Soil – Structure Interaction of Cylindrical Tank of Variable Wall Thickness under the Thermal Gradient Conditions

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Abstract. The paper presents an effective method for the analysis of soil-structure interaction including the behaviour of cylindrical storage tank with varying wall thickness under the thermal gradient conditions. Elastic half-space and the Winkler model have been used for the description of subsoil. The soil-structure interaction is described by using the power series. A computational example of reinforced concrete tank loaded with temperature difference across the wall thickness is given. The analysis of a hydrostatically loaded cylindrical tank, performed for the model, incorporating elastic half-space shows decrease of radial displacements, as well as substantial changes in the distribution of bending moments compared to the Winkler foundation. A local increase of subsoil reaction around the slab circumference is observed for the case of elastic half-space, contrary to the Winkler model. In the case of a tank loaded with the thermal gradient, however, the solutions for both subsoil models do not differ significantly.

1. Introduction

In thin-walled reinforced concrete cylindrical structures subjected to temperature differences across the wall thickness, the development of vertical cracks caused by relatively small temperature gradients is often observed. In the paper, a method for calculating unfavourable thermal stresses and displacements in cylindrical reservoir structures with walls of varying thickness, fixed at their base in the ground slab and subjected to axisymmetric thermal gradient loading is considered.

In the paper, the analytical method for the soil-structure interaction, which allows to describe the behaviour of cylindrical storage silo structures of variable wall thickness based on the two subsoil models, such as the perfectly elastic half-space and the Winkler springs models is presented. The method of Borowicka [1] and Gorbunov-Posadov [2] employed here for the analysis of the ground slab of the cylindrical silo structure, enables the description of subsoil as a perfectly elastic half-space. In order to account for the interaction model, the analysis was carried out using the power series expansion method. This approach gives the advantage of being relatively simple and accurate, when compared to analysis methods based on different representations of soil-structure interaction. In the available literature, however, this method has not been widely used. Alternative approaches require using Hankel transforms and Bessel function series (Hensley [3, 4]) or numerical methods (Melerski [5–9], Horvath & Colasanti [10]), leading to complex algorithms or approximate results. Kukreti & Siddiqi [11] used quadrature methods involving polynomial functions with weights at selected points.
2. Interaction of cylindrical shells of linearly variable thickness with ground slabs

In the analysis of the soil-structure interaction of the structure of cylindrical reservoirs with walls of variable thickness, axisymmetrically loaded and jointed monolithically with the ground slab, the theory of boundary perturbations was used (described e.g. in the textbook [12]). The conditions of applicability of the theory of boundary perturbations (see [12]) in the considered case are fulfilled.

In the description of bending of the cylindrical shells of linearly variable thickness the formulation introduced in monograph [13] was applied. As the thickness of the shell changes linearly, in proportion to variable $x$, such formulated question cannot be reduced to the problem of the behaviour of a shell of constant thickness. The linear function of the change of thickness of the shell is given by the equation:

$$ h = h(x) = \mu \cdot x $$

(1)

The scheme of interaction of tank elements with elastic half-space together with hyperstatic values is shown in the figure 1. The static scheme of the tank is shown together with hyperstatic values.

![Figure 1. Scheme of interaction of tank elements and circular plate with elastic half-space. From the top: scheme of the cylindrical shell of linearly variable wall thickness, scheme of the circular plate](image)

The problem of temperature loading of uneven distribution through the wall thickness leads to the heterogeneous displacement equation of the shell.

Let us use the physical relationships of Novozhilov (see [12]) which include the components that can describe the deformation caused by the difference of temperatures of the value $\Delta T$ (given below):

$$ M_\theta = \frac{E \mu^3 x^3}{12(1-\nu^2)} \left( -\nu \frac{d^2 w}{dx^2} - (1+\nu) \alpha T \frac{\Delta T}{\mu x} \right), $$

(2)

$$ M_x = \frac{E \mu^3 x^3}{12(1-\nu^2)} \left( \frac{d^2 w}{dx^2} - (1+\nu) \alpha T \frac{\Delta T}{\mu x} \right), $$

(3)
Substituting the constitutive equations (equations (2), (3) and those for meridional and latitudinal shell forces) into the equilibrium equations we get the inhomogeneous displacement equation. First, it is necessary to obtain the solution of the homogeneous equation, as it is widely known from the available literature (see e.g. the monograph [14]). The solution of the homogeneous equation can be reduced to the solution of the Bessel-type equation. The general integral of the homogeneous equation can be expressed through derivatives of the Kelvin functions in the form:

$$w_0(x) = \frac{1}{\sqrt{x}} \left( A_1 \text{ber}'(2\lambda\sqrt{x}) + A_2 \text{bei}'(2\lambda\sqrt{x}) + A_3 \text{ker}'(2\lambda\sqrt{x}) + A_4 \text{kei}'(2\lambda\sqrt{x}) \right),$$

where

$$\lambda = \left( \frac{12(1-v^2)}{\mu^2 R^2} \right)^{1/4},$$

The solution $w_0(x)$ is a function of radial displacement of the shell, where $R$ – the shell and slab radius. Subsequently, the constants $A_1 - A_4$ of the solution (4) are determined according to the boundary conditions and the specific solutions of the inhomogeneous equation are sought.

The influence of the temperature difference in the cylindrical shell of the tank ($\Delta T$) can be determined, specifying the particular integral of the displacement equation of the shell, using the prediction method, in the form (the higher temperature prevails inside the shell):

$$w_r = \frac{\alpha \Delta T \mu R^2}{6(1-\nu)x},$$

The stress state of the shell results also from the load of its lower edge with unknown support reactions: the transverse force $Q_0 = X_1$ and the bending moment $M_0 = X_2$ (see figure 3). In the boundary conditions, the upper edge of the shell was assumed to be free.

3. Analysis of elastic half-space loaded by the pressure transmitted from ground slab paper

The displacement of the elastic and isotropic half-space loaded with a circular slab is described by means of the Green function method. The Boussinesq solution to the problem [15] of the isotropic half-space loaded with a concentrated load is utilised to construct the Green function.

The pressure exerted by the circular slab on the half-space is represented by the axisymmetric loading $p(\rho)$, as shown in figures 1 and 2. Vertical displacements of the ground surface (i.e. elastic half-space boundary), $v(\rho)$, are expressed in terms of non-dimensional coordinates:

- $\rho$ – non-dimensional distance from the centre of the plate to the point of the surface of the subsoil, in which the displacement is evaluated,
- $\bar{\rho}$ – non-dimensional distance from the centre of the plate to the point of the application of the loading element,
- $\alpha$ – non-dimensional radius of the loading $q$, $\alpha = a / R$, where: $a$ – the loading radius (figure 2), $\chi$ – the arbitrary parameter of the integration.

The Green function is integrated over the surface of the circular slab. Finally, the elastic subsidence of half-space, $v(\rho)$, due to the loading $p(\rho)$ transmitted by the circular slab, can be presented in the form:
The function $p(\rho)$, describing the interaction between the tank foundation and the elastic half space, as well as the slab deflection, are assumed in the form of power series expansion.

4. Axisymmetric bending of plate on elastic subsoil

The differential equation of the bending surface of the circular plate resting on the elastic half-space and undergoing loading $q$, uniformly distributed on the surface of the circle of radius $a$ (figure 2), after introducing the non-dimensional polar coordinates takes the form (see Gorbunov-Posadov [2]):

$$\frac{d^4w}{dp^4} + \frac{2}{\rho} \frac{d^3w}{dp^3} - \frac{1}{\rho^2} \frac{d^2w}{dp^2} - \frac{1}{\rho^3} \frac{dw}{dp} = \frac{R^4}{D} [q - p(\rho)]$$

where: $D$ – bending stiffness of plate.

For the purpose of evaluating the bending of the plate $w(\rho)$ the relation between the displacements of the plate and the subsoil should be established. Therefore, the displacements of the surface of the elastic half-space and the displacements of the plate must be identically equal: $\nu(\rho) = w(\rho)$. Function $p(\rho)$, defined as the function describing the interaction between the mat foundation and the elastic half-space, can be assumed in the form of the polynomial containing only the components of even powers, assuming an appropriately selected number of them, aiming to attain the requested accuracy of solution.

**Figure 2** Scheme of circular plate on elastic half-space

For the purpose of evaluating the integral of equation (8), at first, the homogeneous equation can be considered and the solution of biharmonic equation using the condition of axial symmetry can be obtained. Then the particular integral of equation (8) can also be sought in the form of the polynomial involving only the components of even powers. The coefficients of this polynomial may be expressed by the components of a series describing the function $p(\rho)$, by substituting both series in equation (8).
and comparing the components of identical powers. The full solution can be obtained by adding the particular integral to the general one. Eventually, the deflection of the plate in the area of the action of loading \( q \) (which is denoted by index \( I \)) is expressed by the equation:

\[
w_I = C_{1,I} \rho^2 \ln \rho + C_{2,I} \rho^2 + C_{3,I} \ln \rho + C_{4,I} \rho + \frac{R^4}{64 D} (q - a_0) \rho^4 - \frac{R^4}{16 D} \sum_{n=1}^{\infty} \frac{a_{2n}}{(n + 2)^2 (n + 1)^2} \rho^{2n+4}, \tag{9}
\]

Equation (9) holds also in the unloaded area (of index \( II \)) – by the substitution \( q = 0 \). The constants \( C_{1,\ldots,4} \) can be determined from the conditions in the centre and at the boundaries of the plate. Moreover, substituting the expression for the function describing the interaction of the subsoil and the ground slab, \( p(\rho) \), into the equation of the vertical displacement of the subsoil surface (7), expressing both elliptic integrals by the hypergeometric series, and then performing the multiplication and integration of the series, the equation for the vertical displacements of the elastic subsoil can be received also in the form of a power series.

\[
v(\rho) = \frac{2(1-\nu_c^2) R}{E_0} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \prod_{k=0}^{n} (2k - 1) \right] \frac{a_{2n}}{2n - 2m + 1} \rho^{2n} \right\}, \tag{10}
\]

By identifying \( v(\rho) = w(\rho) \), the coefficients of the series at the equal powers of \( \rho \) can be compared, which leads to a set of algebraic equations of \( N \) unknowns, allowing for the evaluation of the coefficients \( a_{2n} \). In reference to the function describing the values of reciprocal actions between the mat foundation and the elastic half-space, \( p(\rho) \), it can be noticed, that in the surrounding of the circle of radius \( r = R \), stress concentration can be encountered (as it is a contact problem), and therefore this function can pursue infinity. Thus, if this function were expressed by the infinite power series, this would have been divergent at this point.

For the comparative calculations of the ground slab of the tank, the model of the circular plate resting on the two-parameter Winkler springs model (taking into account the vertical stiffness \( k_1 = K_z \) and the horizontal one \( k_2 = K_t \)) was used here. In the calculations, the mathematical description of the problem was used in a similar way to monograph [16]. In the two-parameter model the horizontal reactions at the contact plane between a plate and a springy layer are not neglected.

5. Results and discussion

Computation was performed and the plots were drawn taking into account \( N = 100 \) first terms of the expansion series of the function describing the interaction of the ground slab with soil. This assumption ensures sufficient accuracy (Lewiński and Rak [17]). It was assumed that the tank is a structure made of cast concrete. For the calculation, the following data were assumed: modulus of elasticity of concrete: \( E_c = 31000 \) MPa, Poisson’s ratio of concrete: \( \nu_c = 0.2 \), the modulus of elasticity of the subsoil – an elastic half-space: \( E_0 = 240 \) MPa, Poisson’s ratio of the subsoil : \( \nu_0 = 0.3 \), the coefficient of linear thermal expansion \( \alpha_T = 10 \cdot 10^{-6} \), the coefficient of vertical Winkler springs \( K_z = 22000 \) kN/m\(^2\), the coefficient of horizontal springs \( K_t = 15000 \) kN/m\(^2\), concrete gravity \( \gamma_c = 25 \) kN/m\(^3\). The geometric parameters of the shell were assumed: radius \( R = 7.0 \) m, height \( H = 5.0 \) m. Cylindrical wall thickness varied from 20 cm at the top to 30 cm at the bottom, the thickness of the circular ground slab: \( h_g = 30 \) cm, the loading parameters; water pressure: gravity \( \gamma_w = 10 \) kN/m\(^3\). The imposed difference of temperatures across the shell thickness was assumed: \( \Delta T = 15^\circ \)C.

The case of hydrostatic loading was considered first. The results obtained for the ground slab are shown in a coordinate system with the beginning at the slab centre, while the abscissa for the shell
begins at the upper edge of the cylinder. The analytical results for the cylindrical tank with a ground slab resting on the elastic half-space are compared with the solution for the mat foundation resting on the subsoil modelled by the two-parameter Winkler springs (see figures 3 and 4). Diversified results for the radial moments of the circular ground slab under hydrostatic loading for two subsoil models: the elastic half-space and two-parameter Winkler model are the result of the different distributions of the subsoil reaction under the tank for both subsoil models (see figure 3). In both cases, the slab is bent from the hydrostatic pressure of the liquid and the self-weight of its wall. The self-weight of the slab is not taken into account because the bending of the cast in situ slab, due to its self-weight, occurs before the concrete is set. The different edge rotations of the circular ground slab affected the different distributions of the radial moments in the slab (see figure 4) and, in consequence, different distributions of the meridional moments in the tank wall. In the case of the model of elastic half-space a local unlimited growth of subsoil reaction around the slab perimeter is observed, in contrast to the Winkler model (figure 4). In fact, the local plasticisation of the soil may occur in this area [18].

Figure 3. Distributions of the subsoil reaction under the tank under hydrostatic pressure (and self-weight) for two subsoil models: elastic half-space and two-parameter Winkler model

Figure 4. Distributions of the radial moments in the tank slab due to horizontal pressure for two subsoil models: elastic half-space and two-parameter Winkler model

Then let us consider the case of the imposed difference of temperatures across the shell thickness $\Delta T = 15 ^\circ C$, making analogous comparisons (see figures 5–11). In this case, the slab is loaded only by the self-weight of the wall, however, slab-shell interaction is also taken into account (as before). Positive radial displacement of the shell is assumed to be inward. Vertical deflections of the circular ground slab due to the thermal gradient of the shell $\Delta T = 15 ^\circ C$ (and self-weight) for two subsoil models: elastic half-space and two-parameter Winkler model have the same character (figure 5). Distributions of the bending moments in the tank slab due to the thermal gradient of the shell $\Delta T = 15$
°C for the two above subsoil models are quite similar (figures 6 and 7). Conversely, the distribution of internal forces in the tank wall and the horizontal displacements of the tank wall for the considered thermal gradient for the two above soil models are nearly overlapping (figures 8, 9, 10 and 11).

**Figure 5.** Vertical deflections of the circular ground slab under the thermal gradient of the shell $\Delta T = 15^\circ$C (and self-weight) for two subsoil models: elastic half-space and two-parameter Winkler model

**Figure 6.** Distributions of the radial moments in the tank slab due to the thermal gradient of the shell $\Delta T = 15^\circ$C for two subsoil models: elastic half-space and two-parameter Winkler model
Figure 7. Distributions of the circumferential moments in the tank slab due to the thermal gradient of the shell $\Delta T = 15 \, ^\circ C$ for two subsoil models: elastic half-space and two-parameter Winkler model.

Figure 8. Circumferential forces in the tank wall due to the thermal gradient of the shell $\Delta T = 15 \, ^\circ C$ (and self-weight) for subsoil models of elastic half-space and two-parameter Winkler model.

Figure 9. Horizontal displacements of the tank wall due to the thermal gradient of the shell $\Delta T = 15 \, ^\circ C$ (and self-weight) for subsoil models of elastic half-space and two-parameter Winkler model.
Figure 10. Meridional moments in the tank wall due to the thermal gradient of the shell $\Delta T = 15^\circ C$ (and self-weight) for subsoil models of elastic half-space and two-parameter Winkler springs

Figure 11. Latitudinal moments in the tank wall due to the thermal gradient of the shell $\Delta T = 15^\circ C$ (and self-weight) for subsoil models of elastic half-space and two-parameter Winkler springs

Similar values of bending moments at the joint of the slab with the cylindrical shell result in similar behaviour of the cylindrical shell in both cases.

6. Conclusions

The approach presented in the paper seems to be relatively simple and accurate when compared to analysis methods based on different descriptions of soil-structure interaction and in comparison with the existing analyses of fluid reservoirs including subsoil-structure interaction [19, 20, 21, 22]. A significant difference between the results obtained for both subsoil models can be observed in the tank under hydrostatic pressure. The slab supported on the elastic half-space can be bent due to the uniformly distributed load, while the same structure assumed to be supported on Winkler springs is bent only due to edge forces, which results in opposite values of the moments at the joint of the slab with the cylindrical shell (figure 4). To the contrary, in the case of the imposed temperature gradient across the shell thickness, the distribution of internal forces in the tank and the horizontal displacements of the tank wall for the two above ground models are quite similar (figures 6–11). The analytical results indicate that the thermal stresses originated only from the bending moments (figures 10 and 11) in the given example can significantly exceed 2.6 MPa (the mean tensile strength of concrete of the class C25/30 assumed in the presented example), so that a moderate thermal gradient across the shell thickness can lead to severe cracking.
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