Conformastat eletrovacuum spacetimes

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Abstract. We present the first step towards the complete solution of the Einstein-Maxwell equations outside the sources of static spacetimes in which the space of orbits is conformally flat. Under the assumption of functional dependence between the two potentials that define the geometry and the electromagnetic field, the solution consists only of the well known class which depends on a function satisfying the Laplace equation in flat 3-space plus four explicit bi-parametric families of line-elements.

1. Conformastationary spacetimes
The line element of a stationary spacetime \((M, g_{\mu\nu})\) in adapted coordinates \(\{t, x^a\}\) reads \([1]\)
\[
ds^2 = -e^{2U}(dt + A_adx^a)^2 + e^{-2U}\hat{h}_{ab}dx^a dx^b , \tag{1}\]
where \(U, A_a\) and \(\hat{h}_{ab}\) do not depend on \(t\). For a static spacetime \(A_a = 0\). A conformastationary spacetime is a stationary spacetime where \((\Sigma_3, \hat{h}_{ab})\) is conformally flat, i.e. the York tensor density vanishes \([1]\)
\[
Y_a^e = \hat{\eta}^{bec} \left( 2\nabla_c \hat{R}_{ba} - \frac{1}{2} \hat{h}_{ab} \nabla_c \hat{R} \right) = 0. \tag{2}\]
Here \(\hat{R}_{ab}\) and \(\nabla\) are the Ricci tensor and covariant derivative relative to \(\hat{h}_{ab}\), and \(\hat{\eta}_{abc}\) is the volume form of \((\Sigma_3, \hat{h}_{ab})\). The York tensor density satisfies \(Y_{ae} = Y_{ea}\) and \(Y_a^a = 0\). Conformastat spacetimes are conformastationary spacetimes which are static.

We will restrict ourselves to Maxwell fields \(F_{\alpha\beta}\) for which \(\mathcal{L}_\xi F = 0\) with \(\xi \equiv \partial_t\). The Einstein-Maxwell equations outside the sources reduce to the equations (see e.g. \([1, 2]\))
\[
\hat{R}_{ab} = G_a G_b + \nabla_a G_b - (H_a \nabla_b + \nabla_a H_b), \tag{3}\]
\[
\hat{\nabla}^a H_a + \frac{1}{2} \hat{G} \cdot H - \frac{3}{2} G \cdot H = 0, \tag{4}\]
\[
\hat{\nabla}^a G_a - \overline{H} \cdot H - (G - \overline{G}) \cdot G = 0, \tag{5}\]
for \(H_a \equiv (\Re \mathcal{E} + \Phi \overline{\Phi})^{-1/2} \Phi_a\) and \(G_a \equiv 1/2(\Re \mathcal{E} + \Phi \overline{\Phi})^{-1}(\mathcal{E}_a + 2\overline{\Phi} \Phi_a)\), where \(\Phi(x^a)\) is the electromagnetic potential and \(\mathcal{E}(x^a)\) the Ernst potential. The integrability conditions for the two complex potentials are, in fact, \(dH = H \wedge \Re \mathcal{G}\) and \(dG = G \wedge \overline{\mathcal{G}} + \nabla \wedge H\). 

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The metric function $U$ and the 1-form $A_a$ are determined by
\[ e^{2U} = \Re \mathcal{E} + \Phi \mathcal{F}, \quad dA_{ab} = 2e^{-4U} \hat{\eta}_{abc} \mathcal{F}^c. \]
The electromagnetic field, $\mathcal{F} = F + i * F$, is recovered by
\[ \mathcal{F} = -e^{-U} [H \wedge \xi + i * (H \wedge \xi)], \]
where the 1-form $H_\mu$ in $(M, g_{\mu \nu})$ is given by $H_\mu = (0, H_a)$ in the above coordinates adapted to $\xi$.

Note that $\xi_\mu = -e^{2U} (1, A_a)$. The real and imaginary parts of $H_\mu$ correspond to the electric and magnetic fields with respect to the observer defined by $u \equiv e^{-U} \xi$, this is $H_\mu = E_\mu + iB_\mu = \mathcal{F}_{\mu \nu} u^\nu$.

The intrinsic definition of $G_\mu$ in $(M, g_{\mu \nu})$ is given by $G_\mu \equiv u^\nu (\nabla_\nu u_\mu + i * \nabla_\nu u_\mu)$ and its real and imaginary parts $G_\mu = a_\mu + i b_\mu$ correspond to the acceleration and twist vectors of the congruence $u$.

For the **Vacuum Case** we have $\Phi = 0$, so that $H_a = 0$ and hence the integrability condition reads simply $dG = G \wedge \bar{G}$, while (3)-(5) reduce to
\[ \bar{R} = G_a \bar{G}_b + \bar{G}_a G_b, \quad \bar{\nabla}^a G_a - (G - \bar{G}) \cdot G = 0. \]

On the other hand, for the **Static Case** we have $G - \bar{G} = 0$. Then $dG = 0$ (in fact $G_a = U_a$), $dH = H \wedge G$ and $H \wedge \bar{H} = 0$. This leads eventually to $\bar{H} = e^{-2i\theta} H$, with $\theta$ constant. We define $X_a \equiv e^{-i\theta} H_a$, so that $E_a = \cos \theta X_a$ and $B_a = \sin \theta X_a$, and consider the real potential $\Psi = e^{-i\theta} \Phi$. Therefore $X_a = e^{-U} \Psi_a$ and equations (3)-(5) reduce to
\[ \bar{R} = 2(G_a G_b - X_a X_b), \quad \bar{\nabla}^a X_a - G \cdot X = 0, \quad \bar{\nabla}^a G_a - X \cdot X = 0. \]

### 2. A common framework

We define $j$ satisfying $j^2 = 1$ and a “conjugation” $\bar{j} = -j$. Given any two real objects $f$ and $g$ we define a $j$-object $F = f + jg$ and its conjugate $\bar{F} = f - jg$, so that $f = 1/2 (F + \bar{F})$ and $g = 1/2 (F - \bar{F})$. The equation $F = 0$ holds if and only if $\bar{F} = 0$ and therefore $F = 0 \iff f = 0 \iff g$. Considering the static case, we are now ready to define
\[ \Sigma_a \equiv \frac{1}{2} (G_a + jX_a), \]
in terms of which equations (6) read
\[ \bar{R}_{ab} = 4(\Sigma_a \bar{\Sigma}_b + \bar{\Sigma}_a \Sigma_b), \quad \bar{\nabla}^a \Sigma_a - (\Sigma - \bar{\Sigma}) \cdot \Sigma = 0, \]
and the integrability conditions satisfied by $G$ and $X$ now simply reduce to $d\Sigma = \Sigma \wedge \bar{\Sigma}$. The equations for the hyperbolic complex $\Sigma_a$ in the static case and the complex $G_a$ in the vacuum case are analogous except for the number appearing in the corresponding equations for the Ricci tensor. We can thus set a common problem by denoting by $i$ any of $i$ and $j$, so that $i^2 = \pm 1$ accordingly, a general conjugation by $\tilde{\cdot}$, so that $\tilde{i} = -i$, and call $f + ig$ a composed object. Consider then a composed vector field $\mathcal{V}_a$ which satisfies the system of equations
\[ \bar{\nabla}^a \mathcal{V}_a - (\mathcal{V} - \bar{\mathcal{V}}) \cdot \mathcal{V} = 0, \]
and the metric $\hat{h}_{ab}$ satisfying
\[ \hat{R}_{ab} = N(\mathcal{V}_a \bar{\mathcal{V}}_b + \bar{\mathcal{V}}_a \mathcal{V}_b). \]
The vacuum case is recovered by taking \( N = 1 \), \( \mathcal{Y}_a = G_a \), a complex 1-form, and the conjugate being the complex conjugate. The static case corresponds to \( N = 4 \), \( \mathcal{Y}_a = \Sigma_a \), a \( j \)-1-form and the conjugate being the \( j \)-conjugation.

Conformastationarity demands the vanishing of the York tensor density of \( \tilde{h}_{ab} \). Let us introduce first the vector

\[
L \equiv \ast(\mathcal{Y} \wedge \tilde{\mathcal{Y}}),
\]

where \( \ast \) denotes the Hodge-dual in \((\Sigma^3, \tilde{h}_{ab})\), i.e. \( L_a = \mathcal{Y}^b \tilde{\mathcal{Y}}^c \tilde{h}_{bc} \). By construction we have \( \tilde{L} = -L \) and \( L \cdot \mathcal{Y} = L \cdot \tilde{\mathcal{Y}} = 0 \). Introducing (10) into (2) one obtains the real equation

\[
(\mathcal{Y}_a - \tilde{\mathcal{Y}}_a)L^e + \tilde{\eta}^{bce}(\tilde{\mathcal{Y}}_b \nabla_c \mathcal{Y}_a + \mathcal{Y}_b \nabla_c \tilde{\mathcal{Y}}_a) - \frac{1}{2} \hat{h}_{ab} \tilde{\eta}^{bce} \nabla_e (\mathcal{Y} \cdot \mathcal{Y}) = 0.
\]

Since \( \mathcal{Y}_a^* = 0 \) this equation contains at most 5 independent components. Two very different situations arise in the study of this system of equations for \( \mathcal{Y} \): the class of solutions for which \( L \neq 0 \) and those for which \( L = 0 \). Nevertheless, before entering into the study of the two classes one has to consider the case \( \mathcal{Y} \cdot \mathcal{Y} = 0 \). In the static case \( \mathcal{Y} = \Sigma \) is \( j \)-composed and \( \Sigma \cdot \Sigma = 0 \) implies, in particular, \( G \cdot G + X \cdot X = 0 \), which leaves us only with the trivial case \( G = X = 0 \). However, in the vacuum case \( \mathcal{Y} = G \) is complex and one can have, in principle, fields for which \( G \cdot G = 0 \). The study of these null fields was performed in [2], where it was proven that no null conformastationary vacuum spacetimes exist apart from the trivial case of flat spacetime. We can thus take \( \mathcal{Y} \cdot \mathcal{Y} = 0 \) without loss of generality.

3. The class \( L = 0 \)

The vanishing of \( L \) means, first, that \( \mathcal{Y} \) and \( \tilde{\mathcal{Y}} \) are parallel. Secondly, because of (9) \( \mathcal{Y} \) is the gradient of some composed potential. As a result, the real and “imaginary” parts of the potential must be functionally dependent. In the vacuum case this means that \( G \wedge \tilde{G} = 0 \), and therefore that \( E_{[a} \tilde{E}_{b]} = 0 \), so that \( E = \tilde{E} (\tilde{\tilde{E}}) \). This case was studied in [2], and the solution consists only of three explicit bi-parametric families of line-elements. In the static case \( \mathcal{Y} \wedge \tilde{\mathcal{Y}} = 0 \) stands for \( \Sigma \wedge \tilde{\Sigma} = 0 \), this is \( G \wedge X = 0 \), which is in turn equivalent to a functional relationship \( U = U(\Psi) \). The divergence equation (8) firstly fixes this relationship to be (see e.g. [1])

\[
e^{2U} = 1 - 2k\Psi + \Psi^2,
\]

with \( k = \text{const} \), which can be rewritten in parametric form as

\[
k^2 = 1 : \Psi = k - 1/V, \quad e^{2U} = V^{-2},
\]

\[
k^2 > 1 : \Psi = k - \sqrt{k^2 - 1} \coth V, \quad e^{2U} = (k^2 - 1) \sinh^{-2} V,
\]

\[
k^2 < 1 : \Psi = k - \sqrt{1 - k^2} \cot V, \quad e^{2U} = (1 - k^2) \sin^{-2} V,
\]

and secondly implies \( \tilde{\nabla}^2 V = 0 \) in all cases. The Ricci equations (10) reduce now to

\[
k^2 = 1 : \tilde{R}_{ab} = 0, \quad k^2 > 1 : \tilde{R}_{ab} = 2V_{,a}V_{,b}, \quad k^2 < 1 : \tilde{R}_{ab} = -2V_{,a}V_{,b},
\]

and the remaining equation that \( V_a \) has to satisfy is encoded in (12).

The solutions for \( k^2 = 1 \) correspond to the well known Majumdar-Papapetrou class of solutions, which depend on functions satisfying the Laplace equation in flat \( \tilde{h}_{ab} \) 3-space [1]. The important point to note for case \( k^2 > 1 \) is that in the conformastation vacuum case \( \Phi = 0 \) and \( G_a = u_{,a} \), so that the Einstein-Maxwell equations reduce to

\[
\tilde{R}_{ab} = 2u_{,a}u_{,b}, \quad \tilde{\nabla}^2 u = 0.
\]
These equations together with equation (12) for $u_a$ constitute a problem for $u$ and $\hat{h}_{ab}$ which is equivalent to that for $V$ and $\hat{h}_{ab}$ in the conformastat electrovacuum case for $k^2 > 1$. Accordingly, given a conformastat vacuum solution, a conformastat electrovacuum solution with $k^2 > 1$ is obtained by means of the substitution

$$V = u,$$ \hfill (16)

while keeping $\hat{h}_{ab}$. Now, the general solution of the conformastat vacuum problem was found in [3, 4] (also included in [2]), and consists of three explicit one-parameter families of line-elements, plus the trivial solution, the flat case. This last case has to be considered in principle, but it does not produce a different spacetime eventually. By using those three “seeds” we can construct the conformastat electrovacuum line-elements with $k^2 > 1$ as

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left\{dr^2 + e^{2V}r^2\left\{d\vartheta^2 + f^2(\vartheta)d\varphi^2\right\}\right\},$$ \hfill (17)

where $e^{2U}$ and $\Psi$ are computed by means of expression (14) from each one of the following four possibilities

$$V(r) = \frac{1}{2}\ln\left(\frac{b}{r}\right), \quad f(\vartheta) = \vartheta,$$

$$V(r) = \frac{1}{2}\ln\left(\frac{r-2b}{r}\right), \quad f(\vartheta) = \sin \vartheta,$$

$$V(r) = \frac{1}{2}\ln\left(\frac{2b-r}{r}\right), \quad f(\vartheta) = \sinh \vartheta,$$

where $b$ is a real parameter. One thus obtains three explicit bi-parametric families of line-elements for the corresponding electromagnetic potentials computed from (14). Finally, one can show that the class $k^2 < 1$ contains only one bi-parametric family, whose explicit form and study will be presented elsewhere.

4. Concluding Remarks

Conformastat electrovacuum solutions with a functional relationship $U = U(\Psi)$ consist only of the well known class of solutions which depend on a function satisfying the Laplace equation in a flat 3-space, plus four explicit bi-parametric families of line elements, three of which are presented above. At present we are studying conformastat electrovacuum solutions without a functional relationship between the potentials $U$ and $\Psi$. A preliminary result is that the spacetime $(\mathcal{M},g_{\mu\nu})$ admits a further spacelike isometry.

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