FROM EULER’S ELASTICA TO THE MKDV HIERARCHY, THROUGH THE FABER POLYNOMIALS

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Abstract. The modified Korteweg-de Vries hierarchy (mKdV) is derived by imposing isometry and isoenergy conditions on a moduli space of plane loops. The conditions are compared to the constraints that define Euler’s elastica. Moreover, the conditions are shown to be constraints on the curvature and other invariants of the loops which appear as coefficients of the generating function for the Faber polynomials.

1. Introduction

When the modified KdV (mKdV) equation was added to the unsystematically growing number of “completely integrable PDEs”, the ad hoc explanation given was Miura’s powerful idea: starting with the celebrated KdV equation

\[ u_t + 6uu_x + u_{xxx}, \]

“the simplest modification of the nonlinear term” becomes mKdV; what is now called a “Miura transformation” [14] \( u = v_x - v^2 \) factors, with the mKdV equation as the right-hand factor, \((-2v + \partial_x)(v_t - 6v^2v_x + v_{xxx}) = 0\). Solutions to mKdV thus transform into solutions of KdV, though the converse does not always hold.

However, there appeared to be no direct motivation for mKdV, as opposed to the wave motion in a shallow canal that was modeled by KdV in the nineteenth century, even though Miura (loc. cit.) does refer to anharmonic lattices. Goldstein and Petrich[4, 5] showed that going from the real \((x, y)\) to the complex \(z = x + \sqrt{-1}y\) representation of a number on the plane, takes KdV to mKdV, and they obtained the latter equation by the description of a plane region which moves in time with conserved area and perimeter: the curvature satisfies the mKdV equation (up to rescaling).

The first-named author et al. in a series of papers[8, 9, 10, 11, 12], defined “quantized elastica”, based on Euler’s Elastica theory, and derived mKdV. Euler’s Elastica have a long history, spanning from Pappus of Alexandria around 300 A.D., to splines and computer vision today, through the discovery of the addition theorem for elliptic functions, which give the closed-form solution, as thoroughly surveyed in a recent Ph.D. thesis[7, 6], as well as Truesdell’s classic works[17] on the history of mechanics, particularly Euler’s works. Put in modern terms, the elastica are curves that solve an isoperimetric variational problem, by minimizing a potential energy, but there are many equivalent formulations; in particular, by writing the equivalent dynamics of the simple swinging pendulum, the general solution can be parametrized by one number, a Lagrange multiplier, representing the force of gravity in a suitable normalization[6]. “Quantized elastica” replace the Euler-Bernoulli energy functional[17] (cf. Section 2 for the meaning of the symbols),

\[ \mathcal{E}[Z] := \int \{Z, z\} \text{sd} s = \frac{1}{2} \int k^2 \text{d} s \]
with a ‘quantized’ version
\[ Z[\beta] = \int_M DZ \exp(-\beta E[Z]), \]
the partition function\[15\] of the temperature $\beta$. The mKdV hierarchy then appears by deforming the coefficients of the expansion of the integral at its extrema\[8\].

In this paper, we derive the mKdV hierarchy by introducing time parameters over a suitable moduli space of plane loops; we use the setting of Brylinski\[2\] to construct differential flows on several related moduli spaces, and we partition the appropriate moduli space by the mKdV orbits under a commuting hierarchy of isometric and isoenergy flows. We present the invariant that defines the elastica as the integral of a coefficient in the expansion of a holomorphic function used by Tjurin\[16\] to represent projective connections on a Riemann surface, so that the setting is that of the moduli space of curves; both the isometric and isoenergy conditions are therefore constraints on the coefficients of said expansion, which points out a relationship with modular forms and Faber polynomials (Subsection 6.5).

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2. Setting and review

We work in the free loop space $LM$ of smooth (i.e., real analytic) maps $S^1 \to M$ (cf. Section 3.1 in Ref. \[2\]), where the (smooth, paracompact, oriented) manifold $M$ is the real plane. In other words, our space $LM$ is the set of real-analytic immersions of $S^1$ in $\mathbb{C}$, $Z : S^1 \hookrightarrow \mathbb{C}, Z(s) \in C^\omega(S^1, \mathbb{C})$, which we parametrize by arclength; moreover, we fix the length and take the quotient space under Euclidean motion:
\[ M := \{ Z : S^1 \hookrightarrow \mathbb{C} | Z(s) \in C^\omega(S^1, \mathbb{C}), |dZ/ds| = 1, \oint |dZ| = 2\pi \}, \quad \mathbb{M} := M/\sim. \]

The condition $|dZ/ds| = 1$, namely $dZ/ds \in U(1)$, allows us to define an action $g_{s_0}Z(s) = Z(s - s_0)$ (where we identified $S^1$ with $\mathbb{R}$ modulo the integers); we call “Euclidean motion” a transformation with this property. We set:
\[ \mathfrak{M} := \mathbb{M}/U(1). \]

We denote the natural projections by $pr_1 : M \to \mathbb{M}$ and $pr_2 : \mathbb{M} \to \mathfrak{M}$. For $Z \in M$, which we identify with $pr_1(Z)$,
\[ \partial_s Z(s) = e^{\sqrt{-1}\varphi(s)} \]
where $\varphi$ is a real-analytic function. The curvature of the loop is given by:
\[ k(s_0) := \frac{1}{\sqrt{-1} \frac{Z''(s_0)}{Z'(s_0)}} = \partial_s \varphi(s_0), \]
so that, by equating separately the real and imaginary part\[4\], the Frénet-Serret relation can be expressed as,
\[ \partial_s(\partial_s Z) = k\sqrt{-1}(\partial_s Z). \]
Moreover, the integral of $k$ with respect to $ds$

$$W[Z] := \frac{1}{\sqrt{-1}} \oint \frac{Z''(s)}{Z'(s)} ds = \oint k ds$$

is the winding number of the loop.

The integral

$$\mathcal{E}[Z] := \oint \{Z, s\}_{SD} ds = \frac{1}{2} \oint k^2 ds,$$  \hspace{0.5cm} (2.1)

where the subscript SD denotes the Schwarzian derivative (2.3), is the Euler-Bernoulli energy functional—here we return to the representation of $s$ as a complex number of norm 1.

To address applications to biology and other sciences, the first-named author[12] replaced the Euler-Bernoulli functional with:

$$Z[\beta] = \int M DZ \exp(-\beta \mathcal{E}[Z])$$

on a measure space $(\mathfrak{M}, \mathfrak{F})$ with measure $DZ$.

We note that these invariants appear in an expansion that was used by Tjurin[16] to encode the projective structure of a Riemann surface, but we point out that in our case it is a formal expansion because, while $s$ is viewed as a complex number by Tjurin (and in the expression of the Schwarzian derivative below), it must be viewed as a real number when taking differences of the type $s - s_0$.

$$\frac{1}{2} \log \frac{Z(s_2) - Z(s_1)}{s_2 - s_1} = \frac{1}{2} \log Z'(s_0) + \frac{1}{2} \frac{Z''(s_0)}{Z'(s_0)}[(s_1 - s_0) + (s_2 - s_0)]$$

$$+ \frac{1}{2} \frac{Z''(s_0)}{Z'(s_0)} \left[ \frac{3}{4} \left( \frac{Z''(s_0)}{Z'(s_0)} \right)^2 \right] [(s_2 - s_0)^2 + (s_1 - s_0)^2]$$

$$+ \frac{1}{2} \frac{Z''(s_0)}{Z'(s_0)} \left[ \frac{3}{2} \left( \frac{Z''(s_0)}{Z'(s_0)} \right)^2 \right] (s_1 - s_0)(s_2 - s_0)$$

+ \ldots \hspace{0.5cm} (2.2)

Using the representation

$$\partial_s Z(s) = e^{\sqrt{-1} \varphi(s)},$$

the first term of the expansion is $\sqrt{-1} \varphi(s_0)/2$, the coefficient of the second term is the curvature $k(s_0)$, the coefficients of the third and fourth term are expressed by:

$$\left[ \frac{Z''(s_0)}{Z'(s_0)} - \frac{3}{4} \left( \frac{Z''(s_0)}{Z'(s_0)} \right)^2 \right] = - \left[ \frac{1}{4} k^2 - \sqrt{-1} \partial_s k \right]$$

$$\frac{1}{2} \left[ \frac{Z''(s_0)}{Z'(s_0)} - \frac{3}{2} \left( \frac{Z''(s_0)}{Z'(s_0)} \right)^2 \right] = \left[ \frac{1}{4} k^2 + \sqrt{-1} \partial_s k \right].$$

We write them together:

$$w^2 \pm \sqrt{-1} \partial_s w = \left[ \frac{1}{4} k^2 \pm \sqrt{-1} \partial_s k \right]$$
where \( w = k/2 \). The fourth coefficient is the Schwarzian derivative, namely the operator,

\[
\{f, z\}_{SD} = \left[ \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right],
\]

(2.3)
in our case

\[
\{Z, s\}_{SD} = \frac{\partial}{\partial s_1} \frac{\partial}{\partial s_2} \frac{1}{2} \log \frac{Z(s_2) - Z(s_1)}{s_2 - s_1} \bigg|_{s_2 = s_1 = s}.
\]

The integrals of the coefficients of both the third and fourth terms are therefore proportional to the Euler-Bernoulli energy functional (2.1).

For an element \( Z \in \mathbb{M} \), \( Z(0) \) vanishes up to Euclidean motion; assuming this to be the case, we introduce the expansion,

\[
Z(s) = s + a_1 s^2 + a_2 s^3 + a_3 s^4 + \cdots
= \cdots + c_{-2} q^{-2} + c_{-1} q^{-1} + c_1 q^1 + c_2 q^2 + c_3 q^3 + \cdots,
\]

where \( q = e^{\sqrt{-1}s} \) and will use the coefficients \( a_i \) as moduli of the loops; we introduce a dependence of the \( a_i \)'s on parameters \( t = (t_2, t_3, \ldots) \in [0, \varepsilon) \), where \( \varepsilon \) is a small positive number, to be viewed as deformations independent of each other,

\[
[\partial_{t_i}, \partial_{t_j}] = 0, \quad (i, j = 2, 3, \ldots),
\]

(2.4)
and we require that they also satisfy the condition

\[
[\partial_s, \partial_{t_i}] = 0, \quad (i = 2, 3, \ldots),
\]

(2.5)
which we call “isometric”; we think of \( s \) as the stationary flow \( t_1 \), cf. Section 5. We will derive the mKdV hierarchy by imposing an extra condition on the loops

\[
Z = Z(s, t_2, t_3, \ldots), \quad a_i = a_i(t_2, t_3, \ldots).
\]

We denote by \( \mathcal{A}_{S^1}^p(\mathbb{R}) \) the real-analytic \( p \)-forms on \( S^1 \) with \( \mathbb{R} \)-valued coefficients, and by \( \mathcal{A}_{S^1}^0(\mathbb{C}) \) the real-analytic \( p \)-forms on \( S^1 \) with \( \mathbb{C} \)-valued coefficients.

**Lemma 2.1.** The kernel of the \( \mathbb{R} \)-linear map \( f : \mathcal{A}_{S^1}^1(\mathbb{R}) \to \mathbb{R} \) is equal to \( d\mathcal{A}_{S^1}^0(\mathbb{R}) \).

**Proof.** For \( \eta \in \mathcal{A}_{S^1}^1(\mathbb{R}) \), there exists \( F \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \) such that \( \eta := dF \) since \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \). When \( \eta \) is in the kernel of \( f \), \( F(2\pi) = F(0) \), so \( F \) belongs to \( \mathcal{A}_{S^1}^0(\mathbb{R}) \). \( \square \)

For a given \( Z \in \mathbb{M} \), the tangential angle \( \varphi \in \mathcal{A}_{S^1}^0(\mathbb{R}) \) and the curvature \( k \in \mathcal{A}_{S^1}^0(\mathbb{R}) \) are defined as

\[
\varphi(s) := \varphi[Z](s) := \frac{1}{\sqrt{-1}} \log \partial_s Z(s) \in \mathcal{A}_{S^1}^0(\mathbb{R}),
\]

\[
k(s) := k[Z](s) := \partial_s \varphi[Z](s),
\]
consistent with the terminology at the beginning of this Section. Since \( |\partial_s Z(s)| = 1 \), for fixed parameters \( t_i \), we view the curvature \( k \) as an element of \( \mathcal{A}_{S^1}^0(\mathbb{R}) \) and \( k ds \) as an element of \( \mathcal{A}_{S^1}^1(\mathbb{R}) \).
3. Isometric Deformations

We consider the tangent space of the isometric deformations in $\mathbb{M}$, namely a fiber space $T_Z\mathbb{M}$ over the tangent space $T\mathbb{M}$ of $\mathbb{M}$ at $Z$ in $\mathbb{M}$. Regarding the parameter $s$ in $Z$ as fixed, the fiber is given by the isometric deformations of $\mathbb{M}$. We let $\text{IDiff}(\mathbb{M})$ be the group acting on $\mathbb{M}$ that gives isometric deformations, and we consider the action of its Lie algebra $\text{idiff}(\mathbb{M})$ at a point $Z$ in $\mathbb{M}$. We identify the infinitesimal orbit of $\text{idiff}(\mathbb{M})$, or the infinitesimal deformations, and the elements of $T_Z\mathbb{M}$ at $Z \in \mathbb{M}$. We do not need to prove the existence of this, say, analytic Lie group, because it will consists of the flows that we construct concretely. We take the limit of the flow parametrized by $t \in [0,\varepsilon)$ at $Z \in \mathbb{M}$ as $\varepsilon$ approaches zero, in order to investigate the fiber $T_Z\mathbb{M}$. As a result, this limit gives us tangent vectors and differential operators belonging to the fiber, without a need to introduce a connection: for our purposes, we only need local calculations for given coordinates.

A deformation is defined by:

$$\partial_t Z(s) = v(s)\partial_s Z(s), \quad s \in S^1,$$

where $v$ belongs to $A^0_{S^1}(\mathbb{C})$ and

$$v(s) = v^{(t)}(s) + \sqrt{-1}v^{(i)}(s),$$

where $v^{(t)}$ and $v^{(i)}$ denote real and imaginary parts respectively, so that

$$v^{(t)}, v^{(i)} \in A^0_{S^1}(\mathbb{R}) := C^\omega(S^1, \mathbb{R}).$$

Recalling that the “isometric” condition means that $s, t$ are independent variables, namely $\partial_t \partial_s Z = \partial_s \partial_t Z$, we check that:

**Proposition 3.1.** For a point $Z \in \mathbb{M}$, the isometric condition $\partial_t \partial_s Z = \partial_s \partial_t Z$ is reduced to

$$\partial_t \partial_s \phi(s) = \partial_s \partial_t \phi(s), \quad s \in S^1,$$

where $\phi = \phi[Z]$.

**Proof.** The statement follows from the calculations, $\partial_t \frac{\partial^2 Z}{\partial_s Z} = \frac{\partial_t \partial_s^2 Z}{\partial_s Z} - \frac{\partial_s^2 Z}{(\partial_s Z)^2} \partial_t \partial_s Z$ and $\partial_s \frac{\partial_t \partial_s Z}{\partial_s Z} = \frac{\partial_s \partial_t \partial_s Z}{\partial_s Z} - \frac{\partial_s^2 Z}{(\partial_s Z)^2} \partial_t \partial_s Z$.

An isometric deformation of $\partial_s Z = e^{\sqrt{-1}t\varphi}$ belongs to $C^\omega(S^1, U(1))$, and its derivative along $t \in [0, \varepsilon)$ is given by

$$\partial_t \partial_s Z = \sqrt{-1} \partial_t \varphi(s) \partial_s Z.$$

By applying $\partial_s$ to both sides of (3.1), we have

$$\partial_s \partial_t Z = \partial_s (v \partial_s Z).$$

$$\sqrt{-1}(\partial_t \varphi) \partial_s Z = (\partial_s v + v \sqrt{-1} k) \partial_s Z.$$

Since $\partial_t \varphi$ is real-valued, we obtain two conditions by equating the real and imaginary parts:
Proposition 3.2. For a point $Z \in \mathbb{M}$, the isometric condition $[\partial_s, \partial_t]Z = 0$ is reduced to two conditions\cite{4, 5}:

\begin{align*}
\partial_t k &= \partial_s (\partial_s \frac{1}{k} \partial_s + k)v^{(r)} = \partial_s (\partial_s + k \partial_s^{-1} k)v^{(i)}, \\
kv^{(i)} &= \partial_s v^{(r)}.
\end{align*}

\begin{equation}
(3.2)
\end{equation}

In the limit as $\varepsilon$ approaches zero, these conditions hold on the fiber of the tangent space. Therefore we can define the following length-preserving linear transformation on the tangent space, which we identified with $\text{diff}^1(\mathbb{M})$,

\begin{equation}
\partial_s \left( \begin{array}{c} v^{(i)} \\ v^{(r)} \end{array} \right) = \left( \begin{array}{cc} 0 & \partial_s^{-1} \partial_s \\ k & 0 \end{array} \right) \left( \begin{array}{c} v^{(i)} \\ v^{(r)} \end{array} \right), \quad \left( \begin{array}{cc} \partial_s & -\partial_s^{-1} \partial_s \\ -k & \partial_s \end{array} \right) \left( \begin{array}{c} v^{(i)} \\ v^{(r)} \end{array} \right) = 0.
\end{equation}

\begin{equation}
(3.3)
\end{equation}

We derive some consequences from the second condition, (3.3). Let us consider the map,

\begin{equation}
\ell_d : A^0_{S^1}(\mathbb{R}) \to A^1_{S^1}(\mathbb{R}), \quad \ell_d(v^{(i)}) = kv^{(i)}ds,
\end{equation}

and denote the inverse image of $dA^0_{S^1}(\mathbb{R}) \subset A^1_{S^1}(\mathbb{R})$ by $\widehat{A}^0_{S^1}(\mathbb{R}) := \ell_d^{-1}dA^0_{S^1}(\mathbb{R})$, together with the map,

\begin{equation}
\widehat{\ell}_d : \widehat{A}^0_{S^1}(\mathbb{R}) \to dA^0_{S^1}(\mathbb{R}) \subset A^1_{S^1}(\mathbb{R}), \quad \widehat{\ell}_d(v^{(i)}) = kv^{(i)}ds = \partial_s v^{(r)}ds = dv^{(r)}.
\end{equation}

We define $\ell_r : \widehat{A}^0_{S^1}(\mathbb{R}) \to A^0_{S^1}(\mathbb{R})/\mathbb{R}$ where $\ell_r(v^{(i)}) = \int_0^s kv^{(i)}ds = \int_0^s \partial_s v^{(r)}ds$. We will need (cf. Lemma 5.1) only deformations whose real part $v^{(r)}$ is unique up to multiplicative constant, so we fix a representative for the image in the quotient $\ell_r^0 : \widehat{A}^0_{S^1}(\mathbb{R}) \to A^0_{S^1}(\mathbb{R})$ of $\ell_r$ so that $\ell_r^0(v^{(i)}) = 0$, by requiring

\begin{equation}
\oint ds \ell_r^0(f) = 0.
\end{equation}

\begin{equation}
(3.4)
\end{equation}

We have an induced map,

\begin{equation}
\ell : \widehat{A}^0_{S^1}(\mathbb{R}) \to A^0_{S^1}(\mathbb{C}), \quad \ell(f) = \ell_r^0(f) + \sqrt{-1}f,
\end{equation}

which is a Euclidean motion,

\begin{equation}
\partial_t Z|_{s=0} = \ell(v^{(i)}) \partial_s Z|_{s=0} = C.
\end{equation}

By introducing a loop of constant derivative,

\begin{equation}
\partial_t Z_0 = C,
\end{equation}

we can arrange for

\begin{equation}
\partial_t (Z - Z_0)|_{s=0} = C - C = 0;
\end{equation}

similarly, we can add a constant to $\partial_t \phi(s)$ by using rotation as the Euclidean motion. We will use this choice of representative in $\mathbb{M}$ for an element $Z \in \mathcal{M}$.

The following proposition is essentially the same as Proposition 3.3.4 (i) in Ref. [2], following from (3.3) for the map $\ell$.  

\begin{equation}
6
\end{equation}
Proposition 3.3. For $v \in \hat{A}_{S_1}^0(\mathbb{R})$ and $\tilde{Z} \in \mathcal{M}$, $\ell$ induces the bijection $\ell^\sharp$ and the surjection $\ell^\flat$,

$$
\begin{array}{c}
\hat{A}_{S_1}^0(\mathbb{R}) \\
\downarrow \ell^\sharp
\end{array}
\xrightarrow{\ell^\flat}
\begin{array}{c}
T_{\tilde{Z}}(\mathcal{M}) \\
\downarrow \text{pr}_{1*}
\end{array}
\begin{array}{c}
T_{\text{pr}_1(\tilde{Z})(\mathbb{M})}
\end{array}
$$

in the sense that, for every element $v$ in the vector space $T_{\tilde{Z}}(\mathbb{M})$, where $Z$ has the representative $\tilde{Z}$ in $\mathcal{M}$, we have an element $v^{(i)} \in \hat{A}_{S_1}^0(\mathbb{R})$ such that $v = \ell(v^{(i)})$.

Given this result, it will be more convenient and sufficient for our purposes to deal with elements of $\mathbb{M}$ rather than $\mathcal{M}$.

For the first condition, (3.2), let us introduce the differential operators,

$$
\Omega^{(I)} := \partial_s (\partial_s - \frac{1}{k} \partial_s + k), \quad \Omega^{(II)} := \partial^2_s + \partial_s (k \partial_s^{-1} k),
$$

where $\Omega^{(II)}$ is the recursion operator for the mKdV hierarchy.

Lemma 3.4. The infinitesimal isometric deformation (3.1) is reduced to

$$
\partial_t k = -\Omega^{(II)} v^{(i)}.
$$

as an element in the fiber of $T_{\tilde{Z}}(\mathbb{M})$.

4. ISOENERGY DEFORMATIONS

We will now impose, in addition to the isometric condition, that the Euler-Bernoulli energy functional $E$ is preserved by the deformation. We call the Lie group of isometric and isoenergy motions acting on $\mathbb{M}$ $\text{idiff}(\mathbb{M}) \subset \text{Diff}(\mathbb{M})$, and its Lie algebra acting at a point $Z$ in $\mathbb{M}$, $\text{iidiff}(\mathbb{M})$. We identify the infinitesimal orbit of an element of $\text{iidiff}(\mathbb{M})$ with the infinitesimal deformation itself and with the corresponding element of $T_{\tilde{Z}}(\mathbb{M})$ of $Z \in \mathbb{M}$.

Definition 4.1. We define

$$
\mathbb{M}_E := \{ Z \in \mathbb{M} \mid E[Z] = E \text{ is preserved}\},
$$

and denote by $\text{pr}_{E}$ the natural projection $\mathbb{M}_E \to \mathbb{M}_E := \mathbb{M}_E/U(1)$.

From Lemma 2.1, we have the following result:

Proposition 4.2. For $Z \in \mathbb{M}$, $\partial_t E(Z)$ vanishes iff $k \partial_t k ds \in d\mathbb{A}_{S_1}^0(\mathbb{R})$, i.e., there exists a function $f \in \mathbb{A}_{S_1}^0(\mathbb{R})$ satisfying $k \partial_t k = \partial_s f$.

Note that (3.3), one of the isometric conditions, says that $k \partial_t k$ is an exact differential, akin to the isoenergy condition.

We now construct an isometric deformation: let $v^{(i)} := \partial_t k$ and $v^{(r)} := \ell_r(v^{(i)})$, which satisfies $k \partial_t k = kv^{(i)} = \partial_s v^{(r)}$. This yields the isometric condition,

$$
\partial_t v Z = (\ell(v^{(i)})) \partial_s Z,
$$

or

$$
\partial_t v k = \Omega^{(I)} v^{(i)}, \quad k v^{(i)} = \partial_s v^{(r)}.
$$

The following proposition and corollary use this construction:
Proposition 4.3. If two isometric deformations, \( v^{(i)}, v'^{(i)} \in \hat{A}^{0}_{S^1}(\mathbb{R}) \),
\[
\partial_t Z = (\ell(v^{(i)})) \partial_s Z \quad \text{or} \quad \partial_t k = \Omega^{(\Pi)} v^{(i)},
\]
\[
\partial_t Z = (\ell(v'^{(i)})) \partial_s Z \quad \text{or} \quad \partial_t k = \Omega^{(\Pi)} v'^{(i)},
\]
satisfy the relation \( \partial_t k = v'^{(i)} \), then:

1. the deformation \( \partial_t Z \) is isoenergy, i.e., \( \partial_t E[Z] = 0 \), and
2. \( \partial_t k = \Omega^{(\Pi)} v'^{(i)} = \Omega^{(\Pi)^2} v^{(i)} \).

The converse also holds:

Corollary 4.4. If the isometric deformation,
\[
\partial_t Z = (\ell(v^{(i)})) \partial_s Z \quad \text{for which} \quad \partial_t k = \Omega^{(\Pi)} v^{(i)},
\]
is isoenergy, then there is another isometric flow given by \( \ell(v'^{(i)}) \) satisfying \( \partial_t k = v'^{(i)} \).

Remark 4.5. Proposition 4.3 (2) gives rise to a recursive construction: indeed, if \( k \partial_t k \) belongs to \( \hat{d}_A^{0}_{S^1}(\mathbb{R}) \), there is another isometric deformation \( k \partial_t k = \Omega^{(\Pi)} v^{(i)} \), satisfying \( \partial_t k = v'^{(i)} \); then the deformation \( \partial_t Z \) is also isoenergy, and
\[
\partial_t k = \Omega^{(\Pi)^3} v^{(i)}
\]
holds.

It follows from this remark that,

Proposition 4.6. If \( v^{(i)} \in A^{0}_{S^1}(\mathbb{R}) \) is given such that each element of the sequence \( \{\Omega^{(\Pi)^n} v^{(i)}\}_{n=0,1,2,...} \) belongs to \( \hat{A}^{0}_{S^1}(\mathbb{R}) \), by introducing parameters \( (\tilde{t}_1, \tilde{t}_2, \ldots) \in [0, \varepsilon) \), we obtain infinitesimal isometric and isoenergy deformations satisfying
\[
\partial_{\tilde{t}_1} k = \Omega^{(\Pi)} v^{(i)},
\]
\[
\partial_{\tilde{t}_2} k = \Omega^{(\Pi)} \partial_{\tilde{t}_1} k = \Omega^{(\Pi)^2} v^{(i)},
\]
\[
\partial_{\tilde{t}_3} k = \Omega^{(\Pi)} \partial_{\tilde{t}_2} k = \Omega^{(\Pi)^2} \partial_{\tilde{t}_1} k = \Omega^{(\Pi)^3} v^{(i)},
\]
\[
\partial_{\tilde{t}_4} k = \Omega^{(\Pi)} \partial_{\tilde{t}_3} k = \Omega^{(\Pi)^2} \partial_{\tilde{t}_2} k = \Omega^{(\Pi)^3} \partial_{\tilde{t}_1} k = \Omega^{(\Pi)^4} v^{(i)},
\]
\[
\vdots
\]

However, the problem remains of finding an initial isometric and isoenergy deformation such that each element of the sequence \( \{\Omega^{(\Pi)^n} v^{(i)}\}_{n=0,1,2,...} \) belongs to \( \hat{A}^{0}_{S^1}(\mathbb{R}) \). We will construct a natural one in the next Section.

5. The stationary deformation

Since \( Z(s) \) and \( Z(s - s_0) \) have the same image loop, we call this the “stationary” deformation, in accordance with the terminology for the mKdV equation, when the solution \( u(x, t_2, \ldots, t_n, \ldots) \) is viewed as a function of \( x \) only (N.B. In previous work by the first-named author (Section 3 in Ref. [8]), this is called the “trivial” deformation).

Lemma 5.1. For any given real number \( c \in \mathbb{R} \) and \( Z \in \mathcal{M} \), \( Z(s + ct) \), the stationary deformation, is a solution of
\[
\partial_t Z = cZ.
\]
Proposition 5.2. The stationary deformation is compatible with the map \( \text{pr}_2 : \mathcal{M} \to \mathcal{M} \) and with its lifting map \( \text{id} : \mathcal{E} \to \mathcal{E} \), the identity.

Remark 5.3. The stationary deformation is isometric and isoenergy, according to the equation,
\[
\partial t_1 k = \Omega^{(II)} \partial_s k = \partial_s k.
\]

Proposition 5.4. For \( Z \in \mathcal{M} \) and \( k := k[Z] \), starting with the stationary deformation:
\[
\partial t_1 k = \partial_s k,
\]
each element of the sequence \( \{\Omega^{(II)} \partial_s k\}_{n=0,1,2,\ldots} \) belongs to \( \hat{\mathcal{A}}^0_{S_1}(\mathbb{R}) \), yielding a sequence of isometric and isoenergy relations,
\[
\partial t_2 k = \Omega^{(II)} \partial_s k = \Omega^{(II)} \partial_t k = \Omega^{(II)} \partial_s k,
\]
\[
\partial t_3 k = \Omega^{(II)} \partial_s k = \Omega^{(II)} \partial_s k = \Omega^{(II)} \partial_s k,
\]
\[
\partial t_4 k = \Omega^{(II)} \partial_s k = \Omega^{(II)} \partial_s k = \Omega^{(II)} \partial_s k,
\]
\[
\vdots
\]
This recursive sequence is known as the modified KdV hierarchy.

Proof. We let \( v^{(i)} \) in Proposition 4.6 correspond to the stationary deformation \( v^{(i)} = \partial_s k \), and \( \tilde{t}_i = t_{i-1} \) for \( i = 2, 3, \ldots \). The equalities hold because the sequence of \( t_i \) are the higher mKdV flows, since \( \Omega^{(II)} \) is the mKdV recursion operator. \( \square \)

Remark 5.5. For the stationary deformation, we can write:
\[
\partial_s k = \Omega^{(II)} = (\partial_s^2 + \partial_s k \partial_s^{-1} k) \cdot 0 = \partial_s k \partial_s^{-1} k \cdot 0
\]
by choosing the factor, \( \partial_s^{-1} 0 = 1 \), and thus write,
\[
\partial t_1 k = \Omega^{(II)} 0.
\]
Then we can take \( v^{(i)} \) in Proposition 4.6 equal to 0. Then, with \( v^{(i)} = 0, \tilde{t}_i = t_i \) for \( i = 1, 2, 3, \ldots \).

By letting \( w_j = \partial_{y_j} \varphi \), we also have a matrix format:
\[
\partial_s \begin{pmatrix}
1 \\
w_1 \\
w_2 \\
\vdots \\
w_\ell
\end{pmatrix} = \begin{pmatrix}
\Omega^{(II)} \\
\Omega^{(II)} \\
\vdots \\
\Omega^{(II)}
\end{pmatrix} \partial_s \begin{pmatrix}
1 \\
w_1 \\
w_2 \\
\vdots \\
w_\ell
\end{pmatrix}.
\]

It is well-known that the mKdV hierarchy is integrable and its orbits are well-defined, thus the Lie-group and Lie-algebra actions we used, for isometric and isoenergy diffeomorphism IIDiff(\( \mathcal{M} \)) on \( \mathcal{M} \), make sense.

If the curvature \( k \) is given, \( \varphi \) is determined modulo a constant of integration. If \( \varphi \) is given, \( Z \) is determined modulo Euclidean motion. Therefore, the mKdV hierarchy acts on the space \( \mathcal{M} \) rather than \( \mathbb{M} \):

Proposition 5.6. There exists an action IIDiff(\( \mathcal{M} \)) on \( \mathcal{M} \) given by the mKdV hierarchy.
6. Related constructions, finite orbits, Comments

In this section, we review previous results (Prop. 3.11 in Ref. [10]), to relate the KdV flow with the deformations we described; we give equations for a class of finite mKdV orbits; and conclude with comments.

6.1. Connection between KdV flow and mKdV flow. In this subsection, we show the connection between the KdV flow (Prop. 3.11 in Ref. [10]) and the mKdV flow described above.

For a solution \( \psi \) of

\[
\left( -\partial_s^2 - \frac{1}{2}\{ Z, s \}_{SD} \right) \psi = 0,
\]

the deformation \( \partial_t Z = v \partial_s Z \) induces the deformation,

\[
\partial_t \psi = -\frac{1}{2} (\partial_s v) \psi + v \partial_s \psi
\]

by direct computation. When \( v^{(i)} = \partial_s k \) is the case, \( v = \frac{1}{2} k^2 + \sqrt{-1} \partial_s k \). Since this deformation gives the mKdV hierarchy, \( v \) induces the KdV hierarchy, via the Miura map.

We have the natural symplectic structure on \( T\mathcal{M} \),

\[
\langle Y_1, Y_2 \rangle = \frac{1}{2} \int_{S^1} \left( \int_0^s (Y_2(s)Y_1(s') - Y_1(s)Y_2(s')) ds' \right) ds,
\]

under which the KdV flow is Hamiltonian. Moreover, the KdV hierarchy has a bi-Hamiltonian structure, one of the two compatible structures being associated to the Hamiltonian flow which preserves the Euler-Bernoulli energy of the elastica.

6.2. Symplectic structures. Besides the symplectic structure for the mKdV flow associated with (6.1) through the correspondence between KdV and mKdV, we have a sequence of natural symplectic structures, as follows.

For two deformations \( \partial_t Z = u \partial_s Z \) and \( \partial_t' Z = u \partial_s Z \), \( (v(s) = v^{(i)}(s) + \sqrt{-1} v^{(r)}(s), u(s) = u^{(r)}(s) + \sqrt{-1} u^{(i)}(s)) \), we define:

\[
\langle u, v \rangle_\ell = \frac{1}{2} \int \text{Im} \left( \partial_t' \partial_t Z \right) k^\ell ds
\]

\[
= \frac{1}{2} \int \left( (u^{(r)}(s)v^{(i)}(s) - u^{(i)}(s)v^{(r)}(s)) \right) k^\ell(s) ds,
\]

where \( \ell \) is a natural number. Clearly, \( \langle u, v \rangle_\ell = -\langle v, u \rangle_\ell \). Moreover, for \( u \) and \( v \) giving isometric deformations,

\[
\langle u, v \rangle_\ell = \int k^{\ell-1}(s) \left( v^{(r)}(s) \partial_s u^{(i)}(s) \right) ds
\]

by straightforward computation. The non-degeneracy is also clear: for a point \( u \) of \( T\mathcal{M} \) such that \( \langle u, v \rangle_\ell = 0 \) for every \( v \) of \( T\mathcal{M} \), \( u = 0 \). For the \( \ell = 1 \) case, this gives the natural symplectic structure of \( T\mathcal{M} \).
6.3. Finite-dimensional orbits of a point \( Z \). The (disjoint) orbits of the action \( \text{IIDiff}(M) \) on \( M \) give a decomposition of \( M \).

The orbits are finite-dimensional when the hierarchy defined recursively contains a finite number of linearly-independent flows. This happens, in particular, when the deformations \( \partial_t \) are stationary for \( i > g + 1 \), although this is not the only case: as an analogy, the flows of a finite-dimensional orbit of the KdV equation span the Jacobian of a spectral curve, of given genus \( g \); they are coordinatized by the coefficients of the expansion of a basis of holomorphic differentials in (the inverse of) a local parameter \( t \) on the curve, \( \omega_i = \sum_{j=0}^{\infty} c_j^i t^j \), \( i = 1, ..., g \), and of course all vectors \( (c_1^j, ..., c_g^j) \) could be non-zero. We call the union of the particular finite-dimension orbits for which the flows become stationary exactly at the \( g \) stage \( M_g \), although the orbit in fact could have any smaller dimension, equal to the dimension of the span of the first \( g - 1 \) flows:

\[
M_g := \{ Z \in M \mid \text{for } k = 1, ..., g, \Omega^{(II)}k = 0 \},
\]

\[
M_g := \{ Z \in M \mid \text{for } k = 1, ..., g, \Omega^{(II)}k = 0 \} = \text{pr}_2(M_g),
\]

giving a filtration:

\[
M_g \subset M_{g+1}, \quad \text{and} \quad M_g \subset M_{g+1}.
\]

The \( g = 1 \), non-stationary case corresponds to Euler’s elastica[12].

**Proposition 6.1.** The elements of the subspace \( M_g \) are defined by the condition,

\[
\partial_s \begin{pmatrix}
1 \\
w_1 \\
w_2 \\
\vdots \\
w_{g-1} \\
w_g
\end{pmatrix} = \begin{pmatrix}
\Omega^{(II)} & \Omega^{(II)} & \cdots & \Omega^{(II)} \\
\Omega^{(II)} & \cdots & \Omega^{(II)} \\
\cdots & \cdots & \cdots & \cdots \\
\Omega^{(II)} & \cdots & \cdots & \cdots
\end{pmatrix} \partial_s \begin{pmatrix}
1 \\
w_1 \\
w_2 \\
\vdots \\
w_{g-1} \\
w_g
\end{pmatrix},
\]

where the multiplicative-constant ambiguity of each \( \Omega^{(II)} \) is adjusted by rescaling the corresponding \( t_j \).

6.4. **Action on cohomology.** As in Ref. [10], we can interpret the hierarchy of flows on the filtration by orbits, by viewing \( M \), the loops in the Euclidean plane, as the topological space obtained by compactifying \( \mathbb{C} \) to \( \mathbb{P} \). We recall the result (Section III.16 in Ref. [1]):

**Theorem 6.2.** The cohomology of the loop space \( \Omega S^n \) over \( S^n \) is given by

\[
H^p(\Omega S^n, \mathbb{R}) = \mathbb{R} \delta_{|p \text{ mod}(n-1)|, 0}.
\]

As for the ring structure, write:

\[
H^*(\Omega S^n, \mathbb{R}) = \mathbb{R} + \mathbb{R}x + \mathbb{R}e + \mathbb{R}xe + \mathbb{R}e^2 + \mathbb{R}xe^2 + \cdots;
\]

then, for the \( n = 2 \) case, the ring structure is given by

\[
H^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[e],
\]

where \( \text{degree}(e) = 2 \) and \( \text{degree}(x) = 1 \).

On the other hand, in Prop. 7.1 of Ref.[10], the following is proved,
**Theorem 6.3.** For the forgetful functor from the category of differential geometry to that of topological spaces, $F : \text{Diff} \to \text{Top}$, we have

$$H^*(\Omega S^2, \mathbb{R}) = H^*(F(M), \mathbb{R})$$

i.e., for $H^*(\Omega S^2, \mathbb{R}) = \mathbb{R}[x]/(x^2) \cdot \mathbb{R}[\epsilon] = H^*(F(M), \mathbb{R}) = \Lambda_R[dt_1, \epsilon]$, where $\Lambda_R[dt_1, \epsilon]$ is a ring generated by $dt_1$ and

$$\epsilon = dt_1 + dt_2 \wedge (dt_1i_{\delta_1}) + dt_3 \wedge (dt_1i_{\delta_1}) + \cdots$$

with the wedge product and the degree: $\text{degree}(dt_i) = 1$,

$$H^*(F(M), \mathbb{R}) = \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}\epsilon + \mathbb{R}dt_1 + \mathbb{R}\epsilon^2 + \mathbb{R}\epsilon^2dt_1 + \cdots.$$  

**Proof.** Since $\epsilon \cdot 1 = dt_1$, and $\epsilon^{n-1} \cdot dt_1 = \epsilon^n \cdot 1 = dt_n \wedge dt_{n-1} \wedge \cdots \wedge dt_2 \wedge dt_1$, we have

$$\Lambda_R[dt_1, \epsilon] = \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}\epsilon + \mathbb{R}dt_1 + \mathbb{R}\epsilon^2 + \mathbb{R}\epsilon^2dt_1 + \cdots$$

$$= \mathbb{R} + \mathbb{R}dt_1 + \mathbb{R}dt_1 \wedge dt_2 + \mathbb{R}dt_1 \wedge dt_2 \wedge dt_3 + \cdots.$$  

The Bäcklund transformation acts on $M$ as a telescopic-type topological space according to the genera. In conclusion:

$$H^*(F(M), \mathbb{R}) = \Lambda_R[dt_1, \epsilon].$$

□

6.5. **Comments.** The derivation of the mKdV hierarchy in this paper is based on the conservation of geometric and energy properties of “soliton elastica”. Moreover, the conserved quantities appear in the expansion (2.2) of the generating function $\log \frac{Z(s) - Z(s')}{s - s'}$ of the Faber polynomials[3, 13]. Indeed, the Faber polynomials $P_{f,n}$ for a function

$$f(q) = \frac{1}{q} + a_1q + a_2q^2 + a_3q^3 \cdots$$

are defined as

$$\log \left( q(f(q) - f(p)) \right) = \log \left( 1 - f(p)q + a_1q^2 + a_2q^3 + a_3q^4 \cdots \right)$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n} P_{f,n}(f(p))q^n$$

and $P_{f,0}(f(p)) = 1$, so that

$$\log \left( pq \frac{f(p) - f(q)}{p - q} \right) = \sum_{n=1}^{\infty} \frac{1}{n} \left( P_{f,n}(f(p))q^n - \left( \frac{q}{p} \right)^n \right).$$

Notice that, setting $Z = Z(s)$, $Z' = Z(s')$, $p = 1/Z$, $q = 1/Z'$, and calling the inverse functions $s = g(p)$, $s' = g(q)$, then:

$$\log \frac{Z(s) - Z(s')}{s - s'} = -\log \left( pq \frac{g(p) - g(q)}{p - q} \right).$$

This seems to be an important connection with the problem of identifying and acting on the “replicable functions” $f(q)$, for which,

$$qp \frac{f(q) - f(p)}{q - p} = \exp \left( -\sum_{n,m \geq 1} h_{m,n}p^mq^n \right).$$
where $h_{m,n}$ is the Grunsky coefficient:

$$
\sum_{m,n} h_{m,n} p^m q^n := \log \left( \frac{pq f(p) - f(q)}{p - q} \right) = \log \left( \frac{f(p) - f(q)}{1 - \frac{1}{q} - \frac{1}{p}} \right),
$$

and

$$
\{ f, g \}_{\text{SD}} = 6 \sum_{n,m \geq 1} mnh_{m,n} q^{m+n-2}.
$$

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