ON THE EXISTENCE AND CUSP SINGULARITY OF SOLUTIONS TO SEMILINEAR GENERALIZED TRICOMI EQUATIONS WITH DISCONTINUOUS INITIAL DATA

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Abstract

In this paper, we are concerned with the local existence and singularity structure of low regularity solutions to the semilinear generalized Tricomi equation $\partial_t^2 u - t^m \Delta u = f(t, x, u)$ with typical discontinuous initial data $(u(0, x), \partial_t u(0, x)) = (0, \varphi(x))$; here $m \in \mathbb{N}$, $x = (x_1, \ldots, x_n)$, $n \geq 2$, and $f(t, x, u)$ is $C^\infty$ smooth in its arguments. When the initial data $\varphi(x)$ is a homogeneous function of degree zero or a piecewise smooth function singular along the hyperplane $\{t = x_1 = 0\}$, it is shown that the local solution $u(t, x) \in L^\infty([0, T] \times \mathbb{R}^n)$ exists and is $C^\infty$ away from the forward cuspidal cone $\Gamma_0 = \{(t, x): t > 0, |x|^2 = \frac{4t^{m+2}}{(m + 2)^2}\}$ and the characteristic cuspidal wedge $\Gamma_1^+ = \{(t, x): t > 0, x_1 = \pm \frac{2t^{\frac{m+1}{2}}}{m + 2}\}$, respectively. On the other hand, for $n = 2$ and piecewise smooth initial data $\varphi(x)$ singular along the two straight lines $\{t = x_1 = 0\}$ and $\{t = x_2 = 0\}$, we establish the local existence of a solution $u(t, x) \in L^\infty([0, T] \times \mathbb{R}^2) \cap C([0, T], C^{\frac{m+n}{2}, \frac{m+n}{2}}(\mathbb{R}^2))$ and show further that $u(t, x) \not\in C^2([0, T] \times \mathbb{R}^2 \setminus (\Gamma_0 \cup \Gamma_1^+ \cup \Gamma_2^+))$ in general due to the degenerate character of the equation under study; here $\Gamma_2^+ = \{(t, x): t > 0, x_2 = \pm \frac{2t^{\frac{m+1}{2}}}{m + 2}\}$. This is an essential difference to the well-known result for solutions $v(t, x) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \setminus (\Sigma_0 \cup \Sigma_1^+ \cup \Sigma_2^+))$ to the 2-D semilinear wave equation $\partial_t^2 v - \Delta v = f(t, x, v)$ with $(v(0, x), \partial_t v(0, x)) = (0, \varphi(x))$, where $\Sigma_0 = \{t = |x|\}$, $\Sigma_1^+ = \{t = x_1\}$, and $\Sigma_2^+ = \{t = \pm x_2\}$.

Keywords: Generalized Tricomi equation, confluent hypergeometric function, hypergeometric function, cusp singularity, tangent vector fields, conormal space

Mathematical Subject Classification 2000: 35L70, 35L65, 35L67, 76N15

§1. Introduction

In this paper, we will study the local existence and the singularity structure of low regularity solution to the following $n$-dimensional semilinear generalized Tricomi equation

$$
\begin{cases}
\partial_t^2 u - t^m \Delta u = f(t, x, u), & (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
u(0, x) = 0, & \partial_t u(0, x) = \varphi(x),
\end{cases}
$$

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where $m \in \mathbb{N}$, $x = (x_1, ..., x_n)$, $n \geq 2$, $\Delta = \sum_{i=1}^{n} \partial_i^2$, $f(t, x, u)$ is $C^\infty$ smooth on its arguments and has a compact support on the variable $x$, and the typical discontinuous initial data $\varphi(x)$ satisfies one of the assumptions:

(A1) $\varphi(x) = g(x, \frac{x}{|x|})$, here $g(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and has a compact support in $B(0, 1) \times B(0, 2)$;

(A2) $\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x_1 < 0, \\ \varphi_2(x) & \text{for } x_1 > 0, \end{cases}$ with $\varphi_1(x), \varphi_2(x) \in C^\infty_0(\mathbb{R}^n)$ and $\varphi_1(0) \neq \varphi_2(0)$;

(A3) For $n = 2$, $\varphi(x) = \begin{cases} \psi_1(x) & \text{for } x_1 > 0, x_2 > 0, \\ \psi_2(x) & \text{for } x_1 < 0, x_2 > 0, \\ \psi_3(x) & \text{for } x_1 < 0, x_2 < 0, \\ \psi_4(x) & \text{for } x_1 > 0, x_2 < 0, \end{cases}$ with $\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x) \in C^\infty_0(\mathbb{R}^n)(1 \leq i \leq 4)$ and $\psi_i(0) \neq \psi_j(0)$ for some $i \neq j (1 \leq i < j \leq 4)$.

It is noted that $\varphi(x) = \psi(x) \frac{x_1}{|x|}$ with $\psi(x) \in C^\infty_0(\mathbb{B}(0, 1))$ is a special function satisfying (A1), which has a singularity at the origin.

Under the assumptions (A1) – (A3), we now state the main results in this paper.

**Theorem 1.1.** There exists a constant $T > 0$ such that

(i) Under the condition (A1), (1.1) has a unique solution $u(t, x) \in C^2([0, T], H^{\frac{m+4}{m+2}}(\mathbb{R}^n)) \cap C([0, T], H^{\frac{m+4}{m+2} -}(\mathbb{R}^n))$ and $u(t, x) \in C^\infty((0, T) \times \mathbb{R}^n \setminus \Gamma_0)$, here $\Gamma_0 = \{(t, x) : t > 0, |x|^2 = \frac{m+2}{m+2} \}$.

(ii) Under the condition (A2), (1.1) has a unique solution $u(t, x) \in L^\infty([0, T] \times \mathbb{R}^n) \cap C([0, T], H^{\frac{m+4}{m+2} -}(\mathbb{R}^n)) \cap C^\infty((0, T) \times \mathbb{R}^n \setminus \Gamma_1 \cup \Gamma^\pm_1)$, here $\Gamma_1 = \{ (t, x) : t > 0, x_1 = \pm \frac{2t^{\frac{m+1}{m}}}{m+2} \}$.

(iii) For $n = 2$, under the condition (A3), if $m \leq 9$, then (1.1) has a unique solution $u(t, x) \in L^\infty([0, T] \times \mathbb{R}^2) \cap C([0, T], H^{\frac{m+4}{m+2} -}(\mathbb{R}^2)) \cap C^\infty((0, T) \times \mathbb{R}^2 \setminus \Gamma_0 \cup \Gamma_1^\pm \cup \Gamma_2^\pm)$, here $\Gamma_0$ and $\Gamma_1^\pm$ have been defined in (i) and (ii) respectively, and $\Gamma_2^\pm = \{(t, x) : t > 0, x_2 = \pm \frac{2t^{\frac{m+1}{m}}}{m+2} \}$.

**Remark 1.1.** In order to prove the $C^\infty$ property of solution in Theorem 1.1(i) and (ii), we will show that the solution of (1.1) is conormal with respect to the cusp characteristic conic surface $\Gamma_0$ or the cusp characteristic surfaces $\Gamma^\pm_1$ respectively in §6 below. And the definitions of conormal spaces will be given in §4.

**Remark 1.2.** Since we only focus on the local existence of solution in Theorem 1.1, it does not lose the generality that the initial data $\varphi(x)$ in (A1) – (A3) are assumed to be compactly supported. In addition, the initial data $u(0, x), \partial_t u(0, x) = (0, \varphi(x))$ in (1.1) can be replaced by the general forms $u(0, x), \partial_t u(0, x) = (\phi(x), \varphi(x))$, where $D_\psi \varphi(x)$ satisfies $\varphi_1(x)$ when $\varphi(x)$ satisfies $\varphi_1$, $\phi(x)$ is $C^1$ piecewise smooth along $\{ t = x_1 = 0 \}$ when $\varphi(x)$ satisfies $\varphi_1$, and $\phi(x)$ is $C^1$ piecewise smooth along the lines $\{ t = x_1 = 0 \}$ and $\{ t = x_2 = 0 \}$ when $\varphi(x)$ satisfies $\varphi_3(x)$, respectively.

**Remark 1.3.** The initial data problem (1.1) under the assumptions (A2) and (A3) is actually a special case of the multidimensional generalized Riemann problem for the second order semilinear degenerate hyperbolic equations. For the semilinear $N \times N$ strictly hyperbolic systems of the form $\partial_t U + \sum_{j=1}^{n} A_j(t, x) \partial_j U = F(t, x, U)$ with the piecewise smooth or conormal initial data along some hypersurface $\Delta_0 \subset \{ (t, x) : t = 0, x \in \mathbb{R}^n \}$ (including the Riemann discontinuous initial data), the authors in [19-20] have established the local well-posedness of piecewise smooth or bounded conormal solution with respect to the $N$ pairwise transverse characteristic surfaces $\Sigma_j$ passing through $\Delta_0$. With respect to the Riemann problem of higher order semilinear
Remark 1.4. The reason that we pose the restriction on \( m \leq 9 \) in (iii) of Theorem 1.1 is due to the requirement for utilizing the Sobolev’s imbedding theorem to derive the boundedness of solution (one can see details in (5.7) of [50] below), otherwise, it seems that we have to add some other conditions on the nonlinear function \( f(t,x,u) \) since the solution \( w(t,x) \in L^\infty([0,T] \times \mathbb{R}^2) \) does not hold even if \( w(t,x) \) satisfies a linear equation \( \partial^2_t w - t^m \Delta w = g(t,x) \) with \( (w(0,x), \partial dw(0,x)) = (0,0) \) and \( g(t,x) \in L^\infty([0,T] \times \mathbb{R}^2) \cap L^s([0,T], H^s(\mathbb{R}^2)) \) with \( 0 \leq s < 1 \) for large \( m \). Firstly, this can be roughly seen from the following explicit formula of \( w(t,x) \) in Theorem 3.4 of [24]:

\[
w(t,x) = \frac{1}{\pi} \frac{4}{(m+2)^{m+2}} \int_0^t \int_0^1 \frac{\phi(t) - \phi(\tau)}{\sqrt{r^2 - |x-y|^2}} dr d\gamma \left( r \right) \left( r_1 + \phi(t) + \phi(\tau) \right)^{-1} F \gamma, \gamma; 1; \left( \frac{-r_1 + \phi(t) - \phi(\tau)}{\left( r_1 + \phi(t) - \phi(\tau) \right) \left( r_1 - \phi(t) - \phi(\tau) \right)} \right),
\]

where \( \phi(t) = \frac{2t^{\frac{m+1}{m+2}}}{m+2}, \gamma = \frac{m}{2(m+2)}, \) and \( F(a,b;c;z) \) is the hypergeometric function. It is noted that

\[
\partial_r(\int_{B(x,r)} \frac{g(r,y)}{\sqrt{r^2 - |x-y|^2}} dy) = \partial_r(\int_0^2 \int_{\Sigma} g(r,x_1 + \sqrt{r^2 - q^2 \cos \theta}, x_2 + \sqrt{r^2 - q^2 \sin \theta}) dq d\theta)
\]

holds and thus the \( L^\infty \) property of \( w(t,x) \) is closely related to the integrability of the first order derivatives of \( g(t,x) \), which is different from the case in 2-D linear wave equation. On the other hand, the regularity of \( w(t,x) \) is in \( C([0,T], H^{\frac{m}{2}+\frac{n}{2}}(\mathbb{R}^2)) \nsubseteq L^\infty([0,T] \times \mathbb{R}^2) \) for large \( m \) by Proposition 3.3 below and Sobolev’s imbedding theorem.

Remark 1.5. By \( u(t,x) \notin C^2((0,T] \times \mathbb{R}^2 \setminus \Gamma_0 \cup \Gamma_1 \cup \Gamma_2) \) in Theorem 1.1.(iii), we know that there exists an essential difference on the regularity of solutions between the degenerate hyperbolic equation and strictly hyperbolic equation with the same initial data in \( (A_2) \) since \( v(t,x) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \setminus \Sigma_0 \cup \Sigma_1 \cup \Sigma_2) \) will hold true if \( v(t,x) \) is a solution to the 2-D linear wave equation \( \partial^2_t v - \Delta v = f(t,x,v) \) with \( (v(0,x), \partial v(0,x)) = (0, \varphi(x)) \), where \( \Sigma_0 = \{ t = |x| \}, \Sigma_1 = \{ t = \pm x_1 \}, \) and \( \Sigma_2 = \{ t = \pm x_2 \} \). The latter well-known result was established in the references [3-8] and [10-12] respectively under some various assumptions.

For \( m = 1, n = 1 \) and \( f(t,x,u) \equiv 0 \), the equation in (1.1) becomes the classical Tricomi equation which arises in transonic gas dynamics and has been extensively investigated in bounded domain with suitable boundary conditions from various viewpoints (one can see [8], [18], [21-24] and the references therein). For \( m = 1 \) and \( n = 2 \), with respect to the equation \( \partial^2_t u - t u_{\Delta u} = f(t,x,u) \) together with the initial data of higher \( H^s(\mathbb{R}^n) \)-regularity \( s > \frac{3}{2} \). M. Beals in [3] show the local existence of solution \( u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^{1}([0,T], H^{s-\frac{3}{2}}(\mathbb{R}^n)) \cap C^2([0,T], H^{s-\frac{5}{2}}(\mathbb{R}^n)) \) for some \( T > 0 \) under the crucial assumption that the support of \( f(t,x,u) \) on the variable \( t \) lies in \( \{ t \geq 0 \} \). Meanwhile, the conormal regularity of \( H^s(\mathbb{R}^n) \) solution \( u(t,x) \) with respect to the characteristic surfaces \( x_1 = \pm \frac{2}{3} t^2 \) is also established in [3].

With respect to more general nonlinear degenerate hyperbolic equations with higher order regularities, the authors in [10-11] studied the local existence and the propagation of weak singularity of classical solution. For the linear degenerate hyperbolic equations with suitable initial data, so far there have existed some interesting results on the regularities of solution when Levi’s conditions are posed (one can see [12], [14-15] and the references therein). In the present paper, we focus on the low regularity solution problem for the second order semilinear degenerate equation with no much restrictions on the nonlinear function \( f(t,x,u) \) in (1.1) and typical discontinuous initial data.

We now comment on the proof of Theorem 1.1. In order to prove the local existence of solution to (1.1) with the low regularity, at first we should establish the local \( L^\infty \) property of solution \( v(t,x) \) to the linear problem \( \partial^2_t v - t^m \Delta v = F(t,x) \) with \( (v(0,x), \partial v(0,x)) = (\varphi_0(x), \varphi_1(x)) \) so that the composition function \( f(t,x,v) \) makes sense. In this process, we have to make full use of the special structure of the piecewise smooth initial data and the explicit expression of solution \( v(t,x) \) established in [23-24] since we can not apply for the Sobolev imbedding theorem directly to obtain \( v(t,x) \in L^\infty_{loc} \) due to its low regularity (for examples,
in the cases of \((A_2) - (A_3)\), the initial data are only in \(H^{1/2-}(\mathbb{R}^n)\). Based on such \(L^\infty\)-estimates, together with the Fourier analysis method and the theory of confluent hypergeometric functions, we can construct a suitable nonlinear mapping related to the problem \((1.1)\) and further show that such a mapping admits a fixed point in the space \(L^\infty(0, T) \times \mathbb{R}^n) \cap C([0, T], H^s(\mathbb{R}^n))\) for suitable \(T > 0\) and some number \(s_0 > 0\), and then the local solvability of \((1.1)\) can be shown. Next, we are concerned with the singularity structures of solution \(u(t, x)\) of \((1.1)\). It is noted that the initial data are suitably conormal under the assumptions \((A_1)\) and \((A_2)\), namely, \(\Pi_{1 \leq i,j \leq n}(x_i \partial_j)^{k_{ij}} \varphi(x) \in H^{3/2-}(\mathbb{R}^n)\) for any \(k_{ij} \in \mathbb{N} \cup \{0\}\) in the case of \((A_1)\), and \((x_1 \partial_1)^{k_1} \Pi_{2 \leq i \leq n} \partial_i^{k_i} \varphi(x) \in H^{3/2-}(\mathbb{R}^n)\) for any \(k_i \in \mathbb{N} \cup \{0\}\) (\(i = 1, ..., n\)) in the case of \((A_2)\). Then, we intend to use the commutator arguments as in \([5-6]\) to prove the conormality of solution \(u(t, x)\) to \((1.1)\). However, due to the cusp singularities of surfaces \(\Gamma_0, \Gamma_1^{\pm}\) together with the degeneracy of equation, it seems that it is difficult to choose the smooth vector fields \(\{Z_1, ..., Z_k\}\) tangent to \(\Gamma_0\) or \(\Gamma_1^{\pm}\) as in \([5-6]\) to define the conormal space and take the related analysis on the commutators \([\partial_i^2 - t^m \Delta, Z_1^i \cdots Z_k^i]\) since this will lead to the violation of Levi’s condition and bring the loss of regularity of \(Z_1^i \cdots Z_k^i u\) (more detailed explanations can be found in §4 below). To overcome this difficulty, motivated by \([2-3]\) and \([18]\), we will choose the nonsmooth vector fields and try to find the extra regularity relations provided by the operator itself and some parts of vector fields to yield full conormal regularity of \(u(t, x)\) together with the regularity theory of second order elliptic equation and further complete the proof of Theorem 1.1.(i) and (ii), here we point out that it is nontrivial to find such crucial regularity relations. On the other hand, in the case of \(n = 2\) and assumption \((A_3)\), due to the lack of the strong Huyghen’s principle, we can derive that the solution \(u(t, x) \not\in C^2([0, T] \times \mathbb{R}^2 \setminus \Gamma_0 \cup \Gamma_1^{\pm} \cup \Gamma_2^{\pm})\) of \((1.1)\), which yields a different phenomenon from that in the case of second order strict hyperbolic equation as pointed out in Remark 1.5.

This paper is organized as follows. In §2, for later uses, we will give some preliminary results on the regularities of initial data \(\varphi(x)\) in various assumptions \((A_1) - (A_3)\) and establish the \(L^\infty\) property of solution to the related linear problem. In §3, by the partial Fourier-transformation, we can change the linear generalized Tricomi equation into a confluent hypergeometric equation, and then some weighted Sobolev regularity estimates near \(\{t = 0\}\) are derived. In §4, the required conormal spaces are defined and some crucial commutator relations are given. In §5, based on the results in §2-§3, the local solvability of \((1.1)\) is established. In §6, we complete the proof on Theorem 1.1 by utilizing the concepts of conormal spaces and commutator relations in §4 and taking some analogous analysis in Lemma 2.4 of §2 respectively.

In this paper, we will use the following notation:

\[
H^{s-}(\mathbb{R}^n) = \{w(x) : w(x) \in H^{s-\delta}(\mathbb{R}^n) \text{ for any fixed constant } \delta > 0.\}
\]

\section{Some preliminaries}

In this section, we will give some basic lemmas on the regularities of initial data \(\varphi(x)\) in the assumptions \((A_1) - (A_3)\) and establish some \(L^\infty\) property of solution to the linear problem \(\partial_t^2 u - t^m \Delta u = f(t, x)\) with suitably piecewise smooth initial data.

With respect to the functions \(\varphi(x)\) given in \((A_1) - (A_3)\) of §1, we have the following regularities in Sobolev space.

**Lemma 2.1.** (i) If \(\varphi(x) = g(x, \frac{x}{|x|})\), here \(g(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)\) and has a compact support in \(B(0, 1) \times B(0, 2)\), then \(\varphi(x) \in H^{3/2-}(\mathbb{R}^n)\).

(ii) If \(n = 2\) and \(\varphi(x) = \begin{cases} \psi_1(x) & \text{for } x_1 > 0, x_2 > 0, \\ \psi_2(x) & \text{for } x_1 < 0, x_2 > 0, \\ \psi_3(x) & \text{for } x_1 < 0, x_2 < 0, \\ \psi_4(x) & \text{for } x_1 > 0, x_2 < 0, \end{cases}\) where \(\psi_i(x) \in C_0^\infty(\mathbb{R}^2)\) (\(1 \leq i \leq 4\)), then \(\varphi(x) \in H^{3/2-}(\mathbb{R}^2)\).

(iii) If \(\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x_1 < 0, \\ \varphi_2(x) & \text{for } x_1 > 0, \end{cases}\) where \(\varphi_1(x), \varphi_2(x) \in C_0^\infty(\mathbb{R}^n)\), then \(\varphi(x) \in H^{3/2-}(\mathbb{R}^n)\) and \(x_1 \varphi(x) \in \)
that \( (1 + |\xi|)^{-\delta} \hat{\varphi}(\xi) \in L^2(\{|\xi| \leq 1\}) \).

For \(|\xi| > 1\), we decompose \(\hat{\varphi}\) into two parts

\[
\hat{\varphi}(\xi) = \int_{|x| < \frac{1}{\xi}} e^{-ix\cdot\xi} \varphi(x) dx + \int_{\frac{1}{\xi} \leq |x| \leq 1} e^{-ix\cdot\xi} \varphi(x) dx
\]

\[
= I + II = I + \sum_{\ell=1}^{n} \chi_{\ell}(\xi)II,
\]

where \(\{\chi_{\ell}\}_{\ell=1}^{n}\) is a \(C^\infty\) conic decomposition of unity corresponding to the domain \(\{\xi \in \mathbb{R}^n : |\xi| \geq 1\}\), moreover \(\xi_{\ell} \neq 0\) in \(\text{supp} \chi_{\ell}\).

Obviously, the term \(I\) can be dominated by the multiplier of \(|\xi|^{-n}\). On the other hand, for any \(1 \leq \ell \leq n\),

\[
\chi_{\ell}(\xi)II = \frac{\chi_{\ell}(\xi)}{|\xi|^{n}} \int_{1 \leq |x| \leq |\xi|} e^{-ix\cdot\xi} \frac{g(\frac{x}{|\xi|}, \frac{x}{|x|})}{|x|} dx
\]

\[
= \frac{\chi_{\ell}(\xi)}{|\xi|^{n}} \int_{1 \leq |x| \leq |\xi|} e^{-ix\cdot\xi} \partial_{\xi}^k \left( g\left(\frac{x}{|\xi|}, \frac{x}{|x|}\right) \right) dx
\]

\[
+ \frac{\chi_{\ell}(\xi)}{|\xi|^{n}} \int_{|x|=1} e^{-ix\cdot\xi} \partial_{\xi}^n \left( g\left(\frac{x}{|\xi|}, \frac{x}{|x|}\right) \right) \cos(n_\xi n_{\xi}) dS
\]

\[
= \chi_{\ell}(\xi) \int_{1 \leq |x| \leq |\xi|} e^{-ix\cdot\xi} \partial_{\xi}^m \left( g\left(\frac{x}{|\xi|}, \frac{x}{|x|}\right) \right) \cos(n_\xi n_{\xi}) dS
\]

\[
= III + IV.
\]

Due to

\[
|\partial_{\xi}^k \left( g\left(\frac{x}{|\xi|}, \frac{x}{|x|}\right) \right)| \leq \sum_{j=0}^{k} \sum_{|\alpha| \leq k-j} C_{\alpha j} |(\partial_{\xi}^j \partial_{\xi}^\alpha g)(\frac{x}{|\xi|}, \frac{x}{|x|})| |x|^{-(k-j)} |\xi|^{-j},
\]

then from (2.1), \(IV\) is dominated by the multiplier of \(|\xi|^{-n}\), and moreover,

\[
|III| \leq \frac{C}{|\xi|^{n}} \int_{1 \leq |x| \leq |\xi|} |x|^{-\beta} dx
\]

\[
\leq \begin{cases} 
\frac{C}{|\xi|^{n}} \sum_{\alpha + \beta = m} C_{\alpha \beta} \left( \frac{1}{|\xi|^{m}} + \frac{1}{|\xi|^{n+\alpha}} \right) & \text{if } \beta \neq n; \\
\frac{C}{|\xi|^{n} ln|\xi|} & \text{if } \beta = n.
\end{cases}
\]

Therefore, for \(m \geq n\) and \(|\xi| \geq 1\), we have \(|\hat{\varphi}(\xi)| \leq \frac{C(1 + ln|\xi|)}{|\xi|^{n}}\) by (2.2)-(2.4), which derives \((1 + |\xi|)^{-\delta} \hat{\varphi}(\xi) \in L^2(\{|\xi| \geq 1\})\) for any \(\delta > 0\), and further completes the proof of (i).
(ii) Without loss of generality, we assume \( \text{supp} \, \psi_1(x) \subset [-1, 1; -1, 1] \) \( (1 \leq i \leq 4) \).

Since

\[
|\hat{\varphi}(\xi)| = \left| \int_{0}^{1} \int_{0}^{1} \psi_1(x)e^{-i\xi \cdot x}dx + \int_{-1}^{0} \int_{0}^{1} \psi_2(x)e^{-i\xi \cdot x}dx \\
+ \int_{-1}^{0} \int_{-1}^{0} \psi_3(x)e^{-i\xi \cdot x}dx + \int_{0}^{1} \int_{-1}^{0} \psi_4(x)e^{-i\xi \cdot x}dx \right|
\]

\[
\leq \begin{cases} 
\frac{C}{|\xi_1|} & \text{for } |\xi_1| \geq 1, |\xi_2| \geq 1; \\
\frac{C}{|\xi_1|} & \text{for } |\xi_1| \geq 1, |\xi_2| < 1; \\
\frac{C}{|\xi_2|} & \text{for } |\xi_1| < 1, |\xi_2| \geq 1; \\
C & \text{for } |\xi_1| < 1, |\xi_2| < 1,
\end{cases}
\]

then from the fact \( 1 + |\xi| \leq (1 + |\xi_1|)(1 + |\xi_2|) \) one has for any \( 0 < \delta \leq \frac{1}{2} \)

\[
\int_{\mathbb{R}^2} (1 + |\xi|)^{1-\delta} |\hat{\varphi}(\xi)|^2d\xi \\
\leq C \sum_{i=1}^{2} \int_{1}^{\infty} \frac{(1 + |\xi_i|)^{1-\delta}}{|\xi_i|^2}d\xi_i + C \sum_{i=1}^{2} \int_{1}^{\infty} \frac{(1 + |\xi_i|)^{1-\delta}}{|\xi_i|^2}d\xi_i + C \\
\leq C.
\]

Thus, the proof of (ii) is completed.

(iii) The proof procedure is similar to that in (ii), we omit it here. \( \square \)

**Remark 2.1.** By the similar proof procedure as in Lemma 2.1.(i), we can also prove: If \( f(x) \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) and has compact support, moreover, \( |\partial^\alpha f(x)| \leq C_n |x|^{-|\alpha|} \) for \( x \neq 0 \) and \( r > -\frac{n}{2} \), then \( f(x) \in H^{\frac{n}{2}+r-}(\mathbb{R}^n) \).

**Remark 2.2.** Under the assumption (A2), for any \( \alpha \in (\mathbb{N} \cup \{0\})^{n-1} \), we can also have that \( \partial^\alpha \varphi(x) \in H^{\frac{n}{2}-\delta}(\mathbb{R}^n) \) for any \( \delta > 0 \) small, here \( x' = (x_2, \ldots, x_n) \). Thus, \( (1 + |\xi|)^{\frac{n}{2}-\delta}(1 + |\xi'|)^{|\alpha|}\hat{\varphi}(\xi) \in L^2(\mathbb{R}^n) \), where \( \xi' = (\xi_2, \ldots, \xi_n) \).

**Lemma 2.2.** If \( u(t,x) \in C([0,T], H^{\frac{n}{2}-}(\mathbb{R}^n)) \) is a solution of the following linear equation

\[
\begin{cases} 
\partial_t^2 u - t^m \Delta u = 0, \quad (t,x) \in [0, +\infty) \times \mathbb{R}^n, \\
u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \varphi(x),
\end{cases}
\]

(2.5)

where \( \varphi(x) \) satisfies the assumption (A2), \( \partial_x^\alpha \varphi(x) \in H^{\frac{n}{2}-}(\mathbb{R}^n) \) for all \( 0 \leq |\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1 \), then \( u(t,x) \in L^\infty([0,T] \times \mathbb{R}^n) \).

**Proof.** Set \( y(t,\xi) = \int_{\mathbb{R}^n} u(t,x)e^{-i\xi \cdot x}dx \) with \( \xi \in \mathbb{R}^n \) and \( y''(t,\xi) \equiv \partial_x^2 y(t,\xi) \), then it follows from the equation of (2.5) that

\[
y''(t,\xi) + t^m |\xi|^2 y(t,\xi) = 0.
\]

(2.6)

Let \( \tau = \frac{2t^{\frac{n}{2}+1} |\xi|}{m+2} \) and \( v(\tau) \equiv y(t,|\xi|) \), then

\[
\frac{d^2 v}{d\tau^2} + \frac{m}{(m+2)\tau} \frac{dv}{d\tau} + v = 0.
\]

(2.7)
As in [25], taking $z \equiv 2i\tau = \frac{4i}{m+2} \frac{m+2}{m+2} |\xi|$ and $w(z) = v\left(\frac{z}{2i}\right) e^z$ yields for $t > 0$ and $|\xi| \neq 0$

$$z w''(z) + \left(\frac{m}{m+2} - z\right) w'(z) - \frac{m}{2(m+2)} w(z) = 0. \quad (2.8)$$

(2.8) has two linearly independent solutions $w_1(z) = \Phi\left(\frac{m}{2(m+2)}, \frac{m+2}{m+2}, z\right)$ and $w_2(z) = z^{\frac{m}{m+2}} \Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, z\right)$ by [13], which are called the confluent hypergeometric functions.

By (2.6)-(2.8) and [23], we have for $t \geq 0$ and $\xi \in \mathbb{R}^n$

$$y(t, \xi) = V_1(t, |\xi|) \psi^\wedge(\xi) + V_2(t, |\xi|) \varphi^\wedge(\xi)$$

$$\equiv y_1(t, \xi) + y_2(t, \xi) \quad (2.9)$$

with

$$\begin{align*}
V_1(t, |\xi|) &= e^{\frac{m}{m+2} t} \Phi\left(\frac{m}{2(m+2)}, \frac{m+2}{m+2}, z\right), \\
V_2(t, |\xi|) &= te^{\frac{m}{m+2} t} \Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, z\right).
\end{align*} \quad (2.10)$$

Since $\Phi\left(\frac{m}{2(m+2)}, \frac{m+2}{m+2}, z\right)$ and $\Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, z\right)$ are analytic functions of $z$, then $|\Phi\left(\frac{m}{2(m+2)}, \frac{m+4}{m+2}, z\right)|$ and $|\Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, z\right)| \leq C_M$ for $|z| \leq M$. For sufficiently large $|z|$, we have from formula (9) in pages 253 of [13] that

$$|\Phi\left(\frac{m}{2(m+2)}, \frac{m+4}{m+2}, z\right)| \leq C |z|^{-\frac{m}{2(m+2)}} (1 + O(|z|^{-1})), \quad |\Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, z\right)| \leq C |z|^{-\frac{m+4}{2(m+2)}} (1 + O(|z|^{-1})).$$

(2.11)

From Remark 2.2, we have that for $0 \leq |\alpha| \leq [\frac{n}{2}] + 1$ and $0 < \delta < \frac{1}{2}$

$$\varphi(\xi) = \frac{g_\alpha(\xi)}{(1 + |\xi|)^{\frac{1}{2}}} \Phi\left(\frac{m+2}{2}, \frac{m+2}{m+2}, 2\alpha\right),$$

(2.12)

where $g_\alpha(\xi) \in L^2(\mathbb{R}^n)$, $\xi' = (\xi_2, \ldots, \xi_n)$.

Therefore, for any $t \in (0, T]$, we have

$$\int_{\mathbb{R}^n} |y_2(t, \xi)| d\xi \leq C t \int_{\mathbb{R}^n} \left| e^{\frac{m}{m+2} t} \Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, \frac{4i}{m+2} \frac{m+2}{m+2} |\xi|\right) \varphi^\wedge(\xi) \right| d\xi$$

$$\leq C t \left(\frac{m+2}{2t \frac{m+2}{m+2}} \right)^n \left(\int_{\mathbb{R}^n} \Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, \left|\frac{4i}{m+2} \frac{m+2}{m+2} \xi\right|\right) \varphi^\wedge(\xi) d\xi\right)$$

$$\leq C t \left(\frac{m+2}{2t \frac{m+2}{m+2}} \right)^n \left(\int_{\mathbb{R}^n} \left|\frac{1}{(1 + |\eta|^2)^{\frac{m+4}{m+2}}} (1 + \frac{|\eta|^2}{m+2})^\frac{1}{2} \frac{1}{2i \frac{m+2}{m+2}} |\varphi^\wedge(\xi)\right|^2 d\eta \right)^\frac{1}{2} \quad (by \ (2.11))$$

$$\leq C t \left(\frac{m+2}{2t \frac{m+2}{m+2}} \right)^n \left(\int_{\mathbb{R}^n} \left|\frac{1}{(1 + |\eta|^2)^{\frac{m+4}{m+2}}} (1 + \frac{|\eta|^2}{m+2})^\frac{1}{2} \frac{1}{2i \frac{m+2}{m+2}} |\varphi^\wedge(\xi)\right|^2 d\eta \right)^\frac{1}{2} \quad (by \ (2.12))$$

$$\leq C t \left(\frac{m+2}{2t \frac{m+2}{m+2}} \right)^n \left(\int_{\mathbb{R}^n} \left|\frac{1}{(1 + |\eta|^2)^{\frac{m+4}{m+2}}} (1 + \frac{|\eta|^2}{m+2})^\frac{1}{2} \frac{1}{2i \frac{m+2}{m+2}} |\varphi^\wedge(\xi)\right|^2 d\eta \right)^\frac{1}{2} \quad (choosing \ |\alpha| = \left[\frac{n}{2}\right] + 1 > \frac{n}{2})$$

$$\leq C t \left(\frac{m+2}{2t \frac{m+2}{m+2}} \right)^n \left(\int_{\mathbb{R}^n} \left|\frac{1}{(1 + |\eta|^2)^{\frac{m+4}{m+2}}} (1 + \frac{|\eta|^2}{m+2})^\frac{1}{2} \frac{1}{2i \frac{m+2}{m+2}} |\varphi^\wedge(\xi)\right|^2 d\eta \right)^\frac{1}{2} \quad (choosing \delta < \frac{m+4}{3(m+2)})$$

$$\leq C t \quad (choosing \delta < \frac{4}{3(m+2)}).$$
Similarly,
\[
\int_{\mathbb{R}^n} |g_1(t, \xi)|d\xi \leq C\left(\frac{m+2}{2t^{m+2}}\right)^{\alpha} \int_{\mathbb{R}^n} \left|\Phi\left(\frac{m}{2(m+2)} \cdot \frac{m}{m+2} \cdot 2i|\eta|\right)\right| |f^\wedge(\frac{m+2}{2t^{m+2}})| d\eta
\]
\[
\leq C_\alpha \left(\frac{m+2}{2t^{m+2}}\right)^{\alpha} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\eta_1|)^{3-2\delta}} (1+|\eta|^2)^{2|\alpha|} d\eta\right)^{\frac{1}{2}}
\]
\[
\leq C_\alpha \left(\frac{1}{(1+|\eta_1|)^{3-2\delta}} (1+|\eta|^2)^{2|\alpha|} d\eta\right)^{\frac{1}{2}}
\]
\[
\leq C_T \text{ (choosing } |\alpha| = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } 0 < \delta < 1).}

Thus, \(|u(t, x)| \leq \int_{\mathbb{R}^n} |y(t, \xi)|d\xi \leq \int_{\mathbb{R}^n} |g_1(t, \xi)|d\xi + \int_{\mathbb{R}^n} |g_2(t, \xi)|d\xi \leq C_T \text{ for } (t, x) \in (0, T) \times \mathbb{R}^n\), and then Lemma 2.2 is proved. □

**Lemma 2.3.** If \(f(t, x) \in C([0, T], H^s(\mathbb{R}^n))\) and \(\partial^s_\tau f(t, x) \in L^\infty([0, T], H^s(\mathbb{R}^n))\) with \(s > \frac{1}{2}\) and \(|\alpha| \leq \left\lfloor \frac{m}{2} \right\rfloor + 1\), \(v(t, x)\) is a solution to the following problem

\[
\begin{cases}
\partial^s_\tau u - t^m \Delta u = f(t, x), \\
u(0, x) = \partial_\tau v(0, x) = 0,
\end{cases}
\]

(2.13)

then \(u(t, x) \in L^\infty([0, T] \times \mathbb{R}^n)\).

**Proof.** By the assumptions on \(f(t, x)\), we have

\[
f^\wedge(t, \xi) = \frac{g_\alpha(t, \xi)}{(1 + |\xi_1|)^{\alpha}(1 + |\xi|^2)|\alpha|},
\]

where \(g_\alpha(t, \xi) \in L^\infty([0, T], L^2(\mathbb{R}^n))\) and \(|\alpha| = \left\lfloor \frac{m}{2} \right\rfloor + 1\).

From (2.13), we have

\[
u(t, x) = \left(\int_0^t (V_2(t, |\xi|)V_1(\tau, |\xi|) - V_1(t, |\xi|)V_2(\tau, |\xi|))f^\wedge(\tau, \xi)d\tau\right)^\vee (t, x),
\]

where the expressions of \(V_1(t, |\xi|)\) and \(V_2(t, |\xi|)\) are given in (2.10).

It is noted that

\[
|u^\wedge(t, \xi)| \leq \int_0^t |V_2(t, |\xi|)V_1(\tau, |\xi|)f^\wedge(\tau, \xi)|d\tau + \int_0^t |V_1(t, |\xi|)V_2(\tau, |\xi|)f^\wedge(\tau, \xi)|d\tau
\]

(2.14)

\[
\equiv I + II.
\]

Set \(\eta = \frac{2}{m+2}t^{\frac{m+2}{2}}\xi\), we have

\[
|I| \leq C t \int_0^t \left|\Phi\left(\frac{m+4}{2(m+2)} \cdot \frac{m+4}{m+2} \cdot 2i|\eta|\right)\Phi\left(\frac{m}{2(m+2)} \cdot \frac{m}{m+2} \cdot 2i(t)\right)\right| |f^\wedge(\tau, (m+2)t^{\frac{m+2}{2}})|d\tau
\]

\[
\leq C_\alpha t \int_0^t \left(1 + |\eta|\right)^{-\frac{m+2}{m+4}} \left(1 + (\frac{\tau}{t})^{\frac{m+2}{2}} |\eta|\right)^{-\frac{m}{m+4}} |g_\alpha(\tau, \frac{(m+2)t^{\frac{m+2}{2}}}{2\tau^{\frac{m+2}{2}}})| d\tau
\]

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and thus
\[
\int_{\mathbb{R}^n} |I|d\xi \leq C_n t^{1 - \frac{m+2}{m+2}} \int_0^t dt \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |\eta|)^{\frac{m+2}{2s}}(1 + |\eta'|^{2s})(1 + \frac{|\eta|}{\sqrt{t}})^2|\alpha|} d\eta \right)^\frac{1}{2} \leq C_n t \int_0^t dt \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |\eta|)^{\frac{m+2}{2s}}(1 + |\eta'|^{2s})(1 + \frac{|\eta|}{\sqrt{t}})^2|\alpha|} d\eta \right)^\frac{1}{2} \leq C_n t^2.
\] (2.15)

On the other hand, due to
\[
|II| \leq C_n \int_0^t (1 + |\eta|)^{-\frac{m}{2s}}(1 + (\frac{\tau}{t})^{\frac{m+2}{2s}}|\eta|)^{-\frac{m+4}{2s}} \frac{\tau|g_\alpha(\tau, (m+2)|\eta|)}{1 + \frac{|\eta|}{\sqrt{\tau}}} d\tau,
\]
then we can obtain as in (2.15)
\[
\int_{\mathbb{R}^n} |II|d\xi \leq C_l t^2.
\] (2.16)

Substituting (2.15) and (2.16) into (2.14) yields
\[
\int_{\mathbb{R}^n} |u^\wedge (t, \xi)|d\xi \leq C_l t^2.
\]

Consequently, \(|u(t, x)| \leq \int_{\mathbb{R}^n} |u^\wedge (t, \xi)|d\xi \leq C_l t^2\), and the proof on Lemma 2.3 is completed. \(\square\)

Finally, we study the \(L^\infty\) property of solution to the 2-D linear problem (1.1) under the assumption \((A_3)\).

**Lemma 2.4.** If \(u(t, x) \in C([0, T], H^\frac{1}{2} - (\mathbb{R}^2))\) is a solution of the following linear problem
\[
\left\{ \begin{array}{ll}
\partial^2_t u - tm \Delta u = 0, & (t, x) \in [0, T] \times \mathbb{R}^2, \\
u(0, x) = 0, & \partial_t u(0, x) = \varphi(x),
\end{array} \right
\] (2.17)

where \(\varphi(x)\) satisfies the assumption \((A_3)\), then \(u \in L^\infty([0, T] \times \mathbb{R}^2)\).

**Remark 2.3.** Due to \(\varphi(x) \in H^\frac{1}{2} - (\mathbb{R}^2)\) by Lemma 2.1.(ii), then the optimal regularity of the solution \(u(t, x)\) to (2.17) is \(L^\infty([0, T], H^\frac{1}{2} - \frac{m+2}{m+2} - (\mathbb{R}^2))\) (see Proposition 3.3 in §3 below). Thus, for \(m \geq 2\), we can not derive \(u(t, x) \in L^\infty([0, T] \times \mathbb{R}^2)\) directly by the Sobolev imbedding theorem. On the other hand, the proof procedure on Lemma 2.4 will be rather useful in analyzing the singularity structure of \(u(t, x)\) in §6 below.

**Proof.** In terms of Corollary 3.5 in [24], we have the following expression for the solution of (2.17)
\[
u(t, x) = 2tC_m (\phi(1))^{\phi(1)} F(\gamma, \gamma; 1; 1) \int_0^1 (1 - s^2)^{-\gamma} (\partial_t \nu)(s\phi(t), x)ds,
\] (2.18)

where \(C_m = \left(\frac{2}{m+2}\right)^{m+2} \frac{2}{m+2}, \gamma = \frac{m}{2(m+2)}, F(\gamma, \gamma; 1; 1) = F(\gamma, \gamma; 1; z)|_{z=1}\) with \(F(\gamma, \gamma; 1; z)\) a hypergeometric function, which satisfies \(z(1-z)\omega''(z) + (1 - (2\gamma + 1)z)\omega'(z) - \gamma^2 \omega(z) = 0\), and \(v(t, x)\) is a solution to the following linear wave equation
\[
\partial^2_t v - \Delta v = 0, \quad v(0, x) = 0, \quad \partial_t v(0, x) = \varphi(x).
\] (2.19)

From (2.19), we have
\[
v(t, x) = \frac{1}{2\pi} \int_{B(x, t)} \frac{\varphi(\xi)}{\sqrt{t^2 - (x_1 - \xi_1)^2 - (x_2 - \xi_2)^2}} d\xi.
\] (2.20)
Let $v_i(t, x)$ be the smooth solution to the linear wave equation $\partial_t^2 v_i - \Delta v_i = 0$ with the initial data $(v_i(0, x), \partial_t v_i(0, x)) = (0, v_i(x))$. Then it follows from (2.20) and a direct computation that for $t > 0$ and $x_1 > 0, x_2 > 0$ (in other domains, the expressions are completely analogous)

$$v(t, x) = \begin{cases} v_1(t, x) & \text{for } \frac{x_1}{t} \geq 1, \frac{x_2}{t} \geq 1; \\
v_1(t, x) + I_1(t, x) & \text{for } \frac{x_1}{t} \leq 1, \frac{x_2}{t} \geq 1; \\
v_1(t, x) + I_2(t, x) & \text{for } \frac{x_2}{t} \leq 1, \frac{x_1}{t} \geq 1; \\
v_1(t, x) + I_1(t, x) + I_2(t, x) & \text{for } 0 < x_1 < t, 0 < x_2 < t, |x| > t; \\
v_1(t, x) + I_1(t, x) + I_2(t, x) + I_3(t, x) & \text{for } x_1 > 0, x_2 > 0, |x| < t \end{cases}$$

(2.21)

with

$$I_1(t, x) = \frac{1}{2\pi} \int_{t_1}^t r \, dr \int_{-\arccos(\frac{x_1}{t})}^{\arccos(\frac{x_2}{t})} \frac{(\psi_2 - \psi_1)(x - r\omega)}{\sqrt{t^2 - r^2}} d\theta,$$

$$I_2(t, x) = \frac{1}{2\pi} \int_{t_2}^t r \, dr \int_{-\arccos(\frac{x_2}{t})}^{\arccos(\frac{x_1}{t})} \frac{(\psi_4 - \psi_3)(x - r\omega)}{\sqrt{t^2 - r^2}} d\theta,$$

$$I_3(t, x) = \frac{1}{2\pi} \int_{t_1}^t r \, dr \int_{-\arccos(\frac{x_1}{t})}^{\arccos(\frac{x_2}{t})} \frac{(\psi_1 + \psi_3 - \psi_2 - \psi_4)(x - r\omega)}{\sqrt{t^2 - r^2}} d\theta,$$

where $\omega = (\cos \theta, \sin \theta)$, $r = \sqrt{|x_1 - \xi_1|^2 + |x_2 - \xi_2|^2}$ and $(x_1 - \xi_1, x_2 - \xi_2) = (r \cos \theta, r \sin \theta)$.

Due to $\varphi(x) \in H_t^\infty(\mathbb{R}^2)$ by Lemma 2.1.(ii), then it follows from the regularity theory of solution to linear wave equation that

$$v(t, x) \in C([0, T], H_t^\infty(\mathbb{R}^2)) \cap C^1([0, T], H_t^\infty(\mathbb{R}^2)) \subset W^{1,1}([0, T] \times \mathbb{R}^2).$$

Thus, we can take the first order derivative $\partial_t v$ piecelywise for $t > 0$ and $x_1 > 0, x_2 > 0$ as follows

$$\partial_t v(t, x) = \begin{cases} \partial_t v_1(t, x) & \text{for } \frac{x_1}{t} \geq 1, \frac{x_2}{t} \geq 1; \\
\partial_t v_1(t, x) + \partial_t I_1(t, x) & \text{for } \frac{x_1}{t} \leq 1, \frac{x_2}{t} \geq 1; \\
\partial_t v_1(t, x) + \partial_t I_2(t, x) & \text{for } \frac{x_2}{t} \leq 1, \frac{x_1}{t} \geq 1; \\
\partial_t v_1(t, x) + \partial_t I_1(t, x) + \partial_t I_2(t, x) & \text{for } 0 < x_1 < t, 0 < x_2 < t, |x| > t; \\
\partial_t v_1(t, x) + \partial_t I_1(t, x) + \partial_t I_2(t, x) + \partial_t I_3(t, x) & \text{for } x_1 > 0, x_2 > 0, |x| < t \end{cases}$$

(2.22)

Here we only treat the term $\partial_t I_3$ in (2.22) since the treatments on $\partial_t I_1$ and $\partial_t I_2$ are analogous or even simpler in their corresponding domains.

If we set $\varphi = \psi_1 + \psi_3 - \psi_2 - \psi_4$, then it follows from a direct computation that for $x_1 > 0, x_2 > 0$ and $|x| < t$

$$I_3(t, x) = \int_0^d f_1(x) \int_0^d f_2(x) \frac{\psi(x)}{\sqrt{t^2 - |x|^2}} d\xi_2.$$
which derives
\[ \partial_1 I_3(t, x) = (\partial_1 J)(t, \frac{x}{t}), \quad \partial_2 I_3(t, x) = (\partial_2 J)(t, \frac{x}{t}) \]
and thus
\[
\partial_1 I_3(t, x) = \frac{I_3(t, x)}{t} + \frac{1}{t} \int_{x_1 - \sqrt{t^2 - x_2^2}}^{0} d\xi_1 \int_{x_2 - \sqrt{t^2 - (x_1 - \xi_1)^2}}^{0} \frac{\xi \cdot \nabla \psi(\xi)}{\sqrt{t^2 - |x - \xi|^2}} d\xi_2 - \frac{x \cdot \nabla J(t, x)}{t}.
\] (2.24)

It is noted that for \( x_1 > 0, x_2 > 0 \) and \( |x| < t \),
\[
|\partial_1 I_3(t, x)| = |\lim_{h \to 0} \frac{I_3(t, x + h, x_2) - I_3(t, x_1, x_2)}{h}|
\]
\[
= \left| \int_{x_1 - \sqrt{t^2 - x_2^2}}^{0} d\xi_1 \int_{x_2 - \sqrt{t^2 - (x_1 - \xi_1)^2}}^{0} \frac{\partial_1 \psi(\xi)}{\sqrt{t^2 - |x - \xi|^2}} d\xi_2 - \frac{\psi(0, \xi_2)}{\sqrt{t^2 - x_1^2}} \right|
\]
\[
= \left| \int_{x_1 - \sqrt{t^2 - x_2^2}}^{0} d\xi_1 \int_{x_2 - \sqrt{t^2 - (x_1 - \xi_1)^2}}^{0} \frac{\partial_1 \psi(\xi)}{\sqrt{t^2 - |x - \xi|^2}} d\xi_2 + \int_{1}^{\sqrt{t^2 - x_1^2}} \psi(0, x_2 - s \sqrt{t^2 - x_1^2}) d(\arcsin s) \right|
\]
\[
= \left| \int_{x_1 - \sqrt{t^2 - x_2^2}}^{0} d\xi_1 \int_{x_2 - \sqrt{t^2 - (x_1 - \xi_1)^2}}^{0} \frac{\partial_1 \psi(\xi)}{\sqrt{t^2 - |x - \xi|^2}} d\xi_2 + \psi(0, 0) \arcsin \left( \frac{x_1}{\sqrt{t^2 - x_2^2}} \right) - \frac{\pi}{2} \psi(x_1 - \sqrt{t^2 - x_2^2}, 0) + \sqrt{t^2 - x_2^2} \int_{1}^{\sqrt{t^2 - x_1^2}} \partial_1 \psi(x_1 - s \sqrt{t^2 - x_2^2}, 0) \arcsin ds \right|
\]
\[
\leq C_T \left( 1 + \int_{x_1 - \sqrt{t^2 - x_2^2}}^{0} d\xi_1 \int_{x_2 - \sqrt{t^2 - (x_1 - \xi_1)^2}}^{0} \frac{1}{\sqrt{t^2 - |x - \xi|^2}} d\xi_2 \right)
\]
\[
\leq C_T (1 + t)
\] (2.25)

and
\[
|\partial_2 I_3(t, x)| = \left| \int_{x_1 - \sqrt{t^2 - x_2^2}}^{0} d\xi_1 \int_{x_2 - \sqrt{t^2 - (x_1 - \xi_1)^2}}^{0} \frac{\partial_2 \psi(\xi)}{\sqrt{t^2 - |x - \xi|^2}} d\xi_2 + \frac{\psi(0, 0) \arcsin \left( \frac{x_1}{\sqrt{t^2 - x_2^2}} \right)}{\sqrt{t^2 - x_2^2}} \right|
\]
\[
- \frac{\pi}{2} \psi(x_1 - \sqrt{t^2 - x_2^2}, 0) + \sqrt{t^2 - x_2^2} \int_{1}^{\sqrt{t^2 - x_1^2}} \partial_1 \psi(x_1 - s \sqrt{t^2 - x_2^2}, 0) \arcsin ds \right|
\]
\[
\leq C_T (1 + t)
\] (2.26)

On the other hand, analogous computation yields for \( x_1 > 0, x_2 > 0 \) and \( |x| < t \leq T \)
\[
\left| \frac{I_3(t, x)}{t} \right| \leq C_T \quad \text{and} \quad \left| \frac{1}{t} \int_{x_1 - \sqrt{t^2 - x_2^2}}^{0} d\xi_1 \int_{x_2 - \sqrt{t^2 - (x_1 - \xi_1)^2}}^{0} \frac{\xi \cdot \nabla \psi(\xi)}{\sqrt{t^2 - |x - \xi|^2}} d\xi_2 \right| \leq C_T.
\] (2.27)

Therefore, \( \partial_1 I_3(t, x) \in L^\infty \) in the domain \( \{(t, x) : x_1 > 0, x_2 > 0, |x| < t \leq T \} \) by (2.24). Similarly, we can obtain \( \partial_2 I_3(t, x) \in L^\infty \) in the related domains, and thus \( \partial_1 v(t, x) \in L^\infty ([0, T] \times \mathbb{R}^2) \). These, together with (2.18), yield
\[
u(t, x) \in L^\infty ([0, T] \times \mathbb{R}^2) \quad \text{and} \quad \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C_T \sum_{i=1}^{4} ||\psi_i(x)||_{C^1}.
\] (2.28)

Consequently, we complete the proof of Lemma 2.4. \( \square \)

**Remark 2.4.** It is not difficult that by the expression (2.21) of \( \psi(t, x) \), one can get \( \psi(t, x) \in C^\infty ([0, T] \times \mathbb{R}^2 \setminus \Sigma_0 \cup \Sigma_1^+ \cup \Sigma_2^z) \), where \( \Sigma_0 = \{(t, x) : t > 0, |x| = t\} \), \( \Sigma_1^+ = \{(t, x) : t > 0, x_1 = \pm t\} \) and \( \Sigma_2^z = \{(t, x) : x_2 = 0, x_1 = \pm t\} \).
\( t > 0, x_2 = \pm t \). On the other hand, \( v(t, x) \notin C^2((0, T] \times \mathbb{R}^2) \) since \( v(t, x) \) has a strong singularity when the variables \((t, x)\) go across \( \Sigma_0 \cup \Sigma_1^+ \cup \Sigma_2^+ \). Indeed, for example, it follows from \((2.25)\) and a direct computation that for \( x_1 > 0, x_2 > 0 \) and \( |x| < t \)

\[
\partial^2_{t^2} I_3(t, x) = \int_{x_1 - \sqrt{t^2 - x_2}}^{x_1 + \sqrt{t^2 - x_2}} d\xi_1 \int_{x_2 - \sqrt{t^2 - x_1}}^{x_2 + \sqrt{t^2 - x_1}} d\xi_2 + \frac{3\psi(0, 0)}{\sqrt{t^2 - |x|^2}} + \text{bounded terms.} \tag{2.29}
\]

Thus, \((2.29)\) implies \( \partial^2_{t^2} I_3(t, x) \to \infty \) as \((t, x) \to \Sigma_0 \) since \( \psi(0) \neq 0 \) can be assumed without loss of generality (this is due to the assumption of \( \psi_i(0) \neq \psi_j(0) \) for some \( i \neq j \) and \( 1 \leq i < j \leq 4 \) in \((A_1)\) and the different expressions of \( \psi(x) \) in the related domains \((\{t, x\}: t > 0, x_1 > 0, \pm x_2 > 0 \}) \) respectively). In addition, by an analogous computation, we can derive that \( \partial^2_{t^2} I_2(t, x) \) and \( \partial^2_{t^2} I_2(t, x) \) are bounded for \( x_1 > 0, x_2 > 0 \) and \( |x| < t \). Hence \( \partial^2_{t^2} v(t, x) \to \infty \) as \((t, x) \to \Sigma_0 \) and further \( v(t, x) \notin C^2((0, T] \times \mathbb{R}^2) \) is proved. However, by the expression \((2.18)\) and due to the lack of strong Huyghens’ principle for the Tricomi-type equations, we can show that the solution \( u(t, x) \notin C^2((0, T] \times \mathbb{R}^2 \setminus \Gamma_0 \cup \Gamma_1^+ \cup \Gamma_2^+) \) of \((2.17)\) holds true in \( \mathbb{S}_6 \) below, which implies an essential difference between the degenerate equation and the strict hyperbolic equation.

\section*{§3. Some regularity estimates on the solutions to linear generalized Tricomi equations}

At first, we list some results on the confluent hypergeometric functions for our computations later on. The confluent hypergeometric equation is

\[
zw''(z) + (c - z)w'(z) - aw(z) = 0, \tag{3.1}
\]

where \( z \in \mathbb{C} \), \( a \) and \( c \) are constants. The solution of \((3.1)\) is called the confluent hypergeometric function. When \( c \) is not an integer, \((3.1)\) has two linearly independent solutions:

\[
w_1(z) = \Phi(a, c; z), \quad w_2(z) = z^{1-c} \Phi(a+c+1, 2-c; z).
\]

Below are some crucial properties of the confluent hypergeometric functions.

\textbf{Lemma 3.1.}

(i) (pages 278 of [13]). For \( -\pi < \arg z < \pi \) and large \( |z| \), then

\[
\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a - c)} (e^{\pi\epsilon z} z^{-1})^a \sum_{n=0}^{M} \frac{(a)_n (a - c + 1)_n}{n!} (-z)^{-n} + O(|z|^{-a-M-1})
\]

\[
+ \frac{\Gamma(c)}{\Gamma(a)} z^a e^{-\epsilon z} \sum_{n=0}^{N} \frac{(c-a)_n (1-a)_n}{n!} z^{-n} + O(|z|^{-a-c-N-1}), \tag{3.2}
\]

where \( \epsilon = 1 \) if \( \text{Im} z > 0 \), \( \epsilon = -1 \) if \( \text{Im} z < 0 \), \((a)_0 \equiv 1, (a)_n \equiv a(a+1) \cdots (a+n-1), \) \( a = 0, 1, 2, 3, \ldots \)

(ii) (page 253 of [13]). \( \Phi(a, c; z) = e^{\epsilon z} \Phi(c-a, c; -z) \).

(iii) (page 254 of [13]).

\[
\frac{d^n}{dz^n} \Phi(a, c; z) = \frac{(a)_n}{(c)_n} \Phi(a+n, c+n; z) \tag{3.3}
\]
and
\[
\frac{d}{dz} \Phi(a, c; z) = \frac{1 - c}{z} \left( \Phi(a, c; z) - \Phi(a, c - 1; z) \right). 
\tag{3.4}
\]

For such a problem
\[
\begin{align*}
&\{ \begin{array}{ll}
\partial_t^2 u - t^m \Delta u = 0, & (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
u(0, x) = \phi_1(x), & \partial_{t} u(0, x) = \phi_2(x),
\end{array} \}
\tag{3.5}
\]
by the results in [23], one has for \( t \geq 0 \)
\[
u^\wedge(t, \xi) = V_1(t, |\xi|)\phi_1^\wedge(\xi) + V_2(t, |\xi|)\phi_2^\wedge(\xi),
\tag{3.6}
\]
where the expressions of \( V_1(t, |\xi|) \) and \( V_2(t, |\xi|) \) have been given in (2.10).

In order to analyze the regularities of \( \nu^\wedge(t, \xi) \) in (3.6) under some restrictions on \( \phi_i(x) (i = 1, 2) \), we require to establish the following estimates:

**Lemma 3.2.** For \( 0 \leq s_1 \leq \frac{m}{m+2}, \ 0 \leq s_2 \leq \frac{m+4}{2(m+2)} \) and some fixed positive constant \( T \), if \( g(x) \in H^s(\mathbb{R}^n) \) with \( s \in \mathbb{R} \), then we have for \( 0 < t \leq T \):

\[
\begin{align*}
(i) \quad &\left\| V_1(t, |\xi|)g^\wedge(\xi) \right\|_{H^{s_1+2}} \leq C t^{-\frac{m}{2(m+2)} - \frac{s_1}{2}} \| g \|_{H^s}, \\
&\left\| V_2(t, |\xi|)g^\wedge(\xi) \right\|_{H^{s_1+2}} \leq C t^{-\frac{m+4}{2(m+2)} - \frac{s_1}{2}} \| g \|_{H^s}, \\
(ii) \quad &\left\| \partial_t V_1(t, |\xi|)g^\wedge(\xi) \right\|_{H^{s_1-\frac{m+4}{2(m+2)} - \frac{s}{2}}} \leq C \| g \|_{H^s}, \\
&\left\| \partial_t V_2(t, |\xi|)g^\wedge(\xi) \right\|_{H^{s_1-\frac{m+4}{2(m+2)} - \frac{s}{2}}} \leq C \| g \|_{H^s}.
\end{align*}
\tag{3.7}
\]

**Proof.** (i) First, we fix \( t = (\frac{m}{m+2})^{-\frac{1}{m+2}} \) to show (3.7) (in this case, the corresponding variable \( z \) in (2.10) becomes \( z = 2|\xi| \)). Subsequently, for the variable \( t \), as in [25] and so on, we can use the scaling technique to derive (3.7).

Since \( \Phi(a, c; z) \) is an analytic function of \( z \), then \( \Phi(\frac{m}{2(m+2)}, \frac{m}{m+2}; 2i|\xi|) \) and \( \Phi(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; 2i|\xi|) \) are bounded for \( |\xi| \leq C \). On the other hand, it follows from (3.2) that for large \( |\xi| \)
\[
|\Phi(\frac{m}{2(m+2)}, \frac{m}{m+2}; 2i|\xi|)| \leq C(1 + |\xi|^2)^{-\frac{m}{m+2}}
\]
and
\[
|\Phi(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; 2i|\xi|)| \leq C(1 + |\xi|^2)^{-\frac{m+4}{2(m+2)}}.
\]

Thus, for any \( s_1 \in [0, \frac{m}{2(m+2)}] \) and \( s_2 \in [0, \frac{m+4}{2(m+2)}] \), by a direct computation, we arrive at

\[
\begin{align*}
&\left\| (V_1(\frac{m+2}{2})^\wedge, |\xi|)g^\wedge(\xi) \right\|_{H^{s_1+2}} \\
= &\| (1 + |\xi|^2)^{-\frac{m}{2(m+2)} + \frac{m+2}{2} - |\xi|^2|\xi|} \phi(\frac{m}{2(m+2)}, \frac{m}{m+2}; 2i|\xi|)g^\wedge(\xi) \|_{L^2} \\
\leq &\| (1 + |\xi|^2)^{-\frac{m}{2(m+2)} + \frac{m+2}{2} - |\xi|^2|\xi|} \|_{L^\infty} \| (1 + |\xi|^2)^{\frac{m+4}{2(m+2)}} g^\wedge(\xi) \|_{L^2} \\
\leq &C \| g \|_{H^s}.
\end{align*}
\tag{3.9}
\]

and
\[
\left\| (V_2(\frac{m+2}{2})^\wedge, |\xi|)g^\wedge(\xi) \right\|_{H^{s_1+2}} \leq C \| g \|_{H^s}.
\tag{3.10}
\]
Next we treat $\|(V_1(t, |\xi|)g^\wedge(\xi))^\wedge\|_{H^{s+1}}$ and $\|(V_2(t, |\xi|)g^\wedge(\xi))^\wedge\|_{H^{s+2}}$. To this end, we introduce the following transformation
\[
\eta = \frac{2}{m+2} \frac{m+4}{m+2} \xi,
\]
and then we have
\[
\|(V_1(t, |\xi|)g^\wedge(\xi))^\wedge\|_{H^{s+1}}
= \left( \int_{\mathbb{R}^n} \left| 1 + |\xi|^2 \right|^{\frac{m+2}{2}} e^{-\frac{m+4}{m+2}|\xi|^2} \Phi \left( \frac{m}{2(m+2)} \frac{m}{m+2} \frac{4i}{m+2} |\xi|^2 + \frac{m+2}{m+2} |\xi|^2 \right)^2 d\xi \right)^{\frac{1}{2}}
= \left( \frac{m+2}{2t^{\frac{m+2}{2}}} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \left| 1 + \frac{((m+2)|\eta|^2)^2}{2t^{\frac{m+2}{2}}} \right|^{\frac{m+2}{2}} e^{-\frac{m+4}{m+2}|\eta|^2} \Phi \left( \frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{2i|\eta|}{m+2} \right)^2 G^\wedge(\eta)^2 d\eta \right)^{\frac{1}{2}},
\] (3.11)
and
\[
\|(V_2(t, |\xi|)g^\wedge(\xi))^\wedge\|_{H^{s+2}}
= \left( \int_{\mathbb{R}^n} \left| 1 + \frac{((m+2)|\eta|^2)^2}{2t^{\frac{m+2}{2}}} \right|^{\frac{m+2}{2}} e^{-\frac{m+4}{m+2}|\eta|^2} \Phi \left( \frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{2i|\eta|}{m+2} \right)^2 G^\wedge(\eta)^2 d\eta \right)^{\frac{1}{2}},
\] (3.12)
here and below the notation $G^\wedge(\eta)$ is defined as
\[
G^\wedge(\eta) = \left( 1 + \frac{((m+2)|\eta|^2)^2}{2t^{\frac{m+2}{2}}} \right)^{\frac{m+2}{2}} g^\wedge(\frac{(m+2)|\eta|^2}{2t^{\frac{m+2}{2}}}).
\]
It is noted that
\[
\|G^\wedge(\eta)\|_{L^2} = \left( \int_{\mathbb{R}^n} \left| 1 + |\xi|^2 \right|^{\frac{m+2}{2}} e^{-\frac{m+4}{m+2}|\xi|^2} \Phi \left( \frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{2i|\xi|}{m+2} \right)^2 d\xi \right)^{\frac{1}{2}} \leq C t^{-\alpha(m+2)} \|g\|_{H^s}.
\] (3.13)
Additionally, for $0 < t \leq T$ and $\alpha \geq 0$, we have
\[
\left( 1 + \frac{((m+2)|\eta|^2)^2}{2t^{\frac{m+2}{2}}} \right)^{\alpha} < C t^{-\alpha(m+2)} (1 + |\eta|^2)^{\alpha}.
\] (3.14)
Thus, we obtain from (3.11)-(3.14) that for $0 < t \leq T$
\[
\|(V_1(t, |\xi|)g^\wedge(\xi))^\wedge\|_{H^{s+1}}
\leq C t^{-\frac{\alpha(m+2)}{2}} \left( \int_{\mathbb{R}^n} \left| 1 + |\eta|^2 \right|^{\frac{m+2}{2}} e^{-\frac{m+4}{m+2}|\eta|^2} \Phi \left( \frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{2i|\eta|}{m+2} \right)^2 G^\wedge(\eta)^2 d\eta \right)^{\frac{1}{2}}
= C t^{-\frac{\alpha(m+2)}{2}} \|G^\wedge(\eta)\|_{L^2}
\leq C t^{-\frac{\alpha(m+2)}{2}} \|g\|_{H^s},
\] (3.15)
and
\[
\|(V_2(t, |\xi|)g^\wedge(\xi))^\wedge\|_{H^{s+2}} \leq C t^{1-\frac{\alpha(m+2)}{2}} \|g\|_{H^s}.
\] (3.16)
Consequently, we complete the proof of Lemma 3.2.(i).
(ii). It follows from a direct computation and (3.3)-(3.4) that
\[
\partial_t V_1(t, |\xi|)
= 2i \frac{(m+2)}{4t} \frac{m+2}{m+2} |\xi|^{\frac{m+2}{2}} e^{-\frac{m+4}{m+2}|\xi|^2} \left( - \frac{1}{2} e^{-\frac{2m+4}{m+2}|\xi|^2} \Phi \left( \frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{2i|\xi|}{m+2} \right)^2 \right)
- i \frac{(m+2)}{4t} \frac{m+2}{m+2} |\xi|^{\frac{m+2}{2}} \frac{m+2}{m+2} \frac{m+4}{m+2} \frac{2i|\xi|}{m+2} e^{-\frac{m+4}{m+2}|\xi|^2} \left( \Phi \left( \frac{3m+4}{2(m+2)} \frac{2(m+1)}{m+2} \frac{2i|\xi|}{m+2} \right)^2 \right)
\] (3.17)
\[\partial_t V_2(t, |\xi|) = e^{-\frac{1}{2}} \left( \Phi\left( \frac{m + 4}{2(m + 2)}, \frac{2}{m + 2}, z \right) - \frac{(m + 2)z}{4} \Phi\left( \frac{m + 4}{2(m + 2)}, \frac{m + 4}{m + 2}, z \right) \right). \]  

(3.18)

Thus, in terms of (3.2), we have for large \(|z|\)

\[|\partial_t V_1(t, |\xi|)| \leq i \left( \frac{m+2}{4i} \right)^{\frac{m}{2(m+2)}} |\xi|^2 e^{-\frac{1}{2}} \left( \left| \Phi\left( \frac{3m+4}{2(m+2)}, \frac{2(m+1)}{m+2}, z \right) \right| + \left| \Phi\left( \frac{m}{2(m+2)}, \frac{m}{m+2}, z \right) \right| \right) \leq C |\xi|^2 e^{-\frac{1}{2}} \left( |\xi|^2 + \frac{1}{m+2} \right) \]

(3.19)

and

\[|\partial_t V_2(t, |\xi|)| \leq e^{-\frac{1}{2}} \left( \left| \Phi\left( \frac{m+4}{2(m+2)}, \frac{2}{m+2}, z \right) \right| + \left| \gamma \Phi\left( \frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, z \right) \right| \right) \leq C \left( |\xi|^2 + \frac{1}{m+2} \right). \]

(3.20)

Next it suffices to estimate \(\| (\partial_t V_2(t, |\xi|) g^\gamma(\xi))^\gamma \|_{H^r_{\kappa}} \) since the treatment on \( (\partial_t V_1(t, |\xi|) g^\gamma(\xi))^\gamma \) is completely analogous.

As in (i), we fix \(t = \left( \frac{m+2}{2} \right)^{\frac{m}{2(m+2)}} \). In this case, by the analytic property of \(\Phi(a, c; z)\) and (3.20), we arrive at

\[\| (1 + |\xi|^2)^{-\frac{m}{2(m+2)}} \partial_t V_2\left( \left( \frac{m+2}{2} \right)^{\frac{m}{m+2}}, |\xi| \right) \|_{L^2} \leq \| (1 + |\xi|^2)^{-\frac{m}{2(m+2)}} \partial_t V_2\left( \left( \frac{m+2}{2} \right)^{\frac{m}{m+2}}, |\xi| \right) \|_{L^\infty} \| (1 + |\xi|^2)^{\frac{m}{m+2}} g^\gamma \|_{L^2} \leq C \| g \|_{H^r}. \]

(3.21)

For any \(t > 0\), we have

\[\| (\partial_t V_2(t, |\xi|) g^\gamma(\xi))^\gamma \|_{H^r_{\kappa}} \leq \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{-\frac{m}{2(m+2)}} \partial_t V_2\left( \left( \frac{m+2}{2} \right)^{\frac{m}{m+2}}, |\xi| \right) \right|^2 \frac{4i}{m+2} \left| \Phi\left( \frac{m+4}{2(m+2)}, \frac{2}{m+2}, \frac{m+4}{m+2}, |\xi| \right) \right|^2 d\xi \right)^{\frac{1}{2}} \]

\[= \left( \frac{m+2}{2e^{\frac{m}{m+2}}} \right)^{\frac{m}{m+2}} \left( \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{-\frac{m}{2(m+2)}} \partial_t V_2\left( \left( \frac{m+2}{2} \right)^{\frac{m}{m+2}}, |\xi| \right) \right|^2 \frac{4i}{m+2} \left| \Phi\left( \frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, 2i|\eta| \right) \right|^2 d\xi \right)^{\frac{1}{2}} \]

\[= \left( \frac{m+2}{2e^{\frac{m}{m+2}}} \right)^{\frac{m}{m+2}} \left( \left( \int_{\mathbb{R}^n} \left| (1 + |(m+2)|\eta|^2)^{-\frac{m}{2(m+2)}} \partial_t V_2\left( \left( \frac{m+2}{2} \right)^{\frac{m}{m+2}}, |\xi| \right) \right|^2 \frac{4i}{m+2} \left| \Phi\left( \frac{m+4}{2(m+2)}, \frac{m+4}{m+2}, 2i|\eta| \right) \right|^2 d\xi \right)^{\frac{1}{2}} \]

(3.22)

Substituting the estimates (3.13) and (3.21) into the expression (3.22) yields for \(0 < t \leq T\)
Proposition 3.3 can be shown.

As in (2.14) of Lemma 2.3, one has

\[ \left\| \phi_1 \in H^s(\mathbb{R}^n) \text{ and } \phi_2 \in H^{s - \frac{m}{2(2m+2)}}(\mathbb{R}^n) \right\| \]

Proof. In Lemma 3.2, if we take \( s_1 = 0 \) or \( s_1 = \frac{m}{2(2m+2)} \) and \( s_2 = 0 \) or \( s_2 = \frac{m+4}{2(2m+2)} \) respectively, then Proposition 3.3 can be shown. □

Based on Lemma 3.2, we can derive the following estimates for the solution of (3.5).

**Proposition 3.3.** If \( \phi_1 \in H^s(\mathbb{R}^n) \) and \( \phi_2 \in H^{s - \frac{m}{2(2m+2)}}(\mathbb{R}^n) \) with \( s \in \mathbb{R} \), then (3.5) has a solution \( u(t, x) \in C([0, T], H^s(\mathbb{R}^n)) \) which admits the following estimates for \( 0 < t \leq T \)

\[ \left\| u(t, \cdot) \right\|_{H^s(\mathbb{R}^n)} + t^{\frac{m}{2}} \left\| u(t, \cdot) \right\| + \left\| \partial_t u(t, \cdot) \right\|_{H^{s - \frac{m}{2(2m+2)}}(\mathbb{R}^n)} \leq C \left( \left\| \phi_1 \right\|_{H^s(\mathbb{R}^n)} + \left\| \phi_2 \right\|_{H^{s - \frac{m}{2(2m+2)}}(\mathbb{R}^n)} \right). \]

**Proof.** In Lemma 3.2, if we take \( s_1 = 0 \) or \( s_1 = \frac{m}{2(2m+2)} \) and \( s_2 = 0 \) or \( s_2 = \frac{m+4}{2(2m+2)} \) respectively, then Proposition 3.3 can be shown. □

Next, we consider the following inhomogeneous problem

\[
\begin{aligned}
\partial^2_t u - t^m \Delta u &= f(t, x) \\
u(0, x) &= 0, \quad u_t(0, x) = 0.
\end{aligned}
\]

As in (2.14) of Lemma 2.3, one has

\[ u^\wedge(t, \xi) = \int_0^t (V_2(t, |\xi|) V_1(\tau, |\xi|) - V_1(t, |\xi|) V_2(\tau, |\xi|)) f^\wedge(\tau, \xi) d\tau. \]

Based on Lemma 3.1-Lemma 3.2, we can establish

**Lemma 3.4.** If \( f(t, \cdot) \in C([0, T], H^s(\mathbb{R}^n)) \) with \( s \in \mathbb{R} \) and \( T \) a fixed positive constant, then for \( t \in (0, T] \)

\[
\begin{aligned}
\left\| u(t, \cdot) \right\|_{H^{s+p_1}} &\leq C t^2 - \frac{m}{2(2m+2)} \left\| f(t, x) \right\|_{L^\infty([0, T], H^s)}, \\
\left\| \partial_t u(t, \cdot) \right\|_{H^{s - \frac{m}{2(2m+2)} + p_2}} &\leq C p_2 t^1 - \frac{m+2}{2} p_2 \left\| f(t, x) \right\|_{L^\infty([0, T], H^s)},
\end{aligned}
\]

where \( 0 \leq p_1 < p_1(m) = \begin{cases} \frac{m + 8}{2(m + 2)} & \text{for } m \geq 4 \\ 1 & \text{for } m \leq 4 \end{cases} \) and \( p_2 < p_2(m) = \min \left\{ \frac{2}{m+2}, \frac{m}{2(m+2)} \right\} \).
Proof. It follows from the Minkowski inequality and (3.24) that

\[
\|u(t, \cdot)\|_{H^{s_1}+p_1} \leq \int_0^t \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s_1}{2} + \frac{p_1}{4}} (V_2(t, |\xi|) V_1(\tau, |\xi|) - V_1(t, |\xi|) V_2(\tau, |\xi|)) f^\wedge(\tau, \xi) \right|^2 d\xi \right)^{\frac{1}{2}} d\tau \\
\leq \int_0^t \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s_1}{2} + \frac{p_1}{4}} V_2(t, |\xi|) V_1(\tau, |\xi|) f^\wedge(\tau, \xi) \right|^2 d\xi \right)^{\frac{1}{2}} d\tau \\
+ \int_0^t \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s_1}{2} + \frac{p_1}{4}} V_1(t, |\xi|) V_2(\tau, |\xi|) f^\wedge(\tau, \xi) \right|^2 d\xi \right)^{\frac{1}{2}} d\tau \\
\equiv I_1 + I_2. 
\]

(3.27)

Let \( p_1 = s_1 + s_2 \) with \( 0 \leq s_1 < \min\left\{ \frac{m}{2(m+2)}, \frac{2}{m+2} \right\} \) and \( 0 \leq s_2 \leq \frac{m+4}{2(m+2)} \), then we have by Lemma 3.2

\[
I_1 \leq Ct^{\frac{s_1(m+2)}{2}} \int_0^t \left( 1 + |\xi|^2 \right)^{\frac{s_1}{2} + \frac{p_1}{2}} \left\| (1 + |\xi|^2)^{\frac{s_1}{2} + \frac{p_1}{4}} V_1(\tau, |\xi|) f^\wedge(\tau, \xi) \right\|_{L^2} d\tau \\
\leq Ct^{\frac{s_1(m+2)}{2}} \int_0^t \tau^{-\frac{s_1(m+2)}{2}} \left\| f(\tau, \cdot) \right\|_{H^s} d\tau \\
\leq Ct^{\frac{p_1(m+2)}{2}} \left\| f \right\|_{L^\infty([0,T], H^s)}. 
\]

(3.28)

On the other hand, if we set \( p_1 = \tilde{s}_1 + \tilde{s}_2 \) with \( 0 \leq \tilde{s}_1 \leq \frac{m}{2(m+2)} \) and \( 0 \leq \tilde{s}_2 < \min\left\{ \frac{m+4}{2(m+2)}, \frac{1}{m+2} \right\} \), then we have by Lemma 3.2

\[
I_2 \leq Ct^{\frac{\tilde{s}_1(m+2)}{2}} \int_0^t \tau^{1 - \frac{\tilde{s}_1(m+2)}{2}} \left\| f(\tau, \cdot) \right\|_{H^s} d\tau \\
\leq Ct^{\frac{\tilde{s}_1(m+2)}{2}} \left\| f \right\|_{L^\infty([0,T], H^s)}. 
\]

(3.29)

Substituting (3.28)-(3.29) into (3.27) yields (3.25) for \( 0 \leq p_1 < p_1(m) \).

Next, we show (3.26).

Due to

\[
\partial_t u^\wedge(t, \xi) = \int_0^t \left( \partial_t V_2(t, |\xi|) V_1(\tau, |\xi|) - \partial_t V_1(t, |\xi|) V_2(\tau, |\xi|) \right) f^\wedge(\tau, \xi) d\tau,
\]

one has by Minkowski inequality

\[
\left\| \partial_t u(t, \cdot) \right\|_{H^{s_1}+p_2} \\
\leq \int_0^t \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s_1}{2} + \frac{p_2}{4}} \left( \partial_t V_2(t, |\xi|) V_1(\tau, |\xi|) - \partial_t V_1(t, |\xi|) V_2(\tau, |\xi|) \right) f^\wedge(\tau, \xi) \right|^2 d\xi \right)^{\frac{1}{2}} d\tau \\
\leq \int_0^t \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s_1}{2} + \frac{p_2}{4}} \partial_t V_2(t, |\xi|) V_1(\tau, |\xi|) f^\wedge(\tau, \xi) \right|^2 d\xi \right)^{\frac{1}{2}} d\tau \\
+ \int_0^t \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{s_1}{2} + \frac{p_2}{4}} \partial_t V_1(t, |\xi|) V_2(\tau, |\xi|) f^\wedge(\tau, \xi) \right|^2 d\xi \right)^{\frac{1}{2}} d\tau \\
\equiv I_3 + I_4.
\]

(3.30)

Applying for Lemma 3.2 yields for \( 0 < t \leq T \) and \( 0 < \tau \leq T \)

\[
I_3 \leq C \int_0^t \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{\frac{p_2}{4}} V_2(t, |\xi|) (1 + |\xi|^2)^{\frac{p_2}{4}} f^\wedge(\tau, \xi) \right|^2 d\xi \right)^{\frac{1}{2}} d\tau \\
\leq C \int_0^t \tau^{\frac{m+4}{2(p_2)}} \left\| f(\tau, \cdot) \right\|_{H^s} d\tau \\
\leq C_{p_2} t^{1 - \frac{m+4}{2(p_2)}} \left\| f \right\|_{L^\infty([0,T], H^s)}. 
\]

(3.31)
and
\[
I_k \leq C \int_0^t \| (1 + |\xi|^2)^{\frac{1}{2}} \min_{\frac{m+2}{m+1}} \xi_\tau - \xi_\tau^\ast \| V_\xi(\tau, |\xi|)(1 + |\xi|^2) \| L^2 \| d\tau
\]
\[
\leq C \int_0^t \tau^{1-\frac{m+2}{p_2}} \| f(\tau, \cdot) \|_{H^s} \| d\tau
\]
\[
\leq C p_2^{1-\frac{m+2}{p_2}} \| f \|_{L^\infty([0, T], H^s)}.
\]
(3.32)

Substituting (3.31)-(3.32) into (3.30) yields (3.26).
Consequently, Lemma 3.4 is proved. \(\square\)

§4. Conormal spaces and commutator relations

In this section, we will give the definitions of conormal spaces related to our problems. To this end, as the first step, we look for the basis of vector fields tangent to some surface (or surfaces).

**Lemma 4.1.** Let \(\Gamma_0 = \{(t, x) : t \geq 0, |x|^2 = \frac{4t^{m+2}}{(m+2)^2}\}, \) then a basis of the \(C^\infty\) vector fields tangent to \(\Gamma_0\) is given by
\[
L_0 = 2t \partial_t + (m + 2)(x_1 \partial_1 + \cdots + x_n \partial_n);
\]
\[
L_i = 2t^{m+1} \partial_i + (m + 2)x_i \partial_t, \quad i = 1, 2, \cdots, n;
\]
\[
L_{ij} = x_i \partial_j - x_j \partial_i, \quad 1 \leq i < j \leq n.
\]

**Proof.** At first, we prove such an assertion:
*Given a smooth function \(c(t, x)\) vanishing on \(\Gamma_0\), then there exists a smooth function \(d(t, x)\) such that
\[
c(t, x) = d(t, x)(|x|^2 - \frac{4t^{m+2}}{(m+2)^2}).
\]
(4.1)

Indeed, it follows from Malgrange Preparation Theorem (see Theorem 7.5.6 of [16]) that there exist smooth functions \(c_1(t, \tilde{x}), c_2(t, \tilde{x})\) with \(\tilde{x} = (x_1, \ldots, x_{n-1})\) and \(d(t, x)\) such that
\[
c(t, x) = d(t, x)(|x|^2 - \frac{4t^{m+2}}{(m+2)^2}) + x_n c_1(t, \tilde{x}) + c_2(t, \tilde{x}).
\]

For \((t, x) \in \Gamma_0\), we have
\[
0 = \pm c_1(t, \tilde{x}) \sqrt{\frac{4t^{m+2}}{(m+2)^2} - |\tilde{x}|^2 + c_2(t, \tilde{x}).
\]

This yields \(c_1(t, \tilde{x}) = c_2(t, \tilde{x}) = 0\). Hence, we complete the proof on (4.1).
Next we use the induction method and (4.1) to prove Lemma 4.1.

For \(n = 1\), we assume that the vector field \(L = a(t, x_1) \partial_1 + b(t, x_1) \partial_t\) is tangent to \(\Gamma_0^1 \equiv \{x_1^2 = \frac{4t^{m+2}}{(m+2)^2}\}, \) which means
\[
L(x_1^2 - \frac{4t^{m+2}}{(m+2)^2}) = 2a(t, x_1)x_1 - \frac{4b(t, x_1)}{m+2} t^{m+1} \equiv 0 \quad \text{on} \quad \Gamma_0^1.
\]
(4.2)

By (4.1), we know that there exists a smooth function \(d(t, x_1)\) such that
\[
2a(t, x_1)x_1 - \frac{4b(t, x_1)}{m+2} t^{m+1} = d(t, x_1)(x_1^2 - \frac{4t^{m+2}}{(m+2)^2}).
\]
(4.3)
This derives
\[ b(t, 0) = \frac{d(t, 0)t}{m + 2}. \]  
\hspace{1cm} (4.4)

On the other hand, it follows from (4.3) that there exist two smooth functions \( b_1(t, x_1) \) and \( d_1(t, x_1) \) such that
\[ 2a(t, x_1)x_1 - \frac{4b(t, 0) + 4b_1(t, x_1)x_1}{m + 2} + \frac{4t^{m+2}}{(m + 2)^2} = (d(t, 0) + d_1(t, x_1)x_1) \left( x_1^2 - \frac{4t^{m+2}}{(m + 2)^2} \right). \]  
\hspace{1cm} (4.5)

Substituting (4.4) into (4.5) yields
\[ a(t, x_1) = \frac{2b_1(t, x_1)}{m + 2} + \frac{d(t, 0)}{2} x_1 + \frac{d_1}{2} \left( x_1^2 - \frac{4t^{m+2}}{(m + 2)^2} \right). \]  
\hspace{1cm} (4.6)

Therefore,
\[ L = a(t, x_1)\partial_t + (b(t, 0) + b_1(t, x_1)x_1)\partial_t \]
\[ = \frac{b_1}{m + 2}L + \frac{d(t, 0)}{2(m + 2)}L_0 + \frac{d_1}{2(m + 2)^2} \left( (m + 2)x_1L_0 - 2tL_1 \right) \]
\[ = \left( \frac{b_1}{m + 2} - \frac{d_1t}{(m + 2)^2} \right)L_1 + \left( \frac{d_1x_1}{2(m + 2)} + \frac{d(t, 0)}{2(m + 2)} \right)L_0. \]

This yields the case of \( n = 1 \) in Lemma 4.1.

By the induction hypothesis, we assume that Lemma 4.1 holds for \( n - 1 \).

We now show the case of \( n \).

Assume that \( L = a_0(t, x)\partial_t + \sum_{i=1}^{n} a_i(t, x)\partial_i \) is tangent to \( \Gamma_0 \), and we rewrite \( L \) as
\[ L = \sum_{i=0}^{n} a_i(t, \tilde{x}, 0)\partial_i + \sum_{i=0}^{n} b_i(t, \tilde{x}, x_n)x_n\partial_i \]
\[ \equiv M_{n-1} + a_n(t, \tilde{x}, 0)\partial_n + \sum_{i=0}^{n} b_i(t, \tilde{x}, x_n)x_n\partial_i, \]
where \( \tilde{x} = (x_1, \ldots, x_{n-1}) \), \( \partial_0 = \partial_t \), and \( M_{n-1} = \sum_{i=0}^{n-1} a_i(t, \tilde{x}, 0)\partial_i \).

We can assert that \( M_{n-1} \) is tangent to the surface \( \{ |\tilde{x}|^2 = \frac{4t^{m+2}}{(m + 2)^2} \} \).

Indeed, due to
\[ L(|\tilde{x}|^2 - \frac{4t^{m+2}}{(m + 2)^2}) = M_{n-1}(|\tilde{x}|^2 - \frac{4t^{m+2}}{(m + 2)^2}) + a_n(2a_n(t, \tilde{x}, 0) + \sum_{i=0}^{n} b_i(|\tilde{x}|^2 - \frac{4t^{m+2}}{(m + 2)^2})], \]
and \( L(|\tilde{x}|^2 - \frac{4t^{m+2}}{(m + 2)^2}) \equiv 0 \) on \( \{ |\tilde{x}|^2 = \frac{4t^{m+2}}{(m + 2)^2} \} \), then \( L(|\tilde{x}|^2 - \frac{4t^{m+2}}{(m + 2)^2}) \equiv 0 \) holds true on \( \{ |\tilde{x}|^2 - \frac{4t^{m+2}}{(m + 2)^2} = 0, x_n = 0 \} \) and further \( M_{n-1}(|\tilde{x}|^2 - \frac{4t^{m+2}}{(m + 2)^2}) \equiv 0 \) on \( \{ |\tilde{x}|^2 = \frac{4t^{m+2}}{(m + 2)^2} \} \) is derived.

By the induction hypothesis, we know that \( M_{n-1} \) can be expressed as a linear combination of \( L_0^{(n-1)} = 2t\partial_t + (m + 2)(x_1\partial_1 + \cdots + x_{n-1}\partial_{n-1}) \), \( L_i^{(n-1)} = 2t^{m+1}\partial_i + (m + 2)x_i\partial_t \) \( (i = 1, 2, \cdots, n - 1) \) and \( L_{ij}^{(n-1)} = x_i\partial_j - x_j\partial_i \)
(1 ≤ i < j ≤ n − 1). On the other hand, we have $x_n \partial_i = \frac{1}{m+2} t_0^{(n)} - \frac{2}{m+2} t^{m+1} \partial_n$ and $x_n \partial_i = L_{ni}^{(n)} + x_i \partial_n$. Thus, we can arrive at

$$L = \bar{a}(t, x) \partial_n + p(L_{0}^{(n)}, L_{i}^{(n)}, L_{ij}^{(n)})_{1 \leq i < j \leq n},$$

(4.7)

where $\bar{a}(t, x)$ is a smooth function, $p(L_{0}^{(n)}, L_{i}^{(n)}, L_{ij}^{(n)})_{1 \leq i < j \leq n}$ represents a first order polynomial of $L_{0}^{(n)}$, $L_{i}^{(n)}$, $L_{ij}^{(n)}$ with $1 \leq i \leq n$ and $1 \leq i < j \leq n$ respectively.

Since $L$ and $L_{0}^{(n)}, L_{i}^{(n)}, L_{ij}^{(n)}$ are all tangent to $\Gamma_0$, then one has $\bar{a}(t, x) \equiv 0$ on $\Gamma_0$. This yields that there exists a smooth function $d(t, x)$ such that $\bar{a}(t, x) = d(t, x)((m + 2)|x|^2 - 4t^{m+2})$. It is noted that $((m + 2)^2|x|^2 - 4t^{m+2})\partial_n = (m + 2)x_n L_{0}^{(n)} - 2t L_{n}^{(n)} - (m + 2)^2 \sum_{i=1}^{n-1} x_i L_{ni}^{(n)}$. This, together with (4.7), yields the proof on Lemma 4.1.

In order to apply for the commutator argument to treat our degenerate equation whose characteristic cone and characteristic surfaces have cusp singularities, we will use the following revised vector fields tangent to $\Gamma_0$:

$$L_0 = 2t \partial_1 + (m + 2)(x_1 \partial_1 + \cdots + x_n \partial_n);$$

$$L_i = 2t \frac{x_i}{t^{m+1}} + (m + 2) \frac{x_i}{t^{m+1}} \partial_1; \quad i = 1, 2, \cdots, n;$$

$$L_{ij} = x_i \partial_j - x_j \partial_i, \quad 1 \leq i < j \leq n.$$

Let $[A, B] = AB - BA$ denote the commutator of $A$ and $B$. By a direct computation, one has

**Lemma 4.2.** For $1 \leq i \leq n$ and $1 \leq i < j \leq n$,

$$[L_0, L_i] = 0; \quad [L_0, L_{ij}] = 0;$$

$$[L_i, L_j] = 2(m + 1)(m + 2)L_{ij} + \frac{m(m + 2)}{2} \left( \frac{x_j}{t^{m+1}} L_i - \frac{x_i}{t^{m+1}} L_j \right); \quad [L_i, L_{ij}] = L_j;$$

$$[L_k, L_{ij}] = 0 \quad \text{for } k \neq i \text{ and } k \neq j;$$

$$[L_{ij}, L_{kl}] = 0 \quad \text{for } 1 \leq k < l \leq n, k \neq i, l \neq j; \quad [L_{ij}, L_{ik}] = L_{kj} \quad \text{for } k \neq j.$$

In addition, let $P = \partial_t^2 - t^m \Delta$, then

$$[P, L_0] = 4P; \quad [P, L_{ij}] = 0; \quad [P, L_i] = -m(m + 2) \frac{x_i}{t^{m+1}} P + \frac{m(m + 2)}{4t^2} L_i.$$

**Remark 4.1.** If we choose the smooth vector fields $L_i = 2t^{m+1} \partial_i + (m + 2)x_i \partial_1$ instead of $\bar{L}_i$ in Lemma 4.2, then a direct computation yields that for $1 \leq i \leq n$

$$[P, L_i] = m t^{m-1} \left( \partial_t L_0 + (m + 2) \sum_{j \neq i} \partial_j L_{ij} + (n(m + 2) - 2) \partial_i \right)$$

and

$$[P, L_i^m] \text{ will include the term } ((m + 2)x_i)^{m-1} m! \left( \partial_t L_0 + (m + 2) \sum_{j \neq i} \partial_j L_{ij} + (n(m + 2) - 2) \partial_i \right).$$

When the conormal regularities of solution $u(t, x)$ to problem (1.1) are studied, we will meet the following problem

$$P L_i^m u = L_i^m (f(t, x, u)) + [P, L_i^m] u.$$
By Proposition 3.3, under the assumptions \((A_2) - (A_3)\), one can only expect \(u(t, x) \in C([0, T], H^{\frac{1}{2} + \frac{m}{2} - s} - \frac{1}{2} - \frac{1}{2} \mathbb{R}^n))\) which implies \(PL^m u = L^m(f(t, x, u)) + \text{some terms of } C([0, T], H^{\frac{1}{2} - \frac{1}{2}}(\mathbb{R}^n))\) and thus only \(L^m u \in C([0, T], H^{\frac{1}{2} - \frac{1}{2}}(\mathbb{R}^n))\) can be expected by Lemma 3.4. Hence, for large \(m\), one just only obtains \(L^m u \in C([0, T], H^{-\frac{1}{2}}(\mathbb{R}^n))\), which leads to the loss of regularities of \(L^m u\) and more losses of regularities of \(\{L^m u\}_{t>m}\) can be produced. It is noted that \(L^m u \in C([0, T], L^2(\mathbb{R}^n))\) with \(k \geq m\) should be obtained in order to show Theorem 1.1. Hence, one can not use the smooth vector fields and commutator arguments directly to show Theorem 1.1.

Next, we introduce the vector fields tangent to \(\Gamma^+_k = \{(t, x) \in \mathbb{R} \times \mathbb{R} : x_1 = \pm \frac{2m+2}{m+2}\} \times \mathbb{R}^n\). Let

\[
\begin{align*}
\bar{L}_0 &= 2t\partial_t + (m + 2)x_1\partial_1, \\
\bar{L}_1 &= 2t\bar{\partial}_t + (m + 2)x_1\bar{\partial}_1, \quad R_k = \partial_k, 2 \leq k \leq n.
\end{align*}
\]

Moreover, we have the following commutator relations:

\[
\begin{align*}
[\bar{L}_0, \bar{L}_1] &= 0, \\
[\bar{L}_0, R_k] &= -(m + 2)R_k \quad \text{for } k \geq 2, \\
[\bar{L}_1, R_k] &= 0 \quad \text{for } k \geq 2, \\
[P, \bar{L}_0] &= 4P + 2mR_k \sum_{i=2}^n \partial_i R_i, \\
[P, \bar{L}_1] &= -m(m + 2)x_1\bar{\partial}_1 P + \frac{m(m + 2)}{4t^2}\bar{L}_1, \\
[P, R_k] &= 0 \quad \text{for } k \geq 2.
\end{align*}
\]

**Proof.** This can be verified directly, we omit it here. \(\square\)

**Remark 4.2.** In the expression of commutator \([P, \bar{L}_1]\), there appear a singular factor \(\frac{1}{t^2}\) before \(\bar{L}_1\). This will produce such an equation on \(\bar{L}_1 u\) from (1.1):

\[
PL_1 u - \frac{m(m + 2)}{4t^2} \bar{L}_1 u - f'(t, x, u)\bar{L}_1 u = (\bar{L}_1 f)(t, x, u) - m(m + 2)x_1\bar{\partial}_1 f(t, x, u).
\]

Such a degenerate equation with a singular coefficient \(\frac{1}{t^2}\) has a bad behavior near \(t = 0\), and thus it is not suitable to use the commutator argument on \(\bar{L}_1 u\) (or more generally \(\bar{L}_k u\)) to derive the regularity of \(\bar{L}_k u\)

Based on the preparations above, we will define the conormal spaces which will be required later on. To this end, such terminologies as in [2]-[3] are introduced:

\[
\{M_1, \cdots, M_k\}
\]

stands for a collection of vector fields with bounded coefficients on an open set \(\Omega \subset \mathbb{R}^n\) such that all commutators \([M_i, M_j]\) are in the linear span over \(C^\infty(\Omega)\) of \(M_1, \cdots, M_k\).

**Definition 4.1 (Admissible function)** A function \(h(x) \in L^\infty(\Omega) \cap C^\infty(\Omega)\) is called admissible with respect to \(\{M_1, \cdots, M_k\}\) if \(M_1 h, \cdots, M_k h \in L^\infty(\Omega) \cap C^\infty(\Omega)\) for all \((j_1, \cdots, j_k)\).

Obviously, the linear span of \(\{M_1, \cdots, M_k\}\) with admissible coefficients is a Lie algebra of vector fields on \(\Omega\).

We now define the admissible tangent vector fields related to the surface \(\Gamma_0\).

**Definition 4.2 (Admissible tangent vector fields of \(\Gamma_0\))**

1. Let \(\Omega_1\) be a region of the form \(\{(t, x) : 0 < t < C|x| \leq \varepsilon\}\) and \(S_1\) be the Lie algebra of vector fields with admissible coefficients on \(\Omega_1\) generated by \(\{x|\partial_t| t^m \partial_t|x|\partial_t, L_{ij}, i, j = 1, 2, \cdots, n\}\).

2. Let \(\Omega_2\) be a region of the form \(\{(t, x) : |x| \leq C \varepsilon\} \cap \{(t, x) : |x| - \frac{\varepsilon^m}{m+2} | < C \varepsilon^{\frac{m+2}{2}}\}\) and \(S_2\) be the Lie algebra of vector fields with admissible coefficients on \(\Omega_2\) generated by \(\{L_0, L_i, L_{ij}, i, j = 1, 2, \cdots, n\}\) in Lemma 4.2.

3. Let \(\Omega_3\) be a region of the form \(\{(t, x) : |x| < C \varepsilon\} \cap \{(t, x) : t \varepsilon^{-m} < C|x| - \frac{\varepsilon^{m+2}}{m+2}\}\) and \(S_3\) be the Lie algebra of vector fields with admissible coefficients on \(\Omega_3\) which generated by \(\{t\partial_t, t^{m+1}\partial_t, L_{ij}, i, j = 1, 2, \cdots, n\}\).

Next, the conormal space \(P^s H^s(\Gamma_0)\) with \(0 \leq s < \frac{\varepsilon}{4}\) is defined.
Definition 4.3 (Conormal space $I^{\infty}H^s(\Gamma_0)$). Define the function $u(t,x) \in I^{\infty}H^s(\Gamma_0)$ in $\{(t,x) : 0 \leq t \leq T, x \in \mathbb{R}^n\}$ if, away from $\{|x| = t = 0\}$, $Z_1 \cdots Z_k u \in L^\infty([0,T], H^s(\mathbb{R}^n))$ for any $k \in \mathbb{N} \cup \{0\}$ and all smooth vector fields $Z_1, \ldots, Z_j \in \{L_0, L_i, L_{ij}, i,j = 1,2,\ldots,n\}$, and near $\{|x| = t = 0\}$, the following properties hold:

1) If $h_1(t,x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $\Omega_1 = \{(t,x) : 0 \leq t < C|x| \leq \varepsilon\}$, then $Z_1 \cdots Z_k (h_1(t,x) u(t,x)) \in L^\infty([0,T], H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in S_1$.

2) If $h_2(t,x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{(t,x) : |x| < Ct \leq \varepsilon\}$ and $\chi(\theta) \in C^\infty$ has compact support near $\{\theta = 1\}$, then $Z_1 \cdots Z_k (h_2(t,x) \chi(\frac{(m+2)|x|}{2t^{m+1}}) u) \in L^\infty([0,T], H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in S_2$.

3) If $h_3(t,x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{(t,x) : |x| < Ct \leq \varepsilon\}$ and $\chi_0(\theta) \in C^\infty$ has compact support near $\{\theta = 1\}$, then $Z_1 \cdots Z_k (h_3(t,x) \chi_0(\frac{(m+2)|x|}{2t^{m+1}}) u) \in L^\infty([0,T], H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_j \in S_3$.

It is noted that $h_1(t,x), h_2(t,x) \chi(\frac{(m+2)|x|}{2t^{m+1}})$, and $h_3(t,x) \chi_0(\frac{(m+2)|x|}{2t^{m+1}})$ are admissible functions on domains $\Omega_1, \Omega_2$, and $\Omega_3$ respectively, moreover are in the space $L^\infty(0,\infty, H^s(\mathbb{R}^n))$.

Because some vector fields (for examples, $\tilde{L}_i, i = 1, \ldots, n$) in Definition 4.3 has no good commutator relations (i.e., the coefficients of commutator are not admissible, one can also see the explanations in Remark 4.2) with $P = \partial_t^2 - t^m \Delta$, we have to look for some auxiliary relations among those vector fields which possess good commutator relations (for examples, $L_0, L_1, \ldots, L_n$ in Lemma 4.2, and $L_0, R_2(2 \leq j \leq n)$ in Lemma 4.3) and $P$.

Formally, it follows from a direct computation that

$$
\begin{align*}
\partial_t & = \frac{(4t^{m+2} - (m+2)^2 \sum_{j \neq i} x_j^2 \tilde{L}_i + (m+2)^2 \sum_j x_j x_i \tilde{L}_j - 2(m+2)x_i t^{m+1} \bar{L}_0}{2t^{m+1}(4t^{m+2} - (m+2)^2 |x|^2)}, \quad i = 1, \ldots, n;
\end{align*}
$$

$$
\begin{align*}
2t^{m+1} \bar{L}_0 - (m+2)^2 \sum_{i=1}^n x_i t^{m+1} \bar{L}_i
\end{align*}
$$

According to this and some crucial observations, we have

Lemma 4.4. Let $\Omega_i (i = 1,2,3)$ be given in Definition 4.2, one has

1) On $\Omega_1$, set $N_0^i = |x| \partial_i$, $N_1^i = t^{\frac{m}{2}} |x| \partial_i$ with $i = 1, \ldots, n$, then

$$
\begin{align*}
(N_0^i)^2 & = \frac{1}{4t^{m+2} - (m+2)^2 |x|^2} \left(-4|x|^2 t^{m+2} P - |x|^2 t^m \sum_{j=1}^n \tilde{L}_j + 4|x| t^{m+1} \bar{L}_0 \end{align*}
$$

$$
+ (m+1)|x|^2 t^m \bar{L}_0 + (2nm + 2n - 2m - 2)m+1 |x| - \frac{m(m+2)^2 |x|^3}{2t^3} \right) N_0^i \right);$$

$$
\begin{align*}
(N_1^i)^2 & = \frac{1}{4t^{m+2} - (m+2)^2 |x|^2} \left(-|x|^2 (4t^{m+2} - (m+2)^2 \sum_{j \neq i} x_j^2) P + t^m |x|^2 \bar{L}_0 - |x|^2 t^{m+2} \sum_{j \neq i} \tilde{L}_j \end{align*}
$$

$$
- 2(m+2)x_i |x| t^{m+1} \bar{L}_i + \left((n-1)(m+2) - 2|m|^2 \sum_{j \neq i} x_j^2 \right)|x|^2 t^m \bar{L}_0 \right)$$

$$
+ a_i^i N_1^i + \sum_{j \neq i} b_i^j N_1^j,$$

where $a_i^i, b_i^j$ are admissible on $\Omega_1$. 22
(2) On $\Omega_2$, set $N_2^i = (|x| - \frac{2}{m+2} t^{\frac{m+2}{2}}) \partial_i$ with $i = 1, \ldots, n$, then

$$
\bar{L}_i = \frac{1}{2t^{\frac{m+2}{m+1}}} \left( (m+2)x_iL_0 - (m+2)^2 \sum_{k \neq i} x_kL_{ik} - (m+2)((m+2)|x| + 2t^{\frac{m+1}{m+1}})N^i_2 \right)
$$

(4.9)

and

$$
(N^2_2)^2 = \frac{1}{(m+2)((m+2)|x| + 2t^{\frac{m+2}{m+1}})} \left( - (|x| - \frac{2}{m+2} t^{\frac{m+2}{2}})(4t^2 - (m+2)^2 \sum_{j \neq i} x_j^2)P + (|x| - \frac{2}{m+2} t^{\frac{m+2}{2}}) \sum_{j \neq i} \bar{L}_j^2 - 2(m+2)x_iN^2_2L_0 + (|x| - \frac{2}{m+2} t^{\frac{m+2}{2}}) \left( (n-1)(m+2) - 2 - \frac{m(m+2)^2}{4tm+2} \sum_{j \neq i} x_j^2 \right) L_0 \right) + a^i_2 N^i_2 + \sum_{j \neq i} b^j_2 N^j_2,
$$

(4.10)

where $a^i_2$ and $b^j_2$ are admissible on $\Omega_2$.

Thus, one has from (4.9) and (4.10) that for $i = 1, \ldots, n$,

$$
(N^2_2)^2 = a_0 P + a_1 L_0^2 + \sum_{1 \leq i \neq k \leq n, 1 \leq m < n} a^m_{ik} L_{ik} L_{ml} + \sum_{1 \leq i \leq n} b_{ik} L_0 L_{ik} + \sum_{1 \leq i \leq n} b_i N^i_2 L_0 + \sum_{1 \leq i \leq n, 1 \leq m < l \leq n} b^m_{i} N^i_2 L_{ml} + \sum_{1 \leq i \leq n} c_{0i} N^i_2 + c L_0,
$$

(4.11)

where the coefficients $a_0, a_1, a^m_{ik}, b_{ik}, b_i, b^m_{i}, c_{0i}$ and $c$ are admissible on $\Omega_2$.

(3) On $\Omega_3$, set $N_3^i = t \partial_i, N_3^i = \frac{t^{m+1}}{t^{m+1}} \partial_i$ with $i = 1, \ldots, n$, then

$$
(N_3^0)^2 = \frac{1}{4tm^2 - (m+2)^2|x|^2} \left( - 4t^{m+4} + t^{m+2} \sum_{j=1}^{m} \bar{L}_j^2 + 4t^{m+2}N_3^0 L_0 + (m+2)t^{m+2} L_0 + \left( 2(n-1)(m+2)t^{m+2} - \frac{(4+m)(m+2)^2|x|^2}{2} \right) N_3^0 \right)
$$

and

$$
(N_3^i)^2 = \frac{1}{(m+2)^2|x|^2 - 4tm^2} \left( t^{m+2}(4tm^2 - (m+2)^2 \sum_{j \neq i} x_j^2)P - t^{2(m+1)} \bar{L}_0^2 + t^{2m+2} \sum_{j \neq i} \bar{L}_j^2 + 2(m+2)x_i \frac{t^{m+2}}{4tm^2} N_3^0 L_0 \left( (2 - (m+2)(n-1))t^{m+2} + \frac{m(m+2)^2}{4tm} \sum_{j \neq i} x_j^2 \right) L_0 \right) + a^i_3 N_3^i + \sum_{j \neq i} b^j_3 N^j_3
$$

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where $a^i_3, b^i_3$ are admissible on $\Omega_3$.

**Remark 4.3.** It is also easy to verify that the coefficients on each region $\Omega_i$ ($i = 1, 3$) in Lemma 4.4 are admissible with respect to the vector fields $S_i$, respectively.

We start to define the admissible tangent vector fields related to the surfaces $\Gamma^+_1$ and $\Gamma^-_1$.

**Definition 4.4 (Admissible tangent vector fields of $\Gamma^+_1$)**

1. Let $W_1$ be a region of the form $\{(t, x) \colon 0 < t < C|x_1| \leq \varepsilon \}$ and $M_1$ be the Lie algebra of vector fields with admissible coefficients on $W_1$ generated by $\{x_1 \partial_t, x_3 \partial_3, \partial_3, i = 2, \ldots, n\}$.

2. Let $W_{2, \pm}$ be a region of the form $\{(t, x) \colon |x_1| < C \varepsilon \} \cap \{(t, x) : |x_1 + \frac{2m+2}{m+2}t^{\frac{m+2}{2}}| < C t^{\frac{m+2}{2}} \}$ and $M_{2, \pm}$ be the Lie algebra of vector fields with admissible coefficients on $W_{2, \pm}$ generated by $\{L_0, \tilde{L}_1, R_2, \ldots, R_n\}$ in Lemma 4.3.

3. Let $W_3$ be a region of the form $\{(t, x) : |x_1| < C \varepsilon \} \cap \{(t, x) : t^{\frac{m+2}{2}} < C|x_1 + \frac{2m+2}{m+2}t^{\frac{m+2}{2}}| \}$ and $M_3$ be the Lie algebra of vector fields with admissible coefficients on $W_3$ generated by $\{t \partial_t, t^{\frac{m+2}{2}} \partial_1, \partial_1, i = 2, \ldots, n\}$.

**Remark 4.4.** On $W_{2, \pm}$, for the convenience of computation, we sometimes use the equivalent vector fields $M_{2, \pm} = \{L_0, \tilde{L}_1, R_2, \ldots, R_n\}$ with $N_{2, \pm} = (x_1 + \frac{2m+2}{m+2}t^{\frac{m+2}{2}}) \partial_t$ instead of $\{L_0, \tilde{L}_1, R_2, \ldots, R_n\}$ from now on. The equivalence comes from the following facts:

\[
N_{2, \pm} = \frac{t^{\frac{m+1}{2}}}{(m+2)^2 x_1 + \frac{2m+2}{m+2}t^{\frac{m+2}{2}}} \left( \frac{(m+2)x_1 L_0}{t^{\frac{m+1}{2}}} - 2 L_1 \right),
\]

\[
L_0 = \frac{t^{\frac{m+1}{2}}}{(m+2)^2 x_1} \left( \frac{(m+2)^2 x_1 + \frac{2m+2}{m+2}t^{\frac{m+2}{2}}} {t^{\frac{m+1}{2}}} N_{2, \pm} + 2 L_1 \right),
\]

where all related coefficients are admissible on $W_{2, \pm}$.

Similarly, we define the conormal space $I^\infty H^s(\Gamma^+_1 \cup \Gamma^-_1)$ with $0 \leq s < \frac{m}{2}$.

**Definition 4.5 (Conormal space $I^\infty H^s(\Gamma^+_1 \cup \Gamma^-_1)$)**. Define the function $u(t, x) \in I^\infty H^s(\Gamma^+_1 \cup \Gamma^-_1)$ in $t \geq 0$, if away from $\{x_1 = t = 0\}$, $Z_1 \cdots Z_k u \in L^\infty([0, T], H^s(\mathbb{R}^n))$ for all smooth vector fields $Z_1, \ldots, Z_k \in \{L_0, \tilde{L}_1, R_2, \ldots, R_n\}$ in Lemma 4.3, and near $\{x_1 = t = 0\}$, the following properties hold:

1. If $h_1(t, x_1) \in C^\infty(\mathbb{R} \setminus \{0\})$ is homogeneous of degree zero and supported on $W_1 = \{(t, x) : 0 \leq t < C|x_1| \leq \varepsilon \}$, then $Z_1 \cdots Z_k (h_1(t, x_1) u(t, x)) \in L^\infty([0, T], H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in M_1$.

2. If $h_2(t, x_1) \in C^\infty(\mathbb{R} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{(t, x_1) : |x_1| < C \varepsilon \}$ and $\chi_\pm(\theta) \in C^\infty$ has compact support near $\{\theta = \pm 1\}$, then $Z_1 \cdots Z_k (h_2(t, x_1) \chi_\pm(t^{\frac{m+2}{2}}) u) \in L^\infty([0, T], H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in M_{2, \pm}$.

3. If $h_3(t, x_1) \in C^\infty(\mathbb{R} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{(t, x_1) : |x_1| < C \varepsilon \}$ and $\chi_0(\theta) \in C^\infty$ has compact support away $\{\theta = \pm 1\}$, then $Z_1 \cdots Z_k (h_3(t, x_1) \chi_0(t^{\frac{m+2}{2}}) u) \in L^\infty([0, T], H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in M_3$.

Obviously, the cutoff functions $h_1(t, x_1), h_2(t, x_1) \chi_\pm(t^{\frac{m+2}{2}}), h_3(t, x_1) \chi_0(t^{\frac{m+2}{2}})$ are admissible on domains $W_1, W_{2, \pm}$ and $W_3$ respectively, moreover are in the space $L^\infty([0, \infty), H^{\frac{m}{2}}(\mathbb{R}^n))$.

Similar to Lemma 4.4 and by some crucial observations, we have

**Lemma 4.5.** Let $W_1, W_{2, \pm}$ and $W_3$ be given in Definition 4.4, one has

1. On $W_1$, set $N_1 = x_1 \partial_3$, then

\[
N_1^2 = \frac{1}{(m+2)^2 x_1^2 - 4 t^{m+2}} \left( (m+2)^2 x_1^4 P + x_1^2 t^{m+2} \tilde{L}_0^2 - 4 x_1 t^{m+1} N_1 \tilde{L}_0 
+ (m+2)^2 x_1^4 \sum_{i=2}^n R_i^2 - (m+2) x_1^2 t^{m+1} \tilde{L}_0 + 2 (m+4) x_1 t^{m+1} N_1 \right).
\]
(2) On $W_{2,\pm}$, set $N_{2,\pm} = (x_1 + \frac{2}{m+2} t^\frac{m+2}{m+2}) \partial_1$, then
\[
N_{2,\pm}^2 = \frac{x_1 \mp \frac{2}{m+2} t^\frac{m+2}{m+2}}{(m+2)^2(x_1 \pm \frac{2}{m+2} t^\frac{m+2}{m+2})} \left( 4t^2 P - \tilde{L}_0^2 + 4t^m + 2 \sum_{i=2}^n R_i^2 + 2 L_0 \right)
+ \frac{2x_1}{(m+2)(x_1 \pm \frac{2}{m+2} t^\frac{m+2}{m+2})} N_{2,\pm} L_0 - \frac{2(x_1 \pm \frac{m+2}{m+2})}{(m+2)(x_1 \pm \frac{2}{m+2} t^\frac{m+2}{m+2})} N_{2,\pm}.
\]

(3) On $W_3$, set $N_3 = t \partial_t, N_{3y} = t^\frac{m+2}{m+2} \partial_1$, then
\[
N_3^2 = \frac{1}{(m+2)^2 x_1^2 - 4t^m} \left( (m+2)^2 x_1^2 t^2 P + t^m L_0^2 - 4t^m N_3 L_0 \right)
+ (m+2)^2 x_1^2 t^m \sum_{i=2}^n R_i^2 - (m+2)^2 t^m L_0 + ((m+2)^2 x_1^2 + 2(m+2)^2 t^m) N_3
\]
and
\[
N_{3y}^2 = \frac{1}{(m+2)^2 x_1^2 - 4t^m} \left( 4t^{m+4} P - t^m L_0^2 + 2(m+2)x_1 t^\frac{m+2}{m+2} N_{3y} L_0 \right)
+ 4t^{2(m+2)} \sum_{i=2}^n R_i^2 + 2t^m L_0 - 3(m+2)^2 x_1 t^\frac{m+2}{m+2} N_{3y}.
\]

**Remark 4.5.** As in Remark 4.3, one can easily verify that the coefficients on each domain in Lemma 4.5 are admissible with respect to the corresponding vector fields.

Finally, we define the conormal space $I^k L_{\infty}^\infty \Gamma_0, \Gamma_1^\pm \cup \Gamma_2^\pm$) of order $k$, which are related to the surfaces $\Gamma_0, \Gamma_1^\pm$, and $\Gamma_2^\pm$ in $\mathbb{R}_+ \times \mathbb{R}^2$. For this end, at first we will introduce the admissible vector fields as in Definition 4.2 and Definition 4.4.

Set
\[ t_i^\pm = \Gamma_i^\pm \cap \Gamma_0 \quad \text{for} \quad i = 1, 2, \quad t_3^\pm = \Gamma_1^\pm \cap \Gamma_2^\pm. \]

For small fixed constant $\delta > 0$ we define the following domains:

\[ \Omega_1^\pm = \{(t, x) : t > 0, |x_1| \leq \frac{2t^\frac{m+2}{m+2}}{m+2} \delta^\frac{m+2}{m+2}, |x_2| < \delta t\frac{m+2}{m+2} \}, \]
\[ \Omega_2^\pm = \{(t, x) : t > 0, |x_2| \leq \frac{2t^\frac{m+2}{m+2}}{m+2} \delta^\frac{m+2}{m+2}, |x_1| < \delta t\frac{m+2}{m+2} \}, \]
\[ \Omega_3^\pm = \{(t, x) : t > 0, |x_1| \leq \frac{2t^\frac{m+2}{m+2}}{m+2} \delta^\frac{m+2}{m+2}, |x_2| \leq \frac{2t^\frac{m+2}{m+2}}{m+2} \delta^\frac{m+2}{m+2} \}, \]
\[ \Omega_i^\pm = \{(t, x) : t > 0, |x_i| \leq \frac{2t^\frac{m+2}{m+2}}{m+2} \delta^\frac{m+2}{m+2} \}, \quad \text{away from the lines} \ t_i^\pm \text{ and } t_3^\pm \}, \quad i = 1, 2. \]

In addition, $\Omega_1^\pm \cup \Omega_2^\pm \cup \Omega_3^\pm \cup \Omega_i^\pm \cup \Omega_3^\pm \cup (\cup_{j=1}^\infty \Omega_j)$ is an open cusp conic covering of $\mathbb{R}_+^3 \setminus \{O\}$ such that $\Omega_j$ ($1 \leq j \leq N$) intersects at most one surface in $\{\Gamma_0, \Gamma_1^\pm, \Gamma_2^\pm\}$.  

In $\Omega_1^\pm$, set $\mathcal{M}_1^\pm = \{L_0, L_1, M_i^\pm \}$, here $M_i^\pm = \frac{x_2}{l^2}(\partial_i \pm i \bar{t} \partial_1) + \left( \frac{2t^{\frac{m+2}{m+2}}}{m+2} \partial_1 \mp x_1 \right) \partial_2$;  

In $\Omega_2^\pm$, set $\mathcal{M}_2^\pm = \{L_0, L_2, M_i^\pm \}$, here $M_i^\pm = \frac{x_1}{l^2}(\partial_i \pm i \bar{t} \partial_2) + \left( \frac{2t^{\frac{m+2}{m+2}}}{m+2} \partial_1 \mp x_2 \right) \partial_1$;  

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In $\Omega_3^{\pm,\pm}$, set $\mathcal{M}_3^{\pm,\pm} = \{L_0, N_1^{\pm,\pm}, N_2^{\pm,\pm}\}$, here $N_1^{\pm,\pm} = (\pm x_1 \mp x_2)/t^{\pm 1} \partial_1 \pm (2t^{\pm 1}/m + 2 \mp x_2) \partial_2 \pm (\pm x_1 - 2t^{\pm 1}/m + 2) \partial_2$, $N_2^{\pm,\pm} = t \partial_1 + t^{\pm 1}(\pm \partial_1 \mp \partial_2)$;

In $\Omega_{i+3}^\pm$ (i=1,2), set $\mathcal{M}_i^{\pm} = \{L_0, R_i^{\pm,1}, R_i^{\pm,2}\}$, here $R_i^{\pm,1} = (x_i \mp t^{\pm 1}/m + 2) \partial_i$, $R_{2,1} = t \partial_2$ or $R_{2,2} = t \partial_1$;

In $\Omega_j$ (j = 1, ..., N), set $M_0 = \{L_0, L_1, L_2, L_{12}\}$.

Those vector fields with admissible coefficients generated by $\mathcal{M}_1^+, \mathcal{M}_2^+, \ldots, \mathcal{M}_0$ respectively are called the admissible vector fields of surface variety $\{\Gamma_0^+, \Gamma_1^+, \Gamma_2^+\}$.

Next, we give the definition of conormal space $I^kL_0^\infty_0(\Gamma_0 \cup \Gamma_1^+ \cup \Gamma_2^+)$.

**Definition 4.6 (Conormal space $I^kL_0^\infty_0(\Gamma_0 \cup \Gamma_1^+ \cup \Gamma_2^+)$)** We call a function $u(t, x) \in I^kL_0^\infty_0(\Gamma_0 \cup \Gamma_1^+ \cup \Gamma_2^+)$ if $Z^\alpha \left(\chi\left(\frac{(m + 2)x}{2t^{-\alpha}}\right) u(t, x)\right) \in L_0^\infty([0, T] \times \mathbb{R}^2)$ holds for any $|\alpha| \leq k$ and the homogeneous cut-off function $\chi\left(\frac{(m + 2)x}{2t^{-\alpha}}\right)$ of degree zero whose support lies in some fixed conic neighborhood of $\Omega_i^+(i = 1, 2), \Omega_3^{\pm,\pm}, \Omega_{i+3}^+(i = 1, 2), \Omega_j(j = 1, ..., N)$, and $Z$ represents the admissible tangent vector in related domains.

§5. Local existence of solution to problem (1.1)

In this section, we will show the local existence of the low regularity solution to (1.1) in Theorem 1.1. At first, we study the 2-D case under the condition (A3) since the case of condition (A1) is completely analogous and even simpler.

**Theorem 5.1.** Under the assumptions (A3) and $m \leq 9$, there exists a constant $T > 0$ such that (1.1) has a local solution $u \in L^\infty([0, T] \times \mathbb{R}^2) \cap C([0, T], H^{\frac{m+6}{m+2}} - \mathbb{R}^2) \cap C([0, T], H^{\frac{m+8}{m+2}} - \mathbb{R}^2) \cap C^1([0, T], H^{\frac{m+8}{m+2}} - \mathbb{R}^2))$.

**Proof.** Set

$$
\begin{cases}
\partial_t^2 u_1 - t^m \Delta u_1 = 0, \\
u_1(0, x) = 0, \\
\partial_t u_1(0, x) = \psi(x).
\end{cases}
$$

(5.1)

Due to $\varphi(x) \in H^{\frac{m-4}{2}} - \mathbb{R}^2$ by Lemma 2.1.(ii), then it follows from Lemma 2.4 and Proposition 3.3 that for any fixed $0 < \delta < \frac{m}{m+2}$

$$u_1(t, x) \in L^\infty([0, 1] \times \mathbb{R}^2) \cap C([0, 1], H^{\frac{m+6}{m+2}} - \mathbb{R}^2) \cap C([0, 1], H^{\frac{m+8}{m+2}} - \mathbb{R}^2) \cap C^1([0, 1], H^{\frac{m+8}{m+2}} - \mathbb{R}^2),$$

which satisfies for $t \in [0, 1]$

$$\|u_1(t, \cdot)|_{L^\infty(\mathbb{R}^2)} + \|u_1(t, \cdot)|_{H^{\frac{m+8}{m+2}} - \mathbb{R}^2} + t^\frac{m+6}{m+2} \|u_1(t, \cdot)|_{H^{\frac{m+8}{m+2}} - \mathbb{R}^2} + \|\partial_t u_1(t, \cdot)|_{H^{\frac{m+8}{m+2}} - \mathbb{R}^2} \leq C(\delta).$$

(5.2)

Let

$$
\begin{cases}
\partial_t^2 u_2 - t^m \Delta u_2 = f(t, x, 0), \\
u_2(0, x) = \partial_t u_2(0, x) = 0.
\end{cases}
$$

(5.3)

Since $f(t, x, 0) \in C^\infty([0, \infty) \times \mathbb{R}^2)$ has a compact support on $x$, then (5.3) has a $C^\infty([0, \infty) \times \mathbb{R}^2)$ solution $u_2(t, x)$ which possesses a compact support with respect to the variable $x$ when $t \geq 0$ is fixed.

Set $v(t, x) = u(t, x) - u_1(t, x) - u_2(t, x)$, then it follows from (1.1), (5.1) and (5.3) that

$$
\begin{cases}
\partial_t^2 v - t^m \Delta v = f(t, x, u_1 + u_2 + v) - f(t, x, 0), \\
v(0, x) = \partial_t v(0, x) = 0.
\end{cases}
$$

(5.4)

For $w(t, x) \in C([0, T], H^{\frac{m+6}{m+2} + p_0(m) - \delta}) \cap C([0, T], H^{\frac{m+8}{m+2} + p_1(m) - \delta}) \cap C^1([0, T], H^{\frac{m+8}{m+2} + p_2(m) - \delta})$ satisfying for $t \in [0, T]$

$$\|w(t, \cdot)|_{H^{\frac{m+6}{m+2} + p_0(m) - \delta}(\mathbb{R}^2)} + t^{\frac{m+2}{2}} \|w(t, \cdot)|_{H^{\frac{m+8}{m+2} + p_1(m) - \delta}(\mathbb{R}^2)}$$

$$+ \|\partial_t w(t, \cdot)|_{H^{\frac{m+8}{m+2} + p_2(m) - \delta}(\mathbb{R}^2)} < \infty,$$

(5.5)
where \( p_0(m) = \begin{cases} 4 & \text{for } m \geq 2 \\ 1 & \text{for } m \leq 2 \end{cases} \), \( p_1(m) \) and \( p_2(m) \) have been defined in Lemma 3.4, and 0 < \( T \leq 1 \), we define the set \( G \) as follows

\[
G \equiv \left\{ w \in C([0, T], H^{m+6}) \cap C([0, T], H^{m+6}+p_1(m)-\delta) \cap C^2([0, T], H^{m+6}+p_2(m)-\delta) : \sup_{t \in [0, T]} \|w(t, \cdot)\| \leq 1 \right\}.
\]

Denote by

\[
E(f(t, x, u) - f(t, x, 0)) \equiv 
\left( \int_0^t (V_2(t, |\xi|)V_1(\tau, |\xi|) - V_1(t, |\xi|)V_2(\tau, |\xi|))(f(\tau, x, u(\tau, x)) - f(\tau, x, 0))d\tau \right)^\vee (t, x)
\]

and define a nonlinear mapping \( F \) as follows

\[
F(w) = E(f(t, x, u_1 + u_2 + w) - f(t, x, 0)).
\] (5.6)

We now show that the mapping \( F \) is from \( G \) into itself and is contractible for small \( T \).

By (3.25) in Lemma 3.4 (taking \( s_0 = p_0(m) - \frac{\delta}{2} \)) and (5.2), we have for \( w \in G \)

\[
\|F(w)(t, \cdot)\|_{H^{\frac{m+6}{2(m+2)}+p_0(m)-\delta}} \leq C T^{\frac{m+6}{2(m+2)}+p_0(m)-\frac{\delta}{2}} \|f(t, \cdot, u_1(t, \cdot) + u_2(t, \cdot) + w(t, \cdot) - f(t, \cdot, 0)\|_{H^{\frac{m+6}{2(m+2)}+p_0(m)-\frac{\delta}{2}}}
\]

\[
\leq C T^{\frac{m+6}{2(m+2)}+p_0(m)-\frac{\delta}{2}} \|u_1(t, \cdot) + u_2(t, \cdot) + w(t, \cdot)\|_{H^{\frac{m+6}{2(m+2)}+p_0(m)-\frac{\delta}{2}}}
\]

\[
\leq C(\delta) T^{\frac{m+6}{2(m+2)}+p_0(m)-\frac{\delta}{2}}.
\] (5.7)

Here we have used the following facts:

- \( f(u) \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^\infty([0, T], H^s(\mathbb{R}^n)) \) if \( u \in L^\infty([0, T] \times \mathbb{R}^n) \cap L^\infty([0, T], H^s(\mathbb{R}^n)) \) with \( f \in C^\infty \), \( f(0) = 0 \) and \( s \geq 0 \);

- Sobolev’s imbedding theorem of \( L^\infty([0, T], H^{\frac{m+6}{2(m+2)}+p_0(m)-\delta}(\mathbb{R}^2)) \subset L^\infty([0, T] \times \mathbb{R}^2) \) for small \( \delta > 0 \) and \( m \leq 9 \).

For small \( T \), one can derive from (5.7) that

\[
\|F(w)(t, \cdot)\|_{H^{\frac{m+6}{2(m+2)}+p_0(m)-\delta}} \leq \frac{1}{3}.
\] (5.8)

On the other hand, by (3.25) in Lemma 3.4 (taking \( s_0 = p_1(m) - \frac{\delta}{2} \)), we have

\[
\|F(w)(t, \cdot)\|_{H^{\frac{m+6}{2(m+2)}+p_1(m)-\delta}} \leq C T^{\frac{m+6}{2(m+2)}+p_1(m)-\frac{\delta}{2}} \|f(t, \cdot, u_1(t, \cdot) + u_2(t, \cdot) + w(t, \cdot) - f(t, \cdot, 0)\|_{H^{\frac{m+6}{2(m+2)}+p_1(m)-\frac{\delta}{2}}}
\]

\[
\leq C T^{\frac{m+6}{2(m+2)}+p_1(m)-\frac{\delta}{2}} \|u_1(t, \cdot) + u_2(t, \cdot) + w(t, \cdot)\|_{H^{\frac{m+6}{2(m+2)}+p_1(m)-\frac{\delta}{2}}},
\]

which yields for small \( T \)

\[
t^{\frac{m+6}{2(m+2)}+p_1(m)-\delta} \|F(w)(t, \cdot)\|_{H^{\frac{m+6}{2(m+2)}+p_1(m)-\delta}(\mathbb{R}^2)} \leq \frac{1}{3}.
\] (5.9)

If we take \( p_2 = p_2(m) - \delta \) in (3.26) of Lemma 3.4, then we have for small \( T \)

\[
\|\partial_t F(w)(t, \cdot)\|_{H^{\frac{m+6}{2(m+2)}+p_2(m)-\delta}} \leq C T^{\frac{m+6}{2(m+2)}+p_2(m)-\delta} \|f(t, \cdot, u_1(t, \cdot) + u_2(t, \cdot) + w(t, \cdot) - f(t, \cdot, 0)\|_{H^{\frac{m+6}{2(m+2)}+p_2(m)-\delta}}
\]

\[
\leq C T^{\frac{m+6}{2(m+2)}+p_2(m)-\delta}
\]

\[
\leq \frac{1}{3}.
\] (5.10)
Collecting (5.8)-(5.10) yields for small T

$$\sup_{t \in [0, T]} \| F(w)(t, \cdot) \| \leq 1, \quad (5.11)$$

which means $F$ maps $G$ into $G$.

Next we prove that the mapping $F$ in contractible for small $T$.

For $w_1, w_2 \in G$, due to $f(\tau, x, u_1 + u_2 + w_1) - f(\tau, x, u_1 + u_2 + w_2) = \int_0^1 f'(\tau, x, u_1 + u_2 + \theta w_1 + (1 - \theta)w_2)(w_1 - w_2)d\theta$, then a direct computation yields as in (5.8)-(5.10) for $t \in [0, T]$ and small $T$

$$\sup_{t \in [0, T]} \| F(w_1)(t, \cdot) - F(w_2)(t, \cdot) \|$$

$$= \sup_{t \in [0, T]} \| E(f(t, x, u_1 + u_2 + w_1(\cdot)) - E(f(t, x, u_1 + u_2 + w_2(\cdot)) \|$$

$$\leq C(T^{\frac{\alpha}{2} + 2} + T^{\frac{1}{2} + \frac{1}{2}}) \sup_{t \in [0, T]} \| w_1 - w_2 \|$$

$$\leq \frac{1}{\sqrt{2}} \sup_{t \in [0, T]} \| w_1 - w_2 \|. \quad (5.12)$$

Therefore, by the fixed point theorem and (5.11)-(5.12), we complete the proof of Theorem 5.1.

Under the assumption (A1), we have

**Theorem 5.2.** Under the assumption (A1), there exists a constant $T > 0$ such that (1.1) has a local solution $u \in C([0, T], H^{\frac{m+6}{2m+2} - \epsilon} (\mathbb{R}^n)) \cap C([0, T], H^{\frac{m+6}{2m+2} - \epsilon} (\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m+6}{2m+2} - \epsilon} (\mathbb{R}^n))$.

**Proof.** Since $C([0, T], H^{\frac{m+6}{2m+2} - \epsilon} (\mathbb{R}^n)) \subset L^\infty([0, T] \times \mathbb{R}^n)$, then Theorem 5.2 can be shown by the same procedure as in Theorem 5.1, we omit it here. \(\square\)

Finally, we prove the local existence of solution to (1.1) under the condition (A2).

**Theorem 5.3.** Under the assumption (A2), there exists a constant $T > 0$ such that (1.1) has a local solution $u \in L^\infty([0, T] \times \mathbb{R}^n) \cap C([0, T], H^{\frac{m+6}{2m+2} - \epsilon} (\mathbb{R}^n)) \cap C([0, T], H^{\frac{m+6}{2m+2} - \epsilon} (\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m+6}{2m+2} - \epsilon} (\mathbb{R}^n))$.

**Proof.** Let $u_1(t, x)$ satisfy

$$\begin{align*}
\partial_t^2 u_1 - t^m \Delta u_1 &= 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
u_1(0, x) &= 0, \quad \partial_t u_1(0, x) = \varphi(x),
\end{align*} \quad (5.13)$$

where $\varphi(x)$ satisfies the assumption (A2), then by Lemma 2.2 and Proposition 3.3 we know that for any fixed $\delta > 0$ with $\delta < \frac{1}{2(m + 2)}$

$$u_1(t, x) \in L^\infty([0, 1] \times \mathbb{R}^n) \cap C([0, 1], H^{\frac{m+6}{2m+2} - \delta} (\mathbb{R}^2)) \cap C([0, 1], H^{\frac{m+6}{2m+2} - \delta} (\mathbb{R}^2)) \cap C^1([0, 1], H^{\frac{m+6}{2m+2} - \delta} (\mathbb{R}^2)),$$

which satisfies for $t \in [0, 1]$

$$\| u_1(t, \cdot) \|_{L^\infty(\mathbb{R}^n)} + \| u_1(t, \cdot) \|_{H^{\frac{m+6}{2m+2} - \delta} (\mathbb{R}^2)} + t^{\frac{\alpha}{2}} \| u_1(t, \cdot) \|_{H^{\frac{m+6}{2m+2} - \delta} (\mathbb{R}^2)} + \| \partial_t u_1(t, \cdot) \|_{H^{\frac{m+6}{2m+2} - \delta} (\mathbb{R}^2)} \leq C(\delta). \quad (5.14)$$

Next, we establish the more regularities of $u_1(t, x)$ in the directions $x' = (x_2, ..., x_n)$.

It is noted that for $|\alpha| \geq 1$

$$\begin{align*}
\partial_{x'}^\alpha u_1 - t^m \Delta \partial_{x'}^\alpha u_1 &= 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
\partial_t \partial_{x'}^\alpha u_1(0, x) &= 0, \quad \partial_{x'} \partial_{x'}^\alpha u_1(0, x) = \partial_{x'}^\alpha \varphi(x),
\end{align*} \quad (5.15)$$
This derives
\[ \partial_{x}^{\alpha} u_1(t, x) \in L^{\infty}([0, 1] \times \mathbb{R}^n) \cap C([0, 1], H^{m+\frac{\alpha}{2}}(\mathbb{R}^n)) \cap C([0, 1], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \cap C^1([0, 1], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \]
and satisfies for \( t \in [0, 1] \)
\[ \| \partial_{x}^{\alpha} u_1(t, \cdot) \|_{L^{\infty}(\mathbb{R}^n)} + \| \partial_{x}^{\alpha} u_1(t, \cdot) \|_{H^{m+\frac{\alpha}{2}}(\mathbb{R}^n)} + t^{\frac{\alpha}{2}} \| \partial_{x}^{\alpha} u_1(t, \cdot) \|_{H^{m+\frac{3}{2}}(\mathbb{R}^n)} \leq C_{\alpha}(\delta). \] (5.16)

Set \( v = u - u_1 - u_2 \), where \( u_2 \) is defined as in (5.3), then we have from (1.1)
\[ \begin{align*}
\partial_{x}^{2} v - t^{m} \Delta v &= f(t, x, u_1 + u_2 + v) - f(t, x, 0), \\
v(0, x) &= \partial_{x} v(0, x) = 0. \end{align*} \] (5.17)

In order to solve (1.1), it only suffices to solve (5.17). This requires us to establish the a priori \( L^{\infty} \) bound of \( \partial_{x}^{\alpha} v \) in (5.17) for \( |\alpha| \leq \left[ \frac{n}{2} \right] + 1 \). For this end, motivated by Lemma 2.2 and Lemma 2.3, we should establish \( \partial_{x}^{\alpha+\beta} v \in L^{\infty}([0, T], H^{s}(\mathbb{R}^n)) \) with \( s > \frac{1}{2} \) and \( |\beta| \leq \left[ \frac{n}{2} \right] + 1 \).

Taking \( \partial_{x}^{\gamma} ((|\gamma| \leq 2 \left[ \frac{n}{2} \right] + 2) \) on two hand sides of (5.17) yields
\[ \begin{align*}
\partial_{x}^{2} \partial_{x}^{\gamma} v - t^{m} \Delta \partial_{x}^{\gamma} v &= F_{\gamma}(t, x, \partial_{x}^{\gamma} v)_{|\alpha| \leq |\gamma|} = \sum_{|\beta| + |\gamma| \leq |\gamma|} C_{\beta \gamma} (\partial_{x}^{\beta} f)(t, x, u_1 + u_2 + v) - (\partial_{x}^{\beta} f)(t, x, 0) \\
\times \partial_{x}^{\gamma} f(t, x, u_1 + u_2 + v) \Pi_{1 \leq k \leq l} \partial_{x}^{\beta_{k}} (u_1 + u_2 + v), \\
\partial_{x}^{\gamma} v(0, x) &= \partial_{x} \partial_{x}^{\gamma} v(0, x) = 0. \end{align*} \] (5.18)

If
\[ \sum_{|\alpha| \leq \left[ \frac{n}{2} \right] + 1} \| \partial_{x}^{\alpha} v \|_{L^{\infty}([0, T] \times \mathbb{R}^n)} + \sum_{|\gamma| \leq 2 \left[ \frac{n}{2} \right] + 2} \| \partial_{x}^{\gamma} v \|_{L^{\infty}([0, T], H^{s}(\mathbb{R}^n))} \leq 2 \] with \( s > \frac{1}{2} \) and \( T \leq 1 \), then by Lemma 2.3 and (5.16), we have from (5.18) that for small \( T \)
\[ \sum_{|\alpha| \leq \left[ \frac{n}{2} \right] + 1} \| \partial_{x}^{\alpha} v \|_{L^{\infty}([0, T] \times \mathbb{R}^n)} \leq 1. \] (5.19)

Based on the preparations above, we will use the fixed point theorem to show Theorem 5.3.

For \( w \in L^{\infty}([0, T] \times \mathbb{R}^n) \cap C([0, T], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \cap C^1([0, T], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \)
\[ \cap C^1([0, T], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \cap C(0, T, H^{m+\frac{3}{2}}(\mathbb{R}^n)) \cap C([0, T], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \]
we define
\[ ||| w(t, \cdot) ||| = \sum_{|\alpha| = 0} \| \partial_{x}^{\alpha} w(t, x) \|_{L^{\infty}([0, T] \times \mathbb{R}^n)} + \sum_{|\gamma| = 0} \| \partial_{x}^{\gamma} w(t, \cdot) \|_{H^{m+\frac{3}{2}}(\mathbb{R}^n)} \]
\[ + t^{\frac{m+2p_{1}(m)}{2}} \sum_{|\gamma| = 0} \| \partial_{x}^{\gamma} w(t, \cdot) \|_{H^{m+\frac{3}{2}+p_{1}(m)-\delta}(\mathbb{R}^n)} + \sum_{|\gamma| = 0} \| \partial_{x}^{\gamma} w(t, \cdot) \|_{H^{m+\frac{3}{2}+p_{2}(m)-\delta}(\mathbb{R}^n)} \]

A set \( Q \) is defined as follows
\[ Q \equiv \left\{ w \in L^{\infty}([0, T] \times \mathbb{R}^n) \cap C([0, T], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \cap C((0, T], H^{m+\frac{3}{2}}(\mathbb{R}^n)) \cap C^1([0, T], H^{m+\frac{3}{2}}(\mathbb{R}^n)) : \right. \]
\[ \left. \sup_{t \in [0, T]} ||| w(t, \cdot) ||| \leq 2 \right\}.

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Let us define a nonlinear mapping \( \mathcal{F} \) as follows
\[
\mathcal{F}(w) = E(f(t, x, u_1 + u_2 + w) - f(t, x, 0)),
\]
where the meaning of the operator \( E \) is given in (5.5).

As in the proof procedure of Theorem 5.1, we now show that the mapping \( \mathcal{F} \) is from \( Q \) into itself and is contractible for small \( T \).

At first, \( \mathcal{F}(w) \) solves the following problem
\[
\begin{cases}
(\partial_t^2 - t^m \Delta) \mathcal{F}(w) = f(t, x, u_1 + u_2 + w) - f(t, x, 0), \\
\mathcal{F}(w)|_{t=0} = \partial_t \mathcal{F}(w)|_{t=0} = 0.
\end{cases}
\]

By (5.19), we can derive that for small \( T \)
\[
\sum_{|\alpha| = 0}^{[\frac{n}{2}]+1} \| \partial^\alpha_\nu \mathcal{F}(w, x) \|_{L^\infty([0, T] \times \mathbb{R}^n)} \leq 1. \tag{5.21}
\]

Similar to the proof as in Theorem 5.1, one has for small \( T \) and \( t \in [0, T] \)
\[
\begin{align*}
&\sum_{|\gamma| = 0}^{[\frac{n}{2}]+2} \| \partial^\gamma_\nu w(t, \cdot) \|_{H^{\frac{m}{2}+\frac{m}{2}+p_0(m)\delta}(\mathbb{R}^n)} + t^{\frac{(m+1)p_1(m)}{2}} \sum_{|\gamma| = 0}^{[\frac{n}{2}]+2} \| \partial^\gamma_\nu w(t, \cdot) \|_{H^{\frac{m}{2}+\frac{m}{2}+p_1(m)\delta}(\mathbb{R}^n)} \\
&+ \sum_{|\gamma| = 0}^{[\frac{n}{2}]+2} \| \partial_t \partial^\gamma_\nu w(t, \cdot) \|_{H^{\frac{m}{2}+\frac{m}{2}+p_2(m)\delta}(\mathbb{R}^n)} \leq 1 \tag{5.22}
\end{align*}
\]

and
\[
\| \mathcal{F}(w_1) - \mathcal{F}(w_2) \| \leq \frac{1}{2} \| w_1 - w_2 \|, \tag{5.24}
\]

where \( w_1, w_2 \in Q \).

Combining (5.21) with (5.22) yields
\[
\| \mathcal{F}(w) \| \leq 2,
\]

which means \( \mathcal{F} \) maps \( Q \) into itself. Therefore, it follows from the fixed point theorem that we complete the proof of Theorem 5.3.

\section*{§6. Proof on Theorem 1.1}

Based on the results in \( \S 2-\S 5\), we now start to prove Theorem 1.1. At first, under the assumptions \( (A_3) \) and \( (A_1) \), we establish the following conclusions on the conormal regularities of the local solution \( u(t, x) \) obtained in Theorem 5.2 and Theorem 5.3, respectively.

**Theorem 6.1.** (i) For the solution \( u(t, x) \) in Theorem 5.2, we have \( u(t, x) \in I^\infty H^{\frac{m}{2}+\frac{m}{2}+p_0(m)\delta}(\Gamma_0); \)
(ii) For the solution \( u(t, x) \) in Theorem 5.3, then \( u(t, x) \in I^\infty H^{\frac{m}{2}+\frac{m}{2}+p_1(m)\delta}(\Gamma_1^+ \cup \Gamma_1^-). \)

**Proof.** (i) By the commutator relations in Lemma 4.2 and a direct computation, we have from (1.1)
\[
\begin{cases}
\partial_t^2 U_k - t^m \Delta U_k = \sum_{\beta_0 + l_0 \leq k_0} C_{\beta_0} ( \sum_{l_0 \leq j_0} L_{i,j}^{\beta_0} \partial_{u_j} f(t, x, u) \Pi_{1 \leq i \leq j} L_{i,j}^{\beta_0} u), \\
U_k(0, x) \in H^{\frac{m}{2}+1}(\mathbb{R}^n), \quad \partial_t U_k(0, x) \in H^{\frac{m}{2}}(\mathbb{R}^n), \tag{6.1}
\end{cases}
\]
here $U_k = \{ L_0^{k_0} \Pi_{1 \leq i < j \leq n} t_{ij}^{k_{ij}} u \}_{k_0 + \sum k_{ij} = k}$ for $k \in \mathbb{N} \cup \{0\}$, and in the process of deriving the regularities of $U_k(0, x)$ and $\partial_t U_k(0, x)$ we have used that facts of $\Pi_{1 \leq i < j \leq n}(x, \partial_j)^{k_{ij}} \varphi(x) \in H^{\frac{m}{m+21}}(\mathbb{R}^n)$ and $w_1(x)w_2(x) \in H^{\frac{m}{21}}(\mathbb{R}^n)$.

Next we use the induction method to prove

$$U_k(t, x) \in C((0, T), H^{\frac{m}{m+21}}(\mathbb{R}^n)) \cap C((0, T), H^{\frac{m}{m+21}}(\mathbb{R}^n)) \cap C^{1}([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n))$$

(6.2) which satisfies for any small fixed $\delta > 0$

$$\|U_k\|_{C([0, T], H^{\frac{m}{m+21}})} + \|U_k(t, \cdot)\|_{H^{\frac{m}{m+21}}} + \|\partial_t U_k\|_{C([0, T], H^{\frac{m}{m+21}})} \leq C_k(\delta).$$

(6.3)

It is noted that (6.2)-(6.3) has been shown in Theorem 5.2 in the case of $k = 0$. Assume that (6.2)-(6.3) hold for the case up to $k - 1$, then one has by (6.1)

$$\begin{cases}
\partial_t^2 U_k - t^m \Delta U_k - (\partial_u f)(t, x, u) U_k = F_k(t, x), \\
U_k(0, x) \in H^{\frac{m}{m+21}}(\mathbb{R}^n), \quad \partial_t U_k(0, x) \in H^{\frac{m}{m+21}}(\mathbb{R}^n),
\end{cases}$$

(6.4)

where $F_k(t, x) \in C([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n))$.

This, together with Proposition 3.3 and Lemma 3.4, yields (6.2)-(6.3) in the case of $k$.

We now prove $u(t, x) \in \mathcal{I}^{m} H^{\frac{m}{m+21}}(\Gamma_0)$.

It is noted that for $i = 1, \cdots, n$, by (6.3) and Remark 2.1,

$$N^2_i(h_2(t, x) \chi)^{(m+2)|x|\over 2m^2} u = N^2_i(h_2 \chi)u + \frac{2}{m+2}(\frac{(m+2)|x|}{2m^2} - 1)h_2 \chi t^{\frac{m+2}{m}} \partial_t u \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n)),$$

where the definitions of $h_2(t, x)$ and $\chi^{(m+2)|x|\over 2m^2}$ are given in Definition 4.3. Furthermore, by (4.11) in Lemma 4.4 and (6.3), we can obtain for any $k_i, k_0, k_{ij} \in \mathbb{N} \cup \{0\}$

$$(N^2_i)^{k_i} L_0^{k_0} \Pi_{1 \leq i < j \leq n} L^{k_{ij}}(h_2 \chi)u \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n)).$$

(6.5)

This, together with (4.9) in Lemma 4.4, yields

$$L^{k_i}_i L_0^{k_0} \Pi_{1 \leq i < j \leq n} L^{k_{ij}}(h_2 \chi)u \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n)).$$

(6.6)

In order to show $u(t, x) \in \mathcal{I}^{m} H^{\frac{m}{m+21}}(\Gamma_0)$, we need to prove

$$\Pi_{1 \leq i < j \leq n} L^{k_i}_i L_0^{k_0} \Pi_{1 \leq i < j \leq n} L^{k_{ij}}(h_2 \chi)u \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n))$$

or equivalently

$$\Pi_{1 \leq i \leq n} (N^2_i)^{k_i} L_0^{k_0} \Pi_{1 \leq i < j \leq n} L^{k_{ij}}(h_2 \chi)u \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n)).$$

(6.7)

For this end, by the commutator relations in Lemma 4.2 and (4.11), it suffices to prove

$$N^2_1 N^2_2 \cdots N^2_n(h_2 \chi)u \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n))$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $2 \leq k \leq n$

(6.8)

since the proof on $N^2_1 N^2_2 \cdots N^2_n L_0^{k_0} \Pi_{1 \leq i < j \leq n} L^{k_{ij}}(h_2 \chi)u \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n))$ is completely similar.

Indeed, by the expression of $N^2_1$ $a(t, x) \partial_t$ with $a(t, x) = |x| - \frac{2}{m+2} t^{\frac{m+2}{m}}$ and (6.5), we have for $1 \leq i \leq n$

$$\partial_t^2(\partial_t^2(t, x)h_2 \chi u) = (a \partial_t)^2(h_2 \chi u) + \partial_t (a \partial_t h_2 \chi u) + 2a(\partial_t^2 h_2 \chi u) + 2(\partial_t h_2 \chi u) = (N^2_i)^2(h_2 \chi u) + (\partial_t h_2 \chi u) + 2a(\partial_t^2 h_2 \chi u) + 2(\partial_t h_2 \chi u) \in L^\infty([0, T], H^{\frac{m}{m+21}}(\mathbb{R}^n)),$$

(6.9)
here we use the facts of \( \frac{x_i}{|x|} \in H_{\text{loc}}^{\frac{n}{2}}(\mathbb{R}^n) \) and \( w_1(x)w_2(x) \in H^{\min\{s_1, s_2, s_1 + s_2 - \frac{m}{2}\}}(\mathbb{R}^n) \) if \( w_1(x) \in H^{s_1}(\mathbb{R}^n) \) and \( w_2(x) \in H^{s_2}(\mathbb{R}^n) \) with \( s_1, s_2 \geq 0 \).

From (6.9), we have
\[
\Delta \left( a^2(t, x)h_2 \chi u \right) \in L^\infty([0, T], H^{\frac{n}{2}}(R^m, \mathbb{R})) \tag{6.10}
\]
which derives by the regularity theory of second order elliptic equation

\[
\partial_{ij}^2 (a^2(t, x)h_2 \chi u) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}(R^m, \mathbb{R})) \quad \text{for any } 1 \leq i < j \leq n
\]
or equivalently
\[
N_i^j N_j^i (h_2 \chi u) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}(R^m, \mathbb{R})) \quad \text{for any } 1 \leq i < j \leq n.
\]
(6.12)

Analogously, we can get for any \( 1 \leq i, k \leq n \)
\[
\partial_i^2 \left( a^3 \partial_k (h_2 \chi u) \right) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}(R^m, \mathbb{R}))
\]
and
\[
\Delta (a^3 \partial_k (h_2 \chi u)) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}(R^m, \mathbb{R}))
\]
which derives
\[
\partial_i^2 (a^3 \partial_k (h_2 \chi u)) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}(R^m, \mathbb{R}))
\]
and further by (6.12),
\[
N_i^j N_j^i (h_2 \chi u) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}(R^m, \mathbb{R})) - (6.13)
\]

By induction method, we can complete the proof on (6.8).

Consequently, we have
\[
L_0^{k_0} \Pi_{1 \leq i \leq n} L_i^{k_i} \Pi_{1 \leq i < j \leq n} T_{ij}^{k_{ij}} (h_2 \chi u) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}(R^m, \mathbb{R})) \tag{6.14}
\]

Similarly, by (1) and (3) in Lemma 4.4 (noting that \( L_i \) can be expressed as a linear combination of \( \tilde{L}_0 \) and \( L_{jk} \) with admissible coefficients in \( \Omega_{1} \) and \( \Omega_{3} \) respectively), we can arrive at
\[
Z_1 \cdots Z_k (h_1 u) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}) \quad \text{for all } Z_1, \ldots, Z_k \in S_1,
\]
and
\[
Z_1 \cdots Z_k \left( h_3 \chi_0 \left( \frac{(m + 2)|x|}{2^{m+2}} \right) u \right) \in L^\infty([0, T], H^{\frac{n}{m} - \frac{m}{2(n+m)}}) \quad \text{for all } Z_1, \ldots, Z_k \in S_3,
\]
where the functions \( h_1, h_3 \) and \( \chi_0 \) are given in Definition 4.3.

Therefore,
\[
u(t, x) \in L^\infty H^{\frac{n}{m} - \frac{m}{2(n+m)}}(\Gamma_0).
\]

(ii) By the commutator relations in Lemma 4.3 and the equation (1.1), we have for \( k \geq 2 \) and \( j \geq 1 \)
\[
\left\{ \begin{array}{l}
\partial_k^2 U_k - t^m \Delta U_k = \sum_{\beta_0 + \delta_0 + k_0 = k_0} C_{\beta_0} (\tilde{L}_0^{\beta_0} \Pi_{2 \leq i \leq n} R_i^{\delta_0} \partial_i f)(t, x, u) \Pi_{1 \leq s \leq (L_0^{\delta_0} \Pi_{2 \leq i \leq n} R_i^{\delta_0})} u, \\
U_k(0, x) \in W^{1, \infty}(\mathbb{R}^n) \cap H^{\frac{n}{2}}(\mathbb{R}^n), \quad \partial_t U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{\frac{n}{2}}(\mathbb{R}^n),
\end{array} \right. \tag{6.15}
\]
here \( U_k = \{ \tilde{L}_0^{\delta_0} \Pi_{2 \leq i \leq n} R_i^{\delta_0} u \}_{k_0 + \sum \delta_0 = k} \) for \( k \in \mathbb{N} \cup \{0\} \), the definitions of \( \tilde{L}_0, R_i(2 \leq i \leq n) \) see Lemma 4.3, and \( (x_1 \partial_1)^{\gamma_0} \partial_x^\gamma U_k(0, x) \in W^{1, \infty}(\mathbb{R}^n) \cap H^{\frac{n}{2}}(\mathbb{R}^n), \quad (x_1 \partial_1)^{\gamma_0} \partial_x^\gamma \partial_t U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{\frac{n}{2}}(\mathbb{R}^n) \) for any multiple indices \( (\gamma_0, \gamma) \).
By Lemma 2.2-Lemma 2.3, Proposition 3.3 and Lemma 3.4, we know from (6.15) that
\[ U_k(t, x) \in L^\infty([0, T] \times \mathbb{R}^n) \cap C([0, T], H^{\frac{m+6}{2(k+1)}}(\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m+4}{2(k+1)}}(\mathbb{R}^n)) \] and satisfies for \( t \in [0, T] \)
\[ \|U_k(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} + \|U_k(t, \cdot)\|_{H^{\frac{m+6}{2(k+1)}}(\mathbb{R}^n)} + t + \|U_k(t, \cdot)\|_{H^{\frac{m+4}{2(k+1)}}(\mathbb{R}^n)} \leq C_k(\delta). \quad (6.16) \]

Due to \( \mathbb{N}_{2, \pm}(h_2(t, x_1)\chi_{\pm}((m+2)x_1)u) = \mathbb{N}_{2, \pm}(h_2\chi)u + \left(\frac{m+2}{2}\right)h_2\chi + 1 \right) + \partial_1 u \in L^\infty([0, T], H^{\frac{m+6}{2(k+1)}}(\mathbb{R}^n)) \)
by (6.16), here the functions \( h_2 \) and \( \chi_{\pm} \) are defined in Definition 4.5. Furthermore, applying for the relations in (2) of Lemma 4.5 together with (6.16) yields
\[ \mathcal{N}^{k_1}_{\mathcal{L}_1} = L^{k_1}_{\mathcal{L}_1}\mathcal{S}_{2, \pm}(h_2\chi u) \in L^\infty([0, T], H^{\frac{m+6}{2(k+1)}}(\mathbb{R}^n)). \quad (6.17) \]

Analogously, by (1) and (3) in Lemma 4.5 and the same proof procedure of (6.17), one can obtain
\[ Z_1 \cdots Z_k u(t, x) \in L^\infty([0, T], H^{\frac{m+6}{2(k+1)}}(\mathbb{R}^n)); \quad \forall \mathcal{L}_i \text{ for all } Z_1, \cdots, Z_k \in \mathcal{M}, \quad i = 1, 3. \]

Therefore,
\[ u(t, x) \in I^\infty H^{\frac{m+6}{2(k+1)}}(\mathcal{L} \cup \mathcal{L}^\pm). \]

We have completed the proof of Theorem 6.1. \( \square \)

Next, we start to illustrate \( u \notin C^2((0, T] \times \mathbb{R}^2 \setminus \mathcal{L}_1 \cup \mathcal{L}^\pm \cup \mathcal{L}^\pm) \) in (iii) of Theorem 1.1. Especially, we assume that the problem (1.1) is a 2-D linear degenerate equation with Riemann discontinuous initial data as follows
\[ \begin{cases} \partial_t^2 u - t^m \Delta u = 0, & (t, x) \in [0, +\infty) \times \mathbb{R}^2, \\ u(0, x) = 0, & \partial_1 u(0, x) = \varphi_0(x), \end{cases} \quad (6.18) \]
where \( \varphi_0(x) = \begin{cases} C_1 & \text{for } x_1 > 0, x_2 > 0, \\ C_2 & \text{for } x_1 < 0, x_2 > 0, \\ C_3 & \text{for } x_1 < 0, x_2 < 0, \\ C_4 & \text{for } x_1 > 0, x_2 < 0, \end{cases} \) with \( C_1 \neq C_j \) for any \( 1 \leq i < j \leq 4 \) and \( C_1 + C_3 - C_2 - C_4 \neq 0. \)

**Theorem 6.2.** For the solution \( u(t, x) \) of (6.18), then \( u(t, x) \notin L^\infty_{lo}((\mathcal{L}_1 \cup \mathcal{L}^\pm \cup \mathcal{L}^\pm) \cup ((x, x) \cup (\mathcal{L}_1 \cup \mathcal{L}^\pm \cup \mathcal{L}^\pm)) \) with \( k = 2. \)

**Proof.** For convenience to write, we set \( \phi(t) = \frac{k+1}{k+1} \) and \( \gamma = \frac{k}{2(k+1)} \) with \( k = \frac{m}{2} \). Then as in Lemma 2.4, the solution of (6.18) can be expressed as
\[ u(t, x) = C_0 \int_0^1 (1 - s^2)^{-\gamma} V(s\phi(t), x) ds, \quad (6.19) \]
where \( C_0 > 0 \) is some fixed constant, and
\[ V(\tau, x) = \begin{cases} \varphi_0(x) & \text{if } |x_1| \geq \tau, |x_2| \geq \tau, \\ C_1 + C_2 & \text{if } |x_1| \leq \tau, x_2 \geq \tau; \\ C_3 + C_4 & \text{if } |x_1| \leq \tau, x_2 \geq \tau; \\ C_2 + C_4 & \text{if } |x_1| \leq \tau, x_2 \geq \tau; \\ C_1 + C_3 & \text{if } |x_1| \leq \tau, x_2 \leq \tau; \\ C_2 + C_4 & \text{if } x_1^2 + x_2^2 \geq \tau^2, \text{ and } 0 \leq x_1 \leq \tau, 0 \leq x_2 \leq \tau \text{ or } -\tau \leq x_1 \leq 0, -\tau \leq x_2 \leq 0; \\ C_1 + C_3 & \text{if } x_1^2 + x_2^2 \geq \tau^2, \text{ and } -|x_1| \leq 0, 0 \leq x_2 \leq \tau \text{ or } 0 \leq x_1 \leq \tau, -\tau \leq x_2 \leq 0; \\ C_2 + C_4 & \text{if } x_1^2 + x_2^2 \geq \tau^2, \text{ and } -\tau \leq x_1 \leq 0, 0 \leq x_2 \leq \tau \text{ or } 0 \leq x_1 \leq \tau, -\tau \leq x_2 \leq 0; \\ C_1 + C_3 & \text{if } \phi_1 + C_1 - C_2 - C_4 \partial_1 \phi(t, x) \text{ if } x_1^2 + x_2^2 \leq \tau^2 \text{ and } x_1 x_2 > 0; \\ C_1 + C_3 & \text{if } \phi_1 + C_1 - C_2 - C_4 \partial_1 \phi(t, x) \text{ if } x_1^2 + x_2^2 \leq \tau^2 \text{ and } x_1 x_2 < 0. \end{cases} \]
Due to

with

here

and further for \( (t, x) \in \Omega^+ \cap \{ |x| < \phi(t) \} \)

It follows from (6.19), Remark 2.4 and a direct computation that for \( (t, x) \in \Omega^+ \cap \{ |x| < \phi(t) \} \)

Thus, in domain \( \Omega^+ = \{(t, x) : 0 < t < 1, |x_1 - \phi(t)| < \delta \phi(t), |x_2| < \delta \phi(t) \} \)

In addition, for \( |x| \leq \tau \) and by (6.20), one has

and further for \( (t, x) \in \Omega^+ \cap \{ |x| < \phi(t) \} \) and \( |x| \leq \tau \)

\[
\begin{align*}
\tau \partial_\tau V + x_1 \partial_1 V + x_2 \partial_2 V &= 0, \\
|x_1 \partial_1 V + \tau \partial_\tau V| &= \frac{|x_2| \sqrt{\tau^2 - |x|^2}}{\tau^2 - x_2^2} \leq \frac{|x_2| \sqrt{\tau^2 - |x|^2}}{x_1^2} \in L^\infty, \\
\left| (x_1 \partial_1 V + \tau \partial_\tau V)^2 \right| &= \frac{2x_1 |x_2| \tau \sqrt{\tau^2 - |x|^2}}{(\tau^2 - x_2^2)^2} \in L^\infty, \\
\partial_1 (x_1 \partial_1 V + \tau \partial_\tau V) &= \frac{x_1 x_2}{(\tau^2 - x_2^2)^2 (\tau^2 - |x|^2)^{3/2}} \\
\partial_\tau^2 V &= \frac{x_1 x_2 (3\tau^2 - 2|x|^2 - x_2^2)}{(\tau^2 - x_2^2)^2 (\tau^2 - |x|^2)^{3/2}}.
\end{align*}
\]
From (6.23), we can arrive at $L_0 u = 0$ and further $L_0^l u = 0$ for any $l \in \mathbb{N} \cup \{0\}$. We now show $L_2 u \not\in L^\infty(\Omega_2^+ \cap \{(t, x) : |x| < \phi(t)\})$ but $\tilde{L}_2 u \not\in L^\infty(\Omega_2^+ \cap \{(t, x) : |x| < \phi(t)\})$.

By the expression of $\tilde{L}_2 u$ and (6.23), it suffices to prove

$$t \phi(t) \int_0^1 (1 - s^2)^{1-\gamma} \partial_t V(s \phi(t), x) ds \in L^\infty(\Omega_1^+).$$

By the expression of $V(\tau, x)$, we only require to take care of $\partial_t V(s \phi(t), x)$ in the domain $\{(t, x) : |x| \leq \phi(t)s\}$ in (6.24). At this time, a direct computation yields for $(t, x) \in \Omega_1^+ \cap \{(t, x) : |x| < \phi(t)\}$

$$|t \phi(t) \int_0^1 (1 - s^2)^{1-\gamma} \partial_t V(s \phi(t), x) ds| \leq t \int_0^1 \frac{|s_2| \phi^2(t)s}{\sqrt{\phi^2(t)s^2 - |x|^2 \phi^2(t)s^2 - x_1^2}} ds \equiv A_1(t, x).$$

We can assert

$$A_1(t, x) \in L^\infty(\Omega_1^+ \cap \{(t, x) : |x| < \phi(t)\}).$$

Indeed, if we set $a = \frac{\sqrt{\phi(t) - |x|}}{\phi(t) + |x|}$ and $\xi = \frac{\phi(t)s - |x|}{\phi(t)s + |x|}$ then $A_1(t, x)$ can be estimated as follows

$$|A_1(t, x)| = 2t |s_2||x||\int_0^a \frac{x_2 |x|(1 + \xi^2)}{4x_1^2 \xi^2 + x_2^2 (1 + \xi^2)^2} d\xi|$$

$$= 2t |s_2| |x| \left\{ \int_0^a \frac{x_2^2 (1 + \xi^2 + \frac{2x_1^2 + 2x_1 |x|}{x_2^2})(1 + \xi^2 + \frac{2x_1^2 - 2x_1 |x|}{x_2^2})}{x_2^2 (1 + \xi^2 + \frac{2x_1^2 + 2x_1 |x|}{x_2^2})(1 + \xi^2 + \frac{2x_1^2 - 2x_1 |x|}{x_2^2})} d\xi \right\}$$

$$= \frac{t |s_2|}{2x_1} \left\{ \int_0^a \left( \frac{\frac{2x_1^2 + 2x_1 |x|}{x_2^2} - \frac{2x_1^2 - 2x_1 |x|}{x_2^2}}{1 + \xi^2 + \frac{2x_1^2 + 2x_1 |x|}{x_2^2}} \right) d\xi \right\}$$

$$= t \left| \text{arctan} [\frac{2x_1^2 - 2x_1 |x|}{x_2^2}]_{\xi=0}^{\xi=\frac{2x_1^2 + 2x_1 |x|}{x_2^2}} \right| \leq Ct.$$

Consequently, (6.26) holds true.

Next we show that

$$\tilde{L}_1^2 u \not\in L^\infty(\Omega_1^+ \cap \{(t, x) : |x| < \phi(t)\}).$$

For $(t, x) \in \Omega_1^+ \cap \{(t, x) : |x| < \phi(t)\}$, we write

$$t \phi^2(t) \int_0^1 (1 - s^2)^{1-\gamma} \partial_\tau^2 V|_{\tau = s \phi(t)} ds \equiv B_1(t, x) + B_2(t, x),$$

where

$$B_1(t, x) = t \phi^3(t) \int_0^1 s (1 - s^2)^{1-\gamma} \frac{3x_1 x_2}{(\phi^2(t)s^2 - x_1^2)^2 \sqrt{\phi^2(t)s^2 - |x|^2}} ds$$

$$B_2(t, x) = t \phi^3(t) \int_0^1 s (1 - s^2)^{1-\gamma} \frac{x_1 x_2^2}{(\phi^2(t)s^2 - x_1^2)^2 (\phi^2(t)s^2 - |x|^2)^2} ds$$
As in the process to treat $A_1(t, x)$, we set $a = \sqrt{\frac{\phi(t) - |x|}{\phi(t) + |x|}}$ and $\xi = \sqrt{\frac{\phi(t)s - |x|}{\phi(t)s + |x|}}$, then

$$|B_1(t, x)| = x_1|x_2||t\varphi(t)\int_0^a (1 + \xi^2)
\left(1 - \left(\frac{|x|}{\phi(t)} + \xi^2\right)^2\right)^{2-\gamma}\frac{(1 - \xi^2)^2}{(x_2^2 + 2(x_1^2 + |x|^2)\xi^2 + x_2^2\xi^4)^2}d\xi$$

$$< + \infty,$$

$$|B_2(t, x)| = \frac{Cx_1|x_2|^3\varphi(t)}{|x|}\int_0^a (1 + \xi^2)
\left(1 - \left(\frac{|x|}{\phi(t)} + \xi^2\right)^2\right)^{2-\gamma}\frac{(1 - \xi^2)^4}{(x_2^2 + 2(x_1^2 + |x|^2)\xi^2 + x_2^2\xi^4)^2}d\xi$$

$$= + \infty. \quad \text{(Due to } \xi = 0 \text{ is a singularity point, the integrand behaves like } \frac{1}{\xi^2} \text{ near } \xi = 0)$$

(6.28)

(6.29)

It is noted that the integrand in $B_2(t, x) > 0$ is positive for $x_2 > 0$ or negative for $x_2 < 0$ respectively, then by the definition of partial derivatives together with Fatau’s lemma (i.e., for $G(y) = \int_a^b g(s, y)ds$ and $\partial_y g(s, y) > 0$, then $\lim_{h \to 0^+} G(y + h) - G(y) \geq \int_a^b \partial_y g(s, y)ds$), we have from a direct computation that for $(t, x) \in \Omega^+ \cap \{(t, x) : |x| < \varphi(t)\}$ and if $u \in C^2([0, T] \times \mathbb{R}^2 \setminus \Gamma_0 \cup \Gamma_1^\pm \cup \Gamma_2^\pm)$

\[ |L^2 u| \geq |Ct\varphi^2(t)|\int_{\mathbb{R}^2}^{1} s(1 - s^2)^{1-\gamma}\partial_1^2 V(\tau, x)|_{\tau = \varphi(t)}ds| - L^\infty \text{ terms} \]

\[ - C\varphi(t)\int_{\mathbb{R}^2}^{1} s(1 - s^2)^{-\gamma}|(\partial_1(x_1\partial_1 + \tau\partial_1)V)|_{\tau = \varphi(t)}ds| \]

\[ = |Ct\varphi^2(t)|\int_{\mathbb{R}^2}^{1} 1 - s^2)^2\gamma|\partial_1^2 V|_{\tau = \varphi(t)}ds| - L^\infty \text{ terms} - C\partial A_1(t, x) \]

\[ \geq |CB_1(t, x) + CB_2(t, x) - L^\infty \text{ terms}| \quad \text{(here we have used (6.26))} \]

\[ = + \infty \quad \text{(here we have used (6.28)-(6.29))} \]

Therefore, (6.27) is proved, and $u(t, x) \notin L^\infty_{loc}(\Gamma_0 \cup \Gamma_1^\pm \cup \Gamma_2^\pm)$ with $k = 2$. \qed

Finally, we can complete the proof on Theorem 1.1.

**Proof of Theorem 1.1.**

(i) Combining Theorem 5.2 with Theorem 6.1.(i) yields its proof.

(ii) Its proof comes from Theorem 5.3 and Theorem 6.1.(ii).

(iii) Based on Theorem 5.1 and Theorem 6.2, the proof can be completed.

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