Thermodynamics of Ricci-Gauss-Bonnet Dark Energy

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Abstract

We investigate the validity of generalized second law of thermodynamics of a physical system comprising of newly proposed dark energy model called Ricci Gauss-Bonnet and cold dark matter enveloped by apparent horizon and event horizon in flat Friedmann-Robertson-Walker (FRW) universe. For this purpose, Bekenstein entropy, Renyi, logarithmic and power law entropic corrections are used. It is found that this law exhibits the validity on both apparent and event horizons except for the case of logarithmic entropic correction at apparent horizon. Also, we check the thermodynamical equilibrium condition for all cases of entropy and found its vitality in all cases of entropy.

1 Introduction

The revelation of black holes thermodynamics motivated the physicist to examine the thermodynamics of cosmological models in accelerated expanding universe \cite{1,2,3}. Bekenstein and Hawking determined that the entropy of black hole is proportional to its event horizon \cite{4,5} which leads to important law named as generalized second law of thermodynamics (GSLT) for

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black hole physics. This law can be defined as the entropy of black hole and its exterior is always increasing. The primitive level of thermodynamics properties of horizons are exhibited by considering Einstein field equations as alternate of first law of thermodynamics [6, 7]. Gibbons and Hawking developed the Bekenstein’s idea for cosmological system by exhibiting that the entropy of cosmological event horizon is proportional to horizon area [8]. They represented the equality of apparent horizon and event horizon for de Sitter universe. The validity of GSLT was deeply studied later [9]-[11]. GSLT in cosmological scenario implies that the rate of change of entropy of horizon along with that of fluid inside it will always greater than or equal to zero. Its mathematical expression is

\[
\frac{dS_{\text{horizon}}}{dt} + \frac{dS_{\text{inside}}}{dt} \geq 0.
\]  

(1)

In addition, the holographic dark energy (HDE) is an interesting effort in exploring the nature of dark energy in the framework of quantum gravity. This model is motivated from the fundamental holographic principle, that arises from black hole thermodynamics and string theory [12]-[15]. HDE fascinated a large amount of research despite of some objections [16, 17]. The choice of the length scale \( L \) appearing in the holographic dark energy density \( \rho_{de} = 3M_{pl}L^{-2} \) gives rise to different dark energy models. One of the its crucial model is holographic Ricci dark energy model which is developed by assuming IR length scale as the average radius of Ricci scalar curvature, \( R^{-1/2} \) [18]-[20]. Moreover, its modified form is also presented and discussed widely [21]-[23].

Further, Wang et al. [24] observed that GSLT is verified at apparent horizon but not at event horizon for a specific model of dark energy. In case of new holographic dark energy, GSLT is valid fully on apparent horizon but partially on event horizon of universe [25]. The breakdown of GSLT was argued in case of event horizon enveloping the universe as compared to apparent horizon [26]. Setare [27] has derived the constraints on deceleration parameter in order to fulfill GSLT in case of non-flat universe enveloped by event horizon. The GSLT of thermodynamics has also been analyzed in case of Braneworld [28, 29] and generally Levelock gravity [30].

Moreover, modified matter part of Einstein Hilbert action results dynamical models such as cosmological constants, quintessence, k-essence, Chaplygin gas and holographic dark energy (HDE) models [31]-[37]. Moreover, several modified theories of gravity are \( f(R) \), \( f(T) \) [38], \( f(R,T) \) [39], \( f(G) \) [40],
For clear review of DE models and modified theories of gravity, see the reference [37]. Some authors [44]-[47] have also discussed various DE models in different frameworks and found interesting results.

Recently, Saridakis [48] Ricci-Gauss-Bonnet holographic dark energy in which Infrared cutoff is determined by both Ricci scalar and the Gauss-Bonnet invariant. Such a construction has the significant advantage that the Infrared cutoff, and consequently the HDE density, does not depend on the future or the past evolution of the universe, but only on its current features, and moreover it is determined by invariants, whose role is fundamental in gravitational theories. This model has IR cutoff form as

\[ \frac{1}{L^2} = -\alpha R + \beta \sqrt{|G|} \]

where \( \alpha \) and \( \beta \) are model parameters. In flat FRW geometry, the Ricci scalar \( R \) and the Gauss-Bonnet invariant \( G \) are given as

\[ R = -6(2H^2 + \ddot{H}) \]
\[ G = 24H^2(H^2 + \dot{H}) \]

In the present work, we examine the validity of GSLT by assuming various forms of entropy on apparent and event horizons. We have also examined whether each entropy attains maximum (thermodynamic equilibrium) by satisfying the condition \( \dot{S}_{\text{tot}} < 0 \). The plan of the paper is as follows: In sections 2 and 3, we have examined the validity of GSLT as well as thermal equilibrium condition at apparent and event horizons, respectively. The results are summarized in the last section.

## 2 Generalized Second Law of Thermodynamics at Apparent Horizon

According to GSLT, the entropy of horizon and entropy of matter resources inside horizon does not decrease with respect to time. Following Eq. (1), we can write

\[ \dot{S}_{\text{tot}} = \dot{S}_h + \dot{S}_{\text{in}} \geq 0. \]  

(2)

Here \( \dot{S}_h \) gives entropy of horizon and entropy of matter inside horizon is represented by \( \dot{S}_{\text{in}} \). Now considering spatially flat FRW universe, the first Friedmann equation is

\[ H^2 = \frac{\kappa^2}{3}(\rho_{\text{eff}} + P_{\text{eff}}). \]  

(3)
Here $\rho_{\text{eff}}$ and $P_{\text{eff}}$ are effective density and pressure, respectively. We have made the following two assumptions (i) an entropy is associated with the horizon in addition to the entropy of the universe inside the horizon, (ii) according to the local equilibrium hypothesis, there is no spontaneous exchange of energy between the horizon and fluid inside. Moreover, Gibb’s equation can be written as

$$TdS_{\text{in}} = P_{\text{eff}}dV + dE_{\text{in}}.$$  \hspace{1cm} (4)

Here $E_{\text{in}} = \rho_{\text{eff}}V$, $V = \frac{4\pi}{3}R_h^3$ and $T = \frac{1}{2\pi R_h}$ which modified above equation as follows

$$\dot{S}_{\text{in}} = 8\pi^2 R_h^3(\rho_{\text{eff}} + P_{\text{eff}})(\dot{R}_h - H R_h).$$  \hspace{1cm} (5)

For flat FRW universe, the Hubble horizon can be defined as

$$R_h = \frac{1}{H}.$$  \hspace{1cm} (6)

By utilizing above horizon, $P_{\text{eff}} = p_{de}$ (for cold dark matter $p_m = 0$) and $\rho_{\text{eff}} = \rho_d + \rho_m$ in Eq.(5), we can get

$$\dot{S}_{\text{in}} = 8\pi^2 R_h^3(\rho_{\text{eff}} + \rho_d \omega_{de})(\dot{R}_h - 1).$$  \hspace{1cm} (7)

From conservation equation, one can obtain

$$\dot{\rho}_d = 3H(1 + \omega_d)\rho_d \Rightarrow \omega_d = -1 - \frac{\dot{\rho}_d}{3H\rho_d}.$$  \hspace{1cm} (8)

Substituting the value of $\omega_{de}$ in Eq.(7), we get

$$\dot{S}_{\text{in}} = 8\pi^2 R_h^3\left(\rho_m - \frac{\dot{\rho}_d}{3H}\right)(\dot{R}_h - 1).$$  \hspace{1cm} (9)

Moreover, Ricci-Gauss Bonnet dark energy can be defined as follows

$$\rho_d = 3\left(6\alpha(2H^2 + \dot{H}) + 2\sqrt{3}\beta H \sqrt{|H^2 + \dot{H}|}\right).$$  \hspace{1cm} (10)

Here $\alpha$ and $\beta$ are the model parameters. Standard Ricci dark energy can be obtained by substituting $\beta = 0$ and $\alpha = 0$ yields a pure Gauss-Bonnet HDE. The density parameters can be introduced as

$$\Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_d = \frac{\rho_d}{3H^2}.$$  \hspace{1cm} (11)
According to first Friedman equation, we can obtain

$$\Omega_d + \Omega_m = 1. \quad (12)$$

Also, $\rho_m$ can be evaluated by using conservation equation as follows

$$\rho_m = \frac{\rho_{m_0}}{a^3}, \quad (13)$$

with $\rho_{m_0} = 3H_0^2\Omega_{m_0}$. By using this value of $\rho_m$, $\Omega_m$ takes the following form

$$\Omega_m = \frac{\Omega_{m_0}H_0^2}{a^3H^2}. \quad (14)$$

Using Eqs. (12) and (14), we can find $H$ as

$$H = \frac{H_0\sqrt{\Omega_{m_0}}}{a^3(1 - \Omega_d)}. \quad (15)$$

Differentiating $H$, we obtain

$$\dot{H} = -\frac{H^2}{2}\left(3 - \frac{\Omega_d}{1 - \Omega_d}\right). \quad (16)$$

where prime denotes the differentiation with respect to $x = \ln a$. Also, differentiation of $R_A$ with respect to $t$ leads to

$$\dot{R}_h = -\frac{\dot{H}}{H^2} = \frac{1}{2}\left(3 - \frac{\Omega_d}{1 - \Omega_d}\right). \quad (17)$$

We get the following value of $\dot{\Omega}_d$ by differentiating Eq. (11)

$$\dot{\Omega}_d = \left(\frac{\dot{\rho}_d}{3H^2} + \frac{\rho_d}{H^2}\right) \left(1 + \frac{\rho_d}{3H^2(1 - \Omega_d)}\right)^{-1}. \quad (18)$$

Now, $\dot{R}_h$ takes the form

$$\dot{R}_h = \frac{1}{2}\left(3 - \left(\frac{1}{1 - \Omega_d}\right) \left(\frac{\dot{\rho}_d}{3H^2} + \frac{\rho_d}{H^2}\right) \left(1 + \frac{\rho_d}{3H^2(1 - \Omega_d)}\right)^{-1}\right). \quad (19)$$

Also, Friedman first equation gives $\rho_m = 3H^2 - \rho_d$ and hence we can write

$$\dot{S}_m = 8(\pi)^2R_h^3 \left(3H^2 - \rho_d - \frac{\dot{\rho}_d}{3H}\right)(\dot{R}_h - 1). \quad (20)$$
By inserting Eq. (6) in above equation, we have

\[
\dot{S}_{in} = \frac{8(\pi)^2}{H^3} \left( 3H^2 - \rho_d - \frac{\dot{\rho}_d}{3H} \right) (\dot{R}_h - 1).
\] (21)

By using value of \( \dot{R}_h \) from Eq. (19), we get

\[
\dot{S}_{in} = \frac{8\pi^2}{H^3} \left( 3H^2 - \rho_d - \frac{\dot{\rho}_d}{3H} \right) \left( \frac{1}{2} - \left( \frac{1}{2(1 - \Omega_d)} \right) \left( \frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2} \right) \right)
\times \left( 1 + \frac{\rho_d}{3H^2(1 - \Omega_d)} \right)^{-1}.
\] (22)

Next, we will discuss the various expressions of entropy-area relations in order analyze the validity of GSLT on Hubble horizon.

### 2.1 Bekenstein Entropy

The Bekenstein entropy is given by

\[
S_h = \frac{A}{4G}.
\] (23)

By using \( G = 1, \quad c = 1 \) and \( A = 4\pi R_h^2 \) being the area of horizon, we get

\[
S_h = \pi R_h^2 \quad \Rightarrow \quad \dot{S}_h = 2\pi R_h \dot{R}_h.
\] (24)

By using the expressions of \( R_h \) and \( \dot{R}_h \), we have

\[
\dot{S}_h = \frac{\pi}{H} \left( 3 - \left( \frac{1}{1 - \Omega_d} \right) \left( \frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2} \right) \left( 1 + \frac{\rho_d}{3H^2(1 - \Omega_d)} \right)^{-1} \right).\] (25)

Equations (22) and (25) join to form

\[
\dot{S}_{tot} = \frac{8\pi^2}{H^3} \left( \frac{1}{2} - \left( \frac{1}{2(1 - \Omega_{de})} \right) \left( 3 - \frac{\rho_{de}}{H^2} \right) \left( 1 + \frac{\rho_{de}}{3H^2(1 - \Omega_{de})} \right)^{-1} \right)
\times \left( 6H^2 - \rho_{de} \right) + \frac{\pi}{H} \left( 3 - \left( \frac{1}{1 - \Omega_{de}} \right) \left( 1 + \frac{\rho_{de}}{3H^2(1 - \Omega_{de})} \right)^{-1} \right)
\times \left( 3 - \frac{\rho_{de}}{H^2} \right).
\] (26)
where $\dot{S}_{\text{tot}}$ represents the total entropy, i.e., $\dot{S}_{\text{tot}} = \dot{S}_{\text{in}} + \dot{S}_{h}$.

Now, we assume the power law form of scale factor, i.e., $a = a_0 t^n$, where $n$ and $a_0$ appear as constant parameters. Under this assumption, the values of $H$ and $R_h$ turns out to be $\frac{4}{n}, \frac{1}{n}$ respectively. In this way, $S_{\text{tot}}$ reduces to

$$S_{\text{tot}} = \frac{8\pi^2 t}{n^3} \left(3n^2 + U \left(\frac{2}{3n} - 1\right)\right) \left(\frac{1}{2} + \frac{U}{2n^2} \left(\frac{2}{3n} - 1\right)\right)$$

$$+ \ \frac{\pi t}{n} \left(3 + \frac{U}{n^2} \left(\frac{2}{3n} - 1\right)\right).$$

(27)

where $U = 3 \left(6\alpha(2n^2 - n) + 2\sqrt{3}\beta n\sqrt{n^2 - n}\right)$. In order to analyze the clear picture of validity of GSLT for this entropy on the Hubble horizon, we plot $\dot{S}_{\text{tot}}$ against cosmic time ($t$) by fixing constant parameters as $\alpha = 0.2$, $\beta = 0.001$ and $n = 4$ as shown in Figure 1. This shows that $\dot{S}_{\text{tot}}$ remains positive with increasing value of $t$ which confirms the validity of GSLT at apparent horizon with Bekenstein entropy.

To examine the thermodynamic equilibrium, we differentiate Eq.(27) to get $\ddot{S}_{\text{tot}}$ given below

$$\ddot{S}_{\text{tot}} = \frac{8\pi^2 t}{n^3} \left(3n^2 + U \left(\frac{2}{3n} - 1\right)\right) \left(\frac{1}{2} + \frac{U}{2n^2} \left(\frac{2}{3n} - 1\right)\right)$$

$$+ \ \frac{\pi t}{n} \left(3 + \frac{U}{n^2} \left(\frac{2}{3n} - 1\right)\right).$$

(28)

We plot $X = \dot{S}_{\text{tot}}$ versus $n$ in Figure 2 which shows that $\dot{S}_{\text{tot}} < 0$ for the selected range of $n$. Hence, thermal equilibrium condition is satisfied for Bekenstein entropy at apparent horizon.

### 2.2 Logarithmic corrections to entropy

Logarithmic corrections arises from loop quantum gravity due to thermal equilibrium and quantum fluctuations [49-55]. The entropy on apparent horizon can be defined as follows

$$S_h = \frac{A}{4G} + \eta \ln \left[ \frac{A}{4G} \right] - \xi \frac{4G}{A} + \gamma,$$

(29)
here $\eta$, $\xi$ and $\gamma$ are dimensionless constants. Differentiating with respect to $t$, we get

$$\dot{S}_h = \left(\frac{2\pi}{H} + 2\eta H + \frac{2\xi H^3}{\pi}\right)\dot{R}_h,$$

which takes the following form by inserting value of $\dot{R}_h$ from Eq.(19)

$$\dot{S}_h = \left(\frac{\pi}{H} + \eta H + \frac{\xi H^3}{\pi}\right)\left(3 - \left(\frac{1}{1 - \Omega_d}\right)\left(\frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2}\right)\left(1 + \frac{\rho_d}{3H^2(1 - \Omega_d)}\right)^{-1}\right).$$

In the presence of logarithmic entropy, $\dot{S}_{tot}$ can be obtained by using Eq.(22) and Eq.(31)

$$\dot{S}_{tot} = \frac{8\pi^2}{H^3} \left(3H^2 - \rho_d - \frac{\dot{\rho}_d}{3H}\right) \left(\frac{1}{2} - \left(\frac{1}{2(1 - \Omega_d)}\right)\right) \times \left(\frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2}\right) \left(1 + \frac{\rho_d}{3H^2(1 - \Omega_d)}\right)^{-1}\left(\frac{\pi}{H} + \eta H + \frac{\xi H^3}{\pi}\right) \times \left(3 - \left(\frac{1}{1 - \Omega_d}\right)\left(\frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2}\right)\left(1 + \frac{\rho_d}{3H^2(1 - \Omega_d)}\right)^{-1}\right).$$

By substituting value of scale factor, the above equation reduces to

$$\dot{S}_{tot} = \frac{8\pi^2t}{n^3} \left(3n^2 + U\left(\frac{2}{3n} - 1\right)\right) \left(\frac{1}{2} + \frac{U}{2n^2}\left(\frac{2}{3n} - 1\right)\right).$$
Differentiating above equation, we get

\[
\ddot{S}_{\text{tot}} = \frac{8\pi^2}{n^3} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) \left( \frac{1}{2} + \frac{U}{2n^2} \left( \frac{2}{3n} - 1 \right) \right) + \left( \frac{\pi}{n} - \frac{\eta}{t^2} - \frac{3\xi n^3}{\pi t^4} \right) \left( 3 + \frac{U}{n^2} \left( \frac{2}{3n} - 1 \right) \right). \tag{34}
\]

Figure 3 presents the plot of \(\dot{S}_{\text{tot}}\) at apparent horizon by taking logarithmic entropy as entropy at apparent horizon, where time is measured in second.

Figure 3: Plot of \(\dot{S}_{\text{tot}}\) by taking Logarithmic entropy as entropy at apparent horizon, where time is measured in second.

Entropy at apparent horizon. Here we have taken \(\eta = 3.8\) and \(\xi = 3\) along with same values of \(\alpha, \beta\) and \(n\) as in above mentioned case. Here \(\dot{S}_{\text{tot}}\) remains negative for \(t < 1.5\) while it moves in positive direction \(t \geq 1.5\). Hence, validity of GSLT is verified for \(t \geq 1.5\) at apparent horizon with logarithmic entropy. Figure 4 shows that \(X = \ddot{S}_{\text{tot}} < 0\) with increasing value of \(t\) and \(n = 1.5\). Hence, for logarithmic entropy at apparent horizon, the condition of thermal equilibrium is satisfied.

\[\text{Figure 4: Plot of } X = \ddot{S}_{\text{tot}} \text{ by taking Logarithmic entropy as entropy at apparent horizon, where time is measured in second.}\]

\[\text{2.3 Renyi Entropy}\]

A novel type of Renyi entropy was recommended by Biro and Czinner [56] on black hole horizons by considering Bekenstein-Hawking entropy as non
extensive Tsallis entropy. The modified Renyi entropy can be defined as \[ S_h = \frac{1}{\lambda} \ln \left[ 1 + \lambda \frac{A}{4G} \right]. \] (35)

it behaves as Bekenstein entropy for \( \lambda = 0 \). Differentiating with respect to \( t \), we obtain
\[
\dot{S}_h = \frac{2\pi H}{H^2 + \lambda \pi} \dot{R}_h. \tag{36}
\]

Using Eq. (19) in above equation, we get
\[
\dot{S}_h = \frac{\pi H}{H^2 + \lambda \pi} \left( 3 - \left( \frac{1}{1 - \Omega_d} \right) \left( \frac{\dot{\rho}_d}{3H^3} + \frac{\rho_{de}}{H^2} \right) \left( 1 + \frac{\rho_d}{3H^2(1 - \Omega_d)} \right)^{-1} \right). \tag{37}
\]

Combining Eqs. (22) and (37) to get
\[
\dot{S}_{\text{tot}} = \frac{8(\pi^2)}{H^3} \left( 3H^2 - \rho_d - \frac{\dot{\rho}_d}{3H} \right) \left( \frac{1}{2} - \left( \frac{1}{2(1 - \Omega_d)} \right) \left( \frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2} \right) \right)
\times \left( 1 + \frac{\rho_d}{3H^2(1 - \Omega_d)} \right)^{-1} + \frac{\pi H}{H^2 + \lambda \pi} \left( 3 - \left( \frac{1}{1 - \Omega_d} \right) \right)
\times \left( \frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2} \right) \left( 1 + \frac{\rho_d}{3H^2(1 - \Omega_d)} \right)^{-1}. \tag{38}
\]

For power law scale factor, we obtain
\[
\dot{S}_{\text{tot}} = \frac{8\pi^2 t}{n^3} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) \left( \frac{1}{2} + \frac{U}{2n^2} \left( \frac{2}{3n} - 1 \right) \right)
\times \left( \frac{n\pi t}{n^2 + \pi\lambda t^2} \right) \left( 3 + \frac{U}{n^2} \left( \frac{2}{3n} - 1 \right) \right). \tag{39}
\]

The plot of \( \dot{S}_{\text{tot}} \) by taking Renyi entropy at apparent horizon is presented by Figure 5. Here \( \alpha, \beta \) and \( n \) has same values like previous case and \( \lambda = 1.5 \). In this case, \( \dot{S}_{\text{tot}} \) behaves positively with the passage of time which verifies the validity of GSLT for the present case. Further, differentiating above equation, we get
\[
\ddot{S}_{\text{tot}} = \frac{8\pi^2}{n^3} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) \left( \frac{1}{2} + \frac{U}{2n^2} \left( \frac{2}{3n} - 1 \right) \right)
\times \left( \frac{n\pi t}{n^2 + \pi\lambda t^2} \right) \left( 3 + \frac{U}{n^2} \left( \frac{2}{3n} - 1 \right) \right). \]
The plot of this expression is shown in Figure 6 which shows that $\ddot{S}_{\text{tot}} < 0$ for $n = 1.5$ with the passage of time. Hence, the condition for thermal equilibrium is satisfied in case of Renyi entropy at apparent horizon.

![Figure 5: Plot of $S'_{\text{tot}}$ by taking Renyi entropy as entropy at apparent horizon, where time is measured in second.](image)

![Figure 6: Plot of $X = \ddot{S}_{\text{tot}}$ by taking Renyi entropy as entropy at apparent horizon, where time is measured in second.](image)

### 2.4 Power Law Entropic correction

The power law corrections to entropy appear in dealing with entanglement of quantum fields in and out of the horizon [58]. The corrected entropy takes the form [59]

$$S_h = \frac{A}{4G} \left(1 - k \mu A^{1-\frac{3}{2}}\right),$$  

with $k = \frac{\mu(4\pi)^{\frac{3}{2}}}{(4-\mu) r_c^{2-\mu}}$, $r_c$ is crossover length and $\mu$ appears as a constant.

$$\dot{S}_h = \frac{\pi \dot{R}_h}{H} \left(2 - k \mu (4 - \mu) \left(\frac{4\pi}{H^2}\right)^{1-\frac{3}{2}}\right).$$

Utilization of Eq.(19) in above equation leads to

$$\dot{S}_h = \frac{\pi}{2H} \left(2 - k \mu (4 - \mu) \left(\frac{4\pi}{H^2}\right)^{1-\frac{3}{2}}\right) \left(3 - \left(\frac{1}{1-\Omega_d}\right) \left(\frac{\rho_d}{3H^3} + \frac{\rho_d}{H^2}\right)\right)$$
Joining Eqs. (22) and (43) to obtain

\[
\dot{S}_{\text{tot}} = \frac{8(\pi)^2}{H^3} \left( 3H^2 - \rho_d - \frac{\dot{\rho}_d}{3H} \right) \left( \frac{1}{2} - \left( \frac{1}{2} - \frac{1}{1 - \Omega_d} \right) \left( \frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2} \right) \right) \times \left( 1 + \frac{\rho_d}{3H^2(1 - \Omega_d)} \right)^{-1} \times \left( 3 - \left( \frac{1}{1 - \Omega_d} \right) \left( \frac{\dot{\rho}_d}{3H^3} + \frac{\rho_d}{H^2} \right) \left( 1 + \frac{\rho_d}{3H^2(1 - \Omega_d)} \right)^{-1} \right).
\]

(44)

In the presence of scale factor, the above expression turns out to be

\[
\dot{S}_{\text{tot}} = \frac{8\pi^2 t}{n^3} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) \left( 1 + \frac{U}{2n^2} \left( \frac{2}{3n} - 1 \right) \right) + \left( \frac{\pi t}{2n} \left( 2 - \frac{\mu}{(nr_c)^{2-\mu}} \right) \left( 3 + \frac{U}{n^2} \left( \frac{2}{3n} - 1 \right) \right) \right).
\]

(45)

By taking power Law entropy at apparent horizon, \( \dot{S}_{\text{tot}} \) is plotted at apparent horizon as shown in Figure 7. With same values for \( \alpha, \beta \) and \( n \), we have taken \( \mu = 5 \) and \( r_c = 2 \). Here the effectiveness of GSLT at apparent horizon is certified by positive moves of \( \dot{S}_{\text{tot}} \) with increasing \( t \). Differentiating above equation, we get

\[
\ddot{S}_{\text{tot}} = \frac{8\pi^2 t}{n^3} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) \left( 1 + \frac{U}{2n^2} \left( \frac{2}{3n} - 1 \right) \right) \times \left( \frac{\pi t}{n} - \frac{\pi \mu}{2n} (nr_c)^{(2-\mu)\frac{t}{(\mu-1)}} \right) \left( 1 + \frac{U}{n^2} \left( \frac{2}{3n} - 1 \right) \right) + \left( \frac{\pi t}{2n} \left( 2 - \frac{\mu}{(nr_c)^{2-\mu}} \right) \left( 3 + \frac{U}{n^2} \left( \frac{2}{3n} - 1 \right) \right) \right).
\]

(46)

Just like above mentioned three cases, in case of power law entropy at apparent horizon, the condition for thermal equilibrium is satisfied with the passage of cosmic time as shown in Figure 8.

3 Generalized Second Law of Thermodynamics at event horizon

In this section, we study GSL of thermodynamics at event horizon which is defined as \( R_h = a(t) \int_0^\infty \frac{d\dot{\rho}}{a(t)} \). Its derivative with respect to time is given by
Figure 7: Plot of $\dot{S}_{\text{tot}}$ by taking Power Law entropy as entropy at apparent horizon, where time is measured in second.

Figure 8: Plot of $X = S_{\text{tot}}^{\dot{}}$ by taking Power Law entropy as entropy at apparent horizon, where time is measured in second.

$\dot{R}_h = HR_h - 1$. The temperature we used in this section is $T = \frac{bH}{2\pi}$, where $b$ is a constant. For the present case, rewriting Eq. (4) by using value of $T$ and $\dot{R}_h$, we have following equation for entropy inside horizon

$$\dot{S}_m = -\frac{8\pi^2}{bH} R_h^2 \left(3H^2 - \rho_d - \frac{\dot{\rho}_d}{3H}\right).$$

(47)

### 3.1 Bekenstein Entropy

Under this scenario, Eq. (24) can be written as

$$\dot{S}_h = 2\pi R_h (HR_h - 1).$$

(48)

The equation for $\dot{S}_{\text{tot}}$ can be obtained by using Eqs. (43) and (48) as follows

$$\dot{S}_{\text{tot}} = -\frac{8\pi^2}{bH} R_h^2 \left(3H^2 - \rho_{de} - \frac{\dot{\rho}_{de}}{3H}\right) + 2\pi R_h (HR_h - 1).$$

(49)

By putting values of scale factor and $R_h$ in above equation, we have

$$\dot{S}_{\text{tot}} = -\frac{8\pi^2 t}{n(n-1)^2 b} \left(3n^2 + U \left(\frac{2}{3n} - 1\right)\right) + 2\pi t.$$  

(50)

Differentiating above equation with respect to $t$, we get

$$\dot{S}_{\text{tot}} = -\frac{8\pi^2}{n(n-1)^2 b} \left(3n^2 + U \left(\frac{2}{3n} - 1\right)\right) + 2\pi.$$  

(51)
Figure 9 contains the plot of $\dot{S}_{tot}$ by taking Bekenstein entropy at event horizon. Here we have taken $\alpha = 0.2$, $\beta = 0.001$ and $n = 4$. Its clear from figure that $\dot{S}_{tot}$ remains positive with increasing value of $t$. This confirms the validity of GSLT at event horizon with Bekenstein entropy. Figure 10 shows that $\ddot{S}_{tot} < 0$ for increasing values of $n$. Hence, at event horizon, the Bekenstein entropy fulfilled the condition of thermodynamic equilibrium.

### 3.2 Logarithmic Entropy

For this entropy at event horizon, Eq.(29) leads to

$$\dot{S}_h = \left( \frac{2\pi}{H} + 2\eta H + \frac{2\xi H^3}{\pi} \right) \dot{R}_h. \quad (52)$$

By using Eqs.(47) and (52), the expression of $\dot{S}_{tot}$ can be written as

$$\dot{S}_{tot} = -\frac{8\pi^2}{bH} \dot{R}_h \left( 3H^2 - \rho_{de} - \frac{\dot{\rho}_{de}}{3H} \right) + \left( \frac{2\pi}{H} + 2\eta H + \frac{2\xi H^3}{\pi} \right) \dot{R}_h. \quad (53)$$

The following equation is obtained by using values of scale factor and $R_h$

$$\dot{S}_{tot} = -\frac{8\pi^2 t}{n(n-1)^2 b} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) + \frac{2\pi t}{(n-1)^2} + \frac{2\eta}{t} + \frac{2\xi(n-1)^2}{\pi t^3}. \quad (54)$$
Differentiating above equation with respect to $t$, we obtain

$$S_{tot}^{\ddot{}} = -\frac{8\pi^2}{n(n-1)^2b} \left( 3n^2 + U\left(\frac{2}{3n} - 1\right) \right) + \frac{2\pi}{(n-1)^2} - \frac{2\eta}{t^2} - \frac{6\xi(n-1)^2}{\pi t^4}. \quad (55)$$

Figure 11: Plot of $S_{tot}$ by taking logarithmic entropy as entropy at event horizon, where time is measured in second.

Figure 12: Plot of $X = S_{tot}^{\ddot{}}$ by taking logarithmic entropy as entropy at event horizon, where time is measured in second.

Figure 11 presents the plot of $S_{tot}^{\ddot{}}$ by taking logarithmic entropy at event horizon. Here we have taken $\eta = 4$ and $\xi = 6$ along with same values of $\alpha$, $\beta$ and $n$ as in above mentioned case. Clearly, $S_{tot}^{\ddot{}}$ moves in positive direction as value of $t$ increases. The validity of GSLT is verified at event horizon in the presence of logarithmic entropy. From Figure 12, we can see that $S_{tot}^{\ddot{}} < 0$ for $n = 1.5$. Hence, for this case, thermodynamic equilibrium condition holds.

### 3.3 Renyi Entropy

The following form is obtained from Eq. (34), by substituting value for $\dot{R}_h$

$$\dot{S}_h = \frac{2\pi H}{H^2 + \lambda \pi} (H R_h - 1). \quad (56)$$

Joining Eqs. (47) and (56), we get

$$S_{tot}^{\ddot{}} = -\frac{8\pi^2}{bH} R_h^2 \left( 3H^2 - \rho_{de} - \frac{\dot{\rho}_{de}}{3H} \right) + \frac{2\pi H}{H^2 + \lambda \pi} (H R_h - 1). \quad (57)$$
By using values of scale factor and $R_h$, above equation reduces to
\[
\dot{S}_{\text{tot}} = -\frac{8\pi^2 n}{n(n-1)^2 b} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) + \frac{2\pi t}{(n-1)^2 + \lambda \pi t^2}. \tag{58}
\]
Differentiating above equation with respect to $t$, we get
\[
\ddot{S}_{\text{tot}} = -\frac{8\pi^2 n}{n(n-1)^2 b} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) + \frac{2\pi (n-1)^2 - 2\pi^2 \lambda t^2}{((n-1)^2 + \lambda \pi t^2)^2}. \tag{59}
\]
The plot of $\dot{S}_{\text{tot}}$ for Renyi entropy at event horizon is presented in Figure 13.

![Figure 13: Plot of $\dot{S}_{\text{tot}}$ by taking Renyi entropy as entropy at event horizon.](image1)

![Figure 14: Plot of $X = \ddot{S}_{\text{tot}}$ by taking Renyi entropy as entropy at event horizon.](image2)

Here $\alpha$, $\beta$ and $n$ has same values like previous case while $\lambda = 1.5$. In this case, $\dot{S}_{\text{tot}}$ behaves positively with the passage of time which verifies the validity of GSLT. Figure 14 shows that the trajectories of $\ddot{S}_{\text{tot}}$ remains negative for increasing of $t$ with $n = 1.5$. This means that the present scenario obeys the condition for thermodynamic equilibrium.

### 3.4 Power law Entropy

Under conditions of present section, Eq. (41) reduces to
\[
\dot{S}_h = \frac{\pi}{H} \left( 2 - k_\mu (4 - \mu) \left( \frac{4\pi}{H^2} \right)^{1-\frac{\gamma}{2}} \right) (HR_h - 1). \tag{60}
\]
Joining Eqs. (47) and (60) to get the following equation
\[
\dot{S}_{\text{tot}} = -\frac{8\pi^2}{bH} R_h^2 \left( 3H^2 - \rho_{de} - \dot{\rho}_{de} \right) + \frac{\pi}{H} \left( 2 - k_\mu (4 - \mu) \left( \frac{4\pi}{H^2} \right)^{1-\frac{\gamma}{2}} \right) (HR_h - 1). \tag{61}
\]
Inserting conditions for scale factor and \( R_h \) in above equation, we get

\[
\dot{S}_{tot} = -\frac{8\pi^2 t}{n(n-1)^2 b} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) + \left( 2 - \mu \left( \frac{t}{r_c(n-1)} \right)^{2-\mu} \right) \frac{\pi t}{(n-1)^2}.
\]  

(62)

The plot of this expression is displayed in Figure 15 with same values for \( \alpha \), \( \beta \) and \( n \) while \( \mu = 5 \) and \( r_c = 2 \). Here the effectiveness of GSLT at event horizon is certified by positive moves of \( S_{tot}' \) with increasing \( t \). Differentiating with respect to \( t \), we obtain

\[
\ddot{S}_{tot} = -\frac{8\pi^2}{n(n-1)^2 b} \left( 3n^2 + U \left( \frac{2}{3n} - 1 \right) \right) + \frac{2\pi}{(n-1)^2} - \frac{\mu \pi (3 - \mu) t^{2-\mu}}{(r_c(n-1))^{2-\mu}(n-1)^2}.
\]  

(63)

Figure 16 shows that the present scenario fulfils the thermodynamic equilibrium condition for power law entropy at event horizon.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig15.png}
\caption{Plot of \( S_{tot}' \) by taking Power Law entropy as entropy at event horizon.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig16.png}
\caption{Plot of \( X = \ddot{S}_{tot} \) by taking Power Law entropy as entropy at event horizon.}
\end{figure}

4 Conclusion

The concept of thermodynamics in cosmological system originates through black hole physics. It was suggested [60] that the temperature of Hawking radiations emitting from black holes is proportional to their corresponding surface gravity on the event horizon. Jacobson [61] found a relation between thermodynamics and the Einstein field equations. He derived this relation
on the basis of entropy-horizon area proportionality relation along with first
law of thermodynamics (also called Clausius relation) \( dQ = T dS \), where
\( dQ \), \( T \) and \( dS \) indicate the exchange in energy, temperature and entropy
change for a given system. It was shown that the field equations for any
spherically symmetric spacetime can be expressed as \( T dS = dE + P dV \) (\( E \), \( P \)
and \( V \) represent the internal energy, pressure and volume of the spherical
system) for any horizon \([62]\). By utilizing this relation, GSLT has been
studied extensively in the scenario of expanding behavior of the universe. In
order to discuss GSLT, horizon entropy of the universe can be taken as one
quarter of its horizon area \([63]\) or power law corrected \([64]\) or logarithmic
corrected \([65]\) forms. Many people have explored the validity of GSLT of
different systems including interaction of two fluid components like DE and
dark matter \([66]\), as well as interaction of three components of fluid \([67]\) in
the FRW universe by using simple horizon entropy of the universe. The
thermodynamical analysis widely performed in modified theories of gravity
\([68]\).

Motivated by above mentioned works, we have considered a newly pro-
posed DE model named as Ricci Gauss-Bonnet DE in flat FRW universe.
We have developed thermodynamical quantities and analyzed the validity of
GSLT and thermodynamic equilibrium. For dense elaboration of thermody-
namics of present DE model, we have assumed various entropy corrections
such as Bekenstein entropy, Logarithmic corrected entropy, Renyi entropy
and power law entropy at apparent horizon as well as event horizon of the
universe. We have found that GSLT holds for all cases of entropies as well
as horizons. Also, thermal equilibrium condition satisfied under certain con-
ditions on constant parameters. The detailed of results are as follows:

**On Apparent Horizon**

By utilizing usual entropy, GSLT on the apparent horizon has shown in Fig-
ure 1 which shows that \( S_{tot} \) remains positive with increasing value of \( t \) and
confirms its validity. Figure 2 has also indicated that thermal equilibrium
condition is satisfied for Bekenstein entropy at apparent horizon. For loga-
rithmic corrected entropy, GSLT on apparent horizon has displayed in Figure
3 which exhibits that GSLT remains valid for \( t \geq 1.5 \). However, Figure 4
shows that \( X = S_{tot} < 0 \) with increasing value of \( t \) and \( n = 1.5 \). Hence, for
logarithmic entropy at apparent horizon, the condition of thermal equilib-
rium is satisfied.
The plot of \( \dot{S}_{\text{tot}} \) by taking Renyi entropy at apparent horizon has displayed in Figure 5 which behaves positively with the passage of time and exhibits the validity of GSLT. Also, for this entropy, the condition for thermal equilibrium has been satisfied in case of Renyi entropy at apparent horizon (Figure 6). By taking power Law entropy at apparent horizon, \( \dot{S}_{\text{tot}} \) is plotted at apparent horizon as shown in Figure 7. Here the effectiveness of GSLT at apparent horizon is certified by positive moves of \( \dot{S}_{\text{tot}} \) with increasing \( t \). Just like above mentioned three cases, in case of power law entropy at apparent horizon, the condition for thermal equilibrium is satisfied with the passage of cosmic time as shown in Figure 8.

**On Event Horizon**

It has been observed from Figure 9 that GSLT remains valid at event horizon with Bekenstein entropy. Also, at event horizon, the Bekenstein entropy fulfilled the condition of thermodynamic equilibrium (Figure 10). The validity of GSLT is verified at event horizon in the presence of logarithmic entropy (Figure 11). From Figure 12, we can see that \( \ddot{S}_{\text{tot}} < 0 \) for \( n = 1.5 \) which leads to the validity of thermal equilibrium condition.

The plot of \( S_{\text{tot}} \) for Renyi entropy at event horizon is presented in Figure 13. It is observed that \( \dot{S}_{\text{tot}} \) behaves positively with the passage of time which verifies the validity of GSLT. Figure 14 shows that the trajectories of \( \ddot{S}_{\text{tot}} \) remains negative for increasing of \( t \) with \( n = 1.5 \). This means that the present scenario obeys the condition for thermodynamic equilibrium. The plot of \( \dot{S}_{\text{tot}} \) for power law corrected entropy is displayed in Figure 15 and observe that GSLT holds in this case. Figure 16 shows that the present scenario fulfils the thermodynamic equilibrium condition for power law entropy at event horizon.

The authors declare that there is no conflict of interest regarding the publication of this paper.

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