Ground States for Infrared Renormalized Translation-Invariant Non-Relativistic QED

David Hasler*    Oliver Siebert†

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We consider a translation-invariant Pauli-Fierz model describing a non-relativistic charged quantum mechanical particle interacting with the quantized electromagnetic field. The charged particle may be spinless or have spin one half. We decompose the Hamiltonian with respect to the total momentum into a direct integral of so called fiber Hamiltonians. We perform an infrared renormalization, in the sense of norm resolvent convergence, for each fiber Hamiltonian, which has the physical interpretation of removing an infinite photon cloud. We show that the renormalized fiber Hamiltonians have a ground state for almost all values for the total momentum with modulus less than one.

1. Introduction

The infrared problem is a technical obstacle in describing scattering of charged particles with a quantized radiation field, which has been thoroughly investigated in the literature over the last decades. As already pointed out in the seminal paper by Bloch and Nordsieck [BN37] it is an artifact of the emergence of a cloud of infinitely many photons with a finite amount of total energy in the asymptotic scattering states. While there appeared several perturbative methods like the one by Fadeev and Kulish [KF70] for the computation of divergence-free scattering transition probabilities, a mathematical precise theory was not available for a long time. A first rigorous description of so called dressed one-electron states, i.e., the ‘plane-wave’ states of an electron traveling with a

*Institut für Mathematik, Friedrich-Schiller-Universität Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany, david.hasler@uni-jena.de
†FB Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany, oliver.siebert@uni-tuebingen.de
cloud of photons at some fixed total momentum, was given by Kibble \cite{Kib68} based on ideas of Chung \cite{Chu65}. The basic principle behind it is, that due to the infinite number of photons, the states cease to exist in the original Hilbert space of the Fock representation. Instead, one has to use non-Fock coherent representations of the canonical commutation relations (CCR). A significant problem is that different total momenta lead to different inequivalent representations, making it hard to construct scattering states, cf. \cite{Fro73} and for recent work on this subject \cite{CD19}.

As a first step towards a more realistic model Fröhlich discussed so called dressed one-electron states in the Nelson model \cite{Fro73,Fro74}, where a charged particle like an electron interacts with a bosonic field via a coupling term which is linear in the creation and annihilation operators. He shows that it is impossible to implement one-electron states for different total momenta in a joint Hilbert space, and he gave a non-constructive proof for the existence of the corresponding coherent states as a weak-* limit where the infrared regularization is removed. Later Pizzo accomplished an explicit perturbative construction of such states in a Hilbert space by means of a dressing transformation and an iterative algorithm \cite{Piz03}, which he then used for the construction of asymptotic scattering states \cite{Piz03}. Related results for the weak coupling model of non-relativistic QED, which is technically more involved due to the interaction being quadratic in the creation and annihilation operators were obtained in \cite{CF07,CFP10,CFP09,Bac+07,Che08}. Furthermore, in the last years there appeared a series of papers for Coulomb scattering in the Nelson model between two electrons starting with \cite{DP14} and between an electron and atom \cite{DP22} in the presence of a quantized radiation field. Recently, there was also a novel approach developed for the construction of scattering states in the Nelson model without any infrared limiting procedure \cite{BDG21}.

Due to its significance for scattering theory, a lot of progress was made to investigate such one-electron states without infrared regularization in different models, e.g., apart from the Nelson model and non-relativistic QED also for the UV-renormalized Nelson model \cite{BDP12} and the semi-relativistic Pauli-Fierz model \cite{KM14}. Those states can be described mathematically as ground states of a (renormalized) fiber Hamiltonian corresponding to fixed total momentum.

In this paper we show the existence of one-electron states in the non-relativistic Pauli-Fierz model for almost all momenta with modulus less than one and for all values of the coupling constant. The charged particle may be spinless or have spin one half, where in the latter case have to assume an energy inequality, cf. (2.12). Moreover, we show that they are ground states of fiber-wise renormalized Hamiltonians. A similar result has been obtained for the UV renormalized translation invariant Nelson model in \cite{BDP12} following the strategy developed in \cite{Piz03}. Moreover, we want to mention a related result for the Nelson model in \cite{Ara01}. We use a compactness argument, originally tracing back to \cite{GLL01}, in the same way as in \cite{HS20,HHS21}, where we showed the existence of ground states at zero momentum. The proof is non-constructive but also non-perturbative. The infrared renormalization is done via a Gross transformation similarly as performed in \cite{Nel64,Piz03,Piz05} for the Nelson model or for non-relativistic QED in \cite{CF07}.

The main results are presented in Section 3. In Section 4 we outline the strategy
of the proof. The renormalization of the fiber Hamiltonians is performed in Section 5. Key elements of the proof of the main result, stated in Theorem 3.5, are an argument based on second order perturbation theory in the total momentum and a photon number bound together with a bound on derivatives of the photon momenta, which can be found in Section 6 and Section 7 respectively. Finally, Section 8 contains the proof of the main result, where the compactness argument is presented and previous estimates are used to establish indeed compactness.

2. Model

Let $\mathfrak{h}$ be a complex Hilbert space. We define the symmetric Fock space

$$\mathcal{F}(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \mathfrak{h}^{(n)},$$

with $\mathfrak{h}^{(0)} := \mathbb{C}$ and $\mathfrak{h}^{(n)} := \mathcal{P}_n(\bigotimes_{k=1}^{n} \mathfrak{h})$, $n \in \mathbb{N}$, with $\mathcal{P}_n$ denoting the orthogonal projection onto the subspace of totally symmetric tensors. Thus we can identify $\psi \in \mathcal{F}(\mathfrak{h})$ with the sequence $(\psi_n)_{n \in \mathbb{N}_0}$ with $\psi_n \in \mathfrak{h}^{(n)}$. The vacuum is the vector $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}(\mathfrak{h})$. For a linear subspace $\mathfrak{v}$ of the Hilbert space $\mathfrak{h}$ we define $\mathcal{F}_{\mathfrak{lin}}(\mathfrak{v}) \subset \mathcal{F}(\mathfrak{h})$ as the vector space of finite linear combinations of $\Omega$ and vectors of the form $\mathcal{P}_n(v_1 \otimes \cdots \otimes v_n)$ with $v_1, \ldots, v_n \in \mathfrak{v}$ and $n \in \mathbb{N}$.

We define for $f \in \mathfrak{h}$ the creation operator $a^*(f)$ acting on vectors $\psi \in \mathcal{F}(\mathfrak{h})$ by

$$(a^*(f)\psi)_{(n)} = \sqrt{n}\mathcal{P}_n(f \otimes \psi_{(n-1)})$$

with domain $D(a^*(f)) := \{\psi \in \mathcal{F}(\mathfrak{h}) : a^*(f)\psi \in \mathcal{F}(\mathfrak{h})\}$. This yields a densely defined closed operator. For $f \in \mathfrak{h}$ we define the annihilation $a(f)$ as the adjoint of $a^*(f)$, i.e.,

$$a(f) = [a^*(f)]^*.$$

It follows from the definition that $a(f)$ is anti-linear, and $a^*(f)$ is linear in $f$. Creation and annihilation operators are well known to satisfy the so called canonical commutation relations

$$[a^*(f), a^*(g)] = 0 \quad , \quad [a(f), a(g)] = 0 \quad , \quad [a(f), a^*(g)] = \langle f, g \rangle_{\mathfrak{h}}, \quad (2.1)$$

where $f, g \in \mathfrak{h}$, $[\cdot, \cdot]$ stands for the commutator, and $\langle f, g \rangle_{\mathfrak{h}}$ denotes the inner product of $\mathfrak{h}$. For $f \in \mathfrak{h}$ we introduce the following notation for the field operator and the conjugate field operator

$$\phi(f) = \text{closure of} \quad \frac{1}{\sqrt{2}}(a(f) + a^*(f)), \quad \pi(f) = \text{closure of} \quad \frac{-i}{\sqrt{2}}(a(f) - a^*(f)).$$

We note that $\pi(f) = \phi(if)$. 

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For a self-adjoint operator $A$ in $\mathfrak{h}$ we define the operator $d\Gamma(A)$ as follows. In $\mathfrak{h}^{(n)}$ we set

$$A^{(n)} := A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes A, \quad n \in \mathbb{N},$$

in the sense of [RS81, VIII.10] and $A^{(0)} := 0$. By definition $\psi \in \mathcal{F}(\mathfrak{h})$ is in the domain of $d\Gamma(A)$ if $\psi(n) \in D(A^{(n)})$ for all $n \in \mathbb{N}_0$ and

$$(d\Gamma(A)\psi)(n) = A^{(n)}\psi(n), \quad n \in \mathbb{N}_0,$$

is a vector in $\mathcal{F}(\mathfrak{h})$, in which case $d\Gamma(A)\psi$ is defined by (2.2). The operator $d\Gamma(A)$ is self-adjoint, see for example [RS81, VIII.10].

Henceforth, we shall consider specifically

$$\mathfrak{h} := L^2(\mathbb{Z}_2 \times \mathbb{R}^3) \cong L^2(\mathbb{R}^3; \mathbb{C}^2)$$

and write $\mathcal{F}$ for $\mathcal{F}(\mathfrak{h})$. The Hilbert space $\mathfrak{h}$ describes so called transversally polarized photons. By physical interpretation the variable $(\lambda, k) \in \mathbb{Z}_2 \times \mathbb{R}^3$ consists of the wave vector $k$ and the polarization label $\lambda$. Because of (2.3), the elements $\psi \in \mathcal{F}_0$ can be identified with sequences $(\psi(n))_{n=0}^{\infty}$ of so called $n$-photon wave functions, $\psi(n) \in L^2_{\text{sym}}((\mathbb{Z}_2 \times \mathbb{R}^3)^n)$, where the subscript “sym” stands for the subspace of functions which are totally symmetric in their $n$ arguments. Henceforth, we shall make use of this identification without mention. The Fock space inherits a scalar product from $\mathfrak{h}$, explicitly

$$\langle \psi, \varphi \rangle = \overline{\psi(0)}\varphi(0) + \sum_{n=1}^{\infty} \sum_{\lambda_1, \ldots, \lambda_n \in \{1, 2\}} \int \psi(n)(\lambda_1, k_1, \ldots, \lambda_n, k_n)\varphi(n)(\lambda_1, k_1, \ldots, \lambda_n, k_n)dk_1 \cdots dk_n.$$

For $m \geq 0$ the field energy operator denoted by $H_{f,m}$ is defined by

$$H_{f,m} = d\Gamma(\omega_m),$$

where $\omega_m: \mathbb{Z}_2 \times \mathbb{R}^3 \to \mathbb{R}, \quad \omega_m(\lambda, k) := \omega_m(k) := \sqrt{m^2 + k^2}$. The operator of momentum $P_i$ is defined as a three dimensional vector of operators, where the $j$-th component is given by

$$(P_i)_j := d\Gamma(K_j),$$

with $K_j: \mathbb{Z}_2 \times \mathbb{R}^3 \to \mathbb{R}, \quad K_j(\lambda, k) = k_j$.

The Hilbert space describing the system composed of a charged particle with spin $s \in \{0, \frac{1}{2}\}$ and the quantized field is

$$\mathcal{H}_{\text{full}} := L^2(\mathbb{R}^3 \times \mathbb{Z}_{2s+1}) \otimes \mathcal{F}.$$  

We consider the Hamiltonian

$$H_m = \frac{1}{2} (p + eA(x))^2 + eS \cdot B(x) + H_{f,m},$$

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\[ A_j(x) = \phi(f_{A,j}E_x), \quad B_j(x) = \phi(f_{B,j}E_x), \quad j = 1, 2, 3, \quad (2.5) \]

where we defined the following functions \( E_y : \mathbb{R}^3 \to \mathbb{R}^3 \) by \( k \mapsto e^{-ik \cdot y} \) for \( y \in \mathbb{R}^3 \),

\[ f_{A,j}(k, \lambda) := \frac{\rho(k)}{\sqrt{|k|}} \varepsilon_{\lambda}(k), \quad f_{B,j}(k, \lambda) := -i[k \wedge f_{A,j}(k, \lambda)]_j, \quad k \in \mathbb{R}^3, \]

where the \( \varepsilon_{\lambda}(k) \in \mathbb{R}^3 \) are so called polarization vectors, depending measurably on \( \hat{k} = k/|k| \), such that \((\hat{k}, \varepsilon_1(k), \varepsilon_2(k))\) forms an orthonormal basis. For the proof we shall make an explicit choice of the polarization vectors in (7.21), below. We note that the integral \( \int f \) denotes the operator of multiplication with the position coordinates of the first component of the tensor product \( (2.4) \). Mathematically this amounts to the following definition. By means of the unitary isomorphism \( \mathcal{H}_{\text{full}} \cong L^2(\mathbb{R}^3 \times \mathbb{Z}_{2s+1}; \mathcal{F}) \) one has the identity

\[ [(A_j(x)\psi)(x, s) = \phi(f_{A,j}E_x)\psi(x, s), \quad (x, s) \in \mathbb{R}^3 \times \mathbb{Z}_{2s+1}, \quad (2.6) \]

for \( \psi \in L^2(\mathbb{R}^3 \times \mathbb{Z}_{2s+1}; \mathcal{F}) \) such that \( \psi(x, s) \in D(\phi(f_{A,j}E_x)) \) for all \((x, s) \in \mathbb{R}^3 \times \mathbb{Z}_{2s+1} \) and \( (2.6) \) defines again an element in \( L^2(\mathbb{R}^3 \times \mathbb{Z}_{2s+1}; \mathcal{F}) \). Likewise we define \( B_j(x) \). In the physics literature the operators \( (2.5) \) are often written in terms of so called operator valued distributions. This is outlined in the following remark.

**Remark 2.1.** Introducing operator valued distributions \( a_\lambda(k) \) and \( a^*_\lambda(k) \) satisfying the so called canonical commutation relations, [RS75, X.7], one can write

\[ A(x) = \sum_{\lambda=1,2} \int \frac{\varepsilon_{\lambda}(k)}{2|k|} \left( \rho(k)a_\lambda(k)e^{ik \cdot x} + \rho(k)a^*_\lambda(k)e^{-ik \cdot x} \right) dk, \quad (2.7) \]

\[ B(x) = \sum_{\lambda=1,2} \int \frac{ik \wedge \varepsilon_{\lambda}(k)}{2|k|} \left( \rho(k)a_\lambda(k)e^{ik \cdot x} - \rho(k)a^*_\lambda(k)e^{-ik \cdot x} \right) dk, \quad (2.8) \]

where the integrals are understood as weak integrals on a suitable dense subspace.

We shall adopt the standard convention that for \( \nu = (v_1, v_2, v_3) \) we write \( \nu^2 := \sum_{j=1}^3 v_j v_j. \) By \( x \) we denote the position of the electron and its canonically conjugate momentum by \( p = -i\nabla_x. \) If \( s = 1/2 \), let \( S = (\sigma_1, \sigma_2, \sigma_3) \) denote the vector of Pauli-matrices. If \( s = 0 \), let \( S = 0. \) The number \( e \in \mathbb{R} \) is called the coupling constant. The so called form factor \( \rho : \mathbb{R}^3 \to \mathbb{C} \) is a measurable function for which we shall assume the following hypothesis for the main theorem. We assume that for some \( \Lambda \in (0, \infty) \) we have

\[ \rho(k) = \frac{1}{(2\pi)^{3/2}} \mathbf{1}_{[0,\Lambda]}(|k|), \quad k \in \mathbb{R}^3. \quad (2.9) \]

where \( \mathbf{1}_{[0,\Lambda]} \) denotes the characteristic function of the set \([0, \Lambda] \). One can show that \( H_m \) is self-adjoint on \( D(\Delta \otimes 1) \cap D(1 \otimes H_{\ell,m}) \), cf. [HH08b, Hir02].
The Hamiltonian is translation invariant and commutes with the generator of translations, i.e., the operator of total momentum

\[ P_{\text{tot}} = p + P_t . \]

Let

\[ W = \exp(\text{i}x \cdot P_t) . \]

Note \( WP_{\text{tot}}W^* = p \) so that in the new representation \( p \) is the total momentum. One easily computes

\[ WH_m W^* = \frac{1}{2} (p - P_t + eA)^2 + eS \cdot B + \mathcal{H}_{f,m} , \]

where we set \( A := A(0) \) and \( B := B(0) \). Let \( F \) be the Fourier transform in the electron variable \( x \), i.e., on \( L^2(\mathbb{R}^3) \),

\[ (F\psi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} \psi(x) \, dx . \] (2.10)

Then the composition \( U = FW \) is a unitary operator

\[ U : \mathcal{H}_{\text{full}} \to L^2(\mathbb{R}^3 \times \mathbb{Z}_{2s+1}) \otimes \mathcal{F} \cong L^2(\mathbb{R}^3; C^{2s+1} \otimes \mathcal{F}) = \int_{\mathbb{R}^3} C^{2s+1} \otimes \mathcal{F} d\xi , \]

yielding the so called fiber decomposition of the Hamiltonian,

\[ U H_m U^* = \int_{\mathbb{R}^3} H_m(\xi) d\xi , \]

where

\[ H_m(\xi) = \frac{1}{2} (\xi - P_t + eA)^2 + eS \cdot B + \mathcal{H}_{f,m} \] (2.11)

is an operator in the so called reduced Hilbert space

\[ \mathcal{H} := C^{2s+1} \otimes \mathcal{F} , \]

cf. [RS78, Spe04]. The following result shows that the operator (2.11) is well defined, cf. [Hir07, HL08, LMS07] and [HS20].

**Theorem 2.2.** For all \( m \geq 0 \), \( \xi \in \mathbb{R}^3 \), and \( e \in \mathbb{R} \) the operator \( H_m(\xi) \) is bounded from below and self-adjoint on the natural domain of \( P_t^2 + \mathcal{H}_{f,m} \).

We call

\[ E_m(\xi) := \inf \sigma(H_m(\xi)). \]

the ground state energy, which in general does not need to be an eigenvalue. Let us collect a few elementary properties of the ground state energy as a function of \( \xi \).

**Lemma 2.3.** The following holds.

(i) The function \( t_m : \xi \mapsto \frac{e^2}{2} - E_m(\xi) \) is convex.
(ii) The function $\xi \mapsto E_m(\xi)$ is almost everywhere differentiable.

(iii) The function $\xi \mapsto E_m(\xi)$ is rotationally invariant.

(iv) If $E_m$ is differentiable in $\xi$ we have $|\nabla E_m(\xi)| \leq |\xi|$

Proof. (i) This follows since the supremum of convex functions is convex. (ii) Convex functions are locally Lipschitz, see for example Lemma C.2 in the Appendix. Lipschitz functions are almost everywhere differentiable by Rademacher’s theorem, see for example Theorem C.4 in the Appendix. So (ii) follows from (i). (iii) Follows from well known transformation properties of the field operators and the rotation invariance of $\rho$. (iv) By restricting the function $t_m$ onto a straight line through the origin the claim follows from Lemma C.3 in the appendix, where the symmetry assumption follows from (iii). \qed

Henceforth, we shall write $H_f$, $H$, $H(\xi)$, and $E(\xi)$ for $H(0)$, $H_0(\xi)$, and $E_0(\xi)$, respectively. For massless photons, which corresponds to the case $m = 0$, the fiber Hamiltonian does not have a ground state for momenta $\xi$ for which $\nabla E(\xi) \neq 0$. This is the content of the following theorem shown in [HH08a].

**Theorem 2.4.** Let $e \neq 0$. If $E(\cdot)$ is differentiable at $\xi$ and has a nonzero derivative, then $H(\xi)$ does not have a ground state.

The physical interpretation of Theorem 2.4 is that charged particles with nonzero velocity $\nabla E(\xi) \neq 0$, acquire an infinite photon cloud which ceases to be square integrable. This will be made mathematically more precise below and can be viewed as a manifestation of the so called infrared catastrophe. On the other hand, if one considers the case with zero momentum $\xi = 0$ one can show in fact that the fiber Hamiltonian has a ground state. This has been established for small values of the coupling constant in [Che08] and recently [HS20], for all values of the coupling constant under an energy inequality assumption (2.12), which is discussed below.

To control the infrared divergence for $\xi \neq 0$ one typically introduces an infrared regularization. One way to achieve this, is to introduce a positive photon mass, i.e., one considers $H_m(\xi)$ for $m > 0$. One can shown that the operator $H_m(\xi)$ has a ground state for all $m > 0$ and $|\xi|$ sufficiently small. Such a result has been obtained in various situations [Frö74, Frö73, Spo04, GLL01, LMS07]. In the following we will work with the result from [HS20], since it suits our specific situation. To formulate it, we will use the following energy inequality, which reads as follows. We say that the energy inequality holds for $e \in \mathbb{R}$ and $m \geq 0$ if

$$E_m(\xi) \geq E_m(0), \quad \forall \xi \in \mathbb{R}^3.$$  \hspace{1cm} (2.12)

This inequality has been intensively investigated in the literature. In the spinless case $s = 0$ it has been shown to hold for all values $m \geq 0$ and $e \in \mathbb{R}$ using functional integration, [Gro72, Spo04, Hir07, LMS07]. This is the content of the following theorem.

**Theorem 2.5.** In the spin-less case, $s = 0$, (2.12) holds for all $e \in \mathbb{R}$ and $m \geq 0$. 

In case of spin one half it has been shown to hold, but only in a limited range of parameters. That is, for \( s = 1/2 \) Inequality (2.12) has to the best of our knowledge not yet been shown by means of functional integration. For \( s = 1/2 \) Inequality (2.12) follows for small \(|e|\) from the main theorem stated in [Che08], which in turn is based on perturbative arguments. Now let us state the result about existence of ground states for positive photon mass from [HS20], which will be used in this paper, see also [Frö73; Frö74; Spo04; GLL01; LMS07].

**Theorem 2.6.** Let \( e \in \mathbb{R} \) and \( m > 0 \) and suppose the energy inequality (2.12) holds. If \(|\xi| \leq 1\), then \( E_m(\xi) \) is an eigenvalue of \( H_m(\xi) \) isolated from the essential spectrum.

Henceforth, we let \( \psi_m(\xi) \) denote a normalized eigenvector of \( H_m(\xi) \) with eigenvalue \( E_m(\xi) \), whose existence is granted by Theorem 2.6. In Proposition 4.1, in Section 4, we will show that \( E_m(\xi) \) is monotonically decreasing as \( m \downarrow 0 \) and

\[
E_m(\xi) \to E(\xi). \tag{2.13}
\]

As the positive mass is removed, the ground state \( \psi_m(\xi) \) acquires an infinite photon cloud which ceases to be square integrable and hence does not converge in the Hilbert space. However, if one removes this diverging photon cloud by a Bogoliubov transformation, one can show that the transformed ground states converge to a nonzero vector in Hilbert space. Such a picture has been established in various situations [Frö73; Frö74; Piz05; CF07; CFP09]. To describe this in more detail we define for \( m \geq 0 \) the function

\[
h_{m,\xi}(\lambda, k) = e \varepsilon(\lambda(k)) \cdot \nabla_\xi E_m(\xi) \frac{\rho(k)}{\sqrt{|k|} \omega_m(k)} \frac{1}{\omega_m(k) - k \cdot \nabla_\xi E_m(\xi)}, \tag{2.14}
\]

where \((\lambda, k) \in \{1, 2\} \times \mathbb{R}^3\). In Remark 2.7 we sketch a heuristic argument for (2.14).

**Remark 2.7.** Following [Piz03, Section 2] and references therein we give a heuristic and formal argument for the choice (2.14). Assuming that \( \psi_m(\xi) \) should be “coherent in the infrared region” we make the following “Ansatz” as \( k \to 0 \)

\[
a_\lambda(k) \psi_m(\xi) \approx -g_{m,\xi}(\lambda, k) \psi_m(\xi) \tag{2.15}
\]

for some function \( g_{m,\xi} \) with values in the linear maps of \( \mathbb{C}^{2s+1} \). By a formal application of the virial theorem we have

\[
\langle \psi_m(\xi), [H_m(\xi), a_\lambda(k)] \psi_m(\xi) \rangle = 0. \tag{2.16}
\]

Calculating the formal commutator \([H_m(\xi), a_\lambda(k)]\) by means of the so called pull-through formula, c.f. [BFS98, Lemma A.1], inserting the ansatz (2.15) into (2.16), using the Feynman-Hellmann formula, c.f. Lemma 6.1, and solving for \( g_{m,\xi} \) we find

\[
g_{m,\xi}(\lambda, k) \approx e (\nabla_\xi E_m(\xi) \cdot f_A(\lambda, k) + S \cdot f_B(\lambda, k)) \frac{1}{\omega_m(k) - k \cdot \nabla_\xi E_m(\xi) + \frac{1}{2} k^2}
\]

as \( k \to 0 \). Dropping the higher order terms involving \( k^2 \) and \( f_B \) we arrive at (2.14).
From Lemma 2.3 we see that the derivative of $E_m$ exists almost everywhere. Now if the derivative exists, then $h_{m,\xi} \in \mathfrak{h}$ whenever $m > 0$ and $|\nabla E_m(\xi)| < 1$, where the latter inequality always holds if $|\xi| < 1$ by Lemma 2.3. However in case $m = 0$ we have $h_{0,\xi} \notin \mathfrak{h}$. Whenever $h_{m,\xi} \in \mathfrak{h}$, we can define

$$U_m(\xi) = e^{i\pi(h_{m,\xi})}. \quad (2.17)$$

Working with $U_m(\xi)$ will require control of the derivatives of the ground state energies. In Proposition 4.2 at the end of Section 4 we show that for any sequence $(m_j)_{j \in \mathbb{N}}$ of nonnegative numbers which tend to zero, we have

$$\nabla E_{m_j}(\xi) \xrightarrow{j \to \infty} \nabla E(\xi) \quad (2.18)$$

for almost all $\xi \in \mathbb{R}^3$. This will allow us to study strong limits of $U_m(\xi)$ as $m \downarrow 0$.

3. Statement of Main Results

As a first result we will show in Section 8 the following theorem, which describes explicitly the removal of the divergent photon cloud of the ground state as the positive photon mass tends to zero.

**Theorem 3.1.** Let $e \in \mathbb{R}$ and suppose there exists an $m_0 > 0$ such that the energy inequality (2.12) holds for all $m \in (0, m_0)$. Then for almost all $\xi \in \mathbb{R}^3$ with $|\xi| < 1$, there exists a sequence $(m_j)_{j \in \mathbb{N}}$ of positive numbers converging to zero such that $U_{m_j}(\xi)\psi_{m_j}(\xi)$ converges to a nonzero vector, $\tilde{\psi}_0(\xi)$, in the Hilbert space $\mathcal{H}$.

**Remark 3.2.** We note that Theorem 3.1 is similar to a result obtained in [CFP09], where convergence is shown for a spinless charged particle. The proof given in that paper is constructive, and holds for all $\xi$ with $|\xi| < 1/3$ and small values of $|e|$. Theorem 3.1 extends that result for almost all $\xi$ with $|\xi| < 1$ to all values of the coupling constant $e \in \mathbb{R}$, see also Remark 3.6.

Next we address the question whether the nonzero vector $\tilde{\psi}_0(\xi)$ given in Theorem 3.1 can be expressed as the ground state of an infrared renormalized Hamiltonian. The main result of this paper is, that this is indeed the case. For this we will show in Theorem 3.4 below, that as $m_j$ tends to zero the operators

$$U_{m_j}(\xi)H(\xi)U_{m_j}(\xi)^\dagger \quad (3.1)$$

coverge in resolvent sense for almost all $\xi$ with $|\xi| < 1$, to a self-adjoint operator, which will be given explicitly in (3.3) below, and which we will refer to as the infrared renormalized fiber Hamiltonian. To capture this limit in explicit terms we introduce first for measurable functions $f, g: \mathbb{Z}_2 \times \mathbb{R}^3 \to \mathbb{C}$ with $f \in L^1(\mathbb{Z}_2 \times \mathbb{R}^3)$ the sesquilinear form

$$s(f, g) = \frac{1}{2} \sum_{\lambda=1}^2 \int \mathcal{F}(\lambda, k)g(\lambda, k)dk.$$
Second for a measurable function \( f : \mathbb{Z}_2 \times \mathbb{R}^3 \to \mathbb{R} \) we define the following formal operator
\[
T_m(f; \xi) := \frac{1}{2} \sum_{j=1}^{3} (\xi_j - P_{f,j} + e A_j - \phi(K_j f) - 2e \text{ Re } s(f, f_{A,j}) - s(f, K_j f))^2 \\
+ e \sum_{j=1}^{3} S_j (B_j - 2 \text{ Re } s(f, f_{B,j})) + H_{f,m} + \phi(\omega_m f) + s(f, \omega_m f). \tag{3.2}
\]

We will show in in Section 5, Lemma 5.1, that this operator is \( P_l^2 + H_{f,m} \) bounded, provided the function \( f \) satisfies certain properties. Specifically, using the next lemma, following from Proposition 5.5 in Section 5, we can give an explicit definition of the infrared renormalized fiber Hamiltonian.

**Lemma 3.3.** Let \( E \) be differentiable in \( \xi \) and suppose \( |\nabla_\xi E(\xi)| < 1 \). Then
\[
\hat{H}(\xi) := T_0(h_{0,\xi}; \xi) \tag{3.3}
\]
can be realized as a selfadjoint operator with domain \( D(P_l^2 + H_l) \).

We now state the first main result of this paper, which follows from Proposition 5.5 in Section 5 as well.

**Theorem 3.4.** Let \( E \) be differentiable in \( \xi \) and suppose \( |\nabla_\xi E(\xi)| < 1 \). Let \( (m_j)_{j \in \mathbb{N}} \) be a sequence of positive numbers converging to zero and \( \nabla E_{m_j}(\xi) \to \nabla E(\xi) \). Then the operators
\[
U_{m_j}(\xi) H(\xi) U_{m_j}(\xi)^*,
\]
converge as \( j \to \infty \) in norm resolvent sense to \( \hat{H}(\xi) \).

We note that in view of Lemma 2.3 and (2.18) the assumptions of Theorem 3.4 are valid for almost all \( \xi \in \mathbb{R}^3 \) with \( |\xi| < 1 \). We now state the second main result of this paper, which will be shown in Section 8 and which covers charged particles without spin and charged particles with spin one half.

**Theorem 3.5.** Let \( e \in \mathbb{R} \). Suppose that one of the following two assumptions is satisfied:

(i) \( s = 0 \),

(ii) \( s = 1/2 \) and there exists an \( m_0 > 0 \) such that the energy inequality (2.12) holds for all \( m \in (0, m_0) \).

Then for almost all \( \xi \in \mathbb{R}^3 \) with \( |\xi| < 1 \) the following holds.

(a) The function \( E \) is differentiable at \( \xi \), \( |\nabla E(\xi)| < 1 \), and the operator \( \hat{H}(\xi) \) has a ground state, i.e., \( E(\xi) \) is an eigenvalue of \( \hat{H}(\xi) \).

(b) There exists a sequence \( (m_j)_{j \in \mathbb{N}} \) of positive numbers converging to zero such that \( U_{m_j}(\xi) \psi_{m_j}(\xi) \) converges to the ground state.
The fact that the assertion of Theorem 3.5 only holds for almost all $\xi$ with length less than one, does not affect the fiber direct integral, since sets of measure zero do not play a role in integration.

**Remark 3.6.** We will determine in Theorem 8.1, below, the set of $\xi$’s, for which the assertions of Theorem 3.1 as well as Theorem 3.5 can be shown, more precisely. That set only depends on regularity properties of the energies $E_m$. We note that regularity properties for such models have been studied using various methods. In this regard we want to mention well established methods from renormalization, iterated perturbation, or statistical mechanics, cf. [Che08; Bac+07; Piz03; AH12].

### 4. Outline of the Proofs

Theorem 3.4 will be shown in Section 5 using well established estimates involving creation and annihilation operators. To prove the existence of a ground state for the renormalized Hamiltonian $\hat{H}(\xi)$, i.e., Theorem 3.5, we follow closely ideas given in [GLL01] and [LMS07] combined with a regularization procedure used in [Fro73; Fro74; Piz05; CFP09].

The basic idea of the proof is to regularize the Hamiltonian by adding a positive mass to the photons and to study ground state properties when this regularization is removed. That is, we consider the Hamiltonian $H_m(\xi)$ for $m > 0$. Recall that the operator $H_m(\xi)$ has a normalized ground state for all $m > 0$ and $|\xi|$ sufficiently small, see Theorem 2.6 which we denote by $\psi_m(\xi)$. In Proposition 4.1 we will show that $E_m(\xi)$ is monotonically decreasing as $m \downarrow 0$ and

$$E_m(\xi) \to E(\xi).$$

The key idea for proving the existence of a ground state is to show that $U_m(\xi)\psi_m(\xi)$ has a convergent subsequence. This is achieved by showing that the sequence of these

Using (4.1) we will show in Proposition 5.5 in Section 5 that

$$U_m(\xi)H(\xi)U_m(\xi)^* \to \hat{H}(\xi)$$

in norm resolvent sense for almost all $\xi$ with $|\xi| < 1$. We then show in Proposition 5.6 using (2.13) and standard estimates involving creation and annihilation operators that $U_{m_j}(\xi)\psi_{m_j}(\xi)$ is a minimizing sequence for $\hat{H}(\xi)$, i.e.,

$$0 \leq \langle U_{m_j}(\xi)\psi_{m_j}(\xi), (\hat{H}(\xi) - E(\xi))U_{m_j}(\xi)\psi_{m_j}(\xi) \rangle \to 0.$$ 

(4.2)

The key idea for proving the existence of a ground state is to show that $U_{m_j}(\xi)\psi_{m_j}(\xi)$ has a convergent subsequence. This is achieved by showing that the sequence of these
vectors lies in a compact subset of the Hilbert space. To this end, we make use of the
so called pull-through formula, which is shown in Lemma 7.2,

\[
a_\lambda(k)\psi_m(\xi) = \frac{e\rho(k)}{\sqrt{2|k|}} R_m,\xi(k) (-\varepsilon_\lambda(k) \cdot v(\xi) + S \cdot (ik \wedge \varepsilon_\lambda(k))) \psi_m(\xi),
\]

where we defined

\[
R_m,\xi(k) := (H_m(\xi - k) + \omega_m(k) - E_m(\xi))^{-1},
\]

\[
v(\xi) := \xi - P_f + eA.
\]

To estimate the resolvent occurring on the right hand side of (4.3), we will relate it to
the second order derivative of the ground state energy as a function of \(\xi\). In Proposition
4.2 estimates on the second order derivatives of the ground state energy are established
using convexity properties, where it is shown that for almost all \(\xi\) the second order
derivatives \(\partial^2 E_{m,j}(\xi)\) exist and

\[
\lim \inf_j (\partial^2 E_{m,j}(\xi)) < \infty
\]

for every \(l = 1, 2, 3\). In Section 6 it will be shown that these bounds on the second order
derivative of the energy yield bounds on the resolvent. Using these resolvent bounds in
the pull-through formula (4.3) we will derive in Propositions 7.4 and 7.5 of Section 7
estimates on

\[
a_\lambda(k)U_{m,j}(\xi)\psi_{m,j}(\xi) \text{ and } \nabla_k a_\lambda(k)U_{m,j}(\xi)\psi_{m,j}(\xi).
\]

With the help of these estimates we will prove a fractional derivative bound in Lemma
7.6. In Section 8 we use this fractional derivative bound to show that the vectors
\(U_{m,j}(\xi)\psi_{m,j}(\xi), j \in \mathbb{N}\), lie indeed in a compact set. Thus by compactness there exists
a strongly convergent subsequence which converges to a nonzero vector, say \(\hat{\psi}_0(\xi)\). We
recall that this is the content of Theorem 3.1, which will be proven in Section 8.

Using lower semicontinuity of nonnegative quadratic forms [Sim78] (or alternatively
the spectral theorem and Fatou’s Lemma), we will conclude from (4.2) and Theorem 3.1
that for almost all \(\xi\) with \(|\xi| < 1\)

\[
0 \leq \langle \hat{\psi}_0(\xi), (\hat{H}(\xi) - E(\xi))\hat{\psi}_0(\xi) \rangle \\
\leq \lim \inf_j \langle U_{m,j}(\xi)\psi_{m,j}(\xi), (\hat{H}(\xi) - E(\xi))U_{m,j}(\xi)\psi_{m,j} \rangle = 0,
\]

i.e., that \(\hat{\psi}_0(\xi)\) is a ground state of \(\hat{H}(\xi)\). As outlined in Section 8 this will then establish
Theorem 3.5.

In the remaining part of this section we derive further properties of the ground state
energies \(E_{m}(\xi)\).

**Proposition 4.1.** Let \(\xi \in \mathbb{R}^3\). Whenever \(m_1 \geq m_2 \geq 0\) we have \(E_{m_1}(\xi) \geq E_{m_2}(\xi) \geq E(\xi)\), and

\[
E(\xi) = \lim_{m \downarrow 0} E_m(\xi).
\]

(4.6)
Proof. For $0 \leq m_2 \leq m_1$ we have $\omega \leq \omega_{m_2} \leq \omega_{m_1}$ and hence $H(\xi) \leq H_{m_2}(\xi) \leq H_{m_1}(\xi)$. This implies the first statement. Moreover, it implies the existence of the limit

$$E(\xi) \leq \lim_{m \downarrow 0} E_m(\xi). \quad (4.7)$$

To show the opposite inequality we argue as follows. From Theorem 2.2 it follows that any core for $P^2_t + H^t$ is a core for $H(0)$. Thus for any $\varepsilon > 0$, there exists a vector $\phi \in D(N) \cap D(P^2_t + H^t)$ such that

$$\langle \phi, H(\xi) \phi \rangle \leq E(\xi) + \varepsilon.$$

On the other hand, since $H_m(\xi) \leq H(\xi) + m d \Gamma(1)$, it follows that for any $m$,

$$E_m(\xi) \leq \langle \phi, H_m(\xi) \phi \rangle \leq \langle \phi, H(\xi) \phi \rangle + m \langle \phi, d\Gamma(1) \phi \rangle \leq E(\xi) + \varepsilon + m \langle \phi, d\Gamma(1) \phi \rangle.$$

Hence

$$\lim_{m \downarrow 0} E_m(\xi) \leq E(\xi) + \varepsilon. \quad (4.8)$$

Since $\varepsilon > 0$ is arbitrary, (4.6) follows from (4.7) and (4.8).

Proposition 4.2. Let $(m_j)_{j \in \mathbb{N}}$ be a sequence of nonnegative numbers which converges to zero. Then there exists a set $D$ such that $\mathbb{R}^3 \setminus D$ has Lebesgue measure zero and the following holds. The functions $E_{m_j}$, $j \in \mathbb{N}$, and $E$ are differentiable at all points in $D$, and for all $\xi \in D$ we have

(a) $\nabla E_{m_j}(\xi) \xrightarrow{j \to \infty} \nabla E(\xi)$,

(b) the second partial derivatives $\partial^2_l E_{m_j}(\xi)$ exist and satisfy for every $l = 1, 2, 3$ that

$$\lim \inf_j (-\partial^2_l E_{m_j}(\xi)) < \infty.$$

Proof. We note that $t_m : \xi \mapsto \frac{1}{2} \xi^2 - E_m(\xi)$ are convex functions by Lemma 2.3. Together with Proposition 4.1 we see that (a) and (b) follow from Lemmas C.6 and C.7 in the appendix, respectively.

5. Transformation of the Hamiltonian

We will now look at unitary transformations of the form

$$e^{i\pi f} H_m(\xi) e^{-i\pi f}, \quad f \in \mathfrak{h},$$

and limits of such objects. Some background is collected in Appendix B.

Before we state the lemma we define

$$\|f\|_{(m)} = \|f\| + \|\omega_m^{-1/2} f\|$$

for all measurable functions $f : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{C}$, where $\| \cdot \|$ denotes the $L^2$-norm in $L^2(\mathbb{Z}_2 \times \mathbb{R}^3)$. We write $f \in L^2_{(m)}(\mathbb{Z}_2 \times \mathbb{R}^3)$ if $\|f\|_{(m)} < \infty$. 

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Lemma 5.1. Let $m \geq 0$ and $f: \mathbb{Z}_2 \times \mathbb{R}^3 \to \mathbb{R}$ be a measurable function, with
\begin{align}
K_j f, K_j^2 f, \omega_m f & \in L^2_{(m)}(\mathbb{Z}_2 \times \mathbb{R}^3), \\
\omega_m |f|^2, K_j |f|^2, e^2 J_{A,j}, e^2 J_{B,j} & \in L^4(\mathbb{Z}_2 \times \mathbb{R}^3).
\end{align}

Then $T_m(f; \xi)$, given in (3.2), defines a $P^2 + H_{t,m}$ bounded operator on $D(P^2 + H_{t,m})$. In particular, there exists a constant $C$ such that for all $\psi \in D(P^2 + H_{t,m})$ we have
\begin{align}
\|T_m(f; \xi)\psi\| \leq C \sum_j \left\{ \|e^{\psi}_{f,B,j}\|_{(m)} + \|e^{\psi}_{f,B,j}\|_1 + \|\omega_m f\|_{(m)} + \|f\|^2 \omega_m\right\} \\
+ (\|e_{f,A,j}\|_{(m)} + \|K_j f\|_{(m)} + \|c_j(f)\| + 1)^2 \\
+ \|K^2 f\|_{(m)} + \|e K_j f\|_{(m)}) \|P^2 + H_{t,m} + 1\|\psi||,
\end{align}
where
\begin{align}
c_j(h) := \xi_j - 2e \text{Res}(h, f_{A,j}) - s(h, K_j h).
\end{align}

Proof. From the assumptions (5.2) it follows that all expressions involving the sesquilinear form $s$ are finite. Multiplying out the square in (3.2) we will show that each term is $P^2 + H_{t,m}$ bounded. For $\psi \in F_{m}(C_c(\mathbb{Z}_2 \times \mathbb{R}^3))$ we find that
\begin{align}
T_m(f; \xi)\psi \\
= \left(\frac{1}{2} P^2 + H_{t,m}\right) \psi \\
- \frac{1}{2} \sum_j P_{t,j} \left( \phi(e_{f,A,j} - K_j f) + c_j(f) \right) \psi \\
- \frac{1}{2} \sum_j \left( \phi(e_{f,A,j} - K_j f) + c_j(f) \right) P_{t,j} \psi \\
+ \frac{1}{2} \sum_j \left( \phi(e_{f,A,j} - K_j f) + c_j(f) \right)^2 \psi \\
+ e \sum_j S_j \left( \phi(f_{B,j}) - 2\text{Res}(f, f_{B,j}) \right) \psi + \left( \phi(\omega_m f) + s(f, \omega_m f) \right) \psi.
\end{align}

From the first identity in Lemma A.2 see that the terms in (5.9) are $H_{t,m}$ bounded, since $\omega_m f, f_{B,j} \in L^2_{(m)}$, i.e.,
\begin{align}
\|T_m(f; \xi)\| \leq C \left( \sum_j (\|e_{f,B,j}\|_{(m)} + \|e^{\psi}_{f,B,j}\|_1 + \|\omega_m f\|_{(m)} + \|f\|^2 \omega_m) \right) \|(H_{t,m} + 1)^{1/2} \psi\|
\end{align}

Now let us estimate the terms in (5.8). Clearly $(\phi(K_j f) + e A_j)^2$ is $H_{t,m}$ bounded by Lemma A.2, since $K_j f, f_{A,j} \in L^2_{(m)}(\mathbb{Z}_2 \times \mathbb{R}^3)$ and so
\begin{align}
\|T_m(f; \xi)\| \leq C \sum_j (\|e_{f,A,j}\|_{(m)} + \|K_j f\|_{(m)} + \|c_j(f)\| + 1)^2 \|(H_{t,m} + 1) \psi\|
\end{align}
Similarly, the term in (5.7) can be bounded by Lemma A.2 as
\[ \| 5.7 \| \leq C \sum_j (\| \epsilon_{fA,j} \|_{(m)} + \| K_j f \|_{(m)} + |c_j(f)|) \| (H_{t,m} + 1)^{1/2} P_{t,j} \psi \| . \]

For (5.6) observe that Lemma B.1 yields on \( F \)
\[ \| 5.6 \| \leq C \sum_j (\| \epsilon_{fA,j} \|_{(m)} + \| K_j f \|_{(m)} + |c_j(f)|) \| (H_{t,m} + 1)^{1/2} P_{t,j} \psi \| . \]

For (5.7) observe that Lemma B.1 yields on \( F_{\text{fin}}(C_c(Z_2 \times \mathbb{R}^3)) \)
\[ P_{f,j} \phi(K_j f + \epsilon_{fA,j}) = \phi(K_j f + \epsilon_{fA,j}) P_{f,j} - i \phi(i K_j^2 f + i \epsilon K_j f_{A,j}). \] (5.10)

Now each term in (5.10) is estimated using Lemma A.2 and we find for \( \psi \in F_{\text{fin}}(C_c(Z_2 \times \mathbb{R}^3)) \)
\[ \| 5.8 \| \leq C \sum_j (\| \epsilon_{fA,j} \|_{(m)} + \| K_j f \|_{(m)} + |c_j(f)|) \| (H_{t,m} + 1)^{1/2} P_{t,j} \psi \| . \] (5.11)

Then (5.11) extends to all \( \psi \in D(P_t^2 + H_{t,m}), \) since \( F_{\text{fin}}(C_c(Z_2 \times \mathbb{R}^3)) \) is a core for \( P_t^2 + H_{t,m}. \) Finally, collecting above estimates shows (5.13) and hence the claim follows.

We note that if the assumptions of Lemma 5.1 hold, we will define \( T_{m}(f; \xi) \) according to that lemma as the operator with domain \( D(P_t^2 + H_{t,m}). \) The following lemma will be used for the proof of Proposition 5.3 below, to establish self-adjointness. We note that the self-adjointness could alternatively be shown directly using alternative methods, cf. [HS20, LMS07, Hir02] and references therein.

**Lemma 5.2.** There exists a constant \( c_0 \) such that for \( m \geq 0 \) and \( f \in \mathfrak{h} \) with \( \omega^2 f \in L^2_{(m)}(Z_2 \times \mathbb{R}^3) \) we have
\[ \left\| \left( \frac{1}{2} P_t^2 + H_{t,m} \right) e^{i \pi(f)} \psi \right\| \leq C(f) \| (P_t^2 + H_{t,m} + 1) \psi \| , \] (5.12)
where
\[ C(f) = c_0 \left( \left[ 1 + \| \omega_{m}^{1/2} f \| + \| \omega_m f \|_{(m)} \right]^2 + \| \omega^2 f \|_{(m)} \right) . \]

In particular, \( e^{i \pi(f)} D(P_t^2 + H_{t,m}) = D(P_t^2 + H_{t,m}). \)

**Proof.** We apply Lemma B.5 to the operators \( P_{t,j} \) and \( H_{t,m}, \) and find on \( F_{\text{fin}}(C_c(Z_2 \times \mathbb{R}^3)) \subset D(P_t^2 + H_{t,m}) \)
\[ e^{-i \pi(f)} (P_t^2 + H_{t,m}) e^{i \pi(f)} \]
\[ = \frac{1}{2} e^{-i \pi(f)} P_t e^{i \pi(f)} \cdot e^{-i \pi(f)} P_t e^{i \pi(f)} + e^{-i \pi(f)} H_{t,m} e^{i \pi(f)} \]
\[ = \frac{1}{2} \sum_{j=1}^3 (P_{f,j} + \phi(K_j f) + \frac{1}{2} \bar{s}(f, K_j f))^2 + H_{t,m} + \phi(\omega_m f) + \frac{1}{2} \bar{s}(f, \omega_m f) . \]

Now the bound (5.12) follows from Lemma 5.1 with \( e = 0, \) \( \xi = 0, \) and \( s = 0 \) using the fact that \( F_{\text{fin}}(C_c(Z_2 \times \mathbb{R}^3)) \) is a core for \( P_t^2 + H_{t,m}. \) This implies that \( e^{i \pi(f)} D(P_t^2 + H_{t,m}) \subset D(P_t^2 + H_{t,m}). \) Finally, the unitarity of \( e^{i \pi(f)} \) implies that we have in fact equality.
Proposition 5.3. Let \( m \geq 0 \), and let \( f \in \mathfrak{h} \) satisfy \( K_j^2 f \in L^2_{(m)}(\mathbb{Z}_2 \times \mathbb{R}^3) \), \( j = 1, 2, 3 \). Then we have on \( D(P_t^2 + H_{t,m}) \)

\[
e^{-i \pi(f)} H_m(\xi) e^{i \pi(f)} = T_m(f; \xi)
\] (5.14)

and the operator is self-adjoint on \( D(P_t^2 + H_{t,m}) \).

Proof. We apply Lemma B.4 to the field operators \( A \) and \( B \) and Lemma B.3 to the operators \( P_{t,j} \) and \( H_{t,m} \), and find on \( \mathcal{F}_{\text{fin}}(C_c(\mathbb{Z}_2 \times \mathbb{R}^3)) \subset D(P_t^2 + H_{t,m}) \)

\[
e^{-i \pi(f)} H_m(\xi) e^{i \pi(f)} = \frac{1}{2} e^{-i \pi(f)}(\xi - P_t + eA) e^{i \pi(f)} \cdot e^{-i \pi(f)}(\xi - P_t + eA) e^{i \pi(f)} + eS \cdot e^{i \pi(f)} Be^{i \pi(f)}
\]

\[
+ e^{-i \pi(f)} H_{t,m} e^{i \pi(f)}
\]

\[
= \frac{1}{2} \sum_{j=1}^3 \left( \xi_j - P_{f,j} + eA_j - \phi(K_j f) - 2e \text{Res}(f, f_{A,j}) - s(f, K_j f) \right)^2
\]

\[
+ e \sum_{j=1}^3 S_j (B_j - 2 \text{Res}(f, f_{B,j})) + H_{t,m} + \phi(\omega_m f) + s(f, \omega_m f)
\]

\[
= T_m(f; \xi),
\]

where the last identity is simply the definition (3.2) of \( T_m(f; \xi) \). By Theorem 2.2 we know that \( H_m(\xi) \) is self-adjoint on \( D(P_t^2 + H_t) \). By unitarity of \( e^{-i \pi(f)} \) it follows that \( e^{-i \pi(f)} H_m(\xi) e^{i \pi(f)} \) is self-adjoint on the domain \( e^{-i \pi(f)} D(P_t^2 + H_t) \). But this domain equals \( D(P_t^2 + H_t) \) by Lemma 5.2.

Next we want to consider limits of expressions of the form (3.1). For this we prove the following lemma.

Lemma 5.4. Suppose \( f \) satisfies the assumptions of Lemma 5.1 and \( g \in \mathfrak{h} \). Then for all \( \psi \in D(P_t^2 + H_t) \) we have

\[
\| (T_0(f; \xi) - e^{-i \pi(g)} H(\xi) e^{i \pi(g)}) \psi \| \leq C(g, f; \xi) \| (H_t + \frac{1}{2} P_t^2 + 1) \psi \|, \quad (5.16)
\]

where

\[
C(g, f; \xi) = |D(f, g)| + \sqrt{2} \| \omega(f - g) \|_{(0)}
\]

\[
+ \sum_{j=1}^3 \left\{ (|d_j(f, g)| + \sqrt{2} \| K_j(f - g) \|_{(0)})
\]

\[
\times (2 + \sqrt{2}(\| e f_{A,j} - K_j f \|_{(0)} + \| e f_{A,j} - K_j g \|_{(0)}) + |c_j(f)| + |c_j(g)|) \right\}
\]

\[
+ \sum_{j=1}^3 \sqrt{2} \| K_j^2(f - g) \|_{(0)},
\]

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with

\[ c_j(h) := \xi_j - 2e \text{Res}(h, f_{A,j}) - s(h, K_j h), \]

\[ D(f, g) := -2 \sum_{j=1}^{3} e^{j} \text{Res}(f - g, f_{B,j}) + s(f - g, \omega f) + s(g, \omega(f - g)), \]

\[ d_j(f, g) := c_j(f) - c_j(g) = -2e \text{Res}(f - g, f_{A,j}) - s(f - g, K_j f) - s(g, K_j(f - g)). \]

Proof. First we take the difference of the operators given in (5.14). Using Proposition 5.3 and (3.2) we find on \( D(P^2_l + H_l) \)

\[ T_0(f; \xi) - e^{-i\pi(g)} H(\xi) e^{i\pi(g)} = T_0(f; \xi) - T_0(g; \xi) = \frac{1}{2} \sum_{j=1}^{3} \{ F_j(f)^2 - F_j(g)^2 \} + \Phi(\omega(f - g)) + D(f, g) \]

\[ = \frac{1}{2} \sum_{j=1}^{3} \{ F_j(f)(F_j(f) - F_j(g)) + (F_j(f) - F_j(g))F_j(g) \} \quad (5.18) \]

\[ + \Phi(\omega(f - g)) + D(f, g) \quad (5.19) \]

where we defined for \( j = 1, 2, 3, \)

\[ F_j(h) := -P_{l,j} + eA_j - \Phi(K_j h) + c_j(h). \]

Now the terms in the line (5.19) are easily estimated using Lemma A.2 and contribute (5.17) to the constant. Now let us estimate the term in line (5.18). Taking the difference we find

\[ F_j(f) - F_j(g) = -\Phi(K_j(f - g)) + d_j(f, g). \]

Recall the notation \( A_j = \Phi(f_{A,j}). \) First we estimate the first term in the sum in line (5.18). By the triangle inequality

\[ \| F_j(f)(F_j(f) - F_j(g)) \psi \| \leq \| P_{l,j}(F_j(f) - F_j(g)) \psi \| \]

\[ + \| \Phi(ef_{A,j} - K_j f)(F_j(f) - F_j(g)) \psi \| \]

\[ + \| c_j(f)(F_j(f) - F_j(g)) \psi \|. \]

Now we estimate the terms on the right hand side. Using Lemma A.2 and Lemma B.1 we find

\[ (5.20) \leq \left\{ |d_j(f, g)| + \sqrt{2} \| K_j(f - g) \| (0) \right\} \| (H_l + 1)^{1/2} P_{f, j} \psi \|

+ \sqrt{2} \| K_{j}^2(f - g) \| (0) \| (H_l + 1)^{1/2} \psi \|. \]

(5.23)
Again by Lemma A.2 we have

\[\begin{align*}
(5.21) & \leq (|d_j(f, g)| + \sqrt{2}\|K_j(f - g)\|_{(0)})\sqrt{2}\|e f_{A, j} - K_j g\|_{(0)}((H_t + 1)\psi), \\
(5.22) & \leq (|d_j(g, f)| + \sqrt{2}\|K_j(f - g)\|_{(0)})c_j(f)\|((H_t + 1)^{1/2}\psi). 
\end{align*}\]

The second term in the sum of (5.18) is estimated analogously with \(g\) and \(f\) interchanged, with the only difference that we do not need the term in line (5.22). The lemma now follows by collecting estimates and observing that \(d_j(g, f) = -d_j(f, g)\). □

Now Lemma 3.3 will follow as an immediate consequence of the following proposition. Furthermore, we will show below that it also implies Theorem 3.4.

**Proposition 5.5.** Let \(E\) be differentiable in \(\xi\) and \(|\nabla E(\xi)| < 1\). If \(m > 0\) and \(|\nabla E_m(\xi)| < 1\), then \(h_{m, \xi} \in \mathfrak{h}\). Furthermore, assume \((m_j)_{j \in \mathbb{N}}\) is a sequence of positive numbers converging to zero and \(\nabla E_{m_j}(\xi) \to \nabla E(\xi)\). Then

\[
(e^{-i\pi(h_{m_j}, \xi)}H(\xi)e^{i\pi(h_{m_j}, \xi)} - T_0(h_0, \xi))(P_t^2 + H_t + 1)^{-1} \to 0, \quad (j \to \infty). \tag{5.24}
\]

**Proof.** For \(m > 0\) and \(|\nabla E_m(\xi)| < 1\) we have \(h_{m, \xi} \in \mathfrak{h}\), since \(\omega_m(k) - k\nabla E_m(\xi) > \omega_m(k)(1 - |\nabla E_m(\xi)|)\). Statement (5.24) follows by inserting \(h_0\) for \(f\) and \(h_{m, \xi}\), \(m > 0\), for \(g\) in Lemma 5.4 and by observing that the constant \(C(h_{m, \xi}, h_0, \xi; \xi)\) given in (5.17) tends to zero as \(m \downarrow 0\). This can be seen by looking at the explicit expressions and using dominated convergence. □

**Proof of Lemma 5.2.** The operators \(e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})}\) are self-adjoint on \(D(P_t^2 + H_t)\) by Proposition 5.3. On the other hand \(T_0(h_0, \xi)\) is \(P_t^2 + H_t\) bounded by Lemma 5.4. So the self-adjointness of \(T_0(h_0, \xi)\) follows in view of Kato-Rellich, and the following estimate

\[
\begin{align*}
\|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} - T_0(h_0, \xi)\|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} + i)^{-1} & \\
& \leq \|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} - T_0(h_0, \xi)\|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} + i)^{-1} \\
& \times \|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} + i)^{-1} \\
& \leq \|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} - T_0(h_0, \xi)\|e^{-i\pi(h_{m, \xi})}H(\xi) + i)^{-1} \\
& \times \|e^{-i\pi(h_{m, \xi})}H(\xi) + i)^{-1} \\
& \leq \|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} - T_0(h_0, \xi)\|e^{-i\pi(h_{m, \xi})}H(\xi) + i)^{-1} \\
& \times \|e^{-i\pi(h_{m, \xi})}H(\xi) + i)^{-1} \\
& \leq \|e^{-i\pi(h_{m, \xi})}H(\xi)e^{i\pi(h_{m, \xi})} - T_0(h_0, \xi)\|e^{-i\pi(h_{m, \xi})}H(\xi) + i)^{-1} \\
& \times \|e^{-i\pi(h_{m, \xi})}H(\xi) + i)^{-1}.
\end{align*}\]

Now the last term on the right hand side of (5.25) is bounded by Theorem 2.2, the second term on the right hand side is bounded uniformly in \(m \geq 0\) in view of the estimate (5.12) in Lemma 5.2, and the first term on the right hand side of (5.25) tends to zero as \(m \downarrow 0\) by Proposition 5.5. □
Proof of Theorem 3.4. By Proposition 5.5 (a) the resolvent \((T_0(h_0,ξ; ξ) + i)^{-1}\) is well-defined and we obtain
\[
\| (e^{-iπ(h_m,ξ)} H(ξ)e^{iπ(h_m,ξ)} - T_0(h_0,ξ; ξ))(T_0(h_0,ξ; ξ) + i)^{-1} \| \\
\leq \| (e^{-iπ(h_m,ξ)} H(ξ)e^{iπ(h_m,ξ)} - T_0(h_0,ξ; ξ))(H + P_t^2 + 1)^{-1} \| \\
\times \| (H_t + P_t^2 + 1)(T_0(h_0,ξ; ξ) + i)^{-1} \|. 
\] (5.26)

Now the right hand side of (5.26) tends to zero as \(m \downarrow 0\) by (b) of Proposition 5.5. This implies norm resolvent convergence. We recall that for a sequence of self-adjoint operators \((A_n)_{n ∈ N}\) to converge in norm resolvent sense to a selfadjoint operator \(A\), it suffices to show that \(\| (A_n - A)(A + i)^{-1} \| \to 0 \) as \(n \to ∞\), which can be seen from [RS81, Theorem VIII.19] and the first resolvent identity.

In the following proposition we will establish that \(U_{m_j}(ξ)ψ_{m_j}(ξ)\) is a minimizing sequence of \(\tilde{H}(ξ)\).

Proposition 5.6. Suppose \(|ξ| < 1\). Furthermore, assume \((m_j)\) is a sequence of positive numbers converging to zero and \(∇E_{m_j}(ξ) → ∇E(ξ)\). Then
\[
0 \leq \langle U_{m_j}(ξ)ψ_{m_j}(ξ), (\tilde{H}(ξ) - E(ξ))U_{m_j}(ξ)ψ_{m_j}(ξ) \rangle \to 0
\]
in the limit \(j → ∞\).

Proof. We recall that the assumption \(|ξ| < 1\) guarantees by Lemma 2.3 that always \(|∇E_m(ξ)| < 1\) and that, for \(m > 0\), \(U_m(ξ)\) is a well defined unitary operator. Therefore we know that \(h_{m,ξ} ∈ ℱ\). For the proof we define
\[
\tilde{H}(ξ) := U_m(ξ)H(ξ)U_m(ξ)^*.
\]

Then we have by definition
\[
\langle U_m(ξ)ψ_m, (\tilde{H}(ξ) - E(ξ))U_{m_j}(ξ)ψ_{m_j} \rangle \\
= \langle U_m(ξ)ψ_m, (\tilde{H}(ξ) - \tilde{H}_{m_j}(ξ) + \tilde{H}_{m_j}(ξ) - E(ξ))U_{m_j}(ξ)ψ_{m_j} \rangle \\
= \langle U_m(ξ)ψ_m, (\tilde{H}(ξ) - \tilde{H}_{m_j}(ξ))U_{m_j}(ξ)ψ_{m_j} \rangle + \langle ψ_m, (H(ξ) - E(ξ))ψ_m \rangle. \quad (5.27)
\]

Now we use Proposition 5.5 to control the first term of (5.27) along the sequence \((m_j)\)
\[
(\tilde{H}(ξ) - \tilde{H}_{m_j}(ξ))^*(\tilde{H}(ξ) - \tilde{H}_{m_j}(ξ)) \leq C_j^2(H_t + P_t^2 + 1)^2,
\]
with \(C_j → 0\) as \(j → ∞\). Taking the square root yields
\[
\tilde{H}(ξ) - \tilde{H}_{m_j}(ξ) \leq |\tilde{H}(ξ) - \tilde{H}_{m_j}(ξ)| \leq |C_j|(H_t + P_t^2 + 1).
\]

Inserting this above we find
\[
\langle U_{m_j}(ξ)ψ_{m_j}, (\tilde{H}(ξ) - \tilde{H}_{m_j}(ξ))U_{m_j}(ξ)ψ_{m_j} \rangle \leq |C_j|\langle U_{m_j}(ξ)ψ_{m_j}, (H_t + P_t^2 + 1)U_{m_j}(ξ)ψ_{m_j} \rangle.
\]
The second term in \((5.27)\) can be bounded by means of \(H(\xi) \leq H_m(\xi)\)
\[
\langle \psi_m, (H(\xi) - E(\xi))\psi_m \rangle \leq \langle \psi_m, (H_m(\xi) - E(\xi))\psi_m \rangle = E_m(\xi) - E(\xi).
\]
Now, as \(H(\xi)\) is closed on \(D(P_t^2 + H_f)\) we find with Lemma 5.2 for some constant \(C\) independent of \(m \geq 0\)
\[
\langle U_m(\xi)\psi_m, (H_t + P_t^2 + 1)U_m(\xi)\psi_m \rangle \leq C\langle \psi_m, (H(\xi) + 1)\psi_m \rangle
\leq C\langle \psi_m, (H_m(\xi) + 1)\psi_m \rangle = C(E_m(\xi) + 1),
\]
which is in fact bounded by Proposition 4.1. Collecting estimates and using that \(E_m(\xi) \to E(\xi)\) by Proposition 4.1, the claim follows.

6. Perturbation Theory

In this section we use perturbation theory to obtain resolvent bounds. More precisely, we use second order perturbation theory to determine the second derivative of the energy in terms of the resolvent. Then using bounds on the second derivative of the energy, we find bounds on the resolvent. These bounds will be used to obtain infrared bounds for the ground states of the massive Hamiltonians.

If \(|\xi| \leq 1\), then we know by Theorem 2.6 that \(E_m(\xi)\) is an eigenvalue of \(H_m(\xi)\) isolated from the essential spectrum. Let \(P_m(\xi)\) denote the projection onto the kernel of \(H_m(\xi)\), which is finite dimensional. We recall the definition of \(v(\xi)\) in \((4.5)\).

Lemma 6.1. Let \(e \in \mathbb{R}\) and \(m > 0\), and suppose \((2.12)\) holds. Let \(|\xi| \leq 1\) and suppose that all first order partial derivatives of \(E_m(\cdot)\) exist at \(\xi\). Then the following holds.

(a) \(P_m v(\xi) P_m = \nabla_\xi E_m(\xi) P_m\).

(b) \(E_m(\cdot)\) is twice partially differentiable at \(\xi\) and for \(j = 1, 2, 3\),
\[
\partial_j^2 E_m(\xi)
\leq 1 - 2P_m(\xi) (v(\xi) - \nabla E_m(\xi))_j \frac{1}{H_m(\xi) - E_m(\xi)} (v(\xi) - \nabla E_m(\xi))_j P_m(\xi)
\]
as inequality in the sense of operators on \(\text{ran} P_m(\xi)\).

Remark 6.2. We note that by (a) the vectors in \(\text{ran}(v(\xi) - \nabla E_m(\xi))_j P_m(\xi)\) are orthogonal to \(P_m(\xi)\) and hence the resolvent in (b) is well defined.

Proof. For the proof we use analytic perturbation theory. For details we refer the reader to [Kat80; RS78]. On \(\mathbb{C}^3\) the operator valued function \(\xi \mapsto H_m(\xi + \frac{\xi}{2} v(\xi) + \frac{1}{2} \xi^2)\) is an analytic family of type (A) in each component. By Theorem 2.6 we know that \(E_m(\xi)\) is an eigenvalue isolated from the essential spectrum, and thus with finite multiplicity \(n(\xi) = \dim \text{ran} P_m(\xi)\).
By degenerate perturbation theory, cf. [RS78, Theorem XII.13], we know that there exist \( n(\xi) \) complex analytic functions \( e_s, s = 1, \ldots, n(\xi) \), in a neighborhood of zero such that
\[
e_s(0) = E_m(\xi), \quad s = 1, \ldots, n(\xi),
\]
and \( e_s(\zeta) \) is an eigenvalue of \( H_m(\xi + \zeta) \). The functions \( e_s, s = 1, \ldots, n(\xi) \), are real for real \( \zeta \), and
\[
\partial_j e_s(0), \quad s = 1, \ldots, n(\xi),
\]
are the eigenvalues of
\[
P_m(\xi) v_j(\xi) P_m(\xi),
\]
which can be seen from a power series expansion and comparison of coefficients. Since \( x \mapsto E_m(\xi + x) = \inf_{s=1,\ldots,n} e_s(x) \) is differentiable at \( x = 0 \), by assumption, and \( e_s(0) = E_m(\xi) \), it can be seen from a first order Taylor expansion that all derivatives must be equal at the origin and equal \( \partial_j E(\xi) \), i.e.,
\[
\partial_j e_s(0) = \partial_j E(\xi), \quad s = 1, \ldots, n(\xi).
\]
Thus, \( \nabla E_m(\xi) P_m(\xi) = P_m(\xi) v(\xi) P_m(\xi) \). This shows (a). Now by second order degenerate perturbation theory, we find
\[
\partial^2_j e_s(0), \quad s = 1, \ldots, n(\xi),
\]
are the eigenvalues of
\[
P_m(\xi) - 2P_m(\xi) (v(\xi) - \nabla E(\xi))(H_m(\xi) - E_m(\xi))^{-1} (v(\xi) - \nabla E(\xi)) P_m(\xi),
\]
which can again be seen from a power series expansion. By (6.2) and (6.1) and by the definition of \( E_m(\xi) \) as an infimum, it can be seen from a second order Taylor expansion that \( E_m \) is twice differentiable at \( \xi \) and that
\[
\partial^2_j E_m(\xi) = \inf_s \partial^2_j e_s(0).
\]
This shows (b).

7. Infrared Bounds

In this section we derive infrared bounds. These will then be used in the next section to show that infrared regularized ground states for a fixed momentum lie in a compact set. For \( \xi \in \mathbb{R}^3 \) and \( m \geq 0 \) we define the following quantity
\[
\Delta_m(\xi) = \inf_{k \in \mathbb{R}^3} \{ E_m(\xi - k) - E_m(\xi) + \omega_m(k) \},
\]
which we shall use to estimate the resolvent of the Hamiltonian. We will use the notation
\[
v := v(0) = -P_t + eA.
\]
The following proposition tells us when this resolvent defined in (1.4) is well defined.
Proposition 7.1. Let $e \in \mathbb{R}$ and $m > 0$, and suppose the energy inequality (7.12) holds. Then the following holds.

(a) We have $\Delta_m(\xi) > 0$, whenever $|\xi| \leq 1$.

(b) For $|\xi| \leq 1$ and all $k \in \mathbb{R}^3$ we have

$$H_m(\xi - k) + \omega_m(k) - E_m(\xi) > 0.$$ 

(c) For $|\xi| < 1$ and all $k \in \mathbb{R}^3$

$$\|R_{m,\xi}(k)\| \leq \frac{1}{1 - |\xi| |k|}.$$  

(7.1)

Proof. First we show that the function $\xi \mapsto E_m(\xi)$ satisfies the assumptions (i)–(iii) of Proposition C.1 in the Appendix. Property (i) is simply the energy inequality (2.12). From Lemma 2.3 we know on the one hand that $t_m(\xi) = \frac{1}{2}k^2 - E_m(\xi)$ is convex, i.e., (iii) and on the other hand that $E_m(-\xi) = E_m(\xi)$, which implies $t_m(-\xi) = t_m(\xi)$. Thus by convexity $t_m(0) \leq t_m(\xi)$, which implies (ii). It therefore follows from Proposition C.1

$$E_m(\xi - k) - E_m(\xi) \geq \left\{ \begin{array}{ll} -|k||\xi| + \frac{1}{2}k^2, & \text{if } |k| \leq |\xi|, \\ -\frac{1}{2}k^2, & \text{if } |k| \geq |\xi|. \end{array} \right.$$  

(7.2)

(a) We find using $\omega(k) > |k|$ in (7.2)

$$E_m(\xi - k) - E_m(\xi) + \omega(k) \geq \left\{ \begin{array}{ll} -|k||\xi| + |k|, & \text{if } |k| \leq |\xi|, \\ -\frac{1}{2}k^2 + |\xi|, & \text{if } |k| \geq |\xi|, \end{array} \right.$$  

$$\geq 0,$$

provided $|\xi| \leq 1$.

(b) From (a) we have $H_m(\xi - k) + \omega_m(k) - E_m(\xi) \geq E_m(\xi - k) + \omega_m(k) - E_m(\xi) \geq \Delta_m(\xi) > 0$.

(c) From (7.2) we find

$$H_m(\xi - k) + \omega_m(k) - E_m(\xi) \geq E_m(\xi - k) - E_m(\xi) + |k|$$

$$\geq \left\{ \begin{array}{ll} |k|(1 - |\xi|), & \text{if } |k| \leq |\xi|, \\ |k| - |\xi|^2/2, & \text{if } |k| \geq |\xi|. \end{array} \right.$$  

Observing that $|k| - |\xi|^2/2 \geq |k| - \frac{|k||\xi|}{2} \geq (1 - |\xi|)|k|$ if $|k| \geq |\xi|$ shows the claim. \hfill \Box

Now we use the following relation, which we shall refer to as the pull-through resolvent identity. To formulate it we shall make use of the physics notation of the pointwise annihilation operator. One defines for $(\lambda, k) \in \mathbb{Z}_2 \times \mathbb{R}^3$ and $\psi \in \mathcal{F}$

$$[a_\lambda(k)\psi_{(n)}](\lambda_1, k_1, \ldots, \lambda_n, k_n) = \sqrt{n + 1}a_{(\lambda_1, k_1, \ldots, \lambda_n, k_n)}(\lambda, k, \lambda_1, k_1, \ldots, \lambda_n, k_n),$$  

(7.3)
Note that by the Fubini-Tonelli theorem $a_{\lambda}(k)\psi_{n+1} \in h^{(n)}$ for almost every $k$. If $\psi \in D(H_1)$ and $f \in h$ with $\|f\|_0 < \infty$, then a straightforward calculation using the definitions shows that in the sense of weak integrals

$$a(f)\psi = \sum_{\lambda = 1,2} \int dk f(\lambda,k)a_{\lambda}(k)\psi. \quad (7.4)$$

In fact, the integral on the right hand side $|bf|$ exists as a Bochner integral. This can be seen from the following estimate

$$\sum_{\lambda = 1,2} \int \|f(\lambda,k)a_{\lambda}(k)\psi\|dk = \sum_{\lambda = 1,2} \int dk |f(\lambda,k)\omega(k)^{-1/2}||\omega(k)^{1/2}a_{\lambda}(k)\psi|dk$$

$$\leq \|\omega^{-1/2}f\| \left( \sum_{\lambda = 1,2} \int dk \langle a_{\lambda}(k)\psi, \omega(k)a_{\lambda}(k)\psi \rangle \right)^{1/2}$$

$$= \|\omega^{-1/2}f\| \|H^{1/2}_1\psi\|.$$

**Lemma 7.2** (Pull-through resolvent identity). Let $e \in \mathbb{R}$ and $m > 0$, and suppose the energy Inequality (2.12) holds. Suppose $|\xi| < 1$. Then for all $k \in \mathbb{R}^3$ and $\lambda \in \{1, 2\}$ we have

$$a_{\lambda}(k)\psi_m(\xi) = e^{\rho(k)}R_{m,\xi}(k) (-\varepsilon_{\lambda}(k)e(\xi) + S \cdot (ik \land \varepsilon_{\lambda}(k)))\psi_m(\xi). \quad (7.5)$$

**Sketch of proof.** For details we refer to [CF07, Lemma 6.1] and [HH08a, Lemma 7]. As $\psi_m \in D(H_1)$, we have

$$\sum_{n=0}^{\infty} \sum_{\lambda} \int |k| \|(a_{\lambda}(k)\psi_m(n))\|^2 dk = \langle \psi_m, H_t\psi_m \rangle < \infty,$$

which implies $\|a_{\lambda}(k)\psi_m\| < \infty$ almost everywhere.

The pull-through formula yields, e.g. [BFS98, Lemma A.1], for a.e. $k \in \mathbb{R}^3$, $\lambda \in \{1, 2\}$, in the sense of measurable functions

$$a_{\lambda}(k)H_m(\xi)\psi = \left(\frac{1}{2}(\xi - k - P_i + eA)^2 + eS \cdot B + H_{t,m} + \omega_m(k)\right)a_{\lambda}(k)\psi$$

$$+ (ef_A(k) \cdot v(\xi) + eS \cdot f_B(k))\psi$$

for any $\psi \in F$. Therefore,

$$a_{\lambda}(k)E_m(\xi)\psi_m(\xi) = a_{\lambda}(k)H_m(\xi)\psi_m(\xi)$$

$$= (((H_m(\xi - k) + \omega_m(k))a_{\lambda}(k) + ef_A(k) \cdot v(\xi) + eS \cdot f_B(k))\psi_m(\xi),$$

or in other words,

$$((H_m(\xi - k) - E_m(\xi) + \omega_m(k))a_{\lambda}(k)\psi_m(\xi) = -(ef_A(k) \cdot v(\xi) + eS \cdot f_B(k))\psi_m(\xi).$$

Multiplying with $R_{m,\xi}(k)$ yields the desired formula. $\square$

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The next lemma is needed to estimate the resolvent occurring in (7.5). It uses the estimate obtained by second order perturbation theory stated in Lemma 6.1.

**Lemma 7.3.** There exists a constant $C$ such that the following holds. Let $e \in \mathbb{R}$ and $m > 0$, and suppose the energy Inequality (2.12) holds. Then for all $\xi \in \mathbb{R}^3$ with $|\xi| < 1$, such that $\nabla E_m(\xi)$ exists, and all $k \in \mathbb{R}^3$ and $i = 1, 2, 3$, we have

\[
\begin{align*}
(a) \quad & \| R_{m,\xi}(k)(v_i(\xi) - \partial_i E_m(\xi))\psi_m(\xi) \| \leq C \frac{1}{|\xi|} \frac{\partial^2 E_m(\xi)}{1+|E_m(\xi)|} (|k|^{-1/2} + 1), \\
(b) \quad & \| R_{m,\xi}(k)v_i(\xi) \| \leq C (1 - |\xi|)^{-1} (1 + |E_m(\xi)|)^{1/2} (|k|^{-1} + 1).
\end{align*}
\]

**Proof.** To simplify the notation we drop the $\xi$ label in $\psi_m(\xi)$, $R_{m,\xi}(k)$, and $v(\xi)$, and we write

\[ h_m := H_m(\xi), \quad e_m := E_m(\xi). \]

(a) We start with the product inequality

\[ \| R_m(k)v_j\psi_m \| \leq \| R_m(k)(h_m - e_m)^{1/2} \| (h_m - e_m)^{-1/2} (v_j - \partial_j E_m(\xi))\psi_m \|. \]  

(7.6)

By Lemma 6.1 the second factor on the right hand side can be estimated using

\[ \| (h_m - e_m)^{-1/2} (v_j - \partial_j E_m(\xi))\psi_m \| \leq \left| \frac{1}{2} - \partial_j^2 E_m(\xi) \right|^{1/2}. \]

It remains to estimate the first factor in (7.6). First we use the trivial identity

\[ h_m - e_m = \frac{1}{2} (v - k)^2 + H_{f,m} - e_m + \frac{1}{2} k^2 + (v - k) \cdot k. \]

Estimating the last term using

\[ (v - k) \cdot k \leq \frac{1}{2} |k| + \frac{1}{2} |k| (v - k)^2, \]

we find with $\frac{1}{2} (v - k)^2 \leq H_m(\xi - k)$ that

\[ h_m - e_m \leq (1 + |k|)(H_m(\xi - k) + \omega_m(k) - e_m) + \frac{1}{2} (|k| + k^2) + |k|e_m. \]  

(7.7)

Now multiplying this inequality on both sides with the self-adjoint operator $R_m(k)$ we obtain

\[ R_m(k)(h_m - e_m) R_m(k) \leq (1 + |k|) R_m(k) + \left( \frac{1}{2} (|k| + k^2) + |k|e_m \right) R_m(k)^2. \]

Using this, we estimate

\[ \| R_m(k)(h_m - e_m)^{1/2} \|^2 \leq \| (h_m - e_m)^{1/2} R_m(k) \|^2 \]

\[ = \sup_{\|\phi\|=1} \langle \phi, R_m(k)(h_m - e_m) R_m(k) \phi \rangle \]

\[ \leq (1 + |k|) \| R_m(k) \|^2 + \left( \frac{1}{2} (|k| + k^2) + |k|e_m \right) \| R_m(k) \|^2. \]  

(7.8)

(7.9)
Now using Inequality (7.1) this implies the bound stated in (a).

(b) Using that \( v^2 \leq h_m \) we see from (7.1) that

\[
v^2 \leq (1 + |k|)(H_m(\xi - k) + \omega_m(k) - e_m) + \frac{1}{2}(|k| + k^2) + (1 + |k|)e_m.
\]

This implies

\[
||R_m(k)v_i||^2 \leq (1 + |k|)||R_m(k)|| + \left( \frac{1}{2}(|k| + k^2) + (|k| + 1)e_m \right) ||R_m(k)||^2.
\]

Now using Inequality (7.1) this implies the bound in (b).

We note that the assumption \(|\xi| < 1\) guarantees by Lemma 2.3 that \(|\nabla E_m(\xi)| < 1\) always holds, provided the derivative exists. Therefore we know that \(h_{m,\xi} \in \mathfrak{h}\) and the following two expressions are well defined and positive,

\[
D_{m,\xi}(k) := (\omega_m(k) - k \cdot \nabla E_m(\xi))^{-1}, \quad (7.10)
\]

\[
\tilde{D}_{m,\xi}(k) := (1 - \omega_m(k)^{-1}k \cdot \nabla E_m(\xi))^{-1}. \quad (7.11)
\]

To write the following estimates in compact notation we define

\[
F_{m,\xi} := \sum_{i=1}^{3} \left( 1 + \left| \frac{1}{2} - \partial_i E_m(\xi) \right|^{1/2} \right) (1 + |E_m(\xi)|)^{1/2}. \quad (7.12)
\]

Recall that if \(\nabla E_m(\xi)\) exists, then also the second derivative, cf. Lemma 6.1.

**Proposition 7.4.** There exists a constant \(C\) such that the following holds. Let \(e \in \mathbb{R}\) and \(m > 0\), and suppose the energy Inequality (2.12) holds. Let \(\xi \in \mathbb{R}^3\) with \(|\xi| < 1\), such that \(\nabla E_m(\xi)\) exists. Then for a.e. \(k \in \mathbb{R}^3\)

\[
\|a_\lambda(k)U_m(\xi)\psi_m(\xi)\| \leq \frac{C|e\rho(k)|F_{m,\xi}(\tilde{D}_{m,\xi}(k) + 1)(|k|^{-1} + |k|^{-1/2}).
\]

**Proof.** Using the convergence of (7.4) as a Bochner integral and Lemma 3.4 we find for any \(f \in \mathfrak{h}\) with \(\|f\|_0 < \infty\) and \(\varphi \in \mathcal{F}_{\text{fin}}(\mathfrak{h})\) that

\[
\sum_{\lambda=1,2} \int dk f(\lambda, k)\langle \varphi, U_m(\xi)\ast a_\lambda(k)U_m(\xi)\psi_m(\xi) \rangle = \langle \varphi, U_m(\xi)\ast a(f)U_m(\xi)\psi_m(\xi) \rangle
\]

\[
= \sum_{\lambda=1,2} \int dk f(\lambda, k)\langle \varphi, a_\lambda(k)\psi_m(\xi) \rangle
\]

\[
+ \sum_{\lambda=1,2} \int dk f(\lambda, k)e\varepsilon(\lambda)(k) \cdot \nabla \xi E_m(\xi) \frac{\rho(k)}{\sqrt{2|k|\omega_m(k) - k\nabla \xi \cdot E_m(\xi)}} \langle \varphi, \psi_m(\xi) \rangle.
\]
Since \( f \in \mathfrak{h} \) with \( \| f \|_{(0)} < \infty \) and \( \varphi \in F_{\text{fin}}(\mathfrak{h}) \) are arbitrary it follows from (7.13) and density of \( F_{\text{fin}}(\mathfrak{h}) \) that for a.e. \( k \in \mathbb{R}^3 \)

\[
U_m(\xi)^* a_\lambda(k) U_m(\xi) \psi_m(\xi) = a_\lambda(k) \psi_m(\xi) + e \varepsilon_\lambda(k) \nabla_\xi E_m(\xi) \frac{\rho(k)}{\sqrt{2 |k|}} \frac{1}{\omega_m(k) - k \nabla_\xi E_m(\xi)} \psi_m(\xi). \tag{7.14}
\]

The goal is to show that as \( k \to 0 \) the leading order contribution of the first term in (7.14) cancels the second term. To show this we use the pull-through resolvent identity in Lemma 7.2 and can write

\[
a_\lambda(k) \psi_m(\xi) = (I) + (II), \tag{7.15}
\]

where

\[
(I) := R_{m,\xi}(k) \frac{e \rho(k)}{\sqrt{2 |k|}} S \cdot (ik \wedge \varepsilon_\lambda(k)) \psi_m(\xi),
\]

\[
(II) := -R_{m,\xi}(k) \frac{e \rho(k)}{\sqrt{2 |k|}} \varepsilon_\lambda(k) \cdot v(\xi) \psi_m(\xi).
\]

The first term, \((I)\), can be estimated using (7.1)

\[
\| (I) \| \leq \frac{|e \rho(k)|}{1 - |\xi|} \frac{1}{\sqrt{2 |k|}}.
\]

Furthermore, we write the second term

\[
(II) = (II)_1 + (II)_2
\]

by dividing it into the following two parts

\[
(II)_1 := -R_{m,\xi}(k) \frac{e \rho(k)}{\sqrt{2 |k|}} \varepsilon_\lambda(k) \cdot (v(\xi) - \nabla E_m(\xi)) \psi_m(\xi),
\]

\[
(II)_2 := -R_{m,\xi}(k) \frac{e \rho(k)}{\sqrt{2 |k|}} \varepsilon_\lambda(k) \cdot \nabla E_m(\xi) \psi_m(\xi).
\]

We have

\[
\| (II)_1 \| \leq C \frac{|e \rho(k)|}{\sqrt{2 |k|}} F_{m,\xi}(1 - |\xi|)^{-1}(|k|^{-1/2} + 1)
\]

by Lemma 7.3. To estimate \((II)_2\), we proceed as follows. Similarly to [HH08a] we introduce the operator

\[
R_{m,\xi}^{(0)}(k) := (H_m(\xi) + \omega_m(k) - E_m(\xi))^{-1}.
\]
Using the second resolvent identity, we find
\[
R_{m,\xi}(k)\psi_m(\xi) = R_{m,\xi}^{(0)}(k)\psi_m + R_{m,\xi}(k)(-\frac{1}{2}k^2 + k \cdot v(\xi))R_{m,\xi}^{(0)}(k)\psi_m
\]
\[
= \omega_m(k)^{-1}\psi_m + \omega_m(k)^{-1}R_{m,\xi}(k)(k \cdot v(\xi) - \frac{1}{2}k^2)\psi_m
\]
\[
= \omega_m(k)^{-1}(\psi_m + k \cdot \nabla E_m(\xi)R_{m,\xi}(k)\psi_m) + (\mathrm{III})_1 + (\mathrm{III})_2, \quad (7.16)
\]
where we defined
\[
(\mathrm{III})_1 := -\frac{1}{2}k^2 R_{m,\xi}(k)\psi_m(\xi),
\]
\[
(\mathrm{III})_2 := R_{m,\xi}(k)(k \cdot (v(\xi) - \nabla E(\xi))\psi_m(\xi).
\]
Hence, multiplying out (7.16) and bringing the term \(\omega_m(k)^{-1}k \cdot \nabla E_m(\xi)R_{m,\xi}(k)\psi_m(\xi)\) to the left, we arrive at
\[
R_{m,\xi}(k)\psi_m(\xi) = D_{m,\xi}(k)(\psi_m(\xi) + (\mathrm{III})_1 + (\mathrm{III})_2). \quad (7.17)
\]
By Inequality (7.1) and by Lemma 7.3 respectively, we find
\[
\| (\mathrm{III})_1 \| \leq \frac{C|k|}{1 - |\xi|}, \quad (7.18)
\]
\[
\| (\mathrm{III})_2 \| \leq \frac{C|F_{m,\xi}|}{1 - |\xi|}(|k|^{1/2} + |k|). \quad (7.19)
\]
Now inserting the above, we arrive at
\[
(\mathrm{II})_2 = -\varepsilon_\lambda(k) \cdot \nabla E_m(\xi) \frac{e^\rho(k)}{\sqrt{2|k|}} \frac{1}{\omega_m(k) - k \cdot \nabla E_m(\xi)} \psi_m(\xi)
\]
\[
- \varepsilon_\lambda(k) \cdot \nabla E_m(\xi) \frac{e^\rho(k)}{\sqrt{2|k|}} \frac{1}{\omega_m(k) - k \cdot \nabla E_m(\xi)}((\mathrm{III})_1 + (\mathrm{III})_2). \quad (7.20)
\]
Using (7.18) and (7.19) we can estimate the second term on the right hand side of (7.20) and find that it is of order \(|k|^{-1}\). The claim now follows by collecting estimates and using that the first term on the right hand side of (7.20) exactly cancels the second term on the right hand side of (7.14).

We still need an estimate involving derivatives. To this end, we shall henceforth make an explicit choice of the polarization vectors. After a possible unitary transformation on Fock space we can always achieve that the polarization vectors are given by
\[
\varepsilon_1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad \varepsilon_2(k) = \frac{k}{|k|} \wedge \varepsilon_1(k). \quad (7.21)
\]
Proposition 7.5. There exists a constant $C$ (depending on $\Lambda$) such that the following holds. Let $e \in \mathbb{R}$ and $m > 0$, and suppose the energy Inequality \((7.12)\) holds. Let $\xi \in \mathbb{R}^3$ with $|\xi| < 1$ such that $\nabla E_m(\xi)$ exists. Then for almost all $k$ with $|k| < \Lambda$,

$$
\|\nabla_k (a_\lambda(k) U_m(\xi) \psi_m(\xi))\| \leq \frac{CF^2_{m,\xi}(e\rho(k))|\mathcal{D}_{m,\xi}(k)| + 1}{(1 - |\xi|)^2 |k| \sqrt{k^2_1 + k^2_2}}.
$$

Proof. First we note that for all nonzero $k \in \mathbb{R}^3$ and $\lambda = 1, 2$, we have

$$
|\frac{\partial}{\partial k_j} \varepsilon_\lambda(k)| \leq \frac{1}{\sqrt{k^2_1 + k^2_2}}, \quad j = 1, 2, 3. \tag{7.22}
$$

Moreover, as $\rho$ is constant for $|k| < \Lambda$, we do not need to take derivatives of $\rho$ into account. Since $|k| < \Lambda$ it suffices to consider the leading order contributions as $|k|$ is small. From differentiating \((7.14)\) we find

$$
U_m(\xi) \nabla_k a_\lambda(k) U_m(\xi) \psi_m(\xi) = \nabla_k U_m(\xi) a_\lambda(k) U_m(\xi) \psi_m(\xi) \tag{7.23}
$$

where the derivative is understood in the strong sense. First we calculate the derivative of the first term using the pull-through resolvent identity in Lemma \((7.22)\)

For $k$ with $|k| < \Lambda$ we find by the product rule

$$
\nabla_k a_\lambda(k) \psi_m(\xi)
$$

$$
= \nabla_k \left( - \frac{e\rho(k)}{\sqrt{2|k|}} R_{m,\xi}(k) (\varepsilon_\lambda(k) \cdot v(\xi) + \sigma \cdot (k \wedge \varepsilon_\lambda(k)) \psi_m(\xi) \right)
$$

$$
= J_1 + J_2, \tag{7.24}
$$

where we introduced

$$
J_1 := R_{m,\xi}(k) (k - v(\xi) + \nabla_k \omega_m(k)) R_{m,\xi}(k)
$$

$$
\times \frac{e\rho(k)}{\sqrt{2|k|}} (\varepsilon_\lambda(k) \cdot v(\xi) + \sigma \cdot (k \wedge \varepsilon_\lambda(k)) \psi_m(\xi)
$$

and

$$
J_2 := J_{2,1} + J_{2,2}
$$

with

$$
J_{2,1} := -R_{m,\xi}(k) \nabla_k \left( \frac{e\rho(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \cdot v(\xi) \psi_m(\xi) \right),
$$

$$
J_{2,2} := -R_{m,\xi}(k) \nabla_k \left( \frac{e\rho(k)}{\sqrt{2|k|}} S \cdot (k \wedge \varepsilon_\lambda(k)) \psi_m(\xi) \right).
$$

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To estimate $J_{2,2}$ we use (7.11) and (7.22), which yields

$$
\|J_{2,2}\| \leq \frac{C|\varepsilon(k)|}{1-|\xi|} \frac{1}{|k|\sqrt{k_1^2 + k_2^2}}.
$$

(7.25)

Next we consider $J_{2,1}$ and write

$$
v(\xi) = \nabla E_m(\xi) + (v(\xi) - \nabla E_m(\xi))
$$

(7.26)

and use Lemma 7.3(a) and (7.22) to find

$$
\left\| J_{2,1} + R_{m,\xi}(k) \nabla_k \left( \frac{e^{\rho(k)}}{\sqrt{2|k|}} \varepsilon_{\lambda}(k) \cdot \nabla E_{m}(\xi) \psi_{m}(\xi) \right) \right\| \leq \frac{C F_{m,\xi}|e^{\rho(k)}|}{(1-|\xi|)|k|\sqrt{k_1^2 + k_2^2}}.
$$

(7.27)

Now we use (7.17)–(7.19) to replace the resolvent $R_{m,\xi}$ by $D_{m,\xi}$ and get (noting that $|\nabla E_m(\xi)| < 1$)

$$
\left\| J_{2,1} + D_{m,\xi}(k) \nabla_k \left( \frac{e^{\rho(k)}}{\sqrt{2|k|}} \varepsilon_{\lambda}(k) \cdot \nabla E_{m}(\xi) \psi_{m}(\xi) \right) \right\| \leq \frac{C F_{m,\xi}|e^{\rho(k)}| (\widetilde{D}_{m,\xi}(k) + 1)}{(1-|\xi|)|k|\sqrt{k_1^2 + k_2^2}}.
$$

(7.27)

Finally let us consider $J_1$. The term involving the spin has a lower order singularity as well as the term involving the $-k$. Thus we find with (7.11) and Lemma 7.3(b)

$$
\left\| J_1 - R_{m,\xi}(k)(\nabla_k \omega_m(k) - v(\xi))R_{m,\xi}(k) \frac{e^{\rho(k)}}{\sqrt{2|k|}} \varepsilon_{\lambda}(k) \cdot v(\xi) \psi_m(\xi) \right\| \leq \frac{C F_{m,\xi}|e^{\rho(k)}|}{(1-|\xi|)^2|k|^{3/2}}.
$$

Using again (7.26) to replace the second $v(\xi)$ by $\nabla E_m(\xi)$, we find with Lemma 7.3(a) and Lemma 7.3(b) that

$$
\left\| J_1 - R_{m,\xi}(k)(\nabla_k \omega_m(k) - v(\xi))R_{m,\xi}(k) \frac{e^{\rho(k)}}{\sqrt{2|k|}} \varepsilon_{\lambda}(k) \cdot \nabla E_m(\xi) \psi_m(\xi) \right\| \leq \frac{C F^2_{m,\xi}|e^{\rho(k)}|}{(1-|\xi|)^2|k|^2}.
$$

Now we use as before (7.17)–(7.19) to replace the second resolvent $R_{m,\xi}$ by $D_{m,\xi}$

$$
\left\| J_1 - R_{m,\xi}(k)(\nabla_k \omega_m(k) - v(\xi))D_{m,\xi}(k) \varepsilon_{\lambda}(k) \cdot \frac{e^{\rho(k)}}{\sqrt{2|k|}} \nabla E(\xi) \psi_m(\xi) \right\| \leq \frac{C F^2_{m,\xi}|e^{\rho(k)}| (\widetilde{D}_{m,\xi}(k) + 1)}{(1-|\xi|)^2|k|^2}.
$$

(7.28)

Repeating the above, that is, using again first (7.26) to replace $v(\xi)$ by $\nabla E_m(\xi)$ and then (7.17)–(7.19) to replace the first resolvent $R_{m,\xi}$ by $D_{m,\xi}$, we arrive at

$$
\left\| J_1 - (\nabla_k \omega_m(k) - \nabla E_m(\xi)) [D_{m,\xi}(k)]^2 \varepsilon_{\lambda}(k) \cdot \frac{e^{\rho(k)}}{\sqrt{2|k|}} \nabla E(\xi) \psi_m(\xi) \right\| \leq \text{r.h.s. of (7.28)}.
$$

(7.29)
Finally, note that the derivative of the second term in (7.23) is
\[ \nabla_k \left( \varepsilon_\lambda(k) \cdot \nabla_\xi E_m(\xi) \frac{e^\rho(k)}{\sqrt{2|k|}} D_m,\xi(k)\psi_m(\xi) \right) \]
\[ = D_m,\xi(k)\nabla_k \left( \frac{e^\rho(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \cdot \nabla E_m(\xi)\psi_m(\xi) \right) - D_m,\xi(k)^2(\nabla_k \omega_m(k) - \nabla E_m(\xi)) \frac{e^\rho(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \cdot \nabla E_m(\xi)\psi_m(\xi). \] (7.30)

(7.31)

Thus, considering the decomposition (7.24) for the first term and combining (7.27) with (7.30), noting (7.25), as well as (7.29) with (7.31) we obtain the desired upper bound.

\[ \square \]

**Lemma 7.6** ($y$-Bound). Let $e \in \mathbb{R}$. Suppose there exists an $m_0 > 0$ such that (2.12) holds for all $m \in (0, m_0)$. Let $|\xi| < 1$. Then for every $M > 0$ there exists a constant $C$, and a $\delta > 0$ such that for all $m \in (0, m_0)$ and all $n \in \mathbb{N}$,
\[ \sum_{\lambda_1, \ldots, \lambda_n} \int \sum_{i=1}^n n^{-1}|y_i|^\delta \| \hat{F}(U_m(\xi)\psi_m(\xi))_n(\lambda_1, y_1, \ldots, \lambda_n, y_n) \|^2 dy_1 \ldots dy_n \leq C, \]
whenever $|E_m(\xi)| < M$ and $\max\{|\partial_i^2 E_m(\xi)| : i = 1, 2, 3\} < M$. Here $\hat{F}(U_m(\xi)\psi_m(\xi))_n$ denotes the Fourier transform of the $n$-photon component of $U_m(\xi)\psi_m(\xi)$.

**Proof.** We drop the subscript $m$. We write $\hat{\psi} = U_m(\xi)\psi_m(\xi)$. Thus, $\hat{F}(\hat{\psi})_n$ denotes the Fourier transform of $\hat{\psi}(n)$ in all its $n$-components. We define the functions
\[ \hat{\psi}(n)(k) : (\lambda, k_1, \lambda_1, \ldots, k_{n-1}, \lambda_{n-1}) \mapsto \hat{\psi}(n)(k, \lambda, k_1, \lambda_1, \ldots, k_{n-1}, \lambda_{n-1}), \]
\[ \hat{F}(\hat{\psi})(y) : (\lambda, y_1, \lambda_1, \ldots, y_{n-1}, \lambda_{n-1}) \mapsto \hat{F}(\hat{\psi}_n)(y, \lambda, y_1, \lambda_1, \ldots, y_{n-1}, \lambda_{n-1}). \]

**Step 1:** There exists a $\delta > 0$ and a constant $C$ such that for all $a \in \mathbb{R}^3$,
\[ \int |1 - e^{-iay}|^2 \| \hat{F}(\hat{\psi})(y) \|^2 dy \leq \begin{cases} C|a|^\delta & \text{if } |a| < \frac{1}{2}\Lambda, \\ C & \text{if } |a| \geq \frac{1}{2}\Lambda. \end{cases} \]

The claim follows easily for $|a| \geq \frac{1}{2}\Lambda$, since $\hat{\psi}$ is a normalized state in Fock space and $|1 - e^{-iay}| \leq 2$. Now let’s consider the case $|a| < \frac{1}{2}\Lambda$. By the Fourier transform, we have the identity
\[ \int |1 - e^{-iay}|^2 \| \hat{F}(\hat{\psi})(y) \|^2 dy \]
\[ = \int \| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \|^2 dk \]
\[ = \int_{|k| < \Lambda - |a|} \| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \|^2 dk + \int_{\Lambda - |a| \leq |k|} \| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \|^2 dk. \] (7.32)
To estimate the second integral we use Proposition 7.4 and observe that the integrand vanishes for $|k| > \Lambda + |a|$, 
\[
\int_{|k| \leq |a|} \left\| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \right\|^2 dk \leq \text{const.} \int_{|k| \leq \Lambda + |a|} \left( \frac{1}{|k + a|^2} + \frac{1}{|k|^2} \right) dk 
\leq \text{const.} \int_{|k| \leq |a|} \frac{1}{|k|^3} dk,
\]
where const. denotes a numerical constant changing from line to line. Next we estimate the first integral and assume $|k| < \Lambda - |a|$. Using Proposition 7.5 we find 
\[
\| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \| = \left\| \int_0^1 \left( \frac{d}{dt} \hat{\psi}_n(k + ta) \right) dt \right\| 
\leq |a| \int_0^1 \| \nabla_k \hat{\psi}_n(k + ta) \| dt 
\leq \text{const.} |a| \int_0^1 \frac{\rho(k + ta)}{|k + ta| |\pi_3(k + ta)|} dt,
\]
where $\pi_3$ denotes the projection in $\mathbb{R}^3$ along the 3-axis and const. denotes a finite constant independent of $n$. Let $\pi_a$ denote the projection in $\mathbb{R}^3$ along the vector $a$ and let $\pi_{3,a}$ denote the projection in the (1, 2)-plane along $\pi_3 a$ (with convention that $\pi_{3,a} = \pi_3$, if $\pi_3 a = 0$). We find from (7.34)
\[
\| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \| \leq \text{const.} \frac{|a|}{|\pi_a(k)||\pi_{3,a}(k)|},
\]
(7.35)
On the other hand using Proposition 7.4 we obtain 
\[
\| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \| \leq \text{const.} \left( \frac{\rho(k + a)}{|k + a|} + \frac{\rho(k)}{|k|} \right).
\]
(7.36)
Introducing Inequalities (7.35) and (7.36) into the second integral of (7.32), we find for any $\theta$ with $0 \leq \theta \leq 1$,
\[
\int_{|k| \leq |a|} \left\| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \right\|^2 dk = \text{const.} |a|^{2p} \int_{|k| \leq |a|} \frac{1}{|\pi_a(k)|^{2p} |\pi_{3,a}(k)|^{2p}} \left( \frac{\rho(k + a)}{|k + a|} + \frac{\rho(k)}{|k|} \right)^{2(1-\theta)} dk.
\]
Now we use Young’s inequality: $bc \leq b^p/p + c^q/q$, whenever $p, q > 1$ and $p^{-1} + q^{-1} = 1$; and the convexity of $x \mapsto x^{2(1-\theta)q}$ on $\mathbb{R}_+$, for $0 < \theta < 1/2$. Thus for $0 < \theta < 1/2$,
\[
\int_{|k| \leq |a|} \left\| \hat{\psi}_n(k + a) - \hat{\psi}_n(k) \right\|^2 dk \leq |a|^{2p} \text{const.} \int_{|k| \leq \Lambda} \left( \frac{1}{|\pi_a(k)|^{4p} |\pi_{3,a}(k)|^{4p}} + \frac{1}{|k + a|} + \frac{1}{|k|} \right)^{2(1-\theta)q} dk.
\]
(7.37)
For any $q$ with $1 < q \leq 3/2$, we can choose $\theta > 0$ sufficiently small such that the right hand side is finite. Inserting (7.33) and (7.37) into (7.32) we obtain the desired estimate.

**Step 2:** Step 1 implies the statement of the Lemma.

From Step 1 we know that there exists a finite constant $C$ such that

$$\int \left|1 - e^{-ay}\right|^2 \left\|F_\psi(n)(y)\right\|^2 \frac{da}{|a|^3} \leq C.$$  

After interchanging the order of integration and a change of integration variables $b = |y/a|$, we find

$$C \geq \int \left\|F_\psi(n)(y)\right\|^2 \int \frac{1 - e^{-iyb}2}{|a|^{\delta/2}} \frac{da}{|a|^3} dy = \int \left\|F_\psi(n)(y)\right\|^2 |y|^{\delta/2} \int \frac{1 - e^{-ibb/|y|^2}2}{|b|^{\delta/2}} \frac{db}{|b|^3} dy,$$

where $c$ is nonzero and does not depend on $y$. 

### 8. Compactness Argument

Using a compactness argument we show Theorem 3.1. It states that for suitable $\xi$ there exists a sequence $(m_j)$ of positive numbers converging to zero such that $U_{m_j}(\xi) \psi_{m_j}(\xi)$ converges to a vector in the Hilbert space, say $\hat{\psi}_0$. Then we prove the main Theorem 3.5, which states that this vector $\hat{\psi}_0$ is indeed the ground state of the renormalized Hamiltonian using a semi-continuity argument of quadratic forms.

**Theorem 8.1.** Let $e \in \mathbb{R}$, and suppose there exists an $m_0 > 0$ such that the energy inequality (2.12) holds for all $m \in (0, m_0)$. Let $\xi \in \mathbb{R}^3$ with $|\xi| < 1$ such that $E$ is differentiable at $\xi$. Suppose there exists a sequence $(m_j)_{j \in \mathbb{N}}$ of positive numbers converging to zero such that

(i) $E_{m_j}$ is differentiable at $\xi$,

(ii) $\nabla E_{m_j}(\xi) \xrightarrow{j \to \infty} \nabla E(\xi)$,

(iii) for every $l = 1, 2, 3$ the partial derivatives $\partial_l^2 E_{m_j}(\xi)$ exist and satisfy

$$\sup_j (-\partial_l^2 E_{m_j}(\xi)) < \infty.$$  

Then $E(\xi)$ is an eigenvalue of $\tilde{H}(\xi)$, and there exists a subsequence of $(U_{m_j}(\xi)\psi_{m_j}(\xi))_{j \in \mathbb{N}}$ converging to the eigenvector.
Proof. Step 1: The sequence of vectors  
\[ \hat{\psi}_{m_j}(\xi) := U_{m_j}(\xi)\psi_{m_j}(\xi), \quad j \in \mathbb{N}, \]
lies in a compact subspace of the reduced Hilbert space  \( \mathcal{H} = \mathbb{C}^{2s+1} \otimes \mathcal{F}. \)

Let  \( L \) be the self-adjoint operator associated to the nonnegative and closed quadratic form  \( q \) in  \( \mathcal{H} \) defined by
\[
q(\phi) := \langle \phi, d\Gamma(1)\phi \rangle + \sum_{n=1}^{\infty} n^{-3}\langle \phi_n, \sum_{i=1}^{n} |y_i|^3 \phi_i \rangle + \langle \phi, H_f\phi \rangle
\]
on the natural form domain  \( D(q) \). We choose  \( \delta > 0 \) such that Lemma 7.6 holds. By this and Propositions 4.2 and 7.4, there exists a finite  \( C \) such that for all  \( m \) with  \( 0 < m < m_0 \),
\[ \psi_m \in \mathcal{K} := \{ \phi \in D(q) : \| \phi \| \leq 1, q(\phi) \leq C \}. \]
The set  \( \mathcal{K} \) is a compact subset of  \( \mathcal{H} \), provided  \( L \) has compact resolvent \( [\text{RS78}, \text{Theorem XIII.64}] \). Hence it remains to show that  \( L \) has compact resolvent. The operator  \( L \) preserves the  \( n \)-photon sectors. Let  \( L_n \) denote the restriction of  \( L \) to the  \( n \)-photon sector. From Rellich’s criterion \( [\text{RS78}, \text{Theorem XIII.65}] \) it follows that  \( L_n \) has compact resolvent. Therefore  \( \mu_l(L_n) \to \infty \) as  \( l \) tends to infinity, where  \( \mu_l \) denotes the  \( l \)-th eigenvalue obtained by the min-max principle. Moreover, since  \( \mu_l(L_n) \geq n \) for all  \( l, n \), it follows that  \( \mu_l(L) \to \infty \) as  \( l \to \infty \). Hence  \( L \) has a compact resolvent.

Step 2: The sequence in Step 1 has a subsequence which converges to a normalized vector  \( \hat{\psi}_0 \in \mathcal{H}. \)

This follows directly from Step 1 and the property of compact sets.

Step 3: The vector  \( \hat{\psi}_0 \) is an eigenvector of the renormalized fiber Hamiltonian  \( \widehat{H}(\xi) \) with eigenvalue  \( E(\xi). \)

Using lower semicontinuity of nonnegative quadratic forms it follows from Step 2 and Proposition 5.6 that for almost all  \( \xi \) with  \( |\xi| < 1 \)
\[
0 \leq \langle \hat{\psi}_0(\xi), (\widehat{H}(\xi) - E(\xi))\hat{\psi}_0(\xi) \rangle \\
\leq \lim\inf_{i \to \infty} \langle U_{m_i}(\xi)\psi_{m_i}(\xi), (\widehat{H}(\xi) - E(\xi))U_{m_i}(\xi)\psi_{m_i} \rangle = 0,
\]
i.e., that  \( \hat{\psi}_0(\xi) \) is a ground state of  \( \widehat{H}(\xi). \)

Now the above theorem implies together with Proposition 4.2 Theorems 3.1 and 3.5.

Proof of Theorems 3.1 and 3.5: Let  \( e \in \mathbb{R} \). By Theorem 2.5 in case  \( s = 0 \) or by assumption in case  \( s = 1/2 \), there exists an  \( m_0 > 0 \), such that (2.12) holds for all  \( m \in (0, m_0) \). Let  \( (m_j)_{j \in \mathbb{N}} \) be any sequence in  \( (0, m_0) \) which converges to zero. Then by Proposition 4.2 there exists a set  \( D \subset \mathbb{R}^3 \) of full Lebesgue measure with the following property: For all  \( \xi \in D \) the functions  \( E_{m_j}, j \in \mathbb{N}, \) and  \( E \) are differentiable and
(a) \( \nabla E_{m_j}(\xi) \xrightarrow{j \to \infty} \nabla E(\xi) \),

(b) the second partial derivatives \( \partial_{ll}^2 E_{m_j}(\xi) \) exist and satisfy for every \( l = 1, 2, 3 \) that
\[
\liminf_j (-\partial_{ll}^2 E_{m_j}(\xi)) < \infty.
\]

Now pick any \( \xi \in D \) with \( |\xi| < 1 \) and fix it. Then the assumptions of Theorem 8.1 hold. Therefore, Theorem 3.5 and Theorem 3.1 follow directly from Theorem 8.1.

\[\square\]

**A. Energy Estimates**

In this section we collect a few well known properties related to the canonical commutation relations, which we need in particular in Section 5.

**Lemma A.1.** Let \( f_1, \ldots, f_n \in L^2_{(\mathbb{R}^3 \times \mathbb{Z}_2)} \). Then for any \( \psi \in \mathcal{F} \) we have

\[
\|a(f_1) \cdots a(f_n)\psi\| \leq \left( \prod_{j=1}^n \|f_j\omega^{-1/2}_m\| \right) \|H_{l,m}^{n/2}\psi\|.
\]

**Proof.** In the following we use the notation \( k = (\lambda, k) \) and \( \int (\cdots)dk = \sum_{\lambda=1,2} \int (\cdots)dk \). By the definition of the annihilation operator, by Cauchy-Schwarz and Fubini, we find

\[
\|a(f_1) \cdots a(f_n)\psi\| \\
\leq \int |f_1(k_1) \cdots f_n(k_n)||a(k_1) \cdots a(k_n)\psi|dk_1 \cdots dk_n \\
\leq \left( \prod_{j=1}^n \|f_j\omega^{-1/2}_m\| \right) \left( \int \omega_1(k_1) \cdots \omega_n(k_n)||a(k_1) \cdots a(k_n)\psi|^2dk_1 \cdots dk_n \right)^{1/2}.
\]

To estimate the second factor we use

\[
\int \omega_1(k_1) \cdots \omega_n(k_n)||a(k_1) \cdots a(k_n)\psi|^2dk_1 \cdots dk_n \\
= \int \omega_1(k_2) \cdots \omega_n(k_n)||H_{l,m}^{1/2}a(k_2) \cdots a(k_n)\psi|^2dk_2 \cdots dk_n \\
= \int \omega_1(k_2) \cdots \omega_n(k_n)||H_{l,m}^{1/2}a(k_2) \cdots a(k_n)H_{l,m}^{-1/2}H_{l,m}^{1/2}\psi|^2dk_2 \cdots dk_n \\
= \int \omega_1(k_2) \cdots \omega_n(k_n)||H_{l,m}^{1/2}(H_{l,m} + \sum_{j=2}^n \omega_m(k_j))^{-1/2}a(k_2) \cdots a(k_n)H_{l,m}^{1/2}\psi|^2dk_2 \cdots dk_n \\
\leq \int \omega_1(k_2) \cdots \omega_n(k_n)||a(k_2) \cdots a(k_n)H_{l,m}^{1/2}\psi|^2dk_2 \cdots dk_n \\
\vdots \\
\leq \|H_{l,m}^{n/2}\psi\|^2,
\]

where we used the pull-through formula, cf. [BFS98, Lemma A.1], and that \( H_l\varphi = 0 \) implies \( a(f)\varphi = 0 \).
Lemma A.2. For any \( f \in L^2_{(m)}(\mathbb{Z}_2 \times \mathbb{R}^3) \) and \( \psi \in \mathcal{F} \)

\[
\| \phi(f)\psi \| \leq \sqrt{2}\|f\|_{(m)}\|(H_{f,m} + 1)^{1/2}\psi\|,
\]

\[
\|\phi(f_1)\phi(f_2)\psi\| \leq 2\|f_1\|_{(m)}\|f_2\|_{(m)}\|(H_{f,m} + 1)\psi\|.
\]

Note that the proof of Lemma A.2 can be found in [Ara18, Theorem 5.18], but for the convenience of the reader we give a proof below.

Proof. From the canonical commutation relations we get

\[
\|a(f)^*\psi\|^2 = \langle a^*(f)\psi, a^*(f)\psi \rangle = \|f\|^2\|\psi\|^2 + \|a(f)\psi\|^2.
\]

Now the first identity follows from the triangle inequality and Lemma A.1.

For the second identity we use again the canonical commutation relations and find

\[
\|a^*(f_1)a^*(f_2)\psi\|^2 = \langle a^*(f_1)a^*(f_2)\psi, a^*(f_1)a^*(f_2)\psi \rangle
\]

\[
\leq \|a(f_1)a(f_2)\psi\|^2 + 2\|f_1\|^2\|a(f_2)\psi\|^2 + 2\|f_2\|^2\|a(f_1)\psi\|^2 + 2\|f_1\|^2\|f_2\|^2,
\]

and

\[
\|a^*(g_1)a(g_2)\psi\|^2 = \langle a^*(g_1)a(g_2)\psi, a^*(g_1)a(g_2)\psi \rangle
\]

\[
\geq \|a(g_1)a(g_2)\psi\|^2 + 2\|g_1\|^2\|a(g_2)\psi\|^2 - 2\|f_1\|^2\|f_2\|^2 - \|f_1\|^2\|f_2\|^2
\]

The second inequality follows now by collecting estimates, the triangle inequality and Lemma A.1.

\[\square\]

B. Statements about CCR Algebras

Lemma B.1. Let \( A \) be a self-adjoint operator in \( \mathfrak{h} \) and \( f \in D(A) \subset \mathfrak{h} \). Then we have the relations

\[
d\Gamma(A)a^*(f) = a^*(f)d\Gamma(A) + a^*(Af),
\]

\[
d\Gamma(A)a(f) = a(f)d\Gamma(A) - a(Af),
\]

\[
d\Gamma(A)\phi(f) = \phi(f)d\Gamma(A) - i\phi(iAf)
\]
on \( \mathcal{F}_{im}(D(A)) \).

Proof. The first identity follows directly from the definition of the creation operators and \( d\Gamma(A) \). The second identity follows from the first by taking adjoints. The last identity follows from the first two. For details see for example [Ara18, Proposition 5.10]. \[\square\]

Lemma B.2. Let \( f, g \in \mathfrak{h} \). Then for every \( \psi \in D(d\Gamma(1)) \) one has

\[
(\phi(f)\phi(g) - \phi(g)\phi(f))\psi = i\text{Im}(f,g)\psi.
\]
Proof. This follows directly from the canonical commutation relations \([2.1]\). For details see for example \([\text{Ara18}, \text{Proposition 5.14}]\). \(\square\)

For \(f \in \mathfrak{h}\) we define

\[
W(f) := \exp(i\phi(f)) = \exp(i\pi(if)).
\]

Recall that \(\mathcal{F}_\text{fin}(\mathfrak{h})\) denotes the subspace of elements \(\psi \in \mathcal{F}(\mathfrak{h})\) such that \(\psi_n = 0\) for all but finitely many \(n\). For the definition of an analytic vector for an operator we refer the reader to \([\text{RS75}]\).

**Lemma B.3.** Let \(f, g \in \mathfrak{h}\) and \(\psi \in \mathcal{F}_\text{fin}(\mathfrak{h})\). Then for all \(t \in \mathbb{C}\) we have

\[
\sum_{n=0}^{\infty} \frac{||\phi(f)^n\psi||}{n!}|t|^n < \infty.
\]

In particular, \(\mathcal{F}_\text{fin}(\mathfrak{h})\) is a dense set of analytic vectors for \(\phi(f)\).

**Proof.** See the proof of \([\text{BR03, Proposition 5.2.3}]\). \(\square\)

**Lemma B.4.** Let \(f, g \in \mathfrak{h}\).

(a) Then \(W(f)D(\phi(g)) = D(\phi(g))\) and

\[
W(f)\phi(g)W(f)^* = \phi(g) - \text{Im}(f, g).
\]

(b) Then \(W(f)D(a^\#(g)) = D(a^\#(g))\) and

\[
\begin{align*}
W(f)a(g)W(f)^* &= a(g) - i2^{-1/2}\langle g, f \rangle, \\
W(f)a^*(g)W(f)^* &= a^*(g) + i2^{-1/2}\langle f, g \rangle.
\end{align*}
\]

The proof of the lemma before can be found for example in \([\text{BR03, Proposition 5.2.4}]\) and \([\text{Ara18, Corollary 5.12}]\). For the convenience of the reader we sketch a proof below.

**Sketch of proof.** (a) By Lemma \([\text{B.3}]\) every \(\psi \in \mathcal{F}_\text{fin}(\mathfrak{h})\) is analytic for \(\phi(f)\). Thus one can define \(\phi(g)W(f)^*\) on \(\psi\) by a power series expansion, which yields the identity

\[
\phi(g)W(f)^*\psi = W(f)^*\phi(g) - \text{Im}(f, g)\psi.
\]

Since \(\mathcal{F}_\text{fin}(\mathfrak{h})\) is an operator core for \(\phi(g)\), the claim now follows since \(\phi(g)\) is by definition a closed operator. For details we refer the reader to \([\text{BR03, Proposition 5.2.4}]\).

Part (b) is shown similarly. \(\square\)

**Lemma B.5.** Let \(A\) be a self-adjoint operator in \(\mathfrak{h}\) and \(f \in D(A)\). Then we have

\[
W(f)D(d\Gamma(A)) \supset D(d\Gamma(A)) \cap D(\phi(iAf))
\]

and as an identity on the latter

\[
W(f)d\Gamma(A)W(f)^* = d\Gamma(A) - \phi(iAf) + \frac{1}{2}(f, Af).
\]

\[\text{(B.1)}\]
Proof. First we note that by definition of $d\Gamma(A)$ we have on $\mathcal{F}_{\text{fin}}(D(A))$

$$e^{i\Gamma(A)t}f^n = \phi(e^{At}f)^n e^{i\Gamma(A)t}.$$ 

On the same domain we can differentiate with respect to $t$, and find at $t = 0$

$$\Gamma(A)\phi(f)^n = \phi(f)^n\Gamma(A) + \sum_{l=0}^{n-1} \phi(f)^l\phi'(fA)\phi(f)^{n-1-l}$$

$$= \phi(f)^n\Gamma(A) + n\phi(f)^n\phi'(fA) - \frac{n(n-1)}{2}f^n(f,fA),$$

where we used Lemma \ref{lemma:B.2} for the last identity. Multiplying with $(-i)^n(n!)^{-1}$ and summation over $n \in \mathbb{N}_0$ yields on $\mathcal{F}_{\text{fin}}(D(A))$

$$\Gamma(A)W(f)^* = iW(f)^*\Gamma(A) - iW(f)^*\phi(iA)f + \frac{1}{2}W(f)^*(f,fA).$$

Thus, it follows that \ref{eq:B.1} holds on $\mathcal{F}_{\text{fin}}(D(A))$. If $\psi \in D(\Gamma(A)) \cap D(\phi(iA))$, then $\psi_n = 1_{N \leq n}\psi \in \mathcal{F}_{\text{fin}}(D(A))$. Now $(\Gamma(A) - \phi(iA)f + \frac{1}{2}(f,fA))\psi_n \to (\Gamma(A) - \phi(iA)f + \frac{1}{2}(f,fA))\psi$. Therefore, $\psi \in D(W(f)d\Gamma(A)W(f)^*) = W(f)D(\Gamma(A))$ and by closedness \ref{eq:B.1} holds for $\psi$. \hfill $\Box$

C. Some Statements about Convex Functions

Proposition C.1. Let $F: \mathbb{R}^n \to \mathbb{R}$ be a function that satisfies the following conditions:

(i) $F(0) \leq F(x)$,

(ii) $F(x) \leq \frac{x^2}{2} + F(0)$,

(iii) $x \mapsto \frac{x^2}{2} - F(x)$ is a convex function.

Then we have

$$F(x - k) - F(x) \geq \begin{cases} -|k||x| + k^2/2, & \text{if } |k| \leq |x|, \\ -x^2/2, & \text{if } |k| \geq |x|. \end{cases}$$

Proof. A proof is given in \cite[Appendix A]{LMS07}.

Lemma C.2. Let $g: (a, b) \to \mathbb{R}$ be convex. Then for any compact interval $[c, d] \subset (a, b)$, the function $g$ is Lipschitz continuous on $[c, d]$ with Lipschitz constant $K$ bounded by $\max\{\varepsilon^{-1}|g(c) - g(c - \varepsilon)|, \varepsilon^{-1}|g(d + \varepsilon) - g(\varepsilon)|\}$ for any $\varepsilon > 0$ such that $[c - \varepsilon, d + \varepsilon] \subset (a, b)$.

Proof. If $F$ is convex, then for all $s, t, s', t' \in (a, b)$ such that $s \leq s' < t'$ and $s < t \leq t'$,

$$\frac{F(t) - F(s)}{t - s} \leq \frac{F(t') - F(s')}{t' - s'},$$

see for example \cite{Fol99}. The claim now follows from the above inequality. \hfill $\Box$
Lemma C.3. Let \( f: \mathbb{R} \to \mathbb{R} \) be a convex function with \( f(x) = f(-x) \) for all \( x \in \mathbb{R} \). Then \( f \) has a global minimum in \( 0 \) and in points \( x \geq 0 \) where \( f \) is differentiable we have \( f'(x) \geq 0 \).

Proof. By convexity and symmetry \( f(0) = f(\frac{1}{2}x - \frac{1}{2}x) \leq \frac{1}{2}(f(x) + f(-x)) = f(x) \). If \( f \) is differentiable at \( x > 0 \), then by elementary convexity we conclude that \( 0 \leq \frac{f(x) - f(0)}{x - 0} \leq f'(x) \). The case when \( x = 0 \) follows from the minimality property. \( \square \)

Let us state the following theorem, which implies in view of the previous lemma that every convex function is almost everywhere differentiable.

**Theorem C.4** (Rademacher). Let \( U \subset \mathbb{R}^n \) be open and \( f: U \to \mathbb{R} \) be Lipschitz continuous. Then \( f \) is almost everywhere in \( U \) differentiable.

A proof can be found in [Eva18]. In fact, for convex functions the second derivative exists almost everywhere. This is the statement of the so-called Alexandrov theorem, [Ale39].

**Theorem C.5** (Alexandrov). Let \( U \subset \mathbb{R}^n \) be open and \( f: U \to \mathbb{R} \) convex. Then \( f \) has a second derivative almost everywhere.

In the following two lemmas we study sequences of convex functions which converge pointwise. This will be needed to control properties of the ground state energy as the mass regularization is removed.

**Lemma C.6.** Let \( f, (f_n)_{n \in \mathbb{N}} \) be convex functions from \( \mathbb{R} \) to \( \mathbb{R} \) with \( f_n \xrightarrow{n \to \infty} f \) pointwise. Then there exists a set \( D \) with \( \mathbb{R} \setminus D \) a Lebesgue null set such that on \( D \) the functions \( f \) and \( f_n, n \in \mathbb{N} \), are differentiable and pointwise \( f'_n \to f' \).

Proof. By Lemma C.2 we see that each convex function is locally Lipschitz continuous, hence almost everywhere differentiable by Rademacher’s theorem, Theorem C.4. Hence there exists a set \( D \), where \( f_n, n \in \mathbb{N} \), and \( f \) are differentiable such that \( \mathbb{R}^d \setminus D \) is a set of Lebesgue measure zero. By pointwise convergence and again Lemma C.2 we see that \( f_n \) is a family of uniformly equicontinuous functions on any compact interval. Now uniform equicontinuity and pointwise convergences imply uniform convergence (see for example [RS81, Theorem I.27]). So let \( x \in D \) be arbitrary. Then we have that \( \delta_n := \sup_{y \in [x-1, x+1]} |f(y) - f_n(y)| \) satisfies \( \delta_n \xrightarrow{n \to \infty} 0 \). Let \( n_0 \in \mathbb{N} \) be such that \( \delta_n \leq 1 \) for all \( n \geq n_0 \). Since \( f_n \) is convex and differentiable in \( x \in D \), we have

\[
\frac{f'_n(x)}{\sqrt{\delta_n}} \leq \frac{f_n(x + \sqrt{\delta_n}) - f_n(x)}{\sqrt{\delta_n}} = f_n(x) + \sqrt{\delta_n} - f_n(x)
\]

Therefore, for \( n \geq n_0 \), we get

\[
f'_n(x) \leq \frac{f(x + \sqrt{\delta_n}) - f(x) + 2\delta_n}{\sqrt{\delta_n}} \xrightarrow{n \to \infty} f'(x).
\]

This implies \( \limsup_{n \to \infty} f'_n(x) \leq f'(x) \). Analogously, we obtain \( \liminf_{n \to \infty} f'_n(x) \geq f'(x) \). This shows that \( f'_n(x) \xrightarrow{n \to \infty} f'(x) \). \( \square \)
Lemma C.7. Let $f$ and $f_n$, $n \in \mathbb{N}$, be convex functions from $\mathbb{R}$ to $\mathbb{R}$ with $f_n \to f$ pointwise. Then there exists a set $D \subset \mathbb{R}$ such that $\mathbb{R} \setminus D$ has Lebesgue measure zero and the following holds.

(a) For all $x \in D$ and $n \in \mathbb{N}$ the function $f_n$ is twice differentiable in $x$ and $f_n''(x) \geq 0$.

(b) For all $x \in D$ we have $\liminf_n f_n''(x) < \infty$.

Remark C.8. We note that (a) follows directly from Alexandrov’s theorem, Theorem C.5. However, in the proof of (b), which we will present below, Part (a) will follow as an intermediate step.

Proof. Assume that $g$ is a convex function. Then it is locally Lipschitz continuous by Lemma C.2. Thus it is absolutely continuous and therefore by the fundamental theorem of calculus for Lebesgue integrals (see for example [Fol99] Theorem 3.35), $g$ is almost everywhere differentiable, $g'$ is locally in $L^1$, and for every $x \in \mathbb{R}$ we have

$$g(x) = \int_0^x g'(\xi)d\xi.$$  

By convexity $g'$ is monotone on the set of points where $g$ is differentiable. By possibly extending $g'$ to a monotone function on $\mathbb{R}$, we can assume without loss that $g'$ is a monotone function on $\mathbb{R}$. Since monotone functions have at most countably many discontinuities (see for example [Fol99] Theorem 3.23), we can assume furthermore that $g'$ is right continuous. Thus, there exists a unique measure $\mu$ on $\mathbb{R}$ such that for all $a, b \in \mathbb{R}$

$$\mu((a,b]) = g'(b) - g'(a)$$

(see for example [Fol99, Theorems 1.16 and 1.18]). Let $\mu = hd\lambda + \rho$ be its Lebesgue-Radon-Nykodim representation, where $\rho$ is mutually singular to the Lebesgue measure $\lambda$ and $h \geq 0$ is Borel measurable (see for example [Fol99, Theorem 3.8]). Then by taking suitable families shrinking nicely to $x$ we see that (see for example [Fol99, Theorem 3.22]) $g'$ is almost everywhere differentiable and $g'' = h$. This shows (a). Furthermore, we find

$$g'(b) - g'(a) = \mu((a,b]) = \int_a^b g''(x)dx + \rho((a,b]) \geq \int_a^b g''(x)dx. \quad (C.1)$$

By Lemma C.6 there exist for each $m \in \mathbb{N}$ two numbers $a \leq -m, m \leq b$ for which $f_n'(b)$ and $f_n'(a)$ converge as $n \to \infty$. Thus, inserting $f_n$ into (C.1), and using the Lemma of Fatou, we obtain

$$\lim_n(f_n'(b) - f_n'(a)) \geq \int_a^b \liminf_n f_n''(x)dx.$$  

It follows that $\liminf_n f_n''(x) < \infty$ almost everywhere. This shows (b).
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