Minimum-gain Pole Placement with Sparse Static Feedback
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Abstract—The minimum-gain eigenvalue assignment/pole placement problem (MGEAP) is a classical problem in LTI systems with static state feedback. In this paper, we study the MGEAP when the state feedback has arbitrary sparsity constraints. We formulate the sparse MGEAP problem as an equality-constrained optimization problem and present an analytical characterization of its solution in terms of eigenvector matrices of the closed loop system. This result is used to provide a geometric interpretation of the solution of the non-sparse MGEAP, thereby providing additional insights for this classical problem. Further, we develop an iterative projected gradient descent algorithm to solve the sparse MGEAP using a parametrization based on the Sylvester equation. We present a heuristic algorithm to compute the projections, which also provides a novel method to solve the sparse EAP. Also, a relaxed version of the sparse MGEAP is presented and an algorithm is developed to obtain approximately sparse solutions to the MGEAP. Finally, numerical studies are presented to compare the properties of the algorithms, which suggest that the proposed projection algorithm converges in almost all instances.

I. INTRODUCTION
The Eigenvalue/Pole Assignment Problem (EAP) using static state feedback is one of the central problems in the design of Linear Time Invariant (LTI) control systems. It plays a key role in system stabilization and shaping its transient behavior. Given the following LTI system

\[ \dot{x}(k) = Ax(k) + Bu(k), \]  
\[ u(k) = F x(k), \]

where \( x \in \mathbb{R}^n \) is the state of the LTI system, \( u \in \mathbb{R}^m \) is the control input, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \), and \( D \) denotes either the continuous time differential operator or the discrete-time shift operator, the EAP involves finding a real feedback matrix \( F \in \mathbb{R}^{m \times n} \) such that the eigenvalues of the closed loop matrix \( A_c(F) \triangleq A + BF \) coincide with a given set \( S = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) that is closed under complex conjugation.

It is well known that the existence of \( F \) depends on the controllability properties of the pair \( (A, B) \). Further, for single input systems \((m = 1)\), the feedback vector that assigns the eigenvalues is unique and can be obtained using the Ackermann’s formula \([1]\). On the other hand, for multi-input systems \((m > 1)\), the feedback matrix is not unique and there exists a flexibility to choose the eigenvectors of the closed loop system. This flexibility can be utilized to choose a feedback matrix that satisfies some auxiliary control criteria in addition to assigning the eigenvalues. For instance, the feedback matrix can be chosen to minimize the sensitivity of the closed loop system to perturbations in the system parameters, thereby making the system robust. This is known as Robust Eigenvalue Assignment Problem (REAP). Alternatively, one can choose the feedback matrix with minimum gain, thereby reducing the overall control effort. This is known as Minimum Gain Eigenvalue Assignment Problem (MGEAP). Both of these problems have been studied extensively \([2], [3]\).

Recently, considerable attention has been given to the study and design of sparse feedback control systems, where certain entries of the matrix \( F \) are required to be zero. Feedback sparsity typically arises in large scale systems in which the controllers do not have access to all the states of the system. Sparsity may also be a result of the special structure of the control system which prohibits feedback from some states to the controllers.

The feedback design problem with sparsity constraints is considerably more difficult than the unconstrained case. There have been numerous studies to determine the optimal feedback control law for H2/LQR/LQG problems with sparsity, particularly when the controllers have access to only local information (see \([4], [7]\) and the references therein). While the optimal H2/LQR/LQG design problems with sparsity have a rich history, studies on the REAP/MGEAP in the presence of arbitrary sparsity constrains are lacking. Even the problem of finding a particular (not necessary optimal) sparse feedback matrix that solves the EAP is not well studied. In this paper, we study the EAP and MGEAP with arbitrary sparsity constraints on the feedback matrix \( F \). We provide analytical characterization for the solution of sparse MGEAP and provide iterative algorithms to solve the sparse EAP and MGEAP.

Related work There have been numerous studies on the optimal pole placement problem without sparsity constraints. For the REAP, authors have considered optimizing different metrics which capture the sensitivity of the eigenvalues, such as the condition number of the eigenvector matrix \([2], [8]–[11]\), departure from normality \([12]\) and others \([13], [14]\). Most of these methods use gradient-based iterative procedures to obtain the solutions. For surveys and performance comparisons of these REAP methods, see \([9], [15], [16]\) and the references therein.

Early works for MGEAP, including \([17], [18]\), presented approximate solutions using low rank feedback and successive pole placement techniques. Simultaneous robust and minimum gain pole placement were studied in \([17], [19]–[21]\). For a survey and performance comparison of these MGEAP studies, see \([3]\) and the references therein. The regional pole placement problem was studied in \([22], [23]\), where the eigenvalues were assigned inside a specified region. While all these studies have provided useful insights on REAP/MGEAP, they do not consider sparsity constraints on the feedback matrix. Further, the techniques in these studies cannot be used or simply
extended to obtain a sparse solution. In contrast, we study the sparse EAP/MGEAP by explicitly including the sparsity constraints into the problem.

There have also been numerous studies on EAP with sparse dynamic LTI feedback. The concept of decentralized fixed modes (DFMs) was introduced in [24] and later refined in [25]. Decentralized fixed modes are those eigenvalues of the system which cannot be shifted using a static/dynamic feedback with fully decentralized sparsity pattern (i.e. the case where controllers have access only to local states). The remaining eigenvalues of the system can be arbitrarily assigned. However, this cannot be achieved in general using a static decentralized controller and requires the use of dynamic decentralized controller [24]. Other algebraic characterizations of the DFM were presented in [26], [27]. The notion of DFM was generalized for an arbitrary sparsity pattern and the concept of structurally fixed modes (SFMs) was introduced in [28]. Graph theoretical characterizations of structurally fixed modes were provided in [29], [30]. As in the case of DFM, assigning the non-SFM also requires dynamic controllers. These studies on DFM and SFM present feasibility conditions and analysis methods for the EAP problem with sparse dynamic feedback. In contrast, we study both EAP and MGEAP with sparse static controllers, assuming the sparse EAP is feasible. We remark that EAP with sparsity and static feedback controller is in fact important for several network design and control problems, and easier to implement than its dynamic counterpart.

Recently, there has been a renewed interest in studying linear systems with sparsity constraints. Using a different approach than [29], the original results regarding DFM in [24] were generalized for an arbitrary sparsity pattern by the authors in [31], [32], where they also present a sparse dynamic controller synthesis algorithm. Further, there have been many recent studies on minimum cost input/output and feedback sparsity pattern selection such that the system has no structurally fixed modes (see [33], [34] and the references therein). In contrast, we consider the problem of finding a static minimum gain feedback with a given sparsity pattern that solves the EAP.

**Contribution** The contribution of this paper is three-fold. First, we study the MGEAP with static feedback and arbitrary sparsity constraints (assuming feasibility of sparse EAP). We formulate the sparse MGEAP as an equality constrained optimization problem and present an analytical characterization of an optimal sparse solution. As a minor contribution, we use this result to provide a geometric insight for the non-sparse MGEAP solutions. Second, we develop two heuristic iterative algorithms to obtain a solution of the sparse EAP. The first algorithm is based on repeated projections on linear subspaces. The second algorithm is developed using the Sylvester equation based parametrization and it obtains a solution via projection of a non-sparse feedback matrix on the space of sparse feedback matrices that solve the EAP. Third, using the latter EAP projection algorithm, we develop a projected gradient descent method to solve the sparse MGEAP. We also formulate a relaxed version of the sparse MGEAP using penalty based optimization method and develop an algorithm to obtain approximately-sparse solutions.

**Paper organization** The remainder of the paper is organized as follows. In Section II we formulate the sparse MGEAP optimization problem. In Section III we obtain the solution of the optimization problem using the Lagrangian theory of optimization. We also provide a geometric interpretation for the optimal solutions of the non-sparse MGEAP. In Section IV we present two heuristic algorithms for solving the sparse EAP. Further, we present a projected gradient descent algorithm to solve the sparse MGEAP and also an approximately-sparse solution algorithm for a relaxed version of the sparse MGEAP. Section V contains numerical studies and comparisons of the proposed algorithms. Finally, we conclude the paper in Section VI.

### II. SPARSE MGEAP FORMULATION

#### A. Mathematical Notation and Properties

We use the following notation throughout the paper:

| Symbol | Description |
|--------|-------------|
| \(\|\cdot\|_2\) | Spectral norm |
| \(\|\cdot\|_F\) | Frobenius norm |
| \(<\cdot,\cdot>_F\) | Inner (Frobenius) product |
| \(|\cdot|\) | Cardinality of a set |
| \(\Gamma(\cdot)\) | Spectrum of a matrix |
| \(\text{tr}(\cdot)\) | Trace of a matrix |
| \((\cdot)^+\) | Moore-Penrose pseudo inverse |
| \((\cdot)^T\) | Transpose of a matrix |
| \(\mathcal{R}(\cdot)\) | Range of a matrix |
| \(A > 0\) | Positive definite matrix |
| \(\circ\) | Hadamard (element-wise) product |
| \(\otimes\) | Kronecker product |
| \((\cdot)^*\) | Complex conjugate |
| \((\cdot)^H\) | Conjugate transpose |
| \(\text{supp}(\cdot)\) | Support of a vector |
| \(\text{vec}(\cdot)\) | Vectorization of a matrix |
| \(\text{diag}(a)\) | \(n \times n\) Diagonal matrix with diagonal elements given by \(n\)-dim vector \(a\) |
| \(\text{Re}(\cdot)\) | Real part of a complex variable |
| \(\text{Im}(\cdot)\) | Imaginary part of a complex variable |
| \(1_{n}(0_{n})\) | \(n\)-dim vector of ones (zeros) |
| \(1_{n \times m}(0_{n \times m})\) | \(n \times m\)-dim matrix of ones (zeros) |
| \(I_{n}\) | \(n\)-dim identity matrix |
| \(e_{i}\) | \(i\)-th canonical vector |
| \(T_{m,n}\) | Permutation matrix that satisfies \(\text{vec}(A^T) = T_{m,n}\text{vec}(A), A \in \mathbb{R}^{m \times n}\). |

The Kronecker sum of two square matrices \(A\) and \(B\) with dimensions \(n\) and \(m\), respectively, is denoted by

\[ A \oplus B = (I_{m} \otimes A) + B \otimes I_{n}. \]

Further, we use the following properties for deriving our results [35], [36]:

- **P.1** \(\text{tr}(A) = \text{tr}(A^T)\) and \(\text{tr}(ABC) = \text{tr}(CAB)\),
- **P.2** \(\|A\|_F^2 = \text{tr}(A^TA) = \text{vec}(A)^T\text{vec}(A)\),
- **P.3** \(\text{vec}(AB) = (I \otimes A)\text{vec}(B) = (B^T \otimes I)\text{vec}(A)\),
- **P.4** \(\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)\),
- **P.5** \((A \otimes B)^T = A^T \otimes B^T\) and \((A \otimes B)^H = A^H \otimes B^H\),
- **P.6** \(1_{n}(A \otimes B)1_{n} = \text{tr}(A^TB)\),
- **P.7** \(A \circ B = B \circ A\) and \(A \circ (B \circ C) = (A \circ B) \circ C\),
- **P.8** \(\text{vec}(A \circ B) = \text{vec}(A) \circ \text{vec}(B), (A \circ B)^T = A^T \circ B^T\).
The closed loop matrix $X$ slight abuse of notation, c.f. (1a)) let

The complementary sparsity structure matrix. Further, (with a

$F \in \{0,1\}^{m \times n}$ denote a binary matrix that
specifies the sparsity structure of the feedback matrix $F$. If

where $\ast$ denotes a real number. Let $\bar{F}^c \triangleq 1_{m \times n} - \bar{F}$ denote
the complementary sparsity structure matrix. Further, (with a
slight abuse of notation, c.f. (1a)) let $X \triangleq [x_1, x_2, \cdots, x_n] \in
\mathbb{C}^{n \times n}, x_i \neq 0$, denote the non-singular eigenvector matrix of
the closed loop matrix $A_0(F) = A + BF$.

The MGEAP can be mathematically stated as the following
optimization problem

\begin{equation}
\begin{aligned}
\text{min}_{F, \Lambda} & \quad \frac{1}{2} \|F\|_F^2 \\
\text{s.t.} & \quad (A + BF)X = XL, \quad (2a) \\
x_i^Hx_i = 1 & \quad \forall i = 1, 2, \cdots, n \quad \text{and} \quad (2b) \\
\bar{F}^c \circ F & = 0_{m \times n}, \quad (2c)
\end{aligned}
\end{equation}

where $\Lambda = \text{diag}([\lambda_1, \lambda_2, \cdots, \lambda_n]^T)$ is the diagonal matrix
constructed from the entries of the desired eigenvalue set $S$. Equations
(2a) and (2c) represent the eigenvalue assignment and sparsity constraints, respectively. Further, the normalization
constraint in (2b) is included to uniquely determine the
eigenvector matrix $X$.

The constraint (2c) is not convex in $(F, X)$ and, therefore,
the optimization problem (2) is non-convex. Consequently,
multiple local minima may exist. This is a common feature
in various minimum distance and eigenvalue assignment prob-
lems [37], including the non-sparse MGEAP.

Remark 1. (Choice of norm) The Frobenius norm explicitly
measures the element-wise gains of a matrix, which is informa-
tive in sparsity constrained problems arising, for instance,
in network control problems. Further, it is also convenient
for the analysis, as it allows us to analytically compute the
derivatives of the cost function.

Definition 1. (Fixed modes [24], [31]) The fixed modes of
$(A, B)$ with respect to the sparsity constraints $F$ are those
eigenvalues of $A$ which cannot be changed using LTI static
(and also dynamic) state feedback, and are denoted by

$$
\Gamma_f(A, B, F) \triangleq \bigcap \Gamma(A + BF).
$$

We make the following assumptions regarding the fixed modes and feasibility of the optimization problem [2].

A1: The fixed modes of the triplet $(A, B, F)$ are included in
the desired eigenvalue set $S$, i.e., $\Gamma_f(A, B, F) \subseteq S$.

A2: There exists at least one feedback matrix $F$ that satisfies constraints (2a)-(2c) for the given $S$.

Assumption A1 is clearly necessary for the feasibility of
the optimization problem [2]. Assumption A2 is restrictive
because, in general, it is possible that a static feedback matrix
with a given sparsity pattern cannot assign the closed loop
eigenvalues to arbitrary locations (i.e. for an arbitrary set $S$
satisfying Assumption A1)\footnote{Although a sparse dynamic feedback law can assign the eigenvalues to arbitrary locations under Assumption A1 \cite{32}.}
In such cases, only a few ($< n$) eigenvalues can be assigned independently and other
remaining eigenvalues are a function of them. To the best
of our knowledge, there are no studies on characterizing
conditions for the existence of a static feedback matrix for an
arbitrary sparsity pattern $F$ and eigenvalue set $S$ [38] (although
such characterization is available for dynamic feedback laws
with arbitrary sparsity pattern [28], [31], [32], and static output
feedback for decentralized sparsity pattern [39]). Thus, for
the purpose of this paper, we focus on finding the optimal
feedback matrix assuming that at least one such feedback
matrix exists.

III. SOLUTION TO THE SPARSE MGEAP

In this section we present the solution to the optimization
problem (2). We use the theory of Lagrangian multipliers for
equality constrained minimization problems to derive the op-
timality conditions. We begin with a remark on the formalism
involving complex variables.

Remark 2. (Complex variables) If the desired eigenvalue set $S$
contains complex entries, the corresponding eigenvectors
in $X$ will also be complex. Thus, the eigenvalue-eigenvector
constraint in (2a) is a complex-valued constraint, and it
induces the following conjugate constraint:

$$
(A + BF)X^* = X^*\Lambda^*.
$$

In the optimization problem, we use the formalism wherein a
complex number and its conjugate are treated as independent
variables [40], [41] and, thus, we treat $X$ and $X^*$ as
independent variables.

Remark 3. (Conjugate eigenvectors) We use the convention
that the right eigenvectors $(x_i, x_j)$ corresponding to two
conjugate eigenvalues $(\lambda_i, \lambda_j)$ are also conjugate. Thus, if
$\lambda_i = \lambda_j^*$, then $x_i = x_j^*$. Similar convention is also used
for the left eigenvectors.

$$
\begin{aligned}
\text{max}_{x_i, x_j} & \quad \text{Re}(x_i^H A x_j) \\
\text{s.t.} & \quad x_i^H x_i = 1 \\
& \quad x_j^H x_j = 1 \\
& \quad x_i, x_j \text{ are complex}.
\end{aligned}
$$

In the optimization problem, we use the formalism wherein a
complex number and its conjugate are treated as independent
variables [40], [41] and, thus, we treat $X$ and $X^*$ as
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$\lambda_i = \lambda_j^*$, then $x_i = x_j^*$. Similar convention is also used
for the left eigenvectors.
In the theory of equality constrained optimization, the first-order optimality conditions are meaningful only when the optimal point satisfies the following regularity condition: the Jacobian of the constraints, defined by $J_b$, is full rank. This regularity condition is mild and usually satisfied for most classes of problems \[\text{(22)}.\] Before presenting the main result, we derive the Jacobian and state the regularity condition for the optimization problem \[\text{(2)}.\]

Computation of $J_b$ requires vectorization of the matrix constraints \[\text{(2a)-(2c)}.\] For this purpose, let $x \triangleq \text{vec}(X) \in \mathbb{C}^{n^2}$, $f \triangleq \text{vec}(F) \in \mathbb{R}^{mn}$. Recalling Remark \[\text{2},\] let $z \triangleq [x^T, f^T]^T$ be the vector containing all the independent variables of the optimization problem. Further, let $n_s$ denote the total number of feedback sparsity constraints (i.e., number of 1’s in $F^c$):

$$n_s = \{((i,j) : f_{ij}^c = 1, f_{ij}^c = 1]}.$$

Finally, note that the constraint \[\text{(2c)},\] can be equivalently written as (see the proof of Lemma \[\text{III.1}.\]) for a formal definition of $Q$)

$$Qf = 0, \quad \text{(4)}$$

for some binary matrix $Q \in \{0, 1\}^{n_s \times mn}$.

**Lemma III.1. (Jacobian of the constraints) The Jacobian of the equality constraints \[\text{(2a)-(2c)}\] is given by

$$J_b(z) = \begin{bmatrix} A_c(F) \otimes (-\Lambda^T) & 0_{n_s \times n^2} & X^T \otimes B \\ 0_{n_s \times n^2} & A_c(F) \otimes (-\Lambda^H) & X^H \otimes B \\ 0_{n_s \times n^2} & 0_{n_s \times n^2} & Q \end{bmatrix}, \quad \text{(5)}$$

where $\bar{x} \triangleq \left(1_1 \otimes x^H\right) \circ \left(I_n \otimes 1_n\right)$.

**Proof.** We construct the Jacobian $J_b$ by rewriting the constraints \[\text{(2a)-(2c)},\] in vectorized form and taking their derivatives with respect to $z$. Constraint \[\text{(2a)}\] can be vectorized in the following two different ways (using \[\text{P.3} \text{ and P.4}); \]

$$[(A + BF) \otimes (-\Lambda^T)]x = 0, \quad \text{(6a)}$$

$$[A + (-\Lambda^T)]x + (X^T \otimes B)f = 0. \quad \text{(6b)}$$

Differentiating \[\text{(6a)}\] w.r.t. $x$ and \[\text{(6b)}\] w.r.t. $f$ yields the first (block) row of $J_b$. Similar vectorizations of the conjugate eigenvalue constraint \[\text{(3)}\] and differentiation w.r.t. $z$ yields the second (block) row of $J_b$. Differentiating constraint \[\text{(2b)}\] for $i = 1, 2, \ldots, n$, w.r.t. $x$ and $x^*$ yields

$$\frac{d}{dx} \begin{bmatrix} x_i^H x_i - 1 \\ x_i^H x_{i+1} - 1 \\ : \\ x_i^H x_n - 1 \end{bmatrix} = \begin{bmatrix} x_i^H & 0^T & \ldots & 0^T \\ 0^T & x_{i+1}^H & \ldots & 0^T \\ : & : & \vdots & : \\ 0^T & 0^T & \ldots & x_n^H \end{bmatrix} = \bar{x}_i,$$

and $\bar{x}^*$, respectively. Finally, \[\text{(2c)}\] consists of $n_s$ non-trivial sparsity constraints, which can be written as \[\text{(4)}\] where $Q = [e_{q_1}, e_{q_2}, \ldots, e_{q_{n_s}}]^T$ and \[\{q_1, \ldots, q_{n_s}\} \supseteq \text{supp} (\text{vec}(F^c))\] is the set of indices indicating the ones in $\text{vec}(F^c)$. Differentiating \[\text{(4)}\] w.r.t. $z$ yields the fourth (block) row of $J_b$, thus completing the proof.

Next, we provide the optimality conditions for the optimization problem \[\text{(2)}.\]

**Theorem III.2. (Optimality conditions)** Let $(\hat{X}, \hat{F})$ (equivalently $\hat{z} = [\hat{x}^T, \hat{x}^H, \hat{f}^T]^T$) satisfy the constraints \[\text{(2a)-(2c)}.\] Then, $(\hat{X}, \hat{F})$ is a local minimum of the optimization problem \[\text{(2)}\] if and only if

$$\hat{F} = -\hat{\mathcal{F}} \circ (B^T L \hat{X}^T), \quad \text{(7a)}$$

where $\hat{X}$ and $\hat{L}$ are the right and left eigenvector matrices of $A_c(\hat{F})$, respectively, and satisfy

$$\begin{align*}
(A + B \hat{F}) \hat{X} &= \hat{X} \Lambda^T & \text{(7b)} \\
(A + B \hat{F})^T \hat{L} &= \hat{L} \Lambda^T & \text{(7c)} \\
J_b(\hat{z}) &= \text{full rank} & \text{(7d)} \\
P(\hat{z}) \hat{D} P(\hat{z}) > 0, & \text{(7e)}
\end{align*}$$

where $\hat{D}$ is the Hessian defined as

$$\hat{D} \triangleq \begin{bmatrix} 0_{n_s \times n^2} & 0_{n_s \times n^2} & \hat{L}^H \\
0_{n_s \times n^2} & 0_{n_s \times n^2} & \hat{L}^T \\
\hat{L} & \hat{L}^* & 2I_{mn} \end{bmatrix}, \quad \text{(8)}$$

where $\hat{L} \triangleq \text{vec}(B^T \hat{L} \otimes I_n)$, and $P(z)$ is the projection matrix of $J_b(z)$ defined as

$$P(z) = I_{2n^2 + mn} - J_b^T(z)J_b(z). \quad \text{(9)}$$

**Proof.** We prove the result using the Lagrange theorem for equality constrained minimization. Let $L \in \mathbb{C}^{n \times n}, L^*, M \in \mathbb{R}^{m \times n}$ and $h \in \mathbb{R}^m$ be the Lagrange multipliers associated with constraints \[\text{(2a)-(2c)},\] and \[\text{(2d)},\] respectively. Further, let $H = \text{diag}(h)$. The Lagrange function for the optimization problem \[\text{(2)}\] is given by

$$\mathcal{L} = \frac{1}{2} \text{tr}(F^T F) + \frac{1}{2} \text{tr}([L \circ (A_c(F)X - XL)]1_n + \text{tr}([H X - I_n])H) + \text{tr}([M \circ (\hat{F}^c \circ F)]1_n) + \frac{1}{2} \text{tr}([L^T (A_c(F)X - XL)]) + \frac{1}{2} \text{tr}([H^T (A_c(F)X - XL^*)] \circ (H^T X - I_n))H) + \text{tr}([M \circ (\hat{F}^c \circ F)]^T F).$$

We next derive the first-order necessary condition for a stationary point. Differentiating l w.r.t. $X$ and setting to 0, we get

$$\frac{d}{dX} \mathcal{L} = \frac{1}{2} [A_c^T(\hat{F})L - L \Lambda^T] + X^*H = 0. \quad \text{(10)}$$

Pre-multiplying \[\text{(10)}\] by $X^T$, we get

$$\frac{1}{2} [X^T A_c^T(\hat{F})L - X^T L \Lambda^T] + X^* X^*H = 0. \quad \text{(11)}$$

Since $\Lambda$ is diagonal and $x_i^H x_i = 1$, the vector containing the diagonal elements of the left side of \[\text{(11)}\] is $h$. Therefore, $h = 0_n$ (and $H = 0_{n \times n}$) and from \[\text{(10)}\], we get \[\text{(7c)}.\] Equation \[\text{(7b)}\] is a restatement of \[\text{(2a)}\] for the optimal $(\hat{F}, \hat{X})$. \[\square\]
Differentiating $\mathcal{L}$ w.r.t. $F$, we get
\[ \frac{d}{dF} \mathcal{L} \] 
\[ = F + \text{Re}(B^T LX^T) + M \circ \mathbb{F}^c = 0_{m \times n}. \]  
\[ (12) \]

Taking the Hadamard product of (12) with $\mathbb{F}^c$ and using (2c), we get (since $\mathbb{F}^c \circ \mathbb{F}^c = \mathbb{F}^c$)
\[ \mathbb{F}^c \circ \text{Re}(B^T LX^T) + M \circ \mathbb{F}^c = 0_{m \times n}. \]  
\[ (13) \]

Replacing $M \circ \mathbb{F}^c$ from (13) into (12), we get
\[ F = -\mathbb{F} \circ \text{Re}(B^T LX^T). \]

Since the eigenvectors in $X$ (and $L \triangleq [l_1, l_2, \ldots, l_n]$) are closed under complex conjugation (c.f. Remark 3), the product $LX^T = \sum_{i=1}^{n} l_i x_i^T$ is real and (7a) follows. Equation (7d) is the necessary regularity condition and follows from Lemma III.1.

Next, we derive the second-order sufficient condition for a local minimum by calculating the Hessian of $\mathcal{L}$. Taking the differential of $\mathcal{L}$ twice, we get
\[ d^2 \mathcal{L} = \text{tr}((dF)^T dF) + \text{tr}(L^T BdFdX) + \text{tr}(L^T BDdX^*) + 2d \text{tr}(dH^T dX) \]
\[ = \frac{1}{2} \text{tr}(dH^T dX) + \text{tr}(dF^T dF) + \text{tr}(dF^T LdX) + \text{tr}(dF^T L^* dX^*) 
\]
\[ = \frac{1}{2} [dH^T, dX^T, df^T] \begin{bmatrix} dx^T & dx^T \end{bmatrix} \] 
\[ (P_{12}, P_{13}, P_{23}) \]

where $D$ is the Hessian (c.f. [12]) defined in [9]. The sufficient second-order optimality condition for the optimization problem requires the Hessian to be positive definite in the kernel of the Jacobian at the optimal point [45]. That is, $y^T D y > 0, \forall y: J_0(y) = 0$. This condition is equivalent to $P(z)DP(z) > 0$, since $J_0(y) = 0$ if and only if $y = P(z)s$ for a $s \in \mathbb{R}^{2n^2 + mn}$ [42]. Since the projection matrix $P(z)$ is symmetric, (7c) follows, and this concludes the proof. \[ \square \]

**Remark 4. (Eigenvector normalization constraints)** From the proof of Theorem III.2, we observe that the Lagrange multipliers associated with the eigenvector normalization constraints in (2b) are zero ($h = 0_n$). This implies that these constraints are redundant. Further, observe from (7a) that any pair of left and right eigenvector matrices $\{LD^{-1}, XD\}$ where $D$ is any diagonal matrix with non-zero (and possibly complex) diagonal entries, will result in the same feedback matrix $F$. Thus, we can always choose a suitable $D$ to normalize $X$. Therefore, we ignore the eigenvector normalization constraints in the remainder of the paper. \[ \square \]

Observe that the Hadamard product in (7a) guarantees that the feedback matrix satisfies the sparsity constraints given in (2c). However, note that the optimal sparse feedback $\hat{F}$ cannot be obtained by sparsification of the optimal non-sparse feedback. The feedback optimality condition (7a) is an implicit condition in terms of the closed loop right and left eigenvector matrices. Next, we provide an explicit optimality condition in terms of $\{\hat{L}, \hat{X}\}$.

**Corollary III.3. (Stationary point equation)** \[ \hat{Z} \triangleq [\hat{X}^T, \hat{L}^T]^T \]
is a stationary point of the optimization problem (2) if and only if
\[ \hat{A}\hat{Z} - \hat{Z}\Lambda = \hat{B}_1[F \circ (B_1^T \hat{X} \hat{Z}^T \hat{B}_2)]B_2^T \hat{Z}, \]  
\[ (14) \]

where,
\[ \hat{A} \triangleq \begin{bmatrix} A & 0_{n \times n} \\ 0_{n \times n} & A^T & \end{bmatrix}, \quad \hat{B}_1 \triangleq \begin{bmatrix} B & 0_{n \times n} \\ 0_{n \times n} & I_n & \end{bmatrix}, \quad \hat{B}_2 \triangleq \begin{bmatrix} I_n \\ 0_{n \times n} \end{bmatrix}, \quad \hat{F} \triangleq \begin{bmatrix} \bar{F} & 0_{n \times m} \\ 0_{n \times m} & \hat{F}^T \end{bmatrix}, \quad \text{and} \]
\[ \hat{I} \triangleq \begin{bmatrix} 0_{n \times n} & I_n \\ I_n & 0_{n \times n} \end{bmatrix}. \]

**Proof.** Combining (7b) and (7c) and using $\Lambda^T = \Lambda$, we get
\[ \begin{bmatrix} A\hat{X} - \hat{X} \Lambda \\ A^T \hat{L} - \hat{L} \Lambda \end{bmatrix} = - \begin{bmatrix} B\hat{F} \hat{X} \\ \hat{F}^T B\hat{T} \end{bmatrix} \]
\[ \Rightarrow \hat{A}\hat{Z} - \hat{Z}\Lambda = - \hat{B}_1 \begin{bmatrix} \hat{F} & 0 \end{bmatrix} \begin{bmatrix} B_1^T \hat{X}^T \\ 0 \end{bmatrix} \hat{B}_2^T \hat{Z} 
\]
\[ = \hat{B}_1 \begin{bmatrix} \hat{F} \circ (B_1^T \hat{X}^T) \\ 0 \end{bmatrix} \begin{bmatrix} \hat{F}^T \circ (\hat{X}^T \hat{F}) \\ \hat{F}^T \end{bmatrix} \begin{bmatrix} B_1 \hat{L}^T \hat{F} \\ 0 \end{bmatrix} \begin{bmatrix} B_2 \hat{F} \\ 0 \end{bmatrix} \hat{B}_2 \hat{L} \hat{Z} 
\]
\[ = \hat{B}_1 \begin{bmatrix} \hat{F} \circ (B_1^T \hat{X} \hat{Z}^T \hat{B}_2) \end{bmatrix} \hat{B}_2 \hat{Z}, \]
\[ (P_{12}, P_{13}, P_{23}) \]

where the above equations follow from direct manipulations using definition of Hadamard product. \[ \square \]

**Remark 5. (Partial spectrum assignment)** The results of Theorem III.2 and Corollary III.2 are also valid when only $p$ ($p < n$) eigenvalues need to be assigned and the remaining eigenvalues are unspecified. In this case, $\Lambda \in \mathbb{C}^{p \times p}$, $\hat{X} \in \mathbb{C}^{n \times p}$ and $\hat{L} \in \mathbb{C}^{n \times p}$. Partial assignment is useful when the sparsity constraints allow only a limited number of eigenvalues to be assigned independently and the remaining eigenvalues are functionally related to them (c.f. discussion below Assumption A2). Nevertheless, in this paper we focus on assigning all the closed loop eigenvalues. \[ \square \]

**Remark 6. (General eigenstructure assignment)** Although the optimization problem (2) is formulated by considering $\Lambda$ to be diagonal, the result in Theorem III.2 is valid for any general $\Lambda$ satisfying $\Gamma(\Lambda) = S$. For instance, we can choose $\Lambda$ in a Jordan canonical form. However, note that for a general $\Lambda$, $\hat{X}$ and $\hat{L}$ will cease to be an eigenvector matrix. Thus, in such cases, we need to omit the normalization constraints (2b) in the optimization problem. \[ \square \]

A solution of the optimization problem (2) can be obtained by numerically/iteratively solving the matrix equation (14), which resembles a Sylvester type equation with a non-linear right side, and using (7d) to compute the feedback matrix. The regularity and local minimum of the solution can be verified using (7d) and (14), respectively. Since the optimization problem is not convex, only local minima can be obtained.
via this procedure. To improve upon the local solutions, the above procedure can be repeated for several different initial conditions (for solving (14)). However, convergence to the global minimum is not guaranteed.

In this paper, instead of solving (14) directly, we use a different approach based on the gradient descent procedure to obtain a local solution. Details of this approach and corresponding algorithms are presented in Section IV.

A. Results for non-sparse MGEAP

In this subsection, we present some results specific to the case when the optimization problem (2) does not have any sparsity constraints (i.e. $F = I_{m \times n}$). Although the non-sparse MGEAP has been studied previously, these results are novel and further illustrate the properties of an optimal solution.

We begin by presenting a geometric interpretation of the optimality conditions provided in Theorem III.2 under the assumption that $B = I_n$, i.e. all the entries of $A$ can be perturbed independently. In this case, the optimization problem (2) can be written as (while ignoring the eigenvector normalization constraints; c.f. Remark 4):

$$\min_{X} \frac{1}{2} \|A - XAX^{-1}\|_F^2. \quad (15)$$

Since $A$ and $R(X) = XAX^{-1}$ are elements (or vectors) of the matrix inner product space with Frobenius norm, a solution of the optimization problem (15) is given by the projection of $A$ on the manifold $M \triangleq \{ R(X) : X \text{ is non-singular} \}$. This projection can be obtained by solving the normal equation, which states that the optimal error vector $\tilde{F} = A - XAX^{-1}$ should be orthogonal to the tangent plane of the manifold $M$ at the optimal point $\hat{X}$. The next result shows that the optimality conditions derived in Theorem III.2 are in fact the normal equations for the optimization problem (15).

**Lemma III.4. (Geometric interpretation)** Let $F = I_{m \times n}$ and $B = I_n$. Then, Equations (7a)–(7c) are equivalent to the following normal equation:

$$\tilde{F}^T T_M(\hat{X}) \tilde{F} = 0, \quad (16)$$

where $T_M(X)$ denotes the tangent space of $M$ at $X$.

**Proof.** We begin by characterizing the tangent space $T_M(X)$, which is given by the first order approximation of the matrix function $R(X)$:

$$R(X + dX) = (X + dX)\Lambda(X + dX)^{-1} \quad \text{Pf.1}$$

+ higher order terms.

Thus, the tangent space is given by

$$T_M(X) = \{ YAX^{-1} - XAX^{-1}YX^{-1} : Y \in \mathbb{C}^{n \times n} \}.$$

**Necessity:** Using $\tilde{F}$ given by (7a), we get

$$\tilde{F}^T T_M(\hat{X}) \tilde{F} = \text{tr}(\hat{F}^T (YAX^{-1} - \hat{X}AX^{-1}Y\hat{X}^{-1}))$$

$$= \text{tr}(\hat{F}^T Y\hat{X}^{-1} - \hat{X}AX^{-1}Y\hat{X}^{-1})$$

$$= \text{tr}(\hat{F}^T Y\hat{X}^{-1} - \hat{X}AX^{-1}Y\hat{X}^{-1})$$

$$= 0,$$

where (a) follows from the fact that $\Lambda$ and $\hat{L}^T \hat{X}$ commute.

**Sufficiency:** From (16), we get

$$\text{tr}(\hat{F}^T (YAX^{-1} - \hat{X}AX^{-1}Y\hat{X}^{-1})) = 0$$

$$\Rightarrow \text{tr}(\hat{L}^T Y\Lambda - \hat{X}AX^{-1}Y\hat{X}^{-1}) = 0.$$

Since the above equation is true for all $Y \in \mathbb{C}^{n \times n}$, we get

$$\Lambda\hat{X}^{-1} - \hat{X}^{-1}\hat{F}^T \hat{X} \Lambda^{-1} = 0_{n \times n}$$

$$\Rightarrow \Lambda\hat{X}^{-1} = \hat{X}^{-1}\hat{F}^T \hat{X} \Lambda^{-1}$$

$$\Rightarrow A_c(\hat{F}) = \hat{F}^T A_c(\hat{F}).$$

Thus, $A_c(\hat{F})$ and $\hat{F}^T$ commute and therefore, have common left and right eigenspaces [44], i.e. $\hat{F}^T = -\hat{X}\hat{G}\hat{X}^{-1} = -\hat{X}\hat{L}^T$, where $G$ is a diagonal matrix. This completes the proof.

Next, we show the equivalence of the non-sparse MGEAP for two orthogonally similar systems.

**Lemma III.5. (Invariance under orthogonal transformation)** Let $\tilde{F} = I_{m \times n}$ and $(A_1, B_1), (A_2, B_2)$ be two orthogonally similar systems such that $A_2 = P A_1 P^{-1}$ and $B_2 = P B_1$, with $P$ being an orthogonal matrix. Let an optimal solutions of (2) for the two systems be denoted by $(\hat{X}_1, \hat{L}_1, \hat{F}_1)$ and $(\hat{X}_2, \hat{L}_2, \hat{F}_2)$, respectively. Then

$$\hat{X}_2 = P\hat{X}_1, \quad \hat{L}_2 = P\hat{L}_1, \quad \hat{F}_2 = \hat{F}_1 P^T,$$

$$||\hat{F}_1||_F = ||\hat{F}_2||_F. \quad (17)$$

**Proof.** From (7b), we have

$$(A_2 + B_2\hat{F}_2)\hat{X}_2 = \hat{X}_2\Lambda$$

$$\Rightarrow (PA_1 P^{-1} + PB_1 \hat{F}_1 P^T) P\hat{X}_1 = P\hat{X}_1\Lambda$$

$$\Rightarrow (A_1 + B_1 \hat{F}_1)\hat{X}_1 = \hat{X}_1\Lambda.$$

Similar relation can be shown between $\hat{L}_1$ and $\hat{L}_2$ using (7c). Next, from (7a), we have

$$\hat{F}_2 = -B_2^T L_2 \hat{X}_2^T = -B_1^T \hat{L}_1 \hat{X}_1^T P^T = \hat{F}_1 P^T.$$

Finally, $||\hat{F}_1||_F^2 = \text{tr}(\hat{F}_1^T \hat{F}_1)$ and $||\hat{F}_2||_F^2 = \text{tr}(\hat{F}_2^T \hat{F}_2)$, thereby completing the proof.

Recall from Remark 5 that Theorem III.2 is also valid for MGEAP with partial spectrum assignment. Next, we consider the case when only one real eigenvalue needs to assigned for the MGEAP while the remaining eigenvalues are unspecified. In this special case, we can explicitly characterize the global minimum of (2) as shown in the next result.

**Corollary III.6. (One real eigenvalue assignment)** Let $\tilde{F} = I_{m \times n}$, $A \in \mathbb{R}$, and $B = I_n$. Then, the global minima of the optimization problem (2) is given by $\hat{F}_{gl} = -\sigma_{min}(A - \Lambda I_n) u v^T$, where $u$ and $v$ are unit norm left and right singular vectors, respectively, corresponding to $\sigma_{min}(A - \Lambda I_n)$. Further, $||\hat{F}_{gl}||_F = \sigma_{min}(A - \Lambda I_n)$. 
Proof. Since $\Lambda \in \mathbb{R}$, $\hat{X} \in \mathbb{R}^n \triangleq \hat{x}$ and $\hat{L} \in \mathbb{R}^n \triangleq \hat{l}$. Let
\[
\hat{l} = \beta \hat{l}
\]
where $\beta \triangleq \|\hat{l}\|_2 > 0$. Substituting $\hat{F} = -\hat{l}\hat{x}^T$ from (7a) into (7b)-(7c), we get
\[
(A - \hat{l}\hat{x}^T)\hat{x} = \hat{x}\Lambda \quad \text{(7a)}
\]
and,
\[
(A^T - \hat{x}\hat{l}^T)\hat{l} = \hat{l}\Lambda \Rightarrow (A - \Lambda\hat{l})^T\hat{l} = \beta\hat{x}.
\]
The above two equations imply that the unit norm vectors $\hat{x}$ and $\hat{l}$ are left and right singular vectors of $A - \Lambda\hat{l}$ associated with the singular value $\beta$. Since $\|\hat{F}\|_F = \text{tr}(\hat{F}\hat{F}^T) = \text{tr}(\hat{x}\hat{l}^T\hat{l}\hat{x}^T) = \beta^2$, we pick $\beta$ as the minimum singular value of $A - \Lambda\hat{l}$, and the proof is complete.

We conclude this subsection by presenting a brief comparison of the non-sparse MGEAP solution with deflation techniques for eigenvalue assignment. For $B = I_n$, an alternative method to solve the non-sparse EAP is via the Wielandt deflation technique [45]. Wielandt deflation achieves pole assignment by modifying the matrix $A$ in $n$ steps $A \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n$. Step $i$ shifts one eigenvalue of $A_{i-1}$ to a desired location $\lambda_i$, while keeping the remaining eigenvalues of $A_{i-1}$ fixed. This is achieved by using the feedback $F_{df}^i = -(\mu_i - \lambda_i)v_i z_i^T$, where $\mu_i$ and $v_i$ are any eigenvalue and right eigenvector pair of $A_{i-1}$, and $z$ is any vector such that $z_i^T v_i = 1$. Thus, the overall feedback that solves the EAP is given as $F_{df} = \sum_{i=1}^n F_{df}^i$.

It is interesting to compare the optimal feedback expression in (7a), $\hat{F} = -\sum_{i=1}^n i\hat{l}\hat{x}^T$, with the deflation feedback. Both feedbacks are sum of $n$ rank–1 perturbations. However, the Wielandt deflation has an inherent special structure and a restrictive property that, in each step, all except one eigenvalue remain unchanged. Furthermore, each rank–1 term in $\hat{F}$ and $F_{df}$ involves the right/left eigenvectors of the closed and open loop matrix, respectively. Due to these reasons, it clearly follows that $\|\hat{F}\|_F \leq \|F_{df}\|_F$.

IV. Solution Algorithms

In this section, we present an iterative algorithm to obtain a solution to the sparse MGEAP in (2). To develop the algorithm, we first present two algorithms for computing non-sparse and approximately-sparse solutions to the MGEAP, respectively. Next, we present two heuristic algorithms to obtain a sparse solution of the EAP (i.e. any sparse solution, which is not necessarily minimum-gain). Finally, we use these algorithms to develop the algorithm for sparse MGEAP. Note that although our focus is to develop the sparse MGEAP algorithm, the other algorithms presented in this section are novel in themselves to the best of our knowledge.

We make the following assumptions:

A1: The triplet $(A, B, F)$ has no fixed modes, i.e., $\Gamma_f(A, B, F) = \emptyset$.

A2: The open and closed loop eigenvalue sets are disjoint, i.e., $\Gamma(A) \cap \Gamma(L) = \emptyset$.

Assumption A2 is not restrictive since if there are any common eigenvalues in $A$ and $\Lambda$, we can use a preliminary sparse feedback $F_p$ to shift the eigenvalues of $A$ to some other locations such that $\Gamma(A + BF_p) \cap \Gamma(L) = \emptyset$. Since the system has no fixed modes (c.f. Assumption A3), such a $F_p$ always exists. Then, we can solve the modified MGEAP with parameters $(A + BF_p, B, L, F)$. If $F$ is the sparse solution of this modified problem, then the solution of the original problem is $F_p + F$.

To avoid complex domain calculations in the algorithms, we use the real counterpart of (2a). For two complex conjugate eigenvalues $(\lambda_i, \lambda_i^*)$ and corresponding eigenvectors $(x_i, x_i^*)$, the following complex equation
\[
(A + BF)[x_i \quad x_i^*] = [x_i \quad x_i^*][\lambda_i \quad 0 \quad 0 \quad \lambda_i^*]
\]
is equivalent to the following real equation
\[
(A + BF)[\text{Re}(x_i) \quad \text{Im}(x_i)] = [\text{Re}(x_i) \quad \text{Im}(x_i)][\text{Re}(\lambda_i) \quad \text{Im}(\lambda_i); -\text{Im}(\lambda_i) \quad \text{Re}(\lambda_i)]
\]
Thus, for each complex eigenvalue, the columns $[x_i \quad x_i^*]$ of $X$ are replaced by $[\text{Re}(x_i) \quad \text{Im}(x_i)]$ to obtain a real $X'$ and the sub-matrix $[\lambda_i \quad 0; 0 \quad \lambda_i^*]$ of $\Lambda$ is replaced by $[\text{Re}(\lambda_i) \quad \text{Im}(\lambda_i); -\text{Im}(\lambda_i) \quad \text{Re}(\lambda_i)]$ to obtain a real $\Lambda'$. Clearly, $X'$ is not the eigenvector matrix of $A + BF$ (c.f. Remark 5), and the latter can be easily obtained by combining the columns of $X'$. Also, the invertibility of $X$ is equivalent to the invertibility of $X'$. In the remainder of this section, we assume (2a) to be in the real form.

A. Algorithms for non-sparse MGEAP

In this subsection, we present two iterative gradient based descent algorithms to obtain non-sparse and approximately-sparse solutions to the MGEAP, respectively. To develop the algorithms, we use the Sylvester equation based parametrization [11], [46]. In this parametrization, instead of defining $(F, X)$ as free variables, we define a parameter $G \triangleq FX \in \mathbb{R}^{m \times m}$ as the free variable. With this parametrization, the non-sparse MGEAP is stated as (while ignoring the eigenvector normalization constraints; c.f. Remark 4):

\[
\min_G \quad J = \frac{1}{2} \|F\|_F^2 \tag{18a}
\]
\[
\text{s.t.} \quad AX - XL + BG = 0, \tag{18a}
\]
\[
F = GX^{-1}. \tag{18b}
\]

Note that, for any given $G$, we can solve the Sylvester equation (18a) to obtain $X$. Assumption A4 guarantees that (18a) has a unique solution [47]. Further, we can use (18b) to obtain a non-sparse feedback matrix $F$. Thus, (18) is an unconstrained optimization problem in the free parameter $G$.

The Sylvester equation based parametrization requires the unique solution $X$ of (18a) to be non-singular, which holds generically if (i) $(A, -BG)$ is controllable and (ii) $(\Lambda, -BG)$ is observable [18]. Since the system has no fixed modes (c.f. Assumption A3), $(A, B)$ is controllable. Therefore, the above two conditions are mild and are satisfied for almost all instances as confirmed in our simulations (see Section V).

The next result provides the gradient and Hessian of the cost $J$ w.r.t. to the parameter $g \triangleq \text{vec}(G)$.

Although the minimization cost of the modified MGEAP is $0.5\|F_p + F\|_F^2$, it can be solved using techniques similar to solving MGEAP in (2).
Lemma IV.1. (Gradient and Hessian of $J$) The gradient and
Hessian of the cost $J$ in (18) with respect to $g$ is given by
\[
\frac{dJ}{dg} = \left( X^{-1} \otimes I_m \right) + \left( I_n \otimes B^T \right) \bar{A}^{-T} \left( X^{-1} \otimes F^T \right) f , \quad \frac{d^2 J}{dg^2} \triangleq Z(F, X) \]

or
\[
\frac{d^2 J}{dg^2} \triangleq H(F, X) = Z(F, X) Z^T(F, X) + Z(F, X) Z^T(F, X), \quad \text{(20)}
\]
where $Z_1(F, X) \triangleq \left( I_n \otimes B^T \right) \bar{A}^{-T} \left( X^{-1} F^T \otimes I_n \right) T_{m,n}$, and
\[ \bar{A} \triangleq A \otimes (-\bar{A}^T). \]

Proof. Vectorizing (18a), we get
\[
\bar{A} x + \left( I_n \otimes B \right) g = 0 \Rightarrow \bar{A}^{-1} \left( I_n \otimes B \right) dg = 0.
\]
Note that due to Assumption A4, $\bar{A}$ is invertible. Taking the differential of (18b) and vectorizing, we get
\[
dF = \left( X^{-T} \otimes I_n \right) dg - (X^{-T} \otimes F) dx \quad \text{and} \quad Z(F, X) = \left( X^{-T} \otimes I_n \right) \bar{A}^{-1} \left( I_n \otimes B \right) dg.
\]

The differential of cost $J$, \( \frac{dJ}{f} \), is given by \( dJ = f^T df \).

Using (25) and (17), we get (19). To derive the Hessian, we compute the second-order differentials of the involved variables. Note that since $g(G)$ is an independent variable, \( d^2 g = 0 \) \( (d^2 G = 0) \). Further, the second-order differential of (18a) yields \( d^2 \bar{X} = 0 \). Rewriting (23) as $dF = dG - F dX$, and taking its differential and vectorization, we get
\[
\frac{d^2 F}{dX} X + (dF)(dX) = -(dF)(dX) \Rightarrow \frac{d^2 F}{dX^2} = -2(dF)(dX) X^{-1} = -2X^{-T} \otimes (dF)(dX).
\]

Taking the second-order differential of $J$ (omitting the parameter dependence notation), we get
\[
\frac{d^2 J}{df^T df} + f^T \left( \frac{d^2 f}{df^T df} \right) f = (df)^T df + \left( \frac{d^2 f}{df^T df} \right) f \]
\[
= \left( X^{-1} \otimes I_n \right) \dot{\bar{A}}^{-T} \left( X^{-1} \otimes (df)^T \right) f \quad \text{and} \quad \frac{d^2 J}{dg^2} \dot{\bar{A}}^{-T} \left( X^{-1} \otimes I_n \right) T_{m,n} \dot{g}
\]
\[
\frac{d^2 J}{dg^2} \dot{\bar{A}}^{-T} \left( X^{-1} \otimes I_n \right) T_{m,n} Z \dot{g} + \frac{d^2 J}{dg^2} \dot{\bar{A}}^{-T} \left( X^{-1} \otimes I_n \right) T_{m,n} Z \dot{g} \frac{d^2 J}{dg^2} \dot{\bar{A}}^{-T} \left( X^{-1} \otimes I_n \right) T_{m,n} Z \dot{g}.
\]

Clearly, the first-order optimality condition of the unconstrained optimization problem (18) is given by $\frac{dJ}{dg} = Z(F, X) f = 0$. The next result shows that this condition is equivalent to the first-order optimality conditions of Theorem III.2 without sparsity constraints.

Corollary IV.2. (Equivalence of first-order optimality conditions) Let $\bar{F} = \bar{I}_{m \times n}$. Then, the first-order optimality condition $Z(\bar{F}, \bar{X}) \dot{f} = 0$ of (18) is equivalent to (7a)–(7c), where
\[
\dot{f} \triangleq \text{vec}(\dot{L}) = \bar{A}^{-T} (\bar{X}^{-1} \otimes \bar{F}^T) \dot{f}.
\]

Proof. The optimality condition (7b) follows from (18a)–(18b). Equation (7a) can be rewritten as $\bar{F} \bar{X}^{-T} + \bar{B} \bar{L} = 0$ and its vectorization using (23) yields $\text{vec}[(A + B \bar{F})^T \bar{L} - \bar{L} \bar{A}^T] = \text{vec}[A^T \bar{L} - \bar{L} \bar{A}^T + (B \bar{F})^T \bar{L}]$.

Using Lemma [V.1] we next present a steepest/Newton descent algorithm to solve the non-spars MGEAP (18) [42]. In the algorithms presented in this section, we interchangeably use the matrices $(G, F, X)$ and their respective vectorizations $(g, f, x)$. The conversion of a matrix to the vector (and vice-versa) is not specifically stated in the steps of the algorithm and is assumed wherever necessary.

Algorithm 1: Non-spars solution to the MGEAP

Input: $A, B, \Lambda, G_0$. 
Output: Local minimum $(F, \bar{X})$ of (18).

Initialize: $G_0, X_0 \leftarrow$ Solution of (18a), $F_0 \leftarrow G_0 X_0^{-1}$

repeat

1. $\alpha \leftarrow$ Compute step size (see below);
2. $g \leftarrow g - \alpha Z(F, X) f$ or;
3. $g \leftarrow g - \alpha [H(F, X) + V(F, X)^{-1}] Z(F, X) f$;

$X \leftarrow$ Solution of Sylvester equation (18a);

until convergence;

return $(F, \bar{X})$.

Steps 2 and 3 of Algorithm 1 represent the steepest and (damped) Newton descent steps, respectively. Since, in general, the Hessian $H(F, X)$ is not positive-definite, the Newton descent step may not result in a decrease of the cost. Therefore, we add a Hermitian matrix $V(F, X)$ to the Hessian to make it positive definite [42]. We will comment on the choice of $V(F, X)$ in Section [V]. In step 1, the step size $\alpha$ can be determined by backtracking line search or Armijo’s rule [42]. For a detailed discussion of the steepest/Newton descent methods, the interested reader is referred to [42].
Sylvester equation (18a) and evaluating the inverses of $X$ and $H + V$.

Next, we present a relaxation of the optimization problem (2) and a corresponding algorithm that provides approximately-sparse solutions to the MGEAP. We remove the explicit feedback sparsity constraints (2c) and modify the cost function to penalize it when these sparsity constraints are violated. Using the Sylvester equation based parametrization, the relaxed optimization problem is stated as (while ignoring the eigenvector normalization constraints; c.f. Remark [4]):

$$
\min_{g} \ J_w = \frac{1}{2} \| W \circ F \|^2_F
$$

subject to (18a) and (18b) hold true,

where $W \in \mathbb{R}^{m \times n}$ is a weighing matrix that penalizes the cost for violation of sparsity constraints, and is given by

$$W_{ij} = \begin{cases} 
1 & \text{if } F_{ij} = 1, \\
\geq 1 & \text{if } F_{ij} = 0.
\end{cases}$$

As the penalty weights of $W$ corresponding to the sparse entries of $F$ increase, an optimal solution of (27) becomes more sparse and approaches towards the optimal solution of (2). Note that the relaxed problem (27) corresponds closely to the non-sparse MGEAP (18). Thus, we use a similar gradient based approach to obtain its solution.

**Lemma IV.3. (Gradient and Hessian of $J_W$)** The gradient and Hessian of the cost $J_W$ in (27) with respect to $g$ is given by

$$
dJ_w = Z(F, X)\tilde{W}f,
$$

$$
d^2J_w \equiv H_W(F, X) = Z(F, X)\tilde{W}Z^T(F, X) + Z_1W(F, X)Z^T_1W(F, X),
$$

where $\tilde{W} \equiv \text{diag}(\text{vec}(W \circ W))$ and,$Z_1W(F, X) \equiv (I_3 \otimes B_3)\tilde{A}^{-1}X^{-1}(W \circ W \circ f)^T \otimes I_n)\text{vec}(W \circ W \circ f) = \frac{1}{2} \text{vec}(W \circ W \circ f)^T \text{vec}(W \circ W \circ f)$.

**Proof.** Since the constraints of optimization problems (18) and (27) are same, Equations (21)-(24) from Lemma IV.1 also hold true for problem (27). Now, $J_w \Rightarrow P_1P_2P_3P_4 \Rightarrow \frac{1}{2} \text{vec}(W \circ W \circ f)^T \text{vec}(W \circ W \circ f) = \frac{1}{2} \tilde{W}f$. Thus, $dJ_w = \tilde{W}f$ and

$$
d^2J_w = (df)^T \tilde{W}f + \tilde{W}df.$$

Using the relation $\text{vec}(W \circ W \circ f) = \tilde{W}f$, the remainder of the proof is similar to proof of Lemma IV.1.

Using Lemma IV.3, we next present an algorithm to obtain an approximately-sparse solution to the MGEAP.

The step size rule and modification of the Hessian in Algorithm 2 is similar to Algorithm 1.

**Algorithm 2:** Approximately-sparse solution to the MGEAP

**Input:** $A, B, X, W, G_0$

**Output:** Local minimum $(\tilde{F}, \tilde{X})$ of (27).

**Initialize:** $G_0, X_0 \leftarrow$ Solution of (18a), $F_0 \leftarrow G_0X_0^{-1}$

repeat

1. $\alpha \leftarrow \text{Update step size};$
2. $g \leftarrow g - \alpha Z(F, X)\tilde{W}f$ or;
3. $g \leftarrow g - \alpha [H_W(F, X) + V_W(F, X)]^{-1}Z(F, X)\tilde{W}f$;
4. $X \leftarrow$ Solution of Sylvester equation (18a);
5. $F \leftarrow GX^{-1};$

until convergence;

return $(F, X)$

**Lemma IV.4. (Feasibility of $(F, X)$)** An invertible matrix $X \in \mathbb{R}^{n \times n}$ satisfies (2a) and (2c) if and only if

$$\tilde{a}(x) \in \mathcal{R}(\tilde{B}(X)) \text{ where, (30)}$$

$$\tilde{a}(x) \triangleq \tilde{A}x, \tilde{B}(X) \triangleq -(X^T \otimes B)P_\tilde{F}, P_\tilde{F} \triangleq \text{diag}(\text{vec}(\tilde{F})).$$

Further, if (30) holds true, then the set of sparse feedback matrices that satisfy (2a) and (2c) is given by

$$\mathcal{F}_X = \{P_{f_n} : \tilde{B}(X)f_{n\in n} = \tilde{a}(x), f_{n\in n} \in \mathbb{R}^{m\times n}\}.$$

**Proof.** Any feedback $f$ which satisfies the sparsity constraint (2c) can be written as $f = P_{f_n}f_{ns}$ where $f_{ns} \in \mathbb{R}^{m\times n}$ is a non-sparse vector. Vectorizing (2a), substituting $f = P_{f_n}f_{ns}$, we get

$$\tilde{A}x = -(X^T \otimes B)P_{f_n}f_{ns},$$

from which (30) and (31) follow.

Based on Lemma IV.4, we develop a heuristic iterative algorithm to obtain a sparse solution to the EAP. The algorithm starts with a non-sparse EAP solution $(F_0, X_0)$ that does not satisfy (2c) and (30). Then, it takes repeated projections of $\tilde{a}(x)$ on $\mathcal{R}(\tilde{B}(X))$ to update $X$ and $F$, until a sparse solution is obtained.

**Algorithm 3:** Sparse solution to EAP

**Input:** $A, B, \Lambda, \tilde{F}, G_0$, $\text{iter}_{max}$

**Output:** $(F, X)$ satisfying (2a) and (2c).

**Initialize:** $G_0, X_0 \leftarrow$ Solution of (18), $F_{ns, 0} \leftarrow G_0X_0^{-1}, i \leftarrow 0$

repeat

1. $\tilde{a}(x) \leftarrow \tilde{B}(X)[\tilde{B}(X)]^{-\dagger}\tilde{a}(x);$
2. $x \leftarrow \tilde{A}^{-1}\tilde{a}(x);$
3. $X \leftarrow \text{Normalize } X;
4. $i \leftarrow i + 1$

until convergence or $i \geq \text{iter}_{max}$

return $(f \in \mathcal{F}_X : (31))$.

In step 1 of Algorithm 3, we update $\tilde{a}(x)$ by projecting it on $\mathcal{R}(\tilde{B}(X))$. Step 2 computes $x$ from $\tilde{a}(x)$ using the fact that $\tilde{A}$

Since $f$ satisfies (30), it can also be characterized as $f = (I_{m_n} - Q^+Q)f_{ns}$, and thus $P_{f_n} = I_{m_n} - Q^+Q$. 


is invertible (c.f. Assumption A4). Finally, the normalization in step 3 is performed to ensure invertibility of $X^T$.

Next, we develop a second heuristic algorithm for solving the sparse EAP problem using the non-sparse MGEAP solution in Algorithm [1]. The algorithm starts with a non-sparse EAP solution $(F_0, X_0)$. In each iteration, it sparsifies the feedback to obtain $f = F_0 f_{ns}$ (or $F = F \circ F_{ns}$), and then solves the following non-sparse MGEAP

$$\min_{F_{ns}, X} \frac{1}{2} \|F_{ns} - F\|_F^2,$$

s.t. $(A + BF_{ns})X = NX$.

(33a)

to update $F_{ns}$ that is close to the sparse $F$. Thus, using the heuristics of repeated sparsification of the solution of non-sparse MGEAP in (33), the algorithm obtains a sparse solution. Note that a solution $F_{ns}$ of the optimization problem (33) with parameters $(A, B, \Lambda, F)$ satisfies $F_{ns} = F + \bar{K}_{ns}$, where $\bar{K}_{ns}$ is a solution of the optimization problem (18) with parameters $(A + BF, B, \Lambda)$. Thus, we can use Algorithm [1] to solve (33).

Algorithm 4: Projection-based sparse solution to EAP

| Input: | $A, B, \Lambda, F, G_0, \text{iter}_{max}$. |
|---------|-----------------------------------|
| Output: | $(F, X)$ |
| Initialize: | $G_0, X_0 \leftarrow \text{Solution of (18a)}$, $F_{ns,0} \leftarrow G_0 X_0^{-1}$, $i \leftarrow 0$. |
| repeat | |
| $F \leftarrow F \circ F_{ns}$; $i \leftarrow i + 1$; |
| $(K_{ns}, X) \leftarrow \text{Algorithm [1]}(A + BF, B, \Lambda)$; $F_{ns} \leftarrow F + \bar{K}_{ns}$; |
| until convergence or $i > \text{iter}_{max}$; |
| return $(F, X)$. |

Remark 7. (Comparison of EAP Algorithms [3] and [4])

1. Projection property: In general, Algorithm [3] results in a sparse EAP solution $F$ that is considerably different from the initial non-sparse $F_{ns,0}$. In contrast, Algorithm [4] provides a sparse solution $F$ that is close to $F_{ns,0}$. This is due to the fact that Algorithm [4] updates the feedback by solving the optimization problem (33), which minimizes the deviations between successive feedback matrices. Thus, Algorithm [4] provides a good (although not necessarily orthogonal) projection of a given non-sparse EAP solution $F_{ns,0}$ on the space of sparse EAP solutions.

2. Complexity: The computational complexity of Algorithm [4] is considerably larger than that of Algorithm [3]. This is because Algorithm [4] requires a solution of a non-sparse MGEAP problem in each iteration. In contrast, Algorithm [3] only requires projections on the range space of a matrix in each iteration. Thus, Algorithm [3] is considerably faster as compared to Algorithm [4].

3. Convergence: Although we do not formally prove the convergence of heuristic Algorithms [3] and [4] in this paper,

4Since $X$ is not an eigenvector matrix, we compute the eigenvectors from $X$, normalize them, and then recompute real $X$ from the normalized eigenvectors.

a comprehensive simulation study in Subsection [V-B] suggests that Algorithm [4] converges in almost all instances. In contrast, Algorithm [3] converges in much fewer instances and its convergence deteriorates considerably as the number of sparsity constraints increase (see Subsection [V-B]).

C. Algorithm for sparse MGEAP

In this subsection, we present an iterative projected gradient algorithm to compute the sparse solutions of the MGEAP in (2). The algorithm consists of two loops. The outer loop is same as the non-sparse MGEAP Algorithm [1] (using steepest descent) with an additional projection step, which constitutes the inner loop. Figure [1] represents one iteration of the algorithm. First, the gradient $Z(F_k, X_k) f_k$ is computed at a current point $G_k$ (equivalently $(F_k, X_k)$, where $F_k$ is sparse). Next, the gradient is projected on the tangent plane of the sparsity constraints (4), which is given by

$$T_F = \left\{ y \in \mathbb{R}^{mn} : \begin{bmatrix} d(Qf) \hline dg \end{bmatrix}^T y = 0 \right\} = \left\{ y \in \mathbb{R}^{mn} : QZ^T(F, X)y = 0 \right\}. \quad \text{(34)}$$

From (34), the projection of the gradient on $T_F$ is given by $P_{T_F} Z(F_k, X_k) f_k$, where $P_{T_F} = I_{mn} - [QZ^T(F_k, X_k)]^T QZ^T(F_k, X_k)]$. Next, a move is made in the direction of the projected gradient to obtain $G_{ns,k}(F_{ns,k}, X_{ns,k})$. Finally, the orthogonal projection of $G_{ns,k}$ is taken on the space of sparsity constraints to obtain $G_{ns,k}(F_{ns,k}, X_{ns,k})$. This orthogonal projection is equivalent to solving (33) with sparsity constraints (2c), which in turn is equivalent to the original sparse MGEAP (2). Thus, the orthogonal projection step is as difficult as the original optimization problem. To address this issue, we use the heuristic Algorithm [4] to compute the projections. Although the projections obtained using Algorithm [4] are not necessarily orthogonal, they are typically good (c.f. Remark 7).

Algorithm 5 is computationally intensive due to the use of Algorithm [4] in step 1 to compute the projection on the space of sparse matrices. In fact, the computational complexity of Algorithm [5] is one order higher than that of non-sparse MGEAP Algorithm [1]. However, a way to considerably reduce the number of iterations of Algorithm [5] is to initialize it using the approximately-sparse solution obtained by Algorithm [2]. In this case, Algorithm [5] starts near the the local minimum and, thus, its convergence time reduces considerably.
Algorithm 5: Sparse solution to the MGEAP

Input: $A, B, \Lambda, \bar{F}, G_0, \iota_{\text{iter max}}$.
Output: Local minimum $(\bar{F}, \bar{X})$ of (2).

Initialize:

$(F_0, X_0) \leftarrow \text{Algorithm 4}(A, B, \Lambda, \bar{F}, G_0, \iota_{\text{iter max}})$,
$G_0 \leftarrow F_0 X_0$, $i \leftarrow 0$
repeat

$\alpha \leftarrow \text{Update step size};$
$g_{ns} \leftarrow g - \alpha \frac{dJ}{dg}$;
$X_{ns} \leftarrow \text{Solution of Sylvester equation (18a)}$;
$F_{ns} \leftarrow G_{ns} X_{ns}^{-1}$;
$(F, X) \leftarrow \text{Algorithm 4}(A, B, \Lambda, \bar{F}, G_{ns}, \iota_{\text{iter max}})$;
$G \leftarrow F X$;
$i \leftarrow i + 1$;
until convergence or $i > \iota_{\text{iter max}}$;
return $(F, X)$

V. SIMULATION STUDIES

In this section, we present the implementation details of the algorithms developed in Section IV and provide numerical simulations to illustrate their properties.

A. Implementation aspects of the algorithms

In the Newton descent step (step 3) of Algorithm 1 we need to choose (omitting the parameter dependence notation) $V$ such that $H + V$ is positive-definite. We choose $V = \delta I_{mn} - Z_1 Z^T - Z Z^T$ where, $0 < \delta \ll 1$. Thus, from (20), we have: $H + V = Z Z^T + \epsilon I_{mn}$. Clearly, $Z Z^T$ is positive-semidefinite and the small additive term $\epsilon I_{mn}$ ensures that $H + V$ is positive-definite. Note that other possible choices of $V$ also exist. In step 1 of Algorithm 1 we use the Armijo rule to compute the step size $\alpha$. Finally, we use $\|\frac{dJ}{dg}\|_2 < \epsilon$, $0 < \epsilon \ll 1$ as the convergence criteria of Algorithm 1. For Algorithm 2 we analogously choose $V_W = \delta I_{mn} - Z_1 W Z^T - Z Z^T_{1W}$ and the same convergence criteria and step size rule as Algorithm 1. In both algorithms, if we encounter a scenario in which the solution $X$ of (18a) is singular (c.f. paragraph below (18b)), we perturb $G$ slightly such that the new solution is non-singular, and then continue the iterations. We remark that such instances occur extremely rarely in our simulations.

For the sparse EAP Algorithm 3 we use the convergence criteria $e_X = \|\| I_{nz} - B(X) (B(X))^+ \|_{2} \| < \epsilon$, $0 < \epsilon \ll 1$. For Algorithm 4 we use the convergence criteria $e_F = \|F - \bar{F} \circ \bar{F}\|_F < \epsilon$, $0 < \epsilon \ll 1$. Thus, the iterations of these algorithms stop when $x$ lies in a certain subspace and when the sparsity error becomes sufficiently small (within the specified tolerance), respectively. Further, note that Algorithm 4 uses Algorithm 1 in step 1 without specifying an initial condition $G_0$ for the latter. This is because in step 1 we effectively run Algorithm 1 for multiple initial conditions in order to capture its global minima. We remark that the capture of global minima by Algorithm 1 is crucial for convergence of Algorithm 4.

As the iterations of Algorithm 4 progress, the sparse matrix $F$ achieves eigenvalue assignment with increasing accuracy. As a result, near the convergence of Algorithm 4 the eigenvalues of $A + BF$ and $\Lambda$ are very close to each other. This creates numerical difficulties when Algorithm 1 is used with parameters $(A + BF, B, \Lambda)$ in step 1 (see Assumption A4). To avoid this issue, we run Algorithm 1 using a preliminary feedback $F_p$, as explained below Assumption A4.

Finally, for Algorithm 5 we use the following convergence criteria: $\|P_F \frac{dJ}{dg}\|_2 < \epsilon$, $0 < \epsilon \ll 1$. We choose stopping tolerance $\epsilon$ typically between $10^{-6}$ and $10^{-5}$ for all the algorithms.

B. Numerical study

We begin this subsection with the following example:

$$A = \begin{bmatrix} -3.7653 & -2.1501 & 0.3120 & -0.2484 \\ 1.6789 & 1.0374 & -0.5306 & 1.3987 \\ -2.1829 & -2.5142 & -1.2275 & 0.2833 \\ -13.6811 & -9.6804 & -0.5242 & 2.9554 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 3 & 4 & 2 \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$S = \{-2, -1, -0.5 \pm j\}.$$

The eigenvalues of $A$ are $\Gamma(A) = \{-2, -1, 1 \pm 2j\}$. Thus, the feedback $F$ is required to move two unstable eigenvalues into the stable region while keeping the other two stable eigenvalues fixed. Table I shows the non-sparse, approximately-sparse and sparse solutions obtained by Algorithms 1, 2 and 5 respectively, and Figure 2 shows a sample iteration run of these algorithms. Since the number of iterations taken by the algorithms to converge depends on their starting points, we report the average number of iterations taken over 1000 random starting points. Further, to obtain approximately-sparse solutions, we use the weighing matrix with $W_{ij} = 0$ if $\bar{F}_{ij} = 1$. All the algorithms obtain three local minima, among which the first is the global minimum. The second column in $X$ and $\bar{L}$ is conjugate of the first column (c.f. Remark 3). It can be verified that the non-sparse and sparse solutions satisfy the optimality conditions of Theorem III.2.

For the non-sparse solution, the number of iterations taken by Algorithm 1 with steepest descent are considerably larger than the Newton descent. This is because the steepest descent converges very slowly near a local minimum. Therefore, we use Newton descent steps in Algorithms 1 and 2. Next, observe that the entries at the sparsity locations of the local minimum feedbacks obtained by Algorithm 2 have small magnitude. Further, the average number of Newton descent iterations for convergence and the norm of the feedback obtained of Algorithm 2 is larger as compared to Algorithm 1. This is because the approximately-sparse optimization problem (27) is effectively more restricted than its non-sparse counterpart (18).

Finally, observe that the solutions of Algorithm 5 are sparse. Note that Algorithm 5 involves the use of projection Algorithm 4 which in turn involves running Algorithm 1 multiple times. Thus, for a balanced comparison, we present the total
Convergence instances = 220
Convergence instances = 53
Average Convergence instances = 427

observe that Algorithm 5 involves considerably more Newton
iterations than Algorithms 1 and 2. This is due to the increased
complexity of the algorithm and the need for more
iterations to ensure convergence.

Table II

| Algorithm | Convergence instances | Average Over 1000 runs |
|-----------|-----------------------|------------------------|
| 1         | 220                   | 12                     |
| 2         | 53                    | 8                      |

Although the number of outer iteration of Algorithm 5 are considerably
less, for instance, 20 in Figure 2.

projection error \( e_X \) and the sparsity error \( e_F \) decrease as the respective
algorithms progress.

Next, we provide an empirical verification of the converg-
ence of heuristic Algorithms 3 and 4. Let the sparsity ratio
(SR) be defined as the ratio of the number of sparse
entries to the total number of entries in \( F \) (i.e., \( SR = \frac{\text{Number of 0's in } F}{\text{Total number of entries}} \)).

We perform 1000 random executions of both the
algorithms. In each execution, \( n \) is randomly selected
between 4 and 20 and \( m \) is randomly selected between 2 and 100. Then, matrices
(\( A, B \)) are randomly generated with appropriate
dimensions. Next, a binary sparsity pattern matrix \( \bar{F} \) is randomly
generated with the number of sparsity entries given by \( \lfloor SR \times mn \rfloor \),
where \( \lfloor \cdot \rfloor \) denotes rounding to the next lower integer. To
ensure feasibility (c.f. Assumption A2 and discussion below),
we pick the desired eigenvalue set \( S \) randomly as follows:
we select a random \( F' \) which satisfies the selected sparsity pattern
\( \bar{F} \), select \( S = \Gamma(A + BF_0) \). Finally, we set \( \text{iter}_{max} = 1000 \)
and select a starting point \( [G_0, X_0] \), and run both
algorithms from the same starting point. Let \( F_{sol} \) denote the
feedback solution obtained by Algorithms 3 and 4
respectively, and let \( d_{F_{sol}, F_0} = \| F_{sol} - F_0 \|_F \) denote the distance
between the starting point \( F_0 \) and the final solution.

Table II

| Algorithm | SR | Convergence instances | Average Over 1000 runs |
|-----------|----|-----------------------|------------------------|
| 1         | 1/4| 220                   | 8.47                   |
| 2         | 1/2| 53                    | 6.12                   |
| 3         | 2/3| 53                    | 6.12                   |

Fig. 2: Optimization costs for a sample run of Algorithms 1, 2 and 3 (the
algorithms converge to their global minima).

Fig. 3: Projection and sparsity errors for a sample run of (a) Algorithm 5
and (b) Algorithm 4 respectively.
Table II shows the convergence results of Algorithms 3 and 4 for three different sparsity ratios. While the convergence of Algorithm 3 deteriorates as \( F \) becomes more sparse, Algorithm 4 converges in almost all instances. This implies that in the context of Algorithm 3, Algorithm 4 provides a valid projection in step 1 in almost all instances. We remark that in the rare case that Algorithm 4 fails to converge, we can reduce the step size \( \alpha \) to obtain a new \( G_{\alpha} \) and compute its projection. Further, we compute the average of distance \( d_{F_{\alpha i}, F_\theta} \) over all executions of the Algorithms 3 and 4 that converge. Observe that the average distance for Algorithm 4 is smaller than Algorithm 3. This shows that Algorithm 4 provides considerably better projection of \( F_\theta \) in the space of sparse matrices as compared to Algorithm 3 (c.f. Remark 7).

VI. CONCLUSION
In this paper we studied the MGEAP for LTI systems with arbitrary sparsity constraints on the static feedback matrix. We presented an analytical characterization of its locally optimal solutions, thereby providing explicit relations between an optimal solution and the eigenvector matrices of the associated closed loop system. We also provided a geometric interpretation of an optimal solution of the non-sparse MGEAP. Using the Sylvester equation based parametrization, we developed a projected gradient descent algorithm to solve the MGEAP. We also presented two novel algorithms for solving the sparse EAP and an algorithm to obtain approximately sparse solution to the MGEAP. Various convergence properties of the algorithms were discussed using numerical studies.

The analysis in the paper is developed under the assumption that the sparse EAP problem with static feedback is feasible. A future direction of research include finding necessary/sufficient conditions for the feasibility of the sparse EAP, and formulating a convex relaxation of the sparse MGEAP with guaranteed distance from optimality.

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