Abstract

It is shown that the Lorentz group plays prominent roles in at least two areas in condensed matter physics, namely in the Bogoliubov transformation and optical filters. It is pointed out that the underlying symmetry of the Bogoliubov transformation is that of two coupled oscillators, and that the underlying symmetry of two coupled oscillators in that of the group $O(3,3)$. The Lorentz group is also shown to be the underlying symmetry group for the Jones matrix formalism which is standard language for optical filters.

1 Introduction

The Lorentz group is definitely the language of Lorentz transformations,[1, 2] and is therefore an indispensable tool in high-energy particle physics which deals with particles moving with speed very close to the speed of light. I have built up my background in the Lorentz group because I am a particle physicist. However, this group provides the underlying mathematics in many other branches of physics. Quantum optics is a case in point. In connection with two-photon coherent states, our colleagues in optics produced a new scientific word “squeezed states” of light.[3, 4, 5]

The Lorentz group is useful for squeezed states because Lorentz boosts are squeeze transformations.[6, 7] Indeed, there are many other squeeze transformations in physics, and the Lorentz group serves as the hidden symmetry group in many branches of physics. For instance, linear canonical transformations in both quantum and classical mechanics contain the symmetry of the Lorentz group.[8] While there are many other possibilities, we should not overlook condensed matter physics.

In this report, we shall discuss applications of the Lorentz group in two distinct areas of condensed matter physics. The first area is the Bogoliubov transformation in superconductivity, and the other area is the Jones matrix formalism governing symmetries of optical filters. It is well known that the Bogoliubov transformation is a problem of two coupled oscillators. It is also known that the transformation is like a Lorentz
boost. What is not well known is that the coupled oscillator problem is a problem in the Lorentz group.

We can study much of crystal structures from optical properties of matter. Often we use them to construct optical filters. The standard mathematical tool for the filters is called the Jones matrix formalism. However, what is unknown is the fact that the Jones matrix formalism is a representation of the Lorentz group. In this note, we would like to point out that the bilinear representation of the six-parameter Lorentz group is the natural language for polarization of light waves.

In Sec. 2, we discuss the symmetry of two coupled oscillators, which forms the basis for the Bogoliubov transformation. It is pointed out that the symmetry is as rich as that of the Lorentz group with three space-like dimensions and three time-like directions. In Sec. 3, it is shown that the two-by-two matrix formalism of the Lorentz group is the natural language for optical filters. This formalism reproduces the traditional mathematical device called the Jones matrix formalism.

2 Construction of the O(3,3) Symmetry Group from Two Coupled Oscillators

The Bogoliubov transformation is a well-established language in condensed matter physics, and it does not seem to appear necessary to explain what how this transformation works in the theory of superconductivity. On the other hand, it is interesting to note that the transformation is basically the problem of diagonalizing a system of two coupled oscillators. Then we can raise the following question.

The coupled-oscillator problem is discussed in Goldstein’s textbook on classical mechanics whose second edition was published in 1980. Most of us have an impression that we can solve the coupled oscillator problem using a rotation matrix, even though Goldstein states in his book that the rotation alone is not enough. It has been shown recently that Han et al. that it requires the O(3,3) symmetry in order to understand fully the coupled oscillator problem. This section is based on the work of Han et al.

We assume here that the reader is familiar with the basic fact that one isolated oscillator has an Sp(2) symmetry in phase space. Thus, we should start with two independent Sp(2) symmetries. It will be shown that an attempt to couple them by introducing a rotation matrix leads to the Sp(4) symmetry which Dirac discovered in 1963. However, the Sp(4) symmetry deals only with canonical transformations. If the size of the phase space is enlarged or contracted, the transformation is not canonical. In this case, we need a large symmetry, and this symmetry group is locally isomorphic to O(3,3) or the Lorentz group with three space-like and three time-like directions.

Let us consider a system of two coupled harmonic oscillators. The Hamiltonian for
this system is

\[ H = \frac{1}{2} \left\{ \frac{1}{m_1} p_1^2 + \frac{1}{m_2} p_2^2 + A x_1^2 + B x_2^2 + C x_1 x_2 \right\}. \]

(1)

where

\[ A > 0, \quad B > 0, \quad 4AB - C^2 > 0. \]

(2)

We are interested in transformations which will uncouple the above Hamiltonian, or conversely those which will bring to the above coupled form from two uncoupled one-oscillator Hamiltonian. For the two uncoupled oscillators, we can start with the coordinate system:

\[ (\eta_1, \eta_2, \eta_3, \eta_4) = (x_1, p_1, x_2, p_2). \]

(3)

This coordinate system is different from the traditional coordinate system where the coordinate variables are ordered as \((x_1, x_2, p_1, p_2)\). This unconventional coordinate system does not change the physics or mathematics of the problem, but is convenient for studying the uncoupled system as well as expanding and shrinking phase spaces.

Since the two oscillators are independent, it is possible to perform linear canonical transformations on each coordinate separately. The canonical transformation in the first coordinate system is generated by

\[ A_1 = \frac{1}{2} \left( \begin{array}{cc} \sigma_2 & 0 \\ 0 & 0 \end{array} \right), \quad B_1 = \frac{i}{2} \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & 0 \end{array} \right), \quad C_1 = \frac{i}{2} \left( \begin{array}{cc} \sigma_1 & 0 \\ 0 & 0 \end{array} \right). \]

(4)

These generators satisfy the Lie algebra:

\[ [A_1, B_1] = iC_1, \quad [B_1, C_1] = -iA_1, \quad [C_1, A_1] = iB_1. \]

(5)

It is also well known that this set of commutation relations is identical to that for the \((2 + 1)\)-dimensional Lorentz group. Linear canonical transformations on the second coordinate are generated by

\[ A_2 = \frac{1}{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & \sigma_2 \end{array} \right), \quad B_2 = \frac{i}{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & \sigma_3 \end{array} \right), \quad C_2 = \frac{i}{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & \sigma_1 \end{array} \right). \]

(6)

These generators also satisfy the Lie algebra of Eq.(5). We are interested here in constructing the symmetry group for the coupled oscillators by soldering two \(Sp(2)\) groups generated by \(A_1, B_1, C_1\) and \(A_2, B_2, C_2\) respectively.

It will be more convenient to use the linear combinations:

\[ A_+ = A_1 + A_2, \quad B_+ = B_1 + B_2, \quad C_+ = C_1 + C_2, \]

\[ A_- = A_1 - A_2, \quad B_- = B_1 - B_2, \quad C_- = C_1 - C_2. \]

(7)

These matrices take the form

\[ A_+ = \frac{1}{2} \left( \begin{array}{cc} \sigma_2 & 0 \\ 0 & \sigma_2 \end{array} \right), \quad A_- = \frac{1}{2} \left( \begin{array}{cc} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{array} \right), \]

3
$B_+ = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad B_- = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$

$C_+ = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad C_- = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \quad (8)$

The sets $(A_+, B_+, C_+)$ and $(A_+, B_-, C_-)$ satisfy the Lie algebra of Eq.(4). The same is true for $(A_-, B_+, C_-)$ and $(A_-, B_-, C_+)$. 

Next, let us couple the oscillators through a rotation generated by

$A_0 = \frac{i}{2} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \quad (9)$

In view of the fact that the first two coordinate variables are for the phase space of the first oscillator, and the third and fourth are for the second oscillator, this matrix generates parallel rotations in the $(x_1, x_2)$ and $(p_1, p_2)$ coordinates. As the coordinates $(x_1, x_2)$ are coupled through a two-by-two matrix, the coordinate $(p_1, p_2)$ are coupled through the same two-by-two matrix.

Then, $A_0$ commutes with $A_+, B_+, C_+$, and the following commutation relations generate new operators $A_3, B_3$ and $C_3$:

$[A_0, A_+] = iA_3, \quad [A_0, B_+] = iB_3, \quad [A_0, C_+] = iC_3, \quad (10)$

where

$A_3 = \frac{1}{2} \begin{pmatrix} 0 & \sigma \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad B_3 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad C_3 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \quad (11)$

In this section, we started with the generators of the symmetry groups for two independent oscillators. They are $A_1, B_1, C_1$ and $A_2, B_2, C_2$. We then introduced $A_0$ which generates coupling of two oscillators. This processes produced three additional generators $A_3, B_3, C_3$. It is remarkable that $C_3, B_3$ and $A_+$ form the set of generators for another $Sp(2)$ group. They satisfy the commutation relations

$[B_3, C_3] = -iA_+, \quad [C_3, A_+] = iB_3, \quad [A_+, B_3] = iC_3. \quad (12)$

The same can be said about the sets $A_+, B_1, C_1$ and $A_+, B_2, C_2$. These $Sp(2)$-like groups are associated with the coupling of the two oscillators.

For a dynamical system consisting of two pairs of canonical variables $x_1, p_1$ and $x_2, p_2$, we have introduced the coordinate system $(\eta_1, \eta_2, \eta_3, \eta_4)$ defined in Eq.(3). The transformation of the variables from $\eta_i$ to $\xi_j$ is canonical if

$MJ\tilde{M} = J, \quad (13)$

where

$M_{ij} = \frac{\partial}{\partial \eta_j} \xi_i,$

$\eta_1 = \frac{\partial}{\partial \eta_1} \xi_1,$

$\eta_2 = \frac{\partial}{\partial \eta_2} \xi_2,$

$\eta_3 = \frac{\partial}{\partial \eta_3} \xi_3,$

$\eta_4 = \frac{\partial}{\partial \eta_4} \xi_4.
and
\[ J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}. \quad (14) \]

This form of the J matrix appears different from the traditional literature, because we are using the new coordinate system. In order to avoid possible confusion and to maintain continuity with our earlier publications, we give in the Appendix the expressions for the J matrix and the ten generators of the \( Sp(4) \) group in the traditional coordinate system. There are four rotation generators and six squeeze generators in this group.

In this new coordinate system, the rotation generators take the form
\[ L_1 = -\frac{1}{2} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad L_2 = \frac{i}{2} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \]
\[ L_3 = -\frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad S_3 = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \quad (15) \]

The squeeze generators become
\[ K_1 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad K_2 = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad K_3 = -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \]
\[ Q_1 = \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad Q_2 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad Q_3 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \quad (16) \]

There are now ten generators. They form the Lie algebra for the \( Sp(4) \) group:
\[ [L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, S_3] = 0, \]
\[ [L_i, K_j] = i\epsilon_{ijk}K_k, \quad [L_i, Q_j] = i\epsilon_{ijk}Q_k, \]
\[ [K_i, K_j] = [Q_i, Q_j] = -i\epsilon_{ijk}L_k, \quad [K_i, Q_j] = -i\delta_{ij}S_3, \]
\[ [K_i, S_3] = -iQ_i, \quad [Q_i, S_3] = iK_i. \quad (17) \]

Indeed, these matrices can be identified with the \( A, B, \) and \( C \) matrices derived from the coupled oscillators in the following manner.
\[ A_+ = S_3, \quad A_- = -L_3, \quad A_3 = -L_1, \quad A_0 = L_2, \]
\[ B_+ = K_2, \quad B_- = -Q_1, \quad B_3 = Q_3, \]
\[ C_+ = Q_2, \quad C_- = K_1, \quad C_3 = -K_3. \quad (18) \]

In this section, we started with the \( Sp(2) \) symmetry for each of the two oscillator, and introduced the parallel rotation to couple the system. It is interesting to note that this process leads to the \( Sp(4) \) symmetry.
The $A_0$ matrix given in Eq.(9) generates the coupling of two phase spaces by rotation. Within this coordinate system, we are interested in relative adjustments of the sizes of the two phase spaces. By making this adjustment, we are changing the relative size of the two phase spaces. This is not a canonical transformation, but is quite relevant to the physics with dissipation or with input energy from external sources.

For this purpose, we need the generators of the form

$$G_3 = \frac{i}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$ (19)

This matrix generates scale transformations in phase space. The transformation leads to a radial expansion of the phase space of the first coordinate and contracts the phase space of the second coordinate. What is the physical significance of this operation? Classically, it is a routine procedure. In quantum mechanics, the expansion of phase space leads to an increase in uncertainty and entropy. Mathematically speaking, the contraction of the second coordinate should cause a decrease in uncertainty and entropy. Can this happen? The answer is clearly No, because it will violate the uncertainty principle. This problem requires further research.

In the meantime, let us study what happens when the matrix $G_3$ is introduced into the set of matrices given in Eq.(15) and Eq.(16). It commutes with $S_3, L_3, K_3, K_2, Q_1,$ and $Q_2$. However, its commutators with the rest of the matrices produce four more generators:

$$[G_3, L_1] = iG_2, \quad [G_3, L_2] = -iG_1,$$
$$[G_3, K_3] = iS_2, \quad [G_3, Q_3] = -iS_1,$$ (20)

with

$$G_1 = \frac{i}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad G_2 = \frac{1}{2} \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix},$$
$$S_1 = -\frac{i}{2} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad S_2 = \frac{i}{2} \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.$$ (21)

If we take into account the above five generators in addition to the ten generators of $Sp(4)$, there are fifteen generators. These generators satisfy the following set of commutation relations.

$$[L_i, L_j] = i \epsilon_{ijk} L_k, \quad [S_i, S_j] = i \epsilon_{ijk} S_k, \quad [L_i, S_j] = 0,$$
$$[L_i, K_j] = i \epsilon_{ijk} K_k, \quad [L_i, Q_j] = i \epsilon_{ijk} Q_k, \quad [L_i, G_j] = i \epsilon_{ijk} G_k,$$
$$[K_i, K_j] = [Q_i, Q_j] = [Q_i, Q_j] = -i \epsilon_{ijk} L_k,$$
$$[K_i, Q_j] = -i \delta_{ij} S_3, \quad [Q_i, G_j] = -i \delta_{ij} S_1, \quad [G_i, K_j] = -i \delta_{ij} S_2.$$ 6
\[ [K_i, S_3] = -iQ_i, \quad [Q_i, S_3] = iK_i, \quad [G_i, S_3] = 0, \]
\[ [K_i, S_1] = 0, \quad [Q_i, S_1] = -iG_i, \quad [G_i, S_1] = iQ_i, \]
\[ [K_i, S_2] = iG_i, \quad [Q_i, S_2] = 0, \quad [G_i, S_2] = -iK_i. \] 

(22)

Indeed, the ten \( Sp(4) \) generators together with the five new generators form the Lie algebra for the group \( SL(4, r) \). This group is known to be locally isomorphic to the Lorentz group \( O(3, 3) \) with three space variables and three time variables. Indeed, we can study the coupled oscillator problem in terms of the Lorentz group. Conversely, we can use the oscillator problem in order to understand the Lorentz group. It would be a challenging problem to see whether there is a larger symmetry in superconductivity than what is known today.

In this connection, we should note that there have been many papers in recent on squeezed states of light which deal with coupling of two-particle systems, and the physics of squeezed states is strikingly similar to the physics of superconductivity.\[17, 18\] As in the case of squeezed states, the basic symmetry of the Bogoliubov transformation is that of the \( Sp(4) \) group.\[19\] We should study more along this direction.

3 Lorentz Group for Optical Filters

There is another area of condensed matter physics where the Lorentz group serves as the basic language. Anisotropic optical filters are due to crystal structures. Indeed for optical filters, a systematic mathematics has been developed.\[9\] Since Jones was the one who initiated this process,\[8\] the formalism is called the Jones matrix formalism. There also have been attempts to show that this formalism constitutes a representation of the Lorentz group.\[20\]

Recently, Han et al.\[21\] used the algebra of squeezed states\[3\] to reproduce the Jones matrix formalism. The algebra Han et al. used in their paper is a two-by-two representation of the six-parameter Lorentz group. They did not realize that what they did was the Jones matrix formalism, but they got all the ingredient to reach this conclusion in their paper. They even went further to show that the bilinear representation of the Lorentz group is the most appropriate language for the Jones matrix formalism. In this section, we shall review the work done by Han et al.

In studying polarized light propagating along the \( z \) direction, the traditional approach is to consider the \( x \) and \( y \) components of the electric fields. Their amplitude ratio and the phase difference determine the degree of polarization. Thus, we can change the polarization either by adjusting the amplitudes, by changing the relative phase shift, or both. For convenience, we call the optical device which changes amplitudes an “attenuator,” and the device which changes the relative phase a “phase shifter.”
Let us write the electric field vector as

\[ E_x = A \cos (kz - \omega t + \phi_1), \]
\[ E_y = B \cos (kz - \omega t + \phi_2), \tag{23} \]

where \( A \) and \( B \) are the amplitudes which are real and positive numbers, and \( \phi_1 \) and \( \phi_2 \) are the phases of the \( x \) and \( y \) components respectively. This form is useful not only in classical optics but also applicable to coherent and squeezed states of light.\[5\]

The traditional language for this two-component light is the Jones matrix formalism which is discussed in standard optics textbooks.\[22\] In this formalism, the above two components are combined into one column matrix with the exponential form for the sinusoidal function.

\[
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
= \begin{pmatrix}
A e^{i(kz - \omega t + \phi_1)} \\
B e^{i(kz - \omega t + \phi_2)}
\end{pmatrix}. \tag{24}
\]

The content of polarization is determined by the ratio:

\[
\frac{E_y}{E_x} = \left( \frac{B}{A} \right) e^{i(\phi_2 - \phi_1)}. \tag{25}\]

which can be written as one complex number:

\[
w = re^{i\phi} \tag{26}\]

with

\[ r = \frac{B}{A}, \quad \phi = \phi_2 - \phi_1. \]

The degree of polarization is measured by these two real numbers, which are the amplitude ratio and the phase difference respectively.

The purpose of this paper is to discuss the transformation properties of this complex number \( w \). The transformation takes place when the light beam goes through an optical filter whose transmission properties are not isotropic. There are two transverse directions which are perpendicular to each other. The absorption coefficient in one transverse direction could be different from the coefficient along the other direction. Thus, there is the “polarization” coordinate in which the absorption can be described by

\[
\begin{pmatrix}
e^{-\eta_1} & 0 \\
0 & e^{-\eta_2}
\end{pmatrix}
= e^{-(\eta_1+\eta_2)/2}
\begin{pmatrix}
e^{\eta/2} & 0 \\
0 & e^{-\eta/2}
\end{pmatrix}. \tag{27}\]

with \( \eta = \eta_2 - \eta_1 \). This attenuation matrix tells us that the electric fields are attenuated at two different rates. The exponential factor \( e^{-(\eta_1+\eta_2)/2} \) reduces both components at the same rate and does not affect the degree of polarization. The effect of polarization is solely determined by the squeeze matrix

\[
S(0, \eta) = \begin{pmatrix}
e^{\eta/2} & 0 \\
0 & e^{-\eta/2}
\end{pmatrix}. \tag{28}\]
This type of mathematical operation is quite familiar to us from squeezed states of light, if not from Lorentz boosts of spinors. For convenience, we call the above matrix an attenuator.

If the polarization coordinate is the same as the \( xy \) coordinate where the electric field components take the form of Eq. (23), the above attenuator is directly applicable to the column matrix of Eq. (24). If the polarization coordinate is rotated by an angle \( \theta/2 \), or by the matrix

\[
R(\theta) = \begin{pmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{pmatrix},
\]

then the squeeze matrix becomes

\[
S(\theta, \eta) = R(\theta) S(0, \eta) R(-\theta),
\]

where

\[
S(\theta, \eta) = \begin{pmatrix}
e^{\eta/2} \cos^2(\theta/2) + e^{-\eta/2} \sin^2(\theta/2) & (e^{\eta/2} - e^{-\eta/2}) \cos(\theta/2) \sin(\theta/2) \\
(e^{\eta/2} - e^{-\eta/2}) \cos(\theta/2) \sin(\theta/2) & e^{-\eta/2} \cos^2(\theta/2) + e^{\eta/2} \sin^2(\theta/2)
\end{pmatrix}.
\]

Another basic element is the optical filter with two different values of the index of refraction along the two orthogonal directions. The effect of this filter can be written as

\[
\begin{pmatrix}
e^{i\lambda_1} & 0 \\
0 & e^{i\lambda_2}
\end{pmatrix} = e^{-i(\lambda_1 + \lambda_2)/2} \begin{pmatrix}
e^{-i\lambda/2} & 0 \\
0 & e^{i\lambda/2}
\end{pmatrix},
\]

with \( \lambda = \lambda_2 - \lambda_1 \). In measurement processes, the overall phase factor \( e^{-i(\lambda_1 + \lambda_2)/2} \) cannot be detected, and can therefore be deleted. The polarization effect of the filter is solely determined by the matrix

\[
P(0, \lambda) = \begin{pmatrix}
e^{-i\lambda/2} & 0 \\
0 & e^{i\lambda/2}
\end{pmatrix}.
\]

This phase-shifter matrix appears like a rotation matrix around the \( z \) axis in the theory of rotation groups, but it plays a different role in this paper. We shall hereafter call this matrix a phase shifter.

Here also, if the polarization coordinate makes an angle \( \theta \) with the \( xy \) coordinate system, the phase shifter becomes

\[
P(\theta, \lambda) = R(\theta) P(0, \lambda) R(-\theta)
\]

which takes the form

\[
\begin{pmatrix}
e^{-i\lambda/2} \cos^2(\theta/2) + e^{i\lambda/2} \sin^2(\theta/2) & (e^{-i\lambda/2} - e^{i\lambda/2}) \cos(\theta/2) \sin(\theta/2) \\
(e^{-i\lambda/2} - e^{i\lambda/2}) \cos(\theta/2) \sin(\theta/2) & e^{i\lambda/2} \cos^2(\theta/2) + e^{-i\lambda/2} \sin^2(\theta/2)
\end{pmatrix}.
\]

Since we are interested in repeated applications of these two different kinds of matrices with different parameters, we shall work with the generators of these transformations. Let us introduce the Pauli spin matrices of the form

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

These matrices are written in a different convention. Here \( \sigma_3 \) is imaginary, while \( \sigma_2 \) is imaginary in the traditional notation. Also in this convention, we can construct three rotation generators

\[
J_i = \frac{1}{2} \sigma_i,
\]

with
which satisfy the closed set of commutation relations

\[ [J_i, J_j] = i\epsilon_{ijk}J_k. \]  

We can also construct three boost generators

\[ K_i = \frac{i}{2}\sigma_i, \]

which satisfy the commutation relations

\[ [K_i, K_j] = -i\epsilon_{ijk}J_k. \]  

The \( K_i \) matrices alone do not form a closed set of commutation relations, and the rotation generators \( J_i \) are needed to form a closed set:

\[ [J_i, K_j] = i\epsilon_{ijk}K_k. \]

The six matrices \( J_i \) and \( K_i \) form a closed set of commutation relations, and they are like the generators of the Lorentz group applicable to the (3 + 1)-dimensional Minkowski space. The group generated by the above six matrices is called \( SL(2,c) \) consisting of all two-by-two complex matrices with unit determinant.

If we consider only the phase shifters, the mathematics is basically repeated applications of \( J_1 \) and \( J_2 \), resulting in applications also of \( J_3 \). Thus, the phase-shift filters form an \( SU(2) \) or \( O(3) \)-like subgroup of the group \( SL(2,c) \). On the other hand, if we consider only the attenuators, the mathematics consists of repeated applications of \( K_1 \) and \( J_3 \), resulting in applications also of \( K_2 \). This is evident from the commutation relation

\[ [J_3, K_1] = iK_2. \]

Indeed, \( J_3, K_1 \) and \( K_2 \) form a closed set of commutation relations for the \( Sp(2) \) or \( O(2,1) \)-like subgroup of \( SL(2,c) \).[23] This three-parameter subgroup has been extensively discussed in connection with squeezed states of light.[3, 5]

If we use both the attenuators and phase shifters, the result is the full \( SL(2,c) \) group with six parameters. The transformation matrix is usually written as

\[ L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \]

with the condition that its determinant be one: \( \alpha\delta - \gamma\beta = 1 \). The repeated application of two matrices of this kind results in

\[ \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2\alpha_1 + \beta_2\gamma_1 & \alpha_2\beta_1 + \beta_2\delta_1 \\ \gamma_2\alpha_1 + \delta_2\gamma_1 & \gamma_2\beta_1 + \delta_2\delta_1 \end{pmatrix}. \]  

The most general form of the polarization transformation is the application of this algebra to the column matrix of Eq.(24).
We can obtain the same algebraic result by using the bilinear transformation:

$$w' = \frac{\delta w + \gamma}{\beta w + \alpha}. \quad (43)$$

The repeated applications of these two transformations can be achieved from

$$w_1 = \frac{\delta_1 w + \gamma_1}{\beta_1 w + \alpha_1}, \quad w_2 = \frac{\delta_2 w_1 + \gamma_2}{\beta_2 w_1 + \alpha_2}. \quad (44)$$

Then, it is possible to write $w_2$ as a function of $w$, and the result is

$$w_2 = \frac{(\gamma_2 \beta_1 + \delta_2 \delta_1)w + (\gamma_2 \alpha_1 + \delta_2 \gamma_1)}{(\alpha_2 \beta_1 + \beta_2 \delta_1)w + (\alpha_2 \alpha_1 + \beta_2 \gamma_1)}. \quad (45)$$

This is a reproduction of the algebra given in the matrix multiplication of Eq.(42). The form given in Eq.(43) is the bilinear representation of the Lorentz group.[2]

Let us go back to physics. If we apply the matrix $L$ of Eq.(41) to the column vector of Eq.(24), then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \alpha E_x + \beta E_y \\ \gamma E_x + \delta E_y \end{pmatrix}, \quad (46)$$

which gives

$$\frac{E'_y}{E'_x} = \frac{\gamma E_x + \delta E_y}{\alpha E_x + \beta E_y}. \quad (47)$$

In term of the physical quantity $w$ defined in Eq.(26), this formula becomes

$$w' = \frac{\gamma + \delta w}{\alpha + \beta w}. \quad (48)$$

This equation is identical to the bilinear form given in Eq.(43), and the ratio $w$ can now be identified with the $w$ variable defined as the parameter of the bilinear representation of the Lorentz group in the same equation. As we stated before, the purpose of this paper was to derive the transformation property of this complex number, and the purpose has now been achieved. The bilinear representation of the Lorentz group is clearly the language of optical filters and resulting polarizations.

Here we restricted ourselves to the cases where the polarization axes are are orthogonal and the polarization planes is perpendicular to the direction of propagation. But this is not always true, and the physics becomes more complicated if more complex crystals are taken into consideration. If we combine the group theory of polarization and the group theory of crystals, the result will be a group extension. This will introduce many new symmetry problems to condensed matter physics.
References

[1] E. P. Wigner, *Ann. Math.* **40**, 149 (1939).

[2] V. Bargmann, *Ann. Math.* **48**, 568 (1947).

[3] H. P. Yuen, *Phys. Rev. A* **13**, 2226 (1976).

[4] C. M. Caves and B. L. Schumaker, *Phys. Rev. A* **31**, 3068 (1985); B. L. Schumaker and C. M. Caves, *Phys. Rev. A* **31**, 3093 (1985).

[5] Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).

[6] P. A. M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949).

[7] H. Yukawa, *Phys. Rev.* **91**, 416 (1953).

[8] R. C. Jones, *J. Opt. Soc. Am.* **31**, 488, (1941).

[9] W. Swindell, *Polarized Light* (Dowden, Hutchinson, and Ross, Inc., Stroudsburg, PA, 1975).

[10] N. N. Bogoliubov, *Nuovo Cimento* **7**, 843 (1958); *Zh. Eksperim. i Teor. Fiz* **34**, 58 (1958) [Soviet Phys. - JETP **7**, 41 (1958)].

[11] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw–Hill, New York, 1971).

[12] M. Tinkham, *Introduction to Superconductivity* (Krieger, Malabar, FL, 1975).

[13] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1980).

[14] D. Han, Y. S. Kim, and M. E. Noz, *J. Math. Phys.* **36** 3940 (1995).

[15] P. A. M. Dirac, *J. Math. Phys.* **4**, 901 (1963).

[16] Y. S. Kim and M. Li, *Phys. Lett. A* **139**, 445 (1989).

[17] R. F. Bishop and A. Vourdas, *Z. Physik B* **71**, 527 (1988).

[18] A. Vourdas, *Phys. Rev. A* **46**, 443 (1992).

[19] S. K. Kim and J. L. Birman, *Phys. Rev. B* **38**, 4291 (1988).

[20] G. P. Parent and P. Roman, *Nuovo Cimento* **15**, 370 (1960); H. Takenaka, em Nouvelle Revue d’Optique **4**, 37 (1973); S. R. Cloude, *Optik* **75**, 26 (1986); P. Pellat-Finet, *Optik (Stuttgart)* **87**, 27 (1991); P. Pellat-Finet and M. Buasset, *ibid.* **90**, 101 (1992); T. Opartrny and J. Perina, *Phys. Lett. A* **181**, 199 (1993).
[21] D. Han, Y. S. Kim, and M. E. Noz, Phys. Lett. A 219, 21 (1996).

[22] F. L. Pedrotti and L. S. Petrotti, Introduction to Optics, 2nd Ed. (Prentice Hall, Englewood Cliff, New Jersey, 1993).

[23] M. Kitano and T. Yabuzaki, Phys. Lett. A 142, 321 (1989).