RESOLUTIONS OF INITIAL IDEALS OF CLOSED DETERMINANTAL FACET IDEALS

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Abstract. We study resolutions of initial ideals of closed determinantal facet ideals with respect to standard lexicographic order. We show that the multigraded Betti numbers of these ideals are always 0 or 1, regardless of the characteristic of the field. In addition, we show that the standard graded Betti numbers of closed determinantal facet ideals and their initial ideals coincide when generators of the ideal come from maximal minors of a generic \( n \times m \) matrix with \( n > 2 \). Next, we give lower bounds on the Betti numbers of certain classes of ideals of initial terms of the generators of determinantal facet ideals with respect to arbitrary term orders. We give an explicit minimal free resolution of the initial ideal of the ideal of maximal minors with respect to standard lexicographic order. We show that the Betti numbers of a certain closed determinantal facet ideal and its initial ideal coincide, verifying a conjecture of Ene, Herzog, and Hibi in a new case. We give explicit differentials for the linear strand of the initial ideal with respect to standard lexicographic order of an arbitrary closed determinantal facet ideal.

1. Introduction

Let \( R = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m] \) where \( k \) is any field, and \( M \) be a generic \( n \times m \) matrix of indeterminates. The study of the ideal generated by all minors of a given size of \( M \) has a long history, and is well known (see, for instance, \([6]\)). In a similar vein, one can instead consider the ideal generated by some of the minors of a given size of \( M \); these are known as determinantal facet ideals and were introduced Ene, Herzog, Hibi, and Mohammadi in \([10]\). This problem turns out to be much more subtle and has seen comparably less attention, even though such ideals arise naturally in algebraic statistics (see \([8]\) and \([12]\)). In \([13]\), the linear strand of determinantal facet ideals is made explicit; in particular, the linear Betti numbers of such ideals may be computed in terms of the \( f \)-vector of an associated simplicial complex. Likewise, in \([19]\), explicit Betti numbers of certain classes of determinantal facet ideals are computed in all degrees; for arbitrary determinantal facet ideals, higher degree Betti numbers have proven to be nontrivial to compute.

Determinantal facet ideals for the case \( n = 2 \) were originally introduced as binomial edge ideals independently by Ohtani \([15]\) and Herzog, et. al. \([12]\); this generalized work of Diaconis, Eisenbud, and Sturmfels in \([8]\). To study binomial edge ideals, one can associate each column of \( M \) with a vertex of a graph \( G \), and one can associate a minor of

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$M$ involving two columns $i$ and $j$ with an edge $(i, j)$ in the graph. For example, the ideal generated by all maximal minors of a $2 \times m$ matrix corresponds to a complete graph on $m$ vertices. The relationship between homological invariants of ideals generated by some maximal minors of $M$ and combinatorial invariants of the associated graph $G$ has been widely studied; see the survey paper [14] for a compilation of such results. Determinantal facet ideals naturally extend this idea by instead associating a simplicial complex $\Delta$ on $m$ vertices to the ideal $I$, where each $(n - 1)$-dimensional facet of $\Delta$ corresponds to a maximal minor in the set of generators of $I$.

One method of studying an ideal $I$ is to reduce to the study of its initial ideal, $\text{in}_<(I)$, with respect to some term order $<$. This has the advantage of gaining access to the combinatorial and topological techniques developed for the purposes of studying monomial ideals. A particularly interesting class of determinantal facet ideals is those for which the generators form a Gröbner basis. It is well-known that the ideal of maximal minors of a matrix is a Gröbner basis under any monomial order $[17, 2]$. In the case where the generators of a determinantal facet ideal form a Gröbner basis under lex monomial order, we say that the ideal (or its corresponding simplicial complex) is closed.

It is well-known that the Betti numbers of an initial ideal $\text{in}_<(I)$ under any term order $<$ is an upper bound for the Betti numbers of $I$, and it is rare that the Betti numbers of $\text{in}_<(I)$ and $I$ coincide. Ene, Herzog, and Hibi [9] conjecture that the Betti numbers of both a closed binomial edge ideal and its initial ideal with respect to standard lexicographic order coincide. Conca and Varbaro [7] show that for any ideal $I$ in a standard graded homogeneous polynomial ring with a squarefree initial ideal $\text{in}_<(I)$ with respect to some term order $<$, the extremal Betti numbers (see [1]) of $I$ and $\text{in}_<(I)$ coincide. In particular, the regularity and the projective dimension are the same for both $I$ and its squarefree initial ideal $\text{in}_<(I)$.

In this paper, we turn our attention to the case of closed determinantal facet ideals and study properties of resolutions of their initial ideals with respect to $<$, where $<$ denotes standard lexicographic order. In the case where $I$ corresponds to a closed determinantal facet ideal, $\text{in}_<(I)$ is squarefree and corresponds exactly to the leading terms of the generators of $I$. This enables us to use tools from combinatorics and topology to study the Stanley-Reisner complex of $\text{in}_<(I)$ and conclude that all the $\mathbb{Z}^{nm}$-graded Betti numbers of $\text{in}_<(I)$ are either 0 or 1. We observe that consecutive cancellations among the Betti numbers of $\text{in}(I)$ are never possible when $n > 2$, implying that the Betti numbers of $J_\Delta$ and $\text{in}(J_\Delta)$ coincide in these cases.

Next, we employ the construction of trimming complexes as introduced in [19] to so-called sparse Eagon-Northcott complexes, which were introduced by Boocher in [4]. In particular, we obtain explicit lower bounds on the Betti numbers for certain classes of initial ideals of determinantal facet ideals with respect to any term order. We then construct an explicit minimal free resolution for the initial ideal of the ideal of all maximal minors with respect to standard lexicographic order for an arbitrary $n \times m$ matrix. This allows us to verify the previously mentioned conjecture of Ene, Herzog, and Hibi when removing a single generator from the ideal generated by all maximal minors.
Using a slight generalization of the above resolution, we are able to obtain an explicit linear strand for the initial ideal in_<(I) of a closed determinantal facet ideal I where < is standard lexicographic order. In particular, this allows us to show that the Betti numbers of the linear strand for both I and in_<(I) agree, for all n ≥ 2.

The paper is organized as follows. In Section 2, we recall the construction of trimming complexes. In Section 3, we approach the problem of computing graded Betti numbers of a closed determinantal facet ideal I by reducing to the study of its initial ideal in_<(I) with respect to standard lexicographic order <. In particular, we show that the \( \mathbb{Z}^{nm} \)-graded Betti numbers of in_<(I) are always either 0 or 1 (see Theorem 3.18). As an application, it is shown that in general, the Betti numbers of the linear strand of a closed determinantal facet ideal I and its initial ideal in_<(I) agree. Even more, we show that when \( \Delta \) is a closed and pure \((n-1)\)-dimensional simplicial complex and \( n > 2 \), the Betti numbers of \( J_{\Delta} \) and in(\( J_{\Delta} \)) coincide.

In Section 4, we reformulate certain types of complexes considered in [4] for the purposes of using the trimming complex construction. In particular, we compute explicit \( q_i \)-maps for sparse Eagon-Northcott complexes with respect to any arbitrary monomial ordering and hence are able to bound their rank. Combining these bounds with Corollary 2.5 yields lower bounds on the Betti numbers of the initial ideal of certain classes of determinantal facet ideals with respect to any term order.

In Section 5, we construct an explicit minimal free resolution of the initial ideal of the ideal of maximal minors of a generic \( n \times m \) matrix with respect to standard lexicographic order. In particular, this also gives a minimal free resolution of the box polarization of any power of the standard graded maximal ideal. Moreover, via specialization, we obtain novel minimal free resolutions of both the ideal of all squarefree monomials of a specified degree and arbitrary powers of the homogeneous maximal ideal. We use this resolution to verify the conjecture of Ene, Herzog, and Hibi in the case where \( n = 2 \) when removing a single generator.

In Section 6, we pursue a slight strengthening of a result in Section 3. More precisely, we can generalize the complex introduced in Section 5 in order to deduce an explicit linear strand for the initial ideal in_<(I) of a closed determinantal facet ideal I, where < is standard lexicographic order. The proofs in this section follow closely that of [13], where a different complex is used in place of the so-called generalized Eagon-Northcott complex. In particular, we recover Corollary 3.20 using different methods.

2. Trimming Complexes

We recall the construction of trimming complexes. All proofs of the following results may be found in Section 2 and 3 of [19].

**Setup 2.1.** Let \( R = k[x_1, \ldots, x_n] \) be a standard graded polynomial ring over a field \( k \). Let \( I \subseteq R \) be a homogeneous ideal and \((F_\bullet, d_\bullet)\) denote a homogeneous free resolution of \( R/I \).
Write $F_1 = F'_1 \oplus \left( \bigoplus_{i=1}^{m} Re_0^i \right)$, where each $e_0^i$ generates a free direct summand of $F_1$. Using the isomorphism

$$\text{Hom}_R(F_2, F_1) = \text{Hom}_R(F_2, F'_1) \oplus \left( \bigoplus_{i=1}^{m} \text{Hom}_R(F_2, Re_0^i) \right)$$

write $d_2 = d'_2 + d_0^1 + \cdots + d_m^i$, where $d'_2 \in \text{Hom}_R(F_2, F'_1)$, $d_0^i \in \text{Hom}_R(F_2, Re_0^i)$. Let $a_i$ denote any homogeneous ideal with

$$d_0^i(F_2) \subseteq a_i e_0^i,$$

and $(G^i_*, m^i_*)$ be a homogeneous free resolution of $R/a_i$.

Use the notation $K := \text{im}(d_1|_{F'_1} : F'_1 \to R)$, $K_0^i := \text{im}(d_1|_{Re_0^i} : Re_0^i \to R)$, and let $J := K' + a_1 \cdot K_0^1 + \cdots + a_m \cdot K_0^m$.

**Proposition 2.2.** Adopt notation and hypotheses of Setup 2.1. Then for each $i = 1, \ldots, m$ there exist maps $q^i_1 : F_2 \to G^1_i$ such that the following diagram commutes:

\[
\begin{array}{ccc}
F_2 & \xrightarrow{q^i_1} & G^1_i \\
\downarrow d'_0 & & \downarrow a_i \\
G^1_i & \xrightarrow{m^i_i} & a_i \\
\end{array}
\]

where $a_0^i : F_2 \to R$ is the composition

\[
F_2 \xrightarrow{d_0^i} Re_0^i \xrightarrow{e_0^i} R,
\]

the second map sending $e_0^i \mapsto 1$.

**Proposition 2.3.** Adopt notation and hypotheses as in Setup 2.1. Then for each $i = 1, \ldots, m$ there exist maps $q^i_k : F_{k+1} \to G^i_k$ for all $k \geq 2$ such that the following diagram commutes:

\[
\begin{array}{ccc}
F_{k+1} & \xrightarrow{d_{k+1}} & F_k \\
\downarrow q^i_k & & \downarrow q^i_{k-1} \\
G^i_k & \xrightarrow{m^i_k} & G^i_{k-1} \\
\end{array}
\]
Theorem 2.4. Adopt notation and hypotheses as in Setup 2.1. Then the mapping cone of the morphism of complexes

\[
\begin{array}{cccccc}
\cdots & \rightarrow & F_k & \rightarrow & \cdots & \rightarrow F_1' \\
\bigg| & & \bigg| & & \bigg| & \downarrow & d_1 \\
\oplus m_i & \oplus m_i & \oplus m_i & \oplus m_i & \oplus m_i & R \\
\cdots & \rightarrow & G_{k-1} & \rightarrow & \cdots & \rightarrow G_1 \\
\end{array}
\]

is acyclic and forms a resolution of the ideal \( K' + a_1 \cdot K_0^1 + \cdots + a_m \cdot K_0^m \).

As an immediate consequence, one obtains:

Corollary 2.5. Adopt notation and hypotheses of Setup 2.1. Assume furthermore that the complexes \( F_\bullet \) and \( G_\bullet \) are minimal. Then for \( i \geq 2 \),

\[
\dim_k \text{Tor}^R_i(R/J, k) = \text{rank } F_i + \sum_{j=1}^m \text{rank } G_i^j - \text{rank } \left( \begin{pmatrix} q_i^1 \\ \vdots \\ q_i^m \end{pmatrix} \otimes k \right) - \text{rank } \left( \begin{pmatrix} q_{i-1}^1 \\ \vdots \\ q_{i-1}^m \end{pmatrix} \otimes k \right).
\]

Similarly,

\[
\mu(J) = \mu(K) - m + \sum_{j=1}^m \mu(a_j) - \text{rank } \left( \begin{pmatrix} q_i^1 \\ \vdots \\ q_i^m \end{pmatrix} \otimes k \right).
\]

The resolution of Theorem 2.4 may be used to construct resolutions of subsets of generating sets of an ideal by the following observation.

Observation 2.6. Adopt notation and hypotheses as in Setup 2.1 with \( m = 1 \). If \( d_0^1(F_2) = a_1 e_0' \), then the resolution of Theorem 2.4 resolves \( K' \).

With this observation, the following can be shown.

Theorem 2.7 ([19], Theorem 5.6). Let \( R = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m] \) where \( k \) is any field and \( M = (x_{ij}) \) denote a generic \( n \times m \) matrix, with \( n \leq m \). Choose indexing sets \( I_j = (i_{j1}, \ldots, i_{jn}) \) for \( j = 1, \ldots, r \) pairwise disjoint; that is, \( I_i \cap I_j = \emptyset \) for \( i \neq j \) (this intersection is taken as sets). Define

\[
rk_{\ell} := \binom{n + \ell - 1}{\ell} \cdot \sum_{i=1}^r (-1)^{i+1} \binom{r}{i} \binom{m - in}{\ell - (i-1)n}.
\]

If \( J = (j_1, \ldots, j_n) \) with \( j_1 < \cdots < j_n \), let \( \Delta_J \) denote the determinant of the matrix formed by columns \( j_1, \ldots, j_n \) of \( M \). Then the ideal

\[
K' := (\Delta_J \mid J \neq I_j, j = 1, \ldots, r)
\]
has Betti table

|       | 0   | 1   | \cdots | \ell | \cdots | n(m-n)-1 | n(m-n) |
|-------|-----|-----|--------|------|--------|----------|--------|
| 0     | 1   | 0   | \cdots | 0    | \cdots  | 0        | 0      |
| n-1   | 0   | (m) | r      | (n+\ell-2) | (m) | - rk_{\ell-1} | \cdots | 0     |
| n     | 0   | 0   | \cdots | r \cdot (n^{m-n}) | - rk_{\ell} | \cdots | r \cdot n(m-n) | r     |

In particular, \( \text{pd}_R R/K' = n(m-n) \).

## 3. Multigraded Betti Numbers of Closed Determinantal Facet Ideals

In this section we study the initial ideals generated by arbitrary collections of maximal minors of an \( n \times m \) matrix of indeterminates in the case where the set of generators forms a Gröbner basis for the ideal. In this case, the initial ideal is of degree \( n \) and squarefree, so that Stanley-Reisner theory may be employed to compute the \( \mathbb{Z}^{nm} \)-graded Betti numbers.

**Setup 3.1.** Let \( \tilde{X}_i = \{x_{i1}, \ldots, x_{im}\} \) for all \( 1 \leq i \leq n \), and let \( S = k[\tilde{X}_1, \ldots, \tilde{X}_n] \) be a polynomial ring in the variables of the \( \tilde{X}_i \) over a field \( k \). Let \( M \) be an \( n \times m \) matrix of variables in \( S \), where the variables of \( \tilde{X}_i \) are in row \( i \) of the matrix. Let \( < \) denote standard lexicographic order in \( S \); that is, lexicographic order with \( x_{11} < \cdots < x_{1m} < x_{21} < \cdots < x_{nm} \). Denote by \([a] = [a_1, \ldots, a_n]\) the determinant of the maximal minor corresponding to columns \( a_1, \ldots, a_n \), where \( 1 \leq a_1 < a_2 < \cdots < a_n \leq m \). Let \( \Delta \) be a pure \((n-1)\)-dimensional simplicial complex on the vertex set \([m]\).

**Definition 3.2.** Adopt notation and hypotheses as in Setup 3.1. For a simplicial complex \( \Delta \) and an integer \( i \), the \( i \)-th skeleton \( \Delta^{(i)} \) of \( \Delta \) is the subcomplex of \( \Delta \) whose faces are those faces of \( \Delta \) whose dimension is at most \( i \). Let \( S \) denote the set of simplices \( \Gamma \) with vertices in \([m]\) with \( \text{dim}(\Gamma) \geq n-1 \) and \( \Gamma^{(n-1)} \subset \Delta \).

Let \( \Gamma_1, \ldots, \Gamma_r \) be maximal elements in \( S \) with respect to inclusion, and let \( \Delta_i := \Gamma_i^{(n-1)} \). Each \( \Gamma_i \) is called a clique. The simplicial complex \( \Delta^{\text{clique}} \) whose facets are the cliques of \( \Delta \) is called the clique complex associated to \( \Delta \). The decomposition \( \Delta = \Delta_1 \cup \cdots \cup \Delta_r \) is called the clique decomposition of \( \Delta \).

**Definition 3.3.** Adopt notation and hypotheses as in Setup 3.1. A determinantal facet ideal \( J_\Delta \subseteq S \) is the ideal generated by determinants of the form \([a]\) where \( a \) supports an \( n-1 \) face of \( \Delta \); that is, the columns of \([a]\) correspond to the vertices of some facet \( \sigma \in \Delta \).

**Remark 3.4.** Let \( I \) be an ideal generated by a subset of minors of an \( n \times m \) matrix \( M \). The simplicial complex \( \Delta \) associated to a determinantal facet ideal can be viewed as a combinatorial tool to keep track of the generators of such an ideal, since each facet corresponds to a minor in the generating set of \( I \). The clique decomposition of \( \Delta = \bigcup_{i=1}^r \Delta_i \) keeps track of the largest submatrices \( M_i \) of \( M \) where the ideal of maximal minors of \( M_i \) is contained in \( I \).

**Definition 3.5.** A simplicial complex \( \Delta \) is said to be closed (with respect to a given labeling) if it satisfies any one of the following equivalent conditions:
(1) For any two facets $F = \{a_1 < \cdots < a_n\}$ and $G = \{b_1 < \cdots < b_n\}$ with $a_i = b_i$ for some $i$, the \((n-1)\)-skeleton of the simplex on the vertex set $F \cup G$ is contained in $\Delta$.

(2) For all $i \neq j$ and all $F = \{a_1 < a_2 < \cdots < a_n\}$ in $\Delta_i$ and $G = \{b_1 < b_2 < \cdots < b_n\}$ in $\Delta_j$, we have $a_\ell \neq b_\ell$ for all $\ell$.

(3) For all $i \neq j$ and all $F = \{a_1 < a_2 < \cdots < a_n\}$ in $\Delta_i$ and $G = \{b_1 < b_2 < \cdots < b_n\}$ in $\Delta_j$, the monomials in $<[a_1, \cdots, a_n]$ and in $<[b_1, \cdots, b_n]$ are relatively prime.

**Proposition 3.6** ([10], Theorem 1.1). Adopt notation and hypotheses as in Setup 3.11.

A simplicial complex $\Delta$ is closed if and only if the minors generating the determinantal facet ideal $J_\Delta$ form a Gröbner basis with respect to lexicographic order.

The generators of $J_\Delta$ are squarefree, so one can turn to Stanley-Reisner theory to study it. For a survey of Stanley-Reisner theory see, for instance, Chapter 5 of [5].

**Definition 3.7.** For a squarefree monomial ideal $I$, the Stanley-Reisner complex of $I$ is the simplicial complex with faces supported on squarefree monomials not contained in $I$. For each $\sigma \subseteq [m]$, define the restriction of $\Delta$ to $\sigma$ by

$$\Delta|_\sigma = \{\tau \in \Delta \mid \tau \subseteq \sigma\}.$$ 

**Definition 3.8.** The simplicial join of simplicial complexes $\Gamma_1$ and $\Gamma_2$, denoted $\Gamma_1 \ast \Gamma_2$, is the simplicial complex with faces $\sigma_1 \cup \sigma_2$ where $\sigma_1 \in \Gamma_1$ and $\sigma_2 \in \Gamma_2$. The cone of a simplicial complex $\Gamma$, denoted $cone(\Gamma)$, is the join $\Gamma \ast v$ of $\Gamma$ with some vertex $v$ not in $\Gamma$.

**Proposition 3.9** (Hochster’s Formula). Let $\Gamma$ be a simplicial complex on $V = \{x_1, \ldots, x_n\}$ and let $I$ be the associated Stanley-Reisner ideal in $S = k[x_1, \ldots, x_n]$. Then the nonzero Betti numbers of $I \Gamma$ lie only in squarefree degrees $\sigma$, and

$$\beta_{i-1,\sigma}(I) = \beta_{i,\sigma}(S/I) = \dim_k \tilde{H}^{[\sigma]-i-1}(\Gamma|_\sigma; k).$$

**Notation 3.10.** Let $\mathcal{A}$ be a set of pairs $(i_j, \tau_j)$ such that $\tau = [\tau_1, \ldots, \tau_k] \in \Delta^{clique}$ and $k \geq n$. Associate to $\mathcal{A}$ a monomial $m_{\mathcal{A}}$ where

$$m_{\mathcal{A}} = \prod_{(i_j, \tau_j) \in \mathcal{A}} x_{i_j, \tau_j}.$$ 

Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Define $|a| := \sum_{i=1}^n a_i$. Define

$$a_{\leq i} := (a_1, \ldots, a_i),$$

where $a_{\leq i} = \emptyset$ if $i \leq 0$ and $a_{\leq i} = a$ if $i \geq n$.

**Setup 3.11.** Let $\Delta$ be a closed pure \((n-1)\)-dimensional simplicial complex. Define the set $\mathcal{A}(a; \tau) = \{(i, \tau_j) \mid i \in [n], |a_{\leq i-1}| < j \leq |a_{\leq i}|\}$, where $\tau$ is a \((k-1)\)-face of $\Delta^{clique}$ with $k \geq n$, and $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $|a| = k$. Set $m_k(a; \tau) := m_{\mathcal{A}(a; \tau)}$.

**Example 3.12.** Let $\Delta$ be the 2-dimensional simplicial complex with clique decomposition $\{1, 2, 3, 4\} \cup \{2, 3, 4, 5\}$, so $J_\Delta$ is generated by the maximal minors of a generic $3 \times 5$ matrix that are contained in the submatrices given by the first four columns and the last four columns. If $\sigma = \{1, 2, 3, 4\}$ and $a = (1, 2, 1)$, then $m_4(a; \sigma) = x_{1,1}x_{2,2}x_{2,3}x_{3,4}$. 


**Proposition 3.13.** Adopt notation and hypotheses as in Setups 3.1 and 3.11. Let $\Gamma$ be the associated Stanley-Reisner complex of the squarefree monomial ideal $\text{in}(J_\Delta)$ and let $u = m_k(a; \tau)$ for some $\tau \in \Delta^{\text{clique}}$ and $a \in \mathbb{N}^n$ such that $|a| = k$ and $a_i \neq 0$ for $1 \leq i \leq n$. Then $\Gamma|_u$ is homotopy equivalent to $S^{n-2}$.

The proof of this proposition requires some notions from simplicial topology. For further reference, see, for example, [3].

**Definition 3.14.** The nerve of a finite set of simplicial complexes $\{\Lambda_i\}_{i \in \mathcal{A}}$ is the simplicial complex $N$ on vertex set $\mathcal{A}$ and with faces

$$\{\sigma \subseteq \mathcal{A} | \cap_{i \in \sigma} \Lambda_i \neq \emptyset\}.$$

**Lemma 3.15** (Nerve Theorem). Let $\Delta$ be a finite simplicial complex and $\{\Lambda_i\}_{i \in \mathcal{A}}$ be a finite cover (that is, a set of subcomplexes such that $\cup_{i \in \mathcal{A}} \Lambda_i = \Delta$). Suppose that every non-empty intersection $\cap_{i \in \sigma} \Lambda_i$ is contractible. Then $\Delta$ and the nerve $N$ are homotopic.

**Proof of Proposition 3.13.** Write $u = u_1 \cdots u_n$ where

$$u_i := \prod_{(i, \tau_j) \in \mathcal{A}(a; \tau)} x_{i, \tau_j}$$

for a fixed $i$. Let $\Gamma$ be the Stanley-Reisner complex of $\text{in}(J_G)$. Let $\tilde{u}_i = \frac{u_i}{u_i}$, and let $\Lambda_i = \Gamma|_{\tilde{u}_i}$. Each $\Lambda_i$ is a simplex, since there is no monomial of $\text{in}(J_\Delta)$ that does not contain at least one variable from every set of variables $X_i$. The set $\{\Lambda_i\}_{i \in [m]}$ is a simplicial cover for $\Gamma|_u$.

Now consider the nerve $N$ of the simplicial cover $\{\Lambda_i\}$. Every non-empty intersection $\cap_{i \in \sigma} \Lambda_i$ is a simplex (since every nonempty intersection of simplices is a simplex), hence contractible. By the Nerve Lemma 3.15, $N$ and $\Gamma|_u$ are homotopic.

Therefore, $N$ has $n$ vertices corresponding to each $\Lambda_i$. The intersection of any $n - 1$ of the $\Lambda_i$ is nonempty, which will correspond to a simplex on the variables in some monomial $u_k$, but the intersection of all $n$ simplices in $\{\Lambda_i\}$ is empty. So $N$ is homotopic to the boundary of an $n - 1$ simplex, and in particular $\Gamma|_u \cong N \cong S^{n-2}$. $\square$

**Lemma 3.16.** Adopt notation and hypotheses as in Setups 3.1 and 3.11. Let $w$ be a monomial that can be written as $w = w_1 \cdots w_\ell$ such that each $w_i$ satisfies the following:

1. Each $w_i = m_{a_i}$ where $\mathcal{A}_i = \{(k_{i,j}, \tau_{i,j}) | \tau^i \in \Delta^{\text{clique}} 	ext{ and } k_{i,j} \in \mathbb{Z}_{\geq 0}\}$,
2. $\tau^i \cup \tau^j \notin \Delta^{\text{clique}}$ for $i \neq j$, and
3. For all $w_i$ and $w_j$ with $i \neq j$, $w_i$ and $w_j$ are relatively prime.

Then $\Gamma|_w = \Gamma|_{w_1} \ast \cdots \ast \Gamma|_{w_\ell}$.

**Proof.** Proceed by induction on $\ell$. $\ell = 1$ is clear.

Let $w' = w_1 \cdots w_{\ell-1}$, and assume by induction that $\Gamma|_{w'} = \Gamma|_{w_1} \ast \cdots \ast \Gamma|_{w_{\ell-1}}$. We wish to show that $\Gamma|_w = \Gamma|_{w'} \ast \Gamma_\ell$.

Take a face $\sigma \in \Gamma|_{w'}$ and a face $\sigma' \in \Gamma_\ell$. It suffices to show that $\rho = \sigma \cup \sigma'$ is a face of $\Gamma$. Suppose, seeking contradiction, that $\rho$ is not a face of $\Gamma$. Then the monomial corresponding to $\rho$ is in the ideal $\text{in}(J_\Delta)$, so it is divisible by some generator $x_{i_1,i_1}x_{i_2,i_2} \cdots x_{i_n,i_n}$ where $\{i_1 < i_2 < \cdots < i_n\}$ is a facet of $\Delta$. This implies $\rho$ is a face
of $\Delta_{\text{clique}}$, contradicting the assumption that the monomial $\tau^r \cup \tau^i \notin \Delta_{\text{clique}}$ for any $i < r$.

**Setup 3.17.** Adopt notation and hypotheses of Setups 3.1 and 3.11. Let $\Delta = \Delta_1 \cup \Delta_2$ be a clique decomposition of $\Delta$ with the total order defined by $\Delta_1 < \Delta_2 < \cdots < \Delta_r$ given by $\min(V(\Delta_1)) < \min(V(\Delta_2)) < \cdots < \min(V(\Delta_r))$. Observe that this is well-defined, since distinct cliques of a closed simplicial complex must have distinct minimum indexed vertices.

Let $w$ be a product of monomials $w = w_1 \cdots w_\ell$ where each $w_i = m_{ki}(a^i; \tau^i)$ is a monomial satisfying the hypotheses of Setup 3.11 such that:

1. For any $i \neq j$, $w_i$ and $w_j$ are relatively prime.
2. If $i < j$, then $\tau^i \cup \tau^j$ is not a face of $\Delta_{\text{clique}}$, and any clique that contains $\tau_i$ is strictly less than any clique containing $\tau_j$ in the total ordering on the cliques.
3. $|a^i| = k$ for all $i$, and $a^i_j \neq 0$ for any $i, j$.
4. Let $v \in \tau^j$ and let $(k_v, v)$ be the pair associated to $v$ in $\mathcal{A}(a^i; \tau^j)$. If $v \cup \tau^i \notin \Delta_{\text{clique}}$ for some $i < j$, then there is some $(p_1, q_1) \in \tau^i$ such that that $q_1 > v$ but $p_1 < k_v$.

**Theorem 3.18.** Adopt notation and hypotheses as in Setups 3.1 and 3.17. Let $w$ be a product of monomials $w = w_1 \cdots w_\ell$ satisfying the conditions of Setup 3.17. Then

$$\beta_{i,w}(S/\text{in}(J_\Delta)) = \begin{cases} 1 & \text{if } i = |w| - \ell(n-1) \\ 0 & \text{otherwise.} \end{cases}$$

For any monomial $m$ that cannot be written as a product of monomials as in Setup 3.17, $\beta_{i,m}(S/\text{in}(J_\Delta)) = 0$ for all $i$. In particular, the $\mathbb{Z}^{nm}$-graded Betti numbers for $S/\text{in}(J_\Delta)$ are either 0 or 1.

**Example 3.19.** Let $G$ be the graph on 4 vertices with edge set

$$E(G) = \{(1, 2), (2, 3), (2, 4), (3, 4)\}$$

and let $J_G$ be its corresponding binomial edge ideal. Observe that the monomial $u = x_{1,1}x_{1,2}x_{2,3}x_{2,4}$ can never be written as a product of monomials satisfying the conditions of Setup 3.17. For example, if one writes $w_1 = m(\{1, 1\}; \{1, 3\}) = x_{1,1}x_{2,3}$ and $w_2 = m(\{1, 1\}; \{2, 4\}) = x_{1,2}x_{2,4}$, observe that $w_1$ an $w_2$ do not satisfy condition (4) above because $x_{1,2}$ could instead be placed in $w_1$. If $w'_1 = m(\{2, 1\}; \{1, 2, 3\}) = x_{1,1}x_{1,2}x_{2,4}$, then $w'_2 = m(\{0, 1\}; \{4\}) = x_{2,4}$, which does not satisfy condition (3). Therefore, $\beta_{i,u}(S/\text{in}(J_G)) = 0$ for all $i$.

In contrast, one can write the monomial $v = x_{1,1}x_{1,2}x_{1,3}x_{2,2}x_{2,3}x_{2,4}$ as a product of monomials $w_1 = m(\{1, 2\}; \{1, 2, 3\}) = x_{1,1}x_{2,2}x_{2,3}$ and $w_2 = m(\{2, 1\}; \{2, 3, 4\}) = x_{1,2}x_{1,3}x_{2,4}$ satisfying the conditions of Setup 3.17 so

$$\beta_{i,v}(S/\text{in}(J_\Delta)) = \begin{cases} 1 & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof of Theorem 3.18.** By Lemma 3.16, $\Gamma|_w \cong \Gamma_1 \ast \Gamma_2 \ast \cdots \ast \Gamma_\ell$. By the proof of Proposition 3.13, the nerve of each $\Gamma_i$ is homotopy equivalent to the boundary of an $(n-1)$-simplex, so it is a homology $(n-2)$-sphere. It is well known that the join of $\ell$
homology \((m - 2)\)-spheres is a homology \((\ell(m - 1))\)-sphere (see, for example, [3, 9.12]). So \(\Gamma_{1,w} \cong S^{(n-1)+1}\), and we obtain the desired result by Hochster’s formula.

Now it remains to check that for any monomial \(m\) not satisfying the properties above, \(\beta_{i,m}(S/\in(J_\Delta)) = 0\) for all \(i\). First, observe that since \(\in(J_\Delta)\) is a squarefree monomial ideal, \(\beta_{i,m}(S/\in(J_\Delta)) = 0\) for all \(i\) if \(m\) is not squarefree.

Let \(m\) be a squarefree monomial where the second indices of all the variables correspond to a face of \(\Delta_{\text{clique}}\), but there does not appear one variable from each color class \(X_i\) (i.e. from every row of \(M\)). Then this monomial is not divisible by any generator of \(\in(J_\Delta)\), so it is not in \(\in(J_\Delta)\) and \(\Gamma_{1,m}\) corresponds to a contractible face of \(\Gamma\). Similarly, if \(m\) is divisible by \(x_{i,j}x_{k,\ell}\) where \(i < k\) but \(j > \ell\) and \((j,\ell)\) is a face of \(\Delta\), then \(m\) cannot be in \(\in(J_\Delta)\) and therefore \(\Gamma_{1,m}\) must be a contractible face of \(\Gamma\).

Let \(u\) be a product of monomials \(u = u_1 \cdots u_r\) where each \(u_i\) is a monomial satisfying the hypotheses of Setup 3.11 in a distinct maximal clique from \(u_j\) for \(i \neq j\), except for \(u_\ell\). Then \(u_\ell\) corresponds to a contractible face of \(\Gamma\). By Lemma 3.16, \(\Gamma_{1,u} \cong \Gamma_{1,u_1} \cdots \Gamma_{1,u_{\ell-1}}\). But since \(u_\ell\) is contractible, \(\Gamma_{1,u} \cong \text{cone}(\Gamma_{1,u_1} \cdots \Gamma_{1,u_{\ell-1}})\). Since the cone of a simplicial complex is always contractible, \(\Gamma_{1,u}\) is contractible.

By summing over all possible monomials of the form in Setup 3.11 and applying Proposition 3.13, we obtain the following result.

**Corollary 3.20.** Adopt notation and hypotheses as in Setup 3.1 and suppose \(\Delta\) is a pure \((n - 1)\)-dimensional simplicial complex which is closed. Then

\[ \beta_{i,i+\ell}(S/\in(J_\Delta)) = \beta_{i,i+\ell}(S/J_\Delta) = \binom{\ell - i + 1}{\ell - 1} f_{\ell+i-1}(\Delta_{\text{clique}}) \]

where \(f(\Delta_{\text{clique}})\) is the \(f\)-vector of \(\Delta_{\text{clique}}\). In particular, the Betti numbers in the linear strand of \(J_\Delta\) and \(\in(J_\Delta)\) coincide.

As an application of Theorem 3.18, we prove the following result.

**Theorem 3.21.** Let \(\Delta\) be a pure \((n - 1)\)-dimensional simplicial complex which is closed. When \(n > 2\), the standard graded Betti numbers of \(S/J_\Delta\) and \(S/\in(J_\Delta)\) coincide.

Before proving the theorem above, we recall the following definition.

**Definition 3.22 ([16]).** A sequence \(q_{i,j}\) of numbers is obtained from a sequence \(p_{i,j}\) by a consecutive cancellation if there exist indexes \(s\) and \(r\) such that

\[ q_{s,r} = p_{s,r} - 1, \quad q_{s+1,r} = p_{s+1,r} - 1, \]

\[ q_{i,j} = p_{i,j} \quad \text{for all other values of } i, j. \]

**Proof of Theorem 3.21.** Observe that the Betti numbers of \(S/J_\Delta\) can be obtained from the Betti numbers of \(S/\in(J_\Delta)\) by consecutive cancellations (see [16, Theorem 22.12]). By Theorem 3.18, the basis elements of the modules in the free resolution of \(S/\in(J_\Delta)\) are given by monomials of the form

\[ m_{k_1}(a^1, \tau^1) \cdots m_{k_\ell}(a^\ell, \tau^\ell) \]

which have homological degree \(\sum k_i - \ell(n - 1)\) and internal degree \(\sum k_i\).
A consecutive cancellation is only possible when both $β_{i,j}(S/\in(J_Δ))$ and $β_{i+1,j}(S/\in(J_Δ))$ are nonzero. Assume that $β_{i,j}(S/\in(J_Δ))$ is nonzero. Then it has basis elements above where $j = \sum k_i$ and $i = \sum k_i - \ell(n-1) = j - \ell(n-1)$; that is, $j - i = \ell(n-1)$. However, if $β_{i+1,j}(S/\in(J_Δ))$ is also nonzero, then $j - i - 1$ must also be divisible by $n - 1$. Since two consecutive numbers cannot be divisible by $n - 1$ unless $n = 2$, $β_{i+1,j}(S/\in(J_Δ)) = 0$ and no consecutive cancellations are possible. Therefore, the Betti numbers of $S/J_Δ$ and $S/\in(J_Δ)$ coincide for $n > 2$. 

Remark 3.23. Observe that consecutive cancellations may still be possible in the case when $n = 2$. However, it is conjectured that even in the case when $n = 2$, the Betti numbers of $J_Δ$ and $\in(J_Δ)$ coincide; see [9].

4. Sparse Eagon-Northcott Complexes

Definition 4.1. Let $P$ be a logical statement outputting the values true or false. Define

$$\chi(P) = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false}. \end{cases}$$

Example 4.2. Let $S = \{1, 2, 3\}$. Then $\chi(1 \in S) = 1$ and $\chi(5 \in S) = 0$.

Definition 4.3. Let $φ : F → G$ be a homomorphism of free modules of ranks $f$ and $g$, respectively, with $f ≥ g$. Via the isomorphism $\text{Hom}_R(F, G) = F^* \otimes G$, $φ$ induces an element $c_φ \in F^* \otimes G$. The Eagon-Northcott complex is the complex

$$0 \rightarrow D_{f−g}(G^*) \otimes \bigwedge^f F \rightarrow D_{f−g−1}(G^*) \otimes \bigwedge^f F \rightarrow \cdots \rightarrow \bigwedge^g F \rightarrow \bigwedge^g G$$

with differentials in homological degree $≥ 2$ induced by multiplication by the element $c_φ \in F^* \otimes G$, and the map $\bigwedge^g F → \bigwedge^g G$ is $\bigwedge^g φ$.

Setup 4.4. Let $R = k[x_{ij} \mid 1 ≤ i ≤ n, 1 ≤ j ≤ m]$ and $M = (x_{ij})_{1 ≤ i ≤ n, 1 ≤ j ≤ m}$ denote a generic $n \times m$ matrix, where $n ≤ m$. View $M$ as a homomorphism $M : F → G$ of free modules $F$ and $G$ of rank $m$ and $n$, respectively.

Let $f_i$, $i = 1, \ldots , m$, $g_j$, $j = 1, \ldots , n$ denote the standard bases with respect to which $M$ has the above matrix representation. Let $w$ be an integer weight order on the variables of $R$ and let $E^*_\bullet$ denote the Eagon-Northcott complex resolving $R/I_n^h(M)$ where $I_n^h(M)$ denotes the ideal of maximal minors of $M$.

Consider the ring $R[t]$, where $t$ is an arbitrary variable of degree 1. Let $I_n^h(M) ⊆ R[t]$ denote the ideal generated by the homogenization $Δ^h_i$ of each minor $Δ_i$ with respect to the variable $t$. There is an induced homogenization of the Eagon-Northcott complex, denoted $E^h_\bullet$, which is a complex of free $R[t]$-modules.

Proposition 4.5 ([1], Proposition 3.8 and Corollary 3.9). Adopt notation and hypotheses of Setup 4.4. Then $E^h_\bullet$ is a minimal free resolution of $R[t]/I_n^h(M)$.

Moreover, to obtain the minimal free resolution of $R/\in_w(I_n(M))$, simply set $t = 0$ in the resolution $E^h_\bullet$. 
Definition 4.9. Adopt notation and hypotheses as in Setup 4.12. Define the indexing set \( L \) appearing in Proposition 4.7 will be called the sparse Eagon-Northcott complex. Enumerate the set \( A \).

Definition 4.10. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_i \geq 0 \) and \( |\alpha| = \ell \) and \( I = (i_1, \ldots, i_{n+\ell}) \) with \( i_1 < \cdots < i_{n+\ell} \).

The notation \( f_i \) will denote \( f_{i_1} \wedge \cdots \wedge f_{i_{n+\ell}} \in \bigwedge^{n+\ell} F \), and \( g^{(\alpha)} \) will denote \( g_1^{(\alpha_1)} \cdots g_n^{(\alpha_n)} \in D_\ell(G^*) \). The notation \( \epsilon_i \) denotes the vector with a 1 in the \( i \)th spot and 0’s elsewhere.

The following is a translation of Proposition 4.5 to a form that will be convenient.

Proposition 4.7. Adopt notation and hypotheses of Setup 4.4. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_i \geq 0 \) and \( |\alpha| = \ell \) and \( I = (i_1, \ldots, i_{n+\ell}) \) with \( i_1 < \cdots < i_{n+\ell} \). Then the minimal free resolution \( E_\alpha^* \) of \( \text{in}_<(I_n(M)) \) over \( R \) is such that \( E_\alpha^\ell = D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \), and the differential \( E_\alpha^\ell \to E_\alpha^{\ell-1} \) takes the form

\[
g^{(\alpha)} \otimes f_i \mapsto \sum_{i,j} (-1)^{i+j+\ell} \chi((i, j) \in I_\alpha(\alpha, I)) x_{ij} g^{(\alpha-\epsilon_i)} \otimes f_{I\setminus j},
\]

where \( I_\alpha(\alpha, I) \subseteq \{ i \mid \alpha_i > 0 \} \times I \) is some subset depending on \( \alpha \) and \( I \).

For convenience, we use the above Proposition to define sparse Eagon-Northcott complexes and the indexing sets \( I_\alpha(\alpha, I) \).

Definition 4.8. Adopt notation and hypotheses of Setup 4.4. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_i \geq 0 \) and \( |\alpha| = \ell \) and \( I = (i_1, \ldots, i_{n+\ell}) \) with \( i_1 < \cdots < i_{n+\ell} \). The complex \( E_\alpha^* \) appearing in Proposition 4.7 will be called the sparse Eagon-Northcott complex.

The indexing set \( I_\alpha(\alpha, I) \subseteq \{ i \mid \alpha_i > 0 \} \times I \) is defined via the nonvanishing terms in the differential of the sparse Eagon-Northcott complex, as in Proposition 4.7.

Definition 4.9. Adopt notation and hypotheses as in Setup 4.12. Define the indexing set \( L_I \) to be the indexing set such that

\[
\alpha = (x_{ij} \mid (i, j) \in L_I).
\]

Enumerate the set \( L_I = \{(i_1, j_1), \ldots, (i_N, j_N)\} \), where \( N \) is some integer. Consider the vector \( A_{L_I} \in \mathbb{Z}^n \) defined by setting

\[
(A_{L})_k := |\{ j \mid i_j = k \}|.
\]

Definition 4.10. Let \( J = (j_1, \ldots, j_\ell) \) be an indexing set of length \( \ell \) with \( j_1 < \cdots < j_\ell \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_i \geq 0 \) for each \( i \). Define \( \mathcal{L}(\alpha, J) \) to be the subset of size \( \ell \) subsets of the cartesian product

\[
\{ i \mid \alpha_i \neq 0 \} \times J,
\]

where \( \{(r_1, j_1), \ldots, (r_\ell, j_\ell)\} \in \mathcal{L}(\alpha, J) \) if \( |\{ i \mid r_i = j \}| = \alpha_j \).

Assume that \( w \) is any weight vector and fix and indexing set \( I = (i_1, \ldots, i_n) \) with \( i_1 < \cdots < i_n \). Define the indexing set \( L_w^I(\alpha, J) \) to be all elements \( L \in \mathcal{L}(\alpha, J) \) with \( L \subseteq I_\alpha(\alpha, I \cup J) \cap L_I \).

Lemma 4.11. Adopt notation and hypotheses of Setup 4.4. Let \( I = (i_1, \ldots, i_n) \) and \( J = (j_1, \ldots, j_\ell) \) with \( i_1 < \cdots < i_n \) and \( j_1 < \cdots < j_\ell \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_i \geq 0 \) for each \( i \). Use the notation \( \alpha^i := (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n) \). Assume that \( (i, j_k) \in I_\alpha(\alpha, I \cup J) \cap L_I \). Then any \( L' \in L_w^I(\alpha^i, J \setminus j_k) \) is contained in a unique element \( L \in L_w^I(\alpha, J) \).
Proof. Given $L'$, simply take $L := L' \cup (i, j_k)$, ordered appropriately. This element is unique because the second coordinate must be $j_k$, and since $\alpha^i$ differs from $\alpha$ by 1 in the $i$th spot, the first coordinate must be $i$. Moreover, this is well defined since $(i, j_k) \in I_w(\alpha, I \cup J) \cap L_I$, whence $L \in \mathcal{L}_w(I)$ by definition. \qed

Setup 4.12. Adopt notation and hypotheses as in Setup 4.12. Fix an indexing set $I = (i_1, \ldots, i_n)$ with $i_1 < \cdots < i_n$ and consider the map $G^* \otimes \bigwedge^{n+1} F \to Rf_I$ induced by the differentials of the sparse Eagon-Northcott complex $E'$.

Let $\mathfrak{a}$ be the ideal such that $d_0(E'_2) = \mathfrak{a}f_I$. Observe that $\mathfrak{a}$ is a complete intersection since it is generated by a subset of the generating set for the row of the standard Eagon-Northcott complex. Let $K_* = \bigwedge^* U$ denote the Koszul complex resolving $R/\mathfrak{a}$, where $U$ is a free $R$-module with basis $e_1, \ldots, e_N$.

Define $q_1 : G^* \otimes \bigwedge^{n+1} F \to U$ by sending all basis elements of the form $g_i \otimes f_{IJ} \mapsto e_L$, where $L \in \mathcal{L}_w(i, j)$, and all other basis elements to 0. By construction, the following diagram commutes:

\[
\begin{array}{ccc}
G^* \otimes \bigwedge^{n+1} F & \xrightarrow{q_1} & U \\
\downarrow & & \downarrow \\
\mathfrak{a} & \xleftarrow{d_0} & \mathfrak{a}
\end{array}
\]

Proposition 4.13. Let $R = k[x_1, \ldots, x_m]$ and let $I_m^n$ denote the ideal generated by all squarefree monomials of degree $n$ in $R$. Observe that the rows of any minimal presenting matrix $N$ for $I_m^n$ is indexed by all indexing sets $I = (i_1 < \cdots < i_n)$, corresponding to the generator $x_{i_1} \cdots x_{i_n}$.

Then, the $I$th row of $N$ generates the complete intersection 

\[ (x_j \mid j \in [m]\setminus I). \]

Proof. Let

\[ K' : (x_{a_1} \cdots x_{a_n} \mid a_1 < \cdots < a_n, (a_1, \ldots, a_n) \neq (i_1, \ldots, i_n)). \]

The $I$th row of $N$ generates to ideal $(K' : x_{i_1} \cdots x_{i_n})$, and it is immediate that this is equal to the ideal in the statement of the Proposition. \qed

Proposition 4.14. Adopt notation and hypotheses as in Setup 4.12. Assume that $w$ corresponds to a proper monomial order $<$. Then $\mathfrak{a}$ is a complete intersection on at least $m - n$ elements.

Moreover, if $\mathfrak{a} = (x_{ij} \mid (i, j) \in L)$, where $L$ is some indexing set, then the set of $j$ such that $(i, j) \in L$ for some $1 \leq i \leq n$ is precisely $[m]\setminus I$.

Proof. Let $E'_* \otimes R/(\sigma)$ denote the sparse Eagon-Northcott complex of Definition 4.8. By the proof of Theorem 3.1 in [4], $E'_* \otimes R/(\sigma)$, where

\[
\sigma = \{x_{11} - x_{21}, x_{11} - x_{31}, \ldots, x_{11} - x_{n1}\} \cup \{x_{12} - x_{22}, \ldots, x_{12} - x_{n2}\} \cup \ldots \cup \{x_{1m} - x_{2m}, \ldots, x_{1m} - x_{nm}\},
\]

is a minimal free resolution of the ideal of all monomials of degree $n$ in $m$ variables. Each row of this specialized presenting matrix is a complete intersection on precisely $m - n$ elements by Proposition 4.13, implying that the original presenting matrix had
Lemma 4.15. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a vector with \(|\alpha| = d \) and let \( \ell \leq d \). Let
\[
\alpha' := \{ \alpha' \mid |\alpha'| = \ell \}.
\]
Then,
\[
n_{\alpha}^{\ell} = \sum_{S \subseteq [n]} (-1)^{|S|} \left( n + \ell - \sum_{j \in S} (\alpha_j + 1) - 1 \right).
\]

Proof. This is the “balls in bins with limited capacity” counting problem. \( \Box \)

If \( w \) corresponds to a proper monomial order \( < \), Proposition 4.13 implies that \( L_I \in \mathcal{L}(A_{L_I}, [m]/I) \), where \( \mathcal{L}(A_{L_I}, [m]/I) \) is as in Definition 4.10. The following definition combines definitions 4.8, 4.10, and 4.9 and will be needed to define the \( q_i \) maps of Proposition 2.3.

Definition 4.16. Adopt notation and hypotheses of Setup 4.4. Let \( I = (i_1, \ldots, i_n) \) and \( J = (j_1, \ldots, j_\ell) \) with \( i_1 < \cdots < i_n \) and \( j_1 < \cdots < j_\ell \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_i \geq 0 \) for each \( i \) and \(|\alpha| = \ell \). Define the indexing set \( \mathcal{L}_w^I(\alpha, J) \subseteq \mathcal{L}(\alpha, J) \) as all \( L \in \mathcal{L}(\alpha, I) \) such that \( L \subseteq \mathcal{I}_w(\alpha, I \cup J) \cap L_I \).

Lemma 4.17. Define
\[
q_\ell : D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \rightarrow \bigwedge^\ell U
\]
by sending
\[
g^*(\alpha) \otimes f_{J,I} \mapsto \sum_{L \in \mathcal{L}_w^I(\alpha, J)} e_{L_1} \wedge \cdots \wedge e_{L_\ell},
\]
where \( J = (j_1, \ldots, j_\ell) \), \( j_1 < \cdots < j_\ell \), and all other basis vectors to 0. Then the following diagram commutes:
\[
\begin{array}{ccc}
D_\ell(G^*) \otimes \bigwedge^{n+\ell} F & \xrightarrow{d_{k+1}} & D_{\ell-1} \otimes \bigwedge^{n+\ell-1} F \\
\downarrow q_\ell & & \downarrow q_{\ell-1} \\
\bigwedge^\ell U & \longrightarrow & \bigwedge^{\ell-1} U
\end{array}
\]

Proof. Compute the image of \( g^*(\alpha) \otimes f_{J,I} \) going clockwise:
\[
g^*(\alpha) \otimes f_{J,I} \mapsto \sum_{\{i \mid \alpha_i \neq 0\}} (-1)^{j_1+1} \chi((i, j) \in \mathcal{I}_w(\alpha, I \cup J)) x_{i,j} g^{*(\alpha - \epsilon_i)} \otimes f_{J \setminus j,I}
\]
Proposition 4.18. Adopt notation and hypotheses as in Setup 4.12. Assume that \( w \) corresponds to a proper monomial order \(<\). Let

\[
q_{\ell} : D_{\ell}(G^*) \otimes \bigwedge^{n+\ell} F \to \bigwedge^{\ell} U
\]

be the maps of Lemma 4.11. If \( q_{\ell}(g^{(\alpha)} \otimes f_{J,I}) \neq 0 \), then \( \alpha \leq A_{L_i} \). In particular,

\[
\text{rank}(q_{\ell} \otimes k) \leq n^\ell_{A_{L_i}} \cdot \binom{m-n}{\ell}
\]

Proof. It is clear that if \( q_{\ell}(g^{(\alpha)} \otimes f_{J,I}) \neq 0 \), then \( \mathcal{L}_w(\alpha, J) \neq \emptyset \). Thus, \( \alpha \leq A_{L_i} \) and \( J \subseteq [m] \setminus I \).

Observe that \( \mathcal{L}(\alpha, J) \cap \mathcal{L}(\alpha', J) = \emptyset \) for \( \alpha \neq \alpha' \). This means that \( \text{rank}(q_{\ell} \otimes k) \) is the number of \( g^{(\alpha)} \otimes f_{J,I} \in D_{\ell}(G^*) \otimes \bigwedge^{n+\ell} F \) with nonzero image. By the above, there are at most \( n^\ell_{A_{L_i}} \cdot \binom{m-n}{\ell} \) such elements.

Setup 4.19. Let \( R = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m] \) denote a generic \( n \times m \) matrix, where \( n \leq m \). View \( M \) as a homomorphism \( M : F \to G \) of free modules \( F \) and \( G \) of rank \( m \) and \( n \), respectively.

Choose indexing sets \( I_j = (i_{j1}, \ldots, i_{jn}) \) for \( j = 1, \ldots, r \) pairwise disjoint; that is, \( I_i \cap I_j = \emptyset \) for \( i \neq j \) (this intersection is taken as sets).
Let \( f_i, i = 1, \ldots, m \), \( g_j, j = 1, \ldots, n \) denote the standard bases with respect to which \( M \) has the above matrix representation. Write
\[
\bigwedge^n F = \bigwedge^n F' \oplus Rf_{I_1} \oplus \cdots \oplus Rf_{I_r},
\]
where the notation \( f_{I_j} \) denotes \( f_{i_1} \wedge \cdots \wedge f_{i_n} \). Recall that the sparse Eagon-Northcott complex of Definition 4.8 resolves the initial ideal of \( n \times n \) minors of \( M \).

Observe that the sparse Eagon-Northcott differential \( d_2 : G^* \otimes \bigwedge^{n+1} F \to \bigwedge^m F \) induces homomorphisms \( d_2^\ell : G^* \otimes \bigwedge^{n+1} F \to Rf_{I_j} \) by sending
\[
g_i^* \otimes f_{i(\ell), I_j} \mapsto x_i^\ell f_{I_j},
\]
and all other basis elements to 0. In the notation of Setup 2.1, this means we are considering the family of ideals
\[
a_j = \langle x_i^\ell | (i, \ell) \in I_j \rangle.
\]
For each \( j = 1, \ldots, r \), \( a_j \) is a complete intersection generated by at least \( m-n \) elements (by Proposition 4.14), hence resolved by the Koszul complex. Let
\[
U_j = \bigoplus_{(i, \ell) \in L_{I_j}} R e_i^\ell
\]
with differential induced by the homomorphism \( m_\ell^J : U_j \to R \) sending \( e_i^\ell \mapsto x_i^\ell \). If \( L = (i, j) \) is a 2-tuple, then the notation \( e_L \) will denote \( e_{ij} \).

Lemma 4.20. Adopt notation and hypotheses of Setup 4.19. Define
\[
q_\ell^J : D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \to \bigwedge^\ell U_j
\]
by sending
\[
g_i^*(\alpha) \otimes f_{J, I_j} \mapsto \sum_{L \in L_{(J,I_j)}} e_{Li} \wedge \cdots \wedge e_{L\ell},
\]
where \( J = (j_1, \ldots, j_\ell), j_1 < \cdots < j_\ell \), and all other basis vectors to 0. Then the following diagram commutes:
\[
\begin{array}{ccc}
D_\ell(G^*) \otimes \bigwedge^{n+\ell} F & \xrightarrow{d_{\ell+1}} & D_{\ell-1} \otimes \bigwedge^{n+\ell-1} F \\
\bigwedge^\ell U & \downarrow q_\ell^J & \bigwedge^{\ell-1} U \\
\end{array}
\]

Notation 4.21. Let \( A_{I_j} \) be the vector associated to the indexing set \( I_j \) as in Definition 4.9. Define the vector \( \min_j \{ A_{I_j} \} \) to be the vector with \( k \)th component equal to
\[
\min_j \{(A_{I_j})_k\},
\]
for all \( 1 \leq k \leq n \). Let \( \mathcal{P}_i([m]) \) denote all size \( i \) subsets of \([m]\).
Theorem 4.22. Adopt notation and hypotheses as in Setup 4.19 and let \( q_i : D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \to \bigwedge^\ell U_j \) be as in Lemma 4.20. Then,

\[
\text{rank} \left( \left[ \begin{array}{c} q_1^1 \\ \vdots \\ q_r^\ell \end{array} \right] \otimes k \right) \leq \sum_{i=1}^{r} (-1)^{i+1} \left( \sum_{T \in \mathcal{P}_i([r])} n_{\min_j \in T(A_j)}^\ell \binom{m-in}{\ell -(i-1)n} \right)
\]

Proof. The proof employs the inclusion-exclusion principle. We first count all \( g_{\alpha}^{(a)} \otimes f_j \in D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \) with nonzero image under precisely \( i \) of the \( q_i^j \) maps. It is easy to see that if such an element has nonzero image under precisely \( i \) maps, then \( \alpha \leq \min_{j \in T(A_j)} \) for some index \( i \) subset \( T \subseteq [r] \), and \( J = J' \cup \left( \bigcup_{j \in T I_j} \right) \).

There are precisely \( n_{\min_j \in T(A_j)}^\ell \) such \( \alpha \) and \((\ell - (i-1)n)\) such indexing sets \( J' \). Summing over all \( T \in \mathcal{P}_i([r]) \) gives the number of \( g_{\alpha}^{(a)} \otimes f_j \in D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \) with nonzero image under precisely \( i \) of the \( q_i^j \) maps. By the inclusion exclusion principle, the number of \( g_{\alpha}^{(a)} \otimes f_j \in D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \) with nonzero image under some collection of \( q_i^j \) is bounded by

\[
\sum_{i=1}^{r} (-1)^{i+1} \left( \sum_{T \in \mathcal{P}_i([r])} n_{\min_j \in T(A_j)}^\ell \binom{m-in}{\ell -(i-1)n} \right).
\]

Moreover, since \( L(\alpha, J) \cap L(\alpha', J) = \emptyset \) for \( \alpha \neq \alpha' \), the above is an upper bound of \( \text{rank}_k \left( \left[ \begin{array}{c} q_1^1 \\ \vdots \\ q_r^\ell \end{array} \right] \otimes k \right) \).

Example 4.23. Observe that we do not necessarily have equality in the above ranks, as the following example shows. Let \( M \) be a generic \( 3 \times 5 \) matrix, and consider removing the generator \( x_{11}x_{33}x_{34} \in \text{in}_< I_3(M) \), where \( < \) denotes standard lexicographic order. The corresponding ideal \( \mathfrak{a} \) is \( (x_{12}, x_{22}, x_{35}) \), so that \( A_{L(1,3,4)} = (1,1,1) \). However, it can be verified in Macaulay2 that \( q_2 \otimes k \) has rank \( 2 < n_{(1,1,1)}^2 \cdot \binom{2}{2} = 3 \).

For the sake of providing more examples, retain the setting above. If we remove the generator \( x_{11}x_{22}x_{35} \in \text{in}_< I_3(M) \), the corresponding ideal \( \mathfrak{a} \) is \( (x_{23}, x_{24}, x_{33}, x_{34}) \). This yields \( A_{L(1,2,3)} = (0,2,2) \), and Macaulay2 shows that \( \text{rank}(q_2 \otimes k) = 3 = n_{(0,2,2)}^2 \cdot \binom{3}{2} \).

If we remove the generator \( x_{11}x_{23}x_{35} \in \text{in}_< I_3(M) \), then \( \mathfrak{a} = (x_{12}, x_{22}, x_{24}, x_{34}) \). In this case, \( A_{L(1,3,5)} = (1,2,1) \), and Macaulay2 shows that \( \text{rank}(q_2 \otimes k) = 4 = n_{(1,2,1)}^2 \cdot \binom{3}{2} \).

5. The Case for Standard Lex Order

In this section, we specialize to the case of standard lexicographic order. Our first goal is to produce an explicit resolution of the initial ideal of maximal minors of an \( n \times m \) matrix under standard lexicographic order. We then employ Theorem 2.4 to verify a conjecture of Ene, Herzog, and Hibi in a previously unknown case.

Lemma 5.1. Adopt notation and hypotheses as in Setup 4.4 and assume that \( w \) corresponds to the standard lexicographic order \( < \). Let \( N \) denote any minimal presenting matrix for \( \text{in}_<(I_n(M)) \), whose rows are indexed by indices \( I = (i_1, \ldots, i_n) \) with
Let \(i_1 < \cdots < i_n\). Then the \(I\)th row of \(N\) generates the ideal
\[
\left( x_{k\ell_k} \mid \begin{array}{c}
i_{k-1} < \ell_k < i_{k+1}, \\
i_{n-1} < \ell_n \leq m, \\
i_{n} \neq i_n
\end{array} \right).
\]

In particular, the \(I\)th row of \(N\) generates a complete intersection on \(i_n - i_1 + 1 + m - 2n\) elements.

**Proof.** Let
\[
K' := (x_{i_1 \cdots x_{i_n}} \mid a_1 < \cdots < a_n, (a_1, \ldots, a_n) \neq (i_1, \ldots, i_n)).
\]
It is clear that the \(I\)th row of \(N\) generates the ideal \((K' : x_{i_1 \cdots x_{i_n}})\); since \(K'\) is a monomial ideal, it is straightforward to check that this colon ideal is the ideal of the statement of the Lemma. This colon ideal is generated by distinct variables, so it is a complete intersection, and a direct count shows that the number of variables is precisely \(i_n - i_1 + 1 + m - 2n\).

**Notation 5.2.** Let \(\alpha = (\alpha_1, \ldots, \alpha_n)\). Define
\[
\alpha_{\leq i} := (\alpha_1, \ldots, \alpha_i),
\]
where \(\alpha_{\leq i} = \emptyset\) if \(i \leq 0\) and \(\alpha_{\leq i} = \alpha\) if \(i \geq n\).

**Definition 5.3.** Let \(\alpha = (\alpha_1, \ldots, \alpha_n)\) with \(|\alpha| = \ell\) and \(I = (i_1 < \cdots < i_{n+\ell})\). Define the indexing set
\[
I_{\prec}(\alpha, I) := \{(i, I_{i+j}) \mid i \in \{k \mid \alpha_k > 0\}, \ |\alpha_{\leq i-1}| \leq j \leq |\alpha_{\leq i}|\}
\]

**Example 5.4.** One easily computes:
\[
I_{\prec}((1, 1, 1), (1, 2, 3, 4, 5, 6)) = \{(1, 1), (1, 2), (2, 3), (2, 4), (3, 5), (3, 6)\}
\]
\[
I_{\prec}((1, 0, 2), (1, 2, 3, 4, 5, 6)) = \{(1, 1), (1, 2), (3, 4), (3, 5), (3, 6)\}
\]
\[
I_{\prec}((2, 1), (1, 2, 3, 4, 5, 6)) = \{(1, 1), (1, 2), (1, 4), (2, 5), (2, 6)\}
\]

**Definition 5.5.** Adopt notation and hypotheses as in Setup 4.4 and assume that \(w\) corresponds to the standard lexicographic order \(<\). Let \(E'_\bullet\) denote the sequence of module homomorphisms with
\[
E'_\ell = \begin{cases} \bigwedge^n G & \text{if } \ell = 0 \\ D_{\ell}(G^*) \otimes \bigwedge^{n+\ell} F & \text{otherwise}, \end{cases}
\]
and first differential \(d'_1 : \bigwedge^n F \to \bigwedge^n G\) sending \(f_I \mapsto \text{in}_{\prec}(M(f_I))\). For \(\ell \geq 2\), \(d'_\ell : D_{\ell-1}(G^*) \otimes \bigwedge^{n+\ell-1} F \to D_{\ell-2}(G^*) \otimes \bigwedge^{n+\ell-2} F\) is the sparse Eagon-Northcott differential
\[
d_{\ell}(g^{s(\alpha)} \otimes f_I) = \sum_{\{i|\alpha_i>0\}} \sum (-1)^{j+1} \chi((i, I_j) \in I_{\prec}(\alpha, I)) x_{i_1} \cdots x_{i_j} g^{s(\alpha - \epsilon_i)} \otimes f_{I \setminus I_j}.
\]

**Proposition 5.6.** The sequence of homomorphisms \(E'_\bullet\) of Definition 5.5 forms a complex.
Proof. Observe first that the map \( d'_1 : \bigwedge^n F \to \bigwedge^n G \) sends

\[ f_I \mapsto x_{1I_1} \cdots x_{nI_n} g[n]. \]

We first verify that \( d'_1 \circ d'_2 = 0 \). Let \( g_k^* \otimes f_I \in G^* \otimes \bigwedge^{n+1} F \); then:

\[
d'_1 \circ d'_2(g_k^* \otimes f_I) = d'_1((-1)^{k+1} x_{kI_k} f_{I \setminus I_k} + (-1)^k x_{kl_{k+1}I_k} f_{I_{k+1} \setminus I_k})
\]

\[
= (-1)^{k+1} x_{kI_k} (x_{1I_1} \cdots x_{kI_k} x_{kl_{k+1}} \cdots x_{nI_n}) g[n]
\]

\[
+ (-1)^k x_{kl_{k+1}} (x_{1I_1} \cdots x_{kI_k} x_{kl_{k+1}} \cdots x_{nI_n}) g[n]
\]

\[= 0. \]

Assume now that \( \ell \geq 1 \); the fact that \( d'_{\ell+1} \circ d'_{\ell+2} = 0 \) is a nearly identical computation to that of the standard Eagon-Northcott differential, where one uses the fact that

\[ I \prec \epsilon_i, I \setminus I_j = \begin{cases} I \prec \epsilon_i, I \setminus I_j \setminus \{i, I_j\} & \text{if } \alpha_i > 1 \\ I \prec \epsilon_i, I \setminus I_j \setminus \{i', I_j'\} & \text{if } \alpha_i = 1. \end{cases} \]

\[ \square \]

Theorem 5.7. Adopt notation and hypotheses as in Setup 4.4 and assume that \( \omega \) corresponds to the standard lexicographic order \(<\). Then the complex of Definition 5.5 is a minimal free resolution of the ideal \( \mathfrak{m} \). The proof of Theorem 5.7 will follow after a series of Lemmas. The idea for the proof uses Theorem 5.8, which is inspired by the proof of acyclicity of the complexes constructed in [11]. The proof is given in [18].

Theorem 5.8 ([18]). Let \( R \) be a commutative ring. Let \((F_\bullet, d_\bullet)\) be an \( n \)-linear complex of free \( R \)-modules such that for all \( i \geq 1 \),

\[ \text{rank}(F_i)_{i+n} = \beta_{i,i+n}(H_0(F_\bullet)). \]

If for all \( i \geq 1 \), the map

\[ (d_i)_{i+n} : (F_i)_{i+n} \to (F_{i-1})_{i+n} \]

is left invertible, then \( F_\bullet \) is acyclic.

Lemma 5.9. Adopt notation and hypotheses of Setup 4.4. Then for all \( \ell, j \),

\[ (d_\ell)_j : (E_\ell)_j \to (E_{\ell-1})_j \]

is left invertible, where \( d_\ell \) denotes the standard Eagon-Northcott differential.

Proof. We will prove that the matrix representation of \((d_\ell)_{n+\ell}\) with respect to the standard bases is such that every column has a nonzero entry and every row has at most 1 nonzero entry, whence \((d_\ell)_{n+\ell}\) contains a full rank permutation matrix as a submatrix and is hence left invertible.

The fact that every column contains a nonzero entry is the statement that \( d_\ell(g^*(\alpha) \otimes f_I) \neq 0 \) for all \( I, \alpha \), which is trivial. Similarly, given any \( x_{ij} g^*(\alpha) \otimes f_I \) with \( j \notin I \), the row corresponding to this basis element has entry \( \pm 1 \) only in the column corresponding to \( g^*(\alpha+\epsilon_i) \otimes f_{I \cup \{j\}} \), just by definition of the Eagon-Northcott differential. If \( j \in I \), then all entries corresponding to this row are 0. This proves the statement. \( \square \)
Corollary 5.10. Adopt notation and hypotheses of Setup 4.4. Then the restriction of the differentials of Definition 5.5 to degree \( n + \ell \)
\[
(d_\ell')_{n+\ell} : (E'_\ell)_{n+\ell} \to (E'_{\ell-1})_{n+\ell}
\]
are left invertible.

Proof. By construction, the differentials \( d_\ell' \) satisfy the same property as in the proof of Lemma 5.9 and are hence left invertible. \( \square \)

Proof of Theorem 5.7. By [4, Theorem 3.1], the minimal free resolution of \( \text{in}_<(I_n(M)) \) is \( n \)-linear with ranks precisely the ranks of the Eagon-Northcott complex. Thus, combining Theorem 5.8 with Corollary 5.10, the complex of Definition 5.5 is acyclic. \( \square \)

Corollary 5.11. Adopt notation and hypotheses as in Setup 4.4, and assume that \( \omega \) corresponds to the standard lexicographic order \( < \). Let \( E'_\omega \) denote the minimal free resolution of \( \text{in}_<(I_n(M)) \). Then \( E'_\omega \otimes R/\sigma \) is a minimal free resolution of the ideal of all squarefree monomials of degree \( n \) in \( m \) variables, where
\[
\sigma = \{ x_{11} - x_{21}, x_{11} - x_{31}, \ldots, x_{11} - x_{n1} \} \cup \{ x_{12} - x_{22}, \ldots, x_{12} - x_{n2} \} \cup \ldots \cup \{ x_{1m} - x_{2m}, \ldots, x_{1m} - x_{nm} \}
\]

Corollary 5.12. Adopt notation and hypotheses as in Setup 4.4, and assume that \( \omega \) corresponds to the standard lexicographic order \( < \). Let \( E'_\omega \) denote the minimal free resolution of \( \text{in}_<(I_n(M)) \). Then, under the relabelling
\[
x_{ij} \mapsto x_{j-i+1,i},
\]
\( E'_\omega \) is a minimal free resolution of the box polarization of \((x_1, \ldots, x_m - n + 1)^n\).

In particular, with the above relabelling, \( E'_\omega \otimes R/\sigma \) is a minimal free resolution of \((x_1, \ldots, x_m - n + 1)^n\), where
\[
\sigma = \{ x_{11} - x_{12}, x_{11} - x_{13}, \ldots, x_{11} - x_{1n} \} \cup \{ x_{21} - x_{22}, \ldots, x_{21} - x_{2n} \} \cup \ldots \cup \{ x_{m-n+1,1} - x_{m-n+1,2}, \ldots, x_{m-n+1,1} - x_{m-n+1,n} \}
\]

Remark 5.13. Theorem 5.7 and its corollaries presents a novel construction of explicit minimal free resolutions of powers of the graded maximal ideal and the ideal generated by all squarefree monomials. Minimal resolutions for both classes of ideals were already well known, however, the resolutions constructed above have the advantage of being built up by simpler free modules with differentials explicitly computable without the use of any kind of straightening algorithms for Young tableaux.

Theorem 5.14. Adopt notation and hypotheses as in Theorem 2.7 and let \( \omega \) denote standard lexicographic order. If \( n > 2 \), then the ideals \((\Delta_J \mid J \neq I_j, j = 1, \ldots, r)\) and \((\text{in}_<(\Delta_J) \mid J \neq I_j, j = 1, \ldots, r)\) have different Betti tables.

Proof. By Lemma 5.1, the second differential in the minimal free resolution of \( \text{in}_n(I_n(M)) \) over \( R \) is a complete intersection on at most \( 2(m - n) \) elements. In the notation of Observation 2.6, this implies that the ideal \( a \) corresponding to any one of the removed generators has \( \text{pd}_R R/a \leq 2(m - n) \). By Theorem 2.7, the ideal \((\text{in}_<(\Delta_J) \mid J \neq I_j, j = 1, \ldots, r)\) also has projective dimension \( \leq 2(m - n) \). Since \( n > 2 \), this implies
\[
\text{pd}_R R/(\text{in}_<(\Delta_J) \mid J \neq I_j, j = 1, \ldots, r) \leq 2(m - n) < n(m - n) = \text{pd}_R R/I_n(M),
\]
where the rightmost equality is by Theorem 2.7. Since the ideals \((\Delta_J \mid J \neq I_j, \ j = 1, \ldots, r)\) and \((\in_{<}(\Delta_J) \mid J \neq I_j, \ j = 1, \ldots, r)\) have different projective dimensions, the conclusion follows.

**Corollary 5.15.** Adopt notation and hypotheses of Setup 4.12 with \(n = 2\) and assume that \(w\) corresponds to standard lexicographic order \(<\). Let \(K' = (\in_{<}(\Delta_J) \mid J \neq (1, m))\). Then \(K'\) has Betti table

\[
\begin{array}{ccccccccc}
0 & 1 & \cdots & \ell & \cdots & 2(m-2) - 1 & 2(m-2) \\
0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
1 & 0 & (m-1) & \cdots & \ell \left( \frac{m}{\ell+1} - \frac{m-2}{\ell-1} \right) & \cdots & 0 & 0 \\
2 & 0 & 0 & \cdots & (2(m-2)) - (\ell + 1)(m-2) & \cdots & 2(m-2) & 1.
\end{array}
\]

**Proof.** Let \(E'_s\) denote the sparse Eagon-Northcott complex resolving \(\in_{<} I_n(M)\). The row corresponding to the initial term of the minor \((1, m)\) of the second differential of \(E'_s\) is a complete intersection on \(m - 1 + 1 + m - 2 \cdot 2 = 2(m-2)\) elements by Lemma 5.1.

This implies that \(A_{L(1,m)} = (m-2, m-2)\) (where \(A_{L(1,m)}\) is as in Definition 4.9), and hence for all \(\ell \leq m-2\), \(n_{A_{L(1,m)}} = (2+\ell-1) = \ell+1\). By Proposition 4.18, \(\text{rank}(q_\ell \otimes k) \leq (\ell + 1)(m-2)\). To prove the reverse inequality, it suffices to show that for all \(\alpha, J\) with \(|\alpha| = \ell\), \(J = (j_1 < \cdots < j_\ell)\), \(j_1 \neq 1, j_\ell \neq m\), there exists \(L \in \mathcal{L}(\alpha, J)\) with \(L \subseteq \mathcal{I}_{<}(\alpha, (1 < j_1 < \cdots < j_\ell < m))\).

Since \(n = 2\), \(\alpha = (p, \ell - p)\) for some \(0 \leq p \leq \ell\). Then, consider \(L = ((1, j_1), \ldots, (1, j_p), (2, j_{p+1}), \ldots, (2, j_\ell))\).

This is clearly an element of \(\mathcal{L}((p, \ell - p), J)\). Moreover, by definition, if \(K = (1, j_1, \ldots, j_\ell, m)\), then \((1, j_s) \in \mathcal{I}_{<}(\alpha, K)\). This is simply because \(j_s = K_{s+1}\), and by construction \((1, j_s) \in L \iff 0 \leq s \leq p\) and \((2, j_s) \in L \iff p + 1 \leq s \leq \ell\).

The Betti table is an immediate consequence of Corollary 2.5. \(\square\)

**Corollary 5.16.** Adopt notation and hypotheses of Setup 4.4 with \(n = 2\) and assume that \(w\) corresponds to standard lexicographic order \(<\). Let \((i, j)\) be an arbitrary indexing set with \(1 \leq i < j \leq m\). Then the ideals \((\Delta_J \mid J \neq (i, j))\) and \((\in_{<}\Delta_J \mid J \neq (i, j))\) have the same Betti table if and only if \((i, j) = (1, m)\).

**Proof.** The converse is Corollary 5.15 combined with Theorem 2.7. For the forward implication, observe that \(R/(\Delta_J \mid J \neq (i, j))\) has projective dimension \(2(m-n)\) and \(R/(\in_{<}\Delta_J \mid J \neq (i, j))\) has projective dimension \(j - i + 1 + m - 2n\). If \(j - i < m - 1\), then these ideals have different projective dimensions and hence have different Betti tables. \(\square\)

**Corollary 5.17.** Adopt notation and hypotheses of Setup 4.4. Then the generators of \((\Delta_J \mid J \neq I_j, \ j = 1, \ldots, r)\) do not form a Gröbner basis if \(n > 2\).
Proof. The projective dimension of \( \text{in}_<(\Delta_J \mid J \neq I_j, j = 1, \ldots, r) \) is an upper bound of the projective dimension of \( (\Delta_J \mid J \neq I_j, j = 1, \ldots, r) \), but \( (\text{in}_<(\Delta_J \mid J \neq I_j, j = 1, \ldots, r)) \) has strictly smaller projective dimension for \( n > 2 \) by the proof of Theorem 5.14. \( \square \)

6. Linear Strand of the Minimal Free Resolution of Lex-Initial Determinantal Facet Ideals

Setup 6.1. Let \( R = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m] \) and \( M = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} \) denote a generic \( n \times m \) matrix, where \( n \leq m \). View \( M \) as a homomorphism \( M : F \to G \) of free modules \( F \) and \( G \) of rank \( m \) and \( n \), respectively.

Let \( f_i, i = 1, \ldots, m, g_j, j = 1, \ldots, n \) denote the standard bases with respect to which \( M \) has the above matrix representation. Let \( < \) denote standard lexicographic order on \( R \) and \( \text{in}_< I_n(M) \) the initial ideal with respect to \( < \) of the ideal of maximal minors of \( M \).

Theorem 6.2 ([13], Theorem 1.1). Let \( R \) be a standard graded polynomial ring over a field \( k \). Let \( G_\bullet \) be a finite linear complex of free \( R \)-modules with initial degree \( n \). Then the following are equivalent:

1. The complex \( G_\bullet \) is the linear strand of a finitely generated \( R \)-module with initial degree \( n \).
2. The homology \( H_i(G_\bullet)_{i+n+j} = 0 \) for all \( i > 0 \) and \( j = 0, 1 \).

Proposition 6.3 ([13], Corollary 1.2). Let \( R \) be a standard graded polynomial ring over a field \( k \). Let \( G_\bullet \) be a finite linear complex of free \( R \)-modules with initial degree \( n \) such that \( H_i(G_\bullet)_{i+n+j} = 0 \) for all \( i > 0, j = 0, 1 \).

Let \( N \) be a finitely generated \( R \)-module with minimal graded free resolution \( F_\bullet \). Assume that there exist isomorphisms making the following diagram commute:

\[
\begin{array}{ccc}
G_1 & \rightarrow & G_0 \\
\sim & & \sim \\
F_1^{\text{lin}} & \rightarrow & F_0^{\text{lin}}
\end{array}
\]

Then \( G_\bullet \cong F_\bullet^{\text{lin}} \).

Definition 6.4. Adopt notation and hypotheses as in Setup 6.1. Let \( \Delta \) denote a simplicial complex on the vertex set \([m]\). Define \( C_0^<(\Delta, M) := \bigwedge^n G \). For \( i \geq 1 \), let \( C_i^<(\Delta, M) \subseteq D_{i-1}(G^*) \otimes \bigwedge^{n+i-1} F \) denote the free submodule generated by all elements of the form

\[ g^{*(\alpha)} \otimes f_{\sigma}, \]

where \( \sigma \in \Delta \) with \( |\sigma| = n + i - 1 \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( |\alpha| = i - 1 \).

Let \( C_\bullet^<(\Delta, M) \) denote the complex induced by the differentials of Definition 5.5 on the free submodules defined above.

Definition 6.5. Let \( \Delta \) be a simplicial complex. Then an \( i \)-nonface \( \sigma \) is an element \( \sigma \notin \Delta \) such that for some \( j \geq 1 \), \( \sigma \setminus \sigma_{j+k} \in \Delta \) for all \( k = 0, \ldots, i \).
Example 6.6. Consider the following graph $G$ on vertices $\{1, 2, 3, 4\}$:

```
  3   2
  |   |
  4   1.
```

Observe that the associated clique complex has facets $\{1, 2, 4\}$ and $\{1, 3, 4\}$, and no minimal nonfaces. However, $\{1, 2, 3, 4\}$ is a 1-nonface of the clique complex, since $\{1, 2, 4\}$ and $\{1, 3, 4\}$ are both facets.

If we instead consider the graph

```
  3   2
  |   |
  4   1.
```

then the clique complex has facets $\{1, 2, 3\}$ and $\{2, 3, 4\}$. The set $\{1, 2, 3, 4\}$ is not a 1-nonface. Likewise, there are no 1-nonfaces of cardinality 3. In the graph

```
  3   2
  |   |
  4   1.
```

the associated clique complex has facets $\{1, 2, 4\}$ and $\{2, 3, 4\}$, and has no 1-nonfaces of cardinality 4. However, $\{1, 3, 4\}$ is a 1-nonface of cardinality 3 since $\{1, 4\}$ and $\{3, 4\}$ are vertices of $G$.

Remark 6.7. Notice that a minimal nonface $\sigma$ is a $\dim \sigma$-nonface in the above definition. Moreover, any $i$-nonface is a $k$-nonface for all $k \leq i$.

In the proofs of the results in the remainder of this section, notice that we have chosen to augment our complexes with the ring $R$. This means that we are resolving the quotient ring $R/I$ as opposed to the module $I$; this has the effect of shifting the indexing in the statements of Theorem 6.2 and Proposition 6.3.

Lemma 6.8. Adopt notation and hypotheses as in Setup 6.1. If the simplicial complex $\Delta$ has no 1-nonfaces of cardinality $\geq n + 1$, then the complex $C_\leq^i(\Delta, M)$ is the linear strand of a finitely generated graded $R$-module with initial degree $n$.

Proof. Employ Theorem 6.2. To avoid trivialities, assume $n \leq \dim(\Delta) + 1$. Observe first that $H_i(C_\leq^i(\Delta, M))_{i+n-1} = 0$ for all $i \geq 1$ trivially.

To finish the proof, we show that

$$H_i(C_\leq^i(\Delta, M))_{i+n} \neq 0 \text{ for all } i \geq 1$$

$$\implies$$

there exists a 1-nonface of cardinality $n + i$, for all $i \geq 1$.

For convenience, use the notation $C_\leq^i(M) := E'_i$, where $E'_i$ is as in Definition 5.5.

Assume $H_i(C_\leq^i(\Delta, M))_{i+n} \neq 0$. Let $z \in C_\leq^i(\Delta, M)$ be a cycle that is not a boundary; without loss of generality, assume $z$ is multihomogeneous. The complex $C_\leq^i(M)$ is exact,
whence \( z = d(y) \) for some \( y \in C_{i+1}(\Delta,M) \). By multihomogeneity, \( y = \lambda g^*(\alpha) \otimes f_\sigma \) for some \( \lambda \in k^\times \) with \( |\sigma| = n + i \), \( |\alpha| = i \). The assumption that \( z \) is not a boundary implies that \( \sigma \notin \Delta \), since otherwise \( y \in C_{i+1}(\Delta,M) \). By definition of the differential of \( C_\bullet(S) \),

\[
z = \lambda \sum_{\ell \sigma_j > 0} \sum_j (-1)^{j+1} \chi((\ell,\sigma_j) \in I_{\Delta}(\alpha,\sigma)) x_{\ell \sigma_j} g^*(\alpha-\epsilon_\ell) \otimes f_\sigma|_{\sigma_j}.
\]

Since \( z \neq 0 \), \((\ell,\sigma_j) \in I_{\Delta}(\alpha,\sigma)\) for some \( \ell, j \). By definition of \( I_{\Delta}(\alpha,\sigma) \), this means \((\ell,\sigma_k) \in I_{\Delta}(\alpha,\sigma)\) for all \( |\alpha| \leq |\alpha| \leq |\alpha| \). This translates to the fact that \( \sigma \) is an \( \alpha_\ell \)-nonface of cardinality \( n + i \). Since \( \alpha_\ell \geq 1 \), the result follows. \( \square \)

**Remark 6.9.** The proof of Lemma 6.8 allows one to construct explicit examples of nonzero homology on the complex \( C_\bullet(\Delta,M) \). Let \( \Delta^{\text{clique}} \) be the simplicial complex associated to the first graph of Example 6.6. Then the element

\[
z = x_{22} f_{1,3,4} - x_{23} f_{1,2,4}
\]

is a cycle which is not a boundary \( n C_\bullet(\Delta^{\text{clique}},M) \).

**Lemma 6.10.** Adopt notation and hypotheses as in Setup 6.1. Then the following are equivalent:

1. \( H_1(C_\bullet(\Delta,M)) = 0 \),

2. There are no 1-nonfaces of cardinality \( n + 1 \).

**Proof.** The implication (2) \( \implies \) (1) is Lemma 6.8. Conversely, assume that \( \sigma \notin \Delta \) is a 1-nonface of cardinality \( n + 1 \). By definition, there exists some \( j \) such that \( \sigma_j \) and \( \sigma_j |_{\sigma_j+1} \in \Delta \). This means that \( z = (-1)^{j+1}(x_{\sigma_j} f_\sigma|_{\sigma_j} - x_{\sigma_j+1} f_\sigma|_{\sigma_j+1}) \) is a cycle in \( C_1(\Delta,M) \) that is not a boundary, since \( z = d_2(g_\rho^* f_\sigma) \), and \( g_\rho^* f_\sigma \notin C_2(\Delta,M) \) by construction. \( \square \)

**Lemma 6.11.** Let \( \Delta \) be a pure \((n-1)\)-dimensional simplicial complex on the vertex set \([m]\). If the simplicial complex \( \Delta \) is closed, then the associated clique complex \( \Delta^{\text{clique}} \) has no 1-nonfaces of cardinality \( \geq n + 1 \).

**Proof.** Assume that \( \Delta^{\text{clique}} \) has a 1-nonface \( \sigma \notin \Delta^{\text{clique}} \) of cardinality \( \geq n + 1 \). By definition, there exists \( j \geq 1 \) such that \( \sigma_j, \sigma_j |_{\sigma_j+1} \in \Delta^{\text{clique}} \). Let \( \Gamma_1, \ldots, \Gamma_r \) be the cliques of \( \Delta \); then there are two cases:

1. Case 1: \( \sigma_j, \sigma_j |_{\sigma_j+1} \in \Gamma_i \) for some \( i \). Since \( \Gamma_i \) is a simplex, \( |\sigma| = |\Gamma| + 1 \). Thus there is only 1 element \( \sigma \) not contained in \( \Gamma_i \), in which case there are obviously no 1-nonfaces, since if \( \sigma_j \notin \Gamma_i \), then \( \sigma_j |_{\sigma_j+1} \notin \Gamma_i \) (notice that this case is impossible, regardless of closedness).

2. Case 2: \( \sigma_j |_{\sigma_j} \in \Delta_k \), \( \sigma_j |_{\sigma_j+1} \in \Gamma_i \) for some \( i \neq k \). Without loss of generality, we may assume that \( |\sigma| = n + 1 \) by taking an appropriate subset of \( \sigma \). But then \( \sigma_j |_{\sigma_j} \in \Delta_k \), \( \sigma_j |_{\sigma_j+1} \in \Delta_i \), and it is clear that all but the \( j \)th entries of \( \sigma_j \) and \( \sigma_j |_{\sigma_j+1} \) are equal, whence \( \Delta \) is not closed. \( \square \)

Recall that the standard Eagon-Northcott complex inherits a \( \mathbb{Z}^n \times \mathbb{Z}^m \)-grading, as described in Section 3 of 13. Since the sparse Eagon-Northcott complexes of Section 4 are obtained by simply setting certain entries in the differentials equal to 0, these maps
will remain multigraded in an identical manner. We tacitly use this multigrading for
the remainder of this section.

**Theorem 6.12.** Adopt notation and hypotheses as in Setup 6.1. Assume that \( \Delta \) is an
\((n - 1)\)-pure closed simplicial complex. Let \( \text{in}_{\prec} J_\Delta \) denote the initial ideal with respect
to standard lexicographic order of the determinantal facet ideal associated to
Theorem 6.12.

Let \( F_* \) denote the minimal graded free resolution of \( \text{in}_{\prec} J_\Delta \); then

\[
F^\text{lin}_* \cong C_*^{\prec} (\Delta^{\text{clique}}, M)
\]

**Proof.** Let \( Z^\text{lin} := (\ker d_1)_{n+1} \), where \( d_1 \) is the first differential of the complex \( C_*^{\prec} (\Delta^{\text{clique}}, M) \).

By construction, \( C_*^{\prec} (\Delta^{\text{clique}}, M) \) is generated in degree \( n + 1 \) and hence induces a homogenous map

\[
\partial : C_*^{\prec} (\Delta^{\text{clique}}, M) \rightarrow Z^\text{lin}.
\]

Let \( 0 \neq \lambda \in Z^\text{lin} \) be an element of multidegree \((\epsilon_s + 1, \epsilon_i + \cdots + \epsilon_{i_{n+1}})\) (where \( 1 \) denotes the appropriately sized vector of all 1’s). Set \( \tau := \{i_1 < \cdots < i_{n+1}\} \); by multihomogeneity, there are constants \( \lambda_k \in k \) such that

\[
\lambda \sum_{k=1}^{n+1} \lambda_k x_{si_k} f_{\tau \setminus i_k}.
\]

Since \( \lambda \) is a cycle of \( C_*^{\prec} (M) \) (where \( C_*^{\prec} (M) := E_1^* \) is as in Definition 5.5), there exists \( y \in C_*^{\prec} (M) \) such that \( d_2(y) = \lambda \). By multihomogeneity, \( y = \lambda g_{\tau} \otimes f_{\tau} \) for some constant \( \lambda \), whence \( z = \lambda(1)^{s+1}(x_{si_s} f_{\tau \setminus i_s} - x_{si_{s+1}} f_{\tau \setminus i_{s+1}}) \). This implies that \( \sigma \in \Delta^{\text{clique}}, \) since otherwise \( \Delta^{\text{clique}} \) would have a 1-nonface of cardinality \( n + 1 \), contradicting Lemma 6.11.

Thus \( Z^\text{lin} \) is generated by

\[
\{r_s(\sigma) := (-1)^{s+1}(x_{si_s} f_{\tau \setminus i_s} - x_{si_{s+1}} f_{\tau \setminus i_{s+1}}) \mid 1 \leq s \leq n, \sigma \in \Delta^{\text{clique}}, |\sigma| = n + 1 \}.
\]

Moreover, since \( \text{mdeg}(r_s(\sigma)) \neq \text{mdeg}(r_s(\sigma')) \) for \( s \neq s', \sigma \neq \sigma' \), the above is a basis. Finally, \( d_2(g_{\tau} \otimes f_{\sigma}) = r_s(\sigma) \), whence the induced map \( \partial \) is an isomorphism of vector spaces.

**Remark 6.13.** Let \( \Delta \) be an \((n - 1)\)-pure closed simplicial complex. Then \( \Delta^{\text{clique}} \) has no minimal nonfaces in cardinality \( \geq n + 1 \), since any minimal nonface is in particular a 1-nonface. This means that \( \Delta^{\text{clique}} \) satisfies the hypotheses of Theorem 3.1 of [13].

**Theorem 6.14.** Adopt notation and hypotheses as in Setup 6.1. Assume that \( \Delta \) is an
\((n - 1)\)-pure closed simplicial complex. Then for all \( i \geq 1, \)

\[
\beta_{i,n+i}(J_\Delta) = \beta_{i,n+i} (\text{in}_{\prec} J_\Delta).
\]

**Proof.** Notice that the linear strand of \( J_\Delta \) is \( C_* (\Delta^{\text{clique}}, M) \) where \( C_* \) is the generalized Eagon-Northcott complex of [13]. Then, \( C_* \) and \( C_*^{\prec} \) have the same underlying free modules, so the result follows.

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