Graphs with Sudoku number $n - 1$

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Abstract

Recently Lau-Jeyaseeli-Shiu-Arumugam introduced the concept of the “Sudoku colourings” of graphs — partial $\chi(G)$-colourings of $G$ that have a unique extension to a proper $\chi(G)$-colouring of all the vertices. They introduced the Sudoku number of a graph as the minimal number of coloured vertices in a Sudoku colouring. They conjectured that a connected graph has Sudoku number $n - 1$ if, and only if, it is complete. In this note we prove that this is true.

1 Introduction

All colourings in this note are vertex-colourings. A vertex colouring of a graph is proper if adjacent vertices receive different colours. The chromatic number, $\chi(G)$ is the minimal number of colours in a proper colouring of $G$. A partial proper colouring of a graph is a colouring of some subset of $V(G)$ which doesn’t give adjacent vertices the same colour. We say that a colouring $\phi$ extends a partial colouring $\psi$ if all vertices coloured in $\psi$ receive the same colour in $\phi$. A variety of tasks can be encoded as taking a partial proper colouring in a graph and then extending it to a full proper colouring of all the vertices.

For example the well known “Sudoku puzzle” can be encoded in this form. Consider a graph $G_{\text{Sudoku}}$ on 27 vertices that are identified with the cells in a $9 \times 9$ grid. Add an edge between any two vertices in the same column, between any two vertices in the same row, and between any two vertices in the same $3 \times 3$ box. It is easy to see that a proper 9-colouring of $G_{\text{Sudoku}}$ exactly corresponds to filling in a $9 \times 9$ array according to the rules of Sudoku puzzles. Thus a Sudoku puzzle can be summarized as “you are given a partial colouring of $G_{\text{Sudoku}}$ and need to complete it to a proper colouring of all the vertices of $G_{\text{Sudoku}}$.”

One of the conventions for designing Sudoku puzzles is that there should always be precisely one way of filling in the $9 \times 9$ array i.e. there should always exist one solution, and there shouldn’t exist multiple solutions. This motivates the definition of a Sudoku colouring of a graph. Lau-Jeyaseeli-Shiu-Arumugam defined it as follows in [1].

Definition 1. A Sudoku colouring of a graph $G$ is a partial proper $\chi(G)$-colouring of $G$ which has precisely one extension to a $\chi(G)$-colouring of $G$.

Sudoku colourings of $G_{\text{Sudoku}}$ are thus in one-to-one correspondence with uncompleted Sudoku puzzles (that have unique completion). But we can also investigate Sudoku colourings of general graphs.

Lau-Jeyaseeli-Shiu-Arumugam [1] defined the Sudoku number of $G$, $sn(G)$ as the smallest number of coloured vertices in a Sudoku colouring of $G$. The motivation for this is that $sn(G_{\text{Sudoku}})$ now asks for the minimum number of clues (i.e. non-blank entries) in a Sudoku puzzle with unique solution. This number has been determined as $sn(G_{\text{Sudoku}}) = 17$ by McGuire, Tugemann, and Civario using a computer-assisted proof [2].

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For general graphs, Lau-Jeyaseeli-Shiu-Arumugam [1] determined $sn(G)$ for various classes of graphs $G$ and obtained bounds for other classes. For example, they showed that $sn(G) = 1$ if, and only if, $G$ is connected and bipartite. On the other extreme, they showed that $sn(G) \leq |G| - 1$ for all graphs and conjectured that for connected graphs, equality holds if, and only if, $G$ is complete. Here we show that this is the case.

**Theorem 2.** A connected graph has $sn(G) = |G| - 1$ if, and only if, $G$ is complete.

The backwards direction already appears in [1] (see Corollary 3.4), so we focus on proving the statement "let $G$ be a connected graph with $sn(G) = |G| - 1$. Then $G$ is complete". This amounts to showing that non-complete graphs have partial proper $\chi(G)$-colourings with $\geq 2$ uncoloured vertices which have a unique extension to a full proper $\chi(G)$-colouring of $G$.

## 2 Proofs

In a $k$-coloured graph our set of colours will always be $|k| = \{1, \ldots, k\}$. For a colouring $\phi$ and vertices $u_1, \ldots, u_k$, we use $\phi - u_1 - \cdots - u_k$ to mean the partial colouring formed by uncolouring the vertices $u_1, \ldots, u_k$. For a partial colouring $\phi$ and a set of vertices $S$, we define $\phi(S) := \{\phi(s) : s \in S\}$ to mean the set of colours appearing in $S$. We use $N(v)$ to denote the set of vertices connected to $v$ by an edge (our graphs are simple, so this never includes $v$ itself).

The following is the tool we use for constructing Sudoku colourings in this note. It gives two kinds of Sudoku colourings with two uncoloured vertices.

**Lemma 3.** Let $\psi$ be a partial proper $\chi(G)$-colouring with exactly two uncoloured vertices $u, v$. Suppose that either of the following holds:

(i) $uv$ is a nonedge and $|\phi(N(u))| = |\phi(N(v))| = \chi(G) - 1$.

(ii) $uv$ is an edge, $|\phi(N(u))| = \chi(G) - 1$, $|\phi(N(v))| = \chi(G) - 2$, and $\phi(N(v)) \subset \phi(N(u))$.

Then $\psi$ is a Sudoku colouring.

**Proof.** In case (i), there is precisely one colour missing from $N(u)$ and precisely one colour missing from $N(v)$. In a proper $\chi(G)$-colouring, $u$ and $v$ must receive exactly these colours, and so the extension is unique.

In case (ii), there are two colours $c, d$ missing from $N(v)$ and one of these colours (say $c$), is missing from $N(u)$. To complete the colouring $u$ must receive colour $c$, and $v$ must receive colour $d$, so the extension is unique. □

The following definition is crucial for us.

**Definition 4.** Let $\phi$ be a proper $\chi(G)$-colouring of $G$. A vertex is **full** if it is adjacent to vertices of all colours (aside from its own) i.e. if $\phi(N(v)) = [\chi(G)] \setminus \phi(v)$ (or equivalently if $|\phi(N(v))| = \chi(G) - 1$).

In graphs with Sudoku number $|G| - 1$, it turns out that the full vertices form a complete subgraph.

**Lemma 5.** Let $sn(G) = |G| - 1$ and let $\phi$ be a proper $\chi(G)$-colouring of $G$. Then any two full vertices are connected by an edge.

**Proof.** Let $u, v$ be full and suppose for contradiction that $uv$ is not an edge in $G$. Consider the partial colouring $\psi := \phi - u - v$. Since $u, v$ are full, we have $|\phi(N(u))| = |\phi(N(v))| = \chi(G) - 1$. Since $uv$ is a non-edge, the neighbours of $u, v$ all remain coloured in $\psi$ and so $|\psi(N(u))| = |\psi(N(v))| = \chi(G) - 1$. Thus, by Lemma 3 (i), $\psi$ is a Sudoku colouring. It has two uncoloured vertices, contradicting $sn(G) = |G| - 1$. □
We say that a proper \( k \)-colouring of \( G \) is \( c \)-minimal if it has as few colour \( c \) vertices as possible (for a \( k \)-colouring of \( G \)). The following lemma shows that there are a lot of full vertices around all \( c \)-coloured vertices in a \( c \)-minimal colouring.

**Lemma 6.** Let \( \phi \) be a 1-minimal proper \( \chi(G) \)-colouring of \( G \). Let \( v \) be a vertex with \( \phi(v) = 1 \). Then for every colour \( c = 2, \ldots, \chi(G) \), the vertex \( v \) has at least one colour \( c \) neighbour \( u \) with \( u \) full. In particular, \( v \) is full.

**Proof.** First notice that it is impossible that \( v \) has no colour \( c \) neighbours -- indeed otherwise, we could recolour \( v \) with colour \( c \) to get a proper colouring with one fewer colour 1 vertex (contradicting 1-minimality). This proves the “in particular \( v \) is full” part — since we’ve shown that every colour other than \( \phi(v) = 1 \) appears on \( N(v) \).

Now let the set of colour \( c \) neighbours of \( v \) be \( \{u_1, \ldots, u_k\} \). Suppose for contradiction that none of these are full — equivalently there are colours \( c_1, \ldots, c_k \in [\chi(G)] \setminus c \) with \( c_i \) missing from \( N(u_i) \). Note that \( c_i \neq 1 \) for all \( i \), since \( v \in N(u_i) \) and \( \phi(v) = 1 \). Note that \( \{u_1, \ldots, u_k\} \) is an independent set since all these vertices have colour \( c \) and the colouring is proper.

Now recolour \( u_i \) by \( c_i \) for each \( i \) and recolour \( v \) by \( c \). Notice that this colouring is proper. To show this, we need to check that the recoloured vertices \( v, u_1, \ldots, u_k \) have different colours to all their neighbours (everywhere else the colouring remains proper just because \( \phi \) was proper).

Indeed \( v \) has no colour \( c \) neighbours since \( \{u_1, \ldots, u_k\} \) was the set of all colour \( c \) neighbours of \( v \) and these have all been recoloured by colours \( c_i \neq c \). Vertex \( u_i \) has no colour \( c_i \) neighbour since it initially had no colour \( c_i \) neighbours and the only neighbour of \( u_i \) that was recoloured was \( v \) (which received colour \( c \neq c_i \)).

But the new colouring we have has one fewer colour 1 vertex, contradicting 1-minimality.

Applying the above lemma to a graph with Sudoku number \( |G| - 1 \) gives even more structure in a minimal colouring.

**Lemma 7.** Let \( sn(G) = |G| - 1 \) and let \( \phi \) be a 1-minimal proper \( \chi(G) \)-colouring of \( G \). Then there is precisely one colour 1 vertex. Additionally, this vertex \( v \) is full and has \( |N(v)| = \chi(G) - 1 \).

**Proof.** Let \( v \) be a colour 1 vertex. By Lemma 6 \( v \) is full. There can’t be another colour 1 vertex \( z \) — because otherwise \( z \) would also be full, which would give two disconnected full vertices (contradicting Lemma 5).

Suppose that \( |N(v)| > \chi(G) - 1 \). Then, by the pigeonhole principle there must be some colour \( c \) which occurs more than once on \( N(v) \). By Lemma 6 \( v \) has some full colour \( c \) neighbour \( u \). Let \( w \) be some other colour \( c \) neighbour of \( v \). Now let \( \psi := \phi - u - v \). Note that \( \psi(N(v)) = \phi(N(v)) = [\chi(G)] \setminus 1 \) (the first equality holds because precisely one neighbour \( u \) of \( v \) was uncoloured, but that neighbour \( u \) had colour \( c \) which is still present at \( w \). The second equality holds because \( v \) is full and has colour 1). Also \( \psi(N(u)) = \phi(N(u)) \setminus 1 = [\chi(G)] \setminus \{1, c\} \) (the first equality holds because precisely one neighbour \( v \) or \( u \) was uncoloured, and that neighbour had colour 1 which isn’t present anywhere else in the graph. The second equality holds because \( \phi(N(u)) = [\chi(G)] \setminus c \) since \( u \) is full and has colour \( c \)). Thus by Lemma 3 (ii), \( \psi \) is a Sudoku colouring with two uncoloured vertices, contradicting \( sn(G) = |G| - 1 \).

We are ready to prove our main theorem.

**Proof of Theorem 2.** The backwards direction already appears in [1] (see Corollary 3.4), so it remains to prove that every connected \( G \) with \( sn(G) = |G| - 1 \) is complete. To that end, let \( G \) be connected with \( sn(G) = |G| - 1 \).

Consider a 1-minimal proper \( \chi(G) \)-colouring of \( G \). By Lemma 7, there is precisely one colour 1 vertex, call it \( v \). Also by Lemma 7, \( v \) is full and \( |N(v)| = \chi(G) - 1 \) — which, using the definition of “full”, implies that all neighbours of \( v \) have different colours. By Lemma 6, we have that all neighbours of \( v \) are full. Now we have that all vertices in \( v \cup N(v) \) are full, and so Lemma 5 tells us that \( v \cup N(v) \) is complete.
Unless $G$ is complete, then, by connectedness, there is some vertex $w$ outside $v \cup N(v)$ with a neighbour $u$ in $N(v)$. Now construct a colouring $\psi$ by recolouring $w$ by colour 1 and uncolouring $u, v$. First notice that this is a proper (partial) colouring — indeed $w$ has no colour 1 neighbours since $v$ is the unique colour 1 vertex and $w \notin N(v)$. We have that $\psi(N(v)) = \phi(N(v)) \setminus \phi(u) = [\chi(G)] \setminus \{1, \phi(u)\}$ (the first equality holds because the neighbour $u$ of $v$ was uncoloured and $\phi(u)$ doesn’t appear anywhere else on $N(v)$ since the neighbours of $v$ have different colours. The second equality holds because $\phi(N(v)) = [\chi(G)] \setminus 1$ since $v$ is full and has colour 1). Also $\psi(N(u)) = [\chi(G)] \setminus \{\phi(u)\}$ (colour 1 is present on $N(u)$ in $\psi$ because $\psi(w) = 1$ and $w \in N(u)$. All colours in $[\chi(G)] \setminus \{1, \phi(u)\}$ are present on $N(u)$ in $\psi$ because the vertices of $N(v) \setminus u$ have exactly these colours in $\phi$, $u$ is connected to all of them, and their colours don’t change going from $\phi$ to $\psi$). Thus by Lemma 3 (ii), $\psi$ is a Sudoku colouring with two uncoloured vertices, contradicting $sn(G) = |G| - 1$.

References

[1] Gee-Choon Lau, J. Maria Jeyaseeli, Wai-Chee Shiu, and S. Arumugam. Sudoku number of graphs. arXiv:2206.08106, 2022.

[2] Gary McGuire, Bastian Tugemann, and Gilles Civario. There is no 16-clue sudoku: Solving the sudoku minimum number of clues problem via hitting set enumeration. Experimental Mathematics, 23(2):190–217, 2014.