NEW STRUCTURAL PROPERTIES OF INVENTORY MODELS WITH POLYA FREQUENCY DISTRIBUTED DEMAND AND FIXED SETUP COST

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Abstract. We study a stochastic inventory model with a fixed setup cost and zero order lead time. In a finite-horizon lost sales model, when demand has a Polya frequency distribution ($PF_n$), we show that there are no more than a predetermined number of minima of the cost function. Consequently, depending on the relative cost of lost sales and inventory holding cost, there can be as few as one local minimum. These properties have structural implications for the optimal policies and cost functions. A necessary condition for the results to hold for the backordered model has been explained. We further conduct a numerical study to validate our structural results.

1. Introduction. For a stochastic inventory system with a unit variable ordering cost $c$, a fixed setup cost $K$, and zero order lead time, it is well known in inventory literature that an $(s, S)$ policy is optimal. Under this policy, if the inventory position falls below a threshold $s$, an order is placed to bring the inventory position up to the level $S$. The values of $s$ and $S$ are chosen so that the expected cost for the inventory system is minimized.

The literature on this inventory management problem with fixed setup cost falls into two main streams. The first stream of research focus on providing technical conditions in establishing the optimality of the $(s, S)$ policy. Literature along this line can be traced back to the seminal work of Scarf [18], who introduces the concept of $K$-convexity and proves the optimality of the $(s, S)$ policy for the finite horizon case with backorders. Later, Iglehart [14] extends Scarf’s results [18] to the infinite horizon case with stationary demand and cost parameters, and Shreve [20] derives similar structural results under the lost sales assumption. Using different technical conditions, Veinott [24] and Porteus [16] establish the optimality of the $(s, S)$ policy...

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by showing that the cost function is non-$K$-decreasing; specifically, the later work assumes that the demand has a one-sided Polya density.

Both the ideas of $K$-convexity and non-$K$-decreasing are widely used by many other researchers in showing the optimality of the $(s, S)$ policy for different variants of inventory problems with fixed setup cost. For examples, Sobel and Zhang [22] consider a finite horizon periodic review inventory system with demand arriving simultaneously from a deterministic source and a random source. They prove that, under certain conditions, a modified $(s, S)$ policy is optimal. Later, Chen and Xu [5] relax some technical conditions in Sobel and Zhang [22] while still obtaining the optimality of the $(s, S)$ policy. Chao and Zipkin [4] study an inventory system with a fixed cost when the order quantity exceeds a contracted capacity level. For this system, they partially characterize the optimal policy by three parameters, $s$, $u$ and $S$. Huggins and Olson [13] consider an infinite-horizon periodic-review inventory problem with generalized expediting. For the special case where the expediting cost function is concave, they show that the optimal expediting policy is of generalized $(s, S)$ form.

More recently, Alisaka et al. [1] study a periodic-review inventory system where the retailer can order from one reliable supplier and/or from an unreliable supplier with lower unit purchase cost. For all four possible scenarios, $(s, S)$-type policies characterize the optimal ordering decisions when a fixed setup cost is present. Bijnant et al. [2] consider an inventory system with lost sales and fixed order cost. They propose a modified $(s, S)$ policy with an upper limit on order quantity when it is possible to have multiple orders outstanding due to positive lead time. Numerical analysis shows that such a policy performs close to the optimal. Chen et al. [6] generalize the traditional stochastic inventory models to incorporate uncertainties in discount factors and operational costs. They prove under certain conditions that the state-dependent $(s, S)$ policies are optimal for three stockout protocols.

While $K$-convexity or non-$K$-decreasing are extremely useful in showing the optimality of $(s, S)$ policy, the computation of optimal parameters remains very challenging. As noted in Li and Yu [15], the main reason is that a $K$-convex or non-$K$-decreasing function might have many local minimizers, and it is typically unknown how many it has and where they are. This sometimes hinders the full characterization of the optimal policies. Therefore, in the second stream of research, progresses are made in finding the optimal reorder point $s$ and the optimal order-order-up level $S$. On the computation of $(s, S)$ policies, Iglehart [14] and Veinott [24] have established bounds for the finite horizon case, whereas Veinott and Wagner [25] develop an algorithm to compute the optimal values of $s$ and $S$ for the infinite horizon case. Heuristic methods on $(s, S)$ policies are due to Sivazlian [21], Ehrhardt [8, 9], Freeland and Porteus [12], Tijms and Groenevelt [23], and Bollapragada and Morton [13]. For discrete demand distributions, efficient algorithms are developed by Federgruen and Zipkin [10] and Zheng and Federgruen [28]. Feng and Xiao [11] have provided additional insights into these algorithms.

Recently, Xu [26], and Xu et al. [27] have developed additional structural properties for the lost sales model under the $(s, S)$ policy; specifically, the later derive new bounds on the best reorder level and order-up-to level. Li and Yu [15] provide new insights on the optimal $(s, S)$ policies when there are extra constraints; they show that the objective function is quasi-concave under certain technical conditions including $PF_2$ demand distribution (Polya frequency function of order 2).
Motivated by Li and Yu [15] and other papers in the second stream of research on \((s,S)\) policy, in this paper, we focus on providing bounds for the number of possible local minimums that have to be evaluated to find the optimal value of \(S\). Essentially, the finiteness or bounds for the number of local minimums of the cost function has theoretical and practical merit for developing efficient algorithm to find the optimal order-up-to level \(S\).

We address the finite horizon problems with lost sales as well as backorders. As in Porteus [16] and Li and Yu [15], our analysis focuses on the case when demand is modeled by a Polya frequency distribution (\(PF_n\)), which, as discussed by Porteus [17] have powerful smoothing properties. Specifically, we show that for the lost sales model, when the cost of lost sales is low relative to the holding cost, then the cost function for each period in the resulting dynamic program is unimodal for \(PF_2\) distributions, which is the widest class of this family. As the relative cost of lost sales increases, the number of modes may increase and the results hold for somewhat smaller subsets of this family.

Note that Li and Yu [15] also show that the objective cost function is unimodal under \(PF_2\) demand distribution (condition 1(a) therein); however, our condition on cost parameters are different from Li and Yu [15] in the sense that, our condition on cost parameters does not imply and are not implied by condition 1(c) in Li and Yu [15]. Another difference between our work and Li and Yu [15] lies in that our results hold for both lost sales and backorder models, while Li and Yu [15] focus on the case of lost sales only.

In Section 2, we introduce notation and derive the cost function for the lost sales case. We also introduce new structural properties that facilitate subsequent analysis of the problem. In Section 3, these properties are applied to the finite horizon problem when demand is \(PF_n\); we also conduct a numerical study in subsection 3.1 to validate the structural properties. Analogous results for the backorder case are considered in Section 4. Section 5 concludes the paper.

2. The lost sales case: Problem formulation and preliminary results. We consider the lost sales model over a finite horizon with stationary parameters and zero lead time. In our problem, there are \(T\) periods which, starting with the first period, are chronologically labeled as \(T,T-1,\ldots,1\). The demand in each period is an independently and identically distributed continuous random variable. In formulating the problem, we use the following notation:

| Cost and Model Parameters |
|---------------------------|
| \(K\) = fixed setup cost  |
| \(c\) = unit variable ordering cost |
| \(h\) = unit inventory holding cost |
| \(l\) = unit lost sales cost (\( l > c \)) |
| \(b\) = unit backorder cost |
| \(\alpha\) = discount factor (\(0 < \alpha \leq 1\)) |
| \(T\) = time horizon |

| Demand Information |
|--------------------|
| \(\xi_t\) = the random observation of demand in period \(t\), \(t = 1,2,\ldots,T\) |
| \(f(\cdot)\) = the probability density function (PDF) of demand in each period |
| \(F(\cdot)\) = the cumulative distribution function (CDF) of demand in each period |
Decision Variables
\[ s_t = \text{optimal reorder level in period } t \]
\[ S_t = \text{optimal order-up-to level in period } t \]

Cost Functions
\[ L(\cdot) = \text{one period inventory holding and shortage penalty cost function} \]
\[ V_t(x) = \text{total minimal expected cost from period } t \text{ onwards} \]
\[ (t-1, \ldots, 2, 1), \text{ given that the on-hand inventory at the beginning of period } t \text{ is } x \]
\[ G_t(y) = \text{total expected cost from period } t \text{ onwards} \]
\[ \text{after inventory level is increased to } y \]

Other Useful Functions
\[ \delta(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \end{cases}, \text{ the indicator function for ordering decisions} \]
\[ x^+ = \max\{x, 0\} \]

Since all excess demand is lost, the state variable \( x \) at time 0 must be non-negative. Therefore, to settle accounts at the end of the planning horizon, we assume that leftover inventory is sold at unit value \( w \) (\( w \leq c \)). If \( w \) is positive the inventory is salvaged, and, if \( w \) is negative, disposal costs are incurred. Hence, we can write that:
\[ V_0(x) = -wx \quad (x \geq 0). \quad (1) \]

In order to formulate the problem as a dynamic program, we first define the one period cost, which is time independent because all parameters are stationary. If an order is placed and received to bring the inventory level up to \( y \), the one period cost can be written as:
\[ L(y) = h \int_0^y (y - \xi)f(\xi)d\xi + l \int_y^{+\infty} (\xi - y)f(\xi)d\xi. \quad (2) \]

Since an optimal policy is followed for the remaining periods, the expected discounted cost with \( y \) units of inventory after ordering in period \( t \) is given by:
\[ G_t(y) = cy + L(y) + \alpha E[V_{t-1}(y - \xi)^+], \quad t = 1, 2, \ldots, T, \quad (3) \]

where, the first term on the right-hand-side represents the direct cost (exclusive of \( K \)) of the inventory \( y \); the second term is given by \[2\], and the third term represents the optimal total discounted cost from period \( t - 1 \) to the end of the horizon, given \((y - \xi_t)^+\) is left at the end of period \( t \). Thus,
\[ V_t(x) = -cx + \min\{K\delta(y - x) + G_t(y) : y \geq x\}, \quad (4) \]
represents the minimal total discounted cost from period \( t \) to the last period when the initial on-hand inventory at the beginning of period \( t \) equals \( x \). From Shreve \[20\], we know that both \( G_t(\cdot) \) and \( V_t(\cdot) \) are \( K \)-convex; hence there exist \((s_t, S_t)\) which minimize \( V_t(x) \): if the initial on-hand inventory \( x \leq s_t \), the retailer will order \( S_t - x \); if \( x > s_t \), no order is placed.

To facilitate the analysis of \( G_t(y) \), we define
\[ G_0(y) = cy + L(y) - \alpha c \int_0^y (y - \xi)f(\xi)d\xi, \quad (5) \]
which can be interpreted as a single-period newsvendor model, where all leftovers are fully refunded with purchase cost \( c \). Since \( G_0(\cdot) \) is convex, there exists a unique myopic policy \( S_0 \) at which \( G_0(\cdot) \) attains its local minimum.
Now that we have developed the lost sales model, we are ready to present some key structural properties of the cost functions, which play a critical role in establishing the results of the next section.

**Lemma 2.1.** ([Xu et al. 2010]) In a T-period problem, for period $t = 1, 2, \ldots, T$, we have: $G_t'(y) \geq c - l, \forall y > 0$.

The above lower bound on $G_t'(\cdot)$ will be used to build the connection between the number of local minima and the relationship among cost parameters $c, h$ and $l$. To proceed with our analysis, we will develop the following properties for $G_t'(\cdot)$ and its linear transformation:

**Lemma 2.2.** In a T-period problem, for period $t = 2, \ldots, T$, we have

1) $G_t'(y) = \begin{cases} \int_{-\infty}^{y} H_t(x) \cdot f_t(y-x) \, dx & \text{if } y \geq s_{t-1} \\ G_0'(y) & \text{if } y < s_{t-1}, \end{cases}$

where $H_t(x) = \begin{cases} c - l & \text{for } x \leq 0, \\ h + c(1 - \alpha) & \text{for } 0 < x \leq s_{t-1}, \\ h + c(1 - \alpha) + \alpha G_{t-1}'(x) & \text{for } s_{t-1} < x \leq y, \end{cases}$

and $G_0(y)$ is as defined in [2].

2) For arbitrary nonnegative $b$ and $d$, we have

$b + dG_t'(y) = \begin{cases} \int_{-\infty}^{y} [b + dH_t(x)] \cdot f_t(y-x) \, dx & \text{if } y \geq s_{t-1}; \\ b + dG_0'(y) & \text{if } y < s_{t-1}; \end{cases}$

If $\int_{-\infty}^{y} [b + dH_t(x)] \cdot f_t(y-x) \, dx$ changes sign $2n-1$ times on $[s_{t-1}, +\infty)$, then $b + dG_t'(y)$ changes sign $2n$ times on $[0, +\infty)$;

3) For arbitrary nonnegative $b$ and $d$, we have

$b + dH_t(x) = \begin{cases} d(c - l) + b & \text{for } x \leq 0, \\ d[h + c(1 - \alpha)] + b & \text{for } 0 < x \leq s_{t-1}, \\ d[h + c(1 - \alpha) + \alpha G_{t-1}'(x)] + b & \text{for } s_{t-1} < x \leq y; \end{cases}$

If $d[h + c(1 - \alpha) + \alpha G_{t-1}'(x)] + b$ changes sign $n$ times on $(s_{t-1}, y)$, then $b + dH_t(x)$ changes sign at most $n+2$ times on $(-\infty, y)$.

**Proof.** Before proving the results in Lemma 2.2, we derive a few relationships that will facilitate the proofs. First, note that

$G_1(y) = cy + L(y) + \alpha E \left[ V_0 \left( (y - \xi)^+ \right) \right] = cy + L(y) - \alpha w \int_{0}^{y} (y - \xi) f(\xi) d\xi, \quad (6)$

which is the cost function of a single-period newsvendor model, hence is convex. Second, from the properties of $K$-convexity, we know

$V_{m+1}(x) = \begin{cases} -cx + G_{m+1}(S_{m+1}) + K & \text{if } 0 \leq x \leq s_{m+1}, \\ -cx + G_{m+1}(x) & \text{if } x > s_{m+1}, \end{cases}$

for $m = 0, 1, \ldots, T - 1$.

Moreover, if $0 \leq y \leq s_m$,

$G_{m+1}(y) = cy + L(y) + \alpha E \left[ V_m \left( (y - \xi)^+ \right) \right] = cy + L(y) + \alpha \left[ K + G_m(S_m) - c \int_{0}^{y} (y - \xi) f(\xi) d\xi \right] = G_0(y) + \alpha [K + G_m(S_m)]. \quad (7)$

(7) follows from the fact that: if $y \leq s_m$, then $(y - \xi)^+ \leq s_m$; consequently, an order of $S_m - (y - \xi)^+$ would be placed in period $m$.
And, if $y \geq s_m$ then

$$G_{m+1}(y) = cy + L(y) + \alpha E[V_m ((y - \xi)^+)]$$

$$= cy + L(y) + \alpha \int_0^{y-s_m} [G_m(y - \xi) - c(y - \xi)] f(\xi)d\xi$$

$$+ \alpha \int_y^{y-s_m} f(\xi)d\xi - c \int_{y-s_m}^y (y - \xi)f(\xi)d\xi$$

$$+ \alpha \int_y^{+\infty} f(\xi)d\xi$$

$$= G_0(y) + \alpha \int_0^{y-s_m} G_m(y - \xi)f(\xi)d\xi$$

$$+ \alpha (K + G_m(S_m))(1 - F(y - s_m)).$$

(8) is derived from three contingencies: 1) if $s_m \leq (y - \xi)$, then no order will be placed in period $m$; 2) if $0 < (y - \xi) < s_m$, an order of $S_m - (y - \xi)$ will be placed in period $m$; and 3) if $(y - \xi) \leq 0$, or $(y - \xi)^+ = 0$, an order of $S_m$ will be placed in period $m$). Subsequently, taking derivatives of (7) and (8), we get

$$G'_{m+1}(y) = G'_0(y), \quad \text{if } 0 \leq y \leq s_m, \quad (9)$$

$$G'_{m+1}(y) = G'_0(y) + \alpha \int_0^{y-s_m} G'_m(y - \xi)f(\xi)d\xi, \quad \text{if } y \geq s_m. \quad (10)$$

Now we will prove the results in Lemma 2.2

1) For $y \geq s_{t-1}$, for period $t, t = 2, 3, \ldots, T$, from (10) we have

$$G'_t(y) = (c - l) \int_0^{+\infty} f(\xi)d\xi + (h + l - \alpha c) \int_y^{+\infty} f(\xi)d\xi$$

$$+ \alpha \int_0^{y-s_{t-1}} [G'_{t-1}(y - \xi)]f(\xi)d\xi$$

$$= (c - l) \int_{-\infty}^y f(y - x)dx + (h + l - \alpha c) \int_y^{+\infty} f(y - x)dx$$

$$+ \alpha \int_y^{y-s_{t-1}} [G'_{t-1}(x)]f(y - x)dx$$

$$= (c - l) \int_{-\infty}^0 f(y - x)dx + (h + c(1 - \alpha) + \alpha G'_{t-1}(x)]f(y - x)dx$$

$$+ \int_{s_{t-1}}^y [h + c(1 - \alpha) + \alpha G'_{t-1}(x)]f(y - x)dx$$

$$= \int_{-\infty}^y H_t(x) \cdot f(y - x)dx,$$

where

$$H_t(x) = \begin{cases} 
  c - l & \text{for } x \leq 0 \\
  h + c(1 - \alpha) & \text{for } 0 < x \leq s_{t-1} \\
  h + c(1 - \alpha) + \alpha G'_{t-1}(x) & \text{for } s_{t-1} < x \leq y.
\end{cases}$$

And, for $y < s_{t-1}$, from (9) and (10) we have $G'_t(y) = G'_0(y) = c + L'(y) - \alpha F(y)$. 
2) First part follows immediately from Lemma 2.2.1. For the second part, since $2n - 1$ is an odd number, if $b + dG'_\ell(y) = \int_0^y \frac{dH_t(x)}{c} \cdot f_t(y - x) dx$ changes sign $2n - 1$ times on $[s_{t-1}, +\infty)$, we must have $b + dG'_\ell(s_{t-1}) < 0$, because $b + dG'_\ell(\infty) = b + d[c - \alpha^t w + h(1 + \alpha + \ldots + \alpha^{t-1})] > 0$; while for $y < s_{t-1}$, we have $b + dG'_\ell(y) = b + dG'_0(y)$ which is monotone increasing since $G_0(\cdot)$ is convex; thus $b + dG'_\ell(s_{t-1}) < 0$ implies that $b + dG'_\ell(y) < 0$ in $[0, s_{t-1})$, hence does not change sign on $[0, s_{t-1})$. Therefore, $b + dG'_\ell(y)$ changes sign $2n - 1$ times on $[0, +\infty)$.

3) First part follows from Lemma 2.2.1. For the second part, $b + dH_t(x)$ is a piecewise function defined on three disjoint intervals: $(-\infty, 0], (0, s_{t-1})$ and $[s_{t-1}, y)$. $d(c - l) + b$ and $d[h + c(l - \alpha)] + b$ are two constant functions on $(-\infty, 0]$ and $(0, s_{t-1})$ respectively, which means that $b + dH_t(x)$ does not change sign on each of $(-\infty, 0)$ and $(0, s_{t-1})$, but could change sign at two segmentation points $x = 0$, and $x = s_{t-1}$; hence $b + dH_t(x)$ could change sign at most twice on $(-\infty, s_{t-1})$. Given that $d[h + c(l - \alpha)] + \alpha G'_{t-1}(x) + b$ changes sign $n$ times on $(s_{t-1}, y)$, we conclude that $b + dH_t(x)$ could change sign at most $n + 2$ times on $(-\infty, y)$.

As will be seen next, to characterize the cost function when demand has a Polya frequency distribution ($PF_n$), it will be sufficient to focus on the structure of $H_t(x)$.

3. The lost sales model with $PF_n$ demand. While the theory of $K$-convexity facilitates establishing the optimality and structural properties of $(s, S)$ policies, it does not resolve the question about how many local minima have to be evaluated to find the optimal $S$. In a $K$-convex cost function, potentially an infinite number of local minima can arise, setting a challenge to the development of efficient or even feasible computation schemes. However, as we will show in this section, significant simplification is possible when demand has a Polya frequency distribution ($PF_n$).

Such a distribution is defined as:

**Lemma 3.1.** (Schoenberg [19]) A Polya distribution consists solely of translations of exponential distributions and convolutions of exponentials, reflected exponentials, and normals. A Polya random variable consists of the sum of a translated normal, exponentials, and reflected exponentials. A positive Polya random variable consists of a positive translation of the sum of exponentials.

Thus, by Lemma 3.1 $PF_n$ distributions include exponential, reflected exponential, uniform, Erlang, normal, truncated normal, and all translations and convolutions of such distributions (Porteus [17]). And the variation diminishing property of distributions is as follows:

**Lemma 3.2.** (Schoenberg [19]) If $f : R \to R$ changes sign $j < n$ times, $X$ is a $PF_n$ random variable, and $g(x) := Ef(x - X)$, then $g$ also changes sign at most $j$ times. If $g$ changes sign exactly $j$ times, then the changes must occur in exactly the same order as $f$.

As discussed by Porteus [17], $PF_n$ distributions have properties that smooth out the cost function under the expectation operator. Thus, it may significantly reduce fluctuations in cost functions so that there are fewer local optima in $g$ than in $f$. To understand how this is operationalized, in the following proposition we consider the $H_t(x)$ and $G'_\ell(y)$ functions which are defined in the previous section.

**Proposition 1.** If the density function of period $t$ is $PF_{2t}$, $t = 2, \ldots, T$, then both $b + dH_t(x)$ and $b + dG'_\ell(y)$ could change sign at most $2t - 1$ times on their domains.
Proof. (By induction) For $t = 2$, from Lemma 2.2.3, for $y \geq s_1$ we have

\[ b + dH_2(x) = \begin{cases} 
  d(c - l) + b & \text{for } x \leq 0, \\
  d[h + c(1 - \alpha)] + b & \text{for } 0 < x \leq s_1, \\
  d[h + c(1 - \alpha) + \alpha G'_1(x)] + b & \text{for } s_1 < x \leq y.
\end{cases} \]

Since $G_1(x)$ is convex, $d[h + c(1 - \alpha) + \alpha G'_1(x)] + b$ is non-decreasing in $x$. Hence, it could change sign at most once on $(s_1, y)$. Therefore, by Lemma 2.2.3, $b + dH_2(x)$ could change sign at most three times on $(-\infty, y)$.

Next, by Lemma 2.2.2,

\[ b + dG'_{m+1}(y) = \begin{cases} 
  \int_{-\infty}^{y} [b + dH_{m+1}(x)] \cdot f_{m+1}(y-x) dx & \text{if } y \geq s_m, \\
  b + dG'_0(y) & \text{if } y < s_m.
\end{cases} \]

From the above analysis, we know that $b + dH_2(x)$ could change sign at most three times. Hence, if $f_2(\cdot)$ is $PF_4$, by Lemma 3.2 we conclude that $b + dG'_2(y)$ could change sign at most three times on $(s_1, +\infty)$. Therefore, by Lemma 2.2.2, $b + dG'_2(y)$ could change sign at most three times on $[0, +\infty)$.

Assume this is true for $t = m$ ($m > 2$), namely, both $b + dH_m(x)$ and $b + dG'_m(y)$ could change sign at most $2m - 1$ times when the density function of period $m$ is $PF_{2m}$ for arbitrary nonnegative $b$ and $d$ (the induction hypothesis). For $t = m + 1$, by Lemma 2.2.3, for $y \geq s_m$ we have

\[ b + dH_{m+1}(x) = \begin{cases} 
  d(c - l) + b & \text{for } x \leq 0, \\
  d[h + c(1 - \alpha)] + b & \text{for } 0 < x \leq s_m, \\
  d[h + c(1 - \alpha) + \alpha G'_m(x)] + b & \text{for } s_m < x \leq y.
\end{cases} \]

Given that $d[h + c(1 - \alpha) + \alpha G'_m(x)] + b$ changes sign at most $2m - 1$ times (by the induction hypothesis), by Lemma 2.2.3, we know that $b + dH_{m+1}(x)$ changes sign at most $2m + 1$ times.

Now, by Lemma 2.2.2, we have

\[ b + dG'_{m+1}(y) = \begin{cases} 
  \int_{-\infty}^{y} [b + dH_{m+1}(x)] \cdot f_{m+1}(y-x) dx & \text{if } y \geq s_m, \\
  b + dG'_0(y) & \text{if } y < s_m.
\end{cases} \]

From the analysis above, $b + dH_{m+1}(x)$ changes sign at most $2m + 1$ times. Hence, if $f_{m+1}(\cdot)$ is $PF_{2m+2}$, by Lemma 3.2 we conclude that $b + dG'_{m+1}(y)$ could change sign at most $2m + 1$ times on $(s_m, +\infty)$. Therefore, by Lemma 2.2.2, $b + dG'_{m+1}(y)$ could change sign at most $2m + 1 = 2(m + 1) - 1$ times on $[0, +\infty)$. This completes the induction proof.

Since $c - l$ is negative and $h + c(1 - \alpha) > 0$, $H_t(x)$ changes sign exactly once when $x \leq s_{i-1}$. Also from Lemma 2.1 we know that $G'_{i-1} \geq c - l$, for any $y > 0$, so that we have $h + c(1 - \alpha) + \alpha G'_{i-1}(x) \geq h + c(1 - \alpha) + \alpha(c - l)$. Thus, if $l \leq c + \frac{h + c(1 - \alpha)}{\alpha}$, $H_t(x)$ has exactly one sign change. Thus, it follows from Lemma 3.2 that in this case $G'_t(y)$ changes sign exactly once when demand is $PF_2$, that is, $G_t(y)$ is unimodal when demand is $PF_2$. Now that we have presented an intuitive explanation of our results we formally state:

**Theorem 3.3.** In a $T$-period problem, for period $t, t = 2, \ldots, T$, we have

1) If $l \leq c + \frac{h + c(1 - \alpha)}{\alpha}$, and $f(\cdot)$ is $PF_2$, then $G_t(y)$ is unimodal;
2) If $c + [h + c(1 - \alpha)] \sum_{i=1}^{m-1} \alpha^{-i} \leq l \leq c + [h + c(1 - \alpha)] \sum_{i=1}^{m} \alpha^{-i} (1 \leq m < t)$, $f_{t-j}(\cdot)$ is $PF_{2(m-j)} (0 \leq j \leq m)$, then $G'_t(y)$ could change sign at most $(2m - 1)$ times; and,
3) If the density function of period $t$ is $PF_{2t}$, then $G_t'(y)$ could change sign at most $(2t - 1)$ times, with $t$ local minima and $(t - 1)$ local maxima.

Proof. 1) The proof is as given above.

2) By Lemma 2.2.1, we have

$$G_t'(y) = \begin{cases} \int_{-\infty}^{y} H_t(x) \cdot f_t(y - x) \, dx & \text{if } y \geq s_{t-1} \\ G_0'(y) & \text{if } y < s_{t-1}, \end{cases}$$

where

$$H_t(x) = \begin{cases} c - l & \text{for } x \leq 0 \\ h + c(1 - \alpha) & \text{for } 0 < x \leq s_{t-1} \\ h + c(1 - \alpha) + \alpha G_{t-1}'(x) & \text{for } s_{t-1} < x \leq y. \end{cases}$$

By Lemmas 2.2.2 and 2.2.3, we have

$$h + c(1 - \alpha) + \alpha G_{t-1}'(y)$$

$$= \begin{cases} \int_{-\infty}^{y} \left[ [h + c(1 - \alpha)](1 + \alpha) + \alpha^2 H_{t-2}(y) \right] f_{t-2}(y - x) \, dx & \text{if } y \geq s_{t-3}, \\ [h + c(1 - \alpha)](1 + \alpha) + \alpha^2 G_0'(y) & \text{if } y < s_{t-3}, \end{cases}$$

where

$$[h + c(1 - \alpha)](1 + \alpha) + \alpha^2 G_0'(y)$$

$$= \begin{cases} \alpha^2(c - l) + [h + c(1 - \alpha)](1 + \alpha) & \text{for } x \leq 0, \\ [h + c(1 - \alpha)](1 + \alpha) & \text{for } 0 < x \leq s_{t-3}, \\ [h + c(1 - \alpha)](1 + \alpha + \alpha^2) + \alpha^3 G_{t-3}'(x) & \text{for } s_{t-3} < x \leq y. \end{cases}$$

Repeating this iteration for $m - 1$ ($m > 1$) steps, we will get

$$[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^{m-1} G_{t-m+1}'(y)$$

$$= \begin{cases} \int_{-\infty}^{y} \left[ [h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^{m-1} H_{t-m+1}(x) \right] f_{t-m+1}(y - x) \, dx & \text{if } y \geq s_{t-m}, \\ [h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^{m-1} G_0'(y) & \text{if } y < s_{t-m}, \end{cases}$$

where

$$[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^{m-1} H_{t-m+1}(x)$$

$$= \begin{cases} \alpha^2(c - l) + [h + c(1 - \alpha)](1 + \alpha) & \text{for } x \leq 0, \\ [h + c(1 - \alpha)](1 + \alpha) & \text{for } 0 < x \leq s_{t-3}, \\ [h + c(1 - \alpha)](1 + \alpha + \alpha^2) + \alpha^3 G_{t-3}'(x) & \text{for } s_{t-3} < x \leq y. \end{cases}$$
\[
\begin{align*}
\alpha^{m-1}(c - l) + [h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i & \quad \text{for } x \leq 0, \\
[h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i & \quad \text{for } 0 < x \leq s_{t - m}, \\
[h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m G_{t - m}'(x) & \quad \text{for } s_{t - m} < x \leq y.
\end{align*}
\]

By Lemma 2.1

\[
[h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m G_{t - m}'(x) \geq [h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m(c - l).
\]

Now, if \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m(c - l) \geq 0\), or \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m(c - l) \leq 0\), then we have \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m G_{t - m}'(x) \geq 0\). If \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m G_{t - m}'(x) \leq 0\), then \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m G_{t - m}'(x)\) does not change sign on \((s_{t - m}, y)\); moreover, \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i > 0\), which implies that \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m H_{t - m + 1}(x)\) does not change sign on \((0, y)\). However, if \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i < 0\), or \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i \leq 0\), we can conclude that \([h + c(1 - \alpha)] \sum_{i=0}^{m-1} \alpha^i + \alpha^m H_{t - m + 1}(x)\) changes sign exactly once (at \(x = 0\)) on \((-\infty, y)\).

Thus, for

\[
[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m H_{t - m + 1}(x)\]

\[
= \int_{-\infty}^{y} \left\{ [h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m H_{t - m + 1}(x) \right\} f_{t - m + 1}(y - x) dx
\]

\[
\text{if } y \geq s_{t - m},
\]

\[
[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m G_{t - m}'(y)
\]

\[
\text{if } y < s_{t - m};
\]

if \(f_{t - m + 1}(\cdot)\) is \(PF_2\), then by Lemma 3.2

\[
[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m G_{t - m}'(y) = \int_{-\infty}^{y} \left\{ [h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m H_{t - m + 1}(x) \right\} f_{t - m + 1}(y - x) dx
\]

changes sign only once on \((s_{t - m}, +\infty)\). Hence, by Lemma 2.2, for \([h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m G_{t - m}'(y)\) changes sign only once on \([0, +\infty)\) if \(c + [h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m G_{t - m}'(y) < 0\).

Next we go backward to

\[
[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^m G_{t - m + 2}(y)
\]

\[
= \int_{-\infty}^{y} \left\{ [h + c(1 - \alpha)] \sum_{i=0}^{m-3} \alpha^i + \alpha^m H_{t - m + 2}(x) \right\} f_{t - m + 2}(y - x) dx
\]

\[
\text{if } y \geq s_{t - m + 1},
\]

\[
[h + c(1 - \alpha)] \sum_{i=0}^{m-3} \alpha^i + \alpha^m G_{0}(y)
\]

\[
\text{if } y < s_{t - m + 1};
\]
where
\[
[h + c(1 - \alpha)] \sum_{i=0}^{m-3} \alpha^i + \alpha^{m-2}H_{t-m+2}(x)
\]

\[
= \begin{cases} 
\alpha^{m-2}(c - l) + [h + c(1 - \alpha)] \sum_{i=0}^{m-3} \alpha^i & \text{for } x \leq 0, \\
[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i & \text{for } 0 < x \leq s_{t-m}, \\
[h + c(1 - \alpha)] \sum_{i=0}^{m-2} \alpha^i + \alpha^{m-1}G'_{t-m+1}(x) & \text{for } s_{t-m} < x \leq y.
\end{cases}
\]

By Lemma 2.2.3 and the above analysis, \([h + c(1 - \alpha)] \sum_{i=0}^{m-3} \alpha^i + \alpha^{m-2}H_{t-m+2}(x)\) changes sign at most three times on \((-\infty, y)\); and if \(f_{t-m+2}(\cdot)\) is \(PF_4\), then by Lemma 3.2 \([h + c(1 - \alpha)] \sum_{i=0}^{m-3} \alpha^i + \alpha^{m-2}G'_{t-m+2}(y)\) could change sign at most three times on \((s_{t-m+1}, +\infty)\). Therefore, by Lemma 2.2.2, \([h + c(1 - \alpha)] \sum_{i=0}^{m-3} \alpha^i + \alpha^{m-2}G'_{t-m+2}(y)\) could change sign at most three times on \([0, +\infty)\). If we go backward further, with similar argument as above, we can show that \([h + c(1 - \alpha)] \sum_{i=0}^{m-4} \alpha^i + \alpha^{m-3}H_{t-m+3}(x)\) changes sign at most five times on \((-\infty, y)\); and given that \(f_{t-m+1}(\cdot)\) is \(PF_6\), \([h + c(1 - \alpha)] \sum_{i=0}^{m-4} \alpha^i + \alpha^{m-3}G'_{t-m+3}(y)\) changes sign at most five times on \([0, +\infty)\). And repeat this argument till we reach period \(t - 1\),

\[
h + c(1 - \alpha) + \alpha G'_{t-1}(y) = \begin{cases} 
f_{y}^{-\infty} \{[h + c(1 - \alpha)] + \alpha H_{t-1}(x)\} f_{t-1}(y - x) \, dx & \text{if } y \geq s_{t-2}, \\
h + c(1 - \alpha) + \alpha G''_{t}(y) & \text{if } y < s_{t-2};
\end{cases}
\]

where

\[
[h + c(1 - \alpha)] + \alpha H_{t-1}(x) = \begin{cases} 
\alpha(c - l) + [h + c(1 - \alpha)] & \text{for } x \leq 0, \\
[h + c(1 - \alpha)] (1 + \alpha) & \text{for } 0 < x \leq s_{t-2}, \\
[h + c(1 - \alpha)] (1 + \alpha) + \alpha^2 G'_{t-2}(x) & \text{for } s_{t-2} < x \leq y.
\end{cases}
\]

By Lemma 2.2.3, we know \([h + c(1 - \alpha)] + \alpha H_{t-1}(x)\) could change sign at most \(2m - 3\) times. If \(f_{t-1}(\cdot)\) is \(PF_{2m-2}\), then by Lemma 2.2.2 and Lemma 7.2 we can conclude that \(h + c(1 - \alpha) + \alpha G'_{t-1}(y)\) changes sign at most \(2m - 3\) times. Finally, in period \(t\), with similar arguments as above, \(G'_{t}(y)\) changes sign at most \(2m - 1\) times if \(f_{t}(\cdot)\) is \(PF_{2m}\). Altogether, we can conclude that if \(c + [h + c(1 - \alpha)] (\alpha^{-1} + \alpha^{-2} + \cdots + \alpha^{-m+1}) < l \leq c + [h + c(1 - \alpha)] (\alpha^{-1} + \alpha^{-2} + \cdots + \alpha^{-m})\), and \(f_{t-j}(\cdot)\) is \(PF_{2(m-j)}\) \((0 \leq j \leq m)\), then \(G'_{t}(y)\) could change sign at most \(2m - 1\) times.

3) Theorem 3.3.3 is immediately obtained by choosing \(b = 0\) and \(d = 1\) in Proposition 1.

As stated above, Theorem 3.3.1 shows that when the unit lost sales cost \(l\) is not quite high compared with the unit holding cost \(h\), then unimodality of the cost function holds under the most general class of \(PF_2\) demand distributions; as \(l\) increases, from Theorem 3.3.2 we see that the number of minima may increase slowly, concomitantly with a reduction in the set of admissible members.

Theorem 3.3 has an attractive interpretation that is best seen by focusing on Theorem 3.3.2, for which the condition is equivalent to \(\alpha^{m+1}l \leq c + h \sum_{q=0}^{m} \alpha^q\) and \(\alpha^m l > c + h \sum_{q=0}^{m-1} \alpha^q\). For the first inequality, the right-hand-side represents the discounted cost of purchasing one unit and holding it in inventory to save the
discounted lost sales cost of period $m+1$ in the future, which is the left-hand-side, and the second inequality has similar interpretation. This indicates that if the discount rates are relatively low, $m$ would tend to be small, so is the number of sign change of $G'_t(y)$; also this property guarantees that the number of sign change of $G'_t(y)$ will be bounded as shown by Theorem 3.3.3.

3.1. **Numerical studies.** To examine how sensitive Theorem 3.3 is with respect to $t$, we conduct a numerical study. In particular, we consider the case of Erlang demand distributions which are a member of $PF_\infty$. In general, a random variable $X$ has an Erlang-$r$ distribution, when its probability density function $f(x) = \frac{\lambda^r x^{r-1}}{(r-1)!} e^{-\lambda x}, \lambda > 0$, $r$ positive integer and the cumulative distribution function $F(x) = 1 - \sum_{j=0}^{r-1} \frac{(\lambda x)^j}{j!}$. In the five-period numerical examples presented in Tables 1 and 2, we choose $\lambda = 1$ and $r = 2$ for the parameters of the demand distribution, $c = 1$, $h = 0.1$ and $w = 0.5$ for the cost parameters and $\alpha = 0.9$ for the discount factor. In Table 1, the cost of a lost sales ($l = 2$) is low while it is relatively high in Table 2 ($l = 10$). We work recursively to find the global optimum for each period. In all the cases considered, there is exactly one local minimum, which suggests that the bounds in Theorems 3.3.2 and 3.3.3 are not tight. In addition, notice that no additional monotonicity properties can be discerned as the planning horizon and/or setup cost increases. And there is no discernible pattern in the observed optimal policies since the cost functions are $K$-convex.

| Table 1. Optimal Solutions for the Case with Unit Lost Sales Cost $l = 2$ |
|---|---|---|---|
| $K$ | $t$ | Optimal Reorder Point ($s_t$) | Optimal Order-up-to Level ($S_t$) | $\Delta_t = S_t - s_t$ | Optimal Cost $G_t(S_t)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1.401973349 | 2.045088007 | 0.643114721 | 2.892765731 |
| 2 | 2.279867486 | 3.144352495 | 0.845991005 | 5.161645538 |
| 3 | 2.29361490 | 4.535000000 | 1.155138510 | 7.213139600 |
| 4 | 2.28463504 | 3.447230000 | 1.162594496 | 9.06221220 |
| 5 | 2.287417275 | 3.449250000 | 1.161832725 | 10.725835800 |
| 0.1 | 1 | 0.666145602 | 2.045088007 | 1.478942404 | 2.892765731 |
| 2 | 1.63133899 | 3.43334720 | 1.803200821 | 5.280907719 |
| 3 | 1.589211420 | 4.215151800 | 2.632846666 | 7.499670690 |
| 4 | 1.522671338 | 4.399540000 | 2.876868662 | 9.526311140 |
| 5 | 1.541934791 | 4.351150000 | 2.809215209 | 11.341494260 |
| 0.5 | 1 | 0.107558637 | 2.045088007 | 1.937529370 | 2.892765731 |
| 2 | 1.245066678 | 3.623702424 | 2.378635746 | 5.361056167 |
| 3 | 1.242995200 | 4.699700000 | 3.456704800 | 7.499670690 |
| 4 | 1.118285145 | 5.263110000 | 4.144824855 | 9.873385400 |
| 5 | 1.109158173 | 5.243850000 | 4.13491827 | 11.838495150 |

Our primary objective of these numerical studies is to examine how the number of modes grows with problem size. To this end, we choose the parameters in such a way that Theorem 3.3.3 applies. Thus, there can be up to $m$ minima in period $m$ of a $T$-period problem. Yet, in all problems examined there is exactly one minima/mode in each sub-period of the planning horizon. This remarkable finding suggests that Theorem 3.3 may not be as loose as stated.
Table 2. Optimal Solutions for the Case with Unit Lost Sales

| K | Cost | Reorder Point (s_t) | Optimal Order-up-to Level (S_t) | ∆_t = S_t - s_t | Optimal Cost | G_t(S_t) |
|---|---|---|---|---|---|---|
| 1 | 0.1 | 3.810323346 | 4.380527192 | 0.570203846 | 4.318148680 | 1.477580090 |
| 2 | 4.797805509 | 5.568997000 | 0.771191491 | 6.934754800 |
| 3 | 4.844958018 | 5.924312000 | 1.079353982 | 11.416826560 |
| 4 | 4.848185556 | 5.926906000 | 1.078720444 | 13.328890410 |
| 5 | 4.817885556 | 5.922066000 | 1.078720444 | 14.318148680 |

4. The backorder case. In the previous section, we assumed that the unmet demand is lost; if instead, unsatisfied demand is backordered, while the bulk of the results continue to apply, some technical changes must be made. Now the mechanism for settling accounts at the end of the planning horizon is modified to:

\[ V_0(x) = \begin{cases} -wx & x \geq 0, \\ -cx & x < 0. \end{cases} \]  

(11)

Here, we have made the standard assumption that all backorders are cleared at the unit cost c without incurring setup costs, and w is the salvage value for any leftovers. Now the one period cost becomes:

\[ L(y) = h \int_0^y (y - \xi) f(\xi) d\xi + b \int_y^{+\infty} (\xi - y) f(\xi) d\xi. \]  

(12)

Here the second integral now represents the expected backorders. Consequently,

\[ \begin{cases} G_t(y) = cy + L(y) + \alpha E[V_{t-1}(y - \xi_t)], \\ V_t(x) = -cx + \min\{Kd(y - x) + G_t(y) : y \geq x \}. \end{cases} \]  

(13)

With (12) and (13), we solve S_t from

\[ G'_t(y) = 0 \Rightarrow c(1 - \alpha) - b + (h + b + ac - \alpha w) F(y) = 0 \Rightarrow S_t = F^{-1} \left[ \frac{b - c(1 - \alpha)}{h + b + ac - \alpha w} \right]. \]

Here, we must assume that b > (1 - \alpha)c; otherwise, it would be less costly to incur the shortage penalty cost b plus the discounted terminal cost ac than to place an order, making it optimal not to order anything.
Then, as in the lost sales model, we define the one-period cost function \( G_0(y) \) as

\[
G_0(y) = cy + L(y) - \alpha cE[y - \xi]
\]  

(14)

Clearly, \( G_0(y) \) is convex. This cost function can be explained by noting that any unsatisfied demand would be satisfied with an expedited purchase cost \( c \) (without the fixed setup cost \( K \)), and any leftovers would be sold with purchase cost \( c \).

With these modifications, we get the analog of Theorem 3.3 but with the proviso that Lemma 2.1 only holds if it is assumed that \( s_1 \geq 0 \) or just \( s_1 \geq 0 \). Hence, this qualification is also needed for subsequent results regarding \( PF_n \) distributed demands. The modification required is that \( l \) be replaced by \( b + \alpha c \) in Lemma 2.1, Theorems 3.3.1 and 3.3.2, which is the cost of backlogging one unit and fulfilling it in the next period, while in the lost sales case, the retailer will incur unit lost sales cost \( l \) only.

5. **Conclusion.** In this paper we have studied the inventory problem for which \((s,S)\) policies are optimal. We considered the finite horizon version of the lost sales and backorder models. New structural results are established for the optimal policies and the cost functions. We have shown that the specialization of the cost function to \( PF_n \) distributions can lead to finding upper bounds on the number of local minima.

Although it is an important question to investigate whether the number of local minima that need to be evaluated to find the optimal \( S \) is finite (or bounded) in an \((s,S)\) inventory system, it's not been addressed in the existing inventory theory. In the literature, Schoenberg [19] has addressed preservation of number of sign changes and Porteus [16, 17] has addressed preservation of quasi-\( K \)-convexity, for the expected cost function under translation by a Polya random variable. However, their results do not answer the question of how many local minima need to be evaluated to find the optimal policy. This is what we studied in this paper. Our result implies that in a finite horizon problem, for \( PF_n \) distributed demands, due to the boundedness of the number of local minima, it is possible to find the global optimum for each period by implementing a search procedure. Our numerical studies indicate that the optimal search could be very efficient since the cost functions are unimodal for a wide range of cost parameters.

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