A greedy chip-firing game

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Abstract
We introduce a deterministic analogue of Markov chains that we call the hunger game. Like rotor-routing, the hunger game deterministically mimics the behavior of both recurrent Markov chains and absorbing Markov chains. In the case of recurrent Markov chains with finitely many states, hunger game simulation concentrates around the stationary distribution with discrepancy falling off like $N^{-1}$, where $N$ is the number of simulation steps; in the case of absorbing Markov chains with finitely many states, hunger game simulation also exhibits concentration for hitting measures and expected hitting times with discrepancy falling off like $N^{-1}$ rather than $N^{-1/2}$. When transition probabilities in a finite Markov chain are rational, the game is eventually periodic; the period seems to be the same for all initial configurations and the basin of attraction appears to tile the configuration space (the set of hunger vectors) by translation, but we have not proved this.

KEYWORDS
chip-firing, recurrence, stationary distribution

1 | INTRODUCTION

In the 1970s, Engel [5, 6] introduced a “stochastic abacus” that deterministically mimics many aspects of the behavior of finite-state Markov chains with rational probabilities. Unaware of Engel’s work, various mathematicians and physicists invented and studied the abelian sandpile model [4] and the chip-firing game [1] which in many respects embody the same core idea as Engel’s abacus but with different motivations. For books surveying chip-firing, we refer readers to [2, 11].
In the 2010s, inspired by Engel’s work, the second author of this article in collaboration with Ander Holroyd [8] introduced a different way to deterministically mimic Markov chains via “rotor-routing” (unaware that physicists were already studying the process under the name “the Eulerian walkers model” [13]). The emphasis of much of this work on the rotor-router model, inspired by discrepancy theory and quasi-Monte Carlo methods, was on the fidelity of the deterministic process to the associated Markov chain. Specifically, it was shown that for certain asymptotically defined quantities associated with Markov chains (e.g., the proportion of the time that the chain spends in a specific state), rotor-router simulation has the same limiting behavior as the Markov chain, typically with faster convergence than the Markov chain itself would typically exhibit. For a comprehensive background on chip-firing and rotor-routing and the relationship between them, we refer readers to [7].

Here we discuss another way to derandomize Markov chains which we call the hunger game. It can be applied to any discrete-state discrete-time Markov chain, whether or not the transition probabilities are rational. (Rotor-routing can be extended to this regime—see the discussion of stack walks in [8]—though we know of no way to extend chip-firing in this direction.) A key difference between rotor-routing and the hunger game is that the frequency with which state $i$ is followed by state $j$ in rotor-router simulation converges to the transition probability $P_{ij}$; this is not the case for the hunger game. That is, the hunger game does not exhibit fidelity with regard to transition-frequencies. However, we believe this may be a virtue rather than a vice, as we will explain in the concluding section.

In Section 2, we define the hunger game and the chip addition operators associated with it. In Section 3, we prove fundamental results on the boundedness of hunger, and in Section 4, we prove basic results on the behavior of chip addition operators. In Section 5, we prove the main result of this article, demonstrating that the normalized firing vector of a hunger game process converges to the unique stationary distribution of an irreducible Markov chain with a discrepancy bound inversely proportional to the number of time steps. We apply this result to show how the hunger game process can calculate hitting probability distributions, escape probabilities, expected absorption times, and expected return times in Section 6. In Section 7, we focus on finite Markov chains with rational transition probabilities. We introduce the notion of a recurrent hunger vector (a vector that returns to itself under the hunger game) and study the properties of the basin of attraction (the set of recurrent vectors). In the case where all transition probabilities are rational, we prove the zero vector is always recurrent and determine its period, and conjecture that all the periods for a given Markov chain are the same. We conclude with Section 8, comparing the discrepancies of the rotor-router model and the hunger game.

2 | PRELIMINARIES

See [9] or [12] for basic facts about Markov chains. Throughout this article, except where otherwise noted, Markov chains are assumed to have a finite state space indexed by positive integers $1, \ldots, n$ for some $n$; when we consider Markov chains with countably infinite state spaces, we will assume that for each state $i$ there are only finitely many states $j$ such that the transition probability $P_{ij}$ from state $i$ to state $j$ is positive. We can represent a Markov chain by a weighted directed graph $G = (V, E)$ whose vertices $v_i \in V$ are the allowed states and the weight of the directed edge $(v_i, v_j) \in E$ is $P_{ij}$. For brevity, we will use the words state and vertex to refer to both the state in the Markov chain and the corresponding vertex in $G$, passing back and forth between the abstract Markov chain and its concrete embodiment as a random walk on $G$. We say that state $i$ is an absorbing state if $P_{ii} = 1$ (equivalently if $G$ has a loop at $v_i$ with weight 1). Notice that for every vertex $v \in G$, the sum of the weights of all edges $(v, w)$ is 1.
Let $H = P - I$, where $I$ is the $n$-by-$n$ identity matrix. Let $H_i$, $P_i$, and $I_i$ denote the $i$th rows of the matrices $H$, $P$, and $I$ respectively. Observe that $I_i = e_i$ (the $i$th unit vector) and that $-H$ is the Laplacian of the graph. The Markov chain admits at least one stationary measure $\pi$ for which the vector $\nu = [\pi(1), \ldots, \pi(n)]$ satisfies $\nu P = \nu$ (this follows from the assumption that the Markov chain is finite; see, e.g., [12]), so 1 is an eigenvalue of $P$ and 0 is an eigenvalue of $H$. All eigenvalues of $P$ have magnitude at most 1. When the Markov chain is irreducible (i.e., when every state can be reached from every other state in some finite number of steps), 1 is a simple eigenvalue of $P$ and 0 is a simple eigenvalue of $H$, so that $H$ has rank $n - 1$, and the rows of $H$, taken with integer coefficients, generate an $n - 1$-dimensional sublattice of the space of vectors with entries summing to zero; in this case, there is a unique stationary measure $\pi$ satisfying $\pi(1) + \cdots + \pi(n) = 1$.

We now informally introduce the hunger game on the weighted directed graph $G$ by comparing it to the chip-firing model and the rotor-router model before offering a more technical definition.

The “goodness” of a deterministic analogue of a random process can be assessed by the notion of discrepancy. If some numerical characteristic of the deterministic process converges to a corresponding numerical characteristic of the random process as simulation time goes to infinity, one can try to determine the rate of convergence.

The simplest deterministic analogues of Markov chains were invented by Engel [5, 6], under the name the stochastic abacus (though the term chip-firing is more common nowadays). Suppose that all the transition probabilities $P_{ij}$ are rational, and that we have positive integers $d_1, \ldots, d_n$ such that $d_iP_{ij}$ is an integer for all $i, j$. Assume that the Markov chain is irreducible. Define a chip-configuration as an $n$-tuple $c = (c_1, \ldots, c_n)$; say that a chip-configuration is stable if $c_i < d_i$ whenever $i$ is a nonabsorbing state. If $c_i \geq d_i$, then we obtain another chip-configuration by firing at $i$, replacing $c$ by $c' = c + d_iH_i$; if we represent the Markov chain by drawing a graph with vertices $v_1, \ldots, v_n$ and we represent the chip-configuration $c$ by putting $c_j$ chips at $v_j$ for all $j$, then firing means sending $d_iP_{ij}$ chips from $i$ to $j$ for each $j \neq i$. Chip-firing can be used to find the stationary probability measure of an irreducible Markov chain as follows. Put sufficiently many chips on the state-graph so that stabilization is impossible no matter how many firings are performed, and start performing firings however one wishes; when we encounter a chip-configuration we have seen before (as must happen eventually), the vector that records the number of times each state has fired will be a stationary vector.

**Example 2.1.** Suppose we have the Markov chain given by the Markov matrix

\[
\begin{bmatrix}
1 & \frac{1}{2} & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix},
\]

representing a doubly reflecting random walk. Its corresponding graph is shown in Figure 1. If we place 1 chip at state 1, 2 chips at state 2, and 1 chip at state 3, then state 2 is unstable, so we may fire at 2, turning the chip-configuration $(1, 2, 1)$ into the chip-configuration $(2, 0, 2)$. Then firing at 1 and at 3 brings us back to $(1, 2, 1)$. The vector that records the number of times each state fired is $(1, 1, 1)$, which is indeed a stationary vector for this Markov chain.

The rotor-router model [8] is a different scheme for imitating Markov chains deterministically. Assume as above that the Markov chain has rational transition probabilities, with $d_i$ as above. Represent the Markov chain using a directed graph with $d_iP_{ij}$ parallel edges from $v_i$ to $v_j$, so that $v_i$ has outdegree $d_i$. Each vertex distributes arriving chips along its outgoing edges in a cyclic manner, not sending a chip along any edge for a second time until it has sent a chip along every edge at least once, and thereafter
always sending the next chip along the edge along which it has sent a chip least recently. Inasmuch as
the vertex with the chip gets to decide where the chip goes next, we call this “supply-side” management
of the chip’s movement. Assume that the Markov chain is irreducible. It can be shown that once the
chip enters an infinite loop (as must happen eventually), the fraction of the time that the chip spends
at vertex $v_i$ is proportional to the steady-state $\pi(i)$.

**Example 2.2.** We use the same Markov chain as Example 2.1. Suppose we start with the chip at $v_1$
and begin the game by sending the chip to $v_2$, then $v_3$, then $v_3$ again. Since at this point the chip has
already traveled along the edge sending $v_3$ to $v_1$, it now travels along the edge from $v_3$ to $v_2$. As the
chip has already gone from $v_2$ to $v_3$, the rotor-router protocol dictates that it must now travel along the
other edge from $v_2$ and go to $v_1$. Similarly, as it has already gone from $v_1$ to $v_2$, now it must travel from
$v_1$ to $v_1$. Thereafter the process cycles forever. Since within each cycle the chip spends 2 steps at each
vertex, $(2,2,2)$ is a stationary vector.

The **hunger game** introduced in this article can be seen as a “demand-side” management system:
each vertex has a “hunger” for chips, determined by its expectation of receiving chips from neighboring
vertices that have been previously visited. When a vertex $v$ receives a chip, the neighboring vertices’
hunger increases in accordance to the transition probabilities from $v$ to those vertices, and $v$ sends its
chip to the vertex with highest hunger (regardless of whether that chip is a neighbor of $v$).

Now we give a more formal definition of the **hunger game**. It is simplest to start with the situation
in which the chain runs forever without restarts (in contrast to chains that will be restarted when they
enter an absorbing state). We also start with the case in which the state space is finite, with $|V| = n$,
deferring discussion of infinite-state spaces until later. The **hunger vector** $h \in \mathbb{R}^n$ represents the hunger
at each vertex in $V$; we will sometimes call it the **hunger state** to emphasize its interpretation as a
state of the hunger game system. At each step, whichever vertex has the highest hunger receives the
chip; if vertex $v_i$ receives the chip, then the hunger vector $h$ is updated by adding $H_i = P_i - I_i$ to it,
corresponding to the increase in vertices’ hunger from the presence of this chip at $v_i$ but also the
satiation of $v_i$ after receiving a chip. If multiple states are tied for the highest hunger, we break the
tie by choosing the lowest-indexed such vertex. Since each row of $H$ has entries summing to 0, total
hunger never changes.

Each relocation of the chip is referred to as **firing** the chip; specifically, when the chip is relocated
at $i$, we say the chip fires to $i$. Unlike rotor-router or chip-firing, under the hunger game rules a chip
does not necessarily have to be fired to a vertex adjacent to its current location; see Example 2.3.
In fact, the determination of the chip’s next location depends only on the hunger state $h$ and not the
current location of the chip. The step can be described purely in terms of the matrix $H$ and the vector $h$
without any reference to chips, via the rule “Replace $h$ by $h' = h + H_i$ where $i$ maximizes $h_i$, choosing
the smallest such $i$ in the event of a tie.”

**Example 2.3.** We use the same Markov chain as Example 2.1. Starting with $h = 0$, as shown in
Figure 2A, regardless of the initial location of the chip, we fire the chip to $v_1$ under the tie-breaking
rule, yielding $h = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ as shown in Figure 2B. After this, we fire to $v_2$, as shown in Figure 2C,
and then fire to $v_3$, as shown in Figure 2D. Notice that we have returned back to the initial hunger state $h = 0$, so this process repeats, visiting states 1, 2, 3, then back to 1, and so on. Notice that we fire the chip from 3 to 1 even though $P_{31} = 0$.

If one ignores the chip and focuses on the entries of the hunger vector, the hunger game can be viewed as a greedy variant of chip-firing. Specifically, if all entries of $h$ and $P$ are rational, we can without loss of generality assume they are all non-negative integers (since the firing rule is invariant under affine transformation of the hunger vector) and create a chip-configuration of the customary kind in which the number of chips at $i$ is $h_i$. Then the firing rule, translated to the new context, tells us to fire the vertex $i$ that has the most chips, with ties resolved as before.

When the state space is (countably) infinite, our hunger vectors are infinite sequences. We restrict ourselves to sequences that are bounded and take on only finitely many distinct values; this ensures that there exists at least one $i$ for which $h_i$ equals $\sup_i h_i$, from which it follows that a smallest such $i$ exists. As each vertex has only finitely many outgoing edges, this assumption ensures that after any finite number of steps in the hunger game there are only finitely many distinct values of hunger, so a vertex of highest hunger can be found and the lowest-indexed one can be chosen, ad infinitum.

For Markov chains with absorbing states we vary the procedure slightly. We start with a graph devoid of chips, add a chip at an initial vertex, and then follow the rule for moving the chip described above, with the extra stipulation that when the chip reaches an absorbing vertex, it gets removed from the graph. In more detail, we define chip addition operators $E_i$ as follows: Given an initial hunger vector $h$, we place a chip at $v_i$ (increasing the hunger of the neighbors of $v_i$) and add $P_i$ to $h$, and we then repeatedly move the chip to the currently hungriest vertex (the lowest-indexed one, in the event of a tie), simultaneously incrementing the hunger vector by the row of $H$ corresponding to the chip’s new location, until we arrive at an absorbing vertex $v_k$, at which point we subtract $P_k = I_k$ from the current hunger vector and remove the chip from $v_k$. We define $E_i(h)$ to be the final hunger vector, and call $E_i$ the chip addition operator at $i$. It is possible for $E_i(h)$ to be undefined, in the event that the process never arrives at an absorbing state, but we will show in Lemma 4.1 that for finite absorbing chains the process must terminate so that $E_i$ is well-defined; moreover, each chip addition operator preserves
FIGURE 3 A graph $G$ corresponding to an absorbing Markov chain

FIGURE 4 The hunger game on $G$ from Figure 3 after inserting a chip at $v_3$. (A) The effect of inserting a chip at $v_3$, shown in blue. States with updated hungers are shown in yellow. (B) $h$ as the chip fires successively to $v_2$, $v_4$, and $v_5$, shown in blue. Updated hungers are shown in yellow. (C) $h$ after removing chip from $v_5$, shown in yellow.

total hunger, since the sum of the entries increases by 1 when $P_i$ is added, stays the same each time a row of $H$ is added, and decreases by 1 when $P_k$ is subtracted.

As in the previous situation, the process can be described purely in terms of vector and matrix operations without reference to $G$ or a chip. Given a vector $h \in \mathbb{R}^n$, add row $P_i$ to $h$. Thereafter, if $j$ is the unique value such that $h_{j'} < h_j$ for all $j' < j$ and $h_{j'} \geq h_j$ for all $j' > j$, add $H_j$ to $h$, unless $j$ is an absorbing state (call it $k$), in which case subtract $P_k$ from the hunger vector and stop, calling the result $E_i(h)$.

Example 2.4. Suppose we have the Markov chain represented by the graph in Figure 3. It has absorbing states $v_1$ and $v_5$. Let us compute $E_3(0)$. Starting with $h = 0$, for our first step we add a chip to $v_3$ to yield $h = [0, 0.6, 0, 0.4, 0]$, as shown in Figure 4A. After this, we follow additional steps of the hunger game process, firing the chip successively to $v_2$, $v_4$, and $v_5$, as shown in Figure 4B. Having reached an absorbing state, the final step is to remove the chip from $v_5$, resulting in $E_3(0) = [0.2, 0.2, 0.4, -0.6, -0.2]$, as shown in Figure 4C.

Notice that the total hunger is 0 at the start, increases to 1 when the chip is added at $v_3$, stays 1 as the chip moves through $G$, and decreases to 0 when the chip is removed at the end.

Since increasing the hunger at every vertex by the same amount has no effect on the dynamics of the hunger game, when our Markov chain is finite we will often assume that total hunger is 0.
3 | BOUNDEDNESS OF HUNGER

The following lemmas will be useful in our discussion of the hunger game. The first lemma shows that on a countable Markov chain, hunger stays uniformly bounded from below under the hunger game process, including the removal and reinsertion of the chip if it reaches an absorbing state.

**Lemma 3.1.** Suppose we play the hunger game for a countable Markov chain, with an initial hunger vector \( h^{(0)} \) that is bounded below by \( h_{\min} \in \mathbb{R} \), meaning \( h^{(0)}_i \geq h_{\min} \) for all \( i \). Then hunger remains uniformly bounded below by \( h_{\min} - 1 \) under iteration of the hunger game process. That is, suppose we have a sequence of hunger states \( h^{(1)}, h^{(2)}, \ldots \) such that for each \( k \geq 1 \), \( h^{(k)} \) is reached from \( h^{(k-1)} \) by either firing the chip if it is not at an absorbing state or removing and reinserting the chip if it is. Then \( h^{(k)}_i \geq h_{\min} - 1 \) holds for all \( i \) and \( k \).

**Proof.** Without loss of generality, we may take \( h_{\min} \) to be 0, so that \( h^{(0)}_i \geq 0 \) for all \( i \). Suppose our claim is false, and let \( k \) be the smallest index for which the claim fails, so that \( h^{(k)}_i < -1 \) for some \( i \). Consider the set \( S \) of states whose hunger changed during the hunger game process from \( h^{(0)} \) to \( h^{(k)} \); this set must be finite, for each of the \( k \) steps can only change the hunger of finitely many states, and must contain \( i \). The total hunger of this set must be constant during this hunger game process if we combine the process of removing a chip and inserting the next chip into a single operation. Let \( h \) be the total hunger of \( S \) throughout the process. As the hunger of all states was originally bounded below by 0, we have \( h \geq 0 \). Since in going from \( h^{(k-1)} \) to \( h^{(k)} \) we fired the chip to \( i \), \( v_i \) must have been one of the hungriest vertices. But in \( h^{(k-1)} \), \( v_i \) must have had hunger \( < 0 \) (otherwise it could not have reached hunger \( < -1 \) in a single firing). Since the hungers of the vertices in \( S \) have nonnegative sum, at least one of them must be nonnegative, and hence \( v_i \) could not have been one of the hungriest vertices. This contradiction shows that hunger is bounded from below, and more specifically that \( h_{\min} - 1 \) is a lower bound.  

The lemma remains true if one replaces \( \geq \) by \( > \) in both the hypothesis and the conclusion.

**Lemma 3.1** only provides a uniform lower bound on hunger; a uniform upper bound on hunger does not exist in general for countable Markov chains. One may construct Markov chains where depending on the location of chip insertion, hunger can grow arbitrarily large at a certain state. As a specific example, consider the countably infinite Markov chain shown in Figure 5. We have states \( v_{i,j} \) for all integers \( i \geq j \geq 0 \), where \( v_{0,0} \) is an absorbing state. A walker at vertex \( v_{i,j} \) for \( i > j \) moves with equal probability to any vertex \( v_{i,k} \) with \( k > j \), in other words with probability \( \frac{1}{i-j} \) to \( v_{i,k} \) for all \( j+1 \leq k \leq i \), and a walker at \( v_{i,j} \) moves with probability 1 to the absorbing vertex \( v_{0,0} \). Starting with \( h = 0 \), inserting a chip at \( v_{n,0} \) for \( n > 0 \) causes the chain to move from \( v_{n,0} \) to \( v_{n,n} \), increasing the second coordinate by 1 each step (i.e., moving one position to the right at each step). When the chip fires to \( v_{n,n-1} \), state \( v_{n,n} \) has hunger \( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \), the \( n \)th partial sum of the harmonic series. As the harmonic series diverges, by picking sufficiently large \( n \), hunger can become arbitrarily large at \( v_{n,n} \), and is thus not uniformly bounded under the hunger game process, when chip removal and insertion are allowed.

This example used a countably infinite state space, which is necessary for this unbounded behavior to occur. As shown in the following lemma, for a finite Markov chain, hunger stays uniformly bounded from both sides under the hunger game process, including the removal and reinsertion of the chip if it reaches an absorbing state.

**Lemma 3.2.** For a given hunger state \( h^{(0)} \) on a finite Markov chain, hunger remains uniformly bounded under the hunger game process. In other words, there exist \( a, b \in \mathbb{R} \) such that for any sequence
of hunger states \( h^{(1)}, h^{(2)}, \ldots \), where for each \( k \geq 1 \), \( h^{(k)} \) is reached from \( h^{(k-1)} \) by either firing the chip if it is not at an absorbing state or removing and reinserting the chip if it is, the inequality \( a \leq h^{(k)} \leq b \) holds for all \( i \) and \( k \).

**Proof.** The existence of a lower bound \( a \) follows directly from Lemma 3.1, as any hunger state on a finite Markov chain must be bounded.

As hunger is bounded from below by \( a \) and total hunger \( h \) is constant during the hunger game process, when we conceive of removing a chip and inserting the next chip as occurring simultaneously, the hunger of any single vertex cannot exceed \( h - (n - 1)a \), and thus hunger is bounded from above and \( b \) exists.

**Remark 3.3.** When all transition probabilities are rational and the Markov chain has no absorbing states, so that we never remove and reinsert the chip—which introduces choice into the hunger game—we claim Lemma 3.2 implies that the hunger game is eventually periodic. Let \( d \) be the least common multiple of the denominators of all the transition probabilities. For any \( h^{(0)} \), as each entry is uniformly bounded and may only change by a multiple of \( \frac{1}{d} \), there are only finitely many values that \( h^{(k)} \) may take on as \( k \) varies. This implies that eventually we must have \( h^{(j)} = h^{(i)} \) with \( j > i \), so that the hunger game has become periodic with period dividing \( j - i \). If we define \( v \) as the vector that counts the number of visits to each state of the chain from time \( i + 1 \) to time \( j \), we have \( vH = 0 \), implying that \( v \) is a stationary vector and that \( \frac{1}{j-i}v \) is the stationary probability measure for the Markov chain. Theorem 5.1 extends this claim, in a suitable asymptotic sense, to situations in which the transition probabilities are not all rational.

### 4 | TERMINATION OF STABILIZATION PROCESSES

An absorbing Markov chain is a (not necessarily finite) Markov chain where each vertex has a path of finite length to an absorbing state. For example, the infinite Markov chain shown in Figure 5 is
absorbing, as a walker at any vertex $v_{ij}$ can move to absorbing vertex $v_{00}$ within 2 moves. We claim that in a finite absorbing Markov chain, the chip addition process at any vertex $v$ in the corresponding graph $G$ is guaranteed to terminate, regardless of the initial hunger state, and thus the chip addition operators $E_i$ are well-defined for finite absorbing Markov chains. This follows immediately from the finiteness of $G$ and the following lemma:

**Lemma 4.1.** For a finite absorbing Markov chain, each vertex $v$ gets visited only finitely often in the hunger game before the chip reaches an absorbing vertex.

**Proof.** The assumption that the Markov chain is absorbing guarantees that for each $v$ there is a finite path $w_0, w_1, \ldots, w_m$ such that $w_0$ is $v$, $w_m$ is an absorbing vertex, and the transition probability from $w_i$ to $w_{i+1}$ is positive for all $0 \leq i \leq m-1$. We prove the claim by induction on $m$. The case $m = 0$ is trivial, as once the chip visits an absorbing vertex it is removed. For a less trivial case, consider $m = 1$. For sake of contradiction, suppose $v = w_0$ is visited infinitely often. Each time $w_0$ is visited, the hunger of $w_1$ increases by a fixed amount. As $w_1$ is an absorbing vertex, we cannot visit it, as otherwise we would only visit $w_0$ finitely often before reaching an absorbing vertex. This implies the hunger of $w_1$ increases unboundedly due to $w_0$ being visited infinitely often, contradicting the existence of an upper bound on hunger demonstrated in Lemma 3.2. Hence $w_0$ is visited finitely often, proving the case $m = 1$.

The inductive step follows similar reasoning to the $m = 1$ argument. Suppose now that the claim is true for $m - 1$, so that $w_1$ is visited only finitely often. Assume for sake of contradiction that $w_0$ is visited infinitely often, where each visit increases the hunger of $w_1$ by a fixed amount. After $w_1$ is visited for the last time, its hunger must grow without bound due to $w_0$ being visited infinitely often; this contradicts Lemma 3.2. Hence the claim holds for all vertices. □

Since $G$ has only finitely many vertices, each of which can fire only finitely many times before the chip is absorbed, the chip must eventually be absorbed, as claimed.

A weakened version of this lemma applies to countable absorbing Markov chains.

**Proposition 4.2.** On a countable absorbing Markov chain, if we add a chip and then perform repeated firing, then either the chip eventually gets absorbed or the chip is not confined within any finite subset of the state space.

**Proof.** If the Markov chain is finite, the result immediately follows from Lemma 4.1, so assume the state space is countably infinite. Suppose there is a finite subset $U$ of the state space $V$ such that the chip is always in $U$. Consider the set $S$ given by

$$S = U \cup \{v \in V | \exists i \in U : P_{ij} > 0\}.$$ 

Because $U$ is finite and each vertex has finitely many outgoing edges, $S$ is also finite. Crucially, as the chip is bounded within $U$, the only states whose hunger can change during the chip addition process are the states in $S$. Hence, we may equivalently view the hunger game as acting upon the induced subgraph of $G$ formed from using $S$ as the vertex set, and from Lemma 4.1 the result follows. □

**Remark 4.3.** For infinite absorbing Markov chains, it is possible for the chip to wander off without being confined to any finite subset of the state space. As a simple example, based on the goldbug system introduced by the second author of this article as described in Kleber [10], take $\mathbb{N} \cup \{-1\}$ as the state space where states $-1$ and 0 are absorbing, and for $i \geq 1$, a walker at vertex $i$ moves to vertex $i - 2$ and $i + 1$ each with probability $\frac{1}{2}$, as shown in Figure 6. Using the hunger state $h$ where
FIGURE 6  The goldbug system

FIGURE 7  A hunger state on the goldbug system where a chip inserted at $v_1$ goes to infinity. (A) The initial hunger state $\mathbf{h}$. (B) $\mathbf{h}$ after inserting a chip at $v_1$, shown in blue. States with updated hungers are shown in yellow. (C) $\mathbf{h}$ as the chip fires successively to $v_2$, $v_3$, and $v_4$, shown in blue. Updated hungers are shown in yellow.

$\mathbf{h}_0 = \mathbf{h}_1 = -\frac{1}{2}$ and $\mathbf{h}_i = 0$ for $i > 1$, as shown in Figure 7A, a chip inserted at state 1, as shown in Figure 7B, will move rightwards one state at a time to infinity; the first few steps of the process are shown in Figure 7C.

5 | STATIONARY DISTRIBUTIONS

We now consider the hunger game process for nonabsorbing Markov chains, so that chips are never inserted or removed.
Say that a vector $v$ is stationary under $P$ if $vP = v$, or equivalently $vH = v(P - I) = 0$. Let $E$ be the space of vectors that are stationary under $P$, or equivalently the nullspace of $H$. When the Markov chain is irreducible, so that there is a unique stationary distribution $\pi$, $E$ is the 1-dimensional subspace of $\mathbb{R}^n$ consisting of multiples of $[\pi(1), \ldots, \pi(n)]$.

Define the firing vector $v$ after $N$ steps of a hunger game process to be the vector whose $i$th component $v_i$ is the number of times $H_i$ was added to $h$, that is, the number of times the chip fired to vertex $v_i$. The following theorem demonstrates that the normalized firing vector $\frac{1}{N}v$ approximates the unique stationary distribution $\pi$ of an irreducible finite Markov chain within a distance proportional to $N^{-1}$.

**Theorem 5.1.** Given a hunger game process on an irreducible finite Markov chain, let $v(N)$ be the firing vector after $N$ steps. Then the sequence of normalized firing vectors $\left\{ \frac{1}{N}v(N) \right\}$ converges to the unique stationary distribution $\pi$, where there exists a constant $C$ such that for all $N$, the normalized firing vector is within distance $\frac{C}{N}$ of $\pi$ in the $L^1$ metric.

**Proof.** Since the Markov chain is irreducible, 0 is a simple eigenvalue of $H$. Let $c$ be the maximum of $1/|\lambda|$ where $\lambda$ ranges over the nonzero eigenvalues of $H$. For $d \in \mathbb{R}$, let $U_d = \{x|x_1 + \cdots + x_n = d\}$, where there are $n$ states in the Markov chain. The fact that 0 is a simple eigenvalue of $H$ and that every other eigenvalue has norm at least 1 implies that the restriction of $H$ to an affine map from $U_1$ to $U_0$ is invertible, and that if two points $p^{(1)}, p^{(2)} \in U_0$ are within distance $\varepsilon$, then their two preimages $x^{(1)}, x^{(2)} \in U_1$ are within distance $c\varepsilon$ of each other.

It follows that if we have a point $p \in U_0$ within distance $\varepsilon$ of 0, the preimages $v, \pi \in U_1$ of $p$ and 0, respectively, are within distance $c\varepsilon$ of each other. As a result, $v$ is within distance $c\varepsilon$ of the stationary distribution $\pi$.

By Lemma 3.2, during any hunger game process hunger remains bounded, so the change in hunger after $N$ steps, given by $v(N)H$, is within a bounded distance, say $b$, of 0. This implies $\left( \frac{1}{N}v(N) \right)H$ is within distance $\frac{b}{N}$ of 0, which implies the normalized firing vector $\frac{1}{N}v(N)$ is within distance $\frac{bc}{N}$ of a stationary distribution $\pi$. Setting $C = bc$ yields our desired result. 

**Remark 5.2.** As the only time irreducibility was assumed in Theorem 5.1 was when we claimed the Markov chain had a unique stationary distribution $\pi$, the result holds for any finite Markov chain with a unique stationary distribution. Namely, it also works for absorbing Markov chains with a single absorbing state $v_k$, where $\pi$ is the vector with a 1 corresponding to state $v_k$ and 0s elsewhere.

When a unique stationary distribution exists, the stationary distribution represents, in the long run, the probability distribution of the chain being at each particular state, or the occupation frequency distribution. As a result, Theorem 5.1 illustrates that the normalized firing vector, which counts how many times the chip arrives at each state in the hunger game, approximates the occupation frequency distribution within a discrepancy proportional to $N^{-1}$, better than the $N^{-1/2}$ discrepancy expected from repeated samplings of the corresponding random process.

**Example 5.3.** The Markov chain described in Example 2.3 is irreducible, and has unique stationary distribution $\pi = \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right]$, as shown in Figure 8. From Example 2.3 we know that the states are visited periodically in the order 1, 2, and 3, so the firing vector starting at $h = 0$ is given by

$$v(N) = \left[ \left\lfloor \frac{N + 2}{3} \right\rfloor, \left\lfloor \frac{N + 1}{3} \right\rfloor, \left\lfloor \frac{N}{3} \right\rfloor \right].$$
The unique stationary distribution $\pi$ of a doubly reflecting random walk

or equivalently

$$v^{(3M)} = [M, M, M],$$

$$v^{(3M+1)} = [M + 1, M, M],$$

$$v^{(3M+2)} = [M + 1, M + 1, M].$$

Theorem 5.1 guarantees the existence of some constant $C$ such that the normalized firing vector $\frac{1}{N}v^{(N)}$ is within $C/N$ of $\pi$ for all $N$. In fact, we will show $C = \frac{4}{3}$ is the minimal such constant. Then for $N = 3M$, the normalized firing vector is $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, so there is no discrepancy, while for $N = 3M + 1$ the normalized firing vector is $[\frac{M+1}{3M+1}, \frac{M}{3M+1}, \frac{M}{3M+1}]$, whose discrepancy from $\pi = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ is $\frac{4/3}{3M+1} = C/N$.

Similarly for $N = 3M + 2$.

In general, however, unlike in this example, the sequence of states visited need not be periodic, as the transition probabilities can be irrational.

6 | HITTING PROBABILITIES AND ABSORPTION TIMES

Let $X_0, X_1, \ldots$ be a Markov chain on a finite state space $V$, and let $P_v$ denote the law of the Markov chain initialized at state $v$, or in other words $X_0 = v$. Define $T_u$ to be the hitting time of vertex $u$ by the Markov chain, given by

$$T_u = \min\{t \geq 0 : X_t = u\},$$

where if no such $t$ exists then $T_u = \infty$.

For an absorbing Markov chain, let the set of absorbing states be $U \subseteq V$. If our starting vertex $v$ belongs to $U$, then the behavior of the chain is trivial, so we will assume henceforth that $v \notin U$. We define the hitting probability of $u \in U$ by

$$h_u(v) = P_v(X_t = u \text{ for all } t \text{ sufficiently large}),$$

where $P_v$ denotes the law of the Markov chain initiated at $v$. The hitting probability is equivalently

$$h_u(v) = P_v(\forall u' \in U, \ T_u \leq T_{u'}),$$

which will give us the flexibility later on to modify the Markov chain so that the states in $U$ are no longer absorbing. As an absorbing Markov chain reaches an absorbing state with probability 1, for any $v$ we have

$$\sum_{u \in U} h_u(v) = 1.$$
words, for each \( u \in U \) we replace \( P_u \), the row of \( P \) corresponding to state \( u \), with \( e_v \), the unit vector corresponding to \( v \). Crucially, this modification does not change the hitting probabilities \( h_u(v) \) as given by our second definition. Additionally, we remove any states in \( V \) that cannot be reached from \( v \) with positive probability; this does not alter the hitting probabilities. We will refer to this modified Markov chain as the rerouted Markov chain associated with \( v \). Due to the removal of unreachable states, a rerouted Markov chain of an absorbing Markov chain is always irreducible.

**Example 6.1.** Consider the absorbing Markov chain given in Figure 9A, which has absorbing states \( v_1 \) and \( v_4 \). The rerouted Markov chain associated with nonabsorbing state \( v_2 \) is shown in Figure 9B, where since \( v_5 \) cannot reach \( v_2 \) with positive probability, it is removed from the system; this ensures that the Markov chain is irreducible. Hence, it has a unique stationary distribution; this applies for all rerouted Markov chains of any finite absorbing Markov chain.

The following lemma will be useful for the upcoming result, Theorem 6.3.

**Lemma 6.2.** Let \( a \) and \( b \) be positive real numbers and let \( a' \) and \( b' \) be nonnegative real numbers. For a given \( \epsilon \) satisfying \( 0 \leq \epsilon < \frac{a+b}{2} \), if \( |a' - a| \leq \epsilon \) and \( |b' - b| \leq \epsilon \), then

\[
\frac{a - \epsilon}{a + b} \leq \frac{a'}{a' + b'} \leq \frac{a + \epsilon}{a + b}.
\]

**Proof.** Let \( a' = a + \delta_1 \) and \( b' = b + \delta_2 \), where \( |\delta_1|, |\delta_2| \leq \epsilon \). First, notice that \( \frac{a'}{a' + b'} \) is well-defined, as \( a' + b' = a + b + \delta_1 + \delta_2 \geq a + b - 2\epsilon > 0 \). Since \( a' \geq 0 \), \( \frac{a'}{a' + b'} \geq 0 \). Holding \( a \) and \( b \) fixed, we see that the quantity \( \frac{a'}{a' + b'} = 1/(1 + \frac{b'}{a'}) = 1/(1 + \frac{b + \delta_2}{a + \delta_1}) \) is weakly increasing with respect to \( \delta_1 \) and weakly decreasing with respect to \( \delta_2 \). (Technically this argument requires \( a + \delta_1 > 0 \) and therefore does not handle the case \( a' = 0 \) but this case is easily dealt with separately.) Hence, within the bounds for \( \delta_1 \) and \( \delta_2 \), \( \frac{a'}{a' + b'} \) is minimized when \( \delta_1 = -\epsilon \) and \( \delta_2 = \epsilon \), yielding

\[
\frac{a'}{a' + b'} = \frac{a + \delta_1}{a + \delta_1 + \delta_2} \geq \frac{a - \epsilon}{a + b - \epsilon + \epsilon} = \frac{a - \epsilon}{a + b},
\]

and \( \frac{a'}{a' + b'} \) is maximized when \( \delta_1 = \epsilon \) and \( \delta_2 = -\epsilon \), yielding

\[
\frac{a'}{a' + b'} = \frac{a + \delta_1}{a + \delta_1 + \delta_2} \leq \frac{a + \epsilon}{a + b + \epsilon - \epsilon} = \frac{a + \epsilon}{a + b},
\]

which completes the proof.\[\square\]
The following theorem demonstrates how the hitting probabilities of a finite absorbing Markov chain are approximated by the firing vector of the rerouted Markov chain.

**Theorem 6.3.** Given a finite absorbing Markov chain, let \( \mathbf{v}^{(N)} \) be the firing vector after \( N \) steps of a hunger game process on the rerouted Markov chain associated with state \( v \). Then the sequence \( \{a_N\} \) defined by

\[
a_N = \frac{\mathbf{v}^{(N)}_u}{\sum_{u' \in U} \mathbf{v}^{(N)}_{u'}},
\]

where we define \( a_N \) to be 0 when the denominator equals 0, converges to the hitting probability \( h_u(v) \) with discrepancy \( O(1/N) \); that is, there exists a constant \( C \) such that \( a_N \) differs from \( h_u(v) \) by at most \( C/N \) for all \( N \).

**Proof.** In the rerouted Markov chain, \( v \) can reach every state in \( V \), and every state in \( V \) can reach an absorbing state (as the original Markov chain was absorbing), which in the rerouted Markov chain moves back to \( v \) with probability 1; as a result, the rerouted Markov chain is irreducible. By Theorem 5.1, the normalized firing vector \( \frac{1}{N} \mathbf{v}^{(N)} \) converges to the unique stationary distribution \( \mathbf{\pi} \) of the rerouted Markov chain within distance \( \frac{C}{N} \) for some constant \( c \).

It is a standard fact that the expected number of visits to state \( w \) before returning to \( v \) is given by \( \frac{\mathbf{\pi}_w}{\mathbf{\pi}_v} \); see, for example, [12, Theorem 1.7.6]. But, as after visiting some \( u' \in U \) the next state visited can be visited at most once before returning to \( v \), the expected number of visits to any such \( u' \) equals the probability of visiting \( u' \) before returning to \( v \). As a result, for all \( u \in U \), the hitting probability \( h_u(v) \) is proportional to \( \mathbf{\pi}_u \), and given that an absorbing Markov chain reaches an absorbing state with probability 1, we have

\[
h_u(v) = \frac{\mathbf{\pi}_u}{\sum_{u' \in U} \mathbf{\pi}_{u'}}.
\]

As \( \sum_{u' \in U} \mathbf{\pi}_{u'} \) is a positive constant, there exists a finite \( M \) such that for all \( N > M \) we have

\[
\frac{C}{N} < \frac{1}{2} \sum_{u' \in U} \mathbf{\pi}_{u'}
\]

with \( c \) as above. As \( a_N \in [0, 1] \) is bounded, there exists a constant \( C_1 \) such that \( a_N \) is within \( \frac{C_1}{N} \) of \( h_u(v) \in [0, 1] \) for all \( N \leq M \); in particular, \( C_1 = M \) suffices.

For \( N > M \), notice that

\[
a_N = \frac{\mathbf{v}^{(N)}_u}{\sum_{u' \in U} \mathbf{v}^{(N)}_{u'}} = \frac{\frac{1}{N} \mathbf{v}^{(N)}_u}{\frac{1}{N} \mathbf{v}^{(N)}_u + \sum_{u' \in U \setminus \{u\}} \frac{1}{N} \mathbf{v}^{(N)}_{u'}}.
\]

As \( \frac{1}{N} \mathbf{v}^{(N)} \) is within distance \( \frac{c}{N} \) of \( \mathbf{\pi} \) in the \( L^1 \) metric, we know both \( \left| \frac{1}{N} \mathbf{v}^{(N)}_u - \mathbf{\pi}_u \right| \leq \frac{c}{N} \) and

\[
\left| \sum_{u' \in U \setminus \{u\}} \frac{1}{N} \mathbf{v}^{(N)}_{u'} - \sum_{u' \in U \setminus \{u\}} \mathbf{\pi}_{u'} \right| \leq \frac{c}{N}.
\]
Due to the rerouted Markov chain being irreducible, we also know that $\pi_v > 0$ for all states $v \in V$; in addition, each component of the visit vector $v^{(N)}$ is a nonnegative integer. Lastly, by definition of $M$, we have

$$0 \leq \frac{c}{N} < \frac{1}{2} \left( \pi_u + \sum_{u' \in U \setminus \{u\}} \pi_{u'} \right).$$

Letting $\varepsilon = \frac{c}{N}$, we apply Lemma 6.2 with the values

$$a = \pi_u > 0, \quad b = \sum_{u' \in U \setminus \{u\}} \pi_{u'} > 0, \quad a' = \frac{1}{N} v_u^{(N)} \geq 0, \quad b' = \sum_{u' \in U \setminus \{u\}} \frac{1}{N} v_{u'}^{(N)} \geq 0$$

to find

$$\frac{\pi_u - \frac{c}{N}}{\pi_u + \sum_{u' \in U \setminus \{u\}} \pi_{u'}} \leq \frac{\frac{1}{N} v_u^{(N)}}{\frac{1}{N} v_u^{(N)} + \sum_{u' \in U \setminus \{u\}} \frac{1}{N} v_{u'}^{(N)}} \leq \frac{\pi_u + \frac{c}{N}}{\pi_u + \sum_{u' \in U \setminus \{u\}} \pi_{u'}}.$$

or equivalently

$$h_u(v) - \frac{1}{N} \cdot \frac{c}{\sum_{u' \in U} \pi_{u'}} \leq a_N \leq h_u(v) + \frac{1}{N} \cdot \frac{c}{\sum_{u' \in U} \pi_{u'}}.$$

Thus, the deviation of $a_N$ from $h_u(v)$ is at most

$$\frac{1}{N} \cdot \frac{c}{\sum_{u' \in U} \pi_{u'}}.$$

The second factor is a constant, which we will denote $C_2$, and thus taking $C = \max(C_1, C_2)$ we find that for all $N$, $a_N$ differs from $h_u(v)$ by at most $\frac{C}{N}$.

Remark 6.4. We can apply Theorem 6.3 to deterministically approximate the escape probability of $u$ from $v$ on an irreducible Markov chain, or the probability that a chain starting at state $v$ reaches state $u$ before returning to $v$. We do this by splitting vertex $v$ into vertices $v_0$ and $v_1$ where all outgoing edges of $v$ now emanate from $v_0$ and all inbound edges of $v$ now end at $v_1$. Furthermore, we may remove all outgoing edges from vertex $u$. The resulting Markov chain is absorbing with the set of absorbing states $U = \{u, v\}$, and thus one may use Theorem 6.3 on the rerouted Markov chain associated with state $v_0$ to approximate the hitting probability $h_{v_1}(v_0)$. Notice that $h_{v_1}(v_0)$ equals the escape probability of $u$ from $v$ in the original irreducible Markov chain, and thus Theorem 6.3 enables us to calculate escape probabilities.

We now consider the absorption time of an absorbing Markov chain whose set of absorbing states is $U \subseteq V$, given by

$$T_U = \min\{t \geq 0 : X_t \in U\}.$$

At the absorption time, the chain enters an absorbing state, where it remains forever.

For example, starting at state $v_2$ in the absorbing Markov chain given in Figure 9A, with probability $\frac{1}{2}$ we enter absorbing state $v_1$ in the first step, and otherwise move to state $v_3$, which has the same 50-50 split between moving to an absorbing state ($v_4$) or not ($v_2$). Thus $T_{v_1}$ in this case is a geometric random variable with success probability $\frac{1}{2}$, which has expected value 2.
The expected absorption time is $E_v T_U$, where $E_v$ denotes the expected value for the law of the Markov chain initialized at state $v$. Then, when using the rerouted Markov chain, a chain can naturally be divided into epochs each of which ends with an occurrence of a state in $U$, which is then rerouted in the next time step to $v$. Hence, the expected absorption time $E_v T_U$ can be calculated from the stationary distribution $\pi$ of the rerouted Markov chain associated with $v$ to be

$$E_v T_U = \frac{1}{\sum_{u \in U} \pi_u} - 1.$$ 

Notice that if $v \in U$, then $E_v T_U = 0 = \frac{1}{1} - 1$; the $-1$ term corresponds to the extra time step needed to reroute from an absorbing state to $v$ for each epoch.

The following theorem shows how the expected absorption time of an absorbing Markov chain can be approximated using the normalized firing vector of the rerouted Markov chain.

**Theorem 6.5.** Given an absorbing Markov chain, let $v^{(N)}$ be the firing vector after $N$ steps of the hunger game process on the rerouted Markov chain associated with state $v$. Then the sequence $\{b_N\}$ defined by

$$b_N = \frac{N}{\sum_{u \in U} v_u^{(N)}} - 1,$$

where we define $b_N$ to be 0 when the denominator equals 0, converges to the expected absorption time $E_v T_U$, where there exists a constant $C$ such that $b_N$ is within distance $\frac{C}{N}$ of $E_v T_U$ for all $N$.

**Proof.** From the proof of Theorem 6.3, we see that the rerouted Markov chain has a unique stationary distribution $\pi$, and thus the normalized firing vector $\frac{1}{N} v^{(N)}$ converges to $\pi$ within distance $\frac{c}{N}$ for some constant $c$.

As $\sum_{u \in U} \pi_u$ is a positive constant, there exists a finite $M$ such that for all $N > M$ we have

$$\frac{c}{N} < \frac{1}{2} \sum_{u \in U} \pi_u.$$

As $b_N$ is finite, there exists a constant $C_1$ such that $b_N$ is within $\frac{C_1}{N}$ of $E_v T_U$ for all $N \leq M$. For $N > M$, it suffices to show the existence of a constant $C_2$ such that for all $N > M$,

$$\left| \frac{1}{\sum_{u \in U} \pi_u^{(N)}} - \frac{1}{\sum_{u \in U} \pi_u} \right| < \frac{C_2}{N}.$$

The furthest away $b_N$ can be from $E_v T_U$ results in the left hand side becoming

$$\frac{1}{\sum_{u \in U} \pi_u} - \frac{1}{\sum_{u \in U} \pi_u} = \frac{1}{N} \left( \frac{c}{N} + \sum_{u \in U} \pi_u \right) \frac{c}{\left( \sum_{u \in U} \pi_u \right)^2}.$$

As $N > M$, we have

$$\frac{c}{N} + \sum_{u \in U} \pi_u > \frac{1}{2} \sum_{u \in U} \pi_u > 0,$$
so in the worst case the deviation from $E_v T_U$ is less than

$$\frac{1}{N} \cdot \frac{2c}{(\sum_{u \in U} \nu_u)^2}.$$  

The second factor is a constant, which we will denote $C_2$, and thus taking $C = \max(C_1, C_2)$ yields that $b_N$ is at most $\frac{C}{N}$ away from $E_v T_U$ for all $N$.  

**Remark 6.6.** We can apply Theorem 6.5 to calculate the expected return time for state $v$ in an irreducible Markov chain. We do this in a similar method to Remark 6.4 by splitting vertex $v$ into vertices $v_0$ and $v_1$ where all outgoing edges of $v$ now emanate from $v_0$ and all inbound edges of $v$ now end at $v_1$. The resulting Markov chain is absorbing with unique absorbing state $v_1$, and thus one may use Theorem 6.5 on the rerouted Markov chain associated with state $v_0$ to approximate the expected absorption time. As $v_1$ is the only absorbing state, this calculates the expected hitting time from $v_0$ to $v_1$, which is the expected return time from $v$ in the original irreducible Markov chain.

**7 | RECURRENT STATES AND THE BASIN OF ATTRACTION**

In this section, we restrict our attention to irreducible finite Markov chains with no absorbing state. Additionally we assume that all transition probabilities are rational.

From the argument in Remark 3.3, all hunger vectors $h \in \mathbb{R}^n$ are preperiodic (i.e., eventually enter a cycle). We say that a vector that is part a cycle (i.e., returns to itself after a finite number of steps) is recurrent, and we define the basin of attraction to be the set of recurrent vectors on the hyperplane $Z = \{ h | h \cdot 1 = 0 \}$ (the hyperplane of hunger vectors with total hunger 0). The basin of attraction, being the set of recurrent vectors, is thus in some sense analogous to the critical group $\mathbb{Z}^n / \mathbb{Z}^n \Delta$ in the context of chip-firing.

**Example 7.1.** Consider the finite rational irreducible Markov chain with hunger matrix

$$H = \begin{bmatrix} -0.2 & 0.2 & 0 \\ 0.2 & -0.6 & 0.4 \\ 0.6 & 0.4 & -1 \end{bmatrix},$$

whose unique stationary distribution is $\left(\frac{11}{18}, \frac{5}{18}, \frac{2}{18}\right)$. Its basin of attraction is given in Figure 10. The basin is partitioned into four colors based on the firing order, up to cyclic shift. Red corresponds to the cyclic firing order 23112112311211211, that is, firing state 2, then state 3, then state 1, and so on. Green corresponds to 3121121231121121121, cyan corresponds to 231112123111211211, and violet corresponds to 31211212311121121. As the visit vector for each of these 18-step cycles must satisfy $vH = 0$, that is, be a multiple of the stationary distribution, we find each cyclic firing order consists of state 1 firing eleven times, state 2 firing five times, and state 3 firing two times.

**Remark 7.2.** Numerous observations can be made from Example 7.1 and Figure 10. First, the partitioning based on cyclic firing order appears to yield discrete congruent pieces; in Figure 10, the red and violet colors are split into congruent parallelograms, and the cyan and green colors are each split into congruent triangles. Each color corresponds to a cyclic firing order, and if this cycle has period $p$, then the set of vectors of this particular color can be partitioned into $p$ sets based on the firing order of
The basin of attraction for a finite rational irreducible Markov chain with three states. The basin lies in the hyperplane \( x + y + z = 0 \), so the third coordinate is omitted in order to plot it on the \( x, y \)-plane. For example, the point \((0.8, -0.1)\) in the rightmost violet region corresponds to \( h = (0.8, -0.1, -0.7) \). The basin is partitioned into four colors based on the order of firings; for example, the vectors in the violet portions of the basin all follow the same sequence of firings, up to cyclic shift, that is, choosing a starting index for the periodic firing sequence. So, the four colors correspond to four distinct cyclic firing orders.

their cycle, no longer up to cyclic shift. Each of these sub-pieces are mapped to each other in a cycle under the hunger game process, so clearly the sub-pieces of a given color are all congruent.

While evidently the entire basin of attraction need not be convex, each sub-piece of the basin of attraction, for example, the parallelograms and triangles of Figure 10, is convex; this holds for all finite rational Markov chains. To see why this is the case, notice that each sub-piece corresponds to a unique firing order that returns a hunger vector to itself. The set of vectors that will fire to a given state next under the hunger game process is given by the intersection of various linear inequalities, namely the inequalities that ensure this state had the highest hunger. Hence, the set of vectors that follow the firing order corresponding to a given sub-piece is given by the intersection of various linear inequalities, one set of inequalities for each firing step of the hunger game process. As the intersection of linear inequalities is convex, we find each sub-piece is convex.

Intuitively, the basin of attraction should be near the origin, for the hunger game naturally attempts to equilibrate the hunger vector so that each state has approximately the same hunger. The following result supports this intuition by demonstrating that the origin is always in the basin of attraction for any finite rational irreducible Markov chain.

**Proposition 7.3.** For any finite rational irreducible Markov chain, the zero vector \( 0 \) is a recurrent hunger state. Moreover, the number of steps needed to return to \( 0 \) is the least common denominator of the stationary probabilities of the Markov chain.

**Proof.** Let \( d \) be the least common denominator of the transition probabilities of the Markov chain. Viewing the Markov chain as a weighted digraph \( G \) with each vertex having outgoing weights summing to 1, we can multiply all weights by \( d \) so that all weights are now positive integers, and convert weighted edges into multiedges to obtain a directed multigraph \( G' \) in which every vertex has outdegree \( d \).
A finite irreducible Markov chain with rational transition probabilities has a unique stationary distribution \( \pi \), all of whose entries are rational. Let \( n \) be the least common denominator of \( \pi \), so that \( n\pi \) is the unique multiple of \( \pi \) that has nonnegative, mutually coprime integer entries, and let \( p = n\pi \). Since the sum of the entries of \( \pi \) is 1, the sum of the entries of \( p \) is \( n \). As \( \pi H = 0 \), we have \( p H = 0 \).

Starting at \( 0 \), consider the first vertex \( v \) that fires more than \( p_v \) times, and consider the hunger state just before this vertex fires for the \((p_v + 1)\)th time. Let the firing vector at this step be \( x \), so that \( x_u \) is the number of times state \( u \) received the chip, and thus \( xH \) is the current hunger state. Because \( x_v = p_v \) and because for all \( u \neq v \) state \( u \) has fired at most \( p_u \) times, \( w := p - x \) satisfies \( w_v = 0 \) and \( w_u \geq 0 \) for all \( u \neq v \). Firing vertices \( u \neq v \) can only (weakly) increase the hunger of \( v \), so \( (wH)_v \geq 0 \). To prove that last assertion more formally, recall that \( w_v = 0 \), so we can write \( w = \sum_{u \neq v} c_u e^{(u)} \) with \( c_u \geq 0 \) for all \( u \), where \( e^{(u)} \) is the elementary basis vector with a 1 at index \( u \) and 0 otherwise. Then we have

\[
(wH)_v = \sum_{u \neq v} c_u (e^{(u)}H)_v = \sum_{u \neq v} c_u H_{uv} = \sum_{u \neq v} c_u p_{uv} \geq 0,
\]

as claimed. This yields that

\[
(xH)_v = (pH - wH)_v = -(wH)_v \leq 0.
\]

Yet with a hunger state of \( xH \), vertex \( v \) received the chip, meaning it has the highest hunger. This means every state has hunger at most \((xH)_v \leq 0 \), so total hunger is at most 0. However, total hunger is invariant under the hunger game process; since the initial hunger vector was \( 0 \) with total hunger 0, total hunger must still be equal to 0. This equality case requires that \( xH = 0 \), implying \( x \) is a multiple of \( \pi \). Since \( x_v = p_v \) and \( p = n\pi \), we deduce \( x = p \).

Thus the hunger state is \( xH = 0 \), so \( 0 \) is recurrent. Moreover, the number of steps needed to return to \( 0 \) is the sum of the entries of \( x = p = n\pi \), which is \( n \), as claimed.

From empirical observations, we conjecture that the periods of all cycles in the hunger game for a given chain are equal, and specifically are equal to the \( n \) that was shown in Proposition 7.3 to be the period of the hunger vector \( 0 \).

**Conjecture 7.4.** The period of every cycle in the hunger game for a given finite rational irreducible Markov chain is the least common denominator of the stationary probabilities of the Markov chain.

Visually, Conjecture 7.4 applied to the Markov chain from Example 7.1 corresponds to the observation that each color in Figure 10 consists of the same number of congruent pieces, namely 18.

The basin of attraction in Figure 10 tiles the hyperplane \( Z \), as illustrated in Figure 11; in particular, one can observe that its concave boundaries fit complementarily with the opposite side of the basin of attraction. The following conjecture formalizes this observation and poses it for general Markov chains.

**Conjecture 7.5.** For a finite rational irreducible Markov chain, the basin of attraction tiles the hyperplane \( Z \) by translation, and the translation vectors that relate each tile with each other forms an \((n - 1)\)-dimensional sublattice in \( Z \) of \( \mathbb{R}^n \) with basis vectors \( H_1 - H_2, H_2 - H_3, \ldots, H_{n-1} - H_n \).

In fact, Conjecture 7.5 implies Conjecture 7.4, as seen in the following proposition.

**Proposition 7.6.** For a finite rational irreducible Markov chain, if the basin of attraction tiles the hyperplane \( Z \) with lattice of translation vectors generated by \( H_1 - H_2, \ldots, H_{n-1} - H_n \), then the
period of any cycle under the hunger game process is the least common denominator of the stationary probabilities of the Markov chain.

Proof. By using Proposition 7.3, it suffices to show that the tiling condition implies that all periods in the basin of attraction are the same. Suppose the lattice in the statement of the proposition is $L$. We say that two points in $\mathbb{Z}$ are equivalent mod $L$ if they differ by an element of $L$, and we write the set of equivalence classes as $\mathbb{Z}/L$. Now view the hunger game process, which naturally acts upon $\mathbb{Z}$, as acting instead on $\mathbb{Z}/L$, where the tiling condition implies that the basin of attraction can serve as a set of coset representatives for $\mathbb{Z}/L$. Initially this might seem like nonsense since the choice of which state to fire depends on inequalities relating the elements of the hunger vector and these inequalities get washed out when we mod out by $L$, but all rows of $H$ are equivalent mod $L$, that is, each row of $H$ has the same image under the projection map $\mathbb{Z} \to \mathbb{Z}/L$, so the choice of which state gets fired is moot; we may as well suppose that the $H_1$ coset is added at each stage, as if state 1 were firing repeatedly. Using the basin of attraction as a set of coset representatives and observing that a vector inside the basin must stay within the basin under the hunger game process, we find the hunger game process on $\mathbb{Z}/L$ is equivalent to the original hunger game on $\mathbb{Z}$ for any vector in the basin. If the zero vector $0$ has period $p$, then $pH_1 = 0$ in $\mathbb{Z}/L$. Then let $q$ be the minimum of the set of positive integers $q'$ such that $q'H_1 = 0$ in $\mathbb{Z}/L$. After $q$ steps of the hunger game process, every vector in the basin thus returns to itself, so we find $p = q$ and every vector in the basin of attraction must have period $p$, which completes the proof.

We provide a partial result toward Conjecture 7.5, proving that the basin of attraction translated under the stated lattice covers the hyperplane $\mathbb{Z}$; to prove Conjecture 7.5 and thus Conjecture 7.4, it remains to show this covering has no overlap.

**Proposition 7.7.** For a finite rational irreducible Markov chain, let $L \subset \mathbb{Z}$ be the sublattice in $\mathbb{Z}$ of $\mathbb{R}^n$ with basis vectors $H_1 - H_2, \ldots, H_{n-1} - H_n$. Then for any vector $h \in \mathbb{Z}$, there exists a $u \in L$ such that $h - u$ lies in the basin of attraction.
As $h$ is preperiodic, after some finite number of steps it will reach some vector in the basin, say after $t_0$ steps. Additionally, every vector in the basin stays in the basin under the hunger game process, so for all integers $t \geq t_0$, applying the hunger game process $t$ times to $h$ yields a vector in the basin. Let $p$ be the least common denominator of the stationary probabilities of the unique stationary distribution $\pi$ of our Markov chain, and fix $t$ to be the minimum nonnegative integer $\geq t_0$ such that $t$ is a multiple of $p$, say $kp$ for positive integer $k$. Then applying the hunger game process $t$ times to $h$ yields a vector $x \in Z$ in the basin of attraction, say after firing state $i$ a total of $v_i$ times for each $i$. Constructing vector $v$ from these values, as $x$ was reached after firing $h$ a total of $t$ times, we have $x = h + vH$ and $v \cdot 1 = t = kp$. Notice that $p\pi$ is the primitive integer vector in the direction of the stationary distribution, so we have $p\pi H = 0$. Hence $x = h + (v - kp\pi)H$, where $(v - kp\pi) \cdot 1 = 0$. As $L = \{wh | w \cdot 1 = 0\}$, letting $u = (kp\pi - v)H \in L$ yields $x = h - u$, as desired. □

8 | COMMENTS

The hunger game achieves high fidelity (i.e., low discrepancy) for the frequency with which a specified state occurs, but not for the frequency with which two specified states occur in succession; the rotor-router game achieves fidelity for the frequency with which two specified states occur in succession, but not for the frequency with which three specified states occur in succession. One could use a block-encoding trick (with new states encoding pairs of old states) to construct a simulation scheme that ensures fidelity for the frequency with which three specified (old) states occur in succession, but at a price: the resulting rotor-router network will have more rotors, and hence the quantities that ensure fidelity for the frequency with which three specified (old) states occur in succession, may allow the hunger game to outperform the rotor-router in convergence.
For example, consider the Markov chain with transition matrix
\[
\begin{pmatrix}
1 - q & q \\
q & 1 - q
\end{pmatrix}
\]
with \(q\) small, as shown in Figure 12. Then under rotor-routing the itinerary of the chip will be periodic with repeating pattern 1, 1, \ldots, 1, 2, 2, \ldots, 2, while under the hunger game the itinerary will simply alternate between 1 and 2, yielding lower discrepancy for the visit-frequencies of the two states.

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