ASYMPTOTIC PROPERTIES OF THE PLANE SHEAR 
THICKENING FLUIDS WITH BOUNDED ENERGY INTEGRAL

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ABSTRACT. In this note we investigate the asymptotic behavior of plane shear 
thickening fluids around a bounded obstacle. Different from the Navier-Stokes case 
considered by Gilbarg-Weinberger in [18], where the good structure of the vorticity 
can be exploited and weighted energy estimates can be applied, we have to overcome 
the nonlinear term of high order. The decay estimates of the velocity was obtained 
by combining Point-wise Behavior Theorem in [16] and Brezis-Gallouet inequality in 
[7] together, which is independent of interest.

1. Introduction

As Ladyzhenskaya suggests in her monograph in [20], it is interesting to investigate 
"new equations for the description of the motion of viscous incompressible fluids", 
which roughly speaking means to consider viscosity coefficients, which depend on the 
modulus of the symmetric gradient,

\[ \varepsilon(u) = \frac{1}{2}(Du + (Du)^T) = \frac{1}{2}(\partial_i u_k + \partial_k u_i)_{1 \leq i, k \leq 2} \]

of the velocity field \( u \), for example, in a monotonically increasing way (shear thick-
ening case). In this note we will consider this problem in a very special situation 
restricting ourselves to stationary flows through an exterior domain \( \Omega \subset \mathbb{R}^2 \) with 
smooth boundary \( \partial \Omega \). More precisely, consider the solution \( u: \Omega \to \mathbb{R}^2, \pi: \Omega \to \mathbb{R} \)
of the following system

\[
\begin{align*}
- \text{div}[T(\varepsilon(u))] + u_k \partial_k u + D\pi &= 0, \quad \text{in } \Omega, \\
\text{div } u &= 0, \quad \text{in } \Omega,
\end{align*}
\]

(1.1)

where the \( \Omega = \mathbb{R}^2 \setminus B_{R_0}(0) \). More details on viscous incompressible flow, we refer to 
[3, 9, 10, 12, 14, 15, 20] and the references therein.

The system (1.1) describes the stationary flow of an incompressible generalized 
Newtonian fluid, where \( u \) is the velocity field, \( \pi \) is the pressure function, \( u_k \partial_k u \) is 
the convective term, and \( T \) represents the stress deviator tensor. And we use \( \varepsilon(u) \) 
to stand for the symmetric part of the differential matrix \( Du \) of \( u \). We assume that

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the stress tensor $T$ is the gradient of a potential $H : S^{2 \times 2} \to \mathbb{R}$ defined on the space $S^{2 \times 2}$ of all symmetric $2 \times 2$ matrices of the following form

$$H(\varepsilon) = h(|\varepsilon|),$$

where $h$ is a nonnegative function of class $C^3$. Thus

$$T(\varepsilon) = DH(\varepsilon) = \mu(|\varepsilon|)\varepsilon, \quad \mu(t) = \frac{h'(t)}{t}. \quad (1.2)$$

Note that the Navier-Stokes equations for incompressible Newtonian fluids follow from the system (1.1) if $\mu$ is a constant. If $\mu$ is not a constant, it means that the viscosity coefficient depends on $\varepsilon$, and system (1.1) describes the motion of continuous media of generalized Newtonian fluids.

As in [11], assume that the potential $h$ satisfies the follow conditions:

1. $h$ is strictly increasing and convex together with $h''(0) > 0$ and $\lim_{t \to 0} \frac{h(t)}{t} = 0$; (A1)
2. there exist a constant $a \geq 1$ such that $h(2t) \leq ah(t)$ for all $t \geq 0$; (A2)
3. $\frac{h'(t)}{t} \leq h''(t)$ for any $t \geq 0$. (A3)

Let us sketch some progress on the system (1.1). First, the existence of strong solutions is proved in a bounded domain by Fuchs in [10]. The existence of Dirichlet energy solutions satisfying the boundary condition at infinity in an exterior is very difficult, even if for the Navier-Stokes equations, which is related to Leray’s question; for example see Leray [21], Amick [1], Russo [24], Pileckas-Russo [23] and the references therein. Different from the Navier-Stokes case, the regularity is also unknown for general form $h(t)$ as in (1.2). Bildhauer-Fuchs-Zhong [6] proved the weak solution is $C^{1,\alpha}$ by assuming $h(t) = t^2(1+t)^m$, see also recent improved result by Jin-Kang in [19]. The Liouville property of (1.1) was started by Fuchs in [11], and later studied by Zhang in [25] and [26], where they obtained the trivial property of the solution with the help of $u \in L^\infty$ or $\int_{\Omega} h(|\nabla u|) < \infty$. The degenerate case $h(t) = t^p$ was also considered by Bildhauer-Fuchs-Zhang in [5] by assuming that $\int_{\Omega} |\nabla u|^p < \infty$. More developments, we refer to [5, 10] and the references therein.

In this note, motivated by the work of Gilbarg-Weinberger [18], we investigate the asymptotic properties of the solutions of (1.1). In [18], they showed that pressure function $\pi$ has a limit at infinity, $u(z) = o(\ln^{\frac{1}{2}} r)$, and $|Du| \leq o(r^{-\frac{1}{2}} (\ln r)^\frac{1}{2})$ provided that the Dirichlet energy is bounded in an exterior domain, i.e., $\int_{\Omega} |Du|^2 dx < \infty$. Their proof relies on the fact that the vorticity of the 2D Navier-Stokes equations satisfies a nice elliptic equation, to which the maximum principle applies. Here we consider the case of shear thickening fluids, for $h$ satisfying the (A1)-(A3); however, it’s difficult to exploit the good structure of the vorticity and apply weighted energy estimates, since the main part in (1.1) is nonlinear. Inspired by Point-wise Behavior Theorem in [16] and Brezis-Gallouet inequality (for example, see [7] or [8], we obtain
the higher energy estimates, which imply the decay estimates by combining point-wise behavior theorem and Brezis-Gallouet inequality together.

As shown in [5], the following properties of functions $h$ follows from (A1)-(A3).

(i) $\mu(t) = \frac{h(t)}{t}$ is an increasing function.
(ii) We have $h(0) = h'(0) = 0$ and

$$h(t) \geq \frac{1}{2} h''(0)t^{2}, \quad h''(0) > 0$$

(iii) There exists a constant $a > 0$ such that the function $h$ satisfies the balancing condition,

$$\frac{1}{a} t h'(t) \leq h(t) \leq t h'(t), \quad t \geq 0$$

(iv) From the assumptions on $h$, we know the system satisfies the following elliptic condition, $\forall \varepsilon, \sigma \in S^2$,

$$\frac{h'(|\varepsilon|)}{|\varepsilon|} |\sigma|^2 \leq D^2 H(\varepsilon)(\sigma, \sigma) \leq h''(|\varepsilon|)|\sigma|^2,$$

from which, together with (1.3) and (1.4), it follows that

$$D^2 H(\varepsilon)(\sigma, \sigma) \geq \frac{1}{2} h''(0) |\sigma|^2.$$  

Let $T_r = B_r(0) \setminus B_{R_0}(0)$ for any $r > R_0 > 0$. Our first result is to estimate the $L^2$ norm of $D^2 u$.

**Proposition 1.1.** For $\Omega = \mathbb{R}^2 \setminus B_{R_0}(0)$, let $u \in C^2(\Omega, \mathbb{R}^2)$ be a solutions of (1.1). Then there hold $\int_{\Omega} |D^2 u|^2 dx < \infty$,

$$\int_{T_{\frac{3}{2}r} \setminus T_{\frac{1}{2}r}} |D^2 u|^2 dx \leq C \frac{\log r}{r},$$

and

$$|u(x)| = o(\sqrt{\ln(|x|)}),$$

provided that $r = |x|$ is large enough and $\int_{\Omega} h(|Du|) dx < \infty$.

**Remark 1.1.** The $L^2$ norm of $D^2 u$ was obtained in [24] in the whole space $\mathbb{R}^2$, 

$$\int_{B_R} |D^2 u|^2 dx \leq C + C \frac{1}{R^3} \int_{B_{2R}} |u|^2 dx$$

Here we refine the estimate, especially for the exterior domain. One observation is to apply the Wirtinger’s inequality in $L^3$ norm and another technique is to use Poincaré-Sobolev inequality in a circular region. The estimate (1.8) is the same as the Navier-Stokes case in [18].
In order to obtain the decay estimate of $Du$, we need to estimate the norm of $D^3 u$. Generally, it’s more difficult. Motivated by the anisotropic variational problems in [2, 4], we give the following assumption.

(v) Assume that

$$1 + C_1 t^{\gamma_1} \leq h''(t) \leq 1 + C_2 t^{\gamma_2}, \quad t \geq 0,$$

where $\gamma_2 \geq \gamma_1 \geq 1$ and $C_2 > C_1 > 0$. From this, it follows that

$$t^2 + C_1' t^{2+\gamma_1} \leq h(t) \leq t^2 + C_2' t^{2+\gamma_2},$$

$$t + C_3' t^{1+\gamma_1} \leq h'(t) \leq t + C_4' t^{1+\gamma_2}, \quad t \geq 0,$$

where $C_1', C_2', C_3', C_4'$ depend on $C_1, C_2, \gamma_1$ and $\gamma_2$.

(vi) Assume that

$$h'''(t) \leq C_3 + C_4' t^{\gamma_0}, \quad \gamma_0 \geq 0, \quad t \geq 0,$$

where $C_3, C_3' \geq 0$.

\textbf{Proposition 1.2.} Let $\Omega = \mathbb{R}^2 \setminus B_{R_0}(0)$, $u \in C^3(\Omega, \mathbb{R}^2)$ is a solutions of (1.1), if $|\varepsilon(u)| \leq C$, then there exists a constant $r_0 > 0$ such that

$$\int_{T_2^r \setminus T_2^r} |D^3 u|^2 dx \leq C,$$

for any $r > r_0$, provided that $\int_{\Omega} h(|Du|) dx < \infty$.

\textbf{Remark 1.2.} The conditions (v) and (vi) are used to estimate the term $J_3$ in Sec. 4:

$$\int_{\Omega} \partial_n \left( \frac{h''(|\varepsilon(u)||\varepsilon(u)| - h'(|\varepsilon(u)|)}{|\varepsilon(u)|^3} \right) (\varepsilon(u) : \varepsilon(\partial_k u)) (\varepsilon(u) : \varepsilon(\partial_k u)) \zeta dx,$$

which need the smallness of the error of $\theta''(t) - h'(t)$ and the growth control of $h'''(t)$.

With the help of Proposition 1.1 and 1.2, an argument by Brezis-Gallouet inequality yields that

\textbf{Theorem 1.1.} Suppose that $u \in C^3(\Omega, \mathbb{R}^2)$ is a solution of (1.1) and $|\varepsilon(u)| \leq C$, then there exists a constant $r_1 > 0$ such that

$$||Du||_{L^\infty(T_2^r \setminus T_r)} \leq C r^{-\frac{3}{4}} (\log r)^{\frac{3}{4}},$$

for all $r > r_1$, provided that $\int_{\Omega} h(|Du|) dx < \infty$.

Our paper is organized as follows: in Sec.2 we prove the proof of Theorem 1.1 under the assumptions of Proposition 1.1 and 1.2 by Brezis-Gallouet inequality. In Sect.3 we are aimed to proving Proposition 1.1 by Point-wise Behavior Theorem in [16] and in the last section we complete the proof of Proposition 1.2.

Throughout this paper we adopt the Einstein summation convention, which means that the sum is taken with respect to indices repeated twice. Moreover, throughout
the remaining section, we denote by $C$ a general positive constant which depends only on known constant coefficients or norms and may be different from line to line.

2. Proof of Theorem 1.1

Under the assumptions of Proposition 1.1 and 1.2, we complete the proof of the main theorem.

Proof. Assume that $R_0 = 1$ without loss of generality, and define

$$
\tilde{u}(\tilde{x}) = ru(r\tilde{x}) = ru(x),
$$

where $x \in T_{2r} \setminus T_r$ and $r$ is large enough. By Lemma 5.4, we have

$$
\|D\tilde{u}\|_{L^\infty(T_2 \setminus T_1)} \leq C(1 + \|D\tilde{u}\|_{H^1(T_2 \setminus T_1)}) \sqrt{\log(e + \|D^3\tilde{u}\|_{L^2(T_2 \setminus T_1)})},
$$

And due to scaling, we have

$$
\|D\tilde{u}\|_{L^\infty(T_2 \setminus T_1)} = r^2\|Du\|_{L^\infty(T_{2r} \setminus T_r)},
\|D^2\tilde{u}\|_{L^2(T_2 \setminus T_1)} = r^2\|D^2u\|_{L^2(T_{2r} \setminus T_r)},
\|D^3\tilde{u}\|_{L^2(T_2 \setminus T_1)} = r^3\|D^3u\|_{L^2(T_{2r} \setminus T_r)}.
$$

Hence

$$
r^2\|Du\|_{L^\infty(T_{2r} \setminus T_r)} \leq C(1 + Cr\|Du\|_{L^2(T_{2r} \setminus T_r)} + r^2\|D^2u\|_{L^2(T_{2r} \setminus T_r)})
\sqrt{\log(e + r^3\|D^3u\|_{L^2(T_{2r} \setminus T_r)})},
$$

(2.1)

Due to the Proposition 1.1, there holds

$$
\int_{T_{2r} \setminus T_r} |D^2u|^2 dx \leq C\frac{\sqrt{\log r}}{r}.
$$

Besides, due to Proposition 1.2 we also get

$$
\int_{T_{2r} \setminus T_r} |D^3u|^2 dx \leq C.
$$

Then using (2.1), one can get

$$
r^2\|Du\|_{L^\infty(T_{2r} \setminus T_r)} \leq C(1 + Cr\|Du\|_{L^2(T_{2r} \setminus T_r)} + r^2\|D^2u\|_{L^2(T_{2r} \setminus T_r)})
\times \sqrt{\log(e + r^3\|D^3u\|_{L^2(T_{2r} \setminus T_r)})}
\leq C(1 + Cr + Cr^\frac{3}{2}(\log r)^\frac{1}{2})\sqrt{\log(e + Cr^3)}
\leq Cr^\frac{3}{2}(\log r)^\frac{1}{4}\sqrt{\log r},
$$

(2.2)

which implies

$$
\|Du\|_{L^\infty(T_{2r} \setminus T_r)} \leq Cr^{-\frac{1}{2}}(\log r)^{\frac{3}{4}}.
$$

The proof of Theorem 1.1 is complete.
3. Proof of Proposition 1.1

Assume that \( R_0 = 1 \) without loss of generality and \( \Omega = \mathbb{R}^2 \setminus \overline{B_1} \). Moreover, \( T_r = B_r \setminus \overline{B_1} \) for any \( r > 1 \).

To prove the decay of \( D^2 u \), on one hand we explore the weak maximum principle of the equation (1.1), which is similar to the vorticity form of the Navier-Stokes equations; on the other hand, an obstacle is to deal with the exterior domain and it seems that one can’t apply the embedding theorem with scaling domain and the iterative lemma 5.3 simultaneously. Our main idea is to apply the embedding theorem and the iterative lemma 5.3 in different domains. More precisely, we use the embedding theorem in a circular domain with a proportional boundary via weak maximum principle of the equation (1.1), but the iterative lemma is applies in the whole domain.

The proof is divided into two steps, at first we want to prove the \( L^2 \) norm of \( D^2 u \) is bounded and at last we prove the decay of this norm.

First, we introduce the cut-off functions, which will be used in the next proof.

Case 1. The choosing of \( \phi \). Assume that \( \phi(x) \in C_0^\infty(\Omega) \) with \( 0 \leq \phi \leq 1 \), satisfying that

i) for \( r > 10, \rho > 0 \) and \( \tau > 0 \), there holds \( \frac{3}{4} r \leq \rho < \tau \leq r \);

ii) \[
\phi(x) = \phi(|x|) = \begin{cases} 1, & \text{in } T_\rho \setminus T_3; \\ 0, & \text{in } T_2, \text{ and } \Omega \setminus T_r \end{cases}
\] \hspace{1cm} (3.1)

iii) \[
|D\phi| \leq \frac{C}{\tau - \rho}, \quad |D^2\phi| \leq \frac{C}{(\tau - \rho)^2}, \quad \text{as } x \in T_\tau \setminus T_\rho;
\]

\[
|D\phi| \leq C, \quad |D^2\phi| \leq C, \quad \text{as } x \in T_3 \setminus T_2.
\]

Case 2. The choosing of \( \psi \). Assume that \( \psi(x) \in C_0^\infty(\Omega) \) with \( 0 \leq \psi \leq 1 \), satisfying that

i) for \( r \gg 10 \), there holds

\[
\psi(x) = \psi(|x|) = \begin{cases} 1, & \text{in } T_{2r} \setminus T_r; \\ 0, & \text{in } T_{2r}, \text{ and } \Omega \setminus T_{3r} \end{cases}
\] \hspace{1cm} (3.2)

ii) \[
|D\psi| \leq \frac{C}{r}, \quad |D^2\psi| \leq \frac{C}{r^2}.
\]

Proof. First, we want to obtain Caccioppoli-type inequality by following the same route as in [11] or [25]. Then we estimate the crucial items in more delicate analysis.
Choose the test function $\varphi_k = \partial_k u \eta^2$, where the cut-off function $\eta \in C_0^\infty(\Omega)$ with $0 \leq \eta \leq 1$. Multiply (1.1) with $\partial_k \varphi_k$, and integration by parts yields
\[
\int_\Omega \partial_k \sigma : \varepsilon(\varphi_k) dx = \int_\Omega D\pi \cdot \partial_k \varphi_k dx - \int_\Omega u_i \partial_i u \cdot \partial_k \varphi_k dx = 0.
\]
where $\sigma = DH(\varepsilon(u)) = \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} \varepsilon(u)$.

Using integration by parts again, we obtain
\[
\int_\Omega \partial_k \sigma : \varepsilon(\partial_k u) \eta^2 dx = \int_\Omega \sigma : \partial_k(D\eta^2 \otimes \partial_k u) dx + \int_\Omega \partial_k \pi \text{div}(\varphi_k) dx
\]
\[
\quad + \int_\Omega u_i \partial_i u \cdot \partial_k \varphi_k \doteq I_1 + I_2 + I_3,
\]
(3.3)
where $\otimes$ is the tensor product of vectors.

For $I_1$, noticing the relation $|D^2 u(x)| \leq 2|D \varepsilon(u)(x)|$, by Young’s inequality we have
\[
I_1 = \int_\Omega \sigma : \partial_k(D\eta^2 \otimes \partial_k u) dx
\]
\[
\leq C \left\{ \int_\Omega \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |Du||D\eta|^2 + \eta|D^2 \eta||D\pi||Du| \eta dx \right\}
\]
\[
\leq \delta \int_\Omega \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |D\varepsilon(u)|^2 \eta^2 dx + C(\delta) \int_\Omega h'(|\varepsilon(u)|)|\varepsilon(u)||D\eta|^2 dx
\]
\[
+ C \int_\Omega h'(|\varepsilon(u)|) |Du||D\eta|^2 + \eta|D^2 \eta||D\pi||Du| \eta dx
\]
\[
\leq \delta \int_\Omega D^2 H(\varepsilon(u))(\varepsilon(\partial_k u), (\varepsilon(\partial_k u)) \eta^2 dx + C(\delta) \int_\Omega h(|\varepsilon(u)|)|D\eta|^2 dx
\]
\[
+ C \int_\Omega h'(|\varepsilon(u)|) |Du||D\eta|^2 + \eta|D^2 \eta||D\pi||Du| \eta dx.
\]
where $\delta > 0$, to be decided, and we used the estimates (1.5) and (1.4) in the last step.

For $I_3$, we have
\[
I_3 = \int_\Omega u_i \partial_i u_j \partial_k (\partial_k u_j \eta^2) dx = - \int_\Omega \partial_k (u_i \partial_i u_j) \partial_k u_j \eta^2 dx
\]
\[
= - \int_\Omega \partial_k u_i \partial_i u_j \partial_k u_j \eta^2 dx - \int_\Omega u_i \frac{1}{2} \partial_i (|\partial_k u_j|^2) \eta^2 dx
\]
\[
= \frac{1}{2} \int_\Omega |Du|^2 u \cdot D\eta^2 dx,
\]
where we use $\partial_k u_i \partial_i u_j \partial_k u_j = 0$ for divergence free vector $u$ in 2D.
Next, we deal with the first term of the right hand. Let
\[
I_2 = \int_{\Omega} \partial_k \pi \text{div}(\varphi_k) dx = \int_{\Omega} \partial_k \pi \partial_k u \cdot D\eta^2 dx
\]
\[
= - \int_{\Omega} \sigma_{ik} \partial_i (\partial_k u \cdot D\eta^2) dx - \int_{\Omega} u_i \partial_i u_k \partial_k u \cdot D\eta^2 dx. \tag{3.4}
\]
We estimate (3.4) in the same way as $I_1$ and $I_3$:
\[
I_2 \leq \delta \int_{\Omega} D^2 H(\varepsilon(u)) (\varepsilon(\partial_k u), (\varepsilon(\partial_k u)) \eta^2 dx + C(\delta) \int_{\Omega} h(\varepsilon(\bar{u})) |D\eta|^2 dx
\]
\[
+ C \int_{\Omega} h'(\varepsilon(\bar{u})) |D\eta|^2 dx + C \int_{\Omega} |Du|^2 |u| |D\eta| \eta dx.
\]
Finally, we observe that
\[
\partial_k \sigma : \varepsilon(\partial_k u) = D^2 H(\varepsilon(u))(\varepsilon(\partial_k u), \varepsilon(\partial_k u)).
\]
Recall (3.3) and collect the estimates of $I_1, \cdots, I_3$, and by choosing $\delta$ small enough we deduce
\[
\int_{\Omega} D^2 H(\varepsilon(u))(\varepsilon(\partial_k u), \varepsilon(\partial_k u)) \eta^2 dx \leq C \int_{\Omega} h(\varepsilon(\bar{u})) |D\eta|^2 dx
\]
\[
+ C \int_{\Omega} h'(\varepsilon(\bar{u})) |D\eta|^2 dx + C \int_{\Omega} |Du|^2 |u| |D\eta| \eta dx. \tag{3.5}
\]
Note that (1.6), $|\varepsilon(\bar{u})| \leq |Du|$ and (1.4) and we get
\[
\int_{\Omega} (\varepsilon(\partial_k u))^2 \eta^2 dx \leq C \int_{\Omega} h(\varepsilon(\bar{u})) |D\eta|^2 dx
\]
\[
+ C \int_{\Omega} h(|Du|) (|D\eta|^2 + |Du||D^2\eta|) dx + C \int_{\Omega} |Du|^2 |u| |D\eta| \eta dx. \tag{3.5}
\]

**Step I. The bounded estimate.**

In this step, we choose the cut-off function $\eta = \phi$. Note that (3.1) (3.5), and the energy bounded assumption, then we deduce that
\[
\int_{T_\tau \setminus T_2} |\varepsilon(\partial_k u)|^2 dx \leq C \frac{1}{(\tau - \rho)^2} + C \frac{1}{\tau - \rho} \int_{T_\tau \setminus (T_\rho \cup T_2 \setminus T_2)} |Du|^2 |u| \phi dx
\]
\[
\leq I_1^* + I_2^*.
\]

For the term $I_2^*$, noting that $\frac{\tau}{2} \leq \rho$ and $Du$ is bounded in $T_3 \setminus T_2$, we have
\[
I_2^* \leq C \frac{1}{\tau - \rho} \int_{T_\tau \setminus T_2} |Du|^2 |u| \phi dx + C \frac{1}{\tau - \rho} \leq I_3^* + C \frac{1}{\tau - \rho}
\]
Next, we deal with the first term of the right hand. Let
\[
\bar{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta,
\]
then by Wirtinger’s inequality (for example, for \( p = 2 \) see Ch II.5 [16]) we have
\[
\int_0^{2\pi} |f - \bar{f}|^3 d\theta \leq C \int_0^{2\pi} |\partial_\theta f|^3 d\theta.
\] (3.6)

By Hölder inequality,
\[
I_3' \leq C \frac{1}{\tau - \rho} \left( \int_{T_\tau \setminus T_{2\tau}} |Du|^3 d\theta \right) \left( \int_{T_\tau \setminus T_{2\tau}} \left| u - \bar{u} \right|^3 d\theta \right)^{\frac{1}{3}}.
\]

Using (3.6) and Lemma 5.1 we derive
\[
\int_{T_\tau \setminus T_{2\tau}} \left| u \right|^3 dx \leq \left( \int_{r < r' < \tau} \int_0^{2\pi} |u(r', \theta) - \bar{u}|^3 d\theta r' dr' \right) + \int_{T_\tau \setminus T_{2\tau}} \left( \int_0^{2\pi} u(r, \theta) d\theta \right)^3 dx
\]
\[
\leq Cr^3 \left( \int_{r < r' < \tau} \frac{1}{r'^3} \int_0^{2\pi} \left| \partial_\theta u \right|^3 d\theta r' dr' \right) + C(\ln r)^{\frac{3}{2}} r^2
\]
\[
\leq Cr^3 \int_{T_\tau \setminus T_{2\tau}} |Du|^3 dx + C(\ln r)^{\frac{3}{2}} r^2 \tag{3.7}
\]
for \( r > 2r_0(r_0) \) is a constant in Lemma 5.1, since (1.3).

Hence, by using \((\ln r)^{\frac{3}{2}} \leq Cr^\frac{1}{2}\) there holds
\[
I_3' \leq C \frac{r}{\tau - \rho} \int_{T_\tau \setminus T_{2\tau}} |Du|^3 dx + Cr^{-\frac{1}{2}}
\]

Recall that the following Poincaré-Sobolev inequality holds (see, for example, Theorem 8.11 and 8.12 [22])
\[
\|w\|_{L^3(B_r \setminus B_{r/2})} \leq C \|Dw\|_{L^2(B_r \setminus B_{r/2})} \|w\|_{L^2(B_r)} + C \tau^{-\frac{3}{2}} \|w\|_{L^2(B_r \setminus B_{r/2})},
\] (3.8)
which implies that
\[
I_3' \leq C \frac{r}{\tau - \rho} \left( \int_{T_\tau \setminus T_{2\tau}} |D^2u|^2 dx \right)^{\frac{1}{2}} \left( \int_{T_\tau \setminus T_{2\tau}} |Du|^2 dx \right)^{\frac{1}{2}} + C \frac{1}{\tau - \rho} \left( \int_{T_\tau \setminus T_{2\tau}} |Du|^2 dx \right)^{\frac{3}{4}}
\]
\[
\leq \frac{1}{16} \left( \int_{T_{\tau} \setminus T_{2\tau}} |D^2u|^2 dx \right) + C \frac{r^2}{(\tau - \rho)^2}
\]

Collecting the estimates of \( I_1', \ldots, I_3' \), we get
\[
\int_{T_{\tau} \setminus T_2} |\varepsilon(\partial_\theta u)|^2 dx \leq \frac{1}{16} \left( \int_{T_{\tau} \setminus T_2} |D^2u|^2 dx \right) + C \frac{1}{(\tau - \rho)} + C \frac{r^2}{(\tau - \rho)^2}
\]
Then by applying \( |D^2 u(x)| \leq 2|D\varepsilon(u)(x)| \) again, Lemma 5.3 yields
\[
\int_{T_3 \setminus T_2} |D^2 u|^2 \, dx \leq C.
\]
Finally, by taking \( r \to \infty \), we arrive at
\[
\int_{\Omega \setminus T_2} |D^2 u|^2 \, dx \leq C. \tag{3.9}
\]

**Step II. The decay estimate.**

In this step, we choose the cut-off function \( \eta = \psi \). Note that (3.2)–(3.5), and the energy bounded assumption, then we deduce that
\[
\int_{T_2 \setminus T_r} |\varepsilon(\partial_k u)|^2 \, dx \leq C \left( \frac{1}{r^2} + \frac{1}{r} \int_{T_3 \setminus T_{r/2}} |Du|^2 |u| \phi \, dx \right)
\]
\[
= C \left( \frac{1}{r^2} + I_4' \right).
\]
Due to \( Du \in L^2(\Omega) \) and (3.9), it follows from Gagliardo-Nirenberg inequality that
\[
\|Du\|_{L^p(\Omega \setminus T_2)} < \infty, \quad \forall \, p > 2. \tag{3.10}
\]
Thus with the help of Lemma 5.2 we have
\[
|u(x)| \leq \sqrt{\ln(|x|)} \tag{3.11}
\]
for a sufficient large constant, still denoted by \( r_0 \), and \( |x| \geq r_0 \). Consequently,
\[
\int_{T_2 \setminus T_r} |\varepsilon(\partial_k u)|^2 \, dx \leq C \frac{\sqrt{\ln r}}{r},
\]
which implies the required inequality (1.7).

\[\Box\]

4. Proof of Proposition 1.2

In this section, we introduce another cut-off functions, which will be used in the next proof.

**Case III. The choosing of \( \zeta \).** First, we introduce a cut-off function \( \zeta \in C_0^\infty(\Omega) \) with \( 0 \leq \zeta \leq 1 \), satisfying that
i) for \( r > 2r_0, \rho > 0 \) and \( \tau > 0 \), there holds \( \frac{3}{4} r \leq \rho < \tau \leq r; \)
ii) \[
\zeta(x) = \zeta(|x|) = \begin{cases} 1, & \text{in } T_{2\rho} \setminus T_{r + \frac{\tau - r}{2}}; \\ 0, & \text{in } T_{r + \frac{\tau - r}{2}}, \text{ and } \Omega \setminus T_{2r} \end{cases} \tag{4.1}
\]
iii) \( |D\zeta| \leq \frac{C}{\tau - \rho}, \ |D^2 \zeta| \leq \frac{C}{(r - \rho)^2}. \)
Proof. For any $\zeta \in C_0^\infty(\Omega)$ with $0 \leq \zeta \leq 1$, letting $\varphi_k = \partial_k \Delta u \zeta^3$ with $k = 1, 2$, we multiply (4.1) with $\partial_k \varphi_k$ and use integration by parts to obtain

$$\int_\Omega \partial_k \sigma : \varepsilon(\varphi_k) dx - \int_\Omega \pi \cdot \partial_k \varphi_k dx - \int_\Omega \varepsilon \partial_k u \cdot \partial_k \varphi_k dx = 0,$$

where $\sigma := DH(\varepsilon(u)) := \frac{h'(\|\varepsilon(u)\|)}{\|\varepsilon(u)\|} \varepsilon(u)$.

Using integration by parts again, by $\varphi_k = \partial_k \Delta u \zeta^3$ we get

$$\int_\Omega \partial_k \sigma : \varepsilon(\partial_k \Delta u) \zeta^3 dx = \int_\Omega \sigma : \partial_k (D\zeta^3 \otimes \partial_k \Delta u) dx$$

$$+ \int_\Omega \partial_k \pi \text{div}(\varphi_k) dx + \int_\Omega \varepsilon \partial_k u \cdot \partial_k \varphi_k dx. \quad (4.2)$$

Next we deal with every term of (4.2):

$$\int_\Omega \partial_k \sigma : \varepsilon(\partial_k \Delta u) \zeta^3 dx$$

$$= \int_\Omega \frac{h''(\|\varepsilon(u)\|)\varepsilon(u)}{|\varepsilon(u)|^2} \varepsilon(\varphi_k) dx - \int_\Omega \frac{h'(\|\varepsilon(u)\|)\varepsilon(u)}{|\varepsilon(u)|} \varepsilon(\varphi_k) dx$$

$$+ \int_\Omega \frac{h'(\|\varepsilon(u)\|)}{|\varepsilon(u)|} \varepsilon(\partial_k \varphi_k) dx$$

$$= -\int_\Omega \frac{h''(\|\varepsilon(u)\|)\varepsilon(u)}{|\varepsilon(u)|^2} \varepsilon(\varphi_k) dx$$

$$+ \int_\Omega \frac{h'(\|\varepsilon(u)\|)}{|\varepsilon(u)|} \varepsilon(\partial_k \varphi_k) dx$$

$$- \int_\Omega \frac{h'(\|\varepsilon(u)\|)}{|\varepsilon(u)|} \varepsilon(\partial_k \varphi_k) dx$$

$$- \int_\Omega \frac{h''(\|\varepsilon(u)\|)\varepsilon(u)}{|\varepsilon(u)|^3} \varepsilon(\partial_k \varphi_k) dx$$

$$+ \varepsilon(\partial_k \varphi_k) dx$$

$$= -J_1 - \cdots - J_8$$
Note that (1.5), and we have \( J_1 \geq 0 \). By (1.9), we have
\[
J_2 \geq \int_{\Omega} |\varepsilon(\partial_{kn} u)|^2 \zeta^3 dx
\]
Using \(|D^3 u(x)| \leq 2|\varepsilon(D^2 u)(x)|\), we have
\[
J_2 \geq \frac{1}{4} \int_{\Omega} |D^3 u|^2 \zeta^3 dx
\]
Then
\[
\frac{1}{4} \int_{\Omega} |D^3 u|^2 \zeta^3 dx \leq \int_{\Omega} \partial_k \sigma : (D\zeta^3 \otimes \partial_k \Delta u) dx \\
- \int_{\Omega} \partial_k \pi \text{div}(\varphi_k) dx - \int_{\Omega} u_i \partial_i u : \partial_k \varphi_k dx - J_3 - \cdots - J_8.
\]
For \( J_3 \), using (1.11), Young’s inequality and \(|\varepsilon(u)| \leq C\), we have
\[
J_3 = \int_{\Omega} \partial_n \left( \frac{h''(|\varepsilon(u)|) \varepsilon(u) - h'(|\varepsilon(u)|)}{|\varepsilon(u)|^2} \right) \varepsilon(u) : \varepsilon(\partial_k u) (\varepsilon(u) : \varepsilon(\partial_{kn} u)) \zeta^3 dx
\]
\[
\leq C \int_{\Omega} (|\varepsilon(u)|^7 + 1 + |\varepsilon(u)|^{21} + |\varepsilon(u)|^{17}) |D^2 u|^2 |D^3 u| \zeta^3 dx
\]
\[
\leq C \int_{\Omega} |D^2 u|^4 \zeta^4 dx + \frac{1}{64} \int_{\Omega} |D^3 u|^2 \zeta^2 dx.
\]
where we used
\[
\left| \frac{h''(|\varepsilon(u)|) \varepsilon(u) - h'(|\varepsilon(u)|)}{|\varepsilon(u)|^2} \right| \leq C(|\varepsilon(u)|^{21} + |\varepsilon(u)|^{17})
\]
since (1.10).
Similarly, \( J_6 \) can be estimated immediately as
\[
J_6 = \int_{\Omega} h''(|\varepsilon(u)|) \varepsilon(u) : \varepsilon(\partial_k u) \zeta^3 dx
\]
\[
\leq C \int_{\Omega} |D^2 u|^4 \zeta^4 dx + \frac{1}{64} \int_{\Omega} |D^3 u|^2 \zeta^2 dx.
\]
For \( J_4 \) and \( J_5 \), in the same way we have
\[
J_4 + J_5 \leq C \int_{\Omega} (|\varepsilon(u)|^{21} + |\varepsilon(u)|^{17}) |D^2 u|^2 |D^3 u| \zeta^3 dx
\]
\[
\leq C \int_{\Omega} |D^2 u|^4 \zeta^4 dx + \frac{1}{64} \int_{\Omega} |D^3 u|^2 \zeta^2 dx.
\]
For \( J_7 \) and \( J_8 \), using (4.4) and Young inequality again, we get
\[
J_7 + J_8 \leq C \int_{\Omega} (1 + |\varepsilon(u)|^2 + |\varepsilon(u)|^7) |D^2 u||D^3 u|^2 |D\zeta| dx
\]
\[ \leq C \int_{\Omega} |D^2 u|^2 \zeta^2 |D\zeta|^2 dx + \frac{1}{64} \int_{\Omega} |D^3 u|^2 \zeta^2 dx. \]

Next, we estimate the first three terms of the right hand in (4.3). Firstly, using (1.9) and (1.10) we get
\[ A_1 = \int_{\Omega} \partial_k \sigma : (D\zeta^3 \otimes \partial_k \Delta u) dx \]
\[ = \int_{\Omega} \partial_k \left( \frac{h'(|\varepsilon(u)|)}{\varepsilon(u)} \varepsilon(u) \right) : (D\zeta^3 \otimes \partial_k \Delta u) dx \]
\[ \leq C \int_{\Omega} (1 + |\varepsilon(u)|^2) |D^2 u| \zeta^2 |D\zeta||D^3 u| dx \]
\[ \leq C \int_{\Omega} |D^2 u|^2 \zeta^2 |D\zeta|^2 dx + \frac{1}{64} \int_{\Omega} |D^3 u|^2 \zeta^2 dx. \]

(4.5)

Besides,
\[ A_2 = \int_{\Omega} u_i \partial_i u \cdot \partial_k \varphi_k dx \]
\[ = - \int_{\Omega} \partial_k u_i \partial_i u_j \partial_k \Delta u_j \zeta^3 dx - \int_{\Omega} u_i \partial_k u_j \partial_k \Delta u_j \zeta^3 dx \]
\[ \leq C \int_{\Omega} |Du|^2 |D^3 u| \zeta^3 dx + C \int_{\Omega} |u| |D^2 u||D^3 u| \zeta^3 dx \]
\[ \leq \frac{1}{64} \int_{\Omega} |D^3 u|^2 \zeta^2 dx + C \int_{\Omega} |u|^2 |D^2 u|^2 \zeta^4 dx + C \int_{\Omega} |Du|^4 \zeta^4 dx. \]

(4.6)

Finally, we estimate \( A_3 = \int_{\Omega} \partial_k \pi \text{div}(\varphi_k) dx \). According the equation of (1.1),
\[ \int_{\Omega} \partial_k \pi \text{div}(\varphi_k) dx = \int_{\Omega} \partial_k \pi \partial_k \Delta u \cdot D\zeta^3 dx \]
\[ = \int_{\Omega} \partial_k \sigma_{ik} \partial_k \Delta u \cdot D\zeta^3 dx - \int_{\Omega} u_i \partial_k u_k \partial_k \Delta u \cdot D\zeta^3 dx, \]
whose estimates are similar to \( A_1 \) and \( A_2 \), i.e.
\[ A_3 \leq \frac{1}{64} \int_{\Omega} |D^3 u|^2 \zeta^2 dx + C \int_{\Omega} |D^2 u|^2 \zeta^2 |D\zeta|^2 dx \]
\[ + C \int_{\Omega} |u|^2 |Du|^2 |D\zeta|^2 \zeta^2 dx. \]

(4.7)

Recalling (4.3), and combining \( J_3, \ldots, J_8, (4.5), (4.6) \) and (4.7), we have
\[ \frac{1}{4} \int_{\Omega} |D^2 u|^2 \zeta^2 dx \leq \frac{1}{8} \int_{\Omega} |D^3 u|^2 \zeta^2 dx + C \int_{\Omega} |u|^2 |D^2 u|^2 \zeta^4 dx \]
\[ + C \int_{\Omega} |Du|^4 \zeta^4 dx + C \int_{\Omega} |D^2 u|^2 \zeta^2 |D\zeta|^2 dx \]
\[
+ C \int_{\Omega} |D^2u|^4 \zeta^4 \, dx + C \int_{\Omega} |u|^2 |Du|^2 |D\zeta|^2 \zeta^2 \, dx.
\]

Note that the bounded energy integral and (3.10), i.e. \( Du \in L^p(\Omega) \) for any \( p \geq 2 \), and we derive
\[
\frac{1}{4} \int_{\Omega} |D^3u|^2 \zeta^3 \, dx \leq \frac{1}{8} \int_{\Omega} |D^3u|^2 \zeta^2 \, dx + C \left( \frac{\ln r}{r} \right)^{\frac{3}{2}}
+ C + C \frac{\sqrt{\ln r}}{r(\tau - \rho)^2} + C \frac{\ln r}{r(\tau - \rho)^2}
+ C \int_{\Omega} |D^2u|^4 \zeta^4 \, dx \tag{4.8}
\]
where we also used the decay estimate in Proposition 1.1. Next we deal with the last term in the right hand. Due to Gagliardo-Nirenberg inequality,
\[
C \int_{\Omega} |D^2u\zeta|^4 \, dx \leq C_0 \|D^2u\zeta\|_{L^2(\Omega)}^2 \|D(D^2u\zeta)\|_{L^2(\Omega)}^2,
\]
where \( C_0 \) is an absolute constant. Using Proposition 1.1 again, we have
\[
\int_{\Omega} |D^2u\zeta|^4 \, dx \leq C_0 C \frac{\sqrt{\ln r}}{r} \int_{\Omega} |D^3u|^2 \zeta^2 \, dx + C \frac{\ln r}{r^2(\tau - \rho)^2}
\leq \frac{1}{16} \int_{\Omega} |D^3u|^2 \zeta^2 \, dx + C \frac{\ln r}{r^2(\tau - \rho)^2} \tag{4.9}
\]
provided that \( r_0 \) is large enough.

Combining (4.8) and (4.9), by (4.1) we deduce
\[
\int_{T_{2r} \setminus T_{r + \frac{r - \rho}{2}}} |D^3u|^2 \, dx \leq \frac{3}{4} \int_{T_{2r} \setminus T_{r + \frac{r - \rho}{2}}} |D^3u|^2 \, dx + C \left( \frac{\ln r}{r} \right)^{\frac{3}{2}}
+ C + C \frac{\sqrt{\ln r}}{r(\tau - \rho)^2} + C \frac{\ln r}{r(\tau - \rho)^2} + C \frac{\ln r}{r^2(\tau - \rho)^2}
\]
Applying Lemma 5.3, we have
\[
\int_{T_{2r} \setminus T_{r + \frac{r - \rho}{2}}} |D^3u|^2 \, dx \leq C \left( \frac{\ln r}{r} \right)^{\frac{3}{2}}
+ C \frac{\sqrt{\ln r}}{r(\tau - \rho)^2} + C \frac{\ln r}{r(\tau - \rho)^2} + C \frac{\ln r}{r^2(\tau - \rho)^2},
\]
by taking \( \rho = \frac{3}{4} r \) and \( \tau = r \), which implies that
\[
\int_{T_{\frac{3}{4}r} \setminus T_{\frac{1}{2}r}} |D^3u|^2 \, dx \leq C \left( \frac{\ln r}{r} \right)^{\frac{3}{2}} + C \frac{\sqrt{\ln r}}{r^3} + C \frac{\ln r}{r^2} + C
\]
\[ \leq C \]

which implies the required inequality (1.2).

\[ \square \]

5. Appendix

In the proof of Theorem, we need the following known lemmas.

First, let us recall a result of Gilbarg-Weinberger in [18] about the decay of functions with finite Dirichlet integrals.

**Lemma 5.1** (Lemma 2.1, 2.2, [18]). Let a \( C^1 \) vector-valued function \( f(x) = (f_1, f_2)(x) = f(r, \theta) \) with \( r = |x| \) and \( x_1 = r \cos \theta \). There holds finite Dirichlet integral in the range \( r > r_0 \), that is

\[ \int_{r>r_0} |Df|^2\,dxdy < \infty. \]

Then, we have

\[ \lim_{r \to \infty} \frac{1}{\ln r} \int_0^{2\pi} |f(r, \theta)|^2d\theta = 0. \]

If, furthermore, we assume \( Df \in L^p(\mathbb{R}^2) \) for some \( 2 < p < \infty \), then the above decay property can be improved to be point-wise uniformly. More precisely, we have

**Lemma 5.2** (Point-wise Behavior Theorem, Theorem II.9.1 [16]). Let \( \Omega \subset \mathbb{R}^2 \) be an exterior domain and let

\[ Df \in L^2 \cap L^p(\Omega), \]

for some \( 2 < p < \infty \). Then

\[ \lim_{|x| \to \infty} \frac{|f(x)|}{\sqrt{\ln(|x|)}} = 0, \]

uniformly.

We also need a Giaquinta’s iteration lemma [17, Lemma 3.1].

**Lemma 5.3** (Lemma 3.1 [17]). Let \( f(r) \) be a non-negative bounded function on \( [R_0, R_1] \subset \mathbb{R}_+ \). If there are negative constants \( A, B, D \) and positive exponents \( b < a \) and a parameter \( \theta \in (0, 1) \) such that for all \( R_0 \leq \rho < \tau \leq R_1 \)

\[ f(\rho) \leq \theta f(\tau) + \frac{A}{(\tau - \rho)^a} + \frac{B}{(\tau - \rho)^b} + D, \]

then for all \( R_0 \leq \rho < \tau \leq R_1 \)

\[ f(\rho) \leq C(a, \theta) \left[ \frac{A}{(\tau - \rho)^a} + \frac{B}{(\tau - \rho)^b} + D \right]. \]
At last, we introduce the Brezis-Gallouet inequality (see Lemma 2 in [7], or Lemma 3.1 in [8]).

**Lemma 5.4.** Let \( f \in H^2(\Omega) \) where \( \Omega \) is a bounded domain or an exterior domain with compact smooth boundary. Then there exists a constant \( C_\Omega \) depending only on \( \Omega \), such that

\[
\| f \|_{L^\infty(\Omega)} \leq C_\Omega \| f \|_{H^1(\Omega)} \ln \left( \frac{1}{2} \left( e + \| D^2 f \|_{L^2(\Omega)} \| f \|_{H^1(\Omega)} \right) \right),
\]

or

\[
\| f \|_{L^\infty(\Omega)} \leq C_\Omega (1 + \| f \|_{H^1(\Omega)}) \ln \left( \frac{1}{2} \left( e + \| D^2 f \|_{L^2(\Omega)} \right) \right).
\]

Note that the second inequality can be obtained immediately from the first one by arguments whether \( \| f \|_{H^1(\Omega)} < 1 \), and we omitted it.

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