SMOOTH CONTRACTIBLE THREEFOLDS WITH HYPERBOLIC $\mathbb{G}_m$-ACTIONS VIA PS-DIVISORS

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Abstract. The aim of this note is to give an alternative proof of the theorem 4.1 of [KR4], that is, a characterization of smooth contractible affine varieties endowed with a hyperbolic action of the group $\mathbb{G}_m \simeq \mathbb{C}^*$, using the language of polyhedral divisors developed in [A-H] as generalization of $\mathbb{Q}$-divisors.

In [KR4], Koras and Russell provided a characterization of smooth contractible affine varieties endowed with a hyperbolic action of the group $\mathbb{G}_m \simeq \mathbb{C}^*$. This is an important step in the proof of the linearization conjecture in dimension three [KaKMLR]. The conjecture states that every $\mathbb{G}_m$-action on the affine $n$-space is linearizable, that is, up to a conjugation by an automorphism of $\mathbb{A}^n$, we can assume that the action is linear. The case $n = 1$ is trivial and corresponds to the simplest toric case, $\mathbb{G}_m$ acting on $\mathbb{A}^1$, for $n = 2$ the original proof is due to Gutwirth [G], the case $n = 3$ is more difficult and obtained after a long series of articles initiated by Kambayashi and Russell [KamR] continued with Koras [KR1, KR2, KR3, KR4] and achieved with the contribution of Kaliman and Makar-Limanov [KaML, KaKMLR]. In higher dimension, that is for $n > 3$, no general results are known.

On the other hand, the case of $\mathbb{G}_m$-action on $\mathbb{A}^3$ can be also viewed as the first case of a linearization conjecture of complexity two, that is, algebraic torus actions of dimension $n - 2$ acting effectively on $\mathbb{A}^n$. The linearization conjecture of complexity zero corresponds to the toric case and it is true, as the linearization conjecture of complexity one by a result of Białynicki-Birula [BB]. Here, once again, in higher dimension no general results are known.

The purpose is to use a geometrico-combinatorial presentation of normal varieties endowed with an effective algebraic torus action, developed by Altmann and Hausen [A-H] several years after the result of Koras and Russell. This presentation is a generalization of two presentations, the first one is the presentation of affine toric varieties via cones in lattices [CLS] and the second is the presentation of $\mathbb{G}_m$-surfaces via $\mathbb{Q}$-divisors [De, FZ]. A recollection in our particular case will be made in the section one. The presentation of a smooth affine variety $X$ endowed with a hyperbolic action of $\mathbb{G}_m$ consists of a pair $(Y, D)$ such that $Y$ is a normal variety of dimension $n - 2$ playing essentially a role of quotient and $D$ is a divisor on $Y$ with particular coefficients attached on each irreducible components. As it can be expected, this presentation is particularly adapted to give a shorter proof of the characterisation given initially by Theorem 4.1 in [KR4], this is realized in section two. We hope that this note serves to clarify and condense existing results in the field and can be the first step for possible generalizations.

1. Hyperbolic $\mathbb{G}_m$-actions on smooth affine varieties

Let $X = \text{Spec}(A)$ be an affine smooth variety endowed with a hyperbolic $\mathbb{G}_m$-action. First, recall that the coordinate ring $A$ of $X$ is $\mathbb{Z}$-graded in a natural way by its subspaces of semi-invariants of weight $n$, $A_n$, for the effective $\mathbb{G}_m$-action on $X$:

$$A_n := \{ f \in A / f(\lambda \cdot x) = \lambda^n f(x), \forall \lambda \in \mathbb{G}_m \}.$$
The action is said to be hyperbolic if there is at least one $n_1 < 0$ and one $n_2 > 0$ such that $A_{n_1}$ and $A_{n_2}$ are nonzero. In particular $A_0$ is the ring of invariant functions on $X$, thus $q : X \to Y_0(X) := X/G_m = \text{Spec}(A_0)$ is the algebraic quotient to the $G_m$-action on $X$.

**Definition 1.** The A-H quotient $Y(X)$ of $X$ is the blow-up $\pi : Y(X) \to Y_0(X)$ of $Y_0(X)$ with center at the closed subscheme defined by the ideal $I = \langle A_d \cdot A_{-d} \rangle$, where $d > 0$ is chosen such that $\bigoplus_{n \in \mathbb{Z}} A_{dn}$ is generated by $A_0$ and $A_{\pm d}$.

*Remark.* If $X$ admits a unique fixed point $x_0$, then the center of the blow-up is supported at the image of $x_0$ by the algebraic quotient morphism $q$.

**Definition 2.** A segmental divisor $D$ on an algebraic variety $Y$ is a formal finite sum $D = \sum [a_i, p_i] \otimes D_i$, where $D_i$ are prime Weil divisors on $Y$ and $[a_i, p_i]$ are closed intervals with rational bounds $a_i \leq p_i$. Every element $n \in \mathbb{Z}$ determines a map from segmental divisors to the group of Weil $\mathbb{Q}$-divisors on $Y$:

$$D = \sum [a_i, p_i] \otimes D_i \to D(n) = \sum (\min[n a_i, n p_i]) D_i = \sum q_i D_i.$$

**Definition 3.** A proper-segmental divisor (ps-divisor) $D$ on a variety $Y$ is a segmental divisor on $Y$ such that for every $n \in \mathbb{Z}$, $D(n)$ satisfies the following properties:

(i) $D(n)$ is a $\mathbb{Q}$-Cartier divisor on $Y$.

(ii) $D(n)$ is semi-ample, that is, for some $p \in \mathbb{Z}_{>0}$, $Y$ is covered by complements of supports of effective divisors linearly equivalent to $D(pn)$.

(iii) $D(n)$ is big, that is, for some $p \in \mathbb{Z}_{>0}$, there exists an effective divisor $D$ linearly equivalent to $D(pn)$ such that $Y \setminus \text{Supp}(D)$ is affine.

**Definition 4.** A variety $Y$ is said to be semi-projective if its ring of regular functions $\Gamma(Y, \mathcal{O}_Y)$ is finitely generated and $Y$ is projective over $Y_0 = \text{Spec}(\Gamma(Y, \mathcal{O}_Y))$.

Considering a ps-divisor $D$ on a semi-projective variety $Y$ the $\mathbb{Z}$-graded algebra

$$A = \bigoplus_{n \in \mathbb{Z}} A_n = \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(D(n))),$$

is finitely generated. The associated affine variety $X = \text{Spec}(A)$ is therefore a $G_m$-variety. In the case of hyperbolic $G_m$-action, the main theorem of [A-H] can be reformulated as follows:

**Theorem 5.** For any ps-divisor $D$ on a normal semi-projective variety $Y$ the scheme

$$\mathbb{S}(Y, D) = \text{Spec} \left( \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(D(n))) \right)$$

is a normal affine variety of dimension $\dim(Y) + 1$ endowed with an effective hyperbolic $G_m$-action, whose A-H quotient $Y(\mathbb{S}(Y, D))$ is birationally isomorphic to $Y$.

Conversely any normal affine variety $X$ endowed with an effective hyperbolic $G_m$-action is isomorphic to $Y(X), D)$ for a suitable ps-divisor $D$ on $Y(X)$.

A ps-divisor $D$ such that $X \simeq \mathbb{S}(Y, D)$ can be obtained by the following downgrading (see [A-H, section 11]):

Consider $X$ embedded as a $G_m$-stable subvariety of an affine affine toric variety. The calculation is then reduced to the toric case by considering an embedding in $\mathbb{A}^n$ endowed with a linear action of a torus $\mathbb{T}$ for a sufficiently large $n$. By this way, the inclusion of $\mathbb{G}_m \hookrightarrow \mathbb{T}$ corresponds to an inclusion of the lattice $\mathbb{Z}$ of one parameter subgroups of $\mathbb{G}_m$ in the lattice $\mathbb{Z}^n = N$ of one parameter subgroups of $\mathbb{T}$. We obtain the exact sequence:

$$0 \longrightarrow \mathbb{Z} \overset{s}{\longrightarrow} N = \mathbb{Z}^n \overset{p}{\longrightarrow} N' = \mathbb{Z}^n / \mathbb{Z} \longrightarrow 0,$$
where $F$ is given by the induced action of $G_m$ on $\mathbb{A}^n$ (it is the matrix of weights) and $s$ is a section of $F$. Let $v_i$, for $i = 1, \ldots, n$, be the first integral vectors of the unidimensional cone generated by the i-th column vector of $F$ considered as rays in the lattice $N' \simeq \mathbb{Z}^{n-1}$. Let $Z$ be the toric variety, of maximal dimension $(n-1)$, determined by the coarsest fan containing all cones generated by subsets of $\{v_1, \ldots, v_n\}$ in $N'$. Then each $v_i$ corresponds to a $\mathbb{T}^r$-invariant divisor where $T' = \text{Spec}(\mathbb{C}[N'^\vee])$, for $i = 1, \ldots, n$. By [A-H, section 11], $Z$ contains the A-H quotient $Y$ of $X$, as sub-variety, and the support of $D_i$ is obtained by restricting the $\mathbb{T}^r$-invariant divisor corresponding to $v_i$ to $Y$. The segment associated to the divisor $D_i$ is equal to $s(\mathbb{R}_{>0}^n \cap P^{-1}(v_i))$, it can occur that the segment is a point. If $X$ is the affine space endowed with a linear action of $G_m$, the embedding of $X$ as a $G_m$-stable subvariety of an affine toric variety is reduced to the identity. In this case, the A-H quotient of $\mathbb{A}^n$ is $Z$ itself.

Linear hyperbolic $G_m$-actions on $\mathbb{A}^3$ have been fully characterized by this method, see for instance [L-P]. Let $F = t(a_1, a_2, a_3)$ where $-a_1$, $a_2$ and $a_3$ are strictly coprime positive integers, and let $s = (\alpha, \beta, \gamma)$ where $\alpha$, $\beta$, $\gamma$ are integers such that $s \circ F = 1$. Then $\rho(i, j) = \gcd(i, j)$ for $i, j = a_1, a_2, a_3$ and $\delta = \gcd(\alpha, \beta, \gamma, \rho(a_1, a_2), \rho(a, a_3))$. In the case of linear hyperbolic $G_m$-actions on $\mathbb{A}^3$, [L-P, Proposition 11] gives:

**Proposition 6.** Let $X \simeq \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, z, t])$ endowed with a linear hyperbolic $G_m$-action. Then $\mathbb{A}^3$ is the $G_m$-equivariant cyclic quotient of a variety $Y$ isomorphic to $\mathbb{A}^3$ and such that $X$ is equivariantly isomorphic to the $G_m$-variety $Y$ with $Y$ and $D$ defined as follows:

(i) $Y$ is isomorphic to a blow-up $\pi : \mathbb{A}^2 \to \mathbb{A}^2$ at the origin.

(ii) $D$ is of the form:

$$D = \left\{ \frac{\alpha \rho(a_1, a_2)}{-a_1} \right\} \otimes D_2 + \left\{ \frac{\beta \rho(a_1, a_3)}{-a_1} \right\} \otimes D_3 + \left[ \frac{\gamma}{\delta} + \frac{1}{-\delta a_1} \right] \otimes E,$$

with $D_2$, $D_3$ are the strict transforms of the coordinate axes, $E$ is the exceptional divisor of $\pi$ and the linear $G_m$-action on $X$ is given by its matrix of weights $F = t(a_1, a_2, a_3)$ with $-a_1$, $a_2$ and $a_3$ strictly positive integers.

Let $X$ be equivariantly isomorphic to $S(Y, D)$. Then segmental prime divisors $[a_i, b_i] \otimes D_i$ occurring in the ps-divisor $D = \sum_{i=1}^n [a_i, b_i] \otimes D_i$ encode, according to their coefficients, two distinct facts (see [P] and [FZ, Theorem 4.18]):

(i) For any $i$ such that $[a_i, b_i]$ is reduced to a rational point $\{b_i\}$, the divisor $\{b_i\} \otimes D_i$ encodes isotropy subgroup of finite order, or equivalently the fact that $X$ is the finite cyclic cover of an other $G_m$-variety having a A-H quotient isomorphic to that of $X$ and where $D_i$ does not appear in the presentation.

(ii) For any $i$ such that $[a_i, b_i]$ is an interval with non-empty interior, the divisor $[a_i, b_i] \otimes D_i$ encodes isotropy subgroup of infinite order, equivalently fixed points.

In particular as we have assumed that $X$ admits a unique fixed point, the only divisor whose coefficient is an interval with non-empty interior is the exceptional divisor of the blow-up $\pi : Y \to Y_0$.

The smoothness of the $G_m$-threefold $X$ can be checked using [L-P, Theorem 7 and proposition 11]:

(iii) If $X = S(Y, D)$ is an affine $G_m$-variety of dimension $n$, then, $X$ is smooth if and only if the combinatorial data $(Y, D)$ is locally isomorphic in the étale topology to the combinatorial data of the affine space endowed with a linear $G_m$-action.

**Example.** Let $X \simeq \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ endowed with the $G_m$-action with matrix of weights $F = t(2, 3, -6)$. Then $\mathbb{A}^3$ is equivariantly isomorphic to $S(\mathbb{A}^2, D)$ where $\pi : \mathbb{A}^2 \to \mathbb{A}^2$ is the blow-up of the algebraic quotient $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v]) = \text{Spec}(\mathbb{C}[x^3, y^2z])$ with center at the closed subscheme defined by the ideal $I = \langle u, v \rangle$, the p-s divisor $D$ is of the form

$$D = \left\{ \frac{1}{3} \right\} \otimes D_1 + \left\{ \frac{1}{2} \right\} \otimes D_2 + \left[ 0, \frac{1}{6} \right] \otimes E,$$

where $D_1$, $D_2$ are the strict transforms of the coordinates axes $\{u = 0\}$ and $\{v = 0\}$, and $E$ is the exceptional divisor of the blow-up $\pi$. 
\( \mathbb{A}^3 \) admits commuting actions of the groups \( \mu_2 \) and \( \mu_3 \) of square and cubic roots of the unity defined respectively by \( \epsilon \times (x, y, z) \to (\epsilon x, y, \epsilon z) \) and \( \iota \times (x, y, z) \to (x, \epsilon y, \epsilon^2 z) \). These commute with the \( \mathbb{G}_m \)-action and \( X \simeq \mathbb{A}^3 \) has a structure of a \( \mathbb{G}_m \)-equivariant bi-cyclic cover of \( X/((\mu_2 \times \mu_3) \simeq \mathbb{A}^3 \) with matrix of weights \( F = (1, 1, -1) \). Applying the main Theorem of [P], \( X/((\mu_2 \times \mu_3) \) is equivariantly isomorphic to \( \mathbb{S}(Y, [-1, 0] \otimes E) \). The ring of regular functions of \( X \) is \( \mathbb{Z} \)-graded via \( M \) the character lattice of \( \mathbb{G}_m \) and the ring of regular functions of \( X/((\mu_2 \times \mu_3) \) is \( \mathbb{Z} \)-graded via a sublattice \( M' \subset M \) of index 6, thus A-H presentation of \( X/((\mu_2 \times \mu_3) \) is obtained considering the same A-H quotient \( Y \) and a multiple of \( D \), namely, \( 6D \sim [-1, 0] \otimes E \). The structure of \( \mathbb{G}_m \)-equivariant bi-cyclic covering implies that the induced action of \( \mu_2 \times \mu_3 \) on \( Y \) is trivial so that \( Y \simeq Y(X/((\mu_2 \times \mu_3)) \). So, only the \( \text{ps-divisors} \) change as the lattice changes (see [P]). This is illustrated by the following diagram,

\[
\begin{align*}
X \simeq \mathbb{A}^3 & \simeq \mathbb{S}(Y, D) \\
X/\mu_3 \simeq \mathbb{A}^3 & \simeq \mathbb{S}(Y, 3D) \\
X/\mu_2 \simeq \mathbb{A}^3 & \simeq \mathbb{S}(Y, 2D) \\
X/((\mu_2 \times \mu_3) & \simeq \mathbb{A}^3 \simeq \mathbb{S}(Y, 6D) \\
\end{align*}
\]

One way to constructs many examples of varieties endowed with hyperbolic \( \mathbb{G}_m \)-action is the following (see [T, Z]):

**Definition 7.** Let \( X = \{ f(u_2, . . . , u_n) \} \subset \mathbb{A}^{n-1} \) be a smooth hypersurface defined by the zeros of the polynomial \( f \). The *hyperbolic modification* of \( X \) is obtained blowing-up the variety \( \mathbb{A}^3 \times X \subset \mathbb{A}^3 \times \mathbb{A}^{n-1} \) with center at the origin and removing the proper transform of \( \{ 0 \} \times X \). By this way we obtain a variety \( X_f = \{ f(x_1 x_2, . . . , x_1 x_n)/x_1 = 0 \} \subset \mathbb{A}^n \) which is stable for the following hyperbolic \( \mathbb{G}_m \)-action on \( \mathbb{A}^n \), \( \lambda \cdot (x_1, x_2, . . . , x_n) = (\lambda^{-1} x_1, \lambda x_2, . . . , \lambda x_n) \).

Each of these hyperbolic modifications will be characterized by the following A-H presentation: \( X_f \) is equivariantly isomorphic to \( \mathbb{S}(X, [-1, 0] \otimes E) \) where \( \pi : \tilde{X} \to X \) is the blow-up of \( X \) with center at the closed subscheme defined by the ideal \( I = (u_2, . . . , u_n) = (x_1 x_2, . . . , x_1 x_n) \), and \( E \) is the exceptional divisor of the blow-up.

**Constructions of Koras-Russell threefolds and \( \mathbb{A}^3 \) via segmental divisors**

In this section we will consider an approach to the classification given by Koras and Russell using segmental divisors in an étale neighborhood of a fixed point to determine all possible configurations for the threefolds.

The tangent space of a smooth \( \mathbb{G}_m \)-variety at the fixed point is an affine three-space with a linear \( \mathbb{G}_m \)-action. The action \( \mathbb{G}_m \times \mathbb{A}^3 \to \mathbb{A}^3 \) is given by \( \lambda \cdot (x, y, z) = (\lambda^{a_1} x, \lambda^{a_2} y, \lambda^{a_3} z) \) and characterized by its matrix of weight \( F = (a_1, a_2, a_3) \).

In [K], Koras proves that if \( \mathbb{A}^3 \) is endowed with an algebraic \( \mathbb{G}_m \)-action such that the algebraic quotient is of dimension 2, then the quotient is isomorphic to \( \mathbb{A}^2/\mu \), where \( \mu \) is a finite cyclic group. If a variety \( X \) has such algebraic quotient by \( \mathbb{G}_m \)-action it was said that \( X \) has quotient as expected for a \( \mathbb{G}_m \)-action on \( \mathbb{A}^3 \).

If \( X \) is endowed with an action of \( \mu_i \) of \( i \)-th roots of the unity commuting with the \( \mathbb{G}_m \)-action, then its algebraic quotient \( X/\mathbb{G}_m \) inherits also of a \( \mu_i \)-action, possibly trivial.

**Theorem 8.** A smooth, contractible, affine threefolds \( X = \text{Spec}(A) \) with a hyperbolic \( \mathbb{G}_m \)-action, an unique fixed point and an algebraic quotient isomorphic to \( \mathbb{A}^2/\mu \), where \( \mu \) is a finite cyclic group, is determined by the following data:

(a) A triple of weights \( a_1, a_2, a_3 \) with \(-a_1, a_2, a_3 > 0\). These define a hyperbolic \( \mathbb{G}_m \)-action on the tangent space of the fixed point \( \mathbb{A}^3 \) via the matrix weight \( F = (a_1, a_2, a_3) \).
(b) A triple of positive integers \( s = (\alpha, \beta, \gamma) \) such that \( s \circ F = 1 \).

(c) Curves \( C_2, C_3 \) in \( k^2 \) satisfying:

(i) \( C_1 \) is defined by an \( \mu_{a_1} \)-homogeneous polynomial.

(ii) \( C_1 \simeq A^1 \).

(iii) \( C_2 \) and \( C_3 \) meet normally in \((0, 0)\) and \( d - 1 \) additional points.

Proof. Let \( X \) be an affine three space endowed with a hyperbolic \( G_m \)-action, we assume that \( X \) is equivariantly isomorphic to \( S(Y, D) \) where the pair is minimal constructed as in section one.

First step: the A-H quotient.

By assumption the fixed point set \( X^{G_m} \) of \( X \) is of dimension 0, and by [BB] it is non-empty and connected and therefore reduced to a unique point, \( x_0 \in X \). Using the result of [K], the algebraic quotient \( Y_0 \) of \( X \) by the \( G_m \)-action is isomorphic to \( \mathbb{A}^2//\mu \) where \( \mu \) is a finite cyclic group. So the A-H quotient \( Y \) of \( X \) is isomorphic to a blow-up of \( Y_0 \simeq \mathbb{A}^2//\mu \) supported on the image of \( x_0 \) by the quotient morphism \( q: X \to Y_0 \simeq \mathbb{A}^2//\mu \). Since \( X \) is smooth, it follow from [L-P] that there exists an étale \( G_m \)-invariant neighborhood \( U \) of \( x_0 \), which is equivariantly isomorphic to an étale neighborhood of a fixed point in \( \mathbb{A}^3 \) endowed with a linear hyperbolic \( G_m \)-action of the form, \( \lambda \cdot (x, y, z) = (\lambda^{a_1} x, \lambda^{a_2} y, \lambda^{a_3} z) \) with \(-a_1, a_2, a_3 > 0\). So \( Y(X) \) is isomorphic to the A-H quotient of the previous \( \mathbb{A}^3 \) and determined by the triple of reduced weights \( a_1, a_2, a_3 \).

Considering an embedding of \( X \) as a \( G_m \)-stable subvariety of an affine space endowed with a linear \( G_m \)-action, we can always construct a finite cyclic cover along coordinate axes of the ambient space such that the new variety obtained by this finite ramified morphism admits an algebraic quotient isomorphic to the complex plane \( \mathbb{A}^2 \). By this way and using [P], we can assume that the A-H presentation of \( X \) is completely determined by that of the cyclic cover and the data of the cyclic group.

Second step: the ps-divisor \( D \) in an étale neighborhood of the exceptional divisor.

In terms of ps-divisor as \( G_m \)-variety, the smoothness criterion gives us that there exists \( \mathcal{V} \), in \( Y \), an étale neighborhood of the exceptional divisor such that \( S(\mathcal{V}, D|_{\mathcal{V}}) \) is equivariantly isomorphic to an étale neighborhood of a fixed point in \( \mathbb{A}^3 \) endowed with a linear hyperbolic \( G_m \)-action.

Moreover as \( X \) is smooth in the étale neighborhood of the fixed point, by [L-P], \( D|_{\mathcal{V}} \) is of the form:

\[
D_{|\mathcal{V}} = \left\{ \frac{\alpha \rho(a_1, a_2)}{-a_1} \right\} \otimes D_2 + \left\{ \frac{\beta \rho(a_1, a_3)}{-a_1} \right\} \otimes D_3 + \left[ \frac{\gamma}{\delta} + \frac{1}{\mu a_1} \right] \otimes E,
\]

where Proposition 6 provides the coefficients and the supports of the divisors in the étale neighborhood.

Third step: the number of irreducible component of the support of \( D \).

The ps-divisor \( D \) is completely determined by \( D_{|\mathcal{V}} \). Indeed as \( X \) admits a unique fixed point by the first section, \( E \) is the unique divisor whose coefficient is an interval with non-empty interior. Now suppose that

\[
D = \left\{ \frac{\alpha \rho(a_1, a_2)}{-a_1} \right\} \otimes D_2 + \left\{ \frac{\beta \rho(a_1, a_3)}{-a_1} \right\} \otimes D_3 + \left[ \frac{\gamma}{\delta} + \frac{1}{\mu a_1} \right] \otimes E + \sum_{i=4}^p \{ q_i \} \otimes D_i,
\]

with \( D_{|\mathcal{V}} = \left\{ \frac{\alpha \rho(a, c)}{c} \right\} \otimes D_{2|\mathcal{V}} + \left\{ \frac{\beta \rho(b, c)}{c} \right\} \otimes D_{3|\mathcal{V}} + \left[ \frac{\gamma}{\delta} + \frac{1}{\mu c} \right] \otimes E.

By the first section, this implies that there exists a \( G_m \)-variety \( \hat{X} \) such that \( c: X \to \hat{X} \) is an equivariant cyclic cover of \( \hat{X} \) along divisors whose images in the algebraic quotient do not intersect the image of the fixed point and thus \( X \) does not admit a fixed point. This contradicts the assumption and \( X \) is equivariantly isomorphic to \( S(Y, D) \) where \( Y \) is isomorphic to the A-H quotient of an \( \mathbb{A}^3 \) endowed with a hyperbolic linear \( G_m \)-action and \( D \) is of the form:

\[
D = \left\{ \frac{\alpha \rho(a_1, a_2)}{-a_1} \right\} \otimes D_2 + \left\{ \frac{\beta \rho(a_1, a_3)}{-a_1} \right\} \otimes D_3 + \left[ \frac{\gamma}{\delta} + \frac{1}{\mu a_1} \right] \otimes E,
\]
where $E$ is the exceptional divisor of $\pi : Y \to Y_0$ and $D_2, D_3$ are the strict transform of the supports of two irreducible curves $C_2$ and $C_3$ defined by the zeros of $f \in \mathbb{C}[u,v]$ and $g \in \mathbb{C}[u,v]$ respectively.

Fourth step: the topology of the support of $\mathcal{D}$.

Once again by the smoothness criterion the support of $\mathcal{D}$ has to be a SNC-divisor, in particular each irreducible component of the divisors is smooth. Moreover using the previous presentation and [P], $X$ admits two actions of finite cyclic group $\mu_\zeta$ and $\mu_\xi$ which factorize by the $\mathbb{G}_m$ action. In particular $X$ can be viewed as a bi-cyclic cover of $\mathbb{A}^3$ of order $\zeta$ and $\xi$, and it admits the following presentation:

$$X = \text{Spec} \left( \mathbb{C}[x_1, x_2, x_3, x_4, x_5]/\left( x_1^2 = f(x_1x_2, x_1x_3)/x_1 \right) \right).$$

Thus we obtain the following tower of threefolds where $X/\langle \mu_\zeta \times \mu_\xi \rangle$ is equivariantly isomorphic to $\mathbb{A}^3$ with a linear $\mathbb{G}_m$-action:

$$X/\langle \mu_\zeta \rangle \to X/\langle \mu_\xi \rangle \to X/\langle \mu_\zeta \times \mu_\xi \rangle \simeq \mathbb{A}^3$$

As $X$ is contractible, so are $X/\langle \mu_\zeta \rangle$ and $X/\langle \mu_\xi \rangle$ by [KrPRa, Theorem B]. These varieties are obtained by two modifications: first a hyperbolic modification of $f$ (or $g$ respectively) see Definition 7. By this way we obtain two varieties $X_f = \{ f(x_1x_2, x_1x_3)/x_1 = 0 \} \subset \mathbb{A}^3$ and $X_g = \{ g(x_1x_2, x_1x_3)/x_1 = 0 \} \subset \mathbb{A}^3$ which are stable for the following hyperbolic $\mathbb{G}_m$-action on $\mathbb{A}^3$, $\lambda \cdot (x_1, x_2, x_3) = (\lambda^{-a_1} x_1, \lambda^{a_2} y, \lambda^{a_3} z)$.

The second modification is a cyclic cover of order $\zeta$ (or $\xi$ respectively) of $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x_1, x_2, x_3])$ along the variety $\{ f(x_1x_2, x_1x_3)/x_1 = 0 \}$ (or $\{ g(x_1x_2, x_1x_3)/x_1 = 0 \}$ respectively). By [Ka, Theorem A], as the varieties $X/\langle \mu_\zeta \rangle$ and $X/\langle \mu_\xi \rangle$ are contractible, the sub-varieties $\{ f(x_1x_2, x_1x_3)/x_1 = 0 \}$ and $\{ g(x_1x_2, x_1x_3)/x_1 = 0 \}$ of $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x_1, x_2, x_3])$ have to be $\mathbb{Z}_k$-acyclic for almost every $k$, that is the $\mathbb{Z}_k$-homology of the point for almost every $k$. By a classical classification result, this is possible if and only if the smooth curves defined by $\{ f(u, v) = 0 \}$ and $\{ g(u, v) = 0 \}$ in $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v])$ are acyclic and so, are two copies of the affine line $\mathbb{A}^1$.

To summarize, every $\mathbb{A}^3$ endowed with a hyperbolic $\mathbb{G}_m$-action is characterized by a pair $(Y, \mathcal{D})$ such that $\mathbb{A}^3 \simeq S(Y, \mathcal{D})$. The A-H quotient $Y$ of $\mathbb{A}^3$ is also the A-H quotient of an $\mathbb{A}^3$ endowed with a hyperbolic and linear $\mathbb{G}_m$-action. The coefficient of $\mathcal{D}$ is the same as those used in the presentation of the $\mathbb{A}^3$ endowed with a hyperbolic linear $\mathbb{G}_m$-action and the support of $\mathcal{D}$ is the union of the exceptional divisor, the strict transform of two curves, $C_2$ and $C_3$, both isomorphic to a line in $\mathbb{A}^2$, given by polynomials of weights $a_2, a_3$ respectively and intersecting normally at the origin and in $(n - 1)$ other points, where $n \geq 1$.

Now as the algebraic quotient of $X$ is not necessarily $\mathbb{A}^2$, it has been shown that there exists a cyclic group of order $\mu$ such that the algebraic quotient $\mathbb{A}^2/\mu$. The tangent space of $X$ at the fixed point is an affine three-space with a linear hyperbolic $\mathbb{G}_m$-action given by $\lambda \cdot (x, y, z) = (\lambda^{-a_1} x, \lambda^{a_2} y, \lambda^{a_3} z)$ with $a_1, a_2$ and $a_3$ positive integers. By construction the order of the finite cyclic group $\mu$ can be chosen as to be $a_1$. The general case is obtained as a quotient of the previous case by the cyclic group. In particular as proved in [P], the two divisor $D_2$ and $D_3$ have to be invariant for the induced (and possibly trivial) action on the A-H quotient, so $C_2$ and $C_3$ are homogeneous for $\mu_{a_1}$. □

All possible varieties that can be built, and that verify the previous Theorem are not necessary $\mathbb{A}^3$. There is a dichotomy, obtained in [KaML], using the additive group action $\mathbb{G}_a$ on them, in particular they prove that the one class of varieties obtained by Koras and Russell in [KR4] which are indeed $\mathbb{A}^3$ are those which are “obviously” $\mathbb{A}^3$ with a linear hyperbolic $\mathbb{G}_m$-action. This corresponds to the case where $D_2$ and $D_3$ are the two coordinate axes in the same coordinate system in Theorem 8. The,
remaining varieties are called Koras-Russell threefolds and classified in three types according to the richness of the $\mathbb{G}_m$-actions on them. The A-H presentation of these varieties has been computed in [P].

**Example 9.** Let $X$ be the $\mathbb{G}_m$-variety equivariantly isomorphic to $\mathbb{G}_m^2 = \text{Spec}(\mathbb{C}[u, v])$ with center at the closed subscheme defined by the ideal $I = (u, v^d)$. The $p$-s divisor $D$ is of the form

$$D = \left\{ \frac{1}{3} \right\} \otimes D_1 + \left\{ \frac{-3}{5} \right\} \otimes D_2 + \left[ 0, \frac{1}{15} \right] \otimes E,$$

where $E$ is the exceptional divisor of the blow-up $\pi$ and $D_1$, $D_2$ are the strict transforms of $\{ u = 0 \}$ and $\{ v + (u + v^2)^3 = 0 \}$, then $X$ is a smooth contractible threefold endowed with an hyperbolic $\mathbb{G}_m$-action but it is not $\mathbb{A}_3$. In particular

$$X = \{ x + y^5(x^2 + z^3)^3 + t^5 = 0 \} \subset \mathbb{A}_4 = \text{Spec}(\mathbb{C}[x, y, z, t]).$$

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