The Hilbert–Schmidt Analyticity Associated with Infinite-Dimensional Unitary Groups

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Abstract. The article is devoted to the problem of Hilbert–Schmidt type analytic extensions in Hardy spaces over the infinite-dimensional unitary group endowed with an invariant probability measure. Reproducing kernels of Hardy spaces, integral formulas of analytic extensions and their boundary values are considered.

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1. Introduction

The paper deals with the problem of Hilbert–Schmidt type analytic extensions in the Hardy space \( H_2^\chi \) of complex functions over the infinite-dimensional group \( U(\infty) = \bigcup \{ U(m) : m \in \mathbb{N} \} \) endowed with an invariant probability measure \( \chi \) where \( U(m) \) are subgroups of unitary \( m \times m \)-matrices. The measure \( \chi \) is defined as a projective limit \( \chi = \lim \leftarrow \chi_m \) of the Haar probability measures \( \chi_m \) on \( U(m) \). Moreover, \( \chi \) is supported by a projective limit \( \mathcal{U} = \lim \leftarrow U(m) \) and is invariant under the right action of \( U^2(\infty) := U(\infty) \times U(\infty) \) on \( \mathcal{U} \).

A goal of this work is to find integral formulas for Hilbert–Schmidt analytic extensions of functions from \( H_2^\chi \) and to describe their radial boundary values on the open unit ball in a Hilbert space \( E \) where \( U(\infty) \) acts irreducibly.

The measure \( \chi \) on \( \mathcal{U} \) was described by Olshanski [13] and Neretin [12]. The notion \( \mathcal{U} \) is related to Pickrell’s space of a virtual Grassmannian [16]. Hardy spaces in infinite-dimensional settings were discussed in the works of Cole and Gamelin [5], Ørsted and Neeb [14]. Spaces of analytic functions of Hilbert–Schmidt holomorphy types were considered by Dwyer III [6] and Petersson [15].
More general classes of analytic functions associated with coherent sequences of polynomial ideals were described by Carando et al. [4]. Integral formulas for analytic functions employing Wiener measures on infinite-dimensional Banach spaces were suggested by Pinasco and Zalduendo [17].

Note that spaces of integrable functions with respect to invariant measures over infinite-dimensional groups have been widely applied in stochastic processes [2,3], as well as in other areas.

This paper presents the following results. In Theorem 3.2, we describe an orthogonal basis in the Hardy space $H^2_\chi$ indexed by means of Yang diagrams, consisting of $\chi$-essentially bounded functions. Using this basis, in Theorem 4.2 the reproducing kernel of $H^2_\chi$ is calculated. It also allows us to define an antilinear isometric isomorphism $\mathcal{J}$ between $H^2_\chi$ and the symmetric Fock space $\Gamma$ generated by $E$. This isomorphism equips $H^2_\chi$ with a suitable infinite-dimensional analytic structure. By means of $\mathcal{J}$, we establish in Theorem 6.2 an integral formula for Hilbert–Schmidt analytic extensions of functions from $H^2_\chi$ on the open unit ball $B \subset E$. The radial boundary values of these analytic extensions are described in Theorem 7.1.

2. Background on Invariant Measure

Let $U(m)$ ($m \in \mathbb{N}$) be the group of unitary $(m \times m)$-matrices. We endow $U(\infty) = \bigcup U(m)$ with the inductive topology under every continuous inclusion $U(m) \ni u \mapsto \rho(u)\in U(\infty)$ which assigns to any $u_m \in U(m)$ the matrix $\begin{bmatrix} u_m & 0 \\ 0 & 1 \end{bmatrix} \in U(\infty)$. The right action over $U(\infty)$ is defined via

$$u.g = w^{-1}uw, \quad u \in U(\infty), \quad g = (v, w) \in U^2(\infty) \quad (2.1)$$

(the right action over $U(m)$ is defined similarly with $u \in U(m)$ and $g = (v, w) \in U^2(m)$ where $U^2(m) := U(m) \times U(m)$).

Following [12,13], every $u_m \in U(m)$ with $m > 1$ can be written as $u_m = \begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix}$ so that $z_{m-1}$ is a $(m-1) \times (m-1)$-matrix and $t \in \mathbb{C}$. It was proven that the Livšic-type mapping (which is not a group homomorphism)

$$\pi^m_{m-1}: u_m \longmapsto u_{m-1} := \begin{cases} z_{m-1} - [a(1 + t)^{-1}b] : t \neq -1 \\ z_{m-1} : t = -1 \end{cases} \quad (2.2)$$

from $U(m)$ onto $U(m-1)$ is Borel and surjective.

Consider the projective limit $\mathfrak{U} = \varprojlim U(m)$ taken with respect to $\pi^m_{m-1}$. The embedding $\rho: U(\infty) \ni u \mapsto \mathfrak{U}$ assigns to every $u_m \in U(m)$ the stabilized sequence $u = (u_k)_{k \in \mathbb{N}}$ (see [13, n.4]) so that
\[ \rho : U(m) \ni u_m \mapsto (u_k) \in \mathfrak{U}, \quad u_k = \begin{cases} \pi^m_k(u_m) : k < m, \\ u_m : k = m, \\ \left[ \begin{array}{c} u_m \\ 0 \\ 0 \\ 1 \end{array} \right] : k > m \end{cases} \tag{2.3} \]

where the projections \( \pi^m_k : \mathfrak{U} \ni u \mapsto u_m \in U(m) \) such that \( \pi^m_{m-1} \circ \pi^m_k = \pi^m_{m-1} \) are surjective and \( \pi^m_k := \pi^m_{k+1} \circ \cdots \circ \pi^m_{m-1} \) for \( k < m \). Using (2.1), the right action of \( U^2(\infty) \) over \( \mathfrak{U} \) can be defined as

\[ \pi^m(u.g) = w^{-1} \pi^m(u)v, \quad u \in \mathfrak{U} \tag{2.4} \]

where \( m \) is so large that \( g = (v, w) \in U^2(m) \) (see [13, Def 4.5]).

We endow every group \( U(m) \) with the probability Haar measure \( \chi_m \). It is known [12, Thm 1.6] that the pushforward of \( \chi_m \) to \( U(m-1) \) under \( \pi^m_{m-1} \) is the probability Haar measure \( \chi_{m-1} \) on \( U(m) \). Let \( U'(m) \) be the subset in \( U(m) \) of matrices which do not have \(-1\) as an eigenvalue. Then \( U'(m) \) is open in \( U(m) \) and \( U(m) \setminus U'(m) \) is \( \chi_m \)-negligible. Moreover, the restriction \( \pi^m_{m-1} : U'(m) \to U'(m-1) \) is continuous and surjective [13, Lem. 3.11].

Following [13, Lem. 4.8], [12, n.3.1], via of the Kolmogorov consistency theorem we uniquely define on \( \mathfrak{U} \) the probability measure \( \chi \) which is the projective limit under the mapping (2.2), i.e., we put

\[ \chi = \lim \chi_m \quad \text{with} \quad \chi_m = \chi \circ \pi^m_{m-1} \quad \text{for all} \quad m \in \mathbb{N}. \tag{2.5} \]

If \( \mathfrak{U}' = \lim U'(m) \) is the projective limit with respect to \( \pi^m_{m-1} |_{U'(m)} \) then \( \mathfrak{U} \setminus \mathfrak{U}' \)

is \( \chi \)-negligible, because \( \chi_m \) is zero on \( U(m) \setminus U'(m) \) for any \( m \).

A complex-valued function on \( \mathfrak{U} \) is called cylindrical if it has the form \( f = f_m \circ \pi_m \) for a certain \( m \in \mathbb{N} \) and a complex function \( f_m \) on \( U(m) \) [13, Def. 4.5]. By \( L^\infty_\chi \) we denote the closed linear hull of all cylindrical \( \chi \)-essentially bounded Borel functions endowed with the norm \( \| f \|_{L^\infty_\chi} = \esssup_{u \in \mathfrak{U}} | f(u) | \).

The measure (2.5) is a probability measure and is \( U^2(\infty) \)-invariant under the right actions (2.4) over \( \mathfrak{U} \) [12, Prop. 3.2]. Moreover, this measure is Radon so that

\[ \int_{\mathfrak{U}} f(u.g) \, d\chi(u) = \int_{\mathfrak{U}} f(u) \, d\chi(u), \quad g \in U^2(\infty), \quad f \in L^\infty_\chi \tag{2.6} \]

and it satisfies the property: \( (\chi \circ \pi^m_{m-1})(K) = \chi_m(K) \) for any compact set \( K \) in \( U(m) \) [11, Lem. 1]. Using the invariance property (2.6) and the Fubini theorem (see [11, Lem. 2]), we obtain

\[ \int_{\mathfrak{U}} f \, d\chi = \int_{\mathfrak{U}} d\chi(u) \int_{U^2(m)} f(u.g) \, d(\chi_m \otimes \chi_m)(g), \tag{2.7} \]
\[ \int_{\mathfrak{U}} f \, d\chi = \frac{1}{2\pi} \int_{\mathfrak{U}} d\chi(u) \int_{-\pi}^{\pi} f(\exp(i\theta)u) \, d\theta \tag{2.8} \]
for all $f \in L^\infty_X$. The closed linear hull of cylindrical complex functions endowed with the norm $\|f\|_{L^2_X} = \left( \int_X |f|^2 \, d\chi \right)^{1/2}$ is denoted by $L^2_X$. It is clear that $L^\infty_X \ni L^2_X$ and $\|f\|_{L^2_X} \leq \|f\|_{L^\infty_X}$ for all $f \in L^\infty_X$.

3. Hardy Spaces

Throughout the paper $E$ is a separable complex Hilbert space with an orthonormal basis $\{e_k : k \in \mathbb{N}\}$, scalar product $\langle \cdot \mid \cdot \rangle$ and norm $\| \cdot \| = \langle \cdot \mid \cdot \rangle^{1/2}$. So, for any element $x \in E$ the following Fourier decomposition holds,

$$x = \sum e_k \hat{x}_k, \quad \hat{x}_k = \langle x \mid e_k \rangle. \quad (3.1)$$

In what follows, let $B = \{x \in E : \|x\| < 1\}$ and $S = \{x \in E : \|x\| = 1\}$.

Let $E^\otimes n$ be the complete $n$th tensor power of $E$ endowed with the scalar product and norm

$$\langle \psi \mid \phi \rangle = \langle x_1 \mid y_1 \rangle \cdots \langle x_n \mid y_n \rangle, \quad \|\psi\| = \langle \psi \mid \psi \rangle^{1/2}$$

for all $\psi = x_1 \otimes \cdots \otimes x_n$, $\phi = y_1 \otimes \cdots \otimes y_n \in E^\otimes n$ with $x_i, y_i \in E$ ($i = 1, \ldots, n$). As $\sigma : \{1, \ldots, n\} \mapsto \{\sigma(1), \ldots, \sigma(n)\}$ runs through all $n$-elements permutations, the symmetric complete $n$th tensor power $E^\otimes n$ is defined to be a codomain of the orthogonal projector

$$E^\otimes n \ni \psi \longmapsto x_1 \otimes \cdots \otimes x_n := \frac{1}{n!} \sum_{\sigma} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \in E^\otimes n.$$ 

Note that $x^\otimes n = x \otimes \cdots \otimes x = x \otimes \cdots \otimes x = x^\otimes n$. Put $E^\otimes 0 = E^\otimes 0 = \mathbb{C}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$ be a partition of an integer $n \in \mathbb{N}$ with $m \leq n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$, i.e., $|\lambda| = n$ where $|\lambda| := \lambda_1 + \cdots + \lambda_m$. We identify partitions with Young diagrams. By $\ell(\lambda) = m$ we denote the length of $\lambda$ defined as the number of rows in $\lambda$. Let $\mathbb{Y}$ denote all Young diagrams and $\mathbb{Y}_n := \{\lambda \in \mathbb{Y} : |\lambda| = n\}$. Assume that $\mathbb{Y}$ includes the empty partition $\emptyset = (0, 0, \ldots)$.

An orthogonal basis in $E^\otimes n$ is formed by the system of symmetric tensor products (see e.g. [1, Sec. 2.2.2])

$$e_{\otimes \mathbb{Y}_n} = \bigcup_{\lambda \in \mathbb{Y}_n} \{e_{\otimes \lambda} := e_{i_1}^\otimes \lambda_1 \otimes \cdots \otimes e_{i_m}^\otimes \lambda_m : i \in \mathbb{N}^m, m = \ell(\lambda)\}, \quad e_{\otimes \emptyset} = 1$$

where $\mathbb{N}^m := \{i = (i_1, \ldots, i_m) \in \mathbb{N}^m : i_j \neq i_k, \forall j \neq k\}$. As is well known,

$$\|e_{i}^\otimes \lambda\|^2 = \frac{\lambda!}{|\lambda|!}, \quad \lambda := \lambda_1! \cdots \lambda_m!.$$ (3.2)

In what follows, we will use the fact that for every $\psi \in E^\otimes n$ one can uniquely define the so-called Hilbert–Schmidt $n$-homogenous polynomial

$$\psi^*(x) := \langle x^\otimes n \mid \psi \rangle, \quad x \in E.$$
In fact, the polarization formula for symmetric tensor products (see [8, 1.5])

$$z_1 \odot \cdots \odot z_n = \frac{1}{2^n n!} \sum_{\theta_1, \ldots, \theta_n = \pm 1} \theta_1 \cdots \theta_n x^{\otimes n}, \quad x = \sum_{k=1}^n \theta_k z_k \quad (3.3)$$

$(z_1, \ldots, z_n \in E)$ implies that the $n$-homogenous polynomial $\langle x^{\otimes n} \mid \psi \rangle$ is uniquely determines $\psi$, because the set of all $z_1 \odot \cdots \odot z_n$ is total in $E^{\otimes n}$.

Using the embedding (2.3), we define the $E$-valued mapping

$$\zeta: \mathcal{U} \ni u \longmapsto \rho^{-1}(u) e_1$$

which do not depend on the choice of $e_1$ in

$$S(\infty) := \{ \zeta(u) : u \in \mathcal{U} \} = \bigcup \{ S(m) : m \in \mathbb{N} \}$$

where $S(m)$ is the $m$-dimensional unit sphere. In fact, for each stabilized sequence $u = (u_k) \in \mathcal{U}$ there exists an index $m$ such that $\rho^{-1}(u)e_1 = u_k e_1$ belongs to $S(m)$ for all $k \geq m$. On the other hand, for each $e \in S(k)$ there exists $v \in U(k)$ such that $ve = e_1$. Defining $u,g \in \mathcal{U}$ with $g = (1,v) \in U^2(k)$ by means of (2.3)–(2.4), we have $\rho^{-1}(u,g)e = \pi_k(u,g)e = \pi_k(e_1) = \rho^{-1}(u)e_1$.

Consider the following system of cylindrical Borel functions

$$\varepsilon_k(u) := \langle \zeta(u) \mid e_k \rangle, \quad k \in \mathbb{N}$$

where $\varepsilon_k := e_k^* \circ \zeta$. Using $\zeta$, we may define the $E^{\otimes n}$-valued Borel mapping

$$\zeta^{\otimes n}: \mathcal{U} \ni u \longmapsto \zeta(u) \otimes \cdots \otimes \zeta(u)$$

The following assertion, which is a consequence of the polarization formula (3.3), is proved in [11, Lem. 3].

**Lemma 3.1.** The equality $S(\infty) = \{ \zeta(u) : u \in \mathcal{U} \}$ holds. As a consequence, to every $\psi \in E^{\otimes n}$ there uniquely corresponds the function in $L^\infty$

$$\psi_*(u) := \langle \zeta^{\otimes n}(u) \mid \psi \rangle, \quad u \in \mathcal{U}$$

given by continuous restriction to $\mathcal{U}$. In particular, to every $e^{\otimes \lambda}_i \in E^{\otimes Y_n}$ there corresponds in $L^\infty$ the cylindrical function in the variable $u \in \mathcal{U}$,

$$\varepsilon^\lambda(u) := \langle \zeta^{\otimes n}(u) \mid e^{\otimes \lambda}_i \rangle = \prod_{k=1}^{\ell(\lambda)} \langle \zeta(u) \mid e_{i_k} \rangle^{\lambda_k}. \quad (3.4)$$

Lemma 3.1 straightforwardly implies that the system $e^{\otimes Y}_i := \bigcup e^{\otimes Y_n}_i$ of tensor products $e^{\otimes \lambda}_i = e^{\otimes \lambda_1}_i \cdots e^{\otimes \lambda_m}_i$, indexed by $\lambda = (\lambda_1, \ldots, \lambda_m) \in Y$ and $i = (i_1, \ldots, i_m) \in \mathbb{N}_m^m$ with $m = \ell(\lambda)$, uniquely defines the appropriate system

$$e^{\otimes Y}_i := \prod_{\lambda \in Y} \{ \varepsilon^{\lambda}_i := \varepsilon^{\lambda_1}_i \cdots \varepsilon^{\lambda_m}_i : i \in \mathbb{N}_m^m, \ m = \ell(\lambda) \}, \quad \varepsilon^{\otimes \lambda}_i \equiv 1,$$

of $\chi$-essentially bounded cylindrical functions in the variable $u \in \mathcal{U}$ that possess continuous restrictions to $\mathcal{U}'$. 
Theorem 3.2. For any $\iota \in \mathbb{N}_+^m$ and $\psi, \phi \in E_1^{\otimes n}$, the following equality holds,
\[
\left( n + m - 1 \right) \int_\Omega \phi_\zeta \tilde{\psi}_\zeta d\chi = \langle \psi | \phi \rangle. \tag{3.5}
\]
As a consequence, given $(\lambda, \iota) \in \mathbb{Y} \times \mathbb{N}_+^m$ with $m = \ell(\lambda)$, the system $\varepsilon^\lambda_1$ of functions $\varepsilon^\lambda_1$ is orthogonal in the space $L^2_\chi$ and
\[
\| \varepsilon^\lambda_1 \|_{L^2_\chi} = \left( \frac{(m - 1)! \lambda!}{(m - \lambda + 1)!} \right)^{1/2}. \tag{3.6}
\]
Proof. Let $E_1$ with $\iota = (\iota_1, \ldots, \iota_m) \in \mathbb{N}_+^m$ be the $m$-dimensional subspace in $E$ spanned by $\{e_{\iota_1}, \ldots, e_{\iota_m}\}$ and $U(\iota)$ be the unitary subgroup of $U(\infty)$ acting in $E_1$. The symbol $E^{\otimes n}_1$ means the $n$th symmetric tensor power of $E_1$. Briefly denote $\psi_1[v\zeta(u)] := \langle ([v\rho^{-1}(u)]e_1)^{\otimes n} | \psi \rangle$ with $\psi \in E^{\otimes n}_1$ for all $v \in U(\iota)$ and $u \in \Omega$. Using (2.7) with $U(\iota)$ instead of $U(m)$, we have
\[
\int_\Omega \phi_\zeta \tilde{\psi}_\zeta d\chi = \int_\Omega d\chi(u) \int_{U(\iota)} \phi_\zeta[v\zeta(u)] \cdot \tilde{\psi}_\zeta[v\zeta(u)] d\chi_1(v) \tag{3.7}
\]
for all $\psi, \phi \in E^{\otimes n}_1$. It is clear that
\[
\left| \int_{U(\iota)} \phi_\zeta \tilde{\psi}_\zeta d\chi_1 \right| \leq \sup_{v \in U(\iota)} \left| \phi_\zeta[v\zeta(u)] \cdot \tilde{\psi}_\zeta[v\zeta(u)] \right| \leq \| \phi \| \| \psi \|
\]
for all $u \in \Omega$. Hence, the corresponding sesquilinear form in (3.7) is continuous on $E^{\otimes n}_1$. Thus, there exists a linear bounded operator $A$ over $E^{\otimes n}_1$ such that
\[
\langle A\psi | \phi \rangle = \int_{U(\iota)} \phi_\zeta \tilde{\psi}_\zeta d\chi_1.
\]
Next we show that $A$ commutes with all operators $w^{\otimes n} \in \mathcal{L}(E^{\otimes n}_1)$ with $w \in U(\iota)$ acting as $w^{\otimes n}, x^{\otimes n} = (wx)^{\otimes n}$, $(x \in E_1)$. Invariance properties (2.6) of $\chi_1$ under the right action (2.4) yield
\[
\langle (A \circ w^{\otimes n})\psi | \phi \rangle
\]
\[
= \int_{U(\iota)} \langle [v\zeta(u)]^{\otimes n} | \phi \rangle \overline{\langle [v\zeta(u)]^{\otimes n} | w^{\otimes n}\psi \rangle} d\chi_1(v)
\]
\[
= \int_{U(\iota)} \langle [w^{-1}v\zeta(u)]^{\otimes n} | (w^{-1})^{\otimes n}\phi \rangle \overline{\langle [w^{-1}v\zeta(u)]^{\otimes n} | \psi \rangle} d\chi_1(v)
\]
\[
= \int_{U(\iota)} \langle [v\zeta(u)]^{\otimes n} | (w^{-1})^{\otimes n}\phi \rangle \overline{\langle [v\zeta(u)]^{\otimes n} | \psi \rangle} d\chi_1(v)
\]
\[
= \langle A\psi | (w^{-1})^{\otimes n}\phi \rangle = \langle (w^{\otimes n} \circ A)\psi | \phi \rangle,
\]
where $w^{-1} \in U(\iota)$ is the hermitian adjoint matrix of $w$. Hence, the equality
\[
A \circ w^{\otimes n} = w^{\otimes n} \circ A, \quad w \in U(\iota) \tag{3.8}
\]
holds. Let us check that the operator $A$, satisfying the condition (3.8), is proportional to the identity operator on $E^\otimes_i$. To this end we form the $n$th tensor power of the unitary group $U(i)$,

$$[U(i)]^\otimes_n = \{ w^\otimes_n \in \mathcal{L}(E^\otimes_i) : w \in U(i) \}, \quad [U(i)]^\otimes_0 = 1.$$ 

Clearly, $[U(i)]^\otimes_n$ is a unitary group over $E^\otimes_i$. Let us check that the corresponding unitary representation

$$U(i) \ni w \mapsto w^\otimes_n \in \mathcal{L}(E^\otimes_i)$$

is irreducible. This means that there is no subspace in $E^\otimes_i$ other than $\{0\}$ and the whole space which is invariant under the action of $[U(i)]^\otimes_n$.

Suppose, on the contrary, that there is an element $\psi \in E^\otimes_i$ such that the equality $\langle (w\rho^{-1}(u))\epsilon_1^\otimes_n \mid \psi \rangle = 0$ holds for all $w \in U(i)$ and $u \in U(\infty)$. By Lemma 3.1 the elements $w\rho^{-1}(u)$ act transitively on $S(\infty)$. Hence, by $n$-homogeneity, we obtain $\langle x^\otimes_n \mid \psi \rangle = 0$ for all $x \in E_i$. Applying the polarization formula (3.3), we get $\psi = 0$. Hence, (3.9) is irreducible.

Thus, we can apply to (3.9) the Schur lemma [10, Thm 21.30]: a non-zero matrix which commutes with all matrices of an irreducible representation is a constant multiple of the unit matrix. As a result, we obtain that the zero matrix which commutes with all matrices of an irreducible representation

$$\psi \equiv 0.$$ 

By Lemma 3.1 the elements $w\rho^{-1}(u)$ act transitively on $S(\infty)$. Hence, by $n$-homogeneity, we obtain $\langle x^\otimes_n \mid \psi \rangle = 0$ for all $x \in E_i$. Applying the polarization formula (3.3), we get $\psi = 0$. Hence, (3.9) is irreducible.

Thus, we can apply to (3.9) the Schur lemma [10, Thm 21.30]: a non-zero matrix which commutes with all matrices of an irreducible representation is a constant multiple of the unit matrix. As a result, we obtain that the operator $A$, satisfying (3.8), is proportional to the identity operator on $E^\otimes_i$ i.e. $A = \alpha_{(n,i)} \mathbb{1}_{E^\otimes_i}$ with a constant $\alpha_{(n,i)} > 0$. It follows that

$$\int_{U(i)} \phi^\dag \bar{\psi}^\dag \, d\chi_i = \alpha_{(n,i)} \langle \psi \mid \phi \rangle, \quad \phi, \psi \in E^\otimes_i.$$ 

(3.10)

In particular, the subsystem of cylindrical functions $\varepsilon_1^\lambda$ with a fixed $i \in \mathbb{N}^m_\ast$ is orthogonal in $L^2_{\lambda_i}$, because the corresponding system of tensor products $\epsilon_i^\otimes_\lambda$ indexed by $\lambda \in \mathbb{N}_n$ with $\ell(\lambda) = m$ forms an orthogonal basis in $E^\otimes_i$.

It remains to note that the set of all indices $i = (i_1, \ldots, i_m) \in \mathbb{N}_\ast^m$ with all $m = \ell(\lambda)$ is directed with respect to the set-theoretic embedding, i.e., for any $i, i'$ there exists $i''$ so that $i \cup i' \subset i''$. This fact and the above reasoning imply that the whole system $\varepsilon^\lambda_i$ is also orthogonal in $L^2_{\lambda_i}$.

Taking into account (3.2), we can choose $\phi_n = \psi_n = \varepsilon_1^\lambda \sqrt{n!/\lambda!}$ in (3.10). As a result, we obtain

$$\alpha_{(n,i)} = \frac{n!}{\lambda!} \int_{U(i)} |\varepsilon_1^\lambda|^2 \, d\chi_i = \frac{n!}{\lambda!} \left(\frac{\lambda!}{(n + m - 1)!}\right).$$

The well known formula [18, 1.4.9] for the unitary $m$-dimensional group gives

$$\int_{U(i)} |\varepsilon_1^\lambda|^2 \, d\chi_i = \frac{\lambda!(m - 1)!}{(n + m - 1)!}, \quad |\lambda| = n, \quad \ell(\lambda) = m.$$ 

Using the last two formulas, we arrive at the relation

$$\alpha_{(n,i)} = \frac{n!}{\lambda!} \int_{U(i)} |\varepsilon_1^\lambda|^2 \, d\chi_i = \frac{n!}{\lambda!} \frac{\lambda!(m - 1)!}{(n + m - 1)!} = \frac{n!(m - 1)!}{(n + m - 1)!}.$$ 

(3.11)
Combining (3.7) and (3.11), we get (3.5) and, as a consequence, (3.6).

**Definition 3.3.** By $H^2_\chi$ we denote the Hardy space over $U(\infty)$ defined as the $L^2$-closure of the complex linear span of the orthogonal system $\varepsilon^Y$.

Let the space $H^{2,n}_\chi$ be the $L^2$-closure of the complex linear span of the subsystem $\varepsilon^{\nu,n} := \{\varepsilon^\lambda_\nu : (\lambda, \nu) \in Y_n \times N_\ell(\lambda)\}$ with a fixed $n \in \mathbb{Z}_+$. 

**Corollary 3.4.** For any positive integers $n \neq k$ the orthogonality $H^{2,n}_\chi \perp H^{2,k}_\chi$ holds in $L^2_\chi$. As a consequence, the following orthogonal decomposition holds, 

$$H^2_\chi = C \oplus H^{2,1}_\chi \oplus H^{2,2}_\chi \oplus \cdots .$$

**Proof.** The orthogonal property $\varepsilon^\mu_\nu \perp \varepsilon^\lambda_\nu$ with $|\mu| \neq |\lambda|$ for any $\nu \in N_\ell(\mu)$ and $j \in N_\ell(\mu)$ follows from (2.8), since

$$\int_{U} \varepsilon^\mu_\nu \bar{\varepsilon}^\lambda_\nu \, d\chi = \int_{U} \varepsilon^\mu_\nu (\exp(\hat{\chi} \vartheta)u) \bar{\varepsilon}^\lambda_\nu (\exp(\hat{\chi} \vartheta)u) \, d\chi(u)$$

$$= \frac{1}{2\pi} \int_{U} \varepsilon^\mu_\nu \bar{\varepsilon}^\lambda_\nu \, d\chi \int_{-\pi}^{\pi} \exp (i(|\mu| - |\lambda|)\vartheta) \, d\vartheta = 0$$

for all $\lambda \in Y$ and $\mu \in Y \setminus \{0\}$. This yields $H^{2,|\mu|}_\chi \perp H^{2,|\lambda|}_\chi$ in the space $L^2_\chi$. \(\square\)

### 4. Reproducing Kernels

Let us construct the reproducing kernel of $H^2_\chi$. We refer to [19] for the basic definitions and properties of reproducing kernels.

**Lemma 4.1.** For every $u, v \in \mathfrak{U}$ there exists a $q \in \mathbb{N}$ such that the reproducing kernel of the subspace $H^{2,n}_\chi$ in $L^2_\chi$ has the form

$$\mathfrak{h}_n(v, u) = \sum_{m \leq q} \binom{n + m - 1}{n} \langle \zeta(v) | \zeta(u) \rangle^n = \sum_{(\lambda, \nu) \in Y_n \times N_\ell(\lambda)} \frac{\varepsilon^\lambda_\nu(v) \bar{\varepsilon}^\lambda_\nu(u)}{\|\varepsilon^\lambda_\nu\|_{L^2_\chi}^2}, \quad u, v \in \mathfrak{U}. \quad (4.1)$$

**Proof.** Note that $\mathfrak{h}_0 \equiv 1$. From (2.3) it follows that for each stabilized sequence $u \in \mathfrak{U}$ there exists $u_m \in U(m)$ with a certain $m = m(u)$ such that $u = \rho(u_m)$. So, the element $\zeta(u) = \rho^{-1}(u)\varepsilon_1$ is located on the $m$-dimensional sphere $S(m)$. It means that its Fourier series $\zeta(u) = \sum \varepsilon_k \varepsilon_k(u)$ has $m(u)$ terms. The tensor multinomial theorem yields the Fourier decomposition

$$[\zeta(u)]^\otimes n = \left( \sum \varepsilon_k \varepsilon_k(u) \right)^\otimes n = \sum_{(\lambda, \nu) \in Y_n \times N_\ell(\lambda)} \frac{n!}{\lambda!} \varepsilon^\otimes \lambda \varepsilon^\lambda_\nu(u)$$
in the space $E^{\otimes n}$. Using the formula (3.2), we obtain
\[
\langle \zeta(v) \mid \zeta(u) \rangle^n = \langle [\zeta(v)]^{\otimes n} \mid [\zeta(u)]^{\otimes n} \rangle
\]
\[
= \sum_{(\lambda,1) \in \mathbb{Y}_n \times N^{(\lambda)}} \frac{n!}{\lambda!} \langle e_{\lambda}^{\otimes \lambda} \mid e_{\lambda}^{\otimes \lambda} \rangle \varepsilon_{\lambda}^\lambda(v) \varepsilon_{\lambda}^\lambda(u)
\]
\[
= \sum_{(\lambda,1) \in \mathbb{Y}_n \times N^{(\lambda)}} \frac{\varepsilon_{\lambda}^\lambda(v) \varepsilon_{\lambda}^\lambda(u)}{\|e_{\lambda}^{\otimes \lambda}\|^2}
\]
where $\langle \zeta(v) \mid \zeta(u) \rangle$ is decomposed into $q = \min\{m(u), m(v)\}$ summands in virtue of orthogonality. Multiplying both sides by $\binom{n+m-1}{n}$ and summing over all $m \leq q$, we get (4.1). It follows that $\int_{\mathfrak{U}} h_n(v, u) \varepsilon_{\lambda}^\lambda(u) d\chi(u) = \varepsilon_{\lambda}^\lambda(v)$ for each $v \in \mathfrak{U}$. Via Theorem 3.1 the system $\varepsilon_{\lambda}^{\otimes n}$ of functions $\varepsilon_{\lambda}^\lambda$ forms an orthogonal basis in $H^2_x \otimes n$. So, the integral operator
\[
\int_{\mathfrak{U}} h_n(v, u) \psi_{\zeta}(u) d\chi(u) = \psi_{\zeta}(v), \quad \psi_{\zeta} \in H^2_x \otimes n
\]
acts identically on $H^2_x \otimes n$. Thus, the kernel (4.1) is reproducing in $H^2_x \otimes n$. \qed

Let us consider the complex-valued kernel
\[
h(z; v, u) = \prod_{m \leq \min\{m(u), m(v)\}} \left[1 - z \langle \zeta(v) \mid \zeta(u) \rangle\right]^{-m}, \quad u, v \in \mathfrak{U}, \quad |z| < 1
\]
where $m(u)$ is the number of terms in the Fourier series $\zeta(u) = \sum e_k e_k(u)$.

**Theorem 4.2.** The expansion $h(z; v, u) = \sum z^n h_n(v, u)$ holds for any $u, v \in \mathfrak{U}$ and $|z| < 1$. The kernel $h(1; v, u) = \sum h_n(v, u)$ is reproducing in $H^2_x$ in the sense that
\[
\int_{\mathfrak{U}} h(1; v, u) f(u) d\chi(u) = f(v), \quad f \in H^2_x, \quad v \in \mathfrak{U}.
\]

**Proof.** Let $q = \min\{m(u), m(v)\}$ and $m \leq q$. As is well known [18, 1.4.10],
\[
\left[1 - z \langle \zeta(v) \mid \zeta(u) \rangle\right]^{-m} = \sum_{n \in \mathbb{Z}_+} \binom{n+m-1}{n} \langle z \zeta(v) \mid \zeta(u) \rangle^n
\]
for all $|z| < 1$. By the Vandermonde identity, we have
\[
\binom{n+m-1}{n} \langle z \zeta(v) \mid \zeta(u) \rangle^n = \binom{r+k+p+l-2}{r+k} (z \zeta(v) \mid \zeta(u))^{r+k}
\]
\[
= \sum_{r=0}^{\infty} \binom{r+p-1}{r} \binom{n-r+l-1}{n-r} (z \zeta(v) \mid \zeta(u))^{r+k}
\]
for all \( n = r + k \) and \( m = p + l - 1 \). Applying recursively this identity to the series (4.4) with any \( m \leq q \) and using Lemma 4.1, we obtain
\[
b_h(z; v, u) = \prod_{m \leq q} \sum_{n \in \mathbb{Z}_+} \binom{n + m - 1}{n} \langle z \zeta(v) | \zeta(u) \rangle^n
\]
\[
= \sum_{n \in \mathbb{Z}_+} z^n \sum_{(\lambda, r) \in \mathbb{N}_+ \times \mathbb{N}_+} \frac{\varepsilon^\lambda_{r}(v) \varepsilon^\lambda_{r}(u)}{\lVert \varepsilon^\lambda_{r} \rVert^2_{L^2_\chi}} = \sum_{n \in \mathbb{Z}_+} z^n h_n(v, u).
\]
Hence, the required expansion holds. By (3.12) we have \( f = \sum_n f_n \) for any \( f \in H^2_\chi \) where \( f_n \in H^2_\chi, n \) is the orthogonal projection of \( f \). Observing that \( h_k(z; \cdot, u) \perp f_n(\cdot) \) with \( n \neq k \) holds in \( L^2_\chi \), we obtain
\[
\int_\mathcal{U} h(1; v, u) f(u) d\chi(u) = \sum \int_\mathcal{U} h_n(v, u) f_n(v) d\chi(u) = \sum f_n(v) = f(v)
\]
for all \( v \in \mathcal{U} \) and \( f \in H^2_\chi \). Hence, (4.3) is valid.

5. The Hilbert–Schmidt Analyticity

Recall (see e.g. [7]) that a function \( f \) on an open domain in a Banach space is said to be analytic if it is Gâteaux analytic and norm continuous. Similarly to [6,15], we say that \( f \) is Hilbert–Schmidt analytic if its Taylor coefficients are Hilbert–Schmidt polynomials. Now we describe a space \( H^2 \) of Hilbert–Schmidt analytic complex functions on the open ball \( B \).

The symmetric Fock space is defined to be the orthogonal sum
\[
\Gamma = \bigoplus_{n \in \mathbb{Z}_+} E^{\otimes n}, \quad \langle \psi | \phi \rangle = \sum_{n \in \mathbb{Z}_+} \langle \psi_n | \phi_n \rangle
\]
for all elements \( \psi = \bigoplus_n \psi_n, \phi = \bigoplus_n \phi_n \in \Gamma \) with \( \psi_n, \phi_n \in E^{\otimes n} \). The subset \( \{ x^{\otimes n} : x \in B \} \) is total in \( E^{\otimes n} \) by virtue of (3.3). This provides the total property of the subsets \( \{ (1 - x)^{-\otimes 1} : x \in B \} \) in \( \Gamma \) where we denote
\[
(1 - x)^{-\otimes 1} := \sum x^{\otimes n}, \quad x^{0} = 1.
\]
The \( \Gamma \)-valued function \( (1 - x)^{-\otimes 1} \) in the variable \( x \in B \) is analytic, since
\[
\lVert (1 - x)^{-\otimes 1} \rVert^2 = \sum \lVert x \rVert^{2n} = (1 - \lVert x \rVert^2)^{-1} < \infty. \tag{5.1}
\]

Let us define the Hilbert space of analytic complex functions in the variable \( x \in B \), associated with the Fock space \( \Gamma \), as follows
\[
H^2 = \{ \psi^*(x) = \langle (1 - x)^{-\otimes 1} | \psi \rangle : \psi \in \Gamma \}, \quad \lVert \psi^* \rVert_{H^2} := \lVert \psi \rVert
\]
for all \( x \in B \). This description is correct, because each function \( \psi^* \) in the variable \( x \in B \) is analytic by virtue of [9, Prop. 2.4.2], as a composition of the analytic \( \Gamma \)-valued function \( (1 - x)^{-\otimes 1} \) in the variable \( x \in B \) and the linear functional \( \langle \cdot | \psi \rangle \) on \( \Gamma \).
Similarly, we define the closed subspace in $H^2$ of $n$-homogenous Hilbert–Schmidt polynomials $\psi_n^*$ in the variable $x \in E$ as

$$H_n^2 = \{ \psi_n^*(x) = \langle x^\otimes n \mid \psi_n \rangle : \psi_n \in E^\otimes n \}.$$  

Differentiating at zero any function $\psi^* = \bigoplus \psi_n^* \in H^2$ with $\psi_n^* \in H_n^2$, we obtain that its Taylor coefficients at zero $(n!)^{-1} d_0^n \psi^* = \psi_n^*$ are Hilbert–Schmidt polynomials. Hence, every function from $H^2$ is Hilbert–Schmidt analytic. Clearly, the following orthogonal decomposition holds,

$$H^2 = \mathbb{C} \oplus H^2_1 \oplus H^2_2 \oplus \cdots.$$  \hspace{1cm} (5.2)

One can show that $\langle H_n^2 \rangle$ is a coherently sequence of polynomial ideals over $E$ in the meaning of \cite[Def. 1.1]{4}.

For each pair $(\lambda, \iota) \in \mathcal{Y}_n \times \mathbb{N}_*^{(\lambda)}$, we can uniquely assign the Hilbert–Schmidt $n$-homogenous polynomial

$$\hat{x}_i^\lambda := \langle x^\otimes n \mid e_i^\lambda \rangle, \quad x \in E,$$

defined via the Fourier coefficients $\hat{x}_k := e_k^\iota(x) = \langle x \mid e_k \rangle$ of an element $x \in E$. Taking into account (3.2), the tensor multinomial theorem yields the following orthogonal decompositions with respect to the basis $e_i^\mathcal{Y}$ in $\Gamma$,

$$(1 - x)^{-\otimes 1} = \sum_{(\lambda, \iota) \in \mathcal{Y} \times \mathbb{N}_*^{(\lambda)}} \frac{\hat{x}_i^\lambda e_i^\otimes \lambda}{\| e_i^\otimes \lambda \|^2}, \quad x \in B. \hspace{1cm} (5.3)$$

Hence, any function $\psi^* \in H^2$ has the orthogonal expansion

$$\psi^*(x) = \langle (1 - x)^{-\otimes 1} \mid \psi \rangle = \sum_{(\lambda, \iota) \in \mathcal{Y} \times \mathbb{N}_*^{(\lambda)}} \hat{\psi}_{(\lambda, \iota)} \hat{x}_i^\lambda, \quad x \in B$$

(5.4)

where $\hat{\psi}_{(\lambda, \iota)} := \langle e_i^\otimes \lambda \mid \psi \rangle \| e_i^\otimes \lambda \|^2$ are the Fourier coefficients of $\psi \in \Gamma$ with respect to the basis $e_i^\mathcal{Y}$ and, moreover, $\| \psi^* \|^2_{H^2} = \sum_{(\lambda, \iota)} |\langle e_i^\otimes \lambda \mid \psi \rangle|^2 \| e_i^\otimes \lambda \|^2$. Thus, $\| \psi^* \|^2_{H^2}$ is a Hilbert–Schmidt type norm on $H^2$.

6. Integral Formulas

The one-to-one correspondence $e_i^\otimes \lambda \leftrightarrow e_i^\lambda$ allows us to construct an antilinear isometric isomorphism $\mathcal{J} : \Gamma \rightarrow H^2_\mathcal{X}$ and its adjoint $\mathcal{J}^* : H^2_\mathcal{X} \rightarrow \Gamma$ by the following change of orthonormal bases

$$\mathcal{J} : \Gamma \ni e_i^\otimes \lambda \| e_i^\otimes \lambda \|^{-1} \mapsto e_i^\lambda \| e_i^\lambda \|^{-1} \in H^2_\mathcal{X}, \quad \lambda \in \mathcal{Y}, \quad \iota \in \mathbb{N}_*^{(\lambda)}.$$  

Clearly, $\mathcal{J}^* : e_i^\lambda \| e_i^\lambda \|^{-1} \mapsto e_i^\otimes \lambda \| e_i^\otimes \lambda \|^{-1}$, because $\langle \mathcal{J} e_i^\otimes \lambda \mid f \rangle_{L_2^\mathcal{X}} = \langle e_i^\otimes \lambda \mid \mathcal{J}^* f \rangle$ for any $f \in H^2_\mathcal{X}$. Using Theorem 3.2, for any element $\psi \in \Gamma$ with the Fourier coefficients $\hat{\psi}_{(\lambda, \iota)} = \langle e_i^\otimes \lambda \mid \psi \rangle \| e_i^\otimes \lambda \|^2$, we obtain
Theorem 6.2. Each Hilbert–Schmidt analytic function and its Taylor coefficients at zero have the form

\[ J_\psi = \sum_{(\lambda, t) \in Y \times N_{\ell}^\ell(\lambda)} \hat{\psi}_{(\lambda, t)} \| \varepsilon_i^{\ell_\lambda} \|_2^{2} \varepsilon_i^{\ell_\lambda} \text{ where } \| \varepsilon_i^{\ell_\lambda} \|_2^{2} = \frac{(\ell(\lambda) - 1 + |\lambda|)!}{(\ell(\lambda) - 1)!|\lambda|!}. \]

In particular, \( J_x = \sum \hat{x}_k \varepsilon_k \) for any elements \( x \in E \) with the Fourier coefficients \( \hat{x}_k = \langle x | \varepsilon_k \rangle. \) Moreover, \( \| J_x \|^2_{L^2_X} = \sum \| \hat{x}_k \|^2 = \| x \|^2. \)

In what follows, we assign to each \( x \in E \) the \( L^2_X \)-valued function

\[ x_\beta : U \ni u \rightarrow (Jx)(u). \]

Lemma 6.1. The function \( J(1 - x)^{-\otimes 1} = (1 - x_\beta)^{-1} \) in the variable \( u \in \mathcal{U} \) takes values in \( L^2_X \) for all \( x \in \mathcal{B}. \)

Proof. Applying \( J \) to the decompositions (3.1) and (5.3), we obtain

\[ J(1 - x)^{-\otimes 1} = \sum_{(\lambda, t) \in Y \times N_{\ell}^\ell(\lambda)} \frac{\hat{x}_k^{\ell_\lambda}}{\| \varepsilon_i^{\ell_\lambda} \|_2} \]

\[ = \sum_{n \in \mathbb{Z}_+} \left( \sum_{k \in \mathbb{N}} \hat{x}_k \varepsilon_k \right)^n = (1 - x_\beta)^{-1} \tag{6.1} \]

where the following orthogonal series with a fixed \( n \in \mathbb{N}, \)

\[ x_\beta^n = \left( \sum_{k \in \mathbb{N}} \hat{x}_k \varepsilon_k \right)^n = \sum_{(\lambda, t) \in Y_n \times N_{\ell}^\ell(\lambda)} \frac{\hat{x}_k^{\ell_\lambda}}{\| \varepsilon_i^{\ell_\lambda} \|_2}, \tag{6.2} \]

is convergent in \( L^2_X. \) Moreover, taking into account the orthogonality, we get

\[ \| (1 - x_\beta)^{-1} \|^2_{L^2_X} = \sum_{n \in \mathbb{Z}_+} \left( \sum_{k \in \mathbb{N}} |\hat{x}_k|^2 \right)^n = (1 - \| x \|^2)^{-1}. \]

Hence, the function \( (1 - x_\beta)^{-1} \) with \( x \in \mathcal{B} \) takes values in \( L^2_X. \) \( \square \)

Let \( f = \sum_n f_n \in H^2_X \) with \( f_n \in H^2_{X,n}. \) Then \( J f \in \Gamma \) and \( J^* f_n \in E^\otimes n. \)

Briefly denote \( \tilde{f} := (J^* f)^* \in H^2_n \) and \( \tilde{f}_n := (J^* f_n)^* \in H^2. \) Thus,

\[ \tilde{f}(x) = \langle (1 - x)^{-\otimes 1} | J^* f \rangle, \quad x \in \mathcal{B}, \]

\[ \tilde{f}_n(x) = \langle x^\otimes n | J^* f_n \rangle, \quad x \in \mathcal{E}. \]

Theorem 6.2. Each Hilbert–Schmidt analytic function \( \tilde{f} \in H^2 \) has the integral representation

\[ \tilde{f}(x) = \int_{\mathcal{U}} f \frac{d\chi}{1 - x_\beta}, \quad x \in \mathcal{B} \tag{6.3} \]

and its Taylor coefficients at zero have the form

\[ \frac{d^n_0 \tilde{f}(x)}{n!} = \int_{\mathcal{U}} x^n_\beta f_n d\chi, \quad x \in \mathcal{E}. \tag{6.4} \]
The mapping \( f \mapsto \tilde{f} \) produces a linear isometry \( H_x^2 \simeq H^2 \).

**Proof.** Consider the Fourier decomposition of \( f \) with respect to the basis \( \varepsilon_{Y} \) and its \( J^* \)-image, respectively

\[
f = \sum_{(\lambda, s) \in Y \times N_s} \hat{f}_{(\lambda, s)} \varepsilon_{\lambda}, \quad J^* f = \sum_{(\lambda, s) \in Y \times N_s} \tilde{\hat{f}}_{(\lambda, s)} \| \varepsilon_{\lambda} \|^2 \| e_{\lambda} \|^2 \chi \varepsilon_{\lambda}
\]
where \( \hat{f}_{(\lambda, s)} = \| \varepsilon_{\lambda} \|^2 T \int f \varepsilon_{\lambda} d\chi \). Substituting \( \hat{f}_{(\lambda, s)} \) to \( \tilde{f} = (J^* f)^* \) and using the orthogonal property and the relations (5.3) and (6.1), we obtain

\[
\tilde{f}(x) = \sum_{(\lambda, s) \in Y \times N_s} \hat{f}_{(\lambda, s)} \hat{x} \lambda \langle \varepsilon_{\lambda}^* | \varepsilon_{\lambda} \rangle \| e_{\lambda} \|^2 \chi \| \varepsilon_{\lambda} \|^2 \| e_{\lambda} \|^2 \\
= \int_{K} \sum_{(\lambda, s) \in Y \times N_s} \hat{x} \lambda \varepsilon_{\lambda} \| \varepsilon_{\lambda} \|^2 \| e_{\lambda} \|^2 f d\chi = \int_{K} f d\chi.
\]

Hence, (6.3) holds. Using (6.2), we similarly obtain

\[
\tilde{f}_n(x) = \langle x^\otimes n | J^* f \rangle = \int_{K} x^n f d\chi.
\]

Taking into account (6.5) and the orthogonal decomposition (3.12), we get

\[
\tilde{f}(\alpha x) = \langle (1 - \alpha x)^{-\otimes 1} | J^* f \rangle = \sum \alpha^n \int_{K} x^n f d\chi, \quad |\alpha| \leq 1.
\]

Note that \( \tilde{f}(\alpha x) \) is analytic in \( \alpha \) for all \( x \in B \). Differentiating \( \tilde{f}(\alpha x) \) at \( \alpha = 0 \) and using the \( n \)-homogeneity of derivatives, we obtain

\[
\frac{d^n}{d\alpha^n} \sum \alpha^n \int_{K} x^n f d\chi \bigg|_{\alpha=0} = n! \int_{K} x^n f d\chi.
\]

Hence, the functions (6.4) coincide with the Taylor coefficients at zero of \( \tilde{f} \).

Finally, since the image of \( \varepsilon_{Y} \) under \( J^* \) coincides with \( e_{\otimes Y} \), the mapping \( H_x^2 \ni f \mapsto \tilde{f} \in H^2 \) is an isometry.

\[\square\]

### 7. Radial Boundary Values

Using (6.3), for each \( f = \sum_n f_n \in H_x^2 \) with \( f_n \in H_{x}^{2,n} \) we can rewrite (6.6) as

\[
\tilde{f}(rx) = \langle (1 - rx)^{-\otimes 1} | J^* f \rangle = \int_{K} \frac{f d\chi}{1 - rx^3}, \quad x \in K, \quad r \in [0,1)
\]
where \( K = \{ x \in E : \| x \| \leq 1 \} \).

**Theorem 7.1.** The integral transform \( \mathcal{C}_r : f \mapsto \mathcal{C}_r[f] \), defined as

\[
\mathcal{C}_r[f](x) := \int_{K} \frac{f d\chi}{1 - rx^3}, \quad x \in K, \quad r \in [0,1),
\]

is an isometry.
belongs to the space of bounded linear operators $\mathcal{L}(H^2, H^2)$. The radial boundary values of $C_r[f] \in H^2$ are equal to $\tilde{f} \in H^2$ in the following sense:

$$\lim_{r \nearrow 1} \|C_r[f] - \tilde{f}\|_{H^2} = 0. \quad (7.2)$$

Moreover, the following equality holds,

$$\|\tilde{f}\|_{H^2}^2 = \sup_{r \in [0, 1)} \|C_r[f]\|_{H^2}^2. \quad (7.3)$$

Proof. Theorem 6.2 and (7.1) imply the equality $C_r[f] = \sum r^n \tilde{f}_n$ for any $r \in [0, 1)$. By (5.2), we have $\tilde{f}_k \perp \tilde{f}_n$ as $n \neq k$ in $H^2$. It follows that

$$\|C_r[f]\|_{H^2}^2 = \left|\sum r^n \tilde{f}_n\right|_{H^2}^2 = \sum r^{2n} \|\tilde{f}_n\|_{H^2}^2 = \sum r^{2n} \|f_n\|_{L^2_\chi}^2,$$

since $\mathcal{J}^*$ acts isometrically from $H^{2,n}_\chi$ onto the space $E^{\ominus n}$ which is antilinear isometric to $H^2_n$ by definition. Similarly, we obtain that

$$\|C_r[f] - \tilde{f}\|_{H^2}^2 = \sum (r^{2n} - 1) \|f_n\|_{L^2_\chi}^2 \to 0, \quad r \to 1.$$

Moreover, the Cauchy–Schwarz inequality implies that

$$\|C_r[f]\|_{H^2}^2 \leq \frac{1}{(1 - r^2)^{1/2}} \left(\sum \|f_n\|_{L^2_\chi}^2\right)^{1/2} = \frac{\|\tilde{f}\|_{L^2_\chi}}{(1 - r^2)^{1/2}}$$

for all $f \in H^2_\chi$. Hence, the operator $C_r$ belongs to $\mathcal{L}(H^2_\chi, H^2)$ for all $r \in [0, 1)$.

Finally, the equalities

$$\sup_{r \in [0, 1)} \|C_r[f]\|_{H^2}^2 = \sup_{r \in [0, 1)} \sum r^{2n} \|\tilde{f}_n\|_{H^2}^2 = \sum \|\tilde{f}_n\|_{H^2}^2 = \|\tilde{f}\|_{H^2}^2$$

give the required formula (7.3). \hfill \Box

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