ℓ-ADIC TAUTOLOGICAL SYSTEMS

LEI FU, AN HUANG, BONG LIAN, SHING-TUNG YAU, DINGXIN ZHANG, XINWEN ZHU

Abstract. Tautological systems was introduced in [13] as the system of differential equations satisfied by period integrals of hyperplane sections of some complex projective homogenous varieties. We introduce the ℓ-adic tautological systems for the case where the ground field is of characteristic p.

Key words: multiplicative sheaf; Deligne-Fourier transform.
Mathematics Subject Classification: 14F20.

INTRODUCTION

Let $G$ be an algebraic group over $\mathbb{C}$, $\mathfrak{g}$ the Lie algebra of $G$, $\beta : \mathfrak{g} \to \mathbb{C}$ a character, $V$ a finite dimensional complex representation of $G$, and $\bar{X}$ a closed subscheme of $V$ invariant under the action of $G$. For any $\theta \in \mathfrak{g}$, let $L_\theta$ be the vector field on $V$ defined by

$$L_\theta f = \beta(\theta) f,$$

$$g\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}\right) f = 0,$$

where the first equation is taken for all $\theta \in \mathfrak{g}$, and the second equation is taken for all $g(\xi_1, \ldots, \xi_N) \in I(\bar{X})$. In [13], it is proved that under some conditions, the tautological systems are satisfied by period integrals of hyperplane sections of some complex projective homogenous varieties. The $D$-module corresponding to this systems is introduced in [8]. As $f(x_1, \ldots, x_N)$ is a function on $V^\vee$, the vector field $L_\theta$ in the first equation is the vector field on $V^\vee$ for the contragredient action of $G$ on $V^\vee$. This equation can be written as

$$\frac{d}{dt} f(\exp(\theta t) \cdot \xi) = \beta(\theta) f(\exp(\theta t) \cdot \xi),$$

for any $\theta \in \mathfrak{g}$ and $\xi \in V^\vee$. Solving this ordinary differential equation, we get

$$f(\exp(\theta t) \cdot \xi) = e^{\beta(\theta)t} f(\xi),$$

We thank Quentin Guignard and Haoyu Hu for helpful discussions. The main part of this work was done while Lei Fu visited the Center of Mathematical Sciences and Applications (CMSA) at Harvard University in 2015. He would like to thank CMSA for the hospitality. The research of Lei Fu is supported by NSFC 11531008.
that is, $f$ is homogeneous of weight $\beta$ with respect to the group action of $G$ on $V^\vee$. Suppose $G$ is connected. Then $e^\theta$ ($\theta \in \mathfrak{g}$) generate a dense subgroup of $G$. Suppose furthermore that $f$ is continuous. Then for any $\xi \in V^\vee$, the restriction of $f$ to the closure of the orbit $G\xi$ is uniquely determined by the value $f(\xi)$.

Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be the Lie algebra homomorphism defined by the representation $V$ of $G$, and write

$$\rho(\theta)(x_i) = \sum_{j=1}^N \theta_{ij} x_j.$$  

Then the equation $L_\theta f = \beta(\theta)f$ can be written as

$$\sum_{i,j} \theta_{ij} x_i \frac{\partial f}{\partial x_j} + \beta(\theta)f = 0.$$  

Define formally the Fourier transform $\hat{f}$ of $f$ by

$$\hat{f}(\xi_1, \ldots, \xi_N) = \int f(x_1, \ldots, x_N) e^{-(x_1\xi_1 + \cdots + x_N\xi_N)} dx_1 \cdots dx_N.$$  

Formally we have

$$\widehat{(x_i f)} = -\frac{\partial}{\partial \xi_i} \hat{f}, \quad \left(\frac{\partial}{\partial x_k} \hat{f}\right) = \xi_k \hat{f}.$$  

Taking the Fourier transform of the tautological system, we get

$$\sum_{i,j} \theta_{ij} \xi_j \frac{\partial \hat{f}}{\partial \xi_i} + (\text{Tr}(\rho(\theta)) - \beta(\theta)) \hat{f} = 0 \text{ for all } \theta \in \mathfrak{g},$$  

$$g(\xi_1, \ldots, \xi_N) \hat{f} = 0 \text{ for all } g \in I(\hat{X}).$$  

The first system of equations is just

$$L_\theta \hat{f} = (\beta(\theta) - \text{Tr}(\rho(\theta))) \hat{f} \text{ for all } \theta \in \mathfrak{g},$$  

and it means that $\hat{f}$ is homogeneous of weight $\beta - \text{Tr}(\rho)$ with respect to the action of $G$ on $V$.

The second system of equations means that $\hat{f}$ is supported in $\hat{X}$. For a rigorous treatment via the Fourier transform, see [11 §3]. By the Fourier inversion formula, to get a solution for the tautological system, we may start with a function on $X$ supported on $\hat{X}$ and homogenous of weight $\beta - \text{Tr}(\rho)$ with respect to the action of $G$, and take its inverse Fourier transform. This is the method that we are going to use. In this paper, we work out a theory of tautological systems when the ground field is of characteristic $p$, a problem posed in [13 §10]. In characteristic $p$, differential equations or $D$-modules do not behave well. Instead, we work with $\ell$-adic local systems, and more generally objects in the derived category of $\mathcal{O}_X$-sheaves. The idea is that the tautological system is the Deligne-Fourier transform of a homogeneous object.

**Remark 0.1.** Let $G_0$ be an algebraic group acting on a smooth projective variety $X$ of dimension $d$, $\mathcal{L}$ a very ample $G_0$-linearized invertible sheaf, and $V = \Gamma(X, \mathcal{L})^\vee$. Then $G_0$ acts on $V$. Let $G = G_0 \times \mathbb{G}_m$, let $\mathbb{G}_m$ act trivially on $X$, and act on $V$ by scalar multiplication. We have a $G$-equivariant embedding $X \to \mathbb{P}(V)$. Let $\hat{X}$ be the cone over $X$, and let $\beta : \mathfrak{g}_0 \oplus \mathbb{C} \to \mathbb{C}$ be a character, where $\mathfrak{g}_0$ is the Lie algebra of $G_0$. The tautological system $\tau(G, \beta, V, \hat{X})$

$$L_\theta f = \beta(\theta)f \quad (\theta \in \mathfrak{g}_0 \oplus \mathbb{C}),$$

$$g\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)f = 0 \quad (g \text{ lies in the ideal sheaf of } \hat{X}),$$

with
is studied in detail \[9\ [13\]. Under certain conditions, period integrals for the family of hyperplane sections of \(X\) are solutions of this system. More precisely, let \(B\) be the Zariski open subset of \(V^\vee = \Gamma(X, \mathcal{L})\) such that for any \(\phi \in B\), the hypersurface \(Y_\phi = \{\phi = 0\}\) of \(X\) is smooth, let
\[
\mathcal{Y} = \{(x, \phi) \in X \times B|\phi(x) = 0\},
\]
let \(\pi : \mathcal{Y} \to B\) be the projection, let \(\omega_X\) and \(\omega_{Y_\phi}\) be the invertible sheaves of top degree holomorphic forms on \(X\) and \(Y_\phi\), respectively, and let \(i : Y_\phi \to X\) be the close immersion. By the adjunction formula, we have
\[
i^*(\mathcal{L} + \omega_X) \cong \omega_{Y_\phi}.
\]
We thus have a restriction map
\[
R_\phi : H^0(X, \mathcal{L} + \omega_X) \to H^0(Y_\phi, \omega_{Y_\phi}).
\]
Let \(\tau \in H^0(X, \mathcal{L} + \omega_X)\) be an eigenvector for the action of \(g_0\), that is, there exists a character \(\beta_0 : g_0 \to \mathbb{C}\) with the property
\[
\theta \cdot \tau = \beta_0(\theta) \tau
\]
for any \(\theta \in g_0\). Let \(\beta : g_0 \otimes \mathbb{C} \to \mathbb{C}\) be the character defined by \(\beta(\theta, \lambda) = \beta_0(\theta) + \lambda\) and let \(\gamma_\phi \in H_{d-1}(Y_\phi, \mathbb{Z})\) be locally constant homology \((d - 1)\)-cycles. Then by \([13\, Theorem\, 8.8]\), the period integral
\[
\int_{\gamma_\phi} R_\phi(\tau)
\]
as a function of \(\phi\) is a solution of the tautological system \(\tau(G, \beta, V, X)\), provided that there exists a principal \(G\)-equivariant \(H\)-bundle \(M \to X\) with a CY-structure and a character \(\chi : H \to \mathbb{C}^*\) such that \(\mathcal{L}\) is the invertible sheaf corresponding to the line bundle \(M \times_H \mathbb{C} \to X\), where \(H\) acts on the factor \(\mathbb{C}\) through the character \(\chi\). Here a CY-structure is a nowhere vanishing holomorphic form \(\omega_M\) on \(M\) of top degree which is an eigenvector of the action of \(H\), that is \(h \cdot \omega_M = \chi_M(h) \omega_M\) for some character \(\chi_M : H \to \mathbb{C}^*\).

We first construct the homogenous objects. Let \(k\) be a field, and let \(\ell\) be a prime number distinct from the characteristic of \(k\). For any \(k\)-scheme \(S\) of finite type, let \(D^b_c(S, \overline{\mathbb{Q}}_\ell)\) be the derived category of \(\overline{\mathbb{Q}}_\ell\)-sheaves on \(S\). Let \(G'\) be a commutative algebraic group defined over \(k\), and let
\[
m : G' \times_k G' \to G', \quad p_1 : G' \times_k G' \to G', \quad p_2 : G' \times_k G' \to G'
\]
be the multiplication on \(G'\) and the projections. A multiplicative sheaf on \(G'\) is a rank 1 lisse \(\overline{\mathbb{Q}}_\ell\)-sheaf \(\mathcal{L}\) together with an isomorphism
\[
\theta : p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \cong m^* \mathcal{L}
\]
such that the following conditions hold:

1. Symmetry: Let \(\sigma : G' \times_k G' \to G' \times_k G'\) and \(\sigma' : p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \to p_2^* \mathcal{L} \otimes p_1^* \mathcal{L}\) be the morphisms defined by permuting factors. The following diagram commutes:
\[
\begin{array}{ccc}
p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} & \xrightarrow{\theta} & m^* \mathcal{L} \\
\sigma' \downarrow & & \uparrow \cong \\
p_2^* \mathcal{L} \otimes p_1^* \mathcal{L} & \cong & \sigma^*(p_1^* \mathcal{L} \otimes p_2^* \mathcal{L}) \xrightarrow{\sigma'^*(\theta)} \sigma^* m^* \mathcal{L}
\end{array}
\]

2. Associativity: Let \(q_i : G' \times_k G' \times_k G' \to G'\) (resp. \(q_{ij} : G' \times_k G' \times_k G' \to G' \times_k G'\)) be the projections to the \(i\)-th factor (resp. \((i, j)\)-th factor) for any \(1 \leq i \leq 3\) (resp. \(1 \leq i < j \leq 3\), and
Let \( m_3 : G' \times_k G' \times_k G' \to G' \) be the multiplication. The following diagram commutes:

\[
\begin{array}{ccc}
q_1^* \mathcal{L} \otimes q_2^\ast \mathcal{L} \otimes q_3^\ast \mathcal{L} & \xrightarrow{\text{id} \otimes q_2^\ast (\theta) \otimes \text{id}} & q_1^* \mathcal{L} \otimes q_2^\ast \mathcal{L} \\
q_1^* \mathcal{L} \otimes q_2^\ast m^\ast \mathcal{L} & \xrightarrow{\left(q_1 \times q_2 \right)^\ast (\theta)} & q_3^\ast m^\ast \mathcal{L}
\end{array}
\]

The isomorphism \( \theta \) induces an isomorphism

\[ p_i^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes m^\ast \mathcal{L}^{-1} \cong \overline{Q}_\ell. \]

Restricting this isomorphism to \((e, e)\), where \( e : \text{Spec} \, k \to G' \) is the unit section of \( G' \), we get an isomorphism

\[ e^* \mathcal{L} \cong \overline{Q}_\ell. \]

We can obtain multiplicative \( \overline{Q}_\ell \)-sheaf on \( G' \) as is follows: Let

\[ 0 \to N_i \to G'_i \to G' \to 0 \quad (i \in I) \]

be the inverse system of all extensions of \( G' \) by finite discrete abelian groups \( N_i \). Note that each \( G'_i \to G' \) is a Galois étale isogeny with Galois group \( N_i \). We may regard the projective system \( \{G'_i\}_{i \in I} \) as the universal covering space classifying étale isogenies of \( G' \). Set

\[ \pi_1^{\text{isogeny}}(G') = \lim_{i \in I} N_i. \]

It is a quotient of the étale fundamental group \( \pi_1(G') \) of \( G' \), and coincides with \( \pi_1(G') \) if \( k \) is algebraically closed of characteristic 0. Let \( \beta : \pi_1^{\text{isogeny}}(G') \to \overline{Q}_\ell \) be a character. It induces a character on the étale fundamental group of \( G' \). Denote by \( \mathcal{L}_\beta \) the corresponding \( \overline{Q}_\ell \)-sheaf associated to this Galois representation. Then \( \mathcal{L}_\beta \) is multiplicative. Any multiplicative sheaf on \( G' \) is of this form. See [14] 2.3.10 and [6] 2.9 for a proof.

Suppose \( k = \mathbb{F}_q \) is a finite field with \( q \) elements of characteristic \( p \). Let \( F \) be the Frobenius morphism, that is, \( F \) is identity on the underlying topological space of \( G' \), and \( F \) maps any section of \( \mathcal{O}_G \) to its \( q \)-th power. Let

\[ L : G' \to G', \quad x \mapsto F(x) \cdot x^{-1} \]

be the Lang isogeny. Its kernel is the group \( G'(\mathbb{F}_q) \) of \( \mathbb{F}_q \)-points of \( G' \), and it is a Galois étale covering with Galois group \( G'(\mathbb{F}_q) \). By [16] VI §1 Proposition 6, any Galois isogeny over \( G' \) is a quotient of the Lang isogeny. It follows that

\[ \pi_1^{\text{isogeny}}(G') \cong G'(\mathbb{F}_q). \]

Let \( \beta : G'(\mathbb{F}_q) \to \overline{Q}_\ell \) be a character. The multiplicative sheaf \( \mathcal{L}_\beta \) is called the Lang sheaf associated to \( \beta \). For any \( x \in G'(\mathbb{F}_q) \), we have

\[ (\text{Frob}_x, \mathcal{L}_\beta, \bar{x}) = \beta(x'), \]

where \( \text{Frob}_x \) is the geometric Frobenius element in \( \pi_1(G') \).

**Remark 0.2.** Let \( G' \) be a connected commutative complex algebraic group, let \( g' \) be its Lie algebra, and let \( \beta : g' \to \mathbb{C} \) be a character. The exponential map

\[ g' \to G', \quad \theta \mapsto \exp(\theta) \]

is an epimorphism. It is the universal covering space of \( G' \), and its kernel is isomorphic to the fundamental group \( \pi_1(G') \) of \( G' \). Restricting \( e^\beta \) to this kernel, we get a character \( e^\beta : \pi_1(G') \to \mathbb{C}^* \).
The local system $L_{\alpha}$ associated to this character is a multiplicative sheaf. For any holomorphic function $f$ on $G'$ and any invariant vector field $\theta \in g$ on $G'$, let

$$\nabla_\theta(f) = \theta(f) - \beta(\theta)f.$$ 

Then $\nabla$ is an integrable connection on the trivial vector bundle over $G'$. Horizontal sections are those holomorphic functions $f$ such that

$$\theta(f) = \beta(\theta)f,$$

that is, $f$ are homogenous of weight $\beta$. The multiplicative local system $L_{\alpha}$ is isomorphic to the local system of horizontal sections of this connection.

From now on, we assume $k$ is a perfect field of characteristic $p$ unless otherwise stated. Let $G$ be an algebraic group defined over $k$, and let $G \to \text{GL}(V)$ be a representation, where $V$ is a finite dimensional vector space over $k$. By abuse of notation, we also denote by $V$ the affine $k$-scheme $\text{Spec}(\text{Sym}(V^\vee))$. Let $v$ be a nonzero vector in $V$, and let $Q$ be the connected component of the stabilizer of $v$. Consider the morphism

$$\iota : G/Q \to V, \quad gQ \mapsto gv.$$ 

It is quasi-finite. Let $G' = G/Q[G,G]$, which is a commutative algebraic group. Fix a multiplicative sheaf $L_\beta$ on $G'$. Denote its inverse image by the canonical morphism $G/Q \to G'$ also by $L_\beta$. Note that $\iota_!L_\beta$ and $R_\iota_!L_\beta$ are homogeneous objects in $D^b(V,\overline{Q}_\ell)$ supported in the $G$-invariant subset $Gv$. By homogeneity, we mean that

$$\mu^*\iota_!L_\beta \cong L_\beta \otimes \mu_!L_\beta, \quad \iota^*R_\iota_!L_\beta \cong L_\beta \otimes R_\iota_!L_\beta,$$

where $\mu : G \times V \to V$ is the action of $G$ on $V$, and $L'_\beta$ is the inverse image of $L_\beta$ by the morphism $G \to G/Q$. (Confer the proof of Lemma [11]). Fixed a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{Q}_\ell$, and let $\mathcal{F}_\psi : D^b_c(V,\overline{Q}_\ell) \to D^b_c(V^\vee,\overline{Q}_\ell)$ be the Deligne-Fourier transform defined by $\psi$. We define the $\ell$-adic tautological sheaves to be

$$\mathcal{T}_!(G,\beta,V,v,\psi) = \mathcal{F}_\psi(\iota_!L_\beta[\dim G/Q]),$$

$$\mathcal{T}_*(G,\beta,V,v,\psi) = \mathcal{F}_\psi(R_\iota_!L_\beta[\dim G/Q]).$$

Recall that for any vector bundle $E$ of rank $r$ over a $k$-scheme $S$ of finite type, the Deligne-Fourier transform is the functor

$$\mathcal{F}_\psi : D^b_c(E,\overline{Q}_\ell) \to D^b_c(E^\vee,\overline{Q}_\ell), \quad K \mapsto R\text{pr}^\vee_r(\text{pr}^*K \otimes \langle , \rangle^*L_\psi)[r],$$

where $E^\vee \to S$ is the dual vector bundle of $E$, $\text{pr} : E \times_k E^\vee \to E$ and $\text{pr}^\vee : E \times_k E^\vee \to E^\vee$ are the projections, and $\langle , \rangle : E \times_k E^\vee \to \mathbb{A}^1$ is the pairing, and $L_\psi$ is the Artin-Schreier sheaf on $\mathbb{A}^1$ associated to the nontrivial additive character $\psi : \mathbb{F}_p \to \overline{Q}_\ell$, that is, the pulling back to $\mathbb{A}^1$ of the Lang sheaf on the algebraic group $\mathbb{A}^1_p$, associated to the character $\psi : \mathbb{A}^1(\mathbb{F}_p) \to \overline{Q}_\ell$. See [12] for properties of the Fourier transform. For any $k$-scheme $a : S \to \text{Spec} k$ of finite type, denote by $D_S : D^b_c(S,\overline{Q}_\ell) \to D^b_c(S,\overline{Q}_\ell)$ the Verdier dual functor $D_S = R\text{Hom}()$. 

**Proposition 0.3.** Let $n = \dim G/Q$ and $N = \dim V$.

(i) We have $D_{V^\vee}(\mathcal{T}_!(G,\beta,V,v,\psi)) \cong \mathcal{T}_!(G,\beta^{-1},V,v,\psi^{-1})(n+N)$

(ii) In the case where $k$ is a finite field, $\mathcal{T}_!(G,\beta,V,v,\psi)$ (resp. $D_{V^\vee}(\mathcal{T}_!(G,\beta,V,v,\psi))$) is mixed of weights $\leq n+N$ (resp. $\leq -(n+N)$).
Proof. (i) By [23 1.3.2.2], we have
\[
D_{V^\vee}(T_v(G, \beta, V, v, \psi)) = D_{V^\vee}(\mathcal{F}_\psi(R_{t*}\mathcal{L}_\beta[n])) \\
\cong \mathcal{F}_{\psi^{-1}}(D_{V}(R_{t*}\mathcal{L}_\beta[n]))(N) \\
\cong \mathcal{F}_{\psi^{-1}}(tD_G/Q(\mathcal{L}_\beta[n]))(N) \quad (i) \\
\cong \mathcal{F}_{\psi^{-1}}(tL_{\beta^{-1}}[n])(n + N) \quad (ii) \\
\cong \mathcal{T}_\beta(G, \beta^{-1}, V, v, \psi^{-1})(n + N).
\]

(ii) \( nL_\beta[n] \) is mixed of weights \( \leq n \). Its Deligne-Fourier transform \( \mathcal{T}_\beta(G, \beta, V, v) \) is mixed of weights \( \leq n + N \) by [11 Théorème 3.3.1]. Then by (i), \( D_{V^\vee}(T_v(G, \beta, V, v, \psi)) \) is mixed of weights \( \leq -(n + N) \). \( \Box \)

Example 0.4. Let \((w_{ij})\) be an \( n \times N \) matrix of rank \( n \) with integer entries. Take \( G \) to be the torus \( G = \mathbb{G}_{m,k}^n \). Consider the representation \( V \) of \( G \) defined by the action
\[
\mathbb{G}_{m,k}^n \times_k \mathbb{A}_k^N \to \mathbb{A}_k^N, \\
(t_1, \ldots, t_n), (x_1, \ldots, x_N) \mapsto (t_1^{w_{11}}x_1, \ldots, t_n^{w_{nN}}x_N).
\]
The connected component of the stabilizer \( v = (1, \ldots, 1) \in V \) is trivial. Let \( \hat{X} \) the closure of the orbit of \( v \). Then \( \hat{X} \) is the closed subscheme of \( \mathbb{A}_k^N \) defined by \( \prod_{a_j \geq 0} \xi_j^{a_j} - \prod_{a_j < 0} \xi_j^{-a_j} = 0 \), where \((a_1, \ldots, a_N) \in \mathbb{Z}^N \) goes over the family of integer linear relations
\[
\sum_{j=1}^N a_jw_j = 0
\]
among \( w_1, \ldots, w_N \) with \( w_j = (w_{1j}, \ldots, w_{nj}) \in \mathbb{Z}^n \).

Suppose \( k \) is the complex field \( \mathbb{C} \). Let \( \beta \) the character
\[
\beta : \mathbb{C}^n \to \mathbb{C}, \quad (\lambda_1, \ldots, \lambda_n) \mapsto \gamma_1\lambda_1 + \cdots + \gamma_n\lambda_n,
\]
where \( \gamma_1, \ldots, \gamma_n \in \mathbb{C} \). The tautological system \( \tau(G, \beta, V, \hat{X}) \) is exactly the Gelfand-Kapranov-Zelevinsky (GKZ) hypergeometric system
\[
\sum_{j=1}^N w_{ij}x_j \frac{\partial f}{\partial x_j} + \gamma_if = 0 \quad (i = 1, \ldots, n), \\
\prod_{a_j \geq 0} \left( \frac{\partial}{\partial x_j} \right)^{a_j} f = \prod_{a_j < 0} \left( \frac{\partial}{\partial x_j} \right)^{-a_j} f
\]
introduced in [7]. It is the system of differential equations satisfied by the exponential integral (period integral)
\[
f(x_1, \ldots, x_n) = \int_\sigma f_{t_1}^{x_1} \cdots t_n^{x_n} e^{\sum_{j=1}^N x_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n},
\]
where \( \sigma \) is any real \( n \)-dimensional cycles in \((\mathbb{C}^*)^n\).

Suppose \( k = \mathbb{F}_q \) is a finite field with \( q \) elements. Let \( \chi_1, \ldots, \chi_n : \mathbb{F}_q^* \to \mathbb{Q}_p^* \) be multiplicative characters. They define a character
\[
\beta : \mathbb{G}_{m,k}^n(\mathbb{F}_q) \to \mathbb{Q}_p^*, \quad (t_1, \ldots, t_n) \mapsto \chi_1(t_1) \cdots \chi_n(t_n).
\]
The the tautological system $T(G, \beta, V, v, \psi)$ is the GKZ hypergeometric sheaf studied in [5]. For any $\mathbb{F}_q$-rational point $x = (x_1, \ldots, x_N)$ of $V^\vee$, we have
\[
\text{Tr}(\text{Prob}_x, T(G, \beta, V, v, \psi)) = \sum_{t_1, \ldots, t_n \in \mathbb{F}_q^*} \chi_1(t_1) \cdots \chi_n(t_n) \psi \left( \text{Tr}_{\mathbb{F}_q / \mathbb{F}_p} \left( \sum_{j=1}^N t_1^{w_{1j}} \cdots t_n^{w_{nj}} \right) \right).
\]
The exponential sum on the righthand side is the arithmetic counterpart of the above exponential integral.

We will give more examples of tautological systems in §2. In the following, we focus on the tautological systems constructed from a quasi-projective homogenous variety embedded equivariantly in $\mathbb{P}(V)$. More precisely, let $G_0$ be an algebraic group, let $G_0 \to \text{GL}(V)$ be a representation, and let $G = G_0 \times \mathbb{G}_m$. Let $G$ act on $V$ so that the factor $\mathbb{G}_m$ acts on $V$ by scalar multiplication. For any nonzero vector $x \in V$, denote by $[x]$ the corresponding point in the projective space $\mathbb{P}(V) = \text{Proj}(\text{Sym}(V^\vee))$, and denote by $L_{[x]}$ the line in $V$ spanned by $x$. Let $L$ be the tautological line bundle over $\mathbb{P}(V) = \text{Proj}(\text{Sym}(V^\vee))$. The fiber of $L$ at $[x]$ can be identified with the line $L_{[x]}$. Note that $G$ acts on $\mathbb{P}(V)$ and on $L$, and the factor $\mathbb{G}_m$ acts trivially on $\mathbb{P}(V)$ and acts by scalar multiplication on fibers of $L$. Let $L^\vee$ be the dual line bundle of $L$. It is provided with the contragredient action by $G$. The fiber of $L^\vee$ over $[x]$ consists of linear functionals on the line $L_{[x]}$. The canonical morphisms $L \to \mathbb{P}(V)$ and $L^\vee \to \mathbb{P}(V)$ are $G$-equivariant.

We have a canonical embedding
\[ i : L \to V \times_k \mathbb{P}(V) \]
of $L$ into the trivial vector bundle $V \times_k \mathbb{P}(V) \to \mathbb{P}(V)$. The fiber of this morphism over $[x] \in \mathbb{P}(V)$ is the inclusion of the line $L_{[x]}$ into $V$. The transpose of $i$ is the evaluation morphism
\[ ev : V^\vee \times_k \mathbb{P}(V) \to L^\vee. \]
For any $\phi \in V^\vee$, the fiber of $ev$ over $[x]$ maps a point $(\phi, [x])$ in $V^\vee \times_k \mathbb{P}(V)$ to the restriction $\phi|_{L_{[x]}} \in L^\vee_{[x]}$ of $\phi$ to the subspace $L_{[x]}$. Let $L^o$ be the complement of the zero section of $L$. We have an isomorphism
\[ V - \{0\} \xrightarrow{\cong} L^o \]
which maps every nonzero vector $x$ to the point $x$ considered as an element in the line $L_{[x]}$. $L^o$ is invariant under the action of $G$.

Let $v \in V$ be a nonzero vector vector, let $Q \subset G$ be the connected component of the stabilizer of $v$, and let $P = QG_m$. Note that $P$ stabilizes $[v] \in \mathbb{P}(V)$. Regard $v$ as an element in $L_{[v]}$. Let $v^* \in (L^\vee)^o_{[v]}$ define by $v^*(v) = 1$. Note that $Q$ is also the connected component of the stabilizer of $v^*$. Let
\[ L_{G/P} = L \times_{\mathbb{P}(V)} G/P, \quad L_{G/P}^\vee = L^\vee \times_{\mathbb{P}(V)} G/P, \quad i_{G/P} : L_{G/P} \to V \times_k G/P, \quad ev_{G/P} : V^\vee \times_k G/P \to L_{G/P}^\vee \]
the base changes of $L$, $L^\vee$, $i$ and $ev$ respectively with respect to the morphism
\[ G/P \to \mathbb{P}(V), \quad gP \mapsto g[v]. \]
Let $\kappa : G/Q \to L_{G/P}^o$ (resp. $\kappa^\vee : G/Q \to L_{G/P}^{o \vee}$) be the morphism $gQ \mapsto gv$ (resp. $gQ \mapsto g v^*$), where we regard $v$ (resp. $v^*$) as an element in the fiber $L_{G/P, eP} \cong L_{[v]}$ (resp. $L_{G/P, eP}^{o \vee} \cong L_{[v]}^{o \vee}$).

**Lemma 0.5.** $\kappa : G/Q \to L_{G/P}^o$ and $\kappa^\vee : G/Q \to L_{G/P}^{o \vee}$ are isomorphisms.
Proof. Note that $Q$ is exactly the stabilizer of $(v,eP)$ considered as a point in $L^\circ_{G/P}$ with respect to the action of $G$ on $L^\circ_{G/P}$. Moreover, the action of $G$ on $L^\circ_{G/P}$ is transitive. We claim that the morphism

$$G \to L^\circ_{G/P}, \quad g \mapsto g \cdot (v,eP)$$

is smooth. It suffices to show it induces an epimorphism on tangent spaces

$$T_e G \to T_{(v,eP)}(L^\circ_{G/P}).$$

The tangent space $T_{(v,eP)}(L^\circ_{G/P})$ is isomorphic to the direct sum of $T_e G(P/G)$ and the one dimensional space corresponding to the fiber of the line bundle $L_{G/P}$ over $eP$. The canonical homomorphism $T_e G \to T_e G(P/G)$ is surjective. The tangent vectors along the direct factor $G_m$ of $G$ are mapped surjectively onto the above one dimensional subspace. This proves our claim. By $[16, 5.5.4$ and 5.5.5], $L^\circ_{G/P}$ is isomorphic to $G/Q$. Similarly, $L^\lor_{G/P}$ is also isomorphic to $G/Q$. □

We use the isomorphism in the above lemma to identify $G/Q$ with $L^\circ_{G/P}$ (resp. $L^\lor_{G/P}$). By abuse of notation, we denote the sheaf $\kappa_! L_\beta$ (resp. $\kappa_!^\lor L_\beta$) on $L^\circ_{G/P}$ (resp. $L^\lor_{G/P}$) also by $L_\beta$. Let $j_{L/G_P} : L^\circ_{G/P} \to L_{G/P}$ and $j_{L_{G/P}} : L^\lor_{G/P} \to L_{G/P}$ be the open immersions of the complements of zero sections, and let $pr : V^\lor \times_k G/P \to V^\lor$ be the projection. Let $L_{\beta|G_m}$ (resp. $L_\psi|G_m$) be the inverse image of $L_\beta$ (resp. $L_\psi$) under the canonical morphism $G_m \to G' = G/Q[G,G]$ (resp. $G_m \to A^1$). Let $G_1(\beta,\psi)$ and $G_*(\beta,\psi)$ be the sheaves on Spec $k$ defined by the following Galois representations:

$$G_1(\beta,\psi) = \begin{cases} H^1_c(G_m,k, L_\beta|G_m \otimes L_\psi|G_m) & \text{if } L_\beta|G_m \otimes L_\psi|G_m \text{ is nontrivial}, \\ \text{otherwise}, \end{cases}$$

$$G_*(\beta,\psi) = \begin{cases} H^1(G_m,k, L_\beta|G_m \otimes L_\psi|G_m) & \text{if } L_\beta|G_m \otimes L_\psi|G_m \text{ is nontrivial}, \\ \mathbb{Q}(-1) & \text{otherwise}, \end{cases}$$

By Poincaré duality, we have

$$G_*(\beta,\psi) \cong G_1(\beta,\psi)^\lor(-1).$$

Actually if $L_\beta|G_m$ is nontrivial, then

$$H^1_c(G_m,k, L_\beta|G_m \otimes L_\psi|G_m) \cong H^1(G_m,k, L_\beta|G_m \otimes L_\psi|G_m)$$

and hence $G_1(\beta,\psi) = G_*(\beta,\psi)$ in this case. We will show that $G_*(\beta,\psi)$ and $G_1(\beta,\psi)$ are of rank 1. (Confer Lemma $[16, 5.7]$.) Denote the inverse images of $G_1(\beta,\psi)$ and $G_*(\beta,\psi)$ on any $k$-scheme by the same notation. In the case where $k = F_q$ is a finite field with $q$ elements and $L_\beta|G_m$ is nontrivial, by the Grothendieck trace formula, $G_*(\beta,\psi)$ and $G_*(\beta,\psi)$ is defined by the Galois representation

$$\text{Gal}(\overline{F}_q/F_q) \to \overline{Q}_\mathbb{L}, \quad \text{Frob}_q \mapsto - \sum_{t \in \mathbb{G}_m} \beta(t)^q(t),$$

where $\text{Frob}_q$ is the geometric Frobenius element in $\text{Gal}(\overline{F}_q/F_q)$, and the righthand side is nothing but the Gauss sum.

Let $U = ev^{-1}_{G/P}(L^\lor_{G/P})$ and let $H$ be the complement of $U$ in $V^\lor \times_k G/P$. Then $U$ (resp. $H$) is an open (resp. closed) subset of $V^\lor \times G/P$, and a rational point $(\phi, qP)$ of $V^\lor \times G/P$ lies in $U$ (resp. $H$) if and only if $\phi(q(v)) \neq 0$ (resp. $\phi(q(v)) = 0$). For any rational point $\phi$ of $V^\lor$, the fiber $H_\phi$ of $H$ over $\phi$ is the hyperplane section $\{ gP \in G/P | \phi(gv) = 0 \}$ in $G/P$, and the fiber $U_\phi$
Theorem 0.6. Notation as above. Suppose $Q$ is geometrically connected.

(i) We have

$$
\mathcal{T}(G, \beta, V, v, \psi) \cong R\text{pr}_! (\mathcal{L}_{\phi, n + 1}(\mathcal{L}_{\beta} T) [n + N - 1] \otimes G_{*}(\beta, \psi)),
$$

and we have a distinguished triangle

$$
R\text{pr}_! (\mathcal{L}_{\phi, n + 1}(\mathcal{L}_{\beta} T) [n + N - 1] \otimes G_{*}(\beta, \psi)) \rightarrow \mathcal{T}(G, \beta, V, v, \psi) \rightarrow 0,
$$

where $\mathcal{L}_{\phi, n + 1}(\mathcal{L}_{\beta} T)$ are the inverse images of the sheaf $\mathcal{L}_{\beta}$ on $G/P$.

(ii) Suppose furthermore that there exists a multiplicative sheaf $\mathcal{L}_{\beta}$ on $G/P$. Then any point $\phi$ in $V^\vee$, we have

$$
\mathcal{T}(G, \beta, V, v, \psi) \cong R\text{pr}_! (\mathcal{L}_{\beta} T)[n + N - 1] \otimes G_{*}(\beta, \psi).
$$

Proof. Follows from the formula for $\mathcal{T}(G, \beta, V, v, \psi)$ in Theorem 0.6 and the proper base change theorem.

}\square

Remark 0.8. Suppose $k = \mathbb{F}_q$ is a finite field. For any $\mathbb{F}_q$-point $x$ of $G/P$, the fiber $\pi^{-1}(x)$ of the projection $\pi : G \to G/P$ is a principal $P$-homogenous space. By [16] VI §1 Corollary 1, $\pi^{-1}(x)$ has a $\mathbb{F}_q$-point. Thus any $\mathbb{F}_q$-point of $G/P$ is the image of a $\mathbb{F}_q$-point of $G$. For any $\mathbb{F}_q$-point $g$ of $G$, denote its image in $G/P$ by $gP$. Suppose $L_{\phi}$ is the Lang sheaf defined by a character $\beta : (G/Q, G/G)(\mathbb{F}_q) \to \mathbb{G}_{L}$. Then by the Grothendieck trace formula and Corollary 0.7, for any $\mathbb{F}_q$-point $\phi$ of $V^\vee$, we have

$$
\text{Tr}(\text{Frob}_\phi, \mathcal{T}_\phi(G, \beta, V, v, \psi)) = (-1)^{n + N - 1} G_{*}(\beta, \psi) \sum_{z \in (G/P)(\mathbb{F}_q), \phi(gzv) \neq 0} \beta(\phi(gzv)^{-1} \cdot gz),
$$
where for any \( F_q \)-point \( x \) of \( G/P, g_x \) is a \( F_q \)-point of \( G \) in the fiber \( \pi^{-1}(x) \), \( \phi(g_x) \)-1 is considered as a \( F_q \)-point of \( G_m \subset G \), \( \beta(\phi(g_x)^{-1}g_x) \) is the value of \( \beta \) at the image of the \( F_q \)-point \( \phi(g_x)^{-1}g_x \) in \( G/Q[G,G] \), and \( G_*(\beta, \psi) \) is the Gauss sum. Note that the value \( \beta(\phi(g_x)^{-1}g_x) \) is independent of the choice of the \( F_q \)-points \( g_x \) in the fiber \( \pi^{-1}(x) \). If there exists a character \( \beta_0 : (G/P[G,G])(\mathbb{F}_q) \to \mathbb{Q}_\ell \) such that \( \beta \) is the composite
\[
(G/Q[G,G])(\mathbb{F}_q) \to (G/P[G,G])(\mathbb{F}_q) \xrightarrow{\beta_0} \mathbb{Q}_\ell.
\]

Then
\[
\text{Tr}(\text{Frob}_q, T_\ast(G, \beta, V, v, \psi)_\beta) = (-1)^{n+1} q \sum_{x \in (G/P)(\mathbb{F}_q), \phi(g_x) \neq 0} \beta_0(g_x),
\]
\[
\text{Tr}(\text{Frob}_q, T(G, \beta, V, v, \psi)_\beta) = (-1)^{n+1} q \sum_{x \in (G/P)(\mathbb{F}_q)} \beta_0(g_x) - (-1)^{n+1} q \sum_{x \in (G/P)(\mathbb{F}_q), \phi(g_x)=0} \beta_0(g_x).
\]

The second equation follows from the distinguished triangle in Theorem 0.6 (ii).

1. **Proof of Theorem 0.6**

Let \( \kappa_0 : G/Q \to V - \{0\} \) be the morphism \( gQ \to gv \), let \( j_0 : V - \{0\} \hookrightarrow V \) be the open immersion, and let \( \pi : V \times G/P \to V \) be the projection. The following diagram commutes:

\[
\begin{align*}
\begin{array}{ccc}
G/Q & & \kappa_0 \gtrless \kappa \gtrless \kappa_0 \\
\mathbb{L}_{G/P} \downarrow & & \downarrow j_0 \downarrow j_0 \downarrow j_0 \\
\mathbb{L}_{G/P} & \xrightarrow{i_{G/P}} & V \times_k (G/P) \xrightarrow{\pi} V \\
& \gtrless & \downarrow G/P \to \text{Spec } \mathbb{F}_q
\end{array}
\end{align*}
\]

We have \( \iota = j_0 \kappa_0 \), and hence
\[
T_!(G, \beta, V, v, \psi) = \mathcal{F}_\psi(j_0 \kappa_0 \mathcal{L}_\beta[n]) \cong \mathcal{F}_\psi(R\pi_! R\pi_{G/P}! j_{G/P}^* \mathcal{L}_\beta[n]).
\]

By [12] 1.2.3.5 and 1.2.2.4, we have
\[
\mathcal{F}_\psi R\pi_! \cong R\text{pr}_* \mathcal{F}_\psi V \times_k (G/P), \quad \mathcal{F}_\psi V \times_k (G/P) R\pi_{G/P}! \cong \text{ev}^*_{G/P} \mathcal{F}_\psi L_{G/P}[N - 1],
\]
where \( \mathcal{F}_\psi V \times_k (G/P) \) is the Deligne-Fourier transform for the trivial vector bundle \( V \times_k (G/P) \to G/P \) and \( \mathcal{F}_\psi L_{G/P} \) is the Deligne-Fourier transform for the vector bundle \( \mathbb{L}_{G/P} \to G/P \). We thus have
\[
T_!(G, \beta, V, v, \psi) \cong \mathcal{F}_\psi(R\pi_! R\pi_{G/P}! j_{G/P}^* \mathcal{L}_\beta[n]) \cong R\text{pr}_* \text{ev}^*_{G/P} \mathcal{F}_\psi L_{G/P}(j_{G/P}^* \mathcal{L}_\beta)[n + N - 1].
\]

The assertion for \( T_!(G, \beta, V, v) \) in Theorem 0.6 (i) then follows from Lemma 1.8 below.

Using [12] 1.3.1.1, the smooth base change theorem, and the same proof as [12] 1.2.3.5, one can show
\[
\mathcal{F}_\psi R\pi_* = R\text{pr}_* \mathcal{F}_\psi V \times_k (G/P).
\]

Since \( i_{G/P} \) is a closed immersion we have \( R\pi_{G/P}^* = i_{G/P}^* \), and hence by [12] 1.2.2.4, we have
\[
\mathcal{F}_\psi V \times_k (G/P) R\pi_{G/P}^* \cong \text{ev}^*_{G/P} \mathcal{F}_\psi L_{G/P}[N - 1].
\]
So we have
\[ T_\ast(G, \beta, V, v, \psi) \cong \mathcal{F}_\psi(R_{j_0*}R_{\rho_0*}L_\beta[n]) \]
\[ \cong \mathcal{F}_\psi(R\pi_\ast R\pi_\ast Rj_{L_{G/P}^\ast}L_\beta[n]) \]
\[ \cong Rpr_\ast \text{ev}_{G/P}^\ast \mathcal{F}_\psi \log_{G/P}(Rj_{L_{G/P}^\ast}L_\beta)[n + N - 1]. \]

The assertion for \( T_\ast(G, \beta, V, v, \psi) \) in Theorem 0.6(i) also follows from Lemma 1.8 below and the proper base change theorem.

Suppose furthermore that there exists a multiplicative sheaf \( L_{\beta_0} \) on \( G/P[G, G] \) such that \( L_\beta \) is isomorphic to its inverse image under the canonical morphism \( G/Q[G, G] \rightarrow G/P[G, G] \). Then we have
\[ G_\ast(\beta, \psi) \cong \mathcal{U}_\ell, \quad G_1(\beta, \psi) \cong \mathcal{U}_\ell(-1). \]

Moreover, by (i) and the proper base change theorem, we have
\[ T_\ast(G, \beta, V, v, \psi) \cong Rpr_\ast \text{ev}_{G/P}^\ast j_\ast L_{G/P}^\ast(\text{ev}_{G/P}^\ast L_{\beta_0} |_{L_{G/P}^\ast})(-1)[n + N - 1] \]
\[ \cong Rpr_\ast j_\ast \text{ev}_{G/P}^\ast(L_{\beta_0} |_{L_{G/P}^\ast})(-1)[n + N - 1] \]
\[ \cong Rpr_\ast j_\ast(L_{\beta_0} |_{V})(-1)[n + N - 1]. \]

Let \( i : G/P \rightarrow L_{G/P}^\ast \) be the zero section. We have
\[ R\tilde{i}_\ast(L_{\beta_0} |_{L_{G/P}^\ast}) \cong L_{\beta_0}(-1)[-2] \]
and hence we have a distinguished triangle
\[ i_\ast L_{\beta_0}(-1)[-2] \rightarrow L_{\beta_0} |_{L_{G/P}^\ast} \rightarrow Rj_{L_{G/P}^\ast}^\ast(L_{\beta_0} |_{L_{G/P}^\ast}) \to . \]

Applying ev_{G/P}^\ast, we get a distinguished triangle
\[ i_\ast L_{\beta_0}(-1)[-2] \rightarrow (L_{\beta_0} |_{V \times x G/P}) \rightarrow \text{ev}_{G/P}^\ast Rj_{L_{G/P}^\ast}^\ast(L_{\beta_0} |_{L_{G/P}^\ast}) \to . \]

Applying Rpr_\ast, we get the distinguished triangle
\[ Rpr_\ast i_\ast(L_{\beta_0} |_H)(-1)[n + N - 3] \rightarrow Rpr_\ast(L_{\beta_0} |_{V \times x G/P})[n + N - 1] \to T_\ast(G, \beta, V, v, \psi) \to . \]

In the rest of this section, we prove Lemma 1.1. which gives formulas for \( \mathcal{F}_\psi \log_{L_{G/P}^\ast}(L_{\beta_0}^\ast L_\beta) \) and \( \mathcal{F}_\psi \log_{L_{G/P}^\ast}(L_{\beta_0}^\ast L_\beta) \)

Lemma 1.1. Let \( m : G \times_k \mathbb{L}_{G/P} \rightarrow \mathbb{L}_{G/P} \) be the morphism \( (g, x) \mapsto gx \). Denote the inverse image of \( L_\beta \) by the morphism \( G \rightarrow G/Q \) by \( L_\beta^\prime \). We have
\[ m^* j_{L_{G/P}^\ast}^\ast L_\beta \cong L_\beta^\prime \boxtimes j_{L_{G/P}^\ast}^\ast L_\beta, \quad m^* Rj_{L_{G/P}^\ast}^\ast L_\beta \cong L_\beta^\prime \boxtimes Rj_{L_{G/P}^\ast}^\ast L_\beta. \]

Proof. We prove the second statement. The proof for the first statement is similar.

Let \( m' : G \times_k G/Q \rightarrow G/Q \) be the morphism \( (g, x) \mapsto gx \). We have
\[ m'^* L_\beta \cong L_\beta^\prime \boxtimes L_\beta. \]

Let \( \phi : G \times_k \mathbb{L}_{G/P} \rightarrow G \times_k \mathbb{L}_{G/P} \) be the isomorphism defined by \( (g, x) \mapsto (g, gx) \) and fix notation by the following commutative diagram:
\[
\begin{array}{cccccc}
G & \xrightarrow{\pi_1} & G \times_k G/Q & \xrightarrow{\phi} & G \times_k G/Q & \xrightarrow{\pi_2} & G/Q \\
\| & id_{G \times_k (j_{L_{G/P}^\ast}^\ast L_\beta)} & \downarrow & id_{G \times_k (j_{L_{G/P}^\ast}^\ast L_\beta)} & \downarrow & j_{L_{G/P}^\ast}^\ast L_\beta & \downarrow \\
G & \xrightarrow{\pi_1} & G \times_k \mathbb{L}_{G/P} & \xrightarrow{\phi} & G \times_k \mathbb{L}_{G/P} & \xrightarrow{\pi_2} & \mathbb{L}_{G/P}.
\end{array}
\]
Lemma 1.2. Let \( m = \pi_2 \phi \). It follows that \( m \) is smooth. By the smooth base change theorem and the projection formula, we have
\[
m^* R(j_{G/\mathcal{O}_S}^* \kappa)_s \mathcal{L}_\beta \cong R(id_G \times (j_{G/\mathcal{O}_S}^* \kappa))_s m^* \mathcal{L}_\beta \\
\cong R(id_G \times (j_{G/\mathcal{O}_S}^* \kappa))_s (\mathcal{L}_\beta \boxtimes \mathcal{L}_\beta) \\
\cong R(id_G \times (j_{G/\mathcal{O}_S}^* \kappa))_s ((id_G \times (j_{G/\mathcal{O}_S}^* \kappa))^* \pi_1^* \mathcal{L}_\beta \otimes \pi_2^* \mathcal{L}_\beta) \\
\cong \pi_1^* \mathcal{L}_\beta \otimes R(id_G \times (j_{G/\mathcal{O}_S}^* \kappa))_s \pi_2^* \mathcal{L}_\beta \\
\cong \pi_1^* \mathcal{L}_\beta \otimes \pi_2^* R(j_{G/\mathcal{O}_S}^* \kappa)_s \mathcal{L}_\beta.
\]

This proves the second assertion. \( \square \)

**Lemma 1.2.** Let \( S \) and \( T \) be \( k \)-schemes of finite type, \( E \to S \) and \( F \to T \) vector bundles over \( S \) and \( T \) of ranks \( r_1 \) and \( r_2 \), respectively, and \( f : S \to T \) an isomorphism. Let \( \pi : F \times_T S \to S \) be the projection. Let \( g : E \to F \times T \) be the composite of the morphism \( g \) and the projection \( \pi : F \times_T S \to S \). Let \( f^\vee : F^\vee \to E^\vee \) be the transpose of \( f \). Then for any object \( K \in \text{ob} D^b_c(E, \mathbb{Q}_l) \), we have
\[
\mathcal{F}_\psi(Rf_! K) \cong f^\vee \mathcal{F}_\psi(K)[r_2 - r_1].
\]

Let's describe the transpose \( f^\vee : F^\vee \to E^\vee \). We have commutative diagrams
\[
\begin{array}{cccccc}
E & \xrightarrow{f} & F & \xrightarrow{f^\vee} & F^\vee & \xrightarrow{\phi^\vee} & E^\vee \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{\phi} & T, & \phi^{-1}(T) & \xrightarrow{\phi^{-1} \circ \phi} & S.
\end{array}
\]

Let \( \mathcal{E} \) and \( \mathcal{E}^\vee \) be the \( \mathcal{O}_S \)-modules of sections of \( E \to S \) and \( E^\vee \to S \), respectively, and let \( \mathcal{F} \) and \( \mathcal{F}^\vee \) be the \( \mathcal{O}_T \)-modules of sections of \( F \to T \) and \( F^\vee \to T \), respectively. The morphism \( f : E \to F \) is completely characterized by the homomorphism of \( \mathcal{O}_T \)-modules \( f^t : \phi_* \mathcal{E} \to \mathcal{F} \). For any open subset \( V \) of \( T \), let \( U = \phi^{-1}(V) \). The homomorphism \( f^t_U : \mathcal{E}(U) \to \mathcal{F}(V) \), maps a section \( s : U \to E \) of the vector bundle \( E \to S \) to the section \( \pi s \phi^{-1} : V \to F \) of the vector bundle \( F \to T \). Similarly, the morphism \( f^\vee : F^\vee \to E^\vee \) is completely characterized by the homomorphism of \( \mathcal{O}_S \)-modules \( f^{\vee t}_U : \phi_* \mathcal{E}^\vee \to \mathcal{E}^\vee(U) \), and the homomorphism \( f^{\vee t}_U : \mathcal{F}(V) \to \mathcal{F}^\vee(U) \) maps a section \( t^* : V \to F^\vee \) of the vector bundle \( F^\vee \to T \) to the section \( f^\vee t^* \phi \) : \( U \to E^\vee \) of the vector bundle \( E^\vee \to S \). For any \( s \in \mathcal{E}(U) \) and \( t^* \in \mathcal{F}^\vee(V) \), we have
\[
\phi^\vee(U)(f^{\vee t}(s), t^*) = (s, f^{\vee t}(t^*)),
\]
where \( (, ) \) denotes the pairings \( \mathcal{E} \times \mathcal{E}^\vee \to \mathcal{O}_S \) and \( \mathcal{F} \times \mathcal{F}^\vee \to \mathcal{O}_T \), and \( \phi^\vee : \mathcal{O}_T \to \phi_* \mathcal{O}_S \) is the morphism on structure sheaves coming from the morphism \( \phi : S \to T \).

**Proof of Lemma 1.2.** Let \( g^\vee : F^\vee \times_T S \to E^\vee \) be the transpose of \( g \), and let \( \pi^\vee : F^\vee \times_T S \to F^\vee \) be the projection. Note that \( \pi^\vee \) is an isomorphism and \( g^\vee = f^\vee \pi^\vee \). By [12, 1.2.3.5 and 1.2.2.4], we have
\[
\mathcal{F}_\psi(Rf_! K) \cong \mathcal{F}_\psi(R\pi_1 R\pi_2 K) \\
\cong R\pi_1^* \mathcal{F}_\psi(Rg_! K) \\
\cong R\pi_1^* g^\vee \mathcal{F}_\psi(K)[r_2 - r_1] \\
\cong f^\vee \mathcal{F}_\psi(K)[r_2 - r_1].
\]

\( \square \)
Lemma 1.3. Let $X$ be an $k$-scheme of finite type, $E \to X$ a vector bundle, and $G$ an algebraic group over $k$ acting on $X$ and acting equivariantly and linearly on the vector bundle $E \to X$. Let $m : G \times_k E \to E$ be the action of $G$ on $E$, and let $m^\vee : G \times_k E^\vee \to E^\vee$ be the contragredient action of $G$ on $E^\vee$. For any $K \in \text{ob} \, D^b_c(E, \mathbb{Q}_k)$, we have
\[ m^\vee \mathfrak{F}_\psi(K) \cong \mathfrak{F}_\psi(m^*K), \]
where the righthand side is the Deligne-Fourier transform for the vector bundle $G \times_k E \to G \times_k X$.

Proof. Let $f : G \times_k E \to G \times_k E$ be the isomorphism $(g, x) \mapsto (g, g^{-1}x)$, and let $\phi : G \times_k X \to G \times_k X$ be the isomorphism defined in the same way. The transpose of $f$ as described in Lemma 1.2 is exactly the morphism
\[ f^\vee : G \times_k E^\vee \to G \times_k E^\vee, \quad (g, x) \mapsto (g, gx). \]
Let $\pi : G \times_k E \to E$ and $\pi^\vee : G \times_k E^\vee \to E^\vee$ be the projections. Since $m^\vee = \pi^\vee f^\vee$, we have
\[ m^\vee \mathfrak{F}_\psi(K) \cong f^\vee \mathfrak{F}_\psi(\pi^\vee K). \]
Since the Deligne-Fourier transform commutes with base change ([12 1.2.9]), we have
\[ f^\vee \mathfrak{F}_\psi(\pi^* K) \cong \mathfrak{F}_\psi(Rf_! \pi^* K). \]
By Lemma 1.2 we have
\[ f^\vee \mathfrak{F}_\psi(\pi^* K) \cong \mathfrak{F}_\psi(Rf_! \pi^* K). \]
We thus have
\[ m^\vee \mathfrak{F}_\psi(K) \cong \mathfrak{F}_\psi(Rf_! \pi^* K). \]
Since $f$ is an isomorphism and $\pi f^{-1} = m$, we have
\[ Rf_! \pi^* K \cong m^* K. \]
Our assertion follows. □

Lemma 1.4. Let $S$ be an $k$-scheme of finite type, let $E \to S$ be a vector bundle, $Y \to S$ a morphism of finite type, and $K \in \text{ob} \, D^b_c(E, \mathbb{Q}_k)$ and $L \in \text{ob} \, D^b_c(Y, \mathbb{Q}_k)$. Then we have
\[ \mathfrak{F}_\psi(K \boxtimes L) \cong \mathfrak{F}_\psi(K) \boxtimes \mathfrak{F}_\psi(L), \]
where on the lefthand side the Deligne-Fourier transform is for the vector bundle $E \times_S Y \to Y$, and on the righthand side it is for the vector bundle $E \to S$.

Proof. This follows from the definition of the Deligne-Fourier transform and the Künneth formula. □

Lemma 1.5. (i) Any multiplicative sheaf $\mathcal{L}$ on $\mathbb{G}_{m,k}$ is tamely ramified at $0$ and $\infty$.

(ii) If a multiplicative sheaf on $\mathbb{G}_{m,k}$ is trivial on $\mathbb{G}_{m,k}$, then it is trivial.

Proof. (i) This is proved in [8 Lemma 3.5]. For completeness, we include another proof, which is communicated to me by Haoyu Hu. By base change, we may assume $k$ is algebraically closed. Let $\bar{\eta}$ be a geometric generic point of $\mathbb{G}_m$, let $\rho : \pi_1(\mathbb{G}_m, \bar{\eta}) \to \mathbb{Q}_l$ be the character defined by $\mathcal{L}$, and let $P$ be the wild inertia subgroup at $0$ or at $\infty$. The group $P$ is a pro-$p$-group and $\rho(P)$ is finite. Let $p^n$ be the number of elements in $\rho(P)$. Then we have $\rho^{p^n} |_P = 1$. Let $m_n : \mathbb{G}_{m,k}^n \to \mathbb{G}_{m,k}$ be the morphism defined by multiplication. We have
\[ m_n^* \mathcal{L} \cong \boxtimes^n \mathcal{L}. \]
Pulling back this isomorphism by the diagonal morphism \( \mathbb{G}_{m, k} \to \mathbb{G}_{m, k}^n \), we get
\[
[n]^* \mathcal{L} \cong \mathcal{L}^\otimes n,
\]
where \([n] : \mathbb{G}_{m, k} \to \mathbb{G}_{m, k}\) is the morphism \( x \mapsto x^n\). In particular, the character corresponding to the sheaf \([p^n]^* \mathcal{L} = p^n \mathcal{L}\). Since \( p^n \mathcal{L}\) is tamely ramified at 0 and \( \infty \). But \([p^n]^* \mathcal{L}\) is a finite surjective radiciel morphism, and it has no effect on étale topology. So \( \mathcal{L} \) is tame at 0 and \( \infty \).

(ii) Let \( \mathcal{L} \) be a multiplicative sheaf on \( \mathbb{G}_{m, k} \) which is trivial on \( \mathbb{G}_{m, k} \). Then \( \mathcal{L} \) is isomorphic to the inverse image of a sheaf on \( \text{Spec} \, k \). But the restriction of \( \mathcal{L} \) to the unit section \( 1 : \text{Spec} \, k \to \mathbb{G}_{m, k} \) is trivial. So we have \( \mathcal{L} \cong \mathcal{O}_\mathbb{G} \).

\[\text{Lemma 1.6.} \quad R_{jL_{G/P,*}}^* \mathcal{L}_\beta \text{ commutes with any base change } Y \to G/P.\]

\[\text{Proof.}\] Note that \( \mathbb{L}_{G/P}^* \to G/P \) is a principal homogeneous space for the multiplicative group-scheme \( \mathbb{G}_{m, G/P} \) over \( G/P \), and \( \mathbb{L}_{G/P} \to G/P \) is the associated line bundle for the canonical action of \( \mathbb{G}_m \) on \( \mathbb{A}^1 \). Let \( \{U_i\} \) be a Zariski open covering of \( G/P \) such that the line bundle \( \mathbb{L}_{G/P} \) is trivial over each \( U_i \), and we compactify \( \mathbb{L}_{U_i}^* \) to a \( \mathbb{P}^1 \)-bundle by adding the 0 section and the \( \infty \) section. The sheaf \( \mathcal{L}_\beta \) is homogeneous on \( \mathbb{L}_{G/P}^* \) with respect to the \( \mathbb{G}_{m, G/P} \)-action. In particular, when restricted to the generic point \( \eta \) of \( G/P \), \( \mathcal{L}_\beta|_{\mathbb{L}_\eta}^* \) is a multiplicative sheaf on \( \mathbb{G}_{m, \eta} \). By Lemma 1.5, \( \mathcal{L}_\beta|_{\mathbb{L}_\eta}^* \) is tamely ramified at 0 and \( \infty \). This implies that \( \mathcal{L}_\beta|_{\mathbb{L}_{U_i}}^* \) is tamely ramified at the 0 section and the \( \infty \) section. Our assertion then follows from [3, XIII 2.1.10].

Let \( G_1(\beta, \psi) \) and \( G_\ast(\beta, \psi) \) be the sheaf on \( \text{Spec} \, k \) defined in Theorem 0.6.

\[\text{Lemma 1.7.} \quad \text{Regard } v^* \text{ as a } k \text{-point in the fiber of } \mathbb{L}_{G/P,eP}^* \cong \mathbb{L}_{\eta}^*, \text{ and let } 0_v \text{ be the zero element in the line } \mathbb{L}_{G/P,eP}^* \cong \mathbb{L}_{\eta}^*. \text{ We have}
\]
\[
\mathcal{F}_v(j_{L_{G/P,*}}^* \mathcal{L}_\beta[1])|_{v^*} \cong G_1(\beta, \psi)[1],
\]
\[
\mathcal{F}_v(R_jj_{L_{G/P,*}}^* \mathcal{L}_\beta[1])|_{v^*} \cong G_\ast(\beta, \psi)[1],
\]
\[
\mathcal{F}_v(R_jj_{L_{G/P,*}}^* \mathcal{L}_\beta[1])|_{0_v} \cong 0.
\]

Moreover \( G_1(\beta, \psi) \) and \( G_\ast(\beta, \psi) \) are of rank 1.

\[\text{Proof.}\] With respect to the morphism \( \text{Spec} \, k \to G/P \) defined by the rational point \( eP \), the base change of the morphism \( j_{L_{G/P,*}}^* : \mathbb{L}_{G/P}^* \to \mathbb{L}_{G/P}^* \) can be identified with the canonical open immersion \( j : \mathbb{G}_{m, k} \to \mathbb{A}_k^1 \). By Lemma 1.5 the base change of \( R_jj_{L_{G/P,*}}^* \mathcal{L}_\beta[1] \) can be identified with \( R_j\mathcal{L}_\beta|_{\mathbb{G}_{m, k}}[1] \). The Deligne-Fourier transform and \( j_{L_{G/P,*}}^* \mathcal{L}_\beta[1] \) also commute with base change ([12, 1.2.2.9]). So we are reduced to show
\[
\mathcal{F}_v(j_!(\mathcal{L}_\beta|_{\mathbb{G}_{m, k}}[1]))|_1 \cong G_\ast(\beta, \psi)[1],
\]
\[
\mathcal{F}_v(R_j(\mathcal{L}_\beta|_{\mathbb{G}_{m, k}})[1])|_1 \cong G_1(\beta, \psi)[1],
\]
\[
\mathcal{F}_v(R_j(\mathcal{L}_\beta|_{\mathbb{G}_{m, k}})[1])|_0 \cong 0,
\]
where the left-hand side are restrictions to the unit section \( 1 : \text{Spec} \, k \to \mathbb{A}_k^1 \) and the zero section \( 0 : \text{Spec} \, k \to \mathbb{A}_k^1 \).

For convenience, denote \( \mathcal{L}_\beta|_{\mathbb{G}_{m, k}} \) by \( \mathcal{L} \). First consider the case where \( \mathcal{L}|_{\mathbb{G}_{m, k}} \) is nontrivial. In this case, we claim that \( (R_j\mathcal{L})_0 = 0 \) so that we have \( j_! \mathcal{L} \cong R_j \mathcal{L} \). Indeed, by Lemma 1.5 \( \mathcal{L}|_{\mathbb{G}_{m, k}} \) is a nontrivial multiplicative sheaf on \( \mathbb{G}_{m, k} \). It is necessarily isomorphic to a nontrivial Kummer sheaf \( \mathcal{K}_c \) for a character \( \chi : \lim_{\phi(n,p)=1} H_n(k) \to \mathbb{Q}_\ell \). The invariant and coinvariant of the inertia
subgroup \(I_0\) at 0 for the representation of \(\pi_1(\mathbb{G}_{m,k})\) corresponding to \(K_{X}\) are trivial. Our claim follows the calculation in [2] Dualité 1.3. So it suffices to prove
\[
\mathcal{F}_{\psi}(j_!(\mathcal{L}[1])_1) \cong G_t(\beta, \psi)[1], \quad \mathcal{F}_{\psi}(j_!(\mathcal{L}[1])_0) \cong 0.
\]
By definition, we have
\[
\mathcal{F}_{\psi}(j_!(\mathcal{L}[1])_1) \cong R\Gamma_c(\mathbb{G}_{m,k}, \mathcal{L} \otimes \mathcal{L}_{\psi})[2].
\]
We have
\[
H^0_c(\mathbb{G}_{m,k}, \mathcal{L} \otimes \mathcal{L}_{\psi}) = 0
\]
since \(\mathbb{G}_{m,k}\) is an affine curve. We have
\[
H^2_c(\mathbb{G}_{m,k}, \mathcal{L} \otimes \mathcal{L}_{\psi}) \cong H^0(\mathbb{G}_{m,k}, \mathcal{L}^{-1} \otimes \mathcal{L}_{\psi^{-1}})(-1) = 0
\]
by Poincaré duality and the fact that the invariant of the representation of \(\pi_1(\mathbb{G}_{m,k})\) corresponding to \(\mathcal{L} \otimes \mathcal{L}_{\psi}\) is trivial. By the Grothendieck-Ogg-Shafarevich formula and the fact that \(\mathcal{L} \otimes \mathcal{L}_{\psi}\) is lisse on \(\mathbb{G}_{m,k}\), tame at 0 and with Swan conductor 1 at \(\infty\), we have
\[
\chi_c(\mathbb{G}_{m,k}, \mathcal{L} \otimes \mathcal{L}_{\psi}) = -1.
\]
So \(H^1_c(\mathbb{G}_{m,k}, \mathcal{L} \otimes \mathcal{L}_{\psi})\) is of rank 1. Hence
\[
\mathcal{F}_{\psi}(j_!(\mathcal{L}[1])_1) \cong G_t(\beta, \psi)[1],
\]
and \(G_*(\beta, \psi)\) is of rank 1. Similarly one can prove \(H^1_c(\mathbb{G}_{m,k}, \mathcal{L}) = 0\) for all \(i\). Hence \(R\Gamma_c(\mathbb{G}_{m,k}, \mathcal{L}) = 0\).

Next, suppose \(\mathcal{L}_{\mathbb{G}_{m,k}}\) is trivial. By Lemma [15] (ii), we have \(\mathcal{L} \cong \mathfrak{t}_\ell\). So we have
\[
\mathcal{F}_{\psi}(j_!(\mathcal{L}[1])_1) \cong R\Gamma_c(\mathbb{G}_{m,k}, \mathcal{L}_{\psi})[2].
\]
We have a distinguished triangle
\[
R\Gamma_c(\mathbb{G}_{m,k}, \mathcal{L}_{\psi}) \rightarrow R\Gamma_c(A^1_k, \mathcal{L}_{\psi}) \rightarrow \mathcal{L}_{\psi,0} \rightarrow .
\]
By [2] Sommes trig. 2.7, we have
\[
R\Gamma_c(A^1_k, \mathcal{L}_{\psi}) = 0, \quad \mathcal{L}_{\psi,0} \cong \mathfrak{t}_\ell.
\]
So we have
\[
R\Gamma_c(\mathbb{G}_{m,k}, \mathcal{L}_{\psi})[2] \cong \mathfrak{t}_\ell[1].
\]
Therefore
\[
\mathcal{F}_{\psi}(j_!(\mathcal{L}[1])_1) \cong \mathfrak{t}_\ell[1] \cong G_t(\beta, \psi)[1].
\]
Denote by \(D\) the Verdier dual functor on \(A^1_k\). We have
\[
\mathcal{F}_{\psi}(Rj_*\mathfrak{t}_\ell[1]) \cong \mathcal{F}_{\psi}(D(j_!(\mathfrak{t}_\ell[1])))(-1) \cong D(\mathcal{F}_{\psi^{-1}}(j_!(\mathfrak{t}_\ell[1])))(-2).
\]
By [12] 2.3.1.3, we have
\[
\mathcal{F}_{\psi^{-1}}(j_!(\mathfrak{t}_\ell)[1])|_{A^1_k - \{0\}} \cong \mathcal{F}[1]
\]
for a lisse sheaf \(\mathcal{F}\) on \(A^1_k - \{0\}\), and by the formula that we have already proved, we have \(\mathcal{F}|_{1} \cong \mathfrak{t}_\ell\). So
\[
\mathcal{F}_{\psi}(Rj_*\mathcal{L}[1])_1 \cong \mathcal{F}_{\psi}(Rj_*\mathfrak{t}_\ell[1])_1 \cong \text{Hom}(\mathcal{F}_1, \mathfrak{t}_\ell)[(-1)] \cong \mathfrak{t}_\ell(-1)[1] = G_*(\beta, \psi)[1].
\]
We have
\[
\mathcal{F}_{\psi}(Rj_*\mathcal{L}[1])_0 \cong \mathcal{F}_{\psi}(Rj_*\mathfrak{t}_\ell)_0 \cong R\Gamma_c(A^1_k, Rj_*\mathfrak{t}_\ell).
Let $J : \mathbb{G}_{m,k} \hookrightarrow \mathbb{P}^1_k$ be the open immersion. We have a distinguished triangle

$$R\Gamma_*(\mathbb{A}^1_k, Rj_*\overline{\mathbb{Q}}_\ell) \to R\Gamma(\mathbb{P}^1_k, Rj_*\overline{\mathbb{Q}}_\ell) \to R\Gamma(\mathbb{G}_{m,k},\overline{\mathbb{Q}}_\ell) \to (Rj_*\overline{\mathbb{Q}}_\ell)^{\wedge}$$

We claim that

$$R\Gamma(\mathbb{G}_{m,k},\overline{\mathbb{Q}}_\ell) \cong (Rj_*\overline{\mathbb{Q}}_\ell)^{\wedge}.$$

Therefore $R\Gamma_*(\mathbb{A}^1_k, Rj_*\overline{\mathbb{Q}}_\ell) = 0$ and hence $\mathcal{F}_\psi(Rj_*\mathcal{L}[1])_0 = 0$. To prove the claim, let $\tilde{K}_\infty$ be the fraction field of the strict henselization of $\mathbb{G}_{m,k}$ at $\infty$. We have

$$(Rj_*\overline{\mathbb{Q}}_\ell)^{\wedge} \cong R\Gamma(\text{Spec } \tilde{K}_\infty, \overline{\mathbb{Q}}_\ell).$$

It suffices to show

$$H^i(\mathbb{G}_{m,k}, \mu_n) \cong H^i(\text{Spec } \tilde{K}_\infty, \mu_n)$$

for all $(n,p) = 1$. We compute $H^i(\mathbb{G}_{m,k}, \mu_n)$ and $H^i(\text{Spec } \tilde{K}_\infty, \mu_n)$ and compare them using the long exact sequences of cohomology groups associated to the the Kummer short exact sequence. □

**Lemma 1.8.** We have

$$\mathcal{F}_\psi_* L_{G/P}(j_{L_{G/P}}^* J_\beta[1]) \cong R(j_{L_{G/P}}^* J_\beta[1]) \otimes G_*(\beta, \psi),$$

$$\mathcal{F}_\psi_* L_{G/P}(R(j_{L_{G/P}}^* J_\beta[1]) \cong j_{L_{G/P}}^* L_{\beta}[1] \otimes G_*(\beta, \psi).$$

**Proof.** The first assertion can be deduced from the second one by taking Verdier dual. Let’s prove the second assertion.

Let $m : G \times_k \mathbb{L}_{G/P} \to \mathbb{L}_{G/P}$ be the action of $G$ on $\mathbb{L}_{G/P}$, let $m^\vee : G \times_k \mathbb{L}_{G/P}^\vee \to \mathbb{L}_{G/P}^\vee$ be the contragredient action, and let $\phi$ be the composite

$$G \to G/Q \cong \mathbb{L}_{G/P}^\vee.$$

By Lemmas 1.4 and 1.3 we have

$$m^\vee_* \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1]) \cong \mathcal{F}_\psi(m^* R(j_{L_{G/P}}^* J_\beta[1]) \cong \mathcal{F}_\psi(L_\beta^\vee \otimes R(j_{L_{G/P}}^* J_\beta[1]) \cong L_\beta^\vee \otimes \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1])) \cong \phi^* L_\beta \otimes \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1])).$$

Regard $\nu^*$ as a point in the fiber of $\mathbb{L}_{G/P, eP}^\vee \cong (\mathbb{L}^\vee)[v]$. Restricting the isomorphism

$$m^\vee_* \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1]) \cong \phi^* L_\beta \otimes \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1]))$$

to $G \times v^*$, we get

$$\phi^* j_{L_{G/P}}^* \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1]) \cong \phi^* L_\beta \otimes \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1]).$$

By Lemma 1.7 we have

$$\mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1])[v^*] \cong G_*(\beta, \psi)[1].$$

We thus have

$$\phi^* j_{L_{G/P}}^* \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1]) \cong \phi^* (L_\beta[1] \otimes G_*(\beta, \psi)).$$

The morphism $\phi$ is smooth. Locally with respect to étale topology, $\phi$ has sections. The above isomorphism then implies that $j_{L_{G/P}}^* \mathcal{F}_\psi(R(j_{L_{G/P}}^* J_\beta[1])$ is a lisse sheaf on $\mathbb{L}_{G/P}^\vee$. The fibers of $\phi$ are geometrically connected. This implies that if $V \to \mathbb{L}_{G/P}^\vee$ is an étale covering space such that $V$
is connected, then \( G \times L_{G/P} \), \( V \) is also connected. So the homomorphism induced by \( \phi \) on fundamental groups \( \pi_1(G) \rightarrow \pi_1(L_{G/P}) \) is surjective. The isomorphism \( \phi^*j_{L_{G/P}}^*\mathcal{F}_\psi(Rj_{L_{G/P}}^*\mathcal{L}_{1}) \cong \mathcal{P}_{1}(\mathcal{L}_{1} \otimes G_{\ast}(\beta, \psi)) \) thus implies that

\[
j_{L_{G/P}}^*\mathcal{F}_\psi(Rj_{L_{G/P}}^*\mathcal{L}_{1}) \cong \mathcal{L}_{1} \otimes G_{\ast}(\beta, \psi).
\]

By Lemma 1.7, we have

\[
\mathcal{F}_\psi(Rj_{L_{G/P}}^*\mathcal{L}_{1})|_{0_{\mathcal{E}}} = 0.
\]

Restricting the isomorphism \(\mathcal{F}_\psi(Rj_{L_{G/P}}^*\mathcal{L}_{1}) \cong \phi^*\mathcal{L}_{1} \otimes \mathcal{F}_\psi(Rj_{L_{G/P}}^*\mathcal{L}_{1}) \) to \( G \times 0_{\mathcal{E}} \), we see the restriction of \( \mathcal{F}_\psi(Rj_{L_{G/P}}^*\mathcal{L}_{1}) \) to the zero section of \( L_{G/P} \) vanishes. It follows that

\[
\mathcal{F}_\psi(Rj_{L_{G/P}}^*\mathcal{L}_{1}) \cong j_{L_{G/P}}^*\mathcal{L}_{1} \otimes G_{\ast}(\beta, \psi).
\]

This proves our assertion.

2. Tautological Systems Associated to a Family of Representations

Let \( G \) be an algebraic group \( G \) over a perfect field \( k \), let \( \rho_\lambda : G \rightarrow GL(V_\lambda) \) \( (\lambda \in \Lambda) \) be a finite family of representations so that the morphism

\[
i : G \rightarrow \prod_{\lambda \in \Lambda} \text{End}(V_\lambda), \quad g \mapsto (\rho_\lambda(g))
\]

is quasi-finite, and let

\[
V = \prod_{\lambda \in \Lambda} \text{End}(V_\lambda).
\]

We have an action of \( G \) on \( V \) given by

\[
(g, (A_\lambda)_{\lambda \in \Lambda}) \mapsto (\rho_\lambda(g)A_\lambda)_{\lambda \in \Lambda}
\]

for any points \( g \) in \( G \) and \( (A_\lambda)_{\lambda \in \Lambda} \) in \( V \). Take \( v \) to be the \( k \)-point \( v = (id_{V_\lambda})_{\lambda \in \Lambda} \) of \( V \). Then the connected component of the stabilizer of \( v \) is trivial. We have a perfect pairing

\[
\langle \ , \ \rangle : V \times V \rightarrow k, \quad ((A_\lambda)_{\lambda \in \Lambda}, (B_\lambda)_{\lambda \in \Lambda}) \mapsto \sum_{\lambda \in \Lambda} \text{Tr}(A_\lambda B_\lambda).
\]

We identify \( V \) with \( V^\vee \) through this pairing. Let \( \mathcal{L}_{\beta} \) be a multiplicative sheaf on the maximal abelian quotient \( G/[G, G] \) of \( G \). By our assumption, \( \iota \) is both quasi-finite and affine. So \( \iota_{\ast}\mathcal{L}_{\beta}[n] \) and \( R_{\ast}\mathcal{L}_{\beta}[n] \) are perverse sheaves. Hence the \( \ell \)-adic tautological systems \( \mathcal{I}(G, \beta, V, v, \psi) = \mathcal{F}_\psi(\iota_{\ast}\mathcal{L}_{\beta}[n]) \) and \( \mathcal{T}_\ast(G, \beta, V, v, \psi) = \mathcal{F}_\psi(R_{\ast}\mathcal{L}_{\beta}[n]) \) on \( V \) are perverse.

**Lemma 2.1.** Notation as above. Let \( F : G \times_k V \rightarrow A^1_k \) be the morphism defined by

\[
F(g, (A_\lambda)_{\lambda \in \Lambda}) = \sum_{\lambda \in \Lambda} \text{Tr}(\rho_\lambda(g)A_\lambda),
\]

and let

\[
\pi_1 : G \times_k V \rightarrow G, \quad \pi_2 : G \times_k V \rightarrow V
\]

be the projections. Then we have

\[
\mathcal{I}(G, \beta, V, v, \psi) \cong R\pi_2_*(\pi_1^\ast\mathcal{L}_{\beta} \otimes F^\ast L_\psi)[n + N],
\]

where \( n = \dim G \) and \( N = \dim V \).
Proof. We have a commutative diagram
\[ \begin{array}{ccc}
G \times_k V & \xrightarrow{\iota \times \text{id}_V} & V \times_k V \\
\pi_1 \downarrow & & \downarrow \text{pr}_1 \\
G & \xrightarrow{\iota} & V.
\end{array} \]

By the proper base change theorem and the projection formula, we have
\[ T_i(G, \beta, V, v, \psi) \cong R\text{pr}_2^* \left( (\pi_2^* \xi_1 \xi_1 \mathcal{L}_\beta \otimes (\xi \xi^* \xi_1 \mathcal{L}_\psi) \right)[n + N] \]
\[ \cong R\text{pr}_2^* \left( (\iota \times \text{id}_V) \pi_1^* \mathcal{L}_\beta \otimes (\xi \xi^* \xi_1 \mathcal{L}_\psi) \right)[n + N] \]
\[ \cong R(\text{pr}_2 \circ (\iota \times \text{id}_V)) \left( \pi_1^* \mathcal{L}_\beta \otimes (\xi \xi^* \xi_1 \mathcal{L}_\psi) \right)[n + N] \]
\[ \cong R\pi_2^* (\pi_1^* \mathcal{L}_\beta \otimes F^* \mathcal{L}_\psi)[n + N]. \]
\[
\square
\]

We call
\[ \text{Hyp}_\psi(\Lambda, \beta) = R\pi_2^* (\pi_1^* \mathcal{L}_\beta \otimes F^* \mathcal{L}_\psi)[n + N] \]
the hypergeometric sheaf on \( V \). Over the complex number field \( \mathbb{C} \) and in the case where \( G \) is a reductive group and \( \Lambda \) is a finite family of irreducible representations of \( G \), this is introduced in [10]. The GKZ hypergeometric sheaf in Example [0.4] is a special case for the group \( G = \mathbb{G}_m^n \) and the family of representations defined by the character
\[ \beta : (G/G,G)(\mathbb{F}_q) \rightarrow \mathbb{Q}_p^*. \]
Then for any \( \mathbb{F}_q \)-point \( A = (A_\lambda) \lambda \in \Lambda \) of \( V \), we have
\[ \text{Tr}(\text{Frob}_A, \text{Hyp}_\psi(\beta, \Lambda)_{\mathcal{A}}) = (-1)^{n+N} \sum_{g \in G(\mathbb{F}_q)} \beta(g) \psi \left( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \left( \sum_{\lambda \in \Lambda} \text{Tr} (\rho_\lambda(g) A_\lambda) \right) \right) \]
by the Grothendieck trace formula.

Example 1. Let \( G \subset \text{GL}_m \) and let \( \Lambda \) be the set consisting of only the standard representation of \( \rho : G \hookrightarrow \text{GL}_m \). Determinant on \( \text{GL}_m \) induces a homomorphism
\[ G/[G,G] \rightarrow \mathbb{G}_m. \]
Suppose \( k = \mathbb{F}_q \) is a finite field, and let \( \mathcal{L}_\beta \) be the Lang sheaf associated to a character
\[ \beta : (G/G,G)(\mathbb{F}_q) \rightarrow \mathbb{C}^*. \]
Then for any \( \mathbb{F}_q \)-point \( A = (A_\lambda) \lambda \in \Lambda \) of \( V \), we have
\[ \text{Tr}(\text{Frob}_A, \text{Hyp}_\psi(\beta, \Lambda)_A) = (-1)^{n+N} \sum_{g \in G(\mathbb{F}_q)} \beta(g) \psi \left( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \left( \sum_{\lambda \in \Lambda} \text{Tr} (\rho_\lambda(g) A_\lambda) \right) \right) \]
by the Grothendieck trace formula.

Example 2 ([10] Example 6.1]). Let \( L \) and \( N \) be two 1-dimensional \( k \)-vector spaces, and let \( V \) be an \( n \)-dimensional \( k \)-vector space. Set
\[ G = \text{GL}(N) \times_k \text{GL}(L) \times_k \text{GL}(V) \cong \mathbb{G}_m \times_k \mathbb{G}_m \times_k \text{GL}_m. \]
Let \( \Lambda = \{ N \otimes_k N, N \otimes_k L, N \otimes_k V, L \otimes_k V \} \). In this case, we have
\[ V = \text{End}(N \otimes_k N) \oplus \text{End}(N \otimes_k L) \oplus \text{End}(N \otimes_k V) \oplus \text{End}(L \otimes_k V) \]
which is of dimension $2(1+n^2)$. The stabilizer in $G$ of the point $v = (id_{N \otimes E}, \ldots, id_{D \otimes V})$ in $V$ is trivial. Let $\iota : G \to V$ be the morphism

$$\iota : \mathbb{G}_m \times_k \mathbb{G}_m \times_k GL_n \to A^1 \times_k A^1 \times_k gl_l \times_k gl_l, \quad (s, t, g) \mapsto (s^2, st, sg, tg).$$

Its image can be identified with the orbit of $v$. Note that $\bar{k}$-points in the image of $\iota$ can be described by

$$\{(u, v, X, Y) \in \bar{k} \times \bar{k} \times gl_l(\bar{k}) \times gl_l(\bar{k}) : uY - vX = 0\}.$$

We have

$$G/[G, G] \cong \mathbb{G}_m \times_k \mathbb{G}_m \times_k GL_n/[GL_n, GL_n] \cong \mathbb{G}_m \times_k \mathbb{G}_m \times_k \mathbb{G}_m.$$

Suppose $k = \mathbb{F}_q$ is a finite field with $q$ elements. Let $\chi_1, \chi_2, \chi_3 : \mathbb{F}_q^* \to \mathbb{Q}_\ell^*$ be multiplicative characters and let $\beta : (G/[G, G])(\mathbb{F}_q) \to \mathbb{Q}_\ell$ be the character

$$(s, t, g) \mapsto \chi_1(s)\chi_2(g)\chi_3(det(g)).$$

For any $\mathbb{F}_q$-points $x = (a, b, C, D)$ of $V$, we have

$$\text{Tr}(\text{Frob}_x, (\eta(G, \beta, V, v, \psi)_x) = \sum_{s \in \mathbb{F}_q^*, t \in \mathbb{F}_q^*, g \in GL_n(\mathbb{F}_q)} \chi_1(s)\chi_2(t)\chi_3(det(g))\psi(s^2a + stb + s\text{Tr}(gC) + t\text{Tr}(gD)).$$

REFERENCES

[1] P. Deligne, La conjecture de Weil II, Publ. Math. IHES, 52 (1981), 313-428.
[2] P. Deligne, Cohomologie étale (SGA 4 1/2), Lecture Notes in Math. 569, Springer-Verlag (1977).
[3] P. Deligne and N. Katz, Groupes de Mondoromie en Géométrie Algébrique (SGA 7), Lecture Notes in Math. 569, Springer-Verlag (1977).
[4] L. Fu, Gelfand-Kapranov-Zelevinsky hypergeometric sheaves, Proceedings of the 6th International Congress of Chinese Mathematicians Vol. I, 281-295, Adv. Lect. Math. (ALM), 36, Int. Press, Somerville, MA, 2017.
[5] L. Fu, $\ell$-adic Gelfand-Kapranov-Zelevinsky sheaves, Adv. in Math. 298 (2016), 51-88.
[6] Q. Guignard, On the ramified class field theory of relative curves, Algebra and Number Theory 13-6 (2019), 1299-1326.
[7] I. M. Gelfand, A. V. Zelevinsky, M. M. Kapranov, Hypergeometric functions and toric varieties, English translation, Functional Anal. Appl. 23 (1989), 94-106; Correction to the paper “Hypergeometric functions and toric varieties”, English translation, Functional Anal. Appl. 27 (1993), 295.
[8] R. Hotta, Equivariant D-modules, arXiv: math/9805021v1 (1998).
[9] A. Huang, B. H. Lian, X. Zhu, Period integrals of CY and general type complete intersections, Invent. Math. 191 (2013) 35-89.
[10] L. Moret-Bailly, Pinceaux de Variétés de Abéliennes, Astérique 129 (1985).
[11] J.-P. Serre, Algebraic Groups and Class Fields, Springer-Verlag 1988.
[12] T. A. Springer, Linear Algebraic Groups, 2nd edition, Birkhäuser 1998.
DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA
Email address: yau@math.harvard.edu

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
Email address: zhangdingxin03@gmail.com

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA
Email address: xzhu@caltech.edu