A Simple Proof of Thue’s Theorem

on Circle Packing

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Thue’s theorem states that the regular hexagonal packing is the densest circle packing in the plane. The density of this circle configuration is

$$\frac{\pi}{\sqrt{12}} \approx 0.90690.$$ 

In geometry, circle packing refers to the study of the arrangement of unit circles on the plane such that no overlapping occurs, which is the 2-dimensional analog of Kepler’s sphere packing problem proposed in 1611. A circle configuration which refers to the centers of circles is a set of points such that the distance between any two points in the set is greater than or equal to 2. Imagine filling a large container with small unit circles inside. The density of the arrangement is the proportion of the area of the container that is taken up by the circles. In order to maximize the number of circles in the container, you need to find an arrangement with the highest possible density, so that the circles are packed together as closely as possible. Hence, the density of a circle configuration is the asymptotic limit on density with the container getting bigger and bigger. In 1773, Lagrange proved that the minimal density is $\frac{\pi}{\sqrt{12}}$ by assuming that the circle configurations are lattices. In 1831, Gauss proved that the minimal density of sphere packing is $\frac{\pi}{\sqrt{18}}$ by assuming that the sphere configurations are lattices. Without the lattice assumption, the first proof of circle packing problem was made by Axel Thue. However, it is generally believed that Thue’s original proof was incomplete and that the first complete and flawless proof of this fact was produced by L. F. Toth (1940). Later, different proofs were proposed by Segre and Mahler, Davenport, and Hsiang.

A circle configuration is called saturated if it is not a proper subset of another circle configuration. Given a circle configuration $C$, any saturated circle configuration containing $C$ is called a saturation of $C$. Since the density of a circle configuration $C$ is always less than or equal to the density of any saturation of $C$. Hence, we only need to consider the saturated circle configurations instead of all circle configurations.

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In computational geometry, a point set triangulation, i.e., a triangulation $T(C)$ of a discrete set of points $C$ on the plane is a subdivision of the convex hull of the points into triangles such that any two triangles intersect in a common edge or not at all and the set of points that are vertices of the triangles coincides with $C$. The Delaunay triangulation for a set $C$ of points in the plane is a triangulation $DT(C)$ such that no point in the set $C$ is inside the circumcircle of any triangle in $DT(C)$. Delaunay invented such triangulations in 1934. The uniqueness and existence of Delaunay triangulations are both uncertain. For example, there is no Delaunay triangulation for a set of points on a straight line. The Delaunay triangulations for four points on a circle are not unique; it is obvious that there are two possible triangulations for a cocircular quadrilateral splitting into two triangles. However, there always exists a Delaunay triangulation for a saturated...
circle configuration. To find the Delaunay triangulation of a set of points in the plane can be converted to find the convex hull of a set of points in 3-dimensional Euclidean space, by giving every point \( p \) in a saturated circle configuration an extra coordinate equal to \( |p|^2 \), taking the convex hull, and mapping back to the Euclidean plane by forgetting the last coordinate. A facet of the convex hull not being a triangle implies that at least 4 of the original points lay on the same circle, which makes the triangulation not unique.

**Lemma 1** Let \( \theta \) be the largest internal angle of a triangle \( \Delta ABC \) in a Delaunay triangulation for a saturated circle configuration \( C \). Then

\[
\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}.
\]

**Proof:** The largest internal angle of a triangle is always bigger than or equal to \( \frac{\pi}{3} \). The equality only holds for regular triangles.

Suppose that \( \theta \geq \frac{2\pi}{3} \). Let say A to be the smallest internal angle. We have \( \sin A \leq \frac{1}{2} \) and \( BC \geq 2 \). Denote the circumradius of \( \Delta ABC \) by \( R \). By the sine law, we have

\[
2R = \frac{BC}{\sin A} \geq \frac{2}{\sin A} \geq 4.
\]

Then the circumcenter of \( \Delta ABC \) can be added to the circle configuration \( C \) which contradicts the saturated-ness of the circle configuration \( C \). Therefore, we obtain

\[
\theta < \frac{2\pi}{3}. \quad \text{Q.E.D.}
\]

The density of a triangle \( \Delta ABC \) in a Delaunay triangulation for a saturated circle configuration \( C \) is equal to

\[
\frac{\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C}{
\text{the area of } \Delta ABC}
\]

\[
= \frac{\pi/2}{\text{the area of } \Delta ABC}.
\]

**Lemma 2** The density of a triangle \( \Delta ABC \) in a Delaunay triangulation for a saturated circle configuration \( C \) is less than or equal to \( \pi/\sqrt{12} \). The equality holds only for the regular triangle with side-length 2.

**Proof:** Let say that \( B \) is the largest internal angle of \( \Delta ABC \). Then, by the above lemma,

\[
\text{the area of } \Delta ABC = \frac{1}{2} AB \cdot BC \cdot \sin B \geq \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}.
\]

Therefore, we have

\[
\text{the density of } \Delta ABC = \frac{\pi/2}{\text{the area of } \Delta ABC} \leq \frac{\pi}{\sqrt{12}}.
\]
It is obvious from the computation that the equality holds only when $\Delta ABC$ is a regular triangle and side-length of $\Delta ABC$ is 2. Q.E.D.

The density of the union of any finite Delaunay triangles in a saturated circle configuration is a weighted average of the densities of the Delaunay triangles. i.e.

$$\text{the density} = \frac{\sum_{\Delta_i \text{:: Delaunay triangle}} (\text{the area of } \Delta_i) \times (\text{the density of } \Delta_i)}{\sum_{\Delta_i \text{:: Delaunay triangle}} \text{the area of } \Delta_i}.$$ 

Since we have shown that the density of a Delaunay triangle is less than or equal to $\pi/\sqrt{12}$, the density of the union of any finite Delaunay triangles in a saturated circle configuration is also less than or equal to $\pi/\sqrt{12}$. Therefore, we obtain a simple proof of Thue theorem.

**Theorem 3 (Axel Thue)** The hexagonal lattice is the densest of all possible circle packings.

**References**

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