Cluster expansion for continuous particle systems interacting via an attractive pair potential and subjected to high density boundary conditions

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Abstract

We propose a method based on cluster expansion to study the low activity/high temperature phase of a continuous particle system confined in a finite volume, interacting through a stable and finite range pair potential with negative minimum in presence of non free boundary conditions.

1 Introduction

In the area of rigorous statistical mechanics from the very beginning a great effort has been spent in order to outline the possible influence of the boundary conditions on systems confined in a finite (but possibly arbitrarily large) volume. It has been clear soon (see for instance [25, 6] and references therein) that in the regime in which more phases may coexist the presence of suitable boundary conditions may force the system in one of those phases. This has been rigourously established and put on firm ground for a large class of bounded spin systems in a lattice interacting via a finite range potential. A classical example is the nearest-neighbor Ising model in two or more dimensions. Similar results can be obtained for large classes of bounded spin systems with finite range interaction for which a very robust and effective tool, the Pirogov Sinai theory, is available (see, e.g. [29] and references therein). The effect and influence of boundary conditions on spin systems which are unbounded or interact with infinite range potential appears to be a more delicate issue to be treated rigorously and results in the literature are quite rarer. Problems related to existence and uniqueness of the infinite volume measure for unbounded spin-systems has been discussed e.g. in [15, 4] and [20], while analyticity of free energy and correlations for such systems subjected to rather general boundary conditions has been treated via cluster expansion in [1] and [24].

The situation is even less clear as soon as one considers continuous systems formed by classical particles in \( \mathbb{R}^d \) interacting via a pair potential (such as the Lennard-Jones potential, the Morse potential or even much simpler potentials, e.g. finite range). In this case the only phase which has been rigourously analyzed is the low density/high temperature phase and no proof on the existence of phase transitions has been furnished nor a consistent and rigorous treatment of such systems outside the low density/high temperature region has been provided with the sole exception of the result obtained by Mazel, Lebowitz and Presutti in 1999 [17].
The relation between the boundary condition and the macroscopical behavior of continuous classical particle systems could in principle be studied rigorously at least in the low density/high temperature regime where the powerful tool given by the cluster expansion is available. This problem is incidentally mentioned by the classical texts on rigorous statistical mechanics (see e.g. [25, 6]), where however the computations related to the analyticity of the pressure of the gas in the low density/high temperature regime are always performed assuming free boundary conditions. Although it is widely believed that the macroscopic behavior of continuous particle systems in the region of parameters \( \lambda \) and \( \beta \) where the cluster expansion converges is not affected by (reasonably well behaved) boundary conditions, we are not aware of any rigorous result about this issue as soon as one considers pair potentials with a negative (attractive) tail. In particular, the independence of convergence radius of the cluster expansion from (reasonably well behaved) boundary conditions can be established only by assuming that the pair potential \( v \) is non-negative, in addition to stable and regular (see e.g. the remark at the end of pag. 3 in [27]). Of course the assumption \( v \geq 0 \) rules out nearly all physically relevant pair potential which are usually attractive (and thus negative) at large distances. In this respect it is symptomatic to observe that all results on the dynamics of continuous particle systems which use directly or indirectly the convergence of the cluster expansion treat only systems interacting via positive potentials (see e.g. [2, 3, 9, 10, 12, 18, 27, 28]).

Rigorous result on the influence of boundary conditions in classical particle systems has been only of very general nature, such as well-definiteness and existence problems of the infinite volume measure for such systems (see e.g. [13] and references therein). In particular, the problem of uniqueness of the infinite volume measure (in the high temperature/low density region) has been discussed in [16], [21], [14] and references therein. To our knowledge nothing has been published so far about how the analyticity region of the pressure may be affected by the boundary conditions in continuous particle systems interacting via non purely repulsive potentials.

In this paper we consider a system of continuous classical particles in \( d \) dimensions confined in a finite volume \( \Lambda \) in the grand canonical ensemble at fixed inverse temperature \( \beta \) and fixed fugacity \( \lambda \). We assume that these particles interact via a stable pair potential \( v \) which, for simplicity, we impose to be finite range. On the other hand, we allow \( v \) to have a negative tail. In other words the stability constant of \( v \) may be strictly positive. This system would have a fixed mean density \( \rho^0_\Lambda(\lambda, \beta) \) when submitted to free boundary conditions. We then fix a boundary configuration \( \omega \) outside \( \Lambda \) (i.e. in \( \mathbb{R}^d \setminus \Lambda \)) allowing a density \( \rho_\omega \) which may be much larger than \( \rho^0_\Lambda(\beta, \lambda) \) but has to be uniformly bounded. Note that we allow \( \rho_\omega \) to be arbitrary so that we are actually allowing boundary condition with arbitrarily large (but bounded) densities. With these assumptions we show that the Mayer series of the pressure of the system in presence of the boundary condition \( \omega \) can be written as the sum of two terms. The first series, the bulk term, has a radius of analyticity in the activity \( \lambda \) that coincides with the free boundary condition convergence radius. The second series, the boundary term, has an \( \omega \)-dependent radius of analyticity decreasing exponentially with \( \rho_\omega \), but it tends to zero as \( \Lambda \) goes to infinity. Moreover, we show that the bulk term of the finite volume pressure in presence of boundary conditions \( \omega \) tends to the Mayer series of the pressure calculated with free boundary conditions.

### 2 The model and results

We consider a system of classical continuous particles confined in a bounded compact region \( \Lambda \) of \( \mathbb{R}^d \) interacting via a translational invariant pair potential \( v \). We will suppose hereafter that \( \Lambda \) is a
cube of size $2L$ centered at the origin and $\lim \Lambda \to \infty$ means simply that $L \to \infty$. We will denote by $|\Lambda| = (2L)^d$ the volume of $\Lambda$ and in general if $U$ is a compact in $\mathbb{R}^d$ we denote by $|U|$ its volume. We denote by $x_i \in \mathbb{R}^d$ the position vector of the $i$th particle of the system and by $\|x_i\|$ its Euclidean norm.

We will further suppose that our system is subjected to a boundary condition $\omega$ which is typically a locally finite countable set of points in $\mathbb{R}^d$ (not necessarily distinct), representing the positions of a set of fixed particles in $\mathbb{R}^d$. Namely, $\omega$ is a set such that for any compact subset $C \subset \mathbb{R}^d$, $\#(\omega \cap C) < +\infty$ (here $\#(\omega \cap C)$ is the cardinality of the set $\omega \cap C$). As usual, we will suppose that each particle inside $\Lambda$, say at position $x \in \Lambda$, feels the effect of boundary condition $\omega$ through the potential energy generated by the particles of the configuration $\omega$ which are in $\Lambda^c = \mathbb{R}^d \setminus \Lambda$.

We are interested in studying the behavior of the systems in the limit $\Lambda \to \infty$ for a fixed boundary condition $\omega$ and how eventually this limit may be influenced by $\omega$, having in mind that as the volume $\Lambda$ invades $\mathbb{R}^d$ the fixed particles of $\omega$ entering in $\Lambda$ are disregarded and only those boundary particles outside $\Lambda$ influence particles inside $\Lambda$.

**Assumptions on the pair potential**

The pair potential $v$ is supposed to be a translational invariant, symmetric and Lebesgue measurable function. Therefore it is completely defined by a function $v(x)$ in $\mathbb{R}^d$ with values in $\mathbb{R} \cup \{+\infty\}$ such that $v(x) = v(-x)$ for all $x \in \mathbb{R}^d$. We further assume that

1. $v$ is stable: namely, $v$ is such that for some constant $C \geq 0$, for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in \mathbb{R}^d$

   \[ \sum_{1 \leq i < j \leq n} v(x_i - x_j) \geq -Cn. \]  

2. $v$ is finite range: there exists $R > 0$ such that

   \[ v(x) = 0 \quad \text{for all} \quad \|x\| \geq R. \]  

The optimal constant $C$ in (2.2) is called the stability constant of the potential $v$ and will be denote with the symbol $B_v$. Then

\[ B_v = \sup_{n \geq 2, (x_1, \ldots, x_n) \in \mathbb{R}^{dn}} \left\{ -\frac{1}{n} \sum_{1 \leq i < j \leq n} v(x_i - x_j) \right\}. \]

Note that if $B_v > 0$ then there are points $x \in \mathbb{R}^d$ such that $v(x) < 0$. Let us denote by $v^-$ the negative part of $v$, namely, for $r \in [0, +\infty)$,

\[ v^-(r) = \max\{0, -v(r)\}. \]

By (2.2), the potential $v$ is bounded below by $-2B_v$ and hence $v^-$ is bounded above by $2B_v$. Therefore, by (2.1), we have that

\[ \int_{\mathbb{R}^d} v^-(x)dx \leq 2B_v V_d(R), \]  

where $V_d(R)$ is the volume of the $d$-dimensional ball of radius $R$. 

\[ 3 \]
We will suppose hereafter that \( \mathbb{R}^d \) is partitioned in elementary cubes \( \Delta \) of suitable size \( \delta > 0 \). Along the paper we will denote by \( \mathbb{R}^d_\delta \) the set of all these cubes and, given \( x \in \mathbb{R}^d \), we will denote by \( \Delta(x) \) the cube of \( \mathbb{R}^d_\delta \) to which \( x \) belongs. Moreover, for sake of simplicity we assume that \( \Lambda \) is so chosen in such a way \( \Lambda \) and \( \Lambda^c \) are both the union of elementary cubes in \( \mathbb{R}^d_\delta \), (in other words, for any \( \Delta \in \mathbb{R}^d_\delta \), either \( \Delta \subset \Lambda \) or \( \Delta \subset \Lambda^c \)). We denote by \( \Lambda_\delta \) (respectively \( \Lambda^c_\delta \)) the set of elementary cubes whose union is \( \Lambda \) (respectively \( \Lambda^c \)) and of course by construction \( \Lambda_\delta \cup \Lambda^c_\delta = \mathbb{R}^d_\delta \). Given \( \Delta \in \mathbb{R}^d_\delta \) and \( x \in \mathbb{R}^d \) we let \( \text{dist}(\Delta, x) = \inf_{y \in \Delta} \|x - y\| \) and we will suppose that \( \delta \) is chosen suitably small in such a way so that for any \( x \in \mathbb{R}^d_\delta \)

\[
\delta^d \sum_{\Delta \in \mathbb{R}^d_\delta \text{ s.t. } \text{dist}(\Delta, x) \leq R} 1 \leq 2V_d(R).
\]

\((2.4)\)

Assumptions on boundary conditions

Given the partition of \( \mathbb{R}^d \) in elementary cubes \( \Delta \) of size \( \delta > 0 \) described above and given a locally finite boundary condition \( \omega \), we define the density of \( \omega \) as the function \( \rho^\omega_\delta : \mathbb{R}^d \to [0, +\infty) \)

\[
x \mapsto \rho^\omega_\delta(x)
\]

with

\[
\rho^\omega_\delta(x) = \frac{\#(\omega \cap \Delta(x))}{\delta^{-d}},
\]

then, by definition, \( \rho^\omega_\delta(y) \) is constant for all \( y \in \Delta(x) \). Since \( \omega \) is locally finite, \( \rho^\omega_\delta(x) \) is everywhere finite.

Our assumption on the set of allowed boundary conditions is as follows.

(iii) \( \omega \) is admissible: namely, there exists a finite positive number \( \rho_\omega \) such that, for all elementary cubes \( \Delta \subset \mathbb{R}^d_\delta \),

\[
\sup_{\Delta \subset \mathbb{R}^d_\delta} \frac{\#(\omega \cap \Delta)}{|\Delta|} \leq \rho_\omega.
\]

We call \( \Omega_\rho \) the space of all locally finite configurations of particles in \( \mathbb{R}^d \) with maximal density \( \rho \) and we set \( \Omega^\star = \cup_{\rho \geq 0} \Omega_\rho \). Note that the free boundary condition \( \omega = \emptyset \) is obviously in \( \Omega_\rho \), for all \( \rho > 0 \).

Let \( \partial \Lambda \) denotes the boundary of \( \Lambda \) and let us define, for \( x \in \Lambda \) fixed,

\[
d^\Lambda_x = \text{dist}(x, \partial \Lambda) = \inf_{y \in \partial \Lambda} \|x - y\|.
\]

For a fixed volume \( \Lambda \) and a fixed boundary condition \( \omega \), let us define the function \( w^\omega_\Lambda : \Lambda \to \mathbb{R} \) as follows

\[
w^\omega_\Lambda(x) = \sum_{y \in \omega \cap \Lambda^c} v(x - y).
\]

\((2.5)\)

This function represents the potential energy felt by a particle sitting in the point \( x \in \Lambda \) due to the fixed particles of the boundary condition \( \omega \) sitting in points outside \( \Lambda \). Note that by the
assumptions (ii) on the pair potential and (iii) on the admissible boundary conditions we have that $w_{\Lambda}(x)$ is different from zero only in the frame inside $\Lambda$ constituted by the points at distance less than $R$ to the boundary $\partial \Lambda$.

The partition function of the system in the grand canonical ensemble at fixed inverse temperature $\beta > 0$ and fixed fugacity $\lambda > 0$ is given by

$$\Xi_\omega^\Lambda(\lambda, \beta) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} dx_1 \ldots dx_ne^{-\beta \left[ \sum_{1 \leq i < j \leq n} v(x_i - x_j) + \sum_{i=1}^{n} w_{\omega}(x_i) \right]}.$$ (2.6)

It is easy to check that with our assumptions on $v$ and $\omega$ the power series in the l.h.s. of (2.6) is absolutely convergent for all $\lambda \in \mathbb{C}$. Indeed the bulk factor $\sum_{1 \leq i < j \leq n} v(x_i - x_j)$ in the exponent of the integrand in the r.h.s. of (2.6) is greater than $-Cn$ by the stability condition (2.2) and the boundary factor $\sum_{i=1}^{n} w_{\omega}(x_i)$ in the same exponent is also bounded from below by a negative constant times $n$, as shown in the following proposition.

**Proposition 2.1** If $\omega \in \Omega^*$, then for all $x \in \Lambda$ we have

$$w_{\omega}(x) \begin{cases} = 0 & \text{if } d_{\Lambda}(x) \geq R \\ \geq -\kappa \rho_{\omega} & \text{if } d_{\Lambda}(x) < R \end{cases}$$ (2.7)

where $\kappa = 4B_vV_d(R)$.

**Proof.** If $d_{\Lambda}(x) \geq R$, then $v(x - y) = 0$ for all $y \in \Lambda^c$ and therefore, by definition (2.5), $w_{\omega}(x) = 0$. On the other hand, if $d_{\Lambda}(x) < R$ then

$$w_{\omega}(x) = \sum_{y \in \omega \cap \Lambda^c} v(x - y) \geq -\sum_{y \in \omega \cap \Lambda^c} v^-(x - y) = -\sum_{\Delta \in \Lambda_{\omega}^c} \sum_{y \in \omega \cap \Delta} v^-(x - y) \geq -\sum_{\Delta \in \Lambda_{\omega}^c \text{ dist}(\Delta, x) < R} 2B_v \sum_{y \in \omega \cap \Delta} 1 \geq -2B_v \rho_{\omega} |\Delta| \sum_{\Delta \in \Lambda_{\omega}^c \text{ dist}(\Delta, x) < R} 1 \geq -4B_vV_d(R)\rho_{\omega}$$

where in the last inequality we have used (2.4).

The finite volume pressure of the system is given by

$$\beta p_{\omega}^\Lambda(\lambda, \beta) = \frac{1}{|\Lambda|} \log \Xi_\omega^\Lambda(\lambda, \beta)$$ (2.8)

and the thermodynamic limit of the finite volume pressure (if it exists) is

$$\beta p_{\omega}^*(\lambda, \beta) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log \Xi_\omega^\Lambda(\lambda, \beta).$$ (2.9)

The existence of the limit (2.9) when $\omega = \emptyset$ (i.e in presence of free boundary conditions) is a well established fact since the sixties. Namely, it has been proved (see [23, 26] and references therein)
that $\beta p^\rho(\lambda, \beta)$ exists and it is continuous as a function of $\lambda$ and $\beta$ in the whole physics domain $(\lambda, \beta) \in [0, +\infty) \times [0, +\infty)$, as soon as particles interact via a superstable and regular pair potential. We recall that a pair potential $v$ is superstable if it can be written as $v = v_1 + v_2$, where both $v_1$ and $v_2$ are non-negative and such that there exists a positive constant $a$ such that $v_2(x) \geq v_2(a) > 0$ for all $|x| < a$. We also recall that $v$ is regular if the function $f(x) = e^{-v(x)} - 1$ is absolutely summable in $\mathbb{R}^d$.

Later, Georgii [7 8] studied the limit (2.9) when $\omega \neq \emptyset$ and he showed that it exists and it is equal to $\beta p^\rho(\lambda, \beta)$ as soon as $\omega$ varies in a large class of allowed boundary conditions which Georgii called “tempered boundary conditions $\omega$” (see (2.24) in [7] or (2.6) in [8]) provided that the pair potential $v$, beyond superstable and regular, has a hard-core or diverges in a non summable way at short distances. According to Georgii a boundary condition $\omega$ is tempered if, for some finite positive constant $t$, $\limsup_{\lambda \to \infty} |\lambda|^{-1} \sum_{\Delta \in \Lambda_\delta} [\#(\omega \cap \Delta)]^2 \leq t$, where $\Lambda_\delta$ is a collection of cubes $\Delta$ of fixed size $\delta > 0$ forming a partition of $\Lambda$. It is also worth to mention that very recently it has been shown [23] that the limit (2.9) exists and it is equal to $\beta p^\rho(\lambda, \beta)$ assuming just stability and regularity of $v$ and considering boundary conditions $\omega$ with a density $\rho_\omega$ possibly growing to infinity with the distance from the origin at a rate which depends on decay of $v$ at large distances.

While the superstability condition is required by [7], [8], [22] and [23] to show the existence of the limit (2.9) for all positive values of $\lambda$ and $\beta$, in this paper we just need $v$ to be stable once we are only interested in the region of parameters $\lambda$ and $\beta$ such that $|\lambda| < R(\beta)$, where $R(\beta)$ is the convergence radius of the cluster expansion.

Concerning specifically the analyticity of $\beta p^\omega(\lambda, \beta)$ at low densities, it has been shown in the sixties (see e.g. [25] and references therein) that when $\omega = \emptyset$, both the finite volume pressure $\beta p^\rho_\Lambda(\lambda, \beta)$ and the infinite volume pressure $\beta p^\rho(\lambda, \beta)$ can be written in terms of power series in $\lambda$ (the finite volume Mayer series and the infinite volume Mayer series) which are analytic for all complex $\lambda$ in a disc around $\lambda = 0$ as far as the potential $v$ is stable and regular. The radius $R(\beta)$ of the analyticity disc is uniformly bounded below by a constant depending only on the temperature and the potential. The best constant for the lower bound of $R(\beta)$ for continuous particle systems interacting via stable and regular pair potentials has been given recently by Procacci and Yuhjtman in [22] where it is proved that

$$R(\beta) \geq \frac{1}{e^{3B+1}C_v(\beta)}$$

with

$$C_v(\beta) = \int_{\mathbb{R}^d} dx (1 - e^{-|v(x)|}).$$

The technique used to write $\log \Xi^\rho_\Lambda(\lambda, \beta)$ is terms of a convergent series can be naively extended also when $\omega \neq \emptyset$, but in this case the convergence radius depends on $\omega$ and in general tends to shrink to zero if $\rho_\omega \to \infty$ (unless the potential $v$ is non-negative, [27]). In this paper, we will derive a non-naive expansion of $|\Lambda|^{-1} \log \Xi^\rho_\Lambda(\lambda, \beta)$ in terms of powers of $\lambda$ whose coefficients depend on the inverse temperature $\beta$, the volume $\Lambda$ and the boundary condition $\omega$ (the so called Mayer series) and we will analyse the behavior of this series when $\Lambda$ goes to infinity and $\omega$ varies in $\Omega^*$. Our main result can be summarized by the following theorem.
**Theorem 2.1** Let \( v \) satisfies assumption (i) and (ii) and let \( \omega \in \Omega^* \). Let \( \mathcal{D}^\theta \) be the closed disc in the complex plane

\[
\mathcal{D}^\theta = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{e^{\beta B_v + 1} C_v(\beta)} \right\}
\]  

(2.12)

and let \( \mathcal{D}^\omega \) be the closed disc in the complex plane

\[
\mathcal{D}^\omega = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{e^{\kappa \rho_v e^{\beta B_v + 1} C_v(\beta)}} \right\}.
\]  

(2.13)

Then, for all \( \Lambda \) and all \( \omega \in \Omega^* \) the finite volume pressure \( p^\omega_\Lambda(\beta, \lambda) \) of the system is such that

\[
\beta p^\omega_\Lambda(\beta, \lambda) = \eta^\omega_\Lambda(\lambda, \beta) + \xi^\omega_\Lambda(\lambda, \beta),
\]

where

a) \( \eta^\omega_\Lambda(\lambda, \beta) \) is analytic in \( \lambda \) in the disk \( \mathcal{D}^\theta \) where, uniformly in \( \Lambda \), admits the bound

\[
|\eta^\omega_\Lambda(\lambda, \beta)| \leq \frac{8}{7} e^{\beta B_v + 1} |\lambda|.
\]

b) \( \xi^\omega_\Lambda(\lambda, \beta) \) is analytic in \( \lambda \) in the disk \( \mathcal{D}^\omega \) where, uniformly in \( \Lambda \), admits the bound

\[
|\xi^\omega_\Lambda(\lambda, \beta)| \leq |\lambda| e^{\kappa \rho_v e^{\beta B_v + 1}} g(\Lambda)
\]

for some \( g(\Lambda) \) such that

\[
\lim_{\Lambda \to \infty} g(\Lambda) = 0
\]

and thus

\[
\lim_{\Lambda \to \infty} \xi^\omega_\Lambda(\lambda, \beta) = 0.
\]

c) For all \( \lambda \in \mathcal{D}^\theta \) it holds

\[
\lim_{\Lambda \to \infty} \left[ \eta^\omega_\Lambda(\lambda, \beta) - p^\theta_\Lambda(\lambda, \beta) \right] = 0.
\]

3 Proof of Theorem 2.1

3.1 Mayer expansion

We start by rewriting the partition function (2.6) of the system subjected to the boundary condition \( \omega \in \Omega^* \) as follows

\[
\Xi^\omega_\Lambda(\lambda, \beta) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda} dx_1 \cdots \int_{\Lambda} dx_n e^{-\beta \sum_{1 \leq i < j \leq n} v(x_i-x_j)} f^\omega_\Lambda(x_1) \cdots f^\omega_\Lambda(x_n),
\]

(3.1)

where

\[
f^\omega_\Lambda(x) = e^{-\beta w^\omega_\Lambda(x)}
\]

(3.2)

with \( w^\omega_\Lambda(x) \) defined in (2.5).
Remark 3.1 By Proposition 2.1 we have that

\[ f_\Lambda^\omega(x) \begin{cases} = 1 & \text{if } d_\Lambda^\omega(x) \geq R \\ \leq e^{\beta \rho_\omega} & \text{if } d_\Lambda^\omega(x) < R \end{cases} \]

It is then a standard, but not trivial (see [19]), task to show that the logarithm of \( \Xi_\Lambda^\omega(\lambda, \beta) \) can be written as follows

\[ \log \Xi_\Lambda^\omega(\lambda, \beta) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n \Phi^T(x_1, \ldots, x_n) f_\Lambda^\omega(x_1) \ldots f_\Lambda^\omega(x_n) \quad (3.3) \]

with

\[ \Phi^T(x_1, \ldots, x_n) = \begin{cases} \sum_{g \in G_n} \prod_{\{i,j\} \in E_g} e^{-\beta v(x_i - x_j)} - 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases} \quad (3.4) \]

where \( G_n \) is the set of all connected graphs \( g \) with vertex set \([n] = \{1, 2, \ldots, n\}\) and edge set \( E_g \).

The Mayer series of the (finite volume) pressure in presence of non free boundary conditions \( \omega \) is defined as the power series (3.3) divided by \(|\Lambda|\), namely,

\[ \beta P_\Lambda^\omega(\beta, \lambda) = \frac{1}{|\Lambda|} \log \Xi_\Lambda^\omega(\lambda, \beta) = \sum_{n=1}^{\infty} c_n^\omega(\beta, \Lambda) \lambda^n, \quad (3.5) \]

where

\[ c_n^\omega(\beta, \Lambda) = \frac{1}{|\Lambda| n!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n \Phi^T(x_1, \ldots, x_n) f_\Lambda^\omega(x_1) \ldots f_\Lambda^\omega(x_n). \]

Note that we can also write

\[ \log \Xi_\Lambda^\omega(\lambda, \beta) = \lambda \int_\Lambda dx_0 f_\Lambda^\omega(x_0) \Pi_{x_0, \Lambda}^\omega(\beta, \lambda), \quad (3.6) \]

where

\[ \Pi_{x_0, \Lambda}^\omega(\beta, \lambda) = \sum_{n=0}^{\infty} c_n^\omega(x_0, \beta, \Lambda) \lambda^n \quad (3.7) \]

and

\[ c_n^\omega(x_0, \beta, \Lambda) = \frac{1}{(n+1)!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n \Phi^T(x_0, x_1, \ldots, x_n) f_\Lambda^\omega(x_1) \ldots f_\Lambda^\omega(x_n), \quad (3.8) \]

with

\[ \Phi^T(x_0, x_1, \ldots, x_n) = \begin{cases} \sum_{g \in G_n^0} \prod_{\{i,j\} \in E_g} e^{-\beta v(x_i - x_j)} - 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases} \quad (3.9) \]

where \( G_n^0 \) denotes now the set of all connected graphs \( g \) with vertex set \([n]_0 = \{0, 1, 2, \ldots, n\}\) and edge set \( E_g \). We agree that \( c_n^\omega(x_0, \beta, \Lambda) = 1 \) if \( n = 0 \).
Using the above notations, the pressure of the system at finite volume can also be written as

\[
\beta p^\omega_\Lambda (\beta, \lambda) = \frac{\lambda}{|\Lambda|} \int_\Lambda dx f^\omega_\Lambda (x_0) \prod_{x_0, \Lambda} (\beta, \lambda), \quad (3.10)
\]

which is an useful expression for the computations developed ahead. We conclude this section by proving the following inequality concerning the absolute value of the coefficients \(c_n^\omega (x_0, \beta, \Lambda)\) defined above.

**Proposition 3.1** For any \(x \in \Lambda\) and any \(\omega \in \Omega^*\), it holds that

\[
|c_n^\omega (x_0, \beta, \Lambda)| \leq e^{\beta \kappa \rho \omega} \frac{(n + 1)^{n-1}}{(n + 1)!} e^{\beta B_v (n+1)} [C_v (\beta)]^n. \quad (3.11)
\]

with \(C_v (\beta)\) defined in (2.11).

**Proof.** We first recall that the potential \(v\) is stable with stability constant \(B_v\), therefore we can use the bound proved in [22] (see there Proposition 1), namely,

\[
|\Phi^T (x_0, x_1, \ldots, x_n)| \leq e^{\beta B_v (n+1)} \sum_{\tau \in T_0^n} \prod_{i,j \in E_\tau} (1 - e^{-\beta |V(x_i - x_j)|}), \quad (3.12)
\]

where \(T_0^n\) is the set of trees with vertex set \(\{0, 1, 2, \ldots, n\}\). Moreover, for any \(x \in \Lambda\), by Remark 3.1 we have that \(f^\omega_\Lambda (x) \leq e^{\kappa \beta \rho \omega}\). Thus

\[
|c_n^\omega (x_0, \beta, \Lambda)| \leq \frac{1}{(n + 1)!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n |\Phi^T (x_0, x_1, \ldots, x_n)| f^\omega_\Lambda (x_1) \ldots f^\omega_\Lambda (x_n)
\]

\[
\leq e^{\beta \kappa \rho \omega} \left[ \frac{1}{(n + 1)!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n e^{\beta B_v (n+1)} \sum_{\tau \in T_0^n} \prod_{i,j \in E_\tau} (1 - e^{-\beta |V(x_i - x_j)|}) \right]
\]

\[
\leq e^{\beta \kappa \rho \omega} \left[ \frac{e^{\beta B_v (n+1)}}{(n + 1)!} \sum_{\tau \in T_0^n} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n \prod_{i,j \in E_\tau} (1 - e^{-\beta |V(x_i - x_j)|}) \right].
\]

Now, for any \(n \in \mathbb{N}\) and \(\tau \in T_0^n\) we have (see e.g. Lemma 3 in [22])

\[
\int_\Lambda dx_1 \ldots \int_\Lambda dx_n \prod_{i,j \in E_\tau} (1 - e^{-\beta |V(x_i - x_j)|}) \leq [C_v (\beta)]^n. \quad (3.13)
\]

Therefore,

\[
|c_n^\omega (x_0, \beta, \Lambda)| \leq e^{\beta \kappa \rho \omega} \frac{e^{\beta B_v (n+1)}}{(n + 1)!} [C_v (\beta)]^n \sum_{\tau \in T_0^n} 1
\]

\[
\leq e^{\beta \kappa \rho \omega} \frac{e^{\beta B_v (n+1)}}{(n + 1)!} [C_v (\beta)]^n (n + 1)^{n-1}
\]

where in the last line we have used the Cayley formula (see [5]), i.e. \(|T_0^n| = (n + 1)^{n-1}\).

\(\square\)

We stress that bound (3.11) is very crude and it may be quite strongly improved depending on the distance of the point \(x_0\) from the border of \(\Lambda\). We will analyze in some more detail the behaviour of the coefficients \(c_n^\omega (x_0, \beta, \Lambda)\) in the next section.
3.2 On the behavior of $c_n^\omega(x_0, \beta, \Lambda)$

Recall that we are supposing $\Lambda$ to be a cube of size $2L$ centered at the origin. Let us choose a monotonic increasing continuous function $h(L)$ such that $\lim_{L \to \infty} h(L) = \infty$, $\lim_{L \to \infty} h(L)/L = 0$ and define

$$\Lambda_h = \{x \in \Lambda : d_x^\Lambda > h(L)\}$$

and

$$\Lambda_h^* = \Lambda \setminus \Lambda_h$$

so that $\Lambda_h$ is a cube centered at the origin with size $2(L - h(L))$ fully contained in $\Lambda$ and clearly

$$\lim_{\Lambda \to \infty} |\Lambda_h|/|\Lambda| = 1 \quad (3.14)$$

and

$$\lim_{\Lambda \to \infty} |\Lambda_h^*|/|\Lambda| = 0. \quad (3.15)$$

**Remark 3.2** Observe that $\sqrt{L}$ is an example of a function that satisfies the proprieties described above for the function $h(L)$. However, while the properties (3.14) and (3.15) are essential for our task, the function rule of $h(L)$ does not play an important role in the calculations ahead.

Let us now choose $L$ large enough in such a way that $h(L) > R$, so that

$$n_{h(L)} = \left\lfloor \frac{h(L)}{R} - 1 \right\rfloor \quad (3.16)$$

is greater than or equal to one. Observe that

$$\lim_{\Lambda \to \infty} n_{h(L)} = +\infty. \quad (3.17)$$

**Theorem 3.1** Let $x_0 \in \Lambda_h$ and let $\omega \in \Omega^*$. Then, for all $n \leq n_{h(L)}$, we have that

$$c_n^\omega(x_0, \beta, \Lambda) = c_n^0(x_0, \beta, \Lambda). \quad (3.18)$$

Moreover, for all $n \in \mathbb{N} \cup \{0\}$ the following bound holds

$$|c_n^\omega(x_0, \beta, \Lambda)| \leq \frac{(n + 1)^{n-1}}{(n + 1)!} e^{\beta E_v(n + 1)} [C_v(\beta)]^n. \quad (3.19)$$

**Proof.** Let us start by proving identity (3.18). We recall the definition (3.9) of $\Phi^T(x_0, x_1, \ldots, x_n)$. If $g$ is any connected graph with vertex set $[n]_0$, as $v$ is finite range with $v(x) = 0$ if $\|x\| \geq R$, we have that

$$\prod_{\{i,j\} \in E_g} \left[e^{-\beta v(x_i, x_j)} - 1\right] = 0 \quad (3.20)$$
whenever for some $i \in [n]$, $\|x_0 - x_i\| \geq nR$. Indeed, given $g \in G^0_n$, suppose that there exists a vertex $i \in [n]$ of $g$ such that $\|x_0 - x_i\| \geq nR$. Then there exists a path $p_{i,0} = \{0,i_1,i_2,\ldots,i_{k-1},i_k \equiv i\}$ contained in $g$ connecting 0 to $i$, once $g$ is connected. Let $E_{p_{i,0}}$ be the edge set of such a path. Therefore

$$\prod_{\{i,j\} \in E_g} \left[ e^{-\beta v(x_i - x_j)} - 1 \right] = \prod_{\{i,j\} \in E_g \setminus E_{p_{i,0}}} \left[ e^{-\beta v(x_i - x_j)} - 1 \right] \prod_{s=1}^{k} \left[ e^{-\beta v(x_{s-1} - x_s)} - 1 \right].$$

Observe that in any case $k \leq n$, the hypothesis that $\|x_0 - x_i\| \geq nR$ implies that at least for one $s \in [k]$ we have that $\|x_{s-1} - x_s\| \geq R$ and thus $v(x_{s-1} - x_s) = 0$, so that $e^{-\beta v(x_{s-1} - x_s)} - 1 = 0$. In conclusion, if $\|x_0 - x_i\| \geq nR$, then

$$\prod_{s=1}^{k} \left[ e^{-\beta v(x_{s-1} - x_s)} - 1 \right] = 0$$

and thus (3.20) follows.

The discussion above immediately implies that $\Phi^T(x_0, x_1, \ldots, x_n) = 0$ if there exists $i \in [n]$ such that $\|x_i - x_0\| > nR$ and thus we can rewrite $c^\omega_n(x_0, \beta, \Lambda)$ as follows

$$c^\omega_n(x_0, \beta, \Lambda) = \frac{1}{(n+1)!} \int_{x_1 \in A} \int_{\|x_0-x_1\| \leq nR} dx_1 \ldots \int_{x_n \in A} \int_{\|x_0-x_n\| \leq nR} dx_n \Phi^T(x_0, x_1, \ldots, x_n)f^\omega_n(x_1) \ldots f^\omega_n(x_n).$$

Let us now suppose that $n \leq n_{h(L)}$, i.e.,

$$n \leq \frac{h(L)}{R} - 1,$$

whence, as by hypotheses $x_0 \in \Lambda_h$, then $d_{x_0}^\Lambda \geq h(L)$ and we have

$$d_{x_0}^\Lambda \geq (n+1)R. \hspace{2cm} (3.22)$$

Moreover, by the triangular inequality,

$$d_{x_i}^\Lambda \geq d_{x_0}^\Lambda - \|x_i - x_0\|, \hspace{2cm} \text{dist0}$$

hence

$$d_{x_i}^\Lambda \geq (n+1)R - \|x_i - x_0\| \geq (n+1)R - nR \geq R$$

where in the intermediate inequality we used (3.22) and in the last inequality we used that any $n$-uple $(x_1, \ldots, x_n)$ contributing to the integral of the r.h.s. of (3.21) is such that, for any $i \in [n]$, $\|x_i - x_0\| \leq nR$.

In conclusion we have shown that if $n \leq n_{h(L)}$, then for any $n$-uple $(x_1, \ldots, x_n) \in \Lambda^n$ such that $\|x_0 - x_i\| \leq nR$ for all $i \in [n]$, it holds that $d_{x_i}^\Lambda \geq R$. Recalling the Remark 3.1 we have that $f^\omega_\Lambda(x_i) = 1$ for all $i \in [n]$ in formula (3.8) when $n \leq n_{h(L)}$. Therefore, we have the identity

$$c^\omega_n(x_0, \beta, \Lambda) = \frac{1}{(n+1)!} \int_{\Lambda} dx_1 \ldots \int_{\Lambda} dx_n \Phi^T(x_0, x_1, \ldots, x_n) = c^\emptyset_n(x_0, \beta, \Lambda)$$

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for all \(x_0 \in \Lambda_h\) and for all \(n \leq n_h(L)\). Namely, we have proved the statement (3.18).

Let us now prove bound (3.19). Recalling that the potential \(v\) is stable with stability constant \(B_v\), we can once again use the bound (3.12), namely,

\[
|\Phi^T(x_0, x_1, \ldots, x_n)| \leq e^{\beta B_v(n+1)} \sum_{\tau \in T^0_n} \prod_{(i,j) \in E_\tau} (1 - e^{-\beta|V(x_i-x_j)|}),
\]

where \(T^0_n\) is the set of trees with vertex set \(\{0, 1, 2, \ldots, n\}\). Thus

\[
\int_\Lambda dx_1 \ldots \int_\Lambda dx_n |\Phi^T(x_0, x_1, \ldots, x_n)| \leq e^{\beta B_v(n+1)} \sum_{\tau \in T^0_n} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n \prod_{(i,j) \in E_\tau} (1 - e^{-\beta|V(x_i-x_j)|}).
\]

Now, for any \(n \in \mathbb{N}\) and \(\tau \in T^0_n\) we can use again inequality (3.13) and hence we have the upper bound

\[
c^\theta_n(x_0, \beta, \Lambda) \leq \frac{1}{(n+1)!} e^{\beta B_v(n+1)} [C_v(\beta)]^n \sum_{\tau \in T^0_n} 1
\]

\[
= \frac{(n+1)^{n-1}}{(n+1)!} e^{\beta B_v(n+1)} [C_v(\beta)]^n
\]

where in the last line we have once again used the Cayley formula (see [5]). Note that this bound holds also when \(n = 0\), since \(1 \leq e^{\beta B_v}\). This concludes the proof of bound (3.19).

\[\square\]

### 3.3 Conclusion of the proof of Theorem 2.1

Recalling that we chose \(L\) large enough in such a way that \(h(L) > R\), the definitions of \(\Lambda_h\) and \(\Lambda^*_h\) given at the beginning of Section 3.2 and the Identity (3.10), we can rewrite the finite volume pressure \(\beta p^\omega_{\Lambda}(\beta, \lambda)\) of our system as

\[
\beta p^\omega_{\Lambda}(\beta, \lambda) = \frac{\lambda}{|\Lambda|} \left[ \int_{\Lambda_h} dx f^\omega_{\Lambda}(x) \Pi^\omega_{x,\Lambda}(\beta, \lambda) + \int_{\Lambda^*_h} dx f^\omega_{\Lambda}(x) \Pi^\omega_{x,\Lambda}(\beta, \lambda) \right] = \frac{\lambda}{|\Lambda|} \left[ \int_{\Lambda_h} dx \Pi^\omega_{x,\Lambda}(\beta, \lambda) + \int_{\Lambda^*_h} dx f^\omega_{\Lambda}(x) \Pi^\omega_{x,\Lambda}(\beta, \lambda) \right]
\]

where the last identity follows from Remark 3.1.

By Formula (3.18) in Theorem 3.1, we have that for any \(x \in \Lambda_h\)

\[
\Pi^\omega_{x,\Lambda}(\beta, \lambda) = P^{\theta, n_h(L)}_{x,\Lambda}(\beta, \lambda) + Q^{\omega, n_h(L)}_{x,\Lambda}(\beta, \lambda)
\]

where

\[
P^{\theta, n_h(L)}_{x,\Lambda}(\beta, \lambda) = \sum_{n=0}^{n_h(L)} p^\theta_n(x, \beta, \Lambda) \lambda^n
\]

(3.24)
and

\[ Q_{x,\Lambda}^{\omega,n_{h}(L)}(\beta, \lambda) = \sum_{n=n_{h}(L)+1}^{\infty} c_{n}^{\omega}(x, \beta, \Lambda)\lambda^{n}. \]  \hspace{1cm} (3.25)

Therefore, posing

\[ \eta_{\Lambda}(\lambda, \beta) = \lambda \int_{\Lambda_{h}} dx P_{x,\Lambda}^{\theta,n_{h}(L)}(\beta, \lambda) \]  \hspace{1cm} (3.26)

and

\[ \xi_{\Lambda}(\lambda, \beta) = \frac{\lambda}{|\Lambda|} \left[ \int_{\Lambda_{h}} dx Q_{x,\Lambda}^{\omega,n_{h}(L)}(\beta, \lambda) + \int_{\Lambda_{h}} dx f_{x}^{\omega}(x) \Pi_{x,\Lambda}^{\omega}(\beta, \lambda) \right] \]  \hspace{1cm} (3.27)

we have that

\[ \beta p_{\Lambda}^{\omega}(\beta, \lambda) = \eta_{\Lambda}(\lambda, \beta) + \xi_{\Lambda}(\lambda, \beta). \]

Note that \( \eta_{\Lambda}(\lambda, \beta) \) is a polynomial of degree \( \lambda^{n_{h}(L)}. \) Let us also define the function

\[ Q_{x,\Lambda}^{0,n_{h}(L)}(\beta, \lambda) = \sum_{n=n_{h}(L)+1}^{\infty} c_{n}^{0}(x_{0}, \beta, \Lambda)\lambda^{n}. \]  \hspace{1cm} (3.28)

**Theorem 3.2** Let \( D^{\emptyset} \) be the closed disc in the complex plane defined in (2.12)

\[ D^{\emptyset} = \{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{e^{\beta B_{v} + 1}C_{v}(\beta)} \}. \]

Then the functions \( \Pi_{x,\Lambda}^{\emptyset}(\beta, \lambda) \) and \( Q_{x,\Lambda}^{0,n_{h}(L)}(\beta, \lambda) \) are analytic inside the disc \( D^{\emptyset} \) where, uniformly in \( \Lambda \) and \( x, \) they admit the bounds

\[ |\Pi_{x,\Lambda}^{\emptyset}(\beta, \lambda)| \leq (8/7)e^{\beta B_{v} + 1} \]  \hspace{1cm} (3.29)

and

\[ |Q_{x,\Lambda}^{0,n_{h}(L)}(\beta, \lambda)| \leq \frac{e^{\beta B_{v} + 1/2}}{n_{h}(L)}. \]  \hspace{1cm} (3.30)

**Proof.** We start by proving the analyticity of \( \Pi_{x_{0},\Lambda}^{\emptyset}(\beta, \lambda). \) We have straightforwardly that

\[ |\Pi_{x_{0},\Lambda}^{\emptyset}(\beta, \lambda)| \leq \sum_{n=0}^{\infty} |c_{n}^{\emptyset}(x_{0}, \beta, \Lambda)||\lambda|^{n}, \]

then, by Inequality (3.19) in Theorem 3.1, we have

\[ |\Pi_{x_{0},\Lambda}^{\emptyset}(\beta, \lambda)| \leq \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{(n+1)!} e^{\beta B_{v}(n+1)}||\lambda|C_{v}(\beta)||^{n} = e^{\beta B_{v}} \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{(n+1)!}[e^{\beta B_{v}}||\lambda|C_{v}(\beta)||^{n}. \]
Let us now set, for \( r \geq 0 \)

\[
\Theta(\beta, r) = e^{\beta B_v} \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{(n+1)!} [e^{\beta B_v r C_v(\beta)}]^n
\]

and suppose that there exists

\[
r^* = \max \left\{ r \geq 0 : \Theta(\beta, r) < +\infty \right\}
\]

(3.31),

then clearly \( \Pi^{\emptyset}_{x_0, \Lambda}(\beta, \lambda) \) is analytic for all \( |\lambda| \leq r^* \) and \( |\Pi^{\emptyset}_{x_0, \Lambda}(\beta, \lambda)| \) is bounded by \( \Theta(\beta, r^*) \) for all \( \lambda \) in the disc \( |\lambda| \leq r^* \).

To show that this is indeed true, let us just recall the Stirling bound, namely

\[
\sqrt{2\pi nn!} e^{-n} \leq n! \leq e\sqrt{2\pi n} n^{-n/2}
\]

(3.32) for all \( n \in \mathbb{N} \). So that we may bound

\[
e^{\beta B_v + 1} \left[ 1 + \frac{1}{e} \sum_{n=1}^{\infty} \frac{[e^{\beta B_v + 1 r C_v(\beta)}]^n}{(n+1)^2} \right] \leq \Theta(\beta, r) \leq e^{\beta B_v + 1} \left[ 1 + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{[e^{\beta B_v + 1 r C_v(\beta)}]^n}{(n+1)^{3/2}} \right].
\]

The series \( \sum_{n=1}^{\infty} (n+1)^{-5/2}[e^{\beta B_v + 1 r C_v(\beta)}]^n \) converges if \( e^{\beta B_v + 1 r C_v(\beta)} \leq 1 \) and diverges otherwise. Therefore we get that the number \( r^* \) defined in (3.31) does exist and it is equal to

\[
r^* = \frac{1}{e^{\beta B_v + 1 C_v(\beta)}}.
\]

Moreover,

\[
\Theta(\beta, r^*) \leq e^{\beta B_v + 1} \left[ 1 + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{(n+1)^{3/2}} \right] \leq (8/7) e^{\beta B_v + 1}
\]

and thus we have proved that \( \Pi^{\emptyset}_{x_0, \Lambda}(\beta, \lambda) \) is analytic in the closed disc \( D^{\emptyset} \) and its modulus is bounded there by \( (8/7) e^{\beta B_v + 1} \).

Let us now prove the analyticity and boundedness of the function \( Q^{\emptyset, n_{h(L)}}_{x, \Lambda}(\beta, \lambda) \) defined in (3.28) when \( \lambda \) varies in the complex disc \( D^{\emptyset} \). Using once again inequality (3.19) and the Stirling bound (3.32) and assuming that \( |\lambda| \) varies in \( D^{\emptyset} \), we have

\[
|Q^{\emptyset, n_{h(L)}}_{x, \Lambda}(\beta, \lambda)| = \sum_{n = n_{h(L)} + 1}^{\infty} |c^n_{\emptyset}(x_0, \beta, \Lambda)| |\lambda|^n \leq \sum_{n = n_{h(L)} + 1}^{\infty} \frac{(n+1)^{n-1}}{(n+1)!} e^{\beta B_v (n+1)} [C_v(\beta)]^n |\lambda|^n \leq \frac{e^{\beta B_v + 1}}{\sqrt{2\pi}} \sum_{n = n_{h(L)} + 1}^{\infty} \frac{1}{(n+1)^{3/2}} \leq \frac{e^{\beta B_v + 1}}{\sqrt{2\pi}} \sum_{n = n_{h(L)} + 1}^{\infty} \frac{1}{n^{5/2}} \leq \frac{e^{\beta B_v + 1}}{\sqrt{2\pi}} \int_{n_{h(L)}}^{\infty} \frac{1}{u^{5/2}} du = \frac{2e^{\beta B_v + 1}}{3\sqrt{2\pi}} \frac{1}{n_{h(L)}^{3/2}} < \frac{e^{\beta B_v + 1}}{n_{h(L)}^{3/2}}.
\]

Hence we have proved that \( |Q^{\emptyset, n_{h(L)}}_{x, \Lambda}(\beta, \lambda)| \) is analytic in \( D^{\emptyset} \) and bounded there according to (3.30).
Theorem 3.3 Given \( x \in \Lambda \) and \( \omega \in \Omega^* \), let \( \mathcal{D}^\omega \) the closed disc in the complex plane defined in (2.13)

\[
\mathcal{D}^\omega = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{e^{\beta \omega} e^{\beta B_v + 1} C_v(\beta)} \right\}.
\]

Then the function \( \Pi^\omega_{x,\Lambda}(\beta, \lambda) \) defined in (3.7) is analytic in the disc \( \mathcal{D}^\omega \) where, uniformly in \( \Lambda, x \) and \( \omega \)

\[
|\Pi^\omega_{x,\Lambda}(\beta, \lambda)| \leq (8/7)e^{\beta B_v + 1}.
\]

(3.33)

Moreover, let \( n_{h(\Lambda)} \) be the integer defined in (3.16), then the function \( Q^\omega_{x,h(L)}(\beta, \lambda) \) defined in (3.25) is analytic in the disc \( \mathcal{D}^\omega \) where, uniformly in \( \Lambda, x \) and \( \omega \)

\[
|Q^\omega_{x,h(L)}(\beta, \lambda)| \leq \frac{e^{\beta B_v + 1}}{n_{h(L)}}.
\]

(3.34)

Proof. The proof of (3.33) and (3.34) proceeds along the same lines described in the previous theorem. In order to prove bound (3.33), we can use the bound (3.11) on \( |c^n(x_0, \beta, \Lambda)| \) given in Proposition 3.1 together with the Stirling bound (3.32) to get, for all \( \lambda \in \mathcal{D}^\omega \),

\[
|\Pi^\omega_{x,\Lambda}(\beta, \lambda)| = \sum_{n=0}^\infty |c^n_{x_0}(x_0, \beta, \Lambda)||\lambda|^n \leq \sum_{n=0}^\infty \frac{(n+1)^{n-1}}{(n+1)!} e^{\beta B_v(n+1)} C_v(\beta) e^{\kappa B_v \omega} |\lambda|^n \leq e^{\beta B_v + 1} \left[ 1 + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^\infty \left( \frac{1}{n+1/2} \right) \right] \leq (8/7)e^{\beta B_v + 1}.
\]

Concerning now the bound (3.34) we have,

\[
|Q^\omega_{x,h(L)}(\beta, \lambda)| = \sum_{n=n_{h(L)}+1}^\infty |c^n_{x_0}(x_0, \beta, \Lambda)||\lambda|^n \leq \sum_{n=n_{h(L)}+1}^\infty \frac{(n+1)^{n-1}}{(n+1)!} e^{\beta B_v(n+1)} C_v(\beta) e^{\kappa B_v \omega} |\lambda|^n \leq e^{\beta B_v + 1} \left[ \frac{1}{\sqrt{2\pi}} \sum_{n=n_{h(L)}+1}^\infty \left( \frac{1}{n+1/2} \right) \right] \leq e^{\beta B_v + 1} \frac{1}{\sqrt{2\pi}} \int_{n_{h(L)}}^\infty \frac{1}{u^{3/2}} du = \frac{2e^{\beta B_v + 1}}{3\sqrt{2\pi}} \frac{1}{n_{h(L)}^{3/2}} < \frac{e^{\beta B_v + 1}}{n_{h(L)}^{3/2}}.
\]

Proposition 3.2 The function \( \eta^\omega_\Lambda(\lambda, \beta) \) defined in (3.26) is analytic in the whole complex plane and its modulus is bounded as

\[
|\eta^\omega_\Lambda(\lambda, \beta)| \leq (8/7)e^{\beta B_v + 1} |\lambda|
\]

as \( \lambda \) varies in the disc \( \mathcal{D}^\theta \). Moreover for \( \lambda \in \mathcal{D}^\theta(\beta) \) it holds that

\[
\lim_{\Lambda \to \infty} \eta^\omega_\Lambda(\lambda, \beta) = \beta \eta^\theta(\beta, \lambda)
\]

(3.36)
The analyticity of $\eta^c_\Lambda(\lambda, \beta)$ in the whole complex plane follows trivially from the fact that, by definition (3.26), $\eta^c_\Lambda(\lambda, \beta)$ is, as a function of $\lambda$, a polynomial of degree $n_h(L)$. The fact that the modulus $|\eta^c_\Lambda(\lambda, \beta)|$ is bounded by the r.h.s. of (3.35) when $\lambda \in D^\emptyset$ follows trivially from bound (3.29) of Theorem 3.2. Indeed, if $\lambda \in D^\emptyset$, recalling the definition (3.24) of $P^\emptyset_{x,\Lambda}(\beta, \lambda)$, it easily follows from Theorem 3.2 (see the proof of inequality (3.29)) that

$$|P^\emptyset_{x,\Lambda}(\beta, \lambda)| \leq \sum_{n=0}^{n_h(L)} |c^\emptyset_n(x_0, \beta, \Lambda)||\lambda|^n \leq \sum_{n=0}^{\infty} |c^\emptyset_n(x_0, \beta, \Lambda)||\lambda|^n \leq (8/7)e^{3B_{\emptyset}+1}. \quad (3.37)$$

Therefore, if $\lambda \in D^\emptyset$,

$$|\eta^c_\Lambda(\lambda, \beta)| \leq \frac{|\lambda|}{|\Lambda|} \int_{\Lambda} dx|P^\emptyset_{x,\Lambda}(\beta, \lambda)| \leq \frac{\lambda}{|\Lambda|} \int_{\Lambda} (8/7)e^{3B_{\emptyset}+1} dx \leq |\lambda|(8/7)e^{3B_{\emptyset}+1}.$$

In order to prove (3.36), observe that

$$\beta p^\emptyset(\beta, \lambda) = \frac{\lambda}{|\Lambda|} \int_{\Lambda} dx \Pi^\emptyset_{x,\Lambda}(\beta, \lambda) = \frac{\lambda}{|\Lambda|} \int_{\Lambda} dx \left( P^\emptyset_{x,\Lambda}(\beta, \lambda) + Q^\emptyset_{x,\Lambda}(\beta, \lambda) \right),$$

so that

$$\eta^c_\Lambda(\lambda, \beta) - \beta p^\emptyset(\beta, \lambda) = \frac{\lambda}{|\Lambda|} \left[ \int_{\Lambda} dx P^\emptyset_{x,\Lambda}(\beta, \lambda) - \int_{\Lambda} dx P^\emptyset_{x,\Lambda}(\beta, \lambda) - \int_{\Lambda} dx Q^\emptyset_{x,\Lambda}(\beta, \lambda) \right],$$

hence

$$|\eta^c_\Lambda(\lambda, \beta) - \beta p^\emptyset(\beta, \lambda)| \leq \frac{|\lambda|}{|\Lambda|} \left[ \int_{\Lambda} dx|P^\emptyset_{x,\Lambda}(\beta, \lambda)| + \int_{\Lambda} dx|Q^\emptyset_{x,\Lambda}(\beta, \lambda)| \right].$$

Now, if $\lambda \in D^\emptyset$, by (3.37) and (3.30) we have that

$$|P^\emptyset_{x,\Lambda}(\beta, \lambda)| \leq (8/7)e^{3B_{\emptyset}+1}$$

and

$$|Q^\emptyset_{x,\Lambda}(\beta, \lambda)| \leq \frac{e^{3B_{\emptyset}+1}}{n_{h(L)}^{3/2}}.$$

Hence

$$|\eta^c_\Lambda(\lambda, \beta) - \beta p^\emptyset(\beta, \lambda)| \leq |\lambda|e^{3B_{\emptyset}+1} \left[ \frac{(8/7)|\Lambda| |\Lambda_h|}{|\Lambda|} + \frac{1}{n_{h(L)}^{3/2}} \right] = |\lambda|e^{3B_{\emptyset}+1} \left[ \frac{(8/7)|\Lambda_h|}{|\Lambda|} + \frac{1}{n_{h(L)}^{3/2}} \right].$$

Recalling (3.15), we have that for $\lambda \in D^\emptyset$

$$\lim_{\Lambda \to \infty} |\eta^c_\Lambda(\lambda, \beta) - \beta p^\emptyset(\beta, \lambda)| \leq \lim_{\Lambda \to \infty} |\lambda|e^{3B_{\emptyset}+1} \left[ \frac{(8/7)|\Lambda_h|}{|\Lambda|} + \frac{1}{n_{h(L)}^{3/2}} \right] = 0 \quad (3.38)$$
where in the last line we used (3.15) and (3.17).

Of course (3.38) implies that, for all \( \lambda \in \mathcal{D} \), it holds that

\[
\lim_{\Lambda \to \infty} \eta^\omega_\Lambda(\lambda, \beta) = \lim_{\Lambda \to \infty} \beta p^\theta_\Lambda(\beta, \lambda) = \beta p^\theta(\beta, \lambda),
\]

which ends the proof.

\[\square\]

**Proposition 3.3** The function \( \xi^\omega_\Lambda(\lambda, \beta) \) defined in (3.27) is analytic and bounded as far as \( \lambda \in \mathcal{D} \). Moreover, it holds that

\[
\lim_{\Lambda \to \infty} \xi^\omega_\Lambda(\lambda, \beta) = 0.
\]

(3.39)

**Proof.** We recall the definition (3.27) of \( \xi^\omega_\Lambda(\lambda, \beta) \)

\[
\xi^\omega_\Lambda(\lambda, \beta) = \frac{\lambda}{|\Lambda|} \left[ \int_{\Lambda_h} dx Q^\omega_{x,\Lambda}(\beta, \lambda) + \int_{\Lambda_h^*} dx f^\omega_\Lambda(x) \Pi^\omega_{x,\Lambda}(\beta, \lambda) \right].
\]

By Theorem 3.3 both \( Q^\omega_{x,\Lambda}(\beta, \lambda) \) and \( \Pi^\omega_{x,\Lambda}(\beta, \lambda) \) are analytic and bounded in the closed disc \( \mathcal{D} \). This immediately implies that \( \xi^\omega_\Lambda(\lambda, \beta) \) is also analytic (and bounded) in \( \mathcal{D} \).

Concerning the limit (3.39), we can use the bounds (3.33) and (3.34) given in Theorem 3.3 for \( \Pi^\omega_{x,\Lambda}(\beta, \lambda) \) and \( Q^\omega_{x,\Lambda}(\beta, \lambda) \) respectively and, recalling Remark 3.1, we have that \( f^\omega_\Lambda(x) \leq e^{\beta \kappa \rho \omega} \) and so

\[
|\xi^\omega_\Lambda(\lambda, \beta)| \leq \frac{\lambda}{|\Lambda|} \left[ \int_{\Lambda_h} dx |Q^\omega_{x,\Lambda}(\beta, \lambda)| + \int_{\Lambda_h^*} dx |f^\omega_\Lambda(x)| |\Pi^\omega_{x,\Lambda}(\beta, \lambda)| \right]
\]

\[
\leq \frac{\lambda}{|\Lambda|} \left[ \int_{\Lambda_h} dx e^{\beta B_v + \frac{3}{2} n_h(L)} + \int_{\Lambda_h^*} dx e^{\beta \kappa \rho \omega} (8/7) e^{\beta B_v + 1} \right]
\]

\[
\leq (8/7) \lambda e^{\beta \kappa \rho \omega} e^{\beta B_v + 1} \left[ \frac{|\Lambda_h|}{|\Lambda| n_h(L)} + \frac{|\Lambda_h^*|}{|\Lambda|} \right].
\]

Hence

\[
\lim_{\Lambda \to \infty} |\xi^\omega_\Lambda(\lambda, \beta)| \leq (8/7) \lambda e^{\beta \kappa \rho \omega} e^{\beta B_v + 1} \left[ \lim_{\Lambda \to \infty} \frac{|\Lambda_h|}{|\Lambda| n_h(L)} + \frac{|\Lambda_h^*|}{|\Lambda|} \right]
\]

\[
= (8/7) \lambda e^{\beta \kappa \rho \omega} e^{\beta B_v + 1} \left[ \lim_{\Lambda \to \infty} \frac{|\Lambda_h|}{|\Lambda|} \lim_{\Lambda \to \infty} \frac{1}{n_h(L)} + \lim_{\Lambda \to \infty} \frac{|\Lambda_h^*|}{|\Lambda|} \right]
\]

\[
= 0
\]

where in the last line, by the definitions given at the beginning of Section 3.2, we have used that

\[
\lim_{\Lambda \to \infty} \frac{|\Lambda_h|}{|\Lambda|} = 1, \quad \lim_{\Lambda \to \infty} n_h(L) = +\infty, \quad \lim_{\Lambda \to \infty} \frac{|\Lambda_h^*|}{|\Lambda|} = 0.
\]

This ends the proof of Proposition 3.3.

\[\square\]
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