Dynamics and steady state properties of entanglement in periodically driven Ising spin-chain

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We study the dynamics of microscopic quantum correlations, viz., bipartite entanglement and quantum discord, in Ising spin chain with periodically varying external magnetic field along the transverse direction. Depending upon system parameters, local quantum correlations in the evolved states of such systems may get saturated to non-zero values after sufficiently large number of driving cycles. Moreover, we investigate convergence of the local density matrices, from which the quantum correlations under study originate, towards the final steady-state density matrices as a function of driving cycles. We find that the geometric distance between the non-equilibrium and the steady-state reduced density matrices obey power-law scaling. The steady-state quantum correlations corresponding to various initial states in thermal equilibrium are studied as a function of drive time period of a square pulsed field. The steady-state quantum correlations are marked by presence of peaks in the frequency domain. The steady-state features can be further understood by probing band structures of Floquet Hamiltonian. Finally, we compare the steady state values of the local quantum correlations under study with that of the canonical Gibbs ensemble and infer about canonical ergodic properties. We find that depending upon the quantum phases of the initial state and the pathway of the driving Hamiltonian, quantum correlations may go through (canonical) ergodic to (canonical) non-ergodic transitions in the frequency domain.

I. INTRODUCTION

Entanglement [1], in particular, and quantum correlations [2], in general, have continued to gain enormous amount of interests due to their numerous applications in quantum information theory [3] and many-body physics [4–16]. Quantum correlations are key resources for various quantum information processing tasks [17–19] and communication protocols [20] in many-body systems. Quantum correlations in many-body systems often provide new insights about the cooperative phenomena such as quantum phases [4–10], e.g. in context of symmetry breaking phases [4–6] and topological phases [8, 9, 11, 12]. Particularly, quantum criticality and related aspects in such systems have been investigated in great detail using various tools borrowed from quantum information theory [13–16, 21, 22]. Moreover, recent experimental advances have successfully demonstrated effective manipulation of quantum correlations in several physical substrates [5, 23–29].

Dynamics of closed many-body quantum systems has been a subject of intense research in recent years. Study of isolated systems out-of-equilibrium brings new possibilities for exploring physical phenomena, which are, understandably, not within the reach of equilibrium statistical mechanics, and at the same time, provides a unique way for perceiving the emergence of equilibrium statistical properties [30, 31]. Quenching one or several parameters of a many-body system happens to be a common strategy for exploring non-equilibrium dynamics [32, 33]. Ideas originating from quantum information have been incorporated to study wide range of topics in the context of dynamics of closed quantum systems [32–45], such as revival and collapse phenomena of entanglement [39], Kibble-Zurek mechanism [46], thermalization, many-body localization [47], and decoherence of a qubit attached to a spin chain [48]. A widely studied fundamental topic is ergodicity of local quantities that relates the relaxation of local properties of a many-body systems to their corresponding equilibrium value [30, 31, 40–44, 49–53]. Time evolution of microscopic quantum correlations under sudden quenching and their ergodic properties have been addressed in various spin systems [40–44].

Periodically disturbing the parameters of the underlying Hamiltonian provides an interesting route for monitoring the out-of-equilibrium dynamics of many-body systems in multiple of driving-periods [54]. Such dynamics, which are also called as stroboscopic dynamics, has attracted a vast interest in recent years as it offers a possibility of generating an effective Hamiltonian whose properties might be different from the initial Hamiltonians [54–58]. Generation and relaxation of entanglement entropy towards a steady state have been studied in periodic driven integrable spin models [59–61]. An interesting observation from these works is that the entanglement entropy in the long-time steady state does not reach to a value that corresponds to the infinite temperature state, and hence contradicts the common perception of heating-up associated with repeated disturbance [62]. In a recent work, quantum critical scaling under periodic driving has been studied [63].

An important reason for renewed interests in integrable model, such as quantum spin chains, is due to their experimental relevance in present time. Experiments with ultracold atoms trapped in optical lattice have entered very advanced states. Near-perfect isolation from environment and precise control over the trapping geometries and inter-atomic interaction strengths has established cold-atom system as an ideal platform for studying quantum phenomena in various many-body systems [24, 25]. Moreover, there are several other promising physical substrates, where many-body dynamics can be investigated in a controlled manner [26–29]. Particularly, in recent times there have been considerable efforts for realizing periodically driven many-body systems in experiments [64].

In this paper, we study the dynamics of microscopic quantum correlations, i.e., quantum correlation between two-sites, in a periodic driven Ising model, where periodic driving is implemented via external magnetic field. More specifically, start-
ing from a close to zero-temperature initial state, the dynamics is generated by repeated application of the unitary evolution, obtained in one complete driving period \( \tau \), for reaching at the desired time \( t = n\tau \), where \( n \) denotes the number of applied pulses and \( \tau \) is the time-period between successive pulses. We study the relaxation of bipartite entanglement, measured by concurrence, and quantum discord between two nearest-neighbor sites as a function of the driving cycles \( n \) for different choices of driving frequencies, \( \omega = 2\pi/\tau \). We find that both concurrence and quantum discord tend to saturate to steady-state values after sufficient number of driving cycles \( n \). We also calculate the distance \( d \) between the density matrix after \( n \) driving cycles and the density matrix corresponding to the steady state obtained by taking the asymptotic limit \( n \to \infty \). The distance function, which goes to zero in the asymptotic limit, obeys power law scaling of the form \( d \sim n^{-B} \) with respect to \( n \). The fitted data shows that the distance function goes to zero with scaling exponent \( B = 1.5 \) for \( \tau < 2 \) and \( B = 0.5 \) for \( \tau > 2 \). Next, for various choices of initial states, we look into steady-state \( (n \to \infty) \) quantum correlations, and monitor their variation as a function of \( \tau \). The steady state quantum correlations are characterized by presence of sharp peaks or kinks, which can be further understood by looking into the band structure of the Floquet Hamiltonian, \( H_{k,F} \).

Since the local quantum correlations in the long-time steady-state may survive with finite values, it is fundamentally interesting to see if the final steady state value corresponds to a canonical Gibbs ensemble. If such canonical Gibbs states exists, the quantity is termed as canonical ergodic. We consider two distinct cases depending on the choice of driving pathway: (i) repeated driving across the critical point and (ii) repeated driving within a single phase. For case (i), when the initial state is chosen from the ordered phase, the quantum correlations always remain ergodic. On the contrary, and more interestingly, quantum correlations may undergo ergodic to non-ergodic transitions in frequency domain if the system is initialized in the disordered phase. For case (ii), we find situations, where quantum correlations originating from entanglement-separability paradigm, i.e., concurrence, exhibit completely different ergodic behavior than the information-theoretic quantum correlation, i.e., quantum discord. For such cases, although quantum discord are characterized with ergodic to non-ergodic transitions in the frequency domain, concurrence always remains ergodic when the driving is within the disordered phase.

The paper is organized as follows: Section II introduces the spin model under study and the driving protocol employed in this work. Relaxation of bipartite quantum correlations as a function of driving cycles \( n \) and scaling of the distance between the density matrix after \( n \) driving and the steady state are examined in Sec. III. The results for the steady-state quantum correlations as a function of the time-period of the driving protocol is analyzed in Sec. IV. A comparison for the same for the two-site classical correlators and magnetization is also discussed here. Sec. V discusses condition for the non-trivial evolution, and the notion of ergodicity for the periodically driven cases. In this section, we present a detailed study on the ergodicity of microscopic quantum correlations under periodic driving. Finally, Sec. VI concludes and renders future perspectives. Appendix A, B, and C provide definitions of quantum correlation measures considered in this work, details of Floquet Hamiltonian, and derivations of time-evolved correlation functions.

II. THE MODEL

In this paper, we consider one-dimensional Ising model in presence of an square pulsed transverse magnetic field. For academic interest, we sketch out the methodology for evaluating the time-evolved density matrices in appendix C for generalized scenario with quantum \( XY \) model, whose Hamiltonian is given by

\[
H(t) = \sum_{i=1}^{N} \left[ \frac{J}{4} \left( (1 + \gamma)\sigma_x^i \sigma_x^{i+1} + (1 - \gamma)\sigma_y^i \sigma_y^{i+1} \right) - \frac{1}{2} \sum_i h(t)\sigma_z^i \right],
\]

where \( J \) is the pairwise coupling strength between nearest-neighbor spins, \( h(t) \) is time-dependent external transverse magnetic field, and \( \gamma \) is the anisotropy parameter. Ising model corresponds to the case \( \gamma = 1 \). The periodic boundary condition, i.e., \( \sigma_{N+1} = \sigma_1 \), is considered. In time-independent case, the above model undergoes a quantum phase transition at \( \lambda = 1 \), where \( \lambda = h/J \). For \( \lambda > 1 \), the system is in paramagnetic or disordered phase and for \( \lambda < 1 \), the system is in antiferromagnetic or ordered phase. The quantum criticality of the model has been widely studied via various approaches [4, 5, 49, 51]. We consider a non-equilibrium scenario where the system is driven periodically by introducing the time-dependent magnetic field. The time-dependent magnetic field is taken in the form of a square-pulse, such as \( h(t) = a \) for \( t \leq 0 \), and for \( t > 0 \)

\[
h(t) = \begin{cases} a & \text{if } (n-1)\tau \leq t \leq (n-\frac{1}{2})\tau; \\ b & \text{if } (n-\frac{1}{2})\tau \leq t \leq n\tau. \end{cases}
\]

Here \( \tau (>0) \) is the time period between two successive pulses. The Hamiltonian in Eq. (1) is exactly solvable even in the case of periodic driving. The details are presented in Appendix C. For convenience, here we highlight essential key steps, which involve fermionization of the spin Hamiltonian that reduces the original Hamiltonian into the quadratic form in terms of the fermion creation and annihilation operators. In momentum, \( k \), space, the system can be decomposed into non-degenerate subspaces and the density matrix of the entire system is written as the product of the density matrices of individual subspaces. Moreover, in the case of periodic driving, the evolution operator, \( U(\tau, a, b) \), can be written as the exponential of an effective time-independent Hamiltonian over one complete driving period \( \tau \). The effective Hamiltonian is termed as Floquet Hamiltonian, and its form is derived in Appendix C. The non-interacting nature of the initial Hamiltonian in the fermion picture leads to another important simplification in the dynamics of the entire system, as the time-dependent density matrix of the entire system can be written...
as the tensor product of the density matrix of the individual subspace. Moreover, the evolution operators $U(\tau, a, b)$ over a period $\tau$ can be decomposed as the tensor product of the evolution operators acting on the each $k^{th}$ subspace. Therefore, if the dynamics of the system is started from a thermal equilibrium state at $t = 0$ denoted as $\rho_k(0)$, the evolved state of the $k^{th}$ subspace after $n$ driving cycles is obtained as $\rho_k(n\tau) = [U_k(\tau, a, b)]^n \rho_k(0) [U_k^\dagger(\tau, a, b)]^n$, where the unitary operator is given by

$$U_k(\tau, a, b) = \exp[-iH_{k,F}\tau].$$

Here $H_{k,F}$ is the Floquet Hamiltonian of the $k^{th}$ subspace. From the time dependent density matrix, $\rho_k(n\tau)$, one can calculate the reduced density matrix between two-sites, $\rho_{i,j}(n)$, as a function of driving cycles, $n$, in real space (defined in Appendix A), which can further be expressed in terms of the two-point correlation functions derived in Appendix C. From the two-site reduced density matrix the local quantities under study, such as concurrence and quantum discord, are calculated. The variation of these quantities are examined in time, $t$, which is given as $t = n\tau$.

### III. TEMPORAL BEHAVIOR OF LOCAL QUANTITIES AND CONVERGENCE TOWARDS STEADY STATE VALUES

For the one-dimensional Ising model under periodic driving via external magnetic field (see Sec. II), we look into the temporal behavior of the local quantities, single- and two-site correlations, such as magnetization, classical correlators and bipartite quantum correlations as a function of the periodic driving, $n$, and monitor their convergence towards steady state values at long-time for different choices of $\tau$.

In order to estimate quantum correlations, in this work we consider two distinct measures, one of which originates from entanglement-separability paradigm, namely concurrence, and the second one is quantum discord — a information-theoretic quantum correlation measure. The definitions of the quantum correlation measures considered in this work are provided in Appendix B.

As the low-temperature states possess maximum or near-maximum quantum correlations, which gradually erodes away at higher temperature limits, the initial states considered in this work are the close to zero-temperature canonical equilibrium states of the system with magnetic field $a$ and are given by the density matrices at temperature $J/\beta = 20$.

In the following, we demonstrate the convergence of quantum correlations towards the steady-state values with driving cycles for system parameters, $a/J = 1.4$ and $b/J = 0$ for $0 < \tau \leq 2.0$. The dynamics of the system is captured by the Floquet Hamiltonian, $H_{k,F}$, which is obtained by considering the dynamics at one complete driving period, $\tau$ (see Sec. II and appendix C). The Floquet Hamiltonian allows us to monitor the dynamics of the system at times $t = n\tau$ by exploiting the translation property of the unitary evolution operator. We consider various driving frequencies and investigate its effect on the relaxation processes of bipartite quantum correlations.

Figure 1 shows us the relaxations of (a) concurrence and (b) quantum discord for different values of $\tau$. Concurrence and discord are quantified in units of ebits and bits, respectively. In first few cycles, the quantum correlations oscillates with $n$ and then starts converging towards a steady-state value at large $n$. The saturated values depend upon the lengths of the period $\tau$. It can be seen from Fig. 1 (a) that the entanglement saturates to lower values when $\tau$ is increased. For $J\tau/\hbar = 1.5$ bipartite entanglement completely vanishes. Note that the survival of entanglement explicitly depends on the choice of $\tau$, and may again resurrect at $\tau > 2.0$. A detailed analysis of steady-state properties as a function of driving frequency has been carried out in the following section.

Although a similar saturation behavior is observed also for quantum discord shown in Fig. 1(b), unlike entanglement, quantum discord survives with finite value even at $J\tau/\hbar = 1.5$. It is well known by now from several other works with sudden quenching [39, 41–43] that quantum discord is more robust against disturbance in comparison to entanglement, and long-time quantum discord usually survives even when entanglement vanishes.

In Fig. 2, we demonstrate relaxations of magnetizations, $m_z$, and classical correlators, $t^{xx}$, $t^{yy}$ and $t^{zz}$ (see appendix C for definitions), for $0 < \tau \leq 2.0$, and $a/J = 1.4$, $b/J = 0$. Within this frequency window, these quantities require relatively lesser number of driving cycles for approaching respective saturation values as the time period of the square pulsed field increases. Similar to quantum correlations, the steady state value of the magnetization and classical correlators decreases with increas-
measure the distance between the density matrices itself. In quantum information theory, such distances are often used and their properties have been studied in various contexts [3]. For our case, we consider ‘trace distance’, \( d \), as a measure of overlap of information between two-density matrices. \( d \) is defined as

\[
d = \text{Tr} \sqrt{(\Delta \rho_n)^\dagger \Delta \rho_n},
\]

where \( \Delta \rho_n = \rho_{12}(n) - \rho_{12}(\infty) \). Here \( \rho_{12}(n) \) is the reduced density matrix of the bipartite state after \( n \) driving cycles and \( \rho_{12}(\infty) \) is the same in the limit of \( n \to \infty \). Physically, Eq. (4) represents the distinguishability between two normalized density matrices. The maximum value, \( d = 1 \), defines the maximum distinguishability between two states. In Fig. 3(a-d), plot the \( d \) between the bipartite reduced density matrix at \( t = \infty \) and the reduced density matrix of the driven system approaching the steady-state as a function of \( n \). We also fit the data on power law function such as \( d = An^{-B} \). The exponent \( B \) suggests a qualitative change in the relaxation properties of the local quantities at \( \tau = 2 \), as we find the exponent \( B \) to be 1.5 for \( \tau < 2 \) and 0.5 for \( \tau \geq 2 \). It’s worth mentioning here two possible dynamical phases depending on fast and slow periodic driving have been identified in [59] while studying the relaxation process of entanglement entropy.

### IV. QUANTUM CORRELATIONS IN THE LONG-TIME STEADY STATES

In this section, we discuss the steady state behavior of entanglement and quantum discord as function of \( \tau \). Figure 4 shows steady-state entanglement (see Fig. 4(a)) and quantum discord (see Fig. 4(b)) as a function of the driving period for a particular case of square pulsed field with \( a/J = 1.4 \) and \( b/J = 0.0 \). Noticeably, repeated disturbance may not heat up the system indefinitely causing complete destruction of local quantum correlations present in the system – there exist ranges of \( \tau \), where the system possesses non-zero quantum correlations in the asymptotic limit. In fact, for the ranges of \( \tau \), where bipartite entanglement vanishes, quantum discord survives with small but finite values.

Long-time steady-state quantum correlations show a oscillatory pattern in the frequency domain, and are characterized
by the presence of prominent peaks. In low frequency domain, the peaks appear to be equispaced. Interestingly, it turns out that this unique trait is directly linked with Floquet band structures (see Appendix C) corresponding to the unitary operator describing one complete driving cycle of the period $\tau$. Figure 4(c-f) shows Floquet spectrum, $\epsilon_{k,F}$, in momentum space, $k$, for $J\tau/h = 6.5, 10, 20$ and 25. It can be seen from Fig. 4(a) and Fig. 4(b) that whereas $J\tau/h \approx 6.5$ and $J\tau/h \approx 25$ corresponds to the kinks in quantum correlations, $J\tau/h = 10$ corresponds to vanishing entanglement and discord minimum. For $J\tau/h = 20$, entanglement is zero but discord is finite valued (slightly higher than the minimum value). From Figs. 4(c) and 4(f), it’s evident that the peaks in quantum correlations are a consequence of Floquet band crossings. Quantum correlations assumes minimum values when band gap is maximum. We would like to mention here that appearance of kinks in the frequency domain and their connection with Floquet band crossings has been reported earlier in context of block entropy of the periodically driven systems [59].

V. RELAXATION OF QUANTUM CORRELATIONS UNDER PERIODIC DRIVING

In the previous section, we discussed about the behavior of the quantum correlations in the asymptotic $n \to \infty$ state. It would, therefore, be interesting to ask if the value of the quantum correlations in the asymptotic state corresponds to a canonical thermal equilibrium state. In this section, we first describe the corresponding equilibrium state and the related notion of canonical ergodicity.

A. Steady states and canonical ergodicity

We consider that the spin chain is initially subjected to an external magnetic field $a$, and is in thermal equilibrium at temperature $T$ for $t \leq 0$. The initial state, $\rho_{eq}(\beta, a)$, is given by $\exp[-\beta H(a)]$, where $\beta = 1/(k_B T)$ is the inverse of the absolute temperature $T$ and $k_B$ is the Boltzmann constant. The evolved state, $\rho(\beta, a, b, t)$, at any time, $(t > 0)$, is given by $U^\dagger (a, b, t) \rho_{eq}(\beta, a) U(a, b, t)$. In the following, we discuss the ergodic properties within the notion of canonical equilibration, which we refer as ‘canonical ergodicity’. Within this description, the ergodic properties of the system is inferred by comparing the time-evolved state at large time with the canonical equilibrium states [40, 51]. Note that ergodicity within generalized Gibbs ensemble [65], which is constructed by taking into account the conserved quantities, is not considered in this work.

In order to construct a family of canonical equilibrium states suitable for describing the canonical ergodicity in periodic driving cases, we consider average Hamiltonian over one complete driving period [44], say $\bar{H} \equiv \frac{1}{\tau_0} \int_0^{\tau_0} \bar{H}(t) dt$, and assemble the set of canonical equilibrium states parametrized by $\beta$ for all
$t > 0$ as $\rho_G(\tilde{\beta}, \tilde{h}_0) = \exp[-\tilde{\beta} H(\tilde{h}_0)]$, where \( \tilde{\beta} = 1/(k_B T) \) is the inverse temperature of the possible canonical equilibrium state and \( \tilde{h}_0 \) is the average magnetic field over a period \( \tau \) (See Eq. (2)). A quantity is termed ergodic, if the steady state value of the quantity, $Q_S(\tilde{\beta}, a, b, \tau)$, approaches the ensemble average, $Q_G(\tilde{\beta}, \tilde{h}_0)$, i.e., if

$$\text{Tr}[\rho(\beta, a, b, \tau, n \to \infty) \hat{Q}] \approx \text{Tr}[\rho_G(\tilde{\beta}, \tilde{h}_0) \hat{Q}]. \quad (5)$$

If the local quantity does not satisfy Eq. (5), it is called non-ergodic. As we discuss below in subsequent sections, the local quantum correlations may undergo ergodic to non-ergodic transitions. In order to monitor such transitions in a systematic manner, we use so called ergodicity score [41], which indicates the quantity’s departure from ergodicity. The ergodicity score, \( \eta_S^Q \), is defined as

$$\eta_S^Q = \max \left[ 0, Q_S(\tilde{\beta}, a, b, \tau) - \max_{\beta} Q_G(\tilde{\beta}, \tilde{h}_0) \right]. \quad (6)$$

We search for all \( \tilde{\beta} \) and choose the particular one for which $Q_G(\tilde{\beta}, \tilde{h}_0)$ attains maximum possible value for the given set of system parameters \( a, b \). Non-zero (vanishing) ergodicity score signals non-ergodic (ergodic) behavior of \( Q \).

Before we proceed into further analysis, a few more comments are in order. As mentioned before, close to zero-temperature states possesses maximum or close to maximum quantum correlations, which decays quickly in higher temperature limits. Hence, it is sufficient to consider these low-temperature states for investigating the ergodic properties of bipartite quantum correlations. In this work, we look into canonical equilibrium states at $J \beta = 20$ as the initial states, which can be considered to be effectively zero-temperature states as quantum correlations attains saturation in this limit (for $J \beta \gtrsim 20$ to be more specific). Therefore, it is reasonable to discuss the ergodic behavior in terms of the quantum phases that the initial states belong to. In the following, we look into possible situations that arise depending on the initial state and the choice of driving pathway.

### B. Periodic driving across critical point.

Let us first look into the situation when the system is initiated in the disordered phase, and at each cycle, the driving Hamiltonian leads the system to a final state, which would correspond to the ordered phase of the system at equilibrium. For this case, let us stick to the parameters from the previous section, \( a/J = 1.4 \) and \( b/J = 0 \), which evidently belongs to the case of repeated quenching of the system from disordered to ordered phase. The long-time steady-state values of bipartite entanglement and quantum discord for these chosen set of parameters have already been studied in Fig. 2.

Let us first look into the cases for $J \tau/h \leq 2$. In Fig. 5(a), the red solid line shows the bipartite entanglement in the canonical equilibrium state, as a function of inverse temperature \( \beta \). The canonical equilibrium state is the one that correspond to the average magnetic field \( \tilde{h}_0 \), i.e. $\rho_G(\tilde{\beta}, \tilde{h}_0)$. It is clear from the figure that at high temperature, i.e. at small \( \beta \), the bipartite entanglement completely vanishes. However, below certain temperature, $J \beta \approx 2.2$, system possesses finite amount of entanglement. At low-temperature ($J \beta \gtrsim 10$), the entanglement tend to saturate to a value close to 0.07 ebits as the system approaches zero-temperature state. In order to find if the entanglement of the periodically driven system reaches to a state that corresponds to canonical equilibrium state, we plot the entanglement of the long-time steady-state in the same figure. The steady-state values for varied $\tau$ are shown by horizontal lines in Fig. 5(a). We observe that the steady-state entanglement for all $\tau$ intersect the canonical entanglement at different temperatures, implying that entanglement is always ergodic in the frequency domain for this case.

We have already discussed about the correlations in the quantum states that goes beyond entanglement. Therefore it is important to extend the analysis to information-theoretic quantum correlation measures, such as quantum discord, in order to encompass a complete picture about the time-evolved density matrix. The definition of quantum discord is provided in Appendix B. The red solid line in Fig. 5(b) shows the behavior of quantum discord in the canonical equilibrium state as a function of $\beta$. Quantum discord, unlike entanglement shows monotonic behavior with respect to $\beta$. Quantum discord increases with decreasing system temperature and saturates at low enough temperature ($J \beta \gtrsim 10$). The steady state values of quantum discord for different $\tau$ are again shown by horizontal lines in the same plot. We find that for higher values of $\tau$, the steady-state quantum discord intersect the canonical equilibrium quantum discord curve at different temperature. However, surprisingly, below certain critical time-period, $\tau \lesssim \tau_c$, there is no intersection, implying a ergodic to non-ergodic transition of quantum discord in the frequency domain. For this case, we find $J \tau/h \approx 1.5$. For clarity and an estimation of the degree to which the physical quantities under study is possibly non-ergodic, we calculate the ergodicity score (see Eq. (6)) for both entanglement, $\eta_S^D$, and quantum discord, $\eta_S^D$. In the inset of Fig. 5(b), we show $\eta_S^D$ as function of $\tau$. $\eta_S^D$ is finite valued for $J \tau/h \leq 1.5$, beyond which it becomes zero. It may be mentioned that ergodicity score for entanglement is zero through-
shows steady-state concurrence. In fact, entanglement does not survive under this driving scheme. In fact, entanglement and quantum discord for this case are shown in Figs. (c-d). In (c-d), the solid lines represent the equilibrium values. In (c), the horizontal dashed, dash-dotted and dotted lines show steady-state values for $J\tau/h = 0.1, 1.4$ and 2.0, respectively. In (d), the horizontal dashed, dash-dotted, dotted, dash-dash-dotted and dash-dotted-dotted lines show steady-state values for $J\tau/h = 0.1, 1.4, 2, 3, 4$, respectively.

out the range of $\tau$ for this case.

However, the ergodic to non-ergodic transition in frequency domain is not unique to quantum discord. This becomes evident when the system is driven repeatedly with larger time-period. In fact, this information can be extracted from the steady-state values and canonical equilibrium values available in Figs. 4(a-b) and 5(a-b). The ergodicity scores for concurrence and quantum discord for this case are shown in Figs. 5(c) and 5(d). The positive value of the ergodicity score indicates that both entanglement and quantum discord become ergodic for the value of $\tau$.

In order to find out if the ergodic to non-ergodic phenomenon reported here is generic, we perform similar analysis by initiating the system in the disordered phase but with different magnetic fields ($a/J > 1$) and setting different values $b$ in the ordered phase ($0 < b/J < 1$). We observe that for a given $a$, the quantum correlations undergo ergodic to non-ergodic transition if $b < b_c$, beyond which both kinds of quantum correlations become ergodic irrespective of the driving frequency. For, the case discussed above with $a/J = 1.4$, we find $b_c/J \approx 0.8$. $b_c$ decreases with increasing $a$. When initial states are chosen from sufficiently deep disordered phase, $b_c \rightarrow 0$, and both entanglement and quantum discord become ergodic for any $b$ and $\tau$.

Let us now consider another possible scenario, where the system is driven from the ordered phase to the disordered phase at each driving cycle. The steady-state entanglement does not survive under this driving scheme. In fact, entanglement vanishes only after few cycles and stays so for any $n > 0$. As a result, entanglement is trivially ergodic within this driving scheme. However, long-time steady-state quantum discord, as we may expect by now, survives, albeit with small value. Figure 6 shows an example for quantum discord with $a/J = 0$ and $b/J = 1.4$ for canonical equilibrium state as a function of $\beta$. The inset of Fig. 6 shows steady-state quantum discord as a function of $\tau$. It is easy to infer from the figure that, throughout the entire range of $\tau$, steady-state values always correspond to certain thermal equilibrium state. Hence, along with entanglement, quantum discord is also ergodic for this case. Similar analysis is performed for arbitrary choices of $a$ and $b$ within this driving scheme. We find both kinds of quantum correlations always remain ergodic.

C. Periodic driving within same phase.

Finally, we briefly discuss on system’s response if repeated driving is conducted within same phase. Let us first discuss the case when repeated driving is implemented within the ordered phase. For demonstration, a specific example is presented here with $a/J = 0.8$ and $b/J = 0$. For this case, ergodic to non-ergodic transition is noticed in both kinds of quantum correlations (Fig. 7(a-b)). We investigate additional cases within this driving strategy, where time evolution starts from the same initial state ($a/J = 0.8$) at $t = 0$, and steady-state quantum correlations are studied by changing $b$. We find such transitions are not noticed when $0.8 < b/J < 1$ as both concurrence and quantum discord becomes ergodic in the entire frequency range. Moreover, we find that, irrespective of the initial states, quantum correlations behave ergodically whenever the system is driven close to the phase transition point.

Surprisingly, quantum correlation from entanglement-separability paradigm and information-theoretic ones may not behave coherently when repeated driving is simulated within the disordered phase. In this case, although, concurrence always stays ergodic for arbitrary choices of $a$ and $b$, there may exist specific frequency windows, where quantum discord becomes non-ergodic, i.e., ergodic to non-ergodic transitions occur in frequency domain. Figure 7 exhibits one such case with $a/J = 2.4$ and $b/J = 1.2$ for steady-state (c) entanglement and (d) quantum discord. However, although we always find a canonical equilibrium state that would correspond to the long-time steady-state concurrence for any given $\tau$ (cases for few specific choices of $\tau$ are shown by horizontal lines in Fig. 7(c)), same is not true for quantum discord.

VI. CONCLUSION

In this work, we investigated the behavior of microscopic quantum correlations, viz. bipartite entanglement and quantum discord, in a periodically driven Ising model. Staring from a initial state at thermal equilibrium, we studied their relaxations under a square pulsed external magnetic field with driving cycles, $n$, for varied driving frequencies. We observe both,
entanglement and quantum discord, eventually saturates after sufficient number of driving cycles. As it may be expected, the required number of driving cycles for reaching saturation increases with increasing driving frequencies. Further understandings on dynamical relaxations can be obtained by looking into dynamical evolution of the bipartite reduced density matrices itself, from which the local quantities under study originates. We investigate the trace distance between local density matrices of the driven system and the steady-state density matrix. The scaling of the distance measure $d$ shows a power law behavior, $d = An^{-B}$, with respect to the driving cycles $n$. The exponent $B$, which turns out to be 1.5 or 0.5 depending upon the fast or slow periodic driving, indicates a qualitative change in the relaxation processes of the local quantities under study at $\tau = 2$. Next we studied the steady-state quantum correlation with respect to $\tau$. Long-time steady-state quantum correlations are characterized by the presence of prominent peaks in frequency domain. The peaks are identified as a direct consequence of Floquet band crossings.

Finally, we investigated the canonical ergodicity of the quantum correlations under periodic driving. We find that the canonical ergodic properties of quantum correlations crucially depend upon the quantum phase the initial state is chosen from, and the pathway of repeated driving which may or may not cross the equilibrium phases. Particularly, within a repeated driving scheme via a square pulsed field, when the initial state is chosen from the disordered phase and the final field corresponds to the ordered phase, quantum correlations may display (canonical) ergodic to (canonical) non-ergodic transitions in the frequency domain. The possible degree of non-ergodicity is indicated by so called ergodicity score $\eta^Q$, which shows that the system oscillates between two possible situations, i.e., being ergodic or being non-ergodic, with the modulation of $\tau$. Moreover, for this case, we discussed conditions on the system parameters $a$ and $b$, for which such transitions appear. Another choice of across the phase driving scheme, where the initial state belongs to the ordered phase, and the final state belongs to the disordered phase, both concurrence and quantum discord turns out to be ergodic for any arbitrary driving frequency. Noticeably, when initial states are chosen from deep in order phase, i.e., $a \approx 0$, bipartite entanglement completely vanishes only after few driving cycles for this case, although long-time quantum discord survives. Additionally, we have discussed possibilities of ergodic to non-ergodic transitions in frequency domain, when the system is repeatedly driven within same phase. Surprisingly, entanglement and quantum discord behave differently in the entire frequency range if the driving is conducted within the disordered phase. For this case, we find that entanglement remains ergodic for arbitrary frequency of the square pulse, but there exist frequency windows, where quantum discord becomes non-ergodic.

Our work is relevant to current experimental set-ups for studying Floquet dynamics, particularly via ultracold atoms in optical lattice [64]. Many interesting directions emerging from this work require independent attention, particularly, in context of the correspondence between Floquet band gap and peaks of the quantum correlations. This is beyond the scope of current work, and detailed attention will be paid in our future works. Moreover, there is a lot of scope to further explore the dynamical and steady behavior of the quantities studied in this work. For example, how independent are the features on the choice of driving protocol? It will also be interesting to conduct analogous studies in non-integrable models and higher dimensional systems.

VII. ACKNOWLEDGMENTS

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Appendix A: Density matrix of the system

In order to study microscopic quantum correlations, we need to find the corresponding density matrix of the system both in the evolved and Gibbs state. The local density matrix between two-sites is given in terms of the single-site magnetization and two-site correlation functions as follows:

$$\rho_{\ell m}(t) = \frac{1}{4} I_{\ell} \otimes I_{m} + \sum_{\alpha=x,y,z} m_\ell^\alpha (\sigma_\ell^\alpha \otimes I_m) + m_m^\alpha (I_\ell \otimes \sigma_m^\alpha) + \sum_{\alpha, \beta=x,y,z} t_{\ell m}^{\alpha \beta} (\sigma_\ell^\alpha \otimes \sigma_m^\beta),$$

where $m_\ell^\alpha = \text{Tr}[\rho_\ell \sigma_\ell^\alpha]$ is the magnetization of the $\ell^{th}$ site along the $\alpha$-direction with corresponding single-site density matrix $\rho_\ell = \frac{1}{4}(I + \vec{m} \cdot \vec{\sigma})$, and $t_{\ell m}^{\alpha \beta} = \text{Tr}[\rho_{\ell m} (\sigma_\ell^\alpha \otimes \sigma_m^\beta)]$ are the two-site spin-spin correlation functions given in Appendix C. The $m^x$ and $m^y$ are identically zero as discussed in Refs. [41, 51]. Moreover, translation symmetry of the system guarantees that $m_\ell^z = m_m^z$. Here $I$ is the identity matrix in the Hilbert space of the single-site density matrix. In Appendix C, we have given various correlators as a function of initial magnetic field, $a$, final magnetic field, $b$, and the temperature $T = 1/k_B \beta$. Using these correlators one can calculate the two-site density matrix between any site $\ell$ and $m$.

Appendix B: Quantum correlation measures

**Concurrence:** Concurrence [21] is a well known computable measure of entanglement of a bipartite quantum state in $C^2 \otimes C^2$. If $\rho_{AB}$ denotes an arbitrary two-qubit quantum state, then it’s concurrence is given by $C(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where $\lambda_i$‘s are square roots of the eigenvalues of $\rho_{AB} \rho_{AB}^*$ in descending order with $\rho_{AB} = (\sigma_y \otimes \sigma_y) \rho_{AB}^*(\sigma_y \otimes \sigma_y)$. Here $\sigma^y$ is the Pauli matrix and $\rho_{AB}^*$ is the complex conjugate of $\rho_{AB}$ in the same basis. Concurrence vanishes for separable states and attains to unity for maximally entangled states.

**Quantum Discord:** Quantum discord is a measure of quantum correlations beyond entanglement [2, 22, 66]. It utilizes
the fact that the two equivalent definitions of the mutual informations in terms of the classical probabilities are not same when their natural extensions are considered within quantum theory. The mutual information of the quantum state \(\rho_{AB}\) given by \(I = S(\rho_A) + S(\rho_B) - S(\rho_{AB})\) defines total correlation of the state. Here \(S(\rho_A), S(\rho_B),\) and \(S(\rho_{AB})\) are the von Neumann entropies defined as \(S(\rho) = -\text{Tr}(\rho \log_2 \rho)\). The quantity \(I\) can be interpreted as the amount of information shared by the two parties in a quantum state \(\rho_{AB}\).

The second quantum version of mutual information is given by \(J = S(\rho_A) - S(\rho_{AB})\), where \(S(\rho_{AB})\) is the conditional entropy, \(S(\rho_{AB}) = \min_{(B_i)} \sum_i p_i S(\rho_{A|i})\) and the measurement is performed on subsystem \(B\) (in a similar way it can be defined for measurement on subsystem \(A\)). The measurement operators, \(\{B_i\}\), are rank-1 projective operators and \(p_i\)'s are the probabilities obtained after the measurements on subsystem \(B\). The measured state and the probability of output state are given by \(\rho_{A|i} = \frac{1}{p_i} \text{Tr}_B[B_i \rho_{AB} I_A \otimes B_i]\) and \(p_i = \text{Tr}_A[B_i \rho_{AB} I_A \otimes B_i]\), respectively.

Once we have \(I\) and \(J\), the quantum discord is defined as \(D = I - J\), i.e., \(D = S(\rho_{AB}) - S(\rho_{AB}) + S(\rho_{AB})\). It is to be noted that for pure bipartite states quantum discord reduces to the von Neumann entropy of the reduced state.

**Appendix C: Floquet Hamiltonian and Correlation functions**

In this Appendix, we outline the essential steps for obtaining the effective Floquet Hamiltonian and time-dependent spin-spin correlators in the \(XY\) Hamiltonian (see Eq. (1)). We follow the route provided in references [51, 67].

**Momentum space representation:** The first step is to write the lattice Hamiltonian, given in Eq. (1), in \(k\)-space. We define the ladder operators, i.e., the raising, \(a_i^\dagger\), and lowering, \(a_i\), operators in terms of the spin operators as

\[
\sigma_i^+ = a_i + a_i^\dagger; \quad \sigma_i^- = -i(a_i^\dagger - a_i); \quad \sigma_i^z = 2a_i^\dagger a_i - 1. \quad (C1)
\]

The operators \(a_j\) and \(a_j^\dagger\) are further written in terms of Fermi operators \(b_j\) and \(b_j^\dagger\) using the Jordan-Wigner transformation as

\[
a_j = \exp(-i\pi \sum_{k=1}^{j-1} b_k b_k^\dagger) b_j; \quad a_j^\dagger = \exp(i\pi \sum_{k=1}^{j-1} b_k^\dagger b_k) b_j^\dagger\quad (C2)
\]

The next step is to Fourier transform the Fermi operators \(b_j\) and \(b_j^\dagger\) as:

\[
b_j = \frac{1}{N} \sum_{k=-N/2}^{N/2} \exp(-ij\phi_k) c_k; \quad b_j^\dagger = \frac{1}{N} \sum_{k=-N/2}^{N/2} \exp(ij\phi_k) c_k^\dagger, \quad (C3)
\]

where \(\phi_k = \frac{2\pi k}{N}\). Using Eq. (C1-C3) and related algebra on Eq. (1), one can obtain the Hamiltonian \(H(t) = \sum_{k=1}^{N/2} \hat{H}_k(t)\), where \(\hat{H}_k(t)\) is the Hamiltonian of the \(k\)-th subspace given by

\[
\hat{H}_k(t) = \frac{1}{2} [\alpha(t)(c_k^\dagger c_k + c_k c_k^\dagger) + i\delta(t)(c_k^\dagger c_{-k} + c_k c_{-k}^\dagger) + 2\hbar(t)\], \quad (C4)
\]

where \(\alpha(t) = 2[\cos \phi_k - h(t)], \quad \delta(t) = -2\gamma \sin \phi_k\), and \(c_k\) and \(c_{-k}\) are the fermionic creation and annihilation operators in momentum space. In chosen basis of the \(k\)-th subspace \([|0, 0\rangle, |k, -k\rangle, |k, 0\rangle, |0, -k\rangle\), \(H_k(t)\) can be expressed as a \(4 \times 4\) matrix:

\[
\begin{pmatrix}
    h(t) & -i\frac{\delta(t)}{\alpha(t)} & 0 & 0 \\
    i\frac{\delta(t)}{\alpha(t)} & 2\cos \phi_k - h(t) & 0 & 0 \\
    0 & 0 & \cos \phi_k & 0 \\
    0 & 0 & 0 & \cos \phi_k 
\end{pmatrix}. \quad (C5)
\]

**Floquet Hamiltonian:** The dynamics of the system under the periodic driving protocol described in Eq. (2) is monitored via effective Floquet Hamiltonian. Noticing that the non-trivial contribution in system dynamics is originated from the reduced Hilbert space spanned by the basis \([|0, 0\rangle\) and \([|k, -k\rangle\]), it is sufficient to consider corresponding 2 \(\times\) 2 block of \(H_k(t)\), which can be expanded in terms of the Pauli matrices as

\[
H_k(t) = c_0(k, t) I_{2 \times 2} + c_1(k) \sigma_y + c_2(k, t) \sigma_z. \quad (C6)
\]

where \(c_0(k, t), c_1(k),\) and \(c_2(k, t)\) are coefficients of the expansion defined as \(c_0(k, t) = \frac{1}{2}(\cos \phi_k + h(t)), \quad c_1(k) = \gamma \sin \phi_k, \) and \(c_2(k, t) = -\frac{1}{2}(2 \cos \phi_k - h(t))\). We start the dynamics by assuming that the system is initially in a thermal equilibrium state for all \(t \leq 0\). The corresponding equilibrium density matrix of the \(k\)-th subspace is given by

\[
\rho_k(0) = \exp[-\beta \hat{H}_k(0)], \quad (C7)
\]

where \(\hat{H}_k(0)\) is obtained by substituting \(t = 0\) in Eq. (C5). Now we consider the periodic driving via external magnetic field \(h(t)\) given in Eq. (2). The evolved state of the \(k\)-th subspace after one complete driving period \(\tau\) is given by

\[
\rho_k(\tau) = U_k(\tau) \rho_k(0) U_k(\tau)^\dagger. \quad (C8)
\]

The unitary operator in one complete time period \(\tau\) is given by the time-order product of unitaries for each half-cycles:

\[
U_k(\tau, a, b) = \exp \left[-iH_k(b)^{\tau/2}\right] \exp \left[-iH_k(a)^{\tau/2}\right] = \exp \left[-iH_k, F, \tau\right], \quad (C9)
\]

where \(H_k(a) = \sigma^x \hat{c}_k(a) = |\hat{c}_k(a)| |\hat{\eta}_k(a)|; H_k(b) = \sigma^x \hat{c}_k(b) = |\hat{c}_k(b)| |\hat{\eta}_k(b)|\). The components of \(\hat{c}_k(a)\) are given by \(\hat{c}_k(a) = (0, c_1(k), c_2(k, a))\) where \(c_2(k, a) = -\frac{1}{2}(2 \cos \phi_k - a), \hat{\eta}_k(b)\) and its components are defined similarly. The effective Floquet Hamiltonian, \(H_k, F\), can also be written as \(H_k, F = \sigma^x \hat{c}_k, F = |\hat{c}_k, F| |\hat{\eta}_k, F|\). The quasi-energies are obtained as

\[
|\hat{c}_k, F| = \frac{1}{\tau} \text{Arccos} [\cos(|\hat{c}_k(b)|^{\tau/2}) \cos(|\hat{c}_k(a)|^{\tau/2})] - \hat{\eta}_k(a), \hat{\eta}_k(b) \sin(|\hat{c}_k(b)|^{\tau/2}) \sin(|\hat{c}_k(a)|^{\tau/2}) \quad (C10)
\]
and the \( \hat{n}_{k,F} \) is given by
\[
\hat{n}_{k,F} = \frac{\hat{n}_{k}(b) \sin(|\vec{\epsilon}_{k}(b)|\frac{\tau}{2}) \cos(|\vec{\epsilon}_{k}(a)|\frac{\tau}{2})}{\mathcal{N}_{k}} \\
+ \frac{\hat{n}_{k}(a) \sin(|\vec{\epsilon}_{k}(a)|\frac{\tau}{2}) \cos(|\vec{\epsilon}_{k}(b)|\frac{\tau}{2})}{\mathcal{N}_{k}} \\
- \frac{\hat{n}_{k}(b) \times \hat{n}_{k}(a)}{\mathcal{N}_{k}} \sin(|\vec{\epsilon}_{k}(b)|\frac{\tau}{2}) \sin(|\vec{\epsilon}_{k}(a)|\frac{\tau}{2}),
\]
where \( \mathcal{N}_{k} = |\vec{\epsilon}_{k,F}| \sqrt{1 - \cos^2(|\vec{\epsilon}_{k,F}|)} \). Once the Floquet Hamiltonian is obtained, the state of the system after \( n \) driving cycle is simply given by
\[
\rho_{k}(n\tau) = \exp[i n H_{k,F} \tau] \rho_{k}(0) \exp[-i n H_{k,F} \tau]. \tag{C12}
\]

Spin-spin correlators: We now proceed to derivation of the average magnetization \( m^z(n\tau) \) as a function of driving cycles \( n \), defined as \( m^z(n\tau) = \frac{1}{N} \sum_{j=1}^{N} \langle \hat{\sigma}^z_j \rangle_{\rho(n\tau)} \), and spin-spin correlators, \( t^{\alpha\beta}_{\ell,m}(n\tau) \), defined as \( t^{\alpha\beta}_{\ell,m}(n\tau) = \langle \hat{\sigma}^\alpha_{\ell} \hat{\sigma}^\beta_m \rangle_{\rho(n\tau)} \), where the averages are taken in the time-dependent state \( \rho(n\tau) \). It is shown in [67] that the spin-spin correlation functions can be expressed in terms of fermionic operators \( A \)'s and \( B \)'s where \( A_{\ell} = c_{\ell}^\dagger + c_{\ell} \) and \( B_{\ell} = c_{\ell}^\dagger - c_{\ell} \). Following this procedure for the case of periodic driving it is straightforward to write \( t^{\alpha\beta}_{\ell,i}(n\tau) \) as
\[
\begin{align*}
t^{xx}_{\ell,m}(n\tau) &= \langle B_{\ell} A_{\ell+1} B_{\ell+1} \ldots A_{m-1} B_{m-1} A_{m} \rangle_{\rho(n\tau)} \\
t^{yy}_{\ell,m}(n\tau) &= \langle (-1)^{m-m'} (B_{\ell} A_{\ell+1} A_{\ell+1} \ldots B_{m-1} A_{m-1} B_{m}) \rangle_{\rho(n\tau)} \\
t^{zz}_{\ell,m}(n\tau) &= \langle A_{\ell} B_{\ell} A_{\ell} B_{\ell} \rangle_{\rho(n\tau)} \\
t^{xy}_{\ell,m}(n\tau) &= -i \langle B_{\ell} B_{\ell+1} A_{\ell+1} \ldots B_{m-1} A_{m-1} B_{m} \rangle_{\rho(n\tau)}. \tag{C13}
\end{align*}
\]
A further decomposition of the product of four fermionic operators of the form \( \langle A_{\ell} B_{\ell} A_{m} B_{m} \rangle \) via Wick’s theorem eventually allows one to work with terms formed by the product of only two fermionic operators. In order to calculate the two-site correlators, we define \( G_{\ell,m}(n\tau) = \langle B_{\ell} A_{m} \rangle_{\rho(n\tau)} \), \( G^\prime_{\ell,m}(n\tau) = \langle A_{\ell} B_{m} \rangle_{\rho(n\tau)} \), \( Q_{\ell,m}(n\tau) = \langle A_{\ell} A_{m} \rangle_{\rho(n\tau)} \), \( S_{\ell,m}(n\tau) = \langle B_{\ell} B_{m} \rangle_{\rho(n\tau)} \). Utilizing the definition of \( A_{\ell} \) and \( B_{\ell} \), we obtain
\[
G_{\ell,m}(n\tau) = \frac{1}{N} \sum_{k=1}^{N/2} (-2i \sin[\phi_k(m-\ell)]) \text{Tr}[X^k_{\ell,K} \rho_k(n\tau)] + \frac{1}{N} \sum_{k=1}^{N/2} (2 \cos(\phi_k(m-\ell))) \text{Tr}[m^z_{\ell,K} \rho_k(n\tau)], \tag{C14}
\]
\[
G^\prime_{\ell,m}(n\tau) = \frac{1}{N} \sum_{k=1}^{N/2} (-2i \sin[\phi_k(m-\ell)]) \text{Tr}[X^k_{\ell,K} \rho_k(n\tau)] - \frac{1}{N} \sum_{k=1}^{N/2} (2 \cos(\phi_k(m-\ell))) \text{Tr}[m^z_{\ell,K} \rho_k(n\tau)]. \tag{C15}
\]
\[
Q_{\ell,m}(n\tau) = \frac{1}{N} \sum_{k=1}^{N/2} (-2i \sin[\phi_k(m-\ell)]) \text{Tr}[X^k_{\ell,K} \rho_k(n\tau)] + \frac{1}{N} \sum_{k=1}^{N/2} 2 \cos(\phi_k(m-\ell)), \tag{C16}
\]
\[
S_{\ell,m}(n\tau) = \frac{1}{N} \sum_{k=1}^{N/2} (-2i \sin[\phi_k(m-\ell)]) \text{Tr}[X^k_{\ell,K} \rho_k(n\tau)] - \frac{1}{N} \sum_{k=1}^{N/2} 2 \cos(\phi_k(m-\ell)), \tag{C17}
\]
where \( X^k_{\ell,K} = c^\dagger_{\ell} c^{\dagger}_{\ell-k} - c_{\ell} c_{\ell-k}, \quad X^k_{\ell} = c^\dagger_{\ell} c^\dagger_{-\ell-k} + c_{\ell} c_{-\ell-k} - 1 \). Note that in the thermodynamic limit, \( N \to \infty \), the summations in Eq. (C14-C17) can be replaced by integrals:
\[
\frac{1}{N} \sum_{k=1}^{N/2} \to \frac{1}{2\pi} \int_{0}^{\pi} d\phi.
\]
From the expressions given in Eq. (C14-C17), single-site and two-site quantities, required for constructing the two-body density matrices (see Eq. (A1)), can be calculated. For example, the average magnetization \( m^z(\tau) \) can be obtained from \( G_{\ell,m}(n\tau) \) as \( m^z(\tau) = \frac{1}{N} \sum_{k=1}^{N/2} \langle X^k_{\ell,K} \rangle_{\rho_k(n\tau)} \). For the nearest neighbor sites \((\ell, \ell+1)\), the correlators are obtained as \( t^{xx}_{\ell,\ell+1}(n\tau) = G_{\ell,\ell+1}(n\tau); \quad t^{yy}_{\ell,\ell+1}(n\tau) = -G^\prime_{\ell,\ell+1}(n\tau); \quad t^{zz}_{\ell,\ell+1}(n\tau) = m^z_{\ell}(n\tau) + G^\prime_{\ell,\ell+1}(n\tau) G_{\ell,\ell+1}(n\tau) - Q_{\ell,\ell+1}(n\tau) S_{\ell,\ell+1}(n\tau) \). Once the correlators after \( n \) driving cycles are obtained, one can take the limit \( n \to \infty \) in order to obtain steady-state two-site density matrix \( \rho_{12}(\infty) \).

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