LOCAL THEORY FOR SPATIO-TEMPORAL CANARDS
AND DELAYED BIFURCATIONS
DANIELE AVITABILE *, MATHIEU DESROCHES †, ROMAIN VELTZ †, AND MARTIN WECHSELBERGER ‡

Abstract. We present a rigorous framework for the local analysis of canards and slow passages
through bifurcations in a wide class of infinite-dimensional dynamical systems with time-scale sep-
aration. The framework is applicable to models where an infinite-dimensional dynamical system for
the fast variables is coupled to a finite-dimensional dynamical system for slow variables. We prove the ex-
istence of centre-manifolds for generic models of this type, and study the reduced, finite-dimensional
dynamics near bifurcations of (possibly) patterned steady states in the layer problem. Theoretical
results are complemented with detailed examples and numerical simulations covering systems of
local- and nonlocal-reaction diffusion equations, neural field models, and delay-differential equations.
We provide analytical foundations for numerical observations recently reported in literature, such as
spatio-temporal canards and slow-passages through Hopf bifurcations in spatially-extended systems
subject to slow parameter variations. We also provide a theoretical analysis of slow passage through
a Turing bifurcation in local and nonlocal models.

1. Introduction. Many physical and biological systems consist of processes that
evolve on disparate time- and/or length-scales and the observed dynamics in such
systems reflect these multiple-scale features as well. Mathematical models of such
multiple-scale systems are considered singular perturbation problems with two-scale
problems as the most prominent.

In the context of ordinary differential equations (ODEs), singular perturbation
problems are usually discussed under the assumption that there exists a coordinate
system such that observed slow and fast dynamics are represented by corresponding
slow and fast variables globally, i.e., the system of ODEs under consideration is given
in the standard (fast) form

\[
\begin{align*}
\dot{u} &= F(u, v, \mu, \varepsilon) \\
\dot{v} &= \varepsilon G(u, v, \mu, \varepsilon),
\end{align*}
\]

(1.1)

where \((u, v) \in \mathbb{R}^n \times \mathbb{R}^m, F \text{ and } G\) are smooth functions on \(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{>0}\), and
0 < \(\varepsilon \ll 1\) is the timescale separation parameter. The geometric singular perturbation
theory (GSPT) to analyse such finite dimensional singular perturbation problems is
well established. It was pioneered by Neil Fenichel in the 1970s [39] and is based on the
notion of normal hyperbolicity† which refers to a spectral property of the equilibrium
set \(S\) of the layer problem (also known as the fast subsystem) obtained in the limit
\(\varepsilon \to 0\) in (1.1). In GSPT, this set \(S := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : F(u, v, \mu, 0) = 0\} \) is known
as the critical manifold since it is assumed to be an \(m\)-dimensional differentiable
manifold. Normal hyperbolicity refers then to the property that the (point) spectrum
of the critical manifold is bounded away from the imaginary axis (on every compact
subset of \(S\)).

Loss of normal hyperbolicity is a key feature in finite dimensional singular perturbation
problems for rhythm generation as observed, for example, in the famous
van der Pol relaxation oscillator. An important question in this context is how the

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*Department of Mathematics, Vrije Universiteit Amsterdam and MathNeuro Team, Inria Sophia Antipolis (d.avitabile@vu.nl).
†MathNeuro Team, Inria Sophia Antipolis.
‡School of Mathematics and Statistics, The University of Sydney.
†Precise definitions for this and other GSPT concepts can be found in the Appendix.
transition from an excitable to a relaxation oscillatory state occurs in such a system. The answer is (partially) given by the canard phenomenon which was discovered by French mathematicians who studied the van der Pol relaxation oscillator with constant forcing [12]. They showed an explosive growth of limit cycles from small Hopf-like to relaxation-type cycles in an exponentially small interval of the system parameter. This parameter-sensitive behaviour is hard to observe, and the corresponding solutions in phase space resemble (with a good portion of imagination) the shape of a ‘duck’ which explains the origin of the nomenclature. Note, this phenomenon is degenerate since it only exists in one-parameter families of slow-fast vector fields in $\mathbb{R}^2$. Furthermore, it is always associated with a nearby singular Hopf bifurcation, i.e. frequency and amplitude of the Hopf cycles depend on the singular perturbation parameter $\varepsilon \ll 1$.

A seminal work on van der Pol canards [12] was obtained by Eric Benoît, Jean-Louis Callot, Francine and Marc Diener through non-standard analysis methods and soon after similar results were reached using standard matched asymptotics techniques by Eckhaus [35]. A decade later, further results were obtained using geometric desingularization or blow-up by Dumortier and Roussarie [33], and soon after by Krupa and Szmolyan [56, 57, 58].

Fortunately, this degenerate situation does not occur in systems with two (or more) slow variables where canards are generic, i.e. their existence is insensitive to small parameter perturbations. Benoît [11] was the first to study generic canards in $\mathbb{R}^3$. He also observed how a certain class of generic canards (known as canards of folded node type) cause unexpected rotational properties of nearby solutions. Extending geometric singular perturbation theory to canard problems in $\mathbb{R}^3$, Szmolyan and Wechselberger [74] provided a detailed geometric study of generic canards. In particular, Wechselberger [83] then showed that rotational properties of folded node type canards are related to a complex local geometry of invariant manifolds near these canards and associated bifurcations of these canards. Coupling this local canard structure with a global return mechanism can explain complex oscillatory patterns known as mixed-mode oscillations (MMOs); see, e.g., [83, 18, 29]. This is closely related to canards of folded node and folded saddle-node type. Firing threshold manifolds or, more broadly, transient separatrices in slowly modulated excitable systems form another important class of applications [87, 86] which is a different type of canard mechanism which includes folded saddle canards.

A necessary (but not sufficient) condition for canards is the crossing of a real eigenvalue in the point spectrum of $S$ which leads geometrically to a folded critical manifold. Another important singular perturbation phenomenon is the delayed loss of stability through a Hopf bifurcation [65, 66, 46] associated with the crossing of a complex conjugate pair of eigenvalues in the spectrum of $S$. We note that this phenomenon refers to a Hopf bifurcation in the layer problem which is, in general, different to a singular Hopf bifurcation in the full system (1.1) with $0 < \varepsilon \ll 1$, as the two bifurcations are not necessarily related. A prime application of this phenomenon is associated with elliptic bursters in neurons, see for instance [52].

Canard theory has been extended to arbitrary finite dimensions in [84], i.e. dimensional restrictions on the slow variable subspace are not necessary to study these problems. Slow-fast theory for ODEs plays also an important role in the construction of heterogeneous (and possibly relative) equilibria in PDEs posed on the real line. This construction relies on a spatial-dynamic ODE formulation [70]: stationary states and travelling waves are identified with homo- or hetero-clinic connections of an ODE in the spatial variable, which may exhibit spatial-scale separation. These global orbits may display canard segments, and can be constructed using GSPT theory for ODEs...
A further interesting class of intermediate problems between ODEs and the ones considered here arise when the spatial-dynamical system obtained in travelling waves problems is itself infinite-dimensional, or does not have an obvious phase space. In this context, a GSPT may not be readily available, and recent contributions in this area have been provided by Hupkes and Sandstede for lattice equations [50, 49], and by Faye and Scheel for nonlocal equations with convolutions [38].

More generally, the literature on infinite-dimensional slow-fast dynamical systems is less developed than its ODE counterpart. The papers by Bates and Jones [7] and by Bates, Lu, and Zeng [9], include historical background and references to the state-of-the-art literature (as of the end of 1980s and 1990s, respectively) on invariant manifolds for infinite-dimensional systems, including in particular [25, 78, 8, 82, 77, 47]. In addition, Bates, Lou and Zeng [9] provided a seminal contribution for the persistence of invariant manifolds for problems posed on Banach spaces. The theory presented therein is general, albeit the derivation of GSPT for infinite-dimensional equations is problem-dependent (see, for instance, the one derived by Menon and Haller for the Maxwell-Bloch equations [62]).

The construction of orbits displaying canard segments or delayed bifurcations in the infinite-dimensional setting remains an open problem, with two important obstacles: (i) the loss of normal hyperbolicity, and (ii) the connection of slow and fast orbit segments with the view of obtaining a global, possibly periodic, orbit. The present paper addresses the former, and provides a general local theory for infinite-dimensional problems of the following type

\[
\begin{align*}
\dot{u} &= Lu + R(u, v, \mu, \varepsilon) := F(u, v, \mu, \varepsilon) \\
\dot{v} &= \varepsilon G(u, v, \mu, \varepsilon)
\end{align*}
\]

where the fast variables \( u \) belong to a Banach space \( X \), and the slow variables \( v \) to \( \mathbb{R}^m \), for some \( m \in \mathbb{N} \). The vector \( \mu \in \mathbb{R}^p, p \in \mathbb{N} \), refers to a set of (possible) additional control parameters.\(^2\) We say that (3.3) is a system of \( m \)-slow, \( \infty \)-fast differential equations.

As we discuss below, systems of the form (1.2) are prevalent in the limited literature currently available on canards and delayed bifurcations in infinite-dimensional dynamical systems. The first contribution to this topic was given in the early 1990s by Su [73], who studied a system of type (1.2) in which the fast variables \( u = (u_1, u_2) \) evolve according to the FitzHugh–Nagumo model, with diffusivity in the voltage \( u_1 \), and diffusionless recovery variable \( u_2 \), subject to a heterogeneous, slowly-increasing current with amplitude \( v \). For this setup, Su proved the existence of orbits displaying a delayed passage through a Hopf bifurcation. Two decades later, de Maesschalck, Kaper, and Popovic [28] proved existence of trajectories modelled around the canard phenomenon in the nonlinear example

\[
\varepsilon \partial_\mu u = \varepsilon^\mu \partial_x x + V(u, x, t, \varepsilon) u, \quad \mu > 0,
\]

using the specific scaling in conjunction with the method of lower and upper solutions. An intuition of this group was to bypass the difficulties associated to the loss of normal hyperbolicity, by hard-wiring a slow, non-monotonic evolution in the function \( V \), as opposed to prescribing a coupled dynamic for a slow variable \( v \), as we do here.

\(^2\)Precise definitions of the infinite dimensional dynamical system under study will be given in section 3.
In 2015, Tzou, Ward and Kolokolnikov [76] studied slow passages through a Hopf bifurcation in a reaction-diffusion PDE with a slowly-varying parameter, using asymptotic methods which Breina-Medina, Avitabile, Champneys and Ward concurrently adopted to investigate slow-fast orbits connecting patterned, metastable states in plant dynamics [15, 5]. In the same year Chen, Kolokolnikov, Tzou and Gai gave numerical evidence of slow passages through Turing bifurcations in an advection–reaction–diffusion equation with slowly-varying parameters subject to noise, with applications in vegetation patterns [22].

In 2016, Krupa and Touboul studied canards in a delay-differential equation (DDE) with slowly-varying parameters. This work differs from the other mentioned in this literature review, because it does not deal with a spatially-extended system, albeit the fast variable $u$ lives in a Banach space.

In 2017, Avitabile, Desroches and Knobloch introduced canards in spatially extended systems (termed spatio-temporal canards), producing numerical evidence of cycles and transients containing canard segments of folded-saddle and folded-node type. They studied a 2-slow $\infty$-fast system coupled to a neural-field equation posed on the real line and on the unit sphere. Analytical predictions for spatio-temporal canards were derived in a regime where interfacial dynamics provides a dimensionality reduction of the problem, amenable to standard GSPT analysis.

More recently, in 2018, two papers obtained results similar to the ones in Su [73] but on different models, and proposing different techniques: Bilinsky and Baer [13] used a WKB-expansion to study spatially-dependent buffer points (buffer curves) in heterogeneous reaction-diffusion equations; Kaper and Vo [54] found numerical evidence of delayed-Hopf bifurcations in several examples of heterogeneously, slowly-driven reaction-diffusion systems, and give a formal, accurate, asymptotic argument to compute the associated buffer curves.

The main contribution of this article is the derivation of a local theory for canards and slow passages through bifurcations in $m$-fast $\infty$-slow systems: not only is this a natural and necessary step for the construction of global orbits, but it is an achievable target for generic models. A key ingredient of our framework is a centre-manifold reduction of (1.2), which is an infinite-dimensional system. A general centre-manifold theory is available (see a recent literature review by Roberts [69]), but has not been used for problems with time-scale separation. We proceed systematically, as follows:

1. The layer problem $\dot{u} = F(u,v,\mu,0)$ is a differential equation in $X$, with parameters in $\mathbb{R}^{m+p}$. We assume that this system admits a bifurcation of a possibly heterogeneous steady state $u_*$ at $(v,\mu) = (v_*,\mu_*)$. Thus the linear operator $D_uF(u_*,v_*,\mu_*,0)$ has a non-empty centre spectrum and normal hyperbolicity fails.

2. Under the assumption that $D_uF$ has a spectral gap, and its centre spectrum has finite dimension $n \in \mathbb{N}$, we prove the existence of a $(\mu,\varepsilon)$-dependent centre manifold of the original problem (1.2).

3. We perform a centre-manifold reduction: in a neighbourhood of $(u_*,v_*,\mu_*,0) \in X \times \mathbb{R}^m$, the $m$-slow $\infty$-fast system reduces to an $m$-slow $n$-fast one. The latter inherits the slow-fast structure of the former, because our reduction acts trivially on the slow variables $v$.

4. We derive a normal form for the reduced system, which still exhibits loss of normal hyperbolicity at the origin. Existing ODE results can however be used to prove the local existence of canards and slow passages through bifurcations. Centre-manifold theory is at the core of the procedure described above, and our
treatment relies on existing tools on this topic. In particular, we adopt the notation and formalism developed by Vanderbauwhede and Iooss [78], and expanded in great detail in a book by Haragus and Iooss [44]. We build our results around the ones presented in the latter, to which we refer for further reading.

The theory presented here justifies rigorously numerical evidence presented in recent literature on the subject, and provides new results: we do not assume weak diffusivity in the linear operators, nor locality of the linear/nonlinear operators, and we provide theory and numerical examples for generic systems of integro-differential equations, local- and nonlocal-reaction diffusion problems, and DDEs. The centre-manifold reduction can be carried out for generic singularities, and this enables us to present theory for delayed-passage through Turing bifurcations, which had not been studied analytically before, and are specific to partial differential equations (PDEs).

We aim to present content in a format that is hopefully useful to readers working in both finite- and infinite-dimensional slow-fast problems, and this impacts on the material and style of the paper. For instance, centre-manifold reductions for infinite-dimensional systems rely on checking a series of technical assumptions on the layer problem, hence we took some measures to make the material self-contained, and to template our procedure for a relatively large class of problems. We believe that several applications should be covered by our treatment, or can be derived with minor modifications. For these reasons we structured the paper as follows: in section 2 we provide a suite of numerical examples that show canard phenomena and related slow passage through Hopf and Turing bifurcations in PDEs, DDEs and integro-differential equations; in section 3 we introduce the notation and functional-analytic setup used in the following sections; section 4 exposes steps 1–3 of the procedure described above, for generic \( m \)-slow, \( \infty \)-fast systems; section 5 describes step 4 in the procedure for fold, Hopf, and Turing bifurcations in generic systems; in section 6 we go through several cycles of steps 1–4, showing how they can be used in the concrete applications presented in section 2; we conclude in section 7.

2. Numerical examples. We provide numerical examples of system of \( \infty \)-fast, \( m \)-slow variables to which our analytical framework is applicable. In this section we discuss evidence obtained for slow passage through saddle-node, Hopf, and Turing bifurcations in various models. All computations involve numerical bifurcation analysis of steady states or periodic orbits of the layer problem, and time-stepping of the full problem. All computations in spatially extended systems are performed using a MATLAB suite for generic problems developed in [67] (see [2] for a recent tutorial). Computations for the DDE are performed using DDE-BIFTOOL [37].

2.1. Folded saddle and folded node canards in a neural field model. We begin with an integro-differential equation from mathematical neuroscience, namely a neural field model posed on a compact domain \( \Omega \subset \mathbb{R}^d \),

\[
\frac{\partial u}{\partial t} = -u + \int_{\Omega} w(\cdot, y) \theta(u(y, t), v_1) \, d\rho(y) \quad \text{in } \Omega \times \mathbb{R}_{>0},
\]

(2.1)

\[
\dot{v}_1 = \varepsilon (v_2 + c \int_{\Omega} \theta(u(y, t), v_1) \, d\rho(y)) \quad \text{in } \mathbb{R}_{>0},
\]

\[
\dot{v}_2 = \varepsilon (-v_1 + a + b \int_{\Omega} \theta(u(y, t), v_1) \, d\rho(y)) \quad \text{in } \mathbb{R}_{>0},
\]

where \( \theta \) model a firing rate function and \( w \) the synaptic connections (see [26] for a recent review on neural fields). The firing rate function is commonly modeled via a
Fig. 2.1. Examples of spatio-temporal canards in (2.1). (a) Periodic orbit containing a folded-saddle canard segment obtained with \( \theta = \theta_s, w(x, y) = \kappa_1 \exp(-|x - y|)(\kappa_2 + \kappa_3 \cos(y/\kappa_4)), \Omega = \mathbb{R}/2\mathbb{Z}, l = 100, a = 0.5, b = c = 0, \kappa_1 = 0.5, \kappa_2 = 1, \kappa_3 = 0.5, \mu = 50 \). Left: orbit in the \((A, v_1, v_2)\)-space, with \( A \) defined in (2.2), superimposed on the critical manifold \( S_0 \) of the fast subsystem associated to (2.1) in the variables \((A, v_1, v_2)\). Right: corresponding spatio-temporal solution, in which the level sets are shown for illustrative purposes. (b) Examples of an orbit containing a folded-node canard segment, obtained from the example in (a) by setting \( a = c = 1, b = 0 \). Adapted from [3].

sigmoidal or a Heaviside function

\[
\theta_s(u, h) = \frac{1}{1 + \exp(-\mu(u - h))}, \quad \theta_h(u, h) = H(u - h).
\]

In contrast to standard neural field models, the firing-rate threshold \( v_1 \) oscillates slowly and harmonically in time if \( b = c = 0 \), and its evolution is coupled to the fast neural field activity variable \( u \) if \( b \) or \( c \) are nonzero.

In [3], spatio-temporal canards were introduced and analysed using interfacial dynamics, valid in the case of Heaviside firing rate \( \theta_h \) posed on one-dimensional domains. If \( \Omega \subset \mathbb{R} \) and \( \theta = \theta_h \), one can construct even solutions \( u(x, t) = u(-x, t) \) from of their \( v_1 \)-level set on \( \mathbb{R}_{>0} \).

\[
(2.2) \quad A(t) = \{ x \in \mathbb{R}_{>0} : u(x, t) = v_1(t) \}.
\]

If \( A \) has a single connected component, then an exact evolution equation in the variable \((A, v_1, v_2) \in \mathbb{R}^3\) can be derived for the model and studied using GSPT for ODEs.
Solutions \((A(t), v_1(t), v_2(t))\), with canard segments correspond spatio-temporal canards of the original model.

In the present paper we study the case \(\theta = \theta_s\), on generic domains \(\Omega\) for which we report numerical simulations adapted from [3] in Figures 2.1 and 2.2: spatio-temporal canards of folded-saddle and folded-node type are predicted from the theory, are found numerically with a sigmoidal firing rate (Figure 2.1), and they also persist in higher spatial dimensions, where the theory does not apply (Figure 2.2). In the present paper we will introduce a rigorous treatment of this problem, valid for generic firing rates, kernels, and domains.

The functional setting for this problem will be the Banach space of continuous, real-valued functions defined on \(\Omega\). The theory developed in the following sections is also applicable to the Swift–Hohemberg equation subject to slow parameter variation, as studied in [40], albeit a natural functional setting for this problem requires Hilbert spaces. We shall give below several examples for problems on Hilbert spaces.

### 2.2. Slow passage through a Hopf bifurcations in PDEs and DDEs.

A second class of examples pertains slow passages through Hopf bifurcations in infinite-dimensional systems. A first numerical example is given by the following reaction-diffusion PDE with slowly-varying parameters:

\[
\begin{align*}
\partial_t u_1 &= d_1 \partial_x^2 u_1 + u_1 - u_1^3/3 - u_2 + v & \text{in } (0, 2\pi) \times \mathbb{R}_{>0}, \\
\partial_t u_2 &= u_1 + c - bu_2 & \text{in } (0, 2\pi) \times \mathbb{R}_{>0}, \\
\dot{v} &= \varepsilon & \text{in } \mathbb{R}_{>0},
\end{align*}
\]

subject to periodic boundary conditions

\[
u_1(0, t) = u_1(2\pi, t), \quad u_2(0, t) = u_2(2\pi, t), \quad t \in \mathbb{R}_{>0},
\]

where \(v\) plays the role of an external current. In this case we will show that the fast subsystem admits a homogeneous steady state undergoing a Hopf bifurcation, and we will apply our theory to conclude that system (2.3) displays slow passage through a Hopf bifurcation, of which we give numerical evidence in Figure 2.3. This example shows a slow-passage through a bifurcation of a homogeneous steady state, because...
the boundary conditions allow it. We highlight that the theory presented below is equally applicable to slow-passages through heterogeneous (patterned) steady states, of which the neutral field problem is an example.

The analytical treatment of this problem will be done on Sobolev spaces, and the presentation of this example assumes no prior knowledge of Centre Manifold reduction for PDEs, but only basic notions in functional analysis. The theory developed in the next sections is applicable to generic reaction–diffusion PDEs, and in particular to the examples considered in [54].

A separate analytical treatment is instead required for the slow-passage through a Hopf bifurcation in DDEs, for which we consider the model problem
\begin{align}
\dot{u}(t) &= v(t)u(t) - u^3(t) - u(t - \tau) + d & \text{ in } \mathbb{R}_{>0}, \\
\dot{v}(t) &= \varepsilon & \text{ in } \mathbb{R}_{>0}.
\end{align}

The fast subsystem of (2.4) is therefore the one-component delay-differential equation, in which the trivial equilibrium undergoes a first Hopf bifurcation at } \alpha = -3/4. The slow drift on } \alpha \text{ induces the slow passage presented in Figure 2.4.

2.3. Slow passage through Turing bifurcations in local- and nonlocal-reaction diffusion model. Since centre-manifold reductions are possible for generic bifurcations in PDE or integro-differential equations, we consider the case of a slow passage through a Turing bifurcation. In general, we will study problems of the following type
\begin{align}
\partial_t u &= D \partial_x^2 u + f(u, v) & \text{ in } \Omega \times \mathbb{R}_{>0}, \\
\dot{v} &= \varepsilon & \text{ in } \mathbb{R}_{>0},
\end{align}

subject to suitable boundary conditions, where } u \text{ denotes the concentrations of } q \text{ reactants, and } D = \text{diag}\{d_1, \ldots, d_q\} \text{ with } d_i > 0 \text{ for all } i \in \mathbb{N}_q, \text{ is a diffusion matrix.}
While developing a theory for generic models of this type is possible, we concentrate on two main examples, which can easily be generalised.

### 2.3.1. Nonlocal-reaction diffusion model

The first example under consideration is a model with nonlocal reaction, which has been studied for \( \varepsilon = 0 \) in the context of population dynamics [41, 81].

\[
\begin{align*}
\partial_t u &= d \partial_x^2 u + (v - b)u - u \int_\Omega w(\cdot - y)u(y, t) \, dy, \quad \text{in } \Omega \times \mathbb{R}_{>0} \\
\dot{v} &= \varepsilon, \quad \text{in } \mathbb{R}_{>0}
\end{align*}
\]

where \( \Omega = \mathbb{R}/2\mathbb{Z} \), and \( w \) is an interaction kernel. The model undergoes a Turing-like bifurcation of a homogeneous steady state, and we show numerical results of a slow-passage through this bifurcation (see Figure 2.5). We will derive theoretical results for this 1-component, nonlocal problem, and present a strategy that is easily adaptable to generic \( d \)-component reaction diffusion equation subject to Dirichlet, Neumann, or mixed homogeneous boundary conditions, under mild hypotheses on \( f \). This strategy can be used also in reaction-diffusion problems undergoing other bifurcations.

### 2.3.2. Schnakenberg model

As an example of a local-reaction diffusion equation, we consider the following Schnakenberg model, which undergoes a sub- or supercritical Turing bifurcation, as discussed in [16],

\[
\begin{align*}
\partial_t u_1 &= d_1 \partial_x^2 u_1 + vu_1^2 u_2 - (c + r)u_1 + hu_2, \quad x \in (0, l) \\
\partial_t u_2 &= d_2 \partial_x^2 u_2 - vu_1^2 u_2 + cu_1 - hu_2 + b, \quad x \in (0, l) \\
\dot{v} &= \varepsilon,
\end{align*}
\]

subject to Neuann boundary conditions

\[
\begin{align*}
\partial_x u_1(0, t) = \partial_x u_1(l, t) &= 0, \\
\partial_x u_1(0, t) = \partial_x u_1(l, t) &= 0.
\end{align*}
\]

Numerical results for this model are reported in Figure 2.6.
3. General setup for infinite dimensional problems.

3.1. Notation. We will use a unique symbol, $|\cdot|$, to denote the 2-norm in $\mathbb{R}^n$, for any integer $n$. We indicate by $X$, $Y$, $Z$ Banach spaces on the field $\mathbb{R}$ or $\mathbb{C}$, endowed with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$, respectively. Throughout this article, we will assume the continuous embeddings $Z \hookrightarrow Y \hookrightarrow X$.

For a given Banach space $X$, we denote by $C_\eta(\mathbb{R}, X)$ the following weighted space:

\begin{equation}
C_\eta(\mathbb{R}, X) = \{ u \in C^0(\mathbb{R}, X) : \| u \|_{C_\eta(\mathbb{R}, X)} = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \| u(t) \|_X < \infty \}.
\end{equation}

which is a Banach space equipped with the norm $\| \cdot \|_{C_\eta(\mathbb{R}, X)}$.

Furthermore, we indicate by $\mathcal{L}(Z, X)$ the set of bounded linear operators from $Z$ to $X$ and by $Z^*$, $X^*$ the dual spaces of $Z$ and $X$, respectively. The adjoint of a linear operator $L \in \mathcal{L}(Z, X)$ is the operator $L^* \in \mathcal{L}(X^*, Z^*)$ satisfying

\begin{equation}
\langle f, Lz \rangle_X = \langle L^* f, z \rangle_Z \quad \text{for all } z \in Z \text{ and for all } f \in X^*,
\end{equation}

where $\langle \cdot, \cdot \rangle_X : X^* \times X \to \mathbb{R}$ is the canonical duality pairing between $X$ and its dual $X^*$ (a similar definition holds for $\langle \cdot, \cdot \rangle_Z$).

The resolvent set of a linear operator $L : Z \subset X \to X$ is the set of complex numbers $\rho(L) = \{ \lambda \in \mathbb{C} : (\lambda \text{ id} - L) \text{ is invertible and } (\lambda \text{ id} - L)^{-1} \in \mathcal{L}(X) \}$, and the spectrum of $L$ is the complement of the resolvent set $\sigma(L) = \mathbb{C} \setminus \rho(L)$. A complex number $\lambda$ is called an eigenvalue of $L$ if $\ker(\lambda \text{ id} - L) \neq \{0\}$ and the non null elements of this kernel are called the eigenvectors of $L$.

The $k$th Fréchet derivative of a nonlinear operator $R \in C^k(Z, Y)$ will be denoted by $D^kR$. For the Taylor expansions of $R \in C^k(Z, Y)$ around $0 \in Z$ we will use the
Fig. 2.6. Slow passage through a supercritical (a) and subcritical (b) Turing bifurcation of the Schnakenberg model (2.7). In the subcritical setting, the slow passage is through a first Turing bifurcation of the homogeneous steady state, albeit the solution jumps to a stable branch emanating from a second Turing bifurcation, which restabilises after a fold and a secondary bifurcation. Parameters: $c = r = h = b = 1$, $d_2 = 10$, and (a) $d_1 = 0.26$, (b) $d_1 = 0.1497$.

compact notation

$$R(u) = \sum_{n=0}^{k} R_n(u^{(n)}) + o(||u||^2_2)$$

where $u^{(n)} = (u, \ldots, u) \in (Z)^n$ represents $u \in Z$ repeated $n$ times and the $n$-linear operator $R_n \in \mathcal{L}^n(Z, X) = \mathcal{L}(\prod_{j=1}^{n} Z, X) \cong \mathcal{L}(Z, \mathcal{L}(Z, \ldots, \mathcal{L}(Z, X)))$, with $n$ recursions [24, Lemma 1.6], acts as

$$R_n(u^{(n)}) = \frac{1}{n!} D^n R(0) u^{(n)}.$$

3.2. Systems of $m$-slow, $\infty$-fast differential equations. For ease of reference, we recall system (1.2), i.e. we fix $m, p \in \mathbb{N}$, and study differential equations posed on $X \times \mathbb{R}^m$, featuring $p + 1$ control parameters, of the following type:

$$\begin{align*}
\dot{u} &= Lu + R(u, v, \mu, \varepsilon) := F(u, v, \mu, \varepsilon) \\
\dot{v} &= \varepsilon G(u, v, \mu, \varepsilon)
\end{align*}$$

(3.3)
where $L$ is a linear operator while $R$, $F$, and $G$ are nonlinear operators. More precisely we assume $L: Z \to X$, $R: Z \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to Y$, $F: Z \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to X$, and $G: Z \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^m$. We have collected the $p + 1$ control parameters in the vector $(\mu, \varepsilon) \in \mathbb{R}^p \times \mathbb{R}$. This choice reflects the special role played by the parameter $\varepsilon$, which is taken to be small and determines a separation of time scales between the fast variables $u$ and the slow variables $v$. We are interested in applications where $X$ is infinite dimensional, therefore we say that (3.3) is a system of $m$-slow, $\infty$-fast differential equations.

It is sometimes advantageous to write the system (3.3) in a more compact form. To this end, we introduce the spaces $\tilde{X} = X \times \mathbb{R}^m$, $\tilde{Y} = Y \times \mathbb{R}^m$ and $\tilde{Z} = Z \times \mathbb{R}^m$, endowed with the corresponding natural norms and embeddings, and rewrite (3.3) as a differential equation on $\tilde{X}$

\[
\frac{d}{dt} \tilde{u} = \tilde{L} \tilde{u} + \tilde{R}(\tilde{u}, \mu, \varepsilon) := \tilde{F}(\tilde{u}, \mu, \varepsilon).
\]

where $\tilde{u} = (u, v)$ and

\[
\tilde{L} = \begin{pmatrix}
L & D_u R(0, 0, 0, 0) \\
0 & 0
\end{pmatrix}, \quad \tilde{R}(\tilde{u}, \mu, \varepsilon) = \begin{pmatrix}
R(u, v, \mu, \varepsilon) - D_u R(0, 0, 0, 0)v \\
\varepsilon G(u, v, \mu, \varepsilon)
\end{pmatrix}.
\]

**4. Centre manifold reduction.** We aim to perform a centre manifold reduction of (3.3) near bifurcations points of equilibria of the fast subsystem

\[
\dot{u} = F(u, v, \mu, 0) = Lu + R(u, v, \mu, 0)
\]
in which $(v, \mu) \in \mathbb{R}^m \times \mathbb{R}^p$ is fixed. As we shall see below, this in turn will provide a reduction of (3.3) to a set of $m$-slow, $n$-fast ODEs, which can subsequently be studied with standard tools for slow-fast ODEs.

**Definition 4.1 (Solution).** A solution of the differential equation (4.1) is a function $u: I \to Z \to X$ defined on an interval $I \subset \mathbb{R}$ such that $u \in C(I, Z) \cap C^1(I, X)$, and equality (4.1) holds in $X$ for all $t \in I$.

In order to perform the reduction of the fast subsystem, we employ the framework introduced by Haragus and Iooss in [79, 78, 44] We report below a set of hypotheses that are necessary to perform the centre manifold reduction. These hypotheses are adapted (or reproduced) from [44]. We begin with an assumption on the vector field (3.3), and on a subset of its equilibria.

**Hypothesis 4.2 (Vector field).** The operators $L$, $R$ in (3.3) satisfy the following conditions:

(i) $L \in \mathcal{L}(Z, X)$.

(ii) There exist an integer $k \geq 2$ and neighbourhoods $V_u \subset X$, $V_v \subset \mathbb{R}^m$, $V_\mu \subset \mathbb{R}^p$, $V_\varepsilon \subset \mathbb{R}$ of 0 such that $R \in C^k(V_u \times V_v \times V_\mu \times V_\varepsilon, Y)$ and

\[
R(0, 0, 0, 0) = 0, \quad D_u R(0, 0, 0, 0) = 0
\]

Hypothesis 4.2 implies that $0 \in X \times \mathbb{R}^m$ is an equilibrium of (3.4) for $(\mu, \varepsilon) = 0$, that is, $0 \in X$ is an equilibrium of the fast subsystem (4.1) for $(\mu, \varepsilon) = 0$. Note that Hypothesis 4.2.(ii) implies $F(0, 0, 0, 0) = 0$, while in general $G(0, 0, 0, 0) \neq 0$.

**Hypothesis 4.3 (Spectral decomposition).** The spectrum $\sigma(L)$ of $L$ can be written as follows

\[
\sigma(L) = \sigma_u \cup \sigma_e \cup \sigma_s,
\]
where

\[ \sigma_u := \{ \lambda \in \sigma(L) : \text{Re} \lambda > 0 \}, \quad \sigma_c := \{ \lambda \in \sigma(L) : \text{Re} \lambda = 0 \}, \quad \sigma_s := \{ \lambda \in \sigma(L) : \text{Re} \lambda < 0 \}, \]

satisfy the following assumptions

(i) The set \( \sigma_c \) consists of a finite number \( n > 0 \) of eigenvalues with finite algebraic multiplicities.

(ii) There exists a positive constant \( \gamma > 0 \) such that

\[
\inf_{\lambda \in \sigma_u} \text{Re} \lambda > \gamma, \quad \sup_{\lambda \in \sigma_s} \text{Re} \lambda < -\gamma
\]

If Hypotheses 4.2 and 4.3 hold, we can introduce the spectral projector on the centre subspace, \( P_c \in \mathcal{L}(X, Z) \) associated with \( \sigma_c \) and its complementary projector \( P_{su} = \text{id}_X - P_c \in \mathcal{L}(X) \cap \mathcal{L}(Y) \cap \mathcal{L}(Z) \). We therefore have the decomposition

\[ X = E_c \oplus X_{su}, \quad E_c = \text{range } P_c = \ker P_{su} \subset Z, \quad X_{su} = \text{range } P_{su} = \ker P_c \subset X, \]

we set \( Z_{su} = P_{su}Z \subset Z \) and \( Y_{su} = P_{su}Y \subset Y \), and we let \( L_c \in \mathcal{L}(E_c), \ L_{su} \in \mathcal{L}(Z_{su}, Y_{su}) \), be the restrictions of \( L \) to \( E_c \) and \( Z_{su} \), respectively. We are now ready to state one final hypothesis necessary for the centre manifold reduction:

**Hypothesis 4.4** (Linear equation in the hyperbolic subspace). For any \( \eta \in [0, \gamma] \), where \( \gamma \) is the spectral gap defined in Hypothesis 4.3, and any \( f \in C_\eta(\mathbb{R}, Y_{su}) \), the linear problem

\[
(4.2) \quad \dot{u}_{su} = L_{su}u_{su} + f(t),
\]

has a unique solution \( u_{su} = K_{su}f \in C_\eta(\mathbb{R}, Z_{su}) \). The linear map \( K_{su} \) belongs to \( \mathcal{L}(C_\eta(\mathbb{R}, Y_{su}), C_\eta(\mathbb{R}, Z_{su})) \), and there exists \( \Pi \in C([0, \gamma], \mathbb{R}) \) such that

\[
(4.3) \quad \|K_{su}\|_{\mathcal{L}(C_\eta(\mathbb{R}, Y_{su}), C_\eta(\mathbb{R}, Z_{su}))} \leq \Pi(\eta).
\]

Checking this hypothesis in applications may be nontrivial, as it requires to prove that there is a gain in regularity of solution \( u(t) \in Z \) to (4.2), when the forcing term \( f(t) \) belongs to \( Y \). Note that this forcing term mimics the effect of the nonlinearity and, as such, shares it range. In some cases, this hypothesis can be substituted by an estimate on the resolvent set of \( L \), and in the examples treated below we will give pointers to the literature for these cases.

We now proceed to derive a centre manifold existence result for (3.3) in a neighbourhood of \( 0 \in X \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^q \).

**Theorem 4.5** (Parameter-dependent centre manifold). Assume Hypotheses 4.2 to 4.4. Further, assume \( G \in C^k(V_u \times V_c \times V_p \times V_c, \mathbb{R}^m) \). Then the \( m \)-slow, \( \infty \)-fast system (3.3) admits a finite-dimensional centre manifold. More precisely, there exists a map \( \Psi \in C^k(E_c \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}, Z_{su}) \) with

\[
\Psi(0, 0, 0, 0) = 0, \quad D_\mu \Psi(0, 0, 0, 0) = 0,
\]

and a neighbourhood \( O_u \cap X_c \cap O_\mu \cap O_c \) of \( 0 \in Z \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R} \) such that, for all \( (\mu, \varepsilon) \in O_\mu \cap O_c \) the manifold

\[
(4.4) \quad M_c(\mu, \varepsilon) = \{(u_c, v) + (\Psi(u_c, v, \mu, \varepsilon), 0) : u_c \in E_c, \ v \in O_v \}
\]

has the following properties:
(i) \( M_c(\mu, \varepsilon) \) is locally invariant, that is, if \((u, v)\) is a solution of (3.3) such that \((u(0), v(0)) \in M_c(\mu, \varepsilon) \cap (O_u \times O_v)\) for all \(t \in [0, T]\), then \((u(t), v(t)) \in M_c(\mu, \varepsilon)\) for all \(t \in [0, T]\).

(ii) \( M_c(\mu, \varepsilon) \) contains the set of bounded solutions of (3.3) staying in \(O_u \times O_v\) for all \(t \in \mathbb{R}\), that is, if \((u, v)\) is a solution of (3.3) satisfying \((u(t), v(t)) \in O_u \times O_v\) for all \(t \in \mathbb{R}\), then \((u(0), v(0)) \in M_c(\mu, \varepsilon)\).

Proof. We use the compact formulation of the \(\infty\)-fast, \(m\)-slow system (3.3) as a differential equation on \(\tilde{X} = X \times \mathbb{R}^m\),

\[
\frac{d}{dt} \tilde{u} = \tilde{L}\tilde{u} + \tilde{R}((\tilde{u}, \mu, \varepsilon))
\]

where \(\tilde{u} = (u, v)\) and

\[
\tilde{L} = \begin{pmatrix}
L & D_v R(0, 0, 0, 0) \\
0 & 0
\end{pmatrix},
\tilde{R}(\tilde{u}, \mu, \varepsilon) = \begin{pmatrix}
R(u, v, \mu, \varepsilon) - D_u R(0, 0, 0, 0)v \\
\varepsilon G(u, v, \mu, \varepsilon)
\end{pmatrix}.
\]

We aim to show that, if Hypotheses 4.2 to 4.4 hold, then the same hypotheses hold for \(\tilde{L}, \tilde{R}\) on Banach spaces \(\tilde{X}, \tilde{Y}, \tilde{Z}\), upon defining suitable projectors \(\tilde{P}_c, \tilde{P}_{su}\). The assertion is then a consequence of Theorem 3.3 of [44], applied to (4.5). We proceed to verify the hypotheses in 3 steps:

Step 1. Verification of Hypothesis 4.2 for system (4.5). It is immediate to prove that Hypothesis 4.2 and \(G \in C^k(V_u \times V_v \times V_p \times V_\varepsilon, \mathbb{R}^m)\) imply that Hypothesis 4.2 holds also for \(\tilde{L}, \tilde{R}\) in system (4.5), and we omit this proof.

Step 2. Verification of Hypothesis 4.3 for system (4.5). The expression of \(\tilde{L}\) makes it easy to compute its resolvent as function of the one of \(L\). We conclude that \(\sigma_c(\tilde{L}) = \sigma_c(L) \cup \{0\}\). One consequence is that the spectrum of \(\tilde{L}\) admits the decomposition

\[
\sigma(\tilde{L}) = \sigma_u \cup \tilde{\sigma}_c \cup \sigma_s,
\]

where \(\sigma_u, \sigma_s\) are the stable and unstable spectra of \(L\), respectively, and where \(\tilde{\sigma}_c\) contains \(n + m\) purely imaginary eigenvalues, where \(n\) is as in Hypothesis 4.3(i). This guarantees that \(\tilde{\sigma}_c\) contains finitely many eigenvalues. Next, we define the projectors associated with \(\tilde{\sigma}_c\) as follows

\[
\tilde{P}_c = \begin{pmatrix} P_c & 0 \\ 0 & \text{id}_{\mathbb{R}^m} \end{pmatrix}, \quad \tilde{P}_{su} = \begin{pmatrix} P_{su} & 0 \\ 0 & 0_{\mathbb{R}^m} \end{pmatrix},
\]

where \(P_c, P_{su}\) are the spectral projectors associated with the centre eigenspace of \(L\). We have \(\tilde{X} = \tilde{E}_c \oplus \tilde{X}_{su}\), where

\[
\tilde{E}_c = \text{range } \tilde{P}_c, \quad \tilde{X}_{su} = \ker \tilde{P}_c, \quad \dim \tilde{E}_c = \dim E_c + m < \infty.
\]

Since \(\tilde{E}_0\) is finite dimensional, we conclude that the \(\tilde{u}\) eigenvalues in \(\tilde{\sigma}_c\) have finite multiplicities, hence Hypothesis 4.3(i) holds for \(\tilde{L}\). Furthermore, since \(\tilde{L}\) and \(L\) have the same stable and unstable spectra, their spectral gaps coincide, that is, Hypothesis 4.3(ii) for \(L\) implies that the same hypothesis holds for \(\tilde{L}\), with the same \(\gamma\).

Step 3. Verification of Hypothesis 4.4 for system (4.5). We Set

\[
\tilde{Y}_{su} = P_{su} Y = Y_{su} \times \{0_{\mathbb{R}^m}\}, \quad \tilde{Z}_{su} = P_{su} Z = Z_{su} \times \{0_{\mathbb{R}^m}\},
\]

and we denote by \(\tilde{L}_{su}\) the restriction of \(\tilde{L}\) to \(\tilde{Y}_{su}\). To prove the assertion we must show that Hypothesis 4.4 implies that the same hypothesis holds for the following
differential equation on $\tilde{X}_{su}$

\begin{equation}
\frac{d}{dt} \tilde{u}_{su} = \tilde{L}_{su} \tilde{u}_{su} + \tilde{f}(t),
\end{equation}

where $\tilde{f} = (f, 0_{\mathbb{R}^m}) \in C_\eta(\mathbb{R}, \tilde{Y}_{su})$, with $f$ and $\eta$ given in Hypothesis 4.4. In the remainder of the proof, we use the symbol $0$ in place of $0_{\mathbb{R}^m}$ to simplify the notation.

We consider the linear mapping

$$\tilde{K}_{su}: Z_{su} \times \mathbb{R}^m \to Z_{su} \times \mathbb{R}^m, \quad (U, V) \mapsto (K_{su}U, 0).$$

It suffices to prove the following claims:

1. $\tilde{K}_{su} \tilde{f} = (K_{su}f, 0)$ is an element of $C_\eta(\mathbb{R}, \tilde{Z}_{su})$.
2. $\tilde{K}_{su} \tilde{f}$ is the unique solution to (4.7).
3. Let $\Pi(\eta)$ be fixed as in Hypothesis 4.4, then the following bound holds

$$\|\tilde{K}_{su} \tilde{f}\|_{C_\eta(\mathbb{R}, \tilde{Z}_{su})} \leq \Pi(\eta).$$

To prove claim 1, we recall that $\tilde{Z}_{su} = Z_{su} \times \{0\}$, therefore if $U \in Z_{su}$ then $(U, 0) \in Z_{su}$ and $\|U\|_{Z_{su}} = \|(U, 0)\|_{\tilde{Z}_{su}}$. By Hypothesis 4.4, $(K_{su}f)(t) \in Z_{su}$ for all $t \in \mathbb{R}$, and the mapping $t \mapsto (K_{su}f)(t)$ is continuous. This implies $\tilde{K}_{su} \tilde{f} = (K_{su}f, 0) \in C^1(\mathbb{R}, \tilde{Z}) \subset C^0(\mathbb{R}, Z)$ and

$$\|\tilde{K}_{su} \tilde{f}\|_{C(\mathbb{R}, \tilde{Z}_{su})} = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|(K_{su}f)(t)\|_{Z_{su}} = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|K_{su}f(t)\|_{Z_{su}} = \|K_{su}f\|_{C_\eta(\mathbb{R}, Z_{su})},$$

which is finite by Hypothesis 4.4. Therefore $\tilde{K}_{su} \tilde{f} \in C_\eta(\mathbb{R}, \tilde{Z}_{su})$.

In order to prove claim 2 we note that

$$\frac{d}{dt} \tilde{K}_{su} \tilde{f} = \frac{d}{dt} \begin{pmatrix} K_{su}f \\ 0 \end{pmatrix} = \begin{pmatrix} L_{su}K_{su}f + f(t) \\ 0 \end{pmatrix} = L_{su} \tilde{K}_{su} \tilde{f} + \tilde{f}(t),$$

therefore $\tilde{K}_{su} \tilde{f}$ solves (4.7). Uniqueness is proved by showing that any solution $\tilde{U} = (U, V) \in C_\eta(\mathbb{R}, \tilde{Z}_{su})$ to (4.7) is equal to $\tilde{K}_{su} \tilde{f}$. Since $\tilde{Z}_{su} = Z_{su} \times \{0\}$, we have $V(t) \equiv 0$ for all $t \in \mathbb{R}$, hence $\tilde{U} = (U, 0)$. Since $\tilde{U}$ solves (4.7), we have

$$\frac{d}{dt} \tilde{U} = L_{su} \tilde{U} + f(t),$$

therefore $\tilde{U}$ solves (4.2), and Hypothesis 4.4 guarantees $U = K_{su}f$. We conclude $\tilde{U} = (K_{su}f, 0) = \tilde{K}_{su} \tilde{f}$, hence $\tilde{K}_{su} \tilde{f}$ is the unique solution to (4.7).
To prove claim 3, we estimate
\[
\|\tilde{K}_{su}\|_{L(C_0(\mathbb{R},\mathbb{Y}_{su}),C_0(\mathbb{R},\mathbb{Z}_{su}))}
\]
\[
= \sup \{\|\tilde{K}_{su}\|_{C_0(\mathbb{R},\mathbb{Z}_{su})}: \|(U, V)\|_{C_0(\mathbb{R},\mathbb{Y}_{su})} = 1\}
\]
\[
= \sup \{\|K_{su}U\|_{C_0(\mathbb{R},\mathbb{Z}_{su})}: \|(U, V)\|_{C_0(\mathbb{R},\mathbb{Y}_{su})} = 1\}
\]
\[
= \sup \{\|K_{su}U\|_{C_0(\mathbb{R},\mathbb{Z}_{su})}: \|(U, V)\|_{C_0(\mathbb{R},\mathbb{Y}_{su})} = 1\}
\]
\[
\leq \|K_{su}\|_{L(C_0(\mathbb{R},\mathbb{Y}_{su}),C_0(\mathbb{R},\mathbb{Z}_{su}))} \sup \{\|U\|_{C_0(\mathbb{R},\mathbb{Y}_{su})}: \|(U, V)\|_{C_0(\mathbb{R},\mathbb{Y}_{su})} = 1\}
\]
\[
\leq \|K_{su}\|_{L(C_0(\mathbb{R},\mathbb{Y}_{su}),C_0(\mathbb{R},\mathbb{Z}_{su}))} \leq \Pi(\eta),
\]
which completes this step.

Following the 3 steps above, the assertion is a consequence of the parameter-dependent version of the centre-manifold theorem [44, Theorem 3.3] for system (4.5). This theorem guarantees the existence of neighbourhoods \(O_\delta \times O_\mu \times O_\varepsilon\) of \(0 \in \mathbb{Z} \times \mathbb{R}^q \times \mathbb{R}\), a reduction function \(\Psi \in C^k(E_c \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}, Z_{su})\) and a centre manifold
\[
M_c(\mu, \varepsilon) = \{\tilde{u}_c + \Psi(\tilde{u}_c, \mu, \varepsilon): \tilde{u}_c \in \tilde{E}_c\},
\]
that satisfy the assertion. The expression above, however, is seemingly different from (4.4). To recover (4.4) and the assertion in terms of a reduction function \(\Psi \in C^k(E_c \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}, Z_{su})\), we note that \(\Psi\) is with values in \(\tilde{Z}_{su} = Z_{su} \times \{0\}\) hence \(\Psi = (\Psi, 0)\), for some \(\Psi \in C^k(E_c \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}, Z_{su})\), and this completes the proof.

After the existence of a centre manifold has been derived, we proceed to write the reduced equation on the centre manifold, as the following corollary states.

**Corollary 4.6.** Assume the hypotheses of Theorem 4.5. Let \(T > 0\), let \((\mu, \varepsilon) \in O_\mu \times O_\varepsilon\), and let \((u, v)\) be a solution to (3.3) which belongs to \(M_c(\mu, \varepsilon)\) for \(t \in [0, T]\). Then \(u = u_c + \Psi(u_c, v, \mu, \varepsilon), \) with \((u_c, v)\) satisfying
\[
\dot{u}_c = P_cF(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon),
\]
\[
\dot{v} = \varepsilon G(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon).
\]

**Proof.** Let \(\hat{u} = (u, v)\) be a solution to (3.4), the compact formulation of the \(m\) slow \(\infty\)-fast problem, and assume \(\hat{u}\) stays in \(M_c\) for \(t \in [0, T]\). Then
\[
\hat{u} = \left(\begin{array}{c}
u \\ u \end{array}\right) = \left(\begin{array}{c}u_c + \Psi(u_c, v, \mu, \varepsilon) \\ v \end{array}\right), \quad u_c = P_cu.
\]
From (3.4) and the definition of \(P_c\) and \(F\) we obtain
\[
\frac{d}{dt} \left(\begin{array}{c}u_c \\ v \end{array}\right) = \left(\begin{array}{c}P_c \dot{u}_c + \Psi(\hat{u}_c, \mu, \varepsilon) \\ F(u_c + \Psi(u_c, v, \mu, \varepsilon), v) \end{array}\right)
\]
\[
= \left(\begin{array}{c}P_c F(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon) \\ F(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon) \end{array}\right).
\]
Importantly, the evolution equation (4.8) is posed on a finite-dimensional state space, whose dimension is \( \dim E_c + m \). Owing to the particular form of the right-hand side of (3.4), this finite-dimensional dynamical system has \( m \) slow and \( \dim E_c \) fast variables.

**Remark 4.7.** The centre manifold reduction theory outlined here applies directly to the finite dimensional case, i.e. all results have a finite dimensional version for a slow-fast system (1.1). We will highlight these results in later sections where we discuss folds and delayed Hopf (and Turing) bifurcations in detail.

### 4.1. Centre Manifold reduction with symmetries.

When the fast subsystem admits symmetries, it is expected that they influence the centre-manifold reduction. The following result combines a parameter-dependent centre-manifold reduction for infinite dimensional dynamical systems [42, 23]. We give this result without proof, as it is a minor modification of Theorem 4.5.

**Theorem 4.8.** Assume there is a linear operator \( T \in \mathcal{L}(X) \cap \mathcal{L}(Z) \) which commutes with the fast vector field

\[
TL = LT, \quad TR(u, v, \mu, \epsilon) = R(Tu, v, \mu, \epsilon),
\]

and such that the restriction \( T_c \) of \( T \) to the subspace \( E_c \) is an isometry. Under the assumptions in Theorem 4.5, there exists a reduction function \( \Psi \) which commutes with \( T \), that is, \( T \Psi(u_c, v, \mu, \epsilon) = \Psi(T_c u_c, v, \mu, \epsilon) \) for all \( u_c \in X_c \), and such that the vector field in the reduced equation (4.8) commutes with \((T_c, \text{id}).\)

### 5. Reduction around bifurcations of the fast subsystem.

We now specialise the result of the previous section, and find reductions of the full systems around saddle-node, Hopf, and Turing bifurcations of the fast subsystem. We will also derive normal form calculations for all these cases.

#### 5.1. Reduction around a fold of the critical manifold.

Before deriving a normal form around the fold of the fast subsystem, we make additional assumptions that are verified at the saddle-node bifurcation point:

**Hypothesis 5.1** (Fold of the critical manifold). Let \( (e_i)_{i=1}^m \) be the canonical basis in \( \mathbb{R}^m \).

(i) The centre spectrum \( \sigma_c \) of \( L \) is given by \( \sigma_c = \{0\} \). The eigenvalue 0 has multiplicity 1 and eigenvector \( \zeta \in Z \).

(ii) We have \( P_c(D_u R(0,0,0,0)\zeta^{(2)}) \neq 0 \).

(iii) There exists \( 1 \leq j \leq m \) such that \( P_c(D_u R(0,0,0,0)e_j) \neq 0 \).

**Remark 5.2.** When \( L \) has an adjoint operator \( L^* \) (this happens for instance when \( Z \) is dense in \( X \)) then there is a general expression for the centre projector, namely \( P_c(u) = \langle \zeta^*, u \rangle_X \zeta \) where \( \zeta^* \) is the eigenvector of \( L^* \) for the eigenvalue 0 such that \( \langle \zeta^*, \zeta \rangle_X = 1 \).

**Lemma 5.3** (Normal form for fold). Assume Hypothesis 5.1(i) and the hypotheses of Corollary 4.6, and let \( O_{\mu}, O_{\epsilon} \) be defined as in Corollary 4.6. There exist neighbourhoods \( O_A, O_B \) of 0 in \( \mathbb{R} \) and \( \mathbb{R}^m \), respectively, such that, for any \( (\mu, \epsilon) \in O_{\mu} \times O_{\epsilon} \) there is a change of variables, defined in \( O_A \times O_B \) which transforms the reduced equation (4.8) into

\[
\begin{align*}
\dot{A} &= \alpha(B) + \beta A^2 + \gamma(\mu, \epsilon) + O(A^2(\|B\| + |\mu| + |\epsilon|) + (\|B\| + |\mu| + |\epsilon|)^2) \\
\dot{B} &= \epsilon G(\lambda, \Psi(A, B, \mu, \epsilon), B, \mu, \epsilon)
\end{align*}
\]
where \( A(t) \in \mathbb{R} \), \( B(t) \in \mathbb{R}^m \) and

\[
\begin{align*}
\alpha(B) \zeta &= P_c(D_c R(0,0,0,0)B), \\
\beta \zeta &= P_c(D_c^2 R(0,0,0,0)\zeta(2)), \\
\gamma(\mu, \varepsilon) \zeta &= P_c(D_\mu R(0,0,0,0)\mu + D_\varepsilon R(0,0,0,0)\varepsilon).
\end{align*}
\]

If, in addition, Hypothesis 5.1(ii) and Hypothesis 5.1(iii) hold, we have \( \alpha \neq 0 \) and \( \beta \neq 0 \), respectively.

**Proof.** We fix \((\mu, \varepsilon) \in O_\mu \times O_\varepsilon\). From \( X_\varepsilon = \text{span}(\zeta) \), we write \((u_\varepsilon, v) = (A\zeta, B)\), where \( A \in \mathbb{R} \), \( B \in \mathbb{R}^m \). This parametrization is defined in a neighbourhood \( O_A \times O_B \) of \( 0 \in \mathbb{R} \times \mathbb{R}^m \). From Equation (4.8) and the definition of \( F \), we obtain

\[
\begin{align*}
\dot{A} \zeta &= P_c R(A \zeta + \Psi(A \zeta, B, \mu, \varepsilon), B, \mu, \varepsilon) \\
\dot{B} &= \varepsilon G(A \zeta + \Psi(A \zeta, B, \mu, \varepsilon), B, \mu, \varepsilon).
\end{align*}
\]

The second equation in the system above is already written as in Equation (5.1) of the statement, therefore we concentrate henceforth only on the first equation.

In the remainder of this proof we set \( \| \cdot \| = \| \cdot \|_2 \), and with a small abuse of notation we use \( | \cdot | \) for the 2-norm in both \( \mathbb{R}^p \) and \( \mathbb{R}^m \). We consider the Taylor expansion of \( R \) around the origin

\[
R(u, v, \mu, \varepsilon) = \sum_{0 \leq i + j + r + s \leq 2} R_{i j r s}(u^{(i)}, v^{(j)}, \mu^{(r)}, \varepsilon^{(s)}) + O\left( \sum_{i + j + r + s = 3} \| u \|^{i} | u |^{j} | \mu |^{r} | \varepsilon |^{s} \right),
\]

and we recall that \( R_{0000} = 0 \), \( R_{1000}(u) = 0 \) by Hypothesis 4.2, hence

\[
R(u, v, \mu, \varepsilon) = R_{0100}(v) + R_{0010}(\mu) + R_{0001}(\varepsilon) + R_{2000}(u^{(2)}) \\
+ O\left( \sum_{i + j + r + s = 2} \| u \|^{i} | v |^{j} | \mu |^{r} | \varepsilon |^{s} \right).
\]

Using the following identities

\[
\begin{align*}
R_{0100}(v) &= D_v R(0,0,0,0)v, \\
R_{0010}(\mu) &= D_\mu R(0,0,0,0)\mu, \\
R_{0001}(\varepsilon) &= D_\varepsilon R(0,0,0,0)\varepsilon, \\
R_{2000}(u^{(2)}) &= \frac{1}{2} D_\varepsilon^2 R(0,0,0,0)u^{(2)}
\end{align*}
\]

and recalling that \( \Psi = O(\| u \|^2 + | v |^2 + | \mu | + \varepsilon) \), we obtain

\[
\begin{align*}
\dot{A} \zeta &= P_c(D_v R(0,0,0,0)B) + A^2 P_c(D_c^2 R(0,0,0,0)\zeta(2)) \\
&\quad + P_c(D_\mu R(0,0,0,0)\mu + D_\varepsilon R(0,0,0,0)\varepsilon) \\
&\quad + O\left( \sum_{i + j + r + s = 2} \| u \|^{i} | v |^{j} | \mu |^{r} | \varepsilon |^{s} \right)
\end{align*}
\]

which, combined with the second equation in (5.2) proves that (5.1) holds in \( O_A \times O_B \). Hypothesis 5.1(iii) guarantees \( \beta \neq 0 \) and Hypothesis 5.1(iv) implies that \( \alpha \) is nonzero in \( O_B \setminus \{0\} \).
Remark 5.4. Note that, with a further change of coordinates, Equation (5.1) can be transformed into

\[
\frac{d}{dt} A = \alpha \tilde{B}_1 + \beta A^2 + O\left(\sum_{i+j+r+s=2, i \neq 2} A^i \tilde{B}^j |\mu|^r \epsilon^s\right)
\]

\[
\frac{d}{dt} \tilde{B} = \epsilon \tilde{G}(A\zeta + \Psi(A, \tilde{B}, \mu, \epsilon), \tilde{B}, \mu, \epsilon)
\]

The reduction above leads to studying an ODE with 1 fast and \(m\) slow variables, near the fold. This system is amenable to applying existing results on canards in this context. We mention here classical results on canards for \(m = 1\) [12, 33, 56, 58], for \(m = 2\) near a folded singularity [74, 83, 18] and their generalisation to the case of \(m\) slow variables [84].

5.2. Reduction around a Hopf bifurcation in fast subsystem.

Hypothesis 5.5 (Hopf bifurcation). The centre spectrum \(\sigma_c\) of \(L\) consists of a simple pair of complex conjugate purely imaginary eigenvalues, \(\sigma_c = \{i\omega, -i\omega\}, \omega > 0\), with associated eigenvectors \(\{\zeta, \tilde{\zeta}\}\).

Lemma 5.6 (Normal form of Hopf bifurcation). Assume Hypothesis 5.5 and the hypotheses of Corollary 4.6. Then there exist neighborhoods \(W_u, W_v, W_\mu, W_\epsilon\) in \(O_u, O_v, O_\mu, O_\epsilon\), respectively, such that for any \((\epsilon, \mu) \in W_\mu \times W_\epsilon\) there exists a \((\mu, \epsilon)\)-dependent polynomial change of variables

\[
u_c = A\zeta + A\tilde{\zeta} + \Phi(\tilde{A}\zeta + \tilde{A}\tilde{\zeta}, B, \mu, \epsilon)
\]

\(v = B\)

with \(A \in \mathbb{C}, B \in \mathbb{R}^m\), which transforms the reduced equation (4.8) for \((u_c, v) \in W_u \times W_v\) into

\[
\dot{A} = i\omega A + \alpha(B, \mu, \epsilon)A + \beta A|A|^2 + O\left(||B| + |\mu| + \epsilon + |A|^2\right)
\]

\[
\dot{B} = \epsilon \tilde{G}(A\zeta + A\tilde{\zeta} + \Phi(\zeta, \tilde{\zeta}), A, B, \mu, \epsilon, B, \mu, \epsilon)
\]

where \(\alpha : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{C}\) is multilinear and \(\beta \in \mathbb{R}\).

Proof. In this case, Equation (4.8) leads to

\[
\dot{u}_c = L_c u_c + P_c R(u_c + \Psi(u_c, v, \mu, \epsilon), v, \mu, \epsilon),
\]

\[
\dot{v} = \epsilon \tilde{G}(u_c + \Psi(\mu, \epsilon), v, \mu, \epsilon, v, \mu, \epsilon).
\]

with \(L_c u_c \neq 0\). The operators \(L_c\) and \(L\) have an identical centre spectrum, and by Hypothesis 5.5 we have \(E_c = \text{span}\{\zeta, \tilde{\zeta}\}\), hence \(\text{dim} E_c = 2\). As in the case of saddle-node bifurcations, we write a normal form for the first of the two equations above, when \(v\) is fixed. We set \(v = B\) and consider

\[
\dot{u}_c = L_c u_c + P_c R(u_c + \Phi(u_c, B, \mu, \epsilon), B, \mu, \epsilon).
\]

We use a parameter-dependent normal form result [44, Theorem 2.2, Chapter 3] on the 2-dimensional system above, with parameters \((B, \mu, \epsilon) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}\). There exist neighbourhoods \(W_u, W_v, W_\mu, W_\epsilon\) in \(O_u, O_v, O_\mu, O_\epsilon\) respectively, and a polynomial \(\Phi\) such that, for any \((B, \mu, \epsilon) \in W_v \times W_\mu \times W_\epsilon\), the change of variables \(\nu_c = u_c + \Phi(\nu_c, B, \mu, \epsilon)\) transforms the system above into the normal form

\[
\dot{v} = \Lambda v + N(\nu_c, B, \mu, \epsilon) + o(||(B, \mu, \epsilon)||^m)
\]
for any integer \( q \) with \( 2 \leq q \leq k \), where \( k \) is fixed as in Hypothesis 4.2(ii). We aim to recover the normal form for a Hopf bifurcation, hence it is convenient to identify \( \mathbb{R}^2 \) with the diagonal \( \{ (z, \bar{z}) : z \in \mathbb{C} \} \), thereby representing \( \nu_c \) as \( \nu_c(t) = A(t)\zeta + \bar{A}(t)\bar{\zeta} \) and obtaining [44, Lemma 1.7, Chapter 3]

\[
\frac{d}{dt} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} + \left( \alpha(B, \mu, \varepsilon)A + \beta|A|^2 \right) + O\left( (|v| + |\mu| + \varepsilon + |A|^2)^2 \right)
\]

where \( \alpha \) is a linear form and \( \beta \) a real number. We have therefore shown the existence of a change of coordinates of type (5.3) which transforms the reduced equation (5.5) into (5.4).

**Remark 5.7.** We highlight that, in contrast to the reduction around a saddle-node bifurcation, we do not include here explicit expressions for the coefficients of the normal form \( \alpha, \beta \), and we do not include in Hypothesis 5.5 non-degeneracy conditions to guarantee \( \alpha, \beta \neq 0 \). These conditions are known for ODEs [60], and computing the coefficients is strongly problem-dependent, therefore we omit such calculations in the present paper.

The reduction above leads to studying an ODE with 2 fast and \( m \) slow variables. This system is amenable to applying existing results on delayed loss of stability through a Hopf bifurcation. We refer to classical results for \( m = 1 \), e.g., [65, 66, 46].

### 5.3. Reduction around Turing bifurcation

In order to perform a reduction around a Turing bifurcation for the applications, we make some additional assumptions concerning the symmetry of the problem.

**Hypothesis 5.8** (\( O(2) \)-equivariance). There exists a one-parameter continuous family of linear maps \( T_\varphi \) on \( X \), for \( \varphi \in \mathbb{R}/2\mathbb{Z} \), \( l \in \mathbb{R} \), and a symmetry \( S \) on \( X \) such that:

1. \( T_\varphi \circ T_\psi = T_{\varphi + \psi} \) and \( ST_\varphi = T_{-\varphi}S \) for \( \varphi, \psi \in \mathbb{R}/2\mathbb{Z} \)
2. \( T_0 = \text{id} \) and \( S^2 = \text{id} \)
3. The fast vector field \( u \mapsto F(u, v, \mu, \varepsilon) \) commutes with \( T_\varphi, S \).

**Lemma 5.9** (Normal form of Pitchfork \( O(2) \) bifurcation). Assume Hypothesis 5.8, and the hypotheses of Corollary 4.6. Further, assume that 0 is a double eigenvalue of \( L \) and no other eigenvalue is in \( \sigma_c \) for \( (\varepsilon, \mu) = (0, 0) \). Assume that the action of \( L_{\varphi} \) on \( X_c \) is not trivial. Then, there exist neighborhoods \( O_u, O_v, O_\mu, O_\varepsilon \) of 0 in \( \mathbb{R}^2 \), \( \mathbb{R}^m \), \( \mathbb{R}^p \), and \( \mathbb{R} \), respectively, such that for any \( (\varepsilon, \mu) \in O_\mu \times O_\varepsilon \), in a suitable basis \((\zeta, \bar{\zeta})\) of \( \ker L \)

\[
\begin{align*}
u_c &= A\zeta + \bar{A}\bar{\zeta} + \Phi(A\zeta + \bar{A}\bar{\zeta}, B, \mu, \varepsilon) \\
v &= B
\end{align*}
\]

with \( A \in \mathbb{C}, B \in \mathbb{R}^m \), the reduced equation (4.8) for \((u_c, v) \in O_u \times O_v \) has, at third order in \((A, \bar{A})\), the expression

\[
\begin{align*}
\dot{A} &= \alpha(B, \mu, \varepsilon)A + \beta|A|^2 + O\left( (|v| + |\mu| + \varepsilon + |A|^2)^2 \right) \\
\dot{B} &= \varepsilon G(A\zeta + \bar{A}\bar{\zeta} + \Phi(A\zeta + \bar{A}\bar{\zeta}, B, \mu, \varepsilon), B, \mu, \varepsilon)
\end{align*}
\]

where \( \alpha, \beta \) are real multilinear forms.

**Proof.** This proof follows closely the one of Lemma 5.6 except that at the end, we have to use the \( O(2) \)-pitchfork normal formal instead of the Hopf one. As shown in
where $\alpha, \beta$ are real valued.

In this basis, we represent $u$ as $u(t) = A(t)\zeta + \hat{A}(t)\bar{\zeta}$ and obtain at third order

$$\dot{\hat{A}} = \alpha(B, \mu, \varepsilon)A + N_3(A, \hat{A}, \mu, \varepsilon) + O((|v| + |\mu| + \varepsilon + |A|^2)^2).$$

where $N_3$ is a cubic polynomials in $(A, \hat{A})$ which commutes with the action of the symmetry group. Thus $N_3(e^{i\varphi}A, e^{-i\varphi}\hat{A}, \mu, \varepsilon) = e^{i\varphi}N_3(A, \hat{A}, \mu, \varepsilon)$ and $N_3(A, \hat{A}, \mu, \varepsilon) = N_3(\hat{A}, A, \mu, \varepsilon)$. This gives:

$$\dot{\hat{A}} = \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + O((|v| + |\mu| + \varepsilon + |A|^2)^2)$$

where $\alpha, \beta$ are real valued.

This lemma applies to the case of PDE equivariant with respect to translations (which take the role of $L_\varphi$) in one unbounded spatial direction and possesses a reflection symmetry in this direction.

The reduction above leads to studying an ODE with 1 fast and $m$ slow variables. This system is amenable to applying existing results on canards (with certain symmetry). We refer to classical results on slow passage through pitchfork and transcritical bifurcation for $m = 1$, e.g., [57] or [14].

6. Applications. In this section we collect applications of centre-manifold reductions and normal forms to several examples presented in section 2. We highlight that the order in which the material is presented does not follow the one used in section 2, because we have chosen to start from simple examples and work towards more advanced material. In subsection 6.1 we present preparatory results for Hilbert spaces, which will be used in the following calculations. We then work through an example of slow-passage through a Hopf bifurcation in (2.3) in subsection 6.2, assuming no prior knowledge of centre-manifold reductions for infinite-dimensional systems. In subsection 6.3, we discuss slow-passage through a Turing bifurcation in the nonlocal reaction–diffusion equation (2.6), and give pointers for performing the same analysis in local reaction–diffusion problems. The technical treatment of these sections depends on the fact that Hilbert spaces have been used in the accompanying examples. This motivates our presentation of the example in subsection 6.5, which is posed on Banach spaces and discusses canards in neural field models. Towards the end of this section, in subsection 6.6, we discuss slow passage through Hopf in the DDE (2.4).

6.1. Preparatory results. We begin by introducing two lemmas that will be useful in the upcoming calculations.

Lemma 6.1. Let $\Omega = (-l, l)$, for some $l > 0$, and let $X = L^2(\Omega, \mathbb{C})$ and $Z = H^2_{\text{per}}(\Omega, \mathbb{C})$, endowed with the standard norms. Further consider the Laplacian operator $\partial_z^2 \in \mathcal{L}(Z, X)$. If $z \in \rho(\partial_z^2)$, then

$$\|(z - \partial_z^2)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{|\text{Im} z|}.$$

Proof. We denote by $\|\cdot\|_X$ and $\langle \cdot, \cdot \rangle_X$ the standard norm and associated inner product on $X$, respectively. Let $z \in \rho(\partial_z^2)$. For any $f \in X$, there is a unique solution $u \in Z$ to $\partial_z^2 u - zu = f$. Multiplying the equation by the complex conjugate $\bar{u}$ of $u$, integrating over $\Omega$, and using integration by parts and boundary conditions we obtain

$$\|\partial_z u\|^2_X - z \|u\|_X^2 = \langle f, u \rangle_X.$$
Taking the imaginary part of the equation above we obtain the bound

$$\|\text{Im} z\|_X^2 = |\text{Im}(f, u)_X| \leq \|f\|_X \|u\|_X,$$

hence

$$\|u\|_X \leq \frac{\|f\|_X}{|\text{Im} z|}$$

and

$$\left\|(z - \partial_x^2)^{-1}\right\|_{L(X)} = \sup_{\|f\|_X \neq 0} \frac{|(z - \partial_x^2)^{-1} f|_X}{\|f\|_X} \leq \frac{1}{|\text{Im} z|}. \quad \Box$$

**Remark 6.2.** We note that Lemma 6.1 can be derived in a similar fashion also when the problem is subject to Neumann or Dirichlet boundary conditions. We omit these cases for simplicity.

**Lemma 6.3.** Let $A \in \mathcal{L}(Z, X)$, $B \in \mathcal{L}(X)$, and assume there exist $\omega_0, K > 0$ such that $\|(i\omega - A)^{-1}\|_{L(X)} \leq K/|\omega|$ for all $\omega \in \mathbb{R}$ such that $|\omega| > \omega_0$. Then there exists $C > 0$ such that

$$\|(i\omega - A - B)^{-1}\|_{L(X)} \leq \frac{C}{|\omega|} \quad \text{for all } |\omega| > \max(\omega_0, \|B\|_{L(X)} K).$$

**Proof.** Since $i\omega \in \rho(A)$ and $\|B\|_{L(X)} K < |\omega|$, then $\text{id} - B(i\omega - A)^{-1}$ is invertible. We use the following identity

$$(i\omega - A - B)^{-1} = (i\omega - A)^{-1} \left[\text{id} - B(i\omega - A)^{-1}\right]^{-1}$$

and the main hypotheses to derive the bound

$$\|(i\omega - A - B)^{-1}\|_{L(X)} \leq \frac{K}{|\omega|} \frac{1}{1 - \|B\|_{L(X)} K/|\omega|} \leq \frac{C}{|\omega|}. \quad \Box$$

### 6.2. Analytical results for slow-passage through a Hopf bifurcation in the FitzHugh-Nagumo PDE

We now show how to apply our framework to the FitzHugh-Nagumo model

\begin{align}
\partial_t u_1 &= \partial_x^2 u_1 + u_1 - u_1^3/3 - u_2 + v, \quad (x, t) \in (0, 2\pi) \times \mathbb{R}_{>0}, \\
\partial_t u_2 &= u_1 + c - bu_2, \quad (x, t) \in (0, 2\pi) \times \mathbb{R}_{>0}, \\
\dot{v} &= \varepsilon,
\end{align}

subject to periodic boundary conditions

$$u_1(0, t) = u_1(2\pi, t), \quad u_2(0, t) = u_2(2\pi, t), \quad t \in \mathbb{R}_{>0}.$$  

We make the following assumption, in order to guarantee the existence of a homogeneous steady state undergoing a Hopf bifurcation.

**Hypothesis 6.4.** There exist $u_{s1}, u_{s2}, v_s, b_s, c_s \in \mathbb{R}$ such that $u(x, t) \equiv (u_{s1}, u_{s2})$ is an equilibrium of (6.1) for $(v, b, c, \varepsilon) = (v_s, b_c, c_s, 0)$, with $1 - (u_{s1})^2 = b_s \in (0, 1)$.

In order to cast the problem in the framework of the previous sections, we set $u = (u_1, u_2)$, $\mu = (b, c)$, apply the change of variables $u = u_s + \tilde{u}$, $v = v_s + \tilde{v}$, $\mu = \mu_s + \tilde{\mu}$, and obtain, after dropping tildes and setting $a = 1 - u_{s1}^2 = b_s$ for notational convenience,

\begin{align}
\dot{u} &= Lu + R(u, v, \mu, \varepsilon) \\
\dot{v} &= G(u, v, \mu, \varepsilon)
\end{align}
obtain the following bound using Cauchy-Schwarz inequality:

\[ (\partial^2_x + a \text{id}) - a \text{id}, \quad R(u,v,\mu,\varepsilon) = o(||u||), \quad G(u,v,\mu,\varepsilon) = \varepsilon, \]

and \( R \) is the remainder, linear in \( v \) and \( \mu \), and independent of \( \varepsilon \).

System (6.2) has a homogeneous equilibrium \( u = 0 \) at \( (v,\mu,\varepsilon) = (0,0,0) \) and, as we shall see below, Hypothesis 6.4 ensures that \( u = 0 \), considered as an equilibrium of the fast subsystem, undergoes a Hopf bifurcation for these parameter values. We now proceed to check the hypotheses of Theorem 4.5.

### 6.2.1. Choice of function spaces.

We begin by selecting function spaces for the problem. In particular, we set

\[ Z = H^2_{\text{per}}(0, 2\pi) \times L^2_{\text{per}}(0, 2\pi), \quad Y = X = L^2(0, 2\pi) \times L^2(0, 2\pi). \]

where \( L^2(0, 2\pi) \) and \( H^2_{\text{per}}(0, 2\pi) \) are standard Sobolev spaces endowed with the inner-product norms \( \| \cdot \|_{L^2} \) and \( \| \cdot \|_{H^2} \), respectively. We endow \( X \) and \( Z \) with norms

\[ ||(u_1, u_2)||_Z = \left( ||u_1||_{H^2}^2 + ||u_2||_{L^2}^2 \right)^{1/2}, \quad ||(u_1, u_2)||_X = \left( ||u_1||_{L^2}^2 + ||u_2||_{H^2}^2 \right)^{1/2}. \]

However, since \( u_1, u_2 \) are \( 2\pi \)-periodic functions defined on \( (0, 2\pi) \), it will be convenient to write them in terms of their Fourier Series

\[ u_i(x) = \sum_{n \in \mathbb{Z}} u_{i,n} \psi_n(x), \quad i = 1, 2, \quad \psi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{Z}. \]

and use for \( X, Z \) norms which expose the respective Fourier coefficients [1]

\[ ||(u_1, u_2)||_X = \sum_{n \in \mathbb{Z}} ||(u_{1,n}, u_{2,n})||_2^2, \quad ||(u_1, u_2)||_Z \sim \sum_{n \in \mathbb{Z}} n^4 |u_{1,n}|^2 + |u_{2,n}|^2. \]

In passing we note that the latter norm is not equal, but only equivalent, to \( \| \cdot \|_Z \).

### 6.2.2. Checking Hypothesis 4.2.

To check Hypothesis 4.2(i) we prove that \( L \) is a continuous linear operator from \( Z \) to \( X \). We fix \( u \in Z \), let \( f = Lu \), and obtain the following bound using Cauchy-Schwarz inequality:

\[ \|f_1\|_{L^2}^2 \leq \left( \|\partial^2_x u_1\|_{L^2} + a\|u_1\|_{L^2} + \|u_2\|_{L^2} \right)^2 \leq (2 + a^2) \left( \|\partial^2_x u_1\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 \right) := K_1^2 \|u\|_{Z^2}. \]

Similarly we find \( \|f_2\|_{L^2}^2 \leq K_2^2 \|u\|_{Z^2}^2 \), where \( K_1^2 = 1 + a^2 \). Combining the bounds we obtain \( \|Lu\|_X \leq K \|u\|_Z \), where \( K = \sqrt{\max(K_1, K_2)} \). This proves \( L \in \mathcal{L}(Z, X) \).

Further, \( R \) is a cubic polynomial in \( X \) with \( R(0,0,0,0) = 0 \) and \( D\varepsilon R(0,0,0,0) = 0 \). Therefore, owing to the fact that \( Z \) is a Banach algebra, Hypothesis 4.2(ii) holds for any integer \( k \).

### 6.2.3. Checking Hypothesis 4.3.

To check this hypothesis, we determine the resolvent set of \( L \), as in the following statement.

**Proposition 6.5.** Let \( D_n(\lambda) = \lambda^2 + n^2 \lambda + a(n^2 - a) + 1 \), for \( \lambda \in \mathbb{C} \) and \( n \in \mathbb{Z} \). The resolvent set of the operator \( L : Z \to X \) defined in (6.3) is given by

\[ \rho(L) = \{ \lambda \in \mathbb{C} : \lambda \neq -a, D_n(\lambda) \neq 0 \text{ for all } n \in \mathbb{Z} \}. \]
The first equation leads to the bound
\[ |(6.5)\]
the system above we get
Since \(D_n(\lambda) \neq 0\) is the determinant of the matrix above, the system has a unique solution for all \(n \in \mathbb{Z}\). We now proceed to show that \(u = \sum_{n \in \mathbb{Z}} u_n \psi_n(x) \in \mathbb{Z}\). From the system above we get
\[ D_n(\lambda)u_{1,n} = (\lambda + a)f_{1,n} - f_{2,n}, \]
\[ (\lambda + a)u_{2,n} = (u_{1,n} + f_{2,n}). \]
The first equation leads to the bound \(|D_n(\lambda)u_{1,n}|^2 \leq (1 + |\lambda + a|^2)\|f_n\|^2_2\). We note that \(D_n(\lambda) \sim n^2\) as \(n \to \infty\): there exist an integer \(n_0(\lambda)\) and a positive real constant \(C(\lambda)\) such that \(C(\lambda)||D_n(\lambda)|| \geq n^2\) for all \(n > n_0(\lambda)\), from which we deduce
\[ n^4|u_{1,n}|^2 \leq C(\lambda)^{-2}(1 + |\lambda + a|^2)||f_n||^2_2 = K_1(\lambda)||f_n||^2_2, \quad \text{for all } n > n_0(\lambda). \]
Since \(\lambda \neq -a\), the second equation in (6.5) gives the following bound
\[ |u_{2,n}|^2 \leq 2|\lambda + a|^{-2}\left(|u_{1,n}|^2 + |f_{2,n}|^2\right) \]
\[ \leq 2|\lambda + a|^{-2}(K_1(\lambda) + 1)||f_n||^2_2 \]
\[ = K_2(\lambda)||f_n||^2_2 \quad \text{for all } n > n_0(\lambda). \]
therefore \(n^4|u_{1,n}|^2 + |u_{1,n}|^2 \leq \max(K_1(\lambda), K_2(\lambda))\|f_n\|^2_2\) for all \(n > N_1(\lambda)\) and owing to \(f \in X\), we conclude that there exists \(K(\lambda)\) such that \(\|u\|_Z \leq K(\lambda)\|f\|_X\). For all \(n\), \(u_n\) is the unique solution to (6.4), hence the uniqueness of the Fourier series implies uniqueness of \(u\).

**Step 2:** \(E \supseteq \rho(L)\). Equivalently, we prove \(\mathbb{C} \setminus E \subseteq \mathbb{C} \setminus \rho(L) = \sigma(L)\). If there exist \(\lambda \in \mathbb{C}\) and \(n \in \mathbb{Z}\) such that \(D_n(\lambda) = 0\), then (6.4) does not have a solution, hence \(\lambda \in \sigma(L)\). If \(\lambda = -a\), then \((\lambda - L)u = f\) does not have a solution in \(Z\) for all \(f \in X\), because \(\lambda = -a\) implies \(u_1 = -f_2 \in L^2(0, 2\pi)\), hence \(u_1 \notin H^2_{\text{per}}(0, 2\pi)\) in general. We therefore have \(-a \in \sigma(L)\).

**Proposition 6.5** and **Hypothesis 6.4** imply
\[ \sigma(L) = \left\{ \pm i\sqrt{1 - a^2}, -a \right\} \cup \left\{ \frac{-n^2 + \sqrt{(n^2 - 2a)^2 - 4}}{2} : n \in \mathbb{Z} \neq 0 \right\} \]
therefore the homogeneous equilibrium \(u \equiv 0 \in X\) of the fast subsystem of (6.2) undergoes a Hopf bifurcation at \((\epsilon, b, c) = (0, 0, 0)\), with Hopf frequency \(\omega_* = \sqrt{1 - a^2} = \sqrt{1 - (b^*)^2}\). It can be shown (we omit this calculation for brevity)
\[ \text{Re} \sigma_s(L) \subset (-\infty, -a/2), \quad \sigma_s(L) = \{ \pm i\omega_* \}, \quad \sigma_s(L) = \{ 0 \}, \]
therefore **Hypothesis 4.3** holds with \(n = 2\) and \(\gamma = a/2\).
6.2.4. Checking Hypothesis 4.4. Checking this hypothesis directly may be challenging in applications. However, in a wide class of problems, this hypothesis can be replaced by some other conditions, which are easier to check. In particular, since \( X = Y \) and \( X \) is a Hilbert space, it suffices to prove the following condition \([44, \text{Section 2.2.3}].\)

**Proposition 6.6.** Let \( L \in \mathcal{L}(Z, X) \) be the linear operator defined in (6.3). There exist \( \omega_0, K_0 > 0 \) such that, for all \( \omega \in \mathbb{R} \) with \( |\omega| > \omega_0, i\omega \in \rho(L) \) and

\[
\|(i\omega - L)^{-1}\|_{L(X, X)} \leq K_0 / |\omega| \tag{6.6}
\]

Proof. We note that \( i\omega \in \sigma(L) \) if, and only if, \( |\omega| = \sqrt{4 - (n^2 - 2a)^2} / 2 < 1\), hence \( |\omega| > 1 := \omega_1 \) implies \( i\omega \in \rho(L) \). We now fix \( \omega \) with \( |\omega| > \omega_1, f \in X \), and prove that there exist \( K_0, \omega_0 \) such that

\[
\|(i\omega - L)^{-1} f\|_X \leq K_0 \|f\|_X / |\omega| \tag{6.7}
\]

which implies (6.6). Since \( i\omega \in \rho(L) \), there exists a unique \( u \in Z \) such that \( u = (i\omega - L)^{-1} f \), hence

\[
- \partial_x^2 u_1 + \left( i\omega - a + \frac{1}{i\omega + b} \right) u_1 = f_1 - \frac{f_2}{i\omega + b}, \quad u_2 = \frac{u_1 + f_2}{i\omega + b}. \tag{6.8}
\]

We use Lemma 6.1 to bound \( u_1 \): the operator \( \partial_x^2 \) in (6.8) maps from \( H^2_{\text{per}}((0, 2\pi), \mathbb{C}) \) to \( L^2((0, 2\pi), \mathbb{C}) \), and

\[
\left( i\omega - a + \frac{1}{i\omega + b} \right) \in \rho(\partial_x^2), \quad f_1 - \frac{f_2}{i\omega + b} \in L^2((0, 2\pi), \mathbb{C}),
\]

therefore we obtain

\[
|\omega| \left( 1 - \frac{1}{\omega_1^2 + b^2} \right) \|u_1\|_{L^2} \leq \left\| f_1 - \frac{f_2}{i\omega + b} \right\|_{L^2} \leq \left( 1 + \frac{1}{\sqrt{\omega_1^2 + b^2}} \right) \|f\|_X.
\]

We conclude that there exist constants \( K_2, \omega_2 = 1 + \omega_1 > 0 \) such that for all \( \omega \in \mathbb{R} \) with \( |\omega| > \max(\omega_1, \omega_2) \)

\[
\|u_1\|_{L^2} \leq \frac{K_2}{|\omega|} \|f\|_X.
\]

Taking the norm of the second equation in (6.8), we arrive at the following bound, valid for all \( \omega \) with \( |\omega| > \max(\omega_1, \omega_2) \)

\[
\|u_2\|_{L^2} \leq \frac{1}{\sqrt{\omega^2 + b^2}} \left( 1 + \frac{K_2}{|\omega|} \right) \|f\|_X = \frac{1}{|\omega|} \sqrt{|\omega|^2 + b^2} \|f\|_X,
\]

and we conclude that there exist \( K_3, \omega_3 > 0 \) such that, for all \( \omega \in \mathbb{R} \) with \( |\omega| > \max(\omega_1, \omega_2, \omega_3) \)

\[
\|u_2\|_{L^2} \leq \frac{K_3}{|\omega|} \|f\|_X.
\]

Using the bounds found for \( \|u_1\|_{L^2} \) and \( \|u_2\|_{L^2} \) we find, for all \( \omega \in \mathbb{R} \) with \( |\omega| > \max(\omega_1, \omega_2, \omega_3) \),

\[
\|(i\omega - L)^{-1} f\|_X = \left( \|u_1\|_{L^2} + \|u_2\|_{L^2} \right) \leq \frac{\sqrt{K_2^2 + K_3^2}}{|\omega|} \|f\|_X.
\]

Therefore (6.7) holds and, hence, (6.8) hold with \( \omega_0 = \max(\omega_1, \omega_2, \omega_3) \) and \( K_0 = \sqrt{K_2^2 + K_3^2} \). \( \square \)
6.2.5. System of 2-fast, 1-slow variables. The sections above show that the hypotheses of Lemma 5.6 hold for the FitzHugh-Nagumo reaction-diffusion system (6.1). This leads to the finite-dimensional Hopf normal form (5.4) for which the function $G$ in the slow equation is the identity. We obtain the system

\[
\begin{align*}
\dot{A} &= i\omega A + \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + \delta \varepsilon^2 + \text{h.o.t.} \\
\dot{B} &= \varepsilon
\end{align*}
\]

(6.9)

where $A$ and $B$ are one-dimensional complex and real variables, respectively. One can then apply classical results on slow passage through a Hopf bifurcation to this system. Such results were first unveiled by Shishkova [72] and then proved within a general framework by Neishtadt [64, 65, 66]; see also [68, 6, 48] for an independent treatment. We therefore have a theoretical local explanation for the numerical results shown in Figure 2.3. We note that it is straightforward to apply the theory above to any reaction–diffusion system of PDEs, for which we expect to find generically a slow-passage through Hopf bifurcations. Other types of boundary conditions can also be easily included, by changing the underlying function spaces.

6.3. Analytical results for slow passage through Turing bifurcation. We have presented numerical results showing a slow-passage through a Turing bifurcation for a system of reaction diffusion equations, and for a one-component nonlocal reaction-diffusion equation model. In this section, we present in detail only the nonlocal case, which is less well-studied. The computations for the other case follow in a similar way, and we omit them here.

6.4. Slow passage through Turing in a nonlocal reaction-diffusion equation. We recall the model under consideration,

\[
\begin{align*}
\partial_t u(x, t) &= d\partial_x^2 u(x, t) + v(t)u(x, t) - u(x, t)\int_\Omega w(x - y)u(y, t) \, dy, \\
\dot{v} &= \varepsilon
\end{align*}
\]

(6.10)

for $(x, t) \in \Omega \times \mathbb{R}_{>0}$, with $\Omega = \mathbb{R}/2l\mathbb{Z}$, which implies periodic boundary conditions

\[
u(-l, t) = u(l, t), \quad t \in \mathbb{R}_{>0}.
\]

This model is obtained from (2.6) by setting $v \mapsto v - b$, which slightly simplifies the notation. We make some preliminary assumptions on the kernel and on the existence of a homogeneous equilibrium.

**Hypothesis 6.7.** The interaction kernel $w \in L^1(\Omega)$ is an even $2l$-periodic function, with Fourier coefficients $(w_i)_{i \in \mathbb{Z}}$, $w_0 = 1$. Further, there exist $u_*, v_*, d_* \in \mathbb{R}_{>0}$ and $n_* \in \mathbb{Z}_{\neq 0}$ such that $u(x, t) \equiv u_*$ is an equilibrium of (6.10) for $(v, d, \varepsilon) = (v_*, d_*, 0)$ with $u_* = v_*, d_* = -v_*(n_*)^2/(n_*\pi)^2$.

As we shall see below, the requirement on $d_*$ guarantees that $u_*$ undergoes a Turing-like bifurcation to a mode with wavelength $\pi/(ln_*)$. A change of variables, similar to the one used in other examples, leads to the system

\[
\begin{align*}
\partial_t u &= (d_*\partial_x^2 u - u_*W)u + (d\partial_x^2 + v)u + u_*v - uWu \\
\dot{v} &= \varepsilon
\end{align*}
\]

(6.11)

where we dropped the tildes and set $(Wu)(x) = \int_\Omega w(x - y)u(y) \, dy$. The system above is in the form (6.2) with
where \( \mu = d \).

6.4.1. Choice of function spaces. In this problem, we choose \( Z = H^2_{\text{per}}(\Omega) \) and \( Y = X = L^2_{\text{per}}(\Omega) \). We note that \( X = Y \) is a consequence of the fact that the linear operator in the original problem (6.10) contains a Laplacian and is parameter dependent.

6.4.2. Checking Hypothesis 4.2. Since \( w \in L^1(\Omega) \) by Hypothesis 6.7, the Young’s convolution theorem [17, Theorem 4.15] gives

\[
\|Wu\|_X \leq \|w\|_{L^1} \|u\|_X, \quad \text{for all } u \in X, \quad \|Wu\|_Z \leq \|w\|_{L^1} \|u\|_Z, \quad \text{for all } u \in Z.
\]

We conclude that, since \( \partial_x^2 \) and \( W \) are continuous operators from \( Z \) to \( X \), then their linear combination \( L \) is in \( L(Z, X) \). As for the nonlinear function \( R \), we note that the mapping \( u \mapsto uWu \), seen as a nonlinear operator from \( Z \) to \( Z \), is the composition of \( W \) and a product in the Banach algebra \( Z \). Since \( W \in L(Z) \), we conclude that \( R \in C^k(Z \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, X) \) for any integer \( k \).

6.4.3. Checking Hypothesis 4.3. We now study the spectrum of \( L \) which is a closed operator in \( X \) with domain \( Z \). Since \( Z \) is compactly embedded in \( X \) by Sobolev embeddings theorem (see [17], Theorem 8.8), it follows that \( L \) has a compact resolvent. Thus, the spectrum of \( L \) consists of at most a countable sequence of isolated eigenvalues with finite algebraic multiplicity (see [55, Theorem III.6.29]) which can accumulate at 0 or \( \pm \infty \). For the problem under consideration we can directly compute this sequence, and obtain

\[
\sigma(L) = \{-d_s(n\pi/l)^2 - u_n w_n : n \in \mathbb{Z}\}.
\]

Since \( w \in L^1(\Omega) \), the Riemann–Lebesgue Lemma [71, Chapter 6] implies \( w_n \to 0 \) as \( n \to \infty \), hence the spectrum accumulates at infinity. Hypothesis 6.7 guarantees that

\[
\sigma_c(L) = \{0\}, \quad \sigma_u(L) \cup \sigma_s(L) = \{-d_s(n\pi/l)^2 - u_n w_n : n \in \mathbb{Z}_{\neq n_s}\},
\]

and we have a spectral gap for \( L \).

6.4.4. Checking Hypothesis 4.4. Since \( Y = X \), we proceed as in subsection 6.2.4, and we prove a result analogous to Proposition 6.6. We write \( L = A + B \), where \( A = d_s \partial_x^2 \) and \( B = -u_s W \). We note that \( \sigma(A) \) is purely real, hence \( i\omega \in \rho(A) \).

We apply in sequence Lemmas 6.1 and 6.3, and conclude that there exist positive constants \( \omega_0, K_0 \) such that \( \|(i\omega - L)^{-1}\|_{L(X)} \leq K_0/|\omega| \) for all \( \omega \in \mathbb{R} \) with \( |\omega| > \omega_0 \), hence Hypothesis 4.4 holds.

6.4.5. Reduction to 1-slow, 1-fast system. We note that Hypothesis 5.8 holds with \( T_\phi \) and \( S \) representing translations by \( \phi \in \mathbb{R}/2l\mathbb{Z} \) and reflections about the \( x = 0 \) axis, respectively, that is

\[
(T_\phi u)(x) = u(x - \phi), \quad (Su)(x) = u(-x), \quad x, \phi \in \mathbb{R}/2l\mathbb{Z}.
\]

We can then apply Lemma 5.9 and obtain, in a neighbourhood of the origin, a 1-slow, 1-fast system

\[
\dot{A} = \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + O((|v| + |\mu| + \varepsilon + |A|^2)^2),
\]

\[B = \varepsilon,\]
where $\alpha, \beta \in \mathbb{R}$. The reduction above leads to studying an ODE with 1 fast and $m$ slow variables. This system is amenable to applying existing results on canards (with certain symmetry). We refer to classical results on slow passage through pitchfork and transcritical bifurcation for $m = 1$, e.g., [57, 14].

### 6.5. Spatio-temporal canards in neural field model

In this section we study centre-manifold reductions for the nonlocal neural field problem (2.1) where $\Omega$ is a compact subset of $\mathbb{R}^d$. We introduce the integral operators

$$W: u \mapsto \int_{\Omega} w(\cdot, y) u(y) \, d\rho(y), \quad Q: (u, v) \mapsto \int_{\Omega} \theta(u, v)(x) \, d\rho(x),$$

where the integrals are over $\Omega$, and the Nemytskii operator $N: (u, v) \mapsto \theta(u(x), v)$, and rewrite the original model as

$$\dot{u} = -u + WN(u, v_1),$$
$$\dot{v}_1 = \varepsilon (v_2 + cQ(u, v_1)),$$
$$\dot{v}_2 = \varepsilon (-v_1 + a + bQ(u, v_1)).$$

We make the following preliminary assumption:

**Hypothesis 6.8.** The firing rate function $\theta$ is in $C^k_b(\mathbb{R}^2)$, the space of bounded functions defined on $\mathbb{R}^2$ with continuous bounded derivative up to order $k$, for some $k \geq 2$. The synaptic kernel $w$ is in $C(\Omega^2)$. There exists $(u_*, v_{*1}) \in C(\Omega) \times \mathbb{R}$ such that $u_* = WN(u_*, v_{*1})$. In addition, 1 is a simple eigenvalue of $WD_u N(u_*, v_{*1})$, where $D_u N(u_*, v_{*1}): u \mapsto \partial_u \theta(u_*, v_{*1}) u$ is regarded as an operator from $C(\Omega)$ to itself.

The setting of this example is different from the previous ones. Firstly, we note that in this case $u_*$ is not a homogeneous equilibrium of the model, and indeed canards were found in [3] close to saddle-node bifurcations of heterogeneous equilibria. Secondly, in this problem, we aim to provide an example of evolution equation on a Banach space as opposed to a Hilbert space (which is why we demand $u_* \in C(\Omega)$). This functional setting is more challenging, because we can not use resolvent estimates as in Propositions 6.5 and 6.6, and instead Hypothesis 4.4 must be checked directly.

A Hilbert-space setting is however possible, and relies on results in [63, 44].

After setting $u = u_* + \tilde{u}$, $v_1 = v_{*1} + \tilde{v}_1$, and dropping tildes, we obtain

$$\dot{\tilde{u}} = -\tilde{u} + WD_u N(u_*, v_{*1}) \tilde{u} + R(u, v_1)$$
$$\dot{\tilde{v}}_1 = \varepsilon \left( v_2 + cQ(u_*, u, v_{*1} + v_1) \right)$$
$$\dot{\tilde{v}}_2 = \varepsilon \left( -v_{*1} - v_1 + a + bQ(u_*, u, v_{*1} + v_1) \right)$$

which fits in our framework with $\mu = (a, b, c)$.

$$L: u \mapsto -u + WD_u N(u_*, v_{*1}) u,$$

and nonlinear terms $R, G$ given by

$$R(u, v_1) = v_1 \int_{\Omega} w(x, y) \partial_{v_1} \theta(u_*(y), v_{*1}) \, dy + o(|u| + |v_1|)$$
$$G(u, v, \mu) = \left( v_2 + \mu_3 Q(u_*, u, v_{*1} + v_1), -v_{*1} - v_1 + \mu_1 + \mu_2 Q(u_*, u, v_{*1} + v_1) \right)$$
6.5.1. Function spaces and Hypothesis 4.2. For this example we choose $Z = Y = X = C(\Omega)$. For all $u \in Z$ we have, owing to Hypothesis 6.8,
\[
\|Lu\|_X = \|Lu\|_\infty \leq (1 + \text{meas}(\Omega)\|w\|_\infty\|\partial u\|_\infty)\|u\|_\infty := K\|u\|_Z,
\]
where the infinity norms are on $C(\Omega)$ or $C(\Omega^2)$, as appropriate. Therefore $L \in \mathcal{L}(Z,X)$. The nonlinear terms $R$ satisfy Hypothesis 4.2.

6.5.2. Checking Hypothesis 4.3. We first recall the following result:

**Lemma 6.9.** If Hypothesis 6.8 holds the operator $WD_u N \in \mathcal{L}(X)$ is compact.

*Proof.* As $\Omega$ is compact, the Weierstrass theorem provides a sequence of polynomials $(w_k)_k$ which tends to $w$ in $C(\Omega^2)$, and the same is true for the associated integral operators $(W_k)_k$ in $\mathcal{L}(X)$. These operators have finite dimensional range and are thus compact operators. The set of compact operators is closed in $\mathcal{L}(X)$ (see [17]) hence leading to the conclusion that $W$ is compact. As $D_uN$ is a continuous operator on $X$, $WD_u N$ is compact. \[\Box\]

It follows that the spectrum $\sigma(WD_u N)$ of $WD_u N$ contains 0, and $\sigma(WD_u N) \setminus \{0\}$ consists at most of a countable sequence of isolated eigenvalues with finite algebraic multiplicity which may accumulate at 0. Since $L = -\mathrm{id}_X + WD_u N$ and Hypothesis 6.8 holds, Hypothesis 4.3(i) is verified, and there exists a spectral gap $\gamma > 0$ which may be chosen such that Hypothesis 4.3(ii) is satisfied. In addition, the structure of the spectrum of $WD_u N$ implies that $\sigma_u(L)$ is finite and $X_u$ is finite dimensional.

6.5.3. Checking Hypothesis 4.4. As anticipated above, since we are dealing with a quasilinear formulation ($Y = X$) on a Banach space, we can not make use of resolvent estimates and we must verify Hypothesis 4.4 directly (see also [51] for a similar strategy). In the following, we write $T(t) = e^{tL} \in \mathcal{L}(X)$ the semigroup generated by $L$. We initially derive the following useful estimates:

**Proposition 6.10.** Assume Hypothesis 6.8, and let $L$ be the operator defined in (6.14), and let $P_u$, $P_s$ be the associated projectors on $X_u$, $X_s$, respectively. There exists a constant $M > 0$ such that
\[
\|T(t)P_u\| \leq Me^{\beta_u t} \quad \text{for all } t \leq 0, \quad \beta_u \in \left(0, \min_{\lambda \in \sigma_u} \Re \lambda \right),
\]
\[
\|T(t)P_s\| \leq Me^{-\beta_s t} \quad \text{for all } t \geq 0, \quad \beta_s \in \left(0, \sup_{\lambda \in \sigma_s} \Re \lambda \right).
\]

*Proof.* The space $X_u$ is finite dimensional (see subsection 6.5.2), hence $T(t)P_u$ has finite-dimensional range, and a Dunford decomposition yields (6.15). The operator $T(t)P_a$ is bounded on $X$. As a consequence, the spectral mapping theorem [34, Section VII.3.6, Theorem 11] yields $\sigma(T(t)P_a) \setminus \{0\} = e^{\sigma_{a,L}}(t)$. Finally, an application of the Gelfand spectral radius theorem [34] yields (6.16). Note that Hypothesis 6.8 provides the existence of $\beta_u, \beta_s$. \[\Box\]

**Proposition 6.11.** Assume the hypotheses of Proposition 6.10. Then Hypothesis 4.4 holds with
\[
(K_{su}f)(t) = -\int^\infty_t T(t - \tau)P_u f(\tau) d\tau + \int^t_\infty T(t - \tau)P_s f(\tau) d\tau.
\]

*Proof.* Hypothesis 4.4 is stated in terms of the constant $\gamma > 0$, fixed as in Hypothesis 4.3(ii). The constant can be chosen to be $\gamma = \min(\beta_s, \beta_u)/2$, where $\beta_u, \beta_s$ are in
the range specified in Proposition 6.10. Henceforth we fix \( \eta \in [0, \gamma] \), \( f \in C_0(\mathbb{R}, Y_{su}) \), let \( u = K_{su} f \), and we prove that Hypothesis 4.4 holds for \( u \), using 4 steps.

**Claim 1:** \( u \) is exponentially bounded. Owing to the standard properties of the projectors \( P_s, P_u \), we have \( u(t) \in \mathcal{Z}_{su} \) for all \( t \in \mathbb{R} \). We set \( u = u_s + u_u \) with

\[
   u_u(t) = \int_{-\infty}^{t} T(t - \tau)P_u f(\tau) \, d\tau, \quad u_u(t) = -\int_{t}^{\infty} T(t - \tau)P_u f(\tau) \, d\tau,
\]

and use (6.16) to derive the following bound

\[
   \|u_u(t)\| \leq M \|f\|_0 \int_{t}^{\infty} e^{\beta_u (t - \tau) + \eta |\tau|} \, d\tau \leq M \|f\|_0 e^{\eta |t|} \frac{M}{\beta_u - \eta} \|f\|_0 e^{\eta |t|}.
\]

A similar bound can be found for \( u_s \), and we obtain

\[
   \|u(t)\| \leq \left( \frac{1}{\beta_u - \eta} + \frac{1}{\beta_s - \eta} \right) M \|f\|_0 e^{\eta |t|}.
\]

**Claim 2:** \( u \) is in \( C_0(\mathbb{R}, \mathcal{Z}_{su}) \). The above estimate, the continuity of \( f : \mathbb{R} \to X \) and the dominated convergence theorem imply that \( u : \mathbb{R} \to X \) is continuous and \( u \in C_0(\mathbb{R}, \mathcal{Z}_{su}) \). In passing we note that the estimate above also implies \( \|K_{su}\| \leq \Pi(\eta) = M/(\beta_u - \eta) + M/(\beta_s - \eta) \).

**Claim 3:** \( u \) is differentiable in \( X \) and solves (4.2). We treat the term \( u_s(t) \) for some fixed \( t \in \mathbb{R} \), as the result for \( u_u(t) \) follows in a similar way. Using a Taylor expansion of the exponential \( T_u(\tau) = e^{L_u \tau} \), we obtain, for sufficiently small \( \varepsilon \),

\[
   u_s(t + \varepsilon) - u_s(t) = \int_{-\infty}^{t} [T_u(t - \tau + \varepsilon) - T_u(t - \tau)] f(\tau) \, d\tau + \int_{t}^{t+\varepsilon} T_u(t - \tau + \varepsilon) f(\tau) \, d\tau = \varepsilon L_s \int_{-\infty}^{t} T_u(t - \tau) f(\tau) \, d\tau + \varepsilon L_s \int_{t}^{t+\varepsilon} T_u(t - \tau) f(\tau) \, d\tau + O(\varepsilon^2)
\]

Hence, \( u_s \) is differentiable at \( t \) and \( \dot{u}_s(t) = P_s f(t) + L_s u_s(t) \). It follows that \( u_s \) is \( C^1 \) in \( X \). The claim is proved using a similar argument for \( u_u \).

**Claim 4:** \( u \) is unique. Assume there are two solutions \( u, \tilde{u} \) to (4.2). Then \( \nu = u - \tilde{u} \) solves \( \dot{\nu} = L_{su} \nu \) and belongs to \( C_0(\mathbb{R}, \mathcal{X}_{su}) \). This implies \( \nu(0) \in \mathcal{X}_s \cap \mathcal{X}_u = \{0\} \), hence \( \nu(t) \equiv 0 \) and the uniqueness of \( u \). □

### 6.5.4. Reduction to 1-fast 2-slow system. We now apply Lemma 5.3 to (6.13). We note that \( \gamma(\mu, \varepsilon) = 0 \) because \( R \) is independent of \( (\mu, \varepsilon) \); in addition, \( R \) does not depend on \( v_2 \), hence \( \alpha(B) \) and higher order terms depend solely on \( B_1 \). In a neighbourhood of the origin we obtain the reduction

\[
\begin{align*}
   \dot{A} &= \alpha B_1 + \beta A^2 + \omega(A, B_1, \mu, \varepsilon) \\
   \dot{B}_1 &= \varepsilon (B_2 + \mu_3 H(A, B_1, B_2, \mu, \varepsilon)) \\
   \dot{B}_2 &= \varepsilon (- v_{s1} - B_1 + \mu_1 + \mu_2 H(A, B_1, B_2, \mu, \varepsilon)),
\end{align*}
\]

where \( H : \mathbb{R}^3 \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R} \) is defined by

\[
H(A, B_1, B_2, \mu, \varepsilon) = Q(u_* + A\zeta + \Psi(A, (B_1, B_2), \mu, \varepsilon), v_{s1} + B_1),
\]
and
\[ \omega(A, B_1, \mu, \varepsilon) = O(A^2(|B_1| + |\mu| + |\varepsilon|) + (|B_1| + |\mu| + |\varepsilon|)^2). \]

**Lemma 6.12.** Assume \( \alpha, \beta \neq 0 \), and let \( O_\mu \) be defined as in Lemma 5.3. There exists an open subset \( U_\mu \subset O_\mu \) and a function \( \xi : U_\mu \to \mathbb{R} \) such that system (6.17) admits a folded singularity at \((A, B_1, B_2, \mu) = (0, 0, \xi(\mu), \mu)\) for all \( \mu \in U_\mu \). The folded singularity is

- a folded saddle if \( J_{12} J_{21} > 0 \),
- a folded node if \( J_{11} < 0 \) and \( J_{11}^2 + 4J_{12} J_{21} > 0 \),
- a folded saddle-node if \( J_{12} J_{21} = 0 \),

where
\[
\begin{align*}
J_{11} &= \mu_3 \beta / \alpha \left[ \partial_A H(0, 0, \xi(\mu), \mu, 0) + \partial_{B_1} H(0, 0, \xi(\mu), \mu, 0) \partial_A \eta(0, \mu) \right], \\
J_{12} &= \beta / \alpha \left[ 1 + \mu_3 \partial_{B_1} H(0, 0, \xi(\mu), \mu, 0) \right], \\
J_{21} &= -2(-v_{A1} - B_1 + \mu_1 + \mu_2 H(A, B_1, B_2, \mu, \varepsilon)).
\end{align*}
\]

**Proof.** By setting \( \hat{t} = t/\beta, \hat{\varepsilon} = \varepsilon/\beta, \kappa = \beta/\alpha \) (and dropping the tilde in \( \varepsilon \)) we cast system (6.17) as
\[
\begin{align*}
\dot{A} &= \kappa B_1 + A^2 + \omega(A, B_1, \mu, \varepsilon), \\
\dot{B}_1 &= \varepsilon (B_2 + \mu_3 H(A, B_1, B_2, \mu, \varepsilon)), \\
\dot{B}_2 &= \varepsilon (-v_{A1} - B_1 + \mu_1 + \mu_2 H(A, B_1, B_2, \mu, \varepsilon)).
\end{align*}
\]

After passing to the slow time \( \tau = \varepsilon t \) in (6.19) we obtain, at \( \varepsilon = 0 \), the slow subsystem
\[
\begin{align*}
0 &= \kappa B_1 + A^2 + \omega(A, B_1, \mu, 0), \\
B_1' &= (B_2 + \mu_3 H(A, B_1, B_2, \mu, 0)), \\
B_2' &= (-v_{A1} - B_1 + \mu_1 + \mu_2 H(A, B_1, B_2, \mu, 0)).
\end{align*}
\]

The critical manifold is a graph over \( A \) in the neighbourhood of the origin; the Implicit Function Theorem guarantees the existence of a subset \( V_A \times V_\mu \) of \( O_A \times O_\mu \), and a unique function \( \eta : V_A \times V_\mu \to \mathbb{R} \) such that
\[
0 = \kappa \eta(A, \mu) + A^2 + \omega(A, \eta(A, \mu), \mu, 0) \quad (A, \mu) \in V_A \times V_\mu.
\]
The desingularised reduced system associated to (6.20) is given by
\[
\begin{align*}
\hat{A} &= [\kappa + \partial_{B_1} \omega(A, \eta(A, \mu), \mu, 0)] \left[ B_2 + \mu_3 H(A, \eta(A, \mu), B_2, \mu, 0) \right], \\
\hat{B}_2 &= \left[ -2A - \partial_A \omega(A, \eta(A, \mu), \mu, 0) \right] (-v_{A1} - B_1 + \mu_1 + \mu_2 H(A, \eta(A, \mu), B_2, \mu, 0)).
\end{align*}
\]
Since by the Implicit Function Theorem \( \kappa + \partial_{B_1} \omega(A, \eta(A, \mu), \mu, 0) \) does not vanish in \( V_A \times V_\mu \), then a folded singularity of (6.17) is an equilibrium of (6.21) satisfying
\[
\begin{align*}
0 &= B_2 + \mu_3 H(A, \eta(A, \mu), B_2, \mu, 0), \\
0 &= -2A - \partial_A \omega(A, \eta(A, \mu), \mu, 0).
\end{align*}
\]
The second equation holds for \( A = 0 \). A further application of the Implicit Function Theorem to the first equation with \( A = 0 \) then yields the existence of \( U_\mu \subset V_\mu \) a function \( \xi : U_\mu \to \mathbb{R} \) such that \((A, B_1, B_2) = (0, 0, \xi(\mu))\) is a folded singularity. The statement follows from the fact that the Jacobian of (6.21) at the folded singularity is the matrix
\[
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & 0
\end{bmatrix}.
\]

\[ \square \]
The previous lemma implies that solutions to the original neural field model near saddle-node bifurcation points of the fast subsystem, which are patterned states, are expected to display canard segments. The existence of these structures was predicted analytically in [3] in the case of Heaviside firing rate $\theta$ and is therefore valid for generic smooth, bounded firing rates. Simulations have been reproduced from that paper in Figures 2.1 and 2.2. We note that, while Lemma 6.12 does not directly apply when $\Omega$ is a ring or a sphere, similar dynamics is observable on Neural Field models with a heterogeneous external input, which are not equivariant with respect to the action of a Lie group, and to which Lemma 6.12 applies.

6.6. Delay-differential equation. We now present results for the DDE (2.4). The treatment of this case is separate to the others, as it requires different technical tools. We provide pointers to specialised literature on the topic further below in the section.

We denote by $u_*$ an equilibrium of the fast subsystem associated to (2.4), set $u = u_* + x$, and obtain

$$
\begin{align*}
\dot{x}(t) &= vx(t) - x(t - \tau) + u_*^3 - (u_* + x(t))^3, \quad t \geq 0 \\
x(t) &= \varphi(t), \quad t \in [-\tau, 0].
\end{align*}
$$

The analysis of delay differential equations is technical and relies on tools from functional analysis and semigroup theory [36]. In this section we aim to provide a self-contained treatment for the analysis of (6.22), and we refer the reader to the textbooks [43, 31] for an exhaustive treatment of the subject.

To make sense of the right hand side of (6.22), $x$ must be known on a time interval of length $\tau$, typically a history time interval $[t - \tau, t]$. This implies that the state space of (6.22) is infinite dimensional and included in some function space from $[-\tau, 0]$ to $\mathbb{R}$. The choice of this function space affects the analysis of (6.22). For example, if we were to choose $X = C^0([-\tau, 0], \mathbb{R})$, it would appear that the operator $L$ in (4.1) encodes the delay differential equation in its domain $Z$ (see [43] and Remark 6.13). We therefore could not apply the centre-manifold results presented in the previous sections, as the space $Z$ would depend on the parameter $v$.

The standard formalism aimed to tackle this difficulty is that of the sun-star calculus [30, 31] which seeks solutions in a larger state space of less regular functions. This theory can also be applied to centre-manifold reductions [31]. We aim at showing a self-contained example that does not necessarily require a deep knowledge of sun-star calculus, hence we proceed as follows:

1. We select the state spaces $Z, Y, X$ as suggested by the sun-star theory and check that they are suitable for our problem (see also [10] for a way to bypass the use of sun-star calculus).

2. We do not apply the center manifold reduction exposed in [30, 31] because that approach looks for integral solutions to the problem and thus solves (4.2) in $C^0_\eta(\mathbb{R}, \mathbb{R})$ instead of $C^0_\eta(\mathbb{R}, Z)$, which is required by our formalism. It is possible to adapt the proof in [30, 31] to fit our framework, but we choose a direct and self-contained approach. We highlight that we will exploit (in Lemma 6.17) the fact that the nonlinearity $R$ has a finite-dimensional range, as it is done in the sun-star references given above.

Before proceeding we introduce, for any $t \geq 0$ the history function $w_1 : [-\tau, 0] \to X$ of $w : [-\tau, \infty) \to X$ defined by $w_1(\theta) := w(t + \theta)$.

6.6.1. Function spaces and Hypothesis 4.2. A first step in the presentation is to rewrite (6.22) as a Cauchy problem. We chose a Hilbert space setting with $X =
However, where \( \Phi = \pi_1, \pi_2 \) the canonical projections from \( X \) onto \( \mathbb{R} \) and \( L^2((-\tau, 0), \mathbb{R}) \), respectively. We set \( Z = \{ u \in \mathbb{R} \times W^{1,2}((-\tau, 0), \mathbb{R}) \mid (\pi_2 u)(0) = \pi_1 u \} \) and note that \( Z \) is continuously embedded in \( X \). This allows us to define the following operators (see [10]):

\[
L = \begin{pmatrix} v \\ 0 \end{pmatrix} \frac{d}{d\theta} \in L(Z, X), \quad R(u) = \begin{pmatrix} u^3 - (u_1 + \pi_1(u))^3 \\ 0 \end{pmatrix}
\]

where \( \Phi = u_2 \to -u_2(-\tau) \in L(W^{1,2}((-\tau, 0); \mathbb{R}), \mathbb{R}) \). Note that \( R(u) \notin Z \) for \( u \in Z \). However, \( R \in C^\infty(Z, X) \) because \( Z \) is a Banach algebra.

In conclusion, we consider the Cauchy problem

\[
\dot{u} = Lu + R(u)
\]

with initial condition in \( Z \). Although it may appear mysterious at first, one can check, using the first component, that (6.22) is indeed encoded by the above Cauchy problem and this provides Hypothesis 4.2.

**Remark 6.13.** A possible alternative choice [43, 31] is to define

\[
X = C^0([-\tau, 0], \mathbb{R}), \quad Z = \left\{ \varphi \in C^1([-\tau, 0], \mathbb{R}) \mid \frac{d}{d\theta} \varphi(0) = v \varphi(0) - \varphi(-\tau) \right\},
\]

and then

\[
L : Z \to X, \quad u \mapsto \frac{d}{d\theta} u.
\]

With this choice \( Z \) depends on the parameter \( v \), which is not suitable for our analysis because the spaces \( Z, Y, X \) cannot depend on a varying parameter. One then uses the sun-star formalism [30, 31, 36], akin to a double dual operation, to rewrite the problem in the space \( X = \mathbb{R} \times L^\infty((-\tau, 0), \mathbb{R}) \) obtaining a form similar to (6.23).

### 6.6.2. Checking Hypothesis 4.3.

We now check hypotheses on the spectrum of \( L \) in (6.23), which is characterised in the following lemma:

**Lemma 6.14.** Fix \( \tau > 0, v \in \mathbb{R} \). The spectrum of \( L \) is composed solely of eigenvalues and is given by \( \sigma(L) = \{ \lambda \in \mathbb{C} \mid v - e^{-\lambda \tau} - \lambda = 0 \} \). An eigenvector associated to the eigenvalue \( \lambda \in \sigma(L) \) is \( (1, e^{\lambda \theta}) \in Z \).

**Proof.** In this proof, we write \( L^2 = L^2((-\tau, 0), \mathbb{R}) \) and \( W^{1,2} = W^{1,2}((-\tau, 0), \mathbb{R}) \) to simplify notations. Let \( E = \{ \lambda \in \mathbb{C} \mid v - e^{-\lambda \tau} - \lambda \neq 0 \} \). We prove that \( E \supset \rho(L) \) by showing that \( \mathbb{C} \setminus E \subset \sigma(L) \). For each \( \lambda \) solution of \( v - e^{-\lambda \tau} - \lambda = 0 \), \( (1, e^{\lambda \theta}) \in Z \) is an eigenvector showing that \( \lambda - L \) is not invertible.

Conversely, we consider \( \lambda \in E \). We show that the equation \( Lu = \lambda u + w \) has a unique solution \( u \in Z \) for any \( w \in X \). Writing \( u = (u_1, u_2) \), one finds

\[
\frac{d}{d\theta} u_2 = \lambda u_2 + w_2, \quad u_2(\theta) = e^{\lambda \theta} u_1 + \int_0^\theta e^{\lambda (\theta - s)} w_2(s) \, ds.
\]

The implies that \( u_2 \) is of Sobolev regularity. One also has \( vu_1 - u_2(-\tau) = \lambda u_1 + w_1 \) which gives

\[
(v - e^{-\lambda \tau} - \lambda) u_1 = w_1 + \int_0^{\tau} e^{\lambda (-\tau - s)} w_2(s) \, ds.
\]
This equation has a unique solution \( u_1 \in \mathbb{R} \) because \( \lambda \in E \) and this provides the unique solution to \( Lu = \lambda u + w \). Finally, \( \|u\|_2 = |u_1| + \|u_2\|_{W^{1,2}} \). From the Cauchy-Schwartz inequality
\[
|u_1| \leq \frac{1}{v - e^{-\lambda r} - \lambda} (|w_1| + \|e^{Re \lambda} \|_{L^2} \|w_2\|_{L^2}) = O (\|w\|_X).
\]
From the definition of \( u_2 \), we find
\[
\|u_2\|_{L^2} \leq e^{\lambda \cdot \|x\|_\infty} (|u_1| + \sqrt{\tau} \|w_2\|_{L^2}) = O (\|w\|_X).
\]
Finally,
\[
\left\| \frac{d}{d\theta} u_2 \right\|_{L^2} \leq |\lambda| \cdot \|u_2\|_{L^2} + \|w_2\|_{L^2} = O (\|w\|_X).
\]
This implies that \( \|u\|_{W^{1,2}} = O (\|w\|_X) \). We have shown that
\[
\sigma(L) = \{ \lambda \in \mathbb{C} : v - e^{-\lambda r} - \lambda = 0 \}.
\]
We note that the spectrum is composed of eigenvalues \( \lambda \) because the vector \( (1, e^{\lambda \cdot \cdot}) \in Z \) is an associated eigenvector.

The spectrum of \( L \) is composed of the zeros of an holomorphic function, hence the (at most countable) spectral elements are isolated. Therefore Hypothesis 4.3 holds and \( \sigma_u \) is finite. As a consequence \( X_u \) is finite dimensional. Note that the Dunford projectors \( P_u, P_s \) are easily expressed using the expression of the eigenvectors and the scalar product on \( X \).

**Remark 6.15.** Using the different branches \( (W_k)_{k \in \mathbb{Z}} \) of the Lambert function \( W \) (see [27]) which satisfies \( W(z)e^{W(z)} = z, z \in \mathbb{C} \), it is possible to compute all the eigenvalues
\[
\lambda_k = v + \frac{1}{\tau} W_k (-\tau e^{-\tau v}), k \in \mathbb{Z}.
\]

**6.6.3. Checking Hypothesis 4.4.** We now check the last hypothesis required to have a center manifold, namely Hypothesis 4.4 which is more challenging than the previous steps. The case we study is a nontrivial example for which the conditions stated in \([44, \text{Section 2.2.3}]\) do not hold, whereas Hypothesis 4.4 is satisfied. As for the neural field example, we check directly that the solution to (4.2) is
\[
(K_{su}f)(t) = -\int_t^\infty T(t-r)P_u f(r)dr + \int_{-\infty}^t T(t-r)P_s f(r)dr
\]
where \( (T(t)) \) is the (strongly continuous) semigroup of solutions generated by \( L \). This semigroup can be directly expressed using the method of steps. We first give the expression of \( (T(t)) \) which allows to properly define \( K_{su} \).

**Lemma 6.16.** Let \( (x, \varphi) \in X \). For each \( t \in [0, \tau] \) we have
\[
\pi_1 (T(t) (x, \varphi)) = e^{vt} x - \int_0^t e^{v(t-s)} \varphi(s - \tau) ds := u_1(t)
\]
\[
\pi_2 (T(t) (x, \varphi)) = (u_1)_t
\]
where \( (u_1)_t \) is the history function of \( u_1 \). In addition, \( \text{range}(T(\tau)) \subset Z \).
Proof. Let us compute the linear flow $u(t) = T(t)(x, \varphi)$ for $(x, \varphi) \in X$. We write $u_i = \pi_i(u)$, for $i = 1, 2$. We start with $(x, \varphi) \in D(L) = Z$. Proposition 3.9 in [10] shows that the second component $\pi_2$ above analytical expression shows that the continuity of $K$ in $X$, it because the spaces $a$ specific linear subspace of $X$, proved [80, 43] by showing that $K_1$ is solution of $u$ in $X$ and Proposition 6.11 holds for the above semigroup, this is usually given $u \in \mathbb{R}$. Proposition 6.17. Let us consider a general $t$ that belong to $Z$, $X$. We thus have found the expression of $u_1(t) = (u_1)_t$. We have $u_1(t) = v u_1(t) - u_1(t - \tau)$ and for $t \in [0, \tau]$, we get

$$
(6.25) \quad u_1(t) = e^{ut} x - \int_0^t e^{v(t-s)} \varphi(s - \tau) ds.
$$

We thus have the expression of $u(t) = T(t)(x, \varphi)$ for $t \in [0, \tau]$ on $Z$. Further, as $Z$ is dense in $X$, we can consider a sequence such that $Z \ni (x_n, \varphi_n) \to (x, \varphi) \in X$. The above analytical expression shows that the $\pi_1(T(t)(x_n, \varphi_n))$ converges to the right hand side of (6.25) in $X$, namely

$$
\pi_1(T(t)(x, \varphi)) = e^{ut} x - \int_0^t e^{v(t-s)} \varphi(s - \tau) ds.
$$

Proposition 3.11 in [10] shows that the second component $\pi_2(T(t)(x, \varphi))$ is actually $(u_1)_t$. We thus have found the expression of $T(t)$ for $t \in [0, \tau]$ in the whole space $X$. The last statement is straightforward to check, using an approach similar to the proof of Lemma 6.14.

Let us consider a general $t = n \tau + s$ with $s \in [0, \tau)$, the semigroup property gives $T(t) = T(\tau)^n T(s)$ and we thus have the expression of $T(t)$ for any $t \geq 0$. We also assume that Proposition 6.10 holds for the above semigroup, this is usually proved [80, 43] by showing that $T$ is eventually norm continuous. Therefore, we have everything at hand to define $K_{su}$ as above.

In order to check Hypothesis 4.4, we need to:
1. show that $K_{su} \in \mathcal{L}(C^\eta(R, X_{su}), C^\eta(R, Z_{su}))$,
2. show that $K_{su} f \in C^1(R, X)$,
3. show that equality (4.2) holds in $X$ for all $t \in I$.

We will not complete all the steps above, as this is laborious and parallels the proof of [80, Proposition C4]. Instead, we underline some salient points which are different from the previous examples, the most notable one being the gain of regularity of the solution, that is, $K_{su} f$ belongs to $Z$, and is differentiable in $X$.

An additional technical point which proves useful is the following. From the proof of the center manifold theorem in [44], it can be noted that the linear operator $K_{su}$ is always applied to vectors such as $P_{su} R(u), P_{su} D R(u), \cdots$. Given the particular form of these vectors $P_{su} (\cdot, 0)$, we only have to solve (4.2) for functions $f$ that belong to a specific linear subspace of $X$, i.e., the space $P_{su}(R \times \{0\})$.

We would wish to proceed as in the proof of Proposition 6.11, but we must adapt it because the spaces $Z, X$ are different in the present case and this requires to show the gain of regularity. This gain of regularity from $X$ to $Z$ is provided below. For the continuity of $K_{su}$ from $C^\eta(R, X_{su})$ in $C^\eta(R, Z_{su})$, we refer to [80, Proposition C4].

Lemma 6.17. Let $f \in C^\eta(R, R)$ and consider the stable component of $K_{su}$:

$$
u(t) = \int_{-\infty}^t T(t - r) P_s(f(r), 0) dr.$$
Then for all \( t \in \mathbb{R} \) \( u(t) \in Z \), \( u \) is differentiable in \( X \) and \( u(t) = Lu(t) + P_s(f(t), 0) \).

Proof. Proposition 6.10 allows to give a meaning to the expression of \( u \) by showing that it is well defined and finite. Here we first show that \( u(t) \in Z \). Using a change of variables, we find

\[
u(t) = \int_{t-\tau}^{t} T(t-r)P_s(f(r), 0) \, dr + T(\tau) \int_{-\infty}^{t} T(t-r)P_s(f(r-\tau), 0) \, dr.
\]

By Lemma 6.16, the second term is in \( Z \), we therefore focus on the first term, which we denote by \( u^1 \). We fix \( r \in \mathbb{R} \) and introduce

\[
h(t,r) = \begin{cases} \pi_1 T(t)P_s(f(r), 0), & \text{if } t \geq 0, \\ f_2(r)(t) = 0, & \text{if } t \in [\tau, 0]. \end{cases}
\]

We know from the previous lemma Lemma 6.16 that

\[
\pi_2 T(t)P_s(f(r), 0)(\theta) = h(t + \theta, r) \quad \text{for all } \theta \in [\tau, 0].
\]

Owing to Lemma 6.16, the only point left to show is \( \pi_2(u^1(t)) \in W^{1,2}((\tau, 0), \mathbb{R}), \mathbb{R}) \). For all \( \theta \in [\tau, 0] \) we have

\[
\pi_2(u(t))\theta = \int_{t-\tau}^{t+\theta} h(t-r + \theta, r) dr = \int_{t-\tau}^{t+\theta} e^{v(t-r+\theta)} f(r) dr.
\]

Since \( f \in C(\mathbb{R}, \mathbb{R}) \), it follows that \( \pi_2(u^1(t)) \in W^{1,2}((\tau, 0), \mathbb{R}), \mathbb{R}) \) and \( u(t) \in Z \).

Finally, we address the differentiability of the solution. For a fixed \( t \in \mathbb{R} \) and \( \epsilon \), one finds

\[
\frac{1}{\epsilon}(u(t + \epsilon) - u(t)) = \frac{1}{\epsilon}(T(\epsilon)u(t) - u(t)) + \frac{1}{\epsilon} \int_{t}^{t+\epsilon} T(t + \epsilon - r)P_s(f(r), 0) \, dr.
\]

By continuity of \( f \), the last term converges to \( P_s(f(r), 0) \) in \( X \) as \( \epsilon \) tends to zero. As \( u(t) \in Z \) which is the domain of \( L \),

\[
\frac{T(\epsilon)u(t) - u(t)}{\epsilon} \to Lu(t) \quad \text{as } \epsilon \to 0.
\]

Hence, \( u \) is differentiable in \( X \) and satisfies (4.2) in \( X \).

6.7. Slow-passage through Hopf bifurcation. Looking for purely imaginary eigenvalues \( \pm \omega \) to the equation \( v - i\omega = e^{-i\omega \tau}, \) one finds that this requires \( |v| \leq 1 \) by taking the real part. By taking the norm, we find \( \omega = \pm \sqrt{1 - v^2} \). Then the Hopf bifurcation point \( v_H = \cos(\tau \sqrt{1 - v_H^2}) \). We now consider such value of \( v \) for which \( \sigma_v = \{ \pm \omega \} \) with \( \omega > 0 \). We can henceforth proceed as in section 6.2.5 because the normal form and the slow system is identical to the present one.

7. Conclusions. In this paper, we have provided new local results on the outstanding problem of proving the existence of canard solutions in infinite-dimensional slow-fast dynamical systems as well as delayed bifurcation scenarios in this context. Namely, we have addressed the general case of systems with \( m \) slow variables and infinitely many fast variables, that is, systems for which the fast component lives in a Banach space. In this general context, we have proven center manifold reductions near points where normal hyperbolicity was lost as these correspond to where canard
dynamics can emerge. This effectively enabled us to find local coordinates in which
the original infinite-dimensional problem reduces to an \( m \)-slow/\( n \)-fast system, where
standard results from canard theory and delayed bifurcations of ODEs apply. There-
fore, we have obtained the existence of local canard segments as well as slow passages
through bifurcations in the general framework of dynamical systems with \( m \) slow and
infinitely many fast variables. We then gave a complete rigorous description of such
results near a fold bifurcation of the original fast subsystem as well as in the slow
passage through a Hopf bifurcation. We also provided the main steps of the proofs
in other cases, like in the slow passage through a Turing bifurcation, which had not
been analysed before. Finally, we brought new results along similar lines in slow-fast
delay-differential equations. Every case that has been covered theoretically was also
accompanied by a computational example. Future work will include connecting our
local results to global ones, in order to be able to fully describe canard solutions,
in particular canard cycles, in infinite-dimensional slow-fast dynamical systems, first
with finitely many slow variables and then in systems where both fast and slow vari-
ables are infinite dimensional, which is a substantially more difficult problem. Yet the
present work constitutes a necessary rigorous initial first step towards achieving this
research programme.

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Appendix A. Slow-fast concepts for ODEs. Here we recall basic notions
about slow-fast (finite-dimensional) dynamical systems and elements of geometric
singular perturbation theory (GSPT) [39, 53, 59, 85].

A.1. The standard GSPT setting. For ease of reference, we recall the singu-
lar perturbation problem (1.1),

\[
\begin{align*}
\dot{u} &= F(u, v, \mu, \varepsilon) \\
\dot{v} &= \varepsilon G(u, v, \mu, \varepsilon),
\end{align*}
\]

(A.1)

where \((u, v) \in \mathbb{R}^n \times \mathbb{R}^m\), \(F\) and \(G\) are smooth functions on \(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}\), and
\(0 < \varepsilon \ll 1\) is the timescale separation parameter. The overdot \(\dot{\cdot} = d/dt\) refers to
the fast timescale \(t\) and, hence, we refer to (A.1) as the fast system. Alternatively,
we may introduce the time transformation \(d\tau = \varepsilon dt\) which transforms (A.1) to the
equivalent slow system

\[
\begin{align*}
\varepsilon \dot{u}' &= F(u, v, \mu, \varepsilon) \\
v' &= G(u, v, \mu, \varepsilon),
\end{align*}
\]

(A.2)

where prime \(\dot{\cdot} = d/d\tau\) refers now to the slow timescale \(\tau\). System (A.1) respectively
(A.2) are topologically equivalent and solutions often consist of a mix of slow and
fast segments reflecting the dominance of one time scale or the other. As \(\varepsilon \to 0\), the
trajectories of (A.1) converge during fast segments to solutions of the \(n\)-dimensional
layer problem

\[
\begin{align*}
\dot{u} &= F(u, v, \mu, 0) \\
\dot{v} &= 0,
\end{align*}
\]

(A.3)

while during slow segments, trajectories of (A.2) converge to solutions of

\[
\begin{align*}
0 &= F(u, v, \mu, 0) \\
v' &= G(u, v, \mu, 0),
\end{align*}
\]

(A.4)
which is a \( m \)-dimensional differential-algebraic problem called the \textit{reduced problem}. GSPT uses these lower-dimensional subsystems (A.4) and (A.3) to predict the dynamics of the full \((n + m)\)-dimensional system (A.2) or (A.1) for \( \varepsilon > 0 \).

\textbf{Definition A.1.} The set
\begin{equation}
S := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m \mid F(u, v, \mu, 0) = 0\}
\end{equation}
is the set of equilibria of (A.3). In general, this set \( S \) defines a regular \( m \)-dimensional differentiable manifold, i.e. the Jacobian \( D_{(u,v)} F \) evaluated along \( S \) has full (row) rank \( n \). We refer to \( S \) as the \textit{critical manifold}.

\textbf{Remark A.2.} In general, the set of singularities \( S \) could be a union of disjoint manifolds or a union of manifolds intersecting along lower dimensional submanifolds. The theory exists for these cases as well.

The existence of an \( m \)-dimensional manifold \( S \) in (A.3) implies that its Jacobian evaluated along \( S \) has at least \( m \) zero eigenvalues corresponding to the \( m \)-dimensional tangent space \( T_{(u,v)} S \) of \( S \) at each point \((u, v) \in S\). We call these \( m \) eigenvalues \textit{trivial} associated with the \( \dot{v} = 0 \) subsystem in (A.3). The remaining \( n \) eigenvalues are called \textit{nontrivial} associated with the \( n \)-dimensional submatrix \( D_u F|_S \).

\textbf{A.2. Normal hyperbolicity.}

\textbf{Definition A.3.} A subset \( S_h \subseteq S \) is called \textit{normally hyperbolic} if all \((u, v) \in S_h\) are hyperbolic equilibria of the layer problem (A.3), i.e., the non-trivial eigenvalues have all nonzero real part.

\begin{itemize}
  \item We call a normally hyperbolic subset \( S_a \subseteq S \) attracting if all non-trivial eigenvalues have negative real parts for \((u, v) \in S_a\).
  \item \( S_r \subseteq S \) is called repelling if all non-trivial eigenvalues have positive real parts for \((u, v) \in S_r\).
  \item If \( S_s \subseteq S \) is normally hyperbolic and neither attracting nor repelling we say it is of saddle type.
\end{itemize}

Normal hyperbolicity induces a (unique) splitting of the corresponding tangent space along \( S \), i.e.
\begin{equation}
T_{(u,v)}\mathbb{R}^{n+m} = T_{(u,v)} S \oplus N_{(u,v)}, \quad \forall(u, v) \in S_h,
\end{equation}
where \( T_{(u,v)} S \) and \( N_{(u,v)} \) are invariant subspaces under the Jacobian
\[
J = \begin{pmatrix}
D_u F & D_v F
\end{pmatrix}
\]
of (A.3). More precisely, \( T_{(u,v)} S \) is in the kernel of \( J \) and, hence, corresponds to the \( m \) trivial eigenvalues. This induces a linear map on the \( n \)-dimensional quotient space \( N_{(u,v)} = T_{(u,v)}\mathbb{R}^{n+m}/T_{(u,v)} S \) corresponding to the \( n \) non-trivial eigenvalues of \( J \). The disjoint union of all \( T_{(u,v)} S \), \( TS = \cup T_{(u,v)} S \), forms the \textit{tangent bundle} along \( S \), while the disjoint union of all \( N_{(u,v)} \), \( N = \cup N_{(u,v)} \), forms the corresponding \textit{fast fibre bundle}.

The reduced problem (A.4) is a differential algebraic problem and describes the evolution of the slow variables \( v \) constrained to the critical manifold \( S \). As a consequence, \( S \) defines an interface between the two sub-systems (A.3) and (A.4). In the case of normal hyperbolicity, the reduced vector field can be defined as a dynamical
system by appealing to the (unique) splitting (A.6) which defines a unique projection operator
\[ \Pi^{S_h} : T\mathbb{R}^{n+m}|_{S_h} = TS_h \oplus \mathcal{N} \rightarrow TS_h, \]
a map from \( T\mathbb{R}^{n+m} \) onto the base space \( TS_h \) along \( \mathcal{N} \). This projection operator can be calculated explicitly, i.e. the reduced problem (A.4) along \( S \) is given by the following dynamical system
\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} = \Pi^{S_h} \begin{pmatrix}
F_1(u,v,\mu,0) \\
G(u,v,\mu,0)
\end{pmatrix} = \begin{pmatrix}
0 & -(D_u F)^{-1} D_v F \\
0 & 1
\end{pmatrix} \begin{pmatrix}
F_1(u,v,\mu,0) \\
G(u,v,\mu,0)
\end{pmatrix} = \begin{pmatrix}
-(D_u F)^{-1} D_v F G(u,v,\mu,0) \\
G(u,v,\mu,0)
\end{pmatrix},
\]
where \( F_1(u,v,\mu,0) \) is the \( O(\varepsilon) \) term in the power series expansion of the first component \( F(u,v,\mu,\varepsilon) \) with respect to \( \varepsilon \) of the vector field (A.1); \( G(u,v,\mu,0) \) is the corresponding other (leading order) slow component. Note that \( D_u F \) is a regular square matrix due to normal hyperbolicity. This regularity implies via the implicit function theorem that the critical manifold has a graph representation \( S = \{(u,v) \in \mathbb{R}^{n+m} : u = h(v)\} \). Hence, in the case of normal hyperbolicity, the reduced problem (A.4) can be studied in this coordinate chart \( v \) given by
\[
\dot{v} = G(h(v),v,\mu,0).
\]

The notion of normal hyperbolicity is central to the study of systems of the form (A.1) as key assumption for the persistence of segments of \( S \), which are invariant for \( \varepsilon = 0 \), as local invariant slow manifold \( S_\varepsilon \) for small enough \( \varepsilon > 0 \). These major results have been obtained in the mid-1970s by Neil Fenichel [39].

A.3. Loss of normal hyperbolicity. Geometrically, loss of normal hyperbolicity occurs (generically) along codimension-one submanifold(s) of \( S \) where nontrivial eigenvalue(s) of the layer problem crosses the imaginary axis. Within this set \( S\setminus S_h \), we distinguish two subsets \( \mathcal{AH} \) and \( \mathcal{F} \).

**Definition A.4.** Consider the two cases of loss of normal hyperbolicity associated with the two generic codimension-one bifurcations in the layer problem (A.3):
- the set \( \mathcal{AH} \) of Andronov-Hopf points associated with the crossing of a pair of complex conjugate eigenvalues (with nonzero imaginary part),
- the set \( \mathcal{F} \) of folds or saddle-node points associated with the crossing of a real eigenvalue.

The layer problem is thus considered a bifurcation problem; see, e.g., [60]. The main question is what happens to this bifurcation structure as the singular bifurcation parameter \( 0 < \varepsilon \ll 1 \) is turned on, i.e. when the ‘bifurcation parameter’ \( v \) starts to evolve slowly. To answer this question, we need to understand the reduced problem in a neighbourhood of these codimension-one subsets.

In the case \( \mathcal{AH} \), we note that \( D_u F \) is still a regular square matrix, i.e. \( \det(D_u F) \neq 0 \) everywhere along \( S \) including \( \mathcal{AH} \). Hence the reduced problem can be still studied in the slow coordinate chart given by (A.8). In general, the reduced vector field obeys \( G(h(v),v,\mu,0) \neq 0 \) along \( \mathcal{AH} \subset S \). Thus the reduced flow crosses from one normally hyperbolic branch of \( S \) via \( \mathcal{AH} \subset S \) to another normally hyperbolic branch of \( S \). Assuming this reduced flow is from an attracting branch \( S_a \) to a repelling branch \( S_{r/s} \), this leads to the phenomenon of delayed loss of stability through an Andronov-Hopf bifurcation; see [65, 66, 46] for details.
In the case $F$, the matrix $D_u F$ is singular, i.e. $\det(D_u F) = 0$ along $F$. So, we cannot use (A.8) to describe the reduced flow. While system (A.7) is also not well defined near fold(s) $F \subset S$, we can make an equivalence transformation as follows: First, we use the identity $(D_u F)^{-1} = \text{adj}(D_u F)/(\det D_u F)$ where $\text{adj}(D_u F)$ denotes the adjoint (or adjugate) of the matrix $D_u F$, i.e. the transpose of the co-factor matrix, which gives

\begin{equation}
(A.9) \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -1/\det(D_u F) & \text{adj}(D_u F)/\det(D_u F) \\ G(u, v, \mu, 0) & -\det(D_u F) G(u, v, \mu, 0) \end{pmatrix} .
\end{equation}

This isolates the singularity into the scaler function $\det(D_u F)$. We note that the matrix $\text{adj}(D_u F)$ is still well defined and of rank one, when $D_u F$ has rank deficiency of one (along $F$). Now, we can remove the scalar singularity (pole) by the time transformation

$$d\tau = -\det(D_u F) d\tau_1$$

which gives the corresponding desingularised system

\begin{equation}
(A.10) \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \text{adj}(D_u F) D_v F G(u, v, \mu, 0) \\ -\det(D_u F) G(u, v, \mu, 0) \end{pmatrix} ,
\end{equation}

where with a slight abuse of notation the overdot ($\dot{\cdot} = d/d\tau_1$) refers now to the new slow time $\tau_1$. Importantly, (A.10) represents a dynamical system without singularities that can be analysed by standard dynamical systems tools even in a neighbourhood of $F \subset S$. Note that systems (A.9) and (A.10) are equivalent on normally hyperbolic branches where $\det(D_u F) > 0$, and these systems are also equivalent up to an orientation change on normally hyperbolic branches where $\det(D_u F) < 0$.

**Definition A.5.** Any point $(u, v) \in F$ where

\begin{equation}
(A.11) \quad \text{adj}(D_u F) D_v F G(u, v, \mu, 0) \neq 0
\end{equation}

is called a regular jump point.

Note, solutions of the reduced problem (A.9) in a neighbourhood of $F$ approach regular jump points (A.11) in forward or backward time and they cease to exist at regular jump points due to a finite time blow-up, i.e., the reduced problem (A.9) has a pole at jump points. This finite time blow-up of the reduced flow at a jump point happens exactly in the eigendirection of the defect of the layer problem which defines the locus (and direction) where a switch from slow to fast dynamics (or vice versa) in the full system is possible. This, together with an adequate global return mechanism via the layer problem may provide the seed for a singular relaxation cycle, a concatenation of slow and fast orbit segments that form a loop. For persistence results of such relaxation cycles under small perturbations $\varepsilon \ll 1$ we refer the reader to, e.g., [58, 75].

**Definition A.6.** Any point $(u, v) \in F$ where

\begin{equation}
(A.12) \quad \text{adj}(D_u F) D_v F G(u, v, \mu, 0) = 0
\end{equation}

is called a folded singularity.

Folded singularities are equilibria of the desingularised problem (A.10) but not necessarily of the corresponding reduced problem (A.9) itself. At such a folded singularity,
the vector field of \((A.9)\) contains indeterminate forms which could prevent a finite
time blow-up of certain solutions of the reduced problem approaching a folded
singularity. Such special solutions of the reduced problem that are able to pass in finite
time through such a folded singularity from one branch of \(S\) to another are called
canards, and they play an important role in understanding the genesis of relaxation
oscillations as well as in creating more complex oscillatory patterns in singular
perturbation problems. The terminology ‘canard’ was introduced in [12], and we refer
the interest reader to important persistence results in, e.g., [33, 58, 11, 74, 83, 84].

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