Quantum quenches in the Luttinger model and its close relatives

M A Cazalilla$^1$ and Ming-Chiang Chung$^2$

$^1$ Department of Physics, National Tsing Hua University, and National Center for Theoretical Sciences (NCTS), Hsinchu City, Taiwan
$^2$ Department of Physics, National Chung Hsing University, Taichung, Taiwan
E-mail: miguel.cazalilla@gmail.com and mingchiangha@phys.nchu.edu.tw

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Abstract. A number of results on quantum quenches in the Luttinger and related models are surveyed with emphasis on post-quench correlations. For the Luttinger model and initial gaussian states, we discuss both sudden and smooth quenches of the interaction and the emergence of a steady state described by a generalized Gibbs ensemble. Comparisons between analytics and numerics, and the question of universality or lack thereof are also discussed. The relevance of the theoretical results to current and future experiments in the fields of ultracold atomic gases and mesoscopic systems of electrons is also briefly touched upon. Wherever possible, our approach is pedagogical and self-contained. This work is dedicated to the memory of our colleague Alejandro Muramatsu.

Keywords: Luttinger liquids, quantum quenches, bosonisation, entanglement in extended quantum systems
1. Introduction

The study of non-equilibrium dynamics in isolated many-particle systems has become a very active research area in recent years [1, 2, 4]. Many review articles have been devoted to various aspects of it [2, 4]. This article focuses on a specific topic which is concerned with the quench dynamics of the Luttinger and related models. Even with this constraint in mind, the number of results that have continued to appear since one of us [28] studied the quench of the interaction in this model in 2006 is fairly large.

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Almost a decade has gone by, and with a certain perspective, we have tried to provide a historical and personal account of how some of the ideas developed and what the concerns of the community at the time were. We have also attempted to survey some of the most interesting developments in recent years, while providing a (hopefully) pedagogical introduction to the subject. This has forced us to make many choices in order to render the article as self-contained and coherent as possible. For this reason, we would like to stress from the beginning that this work is far from perfect and cannot be considered comprehensive. As humans facing space and time constraints, we may have ended up leaning towards what we know and understand best. Not surprisingly, this also overlaps strongly with our own work in the field. Let us also state that neither of us is an expert in conformal field theory or integrable systems. The scope of this article is the quench dynamics of the Luttinger model, for which fortunately not much background in those topics is required. Furthermore, in this same special issue, readers will find several review articles on non-equilibrium dynamics written by leading experts in integrable systems and conformal field theory \[5–11\]. With such constraints and imperfections, we hope that this article will serve as a good starting point (and perhaps even as inspiration) for those students and nonexperts willing learn about this fascinating subject. At the same time, we apologize in advance to the experts who, after going through the manuscript, find that we did not properly represent the most interesting aspects of their work, or those whose work has been omitted.

Without further ado, let us get started. The rest of the paper is divided into seven sections. Section 2 contains a brief historical review of the Luttinger model, which also introduces the notation that is used in the rest of the article. In the next two sections (3 and 4), we deal with the dynamics of the post-quench correlations in sudden and smooth quantum quenches. Section 5 discusses the generalized Gibbs ensemble that describes the asymptotic long-time correlations of the Luttinger model following a (sudden) quench as well as some interesting connections to the theory of quantum entanglement. In section 6, we briefly survey some of the results obtained for other models that are related to the Luttinger model. Section 7 discusses the relevance of the results for experiments both with quantum gases and in mesoscopic physics. Finally, in section 8, we present our conclusions and an outlook. From this point on, we shall work in units where $\hbar = k_B = 1$.

2. The Luttinger model in equilibrium: a brief history

Luttinger [12] introduced the model that bears his name in 1963 as an example of an exactly solvable model of interacting spinless fermions. However, the solution that he obtained for his own model was not entirely correct, as it was shown shortly thereafter by Mattis and Lieb [13].

Luttinger’s assumptions included a linear dispersion for the fermions [12]. However, since such a dispersion can take arbitrarily large negative values, in order to obtain a physically sensible model with a spectrum bounded from below, Luttinger had to occupy all single-particle levels with negative kinetic energy with an infinite number of fermions. In other words, the ground state of Luttinger’s model is a ‘Dirac sea’. This
was quite a departure from the non-relativistic models studied in the theory of quantum many-particle systems up to that point. In those models, such as the gas of interacting fermions with a parabolic dispersion, the single-particle dispersion is bounded from below. Thus, the ground state is a Fermi sea containing a finite number of fermions. By contrast, the number of particles in the Dirac sea is infinite, meaning that Luttinger’s model is indeed a quantum field theory in disguise. Indeed, in particle physics, the Lorentz-invariant version of Luttinger’s model is known as the Thirring model.

The above observations were made by Mattis and Lieb [13], who emphasized that the Dirac-sea character of the ground state has deep consequences for the structure of the Hilbert space and its operator content. In particular, Luttinger had used a transformation to map the interacting model onto the non-interacting one [12]. The transformation appears to be canonical but in reality is not [13]. Indeed, according to an earlier observation by Schwinger [16], the requirement of a Dirac sea makes the commutation relation of certain operators, such as the density, non-vanishing i.e. ‘anomalous’. This happens independently of whether such operators commute in their first quantized form that applies to systems consisting of a finite number of particles.

After discussing the structure of the (non-interacting) ground state, let us consider the form of the Hamiltonian. In the notation that we shall be following in the rest of the article, the second quantized Hamiltonian can be written as the sum of three terms, i.e. $H_{LM} = H_0 + H_2 + H_4$, where $H_0$ is the kinetic energy of the fermions with linear dispersion:

$$H_0 = \sum_p v_F p \left[ : \psi_R^\dagger \psi_R (p) + : \psi_L^\dagger \psi_L (p) : \right].$$

In this expression $v_F$ is the Fermi velocity. The term $H_2 + H_4$ describes the interactions:

$$H_2 = \frac{2\pi}{L} \sum_{pq} g_2(q) : \rho_R(q) \rho_L(q) : ;$$

$$H_4 = \frac{\pi}{L} \sum_{pq} g_4(q) \left[ : \rho_R(q) \rho_R(-q) + : \rho_L(q) \rho_L(-q) : \right].$$

In the above equations, the operators $\psi_\alpha(p)$ ($\psi_\alpha^\dagger(p)$) annihilate (create) fermions with momentum $p$ and chirality (i.e. direction of motion) $\alpha = R, L$, and obey $\{\psi_\alpha(p), \psi_\beta^\dagger(p')\} = \delta_{\alpha,\beta} \delta_{p,p'}$, anti-commuting otherwise. For use further below, it is also useful to define the Fermi field operator:

$$\psi_\alpha(x) = \frac{1}{\sqrt{L}} \sum_p e^{i p x} \psi_\alpha(p),$$

where $s_R = -s_L = 1$. In order to avoid a degenerate ground state, we assume the field operators to obey anti-periodic boundary conditions, i.e. $\psi_\alpha(x + L) = -\psi_\alpha(x)$, i.e. $p = \frac{2\pi}{L} \left( n + \frac{1}{2} \right)$, $n$ being an integer. The normal ordering of the operator $O$ is defined as: $O = O - \langle 0 | O | 0 \rangle$, where $| 0 \rangle$ is the ground state of the non-interacting system. In Luttinger’s original model, the functions $g_2(q)$ and $g_4(q)$ were taken to be equal. However, in modern literature it has become standard to treat them as different. It is also assumed
that the interactions have a characteristic range, $R$, beyond which they decay to zero in real space. In terms of the Fourier components $g_2(q)$ and $g_4(q)$, this means that these functions rapidly vanish for $|q| \gg R^{-1}$. For the moment, we shall also assume that they are free of singularities as $q \to 0$.

It was pointed out by Mattis and Lieb that the second-quantized density operators
\[
\rho_{\alpha}(q) = \sum_p \psi^\dagger_{\alpha}(p+q) \psi_{\alpha}(p)
\]
satisfy the following algebra [13, 24, 25]:
\[
[\rho_{\alpha}(q), \rho_{\alpha'}(q')] = \frac{qL}{2\pi} \delta_{q+q',0} \delta_{\alpha,\alpha'}.
\]

In modern literature, this algebra is known as the Abelian [U(1)] Kac-Moody (KM) algebra. It was Mattis and Lieb’s realization that the KM algebra is the key to the exact solubility of the model. This is because it is possible to rewrite the KM algebra in terms of the operators:
\[
\pi \theta \rho \theta\pi = \left| q \right| - \left| q \right| L_{\pi,0,0,0}
\]
\[
\pi \theta \rho \theta \dagger = \left| q \right| - \left| q \right| L_{\pi,0,0,0}
\]
\[
\delta = \left( a \right) \left( a^\dagger \right)
\]
such that $[a(q), a^\dagger(q')] = \delta_{q,q'}$ and commute otherwise, as corresponds to canonical bosons. In addition, Mattis and Lieb rediscovered an exact result obtained by Jordan [14] in the context of his neutrino theory of light, which states that the kinetic energy of the fermions with linear dispersion can be written as:
\[
H_0 = \sum_{q \neq 0} v_F |q| a^\dagger(q)a(q).
\]
Thus, since the interactions $H_2$ and $H_4$ are quadratic in the density operators, which means they are also quadratic in $a(q)$ and $a^\dagger(q)$, we obtain a quadratic Hamiltonian in the bosonic basis, which can be diagonalized by means of a canonical (Bogoliubov) transformation:
\[
a(q) = \cosh \varphi(q) b(q) + \sinh \varphi(q) b^\dagger(-q),
\]
where $[b(q), b^\dagger(q')] = \delta_{q,q'}$, commuting otherwise. The Bogoliubov angle $\varphi(q)$ is determined from the equation:
\[
\tanh(2\varphi(q)) = \frac{g_2(q)}{v_F + g_4(q)}.
\]
Therefore, the Hamiltonian of the interacting system, $H$, is diagonal in terms of the new bosonic operator basis $\{b(q), b^\dagger(q)\}_{q \neq 0}$:
\[
H_{LM} = \sum_{q \neq 0} v(q)|q| b^\dagger(q)b(q).
\]
where the boson velocity is given by:

\[ v(q) = \sqrt{(v_F + g_1(q))^2 - (g_2(q))^2}. \]  

(12)

Mattis and Lieb’s solution of the Luttinger model (LM) provided the first concrete example of an interacting Fermi system exhibiting an excitation spectrum that strongly deviates from Landau’s normal ‘Fermi-liquid’ paradigm. The spectrum of the model, as shown in equation (11), consists of collective, plasmon-like, bosonic modes known as Tomonaga bosons. These bosonic elementary excitations are quite unlike the fermionic quasi-particles that describe the low-lying states of Fermi liquids.

Perhaps the most striking signature of the failure of the LM to conform to the framework of normal Fermi liquids can be observed in the momentum distribution. Mattis and Lieb noticed that, in the thermodynamic limit (i.e. for \( L \rightarrow +\infty \)) instead of the characteristic discontinuity at the Fermi momentum \( \rho_F \), the momentum distribution of the LM exhibits a much weaker, power-law singularity:

\[ n(p) = \int dx e^{-ipx} C^GS_{\psi_R}(x) \approx \frac{1}{2} + \left| p - p_F \right|^{\gamma_\phi} \text{sgn}(p - p_F), \]  

(13)

for \( p \approx p_F \); the exponent \( \gamma_\phi^2 = \cosh(2\varphi(q = 0)) - 1 \) depends on the details of the interaction; \( C^GS_{\psi_R}(x) = \langle \psi_R^+(x)\psi_R(0) \rangle \) is the single-particle density matrix (the expectation value \( \langle ... \rangle \) is taken over the ground state of the interacting system). Mattis and Lieb were able to obtain this result by a method equivalent to bosonization \([23, 24]\). Here, we shall recall the main identities and results of this method, referring the interested readers to the vast available literature on the subject \([24, 25, 36]\). The method relies on the following identity:

\[ \psi_\alpha(x) = \frac{\eta_\alpha}{\sqrt{2\pi a_0}} e^{is_0\phi_\alpha(x)}. \]  

(14)

This allows to express the Fermi fields in terms of the boson field \((s_R = -s_L = 1)\):

\[ \phi_\alpha(x) = s_\alpha \phi_0 + \frac{2\pi}{L} N_\alpha + \Phi_\alpha(x) + \Phi_\alpha^d(x), \]

\[ \Phi_\alpha(x) = \sum_{q > 0} \left( \frac{2\pi}{qL} \right)^{1/2} e^{-a_0q/2} e^{ipx} a(s_\alpha q), \]  

(15)

where \( a_0 \) is a short distance cut-off and \( \{ \eta_\alpha, \eta_\alpha' \} = \delta_{\alpha\alpha'} \), which ensures the anti-commutation between fermions of different chirality. The operators \( \phi_0 \) and \( N_\alpha = \sum_p : \psi_\alpha^+(p)\psi_\alpha(p) : \) are a canonically conjugate pair (i.e. \([N_\alpha, \phi_0 \alpha'] = i\delta_{\alpha\alpha'} \)). Thus,

\[ C^GS_{\psi_R}(x) = \frac{1}{2\pi a_0} (e^{i\phi_0 (q)}e^{-i\phi_0 (0)}) = Z^GS(x) C^G(0)(x), \]  

(16)

where

\[ C^G(0)(x) = \frac{e^{ipx}}{2i L \sin\left[ \pi(x + ia_0) / L \right]}, \]  

(17)
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\begin{equation}
Z^{\text{GS}}(x) = \left( \frac{R}{d(x|L)} \right)^{\gamma_{\text{eq}}} \tag{18}
\end{equation}

In the above expressions, \( C^{(0)}_{\phi}(x) \) is the non-interacting single-particle density matrix and \( d(x|L) = L |\sin(\pi x/L)|/\pi \) the chord function and \( \gamma_{\text{eq}}^2 \) is the equilibrium exponent that has been introduced under equation (13) above. Another correlation function of interest is the density correlation function. In terms of the boson field, the density operator \( \rho(x) = \partial_x \phi(x)/2\pi \), and therefore

\begin{equation}
C^{\text{GS}}_{\rho}(x) = \langle \rho(x)\rho(0) \rangle = -\frac{e^{-2x(0)}}{4\pi^2} \left[ \frac{1}{d(x|L)} \right]^2, \tag{19}
\end{equation}

which also exhibits an algebraic decay with distance, but with an exponent that is independent of the interaction (although the pre-factor is not).

As pointed out by Mattis and Lieb [13], the solution of the LM shares many interesting properties with the approximate solution of the one-dimensional electron gas obtained by Tomonaga [15] in 1950. The striking resemblance was to become more and more important in the course of time. Indeed, after Luttinger’s and Mattis and Lieb’s seminal contributions, the exotic properties of the model turned out to be more than just a mathematical curiosity. Beginning in the 1970s, the study of one-dimensional interacting systems started to attract an increasingly large amount of attention motivated by the advances in materials synthesis and spurred by Little’s proposal [17] for a new class of organic high-temperature superconductors based on highly anisotropic materials made up of quasi-one dimensional metallic molecules. Ground breaking work along in this direction was done by Luther and Emery [18] by extending the Luttinger model to spinful fermions and finding that, when backscattering processes are taken into account, the system develops a spectral gap for spin excitations but remains gapless for charge excitations. Such system, currently known as the Luther-Emery liquid, is the canonical example of a one-dimensional superconductor. In addition, Luther in collaboration with Peschel [19] provided the first crucial insights into the fundamental observation that the low-temperature behavior of the LM is universal and applies to a general class of 1D models. Building upon the earlier work by Luther and Peschel on the anisotropic Heisenberg XYZ spin-chain model, and in the spirit of Landau’s Fermi liquid theory, Haldane [20, 21] coined the name ‘Luttinger liquids’ for this new universality class, which encompasses a large class of one-dimensional models of fermions [13], bosons [22], and spins [19, 20].

Indeed, in the language of the renormalization group [24, 25, 37, 38], the LM turns out to be a fixed point Hamiltonian for this universality class. The fixed point Hamiltonian of a TLL can be written in terms of the (total) density \( \phi = (\phi_R + \phi_L)/2 \) and phase \( \theta = (\phi_R - \phi_L)/2 \) fields, as follows:

\begin{equation}
H_{\text{TLL}} = \frac{v}{2\pi} \int dx \left[ K^{-1} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right], \tag{20}
\end{equation}
where the Tomonaga boson velocity is $v = v(q = 0)$ (see equation (12)), $K = e^{-2\varphi(q=0)}$ is the the so-called Luttinger parameter in terms of Bogoliubov rotation angle at $q = 0$ (see equation (10)). Note that the density stiffness is proportional to $vK^{-1}$ and the phase stiffness is related to $vK$. In Galilean-invariant systems $vK = \rho_0/m$, where $\rho_0$ is the particle density [22, 25, 39]. The Hamiltonian in equation (20) can be diagonalized and brought to a form similar to equation (54), with $v(q)$ replaced by $v = v(q = 0)$. Thus, the low-energy excitation spectrum is completely exhausted by the Tomonaga bosons. In recognition to Tomonaga’s pioneering contributions, the universality class has been renamed as ‘Tomonaga–Luttinger liquids’ (TLLs). The defining properties of the systems in the TLL universality class are the linear dispersion of the Tomonaga bosons and the power-law correlations at zero temperature. The exponents of the power-laws are parametrized by $K$. Notice that the previous observations apply to systems for which the interactions at $q = 0$ are not singular, which implies that both $v$ and $K$ are finite as $q \rightarrow 0$. This is not the case when the interactions are long range such like the case of the Coulomb interaction. We refer the interested readers to e.g. [25] (and references therein) for an account of how the correlations are modified in this case.

3. Dynamics after a sudden quench

3.1. Introduction and historical context

In 2006, one of us (MAC) attended a workshop at the Max Planck Institute for Complex Systems in Dresden. The workshop, co-organized by Alejandro Mumamatsu, was devoted to non-equilibrium dynamics in interacting systems. At the workshop, Rigol reported on the results of his ground-breaking work with Dunjko, Yurovskii, and Olshanii [26], motivated by experiments performed in Weiss’ group [27] at Penn State. In their work [26], Rigol and coworkers showed that a gas of lattice hardcore bosons in 1D (also known as the XX model) does not thermalize to the standard Gibbs ensemble when prepared in an initial state that is not an eigenstate of the Hamiltonian. This study [26] and subsequent ones [89] provided convincing numerical evidence that the XX model relaxes to a ‘generalized’ Gibbs ensemble that is constructed from a particular set of integrals of motion of the system (see section 5 for a description, and also [5, 7, 102]).

Following Rigol’s report at the workshop, ensuing discussions with Muramatsu and Rigol provided enough motivation for one of us (MAC) to search for an analytically-solvable example of such peculiar behavior. Out of the several possible candidates, the Luttinger model seemed the most natural choice. However, several obstacles had to be surmounted. In Rigol et al’s work, two kinds of initial states had been considered: In one of them, it was assumed the hardcore boson gas is initially trapped in a smaller box and suddenly released into a larger box. The second choice was motivated by one of the experiments reported in [27], and considered that a bi-periodic potential is applied in the initial state of the lattice boson gas and suddenly switched off at start of the evolution [26].
Neither of the above choices of initial states considered seemed analytically tractable in the case of the LM because they break translational invariance (however, see remarks at the end of section 5.2). Instead, what seemed most natural and amenable to analytical calculation was to assume that the interaction between the fermions, described by the terms $H_0 + H_4$ in the Hamiltonian, is suddenly switched-on at the start of the time evolution. An attractive feature of such a quench is that the eigenstates of the non-interacting system can be still described in terms of fermionic eigenmodes. On the other hand, the eigenstates of $H_{LM} = H_0 + H_2 + H_4$ are described in terms of Tomonaga bosons. Thus, the quench of the interaction allows to study how the fermionic features of the spectrum are dynamically destroyed.

Assuming that the system is prepared in the non-interacting ground state, $|0\rangle$ and it evolves according to $H_{LM} = H_0 + H_2 + H_4$ for all times $t > 0$ is equivalent to a quantum quench of the interaction. The term ‘quantum quench’ was introduced by Cardy and Calabrese [31] in their 2006 pioneering work (see also this special issue) where they studied the evolution of correlations when the system is suddenly driven from a off-critical to a critical state. The main difference with situation considered by Cardy and Calabrese is that the initial state in [28] lacks any characteristic length scale, i.e. it is a critical state.

3.2. Dynamics of the Luttinger model following a quench of the interaction

The solution to the quench of the interaction in the LM can be obtained with minimal use of formalism in the operator representation3. The solution takes the form of a canonical transformation relating the bosonic operators at times $t = 0$ and $t > 0$:

$$a(q, t) = e^{iHt}a(q)e^{-iHt} = f(q, t)a(q) + g^*(q, t)a^f(-q), \quad (21)$$

where the time-dependent (Bogoliubov) coefficients read:

$$f(q, t) = \cos(\nu(q)|q|t) - i \cosh(2,\varphi(q)) \sin(\nu(q)|q|t) \quad (22)$$

$$g(q, t) = i \sinh(2,\varphi(q)) \sin(\nu(q)|q|t). \quad (23)$$

Note that $f(q, t = 0) = 1$ and $g(q, t = 0) = 0$, in agreement with the initial state of the system being the non-interacting ground state, $|0\rangle$, which is annihilated by $a(q)$ for all $q \neq 0$. In addition, equation (21) shows that the time evolution with the interacting Hamiltonian $H_{LM}$ alters, in a time-dependent way, the entanglement of the modes at opposite momenta $q$ and $-q$, which is a consequence of the translational invariance of both the Hamiltonian and the initial state. We shall return to this point in section 5. In order to obtain the time evolution of correlations from the initial (non-interacting) state and study how the fermionic features of the non-interacting LM are wiped out, let us consider the instantaneous single-particle density matrix [28]:

$$C_{\psi R}(x, t) = \langle 0|e^{iH_{LM}t}\psi_R^\dagger(x)\psi(0)e^{-iH_{LM}t}|0\rangle = \langle 0|\psi_R^\dagger(x, t)\psi_R(0, t)|0\rangle, \quad (24)$$

3 Keldysh aficionados are referred to e.g. [29, 113], where the same solution is obtained using the non-equilibrium Green’s function formalism.

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which can be computed using the bosonization formula, equation (14), together with equation (21). The calculation proceeds pretty much along the lines of the calculation in the equilibrium case [24, 25, 36, 39], and yields the following result in the scaling limit (i.e. for $d(x/L), d(2vt/L) \gg R$):

$$C_{\psi}(x, t) = Z(x, t)C_{0\psi}(x),$$

(25)

$$Z(x, t) = \left[ \frac{R}{d(x/L)} \right]^{\gamma^2} \left[ \frac{d(x - 2vt|L)d(x + 2vt|L)}{d(2vt)} \right]^{\gamma^2/2},$$

(26)

Likewise, we can obtain the density correlations:

$$C_{\rho\rho}(x, t) = \langle 0|e^{iH_{LM}t}\rho_t(x)\rho_t(0)e^{-iH_{LM}t}|0 \rangle = \langle 0|\rho_t(x, t)\rho_t(0, t)|0 \rangle$$

(27)

$$= -\frac{1}{4\pi} \left\{ \frac{1 + \gamma^2}{[d(x/L)^2] - \frac{\gamma^2}{2[d(x - 2vt|L)^2]} - \frac{\gamma^2}{2[d(x + 2vt|L)^2]} \right\},$$

(28)

In the above expressions $v = v(q = 0)$ and $\gamma^2 = \sinh^2(2\varphi(q = 0)) = (K^2 + K^{-2} - 2)/4$, where $K = e^{2\pi(q = 0)}$ is the Luttinger parameter. Note that the correlation functions are periodic functions of time and will exhibit periodic recurrences with a period $t_L = L/2v$ [28, 30]. This is the consequence of the finite size of the system. In the thermodynamic limit, $L \to +\infty$, and therefore $t_L \to +\infty$. Thus, the chord functions yield power-laws by the replacement $d(z|L) \to |z|$. For $L \to +\infty$ (or for $|x|, 2vt \ll L$), the behavior of the above correlation functions exhibits two clearly distinct regimes. At short times, i.e. for $t \ll |x|/2v$,

$$C_{\psi}(x, t) \simeq Z(t)C_{0\psi}(x), \quad Z(t) \sim t^{-\gamma^2}$$

(29)

$$C_{\rho\rho}(x, t) \simeq -\frac{1}{4\pi^2x^2} = C_{0\rho\rho}(x).$$

(30)

That is, the correlations take a similar form to those of the non-interacting system. In the case of the single-particle density matrix, it decreases by an overall factor $Z(t) \sim t^{-\gamma^2}$. On the other hand, in the long time regime, i.e. for $t \gg |x|/2v$, the correlations crossover to

$$C_{\psi}(x, t) \simeq C_{0\psi}(x) \left[ \frac{R}{x} \right]^{\gamma^2},$$

(31)

$$C_{\rho\rho}(x, t) \simeq -\frac{1 + \gamma^2}{4\pi^2x^2}.$$

(32)
In this case, the correlations become qualitatively different from the initial state correlations. In particular, to leading order, they do not depend on time, which indicates that for \( t \to +\infty \) the system reaches a steady state. We shall investigate this behavior more in detail further below, in section 5. However, at this point, it is worth pointing out that the existence of the two distinct correlation regimes separated by a time scale \( t_x = |x|/2v \) has to do with the finite propagation velocity of the elementary excitations of the LM. Since the LM is a relativistic model in which the role of the speed of light is played by the boson velocity \( v \), this can be expected. The time scale \( t_x \) is thus related to the time it takes for the excitations in the initial state localized at two points a distance \(|x|\) apart to overlap. This phenomenon has been termed ‘light-cone effect’ by Cardy and Calabrese [31] in their study of quantum quenches starting from an off-critical state and ending in a critical state (see also [5, 7] in this special issue). In passing, it is worth pointing out that the light-cone effect has been experimentally observed by Cheneau et al [32].

The correlations in the steady state are different from those of the initial state and from those in the ground state of the LM. Despite the fact that the initial state is a complicated superposition of eigenstates of \( H_{LM} \), the correlations are not thermal. If they were, they would exhibit an exponential decay beyond a characteristic length scale determined by the final (effective) temperature of the system \( T_f \) (see e.g. [24, 25, 36, 39] and references). To see this, let us consider an ‘infinitesimal’ quench in which a very weak interaction (i.e. \( \varphi(q = 0) \ll 1 \)) is switched on. Thus, the energy of the initial state differs from the energy of the LM ground state (taken to be zero) by an infinitesimal amount,

\[
\frac{\Delta E}{L} = \frac{1}{L} \langle 0 | H_{LM} | 0 \rangle = \frac{1}{L} \sum_{q \neq 0} v(q) |q\rangle \langle q| b^\dagger(q) b(q) |0\rangle \approx \frac{v}{\pi R^2} \sinh^2 \varphi(q = 0) \simeq \frac{v}{\pi R^2} \varphi^2(q = 0),
\]

which means that only low-lying excited states should be involved in the evolution following the quench. Thus, we can use the low-temperature form of the free energy in the canonical Gibbs ensemble for the LM (see e.g. [13]) \( F(T)/L \simeq \pi^2 T_f^2 v/6 \). Hence, the corresponding final temperature, \( T_f \propto \varphi(q = 0) \). However, this is still finite, which means that, if the system relaxes to the canonical Gibbs ensemble, the correlations should decay exponentially at long distances \([24, 36]\), e.g.:

\[
C_\varphi(x, t \to +\infty) \sim e^{-|x|/\xi(T_f)},
\]

for \(|x| \gg \xi(T_f)\) where \( \xi(T_f) \propto v/T_f \). By contrast, the asymptotic \( t \to +\infty \) state following the quantum quench exhibits power-law correlations, as shown above. In other words, the system does not thermalize to the canonical Gibbsian ensemble. We will see in section 5 that this is because the LM is exactly solvable and relaxes instead to a generalized Gibbs ensemble.

Before discussing the properties of the asymptotic state, let us discuss the consequences of the above results for the evolution of the momentum distribution. As mentioned in section 2, the latter is the Fourier transform of the equilibrium single-particle density matrix. By analogy, we can define the instantaneous momentum distribution,
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\begin{align*}
    n(p, t) &= \int dx \ e^{-ipx} C_{\psi}(x, t) = \begin{cases} 
        Z(t) \ \theta(p_F - p) & \text{for finite } t \quad (Z(t) \sim t^{-\gamma}), \\
        \frac{1}{2} \ \text{sgn}(p - p_F) |p_F - p|^{\gamma - 1} & \text{for } t \to +\infty.
    \end{cases} \\
    &= \begin{cases} 
        \int dx \ e^{-ipx} C_{\psi}(x, t) = \begin{cases} 
            Z(0) \ \theta(p_F - p) & \text{for finite } t \quad (Z(t) \sim t^{-\gamma}), \\
            \frac{1}{2} \ \text{sgn}(p - p_F) |p_F - p|^{\gamma - 1} & \text{for } t \to +\infty.
        \end{cases}
    \end{cases}
\end{align*}

(35)

The results on the right-hand side hold for $p \approx p_F$. Figure 1 shows the evolution of the momentum distribution schematically. The discontinuity at the Fermi momentum $p_F$ decreases algebraically and closes for $t \to +\infty$ becoming a weaker (power-law) singularity characterized by an exponent that differs from one in the ground state of the interacting system.

Additional post-quench correlation functions were obtained by Iucci and one of the authors in [33]. Here, we merely quote the most important results for the correlations of the so-called vertex operators starting from the non-interacting ground state. For the ratio of the non-equilibrium to the initial state correlations, the following results hold in the thermodynamic limit:

\begin{align*}
    \frac{C_{V, \psi}(x, t)}{C_{\psi, \psi}(x)} &= \frac{\langle 0 | e^{i2\chi(x, t)} e^{-i2\chi(0, t)} | 0 \rangle}{\langle 0 | e^{i2\chi(x)} e^{-i2\chi(0)} | 0 \rangle} = \mathcal{A}_m^\psi \left( \frac{R}{2vt} \right) \left( \frac{x^2 - (2vt)^2}{x^2} \right)^{m^2(K^2 - 1)^2/2}. \\
    &= \frac{\langle 0 | e^{i\phi(x, t)} e^{-i\phi(0, t)} | 0 \rangle}{\langle 0 | e^{i\phi(x)} e^{-i\phi(0)} | 0 \rangle} = \mathcal{A}_m^\phi \left( \frac{R}{2vt} \right) \left( \frac{x^2 - (2vt)^2}{x^2} \right)^{m^2(K^2 - 1)/8},
\end{align*}

(36)

where $m$ is an integer. In the above expressions [24, 25, 36, 39],

\[ C_{\psi, \psi}(x) = \langle 0 | e^{i2\chi(x)} e^{-i2\chi(0)} | 0 \rangle = \left( \frac{R}{x} \right)^{2m^2}, \]

(38)

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Note that the equilibrium duality relations requiring that if \( \phi \to \theta \) then \( K \to K^{-1} \) also hold for the post-quench correlations. Besides the above results, correlations at finite temperature were also obtained in [33], in particular the momentum distribution in the asymptotic steady state at \( t \to +\infty \). Finally, it is worth mentioning that in [33] a quantum quench in which the interactions are suddenly switched off was also studied. We refer the interested readers to section II.C of [33] for the detailed form of the post-quench correlations in this case.

In the derivation of the previous results, it has been assumed that the interactions are long ranged but not singular. This translates into \( g_2(q) \) and \( g_4(q) \) being regular functions of \( q \) as \( q \to 0 \), which is necessary to ensure that \( K \) and \( v \) are both finite. This is not the case for the Coulomb interaction for which \( g_2(q) = g_4(q) = V(q)/2\pi \). Here \( V(q) = 2e^2K_0(qd) \) is the Fourier transform of the Coulomb potential, \( K_0(x) \) being the modified Bessel function and \( d \) a length scale of the order of the transverse dimensions system (recall that \( K_0(x \ll 1) = \log \left( \frac{2}{\pi x} \right) + \cdots \), where \( \gamma = 0.5772156649 \ldots \) is Euler’s constant). The post-quench correlations for the LM with Coulomb interactions have been obtained by Nessi and Iucci [34]. In what follows, we reproduce here their results for the single-particle density matrix, specializing to the case of a quench from the non-interacting system (which corresponds to setting \( K_i = 1 \) in their expressions).

For the single-particle density matrix \( C_{\psi n}(x, t) \), the following expression for the factor \( Z(x, t) \) in equation (25) was obtained [34]:

\[
Z(x, t) = \exp \left\{ - \int_0^{+\infty} \frac{dq}{q} \sinh^2 [\varphi(q)] \left[ 1 - \cos(2v(q)qt) \right] \left[ 1 - \cos qx \right] \right\}, 
\]

where \( v(q) = \sqrt{v_F^2 + 2V(q)/\pi} \) is the Tomonaga boson velocity, \( v(q \ll 1/d) \sim v_F \log^{1/2}(1/qd) \); the Bogoliugov rotation angle of the LM in the presence of Coulomb interactions follows from \( \tanh 2\varphi(q) = V(q)/(2\pi v_F + V(q)) \). Asymptotic expressions can be obtained from equation (40). For instance, the asymptotic long time limit reads [34]:

\[
C_{\psi n}(x, t \to +\infty) = C_{\psi n}^{(0)}(x)e^{-\frac{2}{4} \log^2(x/d)} \quad (41)
\]

where \( g = e^2/\pi v_F \), \( e \) being the fundamental fermion charge. This form again differs from the equilibrium expression [25, 34]. The intermediate time dynamics, however, is complicated by the divergence of the Tomonaga boson velocity \( v(q) \) as \( q \to 0 \), which leads to a non-linear light-cone effect [34]. Thus, for times fulfilling the condition:

\[
\frac{d}{v_F} \ll t \ll t_x = \frac{x}{2v_F \sqrt{1 + 2g \log(d/l)}} \quad (42)
\]

the single-particle density matrix takes the form:

\[
C_{\psi n}(x, t) = C_{\psi n}^{(0)}(x)e^{-\frac{2}{4} \log^2(2vt/l)} \quad (43)
\]
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with exponential accuracy. Hence, it follows that the discontinuity at \( p_F \) in momentum distribution decreases as \( Z(t) \sim e^{-\frac{1}{2} \log^2(2\pi t/d)} \) instead of the power-law \( \sim t^{-\gamma^2} \) found for the LM with non-singular interactions. Asymptotic forms for other post-quench correlations, such as those of vertex operators, were also obtained in [34], and we refer the interested readers to the original article for the details.

3.3. Quest for universality: quenches in models of the TLL class

In the previous section, we have reviewed the results obtained for a sudden quench of the interaction in the LM. Since sudden quenches can potentially drive the system far from equilibrium, there is no reason to expect that the results discussed above are universal. Indeed, universality applies to the low-temperature, long distance/time correlations of systems in equilibrium and it is borne out on the ideas of the renormalization group [37, 38]. According to the latter, the low-temperature properties of a system are rather insensitive to the microscopic details as it is the structure of the low-lying excited states. Therefore, a sudden quantum quench that involves highly excited states is not likely to yield correlations that are universal.

Nevertheless, since LM is a renormalization-group fixed point for the Tomonaga–Luttinger liquid universality class, there is much interest in investigating to which extent correlations following a sudden quench exhibit universality. Analytical progress in this regard is particularly difficult. Therefore, in order to ascertain whether the correlations are independent or not of the microscopic details of the model, a number of numerical and semi-numerical techniques have been deployed.

In particular, the LM prediction for the dynamics of the discontinuity at \( p_F \) in the momentum distribution reviewed in the previous section has been numerically tested by Karrasch, Rentrop, Schuricht, and Meden (KRSM) \([40]\) using time-dependent DMRG \([35]\). KRSM considered a (sudden) quench of the interaction in the following lattice model:

\[
H = \sum_{m=1}^{L} \left[ -\frac{1}{2} (c_{m+1}^c c_m + c_m^c c_{m+1}) + \sum_n (\Delta n_m n_{m+1} + \Delta_2 n_m n_{m+2}) \right], \tag{44}
\]

from the non-interacting system, i.e. starting from the ground state of the XX model (i.e. \( \Delta = \Delta_2 = 0 \)) to the interacting model with either \( \Delta \neq 0 \) or \( \Delta = 0 \) and \( \Delta_2 \neq 0 \). In the former case, it is known as the XXZ model and it can be solved exactly using the Bethe-ansatz method (see e.g. [25, 39] and references therein). However, when the term proportional to \( \Delta_2 \) is present, the model is no longer integrable in Bethe-ansatz sense. Yet, the results in both cases showed good agreement with the LM predictions.

In addition, KRSM also obtained the evolution of the kinetic energy per unit length, i.e.

\[
e_{\text{imp}}(t) = -\frac{1}{2L} \sum_{m=1}^{L} \langle \Phi_0 | e^{iHt} (c_{m+1}^c c_m + c_m^c c_{m+1}) e^{-iHt} | \Phi_0 \rangle,
\tag{45}
\]

where \( |\Phi_0\rangle \) is the ground state of the non-interacting model (i.e. equation (44) with \( \Delta = \Delta_2 = 0 \)). The time-derivative kinetic energy exhibits a power-law decay \( \sim c(K, \nu)/t^{3} \) whose prefactor \( c(K, \nu) = (K^2 + K^{-2} - 2)v_F/16\pi v^2 \) \([34, 40]\) is a function of the Luttinger

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Parameter $K$ and the Tomonaga-boson velocity $v$, which provides a further test for the universality of the LM predictions. Further signatures of universality of the LM predictions has been found in the spectral functions of the steady state at large $t$, as reported by Kennes and Meden in [41].

More recently, a thorough study of the universality (and lack thereof) of the LM predictions has been undertaken by Collura, Calabrese, and Essler (CCE) [42]. These authors carried out a numerical study of a sudden quench in XXZ spin chain (see equation (44) with $\Delta_2 = 0$, see also equation (65) below for the model in terms of spins). CCE analyzed quenches starting from the ground state with the XX model ($\Delta = 0$) and

In addition to the scaling predicted by the LM, the plot also shows the light-cone effect appearing as a cusp (rounded off by short-distance lattice effects) at $2vt/\ell \approx 1$.

The Luttinger model prediction (dashed line) (see equations (37) and (46)) is $\langle |S_i^x(t)S_j^x(t)\rangle \ell^{\alpha+1/2} \sim t^{-\alpha}[1 - (2vt/\ell)^2]^{\alpha}$.

For $\Delta = 0$, these parameters were obtained numerically by Karrasch et al [40].

$^4$ For the Bethe-ansatz solvable XXZ, analytical expressions are known for $v$ and $K$ at half-filling [24, 25, 39]. For $\Delta_2 \neq 0$, these parameters were obtained numerically by Karrasch et al [40].
considered several values for the (interaction) coupling in the post-quench Hamiltonian: \( \Delta = \pm 0.2, \pm 0.5 \). Using the time-evolving block decimation (TEBD) algorithm \([35]\), they numerically obtained the transverse and longitudinal spin-spin correlations, for which (to leading order) the LM predictions are:

\[
\langle S_{n+\ell}^x(t)S_n^x(t) \rangle \simeq (-1)^\ell C_{\ell^2}(x = \ell a_0, t),
\]

\[
\langle S_{n+\ell}^y(t)S_n^y(t) \rangle \simeq 2C_{\ell^0}(x = \ell a_0, t) + (-1)^\ell C_{\ell^2}(x = \ell a_0, t).
\]

where \( a_0 \) is the lattice parameter, and \( C_{\ell^0}(x, t) \), \( C_{\ell^2}(x, t) \) and \( C_{\ell^2}(x, t) \) are given by equations (37), (28) and (36), respectively. CCE found a fairly good agreement of the LM predictions with the numerics for the transverse spin correlations, \( \langle S_{i+1}^x(t)S_i^x(t) \rangle \) (see figure 2). However, the agreement with the LM predictions for the longitudinal correlations, \( \langle S_{i+1}^z(t)S_i^z(t) \rangle \), was found to be much poorer (see figure 3). CCE convincingly argued that the lack of agreement in the latter case stems from the different character of the \( S_n^z \) spin operators, as compared to \( S_n^x \). Indeed, the spin operator measuring the projection on the z-axis reads: \([24, 25]\):

\[
S_n^z = c_n^\dagger c_n - \frac{1}{2} \cos 2\phi(x = na_0) + \cdots
\]

that is, a rather local operator in terms of the Jordan–Wigner fermion operators \( c_n \) and \( c_n^\dagger \) \([25, 39]\). On the other hand, the spin operator along the x axis,

\[
S_n^x = (c_n + c_n^\dagger) \prod_{l<n} (1 - 2c_l^\dagger c_l) = (-1)^n \cos \theta(x = na_0) + \cdots,
\]
is rather non-local operator due to the Jordan–Wigner string $\prod_{i<j}(1 - 2c_i^\dagger c_j)$ [25, 39]. To fully appreciate the impact of this difference, let us recall that the (initial) state, which is the ground state of the XX model (i.e. $\Delta = 0$), can be written as a non-interacting Fermion sea of the Jordan–Wigner fermions, i.e. $|0\rangle = \prod_{k<\frac{\pi}{L}} c_k^\dagger|\text{vac}\rangle$, where $|\text{vac}\rangle$ is the Fermion vacuum state and $c_k = L^{-1}\sum_n e^{-i\kappa n}c_n$. Similar difficulties concerning the failure of the LM to reproduce the post-quench density correlations were also pointed by Coira, Becca, and Parola [43] who carried out an exact diagonalization study of the fermionic XXZ model (i.e. equation (44) with $\Delta_2 = 0$) in small size systems.

Finally, let us point out that there is a large body of work by the integrable systems community [44–61] on post-quench correlations of spin chains, Lieb-Liniger, sine-Gordon, and sinh-Gordon models. Due to lack of expertise of the authors on this subject, this work will not reviewed here. We refer the interested readers to the relevant review articles [6, 7, 10] in this special issue.

3.4. Pre-thermalization and quench in a 2D Fermi liquid

Pre-thermalization has been discussed by Berges and coworkers in the context of high-energy ion collisions [64]. It refers to a metastable state of a system that has been driven out of equilibrium and rapidly establishes a kinetic temperature based on the average of the kinetic energy. Despite this, the eigenmode distribution of the in the metastable state does not correspond to a Bose–Einstein (for bosons) or a Fermi-Dirac (for fermions) distribution and therefore the system has not relaxed to a thermal state. These ideas have found resonance in the study of non-equilibrium (quench) dynamics of ultracold atomic systems [67–74]. Moeckel and Kehrein [62] discussed them in relation to a two-stage thermalization scenario that should take place following a quantum quench. In their work, Moeckel and Kehrein studied a quench of the interaction in the Hubbard model using the flow equation method [62]. Considering the infinite-dimensional Hubbard model on the Bethe lattice, they obtained the short to intermediate time evolution of the momentum distribution. Their result shows some striking resemblance with the results described earlier for the LM (see figure 1). However, one major difference with the LM is that the discontinuity at the Fermi surface does not close completely. Instead, Moeckel and Keherein found that it saturates at a constant value $Z_{\text{neq}}$ which obeys the relation $(1 - Z_{\text{neq}}) = 2(1 - Z_{\text{eq}})$, where $Z_{\text{eq}} < 1$ is the discontinuity in the momentum distribution in the interacting ground state.

Pre-thermalization has also been discussed in connection to quantum quenches in ultracold bosonic gases in Schmiedmayer’s group in Vienna [65–67] (see also the review article by Langen, Gasenzer, and Schmiedmayer [68] in this special issue). However, in this section we focus on reviewing the results for the pre-thermalization dynamics for a two-dimensional (2D) gas of spinless fermions interacting with long-ranged interactions, which, in a certain sense, can be regarded as the 2D generalization of the LM. The dynamics ensuing a quench of the interaction in this system has been studied by Nessi, Iucci, and Cazalilla (NIC) [75], and it may be of relevance for the study of the non-equilibrium dynamics of dipolar quantum gases [116]. The Hamiltonian of the model studied by NIC reads:
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\[ H = H_0 + H_{\text{int}}, \]  
\[ H_0 = \sum_k \epsilon(k) \psi^\dagger(k) \psi(k), \]  
\[ H_{\text{int}} = \frac{1}{2 \Omega} \sum_{kq} V(q) \psi^\dagger(k + q) \psi^\dagger(p - q) \psi(p) \psi(q), \]  

where \( \psi(k) (\psi^\dagger(k)) \) annihilate (create) fermions with momentum \( k \), where \( k = (k_x, k_y) \) is a two dimensional vector. As in the case of the LM, it was assumed by NIC that the system is prepared in the ground state of the non-interacting Hamiltonian \( H_0 \) and it evolves according to the interacting Hamiltonian \( H \) for \( t \geq 0 \), which is tantamount to a quench of the interaction. The Fourier transform of interaction potential \( V(q) \) is assumed to vanish rapidly for \( |q| \gg q_c \), where \( q_c^\ell \sim R \gg p_F^{-1} \), \( R \) being the range of the interaction and \( p_F = \sqrt{4\pi \rho_0} \) the Fermi momentum (\( \rho_0 \) is the areal particle density). In order to render the expressions analytically tractable, the Fourier transform of the interaction potential is taken to be of form

\[ V(q) = f_0 (q/q_c)\eta e^{-q^\ell/q_c}, \]

where \( f_0 \) parametrizes the interaction strength and \( \eta = \ldots, 0, 1, \ldots \) is a positive or zero integer (see below).

To access the short to intermediate time dynamics of the model, NIC first carried out a perturbative analysis to leading (i.e. second) order in the interaction. Thus, they showed that, the discontinuity at \( p_F \) of the momentum distribution, \( Z(t) \), also exhibits a plateau similar to the one observed by Moeckel and Kehrein [62] in the case of the Hubbard model (see also Eckstein et al [63]). This plateau indicates the existence of a pre-thermalized state. Furthermore, NIC found that, to the leading order in \( f_0 \), the relationship

\[ 1 - Z_{\text{neq}} = 2(1 - Z_{\text{eq}}) \]

also holds for the 2D Fermi gas described by the above model.

In addition, in order to understand the emergence of the pre-thermalized state, NIC resorted to the Fermi surface (FS) bosonization method [76]. This method has been applied in equilibrium and it provides a non-perturbative foundation to Landau’s Fermi liquid theory. Unlike the equilibrium case where the method is applied to a low-energy effective Hamiltonian [76], NIC applied it to the bare Hamiltonian, \( H \) (see equation (50)), which describes the interactions between the bare fermions. Performing a truncation of \( H \) that amounts to neglecting inelastic scattering between the fermions and keeps only forward and exchange interactions, NIC [75] re-wrote \( H \) in terms of the Fourier components of the density operator [76]:

\[ H \approx \sum_{S,T} \sum_{|q| \leq \Lambda} \rho_S(q) \left[ \frac{v_F}{\Omega} + \frac{V(q)}{A} \right] \rho_T(-q), \]  

where \( v_F = |\nabla_k \epsilon(k)| \) is the Fermi velocity and \( A \) is the area of the system. In writing equation (53), it is assumed that a crown of width \( \lambda \) around the FS has been ‘sectorized’ into \( N \) squat patches of transverse size \( \Lambda \), such that \( p_F \gg \Lambda \gg \lambda > q_c \).

The number of patches, \( N \), must be taken to be large but finite, in order to keep under control the divergences in the Cooper channel leading to the Kohn-Luttinger instabilities [37].
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The constant \( \Omega = \Lambda A/(2\pi)^2 \). As in the case of TLLs, the Fourier components of the density, \( \rho_S(q) \) obey a KM algebra \([76]\) \((|q|, |q'| \ll \lambda, \text{ compare with equation (5)})\):

\[
[\rho_S(q), \rho_T(q')] = \Omega (q \cdot \hat{n}_S) \delta_{S,T} \delta_q + q',0.
\]

Figure 4. Time-evolution of the discontinuity at the Fermi momentum following an quench of the interaction in a two-dimensional Fermi liquid with long range interactions \([75]\). The short time behavior is \( Z(t) \simeq 1 - ct^2 \), where \( c \propto f_0^2 \), where \( f_0 \) is the strength of the interaction. The inset shows better detail of the way \( Z(t) \) approaches its constant asymptote as \( t^{-1} \). The dimensionless interaction coupling \( g = f_0\sqrt{p_F q_s/2\pi^2 v_F} \). The horizontal lines indicate the long-time pre-thermal asymptote, \( Z_{\text{neq}} = (Z_{\text{eq}})^2 \), which is obtained from the Fermi-surface bosonization treatment (see section 3.4).

The constant \( \Omega = \Lambda A/(2\pi)^2 \). As in the case of TLLs, the Fourier components of the density, \( \rho_S(q) \) obey a KM algebra \([76]\) \((|q|, |q'| \ll \lambda, \text{ compare with equation (5)})\):

\[
[\rho_S(q), \rho_T(q')] = \Omega (q \cdot \hat{n}_S) \delta_{S,T} \delta_q + q',0.
\]

where \( \hat{n}_S \) is a unit vector normal to the circular FS at the patch position of \( S \). This equation turns the diagonalization of equation (53) into a problem akin to a system of (chiral) Luttinger models (one for each FS patch) coupled by forward-scattering interaction. Like in the case of the LM, the KM algebra allows us to obtain a diagonal representation of equation (54) in terms of a set of bosonic eigenmode operators:

\[
H \simeq \sum_{l,q} \omega_l(q) \hat{b}_l(q) \hat{b}_l(q).
\]

This expressions clearly shows that the short to intermediate-time dynamics described by \( H \) can be approximated by an exactly solvable Hamiltonian, whose dynamics is strongly constrained by the integrals of motion \( I_l(q) = \hat{b}_l(q) \hat{b}_l(q) \) (more on this further below).

In addition, the diagonalization of \( H \) (neglecting inelastic scattering between fermions) allowed NIC to obtain non-perturbative results for the evolution of the post-quench single-particle density matrix. Using similar expressions to the bosonization formula, equation (14) \([76]\), the following results for the behavior of the discontinuity in the momentum distribution at the FS were obtained \([75]\):

\[
Z(t < R/v_F) = \exp \left[ -ct^2 \right],
\]
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\[ Z(t \gg R/v_F) = Z_{\text{neq}} e^{g^2 a_q (v_F q t)^{2\alpha+1}}. \]  \hspace{1cm} (57)

where \( c \) depends on the details of the interaction [75] and \( g = f_0 \sqrt{p_F q / 2\pi^2 v_F} \). The long-time asymptote of \( Z(t) \) obeys \( Z_{\text{neq}} = (Z_{\text{eq}})^2 \), which is the non-perturbative version of Moeckel and Kehrein’s relationship between the equilibrium and pre-thermalized values of the discontinuity at \( p_F \) [62]. The full crossover from the short time limit (which agrees with the perturbative results [75]) to the long time limit of equation (57) was obtained by evaluating the integrals numerically and it is shown in figure 4.

Thus, the explicit construction of the exactly solvable truncation of \( H \) gives access to a non-perturbative solution of the short to intermediate time dynamics (up to times \( t \gtrsim f_0^2 N(0) \) \( N(0) = k_F / 2\pi v_F \) being the density of states at the Fermi level), at which inelastic collisions kick in and should drive the system to a thermal state. In addition, it also clarifies the physical origin of the phenomenon of pre-thermalization by relating it to constrained dynamics of the exactly solvable model of equation (55), which, as it will be discussed in section 5, does not relax to the grand canonical ensemble but to the generalized Gibbs ensemble. This is due to gaussian nature of the initial state and the dephasing between the bosonic FS eigenmodes of (55).

4. Smooth quantum quenches

4.1. Analytical results

In the previous section, we have focused on the dynamics following a sudden quench of the interaction. However, both from the theoretical and experimental point of view, it is interesting to consider the dynamics of a smooth quench. For the LM, this means to study the non-equilibrium dynamics as a function of the rate with which the interaction is turned on, sudden quenches corresponding to the the fastest and the adiabatic limit to the slowest switching rate, respectively. Therefore, the study of the smooth quenches can be used to compare the effects of a sudden quench to an adiabatic evolution and to better understand the mechanisms by which quantum many-body systems are driven out of equilibrium. In addition, how the time scale that controls the change of the Hamiltonian parameters compares to any other characteristic time scales of the problem is an important issue for experiments studying quench dynamics.

Another interesting issue that can be addressed in this context is how much the time-evolved state of the system is reminiscent of the original starting state. In this regard, the Loschmidt echo, or fidelity in the quantum information-theoretic language, measures the overlap between the state of the smoothly quenched system and the initial state. This measurement provides direct insight into the many-body dynamics and the relation between quench dynamics and quantum information.

In the case of a smooth quench of the interaction in the LM, the Hamiltonian becomes explicitly time-dependent:

\[ H(t) = \sum_{q=0} \left\{ \omega(q, t) a_q^\dagger(q) a(q) + \frac{1}{2} g_2(q, t) |q| [a(q)a(-q) + a_q^\dagger(q)a_q^\dagger(-q)] \right\}. \]  \hspace{1cm} (58)
where \( \omega(q, t) = [v_F + g_4(q, t)]|q| \). The time evolution is described by the Heisenberg equation of motion:

\[
\text{i} \partial_t a(q, t) = [a(q, t), H(t)] = \omega(q, t)a(q, t) + g_2(q, t)|q|a^\dagger(-q, t),
\]

(59)

and similarly

\[
\text{i} \partial_t a^\dagger(-q, t) = -\omega(q, t)a^\dagger(-q, t) - g_2(q, t)|q|a(q, t).
\]

(60)

This equation of motion can be solved by a similar ansatz to the one used for the sudden quench:

\[
a(q, t) = f(q, t)a(q) + g^*(q, t)a^\dagger(-q).
\]

(61)

However, in this case the functions \( f(q, t) \) and \( g(q, t) \) obey the Bogoliubov-deGennes (BdG) equations of motion:

\[
\begin{pmatrix}
\text{i} \partial_t f(q, t) \\
\text{i} \partial_t g(q, t)
\end{pmatrix} =
\begin{pmatrix}
\omega(q, t) & g_2(q, t)|q| \\
-g_2(q, t)|q| & -\omega(q, t)
\end{pmatrix}
\begin{pmatrix}
f(q, t) \\
g(q, t)
\end{pmatrix},
\]

(62)

which are supplemented by the initial conditions: \( f(q, 0) = 1 \), and \( g(q, 0) = 0 \) and the constraint \( |f(q, t)|^2 - |g(q, t)|^2 = 1 \), which is required by Bose–Einstein statistics. Thus, \( f(q, t) \) and \( g(q, t) \) contain all the dynamical information about the smooth quench and, as in the case of the sudden quenches, expectation values of the time-dependent observables can be obtained from their knowledge.

Dora, Haque and Zaránd (DHZ) [105] obtained solutions to the BdG equations for a linear ramp of the the interaction assuming that \( \omega(q, t) = \omega(q) = v_0(q)|q| \) is independent of time and

\[
g_2(q, t) = g_2(q)Q(t)
\]

(63)

where \( Q(t) = t\Theta(t - \tau)/\tau + \Theta(t - \tau) \), with \( \Theta(t) \) the Heaviside function. Note that \( Q(t > \tau) = 1 \) and \( Q(t < 0) = 0 \), that is, \( \tau \) determines the characteristic quench time. For \( \tau \to 0 \), a sudden quench is obtained, while the adiabatic limit is approached by letting \( \tau \to \infty \). Assuming a perturbatively small \( g_2(q) = g_2 e^{-|q|R/2} \), DHZ obtained the following asymptotic form for the instantaneous single-particle density matrix (see equation (24)) for \( t \gg \tau \):

\[
\frac{C_{\psi_\infty}(x, t)}{C_{\psi_\infty}^0(x)} \sim \begin{cases} 
A(\frac{\tau}{\tau_0})\left(\frac{R}{\min(|x|, 2v_0\tau)}\right)^{\gamma_{\text{SQ}}} & \text{for } |x| \gg 2v_0\tau \\
\left(\frac{R}{|x|}\right)^{\gamma_{\text{ad}}} & \text{for } |x| \ll 2v_0\tau
\end{cases}
\]

(64)

where \( \gamma_{\text{SQ}} = g_2^2(q = 0)/v_0^2 + O(g_2^3) \) and \( \gamma_{\text{ad}} = g_2^22v_0^2 + O(g_2^3) \) (\( v_0 = v_0(q = 0) \)) are the (perturbative) sudden quench and adiabatic-limit exponents. The prefactor \( A(\tau/\tau_0) \) depends on the speed of the quench: For a sudden quench, \( A(\tau/\tau_0 \ll 1) \sim 1 \), while for smooth quench \( A(\tau/\tau_0 \gg 1) \sim (\tau/\tau_0)^{\gamma_{\text{ad}}} \). The physical explanation for two kinds of behavior displayed in equation (64) lies in the following crossover behavior: When the interaction is quenched at a rate \( \sim \tau^{-1} \), the ‘slow’ excitations of energy \( \omega(q) < \tau^{-1} \) experience it as a

\[\text{doi:10.1088/1742-5468/2016/06/064004}\]
sudden quench. On the other hand, fast excitations with energies $\omega(q) > \tau^{-1}$ can adjust the change of the interaction strength adiabatically. Since high (low) energy excitations determine the short (long) distance correlations, the tail of $C_{\psi_0}(x, t)$ is governed by the sudden quench exponent, whilst its short distance behavior is described by the adiabatic exponent.

The subject of smooth quenches has often been related to the dynamics across critical points (see e.g. [3] and references therein). However, for the quench of the interaction in LM only one (critical) phase is involved. For example, for the XXZ model, only the critical region is related to the LM, and for Bose Hubbard models, the LM is the low-energy Hamiltonian of the superfluid regime. Therefore, the Kibble–Zurek mechanism, which is related to the production of topological defects when quenching a system across a critical point, is not relevant for smooth quenches of the interaction in the LM. Nevertheless, it is possible to discuss the production of quasi-particles in a smooth quench of the interaction, as it was done by Dziarmaga and Tylukti [112]. The average density of excited quasiparticles $n_{\text{ex}}$ scales with $\tau$ as $n_{\text{ex}} \sim \tau^{-1}$, while the more directly measurable excitation energy density scales like $\varepsilon \sim \tau^{-2}$ at zero temperature. On the other hand, at finite temperature $\varepsilon \sim \tau^{-1}$. At zero temperature, Dziarmaga and Tylukti also showed that the production of excitations does not change the algebraic $x^{-2}$ decay of the density correlations (see equation (32)). Instead (relative to the initial state), they only yield an additive correction to the prefactor. This behavior contrasts with the exponential decay of the correlations expected for the Kibble–Zurek mechanism following a quench from a disorder to an ordered phase [112].

### 4.2. Comparison to numerical approaches

As in the case of sudden quenches, the LM predictions for the correlation functions and other quantities (see below) in the case of a smooth quench have been compared to numerical calculations. In this section, we review some of the most important results in this regard.

In order to test the correctness of the LM predictions for smooth quench dynamics, Pollmann, Haque and Dóra (PHD) [148] numerically studied the XXZ spin-chain model in the critical regime by using the infinite time-evolving block decimation algorithm (iTEDB) [35]. They considered the Hamiltonian:

$$H_{\text{XXZ}} = \sum_m \left[ (S^x_{m} S^x_{m+1} + S^y_{m} S^y_{m+1}) + J(t) S^z_{m} S^z_{m+1} \right],$$

(65)

assuming antiferromagnetic exchange interaction, i.e. $J > 0$ and $J(t)$ according to $J(t) = J t/Q(t)$ with $Q(t < \tau) = t/\tau$ and $Q(t > \tau) = 1$. As explained in section 3.3, using the Jordan Wigner transformation (see e.g. [24, 25, 39]) this model can be mapped to an interacting lattice model of spinless fermions in 1D (see equation (44) with $\Delta_2 = 0$). Bosonization [24, 25] then allows to relate this model to a smooth quench in the LM (see (58)) with $\omega(q) = |v|q$ ($v = J$) and $g_2(q, t) = g_2(t) Q(t)$, $g_2(q) = g_2 e^{-|q| R^2}$, being $R \sim a_0$ a short-distance cut-off of the order of the lattice parameter $a_0$. $R$ was numerically obtained by PHD to be $R = 0.5622 a_0$. In addition, the value of $g_2$ is fixed using perturbation theory, which requires that $-1 \ll g_2/2J = J/\pi J \ll 1$. 

\[ \text{doi:10.1088/1742-5468/2016/06/064004} \]
The bosonized expression for the staggered part of the transverse magnetization is given by equation (49). Hence, in the scaling limit where $|x|, \nu \tau \gg R$, the transverse post-quench correlations can be analytically evaluated to yield:

$$C_s(x, \tau) \approx \frac{C(-1)^{|q|}}{|x|} \left( \frac{g_2}{\tau} \right)^{\frac{3}{2}} \exp \left[ -\frac{g_2}{2\nu f(\frac{x}{2\nu \tau})} \right]$$  \hspace{1cm} (66)$$

where $f(y) = 1/2 \sum_{s=\pm 1} s(y - s) \log |y - s|$, and $C = 2^{-1/3} e^{1/2} A^{-6}$, where $A = 1.28243 \ldots$ is Glaisher’s constant. In figure 5, the comparison of this analytical result with the numerics from TEDB is shown. The agreement is excellent for small $J_z/J$, although it worsens slightly for higher values of $J_z$.

In addition to correlation functions, Dora, Pollmann, Fortágh and Zaránd (DPFZ) have studied the Loschmidt echo (LE) \cite{106} as a many-body generalization of the orthogonality catastrophe \cite{107}. The LE is defined as the overlap of two wave functions $|\Psi_0(t)\rangle$ and $|\Psi(t)\rangle$ evolved from the same initial state $|\Psi_0\rangle$ but with different Hamiltonian ($H_0$ and $H$):

$$\mathcal{L}(t) = \left| \langle \Psi_0 | e^{iH_0 t} e^{-iHt} | \Psi_0 \rangle \right|^2.$$  \hspace{1cm} (67)$$

In quantum information theory this quantity is also called fidelity, and it is a measure of the distance between two quantum states which can be used to identify irreversibility and chaos. Moreover, like entanglement measures, this quantity can be used to detect quantum phase transitions (see e.g. \cite{2, 108} and references therein).

In their paper on LE, DPFZ considered a quench between two different values of the interaction, which corresponds to setting $g_2(q, t) = g_q(t) + \Delta g(q, t)$ in equation (58). Here $g_q(t)$ is the initial value of the interaction, $\Delta g(q, t) = [g_q(t) - g_q(t)]Q(t)$ (where $Q(t)$ has been defined above for a linear ramp of the interaction), and $g_q(t)$ is the final value of the interaction. The initial and final quasi-particle spectra are given by
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\[ \omega_{q(f)}(q) = \sqrt{\omega^2(q) - g_{q(f)}^2(q)q^2}, \]

and the Luttinger parameters are characterized by those interaction strengths in the following way:

\[ K_{q(f)} = \frac{\omega(q) - g_{q(f)}(q)q}{\omega(q) + g_{q(f)}(q)q}. \]  

(68)

Thus, DPFZ expressed the LE analytically in terms of the Bogoliubov coefficients of equation (62):

\[ \mathcal{L}(t) = \exp\left(-\sum_{q>0} \log \left[ f(q,t)^2 \right] \right), \]

(69)

which expresses the LE in terms of the number of excited quasi-particles in the final state. Using \( |f_{ad}(q,t)|^2 = 1/2 + (K_t/K_f + K_f/K_t)/4 \) for \( t > \tau \), DHPZ obtained for the LE in the adiabatic limit the following expression [109]:

\[ \mathcal{L}_{ad} = \left[ \frac{1}{2} + \frac{1}{4} \left( \frac{K_t}{K_f} + \frac{K_f}{K_t} \right) \right] \frac{L}{\pi a^2}, \]

(70)

were \( L \) is the system size and and \( a_0 \) is the short-distance cut-off. Therefore, the LE decays exponentially with the system size, \( L \).

On the other hand, for a sudden quench the long time limit of the LE takes a different form:

\[ \mathcal{L}_{SQ}(t \gg \frac{a_0}{2v_t}) = \mathcal{L}_{ad}^2. \]

(71)

This can be understood in the following way: The LE of an adiabatic quench involves only the ground states of the initial and final Hamiltonian, i.e. square of the overlap of the ground states \( \langle G|G_0 \rangle \) where \( |G_0\rangle \) is the ground state of \( H_0(H) \). For a sudden quench, inserting the resolution of the identity operator in terms the eigenstates of the post quench Hamiltonian \( H \) between the two evolution operators in \( \Psi|\Psi\rangle = \langle \Psi|\Psi\rangle \) and after taking into account the dephasing of the energy phase factors of different excited states the ground state contribution remains. Asymptotically in the thermodynamic limit \( \langle \Psi_0|e^{iH_{ad}t}|\Psi_0\rangle = |\langle G|G_0 \rangle|^2 \). Hence, \( \mathcal{L}_{SQ} = |\langle \Psi_0|e^{iH_{ad}t}|\Psi_0\rangle|^2 = |\langle G|G_0 \rangle|^4 = \mathcal{L}_{ad}^2 \) for \( t \to \infty \). These results for the LE were numerically confirmed by DPFZ for the XXZ spin-chain model using matrix-product states [83].

Recently, Bernier and coworkers [110] have considered a smooth quench of the interaction in a 1D Bose gas, which can be described as a TLL [39]. They addressed the question of how the system is driven out of equilibrium by an increasing quench rate [110]. Working in the weakly interacting limit, and using on Galilean invariance which fixes \( \tau(t) = K(t) \) to be independent of time [39] and \( u(t)/K(t) = (v_0/K_0)(1 + \tau/\tau_0) \) \( (\tau_0 = \pi v_0 t / K_0 (q_f - q_i)) \), they obtained solutions to the equations of motion for the Fourier components of the density \( \phi(x,t) \) and phase \( \theta(x,t) \) fields in terms of Bessel functions. This allowed them to evaluate the phase correlation functions of the bosons, \( \langle e^{i\theta(x,t)}e^{-i\theta(0,t)} \rangle \), which they compared with the numerical results obtained using time-dependent DMRG

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[35] for the Bose–Hubbard model [39]. Similarly to Dora et al [105], they found that, at short distances, the exponent determining the decay of the correlations with the distance is given by the results for adiabatic quench, while for long distances, correlations decay with a power-law exponent approaching the sudden quench value. The long distance regime is separated from an intermediate distance using a generalized Lieb-Robinson bound [111], that is, a length scale defined as $\xi_B = 2l_0 \int_0^t dt' v(q = 0, t')$ with $l_0 = v_B^2 \pi T [K_i(g_i - g_j)]$ [110]. This behavior can be regarded as a generalization of the light-cone-effect discussed above for the sudden quenches.

5. Steady state and the generalized Gibbs ensemble

As mentioned in section 3.1, one of the motivations for the study of a quantum quench in the LM was to find an analytically tractable model that exhibits absence of thermalization. Indeed, it turns out that the LM provides an excellent toy model to understand this phenomenon. As we discuss below, the LM also contains the main ingredients that allow to understand the emergence of the generalized Gibbs ensemble (GGE) as an effective description of the long-time correlations.

In order to gain a broader perspective, let us first recall the context in which GGE was originally introduced by Rigol and coworkers [26] (see also the review article by Vidmar and Rigol [102] in this special issue). The dynamics of an integrable model is strongly constrained by the presence of a large number of integrals of motion, which henceforth we denote $I(q)$ ($\{H, I(q)\} = 0$, where $H$ is the post-quench Hamiltonian). Relying on Jaynes’s foundational work on statistical mechanics, Rigol, Dunjko, Yurovsky, and Olshanii [26] proposed that the steady state of integrable models following a quantum quench is described by the following density matrix:

$$\rho_{\text{GGE}} = \frac{e^{-\sum_q \lambda(q) I(q)}}{Z_{\text{GGE}}}. \quad (72)$$

The above density matrix extremizes the von Neumann entropy $S = -\text{Tr} [\rho \log \rho]$ (for $\rho = \rho_{\text{GGE}}$) subject to the constraints imposed by the conservation of $I(q)$. Note that, if the set of integrals of motion includes only the total energy $H$ and the particle number $N$, the resulting density matrix is the Gibbs’ grand canonical ensemble [98]. In such a case, the Lagrange multipliers $\lambda(q)$ correspond to the familiar inverse absolute temperature $\beta = 1/T$ and the ratio of (minus) the chemical potential to the absolute temperature, $-\mu/T$. However, when the number of a priori conserved quantities is larger, as it is the case of integrable systems, more Lagrange multipliers $\lambda(q)$, are needed. The latter are determined from the initial conditions by requiring that

$$\langle I(q) \rangle_{\text{GGE}} = \text{Tr} \rho_{\text{GGE}} I(q) = \text{Tr} \rho_0 I(q) = \langle I(q) \rangle_0, \quad (73)$$

where $\rho_0$ is the density matrix describing the initial state. Rigol and coworkers provided convincing numerical evidence for the validity GGE by studying several types of quenches in the lattice Bose gas model [26, 100, 101]. For such a system, they identified [26] the set of integrals of motion $I(q)$ as the occupation operators of the eigenmodes
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of the system, which correspond to the occupations of the Jordan–Wigner fermions in momentum space [39]. For the LM, an analytical proof that the GGE describes correlations in the asymptotic state of the quenched LM model was provided in [28]. Subsequently, the GGE has been found to describe the asymptotic long-time correlations in many other models for various kinds of observables and types of quenches [53, 90–93, 95, 118, 126] (see also [5, 7, 102] in this special issue).

However, the accuracy of the description provided by the GGE may appear as something rather non-obvious and even striking. \textit{A priori}, it is striking that a density matrix that (generally speaking) corresponds to a mixed state can describe the result of the unitary evolution of a pure state such as the ground state of the non-interacting LM\textsuperscript{6}. It is also not obvious that the number of integrals of motion required to construct the GGE must be chosen from a particular subset of all the possible integrals of motion of the model. Some insights into these questions can be obtained by applying some results of the theory of quantum entanglement to quantum quenches.

As pointed out above, entanglement plays an important role in the physics of quantum quenches. For example, the behavior of the post-quench correlations described in section 3.2 can be regarded as a consequence of the propagation of entangled pairs of quasi-particles. In addition, we will see below that the GGE emerges as a consequence of decoherence caused by the time evolution which erases all but a certain kind of correlations that exist amongst the eigenmodes in the initial state of the system. This observation is applicable to a certain class of initial states, known as gaussian states, which are defined further below in this section. However, before discussing the connections between the GGE and entanglement, it is worth providing a short pedagogical introduction to the most important concepts of entanglement theory, which is undertaken in the following section.

5.1. Entanglement, reduced density matrices, and entanglement spectra

Entanglement is one of the most remarkable features of quantum mechanics. It was introduced by Schrödinger when addressing the paradox pointed out by Einstein, Podolsky and Rosen (EPR) [78]. Schrödinger used the German word ‘\textit{Verschränkung},’ which was translated into English as ‘entanglement’, to describe the correlations between two particles that interact and then become spatially separated. The EPR paradox arose because of the counter-intuitive non-locality of quantum mechanics. It was meant as thought experiment to explicitly demonstrate the incompleteness of the theory. In order to address the controversy that ensued between EPR, on one side, and the Copenhagen school led by Bohr on the other side, Bell derived a set of inequalities [79] which should be obeyed if reality was local and entanglement did not exist. The violation of Bell inequalities by quantum mechanics could thus demonstrated experimentally, which eventually was accomplished in a series of pioneering experiments carried out by the team led by Aspect [82]. Up to now, the majority of experiments show

\textsuperscript{6} Indeed, the description is effective at the level of correlation functions. However, it has been pointed out [33, 77] that the GGE does not contain all the necessary correlations between the eigenmodes to reproduce other quantities such as the energy fluctuations. In fact, there are situations for which the GGE essentially looks like a thermal density matrix [77, 100] and therefore it appears as if the system exhibits thermal correlations. However, the difference with a real thermal state is exhibited by the failure of the system in the steady state to obey the fluctuation-dissipation theorem for the energy fluctuations [77].
the correctness of quantum mechanics and therefore the reality of entanglement. More recently, entanglement has been realized to be an important resource for quantum computation and quantum communication [80].

In the last two decades, quantum information-theoretic concepts have also had a strong influence on many fundamental aspects of condensed matter theory, statistical mechanics, and quantum field theory [81]. To give a few examples, the theory of quantum entanglement has allowed to understand the success of density matrix renormalization group (DMRG) [83] and other methods based on the concept of matrix product states (MPS) [83, 84]. The understanding of entanglement has also provided new insights into critical phenomena, where it has been shown that entanglement can diverge just like the susceptibility at a second-order critical point [87], and the scaling of the entanglement entropy (see below for a definition) can provide a new way to calculate the central charge of conformal field theories [85, 86].

As mentioned above, entanglement theory can be applied to gain deeper understanding of quench dynamics, and in particular, the emergence of the GGE in Luttinger’s and other models [77, 90–93]. Here we follow the approach of [93], which, among others, dealt with the LM of interest here. In order to introduce the main ideas of [93] and their application to quenches in the LM, let us start by reviewing some results about reduced density matrices. For a system that is divided into a subsystem part, $A$, and an environment part, $B$, the density matrix $\rho = |\Psi\rangle\langle\Psi|$ can be obtained from a pure state $|\Psi\rangle$ describing the composite $AB$ system. The reduced density matrix $\rho_A$ is thus obtained by tracing out the environment part $B$, that is,

$$\rho_A = \text{Tr}_B\rho = \text{Tr}_B|\Psi_{AB}\rangle\langle\Psi_{AB}|.$$

The Hermitian operator $\rho_A$ is interesting for various reasons. First, it allows to obtain the von Neumann entanglement entropy of the subsystem $A$ by means of the expression:

$$S_A = -\text{Tr}\rho_A \log \rho_A.$$

This measure of entanglement is central to quantum information theory. One of the reasons is that, as shown by Wooters and coworkers, under local operations and classical communication (LOCC, i.e. a local unitary transformation), an entangled state of a bipartite system can only be transformed into a state with the same or lower entanglement entropy [96]. In addition, reduced density matrices are used to efficiently truncate the Hilbert space basis for the density matrix renormalization group based methods [35, 83]. An important result about the entanglement entropy is the way it scales with the size of the subsystem $A$. The area law [97] states that for a subsystem of dimension $d$, $S_A(d) \propto L^{d-1}$ and for critical systems an important multiplicative logarithmic correction can also appear [85, 86, 94].

Rather than focusing on the entanglement entropy, much more structure can be found in the eigenvalues of the reduced density matrix, namely, the entanglement spectrum. The latter can be obtained by realizing that the reduced density matrix of subsystem $A$ is a hermitian operator and it admits the following representation [88, 99]:

$$\rho_A = \frac{e^{-H_A^{\text{ent}}}}{Z_A},$$

where $Z_A$ is the partition function of subsystem $A$. The entanglement spectrum is given by the eigenvalues of $\rho_A$ and is related to the entanglement entropy through the area law. The eigenvalues of $\rho_A$ give rise to the so-called entanglement spectrum, which can provide new insights into the quantum phase transitions and the nature of critical points.

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where $Z_A$ is a normalization constant ensuring that $\text{Tr}_A \rho_A = 1$. The operator $H_A^{\text{ent}}$ is known as the entanglement Hamiltonian, and its spectrum as the entanglement spectrum. Note that the basis that diagonalizes $H_A^{\text{ent}}$ also diagonalizes $\rho_A$.

In order to make contact with discussion below on the GGE ensemble, let us consider the entanglement spectra of quadratic Hamiltonians. In an appropriate basis of (fermonic or bosonic) creation and annihilation operators, $b_\gamma$ and $b_\gamma^\dagger$, the latter can be generally written as:

$$H_0 = \sum_{\gamma, \delta} A_{\gamma \delta} b_\gamma^\dagger b_\delta + \frac{1}{2} (B_{\gamma \delta} b_\gamma^\dagger b_\delta^\dagger + \text{h.c.})$$  \hspace{1cm} (77)

In the above expression $\gamma, \delta$ are the quasi-particle quantum numbers, which can be coordinates, wave vectors, etc. The reduced density matrix of such models can be obtained by first obtaining a coherent state representation of the the matrix elements of the full density matrix $\rho_0$ and then explicitly integrating out the degrees of freedom of the environment subsystem $B$ [103]. In this way the reduced density matrix $\rho_A$ can be also diagonalized [99, 104]:

$$\rho_A = \frac{1}{Z_A} e^{-\sum_m \lambda_m \alpha_m^\dagger \alpha_m}$$  \hspace{1cm} (78)

where $\alpha_m^\dagger$ ($\alpha_m$) are the creation (annihilation) operators that diagonalize the entanglement Hamiltonian, $H_A^{\text{ent}}$ and $Z_A = \text{Tr} e^{-\sum_m \lambda_m \alpha_m^\dagger \alpha_m}$. The single-particle entanglement spectrum is given by

$$\lambda_m = \log \frac{1\pm \mu_m}{\mu_m},$$  \hspace{1cm} (79)

where $\mu_m$ are the eigenvalues of the correlation function matrix:

$$G_{\gamma \delta} = \text{Tr}_0 b_\gamma^\dagger b_\delta,$$  \hspace{1cm} (80)

where the quantum numbers $\alpha$ and $\beta$ are restricted to the subsystem $A$. The plus (minus) sign in equation (79) applies to bosons (fermions). We will see below, in section 5.2, that the GGE can be written as product of reduced density matrices of the form of equation (78) for which the role of the subsystem $A$ is played by a single eigenmode.

The above results apply only to the case that the state of the system can be described by a gaussian density matrix, $\rho_0$, which in the pure state case can be regarded as the ground state of a quadratic Hamiltonian. Otherwise, the correlation function matrix in equation (80) is not enough to describe the entanglement between the subsystem $A$ and the environment $B$ due to the failure of Wick’s theorem. The results for the entanglement spectrum also apply to mixed gaussian density matrices, such as, for example a thermal density matrix $\rho_0 \propto e^{-H_0/T}$, where $T$ is the absolute temperature and $H_0$ is a quadratic Hamiltonian. However, in this case, the expression for von Neumann entropy of $\rho_A$, equation (75) cannot be used to calculate the entanglement due to the additional thermal contribution to the entropy.
5.2. GGE and the steady state of the Luttinger model

In this section, we shall consider a quantum quench in the Luttinger model (LM) and show how the GGE provides a description of the asymptotic steady state at $t \to +\infty$ from the perspective of entanglement [93]. The initial state of the quench is assumed to be a gaussian state $\rho_0 \propto e^{-H_0/T}$ (the pure state case is obtained by letting $T \to 0$) where $H_0$ is can be generally written as:

$$
H_0 = \sum_{q, q'} [\epsilon_0(q) \delta_{q, q'} + V_0(q, q')] b^\dagger(q) b(q') + \sum_{q, q'} [\Delta_0(q, q') b(q) b(q') + \Delta_0(q, q') b^\dagger(q') b^\dagger(q)].
$$

Note that, in general, $H_0$ and $\rho_0$ break translational invariance and the conservation of the quasi-particle number. This is a consequence of the coupling of different wave numbers $q$ and $q'$ by the potentials $V_0(q, q')$ and $\Delta_0(q, q')$. We assume a sudden quench where at $t = 0$, the Hamiltonian becomes diagonal in $b(q)$ and $b^\dagger(q)$ (see $H_{LM}$ in equation (54)).

For gaussian states like $\rho_0$, Wick’s theorem allows to obtain the correlators of an arbitrary product of $b(q)$ and $b^\dagger(q)$ eigenmode operators from two-point correlation functions, e.g. $\langle b^\dagger(q) b(q') \rangle_0 = \text{Tr} \rho_0 b^\dagger(q) b(q'), \langle b(q) b(q') \rangle_0$ and $\langle b^\dagger(q) b^\dagger(q') \rangle$, etc. In addition, another useful property of such gaussian states, which was described in the previous section, is that the reduced density matrices of an arbitrary partition of the system are also gaussian. In particular, if we choose a partition where the subsystem $A$ is one of the modes, say, $q$ and the environment $B$ is the rest $q' \neq q$, then, tracing the environment yields

$$
\rho(q) = \text{Tr}_B \rho_0 = \text{Tr}_{q' \neq q} \rho_0 = \frac{e^{-\lambda(q) I(q)}}{Z(q)},
$$

where $I(q) = b^\dagger(q) b(q)$ is the quasiparticle occupation operator. For this particular partition the entanglement Hamiltonian equals $\lambda(q) I(q)$ and $\lambda(q)$ is the single-mode entanglement spectrum, which related to the occupation number of the density matrix (see equation (79)), $n(q) = \langle I(q) \rangle = \text{Tr} \rho_0 I(q) = \text{Tr} \rho_0 b^\dagger(q) b(q)$ by means of the relation:

$$
\lambda(q) = \log \frac{1 + n(q)}{n(q)}.
$$

We shall argue below that, as a consequence of decoherence, the long-time correlations can be effectively obtained from a product of such single-mode reduced density matrices, i.e.

$$
\rho_{GGE} = \bigotimes_q \rho(q).
$$

This is because for local and non-local operators in terms of the system eigenmodes, decoherence erases the dependence of their correlators on the off-diagonal eigenmode correlations [93] of the type $\langle b^\dagger(q) b(q') \rangle$ (for $q \neq q'$), $\langle b(q) b(q') \rangle$, etc. Thus, if all relevant correlators depend only on the $n(q) = \langle I(q) \rangle$, taking the expectation value over the GGE and over $\rho_0$ yield the same result. However, this point of view of the GGE is tantamount to the mathematical statement that each eigenmode acquires a mode-dependent
effective temperature \( T(q) = \lambda(q)\sqrt{|v(q)|q} \) as a result of its entanglement with other eigenmodes in the initial state of the system.

For the sake of simplicity and in order to make connection with the previous discussion about quenches of the interaction, we shall next consider translational invariant initial states. However, the discussion can be easily generalized to non-translationally invariant cases (see remarks at the end of this section). Thus, equation (81) simplifies to

\[
H_0 = \sum_{q \neq 0} \left\{ v_q(q)|q\rangle\langle q| b^\dagger(q)b(q) + \frac{1}{2} g_0(q)|q\rangle\langle q| \left[ b^\dagger(q)b^\dagger(-q) + b(q)b(-q) \right] \right\}.
\]

For a quench of the interaction starting from the non-interacting system, \( v_0(q) = v_F \cosh 2\varphi(q) \) and \( g_0(q) = v_F \sinh 2\varphi(q) \), as follows from equation (9) (\( \varphi(q) \) is the Bogoliubov angle).

By virtue of bosonization \([24, 25, 36]\), the observables of LM can be expressed in terms of the boson fields \( \phi_{\lambda=\pm L}(x) \) (see equation (15)). Upon rewriting them in terms of the eigenmodes of post-quench Hamiltonian, \( H_{\text{LM}} \), they read:

\[
\Phi_\alpha(x, t) = \sum_{q > 0} \left( \frac{2\pi}{L} \right)^{1/2} e^{i s_\alpha q} \left[ \cosh \varphi(q)e^{-i\alpha(q)}|q\rangle\langle q|b(s_\alpha q) - \sinh \varphi(q)e^{i\alpha(q)}|q\rangle\langle q|b^\dagger(-s_\alpha q) \right],
\]

\[
\phi_\alpha(x) = s_\alpha \phi_\alpha + \frac{2\pi x}{L} N_\alpha + \Phi_\alpha(x) + \Phi_\alpha^\dagger(x).
\]

In addition, thanks to the applicability of Wick’s theorem to gaussian states, the correlation functions of vertex operators (i.e. exponentials of the boson fields) can be expressed in term of two-point correlations of the fields \( \phi_\alpha(x, t) \). The key identity is the following\(^7\):

\[
\langle e^{iA_\alpha(x_1, \ldots, x_n)} \rangle_0 = \text{Tr} \left[ \rho_0 e^{iA_\alpha(x_1, \ldots, x_n)} \right] = e^{-\frac{1}{2} \left( A_\alpha^\dagger(x_1, \ldots, x_n) \right)_0},
\]

where

\[
A_\alpha(x_1, \ldots, x_n, t) = \sum_{i=1}^n p_i \phi_\alpha(x_i, t),
\]

with \( \sum_{i=1}^n p_i = 0 \). The vertex operators describe non-local observables in terms of the eigenmodes of the post-quench Hamiltonian. Examples of local observables are the density operator \( \rho_\alpha(x) = \partial_t \phi_\alpha(x)/2\pi \), whose correlation functions can be also obtained by relying on Wick’s theorem. Note that \( \phi_\alpha(x) \) itself is not an observable. However, observables in the LM are related to correlation functions of \( \rho_\alpha(x) \) and vertex operators. Despite this fact, the correlations of \( \phi_\alpha(x) \) play a central role in the LM, as we have just shown. For example, using equation (88), the two-point correlation function of the right moving Fermi fields

\[
C_{\psi_R}(x, t) = \langle \psi_R^\dagger(x, t)\psi_R(0, 0) \rangle_0 = \text{Tr} \left[ \rho_0 \psi_R^\dagger(x, t)\psi_R(0, 0) \right]
\]

\(^7\) equation (88) can be proven by expanding in series the exponential in the left-hand side in a Taylor series and applying Wick’s theorem to all the terms, which involve powers of \( A(x_1, \ldots, x_n) \). Resuming the resulting series, the right hand-side of equation (88) is obtained.
where $A$ is a cut-off dependent prefactor. Thus, because of Wick’s theorem, it is sufficient to consider the two-point correlations of $\phi(x, t)$, which, using (87), can be written as follows:

$$C_{\phi_R}(x, t) = \langle \phi_R(x, t)\phi_R(0, t) \rangle_0 = D_{\phi_R}(x) + F_{\phi_R}(x, t),$$

(92)

where

$$D_{\phi_R}(x) = \sum_{q=0}^{\pi \frac{\pi}{qL}} \frac{\pi}{qL} \left[ \cosh 2\varphi(q) + \text{sgn}(q) \left[ e^{iqx} + e^{-iqx}n(q)\right] \right]$$

(93)

is the contribution of the diagonal correlations of the eigenmodes in the initial state $n(q) = \langle b^\dagger(q)b(q) \rangle_0$. Furthermore,

$$F_{\phi_R}(x, t) = \sum_{q=0}^{\pi \frac{\pi}{qL}} \frac{\pi}{qL} \left[ \cosh 2\varphi(q) + \text{sgn}(q) \left[ e^{iqx} - 2\text{sgn}(q) |q| t \Delta(q) + e^{-iqx} + 2\text{sgn}(q) |q| t \Delta^*(q) \right] \right],$$

(94)

where $\Delta(q) = \langle b(q)b(-q) \rangle_0$ is related to the anomalous correlations of the eigenmodes in the initial state. Due to the translation invariance, $n(q)$ and $\Delta(q)$ are the only non-vanishing two-point correlations of the eigenmodes in the initial state.

From equations (93) and (94) we can see why the correlation functions for $t \to +\infty$ depend on the diagonal correlations, $n(q)$, only. First notice that, whereas the contribution involving the eigenmode occupations $n(q)$ is time independent, the contribution of the anomalous terms depends on time. Hence, because of dephasing between the different Fourier components (mathematically, by the Riemann–Lebesgue lemma), $F_{\phi_R}(x, t)$ vanishes in the thermodynamic limit as $t \to +\infty$. For instance, this can be checked by explicit evaluation of $F_{\phi_R}(x, t)$ at zero temperature ($T = 0$) for the quench of the interaction in the LM,

$$F_{\phi_R}(x, t) - F_{\phi_R}(0, t) \sim \log \left| \frac{(2vt)^2 - x^2}{(2vt)^2} \right|,$$

(95)

which vanishes as $t \to +\infty$ (provided $x$ is kept finite). This implies that all correlations are asymptotically determined by $D_{\phi_R}(x)$, which depends only on $n(q) = \langle b^\dagger(q)b(q) \rangle_0$. This observation allows to trace out all the $q' \neq q$ eigenmodes since $n(q) = \text{Tr} [\rho_0 b^\dagger(q)b(q)] = \text{Tr} [\rho(q)b^\dagger(q)b(q)] = \text{Tr} \rho_{\text{GGE}} b^\dagger(q)b(q)$. Thus, we arrive at the same result as if we had used the GGE density matrix $\rho_{\text{GGE}} = \bigotimes_q \rho(q)$. Hence, $C_{\phi_R}(x, t \to +\infty) - C_{\phi_R}(x, 0) = C_{\phi_R}^{\text{GGE}}(x) - C_{\phi_R}^{\text{GGE}}(0)$, where $C_{\phi_R}^{\text{GGE}} = \text{Tr} [\rho_{\text{GGE}} \phi_R(x)\phi_R(0)]$, and using equation (91) yields:

$$\lim_{t \to +\infty} C_{\phi_R}(x, t) = C_{\phi_R}^{\text{GGE}}(x).$$

(96)

It is worth noting that the translational invariant initial state implies that eigenmode correlations are bipartite, that is, each mode at $q$ is entangled only with the eigenmode at $-q$. Thus, we can regard the effective temperature $T(q)$ for the
eigenmodes with $q$ as the result of their quantum correlations with the $-q$ eigenmodes and vice versa. However, the translational invariance of initial states is not a necessary condition for the long time correlations to be described by the GGE. If the translational-invariance constraint is relaxed, dephasing will still erase the off diagonal correlations, not only the ‘anomalous’ ones $\langle b(q)b(q')\rangle_0$ but also the normal ones $\langle b^\dagger(q)b(q')\rangle$ ($q' \neq q$) because they always appear in the correlators multiplied by phase factors of the form $e^{i\varepsilon(q)|q'|}$, which oscillate very rapidly for $t \to \infty$ and therefore yield a vanishing contribution. This is essentially the reason why the GGE is so effective in describing the asymptotic state correlations. Furthermore, it also shows why only the occupation operators of the eigenmodes (quasi-particles) of the post-quench Hamiltonian, i.e. $I(q) = b^\dagger(q)b(q)$, are the only integrals of motion required for its construction.

5.3. Non-gaussian initial states

From the discussions above the gaussian initial state $\rho_0$ is needed to prove GGE correct. However, Dinh, Bagrets, and Mirlin (DBM) [113] studied a sudden quench of the interaction in LM assuming a double-step initial momentum distribution function of the fermions (a situation relevant to experiments in the quantum Hall regime of the two-dimensional electron gas, see section 7.2 and references therein). Using non-equilibrium bosonization, they obtained the steady state energy ($\propto$ momentum) distribution. DBM pointed out that the resulting steady state distribution cannot be obtained from the GGE. The latter, at large distances, predicts an exponential decay of single-particle density matrix, i.e. [113]:

$$C_{\psi R}(x) \approx C_{\psi R}^{(0)}(x) \frac{R^{-\alpha}}{x} e^{-\kappa|x|},$$

(97)

where $\alpha$ and $\kappa$ depend on the interaction and details of the initial state. The reason why GGE fails in this case is because the initial state contains non-gaussian correlations amongst the bosonic eigenmodes of the system. Memory of the non-gaussianity of the initial state survives in the steady state at $t \to +\infty$. A similar conclusion has been reached by Sotiriadis for general initial non-gaussian states using conformal field theory methods [114]. In a general framework, it has been recently proved by Sotiriadis and Calabrese [115] that relaxation to the GGE takes place when the initial state satisfies cluster decomposition.

6. Brief survey of Luttinger’s relatives

Various kinds of perturbations to the LM have been considered as well as their various effects on the quench dynamics. The number of possible perturbations is rather large, and given the space constraints, we cannot make justice to all the recent developments in this area. We merely mention the most relevant here.

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6.1. Quenches in the sine-Gordon model

A well known perturbation to the LM is the sine-Gordon model, e.g.

\[ H_{SGM} = H_{TLL} + g(t) \int dx \cos 2\phi(x), \]

where \( H_{TLL} \) is the fixed-point Hamiltonian in equation (20). Assuming \( g(t) = g \theta(t) \), for instance, this model describes the sudden application to the LM of an external periodic potential that is commensurate with half the Fermi wave number \( \pi/p_F \) (recall \( p_F \) is the Fermi momentum) [25, 39]. The model also has a dual version where \( 2\phi \to \theta \) in the cosine term, which describes a quench of the Josephson coupling [25, 39]. In equilibrium (i.e. for a time-independent coupling \( g(t) \)) the cosine perturbation is relevant in the renormalization group sense for \( K \lesssim 2 \) (for infinitesimal \( g \)) [24, 25, 39], which opens a spectral gap. For \( K > 2 \), the perturbation is irrelevant and the low energy spectrum is thus gapless and adiabatically connected to the LM spectrum (up to corrections that rapidly decrease with the excitation energy).

Like the LM, quantum quenches in the sine-Gordon model have also attracted much attention. Iucci and Cazalilla [118] studied a sudden quench of the cosine term where \( g(t) = g \theta(-t) \) in the so-called harmonic limit (holding for \( K \ll 1 \)) and at the so-called Luther-Emery line [18] (corresponding to \( K = 1 \) [39] for equation (98)). The results are entirely consistent with the general results of Cardy and Calabrese [31] for a quench from an off critical to a critical Hamiltonian. Iucci and Cazalilla also showed that in both the harmonic limit and the Luther-Emery line the system relaxes to the GGE. In addition, the reverse quench (from critical to non-critical) was also analyzed in [118]. Applications of the quench of the sine-Gordon to the experiments in Schmiedmayer’s group have also been discussed recently by Dalla Torre, Demler, and Polkovnikov [119], and by Foini and Giamarchi [120]. In addition, the quench dynamics of the sine-Gordon and the related sinh-Gordon model has been investigated in [121–127] by relying on its integrability, which allows to access a wider range of parameters.

Smooth quantum quenches where the coupling \( g(t) \sim t' \) have been studied by De Grandi, Gritsev, and Polkovnikov (GGP) [128], who focused on the dynamics near (i.e. starting from or ending at) the critical point between the gapped and gapless phases. By changing the exponent \( r \), it is possible to interpolate between the sudden quench (\( r \to 0 \)) and the adiabatic quench limit (for \( r \to \infty \)). In between, for the linear quench \( r = 1 \), the Kibble–Zurek mechanism can be studied. Rather than the dynamics of correlations, GGP focused on the production rate of excitations, \( P_{\text{ex}} \), the density of the quasiparticles \( n_{\text{ex}} \), the diagonal entropy \( S_d \) and the heat (the excess energy above the new ground state of the post-quench Hamiltonian) \( Q \). They showed that the scaling of \( P_{\text{ex}}, n_{\text{ex}} \) and \( S_d \) is associated with the singularities of the generalized adiabatic susceptibility \( \chi_{2r+2}(\lambda) \) of order \( 2r + 2 \) defined as

\[ \chi_{n}(\lambda) = \frac{1}{E^d} \sum_{n \neq 0} \frac{\langle \langle n | V | 0 \rangle \rangle^2}{|E_n(\lambda) - E_0(\lambda)|} \]

where a \( d \)-dimensional perturbative Hamiltonian \( H(\lambda) = H_0 + \lambda V \) is considered with the eigenenergy \( E_n \) of the state \( |n\rangle \), while if the quench ends at the critical point the scaling of \( Q \) is related to \( \chi_{2r+1} \) [128]. For a sudden quench, i.e. \( r = 0 \), \( \chi_{2r+2}(\lambda) \) reduces to the fidelity susceptibility \( \chi_2 \equiv \chi_f \). In two exactly solvable limits: the massive bosons (i.e.
the harmonic limit) and the massive fermions, they also obtained results for quenches at finite temperature. Due to the statistics of the quasiparticles, they showed that the structure of the singularity remains the same except that for $n_{ex}$ and $Q$, the dimensionality $d$ is replaced by $d - z$, where $z$ is dynamical exponent for bosons. On the other hand, for fermions $d \to d + z$. The difference stems from the bunching of bosons, which enhances non-adiabatic effects, whereas anti-bunching of fermions suppresses transitions [128].

6.2. Long-ranged hopping models

Other systems that have been attracting much interest in recently in connection with experiments in ion traps are models with long-ranged interactions [132–138]. We have already discussed how long-ranged interactions affect the post-quench correlations of the LM [34]. Other types of interactions may correspond to a long-range hopping of bosons in a lattice, which translates into a long-ranged Heisenberg exchange for spins. Tezuka, García-García and Cazalilla (TGC) studied a quench of the range of the boson hopping, focusing on the dynamics of the condensate [129]. When hopping amplitude decays as a power-law of the distance $r$, i.e. $t_r \sim |r|^{-\kappa}$, the system exhibits long range order at zero temperature for $\kappa < 3$ (up to interaction-induced corrections) [130, 131]. Quenching the power-law tail of the hopping amplitude (or, equivalently, the value of $\kappa$) is tantamount to changing the effective dimensionality of the system [129]. Using bosonization [39], TGC obtained that the condensate fraction (normalized to the initial state fraction) $f(t)$ decays at short times as $f(t) = 1 - b t^2$ and at long times as a stretched exponential $f(t) \sim e^{-c t^{\alpha(\kappa)}}$, where $b$ and $c$ depend on the model parameters like lattice filling, interaction, etc. These predictions were found to be in reasonable agreement [129] with numerical results obtained using time-dependent DMRG, despite the fact that the bosonization treatment does not take into account the possibility of phase slips, which may be required in order to achieve a complete understanding of the dynamical destruction of the condensate following the quench of the hopping range.

6.3. To thermalize or to not thermalize

There is a great deal of evidence that observables of generic, non-integrable isolated systems relax to a state that can described by a standard thermal equilibrium ensemble [139, 140] (see also [4] and references therein). Thus, for such a generic systems, the LM results that we have reviewed above should break down at sufficient long times. Correlations are therefore expected to crossover to their thermal averages (e.g. equation (34)). Thermalization being impossible to avoid in most cases, the question is therefore one of relaxation dynamics and time scales. And the latter can be very long due to the limited available phase space for quasi-particle scattering in especially one-dimensional systems.

In the intermediate time-regime, before the crossover to thermal behavior takes place, the LM predictions should be accurate provided the system is not strongly excited at the outset in the quench process. This expectation is based on the observation that,

8 Of course, this assumes the validity of some kind of quasi-particle picture, as it is the case of the Luttinger model, but may not be case of other critical models.
even if the high excited states of a generic, non-integrable model are highly chaotic [4],
the low-energy part of spectrum may still retain some features that can be captured by
a suitable exactly solvable model like the LM.

It becomes therefore apparent that the results described above should describe some
kind of pre-thermal state, which exists for some time, and which crosses over at a later
time to a fully thermal state. How this happens and what kind of perturbations to the
LM drive such thermalization is still very much under debate (although a number of
important results have emerged recently, see below).

A first attempt to understand how terms that have been neglected by the use
of TLL fixed-point (i.e. the LM) Hamiltonian to describe the quench dynamics was
undertaken by Mitra and Giamarchi [29]. Using the non-equilibrium Green’s function
(Keldysh) formalism, and assuming that a non-linearity in the form of a sine-Gordon
term (see equation (98)) is adiabatically switched-on following an interaction quench,
they showed that the system would eventually reach a thermal state. In other words,
the coupling between the eigenmodes due to the sine-Gordon term introduces quasi-
particle scattering that violates the infinite conservation laws of the LM and relaxes
the system to a thermal state.

Nevertheless, intrinsic sources of scattering between quasi-particles are present in
most models of the TLL class at all times. One of them is the curvature in the fermion
dispersion, which in bosonized form reads [21]:

$$ H_m = \frac{1}{m} \int dx (\partial \theta)^2 \partial^2 \phi. $$

(100)

The effects on the post-quench dynamics of the resonant scattering of Tomonaga
bosons caused by equation (100) have been recently addressed by Buchhold, Heyl, and
Diehl (BHD) [141]. By using non-equilibrium Green’s functions, BHD wrote a quantum
kinetic equation for the Tomonaga bosons in the presence of collisions mediated by
the equation (100). From the numerical solution of the kinetic equation, the following
picture for the fermion single-particle correlation function emerges: Besides the two
regimes that have been discussed in section 3, which are called pre-quench and pre-
thermal by the authors of [141], a new regime, termed ‘thermal’ appears. In the thermal
regime, $C_{\psi_{\theta}}(x, t)$ exhibits an exponential decay with distance. Thus, summarizing
BHD’s results for a quench of the interaction starting from the non-interacting ground
state, the following three distinct regimes exist:

$$ C_{\psi_{\theta}}(x, t) = \begin{cases} 
Z(t)C_{\psi_{\theta}}^{(0)}(x, t), & Z(t) \sim t^{-\gamma_2} \quad 0 < t < t_x = |x|/2v, \\
C_{\psi_{\theta}}^{GGE}(x) \simeq \left[ \frac{R}{x} \right]^{\gamma_2} & t_x < t < t_{th} \\
e^{-|x|/\xi_{th}(T(0))} & t > t_{th}.
\end{cases} $$

(101)

In the pre-quench regime (i.e. for $t < t_x$), correlations are only multiplicatively modified
from their initial state values. The pre-thermalized regime corresponds to $t > t_x = 2vt$.
In this regime, the dynamics is controlled by the exactly solvable truncation of the total
Hamiltonian, i.e. $H_{LM}$. In this regard, $H_{LM}$ plays a similar role to the exactly solvable
truncation of the interacting 2D Fermi gas discussed in section 3.4. In other words, the
exactly solvable models describe a regime of ‘inertial response’ to the quench of the
interaction, in which quasi-particles are formed before they can start to scatter each other.

The final (‘thermal’) regime takes place for $t > t_{th}$. The thermalization time $t_{th} \approx t_0(\alpha(K) \sqrt{2K})$, where $0 < \alpha(K) < 1$ and $\beta(K)$ are functions of the Luttinger parameter, $K$, that need to be determined numerically [141] ($R$ is the interaction range, which effectively plays the role of momentum cut-off, see section 2). The curvature, equation (100), leads to the emergence of a new time scale in the problem $t_0 = R^2/w_0 \sim mR^2$, where $w_0 \sim 1/m\sqrt{K}$ is the strength of the interaction-vertex arising from equation (100).

BHD also pointed that the existence of the thermalized regime and the different scaling of the time (or equivalently, length) scales that determine them, implies the existence of a minimum time or distance below which the pre-thermalized behavior cannot be observed. This distance, $x_m$ is found by equating $2vt_m = \beta(K)R(t/t_0)^\alpha(K)$ [141] and hence $x_m = 2vt_m$, and $|x| < x_m$ the quasi-particles with short wavelength have not formed before they begin to scatter each other. Thus, below this length scale the system correlations will behave either as in the pre-quench or in the thermal regime.

Finally, it is worth pointing out that the thermal regime is a stationary state only asymptotically. This is because the correlation length $\xi_{th}(t) = K/(1 + K^2)(\nu/\pi T(t))$, where the effective temperature $T(t) = T + \Delta(K)(\nu/R)(t/t_0)^\nu$, where $T$ is the final temperature of the system, $\Delta(K)$ is a parameter that is determined numerically, and the exponent $\mu = 2/3$ [141].

7. Relevance to experiments

7.1. Ultracold atomic gases

In [28], the experimental realization that was envisaged was a quantum quench in a dipolar Fermi gas effectively confined to one dimension in a very anisotropic trap. However, cooling Fermi gases to temperatures well below the Fermi temperature, $T_F$, is technically difficult, although some progress has been recently reported in the case of dipolar Fermi gases due to their long-ranged interactions [116]. Currently, the lowest temperatures attainable are $\approx 10\%$ of $T_F$. Thus, even if the dynamics of trapped ultracold gases can be studied for quite some time due to their isolated nature making it extremely quantum coherent, finite temperature effects must be accounted for when comparing with the experiment. Such effects were theoretically addressed in [33] for the momentum distribution, with the conclusion that the latter is probably not the most ideal observable to study quench dynamics. As argued in [33], the reason are the rather small differences between the non-equilibrium steady state momentum distribution and the interacting finite temperature momentum distribution may be hard to discern experimentally. Let us recall that at finite temperatures, the discontinuity in momentum distribution is absent due to entropic effects. In addition, the discontinuity is also very sensitive to other effects, such as inhomogeneity, finite-size, etc. However, it is worth noticing that not all finite-temperature effects are so perverse. Indeed, if

9 Note that, unlike for the Coulomb potential, the Fourier transform of the dipolar potential is regular in one dimension.
Quantum quenches in the Luttinger model and its close relatives

the initial state is a finite temperate state $\rho_0 = e^{-H_0/T}$, then the steady state is reached much faster, $t \approx 1/T$ [28, 33], rather than for $t \to +\infty$.

Cooling problems are less severe for ultracold atomic gases of bosons, and therefore much more progress has been made in studying the non-equilibrium dynamics of such systems. In fact, using a trapped 1D cloud of bosons that is suddenly split longitudinally into two 1D clouds, it has been possible to observe pre-thermalization [65–68] and also find strong evidence for relaxation to the generalized Gibbs ensemble (in the pre-thermalized regime) [117]. As we have mentioned in the previous section, to some extent, these experiments can be interpreted in terms of a quench in a sine-Gordon model that is dual to the one in equation (98), i.e. with the replacement $\phi \to \cos \theta$ [118–120]. In the experiment, the field $\theta$ corresponds to relative phase between the two atomic clouds and the coupling $g(t)$, which describes the Josephson tunneling between the two clouds, is suddenly quenched from a large to a zero value. In other words, the quench proceeds from the gapped to the gapless phase of the sine-Gordon model. Since the initial (Josephson) coupling $g(t < 0)$ is large due to the initial complete overlap of the two clouds, and the atoms in the 1D clouds are weakly interacting, the harmonic approximation where $\theta \approx -\cos 2\theta$ is a good starting point [65, 66, 118–120]. Thus, the quench can be described by a quadratic model and a gaussian initial state, and it is therefore expected to relax to the GGE, as we have discussed in section 5. In the experiment, the post-quench correlations of the (relative) phase are measured by letting the two atomic clouds interfere after releasing them from the trap [65, 66, 117].

7.2. Mesoscopic systems

Despite being ultracold atomic gases a major motivation for the study of quantum quenches and non-equilibrium dynamics, it may appear somewhat striking that some of the above theoretical ideas have found faster applications in the realm of mesoscopic systems of electrons. However, when subjected to intense magnetic fields and low temperatures, the ‘dirty’ 2D gases of interacting electrons in the quantum Hall regime have been for some time the arena for many experiments that have reached the precision standards of atomic physics in interferometry [142], for instance.

Chirality arising from the application of the magnetic field makes the edges of the 2D electron gas in the integer quantum Hall regime behave as clean, non-interacting, conducting channels as far as the transport at small voltage bias is concerned. This is manifested in the perfect conductance quantization of the latter in units of $e^2/h$, which can be experimentally observed at low temperatures. However, this does not mean that the (screened) Coulomb interaction can be neglected. Coulomb interaction is indeed important to account for the spectral properties of the channels. Furthermore, its effect is especially felt when driving the system out of equilibrium. In this regard, Kovrizhin and Chalker [143, 144] first pointed out the interesting analogies between the study of the relaxation mechanisms of integer quantum Hall interferometers and the interaction quench in the LM. In this section, we shall review the work of Milletari and Rosenow [145], which has also found recent experimental confirmation [146].

Using a Hall bar device like the one shown in figure 6 at Landau-level filling $\nu = 2$, it is possible to prepare a double-step non-equilibrium distribution for the fermions in the outer edge channel (denoted by 1 in figure 6). Mathematically,
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\[ n_1R(\epsilon) = a\theta(\mu_1 - \epsilon) + (1 - a)\theta(\mu_2 - \epsilon), \]

where \( \mu_1 = (1 - a)eV \) and \( \mu_2 = -aeV \) \((eV > 0)\), and \( 0 < a < 1 \). The initial state corresponding to the above distribution is prepared by taking the outer channel 1, which comes from a reservoir at chemical potential \( eV \) through the first quantum point contact (QPC1). At this contact, the channel is brought in close contact with its grounded left-moving partner at the other edge of the Hall bar, which allows for local tunneling of electrons with different chemical potentials between the two. For non-interacting electrons, this results in the distribution function displayed in equation (102). After being driven out of equilibrium, channel 1 is allowed to relax by interacting downstream with the grounded inner channel 2 in the shaded region of figure 6. Chirality implies that time and space play the same role and the spatial overlap of the two channels in this region can be regarded as sudden quench of their mutual interaction. In the interaction region the channels are described by the (‘post-quench’) Hamiltonian:

\[
H = \sum_{q > 0}(a_{1R}^\dagger(q) a_{2R}^\dagger(q)) \begin{pmatrix}
    v_1q & g_4q/2 \\
    g_4q/2 & v_2q
\end{pmatrix}
\begin{pmatrix}
    a_{1R}(q) \\
    a_{2R}(q)
\end{pmatrix},
\]

which can be diagonalized by means of the canonical transformation \( a_{1R}(q) = \cos \theta \ b_{1R}(q) - \sin \theta \ b_{2R}(q) \) and \( a_{2R}(q) = \sin \theta \ b_{1R}(q) + \cos \theta \ b_{2R}(q) \). The \( \theta \) is the mixing angle, which is determined from \( \tan 2\theta = \frac{g_4}{v_1 - v_2} \), where \( g_4 \) parametrizes the (screened Coulomb) interaction between the two channels and \( v_1 \) and \( v_2 \) are the outer and inner channel velocities, respectively. Note that the appearance of trigonometric functions here (instead of the hyperbolic functions of the previous sections) is due to the same (right-moving) chirality of the two coupled fermionic channels. Nevertheless, as in the case of the LM,
a solution of the quench dynamics can be obtained by means of the following time-dependent canonical transformation:

\[ a_{1R}(q, t) = f_1(q, t)a_{1R}(q) + g(q, t)a_{2R}(q), \]
\[ a_{2R}(q, t) = g(q, t)a_{1R}(q) + f_2(q, t)a_{2R}(q), \]

where

\[ f_1(q, t) = \frac{1}{2}(e^{-i\tilde{v}_1qt} + e^{-i\tilde{v}_2qt}) + \frac{1}{2}(e^{-i\tilde{v}_1qt} - e^{-i\tilde{v}_2qt})\cos 2\theta, \]
\[ f_2(q, t) = \frac{1}{2}(e^{-i\tilde{v}_1qt} + e^{-i\tilde{v}_2qt}) - \frac{1}{2}(e^{-i\tilde{v}_1qt} - e^{-i\tilde{v}_2qt})\cos 2\theta, \]
\[ g(q, t) = \frac{1}{2}(e^{-i\tilde{v}_1qt} - e^{-i\tilde{v}_2qt})\sin 2\theta. \]

The parameters \( \tilde{v}_{1(2)} = v_{1(2)}\cos^2 \theta + v_{2(1)}\sin^2 \theta \pm \frac{i}{2}g_4\sin 2\theta \) are the eigenmode velocities.

After spatially overlapping, the two channels 1 and 2 are again spatially separated in order to probe the inner channel 2 by taking it through a second quantum point contact at QPC2. There, it comes into contact with its left-moving partner before reaching ground together with outer channel (see figure 6). The shot noise in this channel is thus measured at contact number 3 (see figure 6). Roughly speaking, shot noise measures the (fractional) charge of the carriers and can be obtained from the single-particle Green’s function of the channel (\( \alpha = R, L \)):

\[ G^\leq_{2\alpha}(\tau) = \langle \psi^\dagger_{2\alpha}(x, t + \tau)\psi_{2\alpha}(x, t) \rangle. \]

through the expression:

\[ S = \frac{2e^2}{\hbar} |t_2|^2 \int d\epsilon \left[ G^\leq_{2R}(\epsilon)G^\leq_{2L}(\epsilon) + G^\leq_{2L}(\epsilon)G^\leq_{2R}(\epsilon) \right] \]

where \( t_2 \) is the tunneling amplitude at the second point contact, QPC2. The Green’s function of the inner upstream channel can be written as follows:

\[ G^\leq_{2R}(\tau) = G^{\leq(0)}_{2R}(0, \tau)Z(x, t, \tau), \]
\[ Z(x, t, \tau) = \langle e^{i\chi(x, t, \tau)}e^{-i\chi(x, t, \tau)} \rangle, \]
\[ G^{\leq(0)}_{2R}(0, \tau) = \frac{1}{2\pi} \frac{1}{(-i\tilde{v}_1\tau + a_0)\sin^2 \theta (-i\tilde{v}_2\tau + a_0)\cos^2 \theta}, \]

where \( \chi(x, t, \tau) = \sum_{q>0} \left( \frac{2\pi}{qL} \right)^{1/2} e^{-qa_0/2} e^{iq\tau} [g(q, t + \tau) - g(q, t)] a_{1R}(q) \) and \( a_0 \) is a short-distance cut-off of the order of the magnetic length. The calculation of \( Z(x, t, \tau) \) is complicated.
by non-gaussianity of the initial state, which yields the double-step distribution of equation (102). However, it can be still expressed in terms of a Fredholm determinant of the Toeplitz type and evaluated numerically [145]. The properties of the steady state are accessed by taking the limit $t \to +\infty$ in the above Green’s function. The resulting steady state is not thermal [145], which is a consequence of the constrained dynamics of the model. The shot noise can be then extracted from the steady state Green’s function. The so-called Fano factor, corresponding to the ratio $F = S/S_{\text{ref}}$ where $S_{\text{ref}} = 4epa(1 - a)(e^2/h)V$ ($p = |t_2|^2 \tilde{\nu}_1^2 \tilde{\nu}_2 \sin^2 \theta \cos^2 \theta$ is the reflection probability at the second quantum point contact [145], and $V$ is the voltage bias), is shown in figure 7 as a function of the mixing angle, $\theta$. The experimental determination of its value (with the corresponding error bars) is indicated by the shaded (blue) region.

8. Conclusions and outlook

Having the opportunity to write this article has taught us that the subject of quantum quenches in the Luttinger and related models continues to enjoy tremendous vitality after ten years. There are plenty of new analytical results, often obtained by the combination of powerful techniques which include conformal field theory, Bethe ansatz,
non-equilibrium (Keldysh) Green’s functions, etc. In addition, powerful numerical methods have been applied to various models and are teaching us where the analytical results can be (and cannot be) applied.

Nevertheless, a deeper understanding of the universality of the LM predictions, beyond the rather fragmented knowledge obtained from specific models, is still lacking. In the authors’ view, such a framework still needs to be put in place. This framework should relate to some of the concepts of entanglement that have been surveyed in section 5 and should be able to account for the differences between gaussian and non-gaussian initial states. More importantly, the theoretical framework should also provide quantitative answers to questions about time scales and requirements for thermalization. It should tell which perturbations to the Hamiltonian of an exactly solvable model like Luttinger’s are dominant in driving the system towards thermal equilibrium, which ones subdominant, and what the time scales associated with them are.

On the experimental side, results are coming out at an increased pace. These include the recent observation of relaxation to the generalized Gibbs ensemble in an ultracold gas [117]. However, as theorists who cherish their own (theoretical) pets, we would like to see similar experiments carried out for a more faithful (fermionic) realization of the interaction quench in the Luttinger model. Our hope is that this article may motivate a young experimentalist to take on this challenge. In addition, let us mention that using different probes in the mesoscopic setup discussed in section 7.2 may allow for a more thorough characterization of the non-equilibrium steady state, and whether it is really a steady state or just a pre-thermalized one. Ultracold atomic systems still have a lot to offer as well as trapped ions [132] since possibilities for the control of the system parameters and the preparation of the initial state are larger for these systems.

There are also new venues waiting to be fully explored. There is a zoo of newly discovered topologically-protected phases out there. Many of them are endowed with gapless states at their interfaces with topologically trivial matter. In two dimensions, some of these gapless states can be described as Tomonaga–Luttinger liquids, and the ideas described in section 7.2 can be relevant for the understanding of their non-equilibrium dynamics as well. From a broader perspective, we may also wonder what kind of new manifestations in the non-equilibrium dynamics of interacting systems topological protection can bring about.

Going beyond isolated systems (or systems that can be treated to a large extent as such), the dynamics of the Luttiger model and its relatives coupled to environments is, to the best of our knowledge, a largely uncharted territory. The study of mesoscopic systems is clearly an area that can benefit from such studies since, in the solid state, it is much more difficult to completely isolate fragile quantum systems from their environment. Accounting for the effects of realistic sources of quantum dissipation [130] is therefore an interesting research direction. Furthermore, in ultracold atomic systems, despite being largely isolated, it is also possible to engineer interesting environments [147] and the study of quantum dynamics in those setups should be a promising research topic.

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