MULTIPLE NONCOMMUTATIVE TORI AND HOPF ALGEBRAS

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Abstract. We derive the Kac-Paljutkin finite-dimensional Hopf algebras as finite fibrations of the quantum double torus and generalize the construction for quantum multiple tori.

1. Introduction

Noncommutative geometry is a rich area of research, which extends the classical notions of topology, differential geometry and group theory to noncommutative objects and their symmetries.

The latter topic, which includes the theory of Hopf algebras, Hopf groupoids etc., has been extensively studied in the recent years both in the direction of Lie-type objects as well as finite dimensional objects. One of the ambitious tasks, comparable with the classification of finite groups might be the classification of finite-dimensional semisimple Hopf star algebras. Only partial results for low dimensions and special cases are know so far (see [1, 8] for a review).

One of best-known examples of semisimple finite Hopf algebras is the Kac-Paljutkin algebra, especially its lowest dimensional selfdual example $A_8$ [4].

In this paper we shall demonstrate that they arise from the exact sequences of Hopf algebras built using the double noncommutative torus [5]. We shall generalize this construction and provide other examples.

The paper is organized as follows, first, we briefly review the construction of the double torus and its dual, then we construct Kac-Paljutkin algebras through exact sequences (for the deformations at roots of unity) and in the second part of the paper we generalize the construction and discuss new examples.

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2. The quantum double torus

The noncommutative torus, although it has the same symmetries as the commutative one, is no longer a Hopf algebra. However, it appears that an interesting Hopf algebra structure exists on the direct sum of the deformed and nondeformed torus.

The detailed construction is based on the cocycle deformation of the tensor product of $C(Z_2) \otimes C(T^2)$ \cite{5}, however, here, for simplicity, we shall use a straightforward description with generators.

**Definition 2.1.** Let $DT_q$ be an unital algebra generated by two unitaries $U, V$ and two selfadjoint projectors $P_+, P_-$, satisfying the relations:

\begin{align}
  P_+ + P_- &= 1, \\
P_+ U &= U P_+, \\
P_+ V &= V P_+, \\
V U &= P_+ UV + q P_- UV,
\end{align}

where $|q| = 1$.

In fact, the introduction of $P_\pm$ as generators is not necessary, as we may use the following relations:

\begin{align}
  V U V^{-1} U^{-1} - 1 &= (q - 1) P_-, \\
V U V^{-1} U^{-1} - q &= (1 - q) P_+.
\end{align}

**Proposition 2.2** (see \cite{5}). The algebra $DT_q$ is a Hopf algebra with the following coproduct, counit and antipode:

\begin{align}
  \Delta U &= U \otimes U P_+ + V \otimes U P_-,
  \\
  \Delta V &= V \otimes V P_+ + U \otimes V P_-,
  \\
  \Delta P_+ &= P_+ \otimes P_+ + P_- \otimes P_-,
  \\
  \epsilon(V) &= 1,
  \\
  \epsilon(U) &= 1,
  \\
  \epsilon(P_+) &= 1,
  \\
  S U &= U^{-1} P_+ + V^{-1} P_-, \\
  S V &= V^{-1} P_+ + U^{-1} P_-,
  \\
  S P_+ &= P_+.
\end{align}

Verification of all Hopf algebra properties is left to the reader, equivalently one may use \cite{5} and the identification of the generators $U, V, P_\pm$ with the generators of the tensor algebra $T \otimes C(Z_2)$ \cite{6}.

**Observation 2.3.** $DT_q$ is a $\ast$-Hopf algebra.

\footnote{Note that the parameter $q$ used here corresponds to $q^2$ of \cite{5}.}
Indeed, another simple exercise verifies that for the generators we have:
\[ \Delta(a^*) = a^*_1 \otimes a^*_2. \]

In fact, using the operator \( U_\pm = UP_\pm \) and \( V_\pm = VP_\pm \) we may see that the \( C^* \) algebra generated by \( DT_q \) is a compact matrix quantum group (compact matrix pseudogroup) in the sense of Woronowicz:
\[ \Delta \left( \begin{array}{cc} U_+ & V_- \\ U_- & V_+ \end{array} \right) = \left( \begin{array}{cc} U_+ & V_- \\ U_- & V_+ \end{array} \right) \otimes \left( \begin{array}{cc} U_+ & V_- \\ U_- & V_+ \end{array} \right). \]

We leave the verification of the remaining necessary conditions (see [11]) to the reader.

The theory of unitary representations has been discussed in [5]:

**Proposition 2.4.** The representation of the algebra \( DT_q \) on the Hilbert space \( \mathcal{H} = l_2(\mathbb{Z}^2) \oplus l_2(\mathbb{Z}^2) \) is given by:
\[
\begin{align*}
U|n, m, \pm\rangle &= |n + 1, m, \pm\rangle, \\
V|n, m, +\rangle &= |n, m + 1, +\rangle, \\
V|n, m, -\rangle &= q^n|n, m + 1, -\rangle.
\end{align*}
\]

Of course, the operators \( P_\pm \) act as projections on \( \mathcal{H}_\pm \).

3. **The Dual Hopf Algebra of the Quantum Double Torus.**

To construct the dual Hopf algebra we have to choose the appropriate mathematical setup. Although it is not a problem to define linear functionals on \( DT_q \) and their algebraic structure, the notion of the coproduct requires caution. It seems to us that the definition of multiplier Hopf algebras [3] is best suited for the example. We shall recall the necessary definitions when necessary.

First, let us introduce the basis of the dual algebra generators \( c_{mn}^+ \), \( c_{mn}^- \) defined as linear functionals:
\[ c_{mn}^+ (U_+^k V_+^l) = \delta^{mk} \delta^{nl}, \quad c_{mn}^+ (U_-^k V_-^l) = 0, \quad c_{mn}^- (U_+^k V_-^l) = 0, \quad c_{mn}^- (U_-^k V_+^l) = q^{-\frac{1}{2}mn} \delta^{mk} \delta^{nl}. \]

We have chosen here a nontrivial factor in the duality relation, which corresponds to the rescaling of the generators in order to obtain simpler algebraic relations. Indeed, we could translate the multiplication and comultiplication rules to the dual algebra and obtain:
\[ c_{kl}^{mn} c_{mn}^{kl} = \delta^{km} \delta^{ln} c_{kl}^{mn}, \quad c_{kl}^{mn} c_{mn}^{kl} = \delta^{km} \delta^{ln} c_{kl}^{mn}, \quad c_{kl}^{mn} c_{mn}^{kl} = \delta^{km} \delta^{ln} c_{kl}^{mn}. \]
Then we define the dual algebra \((DT_q)^*\) as an algebra spanned by linear combinations of finitely many generators \(c_{ij}^\pm\). In fact the algebra \((7)\) might be identified with the crossed-product algebra of the group \(\mathbb{Z}_2\) by the algebra of functions of compact support on \(\mathbb{Z}^2\).

Let us recall a definition of comultiplication for the multiplier Hopf algebra \(A\):

**Definition 3.1** (see [3]). A comultiplication is a \(*\)-homomorphism \(\Delta : A \to M(A \otimes A)\) so that \(\Delta(a)(1 \otimes b)\) and \((a \otimes 1)\Delta(b)\) are in \(A \otimes A\) and \(\Delta\) is coassociative in the sense that for every \(a, b, c \in A\)

\((a \otimes 1 \otimes 1)(\Delta \otimes \text{id})(\Delta(b)(1 \otimes c)) = (\text{id} \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)\).

For the algebra \((DT_q)^*\) we have:

**Definition 3.2.** The coproduct is as follows:

\[
\Delta c_{+}^{mn} = \sum_{i+k=m} \sum_{j+l=n} c_{+}^{ij} \otimes c_{+}^{kl},
\]

\[
\Delta c_{-}^{mn} = \sum_{i+k=m} \sum_{j+l=n} q^{\frac{1}{2}(jk-il)} c_{-}^{ij} \otimes c_{-}^{kl},
\]

then the antipode and counit are:

\[
\epsilon(c_\pm^{ij}) = \delta_i^0 \delta_j^0,
\]

\[
S(c_+^{ij}) = c_+^{(-i)(-j)},
\]

\[
S(c_-^{ij}) = c_-^{(-j)(-i)},
\]

Finally, the star structure:

\[
(c_+^{ij})^* = c_+^{ij},
\]

\[
(c_-^{ij})^* = c_-^{ji}.
\]

Clearly the target space of the comultiplication is contained within the multiplier algebra of \((DT_q)^* \otimes (DT_q)^*\), it is also easy to verify that the other conditions are satisfied, for example, we shall demonstrate here that \(\Delta(c_+^{ij})(1 \otimes c_-^{kl})\) is in \((DT_q)^* \otimes (DT_q)^*\):

\[
\Delta(c_+^{ij})(1 \otimes c_-^{kl}) = \sum_{r+s=i} \sum_{t+w=j} (c_+^{rt} \otimes c_+^{sw})(1 \otimes c_-^{kl}) = \]

\[
= c_+^{(i-t)(j-k)} \otimes c_-^{kl}.
\]

In fact, one can easily verify that \((DT_q)^*\) is a *discrete quantum group* in the sense of Van Daele’s definition [4], as it is a direct sum of full matrix algebras with a multiplier Hopf algebra structure.
Since the algebra is contained in its multiplier we might try to extend the definition of the comultiplication to \( M((DT_q)^*) \). Let us define three elements \( e_1, e_2 \) and \( \sigma \) from \( M((DT_q)^*) \):

\[
e_1 = \sum_{n,m} nc_{+}^{nm}, \quad e_2 = \sum_{n,m} mc_{+}^{nm},
\]

\[
\sigma = \sum_{n,m} c_{-}^{nm}.
\]

These elements must be understood as belonging to \( M((DT_q)^*) \), i.e. through their multiplication on the elements of \( (DT_q)^\). Notice that \( \sigma^2 \) is the identity in \( M((DT_q)^*) \) and all elements \( c_{-}^{nm} \) could be expressed as a product of \( \sigma \) and \( e_{nm}^{+} \):

\[
\sigma c_{-}^{nm} = c_{-}^{mn}, \quad c_{+}^{mn} = c_{-}^{mn}.
\]

Let us examine the algebra generated by \( e_1, e_2 \) and \( \sigma \) and the extension of the comultiplication on them:

\[
[e_1, e_2] = 0, \quad e_1 \sigma = \sigma e_2, \quad \sigma^2 = 1, \quad \Delta e_i = e_i \otimes 1 + 1 \otimes e_i, \quad \Delta \sigma = (\sigma \otimes \sigma)(q^{\frac{1}{2}}(e_2 \otimes e_1 - e_1 \otimes e_2)).
\]

Note that the expression \( q^{\frac{1}{2}}(e_2 \otimes e_1 - e_1 \otimes e_1) \) makes sense as an element of the multiplier algebra of the tensor product, indeed, taking the example \( c_{ij}^+ \otimes c_{kl}^+ \), for instance, we have:

\[
q^{\frac{1}{2}}(e_2 \otimes e_1 - e_1 \otimes e_1) c_{ij}^+ \otimes c_{kl}^+ = q^{\frac{1}{2}}(j-k-i \otimes j \otimes i) c_{ij}^+ \otimes c_{kl}^+ \in (DT_q)^* \otimes (DT_q)^*.
\]

The elements \( e_1, e_2 \) are self-adjoint. Out of the above relations we might immediately obtain:

**Proposition 3.3.** The algebra defined above is a twist of the cocommutative crossed product of \( u(1) \times u(1) \) with \( \mathbb{CZ}_2 \) by the Cartan element \( q^{\frac{1}{2}}(e_1 \otimes e_2) \).

**Proof:** Indeed, using (17) we have:

\[
q^{\frac{1}{2}}(e_1 \otimes e_2)(\sigma \otimes \sigma) = (\sigma \otimes \sigma)q^{\frac{1}{2}}(e_2 \otimes e_1),
\]

so:

\[
\Delta \sigma = (\sigma \otimes \sigma) \left( q^{\frac{1}{2}}(e_2 \otimes e_1 - e_1 \otimes e_2) \right) = q^{\frac{1}{2}}(e_1 \otimes e_2)(\sigma \otimes \sigma)q^{-\frac{1}{2}}(e_1 \otimes e_2).
\]

Since \( e_1 \) and \( e_2 \) commute with each other their coproduct does not change. \( \blacksquare \)
3.1. **The action of the dual on the quantum torus.** Similarly as in the classical case of the Lie group of the torus we might interpret the dual algebra (and its multiplier) as the symmetry algebra of the $DT_q$, in terms of the action of the Hopf algebra on the algebra of the double quantum torus:

\[
\begin{align*}
  e_1 &\triangleright U_\pm = U_\pm, \quad e_1 \triangleright V_\pm = 0, \\
  e_2 &\triangleright U_\pm = 0, \quad e_2 \triangleright V_\pm = V_\pm, \\
  \sigma &\triangleright U_\pm = V_\mp, \quad \sigma \triangleright V_\pm = U_\mp.
\end{align*}
\]

(18)

4. **Finite extensions of the torus.**

So far, we have been treating $q$ as a generic parameter, $|q| = 1$. An interesting situation occurs, however, when $q$ is a root of unity. In fact, when $q = 1$, $DT_q$ is a commutative algebra, which is the algebra of functions on the twisted double torus:

**Observation 4.1.** Let $G$ be the crossproduct group $T^2 \rtimes \mathbb{Z}_2$ with the nontrivial action of $\mathbb{Z}_2$ on $T^2$ by the flip:

\[
\sigma \left( \begin{pmatrix} z \\ w \end{pmatrix} \right) = \left( \begin{pmatrix} w \\ z \end{pmatrix} \right),
\]

if $\sigma$ is the generator of $\mathbb{Z}_2$.

With the multiplication on $G$

\[
\left( \begin{pmatrix} z \\ w \end{pmatrix}, \sigma \right) \cdot \left( \begin{pmatrix} z' \\ w' \end{pmatrix}, \sigma' \right) = \left( \begin{pmatrix} z \\ w \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} z' \\ w' \end{pmatrix} \right), \sigma \sigma' \right),
\]

we have a group structure on the space, which topologically is a disjoint sum of two tori. The algebra of continuous functions on $G$ is a Hopf algebra and it is identified with the $C^*$ algebra $DT_q=1$.

The proof of both group structure and the identification of the obtained Hopf algebra is an easy exercise and is left to the reader.

If $q$ is a primitive nontrivial root of unity so that $q^N = 1$, $N \geq 2$ we have the following:

**Proposition 4.2.** Let $j$ be a map $j : DT_q \to DT_q$ defined as identity on polynomials of order less than $N$ in $U$ and $V$, and $j(U^N) = 1$ and $j(V^N) = 1$, and extended as the algebra homomorphism on the entire $DT_q$. Then, if $q^N = 1$, $j$ is an Hopf algebra homomorphism.

**Proof:** First, we check that the definition is correct - indeed, since $U^N$ and $V^N$ are invertible elements from the center we might map it to the unit of the algebra. Next, we verify:

\[
\Delta 1 = 1 \otimes 1 = j(U^N) \otimes j(U^N) P_+ + j(V^N) \otimes j(U^N) P_- = \Delta j(U^N),
\]
Corollary 4.3. The image \( j(DT_q) \) is an Hopf algebra \( T^f_q \).

Using the map \( j \) we might construct the exact sequences of Hopf algebras:

\[
\begin{align*}
DT^N_q & \xrightarrow{i} DT_q \xrightarrow{j} T^f_q, \\
\end{align*}
\]

where \( DT^N_q \) denotes the subalgebra generated by \( U^N \) and \( V^N \) (it is easy to demonstrate that it is a sub-Hopf algebra), \( i \) is the inclusion and \( j \) the above mentioned surjection. All maps are, of course, Hopf algebra homomorphisms and the composition \( j \circ i \) gives the counit:

\[
j \circ i(a) = 1 \epsilon(a), 1 \in T^f_q, \forall a \in DT^N_q,
\]

What remains to prove is that the kernel of \( j \) is exactly the Hopf ideal of the form \( DT^N_q \cap (DT^N_q)^+ DT_q \), where \( (DT^N_q)^+ \) is the kernel of the counit map on \( DT^N_q \).

The inclusion is obvious by construction, so let us assume that \( a \in DT_q^N \) is such that \( j(a) = 0 \). We shall restrict ourselves in the proof to the case when \( a \) is a polynomial in \( U, V \), in fact, it would be easier to use \( U_+, U_-, V_+, V_- \), for which the map \( j \) is given as:

\[
j(U_+^N) = P_+, \quad j(U_-^N) = P_-, \quad j(V_+^N) = P_+, \quad j(V_-^N) = P_-.
\]

Since \( P_+ \) and \( P_- \) are projections commuting with the entire algebra we consider them and the corresponding subalgebras separately for a while (the kernel of \( j \) splits as well). For \( P_+ \) we have a completely commutative situation and the kernel of \( j \) (restricted to \( P_+ DT_q \)) is characterized by the condition that a polynomial of two variables \( x, y \) vanishes for every pair of \( N \)-th roots of unity \( x, y \in \mathbb{C}, x^N = y^N = 1 \). Clearly, it must be of the form:

\[
(x^N - 1)p(x, y) + (y^N - 1)r(x, y),
\]

and thus it is an element of the ideal generated by \( i(U^N) - 1 \) and \( i(V^N) - 1 \).

For the \( P_- \) part we have a noncommutative situation. Take a polynomial \( p(U_-, V_-) \) and assume that \( j(p) = 0 \). If \( p = \sum a_{nm} U^n V^m \) then it means that for every \( 0 \leq i, j < N \) we have:

\[
\sum_{r,s} a_{(i+rN)(j+sN)} = 0.
\]

Let us fix \( i, j \). Since only a finite number of \( a_{(i+rN)(j+sN)} \) is different from zero the corresponding part of the polynomial \( p(U_-, V_-) \) could be
written as:
\[
\left( \sum_{r,s} a_{(i+rN)(j+sN)} U_{-rN} V_{-sN} \right) U^i V^j,
\]
where we have used the fact that \( q^N = 1 \) and therefore \( U_{-rN} \) and \( V_{-sN} \) commute with all elements. Note that we have again reduced our problem to the commutative one: we have a polynomial in \( U_{-N} \) and \( V_{-N} \) such that the sum of its all coefficients vanishes. We leave to the reader the verification that it could be written as:
\[
(U^N - 1)p_1(U_{-N}, V_{-N}) + (V^N - 1)p_2(U_{-N}, V_{-N}),
\]
for some polynomials \( p_1, p_2 \), and this clearly belongs to the ideal generated by \( i(U_{-N}) - 1 \) and \( i(V_{-N}) - 1 \).

Since we can view the algebra \( DT_q^N \) as an algebra of functions on the torus (commutative algebra generated by two unitaries), the finite Hopf algebra \( T_q^f \) could be interpreted as functions on the noncommutative “fibre”, which extends the classical torus to the quantum double torus at roots of unity.

**Proposition 4.4.** The algebra \( T_q^f \) is a finite dimensional semisimple \( C^* \) algebra, and as an algebra over \( \mathbb{C} \) is isomorphic to \( M_N(\mathbb{C}) \oplus \mathbb{C}^N \).

**Proof:** The algebra is described through the relations (1) with the additional constraints \( U^N = V^N = 1 \). Clearly, the projections \( P_+ \) and \( P_- \) define two ideals, a commutative and a noncommutative one. When we restrict ourselves to the algebra of the noncommutative ideal, then \( P_- \) becomes a unit and the algebra of this ideal is generated by two \( q \)-commuting, \( q^N = 1 \), unitaries, hence it is a complex matrix algebra of dimension \( N \).

**Proposition 4.5** (see [9], [10]). The algebra \( T_q^f \) for \( q^N = 1 \) is isomorphic to the Kac-Paljutkin finite Hopf algebra of rank \( 2N^2 \).

For the case of the dual Hopf algebra of the quantum double torus we find as well an embedded Hopf subalgebra (since we work with infinite sums the result makes sense only as a multiplier algebra):

**Proposition 4.6.** Let \( w_{ij}^\pm \), \( 0 \leq i, j < N \), be the elements defined as formal series:
\[
w_{ij}^\pm = \sum_{r,s} c_{ij}^{(i+Nr),(j+sN)} \cdot
\]
Then the subalgebra \( \mathcal{W} \) generated by \( w_{ij}^\pm \) is a Hopf subalgebra.
Proof: First, let us notice that the commutation relations will be identical as for $c_{ij}^{\pm}$:

\begin{align}
  w_{kl}^+ w_{mn}^+ &= \delta^{km} \delta^{ln} w_{+}^{mn} \\
  w_{kl}^- w_{mn}^- &= \delta^{km} \delta^{ln} w_{-}^{mn} \\
  w_{kl}^+ w_{mn}^- &= \delta^{km} \delta^{ln} w_{+}^{mn} \\
  w_{kl}^- w_{mn}^+ &= \delta^{km} \delta^{ln} w_{-}^{mn} \\
  (w_{ij}^+)^* &= w_{ij}^+ \\
  (w_{ij}^-)^* &= w_{ji}^-.
\end{align}

(21)

It only remains to prove whether the coproduct closes within $W \otimes W$ (we use the formal expressions within the multiplier of the tensor product):

\begin{align}
  \Delta w_{+}^{mn} &= \sum_r \sum_s \Delta c_{+}^{(m+Nr), (n+sN)} \\
  &= \sum_r \sum_s \sum_{i+sN} \sum_{j+nS} c_{ij}^+ \otimes c_{kl}^+ \\
  &= \sum_{i,j} c_{ij}^+ \otimes w_{+}^{([m-i]) ([n-j])} \\
  &= \sum_{0 \leq i,j < N} w_{+}^{[i][j]} \otimes w_{+}^{([m-i]) ([n-j])} \\
  &= \sum_{[i'+k'] = m} \sum_{[j'+k'] = n} w_{+}^{i'j'} \otimes w_{+}^{k'k'},
\end{align}

where we use $[x]$ to denote the operation modulo $N$ and the last sum is over indices $i', j', k', k'$, which run from 0 to $N - 1$. Of course, a similar calculation could be done for $w_{-}^{ij}$.

The algebra has $2N^2$ generators and the following proposition shows its structure:

**Proposition 4.7.** For fixed $i \neq j$, the linear span of generators $w_{ij}^+, w_{ij}^-$ forms a maximal ideal within $W$, and the $*$-representation of these generators on $\mathbb{C}^2$ is:

\begin{align}
  w_{ij}^+ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & w_{ij}^- &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
  w_{ij}^+ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & w_{ij}^- &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{align}
\textbf{Proposition 4.8.} For $i = j$, the linear span of $w_{ij}^i$ generates a commutative subalgebra, with the following representation:

\[ w_{ij}^i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_{-ij}^i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The proof of both lemmas is straightforward and therefore we skip it.

\textbf{Observation 4.9.} As an algebra, $\mathcal{W}$ is isomorphic to:

\[ M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \cdots \mathbb{C}, \]

with $\frac{1}{2}N(N - 1)$ copies of $M_2(\mathbb{C})$ and $2N$ copies of $\mathbb{C}$.

\textbf{Corollary 4.10 (see again [10, 11]).} The finite-dimensional Hopf algebra $\mathcal{W}$ is the dual to the Kac-Paljutkin algebra.

\section{5. The Multidimensional Generalization} \footnote{Note that in this section $i, j, k, \ldots$ will be multiindices unless stated otherwise, and not integers as in previous section.}

Let $G$ be a finite group, which we treat as a subgroup of a permutation group, so for $h \in G$ and $i \in \mathbb{Z}_n^2$ $h(i)$ will denote the permutation of the multiindex $i$. Let $C^i_g$, $g \in G$, $i \in \mathbb{Z}^n$ be the generators of the associative algebra $\mathcal{T}_G$:

\[ C^i_g C^j_h = \delta^{i(h,j)} C^j_{(gh)}, \tag{22} \]

Now, we can introduce a nontrivial coproduct structure on the algebra (again, valued in the multiplier of the tensor product of $\mathcal{T}_G$):

\[ \Delta C^k_g = \sum_{i + j = k} \alpha^i_g C^i_g \otimes C^j_g, \tag{23} \]

where the complex valued coefficients $\alpha^i_g$ must satisfy:

\[ \alpha^i_g \alpha^{(i+j)k}_g = \alpha^i_g \alpha^{(j+k)j}_g \tag{24} \]

\[ \alpha^i_g = \alpha^i_h \alpha^{h(i)h(j)}_g. \tag{25} \]

Then (24) guarantees that the coproduct is coassociative (in the sense of definition (3.1)) and (25) that it is an algebra homomorphism.

The first condition can be solved immediately using a bilinear form on the $\mathbb{Z}^n$. If we assume:

\[ \alpha^k_g = e^{2\pi i \theta_g(k,l)}, \tag{26} \]
where $\theta_g$ is a bilinear form then (24) is satisfied automatically and (25) leads to:

\begin{equation}
\theta_{gh}(m, n) = \theta_h(m, n) + \theta_g(h(m), h(n)),
\end{equation}

for all $g, h \in G$ and $m, n \in \mathbb{Z}^n$. Clearly, for the neutral element $e \in G$ we have $\theta_e \equiv 0$.

Note that the condition (27) is a cocycle condition valued in the bilinear forms. For simplicity we shall denote the latter space by $B$. Consider for a given $k$ all maps from $G^k$ to $B$, this forms a linear space $C^k$ and let us define a linear map $\delta : C^k \to C^{k+1}$:

\begin{equation}
\delta \phi(g_1, \ldots, g_{k+1}) = \phi(g_2, \ldots, g_{k+1}) + \\
\sum_{i=1}^{k} (-1)^i \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) + (-1)^{k+1} g_{k+1} \triangleright \phi(g_1, \ldots, g_k).
\end{equation}

where $g \triangleright \phi$ denotes the left action of $G$ on $B$. This defines a group cohomology (as $\delta^2 \equiv 0$) with values in the group module $B$ with the trivial right module multiplication and the left multiplication (action) defined by the representation (permutation) of the group $G$.

Thus our condition (27) is nothing but a cocycle condition, for $\theta$ understood as a map $G \to B$:

\[
\delta \theta = 0,
\]

so we can immediately provide at least trivial solutions $\theta = \delta \phi$, where $\phi$ is an element of $B$.

To restrict the freedom depending on the choice of the basis we observe:

**Proposition 5.1.** The map:

\[
C^j_g \mapsto e^{-\pi i \theta_g(j,j)} C^j_g = \tilde{C}^j_g,
\]

is an algebra automorphisms, which changes the coalgebra structure so that:

\begin{equation}
\Delta \tilde{C}^k_g = \sum_{i+j=k} \tilde{\alpha}^{ij}_g \tilde{C}^i_g \otimes \tilde{C}^j_g,
\end{equation}

where

\[
\tilde{\alpha}^{ij}_g = e^{\pi i (\theta_g(i,j) - \theta_g(j,i))},
\]

so $\tilde{\theta}_g^j$ is then antisymmetric for every $g$. 
Hence, without loosing any information, we can always restrict ourselves to antisymmetric cocycles $\theta_g$ and from now on we shall use the notation without tilde keeping in mind that this is already a basis in which $\theta$-s are antisymmetric.

Further, having produced a coproduct we shall demonstrate the existence of the counit and the antipode.

**Proposition 5.2.** With the above coproduct, counit:
\[
\epsilon(C^i_g) = \delta^{i,0},
\]
and the antipode:
\[
SC^k_g = C^{-g(k)}_{g^{-1}},
\]
as well as with the star structure:
\[
(C^i_g)^* = C^{g(i)}_{g^{-1}},
\]
it is a $*$-Hopf algebra provided that
\[
\theta_g(i, j)^* = -\theta^{-1}_g(g(i), g(j)),
\]
however, one can easily notice that this is guaranteed by (27) if the forms $\theta$ are real.

The algebra $T_G$ is a $*$-Hopf algebra (in the sense of multiplier Hopf algebras) and - similarly as for the dual of the double torus - it is an example of a discrete quantum group [4].

**Observation 5.3.** For $\mathbb{Z}_N$ the structure of the coproduct is set by a single bilinear antisymmetric form (matrix) $\Theta = \theta_1$. Its matrix elements have to satisfy (addition is mod $N$):
\[
\sum_{k=0}^{N-1} \Theta_{i+k,j+k} = 0.
\]
for every $0 \leq i, j < N$. This follows directly from (27).

**Example 5.4.** For the group $\mathbb{Z}_2$ the relations for the matrix $\Theta = \theta_1$ are:
\[
\Theta_{12} + \Theta_{21} = 0,
\]
\[
\Theta_{11} + \Theta_{22} = 0.
\]
and they are automatically satisfied for any antisymmetric matrix:
\[
\Theta = \begin{pmatrix}
0 & \theta \\
-\theta & 0
\end{pmatrix},
\]
The solution (36) is a trivial cocycle, it is easy to see that it is $\delta$ of a constant bilinear form and the algebra is the dual of the double torus as described by (7).

**Example 5.5.** Let us consider the simplest nontrivial example – apart from the known $\mathbb{Z}_2$ case – for the group $\mathbb{Z}_3$. As we have observed in (5.3) we are left with the following restriction for the matrix elements of $\Theta$ (we have already used its antisymmetry):

$$\Theta_{01} + \Theta_{12} + \Theta_{20} = 0,$$

and the solution is given by:

$$\Theta = \begin{pmatrix} 0 & \theta & -\rho \\ -\theta & 0 & -\theta + \rho \\ \rho & \theta + \rho & 0 \end{pmatrix}$$

### 5.1. The Lie algebra picture.

Let us define, similarly as for the double torus case (14) a formal construction of the convenient generators within the multiplier algebra:

$$e_j = \sum_{i \in \mathbb{Z}^n} i_j C^i_v, \quad j = 1, 2, \ldots, n.$$

and the elements $\tilde{g}$, labelled by the elements of the group $G$:

$$\tilde{g} = \sum_{i \in \mathbb{Z}^n} C^i_g.$$ 

Using the relations (22) we verify the algebra structure:

$$[e_i, e_j] = 0$$

$$\tilde{g}\tilde{h} = (gh)$$

$$\tilde{g}e_i = e_{g(i)}\tilde{g}$$

However, the structure of the coproduct is more complicated:

$$\Delta e_i = e_i \otimes 1 + 1 \otimes e_i$$

$$\Delta \tilde{g} = e^{2\pi i \Theta_{kl} e_k \otimes e_l} \tilde{g} \otimes \tilde{g},$$

where $\Theta$ is the matrix of the antisymmetric form $\theta$. The formula (42) can arise from a twist only if $\theta = \delta \phi$, so $\theta$ is a trivial cocycle. Then:

$$\Delta \tilde{g} = e^{2\pi i \phi_{kl} e_k \otimes e_l} \tilde{g} \otimes \tilde{g} e^{-2\pi i \phi_{kl} e_k \otimes e_l},$$

We can summarize that for the trivial cocycle the presented deformation is a twist of the cross-product of multiple dual tori Hopf algebras by a finite group.
5.2. The dual algebra. Similarly as in the original \( \mathbb{Z}_2 \)-case we might construct the dual Hopf algebra. We introduce the algebra elements \( U^i_g, \ g \in G, \ i \in \mathbb{Z}^n \) and by duality:
\[
\langle C^i_g, U^j_h \rangle = \delta_{gh} \delta^{ij},
\]
we obtain from (22) and (23) both the algebra and the coalgebra structure. Again, we restrict ourselves only to the case of the antisymmetric matrix \( \theta \):
\[
U^i_g U^j_h = \delta_{gh} e^{2\pi i \theta(i,j)} U^{i+j},
\]

\[
\Delta U^i_j = \sum_{gh=f} U^h_g \otimes U^i_h.
\]

The counit, antipode and the star structure are, respectively, given by:
\[
\epsilon(U^j_g) = \delta_{ge},
\]
\[
S(U^j_g) = U^{-g(j)}_g,
\]
\[
(U^j_g)^* = U^{-j}_g.
\]

where \( e \) is the neutral element of the group and we have, of course, identified \( \sum_{g \in G} U^0_g \) with the unity in the algebra. Of course, each \( U^0_g \) is a central, self-adjoint projector in the algebra.

**Observation 5.6.** The algebra spanned by \( U^i_g \) could be also viewed as generated by unitaries. If \( v_1, \ldots, v_n \) are the basis of \( \mathbb{Z}^n \) then:
\[
U^{v_i} = \sum_{g \in G} U^{v_i}_g,
\]
are unitary elements of the algebra, and:
\[
(U^{v_m})^*(U^{v_n})^* U^{v_m} U^{v_n} = \sum_g e^{4\pi i \theta(v_m, v_n)} U^0_g.
\]

6. Finite subalgebras

Similarly as in the \( \mathbb{Z}_2 \) case, where for \( q \) being a simple root of unity we have constructed a finite dimensional Hopf subalgebra, we might attempt to repeat the procedure here.

Let us assume that the matrices \( \theta \) are in \( M_n(\mathbb{Q}) \) and \( L \) be a lattice in \( \mathbb{Z}^n \), such that for every \( g \in G \) and every \( i \in L \) and arbitrary \( j \in \mathbb{Z}^n \) we have:
\[
\theta_g(i, j) \in \mathbb{Z}.
\]

Clearly, \( L \) must be a subring of \( \mathbb{Z}^n \), which is invariant by the action of \( G \). Then we have:
Proposition 6.1. Let $\mathcal{I}$ be the quotient $\mathcal{I} = \mathbb{Z}^n/L$. Then we might define a Hopf algebra by the following relations:

$$w_g^{[i]} = \sum_{p \in L} C_g^{i+p}, \; [i] \in \mathcal{I}.$$ 

It is an easy exercise to prove that this is a sub algebra of $T_G$:

$$\Delta w_g^{[i]} = \sum_{s} \sum_{k+l=i+s} e^{2\pi i \theta_g(k,l)} C_g^{k} \otimes C_g^{l} = \ldots$$

(46)

For the coaction we have:

$$\Delta w_g^{[i]} = \sum_{s} \sum_{k+l=i+s} e^{2\pi i \theta_g(k,l)} C_g^{k} \otimes C_g^{l} = \ldots$$

now, since we know that $\theta_g(k,l)$ is integer for every $k \in L$ and arbitrary $l$ the factor in front of the tensor product depends only on the class of $k$ and $[l]$ in the quotient, $[k], [l] \in \mathcal{I}$. Therefore:

$$\ldots = \sum_{s} \sum_{t} \sum_{l} e^{2\pi i \theta_g([k],[i+s-k-t])} C_g^{k+t} \otimes C_g^{i+s-k-t} = \ldots$$

summing up over $s$ and $t$ and keeping in mind that $[i+s-k-t] = [i]-[k]$ we finally obtain:

$$\ldots = \sum_{[k]+[l]=[i]} e^{2\pi i \theta_g([k],[l])} C_g^{[k]} \otimes C_g^{[l]},$$

where the sum is over $[k], [l] \in \mathcal{I}$.

As for the dual case, the arguments follow the line of construction in Proposition 4.2. We notice that the elements $U_v^i$, for $v_i \in L$ form a commutative subalgebra and, since $L$ is invariant with respect to the action of the group $G$, it is a Hopf-subalgebra $T^L_G \subset T_G$.

Then we can construct (similarly as in the Proposition 4.2) an exact sequence of Hopf algebras:

$$T^L_G \xrightarrow{i} T_G \xrightarrow{j} F^L_G,$$

(47)

where $F^L_G$ is a finite Hopf algebra, which could be constructed as an image of the map $\tilde{j} : T_G \to T_G$ ($j$ is $\tilde{j}$ with target space restricted to the image $\tilde{j}(T_G)$):

$$\tilde{j}(U_g^i) = U_g^{[i]},$$

where $[i] \in \mathbb{Z}^n/L$, so in particular for every $k \in L$:

$$\tilde{j}(U_g^k) = U_g^0,$$

and

$$\tilde{j}(U_g^k) = 1,$$
where $U^k$ denotes $\sum_g U^k_g$.

It is easy to verify that $\tilde{j}$ is an Hopf algebra homomorphism, so its image is a Hopf algebra. It only remains an easy exercise to verify that the sequence (47) is exact.

6.1. The example of \( \mathbb{Z}_3 \). We shall present here the application of the results obtained earlier for the case of the group \( G = \mathbb{Z}_3 \) acting by cyclic permutation.

As we have already demonstrated in (38) the structure of the Hopf algebra is set by an antisymmetric matrix with two arbitrary parameters $\theta$ and $\rho$ (5.3).

We shall assume now that both $N\theta$ and $N\rho$ are integer for some $N \in \mathbb{Z}$ (we take, of course the smallest possible $N$). Define the lattice $L$ as \( N\mathbb{Z}^3 \), i.e., all integer multiindices, such that every component is divisible by $N$.

Let us denote the generators of our finite Hopf-algebra by $U^0_v$, $U^1_v$ and $U^2_v$, the indices corresponding to the elements of the group $\mathbb{Z}_3$ and $v$ taking values in the three standard basis vectors of $\mathbb{Z}^3$, for simplicity we shall abbreviate the notation and denote $U^i_v$ as $U^i$. We shall also use the projectors on each of the component denoted respectively $P_i$.

Clearly, according to (44) $U^i_j U^w_k = 0$ if $j \neq k$, so we shall have a direct sum of three subalgebras.

The commutation relations within each of them are as follows:

- $g = 0$
  \[ U^0_j U^0_j = U^0_j U^0_i, \quad (U^0_j)^N = P_0. \]

- $g = 1$
  \[ U^1_1 U^2_1 = q^a U^2_1 U^1_1, \]
  \[ U^1_1 U^3_1 = q^b U^3_1 U^1_1, \]
  \[ U^2_1 U^3_1 = q^{b-a} U^3_1 U^2_1, \]
  \[ (U^1_1)^N = P_1, \]
  where $q^N = 1$, $a = N\theta$, $b = -N\rho$.

- $g = 2$
  \[ U^1_2 U^2_2 = q^b U^2_2 U^1_2, \]
  \[ U^1_2 U^3_2 = q^{b-a} U^3_2 U^1_2, \]
  \[ U^2_2 U^3_2 = q^{-a} U^3_2 U^2_2, \]
  \[ (U^2_2)^N = P_2. \]

The first component of the algebra is commutative and is simply isomorphic to $\mathbb{C}^{(N^3)}$. As for the second and the third let us notice that the relations strongly depend on $N$ as well as $a$ and $b$. 
To provide an explicit simplest but new example, which extends the results of the double torus and the Kac-Paljutkin algebras, we shall focus our attention on the cases $N = 2$ and $N = 3$.

6.1.1. $N = 2$ case. In this case $q = -1$ and $a, b$ are either 0 or 1. Note that the second and the third component of the algebra ($g = 2$) are noncommutative (apart from the $a = b = 0$ trivial case) and, for each of the possible situations ($a = 1, b = 0; a = 0, b = 1; a = b = 1$) we obtain the algebra $M_2(\mathbb{C})$ for. Therefore the algebraic structure of the full algebra is $M_2(\mathbb{C})^4 \oplus \mathbb{C}^8$.

6.1.2. $N = 3$ case. In this situation, $q^3 = 1$ and we have, in principle, 9 possibilities for the $g = 1$ (as well as $g = 2$) components of the algebra. Let us analyze the center of the subalgebra in question. Taking a monomial $(U_1^1)^\alpha(U_1^2)^\beta(U_3)^\gamma$, we verify that it is in the center if:

\begin{align*}
\beta a + \gamma b &= 0 \pmod{3}, \\
-\alpha a + \gamma (b - a) &= 0 \pmod{3}, \\
-\alpha b - \beta (b - a) &= 0 \pmod{3}.
\end{align*}

Out of this system of linear equations we immediately get that either $a = b = 0$ or $\alpha + \beta + \gamma$ must be 0 modulo 3. Since we are interested only in the nontrivial case we assume the latter identity. In addition, we have to take one of the equations, for instance $a \beta + b \gamma = 0 \pmod{3}$.

It appears that independently of the values of $a$ and $b$ (apart from the $a = b = 0$ trivial case) the center of the subalgebra is isomorphic to $\mathbb{C}^3$ and, furthermore, the structure of the algebra is $M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$.

Since for the third component $g = 2$ we have almost the same system of equations, with the only exception that $a, b$ are replaced by $b, b - a$ (modulo 3). Therefore the same result applies in this case and finally we could summarize that for the $N = 3$ finite fibration of the ”triple tori” we obtain a Hopf algebra structure for the algebra $\mathbb{C}^{27} \oplus M_3(\mathbb{C})^6$.

7. Final remarks

We have presented in this paper the method of construction of finite semisimple Hopf algebras through the finite fibrations of the double torus, then extending the results to the generalizations - ”quantum multiple tori”.

We have related the construction with the cohomology of finite groups, showing that the Hopf algebra structure is a twist of the crossproduct
of the group by the dual tori only if the cocycle defining the deformation is trivial. It would be an interesting task to provide an explicit example of such deformation and study its finite fibrations.

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