\]

Since \(\Lambda\) and \(\Lambda'\) are connected, replacing \(\Lambda\) by \(\Lambda'\) if necessary, we may assume that \(y > x\) for any \(y \in \Lambda\) and \(x \in \Lambda'\). Then we have

\[
t^{\Lambda \cup \Lambda'} f - t^\Lambda f - t^{\Lambda'} f = \sum_{y > x, y \in \Lambda, x \in \Lambda'} 1_{\{\eta_x = 1, \eta_y = 2\}} = \xi^{(1)}_{\Lambda' \times \Lambda} \xi^{(2)}_{\Lambda'},
\]

where \(\xi^{(1)}\) and \(\xi^{(2)}\) are the conserved quantities defined in Example 1.3 (2). Fix \(y \in \Lambda\) and \(y' \in \Lambda'\). We let \(\eta = (\eta_x) \in S^Y\) be an element such that \(\eta_{y'} = 1\), \(\eta_y = 2\) and is at base state outside \(y\) and \(y'\), and we let \(\eta' = (\eta'_x) \in S^{Y'}\) be an element such that \(\eta'_{y'} = 2\), \(\eta'_y = 1\) and is at base state outside \(y\) and \(y'\). Then we have \(\xi^{(1)}_{\Lambda'}(\eta)\xi^{(2)}_{\Lambda'}(\eta') = 1\) but \(\xi^{(1)}_{\Lambda'}(\eta')\xi^{(2)}_{\Lambda'}(\eta') = 0\), hence we have

\[
t^{\Lambda \cup \Lambda'} f(\eta) - t^\Lambda f(\eta) - t^{\Lambda'} f(\eta) \neq t^{\Lambda \cup \Lambda'} f(\eta') - t^\Lambda f(\eta') - t^{\Lambda'} f(\eta'),
\]

which gives a contradiction as desired.

\(\square\)

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References

[1] M. F. Atiyah and C. T. C. Wall, Cohomology of groups, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 94–115. MR0219512
[2] K. Bannai and M. Sasada, A Decomposition Theorem of Varadhan Type for Co-local Forms on Large Scale Interacting Systems (2021), preprint, available at [arXiv:2105.06043[math.PR]]
[3] K. Bannai and M. Sasada, Varadhan’s Decomposition of Shift-Invariant Closed \(L^2\)-forms for Large Scale Interacting Systems on the Euclidean Lattice (2021), preprint, available at [arXiv:2111.08934[math.PR]]
[4] Daniel E. Cohen, Ends and free products of groups, Math. Z. 114 (1970), 9–18, DOI 10.1007/BF01111864. MR260877
[5] C. Erignoux, Hydrodynamic limit for an active exclusion process (2020), preprint, available at [hal-01350532v3]
[6] T. Funaki, K. Uchiyama, and H. T. Yau, Hydrodynamic limit for lattice gas reversible under Bernoulli measures, Nonlinear stochastic PDEs (Minneapolis, MN, 1994), IMA Vol. Math. Appl., vol. 77, Springer, New York, 1996, pp. 1–40, DOI 10.1007/978-1-4613-8468-7_1. MR1395890
[7] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan, *Nonlinear diffusion limit for a system with nearest neighbor interactions*, Comm. Math. Phys. 118 (1988), no. 1, 31–59. MR954674

[8] C. Kipnis and C. Landim, *Scaling limits of interacting particle systems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 320, Springer-Verlag, Berlin, 1999. MR1707314

[9] C. Kipnis, C. Landim, and S. Olla, *Hydrodynamical limit for a nongradient system: the generalized symmetric exclusion process*, Comm. Pure Appl. Math. 47 (1994), no. 11, 1475–1545, DOI 10.1002/cpa.3160471104. MR1296786

[10] M. Kotani and T. Sunada, *Standard realizations of crystal lattices via harmonic maps*, Trans. Amer. Math. Soc. 353 (2001), no. 1, 1–20, DOI 10.1090/S0002-9947-00-02632-5. MR1783793

[11] Y. Nagahata, *Fluctuation dissipation equation for lattice gas with energy*, J. Statist. Phys. 110 (2003), no. 1-2, 219–246, DOI 10.1023/A:1021022829221. MR1966328

[12] S. Olla and M. Sasada, *Macroscopic energy diffusion for a chain of anharmonic oscillators*, Probab. Theory Related Fields 157 (2013), no. 3-4, 721–775, DOI 10.1007/s00440-012-0469-5. MR3129802

[13] J. Quastel, *Diffusion of color in the simple exclusion process*, Comm. Pure Appl. Math. 45 (1992), no. 6, 623–679, DOI 10.1002/cpa.3160450602. MR1162368

[14] D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996. MR1357169

[15] M. Sasada, *Hydrodynamic limit for two-species exclusion processes*, Stochastic Process. Appl. 120 (2010), no. 4, 494–521, DOI 10.1016/j.spa.2010.01.002. MR2594368

[16] M. Sasada, *Hydrodynamic limit for exclusion processes with velocity*, Markov Process. Related Fields 17 (2011), no. 3, 391–428. MR2895558

[17] T. Sunada, *Topological crystallography*, Surveys and Tutorials in the Applied Mathematical Sciences, vol. 6, Springer, Tokyo, 2013. With a view towards discrete geometric analysis. MR3014418

[18] S. R. S. Varadhan, *Nonlinear diffusion limit for a system with nearest neighbor interactions. II*, Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), Pitman Res. Notes Math. Ser., vol. 283, Longman Sci. Tech., Harlow, 1993, pp. 75–128. MR1354152

[19] S. R. S. Varadhan and H. T. Yau, *Diffusive limit of lattice gas with mixing conditions*, Asian J. Math. 1 (1997), no. 4, 623–678, DOI 10.4310/AJM.1997.v1.n4.a1. MR1621569

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