SOME COHOMOLOGY OPERATORS
IN 2-D FIELD THEORY

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ABSTRACT

It is typical for a semi-infinite cohomology complex associated with a graded Lie algebra to occur as a vertex operator (or chiral) superalgebra where all the standard operators of cohomology theory, in particular the differential, are modes of vertex operators (fields). Although vertex operator superalgebras -with the inherent Virasoro action- are regarded as part of Conformal Field Theory (CFT), a VOSA may exhibit a square-zero operator (often, but not always, the semi-infinite cohomology differential) for which the Virasoro algebra acts trivially in the cohomology. Capable of shedding its CFT features, such a VOSA is called a “topological chiral algebra” (TCA). We investigate the semi-infinite cohomology of the vertex operator Weil algebra and indicate a number of differentials which give rise to TCA structures.

1. Introduction

We will always work with an algebraically closed ground field $K$ of characteristic zero.

1.1. Semi-infinite Cohomology

Designed for tame Lie algebras, that is, $\mathbb{Z}$-graded Lie algebras

$$g = \oplus_n g_n$$

(1)

with $\dim g_n$ finite, and for a large category of $g$ modules, semi-infinite cohomology made its debut in mathematics with B. Feigin’s 1984 paper [Fe]. It was studied independently by physicists as “BRST cohomology”, and further investigated by I. B. Frenkel, H. Garland, and G. J. Zuckerman in [FGZ] (see the references therein for
physics literature). A characterization of the semi-infinite cohomology differential as an associative algebra derivation which is generically square-zero first appeared in [A1]. It is easier to outline this last approach for the special (classical) case \( g = g_0 \).

When the Lie algebra is finite dimensional, a differential complex is just the tensor product of a \( g \) module \( M \) with the exterior algebra on the dual of \( g \), i.e.

\[
M \otimes \wedge g'.
\]  

(2)

The universal enveloping algebra \( \mathcal{U}g \) acts on the first factor and the Clifford algebra \( \mathcal{C}g \) on the second, hence any complex (2) is a natural module over the associative algebra

\[
\mathcal{V}g = \mathcal{U}g \otimes \mathcal{C}g.
\]  

(3)

Furthermore the adjoint action of \( g \) on \( \mathcal{U}g \) and the action on \( \mathcal{C}g \) induced from the adjoint and coadjoint representations give rise to a total action \( \theta \) of \( g \) on \( \mathcal{V}g \) via inner derivations (represented by elements \( \theta(x) \) of \( \mathcal{V}g \) for every \( x \) in \( g \)). Note also that \( \mathcal{C}g \) has a natural supergrading where generators from \( g \) have degree \(-1\) and generators from \( g' \) have degree 1. Then it can be shown that ([A1])

**Theorem 1** There exists a unique inner derivation of \( \mathcal{V}g \) (represented by an element \( d \)) with superdegree 1 which satisfies the Cartan identity

\[
di(x) + i(x)d = \theta(x) \quad \forall x \in g
\]  

(4)

in \( \mathcal{V}g \).

Here \( i(x) \) can be thought of as a generator of \( \mathcal{C}g \) coming from \( g \), or as the substitution (contraction) operator on \( \wedge g' \). The derivation \( d \) is square-zero and acts on any complex (2), enabling us to define the (classical) cohomology of \( g \) with coefficients in \( M \). Passing to infinite dimensional \( g \) and semi-infinite cohomology requires a little more work; the only significant differences are that one dualizes graded tame vector spaces piecewise (all duals are thus “restricted”) , and allows a “completion” of \( \mathcal{V}g \). The process also involves replacing the exterior algebra with the semi-infinite exterior module (to be defined later). A generalization of Theorem 1 holds with the extra condition that the derivation preserves the \( \mathbb{Z} \)-grading induced by that of \( g \) in the completed algebra ([A1]).

We will assume the existence and properties of semi-infinite structures and present only the formulas for various operators when the appropriate notation is introduced.
1.2. Vertex Operator Superalgebras

A vertex operator superalgebra (VOSA), or a chiral superalgebra, is a \( \mathbb{Z} \)-bigraded vector space

\[ V = \bigoplus_{j,n} V^j[n] \]  

with additional structure. We will refer to \( n \) as the degree, or weight (as physicists do). The other grading, \( j \), will be the superdegree (fermion number, ghost number...).

Algebraic properties of VOSA’s have been isolated, refined, and generalized by a number of mathematicians. The Monster book [FLM] and the monographs [FHL], [Li], [FZ], [DL] are good references. First of all a VOSA is equipped with an injective linear map

\[ V \to \text{End}(V)[[z, z^{-1}]] \]

\[ v \mapsto v(z) = \sum v_n z^{-n-1} \]  

where \( v \) is called a state, \( v(z) \) is called a vertex operator (VO), or a field, and \( v_n \) is a mode. We also require that \( V \) be a graded representation of the Virasoro algebra (with respect to \( n \)) such that \( L_0 \) acts semisimply on \( V \) and has eigenvalue \( n \) on \( V^*[n] \). Recall that \( \text{Vir} \) has a basis consisting of \( \{L_n\}_{n \in \mathbb{Z}} \) and a central element \( c \) with relations

\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{m^3-m}{12} \delta_{n+m,0} c. \]  

An important axiom is

\[ L_{-1} v \leftrightarrow \frac{d}{dz} v(z). \]  

Next, there are special elements

\[ 1 \text{ ("vacuum") } \leftrightarrow \text{id} \cdot z^0 \]

\[ \omega \leftrightarrow \sum L_n z^{-n-2} \text{ ("stress-energy field")} \]  

with weights 0, 2 and superdegrees 0, 0 respectively. The main axiom of a VOSA, the Cauchy-Jacobi identity which can be found in [FLM], may be replaced by the following “commutativity” condition as was shown by C. Dong and J. Lepowsky ([DL]): For any homogeneous \( v, w \) in \( V \) there exists \( t \gg 0 \) such that

\[ [v(z_1), w(z_2)](z_1 - z_2)^t = 0 \]  

where [ , ] denotes the supercommutator of two formal series.

Any two elements \( v \) and \( w \) of \( V \) produce infinitely many elements via the multiplications

\[ v_n \cdot w. \]
In this sense, one may ask whether there is a set of naturally chosen elements of \( V \) that generate the whole VOSA. One usually constructs a VOSA from finitely many generators, but it is much more difficult to determine whether a VOSA constructed by other means is finitely generated. Such questions may arise, for example, when the VOSA is a cohomology.

The coefficient of \( z^{-1} \) in (3), namely \( v_0 \in \text{End}(V) \), is called the residue, or charge, of the VO \( v(z) \). It is well-known that

\[
v_0 1 = 0 \tag{12}
\]

and

\[
[v_0, w(z)] = (v_0 \cdot w)(z) \tag{13}
\]

for all \( v, w \) in a VOSA. In particular,

\[
v_0 \cdot w = 0 \text{ in } V \iff [v_0, w_n] = 0 \text{ in } \text{End}(V). \tag{14}
\]

Residues give rise to new VOSA’s, because

**Lemma 1** If \( \omega \) is in \( \text{Ker } v_0 \) for some \( v \in V \), then \( \text{Ker } v_0 \) is a vertex operator subalgebra of \( V \).

**Lemma 2** For a square-zero residue \( v_0 \), \( \text{Im } v_0 \) is an ideal of the VOSA \( \text{Ker } v_0 \). Consequently the cohomology \( H(V, v_0) \) is a VOSA (provided that \( \omega \) is in \( \text{Ker } v_0 \)).

Again these two are widely used results for which short proofs can be found in [A2]. If \( v_0^2 = 0 \), we will call the state \( v \) a reduction element, inspired by Hamiltonian reduction. It is a fact of life that most semi-infinite complexes not only show up as VOSA’s, but also manifest a special reduction element with residue equal to the semi-infinite cohomology differential (which will be denoted by \( Q \) as is customary from now on). Examples are in [LZ].

**2. The Vertex Operator Weil Algebra**

**2.1. Definition and Properties**

The semi-infinite Weil complex \( W^{\infty/2}g \) associated to a tame Lie algebra \( g \) is a generalization of the classical Weil algebra

\[
W \ell = S\ell' \otimes \wedge \ell'
\]
(\ell = \text{finite dimensional Lie algebra}) which is very well understood ([GHV]). As a product of symmetric and exterior algebras, \( W\ell \) has a supercommutative associative algebra structure, as well as two differentials \( d \) and \( h \) (the classical cohomology and Koszul differentials respectively) which are inner superderivations. For an arbitrary \( \ell \), we have
\[
H(W\ell, h) = H(W\ell, d + h) = K
\]
\((d + h \text{ is the Weil differential})\). If \( \ell \) is reductive, it is also known that
\[
H(W\ell, d) = (S\ell')^\ell \otimes (\wedge\ell')^\ell
\]
(17) where superscripts denote subspaces of invariants under the action of \( \ell \) by derivations. It turns out that (17) is a finitely generated supercommutative associative algebra ([GHV]). We expect its analogue to be a (finitely generated?) VOSA.

It is possible to define \( W^{\infty/2}g \) for any tame Lie algebra; B. Feigin and E. Frenkel gave the general definition and computed the semi-infinite cohomology for the case \( g = Vir \) in [FF]. The next logical choice of \( g \) is the loop algebra \( \tilde{\ell} \) of a finite dimensional Lie algebra \( \ell \), defined by
\[
\tilde{\ell} = \ell \otimes K[t, t^{-1}]
\[
[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}.
\]
(18) We will call \( W^{\infty/2}\tilde{\ell} \) a vertex operator Weil algebra (VOWA) since it is naturally a VOSA -for what is called the “standard”, or “physical”, vacuum- and properly contains the classical algebra (15). Even the semi-infinite analogues of \( d \) and \( h \) (now called \( Q \) and \( h \)) restrict to (15) as the classical operators ([A2]). The \( Q \)-cohomology of the VOWA was first investigated in [A2]. We will give full details of the semi-infinite theory below.

Analogues of \( S\ell' \) and \( \wedge\ell' \), namely the semi-infinite symmetric and exterior modules \( S^{\infty/2}\tilde{\ell}' \) and \( \wedge^{\infty/2}\tilde{\ell}' \), can be defined as induced modules over the Weyl and Clifford algebras based on \( \tilde{\ell} \) with its restricted dual
\[
\tilde{\ell}' = \oplus_n Hom(\ell \otimes K t^{-n}, K)
\]
(19) respectively. The resulting tensor product module
\[
W^{\infty/2}\tilde{\ell} = S^{\infty/2}\tilde{\ell}' \otimes \wedge^{\infty/2}\tilde{\ell}'
\]
(20) has appeared under the name “\( bc\beta\gamma \)-system” in physics (although not in connection with semi-infinite cohomology). We will use the alias “VOWA” but keep the notation. Let \( \{u\} \) be a basis for \( \ell \) and \( \{u'\} \) be the dual basis for \( \ell' \). The Clifford algebra on \( \ell \) is generated by the symbols
\[
b_n^u, c_n^{u'}
\]
(21)
which satisfy the relations
\[ c' \ u \ b' + b' \ c = < u', v > \delta_{m,n} \]
\[ b' \ b' + b' \ b' = 0 \]
\[ c' \ c' + c' \ c' = 0. \]  \(22\)

Similarly the Weyl algebra is generated by
\[ \beta_n, \gamma_n' \]  \(23\)

with
\[ \gamma_n' \beta_n - \beta_n \gamma_n' = < u', v > \delta_{n+m} \]
\[ \beta_n \beta_n - \beta_n \beta_n = 0 \]
\[ \gamma_n' \gamma_n' - \gamma_n' \gamma_n' = 0. \]  \(24\)

Assigning a superdegree (fermion number) of \(-1\) to the \(b\)'s, \(+1\) to the \(c\)'s, and \(0\) to the rest of the generators, we will express the relations (22) and (24) in terms of the supercommutator \([,]\) from this point on. We now designate half of the generators as creation operators and the rest as annihilation operators. Let
\[ b_n', \beta_n' \text{ with } n \leq -1, \text{ and} \]
\[ c'_n, \gamma_n' \text{ with } n \leq 0 \]  \(25\)

be the creation operators. Note that operators of either type supercommute among themselves. It is possible to shift the division point (hence choose different vacua) but then \(W^{\infty/2} / \tilde{\ell}\) will not be a VOSA. If \(1_{sym}\) and \(1_{ext}\) denote basis elements for the trivial representations of the subalgebras generated by the annihilation operators, then \(S^{\infty/2} / \tilde{\ell}\) and \(\wedge^{\infty/2} / \tilde{\ell}\) are the induced representations of the Weyl and Clifford algebras, and \(W^{\infty/2} / \tilde{\ell}\) is spanned by strings of creation operators applied to the (standard) vacuum
\[ 1 = 1_{sym} \otimes 1_{ext}. \]  \(26\)

The classical algebra \(13\) sits inside \(20\) as the span of monomials in \(c'_0\) and \(\gamma'_0\).

We define an action of the Virasoro algebra on \(W^{\infty/2} / \tilde{\ell}\) via an action on the (completed) Clifford-Weyl algebra by inner derivations: This action is independent of the Lie algebra structure of \(\ell\) and reflects the Witt = Der \(K[t, t^{-1}]\) action on the second factor of \(18\) (\(Vir\) is the unique central extension of Witt with \(L_n\) corresponding to \(-t^{n+1} \frac{d}{dt}\)). It is easy to check that
\[ L_n \cdot b_n' = -m b_{n+m}' \]
\[ L_n \cdot \beta_n' = -m \beta_{n+m}' \]
\[ L_n \cdot c'_m = (-n - m) c'_{n+m} \]
\[ L_n \cdot \gamma'_m = (-n - m) \gamma'_{n+m} \]  \(27\)
extends to an action of $\text{Vir}$. We have

$$L_n = - \sum_{u, m} m : b_{n+m}^u c_{-m}^{u'} : - \sum_{u, m} m : \beta_{n+m}^u \gamma_{-m}^{u'} :$$

(28)

where the infinite sum of products of operators is well-defined on the module $W^\infty/2\ell$ and the normal ordering $: A :$ merely means that an annihilation operator, if any, should be written on the right (with a proper sign change). For a more detailed survey of completed algebras and normal ordering see [A1]. The central element of $\text{Vir}$ acts trivially on $W^\infty/2\ell$, and

$$L_n \cdot 1 = 0 \iff n \geq -1.$$ 

(29)

Moreover $W^\infty/2\ell$ is graded by the eigenvalues of $L_0$, as any monomial is clearly seen to have a nonnegative $L_0$-degree by (25). The weight zero part is $W_0$.

We need to define a VO $v(z)$ for each basic monomial $v$. First let us define “singles”

$$b_{-n}^u 1 \leftrightarrow b_v(z) = \sum b_{n}^u z^{-n-1},$$

$$\beta_{-1}^u 1 \leftrightarrow \beta_v(z) = \sum \beta_{n}^u z^{-n-1},$$

$$c_0^{u'} 1 \leftrightarrow c_{u'}(z) = \sum c_{n}^{u'} z^{-n},$$

$$\gamma_{0}^{u'} 1 \leftrightarrow \gamma_{u'}(z) = \sum \gamma_{n}^{u'} z^{-n}.$$ 

(30)

Other creation operators are (up to scalars) images of the above under powers of $L_{-1}$, so by (33), we had better have

$$(b_{-n}^u 1)(z) = \frac{1}{(n-1)!} \left( \frac{d}{dz} \right)^{n-1} b_v(z),$$

(31)

and so on. For the most general product

$$v = v^{(1)} \cdots v^{(r)} 1, \ r \geq 1,$$

(32)

$v^{(i)} = \text{a creation operator}$, we define

$$v(z) = : (v^{(1)} 1)(z) \cdots (v^{(r)} 1)(z) :.$$ 

(33)

This normal ordering is defined by induction: For $r = 2,$

$$: v(z) w(z) := v^-(z) w(z) \pm w(z) v^+(z),$$

(34)

$v^+(z)$ indicating half sums over creation and annihilation operators respectively, and the sign in the middle is ($-$) only when both states are odd. Then (33) is defined to be

$$\{v^{(1)}\}^- (z) v^{(2)} (z) \cdots v^{(r)} (z) \pm v^{(2)} (z) \cdots v^{(r)} (z) \{v^{(1)}\}^+(z).$$

(35)
(In [Li] we have the most general definition of normal ordering as

\[ :v(z)w(z) := (v_{-1} \cdot w)(z) \]  

(36) for any \( v, w \) in \( V \).) The Virasoro element \( \omega \) corresponds to the field

\[ \omega(z) = - \sum_u b^u(z) \frac{d}{dz} c^u(z) : - \sum_u \beta^u(z) \frac{d}{dz} \gamma^u(z) : . \]  

(37)

The fact that the \( bc\beta\gamma \)-system thus forms a VOSA was known by some experts but no complete proof was published prior to [A2]. The VOSA structure is implicit, for example, in [KVDL].

2.2. Semi-infinite Structure and Compatibility

So far we made no use of the specific Lie algebra \( \ell \) itself. One defines an action of \( \tilde{\ell} \) on the VOWA via

\[ \theta(u_n) = \sum_{u,v,m} : \beta^{[u,v]}_{n+m} \gamma^v_{-m} : + \sum_{u,v,m} : b^{[u,v]}_{n+m} c^v_{-m} : \]  

(38)

\( (u_n \text{ is } u \otimes t^n) \). The well-known formula for the semi-infinite cohomology operator \( Q \) ([Fe],[A1]) specializes to

\[ Q = \sum_{u,v,i,j} : \beta^{[u,v]}_{i+j} \gamma^v_{-i} c^u_{-j} : + \sum_{u,v,i < j} : b^{[u,v]}_{i+j} \gamma^v_{-i} c^u_{-j} : . \]  

(39)

The semi-infinite version of the Cartan identity ([8]) is

\[ Q b^u_n + b^u_n Q = \theta(u_n) \ \forall u, n. \]  

(40)

Once again, we construct all these operators as inner derivations of the completed Clifford-Weyl algebra ([A2]), and all computations are reduced to supercommutation relations for the generators, such as the last equation and

\[ [Q, c^u_n] = - \frac{1}{2} \sum_{v,m} c^u_{n+m} c^{v'}_{-m} \]

\[ [Q, \beta^u_n] = \sum_{v,m} \beta^{[v,u]}_{n+m} c^{v'}_{-m} \]

\[ [Q, \gamma^u_n] = \sum_{v,m} \gamma^{[v,u']}_{n+m} c^{v'}_{-m} . \]

The superdegree (fermion number) of each operator is easily read from its formula, by subtracting the number of \( b \)'s from the number of \( c \)'s. The supercommutator is an anticommutator only when both arguments are odd. We also have

\[ Q \cdot 1 = 0 \]  

(41)
\[ \theta(u_n) \cdot 1 = 0 \iff n \geq 0. \quad (42) \]

Other commutation relations are
\[
\begin{align*}
[Q, \theta(u_n)] & = 0 \quad \forall u, n \\
[Q, L_n] & = 0 \quad \forall n.
\end{align*}
\quad (43)
\]

At this point we also introduce the semi-infinite *Koszul differential*
\[
h = \sum_{u,n} \gamma_{u-n} b_n^u \quad (44)
\]

([FF]) and a “homotopy operator”
\[
k = \sum_{u,n} n \beta_n^u c_{-n}^u \quad (45)
\]

which is a modified version of the classical one in [GHV].

**Proposition 1** The operators \(Q, h, k\) satisfy
\[
\begin{align*}
[Q, h] & = [Q, k] = 0 \\
[h, k] & = -L_0 \\
[h, L_n] & = 0 \quad \forall n \\
[k, L_0] & = 0 \\
[h, \theta(u_n)] & = 0 \quad \forall u, n \\
[k, \theta(u_0)] & = 0 \quad \forall u \\
Q^2 & = h^2 = k^2 = 0. \quad (46)
\end{align*}
\]

**Proposition 2** The restrictions of the given operators to the degree-zero (classical) subspace of \(W^{\infty/2\ell}\) are
\[
\begin{align*}
Q|_{\text{deg}=0} & = d_{\text{classical}} \\
h|_{\text{deg}=0} & = h_{\text{classical}} \\
k|_{\text{deg}=0} & = 0 \\
\theta(u_0)|_{\text{deg}=0} & = \theta(u). \quad (48)
\end{align*}
\]

Proofs are given in [A2]. Then various cohomologies of \(W^{\infty/2\ell}\) are as follows:
Proposition 3

\[
H(W^{\infty/2} \tilde{\ell}|_{\text{deg}=0}, Q) = H(W \ell, d) \\
H(W^{\infty/2} \tilde{\ell}, h) = H(W \ell, h) = K \quad \text{by (46)} \\
H(W^{\infty/2} \tilde{\ell}, k) = W \ell
\]  

(49)

The acyclicity of the complex \((W^{\infty/2} \tilde{\ell}, Q + h)\) is shown in [FF] by spectral sequences. We include an elementary proof.

Proposition 4

\[ H(W^{\infty/2} \tilde{\ell}, Q + h) = K. \]  

(50)

Proof. Since \([Q + h, k] = -L_0\), the cohomology is restricted to the classical part. By Proposition 3 and (16) the result follows. □

We emphasize that these results generalize to all tame Lie algebras. In the loop algebra case, in order to utilize the powerful VOSA techniques in \(Q\)-cohomology computations, we write the operators above as modes of certain vertex operators (the method of conversion is just trial and error). There may be more than one suitable field, in which case we try to choose one that is in the kernel of \(Q\) or the \(\ell\) action, etc. Here are some:

\[
J(z) = \sum_{u,v} : \beta^{[u,v]}(z) \gamma^u(z) c^v(z) : + \frac{1}{2} \sum_{u,v} : b^{[u,v]}(z) c^{u'}(z) c^{v'}(z) :
\]

(51)

is a familiar sight for mathematical physicists; its residue is \(Q\). The actions of elements of \(\tilde{\ell}\) are represented by modes of the operators

\[
\theta^u(z) = \sum_v : \beta^{[u,v]}(z) \gamma^v(z) : + \sum_v : b^{[u,v]}(z) c^v(z) := \sum_n \theta(u_n) z^{-n-1}.
\]

(52)

The operators \(h\) and \(k\) are \(h_0\) and \(k_1\) for the following VO’s:

\[
h(z) = \sum_u \gamma^u(z) b^u(z), \quad k(z) = \sum_u \beta^u(z) \frac{d}{dz} c^u(z).
\]

(53)

Recall that by our convention the corresponding states are denoted by \(J, \theta^u, h,\) and \(k\).

By Proposition 3 we know that the classical part of the cohomology is always there, and indeed there may not be anything else (it happens for Vir and affine Kac-Moody algebras; see remarks in [A1]). On the other hand for an abelian loop algebra \(\theta\) and \(Q\) are identically zero and we get rather a lot of cohomology. What can be said about the \(Q\)-cohomology for a general loop algebra \(\tilde{\ell}\) outside degree zero?
Theorem 2  The states

\[ h = \sum_u \gamma_0^u b_{-1}^u 1 \]  \hspace{1cm} (54)
\[ k = \sum_u \beta_{-1}^u c_{-1}^u 1 \]  \hspace{1cm} (55)

are both in \( (W^\infty/2\tilde{\ell})^\ell \) and they correspond to nontrivial \( Q \)-cohomology classes. As a result, \( H(W^\infty/2\tilde{\ell}[n], Q) \) is nonzero for every weight \( n \geq 0 \).

Proof. It is straightforward to check

\[ h, k \in (W^\infty/2\tilde{\ell})^\ell \cap \text{Ker } Q \]  \hspace{1cm} (56)

(see [A2]). The first state is nonzero in the cohomology as \( Q \) increases the superdegree by 1 and \( j = -1 \) is already the smallest superdegree possible for weight one. This implies \( L_0 \) -hence \( \text{Vir} \) - acts nontrivially on \( H(W^\infty/2\tilde{\ell}, Q) \). But then \( k \) cannot be exact, either, because \( k = k_1 \) satisfies the nontrivial relation

\[ h k + k h = -L_0 . \]  \hspace{1cm} (57)

Finally, we observe both \( h \) and \( k \) are singular vectors. Therefore we can make use of the relation

\[ L_n L_{-n} - L_{-n} L_n = 2n L_0 \]  \hspace{1cm} (58)

to conclude that \( L_{-n} \cdot h \) with weight \( n + 1 \) represents a nonzero class for each \( n \geq 1 \).

\[ \square \]

Corollary 1  The Virasoro element

\[ \omega = -\sum_u b_{-1}^u c_{-1}^u 1 - \sum_u \beta_{-1}^u \gamma_{-1}^u 1 \]  \hspace{1cm} (59)

represents a nonzero \( Q \)-cohomology class.

Proof. From (13) we know \([Q, L_n] = 0\) for all \( n \), which is equivalent to \( Q \cdot \omega = 0 \) by (14). The mode \( L_0 \) of \( \omega(z) \) is nonzero in the subquotient VOSA by the Theorem, hence \( \omega \) is not \( Q \)-exact.

\[ \square \]

3. The Semi-infinite Cohomology for the Semisimple Case

We will analyze the cohomology space \( H(W^\infty/2\tilde{\ell}, Q) \) for a semisimple Lie algebra \( \ell \) using both linear and VOSA methods, and produce infinitely many explicit nonzero cohomology classes in \( \text{deg} \neq 0 \) for \( \ell = sl(2, K) \).
Even for an arbitrary tame Lie algebra $g$, $W^{∞/2}g$ is a direct sum of finite dimensional subspaces stable under $Q$ and $g_0$, namely those spanned by monomials with fixed $L_0$-degree and fixed symmetric degree (which is the counterpart of fermion number in a VOWA, i.e. the number of $\gamma$’s minus the number of $\beta$’s). For the loop algebra of semisimple $\ell$ we have $g_0 = \ell$ and complete reducibility follows. One remarkable corollary is that every $Q$-cohomology class is representable by an $\ell$ invariant state just like the classical case (see [GHV]).

Recall that $H(W^{∞/2}\tilde{\ell}|_{\deg=0}, Q)$ is an associative algebra generated by finitely many $\ell$ invariants. It is easy to see that the fields corresponding to the classical generators suffice to produce all the deg= 0 cohomology ($c$’s and $\gamma$’s supercommute).

3.1. New Square-zero Operators

The maps $h$ and $k$ make use of the natural correspondence between the $b$-$\beta$ and $c$-$\gamma$ systems. In case of semisimplicity there is one more correspondence to be exploited, namely the (Killing) isomorphism

$$\phi : \ell \to \ell'$$

between the $b$-$\gamma$ and $\beta$-$c$ systems. We add to our list the maps ([A2])

$$r = \sum_{u,n} n \gamma_{-n}^\phi u \ c_n'$$

and

$$t = \sum_{u,n} \beta_{-n}^{\phi^{-1}(u')} b_n'$$

where

$$r = -r_0, \quad t = t_1$$

for the fields

$$r(z) = \sum_u \gamma^\phi(u)(z) \frac{d}{dz} c^u(z)$$

and

$$t(z) = \sum_u \beta^{\phi^{-1}(u')}(z) b^u(z).$$

The states $h, k, r, t$ are all Virasoro-singular reduction elements (they are killed by $L_n$ with $n > 0$). Singular fields are also called “primary”. More supercommutators follow.

**Proposition 5**

$$r^2 = t^2 = 0$$

$$[r, t] = -L_0$$
\[ [Q, r] = [Q, t] = 0 \]  \hspace{1cm} (67)
\[ [r, h] = [r, k] = [t, h] = [t, k] = 0 \]  \hspace{1cm} (68)
\[ [t, \theta(u_n)] = 0 \quad \forall u, n \]  \hspace{1cm} (69)
\[ [r, \theta(u_0)] = 0 \quad \forall u \]  \hspace{1cm} (70)
\[ [r, L_n] = 0 \quad \forall n \]  \hspace{1cm} (71)
\[ [t, L_0] = 0 \]  \hspace{1cm} (72)

**Proof.** See [A2]. \(\square\)

By Propositions 3 and 1 we have a decomposition

\[ (\text{Ker } h \cap \text{Ker } r) \oplus (\text{Ker } h \cap \text{Ker } t) \oplus (\text{Ker } k \cap \text{Ker } r) \oplus (\text{Ker } k \cap \text{Ker } t) \]  \hspace{1cm} (73)

of each subspace \((W^\infty/\tilde{\ell})|_{\text{deg}=\lambda}\) and its \(Q\)-cohomology. This is a useful computational tool, especially since the four subspaces are isomorphic via maps that commute with \(Q\) and \(\ell\).

We have, in addition to Theorem 2,

**Theorem 3** The states

\[ r = \sum_u c_{\alpha} \gamma_{\phi(u)}^0 e_{\gamma}^u 1 \]
\[ t = \sum_u b_{\beta} \rho_{\phi^{-1}(u')}^u b_{\gamma}^u 1 \]  \hspace{1cm} (74)

are both invariant under \(\ell\) (a semisimple Lie algebra) and they represent nontrivial \(Q\)-cohomology classes.

### 3.2. The case \(sl(2, K)\)

Let \(sl(2, K) = \langle x, y, h \rangle\), where

\[ [x, y] = h, \quad [h, x] = 2x, \quad \text{and} \quad [h, y] = -2y. \]  \hspace{1cm} (75)

Then basis elements of \(\tilde{sl}(2, K)\) will be denoted by \(x_n, y_n, h_n\). It is unfortunately more or less standard to denote both the Koszul differential and the basis element in the Cartan subalgebra of \(sl(2, K)\) by \(h\).

The coadjoint action of \(\tilde{\ell} = \tilde{sl}(2, K)\) on \(\tilde{\ell}'\) is given by the table

\[
\begin{align*}
\text{ad}' &\quad x' &\quad y' &\quad h' \\
\gamma_{x_m} &\quad 2h'_{r-m} &\quad 0 &\quad -y'_{r-m} \\
\gamma_{y_m} &\quad 0 &\quad -2h'_{r-m} &\quad x'_{r-m} \\
\gamma_{h_m} &\quad -2x'_{r-m} &\quad 2y'_{r-m} &\quad 0.
\end{align*}
\]  \hspace{1cm} (76)
The Killing form $\mathcal{K} : \ell \times \ell \to K$ is given by

\[
\begin{array}{ccc}
\mathcal{K} & x & y & h \\
x & 0 & \frac{1}{2} & 0 \\
y & \frac{1}{2} & 0 & 0 \\
h & 0 & 0 & 1.
\end{array}
\] (77)

Then $\mathcal{K}$ gives rise to the $\ell$-invariant isomorphisms

\[
\phi : \ell \to \ell', \quad \phi^{-1} : \ell' \to \ell \\
x \mapsto \frac{1}{2}y' \quad x' \mapsto 2y \\
y \mapsto \frac{1}{2}x' \quad y' \mapsto 2x \\
h \mapsto h' \quad h' \mapsto h.
\] (78)

Let us appeal to some classical invariant theory at this stage. As a $\ell$ module, $W^{\infty/\tilde{\ell}}$ is nothing but the tensor product of the symmetric and exterior algebras on countably many copies of $\ell$ and $\ell'$, where we distinguish between adjoint and coadjoint representations, factors to be symmetrized and antisymmetrized, and factors with different $L_0$-degrees just by looking at the symbols and their subscripts. The Lie action doesn’t change any of these subdivisions. To obtain all the associative algebra generators for the $\ell$ invariant subalgebra, all we need to know is the generators of

\[
[S(V) \otimes \wedge(W)]^{\text{sl}(2)}
\] (79)

where $V$ and $W$ denote direct sums of finitely many copies of, say, the adjoint representation. By one of those lucky coincidences that happen at low dimensions the classical $SO(3)$ theory of invariants tells us all ([Ho],[A2]). The generators of (79) are quadratic and cubic, and there is a list of thirty types of infinitely many such generators for the whole invariant subalgebra. We study the corresponding list of VOSA generators and a miracle follows the lucky coincidence.

**Theorem 4** Among the infinitely many purely symmetric $\text{sl}(2)$ invariant quadratics and cubics (i.e. those involving only creation operators in $\beta$ and $\gamma$’s), only the following five are in $\text{Ker} \, Q$:

\[
\begin{align*}
v^{(1)}_1 & = \{ (\beta_{-1}^h)^2 + 4\beta_{-1}^x\beta_{-1}^y \} 1 \\
v^{(2)}_1 & = \{ \beta_{-1}^h \beta_{-2}^h + 2\beta_{-1}^x\beta_{-1}^y + 2\beta_{-1}^y\beta_{-1}^x \} 1 \\
v^{(3)}_1 & = \{ (\gamma_{0}^{\prime h})^2 + \gamma_{0}^{\prime y'} \} 1 \\
v^{(4)}_1 & = \{ \gamma_{-1}^{\prime h} \gamma_{-1}^{\prime y'} + \frac{1}{2} \gamma_{-1}^{\prime x} \gamma_{-1}^{\prime y'} + \frac{1}{2} \gamma_{-1}^{\prime y'} \gamma_{-1}^{\prime x} \} 1 \\
v^{(5)}_1 & = \{ \beta_{-1}^h \gamma_{0}^{\prime h} + \beta_{-1}^x \gamma_{0}^{\prime y'} + \beta_{-1}^y \gamma_{0}^{\prime x} \} 1.
\end{align*}
\] (80-84)

None of the $v^{(6)}_1$ is $Q$-exact.
Proof. First part of the proof is by direct computation and can be found in [A2]. The second assertion follows from

\[
\begin{align*}
\text{h} \cdot v^{(1)} 1 &= 2t \neq 0 \\
L_1 \cdot v^{(2)} 1 &= 2v^{(1)} 1 \neq 0 \\
v^{(3)} 1 &= \text{classical} \\
k \cdot v^{(4)} 1 &= -r \neq 0 \\
h \cdot v^{(5)} 1 &= h \neq 0
\end{align*}
\]

(in the cohomology). □

This drastic reduction gives us hope for a manageable cohomology. It is not difficult to show that the following elements are also closed:

\[
\begin{align*}
\{v^{(1)}\}^m \{v^{(2)}\}^n 1, & \ m, n \geq 0, \\
\{v^{(3)}\}^m \{v^{(4)}\}^n 1, & \ m, n \geq 0, \\
\{v^{(1)}\}^m v^{(5)} 1, & \ m \geq 0, \\
\{v^{(3)}\}^m v^{(5)} 1, & \ m \geq 0.
\end{align*}
\]

Let \((l, j)\) denote the symmetric and super degrees respectively. We have the following specific classes for deg= 0 and deg= 1: The classical part consists of classes

\[
\{(\gamma'_0)^2 + \gamma'_0 \gamma'_0\}^n \{c'_{0' 0' c'_{0' 0'}}\}^a 1
\]

with \(n \geq 0\) and \(a = 0\) or 1. In particular, there is one dimensional cohomology for every pair \((l, j)\) with \(l \geq 0\), even, and \(j = 0\) or 3. All four operators \(h, k, r, t\) act by zero here as they shift \(l\) by \(\pm 1\). As for deg= 1, where it is only possible to get \(j = -1, 0, 1, 2, 3,\) or 4, we can write the following infinitely many classes for the first three values of \(j\):

**Theorem 5** The element

\[
\{v^{(3)}\}^n v^{(4)} 1
\]

is in Ker \(Q\), as well as Ker \(h\) and Ker \(r\), for all \(n \geq 0\). Its isomorphic images in the remaining three subspaces in the decomposition (73) are

\[
\{v^{(3)}\}^n h, \ \{v^{(3)}\}^n v^{(5)} 1, \ \text{and} \ \{v^{(3)}\}^n r.
\]

None of the above four is exact.

Proof. [A2]. □

The first three types of classes in the Theorem account for all of the \(j = -1\) and \(j = 0\) cohomology up to \(l = 10\) (dimensions checked by computer).
4. Topological Chiral Algebras

4.1. Definition and Examples

We take the following definition from [LZ]. A topological chiral algebra (TCA) consists of

(i) A VOSA $V$,

(ii) A weight one even field $F(z)$ whose residue (charge) $F_0$ is the “fermion number operator”,

(iii) A weight one primary (Virasoro-singular) field $J(z)$ with fermion number one and a square-zero charge $J_0 = Q$,

(iv) A weight two primary field $G(z)$ with fermion number $-1$ and satisfying

$$[Q, G(z)] = L(z) \quad (97)$$

where $L(z)$ is the stress-energy field.

Examples of TCA’s are abundant among semi-infinite cohomological complexes associated to the Virasoro algebra (see e.g. [LZ]). It should be pointed out that the above definition is tailored for these examples: The semi-infinite exterior module for $Vir$ is a simple $bc$-system generated by two fields $b(z)$ and $c(z)$, and the total Lie algebra action and the Virasoro action on the VOSA are one and the same. Eqn. (97) is just the Cartan identity in disguise, with $G(z) = b(z)$. This identity always implies that the tame Lie algebra acts trivially in the $Q$-cohomology. In case of $Vir$, it also says the $\omega$ element of the theory is exact.

4.2. The VOWA as a TCA

It is obvious from Corollary 1 that VOWA’s do not enjoy the above characteristics. Since $\omega$ is not exact, we look for differentials other than $Q$ to reduce the complex. It turns out that we have one for every VOWA and an additional one for semisimple $\ell$.

Proposition 6

$$[h, k(z)] = -\omega(z) \quad (98)$$

and

$$[r, t(z)] = -\omega(z). \quad (99)$$

Proof.

$$[h, k(z)] = [h_0, k(z)] = (h_0 \cdot k)(z) \quad (100)$$

and

$$[r, t(z)] = [r_0, t(z)] = (r_0 \cdot t)(z), \quad (101)$$
but
\[ h_0 \cdot k = r_0 \cdot t = -\omega. \quad \Box \quad (102) \]

The fermion numbers of \( h \) and \( r \) are \(-1\) and \(1\) respectively. The field \( F(z) \) exists, as in any respectable theory, and is given by
\[ F(z) = \sum_u : c^u(z)b^u(z) :. \quad (103) \]

Its charge \( F_0 \) counts the superdegrees.

What is the reduction in each case? The \( h \)-cohomology has already been identified as \( K \). The \( r \)-cohomology is in degree zero by virtue of the homotopy relation
\[ rt + tr = -L_0, \quad (104) \]
and for \( \ell = sl(2) \) it can be shown to be the one dimensional space spanned by
\[ c^{h'}_0 c^{x'}_0 c^{y'}_0 1. \quad (105) \]

It may be possible to construct differentials with larger cohomologies, still necessarily confined to weight zero.

4.3. The Big Picture: Say Cheese!

\[
\begin{array}{c}
\text{QFT} & \leftarrow & \text{CQFT} \\
\uparrow & & \uparrow \\
2\text{-D CFT} & \leftarrow & \text{TCFT} & \rightarrow & 2\text{-D TFT} \\
\downarrow & & \downarrow & & \downarrow \\
\text{VOSA} & \leftarrow & \text{TCA} & \rightarrow & \text{DT}
\end{array}
\]

QFT = Quantum Field Theory, CQFT = Cohomological Quantum Field Theory, CFT = Conformal Field Theory, TCFT = Topological Conformal Field Theory, TFT = Topological Field Theory, VOSA = Vertex Operator Superalgebra, TCA = Topological Chiral Algebra, DT = Differential Topology. Arrows roughly indicate inclusions. Some of the notions are mathematically rigorous and some are not. (Courtesy: G. J. Zuckerman)

For more information on the middle column, we recommend [W1], [W2], [LZ], [DVV], and [Ge].
5. Conclusion

The VOWA provides a variety of interesting phenomena and is arguably the simplest VOSA in which we can study them. It has links to both mathematical physics and to classical Lie algebra theory, including the theory of invariants. By analogy to classical results we conjecture that $H(W^{\infty/2\ell}, Q)$ for semisimple $\ell$ is finitely generated as a VOSA, which is supported by the evidence in the special case $\ell = sl(2)$: The fields $h(z), k(z), r(z), t(z),$ and $\Pi c^a(z)$ account for all the cohomology classes computed by direct or indirect methods so far. Producing a canonical set of generators for any $\ell$ is the ultimate goal in this direction. Also the $Vir$ module structure of $H(W^{\infty/2\ell}, Q)$ and the cohomological dimensions are yet to be calculated explicitly.

What are the primary fields? The reduction elements? Can we choose generators to be singular reduction elements (like the five fields above)? Why would we want to do that? Are there any other TCA structures? Eventually, we would like to attach a (pseudo) physical meaning to the VOWA. The context may well be topological conformal field theories.

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