An Assumption-Free Exact Test For Fixed-Design Linear Models With Exchangeable Errors

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Abstract

We propose the cyclic permutation test (CPT) to test general linear hypotheses for linear models. This test is non-randomized and valid in finite samples with exact type-I error $\alpha$ for arbitrary fixed design matrix and arbitrary exchangeable errors, whenever $1/\alpha$ is an integer and $n/p \geq 1/\alpha - 1$. The test applies the marginal rank test on $1/\alpha$ linear statistics of the outcome vectors where the coefficient vectors are determined by solving a linear system such that the joint distribution of the linear statistics is invariant to a non-standard cyclic permutation group under the null hypothesis. The power can be further enhanced by solving a secondary non-linear travelling salesman problem, for which the genetic algorithm can find a reasonably good solution. We show that CPT has comparable power with existing tests through extensive simulation studies. When testing for a single contrast of coefficients, an exact confidence interval can be obtained by inverting the test. Furthermore, we provide a selective yet extensive literature review of the century-long efforts on this problem, highlighting the non-triviality of our test.

keywords: linear model, linear hypothesis, exact test, fixed-design, assumption-free test, marginal rank test, non-linear travelling salesman problem

1 Introduction

In this article, we consider the following fixed-design linear model

\[ y_i = \beta_0 + \sum_{j=1}^{p} x_{ij} \beta_j + \epsilon_i, \quad i = 1, \ldots, n, \]

where $\epsilon_i$’s are stochastic errors and $x_{ij}$’s are treated as fixed quantities. Throughout we will use the following compact notation

\[ y = \beta_0 \mathbf{1} + X \beta + \epsilon, \]

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where \( y = (y_i) \) denote the response vector, \( X = (x_{ij}) \in \mathbb{R}^{n \times p} \) denote the design matrix, \( \epsilon = (\epsilon) \) denote the error terms and \( 1 \in \mathbb{R}^n \) denote the vector with all entries equal to one. Two driving forces in early history of statistics – location problems and analysis of variance (ANOVA) problems – are both special cases of linear models.

Our focus is on testing a general linear hypothesis:

\[
H_0 : R^T \beta = 0, \text{ where } R \in \mathbb{R}^{p \times r} \text{ is a fixed matrix with rank } r.
\]

(3)

Testing linear hypotheses in linear models is ubiquitous and fundamental in numerous areas. One important example is to test whether a particular coefficient is zero, i.e. \( H_0 : \beta_1 = 0 \), a special case where \( R = (1, 0, \ldots, 0)^T \in \mathbb{R}^{p \times 1} \). Another important example is to test the global null, i.e. \( H_0 : \beta = 0 \), equivalent to the linear hypothesis with \( R = I_{p \times p} \). We refer to Chapter 7 of Lehmann and Romano [2006] for an extensive discussion of other examples. By inverting a test with (asymptotically) valid type-I error control, we can obtain a confidence interval/region for \( R\beta \). This is of particular interest when \( r = 1 \), which corresponds to a single linear contrast of the regression coefficient.

This is one of the most fundamental and long-lasting problem in statistics as well as a convenient powerful prototype to generate methodology that works for more complicated statistical problems. In the past century, several categories of methodology were proposed: normal theory based tests [Fisher, 1922, 1924], permutation tests [Pitman, 1937a, 1938], rank-based tests [Friedman, 1937], tests based on regression R-estimates [Hájek, 1962], M-estimates [Huber, 1973] and L-estimates [Bickel, 1973], resampling based tests [Freedman, 1981] and other tests (e.g. median-based tests [Theil, 1950a, Brown and Mood, 1951], symmetry-based tests [Hartigan, 1970] and non-standard tests [Meinshausen, 2015]). we only give the earliest reference we can track for each category to highlight the chronicle of methodology development. We will provide an extensive literature review in Section 4.

For a given confidence level \( 1 - \alpha \), a test is exact if the type-I error is exactly \( \alpha \), in finite samples without any asymptotics. Exact tests are intellectually and practical appealing because they provide strong error control without requirement of large sample or artificial asymptotic regimes. However, perhaps surprisingly, there is no test that is exact under reasonably general assumptions to the best of our knowledge. Below is a brief summary of the conditions under which the existing tests are exact.

- Regression t-tests and F-tests are exact with normal errors;
- Permutation tests are exact for global null or two-way layouts [e.g. Brown and Maritz, 1982];
- Rank-based tests are exact for ANOVA problems;
- Tests based on regression R/M/L-estimates can be exact for global null;
- Hartigan [1970]'s test is exact for certain forms of balanced ANOVA problems with symmetric errors and \( r = 1 \);
- Meinshausen [2015]'s test is exact for rotationally invariant errors with known noise level. Note that if \( \epsilon_i \)'s are i.i.d., rotation invariance implies the normality of \( \epsilon_i \)'s [Maxwell, 1860];
• Other tests are exact either for global null or under unrealistically restrictive assumptions or with infeasible computation.

In this article, we develop an exact test, referred to as cyclic permutation test (CPT), that is valid in finite samples and allows arbitrary fixed design matrix and arbitrary error distributions, provided that the error terms are exchangeable. Exchangeability is weaker than the frequently made i.i.d. assumption. Further, the test is non-randomized if $1/\alpha$ is an integer and $n/(p - r) > 1/\alpha - 1$. The former condition is true for all common choices of $\alpha$, e.g. 0.1, 0.05, 0.01, 0.005. The latter requirement is also reasonable in various applications. For instance, when $\alpha = 0.05$, the condition reads $n/(p - r) > 19$, which is true if $n/p > 19$ or $p - r$ is small. Both are typical in social science applications. We demonstrate the power of CPT through extensive simulation studies and show it is comparable to the existing ones. Admittedly, the exchangeability might not be valid in certain applications. Nevertheless, CPT is the first procedure that is provably exact with reasonable power under such weak assumption. We want to emphasize that the goal of this paper is not to propose a procedure that is superior to existing tests, but to expand the toolbox of exact inference and hopefully to motivate novel methodology for other problems.

The rest of the article is organized as follows: Section 2 discusses the motivation, the implementation and the theoretical property of cyclic permutation tests. In particular, Section 2.6 provides a summary of the implementation of CPT. In Section 3, we compare CPT with five existing tests through extensive simulation studies. Due to the space constraint, we only present partial results and leave others to Appendix A. Section 4 provides a selective yet extensive literature review on this topic. One main goal is to introduce various strategies for this problem demonstrating the difficulty of developing an exact test. We put this long review at the end of the paper to avoid distraction. Section 5 concludes the paper and discusses several related issues. All programs to replicate the results in this article can be found in https://github.com/lihualei71/CPT.

2 Cyclic Permutation Test

2.1 Main idea

Throughout the article we denote by $[n]$ the set $\{1, \ldots, n\}$. First we show that it is sufficient to consider the sub-hypothesis:

$$H_0 : \beta_1 = \ldots = \beta_r = 0.$$  

(4)

In fact, for the general linear hypothesis 3, let $U_R \in \mathbb{R}^{p \times r}$ be an orthonormal basis of the column span of $R$ and $V_R \in \mathbb{R}^{p \times (p-r)}$ be an orthonormal basis of the orthogonal complement. Then $\beta = U_RU_R^T\beta + V_RV_R^T\beta$. Let $\tilde{X} = (XU_R \ XV_R)$ and $\tilde{\beta} = \begin{pmatrix} U_R^T \beta \\ V_R^T \beta \end{pmatrix}$. Then the linear model (2) can be re-formulated as

$$y = \beta_0 \mathbf{1} + XU_R(U_R^T \beta) + XV_R(V_R^T \beta) + \epsilon = \beta_0 \mathbf{1} + \sum_{j=1}^{r} \tilde{X}_j \tilde{\beta}_j + \sum_{j=r+1}^{p} \tilde{X}_j \tilde{\beta}_j + \epsilon.$$  

(5)
On the other hand, since \( R \) has full column rank, the null hypothesis (3) is equivalent to \( H_0: \hat{\beta}_1 = \ldots = \hat{\beta}_r = 0 \), which is typically referred to as a sub-hypothesis [e.g., Adichie, 1978]. For this reason, we will focus on (4) without loss of generality throughout the rest of the paper.

Our idea is to construct a pool of linear statistics \( S = (S_0, S_1, \ldots, S_m) \) such that \( S \) is distributionally invariant under the left shifting operator \( \pi_L \) under the null, in the sense that

\[
S \overset{d}{=} \pi_L(S) \overset{d}{=} \pi_L^2(S) \overset{d}{=} \cdots \overset{d}{=} \pi_L^m(S),
\]

(6)

where

\[
\pi_L^k(S) = (S_k, S_{k+1}, \ldots, S_m, S_0, S_1, \ldots, S_{k-1}), \quad k = 1, 2, \ldots, m.
\]

(7)

Let \( \text{Id} \) denote the identity mapping, then \( G = \{ \text{Id}, \pi_L, \ldots, \pi_L^m \} \) forms a group, which we refer to as the cyclic permutation group (CPG). We say a pool of statistics \( S \) as invariant under CPG if \( S \) satisfies (6). The following trivial proposition describes the main property of CPG invariance.

**Proposition 1.** Assume that \( S = (S_0, S_1, \ldots, S_m) \) is invariant under CPG. Let \( R_0 \) be the rank of \( S_0 \) in descending order, defined as \( R_0 = \{ j \geq 0 : S_j \geq S_0 \} \). Then

\[
R_0 \succeq \text{Unif}([m+1]) \implies p \triangleq \frac{R_0}{m+1} \geq \text{Unif}([0,1])
\]

(8)

where \( \succeq \) denotes stochastic dominance, \( \text{Unif}([0,1]) \) denotes the uniform distribution on \([0,1] \). Furthermore, \( R_0 \sim \text{Unif}([m+1]) \) if \( S \) has no tie with probability 1.

**Proof.** Let \( R_j \) be the rank of \( S_j \) in descending order as defined in (8). Then the invariance of \( S \) implies the invariance of \((R_0, R_1, \ldots, R_m) \). As a result,

\[
R \overset{d}{=} R_1 \overset{d}{=} \cdots \overset{d}{=} R_m.
\]

Then for any \( k \),

\[
\mathbb{P}(R_0 \geq k) = \frac{1}{m+1} \sum_{j=0}^{m} \mathbb{P}(R_j \geq k) = \frac{1}{m+1} \sum_{j=0}^{m} \mathbb{E}(R_j \geq k) = \frac{1}{m+1} \mathbb{E}\left|\{ j \geq 0 : R_j \geq k \}\right|.
\]

Let \( S_{(1)} \geq S_{(2)} \geq \cdots \geq S_{(m+1)} \) be the ordered statistics of \((S_0, \ldots, S_m) \), which may involve ties. Then by definition, \( R_j \geq k \) whenever \( S_j \leq S_{(k-1)} \) and thus,

\[
\left|\{ j \geq 0 : R_j \geq k \}\right| \geq m - k + 1
\]

and thus \( R_0 \geq \text{Unif}([m+1]) \). When there is no tie, the set \( \{R_0, R_1, \ldots, R_m\} \) is always \( \{1, 2, \ldots, m+1\} \) and thus

\[
\mathbb{P}(R_0 \geq k) = \frac{m - k + 1}{m+1}.
\]

\[\square\]

Based on the p-value defined in (8), we can derive a test that rejects the null hypothesis if \( p \leq \alpha \).
We refer to this simple test as *marginal rank test* (MRT). The following trivial proposition shows that MRT is valid in finite samples and can be exact under mild conditions.

**Proposition 2.** Suppose \( S = (S_0, S_1, \ldots, S_m) \) is invariant under CPG under \( H_0 \) and let the p-value be defined as in (8). Then \( \mathbb{P}_{H_0}(p \leq \alpha) \leq \alpha \). If \( 1/\alpha \) is an integer and \( m + 1 \) is divisible by \( 1/\alpha \), then \( \mathbb{P}_{H_0}(p \leq \alpha) = \alpha \).

In practice, the reciprocals of commonly-used confidence levels (e.g. 0.1, 0.05, 0.01, 0.005) are integers. In these cases it is sufficient to set \( m = 1/\alpha - 1 \) to obtain an exact test.

The rank used in MRT only gives one-sided information and may not be suitable for two-sided tests. More concretely, \( S_0 \) may be significantly different from \( S_1, \ldots, S_m \) under the alternative but the sign of the difference may depend on the true parameters. An intuitive remedy is to apply MRT on the following modified statistics

\[
\tilde{S}_j = |S_j - \text{median}\{\{S_j\}_{j=0}^m\}|	ag{9}
\]

Whenever \( S_0 \) is significantly different from \( S_1, \ldots, S_m \), \( \tilde{S}_0 \) is significantly larger than \( \tilde{S}_1, \ldots, \tilde{S}_m \). The following proposition guarantees the validity of the transformation (9). In particular, the transformation in (9) satisfies the condition.

**Proposition 3.** If \( S = (S_0, S_1, \ldots, S_m) \) is invariant under CPG, then \( \tilde{S} = (g(S_0; S), g(S_1; S), \ldots, g(S_m; S)) \) is invariant under CPG for every \( g \) such that

\[
g(x; y) = g(x; \pi_L y).
\]

In this article, we consider linear statistics

\[
S_j = y^T \eta_j, \quad j = 0, 1, \ldots, m,
\]

and apply MRT on \( \tilde{S}_0, \ldots, \tilde{S}_m \) defined in (9). Partition \( X \) into \( (X_{[r]} X_{[-r]}) \) and \( \beta \) into \( (\beta_{[r]}, \beta_{[-r]}) \).

The linear model (2) implies that

\[
y^T \eta_j = (1^T \eta_j) \beta_0 + (X_{[r]}^T \eta_j)^T \beta_{[r]} + (X_{[-r]}^T \eta_j)^T \beta_{[-r]} + \epsilon^T \eta_j.
\]

In the next three subsections we will show how to construct \( \eta_j \)'s to guarantee the type-I error control and to enhance power. Surprisingly, the only distributional assumption on \( \epsilon \) is the exchangeability:

**A1** \( \epsilon \) has exchangeable components, i.e. for any permutation \( \pi \) on \([n]\)

\[
(\epsilon_1, \ldots, \epsilon_n) \overset{d}{=} (\epsilon_{\pi(1)}, \ldots, \epsilon_{\pi(n)}).
\]
2.2 Construction for type-I Error Control

Under $H_0$, (10) can be simplified as

$$y^T \eta_j = (1^T \eta_j) \beta_0 + (X_{[-r]}^T \eta_j)^T \beta_{[-r]} + \epsilon^T \eta_j.$$  \hspace{1cm} (11)

To ensure the distributional invariance of $\{y^T \eta_0, \ldots, y^T \eta_m\}$ to CPG, it is sufficient to construct $\eta_j$'s such that the deterministic parts are identical for all $j$ and the noise parts are invariant under CPG. To match the deterministic part, we can simply set $X_{[-r]}^T \eta_j$ to be independent of $j$.

C1 there exists $\gamma_{[-r]} \in \mathbb{R}^{p-r}$ such that

$$X_{[-r]}^T \eta_j = \gamma_{[-r]} \ j = 0, 1, \ldots, m.$$  

To ensure the invariance of the stochastic part, intuitively $\eta_j$'s should be left shifted transforms of each other. To be concrete, consider the case where $n = 6$ and $m = 2$. Then given any $\eta^* = (\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*, \eta_5^*, \eta_6^*)^T$, the following construction would imply the invariance to CPG:

$$\eta_0 = (\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*, \eta_5^*, \eta_6^*)^T, \quad \eta_1 = (\eta_3^*, \eta_4^*, \eta_5^*, \eta_6^*, \eta_1^*, \eta_2^*)^T, \quad \eta_2 = (\eta_6^*, \eta_0^*, \eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*)^T.$$  

To see this, note that

$$(\epsilon^T \eta_0, \epsilon^T \eta_1, \epsilon^T \eta_2)^T = \begin{pmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \\ \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 \end{pmatrix} \eta^*,$$

and

$$(\epsilon^T \eta_1, \epsilon^T \eta_2, \epsilon^T \eta_0)^T = \begin{pmatrix} \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \end{pmatrix} \eta^*.$$  

By assumption A1,

$$\begin{pmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \\ \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 & \epsilon_1 & \epsilon_2 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \end{pmatrix}.$$  

As a result,

$$(\epsilon^T \eta_0, \epsilon^T \eta_1, \epsilon^T \eta_2) \overset{d}{=} (\epsilon^T \eta_1, \epsilon^T \eta_2, \epsilon^T \eta_0).$$

Using the same argument we can show $(\epsilon^T \eta_0, \epsilon^T \eta_1, \epsilon^T \eta_2) \overset{d}{=} (\epsilon^T \eta_2, \epsilon^T \eta_0, \epsilon^T \eta_1)$ and thus the invariance of $(\epsilon^T \eta_0, \epsilon^T \eta_1, \epsilon^T \eta_2)$ to CPG.

In general, if $n$ is divisible by $m + 1$ with $n = (m + 1)t$, then we can construct $\eta_j$'s as a left shifted
transform of a vector $\eta^*$, i.e.

$$
\eta_j = \pi_{L}^{tj}(\eta^*)
$$

(12)

where $\pi_{L}$ is the left shifting operator defined in (7). More generally, if $n = (m + 1)t + s$ for some integers $t$ and $0 \leq s \leq m$, we can leave the last $s$ components to be the same across $\eta_j$’s while shifting the first $(m + 1)t$ entries as in (12).

C2 there exists $\eta_\ast \in \mathbb{R}^n$ such that

$$
\eta_j = \left[ \pi_{L}^{tj}((\eta^*_1, \ldots, \eta^*_m)\_t), \eta^*_t(m+1), \ldots, \eta^*_n \right]^T,
$$

where $t = \lfloor n/(m + 1) \rfloor$.

**Proposition 4.** Under assumption **A1**, $(y^T \eta_0, \ldots, y^T \eta_m)$ is distributionally invariant under CPG if $(\eta_0, \ldots, \eta_m)$ satisfy **C1** and **C2**.

**Proof.** It is left to prove the invariance of $(\epsilon^T \eta_0, \ldots, \epsilon^T \eta_m)$ to CPG. Further, since the last $n - (m + 1)t$ terms are the same for all $j$, it is left to prove the case where $n$ is divisible by $m + 1$. Let $\tilde{\Pi}$ be the permutation matrix corresponding to $\pi_{L}^{t}$. Then 

$$
\pi_{L}(\epsilon^T \eta_0, \epsilon^T \eta_1, \ldots, \epsilon^T \eta_m) = (\epsilon^T \eta_1, \ldots, \epsilon^T \eta_m, \epsilon^T \eta_0)
$$

$$
= (\epsilon^T \tilde{\Pi} \eta_0, \ldots, \epsilon^T \tilde{\Pi} \eta_m, \epsilon^T \eta_0)
$$

$$
= (\epsilon^T \tilde{\Pi} \eta_0, \ldots, \epsilon^T \tilde{\Pi} \eta_m, \epsilon^T \tilde{\Pi}^{m+1} \eta_0) \quad \text{(Since } \tilde{\Pi}^{m+1} = \text{Id})
$$

$$
= (\epsilon^T \eta_0, \epsilon^T \eta_1, \ldots, \epsilon^T \eta_m). \quad \text{(Since } \tilde{\Pi} \epsilon = \epsilon)
$$

(13)

Repeating (13) for $m - 1$ times, we prove the invariance of $(\epsilon^T \eta_1, \ldots, \epsilon^T \eta_m)$ under CPG. 

Now we discuss the existence of $(\eta_\ast, \gamma_{[-r]})$. Note that $\eta_j$ is a linear transformation of $\eta^*$. Let $\Pi_j \in \mathbb{R}^{n \times n}$ be the matrix such that $\eta_j = \Pi_j \eta^*$. Then **C1** and **C2** imply that

$$
\begin{pmatrix}
-I_{p-r} & X_{[-r]}^T \\
-I_{p-r} & X_{[-r]}^T \Pi_1 \\
\vdots & \vdots \\
-I_{p-r} & X_{[-r]}^T \Pi_m
\end{pmatrix}
\begin{pmatrix}
\gamma_{[-r]} \\
\eta_\ast
\end{pmatrix}
= 0.
$$

(14)

The above linear system has $(m + 1)(p - r)$ equations and $n + p - r$ unknowns. Therefore, a non-zero solution always exists if $(m + 1)(p - r) < n + p - r$.

**Theorem 1.** Under assumption **A1**, 

(a) (14) always has a non-zero solution if 

$$
n/(p - r) > m.
$$

(15)
(b) for any solution \((\gamma_{-r}, \eta^*)\) of (14),

\[
(y^T \eta^*, y^T \Pi_1 \eta^*, \cdots y^T \Pi_m \eta^*)
\]

is invariant under CPG under \(H_0\), where \(\Pi_j \in \mathbb{R}^{n \times n}\) is the coefficient matrix that maps \(\eta^*\) to \(\eta_j\) defined in \(C2\).

Suppose \(\alpha = 0.05\) for illustration and set \(m = 1/\alpha - 1 = 19\). Then the condition (15) reads

\[
n > 19(p - r).
\]

Even when \(r = 1\), this is a mild condition in many applications. On the other hand, when \(r\) is large but \(p - r\) is small, then (15) can still be satisfied even if \(p > n\). This is in sharp contrast to regression F-tests and permutation F-tests that require fitting the full model and thus \(p \leq n\). Furthermore, it is worth emphasizing that Theorem 1 allows arbitrary design matrices. This is fundamentally different from the asymptotically valid tests which always impose regularity conditions on \(X\).

2.3 Construction for high power when \(r = 1\)

To guarantee reasonable power, we need \(y^T \eta_0\) to be significantly different from the other statistics under the alternative. In this subsection we focus on the case where \(r = 1\) to highlight the key idea.

When \(\beta_1 \neq 0\), (10) implies that

\[
y^T \eta_j = (X^T_1 \eta_j) \beta_1 + W_j
\]

where \(W_j = \epsilon^T \eta_j + (1^T \eta_*) \beta_0 + (X^T_{-1} \eta_*)^T \beta_{-1}\) and \((W_1, \ldots, W_m)\) is invariant under CPG by Theorem 1. To enhance power, it is desirable that \(X^T_1 \eta_0\) lies far from \(\{X^T_1 \eta_1, \ldots, X^T_1 \eta_m\}\). In particular, we impose the following condition on \(\eta_j\)'s:

\(C3^{(1)}\) there exists \(\gamma_1, \delta \in \mathbb{R}\), such that

\[
X^T_1 \eta_j = \gamma_1 \ (j = 1, 2, \ldots, m), \quad X^T_1 \eta_0 = \gamma_1 + \delta.
\]

Putting \(C1\), \(C2\) and \(C3^{(1)}\) together, we obtain the following linear system,

\[
\begin{pmatrix}
-1 \quad \cdots 0
\end{pmatrix}
\begin{pmatrix}
A(X)^T
\end{pmatrix}
\begin{pmatrix}
\delta \\
\gamma \\
\eta
\end{pmatrix}
= 0,
\]

where \(e_{1,p(m+1)}\) is the first canonical basis in \(\mathbb{R}^{p(m+1)}\) and

\[
A(X) = \begin{pmatrix}
-I_p & -I_p & \cdots & -I_p \\
X & \Pi_1^T X & \cdots & \Pi_m^T X
\end{pmatrix} \in \mathbb{R}^{(n+p) \times (m+1)p}.
\]

This linear system has \((m + 1)p\) equations and \(n + p + 1\) variables. Thus it always has a non-zero
solution if
\[ n + p + 1 > p(m + 1) \iff n \geq pm. \]
When \( \alpha = 0.05 \) and \( m = 19 \), this condition is still reasonable in many settings.

The normalized gap \( \delta/\|\eta\| \) can be regarded as a proxy of power. Write \( \gamma \) for \( \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{[\gamma]} \end{pmatrix} \) and \( e_1 \) for the first canonical basis vector of \( \mathbb{R}^p \). Putting conditions C1-C3 together, it is natural to consider the following optimization problem:

\[
\begin{aligned}
\max_{\delta \in \mathbb{R}, \gamma \in \mathbb{R}^p, \eta \in \mathbb{R}^n, \|\eta\|_2 = 1} & \delta, \\
\text{s.t.} & \begin{pmatrix}
- e_1, p(m+1) : A(X)^T \\
\end{pmatrix}
\begin{pmatrix}
\delta \\
\gamma \\
\eta
\end{pmatrix} = 0.
\end{aligned}
\]

This linear programming problem can be solved by fitting a linear regression and it permits a closed-form solution. Let \( O^*(X) \) denote the optimal value of the objective function, i.e. maximum achievable value of \( \delta \) in this case.

**Theorem 2.** Assume that \( n \geq pm \). Let

\[
B(X) = \begin{pmatrix}
(I - \Pi_m)^T X & (I - \Pi_m)^T X & \cdots & (I - \Pi_m)^T X
\end{pmatrix} \in \mathbb{R}^{n \times mp}.
\]

Partition \( B(X) \) into \( [B(X)_1 \ B(X)_{-1}] \) where \( B(X)_1 \) is the first column of \( B(X) \). Further let

\[
\tilde{\eta} = (I - H_{[-1]}) B(X)_1, \quad \text{where} \quad H_{[-1]} = B(X)_{[-1]} (B(X)_{[-1]} B(X)_{[-1]}^+) B(X)_{[-1]}^T,
\]

where \( + \) denotes the Moore-Penrose generalized inverse. Then \( O^*(X) = \|\tilde{\eta}\|_2 \) and one global maximizer of (18) is given by

\[
\eta^*(X) = \tilde{\eta}/\|\tilde{\eta}\|_2, \quad \delta^*(X) = \|\tilde{\eta}\|_2.
\]

**Remark 1.** When \( B(X)_{[-1]} \) has full column rank, \( \tilde{\eta} \) is the residual vector by regressing \( B(X)_1 \) on \( B(X)_{[-1]} \) and \( \|\tilde{\eta}\|_2^2 \) is the residual sum of squares. Both can be easily computed using standard softwares. If \( B(X)_{[-1]} \) does not have full column rank, then \( \tilde{\eta} \) is the minimum norm ordinary least squares by regressing \( B(X)_1 \) on \( B(X)_{[-1]} \), which is the limit of ridge estimator when the penalty tends to zero and is the limiting solution of standard gradient descent initialized at zero [e.g. Hastie et al., 2019].

**Proof.** First, (16) can be equivalently formulated as

\[
B(X)^T \eta = \delta e_{1,pm}.
\]

This can be further rewritten as

\[
\delta = B(X)_1^T \eta, \quad B(X)_{[-1]}^T \eta = 0.
\]

For any \( \eta \) satisfying the second constraint,

\[
H_{[-1]} \eta = 0,
\]
and thus
\[ B(X)\eta = B(X)(I - H_{-1})\eta = \eta^T\eta. \]

As a result,
\[ \max_{\eta: \|\eta\|=1} B(X)^T\eta \leq \max_{\|\eta\|=1} \|\eta\|^2 = \|\eta\|^2. \]

In other words, we have shown that \( \delta^*(X) \leq ||\eta||^2 \). On the other hand, the vector \( \tilde{\eta}/||\tilde{\eta}||^2 \) satisfies the constraint (20) and
\[ B(X)^T\tilde{\eta}/||\tilde{\eta}||^2 = ||\tilde{\eta}||^2. \]

This shows that \( \delta^*(X) \geq ||\tilde{\eta}||^2 \). In this case, it is obvious that \( \delta^*(X) = \delta^*(X) \). Therefore, \( \delta^*(X) = ||\tilde{\eta}||^2 \) and one maximizer is \( \eta^*(X) = \tilde{\eta}/||\tilde{\eta}||^2 \).

### 2.4 Construction for high power when \( r > 1 \)

Similar to \( C_3(1) \), we impose the following restriction on \( \eta \).

\( C_3 \) there exists \( \gamma[r], \delta \in \mathbb{R}^r \), such that
\[ X^T[j]\gamma[j] = \gamma[r] \ (j = 1, 2, \ldots, m), \quad X^T[m]\gamma[0] = \gamma[r] + \delta. \]

Combining with (14), we obtain an analogue of (16) as follows.
\[
\begin{pmatrix}
-e_{1,p(m+1)}, \ldots, -e_{r,p(m+1)}^T : A(X)^T
\end{pmatrix}
\begin{pmatrix}
\delta \\
\gamma \\
\eta
\end{pmatrix} = 0, \quad (21)
\]

where \( A(X) \) is defined in (17) and \( \gamma = \begin{pmatrix} \gamma[r] \\ \gamma[-r] \end{pmatrix} \). This linear system involves \( p(m+1) \) equations and \( n + p + r \) variables. Therefore it always has a non-zero solution if
\[ n + p + r > p(m+1) \iff n \geq pm - r + 1. \]

Unlike the univariate case, there are infinite ways to characterized the signal strength since \( \delta \) is multivariate. A sensible class of criteria is to maximize a quadratic form
\[
\max_{\delta \in \mathbb{R}^r, \gamma \in \mathbb{R}^p, \eta \in \mathbb{R}^n, \|\eta\|^2 = 1} \delta^T M \delta \quad \text{s.t.} \quad \begin{pmatrix}
-e_{1,p(m+1)}, \ldots, -e_{r,p(m+1)}^T : A(X)^T
\end{pmatrix}
\begin{pmatrix}
\delta \\
\gamma \\
\eta
\end{pmatrix} = 0. \quad (22)
\]

The following theorem gives the optimal solution given any weighting matrix \( M \). Let \( O^*(X) \) denote the optimal value of the objective function.

**Theorem 3.** Assume that \( n \geq pm - r + 1 \). Let \( B(X) \) be defined in (19). Partition \( B(X) \) into...
(B(X)_{[r]} B(X)_{[-r]}) where B(X)_{[r]} is the matrix formed by the first r columns of B(X). Let
\[ M_r(X) = (I - H_{[-r]})B(X)_{[r]}MB(X)_{[r]}^T(I - H_{[-r]}), \]
where
\[ H_{[-r]} = B(X)_{[-r]}(B(X)_{[-r]}^T B(X)_{[-r]})^+ B(X)_{[-r]}^T. \]
Further let \( \lambda_{\text{max}}(M_r(X)) \) denote the maximum eigenvalue, \( u \) be any eigenvector corresponding to it and \( \eta = (I - H_{[-r]})u \). Then \( O^*(X) = \lambda_{\text{max}}(M_r(X)) \) and
\[ \eta^*(X) = \eta/\|\eta\|_2, \quad \delta^*(X) = B(X)_{[r]}^T \eta^*(X) \]
is an optimal solution of (22).

Proof. Similar to the proof of Theorem 2, we first rewrite (21) as
\[ B(X)_{[r]}^T \eta = \delta, \quad B(X)_{[-r]}^T \eta = 0. \]
As a result, \( \eta \) lies in the row null space of \( B(X)_{[-r]} \) and thus there exists \( \zeta \in \mathbb{R}^n \) such that
\[ H_{[-r]} \eta = 0. \]
Then
\[ \delta^T M \delta = \eta^T (I - H_{[-r]})B(X)_{[r]}MB(X)_{[r]}^T(I - H_{[-r]}) \eta = \eta^T M_r(X) \eta. \]
Since \( \|\eta\|_2 \leq 1 \),
\[ \delta^T M \delta \leq \lambda_{\text{max}}(M_r(X)). \]
On the other hand, for any eigenvector \( u \) of \( M_r(X) \) corresponding to its largest eigenvalue, let \( \tilde{\eta} = (I - H_{[-r]})u \) and \( \eta = \tilde{\eta}/\|\tilde{\eta}\|_2 \), then
\[ \eta^T M_r(X) \eta = \lambda_{\text{max}}(M_r(X)), \quad B(X)_{[-r]} \eta = 0, \quad \|\eta\|_2 = 1. \]
Thus, \( \eta^*(X) = \tilde{\eta}/\|\tilde{\eta}\|_2 \) is an optimal solution. As a result, \( \delta^*(X) = B(X)_{[r]}^T \eta^*(X) \) and \( O^*(X) = \lambda_{\text{max}}(M_r(X)). \)

Although Theorem 3 gives the solution of (22) for arbitrary weight matrix \( M \), it is not clear which \( M \) is the best choice. Since
\[ \eta_j^T y = \delta^T \beta_{[r]} I(j = 0) + \tilde{W}_j \]
where \( \tilde{W}_j = \gamma^T \beta + \eta_j^T \epsilon \) is invariant under CPG. Thus, \( \delta^T \beta_{[r]} \) characterizes the signal strength. In principle, the “optimal” weight matrix should be depend on prior knowledge of \( \beta_{[r]} \). For instance, for a Bayesian hypothesis testing problem with a prior distribution \( Q \) on \( \beta_{[r]} \) under the alternative, the optimal weight matrix is \( M = \mathbb{E}_Q [\beta_{[r]} \beta_{[r]}^T] \).
Given any $X$, we can easily calculate the proxy of signal strength $O^*(X)$ by Theorem 2 and Theorem 3. However, the optimal value is not invariant to row permutation of $X$, that is, for any permutation matrix $\Pi \in \mathbb{R}^{n \times n}$,

$$O^*(X) \neq O^*(\Pi X)$$

in general. Roughly speaking, this is because $\delta^*(X)$ involves the left shifting operator, which depends on the arrangement of the rows of $X$. Figure 1 illustrates the variability of $O^*(\Pi X)$ as a function of $\Pi$ for a fixed matrix with 8 rows and 3 columns, generated with i.i.d. Gaussian entries.

Notably, even in such regular cases the variability is non-negligible. This motivates the following secondary combinatorial optimization problem:

$$\max_{\Pi} O^*(\Pi X).$$

This is a non-linear travelling salesman problem. Note that we aim at finding a solution with reasonably large objective value instead of finding the global maximum of (23), which is NP-hard. For this reason, we solve (23) by Genetic Algorithm (GA), which is generally efficient for moderate $n$ albeit without worst-case convergence guarantee. In a nutshell, GA maintains a population of permutations, generate new permutations by two operations: crossover and mutation, and evolves the population via a mechanism called selection, based on the objective value. We refer the readers to Michalewicz [2013] for more details.

We compare GA with a simple competing algorithm that randomly selects ordering and keeps the one yielding the largest objective value. We refer to this method as Stochastic Search (SS). Although this competitor is arguably too weak and more efficient algorithms may exist (see Section 5), our goal here is simply to illustrate the effectiveness of GA instead of to claim the superiority of GA. We compare the performance of GA and SS on three matrices with $n = 1000$ and $p = 20$ as realizations generated from random one-way ANOVA matrices with exactly one entry in each row at a uniformly random position, random matrices with i.i.d. standard normal entries and random matrices with i.i.d. standard Cauchy entries. The results are plotted in Figure 2 where the y-axis measures $O^*(\Pi X)$, scaled by the maximum achieved by GA and SS for visualization, and the x-axis measures the number
of random samples each algorithm accesses. The population size is set to be 10 for GA in all scenarios. It is clear that GA consistently improves the solution as opposed to SS.

2.6 Implementation of CPT

Based on previous subsections, we summarize the implementation of CPT below:

Step 1 Compute a desirable pre-ordering $\Pi_0$ for the combinatorial optimization problem

$$\max_{\Pi} O^*(\Pi X),$$

where $O^*(\cdot)$ is defined in Theorem 2 when $r = 1$ and is defined in Theorem 3 when $r > 1$;

Step 2 Replace $y$ and $X$ by $\Pi_0y$ and $\Pi_0X$;

Step 3 Compute $\eta^*$ via the formula in Theorem 2 or Theorem 3;

Step 4 Compute $S_j = \eta_j^T y$ for $j = 0, 1, \ldots, m$ where

$$\eta_j = \left[ \pi_x^j \left( \eta_1^*, \ldots, \eta_{(m+1)t}^*, \eta_{(m+1)t+1}^*, \ldots, \eta_n^* \right) \right]^T, \quad t = \lfloor n/(m+1) \rfloor;$$

Step 5 Compute $\tilde{S}_j = |S_j - \text{median}(\{S_j\}_{j=0}^m)|$;

Step 6 Compute the p-value $p = R_0/(m+1)$ where $R_0$ is the rank of $\tilde{S}_0$ in the set $\{\tilde{S}_0, \tilde{S}_1, \ldots, \tilde{S}_m\}$ in descending order;

Step 7 Reject the null hypothesis if $p \leq \alpha$.

The inputs of CPT include the design matrix $X$, the outcome vector $y$, the confidence level $\alpha$, the number of statistics $m + 1$ and a sub-routine to solve Step 1. As the default, we set $m = \lceil 1/\alpha \rceil - 1$ and use Genetic Algorithm, implemented in R package \texttt{gaoptim}, to solve Step 1.
3 Experiments

3.1 Setup

To examine the power of our procedure, we conduct extensive numerical experiments. In all experiments below, we fix the sample size $n = 1000$ and consider three values $25, 33, 40$ for dimension $p$ such that the sample per parameter $n/p \approx 40, 30, 25$. Given a dimension $p$, we consider the three types of design matrices considered in Figure 2. For each type of design matrix, we generate $50$ independent copies. Given each $X$, we generate $3000$ copies of $\epsilon$ with i.i.d. entries from the standard normal distribution and the standard Cauchy distribution.

We consider two variants of CPT – CPT with random ordering and CPT with GA pre-ordering, as well as five competing tests: (1) t/F tests; (2) permutation t/F tests which approximates the null distribution of the t/F statistic by the permutation distribution with $X_{[r]}$ reshuffled; (3) Freedman-Lane test [Freedman and Lane, 1983, Anderson and Robinson, 2001] which approximates the null distribution of the t/F statistic by the permutation distribution with reduced-form regression residuals reshuffled; (4) asymptotic z-test from least absolute deviation (LAD) regression; (5) Group Bound method [Meinshausen, 2015]. For both permutation tests, we calculate the test based on $1000$ random permutations. To further demonstrate the importance of pre-ordering step of CPT, we consider a weaker GA pre-ordering with $1000$ random samples and a stronger GA pre-ordering with $10000$ random samples. Three variants of CPTs are abbreviated as CPTw for CPT with weak pre-ordering, CPTs for strong pre-ordering and CPTr for CPT with random ordering. All tests will be performed with level $\alpha = 0.05$ and the number of statistics $m + 1$ in CPT is set to be $20$.

3.2 Testing for a single coordinate

In the first experiment, we consider testing a single coordinate, i.e. $H_0 : \beta_1 = 0$. Given a design matrix $X$ and an error distribution $F$, we start by computing a benchmark signal-to-noise ratio $\beta_1$ such that the t/F test has approximately $20\%$ power, through Monte-Carlo simulation, when $y$ is generated from

$$y = X_1 \beta_1 + \epsilon, \quad \text{where } \epsilon_i \overset{i.i.d.}{\sim} F.$$

Then all tests are performed on $X$ and the following $18000$ outcome vectors $y^{(b)}_s$, respectively:

$$y^{(b)}_s \triangleq X_1 (s \beta_1) + \epsilon^{(b)}, \quad \text{where } s = 0, 1, \ldots, 5, \quad b = 1, \ldots, 3000.$$

For each $s$, the proportion of rejections among $3000$ $\epsilon$’s is computed. When $s = 0$, this proportion serves as an approximation of the type-I error and should be closed to or below $\alpha$ for a valid test; when $s > 0$, it serves as an approximation of power and should be large for a powerful test. For each of the three types of design matrices, the above experiments were repeated on $50$ independent copies of $X$’s.

Figure 3 displays the type-I error of all tests for three types of design matrices. The box-plots display the variation among $50$ independent copies of design matrices. In all cases, three variants of CPTs are valid as guaranteed by theory and Group Bound method is overly conservative. Permutation test and Freedman-Lane test also appear to be valid in our simulation settings even though there is no
Figure 3: Monte-Carlo type-I error for testing a single coordinate with three types of $X$’s: (top) realizations of random matrices with i.i.d. standard normal entries; (middle) realizations of random matrices with i.i.d. standard Cauchy entries; (bottom) realizations of random one-way ANOVA design matrices.
Figure 4: Median power ratio between each variant of CPT and each competing test for testing a single coordinate with realizations of Gaussian matrices and Gaussian errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.

Theoretical guarantee for heavy-tailed errors. When errors are Gaussian, t-test is valid as guaranteed by theory but can be conservative or anti-conservative with heavy-tailed errors depending on the design matrix. Interestingly, for one-way ANOVA, t-test becomes less valid as the sample size per parameter increases. On the other hand, LAD-based test are anti-conservative when $X$ is a realization of Gaussian matrices with both Gaussian and Cauchy errors, although the validity can be proved asymptotically under regularity conditions that are satisfied by realizations of Gaussian matrices with high probability [e.g. Pollard, 1991]. This makes a case for the fragility of some asymptotic results.

Due to the space constraint, we only show results for the case where the design matrices are realizations of Gaussian matrices and errors are Gaussian in Figure 4 and the case where the design matrices are realizations of Cauchy matrices and errors are Cauchy in Figure 5, respectively. The results for other cases will be presented in Appendix A. All figures plot the median power ratio, from 50 independent copies of $X$'s, between each variant of CPT (CPTw, CPTs and CPTr) and each competing test. First we see that the Group Bound method has zero power in all scenarios and thus
the power ratios are infinite and missing in the plots. Second, the pre-ordering step is significant in raising the power of CPT. Third, the relative power of CPT becomes larger as \( n/p \) increases. In the first case, it is not surprising that t-test is the most powerful ones because it is provably the uniformly most powerful unbiased (UMPU) test for linear models with Gaussian errors. The efficiency loss of CPTs against t-test, permutation t-test and LAD-based test is moderate in general and is low when the sample size per parameter and the signal-to-noise ratio is large. In the second case, CPTs is more powerful than t-test, although it is still less powerful than permutation t-test and LAD-based test. In both cases, CPTs is more powerful than Freedman-Lane test even when \( n/p = 25 \) and the signal-to-noise ratio is small.
Figure 6: Monte-Carlo type-I error for testing five coordinates with three types of $X$’s: (top) realizations of random matrices with i.i.d. standard normal entries; (middle) realizations of random matrices with i.i.d. standard Cauchy entries; (bottom) realizations of random one-way ANOVA design matrices.
3.3 Testing for multiple coordinates

Next we consider testing the first five coordinates $H_0 : \beta_1 = \ldots = \beta_5 = 0$, with a Bayesian alternative hypothesis

$$\beta_{[5]} \sim N(s1_5, \Sigma), \quad \Sigma = \text{diag}(0.2, 0.4, 0.6, 0.8, 1), \quad s \in \{0, 1, \ldots, 5\}$$

All other settings are exactly the same as Section 3.2, except that t-test and permutation t-test are replaced by F-test and permutation F-test. For CPT, we choose the weight matrix $M = \mathbb{E}[\beta_{[5]}\beta_{[5]}^T]$. Figure 6 displays the Monte-Carlo type-I error of all tests. The results are qualitatively the same as the experiment in Section 3.2 except that F-test and LAD-based test become more invalid. Due to the space constraint, all power comparisons are presented in Appendix A.

4 1908-2018: A Selective Review of The Century-Long Effort

Linear model is one of the most fundamental object in the history of statistics and has been developed for over a century. Nowadays it is still among most widely-used models for data analysts to demystify complex data as well as most powerful tools for statisticians to understand complicated methods and expand the toolbox for advanced tasks. It is impossible to exhaust the literature for this long-standing problem. We thus provide a selective yet extensive review to highlight milestones in the past century. In particular, we will focus on the linear hypothesis testing problem, as well as the estimation problem which can yield the former, for vanilla linear models with general covariates and briefly discuss the simplified forms such as location problems and ANOVA problems when necessary. However, we will exclude the topics including high dimensional sparse linear models, selective inference for linear models, linear models with dependent errors, high breakdown regression methods, linear time series, and generalized linear models. We should emphasize that these topics are at least equally important as those discussed in this section. They are exclude simply to avoid digression. Furthermore, as mentioned earlier, one purpose of this review is to highlight the hardness of developing an exact test despite the century-long efforts.

4.1 Normal theory based tests

Motivated by the seminal work by Student [1908a] and Student [1908b] which propose the one-sample and two-sample t-test, Ronald A. Fisher derived the well-known t-distribution [Fisher, 1915] and applied it to testing for a single regression coefficient in homoskedastic Gaussian linear models [Fisher, 1922]. In his 1922 paper, he also derived an equivalent form of F test for testing the global null under the same setting. Later he derived the F-distribution [Fisher, 1924], which he characterized through “z”, the half logarithm of F-statistics, and proposed the F-test for ANOVA, a special case of linear hypothesis in homoskedastic Gaussian linear models. Both tests were elaborated in his insightful book [Fisher, 1925] and the term “F-test” was coined by George W. Snedecor [Snedecor, 1934].

This paramount line of work established the first generation of rigorous statistical test for linear models. They are exact tests of linear hypotheses in linear models with independent and identically distributed normal errors and arbitrary fixed-design matrices. Despite the exactness of the tests
without any assumption on the design matrices, the normality assumption can rarely be justified in practice. Early investigation of the test validity with non-normal errors can be dated back to Egon S. Pearson [Pearson, 1929, Pearson and Adyanthaya, 1929, Pearson, 1931]. Unlike the large-sample theory that is standard nowadays, the early work take an approximation perspective to improve the validity for extremely small sample. It was furthered in the next a few decades [e.g. Eden and Yates, 1933, Bartlett, 1935, Geary, 1947, Gayen, 1949, 1950, David and Johnson, 1951b,a, Box, 1953, Box and Watson, 1962, Pearson and Please, 1975] and it was mostly agreed that the regression t-test is extremely robust to non-normal errors with moderately large sample (e.g. > 30) while the regression F-test is more sensitive to the deviation from normality. It is worth emphasizing that these works were either based on mathematically unrigorous approximation or based on Edgeworth expansion theory that could be justified rigorously [e.g. Esseen, 1945, Wallace, 1958, Bhattacharya and Ghosh, 1978] in the asymptotic regime that the sample size tends to infinity while the dimension of the parameters stays relatively low (e.g. a small constant).

Later on, due to the popularization of rigorous large-sample theory in 1950s [e.g. LeCam, 1953, Chernoff, 1956] pioneered by Doob [1935], Wilks [1938], Mann and Wald [1943], Wald [1949], investigators started to look at the regression test validity in certain asymptotic regimes. This can be dated back to Friedhelm Eicker [Eicker, 1963, 1967], to the best of our knowledge, and developed by Peter J. Huber in his well-known and influential paper [Huber, 1973], which showed that the least square estimate is jointly asymptotically normal if and only if the maximum leverage score tends to zero. This clean and powerful result laid the foundation to asymptotic analysis for t-test and F-test [e.g. Arnold, 1980]. Notably these early works did not assume the dimension p stays fixed, as opposed to simplified arguments in standard textbooks. Before 1990s, the large-sample theory for least squares estimators were well established in the regime where the sample size per parameter n/p grows to infinity, under regularity conditions on the design matrices and on the errors, usually with independent and identically distributed elements and finite moments. It shows that both t-test and F-test are asymptotically valid and can be approximated by z-test and χ²-test, respectively. For t-test, the robustness to non-normality was even established without the typical regularity conditions (e.g. Zellner [1976], Jensen [1979] for spherically invariant errors, Efron [1969], Cressie [1980], Benjamini [1983], Pinelis [1994] for orthant symmetric errors) or beyond the aforementioned regime [e.g. Lei et al., 2018]. By contrast, though similar results exist for F-test [e.g. Zellner, 1976], more non-robustness results were established. For instance, a line of work [e.g. Boos and Brownie, 1995, Akritas and Arnold, 2000, Calhoun, 2011, Anatolyev, 2012] showed that F-test is asymptotically invalid, unless the errors are normal, in the moderate dimensional regime where n/p stays bounded as n approaches infinity, although correction is available under much stronger assumptions on the design matrix or the coefficient vectors. Even with normal errors, Zhong and Chen [2011] showed that the power of F-test diminishes as n/p approaches 1. In a nutshell, there has been tremendous effort over the past century investigating the robustness of regression t-test and F-test and it was agreed that t-test is insensitive to non-normality, high dimension and irregularity of design matrices to certain extent while F-test is less robust in general.
4.2 Permutation tests

Despite the tremendous attention on regression t-test and F-test, other methodologies emerged in parallel as well. The earliest alternative is the permutation test, which justifies the significance of the test through the so-called “permutation distribution”. However, the early attempt to justify permutation tests is through the “randomization model” in contrast to the “population model” that we considered in (2). The “randomization model” was introduced by Jerzy S. Neyman in his master thesis [Neyman, 1923], which is also known as Neyman-Rubin model [Rubin, 1974], or design-based inference (Särndal et al. [1978], in contrast to model-based inference), or “conditional-on-errors” model (Kennedy [1995], in contrast to “conditional-on-treatment” model), and the term was coined by Ronald A. Fisher in 1926 [Fisher, 1926]. The theoretical foundation of permutation test was laid by Edwin J. G. Pitman in his three seminal papers [Pitman, 1937b,a, 1938], where the last two were studied for regression problems, albeit under the “randomization model”. The early works viewed permutation tests as better devices in terms of the logical coherence and robustness to non-normality [e.g. Geary, 1927, Eden and Yates, 1933, Fisher, 1935]. They found that the permutation distribution for “randomization models” mostly agree with the normality-based distribution for “population models”, until 1937 when Li B. Welch disproved the agreement for Latin-squares designs [Welch, 1937]. In the next a few decades, most works on permutation tests were established for “randomization models” without being justified under “population models”. We will skip the discussion for this period and refer to Berry et al. [2013] for a thorough literature review on this line of work, because our paper focuses on the “population model” like (2).

The general theory of permutation tests in “population models” can be dated back to the notable works by Hoeffding [1952] and Box and Andersen [1955] and further developed by Romano [1989, 1990], Chung and Romano [2013]. In regression context, early investigations were done for special cases in ANOVA problems [Mehra and Sen, 1969, Brown and Maritz, 1982, Welch, 1990]. For testing a single regression coefficient, Oja [1987] and Collins [1987] proposed the permutation test on a linear statistic and the F-statistic by permuting the covariate while leaving the others the same. Whereas the procedure can be easily validated for univariate regression, the validity when \( p > 1 \) was only justified in “randomization models”. Manly [1991] proposed permuting the response vector \( y \), which is valid for testing the global null \( \beta = 0 \) but not for general case. Freedman and Lane [1983], Ter Braak [1992], Kennedy and Cade [1996] proposed three different permutation tests on regression residuals. The theory of the aforementioned tests were established in a later review paper by Anderson and Robinson [2001]. The main take-away message is that the permutation test should be performed on asymptotically pivotal statistics. For instance, to test for a single coefficient, the permutation t-test asymptotically valid. This was further confirmed and extended by DiCiccio and Romano [2017] to heteroskedastic linear models with random designs.

4.3 Rank-based tests

Perhaps a bit surprisingly, rank-based methods for linear regression can be dated back to 1936, when Hotelling and Pabst [1936] established the hypothesis testing theory for rank correlation, nowadays known as Spearman’s correlation which was originated from Galton [1894] and developed by Spearman
This work can be regarded as the application of rank-based methods for univariate linear models. Appealed by the normality free nature of rank-based tests, Milton Friedman extended the idea to one-way ANOVA [Friedman, 1937]. It can be identified as the first application of rank-based method for multivariate linear models and was further developed by Kendall and Smith [1939], Friedman [1940]. Friedman’s test transforms continuous or ordinal outcomes into ranks and was widely studied in ANOVA problems, started by the famous Kruskal-Wallis test for one-way ANOVA [Kruskal and Wallis, 1952] and developed by Hodges and Lehmann [1962], Puri and Sen [1966], Sen [1968a], Conover and Iman [1976, 1981], Akritas [1990], Akritas and Arnold [1994], Brunner and Denker [1994], Akritas et al. [1997] for two-way ANOVA problems and factorial designs. Since 1990s, due to the advance of high dimensional asymptotic theory, further progress was made on refining the procedures in presence of large number of factors or treatments [Brownie and Boos, 1994, Boos and Brownie, 1995, Wang and Akritas, 2004, Bathke and Lankowski, 2005, Bathke and Harrar, 2008].

However the aforementioned works are restricted to ANOVA problems (with a few exceptions, e.g. [Sen, 1968b, 1969]) and fundamentally different from the modern rank tests based on regression R-estimates, which are based on ranks of regression residuals. The first R-estimate based test can be dated back to Hájek [1962], which established asymptotically most powerful rank test for univariate regression given the error distribution. Adichie [1967a] extended the idea to testing the intercept and the regression coefficient simultaneously. It was further extended to global testing for multivariate regression [Koul, 1969]. Tests for general sub-hypotheses were first proposed by Koul [1970] and Puri and Sen [1973] for bivariate regression. The general theory of testing sub-hypotheses were independently developed by Srivastava [1972], McKeen and Hettmansperger [1976] and Adichie [1978]. The underlying theory is based on the seminal work by Jana Jurečková [Jureckova, 1969], as a significant generalization of Hodges and Lehmann [1963] for location problems and Adichie [1967b] for univariate regression. Her work was further extended by Jureckova [1971], van Eeden [1972]. However, these approaches are computationally extensive due to the discreteness of ranks. A one-step estimator was proposed by Kraft and Van Eeden [1972], which is asymptotically equivalent to the maximum likelihood estimators if the error distribution is known. Another one-step rank-based estimator, motivated by Bickel [1975] for M-estimators, was proposed by McKeen and Hettmansperger [1978]. On the other hand, Jaeckel [1972] proposed a rank-based objective function, later known as Jaeckel’s dispersion function, that is convex in $\beta$ whose minimizer is asymptotically equivalent to Jurečková’s score-based estimators. Hettmansperger and McKeen [1978] found an equivalent but mathematically more tractable formulation of Jaeckel’s dispersion function as the sum of pairwise difference of regression residuals. A weighted generalization of the dispersion function was introduced by Sievers [1983], which unified Jaeckel’s dispersion function and Kendall’s tau based dispersion function [Sen, 1966b, Sievers, 1978]. Three nice survey papers were written by Adichie [1984], Aubuchon and Hettmansperger [1984] and Draper [1988]. In 1990s, due to the development of quantile regression [Koenker and Bassett, 1978], Gutenbrunner and Jurecková [1992] found an important coincidence between the dual problem of quantile regression and the so-called “rank-score process”, which generalizes the notion introduced by Hájek and Šidák [1967] to linear models. Gutenbrunner et al. [1993] then developed rank-score test for linear hypotheses; see also Koenker [1997] for a review. Over the past decade, there were much fewer works on rank-based tests for linear models [e.g. Feng et al., 2013].
4.4 Tests based on regression M-estimates

Regression M-estimates were introduced by Peter J. Huber in 1964 for location problems [Huber, 1964]. The idea was soon extended to linear models by Relles [1968], who proved the asymptotic theory for Huber’s loss with $p$ fixed and $n$ tending to infinity. The theory was extended to general convex loss functions by Yohai [1972]. Despite the appealing statistical properties, the computation remained challenging in 1970s. Bickel [1975] proposed one-step M-estimates that are computational tractable with the same asymptotic property as full M-estimates. In addition, he proved the uniform asymptotic linearity of M-estimates, which is a fundamental theoretical result that laid the foundation for later works. Based on Bickel [1975]’s technique, Jureckova [1977] established the relation between regression M-estimates and R-estimates. The asymptotic normality of M-estimates directly yields an asymptotically valid Wald-type test for general linear hypotheses. Schrader and Hettmansperger [1980] developed an analogue of likelihood-ratio test based on M-estimators for sub-hypotheses. It was further extended to general linear hypotheses by Silvapulle [1992]. However, both Wald-type tests and likelihood-ratio-type tests involve estimating nuisance parameters. To overcome the extra efforts, Sen [1982] proposed M-test as an analogue of studentized score test M-tests, which is capable to test general linear hypotheses with merely estimates of regression coefficients under the null hypothesis. It is known that Rao’s score test may not be efficient in presence of nuisance parameter. Singer and Sen [1985] discussed an efficient test, which is essentially the analogue of Neyman’s $C(\alpha)$ test based on projected scores [Neyman, 1959], although it brings back nuisance parameters. M-tests were later investigated and generalized in a general framework based on influence functions [e.g. Boos, 1992, Markatou and Ronchetti, 1997].

As with t/F test, the robustness against high dimensionality was investigated extensively for M-estimators in general linear models. In Huber’s 1972 Wald lectures [Huber, 1972], he conjectured that the asymptotic normality of M-estimates proved by Relles [1968] can be extended to the asymptotic regime where $p$ grows with $n$. The conjecture was proved one year later in the regime $\kappa p^2 = o(1)$, where $\kappa$ is the maximum leverage score, which implies $p = o(n^{1/3})$ [Huber, 1973]. This was improved to $\kappa p^{3/2} = o(1)$ by Yohai and Maronna [1979], which implies that $p = o(n^{2/5})$, to $p = o(n^{2/3}/\log n)$ by Portnoy [1985] under further regularity conditions on the design matrix, and to $\kappa n^{1/3}(\log n)^{2/3} = o(1)$, which implies that $p = o(n^{2/3}/(\log n)^{2/3})$. All aforementioned results work for smooth loss functions. For non-smooth loss functions, Welsh [1989] obtained the first asymptotic result in the regime $p = o(n^{1/3}/(\log n)^{2/3})$. It was improved to $p = o(n^{1/2})$ by Bai and Wu [1994]. For a single coordinate, Bai and Wu [1994] showed the asymptotic normality in the regime $p = o(n^{2/3})$. These works prove that the classical asymptotic theory holds if $p \ll n^{2/3}$. However, in moderate dimensions where $p$ grows linear with $n$, the M-estimates are no longer consistent in $L_2$ metric and the risk $\|\hat{\beta} - \beta\|_2^2$ tends to a non-vanishing quantity determined by $p/n$, the loss function and the error distribution through a complicated system of non-linear equations for random designs [El Karoui et al., 2011, Bean et al., 2012, El Karoui, 2013, Donoho and Montanari, 2016, El Karoui, 2018]. This surprising phenomenon marks the failure of classical asymptotic theory for M-estimators. For least squares estimators, Lei et al. [2018] showed that the classical t-test with appropriate studentization is still asymptotically valid under regularity conditions on the design matrix. Cattaneo et al. [2018] proposed a refined test.
for heteroscedastic linear models. However it is unclear how to test general linear hypotheses with
general M-estimators in this regime, even for a single coordinate. Lei et al. [2018] provided the only
fixed-design result for the asymptotic property of a single coordinate for general M-estimates in this
regime. For the purpose of hypothesis testing, the null variance needs to be estimated but there is no
consistent variance estimator, except for special random designs [e.g. Bean et al., 2012].

4.5 Tests based on regression L-estimates

L-estimators constitute an important class of robust statistics based on linear combination of order
statistics. Frederick Mosteller proposed the first L-estimator for Gaussian samples [Mosteller, 1946].
This was further developed by Hastings et al. [1947], Lloyd [1952], Evans and Evans [1955], Jung
[1956], Tukey [1960], Bickel [1965], Gastwirth [1966]. In particular, John W. Tukey advocated the
trimmed mean and Winsorized mean, which he attributed to Charles P. Winsor based on their personal
communication in 1941, in his far-reaching paper [Tukey, 1962]. One year later, the well-known
Hodges and Lehmann estimator was developed [Hodges and Lehmann, 1963], which established the
first connection between R-estimates and L-estimates. For location problems, Bickel and Lehmann
[1975] found the superiority of L-estimates over M-estimates and R-estimates.

Despite the simplicity and nice theoretical property of L-statistics, they are not easy to be gen-
eralized to linear models. The first attempt was made by Bickel [1973], which proposed a one-step
L-estimate for general linear models. However, this estimator is not equivariant to affine transformation
of design matrices. Motivated by this paper, Welsh [1987] proposed a class of one-step L-estimators
that are equivariant to reparametrization of the design matrix. Welsh [1991] further extended the idea
to construct an adaptive L-estimator. Another line of thoughts were motivated by the pinoneering
work of Koenker and Bassett [1978], which introduced the notion of regression quantiles as a natural
analogue of sample quantiles for linear models. Although quantile regression yields an M-estimator,
it had been the driving force for the development of regression L-estimators since 1980s. In this pa-
per, they proposed another class of L-estimators by discrete weighted average of regression quantiles
and derived its asymptotic distribution. This idea was furthered by Koenker and Portnoy [1987] to
L-estimators with continuous weights, by Portnoy and Koenker [1989] to adaptive L-estimators, and
by Koenker and Zhao [1994] to heteroscedastic linear models. Another notable strategy of contructing
L-statistics is based on weighted least squares with “outliers” removed. Ruppert and Carroll [1980]
developed two equivariant one-step estimators as analogues of trimmed mean. Both estimators can be
formulated in the form of weighted least squares where units with extreme residuals are removed and
one is based on regression quantiles. As with Ruppert and Carroll [1980], Jureckova [1983] proposed
an analogue of winsorized mean. The Bahadur representation of trimmed mean least squares esti-
mator was derived by Jurečková [1984]. A nice review article of regression L-estimators was written
by Alimoradi and Saleh [1998]. The asymptotic results of L-estimators induce asymptotically valid
Wald-type tests with consistent estimates of asymptotic variance. Unlike M-estimators, we are not
aware of other types of tests based on L-estimates.
4.6 Resampling based tests

Resampling, marked by jackknife [Quenouille, 1949, 1956, Tukey, 1958] and bootstrap [Efron, 1979], is a generic technique to assess the uncertainty of an estimator. Although both involving resampling, resampling-based tests are fundamentally different from permutation tests, as the former are approximating the sampling distribution under the truth while the latter are approximating the sampling distribution under the null hypothesis, although they are asymptotically equivalent in many cases [e.g. Romano, 1989]. Miller [1974] proposed the first jackknife-based estimate for general linear models. He showed that the estimator is asymptotically normal and the jackknife variance estimator is consistent, thereby the Wald-type test is asymptotically valid. Hinkley [1977] pointed out that Miller’s estimator is less efficient than the least squares estimator and proposed a weighted jackknife estimates to overcome the inefficiency. Wu [1986] proposed a general class of delete-d jackknife estimators for estimating the covariance matrix of the least squares estimator. This was extended by Shao and Wu [1987], Shao [1988, 1989], Peddada and Patwardhan [1992], Liu and Singh [1992].

On the other hand, David A. Freedman first studied bootstrapping procedures for linear models [Freedman, 1981]. He proposed and studied two types of bootstrap: the residual bootstrap, where the regression residuals are resampled and added back to the fitted values, and the pair bootstrap, where the outcome and the covariates are resampled together. In the fixed-\(p\) regime, he showed the consistency of the residual bootstrap for homoscedastic linear models and consistency of pair bootstrap for general “correlation models” including heteroscedastic linear models. Navidi [1989], Hall [1989] and Qumsiyeh [1994] established the higher order accuracy of pair bootstrap for linear models and the results were then presented under a broader framework in the influential monograph of Peter Hall [Hall, 1992]. Wu [1986] found that the residual bootstrap fails in heteroscedastic linear models because its sampling process is essentially homoscedastic. To overcome this, he introduced another type of bootstrapping method based on random re-scaling of regression residuals that match the first and second moments. Liu [1988] introduced a further requirement to match the third moment and improved the rate of convergence. Later Mammen [1993] coined this procedure “wild bootstrap” and proved the consistency for linear least squares estimator under random-design homoscedastic and heteroscedastic linear models. Hu and Zidek [1995] proposed an alternative bootstrap procedure for heteroscedastic linear models that resamples the score function instead of the residuals. A wild bootstrap analogue of score-based bootstrap was proposed by Kline and Santos [2012]. In particular, they developed the bootstrap Wald tests and the bootstrap score tests for general linear hypotheses.

The bootstrap techniques were also widely studied for regression M-estimates. The residual bootstrap was extended to M-estimators with smooth loss functions by Shorack [1982]. Unlike least squares estimators, it requires a debiasing step to obtain distributional consistency. Lahiri [1992] proposed a weighted residual bootstrap that does not require debiasing. He additionally showed the higher order accuracy of the weighted bootstrap and Shorack’s bootstrap for studentized M-estimators. However, this weighted bootstrap is hard to be implemented in general. On the other hand, motivated by Bayesian bootstrap [Rubin, 1981], Rao and Zhao [1992] proposed a bootstrapping procedure by randomly reweighting the objective function. This idea was extended by Chatterjee [1999] in a broader framework called “generalized bootstrap”. It was later re-invented by Jin et al. [2001] and referred to
as “perturbation bootstrap”. The higher order accuracy of perturbation bootstrap was established by Das and Lahiri [2019]. It was pointed out by [Das and Lahiri, 2019] that the perturbation bootstrap coincides with wild bootstrap for least squares estimators. Hu and Kalbfleisch [2000] proposed another estimating function based bootstrap, as essentially an resampling version of Sen [1982]'s M-tests. Wild bootstrap was introduced for quantile regression by Feng et al. [2011].

The robustness of bootstrap methods against high dimension was widely studied in literature. Bickel and Freedman [1983] proved the distributional consistency of residual bootstrap least squares estimators in the regime $p = o(n)$ in terms of the linear contrasts and in the regime $p = o(n^{1/2})$ in terms of the whole vector, for fixed-design linear models with vanishing maximum leverage scores. They further showed the failure of bootstrap in moderate dimensions where $p/n \to c \in (0, 1)$ and the usual variance re-scaling does not help because the bootstrap distribution is no longer asymptotically normal. For M-estimators, Shorack [1982] showed that the debiased residual bootstrap is distributionally consistent in the regime $p = o(n^{1/3})$ in terms of the linear contrasts. The results were extended by Mammen [1989] to the regime $p = o(n^{2/3}/(\log n)^{2/3})$ in terms of the linear contrasts and to the regime $p = o(n^{1/2})$ in terms of the whole vector. For random designs with i.i.d. design points, Mammen [1993] proved the distributional consistency of both pair bootstrap and wild bootstrap, in terms of linear contrasts, in the regime $p = o(n^a)$ for arbitrary $a < 1$. He also proved the consistency for heteroscedastic linear models in the regime $p = o(n^{3/4})$ for pair bootstrap and the regime $p = o(n^{1/2})$ for Wild bootstrap. The was further extended by Chatterjee [1999] to generalized bootstrap, including perturbation bootstrap [Rao and Zhao, 1992], $m$-out-of-$n$ bootstrap [Bickel and Sakov, 2008] and delete-d jackknife [Wu, 1990]. On the other hand, extending Bickel and Freedman [1983]'s negative result, El Karoui and Purdom [2018] showed the failure of various bootstrap procedures for M-estimators in moderate dimensions, including pair bootstrap, residual bootstrap, wild bootstrap and jackknife.

4.7 Other tests

A generic strategy for hypothesis testing is through pivotal statistics. Specifically, if there exists a statistics $S$ of which the distribution is fully known, then the rejection rule $S \in \mathcal{R}$ for any region $\mathcal{R}$ with $P(S \in \mathcal{R}) \geq 1 - \alpha$ yields a finite-sample valid test. For linear models, it is extremely hard to find a pivotal statistics under general linear hypotheses, except for Gaussian linear models for which the $t/F$ statistics are pivotal. However, if the goal is to test all coefficients plus the intercept, i.e. $H_0: \beta_0 = \gamma_0, \beta = \gamma$, then one can recover the stochastic errors as $e_i = y_i - \gamma_0 - \mathbf{x}_i^T \gamma$ under the null and construct pivotal statistics based on $\epsilon$. Taking one step further, given a pivotal statistic, one can invert the above test to obtain a finite-sample valid confidence region $\mathcal{I}$ for $(\beta_0, \beta)$, by collecting all $(\gamma_0, \gamma)$’s to which the corresponding null hypothesis fails to be rejected. This induces a confidence region for $R\beta$ as $\mathcal{I}' = \{R\beta : (\beta_0, \beta) \in \mathcal{I}\}$. Finally, using the duality between confidence interval and hypothesis testing again, the test which rejects the null hypothesis is finite-sample valid for the linear hypothesis $H_0: R\beta = 0$. If $r \ll p$, this seemingly “omnibus test” is in general severely conservative and inferior to the tests discussed in previous subsections. Nonetheless, it stimulates several non-standard but interesting tests in history that are worth discussion.

The most popular strategy to construct pivotal statistics is based on quantiles of $\epsilon_i$’s, especially
the median. Assuming $\epsilon_i$'s have zero median, Fisher [1925] first introduced the sign test for location problems, which was investigated and formalized later by Cochran [1937]. Thirteen years later, Henri Theil proposed an estimator for univariate linear models [Theil, 1950a,b,c], later known as Theil-Sen estimator [Sen, 1968b]. Brown and Mood [1951] proposed a median test for general linear models by reducing the problem into a contingency table and applying the $\chi^2$-tests. The theoretical property of Brown-Mood test was studied by Kildea [1981] and Johnstone and Velleman [1985]. Daniels [1954] proposed a geometry-based test for univariate linear models, which can be regarded as a generalization of Brown-Mood test. It was later connected to the notion of regression depth [Rousseeuw and Hubert, 1999] and applied in deepest regression methods [Van Aelst et al., 2002]. The idea of inverting the sign test was exploited in Quade [1979] and an analogue incorporating Kendall’s tau between the residuals and the covariates was proposed by Lancaster and Quade [1985]. The idea also attracted some attention in engineering literature [e.g. Campi and Weyer, 2005, Campi et al., 2009] and in econometrics literature [e.g. Chernozhukov et al., 2009]. It should be noted that the approach is computationally infeasible even for small dimensions. Assuming further the symmetry of $\epsilon_i$’s, Hartigan [1970] proposed a non-standard test based on an interesting notion of typical values. It was designed for location problems but can be applied to certain ANOVA problems. Furthermore, Siegel [1982] proposed the repeated median estimator and Rousseeuw [1984] proposed the least median squares estimators to achieve high breakdown point.

The pivotal statistics can also be constructed in other ways. Parzen et al. [1994] proposed a bootstrap procedure based on inverting a pivotal estimating function at a random point. This procedure mimics the Fisher’s fiducial inference but can be justified under the common framework. Recently Meinshausen [2015] proposed the Group Bound test for sub-hypotheses, which even works for high-dimensional settings where $p \gg n$. However, the validity is only guaranteed for rotationally invariant errors with known noise level. This requirement is extremely strong as shown by Maxwell [1860]: if $\epsilon_i$’s are further assumed to be i.i.d., then rotation invariance implies the normality of $\epsilon_i$’s.

5 Conclusion and Discussion

In this article, we propose Cyclic Permutation Test (CPT) to test general linear hypotheses for linear models. This test is exact for arbitrary fixed design matrix and arbitrary exchangeable errors, whenever $1/\alpha$ is an integer and $n/(p - r) \geq 1/\alpha - 1$. Extensive simulation studies demonstrate the reasonable performance of CPT.

CPT is non-standard compared to various methodologies developed in the past century. CPT essentially constructs a pivotal statistic in finite samples based on group invariance. This is rare in the territory of distribution-free inference with complex nuisance parameters. Our goal is to expand the toolbox for exact and distribution-free inference and hopefully generate new ideas for more complicated problems. In the following subsections we discuss several extensions and future directions.
5.1 Confidence interval/region by inverting CPT

It is straightforward to deduce a confidence region for $\beta_r$ can be obtained by inverting CPT. Specifically, the inverted confidence region is given by

$$I \triangleq \{ \beta_r : p(y - X\beta) > \alpha \},$$

where $p(y; X)$ is the p-value produced by CPT with a design matrix $X$ and an outcome vector $y$. Under the construction C3,

$$\eta_j^T (y - X\beta) = \eta_j^T y - \gamma^T \beta - \delta^T \beta_r I(j = 0).$$

Thus,

$$\text{median} \left( \{ \eta_j^T (y - X\beta) \}_{j=0}^m \right) = \text{median} \left( \{ \eta_j^T y - \delta^T \beta_r I(j = 0) \}_{j=0}^m \right) - \gamma^T \beta.$$

Then $I$ can be simplified as

$$I = \{ \beta_r : \delta^T \beta_r \in [x_{\min}, x_{\max}] \}$$

(24)

where $x_{\min}$ and $x_{\max}$ are the infimum and the superimun of $x$ such that

$$\frac{1}{m + 1} \left( 1 + \sum_{j=1}^m \mathbb{I} \left( |\eta_0^T y - x - m(x)| \leq |\eta_j^T y - m(x)| \right) \right) > \alpha,$$

and

$$m(x) = \text{median} \left( \{ \eta_j^T y - xI(j = 0) \}_{j=0}^m \right).$$

When $r = 1$, the confidence interval (24) gives a useful confidence interval simply as

$$I = [x_{\min}/\delta, x_{\max}/\delta].$$

However when $r > 1$, the confidence region (24) may not be useful because it is unbounded. More precisely, $\beta_r \in I$ implies that $\beta_r + \xi \in I$ for any $\xi$ orthogonal to $\delta$. We leave the construction of more efficient confidence regions to future research.

5.2 Connection to knockoff based inference

Our test is implicitly connected to the novel idea of knockoffs, proposed by Barber et al. [2015] to control false discovery rate (FDR) for variable selection in linear models. Specifically, they assumed a Gaussian linear model and aimed at detecting a subset of variables that control FDR in finite samples. Unlike the single hypothesis testing considered in this paper, multiple inference requires to deal with the dependence between test statistics for each hypothesis carefully. They proposed an interesting idea of constructing a pseudo design matrix $X$ such that the joint distribution of $(X_1^T y, \ldots, X_p^T y, \hat{X}_r^T y, \ldots, \hat{X}_p^T y)$ is invariant to the pairwise swapping of $X_j^T y$ and $\hat{X}_j^T y$ all for $j$ with $\beta_j = 0$. Then the test statistic for testing $H_{0j} : \beta_j = 0$ is constructed by comparing $X_j^T y$ and $\hat{X}_j^T y$ in an appropriate way, thereby obtaining a valid binary p-value $p_j$ that is uniformly distributed on
The knockoffs-induced p-values marginally resemble the construction of statistics in CPT with $m = 2, \eta_0 = X_j, \eta_1 = \tilde{X}_j$. On the other hand, the validity of knockoffs essentially rests on the distributional invariance of $\epsilon$ to the rotation group while the validity of CPT relies on the distributional invariance of $\epsilon$ to the cyclic permutation group. This coincidence illustrates the charm and the magical power of group invariance in statistical inference.

5.3 More efficient algorithm for pre-ordering

Although GA is able to solve (23) efficiently for problems with moderate size, it is not scalable enough to handle big data. Since the exact minimizer is not required, we can resort to other heuristic algorithms. One heuristic strategy is proposed by Fogel et al. [2013] by relaxing permutation matrices into doubly stochastic matrices, with $\Pi_1 = \Pi_1^T 1 = 0$ and $\Pi_{ij} \geq 0$, and optimizing the objective using continuous optimization algorithms. Taking the case of $r = 1$ for example, by Theorem 2, (23) is equivalent to

$$\min_{\Pi} B(\Pi X)^T (I - B(\Pi X)_{[-1]} (B(\Pi X)_{[-1]}^T B(\Pi X)_{[-1]})^{-1} + B(\Pi X)_{[-1]}^T B(\Pi X)) B(\Pi X)_1.$$ 

By Sherman-Morrison-Woodbury formula, the reciprocal of the above objective is the first diagonal element of $[B(\Pi X)^T B(\Pi X)]^{-1}$. Therefore, (23) is equivalent to

$$\max_{\Pi} e_1^T [B(\Pi X)^T B(\Pi X)]^{-1} e_1.$$ 

Denote by $h(\Pi)$ the above objective function, then the derivative of $h$ with respect to $\Pi$ can be easily calculated as

$$- \frac{\partial h(\Pi)}{\partial \Pi_{ij}} = e_1^T [B(\Pi X)^T B(\Pi X)]^{-1} \left( \frac{\partial}{\partial \Pi_{ij}} [B(\Pi X)^T B(\Pi X)] B(\Pi X) \right) [B(\Pi X)^T B(\Pi X)]^{-1} e_1$$

$$= e_1^T [B(\Pi X)^T B(\Pi X)]^{-1} \left( B(\Pi X)^T \frac{\partial}{\partial \Pi_{ij}} B(\Pi X) + \left( \frac{\partial}{\partial \Pi_{ij}} B(\Pi X) \right)^T B(\Pi X) \right) [B(\Pi X)^T B(\Pi X)]^{-1} e_1$$

$$= e_1^T [B(\Pi X)^T B(\Pi X)]^{-1} \left( B(\Pi X)^T B(e_i X_j^T) + B(e_i X_j^T)^T B(\Pi X) \right) [B(\Pi X)^T B(\Pi X)]^{-1} e_1,$$

where the last line uses the definition of $B(X)$ in (19). The easy gradient computation may suggest an efficient gradient based algorithm. We leave this as a future direction.

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Appendix A  Complementary Experimental Results

In this appendix we present experimental results that complement Section 3. Figure 7 - 10 display the power comparison for testing a single coordinate under the same setting as Section 3.2 for four extra scenarios: realizations of Gaussian matrices + Cauchy errors, realizations of Cauchy matrices + Gaussian errors and realizations of random one-way ANOVA matrices + Gaussian/Cauchy errors.

Figure 11 - 16 display the power results under the same setting as Section 3.3 for six scenarios: realizations of Gaussian matrices + Gaussian/Cauchy errors, realizations of Cauchy matrices + Gaussian/Cauchy errors and realizations of random one-way ANOVA matrices + Gaussian/Cauchy errors.

Figure 7: Median power ratio between each variant of CPT and each competing test for testing a single coordinate with realizations of Gaussian matrices and Cauchy errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 8: Median power ratio between each variant of CPT and each competing test for testing a single coordinate with realizations of Cauchy matrices and Gaussian errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 9: Median power ratio between each variant of CPT and each competing test for testing a single coordinate with realizations of random one-way ANOVA matrices and Gaussian errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 10: Median power ratio between each variant of CPT and each competing test for testing a single coordinate with realizations of random one-way ANOVA matrices and Cauchy errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 11: Median power ratio between each variant of CPT and each competing test for testing five coordinates with realizations of Gaussian matrices and Gaussian errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 12: Median power ratio between each variant of CPT and each competing test for testing five coordinates with realizations of Gaussian matrices and Cauchy errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 13: Median power ratio between each variant of CPT and each competing test for testing five coordinates with realizations of Cauchy matrices and Gaussian errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 14: Median power ratio between each variant of CPT and each competing test for testing five coordinates with realizations of Cauchy matrices and Cauchy errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 15: Median power ratio between each variant of CPT and each competing test for testing five coordinates with realizations of random one-way ANOVA matrices and Gaussian errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.
Figure 16: Median power ratio between each variant of CPT and each competing test for testing five coordinates with realizations of random one-way ANOVA matrices and Cauchy errors. The black solid line marks the equal power. The missing values in the last row correspond to infinite ratios.