Anholonomy and Geometrical Localization in Dynamical Systems

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We characterize the geometrical and topological aspects of a dynamical system by associating a geometric phase with a phase space trajectory. Using the example of a nonlinear driven damped oscillator, we show that this phase is resilient to fluctuations, responds to all bifurcations in the system, and also finds new geometric transitions. Enriching the phase space description is a novel phenomenon of “geometrical localization” which manifests itself as a significant deviation from planar dynamics over a short time interval.

Driven damped nonlinear oscillators, Duffing’s equations for nonlinear mechanical vibrations, the Lorenz equations describing 2D convection in a heated fluid, nonlinear electronic circuit equations, etc., are some well-known examples \cite{4} of Eq. (1).

A phase trajectory of the dynamical system of Eq. (1) can be viewed as a space curve generated by the three-dimensional vector $\mathbf{r}(t)$ parametrized by the time $t$. From Eq. (1), we have $|\mathbf{r}_t|=|v|=v=s_t$, giving $s(t)=\int v\,dt$ as the arc length on the space curve. Thus the unit tangent vector is given by $\mathbf{T}=\mathbf{r}_t/v$. In accordance with the usual formalism \cite{5} for a space curve, we define the orthogonal right-handed triad of unit vectors $(\mathbf{N}, \mathbf{B}, \mathbf{B}_t)$, where $\mathbf{N}$ and $\mathbf{B}$ denote, respectively, the normal and bi-normal unit vectors on the curve. The Frenet-Serret (FS) equations can be written in terms of the variable $t$ as

$$\mathbf{T}_t=v\kappa\mathbf{N}, \quad \mathbf{N}_t=-v\kappa\mathbf{T}+v\tau\mathbf{B}, \quad \mathbf{B}_t=-v\tau\mathbf{N}, \quad (2)$$

where the curvature $\kappa$ and the torsion $\tau$ are functions of $s$ and determine the local geometry of the trajectory. It can be shown that \cite{5}

$$\kappa(s)=|\mathbf{r}_t \times \mathbf{r}_{tt}|/|\mathbf{r}_t|^3, \quad (3)$$

$$\tau(s)=\mathbf{r}_t \cdot (\mathbf{r}_tt \times \mathbf{r}_{ttt})/|\mathbf{r}_t \times \mathbf{r}_{tt}|^2. \quad (4)$$

Intuitively, the curvature measures the departure of a curve from a straight line, while the torsion measures its non-planarity.

To understand how the anholonomy of a trajectory arises \cite{6}, we rewrite Eqs. (2) as $\mathbf{F}_t=\xi \times \mathbf{F}$, where $\mathbf{F}$ stands for $\mathbf{T}, \mathbf{N}$, or $\mathbf{B}$, and $\xi=-v\kappa\mathbf{B}+v\tau\mathbf{T}$. This shows that as one moves on the trajectory $\mathbf{r}(t)$, the FS triad $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ rotates with angular velocities $v\kappa$ and $v\tau$ around $\mathbf{B}$ and $\mathbf{T}$ respectively. In the $(\mathbf{N}, \mathbf{B})$ plane, we may introduce two orthogonal unit vectors $\mathbf{u}$ and $\mathbf{w}$ such that the triad $(\mathbf{T}, \mathbf{u}, \mathbf{w})$ does not rotate around $\mathbf{T}$. This is achieved by using a Fermi-Walker parallel transport of any vector $\mathbf{P}$ moved along the curve according to $\delta \mathbf{P}/\delta t=\mathbf{P}_t-v\kappa(\mathbf{B} \times \mathbf{P})$. Thus, as one moves from $t=0$ to $t=T$, a geometric phase $\Phi_T=\int_0^T v\tau\,dt=\int_0^s \tau\,ds$ develops between its natural frame $(\mathbf{N}, \mathbf{B})$ and the “non-rotating frame” $(\mathbf{u}, \mathbf{w})$. The anholonomy $\Phi_T$ can also be interpreted in another way: Representing the rotation of the triad using Euler angles $(\theta, \varphi, \psi)$ yields \cite{6}

$$\Phi_T=2\pi - \int \sin \theta \,d\theta = \int_0^\psi \varphi \,d\varphi. \quad (5)$$

For a periodic trajectory, the second term is just the solid angle subtended by the area enclosed by the closed path $\gamma(s)$ traced out by the tangent indicatrix \cite{5} on the unit sphere $S^2$. The same result
holds good for a non-periodic trajectory as well, since it can always be closed using a geodesic \cite{7} on the sphere.

As is clear, the anholonomy $\Phi_T$ of a trajectory also characterizes its non-planarity. Since both periodic as well as chaotic trajectories occur in general, a more useful quantity is the geometric phase per unit time, defined over a long time $T$,

$$
\phi_T \equiv \frac{\Phi_T}{T} = \frac{1}{T} \int_0^T v \tau \, dt
$$

where $T$ is much larger than all the natural time scales in the problem. $\phi_T$ as defined above can also be thought of as the mean angular velocity of the FS triad. As we show below, it turns out to be a good order parameter for both periodic and chaotic trajectories.

We now apply this formulation to a driven, damped, bilinear oscillator \cite{9} characterized by two different frequencies $\omega_{1,2}$ for positive and negative displacements respectively:

$$
x_{tt} + 2\beta x_t = -\omega_{1,2}^2 x + f \cos \omega t, \quad (\beta > 0).
$$

This can be written in the form of Eq. (1) by choosing $\mathbf{r} = (x, y, z) = (x, x_t, x_{tt})$. For $\beta = 0$, $f = 0$, the oscillator has a frequency $\omega_{bd} = 2\omega_1\omega_2/(\omega_1 + \omega_2)$. The system is piecewise linear, and analytic solutions can be obtained for $x > 0$ and $x < 0$. The discontinuity at the origin makes it essentially nonlinear, $\alpha = \omega_2/\omega_1$ being the nonlinearity parameter. Without loss of generality, we choose the units of $t$ and $x$ such that $\omega = 1$ and $f = 1$. The two key control parameters are then $\omega_1$ and $\omega_2$, or, equivalently, $\alpha$ and $\omega_1$. The results to be presented are for fixed $\omega_1 = 1.5$, although we have investigated the full two-dimensional parameter space \cite{8}. For a linear damped driven oscillator ($\omega_1 = \omega_2$), it is easy to show that $\tau$ calculated using Eq. (4) is identically zero and hence the attractor is a planar limit cycle.

Our numerical routine uses the analytical solution for $x(t)$ in each half of the phase space. Starting with an initial state $x > 0, x_t = 0$ (so that the oscillator frequency is $\omega_1$), we use very small time increments to determine precisely the time and the velocity when $x$ reaches 0. These are used as the new initial conditions for dynamics in the region $x < 0$ with oscillator frequency $\omega_2$, and the process is repeated. The availability of an analytic solution is particularly useful for computing the torsion of the phase trajectory using Eq. (4).

We have carried out a detailed exploration \cite{8} of the system in the parameter space, and calculated the local phase space variable $x$, the local geometric variable $\tau$, the long-time average anholonomy $\bar{\phi}_T$, as well as $\phi_1$, the average anholonomy over a single period of the driving force.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Bifurcations and geometrical resonances: The attractor at Poincaré points $x_n$ and the geometric phase factors $\phi_1$ and $\Phi_T$ as the nonlinearity parameter $\alpha$ is varied. Plots for $\alpha > 1$ and $\alpha < 1$ are shown separately to highlight the differences in the magnitude of the anholonomy.}
\end{figure}
in $\phi_T$ correspond to a full rotation of the FS triad by $2\pi$ in a full period of the driving force. These transitions can thus be viewed as a resonance effect between the average rotation frequency of the FS triad and the driving frequency. Higher order resonances where the FS triad rotates by $2\pi m$ ($m$ = any integer) were also frequently observed \[8\] for $\alpha < 1$. In the neighborhood of a second-order transition, the mean anholonomy is very large as the FS triad executes many rotations (not necessarily an integer number) during one period of the driving. Such sudden responses in the geometry of the trajectories that are not captured by usual bifurcation diagrams will be referred to as geometrical resonances (GR).

FIG. 2. Geometrical localization near first-order (left) and second-order(right) geometrical transitions. Time series for $\tau$ (thick lines) and $x$ (dashed lines; full x-scale not shown), shown over one driving cycle. The horizontal line corresponds to $x = 0$. Unlike $x$ and $x_t$, $\tau$ is sensitive to the discontinuity in the acceleration at $x = 0$. From top to bottom: on the left, $\alpha = 1.1, 1.2, 1.25, 1.3$; on the right, $\alpha = 0.66, 0.62, 0.61, 0.59$.

Further insight into these intriguing phenomena can be gained by studying the local dynamics of both the phase space and geometrical variables as shown in Fig. 2. As we discuss below, the key to the various effects is the localization characteristics underlying the torsion $\tau$, which we will refer to as the $\tau$-mode. We first discuss the dynamics near the first-order transition, given on the left in Fig. 2. An interesting aspect is the temporal localization of the $\tau$-mode corresponding to large geometrical changes in very short time intervals, as the nonlinearity increases from $\alpha = 1.1$. At a critical parameter value, it becomes a $\delta$-function and then flips, i.e., the localized $\tau$ changes its sign, after which the $\tau$-dynamics that was "bi-directional" (i.e., $\tau$ had both signs) becomes "uni-directional", with the FS triad rotating in one direction alone, as can be inferred from the figure. The geometrical transition appears to be linked to subtle changes in the dynamics. Here the local curvature vanishes as $r_t$ and $r_{tt}$ become collinear. In terms of phase space variables, the force and its two derivatives are very small and the particle is subjected to a small constant force for a long time interval.

One of the interesting aspects of a first-order GR transition is the fact that it is often associated with a period-doubling bifurcation; in that case it is a “twin” transition, where a jump in $\phi_1$ before period-doubling is always accompanied by another jump (of identical magnitude but opposite sign) in $\phi_1$ after the period-doubling. The twin aspect of the GR was seen throughout the two-dimensional parameter space, and may be universal.

In contrast to the first-order case, the dynamics near a second-order transition corresponds to an enhancement in the localization length of the $\tau$-mode, as described on the right in Fig. 2. In addition to this delocalization, the mode also becomes predominantly unidirectional (positive $\tau$) due to the disappearance of the strongly localized (negative) $\tau$. After this stage, the mode becomes bi-directional again. In view of the absence of "flipping", $\phi_T$ remains continuous, while its derivative exhibits a discontinuity.

FIG. 3. Time series for $\tau$ (thick line) and $x$ (thin line) in the chaotic state for $\alpha = 5$. The middle and the bottom plots show the blowups.

Localization of $\tau$ implies that the non-planarity of the attractor is significant only in short intervals of time. This nonlinearity-induced localization of the geometrical variables when the phase space variables continue to exhibit oscillatory dynamics is an interesting new phenomenon that we will refer to as geometrical localization. In addition to their existence near GRs, localized
systems using the FS equations provides a new method which adds geometrical features to the usual phase space description of the complex dynamics of classically chaotic systems. Non-planarity of the attractor is related to anholonomy, which, in turn, is shown to be an appropriate robust order parameter that characterizes the geometry of the attractors. Adding to the richness of the bifurcation diagram is the prediction of first and second-order transitions in the geometry of the attractor. These transitions are related to the localization characteristics of the torsion. This suggests the novel concept of geometrical localization near a transition, which emerges due to the fact that significant deviation from planar dynamics exists only over small time intervals. We thus relate nonlinearity not only to anholonomy but also to localization.

We believe that several of the features we have seen in our geometric description of the bilinear oscillator may be generic and found in other nonlinear systems as well. These were seen throughout the two-dimensional parameter space [8], and also for other choices of phase space variables such as $r = (x, x_1, t)$ and $(x, x_1, \cos \omega t)$. Our formulation can also be extended to $D \geq 4$. For instance, when $D = 4$, the anholonomy can be found from the expression for the angular velocity of a tetrad that can be appropriately defined for the system. We hope that our study will stimulate a new line of research relating geometry and dynamics in nonintegrable systems in general.

The research of IIS is supported by National Science Foundation Grant No. DMR 0072813.

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