Bell-type games on deformable manifolds

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We study bipartite correlations in Bell-type games. We show that in a setup where the information carriers are allowed to locally deform the manifold on which the game is played, stronger correlations may be obtained than those maximally attainable otherwise. We discuss the implications of our results in the context of Bell’s theorem and the Einstein-Podolsky-Rosen paradox.

I. INTRODUCTION

Correlations measure the degree to which physical features fluctuate together. They are responsible for the emergence of complex behavior in physical systems at all scales, and as such play a crucial role in virtually all areas of science [2]. In classical physics, correlations between macroscopic features are understood as emerging from an underlying deterministic microscopic theory, defined on an extended phase-space which includes additional, computationally inaccessible, degrees of freedom. Averaging out these microscopic degrees of freedom leads in turn to a seemingly probabilistic behavior of the remaining degrees of freedom and the correlations thereof. These principles are epitomized in classical statistical physics, which gives rise to a relatively simple description for macroscopic physical systems consisting of a large number of microscopic degrees of freedom [3–5].

Bipartite correlations between physical quantities appearing in the framework of classical statistical physics are often subject to inequality constraints. These constraints stem directly from a result from probabilistic logic known as the Boole-Fréchet inequalities, which places bounds on different combinations of probabilities concerning logical propositions or events that are linked together [6, 7]. Bell’s inequalities [1, 8–13] are celebrated examples of these types of constraints that touch upon the very nature of physical reality as well as on the foundations of quantum mechanics through the renowned Einstein-Podolsky-Rosen (EPR) paradox [14, 15]. Bell’s inequalities restrict the values of certain linear combinations of correlations between pairs of binary degrees of freedom arranged in closed loops. A well-known setup where Bell’s inequalities arise and which is often used to illustrate the distinctive nature of quantum theory is that devised by Clauser, Horne, Shimony and Holt (abbreviated CHSH) [10], which describes a game in which two parties share pairs of identical particles (the ‘information carriers’) and have full control over their respective measurement devices, which detect the particles and produce pairs of binary outcomes. In this paper, we revisit the CHSH-Bell game, exploring a novel scenario in which the curvature of the manifold on which the game is played is deformed by the information carriers shared between the two parties. We find, perhaps counter-intuitively, that the latter setup allows for stronger correlations than those attainable otherwise.

The paper is organized as follows. The general framework is described in detail in Sec. II. In Sec. III, we consider the ‘standard’ CHSH-Bell setup, where the degrees of freedom of the game are embedded in a two-dimensional flat manifold. In Sec. IV, we examine the new setup, in which the curvature of the manifold is deformed by the information carriers and in Sec. V, we discuss the implications of our results, specifically in relation to the CHSH-Bell inequality violations predicted by quantum theory [16]. We offer summary and conclusions in Sec. VI.

II. THE BIPARTITE GAME SETUP

We derive our results using a bipartite CHSH-Bell-type game (or experiment) [10] played in a two-dimensional world that abides by the rules of local and deterministic classical physics. We consider a game played by two parties, Alice and Bob, who are located, respectively, at the two ends A and B of a segment drawn on the two-dimensional flat plane. In our game, each party has a directional detector system, which can be oriented on the plane in a direction chosen by the party. We denote the two-dimensional unit vectors that describe the detectors associated with Alice and Bob by the lowercase letters, \( \vec{a} \) and \( \vec{b} \), respectively, and further denote the angle between them generically by \( \theta \) (we use a convention that if \( \theta \in [−\pi, \pi) \) is the angle of \( \vec{b} \) with respect to \( \vec{a} \) then the relative angle of \( \vec{a} \) with respect to \( \vec{b} \) is \( −\theta \)). We refer to these vectors as the detector’s (or party’s)
direction-vector (or simply as the detector’s vector). In each round of the game, each party can freely and independently (in a way that will also be precisely defined later) orient their detector’s vector along one of two distinct (pre-chosen) directions, \( \vec{a}_1 \) and \( \vec{a}_2 \) for Alice and \( \vec{b}_1 \) and \( \vec{b}_2 \) for Bob (see Fig. 1). In addition, we shall assume the existence of an oven located somewhere along the line connecting the two parties (its precise location will not be of importance). The oven generates pairs of aligned (unit) vectors pointing along some random direction on the two-dimensional manifold (in a precise sense that will be defined later) and transmits one of the copies to Alice and the other to Bob.

The detectors at each location detect the incoming oven-generated vector, and compare it to their own orientation in terms of the azimuthal coordinate \( \Delta \in [-\pi, \pi] \) extended from the detector (see Fig. 2). Each detector then outputs a binary outcome, \( S(\Delta) \), either +1 or −1, according to the response function

\[
S(\Delta) = \text{sign}(\Delta) = \begin{cases} 
-1, & \text{if } \Delta \in [-\pi,0) \\
+1, & \text{if } \Delta \in [0,\pi). 
\end{cases}
\]

The rule given in the above equation is deterministic and local; that is, the output of each detection event is determined uniquely by the relative orientation of the local detector’s vector with respect to the incoming copy of the oven-generated vector, and it is independent of the setting of the detector of the other party.

Upon a (joint) detection event at the two locations, the parties report the bits of information output by their respective detectors, along with the setting of their detector orientation, to an external referee. At the end of the game, after many rounds, the referee uses the information received from the parties to evaluate the bipartite correlations for each of the four joint settings \((\hat{a}_1, \hat{b}_1), (\hat{a}_1, \hat{b}_2), (\hat{a}_2, \hat{b}_1) \) and \((\hat{a}_2, \hat{b}_2) \). For any given setting of the pair of detectors, the (empirical) bipartite correlation is defined as the difference between the probability (or fraction of events) that the two parties report the same output for the shared oven-generated vector and the probability (or fraction of events) that they report opposite outcomes.

We now consider this game as it would be played in two distinct setups, and study the structure of the bipartite correlations and the corresponding CHSH-Bell-type constraints that emerge in each case. In the first scenario, the game is played on the ‘usual’ flat manifold, while in the second we consider a novel situation in which the oven-generated vectors locally curve the (otherwise flat) manifold as they traverse it.

In both scenarios, we shall verify that the following natural symmetry conditions are satisfied, to guarantee that neither party has a preferred orientation and that both parties are treated on an equal footing (i.e., neither one is more special than the other): explicitly:

**Condition 1:** Both parties must observe an expected zero average of their detection outcomes for any chosen orientation of their detector.

**Condition 2:** The bipartite correlations must depend only on the relative orientation of their detectors.

### III. THE BIPARTITE GAME PLAYED ON A FLAT MANIFOLD

We begin by considering the standard setup in which the two parties, Alice and Bob, are located at the ends of a segment drawn on the flat two-dimensional plane, as shown in Fig. 2. As stated in the previous section, each party has a directional detector that can be freely oriented in either of two possible directions, \( \hat{a} \in \{\hat{a}_1, \hat{a}_2\} \) and \( \hat{b} \in \{\hat{b}_1, \hat{b}_2\} \). In each round of the game, the oven located somewhere between the two parties produces two identical copies of a unit vector \((\hat{\lambda}, \hat{\lambda})\), which are then transported to the parties along the path connecting them, one copy per party. Upon receiving their respective copies of the oven-generated vector, each party produces an outcome according to the rule specified in Eq. (1). In this ‘standard’ setup, the affine structure of the flat manifold allows one to refer to the two copies received by the two parties as the same entity \( \hat{\lambda} \). To compare the orientation of the incoming oven-generated vector to the corresponding detector’s direction-vector, we use a planar polar coordinate system, with the angle between any two vectors on the plane defined in the interval \([-\pi, \pi)\) (see Fig. 2). Denoting the relative azimuthal coordinate of \( \hat{a} \) and \( \hat{a} \) as \( \Delta^{(A)} := \Delta(\hat{a}, \hat{\lambda}) \) and, similarly, between \( \hat{\lambda} \) and \( \hat{b} \) as \( \Delta^{(B)} := \Delta(\hat{b}, \hat{\lambda}) \), the output bits of the two parties are explicitly given, following the rule Eq. (1), by

\[
S_A(\Delta^{(A)}) = \begin{cases} 
-1, & \text{if } \Delta^{(A)} \in [-\pi,0), \\
+1, & \text{if } \Delta^{(A)} \in [0,\pi),
\end{cases}
\]

and

\[
S_B(\Delta^{(B)}) = \begin{cases} 
-1, & \text{if } \Delta^{(B)} \in [-\pi,0), \\
+1, & \text{if } \Delta^{(B)} \in [0,\pi).
\end{cases}
\]
Throughout this paper we use $\Delta(\vec{v}, \vec{u}) = -\Delta(\vec{u}, \vec{v})$ to denote the relative azimuthal coordinate (or coordinate difference) of one vector, $\vec{u}$, with respect to another $\vec{v}$. We note that while on a flat manifold the relative azimuthal coordinate between two vectors can be identified with the angle between them, this identification cannot generally be made when curved manifolds are concerned.

We can relate the angle between the detectors, $\theta$, to the two coordinates that determine the outcomes of the detection events, $\Delta^{(A)}$ and $\Delta^{(B)}$, by noting that the orientation of any oven-generated vector $\vec{\lambda}$ with respect to $\vec{a}$ equals the angle between $\vec{\lambda}$ and $\vec{b}$ plus $\theta$, that is,

$$\Delta^{(A)} = \Delta^{(B)} + \theta,$$

where addition is defined here modulo $2\pi$ within the interval $[-\pi, \pi]$ (see top panel of Fig. 3 with $\varphi_A = \Delta^{(A)}$ and $\varphi_B = \Delta^{(B)}$).

To make quantitative predictions, we turn to the symmetry conditions discussed in the previous section. Following Condition 1, for each party and for any direction of their detector, the expected average of the detection events (i.e., bit outcomes) should be zero, i.e.,

$$\langle S_A(\Delta^{(A)}) \rangle = 0 \text{ and } \langle S_B(\Delta^{(B)}) \rangle = 0,$$

for any choice of $\vec{a}$ and $\vec{b}$ on the unit circle. This condition determines the distribution of the oven-generated vectors $\vec{\lambda}$ over the unit circle with respect to $\vec{a}$ and $\vec{b}$. Under the relation given in Eq. (4), to satisfy Condition 1, the oven-generated vectors $\vec{\lambda}$ must be uniformly distributed over the unit circle. As a consistency check, we note that following Eq. (4), if the orientation of $\vec{\lambda}$ with respect to $\vec{a}$, $\Delta^{(A)}$ is distributed uniformly then, since $\theta$ is constant, so is the orientation of $\vec{\lambda}$ with respect to $\vec{b}$, $\Delta^{(B)}$, and thus the requirements of Eq. (5) are satisfied.

Next, let us examine the expected correlation between the two parties in this model, defined by

$$E_{AB}(\theta) = \int \frac{d\Delta^{(A)}}{2\pi} \sin(\Delta^{(A)}) \sin(\Delta^{(B)} - \theta)$$

$$= \int \frac{d\Delta^{(B)}}{2\pi} \sin(\Delta^{(B)}) \sin(\Delta^{(A)} + \theta),$$

where again we make use of the relation given by Eq. (4). The correlation between Alice and Bob’s outputs, given that their detectors are oriented along $\vec{a}$ and $\vec{b}$, can be readily obtained by noting that the response of Bob’s detector changes sign when the oven-generated vector $\vec{\lambda}$ is oriented with respect to Alice’s detector at angles $\Delta^{(A)} = \theta$ and $\Delta^{(A)} = \pm \pi + \theta$ (modulo $2\pi$),

$$\Delta^{(A)} = \theta \iff \Delta^{(B)} = 0,$$

$$\Delta^{(A)} = \pm \pi + \theta \iff \Delta^{(B)} = \pm \pi.$$

Since the expected correlation between detectors’ outputs is given by the probability that they report the same outcome minus the probability they report an opposite outcome (and the probability density is 1/2 between $[0, \pi]$ with respect to both $\vec{a}$ and $\vec{b}$), we can write the correlation between Alice and Bob, Eq. (6), as:

$$E_{AB}^{(\text{flat})}(\theta) = -\int_{-\pi}^{\theta-\pi} \frac{d\xi}{2\pi} + \int_{\theta-\pi}^{\theta} \frac{d\xi}{2\pi} - \int_{\theta}^{\theta+\pi} \frac{d\xi}{2\pi} + \int_{\theta+\pi}^{\pi} \frac{d\xi}{2\pi},$$

for $\theta \in [0, \pi]$, and

$$E_{AB}^{(\text{flat})}(\theta) = \int_{-\pi}^{\theta} \frac{d\xi}{2\pi} - \int_{\theta}^{\theta+\pi} \frac{d\xi}{2\pi} + \int_{\theta}^{\theta+\pi} \frac{d\xi}{2\pi} - \int_{\theta+\pi}^{\pi} \frac{d\xi}{2\pi},$$

for $\theta \in [-\pi, 0]$. Thus, we find that

$$E_{AB}^{(\text{flat})}(\theta) = 1 - \frac{2}{\pi} |\theta|$$
for $\theta \in [-\pi, \pi)$. As expected, the above equation implies
\[
E_{AB}^{(\text{flat})}(0) = 1, \quad \text{(11)}
\]
\[
E_{AB}^{(\text{flat})}\left(\pm \frac{\pi}{2}\right) = 0, \quad \text{and} \quad E_{AB}^{(\text{flat})}(\pm \pi) = -1,
\]
which means that the parties are fully correlated when their detectors are aligned, uncorrelated when their detectors are perpendicular to each other and are fully anti-correlated when their detectors are anti-aligned.

As per Condition 2, the bipartite correlation between Alice and Bob, Eq. (10), depends only on the relative orientation (angle) of the two detectors. It is independent of the (joint) absolute orientation of the detectors (see Fig. 3). The orientations of the two detectors can be described by a ‘difference’ and a ‘sum’ (or average) of their respective angular degrees of freedom: the former is the relative orientation of the detectors, Eq. (4), and the latter is the joint absolute orientation of the two detector-vectors. Unlike the relative angle, which is independent of any external reference, the joint absolute orientation is measured relative to, and depends on, a choice of an external reference vector (hence the bipartite correlations must not depend on this quantity). In particular, upon choosing (for example) the oven-generated vector as a common reference vector, the joint absolute orientation of Alice and Bob’s detectors can be given by
\[
\varphi = -\frac{1}{2}(\Delta^A + \Delta^B). \quad \text{(12)}
\]

Similar to the relative angle, we can define the absolute angle $\varphi$ between pairs of vectors located at different points on the flat manifold. Thus, on a flat manifold the relative angle $\theta$, Eq. (4), and the joint absolute angle $\varphi$, Eq. (12), uniquely determine $\Delta^A$ and $\Delta^B$ (with respect to any external reference chosen to define the absolute angle) and vice versa. In the next section we shall see that similar arguments cannot generally be made for curved (rather, deformable) manifolds.

Finally, one can derive an inequality restricting the bipartite correlations between Bob and Alice by concatenating the four joint settings, $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, and $\hat{b}_2$, of Alice and Bob’s detectors. Specifically, we observe that for any oven-generated vector $\tilde{\lambda}$
\[
\begin{align*}
&\left|S_A(\Delta^{(A_1)})S_B(\Delta^{(B_1)}) + S_B(\Delta^{(B_2)})\right| + \left|S_A(\Delta^{(A_2)})S_B(\Delta^{(B_1)}) - S_B(\Delta^{(B_2)})\right| = 2, \quad \text{(13)}
\end{align*}
\]
where $\Delta^{(A_1)} = \Delta(\hat{a}_i, \tilde{\lambda})$ and $\Delta^{(B_2)} = \Delta(\hat{b}_j, \tilde{\lambda})$ for $i, j = 1, 2$, since $S_B(\Delta^{(B_1)}) - S_B(\Delta^{(B_2)}) = 0$ and $|S_B(\Delta^{(B_1)}) + S_B(\Delta^{(B_2)})| = 2$ whenever $S_B(\Delta^{(B_1)}) = S_B(\Delta^{(B_2)})$, while $S_B(\Delta^{(B_1)}) + S_B(\Delta^{(B_2)}) = 0$ and $|S_B(\Delta^{(B_1)}) - S_B(\Delta^{(B_2)})| = 2$ whenever $S_B(\Delta^{(B_1)}) = -S_B(\Delta^{(B_2)})$.

Therefore, by integration and the use of the triangle inequality, one readily finds that the four bipartite correlations are constrained by
\[
\left|E_{i,1}^{(\text{flat})} + E_{i,2}^{(\text{flat})} + E_{2,1}^{(\text{flat})} - E_{2,2}^{(\text{flat})}\right| \leq 2, \quad \text{(14)}
\]
where $E_{i,j}^{(\text{flat})} := E_{AB}^{(\text{flat})}(\theta_{ij})$, $i, j = 1, 2$, and $\theta_{ij} = \Delta(\hat{a}_i, \hat{b}_j)$ is the angle between $\hat{b}_j$ and $\hat{a}_i$. The above equation is the celebrated CHSH-Bell inequality [10].

At this point, it would be worth noting that Eq. (13), which is the basis for the derivation of the CHSH-Bell inequality, is a statement about four detection events [namely, $S_A(\Delta^{(A_1)}), S_A(\Delta^{(A_2)}), S_B(\Delta^{(B_1)}), S_B(\Delta^{(B_2)})$] concerning one oven-generated vector, $\tilde{\lambda}$. The statement assumes the existence of such a vector, which can participate in all four detection events, despite the fact that in the game itself each round involves only two detection events [$S_A(\Delta^{(A_1)}), S_B(\Delta^{(B_1)}), S_B(\Delta^{(B_2)})$, and $S_B(\Delta^{(B_2)})$].

As already mentioned, in flat space, there exists a global coordinate system (or an external reference frame) according to which all vectors may be uniquely described and which defines in a unique manner the identity of vectors in an absolute way, making it possible to identify a vector $\tilde{\lambda}$ appearing in say an $(\hat{a}_1, \hat{b}_1)$ detection event with a vector appearing in an $(\hat{a}_2, \hat{b}_1)$ detection event and so on, justifying the above

![Fig. 3. Relative and joint absolute angles on a flat manifold.](image-url)
statement. In the next section, we study the CHSH-Bell game played on curved (or more precisely, deformable) manifolds – where there is no such global coordinate system – and study the implications of the absence of such a system to the bipartite correlations and the CHSH-Bell inequality.

As a closing remark for this section, we note that throughout this section we have used the detectors’ direction-vectors as a reference frame for the oven-generated vectors. We could have instead, and without changing the physical predictions of the model, described the game using a reference frame that is attached to and aligned with the oven-generated vectors. From the perspective of the oven-generated vectors, in each round of the game the two detectors’ direction-vectors are ‘generated’ far away at random angles $-\Delta^{(A)}$ and $-\Delta^{(B)}$ distributed uniformly, and then move in closer toward the oven-generated vector located in the origin of the reference frame. The vector is detected and a subsequent response is recorded when the detectors reach the origin [see Fig. 2(b)].

IV. THE GAME PLAYED ON A DEFORMABLE MANIFOLD

In this section, we generalize the previous setup and consider a case wherein the oven-generated vectors locally deform the curvature of the (otherwise flat) manifold in which they are embedded, as they traverse it along the path from the oven towards the detectors [17]. Specifically, we consider the case where the oven-generated vectors break the rotational symmetry of the manifold along the direction in which they are generated but such that far away from the vectors the manifold remains flat (see Fig. 4 for a schematic illustration).

In order to explore the novel features of the current framework it will be convenient, and also illuminating, to describe the bipartite game from a reference frame attached to and aligned with the oven-generated vectors. From the perspective of the oven-generated vector, the manifold is curved near the origin and becomes flatter at increasing distances from the origin. In each round of the game, two detectors’ direction-vectors are generated far away from the vector (where the manifold is flat) oriented at random angles (with a distribution that we precisely describe below), and then move in closer toward the oven-generated vector. Detector responses are recorded as they reach the origin.

Let us now consider the path on the manifold that connects the oven and one of the detectors, say Alice’s. As a convention, we parameterize the path by the continuous variable $z \in [0, \infty)$, with the oven located at $z = 0$ and the detector located (initially) at $z = \infty$, far away from the oven. Let $\Delta$ denote the azimuthal polar coordinate on the local tangent planes along the path. Because we describe the problem from the reference frame of the oven-generated vector, it is convenient to choose the origin of this azimuthal coordinate, $\Delta = 0$, to be the direction of this vector (see Fig. 5); that is, the zero of the azimuthal coordinate is defined along the path as the direction in which a parallel-transported vector, aligned at $z = 0$ with the direction of the oven-generated vector, would point. This is a natural choice in the current model because by (locally) modifying the curvature of the manifold in a directional way, the oven-generated vectors break the rotational symmetry of the otherwise isotropic flat manifold, and introduce a preferred direction on the manifold. We thus use the direction of the oven-generated vector as a preferred reference frame (a ‘beacon’ to define the zero direction, or zero azimuthal coordinate, everywhere along the path) such that all other orientations are measured with respect to it.

Before moving on, it is useful to briefly review a few basic concepts from differential geometry, as we are discussing curved manifolds. On a general curved manifold, vectors at any point are defined on the ‘tangent plane’ associated with that point. Comparing the orientation of two vectors in the tangent planes associated with two distinct locations on the manifold requires ‘transporting’ them toward one another along a defined path. Thus, we shall assume that the manifold is endowed with a ‘metric-
ric’ connection (such as the canonical Levi-Civita connection associated with the metric tensor), which defines the parallel-transport of (tangent) vectors along a path on the manifold. By definition, the scalar product of any two vectors defined on the tangent plane at a given point remains invariant as the vectors are parallel-transported along a (piece-wise) smooth curve drawn on the manifold. Hence, the metric angle \( \cos(\tilde{\Delta}(\tilde{u}, \tilde{v})) = g_{i,j} u^i v^j \),

where \( g_{i,j} \) is the metric tensor at the location of the vectors, also remains invariant. As mentioned in the previous section, while in flat space one may identify the difference between the azimuthal coordinates of any two vectors with the metric angle, this is no longer the case on a curved manifold. In fact, in our model the metric angle between two vectors located at a point \( z \) on the path is taken to be a (non-linear) function of their relative azimuthal coordinate, which also depends on their location on the path, namely, \( \tilde{\Delta} = \mathcal{F}(z, \Delta) \). Since the detector is initially located in a flat region of the manifold (at \( z = \infty \)) the azimuthal coordinate in this region does coincide with the metric angle. Specifically, if the azimuthal coordinate of Alice’s direction-vector is \( \Delta_{0}^{(A)} \), then the metric angle \( \tilde{\Delta}_{0}^{(A)} \) of her detector’s vector is simply given by

\[
\tilde{\Delta}_{0}^{(A)} = \Delta_{0}^{(A)}. \tag{16}
\]

Next, let us consider the parallel transport of vectors along the path connecting Alice’s detector and the oven. Let us denote the metric angle between the detector’s vector and the oven-generated vector at \( z = 0 \) (i.e., when the former reaches the latter) as \( \tilde{\Delta}_{0}^{(A)} \). Since the metric angle between the detector’s vector and the oven-generated vector is conserved along the path, we must have

\[
\tilde{\Delta}_{0}^{(A)} = \tilde{\Delta}_{\infty}^{(A)}. \tag{17}
\]

While in the flat region of the manifold we identified the metric angle with the azimuthal coordinate, in the vicinity of the oven-generated vector, and specifically, at \( z = 0 \), the manifold is curved and the azimuthal coordinate is no longer synonymous with the metric angle. In our model, the oven-generated vector curves space such that the relation between the azimuthal coordinate and the metric angle at \( z = 0 \) is \( \tilde{\Delta} = \Gamma(\Delta) \), where \( \Gamma : [-\pi, \pi) \to [-\pi, \pi) \) is an anti-symmetric bijective function, \( \Gamma(-\Delta) = -\Gamma(\Delta) \), which we shall specify below. Therefore, if \( \Delta_{0}^{(A)} \) is the azimuthal coordinate of Alice’s detector direction-vector with respect to the oven-generated vector at \( z = 0 \), then with this definition we find that the metric angle between the two vectors, \( \tilde{\Delta}_{0}^{(A)} \), is given by

\[
\tilde{\Delta}_{0}^{(A)} = \Gamma(\Delta_{0}^{(A)}). \tag{18}
\]

Combining Eqs. (16)-(18), we arrive at the conclusion that the azimuthal coordinates of the detector’s direction vector at \( z = \infty \) and at \( z = 0 \) (with respect to the orientation of the oven-generated vector) are related through the parallel-transport by

\[
\Delta_{\infty}^{(A)} = \Gamma(\Delta_{0}^{(A)}). \tag{19}
\]

Note that this equation does not depend on the details of the transport (dynamics) along the path connecting the detector and the oven, but only on the final mapping between the azimuthal coordinates at the starting point \( z = \infty \) and the end point \( z = 0 \).

We specify the mapping \( \Gamma \) via its inverse \( \Gamma^{-1} \) as follows

\[
\Gamma^{-1}(\Delta) = \begin{cases} 2\pi \int_{0}^{\Delta} d\xi \rho(\xi), & \text{for } \Delta \in [0, \pi), \\
-2\pi \int_{\Delta}^{0} d\xi \rho(\xi), & \text{for } \Delta \in [-\pi, 0), 
\end{cases} \tag{20}
\]

where \( \rho(\xi) \) is some parity-symmetric, \( \rho(-\xi) = \rho(\xi) = \rho(\pi - \xi) \), non-negative density of probability over the unit circle normalized so that \( \int_{-\pi}^{\pi} d\xi \rho(\xi) = 1 \). We further demand that \( \rho(\xi) \) fulfills all the requirements which would

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**FIG. 5.** The bipartite game played on a locally deformable manifold. The oven-generated vector locally curves space around it. The metric angle is depicted in the color-coded heat map. (a) Away from the oven-generated vector, at \( z = \infty \), space is flat and the metric angle can be identified with the azimuthal coordinates \( \tilde{\Delta}_{\infty} \). (b) At the point of detection, \( z = 0 \), the metric angle is no longer synonymous with the azimuthal coordinate, so the angle between the two detectors’ direction vectors is given by the bijective non-linear function \( \Gamma^{-1} \) acting on \( \theta \), the relative angle between the two detectors. (The observant reader will notice that in contrast to Fig. 2, here the azimuthal coordinates are defined with respect to the oven-generated vector.)
guarantee that $\Gamma^{-1}$ is bijective. Since by definition, $\Gamma^{-1}$ is anti-symmetric, $\Gamma^{-1}(-\Delta) = -\Gamma^{-1}(\Delta)$, $\Gamma$ is bijective and anti-symmetric as well. Therefore, in particular, $\Gamma(0) = 0$ and $\Gamma(\pm \pi) = \pm \pi$. Moreover, one can show that $\Gamma^{-1}$ preserves orthogonality, $\Gamma^{-1}(\pm \pi) = \pm \pi/2$ (and similarly for $\Gamma$). Following Eqs. (19) and (20) we obtain

$$\frac{1}{2\pi} d\Delta^{(A)}_0 = \rho(\Delta^{(A)}_\infty) d\Delta^{(A)}_\infty \text{.}$$

Equation (21) states that in a probabilistic model in which the oven generates random vectors with no preferred direction, the azimuthal coordinate $\Delta^{(A)}_0$ of the detector direction-vectors relative to the oven-generated vector at $z = 0$ will be distributed uniformly, while at $z = \infty$ the azimuthal coordinate $\Delta^{(A)}_\infty$ will be distributed with a density of probability $\rho(\Delta^{(A)}_\infty)$.

The azimuthal coordinates $\Delta^{(A)}_0$ and $\Delta^{(A)}_\infty$, can be given a physically intuitive interpretation as follows. The former can be understood as the ‘proper’ coordinates of the oven-generated vector. In these coordinates the detector’s direction-vectors appear uniformly distributed, since the direction of the sampled oven-generated vectors cannot have any preferred orientation. The latter can be understood as the proper coordinates of the detector’s direction-vector. In these coordinates the oven-generated vectors are distributed with a non-uniform distribution $\rho$ as a result of the deformation of the metric that the oven-generated vectors introduce.

As discussed in Sec. II, the binary response of the detectors is defined by the sign of the azimuthal coordinate of the oven-generated vector with respect to the detector direction-vector (at the point of the detection event, i.e., the origin $z = 0$). In particular, the binary response of Alice’s detector is then

$$S_A(-\Delta^{(A)}_0) = -\text{sign}(\Delta^{(A)}_0) = -\text{sign}(\Delta^{(A)}_\infty) \text{,}$$

where there is an additional minus sign in the above equation since $\Delta^{(A)}_0$ is defined to be the azimuthal coordinate of Alice’s detector direction-vector with respect to the oven-generated vector at $z = 0$, and moreover we used the equality $\text{sign}(\Delta^{(A)}_\infty) = \text{sign}(\Delta^{(A)}_\infty)$, since the function $\Gamma$ is monotonically increasing and $\Gamma(0) = 0$.

Of course, the discussion presented above for Alice’s detector similarly applies to Bob’s detector. Specifically, we may parameterize the path connecting the oven to Bob’s detector by $z \in (-\infty, 0]$ where the oven-generated vector’s location is at $z = 0$ as before and the location of Bob’s detector is initially at $z = -\infty$. Denoting the azimuthal coordinates of Bob’s detector with respect to the oven-generated vector at $z = 0$ and $z = -\infty$ by $\Delta^{(B)}_0$ and $\Delta^{(B)}_\infty$, respectively, and following the same arguments we detailed above, we arrive at

$$\Delta^{(B)}_\infty = \Gamma(\Delta^{(B)}_0) \text{,}$$

and

$$S_B(-\Delta^{(B)}_0) = -\text{sign}(\Delta^{(B)}_0) = -\text{sign}(\Delta^{(B)}_\infty) \text{.}$$

Following Condition 1 in Sec. II, our model should satisfy a zero-mean detection average for both Alice and Bob; that is,

$$\langle S_A(-\Delta^{(A)}_0) \rangle = 0 \text{ and } \langle S_B(-\Delta^{(B)}_0) \rangle = 0 \text{,}$$

for any choice of the detectors’ direction-vector. This condition can alternatively be written as

$$\langle S_A(-\Delta^{(A)}_0) \rangle = 0 \text{ and } \langle S_B(-\Delta^{(B)}_0) \rangle = 0 \text{.}$$

Equation (27) is indeed satisfied in our model since $\Delta^{(A)}_0$ and $\Delta^{(B)}_0$ are both distributed uniformly.

To obtain the correlations between the outcomes of the two detectors, we need to relate, similar to what we did in the previous section, the angle between Alice and Bob’s detectors $\theta$ to the azimuthal coordinates according to which the detection events are recorded. One could try to define this relation as $\Delta^{(A)}_\infty = \Delta^{(B)}_\infty + \theta$. However, this relation would not be consistent with our previous finding that $\Delta^{(A)}_\infty$ and $\Delta^{(B)}_\infty$ are distributed according to $\rho(\Delta^{(A)}_\infty)$ and $\rho(\Delta^{(B)}_\infty)$, respectively. In order to obtain a relation that complies with these findings and that is fully consistent with Eqs. (26) and (27), we begin by writing

$$\Delta^{(A)}_0 = \Delta^{(B)}_0 + C \text{,}$$

for some constant C that we now compute. We recall that the addition of azimuthal coordinates is done modulo $2\pi$ within the interval $[-\pi, \pi]$, so that Eq. (28) is consistent with variables $\Delta^{(A)}_0$ and $\Delta^{(B)}_0$ being both uniformly distributed over the circle and they comply with the zero-mean condition [Eqs. (26) and (27)]. Setting $C = \theta$ is also not consistent with the current model, as $\theta$ is the relative (metric) angle between $\Delta^{(A)}_0$ and $\Delta^{(B)}_0$ are the azimuthal coordinates in a regime where space is curved and hence the difference in azimuthal coordinates between two vectors does not coincide with the metric angle between them. To determine the value of C we rewrite Eq. (28) as

$$\Gamma^{-1}(\Delta^{(A)}_0) = \Gamma^{-1}(\Delta^{(B)}_\infty) + C \text{.}$$

Now, consider the case where the oven-generated vector is aligned with Bob’s detector vector, i.e., when $\Delta^{(B)}_\infty = 0$. In this case, $C = \Gamma^{-1}(\Delta^{(A)}_0)$, and moreover, since Alice’s detector has a relative angle $-\theta$ with respect to Bob’s detector, $\Delta^{(A)}_\infty = -\theta$. Therefore, we find that

$$\Delta^{(A)}_0 = \Delta^{(B)}_0 - \Gamma^{-1}(\theta) \text{,}$$

where the addition of coordinates is defined modulo $2\pi$ in the interval $[-\pi, \pi]$. Equation (30) can be equivalently expressed as

$$\Delta^{(A)}_\infty = \Gamma(\Gamma^{-1}(\Delta^{(B)}_\infty) - \Gamma^{-1}(\theta)) \text{.}$$
Since $\Gamma$ takes a uniformly distributed random variable in $[-\pi, \pi]$ to a zero-mean random variable distributed according to $\rho$, we conclude that Eq. (26) is fulfilled under Eq. (31) for any angle $\theta$, that is, for any choice of Alice and Bob’s detector orientations.

We can now obtain the correlation between Alice and Bob for our model. We already showed that $\Delta_0^{(A)}$ and $\Delta_0^{(B)}$ are distributed uniformly. The correlations between the two parties can then be obtained by noting that, following Eq. (30),

$$
\begin{align*}
\Delta_0^{(A)} &= -\Gamma^{-1}(\theta) \iff \Delta_0^{(B)} = 0, \\
\Delta_0^{(A)} &= \pm \pi - \Gamma^{-1}(\theta) \iff \Delta_0^{(B)} = \pm \pi,
\end{align*}
$$

i.e., the response of Bob’s detector changes when $\Delta_0^{(A)} = -\Gamma^{-1}(\theta)$ and $\Delta_0^{(A)} = \pm \pi - \Gamma^{-1}(\theta)$. We can therefore write the expected correlation between the outcomes of the two parties, Eq. (6), after following the same steps taken in the previous section [c.f., Eq. (10)], as

$$
E_{AB}^{(\text{deform})}(\theta) = 1 - 2 \pi |\Gamma^{-1}(\theta)| = 1 - 4 \int_0^\theta d\xi \rho(\xi). \tag{33}
$$

Note that we were careful to express the correlations as a function of the relative angle $\theta$ between the two detectors, as defined by the bipartite experiment. As desired, the bipartite correlations satisfy

$$
\begin{align*}
E_{AB}^{(\text{deform})}(0) &= 1, \\
E_{AB}^{(\text{deform})}(\pm \pi/2) &= 0, \\
E_{AB}^{(\text{deform})}(\pm \pi) &= -1.
\end{align*}
$$

Similar to the correlations on a flat manifold.

Note also that by properly defining $\rho(\Delta)$, our model is capable of reproducing any bipartite correlation function as long as it fulfills the symmetry constraint $E_{AB}^{(\text{deform})}(\theta) = E_{AB}^{(\text{deform})}(-\theta) = -E_{AB}^{(\text{deform})}(\pi - \theta)$. For example, by considering $\rho(\xi) = 1/2\pi$, that is, a uniform distribution over the unit circle, we recover the bipartite correlations on the flat manifold, which was considered in Sec. III and which satisfies the original CHSH-Bell inequality, Eq. (14). Alternatively, by considering the probability density distribution

$$
\rho(\xi) = \frac{1}{4} |\sin(\xi)|, \tag{35}
$$

we are led to bipartite correlations of the form

$$
E_{AB}^{(\text{deform})}(\theta) = \cos(\theta), \tag{36}
$$

which is the correlation dependence predicted by quantum mechanics for the CHSH-Bell experiment with pairs of maximally entangled particles. In particular, for the specific choice of relative angles, $-\theta_{11} = \theta_{12} = \theta_{21} = \theta_{22}/3 = \pi/4$, we obtain that

$$
\left| E_{1,1}^{(\text{deform})} + E_{1,2}^{(\text{deform})} + E_{2,1}^{(\text{deform})} - E_{2,2}^{(\text{deform})} \right| = 2\sqrt{2}, \tag{37}
$$

in violation of the CHSH-Bell inequality on a flat manifold, Eq. (14), where $E_{ij}^{(\text{deform})} = E_{AB}^{(\text{deform})}(\theta_{ij})$ and $\theta_{ij}$ is the angle between $\tilde{a}_i$ and $\tilde{b}_j$.

V. DISCUSSION

The violation of the CHSH-Bell inequality, Eq. (14), in the model introduced above implies that the inequality does not apply for deformable manifolds. As noted in Sec III, the CHSH-Bell inequality is derived under the premise of an equivalence relation between vectors at different locations on the manifold on which the game is played, which allows one to refer to a vector as a single entity independently of the path along which it is transported. While this assumption is well justified in flat space, we now show that in the current model, where the oven-generated vectors locally curve the manifold on which the game is played, this equivalence does not hold. Specifically, we show that in the current model one cannot use ‘the same’ oven-generated vector to define uniquely and unambiguously four detection events (corresponding to the four settings of the detectors) as one does in the derivation of the CHSH-Bell inequality, c.f., Eq. (13).

For concreteness, let us consider the specific choice of relative angles $-\theta_{11} = \theta_{12} = \theta_{21} = \theta_{22}/3 = \pi/4$ that was discussed above. Consider a detection event where Alice and Bob’s detectors are described by the azimuthal coordinates $\Delta_0^{(A_2)}$ and $\Delta_0^{(B_1)}$ with respect to the oven-generated vector. According to Eq. (30), these azimuthal coordinates are related by

$$
\Delta_0^{(A_2)} = \Delta_0^{(B_1)} - \Gamma^{-1}(\pi/4). \tag{38}
$$

Now suppose that Alice switches her detector’s setting from orientation $\tilde{a}_2$ to $\tilde{a}_1$. Since $\Delta_0^{(A_1)} = \Delta_0^{(B_1)} + \Gamma^{-1}(\pi/4)$, we obtain that

$$
\Delta_0^{(A_2)} = \Delta_0^{(A_1)} - 2\Gamma^{-1}(\pi/4). \tag{39}
$$

Let Bob now change his detector’s orientation from $\tilde{b}_1$ to $\tilde{b}_2$. This leads us to the relation

$$
\Delta_0^{(A_2)} = \Delta_0^{(B_2)} - 3\Gamma^{-1}(\pi/4). \tag{40}
$$

Finally, if Alice changes her detector’s orientation back to $\tilde{a}_2$, we arrive at

$$
\Delta_0^{(A_2)} = \Delta_0^{(A_2)} - 3\Gamma^{-1}(\pi/4) + \Gamma^{-1}(3\pi/4) \tag{41}
$$

$$
= \Delta_0^{(A_2)} - \delta,
$$

where $\delta$ is used here to denote the new azimuthal coordinate after the sequence of mappings has been applied, and $\delta = 3\Gamma^{-1}(\pi/4) - \Gamma^{-1}(3\pi/4)$. Of course $\delta$ will be nonzero in the general case.

Equation (41), also illustrated in Fig. 6, implies that the four concatenated joint two-detector settings considered in the CHSH-Bell inequality [c.f., Eq. (13)] can no
The four concatenated settings, \((\hat{a}_2, \hat{b}_1), (\hat{b}_1, \hat{a}_1), (\hat{a}_1, \hat{b}_2),\) and \((\hat{b}_2, \hat{a}_2)\) it will accumulate a geometric phase \(\delta\), as schematically shown in the figure. From the point of view of the reference frames attached to the detector direction-vectors, the oven-generated vectors acquire a geometric phase \(\delta\) over the closed concatenation of detector settings that enter into the CHSH-Bell inequality. This non-zero geometric phase is however physically permissible because it preserves the statistical distribution \(\rho\) of the oven-generated vectors over the unit circle, and at each round of the experiment, only one of the four possible settings of the two detectors is experimentally accessible.

From the point of view of the reference frame attached to the oven-generated vector, Eq. (41) reveals that in contrast to the case of the flat manifold where one may define an absolute frame of reference with respect to which all vectors on the manifold may be measured against, in the present model, the deformation of the manifold spontaneously breaks the rotational symmetry associated with joint absolute rotations of the two detectors. Since the absolute orientation of the detectors with respect to the oven-generated vectors acquires a global geometric phase through cyclic coordinate transformations, it is not a well-defined physical quantity. Rather, it is a spurious ‘gauge’ degree of freedom that cannot play any role in the physical predictions of the model, as dictated by Condition 2. We note that this spontaneous breaking of rotational symmetry is caused by the information carriers and, therefore, in their absence the symmetry is of course present. Hence, the parties retain the usual symmetries needed to define the settings of the detectors’ direction-vectors prior to the start of the game.

In the presence of a pair of oven-generated vectors traversing the manifold toward the two detectors, breaking the rotational symmetry associated with joint absolute rotations of the two detectors, each party is left with its own set of coordinates, related to each other via Eq. (31). As already noted, all these sets of coordinates are symmetrically equivalent to each other as none of them is preferred over any other.

VI. SUMMARY AND CONCLUSIONS

In this work, we studied the bipartite correlations that emerge in two-party Bell-type games. We showed that by allowing the information carriers shared by the parties (our oven-generated vectors) to locally deform the curvature of the manifold on which the game is played, spontaneously breaking the isotropy of space, classical correlations that violate the usual Bell inequalities associated with the game may appear. We examined a specific example of a local and deterministic Bell-type game in which the bipartite correlations match those predicted by quantum mechanics for actual CHSH-Bell experiments. We showed that nonetheless our model does not contradict CHSH-Bell’s theorem [10] since a setup in which the manifold is deformed by the shared information carriers does not satisfy the assumptions underlying the theorem.

Our model is founded upon the simple realization that the bipartite correlations predicted by quantum mechanics for Bell states depend only on the relative orientation between the two detectors involved in the game, while the angle associated with joint absolute rotations (with respect to the orientation of the information carriers) is a spurious gauge degree of freedom that plays no role in the prediction of physical observables. This gauge degree of freedom allows, in our model, for the appearance of a non-zero geometric phase through cyclically concatenated coordinate transformations, which in turn gives rise to bipartite correlations that evade the constraints imposed by Bell’s theorem. Our local and deterministic model thus proposes a physically intuitive resolution to the EPR paradox and the correlations associated with quantum entanglement.

We note that while we have formulated our model on a two-dimensional surface, our results could have similarly been derived for a higher-dimensional manifold where the planes in which the oven-generated and detector vectors reside are orthogonal to the path along which they are connected. The latter setup would have more closely resembled actual CHSH-Bell experiments wherein the paired photons shared between the parties travel along paths that are perpendicular to the polarization plane in which the information is encoded.

One may wonder whether a ‘static’ curved manifold would have sufficed to reproduce the results of the model
presented above. In the model studied here, in each round of the experiment, the oven-generated vector spontaneously breaks the isotropy of the manifold, yet on average, over the course of many rounds, the symmetry of the manifold is recovered. This important feature would be absent from any model with static curvature as the latter would either have a broken rotational symmetry at each round of the game (as well as after averaging over many rounds) or – if the static curved manifold does not break isotropy – not lead to a violation of the CHSH-Bell inequality.

We believe our work also opens the door to novel interesting research avenues that have the potential to deepen our understanding of physical reality. The framework presented here offers a physically intuitive explanation to the experimentally confirmed violations of the renowned Bell inequalities. Notwithstanding, our model does not single out the ‘choice of nature’ for the bipartite correlations predicted by quantum mechanics. Expressed differently, while our mathematical framework allows the bipartite correlations to go beyond the classical limit, it also allows to reach the maximal algebraically allowed violations of the inequalities and, hence, it does not explain the origin of the Tsirelson bound \cite{18} imposed by the quantum formalism. We leave the exploration of this topic to future work.

The formalism presented in this study generalizes to all other Bell-type experiments with pairs of maximally entangled particles. In future studies, we plan to explore additional Bell-type games that involve the correlations encountered in three-particle systems such as Greenberger–Horne–Zeilinger (GHZ) Bell games. Also worth exploring are experiments that may be sensitive to the curvature of space introduced by the information carriers in actual Bell experiments, as posited in our model.

The relation between the framework developed here and the fundamental principles underlying General Relativity is of great interest as well. Thus another line of research that we believe is worth pursuing is the formulation of our mathematical framework in pseudo-Riemannian Lorentzian manifolds, relating our derivation above directly to the general theory of relativity.

Finally, this work suggests that, very generally, when formulating a statistical model of a physical system involving gauge degrees of freedom, one should take into account the possibility that the system is prone to having the symmetries of its macroscopic phase space spontaneously broken, allowing for the occurrence of geometric phases associated with the gauge degrees of freedom.

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