Scalable quantum control and non-abelian anyon creation in a honeycomb lattice

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The honeycomb spin-1/2 lattice is a model allowing for experimentally implementable quantum computation with topological protection of quantum information. Practical implementation of quantum information processing typically relies on adiabatic, i.e. slow dynamics. Here we show that the restriction to adiabatic dynamics can be overcome with optimal control theory, enabled by an extension of the fermionisation of the honeycomb model to the time-dependent case.

I. INTRODUCTION

Non-abelian anyons are the foundational theoretical tools for topological quantum computation. These generalisations of bosons and fermions, which can only exist in two-dimensional systems [1], allow for a topological form of gate implementation due to their non-trivial braiding statistics [2][4]. Logical qubits are encoded non-locally and anyons may be braided around one another and subsequently fused to carry out computations and measurements respectively as part of the overall implantation of a quantum algorithm [3][5].

The honeycomb lattice model is a notable example of a relatively simply defined system with non-trivial topological order [3][4]. In this system, anyons manifest as vortices introduced into the model which may be fused to create fermionic excitations corresponding to anyonic fusion rules known as Ising anyons [6,7]. Anyonic braiding itself and indeed any form of particles being interchanged is generally assumed to be an adiabatic process [8] so that unwanted excitations may be safely suppressed.

In practice, however, the restriction to adiabatic dynamics is typically conflicting with the requirement to realise all operations on a time-scale that is short compared to the system’s coherence time [9]. Quantum control has proven successful in speeding up adiabatic evolution in a wide range of scenarios [10][14] suggesting its suitability for anyon creation in topological systems. Common optimal control techniques, however, are limited in their applicability to the honeycomb model. Due to the exponential scaling of composite quantum systems, numerical simulations of the time-dependent honeycomb model are only possible for very small system sizes. Fermionisation of the honeycomb model allows for improved scaling and solving larger systems but thus far this has been restricted to systems with time-independent Hamiltonians. The scope of this article is to use quantum control in applying fermionisation within the context of a time-dependent version of the model. This will demonstrate that optimal control does indeed provide access to faster-than-adiabatic anyon creation.

Sec. II of this paper provides a brief overview of anyons in the honeycomb lattice to setup the operators and terminology required for the control problem. Sec. III describes quantum control and pulse optimisation for anyon creation and sets up the key result on fermionisation, which is proven in the appendix. Sec. IV presents explicit numerical results of the optimal control problem defined. An overall summary of results and conclusions are presented in the final section.

II. ANYONS IN THE HONEYCOMB LATTICE

In this section we review one of the numerous methods of solving the Kitaev honeycomb model [15][17]. This is followed by a demonstration of how vortex creation is implemented within the model [3][18] and how this corresponds to creation of non-abelian anyons.

A. Diagonalising the honeycomb

Although diagonalisation of the honeycomb model is not required for solving the time-dependent control problem we define later, we still outline its strategy, as the operators and terminology introduced will also play a role in the time-dependent version. The honeycomb system takes its name from its hexagonal lattice geometry consisting of spin-1/2 particles located at the vertices of hexagonal plaquettes, as shown in Fig. 1. It is defined by the Hamiltonian

\[ H = - \sum_{(j,k) \in N_2} J_{jk} \sigma_j^x \sigma_k^x - K \sum_{(j,k,l) \in N_3} \sigma_j^y \sigma_k^y \sigma_l^y, \]

where \( N_2 \) correspond to honeycomb edges and \( N_3 \) are certain triplets described further below. There are three types \( s = x,y,z \) of two-body nearest neighbour Pauli interactions determined by the position of the edge in the lattice, highlighted in three colours in Fig. 1. The three-body terms act within each hexagonal plaquette in the following way: three adjacent spins contribute to a three-body interaction term with the middle spin interacting through the Pauli operator corresponding to the link pointing outwards from the plaquette, while each of the two remaining spins interact through the Pauli operator corresponding to their link to the middle spin. For example in the plaquette highlighted in Fig. 1 one

\[ \sum_{(j,k,l) \in N_3} \sigma_j^y \sigma_k^y \sigma_l^y, \]
of the three-body interaction terms would be $\sigma^x_j \sigma^z_j \sigma^z_j$, with 5 similar terms following clockwise along the hexagonal plaquette. While the two-body part of the Hamiltonian allows for the model to be solved by a process of Majorana fermionisation, the three-body part preserves the solvability of the model while also breaking time-reversal symmetry and it consequently gives the system non-trivial topological order [3].

The mapped Majorana fermionic Hamiltonian is

$$H_f = \frac{iJ}{2} \sum_{(j,k) \in N_2} \hat{u}_{jk} c_j c_k + \frac{iK}{2} \sum_{\{j,l\},\{k,l\} \in N_2} \hat{u}_{jl} \hat{u}_{kl} c_j c_k,$$

is defined in terms of Majorana operators where we may also define link operators $\hat{u}_{jk} = ib_j b_k^\dagger$ in a system of $L$ total links and $N$ spin qubits. The eigenvalues $\pm 1$ of link operators allow for further partitioning of each vortex sector into link sectors. To this end, we can define a corresponding link sector projector

$$P_u = \prod_{\{j,k\} \in N_2} \frac{1 + u_{jk} \hat{u}_{jk}}{2},$$

which amounts to a tuple of chosen eigenvalues $u_{jk} \in \{\pm 1\}$.

Picking a certain link sector corresponds to fixing a gauge for a specific vortex sector and leads to a quadratic fermionic Hamiltonian $P_u H_f P_u$ that is easily diagonalised [20]. The trivial gauge would consist of setting all link eigenvalues to $u_{jk} = 1$, alongside the constraints imposed by their anti-symmetry $\hat{u}_{jk} = -\hat{u}_{kj}$. This amounts to defining an orientation which for concrete purposes we define as follows: a positive orientation on an $x$-link is directed from the bottom-left qubit to the top-right one ($j$ to $k$), for a $y$-link it is directed from the bottom-right to the top-left, and for a $z$-link it is directed from top to bottom.

The Hamiltonian becomes diagonal in a certain quasiparticle basis $H = \sum_{\omega > 0} \omega_j b_j^\dagger b_j - E_g$ and the ground state is the quasiparticle vacuum state. As previously stated, for physical states of the original Hamiltonian, the states will need to be projected; thus eigenstates of the spin Hamiltonian $|\Psi\rangle$ are related to eigenstates of the quadratic fermionic Hamiltonian $|\psi\rangle$ by

$$|\Psi\rangle = P_D P_u |\psi\rangle.$$

FIG. 1: A honeycomb lattice showing the three kinds of interactions between neighbouring spins which are on the vertices of hexagonal plaquettes. Red, blue and green links indicate $\sigma^x \sigma^x$, $\sigma^y \sigma^y$ and $\sigma^z \sigma^z$ interactions respectively. The hexagonal plaquette operators correspond to many conserved quantities in the system. A generic plaquette whose spin sites have been numbered 1 to 6 is highlighted in grey.

For every hexagonal plaquette, a corresponding plaquette operator $W_p$ may be defined which acts on every spin with the Pauli operator of the outward pointing interaction, so that for example on the numbered grey plaquette in Fig. 1 we have $W_p = \sigma^z_1 \sigma^z_2 \sigma^z_3 \sigma^z_4 \sigma^z_5 \sigma^z_6$. Each plaquette operator squares to the identity so that its eigenvalues adopt the values $\pm 1$ only. Since the plaquette operators all commute with the Hamiltonian and with one another, the system Hilbert space is naturally partitioned into simultaneous eigenspaces of all plaquette operators.

Negative plaquette eigenstates are known as vortices and by a well known theorem [19] it is known that the ground state eigenspace is in the no-vortex sector [3]. Different vortex sectors relate to the presence of anyons localised at the respective vortex plaquettes.

While solving the system remains an intractable exponential problem even after restriction to one vortex sector, a mapping of the problem into a Majorana fermionic Hamiltonian provides a pathway towards diagonalisation. For a more detailed breakdown of diagonalisation see [3] [19]. The mapped Majorana fermionic Hamiltonian is defined by replacing spin qubit sites $j$ with two fermionic sites and their corresponding creation operators $a_{j,1}^\dagger$ and $a_{j,2}^\dagger$. For each site $j$, the real and imaginary parts of the two fermionic modes constitute a total of four Majorana modes per site. The Majorana creation/annihilation operators are defined

$$b_j^x = a_{j,1} + a_{j,1}^\dagger, \quad b_j^y = \frac{1}{i}(a_{j,1} - a_{j,1}^\dagger),$$

$$b_j^z = a_{j,2} + a_{j,2}^\dagger, \quad c_j = \frac{1}{i}(a_{j,2} - a_{j,2}^\dagger).$$

Since with this mapping the Hilbert space is enlarged, a projection is required to obtain vectors that correspond to states in the original Hilbert space of the honeycomb model. This requires the stabiliser projector $[3] [19]

$$P_D = \prod_{j} \left( \frac{1 + D_j}{2} \right)$$

with $D_j = b_j^x b_j^y c_j$.

$\sigma^x$
III. ANYON CREATION AS AN OPTIMAL CONTROL PROBLEM

Let us first describe the usual adiabatic approach. Vortex creation is implemented in terms of the time-dependent Hamiltonian

\[ H(t) = H + \frac{t}{T} H_{\text{control}}^{j,k}, \]

comprised of the original honeycomb Hamiltonian as a drift combined with a control Hamiltonian

\[ H_{\text{control}}^{j,k} = 2J_{jk} \sigma_j^x \sigma_k^x + 2K \sum_{j,k \in \{a,b,c\}} \sigma_a^x \sigma_b^y \sigma_c^z. \]

where \( \{a, b, c\} \) is in \( N_3 \). This amounts to gradually reversing the sign of a specific \( s \)-link, as well as the sign of the nearby three-way interactions, using a linear time-dependence, with steepness and therefore adiabaticity determined by the duration \( T \) of the adiabatic protocol. As we will see in the numerical examples, \( T \) needs to be very large to obtain a good fidelity.

We now want to set up anyon creation as an optimal control problem in the hope that we can obtain similar fidelities in much shorter times compared to the adiabatic evolution. To this end, generalise the time-dependence of Eq. (8) as

\[ H(t) = H + f(t) H_{\text{control}}^{j,k}, \]

where \( f(t) \) is the ramp function defined such that \( f(0) = 0 \) at the initial point in time \( t = 0 \) and such that \( f(T) = 1 \) at the final point in time \( t = T \). Typically, \( f(t) \) is assumed to be piecewise smooth or piecewise constant.

A typical figure of merit to be maximised is the state fidelity \( \mathcal{F} \) defined in terms of an initial state \( |\Phi_0\rangle \), the propagator \( U[f(t),T] \) induced by the time-dependent Hamiltonian \( H(t) \) and a target state that is meant to be created. In the present case, the initial state \( |\Phi_0\rangle \) would usually be the ground state of the honeycomb model whereas the target state \( |\Phi_{\text{target}}\rangle \) is a state with an additional vortex pair created.

While optimising for such a state fidelity is a generally successful approach, it has two flaws when it comes to the honeycomb model. Since topological stability only arises for large lattice size, any practical application of the honeycomb model requires a vast number of qubits. Evaluating the time-evolution operator therefore requires numerics in exponentially large spaces. Secondly, even if the ground state is solvable analytically in the free fermion picture, we would have to translate it back into the spin picture to evaluate \( \mathcal{F} \), which is again exponentially hard. In the following, we will resolve both problems to obtain a scalable optimisation method.

A. Time-dependent fermionic picture

Let us first describe the time-independent case. In the fermionic picture, the quadratic Majorana Hamiltonian can be written in the most general form with a matrix \( J_{jk} \) that incorporates all interaction factors \( J \) as

\[ H = \frac{1}{2} \sum_{j,k} J_{jk} c_j c_k. \]

When written in terms of full fermionic creation and annihilation operators this is:

\[ H = \frac{1}{2} \alpha^\dagger M \alpha \]

where \( \alpha = (a_1, \ldots, a_N, a_1^\dagger, \ldots, a_N^\dagger)^T \) and the Hermitian matrix \( M \)

\[ M = \begin{pmatrix} \mu & \nu \\ -\nu^* & -\mu^* \end{pmatrix} \]

may be defined in terms of a Hermitian matrix \( \mu \) and an antisymmetric matrix \( \nu \).

A canonical transformation \( T \) can then be found so that \( TMT^{-1} = \text{diag}(\omega, -\omega) \) where \( \omega \) is a diagonal 2N-by-2N matrix. This allows for the Hamiltonian to be diagonalised in terms of quasiparticle modes [20].

We now consider how, in the fermionic picture, we may calculate the fidelity between a state evolved from an initialised state by a time-dependent Hamiltonian towards a target state. To this end, we write a ground state of \( H \) as \( |\Phi_0\rangle = A |\text{vac}\rangle \), with the vacuum state \(|\text{vac}\rangle\) satisfying the relation \( a_i |\text{vac}\rangle = 0 \) \( \forall i \). The operator \( A \) is some appropriately chosen function of creation and annihilation operators. In the Appendix, we show that the state fidelity in the fermionic picture is given by

\[ \mathcal{F}(t) = |\langle \Phi_{\text{target}} | k P_D P_u V[f(t),t] A |\text{vac}\rangle|^2 \]

|\langle \Phi_{\text{target}} | k P_D P_u A V[f(t),t] |\text{vac}\rangle|^2. \]

(13)

(14)

Here the projector \( P_u \) is given by Eq. (9), \( V[f(t),t] \) is the evolution operator corresponding to the quadratic Hamiltonian \( P_u H(t) P_u \) and \( P_D \) is given by Eq. (4), while \( k \) is a real number which depends on the specific lattice (see Appendix for specific examples). \( A(t) \) is a Heisenberg picture operator \( A(t) = V[f(t),t] A \) \( V[f(t),t] \)

In analogy to Eqs (11) and (12) it is useful to decompose \( P_u H(t) P_u \) as

\[ P_u H(t) P_u = P_u \frac{1}{2} \alpha^\dagger M(t) \alpha \]

(15)

Since \( A \) depends on annihilation and creation operators, we may write it as \( A(\alpha) \). It can then be shown [20] that

\[ V[f(t),t] A(\alpha) V^\dagger(f,t) = A(W[f(t),t] \alpha) \]

(16)

where \( W(t) \) is the 2N-by-2N the time-ordered product solving the differential equation

\[ \dot{W}[f(t),t] = iM[f(t)] W[f(t),t]. \]

(17)
This solves the problem of an exponentially sized evolution operator, as \( W(t) \) scales linearly in the system size. We will refer to calculations in this picture as the Heisenberg picture, since it is directly obtained from the Heisenberg equations of motions of \( \alpha \). However, the problem of expressing the target and initial state in the spin picture remains. This will be tackled next.

### B. Heisenberg fidelity as optimisation target

In the previous paragraph we showed that the evolution is fully determined by the Heisenberg picture of the quadratic Hamiltonian. If we knew a good target evolution, rather than target state, we could therefore free ourselves from the state picture and obtain all quantities directly in the Heisenberg picture. The key idea here is to get back to the adiabatic evolution to find such good evolution. We phrase such evolution directly in the Heisenberg picture. To this end, let \( W_{\text{ad}} \) be the solution of Eq. (17) for the adiabatic ramp Eq. (8) with some suitably large duration \( T_{\text{ad}} \). This can be computed efficiently without having to refer to states. We define a corresponding Heisenberg fidelity

\[
F_H = \frac{1}{2N} \left| \text{Tr} \left( W_{\text{ad}}^N W[f(t), T] \right) \right| \tag{18}
\]

This quantity obtains its maximum 1 if and only if the evolutions match up to a phase and it can be used for efficient numerical optimisation. In Appendix B, we show that

\[
1 - F_H \geq \frac{1}{32N^3} \left( 1 - \sqrt{F} \right). \tag{19}
\]

This shows that \( F_H \) is a good surrogate for \( F \) and may be optimised instead.

### IV. NUMERICAL RESULTS

Here, we use the QuTiP implementation of GRAPE using L-BFGS as an optimiser with exact gradients. The optimisation takes place over piecewise-constant functions, which means that the number of time-steps becomes an additional parameter of our numerics.

#### A. Optimised non-adiabatic pulse in a simple lattice

While the timescale of anyon creation through adiabatic evolution can be very long, if instead of using linear ramps we use non-linear time-dependence in the Hamiltonian which have been specially designed, then we can achieve high fidelities at shorter timescales. A well-tested gradient-ascent pulse engineering algorithm \([21, 22]\) is used to develop such time-dependent control functions also known as pulses.

The results of using this procedure for a single plaquette of 6 spins are depicted in Fig. [2] showing infidelities as function of the ramp time \( T \). The infidelities obtained with a linear ramp are depicted in blue. There is a slight improvement with increasing ramp time, but the fidelity of about 90% achieved with a ramp time of \( T = 1 \) is only a very small improvement compared to the initial fidelity at time \( T = 0 \). This is consistent with an estimate based on the spectral gap condition that implies ramp times \( T > 0.5 \) are required for high-fidelity operations.

The behaviour with optimised ramps, depicted in orange, is fundamentally different. In the range \( T < 0.2 \) there is a much faster decrease of infidelity with increasing ramp time than in the case of linear ramps. This decrease is a bit less pronounced in the range \( 0.2 < T < 0.4 \), but for ramp time \( T > 0.4 \) this decrease becomes increasingly pronounced with increasing ramp time. For ramp times \( T > 0.8 \), there is a rapid drop in the infidelity, and for ramp times exceeding the threshold value of \( T = T_d \approx 0.85 \), the deviations between the numerically obtained infidelities and the ideal value of 0 are consistent with noise due to finite numerical accuracy.

It is by no means surprising that even with an optimised ramp a finite ramp time is required to reach perfect fidelities. This is due to the constant part in the system Hamiltonian Eq. (10) that defines a natural time-scale of the system. This effect is also referred to as the quantum speed limit \([23, 24]\) and we will refer to the threshold value \( T_d \) of ramp durations at which fidelities drop to values close to their ideal value as the drop time. Apart from limitations imposed by finite-dimensional parametrisation of the ramp function, the numerical optimisation routine and numerical accuracy, this drop time coincides with the minimal duration required to reach perfect fidelity.

The example of a single plaquette with 6 qubits is also a good test-case to compare optimisation of state-fidelity and Heisenberg fidelity. Fig. [3] depicts the Heisenberg fidelity obtained with various ramp functions as function of the ramp time \( T \).

The black triangles represent state fidelity data obtained with linear ramps. Consistently with Fig. [2] there is only a moderate decrease of the infidelity with increasing ramp time. The circles correspond to ramp functions optimised for Heisenberg fidelity, and the different colours correspond to different chosen adiabatic target times with \( T = 100, 200 \) and \( 300 \) for red, orange and green respectively. Similarly to the observations in Fig. [2] there is a clear drop of the infidelities at a drop time \( T_d \approx 3 \). The fact that the numerically observed drop time is essentially the same in all three cases indicates that the drop time is not dependent on the length of the adiabatic target time that is chosen.

The squares depict the state infidelity obtained from the ramp functions that had been optimised for Heisenberg infidelity, in order to see numerical evidence that when implementing our procedure, good fidelity is achieved in the one case which ensures a good infidelity
FIG. 2: Logarithmic comparison of infidelity between initial and adiabatic target state for linear control pulses and optimised control pulses, at various timescales, for the simplest 6-qubit lattice. 100 time steps were used in both cases. We can see more clearly the dramatic improvement in fidelity at approximately $T = 0.8$. The minimum infidelity reached by optimised pulses, on the order of $10^{-9}$, is reached at time $T \approx 0.9$, many orders of magnitude less than the time to reach this infidelity with the linear ramp, which is at $T \approx 1350$.

Here also is a clear drop of the infidelities and it occurs at the same drop time as for Heisenberg infidelities. The fidelities obtained for ramps with a longer ramp time than the drop time $T_d$, however, are not merely limited by numerical accuracy, but they are indeed finite. Their exact value depends on the parametrisation of the ramp function, with finer parametrisations resulting in lower infidelities. Since state fidelity and Heisenberg fidelity are not strictly equivalent, it is not surprising, that a ramp that is optimised for one of these fidelities does not yield the optimally achievable value of the other fidelity. The results in Fig. 2, however, clearly show that ramps optimised for Heisenberg fidelity result in high state fidelities and, in particular, in infidelities that are between 3 and 4 orders of magnitude lower than infidelities obtained with linear ramps.

Since numerical optimisations of state fidelity become rapidly infeasible with increasing system size, the subsequent examples for larger systems feature only Heisenberg fidelities with ramps that are optimised for this Heisenberg fidelity. Fig. 3 shows the Heisenberg infidelity as function of ramp time for linear ramps (blue) and for optimised ramps (orange). Similarly to the cases discussed above, there is a clearly identifiable drop time $T_d$, but its value $T_d \approx 10$ is larger than in the examples of smaller systems. The abscissa depicts that ramp time on the log-scale, highlighting that linear ramps with durations exceeding the drop time by many orders of magnitude are required to achieve any sizeable decrease in infidelity.

The inset depicts a zoomed-in look into the domain around the drop time. It highlights that, on top of the rapid drop of infidelity there is also a finite noise level. When we compare the optimisation results of a system made up of 10 spin-qubits and one with 30-qubits, whose optimised infidelities are shown in Fig. 3, we see again the marked increase in drop time that is achieved.

The observation that the drop time increases with the system size is depicted more systematically in Fig. 4. Generally, the practically achievable drop time depends

FIG. 3: 6-qubit lattice: Heisenberg infidelities based on differing targets (circles) are compared with their corresponding state fidelities (squares). These are compared overall with state fidelities achieved from linear ramp pulses (triangles). Different colours represent different values of $T_\text{ad}$, which are the timescales of 100 (red), 200 (orange) and 300 (green) that define the target unitary $W_\text{ad}$. All optimised pulses are comprised of 200 time steps. There is a significant improvement at times between $T = 2$ and $T = 4$. Thereafter there is effectively perfect fidelity with the presence of numerical noise. It confirms that both Heisenberg and state infidelities improve markedly at the same time $T$ and this is an improvement on the fidelities achieved with a linear ramp pulse.

FIG. 4: 10 qubit lattice: Heisenberg infidelities between optimised unitaries and an optimised target are shown (orange), as compared with infidelities between the target and a unitary defined by a linear ramp control (blue). Each optimised pulse consists of 200 time steps. The drop time is $T \approx 10$, later than for the 6-qubit lattice.
on the parametrisation of the ramp function and the number of free parameters that can be optimised. Only for a ramp function with sufficiently many piecewise constant elements is the drop time independent of the number of free parameters. Fig. 6 depicts that shortest achievable drop time together with the minimal number of piecewise constant elements in the ramp function that is required to achieve this drop time. In both sub-plots the scaling is consistent with a quadratic increase (blue line) with system size.

Within the validity of extrapolation from numerically accessible system sizes, there is thus a very moderate scaling with system size, highlighting that numerical optimisations based on the framework of fermionisation can be performed efficiently.

V. CONCLUSIONS

It is well known that the honeycomb model is a useful system for the encoding of quantum information using non-abelian anyons. Our use of quantum control techniques has allowed us to create anyons not only in adiabatic timescales, but to overcome the necessity for such a restriction through the use of gradient ascent pulse-engineering to construct non-linear ramps whose timescales are many orders of magnitude faster than adiabatic. These ramps achieve very high fidelities at these much faster timescales, and thus are more useful for the encoding of quantum information and quantum algorithm implementation given that decoherence times entail a preference for shorter timescales.

Along with the drawback of requiring long timescales with adiabatic control, the other chief drawback we would encounter with non-adiabatic quantum control is the difficulty in completely determining the dynamics of honeycomb lattices with large numbers of spin qubits. The method we have developed allows us to overcome this obstacle by solving a matrix control problem where the matrices grow linearly in the number of lattice spin-qubits, allowing for the implementation of non-linear pulse-engineering and optimisation of a related matrix or Heisenberg fidelity. This, alongside the analytic expression directly comparing state fidelity and Heisenberg fidelity allows us to be assured of the success of these optimised ramps and allows us to have confidence in the successful implementation of our procedure for us in experimental realisation of the honeycomb lattice. Moreover we have seen that there is indeed sub-exponential growth in the computational difficulty of this control problem with growing system size and so carrying out our methods on lattices on the order of hundreds of qubits is possible, as necessary for scalable topological robustness.
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Appendix A: Dynamics of the time-dependent Hamiltonian in the fermionic picture

Previous work on the honeycomb model considered time-independent Hamiltonians and showed that the model can be drastically simplified through fermionisation. For control we need time-dependent Hamiltonians. Here we show that in regular lattices with open, periodic or half-periodic lattices, fermionisation is still possible.

First let us recall the projections onto a link sector \( u \)

\[
P_u = \prod_{(jk)} \frac{1 + u_{jk} \hat{u}_{jk}}{2},
\]

(A1)

defined by an \( L \)-tuple of link eigenvalues \( u_{jk} \in \{\pm 1\} \); the projector onto a vortex sector \( w \)

\[
P_w = \prod_{j} \frac{1 + w_j \hat{w}_j}{2},
\]

(A2)

defined by a \( P \)-tuple of plaquette eigenvalues \( w_j \in \{\pm 1\} \); and the projector onto the physical subspace of the fermionic space given by

\[
P_D = \prod_{j} \frac{1 + D_j}{2}, \text{ with } D_j = b_j^x b_j^y b_j^z c_j.
\]

(A3)

Since

\[
\hat{w}_j = \prod_{(k,l) \in w,j} \hat{u}_{kl},
\]

(A4)

\( P_u \) and \( P_w \) commute, and moreover \( w \) is fully determined by \( u \). We denote this relationship as \( w = \omega(u) \) and thus have \( P_w P_u = P_u \delta_{u,\omega(u)} P_u \) and

\[
P_w = P_w \sum_u P_u = \sum_{u,\omega(u)=w} P_u.
\]

(A5)

\( P_w \) commutes with \( P_D \), \( P_u \) and the time-dependent fermionic Hamiltonian \( H_f(t) \), so it will suffice to restrict ourselves to a single plaquette sector \( w \). While \( P_D \) is the projection that determines physicality, \( P_u \) will turn the fermionised Hamiltonian into a quadratic (and thereby easy to solve) one. A difficulty arises from the fact that \( P_D \) does not commute with \( P_u \). We can however find another useful relationship between these projectors. Let \( N \) be the number of qubits of the original spin lattice and \( \{\Gamma_k | k = 1, \cdots, 2^N\} \) be the set of all possible products of stabiliser operators \( D_i \) on the qubits, without repetition, where we take an arbitrary but fixed order. For our fixed \( w \) consider the corresponding pre-image \( \omega^{-1}(w) \) of link sectors. We define an equivalence relationship \( \sim \) on this set by \( u \sim v \iff \exists k : P_u = \Gamma_k P_v \Gamma_k \). Let \( \kappa \) be the number of equivalence classes. Let us choose an arbitrary but fixed set of representatives \( u_1, \ldots, u_\kappa \) and define

\[
P_u = \sum_{i=1}^\kappa P_{u_i}.
\]

Upon expanding \( P_D \) in terms of stabilisers, we obtain

\[
P_D P_u P_D = \frac{1}{2^N} \sum_{i=1}^\kappa \sum_{k=1}^\kappa \Gamma_k P_{u_i} \Gamma_k.
\]

(A6)

To understand the right hand side better, we make a counting argument. Firstly, it follows from the anticommutation relationships between link operators and the \( D_j \) that the \( \Gamma_k P_{u_i} \Gamma_k \) are again link projectors. Since we sum over all \( \Gamma_k \) and by definition of the equivalence classes, we know that every \( u \in \omega^{-1}(w) \) appears at least once on the right hand side, and that there are no overlaps between the classes of fixed \( i \). Furthermore the equality \( \Gamma P_{u_i} \Gamma = P_{u_i} \) holds if and only if \( \Gamma = 1 \) or \( \Gamma = \prod_i D_i \). Therefore, there are \( 2^N - 1 \) distinct projectors for each \( i \). It follows that

\[
P_D P_u P_D = \frac{1}{2^N} \sum_{w=\omega(u)=w} P_u = \frac{1}{2^{N-1}} P_D P_w.
\]

(A7)

From the commutativity relation \( [H_f(t), P_D] = 0 \) it also follows that

\[
V(t) P_D P_w = 2^{N-1} V P_D P_u P_D = 2^{N-1} P_D P_u V(t) P_u P_D.
\]

(A8)

where \( V(t) \) is the propagator corresponding to \( H_f(t) \). Hence, the evolution can be computed in the subspace \( P_u \). To conclude the argument, we need to know the value of \( \kappa \), as this determines how many link sectors we need to consider. As long as \( \kappa \) is not exponential, we can efficiently simulate the dynamics.

To this end, we need another counting argument. To simplify the analysis, we only consider three different regular lattice types dubbed open (o), periodic (p) and half-periodic (h), and find their corresponding values of \( \kappa \).

To do this, we first find relationships for the number of link operators \( L \), the number of plaquettes \( P \), and the number of qubits \( N \) for the various lattices. Simple but rather tedious counting of such regular lattices shows that \( L = P = N - 1 \) in the open case and \( L = P = N \) in the other two cases. Next, compute the size of \( \omega^{-1}(w) \). We show in the lemma below that \( |\omega^{-1}(w)|_{o,h} = 2^{L-P} \)
and $|\omega^{-1}(w)|_P = 2^{L-P+1}$. Since each equivalence class has exactly $2^{N-1}$ elements, we have to have

$$|\omega^{-1}(w)| = 2^{N-1}\kappa$$

such that $\kappa_o = 1$, $\kappa_h = 2$, $\kappa_p = 4$.

**Lemma A.1.** For all $w$, $|\omega^{-1}(w)|_{o,h} = 2^{L-P}$ and $|\omega^{-1}(w)|_p = 2^{L-P+1}$.

**Proof.** Consider first the case of a lattice with open boundary conditions. Since there are no boundary constraints, all possible configuration of plaquette eigenvalues $\{\Lambda_j\}$ are possible, and

$$|\{\Lambda_j\}|_o = 2^P. \quad (A9)$$

From the above, for each $u$, we can find $2^{N-1}$ other $v$ with $\omega(v) = \omega(u)$ by conjugation with $\Gamma_k$. Since in the open lattice the number of qubits $N$ follows the relation $N - 1 = L - P$, once can conclude that for each $w$, the inequality $|\omega^{-1}(w)|_o \geq 2^{L-P}$ holds. Since there are by definition $2^L$ different link sectors, we have

$$2^L = \sum_w |\omega^{-1}(w)|_o \geq 2^L = 2^{L-P+1} \quad (A10)$$

so equality holds and the statement follows.

Next, consider a lattice with full periodic boundary conditions. Any link eigenvalue change leads necessarily to exactly two plaquette eigenvalues being flipped. Therefore only even numbers of vortices may ever be present, and the number of plaquette eigenvalue configurations is

$$|\{\Lambda_j\}|_p = 2^{P-1}. \quad (A11)$$

Now, for each $u$, we can find $2^{N-1}$ other $v$ with $\omega(v) = \omega(u)$ by conjugation with $\Gamma_k$, but for each of these we can find 4 inequivalent link sectors. Since $L - P = N$ for periodic boundaries, we obtain

$$2^L = \sum_w |\omega^{-1}(w)|_p \geq 4 \times 2^{L-P-1} = 2^L \quad (A12)$$

and the statement follows again.

Finally, in the half-periodic case we only have 2 inequivalent link sectors, but $|\{\Lambda_j\}| = 2^P$ is twice as big as in the periodic case, and the same counting argument holds.

** FIG. 7:** Simple sets of $y$ (above) and $z$ (below) link flips that preserve the vortex sector while being gauge inequivalent to the trivial link sector.

** FIG. 8:** Topological interpretation of the set of $y$ and $z$ flips that preserve vortex sector but are gauge inequivalent to the trivial link sector which topologically amounts to no loop at all.

sector associated with it which we call $P_{u_0}$. We have already shown that $P_{u_0}$ is gauge equivalent, that is, equivalent up to conjugation by $\Gamma_k$ operators, with $2^{N-1}$ link sector projectors.

Acting on a non-zero eigenstate of $P_{u_0}$ with any stabiliser operator $D_j$ will change three link sector eigenvalues at a time, due to $D_j$ overlapping with the three types of link that contain spin site $j$, the same is therefore also true of $\Gamma_k$ operators. On the other hand acting on spin sites with a Pauli operator such as $Z_j$, which in
operators, as

\[ \tilde{Z}_j = i b^*_j c_j \]  

will flip the sign of only the z-link corresponding to that site and no others. In general a Pauli operator \( \tilde{\sigma}^\alpha \) will only flip the sign of the \( \alpha \)-link containing spin site \( j \). This is due to the fact that

\[ \{ \tilde{\sigma}^\alpha_j, \tilde{\sigma}^\beta_{jk} \} = \{ i b^*_j c_j, i b^*_k b^*_l c_l \} = 0 \]  

(\( \tilde{\sigma}^\alpha_{jk} = \{ i b^*_j c_j, i b^*_k b^*_l c_l \} = 0, \quad j \neq \{k, l\} \). 

Thus we have that

\[ \tilde{Z}_\alpha P_{\alpha j} = P_{\alpha k} \tilde{Z}_\alpha \]  

(A16)

and thus

\[ P_{\alpha j} = \tilde{Z}_\alpha P_{\alpha k} \tilde{Z}_\alpha \]  

(A17)

where in general \( \omega(u_j) \neq \omega(u_k) \).

Flipping a single link eigenvalue will change the vortex sector as a pair of vortices are introduced on adjacent plaquettes. Taking \( \zeta \) to be an arbitrary product of \( \tilde{Z} \) Pauli operators, then in order to have

\[ P_{\alpha j} = \zeta P_{\alpha k} \zeta \]  

(A18)

such that \( \omega(u_j) \) does equal \( \omega(u_k) \), we require \( \zeta \) to consist of a pair of link-flipping \( \tilde{Z} \) operators for each plaquette.

In order to return to the no-vortex sector, vortices need to be annihilated and so the vortex string must form a closed loop. Algebraically speaking, while each \( \tilde{\sigma}^\alpha_j \) will commute with all overlapping and non-overlapping \( D_j \) operators, as

\[ [\tilde{\sigma}^\alpha_j, D_j] = [i b^*_j c_j, b^*_j b^*_l b^*_j c_j] \]

(A19)

\[ = 0 \]

(A20)

and

\[ [\tilde{\sigma}^\alpha_{jk}, D_k] = [i b^*_j c_j, b^*_k b^*_l c_k] \]

(A21)

\[ = 0, \]

(A22)

such Paulis will not commute with overlapping plaquette operators \( W_p \), as for a spin site \( j \), plaquette operators will act with Paulis \( \hat{X} \) or \( \hat{Y} \) and thus there is anticommutation.

![FIG. 9](image)

**FIG. 9:** Clearly \([Z_6, W_p] = [Z_6, Z_1 X_2 Y_3 Z_4 X_5 Y_6] \neq 0 \) but \([Z_6 Z_2, W_p] = [Z_6 Z_2, Z_1 X_2 Y_3 Z_4 X_5 Y_6] = 0 \).

In order for there to be commutation with all plaquette operators, there needs to be \( \tilde{Z}_j \) operators acting on two spin sites per plaquette, as shown in Fig. 9. The simplest example of a product of \( \tilde{Z} \) operators that commutes with all link operators is shown explicitly and schematically in Fig. [7](#) and Fig. [6](#) respectively.

Starting with link sector \( P_{u_0} \) and using only \( \Gamma_k \) operators, it is not possible to flip the sign of only a single row of z-links as in general the action of a \( D_j \) operator flips the sign of all three types of link \( x, y \) and \( z \). In order to keep all \( x \) and \( y \) links unflipped, products of \( D_j \) operators must act on the sites of all spins that correspond to those links, leaving at least a pair of rows with flipped eigenvalues on z-links, as demonstrated in Fig. [10](#). We therefore have found another link sector \( P_u \) whose corresponding vortex sector is the no-vortex sector which cannot be reached from \( P_{u_0} \) by conjugation with \( \Gamma_k \) operators. Algebraically this can be represented by

\[ \hat{\not} \Gamma_k \text{ such that } \Gamma_k P_{u_0} \Gamma_k = P_u \]  

(A23)

where

\[ P_u := \zeta P_{u_0} \zeta \]  

(A24)

and \( \zeta \) is a horizontal, topologically closed loop of Pauli \( \tilde{Z} \) operators. This is because when commuting \( \zeta \) through \( P_{u_0} \), only the signs of a row of z-link eigenvalues are changed. If we call this row of links \( \rho \) then

\[ \zeta P_{u_0} = \zeta \prod_{j k} \frac{1 + u^\alpha_{jk} \tilde{u}^\alpha_{jk}}{2} \]

(A25)

\[ = \prod_{\alpha = z, j k \in \rho} \frac{1 - u^\alpha_{jk} \tilde{u}^\alpha_{jk}}{2} \prod \frac{1 + u^\alpha_{jk} \tilde{u}^\alpha_{jk}}{2} \zeta \]

(A26)

\[ = P_{u_k} \zeta. \]

(A27)

However with \( \Gamma_k \) operators, in order to commute them through \( P_{u_0} \) and have only z-link eigenvalues flipped, this requires a \( \Gamma_k \) consisting of \( D_j \) operators acting on a row of spin sites in the manner of Fig. [10](#) and thus a minimum of two rows of z-link eigenvalues are flipped. Thus for all \( \Gamma_k \)

\[ \Gamma_k P_{u_0} \neq \zeta P_{u_0} \zeta \]  

(A28)

We can now repeat the above process, with the same reasoning, starting with the link sector \( P_{u_+} \) rather than
Us defined between two states with differing dynamics and so we have accounted for all link sectors. There would therefore be two sets of 2 link sector projectors, which satisfies the differential equation

$$\dot{H} = \frac{1}{4d^3} \left(1 - \sqrt{F_s(\Psi_1, \Psi_2)}\right)$$

(A29)

then the four link sector operators

$$P_{u_0} = \zeta P_{u_0} \zeta,$$

(A30)

$$P_{u_z} = \chi P_{u_0} \chi,$$

(A31)

$$P_{u_y} = \zeta \chi P_{u_0} \chi \zeta,$$

(A32)

$$P_{u_x} = \chi \chi P_{u_0} \chi \zeta$$

(A33)

$$(A34)$$

each define gauge-inequivalent link sectors in a particular vortex sector.

**Inequivalent sectors: Half-periodic boundary conditions**

For a system with half-periodic boundary conditions, depending on whether periodicity is in the horizontal or vertical direction, only a single row or ‘column’ of flipped signs would be required to find a link sector projector not reachable from $P_{u_0}$ by actions of $\Gamma_k$ operators. There would therefore be two sets of $2^{N-1}$ link sector projectors corresponding to two equivalence classes in each vortex sector. Accounting for all $2^P$ vortex sectors there would be a total of

$$2^P (2 \times 2^{N-1}) = 2^{N+P} = 2^L$$

(A35)

and so we have accounted for all link sectors.

**Appendix B: Proof of matrix-infidelity state-infidelity bound**

This section contains the proof of Eq. (19) which relates state infidelity and Heisenberg infidelity. State infidelity is defined between two states with differing dynamics $U_j$ ($j = 1, 2$) applied to an initial spin state $|\Psi_I\rangle$. These dynamics are induced by spin Hamiltonians $H_{\alpha,j}$ whose counterparts in the quadratic Majorana fermionic picture $H_{\alpha,j} = \alpha \mathcal{M}_j \alpha$ (see Section 6A) define orthogonal matrices $\tilde{O}_j$ generated by the matrices $\mathcal{M}_j$ in a relation which satisfies the differential equation $\dot{O}_j = i\mathcal{M}_j \tilde{O}_j$.

The relation to be proven reads

$$\mathcal{I}_H(\mathcal{O}_1, \mathcal{O}_2) \geq \frac{1}{4d^3} \left(1 - \sqrt{F_s(\Psi_1, \Psi_2)}\right)$$

(B1)

with the state fidelity

$$F_s(\Psi_1, \Psi_2) = |\langle \Psi_1 | \Psi_2 \rangle|^2 = \left|\langle \Psi_I | U_j^\dagger U_1 | \Psi_I \rangle\right|^2,$$

(B2)

and the Heisenberg infidelity $\mathcal{I}_H = 1 - F_H$ with

$$F_H(\mathcal{O}_1, \mathcal{O}_2) = \frac{1}{d} \left|\text{Tr} \left( \mathcal{O}_1^\dagger \mathcal{O}_2 \right) \right|,$$

(B3)

and the matrix dimension $d$.

The full proof consists of Eq. (B4) and the series of inequalities Eq. (B5) to (B7)

$$\sqrt{2d} \sqrt{\mathcal{I}_H} = \|\mathcal{O}_1 - \mathcal{O}_2\|_F$$

(B4)

$$\geq \frac{1}{2d} \|\Phi_{U_1} - \Phi_{U_2}\|_0$$

(B5)

$$\geq \frac{1}{2d} \min \|U_1 - e^{i\varphi} U_2\|_{\text{op}}$$

(B6)

$$\geq \frac{1}{\sqrt{2d}} \sqrt{1 - \sqrt{F_s}}$$

(B7)

that will be discussed separately in the following subsections.

Eq. (B4) is expressed in terms of the Frobenius norm

$$\|A\|_F := \sqrt{\text{Tr}(A^\dagger A)}$$

(B8)

for any operator $A$. Eq. (B5) is expressed in terms of the operator norm

$$\|A\|_{\text{op}} := \sup \left\{ \frac{\|Ax\|}{\|x\|} : x \in V^d \text{ with } x \neq 0 \right\}.$$  

(B9)

The diamond norm for a quantum channel $\Phi$ in Eq. (B6) is given by

$$\|\Phi\|_\diamond := \max \|\Phi \otimes \mathbb{1}\rho\|_1$$

(B10)

where $\|\circ\|_1 = \text{Tr} \sqrt{\Phi^\dagger \Phi}$ is the trace norm [25], and the maximisation is taken over all density matrices in a space of dimension corresponding to the size of the quantum channel.

1. **[19]**

The Frobenius norm of the difference between two orthogonal operators $\mathcal{O}_1$ and $\mathcal{O}_2$ reduces to

$$\|\mathcal{O}_1 - \mathcal{O}_2\|_F = \sqrt{2d - \text{Tr}(\mathcal{O}_1^\dagger \mathcal{O}_2) - \text{Tr}(\mathcal{O}_2^\dagger \mathcal{O}_1)}$$

(B11)

$$= \sqrt{2d - 2 \text{Re} \{\text{Tr}(\mathcal{O}_1^\dagger \mathcal{O}_2)\}},$$

(B12)

where $d$ is the dimension of $\mathcal{O}_1$ and $\mathcal{O}_2$. It thus depends on the real part of $\text{Tr}(\mathcal{O}_1^\dagger \mathcal{O}_2)$ and not on its absolute value as it is the case for $\mathcal{I}_H$. As we will show in the following, however, in the present case, the object $\text{Tr}(\mathcal{O}_1^\dagger \mathcal{O}_2)$ is real and positive, so that Eq. (B1) is indeed satisfied.
Proof that $\text{Tr}(O^T_1 O_2)$ is real

The orthogonal matrices $O_1$ and $O_2$ satisfy the differential equation $O_j = iM_j O_j$ with generally time-dependent generators $iM_j$. Since the $M_j$ are purely imaginary, the generators $iM_j$ are purely real. Together with the initial condition $O_j(0) = 1$, such that $O_j(0)$ is real, this implies that $O_j(t)$ for $j = 1, 2$ is real for all times. Consequently the overlap $\text{Tr}(O^T_1 O_2)$ is also real.

Proof that $\text{Tr}(O^T_1 O_2)$ is non-negative

Since Majorana fermions move between only one of two pairs of fermionic sites per spin site, the full space that $O_1$ and $O_2$ act on, can be divided into two subspaces $H_x$ and $H_y$ of equal dimension $d/2$. Both $O_1$ and $O_2$ are given as a direct sum of the identity $I$ in $H_x$ and orthogonal operators $\hat{O}$ in $H_y$. The complete trace is thus given as the sum of the two traces $\text{Tr}_x I$ and $\text{Tr}_y O^T_1 O_2$. The latter trace can also be expressed as the sum over the eigenvalues $\lambda_j$ of $O^T_1 O_2$. This results in the relation

$$\text{Tr}(O^T_1 O_2) = \text{Tr}_x I + \text{Tr}_y O^T_1 O_2 = \frac{d}{2} + \sum_{j=1}^{d} \lambda_j.$$  \hspace{1cm} (B13)

Since the trace $\text{Tr}(O^T_1 O_2)$ is purely real, the imaginary parts $\text{Im} \lambda_j$ add up to zero. Because all the eigenvalues $\lambda_j$ are phase factors, i.e. $\lambda_j = \exp(i\varphi_j)$ with $\varphi_j$ real, the real parts of the $\lambda_j$ satisfy the inequality $\text{Re} \lambda_j \geq -1$, so that the relation

$$\text{Tr}(O^T_1 O_2) = \frac{d}{2} + \sum_{j=1}^{d} \lambda_j \geq 0$$  \hspace{1cm} (B14)

is indeed given.

2. from (B4) to (B5)

The Frobenius and operator matrix norms satisfy the inequality $\|A\|_F \geq \|A\|_{op}$ \cite{29}. For the present case, this implies the relation

$$\|O_1 - O_2\|_F \geq \|O_1 - O_2\|_{op}.$$  \hspace{1cm} (B15)

Together with the relation

$$\|O_1 - O_2\|_{op} \geq \frac{1}{2d} \|\Phi_U_1 - \Phi_U_2\|_0,$$  \hspace{1cm} (B16)

that is proven in Eq. (171) in \cite{27} this yields the desired inequality.

3. from (B9) to (B10)

The required inequality

$$\|\Phi_U_1 - \Phi_U_2\|_0 \geq \min_{\varphi} \|U_1 - e^{i\varphi} U_2\|_{op}$$  \hspace{1cm} (B17)

is proven in Eq. (2.1) in \cite{28}.

4. from (B6) to (B7)

Following the definition of the operator norm (Eq. (B9)), the operator norm in Eq. (B6) satisfies the inequality

$$\|U_1 - e^{i\varphi} U_2\|_{op} \geq \|\langle U_1 - e^{i\varphi} U_2\rangle_{\Psi} \|_2$$  \hspace{1cm} (B18)

for any state vector $|\Psi\rangle$ in the spin picture, and, as such, in particular for the initial state $|\Psi_I\rangle$ of the dynamics. That is, the inequality

$$\|U_1 - e^{i\varphi} U_2\|_{op} \geq \|\langle \Psi_1\rangle_{\Psi_2} - e^{i\varphi} \langle \Psi_2\rangle_{\Psi_2} \|_2$$  \hspace{1cm} (B19)

$$= \sqrt{\langle \Psi_1| - e^{-i\varphi} \langle \Psi_2| \rangle \langle \Psi_1| - e^{i\varphi} \langle \Psi_2| \rangle}$$  \hspace{1cm} (B20)

$$= \sqrt{2 - e^{-i\varphi} \langle \Psi_1|\Psi_2\rangle - e^{i\varphi} \langle \Psi_2|\Psi_1\rangle}$$  \hspace{1cm} (B21)

$$= \sqrt{2 - 2|\langle \Psi_1|\Psi_2\rangle| \cos(\varphi + \theta)}$$  \hspace{1cm} (B22)

holds, where the overlap of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ can be defined in terms of state infidelity and a phase

$$\langle \Psi_1|\Psi_2\rangle = |\langle \Psi_1|\Psi_2\rangle| e^{i\theta} = \sqrt{F_s} e^{i\theta}.$$  \hspace{1cm} (B23)

When Eq. (B23) is minimised over all phases $\varphi$ this gives the required result

$$\min_{\varphi} \|U_1 - e^{i\varphi} U_2\|_{op} \geq \sqrt{2} \sqrt{1 - \sqrt{F_s} \langle \Psi_1|\Psi_2\rangle}.$$  \hspace{1cm} (B24)

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