THE LOG CANONICAL THRESHOLD OF
HOMOGENEOUS AFFINE HYPERSURFACES

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1. Introduction

In [CP], I. Cheltsov and J. Park studied the log canonical threshold of singular hyperplane sections of complex smooth, projective hypersurfaces. Let \( X \subset \mathbb{P}^n, n \geq 4 \), be a complex smooth hypersurface of degree \( d \) and \( Y \) a hyperplane section of \( X \) (which has to be irreducible and reduced). I. Cheltsov and J. Park proved that \( Y \) has isolated singular points and they studied the log canonical threshold of the pair \((X, Y)\). They showed that
\[
c(X, Y) \geq \min\{\frac{n-1}{d}, 1\},
\]
and they conjectured that if \( d = n \), then equality holds if and only if \( Y \) is a cone over a (smooth) hypersurface in some \( \mathbb{P}^{n-2} \). Moreover, they showed that their conjecture follows from the Minimal Model Program.

The purpose of this note is to generalize their lower bound in the case of an arbitrary hypersurface in \( \mathbb{P}^{n-1} \) and to give a direct proof of their conjecture in our more general setting. In fact, we get these results also for the affine cone over the hypersurface. The main ingredient is the description of the log canonical threshold in terms of the asymptotic growth of the jet schemes from [Mu2].

Here are our results. Let \( Y \subset \mathbb{P}^{n-1}, n \geq 2 \) be a complex hypersurface of degree \( d \geq 1 \) and let \( Z \subset \mathbb{A}^n \) be the affine cone over \( Y \). Suppose that \( \dim \text{Sing}(Z) = r \).

**Theorem 1.1.** With the above notation, we have the following lower bound for the log canonical threshold of \((\mathbb{A}^n, Z)\):
\[
c(\mathbb{A}^n, Z) \geq \min\{\frac{(n-r)}{d}, 1\}.
\]

**Theorem 1.2.** If in addition \( d \geq n-r+1 \), then \( c(\mathbb{A}^n, Z) = (n-r)/d \) if and only if \( Z = T \times \mathbb{A}^r \), for some hypersurface \( T \) in an \((n-r)\)-dimensional affine space.
Remark 1.3. Since we have
\[ c(P^{n-1}, Y) = c(\mathbb{A}^n \setminus \{0\}, Z \setminus \{0\}) \geq c(\mathbb{A}^n, Z), \]
it follows from Theorem 1.1 that \( c(P^{n-1}, Y) \geq \min\{ (n-r)/d, 1 \} \). Moreover, if \( d \geq n - r + 1 \) and \( c(P^{n-1}, Y) = (n-r)/d \), we deduce from Theorem 1.2 that \( Y \) is the projective cone (with a \( P^{r-1} \) vertex) over a (smooth) hypersurface in some \( P^{n-r-1} \).

The converse is well-known: if \( Y \subset P^{n-1} \) is the projective cone over a smooth hypersurface of degree \( d \geq n - r + 1 \) in \( P^{n-r-1} \), then \( c(P^{n-1}, Y) = (n-r)/d \). However, for completeness, we will include an argument for this assertion in the spirit of this paper in Proposition 2.4 below.

In order to make the connection between the way we stated our results and the results in [CP], we make the following

Remark 1.4. Suppose we are in the situation in [CP]: \( X \subset P^n \) is a smooth hypersurface and \( Y = X \cap H \), where \( H \subset P^n \) is a hyperplane. We have the following equality:
\[ c(H, Y) = c(X, Y). \]
See, for example, Theorem 2.1 below for justification.

Since Cheltsov and Park proved that in this case \( r \leq 1 \), Theorems 1.1 and 1.2 give in particular their lower bound and their conjectured characterization of equality.

2. Jet scheme dimension computations

For the standard definition of the log canonical threshold, as well as for equivalent definitions in singularity theory, we refer to [Ko]. We will take as definition the characterization from [Mu2] in terms of jet schemes.

Recall that for an arbitrary scheme \( W \) (of finite type over \( \mathbb{C} \)), the \( m \)th jet scheme \( W_m \) is a scheme of finite type over \( \mathbb{C} \) characterized by
\[ \text{Hom}(\text{Spec } A, W_m) \simeq \text{Hom}(\text{Spec } A[t]/(t^{m+1}), W), \]
for every \( \mathbb{C} \)-algebra \( A \). Note that \( W_m(\mathbb{C}) = \text{Hom}(\text{Spec } \mathbb{C}[t]/(t^{m+1}), W) \), and in fact, we will be interested only in the dimensions of these spaces.

**Theorem 2.1.** ([Mu2] 3.4) If \( X \) is a smooth, connected variety of dimension \( n \), and \( D \subset X \) is an effective divisor, then the log canonical threshold of \((X, D)\) is given by
\[ c(X, D) = n - \sup_{m \in \mathbb{N}} \frac{\dim D_m}{m + 1}. \]
Moreover, there is \( p \in \mathbb{N} \) such that \( c(X, D) = n - (\dim D_m)/(m+1) \) whenever \( p \mid (m+1) \).

It is easy to write down equations for jet schemes. We are interested in the jet schemes of a hypersurface \( Z \subset \mathbb{A}^n \) defined by a polynomial \( F \in \mathbb{C}[X_i; 1 \leq i \leq n] \). The jet scheme \( Z_m \) is a subscheme of \( \mathbb{A}^{(m+1)n} = \text{Spec } R_m \), where \( R_m = \mathbb{C}[X_i, X'_i, \ldots, X_i^{(m)}] \). If \( D : R_m \to R_{m+1} \) is the unique \( \mathbb{C} \)-derivation such that \( D(X_i^{(j)}) = X_i^{(j+1)} \) for all \( i \) and \( j \), we take \( F^{(p)} := D^p(F) \). The jet scheme \( Z_m \) is defined by the ideal \((F, F', \ldots, F^{(m)})\).

For every \( m \geq 1 \), there are canonical projections \( \phi_m : W_m \to W_{m-1} \) induced by the truncation homomorphisms \( \mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^m) \). By composing these projections we get morphisms \( \rho_m : W_m \to W \).

If \( W \) is a smooth, connected variety, then \( W_m \) is smooth, connected, and \( \dim W_m = (m+1) \dim W \), for all \( m \). It follows from the definition that taking jet schemes commutes with open immersions. In particular, if \( W \) has pure dimension \( n \), then \( \rho_m^{-1}(W_{\text{reg}}) \) is smooth, of pure dimension \((m+1)n\).

For future reference, we record here two lemmas. We denote by \([\cdot]\) the integral part function.

**Lemma 2.2.** ([Mu1] 3.7) If \( X \) is a smooth, connected variety of dimension \( n \), \( D \subset X \) is an effective divisor, and \( x \in D \) is a point with \( \text{mult}_x D = q \), then

\[
\dim \rho_m^{-1}(x) \leq mn - [m/q],
\]

for every \( m \in \mathbb{N} \).

In fact, the only assertion we will need from Lemma 2.2 is that \( \dim \rho_m^{-1}(x) \leq mn - 1 \), if \( m \geq q \), which follows easily from the equations describing the jet schemes.

If we have a family of schemes \( \pi : \mathcal{W} \to S \), we denote the fiber \( \pi^{-1}(s) \) by \( \mathcal{W}_s \). The projection morphism \( (\mathcal{W}_s)_m \to \mathcal{W}_s \) will be denoted by \( \rho_m^{\mathcal{W}_s} \).

**Lemma 2.3.** ([Mu2] 2.3) Let \( \pi : \mathcal{W} \to S \) be a family of schemes and \( \tau : S \to \mathcal{W} \) a section of \( \pi \). For every \( m \in \mathbb{N} \), the function

\[
f(s) = \dim(\rho_m^{\mathcal{W}_s})^{-1}(\tau(s))
\]

is upper semicontinuous on the set of closed points of \( S \).

We give now the proofs of our results.
Proof of Theorem 1.1. If \( \rho_m : Z_m \to Z \) is the canonical projection, then we have an isomorphism
\[
\rho_m^{-1}(0) \simeq Z_{m-d} \times \mathbb{A}^{n(d-1)},
\]
for every \( m \geq d-1 \) (we put \( Z_{-1} = \{0\} \)). Indeed, for a \( C \)-algebra \( A \), an \( A \)-valued point of \( \rho_m^{-1}(0) \) is a ring homomorphism
\[
\phi : \mathbb{C}[X_1, \ldots, X_n]/(F) \to A[t]/(t^{m+1}),
\]
such that \( \phi(X_i) \in (t) \) for all \( i \). Here \( F \) is an equation defining \( Z \). Therefore we can write \( \phi(X_i) = t f_i \), and \( \phi \) is a homomorphism if and only if the classes of \( f_i \) in \( A[t]/(t^{m+1}) \) define an \( A \)-valued point of \( Z_{m-d} \). But \( \phi \) is uniquely determined by the classes of \( f_i \) in \( A[t]/(t^{m+1}) \), so this proves the isomorphism in equation (1).

Recall that we have \( \dim \text{Sing}(Z) = r \). An easy application of Lemma 2.3 shows that for every \( x \in \text{Sing}(Z) \), we have \( \dim \rho_m^{-1}(x) \leq \dim \rho_m^{-1}(0) \). We deduce that if \( m \geq d-1 \), then
\[
\dim Z_m \leq \max\{(m+1)(n-1), r + \dim \rho_m^{-1}(0)\}
= \max\{(m+1)(n-1), \dim Z_{m-d} + n(d-1) + r\}.
\]
A recursive application of this inequality shows that for every \( p \geq 1 \), we have \( \dim Z_{pd-1} \leq pd(n-1) \), if \( d \leq n-r \), and \( \dim Z_{pd-1} \leq p(nd-n+r) \), if \( d \geq n-r \). This implies
\[
\dim Z_{pd-1}/pd \leq \max\{n-1, (nd-n+r)/d\}.
\]

By Theorem 2.1, there is \( p \geq 1 \) such that
\[
c(\mathbb{A}^n, Z) = n - \frac{\dim Z_{pd-1}}{pd},
\]
and we get \( c(\mathbb{A}^n, Z) \geq \min\{1, (n-r)/d\} \).

Proof of Theorem 1.2. We know that \( d \geq n-r+1 \) and \( c(\mathbb{A}^n, Z) = (n-r)/d \). By Theorem 2.1, there is \( k \geq 1 \) such that
\[
(2) \quad \dim Z_{kd-1} = k(nd-n+r).
\]
We first show that if \( k \geq 2 \) and equation (2) holds for \( k \), then it holds also for \( k-1 \).

Since \( c(\mathbb{A}^n, Z) = (n-r)/d \), it follows from Theorem 2.1 that
\[
(3) \quad \dim Z_{(k-1)d-1} \leq (k-1)(nd-n+r).
\]
The isomorphism (1) in the proof of Theorem 1.1 implies
\[
(4) \quad \dim \rho_{kd-1}^{-1}(0) \leq (k-1)(nd-n+r) + nd-n.
\]
On the other hand, we have
\begin{equation}
(5) \quad \dim Z_{kd-1} \leq \max\{kd(n-1), \dim \rho^{-1}_{kd-1}(0) + r\}.
\end{equation}

Since \(kd(n-1) < k(nd - n + r)\), we deduce from \((2)\), \((4)\) and \((5)\) that we have equality in \((4)\), hence in \((3)\). Therefore equation \((2)\) holds also for \(k - 1\).

The above argument shows that equation \((2)\) holds for \(k = 1\), so that we have
\[
\dim Z_{d-1} = dn - n + r > d(n - 1).
\]

Equation \((1)\) in the proof of Theorem \((1)\) gives \(\rho^{-1}(0) \simeq \mathbb{A}^{n(d-1)}\).

Since \(\dim \rho^{-1}_d(x) \leq \dim \rho^{-1}_{d-1}(0)\) for every \(x \in Z\), we deduce that there is a closed subset \(W \subseteq \text{Sing}(Z)\) with \(\dim W = r\) such that \(\dim \rho^{-1}_d(x) = \dim \rho^{-1}_{d-1}(0)\) for all \(x \in W\).

Fix \(x \in W\). If \(\text{mult}_x Z \leq d - 1\), then Lemma \((2)\) would give \(\dim \rho^{-1}_d(x) \leq (d - 1)n - 1\), a contradiction. Therefore we must have \(\text{mult}_x Z \geq d\).

By a linear change of coordinates we may assume that \(x = (1, 0, \ldots, 0)\) and we write the equation \(F\) of \(Z\) as \(F = \sum_{i=0}^d f_i(X_2, \ldots, X_n)X_1^{d-1}\), with \(f_i\) homogeneous of degree \(i\) for all \(i\). Since
\[
\frac{\partial^p F}{\partial X_1^p}(1, 0, \ldots, 0) = 0,
\]
for \(p < d\), we deduce that \(f_i = 0\) for \(i < d\), so that \(Z = T_0 \times \mathbb{A}^1\), where \(T_0\) is the hypersurface defined by \(f_d\) in \(\mathbb{A}^{n-1}\).

Since \(\dim W = r\), by applying inductively the above argument we get a homogeneous hypersurface \(T = T_{r-1} \subset \mathbb{A}^{n-r}\) such that \(Z = T \times \mathbb{A}^r\). Note that our assumption on the singular locus of \(Z\) implies that \(\text{Sing}(Z) = \{0\}\).

The converse is standard, and in fact we will prove a slightly stronger statement in the next proposition.

The following proposition is well-known, but we include a proof for the benefit of the reader.

**Proposition 2.4.** Let \(T \subset \mathbb{A}^{n-r}\) be a hypersurface defined by a homogeneous polynomial of degree \(d \geq n-r\) such that \(\text{Sing}(T) = \{0\}\). If \(Z = T \times \mathbb{A}^r \subset \mathbb{A}^n\) and \(Y \subset \mathbb{P}^{n-1}\) is the projectivization of \(Z\), then \(c(\mathbb{P}^{n-1}, Y) = (n-r)/d\).

**Proof.** An equivalent statement with that of the proposition is that \(c(\mathbb{A}^n \setminus \{0\}, Z \setminus \{0\}) = (n-r)/d\). Since \(Z = T \times \mathbb{A}^r\), we have \(Z_m \simeq T_m \times \mathbb{A}^{mr}\). This implies that every irreducible component of \(Z_m\) dominates \(\mathbb{A}^r\), so that \(c(\mathbb{A}^n \setminus \{0\}, Z \setminus \{0\}) = c(\mathbb{A}^n, Z) = c(\mathbb{A}^{n-r}, T)\).
The fact that this number is \((n-r)/d\) is well-known. To see this using jet schemes we can use the analogue of equation (1) in the proof of Theorem 1.1 together with the fact that
\[
\dim T_m = \max\{(m+1)(n-r-1), \dim (\rho_m^T)^{-1}(0)\},
\]
where \(\rho_m^T\) is the projection corresponding to \(T\). By induction we get
\[
\dim T_{kd-1} = k(n-r)(d-1),
\]
for every \(k \geq 1\), and Theorem 2.1 gives
\[
c(\mathbb{A}^{n-r}, T) = (n-r)/d.
\]

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