Parameterized Complexity of Partial Scheduling

Jesper Nederlof
Eindhoven University of Technology, Combinatorial Optimization, Netherlands
https://www.win.tue.nl/~jnederlo/
j.nederlof@tue.nl, j.nederlof@uu.nl

Céline M. F. Swennenhuis
Eindhoven University of Technology, Combinatorial Optimization, Netherlands
https://research.tue.nl/nl/persons/chC3%A9line-swennenhuis
c.m.f.swennenhuis@tue.nl

Abstract
We study a natural variant of scheduling that we call partial scheduling: In this variant an instance of a scheduling problem along with an integer $k$ is given and one seeks an optimal schedule where not all, but only $k$ jobs have to be processed.

We study the Fixed Parameter Tractability of partial scheduling problems parameterized by $k$ for all variants of scheduling problems that minimize the makespan and involve unit/arbitrary processing times, identical/unrelated parallel machines, release/due dates, and precedence constraints. That is, we investigate whether algorithms with runtimes of the type $O^*(f(k))$ exist, where the $O^*(\cdot)$ notation omits factors polynomial in the input size. We obtain a trichotomy by categorizing each variant to be either in $P$, $NP$-complete and Fixed Parameter Tractable by $k$, or $W[1]$-hard by $k$.

As one of our main technical contributions, we give an $O^*(8^k)$ time algorithm to solve instances of $k$-scheduling problems minimizing the makespan with unit job lengths, precedence constraints and release dates.

2012 ACM Subject Classification General and reference → General literature; General and reference

Keywords and phrases Parameterized Complexity, Fixed-Parameter Tractability, Scheduling, Precedence Constraints

Funding Jesper Nederlof: ERC project no. 617951. and NWO project no. 024.002.003
Céline M. F. Swennenhuis: NWO project no. 613.009.031b, ERC project no. 617951.
1 Introduction

Scheduling is one of the most central application domains of combinatorial optimization. In the last decades, huge combined effort of many researchers led to major progress on understanding the worst-case computational complexity of almost all natural variants of scheduling: By now, for most of these variants it is known whether they are NP-complete or not. Scheduling problems provided the context of some of the most classic approximation algorithms. For example in the standard textbook by Shmoys and Williamson on approximation algorithms \[24\] a wide variety of techniques are illustrated by applications to scheduling problems. These efforts led to a good understanding on how well most natural variants of scheduling problems can be approximated. We refer to the standard textbook on scheduling by Pinedo \[19\] for more background.

Instead of studying approximation algorithms, another natural way to deal with NP-completeness is Parameterized Complexity: Here one identifies one or more parameters of a problem instance that are typically small, and tries to reduce the exponential part of the run time to a function that only depends on this parameter. If this is the case and the parameter is denoted by $k$, we say the problem is Fixed Parameter Tractable (FPT) in $k$.

The application of parameterized complexity theory to the area of scheduling has received considerably less attention than the approximation point of view, but recently its study witnesses explosive growth. For example, many recent results and open problems can be found in a survey by Mnich and van Bevern \[15\], and a workshop on the subject was recently held \[18\].

In this paper we advance this vibrant research direction with a complete mapping of how several standard scheduling parameters influence the parameterized complexity of a natural variant we call partial scheduling.

Partial Scheduling. In many scheduling problems arising in practice, the set of jobs to be scheduled is not predetermined. We refer to this setting as partial scheduling. Partial scheduling is well-motivated from practice. We mention three example settings where it arises naturally:

First, due to uncertainties a close-horizon approach may be employed and thus only relatively few, but still as many as possible, jobs out of a big set of jobs will be scheduled in a short time-frame. Second, in freelance markets typically a large database of jobs is available and a freelancer is interested in selecting only a few of the jobs to work on. Third, the selection of the jobs to process may resemble other choices the scheduler should make, such as to outsource non-processed jobs to various external parties.

Partial scheduling under different names has been studied in the form of maximum throughput scheduling \[20\] (motivated by the first example setting above), job rejection \[21\], job selection \[7, 12, 25\] and its special case interval selection \[5\].

In this paper, we conduct a rigorous study of the parameterized complexity of partial scheduling, parameterized by the number of jobs to be scheduled. We denote this number by $k$. While several isolated result concerning the parameterized complexity of partial scheduling do exist, this parameterization has (somewhat surprisingly) not been rigorously studied yet \[1\]. We address this and study the parameterized complexity of the (arguably) most natural variants of the problem. Thus, the objective is to minimize the makespan

\[1\] We compare the previous works and other relevant studied parameterization in the end of this section.
while scheduling at least \( k \) jobs, for a given integer \( k \). We study all variants featuring any combination of the following well-studied characteristics:
- 1 machine, identical parallel machines or unrelated parallel machines,
- release/due dates, unit/arbitrary processing times, and precedence constraints.

To quickly refer to a variant of the scheduling problem, we use the standard three-field notation by Graham’s et al. \cite{Graham}. See Section 2 for an explanation of this notation. To accommodate our study of partial scheduling, we extend the \( \alpha|\beta|\gamma \) notation as follows:

\begin{definition}
Let \( k\)-sched in the \( \gamma \)-field indicate that we only schedule \( k \) out of \( n \) jobs.
\end{definition}

Unless stated otherwise, we refer to the parameterized complexity of the problem at hand with respect to \( k \). We study the parameterized complexity of all problems \( \alpha|\beta|k\)-sched, \( C_{\text{max}} \), where the options for \( \alpha \in \{1, P, R\} \) and the options for \( \beta \) are all combinations for \( r_j, prec, d_j, p_j \).

**Our Results: A Parameterized Complexity Trichotomy of Partial Scheduling**

We present a trichotomy of the complexity of all aforementioned variants of partial scheduling. Specifically, we classify all variants to be either solvable in polynomial time, to be FPT in \( k \), or to be \( \text{W}[1] \)-hard. The main results to obtain the trichotomy are depicted in Figure 1. In Table 1 in Appendix C we explicitly list the classification. We will now the results of Figure 1 in detail, starting with all results on variants with precedence constraints.

**Precedence Constraints.** A precedence constraint \( a \prec b \) enforces that job \( a \) needs to be finished before job \( b \) can start. The polynomial time algorithms behind result [A] are obtained by a straightforward greedy algorithm: For \( 1|r_j, prec, p_j = 1|k\)-sched, \( C_{\text{max}} \), build the schedule from beginning to end, and schedule an arbitrary job if any is available; otherwise wait until one becomes available.

Our main technical contribution concern result [B] and lie in the following theorem:

\begin{theorem}
\( P|r_j, prec, p_j = 1|k\)-sched, \( C_{\text{max}} \) can be solved in \( O^*(8^k) \) time. \cite{note2}
\end{theorem}

This problem is \( NP \)-complete as it is a generalization of \( P|prec, p_j = 1|C_{\text{max}} \), which was proven to be strongly \( NP \)-complete by Ullman \cite{Ullman}. Theorem 1.2 will be proved in Sections 3.1 to 3.4. The first idea behind the proof is based on a dynamic programming algorithm indexed by antichains of the partial order naturally associated with the precedence constraints. However, evaluating this dynamic program in a naïve way would lead to an \( n^{O(k)} \) time algorithm, where \( n \) denotes the number of jobs in the input.

Our key idea is to only compute a subset of the table entries of this dynamic program guided by a new parameter of an antichain called the depth. Intuitively, the depth of an antichain \( A \) indicates the number of jobs that can be scheduled after \( A \) in feasible schedule without violating the precedence constraints.

We prove Theorem 1.2 by showing we may safely restrict attention in the dynamic program to antichains of depth at most \( k \), and by bounding the number of antichains of depth at most \( k \) indirectly by bounding the number of maximal antichains of depth at most \( k \). We believe our methodology should have more applications for scheduling problems with precedence constraints.

\textsuperscript{2} Here the \( O^*(\cdot) \) notation omits factors polynomial in the input size.
Surprisingly, the positive result of Theorem 1.2 is in stark contrast with the seemingly symmetric case where only deadlines are present: Our next result, indicated as [C] in Figure 1 shows it is much harder:

\textbf{Theorem 1.3.} $P[d_j, prec, p_j = 1]k$-sched, $C_{max}$ is $W[1]$-hard, and cannot be solved in $n^{o(k/\log k)}$ assuming the Exponential Time Hypothesis (ETH).

Theorem 1.3 is a consequence of a reduction outlined in Section 4. Note the $W[1]$-hardness follows from a natural reduction from the $k$-Clique problem (presented originally by Fellows [8]), but this reduction increases the parameter $k$ to $\Omega(k^2)$ and would only exclude $n^{o(\sqrt{k})}$ time algorithms assuming the ETH.

To obtain the tighter bound from Theorem 1.3, we instead provide a non-trivial reduction from the 3-coloring problem based on a new selection gadget. Result [D] follows from a more straightforward reduction from clique similar to the one by Fellows [8]. We refer to Theorem 4.3 for its proof.

\textbf{No Precedence Constraints.} The second half of the trichotomy concerns scheduling problems without precedence constraints, and is easier obtained than the first half. Result [E]
is established by a simple greedy algorithm that always schedules an available job with the earliest deadline. Result [F] is a consequence of Moore’s algorithm [17] that solves the problem \(1\| \sum_j U_j\) in \(O(n \log n)\) time. Notice that this also solves the problem \(1|r_j|k\text{-sched}, C_{\text{max}}\), by reversing the schedule and viewing the release dates as the deadlines. Result [G] is a consequence of color coding as outlined in Appendix [A]. All variants denoted with [G] are NP-complete because \(1|r_j|\sum_j U_j\) is NP-complete [16] and \(P||C_{\text{max}}\) (a special generalization of 3-partition) is NP-complete [10].

**Related Work**

The interest in parameterized complexity of scheduling problems recently witnessed an explosive growth, resulting in e.g. a workshop [14] and a survey by Mnich and van Bevern [15] with a wide variety of open problems.

The parameterized complexity of partial scheduling parameterized by the number of jobs, or the equivalently, the number of jobs ‘on time’ was studied before: Fellows et al. [8] studied a problem called \(k\text{-TASKS ON TIME}\) that is equivalent to \(1|d_j, \text{prec}, p_j = 1|k\text{-sched}, C_{\text{max}}\) and showed that it is W[1]-hard when parameterized by \(k\) and the width of the partial ordered set induced by the precedence constraints. Van Bevern et al. [23] showed that the Job Interval Selection problem, where each job is given a set of possible intervals to be processed on, is FPT in the parameter of jobs that are selected. Bessy et al. [2] consider partial scheduling with a restriction on the jobs called ‘Coupled-Task’, and also remarked the current parameterization is relatively understudied.

Our parameter should be compared to the number of jobs that are *not scheduled*, that also has been studied in several previous works [4, 8, 16]. For example, Mnich and Wiese [16] studied the parameterized complexity scheduling problems with respect to the number of rejected jobs in combination with others variables as parameter. If \(n\) denotes the number of given jobs, this parameter equals \(n - k\). The two parameters are somewhat incomparable in terms of applications: In some settings only few jobs out of many alternatives need to be scheduled, but in other settings rejecting a job is very costly and thus will happen rarely. However, a strong advantage of using \(k\) as parameter is in terms of its computational complexity: If the version of the problem with all jobs mandatory is NP-complete it is trivially NP-complete for \(n - k = 0\), but it may still be FPT in \(k\).

**2 Preliminaries: The three-field notation by Graham et al. [11]**

Throughout this paper we denote scheduling problems using the three-field classification by Graham et al. [11]. Problems are classified by parameters \(\alpha|\beta|\gamma\). The \(\alpha\) describes the machine environment. This paper uses \(\alpha \in \{1, P, R\}\), indicating whether there are one (1), identical (P) or unrelated (R) parallel machines available. Here identical refers to the fact that every job takes a fixed amount of time process independent of the machine, and unrelated means a job could take different time to process per machine. The \(\beta\) field describes the job characteristics, which in this paper can be a combination of the following values: \(\text{prec}\) (precedence constraints), \(r_j\) (release dates), \(d_j\) (deadlines) and \(p_j = 1\) (all processing times are 1). We assume without loss of generality that all release dates and deadlines are integers.

The \(\gamma\) field concerns the optimality criteria. Given a schedule one can compute \(C_j\), the completion time of job \(j\), and \(U_j\), the unit penalty which is 1 if \(C_j > d_j\), and 0 if \(C_j \leq d_j\).

---

3 Our results [C] and [D] build on and improve this result.
In this paper we use the following optimality criteria:

- \( C_{\text{max}} \): minimize the makespan (i.e. the maximum completion time \( C_j \) of any job),
- \( \sum_j U_j \): minimize the number of jobs that finish after their deadline,
- \( k\text{-sched}, C_{\text{max}} \): minimize the makespan of a schedule that allocated at least \( k \) jobs.

A schedule allocates a job if it is processed in the schedule. A schedule is said to be feasible if no constraints (such as deadlines, release dates, precedence constraints) are violated.

3 Precedence Constraints, Release Dates and Unit Processing Times.

In this section we prove that partial scheduling with release dates and unit processing times parameterized by the number \( k \) of scheduled jobs is fixed-parameter tractable (Theorem 1.2). To do so, we present a dynamic programming algorithm based on table entries indexed by antichains in the precedence graph \( G \) describing the precedence relations. Such an antichain describes the maximal jobs already scheduled in a partial schedule. Our key idea is that, to find an optimal solution, it is sufficient to restrict our attention to a subset of all antichains. This subset will be defined in terms of the depth of an antichain.

Notice that the decision variant of this problem asks whether there exists a feasible schedule with makespan at most \( C_{\text{max}} \), for some fixed universal deadline \( C_{\text{max}} \).

This section is organized as follows: In Subsection 3.1 we introduce some notation, in Subsection 3.2 we state the algorithm, and in Subsection 3.3 we analyze its running time. Afterwards we prove the correctness of the algorithm in Section 3.4.

3.1 Notation on Partial Ordered Sets

Any precedence graph \( G \) is a directed acyclic graph and therefore induces a partial order \( \prec \) on \( V(G) \). Indeed, if there is a path from \( x \) to \( y \), we let \( x \preceq y \).

An antichain is a set \( A \subseteq V(G) \) of elements that are pairwise incomparable. We say \( A \) is maximal if there is no antichain \( A' \) with \( A \subseteq A' \). The set of predecessors of an antichain \( A \) is defined as \( \text{pred}(A) = \{ x \in V(G) : \exists a \in A : x \preceq a \} \). The set of comparables of an antichain \( A \) is defined as \( \text{comp}(A) = \{ x \in V(G) : \exists a \in A : x \preceq a \text{ or } x \succeq a \} \). Note that \( \text{comp}(A) = V(G) \) if and only if \( A \) is maximal.

An element \( x \in V(G) \) is a minimal element if \( x \preceq y \) for all \( y \in \text{comp}(\{x\}) \). An element \( x \in V(G) \) is a maximal element if \( x \succeq y \) for all \( y \in \text{comp}(\{x\}) \). Furthermore \( \text{min}(G) = \{ x \mid x \text{ is minimal element in } G \} \) and \( \text{max}(G) = \{ x \mid x \text{ is maximal element in } G \} \).

Notice that \( \text{max}(G) \) is exactly the antichain \( A \) such that \( \text{pred}(A) = V(G) \). We denote the subgraph of \( G \) induced by \( S \) with \( G[S] \). Without loss of generality we assume that \( r_j < r_{j'} \) if \( j \prec j' \) since job \( j' \) will be scheduled later than \( r_j \) in any schedule. To handle release dates, we use the following definition:

Definition 3.1. Let \( G \) be the precedence graph. Then \( G^t \) is the precedence graph restricted to all jobs that can be scheduled on or before time \( t \), i.e. all jobs whose release date \( r_j \) is at most \( t \).

In general, we assume that \( G = G^{C_{\text{max}}} \), since any jobs that have release date greater than \( C_{\text{max}} \) can be ignored.
3.2 Dynamic Program

We now introduce our dynamic programming algorithm for $P|\text{prec}, p_j = 1|k$-sched, $C_{\text{max}}$. Let $m$ be the number of machines available. We start with defining the table entries. For a given antichain $A \subseteq V(G)$ and integer $t$ we define

$$S(A, t) = \begin{cases} 1, & \text{if there exists a feasible schedule of makespan } t \text{ that allocates } \text{pred}(A), \\ 0, & \text{otherwise.} \end{cases}$$

Computing the values of $S(A, t)$ can be done by trying all combinations of scheduling at most $m$ jobs of $A$ at time $t$ and then checking whether all remaining jobs of $\text{pred}(A)$ can be scheduled in makespan $t - 1$. To do so, we also verify that all the jobs in $A$ actually have a release date at or before $t$. Formally, we have the following recurrence for $S(A, t)$:

$\blacktriangleright$ Lemma 3.2.

$$S(A, t) = (A \subseteq V(G^t)) \land \bigvee_{X \subseteq A : |X| \leq m} S(A', t - 1) : A' = \max(\text{pred}(A) \setminus X).$$

**Proof.** First note that when $A \not\subseteq V(G^t)$, then there is a job $j \in A$ with $r_j > t$. So then definitely $S(A, t) = 0$.

For any $X \subseteq A$, $X$ is a set of maximal elements with respect to $G[\text{pred}(A)]$, element-wise incomparable jobs, since $A$ is an antichain. So, we can schedule all jobs from $X$ at time $t$ without violating any precedence constraints. Define $A' = \max(\text{pred}(A) \setminus X)$ as the unique antichain such that $\text{pred}(A) \setminus X = \text{pred}(A')$. Now if $S(A', t - 1) = 1$ and $|X| \leq m$ we can extend the schedule of $S(A', t - 1)$ by scheduling all $X$ at time $t$. This way we get a feasible schedule allocating all jobs of $\text{pred}(A)$ before or at $t$. So if we find such an $X$ with $|X| \leq m$ and $S(A', t - 1) = 1$, we must have $S(A, t) = 1$.

For the other direction, if for all $X \subseteq A$ with $|X| \leq m$, $S(A', t - 1) = 0$, then no matter which set $X \subseteq A$ we try to schedule at time $t$, the remaining jobs cannot be scheduled before $t$. Note that only jobs from $A$ can be scheduled at time $t$, since those are the maximal jobs. Hence, there is no feasible schedule and $S(A, t) = 0$. $\blacktriangleleft$

The above recurrence cannot be directly evaluated since the number of different antichains of a graph can be big: there can be as many as $n^k$ different antichains with $|\text{pred}(A)| \leq k$, for example in the extreme case of an independent set. Even when we restrict our precedence graph to have out degree $k$, there could be $k^k$ different antichains, for example in $k$-ary trees. To circumvent this issue, we restrict our dynamic program only to a specific subset of antichains. To do this, we use the notion of the depth of an antichain.

$\blacktriangleright$ Definition 3.3. Let $A$ be an antichain, the depth (with respect to $t$) of $A$ is

$$d^t(A) = |\text{pred}(A)| + \min(G^t - \text{comp}(A)|.$$

We also denote $d(A) = d^{C_{\text{max}}}(A)$.

The intuition behind this definition is that it quantifies the number of jobs that can be scheduled before (and including) $A$ without violating precedence constraints. See Figure 3 for an example of an antichain and its depth. We restrict the dynamic program to only compute $S(A, t)$ for $A$ satisfying $d^t(A) \leq k$. This ensures that we do not go ‘too deep’ into the precedence graph unnecessarily at the cost of a slow runtime.
= job in antichain

= job in pred(A)

= job in min(G − comp(A))

= job in G − comp(A)

= job in comp(A)

\(d(A) = |\text{pred}(A)| + |\text{min}(G - \text{comp}(A))| = 2 + 2\)

**Figure 2** Example of an antichain and its depth in a perfect 3-ary tree. We see that |\text{pred}(A)| = 2, but \(d(A) = 4\). If \(k = 2\), the dynamic program will not compute \(S(A, t)\) since \(d(A) > k\). The only antichains with depth \(\leq 2\) are the empty set and the root node \(r\) on its own as a set. Indeed \(d(\emptyset) = d(\{r\}) = 1\). Note that for instances with \(k = 2\), a feasible schedule may exist. If so, we will find that \(R(\{r\}, 1) = 1\), which will be defined later. In this way, we can still find the antichain \(A\) as a solution.

Because of this restriction in the depth, it could happen that we check no antichains with \(k\) or more predecessors, while there are corresponding feasible schedules. It is therefore possible that for some antichains \(A\) with \(d'(A) > k\), there is a feasible schedule for all \(t \geq k\) jobs in \(\text{pred}(A)\) before time \(C_{\text{max}}\), but the value \(S(A, C_{\text{max}})\) will not be computed. To make sure we still find an optimal schedule, we also compute the following condition \(R(A, t)\) for all \(t \leq C_{\text{max}}\) and antichains \(A\) with \(d'(A) \leq k\):

\[
R(A, t) = \begin{cases} 
1, & \text{if there exists a feasible schedule with makespan at most } C_{\text{max}} \text{ that} \\
& \text{allocates } \text{pred}(A) \text{ on or before } t \text{ and allocates jobs from } \text{min}(G - \text{pred}(A)) \\
& \text{after } t, \text{ with a total of } k \text{ jobs allocated,} \\
0, & \text{otherwise.}
\end{cases}
\]

By definition of \(R(A, t)\), if \(R(A, t) = 1\) for any \(A\) and \(t \leq C_{\text{max}}\), then we find a feasible schedule that allocates \(k\) jobs on time. The reverse direction is more difficult and postponed to Section 3.3. We now proceed by showing how to compute \(R(A, t)\):

**Lemma 3.4.** There is an \(O(n^2)\) time algorithm \(\text{fill}(A, t)\) that, given an antichain \(A\), integer \(t\), and value \(S(A, t)\), computes \(R(A, t)\).

**Proof.** The algorithm \(\text{fill}(A, t)\) checks if \(S(A, t) = 1\) and if so, greedily schedules jobs from \(\text{min}(G - \text{pred}(A))\) after \(t\) in order of smallest release date. If \(k - |\text{pred}(A)|\) jobs can be scheduled before \(C_{\text{max}}\), it returns ‘true’ \((R(A, t) = 1)\). Otherwise, it returns ‘false’ \((R(A, t) = 0)\).

First we show that if \(\text{fill}(A, t)\) returns ‘true’, it follows that \(R(A, t) = 1\). Since \(S(A, t) = 1\), all jobs from \(\text{pred}(A)\) can be finished at time \(t\). Take that feasible schedule and allocate \(k - |\text{pred}(A)|\) jobs from \(\text{min}(G - \text{pred}(A))\) between \(t\) and \(C_{\text{max}}\). This is possible because \(\text{fill}(A, t)\) is true. All predecessors of jobs in \(\text{min}(G - \text{pred}(A))\) are in \(\text{pred}(A)\) and therefore
allocated before \( t \). Hence, no precedence constraints are violated and we find a feasible schedule with the requirements, i.e. \( R(A, t) = 1 \).

For the other direction, assume that \( R(A, t) = 1 \), i.e. we find a feasible schedule \( \sigma \) where exactly the jobs from \( \text{pred}(A) \) are allocated on or before \( t \) and only jobs from \( \text{min}(G - \text{pred}(A)) \) are processed after \( t \). Thus \( S(A, t) = 1 \). Define \( M \) as the set of jobs processed after \( t \) in \( \sigma \). If \( M \) equals the set of jobs with the smallest release dates of \( \text{min}(G - \text{pred}(A)) \), we can also process the jobs of \( M \) in order of increasing release dates. Then \( \text{fill}(A, t) \) will be ‘true’, since \( M \) has size at least \( k - |\text{pred}(A)| \). However, if \( M \) is not that set, we can replace a job which does not have one of the smallest \( k - |\text{pred}(A)| \) release dates, by one which has and was not in \( M \) yet. This new set can then still be processed between \( t + 1 \) and \( C_{\text{max}} \) because smaller release dates impose weaker constraints. We keep replacing until we end up with \( M \) being exactly the set of jobs with smallest release dates, which is then proved to be schedulable between \( t \) and \( C_{\text{max}} \). Hence, \( \text{fill}(A, t) \) will return ‘true’.

This gives us the algorithm as described in Algorithm 1. It remains to bound its run time and argue its correctness.

**Algorithm 1:** Dynamic Program Algorithm for solving \( P|\text{pred}, p_j = 1|k\text{-sched}, C_{\text{max}} \)

```
foreach \( t = 1, \ldots, C_{\text{max}} \) do
  Enumerate all antichains \( A \) in \( G^t \) with \( d^t(A) \leq k \) using Corollary 3.3
  foreach antichain \( A \) in \( G^t \) with \( d^t(A) \leq k \) do
    Compute \( S(A, t) \) using Lemma 3.2
    if \( \text{fill}(S(A, t), A, t) \) then
      return TRUE
    end
  end
end return FALSE
```

### 3.3 Runtime

To analyze the runtime of the dynamic program, we need the number of different antichains that it checks. Recall that we only check antichains \( A \) with \( d^t(A) \leq k \) for each time \( t \leq C_{\text{max}} \). We will first analyze the number of antichains \( A \) with \( d(A) \leq k \) in any graph and use this to upper bound the number of antichains that are checked at time \( t \).

To analyze the number of antichains \( A \) with \( d(A) \leq k \), we give an upper bound on this number via an upper bound on the number of maximal antichains. Recall from Subsection 3.1 that for a maximal antichain \( A \) we have \( \text{comp}(A) = V(G) \), and therefore \( d(A) = |\text{pred}(A)| \). The following lemma connects the number of antichains and maximal antichains of bounded depth:

**Lemma 3.5.** For any antichain \( A \), there exists a maximal antichain \( A_{\text{max}} \) such that \( A \subseteq A_{\text{max}} \) and \( d(A) = d(A_{\text{max}}) \).

**Proof.** Let \( A_{\text{max}} = A \cup \text{min}(G - \text{comp}(A)) \). By definition, all elements in \( \text{min}(G - \text{comp}(A)) \) are incomparable to each other and incomparable to any element of \( A \). Hence \( A_{\text{max}} \) is an antichain. Since \( \text{comp}(A_{\text{max}}) = V(G) \), \( A_{\text{max}} \) is a maximal antichain. Moreover, \( d(A) = |\text{pred}(A)| + |\text{min}(G - \text{comp}(A))| = |\text{pred}(A_{\text{max}})| = d(A_{\text{max}}) \), since the elements in \( \text{min}(G - \text{comp}(A)) \) have smaller release dates than any other job in \( A_{\text{max}} \).
Corollary 3.6.

◭

Let \( \not \) (but and they can be enumerated in antichains of bounded depth.

\[ \{ A : A \text{ antichain, } d(A) \leq k \} \leq 2^k \{ A : A \text{ maximal antichain, } d(A) \leq k \}. \]

This corollary allows us to restrict attention to only upper bounding the number of \textit{maximal} antichains of bounded depth.

\[ |A| \leq k \]

Lemma 3.7. There are at most \( 2^k \) maximal antichains \( A \) with \( d(A) \leq k \) in any graph \( G \), and they can be enumerated in \( O^*(2^k) \) time.

\[ \text{Proof.} \] Let \( \mathcal{A}_k(G) \) be the set of maximal antichains in \( G \) with depth at most \( k \). We prove that \( |\mathcal{A}_k(G)| \leq 2^k \) for any graph \( G \) by induction. Clearly \( |\mathcal{A}_0(G)| \leq 1 \) for any graph \( G \), since the only antichain with \( d(A) \leq 0 \) is \( A = \emptyset \) if \( G = \emptyset \).

Let \( k > 0 \) and assume \( |\mathcal{A}_j(G)| \leq 2^j \) for \( j < k \) for any graph \( G \). If we have a precedence graph \( G \) with minimal elements \( s_1, ..., s_l \), we partition \( \mathcal{A}_k(G) \) into \( l + 1 \) different sets \( B_1, B_2, ..., B_{l+1} \). The set \( B_i \) is defined as the set of maximal antichains \( A \) of depth at most \( k \) in which \( s_1, ..., s_{i-1} \subseteq A \), but \( s_i \notin A \). If \( s_i \notin A \), then \( s_i \in \text{pred}(A) \) since \( A \) is maximal, so any such maximal antichain has a successor of \( s_i \) in \( A \). If we define \( S_j \) as the set of all successors of \( s_j \) (including \( s_j \)), we see that \( B_i = \mathcal{A}_{k-i} \left( G - \left( \bigcup_{j=1}^{i-1} S_j \cup \{ s_i \} \right) \right) \). Indeed, if \( A \in B_i \), then \( \{s_1, ..., s_{i-1}\} \subseteq A \). Hence we can remove those elements and its successors from the graph, as they are comparable to any such antichain. Moreover, we can also remove \( s_i \) (but \textit{not} its successors) from the graph, since it is in \( \text{pred}(A) \). Thus \( B_i \) is then exactly the set of maximal antichains with depth \( i \) less in the remaining graph. We get the following recurrence relation:

\[ |\mathcal{A}_k(G)| = \sum_{i=1}^{l} |\mathcal{A}_{k-i} \left( G - \left( \bigcup_{j=1}^{i-1} S_j \cup \{ s_i \} \right) \right)| + 1, \]  

(1)

since \( |B_{i+1}| \), the number of antichains satisfying \( \{ s_1, ..., s_l \} \subseteq A \), is exactly one. Notice that we may assume that \( l \leq k \), because otherwise the depth of the antichain will be greater than \( k \). Then if we use the induction hypothesis that \( |\mathcal{A}_i(G)| \leq 2^i \) for \( j < k \) for any graph \( G \), we see by (1) that:

\[ |\mathcal{A}_k(G)| = \sum_{i=1}^{l} |\mathcal{A}_{k-i} \left( G - \left( \bigcup_{j=1}^{i-1} S_j \cup \{ s_i \} \right) \right)| + 1, \]

\[ \leq 2^k \left( \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^k} \right) \]

\[ = 2^k. \]

The Lemma follows since the above procedure can easily be modified in an algorithm to enumerate the antichains.

◭

Going back to (non-maximal) antichains, we find an upper bound on the number of antichains.
Corollary 3.8. There are at most $4^k$ antichains $A$ with $d(A) \leq k$ in any graph $G$, and they can be enumerated within $O^*(4^k)$ time.

We now restrict the number of antichains $A$ in $G'$ with $d'(A) \leq k$. Take $G'$ to be the graph in Corollary 3.8 and notice that $d'(A) = d(A)$ for any antichains $A$ in $G'$. By Corollary 3.8 we obtain:

Corollary 3.9. For any $t$, there are at most $4^k$ antichains $A$ with $d'(A) \leq k$ in any graph $G$, and they can be enumerated within $O^*(4^k)$ time.

Notice that to compute $S(A, t)$, we look at a maximum of $\binom{k}{t}$ different sets of $X$. During this computation, $R(A, t)$ is directly computed in polynomial time. For each time $t \in \{1, ..., C_{\text{max}}\}$, there are at most $4^k$ different antichains $A$ that we compute $S(A, t)$ and $R(A, t)$ for, and there are at most $C_{\text{max}} \leq k$ different values of $t$. Hence, Algorithm 1 runs in time $O^*(8^k)$.

3.4 Correctness of algorithm

We show that the algorithm described in Algorithm 1 indeed returns the correct answer, i.e. finds $R(A, t) = 1$ for some values of $A$ and $t$ if and only if the instance is a yes-instance. To do this, we need one more definition.

Definition 3.10. Let $\sigma$ be a feasible schedule. Then $A(\sigma)$ is the antichain such that $\text{pred}(A(\sigma))$ is exactly the set of jobs that was scheduled in $\sigma$.

Equivalently, if $X$ is the set of jobs allocated by $\sigma$, then $A(\sigma) = \max(G[X])$.

Lemma 3.11. A feasible schedule for $k$ jobs with makespan at most $C_{\text{max}}$ exists if and only if $R(A, t) = 1$ for some $t \leq C_{\text{max}}$ and antichain $A$ with $d'(A) \leq k$.

Proof. Clearly, if $R(A, t) = 1$ for some $t \leq C_{\text{max}}$ and antichain $A$ with $d'(A) \leq k$, we have a feasible schedule with $k$ jobs by definition of $R(A, t)$. Hence, it remains to prove that if a feasible schedule for $k$ jobs exists, then $R(B, t) = 1$ for some $t \leq C_{\text{max}}$ and antichain $B$ with $d'(B) \leq k$. Let $\Sigma^* = \{\sigma|\sigma$ is a feasible schedule that allocates $k$ jobs and has a makespan of at most $C_{\text{max}}\}$, so $\Sigma^*$ is the set of all possible solutions. Define

$$\sigma^* = \arg\min_{\sigma} \{d(A(\sigma))|\sigma \in \Sigma^*\},$$

i.e. $\sigma^*$ is a schedule for which $A(\sigma^*)$ has minimal depth (with respect to $C_{\text{max}}$). We now define $t$ and $B$ such that $R(B, t) = 1$.

- Let $t = \max\{t: \text{job not in } G[\text{pred}(A(\sigma^*))] \text{ was scheduled at time } t\}$, so from $t + 1$ and on, only maximal jobs (with respect to $G[\text{pred}(A(\sigma^*))]$) are scheduled.
- Let $M = \{x: \text{job } x \text{ was scheduled at } t + 1 \text{ or later in } \sigma^* \}$.
- Let $B = \max(\text{pred}(A(\sigma^*) \setminus M))$, so $\text{pred}(B)$ is exactly the set of jobs scheduled on or before time $t$ in $\sigma^*$.

See Figure 3a for an illustration of these concepts. There are two cases to distinguish:
\(d^4(B) \leq k\). In this case we prove that \(R(B, t) = 1\). The feasible schedule we are looking for in the definition of \(R(B, t)\) is exactly \(\sigma^*\). Indeed, all jobs from \(\text{pred}(B)\) were finished at time \(t\). Furthermore, all jobs in \(M\) are maximal, so all their predecessors are in \(\text{pred}(B)\). Hence, \(M \subseteq \min(G - \text{pred}(B))\). So, by definition \(R(B, t) = 1\).

\(d^4(B) > k\). In this case we prove that there is a schedule \(\sigma'\) such that \(d(A(\sigma')) < d(A(\sigma^*))\), i.e. we find a contradiction to that fact that \(d(A(\sigma^*))\) was minimal. This \(\sigma'\) can be found as follows: take schedule \(\sigma^*\) only up until time \(t\). Let \(C\) be a subset of \(\min(G - \text{comp}(B))\) such that \(|C| = k - |\text{pred}(B)|\). This \(C\) can be found since \(d^4(B) \geq k\). Allocate the jobs in \(C\) after time \(t\) in \(\sigma'\). These can all be processed without precedence constraint or release date violations, since their predecessors were already scheduled and \(C \subseteq G^*\). So, we find a feasible schedule that allocates \(k\) jobs, called \(\sigma'\). The choice of \(\sigma'\) is depicted in Figure 3b. Note that \(C \subseteq \min(G - \text{comp}(B))\) and not all jobs of \(\min(G - \text{comp}(B))\) are necessarily allocated in \(\sigma'\).

It remains to prove that \(d(A(\sigma')) < d(A(\sigma^*))\). Define \(D(A) = \text{pred}(A) \cup \min(G - \text{comp}(A))\) for any antichain \(A\). So \(D(A)\) is the set of jobs that contribute to \(d(A)\) and so \(|D(A)| = d(A)\). We will prove that \(D(B) = D(A(\sigma')) \subseteq D(A(\sigma^*))\). This will be done in two steps, first we show that \(D(B) = D(A(\sigma')) \subseteq D(A(\sigma^*))\). In the last step we prove \(D(B) \neq D(A(\sigma^*))\), which gives us \(d(A(\sigma')) < d(A(\sigma^*))\).

Notice that \(C \subseteq D(B)\) since \(C \subseteq \min(G - \text{comp}(B))\), hence \(D(B) = D(B \cup C)\). Since \(A(\sigma') = B \cup C\) it follows that \(D(A(\sigma')) = D(B)\). Next we prove that \(D(B) \subseteq D(A(\sigma^*))\). Clearly, if \(x \in \text{pred}(B)\) then \(x \in \text{pred}(A(\sigma^*))\). It remains to show that \(x \in \min(G - \text{comp}(B))\) implies that \(x \in D(A(\sigma^*))\). If \(x \in \min(G - \text{comp}(B))\), then either \(x \in M\) or \(x \notin M\). If \(x \in M\), then \(x \in A(\sigma^*)\) so \(x \in \text{pred}(A(\sigma^*))\). If \(x \notin M\) then \(x \notin \text{comp}(B \cup M)\) since \(x\) was a minimal element in \(\min(G - \text{comp}(B))\). Since \(A(\sigma^*) \subseteq B \cup M\), and thus \(\text{comp}(A(\sigma^*)) \subseteq \text{comp}(B \cup M)\), we observe that \(x \in \min(G - \text{comp}(A(\sigma^*)))\). We then conclude that \(D(B) \subseteq D(A(\sigma^*))\).

We are left to show that \(D(B) \neq D(A(\sigma^*))\). Remember that \(t\) was chosen such that there is a job allocated at time \(t\) that was not in \(\max(G[\text{pred}(A(\sigma^*))]\)). In other words, there was a job \(x \in B\) in \(A^*\) at time \(t\) with \(y \in M\) such that \(y \succ x\). Note that \(y \notin D(B)\), since \(y \in M\), so \(y\) is not in \(\text{pred}(B)\) and \(y\) is clearly comparable to \(x\). However, \(y \in D(A(\sigma^*))\), so we find that \(d(A(\sigma')) = d(B) < d(A(\sigma^*))\). Hence, we found a schedule with smaller \(d(A(\sigma'))\), which leads to a contradiction.

4 k-scheduling on one machine with precedence constraints

In this section we show that Algorithm [1] cannot be even slightly generalized further: if we allow job-dependent deadlines or non-unit processing times, the problem becomes \(W[1]\)-hard parameterized by \(k\). Even stronger, in the first case we show that the trivial \(n^{O(k)}\) time algorithm cannot significantly improved assuming a standard hypothesis.

Job-dependent deadlines

The fact that combining precedence constraints with job-dependent deadlines makes the problem \(W[1]\)-hard, is a direct consequence from the fact that \(1[p\text{rec}, p_1 = 1] \sum_j U_j\) is \(W[1]\)-hard, parameterized by \(n - \sum_j U_j = k\) where \(n\) is the number of jobs [8]. It is important to notice that the notation of these problems implies that each job can have its own deadline. Hence, we conclude from this that \(1[d, p\text{rec}, p_1 = 1] k\)-sched, \(C_{\text{max}}\) is \(W[1]\)-hard parameterized by \(k\). This is a reduction from \(k\)-clique and therefore we get a lower bound on algorithms.
Figure 3 Visualization of the definitions of $M$ and $B$ and the schedules $\sigma^*$ in the proof of Lemma 3.11 is shown in Figure 3a. Figure 3b depicts the schedule $\sigma'$ as chosen in the subcase $d(B) > k$. The grey boxes indicate which jobs are allocated in the schedules. We will prove that $|D(A(\sigma^*))| < |D(A\sigma^*)|$.

for the problem of $n^{\Omega(\sqrt{k})}$. Based on the Exponential Time Hypothesis, We now sharpen this lower bound with a reduction from 3-coloring:

**Theorem 4.1.** $1|d_j, prec, p_j = 1|k$-sched, $C_{\text{max}}$ is W[1]-hard parameterized by $k$. Furthermore, there is no algorithm solving $1|d_j, prec, p_j = 1|k$-sched, $C_{\text{max}}$ in $2^{o(n)}$ where $n$ is the number of jobs, assuming ETH.

The proof is deferred to Appendix B. A crucial tool in the proof of Theorem 4.1 is a selection gadget that enforces that exactly one job out of a given set of jobs will be allocated in any optimal schedule. Given this gadget, the reduction follows easily by letting the selection correspond to the choices of colorings of vertices.

Note that this bound significantly improves the old lower bound of $2^{\Omega(\sqrt{n})}$ known from the $k$-clique reduction:

**Corollary 4.2.** Assuming ETH, there is no algorithm solving $1|d_j, prec, p_j = 1|k$-sched, $C_{\text{max}}$ in $n^{o(k/\log(k))}$ where $n$ is the number of jobs.

**Proof.** If an algorithm solving it in $n^{o(k/\log(k))}$ time exists, then also a $2^{o(n)}$ algorithm would exist since $k \leq n$. □

**Non-unit processing times**

We will prove that having non-unit processing times combined with precedence constraint makes the problem W[1]-hard parameterized by $k$, even on one machine. This reduction was inspired by the reduction from $k$-clique to $k$-TASKS ON TIME by Fellows [8].

**Theorem 4.3.** $1|prec|k$-sched, $C_{\text{max}}$ is W[1]-hard, parameterized by $k$.

The proof is deferred to Appendix B. So in the case of precedence constraints, even on one machine, if job-dependent deadlines and/or non-unit processing times are present, the $k$-scheduling problems become W[1]-hard parameterized by $k$. 
5 Concluding Remarks

We gave a trichotomy of the parameterized complexity of partial scheduling problems parameterized by the number of jobs to be scheduled.

Our main technical contribution is an $O^*(8^k)$ time algorithm for $P|\text{r}_j,\text{prec},p_j=1|k$-sched, $C_{\text{max}}$, and that the problem is W[1]-hard already when the unit processing time or uniform deadlines are slightly generalized.

We did not attempt to optimize the runtime of our FPT algorithms. An interesting question is whether there exists a $O^*(2^k)$ time (randomized) algorithm for either $R|r_j,d_j,k$-sched|$C_{\text{max}}$ or $P|r_j,\text{prec},p_j=1|k$-sched,$C_{\text{max}}$. For the first problem, we believe representative sets as applied for the weighted $k$-path problems [9, 26] could be used to solve the problem in $O^*(c^k)$ for some $c$ satisfying $2 < c < 2e$. But in Appendix A we opted for a cleaner version based on color-coding. For the second open question, note the number of antichains of depth $k$ can be more than $c^k$ for some $c > 2$, if the precedence constraint graph is i.e. a set of directed paths.

References

1. Noga Alon, Raphael Yuster, and Uri Zwick. Color-coding. J. ACM, 42(4):844–856, 1995.
2. Stéphane Bessy and Rodolphe Giroudeau. Parameterized complexity of a coupled-task scheduling problem. J. Scheduling, 22(3):305–313, 2019.
3. Andreas Björklund, Thore Husfeldt, Petteri Kaski, and Mikko Koivisto. Fourier meets möbius: fast subset convolution. In Proceedings of the thirty-ninth annual ACM symposium on Theory of computing, pages 67–74. ACM, 2007.
4. Hans L. Bodlaender and Michael R. Fellows. W[2]-hardness of precedence constrained k-processor scheduling. Oper. Res. Lett., 18(2):93–97, 1995.
5. Julia Chuzhoy, Rafail Ostrovsky, and Yuval Rabani. Approximation algorithms for the job interval selection problem and related scheduling problems. Math. Oper. Res., 31(4):730–738, 2006.
6. Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. Parameterized algorithms, volume 3, pages 471–473. Springer, 2015.
7. Joonyup Eun, Chang Sup Sung, and Eun-Seok Kim. Maximizing total job value on a single machine with job selection. Journal of the Operational Research Society, 68(9):998–1005, 2017.
8. Michael R. Fellows and Catherine McCartin. On the parametric complexity of schedules to minimize tardy tasks. Theoretical computer science, 298(2):317–324, 2003.
9. Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, and Saket Saurabh. Efficient computation of representative families with applications in parameterized and exact algorithms. J. ACM, 63(4):29:1–29:60, 2016.
10. Michael R. Garey and David S Johnson. “strong”np-completeness results: Motivation, examples, and implications. Journal of the ACM (JACM), 25(3):499–508, 1978.
11. Ron L. Graham, Eugene L. Lawler, Jan Karel Lenstra, and Alexander H. G. Rinnooy Kan. Optimization and approximation in deterministic sequencing and scheduling: a survey. Annals of discrete mathematics, 5(2):287–326, 1979.
12. Christos Koukamas and Shrikant S Panwalkar. A note on combined job selection and sequencing problems. Naval Research Logistics (NRL), 60(6):449–453, 2013.
13. Jan Karel Lenstra, AHG Rinnooy Kan, and Peter Brucker. Complexity of machine scheduling problems. 1:343–362, 1977.
14. Nicole Megow, Matthias Mnich, and Gerhard Woeginger. Lorent Workshop ‘Scheduling Meets Fixed-Parameter Intractability’, 2019.
Parameterized Complexity of Partial Scheduling

Matthias Mnich and René van Bevern. Parameterized complexity of machine scheduling: 15 open problems. *Computers & Operations Research*, 2018.

Matthias Mnich and Andreas Wiese. Scheduling and fixed-parameter tractability. *Math. Program.*, 154(1-2):533–562, 2015.

J Michael Moore. An n job, one machine sequencing algorithm for minimizing the number of late jobs. *Management science*, 15(1):102–109, 1968.

Gerhard Woeginger Nicole Megow, Matthias Mnich. Event report on Lorentz center workshop ‘Scheduling Meets Fixed-Parameter Tractability’, 2019.

Michael L. Pinedo. *Scheduling: Theory, Algorithms, and Systems*. Springer Publishing Company, Incorporated, 3rd edition, 2008.

Jirí Sgall. Open problems in throughput scheduling. In *Algorithms - ESA 2012 - 20th Annual European Symposium, Ljubljana, Slovenia, September 10-12, 2012. Proceedings*, pages 2–11, 2012.

Dvir Shabtay, Nufar Gaspar, and Moshe Kaspi. A survey on offline scheduling with rejection. *Journal of Scheduling*, 16(1):3–28, 2013.

Jeffrey D. Ullman. NP-complete scheduling problems. *Journal of Computer and System sciences*, 10(3):384–393, 1975.

René Van Bevern, Matthias Mnich, Rolf Niedermeier, and Mathias Weller. Interval scheduling and colorful independent sets. *Journal of Scheduling*, 18(5):449–469, 2015.

David P. Williamson and David B. Shmoys. *The Design of Approximation Algorithms*. Cambridge University Press, 2011.

Bibo Yang and Joseph Geunes. A single resource scheduling problem with job-selection flexibility, tardiness costs and controllable processing times. *Computers & Industrial Engineering*, 53(3):420–432, 2007.

Meirav Zehavi. Mixing color coding-related techniques. In *Algorithms - ESA 2015 - 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015, Proceedings*, pages 1037–1049, 2015.

### A k-scheduling without Precedence Constraints

In this section we study partial scheduling without precedence constraints. Notice that the problem on one machine, denoted by $1||k$-sched, $C_{\text{max}}$, can be solved in $O(n \log n)$ time by scheduling the $k$ jobs with the smallest processing times and checking whether they finish in time. However, the problem on parallel machines, $P|k$-sched|$C_{\text{max}}$, is NP-hard since its special case 3-partition is strongly NP-complete [10]. We show that the problem is fixed-parameter tractable in $k$, even in the case of unrelated machines, release dates and deadlines, denoted by $R|r_j, d_j, k$-sched|$C_{\text{max}}$.

**Theorem A.1.** $R|r_j, d_j, k$-sched|$C_{\text{max}}$ is fixed-parameter tractable in $k$ and an instance can be solved in $O^*((2e)^k k^{O(\log k)})$.

**Proof.** We give an algorithm that solves any instance of $R|r_j, d_j, k$-sched|$C_{\text{max}}$ within $O^*((2e)^k k^{O(\log k)})$ time. The algorithm is a randomized algorithm that can be de-randomized using the color coding method, as described in [1]. The algorithm first (randomly) picks a coloring $c : \{1, \ldots, n\} \to \{1, \ldots, k\}$, so each jobs is given one of the $k$ available colors. We then compute whether there is a feasible colorful schedule, i.e. a feasible schedule that processes exactly one job of each color. If this colorful schedule can be found, then it is possible to schedule at least $k$ jobs before $C_{\text{max}}$. 
Given a coloring $c$, we compute whether there exists a colorful schedule in the following way. Define for $1 \leq i \leq m$ and $X \subseteq \{1, ..., k\}$:

$$B_i(X) = \text{minimum makespan of all schedules on machine } i \text{ processing } |X| \text{ jobs, each from a different color in } X.$$ 

Clearly $B_i(\emptyset) = 0$. All other $B_i(X)$ can be computed in $O(2^k n)$ time using dynamic programming by use of the following lemma:

**Lemma A.2.** Let $\min\{\emptyset\} = \infty$. Then

$$B_i(X) = \min_{l \in X} \min_{j:l(j)=i} \{C_j = \max\{r_j, B_i(X \setminus \{l\})\} + p_{ij} : C_j \leq d_j\}.$$ 

**Proof.** In a schedule on one machine with $|X|$ jobs using all colors from $X$, one job should be scheduled as last, defining the makespan. So for all possible jobs $j$, we compute what the minimal end time would be if $j$ was scheduled at the end of the schedule. This $j$ cannot start before its release date or before all other colors are scheduled. Also, the job should complete before its deadline.

Computing all $B_i(X)$ can then be done in $O(2^k n)$ time. Next, define for $1 \leq i \leq m$ and $X \subseteq [k]$:

$$A_i(X) = \begin{cases} 1, & \text{if } B_i(X) \leq C_{\text{max}}, \\ 0, & \text{otherwise.} \end{cases}$$

So $A_i(X) = 1$ if and only if $|X|$ jobs, each from a different color of $X$, can be scheduled on machine $i$ before $C_{\text{max}}$. A colorful feasible schedule exists if and only if there is some partition $X_1, ..., X_m$ of $\{1, ..., k\}$ such that $\Pi_{i=1}^m A_i(X_i) = 1$. The subset convolution of two function is defined as $(A_i * A_{i'})(X) = \sum_{Y \subseteq X} A_i(Y) A_{i'}(X \setminus Y)$. Then $\Pi_{i=1}^m A_i(X_i) = 1$ if and only if $(A_1 * \cdots * A_m)((\{1, ..., k\}) > 0$. The value of $(A_1 * \cdots * A_m)((\{1, ..., k\}) > 0$ can be computed in $2^k k^{O(1)}$ time using fast subset convolution [3].

An overview of the randomized algorithm is given in Algorithm 2. If $k$ jobs that are scheduled in an optimal solution are all in different colors, the algorithm outputs true. By standard analysis, $k$ jobs are all assigned different colors with probability at least $1/e^k$, and thus $e^k$ independent trials are sufficient to boost the error probability of the algorithm to at most $1/2$.

```
1 For a given coloring $c$:
2 foreach $i = 1, ..., m$ do
3     foreach $X \subseteq \{1, ..., k\}$ in order of increasing size do
4         Compute $B_i(X)$ using Lemma A.2
5         Set $A_i(X) = 1$ if $B_i(X) \leq C_{\text{max}}$, set $A_i(X) = 0$ otherwise.
6     end
7 end
8 Compute $(A_1 * \cdots * A_m)((\{1, ..., k\})$ using fast subset convolution [3].
9 if $(A_1 * \cdots * A_m)((\{1, ..., k\}) > 0$ then
10     return $\text{TRUE}$
11 end
```

**Algorithm 2:** Algorithm for solving $R|\overline{r_j}, d_j,k\text{-sched}|C_{\text{max}}$. 

By using the same method as in [1], we can derandomize Algorithm 2. This is done by checking only a family of colorings for which each set of $k$ jobs is colorful in at least one of the colorings. This is called a $k$–perfect hashing family. There are different ways to construct such $k$-perfect hashing family, but there exists families with size $e^{k}O((\log k)n\log n)$ that can be constructed in time $e^{k}O((\log k)n\log n)$ [2 page 118]. Now, if there is a schedule that can schedule $k$ jobs, then in at least one of the coloring these jobs all have different colors. So executing Algorithm 2 for each element in a $k$-perfect family returns the correct answer.

B  Omitted Proofs from Section 4

Proof of Theorem A.1 The proof will be a reduction from 3-coloring, for which no $2^{o(|V|+|E|)}$ algorithm exists under the Exponential Time Hypothesis [3]. Let the graph $G = (V, E)$ be the instance of 3-coloring with $|V| = n'$ and $|E| = m'$. We then create the following instance for $1|\text{prec}, p_j = 1|k\text{-sched}, C_{\text{max}}$.

- For each vertex $v_i \in V$, create 6 jobs:
  - $v_i^1$, $v_i^2$ and $v_i^3$ with deadline $d_{v_i} = i$,
  - $w_i^1$, $w_i^2$ and $w_i^3$ with deadline $d_{w_i} = n' + 2m' + 1 - i$,
  add precedence constraints $v_i^1 < w_i^1$, $v_i^2 < w_i^2$ and $v_i^3 < w_i^3$. These jobs represent which color for each vertex will be chosen (if $v_i^1$ and $w_i^1$ are processed, vertex $i$ gets color 1).

- For each edge $e_j \in E$, create 12 jobs:
  - $e_{j}^{12}$, $e_{j}^{13}$, $e_{j}^{23}$, $e_{j}^{31}$ and $e_{j}^{32}$ with deadline $d_{e_j} = n' + j$,
  - $f_{j}^{12}$, $f_{j}^{13}$, $f_{j}^{23}$, $f_{j}^{31}$ and $f_{j}^{32}$ with deadline $d_{f_j} = n' + m' + 1 - j$,
  add precedence constraints $e_{j}^{ab} < f_{j}^{ab}$. These jobs represent what the colors of the endpoints of an edge will be. So if the jobs $e_{j}^{ab}$ and $f_{j}^{ab}$ are processed for $e = \{u, v\}$, then vertex $u$ has color $a$ and vertex $v$ has color $b$. Since the endpoints should have different colors, the jobs $e_{j}^{aa}$ and $f_{j}^{aa}$ do not exist.

- For each $e_{j}^{ab}$ with $e = \{u, v\}$ add the precedence constraints $a^u < e_{j}^{ab}$ and $b^v < e_{j}^{ab}$.

Set $C_{\text{max}} = k = 2n' + 2m'$.

We prove that the created instance is a yes-instance if and only if the original 3-coloring instance is a yes instance.

Assume that there is a 3-coloring of the graph $G = (V, E)$. Then there is also a feasible schedule. Indeed, for each vertex $v_i$ with color $a$, process the jobs $v_i^a$ and $w_i^a$ at their respective deadlines. For each edge $e_j = \{u, v\}$ with $u$ colored $a$ and $v$ colored $b$, process the jobs $e_{j}^{ab}$ and $f_{j}^{ab}$ exactly at their respective deadlines. Notice that because it is a 3-coloring, each edge has endpoints of different colors, so these jobs exist. Also note that no two jobs were processed at the same time. Exactly $2n' + 2m'$ jobs were processed before time $2n' + 2m'$. Furthermore, no precedence constraints were violated.

For the other direction, assume that we have a feasible schedule in our created instance of $1|\text{prec}, p_j = 1|k\text{-sched}, C_{\text{max}}$. Let $V_i = \{v_i^1, v_i^2, v_i^3\}$, $W_i = \{w_i^1, w_i^2, w_i^3\}$, and let $E_i = \{e_{i}^{12}, e_{i}^{13}, e_{i}^{23}, e_{i}^{31}, e_{i}^{32}\}$ and $F_i = \{f_{j}^{12}, f_{j}^{13}, f_{j}^{23}, f_{j}^{31}, f_{j}^{32}\}$. Then we show that for each of the sets $V_i$, $W_i$, $E_i$ and $F_i$, exactly one job was scheduled at its deadline. We will show this by induction.

Because we have a feasible schedule, this means that at time $2m' + 2n'$, one of the jobs of $W_1$ must be scheduled, since they are the only jobs with a deadline greater than $2n + 2m - 1$. However, if $w_i^1$ was scheduled at time $2m' + 2n'$, then the job $v_i^1$ must be processed at time $1$ because of precedence constraints and since its deadline is $1$. Also, no other job from $V_i$ can be processed in the schedule, since they all have deadline $1$. As a consequence, no other
jobs from $W_i$ can be processed, as they are restricted to precedence constraints. So for $i = 1$ the statement is true.

Now assume that all sets $V_1, ..., V_{i-1}, W_1, ..., W_{i-1}$ have exactly one job scheduled at their respective deadline, and no more can be processed. Since we have a feasible schedule, one job should be scheduled at time $2n' + 2m' - (i - 1)$. However, since no more jobs from $W_i, ..., W_{i-1}$ can be scheduled, the only possible jobs are from $W_i$ since they are the only other jobs with a deadline greater than $2n' + 2m' - i$. However, if $w^i_q$ was scheduled at time $2n' + 2m' - (i - 1)$, then the job $v^i_p$ must be processed at time $i$ because of precedence constraints, its deadline at $i$ and because at times $1, ..., i - 1$ other jobs had to be processed. Also, no other job from $V_i$ can be processed in the schedule, since they all have deadline $i$. As a consequence, no other jobs from $W_i$ can be processed, as they are restricted to precedence constraints. So the statement holds for all set $V_i$ and $W_i$. In the exact same way, one can conclude the same about all sets $E_j$ and $F_j$.

Because of this, we see that each job and each vertex have received a color from the schedule. They must form a 3-coloring, because a job from $E_j$ could only be processed if the two endpoints got two different colors. Hence we find that the 3-coloring instance must have been a yes-instance.

As $k = 2n' + 2m'$ we can therefore conclude that there exists no $2^{o(n)}$ algorithm under the Exponential Time Hypothesis.

Proof of Theorem 4.3 The proof will be a reduction from $k$-clique. We start with $G = (V, E)$, an instance of $k$-clique. For each vertex $v \in V$, create a job $j_v$ with $p_{j_v} = 2$. For each edge $e \in E$, create a job $j_e$ with $p_{j_e} = 1$. Now for each edge $(u, v)$, add the following two precedence relations: $j_u \prec j_e$ and $j_v \prec j_e$, so before one can process a job associated with an edge, both jobs associated with the endpoints of that edge need to be finished. Now let $k' = k + \frac{1}{2}k(k-1)$ and $C_{\text{max}} = 2k + \frac{1}{2}k(k-1)$. We will now prove that $1|\text{prec}|k'-\text{sched}, C_{\text{max}}$ is a yes instance if and only of $k$-clique is a yes instance.

Assume $k$-clique is a yes instance, then process first the $k$ jobs associated with the vertices of the $k$-clique. Next process the $\frac{1}{2}k(k-1)$ jobs associated with the edges of the $k$-clique. In total, $k + \frac{1}{2}k(k-1) = k'$ jobs are now processed with a makespan of $2k + \frac{1}{2}k(k-1)$. Hence, the instance of $1|\text{prec}|k'-\text{sched}, C_{\text{max}}$ is a yes instance.

For the other direction, assume $1|\text{prec}|k'-\text{sched}, C_{\text{max}}$ to be a yes instance, so we have found a feasible schedule. For any feasible schedule, if one schedules $l$ jobs associated with vertices, then at most $\frac{1}{2}(l-1)$ jobs associated with edges can be processed, because of the precedence constraints. However, because $k' = k + \frac{1}{2}k(k-1)$ jobs were done in the feasible schedule before $C_{\text{max}} = 2k + \frac{1}{2}k(k-1)$, at most $k$ jobs associated with vertices can be processed, because they have processing time of size 2. Hence, we can conclude that exactly $k$ vertex-jobs and $\frac{1}{2}k(k-1)$ edge-jobs were processed. Hence, there were $k$ vertices connected through $\frac{1}{2}k(k-1)$ edges, which is a $k$-clique.

C Trichotomy of all Studied Variants
Parameterized Complexity of Partial Scheduling

Table 1 The complexity of all variants of partial scheduling with respect to 1 machine, identical parallel machines or unrelated parallel machines, release/due dates, unit/arbitrary processing times, and precedence constraints. Notice that $p_j = 1$ implies that the machines are identical.