Gravitational Field of Ultrarelativistic Objects with Angular Momentum

Dmitri V. Fursaev
Dubna International University and Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141 980, Dubna, Moscow Region, Russia
E-mail: fursaev@thsun1.jinr.ru

Abstract. A brief review of recently found gyraton metrics which describe the gravitational field of objects having an angular momentum and moving with the velocity of light is given. The gyraton metrics belong to a class of exact plane wave solutions of four and higher dimensional Einstein equations in vacuum or in the presence of a negative cosmological constant.

1. Introduction

A metric describing the gravitational field of a massive object moving with the velocity close to the velocity of light was discovered by Aichelburg and Sexl in 1971 [1]. It can be obtained from the Schwarzschild metric taken in the frame of reference where the source is moving by considering the so called Penrose limit when the velocity of the source approaches the velocity of light while its energy remains finite. The gravitational field of the point-like source in the Penrose limit is "squeezed" in the plane which moves with the source and is orthogonal to the direction of motion. For this reason the corresponding geometry is also called a "plane wave" or a "shock wave". Such gravitational shock waves can be produced by ultrarelativistic elementary particles.

Recently a new class of exact solutions of the Einstein equations which describe the gravitational field of ultrarelativistic objects having an angular momentum was discovered in [3]. Such objects were called gyratons. An example of the gyraton is a narrow beam pulse of a circularly polarized electromagnetic radiation. The problem of finding the gravitational field of pulses of light was first considered long ago by Tolman [2]. The important property of solutions found in [3] is that they are described by stationary metrics which carry the information about the beam’s polarization. Results of [3] hold for axisymmetric gyratons moving in flat spacetimes. They were further generalized in [4] to the case of "deformed gyratons" which have other characteristics in addition to the energy and angular momentum. In [5] exact gyraton solutions were found for Einstein equations with a negative cosmological constant.

There are different motivations for studying the gravitational shock waves. For instance, in the string theory the gravitational shock waves do not receive $\alpha'$ corrections and the theory is exactly solvable [6]-[8]. Another motivation is related to the problem of mini black hole formation in high-energy particle collisions [9]. The gravitational field of each colliding ultrarelativistic spinless particle can be approximated by the Aichelburg-Sexl metric and used to estimate the cross section for black hole production [10]. If the particles have spin, one can expect new effects
connected with spin–orbit and spin–spin interactions. These effects should be studied by using gyraton metrics.

2. Penrose limit

Let us start with a brief description of the Penrose limit for the gravitational field of a non-rotating object in a four-dimensional spacetime. The limit implies that mass $M$ of the source vanishes when its velocity $\beta$ approaches the velocity of light while the energy $E = M\gamma$ remains finite (here $\gamma = (1 - \beta^2)^{-1/2}$ and we put the velocity of light equal to unity). The limit of small mass is equivalent to situation when one stays far from the object at the distance $R \gg MG$.

The metric in this region for the source at rest can be taken in the weak field approximation

$$ds^2 = -dt^2 + dl^2 + \frac{2MG}{R}(dt^2 + dl^2) , \quad dl^2 = d\xi^2 + dx^2 , \quad R = \sqrt{\xi^2 + x^2} . \quad (1)$$

We choose $\xi$ as coordinate along the direction of the motion of the object, $x$ will be called transverse coordinates. Coordinates in the moving frame of reference are $\bar{\xi}, \bar{t}, \bar{x}$. The relation between the coordinates is $\xi = \gamma(\bar{\xi} - \beta \bar{t})$, $t = \gamma(\bar{t} - \beta \bar{\xi})$. One has $-dt^2 + dl^2 = -d\bar{u}d\bar{v} + d\bar{x}^2$, where $\bar{u} = \bar{t} - \bar{\xi}$, $\bar{v} = \bar{t} + \bar{\xi}$. To get result for the entire metric (1) in the Penrose limit we take into account that $\xi \simeq -\gamma u$, $t \simeq \gamma u$ and, hence, $dt^2 + dl^2 \simeq 2\gamma^2 du^2$. One also needs the limiting value of $\gamma/R$ which we denote as $\psi(u, r)$, where $r = \sqrt{x^2}$. It is not difficult to see that

$$\psi(u, r) = \frac{1}{u} - 2\delta(u) \ln r . \quad (2)$$

The first term in the right hand side (r.h.s.) of (2) is a naive limit of $\gamma/R$ based on the asymptotic $R \simeq \gamma u$. The second term in the r.h.s. of (2) takes into account that $1/R$ is a distribution because $\Delta(1/R) = -4\pi\delta(z)\delta(\bar{x})$. If we use (2) in the boosted metric (1) and change $v$ to $v - 4EG\ln u$ the result will be the Aichelburg-Sexl solution

$$ds^2 = -d\bar{u}d\bar{v} + d\bar{x}^2 - 8EG\delta(u)\ln r \ d\bar{u} . \quad (3)$$

Consider now a source with an angular momentum $J$ along the $\xi$-axis. The metric far from the source is given by expression (1) with the additional term $2Jdt(xdy - ydx)R^{-3}$. The value of the $\xi$-component of the angular momentum under the boost does not change. This can be seen, for instance, from the angular momentum of a particle $J_\xi = xp_y - yp_x$ determined only by transverse coordinates and the corresponding momenta $p_x, p_y$, which do not transform under the boost. Thus, in the Penrose limit

$$\frac{2J}{R^3}(xdy - ydx)dt \to \frac{4J}{r^2}\delta(u)(xdy - ydx)du , \quad (4)$$

because $\gamma x/R^3 \to -\partial_\xi \psi(u, r) = 2\delta(u)x/r^2$. This yields the following metric which describes the gravitational field of a gyraton

$$ds^2 = -d\bar{u}d\bar{v} + d\bar{x}^2 - 8EG\delta(u)\ln r \ d\bar{u}^2 + \frac{4J}{r^2}\delta(u)(xdy - ydx)du = -d\bar{u}d\bar{v} + dr^2 + r^2d\varphi^2 + \delta(u)du(-8EG\ln r \ du + 4Jd\varphi) . \quad (5) \quad (6)$$

Here $\varphi$ is an angular coordinate in the polar coordinate system in the transverse plane with the center at $r = 0$. Although we used the weak field approximation, it can be checked that (5) is an exact solution of the vacuum Einstein equations outside the gyraton, i.e., in the region $r > 0, u \neq 0$. 


The gyration metric can be generalized to the case when the gyration has a finite duration $L$ in time
\[ ds^2 = -dt^2 + dx^2 - 8EG\chi(u)\ln r \, du^2 + \frac{4J}{r^2}\chi_1(u)(xy - ydx)du . \] (7)

Here $\chi(u)$ and $\chi_1(u)$ are two arbitrary independent functions which are not zero only in an interval of length $L$. We will assume that integrals $\int \chi(u)du$ and $\int \chi_1(u)du$ are finite and normalized to unity.

3. Electromagnetic gyratons

As was emphasized, the obtained solution is valid only outside the source (gyration). It is singular at the location of the source. One can formally define an effective stress-energy tensor for the source as

\[ T_{\mu\nu} = (8\pi G)^{-1}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) . \]

It can be shown [3] that the only non-trivial components of the stress-energy tensor are

\[ T_{uu} = E\chi(u)\delta(x) + \pi GJ^2L^2\chi_1^2(u)\delta^2(x) , \quad T_{ua} = \frac{J}{4}\chi_1(u)\epsilon_{ab}\partial_b\delta(x) , \] (8)

where $\delta^2(x)$ in $T_{uu}$ indicates that in the presence of spin one must consider spatially distributed sources and allow a finite thickness $\rho$ of the gyration. In this case the second term in $T_{uu}$ is suppressed as compared to the first one by the factor $GJ^2/(EL\rho^2)$, which is small because the thickness $\rho$ is much larger than the Planck length. If $J^2$-term is omitted, $T_{uu}$, $T_{ua}$ take the form of components of the stress-energy tensor of a narrow source with the energy $E$ and the spin tensor $J_{ab} = \frac{1}{2}\epsilon_{ab}J$. Therefore, as follows from (8), functions $\chi(u)$ and $\chi_1(u)$ describe profiles of the energy and the angular momentum distributions along the source.

There may be different kinds of gyratons. A narrow beam pulse of circularly polarized electromagnetic radiation is an example of electromagnetic gyration. Let us consider a monochromatic beam propagating in $\xi$-direction. The components of the vector-potential in the Lorentz gauge are

\[ \mathcal{A}_\mu = \mathcal{A}[e_\mu \exp(-i\omega u) + \bar{e}_\mu \exp(i\omega u)] , \quad \mathcal{A}_u = \mathcal{A}_v = 0 . \] (9)

Here $\mathcal{A}$ is a real amplitude and $e_\mu$ is a complex null vector in the transverse plane $\mathbf{x}$ ($\bar{e}_\mu$ is its complex conjugate). One can choose $e_\mu = e_\mu^+$ (right polarization) or $e_\mu = e_\mu^-$ (left polarization), where $e_\mu^\pm = (\delta_\mu^3 \pm i\delta_\mu^4)/\sqrt{2}$. If $\mathcal{A}$ is constant (9) is exact solution of the Maxwell equations. If $\mathcal{A}$ is a slowly changing function, (9) is an approximate solution provided $\omega$ is large. If $\nabla \mathcal{A} \sim \rho^{-1}\mathcal{A}$ then (9) gives a solution in the geometric optics approximation provided $\lambda/\rho \ll 1$, where $\lambda = \omega^{-1}$ is a wave-length. We assume that the beam-pulse is axisymmetric and localized in the vicinity of $\mathbf{x} = 0$. This means that $\mathcal{A}$ depends only on $r$. For a narrow beam of radius $\rho$ the amplitude vanishes for $r > \rho$.

The metric stress-energy tensor of the electromagnetic field is

\[ t^\nu_\nu = F^{\mu\lambda}F_{\nu\lambda} - \frac{1}{4}\delta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} , \] (10)

where $F_{\mu\nu} = \mathcal{A}_{\mu,\nu} - \mathcal{A}_{\nu,\mu}$. The stress-energy tensor for a monochromatic wave besides a part independent of $u$ contains also a rapidly oscillating contribution $\sim \exp(\pm 2i\omega u)$. This contribution vanishes after averaging over the time interval $L \gg \lambda$ and can be neglected.
It is easy to check that the components $t_{uu}$ and $t_{uv}$ vanish identically. The components of $t_{uv}$, $t_{ab}$ are of the order of $\lambda^2/\rho^2$ and also can be neglected. The leading non-vanishing components of $t_{\mu\nu}$ are

$$t_{uu} = 2\omega^2 A^2 \quad , \quad t_{aa} = i\omega A a_b (\bar{e}_a e_b - e_a \bar{e}_b) \quad .$$

(11)

Since the averaged stress-energy tensor $t_{\mu\nu}$ does not depend on $u$, the energy of the beam is divergent. In order to deal with a realistic system with finite energy it is sufficient to assume that the beam-pulse has a finite duration $L \gg \lambda$ in time. To deal with this situation we assume that during the time interval $u \in (-L/2, L/2)$ the stress-energy tensor is given by (11), and vanishes outside this interval. The corresponding stress-energy tensor can be written as $T_{\mu\nu}(u, \mathbf{x}) = \chi(u) L t_{\mu\nu}(\mathbf{x})$, where $\chi(u) = (\vartheta(u + L/2) - \vartheta(u - L/2))/L$ and $\vartheta(u)$ is the Heaviside step function. The total energy of the beam is

$$E = \int du \, d\mathbf{x} T_{uu} = L \int d\mathbf{x} t_{uu} = 2NL\omega^2 \quad ,$$

(12)

where $N = \int d\mathbf{x} A^2$ is a normalization constant depending on the amplitude $A$. The spin tensor of the beam is

$$J_{ab} = \int d\mathbf{x} M^{ab} u = i \frac{E}{2\omega} (e_a \bar{e}_b - \bar{e}_a e_b) \quad ,$$

(13)

where $M^{\alpha\beta} = x^{\alpha} T^\beta \nu - x^{\beta} T^\alpha \nu$ is the angular momentum tensor. Because the beam-pulse is axisymmetric the components of the spin tensor $J_{aa}, J_{uv}$ vanish. The components $J_{ab}$ may be nontrivial but they are not relevant because their contribution to the gravitational field (the $uv$ component of the metric) is of higher order with respect to the contribution produced by the energy of the beam. The polarization of the beam is

$$J = \varepsilon_{ab} J^{ab} = i \frac{E}{\omega} \varepsilon_{ab} e_a \bar{e}_b = \pm \frac{E}{\omega} \quad ,$$

(14)

where signs $+$ and $-$ stand for the right and left polarization, respectively.

In the limit of the infinitely narrow beam one can put $A^2(r) = N \delta(\mathbf{x})$. Then it follows from (11)–(13) that the stress energy tensor of the beam $T_{\mu\nu}(u, \mathbf{x})$ coincides with the stress energy tensor of the gyraton (8) (with the $J^2$-term omitted), the profiles of the energy and angular momentum are described by a single function $\chi(u)$.

4. General Ricci flat gyraton solutions

Metric (7) suggests the following generalization for a gyraton moving in a flat $D$-dimensional spacetime:

$$ds^2 = -du dv + d\mathbf{x}^2 + \Phi du^2 + 2(A, d\mathbf{x}) du \quad ,$$

(15)

where $\mathbf{x}$ are coordinates in the hyperplane transverse to the direction of motion, index $a$ runs from 1 to $D - 2$. It is assumed in (15) that $\Phi = \Phi(u, \mathbf{x})$, $A_a = A_a(u, \mathbf{x})$ and $(A, d\mathbf{x}) = A_a dx^a$. The metric possesses two remarkable properties established in [4].

First, all the local scalar invariants constructed from the Riemann tensor and its covariant derivatives for the metric (15) vanish off-shell, i.e. even if the metric is not a solution to the vacuum Einstein equations. This property is similar to the property of other plane wave spacetimes which were argued to be exact solutions in the string theory to all orders in the string tension [6]-[8].

Second, the vacuum Einstein equations for (15) are reduced to the following problem in a $(D - 2)$-dimensional Euclidean space

$$\partial_b F_a^b = 0 \quad , \quad \Delta \Phi - 2\partial_u (\partial_a A^a) = \frac{1}{2} F_{ab} F^{ab} \quad ,$$

(16)
where $\Delta = \partial_\mu \partial^\mu$ and indexes $a, b$ are raised and lowered with the flat Euclidean metric. Equations (16) hold outside the point $x = 0$ which corresponds to the location of the gyraton. They are invariant under the "gauge" transformations $\delta A_\mu = \lambda_\mu$, $\delta \Phi = 2\lambda_u$. Its convenient to choose the "Lorentz" gauge $\partial_\mu A^\mu = 0$, and write $\Phi = \phi + \psi$, where

$$\Delta \phi = 0 \quad \Delta \psi = \frac{1}{2} F_{ab} F^{ab}. \tag{17}$$

One, therefore, has linear problems in flat $(D - 2)$-dimensional Euclidean space for a static "electric field" $\phi$ and a "magnetic field" $A$ created by a point-like source. After finding these solutions one can obtain $\psi$ from the second equation in (17).

We describe now a solution of (16) found in [3]. To begin with, let us note that there always exists an orthogonal transformation of coordinates $x$ which brings the spin tensor to a canonical form, where its only nonvanishing components are $J_{12} = -J_{21}$, $J_{34} = -J_{43}$, etc. Each pair of coordinates $((x^1, x^2), (x^3, x^4), \ldots)$ corresponding to the canonical form of $J_{ab}$ determines a bi-plane of rotation. In higher dimensions the number of rotation biplanes $l$ is $(D - 2)/2$ if the spacetime dimension $D$ is even, and $l = (D - 3)/2$ if $D$ is odd. In coordinates where $J_{ab}$ has the canonical form a higher dimensional giraton solution ($D > 4$) is given by metric (15) where [3]

$$\phi = \frac{2m\chi(u)}{r^{D-4}} \quad \psi = \frac{D - 6}{2(D - 4)r^{D-6}} \left[ \frac{p^2}{r^2} + \frac{2p^2}{(D - 3)(D - 6)} \right], \tag{18}$$

$$P^2 = \sum_{i=1}^{l} j_i^2 \chi_i^2(u)(x_i^2 + y_i^2) \quad P^2 = \sum_{i=1}^{l} j_i^2 \chi_i^2(u), \tag{19}$$

$$(A, dx) = \frac{1}{r^{D-2}} \sum_{i=1}^{l} j_i \chi_i(u)(x_i dy_i - y_i dx_i), \quad r^2 = (x, x). \tag{20}$$

Here a pair of components of the vector $x$ corresponding to the $i$-th rotation bi-plane is denoted by $(x_i, y_i)$. Parameter $m$ is proportional to the total energy $E$ of the gyration, while $j_i$ are proportional to the angular momentum $J_i$ of the $i$-th bi-plane,

$$m = \frac{8\pi GE}{\Omega_{D-3}(D - 4)} \quad j_i = \frac{4\pi GJ_i}{\Omega_{D-3}}, \tag{21}$$

where $G$ is the Newton constant, $\Omega_{D-3} = 2\pi^{(D-2)/2}/\Gamma((D - 2)/2)$. The functions $\chi(u)$ and $\chi_i(u)$ describe profiles of distributions of the energy and angular momenta (it is assumed that integrals of $\chi(u)$ and $\chi_i(u)$ over $u$ are finite). The solutions (15), (18)–(20) admit Killing vector fields which generate rotations in the bi-planes $(x_i, y_i)$ around the origin $x_i = y_i = 0$.

The given class of solutions is distinguished by the property that it is uniquely characterized by the total energy $E$ and angular momenta $J_i$. There are, however, more general solutions of (16) which in addition to energy and angular momenta have other characteristics. These solutions were described in [4]. The Euclidean space, where the problem (16) is formulated, is invariant under $O(D - 2)$ rotations around the source at $x = 0$. Thus, in general, fields $\Phi$ and $A$ can be represented as series associated with the Fourier analysis on a hypersphere $S^{D-3}$. From this point of view $\phi$ in (18) is a spherically symmetric "s-mode", while $A$ in (20) is a "dipole mode". Higher multipoles bring in new parameters. These multipoles may also destroy the axial symmetry. For example, the gyration in four dimensions with higher multipoles is described by metric (7) with the $uu$–component given by [4]

$$g_{uu} = -8EG\chi(u) \ln r + \sum_{n=1}^{\infty} r^{-n} (b_n(u) \cos n\varphi + c_n(u) \sin n\varphi),$$

where $b_n(u)$ and $c_n(u)$ are real functions. More non-trivial examples, including explicit metrics for 5-dimensional gyratons can be found in [4].
5. Asymptotically AdS gyraton solutions

It was shown in [5] that exact gyraton solutions also exist for Einstein equations with a negative cosmological constant $\Lambda$. They are described by the following metric which generalizes (15)

$$ds^2 = \frac{L^2}{z^2}(-du dv + dx^2 + \Phi(u, x)du^2 + 2(A(u, x), dx)du).$$

(22)

Here $x$ are coordinates in the hyperplane transverse to the direction of motion of the gyraton, and $z$ is one of these coordinates. The radius $L$ is defined by the relation $\Lambda = -(D - 1)(D - 2)/(2L^2)$. If $\Phi$ and $A$ vanish, (22) coincides with the metric of an anti-de Sitter (AdS) spacetime.

As was shown in [5], (22) possesses two important properties, which are similar to the properties of Ricci flat gyraton geometries. First, all local scalar invariants constructed from the Riemann tensor and its covariant derivatives for the metric (22) are exactly the same as those for a pure AdS spacetime ($\Phi = A = 0$). Second, substitution of (22) in the Einstein equation with the negative cosmological constant results in equations

$$\partial_b F^b_a - \frac{D - 2}{z} F_{az} = 0, \quad \Delta \Phi - 2\partial_u(\partial_a A^a) - \frac{1}{2}F_{ab}F^{ab} - \frac{D - 2}{z}(\partial_z \Phi - 2\partial_a A_z) = 0.$$  

(23)

These equations hold outside the source which is assumed to be located at $z \neq 0$.

It is interesting to note that equations (23) can be rewritten as a problem in a fiducial $(D+2)$-dimensional Euclidean space which is similar to the problem (16). After that the solutions can be obtained by applying the method discussed in the previous section. It is assumed that $\Phi$ and $A$ vanish at spatial infinity, therefore the corresponding metrics are asymptotically AdS. Some explicit AdS gyraton solutions for $D = 4$ and $D = 5$ are presented in [5]. The existence of AdS gyraton solutions may represent an interest in relation to the AdS/CFT correspondence.

References

[1] Aichelburg PC and Sexl RU 1971 Gen. Rel. Grav. 2 303
[2] Tolman RC 1934 Relativity, Thermodynamics and Cosmology (Oxford: Clarendon Press)
[3] Frolov VP and Fursaev DV 2005 Phys. Rev. D71 104034
[4] Frolov VP, Israel W, and Zelnikov A 2005 Phys. Rev. D72 084031
[5] Frolov VP and Zelnikov A 2005 Phys. Rev. D72 104005
[6] Horowitz GT and Steif AR 1990 Phys. Rev. Lett. 64 260
[7] Amati D and Klimcik C 1989 Phys. Lett. B219 443
[8] Sadri D and Sheikh-Jabbari MM 2004 Rev. Mod. Phys. 76 853
[9] Kanti P 2004 Int. J. Mod. Phys. A19 4899
[10] Eardley DM and Giddings SB 2002 Phys. Rev. D66 044011