Sobolev regularity solutions for a class of singular quasilinear ODEs

In this paper, we consider the following singular second-order ordinary differential equations

\[ u_r + \frac{2}{r} u + C^{-1} \left[ (2\mu + \lambda) r^2 u_r + \left( 2r(2\mu + \lambda) - C \right) u_r - 2(2\mu + \lambda) u \right] u^2 + uf(r) = 0, \quad r \in (0, 1), \quad (1.1) \]

with the initial-boundary condition

\[ 0 < u(0) = u_0 < \varepsilon, \quad u(1) = 0, \quad (1.2) \]

and the constraint condition

\[ \int_0^1 u^{-1}(r) dr = C M, \quad \text{with a fixed constant } C, \quad (1.3) \]

where positive constants \( 0 < \varepsilon < 1 \) and \( \mu, \lambda, C > 0 \), \( f(r) \) denotes an external force. Obviously, there is a singular coefficient \( \frac{2}{r} \) in equation (1.1), and it is a class of dissipative-type ODEs with the dissipative term \( \frac{2}{r} u \).

We state the main result of this paper.

**Theorem 1.1.** Let \( \Omega := (0, 1] \). Assume \( f \in H^s(\Omega) \) and \( \|f\|_{H^s} \leq 1 \) for any fixed integer \( s > 1 \). The singular quasilinear ODEs (1.1) with (1.2)-(1.3) admits a positive \( H^s(\Omega) \)-solution.

Above singular quasilinear second-order ordinary differential equations with the constraint condition comes from the three dimensional Navier-Stokes system for compressible fluids in the steady isothermal case in...
a bounded domain \( \Omega \subset \mathbb{R}^3 \):

\[
- \mu \triangle U - (\lambda + \mu) \nabla \text{div} U + \text{div}(\rho U \otimes U) + \nabla P(\rho) = \rho f,
\]

\[\text{div}(\rho U) = 0,\]

(1.4)

where \( U : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3 \) is the fluid velocity and \( \rho : \Omega \subset \mathbb{R}^3 \to \mathbb{R} \) is its density, meanwhile, it satisfies \( \rho \geq 0 \). Moreover, the total mass is described by

\[\int_\Omega \rho \, dx = M > 0.\]

(1.5)

The constant viscosity coefficients \( \mu \) and \( \lambda \) are assumed to satisfy \( \mu > 0 \) and \( \mu + \lambda > 0 \). \( P(\rho) \) denotes the pressure, it satisfies \( P(\rho) = \rho \) for the isothermal case. \( f : \Omega \to \mathbb{R}^3 \) is a given function which models an outer forced density. For simplicity we assume that the domain \( \Omega \) is an unit ball with a radius 1 and smooth boundary \( \partial \Omega \), namely,

\[\Omega = \{ x \in \mathbb{R}^3 ; |x| \leq 1 \}.\]

We treat the case of Dirichlet boundary conditions

\[U = 0 \quad \text{on} \quad \partial \Omega.\]

(1.6)

The spherically symmetric solution \((\rho, U)\) of the problem (1.4) enjoys the form

\[\rho(x) = \rho(r), \quad U(x) = u(r) \frac{x}{r}, \quad \text{with} \quad r = |x|.\]

(1.7)

On one hand, in the spherical coordinates, the original system (1.4) under the assumption (1.7) takes the form

\[
(\rho u)_r + \frac{2\rho u}{r} = 0,
\]

\[-(2\mu + \lambda)(u_r + \frac{2}{r} u) + (\rho u^2 + \rho)_r + \frac{2}{r} \rho u^2 = \rho f,
\]

(1.8)

then multiplying the first equation of (1.8) by \( u \), we derive

\[
\frac{2}{r} \rho u^2 = -(\rho u^2)_r + \rho uu_r,
\]

by which, we reduce the second equation of (1.8) into

\[-(2\mu + \lambda)(u_r + \frac{2}{r} u) + \rho_r + \rho uu_r = pf.\]

(1.9)

On the other hand, we multiply the first equation of (1.8) by \( \rho u \) to get

\[
\frac{d}{dr}(\rho u)^2 + \frac{4(\rho u)^2}{r} = 0,
\]

which gives that

\[\rho u = \frac{C}{r^2}, \quad \text{with} \quad r \in (0, 1],\]

where \( C \) is an arbitrary constant (We will set it to be a big positive constant). Assume that \( u(r) > 0 \), then we have

\[\rho = \frac{C}{r^2 u}.\]

(1.10)

We substitute (1.10) into the second equation of (1.9), then we obtain the singular quasi-linear ODEs (1.1).

Meanwhile, the total mass (1.5) in spherically symmetric case is equivalent to the constraint condition (1.3). Let the pressure law be \( P(\rho) = \rho^\gamma \) in (1.4), when the adiabatic exponent \( \gamma > \frac{2}{3} \), the first existence of weak solution for the system (1.4) was given by Lions [11]. Novotný-Stráškraba employed the concept of oscillation defect measure developed in [4] to prove the existence result for all \( \gamma > \frac{2}{3} \). Frehse-Steinhauer-Weigant [7] got the existence of weak renormalized solutions to problem (1.4) for all \( \gamma > \frac{2}{3} \). After that, Plotnikov-Weigant [16] extended the existence of weak solution of problem (1.4) to the case of \( \gamma > 1 \). Feireisl-Novotný [5] studied
the existence of weak solutions with arbitrarily large boundary data for stationary compressible Navier-Stokes system with general inhomogeneous boundary conditions. Meanwhile, they required the pressure to be given by the standard hard sphere EOS.

When the adiabatic exponent $\gamma = 1$, in two dimensional case, Lions [11] proved the existence of weak solution for this stationary system by means of a slightly modified equation of mass conservation $ap + \text{div}(\rho u) = 0$. After that, Frehse-Steinhauer-Weigant [6] improved this result. In the three dimension case, Lions [11] pointed out that it is an open problem. To our knowledge, there is no result for the three dimensional viscous compressible isothermal stationary Navier-Stokes equations. In this paper, we are devoted to solving this problem by construction of Sobolev regularity solutions for problem (1.4) with the spherically symmetric case.

Notations.
Throughout this paper, let $\Omega = (0, 1]$. we denote the usual norm of $L^2(\Omega)$ and Sobolev space $H^s(\Omega)$ by $\| \cdot \|_{L^2}$ and $\| \cdot \|_{H^s}$, respectively. The symbol $a \lesssim b$ means that there exists a positive constant $C$ such that $a \leq Cb$. The letter $C$ with subscripts to denote dependencies stands for a positive constant that might change its value at each occurrence.

## 2 Proof of Theorem 1.1

In order to solve the dissipative quasi-linear ODE (1.1) with the boundary condition (1.2), we should overcome two difficulties:

1. There is the loss of derivative phenomenon. It means that the classical fixed point theorem can not be used.

2. A positive solution $u(r)$ should be constructed to satisfy the condition (1.3).

Hence, we have to construct a $H^s(\Omega)$-solution for the dissipative quasi-linear ODE (1.1) with the boundary condition (1.2) by using a suitable Nash-Moser iteration scheme. This method has been used in [18–23]. For general Nash-Moser implicit function theorem, one can see the celebrated work of Nash [13], Moser [12] and Hörmander [9], and Rabinowitz [17].

We now introduce a family of smooth operators possessing the following properties.

**Lemma 2.1.** [2, 8] Let $\Omega \subset \mathbb{R}^n$ with dimension $n \geq 1$. There is a family $\{\Pi_\theta\}_{\theta \geq 1}$ of smoothing operators in the space $H^s(\Omega)$ acting on the class of functions such that

\[
\|\Pi_\theta u\|_{H^{k_1}} \leq C \theta^{(k_1 - k_2)}, \quad \|u\|_{H^{k_2}}, \quad k_1, k_2 \geq 0,
\]

\[
\|\Pi_\theta u - u\|_{H^{k_1}} \leq C \theta^{k_1 - k_2} \|u\|_{H^{k_2}}, \quad 0 \leq k_1 \leq k_2,
\]

\[
\frac{d}{d\theta} \|\Pi_\theta u\|_{H^{k_1}} \leq C \theta^{(k_1 - k_2) - 1} \|u\|_{H^{k_2}}, \quad k_1, k_2 \geq 0,
\]

where $C$ is a positive constant and $(s_1 - s_2)_+ := \max(0, s_1 - s_2)$.

In our iteration scheme, we set

\[ \theta = N_m = 2^m, \quad \forall m = 0, 1, 2, \ldots. \]

Then, by (2.1), it holds

\[
\|\Pi_{N_m} u\|_{H^{k_1}} \lesssim N_m^{k_1 - k_2} \|u\|_{H^{k_2}}, \quad \forall k_1 \geq k_2.
\]

We consider the approximation problem of dissipative quasi-linear ODE (1.10) as follows

\[
\mathcal{L}(u) := u_r + \frac{2}{r} u + C^{-1} \Pi_{N_m} \left[ (2 + \lambda) r^2 u_{rr} + \left( 2r(2 + \lambda) - C \right) u_r - 2(2 + \lambda) u \right] u^2 + \Pi_{N_m} uf.
\]

(2.3)
We first show how to construct positive smooth solution for the dissipative quasi-linear ODE (1.1) with the zero boundary condition, then by means of some transformation, the non-zero boundary condition case can be transformed into the zero boundary case (see page 15). One can see [1, 3, 10, 14] for more related results on elliptic-type equations.

### 2.1 The first approximation step \( m = 1 \)

Let constants \( s \geq 1 \) and \( 0 < \varepsilon < 1 \). For any \( r \in \Omega \), we choose the initial approximation function \( u^{(0)}(r) \in H^{s+\varepsilon}(\Omega) \). Meanwhile, for a fixed small constant \( c(\varepsilon) \), it satisfies

\[
\begin{align*}
  u^{(0)}(r) &> c > 0, \quad u^{(0)}(1) = 0, \quad \forall r \in (0, 1), \\
  \int_0^1 \left( u^{(0)}(r) \right)^{-1} dr & = \hat{C} M, \\
  \| u^{(0)} \|_{H^{s+\varepsilon}} & \lesssim \varepsilon, \\
  \| E^{(0)} \|_{H^{s+\varepsilon}} & \lesssim \frac{\varepsilon}{2},
\end{align*}
\]

(2.4)

where \( \hat{C} \) is a positive constant, and \( E^{(0)} \) denotes the error term taking the form

\[
E^{(0)} := \mathcal{L}(u^{(0)}).
\]

We now find the first approximation solution denoted by \( u^{(1)} \) of (2.3). The error step between the initial approximation function and first approximation solution is denoted by

\[
h^{(1)} := u^{(1)} - u^{(0)},
\]

then linearizing nonlinear system (2.3) around \( u^{(0)} \) to get the linearized operators as follows

\[
\begin{align*}
  \mathcal{L}[u^{(0)}]h^{(1)} &:= C^{-1}(2\mu + \lambda)r^2 \Pi_{N_1}(u^{(0)})^2 h_r^{(1)} + \left[ 1 + C^{-1}(2r(2\mu + \lambda) - C) \right] \Pi_{N_1}(u^{(0)})^2 h_r^{(1)} \\
  &+ \left[ 2 - \Pi_{N_1} f + 2C^{-1} \Pi_{N_1} u^{(0)} (2\mu + \lambda)^2 u_r^{(0)} + (2r(2\mu + \lambda) - C) u^{(0)} u_r^{(0)} - 3(2\mu + \lambda) u^{(0)} \right] h_r^{(1)},
\end{align*}
\]

(2.5)

We consider the linear system

\[
\begin{align*}
  \mathcal{L}[u^{(0)}]h^{(1)} &= E^{(0)}, \\
  h^{(1)}(0) &= h^{(1)}(1) = 0, \quad h_r^{(1)}(0) = h_r^{(1)}(1) = 0,
\end{align*}
\]

(2.6)

from which, the solution of it gives the first approximation solution of dissipative quasi-linear ODE (1.1). Thus some priori estimates are needed. We first give \( L^2 \)-estimate of solution for (2.6).

**Lemma 2.2.** Let the initial approximation function \( u^{(0)} \) satisfy (2.4). Assume that \( f \in H^1(\Omega) \) and \( \| f \|_{H^1} \leq 1 \). The solution \( h^{(1)} \) of the linear system (2.6) satisfies

\[
\int_0^1 \left( h^{(1)} \right)^2 + \left( h_r^{(1)} \right)^2 \, dr \lesssim \int_0^1 (E^{(0)})^2 \, dr.
\]

**Proof.** Multiplying both sides of the first equation (2.6) by \( h^{(1)} \) and \( h_r^{(1)} \), respectively, it holds

\[
\begin{align*}
  \frac{d}{dr} \left[ \left( \frac{1}{2} + \frac{1}{r^2} - C^{-1}(2\mu + \lambda) \Pi_{N_1} ru^{(0)}(u^{(0)} + ru_r^{(0)}) + 2 C^{-1} \left( 2r(2\mu + \lambda) - C \right) \Pi_{N_1} (u^{(0)})^2 \right) (h^{(1)})^2 \\
  + C^{-1}(2\mu + \lambda)r^2 \Pi_{N_1} (u^{(0)})^2 h_r^{(1)} h_r^{(1)} \\
  + \left[ \frac{2}{r} + \Pi_{N_1} f + C^{-1} \Pi_{N_1} u^{(0)} (2\mu + \lambda)^2 u_r^{(0)} + (2r(2\mu + \lambda) - C) u^{(0)} u_r^{(0)} \\
  - 7(2\mu + \lambda) u^{(0)} \right] \right] (h^{(1)})^2 &= E^{(0)} h^{(1)},
\end{align*}
\]

(2.7)
and
\[
\frac{d}{dt} \left[ \frac{1}{2} C^{-1} (2\mu + \lambda) \Pi_N (u^{(0)})^2 (h_r^{(1)})^2 + \left( \frac{1}{r} + \Pi_N f \right) (h^{(1)})^2 \right.
\]
\[
+ C^{-1} \Pi_N u^{(0)} \left( (2\mu + \lambda) r^2 u_r^{(0)} + (2r(2\mu + \lambda) - C) u^{(0)} u_r^{(0)} - 3(2\mu + \lambda) u^{(0)} \right) (h_r^{(1)})^2 \]
\[
= \left[ 1 + C^{-1} \left( 2r(2\mu + \lambda) - C \right) \Pi_N (u^{(0)})^2 - \frac{1}{2} C^{-1} (2\mu + \lambda) \Pi_N ((ru^{(0)})^2)_r \right] (h_r^{(1)})^2
\]
\[
+ \left[ \frac{1}{2} \Pi_N f_r - C^{-1} \Pi_N \left( (2\mu + \lambda) r^2 u_r^{(0)} + (2r(2\mu + \lambda) - C) u^{(0)} u_r^{(0)} - 3(2\mu + \lambda) u^{(0)} \right) \right] (h_r^{(1)})^2 \]
\[
\left( \frac{2}{r^2} \right). \tag{2.8}
\]

where we use
\[
r^2 (u^{(0)})^2 h_r^{(1)} h^{(1)} = \frac{d}{dr} \left( (r^2 (u^{(0)})^2 h_r^{(1)} h^{(1)}) - r (u^{(0)})^2 (h^{(1)})^2 - r^2 (u^{(0)})^2 (h_r^{(1)})^2 \right)
\]
\[
+ (h^{(1)})^2 \left( r (u^{(0)})^2 + r^2 (u^{(0)})^2 \right)_r - r^2 (u^{(0)})^2 (h_r^{(1)})^2.
\]

We sum up (2.7) and (2.8) to get
\[
\frac{d}{dt} G(h^{(1)}) = (h^{(1)})^2 + \left( \frac{2}{r} + \frac{1}{r^2} \right) (h^{(1)})^2 = I_1(h_r^{(1)})^2 + I_2(h^{(1)})^2 + E^{(0)} \left( h^{(1)} + h_r^{(1)} \right), \tag{2.9}
\]
where
\[
G(h^{(1)}) := \left[ \frac{1}{2} + \frac{1}{r^2} + \Pi_N f - C^{-1} (2\mu + \lambda) \Pi_N r u_r^{(0)} + (u^{(0)})^2 (u^{(0)} + ru_r^{(0)}) + \left( C^{-1} (2\mu + \lambda) - \frac{1}{2} \right) \Pi_N (u^{(0)})^2 \right.
\]
\[
+ C^{-1} \Pi_N u^{(0)} \left( (2\mu + \lambda) r^2 u_r^{(0)} + (2r(2\mu + \lambda) - C) u^{(0)} u_r^{(0)} - 3(2\mu + \lambda) u^{(0)} \right) \left( h_r^{(1)} \right)^2 \]
\[
+ C^{-1} (2\mu + \lambda) r^2 \Pi_N (u^{(0)})^2 h_r^{(1)} h^{(1)} + \frac{1}{2} C^{-1} (2\mu + \lambda) r^2 \Pi_N (u^{(0)})^2 (h_r^{(1)})^2,
\]
\[
I_1 := -C^{-1} \left( 2r(2\mu + \lambda) - C \right) \Pi_N (u^{(0)})^2 + C^{-1} (2\mu + \lambda) \Pi_N \left( \frac{1}{2} (ru^{(0)})^2 \right)_r + (u^{(0)})^2, \tag{2.11}
\]
\[
I_2 := \frac{1}{2} \Pi_N f_r - \Pi_N f - C^{-1} \Pi_N u^{(0)} \left( 2(2\mu + \lambda) r^2 u_r^{(0)} + (2r(2\mu + \lambda) - C) u^{(0)} u_r^{(0)} - 7(2\mu + \lambda) u^{(0)} \right)
\]
\[
+ C^{-1} \Pi_N \left( (2\mu + \lambda) r^2 u_r^{(0)} + (2r(2\mu + \lambda) - C) (u^{(0)})^2 u_r^{(0)} - 3(2\mu + \lambda) (u^{(0)})^2 \right)_r. \tag{2.12}
\]

Note that the boundary condition given in (2.6). We integrate equality (2.9) over Ω, it holds
\[
\int_0^1 \left( (h_r^{(1)})^2 + \left( \frac{2}{r} + \frac{1}{r^2} \right) (h^{(1)})^2 \right) dr = \int_0^1 \left( I_1(h_r^{(1)})^2 + I_2(h^{(1)})^2 \right) dr + \int_0^1 E^{(0)} \left( h^{(1)} + h_r^{(1)} \right) dr. \tag{2.13}
\]

On one hand, we notice that the first approximation function $u^{(0)}$ satisfy (2.4). So by (2.1), Young’s inequality and $\|f\|_{H^1} \leq 1$, there exists a sufficient big positive constant $C$ such that
\[
\|I_1\|_{L^\infty} \leq C_{\epsilon, C},
\]
\[
\|I_2\|_{L^\infty} \leq \|\Pi_N f\|_{L^\infty} + \frac{1}{2} \|\Pi_N f_r\|_{L^\infty} + C_{\epsilon, C} \leq C_{\epsilon, C}, \tag{2.14}
\]
where $C_{\epsilon, C}$ is a positive constant depending on $\epsilon$ and $C$, it will be small as $\epsilon$ small.

On the other hand, by Young’s inequality, it holds
\[
\left| \int_0^1 E^{(0)} \left( h_r^{(1)} + h^{(1)} \right) dr \right| \leq \int_0^1 (E^{(0)})^2 dr + \frac{1}{2} \left( \|h_r^{(1)}\|_{L^2}^2 + \|h^{(1)}\|_{L^2}^2 \right). \tag{2.15}
\]

Thus, by (2.14)-(2.15), it follows from (2.13) that
\[
\int_0^1 \left[ \left( \frac{1}{2} - C_{\epsilon, C} \right) (h_r^{(1)})^2 + \left( \frac{2}{r} + \frac{1}{r^2} - 2 - C_{\epsilon, C} \right) (h^{(1)})^2 \right] dr \leq \int_0^1 (E^{(0)})^2 dr. \tag{2.16}
\]
Note that \( r \in (0, 1) \) and constant \( C_{e,C} \) being small as \( \varepsilon \) small. Thus there exists a positive constant \( C_1 \) such that

\[
\begin{align*}
\frac{1}{r^2} - C_{e,C} &\geq C_1 > 0, \\
\frac{2}{r} + \frac{1}{r^2} - 2 - C_{e,C} &\geq C_1 > 0.
\end{align*}
\]

Hence, it follows from (2.16) that

\[
\int_0^1 \left( (h^{(1)})^2 + (h^{(1)})^2 \right) dr \lesssim \frac{1}{r} \int_0^1 (E^{(0)})^2 dr.
\]

Furthermore, we derive higher order derivatives estimates. For a fixed integer \( s \geq 1 \), we apply \( D^s := \frac{d^s}{dr^s} \) to both sides of (2.6), it holds

\[
\begin{align*}
C^{-1}(2\mu + \lambda) r^2 \Pi_{N_i} (u^{(0)})^2 D^{s+2} h^{(1)} &+ \left[ 1 + C^{-1} \Pi_{N_i} \left( 2r(2\mu + \lambda) - \lambda \right) (u^{(0)})^2 \right] D^{s+1} \Pi_{N_i} (u^{(0)})^{(2)} + \left[ \frac{2}{r} + \Pi_{N_i} f + 2 C^{-1} \Pi_{N_i} u^{(0)} \left( (2\mu + \lambda) r^2 u^{(0)} \right) + \left( 2r(2\mu + \lambda) - \lambda \right) u^{(0)} - 3(2\mu + \lambda) u^{(0)} \right] D^s h^{(1)} \\
+ F &:= D^s E^{(0)},
\end{align*}
\]

with the boundary condition

\[
D^k h^{(1)}(0) = D^k h^{(1)}(1) = 0, \quad D^{k+1} h^{(1)}(0) = D^{k+1} h^{(1)}(1) = 0,
\]

where the integer \( 0 \leq k \leq s \), and

\[
\begin{align*}
F := C^{-1}(2\mu + \lambda) &\sum_{i_1 + i_2 = s, \ 0 \leq i_3 \leq s-1} \Pi_{N_i} \left( r^2 \Pi_{N_i} (u^{(0)})^2 \right) D^{i_2+1} h^{(1)} \\
+ &\sum_{i_1 + i_2 = s, \ 0 \leq i_3 \leq s-1} \Pi_{N_i} \left[ 1 + C^{-1} \Pi_{N_i} \left( 2r(2\mu + \lambda) - \lambda \right) (u^{(0)})^2 \right] D^{i_2} \Pi_{N_i} (u^{(0)})^{(2)} \\
+ &\sum_{i_1 + i_2 = s, \ 0 \leq i_3 \leq s-1} D^{i_2} \left[ \frac{2}{r} + \Pi_{N_i} f + 2 C^{-1} \Pi_{N_i} u^{(0)} \left( (2\mu + \lambda) r^2 u^{(0)} \right) + \left( 2r(2\mu + \lambda) - \lambda \right) u^{(0)} - 3(2\mu + \lambda) u^{(0)} \right] D^{i_1} h^{(1)}.
\end{align*}
\]

Here we notice this term \( \sum_{i_1 + i_2 = s, \ 0 \leq i_3 \leq s-1} (D^{i_1} (\frac{2}{r}) D^{i_2} h^{(1)}) \) can cause some troubles when we carry out energy estimates due to

\[
\sum_{i_1 + i_2 = s, \ 0 \leq i_3 \leq s-1} (D^{i_1} (\frac{2}{r}) D^{i_2} h^{(1)}) = (-1)^s 2r^{-s+1} h^{(1)} + \sum_{k=1}^{s-1} (-1)^{s-k-1} \frac{(s-1)!}{k!} r^{-(s-k)} (h^{(1)})^{(k)}.
\]

Next we derive higher derivative estimate of solution for (2.6).

**Lemma 2.3.** Let the initial approximation function \( u^{(0)} \) satisfy (2.4). Assume that \( f \in H^s(\Omega) \) and \( ||f||_{H^s} \leq 1 \) for any fixed \( s \geq 1 \). The solution \( h^{(1)} \) of the linear system (2.6) satisfies

\[
\int_0^1 (D^s h^{(1)})^2 dr \lesssim \sum_{k=1}^{s} \int_0^1 (D^k E^{(0)})^2 dr.
\]

**Proof.** This proof is based on the induction. Let \( s = 1, \) by (2.17)-(2.18), it holds

\[
\begin{align*}
C^{-1}(2\mu + \lambda)^2 &\Pi_{N_i} (u^{(0)})^2 h^{(1)} + \left[ 1 + C^{-1} \left( 2r(2\mu + \lambda) - \lambda \right) \Pi_{N_i} (u^{(0)})^2 \right] h^{(1)} \\
+ &\sum_{i_1 + i_2 = 1, \ 0 \leq i_3 \leq 0} \Pi_{N_i} \left[ \frac{2}{r} + \Pi_{N_i} f + 2 C^{-1} \Pi_{N_i} u^{(0)} \left( (2\mu + \lambda) r^2 u^{(0)} \right) + \left( 2r(2\mu + \lambda) - \lambda \right) u^{(0)} - 3(2\mu + \lambda) u^{(0)} \right] h^{(1)} \\
eq & E^{(0)}.
\end{align*}
\]
with the boundary condition
\[
D^k h^{(1)}(0) = D^k h^{(1)}(1) = 0, \quad D^{k+1} h^{(1)}(0) = D^{k+1} h^{(1)}(1) = 0, \quad \text{for} \quad k = 0, 1. \tag{2.20}
\]

Multiplying both sides of equation (2.19) by \( h^{(1)}_r + \frac{1}{2} h^{(1)}_r \), it holds
\[
\frac{d}{dr} G(h^{(1)}) + \frac{1}{2} (h^{(1)}_r)^2 + \left( \frac{2}{r} + \frac{1}{2r^2} \right) (h^{(1)}_r)^2 + \left( \frac{3}{r^4} - \frac{2}{r^3} \right) (h^{(1)})^2 + E_r^0 \left( h^{(1)} + \frac{1}{2} h^{(1)}_r \right), \tag{2.21}
\]
where
\[
G(h^{(1)}) := a_0(r) h^{(1)}_r + a_1(r) h^{(1)}_r + a_2(r) h^{(1)}_r + a_3(r) h^{(1)}_r + a_4(r) h^{(1)}_r,
\]
with the coefficients
\[
a_0(r) := C^{-1}(2\mu + \lambda)^2 \Pi_N \left( u^{(0)} \right)^2, \\
a_1(r) := \frac{1}{4} C^{-1}(2\mu + \lambda)^2 \Pi_N \left( u^{(0)} \right)^2, \\
a_2(r) := \frac{1}{2} + \frac{1}{4} \Pi_N f + \frac{1}{2} C^{-1}(2\mu + \lambda)^2 \Pi_N \left( u^{(0)} \right)^2 + \frac{1}{2} C^{-1}(2\mu + \lambda)(1 - r^2) \Pi_N \left( u^{(0)} \right)^2, \\
a_3(r) := \frac{1}{2} + \frac{1}{4} \Pi_N f + \frac{1}{2} C^{-1}(2\mu + \lambda)^2 \Pi_N \left( u^{(0)} \right)^2 + \frac{1}{2} \left( 2r(2\mu + \lambda) - C \right) u^{(0)} (u^{(0)}), \\
a_4(r) := \frac{1}{2} \Pi_N f + \frac{1}{2} \Pi_N \left( 2(2\mu + \lambda)^2 \Pi_N \left( u^{(0)} \right)^2 + (2r(2\mu + \lambda) - C) u^{(0)} - 3(2\mu + \lambda)(u^{(0)})^2, \\
\]
and
\[
A_0(r) := \frac{1}{2} C^{-1} (r(r-2)(2\mu + \lambda) - C) \Pi_N \left( u^{(0)} \right)^2 + \frac{1}{2} C^{-1}(2\mu + \lambda)(1 - r^2) \Pi_N \left( u^{(0)} \right)^2, \\
A_1(r) := \Pi_N f - \frac{1}{4} \Pi_N f_r + \frac{1}{2} C^{-1}(2\mu + \lambda) \left( r^2 \Pi_N \left( u^{(0)} \right)^2 \right)_r - \frac{1}{2} C^{-1} \left( 2r(2\mu + \lambda) - C \right) \Pi_N \left( u^{(0)} \right)^2 + 2C^{-1} \Pi_N \left( 2(2\mu + \lambda)^2 \Pi_N \left( u^{(0)} \right)^2 + (2r(2\mu + \lambda) - C) u^{(0)} - 3(2\mu + \lambda)(u^{(0)})^2, \\
A_2(r) := \frac{1}{4} \Pi_N f_{rr} - \frac{1}{2} \Pi_N f - \Pi_N \left( 2(2\mu + \lambda)^2 \Pi_N \left( u^{(0)} \right)^2 + (2r(2\mu + \lambda) - C) u^{(0)} - 3(2\mu + \lambda)(u^{(0)})^2, \\
\]
Note that the boundary condition (2.20). We integrate equality (2.21) over \( \Omega \), it holds
\[
\int_0^1 \left[ \frac{1}{2} (h^{(1)}_r)^2 + \left( \frac{2}{r} + \frac{1}{2r^2} \right) (h^{(1)}_r)^2 + \left( \frac{3}{r^4} - \frac{2}{r^3} \right) (h^{(1)})^2 + E_r^0 \left( h^{(1)} + \frac{1}{2} h^{(1)}_r \right) \right] dr \tag{2.22}
\]
We now estimate each term in the right hand side of (2.22). Let constant \( 0 < \varepsilon < \infty \). Since we have chosen the first approximation function \( u^{(0)} \) satisfying (2.4), by Young’s inequality and (2.1), \( ||f||_{H^1} \leq 1 \) and \( r \in (0, 1) \), for a sufficient big \( C \), it holds
\[
||A_0(r)||_{L^\infty} \leq \frac{C}{\varepsilon}, \\
||A_1(r)||_{L^\infty} \leq \frac{5}{4} + \frac{C}{\varepsilon}, \\
||A_2(r)||_{L^\infty} \leq \frac{3}{4} + \frac{C}{\varepsilon}, \tag{2.23}
\]
where $\overline{C}_{e,C}$ is a positive constant depending on $\varepsilon$ and $C$, which can be small as $\varepsilon$ small.

Thus by (2.23), we derive

$$\int_{0}^{1} \left[ A_0(r)(h^{(1)}_{rr})^2 + A_1(r)(h^{(1)}_{r})^2 + A_2(r)(h^{(1)})^2 \right] dr \leq \int_{0}^{1} \left[ \overline{C}_{e,C}(h^{(1)}_{rr})^2 + \left( \frac{5}{4} + \overline{C}_{e,C} \right)(h^{(1)}_{r})^2 + \left( \frac{3}{4} + \overline{C}_{e,C} \right)(h^{(1)})^2 \right] dr. \quad (2.24)$$

On the other hand, by Young’s inequality, we integrate by part to get

$$\int_{0}^{1} E_r^{(0)} \left( h^{(1)}_{r} + h^{(1)} \right) dr \leq \frac{9}{2} \int_{0}^{1} (E_r^{(0)})^2 dr + \frac{1}{2} \int_{0}^{1} \left( h^{(1)}_{r} \right)^2 + \frac{1}{2} \int_{0}^{1} \left( h^{(1)} \right)^2 dr. \quad (2.25)$$

Hence, by (2.24)-(2.25), we can reduce (2.22) into

$$\int_{0}^{1} \left[ \left( \frac{1}{4} - \overline{C}_{e,C} \right)(h^{(1)}_{rr})^2 + \left( \frac{2}{r} \right) + \frac{1}{2} \left( \frac{1}{2} \right) - \frac{7}{4} - \overline{C}_{e,C} \right)(h^{(1)}_{r})^2 + \left( \frac{3}{r^3} \right) - \frac{2}{r} - \frac{3}{4} - \overline{C}_{e,C} \right)(h^{(1)})^2 \right] dr \leq \frac{1}{2} \int_{0}^{1} (E_r^{(0)})^2 dr. \quad (2.26)$$

Note that $r \in (0, 1]$ and constant $\overline{C}_{e,C}$ being small as $\varepsilon$ small. Thus there exists a positive constant $\overline{C}_1$ such that

$$\frac{1}{4} - \overline{C}_{e,C} \geq \overline{C}_1 > 0,$$

$$\frac{2}{r} + \frac{1}{2} \left( \frac{1}{2} \right) - \frac{7}{4} - \overline{C}_{e,C} \geq \overline{C}_1 > 0,$$

$$\frac{3}{r^3} - \frac{2}{r} - \frac{3}{4} - \overline{C}_{e,C} \geq \overline{C}_1 > 0,$$

which combining with (2.26) gives that

$$\int_{0}^{1} \left( h^{(1)}_{r} \right)^2 dr \leq \int_{0}^{1} \left( h^{(1)}_{r} \right)^2 + \left( h^{(1)} \right)^2 dr \leq \int_{0}^{1} (E_r^{(0)})^2 dr.$$

Assume that the $(s - 1)$th derivative case holds, i.e.,

$$\int_{0}^{1} (D^{s-1} h^{(1)})^2 dr \leq \sum_{k=1}^{s-1} \int_{0}^{1} (D^k E^{(0)})^2 dr. \quad (2.27)$$

We now prove the $s$th derivative case holds. Obviously, equation (2.17) can be written as

$$C^{-1}(2\mu + \lambda)^2 \Pi_N(u^{(0)})^2 D^{s+2} h^{(1)} + \left[ 1 + C^{-1} \Pi_N \left( 2r(2\mu + \lambda) - C \right) (u^{(0)})^2 \right.\left. + C^{-1}(2\mu + \lambda) \left( r^2 \Pi_N (u^{(0)})^2 \right)_r \right] D^{s+1} h^{(1)}$$

$$+ \left[ \frac{2}{r} + \Pi_N f + 2C^{-1} \Pi_N u^{(0)} \left( (2\mu + \lambda)^2 u^{(0)} r^2 u^{(0)} - \left( 2r(2\mu + \lambda) - C \right) u^{(0)} - 3(2\mu + \lambda) u^{(0)} \right) \right]$$

$$+ C^{-1}(2\mu + \lambda) \left( r^2 \Pi_N (u^{(0)})^2 \right)_r - 4C^{-1} \left( (2\mu + \lambda) - C \right) \Pi_N (u^{(0)})^2 \right] D^s h^{(1)}$$

$$+ \sum_{i_1 + i_2 = s, \ os i_3 = s-1} D^{i_1} \left( \frac{2}{r} \right) D^{i_2} h^{(1)} + F = D^s E^{(0)}, \quad (2.28)$$
with
\[
\mathcal{F} := C^{-1}(2\mu + \lambda) \sum_{i_1 + i_2 = s, 0 \leq i_2 \leq s - 3} D^{i_1} \left( r^2 \Pi_{N_i}(u^{(0)})^2 \right) D^{i_2} h^{(1)} + \\
+ \sum_{i_1 + i_2 = s, 0 \leq i_2 \leq s - 2} D^{i_1} \left[ 1 + C^{-1} \Pi_{N_i} \left( 2r(2\mu + \lambda) - C \right) (u^{(0)})^2 \right] D^{i_2} h^{(1)} + \\
+ \sum_{i_1 + i_2 = s, 0 \leq i_2 \leq s - 1} D^{i_1} \left[ \Pi_{N_i} f + 2C^{-1} \Pi_{N_i} u^{(0)} \left( (2\mu + \lambda) + \left( 2r(2\mu + \lambda) - C \right) u^{(0)} \right) \\
- 3(2\mu + \lambda) u^{(0)} \right] D^{i_2} h^{(1)}.
\]

(2.29)

Multiplying both sides of equation (2.28) by \( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \), it holds
\[
\frac{d}{dr} \mathcal{G}(h^{(1)}) + \frac{1}{2} (D^{s+1} h^{(1)})^2 + \left( \frac{2}{r} + \frac{1}{2r^2} \right) (D^s h^{(1)})^2 + \mathcal{A}_0(r)(D^{s+1} h^{(1)})^2 + \mathcal{A}_1(r)(D^s h^{(1)})^2 \\
+ \left( \sum_{i_1 + i_2 = s, 0 \leq i_2 \leq s - 1} D^{i_1} u^{(0)} \left( \frac{2}{r} D^{i_2} h^{(1)} \right) \right) \left( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \right) \\
+ \mathcal{F} \left( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \right) = D^s E^{(0)} \left( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \right),
\]

(2.30)

where
\[
\mathcal{G}(h^{(1)}) := C^{-1}(2\mu + \lambda) r^2 \Pi_{N_i}(u^{(0)})^2 D^s h^{(1)} D^{s+1} h^{(1)} + \frac{1}{4} C^{-1}(2\mu + \lambda) \left( r^2 \Pi_{N_i}(u^{(0)})^2 \right) (D^s h^{(1)})^2 \\
\frac{1}{2} \left[ 1 + C^{-1} \Pi_{N_i} \left( 2r(2\mu + \lambda) - C \right) (u^{(0)})^2 \right] (D^s h^{(1)})^2 + \\
+ \frac{1}{2} \left( \sum_{i_1 + i_2 = s, 0 \leq i_2 \leq s - 1} D^{i_1} u^{(0)} \left( \frac{2}{r} D^{i_2} h^{(1)} \right) \right) \left( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \right) \\
+ \Pi_{N_i} f + 2C^{-1} \Pi_{N_i} u^{(0)} \left( (2\mu + \lambda) + \left( 2r(2\mu + \lambda) - C \right) u^{(0)} \right) \\
- 3(2\mu + \lambda) u^{(0)} \right].
\]

By noticing the boundary condition (2.18), we integrate (2.30) over \( (0, 1) \) to get
\[
\int_0^1 \left[ \frac{1}{2} (D^{s+1} h^{(1)})^2 + \left( \frac{2}{r} + \frac{1}{2r^2} \right) (D^s h^{(1)})^2 \right] dr = \\
- \left( \sum_{i_1 + i_2 = s, 0 \leq i_2 \leq s - 1} D^{i_1} u^{(0)} \left( \frac{2}{r} D^{i_2} h^{(1)} \right) \right) \left( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \right) \\
- \mathcal{F} \left( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \right) = D^s E^{(0)} \left( D^s h^{(1)} + \frac{1}{2} D^{s+1} h^{(1)} \right),
\]

(2.31)

We now estimate each of term in the right hand side of (2.31). We notice that the first approximation function \( u^{(0)} \) satisfy (2.4). So by (2.1), Young’s inequality and \( \|f\|_{L^r} \leq 1 \), for a sufficient big \( C \), it holds
\[
\|\mathcal{A}_0(r)\|_{L^\infty} \leq C_{r,C}, \quad \|\mathcal{A}_1(r)\|_{L^\infty} \leq \frac{5}{4} + C_{r,C},
\]

(2.32)
where $C_{c,c}$ is a positive constant depending on $\varepsilon$ and $C$, it will be small as $\varepsilon$ small.

Thus we derive

$$
\int_0^1 \left( \overline{\mathcal{A}}_0(r)(D^{s+1}h^{(1)}_r)^2 + \overline{\mathcal{A}}_1(r)(D^s h^{(1)}_r)^2 \right) dr \leq \int_0^1 \left( C_{c,c}(D^{s+1}h^{(1)}_r)^2 + \left( \frac{5}{4} + C_{c,c} \right) (D^s h^{(1)}_r)^2 \right) dr.
$$

(2.33)

On one hand, we use (2.35) and Young’s inequality to derive

$$
\int_0^1 \left( \sum_{i_1+i_2=s, 0 \leq i_2 < s-1} D^{i_1} \left( \frac{2}{r} \right) D^{i_2} h^{(1)}_r \right) \left( D^s h^{(1)}_r + \frac{1}{2} D^{s+1} h^{(1)}_r \right) dr

\lesssim \frac{1}{16} \int_0^1 \left( (D^s h^{(1)}_r)^2 + (D^{s+1} h^{(1)}_r)^2 \right) dr + C_s \sum_{k=1}^{s-1} \int (D^k h^{(1)}_r)^2 dr,
$$

(2.34)

where we should divide $(0, 1)$ into $(0, \frac{1}{3})$ and $[\frac{1}{3}, 1]$, then if $r \in (0, \frac{1}{3})$, by the boundary condition $D^k h^{(1)}_r(0) = 0 \ (0 \leq k \leq s)$, we can apply de l’Hôpital’s rule to get

$$
\lim_{r \to 0^+} D^{s+1} h^{(1)}_r = \lim_{r \to 0^+} D^s h^{(1)}_r = 0,
$$

which means that for a fixed integer $s > 1$, it holds

$$
D^{s+1} h^{(1)}_r \sim O(\left(\frac{1}{s^4}\right)), \quad \forall r \in (0, \frac{1}{3}).
$$

(2.35)

Thus we use (2.35) and Young’s inequality to get (2.34). If $r \in [\frac{1}{3}, 1]$, (2.34) can be obtained by Young’s inequality directly. Here $C_s$ is a positive constant depending on the fixed integer $s$.

On the other hand, note that (2.29) and $\|f\|_{H^{s+2}} \leq 1$, we again use Young’s inequality to derive

$$
\int_0^1 \overline{F} \left( D^s h^{(1)}_r + \frac{1}{2} D^{s+1} h^{(1)}_r \right) dr \lesssim C_{c,c} \sum_{k=1}^{s-1} \int (D^k h^{(1)}_r)^2 dr + \frac{1}{16} \int \left( (D^s h^{(1)}_r)^2 + (D^{s+1} h^{(1)}_r)^2 \right) dr,
$$

(2.36)

\[
\int_0^1 D^s E^{(0)} \left( D^s h^{(1)}_r + \frac{1}{2} D^{s+1} h^{(1)}_r \right) dr \lesssim \int_0^1 \left( \frac{1}{2} (D^s h^{(1)}_r)^2 + \frac{1}{4} (D^{s+1} h^{(1)}_r)^2 + \frac{9}{2} (D^s E^{(0)})^2 \right) dr.
\]

Hence, by (2.33)-(2.36), it follows from (2.31) that

$$
\int_0^1 \left[ \left( \frac{1}{8} - C_{c,c} \right) (D^{s+1} h^{(1)}_r)^2 + \left( \frac{2}{r} + \frac{1}{2r^2} - \frac{5}{8} - C_{c,c} \right) (D^s h^{(1)}_r)^2 \right] dr

\lesssim \frac{9}{2} \int (D^s E^{(0)})^2 dr + C_s \sum_{k=1}^{s-1} \int (D^k h^{(1)}_r)^2 dr.
$$

(2.37)

Note that $r \in (0, 1]$ and constant $C_{c,c}$ being small as $\varepsilon$ small. Thus there exists a positive constant $C_2$ such that

$$
\frac{1}{8} - C_{c,c} \geq C_2 > 0,
$$

$$
\frac{2}{r} + \frac{1}{2r^2} - \frac{5}{8} - C_{c,c} \geq C_2 > 0.
$$

Therefore, by (2.37), we obtain

$$
\int_0^1 \left( (D^{s+1} h^{(1)}_r)^2 + (D^s h^{(1)}_r)^2 \right) dr \lesssim \sum_{k=1}^{s} \int (D^k E^{(0)})^2 dr.
$$

\[\square\]
Proposition 2.1. Let the initial approximation function \( u^{(0)} \) satisfy (2.4). Assume that \( f \in H^s(\Omega) \) and \( \|f\|_{H^s} \leq 1 \) for any fixed integer \( s \geq 1 \). The linear problem (2.6) admits a solution \( h^{(1)}(r) \in H^s(\Omega) \). Moreover, it holds
\[
\|h^{(1)}\|_{H^s} \lesssim \|E^{(0)}\|_{H^s}. \tag{2.38}
\]

Proof. We use the standard fixed point iteration process to solve the linear problem (2.6). Let \( v^{(1)} = (h^{(1)}, h_r^{(1)}) \). Then linearized equation (2.4) can be rewritten as
\[
\frac{d}{dr}v^{(1)} + \mathcal{A}^{-1}(r) \mathcal{B}(r)v^{(1)} = \mathcal{A}^{-1}(r)G_0(r),
\]
where \( G_0(r) = \begin{pmatrix} 0 \ E^{(0)} \end{pmatrix}^T \) and the matrix \( \mathcal{A}(r) \) is
\[
\mathcal{A}(r) := \begin{pmatrix} 1 & 0 \\ 0 & C^{-1}(2\mu + \lambda)r^2 \Pi_N_i(u^{(0)})^2 \end{pmatrix}, \quad \mathcal{B}(r) := \begin{pmatrix} 0 & 1 \\ a_0(r) & a_1(r) \end{pmatrix},
\]
where the matrix \( \mathcal{A}^{-1}(r) \) is the inverse of matrix \( \mathcal{A}(r) \) due to \( r^2(u^{(0)})^2 > 0 \) by (2.4), and the coefficients
\[
a_0(r) := \frac{2}{r} + \Pi_N_i f + 2C^{-1} \Pi_N_i u^{(0)} \left( (2\mu + \lambda)r^2 u_r^{(0)} - 2(1 - r(2\mu + \lambda))u_r^{(0)} - 3(2\mu + \lambda)u^{(0)} \right),
\]
\[
a_1(r) = 1 - 2C^{-1} \Pi_N_i \left( 1 - r(2\mu + \lambda) \right)(u^{(0)})^2.
\]

Following [24], by the standard fixed point iteration and a priori estimate in Lemma 2.3, we obtain the approximation problem
\[
v^{(1)}(r) = \int_0^r \mathcal{A}^{-1}(y) \left( -\mathcal{B}(y)v^{(1)}(y) + G_0(y) \right) dy
\]
has a Cauchy sequence \( \{v^{(1)}(r)\}_{r \in \mathbb{Z}} \) in \( H^s(\Omega) \) for \( s \geq 1 \), whose limit is \( v^{(1)}(r) \), and it solves the linear problem (2.6) in \( (0, 1] \). Furthermore, summing up both estimates given in Lemma 2.2-2.3, one can derive the estimate (2.38).

\[\square\]

2.2 The \( m \)th approximation step

Let \( R \in (0, 1) \) be a fixed constant. We define
\[
\mathcal{B}_R := \{u^{(k)} : \|u^{(k)}\|_{H^s} \leq R < 1\} \tag{2.39}
\]
with the integers \( 1 \leq k \leq m - 1 \) and \( s \geq 1 \).

Assume that the \( m \)-th approximation solutions of (2.3) is denoted by \( h^{(m)} \) with \( m = 2, 3, \ldots \). Let
\[
h^{(m)} := u^{(m)} - u^{(m-1)},
\]
then it holds
\[
u^{(m)} = u^{(0)} + \sum_{i=1}^{m} h^{(i)}.
\]

Our target is to prove that \( u^{(\infty)} \) is a local solution of nonlinear equations (1.1). It is equivalent to show the series \( \sum_{i=1}^{m} h^{(i)} \) is convergence.

We linearize nonlinear system (1.1) around \( u^{(m-1)} \) to get the linearized system as follows
\[
\mathcal{B}[u^{(m-1)}]h^{(m)} = E^{(m-1)}, \quad \forall r \in \Omega,
\]
with the boundary conditions
\[ h^{(m)}(0) = h^{(m)}(1) = 0, \quad \theta^{(m)}_r(0) = \theta^{(m)}_r(1) = 0, \]
where the error term
\[ E^{(m-1)} := \mathcal{L}[u^{(m-1)}] = \Re(h^{(m)}), \]
and
\[ \Re(h^{(m)}) := \mathcal{L}(u^{(m-1)} + h^{(m)}) - \mathcal{L}(u^{(m-1)}) - \Pi_{N_m} \mathcal{L}[u^{(m-1)}]h^{(m)}, \]
which is also the nonlinear term in approximation problem (2.3) at \( u^{(m-1)} \). The following result is to show how to construct the \( m \)-th approximation solution.

**Proposition 2.2.** Let \( u^{(m-1)} \in \mathbb{B}_R \). Assume that \( f \in H^s(\Omega) \) and \( \|f\|_{H^s} \leq 1 \) for any fixed integer \( s \geq 1 \). Then the linearized problem (2.40) with the boundary condition (2.41) admits a solution \( h^{(m)} \in H^s(\Omega) \), which satisfies
\[ \|h^{(m)}\|_{H^s} \lesssim \|E^{(m-1)}\|_{H^s}, \]
where the error term satisfies
\[ \|E^{(m-1)}\|_{H^s} = \|\Re(h^{(m)})\|_{H^s} \lesssim N_m^2 \|h^{(m)}\|_{H^s}^2. \]

**Proof.** Assume that \( u^{(0)} \) satisfies (2.4). The \( m-1 \)-th approximation solution is
\[ u^{(m-1)} = u^{(0)} + \sum_{i=1}^{m-1} h^{(i)}. \]
Then we will find the \( m \)-th approximation solution \( u^{(m)} \), which is equivalent to find \( h^{(m)} \) such that
\[ u^{(m)} = u^{(m-1)} + h^{(m)}. \]
Substituting (2.46) into (2.3), it holds
\[ \mathcal{L}(u^{(m)}) = \mathcal{L}(u^{(m-1)}) + \Pi_{N_m} \mathcal{L}[u^{(m-1)}]h^{(m)} + \Re(h^{(m)}). \]
Set
\[ \mathcal{L}(u^{(m-1)}) + \Pi_{N_m} \mathcal{L}[u^{(m-1)}]h^{(m)} = 0, \]
we supplement it with the boundary conditions (2.41).

Since we assume \( u^{(m-1)} \in \mathbb{B}_R \), there is the same structure between the linear system (2.6) and the linear system of \( m \)-th approximation solutions. Thus by means of the same proof process in Proposition 2.1, we can show above problem admits a solution \( h^{(m)} \in H^s(\Omega) \). Here we should use (2.2). Furthermore, it holds
\[ \|h^{(m)}\|_{H^s} \lesssim \|E^{(m-1)}\|_{H^s}, \]
where one can see the \( m-1 \)-th error term \( E^{(m-1)} \) such that
\[ E^{(m-1)} := \mathcal{L}(u^{(m-1)}) = \Re(h^{(m)}). \]

\[ \square \]

### 2.3 Convergence of approximation scheme

For a fixed integer \( s > 1 \), let \( 1 \leq \bar{k} < k_0 \leq k \leq s \) and
\[ k_m := \bar{k} + \frac{k - \bar{k}}{2^m}, \quad a_{m+1} := k_m - k_{m+1} = \frac{k - \bar{k}}{2^{m+1}}, \]
which gives that
\[ k_0 > k_1 > \ldots > k_m > k_{m+1} > \ldots \]
**Proposition 2.3.** Assume that \( f \in H^s(\Omega) \) and \( ||f||_{H^s} \leq 1 \) for any fixed integer \( s > 1 \). The dissipative quasilinear ODE
\[
u_r + \frac{2}{T} u + C^{-1} \left[ (2\mu + \lambda) r^2 u_{rr} - 2 \left(1 - r(2\mu + \lambda)\right) u_r - 2(2\mu + \lambda) u \right] u^2 + uf = 0, \tag{2.48}
\]
with the boundary condition
\[
u(0) = u_0 > 0, \quad \nu(1) = 1, \quad \nu(0) = u_0 > 0, \quad \nu(1) = 1,
\]
admits a positive \( H^s \)-solution
\[
u^{(\infty)}(r) = \nu^{(0)}(r) + \sum_{m=1}^{\infty} h^{(m)}(r) + (1 - r)u_0, \quad \forall r \in (0, 1).
\]
Moreover, it holds
\[
\int_0^1 u^{-1}(r) dr = \mathcal{C}M > 1, \text{ with a fixed constant } \mathcal{C}.
\]

**Proof.** The proof is based on the induction. For convenience, we first deal with the case of zero boundary condition, i.e. \( \nu(0) = \nu(1) = 0 \). After that, we discuss the case \( \nu(0) = u_0 > 0 \) and \( \nu(1) = u_1 > 0 \). Note that \( N_m = N_0^m \) with \( N_0 > 1 \). \( \forall m = 1, 2, \ldots \), we claim that there exists a sufficient small positive constant \( \epsilon \) such that
\[
||h^{(m)}||_{H^{m+1}} < \epsilon^m, \quad ||E^{(m-1)}||_{H^m} < \epsilon^{m-1}, \quad u^{(m)} \in \mathcal{B}_\epsilon.
\]

For the case of \( m = 1 \), we recall that the assumption (2.4) on the initial approximation smooth function \( \nu^{(0)} \), i.e.
\[
\nu^{(0)}(r) > c > 0, \quad \text{for a fixed small constant } c < \epsilon,
\]
\[
\int_0^1 \left( \nu^{(0)}(r) \right)^{-1} dr = \mathcal{C}M, \tag{2.50}
\]
\[
||\nu^{(0)}||_{H^{1+1}} \lesssim \epsilon, \quad ||E^{(0)}||_{H^0} \lesssim \frac{\epsilon}{2}.
\]

By (2.44), let \( 0 < \epsilon_0 < N_0^{-8} \epsilon^2 < \frac{\epsilon}{2} \ll 1 \), it derives
\[
||h^{(1)}||_{H^1} \lesssim ||E^{(0)}||_{H^0} \lesssim 2\epsilon_0 < \epsilon.
\]
Moreover, by (2.45) and above estimate, it holds
\[
||E^{(1)}||_{H^1} \lesssim ||\mathcal{R}(h^{(1)})||_{H^1} \lesssim N_2^1 ||h^{(1)}||_{H^1}^2 < 2\epsilon_0 N_2^1 < \epsilon^2,
\]
and
\[
||u^{(1)}||_{H^1} \lesssim ||\nu^{(0)}||_{H^1} + ||h^{(1)}||_{H^1} \lesssim ||\nu^{(0)}||_{H^0} + ||E^{(0)}||_{H^0} \lesssim \epsilon,
\]
which means that \( u^{(1)} \in \mathcal{B}_\epsilon \).
Assume that the case of \( m - 1 \) holds, i.e.
\[
||h^{(m-1)}||_{H^{m-1}} < \epsilon^{m-1}, \quad ||E^{(m-1)}||_{H^{m-1}} < \epsilon^{m-1}, \quad u^{(m-1)} \in \mathcal{B}_\epsilon.
\]

(2.51)
then we prove the case of $m$ holds. Using (2.44), we have
\[
\|h^{(m)}\|_{H^m} \lesssim \|E^{(m-1)}\|_{H^m} < \|E^{(m-1)}\|_{H^{m-1}} < \varepsilon^2,
\]
which combining with (2.45)-(2.47), it holds
\[
\|E^{(m)}\|_{H^m} = \|\mathcal{R}(h^m)\|_{H^m} \\
\lesssim N_{m-1}^2 \|E^{(m-1)}\|_{H^{m-1}}^2 \\
\lesssim N_0^2 \|E^{(m-1)}\|_{H^{m-1}}^2 + \left(\|E^{(m-2)}\|_{H^{m-2}}\right)^2 \\
\lesssim \ldots, \\
\lesssim \left[ N_0^2 \|E^{(0)}\|_{H^0}^2 \right] 2^m.
\]

So by the last condition in (2.50), there is a sufficient small positive constant $\varepsilon_0$ such that
\[
0 < N_0^2 \|E^{(0)}\|_{H^0} < 2 N_0^2 \varepsilon_0 < \varepsilon^2,
\]
which combining with (2.53) gives that
\[
\|E^{(m)}\|_{H^m} < \varepsilon^{2m+1}.
\]

On the other hand, note that $N_m = N_0$, by (2.51)-(2.52), it holds
\[
\|u^{(m)}\|_{H^{m+3}} \lesssim \|u^{(m-1)}\|_{H^{m-1}} + \|h^{(m)}\|_{H^{m+1}} \lesssim \varepsilon + N_m^2 \varepsilon^{2m} \lesssim \varepsilon.
\]

This means that $u^{(m)} \in B_\varepsilon$. Hence we conclude that (2.49) holds.

Furthermore, it follows from (2.49) that the error term goes to 0 as $m \to \infty$, i.e.
\[
\lim_{m \to \infty} \|E^{(m)}\|_{H^m} = 0.
\]

Therefore, equation (2.48) with the zero boundary condition $u(0) = u(1) = 0$ admit a solution
\[
u^{(\infty)}(r) = u^{(0)}(r) + \sum_{m=1}^{\infty} h^{(m)}(r) \in H^k(\Omega),
\]
\[
u^{(0)}(r) + O(\varepsilon) > 0.
\]

Next we discuss the case of non-zero boundary condition
\[
0 < u(0) = u_0 < \varepsilon, \quad u(1) = 0.
\]

We introduce an auxiliary function
\[
f(r) = u(r) - (1 - r)u_0,
\]
then the boundary condition (2.54) is reduced into
\[
\bar{u}(0) = \bar{u}(1) = 0,
\]
and equations (2.48) is transformed into equations of $\bar{u}(r)$. Here we use $u(r)$ to denote $\chi_{(0,1)} u(r)$ for convenience, $\chi_{(0,1)}$ is the character function of the interval $(0, 1)$.

Thus we can follow above iteration scheme to obtain the local existence of $\bar{u}(r)$ for $r \in (0, 1]$. Furthermore, Sobolev regularity solution of equations (2.48) with the boundary condition (2.54) takes the form $\bar{u}(r) + (1 - r)u_0$.

Moreover, since there is
\[
u(r) = u^{(0)}(r) + O(\varepsilon) + (1 - r)u_0 > \varepsilon > 0, \quad r \in (0, 1],
\]
we have

\[ u^{-1}(r) = (u^{(0)}(r))^{-1} \left[ 1 + (u^{(0)}(r))^{-1} \left( \nabla c + (1 - r)u_0 \right) \right]^{-1}, \]

furthermore, we derive

\[
\int_0^1 u^{-1}(r) dr = \int_0^1 (u^{(0)}(r))^{-1} \left[ 1 + (u^{(0)}(r))^{-1} \left( \nabla c + (1 - r)u_0 \right) \right]^{-1} dr
\]

\[
= \left\| 1 + (u^{(0)}(r))^{-1} \left( \nabla c + (1 - r)u_0 \right) \right\|_{L^\infty} \int_0^1 (u^{(0)}(r))^{-1} dr
\]

\[ = C_M. \]

For the density \( \rho \), by (1.10), we know

\[ \rho(r) = \frac{C}{r^2u(r)} = Cr^{-2}(u^{(0)}(r))^{-1} \left[ 1 + (u^{(0)}(r))^{-1} \left( \nabla c + (1 - r)u_0 \right) \right]^{-1}, \]

from which, one can see \( \rho(r) \) is a Sobolev regularity function due to \( u^{(0)}(r) \) being positive smooth function in \( (0, 1) \). This completes the proof.

\[ \square \]

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