Nonlocality, no-signalling, and Bell’s theorem investigated by Weyl conformal differential geometry

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Abstract
The principles and methods of conformal quantum geometrodynamics based on Weyl differential geometry are presented. The theory applied to the case of the relativistic single quantum spin-$\frac{1}{2}$ leads to a novel and unconventional derivation of the Dirac equation. The further extension of the theory to the case of two-spins-$\frac{1}{2}$ in the EPR entangled state and to the related violation of Bell inequalities leads, by an exact non-relativistic analysis, to an insightful resolution of all paradoxes implied by quantum nonlocality.

Keywords: conformal Weyl’s geometry, quantum nonlocality, Bell’s theorem

1. Introduction

Since the 1935 publication of the famous paper by Einstein, Podolsky, and Rosen (EPR), the awkward coexistence within the quantum lexicon of the contradictory terms ‘locality’ and ‘nonlocality’ as primary attributes of quantum mechanics (QM) has been a cause of concern and confusion within the debate over the foundations of this central branch of modern science [1–4]. More recent confirmation by innumerable experiments, following the first one by Alain Aspect and co-workers, of the paradoxical violation of Bell inequalities, emphasizes the dramatic content of the dispute [5–7]. By referring to the implications of relativity with the nonlocal EPR correlations, the philosopher Tim Maudlin writes: ‘One way or another, God has played us a nasty trick. The voice of Nature has always been faint, but in this case it speaks of riddles and mumbles as well...’ [8]. Indeed the violation of Bell inequalities realized by all these experimental tests implied the existence of quite ‘mysterious’ nonlocal correlations linking the outcomes of the measurements carried out over two spatially distant particles. Since ‘correlations cry out for explanations’, according to J S Bell, these experimental results started, about 30 years ago, a theoretical endeavour aimed at discovering the inner dynamics underlying such an enigma [6]. Moreover, it has been recognized that, since any transfer of information through the EPR correlations is forbidden by special relativity, a ‘no-signalling theorem’ must hold. Recently, this theorem was verified experimentally [9].

Aimed at a clarification of such a fundamental and intriguing paradox, the present article is intended to tackle the EPR scheme from a very general, insightful perspective. Driven by an accurate reconsideration of all natural symmetries affecting the dynamics of particles, our theoretical analysis is based on the well-known, standard linear quantum theory. Remarkably, this linear theory has recently been found to be rooted in the Weyl conformal geometric invariance properties affecting the very structure of all physical laws [10, 11]. This concept was well-expressed by Dirac in a 1973 seminal paper [12]: ‘There is a strong reason in support of Weyl’s theory. It appears as one of the fundamental principles of Nature that the equations expressing basic laws should be invariant under the widest possible group of transformations. The confidence that one feels in Einstein [general relativity] theory arises because its equations are invariant under the wide group of transformations of curvilinear coordinates in Riemannian space...’ The passage to Weyl’s geometry is a further step in the direction of widening the group of transformations underlying the physical laws. One has to consider transformations of gauge as well as transformations of curvilinear coordinates and one has to take
one’s physical laws to be invariant under all these transformations, which impose stringent conditions on them... These stringent conditions indeed express the conformal-covariance (co-covariance, or gauge-covariance) of all physical laws, including the ones belonging to electromagnetism and to standard quantum dynamics, as we shall see in the present article. According to Weyl conformal differential geometry, the formal expression of all physical laws, can be expressed in different ‘gauges’, which are related by a conformal mapping preserving the angles between vectors. This theoretical approach, today well-known in the domain of modern general relativity and cosmology [13–15], has never been consistently applied in the past to the analysis of a wide class of low-energy quantum phenomena, including atomic physics [11]. Indeed, the correct application of gauge-covariance (or unit-covariance) to quantum phenomena implies very subtle considerations that were overlooked by Einstein, and later by Weyl himself, at the time (1918) when this elegant abelian gauge theory was first proposed [10, 16]. These considerations imply in the first place the correct choice by definition of several units in terms of which all physical quantities are measured: these units must be mutually independent in the sense that a dimensionless unit cannot be constructed with them. It is conventional in relativistic quantum theory to take c, h, and m_e (the electron mass) to be constant by definition. Other gauges, e.g. by replacing m_e with the gravitational constant G, lead in general to different theories which are mutually connected by conformal mapping [14].

Restricting ourselves to the main topic of the present article, we believe that since Weyl conformal gauge symmetry reflects essential properties of nature it must be rooted in the inner structure of any sensible, complete quantum theory. In the article this general theory will be referred to as ‘conformal quantum geometrodynamics’ (CQG). This one reduces, in a particular gauge, to the well-known, standard linear quantum theory living in Hilbert spaces. In section 2, the principles and methods of CQG are presented extensively. There the fundamental equations of linear quantum dynamics for particles, i.e. the Schrödinger and the Dirac equations, are derived by an exact variational calculus, i.e. with no approximations [17]. In section 3, CQG theory deals with one particle with any spin and then is extended to the case of one and two particles with spin-1/2. Section 4 deals with the key topic of the present article, the EPR scheme, i.e. the nonlocal correlations of two equal entangled particles with spin-1/2. The measurement by two remotely distant stations (Alice (A) and Bob (B)) of the spin of two equal particles (A and B) emerging from two Stern-Gerlach analyzers (SGAs) is thoroughly analyzed by a nonrelativistic approach in the CQG framework, without approximations. There it is shown that CQG theory does indeed naturally violate the Bell inequalities without making recourse to any additional nonlocality assumption. In addition, in section 5, the no-signalling process, i.e. the impossibility of mutual exchange of useful information between A and B for the same two-spin-1/2 particle system is also found to be a natural consequence of CQG. A brief discussion on the perspective of our results in the context of modern physics is contained in section 15.

2. Weyl conformal geometrodynamics

We consider a mechanical system described by n generalized coordinates q_i (i = 1,...,n) spanning the configuration space V_n. The system defines a metric tensor g_ij(q) in V_n, e.g. by its kinetic energy. However, even if the metric is prescribed, the geometrical structure of V_n is fully determined only after the parallel transport law for vectors is also given. We assume an affine transport law given by the connection fields Γ^j_k(q) with zero torsion, i.e. Γ^j_k(q) - Γ^j_k(q) = 0. The connection fields Γ^j_k(q) and their derivatives define in V_n a curvature tensor R^j_ki and, together with the metric tensor, a scalar curvature field R(q) = g^j_k(q).

We introduce the multiple-integral variational principle

\[ \delta \left[ \int d^nq \sqrt{g} \rho \left( g^{ij} \partial_i \sigma \partial_j \sigma + R \right) \right] = 0 \]

where \( g = |\det(\text{det}(g))|, R(q) \) is the scalar curvature, and \( \rho(q) \) and \( \sigma(q) \) are scalar fields.

Variation of \( \rho(q) \) and \( \sigma(q) \) yields, respectively [11]

\[ g^{i}i \partial_i \sigma + R = 0 \quad \left( D_k D^k \sigma + R = 0 \right) \]

and

\[ \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \sigma \right) = 0 \quad \left( D_k D^k \sigma = 0 \right) \]

Variation of (1) with respect to the connections Γ^j_k(q) yields the Weyl conformal connection [10]

\[ \Gamma^j_k(q) = -\left( \begin{array}{c} \frac{i}{jk} \end{array} \right) + \delta^j_k \phi_k + \delta^j_k \phi_j + g^i_k \phi_i \]

where \( \left( \begin{array}{c} \frac{i}{jk} \end{array} \right) \) are the Christoffel symbols out of the metric g_{ij}, \( \phi_i = g^{ij} \phi_i \), and \( \phi_i \) is the Weyl vector given by [11]

\[ \phi_i = -\frac{1}{n-2} \frac{\partial \rho}{\rho} \left( D_k \rho = 0 \right) \]

The curvature tensor R^j_ki and the scalar curvature R derived from the connections (4) are named the ‘Weyl curvature tensor’ and the Weyl scalar curvature, respectively. Moreover, equation (5) shows that the Weyl vector \( \phi_i \) is a gradient, so that the Weyl connection (4) is integrable and we may take \( \rho \) as the Weyl potential. Inserting equation (5) into the well-known expression of Weyl scalar curvature [10], we obtain

\[ R = \mathcal{R} + \left( \frac{n-1}{n-2} \right) \left[ \frac{g^{ij} \partial_i \rho \partial_j \rho}{\rho^2} - \frac{2D_k \left( \sqrt{g} g^{ij} \partial_j \rho \right)}{\rho \sqrt{g}} \right] \]

where \( \mathcal{R} \) is the Riemannian curvature of V_n calculated from the Christoffel symbols of the metric g_{ij}. The connections (4)
are invariant under the Weyl conformal gauge transformations [10]

\[ g_{ij} \rightarrow \lambda g_{ij} \]  
\[ \phi_i \rightarrow \phi_i - \frac{\partial \lambda}{\partial x^i} . \]  

(7)

(8)

The fields \( T(q) \) which under Weyl gauge transform as

\[ T \rightarrow 2w(T)T \]  
are said to transform simply and the exponent \( w \) (\( T \)) is the Weyl ‘weight’ of \( T \). Examples are \( w(g_{ij}) = 1 \), \( w(g^{ij}) = -1 \), \( w(\sqrt{g}) = n/2 \), and \( w(R) = -1 \). The Weyl vector \( \phi_i \) does not transform simply, as shown by equation (8). We see that principle (1) is Weyl-gauge invariant provided \( w(\sigma) = 0 \) and \( w(\rho) = -(n-2)/2 \). In Weyl geometry it is convenient to introduce the Weyl conformally-covariant (co-covariant) derivative \( D_i \) so that the metric tensor is constant, i.e. \( D_i g_{jk} = 0 \). For a tensor field \( T \) of weight \( w(T) \), we have \( D_i T = V^i_j T - 2w(T)\phi_j T \), where \( V^i_j \) is the covariant derivative of the connections (4). The Weyl covariant derivative leaves \( w \) unchanged, i.e. \( w(D_i T) = w(T) \). Because \( D_i g_{jk} = 0 \), summation indices can be raised and lowered using the metric, as usually is made in Riemannian geometry where the covariant derivative is \( \nabla_i \). In the parenthesis of equations (2), (3), and (5) are the same expressions in the co-covariant form thus making the Weyl-gauge covariance of the theory explicit. We notice, in particular, that \( \rho \) is constant with respect to the co-covariant derivative. The field equations (2), (3), (5), and (6) are the main equations of the theory.

### 3. The mechanical interpretation

The field theory based on the variational principle (1) has a straightforward mechanical interpretation. In fact, field equation (2) has the form of the Hamilton–Jacobi equation (HJE) of mechanics for the action function \( \sigma(q) \) of a particle subjected to the scalar potential given by the Weyl curvature \( R \). Alternatively, we may derive equation (2) from the single-integral variational problem \( \delta \int L dt = 0 \) with the homogeneous Lagrangian

\[ L(q, \dot{q}) = -\sqrt{-g} g_{ij}(q) \dot{q}^i \dot{q}^j . \]  

(9)

This Lagrangian (and the associated HJE) have the same form of the Lagrangian of a relativistic particle moving in space–time with mass constant replaced by the curvature field \( R(q) \). Any solution \( \sigma(q) \) of the HJE defines a bundle of (time-like) trajectories in \( V^*_n \) given by \( \dot{q}^i = \gamma^i_{\phi j} \partial_\sigma \), corresponding to possible trajectories of the system in the configuration space, when the system is in the dynamical state defined by \( \sigma(q) \). Each trajectory of the bundle obeys the Euler–Lagrange equations derived from \( L \), so that along its motion the system is subjected to a Newtonian force proportional to the gradient of the Weyl curvature \( R \). However, as said above, the dynamics described by \( \sigma(q) \) must be compatible with the affine connections of \( V_n \), and, hence, the curvature potential \( R \) as well as \( \sigma \) must be simultaneous solutions of equations (2) and (3). Once these two equations are solved, the field \( \sigma(q) \) fixes the dynamics and the field \( \rho(q) \) fixes the affine connections from equations (4) and (5), and the curvature from equation (6). In addition, field equation (3) has a simple mechanical interpretation as a continuity equation (\( \partial_\rho j^i = 0 \)) for the current density

\[ j^i = \sqrt{g} \rho g^{ij} \partial_\sigma \]  

(10)

It is worth noting that the current density \( j^i \) has \( w(j^i) = 0 \) and is therefore Weyl-gauge invariant (co-covariant). This is an important point in a consistent conformally invariant approach, because it is expected that only gauge-invariant quantities have definite physical meaning and can be measured experimentally. We will return to the measurement issue in the final part of the paper. Here we conclude by observing that the continuity equation (3) could also describe the motion of a fluid of density \( \rho \) conveyed along the bundle of trajectories defined by \( \sigma \) according to the hydrodynamical picture of quantum mechanics [18]. Moreover, the last term on the right of equation (6) has the same mathematical form as the ‘quantum potential’ introduced ad hoc by Bohm in order to derive the Schrödinger equation [19, 20]. The quantum potential’s gradient acts as a Newtonian force on the particle. According to the present CQG theory, the active potential originates from geometry, as does gravitation, and arises from the space curvature due to the presence of the non-trivial affine connections of Weyl conformal geometry. Furthermore, it is also worth noting that the conformal invariance requires that Riemannian scalar curvature contributes to the potential, a contribution which is absent in Bohm’s approach.

### 4. The scalar wavefunction

We may exploit this formal analogy to simplify the nonlinear problem implied by equations (2) and (3) by introducing the complex field \( \psi(q) \) given by

\[ \psi(q) = \sqrt{\rho} e^{i\sigma} \]  

(11)

with \( S(q) = \xi \partial_\sigma \), and

\[ \xi = \sqrt{n-2 \over 4(n-1)} . \]  

(12)

With the ansatz (11), field equations (2) and (3) can be grouped in the single linear wave equation for the complex field \( \psi(q) \) given by

\[ \Delta_c \psi \equiv \left( V_{n} V^k - \xi^2 \bar{R} \right) \psi = 0 , \]  

(13)

where the \( \Delta_c \) is the conformal Laplace operator, \( V_n V^k \) is the Laplace–Beltrami operator, and \( \bar{R} \) is the Riemannian scalar curvature of \( V_n \) calculated by the metric tensor \( g_{ij} \). A striking circumstance follows from this approach. Namely, although equation (13) is mathematically equivalent to equations (2)
and (3), any direct reference to Weyl geometric structure of $V_o$ formally disappears in a theory based on equation (13). This remarkable feature, which affects all quantum equations obtained by CGQ, i.e. Schrödinger or Dirac, may explain why this or a similar theory based on Weyl results was never previously formulated. In fact, equation (13) can be written directly once the metric tensor is known, without any reference to the underlying affine connections (4) and curvature (6). The form of equation (13) is the same in all conformal gauges provided $w(\psi) = -(a - 2)/4$ (or $\psi \to \psi' = \lambda^{-(a-2)/4}\psi$), as it can be easily checked from the well-known transformation law of Riemannian scalar curvature under the conformal change $g_{ij} \to \tilde{g}_{ij} = \lambda g_{ij}$ of the metric [21]. In other words, all information about the Weyl structure of the configuration space is lost in the ensuing theory if the wave equation (13) is taken as the starting point of the theory: a full knowledge of the dynamical features of the system may be gained only by making recourse to the full set made by equations (2) and (3), and any direct reference to Weyl geometric structure of $V_o$ formally disappears in a theory based on equation (13).

5. Including external electromagnetic fields

External electromagnetic fields are easily introduced in the theory by the rule $\partial_\sigma \to \partial_\sigma - a_i$ applied to equations (2), (3), and (10), and by adding the term $a_i(q)q^i$ to the Lagrangian $L$ in equation (9). In this way, invariance is gained also with respect to the electromagnetic gauge changes $a_i \to a_i + \partial_\sigma \chi$ and $\sigma \to \sigma + \chi$. Finally, Weyl conformal invariance requires $w(a_i) = 0$. When the ansatz (11) is used, the wave equation (13) is changed into

$$g^{ij}\left[\left(\partial_i - \frac{e}{c}A_i\right)\left(\partial_j - \frac{e}{c}A_j\right)\psi + \hbar^2 \xi^2 R \psi = 0\right]$$

(14)

where we may set $a_i = -\frac{e}{mc}A_i$, $\partial_\xi = -i\hbar V_0$ in order to obtain a more familiar appearance of the wave equation as an $n$-dimensional Klein–Gordon equation with the mass term replaced by the Riemannian scalar curvature of $V_o$. With the same notations, the dynamical equations (2) and (3) become

$$g^{ij}\left(\partial_i S - \frac{e}{c}A_i\right)\left(\partial_j S - \frac{e}{c}A_j\right) + \hbar^2 \xi^2 R = 0$$

(15)

$$\frac{1}{\sqrt{g}}\partial_\xi \left[\sqrt{g} g^{ij}\left(\partial_i S - \frac{e}{c}A_i\right)\right] = 0,$$

(16)

where all quantum effects are accounted for by the Weyl curvature term in equation (15), which vanishes in the classical limit: $\hbar \to 0$.

6. The relativistic spinning particle

Spin is one of the cornerstones of quantum mechanics. Consequently, spin being a peculiar feature of the quantum world, any attempt to find a classical system behaving as a spinning quantum particle is generally considered hopeless. Equations (15) and (16) have a classical structure and the wave equation (14) has only the look of a quantum equation. Since the last equation has the typical ‘bosonic’ form, it is not very surprising that equations (15), (16), and (14) may reproduce all details of the behavior of a quantum integer spin. However, it may be indeed surprising that even the half-integer spin may be accounted for by (14). The proof of this statement is the subject of the present section.

We start from the model by Bopp and Haag [22] of a relativistic top described by six Euler angles $\theta^A (A = 1,\ldots, 6)$. We may visualize this top as a rigid fourleg with the top gyration radius with some abuse of language, we may say that the coordinates $x^i$ belong to the center of mass of the top and that the angles $\theta^A$ yield the top orientation in space–time, even if the vector $e_0^a$ of the fourleg is time-like. We assume also the time component $e_0^0$ of $e_0^a$ as positive, so that the matrix $A = \{e_0^a\}$ is an orthochronous proper Lorentz matrix. The motion of the fourleg is described by the world line $x^\mu(\tau)$ of its center of mass and by the motion of the four vectors $e_\nu^a(\tau)$ described by the six functions $A^a(\tau)$. The parameter $\tau$ is arbitrary, but sometimes it is convenient to take as parameter the space–time arc element $ds$ given by $-ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The four-velocity of the center of mass is given by $u^\mu = dx^\mu/ds$ and the angular velocity of the fourleg $\epsilon_{\mu\nu}^a$ is given by the tensor $\omega_{\mu\nu}^a$ defined by $dx^\mu/ds = \omega_{\mu\nu}^a e_\nu^a$. If the parameter $\tau$ is chosen gauge invariant, we have $w(\omega_{\mu\nu}^a) = 0$. From normalization we obtain $u^\mu u_\mu = -1$ and $\omega_{\mu\nu}^a + \omega_{\nu\mu}^a = 0$, i.e. $\omega_{\mu\nu}^a = g^{\alpha\beta}\omega_{\mu\nu}^\alpha$. The parameter $\tau$ is chosen gauge invariant, we have $w(\omega_{\mu\nu}^a) = 0$. From normalization we obtain $u^\mu u_\mu = -1$ and $\omega_{\mu\nu}^a + \omega_{\nu\mu}^a = 0$, i.e. $\omega_{\mu\nu}^a = g^{\alpha\beta}\omega_{\mu\nu}^\alpha$. The parameter $\tau$ is chosen.
w (a^2) = 1, e is the top charge, A_μ is the electromagnetic four potential, F_μν = ∂_μ A_ν − ∂_ν A_μ is the electromagnetic tensor and, finally, κ is a numerical coupling constant [17, 23]. When written in full as a function of the 10 generalized coordinates q^I = {q^μ, θ^A} and their derivatives, the Lagrangian (17) reduces to the canonical form (9) with the addition of the electromagnetic term a_i(q)q^I and vector a_i(q) = \{a_i(x), a_i(μ, θ)\} = (\partial_μ A_ρ, \frac{ω_μ}{ω_ρ} F_μρ a_ρ(x)^a). Therefore, the dynamical equation (15) and ((16)), the ansatz (11), and the wave equation (14) apply. Unlike Minkowski space–time, which is flat, the configuration space V_10 is curved and has a constant Riemannian curvature R = 6/a^2. We see, therefore, that a constant mass appears in the wave equation (14) of the spinning particle. However, equation (14) still has its bosonic character. To gain a connection with the spinorial description adopted in traditional quantum mechanics, we seek solutions ψ(q) of equation (14) in the mode expansion form

\[ ψ_{\mu}(q) = D^{(\mu,ν)}(Λ(\theta))_ν ψ^\nu(x) + D^{(ν,μ)}(Λ(\theta))_μ ψ^\nu(x) \quad (μ ≤ ν) \]  

where D^{(μ,ν)}(Λ(θ))_ν is the first row of the (2ν + 1) × (2ν + 1) matrix representing the Lorentz transformation Λ(θ) = [e^μ_ρ(θ)] in the irreducible representation labeled by the two numbers μ, ν given by 2μ, 2ν = 0, 1, 2, ..., and the ψ^{\nu}(x) and ψ^\nu(x) are expansion coefficients depending on the space–time coordinates x^μ only. The matrices D^{(μ,ν)}(Λ(θ)) and D^{(ν,μ)}(Λ(θ)) depend on the Euler angles θ^A only, and provide conjugate representations of the Lorentz transformations4. As suggested by the notation, the invariance of ψ_{\mu}(q) under Lorentz transformations implies that ψ^\nu(x) and ψ^{\nu}(x) change as undotted and dotted contravariant spinors, respectively5. Insertion of the expansion (18) into the wave equation (14) yields the following equation for the coefficients ψ^{\nu}(x) and ψ^\nu(x)

\[ \begin{bmatrix} g^{\alpha\beta} & \mathbf{h} \frac{e}{c} \mathbf{A}_\alpha \\ \mathbf{h} \frac{e}{c} \mathbf{A}_\beta & \mathbf{h}^2 + \mathbf{R} \end{bmatrix} \psi^{\nu} + \Delta_j \psi^{\nu} = 0 \]  

where \[ \mathbf{R} = 6/a^2, \psi^{\nu}(x) \] denotes either ψ^{\nu}(x) or ψ^{\nu}(x) and \[ \Delta_j \] is a (2ν + 1) × (2ν + 1) matrix depending on the space–time coordinates x^μ only, given by

\[ \Delta_j = \left[ \frac{\hbar}{a} \mathbf{J} - \frac{ke}{2c} \mathbf{H} \right]^2 - \left[ \frac{\hbar}{a} \mathbf{K} - \frac{ke}{2c} \mathbf{E} \right]^2. \]  

Here \[ \mathbf{J} \] and \[ \mathbf{K} \] are the generators of the Lorentz group in the undotted (or dotted) conjugate representation, corresponding to ψ^{\nu}(x) (or to ψ^\nu(x)). We notice that the motion of the rotating fourleg described by the HJE (15) is in the group SO (3,1) of proper Lorentz transformations, while the evolution of the spinors ψ^{\nu}(x) and ψ^\nu(x) is in the group of complex D-matrices. This last motion, however, has only an auxiliary role in the present approach, where the physics is ascribed to the fourleg dynamics.

Before concluding this section, we observe that the choice of the Minkowski space–time metric g_{μν} = diag(−1,1,1,1) can be made only in one gauge, which we can call the ‘Minkowski gauge’. Only in this gauge the comparison of CQG and standard quantum mechanics be made and only in this gauge can the quantum effects and gravitational effects be ascribed to independent geometric concepts: vector parallel transport and vector length, respectively. In other gauges, a clear separation is impossible and a different picture may emerge. This happens, for example, in the gauge used originally by Weyl in approaching electromagnetism, where the Weyl curvature R is constant (we may call this the ‘Weyl gauge’). It is worth noting however that a gauge also exists where both quantum and gravitational phenomena share the same origin in the metric only. In this particular gauge, which we can call the ‘Riemannian gauge’, the geometry is pure Riemannian and is entirely governed by the metric tensor. In the Riemannian gauge, we have ρ = const., the Weyl vector vanishes, and the Weyl curvature reduces to the Riemannian curvature built from the the metric \[ \tilde{g}_{ij} \] given by \[ \tilde{g}_{ij} = [Ψ(\theta)]^{−1} g_{ij}. \] Notice, however, that the space–time upper diagonal block of the metric \[ g_{ij} \] depends now on the space–time coordinates and on the Euler angles as well.

7. The relativistic spin $^{\frac{1}{2}}$

Equation (19) is written for any spin. Spin-$^{\frac{1}{2}}$ is obtained by setting \[ a = 0 \] and \[ ν = \frac{1}{2} \] in equation (18) so that \[ D^{(0,\frac{1}{2})}(Λ(θ)) \] and \[ D^{(\frac{1}{2},0)}(Λ(θ)) \] in SL(2, C), and ψ^{\nu}(x) and ψ^{\nu}(x) are two-component undotted and dotted Lorentz spinors, respectively. Then, introducing the Dirac four-component spinors \[ \Psi_D = \begin{bmatrix} ψ^{\nu} \\ ψ^\nu \end{bmatrix} \] and \[ \Phi_D = \begin{bmatrix} D(\theta)^{\nu} \\ D(\theta)^\nu \end{bmatrix} \] where \[ D(\theta)^{\nu} \] and \[ D(\theta)^\nu \] are the first column of the matrices \[ D^{(0,\frac{1}{2})}(Λ(θ)) \] and \[ D^{(\frac{1}{2},0)}(Λ(θ)) \], respectively, equation (18) can be written as the Dirac product \[ ψ(q) = \mathcal{B}_D(q) \Psi_D(q) = Φ_D(θ)^{\nu} \Psi_D(x) \], where \[ γ^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \] is the Dirac matrix in the spinor representation.

4 The two matrices are related by \[ D^{(0,\nu)}(Λ) = D^{(\nu,0)}(Λ)^{-1}. \]

5 The spinors \[ ψ^{\nu}(x) \] and \[ ψ^\nu(x) \] have a second Lorentz invariant lower index \[ ρ \] and \[ σ \], respectively, related to the spin component \[ χ \] along the top moving axis \[ ζ \]. With no loss of generality we may orient the axis so as to have \[ χ \] fixed and omit \[ σ \] and \[ σ \].

6 The existence of the Riemannian gauge is due to the fact that in our case the Weyl connections are integrable.
Moreover, setting $\kappa = 2$ for the electron, equation (19) yields:

\[
\begin{aligned}
& g^{\mu\nu} \left( \tilde{\mathbf{p}}_\mu - \frac{e}{c} A_\mu \right) \left( \tilde{\mathbf{p}}_\nu - \frac{e}{c} A_\nu \right) \\
& - \frac{e \hbar}{c} \left( \Sigma \cdot \mathbf{H} - i a \mathbf{E} \right) \left[ \frac{3h^2}{2a^2} (1 + 4x^2) \right] \mathcal{V}_D^+ \\
& + \left[ \frac{e^2 \hbar}{c^3} \left( \mathbf{H}^2 - E^2 \right) \right] \mathcal{V}_D = 0, \quad (21)
\end{aligned}
\]

where $\Sigma = \{ \sigma \ 0 \ 0 \ \alpha \}, \alpha = \{ \sigma \ 0 \ 0 \ -\sigma \}$ and $\sigma = \{ \sigma_\alpha, \sigma_\beta, \sigma_\gamma \}$ are the usual Pauli matrices. By setting

\[
a = \left( \frac{\hbar}{mc} \right) \sqrt{3(1 + 4x^2)} / 2, \quad (22)
\]

where $m$ is the electron mass, and by neglecting the term $(ea/c)^2 (H^2 - E^2)$, equation (21) reduces to the second-order (squared) Dirac equation in the spinor representation (see, for example, [24], equation (32, 7a)). A more compact form of equation (21) is [24]

\[
\begin{aligned}
& \gamma^\mu \gamma^\nu \left( \tilde{\mathbf{p}}_\mu - \frac{e}{c} A_\mu \right) \left( \tilde{\mathbf{p}}_\nu - \frac{e}{c} A_\nu \right) - m^2 c^2 \mathcal{V}_D = 0, \quad (23)
\end{aligned}
\]

where $\gamma^\mu$ are Dirac's matrices in the spinor representation. As is well-known, equation (23) can be written as $D_\mu \mathcal{D}_\nu \mathcal{V}_D = \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{V}_D = 0$, where $D_\mu = \gamma^\mu (\tilde{\mathbf{p}}_\mu - (e/c) A_\mu) \pm m$ are first-order Dirac operators with positive and negative mass $m$, respectively. Any solution $\mathcal{V}_D$ of the second-order equation (23) can be written as a linear superposition of a solution $\Psi$, of the first order Dirac equation $\mathcal{D}_\mu \mathcal{D}_\nu \Psi = 0$ with positive mass $m$ and a solution of the first-order equation $\mathcal{D}_\mu \mathcal{D}_\nu \Psi = 0$ with negative mass. To have full correspondence with the first-order Dirac equation, negative mass solutions of equation (23) must be disregarded as unphysical because they correspond to particles affected by an improper boost (negative determinant) from rest-frame. A systematic way to drop out the unphysical negative mass solutions is to start from arbitrary four-component solution $\mathcal{V}_D$ of the second-order equation (23) and define the field $\Phi_D = \mathcal{D}_\mu \mathcal{D}_\nu \Phi_D$. Then $\Phi_D$, besides being a solution of equation (23), is also a solution of the first-order Dirac equation (see [24], section 32). The occurrence of second-order Dirac equation (23) is expected in the present approach because of the bosonic character of equation (14). We introduced here four-component Dirac spinors because we required invariance under parity transformation. However, it is worth noting that wave equation (14) also has chiral solutions. In fact each one of the two terms on the right of equation (18) obeys equation (14). These solutions correspond to two-component Lorentz spinors with opposite chirality and may have a role in no-parity-preserving interactions. Moreover, as shown by Brown [25], the two-component solutions, besides reproducing the same physical results of the Dirac equation when parity is restored, are also computationally easier to work with. Finally, we notice the presence of the last term on the right of equation (21), which is absent in the standard second-order Dirac equation (23). This term quadratic in the applied fields is needed to preserve the Weyl conformal invariance of the underlying theory and cannot be suppressed. However, the contribution of this term in the equation is negligibly small. In fact, equation (22) shows that $a$ is of the order of the electron Compton wavelength $\lambda_c$. We may then estimate the field $E$ required to render the quadratic term in equation (21) comparable to the linear one. We find: $E \approx 10^{10} \text{ V m}^{-1}$. To have an idea of how large this field is, an electron at rest is accelerated by such a field up to $10^9 \text{ GeV}$ in a linear accelerator 1 m long. Similarly, the term quadratic in the magnetic field becomes comparable to the linear one for the extremely large field: $H \approx 10^9 \text{ T}$.

8. The nonrelativistic limit

As we have seen, any positive mass solution of the second-order Dirac equation (21) provides the coefficients $\psi^u(x)$ and $\psi^d(x)$ in the mode expansion (18) of the wavefunction $\psi(q)$. Taking modulus and phase of $\psi(q)$, we can find the corresponding solution of our main equations (15) and (16) which fix the dynamics of the system and the compatible Weyl geometry of the configuration space. The HJE (15), in particular, defines a bundle of paths $\{x^0(\tau), e^\mu(\tau)\}$ in the configuration space. The curves $x^0(\tau)$ correspond to the world lines described by the center of mass of the particle with four-velocity $u^\mu = \{u^0, u^1, u^2, u^3\}$. A more compact form of the fourleg $e^\mu(\tau)$ defines a rotation of the three space-like unit vectors $\{e^\mu_0, e^\mu_1, e^\mu_2\}$ along the orthogonal axes $\xi, \eta, \zeta$ co-moving with the particle, while the time-like vector $e^\mu_3(\tau)$ describes the world line $y^u(\tau)$ of the particle center of energy with four-velocity given by $v^\mu = \frac{dy^u}{d\tau}(\tau) = e^\mu_3(\tau)$. In general, $u^\mu$ and $v^\mu$ are different, a phenomenon known as zitterbewegung. The dynamics of such a classical rotating object described by six Euler angles can be found, e.g., in Sudarshan and Mukunda [26], chapter 20. However, detailed study of this motion and of zitterbewegung is beyond the scope of the present work and will be left for future work. Here we limit study to the nonrelativistic limit of the theory when velocities are much lower than the speed of light. To this purpose, it is convenient to factorize the Lorentz transformation $\Lambda(\vec{\theta}) = \{e^\mu_3(\vec{\theta})\}$ associated with the particle fourleg as $\Lambda(\vec{\theta}) = B(e_0) R(\alpha, \beta, \gamma)$ where $R(\alpha, \beta, \gamma)$ is a rotation matrix in $\mathbb{SO}(3)$ depending on the three Euler angles $\{\alpha, \beta, \gamma\}$ and $B(e_0)$ is the boost associated with the time-like vector $e^\mu_3(\vec{\theta})$ of the particle fourleg. The rotation $R(\alpha, \beta, \gamma)$ belongs to the little Poincaré group around $e^\mu_3$, and in a Lorentz transformation $\Lambda$ the angles $\{\alpha, \beta, \gamma\}$ transform according to the Wigner rotation $B^{-1}(e_0)\Lambda_0 B(e_0)$, where $\vec{e}_3 = \vec{e}_0 e_3^0$. When the factorization $\Lambda(\vec{\theta}) = B(e_0) R(\alpha, \beta, \gamma)$ is inserted into equation (18) and spin-$\frac{1}{2}$ is considered, we obtain

\[
\begin{aligned}
\psi(q) &= \left[ D(R^{-1}(\alpha, \beta, \gamma)) \psi(R^{-1}(e_0)) \right]_{\alpha, \beta, \gamma} \\
& + \left[ D(R^{-1}(\alpha, \beta, \gamma)) D(B(e_0)) \right] \psi^d, \quad (24)
\end{aligned}
\]

where $D(R(\alpha, \beta, \gamma)) \in \mathbb{SU}(2)$ and the boost $D(B(e_0))$ is given.
by $D^2(B(e^\theta)) = e^{i\theta}\sigma_\theta$ with $\sigma_\theta = [1, \sigma]$ the four-vector of Pauli’s matrices. The nonrelativistic limit is obtained from equation (24) by setting $\psi^N(x) \approx \psi^R(x) = w^0(r, t)$, where $w^0(r, t)$ is a rotation two-component spinor, and setting $\psi_0 = [e_0^0] e^\theta \approx (1, 0, 0, 0)$ because the center of mass velocity $v_0 = \frac{e^\theta e}{\sqrt{\epsilon}} < c$ and the center of energy velocity $v_1 = \frac{e^\theta e}{\sqrt{\epsilon}} < c$.

Then, the nonrelativistic limit of equation (24) is:

$$\psi(q) = D(\alpha, \beta, \gamma)w_1(r, t) = D_1(\alpha, \beta, \gamma)w_1(r, t) + D_1(\gamma, \beta, \gamma)w_1(r, t) = e^{i\frac{\gamma}{2}} \left[ e^{\frac{\beta}{2}} \cos \frac{\beta}{2} w_1(r, t) + e^{-\frac{\beta}{2}} \sin \frac{\beta}{2} w_1(r, t) \right],$$

(25)

where, for brevity, we posed $D(\alpha, \beta, \gamma) = D(R^{-1}(\alpha, \beta, \gamma))$. At the same time, equation (14) reduces to the nonrelativistic Schrödinger equation for the two-component spinor $w^0(r, t) = \{w_1(r, t), w_2(r, t)\}$ with components corresponding to spin up or down along the fixed $z$-axis, respectively. The configuration space is then reduced to the space $V_\theta$, spanned by the position coordinates $r$ and Euler angles $\{\xi^\theta\} = \{\alpha, \beta, \gamma\}$ ($\alpha = 1, 2, 3$). To better see the role played by the wavefunction in the present approach, we consider the simple case of the spin-up state. From equation (25) with $w_1 = 0$ we calculate the mechanical action $S$ and the Weyl curvature $R$ when the spin is up:

$$S(r, t, \xi) = \frac{\hbar}{2} (\gamma + \alpha) + \arg(w_1(r, t)),$$

(26)

$$R(r, t, \xi) = -\frac{5}{2a^2(1 + \cos \beta)} + R_1(r, t) + \text{const.}$$

(27)

where $R_1(r, t)$ is the contribution of $w_1(r, t)$ to Weyl curvature. From equation (26) we see that the $\beta$ coordinate is cyclic, hence $\beta$ is a constant of motion. From equation (27) we see that the particle is not free, but is subjected to a self-force proportional to the gradient of the Weyl curvature. This self-force has a geometric origin and cannot be eliminated since it is needed to assure the Weyl’s gauge-invariance. However, its existence is hidden in the standard quantum mechanics based on the space–time spinor $w_1(r, t)$, which obeys the Schrödinger equation for the free particle. Similar considerations can be done for the spin-down state. The nonrelativistic limit is much simpler to handle, so we will use equation (25) to investigate the intriguing problem raised by Einstein, Podolsky, and Rosen in 1935, i.e. the famous, striking phenomenon of ‘quantum nonlocality’ [1].

9. The two identical spin $\frac{1}{2}$ particles

Following the EPR approach [1], we consider here two identical spin-$\frac{1}{2}$ nonrelativistic particles in the absence of external fields in the nonrelativistic limit. The calculation to obtain equations (26) and (27) from the wavefunction (25) can be repeated when two identical spin-$\frac{1}{2}$ particles are considered [27]. The configuration space is now the product space spanned by the 12 coordinates given by the 6 space coordinates and 6 angular coordinates of the two particles. To clarify the source of EPR quantum correlations, we consider here two cases: (a) the two particles have opposite spin along the $z$-axis; (b) the two particles are in the EPR state. In the quantum notation, case (a) corresponds to the spin product state $|\uparrow\downarrow\rangle\downarrow\downarrow$ and case (b) to the entangled state $(1/\sqrt{2})(|\uparrow\downarrow\rangle\downarrow\downarrow - 1 \downarrow\downarrow\rangle\downarrow\downarrow)$. 

10. The product state of two opposite spins

The wavefunction of the state $|\uparrow\downarrow\rangle\downarrow\downarrow$ is easily written by taking the product of the two terms on the right of equation (25) and $S$ and $R$ are then calculated from modulus and phase of this wavefunction (for details see [11]). The result is

$$\psi_{11}(q) = D_1(\alpha_A, \beta_A, \gamma_A)D_1(\alpha_B, \beta_B, \gamma_B)w_1(r_A, t)w_1(r_B, t)$$

(28)

where $S^{A\{B\}}(r_A, t, \xi_A)$ and $R^{A\{B\}}(r_A, t, \xi_A)$ are given respectively by equations (26) and (27) calculated for particle $A$ and $B$ separately. From equation (28) we see that in this case the particles have independent motions. In particular, the Weyl curvature reduces to the sum of the two Weyl curvatures so that each particle is affected only by its own geometric self-force.

11. The entangled two-spin EPR state

The same procedure can be applied to the EPR singlet wavefunction of the two spins given by

$$\psi_{AA}(q) = \frac{1}{\sqrt{2}} (\psi_{11}(q) - \psi_{1\bar{1}}(q))$$

(31)

where $\psi_{11}(q)$ is given by equation (28) and $\psi_{1\bar{1}}(q)$ is obtained from this by exchanging the up and down arrows. The result is [11]

$$S = \hbar \left[ \frac{\gamma_A + \gamma_B}{2} + \arctan \left( \csc \frac{\beta_A - \beta_B}{2} \sin \frac{\beta_A + \beta_B}{2} \times \tan \frac{\alpha_B - \alpha_A}{2} \right) + \arg(w_i(A)(r_A, t)) + \arg(w_i(B)(r_B, t)) \right]$$

(32)

and

$$R = \frac{5a^2}{2}(1 - \cos \beta_A \cos \beta_B - \cos \Delta \sin \beta_A \sin \beta_B) + R^{A\{B\}}(r_A, t) + R^{B\{A\}}(r_B, t).$$

(33)

In this case, although the particle motions over the spatial...
external variables \( \{ x_i \} \) are independent, the particles are still coupled by the Weyl curvature through the angular internal variables, \( \{ \psi^a, \psi^b \} \) \((a, b = 1, 2, 3) \) to the self-force; each one of them exerts a force on the other. We conjecture, from our present limited nonrelativistic standpoint that the space–time superluminality of the nonlocal correlations comes from the geometrical independence, i.e. disconnectedness, of the two \( \{ x_i \} \) and \( \{ \psi^a, \psi^b \} \) manifolds. The superluminality issue indeed requires a fully relativistic future analysis.

In the next two sections, we will consider in detail the behavior of the two particles prepared in the EPR state (31), analyzed by a couple of equal Stern–Gerlach apparatus (SGAs). We will show that this geometrical interaction among Euler’s angles reproduces exactly all results of standard quantum mechanics leading, in particular, to the violation of Bell’s inequalities.

### 12. The meaning of the quantum measurement

Any experimental apparatus designed to measure some physical property of a quantum particle is made of two parts: (1) a filtering device which addresses the particle to the appropriate detector channel according the possible values of the quantity to be measured (a spin component, in our case), (2) one (or more) detectors able to register the arrival of the particle over each channel. To fix the ideas, we consider here the particular case of the measure of a spin-\( \frac{1}{2} \) particle by SGA. The spin component along the SGA axis can have two values, so we need two detectors \( D_\uparrow \) and \( D_\downarrow \) coupled to the ‘up’ and ‘down’ output channels of the orientable SGA. Each detector measures the flux \( \Phi \) of particles entering its acceptance area \( A \). Let us assume single particle detection. Then this flux is given by

\[
\Phi = \int_A \mathbf{j} \cdot \mathbf{n} dA = \int_A \rho \sqrt{g} \mathbf{g}^{\alpha \beta} \partial_\alpha S_n d\Sigma \tag{34}
\]

extended to the hypersurface \( \Sigma \) in the particle configuration space \( V_6 \) with normal unit vector \( n_i = \{ n, 0, 0, 0 \} \) where \( n \) is the usual 3D-normal to the detector area \( A \). Let us assume that the scalar wavefunction of the particle at the detector location has its space–time and angular parts factorized, i.e. \( \psi = \psi_1(x, y, z, t)\psi_2(\alpha, \beta, \gamma) \). Then

\[
\rho = \rho_1(x, y, z, t)\rho_2(\alpha, \beta, \gamma), \quad S_1(x, y, z, t) + S_2(\alpha, \beta, \gamma),
\]

\[
\Phi = \int_A \mathbf{j} \cdot \mathbf{n} dA \int_A \rho_2(\alpha, \beta, \gamma) d\mu(\alpha, \beta, \gamma),
\tag{35}
\]

where \( \mathbf{j} = \rho_1(x, y, z, t) V_S \) and \( d\mu(\alpha, \beta, \gamma) = \sin \beta d\alpha d\beta d\gamma \). The particle flux \( \Phi \) is the only quantity directly accessible to the detector and depends only on the space–time part \( \psi_1(x, y, z, t) \) of the wavefunction. As shown in equation (35), the Euler angles are integrated away for the simple reason that the detector is located in the physical space and it is insensitive to the particle orientation. It is worth noting that the current density \( \mathbf{j} \), hence the flux \( \Phi \) is Weyl-gauge invariant as it must be for any quantity having a measurable value.

Let us consider now the role played by the filtering apparatus. Unlike the detector, whose role is just to count particles, the filtering stage of the experimental setup must be tailored on the quantity to be measured. In the case of the SGA, the filtering device is the spatial orientation of the inhomogeneous magnetic field crossed by the particle beam. In an ideal filtering apparatus no particle is lost, so its action on the particle’s wavefunction is unitary. The role of the filter is to correlate the space–time path of the particle with the quantity to be measured (the spin component, in our case), so as to extract from the incident beam all particles with a given value of the quantity (spin up, for example) by addressing them to the appropriate detector. The filter acts on the particle motion in space–time only. But, as said before, there is a feedback between the particle motion and the geometric curvature of the configuration space, so that the insertion of the filter changes not only the particle path in space–time but also the overall geometry of the particle configuration space, because it modifies its Weyl curvature \( R \). A similar mechanism is at the core of general relativity: the change in the motion, and/or the addition of a massive body, changes the geometry of the whole surrounding space. In our present approach, both particle motion and space geometry are encoded in the scalar wavefunction, which indeed changes under the action of the unitary, i.e. lossless, transformation introduced by the SGA filter. Solving the full dynamical and geometric problem inside the SGA is a difficult problem, but the asymptotic behavior of the scalar wavefunction far from the SGA is easily found. In this ‘far-field scattering approximation’, a uniformly polarized particle beam is transformed by a SGA rotated at angle \( \theta \) with respect to the \( z \)-axis as follows:

\[
\left[ aD_\uparrow(\alpha, \beta, \gamma) + bD_\downarrow(\alpha, \beta, \gamma) \right] \psi(r, t) \rightarrow
\]

\[
\left( a \cos \frac{\theta}{2} + b \sin \frac{\theta}{2} \right) D_\uparrow(\alpha, \beta, \gamma) \cos \frac{\theta}{2} \psi(r_\uparrow, t) + \left( a \sin \frac{\theta}{2} - b \cos \frac{\theta}{2} \right) D_\downarrow(\alpha, \beta, \gamma) \sin \frac{\theta}{2} \psi(r_\downarrow, t)
\tag{36}
\]

where \( a, b \) are arbitrary complex constants with \( |a|^2 + |b|^2 = 1 \), and labels ‘\( \uparrow \)’ and ‘\( \downarrow \)’ refer to the positions of the detectors located to the up and down exit channels of the \( \theta \)-oriented SGA. The experimental apparatus is arranged so that the wave packets \( \psi(r_\uparrow, t) \) and \( \psi(r_\downarrow, t) \) have negligible superposition and each detector sees a wavefunction with space and angular parts factorized. Thus, for example, the particle flux detected in the up channel of the SGA is given, according to equation (35), by \( \Phi P_\uparrow(\theta) \), where \( \Phi_\uparrow \) is the particle flux on the detector and \( P_\uparrow(\theta) = \left| a \cos \frac{\theta}{2} + b \sin \frac{\theta}{2} \right|^2 \) is usually interpreted as the probability that the particle in the input wavepacket is found with its spin along the up direction of the SGA. As said above, what the filter does is to correlate
the particle space–time trajectory with the quantity to be measured. In the quantum mechanical language, we may say that the filter introduces a controlled entanglement among the quantity to be measured and the particle space–time path (in the SGA case, the space–time degrees of freedom become entangled with the orientational ones). However, the filter is configured so that the wavepackets arriving on each detector (\(D_\alpha\) and \(D_\beta\) in our case) are not superimposed, and the (approximate) wavefunction seen by each detector has the product form considered above in equation (35). The last requirement ensures that the detected particle flux \(\Phi\) provides a correct measure (in the quantum sense) of the measured quantity\(^8\).

13. The EPR state and Bell inequalities

Two equal spin-\(\frac{1}{2}\) particles \(A\) and \(B\), e.g. two neutrons, propagate in opposite directions along the spatial \(y\)-axis \((y)\) of the laboratory with a velocity \(v \ll c\) towards two spatially separate measurement devices, dubbed Alice and Bob, which measure the spin of \(A\) and \(B\), respectively. Each apparatus, measuring the particle \(A\) or \(B\), consists of a standard SGA device followed by a couple of particle detectors that, being rigidly connected to the SGA, can be oriented with it by a rotation in the \(\overrightarrow{x-z}\) plane at the corresponding angles \(\theta_A\) (or \(\theta_B\)) taken with respect to \(\overrightarrow{z}\). Accordingly, \(\theta_A\) and \(\theta_B\) denote the orientation axes of SGAA and SGAB \([7]\).

We now turn our attention to the joint spin measurements of the EPR entangled particles \(A\) and \(B\) described by equation (31). After leaving the source, particles \(A\) and \(B\) travel towards two SGAs, SGAA and SGAB, respectively, located at Alice’s and Bob’s stations on two distant sites along the \(y\)-axis. As said before, each SGA acts locally, by a unitary transformation, on the particle spatial, i.e. external, degrees of freedom by correlating its exit direction of motion with the direction of its spin with respect to the SGA axis, rotated around the \(y\)-axis at angle \(\theta\), taken with respect to the \(z\)-axis. Since we are dealing with spin-\(\frac{1}{2}\) spins, there are only two exit directions, either up or down, available to each particle to finally be registered by a corresponding detector. Let us refer to the Alice and Bob detectors as \(D_{A\alpha}, D_{A\beta}, D_{B\alpha}, D_{B\beta}\), and let \(\theta_A\) and \(\theta_B\) be the angles of SGAA and SGAB, respectively. Labels ‘u’ or ‘d’ refer to the particle’s exit directions from each of the SGAs. As said above, the presence of the two SGAs changes not only the trajectories of the two particles but also the Weyl curvature of their configuration space. These changes are both encoded in the change of the wavefunction \(\psi_{AB}\) in equation (31). Near the source, that wavefunction remains approximately unchanged, but far beyond the spatial positions of the two SGAs, the paths of the particles acquire different directions according to their spin so that near the locations of the detectors the input wavefunction is transformed according to

\[
\psi_{AB} \rightarrow A_{u,u} \psi_A(r_{Au}, t) \psi_B(r_{Bu}, t) + A_{u,d} \psi_A(r_{Ad}, t) \psi_B(r_{Bd}, t) + A_{d,u} \psi_A(r_{Dd}, t) \psi_B(r_{Bd}, t)
\]

where \(r_{Au}, r_{Ad}, r_{Bu}, r_{Bd}\) are the positions of the detectors and \(A_{u,u}, A_{u,d}, A_{d,u}, A_{d,d}\) are coefficients depending on the two-particle Euler angles and on the angles \(\theta_A\) and \(\theta_B\) of SGAA and SGAB, respectively. The coefficients \(A_{ij}\) \((i, j = u, d)\) can be easily calculated by applying equation (36):

\[
A_{u,u} = \left( D_1 (\alpha_1, \beta_1, \gamma_1) \cos \frac{\theta_A}{2} + D_1 (\alpha_1, \beta_1, \gamma_1) \sin \frac{\theta_A}{2} \right) \times \left( D_1 (\alpha_2, \beta_2, \gamma_2) \cos \frac{\theta_B}{2} + D_1 (\alpha_2, \beta_2, \gamma_2) \sin \frac{\theta_B}{2} \right) \sin \Delta \theta
\]

where \(\Delta \theta = \frac{1}{2} (\theta_B - \theta_A)\).

The coincidence rates are given by the joint particle fluxes intercepted by the detectors

\[
\Phi_{ij}(\theta_A, \theta_B) = \int A_{ij} (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2; \theta_A, \theta_B) d\alpha_1 d\beta_1 d\gamma_1 d\alpha_2 d\beta_2 d\gamma_2
\]

and

\[
\phi_{ij}(\alpha, \beta) = \int \psi_{AB}(r, t) V S_{\alpha, \beta}(r, t) \left( \int_{\mathcal{J}_{ij}} n_i(r) \right) dA
\]

where \(n_i(r)\) is the particle current density at the detectors. A simple calculation shows that if all particles falling into the detectors

\(^8\) It is precisely the lack of this condition which prevents use of the SGA to measure the spin of electrons. A way to overcome this fundamental limitation was proposed very recently \([28]\).
are counted the coincidence fluxes are given by

\[ \Phi_{\text{co}}(\theta_A, \theta_B) = \Phi_{\text{d,co}}(\theta_A, \theta_B) = \frac{1}{2} \sin^2(\Delta \theta) \]  \hspace{1cm} (41)

\[ \Phi_{\text{d,co}}(\theta_A, \theta_B) = \Phi_{\text{d,cal}}(\theta_A, \theta_B) = \frac{1}{2} \cos^2(\Delta \theta), \]  \hspace{1cm} (42)

in full agreement with standard quantum theory [6, 29]. This is the key result of the present article. The coincidence fluxes \( \Phi_{\text{co}} \) are Weyl-gauge-invariant and can be experimentally measured. Moreover, they are equal to the joint probabilities \( P_{ij}(\theta_A, \theta_B) \) associated with the EPR state (31) and lead straightforwardly to the violation of Bell’s inequalities within all appropriate experiments consisting of statistical measurements over several choices of the angular quantity \( \Delta \theta \), as shown by many modern texts. For instance, Redhead considers the inequality

\[ F(\Delta \theta) = |1 + 2 \cos(2\Delta \theta) - \cos(4\Delta \theta)| \leq 2 \]

which is violated for all values of \( \Delta \theta \) between 0 and 45° in a simple experiment [7].

14. The no-signalling theorem

Let us consider the EPR experiment discussed above as a paradigmatic example of quantum nonlocality. The fluxes \( \Phi_{ij} \) given in the preceding section are then identified with the probabilities of having the particle spins \( (s_A, s_B) \) oriented in the directions \( (i, j) \), respectively, with \( i, j = (\uparrow, \downarrow) \) and the integrands \( |A_{ij}(\lambda; \theta_A, \theta_B)|^2 \) as the joint probability distributions \( p_{ij}(i|\theta_A, \theta_B) \) to find \( s_A = i \) and \( s_B = j \) conditioned by Alice’s and Bob’s respective SGA settings \( \theta_A, \theta_B \) for fixed values of the internal variables \( \lambda = (\zeta_A^a, \zeta_B^b) \) spanning all six Euler angles of the two particles A and B. From equation (38) we see that \( p_{ij}(i|\theta_A, \theta_B) = |A_{ij}(\lambda; \theta_A, \theta_B)|^2 \) does not have the factor form \( p_i(\theta_A) p_j(\theta_B) \) required by Bell’s locality assumption. Hence, quantum geometrodynamics provides a nonlocal (in Bell’s sense) hidden variable completion to quantum mechanics, where Bell’s inequalities can be violated (and they are, indeed, as shown above). But what about no-signalling? Violating the no-signalling condition would lead to serious problems against the relativistic causality: information could be transferred between Alice and Bob even if they were at space-like locations. No-signalling condition is not required for the joint probabilities because there is no way to transfer information exploiting spin correlations at Alice’s and Bob’s sides. Quantum teleportation, for example, requires a classical communication channel between Alice and Bob. No-signalling is required, instead, for the marginal probabilities (and averages) measured at Alice and Bob locations. Mathematically, the no-signalling conditions are valid

\[ p_i(\theta_A, \theta_B) = p_A(\theta_A) \]  \hspace{1cm} (43)

\[ p_j(\theta_A, \theta_B) = p_B(\theta_B) \]  \hspace{1cm} (44)

respectively, where \( \zeta_A^a = (\alpha_A, \beta_A, \gamma_A) \) and \( \zeta_B^b = (\alpha_B, \beta_B, \gamma_B) \) and \( d \mu(\zeta) \) are the measures in the respective Euler angle spaces. Now, a direct calculation based on equation (38) and on the explicit expression for the normalized D-functions, yields

\[ p_i(\theta_A, \theta_B) = \frac{1}{4} \left[ 1 \pm (\cos \alpha_A \sin \beta_A \cos \theta_A \right. \]  \hspace{1cm} (45)

\[ + \cos \alpha_A \sin \beta_A \sin \theta_A) \left. \right] \]

\[ (i = \uparrow, \downarrow) \]

\[ p_j(\theta_A, \theta_B) = \frac{1}{4} \left[ 1 \pm (\cos \alpha_B \sin \beta_B \cos \theta_B \right. \]  \hspace{1cm} (46)

\[ + \cos \alpha_B \sin \beta_B \sin \theta_B) \left. \right] \]

\[ (j = \uparrow, \downarrow) \]

We see therefore that the marginal probability at each side of the EPR experiment depends on the particle and SGA orientations at the same side only. There is no way for Bob to send signals to Alice by changing all he can change: the angle \( \theta_B \) of his SGA. The same is true for Alice. Quantum geometrodynamics can then be considered as a nonlocal hidden variables theory which, nevertheless, satisfies the no-signalling condition. As a final point, we notice that the marginal probabilities (45) yield to marginal spin averages that are not of the scalar product form required by Leggett nonlocal hidden variable models [32]. Therefore, quantum geometrodynamics can also violate—and indeed violates—Leggett’s inequalities.

15. Conclusions

The above analysis shows that the ‘enigma’ of quantum nonlocality, generally considered to be epitomized by the violation of the Bell’s inequalities, may be understood on the basis of conformal quantum geometrodynamics. This conclusion was obtained by a rigorous and exact formal procedure, i.e. without any approximation. CQG theory bears several appealing properties and may lead to far-reaching...
consequences in modern physics. We summarize them as follows:

1. The linear structure of the standard first quantization theory is fully preserved in any formal detail and made compatible with the further requirement of full conformal gauge invariance.

2. The conformal gauge can be chosen at will even if the equivalence with standard quantum mechanics can be made only in one gauge where the metric reduces to the Minkowski form. In this gauge, gravitational and quantum effects have different origins, namely the metric and the affine connections, respectively. A gauge exists where quantum and gravitational phenomena both originate from the metric tensor.

3. The quantum wavefunction acquires the precise meaning of a physical quantum Weyl gauge field acting in a curved configurational space. In particular, the square modulus of the wavefunction is identified with the Weyl potential, and its gradient with the Weyl vector.

4. A proper theoretical analysis of any quantum entanglement condition must involve the entire configurational space of the system, including the usual space–time of general relativity as well as the internal coordinates of the system. When entanglement is present and if the internal coordinates are really hidden, i.e. if they are absent in the theory—as they are generally considered in standard quantum mechanics—severe limitations may arise on the actual interpretation of any dynamical problem. The interpretation of physics may even be an impossible task, in principle, and paradoxes may spring up. Indeed, in addition to quantum nonlocality, many counterintuitive concepts of quantum mechanics such as those related to several aspects of quantum indeterminism and quantum counterfactuality may arise from the theoretical limitations due to the incompleteness of a description limited to space–time fields: which are indeed limitations to the human knowledge and understanding.

5. ‘Sinister’, ‘disconcerting’, and ‘discomforting’ aspects of entanglement were expressed right after the publication of the EPR paper by a highly concerned Schrödinger in 1935 [33]. Who also added: ‘I would not call that one but rather the characteristic trait of quantum mechanics, one that enforces the departure from the classical line of thought.’

We believe that our present analysis enlightens from a novel, insightful perspective this highly intriguing aspect of modern physics.

Finally, and more generally, we believe that a quite interesting feature of the present theory consists of its apparent unifying structure, connecting for the first time general relativity, electromagnetism, and quantum mechanics within a unique (abelian) gauge theory. It is also interesting, not to say inspiring, to remark that a similar, non-abelian Yang–Mills gauge theory underlies the electro-weak interactions and belongs to the standard model of the elementary particles [16].

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