The dual of pure non-Abelian lattice gauge theory as a spin foam model

Robert Oeckl* and Hendryk Pfeiffer†

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences, Cambridge CB3 0WA, UK

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Abstract

We derive an exact duality transformation for pure non-Abelian gauge theory regularized on a lattice. The duality transformation can be applied to gauge theory with an arbitrary compact Lie group $G$ as the gauge group and on Euclidean space-time lattices of dimension $d \geq 2$. It maps the partition function as well as the expectation values of generalized non-Abelian Wilson loops (spin networks) to expressions involving only finite-dimensional unitary representations, intertwiners and characters of $G$. In particular, all group integrations are explicitly performed. The transformation maps the strong coupling regime of non-Abelian gauge theory to the weak coupling regime of the dual model. This dual model is a system in statistical mechanics whose configurations are spin foams on the lattice.

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1 Introduction

Besides the electric-magnetic duality of the vacuum Maxwell equations, the first example of a duality transformation relating the strong coupling regime of one field theoretic system with the weak coupling regime of the same or another system was probably the Kramers-Wannier transformation for the Ising model \cite{1}. This transformation was generalized to a

* e-mail: R.Oeckl@damtp.cam.ac.uk
† e-mail: H.Pfeiffer@damtp.cam.ac.uk
wide class of Abelian lattice models (spin models, gauge theories and their higher rank tensor generalizations) in any Euclidean space-time dimension. For a review see e.g. [2].

In this paper, we generalize the transformation to the case of lattice gauge theory with a non-Abelian gauge group $G$ (our proofs are valid for compact Lie groups and finite groups). The resulting dual model generalizes the well-known results for the Abelian case and is described in terms of the finite-dimensional unitary representations of $G$, their representation morphisms (intertwiners) and the character expansion of the Boltzmann weight. In particular, all group integrations are explicitly performed.

The method we use is the Peter-Weyl decomposition of the algebra $C_{\text{alg}}(G)$ of representation functions of $G$, the ‘algebraic functions’ on $G$. This decomposition can be viewed as a generalization of Fourier transformation to functions on a compact non-Abelian Lie group. It is convenient to exploit the Hopf algebra structure of $C_{\text{alg}}(G)$ and to employ a purely algebraic description of the Haar measure. The duality transformation follows the lines of the well-known Abelian case, but some attention and geometric intuition is necessary to make the generalized gauge constraint appear in a local form in the dual model.

The duality transformation establishes the equality of the partition function and the expectation value of the non-Abelian Wilson loop (the generic gauge invariant expression which is given by a spin network) with their corresponding purely algebraic expressions in the dual model. The dual model is found to have a Boltzmann weight of such a form that the strong coupling regime of non-Abelian lattice gauge theory is mapped to the weak coupling regime of the dual model. In addition, the strong coupling expansion can be applied to this reformulation of non-Abelian gauge theory in a systematic way.

The duality relation is stated in Theorems 3.1 and 3.3 which form the main result of this paper. The dual model is a system in statistical mechanics whose configurations are spin foams on the lattice. These configurations are assigned Boltzmann weights and are subject to certain constraints. Spin foams have been introduced in the study of quantum gravity, see e.g. [3–6] and the recent introductory article [7]. The configurations of our dual model are closed spin foams on the lattice according to the definition given in [6]. The transformation thus provides an explicit example for the relation of lattice gauge theory with a particular spin foam model.

The dual model reduces to the known results for non-Abelian gauge theory in 2 dimensions where the partition function is particularly simple, as well as to the known results for Abelian lattice gauge theory in arbitrary dimension, i.e. in the cases $G = U(1)$, $\mathbb{Z}$ and $\mathbb{Z}_n$.

Whereas the duality transformation in the Abelian case is known to work in a similar way for spin models, gauge theories and higher rank tensor models on the lattice [2], this is not the case for the non-Abelian generalization. Even though it can be applied to spin models in a straightforward way, the resulting ‘gauge’ constraint cannot be cast in a local form.

The motivation for deriving a dual description of non-Abelian lattice gauge theory arises from conceptual issues such as the ‘dual superconductor’ picture of confinement and from rig-
orous studies in the framework of constructive quantum field theory as well as from technical
and numerical problems in lattice gauge theory.

At present only a few ways of studying gauge theories in their strong coupling regime
are known — in particular if there are no additional symmetries like supersymmetry. In
the famous paper by Wilson \[8\], the lattice formulation of $U(1)$ gauge theory was used in
conjunction with the high temperature expansion which is known from statistical mechanics
and which plays the role of a strong coupling expansion.

The duality transformation for Abelian lattice gauge theories (see e.g. \[2\]) can be seen on
the one hand as a result of attempts to make this strong coupling expansion systematic. On
the other hand there is the picture of ‘dual superconductivity’ as an explanation of confinement
going back to ideas of t’Hooft and Mandelstam, see e.g. \[3,10\]. Whereas in a superconductor
electrically charged quasi-particles condense and force the magnetic flux into quantized tubes,
the picture is that in a gauge theory with confinement, magnetic monopoles condense. This
leads to the formation of electric flux tubes which are responsible for the linearity of the static
potential between opposite external electric charges.

In this picture the magnetic monopoles appear as collective excitations of the gauge theory.
They are found to be quantized topological defects \[11,12\]. The Abelian duality transfor-
mation allows one to spot these topological degrees of freedom. All group integrations are
performed, and the partition function of Abelian lattice gauge theory is rewritten in new
variables such that the topological degrees of freedom have the form of ordinary expectation
values of the dual fields.

In the Abelian case, the lattice approach to gauge theories in conjunction with the du-
ality transformation and related techniques has lead to a number of remarkable results. We
mention the existence of a phase transition in $U(1)$ lattice gauge theory in $d = 4$ \[13,14\],
the existence of world-lines of magnetic monopoles in the same model which behave like
infra-particles \[11,12\], the fact that these are responsible for the phase transition by conden-
sation of magnetic monopoles, and finally the absence of a deconfinement phase transition
in $d = 3$ \[15\]. The picture of dual superconductivity leading to confinement in the Abelian
case is well established, see e.g. the study of the monopole degrees of freedom in Monte-Carlo
simulations \[16\] and their properties at the deconfining phase transition \[17\] of $U(1)$ lattice
gauge theory in $d = 4$.

Although there are strong conjectures both from lattice studies of QCD and from results
in supersymmetric Yang-Mills theory that confinement in non-Abelian gauge theories can be
described in a similar way by dual superconductivity, no analogous approach is available for
non-Abelian gauge theory. The exact duality transformation presented in this paper is meant
to be a first step in this direction.

Finally, we comment on the relations of this work with other approaches. Firstly, in the
Abelian case, the dual model is again a gauge theory if certain cohomologies of the space-time
lattice are trivial. The case of general topology is studied in \[18\]. Furthermore, for $SU(2)$
lattice gauge theory, there are results by explicit computations which relate this theory to
certain simplex models of gravity \cite{13,20}.

Furthermore, there has been an interesting categorial approach to models which could be
dual to non-Abelian gauge theory \cite{21}. It uses the same colouring of links and plaquettes
of the lattice with representations and intertwiners, respectively, as our dual model does on
the dual lattice in $d = 3$ and seems to enjoy many close similarities. However, the gauge
degrees of freedom proposed in \cite{21} are not in a straightforward way a symmetry of the
dually transformed model as derived in this paper. Nevertheless, the ideas developed there
might prove useful for a further study of the dual model.

We thank M.B. Halpern for bringing the paper \cite{22} to our attention. In \cite{22} a plaquette
formulation of lattice gauge theory is derived emphasizing the non-Abelian Bianchi identity.
Using the mathematical methods that we describe in Sections 2.1 to 2.4, it can be shown that
the approach of \cite{22} can be extended to a full duality transformation similar to the treatment
in our Section 3. The result would be the same as given in Theorem 3.1. Finally, we thank a
referee for drawing our attention to the paper \cite{23} where a duality transformation of classical
Yang-Mills theory in continuous space-time based on the loop approach is suggested.

This paper is organized as follows. In Section 2, we recall all definitions and structures
which are needed to present the duality transformation. In particular these are the structure
of the Hopf algebra of representation functions $C_{\text{alg}}(G)$ of $G$ and the lattice formulation of
gauge theories.

In Section 3, we present the duality transformation in detail first for the partition function,
and then for the non-Abelian generalizations of the Wilson loop. We specialize our result to
the known cases of non-Abelian lattice gauge theory in 2 dimensions and to Abelian lattice
gauge theory in arbitrary dimension in Section 4.

Finally, in Section 5 we indicate how the dual formulation can be used in conjunction with
strong coupling expansion techniques and discuss open questions and directions for further
research.

2 Preliminaries

2.1 The Hopf algebra of representation functions

In this section we collect definitions and basic statements related to the algebra of representation
functions $C_{\text{alg}}(G)$ of $G$. These and the results presented in the next section about the
Peter-Weyl decomposition and the Peter-Weyl theorem are basically textbook knowledge,
see e.g. \cite{24,25}. We recall the basic facts to fix our notation and present a purely algebraic
treatment of the relevant results which we have not found elsewhere in this form.

Let $G$ be a finite group or a compact Lie group. We denote finite-dimensional complex
vector spaces on which $G$ is represented by $V_\rho$ and by $\rho: G \to \text{Aut} V_\rho$ the corresponding group
2.1 The Hopf algebra of representation functions

homomorphisms. Since each finite-dimensional complex representation of $G$ is equivalent to a unitary representation, we select a set $\tilde{\mathcal{R}}$ containing one unitary representation of $G$ for each equivalence class of finite-dimensional representations. The tensor product, the direct sum and taking the dual are supposed to be closed operations on this set. This amounts to a particular choice of representation isomorphisms $\rho_1 \otimes \rho_2 \leftrightarrow \rho_3$ etc., $\rho_j \in \tilde{\mathcal{R}}$, which is implicit in our formulas. We furthermore denote by $\mathcal{R} \subseteq \tilde{\mathcal{R}}$ the subset of irreducible representations.

If $\rho \in \tilde{\mathcal{R}}$, we write $\rho^*$ for the dual representation and denote the dual vector space of $V_\rho$ by $V_\rho^*$. The dual representation is given by $\rho^*: G \mapsto \text{Aut} V_\rho^*$, where

$$\rho^*(g): V_\rho^* \rightarrow V_\rho^*, \quad \eta \mapsto \eta \circ \rho(g^{-1}), \quad (2.1)$$

i.e. $(\rho^*(g)\eta)(v) = \eta(\rho(g^{-1})v)$ for all $v \in V_\rho$.

For the unitary representations $V_\rho, \rho \in \tilde{\mathcal{R}}$, we have standard (sesquilinear) scalar products $\langle \cdot; \cdot \rangle$ and orthonormal bases $(v_j)$ in such a way that the basis $(v_j)$ of $V_\rho$ is dual to the basis $(\eta^j)$ of $V_\rho^*$, i.e. $\eta^j(v_k) = \delta_{jk}$. This means that duality is given by the scalar product,

$$\langle v_j; v_k \rangle = \eta^j(v_k), \quad \langle \eta^j; \eta^k \rangle = \eta^k(v_j), \quad 1 \leq j, k \leq \dim V_\rho. \quad (2.2)$$

There exists a one-dimensional ‘trivial’ representation of $G$ which we denote by $V[1] \cong \mathbb{C}$.

The functions

$$t_{\eta,v}^{(\rho)}: G \rightarrow \mathbb{C}, \quad g \mapsto \eta(\rho(g)v), \quad (2.3)$$

where $\rho \in \tilde{\mathcal{R}}, v \in V_\rho$ and $\eta \in V_\rho^*$, are called the representation functions of $G$. They form a commutative and associative unital algebra over $\mathbb{C}$,

$$\mathcal{C}_{\text{alg}}(G) := \{ t_{\eta,v}^{(\rho)}: \rho \in \tilde{\mathcal{R}}, v \in V_\rho, \eta \in V_\rho^* \}, \quad (2.4)$$

whose operations are given by

$$(t_{\eta,v}^{(\rho)} + t_{\eta,w}^{(\sigma)})(g) := t_{\eta+\sigma,v+w}^{(\rho\otimes\sigma)}(g), \quad (2.5a)$$

$$(t_{\eta,v}^{(\rho)} \cdot t_{\eta,w}^{(\sigma)})(g) := t_{\eta\sigma,v\otimes w}^{(\rho\otimes\sigma)}(g), \quad (2.5b)$$

where $\rho, \sigma \in \tilde{\mathcal{R}}$ and $v \in V_\rho, w \in V_\sigma, \eta \in V_\rho^*, \vartheta \in V_\sigma^*$ and $g \in G$. The zero element of $\mathcal{C}_{\text{alg}}(G)$ is given by $t_{0,0}^{[1]}(g) = 0$ and its unit element by $t_{1,0}^{[1]}(g) = 1$ where we have normalized $\eta(v) = 1$.

The algebra $\mathcal{C}_{\text{alg}}(G)$ is furthermore equipped with a Hopf algebra structure with the coproduct $\Delta: \mathcal{C}_{\text{alg}}(G) \rightarrow \mathcal{C}_{\text{alg}}(G) \otimes \mathcal{C}_{\text{alg}}(G) \cong \mathcal{C}_{\text{alg}}(G \times G)$, the co-unit $\varepsilon: \mathcal{C}_{\text{alg}}(G) \rightarrow \mathbb{C}$ and the antipode $S: \mathcal{C}_{\text{alg}}(G) \rightarrow \mathcal{C}_{\text{alg}}(G)$ which are defined by

$$\Delta t_{\eta,v}^{(\rho)}(g, h) := t_{\eta,v}^{(\rho)}(g \cdot h), \quad (2.6a)$$

$$\varepsilon t_{\eta,v}^{(\rho)} := t_{\eta,v}^{(\rho)}(1), \quad (2.6b)$$

$$St_{\eta,v}^{(\rho)}(g) := t_{\eta,v}^{(\rho)}(g^{-1}), \quad (2.6c)$$
where $\rho \in \tilde{R}$ and $v \in V_\rho$, $\eta \in V_\rho^*$ and $g, h \in G$.

Since $G$ is a finite group or a compact Lie group, all finite-dimensional representations of $G$ are completely reducible. Moreover, all representations of $G \times G$ are tensor products of representations of $G$ such that we have an isomorphism of algebras $C_{\text{alg}}(G \times G) \cong C_{\text{alg}}(G) \otimes C_{\text{alg}}(G)$ which is used in the definition of the coproduct. This tensor product is algebraic, and there is no need for a topology or a completion of the tensor product at this point.

In the standard orthonormal bases, the representation functions are given by the coefficients of representation matrices,

\[
t_{\rho}^{(\rho)}(g)_{mn}(v) = \eta^m(\rho(g)v_n) = \langle v_m; \rho(g)v_n \rangle = \rho(g)_{mn},
\]

such that the coproduct corresponds to the matrix product,

\[
\Delta t_{\rho}^{(\rho)}(g, h) = \sum_{j=1}^{\dim V_{\rho}} t_{\rho}^{(\rho)}(g)_{mj} t_{\rho}^{(\rho)}(h)_{jn},
\]

while the antipode refers to the inverse matrix, $S t_{\rho}^{(\rho)}(g) = (\rho(g)^{-1})_{mn}$, and the co-unit describes the coefficients of the unit matrix, $\varepsilon t_{\rho}^{(\rho)} = \delta_{mn}$. Furthermore, the antipode relates a representation with its dual,

\[
S t_{\rho}^{(\rho)}(g) = \eta^m(\rho(g)^{-1}v_n) = \langle \rho^*(g) \eta^m(v_n) \rangle = \langle \eta^m; \rho^*(g) \eta^m \rangle = t_{\rho}^{(\rho)}(g),
\]

which is just the conjugate representation because on the other hand

\[
S t_{\rho}^{(\rho)}(g) = \langle v_m; \rho(g^{-1})v_n \rangle = \langle \rho(g)v_m; v_n \rangle = \overline{\langle v_n; \rho(g)v_m \rangle} = \overline{t_{\rho}^{(\rho)}(g)}.
\]

Here the bar denotes complex conjugation.

### 2.2 Peter-Weyl decomposition and theorem

The structure of the algebra $C_{\text{alg}}(G)$ can be understood if $C_{\text{alg}}(G)$ is considered as a representation of $G \times G$ by combined left and right translation of the function argument,

\[
(G \times G) \times C_{\text{alg}}(G) \rightarrow C_{\text{alg}}(G), \quad (g_1, g_2, f) \mapsto (h \mapsto f(g_1^{-1}hg_2)).
\]

It can then be decomposed into its irreducible components as a representation of $G \times G$:

**Theorem 2.1 (Peter-Weyl Decomposition).** Let $G$ be a finite group or a compact Lie group.

1. There is an isomorphism

\[
C_{\text{alg}}(G) \cong_{G \times G} \bigoplus_{\rho \in \tilde{R}} (V_\rho^* \otimes V_\rho),
\]

of representations of $G \times G$. Here the direct sum is over one unitary representative of each equivalence class of finite-dimensional irreducible representations of $G$. The direct summands $V_\rho^* \otimes V_\rho$ are irreducible as representations of $G \times G$. 
2.2 Peter-Weyl decomposition and theorem

2. The direct sum in (2.12) is orthogonal with respect to the $L^2$ scalar product on $C_{\text{alg}}(G)$ which is formed using the Haar measure on $G$ on the left hand side, and using the standard scalar products on the right hand side, namely

$$\langle t^{(\rho)}_{\eta,v}, t^{(\sigma)}_{\vartheta,w} \rangle_{L^2} = \int_G t^{(\rho)}_{\eta,v}(g) \cdot t^{(\sigma)}_{\vartheta,w}(g) \, dg = \frac{1}{\dim V_{\rho}} \delta_{\rho\sigma} \langle \eta; \vartheta \rangle \langle v; w \rangle,$$  \hspace{1cm} (2.13)

where $\rho, \sigma \in \mathcal{R}$ are irreducible. The Haar measure is denoted by $\int_G dg$ and normalized such that $\int_G dg = 1$.

**Remark 2.2.**  
1. If $G$ is finite, the Haar measure coincides with the normalized summation over all group elements.

2. The decomposition (2.12) directly corresponds to our notation of the representation functions $t^{(\rho)}_{nm}(g)$ if $\rho \in \mathcal{R}$ is irreducible.

3. Each representation function $f \in C_{\text{alg}}(G)$ has a decomposition according to (2.14),

$$f = \sum_{\rho \in \mathcal{R}} f_{\rho},$$  \hspace{1cm} (2.14)

such that we find for the $L^2$-norm

$$||f||^2_{L^2} = \sum_{\rho \in \mathcal{R}} \frac{1}{\dim V_{\rho}} ||f_{\rho}||^2,$$  \hspace{1cm} (2.15)

where $f_{\rho} \in V^*_{\rho} \otimes V_{\rho}$, $\rho \in \mathcal{R}$, and all except finitely many $f_{\rho}$ are zero. Here $||f_{\rho}||^2$ is the trace norm for the finite-dimensional space $V^*_{\rho} \otimes V_{\rho} \cong \text{End} V_{\rho}$.

The analytical aspects of $C_{\text{alg}}(G)$ can now be stated.

**Theorem 2.3 (Peter-Weyl Theorem).** Let $G$ be a compact Lie group. Then $C_{\text{alg}}(G)$ is dense in $L^2(G)$.

**Remark 2.4.**  
1. We use the Peter-Weyl theorem to complete $C_{\text{alg}}(G)$ with respect to the $L^2$ norm to $L^2(G)$. Functions $f \in L^2(G)$ then correspond to square summable series in (2.14). These series are thus invariant under a reordering of summands, and their limits commute with group integrations. We will make use of these invariances in the duality transformation.

2. If $G$ is a finite group, $C_{\text{alg}}(G)$ is a finite-dimensional vector space such that the corresponding results hold trivially.
2.3 Character decomposition

If \( G \) is a finite group or a compact Lie group, the characters \( \chi^{(\rho)} : G \to \mathbb{C} \) associated with the finite-dimensional unitary representations \( \rho \in \widehat{R} \) of \( G \) are obtained from the representation functions by

\[
\chi^{(\rho)} := \dim V_{\rho} \sum_{j=1}^{\dim V_{\rho}} t_{jj}^{(\rho)}. \tag{2.16}
\]

Each class function \( f \in C_{\text{alg}}(g) \) has a character decomposition

\[
f(g) = \sum_{\rho \in \mathcal{R}} \chi^{(\rho)}(g) \hat{f}_\rho, \quad \text{where} \quad \hat{f}_\rho = \dim V_{\rho} \int_G \chi^{(\rho)}(g) f(g) \, dg. \tag{2.17}
\]

The completion of \( C_{\text{alg}}(G) \) to \( L^2(G) \) is compatible with this decomposition.

2.4 Projector description of the Haar measure

For the duality transformation, it is important to understand the Haar measure on \( G \) in the picture of the Peter-Weyl decomposition \( (2.12) \). We describe the Haar measure in terms of projectors.

**Proposition 2.5.** Let \( G \) be a finite group or a compact Lie group and \( \rho \in \widehat{R} \) be a finite-dimensional unitary representation of \( G \) with the orthogonal decomposition

\[
V_{\rho} \cong \bigoplus_{j=1}^{k} V_{\tau_j}, \quad \tau_j \in \mathcal{R}, k \in \mathbb{N}, \tag{2.18}
\]

into irreducible components \( \tau_j \). Let \( P^{(j)} : V_{\rho} \to V_{\tau_j} \subseteq V_{\rho} \) be the \( G \)-invariant orthogonal projectors associated with the above decomposition. Assume that precisely the first \( \ell \) components \( \tau_1, \ldots, \tau_\ell \), \( 0 \leq \ell \leq k \), are equivalent with the trivial representation. Then the Haar measure of a representation function \( t_{mn}^{(\rho)} \), \( 1 \leq m, n \leq \dim V_{\rho} \), is given by

\[
\int_G t_{mn}^{(\rho)}(g) \, dg = \sum_{j=1}^{\ell} \langle v_m; P^{(j)} v_n \rangle = \sum_{j=1}^{\ell} \langle P^{(j)} v_m; P^{(j)} v_n \rangle = \sum_{j=1}^{\ell} P^{(j)}_m P^{(j)}_n, \tag{2.19}
\]

where \( P^{(j)}_m = \eta(P^{(j)} v_m) \) denotes the matrix elements of the \( j \)-th projector. Here \( \eta \in V_{\rho}^* \) is the normalized linear form which is zero everywhere except on the one-dimensional sub-spaces \( V_{\tau_j} \subseteq V_{\rho} \), \( 1 \leq j \leq \ell \).

**Proof.** The representation function is Peter-Weyl decomposed by inserting \( 1 = \sum_{j=1}^{k} P^{(j)} \) twice into the right hand side of \( t_{mn}^{(\rho)}(g) = \langle v_m; \rho(g) v_n \rangle \). We use hermiticity \( P^{(j)} = P^{(j)} \), \( G \)-invariance \( [P^{(j)}, \rho(g)] = 0 \) and transversality \( P^{(i)} P^{(j)} = \delta_{ij} P^{(j)} \) to obtain

\[
\int_G t_{mn}^{(\rho)}(g) \, dg = \sum_{j=1}^{k} \int_G \langle P^{(j)} v_m; \rho(g) P^{(j)} v_n \rangle \, dg. \tag{2.20}
\]
Since the Haar measure is bi-invariant, all terms vanish except those corresponding to the $\tau_j$, $1 \leq j \leq \ell$, which are equivalent to the trivial representation.

### 2.5 The Lattice formulation of non-Abelian gauge theories

The purpose of this section is to fix a notation in which we can write down the partition function and expectation values of non-Abelian lattice gauge theory and which is suitable to formulate the duality transformation. For all other issues we refer the reader to the standard textbooks on lattice gauge theory, e.g. [26, 27], and references therein.

We consider a regular hyper-cubic lattice corresponding to an Euclidean space-time of dimension $d \geq 2$. The lattice points (vertices) are denoted by tuples of integer numbers

$$\Lambda^0 := \{(i_1, \ldots, i_d) \in \mathbb{Z}^d : \ i_\mu \in \{1, \ldots, N_\mu\}\},$$

(2.21)

where the lattice is of size $N_\mu$ in the $\mu$-th dimension. We denote the unit vectors along the lattice axes by

$$\hat{\mu} := (0, \ldots, 0, \underbrace{1}_{\mu}, 0, \ldots, 0), \quad 1 \leq \mu \leq d.$$  

(2.22)

Thus $i + \hat{\mu}$ refers to the neighbour of the point $i$ in the direction $\mu$. We choose periodic (i.e. toroidal) boundary conditions and identify $i \pm N_\mu \cdot \hat{\mu} \equiv i$ for all $\mu \in \{1, \ldots, d\}$. It is crucial for the existence of various integrals that we work on a finite lattice. Periodic boundary conditions are the most convenient choice for our purpose. The non-trivial homologies introduced by the periodic boundary conditions do not play any role for the duality transformation in the form presented below.

The set of all links (edges) is called $\Lambda^1$, the set of all plaquettes (faces, squares) $\Lambda^2$, and more generally the set of all $k$-cells $\Lambda^k$. These are specified by

$$\Lambda^k := \{(i, \mu_1, \ldots, \mu_k) : \ i \in \Lambda^0, 1 \leq \mu_1 < \cdots < \mu_k \leq d\}.$$  

(2.23)

In particular, the sets $\Lambda^k$, $0 \leq k \leq d$, are all finite. The $k$-cells are considered unoriented, e.g. we do not want to distinguish the plaquette $(i, \mu, \nu)$ from $(i, \nu, \mu)$. In our notation both are represented in the standard way $(i, \mu, \nu)$ where $\mu < \nu$.

The configurations of lattice gauge theory are the maps

$$g : \Lambda^1 \to G, \quad (i, \mu) \mapsto g_{i\mu},$$

(2.24)

which assign a group element $g_{i\mu} \in G$ to each link $(i, \mu) \in \Lambda^1$ of the lattice. These group elements correspond to the parallel transports of the gauge connection along the links.

The path integral measure depends on these configurations only via the plaquette product (see also Figure 1),

$$dg : \Lambda^2 \to G, \quad (i, \mu, \nu) \mapsto dg_{i\mu\nu} := g_{i\mu} \cdot g_{i+\hat{\mu},\nu} \cdot g_{i+\hat{\nu},\mu}^{-1} \cdot g_{i,\nu}^{-1},$$

(2.25)
which is the path ordered product of the link variables around a given plaquette \((i, \mu, \nu) \in \Lambda^2\).

For arbitrary generating functions \(\varphi: \Lambda^0 \to G\), any class function of \(G\), evaluated on \(dg_{i\mu\nu}\), is invariant under the gauge transformation

\[
g_{i\mu} \mapsto g'_{i\mu} := \varphi_i \cdot g_{i\mu} \cdot \varphi_i^{-1},
\]

\((2.26)\)

Let \(G\) be a compact Lie group (or a finite group) and \(\int_G dg\) denote the Haar measure. The path integral integrates over all configurations, i.e. it consists of one integration over \(G\) for each link. We denote this integration by

\[
\int \mathcal{D}g = \left( \prod_{(i, \mu) \in \Lambda^1} \int_G dg_{i\mu} \right) := \int_G dg_{i\mu} \ldots \int_G dg_{i\mu}.
\]

\((2.27)\)

The path integral measure of lattice gauge theory is this integration together with a Boltzmann weight \(\exp(-S(dg))\). Here the (local) action \(S\) is given by a sum over all plaquettes,

\[
S(dg) := \sum_{(i, \mu, \nu) \in \Lambda^2} s(dg_{i\mu\nu}),
\]

\((2.28)\)

where \(s: G \to \mathbb{R}\) is an \(L^2\) integrable class function on \(G\) which is bounded from below. The action is thus manifestly gauge invariant.

The full partition function of lattice gauge theory with gauge group \(G\) finally reads

\[
Z = \int \mathcal{D}g \exp(-S(dg)) = \left( \prod_{(i, \mu) \in \Lambda^1} \int_G dg_{i\mu} \right) \prod_{(i, \mu, \nu) \in \Lambda^2} f(dg_{i\mu\nu}),
\]

\((2.29)\)

where \(f(g) = \exp(-s(g))\) is a positive real and \(L^2\) integrable class function on \(G\). The standard example for the action is the Wilson action,

\[
s(g) = -\frac{\beta}{2 \dim V_{\rho}} (\chi_{(\rho)}(g) + \overline{\chi_{(\rho)}(g)}),
\]

\((2.30)\)
where $\chi(\rho) : G \to \mathbb{C}$ is the character of a unitary matrix representation $\rho$ of $G$, usually the fundamental representation. The inverse temperature $\beta$ encodes the coupling constant.

In (2.29) and in the following we are careful not to waste letters of the alphabet for dummy indices. The indices $i, \mu$ of the first product sign are just there to indicate that group integration is performed for each link of the lattice. We adopt the convention that $i$ and $\mu$ in this case do not have any meaning outside the enclosing brackets. So we can use the same letters again after the closing bracket.

### 2.6 Gauge invariant quantities

Of course, Wilson loops are gauge invariant expressions. However, in non-Abelian lattice gauge theory, not all gauge invariant expressions are given by Wilson loops. The generic gauge invariant expressions are so-called spin networks which include branchings of the Wilson lines. This is familiar, for example, from the expression which is used to determine the static three-quark potential.

Here we formalize this generalization and give the following slightly technical definition:

**Definition 2.6.** Let $\tau : \Lambda^1 \to \mathbb{R}, (j, \kappa) \mapsto \tau_{j\kappa}$, associate a finite-dimensional irreducible unitary representation of $G$ with each link of the lattice. Choose furthermore for each lattice point $j \in \Lambda^0$ an intertwiner

$$Q^{(j)} : \bigotimes_{\mu=1}^d \tau_{j-\hat{\mu}, \mu} \to \bigotimes_{\mu=1}^d \tau_{j\mu},$$

(2.31)

which maps from the tensor product of the representations of the $d$ ‘incoming’ links to the tensor product of the $d$ ‘outgoing’ links at the point $j \in \Lambda^0$.

The **non-Abelian Wilson loop** (or spin network) associated with $\tau$ and $Q^{(j)}$ is defined by

$$W(\{\tau_{j\kappa}\}, \{Q^{(j)}\}) := \left( \prod_{(j,\kappa) \in \Lambda^1} \sum_{a_{j\kappa}, b_{j\kappa}} \right) \left( \prod_{(j,\kappa) \in \Lambda^1} \delta^{(j\kappa)}_{a_{j\kappa}, b_{j\kappa}} (g_{j\kappa}) \right)$$

$$\times \left( \prod_{j \in \Lambda^0} Q^{(j)}_{(b_{j-\hat{1}, \ldots, b_{j-\hat{d}, d}), (a_{j1} \ldots a_{jd})}} \right),$$

(2.32)

The indices $(b_{j-\hat{1}, \ldots, b_{j-\hat{d}, d})}$ of $Q^{(j)}$ refer to the tensor factors of the domain of $Q^{(j)}$ (‘incoming’) while the $(a_{j1} \ldots a_{jd})$ refer to the image (‘outgoing’), cf. (2.31).

In (2.32) we have abbreviated the summation of the vector indices $a_{j\kappa}$ and $b_{j\kappa}$ for all links by

$$\left( \prod_{(j,\kappa) \in \Lambda^1} \sum_{a_{j\kappa}, b_{j\kappa}} \right) = \sum_{a_{j\kappa}, b_{j\kappa}=1}^{\dim V_{\tau_{j\kappa}}} \cdots \sum_{a_{j\kappa}, b_{j\kappa}=1}^{\dim V_{\tau_{j\kappa}}},$$

(2.33)

This notation is frequently used in the duality transformation in Section 3.
Proposition 2.7. The expression \( W(\{\tau_{jk}\}, \{Q^{(j)}\}) \) of the non-Abelian Wilson loop in (2.32) is gauge invariant under the transformation (2.26).

Proof. Consider an arbitrary lattice point \( j \in \Lambda^0 \). The gauge transformation (2.26) multiplies all ‘incoming’ links by \( \phi^{-1}_j \) and all ‘outgoing’ links by \( \phi_j \). Since \( Q^{(j)} \) in (2.31) is an intertwiner, \( W(\{\tau_{jk}\}, \{Q^{(j)}\}) \) is unchanged. This holds for all lattice points \( j \in \Lambda^0 \).

Remark 2.8. 1. Depending on the representations \( \tau_{jk} \), there are situations in which for some \( j \in \Lambda^0 \) the only choice is \( Q^{(j)} = 0 \) and thus \( W(\{\tau_{jk}\}, \{Q^{(j)}\}) = 0 \). This is the case e.g. if a would-be Wilson loop is not properly closed.

2. All links \((j, \kappa)\) for which \( \tau_{jk} \cong V_{[1]} \) is the trivial representation, disappear from the expression (2.32). For an ordinary Wilson loop, for example, all links are labelled with the trivial representation except those links which are part of the loop. These are labelled with the fundamental representation of \( G \). The intertwiners \( Q^{(j)} \) (if non-vanishing) are in this case uniquely determined up to normalization.

3. In Definition 2.6, the requirement that the \( \tau_{jk} \) be irreducible, and that there be only one factor \( t^{(\tau)}_{ab} \) per link, can be imposed without loss of generality. Otherwise the expression \( W(\{\tau_{jk}\}, \{Q^{(j)}\}) \) would decompose into similar expressions involving only irreducible representations and only one function \( t^{(\tau)}_{ab} \) per link, cf. (2.5a) and (2.5b).

4. If \( G \) is Abelian, then \( \mathcal{R} \cong \mathbb{Z} \), and all irreducible representations are one-dimensional. Thus \( W(\{\tau_{jk}\}, \{Q^{(j)}\}) \) can be decomposed into a sum of products of Abelian Wilson loops.

The normalized expectation value of a non-Abelian Wilson loop finally reads
\[
\langle W(\{\tau_{jk}\}, \{Q^{(j)}\}) \rangle = \frac{1}{Z} \int \mathcal{D}g W(\{\tau_{jk}\}, \{Q^{(j)}\}) \exp(-S(dg)),
\]
(2.34)
cf. (2.28) and (2.32).

3 The duality transformation

In this section we present the duality transformation in detail. We start with the transformation of the partition function and then turn to the expectation value of the non-Abelian Wilson loop.

3.1 The dual of the partition function

We start with the partition function of non-Abelian lattice gauge theory (2.29). Since the Boltzmann weight function \( f(g) \) is an \( L^2 \) class function, we can insert its character decompo-
3.1 The dual of the partition function

The partition function thus reads

\[ f (g) = \sum_{\rho \in \mathcal{R}} \hat{\chi}^{(\rho)} (g) = \sum_{\rho \in \mathcal{R}} \hat{\chi}^{(\rho)} \sum_{n=1}^{\dim V_{\rho}} t_{nn}^{(\rho)} (g) . \] (3.1)

The partition function thus reads

\[ Z = \left( \prod_{(i, \mu) \in \Lambda^1} \int dg_{i\mu} \right) \sum_{\rho_{i\mu} \in \mathcal{R}} \hat{\chi}^{(\rho_{i\mu})} \sum_{n_{i\mu\nu}=1}^{\dim V_{\rho_{i\mu\nu}}} t_{ni_{i\mu\nu}n_{i\mu\nu}}^{(\rho_{i\mu\nu})} (g_{i\mu} g_{i+\mu, \nu} g_{i+\mu, \nu}^{-1} g_{i+\mu, \nu}^{-1}) , \] (3.2)

where each plaquette \((i, \mu, \nu) \in \Lambda^2\) is coloured with an irreducible representation \(\rho_{i\mu\nu} \in \mathcal{R}\), and the indices \(n_{i\mu\nu}\) which originate from the traces, are summed once for each plaquette.

The next step is to employ the coproduct and the antipode, see (2.6a) and (2.6c), in order to remove all group products and inverses from the arguments of the representation functions. Furthermore, we reorganize the summations:

\[ Z = \left( \prod_{(i, \mu) \in \Lambda^1} \int dg_{i\mu} \right) \prod_{(i, \mu, \nu) \in \Lambda^2} \sum_{\rho_{i\mu\nu} \in \mathcal{R}} \hat{\chi}^{(\rho_{i\mu\nu})} \prod_{(i, \mu, \nu) \in \Lambda^2} \left[ \hat{\chi}^{(\rho_{i\mu\nu})} \times \right. \]

\[ \times \left. \sum_{n_{i\mu\nu}, m_{i\mu\nu}, p_{i\mu\nu}, q_{i\mu\nu}} t_{ni_{i\mu\nu}m_{i\mu\nu}^{\rho_{i\mu\nu}}} (g_{i\mu}) t_{ni_{i\mu\nu}m_{i\mu\nu}^{\rho_{i\mu\nu}}} (g_{i+\mu, \nu}) S_{n_{i\mu\nu}m_{i\mu\nu}} (g_{i+\mu, \nu}) S_{n_{i\mu\nu}m_{i\mu\nu}} (g_{i+\mu, \nu}) \right] . \]

In all places where we have applied the coproduct, written schematically as

\[ t_{nn}(g_1 g_2 g_3 g_4) = \sum_{m, p, q} t_{nm}(g_1) t_{mp}(g_2) t_{pq}(g_3) t_{qn}(g_4) , \]

new vector indices have entered which are associated with the plaquette \((i, \mu, \nu) \in \Lambda^2\) and denoted by \(m_{i\mu\nu}, p_{i\mu\nu}\) and \(q_{i\mu\nu}\). They are summed over the range \(1 \ldots \dim V_{\rho_{i\mu\nu}}\).

In order to perform the group integrations, we have to reorganize the product in (3.3) such that all representation functions whose argument refers to the same link are grouped together. Therefore we have to find all plaquettes which contain a given link \((i, \mu) \in \Lambda^1\) in their boundary, i.e. all plaquettes which cobound the link. Figure 2 illustrates this situation for \(d = 3\). The reorganized product reads

\[ Z = \left( \prod_{(i, \mu, \nu) \in \Lambda^2} \sum_{\rho_{i\mu\nu} \in \mathcal{R}} \hat{\chi}^{(\rho_{i\mu\nu})} \right) \prod_{(i, \mu, \nu) \in \Lambda^2} \sum_{n_{i\mu\nu}, m_{i\mu\nu}, p_{i\mu\nu}, q_{i\mu\nu}} t_{ni_{i\mu\nu}m_{i\mu\nu}^{\rho_{i\mu\nu}}} (g_{i\mu}) \prod_{\nu = \mu + 1}^{d} \left[ \sum_{\nu = \mu + 1}^{d} \left[ \hat{\chi}^{(\rho_{i\mu\nu})} \times \right. \]

\[ \times \left. \left[ \sum_{n_{i\mu\nu}, m_{i\mu\nu}, p_{i\mu\nu}, q_{i\mu\nu}} t_{ni_{i\mu\nu}m_{i\mu\nu}^{\rho_{i\mu\nu}}} (g_{i\mu}) t_{ni_{i\mu\nu}m_{i\mu\nu}^{\rho_{i\mu\nu}}} (g_{i+\mu, \nu}) S_{n_{i\mu\nu}m_{i\mu\nu}} (g_{i+\mu, \nu}) S_{n_{i\mu\nu}m_{i\mu\nu}} (g_{i+\mu, \nu}) \right] \right] \right] . \]

The expression (3.4) means that each plaquette is coloured with an irreducible representation. There is furthermore a (dual Boltzmann) weight factor \(\hat{\chi}^{(\rho)}\) per plaquette. Since we
have reorganized the product of the representation functions $t_{ij}(g)$ such that those whose argument refers to the same link $(i, \mu) \in \Lambda^1$ are placed next to each other, the group integrations in (3.4) are performed for each link separately. In the integrand, the two products $\prod_{\lambda}$ and $\prod_{\nu}$ enumerate all plaquettes cobounding the link $(i, \mu)$ in arbitrary dimension $d$ and are such that $1 \leq \lambda < \mu < \nu \leq d$ always. In dimension $d$, there are $2(d-1)$ factors for each link direction $\mu$.

Next we eliminate the antipodes using (2.9). The group integrals in (3.4) thus read

$$
\int_G dg_{i\mu} \left[ \prod_{\lambda=1}^{\mu-1} \left( t_{\rho_{i-\lambda,\mu}^r}^{(\rho_{i-\lambda,\mu}^r)} (g_{i\mu}) \cdot t_{\rho_{i,\lambda}^s}^{(\rho_{i,\lambda}^s)} (g_{i\mu}) \right) \prod_{\nu=\mu+1}^{d} \left( t_{\rho_{i-\nu,\mu}^r}^{(\rho_{i-\nu,\mu}^r)} (g_{i\mu}) \cdot t_{\rho_{i,\mu}^s}^{(\rho_{i,\mu}^s)} (g_{i\mu}) \right) \right].
$$

(3.5)

Now the group integrations can be performed using the expression (2.19) in terms of projec-

---

Figure 2: All plaquettes whose boundary contains a given link $(i, \mu) \in \Lambda^1$. This figure shows the situation in $d = 3$. In generic dimension $d$, there are $2(d-1)$ such plaquettes. The coordinate axes are chosen such that $1 \leq \lambda < \mu < \nu \leq d$. We illustrate the numbering of the lattice points $i$, $i - \lambda$, etc., which are needed to describe the relevant plaquettes $(i - \lambda, \lambda, \mu)$, $(i, \mu, \nu)$, etc. with their correct orientations. Furthermore the letters $n$, $m$, $p$, $q$ indicate to which lattice point the vector index summations $n_{i\mu}$, $m_{i\mu}$, $p_{i\mu}$ and $q_{i\mu}$ of a given plaquette $(i, \mu, \nu)$ belong.
3.1 The dual of the partition function

tors. We obtain

\[
\int_G dg_i \mu \left[ \ldots \right] = \sum_{P \in \mathcal{P}} i \mu_P (m_{i \lambda \mu}^1 \ldots) (q_{i \hat{\nu}, \mu, \nu}^1 \ldots) \cdot P (p_{i \hat{\lambda}, \lambda, \mu}^1 \ldots) (p_{i \hat{\nu}, \mu, \nu}^1 \ldots) \cdot \text{(3.6)}
\]

Here the sum is over a complete set \( \mathcal{P}_{i \mu} \) of inequivalent orthogonal projectors onto the trivial one-dimensional components in the decomposition of

\[
(\rho_{i \lambda \mu} \otimes \rho_{i \hat{\lambda}, \lambda, \mu}^* \otimes \cdots \otimes (\rho_{i \hat{\nu}, \mu, \nu}^* \otimes \rho_{i \mu \nu}) \otimes \cdots)
\]

into irreducible components. The dots “…” indicate that there are pairs \( \rho \otimes \rho^* \) of tensor factors for all \( \lambda \in \{1, \ldots, \mu - 1\} \) and pairs \( \rho^* \otimes \rho \) for all \( \nu \in \{\mu + 1, \ldots, d\} \). This gives the correct result in arbitrary dimension and takes into account the orientation of the link \((i, \mu)\) in the boundary of the given plaquette. Opposite orientations of the link correspond to dual representations. Similarly, the dots “…” in (3.6) indicate that there is one pair of indices for each pair \( \rho \otimes \rho^* \) resp. \( \rho^* \otimes \rho \) which appears in (3.7).

With this step we have evaluated all group integrations over the links. As new degrees of freedom for the dual path integral, the colourings of all plaquettes with irreducible representations of \( G \) have emerged:

\[
Z = \left( \prod_{(i, \mu, \nu) \in \Lambda^2} \sum_{\rho_{i \mu \nu} \in \mathcal{R}} \right) \left( \prod_{(i, \mu, \nu) \in \Lambda^2} \hat{f}_{\rho_{i \mu \nu}} \right) \left( \prod_{(i, \mu, \nu) \in \Lambda^2} n_{i \mu \nu} m_{i \mu \nu} g_{i \mu \nu} \right)
\]

\[
\times \prod_{(i, \mu) \in \Lambda^1} \left( \sum_{P \in \mathcal{P}_{i \mu}} P_{(\ldots)(\ldots)} \cdot P_{(\ldots)(\ldots)} \right) \cdot \text{(3.8)}
\]

The last sum has the form as in (3.6) for each link \((i, \mu)\) appearing in the product.

There are still the summations over the vector indices \( n_{i \mu \nu}, \ldots, q_{i \mu \nu} \) for all plaquettes. We call the factors arising from them and from the projectors \textit{gauge constraints} because they generalize the conditions which ensure in the Abelian case that the integer 2-form appearing in the dual model, is co-closed. They seem to form a number of complicated non-local constraints. However, the summations can again be reordered in a suitable way so that the constraints appear only locally in a certain sense. In order to see this, some geometric intuition is necessary.

The projectors which have appeared in the group integration \( \int_G dg_i \mu \) for the link \((i, \mu) \in \Lambda^1 \) and their indices are of the form

\[
P (m_{i \hat{\lambda}, \lambda, \mu}^1 \ldots) (q_{i \hat{\nu}, \mu, \nu}^1 \ldots) \cdot P (p_{i \hat{\lambda}, \lambda, \mu}^1 \ldots) (p_{i \hat{\nu}, \mu, \nu}^1 \ldots) \cdot \text{(3.9)}
\]
3 THE DUALITY TRANSFORMATION

Figure 3: All plaquettes containing a given lattice point \( i \in \Lambda^0 \) in their boundaries. This figure shows the situation in \( d = 3 \). In the generic case, there are \( 2d(d - 1) \) such plaquettes. This figure illustrates the numbering of points \( i - \hat{\mu} - \hat{\nu}, i - \hat{\nu}, i - \lambda - \hat{\mu} \) which are used to specify the relevant plaquettes \( (i - \hat{\mu} - \hat{\nu}, \mu, \nu) \), \( (i - \hat{\nu}, \mu, \nu) \) and \( (i, \mu, \nu) \) in the \( (\mu, \nu) \)-plane and similarly for the other directions, cf. (3.11). The letters \( n, m, p, q \) indicate which lattice points the vector index summations are associated with. The labelling for a given plaquette \( (i, \mu, \nu) \) starts with \( n \) at \( i \) and proceeds counter-clockwise with \( m, p, q \).

i.e. the indices correspond to the plaquettes located at \( i - \lambda, i, \ldots, i - \hat{\nu}, i, \ldots \) for the first projector and similarly for the second projector.

The crucial geometrical observation (Figure 2) is that all vector indices \( m_{i - \lambda, \lambda, \mu, \nu}, m_{i - \lambda, \lambda, \mu, \nu} \) of the second projector correspond to the lattice point \( i \) whereas all vector indices \( p_{i - \lambda, \lambda, \mu, \nu}, p_{i - \lambda, \lambda, \mu, \nu}, p_{i - \lambda, \lambda, \mu, \nu} \) of the second projector correspond to the lattice point \( i + \mu \). Recall that the enumeration of the indices \( n, m, p, q \) for a given plaquette \( (i, \mu, \nu) \in \Lambda^2 \) was done starting with \( n \) at the point \( i \), then proceeding counter-clockwise in the \( (\mu, \nu) \)-plane. It is thus possible to associate the summations over the \( n, m, p, q \) with the lattice points \( i \in \Lambda^0 \). For each link \( (i, \mu) \in \Lambda^1 \), one of the two projectors in (3.1) then belongs to \( i \), the other to \( i + \hat{\mu} \). However, the projectors can be separated only if the summation over the projectors \( \mathcal{P}_{i\mu} \) which is associated with a link rather than a point, can be removed from
3.1 The dual of the partition function

these expressions.

In the partition function, there is in total one such summation over projectors for each link of the lattice. It is thus natural to consider these summations as part of the dual path integral. If this is done, we can reorganize the expressions such that all vector index summations and projectors are associated with the lattice points \( i \in \Lambda^0 \) and such that they involve only data from the neighbouring links and plaquettes. This is what is meant by 'locality'. In Figure 3 we show the lattice point \( i \in \Lambda^0 \) together with the \( 2d \) links and \( 2d(d - 1) \) plaquettes which contain \( i \). The partition function finally reads

\[
Z = \left( \prod_{(i, \mu, \nu) \in \Lambda^2} \sum_{\rho_{i\mu\nu} \in \mathbb{R}} \right) \left( \prod_{(i, \mu, \nu) \in \Lambda^1} \sum_{P(i, \nu) \in \mathcal{P}_{i\mu}} \right) \left( \prod_{i \in \Lambda^0} \hat{f}_{\rho_{i\mu\nu}} \right) \prod_{i \in \Lambda^0} C(i). \tag{3.10}
\]

where \( C(i) \) encompasses the vector index summations and projectors associated with the lattice point \( i \in \Lambda^0 \):

\[
C(i) = \left( \prod_{1 \leq \mu < \nu \leq d} \sum_{p_{i-\mu-\nu, \mu, \nu} = 1} \sum_{q_{i-\nu, \mu, \nu} = 1} \sum_{m_{i-\mu, \mu, \nu} = 1} \sum_{n_{i\mu\nu} = 1} \right) \prod_{\mu = 1}^d P^{(i, \mu)} \left( \prod_{\nu \in \{\mu+1, \ldots, d\}} \frac{\dim V_{i-\mu-\nu, \nu, \mu, \nu} \dim V_{i-\nu, \mu, \nu} \dim V_{i-\mu, \mu, \nu} \dim V_{i\mu\nu}}{\dim V_{i-\mu-\nu, \nu, \mu, \nu} \dim V_{i-\nu, \mu, \nu} \dim V_{i-\mu, \mu, \nu} \dim V_{i\mu\nu}} \right). \tag{3.11}
\]

The first product parameterizes all possible planes by \( \mu < \nu \). The four sums are then associated with the four plaquettes \((i - \mu - \nu, \mu, \nu), (i - \nu, \mu, \nu), (i - \mu, \mu, \nu)\) and \((i, \mu, \nu)\) in the \((\mu, \nu)\)-plane which contain the point \( i \) (Figure 3). The last product enumerates the ‘outgoing’ and ‘incoming’ links and contains the projectors associated with this link and with the point \( i \). Note that the projectors in (3.11) for a given lattice point \( i \in \Lambda^0 \) involve only vector indices whose summation is part of the same \( C(i) \).

The partition function (3.11) consists now of a sum over irreducible representations for all plaquettes and a sum over the projectors onto the trivial components in the tensor product (3.7) for all links. This tensor product involves the representations of all plaquettes which cobound the given link. In particular, if the tensor product does not contain a trivial component, this sum over projectors is empty and does not contribute to the path integral.

These results are summarized in the following theorem:

**Theorem 3.1 (Dual partition function).** Let \( G \) be a compact Lie group or a finite group. The partition function (2.29) of lattice gauge theory with the gauge group \( G \) on a \( d \)-dimension-
al finite lattice with periodic boundary conditions is equal to the expression

\[ Z = \left( \prod_{(i,\mu,\nu) \in \Lambda^2} \sum_{\rho_{i\mu\nu} \in R} \right) \left( \prod_{(i,\mu) \in \Lambda^1} \sum_{P_{i\mu} \in P_{i\mu}} \right) \left( \prod_{(i,\mu,\nu) \in \Lambda^2} \hat{f}_{\rho_{i\mu\nu}} \right) \prod_{i \in \Lambda^0} C(i). \tag{3.12} \]

Here \( R \) denotes a set containing one unitary representation for each equivalence class of finite-dimensional irreducible representations of \( G \). \( P_{i\mu} \) denotes the set of all projectors onto the different one-dimensional trivial components in the decomposition of the tensor product \( (3.7) \) into its irreducible components. \( C(i) \) describes a gauge constraint factor for each lattice point \( i \in \Lambda^0 \) which is given by \( (3.11) \). The coefficients \( \hat{f}_{\rho_{i\mu\nu}} \) are defined by the character decomposition of the original Boltzmann weight \( \exp(-s(g)) \),

\[ \hat{f}_{\rho_{i\mu\nu}} = \dim V_{\rho_{i\mu\nu}} \int_G \chi^{(\rho_{i\mu\nu})}(g) \exp(-s(g)) \, dg. \tag{3.13} \]

Remark 3.2. 1. The dual partition function can be described in words as follows: Colour all plaquettes with finite-dimensional irreducible representations of \( G \) in all possible ways. Colour all links with projectors onto the trivial components in the tensor products \( (3.7) \) (if there are any). The partition function contains a (local) dual Boltzmann weight factor which is the coefficient of the character expansion of the original Boltzmann weight for each plaquette. The partition function contains furthermore a (local) gauge constraint factor \( C(i) \), see \( (3.11) \), for each lattice point.

2. The two main differences to the Abelian case (see e.g. \[3\]) are the following: Firstly, in the Abelian case only objects on a single level, namely the plaquettes, are coloured with integer numbers (which characterize the finite-dimensional irreducible unitary representations). In the non-Abelian case we have to colour the plaquettes with representations and the links with intertwiners. The configurations of the dual model are thus spin foams. Note that the choice of projectors \( P_{i\mu} \) in \( (3.6) \) and \( (3.7) \) agrees up to canonical isomorphisms with the choice of intertwiners in the definition of a closed spin foam as given in \[3\]. Here the assignment of ‘incoming’ and ‘outgoing’ faces has to be made according to our standard orientations of plaquettes and links.

Secondly, the integrand is not just a Boltzmann weight, but contains in addition the factor \( C(i) \) for each lattice point. In the Abelian case, this factor together with the sum over projectors enforces co-closedness of the integer 2-form. The dual of Abelian gauge theory is again a gauge theory because if this 2-form is also co-exact, it can be integrated and gauge degrees of freedom appear. In the non-Abelian case there is no obvious integration which would introduce gauge degrees of freedom.
3.2 The dual of the non-Abelian Wilson loop

The duality transformation of the expectation value of the non-Abelian Wilson loop \((\ref{eq:3.4})\) proceeds along the same lines. However, the expressions become slightly more complicated due to the presence of the additional integrand. In the following description of the transformation, we often refer to the calculations for the partition function in Section 3.1.

The expectation value of the non-Abelian Wilson loop is given by

\[
\langle W(\{\tau_{j\kappa}\}, \{Q^{(j)}\}) \rangle = \frac{1}{Z} \left( \prod_{(i,\mu) \in \Lambda^2} \sum_{\rho_{i\mu} \in \mathbb{R}} \left( \prod_{(i,\mu) \in \Lambda^2} \hat{\rho}_{i\mu} \right) \left( \prod_{(j,\kappa) \in \Lambda^1} \sum_{a_{jk}, b_{jk}} \right) \times \left( \prod_{(j,\kappa) \in \Lambda^1} t^{(\tau_{jk})}_{a_{jk} b_{jk}}(g_{jk}) \right) \left( \prod_{j \in \Lambda^0} Q^{(j)}_{(b_{j-1,1} \ldots b_{j-d}, a_{j1} \ldots a_{jd})} \right) \right). \tag{3.14}
\]

We insert the character decomposition \((3.3)\), employ the coproduct and the antipode and reorganize the factors just as in the calculation for the partition function. The result generalizes \((3.4)\) in which \((3.3)\) has been inserted:

\[
\langle W(\{\tau_{j\kappa}\}, \{Q^{(j)}\}) \rangle = \frac{1}{Z} \left( \prod_{(i,\mu) \in \Lambda^2} \sum_{\rho_{i\mu} \in \mathbb{R}} \left( \prod_{(i,\mu) \in \Lambda^2} \hat{\rho}_{i\mu} \right) \left( \prod_{(j,\kappa) \in \Lambda^1} \sum_{a_{jk}, b_{jk}} \right) \left( \prod_{(j,\kappa) \in \Lambda^1} \sum_{n_{i\mu}, m_{\mu}, q_{i\mu}} \right) \left( \prod_{j \in \Lambda^0} Q^{(j)}_{(b_{j-1,1} \ldots b_{j-d}, a_{j1} \ldots a_{jd})} \right) \times \prod_{(i,\mu) \in \Lambda^2} \left\{ \int_G dg_{i\mu} \left[ \prod_{\lambda = 1}^{\mu-1} t^{(\rho_{i-\lambda,\mu})}_{m_{i-\lambda,\mu}, p_{i-\lambda,\mu}}(g_{i\mu}) \cdot t^{(\rho^*_{\lambda,\mu})}_{n_{i\mu}, q_{i\mu}}(g_{i\mu}) \right] \times \prod_{\nu = \mu+1}^{d} \left[ t^{(\rho^*_{\nu-\mu,\nu})}_{q_{i-\nu,\mu,\nu}, p_{i-\nu,\mu,\nu}}(g_{i\mu}) \cdot t^{(\rho_{\mu})}_{m_{\mu}, q_{\mu}}(g_{i\mu}) \cdot t^{(\tau_{\mu})}_{a_{i\mu}, b_{i\mu}}(g_{i\mu}) \right] \right\} \right). \tag{3.15}
\]

The features which are new compared with \((3.4)\) and \((3.5)\) are the summations over \(a_{jk}\) and \(b_{jk}\) for each link, the product over the intertwiners \(Q^{(j)}\) for each lattice point and the additional factor \(t^{(\tau_{\mu})}_{a_{i\mu}, b_{i\mu}}(g_{i\mu})\) in the integrand for each link \((i, \mu)\).

Now we can apply the projector expression for the Haar measure \((\ref{eq:2.19})\). Compared with \((3.6)\), the additional factor in the integrand produces additional indices \(a_{i\mu}\) and \(b_{i\mu}\) of the projectors,

\[
\int_G dg_{i\mu} \left[ \cdots \right] = \sum_{P \in \mathcal{P}_{t_{\mu}}} P(m_{i-\lambda,\mu, \mu}, n_{i\mu}, \ldots) \cdot P(q_{i-\nu,\mu,\nu}, m_{\mu}, \ldots) \cdot \text{new} \cdot a_{i\mu} \cdot b_{i\mu}. \tag{3.16}
\]
These indices correspond to the additional tensor factor in the following decomposition: The orthogonal projectors $P \in \mathcal{P}_i^\prime$ project onto the distinct one-dimensional trivial components in the decomposition of

$$
\left( \rho_{i-\hat{\lambda},\lambda,\mu} \otimes \rho_{i,\lambda,\mu}^* \right) \otimes \cdots \otimes \left( \rho_{\hat{i},-\hat{\nu},\mu,\nu} \otimes \rho_{\hat{i},\mu,\nu} \right) \otimes \cdots \otimes \tau_{i\mu} \quad (3.17)
$$

into its irreducible components, cf. (3.7).

Similar to the calculation for the partition function, the vector index summations over the $n_{i\mu\nu}, \ldots, q_{i\mu\nu}$ as well as over the $a_{i\kappa}$ and $b_{j\kappa}$, the projectors $P_{\{\cdots\}^{(-)}}$ and the intertwiners $Q^{(j)}$ can be reorganized to form local expressions. This construction is entirely analogous to the derivation of (3.10) and (3.11). We obtain the following result which generalizes Theorem 3.1:

**Theorem 3.3 (Dual non-Abelian Wilson loop).** Let $G$ be a compact Lie group or a finite group and consider lattice gauge theory with gauge group $G$ on a $d$-dimensional finite lattice with periodic boundary conditions. The normalized expectation value of the non-Abelian Wilson loop (2.34) is equal to the expression

$$
\langle W(\{\tau_{i\kappa}\}, \{Q^{(j)}\}) \rangle = \frac{1}{2} \left( \prod_{(i,\mu,\nu) \in \Lambda^2} \frac{\sum_{\rho_{i\mu\nu} \in \mathcal{R}}}{\dim V_{\rho_{i\mu\nu}}} \right) \left( \prod_{(i,\mu,\nu) \in \Lambda^1} \frac{\sum_{P^{(i)}(\mu) \in \mathcal{P}_i^\prime}}{\dim V_{P^{(i)}(\mu)}} \right) \left( \prod_{(i,\mu,\nu) \in \Lambda^2} \tilde{f}_{\rho_{i\mu\nu}} \right) \quad \text{dual Boltzmann weight}
$$

$$
\times \prod_{i \in \Lambda^0} \left( \prod_{\mu=1}^d \sum_{a_{i\mu}=1} \sum_{b_{i-\hat{\mu},\mu}=1} \sum_{n_{i\mu\nu}=1} Q^{(i)}_{(b_{i-\hat{\mu},\mu}=1 \cdots)} \bar{C}(i) \right). \quad (3.18)
$$

Here $\mathcal{P}_i^\prime$ denotes the set of all projectors onto the different trivial components in the decomposition of the tensor product (3.17) into its irreducible components. $\bar{C}(i)$ describes a gauge constraint factor for each lattice point $i \in \Lambda^0$ which is given by

$$
\bar{C}(i) = \left( \prod_{1 \leq \mu < \nu \leq d} \frac{\dim V_{\rho_{i-\hat{\mu},\nu,\mu}} \dim V_{\rho_{i-\hat{\nu},\mu,\nu}} \dim V_{\rho_{i-\hat{\nu},\mu,\nu}} \dim V_{\rho_{i-\hat{\mu},\nu,\mu}}}{\dim V_{\rho_{i-\hat{\mu},\nu,\mu}} \dim V_{\rho_{i-\hat{\nu},\mu,\nu}} \dim V_{\rho_{i-\hat{\nu},\mu,\nu}} \dim V_{\rho_{i-\hat{\mu},\nu,\mu}}} \right) \prod_{\mu=1}^d \frac{P^{(i)}(\mu)}{a_{i\mu}} \frac{(m_{i-\hat{\mu},\lambda,\mu} \cdots)(q_{i-\hat{\nu},\mu,\nu} \cdots) b_{i-\hat{\nu},\mu}}{\lambda \in \{1, \ldots, \mu-1\} \nu \in \{\mu+1, \ldots, d\}} \frac{P^{(i)}(i-\hat{\mu},\mu)}{a_{i\mu}} \frac{(m_{i-\hat{\mu},\lambda,\mu} \cdots)(q_{i-\hat{\nu},\mu,\nu} \cdots) b_{i-\hat{\nu},\mu}}{\lambda \in \{1, \ldots, \mu-1\} \nu \in \{\mu+1, \ldots, d\}}. \quad (3.19)
$$

**Remark 3.4.** The dual of the non-Abelian Wilson loop can be described in words as follows: Just as in the Abelian case, it is not an expectation value under the dual partition function, but looks rather like a modified partition function. In addition to the dual partition function,
3.3 The constraints $C(i)$ on the dual lattice

In the expression of the dual partition function (3.12) and (3.11), the factors $C(i)$ look very complicated. They can be understood most easily on the dual lattice. We explain this idea for the case $d = 3$ where the relevant pictures can be drawn. Analogous constructions can be made for arbitrary $d \geq 2$.

We construct the dual lattice in the standard way which is illustrated in Figure 4 for the case $d = 3$. To each $k$-cell $(i, \mu_1, \ldots, \mu_k), 1 \leq \mu_1 < \cdots < \mu_k \leq d$, of the original lattice, there corresponds a $(d - k)$-cell $(i, \nu_1, \ldots, \nu_{d-k}), 1 \leq \nu_1 < \cdots < \nu_{d-k} \leq d$, of the dual lattice such that

$$\{\mu_1, \ldots, \mu_k\} \cup \{\nu_1, \ldots, \nu_{d-k}\} = \{1, \ldots, d\}.$$  \hspace{1cm} (3.20)

In the dual partition function on the original lattice, the plaquettes are coloured with irreducible representations. The links are assigned projectors in a certain tensor product whose factors are given by the representations belonging to the plaquettes that cobound the link. Conversely, on the dual lattice in $d = 3$, the plaquettes cobounding a given link correspond to the links in the boundary of a plaquette (see Figure 5). Thus we have to colour the links of the dual lattice with irreducible representations. The plaquettes of the dual lattice

Figure 4: The relation between objects on the original lattice (solid) and on the dual lattice (dashed) in $d = 3$. It is convenient to draw the points of the dual lattice shifted by half a lattice constant in all positive directions and to draw its axes with reversed directions. Then the cube dual to a point is centered around this point, the plaquette dual to a link is punctured by it etc.
are then assigned projectors onto the trivial components of some tensor product. This tensor product is the product of the representations belonging to the links in the boundary of the plaquette.

Instead of the projectors onto trivial components, schematically

$$P: \rho_1 \otimes \rho_2 \otimes \rho_3^* \otimes \rho_4^* \to \mathbb{C}$$

we now write intertwiners

$$F: \rho_1 \otimes \rho_2 \to \rho_3 \otimes \rho_4,$$

using the isomorphisms of $G$-modules $\text{Hom}_G(V^*_\rho \otimes V_\tau, \mathbb{C}) \cong G \text{Hom}_G(V_\tau, V_\rho)$. The intertwiners $F$ thus map from two links of a given plaquette to the other two links. Note that the $F$ inherit a normalization from the $P$ coming from the implicit inclusion $\mathbb{C} \subseteq \rho_1 \otimes \rho_2 \otimes \rho_3^* \otimes \rho_4^*$.

The factors $C(i)$ of (3.11) are associated with the cubes of the dual lattice. The expression (3.11), interpreted on the dual lattice in $d = 3$, contains one intertwiner per face of the cube as indicated in Figure 5. In this figure, the intertwiners $F$ are represented by double arrows leading from two links of each plaquette to the other two links. The arrows illustrate how the intertwiners have to be composed to account for the contraction of the indices in (3.11). The remaining indices are then summed over.

A similar visualization is straightforward for the factors $\tilde{C}(i)$ in (3.19).

4 Special cases

4.1 Abelian gauge theory in arbitrary dimension

In this section we show how Theorem 3.1 reduces to the well-known results for $G = U(1)$. Similar calculations are available for $\mathbb{Z}$ or $\mathbb{Z}_n$ (Note that the transformation is applicable to...
4.1 Abelian gauge theory in arbitrary dimension

Z although Z is neither compact nor finite. This is because Z gauge theory is dual to U(1) gauge theory).

We start with the dual partition function (3.12). The unitary finite-dimensional irreducible representations of U(1) are all one-dimensional. They are given by homomorphisms $g \mapsto g^k$ for $g \in U(1)$ and are characterized by integer numbers $k \in \mathbb{Z}$, i.e. $\mathcal{R} \cong \mathbb{Z}$. The dual representation is then given by $g \mapsto g^{-k}$.

Consider the tensor product (3.7) and specify the representations by integer numbers $k_{i\mu\nu} \in \mathbb{Z}$. Since all irreducible representations are one-dimensional, so are their tensor products. The question is therefore just whether or not the tensor product (3.7) is equivalent to the trivial representation. This is the case if and only if

$$
\sum_{\lambda=1}^{\mu-1} (k_{i-\lambda,\lambda,\mu} - k_{i\lambda\mu}) + \sum_{\nu=\mu+1}^d (-k_{i-\nu,\mu,\nu} + k_{i\mu\nu}) = 0. \tag{4.1}
$$

In this case, there is exactly one projector onto a trivial component of the tensor product which is the identity map. If (4.1) does not hold, there is no such projector.

Furthermore, since all irreducible representations are one-dimensional, the summations in the constraints $C(i)$, see (3.11), disappear. Moreover, the projectors $P(i\mu)$ do not have indices and are all equal to 1 if they exist. Thus $C(i) = 1$ if (4.1) holds.

Figure 6: The intertwiners $F: \rho_1 \otimes \rho_2 \rightarrow \rho_3 \otimes \rho_4$ between the representations $\rho_j$ associated with the links of the dual lattice in $d = 3$. This figure indicates how the indices of the intertwiners $F$ are contracted in the factor $C(i)$.
The partition function (3.12) therefore reads for $G = U(1)$,

$$Z = \left( \prod_{(i, \mu, \nu) \in \Lambda^2} \sum_{k_{i\mu\nu} \in \mathbb{Z}} \right) \left( \prod_{(i, \mu, \nu) \in \Lambda^2} \tilde{f}_{k_{i\mu\nu}} \right) \times \left( \prod_{(i, \lambda) \in \Lambda^1} \delta \left( \sum_{\lambda=1}^{\mu-1} (k_{i-\lambda, \lambda, \mu} - k_{i\lambda\mu}) + \sum_{\nu=\mu+1}^{d} (k_{i-\nu, \mu, \nu} + k_{i\mu\nu}) \right) \right).$$  (4.2)

Here we have used the notation $\delta(n) = \delta_{0,n}$ for $n \in \mathbb{Z}$.

The dual path integral thus reduces to the summation over the integer numbers for each plaquette while the dual Boltzmann weight is again given by the character decomposition of the original Boltzmann weight,

$$\tilde{f}_{k_{i\mu\nu}} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik_{i\mu\nu}\varphi} \exp(-s(e^{i\varphi})) \, d\varphi.$$  (4.3)

The $\delta$-constraint ensures that the integer 2-form $k_{i\mu\nu}$ is co-closed. This condition provides the dual model with the properties of a gauge theory. As is well-known, the partition function (4.2) describes the dual of $U(1)$ lattice gauge theory, the so-called $\mathbb{Z}$ gauge theory [2, 16] and is here presented on the original rather than on the dual lattice.

### 4.2 Non-Abelian gauge theory in two dimensions

In this section we demonstrate how Theorem 3.1 reduces to the familiar result for non-Abelian lattice gauge theory in $d = 2$. In this case the partition function is particularly simple.

We start again with the dual partition function (3.12). In $d = 2$, there are only two plaquettes which cobound a given link $(i, \mu) \in \Lambda^1$. Imagine the situation of Figure 2 in $d = 2$. The tensor product (3.7) therefore consists of only two factors. It reads for links $(i, 1) \in \Lambda^1$ in the 1-direction,

$$\rho_{i-2}^* \otimes \rho_i,$$  (4.4)

and for links $(i, 2)$ in the 2 direction,

$$\rho_{i-1}^* \otimes \rho_i.$$  (4.5)

Here we have suppressed the last two indices of $\rho_{i\mu\nu}$ which are always $\mu = 1$ and $\nu = 2$. In both cases (4.4) and (4.5), there are trivial components in the tensor product if and only if

$$\rho_{i-2} \cong \rho_i \quad \text{resp.} \quad \rho_{i-1} \cong \rho_i.$$  (4.6)

Since this holds for all $i \in \Lambda^0$, the only contributions to the partition function are given by configurations which assign the same representation to all plaquettes. We thus have

$$Z = \sum_{\rho \in \mathcal{R}} (\tilde{f}_\rho)^{|\Lambda^2|} \prod_{i \in \Lambda^0} C(i).$$  (4.7)
Observe further that the projectors onto the trivial component in the tensor products (4.4) and (4.5) are both given by the trace, i.e. \( P_{ab}^{(i\mu)} = \frac{1}{\dim V_\rho} \delta_{ab} \). The constraint \( C(i) \) can be easily calculated:

\[
C(i) = \sum_{p_i-\hat{1}} \sum_{q_i-\hat{2}} \sum_{m_i-\hat{1}} \sum_{n_i-\hat{2}} P_{1(1)}^{p_i-\hat{1}m_i-\hat{1}} \cdot P_{1\bar{1}(1)}^{q_i-\hat{2}n_i} \cdot P_{2(2)}^{p_i-\hat{1}n_i} \cdot P_{1\bar{2}(2)}^{p_i-\hat{2}q_i-\hat{2}} = \frac{1}{(\dim V_\rho)^3} \tag{4.8}
\]

Therefore the partition function reads

\[
Z = \sum_{\rho \in \mathcal{R}} (\hat{f}_\rho)^{\Lambda^2} (\dim V_\rho)^{-3|\Lambda^0|} \tag{4.9}
\]

This is the well-known result for lattice gauge theory in two dimensions, see e.g. [28].

5 Discussion

The duality transformation given in Theorems 3.1 and 3.3 is a strong-weak duality. For example, the character decomposition of the Boltzmann weight (3.13) reads for the Wilson action of \( G = U(1) \),

\[
\hat{f}_k = I_k(\beta), \quad k \in \mathbb{Z}, \tag{5.1}
\]

and for the Wilson action of \( G = SU(2) \) using the fundamental representation,

\[
\hat{f}_j = 2(2j + 1) I_{2j+1}(\beta)/\beta, \quad 2j \in \mathbb{N}_0. \tag{5.2}
\]

Here the representations are parameterized by integers \( k \) resp. non-negative half-integers \( j \), and \( I_n(x) \) denote the modified Bessel functions. The coefficients \( \hat{f}_k \) resp. \( \hat{f}_j \) are positive and can thus be written \( \hat{f}_k = \exp(-s^*(k)) \) resp. \( \hat{f}_j = \exp(-s^*(j)) \). The \( \beta \)-dependence of the dual action \( s^* \) is such that high and low temperature regimes are exchanged or, in the language of gauge theory, strong and weak coupling (see e.g. [2,28]). However, the coupling constant does not occur as a prefactor of the interaction terms of the dual model because its interactions do not arise from the dual Boltzmann weight but rather from the selection of projectors \( P_{i\mu} \) and from the factors \( C(i) \), cf. (3.12). For details about the character decompositions of the various common actions in lattice gauge theory, see e.g. [27,28] and references therein.

Of course, it is also possible to define the action of non-Abelian lattice gauge theory in terms of the character decomposition of its Boltzmann weight. For example, the heat kernel action (or generalized Villain action) is given by the choice

\[
\hat{f}_\rho = \dim V_\rho \cdot \exp(-C_\rho/\beta), \tag{5.3}
\]
which makes the strong-weak duality manifest. Here $C_\rho$ denotes the eigenvalue of the quadratic Casimir operator (in a certain normalization) on the irreducible representation $\rho$ of $G$. Since $C_\rho$ is essentially quadratic in the highest weight of the representation $\rho$, it is apparent that higher representations are exponentially suppressed in the dual path integral. The smaller $\beta$ is chosen, the more pronounced is the suppression. The dual expressions (3.12) and (3.18) can therefore serve as generating functions for the strong coupling expansion. For details about strong coupling expansion techniques, see e.g. [28].

Since the duality transformation for non-Abelian lattice gauge theory constructed in this paper generalizes the Abelian case in the form written with an explicit gauge constraint rather than in the form which is integrated and exhibits gauge degrees of freedom, there is no immediate answer to the question whether the dual model has any gauge invariance and how these degrees of freedom could be parametrized.

A non-Abelian generalization of the integration of a closed (and exact) $k$-cocycle to the coboundary of a $(k - 1)$-cocycle up to a gauge freedom is not easy to find. If it exists, this paper might help to assemble information on how a non-Abelian generalization of cohomology might look like. Interesting in this context are the ideas developed in [21].

Degrees of freedom in the dual model which are always present, are related to the choice of unitary representatives $\rho \in \widehat{\mathcal{R}}$ of each class of equivalent irreducible representations of $G$. The Clebsh-Gordan coefficients which enter the analysis extensively as the coefficients of the various projectors, depend on these choices.

In any case, the dual model can be expected to be the appropriate starting point to search for the non-Abelian magnetic degrees of freedom which generalize the magnetic monopoles of $U(1)$ lattice gauge theory, and to provide a framework for a rigorous treatment of their properties.

As far as the strong coupling expansion is concerned, the crucial question is to what extent the limit of this expansion is compatible with the continuum limit, i.e. whether properties derived via the strong coupling expansion hold (at least qualitatively) for all couplings. Recall that the continuum limit of lattice QCD consists of a combination of sending the inverse temperature $\beta \to \infty$ and the lattice spacing $a \to 0$.

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References
REFERENCES

[1] H. A. KRAMERS and G. H. WANNIER: Statistics of the two-dimensional ferromagnet. Part I. Phys. Rev. 60 (1941) 252–262.

[2] R. SAVIT: Duality in field theory and statistical systems. Rev. Mod. Phys. 52, No. 2 (1980) 453–487.

[3] M. P. REISENBERGER: World sheet formulation of gauge theories and gravity. Preprint gr-qc/9412035.

[4] J. IWASAKI: A definition of the Ponzano-Regge quantum gravity model in terms of surfaces. J. Math. Phys. 36 (1995) 6288–6298.

[5] M. P. REISENBERGER and C. ROVELLI: ‘Sum over surfaces’ form of loop quantum gravity. Phys. Rev. D 56 (1997) 3490–3508.

[6] J. C. BAEZ: Spin foam models. Class. Quant. Grav. 15, No. 7 (1998) 1827–1858, gr-qc/9709052.

[7] J. C. BAEZ: An introduction to spin foam models of quantum gravity and BF theory. In Geometry and Quantum Physics, number 543 in Lecture Notes in Physics. Springer, Berlin, 2000, gr-qc/9905087.

[8] K. G. WILSON: Confinement of quarks. Phys. Rev. D 10, No. 8 (1974) 2445–2459.

[9] T. BANKS, R. MYERSON and J. B. KOGUT: Phase transitions in Abelian lattice gauge theories. Nucl. Phys. B 129 (1977) 493–510.

[10] M. E. PESKIN: Mandelstam-t’Hooft duality in Abelian lattice models. Ann. Phys. 113 (1978) 122–152.

[11] J. FRÖHLLICH and P.-A. MARCHETTI: Magnetic monopoles and charged states in four-dimensional, Abelian lattice gauge theory. Europhys. Lett. 2, No. 12 (1986) 933–940.

[12] J. FRÖHLLICH and P.-A. MARCHETTI: Soliton quantization in lattice field theories. Comm. Math. Phys. 112 (1987) 343–383.

[13] A. GUTH: Existence proof of a nonconfining phase in four-dimensional U(1) lattice gauge theory. Phys. Rev. D 21 (1980) 2291–2307.

[14] J. FRÖHLLICH and T. SPENCER: Massles phases and symmetry restoration in Abelian gauge theories and spin systems. Comm. Math. Phys. 83 (1982) 411–454.

[15] M. GÖPFERT and G. MACK: Proof of confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant. Comm. Math. Phys. 82 (1982) 545–606.
[16] L. Polley and U.-J. Wiese: Monopole condensate and monopole mass in $U(1)$ lattice gauge theory. *Nucl. Phys.* B 356 (1991) 629–654.

[17] J. Jersák, T. Neuhaus and H. Pfeiffer: Scaling analysis of the magnetic monopole mass and condensate in the pure $U(1)$ lattice gauge theory. *Phys. Rev.* D 60, No. 5 (1999) 054502, hep-lat/9903034.

[18] S. Jaimungal: Wilson loops, Bianchi constraints and duality in Abelian lattice models. *Nucl. Phys.* B 542 (1999) 441–470, hep-th/9808018.

[19] R. Anishetty, S. Cheluvaraja, H. S. Sharatchandra and M. Mathur: Dual of 3-dimensional pure $SU(2)$ lattice gauge theory and the Ponzano-Regge model. *Phys. Lett.* B 314 (1993) 387–390.

[20] D. Diakonov and V. Petrov: Yang-Mills theory in three dimensions as quantum gravity theory. Preprint hep-th/9912268.

[21] H. Grosse and K.-G. Schlesinger: Duals for non-abelian lattice gauge theories by categorial methods. Erwin Schrödinger Institute Preprint ESI 561 (1998), Vienna.

[22] G. G. Batrouni: Plaquette formulation and the Bianchi identity for lattice gauge theories. *Nucl. Phys.* B 208 (1982) 467–483.

[23] Ch. Hong-Mo, J. Faridani and T. S. Tsun: Generalized dual symmetry for non-Abelian Yang-Mills fields. *Phys. Rev.* D 53, No. 12 (1996) 7293–7305.

[24] T. Bröcker and T. Tom Dieck: Representations of Compact Lie Groups. Number 98 in Graduate Texts in Mathematics. Springer, New York, 1985.

[25] R. Carter, G. Segal and I. Macdonald: Lectures on Lie groups and Lie algebras. Number 32 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1995.

[26] H. J. Rothe: Lattice Gauge Theories — An Introduction. World Scientific, Singapore, 1992.

[27] I. Montvay and G. Münster: Quantum fields on a lattice. Cambridge University Press, Cambridge, 1994.

[28] J.-M. Drouffe and J.-B. Zuber: Strong coupling and mean-field methods in lattice gauge theories. *Phys. Rep.* 102, No. 1,2 (1983) 1–119.