GENERALIZED HARMONIC KOEBE FUNCTIONS

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Abstract. We present a family of sense-preserving harmonic mappings in the unit disk related to the classical generalized (analytic) Koebe functions and analyze their properties. In particular, we prove that these are precisely the mappings that maximize simultaneously the real part of every Taylor coefficient in affine and linear invariant families of complex-valued harmonic functions.

INTRODUCTION

Let $S$ be the family of all univalent (one-to-one) analytic mappings $\varphi$ in the unit disk $\mathbb{D}$ with the normalizations $\varphi(0) = 1 - \varphi'(0) = 0$.

It was as early as 1916 when Bieberbach [1, 2] proved that the second Taylor coefficient of any function in the class $S$ is bounded by 2 and conjectured that the bound for the $n$-th Taylor coefficient of mappings in $S$ should be $n$. The Bieberbach conjecture was proved by De Branges [3] in 1985. The Koebe mapping

$$k(z) = \frac{z}{(1 - z)^2}, \quad z \in \mathbb{D},$$

belongs to $S$ and has Taylor series expansion

$$k(z) = z + \sum_{n=2}^{\infty} nz^n.$$

Hence, the bound $n$ for the $n$-th Taylor coefficient of functions in $S$ is sharp.

Notice that since whenever $\lambda \in \partial \mathbb{D}$ and $\varphi \in S$, the rotation $\varphi_{\lambda}$ defined by $\varphi_{\lambda}(z) = \lambda \varphi(\lambda z) \in S$, the problem of maximizing the modulus of the $n$-th Taylor coefficient of functions in $S$ is equivalent to the problem of maximizing the real part of such coefficient. In the latter

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case (as it is explained in [6, Sec. 2.9]), by the use of a variational technique, Marty [12] proved that the relation

$$\tag{2} (n + 1)a_{n+1} = 2a_2a_n + (n - 1)a_{n-1}$$

must be satisfied by the coefficients of each mapping $\varphi \in S$ whose $n$–th Taylor coefficient has maximum real part. By denoting the $n$-th Taylor coefficient of a function $\varphi \in S$ by $a_n(\varphi)$, we see that the Koebe function (1) has the property that

$$\sup_{\varphi \in S} \Re\{a_n(\varphi)\} = \Re\{a_n(k)\}, \quad \text{for all } n \geq 2.$$ 

Obviously, the Koebe mapping satisfies Marty’s relation (2) for all $n$.

The variational method used by Marty to obtain (2) works more generally in the setting of linear invariant families; this is, families of locally univalent analytic functions $\varphi$ in the unit disk normalized as above and which are closed under the transformation

$$K_\zeta(z) = \frac{f\left(\frac{\zeta + z}{1 + \zeta z}\right) - f(\zeta)}{(1 - |\zeta|^2)f'(\zeta)}, \quad \zeta \in \mathbb{D}.$$ 

Several important properties, such as growth, covering, and distortion are determined by the order of a linear invariant family $F$ defined by

$$\alpha(F) = \sup_{\varphi \in F}|a_2(\varphi)| = \sup_{\varphi \in F} \Re\{a_2(\varphi)\}.$$ 

We refer to the reader to the works [13, 14], where Pommerenke studies and carries through a detailed analysis of linear invariant families. We would also like to mention [9, Ch. 5] as a good reference on the topic.

Explicit examples of linear invariant families are the class $S$ defined above (see the books [6] or [15] for more details related to this class) and also the family $F_M$ of all normalized locally univalent analytic functions in the unit disk satisfying

$$\sup_{|z| < 1} |S\varphi(z)|(1 - |z|^2)^2 \leq M,$$

where $S\varphi$ is the Schwarzian derivative of $\varphi$:

$$S\varphi = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2} \left(\frac{\varphi''}{\varphi'}\right)^2.$$ 

Regarding these families $F_M$, Pommerenke [13] proves that

$$\sup_{\varphi \in F_M} |a_2(\varphi)| = \sqrt{1 + \frac{M}{2}}.$$
It is a straightforward calculation to show that given $M \geq 0$, the function

$$k_a(z) = \frac{1}{2a} \left[ \left( \frac{1+z}{1-z} \right)^a - 1 \right], \quad |z| < 1, \quad a = \sqrt{\frac{M}{2} + 1},$$

belongs to $F_M$ and satisfies

$$\frac{1}{2} |k_a''(0)| = \sqrt{1 + \frac{M}{2}}.$$

These functions $k_a$ are called generalized Koebe functions. Note that $k_2$ coincides with the Koebe function $k$ as in [1]. Also, that if $a$ is a positive real number, the mapping $k_a \in S$ if and only if $0 < a \leq 2$. Moreover, it is easy to check that for all $a > 0$, $k_a(\mathbb{D})$ is a domain convex in the horizontal direction: this is, the intersection of $k_a(\mathbb{D})$ with any horizontal line is connected.

These generalized Koebe functions are known to be extremal for a number of problems (see, for instance, [8, Ch. 11] or [11]). In particular, in the next proposition we point out explicitly one property satisfied by $k_a$. Though the proof is not difficult and the result is probably known to the experts, we have not been able to find an explicit reference. We include it here for the sake of completeness.

**Proposition 1.** Let $F$ be a linear invariant family of analytic functions in the unit disk. Assume that there exists a function $\Phi \in F$ such that

$$\sup_{\varphi \in F} \text{Re}\{a_n(\varphi)\} = \text{Re}\{a_n(\Phi)\}.$$

Then, $\Phi = k_a$, where $a$ is the order of $F$.

**Proof.** In order to simplify the notation, let us use $A_n$ to denote the Taylor coefficients of $\Phi$; this is $A_n = a_n(\Phi)$. We first note that all coefficients of $A_n$ must be non-negative real numbers since, otherwise, we could use a rotation of the form $\Phi_\lambda(z) = \overline{\Phi}(\lambda z)$ (with appropriate $\lambda \in \partial \mathbb{D}$ that could depend on $n$) to obtain another function in $F$ with bigger Taylor coefficient than that of $\Phi$. In fact, the coefficient $A_2$ is strictly positive since it coincides with the order of $F$ and, as Pommerenke showed in [13], the order of any linear invariant family of analytic functions is always greater than or equal to 1.

Now, using the assumption that $\Phi$ maximizes the real part of any Taylor coefficient, we get that the Marty relation [2] holds for all $n \geq 1$. Hence, using also that $A_n \in \mathbb{R}$, we have that for all such $n$,

$$(n + 1)A_{n+1} = 2A_2A_n + (n - 1)A_{n-1}.$$
Therefore, we obtain
\[ \Phi'(z) = 1 + \sum_{n=2}^{\infty} nA_n z^{n-1} \]
\[ = 1 + \sum_{n=1}^{\infty} (n + 1)A_{n+1} z^{n} \]
\[ = 1 + 2A_2 \sum_{n=1}^{\infty} A_n z^{n} + \sum_{n=1}^{\infty} (n - 1)A_{n-1} z^{n} \]
\[ = 1 + 2A_2 \Phi(z) + z^2 \Phi'(z). \]

Thus, \( \Phi \) solves the linear differential equation
\[ (1 - z^2)\phi'(z) = 1 + \alpha \phi(z), \quad z \in \mathbb{D}, \]
with \( \alpha = 2A_2 \). Using basic techniques for solving linear differential equations of first order, it is easy to see that the (unique) solution to (4) with initial data \( \varphi(0) = 0 \) is \( \phi = k_a \) with \( a = \alpha/2 = A_2 \). This ends the proof. \( \square \)

Notice that Proposition 1 can be re-stated as follows:

The unique possible function maximizing the real part of every Taylor coefficient of functions in a linear invariant family of analytic mappings in \( \mathbb{D} \) is a generalized Koebe function of the form (3).

Since the generalized Koebe mappings \( k_a \) are univalent if and only if \( 0 < a \leq 2 \) (with \( k_2 \) equal to the Koebe function \( k \) as in (1)) and \( Re\{a_2(k)\} > Re\{a_2(k_a)\} \) for all \( 0 < a < 2 \), we have, as a corollary of Proposition 1, the well-known fact that the analytic Koebe function is the unique mapping in \( S \) maximizing simultaneously the real part of all Taylor coefficients of functions in \( S \).

We will obtain the analogous result to that stated in the previous proposition in the setting of sense-preserving harmonic mappings \( f \) in the unit disk. Note that since \( \mathbb{D} \) is simply connected, such harmonic mapping \( f \) has a canonical decomposition \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). As is usual, we call \( h \) the analytic part of \( f \) and \( g \) the co-analytic part of \( f \). The harmonic mapping \( f \) is analytic if and only if \( g \) is constant. A harmonic mapping \( f = h + \overline{g} \) is sense-preserving if it has positive Jacobian; this is if \( h' \) does not vanish in the unit disk and the (second complex) dilatation \( \omega = g'/h' \) has the property that \( |\omega| < 1 \) in \( \mathbb{D} \). We refer the reader to the book by Duren [7] for an excellent exposition on harmonic mappings.
For a given sense-preserving harmonic mapping \( f = h + g \), we will use the notation \( a_n(h) \) for the \( n \)-th Taylor coefficient of the analytic part of \( f \) and \( b_n(g) \) for the corresponding coefficient of \( g \). If it is clear what the functions \( h \) and \( g \) are, we just write \( a_n \) and \( b_n \) instead of \( a_n(h) \) and \( b_n(g) \), respectively.

As is usual, the family of all sense-preserving univalent harmonic mappings \( f = h + g \) in the unit disk with the normalization \( h(0) = g(0) = h'(0) - 1 = 0 \) is denoted by \( S_H \). If such univalent function satisfies the further normalization \( g'(0) = 0 \), we say that \( f \in S^0_H \). One important problem in the theory univalent harmonic mappings is to determine

\[
(5) \quad \max_{f = h + g \in S^0_H} |a_2(h)| = \max_{f = h + g \in S^0_H} \text{Re}\{a_2(h)\}.
\]

As Sheil-Small pointed out in [16], by obtaining the values in (5), it would be possible to get sharp forms of various distortion and covering theorems for the class \( S^0_H \) and also the corresponding bound for the second Taylor coefficient of mappings in \( S_H \).

We can also consider the problem of determining, for any given \( n \geq 2 \), the values of

\[
(6) \quad \max_{f = h + g \in S^0_H} |a_n(h)| = \max_{f = h + g \in S^0_H} \text{Re}\{a_n(h)\}
\]

and

\[
(7) \quad \max_{f = h + g \in S^0_H} |b_n(g)| = \max_{f = h + g \in S^0_H} \text{Re}\{b_n(g)\}.
\]

There is a great expectation that the harmonic Koebe function \( K = H + \overline{G} \), where

\[
(8) \quad H(z) = \frac{z - \frac{1}{2} z^2 + \frac{1}{6} z^3}{(1 - z)^3} \quad \text{and} \quad G(z) = \frac{\frac{1}{2} z^2 + \frac{1}{6} z^3}{(1 - z)^3} \quad (z \in \mathbb{D}),
\]

will play the extremal role in all of these problems, much like the role played by the Koebe function in the classical theory of analytic univalent mappings. More concretely, it is conjectured that the solution to (5) is \(5/2\), which equals the value of the second Taylor coefficient \( A_2 \) of the analytic part of \( K \). Also, since the coefficients of \( H \) and \( G \) are

\[
A_n = \frac{1}{6}(2n + 1)(n + 1) \quad \text{and} \quad B_n = \frac{1}{6}(2n + 1)(n + 1),
\]

it is conjectured that these are the values at (6) and (7), respectively. If this conjecture where true, we would obtain that the coefficients of any function \( f \) in \( S^0_H \) satisfy

\[
||a_n| - |b_n|| \leq n, \quad n = 2, 3, \ldots ,
\]
a generalization of the Bieberbach conjecture to the class $S^0_H$. Perhaps we should mention that this conjecture has been verified for typically real functions [5] or for starlike functions [16] but it remains open for the full class $S^0_H$. We refer to the reader to [7, Sec. 5.4] for a more detailed exposition about these problems.

Though we cannot say that we can improve any of the existing upper bounds for the problems (5), (6), or (7), we shall see that, as a consequence of our main Theorem 1 (where we characterize the mappings that maximize the real part of Taylor coefficients in the more general setting of affine and linear invariant families of harmonic mappings), we obtain the following corollary.

Corollary 1. If there is a function $f$ in the family $S^0_H$ that simultaneously solves the problems (6) and (7) for all $n \geq 2$, then $f$ is the harmonic Koebe function $K$.

1. Background

1.1. Affine and linear invariant families of harmonic mappings.

Let $F_H$ be a family of sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$, normalized by $h(0) = g(0) = 0$ and $h'(0) = 1$. The family is said to be affine and linear invariant if it closed under the two operations of Koebe transform and affine change:

$$K_{\zeta}(f)(z) = \frac{f\left(\frac{z + \zeta}{1 + \zeta z}\right) - f(\zeta)}{(1 - |\zeta|^2)h'(\zeta)}, \quad |\zeta| < 1,$$

and

$$A_{\varepsilon}(f)(z) = \frac{f(z) - \varepsilon f(z)}{1 - \varepsilon g'(0)}, \quad |\varepsilon| < 1.$$

The reader should be advise of the fact that we are assuming that all the members in an affine and linear invariant mapping of harmonic functions are sense-preserving harmonic mappings in the unit disk.

For a given affine and linear invariant family of harmonic mappings $F_H$, we use $F^0_H$ to denote the subset of functions $f = h + \overline{g} \in F_H$ satisfying $g'(0) = 0$. Note that $F^0_H$ is not an affine and linear invariant family in general.

Sheil-Small [16] offers an in depth study of affine and linear invariant families $F_H$ of harmonic mappings in $\mathbb{D}$. The order of the affine and linear invariant family, given by

$$\alpha(F_H) = \sup_{f = h + \overline{g} \in F_H} |a_2(h)| = \frac{1}{2} \sup_{f = h + \overline{g} \in F_H} |h''(0)|,$$
plays once more a special role in the analysis.

A special example of affine and linear invariant family is the class $S_H$ of (normalized) sense-preserving harmonic mappings which are univalent in the unit disk.

In [10], the authors introduce a definition for the Schwarzian derivative $S_f$ for locally univalent harmonic mappings. Using this definition for the Schwarzian derivative, it is proved in [4] that the family $F^M_H$ of sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$, with $h(0) = g(0) = 0$, $h'(0) = 1$ and $||S_f|| \leq M$, is affine and linear invariant.

1.2. Harmonic Marty relations. As it is mentioned on [7, p. 101], a slight modification of the Marty variation in the analytic case leads to analogues of the Marty relation (2) for harmonic mappings. We refer to the reader to [7, Sec. 6.5] for a proof of the fact that whenever $F_H$ is an affine and linear invariant family of harmonic mappings and $f = h + \overline{g} \in F_H$, with

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n,$$

maximizes the real part of the $n$-th Taylor coefficient of analytic parts of functions in $F^0_H$, then

$$\begin{align*}
(n + 1)a_{n+1} &= 2a_2a_n + 2b_2b_n + (n - 1)\overline{a_{n-1}} \\
(n + 1)b_{n+1} &= 2b_2b_n + 2a_2a_n + (n - 1)\overline{b_{n-1}}.
\end{align*}$$

Perhaps at this point we should stress that as long as a given function $f = h + \overline{g} \in F^0_H$ (with $h$ and $g$ as above) has a coefficient $a_n$ (resp. $b_n$) of maximum real part, then indeed, $a_n$ (resp. $b_n$) must be a non-negative real number since, otherwise, we can consider the function $f_\lambda$ defined by

$$f_\lambda(z) = \overline{\lambda f(\lambda z)} = \overline{\lambda h(\lambda z)} + \overline{\lambda g(\lambda z)},$$

with an appropriate $|\lambda| = 1$, to get another function in $F^0_H$ with bigger coefficient(s) than those of $f$.

2. Generalized harmonic Koebe functions

Given a positive real number $0 < R \leq 1$, the lens-map $l_R$ is defined by

$$l_R(z) = \frac{(\frac{1+z}{1-z})^R - 1}{(\frac{1+z}{1-z})^R + 1}, \quad |z| < 1.$$
Note that $l_1$ coincides with the identity mapping in the unit disk. Also, that for $0 < R < 1$, the function $l_R$ maps $\mathbb{D}$ onto a domain in the unit disk which is symmetric with respect to the real axes and is bounded by two circular arcs passing through the points $\pm 1$.

For any value of $0 < R \leq 1$, we can establish a relation between the generalized Koebe functions defined by (3) and the lens-maps via the formula

$$l_R = \frac{Rk_R}{1 + Rk_R}.$$  

The harmonic Koebe function $K = H + \overline{G}$, where $H$ and $G$ are given by (8), can be obtained as the horizontal shear of the Koebe mapping $k(z) = z/(1 - z)^2$ with dilatation $\omega(z) = z$. In other words, $H$ and $G$ are the solution of the linear system of equations

$$\begin{cases} H(z) - G(z) = k(z) \\ G'(z)/H'(z) = z \end{cases} \quad (z \in \mathbb{D}),$$

with the conditions $h(0) = g(0) = 0$.

For a positive real number $a$ and $0 < R \leq 1$, define the generalized harmonic mapping $K_{a,R}$ as the function $K_{a,R} = h + \overline{g}$ where $h$ and $g$ solve the linear system of equations

$$\begin{cases} h - g = k_a \\ \omega = g'/h' = l_R \end{cases} \quad h(0) = g(0) = 0.$$  

Here, $k_a$ is the generalized Koebe function defined by (3) and $l_R$ is the lens-map (11).

The solution $(h, g)$ of (12) exists for any value of $a > 0$ and gives rise to a locally univalent harmonic mapping $K_{a,R} = h + \overline{g}$. By using [5, Thm. 5.3] (see also [7, Thm. 1 on p. 37]), it is easy to see that $K_{a,R}$ is univalent in the unit disk if and only if $0 < a \leq 2$. Note that the particular case $a = 2, R = 1$ produces the harmonic Koebe mapping (8).

A straightforward computation shows that we can also define $k_{a,R}$ as the function $k_{a,R} = h + \overline{g}$, where $h$ and $g$ solve the linear system of equations

$$\begin{cases} h + g = k_b \\ \omega = g'/h' = l_R \end{cases}, \quad h(0) = g(0) = 0,$$

with $b = a + R$. In particular, we have that the harmonic Koebe function $K$ can also be written as $K = h + \overline{g}$, where

$$\begin{cases} h + g = k_3 \\ \omega = g'/h' = Id \end{cases} \quad h(0) = g(0) = 0.$$
For harmonic mappings \( f = h + \overline{g} \in S_H \), it is conjectured that the second Taylor coefficient of \( h \) is bounded by 3.

It was proved in [4] that for appropriate values of \( a \) and \( R \), the generalized harmonic Koebe function \( K_{a,R} \) is an extremal function for the problem of maximizing the real part of the second Taylor coefficient of \( h \) for \( f = h + \overline{g} \in (F^M_H)^0 \), where \( F^M_H \) is the family of normalized harmonic mappings with Schwarzian norm bounded by \( M \) mentioned in Section 1.1. In the next, final section we will show that these generalized harmonic Koebe mappings satisfy another property regarding the maximum values of Taylor coefficients of functions in linear invariant families.

3. Main result

Recall that we are using the notation

\[
    h(z) = z + \sum_{n=2}^{\infty} a_n(h)z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n(g)z^n
\]

for the Taylor series expansions of harmonic functions in the canonical decomposition of a sense-preserving harmonic mapping \( f = h + \overline{g} \) in \( \mathbb{D} \) and that we just write \( a_n(h) = a_n \) and \( b_n(g) = b_n \) if there is no ambiguity.

Before stating the main result in this paper, we would like to point out a couple of remarks.

The first one is related to the coefficient \( b_2 \) of sense-preserving harmonic functions in the unit disk \( f = h + \overline{g} \) normalized by \( h(0) = g(0) = g'(0) = 1 - h'(0) = 0 \). Notice that for such mapping \( f \), we have that \( \omega = g'/h' \) is an analytic function in the unit disk that fixes the origin. Hence, we have by the Schwarz lemma that \( |\omega'(0)| \leq 1 \). Since we can write \( g' = \omega h' \), we get

\[
    |b_2| = \frac{1}{2} |g''(0)| = \frac{1}{2} |\omega'(0)h'(0) + \omega(0)h''(0)| = \frac{1}{2} |\omega'(0)| \leq \frac{1}{2}.
\]

Therefore, we have that the real part of the coefficient \( b_2 \) is always less than or equal to 1/2.

The second remark regarding the coefficient \( a_2 \) with maximum real part in the family \( F^0_H \) is stated as a lemma.

**Lemma 1.** Let \( F^0_H \) be an affine and linear invariant family of (sense-preserving) harmonic mappings in \( \mathbb{D} \). Assume that there exists \( f_0 = h_0 + \overline{g_0} \in F^0_H \) such that

\[
    \sup_{f = h + \overline{g} \in F^0_H} Re\{a_2(h)\} = a_2(h_0) \quad \text{and} \quad \sup_{f = h + \overline{g} \in F^0_H} Re\{b_2(g)\} = b_2(g_0).
\]
Then, \( a_2(h_0) > 1/2 \).

**Proof.** Assume first, in order to get a contradiction, that \( a_2(h_0) < 1/2 \). (Recall that since \( a_2(h_0) \) maximizes \( \text{Re}\{a_2(h)\} \) in \( F_H^0 \), we necessarily have that \( a_2(h_0) \) is a positive real number.) Then, we obtain that for any function \( f = h + g \in F_H^0 \), the bound \( \text{Re}\{a_2(h)\} < 1/2 \) holds as well.

Now, let \( \Phi = \Psi + \Gamma \) be any function in \( F_H \). Arguing as on [7, p. 79], we see that there exists \( f \in F_H^0 \) and \( \beta \in \mathbb{D} \) such that \( \Phi = f + \beta \bar{f} \). A straightforward computation shows that \( \Psi = h + \beta g \). Therefore,

\[
\sup_{\Phi = \Psi + \Gamma \in F_H} \text{Re}\{a_2(\Psi)\} = \sup_{f = h + \overline{f} \in F_H^0, \beta \in \mathbb{D}} \frac{\text{Re}\{h''(0) + \beta g''(0)\}}{2} \leq \sup_{f = h + \overline{f} \in F_H^0, \beta \in \mathbb{D}} \frac{\text{Re}\{h''_0(0) + |\beta g''(0)|\}}{2} \leq \text{Re}\{h''_0(0)\} + \sup_{f = h + \overline{f} \in F_H^0} |b_2(g)| < 1.
\]

(13)

Denote by \( \mathcal{H} \) the family of all analytic parts of functions in \( F_H \). Since \( F_H \) is affine and linear invariant, it is clear that \( \mathcal{H} \) a linear invariant family of analytic functions. At it was mentioned before, Pommerenke proved that the supremum of \( \text{Re}\{a_2(\Psi)\} \) for \( \Psi \) in any linear invariant family of analytic functions is always bigger than or equal to 1. This contradicts (13) and proves that \( a_2(h_0) \geq 1/2 \).

To finish the proof, we are to show that \( a_2(h_0) = 1/2 \) is not possible. Again, to get a contradiction, we suppose that \( a_2(h_0) = 1/2 \). We have two different cases.

The first case is that

\[
\sup_{f = h + \overline{f} \in F_H^0} |b_2(g)| = b_2(g_0) < \frac{1}{2}.
\]

Then, arguing as before, we see that the order of the linear invariant family \( \mathcal{H} \) is strictly less than one, which is impossible.

The second case to be analyzed is when

\[
b_2(g_0) = \frac{1}{2}.
\]

As before, denote by \( \mathcal{H} \) the linear invariant family of analytic functions which are analytic parts of mappings in \( F_H \). The order of this family coincides with the order of the closure (with respect to the topology of locally uniform convergence) \( \overline{\mathcal{H}} \) of \( \mathcal{H} \). Our hypotheses show that the order \( \alpha(\overline{\mathcal{H}}) \) of \( \overline{\mathcal{H}} \) is equal to 1. By definition, \( \alpha(F_H) = \alpha(\mathcal{H}) \). Hence, \( \alpha(F_H) = 1 \).
On the other hand, since \( b_2(g_0) = 1/2 \), the dilatation \( \omega = g'_0/h'_0 \) of \( f_0 \) satisfies \( \omega(0) = 2b_2(g_0) = 1 \). By the Schwarz lemma, we conclude that \( \omega_0 \) equals the identity function. In other words, we have seen that there is a harmonic mapping \( f_0 = h_0 + \overline{g_0} \in F^0_H \) with dilatation equal to the identity. Hence, following the same argument as in the proof of [4, Thm. 3], we get that \( \alpha(F_H) \geq 2 \). This gives us the desired contradiction and ends the proof of this lemma. □

Note that we have just proved that if \( f = h + \overline{g} \) maximizes simultaneously the second Taylor coefficients of both analytic and anti-analytic parts of functions in \( F^0_H \), then \( a_2(h) - b_2(g) > 0 \).

Now we state and prove our main theorem.

**Theorem 1.** Let \( F_H \) be an affine and linear invariant family of (normalized) sense-preserving harmonic mappings. Assume that there is an element \( f_0 = h_0 + \overline{g_0} \in F^0_H \) such that
\[
\sup_{f = h + \overline{g} \in F^0_H} \text{Re}\{a_n(h)\} = a_n(h_0) \quad \text{and} \quad \sup_{f = h + \overline{g} \in F^0_H} \text{Re}\{b_n(g)\} = b_n(g_0)
\]
for all \( n \geq 2 \). Then, \( f_0 = K_{a,R} \), where \( K_{a,R} \) is a generalized harmonic Koebe function of the form (12). Moreover, the order of \( F_H \) equals \( \alpha(F_H) = a + R \).

**Proof.** Assume that such an extremal function \( f_0 = h_0 + \overline{g_0} \) exists. In order to simplify the notation, let us denote by \( A_n = a_n(f_0) \) and \( B_n = b_n(f_0) \). Using the normalizations for functions in \( F^0_H \) we have \( A_0 = B_0 = B_1 = 0 \) and \( A_1 = 1 \) so that we can write
\[
f_0(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} B_n z^n.
\]

As it was mentioned in Section 1.2 since (14) holds, all Taylor coefficients \( A_n \) and \( B_n \) are non-negative real numbers for \( n \geq 1 \). Moreover, the Marty relations (9) and (10) hold for all those values of \( n \). Therefore, we have that
\[
(n + 1)A_{n+1} = 2A_2A_n + 2B_2B_n + (n - 1)A_{n-1}
\]
and
\[
(n + 1)B_{n+1} = 2A_2B_n + 2B_2A_n + (n - 1)B_{n-1}
\]
for all \( n \geq 1 \).

Now, note that if we sum up (15) and (16), we get
\[
(n + 1)(A_{n+1} + B_{n+1}) = 2(A_2 + B_2)(A_n + B_n) + (n - 1)(A_{n-1} + B_{n-1})
\]
and hence, we have
\[(h'_0 + g'_0)(z) = 1 + \sum_{n=2}^{\infty} n(A_n + B_n)z^{n-1}\]
\[= 1 + \sum_{n=1}^{\infty} (n + 1)(A_{n+1} + B_{n+1})z^n\]
\[= 1 + 2(A_2 + B_2) \sum_{n=1}^{\infty} (A_n + B_n)z^n\]
\[+ \sum_{n=1}^{\infty} (n - 1)(A_{n-1} + A_{n-1})z^n\]
\[= 1 + 2(A_2 + B_2)(H + G)(z) + z^2(H' + G')(z)\].

Thus, we have that \(h_0 + g_0\) satisfies (4) with \(\alpha = 2(A_2 + B_2)\), so that \(h_0 + g_0\) equals the (analytic) generalized Koebe function \(k_{A_2 + B_2}\). In other words, the functions \(h_0\) and \(g_0\) in the canonical decomposition of \(f_0\) satisfy \(h_0 + g_0 = k_{A_2 + B_2}\) (recall that we always have that \(A_2 - B_2 > 0\) by Lemma 1). Hence, we have
\[1 + \omega(z) = \frac{k'_{A_2 + B_2}(z)}{k'_{A_2 - B_2}(z)} = \left(\frac{1 + z}{1 - z}\right)^{2B_2}\]
or
\[w(z) = \frac{(1+z)^{2B_2} - 1}{(1+z)^{2B_2} + 1} = l_{2B_2}\].

This shows that \(f_0 = K_{A_2 - B_2, 2B_2}\). (Thus, the parameters \(a\) and \(R\) in the statement of the theorem equal \(a = A_2 - B_2\) and \(R = 2B_2\), respectively.)

To prove the second part of the theorem about the order of \(F_H\), first note that \(a + R = A_2 + B_2\), where
\[A_2 = \max_{f = h + \Psi \in F_H^0} \text{Re}\{a_n(h)\} = \max_{f = h + \Psi \in F_H^0} |a_n(h)|\]
and
\[B_2 = \max_{f = h + \Psi \in F_H^0} \text{Re}\{b_n(g)\} = \max_{f = h + \Psi \in F_H^0} |b_n(g)|.\]

As it was mentioned in the proof of Lemma 1 any function \(\Phi = \Psi + \Gamma \in F_H\) can be written as
\[\Phi = f + \beta \overline{f}\].
for some \( f = h + \overline{g} \in F_H^0 \) and \( \beta \in \mathbb{D} \). Indeed, \( \Psi = h + \overline{\beta}g \), so that we get that the second Taylor coefficient of any such \( \Psi \) satisfies
\[
|a_2(\Psi)| = |a_2(h + \overline{\beta}g)| \leq A_2 + B_2,
\]
which proves that \( \alpha(F_H) \leq A_2 + B_2 \). Since the function \( h_0 + g_0 \) belongs to the closure \( \overline{H} \) of the linear family of analytic parts of functions in \( F_H \) and, by definition, \( \alpha(F_H) = \alpha(H)(= \alpha(\overline{H})) \), we get the desired result.
\( \square \)

We finish this section with a proof of Corollary 1.

**Proof.** [Proof of Corollary 1] Since \( S_H \) is an affine and linearly invariant family of harmonic functions, we can apply Theorem 1 to obtain that the function \( f_0 = h_0 + \overline{g_0} \) in the statement of the corollary must be a generalize harmonic Koebe function \( K_{a,R} \) of the form (12). In other words, there exists \( a > 0 \) and \( 0 < R \leq 1 \) such that
\[
\begin{align*}
    h_0 - g_0 &= k_a \\
    \omega &= g'/h' = l_R, \\
    h(0) &= g(0) = 0.
\end{align*}
\]

Since \( f_0 \) is necessarily univalent, we have by [5, Thm. 5.3] that \( 0 < a \leq 2 \). A straightforward computation shows that the second Taylor coefficient of the analytic part of \( K_{a,R} \) equals \( a + R/2 \) which attains its maximum value (equal to 5/2) when \( a = 2 \) and \( R = 1 \); this is, when \( K_{a,R} \) coincides with the harmonic Koebe function \( K = H + \overline{G} \) as in (8). (Note that, as a consequence, we see that the order of \( S_H \) is equal to \( A_2(H) + b_2(G) = 5/2 + 1/2 = 3 \).) \( \square \)

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**References**

[1] L. Bieberbach, Über einige Extremalprobleme im Gebiete der donformen Abbildung, *Math. Ann.* 77 (1916), 153–172.
[2] L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, *S.-B Preuss. Akad. Wiss.*, 1916, 940–955.
[3] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* 154 (1985), 137–152.
[4] M. Chuaqui, R. Hernández, and M. J. Martín, Affine and linear invariant families of harmonic mappings, arXiv:1405.5106 [math.CV].

[5] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A 9 (1984), 3–25.

[6] P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.

[7] P. L. Duren, Harmonic Mappings in the Plane, Cambridge University Press, Cambridge, 2004.

[8] A. W. Goodman, Univalent functions. Vol. II. Mariner Publishing Co., Inc., Tampa, FL, 1983.

[9] I. Graham and G. Kohr, Geometric Function Theory in One and Higher Dimensions, Marcel Dekker, Inc., New York, 2003.

[10] R. Hernández and M. J. Martín, Pre-Schwarzian and Schwarzian derivatives of harmonic mappings, J. Geom. Anal. DOI 10.1007/s12220-013-9413-x. Published electronically on April 13th, 2013.

[11] W. Koepf, Close-to-convex functions and linear-invariant families, Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), 349–355.

[12] F. Marty, Sur le module des coefficients de MacLaurin d’une fonction univalente, C. R. Acad. Sci. Paris 198 (1934), 1569–1571.

[13] Ch. Pommerenke, Linear-invariante Familien analytischer Funktionen I, Math. Ann. 155 (1964), 108–154.

[14] Ch. Pommerenke, Linear-invariante Familien analytischer Funktionen II, Math. Ann. 156 (1964), 226–262.

[15] Ch. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.

[16] T. Sheil-Small, Constants for planar harmonic mappings, J. London Math. Soc. 42 (1990), 237–248.

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