CONSTRUCTION OF WILLMORE TWO-SPHERES VIA HARMONIC MAPS INTO $SO^+(1,N+3)/(SO^+(1,1) \times SO(N+2))$

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Abstract

This paper aims to provide a description of totally isotropic Willmore two-spheres and their adjoint transforms. We first recall the isotropic harmonic maps which are introduced by Hélein, Xia-Shen and Ma for the study of Willmore surfaces. Then we derive a description of the normalized potential (some Lie algebra valued meromorphic 1-forms) of totally isotropic Willmore two-spheres in terms of the isotropic harmonic maps. In particular, the corresponding isotropic harmonic maps are of finite uniton type. The proof also contains a concrete way to construct examples of totally isotropic Willmore two-spheres and their adjoint transforms. As illustrations, two kinds of examples are obtained this way.

Keywords: Willmore surfaces; Isotropic Willmore two-spheres; DPW method; adjoint transform; isotropic harmonic maps.

MSC(2010): 53A30; 58E20; 53C43; 53C35

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1. Introduction

Totally isotropic surfaces was first introduced by Calabi [7] in the study of the global geometry of minimal two-spheres. In the study of Willmore two-spheres, totally isotropic surfaces also play an important role [11, 19, 20]. Recently, Dorfmeister and Wang used the DPW method for the conformal Gauss map to study Willmore surfaces [9]. They obtained the first new Willmore two-sphere in \( S^6 \), which admits no dual surface. Along this way, Wang [25] gives a description of all totally isotropic Willmore two-spheres in \( S^6 \), in terms of the normalized potentials of their conformal Gauss maps.

Although Willmore two-spheres may have no dual surfaces, they do admit another kind of transforms, i.e., adjoint transforms introduced by Ma [15]. The main idea of the adjoint transforms is to find out another Willmore surface located in the mean curvature spheres of the original one and having the same complex coordinates. A somewhat surprising result derived by Ma states that a Willmore surface and its adjoint surface in \( S^{n+2} \) also provide a harmonic map into the Grassmannian \( G_{r_1,1}(\mathbb{R}^{n+3}_{1}) \), which was first discovered by Hélein [13, 14] and generalized by Xia and Shen [27]. This kind of harmonic maps also appeared naturally when Brander and Wang considered the Björling problems of Willmore surfaces [1].

However, such harmonic maps will have singularities in general (See Section 6 for example). So it is very hard to use them to discuss the global geometry of Willmore surfaces. Due to this reason, in [9] Dorfmeister and Wang mainly dealt with the conformal Gauss maps of Willmore surfaces, which are another kind of harmonic maps related to Willmore surfaces globally.

Although it exists locally in general, the harmonic map given by a Willmore surface and its adjoint surface is very simple and provides the Willmore surface and its adjoint surface immediately. So it is natural to use this harmonic map to describe totally isotropic Willmore two-spheres. In particular, this provides a more simple way to derive examples of totally isotropic Willmore two-spheres and their adjoint surfaces, in contrast to using the conformal Gauss maps of Willmore surfaces [25].

In this paper, we will give a characterization of the harmonic map given by a totally isotropic Willmore two-sphere and its adjoint surface. In particular, such harmonic maps are very simple so that one can obtain a concrete algorithm to construct all of them, which is the main topic of this paper. As illustrations, we also derive two kinds of examples.

The main idea of our work are based on the DPW method for harmonic maps [8, 13] and the description of harmonic maps of finite uniton type [4, 12, 10]. The DPW method [8] gives a way to produce harmonic maps in terms some meromorphic 1-forms, i.e., normalized potentials. The work of [4, 12, 10] states that harmonic maps of finite uniton type can be derived in a more convenient way, that is, the normalized potentials must take values in some nilpotent Lie algebra. This permits a way to derive such harmonic maps in an explicit way. On the other hand, due to [22, 4, 12, 10], harmonic maps from two-spheres into an inner symmetric space will always be of finite uniton type. This provides a way to classify all Willmore two-spheres in terms of their conformal Gauss maps [24]. For the harmonic maps used in this paper, a main problem is that they are not globally well-defined in general. So one can not apply the theory to such harmonic maps. But using Wu’s formula and the description of the normalized potentials of harmonic maps of finite uniton type, we are able to show that the harmonic map given by a totally isotropic Willmore two-sphere and its adjoint surface is also of finite uniton type. So far we do not have a clear explain for this phenomena, which may need a detailed discussion on the Iwasawa cells of the corresponding non-compact Loop groups. We hope to continue this study.
in future publication.

This paper is organized as follows: In Section 2, we first recall some basic results about Willmore surfaces and their adjoint transforms. Then in Section 3 we discuss the isotropic harmonic maps given by Willmore surfaces and their adjoint transforms. Section 4 provides a description of the normalized potentials of totally isotropic Willmore two-spheres in terms of the isotropic harmonic maps. The converse part, i.e., that generically such normalized potentials will always produce totally isotropic Willmore surfaces and their adjoint transforms, is the main content of Section 5. Using these results, we also derive some concrete examples in Section 5. Then we end the paper by Section 6, which contain the technical and tedious computations of Section 5.

2. Willmore surfaces and adjoint surfaces

In this section we will first recall the basic surface theory of Willmore surfaces in $S^{n+2}$ in the spirit of the treatment of [5, 16]. Then we will collect the descriptions of the adjoint transforms of Willmore surfaces [15]. We refer to [16, 15, 1] for more details.

2.1. Review of Willmore surfaces in $S^{n+2}$. Let $\mathbb{R}^{n+4}_1$ be the Minkowski space, with a Lorentzian metric $\langle x, y \rangle = -x_1y_1 + \sum_{j=2}^{n+3} x_jy_j = x^t I_{1,n+3} y$, $I_{1,n+3} = \text{diag}(-1, 1, \cdots, 1)$. Let $C^{n+3}_+ := \{ x \in \mathbb{R}^{n+4}_1 | \langle x, x \rangle = 0, x_1 > 0 \}$ be the forward light cone. Let $Q^{n+2} := \{ [x] \in \mathbb{R} P^{n-3} | x \in C^{n+3}_+ \}$ be the projective light cone with the induced conformal metric. Then $Q^{n+2}$ is conformally equivalent to $S^{n+2}$, and the conformal group of $S^{n+2} \cong Q^{n+2}$ is the orthogonal group $O(1,n+3)/\{ \pm 1 \}$ of $\mathbb{R}^{n+4}_1$, acting on $Q^{n+2}$ by $T([x]) = [Tx]$ for any $T \in O(1,n+3)$. Let $SO^+(1,n+3)$ be the connected component of $O(1,n+3)$ containing $I$, i.e.,

$$SO^+(1,n+3) = \{ T \in O(1,n+3) | \det T = 1, T \text{ preserves the time direction of } \mathbb{R}^{n+4}_1 \}.$$ 

Let $y : M \rightarrow S^{n+2}$ be a conformal immersion from a Riemann surface $M$, with $z$ a local complex coordinate on $U \subset M$ and $(y_{\bar{z}}, y_{\bar{z}}) = \frac{1}{2} \bar{z} e^{2\omega}$. The lift $Y : U \rightarrow C^{n+3}_+$ is called a canonical lift of $y$ with respect to $z$, satisfying $|dy|^2 = |dz|^2$. Then there is a bundle decomposition

$$M \times \mathbb{R}^{n+4}_1 = V \oplus V^\perp,$$

with $V = \text{Span}_\mathbb{R}\{ Y, \text{Re} Y_{\bar{z}}, \text{Im} Y_{\bar{z}}, Y_{zz} \}$, $V^\perp \perp V$.

Here $V$ is a Lorentzian rank-4 sub-bundle. This decomposition is independent of the choice of $Y$ and $z$. We denote by $V_C$ and $V_C^\perp$ as their complexifications. There exists a unique section $N \in \Gamma(V)$ such that $\langle N, Y_z \rangle = \langle N, Y_{\bar{z}} \rangle = \langle N, N \rangle = 0, \langle N, Y \rangle = -1$. Noting that $Y_{zz}$ is orthogonal to $Y, Y_z$ and $Y_{\bar{z}}$, there exists a complex function $s$ and a section $\kappa \in \Gamma(V_C^\perp)$ such that $Y_{zz} = -\frac{s}{2} Y + \kappa$. This defines two basic invariants $\kappa$ and $s$ depending on coordinates $z$, the conformal Hopf differential and the Schwarzian of $y$ [5]. Let $D$ denote the normal connection and $\psi \in \Gamma(V_C^\perp)$ any section of the normal bundle. The structure equations can be given as follows:

$$\begin{align*}
Y_{zz} &= -\frac{s}{2} Y + \kappa, \\
Y_{\bar{z}z} &= -\langle \kappa, \bar{\kappa} \rangle Y + \frac{1}{2} N, \\
N_{z\bar{z}} &= -2 \langle \kappa, \bar{\kappa} \rangle Y - 2sY_{\bar{z}} + 2D_{\bar{z}}\kappa, \\
\psi_z &= D_{\bar{z}}\psi + 2\langle \psi, D_{\bar{z}}\kappa \rangle Y - 2\langle \psi, \kappa \rangle Y_{\bar{z}}.
\end{align*}$$

(2.1)
The conformal Gauss, Codazzi and Ricci equations as integrable conditions are:

\[\frac{s_\bar{z}}{2} = 3\langle \kappa, D_\bar{z} \kappa \rangle + \langle D_\bar{z} \kappa, \bar{\kappa} \rangle,\]
\[\text{Im}(D_\bar{z} D_\bar{z} \kappa + \frac{\bar{s}}{2} \kappa) = 0,\]
\[D_\bar{z} D_\bar{z} \psi - D_\bar{z} D_\bar{z} \psi = 2\langle \psi, \kappa \rangle \bar{\kappa} - 2\langle \psi, \bar{\kappa} \rangle \kappa.\]

The conformal Gauss map of \(y\) is defined as follow.

**Definition 2.1.** \([3, 5, 11, 16]\) For a conformally immersed surface \(y : M \to S^{n+2}\), the conformal Gauss map \(Gr(p) : M \to Gr_{3,1}(\mathbb{R}^{n+4}) = SO^+(1, n+3)/(SO^+(1, 3) \times SO(n))\) of \(y\) is defined as

\[Gr(p) := V_p.\]

So locally we have \(Gr = Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_z \wedge Y_\bar{z} \wedge N, \) with \(z = u + iv.\)

Direct computation shows that \(Gr\) induces a conformal-invariant metric \(g := \frac{1}{4}\langle dGr, dGr \rangle = \langle \kappa, \bar{\kappa} \rangle |dz|^2\) on \(M.\) Note \(g\) degenerates at umbilic points of \(y.\) The Willmore functional and Willmore surfaces can be defined by use of this metric.

**Definition 2.2.** The Willmore functional of \(y\) is defined as:

\[W(y) := 2i \int_M \langle \kappa, \bar{\kappa} \rangle dz \wedge d\bar{z}.\]

An immersed surface \(y : M \to S^{n+2}\) is called a Willmore surface if it is a critical surface of the Willmore functional with respect to any variation of the map \(y : M \to S^{n+2}.\)

It is well-known that \([3, 5, 11, 23]\) \(y\) is Willmore if and only if

\[D_\bar{z} D_\bar{z} \kappa + \frac{\bar{s}}{2} \kappa = 0;\]

if and only if the conformal Gauss map \(Gr : M \to Gr_{3,1}(\mathbb{R}^{n+3})\) is harmonic. We refer to \([9]\) for the conformal Gauss map approach for Willmore surface.

### 2.2. Adjoint transforms of a Willmore surface and the second harmonic map related to Willmore surfaces.

Transforms play an important role in the study of Willmore surfaces. For a Willmore surface \(y\) in \(S^3,\) it was shown by Bryant in the seminal paper \([3]\) that they always admit a unique dual surface which may have branch points or degenerate to a point. Hence the dual surface is either degenerate or has the same complex coordinate and the same conformal Gauss map as \(y\) at the points it is immersed. This duality theorem, however, does not hold in general when the codimension is bigger than 1 \([11, 5, 15]\). To characterize Willmore surfaces with dual surfaces, in \([11]\) Ejiri introduced the notion of \(S\)-Willmore surfaces. Here we define it slightly differently to include all Willmore surfaces with dual surfaces:

**Definition 2.3.** A Willmore immersion \(y : M^2 \to S^{n+2}\) is called an \(S\)-Willmore surface if its conformal Hopf differential satisfies

\[D_\bar{z} \kappa |\kappa| = \mu,\]

i.e. there exists some function \(\mu\) on \(M\) such that \(D_\bar{z} \kappa + \frac{\bar{s}}{2} \kappa = 0.\)

A basic result of \([11]\) states that a Willmore surface admits a dual surface if and only if it is \(S\)-Willmore. Moreover the dual surface is also Willmore at the points it is immersed.

To consider the generic Willmore surfaces, Ma introduced the adjoint transform of a Willmore surface \(y\) \([15, 16]\). An adjoint transform of \(y\) is a conformal map \(\hat{y}\) which is located on the mean curvature sphere of \(y\) and satisfies some additional condition. To be concrete we have the following
2.2.1. Adjoint transforms. Let $y: U \to S^{n+2}$ be an umbilic free Willmore surface with canonical lift $Y$ with respect to $z$ as above. Set

$$\hat{Y} = N + \bar{\mu}Y_z + \mu Y_z + \frac{1}{2}|\mu|^2 Y,$$

with $\mu dz = 2(\hat{Y}, Y_z)dz$ a connection 1–form. Direct computation yields \cite{15}

$$\hat{Y}_z = \frac{\mu}{2} \hat{Y} + \theta \left( Y_z + \frac{\bar{\mu}}{2}Y \right) + \rho \left( Y_z + \frac{\mu}{2}Y \right) + 2\zeta$$

with

$$\theta := \mu_z - \frac{\mu^2}{2} - s, \, \rho := \bar{\mu}_z - 2\langle \kappa, \bar{\kappa} \rangle, \, \zeta := Dz \kappa + \frac{\bar{\mu}}{2} \kappa.$$

Now we define the adjoint surface as follow.

**Definition 2.4.** \cite{15} The map $\hat{Y}: U \to S^{n+2}$ is called an adjoint transform of the Willmore surface $Y$ if the following two equations hold for $\mu$:

$$\mu_z - \frac{\mu^2}{2} - s = 0, \, \text{Riccati equation,}$$

$$\langle Dz \kappa + \frac{\mu}{2} \kappa, Dz \kappa + \frac{\bar{\mu}}{2} \kappa \rangle = 0.$$

Note that $\hat{Y}$ is the dual surface of $Y$ if and only if $Dz \kappa + \frac{\mu}{2} \kappa = 0$ \cite{3, 11, 15}.

**Theorem 2.5.** \cite{15} Willmore property and existence of adjoint transform: The adjoint transform $\hat{Y}$ of a Willmore surface $y$ is also a Willmore surface (may degenerate). Moreover,

1. If $\langle \kappa, \kappa \rangle \equiv 0$, any solution to the equation (2.6) is also a solution to the equation (2.7). Hence, there exist infinitely many adjoint surfaces of $y$ in this case.
2. If $\langle \kappa, \kappa \rangle \neq 0$ and $\Omega dz^6 := \langle Dz ^2 \kappa, \kappa \rangle^2 - \langle \kappa, \kappa \rangle \langle Dz \kappa, Dz \kappa \rangle dz^6 \neq 0$, there are exactly two different solutions to equation (2.7), which also solve (2.6). Hence, there exist exactly two adjoint surfaces of $y$ in this case.
3. If $\langle \kappa, \kappa \rangle \neq 0$ and $\langle Dz ^2 \kappa, \kappa \rangle^2 - \langle \kappa, \kappa \rangle \langle Dz \kappa, Dz \kappa \rangle \equiv 0$, there exists a unique solution to (2.7), which also solves (2.6). Hence, there exists a unique adjoint surface of $y$ in this case.

**Remark 2.6.** In \cite{6}, dressing transformations of constrained Willmore surfaces are discussed in details. It stays unclear whether the adjoint transforms can be derived as a special kind of dressing transformations.

2.2.2. Harmonic maps into $SO^+(1, n+3)/(SO^+(1, 1) \times SO(n+2))$ related to Willmore surfaces. A crucial observation by Hélein etc. \cite{13, 14, 27, 15} is that $Y$ and $\hat{Y}$ produce furthermore a second useful harmonic map related to a Willmore surface $y$.

**Theorem 2.7.** Let $[Y]$ be a Willmore surface. Let $\mu$ be a solution to the Riccati equation (2.6) on $U$, defining $\hat{Y}$ as (2.4). Let $\mathcal{F}_h: U \to SO^+(1, n+3)/(SO^+(1, 1) \times SO(n+2))$ be the map taking $p$ to $Y(p) \wedge \hat{Y}(p)$. We have the following results.

1. \cite{13, 14, 27} The map $\mathcal{F}_h$ is harmonic, and is called a half–isotropic harmonic map with respect to $Y$.
2. \cite{15} If $\mu$ also solves (2.7), i.e., $\hat{Y}$ is an adjoint transform of $y$, then $\mathcal{F}_h$ is conformally harmonic, and is called an isotropic harmonic map with respect to $Y$.
At umbilic points it is possible that there exists a limit of \( \mu \) such that holds. Due to the following lemma, the harmonic map \( \mathcal{F}_h \) has no definition when \( \mu \) tends to \( \infty \).

**Lemma 2.8.** [11][9] At the umbilic points of \( Y \), the limit of \( \mu \) goes to a finite number or infinity. When \( \mu \) goes to infinity, \( [\hat{Y}] \) tends to \( [Y] \), and at the limit point we have \( [\hat{Y}] = [Y] \).

Restricting to the isotropic harmonic map, we have the following description.

**Theorem 2.9.** [15], [14], [1] Let \( \mathcal{F}_h = Y \land \hat{Y} \) be an isotropic harmonic map. Set \( e_1, e_2 \in \Gamma(V) \) with \( Y_\alpha + b \frac{\mathbb{Y}}{2} = \frac{1}{2}(e_1 - ie_2) \). Let \( \{\psi_j, j = 1, \cdots, n\} \) be a frame of the normal bundle \( V^\perp \). Assume that \( \kappa = \sum_{j=1}^{n} k_j \psi_j, \zeta = \sum_{j=1}^{n} \gamma_j \psi_j, D_\alpha \psi_j = \sum_{i=1}^{n} b_{ji} \psi_i \), \( b_{ji} + b_{ij} = 0 \). Set

\[
F = \left( \frac{1}{\sqrt{2}}(Y + \hat{Y}), \frac{1}{\sqrt{2}}(-Y + \hat{Y}), e_1, e_2, \psi_1, \cdots, \psi_n \right).
\]

Then the Maurer-Cartan form \( \alpha = F^{-1}dF = \alpha' + \alpha'' \) of \( F \) has the structure:

\[
(2.8) \quad \alpha' = \begin{pmatrix} A_1 & B_1 \\ -B_1 I_{1,1} & A_2 \end{pmatrix} dz,
\]

with

\[
A_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1+\rho & i-i\rho & 2\sqrt{2}\gamma_1 & \cdots & 2\sqrt{2}\gamma_n \\ 2\sqrt{2} & i+i\rho & 2\sqrt{2}\gamma_1 & \cdots & 2\sqrt{2}\gamma_n \\ 2\sqrt{2} & 2\sqrt{2} & -2\sqrt{2}\gamma_1 & \cdots & -2\sqrt{2}\gamma_n \\ 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2}\gamma_1 & \cdots & 2\sqrt{2}\gamma_n \end{pmatrix} = \begin{pmatrix} b'_{1} & b'_{2} \\ b'_{2} & b'_{1} \end{pmatrix},
\]

and

\[
(2.9) \quad B_1 B_1^t = 0.
\]

Conversely, if \( \mathcal{F} = Y \land \hat{Y} : U \to SO^{+}(1, n+3)/(SO^{+}(1, 1) \times SO(n+2)) \) is a conformal harmonic map satisfying [2.9], then \( \mathcal{F} \) is an isotropic harmonic map and \( Y \) and \( \hat{Y} \) form a pair of adjoint Willmore surfaces at the points they are immersed. Moreover, set

\[
B_1 = (b_1 \ b_2)^t \quad \text{with} \quad b_1, b_2 \in \mathbb{C}^{n+2}.
\]

Then \( Y \) is immersed at the points \((b_1' + b_2') (\tilde{b}_1 + \tilde{b}_2) > 0 \) and \( \hat{Y} \) is immersed at the points \((b_1' - b_2') (\tilde{b}_1 - \tilde{b}_2) > 0 \).

3. **Isotropic harmonic maps into \( SO^{+}(1, n+3)/(SO^{+}(1, 1) \times SO(n+2)) \)**

In this section we will recall briefly the DPW construction of harmonic maps and applications to the isotropic harmonic maps related to Willmore surfaces. We refer to [13], [14], [27], [11] for more details.

3.1. **The DPW construction of harmonic maps.**

3.1.1. **Harmonic maps into an inner symmetric space.** Let \( G/K \) be an inner symmetric space with involution \( \sigma : G \to G \) such that \( G^\sigma \supset G \supset (G^\sigma)_0 \). Let \( \pi : G \to G/K \) be the projection of \( G \) into \( G/K \). Let \( \mathfrak{g} = \text{Lie}(G) \) and \( \mathfrak{k} = \text{Lie}(K) \) be their Lie algebras. We have the Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{f} \).

Let \( \mathcal{F} : M \to G/K \) be a conformal harmonic map from a Riemann surface \( M \), with \( U \subset M \) an open connected subset. Then there exists a frame \( F : U \to G \) such that \( \mathcal{F} = \pi \circ F \). So we have the Maurer-Cartan form \( F^{-1}dF = \alpha \), and Maurer-Cartan equation \( d\alpha + \frac{1}{2}[\alpha \land \alpha] = 0 \).

Set \( \alpha = \alpha_0 + \alpha_1, \) with \( \alpha_0 \in \Gamma(\mathfrak{f} \oplus T^*M), \) \( \alpha_1 \in \Gamma(\mathfrak{p} \otimes T^*M) \). Decompose \( \alpha_1 \) further into the \((1,0)\)-part \( \alpha'_1 \) and the \((0,1)\)-part \( \alpha''_1 \). Then set \( \alpha_\lambda = \lambda^{-1}\alpha'_1 + \alpha_0 + \lambda\alpha''_1 \), with \( \lambda \in \mathbb{S}^1 \). We have the well-known characterization of harmonic maps:
Lemma 3.1. ([8]) The map $F : M \to G/K$ is harmonic if and only if
\[
d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \text{ for all } \lambda \in S^1.
\]

As a consequence, for a harmonic map $f$, the equation $dF(z, \lambda) = F(z, \lambda) \alpha_\lambda$ with $F(0, \lambda) = F(0)$, always has a solution, which is called the extended frame of $F$.

3.1.2. Two decomposition theorems. We denote by $SO^+(1, n + 3)$ the connected component of the identity of the linear isometry group of $\mathbb{R}_{1}^{n+4}$. Then
\[
\text{so}(1, n + 3) = g = \{X \in \mathfrak{gl}(n + 4, \mathbb{R}) | X^t I_{1,n+3} + I_{1,n+3}X = 0\}.
\]
Define the involution
\[
\sigma : SO^+(1, n + 3) \to SO^+(1, n + 3) \quad A \mapsto DAD^{-1}, \quad \text{where } D = \begin{pmatrix} -I_2 & 0 \\ 0 & I_{n+2} \end{pmatrix}.
\]
We have $SO^+(1, n + 3) \supset SO^+(1, 1) \times SO(n + 2) = (SO^+(1, n + 3)\sigma)_0$. We also have
\[
g = \left\{ \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,1} & A_2 \end{pmatrix} | A_1^t I_{1,1} + I_{1,1}A_1 = 0, A_2 + A_2^t = 0 \right\} = \mathfrak{k} \oplus \mathfrak{p},
\]
with
\[
\mathfrak{k} = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} | A_1^t I_{1,1} + I_{1,1}A_1 = 0, A_2 + A_2^t = 0 \right\}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,1} & 0 \end{pmatrix} \right\}.
\]
Let
\[
G^C = SO^+(1, n + 3, \mathbb{C}) := \{X \in SL(n + 4, \mathbb{C}) | X^t I_{1,n+3}X = I_{1,n+3}\}, \quad g^C = so(1, n + 3, \mathbb{C}).
\]
Extend $\sigma$ to an inner involution of $G^C$ with fixed point group $K^C = S(O^+(1, 1, \mathbb{C}) \times O(n + 2, \mathbb{C}))$.

Let $\Lambda G^C_\sigma$ denote the group of loops in $G^C = SO^+(1, n + 3, \mathbb{C})$ twisted by $\sigma$. Let $\Lambda^+ G^C_\sigma$ denote the subgroup of loops which extend holomorphically to the unit disk $|\lambda| \leq 1$. We also need the subgroup $\Lambda_B G^C_\sigma := \{\gamma \in \Lambda^+ G^C_\sigma | \gamma|_{\lambda=0} \in \mathfrak{B}\}$, where $\mathfrak{B} \subset K^C$ is defined from the Iwasawa decomposition $K^C = K \cdot \mathfrak{B}$. In this case,
\[
\mathfrak{B} = \left\{ \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \right\}, \quad b_1 = \begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}, \theta \in \mathbb{R} \mod 2\pi \mathbb{Z}, \text{ and } b_2 \in \mathfrak{B}_2.
\]
Here $\mathfrak{B}_2$ is the solvable subgroup of $SO(n + 2, \mathbb{C})$ such that $SO(n + 2, \mathbb{C}) = SO(n + 2) \cdot \mathfrak{B}_2$. We refer to Lemma 4 of [13] for more details.

Theorem 3.2. Theorem 5 of [13], see also [27], [8], [21] (Iwasawa decomposition): The multiplication $\Lambda G_\sigma \times \Lambda_B^+ G^C \to \Lambda G^C_\sigma$ is a real analytic diffeomorphism onto the open dense subset $\Lambda G_\sigma \cdot \Lambda_B^+ G^C \subset \Lambda G^C_\sigma$.

Let $\Lambda^- G^C_\sigma$ denote the loops that extend holomorphically into $\infty$ and take values $I$ at infinity.

Theorem 3.3. Theorem 7 of [13], see also [27], [8], [21] (Birkhoff decomposition): The multiplication $\Lambda^- G^C_\sigma \times \Lambda^+ G^C \to \Lambda G^C_\sigma$ is a real analytic diffeomorphism onto the open subset $\Lambda^- G^C_\sigma \cdot \Lambda^+ G^C$ (the big cell) of $\Lambda G^C_\sigma$. 
3.1.3. The DPW construction and Wu’s formula. Here we recall the DPW construction for harmonic maps. Let \( \mathbb{D} \subset \mathbb{C} \) be a disk or \( \mathbb{C} \) itself, with complex coordinate \( z \).

**Theorem 3.4.** [8]

1. Let \( F : \mathbb{D} \to G/K \) be a harmonic map with an extended frame \( F(z, \bar{z}, \lambda) \in \Lambda G_\sigma \) and \( F(0, 0, \lambda) = I \). Then there exists a Birkhoff decomposition

\[
F_-(z, \lambda) = F(z, \bar{z}, \lambda)F_+(z, \bar{z}, \lambda), \quad \text{with} \quad F_+ \in \Lambda^+ G_\sigma^C,
\]

such that \( F_-(z, \lambda) : \mathbb{D} \to \Lambda^- G_\sigma^C \) is meromorphic. Moreover, the Maurer-Cartan form of \( F_- \) is of the form

\[
\eta = F_-^{-1}dF_- = \lambda^{-1}\eta_1(z)dz,
\]

with \( \eta_1 \) independent of \( \lambda \). The 1-form \( \eta \) is called the normalized potential of \( F \).

2. Let \( \eta \) be a \( \lambda^{-1} \cdot \mathfrak{p} \)-valued meromorphic 1-form on \( \mathbb{D} \). Let \( F_-(z, \lambda) \) be a solution to \( F_-^{-1}dF_- = \eta \), \( F_-(0, \lambda) = I \). Then on an open subset \( \mathbb{D}_\eta \) of \( \mathbb{D} \) one has

\[
F_-(0, \lambda) = \tilde{F}(z, \bar{z}, \lambda) \cdot F_+(z, \bar{z}, \lambda), \quad \text{with} \quad \tilde{F} \in \Lambda G_\sigma, \quad F_+ \in \Lambda^+ G_\sigma^C.
\]

This way, one obtains an extended frame \( \tilde{F}(z, \bar{z}, \lambda) \) of some harmonic map from \( \mathbb{D}_\eta \) to \( G/K \) with \( \tilde{F}(0, \lambda) = I \). Moreover, all harmonic maps can be obtained in this way, since these two procedures are inverse to each other if the normalization at some based point is used.

The normalized potential can be determined in the following way. Let \( f \) and \( F \) be as above. Let \( \alpha_\lambda = F^{-1}dF \). Let \( \delta_1 \) and \( \delta_0 \) denote the sum of the holomorphic terms of \( z \) about \( z = 0 \) in the Taylor expansion of \( \alpha_\lambda'(\frac{\partial}{\partial z}) \) and \( \alpha_\lambda'(\frac{\partial}{\partial \bar{z}}) \).

**Theorem 3.5.** [26] (Wu’s formula) We retain the notations in Theorem 3.4. The the normalized potential of \( F \) with respect to the base point 0 is given by

\[
\eta = \lambda^{-1}F_0(z)\delta_1F_0(z)^{-1}dz,
\]

where \( F_0(z) : \mathbb{D} \to G^C \) is the solution to \( F_0(z)^{-1}dF_0(z) = \delta_0dz, \) \( F_0(0) = I \).

**Lemma 3.6.** We retain the notations in Theorem 3.5. Let \( Q \in K \) and \( QF \) be a transform of \( F \) in \( G/K \). Then the normalized potential of \( QF \) with respect to the base point 0 is

\[
\eta_Q = Q\eta Q^{-1}.
\]

**Proof.** We have now the lift \( QFQ^{-1} \) of \( QF \) with respect to the base point 0. So we have the Birkhoff splitting of \( QFQ^{-1} \) as below

\[
F_-Q = QF_-Q^{-1} = QFQ^{-1}QF_+Q^{-1} \quad \text{since} \quad F_- = FF_+.
\]

Hence we obtain

\[
\eta_Q = (QF_-Q^{-1})^{-1}dQF_-Q^{-1} = Q\eta Q^{-1}.
\]

This lemma shows that we can identify the normalized potentials up to an conjugation of elements in \( K \).

3.2. Potentials of isotropic harmonic maps.
3.2.1. The general case. Let \( \mathbb{D} \) denote the unit disk of \( \mathbb{C} \) or \( \mathbb{C} \) itself. Let \( F : \mathbb{D} \to SO^+(1,n+3)/(SO^+(1,1) \times SO(n+2)) \) be a harmonic map with a lift \( F : \mathbb{D} \to SO^+(1,n+3) \) and the Maurer-Cartan form \( \alpha = F^{-1} dF \). Then
\[
\alpha_0' = \left( \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right) dz, \quad \alpha_1' = \left( \begin{array}{cc} 0 & B_1 \\ -B_1 I_{1,1} & 0 \end{array} \right) dz.
\]

**Theorem 3.7.** ([13, 14, 27, 11]) The normalized potential of an isotropic harmonic map \( F = Y \wedge \hat{Y} \) is of the form
\[
(3.2) \quad \eta = \lambda^{-1} \left( \begin{array}{cc} 0 & \hat{B}_1 \\ -\hat{B}_1 I_{1,1} & 0 \end{array} \right) dz, \text{ with } \hat{B}_1 \hat{B}_1^t = 0.
\]
Moreover, \([Y]\) and \([\hat{Y}]\) forms a pair of dual (S-)Willmore surfaces if and only if \( \text{rank}(\hat{B}_1) = 1 \). Conversely, let \( F = Y \wedge \hat{Y} \) be an harmonic map with normalized potential
\[
\eta = \lambda^{-1} \left( \begin{array}{cc} 0 & \hat{B}_1 \\ -\hat{B}_1 I_{1,1} & 0 \end{array} \right) dz
\]
satisfying \((3.2)\). Then \( F \) is an isotropic harmonic map.

3.2.2. On minimal surfaces in space forms. In [13], there is an interesting description of Willmore surfaces Möbius equivalent to minimal surfaces in space forms. Here we restate it as:

**Theorem 3.8.** ([13, 27]) Let \( F_h = Y \wedge \hat{Y} \) be a non-constant isotropic harmonic map.

1. The map \([Y]\) is Möbius equivalent to a minimal surface in \( \mathbb{R}^{n+2} \) if \( \hat{Y} \) reduces to a point. In this case \( B_1 = (b_1 b_1)^t \).
2. The map \([Y]\) is Möbius equivalent to a minimal surface in \( S^{n+2} \) if \( F_h \) reduces to a harmonic map into \( SO(n+3)/SO(n+2) \). In this case \( B_1 = (0 b_1)^t \).
3. The map \([Y]\) is Möbius equivalent to a minimal surface in \( H^{n+2} \) if \( F_h \) reduces to a harmonic map into \( SO^+(1,n+3)/SO^+(1,n+1) \). In this case \( B_1 = (b_1 0)^t \).

Here \( b_1 \) takes values in \( \mathbb{C}^{n+2} \) and satisfies \( b_1^t b_1 = 0 \).

The converse of the above results also hold. That is, if \( B_1 \) is (up to conjugation) of the form stated above, then \([Y]\) is Möbius equivalent to the corresponding minimal surface where it is an immersion.

3.3. On harmonic maps of finite uniton type. In this subsection we will discuss harmonic maps of finite uniton type.

Loops which have a finite Fourier expansion will be called *algebraic loops* and the corresponding spaces will be denoted by the subscript “alg”, like \( \Lambda_{\text{alg}} G_{\sigma}, \Lambda_{\text{alg}} G_{\sigma}^\mathbb{C}, \Omega_{\text{alg}} G_{\sigma} \). We define
\[
\Omega_{\text{alg}}^k G_{\sigma} := \{ \gamma \in \Omega_{\text{alg}} G_{\sigma} | \text{Ad}(\gamma) = \sum_{|j| \leq k} \lambda^j T_j \}.
\]

Let \( G/K \) be an inner symmetric space (given by the inner involution \( \sigma : G \to G \)). We map \( G/K \) into \( G \) as totally geodesic submanifold via the (finite covering) Cartan map: \( \mathfrak{C} : G/K \to G, \mathfrak{C}(gK) = g\sigma(g)^{-1} \).

**Definition 3.9.** ([22, 4, 9, 10])
(1) Let \( f : M \to G \) be a harmonic map into a real Lie group \( G \) with extended solution \( \Phi(z, \lambda) \in \Lambda G^C_\sigma \). We say that \( f \) has finite uniton number \( k \) if
\[
\Phi(M) \subset \Omega^k_{\text{alg}} G_\sigma, \quad \text{and} \quad \Phi(M) \notin \Omega^{k-1}_{\text{alg}} G_\sigma.
\]

(2) A harmonic map \( f \) into \( G/K \) is said to be of finite uniton number \( k \), if it is of finite uniton number \( k \), when considered as a harmonic map into \( G \) via the Cartan map, i.e., \( f \) has finite uniton number \( k \) if and only if \( \mathcal{C} \circ f \) has finite uniton number \( k \).

It is proved that for harmonic maps into inner symmetric space \( G/K \), it is of finite uniton number if and only if its normalized potential takes value in some nilpotent Lie sub-algebra \([4, 12, 10]\). In Section 4 and Section 5 we will give a characterization of totally isotropic Willmore two-spheres in terms of harmonic maps of finite uniton number at most 2.

4. Totally isotropic Willmore two-spheres and their adjoint transforms

In this section we will first collect the geometric results concerning totally isotropic Willmore two-spheres and their adjoint transforms. Then by the geometric descriptions, we are able to derive the normalized potentials of the isotropic harmonic map given by such Willmore surfaces and their adjoint transforms.

4.1. Totally isotropic Willmore surfaces. Let \( y : M \to S^{2m} \) be a conformal immersion and we retain the notion in Section 2. Then \([7, 11]\) \( y \) is called totally isotropic if and only if all the derivatives of \( y \) with respect to \( z \) are isotropic, or equivalently,
\[
\langle Y^{(j)}_z, Y^{(l)}_z \rangle = 0 \quad \text{for all} \quad j, l \in \mathbb{Z}^+.
\]

Here \( "Y^{(j)}_z" \) denotes taking \( j \) times derivatives of \( Y \) by \( z \). As a consequence a totally isotropic Willmore surface always locates in an even dimensional sphere. Moreover, we can find locally an isotropic frame \( \{E_j\}, j = 1, \cdots, m \), such that
\[
\langle E_j, \bar{E}_l \rangle = 2\delta_{jl}, \quad j, l = 1, \cdots, m,
\]
\[
Y_z \in \text{Span}_\mathbb{C}\{E_1\} \mod Y,
\]
\[
Y^{(j)}_z \in \text{Span}_\mathbb{C}\{E_j\} \mod \text{Span}_\mathbb{C}\{Y, E_1, \cdots, E_{j-1}\}, \quad j = 2, \cdots, m,
\]
\[
\{E_j, \bar{E}_j\}_{j=2, \cdots, m} \text{forms a basis of the normal bundle } V^\perp.
\]

Next we call \( y \) an \( H \)-totally isotropic surface if it satisfies furthermore the following conditions
\[
D_z E_j \in \text{Span}_\mathbb{C}\{E_2, \cdots, E_m\}, \quad \text{for all} \quad j = 2, \cdots, m.
\]

It is direct to verify that this condition is independent of the choice of \( z, Y \) and \( E_j \). Here the notion “\( H \)” comes from two facts. First, this condition is similar to the horizontal conditions for minimal two-spheres in \( S^{2m} \) \([2, 7]\). Second, by a result of \([16]\), we can prove that

**Theorem 4.1.** \([18]\) Let \( y \) be a totally isotropic Willmore two-sphere in \( S^{2m} \). Then \( y \) is an \( H \)-totally isotropic surface.

See \([25]\) for a proof in the case \( m = 3 \). The key point of the proof is an application of the holomorphic forms given by \( \kappa \) and its derivatives, which can be found in Section 5 of \([16]\).

Moreover, totally isotropic surfaces in \( S^{2m} \) may not be Willmore. But \( H \)-totally isotropic surfaces must be Willmore.

**Proposition 4.2.** Let \( y \) be an \( H \)-totally isotropic surface in \( S^{2m} \). Then \( y \) is Willmore.
Proof. By definition of \( E_j \), we have \( \kappa \in \text{Span}_C\{E_2\} \). From (1.2), we have that

\[
D_2 \kappa \in \text{Span}_C\{E_2, \cdots, E_m\}, \quad D_3 \kappa \in \text{Span}_C\{E_2, \cdots, E_m\}.
\]

So \( D_2 D_2 \kappa + \frac{8}{2} \kappa \in \text{Span}_C\{E_2, \cdots, E_m\} \). Hence \( \Im (D_2 D_2 \kappa + \frac{8}{2} \kappa) = 0 \) in (2.2) indicates that the Willmore equation \( D_2 D_2 \kappa + \frac{8}{2} \kappa = 0 \) holds, i.e., \( y \) is Willmore. \( \square \)

Concerning the adjoint surfaces of \( H \)-totally isotropic surfaces, we have

**Proposition 4.3.**  \[ \text{Let } y \text{ be an } H \text{-totally isotropic surface in } S^{2m} \text{ (hence Willmore). Then the adjoint surface of } y \text{ is also } H \text{-totally isotropic surface on the points is is immersed.} \]

This can be easily derived since \( \hat{Y}_z \in \text{Span}_C\{E_1, \cdots, E_m\} \mod \{Y, \hat{Y}\} \) by (2.5).

### 4.2. Normalized potentials of \( H \)-totally isotropic surfaces.

The normalized potentials of \( H \)-totally isotropic surfaces can be derived from Wu’s formula as below

**Theorem 4.4.** Let \( y : \mathbb{D} \to S^{2m} \) be an \( H \)-totally isotropic surface with a local adjoint transform \( \hat{y} = [\hat{Y}] \). Assume that \( F|_{z=0} = I \mod K \). Then up to a conjugation, the normalized potential of \( F = Y \wedge \hat{Y} \) has the form

\[
\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ \hat{B}_1^t \hat{I}_{1,1} & 0 \end{pmatrix} dz,
\]

\[
\hat{B}_1 = \begin{pmatrix} h_{11} & \hat{h}_{11} & \cdots & h_{m1} & \hat{h}_{m1} \\ \hat{h}_{11} & \hat{h}_{11} & \cdots & \hat{h}_{m1} & \hat{h}_{m1} \end{pmatrix},
\]

with \( \{h_{j1}dz, \hat{h}_{j1}dz\mid j = 1, \cdots, m\} \) being meromorphic 1-forms on \( \mathbb{D} \).

**Proof.** We have the following

\[
Y_z = -\frac{\mu}{2} Y + \frac{1}{2} E_1.
\]

We consider the lift \( \tilde{F} \) as below

\[
\tilde{F} = \left( \frac{1}{\sqrt{2}}(Y + \hat{Y}), \frac{1}{\sqrt{2}}(-Y + \hat{Y}), e_1, \psi_2, \cdots, \psi_m, \hat{e}_1, \hat{\psi}_2, \cdots, \hat{\psi}_m \right).
\]

Here we use the frame defined in (4.1) and set

\[
E_1 = e_1 + i\hat{e}_1, \quad E_j = \psi_j + i\hat{\psi}_j, \quad j = 2, \cdots, m.
\]

Set

\[
\kappa = \sum_j k_j (\psi_j + i\hat{\psi}_j), \quad \zeta = \sum_j \gamma_j (\psi_j + i\hat{\psi}_j).
\]

Assume that \( D_z E_j = \sum a_{jl} E_l, \quad D_z \hat{E}_j = \sum \hat{a}_{jl} \hat{E}_l \). By (1.1), we have \( a_{jl} + \hat{a}_{lj} = 0 \), and

\[
D_z \psi_j = \frac{1}{2} \left( \sum (a_{jl} - a_{lj}) \psi_l + \sum i(a_{jl} + a_{lj}) \hat{\psi}_l \right),
\]

\[
D_z \hat{\psi}_j = \frac{1}{2} \left( -\sum i(a_{jl} + a_{lj}) \psi_l + \sum (a_{jl} - a_{lj}) \hat{\psi}_l \right).
\]

So

\[
A_2 = \begin{pmatrix} A_{21} & iA_{22} \\ -iA_{22} & A_{21} \end{pmatrix}
\]

with \( A_{12} + A_{12}^t = 0 \), \( A_{22} = A_{22}^t \), and

\[
A_{21} = \begin{pmatrix} 0 & -k_j a_{jl} \\ k_j & \frac{a_{jl} - a_{lj}}{2} \end{pmatrix}_{m \times m}, \quad A_{22} = \begin{pmatrix} -\frac{\mu}{2} & -k_j \\ k_j & \frac{a_{jl} + a_{lj}}{2} \end{pmatrix}_{m \times m}.
\]
Hence, without lose of generality, we assume that $\tilde{F}(0,0,\lambda) = I$ and let
$$\delta_0 = \left( \begin{array}{cc} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{array} \right), \quad \delta_1 = \left( \begin{array}{cc} 0 & \hat{B}_1 \\ -\hat{B}_1^{*}I_{1,1} & 0 \end{array} \right)$$
be the holomorphic parts of $\tilde{\alpha}'_0(\frac{\partial}{\partial z})$ and $\tilde{\alpha}'_1(\frac{\partial}{\partial z})$ respectively. Then we have
\begin{equation}
\tilde{A}_2 = \left( \begin{array}{cc} \hat{A}_{21} & i\hat{A}_{22} \\ -i\hat{A}_{22} & \hat{A}_{21} \end{array} \right) \quad \text{with} \quad \hat{A}_{21}^{*} + \hat{A}_{21} = 0, \quad \hat{A}_{22}^{*} = \hat{A}_{22}, \tag{4.4}
\end{equation}
and
$$\tilde{B}_1 = \left( \begin{array}{cc} b_1^{*} & ib_1^{*} \\ b_1^{*} & i\hat{b}_1 \end{array} \right).$$
Let $F_0(z) : \mathbb{D} \to G^C$ be the solution to $F_0(z)^{-1}dF_0(z) = \delta_0 dz$, $F_0(0) = I$. We see that
$$F_0 = \left( \begin{array}{cc} F_{01} & 0 \\ 0 & F_{02} \end{array} \right)$$
with $F_{01} = \exp(z\hat{A}_1)$ and
$$F_{02} = \exp(z\hat{A}_2) = \left( \begin{array}{cc} F_{021} & F_{022} \\ -F_{022} & F_{021} \end{array} \right), \quad F_{02}^{-1} = F_{02}^{t} = \left( \begin{array}{cc} F_{021}^{t} & -F_{022}^{t} \\ F_{022}^{t} & F_{021}^{t} \end{array} \right),$$
since $\hat{A}_2$ satisfies (4.4). So the normalized potential has the form by Wu’s formula (3.1)
$$\tilde{\eta} = \lambda^{-1} \left( \begin{array}{cc} 0 & \tilde{B}_1 \\ -\tilde{B}_1^{*}I_{1,1} & 0 \end{array} \right) dz \quad \text{with} \quad \tilde{B}_1 = F_{01}\hat{B}_1 F_{02}^{-1}.$$
So
$$\tilde{B}_1 = F_{01}\hat{B}_1 F_{02}^{-1} = F_{01} \left( \begin{array}{cc} b_1^{*}F_{021} + ib_1^{*}F_{022} & i(b_1^{*}F_{021} + ib_1^{*}F_{022}) \\ b_1^{*}F_{021} + ib_1^{*}F_{022} & i(b_1^{*}F_{021} + ib_1^{*}F_{022}) \end{array} \right) = \left( \begin{array}{cc} b_1^{*} & ib_1^{*} \\ b_1^{*} & i\hat{b}_1 \end{array} \right).$$
By a conjugation of (see Lemma 3.6)
$$Q = \left( \begin{array}{cc} I_2 & 0 \\ 0 & Q_2 \end{array} \right) \quad \text{with} \quad Q_2 = \left( \begin{array}{cccc} 1 & 0 & & \\ & 1 & 0 & \\ & & \cdots & 1 \\ & & & 1 \end{array} \right),$$
we see that the normalized potential $\eta = Q^{-1}\tilde{\eta}Q$ has the desired form (4.3).

The converse part of Theorem 4.4 needs a detailed discussion of the Iwasawa decompositions of $F_-$, see the next Section.

5. POTENTIALS CORRESPONDING TO H-TOTALLY ISOTROPIC SURFACES

In this section we will first give a characterization of H-totally isotropic surfaces in terms of normalized potentials. This also provides a procedure to construct examples. As illustrations we derive two kinds of examples. We will state the main results in this section and leave the computations to the next section.
5.1. The characterization of $H$-totally isotropic surfaces.

**Theorem 5.1.** Let $\eta$ be a normalized potential of the form (4.3). Let $\mathcal{F} = Y \wedge \hat{Y}$ be the corresponding isotropic harmonic map. Then $[Y]$ and $[\hat{Y}]$ are a pair of $H$–totally isotropic adjoint Willmore surfaces on the points they are immersed. And $\mathcal{F}$ is a harmonic map of finite uniton number at most 2.

To prove Theorem 5.1, one need to perform an Iwasawa decomposition. A simple way to do this is to make the potential in (4.3) being of strictly upper-triangle matrices. For this purpose, we will need a Lie group isometry. Then under this isometry, we can write down the Iwasawa decomposition in an explicit way. As a consequence, we can derive some geometric properties of the corresponding Willmore surfaces.

First we define a new Lie group as below

$$G(n, \mathbb{C}) = \{A \in \text{Mat}(n, \mathbb{C}) | A^t J_n A = J_n, \det A = 1\}, \text{ with } J_n = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}. $$

Theorem 5.1 can be derived from the following lemmas.

**Lemma 5.2.** We have the Lie group isometry

$$\mathcal{P} : SO(1, 2m + 1, \mathbb{C}) \to G(2m + 2, \mathbb{C}), $$

with

$$\tilde{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & \\ -i & 1 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & i & 1 & \\ & & & & -i & 1 \end{pmatrix} \quad \text{and} \quad \tilde{P}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Moreover, we have the following results.

1. $\mathcal{P}(SO(1, 2m + 1)) = \{B \in G(2m + 2, \mathbb{C}) | B = S_0 \tilde{B} S_0^{-1}\}$ with

$$S_0 = \tilde{P}_1^{-1} \tilde{P}^{-1} \tilde{P} \hat{P}_1 = \begin{pmatrix} 0 & 0 & J_m \\ 0 & I_2 & 0 \\ J_m & 0 & 0 \end{pmatrix}. $$

So $\mathcal{P}(ASO(1, 2m + 1)) = \{F \in \Lambda G(2m + 2, \mathbb{C}) | \tau(F) = F\}$ with

$$\tau : G(2m + 2, \mathbb{C}) \to G(2m + 2, \mathbb{C}), \quad F \mapsto S_0 \tilde{F} S_0^{-1}. $$

And $\tau(F)^{-1} = \hat{J} \tilde{F}^t \hat{J}^{-1}$ with

$$\hat{J} = \begin{pmatrix} I_m & \\ J_2 & \\ & I_m \end{pmatrix}. $$
Lemma 5.3. Under the isometry of \eqref{5.2}, we have the following results

1. For $\eta_{-1}$ in \eqref{5.3}, one has

\begin{equation}
\mathcal{P}(\eta_{-1}) = \begin{pmatrix}
0 & \tilde{f} & 0 \\
0 & 0 & -J_m \tilde{f} J_2 \\
0 & 0 & 0
\end{pmatrix},
\end{equation}

with $\tilde{f} = \begin{pmatrix}
\tilde{f}_{11} & \tilde{f}_{12} \\
\vdots & \vdots \\
\tilde{f}_{m1} & \tilde{f}_{m2}
\end{pmatrix}$, $\tilde{f}^\sharp := J_m \tilde{f} J_2$,

and

$\tilde{f}_{j1} = i(h_{j1} - \hat{h}_{j1}),$ $\tilde{f}_{j2} = -i(h_{j1} + \hat{h}_{j1}),$ $j = 1, \ldots, m.$

2. Let $H$ be a solution to $H^{-1}dH = \lambda^{-1}\mathcal{P}(\eta_{-1})dz$, $H(0, 0, \lambda) = I$. Then

\begin{equation}
H = I + \lambda^{-1}H_1 + \lambda^{-2}H_2 = \begin{pmatrix}
I & \lambda^{-1}f & \lambda^{-2}g \\
0 & I & -\lambda^{-1}f^\sharp \\
0 & 0 & I
\end{pmatrix},
\end{equation}

with $f^\sharp := J_m f^t J_2$, $f = \int_0^z \tilde{f}dz$ and $g = -\int_0^z f^\sharp dz$.

3. We have the Iwasawa decomposition of $H$ as follows

\begin{equation}
\tilde{F} = H\tau(W)L^{-1} = \begin{pmatrix}
(I - f\bar{u}^2 - gJ\bar{v}J_l)^{-1} & \lambda^{-1}(f + gJ\bar{u})l_0^{-1} & \lambda^{-2}gl_4^{-1} \\
-\lambda(\bar{u}^2 J + f^t J\bar{v})J_l^{-1} & (I - f^t J\bar{u})l_0^{-1} & -\lambda^{-1}f^t l_4^{-1} \\
\lambda J\bar{u}l_0^{-1} & \lambda J\bar{u}l_0^{-1} & l_4^{-1}
\end{pmatrix}.
\end{equation}

Here $W = I + \lambda^{-1}W_1 + \lambda^{-2}W_2$ and $L = diag\{l_1, l_0, l_4\}$ satisfy

$W_0 = \begin{pmatrix}
a & 0 & 0 \\
0 & q & 0 \\
0 & 0 & q
\end{pmatrix} = \tau(L)^{-1}L$, $W_1 = \begin{pmatrix}
0 & u & 0 \\
0 & 0 & -u^2 \\
0 & 0 & 0
\end{pmatrix}$, $W_2 = \begin{pmatrix}
0 & 0 & v \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$.

Here $a, q, d, u, v$ are solutions to the following equation

\begin{align}
(5.7a) & \quad a + uqJ\bar{u}^t + v\bar{q}\bar{v}^t = I, \\
(5.7b) & \quad uq - v\bar{q}\bar{v}^t J = f, \\
(5.7c) & \quad v\bar{q} = g, \\
(5.7d) & \quad q + u^t \bar{g}\bar{v}^t J = I + J\bar{f}^\sharp f, \\
(5.7e) & \quad u^t \bar{q} = f^\sharp - J\bar{f}^\sharp g, \\
(5.7f) & \quad \bar{q} = I + \bar{f}^t J\bar{f}^\sharp + \bar{g}^t g.
\end{align}

4. The $M-C$ form of $\tilde{F}$ has the form

\begin{equation}
\tilde{\alpha}_1' = \begin{pmatrix}
0 & l_1 f^t l_0^{-1} & 0 \\
0 & 0 & -l_0 f^t l_4^{-1} \\
0 & 0 & 0
\end{pmatrix} dz,
\end{equation}

\begin{equation}
\tilde{\alpha}_0 = \lambda^{-1} \begin{pmatrix}
-f^t \bar{u}^2 J - l_1 l_1^{-1} & 0 & 0 \\
0 & -\bar{u}^2 Jf' - f^t J\bar{u} - l_0 l_0^{-1} & 0 \\
0 & 0 & -J\bar{f}^t l_4^{-1} - l_4 l_4^{-1}
\end{pmatrix} dz.
\end{equation}
Lemma 5.4. Under the isometry of $[5.2]$, $y$ is an $H$–totally isotropic surface with an adjoint transform $\hat{y} = [\hat{Y}]$.

Lemma 5.5. Under the isometry of $[5.2]$, $P(\eta_{-1})$ takes value in the nilpotent Lie sub-algebra $\mathfrak{o}_{nil} = \{ X \in \text{Ag}(2m + 2, \mathbb{C}) \mid X = \lambda^{-1} \left( \begin{array}{cccc} 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) + \lambda^{-2} \left( \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \}$. As a consequence, $\mathcal{F} = Y \wedge \hat{Y}$ is of finite uniton number at most $2$.

Here the proof of Lemma $[5.2]$ is a straightforward computation so that we leave it to the interested readers. Lemma $[5.5]$ holds apparently. The proof of Lemma 5.3 and 5.4 will be given in Section 6, since it involves many technical computations.

5.2. Constructions of examples. In this subsection we will provide two kinds of examples. The first one concerns the new Willmore two-sphere derived in $[5.2]$, which is the first example of Willmore two-spheres in $S^6$ admitting no dual surfaces. Here we will derive this surface together with one of its adjoint surface. This also indicates that it can be derived from an adjoint transform of some minimal surface in $\mathbb{R}^6$.

The next example concerns one of the most simple Willmore two-spheres in $S^4$, i.e., the one derived by a holomorphic curve in $\mathbb{C}^2 = \mathbb{R}^4$ with total curvature $-4\pi$.

Theorem 5.6. Let

$$\eta = \lambda^{-1} \eta_{-1} \mathrm{d}z \quad \text{with} \quad \hat{B}_1 = \frac{1}{2} \left( \begin{array}{cccc} -i & 1 & i & -1 \\ i & -1 & i & 1 \\ -2iz & 2iz & -2iz & -2iz \end{array} \right).$$

Then the associated family of the corresponding isotropic harmonic maps is $Y \wedge \hat{Y}$, with

$$Y = -\frac{\sqrt{2}}{2\varsigma} \left( \begin{array}{c} -1 - r^2 - \frac{r^4}{4} - \frac{r^6}{9} \\ 1 - r^2 + \frac{r^4}{4} + \frac{r^6}{9} \\ \frac{r^2}{2} (\lambda^{-1} z - \lambda \bar{z}) \\ -\frac{r^2}{2} (\lambda^{-1} z + \lambda \bar{z}) \\ -i(\lambda^{-1} z - \lambda \bar{z}) \\ \lambda^{-1} z + \lambda \bar{z} \\ \frac{ir^2}{3} (\lambda^{-1} z^2 - \lambda \bar{z}^2) \\ -\frac{ir^2}{3} (\lambda^{-1} z^2 + \lambda \bar{z}^2) \end{array} \right), \quad \hat{Y} = \frac{\sqrt{2}}{2\varsigma} \left( \begin{array}{c} 1 + r^2 + \frac{5r^4}{4} + \frac{4r^6}{9} + \frac{r^{10}}{54} \\ 1 - r^2 - \frac{3r^4}{4} + \frac{4r^6}{9} - \frac{r^{10}}{54} \\ -i(\lambda^{-1} z - \lambda \bar{z})(1 + \frac{r^6}{9}) \\ (\lambda^{-1} z + \lambda \bar{z})(1 + \frac{r^6}{9}) \\ \frac{ir^2}{2} (\lambda^{-1} z - \lambda \bar{z})(1 + \frac{r^6}{9}) \\ -\frac{ir^2}{2} (\lambda^{-1} z + \lambda \bar{z})(1 + \frac{r^6}{9}) \\ -i(\lambda^{-1} z^2 - \lambda \bar{z}^2)(1 + \frac{r^6}{12}) \\ (\lambda^{-1} z^2 + \lambda \bar{z}^2)(1 - \frac{r^6}{12}) \end{array} \right).$$

Here $r = |z|$, $\varsigma = \left| 1 - \frac{r^4}{4} - \frac{2r^6}{9} \right|$. Moreover, we have

1. Set $\hat{Y} = (\hat{y}_0, \ldots, \hat{y}_7)^t$. Then $\hat{y} = \frac{1}{\hat{y}_0}(\hat{y}_1, \ldots, \hat{y}_7)^t = [\hat{Y}]$ is an $H$–totally isotropic, Willmore immersion from $S^2$ to $S^6$, with metric

$$|\hat{y}_2|^2|\mathrm{d}z|^2 = \frac{2(1 + 4r^2 + \frac{r^4}{4} + \frac{2r^6}{9} + \frac{4r^8}{9} + \frac{r^{10}}{54} + \frac{r^{12}}{81})}{\left( 1 + r^2 + \frac{5r^4}{4} + \frac{4r^6}{9} + \frac{r^8}{54} \right)^2} |\mathrm{d}z|^2.$$

$[\hat{Y}]$ has no dual surface.
(2) Set \( Y = (y_0, \cdots, y_7)^t \). Then \( y = \frac{1}{y_0}(y_1, \cdots, y_7)^t = [Y] \) is an \( H \)-totally isotropic, Willmore immersion from \( \mathbb{C} \) to \( S^6 \), with metric
\[
|y_z|^2|dz|^2 = \frac{2(1 + \frac{r^4}{4} + \frac{4r^6}{9})}{1 + r^2 + \frac{r^4}{4} + \frac{4r^6}{9}} |dz|^2.
\]

Note that \([Y]\) is a Willmore map from \( S^2 \), with a branched point \( z = \infty \). Moreover, \( y \) is conformally equivalent to the minimal surface \( x \) in \( \mathbb{R}^8 \):
\[
x = \begin{pmatrix}
i(\lambda^{-1}z - \lambda \bar{z}) \\
-\lambda^{-1}z + \lambda \bar{z}
\end{pmatrix}.
\]

(3) The harmonic map \( Y \land \hat{Y} \) has no definition on the curve \( 1 - \frac{r^4}{4} - \frac{2r^6}{9} = 0 \). But the maps \([Y]\) and \([\hat{Y}]\) are well-defined on the whole two-sphere \( S^2 \).

Remark 5.7.

(1) From this we see that it is possible that although the harmonic map \( Y \land \hat{Y} \) is not globally well-defined, the Willmore surfaces \([Y]\) and \([\hat{Y}]\) are well-defined. This is a very interesting phenomena to be explained, which may be related to the Iwasawa decompositions of the loop group \( AG^C_r \) of the non-compact group \( G = SO^+(1, 2m + 1) \).

(2) One can also derive \( \hat{y} \) from a concrete adjoint transform of the minimal surface \( x \). To ensure \( \hat{y} \) to be immersed, one need some restrictions on the minimal surface \( x \). Our examples here play an important role in the discussions of these conditions. We refer to [17] for more details.

Theorem 5.8. Let
\[
\eta = \lambda^{-1}\eta_{-1}dz \quad \text{with} \quad \hat{B}_1 = \frac{1}{2} \begin{pmatrix} i & -1 & -i & 1 \\ i & -1 & i & -1 \end{pmatrix}.
\]

Then the associated family of the corresponding isotropic harmonic maps is \( Y \land \hat{Y} \), with
\[
Y = \frac{\sqrt{2}}{2\varsigma} \begin{pmatrix}(1 + \frac{r^4}{4})^2 \\ -(1 - \frac{r^4}{2})^2 \\ i(\lambda^{-1}z - \lambda \bar{z}) \\ -\lambda^{-1}z + \lambda \bar{z} \\ -\frac{i}{2}(\lambda^{-1}z - \lambda \bar{z}) \\ \frac{i}{2}(\lambda^{-1}z + \lambda \bar{z}) \end{pmatrix}, \quad \hat{Y} = \frac{\sqrt{2}}{2\varsigma} \begin{pmatrix}(1 + \frac{r^4}{4})^2 \\ (1 - \frac{r^4}{2})^2 \\ -\frac{i}{2}(\lambda^{-1}z - \lambda \bar{z}) \\ -\frac{i}{2}(\lambda^{-1}z + \lambda \bar{z}) \\ -i(\lambda^{-1}z - \lambda \bar{z}) \\ \lambda^{-1}z + \lambda \bar{z} \end{pmatrix}.
\]

Here \( r = |z| \) and \( \varsigma = 1 - \frac{r^4}{4} \). Note that in this case \( Y \) is conformally equivalent to \( \hat{Y} \). Moreover, \( Y \) and \( \hat{Y} \) satisfy the following results.

(1) Set \( Y = (y_0, \cdots, y_5)^t \). Then \( y = \frac{1}{y_0}(y_1, \cdots, y_5)^t = [Y] \) is an \( H \)-totally isotropic, Willmore immersion from \( S^2 \) to \( S^4 \), with metric \( \langle y_z, y_z \rangle|dz|^2 = \frac{2 + \frac{r^4}{4}}{(1 + \frac{r^4}{4})^4}|dz|^2 \).
(2) \([Y]\) is conformally equivalent to the minimal surface \(x\) in \(\mathbb{R}^4\):

\[
x = \left( \frac{2\lambda - 1 - i}{2} \right) - \frac{2\lambda - 1 - i}{2} \left( -i\lambda^{-1} z + i\lambda \bar{z} \right) \lambda^{-1} z + \lambda \bar{z} \right)^T.
\]

(3) The harmonic map \(Y \wedge \hat{Y}\) has no definition on the curve \(1 - \frac{r^4}{4} = 0\). But both \([Y]\) and \([\hat{Y}]\) are Willmore immersions on the whole two-sphere \(S^2\).

Remark 5.9. Set \(\lambda = 1\). On the curve \(\Gamma: 1 - \frac{r^4}{4} = 0\) we have \(z = \sqrt{2}e^{it}\) and hence

\[
(1 - \frac{r^4}{4})Y|_{\Gamma} = \left( \begin{array}{cccc} 2\sqrt{2} & -2\sin t & -2\cos t & 2\sin t \\ 0 & -2\sin t & -2\cos t & 2\cos t \end{array} \right),
\]

\[
(1 - \frac{r^4}{4})\hat{Y}|_{\Gamma} = \left( \begin{array}{cccc} 2\sqrt{2} & -2\sin t & -2\cos t & 2\cos t \end{array} \right).
\]

So \(Y \wedge \hat{Y}\) has no definition on the curve \(\Gamma\).

6. Appendix: Iwasawa decompositions and computations of examples

This section contains two parts: the proof of the technical lemmas in Section 5.1 and the computations of the examples in Section 5.2.

6.1. On the technical lemmas of Section 5.1. To begin with, it is convenient to have the explicit expressions of \(P(A)\) in \((5.2)\). So we will first give this expression and then provide the proofs of Lemma 5.3 and Lemma 5.4.

6.1.1. On \(P(A)\). Set

\[
A = (a_{ij}), \ P(A) = B = (b_{ij}), \ j = 2m + 3 - j \quad \text{and} \quad k = 2m + 3 - k.
\]

Then when \(j = 1, \cdots, m\), we have

\[
(b_{jk}) = \begin{cases} \frac{a_{2j+1,2k+1} - ia_{2j+1,2k+1} + ia_{2j+1,2k+1} + a_{2j+1,2k+1} + a_{2j+1,2k+1}}{2}, & k = 1, \cdots, m; \\ \frac{ia_{2j+1,2k+1} + a_{2j+1,2k+1} + i a_{2j+1,2k+1} + a_{2j+1,2k+1}}{2}, & k = m + 1; \\ - \frac{i a_{2j+1,2k+1} - a_{2j+1,2k+1} + i a_{2j+1,2k+1} + a_{2j+1,2k+1}}{2}, & k = m + 2; \\ - \frac{a_{2j+1,2k+1} + i a_{2j+1,2k+1} + i a_{2j+1,2k+1} + a_{2j+1,2k+1}}{2}, & k = m + 3, \cdots, 2m + 2. \end{cases}
\]

When \(j = m + 1\) we have

\[
(b_{jk}) = \begin{cases} - \frac{ia_{1,2k+1} - ia_{2,2k+1} + a_{1,2k+1} + a_{2,2k+1}}{2}, & k = 1, \cdots, m; \\ \frac{a_{11} + a_{21} + a_{12} + a_{22}}{2}, & k = m + 1; \\ - \frac{a_{11} - a_{21} + a_{12} + a_{22}}{2}, & k = m + 2; \\ \frac{i a_{1,2k+1} + i a_{2,2k+1} + i a_{1,2k+1} + a_{2,2k+1}}{2}, & k = m + 3, \cdots, 2m + 2. \end{cases}
\]
When \( j = m + 2 \) we have
\[
(6.3) \quad b_{jk} = \begin{cases}
\frac{i a_{1,2k+1} - i a_{2,2k+1} - a_{1,2k+2} + a_{2,2k+2}}{2}, & k = 1, \ldots, m; \\
\frac{-a_{11} + a_{21} - a_{12} + a_{22}}{2}, & k = m + 1; \\
\frac{a_{11} - a_{21} - a_{12} + a_{22}}{2}, & k = m + 2; \\
\frac{-i a_{1,2k+1} + i a_{2,2k+1} - a_{1,2k+2} + a_{2,2k+2}}{2}, & k = m + 3, \ldots, 2m + 2.
\end{cases}
\]

When \( j = m + 3, \ldots, 2m + 2 \), we have
\[
(6.4) \quad b_{jk} = \begin{cases}
\frac{-a_{2j} - i a_{2j+2,2k+1} - a_{2j+2} + a_{2j+2,2k+2}}{2}, & k = 1, \ldots, m; \\
\frac{-i a_{2j+1,1} + i a_{2j+1,2} - a_{2j+1} + a_{2j+2}}{2}, & k = m + 1; \\
\frac{i a_{2j+1,1} - a_{2j+1,2} - i a_{2j+1,2} + a_{2j+2}}{2}, & k = m + 2; \\
\frac{-i a_{2j+1,2k+1} + i a_{2j+2,2k+1} - a_{2j+1,2k+2} + a_{2j+2,2k+2}}{2}, & k = m + 3, \ldots, 2m + 2.
\end{cases}
\]

6.1.2. Proof of Lemma [5.3].

1. Assume that \( \eta_{-1} = (\eta_{jk}) \). Then one has
\[
a_{1,2j+1} = a_{2j+1,1} = h_{j1}, \quad a_{1,2j+2} = a_{2j+2,1} = i h_{j1},
\]
\[
a_{2,2j+1} = -a_{2j+1,2} = \hat{h}_{j1}, \quad a_{2,2j+2} = -a_{2j+2,2} = i \hat{h}_{j1},
\]
when \( j = 1, \ldots, m + 1 \), and \( a_{jk} = 0 \) otherwise. Substituting into (6.1)–(6.4), one obtains (6.4).

2. First by definition, \( H(0,0,\lambda) = I \). Next,
\[
H_z = \lambda^{-1} H_{1z} + \lambda^{-2} H_{2z} = \begin{pmatrix}
0 & \lambda^{-1} \tilde{f} & -\lambda^{-2} f \hat{f}^z \\
0 & 0 & -\lambda^{-1} \tilde{f}^z \\
0 & 0 & 0
\end{pmatrix} = H \mathcal{P}(\eta_{-1}).
\]

3. First since \( H(0,0,\lambda) = I \), when \( |z| < \varepsilon \), there exists an Iwasawa Decomposition \( H = \tilde{F} V_+ \), with \( \tilde{F} \in \Delta G_\sigma, V_+ \in \Lambda^* G_\sigma^* \). Next we want to express \( \tilde{F} \) in terms of \( H \). Since \( H = I + \lambda^{-1} H_1 + \lambda^{-2} H_2 \), by the reality condition we see that \( V_+ = V_0 + \lambda V_1 + \lambda^2 V_2 \) with \( V_0, V_1 \) and \( V_2 \) independent of \( \lambda \).

Assume that \( V_+ = V_0 \hat{V}_+ \) such that \( \hat{V}_+|_{\lambda=0} = I \). Then we have
\[
\tilde{F} = H \hat{V}_+^{-1} V_0^{-1}.
\]
Since \( \tau(\tilde{F}) = \tilde{F} \), we obtain \( \tau(H) \tau(V_+^{-1}) \tau(V_0^{-1}) = H \hat{V}_+^{-1} V_0^{-1} \), i.e.,
\[
\tau(H)^{-1} H = \tau(V_+^{-1}) \tau(V_0^{-1}) V_0 \hat{V}_+.
\]

We then assume that
\[
\tau(H)^{-1} H = W W_0 \tau(W)^{-1}
\]
with \( W_0 = \tau(V_0^{-1}) V_0 \), \( W = I + \lambda^{-1} W_1 + \lambda^{-2} W_2 = \tau(\hat{V}_+^{-1}) \) and
\[
W_0 = \begin{pmatrix}
a & 0 & b \\
0 & q & 0 \\
c & 0 & q
\end{pmatrix}, \quad W_1 = \begin{pmatrix}
0 & u & 0 \\
-u^z & 0 & -u^z \\
0 & w_0 & 0
\end{pmatrix}.
\]
Comparing the coefficients of \( \lambda \), we obtain

\[
\begin{aligned}
W_2 W_0 &= H_2, \\
W_1 W_0 + W_2 W_0 \tau(W_1) &= H_1 + \tau(H_1) H_2, \\
W_0 + W_1 W_0 \tau(W_1) + W_2 W_0 \tau(W_2) &= I + \tau(H_1) H_1 + \tau(H_2) H_2.
\end{aligned}
\]  

(6.5)

Direct computation shows that

\[
\tau(H)^{-1} = \begin{pmatrix}
I & 0 & 0 \\
\lambda J f^t & I & 0 \\
\lambda^2 J g^t & -\lambda f^t v & I
\end{pmatrix},
\]

and

\[
\tau(H)^{-1} H = \begin{pmatrix}
I & \lambda^{-1} f & \lambda^{-2} g \\
\lambda J f^t & I + J f^t f & \lambda^{-1} (J f^t - f^t) \\
\lambda^2 J g^t & -\lambda (f^t v + \lambda^2) & I + f^t v + \lambda^2 g
\end{pmatrix}.
\]

(6.6)

From the first two equations of (6.5) we can see that

\[
W_1 W_0 = H_1 + \tau(H_1) H_2 - W_2 W_0 \tau(W_1) = H_1 + \tau(H_1) H_2 - H_2 \tau(W_1)
\]

\[
= \begin{pmatrix}
0 & \cdots & 0 \\
0 & 0 & \cdots \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \cdots \\
0 & 0 & 0
\end{pmatrix} - \begin{pmatrix}
0 & \cdots & 0 \\
0 & 0 & \cdots \\
0 & 0 & 0
\end{pmatrix}.
\]

So

\[
W_1 = \begin{pmatrix}
0 & \cdots & 0 \\
\cdots & 0 & \cdots \\
0 & 0 & 0
\end{pmatrix}, \text{ i.e., } u_0 = 0.
\]

Then from the last equation of (6.5) we see that

\[
W_0 = (I + \tau(H_1) H_1 + \tau(H_2) H_2) - W_1 W_0 \tau(W_1) - W_2 W_0 \tau(W_2)
\]

\[
= \begin{pmatrix}
\cdots & 0 & 0 \\
0 & \cdots & 0 \\
0 & 0 & \cdots
\end{pmatrix} - \begin{pmatrix}
\cdots & 0 & 0 \\
0 & \cdots & 0 \\
0 & 0 & \cdots
\end{pmatrix} - \begin{pmatrix}
\cdots & 0 & 0 \\
0 & \cdots & 0 \\
0 & 0 & \cdots
\end{pmatrix},
\]

i.e., \( b = c = 0 \). So we have that

\[
W_0 = \begin{pmatrix}
a & 0 & 0 \\
0 & q & 0 \\
0 & 0 & \bar{q}
\end{pmatrix}, \quad W = \begin{pmatrix}
I & \lambda^{-1} u & \lambda^{-2} v \\
0 & 0 & -\lambda^{-1} u^2 \\
0 & 0 & I
\end{pmatrix} \quad \text{and} \quad \tau(W)^{-1} = \begin{pmatrix}
I & 0 & 0 \\
\lambda J u^t & I & 0 \\
\lambda^2 J v^t & -\lambda u^t v & I
\end{pmatrix}.
\]

So

\[
WW_0 \tau(W)^{-1} = \begin{pmatrix}
a + u q J u^t + v \bar{q} v^t & \lambda^{-1} (u q - v \bar{q} v^t) & \lambda^{-2} v \bar{q} \\
\lambda J q u^t - u^2 \bar{q} v^t & 0 & \lambda^{-1} u^2 \bar{q} \\
\lambda^2 g v^t & -\lambda q u^t v & 0
\end{pmatrix}.
\]

(6.7)

By (6.6), (6.7) and \( \tau(H)^{-1} H = WW_0 \tau(W)^{-1} \) we obtain (5.7).

Apparently \( W_0 \) has the decomposition

\[
W_0 = \tau(L)^{-1} L, \quad \text{with} \quad L = \text{diag}\{l_1, l_0, l_4\}.
\]
Now set $\hat{F} = H\tau(W)L^{-1}$. We see that $\tau(\hat{F}) = \hat{F}$ and $H = \hat{F}L\tau(W)^{-1}$ is an Iwasawa decomposition of $H$. Substituting
\[
\tau(W) = \begin{pmatrix} I & 0 & 0 \\ -\lambda u^2 J & I & 0 \\ \lambda J u I & \lambda J u & I \end{pmatrix},
\]
$L^{-1}$ and $H$ into $\hat{F} = H\tau(W)L^{-1}$, one obtains (5.6).

(4) Since
\[
\tau(W) = I + \lambda \tau(W_1) + \lambda^2 \tau(W_2), \quad \tau(W)^{-1} = I - \lambda \tau(W_1) + \lambda^2 (\cdots),
\]
So
\[
\tilde{\alpha}_1 = L\mathcal{P}(\eta-1)L^{-1}dz, \quad \tilde{\alpha}_0 = (-\tau(W_1)\mathcal{P}(\eta-1) + \mathcal{P}(\eta-1)\tau(W_1) - L_1L^{-1})dz.
\]
Then [5.8] and [5.9] follow. This finishes the proof of Lemma [5.3].

6.1.3. Proof of Lemma [5.4] Assume that
\[
\hat{F} = \mathcal{P}(F) \quad \text{and} \quad \alpha' = F^{-1}F_zdz = (b_{jk})dz.
\]
So we have $\mathcal{P}(\alpha') = \tilde{\alpha}' + \tilde{\alpha}_0$. Applying (6.1), (6.4), (5.8) and (5.9), we obtain that
\[
\begin{align*}
-\frac{1}{2} b_{2j+1,2k+1} + \frac{1}{2} b_{2j+2,2k+1} + b_{2j+1,2k+2} + b_{2j+2,2k+2} = 0, & \quad 1 \leq j \leq m, m + 3 \leq k \leq 2m + 2; \\
- \frac{1}{2} i b_{1,2k+1} - \frac{1}{2} i b_{2,2k+1} + b_{1,2k+2} + b_{2,2k+2} = 0, & \quad 1 \leq k \leq m; \\
\frac{1}{2} b_{1,2k+1} - \frac{1}{2} i b_{2,2k+1} - b_{1,2k+2} + b_{2,2k+2} = 0, & \quad 1 \leq k \leq m; \\
\frac{1}{2} b_{2j+1,2k+1} - \frac{1}{2} i b_{2j+2,2k+1} - b_{2j+1,2k+2} + b_{2j+2,2k+2} = 0, & \quad m + 3 \leq j \leq 2m + 2, 1 \leq k \leq m.
\end{align*}
\]
Here $j = 2m + 3 - j$, $k = 2m + 3 - k$. Since $b_{jk} = -b_{kj}$ for $j, k > 1$, and $b_{1k} = b_{k1}$ for $k > 1$, we have
\[
\begin{align*}
\begin{cases}
\quad b_{j,2k+2} = i b_{j,2k+1}, & j = 1, 2, 1 \leq k \leq m; \\
\quad b_{2j+1,2k+1} = b_{2j+2,2k+2}, & 1 \leq j, k \leq m; \\
\quad b_{2j+1,2k+2} = -b_{2j+2,2k+1}, & 1 \leq j, k \leq m.
\end{cases}
\end{align*}
\]
Set
\[
F = (e_0, \hat{e}_0, \psi, \hat{\psi}, \cdots, \psi, \hat{\psi}), \quad Y = \frac{\sqrt{2}}{2}(e_0 - \hat{e}_0) \quad \text{and} \quad \hat{Y} = \frac{\sqrt{2}}{2}(e_0 + \hat{e}_0).
\]
We have then
\[
\begin{align*}
\begin{cases}
\quad e_{0z} = b_{21}\hat{e}_0 + \sum_{1 \leq j \leq m} b_{1,2j+1}(\psi_j + i\hat{\psi}_j), \\
\quad \hat{e}_{0z} = b_{21}e_0 - \sum_{1 \leq j \leq m} b_{2,2j+1}(\hat{\psi}_j + i\psi_j), \\
\quad \psi_{jz} = b_{1,2j+1}e_0 + b_{2,2j+1}\hat{e}_0 - \sum_{1 \leq k \leq m} \left( b_{2j+1,2k+1}\psi_k + b_{2j+2,2k+2}\hat{\psi}_k \right), \\
\quad \hat{\psi}_{jz} = b_{1,2j+2}e_0 + b_{2,2j+2}\hat{e}_0 - \sum_{1 \leq k \leq m} \left( b_{2j+2,2k+1}\psi_k + b_{2j+1,2k+2}\hat{\psi}_k \right).
\end{cases}
\end{align*}
\]
So
\[
(\psi_j + i\hat{\psi}_j)_z = - \sum_{1 \leq k \leq m} (b_{2j+1,2k+1} + i b_{2j+2,2k+1})(\psi_k + i\hat{\psi}_k) \mod \{Y, \hat{Y}\}.
\]
(ψ_j + i \hat{ψ}_j)z = - \sum_{1 \leq k \leq m} (b_{2j+1,2k+1} + ib_{2j+2,2k+1}) (ψ_k + i \hat{ψ}_k) \mod \{Y, \hat{Y}\}.

Since

Y_z = -b_{21}Y + \frac{\sqrt{2}}{2} \sum_j (b_{1,2j+1} + b_{2,2j+1}) (ψ_j + i \hat{ψ}_j)

and

\hat{Y}_z = b_{21}\hat{Y} + \frac{\sqrt{2}}{2} \sum_j (b_{1,2j+1} - b_{2,2j+1}) (ψ_j + i \hat{ψ}_j),

it is straightforward to verify that Y and \hat{Y} satisfy (6.1) and (6.2). This finishes the proof of Lemma 5.4.

6.2. Computations on the examples. This subsection is to derive the examples stated in Section 5. To begin with, first we recall the formula of expressing Y and \hat{Y} by elements of H. Then we will apply the formula to derive the examples.

6.2.1. From frame to Willmore surfaces. Suppose that \( F = (e_0, \hat{e}_0, \psi_1, \cdots, \psi_m, \hat{\psi}_m) \), and

\[ Y = \frac{\sqrt{2}}{2} (e_0 - \hat{e}_0), \quad \hat{Y} = \frac{\sqrt{2}}{2} (e_0 + \hat{e}_0) \]

Then y = [Y] and \( \hat{y} = [\hat{Y}] \) are the two Willmore surfaces (which may have branched points) adjoint to each other.

Assume that \( F = (c_{jk}), \hat{F} = (\bar{c}_{jk}) \). Then by (6.1)–(6.4), we have that

\[
\begin{align*}
\bar{c}_{j,m+1} + \bar{c}_{j,m+1} &= c_{2j+2,1} + c_{2j+2,2}, \\
- i(\bar{c}_{j,m+1} - \bar{c}_{j,m+1}) &= c_{2j+1,1} + c_{2j+2,1}, \\
\bar{c}_{m+1,m+1} + \bar{c}_{m+2,m+1} &= c_{21} + c_{22}, \\
\bar{c}_{m+1,m+1} - \bar{c}_{m+2,m+1} &= c_{11} + c_{12},
\end{align*}
\]

So we have

\[
Y = -\frac{\sqrt{2}}{2} \begin{pmatrix}
\bar{c}_{m+1,m+2} - \bar{c}_{m+2,m+2} \\
\bar{c}_{m+1,m+2} + \bar{c}_{m+2,m+2} \\
- i(\bar{c}_{1,m+2} - \bar{c}_{2,m+2}) \\
\bar{c}_{m+2,m+2}
\end{pmatrix}, \quad \hat{Y} = \frac{\sqrt{2}}{2} \begin{pmatrix}
\bar{c}_{m+1,m+1} - \bar{c}_{m+2,m+1} \\
\bar{c}_{m+1,m+1} + \bar{c}_{m+2,m+1} \\
- i(\bar{c}_{1,m+1} - \bar{c}_{2,m+1}) \\
\bar{c}_{m+1,m+1}
\end{pmatrix}.
\]

6.2.2. Proof of Theorem 5.6 Set r = |z|. By 5.4 we have

\[
f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2z \\ 0 \end{pmatrix}.
\]

By integration one has

\[
f = \begin{pmatrix} z & 0 \\ 0 & z \\ z^2 & 0 \end{pmatrix} \quad \text{and} \quad g = - \begin{pmatrix} 0 & \frac{z^2}{2} \\ \frac{2z^3}{3} & 0 \\ 0 & \frac{z^3}{3} \end{pmatrix}.
\]
Substituting into (5.7f), one obtains

\[ q = \left( \begin{array}{ccc} 1 + \frac{4r^6}{9} & r^2z & \frac{r^4z}{3} \\ r^2z & 1 + \frac{r^4}{4} + \frac{r^6}{9} & r^2 \\ \frac{r^4}{3} & r^2 & 1 + \frac{r^4}{4} \end{array} \right), \]

with

\[ g^{-1} = \frac{1}{|g|} \left( \begin{array}{ccc} (1 - \frac{r^4}{4})^2 + \frac{4r^6}{9} & (1 + \frac{r^6}{4}) & -\frac{zr^2(1 - \frac{r^4}{12})}{1 + \frac{r^4}{4} + \frac{r^6}{9}} \\ -zr^2(1 - \frac{12}{r^4}) & 1 + \frac{r^4}{4} + \frac{r^6}{9} & -r^2(1 + \frac{r^6}{9}) \\ \frac{2r^4z}{3}(1 - \frac{r^4}{8} - \frac{r^6}{18}) & -r^2(1 + \frac{r^6}{9}) & (1 - \frac{2r^6}{9})^2 + \frac{r^4}{4}(1 + \frac{4r^6}{9}) \end{array} \right). \]

Here \(|g| = \zeta^2\) and \(\zeta = 1 - \frac{r^4}{4} - \frac{2r^6}{9}\). Then by (5.7d) one obtains

\[ u^g = (f^g - Jf^g)g^{-1} = \frac{z}{\zeta} \left( \begin{array}{ccc} \frac{1}{9} + \frac{r^6}{9} & \frac{r^2}{3} & -\frac{r^2}{3} \\ -\frac{r^2}{3} & 1 + \frac{r^4}{3} & \frac{r^2}{3} \\ \frac{z(1 - \frac{r^4}{12})}{z(1 - \frac{r^4}{12})} & -\frac{r^2}{3} & 1 + \frac{r^6}{9} \end{array} \right). \]

So

\[ u = \frac{z}{\zeta} \left( \begin{array}{ccc} \frac{1}{9} + \frac{r^6}{9} & \frac{r^2}{3} & -\frac{r^2}{3} \\ -\frac{r^2}{3} & 1 + \frac{r^4}{3} & \frac{r^2}{3} \\ \frac{z(1 - \frac{r^4}{12})}{z(1 - \frac{r^4}{12})} & -\frac{r^2}{3} & 1 + \frac{r^6}{9} \end{array} \right). \]

Substituting \(f, g\) and \(u\) into (5.7d), one obtains

\[ q = l_2. \]

So \(l_0 = l_2\). By (5.6), since

\[ J\bar{u} = \frac{z}{\zeta} \left( \begin{array}{ccc} \bar{z}(1 - \frac{r^4}{12}) & -\frac{r^2}{3} \\ -\frac{r^2}{2} & 1 + \frac{r^6}{9} & \frac{r^2}{3} \\ 1 + \frac{r^4}{4} + \frac{r^6}{9} & 1 + \frac{r^4}{4} + \frac{r^6}{9} \end{array} \right), \]

\[ f + gJ\bar{u} = \frac{z}{\zeta} \left( \begin{array}{ccc} 1 + \frac{r^6}{9} & \frac{r^2}{3} \\ -\frac{r^2}{2} & 1 + \frac{r^4}{3} & \frac{r^2}{3} \\ z(1 - \frac{r^4}{12}) & -\frac{r^2}{3} \end{array} \right), \]

\[ I - f^gJ\bar{u} = \frac{1}{\zeta} \left( \begin{array}{ccc} \frac{1}{9} + \frac{r^6}{9} & \frac{r^2}{3} & -\frac{r^2}{3} \\ -\frac{r^2}{3} & 1 + \frac{r^4}{3} & \frac{r^2}{3} \\ \frac{z(1 - \frac{r^4}{12})}{z(1 - \frac{r^4}{12})} & -\frac{r^2}{3} & 1 + \frac{r^6}{9} \end{array} \right), \]

we have

\[ \left( \begin{array}{cc} \bar{c}_{14} & \bar{c}_{15} \\ \bar{c}_{24} & \bar{c}_{25} \\ \bar{c}_{34} & \bar{c}_{35} \\ \bar{c}_{44} & \bar{c}_{45} \\ \bar{c}_{54} & \bar{c}_{55} \\ \bar{c}_{64} & \bar{c}_{65} \\ \bar{c}_{74} & \bar{c}_{75} \\ \bar{c}_{84} & \bar{c}_{85} \end{array} \right) = \frac{1}{\zeta} \left( \begin{array}{ccc} \lambda^{-1}z(1 + \frac{r^6}{9}) & -\lambda^{-1}zr^2 \frac{2}{3} \\ -\lambda^{-1}r^2z(1 + \frac{4r^2}{3}) & \lambda^{-1}z \\ \lambda^{-1}z(1 - \frac{r^4}{12}) & -\lambda^{-1}r^2 \frac{2}{3} \\ 1 + \frac{r^4}{4} + \frac{r^6}{9} & -r^2 \\ \frac{2r^4z}{3}(1 - \frac{r^4}{8} - \frac{r^6}{18}) & -r^2(1 + \frac{r^4}{4} + \frac{r^6}{9}) \\ \lambda \bar{z}(1 + \frac{r^6}{9}) & -\lambda \bar{z}^2 \frac{2}{3} \\ -\lambda \bar{z}^2 \frac{2}{3} & \lambda \bar{z} \\ \lambda \bar{z}(1 + \frac{r^6}{9}) & -\lambda \bar{z} \frac{2}{3} \end{array} \right). \]

Substituting \(\bar{c}_{jk}\) into (5.8), one derives (5.10).
The rest are straightforward computations, except ̂y being branched at z = ∞ and ̃y being unbranched at z = ∞. To this end, we need to use another coordinate. Set ̃z = 1/z and ̃r = \sqrt{|z|}, we have that

\[ |y_{\bar{z}}|^2 |d\bar{z}|^2 = \frac{2\tilde{r}^2 (\tilde{r}^6 + \frac{r^2}{4} + \frac{4}{9})}{(\tilde{r}^6 + \tilde{r}^4 + \frac{r^2}{4} + \frac{4}{9})^2} |d\bar{z}|^2, \]

\[ |\tilde{y}_{\bar{z}}|^2 |d\bar{z}|^2 = \frac{2\tilde{r}^{12} + 8\tilde{r}^{10} + \frac{r^2}{2} + \frac{4\tilde{r}^6}{9} + \frac{8\tilde{r}^4}{9} + \frac{\tilde{r}^2}{2} + \frac{2}{91}}{(\tilde{r}^8 + \tilde{r}^6 + \frac{5\tilde{r}^4}{4} + \frac{4\tilde{r}^2}{9} + \frac{1}{36})^2} |d\bar{z}|^2. \]

At ̃z = 0, |y_{\bar{z}}|^2 |d\bar{z}|^2 = 0 and |\tilde{y}_{\bar{z}}|^2 |d\bar{z}|^2 = 32 |d\bar{z}|^2. This finishes the proof.

6.2.3. Proof of Theorem 5.8

Set r = |z|. By (5.4) we have

\[ \bar{f} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow f = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}, \quad g = \frac{-z^2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Substituting into (5.11), one obtains

\[ g = \begin{pmatrix} 1 + \frac{r^4}{4} & r^2 \\ r^2 & 1 + \frac{r^4}{4} \end{pmatrix}, \quad g^{-1} = \frac{1}{|g|} \begin{pmatrix} 1 + \frac{r^4}{4} & -r^2 \\ -r^2 & 1 + \frac{r^4}{4} \end{pmatrix}. \]

Here |g| = \zeta^2 with \zeta = 1 - \frac{r^4}{4}. Next one computes

\[ u^s = \frac{z}{\zeta} \begin{pmatrix} -r^2 \\ 1 \end{pmatrix}, \quad u = \frac{z}{\zeta} \begin{pmatrix} -r^2 \\ 1 \end{pmatrix}, \quad \text{and} \quad q = I_2. \]

So \(l_0 = I_2\). Then we have

\[ \lambda J\bar{u}l_0^{-1} = \lambda \frac{\bar{z}}{\zeta} \begin{pmatrix} 1 & -r^2 \\ -r^2 & 1 \end{pmatrix}, \quad \lambda^{-1}(f + gJ\bar{u})l_0^{-1} = \lambda^{-1} \frac{\bar{z}}{\zeta} \begin{pmatrix} -r^2 \\ 1 \end{pmatrix} \]

and

\[ (I - f^sJ\bar{u})l_0^{-1} = \frac{1}{\zeta} \begin{pmatrix} 1 + \frac{r^4}{4} & -r^2 \\ -r^2 & 1 + \frac{r^4}{4} \end{pmatrix}. \]

By (5.6), we have

\[
\begin{pmatrix}
\tilde{c}_{13} & \tilde{c}_{14} \\
\tilde{c}_{23} & \tilde{c}_{24} \\
\tilde{c}_{33} & \tilde{c}_{34} \\
\tilde{c}_{43} & \tilde{c}_{44} \\
\tilde{c}_{53} & \tilde{c}_{54} \\
\tilde{c}_{63} & \tilde{c}_{64}
\end{pmatrix} = \frac{1}{\zeta} \begin{pmatrix}
\frac{-\lambda^{-1}z^2}{2} & \lambda^{-1}z \\
\lambda^{-1}z & -\frac{\lambda^{-1}z^2}{2} \\
1 + \frac{r^4}{4} & -r^2 \\
-r^2 & 1 + \frac{r^4}{4} \\
\lambda \tilde{z} & -\frac{\lambda \tilde{z}^2}{2} \\
-\frac{\lambda \tilde{z}^2}{2} & \lambda \tilde{z}
\end{pmatrix}.
\]

Substituting these data into (6.8), one derives (5.11). The rest are straightforward computations, which we will leave to interested readers.

**Acknowledgements** The author was supported by the NSFC Project No. 11571255 and the Fundamental Research Funds for the Central Universities.
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