Lazy states, discordant states and entangled states for 2-qubit systems

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We investigate the lazy states, entangled states and discordant states for 2-qubit systems. We show that many lazy states are discordant, many lazy states are entangled, and many mixed entangled states are not lazy. With these investigations, we provide a laziness-discord-entanglement hierarchy diagram about 2-qubit quantum correlations.

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I. INTRODUCTION

Quantum correlation is one of the most striking features of quantum theory. Entanglement is the most famous kind of quantum correlation, and leads to powerful applications [1]. Discord is another kind of quantum correlation, which captures more correlation than entanglement in the sense that a disentangled state may have no zero discord [2]. Due to the theoretical and applications, discord has been extensively studied [2] and still in active research (for examples see [3–6]).

A bipartite state is called lazy, if the entropy rate of one subsystem is zero under any coupling to the other subsystem. Necessary and sufficient conditions have recently been established for a state to be lazy [7], and it was shown that almost all states are pretty lazy [8]. It is shown that a maximally entangled pure state is lazy [9]. This indicates that the correlation described by lazy states is not the same by entanglement. So we are interested to clarify the question that, whether there are many lazy states which are entangled, and whether there are many entangled states which are lazy. This paper answers this question for the 2-qubit case.

This paper is organized as follows. In Section 2, we briefly review the definitions about entangled states, discordant states and lazy states. In Section 3, we establish a necessary and sufficient condition for 2-qubit lazy states. In Section 4, we show that there are many 2-qubit lazy states which are discordant states. In Section 5, we show that there are many disentangled states which are not lazy. In Section 6, we show that there are many 2-qubit mixed lazy states which are entangled. In section 7, we briefly summary this paper by providing a laziness-discord-entanglement hierarchy diagram to characterize the bipartite quantum correlations.

II. ENTANGLED STATES, DISCORDANT STATES, LAZY STATES

We briefly review the definitions about entangled states, discordant states and lazy states.

Finite-dimensional quantum systems $A$ and $B$ are described by the Hilbert spaces $H^A$ and $H^B$ respectively, the composite system $AB$ is then described by the Hilbert space $H^A \otimes H^B$. Let $n_A = \dim H^A$, $n_B = \dim H^B$. A state $\rho^{AB}$ is called a disentangled state (or separable state) if it can be written in the form

$$\rho^{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B,$$  \hspace{1cm} (1)  

where $p_i \geq 0$, $\sum_i p_i = 1$, $\{\rho_i^A\}_i$ are density operators on $H^A$, $\{\rho_i^B\}_i$ are density operators on $H^B$. If $\rho^{AB}$ is disentangled we then say $E(\rho^{AB}) = 0$.

A state $\rho^{AB}$ is called a zero-discord state with respect to $A$ if it can be written in the form

$$\rho^{AB} = \sum_{i=1}^{n_A} p_i |\psi_i^A\rangle\langle \psi_i^A| \otimes \rho_i^B,$$  \hspace{1cm} (2)  

where $p_i \geq 0$, $\sum_i p_i = 1$, $\{|\psi_i^A\rangle\}_i$ is an orthonormal basis for $H^A$, $\{\rho_i^B\}_i$ are density operators on $H^B$. If $\rho^{AB}$ is in the form Eq.(2) we then say $D_A(\rho^{AB}) = 0$.

Evidently,

$$D_A(\rho^{AB}) = 0 \Leftrightarrow E(\rho^{AB}) = 0. \hspace{1cm} (3)$$

A state $\rho^{AB}$ is called a lazy state with respect to $A$ if [7]

$$C_A(\rho^{AB}) = [\rho^{AB}, \rho^A \otimes I^B] = 0, \hspace{1cm} (4)$$

where $\rho^A = tr_B \rho^{AB}$, $I^B$ is the identity operator on $H^B$. An important physical interpretation of lazy states is that the entropy rate of $A$ is zero in the time evolution under any coupling to $B$,

$$C_A(\rho^{AB}(t)) = 0 \Leftrightarrow \frac{d}{dt} tr_A[\rho^A(t) \log_2 \rho^A(t)] = 0. \hspace{1cm} (5)$$

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\[ D_A(\rho^{AB}) = 0 \quad \text{and} \quad C_A(\rho^{AB}) = 0 \] has the inclusion relation below [9]

\[ D_A(\rho^{AB}) = 0 \Leftrightarrow C_A(\rho^{AB}) = 0. \quad (6) \]

Maximal pure entangled states are the examples of \( C_A(\rho^{AB}) = 0 \) but \( D_A(\rho^{AB}) \neq 0 \) [9].

The direct product states have the form

\[ \rho^{AB} = \rho^A \otimes \rho^B, \quad (7) \]

they are obviously zero-discord states.

### III. THE FORM OF 2-QUBIT LAZY STATES

Any 2-qubit state can be written in the form [10]

\[ \rho^{AB} = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} x_i \sigma_i \otimes I + \sum_{j=1}^{3} y_j I \otimes \sigma_j + \sum_{i,j=1}^{3} T_{ij} \sigma_i \otimes \sigma_j) \quad (8) \]

where \( I \) is the two-dimensional identity operator, \( \{\sigma_i\}_{i=1}^{3} \) are Pauli operators, \( \{x_i\}_{i=1}^{3}, \{y_j\}_{j=1}^{3}, \{T_{ij}\}_{i,j=1}^{3} \) are all real numbers satisfying some conditions (we will explore these conditions when we need them) to ensure the positivity of \( \rho^{AB}, \rho^A \) and \( \rho^B \). We often omit \( I \) for simplicity without any confusion.

**Proposition 1.** The 2-qubit state \( \rho^{AB} \) in Eq.(8) is lazy if and only if

\[ \{x_i\}_{i=1}^{3}, \{y_j\}_{j=1}^{3}, \{T_{ij}\}_{i,j=1}^{3} \quad \text{for} \quad j = 1, 2, 3. \quad (9) \]

**Proof.** For state in Eq.(8),

\[ \rho^A = \frac{1}{2}(I + \sum_{k=1}^{3} x_k \sigma_k \otimes I), \quad (10) \]

\[ [\rho^{AB},\rho^A] = \frac{1}{8} \sum_{i,j,k=1}^{3} T_{ij} x_k [\sigma_i \otimes \sigma_j, \sigma_k \otimes I] \]

\[ = \frac{1}{8} \sum_{i,j,k=1}^{3} T_{ij} x_k [\sigma_i, \sigma_j, \sigma_k \otimes I] \]

\[ = \frac{i}{4} \sum_{i,j,k=1}^{3} T_{ij} x_k \varepsilon_{ikl} \sigma_l \otimes \sigma_j. \quad (11) \]

In the last line, \( \varepsilon_{ikl} \) is the permutation symbol. Let \( [\rho^{AB},\rho^A] = 0 \), then

\[ \sum_{i,j,k=1}^{3} T_{ij} x_k \varepsilon_{ikl} = 0, \quad (12) \]

this evidently leads to Eq.(9). □

### IV. LAZY BUT DIACORDANT 2-QUBIT STATES

It is easy to check that \( C_A(\rho^{AB}) = 0 \) defined in Eq.(4) is invariant under locally unitary transformations for arbitrary \( n_A \) and \( n_B \). Under locally unitary transformations, any 2-qubit state in Eq.(8) can be written in the form [11]

\[ \rho^{AB} = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} x_i \sigma_i \otimes I + \sum_{j=1}^{3} y_j I \otimes \sigma_j + \sum_{i,j=1}^{3} \lambda_i \sigma_i \otimes \sigma_i), \quad (13) \]

where \( 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \) being the singular values of \( \{T_{ij}\}_{ij} \) in Eq.(8). Note that \( \{x_i\}_{i=1}^{3}, \{y_j\}_{j=1}^{3} \) in Eq.(9) are not the same with in Eq.(8).

We now look for the conditions such that \( D_A(\rho^{AB}) = 0 \). Suppose \( D_A(\rho^{AB}) = 0 \), then according to Eq.(2), there exists real vector \( \vec{n} = \{n_1, n_2, n_3\} \) with \( n_1^2 + n_2^2 + n_3^2 = 1 \) such that

\[ \rho^{AB} = \Pi_0 \otimes \rho^{AB} \Pi_0 \otimes I + \Pi_1 \otimes I \rho^{AB} \Pi_1 \otimes I, \quad (14) \]

with

\[ \Pi_0 = \frac{1}{2}(I + \vec{n} \cdot \vec{\sigma}), \quad (15) \]

\[ \Pi_1 = \frac{1}{2}(I - \vec{n} \cdot \vec{\sigma}). \quad (16) \]

It can be check that

\[ \Pi_0 \rho_{ij} \Pi_0 + \Pi_1 \rho_{ij} \Pi_1 = n_i \vec{n} \cdot \vec{\sigma}. \quad (17) \]

Then Eq.(14) becomes

\[ \rho^{AB} = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} x_i n_i \vec{n} \cdot \vec{\sigma} \otimes I + \sum_{j=1}^{3} y_j I \otimes \sigma_j + \sum_{i=1}^{3} \lambda_i n_i \vec{n} \cdot \vec{\sigma} \otimes \sigma_i) \]

\[ = \frac{1}{4}(I \otimes I + \sum_{i,j=1}^{3} x_i n_i \sigma_j \otimes I + \sum_{j=1}^{3} y_j I \otimes \sigma_j + \sum_{i,j=1}^{3} \lambda_i n_i \sigma_j \otimes \sigma_i). \quad (18) \]

Comparing to Eq.(13), then for \( j = 1, 2, 3, \)

\[ \sum_{i=1}^{3} x_i n_i n_j = x_j \Rightarrow \vec{n} \cdot \vec{\sigma}, \quad (19) \]

\[ \lambda_i n_i n_j = \delta_{ij} \lambda_j = \delta_{ij} \delta_{i} \Rightarrow \lambda_i = 0 \quad \text{or} \quad n_i = \pm 1. \quad (20) \]

(i) If \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \), let \( \vec{n} \cdot \vec{\sigma} \), then \( D_A(\rho^{AB}) = 0 \).
(ii) If $0 = \lambda_1 = \lambda_2 < \lambda_3 = 0$, then $\overrightarrow{\rho} = (0,0,\pm 1)$, to satisfy $\overrightarrow{\rho} / / \overrightarrow{x}$, we see that only when $\overrightarrow{x} = (0,0,x_3)$ we have $D_A(\rho^{AB}) = 0$.

(iii) If $0 = \lambda_1 < \lambda_2 < \lambda_3 = 0$, then Eq.(20) can not be satisfied, so $\rho^{AB}$ is discordant.

(iv) If $0 < \lambda_1 < \lambda_2 < \lambda_3 = 0$, then Eq.(20) can not be satisfied, so $\rho^{AB}$ is discordant.

Comparing with Proposition 1, we then get Proposition 2 below.

**Proposition 2.** A 2-qubit state in Eq.(13) is lazy but discordant if and only if $\overrightarrow{x} = 0$ and $0 < \lambda_2 < \lambda_3$.

Since any locally unitary transformation keeps $\overrightarrow{x} = 0$ invariant in Eq.(8), then we rewrite Proposition 2 as Proposition 2’ below.

**Proposition 2’.** A 2-qubit state in Eq.(8) is lazy but discordant if and only if $\overrightarrow{x} = 0$ and the matrix $\{T_{ij}\}_{ij}$ have at least two positive singular values.

We make a note that some constraints about $\{y_j\}_{j=1}^3, \lambda_1, \lambda_2, \lambda_3$ are required to guarantee the positivity of $\rho^{AB}, \rho^A, \rho^B$ in Proposition 2. These constraints are rather complex since there are so many parameters. To show there indeed exist many states described in Proposition 2, we choose some special states. For the state

$$\rho^{AB} = \frac{1}{4} (I \otimes I + \sum_{j=1}^{3} y_j I \otimes \sigma_j + \sum_{i=1}^{3} \lambda_i \sigma_i \otimes \sigma_i), \quad (21)$$

where $0 \leq \lambda_1 \leq \lambda_2, 0 < \lambda_2 < \lambda_3$, we have $\rho^A = I$, and

$$\rho^B = \frac{1}{2} (I + \sum_{j=1}^{3} y_j \sigma_j). \quad (22)$$

$\rho^B$ is positive then

$$\sum_{j=1}^{3} y_j^2 \leq 1. \quad (23)$$

Let $y_2 = y_3 = \lambda_1 = 0$, then the four eigenvalues of $\rho^{AB}$ in Eq.(21) are

$$\frac{1}{4}(1 \pm \sqrt{y_1^2 + (\lambda_3 \pm \lambda_2)^2}). \quad (24)$$

These eigenvalues are all nonnegative then we need

$$0 < \lambda_2 < \lambda_3, \quad (25)$$

$$y_1^2 + (\lambda_3 + \lambda_2)^2 \leq 1. \quad (26)$$

There are many triples $\{y_1, \lambda_3, \lambda_2\}$ satisfy Eqs.(25,26), then the corresponding states in Eq.(21) are lazy but discordant states.

**V. SOME DISENTANGLED BUT NOT LAZY 2-QUBIT STATES**

To show there exist many 2-qubit states which are disentangled but not lazy, we consider the states of the form

$$\rho^{AB} = p|\psi_1^A\rangle\langle\psi_1^A| \otimes \rho_1^B + (1 - p)|\psi_2^A\rangle\langle\psi_2^A| \otimes \rho_2^B, \quad (27)$$

where $p \in (0,1), \{|\psi_i^A\rangle\}_{i=1}^2$ are normalized states in $H^A$ but not necessarily orthogonal, $\{|\psi_i^A\rangle\}_{i=1}^2$ are density operators on $H^B$. Note that $p = 0$ or $p = 1$ leads to direct product states, so we do not consider such cases.

Under locally unitary transformations, we let

$$|\psi_1^A\rangle = \frac{I + (0,0,1) \cdot \overrightarrow{\sigma}}{2}, \quad (28)$$

$$|\psi_2^A\rangle = \frac{I + (\sin \alpha, 0, \cos \alpha) \cdot \overrightarrow{\sigma}}{2}, \quad (29)$$

$$\rho_1^B = \frac{I + a(0,0,1) \cdot \overrightarrow{\sigma}}{2}, \quad (30)$$

$$\rho_2^B = \frac{I + b(\sin \beta, 0, \cos \beta) \cdot \overrightarrow{\sigma}}{2}, \quad (31)$$

where $\alpha, \beta \in [0, \pi], a, b \in [0, 1]$.

Some special states can be apparently specified.

(i). $\alpha = 0, \rho^{AB}$ in Eq.(27) are direct product states;

(ii). $\alpha = \pi, \rho^{AB}$ in Eq.(27) are zero-discord states;

(iii). $a = b = 0, \rho^{AB}$ in Eq.(27) are direct product states.

Now we consider the cases excluding (i), (ii), (iii) above. Taking Eqs.(28-31) into Eq.(27), and using the notations in Eq.(8), we get

$$\overrightarrow{x} = (1 - p) \sin \alpha, 0, p + (1 - p) \cos \alpha), \quad (32)$$

$$\{T_{11}\}_i = (b(1 - p) \sin \alpha \sin \beta, 0, b(1 - p) \cos \alpha \sin \beta), \quad (33)$$

$$\{T_{12}\}_i = (0, 0, 0), \quad (34)$$

$$\{T_{13}\}_i = (b(1 - p) \sin \alpha \cos \beta, 0, ap + b(1 - p) \cos \alpha \cos \beta). \quad (35)$$

From Proposition 1, $\rho^{AB}$ in Eq.(27) is lazy if and only if $\overrightarrow{x} / / \{T_{11}\}_i$ and $\overrightarrow{x} / / \{T_{13}\}_i$. Since $x_1 = (1 - p) \sin \alpha \neq 0$, then $\overrightarrow{x} / / \{T_{11}\}_i$ and $\overrightarrow{x} / / \{T_{13}\}_i$ lead to

$$b \sin \beta = 0. \quad (36)$$

$$a = b \cos \beta. \quad (37)$$

Eqs.(36,37) together correspond to direct product states since $\rho_1^B = \rho_2^B$. Otherwise, there are many states violate Eq.(36) or Eq.(37), so they are not lazy states.

**Proposition 3.** 2-qubit disentangled state $\rho^{AB}$ in Eq.(27), is a direct product state when $|\psi_1^A\rangle = |\psi_1^A\rangle$ or $\rho_1^B = \rho_2^B$, is a zero-discord state when $\langle\psi_1^A|\psi_2^A\rangle = 0$. Otherwise, $\rho^{AB}$ is not lazy.
VI. SOME LAZY BUT ENTANGLED STATES

We know that a bipartite pure state is lazy only if under locally unitary transformations it can be written in the form [7] \( |\psi^A\rangle = \frac{1}{\sqrt{2}} \sum_i |\psi^A_i\rangle |\psi^B_i\rangle \), where \( \{|\psi^A_i\rangle\}_i \) are orthonormal sets in \( H^A \), \( \{|\psi^B_i\rangle\}_i \) are orthonormal sets in \( H^B \), \( s \leq \min\{n_A, n_B\} \). When \( s = \min\{n_A, n_B\} \) it is maximally entangled state. In this section we look for more 2-qubit mixed states which are lazy but entangled.

From Proposition 1, we know the following 2-qubit Bell-diagonal states are lazy

\[
\rho^{AB} = \frac{1}{4} (I \otimes I + \sum_{i=1}^3 \lambda_i \sigma_i \otimes \sigma_i),
\]

(38)

where \( \{\lambda_i\}_{i=1}^3 \) are real numbers satisfying some constraints to ensure the positivity of \( \rho^{AB} \).

In this section, for convenience, we do not assume \( \{\lambda_i\}_{i=1}^3 \) are all nonnegative. We represent the states in Eq.(38) in the \((\lambda_1, \lambda_2, \lambda_3)\) space.

The eigenvalues of \( \rho^{AB} \) in Eq.(38) are

\[
\lambda_1 = \frac{1}{4} \left[ 1 - \lambda_1 + \lambda_2 + \lambda_3, 1 + \lambda_1 - \lambda_2 + \lambda_3, 1 + \lambda_1 + \lambda_2 - \lambda_3, 1 - \lambda_1 - \lambda_2 - \lambda_3 \right],
\]

(39)

Then the positivity of \( \rho^{AB} \) requires that \( \{\lambda_i\}_{i=1}^3 \) are in the tetrahedron (with its boundary) with the vertices \((-1, -1, -1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)\) in the \((\lambda_1, \lambda_2, \lambda_3)\) space [12]. Disentangled states in Eq.(38) are in the octahedron (with its boundary) with the vertices \((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\) [12]. From Proposition 2, we know the zero-discord states in Eq.(38) are only three line segments \((\lambda_1, 0, 0)\) with \( \lambda_1 \in [-1, 1], (0, \lambda_2, 0)\) with \( \lambda_2 \in [-1, 1], (0, 0, \lambda_3)\) with \( \lambda_3 \in [-1, 1] \).

Then the states in the tetrahedron (with its boundary) but not in the octahedron (with its boundary) are lazy but entangled. Among these, only the states at the vertices of tetrahedron are (maximally entangled) pure states.

VII. SUMMARY: A HIERARCHY DIAGRAM

We explored some 2-qubit states, showed that many states are lazy but discordant, many states are lazy but entangled, and many states are disentangled but not lazy. With these investigations, we can surely give a hierarchy diagram (Figure 1) of 2-qubit states, including lazy states, disentangled states and zero-discord states.

This hierarchy diagram enriches the entanglement-discord hierarchy, then provides more understandings about the structures of quantum correlations.

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