Cohomology of topologised monoids

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Abstract
We prove standard results of group cohomology – namely, existence of a long exact sequence, classification of torsors via the first cohomology group, Shapiro’s lemma, the Hochschild-Serre spectral sequence, a decomposition of the cochain complex in the direct product case, and Jannsen’s result on the recovery problem – for cohomology theories such as continuous, analytic, bounded, and pro-analytic cohomology. We also prove these results for certain monoids, as the applications we have in mind concern \((\varphi, \Gamma_1)\)-modules. The cohomology groups considered here all have very concrete interpretations by means of a cochain complex. Therefore, we do not use methods of homological algebra, but explicit calculations on the level of cochains, using techniques dating back to Hochschild and Serre.

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Introduction
There are numerous variants of group cohomology defined via cochains: They can be assumed to be continuous, analytic, bounded etc. In these cases, they however lose their functorial

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properties: The obvious coefficient categories rarely admit quotient objects. The most elegant way of fixing this issue is to admit a larger category of coefficients by taking a sheaf-theoretic point of view (cf. [8]). For many applications though, this point of view simply shifts the issue: Considering only cohomology groups defined via cochains, it is unclear whether something like a Hochschild-Serre spectral sequence exists. If we admit a larger category, the existence of such a spectral sequence is clear, but it is unclear what the objects appearing in said spectral sequence look like.

Sometimes these objects are however of importance. In this article we try to push the cochain point of view as far as we can. On the one hand, this means that the results we will prove hold for all aforementioned topologised group cohomology theories. On the other hand, this means that we generally follow the direct method of Hochschild and Serre (cf. [10]) without appealing to homological arguments. In contrast to arguments from homological algebra, ours will be ad-hoc, combinatorial in nature and very calculation intensive.

While the direct method is ridiculously flexible, it is quite hard to present one streamlined proof that shows off many of its features. For this reason, we will introduce an axiomatic framework that allows us to deal with all of these variants in one go.

One application we have in mind is analytic cohomology of \((\varphi, \Gamma')\)-modules in the sense of [4]. For this reason, we are strictly speaking not setting up a framework for topologised groups, but for topologised monoids. This presents additional difficulties.

In contrast to the approach of Casselman and Wigner (cf. [3]), which is comprehensively laid out in a book of Borel and Wallach (cf. [1]), our results do not depend on the existence of certain types of resolutions, but have conditions that are easy to check in practice.

**Organisation**

We start by introducing a framework of topologised categories and how to define group (and monoid) actions for them in Sects. 1 to 4. While the examples we have in mind are very concrete, the framework itself is rather abstract – especially for the later topic of rigidifications. We encourage the reader to mainly think about the examples listed in example 3.2. After a brief discussion of abstract monoid cohomology in Sect. 5, we fix the setup for the following sections in Sect. 6. Afterwards we show how both the existence of a long exact sequence and the classification of torsors hold in our setting (cf. Sect. 7 and 8). The first glimpse at how much we should appreciate arguments of homological algebra is then given in Sect. 9, where we prove Shapiro’s lemma for topologised groups. (Shapiro’s lemma for topologised monoids will have to wait until Sect. 12.) One of the main motivations to introduce the framework however was the spectral sequence of Hochschild and Serre. In Sect. 10 we follow the arguments of Hochschild’s and Serre’s original article to prove it in our setting. As for applications not just the existence of the spectral sequence, but also the existence of a quasi-isomorphism of complexes is of interest, we also prove a corresponding result in Sect. 11, together with a compatibility with the cup-product.

**1 Topological categories**

There are a number of notions of topological categories in the literature, none of which is standard. For our purposes it is sufficient to have a good notion of discrete spaces.
Definition 1.1 A concrete category is a faithful functor \( \iota : C \to \text{Set} \). One often only says that a category \( C \) is a concrete category, even though the forgetful functor \( \iota \) is an essential part of the datum.

Definition 1.2 A concrete category \( \iota : C \to \text{Set} \) is called a category admitting discrete objects if \( C \) has finite limits and \( \iota \) admits a fully faithful left adjoint \( F \).

We will denote by
\[
C^\delta = \{ X \in C \mid \iota (\text{Hom}_C(X, -)) = \text{Hom}_{\text{Set}}(\iota X, \iota (-)) \}
\]
the discrete objects in \( C \). This terminology is justified as all objects in \( \text{Set} \) give rise to discrete objects, cf. proposition 1.7. By abuse of notation, we will often only say that a category admits discrete objects without specifying the forgetful functor.

We will always denote by \( \bullet \) a singleton set.

Remark 1.3 Note that for a category admitting discrete objects, the forgetful functor is represented by \( F \bullet \).

Proposition 1.4 Let \( C \) be a category admitting discrete objects. Then \( \iota \circ F \simeq \text{id}_{\text{Set}} \).

Proof The isomorphism \( \text{Hom}_{\text{Set}}(X, Y) \cong \text{Hom}_C(FX, FY) = \text{Hom}_{\text{Set}}(\iota FX, \iota FY) \) shows that \( Y \cong \iota FY \).

Proposition 1.5 Let \( C \) be a category admitting discrete objects. Then \( \iota \) maps monomorphisms to monomorphisms.

Proof Let \( X \to Y \) be a monomorphism in \( C \). It suffices to show that for \( \alpha \neq \beta \in \text{Hom}_{\text{Set}}(\bullet, \iota X) \) also their induced maps in \( \text{Hom}_{\text{Set}}(\bullet, \iota Y) \) differ. Assume they did not. \( \alpha \) and \( \beta \) correspond to \( \alpha', \beta' \in \text{Hom}_C(F \bullet, X) \). As \( \iota F \simeq \text{id}_{\text{Set}} \) by proposition 1.4, this implies that
\[
\iota \left( \iota \circ \alpha' \to \iota \circ \beta' \right)
\]
so \( \alpha' = \beta' \) as \( X \to Y \) was assumed to be mono.

Proposition 1.6 Subobjects of discrete objects are discrete.

Proof Let \( X \) be a discrete object in a category admitting discrete objects \( C \), \( D \) a subobject and \( Y \) an arbitrary object in \( C \). We have the following commutative diagram:
\[
\begin{align*}
\text{Hom}_C(X, Y) &\cong \text{Hom}_{\text{Set}}(\iota X, \iota Y) \\
\downarrow \quad &\downarrow \quad \\
\text{Hom}_C(D, Y) &\cong \text{Hom}_{\text{Set}}(\iota D, \iota Y)
\end{align*}
\]
As \( \iota D \to \iota X \) is again a mono by proposition 1.5, the map on the right is surjective and hence so is the bottom one, i.e., \( D \) is discrete.

Proposition 1.7 Let \( C \) be a category admitting discrete objects. Then \( F \) is essentially surjective onto the discrete objects, so \( F : \text{Set} \to C^\delta \) is an equivalence of categories.
Proof Let $S$ be a set. Then

$$\text{Hom}_{\mathcal{C}}(FS, Y) \xrightarrow{i} \text{Hom}_{\mathcal{Set}}(iFS, Y) = \text{Hom}_{\mathcal{C}}(iFS, Y) = \text{Hom}_{\mathcal{C}}(FS, Y),$$

and the identifications imply that the first map is an isomorphism.

On the other hand, if $X$ is discrete, i.e., if

$$\text{Hom}_{\mathcal{C}}(X, Y) = \text{Hom}_{\mathcal{Set}}(iX, Y) = \text{Hom}_{\mathcal{C}}(iX, Y)$$

for all $Y$, then Yoneda implies $X \cong iX$, so $\mathbf{F}$ is essentially surjective onto discrete objects. \qed 

Example 1.8 Categories admitting discrete objects don’t quite behave like topological spaces, as for example singleton objects need not be isomorphic. Consider the following category $\mathcal{C}$ where the objects are tuples $(A, \tau_A)$ with $A$ any set and $\tau_A$ any subset of the power set $2^A$ of $A$. A morphism $(A, \tau_A) \longrightarrow (B, \tau_B)$ in $\mathcal{C}$ is defined as a map $f: A \longrightarrow B$ subject to the condition that for $b \in \tau_B$, $f^{-1}(b) \in \tau_A$.

The forgetful functor has the obvious left adjoint $A \longrightarrow (A, 2^A)$, but singleton sets need not be isomorphic: $(\bullet, \emptyset) \not\cong (\bullet, 2^\bullet)$ and only the latter object is discrete while only $(\bullet, \emptyset)$ is final in $\mathcal{C}$.

Definition 1.9 We say that a category $\mathcal{C}$ admitting discrete objects is topological if the functor $\mathbf{F}: \mathcal{C} \longrightarrow \mathcal{C}$ commutes with finite limits and if for every discrete object $D$ and all objects $X$, $Y$ the natural map

$$\text{Hom}_{\mathcal{C}}(D \times X, Y) \longrightarrow \text{Hom}_{\mathcal{Set}}(iD, \text{Hom}_{\mathcal{C}}(X, Y))$$

is a bijection.

Remark 1.10 If $\mathbf{F}$ commutes with finite limits, it especially maps a final object to a final object.

Remark 1.11 The isomorphism

$$\text{Hom}_{\mathcal{C}}(D \times X, Y) \longrightarrow \text{Hom}_{\mathcal{Set}}(iD, \text{Hom}_{\mathcal{C}}(X, Y))$$

replaces some kind of internal Hom-functor: In the category $\mathbf{CGWH}$ of compactly generated weakly Hausdorff spaces we endow for spaces $X$, $Y$ the set $\text{Hom}_{\mathbf{CGWH}}(X, Y)$ with the compact open topology and call the resulting object $[X, Y]$. This results in a pair of adjoint functors:

$$\text{Hom}_{\mathbf{CGWH}}(Z \times X, Y) \cong \text{Hom}_{\mathbf{CGWH}}(Z, [X, Y]).$$

If $Z$ is furthermore discrete, this reads as

$$\text{Hom}_{\mathbf{CGWH}}(Z \times X, Y) \cong \text{Hom}_{\mathbf{CGWH}}(Z, [X, Y]) = \text{Hom}_{\mathcal{Set}}(iZ, \text{Hom}_{\mathbf{CGWH}}(X, Y)),$$

which is precisely the second requirement we posed for a category admitting discrete objects to be topological.

Example 1.12 Examples of topological categories are: the category of topological spaces, of Hausdorff topological spaces, of metric spaces – all with continuous maps. The category of analytic manifolds (over some arbitrary base) is also topological. In all cases, $i$ is the obvious forgetful functor to $\mathcal{Set}$ and $\mathbf{F}$ maps a set to the same set with the discrete topology. Here we regard discrete sets as zero-dimensional manifolds.

Proposition 1.13 In a topological category, every constant map of sets lifts to a morphism.
Proof A constant map of sets factors as
\[ \begin{array}{ccc}
\mathfrak{X} & \to & \bullet \\
\uparrow & & \uparrow \\
\mathfrak{Y} & \to & \\
\end{array} \]
As F\bullet is terminal in a topological category, this factorisation lifts to morphisms in the topological category.

\[\square\]

2 Topologised groups and monoids

Definition 2.1 A topologised group is a group object in a topological category. Similarly, a topologised monoid will mean a monoid object in a topological category. A morphism of topologised groups is a morphism \( \phi : G \to H \) in the ambient category such that the diagram

\[ \begin{array}{ccc}
G \times G & \xrightarrow{\text{mul}} & G \\
\downarrow^{(\phi, \phi)} & & \downarrow^{\phi} \\
H \times H & \xrightarrow{\text{mul}} & H \\
\end{array} \]

commutes. For a morphism of topologised monoids we furthermore require the commutativity of the following diagram:

\[ \begin{array}{ccc}
G & \xrightarrow{\phi} & H \\
\downarrow^{1} & & \downarrow^{1} \\
1 & \xrightarrow{} & 1 \\
\end{array} \]

Here 1 is the trivial group structure on a final object of the ambient category and the morphisms \( 1 \to H \), \( 1 \to G \) are the inclusions of identity elements.

Remark 2.2 Note that if \( \mathcal{C} \) is a topological category, \( \mathfrak{i} \) maps topologised groups to groups and \( \mathbf{F} \) maps groups to (discrete) group objects in \( \mathcal{C} \). We again have an equivalence between the category of (abstract) groups and discrete topologised groups via \( \mathbf{F} \).

Remark 2.3 If this topological category is the category of (Hausdorff) topological spaces, our notions of topologised groups and monoids coincide with the standard ones of topological groups and monoids. Other important examples are the categories of \( L \)-analytic manifolds where \( L \) is a local field.

Definition 2.4 A morphism \( N \to G \) of topologised groups is called normal if its cokernel exists in the category of topologised groups with kernel exactly \( N \to G \). The cokernel \( G \to C \) will also be called the quotient of \( G \) by \( N \) and we will simply write \( C \cong G/N \). A morphism \( U \to G \) is called an open normal subgroup if it is normal and \( G/U \) is discrete.

Remark 2.5 Note that this definition allows us to avoid the notion of strictness, which is rather cumbersome, cf. remark 2.10. Indeed, consider the bijective morphism \( \mathbb{R}^{d} \to \mathbb{R} \) in the category of locally compact groups, where \( \mathbb{R}^{d} \) carries the discrete topology and \( \mathbb{R} \) the usual one. It is easy to see that the cokernel of this morphism is the trivial morphism \( \mathbb{R} \to 1 \), which has kernel \( \mathbb{R} \to \mathbb{R} \), so \( \mathbb{R}^{d} \to \mathbb{R} \) is not normal.

Proposition 2.6 Let \( U \to G \) be an open normal subgroup of topologised groups. Then

\[ \mathfrak{i}(G/U) \cong \mathfrak{i}G/\mathfrak{i}U. \]
Proof Note that $\iota$ commutes with arbitrary limits and especially with taking kernels. Therefore

$$\text{Hom}_{\text{Grp}}(\iota(G/U), H) = \iota(\text{Hom}_{\text{Grp}}(G/U, FH))$$

$$= \iota\left\{ f \in \text{Hom}_{\text{Grp}}(G, FH) \mid \ker f \supseteq U \right\}$$

$$\subseteq \left\{ f \in \text{Hom}_{\text{Grp}}(\iota(G), H) \mid \ker f \supseteq \iota(U) \right\}$$

$$= \text{Hom}_{\text{Grp}}(\iota(G/U), H),$$

which yields a surjection

$$\iota(G/U) \longrightarrow \iota(G/U),$$

as epimorphisms in the category of groups are exactly the surjective group homomorphisms. On the other hand we also have a natural injection $\iota(G/U) \hookrightarrow \iota(G/U)$ as $\iota$ preserves kernels. As both maps clearly coincide, this proves the proposition.

Proposition 2.7 Let $G$ be a topologised group with open normal subgroup $U$. Then $G \longrightarrow G/U$ admits a section in $\mathbf{C}$.

Proof Consider any section $\iota(G/U) \longrightarrow \iota(G)$, which by proposition 2.6 is a section $\iota(G/U) \longrightarrow \iota(G)$. As $G/U$ is discrete, this lifts to a section in $\mathbf{C}$.

Proposition 2.8 Let $G$ be a topologised group with normal subgroup $N$. If $G \longrightarrow G/N$ admits a section in $\mathbf{C}$, then $\iota(G/N) \cong \iota(G/\iota N)$.

Proof As $\iota$ preserves kernel, we always have an injection $\iota(G/\iota N) \longrightarrow \iota(G/N)$. The existence of a section implies that it is also surjective and hence an isomorphism of groups.

Proposition 2.9 Let $G'$ be a topologised group, $M$ a discrete topologised monoid and $U$ an open normal subgroup of $G'$. Then $U \times 1 \longrightarrow G' \times M$ is the kernel of $G' \times M \longrightarrow G'/U \times M$ and the latter map is the cokernel of the former map in the category of topologised monoids.

Proof That $U \times 1 \longrightarrow G' \times M$ is the kernel is clear, as kernels are stable under taking products. For $G' \times M \longrightarrow G'/U \times M$ being its cokernel, note that in the diagram

$$U \times 1 \longrightarrow G' \times M \longrightarrow G'/U \times M$$

the object $G'/U \times M$ is discrete as $\mathbf{F}$ commutes with finite limits, so the corresponding proposition in the category of (abstract) monoids yields the proposition by proposition 2.6.

Remark 2.10 A morphism is called strict if its image and coimage coincide.

Consider the following notion, which we will call the classical image, which is often simply called the image of a morphism, cf. [13, I.10]: The classical image of a morphism $f : X \longrightarrow Y$ is a monomorphism $CI \leftarrow Y$ and a morphism $X \longrightarrow CI$ such that $f = X \longrightarrow CI \leftarrow Y$ and for every other factorisation $f = X \longrightarrow D \longrightarrow Y$ there is a unique morphism $CI \longrightarrow D$ such that the obvious diagrams commute. We can analogously define the classical coimage of a morphism.
Note that it is easy to see that in the category of topological spaces, the classical image is the set theoretic image with the quotient topology (i.e., $V \subseteq f(X)$ is open if and only if $f^{-1}(V)$ is), while the classical coimage is the set theoretic image with the subspace topology of the codomain.

These notions have to be strictly differentiated from the notions of regular images and regular coimages, which are often simply called the image and coimage of a morphism, cf. [12, definition 5.1.1]: The regular image of a morphism $f : X \to Y$ is defined as the equaliser $\lim Y \rightrightarrows Y \sqcup X Y$ and its regular coimage as the coequaliser $\operatorname{colim} X \times Y X \rightrightarrows X$.

It is again easy to see that in the category of topological spaces, the regular image of a morphism is the set theoretic image with the subspace topology, and that the regular coimage is given by the set theoretic image with the quotient topology, i.e., in the category of topological spaces the classical image is the regular coimage and the classical coimage is the regular image!

Indeed, a number of sources simply call regular coimages images to make the confusion complete. For this reason we decided to avoid the notion altogether.

## 3 Rigidified $G$-modules

Let $\mathcal{C}$ be a topological category and $G$ a topologised monoid in $\mathcal{C}$. Then we can define a $G$-module as an abelian group object $A$ in $\mathcal{C}$ together with a morphism $G \times A \to A$ subject to the usual diagrams. Regrettably this definition is too restrictive for our applications.

**Definition 3.1** Let $\mathcal{C}$ be a topological category and $\mathcal{D}$ a concrete category. A rigidification from $\mathcal{C}$ to $\mathcal{D}$ is a bifunctor

$$h : \mathcal{C}^\circ \times \mathcal{D} \to \text{Set}$$

such that functorially in $X$ and $Y$,

$$h(X, Y) \subseteq \text{Hom}_{\text{Set}}(\mathcal{C}(X, Y), \mathcal{D}(Y, Y))$$

and

$$h(\mathcal{D}(\cdot, \cdot)) \cong \mathcal{D}.$$

**Example 3.2** Even though we haven’t yet defined the notion of $h$-pliant objects, we want to give an overview of the most important examples of rigidifications.

| $\mathcal{C}$                      | $\mathcal{D}$ | $h(X, Y)$                                                      | $h$-pliant objects                                      |
|------------------------------------|----------------|---------------------------------------------------------------|---------------------------------------------------------|
| any topological category           | $\mathcal{C}$  | $\text{Hom}_{\mathcal{C}}(X, Y)$                             | all discrete objects                                    |
| analytic manifolds                 | LF-spaces      | locally analytic maps in the sense of [5, section 5]          | all discrete spaces (considered as zero-dimensional manifolds) |
| topological spaces                 | metric spaces  | bounded continuous maps                                       | finite discrete spaces                                  |
But even this notion of rigidifications is in some cases to restrictive.

**Definition 3.3** Let $\mathbf{C}$ be a topological category. A set with $\mathbf{C}$-rigidification is a set $Y$ and a contravariant functor $h_Y : \mathbf{C}^o \to \mathbf{Set}$ such that functorially in $X \in \mathbf{C}$,

$$h_Y(X) \subseteq \text{Hom}_{\mathbf{Set}}(\mathbf{i}X, Y).$$

We furthermore require that $h_Y(\mathbf{F}) = Y$.

For $f \in h_Y(X)$ we will also write $f : X \to Y$. If for discrete $D$ and all $X$ we have an equality $h_Y(D \times X) = \text{Hom}_{\mathbf{Set}}(\mathbf{i}D, h_Y(X))$, we say that $D$ is $Y$-pliant. It follows that if $D$ is $Y$-pliant, then $h_Y(D) = \text{Hom}_{\mathbf{Set}}(\mathbf{i}D, Y)$.

**Remark 3.4** LF-spaces and induced modules are the main reason we have to consider sets with rigidifications and not just rigidifications: Assume a group $G$ with normal subgroup $N$ was to act on an LF-space $A$ in a suitable sense. Then $A^N$, being a kernel, need not be an LF-space itself, cf. [9]. But we still have an object with $\mathbf{C}$-rigidification in the sense of definition 3.3.

**Remark 3.5** Let $h$ be a rigidification from $\mathbf{C}$ to $\mathbf{D}$. Any object in $\mathbf{D}$ then gives rise to an object with $\mathbf{C}$-rigidification via $Y(\mathbf{i}Y, h\mathbf{i}(-, Y))$.

**Definition 3.6** Let $h$ be a rigidification from $\mathbf{C}$ to $\mathbf{D}$. A discrete object $D$ in $\mathbf{C}$ is called $h$-pliant if for all $Y \in \mathbf{D}$, $D$ is $(Y, h(\mathbf{i}(-, Y)))$-pliant.

**Definition 3.7** Let $\mathbf{C}$ be a topological category and $G$ a topologised monoid in $\mathbf{C}$. A $G$-module with $\mathbf{C}$-rigidification is a set with $\mathbf{C}$-rigidification $(A, h_A)$ together with a $G$-module structure on $A$ such that functorially in $X \in \mathbf{C}$

- $h_A(X)$ is a subgroup of $\text{Hom}_{\mathbf{Set}}(\mathbf{i}X, A)$
- for $f \in h_A(X)$ the induced map

$$\mathbf{i}G \times \mathbf{i}X \xrightarrow{(\mathbf{id}, f)} \mathbf{i}G \times A \xrightarrow{\mu} A$$

lies in $h_A(G \times X)$.

A morphism of $G$-modules with $\mathbf{C}$-rigidification $(A, h_A) \to (B, h_B)$ is a morphism of functors $h_A \to h_B$ such that the induced map $A = h_A(\mathbf{F}) \to h_B(\mathbf{F}) = B$ is a morphism of $\mathbf{i}G$-modules. A sequence

$$(A, h_A) \to (B, h_B) \to (C, h_C)$$

is called a short exact sequence of $G$-modules with $\mathbf{C}$-rigidification if for all $X \in \mathbf{C}$ the sequence of abelian groups

$$0 \to h_A(X) \to h_B(X) \to h_C(X) \to 0$$

is exact.

**Remark 3.8** Let $G$ be a topologised group in a topological category $\mathbf{C}$ and $A$ a $G$-module, i.e., an abelian group object in $\mathbf{C}$ with a morphism $G \times A \to A$ subject to the usual diagrams. Then $(\mathbf{i}A, \text{Hom}_{\mathbf{C}}(-, A))$ is a $G$-module with $\mathbf{C}$-rigidification.

If conversely $A$ is an object in $\mathbf{C}$ and $(\mathbf{i}A, h_A)$ a $G$-module with $\mathbf{C}$-rigidification, then we can in general only recover a morphism $G \times A \to A$ in $\mathbf{C}$ if $h_A = \mathbf{i}(\text{Hom}_{\mathbf{C}}(-, A))$: In this case, the map

$$\mathbf{i}G \times \mathbf{i}A \xrightarrow{(\mathbf{id}, \mathbf{i}\mathbf{id})} \mathbf{i}G \times \mathbf{i}A \xrightarrow{\mu} \mathbf{i}A$$

lies in $\mathbf{i}(\text{Hom}_{\mathbf{C}}(G \times A, A))$. 

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Remark 3.9  The definition of an exact sequence of rigidified $G$-modules has concrete interpretations in practice, as the following proposition shows. Indeed they boil down to the usual requirements as for example in [15, (2.7.2)].

The same arguments also show that in a topological category, a sequence of $G$-modules is exact if it is strict (cf. remark 2.10) and the last morphism admits a section in $\mathbf{C}$.

Proposition 3.10  Let $\mathbf{C}$ be the category of compactly generated weakly Hausdorff spaces and $G$ a group object in $\mathbf{C}$. We fix $G$-modules $A$, $B$, $C$ with corresponding rigidifications $h_A$, $h_B$, and $h_C$. Then the short exact sequences

$$\begin{array}{ccc}
(A, h_A) & \longrightarrow & (B, h_B) & \longrightarrow & (C, h_C) \\
\end{array}$$

are in one-to-one correspondence with exact sequences

$$\begin{array}{cccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0,
\end{array}$$

where all maps are continuous, $A$ carries the subspace topology of $B$ and there is a continuous section $C \longrightarrow B$.

Proof  Let us start with a short exact sequences

$$\begin{array}{ccc}
(A, h_A) & \longrightarrow & (B, h_B) & \longrightarrow & (C, h_C). \\
\end{array}$$

By Yoneda, morphisms $h_A \longrightarrow h_B$ are given by morphisms $A \longrightarrow B$ etc. Evaluating at $F$ hence gives a short exact sequence

$$\begin{array}{cccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

with continuous maps. As by assumption

$$\text{Hom}_C(C, B) = h_B(C) \longrightarrow h_C(C) = \text{Hom}_C(C, C)$$

is surjective and the latter includes the identity, this yields a section.

We also clearly have a continuous bijective map $\iota: A \longrightarrow \iota(A)$, where the latter carries the subspace topology. The inclusion $\iota(A) \subseteq A$ clearly get mapped to zero in $h_C(\iota(A))$, so has to come from an element in $h_A(\iota(A))$, which is the (continuous) inverse to $\iota$.

Conversely, if we start with a short exact sequence

$$\begin{array}{cccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

with all maps continuous, $B \longrightarrow C$ admitting a section and $A$ carrying the subspace topology, it is easy to see that indeed all sequences

$$\begin{array}{cccc}
0 & \longrightarrow & h_A(X) & \longrightarrow & h_B(X) & \longrightarrow & h_C(X) & \longrightarrow & 0
\end{array}$$

are exact. \qed

Definition 3.11  Let $\mathbf{C}$ be a topological category, $G$ a topologised monoid in $\mathbf{C}$ and $(A, h_A)$ a $G$-module with $\mathbf{C}$-rigidification. For a normal subgroup $N \leq G$ we define $A^N = (A^N, h_A^N)$ by

$$h_A^N(X) = \left\{ f \in h_A(X) \mid f(\iota \setminus X) \subseteq A^N \right\}.$$ 

We immediately see that this is again a $G$-module with $\mathbf{C}$-rigidification.
Remark 3.12 Let \( C \) be a topological category and \( G \) a topologised group with normal subgroup \( N \). Let \( A \) be a \( G \)-module in the sense that \( A \) is an abelian group object in \( C \) together with a morphism \( G \times A \rightarrow A \) subject to the usual diagrams.

Then there is a slightly more natural notion of the invariants \( A^N \): Let \( g \in \mathcal{L} G \). Then proposition 1.13 yields a morphism

\[
\mu_g: A \rightarrow \mathcal{F} \bullet \times A \rightarrow G \times A \rightarrow A
\]

that we call multiplication by \( g \). We will also denote \( \mu_g - \text{id}_A \in \text{Hom}_C(A, A) \) by \( g - 1 \).

For a finite set \( R \subseteq \mathcal{L} G \), we denote by

\[
\mathcal{L}(A^R) = \ker A \rightarrow \prod_{g \in R} A.
\]

and clearly

\[
\mathcal{L}(A^R) = \mathcal{L}(A)^R.
\]

If \( C \) admits arbitrary limits, we can analogously define \( A^G \). If there is a finite set \( R \subseteq \mathcal{L} G \) such that \( \mathcal{L}(A)^R = \mathcal{L}(A)^{R^e} \), then we will also call \( A^G = A^R \) and it is an easy exercise that both definitions of \( A^G \) (when applicable) coincide. In this case, the universal property of the kernel yields an action of \( G \) on \( A^N \). In the presence of a section \( G \) \( \rightarrow \mathcal{L} G \) in \( C \), we also get a morphism \( G/N \times A^N \rightarrow A^N \) and we can check on the level of sets that this gives \( A^N \) the structure of a \( G/N \)-module.

It is easy to check that both definitions of invariants coincide, i.e.,

\[
(\mathcal{L}(A^N), \text{Hom}_C(-, A^N)) = ((\mathcal{L}A)^{hN}, h_{A^N}),
\]

where \( h_{A^N} \) is defined as in definition 3.11.

4 The induced module

Let \( C \) be a topologised category, \( G \) a topologised monoid in \( C \) and \( H \) a submonoid of \( G \). Let \( (A, h_A) \) be an \( H \)-module with \( C \)-rigidification.

Definition 4.1

\[
\text{Ind}_H^G(A): C \rightarrow \text{Set}
\]

\[
X \mapsto \{ f \in h_A(X \times G) \mid f(x, hg) = h \cdot f(x, g) \text{ for all } x \in \mathcal{L} X, h \in \mathcal{L} H, g \in \mathcal{L} G \}
\]

is called the induced module of \( A \) from \( H \) to \( G \).

Proposition 4.2 Set \( I = \text{Ind}_H^G(A) \), then \( I \) is a \( G \)-module with \( C \)-rigidification, if we give the set \( I(\mathcal{F} \bullet) \subseteq h_A(G) \) the \( \mathcal{L}G \)-module action of right translation:

\[
(gf)(\sigma) = f(\sigma g) \text{ for } f \in I(\mathcal{F} \bullet), g, \sigma \in \mathcal{L} G.
\]

Proof The only difficulty lies in the formalism.

Note first that for \( g \in \mathcal{L} G \) and \( f \in I(\mathcal{F} \bullet) \), \( gf \) indeed lies in \( h_A(G) \), as right-multiplication by \( g \) is a morphism in \( C \). It then follows immediately that \( gf \in I(\mathcal{F} \bullet) \).

We have to show that for \( f \in I(X) \subseteq \text{Hom}_\text{Set}(\mathcal{L} X, I(\mathcal{F} \bullet)) \) \( \subseteq \text{Hom}_\text{Set}(\mathcal{L} X, h_A(G)) \) the induced map

\[
\mathcal{L} G \times \mathcal{L} X \xrightarrow{(\text{id}, f)} \mathcal{L} G \times I(\mathcal{F} \bullet) \xrightarrow{\mu} I(\mathcal{F} \bullet)
\]

\( \square \) Springer
lies in $I(G \times X) \subseteq h_A(G \times X \times G)$.

For this note that there is a morphism $G \times X \times G \to X \times G$ in $\mathcal{C}$, which on the level of elements is given by $(g, x, g') \mapsto (x, g' g)$. Precomposing with this morphism yields a map

$$h_A(X \times G) \to h_A(G \times X \times G)$$

and it is evident that under this map, the subset $I(X)$ gets sent into $I(G \times X)$, which is precisely the map we need. \hfill $\square$

**Remark 4.3** If $\mathcal{C}$ is the category of analytic manifolds over a non-archimedean field, there is also a less sheafy view on the subject of induced modules: Let $G$ be a group object in $\mathcal{C}$, $H \leq G$ a closed subgroup, and $A$ an analytic representation of $H$. Then there is a natural topology on the induced module $\text{Ind}^G_H(A)$, such that the action of $G$ on $\text{Ind}^G_H(A)$ is itself analytic, cf. [7, Kapitel 4].

We have now set the stage to define the cohomology of topologised monoids with coefficients in a rigidified module. Our aim for the remainder of this article is to prove some standard results of group cohomology in this setting. Namely, we compare cohomology of topologised monoids with their discrete counterparts in proposition 7.1, show the existence of a long exact sequence in theorem 7.2, prove two versions of Shapiro’s lemma in theorem 9.8 and 12.1, and show variants of the classical Hochschild-Serre spectral sequence in theorem 10.25 and 11.6.

## 5 Abstract monoid cohomology

Note first that the cohomology of monoids is trickier than one might expect.

**Definition 5.1** Let $M$ be an abstract monoid. Then we define the standard resolution as follows: Denote by $F_n$ the free $\mathbb{Z}[M]$-module with basis $M^n$ and define the coboundary operator via

$$\partial : F_{n+1} \to F_n,$$

$$(x_1, \ldots, x_{n+1}) \mapsto x_1 \cdot (x_2, \ldots, x_{n+1}) + (-1)^n(x_1, \ldots, x_n) + \sum_{i=1}^{n} (-1)^i (x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+1}, \ldots, x_{n+1}).$$

**Proposition 5.2**

$$\ldots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

is a free resolution of the integers.

**Proof** This works exactly the same way it does for groups. \hfill $\square$

**Proposition 5.3** Let $M$ be an abstract monoid and $A$ an $M$-module. Then the inhomogeneous cochain complex computes the cohomology of $A \leftarrow A^M$.

**Proof** Immediate from proposition 5.2. \hfill $\square$
Remark 5.4 The homogeneous cochains do not necessarily form a free resolution. Indeed, set $M = (\mathbb{Z}/2, \cdot)$. Then the homogeneous complex is given by $F'_n = \mathbb{Z}[M^{n+1}]$ with diagonal action and the usual differential. However, it is not a free resolution of the integers: It is evident that $F'_1$ is not cyclic. But every two elements $e_1, e_2 \in F'_1$ admit a non-trivial combination of zero: Multiplied by the monoid element $(0)$, both are contained in $(0, 0)\mathbb{Z} \subseteq \mathbb{Z}[M^2] = F'_1$, say, $(0) \cdot e_1 = \alpha \cdot (0, 0)$ and $(0) \cdot e_2 = \beta \cdot (0, 0)$. If $\alpha$ or $\beta$ is zero, this is a non-trivial combination of zero, otherwise $\beta \cdot (0) \cdot e_1 - \alpha \cdot (0) \cdot e_2$ will do.

Nonetheless, $F'_1$ is still a projective $\mathbb{Z}[M]$-module. Consider the $\mathbb{Z}$-linear homomorphisms

$$F'_1 \longrightarrow \mathbb{Z}[M]$$

$$A_1: (1, 1) \longmapsto (1); (0, 1), (1, 0), (0, 0) \longmapsto (0)$$

$$A_2: (1, 0) \longmapsto (1) - (0); (1, 1), (0, 1), (0, 0) \longmapsto 0$$

$$A_3: (0, 1) \longmapsto (1) - (0); (1, 1), (1, 0), (0, 0) \longmapsto 0.$$

It is easy to verify that these maps are actually $\mathbb{Z}[M]$-linear and that

$$x \longmapsto (1, 1)A_1(x) + (1, 0)A_2(x) + (0, 1)A_3(x)$$

is the identity on $F'_1$, which by the dual basis theorem is therefore projective. One can analogously show that $F'_\bullet$ is still a projective resolution of $\mathbb{Z}$ in this case.

6 Setup

For the remainder of this article, we fix a topological category $\mathbf{C} = (\mathbf{C}, \mathcal{L}, \mathbf{F})$ in which everything takes place, a topologised group $G'$ with an open normal subgroup $U'$ and abelian topologised monoids $M_1$, $M_2$ of which $M_2$ is assumed to be discrete. Set

$$U = U' \times M_1.$$ We also set $G = G' \times M_1 \times M_2$ and see that $U \longrightarrow G$ has a cokernel $G/U \cong G' \times M_2$, whose kernel is $U$ and which is discrete, cf. proposition 2.9. We will furthermore use the shorthand $M = M_1 \times M_2$.

We let further $N'$ be a normal subgroup of $G'$ and $N = N' \times \prod_i M_i^{e_i}$ with $e_i \in \{0, 1\}$. It is again evident that $N \longrightarrow G$ has a cokernel (which we denote by $G/N$) and that the kernel of this cokernel is precisely $N$. We furthermore require the existence of a section $s: G/N \longrightarrow G$ in $\mathbf{C}$ whose image on the level of sets contains the neutral element. If $N = U$, this exists automatically by proposition 2.7 and everything that we prove for the $N$ will also automatically be true for $U$, but the converse does not hold. The section is of vital importance; without it, statements such as $\mathcal{L}(G/N) \cong \mathcal{L}G/\mathcal{L}N$ need not be true, cf. proposition 2.8.

The projections $G \longrightarrow G/N$ and $G \longrightarrow G/U$ will both be denoted by $\pi$. It will always be clear from the context which map is meant.

The section gives rise to two important morphisms: On the one hand, the choice of a representative morphism $(-)^\ast: G \longrightarrow G$, which is the composition of the projection onto $G/N$ followed by the section $s: G/N \longrightarrow G$, and on the other hand the morphism $(-)_N: G \longrightarrow G$ which on $\mathcal{L}G$ is given by $x \longmapsto (x^\ast)^{-1}x$ with the obvious interpretation of this on the monoid parts (either the identity or constant 1). It is clear that $(-)^\ast$ factors through $N \longrightarrow G$, and that the composition

$$N \longrightarrow G \overset{(-)_N}{\longrightarrow} N$$
Lemma 6.1 Let $f \in C^n$. Then $f \in I^j C^n$ if and only if the last $j$ arguments are $\mathcal{I} N$-invariant, i.e.,

$$f(x_1, \ldots, x_{n-j}, x_{n-j+1}\sigma_1, \ldots, x_n\sigma_j) = f(x_1, \ldots, x_n) \text{ for all } x_i \in \mathcal{I} G, \sigma_i \in \mathcal{I} N.$$ 

Proof The “only if” part of the proposition is clear.

For the “if” part, we only show the case of $j = n = 1$, as the other cases follow completely analogously.

As we have a section $s : G/N \to G$, proposition 2.8 shows that $\mathcal{I} G/\mathcal{I} N \cong \mathcal{I} (G/N)$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Set}}(\mathcal{I} G/\mathcal{I} N, A) & \xrightarrow{\mathcal{I} h_A(\pi)} & \text{Hom}_{\text{Set}}(\mathcal{I} G/\mathcal{I} N, A) \\
\text{id} & & \text{Hom}_{\text{Set}}(\mathcal{I} G, A) & \xrightarrow{\mathcal{I} h_A(\pi)} & \text{Hom}_{\text{Set}}(\mathcal{I} G, A) \\
\text{id} & & \text{Hom}_{\text{Set}}(\mathcal{I} G/\mathcal{I} N, A) & \xrightarrow{\mathcal{I} h_A(s)} & \text{Hom}_{\text{Set}}(\mathcal{I} G/\mathcal{I} N, A) \\
\end{array}
\]

Starting with an $\mathcal{I} N$-invariant $f \in h_A(G)$, we know from the assumption that it comes from an element in $\text{Hom}_{\text{Set}}(\mathcal{I} G/\mathcal{I} N, A)$. But the diagram implies that this element is necessarily identical to $h_A(s)(f)$. \hfill $\Box$

Note that for $f \in X^n$, the induced face maps

$$s_k(f) : (x_i) \to f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_{n-1})$$

lie in $X^{n-1}$.

We will often omit the forgetful functor $\mathcal{I}$, e.g., instead of $x \in \mathcal{I} N$ we will simply write $x \in N$.

Proposition 6.2 The assignment

$$\partial f(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n)$$

$$+ \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{n+1}),$$
induced by definition 5.1, gives rise to well-defined maps

\[ \partial : \mathcal{X}^n \longrightarrow \mathcal{X}^{n+1}, \]
\[ \partial : \mathcal{C}^n \longrightarrow \mathcal{C}^{n+1}, \]
\[ \partial : \mathcal{I}_j \mathcal{C}^j \longrightarrow \mathcal{I}_j \mathcal{C}^{n+1}. \]

**Proof** To see that \( \partial f \in \mathcal{X}^{n+1} \) for \( f \in \mathcal{X}^n \), it suffices to check this for each of the summands, as \( \mathcal{X}^{n+1} \) is an abelian group per definition. All but the first summand stem from a composition

\[ G^{n+1} \longrightarrow G^n \xrightarrow{f} A \]

and hence lie in \( \mathcal{X}^{n+1} \). That the first summand lies in \( \mathcal{X}^{n+1} \) is precisely the second requirement in definition 3.7.

If \( f \) is normalised, then in the expansion of \( \partial f(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_n) \) there are exactly two terms that are not trivially zero. But these terms are identical except for an opposing sign and hence cancel.

If \( f \in \mathcal{I}_j \mathcal{C}^n \), we want to show that the last \( j \) arguments of \( \partial f \) are \( \mathcal{N} \)-invariant. But the last \( j \) arguments of \( \partial f \) only contribute to the last \( j \) arguments of every individual summand in the coboundary expansion of \( \partial f \), which are \( \mathcal{N} \)-invariant. \( \square \)

**Remark 6.3** It might seem artificial to consider cochains whose last \( j \) arguments are \( \mathcal{N} \)-invariant instead of the first \( j \), which will also lead to somewhat counter-intuitive definitions later on. But the equation

\[ \partial f(x, y) = x \cdot f(y) - f(xy) + f(x) \]

shows that if \( f \) is \( \mathcal{N} \)-invariant, only the second argument of \( \partial f \) is \( \mathcal{N} \)-invariant and not necessarily the first.

**Remark 6.4** For \( G' \) we can also form the complex \( \widetilde{\mathcal{X}}^\bullet(G', A) \) given by

\[ \widetilde{\mathcal{X}}^n(G', A) = h_A((G'^a)^{n+1}) \]

with differential

\[ \tilde{\partial}(f)(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}). \]

By the usual arguments (cf. e.g. [15, 14]), we get an isomorphism of complexes

\[ \mathcal{X}^\bullet(G', A) \cong \widetilde{\mathcal{X}}^\bullet(G', A). \]

However, the usual morphism

\[ \phi : \mathcal{X}^n(G', A) \longrightarrow \widetilde{\mathcal{X}}^n(G', A), \]

which is given by

\[ \phi(f)(x_0, \ldots, x_n) = x_0 f(x_0^{-1}x_1, x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x_n) \]

can only be defined for the group object \( G' \) and not for the monoid object \( G \). The example in remark 5.4 shows that both complexes cannot be isomorphic in general. They might still be quasi-isomorphic, however, we were unable to show this.
7 Cohomology of topologised monoids

We will call $H^\bullet(G, A) = H^\bullet(X^\bullet, \partial)$ the (C)-cohomology of $G$ with coefficients in $A$. Note that if $G$ is $A$-pliant, the C-cohomology of $G$ with coefficients in $A$ is just the (abstract) monoid cohomology of $\mathcal{C} G$ with coefficients in $A$. Generally, comparing topological cohomology with abstract cohomology only works well in low degrees.

**Proposition 7.1** For every $n$ there is a natural morphism

$$H^n(G, A) \longrightarrow H^n(\mathcal{C} G, A),$$

which is an isomorphism for $n = 0$ and injective for $n = 1$.

**Proof** Clearly the following diagram commutes

$$
\begin{array}{c}
X^{n-1}(G, A) \xrightarrow{\partial} X^n(G, A) \xrightarrow{\partial} X^{n+1}(G, A) \\
\downarrow \quad \downarrow \quad \downarrow \\
X^{n-1}(\mathcal{C} G, A) \xrightarrow{\partial} X^n(\mathcal{C} G, A) \xrightarrow{\partial} X^{n+1}(\mathcal{C} G, A),
\end{array}
$$

which yields the required comparison morphisms. By definition,

$$X^0(G, A) = X^0(\mathcal{C} G, A),$$

so the morphism is indeed an isomorphism for $n = 0$ and injective for $n = 1$. 

Our definition of an exact sequence of rigidified $G$-modules is custom tailored to admit a long exact sequence of cohomology groups.

**Theorem 7.2** Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of rigidified $G$-modules. Then there is a long exact sequence of abelian groups

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow \cdots$$

**Proof** By definition of exactness of a sequence of rigidified $G$-modules, we have the following commutative diagram with exact rows:

$$
\begin{array}{c}
\vdots \quad \vdots \quad \vdots \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow X^n(G, A) \longrightarrow X^n(G, B) \longrightarrow X^n(G, C) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow X^{n+1}(G, A) \longrightarrow X^{n+1}(G, B) \longrightarrow X^{n+1}(G, C) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\vdots \quad \vdots \quad \vdots
\end{array}
$$
As usual, the snake lemma implies the existence of the long exact sequence as required. □

As in the classical case, the normalised inhomogeneous cochains compute the same cohomology as all inhomogeneous cochains.

**Proposition 7.3** The inclusion $C^\bullet \rightarrow X^\bullet$ is a quasi-isomorphism.

**Proof** The original proof in [6, § 6] works without issues, as for every $f \in X^n$ the map $s_k f : (x_1, \ldots, x_{n-1}) \mapsto f(x_1, \ldots, x_{k-1}, 1, x_k, \ldots, x_{n-1})$ is again in $X^{n-1}$. □

8 Cohomology and extensions

For discrete coefficients, the groups $H^i(G, A)$ have concrete interpretations for $i \leq 2$. We will give two concrete interpretations for $H^1$ also in the topological case. For this matter, we fix in this section another topological category $D$ and a rigidification $h : C^\circ \times D \rightarrow \text{Set}$. In this section, we assume that $A$ is actually an abelian group object in $D$ and that $h_A = h(\cdot, A)$.

**Definition 8.1** An $A$-torsor is an object $X$ in $D$ with a right action from $A$ (i.e., an arrow $\mu : X \times A \rightarrow X$ in $D$ subject to the usual conditions), such that the induced map

$$m : X \times A \rightarrow (\text{id}, \mu) : X \times X$$

is an isomorphism. The composition $\pi_A \circ m^{-1} : X \times X \rightarrow A$ will be denoted by $\setminus$.

An $A$-torsor with $G$-rigidification is an $A$-torsor $X$, such that $(\mathcal{L}X, h(\cdot, X))$ is a $G$-set with $C$-rigidification, and on the level of sets for all $g \in \mathcal{L}G, x \in \mathcal{L}X$, and $a \in \mathcal{L}A$ the following holds:

$$g\mu(x, a) = \mu(gx, ga).$$

An isomorphism of $A$-torsors with $G$-rigidification $j : X \rightarrow Y$ is an isomorphism in $D$ such that on the level of sets, $j$ commutes with both the $A$- and $G$-action.

**Theorem 8.2** $H^1(G, A)$ stands in one-to-one correspondence with isomorphism classes of $A$-torsors with $G$-rigidification.

**Proof** Given a torsor $X$, we construct an element in $H^1(G, A)$ as follows: First choose $x \in \mathcal{L}X$. The definition of a $G$-set, together with the existence of constant maps in topological categories, imply the existence of $\cdot x \in h(G, X)$, which on the level of sets is just given by $g \mapsto g \cdot x$. As $h$ is a bifunctor, we can compose it as follows:

$$G \xrightarrow{\cdot x} X \xrightarrow{(x, \text{id})} X \times X \xrightarrow{c_X} X$$

Note that on the level of sets, $c_X(g)$ is the unique element such that $g \cdot x = xc_X(g)$. The verification that $c_X$ is a well-defined cocycle independent of $x \in \mathcal{L}X$ is standard.
For the other direction, take a cocycle in $H^1(G, A)$, represented by $c : G \to A$. Define $X = A$ and $\mu : X \times A \to X$ as the addition in $A$. We define the $G$-action on $\mathcal{L}X$ via

$$g.x = c(g) + g \cdot x,$$

where $g \cdot x$ is the given action of $G$ on $A$. It is easy to check that this gives a well defined $G$-module with C-rigidification. The verifications that this construction is (up to isomorphism) independent of the class of $c$ and both left and right inverse to the previous construction is again standard. \hfill $\square$

Let $R$ be a ring object in $\mathcal{D}$, which is furthermore commutative and unitary. For the remainder of this section we assume that $A$ is an $R$-module, i.e., we additionally require the existence of a morphism $R \times A \to A$ subject to the usual conditions. We also assume that $G$ operates on $R$ via ring endomorphisms in $\mathcal{D}$ and that on the level of sets, the action on $A$ is $R$-semi-linear, i.e.,

$$g \cdot (r \cdot a) = (g \cdot r) \cdot (g \cdot a)$$

for all $g \in G$, $r \in R$, $a \in A$.

We will call such an object a semi-linear $G$-module over $R$.

An exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of semi-linear $G$-modules is an exact sequence of $G$-modules with C-rigidification in the sense of definition 3.7, where we additionally require the morphisms

$$0 \to h_{E'}(Y) \to h_E(Y) \to h_{E''}(Y) \to 0$$

to be $R$-linear. In the above exact sequence, $E$ will be called an extension of $E''$ by $E'$.

An equivalence of extensions of $E''$ by $E'$, which will be denoted by $E \approx \tilde{E}$, is an isomorphism of functors $h_E \cong h_{\tilde{E}}$ such that for all $Y$ the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \to & h_{E'}(Y) & \to & h_E(Y) & \to & h_{E''}(Y) & \to & 0 \\
& & \downarrow^{\cong} & & \| & & \downarrow^{} & & \\
0 & \to & h_{E'}(Y) & \to & \tilde{h}_E(Y) & \to & h_{E''}(Y) & \to & 0.
\end{array}
$$

The equivalence classes of extensions of $E''$ by $E'$ will be denoted by $\text{Ext}(E'', E')$.

**Theorem 8.3** $H^1(G, A) \cong \text{Ext}(R, A)$.

**Proof** Because of the similarity to theorem 8.2, we only sketch the construction.

Let

$$0 \to A \to E \xrightarrow{p} R \to 0$$

be an extension and denote by $1 \in \mathcal{L}R$ the unit in $R$. We can construct a cochain via

$$g \mapsto g \cdot e - e,$$

where $e \in \mathcal{L}E$ is any preimage of $1$.

On the other hand, for a cochain $c : G \to A$ define a $G$-action on $A \times R$ via

$$g \cdot (a, r) = ((g \cdot r) \cdot c(g) + g \cdot a, g \cdot r),$$

which is a well-defined semi-linear $G$-module over $R$. The universal property of the product yields the exactness. \hfill $\square$
Remark 8.4 An alternative to the construction of theorem 8.3 goes as follows: As D is a topological category and all limits exist, we get an object and a morphism \( X = p(1) \) \( \rightarrow \) \( E \). It follows that we have an action \( X \times A \rightarrow X \) given by addition in \( E \) and that the composition

\[
X \times A \xrightarrow{(\text{id},+)} X \times X \xrightarrow{(\text{id},\times - x_1)} X \times E
\]

factors through a morphism \( X \times A \rightarrow X \times A \) and induces the identity, i.e., \( X \) is a \( A \)-torsor. It is also evident that \( X \) inherits a \( G \)-rigidification from \( E \).

Directly constructing an extension from a torsor \( X \) is regrettably not straight-forward, as it is cumbersome to define the correct \( R \)-module structure on \( X \times R \).

9 Shapiro’s Lemma for topologised groups

We will now prove Shapiro’s lemma for the induction of subgroups, i.e., we will assume in this section that \( M = 1 \). Later on we will also prove Shapiro’s lemma for actual monoids, cf. Sect. 12.

Let \( H \leq G \) be a subgroup of \( G \) and assume the existence of a map \( H(-): G \rightarrow H \) in \( \mathbf{C} \) with the following properties:

- \( H(1) = 1 \),
- \( H(hg) = h \cdot H(g) \) for all \( h \in H, g \in G \)

Remark 9.1 Instead of requiring the existence of such a morphism in \( \mathbf{C} \), one could also construct it as follows: Assume that the push-out \( H \backslash G \) of the following diagram exists:

\[
\begin{array}{ccc}
H \times G & \xrightarrow{\pi_2} & G \\
\downarrow{\mu} & & \downarrow{p} \\
G & \xrightarrow{H} & H \backslash G
\end{array}
\]

\( H \backslash G \) is then the space of right cosets of \( H \) in \( G \). Assume the existence of a section \( s: H \backslash G \rightarrow G \) in \( \mathbf{C} \) with \( p \circ s = \text{id}_{H \backslash G} \) and with the neutral element contained in the image of \( s \). We would then define

\[
H(g) = gs(p(g))^{-1}.
\]

Note the similarity with \( (-)_N \), which was defined as \( (g)_N = s(\pi(g))^{-1}g \). As we wanted to use the same convention for the action of \( G \) on \( \text{Ind}_H^G(A) \) as in the literature, we have to use a different convention here. However, we would then need to check many basic properties of this construction, which wouldn’t shed any additional light on what actually happens.

Example 9.2 These requirements are always satisfied if \( G \) is an analytic group and \( H \) a closed subgroup, cf. [2, section III.1.6].

Definition 9.3 Consider the following maps:

1. \( \alpha_n: C^n(G, \text{Ind}_H^G(A)) \rightarrow C^n(H, A) \) given by

\[
\alpha_n(f)(h_1, \ldots, h_n) = f(h_1, \ldots, h_n, 1).
\]

Note that our formalism ensures that this is a well-defined map.
(2) $\beta_n : C^n(H, A) \longrightarrow C^n(G, \text{Ind}_G^H(A))$ by
\[
\beta_n(f)(g_1, \ldots, g_n, x) = h(x) f(h(x)^{-1} h(xg_1), h(xg_1)^{-1} h(xg_1g_2), \ldots, h(xg_1 \ldots g_{n-1})^{-1} h(xg_1 \ldots g_n)).
\]

As we can express $\beta_n(f)$ in a (very large) diagram in $C$, it again gives a well-defined element in $h_A(G^n \times G)$, and we immediately verify that it does lie in $C^n(G, \text{Ind}_G^H(A))$.

(3) $\kappa_{n+1} : C^{n+1}(G, \text{Ind}_G^H(A)) \longrightarrow C^n(G, \text{Ind}_G^H(A))$ given by
\[
\kappa_{n+1} f(g_1, \ldots, g_n, x) = \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i, (xg_1 \ldots g_i)^{-1} h(xg_1 \ldots g_{i+1}), \ldots, h(xg_1 \ldots g_{n-1})^{-1} h(xg_1 \ldots g_n), x).
\]

The summands of $\kappa_{n+1}$ are given by morphisms in $C$ followed by $f$, so indeed $\kappa_{n+1}(f) \in X^n(G, \text{Ind}_G^H(A))$ and also in $C^n(G, \text{Ind}_G^H(A))$.

Remark 9.4 While we can adapt the definition of $\beta$ to also work in the monoid setting by explaining what the map is supposed to do on the monoid part, this is no longer true for $\kappa$. We will prove later in Sect. 12 that Shapiro’s lemma still holds in the monoid setting.

The proof of Shapiro’s lemma now consists of the following few lemmata which show that $\alpha_\bullet$ and $\beta_\bullet$ are quasi-isomorphisms. Their proofs are routine, excruciatingly unenlightening, and given only for sake of completeness.

Lemma 9.5 $\alpha_\bullet$ and $\beta_\bullet$ are maps of chain complexes, i.e., they commute with $\partial$.

Proof First,
\[
\partial(\alpha_n f(h_1, \ldots, h_{n+1})) = h_1 \alpha_n f(h_2, \ldots, h_{n+1}) + (-1)^{n+1} \alpha_n f(h_1, \ldots, h_n)
\]
\[
= h_1 \sum_{i=1}^{n} (-1)^i \alpha_n f(h_1, \ldots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \ldots, h_{n+1})
\]
\[
= h_1 f(h_2, \ldots, h_{n+1}, 1) + (-1)^{n+1} f(h_1, \ldots, h_n, 1)
\]
\[
= h_1 f(h_2, \ldots, h_{n+1}, 1 \cdot h_1) + (-1)^{n+1} f(h_1, \ldots, h_n, 1)
\]
\[
= f(h_2, \ldots, h_{n+1}, 1 \cdot h_1) + (-1)^{n+1} f(h_1, \ldots, h_n, 1)
\]
\[
= \partial f(h_1, \ldots, h_{n+1}, 1)
\]
\[
= \alpha_{n+1} \partial f(h_1, \ldots, h_{n+1}).
\]
Secondly,
\[ \partial \beta_n f (g_1, \ldots, g_{n+1}, x) \]
\[ = \beta_n f (g_2, \ldots, g_{n+1}, xg_1) + (-1)^n \beta_n f (g_1, \ldots, g_n, x) \]
\[ + \sum_{i=1}^{n} (-1)^i \beta_n f (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \]
\[ = H(xg_1) f \left( H(xg_1)^{-1} H(xg_1 g_2), H(xg_1 g_2)^{-1} H(xg_1 g_2 g_3), \ldots, H(xg_1 \ldots g_n)^{-1} H(xg_1 \ldots g_{n+1}) \right) \]
\[ + \sum_{i=1}^{n} (-1)^i H(x) f \left( H(x)^{-1} H(xg_1), H(xg_1)^{-1} H(xg_1 g_2), \ldots, H(xg_1 \ldots g_n)^{-1} H(xg_1 \ldots g_{n+1}) \right) \]
\[ = H(x) \left( H(x)^{-1} H(xg_1) f (\ldots) + \sum_{i=1}^{n} (-1)^i f (\ldots) + (-1)^n f (\ldots) \right) \]
\[ = H(x) \partial f \left( H(x)^{-1} H(xg_1), H(xg_1)^{-1} H(xg_1 g_2), \ldots, H(xg_1 \ldots g_n)^{-1} H(xg_1 \ldots g_{n+1}) \right) \]
\[ = \beta_n + \partial f (g_1, \ldots, g_{n+1}, x). \]

\[ \square \]

**Lemma 9.6** \( \alpha_n \circ \beta_n = \text{id}_{C^*(H,A)} \).

**Proof** Note that \( H(-) \) is the identity on \( H \). We hence have
\[ \alpha_n \beta_n f (h_1, \ldots, h_n) = \beta_n f (h_1, \ldots, h_n, 1) \]
\[ = H(1) f \left( H(1)^{-1} H(1 \cdot h_1), H(1 \cdot h_1)^{-1} H(1 \cdot h_1 h_2), \ldots, H(1 \cdot h_1 \ldots h_{n-1})^{-1} H(1 \cdot h_1 \ldots h_n) \right) \]
\[ = f (h_1, \ldots, h_n). \]

\[ \square \]

**Lemma 9.7** \( \partial \circ \kappa_n + \kappa_{n+1} \circ \partial = \beta_n \circ \alpha_n - \text{id}_{C^*(G, \text{Ind}^G_h(A))} \), i.e., \( \kappa_n \) is a chain homotopy from \( \beta_n \circ \alpha_n \) to the identity.

**Proof** This is going to be as bad as it looks. Let us first compute \( \partial \circ \kappa_n \) and \( \kappa_{n+1} \circ \partial \), subtract \( (\beta_n \circ \alpha_n - \text{id}) \) from this and show that the sum of the remaining terms is zero. We first compute \( \partial \circ \kappa_n \):
\[ (\partial \circ \kappa_n) f (g_1, \ldots, g_n, x) \]
\[ = \kappa_n (f) (g_2, \ldots, g_n, xg_1) + (-1)^n \kappa_n (f) (g_1, \ldots, g_{n-1}, x) \]
\[ + \sum_{i=1}^{n-1} (-1)^i \kappa_n (f) (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n, x). \]
We will expand these summands one after another. First,

$$\kappa_n f(g_2, \ldots, g_n, xg_1)$$

$$= f((xg_1)^{-1} H(xg_1), H(xg_1)^{-1} H(xg_1g_2),$$

$$H(xg_1)^{-1} H(xg_1g_2), \ldots, H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), xg_1)$$

$$+ \sum_{j=1}^{n} (-1)^j f(g_2, \ldots, g_{j+1}, (xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+1}),$$

$$H(xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+2}), \ldots,$$

$$H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), xg_1).$$

Second,

$$(-1)^n \kappa_n f(g_1, \ldots, g_{n-1}, x)$$

$$= (-1)^n f(x^{-1} H(x), H(x)^{-1} H(xg_1),$$

$$H(xg_1)^{-1} H(xg_1g_2), \ldots, H(xg_1 \ldots g_{n-2})^{-1} H(xg_1 \ldots g_{n-1}, x)$$

$$+ \sum_{j=1}^{n-1} (-1)^{j+n} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1} H(xg_1 \ldots g_j),$$

$$H(xg_1 \ldots g_j)^{-1} H(xg_1 \ldots g_{j+1}), \ldots,$$

$$H(xg_1 \ldots g_{n-2})^{-1} H(xg_1 \ldots g_{n-1}, x).$$

At last

$$(-1)^i \kappa_n (f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n, x)$$

$$= (-1)^i f(x^{-1} H(x), H(x)^{-1} H(xg_1), H(xg_1)^{-1} H(xg_1g_2), \ldots,$$

$$H(xg_1 \ldots g_{i-2})^{-1} H(xg_1 \ldots g_{i-1}),$$

$$H(xg_1 \ldots g_{i-1})^{-1} H(xg_1 \ldots g_{i+1}),$$

$$H(xg_1 \ldots g_{i+1})^{-1} H(xg_1 \ldots g_{i+2}), \ldots,$$

$$H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), x)$$

$$+ \sum_{j=1}^{i-1} (-1)^{i+j} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1} H(xg_1 \ldots g_{j+1}),$$

$$H(xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+2}), \ldots,$$

$$H(xg_1 \ldots g_{i-2})^{-1} H(xg_1 \ldots g_{i-1}),$$

$$H(xg_1 \ldots g_{i-1})^{-1} H(xg_1 \ldots g_{i+1}),$$

$$H(xg_1 \ldots g_{i+1})^{-1} H(xg_1 \ldots g_{i+2})^{-1}, \ldots,$$

$$H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), x)$$

$$+ \sum_{j=i}^{n-1} (-1)^{i+j} f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{j+1},$$

$$(xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+1}),$$

$$H(xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+2}), \ldots,$$
On the other hand,

\[
(k_{n+1} \circ \partial)f(g_1, \ldots, g_n, x) = \partial(f)(x^{-1}H(x), H(x)^{-1}H(xg_1),
\]

\[
H(xg_1)^{-1}H(xg_1g_2), \ldots, H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, x)
\]

\[+
\sum_{j=1}^{n} (-1)^j \partial(f)(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_j),
\]

\[H(xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_{j+1}), \ldots,
\]

\[H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, x)
\]

and

\[(-1)^j \partial(f)(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_j),
\]

\[H(xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_{j+1}), \ldots,
\]

\[H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, x)
\]

\[= (-1)^j f(g_2, \ldots, g_j, (xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_j), \quad (\exists, j)
\]

\[H(xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_{j+1}), \ldots,
\]

\[H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, xg_1)
\]

\[+ \sum_{i=1}^{j-1} (-1)^{i+j} f(g_1, \ldots, g_i, g_{i+1}, \ldots, g_j, \ldots, g_i+1, g_{i+2}, \ldots, g_j, \ldots)
\]

\[(xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_j),
\]

\[H(xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_{j+1}), \ldots,
\]

\[H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, x)
\]

\[+ (-1)^{i+j} f(g_1, \ldots, g_{j-1}, (xg_1 \ldots g_{j-1})^{-1}H(xg_1 \ldots g_j),
\]

\[H(xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_{j+1}), \ldots,
\]

\[H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, x)
\]

\[+ (-1)^{i+j+1} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_{j+1}),
\]

\[H(xg_1 \ldots g_{j+1})^{-1}H(xg_1 \ldots g_{j+2}), \ldots,
\]

\[H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, x)
\]

\[+ \sum_{i=j+2}^{n} (-1)^{i+j} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_j), \ldots
\]

\[H(xg_1 \ldots g_j)^{-1}H(xg_1 \ldots g_{j+1}), \ldots,
\]

\[H(xg_1 \ldots g_{i-3})^{-1}H(xg_1 \ldots g_{i-2}),
\]

\[H(xg_1 \ldots g_{i-2})^{-1}H(xg_1 \ldots g_i),
\]

\[H(xg_1 \ldots g_i)^{-1}H(xg_1 \ldots g_{i+1}), \ldots,
\]

\[H(xg_1 \ldots g_{n-1})^{-1}H(xg_1 \ldots g_n, x)
\]
\[ + (-1)^{j+n+1} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1} H(xg_1 \ldots g_j), \]
\[ H(xg_1 \ldots g_j)^{-1} H(xg_1 \ldots g_{j+1}), \ldots, \]
\[ H(xg_1 \ldots g_{n-2})^{-1} H(xg_1 \ldots g_{n-1}), x). \]

We furthermore expand
\[ \partial(f)(x^{-1}H(x), H(x)^{-1}H(xg_1), H(xg_1)^{-1}H(xg_1g_2), \ldots, \]
\[ H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), x) \]
\[ = f(H(x)^{-1}H(xg_1), H(xg_1)^{-1}H(xg_1g_2), \ldots, \]
\[ H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), H(x)) \]
\[ + (-1)^n f(x^{-1}H(x), H(x)^{-1}H(xg_1), H(xg_1)^{-1}H(xg_1g_2), \ldots, \]
\[ H(xg_1 \ldots g_{n-2})^{-1} H(xg_1 \ldots g_{n-1}), x) \]
\[ - f(x^{-1}H(xg_1), H(xg_1)^{-1}H(xg_1g_2), \ldots, \]
\[ H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), x) \]
\[ + \sum_{i=2}^{n} (-1)^{i} f(x^{-1}H(x), H(x)^{-1}H(xg_1), \ldots, \]
\[ H(xg_1 \ldots g_{i-3})^{-1} H(xg_1 \ldots g_{i-2}), \]
\[ H(xg_1 \ldots g_{i-2})^{-1} H(xg_1 \ldots g_i), \]
\[ H(xg_1 \ldots g_i)^{-1} H(xg_1 \ldots g_{i+1}), \ldots, \]
\[ H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), x). \]

Clearly (\(\phi\)) = \(\beta_n \circ \alpha_n(f)(g_1, \ldots, g_n, x)\) and (\(\ast, n\)) = \(- f(g_1, \ldots, g_n, x)\), so it remains to show that the other summands amount to zero.

Note first that
\[
\begin{align*}
(\zeta, 1) &= -(\|), \\
(\zeta', j) &= -(\zeta, j + 1) \text{ for } j = 1, \ldots, n - 1, \\
(\zeta', n) &= 0,
\end{align*}
\]
so in \(\kappa_{n+1} \circ \partial\) all (\(\zeta\), (\(\zeta'\)), and (\(\|\))-terms cancel. Furthermore it is immediately evident that
\[
(\phi) = -(\phi'),
\]
\[
\sum_{j=1}^{n} (\zeta', j) = -(\zeta'),
\]
\[
\sum_{i=1}^{n-1} (\zeta', i) = -(\zeta), \text{ and}
\]
\[
\sum_{i=1}^{n-1} (\ast, j) = -(*).
\]

It remains to show that
\[
\sum_{i=1}^{n} (\ast, i) = - \sum_{j=1}^{n} (\ast, j)
\]
and
\[ \sum_{i=1}^{n} (\downarrow i) = - \sum_{j=1}^{n} (\uparrow j). \]

Write
\[ F(i, j) = f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1} H(xg_1 \ldots g_{j+1}), \]
\[ H(xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+2}), \ldots, \]
\[ H(xg_1 \ldots g_{i-2})^{-1} H(xg_1 \ldots g_{i-1}), \]
\[ H(xg_1 \ldots g_{i-1})^{-1} H(xg_1 \ldots g_{i+1}), \]
\[ H(xg_1 \ldots g_{i+1})^{-1} H(xg_1 \ldots g_{i+2}), \ldots, \]
\[ H(xg_1 \ldots g_{n-2})^{-1} H(xg_1 \ldots g_n), x), \]
\[ G(i, j) = f(g_1, \ldots, g_i-1, g_i g_{i+1}, g_i+2, \ldots, g_{j+1}, \]
\[ (xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+1}), \]
\[ H(xg_1 \ldots g_{j+1})^{-1} H(xg_1 \ldots g_{j+2}), \ldots, \]
\[ H(xg_1 \ldots g_{n-1})^{-1} H(xg_1 \ldots g_n), x), \]
then
\[ \sum_{i=1}^{n} (\downarrow i) = \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{i+j} F(i, j) = \sum_{1 \leq j < i \leq n} (-1)^{i+j} F(i, j), \]
\[ \sum_{j=1}^{n} (\uparrow j) = \sum_{j=1}^{n} \sum_{i=j+2}^{n} (-1)^{i+j} F(i - 1, j) = \sum_{1 \leq j < i \leq n} (-1)^{i+j+1} F(i, j), \]
\[ \sum_{i=1}^{n} (\downarrow i) = \sum_{i=1}^{n} \sum_{j=i}^{n-1} (-1)^{i+j} G(i, j) = \sum_{1 \leq i \leq j \leq n-1} (-1)^{i+j} G(i, j), \]
and
\[ \sum_{j=1}^{n} (\uparrow j) = \sum_{j=1}^{n} \sum_{i=1}^{j-1} (-1)^{i+j} G(i, j - 1) = \sum_{1 \leq i \leq j \leq n-1} (-1)^{i+j-1} G(i, j), \]
so indeed their sum amounts to zero. \(\square\)

We finally deduce Shapiro’s lemma.

**Theorem 9.8** (Shapiro’s lemma) *In the derived category of abelian groups,*
\[ C^\bullet(G, \text{Ind}^H_G(A)) \cong C^\bullet(H, A). \]

Especially
\[ H^n(G, \text{Ind}^H_G(A)) \cong H^n(H, A) \text{ for all } n. \]

### 10 A Hochschild–Serre spectral sequence

We devote this section to proving a Hochschild–Serre spectral sequence in a rather general fashion. Constructing cochains of course happens in \(\text{Hom}_{\text{Set}}(G^n, A)\) and is done by the
usual tedious calculations. As G is only a monoid, extra care is required. Showing that these
cochains stem from (then necessarily unique) elements in X* poses an additional difficulty.

The spectral sequence will indeed follow from the following spectral sequence attached
to the filtered complex I*C*.

**Definition 10.1** As for all n the filtration I*Cn is a finite filtration and all IjC* form a
subcomplex, there is a E1-spectral sequence

\[ ss(I*C*)^{p,q}_{1} \implies H^{p+q}(G, A), \]

where the E1-terms are defined as

\[ ss(I*C*)^{p,q}_{1} = \ker \left( I^p C^{p+q} / I^{p+1} C^{p+q} \xrightarrow{\partial} I^p C^{p+q+1} / I^{p+1} C^{p+q+1} \right) \]

\[ \text{im} \left( I^p C^{p+q-1} / I^{p+1} C^{p+q-1} \xrightarrow{\partial} I^p C^{p+q} / I^{p+1} C^{p+q} \right), \]

cf. e.g. [15, (2.2.1)].

**Definition 10.2** To simplify reading, we will use the following notational convention: The
first time a variable is used, a superscript will denote the set it belongs to. For example,
instead of “Let \( x \in G^p \). Then define \( f(x) = \ldots \)” we will simply write “Define \( f_G^p \) = \ldots”

### 10.1 The meaning of \( C^j(G/N, C^j(N, A)) \)

In the previous section, we fixed a concrete category C to encapsulate our topological data.
Our setup allowed us to give meaning to \( C^j(N, A) \) for G-modules A. It is however very
unclear how we can give the abstract module \( C^j(N, A) \) again the structure of an object in C.

If C is the category of Hausdorff topological spaces, one can topologise \( C^j(N, A) \) with the
compact-open topology, which if we further restrict to compactly generated spaces, has
somewhat nice properties and can be called canonical. But computing cohomology, we are
presented with the issue that images of differentials need not be closed and one subsequently
loses the Hausdorff property, cf. [3] for a thorough discussion of these issues.

If C is a bit more exotic, e.g., analytic \( \mathbb{Q}_p \)-manifolds, then there is no obvious way to give
\( C^j(N, A) \) the structure of an analytic \( \mathbb{Q}_p \)-manifold compatible with the additional structure
- and especially none that also correctly topologises the cohomology groups.

If the quotient is discrete, we can identify \( C^q(U, A) \) with \( FC^q(U, A) \) and define

\[ C^p(G/U, C^q(U, A)) = C^p(G/U, FC^q(U, A)), \]

but note that while \( C^q(U, A) \) carries the structure of a G-module, in general it does not carry
the structure of a \( G/U \)-module (the action is only \( U \)-invariant after passing to cohomology).

**Definition 10.3** For \( N = U \), let \( f \in I^p C^{p+q}(G, A) \) and define its p-restriction \( r_p(f) \in C^p(G/U, C^q(U, A)) = C^p(G/U, FC^q(U, A)) \) via

\[ r_p(f)(x_1, \ldots, x_p)(y_1, \ldots, y_q) = f(y_1, \ldots, y_q, s(x_1), \ldots, s(x_p)). \]

This is well-defined: The induced map

\[ r_p(f)^{(G/U)^p}_U : U^q \rightarrow A \]
stems from the composition
\[ U^q \xrightarrow{(\text{id}, s(x_1), \ldots, s(x_p))} U^q \times G^p \longrightarrow G^q \times G^p \xrightarrow{f} A, \]
where we use the existence of constant maps (cf. proposition 1.13). As \( G/U \) is discrete, \( r_p(f)(x) \) is indeed in \( C^p(G/U, C^q(U, A)) \).

**Lemma 10.4** If \( p' < p \) and \( f \in I_p C^{p+q} \) then \( r_{p'}(f) = 0 \).

**Proof**
\[
r_{p'}(f)(y_1, \ldots, y_{p+q-p'}) = f(y_1, \ldots, y_q, 1, \ldots, 1, s(x)) = 0,
\]
as \( f \) is normalised by assumption. \( \square \)

### 10.2 Extensions of cochains

Comparing \( C^{p+q}(G, A) \) with \( C^p(G/U, C^q(U, A)) \), we will have to extend maps \( U^q \times (G/U)^p \longrightarrow A \) to maps \( G^q \times (G/U)^p \longrightarrow A \). This extension process also works if we work with \( N \) instead of \( U \).

**Definition 10.5** Let \( g : G^{k-1} \times N^{q-k} \times G^p \longrightarrow A \) be a normalised map (meaning its value being zero if one of the arguments is 1) with \( k \geq 2 \). Then for normalised \( f : G^k \times N^{q-k} \times G^p \longrightarrow A \) we define
\[
\text{ext}_f(g) : G^{k-2} \times G \times G^{N^{q-k-1}} \times G^p \longrightarrow A
\]
\[
\Big( y, w, x, \sigma, z \Big) \longmapsto g(y, wx^*, x_N, \sigma, z) + (-1)^k f(y, w, x^*, x_N, \sigma, z)
\]
It is called the extension of \( g \) along \( f \) and is again normalised. Note that it actually lies in \( h_A \), as the modification of the arguments is done via morphisms in \( C \).

Calling it an extension is due to the following fact which is immediately verified:

**Lemma 10.6** In the setting of definition 10.5 the following diagram commutes:

\[
\begin{array}{ccc}
G^{k-1} \times N \times N^{q-k-1} \times G^p & \xrightarrow{\partial} & A \\
\downarrow^{\text{ext}_f(g)} & & \\
G^{k-1} \times G \times N^{q-k-1} \times G^p & &
\end{array}
\]

**Proof** Vectors in the image of the inclusion have \( x^* = 1 \) and \( x_N = x \). \( \square \)

**Lemma 10.7** In the setting of definition 10.5, the usual coboundary formula gives meaning to the function
\[
\partial(\text{ext}_f(g)) : G^k \times N^{q-k} \times G^p \longrightarrow A
\]
and the following diagram commutes:

\[
\begin{array}{ccc}
G^{k-1} \times N \times N^{q-k} \times G^p & \xrightarrow{\partial(g)} & A \\
\downarrow{\partial(\text{ext}_f(g))} & & \\
G^{k-1} \times G \times N^{q-k} \times G^p & \end{array}
\]

**Proof** Immediate from lemma 10.6.

**Remark 10.8** In the setting of definition 10.5, \( g \) is only defined on \( G^{k-1} \times N \times N^{q-k-1} \times G^p \). For the coboundary formula to make sense, all terms \((x_1, \ldots, x_i, x_{i+1}, \ldots, x_{p+q})\) must lie in \( G^{k-1} \times N \times N^{q-k-1} \times G^p \). This is the reason \( \partial g \) is only defined on \( G^{k-1} \times N \times N^{q-k} \times G^p \).

**Proposition 10.9** Assume \( N = U \). Let \( q \geq 2 \) and \( f \in I^pC^{p+q} \) with \( \partial f \in I^{p+1}C^{p+q+1} \).

Take \( u \in C^p(G/U, C^{q-1}(U, A)) \). Define an element \( g = g(u, f) \) as follows:

- \( g_0(x, \sigma, \gamma) = \text{ext}_f(g_{k-1})(G^k \times U^{q-1-k} \times G^p) \) for \( 2 \leq k \leq q-1 \).
- \( g = g_{q-1} \).

Then the following hold:

1. \( g \in I^pC^{p+q-1} \),
2. \( r_p(g) = u \),
3. if \( r_p(f)(G^{(U)/p}) = \partial(u(x)) \) for all \( x \in (G/U)^p \), then \( f - \partial g \in I^{p+1}C^{p+q} \).

**Proof** The first assertion follows immediately from the definitions, as none of the manipulations touches the last \( p \) arguments.

The second assertion follows inductively, the details being carried out in [10, proof of theorem 1] in slightly different phrasing and in [16, proposition 3.6.9] in this exact phrasing.

**Remark 10.10** The cochain \( g = g(u, f) \) of proposition 10.9 has a rather unwieldy definition. However, if \( f = 0 \) and \( U \) is a direct factor of \( G \), then \( g \) has an explicit description, cf. proposition 11.12.

### 10.3 Comparison of the first page

**Proposition 10.11**

\[
ss(I^pC^*)_{C^p}^{p,D} \cong C^p(G/N, A^N)
\]

for all \( p \).
Proof By definition, \( ss(I^\bullet C^\bullet)_1^{p, 0} = \ker I^p C^p \overset{\partial}{\longrightarrow} I^p C^{p+1} / I^{p+1} C^{p+1} \) and furthermore, \( f \in I^p C^p \) comes from a (necessarily unique) morphism \( (G/N)^p \overset{\sim}{\longrightarrow} A \), which yields an element in \( C^p(G/N, A) \). We first need to show that the image of \( f \) is contained in \( A^N \), i.e.,

\[
(n - 1) f \left( \frac{G^p}{x} \right) = 0.
\]

As \( f \) is in the aforementioned kernel, \( \partial f(n, x) = \partial f(1, x) = 0 \), and as \( f \in I^p C^p \) by assumption, the difference between the coboundary expansions of \( \partial f(n, x) \) and \( \partial f(1, x) \) is exactly \((n - 1) f(x)\) and hence also zero. Injectivity of the map is then clear.

On the other hand, for \( f \in C^p(G/N, A^N) \) coming from \( \tilde{f} \in h_{A^N}((G/N)^p) \), consider the induced morphism

\[
\tilde{g} : G^p \longrightarrow (G/N)^p \overset{\tilde{f}}{\longrightarrow} A^N \longrightarrow A.
\]

It is clear that \( g \in I^p C^p \) and gets mapped to \( f \). As the image of \( g \) lies in \( A^N \), \( \partial g \) is \( \mathcal{L}N \)-invariant for all of its \( p + 1 \) arguments, so lemma 6.1 implies that indeed \( g \in ss(I^\bullet C^\bullet)_1^{p, 0} \).

Remark 10.12 By abuse of notation, even for not necessarily open \( N \) we will refer to the map \( ss(I^\bullet C^\bullet)_1^{p, 0} \longrightarrow C^p(G/N, A^N) \) of proposition 10.11 as a map \( r_p : ss(I^\bullet C^\bullet)_1^{p, 0} \longrightarrow C^p(G/N, H^0(N, A)) \). This is clearly compatible with the previous definition of \( r_p \).

Proposition 10.13 Suppose that \( N = U \). Then

\[
r_p : I^p C^{p+q} \longrightarrow C^p(G/U, C^q(U, A))
\]

induces an isomorphism between the \( E_1 \)-terms:

\[
ss(I^\bullet C^\bullet)_1^{p, q} \cong C^p(G/U, H^q(U, A)).
\]

Proof Recall that by definition,

\[
ss(I^\bullet C^\bullet)_1^{p, q} = \frac{\ker \left( I^p C^{p+q} / I^{p+1} C^{p+q} \overset{\partial}{\longrightarrow} I^p C^{p+q+1} / I^{p+1} C^{p+q+1} \right)}{\text{im} \left( I^p C^{p+q-1} / I^{p+1} C^{p+q-1} \overset{\partial}{\longrightarrow} I^p C^{p+q} / I^{p+1} C^{p+q} \right)} = \frac{\ker \left( I^p C^{p+q} \overset{\partial}{\longrightarrow} C^{p+q+1} / I^{p+1} C^{p+q+1} \right)}{\partial(I^p C^{p+q-1}) + I^{p+1} C^{p+q}}.
\]

We will first prove injectivity. Therefore, take \( f \in I^p C^{p+q} \) with \( \partial f \in I^{p+1} C^{p+q+1} \). Assume that \( f \) is zero in \( C^p(G/U, H^q(U, A)) \), i.e.,

\[
r_p(f)(\frac{G^p U}{x}) = \partial(u(x))
\]

for some \( u \in C^p(G/U, C^{q-1}(U, A)) \). We want to find an \( h \in I^p C^{p+q-1} \) with \( f - \partial(h) \in I^{p+1} C^{p+q} \).

The case of \( q = 0 \) was already dealt with in proposition 10.11.

If \( q = 1 \), then define \( h \in I^p C^{p+1} \) as the normalised cocycle corresponding to \( u \in C^p(G/U, C^0(U, A)) \). Note that by assumption \( u \) has the property

\[
f(x, \frac{G^p u}{y}) = r_p(f)(\frac{G^p y}{x}) = \partial(u(y))(x) = x.u(y) - u(y).
\]

\( \square \) Springer
We want to show that \( f - \partial(h) \in I^{p+1}C^{p+1} \). For that matter we need to show that
\[
(f - \partial(h))(x \cdot \sigma, \gamma) \in I^{p+1}C^{p+1}
\]
is independent of \( \sigma \). As \( \partial f \in I^{p+1}C^{p+2} \), we see that \( \partial(f)(x, \sigma, G^p) = 0 \). Expansion of the coboundary operator hence yields
\[
f(x \sigma, \gamma) = x \cdot f(\sigma, \gamma) + f(x, \gamma),
\]
as \( f \in I^pC^p \), and also
\[
\partial(h)(x \sigma, \gamma) = x \cdot \sigma \cdot h(\gamma) + \text{terms independent of } \sigma,
\]
as \( h \in I^pC^p \). We hence get
\[
(f - \partial(h))(x \sigma, \gamma) = f(x \sigma, \gamma) - x \cdot \sigma \cdot h(\gamma) + \text{terms independent of } \sigma
\]
as \( q > 1 \) we are in the situation of proposition 10.9, which deals with exactly this case.

For surjectivity, take \( u \in C^p(G/U, C^q(U, A)) \) such that \( \partial(u(x)) = 0 \) for all \( x \). For \( q = 0 \) finding a preimage is trivial, for \( q \geq 1 \) proposition 10.9 yields a preimage \( g \in I^{p+1}C^{p+q+1} \) with \( \partial g \in I^{p+1}C^{p+q+1} \) (take \( f = 0 \) in the proposition).

\section{10.4 The shuffling mechanism}

To compare the differential in the spectral sequence attached to \( I^\bullet C^\bullet \), Hochschild and Serre use a process they call \textit{shuffling}.

\begin{lemma}
Every \( x \in G \) induces a conjugation action \( G \to G \) that on \( M \) is trivial and on \( G' \) is the usual conjugation by the \( G' \)-part of \( x \).
\end{lemma}

\begin{proof}
On \( G' \) conjugation is defined via the composition
\[
G' \cong F \times G' \times F \xrightarrow{(\cdot^{-1}, \text{id}, x)} G' \times G' \times G' \xrightarrow{\text{mult}} G',
\]
where we use the existence of constant maps from proposition 1.13.
\end{proof}

\begin{remark}
We will write formulas like \( x^{-1}y \cdot x \) even though \( x \) need not be invertible in \( G \).
As \( M \) is central in \( G \), the usual identities such as \( y(y^{-1}xy) = xy \) still hold.
\end{remark}

\begin{definition}
We will make use of the ordered sets
\[
[n] = \{1, \ldots, n\}
\]
for \( n \in \mathbb{N} \). For every injective morphism of ordered sets
\[
\phi: [p] \to [p + q]
\]
there exists a unique (injective) morphism
\[
\phi^*: [q] \to [p + q]
\]

such that

\[ [p + q] = \text{im } \phi \cup \text{im } \phi^*. \]

We furthermore define

\[ \text{sgn } \phi = (-1)^{\sum_{i=1}^{p} \phi^*(i) - i}. \]

**Lemma 10.17** Let \( \phi : [p] \to [p + q] \) be an injective morphism of ordered sets. Then 
\[ \text{sgn}(\phi) \cdot \text{sgn}(\phi^*) = (-1)^{pq}. \]

**Proof**

\[
\sum_{i=1}^{q} \phi^*(i) - i + \sum_{i=1}^{p} \phi(i) - i - p \cdot q = \frac{(p+q)(p+q+1) - q(q + 1) - p(p + 1)}{2} - pq
\]

\[
= \frac{2pq}{2} - pq = 0.
\]

\( \square \)

**Definition 10.18** Denote by \( F_{p+q} \) the free \( \mathbb{Z}[\xi] \)-module with basis \( \xi G^{p+q} \).

For \( \phi : [p] \to [p + q] \) an injective morphism of ordered sets define

\[ (x_1, \ldots, x_q, y_1, \ldots, y_p)^\phi = (y_1, \ldots, y_{p+q}) \]

with

\[ \gamma_{\phi(i)} = y_i \]

and

\[ \gamma_{\phi^*(i)} = (y_1 \cdots y_{\phi^*(i)-i})^{-1}x_i (y_1 \cdots y_{\phi^*(i)-i}). \]

If we are considering multiple morphisms \( \phi \), we will also write \( \gamma(\phi, k) \) instead of \( \gamma_k \).

Define now

\[ \text{shuffle}_{p+q}^p (\xi G^{p+q}) = \sum_{\phi} \text{sgn}(\phi)z^\phi \in F_{p+q}, \]

where here and in the following an unspecified sum over \( \phi \) denotes the sum over all injective morphisms of ordered sets with \( p \) and \( p + q \) clear from the context.

Every \( g \in C^{p+q} \) gives rise to \( g^\phi \in C^{p+q} \) via

\[ g^\phi(z) = g(z^\phi). \]

Indeed \( g^\phi \in X^{p+q} \) as both conjugation and reordering come from morphisms in \( C \). We can also define \( \text{shuffle}_{p+q}^p g \in C^{p+q} \) via

\[ \text{shuffle}_{p+q}^p (g)(z) = \sum_{\phi} \text{sgn}(\phi)g(z^\phi). \]

We will use the convention that \( \text{shuffle}_{0}^p = \text{id} \).

**Proposition 10.19** Let \( \phi : [p] \to [p + q] \) be the unique injective morphism of ordered sets with \( \phi(1) = q + 1 \). Then \( z = z^\phi \) for all \( z \in G^{p+q} \). If \( g \in \Pi C^{p+q} \), then

\[ \text{shuffle}_{p+q}^p g = g \text{ on } N^q \times G^p. \]
Proof The first assertion is clear from the definitions, as then \( \phi^*(i) = i \) for all \( 1 \leq i \leq q \).

For the second assertion we will show that for all other \( \varphi \), \( g^\varphi = 0 \) on \( N^q \times G^p \). In this case, there exists \( q + 1 \leq i \leq p + q \) with \( i = \varphi^*(k) \) for some \( k \) and \( y_i \) is then equal to a conjugate of \( \alpha_k \), which lies by assumption again in \( N \). As \( g \) was supposed to be normalised and \( N \)-invariant in the last \( p \) components, this implies that \( g^\varphi = 0 \) on \( N^q \times G^p \).

Definition 10.20 For \( p, q \geq 1 \) we define the following two partial coboundary operators \( F_{p+q} \rightarrow F_{p+q-1} \):

\[
\partial_q(x_1, \ldots, x_q, y_1, \ldots, y_p) = x_1(x_2, \ldots, x_q, y) + (-1)^q(x_1, \ldots, x_{q-1}, y)
+ \sum_{i=1}^{q-1} (-1)^i(x_1, \ldots, x_{i-1}, x_ix_{i+1}, x_{i+2}, \ldots, x_q, y)
\]

and

\[
\delta_p(x_1, \ldots, x_q, y_1, \ldots, y_p) = y_1(y_1^{-1}x_1y_1, y_2, \ldots, y_p) + (-1)^p(x, y_1, \ldots, y_{p-1})
+ \sum_{i=1}^{p-1} (-1)^k(x, y_1, \ldots, y_{i-1}, y_1y_{i+1}, y_{i+2}, \ldots, y_p),
\]

where \( x = (x_1, \ldots, x_q) \), \( y = (y_1, \ldots, y_p) \) and \( y_1^{-1}x_1y_1 = (y_1^{-1}x_1y_1, \ldots, y_1^{-1}x_qy_1) \). These formulas also give rise to partial coboundary operators \( \partial_q, \delta_p : C^{p+q-1} \rightarrow C^{p+q} \) by the same arguments as in proposition 6.2.

Proposition 10.21 For \( p, q \geq 1 \) and \( z \in F_{p+q} \) we have

\[
\partial \text{ shuffle}^{p+q}_p(z) = (\text{shuffle}^{p+q-1}_p \partial_q z) + (-1)^q (\text{shuffle}^{p+q-1}_{p-1} \delta_p z).
\]

Consequently, for \( f \in C^{p+q-1} \) the following identity holds:

\[
\text{shuffle}^{p+q}_p(\partial f) = \partial (\text{shuffle}^{p+q-1}_p(f)) + (-1)^q \delta_p (\text{shuffle}^{p+q-1}_{p-1}(f)).
\]

The proof of this is of course a combinatorial nightmare. Details can be found in [10, proposition 2], and even more details in [16, proposition 3.6.22].

10.5 Comparison of the second page

So far, we only considered the groups \( C^p(G/U, H^q(U, A)) \) with the \( E_1 \)-terms corresponding to the spectral sequence attached to \( I^*C^* \). Now we need to give \( C^*(G/U, H^q(U, A)) \) the structure of a complex. For this it suffices to give \( H^q(U, A) \) the structure of a \( \xi(G/U) \)-module, as then \( C^*(G/U, H^q(U, A)) = C^*(G/U, \xi H^q(U, A)) \) is a complex by Sect. 6. The module-structure also exists for non-open subgroups \( N \).

We also need compatibility between our partial coboundary operators \( \partial_q, \delta_p \) and the operators

\[
\delta : C^p(G/U, FC^q(U, A)) \rightarrow C^{p+1}(G/U, FC^q(U, A))
\]

and

\[
\delta : C^p(G/U, FC^q(U, A)) \rightarrow C^{p}(G/U, FC^{q+1}(U, A)).
\]
Proposition 10.22 \( C^q(N, A) \) and \( H^q(N, A) \) carry the structure of a \( \mathfrak{g}G \)-module by the usual conjugation action.

**Proof** Recall that every element in \( y \in \mathfrak{g}G \) induces a conjugation morphism on \( G \) (lemma 10.14), which restricts to a morphism on \( N \). Defining
\[
(y.(f((N^q \cdot x))))(y) = y.(f(y^{-1} x y))
\]
yields an element in \( C^q(N, A) \) because of definition 3.7, so altogether we get a \( G \)-action on \( C^q(N, A) \). As in the classical case, this also gives an action on the cohomology groups. \( \square \)

Lemma 10.23 The diagrams
\[
\begin{array}{ccc}
C^{p+q}(G, A) & \xrightarrow{r_p} & C^p(G/U, C^q(U, A)) \\
\downarrow \delta_{p+1} & & \downarrow \delta \\
C^{p+q+1}(G, A) & \xrightarrow{r_{p+1}} & C^{p+1}(G/U, C^q(U, A))
\end{array}
\]
and
\[
\begin{array}{ccc}
C^{p+q}(G, A) & \xrightarrow{r_p} & C^p(G/U, C^q(U, A)) \\
\downarrow \delta_{q+1} & & \downarrow \delta \\
C^{p+q+1}(G, A) & \xrightarrow{r_p} & C^{p+1}(G/U, C^{q+1}(U, A))
\end{array}
\]
are commutative. As before, \( \delta_{p+1} \) and \( \delta_{q+1} \) are the partial coboundary operators from definition 10.20 and \( \delta \) and \( \partial \) are the respective coboundary operators of \( C^\bullet(G/U, -) \) and \( C^\bullet(U, -) \).

**Proof** Immediate from the definitions. \( \square \)

Proposition 10.24 \( \mathfrak{g}N \) operates trivially on \( H^q(N, A) \).

**Proof** We use proposition 10.21 for the topologised monoid \( N \): For \( p = 1 \) and \( f \in C^q(N, A) \cap \ker \partial \) this reads
\[
0 = \text{shuffle}_{1+q}^q(\partial f) = \partial_q(\text{shuffle}_1^q(f)) + (-1)^q \delta_1(\text{shuffle}_0^q(f))
\]
and hence
\[
\delta_1(f) \in \text{im} \partial_q.
\]
But \( \delta_1(f) \) is explicitly given by
\[
(\delta_1 f)(N^q \cdot x, y) = y.f(y^{-1} x y) - f(x) = (y.f)(x) - f(x),
\]
so \( y.f \) and \( f \) are cohomologous, as \( \partial_q \) is the differential on \( C^\bullet(N, A) \), analogously to lemma 10.23. \( \square \)

Theorem 10.25 There is a convergent \( E_2 \)-spectral sequence
\[
H^p(G/U, H^q(U, A)) \Longrightarrow H^{p+q}(G, A).
\]
Even if \( N \) is not necessarily open, we have the classical five term exact sequence:
\[
0 \rightarrow H^1(G/N, A^N) \rightarrow H^1(G, A) \rightarrow H^1(N, A)^{\mathfrak{g}(G/N)} \rightarrow H^2(G/N, A^N) \rightarrow H^2(G, A).
\]
Proof Consider first the case of $N = U$. By proposition 10.13 it suffices to show that the following diagram commutes:

$$
\begin{array}{cc}
ss(I^\bullet C^\bullet)_{1}^{p,q} & \xrightarrow{\delta} \ss(I^\bullet C^\bullet)_{1}^{p+1,q} \\
\downarrow r_p & \downarrow r_{p+1} \\
C^p(G/U, H^q(U, A)) & \xrightarrow{(-1)^q \delta} C^{p+1}(G/U, H^q(U, A))
\end{array}
$$

\begin{equation}
(*)
\end{equation}

Here $\delta$ denotes the coboundary operator on $C^p$, not on lifted maps $U^q \times G^p \longrightarrow A$. By lemma 10.23

$$\delta \circ r_p = r_{p+1} \circ \delta_{p+1}.$$ 

For $q = 0$, the commutativity follows immediately from the definitions, so assume $q \geq 1$. Take $f \in I^p C^{p+q}$ with $\partial f \in I^{p+1} C^{p+q+1}$ Then by proposition 10.21 and multiple applications of proposition 10.19,

$$r_{p+1}(\partial f) = r_{p+1}(\operatorname{shuffle}_{p+1}^{p+q+1}(\partial f))$$

$$= r_{p+1}\left( \partial_q (\operatorname{shuffle}_{p+1}^{p+q+1} f) + (-1)^q \delta_{p+1}(\operatorname{shuffle}_{p+1}^{p+q} f) \right)$$

$$= r_{p+1}(\partial_q (\operatorname{shuffle}_{p+1}^{p+q} f)) + (-1)^q (\delta_{p+1}(\operatorname{shuffle}_{p+1}^{p+q} f))$$

As clearly $r_{p+1}(\partial_q (\operatorname{shuffle}_{p+1}^{p+q} f)) = 0$ in $C^{p+1}(G/U, H^q(U, A))$ by lemma 10.23, this finishes the proof for open subgroups.

For normal subgroups $N$, the spectral sequence

$$\ss(I^\bullet C^\bullet)_{2}^{p,q} \Longrightarrow H^{p+q}(G, A)$$

still yields a five term exact sequence and we are left with showing that the groups $\ss(I^\bullet C^\bullet)_{2}^{1,0}$, $\ss(I^\bullet C^\bullet)_{2}^{2,0}$ and $\ss(I^\bullet C^\bullet)_{2}^{0,1}$ are precisely the cohomology groups we were looking for. For $q = 0$, the same as above argument works, using proposition 10.11 instead of proposition 10.13. For the case of $p = 0$, $q = 1$, we cannot use diagram $\star$. But the same argument as above yields a commutative diagram

$$
\begin{array}{cc}
\ss(I^\bullet C^\bullet)_{1}^{0,1} & \xrightarrow{\delta} \ss(I^\bullet C^\bullet)_{1}^{1,1} \\
\downarrow r_0 & \downarrow \uparrow \\
H^1(N, A) & \xrightarrow{\delta} C^1(\mathfrak{u}(G/N), H^1(N, A))
\end{array}
$$

where the map on the left is induced by the restriction of $f \in C^1(G, A)$ to $N$. The map on the right, defined analogously to before, is however only injective.

The map on the left is however still bijective, adapting the proof of proposition 10.13: Represent an element of $\ss(I^\bullet C^\bullet)_{1}^{0,1}$ by $f \in C^1(G, A)$. Assume its restriction to $N$ is given by $f(n) = n.a - a$ for some $a \in A$. Consider $h = f - (x \longmapsto x.a - a)$, which is the same as $f$ in

$$\ss(I^\bullet C^\bullet)_{1}^{0,1} = \ker C^1 \xrightarrow{\partial(C^0) + I^1 C^1}.$$
We will show that indeed \( h \in I^1 C^1 \) and that hence \( h \) and therefore \( f \) is zero in \( ss(I^1 C^1)^{\geq 0} \). By assumption, \( \partial f(x, n) = 0 \) for all \( x \in G, n \in \mathbb{N} \), so actually

\[
f(xn) = x.f(n) + f(x)
\]

We immediately find that

\[
h(xn) = f(xn) - xn.a + a = x.(n.a - a) + f(x) - xn.a + a = f(x) - (x.a - a) = h(x),
\]

so \( h \in I^1 C^1 \) by lemma 6.1.

For surjectivity, choose a representative \( \tilde{f} \in C^1(N, \mathbb{A}) \) and simply define \( f \) via \( \tilde{f} \circ (-)_N \). Therefore,

\[
ss(I^1 C^1)^{\geq 0} \cong \ker \delta = \{ f \in H^1(N, \mathbb{A}) \mid g.f - f = 0 \text{ for all } g \in G \},
\]

which is precisely \( H^1(N, \mathbb{A})^{G/N} \), as \( N \) already operates trivially by proposition 10.24. \( \square \)

**Remark 10.26** With all this effort, we still cannot recover the Hochschild-Serre spectral sequence for Hausdorff compactly generated topological groups \( G \) with closed normal subgroup \( N \) and discrete coefficients \( \mathbb{A} \). From the point of view presented above, the spectral sequence

\[
H^p(G/N, H^q(N, \mathbb{A})) \Rightarrow H^{p+q}(G, \mathbb{A})
\]

is actually an anomaly: The category \( \mathbf{C} \) would be the category of compactly generated weakly Hausdorff spaces, which is cartesian closed, where the exponential objects are given by \( \text{Hom}_{\mathbf{C}}(X, Y) \), endowed with the compact-open topology (cf. remark 1.11). The analogue of proposition 10.13, which shows an isomorphism of the \( E_1 \)-page, is then generally only a bijection – but for discrete \( \mathbb{A} \), \( C^q(N, \mathbb{A}) \) (and hence also \( H^q(N, \mathbb{A}) \)) is again discrete and bijectivity then suffices for showing the isomorphism.

In any case, it is much more convenient to derive said spectral sequence from homological algebra and reserve the *direct method* for cases where the homological arguments fail.

### 11 A double complex

This section is devoted to making a precise statement of the following sort and proving it afterwards:

**Prototheorem 11.1** Let \( G \) be a topologised monoid, \( D \) an abelian discrete monoid and \( \mathbb{A} \) a topologised \( D \times G \)-module. Then in the derived category of abelian groups the following holds:

\[
C^\bullet(D \times G, \mathbb{A}) \cong \text{tot} C^\bullet(D, C^\bullet(G, \mathbb{A})).
\]

This is very much related to the previous results: There, we filtered the complex on the left hand side. If we are not looking at a direct product \( D \times G \), there is no double complex on the right hand side – but a hypothetical double complex would have \( C^\bullet(D, H^\bullet(G, \mathbb{A})) \) as the cohomology in one direction. We compared this cohomology with the \( E_1 \)-page of the filtered complex and showed that indeed they coincide.
11.1 Setup and precise statement

For the whole section, we fix:

- a topological category $\mathbf{C}$,
- a topologised monoid $G$ in $\mathbf{C}$ as in Sect. 6,
- a discrete abelian monoid $D$,
- $-^*: D \times G \to D \times G$, the morphism of topologised monoids which on the level of sets is given by $(d, g) \mapsto (d, 1)$,
- $-^G: D \times G \to D \times G$, the morphism of topologised monoids which on the level of sets is given by $(d, g) \mapsto (1, g)$,
- the canonical projections $\pi_D: D \times G \to D$, $\pi_G: D \times G \to G$, and
- such a $D \times G$-module with $\mathbf{C}$-rigidification $A$, that $D$ is $A$-pliant.

As before, $C^n(G, A)$ denotes the set of normalised (inhomogeneous) cochains $G^n \to A$.

We write $C^\bullet$ for $C^\bullet(D \times G, A)$ and denote the boundary operator of Sect. 6 by $\partial$. The filtration $I^\bullet C^n$ will be taken with respect to the submonoid $G$ of $D \times G$.

Lemma 11.2 The assignment

$$\partial^n(d f)(\underbrace{\bar{x}_1, \ldots, \bar{x}_n}_{D^n}) = df(\bar{x}) = df(d^{-1} \bar{x} d)$$

gives $C^n(G, A)$ the structure of a $D$-module.

Proof Clear from proposition 10.22. \qed

Remark 11.3 This only works because of the direct product structure of $D \times G$, cf. Sect. 10.1.

Definition 11.4 Denote by $C^{\bullet \bullet}$ the commutative double complex

$$C^p(D, C^q(G, A))$$

with differentials

$$\delta: C^p(D, C^q(G, A)) \to C^{p+1}(D, C^q(G, A))$$

$$\partial: C^p(D, C^q(G, A)) \to C^p(D, C^{q+1}(G, A))$$

explicitly given by

$$\delta(f)(\underbrace{\bar{y}_1, \ldots, \bar{y}_p}_{\Delta^p}) = y_1 f(y_2, \ldots, y_{p+1}) (y_1^{-1} \bar{x} y_1) + (-1)^{p+1} f(y_1, \ldots, y_p)(\bar{x})$$

$$+ \sum_{i=1}^p f(y_1, \ldots, y_{i-1}, y_i y_{i+1}, y_{i+2}, \ldots, y_{p+1})(\bar{x})$$

and

$$\partial(f)(\underbrace{\bar{y}_1, \ldots, \bar{y}_q}_{\Delta^q}) = x_1 f(y)(x_2, \ldots, x_{q+1}) + (-1)^{q+1} f(y)(x_1, \ldots, x_q)$$

$$+ \sum_{i=1}^q f(y)(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{p+1}).$$
That this is indeed a double complex follows from lemma 11.2 and the previous discussion in Sec. 6. We form the total complex

$$ (\text{tot } C^\bullet)^n = \bigoplus_{p+q=n} C^{p,q} $$

with total differential

$$ \Delta : C^{p,q} \longrightarrow C^{p+1,q} \oplus C^{p,q+1} $$

$$ \Delta = \partial + (-1)^q \delta $$

**Remark 11.5** If $D \cong \mathbb{N}_0$ (or $D \cong \mathbb{Z}$) operates via a single operator $\varphi$, then because of proposition 5.2 and the fact that

$$ 0 \longrightarrow \mathbb{Z}[\varphi] \xrightarrow{-q^{-1}} \mathbb{Z}[\varphi] \longrightarrow \mathbb{Z} \longrightarrow 0 $$

is also a free resolution of the integers, we see that (by abstract nonsense)

$$ \text{tot } C^\bullet \cong \text{tot } \left( C^\bullet(G, A) \xrightarrow{-q^{-1}} C^\bullet(G, A) \right) $$

in the derived category of abelian groups. This immediately generalises to monoids $D \cong \mathbb{N}_0^r \times \mathbb{Z}^s$ by induction.

Our main result of this section can now be stated as follows:

**Theorem 11.6** There is a quasi-isomorphism of complexes

$$ C^\bullet \longrightarrow \text{tot } C^\bullet. $$

The preparations of its proof will span the next couple of pages, which itself is given on page 40.

11.2 The morphism ...

**Definition 11.7** For $f \in C^{p+q}$ denote by $r_p(f) \in C^{p,q}$ the map

$$ r_p(f)(\underleftarrow{y}, \underleftarrow{x}) = f((1, x_1), \ldots, (1, x_q), (y_1, 1), \ldots, (y_p, 1)) $$

and by $\alpha$ the map

$$ \alpha : C^n \longrightarrow \bigoplus_{p+q=n} C^{p,q} $$

$$ \alpha(f) = \bigoplus_{p+q=n} r_p(\text{shuffle}^{p+q}_p(f)). $$

**Proposition 11.8** $\alpha$ is a morphism of complexes $C^\bullet \longrightarrow \text{tot } C^\bullet$.  

**Proof** Consider the diagram

$$ C^{n-1} \longrightarrow \bigoplus_{p+q=n-1} C^{p,q} \quad \text{with } \partial \text{ and } \Delta $$

$$ C^n \longrightarrow \bigoplus_{p+q=n-1} \left( C^{p+1,q} \oplus C^{p,q+1} \right). $$
To show that it commutes, let \( p' + q' = n \). We will compare

\[ r_{p'}(\text{shuffle}_{p'}^n(\partial f)) \]

with the entry of \( \Delta(\alpha(f)) \) in \( C^{p',q'} \). By definition, this entry is equal to

\[ \partial(r_{p'}(\text{shuffle}_{p'}^{p'+q'-1}(f))) + (-1)^q' \delta(r_{p'-1}(\text{shuffle}_{p'-1}^{p'+q'-1}(f))). \]

By lemma 10.23,

\[ \partial \circ r_{p'} = r_{p'} \circ \partial \]

and

\[ \delta \circ r_{p'-1} = r_{p'} \circ \delta, \]

where \( \partial \) and \( \delta \) are the maps from definition 10.20. The claim then follows immediately from the additivity of \( r_{p'} \) and proposition 10.21. \( \square \)

11.3 ... and its quasi-inverse

**Lemma 11.9** Consider the map

\[ (-)\sharp : C^{p,q} \longrightarrow C^{p+q}, \]

\[ f^\sharp((D \times G)^{p+q}) = f(\pi_D(z_{q+1}), \ldots, \pi_D(z_{p+q}))(\pi_G(z_1), \ldots, \pi_G(z_q)). \]

Its image lies in \( I^p C^{p+q} \) and the composition

\[ C^{p,q} \overset{\sharp}{\longrightarrow} I^p C^{p+q} \overset{r_p}{\longrightarrow} C^{p,q} \]

is the identity.

**Proof** Clear from the definitions. \( \square \)

**Proposition 11.10** \((-)\sharp\) induces a map

\[ (-)\sharp : (\text{tot } C^{\bullet,\bullet})^n \longrightarrow C^n \]

with

\[ \alpha \circ (-)\sharp = \text{id}_{(\text{tot } C^{\bullet,\bullet})^n}. \]

**Proof** Let \( f \in C^{p,q} \). lemma 11.9 and proposition 10.19 (with \( N = G \)) imply that

\[ r_p(\text{shuffle}_{p'}^{p+q}(f^\sharp)) = f. \]

It now remains to show that for \( p' \neq p, r_{p'}(\text{shuffle}_{p'}^{p+q}(f^\sharp)) = 0. \)

Let \( \varphi : [p'] \longrightarrow [p + q] \) be an injective map of ordered sets, so

\[ r_{p'}((f^\sharp_{\varphi})((\sum_{k=1}^{p'} \gamma_k^{G^{p+q-p'}}/(\sum_{k=1}^{p'} \gamma_k^{G^{p+q-p'}}))) = (f^\sharp_{\varphi})((1, x_1), \ldots, (1, x_{p+q-p'}), (y_1, 1), \ldots, (y_{p'}, 1)) \]

\[ = f^\sharp_{\varphi}(\gamma_1, \ldots, \gamma_{p+q}), \]

with \( \gamma_k \) a conjugate of one of the \( (1, x_i) \) or one of the \( (y_i, 1) \). Therefore, at least \( p' \) of the \( \gamma_k \) have \( \pi_G(\gamma_k) = 1 \) and at least \( p + q - p' \) of the \( \gamma_k \) have \( \pi_D(\gamma_k) = 1 \). Now

\[ f^\sharp_{\varphi}(\gamma_1, \ldots, \gamma_{p+q}) = f(\pi_D(\gamma_{q+1}), \ldots, \pi_D(\gamma_{p+q}))(\pi_G(\gamma_1), \ldots, \pi_G(\gamma_q)). \]
and all cocycles are normalised, so this can only be non-zero if all $\gamma_k$ with $\pi_D(\gamma_k) = 1$ are among the first $q$, so
\[p + q - p' \leq q\]
and if all $\gamma_k$ with $\pi_G(\gamma_k) = 1$ are among the last $p$, so
\[p' \leq p.\]
But this is impossible if $p' \neq p$. Therefore $r_{p'}((f^z)^\phi) = 0$ and hence also
\[r_{p'}(\text{shuffle}_{p'}^{p+q}(f^z)) = 0.\]
\[\square\]

**Remark 11.11** The map $(-)^z$ of proposition 11.10 is not a map of complexes, so while it is easy to construct preimages in the direct product case, these are not particularly useful. Showing that $\alpha$ is a quasi-isomorphism hence again uses the calculations of Sect. 10.2.

**Proposition 11.12** Let $u \in C^{p,q}$ and $g = g(u,0) \in I^pC^{p+q}$ its extension along 0 from proposition 10.9. Then
\[g(D_{xG}^{x}, D_{xG}^{x}, (D_{xG})^p) = x_1^* \cdots x_q^* \cdot u^*(x_1, \ldots, x_q, y).\]

**Proof** Note first that by definition of $-^u$,
\[u^u(z_1, \ldots, z_q, z_1', \ldots, z_p') = u^u((z_1)_G, \ldots, (z_q)_G, z_1^*, \ldots, z_p^*).\]

Define as in proposition 10.9
\[g_1(D_{xG}^{x}, G^{q-2} \cdot \underbrace{y}_{\Gamma}) = x_1^* \cdot u^u((x_1)_G, \underbrace{\sigma}_{\Gamma}, y).\]
and $g_k = \text{ext}_0(g_{k-1})$ for $2 \leq k \leq q$, so that $g = g_q$. We will inductively show that
\[g_k(D_{xG}^{x}, D_{xG}^{x}, G^{q-k} \cdot \underbrace{y}_{\Gamma}) = x_1^* \cdots x_k^* \cdot u^u(x_1, \ldots, x_k, \underbrace{\sigma}_{\Gamma}, y),\]
which is trivial for $k = 1$. By definition of the extension,
\[g_{k+1}(D_{xG}^{x}, D_{xG}^{x}, D_{xG}^{x}, G^{q-k-1} \cdot \underbrace{y}_{\Gamma}) = g_k(x_1, \ldots, x_{k-1}, x_kx_{k+1}^*, (x_{k+1})_G, \underbrace{\sigma}_{\Gamma}, y),\]
which by induction hypothesis is exactly
\[x_1^* \cdots x_{k-1}^* \cdot (x_kx_{k+1}^*)^u((x_1)_G, \ldots, (x_{k-1})_G, (x_kx_{k+1})_G, (x_{k+1})_G, \underbrace{\sigma}_{\Gamma}, y)\]
As in our case $-^u$ and $-_G$ are homomorphisms with $-_G \circ -^u \equiv 1$, this shows the proposition. \[\square\]

**Corollary 11.13** Let $u \in C^{p,q}$ and define $g = g(u,0)$ as in proposition 10.9. Then
\[g((D_{xG})^q \cdot \underbrace{y}_{\Gamma}) = 0\]
if one of the first $q$ arguments lies in $D$.

**Proof** Clear from proposition 11.12 and the definition of $-^u$. \[\square\]

**Proposition 11.14** Let $u \in C^{p,q}$ and $g = g(u,0) \in I^pC^{p+q}$ its extension along 0 from proposition 10.9. Then $\alpha(g) = (0, \ldots, 0, u', 0, \ldots, 0)$. " Springer
Proof  As \( g \in I^p C^{p+q} \), we have for all \( p' < p \)
\[
\alpha(g)^{p',p+q-p'} = r_{p'}(\text{shuffle}_{p'}^{p+q}(g)) = r_{p'}(g) = 0
\]
by proposition 10.19 and lemma 10.4. By propositions 10.9 and 10.19, \( \alpha(g)^{p,q} = u \), so it remains to show that \( \alpha(g)^{p',p+q-p'} = 0 \) for \( p' > p \), i.e., that
\[
\text{shuffle}_{p'}^{p+q} g \left( \frac{d'}{d}, \frac{g^{p+q-p'}}{y} \right) = 0.
\]
But the definition of the shuffle operator implies that this is the sum of values of the form
\[
\pm g \left( \left( D \times G \right)^{p',p+q-p'} x \right)
\]
where at least \( p' \) arguments lie in \( D \). As \( p' > p \), one of these arguments that lie in \( D \) is in one of the first \( q \) positions, so \( g(y) = 0 \) by corollary 11.13.

\[\square\]

Proposition 11.15  Extension along zero is a morphism of complexes
\[
\text{tot } C^{\bullet,\bullet} \longrightarrow C^{\bullet}.
\]

Proof  We need to show the following: Let \( u \in C^{p,q} \) with \( \Delta u = v + w \) with \( v \in C^{p+1,q} \) and \( w \in C^{p,q+1} \). Call their respective extensions along zero from proposition 10.9
\[
g = g(u, 0) \in I^p C^{p+q},
\]
\[
h = g(v, 0) \in I^{p+1} C^{p+q+1}, \quad \text{and}
\]
\[
h' = g(w, 0) \in I^p C^{p+q+1}.
\]
Then
\[
\partial g = h + h'.
\]

Using proposition 11.12, this is now a straight forward (albeit lengthy) calculation. First of all,
\[
\partial g \left( \left( D \times G \right)^{p+q+1} x \right) = x_1 \cdot g(x_2, \ldots, x_{p+q+1}) + (-1)^{p+q+1} g(x_1, \ldots, x_{p+q})
\]
\[
+ \sum_{i=1}^{p+q} (-1)^i g(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{p+q+1})
\]
\[
= x_1 \cdot x_2 \cdots x_{q+1} \cdot u^x(x_2, \ldots, x_{q+1}, x_{q+2}, \ldots, x_{p+q+1})
\]
\[
+ (-1)^{p+q+1} (x_1 \ldots x_q)^x u^x(x_1, \ldots, x_q, x_{q+1}, \ldots, x_{p+q})
\]
\[
+ \sum_{i=1}^{q} (-1)^i (x_1 \ldots x_{q+1})^x u^x(x_1, \ldots, x_i x_{i+1}, \ldots, x_{p+q+1}) \quad (\Sigma.1)
\]
\[
+ \sum_{i=q+1}^{p+q} (-1)^i (x_1 \ldots x_q)^x u^x(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{p+q+1}) \quad (\Sigma.2)
\]
Expanding $h$ we first get
\[
h^{(D \times G)^{p+q+1}}(X) = (x_1 \ldots x_q)^* v^\ast(x_1, \ldots, x_q, x_{q+1}, \ldots, x_{p+q+1})
\]
\[
= (x_1 \ldots x_q)^* v(x_{q+1}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G).
\]

We can furthermore express $(-1)^q v(x_{q+1}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G)$ as follows:
\[
(-1)^q v(x_{q+1}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G)
\]
\[
= x_{q+1}^* u(x_{q+2}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G)
\]
\[
+ (-1)^{p+1} u(x_{q+1}^*, \ldots, x_{p+q}^*)((x_1)G, \ldots, (x_q)G)
\]
\[
+ \sum_{i=1}^p (-1)^i u(x_{q+1}^*, \ldots, x_{q+i-1}^*, x_{q+i}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G)
\]
\[
= x_{q+1}^* u(x_1, \ldots, x_q, x_{q+2}, \ldots, x_{p+q+1})
\]
\[
+ (-1)^{p+1} u(x_1, \ldots, x_q, x_{q+1}, \ldots, x_{p+q})
\]
\[
+ \sum_{i=1}^p (-1)^i u(x_1, \ldots, x_{q+i-1}, x_{q+i} x_{q+i+1}, x_{q+i+2}, \ldots, x_{p+q+1}).
\]

On the other hand,
\[
h'(D \times G)^{p+q+1}(X) = (x_1 \ldots x_q)^* w^\ast(x_1, \ldots, x_q, x_{q+1}, \ldots, x_{p+q+1})
\]
\[
= (x_1 \ldots x_q)^* w(x_{q+2}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G),
\]
and we can express $w(x_{q+2}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G)$ as follows:
\[
w(x_{q+2}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G)
\]
\[
= (x_1)G . u(x_{q+2}^*, \ldots, x_{p+q+1}^*)((x_2)G, \ldots, (x_{q+1})G)
\]
\[
+ (-1)^q u(x_{q+2}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, (x_q)G)
\]
\[
+ \sum_{i=1}^q (-1)^i u(x_{q+2}^*, \ldots, x_{p+q+1}^*)((x_1)G, \ldots, x_i(x_{i+1})G, x_{i+2}, \ldots, (x_q)G)
\]
\[
= (x_1)G . u^\ast(x_2, \ldots, x_{q+1}, x_{q+2}, \ldots, x_{p+q+1})
\]
\[
+ (-1)^q u^\ast(x_1, \ldots, x_q, x_{q+2}, \ldots, x_{p+q+1})
\]
\[
+ \sum_{i=1}^q (-1)^i u^\ast(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{q+1}, x_{q+2}, \ldots, x_{p+q+1}).
\]

We see at once that $(\Sigma.1)$ appears in our expansion of $h'$ and that $(\Sigma.2)$ appears in our expansion of $h$. The remaining terms are as follows:
\[
(\partial g - h - h')(D \times G)^{p+q+1}(X)
\]
\[
= x_1 \cdot x_2^* \ldots x_{q+1}^* u^\ast(x_2, \ldots, x_{p+q+1}) \tag{1}
\]
\[
+ (-1)^q u^\ast(x_1, \ldots, x_q)^* . u^\ast(x_1, \ldots, x_{p+q}) \tag{2}
\]
\[
- (-1)^q(x_1 \ldots x_q)^* x_{q+1}^* u^\ast(x_1, \ldots, x_q, x_{q+2}, \ldots, x_{p+q+1}) \tag{3}
\]
Proof of theorem 11.6  Indeed $\alpha$ is the required quasi-isomorphism

$$\alpha: C^\bullet \overset{\approx}{\longrightarrow} \text{tot } C^{\bullet,\bullet}.$$ 

Surjectivity on the level of cohomology follows immediately from propositions 11.15 and 11.14.

It remains to see that $\alpha$ is injective on cohomology. For this matter, take $f \in C^n$ with $\partial f = 0$ and $\alpha(f) = \Delta(u)$ for some $u \in (\text{tot } C^{\bullet,\bullet})^{n-1}$. Write $\alpha(f) = (f^{p,q})_{p,q} \in \bigoplus_{p+q=n} C^{p,q}$. We will now modify $f$ step by step by elements of $\partial(C^{n-1})$ such that it lies in higher and higher $I^p C^n$ until it lies in $I^{n+1} C^n = 0$, i.e., $f$ is cohomologous to zero.

Let $\tilde{f} \in C^n$ and $\tilde{u} \in (\text{tot } C^{\bullet,\bullet})^{n-1}$. We call the tuple $(\tilde{f}, \tilde{u})$ better than $(f, u)$ at $p$ if the following hold:

1. $\tilde{f} - f \in \partial(C^{n-1})$,
2. $\tilde{f} \in I^p C^n$,
3. $\alpha(\tilde{f}) = \Delta(\tilde{u})$, and
4. $\tilde{u}^{k,n-1-k} = 0$ for $k < p$.

We will inductively construct an $\tilde{f} \in C^n$, such that $(\tilde{f}, 0)$ is better than $(f, u)$ at $n$. We will afterwards show that this $\tilde{f}$ is already zero and hence $f \in \partial(C^{n-1})$.

Obviously $(f, u)$ itself is better than $(f, u)$ at $0$. If $(\tilde{f}, \tilde{u})$ is better than $(f, u)$ at $p$, we construct a tuple $(\tilde{f}, \tilde{u})$ which is better than $(f, u)$ at $p + 1$ as follows: Note that analogously to proposition 11.14, by proposition 10.19 and 10.4, $\tilde{f}^{p',q} = 0$ for all $p' < p$ and $\tilde{f}^{p,q} = r_p(\tilde{f})$. By assumption, $r_p(\tilde{f}) = \partial \tilde{u}^{p,n-1-p}$.

If $p \leq n - 2$, we can do the following: By proposition 10.9 (with $u = \tilde{u}^{p,n-p-1}$, $\tilde{f} = \tilde{f}$, $\tilde{f} = f$, $p = p$, $q = n - p$) we find $g \in I^p C^{n-1}$ with the following properties:

1. $\alpha(g)^{p',n-1-p'} = \tilde{u}^{p',n-1-p'}$ for all $p' \leq p$,
2. $\tilde{f} - \partial(g) \in I^{p+1} C^n$.

Note that for proposition 10.9 to be applicable, we need the assumption that $p \leq n - 2$.

Now set $\tilde{f} = \tilde{f} - \partial(g)$ and $\tilde{u} = \tilde{u} - \alpha(g)$. To show that $(\tilde{f}, \tilde{u})$ is better than $(f, u)$ at $p + 1$, we only have to show that $\alpha(\tilde{f}) = \Delta(\tilde{u})$.

but this is straight forward:

$$\alpha(\tilde{f}) = \alpha(\tilde{f}) - \alpha(\partial g) = \Delta(\tilde{u}) - \Delta(\alpha(g)) = \Delta(\tilde{u}).$$
Repeating this process, we get a tuple \((\tilde{f}, \tilde{u})\), which is better than \((f, u)\) at \(n - 1\), so

\[
\tilde{u} = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \tilde{u}^{n-1,0}
\]

and

\[
\alpha(\tilde{f}) = 0 \oplus \cdots \oplus 0 \oplus \tilde{f}^{n-1,1} \oplus \tilde{f}^{n,0}.
\]

Now set \(g = (\tilde{u}^{n-1,0})^\sharp \in I^{n-1} C^{n-1}(D \times G, A)\), \(\tilde{f} = \tilde{f} - \partial(g), \tilde{u} = \tilde{u} - \alpha(g)\). (Note that by construction, \(\tilde{u} = 0\).)

It is immediately clear that

\[
\alpha(\tilde{f}) = \Delta(\tilde{u}) = 0.
\]

To see that \((\tilde{f}, 0)\) is better than \((f, u)\) at \(n\), it only remains to show that \(\tilde{f} \in I^n C^n\). As it is clear from the construction that \(\tilde{f} \in I^{n-1} C^n\), we only need to show that

\[
\tilde{f}(d, 1, x) = \tilde{f}(d, 1, x).
\]

The equality \(\tilde{f}^{n-1,1} = \partial \tilde{u}^{n-1,0}\) (together with \(\tilde{f} \in I^{n-1} C^n\)) implies

\[
\tilde{f}(1, x, \sigma) = r_n^{-1}(\tilde{f})(\pi_D(x))(\sigma)
= \tilde{f}^{n-1,1}(\pi_D(x))(\sigma)
= \partial(\tilde{u}^{n-1,0}(\pi_D(x)))(\sigma)
= \sigma \tilde{u}^{n-1,0}(\pi_D(x)) - \tilde{u}^{n-1,0}(\pi_D(x))
= (1, \sigma)(\tilde{u}^{n-1,0})^\sharp(x) - (\tilde{u}^{n-1,0})^\sharp(x).
\]

As \(\partial(\tilde{f}) = 0\) and \(\tilde{f} \in I^{n-1} C^n\), the coboundary expansion of \(\partial \tilde{f}((d, 1), (1, \sigma), x) = 0\) yields

\[
\tilde{f}((d, \sigma), x) = (d, 1) \tilde{f}((1, \sigma), x) + \tilde{f}((d, 1), x)
= (d, 1) \tilde{f}((1, \sigma), x) + \text{terms independent of } \sigma
\]

and analogously

\[
\partial(g)((d, \sigma), x) = (d, \sigma)g(x) + \text{terms independent of } \sigma.
\]

We can hence compute:

\[
\tilde{f}((d, \sigma), x) = (\tilde{f} - \partial(g))((d, \sigma), x)
= (d, 1) \tilde{f}((1, \sigma), x) - (d, \sigma)g(x) + \text{terms independent of } \sigma
= (d, \sigma)(\tilde{u}^{n-1,0})^\sharp(x) - (d, 1)(\tilde{u}^{n-1,0})^\sharp(x) - (d, \sigma)(\tilde{u}^{n-1,0})^\sharp(x)
+ \text{terms independent of } \sigma,
\]

which is independent of \(\sigma\), hence \(\tilde{f} \in I^n C^n\).

We conclude the proof by showing that if \(\tilde{f} \in I^n C^n \cap \ker \alpha \cap \ker \partial\), then \(\tilde{f} \in \partial C^{n-1}\). But indeed such an \(\tilde{f} \in I^n C^n \cap \ker \alpha\) is already zero:

\[
\tilde{f}(d_1, x_1, \ldots, d_n, x_n) = \tilde{r}_n(\tilde{f})(d_1, \ldots, d_n)
= \tilde{r}_n(\tilde{f})(d_1, \ldots, d_n)
\]
\[ \alpha(f)_{n,0}(d_1, \ldots, d_n) = 0. \]

\[ \square \]

### 11.4 Compatibility with cup-products

In addition to the objects fixed so far, we also fix a \( D \times G \)-module with \( \mathcal{C} \)-rigidification \( B \), for which \( D \) is pliant.

We also assume the existence of a cup product, i.e., a \( D \times G \)-module with \( \mathcal{C} \)-rigidification \( A \otimes B \), for which \( D \) is pliant, and which satisfies the following:

- \((A \times B)(\bullet) = A(\bullet) \otimes B(\bullet),\)
- the map

\[ \cup: C^n(G, A) \times C^{m-n}(G, B) \rightarrow C^m(G, A \otimes B) \]

given by the usual formula

\[ (f \cup g)(x_1, \ldots, x_m) = f(x_1, \ldots, x_n) \otimes (x_1 \ldots x_n)g(x_{n+1}, \ldots, x_m) \]

is well-defined for all \( n, m \).

We will consider two cup products in what follows: On the one hand, the topologised cup product whose existence we just assumed, and the discrete cup product

\[ C^p(D, C^{n-p}(G, A)) \times C^{a-p}(D, C^{m-n-a+p}(B)) \]

\[ \rightarrow C^a(D, C^{n-p}(G, A) \otimes C^{m-n-a+p}(B, G)) \]

Following [14, (3.4.5.2)], we can extend the discrete cup product to the level of complexes

\[ \text{tot } C^*(D, C^*(G, A)) \times \text{tot } C^*(D, C^*(G, B)) \rightarrow \text{tot } C^*(D, C^*(G, A) \otimes C^*(G, B)) \]

by twisting the individual (discrete) cup products as follows:

\[ \cup^* = \left((-1)^{p(m-n-a)+p} \right)_{p,a-p}^{n-p,m-n-a+p} \]

(The choice of indices here will be convenient later on.) We can now state the precise meaning of the compatibility of \( \alpha \) with cup products.

**Theorem 11.16** The following diagram commutes:

\[ \begin{array}{ccc}
C^*(D \times G, A) \times C^*(D \times G, B) & \xrightarrow{\alpha \times \alpha} & \text{tot } C^*(D, C^*(G, A)) \times \text{tot } C^*(D, C^*(G, B)) \\
\cup & \downarrow & \text{tot } C^*(D, C^*(G, A) \otimes C^*(G, B)) \\
\cup & \downarrow & \text{tot } C^*(D, C^*(G, A) \otimes C^*(G, B)) \\
C^*(D \times G, A \otimes B) & \xrightarrow{\alpha} & \text{tot } C^*(D, C^*(G, A \otimes B))
\end{array} \]

**Remark 11.17** The proof is of course purely formal. The book keeping is however non-trivial. The main problem is the following: Take a summand \( C^{a,m-a}(A \otimes B) = C^a(D, C^{m-a}(G, A \otimes B)) \) in the bottom right corner. There are many summands in the top right corner mapping to this: Namely all \( C^{p,n-p}(A) \times C^{a-p,m-n-a+p}(B) \) with \( p \) running. To deal with this, we have to partition the lower quasi-isomorphism (which is essentially given by shuffling) into the correct components.
Definition 11.18 Denote the set of injective maps of ordered sets \([a] \to [m]\) by \([a < m]\) for \(\phi \in [a < m]\) we set \(\phi(0) = -\infty\) and \(\phi(a + 1) = \infty\) to simplify a few conditions. We consider the subsets
\[ [a < m]^{p,n} = \{\phi \in [a < m] | \phi(p) \leq n \text{ and } \phi(p + 1) > n\}. \]
As always \(\phi(i) \geq i\), we see that for any \(n \leq m\) we can write \([a < m]\) as a disjoint union:
\[ [a < m] = \bigcup_{0 \leq p \leq a} [a < m]^{p,n}. \]

For \(\phi \in [a < m]^{p,n}\) we define
\[ \text{head}(\phi) = \phi|_p \]
and
\[ \text{tail}(\phi) = \phi(i + p) - n. \]

The aforementioned partitioning is done by means of the following lemma.

Lemma 11.19 The map
\[ (\text{head}(-), \text{tail}(-)) : [a < m]^{p,n} \to [p < n] \times [a - p < m - n] \]
is a bijection.

Proof Its inverse is given by
\[ \text{paste}(\phi_h, \phi_t)(i) = \begin{cases} \phi_h(i) & \text{if } i \leq p \\ \phi_t(i - p) + n & \text{else.} \end{cases} \]

We of course also need to keep track of signs.

Lemma 11.20 For \(\phi \in [a < m]^{p,n}\) we have the following identity of signs:
\[ \text{sgn}(\text{head}(\phi)) \cdot \text{sgn}(\text{tail}(\phi)) = (-1)^{p(m-a-n+p)} \cdot \text{sgn}(\phi). \]

Proof It is easy to see that \((\text{head}(\phi))^* \in [n - p < n]\) is just \(\phi^* |_{[n-p]}\) and that \((\text{tail}(\phi))^* \in [m - n - a + p < m - n]\) is given by
\[ (\text{tail}(\phi))^*(i) = \phi^*(i + n - p) - n, \]
therefore
\[ \text{sgn}(\text{head}(\phi)) = (-1)^{\sum_{i=1}^{n-p} \phi^*(i) - i} \]
and
\[ \text{sgn}(\text{head}(\phi)) = (-1)^{\sum_{i=n-p+1}^{m-a+p} \phi^*(i + n - p) - n - i} \]
\[ = (-1)^{\sum_{i=n-p+1}^{m-a} \phi^*(i) + p - i}. \]
Proof of theorem 11.16 We will focus our attention on these summands in diagram ⋆:

\[
C^n(A) \times C^{m-n}(B) \xrightarrow{\alpha} (C^{p,n-p}(A)) \times C^{a-p,m-n-a+p}(B)) \oplus \ldots \oplus \ldots
\]

\[
\begin{array}{c}
C^m(A \otimes B) \xrightarrow{\alpha} C^{a,m-a}(A \otimes B)) \oplus \ldots
\end{array}
\]

For \( f \in C^n(A) \) and \( g \in C^{m-n}(B) \) the entry in \( C^{a,m-a}(A \otimes B) \) via \( C^m(A \otimes B) \) is given by

\[
\alpha(f \cup g)^{a,m-a}(d_1, \ldots, d_a)(x_1, \ldots, x_{m-a}) = \sum_{\phi \in [a < m]} \text{sgn}(\phi) \cdot (f \cup g)(z^{\phi})
\]

with

\[
z = ((1, x_1), \ldots, (1, x_{m-a}), (d_1, 1), \ldots, (d_a, 1)).
\]

The crucial identity is the following: If \( \phi \in [a < m]^{p,n} \), then

\[
\begin{align*}
\cup \left( r_p(f^{\text{head}}(\phi)) \cup_{p,a-p}^{n-p,m-n-a+p} r_{a-p}(g^{\text{tail}}(\phi)) \right)(d)(x) \\
= r_p(f^{\text{head}}(\phi))(d_1, \ldots, d_p)(x_1, \ldots, x_{n-p}) \otimes d_1 \ldots d_p \cdot x_1 \ldots x_{n-p} \cdot r_{a-p}(g^{\text{tail}}(\phi))(d_{p+1}, \ldots, d_a)(x_{n-p+1}, \ldots, x_{m-a}) \\
= (f \cup g)(z^{\phi}).
\end{align*}
\]

The \( \alpha \circ \cup \)-entry can therefore be expressed as follows:

\[
\alpha(f \cup g)^{a,m-a}(d)(x) = \sum_{\phi \in [a < m]} \text{sgn}(\phi) \cdot (f \cup g)(z^{\phi})
\]

\[
= \sum_{0 \leq p \leq n} \sum_{\phi \in [a < m]^{p,n}} \text{sgn}(\phi) \cdot \cup \left( r_p(f^{\text{head}}(\phi)) \cup_{p,a-p}^{n-p,m-n-a+p} r_{a-p}(g^{\text{tail}}(\phi)) \right)(d)(x)
\]

To this inner sum we apply lemma 11.19, where we write \( \cup' \) instead of \( \cup_{p,a-p}^{n-p,m-n-a+p} \):

\[
\sum_{\phi \in [a < m]^{p,n}} \text{sgn}(\phi) \cdot \cup \left( r_p(f^{\text{head}}(\phi)) \cup_{p,a-p}^{n-p,m-n-a+p} r_{a-p}(g^{\text{tail}}(\phi)) \right)(d)(x)
\]

\[
= \sum_{(\phi_h, \phi_t) \in [p < a] \times [a-p < m-n]} \text{sgn}(\text{paste}(\phi_h, \phi_t)) \cdot \cup \left( r_p(f^{\phi_h}) \cup_{p,a-p}^{n-p,m-n-a+p} r_{a-p}(g^{\phi_t}) \right)(d)(x)
\]

\[
= \sum_{\phi_h, \phi_t} (-1)^{p(m-a-n+p)} \text{sgn}(\phi_h) \text{sgn}(\phi_t) \cdot \cup \left( r_p(f^{\phi_h}) \cup_{p,a-p}^{n-p,m-n-a+p} r_{a-p}(g^{\phi_t}) \right)(d)(x)
\]

\[
= (\cup \circ \cup)^* \left( r_p(\text{shuffle}^a_p(f)), r_{a-p}(\text{shuffle}^{m-n}_a(g)) \right)(d)(x)
\]

Here we used lemma 11.20 and the bilinearity of the cup product for the last two equalities. It follows that indeed

\[
\alpha(f \cup g)^{a,m-a} = ((\cup \circ \cup)^*(\alpha(f), \alpha(g))^{a,m-a},
\]

which finishes the proof. \( \square \)
11.5 On a theorem of Jannsen

The main result of [11] also has a variant in the topological setting.

We first recall the following result:

**Proposition 11.21** ([15, (2.3.4)]) Let $C^{\bullet}, D^{\bullet}$ be complexes of modules over a Dedekind domain $R$. Assume that both complexes are bounded in the same direction or that one of them is bounded above and below. If $C^{\bullet}$ consists of flat $R$-modules, then there is a non-canonical splitting

$$H^n(\text{tot}_R C^{\bullet} \otimes R D^{\bullet}) \cong \bigoplus_{p+q=r} ss(C^{\bullet} \otimes_R D^{\bullet})_2^{p,q},$$

where $ss(C^{\bullet} \otimes_R D^{\bullet})_2^{p,q}$ denotes the $E_2$-terms of the spectral sequence attached to the double complex (cf. e.g. [15, (2.2.3)] for details).

**Proposition 11.22** If $D$ is finite and acts trivially on $A$, then

$$H^n(D \times G, A) \cong \bigoplus_{p+q=n} H^p(D, H^q(G, A)).$$

**Proof** By theorem 11.6 it suffices to show that

$$C^{\bullet, \bullet} \cong C^{\bullet}(D, \mathbb{Z}) \otimes C^{\bullet}(G, A)$$

as double complexes, as we can then employ proposition 11.21 to get the desired result. As $D$ is finite, it is clear that

$$C^p(D, \mathbb{Z}) \otimes C^q(G, A) \longrightarrow C^p(D, C^q(G, A))$$

$$(f, g) \longmapsto \left( \begin{array}{c} dp \\ d \end{array} \begin{array}{c} g \\ f(d) \end{array} \right)$$

is bijective and it is easily verified that it commutes with differentials. \hfill \Box

The assumption of finite $D$ is regrettably crucial in the proof. In [11] the case of compact (but not necessarily discrete) $D$ and discrete $A$ is considered. Every morphism $D \longrightarrow A$ then has finite image, which induces the isomorphism above.

However, for the easiest monoids we also have the following:

**Proposition 11.23** If $D \cong \mathbb{N}_0^r$ (or $D \cong \mathbb{Z}^r$) acts trivially on $A$, then

$$H^n(D \times G, A) \cong \bigoplus_{k=0}^r H^{n-k}(G, A) \oplus (G)^r.$$ 

**Proof** It suffices to show the proposition for $r = 1$, as the general case then follows by induction. By remark 11.5 and theorem 11.6

$$C^{\bullet}(D \times G, A) \cong \text{tot} \left( \begin{array}{c} C^{\bullet}(G, A) \\ 0 \end{array} \right) \cong C^{\bullet}(G, A) \oplus C^{\bullet-1}(G, A),$$

so

$$H^n(D \times G, A) \cong H^n(G, A) \oplus H^{n-1}(G, A).$$

\hfill \Box
12 Shapiro’s Lemma for topologised monoids

The results of the previous section allow us to extend Shapiro’s lemma to monoids.

Theorem 12.1 Let $\mathbf{C}$ be a topological category, $G$ a topologised group in $\mathbf{C}$ and $D$ a discrete monoid. Let $H \leq G$ be a subgroup as in Sect. 9 and $A$ a rigidified $D \times H$-module with $D$ being $A$-pliant. Then

$$C^\bullet(D \times G, \text{Ind}_G^H(A)) \cong C^\bullet(D \times H, A)$$

in the derived category of abelian groups.

**Proof** Let us first note that $D$ is also $\text{Ind}_G^H(A)$-pliant: We need to show that for every $X \in \mathbf{C}$ we have an equality

$$\text{Ind}_G^H(A)(D \times X) = \text{Hom}_{\text{Set}}(D, \text{Ind}_G^H(A)(X)).$$

As $D$ is $A$-pliant, $\text{Ind}_G^H(A)(D \times X)$ are those maps in $\text{Hom}_{\text{Set}}(D, h_A(X \times G))$ which are $H$-linear in the $G$-argument. But that is exactly $\text{Hom}_{\text{Set}}(D, \text{Ind}_G^H(A)(X)).$

We can hence use theorem 11.6 to see that

$$\text{tot}\, C^\bullet(D \times G, \text{Ind}_G^H(A)) \cong \text{tot}\, C^\bullet(D, C^\bullet(G, \text{Ind}_G^H(A))).$$

By proposition 7.3 we have a quasi-isomorphism

$$\text{tot}\, C^\bullet(D, C^\bullet(G, \text{Ind}_G^H(A))) \cong \text{tot}\, X^\bullet(D, C^\bullet(G, \text{Ind}_G^H(A))).$$

As $D$ is discrete, $X^\bullet(D, -) = \text{Hom}_{\mathbb{Z}[D]}(F_\cdot, -)$, where $F_\cdot$ is a complex of free $\mathbb{Z}[D]$-modules, cf. proposition 5.2. Thus $X^\bullet(D, -)$ preserves quasi-isomorphisms. Using these arguments again, together with theorem 9.8, we arrive at quasi-isomorphisms

$$\text{tot}\, X^\bullet(D, C^\bullet(G, \text{Ind}_G^H(A))) \cong \text{tot}\, X^\bullet(D, C^\bullet(H, A)) \cong \text{tot}\, C^\bullet(D, C^\bullet(H, A)) \cong C^\bullet(D \times H, A).$$

\[\square\]

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