Compressible fluid inside a linear oscillator

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Abstract
We use Feireisl-Lions theory to deduce the existence of weak solutions to a system describing the dynamics of a linear oscillator containing a Newtonian compressible fluid. The appropriate Navier-Stokes equation is considered on a domain whose movement has one degree of freedom. The equation is paired with the Newton law and we assume a no-slip boundary condition.

Keywords Fluid-structure interaction Compressible fluids Existence of weak solutions

Mathematics Subject Classification 35Q35 · 76N10

1 Introduction
It is known that the presence of a fluid inside a freely moving body has a tremendous effect on the dynamics of the system as it may produce a significant stabilizing effect at sufficiently large times. For example, the study of a system consisting of a freely moving rigid body with an interior cavity fully filled with a liquid has a long lasting history—here we refer to pioneering works by Stokes [16], Zhoukovski [18], Poincaré [14] and Sobolev [15]. Zhoukovski conjectured that ‘the motion of the system will eventually be rigid motions and, precisely, permanent rotation’.

The rigorous proof of the Zhoukovski conjecture for sufficiently general setting was given quite recently by Disser et al. [2]. It is worthwhile to mention that the compressible fluid inside the rotation body was also examined—see [9] and [13].

This stabilizing effect can be observed also in the case of a fluid inside a pendulum. Here we refer to works by Galdi and Mazzone [7] and Galdi et al. [8] who consider an incompressible fluid inside a pendulum. A system consisting of a compressible fluid inside a pendulum is currently under investigation—we refer to [6].

One may ask whether the stabilizing effect appears also in a system consisting of a linear oscillator filled by a fluid. This question is trivial for an incompressible fluid inside a linear oscillator—see Sect. 1.1. However, a system consisting of
a compressible fluid inside a linear oscillator possesses a non-trivial dynamics and thus it is of some interest. Up to our knowledge, this system has not been investigated yet.

The main aim of the paper is to prove the existence of solution to such system. A container (whose mass is \( m > 0 \)) with a cavity occupying a domain \( \Omega_t \subset \mathbb{R}^3 \) at a time instant \( t \), whose position is expressed by an unknown function \( b : [0, T) \mapsto \mathbb{R}^3 \), \( b(t) \times e_1 = 0, b(0) = 0 \) (i.e. \( \Omega_t = \Omega + b(t) \) for some domain \( \Omega \subset \mathbb{R}^3 \)), is joined by a spring to a point whose position is at \( f(t) \), where \( f : [0, T) \mapsto \mathbb{R}^3, f(t) \times e_1 = 0 \) is given – this means that the oscillations are forced as one end of the spring possesses a motion prescribed by \( f \).

The movement of the fluid inside a container is given by the Navier-Stokes equations, i.e.

\[
\partial_t (\rho w) + \text{div} (\rho w \otimes w) - \text{div} \mathbb{T} = 0 \quad \text{in} \quad \Omega_t,
\]

\[
\partial_t \rho + \text{div} (\rho w) = 0 \quad \text{in} \quad \Omega_t. 
\] (1.1)

Here \( w : [0, T) \times \Omega_t \mapsto \mathbb{R}^3 \) is the velocity of the fluid and \( \rho : [0, T) \times \Omega_t \mapsto \mathbb{R} \) is its density. We assume the no-slip boundary condition, i.e.

\[ w |_{\partial \Omega_t} = b. \]

The stress tensor \( \mathbb{T} \) is given as

\[ \mathbb{T} = \mathbb{T}(w, p) = \mathbb{S}(\nabla w) - \mathbb{I} p \] (1.2)

where

\[ \mathbb{S}(\nabla w) = \mu Dw + (\lambda + \mu) \mathbb{I} \text{div} w, \] (1.3)

\( \mathbb{I} \) is an identity \( 3 \times 3 \) matrix and \( Dw = \frac{1}{2} (\nabla w + (\nabla w)^T) \), \( \mu > 0, \lambda + \frac{2}{3} \mu \geq 0 \), and

\[ p = p(\rho) = a \rho^\gamma \] (1.4)

for some \( a, \gamma \in \mathbb{R}^+ \) specified later.

The force caused by the spring, whose given stiffness is \( k > 0 \), is

\[ \mathbf{F}(t) = -(b(t) - f(t)) k \]

This cause a change in the overall linear momentum of the container and the fluid in the \( e_1 \) direction. By the transport theorem [12, Theorem 1.22] we get (recall \( m > 0 \) denotes the given mass of the container and the movement is allowed only in \( e_1 \) direction)
Compressible fluid inside a linear oscillator

\[ \mathbf{F} = \frac{d}{dt} \left( \mathbf{b} m + \left( \int_{\Omega} \rho \mathbf{w} \, dx \cdot \mathbf{e}_1 \right) \mathbf{e}_1 \right) = \mathbf{b} m + \left( \int_{\Omega} \partial_t (\rho \mathbf{w}) + \text{div} (\rho \mathbf{w} \otimes \mathbf{w}) \, dx \cdot \mathbf{e}_1 \right) \mathbf{e}_1 \]

(1.5)

and we use (1.1) to deduce

\[-(\mathbf{b}(t) - \mathbf{f}(t)) k = \mathbf{F} = \int_{\partial \Omega} \text{div} \, \mathbf{T} \, dS_x + \mathbf{b}(t)m \]

yielding

\[-(\mathbf{b}(t) - \mathbf{f}(t)) k = \left( \int_{\partial \Omega} \mathbf{T} n dS_x \cdot \mathbf{e}_1 \right) \mathbf{e}_1 + \mathbf{b}(t)m. \]

Note that the only interaction between the fluid and the container is due to the boundary of \( \Omega \).

1.1 Rigid body case, incompressible case

First, let examine the case of the rigid body. It is sufficient to take \( \rho \equiv 1 \), \( \mathbf{w} \equiv \mathbf{b} \) and we deduce from (1.5) that

\[ \ddot{\mathbf{b}} + \frac{k}{M} \mathbf{b} = \frac{k}{M} \mathbf{f}(t) \]

(1.6)

where \( M \) is a total mass of the rigid body at the end of the spring (i.e. \( M = m + \int_{\Omega} 1 \, dx \)).

Let assume that \( \mathbf{f}(t) \cdot \mathbf{e}_1 = \sin(\omega t) \). As (1.6) is a second-order linear differential equation with constant coefficients, we may use a standard theory (Duhamel formula) to deduce that for \( \omega^2 \neq \frac{k}{M} \) the solutions have form

\[ \mathbf{b} \cdot \mathbf{e}_1 = c_1 \sin \left( \sqrt{\frac{k}{M}} t \right) + c_2 \cos \left( \sqrt{\frac{k}{M}} t \right) + \frac{k}{k - M \omega^2} \sin(\omega t) \]

where \( c_1, c_2 \in \mathbb{R} \) are arbitrary constants. However, there appears a resonance for \( \omega^2 = \frac{k}{M} \) – the solution in such case is

\[ \mathbf{b} \cdot \mathbf{e}_1 = -\frac{\omega}{2} t \cos(\omega t) + c_1 \sin(\omega t) + c_2 \cos(\omega t) \]

and one can see that such solution is unbounded.

Consider now an incompressible fluid. Let \( \rho \equiv 1 \) and \( \text{div} \mathbf{w} = 0 \) in (1.1). Due to the last constraint we have to consider a stress tensor \( \mathbf{T}(\mathbf{w}, \pi) \) where \( \pi \) is an unknown scalar function. In that case one solution is \( \mathbf{w} = \mathbf{b} \) and \( \nabla \pi = -\ddot{\mathbf{b}} \) and we end up with the same system as in the rigid body case. Roughly speaking, the force acting on the fluid has potential (which can be included in the unknown \( \pi \)) and the fluid is not moving relatively to the container.
However, this phenomena does not happen once the fluid is compressible. Indeed, the assumptions $w = b$ and $\nabla p(\rho) = -\dot{b}$ yield a contradiction—the last relation gives $\rho(t, x) = \rho(t)$ and thus $\partial_t \rho$ is nonzero function of time which cannot be compensated by $\text{div} (\rho w) = \nabla \rho w + \rho \text{div} w$ and thus $(1.1)_2$ is not fulfilled. Consequently, there must appear a non-trivial flux. Therefore, the oscillations are damped (the mechanical energy dissipates due to the viscosity) and one may ask whether is this damping strong enough to protect from resonance (as described above). The long time behavior of this system will be examined in forthcoming papers.

2 Formulation

It is convenient to rewrite (1.1) such that the resulting system is considered in a time-independent domain. Therefore we introduce a new variable $x = y - b(t)$ and a new velocity $u(t, x) = w(t, x + b(t))$. Then

$$
\partial_t (\rho u) + \text{div} (\rho u \otimes v) - \text{div} T = 0 \ \text{in} \ (0, T) \times \Omega
$$

$$
\partial_t \rho + \text{div} (\rho v) = 0 \ \text{in} \ (0, T) \times \Omega
$$

(2.1)

where $v = u - b$. Furthermore, the above system is complemented by the no-slip boundary condition

$$
u_{\partial \Omega} = b
$$

(2.2)

and

$$
-(b(t) - f(t)) k = \left( \int_{\partial \Omega} T n \ dS \cdot e_1 \right) e_1 + \dot{b}(t) m.
$$

(2.3)

Recall that $b = b e_1$ for some real function $b$. Here $T = (u, p)$ is given by (1.2), (1.3) and (1.4) and $n$ denotes the unit outward normal to the fluid boundary. We would like to point out that $T$ itself is a symmetric tensor and it depends on the symmetric part of $\nabla u$. Therefore, $T(u, p) = T(v, p)$ and $T : \nabla u = T : \nabla v$. These identities are used without any further notice.

We formally multiply (2.1)$_1$ by $u$ and we use the identity $\text{div} u = \text{div} v$ in order to obtain

---

1 Note that $(\text{div} (\rho u \otimes v))_j = \partial_j (\rho v u_j)$ where the summation convention is used.
We say, that

\[ 0 = \int_{\Omega} \partial_t (\rho \mathbf{u}) \cdot \mathbf{u} + \text{div} (\rho \mathbf{u} \otimes \mathbf{v}) \cdot \mathbf{u} - (\text{div} \mathbb{T}) \cdot \mathbf{u} \, dx \]

\[ = \int_{\Omega} \frac{1}{2} \partial_t (\rho |\mathbf{u}|^2) + \frac{1}{2} \partial_t \rho |\mathbf{u}|^2 - \frac{1}{2} \rho (\nabla |\mathbf{u}|^2) \cdot \mathbf{v} + \mathbb{S}(\nabla \mathbf{v}) : \nabla \mathbf{v} - a \rho \mathbf{v} \cdot \nabla v \, dx - \int_{\partial \Omega} (\mathbb{T} n) \cdot \mathbf{b} \, dS_x \]

\[ = \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2 \, dx + \int_{\Omega} \mathbb{S}(\nabla \mathbf{v}) : \nabla \mathbf{v} + a \nabla (\rho^\gamma) \cdot \mathbf{v} \, dx + (\mathbf{b}(t) - \mathbf{f}(t)) \cdot \mathbf{b}(t) + \mathbb{b}(t) \cdot \mathbf{b}(t)m \]

\[ = \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2 \, dx + \int_{\Omega} \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} - a \frac{\gamma}{\gamma - 1} \rho^{\gamma - 1} \text{div} (\rho \mathbf{v}) \, dx + \frac{k}{2} \frac{d}{dt} |\mathbf{b}(t)|^2 - k \mathbf{b}(t) \cdot \mathbf{f}(t) \]

and we deduce the energy equality which possesses the following form:

\[ \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2 \, dx + \frac{d}{dt} \int_{\Omega} \frac{a}{\gamma - 1} \rho^\gamma d\mathbf{x} + \int_{\Omega} \mathbb{S}(\nabla \mathbf{v}) : \nabla \mathbf{v} + \frac{m}{2} \frac{d}{dt} |\mathbf{b}(t)|^2 + \frac{k}{2} \frac{d}{dt} |\mathbf{b}(t)|^2 = \mathbf{k}(t) \cdot \mathbf{f}(t). \]

Moreover, we assume the following initial conditions

\[ \rho(0, \cdot) = \rho_0(\cdot), \quad \phi(0, \cdot) = (\rho \mathbf{u})_0(\cdot), \quad \mathbf{b}(0) = \mathbf{b}_0. \quad (2.4) \]

Now, we are ready to define the notion of a weak solution.

**Definition 2.1** We say, that \((\rho, \mathbf{u}) \in L^\infty((0, T), L^r(\Omega)) \times L^2((0, T), W^{1,2}(\Omega))\) is a weak solution to (2.1), (2.2), (2.3), and (2.4) if

- the continuity equation is fulfilled in a weak sense, i.e.

\[ \int_0^T \int_{\Omega} \rho \partial_t \phi + \rho \mathbf{v} \cdot (\nabla \phi) \, dx \, dt + \int_{\Omega} \rho \phi(0, \cdot) \, dx = \int_{\Omega} \phi(T, \cdot) \rho(T, \cdot) \, dx = 0 \]

for all \( \phi \in C^\infty([0, T] \times \overline{\Omega}) \),

- the momentum equation is fulfilled in a weak sense, i.e.

\[ \int_0^T \int_{\Omega} (\rho \mathbf{u} \otimes \mathbf{v}) : \nabla \phi - \mathbb{T} : \nabla \phi \, dx \, dt \]

\[ - \int_0^T \phi \partial_t \mathbf{b} \cdot (\mathbf{b}(t) - \mathbf{f}(t)) \, dt + \int_0^T \mathbb{b}(t) \cdot \mathbf{b}(t)m \, dt \]

\[ + \int_0^T \rho \mathbf{u}_0 \cdot \phi(0, \cdot) \, dx - \int_{\Omega} \phi(T, \cdot) \mathbf{u}(T, \cdot) \cdot \phi(T, \cdot) \, dx + m \phi(0) \cdot \mathbf{b}(0) \]

\[ - m \phi(T) \cdot \mathbf{b}(T) = 0 \]

holds for all \( \phi \in C^\infty([0, T] \times \overline{\Omega}) \) such that there exists a function \( \mathbf{b}_\phi \in C^\infty([0, T]) \) such that \( \phi \downarrow_{\partial \Omega} = \mathbf{b}_\phi \).
• \(\mathbf{u}|_{\partial \Omega} = \mathbf{b}(t)\) for some \(\mathbf{b} \in W^{1,\infty}(0, T), \mathbf{b} \times \mathbf{e}_1 = 0\).

• The energy inequality

\[
\left[ \int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2 \, dx + \int_{\Omega} \frac{\alpha}{\gamma - 1} \rho^\gamma \, dx + \frac{m}{2} |\mathbf{b}(\sigma)|^2 + \frac{k}{2} |\mathbf{b}(\sigma)|^2 \right]'_t + \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{u} \, dx \leq k \mathbf{b}(t) \cdot \mathbf{f}(t).
\]

holds for almost all \(t \in [0, T]\).

Note that this definition is sufficient in the sense that once \((\rho, \mathbf{u})\) is a sufficiently smooth weak solution, it solves (2.1) pointwisely. The main theorem of this paper follows.

**Theorem 2.2** Assume that \(\Omega\) is a domain of class \(C^{2+\nu}\), \(\nu > 0\) and \(\gamma > 3/2\). For any given \(T > 0\), \(\rho_0 \in L^r(\Omega), (\rho \mathbf{u})_0 \in L^{2(\gamma+1)/\gamma}(\Omega), \mathbf{b}_0 \in \mathbb{R}\) and \(\mathbf{f} \in L^1(0, T)\) there exists a weak solution to (2.1), (2.2), (2.3) and (2.4).

This theorem is proven by means of the nowadays standard Feireisl-Lions theory (see [5]). First, we introduce an approximate system possessing certain regularity properties which allows to prove the existence result—this is done in Sect. 3. Then, we tend with all the approximations to zero in order to reconstruct the original system. This is a content of Sect. 4.

### 3 Approximate system

There are two approximations. First, we introduce an artificial viscosity in the continuity equation (this is represented by \(\varepsilon > 0\)) and then we add an artificial pressure to the momentum equation (represented by \(\delta > 0\)). The system we have in mind is

\[
\begin{align*}
\partial_t (\rho \mathbf{u}) + \text{div} \,(\rho \mathbf{u}) & = \varepsilon \Delta \rho \quad \text{in} \, (0, T) \times \Omega \\
\partial_t (\rho \mathbf{u}) + \text{div} \,(\rho \mathbf{u} \otimes \mathbf{v}) - \text{div} \, \nabla \mathbf{v} + \nabla \rho + \delta \nabla \rho^\delta + \varepsilon (\nabla \rho \cdot \nabla) \mathbf{u} & = 0 \quad \text{in} \, (0, T) \times \Omega
\end{align*}
\]

and it is complemented with

\[
\mathbf{u}|_{\partial \Omega} = \mathbf{b}, \quad \frac{\partial \rho|_{\partial \Omega}}{\partial \mathbf{n}} = 0
\]

and

\[
-(\mathbf{b}(t) - \mathbf{f}(t))k = \left( \int_{\partial \Omega} \mathbf{n} \, dS_x \right) \mathbf{e}_1 + \mathbf{b}(t) \mathbf{m}.
\]

Moreover, we assume

\[
\rho(0, \cdot) = \rho_{0, \delta}(\cdot), \quad (\rho \mathbf{u})(0, \cdot) = (\rho \mathbf{u})_0(\cdot)
\]

where \(\rho_{0, \delta}\) is such that
\[ \rho_{0, \delta} \in C^{2+\nu}(\Omega), \ 0 \leq \rho \leq \rho_{0, \delta} \leq \bar{\rho} < \infty \] (3.4)

and

\[ \rho_{0, \delta} \to \rho_0 \ \text{strongly in} \ L'(\Omega) \ as \ \delta \to 0. \]

Of course, constants appearing in (3.4) are supposed to depend on \( \delta \).

**Theorem 3.1** Assume \( \Omega \) is of class \( C^{2+\nu} \) for some \( \nu > 0 \). There exists a weak solution\(^2\) \((\rho, u) \in L^\infty((0,T), L^8(\Omega)) \times L^2((0,T), W^{1,2}(\Omega))\) to (3.1), (3.2) and (3.3) such that there exists \( b \in W^{1,\infty}(0,T) \) fulfilling \( u\mid_{\partial\Omega} = b \). \( v := u - b \in L^2((0,T), W^{1,2}_0(\Omega)) \) and \( \rho \in L^2((0,T), W^{2,2}(\Omega)) \). Moreover, the solution satisfies the energy inequality in the form

\[
\frac{d}{dt} \int_\Omega \frac{1}{2} \rho |u|^2 \, dx + \frac{d}{dt} \int_\Omega \frac{a}{\gamma - 1} \rho^\gamma + \delta \rho^8 + \int_\Omega (\nabla v) : \nabla v \, dx + \int_\Omega \epsilon a \rho \nabla \rho^2 |\nabla \rho|^2 \, dx + \frac{m}{2} \partial_j |b(t)|^2 + \frac{k}{2} \partial_j |b(t)|^2 \leq k b(t) \cdot f(t)
\]

(3.5)

**Proof** Our aim is to solve (3.1) by means of the Galerkin approximations. We consider an \( n \)--dimensional space spanned by eigenvectors of the Laplace operator, i.e.,

\[ X_n = \{ \text{span} \{ \psi_j, j = 1,2, \ldots, n \} \}^3 \]

where

\[ -\Delta \psi_j = \lambda_j \psi_j \ \text{on} \ \Omega, \ \psi_j \mid_{\partial\Omega} = 0. \]

Let \( v_n \in C([0,T^*], X_n) \) be given (with \( T^* > 0 \) specified later). According to [5, Lemma 2.2] there is a mapping \( S : C([0,T^*], X_n) \to C([0,T^*], W^{1,2}(\Omega)) \) such that a function \( \rho_n = S(v_n) \) solves

\[ \partial_t \rho_n + \text{div} (\rho_n v_n) = \epsilon \Delta \rho_n, \ \frac{\partial \rho_n}{\partial n} \mid_{\partial\Omega} = 0, \ \rho_n(0,\cdot) = \rho_0. \]

[5, Lemma 2.2] also yields \( \| \frac{1}{\rho_n} \|_{L^\infty} \leq c \) because of the initial data and presumed smoothness of \( v_n \). Next, there is a solution \( b_n \in C^2([0,T^*]) \) of

\[ \ddot{b}_n(t) + k b_n(t) = k f(t) - \left( \int_{\partial\Omega} T(v_n, p(\rho_n)) n \, dS_x \cdot e_1 \right) e_1 \]

satisfying \( b_n(0) = 0, b_n(0) = b_0 \).

We define \( \mathcal{M}_\rho : X_n \to X_n \) as

\[^2\] Hereinafter we use an abbreviation \( \max\{8, \gamma\} = \beta. \]
\[
M_\rho(v) = w \iff \int_\Omega \rho v \cdot \psi \, dx = \int_\Omega w \cdot \psi \, dx \quad \forall \psi \in X_n.
\]

This mapping is invertible assuming \( \frac{1}{\rho} \in L^\infty(\Omega) \) and it satisfies
\[
\|M_\rho^{-1}\| \leq \left\| \frac{1}{\rho} \right\|_{L^\infty(\Omega)}
\]
\[
M_\rho^{-1} - M_\rho^{-1} = M_\rho^{-1}(M_\rho - M_\rho)M_\rho^{-1}
\]
(for more we refer to [5]). Finally, we construct a function \( w_\rho \in C([0, T^*], X_n) \) as
\[
w_\rho(t, \cdot) = M_{\rho_n(t)}^{-1}\left( q - \mathbf{b}_n + \int_0^t \mathcal{N}(\rho_n(s), \mathbf{b}_n(s), \mathbf{v}_n(s)) \, ds \right)
\]
where \( q \in X_n \) is such that \( \int_\Omega q \cdot \psi \, dx = \int_\Omega (\rho_0 \mathbf{v})_0 \cdot \psi \, dx \) for all \( \psi \in X_n \), \( \mathbf{b}_n' \in X_n \) is such that \( \int_\Omega b_n' \cdot \psi \, dx = \int_\Omega \mathbf{b}_0 \cdot \psi \, dx \) for all \( \psi \in X_n \), and \( \mathcal{N}(\rho_n(s), \mathbf{b}_n(s), \mathbf{v}_n(s)) \) is such function that
\[
\int_\Omega \mathcal{N}(\rho_n(s), \mathbf{b}_n(s), \mathbf{v}_n(s)) \cdot \psi \, dx
= \int_\Omega \left( \text{div} \mathbb{S}(\mathbf{v}_n) - \nabla p(\rho_n) - \delta(\nabla \varphi - \epsilon(\nabla \varphi) \cdot \mathbf{v}_n + \mathbf{b}_n - \text{div} (\rho_n(\mathbf{v}_n + \mathbf{b}_n) \otimes \mathbf{v}_n) \right) \cdot \psi \, dx.
\]
Note that \( w_\rho \) satisfies
\[
\int_\Omega \rho_n(w_\rho + \mathbf{b}_n) \cdot \psi \, dx - \int_\Omega (\rho_0 \mathbf{v})_0 \cdot \psi \, dx
= \int_0^t \int_\Omega \left( \text{div} \mathbb{S}(\mathbf{v}_n) - \nabla p(\rho_n) - \delta \nabla \varphi - \epsilon(\nabla \varphi) \cdot \mathbf{v}_n + \mathbf{b}_n - \text{div} (\rho_n(\mathbf{v}_n + \mathbf{b}_n) \otimes \mathbf{v}_n) \right) \cdot \psi \, dx \, ds
\]
(3.6)
for all \( t \in [0, T^*] \). Thus we constructed a mapping \( T : C([0, T^*], X_n) \rightarrow C([0, T^*], X_n) \) which mapps \( v_n \) to \( w_\rho \). This mapping is a contraction assuming \( T^* > 0 \) is sufficiently small. Indeed, \( S \) is Lipschitz and, moreover, for every \( c > 0 \) there is \( T^* \) such that
\[
\|S(v_n) - S(\tilde{v}_n)\|_{C([0,T],W^{1,2})} \leq c.
\]
Due to the Duhamel formula we deduce the same for \( \mathbf{b}_n \)—i.e. for any positive \( c \) there is \( T^* > 0 \) such that
\[
\|\mathbf{b}_n - \mathbf{b}_n\|_{C^2([0,T^*])} \leq c\|\tilde{v}_n - v_n\|_{C([0,T^*],X_n)}
\]
where \( \tilde{v}_n \) is assigned to \( \tilde{v}_n \) and \( \mathbf{b}_n \) is assigned to \( v_n \). Further, let \( \mathbf{w}_n = T(\tilde{v}_n) \) and \( \mathbf{w}_n = T(v_n) \). We have
and one can use the above inequalities to deduce that the mapping is contraction for $T^*$ sufficiently small. Therefore, the Banach fixed point theorem implies the existence of $\nu$, such that $T(\nu_n) = \nu$.

Further, we differentiate (3.6) with respect to $t$ (with $w_n$ replaced by the fixed point $\nu_n$) and we use $\nu_n(t)$ as a test function to obtain
\[
\frac{d}{dt} \left( \int_\Omega \frac{1}{2} \rho_n |u_n|^2 + \frac{a}{\gamma - 1} \rho_n^{\gamma} + \frac{\delta}{2} \rho_n^2 \, dx \right) + \int_\Omega (\nabla \nu_n)^T : \nabla v_n + (\epsilon a \gamma \rho_n^{\gamma - 2} + 8 \delta \rho_n^3) |\nabla \rho_n|^2 \, dx + \frac{k}{\epsilon} \frac{d}{dt} |b|^2 + \frac{m}{\epsilon} \frac{d}{dt} |b|^2 \leq k\theta \cdot f
\]  

(3.7)

where $u_n = \nu_n + b_n$. Thus, by the Young inequality
\[
\frac{m}{\epsilon} \frac{d}{dt} |b|^2 \leq k\theta + k|b|^2
\]

and we deduce by the Grönwall lemma that $|b| \in L^\infty(0, T)$ and
\[
\sup_t \|\rho_n\|_{L^\beta} \leq \sup_t \|\rho_n\|_{L^\beta} + \sup_t |\nabla \rho_n| + \sup_t |f_n| + \sup_t \|v_n\|_{L^2 W^{1,2}} \leq c
\]

where $\beta = \max\{\gamma, 8\}$. Since all norms on the finite-dimensional space are equivalent we deduce a uniform bound
\[
\sup_t \left( \|u_n(t)\|_{L^\infty} + \|\nabla u_n(t)\|_{L^\infty} \right) \leq c(n, T, \rho_0, (\rho u)_0)
\]

which allow to extend the solution to (3.6) to the whole given interval $(0, T)$.

We are going to pass with $n \to \infty$. We deduce from (3.7) the existence of a constant $c > 0$ independent of $n$ such that
\[
\begin{align*}
\sup_{t \in (0, T)} \|\rho_n\|_{L^\beta} &\leq c \\
\sup_{t \in (0, T)} \|\rho_n u_n\|^2_{L^1} &\leq c \\
\|v_n\|_{L^2 W^{1,2}} &\leq c \\
\epsilon \|\nabla \rho_n\|_{L^2 W^{1,2}} &\leq c \\
\|b_n\|_{W^{1,\infty}(0, T)} &\leq c
\end{align*}
\]  

(3.8)

and, consequently, also these convergences (up to a subsequence)

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3 Hereinafter we use the following notation: $\|\cdot\|_{L^p}$ is a norm in $L^p(\Omega)$, $\|\cdot\|_{W^{k,q}}$ is a norm in $W^{k,q}(\Omega)$, $\|\cdot\|_{L^p L^q}$ is norm in $L^p((0, T), L^q(\Omega))$ and $\|\cdot\|_{L^p W^{k,q}}$ is a norm in $L^p((0, T), W^{k,q}(\Omega))$. 

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The Aubin-Lions lemma together with (3.8) yields \( \rho_n \to \rho \) strongly in, say \( L^4((0, T) \times \Omega) \). This strong convergence then gives

\[
\rho_n u_n \to \rho u \quad \text{weakly* in } L^\infty((0, T), L^{2\beta(\beta+1)}(\Omega)).
\]

Further, [5, Lemma 2.4] yields the existence of \( r > 1 \) and \( q > 2 \) such that

\[
\|\partial_t \rho_n\|_{L^r} + \|\Delta \rho_n\|_{L^r} + \|\nabla \rho_n\|_{L^{2r}} \leq c
\]

with \( c \) independent of \( n \).

Next, Arzéla-Ascoli theorem yields that \( b_n \rightharpoonup b \) and we infer with the help of (3.8) that

\[ u_n \to u \quad \text{weakly in } L^2((0, T), W^{1,2}(\Omega)) \]

up to a subsequence. As a matter of fact, \( \rho \) and \( v \) solves

\[
\partial_t \rho + \text{div}(\rho v) = \varepsilon \Delta \rho
\]

almost everywhere.

We test continuity equations by \( \rho_n \) (\( \rho \) respectively) to obtain

\[
\|\rho_n(t)\|_{L^2}^2 + 2\varepsilon \int_0^t \|\nabla \rho_n\|_{L^2}^2 \, ds = -\int_0^t \int_\Omega \text{div} v_n |\rho_n|^2 \, dx \, dt + \|\rho_0\|_{L^2}^2
\]

and

\[
\|\rho(t)\|_{L^2}^2 + 2\varepsilon \int_0^t \|\nabla \rho\|_{L^2}^2 \, ds = -\int_0^t \int_\Omega \text{div} v |\rho|^2 \, dx \, dt + \|\rho_0\|_{L^2}^2.
\]

This with already mentioned convergences yields

\[
\nabla \rho_n \to \nabla \rho \quad \text{strongly in } L^2((0, T) \times \Omega).
\]

Therefore \((\nabla \rho_n \cdot \nabla u_n) \to (\nabla \rho \cdot \nabla) u\) in the sense of distributions and, consequently, we deduce that the limit functions \( \rho \) and \( u \) satisfy (3.1). More precisely, (3.1) is fulfilled almost everywhere (and also in the renormalized weak sense due to the regularity of \( \rho \) and \( v \) – we refer to [4, Lemma 10.13]) and

\[\footnote{Here \( \rightharpoonup \) denotes the uniform convergence.}\]

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\[ \int_0^T \int_{\Omega} \rho\mathbf{u} \cdot \partial_t \varphi + (\rho \mathbf{u} \otimes \mathbf{v}) : \nabla \varphi - \mathcal{S}(\nabla \mathbf{v}) : \nabla \varphi + (p(\rho) + \delta \rho^8) \text{div} \varphi \\
- \epsilon ((\nabla \rho_n \cdot \nabla) \mathbf{u}_n) \cdot \varphi \, dx \, dt \]
\[ - \int_0^T \varphi \cdot (\mathbf{b}(t) - \mathbf{f}(t)) \, dt + \int_0^T \partial_t \varphi \cdot \mathbf{b}(t) \, dt + \int_{\Omega} (\rho \mathbf{u})_0 \cdot \varphi(0, \cdot) \, dx \]
\[ - \int_{\Omega} \rho(T, \cdot) \mathbf{u}(T, \cdot) : \varphi(T, \cdot) \, dx + m \varphi(0) \cdot \mathbf{b}(0) - m \varphi(T) \cdot \mathbf{b}(T) = 0 \]

for all \( \varphi \in C^\infty([0, T] \times \overline{\Omega}) \) for which there is a function \( \mathbf{b}_\varphi \in C^\infty([0, T]) \), \( \mathbf{b}_\varphi \times \mathbf{e}_1 = 0 \) such that \( \varphi \mid_{\partial \Omega} = \mathbf{b}_\varphi \).

4 Vanishing of the approximations

In this Section we perform (successively) limits \( \epsilon \to 0 \) and \( \delta \to 0 \) in (3.1)–(3.3).

4.1 Limit \( \epsilon \to 0 \)

Let \( \rho_\epsilon \) and \( \mathbf{u}_\epsilon \) be the solution constructed in Theorem 3.1. We integrate (3.5) over \((0, t) \subset (0, T)\) where \( t \in (0, T) \) is arbitrary and we use the Gronwall inequality to obtain the following set of estimates

\[ \text{esssup}_{t \in (0, T)} \| \rho_\epsilon \|_{L^\infty} \leq c \]
\[ \| \mathbf{v}_\epsilon \|_{L^2W^{1,2}} \leq c \]
\[ \| \mathbf{b}_\epsilon \|_{W^{1,\infty}(0,T)} \leq c \]

Moreover, we integrate (3.1) over \( \Omega \) to get

\[ \int_{\Omega} \rho_\epsilon(t, \cdot) \, dx = \int_{\Omega} \rho_{0,\delta} \, dx \]

for all \( t \in (0, T) \). Note also that (4.1)\textsubscript{1,2,4}, (4.2) and [3, Lemma 3.2] yields

\[ \| \mathbf{u}_\epsilon \|_{L^2W^{1,2}} \leq c. \]

We multiply (3.1) by \( \rho \) to deduce

\[ \int_{\Omega} \frac{1}{2} \partial_t |\rho_\epsilon|^2 \, dx \]
\[ + \int_{\Omega} \epsilon |\nabla \rho_\epsilon|^2 \, dx = \int_{\Omega} \rho_\epsilon \mathbf{v}_\epsilon \cdot \nabla \rho_\epsilon \, dx \]

which together with (4.1) and \( \int_{\Omega} \rho_\epsilon \mathbf{v}_\epsilon \cdot \nabla \rho_\epsilon \, dx = -\frac{1}{2} \int_{\Omega} \text{div} \mathbf{v}_\epsilon |\rho_\epsilon|^2 \, dx \) yields
the constants appearing on the right hand side of the estimates (4.1) and (4.3) are independent of \( \varepsilon \). Next, we would like to deduce higher integrability of the density. We use \( \varphi = \mathcal{B}(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) \) as a test function in (3.1). Here \( \mathcal{B} \) stands for the Bogovski operator (see [1] or [11]) and \( (\varphi)_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \varphi \, dx \). We deduce that

\[
\int_0^T \int_\Omega (p(\varphi_\varepsilon) + \delta \theta^\varepsilon_0) \varphi_\varepsilon \, dx \, dt = \int_0^T \int_\Omega (p(\varphi_\varepsilon) + \delta \theta^\varepsilon_0(\varphi_\varepsilon)_{\Omega}) \, dx \, dt \\
- \int_\Omega (\varphi_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) (T, \cdot) - (\varphi \mathbf{u}_0 \cdot \mathcal{B}(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega})(\cdot)) \, dx \\
+ \int_0^T \int_\Omega \varphi_\varepsilon \partial_t B(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) \, dx \, dt \\
+ \int_0^T \int_\Omega \varphi_\varepsilon (\mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) \cdot \nabla B(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) \, dx \, dt + \int_0^T \int_\Omega \nabla \mathbf{v}_\varepsilon \cdot \nabla B(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) \, dx \, dt \\
+ \varepsilon \int_0^T \int_\Omega ((\nabla \varphi_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot B(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) \, dx \, dt.
\]

(4.4)

Recall that \( \mathcal{B} \) is a bounded linear operator which maps \( L^p(\Omega) \rightarrow W^{1,\rho}(\Omega) \) for all \( p \in (1, \infty) \). Since \( \varphi_\varepsilon \in L^\infty((0, T), L^\beta(\Omega)) \) uniformly in \( \varepsilon \) and \( (\varphi_\varepsilon)_{\Omega} \) is a constant due to (4.2), we are able to deduce that the right-hand side of (4.4) is bounded independently of \( \varepsilon \). Indeed, the only troublemaker is the term \( \int_0^T \int_\Omega \varphi_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t B(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) \, dx \, dt \). However, we use the linearity of \( \mathcal{B} \) together with the continuity equation in order to deduce

\[
\int_0^T \int_\Omega \varphi_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t B(\varphi_\varepsilon - (\varphi_\varepsilon)_{\Omega}) \, dx \, dt = \int_0^T \int_\Omega \varphi_\varepsilon \mathbf{u}_\varepsilon \cdot B(\partial_t \varphi_\varepsilon) \, dx \, dt = \\
- \int_0^T \int_\Omega \varphi_\varepsilon \mathbf{u}_\varepsilon \cdot B(\text{div}(\varphi_\varepsilon \mathbf{u}_\varepsilon)) \, dx \, dt \\
\leq \int_0^T \| \varphi_\varepsilon \mathbf{u}_\varepsilon \|_{L^2}^2 \, dt \leq \int_0^T \| \varphi_\varepsilon \|_{L^2}^2 \| \mathbf{u}_\varepsilon \|_{L^6}^2 \leq c
\]

with \( c \) independent of \( \varepsilon \). Consequently,

\[
\| \varphi_\varepsilon \|_{L^{p+1}((0, T) \times \Omega)} \leq c.
\]

(4.5)

Therefore, we may extract a subsequence of solutions such that

\[
\varphi_\varepsilon \rightharpoonup \varphi \text{ weakly* in } L^\infty((0, T), L^\beta(\Omega)) \\
\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \text{ weakly in } L^2((0, T), W^{1,2}(\Omega)) \\
\mathbf{b}_\varepsilon \rightrightarrows \mathbf{b}.
\]

Note also that we immediately obtain also \( \mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \) weakly in \( L^2((0, T), W^{1,2}(\Omega)) \).

Since \( L^\beta(\Omega) \) is compactly embedded in \( W^{-1,2}(\Omega) \) and, from (3.1), we deduce \( \partial_t \varphi \in L^\infty((0, T), W^{-1,2\beta/(\beta+1)}(\Omega)) \) we get (see [12, Lemma 6.2]).
\[
\rho_e \rightarrow \rho \quad \text{strongly} \quad \text{in} \ C((0, T), L^\beta_\text{weak}(\Omega)) \\
\rho_e \rightarrow \rho \quad \text{strongly} \quad \text{in} \ L^\beta((0, T), W^{-1,2}(\Omega))
\] (4.6)

for \( p \in (1, \infty) \) and, consequently
\[
\rho_e u_e \rightarrow \rho u \quad \text{weakly*} \quad \text{in} \ L^\infty((0, T), L^{2\beta/(\beta + 1)}(\Omega)).
\]

Equations (3.1) and (4.1) yield a uniform continuity of \( \rho_e u_e \) in \( W^{-1, (\beta + 1)/\beta}(\Omega) \) and we argue similarly as before to deduce
\[
\rho_e u_e \rightarrow \rho u \quad \text{strongly} \quad \text{in} \ C((0, T), L^{2\beta/(\beta + 1)}(\Omega)) \\
\rho_e u_e \rightarrow \rho u \quad \text{strongly} \quad \text{in} \ L^\beta((0, T), W^{-1,2}(\Omega)).
\] (4.7)

The deduced convergences then yield that \( \rho \) and \( u \) satisfies
\[
\int_0^T \int_\Omega \rho \partial_t \varphi + \rho u \cdot \nabla \varphi \, dx \, dt + \int_{\Omega} \rho_0, \varphi(0, \cdot) \, dx = 0
\]

for all \( \varphi \in C^\infty_c([0, T] \times \bar{\Omega}) \) and
\[
\int_0^T \int_\Omega \rho u \cdot \partial_t \varphi + \rho (u \otimes v) : \nabla \varphi - S(\nabla v) : \nabla \varphi + (p(\rho) + \delta \rho^8) \nabla \varphi \, dx \, dt

- \int_0^T \varphi \nabla \cdot (b(t) - f(t)) \, dt

+ \int_0^T \partial_t \varphi \nabla \cdot b(t) m \, dt + \int \rho(0) \cdot \varphi(0, \cdot) \, dx - \int \rho(T, \cdot) u(T, \cdot) \cdot \varphi(T, \cdot) \, dx

+ m \rho(0) \nabla \cdot b(0) - m \rho(T) \nabla \cdot b(T) = 0
\] (4.8)

for all \( \varphi \in C^\infty_c([0, T] \times \bar{\Omega}) \) such that there exists a function \( b_\varphi \in C^\infty_c([0, T]) \), \( b_\varphi \times e_1 = 0 \) fulfilling \( \varphi \nabla \cdot b_\varphi = b_\varphi \). Here we use the notation \( f(\rho) \) to denote the weak limit of \( f(\rho_e) \).

We are going to prove that
\[
\left( p(\rho) + \delta \rho^8 \right) = p(\rho) + \delta \rho^8.
\] (4.9)

The first step is to prove the following lemma

**Lemma 4.1** It holds that
\[
\int_0^T \int_\Omega \left( p(\rho_e) + \delta\rho^8_e - (\lambda + 2\mu) \nabla \varphi \right) \rho_e \, dx \, dt \rightarrow \int_0^T \int_\Omega \left( \left( p(\rho) + \delta \rho^8 \right) - (\lambda + 2\mu) \nabla \varphi \right) \rho \, dx \, dt
\]

for all \( \varphi \in C^\infty_c((0, T) \times \Omega) \).

**Proof** We take \( \varphi_j = \varphi A_j(\rho_e) \) as a test function in (3.1). Here \( A_j(\rho_e) \) is an inverse to \( \nabla \) and its Fourier symbol is
\[ A_j(\xi) = \frac{-i\xi_j}{|\xi|^2} \]

(roughly, \( A(\rho_\varepsilon) = \nabla \Delta^{-1}(\rho_\varepsilon) \)). Further, \( \varphi \) is a real-valued smooth function with compact support in \( (0, T) \times \Omega \). We also emphasize that \( A \) has a symmetric gradient, namely \( \partial_j A_j = \partial_j A_i \). Moreover, we assume \( \varphi \) is extended by 0 outside of \( \Omega \). First note that all the boundary terms disappear because this particular \( \varphi \) is compactly supported. So there appear only five terms (see Eq. (3.9)) which we denote by \( I_1, I_2, I_3, I_4 \) and \( I_5 \).

Let us start with \( I_4 \)

\[
I_4 = \int_0^T \int_\Omega (p(\rho_\varepsilon) + \delta \rho_\varepsilon^\delta) \text{div}(\varphi A(\rho_\varepsilon)) \, dxdt = \int_0^T \int_\Omega (p(\rho_\varepsilon) + \delta \rho_\varepsilon^\delta) \partial_t \varphi \, dxdt \\
+ \int_0^T \int_\Omega (p(\rho_\varepsilon) + \delta \rho_\varepsilon^\delta) \nabla \varphi \cdot A(\rho_\varepsilon) \, dxdt.
\]

Next, \( I_5 \) is handled as

\[
I_5 = \varepsilon \int_0^T \int_\Omega (\nabla \varphi \cdot \nabla) u_\varepsilon \varphi A(\rho_\varepsilon) \, dxdt.
\]

We use the continuity equation to adjust the first term:

\[
I_1 = \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t (\varphi A(\rho_\varepsilon)) \, dxdt = \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi A(\rho_\varepsilon) \, dxdt \\
+ \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi A(\partial_t \rho_\varepsilon) \, dxdt \\
= \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi A(\rho_\varepsilon) \, dxdt + \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi A(\text{div} (\chi \nabla \rho_\varepsilon)) \, dxdt \\
- \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi A(\text{div} \rho_\varepsilon \mathbf{v}_\varepsilon) \, dxdt.
\]

The term \( I_2 \) may be rewritten as follows

\[
I_2 = \int_0^T \int_\Omega \varrho_\varepsilon (\mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) \cdot \nabla \varphi \, dxdt = \int_0^T \int_\Omega \varrho_\varepsilon (\mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) : (\nabla \varphi \otimes A(\rho_\varepsilon)) \, dxdt \\
+ \int_0^T \int_\Omega \varrho_\varepsilon (\mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) : (\varphi \nabla A(\rho_\varepsilon)) \, dxdt.
\]

Finally,
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\[ I_3 = \int_0^T \int_\Omega \mathbb{S}(\nabla v_\varepsilon) : \nabla \varphi \, dx \, dt \]

\[ = \int_0^T \int_\Omega \left( \frac{\mu}{2} (\nabla v_\varepsilon + (\nabla v_\varepsilon)^T) + (\lambda + \mu) \text{div} \, v_\varepsilon \right) : (\nabla A(\rho_\varepsilon)) \varphi \, dx \, dt \]

\[ + \int_0^T \int_\Omega \mathbb{S}(\nabla v_\varepsilon) : (\nabla \varphi \otimes A(\rho_\varepsilon)) \, dx \, dt, \]

where the first integral on the right hand side is handled in following steps: first,

\[ \int_0^T \int_\Omega (\lambda + \mu) \text{div} \, v_\varepsilon \otimes : (\nabla A(\rho_\varepsilon)) \varphi \, dx \, dt = \int_0^T \int_\Omega (\lambda + \mu) \text{div} \, v_\varepsilon \text{div} \, A(\rho_\varepsilon) \varphi \, dx \, dt \]

\[ = \int_0^T \int_\Omega (\lambda + \mu) \text{div} \, v_\varepsilon \text{div} \rho_\varepsilon \varphi \, dx \, dt, \]

second, since \( \nabla A \) is a symmetric matrix we deduce with the help of integration by parts that

\[ \int_0^T \int_\Omega \left( \frac{\mu}{2} (\nabla v_\varepsilon + (\nabla v_\varepsilon)^T) \right) : (\nabla A(\rho_\varepsilon)) \varphi \, dx \, dt \]

\[ = \int_0^T \int_\Omega \mu \partial_{\varepsilon} v_\varepsilon \partial \partial \partial A(\rho_\varepsilon) \varphi \, dx \, dt \]

\[ = - \int_0^T \int_\Omega \mu \partial_{\varepsilon} v_\varepsilon \partial \partial A(\rho_\varepsilon) \partial \partial \varphi \, dx \, dt - \int_0^T \int_\Omega \mu \partial_{\varepsilon} v_\varepsilon \partial \partial A(\rho_\varepsilon) \partial \partial \varphi \, dx \, dt \]

\[ = \int_0^T \int_\Omega \mu \partial_{\varepsilon} v_\varepsilon \partial \partial A(\rho_\varepsilon) \varphi \, dx \, dt + \int_0^T \int_\Omega \mu v_\varepsilon \partial \partial A(\rho_\varepsilon) \partial \partial \varphi \, dx \, dt - \int_0^T \int_\Omega \mu \partial_{\varepsilon} v_\varepsilon \partial \partial A(\rho_\varepsilon) \partial \partial \varphi \, dx \, dt \]

Thus we arrive at
\[
\int_0^T \int_{\Omega} \varphi(p(\phi_\varepsilon) + \delta \phi_\varepsilon^8 - (\lambda + 2\mu) \text{div } \mathbf{v}_\varepsilon) \varrho \, dx \, dt = \int_0^T \int_{\Omega} (\lambda + \mu) \text{div } \mathbf{v}_\varepsilon (\nabla \varphi \cdot A(\phi_\varepsilon)) \, dx \, dt - \int_0^T \int_{\Omega} (p(\phi_\varepsilon) + \delta \phi_\varepsilon^8) (\nabla \varphi \cdot A(\phi_\varepsilon)) \, dx \, dt \\
+ \frac{\mu}{2} \int_0^T \int_{\Omega} \left( (\nabla \mathbf{v}_\varepsilon + (\nabla \mathbf{v}_\varepsilon)^T) \right) (\nabla \varphi \otimes A(\phi_\varepsilon)) \, dx \, dt - \mu \int_0^T \int_{\Omega} \mathbf{v}_\varepsilon \cdot (\nabla A(\phi_\varepsilon) \nabla \varphi) \, dx \, dt \\
+ \mu \int_0^T \int_{\Omega} \mathbf{v}_\varepsilon \cdot \nabla \varphi \mathbf{\varrho} \, dx \, dt \\
+ \varepsilon \int_0^T \int_{\Omega} (\nabla \phi_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot A(\phi_\varepsilon) \varphi \, dx \, dt - \varepsilon \int_0^T \int_{\Omega} \varphi \mathbf{\varrho}_{\varepsilon} \mathbf{u}_\varepsilon \cdot A \left( \text{div}(\chi_{\Omega} \nabla \phi_\varepsilon) \right) \, dx \, dt \\
- \int_0^T \int_{\Omega} \varphi_\varepsilon (\mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) : (\nabla \varphi \otimes A(\phi_\varepsilon)) \, dx \, dt \\
- \int_0^T \int_{\Omega} \varphi_\varepsilon \mathbf{u}_\varepsilon \cdot A(\phi_\varepsilon) \partial_\varepsilon \varphi \, dx \, dt + \int_0^T \int_{\Omega} \varphi \mathbf{\varrho}_{\varepsilon} (\partial_\varepsilon A J(\phi_\varepsilon) \mathbf{v}_\varepsilon) - \mathbf{v}_\varepsilon \partial_\varepsilon A J(\phi_\varepsilon) \, dx \, dt.
\]

(4.11)

Analogously, we use \( \varphi A(\rho) \) as a test function in (4.8) to deduce

\[
\int_0^T \int_{\Omega} \varphi \left( p(\rho) + \delta \phi_\varepsilon^8 - (\lambda + 2\mu) \text{div } \mathbf{v} \right) \varrho \, dx \, dt = \int_0^T \int_{\Omega} (\lambda + \mu) \text{div } \mathbf{v} (\nabla \varphi \cdot A(\rho)) \, dx \, dt - \int_0^T \int_{\Omega} (p(\rho) + \delta \phi_\varepsilon^8) (\nabla \varphi \cdot A(\rho)) \, dx \, dt \\
+ \frac{\mu}{2} \int_0^T \int_{\Omega} \left( (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \right) : (\nabla \varphi \otimes A(\rho)) \, dx \, dt - \mu \int_0^T \int_{\Omega} \mathbf{v} \cdot (\nabla A(\rho) \nabla \varphi) \, dx \, dt \\
+ \mu \int_0^T \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \mathbf{\varrho} \, dx \, dt \\
- \int_0^T \int_{\Omega} \varphi (\mathbf{v} \otimes \mathbf{u}) (\nabla \varphi \otimes A(\rho)) \, dx \, dt - \int_0^T \int_{\Omega} \varphi \mathbf{u} \cdot A(\rho) \partial_\varepsilon \varphi \, dx \, dt \\
+ \int_0^T \int_{\Omega} \varphi \mathbf{\varrho} \mathbf{u} \cdot (\partial_\varepsilon A J(\rho \mathbf{v}) - \mathbf{v} \partial_\varepsilon A J(\rho)) \, dx \, dt
\]

(4.12)

We compare (4.11) and (4.12) and we use the fact that the Mikhlin-Hörmander multipliers theorem [10, Theorem 5.2.7] together with already known convergencies yield

\[
A(\phi_\varepsilon) \to A(\rho) \text{ in } C([0, T] \times \overline{\Omega}) \\
\nabla A(\phi_\varepsilon) \to \nabla A(\rho) \text{ in } L^\infty((0, T), L^\beta_{\text{weak}}(\Omega)).
\]

Consequently
\[ \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \varphi((\rho_\varepsilon(p(\rho_\varepsilon) + \delta \rho_\varepsilon^8) - (\lambda + 2\mu) \text{div} \mathbf{u}_\varepsilon) - (p(\rho) + \delta \rho^8) - (\lambda + 2\mu) \text{div} \mathbf{u}) \, dx \, dt \]

\[ = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \varphi \rho \mathbf{u}_\varepsilon (\partial_t A_j(\rho_\varepsilon \mathbf{v}_\varepsilon) - \mathbf{v}_\varepsilon \partial_t A_j(\rho_\varepsilon)) \, dx \, dt - \int_0^T \int_\Omega \varphi \rho \mathbf{u}(\partial_t A_j(\rho v)) - \mathbf{v}_r \partial_t A_j(\rho) \, dx \, dt \]

and now it is sufficient to show that the right hand side is 0.

Using Div-Curl lemma ([17]) one obtains

\[ \mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^p(\mathbb{R}^3) \]

\[ \mathbf{w}_n \to \mathbf{w} \text{ weakly in } L^q(\mathbb{R}^3) \]

whenever \( r = \frac{p}{p+q} < 1 \). For details of a proof we refer to [5, Lemma 3.4]. Further, (4.6) and (4.7) yield

\[ \rho_\varepsilon \partial_t A_j(\rho_\varepsilon \mathbf{v}_\varepsilon) - \rho \mathbf{v}_\varepsilon \partial_t A_j(\rho_\varepsilon) \to \rho \partial_t A_j(\rho \mathbf{v}) - \rho \mathbf{v} \partial_t A_j(\rho) \text{ strongly in } L^\infty((0, T), L^{2\beta/(\beta+3)}(\Omega)) \]

and the claim follows due to the compact embedding of \( L^{2\beta/(\beta+3)}(\Omega) \) to \( W^{-1,2}(\Omega) \).

\[ \square \]

Recall that \( \rho_\varepsilon, \mathbf{v}_\varepsilon \) are regular enough to fulfill the renormalized continuity equation (see [4, Theorem 10.29])

\[ \partial_t r(\rho_\varepsilon) + \text{div} (r(\rho_\varepsilon) \mathbf{u}_\varepsilon) + (r'(\rho_\varepsilon) \rho_\varepsilon - r(\rho_\varepsilon)) \text{div} \mathbf{u}_\varepsilon - \varepsilon \Delta r(\rho_\varepsilon) \leq 0 \]

for every \( r \) convex and globally Lipschitz and the renormalized continuity equation is true also for \( \rho \) and \( \mathbf{u} \), namely

\[ \partial_t r(\rho) + \text{div} (r(\rho) \mathbf{u}) + (r'(\rho) \rho - r(\rho)) \text{div} \mathbf{u} = 0. \]

Taking \( r(\varepsilon) = \varepsilon \log \varepsilon \) yields

\[ \int_0^T \int_\Omega \rho_\varepsilon \text{div} \mathbf{u}_\varepsilon \, dx \, dt \leq \int_\Omega \rho_0 \log \rho_0 \, dx - \int_\Omega \varepsilon \log \rho_\varepsilon(\tau, \cdot) \log \rho_\varepsilon(\tau, \cdot) \, dx \]

and

\[ \int_0^T \int_\Omega \rho \text{div} \mathbf{u} \, dx \, dt = \int_\Omega \rho_0 \log \rho_0 \, dx - \int_\Omega \rho(\tau, \cdot) \log \rho(\tau, \cdot) \, dx \]

for every \( \tau \in [0, T] \). Let \( \Phi_n \) be a sequence of smooth compactly supported functions such that \( \Phi_n \to 1 \) in \( L^q(\Omega) \) for sufficiently large \( q \). We deduce
\[
\int_\Omega \rho(\tau, \cdot) \log \rho(\tau, \cdot) - \phi(\tau, \cdot) \log \phi(\tau, \cdot) \, dx \geq \int_0^\tau \int_\Omega \partial_x \nabla u_x - \rho \nabla \rho \, dx \, dt
\]
\[
= \int_0^\tau \int_\Omega \Phi_n \left( \partial_x \nabla u_x - \frac{1}{\lambda + 2\mu} (p(\phi_x) + \delta \phi^8_x) \phi_x + \frac{1}{\lambda + 2\mu} (p(\phi_x) + \delta \phi^8_x) \phi_x - \rho \nabla \rho \right) \, dx \, dt
\]
\[
+ \int_0^\tau \int_\Omega (1 - \Phi_n) (\partial_x \nabla u_x - \rho \nabla \rho) \, dx \, dt
\]
and we use (4.10) in order to get
\[
\int_\Omega \rho(\tau, \cdot) \log \rho(\tau, \cdot) - \phi(\tau, \cdot) \log \phi(\tau, \cdot) \, dx
\]
\[
\geq \frac{1}{\lambda + 2\mu} \int_0^\tau \int_\Omega \left( (p(\rho) - \delta \phi^8) \phi - (p(\rho) - \delta \phi^8) \phi \right) \Phi_n \, dx \, dt + \eta(n) \quad \text{(4.13)}
\]
\[
\geq 0 + \eta(n).
\]
where the last inequality is true because of the monotonicity of the mapping \( \rho \mapsto p(\rho) + \delta \phi^8 \). Since \( \eta(n) \) can be made arbitrarily small and since the mapping \( \rho \mapsto \rho \log \rho \) is convex, we get from (4.13)
\[
\phi_x \rightarrow \rho \quad \text{almost everywhere in } (0, T) \times \Omega
\]
which, together with (4.5) yields (4.9).

Lastly, since (3.5) yields
\[
- \int_0^\tau \left( \int_\Omega \frac{1}{2} \phi_x |u_x|^2 + \frac{a}{\gamma - 1} \phi_x^\gamma + \frac{\delta}{\gamma} \phi^8_x \, dx + \frac{m}{2} |b_x(t)|^2 + \frac{k}{2} |b(t)|^2 \right) \phi \, dt
\]
\[
+ \int_0^\tau \left( \int_\Omega \mathbb{S} \nabla v_x : \nabla v_x + \epsilon \phi x |\nabla \phi_x| \, dx - k \nabla \phi_x \nabla f(t) \right) \phi(t) \, dt
\]
\[
+ \left( \int_\Omega \frac{1}{2} \phi_x |u_x|^2 + \frac{a}{\gamma - 1} \phi_x^\gamma + \frac{\delta}{\gamma} \phi^8_x \, dx \right) \phi(0) \leq 0
\]
for all \( \phi \in C_c^c([0, T]), \phi \geq 0 \), we use the already known convergences to deduce that
\[
\frac{d}{dt} \int_\Omega \frac{1}{2} \phi |u|^2 \, dx + \frac{d}{dt} \int_\Omega \frac{a}{\gamma - 1} \phi^\gamma + \frac{\delta}{\gamma} \phi^8 \, dx + \int_\Omega \mathbb{S} \nabla v : \nabla v \, dx + \frac{m}{2} \frac{d}{dt} |b(t)|^2
\]
\[
+ \frac{k}{2} \frac{d}{dt} |b(t)|^2 \leq k \nabla f(t) \quad \text{(4.14)}
\]

4.2 Limit \( \delta \rightarrow 0 \)

Let \( \phi_x, \ u_x(= v_x + b_x) \) be a solution constructed in the previous chapter corresponding to some positive parameter \( \delta > 0 \). We send \( \delta \) to 0. From (4.14) we deduce

\( \delta \) Springer
where the last one follows from the trace theorem and Arzéla-Ascoli theorem.

The density satisfies the following estimate

\[
\int_0^T \int_{\Omega} \phi^\gamma \, dx \, dt \leq c
\]  

(4.16)

for some \( \theta > 0 \) and with \( c \) independent of \( \delta \). Indeed, we take \( \phi = B(\phi^\delta - (\phi^\delta)_{\Omega}) \) for some \( \theta > 0 \) specified later as a test function in (4.8). We obtain

\[
\int_0^T \int_{\Omega} (p(\phi^\delta) + \delta \phi^\delta_\delta) \phi^\delta_\delta \, dx \, dt = \int_0^T \int_{\Omega} (p(\phi^\delta) + \delta \phi^\delta_\delta)(\phi^\delta_\delta)_{\Omega} \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} \phi^\delta_\delta \cdot B(\delta(\phi^\delta - (\phi^\delta)_{\Omega})) \, dx \, dt - \left[ \int_0^T \int_{\Omega} \phi^\delta \cdot B(\phi^\delta - (\phi^\delta)_{\Omega}) \, dx \, dt \right]_{t=0}^{t=T}
\]

\[
+ \int_0^T \int_{\Omega} (\phi^\delta_\delta \, \nabla \phi^\delta \cdot \nabla \phi^\delta) \, dx \, dt + \int_0^T \int_{\Omega} S(\nabla \phi^\delta) \cdot \nabla B(\phi^\delta - (\phi^\delta)_{\Omega}) \, dx \, dt.
\]

Recall that \( \phi^\delta \in L^\gamma/\theta(\Omega) \) and that \( B : L^\gamma(\Omega) \rightarrow W^{1,\gamma}(\Omega) \). With this and (4.15) at hand one can easily deduce estimates of all terms on the right hand side with one exception – the second term. There one has to use the continuity equation in order to get

\[
\int_0^T \int_{\Omega} \phi^\delta_\delta \cdot B(\delta(\phi^\delta - (\phi^\delta)_{\Omega})) \, dx \, dt
\]

\[
= \int_0^T \int_{\Omega} \phi^\delta_\delta \cdot B(- \text{div} (\phi^\delta_\delta \nabla \phi^\delta) - (\theta - 1) \phi^\delta \text{div} \nabla \phi^\delta + (\text{div} (\phi^\delta_\delta \nabla \phi^\delta) + (\theta - 1) \phi^\delta_\delta \text{div} \phi^\delta_\delta)_{\Omega}) \, dx \, dt
\]

\[
= - \int_0^T \int_{\Omega} \phi^\delta_\delta \cdot B((\theta - 1) \phi^\delta_\delta \text{div} \nabla \phi^\delta) - ((\theta - 1) \phi^\delta_\delta \text{div} \phi^\delta_\delta)_{\Omega} \, dx \, dt.
\]

Estimates (4.15) yields \( \phi^\delta_\delta \text{div} \nabla \phi^\delta \in L^2((0, T), L^{6/4(\gamma+3)}(\Omega)) \) and thus it is enough to choose \( \theta \) sufficiently small to get \( \phi^\delta_\delta \in L^\infty((0, T), L^{6/2(\gamma-3)}(\Omega)) \) and we get the control of the first term on the right hand side. The control of the second term follows easily as \( \phi^\delta \text{div} \nabla \phi^\delta \in L^2((0, T), L^{2(\gamma)}/(\gamma+\theta)(\Omega)) \) and thus (with the help of Sobolev embedding theorem)

\[
B((\theta - 1) \phi^\delta_\delta \text{div} \nabla \phi^\delta - ((\theta - 1) \phi^\delta_\delta \text{div} \phi^\delta_\delta)_{\Omega}) \in L^2((0, T), L^{6/2(\gamma+3)}(\Omega)).
\]
Now, assuming \( \theta \) is sufficiently small, we get the boundedness of the second term on the right hand side and we conclude (4.16).

Similarly as before we claim that
\[
\rho_\delta \to \rho \quad \text{strongly in } L^p((0, T), W^{-1,2}(\Omega))
\]
\[
\rho_\delta u_\delta \to \rho u \quad \text{strongly in } L^p((0, T), W^{-1,2}(\Omega))
\]
for all \( p \in (1, \infty) \). Here we used that \( L^2/(r+1)(\Omega) \) is compactly embedded into \( W^{-1,2}(\Omega) \) assuming \( \gamma \geq \frac{3}{2} \).

As a matter of fact, \( \rho \) and \( u \) satisfy
\[
\int_0^T \int_\Omega \nabla \varphi \cdot (\nabla \mathbf{v} \otimes \mathbf{u} + D\varphi + \bar{p}(\rho) \div \mathbf{v}) \, dx \, dt \quad - \int_0^T \varphi \big|_{\partial \Omega} \cdot (\mathbf{b}(t) - f(t))k \, dt \\
+ \int_0^T \partial_t \varphi \big|_{\partial \Omega} \cdot \mathbf{b}(t)m \, dt + m\varphi(0) \big|_{\partial \Omega} \cdot \mathbf{b}(0) - m\varphi(T) \big|_{\partial \Omega} \cdot \mathbf{b}(T) \\
+ \int_\Omega (\rho u)_0 \cdot \varphi(0, \cdot) \, dx - \int_\Omega \rho(T, \cdot) \mathbf{u}(T, \cdot) \cdot \varphi(T, \cdot) \, dx = 0.
\]

It remains to proof that \( \rho_\delta \to \rho \) almost everywhere as then \( \bar{p}(\rho) = p(\rho) \) and also the energy inequality can be deduced similarly as in the previous chapter.

Let \( k \in \mathbb{N} \setminus \{1\} \). We define
\[
T_k(z) = kT_1\left(\frac{z}{k}\right), \quad z \in \mathbb{R}
\]
where \( T_1 \) is a smooth concave function satisfying
\[
T_1(z) = \begin{cases} 
  z & \text{for } z \leq 1 \\
  2 & \text{for } z \geq 3.
\end{cases}
\]

In what follows, we will use that the following equality
\[
\lim_{\delta \to 0} \int_0^T \int_\Omega \varphi\left(p(\rho_\delta) - (\lambda + 2\mu) \div v_\delta\right) T_k(\rho_\delta) \, dx \, dt \\
= \int_0^T \int_\Omega \varphi\left(\bar{p}(\rho) - (\lambda + 2\mu) \div \mathbf{v}\right) T_k(\rho) \, dx \, dt
\]
holds for all smooth \( \varphi \) with compact support in \( (0, T) \times \Omega \). The proof of this equality is similar to the proof of (4.10) with only exception: one has to use a test function \( \Phi_j = \varphi A_j(T_k(\rho_\delta)) \) instead of \( \Phi_j = \varphi A_j(\rho_\delta) \). Nevertheless, as all arguments are similar, we neglect the proof.

Further, \( \rho, v \) solves the renormalized equation of continuity, namely
\[
\partial_t r(\rho) + \div (r(\rho)v) + (r'(\rho)\rho - r(\rho)) \div v = 0.
\]
in the weak sense for any \( r \in C^1(\mathbb{R}) \) with \( r'(z) = 0 \) for \( z \) sufficiently large.

Indeed, first note that \( (p(\rho) - (\lambda + 2\mu) \text{div} \psi_k)T_k(\rho) \) and \( \left( \frac{\rho(\rho)}{\rho} - (\lambda + 2\mu \text{div} \psi) \right) T_k(\rho) \) belong to \( L^q((0, T) \times \Omega) \) for some \( q > 1 \) and thus we may pass with \( \varphi \to 1 \) in \( L^q((0, T) \times \Omega) \), \( q' = \frac{q}{q-1} \), in (4.17) to deduce

\[
\lim_\delta \int_0^T \int_\Omega \left( p(\rho) - (\lambda + 2\mu \text{div} \psi_k) \right) T_k(\rho) \, dx \, dt = \int_0^T \int_\Omega \left( \frac{\rho(\rho)}{\rho} - (\lambda + 2\mu \text{div} \psi) \right) T_k(\rho) \, dx \, dt
\]

since integrands on both sides are uniformly integrable provided \( k \) is fixed. Next, we have

\[
\int_0^T \int_\Omega \left( \varphi_k' T_k(\rho) - \varphi_k^2 T_k(\rho) \right) \, dx \, dt = \int_0^T \int_\Omega \left( \varphi_k' T_k(\rho) - \varphi_k^2 T_k(\rho) \right) \, dx \, dt + \int_0^T \int_\Omega \varphi_k^2 T_k(\rho) - \varphi_k^2 T_k(\rho) \, dx \, dt
\]

yielding

\[
\lim_{\delta \to 0} \int_0^T \int_\Omega \left( \varphi_k' T_k(\rho) - \varphi_k^2 T_k(\rho) \right) \, dx \, dt \geq \lim_{\delta \to 0} \sup \int_0^T \int_\Omega \left( \varphi_k' - \rho \right) \left( T_k(\rho) - T_k(\rho) \right) \, dx \, dt
\]

\[
\geq \lim_{\delta \to 0} \sup \int_0^T \int_\Omega \left| T_k(\rho) - T_k(\rho) \right|^{r+1} \, dx \, dt.
\]

Further,

\[
\lim_{\delta \to 0} \int_0^T \int_\Omega \text{div} \psi_k T_k(\rho) - \text{div} \psi_k T_k(\rho) \, dx \, dt
\]

\[
= \lim_{\delta \to 0} \int_0^T \int_\Omega \left( \frac{T_k(\rho) - T_k(\rho) + T_k(\rho) - T_k(\rho)}{\rho} \right) \text{div} \psi_k \, dx \, dt
\]

\[
\leq 2 \lim_{\delta \to 0} \| \text{div} \psi_k \|_{L^2((0, T) \times \Omega)} \| T_k(\rho) - T_k(\rho) \|_{L^2((0, T) \times \Omega)}.
\]

With help of (4.19) we get

\[
0 = \lim_{\delta \to 0} \int_0^T \int_\Omega \left( \varphi_k' T_k(\rho) - \varphi_k^2 T_k(\rho) - (\lambda + 2\mu) \left( \text{div} \psi_k T_k(\rho) - \text{div} \psi_k T_k(\rho) \right) \right) \, dx \, dt
\]

\[
\geq \lim_{\delta \to 0} \sup \left( \| T_k(\rho) - T_k(\rho) \|_{L^{r+1}((0, T) \times \Omega)}^{r+1} - 2 \| \text{div} \psi_k \|_{L^2((0, T) \times \Omega)} \| T_k(\rho) - T_k(\rho) \|_{L^2((0, T) \times \Omega)} \right).
\]

This yields

\[
\| T_k(\rho) - T_k(\rho) \|_{L^{r+1}((0, T) \times \Omega)} \leq c
\]

Using a smoothing kernel we may deduce the validity of equation

\[
\partial_t b(\overline{T_k(\rho)}) - \text{div} \left( b(\overline{T_k(\rho)}) \psi \right) + \left( b'(\overline{T_k(\rho)}) \overline{T_k(\rho)} - b(\overline{T_k(\rho)}) \psi \right) \text{div} \psi = b(\overline{T_k(\rho)}) \left( (T_k(\rho) - T_k(\rho)) \psi \text{div} \psi \right)
\]

(4.20)
in the sense of distributions. Next we send \( k \) to \( \infty \). First, we have
\[
T_k(\rho) \to \rho \quad \text{strongly in } L^p((0, T) \times \Omega), \quad 1 \leq p < \gamma.
\]
This follows as
\[
\|T_k(\rho) - \rho\|_{L^p((0, T) \times \Omega)} \leq \liminf_{\delta \to 0} \|T_k(\rho_\delta) - \rho_\delta\|_{L^p((0, T) \times \Omega)} \leq \liminf_{\delta \to 0} 2^p k^{p - \gamma} \|\rho_\delta\|_{L^p((0, T) \times \Omega)}^p \to 0.
\]
for \( k \to \infty \).

Recall that \( b \) is such that \( b' \) has compact support. Thus there is \( M > 0 \) such that \( b'(z) \equiv 0 \) for \( z \geq M \) and we have
\[
\int_0^T \int_\Omega \left| b(T_k(\rho))(T_k(\rho) - T_k(\rho) \rho \div v) \right| \, dx \, dt \leq \sup_{0 \leq z \leq M} |b'(z)| \| v \|_{L^2 L^\gamma} \| T_k(\rho_\delta) - T_k(\rho_\delta) - T_k(\rho_\delta) \|_{L^2(Q_{k,m})},
\]
where \( Q_{k,m} = \{(t, x) \in (0, T) \times \Omega, \ T_k(\rho) \leq M \} \). The Hölder inequality yields
\[
\|T_k'(\rho_\delta) - T_k(\rho_\delta)\|_{L^{2+\gamma}(Q_{k,m})} \leq \|T_k'(\rho_\delta) - T_k(\rho_\delta)\|_{L^{1+\gamma}(0, T) \times \Omega} \|T_k'(\rho_\delta) - T_k(\rho_\delta)\|_{L^{1+\gamma}(Q_{k,m})}.
\]
Further, we have
\[
\|T_k'(\rho_\delta) - T_k(\rho_\delta)\|_{L^{1+\gamma}(0, T) \times \Omega} \leq 2^\gamma k^{1 - \gamma} \sup_{\delta} \|T_k(\rho_\delta)\|_{L^{1+\gamma}(0, T) \times \Omega}^\gamma
\]
and, since \( T_k'(\rho_\delta) \leq T_k(\rho_\delta) \),
\[
\|T_k'(\rho_\delta) - T_k(\rho_\delta)\|_{L^{1+\gamma}(Q_{k,m})} \leq 2 \left( \|T_k(\rho_\delta) - T_k(\rho)\|_{L^{1+\gamma}(0, T) \times \Omega} + \|T_k(\rho)\|_{L^{1+\gamma}(Q_{k,m})} \right)
\]
\[
\leq 2 \left( \|T_k(\rho_\delta) - T_k(\rho)\|_{L^{1+\gamma}(0, T) \times \Omega} + \|T_k(\rho) - T_k(\rho)\|_{L^{1+\gamma}(Q_{k,m})} + \|T_k(\rho)\|_{L^{1+\gamma}(Q_{k,m})} \right) \leq c
\]
The renormalize continuity equation (4.18) follows from (4.20), (4.21), (4.22) and (4.23).

To conclude the proof we introduce
\[
L_k = \begin{cases} 
  z \log z & \text{for } 0 \leq z \leq k, \\
  z \log k + z \int_k^z \frac{1}{s} T_k(s) \, ds & \text{for } z \geq k.
\end{cases}
\]
We use this as \( r \) in the renormalized equation (4.18) to conclude
\[
\partial_t L_k(\rho_\delta) + \div (L_k(\rho_\delta) v_\delta) + T_k(\rho_\delta) \div v_\delta = 0
\]
and
\[
\partial_t L_k(\rho) + \div (L_k(\rho) v) + T_k(\rho) \div v = 0.
\]
Similarly to the previous section we deduce (compare with (4.13))
\[
\int_{\Omega} L_k(\rho(\tau, \cdot)) - \bar{L}_k(\rho(\tau, \cdot)) \, dx \geq 0
\]
for almost all \( \tau \in [0, T] \). We send \( k \to \infty \) to deduce that \( \rho \log \rho = \bar{\rho} \log \bar{\rho} \) which yields \( \rho_\delta \to \rho \) almost everywhere. The proof of Theorem 2.2 is finished.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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