A general family of multi-peakon equations and their properties

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Abstract
A general family of peakon equations is introduced, involving two arbitrary functions of the wave amplitude and the wave gradient. This family contains all of the known breaking wave equations, including the integrable ones: Camassa–Holm equation, Degasperis–Procesi equation, Novikov equation, and FORQ/modified Camassa–Holm equation. One main result is to show that all of the equations in the general family possess weak solutions given by multi-peakons which are a linear superposition of peakons with time-dependent amplitudes and positions. In particular, neither an integrability structure nor a Hamiltonian structure is needed to derive N-peakon weak solutions for arbitrary $N > 1$. As a further result, single peakon travelling-wave solutions are shown to exist under a simple condition on one of the two arbitrary functions in the general family of equations, and when this condition fails, generalized single peakon solutions that have a time-dependent amplitude and a time-dependent speed are shown to exist. An interesting generalization of the Camassa–Holm and FORQ/modified Camassa–Holm equations is obtained by deriving the most general subfamily of peakon equations that possess the Hamiltonian structure shared by the Camassa–Holm and FORQ/modified Camassa–Holm equations. Peakon travelling-wave solutions and their features, including a variational formulation (minimizer problem), are derived for these generalized equations. A final main result is that two-peakon weak solutions are investigated and shown to exhibit several novel kinds of behaviour, including the formation of a bound pair consisting of a peakon and an anti-peakon that have a maximum finite separation.

Keywords: peakon, multi-peakon, nonlinear dispersive wave equation
1. Introduction

The study of dispersive nonlinear partial differential equations (PDEs) of the form

\[ u_t - u_{txx} = F(u, u_x, u_{xx}) + G(u, u_x, u_{xx})u_{xxx} \] (1.1)

for \( u(t,x) \) has attracted growing attention in the last two decades, because particular equations in this family describe breaking waves and some of these equations are integrable systems.

One of the first well-studied equations is the Camassa–Holm (CH) equation

\[ u_t - u_{xx} = -3u u_x + 2u_x u_{xx} + u_{xxx} \] (1.2)

which was shown in 1993 to arise \([1, 2]\) from the theory of shallow water waves by an asymptotic expansion of Euler’s equations for inviscid fluid flow. The CH equation provides a model of wave breaking for a large class of solutions in which the wave slope \( u_x \) blows up in a finite time while the wave amplitude \( u \) remains bounded \([3–6]\). Solitary travelling wave solutions of the CH equation consist of \([7]\) peaked waves \( u = a \exp(-|x - ct|) \), with the amplitude-speed relation \( a = c \), where the wave slope \( u_x \) is discontinuous at the wave peak. These solutions are known as peakons \([1, 8, 9]\), and interactions of an arbitrary number of peakons are described by multi-peakon solutions \([10, 11]\) which have the form of a linear superposition of peaked waves with time-dependent amplitudes and speeds. Peakons are not classical solutions but instead are weak solutions \([12–14]\) satisfying an integral formulation of the CH equation. More remarkably, the CH equation is an integrable system in the sense that it possesses \([1, 15, 16]\) a Lax pair, a bi-Hamiltonian structure, and an infinite hierarchy of symmetries and conservation laws.

The CH equation was the only known peakon equation until 2002, when the Degasperis–Procesi (DP) equation

\[ u_t - u_{xx} = -4uu_x + 3u_x u_{xx} + uu_{xxx} \] (1.3)

was derived by the method of asymptotic integrability \([17, 18]\). This equation arises from the asymptotic theory of shallow water waves \([19]\) and possesses peakon and multi-peakon solutions \([18, 20–22]\), as well as shock-type solutions \([23]\). It is also an integrable system, possessing \([18]\) a Lax pair, a bi-Hamiltonian structure, and an infinite hierarchy of symmetries and conservation laws.

In 1995, the Fokas–Fuchssteiner–Olver–Rosenau–Qiao (FORQ) equation

\[ u_t - u_{xxx} = (-u^3 + uu_x^2 + u_x^2 u_{xx} - u_x^3 u_{xxx})_x, \] (1.4)

was derived by applying a bi-Hamiltonian splitting method \([24–26]\) to the modified Korteweg–de Vries (mKdV) equation. At the same time, this equation also appeared in other work \([27]\) on applications of Hamiltonian methods and was shown to arise from the asymptotic theory of surface water waves \([28]\). The FORQ equation \((1.4)\) is an integrable system. Its bi-Hamiltonian structure was obtained in \([25, 29]\), and its Lax pair along with its single peakon solutions appears in \([30, 31]\). Multi-peakon solutions have been considered in 2013 in \([32]\).

Similarly to the relationship between the mKdV equation and the ordinary KdV equation, the FORQ equation and the CH equation share one of their two Hamiltonian structures in common \([32]\) and are related by a combined Miura–Liouville transformation \([33]\). For these reasons and other similarities, the FORQ equation is also called the modified Camassa–Holm (mCH) equation in recent literature.

Following those discoveries of integrable peakon equations, a major direction of work was to find other integrable equations in the family \((1.1)\). In 2009, a classification of integrable
polynomial generalizations of the CH equation with quadratic and cubic nonlinearities was obtained [34], which produced several new integrable equations. The most interesting of these is the Novikov (N) equation

$$u_t - u_{txx} = -4u^2u_x + 3uu_xu_{xx} + u^3u_{xxx}. \quad (1.5)$$

Its peakon and multi-peakon solutions have been derived recently [35, 36].

In addition to the integrable equations in the family (1.1), there are many non-integrable equations that admit peakons and multi-peakons. Two examples are the $b$-family [18, 37]

$$u_t - u_{txx} = -(b + 1)uu_x + bu_xu_{xx} + uu_{xxx}, \quad b \neq 0 \quad (1.6)$$

which includes both the CH equation (1.2) ($b = 2$) and the DP equation (1.3) ($b = 3$), and the modified $b$-family [38]

$$u_t - u_{txx} = -(b + 1)u^2u_x + bu_xu_{xxx} + u^2u_{xxx}, \quad b \neq 0 \quad (1.7)$$

which includes the N equation (1.5) ($b = 3$). All of the equations in the $b$-family with $1 < b \leq 3$ exhibit wave breaking [39] similarly to the CH equation.

Very recently, two larger unified families of equations have been investigated. The first family [40–42]

$$u_t - u_{txx} = -(b + 1)u^pu_x + bu^{p-1}u_xu_{xx} + u^pu_{xxx}, \quad b \neq 0, p \neq 0 \quad (1.8)$$

contains the CH equation (1.2) ($b = 2, p = 1$), the DP equation (1.3) ($b = 3, p = 1$), and the N equation (1.5) ($b = 3, p = 2$), but not the mCH equation (1.4). The second family is given by [43]

$$u_t - u_{txx} = -(b + 1)u^pu_x + (9a + b + 2c - 3p)u^{p-2}u_x^3 + (3p - 2c)u^{p-1}u_xu_{xx} - 6au^{p-2}u_x^2u_{xx} + (u^p - 3au^{p-2}u_x^2)u_{xxx}, \quad p \neq 0 \quad (1.9)$$

which includes the first family (1.8) when $a = 0$ and $b + 2c = 3p$, and which also includes the mCH equation (1.4) when $a = 1/3, c = 1$, and $p = 2$. Every equation in this larger unified family (1.9) possesses single peakon solutions, but multi-peakon solutions are admitted only when the nonlinearity power is related to the coefficients by $6a + b + 2c = 3p$ [43]. This condition includes all of the equations in the first unified family (1.8), whose multi-peakon solutions were derived independently in [40, 41].

No equations in either of these families (1.8) and (1.9) apart from the CH, DP, N, and mCH equations are believed to be integrable.

Another family of equations generalizing the CH equation (1.2) is given by

$$u_t - u_{txx} - \frac{1}{2} \left( pu^{p-1}(u^2 - u_x^2) + u^p(u - u_{xx}) \right)_x = 0, \quad p \neq 0. \quad (1.10)$$

This peakon equation was derived [44] by generalizing one of the Hamiltonian structures of the CH equation, in a search for analogs of the family of generalized Korteweg–de Vries (gKdV) equations [27]

$$u_t - u^pu_x - u_{xx} = 0, \quad p \neq 0. \quad (1.11)$$

The gKdV equation reduces to the ordinary KdV equation when $p = 1$ and shares one of its two Hamiltonian structures; the same relationship holds between the CH equation (1.2) and the generalized equation (1.10). However, while this generalized CH equation possesses single peakon solutions [44], it does not admit multi-peakon solutions, as will be seen from the results below.
In the present paper, we are motivated to look for the most general nonlinear dispersive wave equations that belong to the family (1.1) and possess single peakon and multi-peakon solutions. Our starting point is the observation that all of the known multi-peakon equations share a similar general form when they are expressed as evolution equations in terms of the momentum variable

$$ m = u - u_{xx} $$

which plays an important role in wave breaking phenomena. Specifically, the family of equations

$$ m_t + f(u, u_x)m + (g(u, u_x))_x = 0 $$

contains the CH equation (1.2), DP equation (1.3), mCH equation (1.4), Novikov equation (1.5), and their different unifications (1.8) and (1.9) that admit multi-peakon solutions, as shown in table 1.

We will call equation (1.13) the \textit{fg-family} and study its properties when \( f(u, u_x) \) and \( g(u, u_x) \) are arbitrary non-singular functions. As a main result, in section 2, \( N \)-peakon solutions for all \( N \geq 2 \) will be derived as weak solutions for the entire \textit{fg-family}, and single peakon travelling-wave solutions will be shown to exist under a simple condition on \( f(u, u_x) \). Interestingly, when this condition fails, more general single peakon solutions that have a time-dependent amplitude and a time-dependent speed are shown to exist.

The importance of these results is that they show multi-peaks exist for nonlinear dispersive wave equations without using or relying on any integrability properties or any Hamiltonian structure. This is a sharp contrast to the situation for multi-soliton solutions of evolution equations, where the existence of an arbitrary number \( N > 2 \) of solitons usually requires that the evolution equation be integrable, and finding the \( N \)-soliton solution usually involves explicit use of the integrability structure (e.g. an inverse scattering transform or a bilinear formulation).

In section 3, the most general form of \( f(u, u_x) \) and \( g(u, u_x) \) is obtained such that the multi-peakon equation (1.13) admits the Hamiltonian structure shared by the CH and mCH equations. This yields a large family of Hamiltonian multi-peakon equations

$$ m_t + u_x f_1(u^2 - u_x^2) + g_1(u^2 - u_x^2)m_x = 0 $$

which involves two arbitrary functions \( f_1 \) and \( g_1 \) of \( u^2 - u_x^2 \). Several properties of this family are studied. First, this family is shown to have conserved momentum, energy, and \( H^1 \) norm for all classical solutions \( u(t, x) \), and conservation laws for weak solutions are discussed.

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**Table 1.** Peakon equations.

| \( f(u, u_x) \) | \( g(u, u_x) \) | Name |
|-----------------|-----------------|------|
| \( u \)         | \( u \)         | Camassa–Holm |
| \( 2u \)        | \( u \)         | Degasperis–Procesi |
| \( uu \)        | \( u^2 \)       | Novikov |
| 0               | \( u^2 - u_x^2 \) | Fokas–Olver–Rosenau–Qiao/modified Camassa–Holm |
| \((b - 1)u \)   | \( u \)         | \( b \)-family |
| \((b - 2)uu \)  | \( u^2 \)       | Modified \( b \)-family |
| \((b - p)u^{p-1}u \) | \( u^p \)       | Unified CH–DP–N family |
| \( u^{p-3}u_x((b - p)u^2) \) | \( u^{p-2}(u^2 - 3 uu_x^2) \) | CH–DP–N–mCH unified family |
| +3a(p - 2)u_x^2 |                 |       |

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In the present paper, we are motivated to look for the most general nonlinear dispersive wave equations that belong to the family (1.1) and possess single peakon and multi-peakon solutions. Our starting point is the observation that all of the known multi-peakon equations share a similar general form when they are expressed as evolution equations in terms of the momentum variable

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The importance of these results is that they show multi-peaks exist for nonlinear dispersive wave equations without using or relying on any integrability properties or any Hamiltonian structure. This is a sharp contrast to the situation for multi-soliton solutions of evolution equations, where the existence of an arbitrary number \( N > 2 \) of solitons usually requires that the evolution equation be integrable, and finding the \( N \)-soliton solution usually involves explicit use of the integrability structure (e.g. an inverse scattering transform or a bilinear formulation).

In section 3, the most general form of \( f(u, u_x) \) and \( g(u, u_x) \) is obtained such that the multi-peakon equation (1.13) admits the Hamiltonian structure shared by the CH and mCH equations. This yields a large family of Hamiltonian multi-peakon equations

$$ m_t + u_x f_1(u^2 - u_x^2) + g_1(u^2 - u_x^2)m_x = 0 $$

which involves two arbitrary functions \( f_1 \) and \( g_1 \) of \( u^2 - u_x^2 \). Several properties of this family are studied. First, this family is shown to have conserved momentum, energy, and \( H^1 \) norm for all classical solutions \( u(t, x) \), and conservation laws for weak solutions are discussed.
Second, all single peakon solutions \( u = a \exp(-|x - ct|) \) are derived and shown to have a speed-amplitude relation \( c = c(a) \) which is nonlinear whenever \( f_1 \) or \( g_1 \) are non-constant functions. This leads to an interesting relation between the properties of peakons with \( a > 0 \) and anti-peakons with \( a < 0 \). Third, a minimizer principle is obtained for these peakon solutions, which provides a starting point for establishing their stability.

In section 4, one-parameter subfamilies of the CH-mCH Hamiltonian family are explored, with \( f_1 \) and \( g_1 \) taken to be general powers of \( u^2 - u_x^2 \). These subfamilies represent nonlinear generalizations of the CH equation and mCH equation given by

\[
m_t + (u^2 - u_x^2)^{p-1}u_x m + (u(u^2 - u_x^2)^{p-1}m)_x = 0, \quad p \geq 1 \tag{1.15}
\]

and

\[
m_t + ((u^2 - u_x^2)^p m)_x = 0, \quad p \geq 1. \tag{1.16}
\]

A unified generalization of these subfamilies

\[
m_t + au_x(u^2 - u_x^2)^k/2 m + (au(u^2 - u_x^2)^k/2 m + b(u^2 - u_x^2)^{(k+1)/2} m)_x = 0, \quad k \geq 0 \tag{1.17}
\]

is also discussed. Each of these multi-peakon equations share the common Hamiltonian structure admitted by the CH and mCH equations, and consequently they describe an analog of the well-known Hamiltonian family of generalized KdV equations.

The effect of higher-power nonlinearities on interactions of peakons is explored in section 5 by studying the behaviour of two-peakon weak solutions for the generalized CH and mCH equations (1.15) and (1.16) when \( p = 2 \), which is compared to the behaviour in the ordinary case \( p = 1 \). As main results, for both of these generalized peakon equations, qualitatively new behaviours are shown to occur in the interaction of two peakons whose amplitudes have opposite signs, namely, an ordinary peakon and an anti-peakon.

In the case of the \( p = 2 \) generalized mCH equation (1.16), the peakon and anti-peakon can form a bound pair which has a maximum finite separation in the asymptotic past and future. The pair evolves by slowly collapsing, such that a collision occurs in a finite time, followed by asymptotically expanding, with the amplitudes being finite for all time.

In the case of the \( p = 2 \) generalized CH equation (1.15), the peakon and anti-peakon can exhibit a finite time blow-up in amplitude, before and after they undergo a collision. Starting at the collision, their separation increases to a finite maximum and then decreases to a limiting non-zero value when the blow-up occurs.

Neither of these types of behaviour have been seen previously in interactions of peakon weak solutions. This indicates that peakons can have a wide variety of interesting interactions for different multi-peakon equations in the general \( fg \)-family (1.13), and that the form of the nonlinearity in these equations has a large impact on how peakons can interact.

Some concluding remarks are made in section 6.

2. Peakon solutions

In the analysis of all of the peakon equations listed in table 1, single and multi peakons are commonly derived as weak solutions [12–14, 32, 39, 45] in the setting of an integral formulation. We will now show that the \( fg \)-family (1.13) has a weak formulation for general functions \( f(u, u_x) \) and \( g(u, u_x) \), subject to mild conditions, and then we will use this formulation to obtain single peakon and multi-peakon solutions.
2.1. Weak formulation

A weak solution \( u(t,x) \) of equation (1.13) is a distribution that satisfies an integral formulation of the equation in some suitable function space. We will derive this formulation by the usual steps for wave equations [46]. First, we multiply equation (1.13) by a test function \( \psi(t,x) \) (which is smooth and has compact support), and integrate over \( -\infty < x < \infty \) and \( 0 \leq t < \infty \). Next, we integrate by parts to remove all derivatives of \( u \) higher than first-order. There are three different terms to consider. The first one is given by

\[
\int_0^\infty \int_{-\infty}^\infty \psi(ut - utxx) \, dx \, dt = \int_0^\infty \int_{-\infty}^\infty (\psi - \psi_{xx}) u_t \, dx \, dt.
\]

(2.1)

The next one consists of

\[
\int_0^\infty \int_{-\infty}^\infty \psi(u - uxx) f(u, u_x) \, dx \, dt = \int_0^\infty \int_{-\infty}^\infty \psi(uf - D_x F + u_x F_u) \, dx \, dt
\]

\[
= \int_0^\infty \int_{-\infty}^\infty (\psi(uf + u_x F_u) + \psi_x F) \, dx \, dt,
\]

(2.2)

where

\[
F(u, u_x) = \int f(u, u_x) \, du_x.
\]

(2.3)

The final one is, similarly,

\[
\int_0^\infty \int_{-\infty}^\infty \psi((u - uxx)g(u, u_x)) \, dx \, dt = -\int_0^\infty \int_{-\infty}^\infty \psi_x(u - uxx)g \, dx \, dt
\]

\[
= -\int_0^\infty \int_{-\infty}^\infty (\psi_x ug + u_x G_u + \psi_{xx} G) \, dx \, dt,
\]

(2.4)

where

\[
G(u, u_x) = \int g(u, u_x) \, du_x.
\]

(2.5)

Then, combining these three terms (2.1), (2.2) and (2.4), we obtain the integral (weak) equation

\[
0 = \int_0^\infty \int_{-\infty}^\infty \left( \psi(ut + fu + F_u u_x) - \psi_x(gu + G_u u_x - F) - \psi_{xx}(G + u_i) \right) \, dx \, dt.
\]

(2.6)

An equivalent weak formulation can be obtained by putting \( \phi = \psi - \psi_{xx} \) in the integral (weak) equation (2.6). Note that \( \psi = \kappa(x) \ast \phi \) is given by a convolution integral using the kernel

\[
\kappa(x) = \frac{1}{2} \exp(-|x|), \quad (1 - \partial_x^2)^{-1} = \kappa(x)\ast
\]

(2.7)

for the inverse of the operator \( 1 - \partial_x^2 \). Then we have

\[
0 = \int_0^\infty \int_{-\infty}^\infty \left( \phi(ut + G + \kappa \ast (fu + F_u u_x - G)) + \kappa_x \ast (gu + G_u u_x - F) \right) \, dx \, dt
\]

(2.8)

where \( \phi(t,x) \) is a test function.
The weak equations (2.6) and (2.8) will hold for all distributions $u(t,x)$ in a suitable function space whenever $f(u, u_x)$ and $g(u, u_x)$ satisfy some mild regularity/growth conditions. In particular, the following result is straightforward to prove.

**Proposition 2.1.** If both $|f(u, u_x)| \leq C_1 |u|^k + C_2 |u_x|^k$ and $|g(u, u_x)| \leq C_1 |u|^k + C_2 |u_x|^k$ hold for some positive integer $k$, then the $fg$-equation (1.13) has a weak formulation (2.6), or equivalently (2.8), for $u(t, x) \in C^1(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R})) \cap C^0(\mathbb{R}^+; W^{1,k+1}_{\text{loc}}(\mathbb{R}))$.

The weak equation (2.6) will be necessary for the derivation of multi-peakon solutions of the $fg$-equation (1.13), but single-peakon solutions can be derived more simply by starting from a reduction of the $fg$-equation to a travelling wave ODE.

### 2.2. Single peakon solutions

A single peakon is a travelling wave of the form

$$u(t, x) = a \exp(-|x - ct|)$$

(2.9)

where the wave speed $c$ is an arbitrary constant while the peak amplitude $a$ is related to $c$ in a specific way that depends on the nonlinearity in the $fg$-equation (1.13). As the wave slope is discontinuous at the wave peak, a peakon is a weak solution rather than a classical solution of the $fg$-equation.

Single peakon solutions can also be derived from the reduction of the $fg$-equation to a travelling-wave ODE, arising through a combined time and space translation symmetry that leaves $x - ct$ invariant. Specifically, the travelling wave ansatz

$$u = U(\xi), \quad \xi = x - ct, \quad c = \text{const.} \neq 0$$

(2.10)

reduces the $fg$-equation (1.13) to the ODE

$$(U - U'')f(U, U') + ((U - U'')(g(U, U') - c))' = 0.$$  

(2.11)

A peakon solution will not be a classical solution of this ODE but instead will be a weak solution. To obtain the weak formulation of this ODE (2.11), corresponding to the weak formulation (2.6) of the $fg$-equation, we multiply the ODE by a test function $\psi$ (which is smooth and has compact support), integrate over $-\infty < \xi < \infty$, and use integration by parts to leave at most first-order derivatives of $U$ in the integral. This yields

$$0 = \int_{-\infty}^{\infty} (\psi(U' + U'(FU - c)) + \psi'(F - U - U'G) + \psi''(cU' - G)) \, d\xi$$

(2.12)

where, now,

$$F(U, U') = \int f(U, U') \, dU', \quad G(U, U') = \int g(U, U') \, dU'.$$

(2.13)

Just as in proposition 2.1, only mild conditions on the functions $f$ and $g$ are needed for this weak equation (2.12) to hold for all $U(\xi)$ in a suitable function space.

To find the single peakon solutions, we substitute a general peakon expression

$$U = a \exp(-|\xi|), \quad a = \text{const.}$$

(2.14)

into the weak equation (2.12), and we split up the integral into the intervals $(-\infty, 0)$ and $(0, \infty)$. Integrating by parts, and using the relations
\[ U' = \begin{cases} U, & \xi < 0 \\ -U, & 0 < \xi \end{cases} \] (2.15)

\[ U(0) = a, \quad U'(0^\pm) = \mp a \] (2.16)

which hold on each interval, we obtain the following result.

**Theorem 2.1.** The weak formulation (2.12) of the travelling wave ODE (2.11) possesses single peakon travelling-wave solutions (2.9) if and only if

\[ c = \frac{G(a, a) - G(a, -a)}{2a} \neq 0 \] (2.17)

and

\[ F(a, a) = F(a, -a) \] (2.18)

where \( a \) is the amplitude and \( c \) is the wave speed. These two conditions (2.18) and (2.17) hold if and if only the coefficient functions \( f \) and \( g \) in the \( fg \)-equation (1.13) have the respective forms

\[ f(u, u_x) = \int_u^{u_0} f_1(u, y) \, dy + u_f(u_x) + f_2(u, u_x), \]

\[ g(u, u_x) = g_0(u) + \int_u^{u_0} g_1(u, y) \, dy + u_g(u_x) + g_2(u, u_x), \] (2.19)

for some function \( g_0 \) of \( u \), and some functions \( f_1, f_2, g_1, g_2 \) of \( u \) and \( u_x \), such that

\[ g_0(u) \neq 0, \quad f_i(u, u_x) + f_i(u, -u_x) = 0, \quad g_i(u, u_x) + g_i(u, -u_x) = 0, \quad i = 1, 2. \] (2.20)

In particular, the speed-amplitude relation for single peakon travelling waves is given by

\[ c = g_0(a). \] (2.21)

**Proof.** It will be useful to introduce the notation

\[ h_+ = h(U, U')|_{\xi<0} = h(U, U), \quad h_- = h(U, U')|_{\xi>0} = h(U, -U) \] (2.22)

for any function \( h(U, U') \).

To proceed, we first rearrange the terms in the weak equation (2.12):

\[ 0 = \int_{-\infty}^{\infty} \left( c(\psi'' - \psi)U' + (\psi' F + \psi F_U U') - (\psi'' G + \psi' G_U U') + (\psi f - \psi' g)U \right) \, d\xi. \] (2.23)

Now we consider each term separately.

Integration by parts twice on the first term in equation (2.23), combined with the relations (2.15) and (2.16), yields
\[ \int_{-\infty}^{\infty} c(\psi'' - \psi) U' \, d\xi = \int_{-\infty}^{0} c(\psi'' - \psi) U \, d\xi - \int_{0}^{\infty} c(\psi'' - \psi) U \, d\xi \]
\[ = 2c\psi'(0)U(0) + c \left( -\int_{-\infty}^{0} (\psi' + \psi) U \, d\xi + \int_{0}^{\infty} (-\psi' + \psi) U \, d\xi \right) \]
\[ = 2ac\psi'(0). \quad (2.24) \]

Next, the second term in equation (2.23) gives
\[ \int_{-\infty}^{\infty} (\psi'F + \psi F_U U') \, d\xi = \int_{-\infty}^{0} (\psi'F_+ + \psi(F_U)_+ U) \, d\xi + \int_{0}^{\infty} (\psi'F_+ - \psi(F_U)_- U) \, d\xi \]
\[ = \psi(0)(F_+ - F_-)|_{\xi=0} - \int_{-\infty}^{0} \left( \psi \left( \frac{dF_+}{d\xi} + (F_U)_+ \right) - \int_{0}^{\infty} \psi \left( \frac{dF_+}{d\xi} - (F_U)_- \right) \right) \, d\xi \]
\[ = -\psi'(0)(G_+ - G_-)|_{\xi=0} - \int_{-\infty}^{\infty} \psi'/g \, d\xi. \quad (2.25) \]

The third term in equation (2.23) similarly yields
\[ -\int_{-\infty}^{\infty} (\psi''G + \psi'G_U U') \, d\xi = -\psi'(0)(G_+ - G_-)|_{\xi=0} + \int_{-\infty}^{\infty} \psi'/g \, d\xi. \quad (2.28) \]

Last, the fourth term in equation (2.23) combines with the previous terms (2.24), (2.27) and (2.28), giving
\[ (2ac - (G_+ - G_-)|_{\xi=0})\psi'(0) + (F_+ - F_-)|_{\xi=0}\psi(0) = 0. \quad (2.29) \]

This equation is satisfied for all test functions \( \psi \) if and only if \( 2ac - (G_+ - G_-)|_{\xi=0} = 0 \) and \( (F_+ - F_-)|_{\xi=0} = 0 \). Substituting the relations (2.16) into these two conditions, we obtain equations (2.17) and (2.18).

The form (2.19) for the functions \( f \) and \( g \) is obtained from the conditions (2.17) and (2.18) as follows. First, we decompose \( F \) and \( G \) into even and odd parts under reflection \( (u, u_i) \rightarrow (u, -u_i) \), by expressing \( F = u_f F_1 + F_2 \) and \( G = u_f G_1 + G_2 \) where \( F_1, F_2, G_1, G_2 \) are reflection-invariant functions of \( u, u_i \). Conditions (2.17) and (2.18) then become \( F_1(u, u_i) = 0 \) and \( G_1(u, a) \neq 0 \), which hold iff \( F_1(u, a) = 0 \) and \( G_1(u, u) \neq 0 \). Next, from relations (2.3) and (2.5), we have \( f = F_{u_f} = F_1 + u_f F_{1u} + F_{2u} \) and \( g = G_{u_f} = G_1 + u_f G_{1u} + G_{2u} \). We now put \( f_1 = F_{1u} \) and \( f_2 = F_{2u} \), and likewise \( g_1 = G_{1u} \) and \( g_2 = G_{2u} \), all of which are odd in \( u_i \). This yields \( F_1 = \int_{u_i}^{u} f_1 \, du_i \) and \( G_1 = \int_{u_i}^{u} g_1 \, du_i + g_0(u) \), using \( F_1(u, a) = 0 \) and \( G_1(u, u) \neq 0 \). Hence, \( f, g \) are given by expressions (2.19) and (2.20), which completes the proof. \( \square \)
Theorem 2.1 establishes that single peakon travelling-wave solutions (2.9) exist for a large class of equations in the $fg$-family (1.13). In particular, sufficient conditions are that $f(u, u_x)$ is odd in $u_x$, and that $g(u, u_x) + g(u, -u_x)$ is non-vanishing in $u$, whereby $F(u, u_x)$ is an even function of $u_x$, and $G(u, u_x) - G(u, -u_x) = g_0(u)$ is a non-vanishing function.

This theorem also shows that there is no restriction on the possible form of the speed-amplitude relation $c = g_0(a)$ for single peakon travelling-waves, since $g_0(u)$ can be an arbitrary function of $u$. If $g_0(u)$ is identically zero, then single peakon travelling-waves will be stationary, $c = 0$. Consequently, it is more natural to regard the peak amplitude $a$ as an arbitrary constant, with the speed then being given in terms of $a$ by the speed amplitude relation.

To distinguish the situations when $a$ is positive versus negative, the travelling wave solution (2.9) is typically called an ordinary peakon when $a$ is positive, and an anti-peakon when $a$ is negative. The speed of an anti-peakon compared to the corresponding ordinary peakon depends on the reflection symmetry properties of the function $g_0(a)$: specifically, $c_+ = g_0(a)$ with $a > 0$ will be equal to $c_- = g_0(-a)$ with $-a < 0$ if and only if $g_0(a)$ is an even function of $a$. This shows that peakons and anti-peakons with the same absolute amplitude $|a|$ will have different speeds, $c_+ \neq c_-$, when (and only when) $g_0(a)$ is not invariant under reflections.

Even more interestingly, when $f(u, u_x)$ and $g(u, u_x)$ fail to satisfy the necessary and sufficient conditions (2.17) and (2.18), a generalized peakon solution still exists but with both the speed and the amplitude being time-dependent, as we will show from the derivation of multi-peakon solutions.

### 2.3. Multi-peakon solutions

A multi-peakon solution is a linear superposition of peaked travelling waves given by

$$u(t, x) = \sum_{i=1}^{N} \alpha_i(t) \exp(-|x - \beta_i(t)|), \quad N \geq 2$$

with time-dependent amplitudes $\alpha_i(t)$ and positions $\beta_i(t)$.

We now derive $N$-peakon solutions for the $fg$-equation (1.13), for all $N \geq 1$, with $f(u, u_x)$ and $g(u, u_x)$ being arbitrary non-singular functions. For convenience, we use the notation

$$u = \sum_i \alpha_i e^{-|x - x_i|}, \quad x_i = x - \beta_i$$

(2.31)

where the summation is understood to go from 1 to $N$. The $x$-derivatives of $u$ are distributions given by

$$u_x = -\sum_i \alpha_i e^{-|x - x_i|} sgn(x_i)$$

(2.32)

and

$$u_{xx} = \sum_i \alpha_i (e^{-|x - x_i|} - 2\delta(x_i))$$

(2.33)

in terms of the sign function

$$sgn(x) = \pm 1, \quad \pm x > 0$$

(2.34)

and the Dirac delta distribution

$$\delta(x) = \frac{d}{dx} \left( \frac{1}{2} sgn(x) \right),$$

(2.35)
Similarly, the $t$-derivatives of $u$ are given by the distributions

$$u_t = \sum_i \left( \dot{\alpha}_i e^{-|x|} + \dot{\beta}_i \alpha_i e^{-|x|} \text{sgn}(x_i) \right)$$

(2.36)

and

$$u_{tx} = -\sum_i \left( \dot{\alpha}_i e^{-|x|} \text{sgn}(x_i) + \dot{\beta}_i \alpha_i \left( e^{-|x|} - 2\delta(x_i) \right) \right).$$

(2.37)

To begin, we substitute the general $N$-peakon expression (2.30) into the weak equation (2.6). There are two ways we can then proceed. One way which is commonly used is to assume $\beta_1 < \beta_2 < \cdots < \beta_N$ at a fixed $t > 0$, split up the integral over $x$ into corresponding intervals, and integrate by parts, similarly to the derivation of the single peakon solution. Another way, which is considerably simpler, is to employ the following result which can be easily proven from distribution theory [47].

Let $h(x)$ be a distribution whose singular support is a set of a finite number of points $x = x_i$ in $\mathbb{R}$, and define its non-singular part

$$\langle h(x) \rangle = \begin{cases} h(x), & x \neq x_i \\ \frac{1}{2}(h(x_i^+) + h(x_i^-)), & x = x_i \end{cases}$$

(2.38)

and its jump discontinuities

$$[h]_{x_i} = h(x_i^+) - h(x_i^-) = \left[ \langle h(x) \rangle \right]_{x_i}.$$  

(2.39)

**Lemma 2.1.** (i) If $h(x)$ is a piecewise smooth, bounded function, then $h'(x)$ is a distribution whose singular support is the set of points $x = \beta_i$ at which $h(x)$ is discontinuous, and $\langle h'(x) \rangle$ is a piecewise smooth, bounded function. The functions $\langle h'(x) \rangle$ and $h(x)$ are related by the integral identity

$$\int_{-\infty}^{\infty} \psi' h \, dx + \int_{-\infty}^{\infty} \psi \langle h' \rangle \, dx = -\sum_i \left( \psi \left[ h \right] \right)_{x = \beta_i}$$

(2.40)

holding for any test function $\psi(x)$. (ii) If $h_1(x)$ and $h_2(x)$ are piecewise smooth, bounded functions, then

$$\int_{-\infty}^{\infty} \psi \langle h_1 h_2 \rangle \, dx = \int_{-\infty}^{\infty} \psi h_1 h_2 \, dx, \quad \int_{-\infty}^{\infty} \psi \langle h_1 h_2' \rangle \, dx = \int_{-\infty}^{\infty} \psi h_1 \langle h_2' \rangle \, dx$$

(2.41)

are identities.

We will now use the identities (2.40) and (2.41) to combine and evaluate all of the terms in the weak equation (2.6).

We start with the term

$$-\int \int_{-\infty}^{\infty} \psi_{xx} u_t \, dx \, dt = -\int \int_{-\infty}^{\infty} \psi \left( \langle u_{xx} \rangle \right) \, dx \, dt$$

$$+ \sum_i \int_{-\infty}^{\infty} \left( \psi_{x} (t, \beta_i) [u]_{\beta_i} - \psi (t, \beta_i) \left[ \langle u_{xx} \rangle \right]_{\beta_i} \right) \, dt.$$  

(2.42)
By using expression (2.37), we see
\[ \langle \langle u_{t\alpha} \rangle \rangle = u_t, \] (2.43)
so then the term (2.42) can be combined with the other term in equation (2.6) involving \( u_t \). These two terms yield
\[ \int \int_{-\infty}^{\infty} (\psi u_t - \psi_{x\alpha} u_{t\alpha}) \, dx \, dt = \sum_i \int_{-\infty}^{\infty} \psi_s(t, \beta_i) \left[ u_t \right]_{\beta_i} - \psi(t, \beta_i) \left[ \langle u_{t\alpha} \rangle \right]_{\beta_i} \, dt. \] (2.44)

We next consider the term
\[ \int \int_{-\infty}^{\infty} \psi_x G \, dx \, dt = -\int \int_{-\infty}^{\infty} \psi_s \langle D_s G \rangle \, dx \, dt - \sum_i \int_{-\infty}^{\infty} \psi_s(t, \beta_i) \left[ G \right]_{\beta_i} \, dt. \] (2.45)
Expanding the total derivative \( D_s G \), we note
\[ \int \int_{-\infty}^{\infty} \psi_x \langle D_s G \rangle \, dx \, dt = \int \int_{-\infty}^{\infty} \psi_s \left[ (G_u) \langle u_s \rangle + (G_{u_s}) \langle u_{ss} \rangle \right] \, dx \, dt \]
\[ = \int \int_{-\infty}^{\infty} \psi_s (G_u u_s + gu) \, dx \, dt \] (2.46)
by using
\[ \langle u_{ss} \rangle = u \] (2.47)
which follows from expression (2.33). Thus, we have
\[ \int \int_{-\infty}^{\infty} \psi_x G \, dx \, dt = -\int \int_{-\infty}^{\infty} \psi_s (G_u u_s + gu) \, dx \, dt - \sum_i \int_{-\infty}^{\infty} \psi_s(t, \beta_i) \left[ G \right]_{\beta_i} \, dt \] (2.48)
which then can be combined with the similar term in equation (2.6), yielding
\[ -\int \int_{-\infty}^{\infty} (\psi_s (gu + G_s u_s) + \psi_{sx} G) \, dx \, dt = \sum_i \int_{-\infty}^{\infty} \psi_s(t, \beta_i) \left[ G \right]_{\beta_i} \, dt. \] (2.49)

Last, we consider the remaining terms in equation (2.6). These terms combine similarly to the previous two terms, which gives
\[ \int \int_{-\infty}^{\infty} (\psi f u + F_s u_{t\alpha} + \psi_{x\alpha} F) \, dx \, dt = -\sum_i \int_{-\infty}^{\infty} \psi_s(t, \beta_i) \left[ F \right]_{\beta_i} \, dt. \] (2.50)
Now, by combining all of the terms (2.44), (2.49) and (2.50), we obtain
\[ 0 = \sum_i \int_{-\infty}^{\infty} \psi_s(t, \beta_i) \left[ [u_t]_{\beta_i} + [G]_{\beta_i} \right] \, dt - \sum_i \int_{-\infty}^{\infty} \psi(t, \beta_i) \left[ [F]_{\beta_i} + [\langle u_{t\alpha} \rangle]_{\beta_i} \right] \, dt. \] (2.51)
This equation (2.51) will hold for all test functions \( \psi \) if and only if
\[ [u_t]_{\beta_i} + [G]_{\beta_i} = 0, \quad [\langle u_{t\alpha} \rangle]_{\beta_i} + [F]_{\beta_i} = 0. \] (2.52)
The jump terms involving \( t \)-derivatives of \( u \) are given by
\[ [u_t]_{\beta_i} = 2\alpha_i, \quad [\langle u_{t\alpha} \rangle]_{\beta_i} = -2\alpha_i \] (2.53)
which can be derived directly from expressions (2.36) and (2.37). For the other jump terms, we first note
\[ u(t, \beta_i^\pm) = u(t, \beta_i) = \sum_j \alpha_j e^{-|\beta_j - \beta_i|} = \alpha_i + \sum_{j \neq i} \alpha_j e^{-|\beta_j - \beta_i|}, \quad (2.54) \]

\[ u_s(t, \beta_i) = -\sum_j \text{sgn}(\beta_i - \beta_j) \alpha_j e^{-|\beta_i - \beta_j|} = -\sum_{j \neq i} \text{sgn}(\beta_i - \beta_j) \alpha_j e^{-|\beta_i - \beta_j|}, \quad (2.55) \]

\[ u_s(t, \beta_i^\pm) = \mp \alpha_i + u_s(t, \beta_i) = \mp \alpha_i - \sum_{j \neq i} \text{sgn}(\beta_i - \beta_j) \alpha_j e^{-|\beta_i - \beta_j|} \quad (2.56) \]

using expressions (2.31) and (2.32), where we now extend the definition (2.34) of the sign function by defining
\[ \text{sgn}(0) = 0. \quad (2.57) \]

Then we have
\[ [F]_{\beta_i} = F(u(t, \beta_i), -\alpha_i + u_s(t, \beta_i)) - F(u(t, \beta_i), \alpha_i + u_s(t, \beta_i)), \]
\[ [G]_{\beta_i} = G(u(t, \beta_i), -\alpha_i + u_s(t, \beta_i)) - G(u(t, \beta_i), \alpha_i + u_s(t, \beta_i)), \quad (2.58) \]

which can be expressed directly in terms of \( f \) and \( g \) through the integrals (2.3) and (2.5).

Consequently, after expressions (2.53), (2.54) and (2.58) are substituted into equation (2.52), we have established the following main result.

**Theorem 2.2.** Every equation (1.13) in the \( fg \)-family possesses \( N \)-peakon weak solutions (2.30) for arbitrary \( N \geq 1 \) in which the time-dependent amplitudes \( \alpha_i(t) \) and positions \( \beta_i(t) \) \( (i = 1, \ldots, N) \) satisfy the system of ODEs
\[ \dot{\alpha}_i = \frac{1}{2} (F(U_i, V_i - \alpha_i) - F(U_i, V_i + \alpha_i)), \quad (2.59) \]
\[ \dot{\beta}_i = \frac{1}{2} (G(U_i, V_i + \alpha_i) - G(U_i, V_i - \alpha_i))/\alpha_i, \quad (2.60) \]

with \( F \) and \( G \) given by expressions (2.3) and (2.5), and with
\[ U_i = \sum_{j=1}^N \alpha_j e^{-|\beta_j|} = u(t, \beta_i), \quad V_i = -\sum_{j=1}^N \text{sgn}(\beta_j) \alpha_j e^{-|\beta_j|} = u_s(t, \beta_i), \quad \beta_{ij} = \beta_i - \beta_j. \quad (2.61) \]

It is straightforward to show that the general ODE system (2.59)–(2.61) reproduces the multi-peakon dynamical systems found in the literature [1, 2, 18, 32, 35] using weak formulations for all of the peakon equations in table 1.

This theorem is very interesting because it shows that all equations in the \( fg \)-family (1.13) admit \( N \)-peaks for arbitrary \( N \geq 1 \). Consequently, the existence of multi-peaks for any given equation (1.13) does not rely on the equation explicitly exhibiting any integrability properties or any Hamiltonian structure. Indeed, according to the integrability classification in [34], there are many non-integrable quadratic and cubic peakon equations belonging to this family (1.13), and no Hamiltonian structure is known for these non-integrable equations other than the \( b \)-family equation (1.6).
As another consequence, all equations in $f_g$-family (1.13) admit a 1-peakon solution, possibly having a time-dependent amplitude and a time-dependent speed, given by the case $N = 1$.

**Corollary 2.1.** Any equation (1.13) in the $f_g$-family possesses a generalized peakon solution of the form

$$u(t, x) = \alpha(t) \exp(-|x - \beta(t)|)$$

(2.62)

whose amplitude $\alpha(t)$ and position $\beta(t)$ are given by the ODEs

$$\dot{\alpha} = \frac{1}{2} (F(\alpha, -\alpha) - F(\alpha, \alpha)), \quad \dot{\beta} = \frac{1}{2} (G(\alpha, \alpha) - G(\alpha, -\alpha))/\alpha,$$

(2.63)

where $F$ and $G$ are given by expressions (2.3) and (2.5). This solution reduces to a travelling wave

$$u(t, x) = a \exp(-|x - ct|)$$

(2.64)

with

$$\alpha = a = \text{const.}, \quad \beta = c = \text{const.} \neq 0$$

(2.65)

if and only if $f$ and $g$ have the forms (2.19) and (2.20).

Finally, the derivation of the $N$-peakon solutions indicates that if we try to allow the functions $f$ or $g$ in the $f_g$-family (1.13) to depend on second-order (or higher) derivatives of $u$ then this will unavoidably lead to the resulting weak equation having products of Dirac delta distributions. In such a situation, there would seem to be no sensible way to show that a $N$-peakon expression is a weak solution. Therefore, the $f_g$-family is arguably the most general family of nonlinear dispersive wave equations of the form (1.1) that can possess multi-peakon weak solutions (2.30).

### 3. Hamiltonian structure

Each of the known integrable peakon equations — CH (1.2), DP (1.3), mCH (1.4), N (1.5) — has a bi-Hamiltonian structure

$$m_t = \mathcal{H}(\delta H/\delta m) = \mathcal{E}(\delta E/\delta m)$$

(3.1)

where $\mathcal{H}$ and $\mathcal{E}$ are a pair of compatible Hamiltonian operators, and $H$ and $E$ are corresponding Hamiltonian functionals. This general structure can be expressed equivalently in terms of a pair of associated Poisson brackets

$$m_t = \{m, H\}_\mathcal{H} = \{m, E\}_\mathcal{E}$$

(3.2)

defined by

$$\{H_1, H_2\}_D = \int_{-\infty}^{\infty} (\delta H_1/\delta m) \mathcal{D}(\delta H_2/\delta m) \, dx$$

(3.3)

on the real line. In particular, an operator $\mathcal{D}$ is a Hamiltonian operator when and only when the bracket (3.3) is a Poisson bracket (namely, it is skew and satisfies the Jacobi identity).

As is well-known, it is interesting that both the CH equation (1.2) and the mCH equation (1.4) share one Hamiltonian structure:
\[ m_t = -u_t m - (um)_x = \mathcal{H}(\delta H_{\text{CH}}/\delta m) \]  \hspace{1cm} (3.4)

\[ m_t = -((u^2 - u_x^2)m)_x = \mathcal{H}(\delta H_{\text{mCH}}/\delta m) \]  \hspace{1cm} (3.5)

with

\[ \mathcal{H} = D_x^3 - D_x = -\Delta D_x = -D_x \Delta, \quad \Delta = 1 - D_x^2 \]  \hspace{1cm} (3.6)

and

\[ H_{\text{CH}} = \int_{-\infty}^{\infty} \frac{1}{2} u(u^2 + u_x^2) \, dx = \int_{-\infty}^{\infty} \frac{1}{2} (u^2 - u_x^2)m \, dx \]  \hspace{1cm} (3.7)

\[ H_{\text{mCH}} = \int_{-\infty}^{\infty} \left( \frac{1}{4} (u^2 + u_x^2)^2 - \frac{1}{3} u_x^4 \right) \, dx = \int_{-\infty}^{\infty} \frac{1}{4} u(u^2 - u_x^2)m \, dx. \]  \hspace{1cm} (3.8)

Note, in this Hamiltonian structure (3.4) and (3.5), \( u \) is viewed as a potential for \( m \) through the relation

\[ m = \Delta u. \]  \hspace{1cm} (3.9)

The variational derivatives with respect to \( m \) then can be formulated in terms of variational derivatives with respect to \( u \) by the identity

\[ \Delta \frac{\delta}{\delta m} = \frac{\delta}{\delta u}. \]  \hspace{1cm} (3.10)

This yields

\[ m_t = -u_t m - (um)_x = -D_x (\delta H_{\text{CH}}/\delta u) \]  \hspace{1cm} (3.11)

\[ m_t = -((u^2 - u_x^2)m)_x = -D_x (\delta H_{\text{mCH}}/\delta u) \]  \hspace{1cm} (3.12)

where \(-D_x\) is a Hamiltonian operator with respect to \( u \), which provides a useful alternative form for the common Hamiltonian structure of the CH and mCH equations.

Motivated by these Hamiltonian formulations (3.11) and (3.12), we will now seek the most general form for the functions \( f(u, u_x) \) and \( g(u, u_x) \) in the \( fg \)-family (1.13) so that

\[ m_t = -f(u, u_x)m - (g(u, u_x)m)_x = -D_x (\delta H/\delta u) = -\mathcal{H}(\delta H/\delta m) \]  \hspace{1cm} (3.13)

possesses a Hamiltonian structure using the Hamiltonian operator (3.6) common to the CH and mCH equations. There are two conditions for this structure (3.13) to exist, as determined by the necessary and sufficient relation \( f(u, u_x)m + (g(u, u_x)m)_x = D_x (\delta H/\delta u) \). The first condition is that \( fm = D_x A \) must hold for some differential function \( A \) of \( u \) and \( x \)-derivatives of \( u \). Then the second condition is that \( A + gm = \delta H/\delta u \) must hold for some functional \( H = \int_{-\infty}^{\infty} B \, dx \) where the density \( B \) is a differential function of \( u \) and \( x \)-derivatives of \( u \). We can formulate these two conditions as determining equations on \( f \) and \( g \) by using some tools from variational calculus [48, 49], in particular, the Euler operator and the Helmholtz operator. This leads to two overdetermined linear systems of equations, which we can straightforwardly solve to find \( f(u, u_x) \) and \( g(u, u_x) \) explicitly. Details are shown in the appendix.

We obtain

\[ f(u, u_x) = uf_1(u^2 - u_x^2) + fo \frac{u}{u^2 - u_x^2}. \]  \hspace{1cm} (3.14)
where \( f_1, g_1 \) are arbitrary functions of \( u^2 - u_x^2 \), and \( f_0 \) is an arbitrary constant. To avoid the occurrence of singularities when \( u^2 = u_x^2 \), we will hereafter put \( f_0 = 0 \) and take \( f_1, g_1 \) to be continuous functions. Then we have the following result.

**Proposition 3.1.** In the \( fg \)-family (1.13), all non-singular peakon equations that share the CH-mCH Hamiltonian structure (3.13) are given by

\[
\begin{align*}
 f(u, u_x) &= u_x f_1(u^2 - u_x^2), \\
 g(u, u_x) &= u_x f_1(u^2 - u_x^2) + g_1(u^2 - u_x^2)
\end{align*}
\]  

(3.16)

where \( f_1 \) is an arbitrary \( C^1 \) function of \( u^2 - u_x^2 \), and \( g_1 \) is an arbitrary \( C^1 \) function of \( u^2 - u_x^2 \). The Hamiltonian structure takes the form

\[
 m_x = -u_x f_1(u^2 - u_x^2)m - ((u_x f_1(u^2 - u_x^2) + g_1(u^2 - u_x^2))m)_x = -D_x \Delta (\delta H(u)/\delta m)
\]  

(3.17)

with

\[
 H(u) = \int_{-\infty}^{\infty} \frac{1}{2} (F_1(u^2 - u_x^2) + u \tilde{G}_1(u^2 - u_x^2)) m \, dx
\]  

(3.18)

where

\[
 F_1 = \int_0^{u^2 - u_x^2} f_1(y) \, dy = (u^2 - u_x^2) \int_0^1 f_1(\lambda(u^2 - u_x^2)) \, d\lambda,
\]

\[
 \tilde{G}_1 = (u^2 - u_x^2)^{-1} \int_0^{u^2 - u_x^2} g_1(y) \, dy = \int_0^1 g_1(\lambda(u^2 - u_x^2)) \, d\lambda.
\]  

(3.19)

This Hamiltonian structure can be reformulated as a variational formulation. We write \( u = \psi_x \) and use the variational relations \( D_x E_u = -E_v \) and \( m_t = -\frac{1}{2} E_{uv}(u_x \psi_x + \psi \psi_x) \) where \( E \) denotes the Euler operator (see the appendix), then we obtain

\[
 0 = -\frac{\delta S(v)}{\delta \psi} = m_t + u_x f_1(u^2 - u_x^2)m + ((u_x f_1(u^2 - u_x^2) + g_1(u^2 - u_x^2))m)_x
\]  

(3.20)

given by the action principle

\[
 S(v) = \int_0^\infty \int_{-\infty}^\infty L \, dx \, dt, \quad L = \frac{1}{2} (u_x \psi_x + \psi \psi_t + (F_1(u^2 - u_x^2) + u \tilde{G}_1(u^2 - u_x^2))m).
\]  

(3.21)

The Hamiltonian family of multi-peakon equations (3.16)–(3.19) reduces to the CH equation (3.11) when \( f_1 = 1, g_1 = 0 \), and also reduces to the mCH equation (3.12) when \( f_1 = 0, g_1 = u^2 - u_x^2 \). Thus, this family both unifies and generalizes these two peakon equations. We will call the subfamily with \( g_1 = 0 \) the CH-type family, and likewise the subfamily with \( f_1 = 0 \) will be called the mCH-type family.

We will now discuss some of the general family’s interesting features: conservation laws for strong and weak solutions; single peakon and anti-peakon solutions; and a minimizer principle.
3.1. Conservation laws

Conservation laws are important for analysis of the Cauchy problem as well as for the study of stability of peakon solutions. For the family of Hamiltonian multi-peakon equations

\[ m_t + u f_1(u^2 - u_1^2)m + ((u f_1(u^2 - u_2^2) + g_1(u^2 - u_1^2))m)_x = 0, \]

we start by considering smooth solutions \( u(t,x) \) on the real line.

The Hamiltonian (3.18) of this family will yield a conserved energy under appropriate asymptotic decay conditions on \( u(t,x) \). In particular, the local energy conservation law is given by the continuity equation

\[ D_t E + D_x \Psi_E = 0 \]

(3.23)

where

\[ E = \frac{1}{2}(F_1(u^2 - u_1^2) + u \tilde{G}_1(u^2 - u_1^2))m \]

(3.24)

is the energy (Hamiltonian) density, and where

\[ \Psi_E = \frac{1}{2}(u_{s}(F_1 + u \tilde{G}_1) - u_{s}u_{s}\tilde{G}_1 + K^2 - K_1^2) \]

(3.25)

is the energy flux, in terms of

\[ K = \Delta^{-1}E_u(E) = \kappa * E_u(E). \]

(3.26)

The flux expression (3.25) is derived by first using the variational identity (see (A.6))

\[ D_t E = u_t E_u(E) + D_x(u_t(E_{u} E_{u} - D_x E_{u} E_{u}) + u_{s}E_{u_{s}}) \]

(3.27)

with

\[ E_u(E) = \frac{1}{2}F_1 + (u f_1 + g_1)m, \]

(3.28)

\[ E_{u_{s}} = -u_{s}(F'_1 + u \tilde{G}'_1)m, \quad E_{u_{s}} = -\frac{1}{2}(F_1 + u \tilde{G}_1), \]

(3.29)

and then applying integration by parts to the term \( u_t(\delta E/\delta u) \) after substituting the Hamiltonian structure (3.13) expressed in the form \( u_t = -\Delta^{-1}(D_x E_u(E)) \).

From integration of the energy conservation law over \(-\infty < x < \infty\), we see that the total energy

\[ H(u) = \int_{-\infty}^{\infty} E \, dx \]

will be conserved

\[ \frac{dH(u)}{dt} = -\Psi_E \bigg|_{-\infty}^{\infty} \]

(3.30)

for smooth solutions \( u(t,x) \) that have vanishing asymptotic flux, \( \Psi_E \to 0 \) as \( |x| \to \infty \).

The Hamiltonian structure (3.13) itself has the form of a local conservation law

\[ D_t m + D_x \Psi_M = 0 \]

(3.31)

for the momentum \( m \), where the momentum flux is given by

\[ \Psi_M = E_u(E) \]

(3.32)

in terms of the energy density (3.24). Consequently, the total momentum \( M(u) = \int_{-\infty}^{\infty} m \, dx \) will be conserved.
for smooth solutions \( u(t, x) \) that have vanishing asymptotic flux, \( \Psi^M \to 0 \) as \( |x| \to \infty \).

Finally, we now show that the Hamiltonian structure also ensures conservation of the \( H^1 \) norm of solutions \( u(t, x) \) with sufficient asymptotic decay.

The time derivative of the \( H^1 \) density \( u^2 + u_x^2 \) is given by \( D_t(u^2 + u_x^2) = 2u u_t + D_x(2u u_x) = 2D_t(u u_x - u E_u(E)) + 2u E_u(E) \) by using the Hamiltonian equation (3.13). The term \( 2u E_u(E) \) can be expressed as a total \( x \)-derivative

\[
\frac{dM(u)}{dt} = -\Psi^M |_{-\infty}^{\infty} \tag{3.33}
\]

A strong solution will be a function \( u \). Classical solutions of this equation (3.38) on the real line require \( u(t, x) \), vanishing asymptotic flux, \( \Psi \to 0 \) as \( |x| \to \infty \).

A more relevant setting for analysis is given by reformulating the family of Hamiltonian multi-peakon equation (3.22) in the convolution form

\[
\tilde{M}(u) = \int_{-\infty}^{\infty} u \, dx, \quad \frac{d}{dt} \tilde{M}(u) = -\tilde{\Psi}^M |_{-\infty}^{\infty}, \quad \tilde{\Psi}^M = K \tag{3.40}
\]

and

\[
du E_u(E) = D_x(u F_1 + (u^2 - u_x^2) G_1) \tag{3.34}
\]
\[ H(u) = \int_{-\infty}^{\infty} \frac{1}{2} \left( uF_1 + (u^2 - u_x^2) \hat{G}_1 + 2u_x(uF_1 + \hat{G}_1) \right) \, dx, \]  
\[ \frac{d}{dt} H(u) = -\psi E \bigg|_{-\infty}^{\infty}, \quad \psi = \frac{1}{2} (K^2 - K_x^2) + (uF_1 + \hat{G}_1)K_x, \]

with

\[ \hat{F}_1 = \int_0^\infty f_1(u^2 - y^2) \, dy, \quad \hat{G}_1 = \int_0^\infty g_1(u^2 - y^2) \, dy. \]  

Similarly, the conservation law (3.37) for the \( H^1 \) norm becomes

\[ \frac{d}{dt} |u|_{H^1}^2 = -\psi \bigg|_{-\infty}^{\infty}, \quad \psi = uF_1 + (u^2 - u_x^2) \hat{G}_1 + u\psi_u. \]  

In all of these conservation laws, the fluxes will vanish since \( u \to 0 \) and \( u_x \to 0 \) for strong solutions as \( |x| \to \infty \), while \( \kappa \to 0 \) as \( |x| \to \infty \). This establishes the following result.

**Proposition 3.2.** In the Hamiltonian family (3.22) of multi-peakon equations, strong solutions (3.39) have conserved energy (3.41), momentum (3.40), and \( H^1 \) norm (3.43).

For solutions with less regularity, such as weak solutions, conservation of the energy and the \( H^1 \) norm will not hold in general. Specifically, weak solutions of the Hamiltonian family satisfy the integral equation

\[ 0 = \int_0^\infty \int_{-\infty}^{\infty} \phi(ut + \kappa_x \ast E_u(E)) \, dx \, dt \]  
(see proposition 2.1) for all test functions \( \phi(t, x) \). Any conservation law holding for all weak solutions must arise directly from this integral equation, by selecting a specific \( \phi(t, x) \) with the right properties. It is clear that the global conservation law for the momentum (3.40) can be obtained by choosing \( \phi = 1 \) in a domain \([0, \tau) \times (-L, L)\) with \( \phi \to 0 \) rapidly outside this domain, and then taking \( L \to \infty \). In contrast, the global conservation laws for the energy (3.41) and the \( H^1 \) norm (3.43) clearly cannot be extracted in this way.

Finally, we remark that this weak formulation (3.44) of the Hamiltonian family of multi-peakon equation (3.22) can be expressed as a variational principle. By first putting \( \phi = \Delta \psi \) and \( u = v_x \), we formally have \( \int_0^\infty \int_{-\infty}^{\infty} \psi(ut + \kappa_x \ast E_u(E)) \, dx \, dt = \int_0^\infty \int_{-\infty}^{\infty} \psi E_v(\tilde{L}) \, dx \, dt \), where \( \tilde{L} = \frac{1}{2} (u_t u_x + v_x v_{xx}) + E \), with

\[ E = \frac{1}{2} (uF_1 + (u^2 - u_x^2) \hat{G}_1 + 2u_x(uF_1 + \hat{G}_1)) \]  

given by the density of the energy integral (3.41). Next we use the variational relation \( \int_0^\infty \int_{-\infty}^{\infty} \psi E_v(\tilde{L}) \, dx \, dt = \int_0^\infty \int_{-\infty}^{\infty} \delta \psi \tilde{L} \, dx \, dt \) obtained via integration by parts, where

\[ \delta \psi \tilde{L} = \psi \partial_t \tilde{L} + \psi_x \partial_{xx} \tilde{L} + \psi_{xx} \partial_{x} \tilde{L} + \psi_3 \partial_3 \tilde{L} + \psi_4 \partial_4 \tilde{L} \]  

is the Fréchet derivative of \( \tilde{L} \). Thus, the weak equation (3.44) is equivalent to the variational principle

\[ 0 = \delta \psi \tilde{S}(\psi), \quad \tilde{S}(\psi) = \int_0^\infty \int_{-\infty}^{\infty} \tilde{L} \, dx \, dt \]  

with \( \tilde{S}(\psi) \) being a modified action principle (analogous to the modified energy integral \( H(u) \)).
This equation (3.47) gives a weak variational principle whose solutions $v(t,x)$ are potentials for weak solutions $u(t,x)$ of the multi-peakon equation (3.22).

### 3.2. Single peakons

The general Hamiltonian family (3.22) with $f_1$ and $g_1$ being arbitrary (non-singular) functions of $u^2 - u_x^2$ possesses single peakon travelling-waves (2.9), as seen by applying the two existence conditions in theorem 2.1.

Specifically, we have $F = \int_0^u uf_1(u^2 - y^2) \, dy = -\frac{1}{2} F_1(u^2 - u_x^2)$ and $G = \int_0^u (uf_1(u^2 - y^2) + g_1(u^2 - y^2)) \, dy = u (uG_1(u,x) + F_1(u,x))$ with $G_1$ and $F_1$ given by expressions (3.42).

Since $F_1$, $F_1$ and $G_1$ are even functions of $u_x$, we see that condition (2.18) on $F'$ holds, while condition (2.17) on $G$ becomes $c = a G_1(a,a) + \hat{F}_1(a,a) \neq 0$ where both $G_1(a,a)$ and $\hat{F}_1(a,a)$ are even functions of $a$. Hence, since $a G_1(a,a)$ is an odd function of $a$, this implies that condition (2.17) is equivalent to having $\hat{G}_1(a,a) \neq 0$ or $\hat{F}_1(a,a) \neq 0$. From these two inequalities, we obtain the following result.

**Proposition 3.3.** The Hamiltonian family (3.22) of multi-peakon equations possesses single peakon travelling-wave solutions $u(a) = ae^{-|x-c|}$ if $f_1$ and $g_1$ are (arbitrary) continuous functions. The speed-amplitude relation (2.21) of these peakons is given by

$$c = ac_1(a) + c_0(a)$$

with

$$c_1(a) = \int_0^1 f_1((1 - \lambda^2)u^2) \, d\lambda, \quad c_0(a) = \int_0^1 g_1((1 - \lambda^2)u^2) \, d\lambda,$$

where the speed $c$ is non-zero if and only if $c_1 \neq 0$ or $c_0 \neq 0$.

The speed-amplitude relation (3.48) is linear iff $f_1$ is a constant and $g_1$ is zero. As a result, in general, this relation is nonlinear, where both $c_1(a)$ and $c_0(a)$ are non-constant even functions of $a$. This has some interesting consequences for the speed properties of peakons ($a = a_+ > 0$) and anti-peakons ($a = a_- < 0$).

When peakons and anti-peakons with the same absolute amplitude $|a| = a_+ = -a_-$ are considered, their respective speeds are $c_+ = |a| c_1(|a|) + c_0(|a|)$ and $c_- = -|a| c_1(|a|) + c_0(|a|)$, whereby $c_+ - c_- = 2|a| c_1(|a|)$ is their speed difference. Therefore, in the case of the CH-subfamily $g_1 = 0$, peakons and anti-peakons have opposite speeds: $c_+ = |a| c_1(|a|) = -c_-$. In contrast, in the case of the mCH-subfamily $f_1 = 0$, peakons and anti-peakons have equal speeds: $c_+ = c_0(|a|) = c_-$. In general, we see that all peakons will move to the right, $c_+ > 0$, if $f_1(y)$ and $g_1(y)$ are non-negative functions for $y > 0$, since this implies both $c_0(a_+)$ and $c_1(a_+)$ are positive, so then $c_+ = a_+ c_1(a_+) + c_0(a_+)$ is a sum of two positive terms. In contrast, the direction of anti-peakons depends on the relative magnitudes of $f_1(y)$ and $g_1(y)$ for $y > 0$, since if $f_1(y)$ and $g_1(y)$ are non-negative functions for $y > 0$ then $c_- = c_0(|a_-|) - |a_-| c_1(|a_-|)$ is a difference of two positive terms.

Because single (anti) peakons are travelling waves, their $H^1$ norm along with their momentum and energy will be trivially conserved. We obtain

$$||u(a)||_{H^1} = \sqrt{2} |a|,$$

(3.50)
\[ \dot{M}(u_{(a)}) = a, \quad \dot{H}(u_{(a)}) = a^2 \int_0^1 \left( g_1(a^2 y) \arctanh(\sqrt{1 - y}) + |a| f_1(a^2 y) \sqrt{1 - y} \right) dy. \] (3.51)

### 3.3. Minimizer principle

The peakon travelling waves obtained in proposition 3.3 are solutions \( u(t, x) = U(x - ct) \) of the weak travelling-wave equation (2.12). This weak ODE is readily verified to have the form

\[ 0 = \int_{-\infty}^{\infty} \left( c U'(\psi - \psi''') + \psi' E_{\psi} + \psi'' E_{\psi'} \right) d\xi \] (3.52)

where \( E \) is the energy density (3.45) evaluated for travelling waves. Here \( \psi(\xi) \) is a test function, with \( \xi = x - ct \). If we write \( \phi = \psi' \), then in equation (3.52) the first term can be expressed as \( \int_{-\infty}^{\infty} c U'(\psi - \psi''') d\xi = -\int_{-\infty}^{\infty} c(U\phi + U'\phi') d\xi = -\frac{1}{2} c \phi ||U||_{H^1}^2 \), while the second and third terms are similarly given by \(-\int_{-\infty}^{\infty} \left( \phi E_{\psi} + \phi' E_{\psi'} \right) d\xi = -\phi \dot{H} \) where \( \dot{H}(U) \) is the energy integral (3.41) evaluated for travelling waves. Thus, the weak travelling-wave equation (2.12) has an equivalent formulation as a weak variational principle

\[ 0 = \delta \phi \left( \frac{1}{2} c ||U||_{H^1}^2 + \dot{H}(U) \right) \] (3.53)

with \( \phi(\xi) \) being an arbitrary test function.

This is a counterpart of the variational principle (3.47) for weak solutions of the Hamiltonian family of multi-peakon equation (3.22). It can be used as a starting point to prove stability of peakon travelling waves. In particular, whenever the nonlinearities \( f_1(u^2 - u_1^2) \) and \( g_1(u^2 - u_1^2) \) are such that the energy integral \( \dot{H}(u) \) is positive definite, the functional \( \frac{1}{2} c ||U||_{H^1}^2 + \dot{H}(U) \) will also be positive definite. If peakon travelling waves are the ground state (namely, minimizers) of this functional, then they will be stable.

A natural conjecture is that peakon travelling waves are the solutions of the minimizer principle

\[ I := \inf_{U(\xi)} ||U||_{H^1} \text{ such that } \dot{H}(U) + \frac{1}{2} c ||U||_{H^1}^2 = h(a) + ca^2 \] (3.54)

where \( h(a) = \dot{H}(u_{(a)}) \) denotes the energy of peakon travelling waves (3.51).

In the case of the CH equation (3.11) when \( f_1 = 1, g_1 = 0 \), this minimizer principle reduces to the minimization problem [14] for Camassa–Holm peakons, for which peakon travelling waves are known to be unique solution (up to translations).

A similar proof that peakon travelling waves are the unique solution of the minimization problem for the general Hamiltonian family (3.22), as well as a proof of stability, will be left for subsequent work.

### 4. One-parameter generalizations of CH and mCH equations

The family of Hamiltonian peakon equation (3.16) involves two arbitrary functions \( f_1 \) and \( g_1 \) of \( u^2 - u_1^2 \). This family provides a wide generalization of both the CH equation (3.11) and the mCH equation (3.12). It can be viewed as an analog of the Hamiltonian family of generalized KdV equations.
\[ u_t = f(u)u_x + u_{xxx} = D_x \Delta (\delta H/\delta u), \quad H = \int_{-\infty}^{\infty} \left( F(u) - \frac{1}{2} u_x^2 \right) \, dx \]  

(4.1)

where \( F' = f \). This generalized KdV family has a scaling invariant subfamily given by the gKdV equation (1.11), which involves a general nonlinearity power \( p \neq 0 \).

We can obtain an analogous scaling invariant subfamily of Hamiltonian peakon equation (3.16) by taking the functions \( f_1 \) and \( g_1 \) to have a power form: \( f_1(u^2 - u_1^2) = a(u^2 - u_1^2)^{p-1} \), \( g_1(u^2 - u_1^2) = b(u^2 - u_1^2)^{q-1} \), where \( a, \ b, \ p, \ q \) are constants. This yields \( m_t + au_x(u^2 - u_1^2)^{p-1}m + (au(u^2 - u_1^2)^{p-1}m + b(u^2 - u_1^2)^{q-1}m) = 0 \) which will be invariant under the group of scaling transformations \( u \rightarrow \lambda u \) and \( t \rightarrow \lambda^{-k-1}t \) (\( \lambda \neq 0 \)) if the nonlinearity powers are related by \( 2(q - p) = 1 \), with \( k = 2p - 2 = 2q - 3 \).

Hence, we have a three-parameter family of scaling-invariant Hamiltonian peakon equations

\[ m_t + au_x(u^2 - u_1^2)^{k/2}m + (au(u^2 - u_1^2)^{k/2}m + b(u^2 - u_1^2)^{(k+1)/2}m) = 0 \]  

(4.2)

which has the Hamiltonian structure (3.16)–(3.18), where the Hamiltonian is given by

\[ H = \int_{-\infty}^{\infty} \left( \frac{a}{k+2} (u^2 - u_1^2)^{1/2} + \frac{b}{k+3} u \right) (u^2 - u_1^2)^{(k+1)/2} m \, dx. \]  

(4.3)

This family (4.2) and (4.3) unifies the CH equation \((k = 0, b = 0)\) and the mCH equation \((k = 1, a = 0)\), up to a scaling of \( u \). It represents a close analog of the gKdV equation (1.11).

### 4.1. Generalized CH equation

By putting \( k = 2p - 2, \ b = 0, \ a = 1 \) in the three-parameter family (4.2) and (4.3), we obtain the one-parameter family of generalized CH equation (1.15). This gCH family can be written equivalently as

\[ m_t + \left( \frac{1}{2p} (u^2 - u_1^2)^{p} + u(u^2 - u_1^2)^{p-1}m \right)_x = 0. \]  

(4.4)

Like the relationship between the gKdV equation (1.11) and the ordinary KdV equation, the gCH equation (1.15) reduces to the CH equation (1.2) when \( p = 1 \) and retains one Hamiltonian structure of the CH equation,

\[ m_t = -D_x \Delta (\delta H_{\text{gCH}}/\delta m), \quad H_{\text{gCH}} = \int_{-\infty}^{\infty} \frac{1}{2p} (u^2 - u_1^2)^{p} m \, dx. \]  

(4.5)

For all \( p \geq 1 \), the gCH equation (1.15) possesses peakon travelling-wave solutions and multi-peakon solutions. Its strong solutions have conserved momentum (3.40), energy (3.41), and \( H^s \) norm (3.43). For setting up the peakon equations, we note that \( f = uf_1 \) and \( g = uf_1 \) with \( f_1 = (u^2 - u_1^2)^{p-1} \).

Single peakon and anti-peakon solutions \( u = ae^{-|x-ct|} \) (with \( a > 0 \) and \( a < 0 \), respectively) are given by the speed-amplitude relation

\[ c(a) = \gamma_p a^{2p-1}, \quad \gamma_p = \frac{\sqrt{\pi}}{2} \frac{\Gamma(p)}{\Gamma(p + 1/2)}, \]  

(4.6)

as obtained from proposition 3.3. We see that \( c(a) \) is an odd function of \( a \), and as a result, peakons and anti-peakons have the same speed but move in opposite directions (peakons to the right, and anti-peakons to the left).
From theorem 2.2, multi peakon and anti-peakon solutions \( u = \sum_{i=1}^{N} \alpha_i(t) e^{-|x - \beta_i(t)|} \) for \( N \geq 2 \) are described by the dynamical system

\[
\begin{align*}
\dot{\alpha}_i &= \frac{1}{2p} (H^+(U_i, V_i) - H^-(U_i, V_i)), \\
\dot{\beta}_i &= \frac{1}{2} (U_i + \alpha_i) (V_i + \alpha_i) \hat{H}^+(U_i, V_i) - (V_i - \alpha_i) \hat{H}^-(U_i, V_i) / \alpha_i, \quad i = 1, 2, \ldots, N, \tag{4.7}
\end{align*}
\]

with

\[
\begin{align*}
H^\pm(U_i, V_i) &= (U_i^2 - (V_i \pm \alpha_i)^2)^p = \sum_{j=0}^{p} (-1)^j \binom{p}{j} U_i^{2(p-j)} (V_i \pm \alpha_i)^{2j}, \\
\hat{H}^\pm(U_i, V_i) &= \int_0^1 (U_i^2 - (V_i \pm \alpha_i)^2) \lambda^j \, d\lambda = \sum_{j=0}^{p} \frac{(-1)^j}{2j+1} \binom{p}{j} U_i^{2(p-j)} (V_i \pm \alpha_i)^{2j}, \tag{4.8}
\end{align*}
\]

where \( U_i \) and \( V_i \) are given in terms of the dynamical variables \((\alpha_i(t), \beta_i(t))\) by expression (2.61). The interaction terms in this dynamical system for \( p \geq 2 \) are considerably more complicated than for \( p = 1 \), and qualitatively new features turn out to occur, which we will investigate in section 5.

### 4.2. Generalized mCH equation

We put \( k = 2p - 1, a = 0, b = 1 \) in the three-parameter family (4.2) and (4.3), which yields a one-parameter family of generalized mCH equation (1.16). This gmCH family reduces to the ordinary mCH equation (1.4) when \( p = 1 \), and retains one of its Hamiltonian structures

\[
m_i = -D_x \Delta (\delta H_{\text{gmCH}} / \delta \dot{m}), \quad H_{\text{gmCH}} = \int_{-\infty}^{\infty} \frac{1}{2(p+1)} u(a^2 - u_i^2)^p m \, dx. \tag{4.9}
\]

For all \( p \geq 1 \), this equation (1.15) possesses peakon travelling-wave solutions and multi-peakon solutions. Its strong solutions have conserved momentum (3.40), energy (3.41), and \( H^1 \) norm (3.43). To set up the peakon equations, we note that \( f = 0 \) and \( g = g_1 = (u^2 - u_i^2)^p \).

Single peakon and anti-peakon solutions \( u = ae^{-|x-c|} \) (with \( a > 0 \) and \( a < 0 \), respectively) are given by the speed-amplitude relation

\[
c(a) = \gamma_p + a^{2p} \tag{4.10}
\]

as obtained from proposition 3.3 (where \( \gamma_p \) is given in equation (4.6)). We see that \( c(a) \) is a positive, even function of \( a \), and hence both peakons and anti-peakons have the same speed and move to the right.

From theorem 2.2, multi peakon and anti-peakon solutions \( u = \sum_{i=1}^{N} \alpha_i(t) e^{-|x - \beta_i(t)|} \) for \( N \geq 2 \) are described by the dynamical system

\[
\begin{align*}
\dot{\alpha}_i &= 0, \\
\dot{\beta}_i &= \frac{1}{2} ((U_i + \alpha_i) \hat{H}^+(U_i, V_i) - (V_i - \alpha_i) \hat{H}^-(U_i, V_i)) / \alpha_i, \quad i = 1, 2, \ldots, N \tag{4.11}
\end{align*}
\]

using the notation (4.8), with \( U_i \) and \( V_i \) being given by expressions (2.61) in terms of the dynamical variables \((\alpha_i(t), \beta_i(t))\). Similarly to the gCH equation, this dynamical system turns out to exhibit qualitatively new features for \( p \geq 2 \) compared to the ordinary mCH equation with \( p = 1 \). We will investigate these features in section 5.
4.3. Unification of generalized CH and mCH equations

We can unify the gCH equation (1.15) and the gmCH equation (1.16) into a single one-parameter family by choosing the coefficients $a$ and $b$ to be suitable functions of $k$ in the three-parameter family (4.2) and (4.3). Specifically, $a(k)$ and $b(k)$ need to satisfy $a(0) \neq 0$, $a(1) = 0$, $b(0) = 0$, $b(1) \neq 0$. For example, simple choices are $a(k) = 1 - k$ and $b(k) = k$.

The resulting unified family

$$m_t + (1 - k)u_x(u^2 - u_0^2)k/2m + ((1 - k)u(u^2 - u_0^2)k/2 + k(u^2 - u_0^2)k(1 + x)/(2k))m_x = 0$$

has the same properties as the gCH and gmCH equations.

5. New behaviour in peakon interactions for higher nonlinearities

The qualitative behaviour of (anti-) peakon solutions of the gCH equation (1.15) and the gmCH equation (1.16) when the nonlinearity power is $p = 1, 2$ will now be investigated. There turns out to be a significant difference in how the (anti-) peakons interact in the case $p = 2$ compared to the case $p = 1$.

To proceed, we will consider the $N = 2$ peakon solutions of both equations, by using the respective dynamical systems (4.7) and (4.11). Specifically, we will look at how the separation between two (anti-) peakons behaves with time $t$.

5.1. Two-peakon solutions of the gCH equation

The $p = 1$ case of the gCH equation (1.15) is the ordinary CH equation. $N = 2$ peakon solutions (2.30) are given by the dynamical system

$$\dot{\alpha}_1 = -\dot{\beta}_2 = \text{sgn}(\beta_{1,2})\alpha_1\alpha_2e^{-|\beta_{1,2}|}, \quad \dot{\beta}_1 = \alpha_1 + \alpha_2e^{-|\beta_{1,2}|}, \quad \dot{\beta}_2 = \alpha_2 + \alpha_1e^{-|\beta_{1,2}|}$$

where $\beta_{1,2} = \beta_1 - \beta_2$ is the separation between the two (anti-) peakons or the peakon and the anti-peakon. This system has two constants of motion $M = \alpha_1 + \alpha_2$ and

$$E = \frac{1}{2}(\alpha_1^2(1 - e^{-|\beta_{1,2}|}) + M^2e^{-|\beta_{1,2}|}) > 0,$$

where $\alpha_{1,2} = \alpha_1 - \alpha_2$ is the amplitude difference between the two (anti-) peakons. These quantities arise from the conserved momentum (3.40) and energy (3.41) evaluated for two (anti-) peakons. Using the constants of motion, we obtain

$$\alpha_{1,2} = \pm \sqrt{2E - M^2e^{-|\beta_{1,2}|}} / \sqrt{1 - e^{-|\beta_{1,2}|}}$$

and

$$\dot{\beta}_{1,2} = \pm \text{sgn}(\beta_{1,2}) \sqrt{1 - e^{-|\beta_{1,2}|}} \sqrt{2E - M^2e^{-|\beta_{1,2}|}}.$$  

This ODE can be straightforwardly integrated. The qualitative behaviour of $\beta_{1,2}(t)$, however, can be more easily found by looking at the collision points, defined by $\beta_{1,2} = 0$, and the turning points, defined by $\dot{\beta}_{1,2} = 0$. There are essentially two different types of behaviour, depending on the constant of motion

$$\mu = \frac{2E}{M^2} - 1.$$  

Note, since $\mu M^2 = -4\alpha_1\alpha_2(1 - e^{-|\beta_{1,2}|})$, then $\mu > 0$ implies $\alpha_1\alpha_2 < 0$, representing a peakon and an anti-peakon, and $\mu < 0$ implies $\alpha_1\alpha_2 > 0$, representing two peakons or two anti-peakons. (The case $\mu = 0$ corresponds to $\alpha_1 = 0$ or $\alpha_2 = 0$, which is trivial.)
When $\mu > 0$, the only turning point is $\beta_{1,2} = 0$ which is also a collision point. The constant of motion $\mu = ((\alpha_{1,2}/M)^2 - 1) (1 - e^{-|\beta_{1,2}|})$ then shows that $\beta_{1,2} \to 0$ iff $|\alpha_{1,2}| \to \infty$. Moreover, from the dynamical ODE (5.3), we see that $\beta_{1,2}(t)$ will reach zero at a finite time. This means that a peakon and an anti-peakon will collide such that their relative amplitude blows up. See figure 1.

A different behaviour occurs when $\mu < 0$. There is an turning point $|\beta_{1,2}| = -\ln(1 + \mu)$, at which $\alpha_{1,2} = 0$. The dynamical ODE (5.3) shows that this turning point will be reached at a finite time, and $|\beta_{1,2}|$ will be increasing before and after this time. Consequently, this represents two (anti-) peakons that are approaching each other, reach a minimum separation given by $|\beta_{1,2}| = -\ln(1 + \mu)$ where their amplitudes are equal, and move away from each other. In particular, their separation goes to infinity in the asymptotic past and future. This behaviour describes an elastic ‘bounce’ interaction. See figure 2.

Similar dynamical behaviour is known to occur for all $N \geq 2$, using the integrability properties of the CH equation [50, 51].

Next we consider the case $p = 2$, which describes a nonlinear generalization of the CH equation. The dynamical system (4.7) for $N = 2$ in this case is given by
\begin{align}
\dot{\alpha}_1 &= -\dot{\alpha}_2 = 2 \text{sgn}(\beta_{1,2}) \alpha_1^2 \alpha_2^2 e^{-2|\beta_{1,2}|}, \\
\dot{\beta}_1 &= (\alpha_1 + \alpha_2 e^{-|\beta_{1,2}|}) (\frac{2}{3} \alpha_1^2 + 2 \alpha_1 \alpha_2 e^{-|\beta_{1,2}|}), \\
\dot{\beta}_2 &= (\alpha_2 + \alpha_1 e^{-|\beta_{1,2}|}) (\frac{2}{3} \alpha_2^2 + 2 \alpha_1 \alpha_2 e^{-|\beta_{1,2}|}).
\end{align}

(5.6)

This system again has the momentum $M = \alpha_1 + \alpha_2$ as a constant of motion, however energy is no longer conserved. Instead, a second constant of motion appears when we look at the dynamical equations for $\alpha_{1,2}$ and $\beta_{1,2}$:
\begin{align}
\dot{\alpha}_{1,2} &= \frac{1}{4} \text{sgn}(\beta_{1,2}) (M^2 - \alpha_{1,2}^2) e^{-2|\beta_{1,2}|}, \\
\dot{\beta}_{1,2} &= \frac{1}{6} \alpha_{1,2} (3M^2 + \alpha_{1,2}^2 + \frac{1}{6} (M^2 - \alpha_{1,2}^2) (4 - 3 e^{-|\beta_{1,2}|}) e^{-|\beta_{1,2}|}).
\end{align}

(5.7)

(5.8)

The transformation $e^{\beta_{1,2}} = 4(M^2 - \alpha_{1,2}^2) y/(3M^2 + \alpha_{1,2}^2)$ leads to a separable Abel equation for $y(\alpha_{1,2})$, which possesses an explicit first integral. The corresponding constant of motion in terms of $\beta_{1,2}$ and $\alpha_{1,2}$ is given by
\begin{align}
C &= \frac{3M^2 + \alpha_{1,2}^2 + 3(M^2 - \alpha_{1,2}^2) e^{-|\beta_{1,2}|}}{(3M^2 + \alpha_{1,2}^2 + (M^2 - \alpha_{1,2}^2) e^{-|\beta_{1,2}|})^3}.
\end{align}

(5.9)

This equation represents a cubic polynomial in $\alpha_{1,2}^2$, which thereby gives $\alpha_{1,2}$ as a function of $\beta_{1,2}$, yielding a dynamical ODE for $\beta_{1,2}(t)$. Rather than work with the resulting ODE, we can determine the qualitative behaviour of $\beta_{1,2}(t)$ directly from the pair of ODEs (5.7) and (5.8), by looking at the collision points and turning points in terms of $\beta_{1,2}$. The various behaviours depend essentially on the constant of motion
\begin{align}
\nu &= 9M^2 C.
\end{align}

(5.10)

It is straightforward to show that $1 > \nu > -\infty$. Hereafter, we will write $B_{1,2} = e^{-|\beta_{1,2}|}$ for convenience, where $0 < B_{1,2} \leq 1$. Note that $(\alpha_{1,2}/M)^2 > 1$ implies $\alpha_1$ and $\alpha_2$ have opposite signs, and that $(\alpha_{1,2}/M)^2 < 1$ implies $\alpha_1$ and $\alpha_2$ have the same sign.
From the ODE (5.8), we find that the turning points $\dot{\beta}_{1,2} = 0$ occur for $\alpha_{1,2} = 0$ and $(\alpha_{1,2}/M)^2 = (3B_{1,2}^2 - 4B_{1,2} - 3)/((1 - B_{1,2})(1 - 3B_{1,2})) \geq 13$. In the first case, $B_{1,2}$ is the root of the cubic equation $\nu(B_{1,2} + 3)^3 = 27(B_{1,2} + 1)$ obtained from combining equations (5.9) and (5.10), with $1 > \nu \geq \frac{27}{22}$ due to $0 < B_{1,2} \leq 1$. In the second case, since $3B_{1,2}^2 - 4B_{1,2} - 3 < 0$ for $0 < B_{1,2} \leq 1$, we have the condition $1 > B_{1,2} > \frac{1}{3}$ along with the cubic equation $48\nu B_{1,2}^3(1 - B_{1,2}) = (1 - 3B_{1,2})^3$ again given by combining equations (5.9) and (5.10), where $\nu < 0$.

Figure 1. Relative position and amplitude for CH two-peakon collision ($\mu > 0$). (a) Separation. (b) Relative amplitude.

Figure 2. Relative position and amplitude for CH two-peakon elastic interaction ($\mu < 0$). (a) Separation. (b) Relative amplitude.
We also find that the collision points \( \beta_{1,2} = 0 \) occur for \( \left( \alpha_{1,2}/M \right)^2 = (27 - 32\nu)/9 \), which requires \( \nu \leq \frac{27}{32} \). The only collision point that coincides with a turning point, \( \dot{\beta}_{1,2} = 0 \), is given by \( \alpha_{1,2} = 0 \) with \( \nu = \frac{27}{32} \).

When \( \frac{27}{32} < \nu < 1 \), there is only a turning point, and from the dynamical ODEs (5.7)–(5.8) we can show that this point is reached at a finite time, with \( \alpha_{1,2} = 0 \). This represents two (anti-) peakons whose separation reaches a minimum given by the root of \( \nu(3 + e^{-|\beta_{1,2}|})^3 = 27(1 + e^{-|\beta_{1,2}|}) \), where their amplitudes are equal, and goes to infinity in the asymptotic past and future. Hence, the behaviour is an elastic ‘bounce’ interaction. See figure 3.

A similar behavior occurs when \( \nu = \frac{27}{32} \), with the ‘bounce’ coinciding with a collision between the (anti-) peakons.

When \( 0 \leq \nu < \frac{27}{32} \), there is a collision point but no turning point. The collision can occur for either two (anti-) peakons or a peakon and an anti-peakon, depending on whether \( \nu \geq \frac{9}{16} \) respectively. If \( \nu > 0 \) then the separation in the asymptotic past and future is increasing such that \( \dot{\beta}_{1,2} \sim \frac{1}{6} \alpha_{1,2}(3M^2 + \alpha_{1,2}^2) \) and \( \alpha_{1,2} \sim 3(\sqrt{\frac{1}{\nu} - 1}) \) are constant, as shown by the dynamical ODEs (5.7) and (5.8). Moreover, in contrast to the case \( p = 1 \), \( \alpha_{1,2} \) will be non-zero at the time of the collision. See figure 4.

But if \( \nu = 0 \), we have \( \left( \alpha_{1,2}/M \right)^2 = 3(1 + B_{1,2})/(3B_{1,2} - 1) \geq 3 \), which requires \( B_{1,2} > \frac{1}{3} \), and hence \( |\beta_{1,2}| < \ln 3 \). Then the dynamical ODE (5.8) shows that \( |\beta_{1,2}| \rightarrow \ln 3 \) in a finite time, and consequently \( |\alpha_{1,2}| \rightarrow \infty \). This behaviour describes a blow up in the relative amplitude of a peakon and an anti-peakon, before and after a collision, as their separation approaches \( |\beta_{1,2}| \rightarrow \ln 3 \) in a finite time. At the collision point, \( |\alpha_{1,2}| = \sqrt{3M} \) is non-zero. See figure 5.

Finally, when \( \nu < 0 \), there is a collision point \( \beta_{1,2} = 0 \), and a turning point \( 0 < |\beta_{1,2}| < \ln 3 \) given by the root of \( 48\nu e^{-2|\beta_{1,2}|}(1 - e^{-|\beta_{1,2}|}) = (1 - 3e^{-|\beta_{1,2}|})^3 \). The behaviour in this case is similar to the case \( \nu = 0 \), except that the separation between the peakon and the anti-peakon...
increases (before and after the collision) to a maximum at the turning point and then decreases to a non-zero limit in a finite time such that $|\alpha_{1,2}| \to \infty$. See figure 6.

These behaviours for $\nu \leq 0$ are strikingly different than the blow-up collisions and the elastic ‘bounces’ that occur in the ordinary CH equation. Further analysis, including explicit solution expressions for $\beta_{1,2}$ and $\alpha_{1,2}$, will be given elsewhere.

5.2. Two-peakon solutions of the gmCH equation

Similarly to the investigation of the gCH equation, we will consider the $N=2$ dynamical system in the cases $p=1$ and $p=2$ and examine how the separation between the two (anti-) peakons behaves with time $t$. Neither of these cases has been extensively explored in the literature to-date.

The case $p=1$ corresponds to the mCH equation (1.4). For $N=2$, the dynamical system (4.11) becomes

$$
\dot{\alpha}_1 = 0, \quad \dot{\alpha}_2 = 0, \quad \dot{\beta}_1 = \frac{2}{3} \alpha_1^2 + 2\alpha_1\alpha_2e^{-|\beta_{1,2}|}, \quad \dot{\beta}_2 = \frac{2}{3} \alpha_2^2 + 2\alpha_1\alpha_2e^{-|\beta_{1,2}|}
$$

(5.11)

where $\beta_{1,2} = \beta_1 - \beta_2$ is the separation between the two (anti-) peakons or the peakon and the anti-peakon. The separation obeys the simple dynamical ODE $\dot{\beta}_{1,2} = \frac{2}{3}(\alpha_1^2 - \alpha_2^2)$.

There are essentially two different types of behaviour, depending on the constant of motion $\gamma = \frac{2}{3}(\alpha_1^2 - \alpha_2^2)$.

When $\gamma \neq 0$, we see that the separation $\beta_{1,2}(t)$ changes linearly in time $t$. Hence, there is a collision at a finite time, and in the asymptotic past and future, $\dot{\beta}_1 \sim \frac{2}{3}\alpha_1^2 = c_1$ and $\dot{\beta}_2 \sim \frac{2}{3}\alpha_2^2 = c_2$ are the asymptotic speeds of the (anti-) peakons. These asymptotic speeds are precisely the speeds of the (anti-) peakons in the absence of any interaction, as shown by the relation (4.10). See figure 7(a).

Figure 4. Relative position and amplitude for $p=2$ gCH two-peakon interaction when $0 < \nu < \frac{27}{32}$. (a) Separation. (b) Relative amplitude.
In contrast, when $\gamma = 0$, the separation $\beta_{1,2}(t)$ is time independent. This describes two (anti-) peakons, or a peakon and an anti-peakon, which have equal amplitudes $|\alpha_1| = |\alpha_2|$ and equal speeds $\dot{\beta}_1 = \dot{\beta}_2 = \alpha^2(\frac{2}{\xi} + 2e^{-|\beta_{1,2}(0)|}) > 0$, where $\alpha = |\alpha_1| = |\alpha_2|$. This special case describes a non-dynamical bound pair.

We remark that bound pairs have also been observed recently [52] for conservative two-peakon solutions of the mCH equation. However, conservative peakons are not weak solutions and instead arise from a different kind of regularization of the mCH equation for distributional

Figure 5. Relative position and amplitude for $p = 2$ gCH two-peakon interaction when $\nu = 0$. (a) Separation. (b) Relative amplitude.

Figure 6. Relative position and amplitude for $p = 2$ gCH two-peakon interaction when $\nu < 0$. (a) Separation. (b) Relative amplitude.
solutions. Indeed, the conservative two-peakon system differs from the system (5.11) by omitting the terms $\frac{2}{7} \alpha_1^2$ and $\frac{2}{7} \alpha_2^2$. As a consequence, the dynamics of conservative two-peakons of the mCH equation is trivial. This is a special instance of a general relationship pointed out in [52]: conservative $N$-peakon solutions with equal amplitudes are related to $N$-peakon weak solutions with equal amplitudes by a shift in the speeds of the peakons. In particular, the two-peakon bound-pair solution in the equal-amplitude case $\gamma = 0$ can be transformed into a corresponding bound-pair of conservative two-peakons with equal amplitudes. More discussion of the differences between conservative and weak solutions can be found in [52, 53].

Next, the case $p = 2$ corresponds to the gmCH equation (1.16), describing a nonlinear generalization of the mCH equation (1.4). For $N = 2$, the dynamical system (4.11) in the case $p = 2$ is given by

$$\dot{\alpha}_1 = 0, \quad \dot{\beta}_1 = \frac{8}{15} \alpha_1^2 (\alpha_1^2 + 5 \alpha_1 \alpha_2 e^{-|\beta_{1,2}|} + 10 \alpha_2^2 e^{-2|\beta_{1,2}|}),$$

(5.12)

$$\dot{\alpha}_2 = 0, \quad \dot{\beta}_2 = \frac{8}{15} \alpha_2^2 (\alpha_2^2 + 5 \alpha_1 \alpha_2 e^{-|\beta_{1,2}|} + 10 \alpha_1^2 e^{-2|\beta_{1,2}|}).$$

(5.13)

The separation $\beta_{1,2} = \beta_1 - \beta_2$ satisfies the dynamical ODE

$$\dot{\beta}_{1,2} = \frac{8}{15} (\alpha_1^4 - \alpha_2^4) + \frac{8}{3} \alpha_1 \alpha_2 (\alpha_1^3 - \alpha_2^3) e^{-|\beta_{1,2}|} = \gamma (1 + \sigma e^{-|\beta_{1,2}|})$$

(5.14)

where

$$\gamma = \frac{8}{15} (\alpha_1^4 - \alpha_2^4), \quad \sigma = 5 \alpha_1 \alpha_2 / (\alpha_1^2 + \alpha_2^2)$$

(5.15)

are constants of motion. It is straightforward to integrate this ODE to obtain $\beta_{1,2}(t)$ explicitly. The qualitative behaviour of $\beta_{1,2}(t)$ depends essentially on both $\gamma$ and $\sigma$. 

![Figure 7. Relative position for $p = 1, 2$ gmCH two-peakon interaction. (a) $p = 1$. (b) $p = 2$ when $\sigma > 0$.](image)
First, when $\gamma = 0$, the amplitudes are equal, $|\alpha_1| = |\alpha_2|$, and the separation $\beta_{1,2}(t)$ is time independent, with the speeds being given by

$$\dot{\beta}_1 = \dot{\beta}_2 = \frac{8}{15} \alpha^4 (1 + 5 e^{-|\beta_{1,2}(0)|} + 10 e^{-2|\beta_{1,2}(0)|}) > 0,$$

where $\alpha = |\alpha_1| = |\alpha_2|$. This is the same qualitative behaviour that occurs for $p = 1$.

Next, when $\gamma \neq 0$, the behaviour is most easily understood by looking at the collision points, defined by $\beta_{1,2} = 0$, and the turning points, defined by $\dot{\beta}_{1,2} = 0$. Note that $\sigma > 0$ implies $\alpha_1$ and $\alpha_2$ have the same sign, and that $\sigma < 0$ implies $\alpha_1$ and $\alpha_2$ have opposite signs. (The case $\sigma = 0$ corresponds to $\alpha_1 = 0$ or $\alpha_2 = 0$, which is trivial.)

When $\sigma > -1$, we have $1 + \sigma e^{-|\beta_{1,2}|} > 0$ which implies there are no turning points. Consequently, the separation $\beta_{1,2}(t)$ increases in the asymptotic past and future, such that asymptotic speeds are $\dot{\beta}_1 \sim \frac{8}{15} \alpha_1^4$ and $\dot{\beta}_2 \sim \frac{8}{15} \alpha_2^4$. At a finite time, the separation will be zero. This behaviour represents either two (anti-) peakons if $\sigma > 0$, or a peakon and an anti-peakon if $0 > \sigma > -1$, that undergo a collision and separate asymptotically to infinity before and after the collision. From the amplitude-speed relation (4.10), we see that their asymptotic speeds are precisely the speeds of the (anti-) peakons in the absence of any interaction. This behaviour is again qualitatively the same as the case $p = 1$. See figure 7(b).

When $\sigma \leq -1$, there is a turning point given by $|\beta_{1,2}(0)| = \ln |\sigma|$. This turning point acts as a critical value for the separation between a peakon and an anti-peakon. If the initial separation $|\beta_{1,2}(0)|$ is less than $\ln |\sigma|$, then the peakon and the anti-peakon form a bound pair in the asymptotic past, with their separation asymptotically given by $\ln |\sigma|$. In a finite time, the pair undergoes a collapse such that the separation goes to zero in a collision, and then the pair reforms in the asymptotic future. But if the initial separation $|\beta_{1,2}(0)|$ is greater than $\ln |\sigma|$, then the peakon and the anti-peakon form a bound pair only in the asymptotic past or the asymptotic future, and this pair breaks apart in the other asymptotic direction, with the separation going to infinity. In the special case when $|\beta_{1,2}(0)|$ is equal to $\ln |\sigma|$, the peakon and the anti-peakon form a bound pair with a constant separation given by the value $\ln |\sigma|$. See figure 8.

The formation of a dynamical bound pair with different amplitudes for the peakon and anti-peakon is a qualitatively novel and striking behaviour that has not been seen in other

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**Figure 8.** Relative position for $p = 2$ gmCH two-peakon interaction when $\sigma < -1$. (a) $|\beta_{1,2}(0)| < \ln |\sigma|$, (b) $|\beta_{1,2}(0)| > \ln |\sigma|$.
non-integrable peakon equations. Further analysis of the bound-pair behaviour, including explicit solution expressions for $\beta_{1,2}$ and $\alpha_{1,2}$, will be given elsewhere.

6. Conclusions

In the $fg$-family (1.13) of nonlinear dispersive wave equations, involving two arbitrary functions $f(u, u_x)$ and $g(u, u_x)$, we have shown that $N$-peakon weak solutions (2.30) exist for arbitrary $N \geq 1$. Neither an integrability structure nor a Hamiltonian structure has been used to derive these multi-peakon solutions, and very likely most of the equations in this family do not possess either type of structure.

We have obtained an explicit dynamical system of ODEs (2.59)–(2.61) for the amplitudes and positions of the $N$ individual peaked waves in a general multi-peakon solution (2.30). For the case $N = 1$, in contrast, we show that a symmetry condition (2.18) and (2.17) on $f(u, u_x)$ and $g(u, u_x)$ is necessary for the solution to describe a single peakon travelling wave (2.9). This condition holds whenever $f(u, u_x)$ is odd in $u_x$ and $g(u, u_x)$ is even in $u_x$. Most interestingly, when the condition is not satisfied, we have shown that the $N = 1$ solution instead describes a generalized dynamical peakon whose amplitude and speed can be time dependent. Further exploration of dynamical peakons is given in [54].

As interesting examples of our general results, a subfamily of peakon equations that possesses the Hamiltonian structure (3.13) shared by both the CH and FORQ/mCH equations has been obtained. We have shown that this Hamiltonian subfamily contains a one-parameter nonlinear generalization of the CH equation and a one-parameter nonlinear generalization of the FORQ/mCH equation, as well as a one-parameter multi-peakon equation (4.2) that unifies these two generalizations. We have derived the single peakon travelling-wave solutions $u = a \exp(-|x - ct|)$ for these equations and discussed the relation between the properties of peakons with $a > 0$ and anti-peakons with $a < 0$.

The generalized CH and FORQ/mCH equations involve an arbitrary nonlinearity power $p \geq 1$, where the ordinary CH and FORQ/mCH equations correspond to the case $p = 1$. For both equations, we have investigated the effect of higher nonlinearity on (anti-) peakon interactions by studying the behaviour of two-peakon weak solutions in case $p = 2$. Qualitatively new behaviours are shown to occur in the interaction between a peakon and an anti-peakon. Specifically, for the $p = 2$ generalized mCH equation (1.16), the peakon and anti-peakon can form a bound pair which has a maximum finite separation in the asymptotic past and future and which undergoes a collapse at a finite time. (Stable bound pairs have been seen recently [52] for the mCH equation, but these pairs have trivial dynamics and arise as conservative solutions which are, in general, different than weak solutions [53].) In contrast, for the $p = 2$ generalized CH equation (1.15), the peakon and anti-peakon can exhibit a finite time blow-up in amplitude, before and after they undergo a collision, where their separation increases to a finite maximum and then decreases to a limiting non-zero value when the blow-up occurs.

The novel behaviours of interactions of peakon and anti-peakon weak solutions studied here indicate that peakons can exhibit a rich variety of dynamics for different multi-peakon equations in the general $fg$-family (1.13), and that the form of the nonlinearity in these equations has a large impact on how peakons can interact.

A study of conservation laws (energy, momentum, $H^1$-norm, etc) for the $fg$-family (1.13) has been carried out in recent work [55]. There are several interesting directions for future work:

- seek subfamilies of the $fg$-family of multi-peakon equations having other types of Hamiltonian structure;
explore the conditions under which a Hamiltonian structure will be inherited by the
dynamical system of ODEs for multi-peakons;
• study the interactions of multi-peakons for high-degree nonlinearities;
• understand the difference between peakon weak solutions and conservative peakon solutions
for cubic and higher-degree nonlinearities;
• study local well-posedness of the $fg$-family of multi-peakon equations;
• for the Cauchy problem, investigate global existence of solutions, wave breaking mech-
isms, and blow-up times;
• look for new integrable equations in the $fg$-family.

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Appendix. Tools from variational calculus

Three useful tools are the Frechet derivative, the Euler operator, and the Helmholtz operator,
which are part of the variational bi-complex [48] for differential forms in the jet space
$J = (x, u, u_x, u_{xx}, \ldots)$. Differential functions refer to functions on $J$ depending on only finitely
many $x$-derivatives of $u$.

The Frechet derivative is defined by

$$\delta P = P \partial u + (D_xP) \partial u_x + (D^2_xP) \partial u_{xx} + \cdots \quad (A.1)$$

where $P$ is an arbitrary differential function. Integration by parts on this operator yields the
adjoint Frechet derivative

$$\delta^*_P = P \partial u - D_x(P \partial u_x) + D^2_x(P \partial u_{xx}) + \cdots \quad (A.2)$$

The Euler operator is given by

$$E_u = \partial u - D_x \partial u_x + D^2_x \partial u_{xx} + \cdots \quad (A.3)$$

It has the property that it annihilates a differential function $P$ iff $P = D_x Q$ is a total $x$-derivative
for some differential function $Q$. Similarly, the Helmholtz operator

$$\delta_P - \delta^*_P = P(\partial_x - E_u) + (D_x P)(\partial_{u_x} + E^{(1)}_u) + (D^2_x P)(\partial_{u_{xx}} - E^{(2)}_u) + \cdots \quad (A.4)$$

has the property that it annihilates a differential function $R$ with $P$ being an arbitrary differential
function, iff $R = E_u(Q)$ is an Euler–Lagrange expression for some differential function $Q$. Here

$$E^{(k)}_u = \partial u - \binom{k + 2}{2} D_x \partial u_x + \cdots, \quad k = 1, 2, \ldots \quad (A.5)$$

denote the higher Euler operators.

A useful identity relating the Frechet derivative and the Euler operator is given by

$$\delta_P R = RE_u(P) + D_x \Theta, \quad \Theta = R E^{(1)}_u(P) + (D_x R) E^{(2)}_u(P) + \cdots \quad (A.6)$$
A.1. Proof of proposition 3.1

Using the Euler operator and the Helmholtz operator, it is straightforward to see that a Hamiltonian structure of the form (3.13) exists for the \( f_g \)-family (1.13) of peakon equations if the functions \( f(u, u_x) \) and \( g(u, u_x) \) satisfy the conditions

\[
E_u(f(u, u_x)(u - u_{xx})) = 0 \tag{A.7}
\]

and

\[
\partial_u(g(u, u_x)(u - u_{xx}) + A) - E_u(g(u, u_x)(u - u_{xx}) + A) = 0, \tag{A.8a}
\]

\[
\partial_u(g(u, u_x)(u - u_{xx}) + A) + E_{u_x}(g(u, u_x)(u - u_{xx}) + A) = 0, \tag{A.8b}
\]

\[
\partial_u(g(u, u_x)(u - u_{xx}) + A) - E_{u_x}(g(u, u_x)(u - u_{xx}) + A) = 0 \tag{A.8c}
\]

where

\[
A = D_x^{-1}(f(u, u_x)(u - u_{xx})). \tag{A.9}
\]

Both of the conditions (A.7) and (A.8a) must hold identically in jet space, and hence each one will split with respect to all \( x \)-derivatives of \( u \) that do not appear in the functions \( f(u, u_x) \) and \( g(u, u_x) \).

From condition (A.7), we get two equations

\[
(u_x f_u + uf_u) - uf_u = 0, \quad (u_x f_x + uf_x) - uf_x = 0, \tag{A.10}
\]

which comprise an overdetermined linear system for \( f(u, u_x) \). Integration with respect to \( u \) and \( u_x \) yields \( uf - u_x (uf_x + uf_x) = f_0 \), where \( f_0 \) is an arbitrary constant. This is a linear first-order PDE whose general solution is expression (3.14). We substitute this expression into equation (A.9) and note

\[
u_m f_1(u^2 - u_0^2) = \frac{1}{2} D_x F_1(u^2 - u_0^2), \quad \frac{um}{u^2 - u_0^2} = D_x \left( \frac{1}{2} \ln \left( \frac{u - u_0}{u + u_0} \right) + x \right) \tag{A.11}
\]

through using the relation \( u_m = \frac{1}{2} D_x (u^2 - u_0^2) \). Hence, we obtain

\[
A = \frac{1}{2} F_1(u^2 - u_0^2) + \frac{1}{2} f_0 \ln \left( \frac{u - u_0}{u + u_0} \right) + f_0 x. \tag{A.12}
\]

Then we substitute this expression into the second condition (A.8a), which reduces to the equations \( 0 = D_x ((u - u_{xx})g_u + A_{xx} + D_x^2 g) \) and \( 0 = (u - u_{xx}) g_u + A_{xx} - 2D_x g \). Clearly, the first equation is a differential consequence of the second equation. Expanding the second equation, we get

\[
u_x g_u + ug_u = u_x F_1 + f_0 \frac{u}{u^2 - u_0^2} \tag{A.13}
\]

which is a linear first-order PDE for \( g(u, u_x) \). Its general solution is expression (3.15).

Hence, the first part of proposition 3.1 has been established. To prove the second part, we need to invert the relation \( gm + A = E_u(B) \) to find the Hamiltonian density \( B \), where

\[
gm + A = \frac{1}{2} F_1(u^2 - u_0^2) + (af_1(u^2 - u_0^2) + g_1(u^2 - u_0^2)) m + f_0 \left( \frac{um}{u^2 - u_0^2} + \frac{1}{2} \ln \left( \frac{u - u_0}{u + u_0} \right) + x \right) \tag{A.14}
\]
is given by expressions (A.12) and (3.15). The form of the Hamiltonian densities (3.7) and (3.8) for the CH and FORQ/mCH equations suggests seeking $B = mC(u, u_1)$. We expand $E_u(B) = E_u((u - u_{xx})C(u, u_1))$, equate it to expression (A.14), and split with respect to $m$. This yields a system of two linear equations. By combining these equations, we get

$$\left(uC - u_1(u_xC_u - uC_{u_1})\right)_u = -uu_1f_1 - u_xg_1 - f_0 \frac{u_1}{u^2 - u_1^2},$$

(A.15)

$$(uC - u_1(u_xC_u - uC_{u_1}))_u = u^2f_1 + \frac{1}{2}F_1 + u_xg_1 + \frac{u}{u^2 - u_1^2} + \frac{1}{2}f_0 \ln \left(\frac{u - u_1}{u + u_1}\right) + f_0x.$$  

(A.16)

This system is straightforward to integrate. Its general solution for $C$ is given by

$$C = u_xC_1(u^2 - u_1^2) + \frac{C_0u}{u^2 - u_1^2} + \frac{1}{2}F_1(u^2 - u_1^2) + \frac{1}{2}uG_1(u^2 - u_1^2) + \frac{1}{2}f_0 \ln \left(\frac{u - u_1}{u + u_1}\right) + f_0x,$$

(A.17)

where $C_1$ is an arbitrary function of $u^2 - u_1^2$, and $C_0$ is an arbitrary constant. Then $B = mC$ consists of the Hamiltonian density (3.18) plus a total $x$-derivative

$$D_x\left(\frac{1}{2}C_1(u^2 - u_1^2) + \frac{1}{2}C_0 \ln \left(\frac{u - u_1}{u + u_1}\right) + C_0x\right),$$

with $C_1' = C_1$.

This completes the proof of proposition 3.1.

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**References**

[1] Camassa R and Holm D D 1993 An integrable shallow water equation with peaked solitons Phys. Rev. Lett. 71 1661–4

[2] Camassa R, Holm D D and Hyman J M 1994 A new integrable shallow water equation Adv. Appl. Mech. 31 1–33

[3] Constantin A 2000 Existence of permanent and breaking waves for a shallow water equation: a geometric approach Ann. Inst. Fourier 50 321–62

[4] Constantin A and Escher J 1998 Wave breaking for nonlinear nonlocal shallow water equations Acta Math. 181 229–43

[5] Constantin A and Escher J 1998 Global existence and blow-up for a shallow water equation Ann. Scuola Norm. Super. Pisa 26 303–28

[6] Constantin A and Escher J 2000 On the blow-up rate and the blow-up set of breaking waves for a shallow water equation Math. Z. 33 75–91

[7] Lenells J 2005 Traveling wave solutions of the Camassa–Holm equation J. Differ. Equ. 217 393–430

[8] Alber M S, Camassa R, Holm D D and Marsden J E 1994 The geometry of peaked solitons and billiard solutions of a class of integrable PDE’s Lett. Math. Phys. 32 137–51

[9] Cao C S, Holm D D and Titi E S 2004 Travelling wave solutions for a class of one-dimensional nonlinear shallow water wave models J. Dyn. Differ. Equ. 16 167–78

[10] Beals R, Sattinger D H and Szmigielski J 1999 Multipeakons and a theorem of Stieltjes Inverse Problems 15 L1–4

[11] Beals R, Sattinger D H and Szmigielski J 2000 Multipeakons and the classical moment problem Adv. Math. 154 229–57

[12] Constantin A and Strauss W A 2000 Stability of peakons Commun. Pure Appl. Math. 53 603–10
[13] Constantin A and Molinet L 2000 Global weak solutions for a shallow water wave equation Commun. Math. Phys. 211 45–61
[14] Constantin A and Molinet L 2001 Orbital stability of solitary waves for a shallow water wave equation Physica D 157 75–89
[15] Fisher M and Schiff J 1999 The Camassa–Holm equation: conserved quantities and the initial value problem Phys. Lett. A 259 371–6
[16] Fuchssteiner B and Fokas A S 1981 Symplectic structures, their Bäcklund transformations and hereditary symmetriesPhysica D 4 47–66
[17] Degasperis A and Procesi M 1999 Asymptotic integrability Proc. Symmetry and Perturbation Theory, 1998 (Singapore: World Scientific) pp 23–37
[18] Degasperis A, Holm D D and Hone A N W 2002 A new integrable equation with peakon solutions Theor. Math. Phys. 133 1463–74
[19] Dullin H R, Gottwald G A and Holm D D 2003 Camassa–Holm, Korteweg–de Vries-5 and other asymptotically equivalent shallow water wave equations Fluid Dyn. Res. 33 73–95
[20] Lundmark H and Szmigielski J 2003 Multi-peakon solutions of the Degasperis–Procesi equation Inverse Problems 19 1241–5
[21] Lundmark H and Szmigielski J 2005 Degasperis–Procesi peakons and the discrete cubic string Int. Math. Res. Pop. 2 53–116
[22] Szmigielski J and Zhou L 2013 Peakon–antipeakon interactions in the Degasperis–Procesi equation Algebraic and Geometric Aspects of Integrable Systems and Random Matrices (Contemporary Mathematics vol 593) (Providence, RI: American Mathematical Society) pp 83–107
[23] Lundmark H 2007 Formation and dynamics of shock waves in the Degasperis–Procesi equation J. Nonlinear Sci. 17 169–98
[24] Fokas A 1995 The Korteweg–de Vries equation and beyond Acta Appl. Math. 39 295–305
[25] Olver P and Rosenau P 1996 Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support Phys. Rev. 53 1900–6
[26] Fokas A S, Olver P J and Rosenau P 1997 A plethora of integrable bi-Hamiltonian equations Algebraic Aspects of Integrable Systems (Progress Nonlinear Differential Equations Applications vol 26) (Boston, MA: Brikhauser) pp 93–101
[27] Fuchssteiner B 1996 Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa–Holm equation Physica D 95 229–43
[28] Fokas A S 1995 On a class of physically important integrable equations Physica D 87 145–50
[29] Qiao Z and Li X Q 2011 An integrable equation with non-smooth solitons Theor. Math. Phys. 267 584–9
[30] Schiff J 1996 Zero curvature formulations of dual hierarchies J. Math. Phys. 37 1928–38
[31] Qiao Z 2006 A new integrable equation with cuspons and W/M-shape peaks solitons J. Math. Phys. 47 112701
[32] Gui G, Liu Y, Olver P J and Qu C 2013 Wave-breaking and peakons for a modified Camassa–Holm equation Commun. Math. Phys. 319 731–59
[33] Kang J, Liu X, Olver P J and Qu C 2016 Liouville correspondence between the modified KdV hierarchy and its dual integrable hierarchy J. Nonlinear Sci. 26 141–70
[34] Novikov V S 2009 Generalizations of the Camassa–Holm equation J. Phys. A: Math. Theor. 42 342002
[35] Hone A N W and Wang J P 2008 Integrable peakon equations with cubic nonlinearity J. Phys. A: Math. Theor. 41 372002
[36] Hone A N W, Lundmark H and Szmigielski J 2009 Explicit multipeakon solutions of Novikov’s cubically nonlinear integrable Camassa–Holm type equation Dyn. PDE 6 253–89
[37] Holm D D and Hone A N W 2005 A class of equations with peakon and pulson solutions J. Nonlinear Math. Phys. 12 380–94
[38] Mi Y and Mu C 2013 On the Cauchy problem for the modified Novikov equation with peakon solutions J. Differ. Equ. 254 961–82
[39] Gui G, Liu Y and Tian L 2008 Global existence and blow-up phenomena for the peakon b-family of equations Indiana Univ. Math. J. 57 1209–33
[40] Grayshan G and Himonas A 2013 Equations with peakon traveling wave solutions Adv. Dyn. Syst. Appl. 8 217–32
[41] Anco S C, da Silva P L and Freire I L 2015 A family of wave-breaking equations generalizing the Camassa–Holm and Novikov equations J. Math. Phys. 56 091506
[42] Himonas A and Mantzavinos D 2016 An ab-family of equations with peakon travelling waves Proc. Am. Math. Soc. 144 3797–811
[43] Himonas A and Mantzavinos D 2016 The Cauchy problem for a 4-parameter family of equations with peakon travelling waves Nonlinear Anal. 133 161–99
[44] Anco S C, Recio E, Gandarias M L and Bruzón M S 2015 A nonlinear generalization of the Camassa–Holm equation with peakon solutions Dynamical Systems, Differential Equations and Applications (Proc. 10th AIMS Int. Conf.) (Spain) pp 29–37
[45] Hakkaev S 2006 Stability of peakons for an integrable shallow water equation Phys. Lett. A 354 137–44
[46] Evans L C 1998 Partial Differential Equations (Providence, RI: American Mathematical Society)
[47] Gel’fand I and Shilov G 1964 Generalized Functions (New York: Academic)
[48] Olver P J 1993 Applications of Lie Groups to Differential Equations (New York: Springer)
[49] Anco S C 2017 Generalization of Noether’s theorem in modern form to non-variational partial differential equations Recent Progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science (Fields Institute Communications vol 79) pp 119–82
[50] Chertock A, Liu J-G and Pendleton T 2015 Elastic collisions among peakon solutions for the Camassa–Holm equation Appl. Numer. Math. 93 30–46
[51] Beals R, Sattinger D H and Szmigielski J 2001 Peakon–antipeakon interaction J. Nonlinear Math. Phys. 8 23–7
[52] Chang X and Szmigielski J 2018 Lax integrability and the peakon problem for the modified Camassa–Holm equation Commun. Math. Phys. 358 295–341
[53] Anco S C and Kraus D 2018 Hamiltonian structure of peakons as weak solutions for the modified Camassa–Holm equation Discrete Continuous Dyn. Syst. A 38 4449–65
[54] Anco S C and Recio E 2018 Accelerating dynamical peakons and their behaviour (arXiv:1902.05171 [math-ph])
[55] Anco S C and Recio E 2018 Conserved norms and related conservation laws for multi-peakon equations J. Phys. A: Math. Theor. 51 065203