Extended 5d Seiberg-Witten Theory
and
Melting Crystal

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Abstract

We study an extension of the Seiberg-Witten theory of 5d $\mathcal{N} = 1$ supersymmetric Yang-Mills on $\mathbb{R}^4 \times S^1$. We investigate correlation functions among loop operators. These are the operators analogous to the Wilson loops encircling the fifth-dimensional circle and give rise to physical observables of topologically-twisted 5d $\mathcal{N} = 1$ supersymmetric Yang-Mills in the $\Omega$ background. The correlation functions are computed by using the localization technique. Generating function of the correlation functions of $U(1)$ theory is expressed as a statistical sum over partitions and reproduces the partition function of the melting crystal model with external potentials. The generating function becomes a $\tau$ function of 1-Toda hierarchy, where the coupling constants of the loop operators are interpreted as time variables of 1-Toda hierarchy. The thermodynamic limit of the partition function of this model is studied. We solve a Riemann-Hilbert problem that determines the limit shape of the main diagonal slice of random plane partitions in the presence of external potentials, and identify a relevant complex curve and the associated Seiberg-Witten differential.

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1 Introduction

The notion of random partitions has played a central role in recent studies on supersymmetric gauge theories. A highlight will be the work of Nekrasov and Okounkov [11] who used random partitions to derive the Seiberg-Witten solution [2] of 4d $\mathcal{N} = 2$ supersymmetric gauge theories. Random partitions emerge therein through Nekrasov’s formula [3] of the instanton sum for the gauge theories in the $\Omega$ background [4]. This statistical model has been extended [5] to investigate the integrability of correlation functions of single-traced chiral observables. Such an extension of the Seiberg-Witten theory also becomes attractive to understand 4d $\mathcal{N} = 1$ supersymmetric gauge theories by providing a powerful tool [6].

The so-called melting crystal model provides a similar statistical model for 5d $\mathcal{N} = 1$ supersymmetric gauge theories [7,8,9] and A-model topological strings [10]. This is a statistical model of random plane partitions, also interpreted as a $q$-deformation of random partitions [11,12], that stems from Nekrasov’s formula [3] for 5d supersymmetric gauge theories in the $\Omega$ background.

Recently, an extension of this melting crystal model with external potentials was proposed, and its integrable structure was elucidated [13,14]. It was argued therein that the external potentials are related to loop operators of a 5d $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (SYM). The goal of this paper is to present the detail of this result along with some implications. In particular, we study the thermodynamic limit of the partition function of this model, and solve a Riemann-Hilbert problem that determines the limit shape of the main diagonal slice of random plane partitions in the presence of external potentials. We can thus eventually identify a relevant complex curve and the associated Seiberg-Witten differential.

This paper is organized as follows. Section 2 starts with a brief review about 5d $\mathcal{N} = 1$ SYM in the $\Omega$ background. We introduce loop operators $\mathcal{O}_k$ ($k = 1, 2, \cdots$) of this theory. Computation of correlation functions among these operators is presented by using the localization technique. Generating function of the correlation functions of $U(1)$ theory is expressed as a statistical sum over partitions and reproduces the partition function of the aforementioned melting crystal model. For an application of the localization theorem, we use equivariant descent equations of the loop operators, which is proved in Appendix B, and a related $T^2$-action on the gauge theory is argued in Appendix A. Section 3 is concerned with a common integrable
structure of $5d \mathcal{N} = 1$ SYM in the $\Omega$ background and melting crystal model. The loop operators are converted to the external potentials of the melting crystal model. This eventually shows that, by regarding the coupling constants $t = (t_1, t_2, \cdots)$ of the loop operators as a series of time variables, the generating function becomes a $\tau$ function of 1-Toda hierarchy. In Section 4, we introduce an energy functional. This functional is a quadratic form, and obtained from the logarithm of the statistical weight in the partition function. The formula is stated in a general setting and is proved in Appendix C. By using the formula, we argue the thermodynamic limit of the melting crystal model or the $q$-deformed random partitions. In Section 5, the thermodynamic limit is reformulated as a Riemann-Hilbert problem to obtain an analytic function on $\mathbb{C}^* - I$, where $\mathbb{C}^*$ is a cylinder and $I$ is an interval in the real axis, and satisfies suitable conditions. In Section 6, we solve the Riemann-Hilbert problem. The relevant complex curve and the associated Seiberg-Witten differential are presented. The vev’s of the loop operators are particularly expressed as residue integrals of the Seiberg-Witten differential. The solution of the Riemann-Hilbert problem can be further analyzed by applying the classical Jensen formula in complex analysis. The case of the single coupling constant $t = (t_1, 0, 0, \cdots)$ is described as an example. The related computation is attached to Appendix D. Section 7 is devoted to conclusion and discussion.

2 Loop operators of $5d \mathcal{N} = 1$ SYM in $\Omega$ background

We first consider an ordinary $5d \mathcal{N} = 1$ SYM on $\mathbb{R}^4 \times S^1$. Let $E$ be the $SU(N)$-bundle on $\mathbb{R}^4$ with $c_2(E) = n \geq 0$. A gauge bundle of the theory is the $SU(N)$-bundle $\pi^* E$ on $\mathbb{R}^4 \times S^1$ pulled back from $\mathbb{R}^4$, where $\pi$ is the projection from $\mathbb{R}^4 \times S^1$ to $\mathbb{R}^4$. All the fields in the vector multiplet are set to be periodic along $S^1$. Among them, the bosonic ingredients are a $5d$ gauge potential $A_M(x, t)dx^M$ and a scalar field $\varphi(x, t)$ taking the value in $su(N)$, where the coordinates $x^M$ represent coordinates $x = (x^1, x^2, x^3, x^4)$ of $\mathbb{R}^4$ and a periodic coordinate $t$ of $S^1$. These bosonic fields describe a $5d$ Yang-Mills-Higgs system. The gauge potential can be separated into two parts $A_\mu(x, t)dx^\mu$ and $A_t(x, t)dt$, respectively the components of the $\mathbb{R}^4$- and the $S^1$-directions. Let $A_E$ be the infinite dimensional affine space consisting of all the gauge potentials on $E$. The $4d$ component $A_\mu(x, t)dx^\mu$ describes a loop $A(t)$ in $A_E$, where the loop is parametrized by the periodic coordinate of the fifth-dimensional circle. As for $A_t(x, t)$, together with $\varphi(x, t)$, the
combination $A_t + i\varphi$ describes a loop $\phi(t)$ in $\Omega^0(\mathbb{R}^4, \text{ad}E \otimes \mathbb{C})$, which is the space of all the sections of $\text{ad}E \otimes \mathbb{C}$, where $\text{ad}E$ is the adjoint bundle on $\mathbb{R}^4$ with fibre $su(N)$. Taking account of the periodicity, the same argument is applicable to the gauginos as well. The vector multiplet thereby describes a loop in the configuration space of the 4d theory. By using such a loop, we may describe 5d $\mathcal{N} = 1$ SYM. In the case of the Yang-Mills-Higgs system, the loop $A(t)$ gives a family of covariant differentials on $E$ as $d_{A(t)} = d + A(t)$. For the loop $\phi(t)$, since it involves $A_t(x,t)$, we conveniently introduce a differential operator $H(t)$ by

$$H(t) \equiv \frac{d}{dt} + \phi(t). \quad (2.1)$$

### 2.1 5d $\mathcal{N} = 1$ SYM in $\Omega$ background

Via the standard dimensional reductions, 6d $\mathcal{N} = 1$ SYM gives lower dimensional Yang-Mills theories with 8 supercharges, including the above theory. Furthermore, the dimensional reductions in the $\Omega$ background provide much powerful tools to understand these theories [4]. The $\Omega$ background is a 6d gravitational background on $\mathbb{R}^4 \times T^2$ described by a metric of the form:

$$ds^2 = \sum_{\mu=1}^{4} (dx^\mu - \sum_{a=5,6} V_a^\mu dx^a)^2 + \sum_{a=5,6} (dx^a)^2,$$

where two vectors $V_5^\mu, V_6^\mu$ generate rotations on two-planes $(x^1, x^2)$ and $(x^3, x^4)$ in $\mathbb{R}^4$. By letting $V_1 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$ and $V_2 = x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4}$, they are respectively the real part and the imaginary part of the combination

$$V_{\epsilon_1, \epsilon_2} \equiv \epsilon_1 V_1 + \epsilon_2 V_2, \quad \epsilon_1, \epsilon_2 \in \mathbb{C}. \quad (2.2)$$

The above combination is expressed in component as $V_{\epsilon_1, \epsilon_2} = \Omega_{\nu}^{\mu} \epsilon_1^{\nu} \frac{\partial}{\partial x^\mu}$.

To see the dimensional reduction in the $\Omega$-background, let us first consider the bosonic part of the 5d SYM. The corresponding Yang-Mills-Higgs system is modified from the previous one. However, the system is eventually controlled by replacing $H(t)$ with

$$H_{\epsilon_1, \epsilon_2}(t) \equiv H(t) + K_{\epsilon_1, \epsilon_2}(t). \quad (2.3)$$

Here $K_{\epsilon_1, \epsilon_2}(t)$ is an another differential operator given by [15]

$$K_{\epsilon_1, \epsilon_2}(t) \equiv V_{\epsilon_1, \epsilon_2}^\mu \partial_{A(t) \mu} + \frac{1}{2} \Omega^{\mu \nu} J_{\mu \nu}, \quad (2.4)$$

where $J_{\mu \nu}$ denote the $SO(4)$ Lorentz generators of the system. The above operator generates a $T^2$-action by taking the commutators with $d_{A(t)}$ and $H(t)$. For instance, we have

$$[d_{A(t)}, K_{\epsilon_1, \epsilon_2}(t)] = -\iota_{V_{\epsilon_1, \epsilon_2}} F_{A(t)}, \quad (2.5)$$

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where the right-hand side means $-1$ times the contraction of the curvature two-form $F_{A(t)} = dA(t) + A(t) \wedge A(t)$ with the vector field $V_{\epsilon_1, \epsilon_2}$. As is argued in Appendix $\text{A}$, the infinitesimal rotation $\delta x^\mu = -V_{\epsilon_1, \epsilon_2}^\mu$ can generate a $T^2$-action on $A_E$. The right-hand side of (2.5) is precisely the infinitesimal form (A.10) of the $T^2$-action.

The supercharges $Q_{\epsilon_1, \epsilon_2}$ and $\bar{Q}_{\dot{\epsilon}_1, \dot{\epsilon}_2}$ are realized in a way different from the case of $\epsilon_1 = \epsilon_2 = 0$. Note that we use the 4$d$ notation such that $\alpha, \dot{\alpha}$ and $\alpha$ denote the indexes of the Lorentz group $SU(2)_L \times SU(2)_R$ and the R-symmetry $SU(2)_I$. According to the argument [16], we may interpret the 5$d$ SYM as a topological field theory [17]. Actually, by regarding the diagonal $SU(2)$ of $SU(2)_R \times SU(2)_I$ as a new $SU(2)_R$, we can extract a supercharge that behaves as a scalar under the new Lorentz symmetry. We write the scalar supercharge as $Q_{\epsilon_1, \epsilon_2}$. The gaugino acquires a natural interpretation as differential forms, $\eta(x, t), \psi_\mu(x, t)$ and $\xi_{\mu \nu}(x, t)$. These give fermionic loops, $\eta(t), \psi(t)$ and $\xi(t)$. The main part of the $Q$-transformation can be read as

\begin{align*}
Q_{\epsilon_1, \epsilon_2}A(t) &= \psi(t), \\
Q_{\epsilon_1, \epsilon_2}\psi(t) &= \left( d_{A(t)}, \mathcal{H}_{\epsilon_1, \epsilon_2}(t) \right), \\
Q_{\epsilon_1, \epsilon_2}\mathcal{H}_{\epsilon_1, \epsilon_2}(t) &= 0,
\end{align*}

where $\psi(t)$ is a fermionic loop in $\Omega^1(\mathbb{R}^4, \text{ad}E)$.

The action of the 5$d$ SYM can be now written in a $Q$-exact form as

\begin{equation}
S_{5d\text{SYM}}^{\epsilon_1, \epsilon_2} = \frac{4\pi^2 n}{g^2} \int_{S^1} dt + \int_{S^1} dt \left\{ Q_{\epsilon_1, \epsilon_2}, \mathcal{W}(t) \right\}. \tag{2.8}
\end{equation}

The above action turns out to be a BRST gauge fixed action of the topological term $\frac{4\pi^2 n}{g^2} \int_{S^1} dt$, where $Q_{\epsilon_1, \epsilon_2}$ plays a role of the BRST charge. In this formulation, the original vector multiplet is splitted to several $Q$-multiplets, including $A(t), \psi(t)$ as the doublet and $\mathcal{H}_{\epsilon_1, \epsilon_2}(t)$ as the singlet. We may integrate out all the multiplets except the two. The integrations impose constraints on the two multiplets. As for the doublet, the constraints turn to involve three equations; $F_{A(t)}^{(+)} = 0$, $(d_{A(t)} \psi(t))^{(+)} = 0$, where the symbol $(+)$ means self-dual part of two-forms on $\mathbb{R}^4$, and $d^*_{A(t)} \psi(t) = 0$. The first equation is the anti-self dual equation for gauge potentials on $E$. The second equation is regarded as a linearization of the first. For the singlet, the constraint leads to the equation

\begin{equation}
\Box_{A(t)} \phi(t) + 2\psi(t)^2 + d^*_{A(t)} \left( \iota_{V_{\epsilon_1, \epsilon_2}} F_{A(t)} + \frac{dA(t)}{dt} \right) = 0, \tag{2.9}
\end{equation}

where $\psi(t)^2 = \psi_\mu(t) \psi^\mu(t)$. In the above, $d^*_{A(t)}$ is the formal adjoint of $d_{A(t)}$ and $\Box_{A(t)} = (-1) d^*_{A(t)} d_{A(t)}$ is the scalar Laplacian.
2.2 Loop operators and their correlation functions

Taking account of the relation \( \phi(x, t) = A_t(x, t) + i \varphi(x, t) \), the following path-ordered integral becomes an analogue of holonomy of the 5d gauge potential:

\[
W^{(0)}(x; t_1, t_2) = \text{P exp} \left( - \int_{t_2}^{t_1} dt \phi(x, t) \right),
\]

(2.10)

where the symbol means path-ordered integration, more precisely, it is defined by the differential equation

\[
\left( \frac{d}{dt_1} + \phi(x, t_1) \right) W^{(0)}(x; t_1, t_2) = 0, \quad W^{(0)}(x; t_2, t_2) = 1.
\]

(2.11)

The trace of the holonomy along the circle defines a loop operator by

\[
\mathcal{O}^{(0)}(x) = \text{Tr} W^{(0)}(x; R, 0),
\]

(2.12)

where \( R \) is the circumference of \( S^1 \). The above operator is an analogue of the Wilson loop operator along the circle. Unlike the case of \( \epsilon_1 = \epsilon_2 = 0 \) [18], it is not \( Q \)-closed except at \( x = 0 \). To see this, note that the \( Q \)-transformations (2.6) and (2.7) imply \( Q_{\epsilon_1, \epsilon_2} \phi(t) = -i \nu_{\epsilon_1, \epsilon_2} \psi(t) \). By using this, we obtain the transformation

\[
Q_{\epsilon_1, \epsilon_2} \mathcal{O}^{(0)}(x) = \int_0^R dt \text{Tr} \left\{ W^{(0)}(x; R, t) i \nu_{\epsilon_1, \epsilon_2} \psi(x, t) W^{(0)}(x; t, 0) \right\}.
\]

(2.13)

Since the right-hand side of the above formula vanishes only at \( x = 0 \), this means that \( \mathcal{O}^{(0)}(x) \) becomes \( Q \)-closed only at \( x = 0 \).

The above property may be explained in terms of the equivariant de Rham theory. To see this, let us first generalize the path-ordered integral (2.10) by exponentiating the combination \( F_{A(t)} - \psi(t) + \phi(t) \) in place of \( \phi(t) \) as

\[
W(x; t_1, t_2) = \text{P exp} \left\{ - \int_{t_2}^{t_1} dt \left( F_{A(t)} - \psi(t) + \phi(t) \right) \right\},
\]

(2.14)

where the right-hand side is the solution of the differential equation

\[
\frac{dW(x; t_1, t_2)}{dt_1} + \left( F_{A(t_1)}(x) - \psi(x, t_1) + \phi(x, t_1) \right) \wedge W(x; t_1, t_2) = 0,
\]

\[
W(x; t_2, t_2) = 0.
\]

(2.15)
This particularly means that \( W \) has several components, according to degrees of differential forms on \( \mathbb{R}^4 \), as \( W = W^{(0)} + W^{(1)} + \cdots + W^{(4)} \), where the indexes denote the degrees. We then generalize the loop operator (2.12) as

\[
O(x) = \text{Tr} \ W(x; R, 0).
\]  

(2.16)

In parallel with \( W \), we have the decomposition \( O = O^{(0)} + O^{(1)} + \cdots + O^{(4)} \). Explicitly, \( O^{(i)} \) are operators of the following form:

\[
O^{(1)}(x) = \int_0^R dt \text{Tr} \left\{ W^{(0)}(x; R, t) \psi(x, t) W^{(0)}(x; t, 0) \right\},
\]

\[
O^{(2)}(x) = \int_0^R dt \text{Tr} \left\{ W^{(0)}(x; R, t)(-F_A(t))(x) W^{(0)}(x; t, 0) \right\}
\]

\[
+ \int_0^R dt_1 \int_0^{t_1} dt_2 \text{Tr} \left\{ W^{(0)}(x; R, t_1) \psi(x, t_1) W^{(0)}(x; t_1, t_2) \psi(x, t_2) W^{(0)}(x; t_2, 0) \right\},
\]

\[\vdots\]  

(2.17)

Eq. (2.13) can be now expressed as \( Q_{\epsilon_1, \epsilon_2} O^{(0)} = \iota_{\nu_{\epsilon_1, \epsilon_2}} O^{(1)} \), which is actually the first equation among a series of the equations that \( O^{(i)} \) obey. Such equations eventually show up by expanding component-wise the identity

\[
(d_{\epsilon_1, \epsilon_2} + Q_{\epsilon_1, \epsilon_2})O(x) = 0,
\]  

(2.18)

where \( d_{\epsilon_1, \epsilon_2} \equiv d - \iota_{\nu_{\epsilon_1, \epsilon_2}} \) is the \( T^2 \)-equivariant differential on \( \mathbb{R}^4 \). The above identity implies in components \( T^2 \)-equivariant descent equations of the form

\[
dO^{(i-1)}(x) + Q_{\epsilon_1, \epsilon_2} O^{(i)}(x) - \iota_{\nu_{\epsilon_1, \epsilon_2}} O^{(i+1)}(x) = 0, \quad 0 \leq i \leq 4,
\]  

(2.19)

where \( O^{(-1)} = O^{(5)} = 0 \). We provide a proof of the formula (2.18) in Appendix B.

We can also consider the loop operators encircling the circle many times. Correspondingly, we introduce

\[
O_k(x) = \text{Tr} \ W(x; kR, 0), \quad k = 1, 2, \cdots.
\]  

(2.20)

These satisfy

\[
(d_{\epsilon_1, \epsilon_2} + Q_{\epsilon_1, \epsilon_2})O_k(x) = 0.
\]  

(2.21)
Let us examine the correlation functions \( \langle \prod_{a} \int_{R^4} O_{k_a} \rangle^{\epsilon_1, \epsilon_2} \). Since the integral \( \int_{R^4} O_{k} = \int_{R^4} O_{k}^{(4)} \) is \( Q \)-closed by virtue of the formula (2.21), the correlation function has an interpretation in the topological field theory. In particular, we may compute the correlation functions by a supersymmetric quantum mechanics (SQM) which is substantially equivalent to the 5d SYM as the topological field theory. Such a SQM turns out to be an equivariant SQM on \( \tilde{\mathcal{M}}_n[^3] \), where \( \mathcal{M}_n \) is the moduli space of the framed \( n \) instantons.

We may regard \( \tilde{\mathcal{M}}_n \) as a Riemannian manifold. Actually, by taking a local gauge slice of the anti-self dual gauge potentials, the metric is induced from \( \mathcal{A}_E \), where \( \mathcal{A}_E \) is endowed with a gauge invariant metric of the form, \( G(\delta A, \delta' A) = \int_{R^4} \text{Tr}A(x) \wedge *\delta' A(x) \). Take a loop \( m(t) = (m^I(t)) \), where \( m^I \) denote local coordinates of \( \tilde{\mathcal{M}}_n \). We also introduce its fermionic partner \( \chi(t) = (\chi^I(t)) \). Both are parametrized by the fifth-dimensional circle. The \( Q \)-transformation (2.6) yields a transformation between them. To describe the transformation, note that the infinitesimal variation \( \delta A = -\iota_{\epsilon_1, \epsilon_2} F_A \) preserves the anti-self dual equation \( F_A^{(+)} = 0 \), thus it gives a vector field on \( \tilde{\mathcal{M}}_n \), which we denote by \( \mathcal{V}_{\epsilon_1, \epsilon_2} \). It is actually a Killing vector. This apparently follows since the aforementioned variation preserves the gauge invariant metric \( G \).

The \( Q \)-transformation (2.6) is eventually converted to the transformation

\[
Q_{\epsilon_1, \epsilon_2} m(t) = \chi(t), \quad Q_{\epsilon_1, \epsilon_2} \chi(t) = -\frac{dm(t)}{dt} + \mathcal{V}_{\epsilon_1, \epsilon_2}(m(t)).
\]  

(2.22)

The supersymmetric Yang-Mills is described effectively in terms of \( m(t) \) and \( \chi(t) \), consequently reduces to a quantum mechanical system on \( \tilde{\mathcal{M}}_n \). Actually, the corresponding action can be obtained from (2.8) by integrating out the irrelevant fields. This yields the action

\[
S_{\epsilon_1, \epsilon_2}^{\text{eff}} = \frac{4\pi^2 n}{g^2} \int_{S^1} dt + \int_{S^1} dt \left\{ Q_{\epsilon_1, \epsilon_2}, -\frac{1}{2} G_{IJ}(m(t)) \chi^J(t) \frac{dm^I(t)}{dt} \right\},
\]  

(2.23)

where \( G_{IJ} \) is the Riemannian metric on \( \tilde{\mathcal{M}}_n \). The above action together with the supersymmetry (2.22) is familiar in the physical proof [19] of the equivariant index formula for Dirac operator. In particular, the partition function becomes eventually the \( T^2 \)-equivariant index for the Dirac operator on \( \tilde{\mathcal{M}}_n[^3] \).

The combination \( F_A(t) - \psi(t) + \phi(t) \) in (2.14) can be identified with a loop space analogue of the \( T^2 \)-equivariant curvature \( \mathcal{F}_{\epsilon_1, \epsilon_2} \) of the universal connection [20], where the universal bundle becomes equivariant by the \( T^2 \)-action on \( \mathcal{A}_E \times R^4 \). In the computation of the correlation function, by virtue of the supersymmetry (2.22), only the constant modes \( m_0, \chi_0 \) contribute
to the observables. For such constant modes, the above combination is precisely identified with the equivariant curvature $F_{\epsilon_1,\epsilon_2}$ [4]. This means that $\mathcal{O}_k(x)$ substantially truncates to the $T^2$-equivariant Chern character $\text{Tr} e^{-kRF_{\epsilon_1,\epsilon_2}}$. Thus we obtain the following finite dimensional integral representation of the correlation functions:

$$\left\langle \prod_a \int_{\mathbb{R}^4} \mathcal{O}_{k_a} \right\rangle_{\text{n-instanton}}^{\epsilon_1,\epsilon_2} = \frac{1}{(2\pi i R)^{\dim \tilde{\mathcal{M}}_n}} \int_{\tilde{\mathcal{M}}_n} \hat{A}_{T^2}(Rt_{\epsilon_1,\epsilon_2}, \tilde{\mathcal{M}}_n) \prod_a \int_{\mathbb{R}^4} \text{Tr} e^{-k_a R F_{\epsilon_1,\epsilon_2}}. \quad (2.24)$$

where $\hat{A}_{T^2}(\cdot, \tilde{\mathcal{M}}_n)$ is the $T^2$-equivariant $\hat{A}$-genus of the tangent bundle of $\tilde{\mathcal{M}}_n$, and $t_{\epsilon_1,\epsilon_2}$ is a generator of $T^2$ that gives the Killing vector $V_{\epsilon_1,\epsilon_2}$.

Introducing coupling constants $t = (t_1, t_2, \cdots)$, the generating function of the correlation functions is given by $Z_{\epsilon_1,\epsilon_2}(t) = \langle e^{\sum k \int f_{\mathbb{R}^4} \mathcal{O}_k} \rangle_{\epsilon_1,\epsilon_2}$. Since $n$-instanton contributes with the weight $(RA)^{2nN}$, where $\Lambda$ is the dynamical scale, letting $Q = (RA)^2$, we can express the generating function as

$$Z_{\epsilon_1,\epsilon_2}(t) = \sum_{n=0}^{\infty} Q^n \left\langle e^{\sum k t_k f_{\mathbb{R}^4} \mathcal{O}_k} \right\rangle_{\text{n-instanton}}^{\epsilon_1,\epsilon_2}. \quad (2.25)$$

2.3 Application of localization technique

The right-hand side of the formula (2.24) is eventually replaced with a statistical sum over partitions. To see their appearance, note that the integration localizes to the fixed points of the $T^2$-action. However, the fixed points in $\tilde{\mathcal{M}}_n$ are small instanton singularities since the variation $\delta A = -t_{\epsilon_1,\epsilon_2} F_A$ vanishes there. These singularities can be resolved by instantons on a non-commutative $\mathbb{R}^4$. By using such a regularization via the non-commutativity, the fixed points get isolated, so that they are eventually labelled by using partitions [21].

A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a sequence of non-negative integers satisfying $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$. Partitions are identified with the Young diagrams in the standard manner. The size is defined by $|\lambda| = \sum_{i \geq 1} \lambda_i$, which is the total number of boxes of the diagram.

Let us examine the formula (2.24) for the $U(1)$ theory. The relevant computation of the localization can be found in [21, 22]. We truncate $\epsilon_{1,2}$ as $-\epsilon_1 = \epsilon_2 = i\hbar$, where $\hbar$ is a positive real parameter. Consequently, the formula becomes a $q$-series by the truncation, where $q = e^{-R\hbar}$. The fixed points in $\tilde{\mathcal{M}}_n$ are labelled by partitions of $n$. The equivariant $\hat{A}$-genus takes the
following form at the partition $\lambda$ of $n$:

\[
(2\pi i R)^{-2n} \hat{A}_{T^2}(R t_{-ih\lambda}, \hat{\mathcal{M}}_n) \bigg|_{\lambda} = (-)^n \left( \frac{\hbar}{2\pi} \right)^{2n} \prod_{s \in \lambda} \left\{ \frac{h(s)}{q^{h(s)/2} - q^{h(s)/2}} \right\}^2,
\]

where $h(s)$ denotes the hook length of the box $s$ of the Young diagram $\lambda$. The right-hand side of the above formula can be expressed by using a Schur function as

\[
(2\pi i R)^{-2n} \hat{A}_{T^2}(R t_{-ih\lambda}, \hat{\mathcal{M}}_n) \bigg|_{\lambda} = (-)^n \left( \frac{\hbar}{2\pi} \right)^{2n} \left( \prod_{s \in \lambda} h(s) \right)^2 q^{\kappa(\lambda)} s_{\lambda}(q^\rho)^2,
\]

where $s_{\lambda}(q^\rho)$ is the Schur function $s_{\lambda}(x_1, x_2, \cdots)$ specialized to $x_i = q^{i-\frac{1}{2}}$, and $\kappa(\lambda) = 2 \sum_{(i,j) \in \lambda} (j-i)$. To obtain (2.27), we have made use of the $q$-hook formula [23]

\[
s_{\lambda}(q^\rho) = q^{n(\lambda) + |\lambda|} \prod_{s \in \lambda} \left( 1 - q^{h(s)} \right)^{-1},
\]

where $n(\lambda) = \sum_{i \geq 1} (i-1) \lambda_i$. Actually, taking account of the formula $2n(\lambda) + |\lambda| = -\kappa(\lambda)/2 + \sum_{s \in \lambda} h(s)$, the $q$-hook formula gives $\prod_{s \in \lambda} \left( q^{-h(s)} - q^{h(s)} \right)^{-1} = q^{\kappa(\lambda)} s_{\lambda}(q^\rho)$. By plugging this into the right-hand side of (2.26), we obtain (2.27).

Similarly, the fixed points in $\hat{\mathcal{M}}_n \times \mathbb{R}^4$ are given as $(\lambda, 0)$, where $\lambda$ is a partition of $n$ and 0 is the origin of $\mathbb{R}^4$. The equivariant Chern character takes the form $\Tr e^{-k R F_{-ih, \delta \theta}} |_{(\lambda, 0)} = \mathcal{O}_k(\lambda)$, where $\mathcal{O}_k(\lambda)$ is given by

\[
\mathcal{O}_k(\lambda) = (1 - q^{-k}) \sum_{i=1}^{\infty} \left\{ q^{k(\lambda_i - i + 1)} - q^{k(-i + 1)} \right\} + 1.
\]

The above functions have been exploited in [5, 24] from the 4d gauge theory viewpoint, with $q$ or $q^k$ being replaced by a generating spectral parameter.

By the localization formula [25], integration in the right-hand side of (2.24) reduces to a summation over contributions from the isolated fixed points of the $T^2$-action on $\hat{\mathcal{M}}_n \times (\mathbb{R}^4 \times \cdots \times \mathbb{R}^4)$. The contribution from the fixed point consists of two factors. One is the value of the integrand at the fixed point, and the other is the Jacobian factor in a change of coordinates in a neighborhood of the fixed point. By multiplying these two factors and summing up them over the fixed points, we obtain the right-hand side of (2.24). The fixed points are of the form $(\lambda, (0, \cdots, 0)) \in \hat{\mathcal{M}}_n \times (\mathbb{R}^4 \times \cdots \times \mathbb{R}^4)$, where $\lambda$ is a partition of $n$ and $(0, \cdots, 0)$ is the origin of $\mathbb{R}^4 \times \cdots \times \mathbb{R}^4$. The values of the integrand at the fixed points can be expressed by using
the formulas (2.27) and (2.29). The Jacobian factor can be further factorized by the product structure of $\tilde{M}_n$ and $\mathbb{R}^4$'s. The Jacobian factor originating in the change of coordinates of $\tilde{M}_n$ in a neighborhood of $\lambda$ takes the form $(2\pi/\hbar)^2n(\prod_{s \in \lambda} h(s))^{-2}$, while the equivariant volume of $\mathbb{R}^4 \times \cdots \times \mathbb{R}^4$ is given by the products of $\hbar^{-2}$ for each $0 \in \mathbb{R}^4$. Therefore, the formula (2.24) becomes eventually the statistical sum over partitions given by

\[
\left< \prod_a \int_{\mathbb{R}^4} O_{ka} \right>_{n\text{-instanton}}^{\text{-ih,ih}} = (-)^n \sum_{\lambda: \text{partition of } n} \frac{\kappa(\lambda)}{q^2} s_\lambda(q^\rho)^2 \prod_a \hbar^{-2} O_{ka}(\lambda),
\]

(2.30)

where the summation is taken over partitions of $n$.

Although we have not taken into account, the Chern-Simon term can be added to a 5d gauge theory, with the coupling constant being quantized, in particular, for the $U(1)$ theory, $m = 0, \pm 1$. It modifies the right-hand side of (2.30) by giving a contribution of the form $(-)^m|\lambda| q^{-\frac{m\kappa(\lambda)}{2}}$, for each $\lambda$ [7]. Hereafter, we consider the case of the $U(1)$ theory having the Chern-Simon coupling, $m = 1$. The corresponding generating function (2.25) becomes

\[
Z_{-i\hbar,\hbar}^{U(1)}(t) = \sum_{\lambda} Q^{\lambda|s_\lambda(q^\rho)^2 e^{\hbar^{-2} \sum_{k=1}^\infty t_k O_k(\lambda)}},
\]

(2.31)

where the summation is taken over all the partitions.

3 Integrability of 5d $\mathcal{N} = 1$ SYM in $\Omega$ background

We can regard the generating function (2.31) as a $q$-deformed random partition. To see this, note that the 4d limit $R \to 0$ makes $q = e^{-R\hbar} \to 1$, therefore, by using (2.28), the Boltzmann weight $Q^{|\lambda|s_\lambda(q^\rho)^2}$ in (2.31) takes at this limit, the form $(\Lambda/\hbar)^{2|\lambda|}(\prod_{s \in \lambda} h(s))^{-2}$, which is the standard weight of a random partition, that is, a Poissonized Plancherel measure of symmetric group. It can be also viewed as a melting crystal model, known as random plane partition. The corresponding model is studied in [13] as a melting crystal model with external potential, where the Chern characters $O_k$ in (2.31) correspond precisely to the external potentials.
3.1 Melting crystal model with external potentials

A plane partition $\pi$ is an array of non-negative integers

\[
\begin{array}{cccccc}
\pi_{11} & \pi_{12} & \pi_{13} & \cdots \\
\pi_{21} & \pi_{22} & \pi_{23} & \cdots \\
\pi_{31} & \pi_{32} & \pi_{33} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]  

satisfying $\pi_{ij} \geq \pi_{i+1,j}$ and $\pi_{ij} \geq \pi_{ij+1}$ for all $i, j \geq 1$. Plane partitions are identified with the $3d$ Young diagrams. The $3d$ diagram $\pi$ is a set of unit cubes such that $\pi_{ij}$ cubes are stacked vertically on each $(i,j)$-element of $\pi$. Diagonal slices of $\pi$ become partitions, as depicted in

![Figure 1](image_url)

Figure 1: The $3d$ Young diagram (a) and the corresponding sequence of partitions (b).

Fig.1. Denote $\pi(m)$ the partition along the $m$-th diagonal slice, where $m \in \mathbb{Z}$. In particular, $\pi(0) = (\pi_{11}, \pi_{22}, \cdots)$ is the main diagonal one. This series of partitions satisfies the condition

\[
\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots ,
\]  

(3.2)

where $\mu \succ \nu$ means the interlace relation; $\mu \succ \nu \iff \mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots$.

The Hamiltonian picture emerges from the above interlace relations, by viewing a plane partition as evolutions of partitions by the discrete time $m$. In particular, transfer matrix formulation using $2d$ complex free fermions is presented in [26], by taking advantage of the well-known realization of partitions in the fermion Fock space. Let $\psi(z) = \sum_{m \in \mathbb{Z}} \psi_m z^{-m-1}$ and
ψ∗(z) = \sum_{m \in \mathbb{Z}} \psi_m^* z^{-m} be complex fermions with the anti-commutation relations, \{\psi_m, \psi_n^*\} = \delta_{m+n,0} and \{\psi_m, \psi_n\} = \{\psi_m^*, \psi_n^*\} = 0. Partitions can be realized as states of the fermion Fock space. For a partition \( \lambda \), the corresponding state reads

\[ |\lambda\rangle = \pm \prod_{i=1}^{\infty} \psi_{i-\lambda_i-1}^* \psi_{i+1} |0\rangle, \] (3.3)

where \( |0\rangle \) denotes the Dirac sea with the vanishing \( U(1) \) charge, and is defined by the conditions

\[ \psi_m |0\rangle = 0 \quad \text{for } \forall m \geq 0, \quad \psi_m^* |0\rangle = 0 \quad \text{for } \forall m \geq 1. \] (3.4)

We may separate the interlace relations (3.2) into two parts, each corresponding to (forward or backward) evolutions of partitions for \( m \leq 0 \) and \( m \geq 0 \) towards \( \lambda = \pi(0) \). These two types of the evolutions are described in the transfer matrix formulation by using operators \( G_\pm \) of the forms

\[ G_\pm = \exp \left\{ \sum_{k=1}^{\infty} \frac{q^k}{k(1-q^k)} J_{\pm k} \right\}, \] (3.5)

where \( J_{\pm k} = \sum_{n \in \mathbb{Z}} :\psi_{\pm k-n}^* \psi_n^* : \) are the modes of the \( U(1) \) current :\( \psi(z) \psi^*(z) := \sum_{m \in \mathbb{Z}} J_m z^{-m-1} \). In accord with the types of the evolutions, \( G_\pm \) generate partitions from the Dirac sea as

\[ \langle 0 | G_+ = \sum_{\lambda} s_\lambda(q^\rho) \langle \lambda |, \quad G_- |0\rangle = \sum_{\lambda} s_\lambda(q^\rho) |\lambda\rangle. \] (3.6)

In the free fermion description, we can convert the loop operators \( O_k \) to fermion bilinear operators \( \hat{O}_k \) of the form

\[ \hat{O}_k = (1 - q^{-k}) \sum_{n = -\infty}^{\infty} q^{kn} :\psi_{-n}^* \psi_n^* : +1, \] (3.7)

Actually, the state (3.3) becomes the simultaneous eigenstate and reproduces the Chern characters as the eigenvalues:

\[ \hat{O}_k |\lambda\rangle = O_k(\lambda) |\lambda\rangle. \] (3.8)

Therefore, taking account of (3.6) and (3.8), we can express the generating function (2.31) in the fermionic representation as

\[ Z_{\text{U}(1)}^{U(1)}(t) = \langle 0 | G_+ Q^{L_0} \exp \left\{ \frac{1}{\hbar^2} \sum_{k=1}^{\infty} t_k \hat{O}_k \right\} G_- |0\rangle, \] (3.9)

where \( L_0 = \sum_{n \in \mathbb{Z}} n :\psi_{-n}^* \psi_n^* : \) is a special element of the Virasoro algebra.
3.2 The integrable structure

The fermion bilinears $\hat{O}_k$ can be regarded as a commutative sub-algebra of the quantum torus Lie algebra realized by the free fermions [13]. The adjoint actions of $G_\pm$ on the Lie algebra generate automorphisms of the algebra. Among them, taking advantage of the shift symmetry, we can eventually convert the representation (3.9) into the expression [13]

$$Z_{-i\hbar, i\hbar}^{U(1)}(t) = \langle 0 | \exp \left\{ \frac{1}{2\hbar^2} \sum_{k=1}^{\infty} (-)^k (1 - q^{-k}) t_k J_k \right\} \times g^{5dU(1)}_s \exp \left\{ \frac{1}{2\hbar^2} \sum_{k=1}^{\infty} (-)^k (1 - q^{-k}) t_k J_{-k} \right\} | 0 \rangle.$$  (3.10)

In the above formula, $g^{5dU(1)}_s$ is the element of $GL(\infty)$ given by

$$g^{5dU(1)}_s = q^W G_- G_+ Q^L G_- G_+ q^{\frac{W}{2}},$$  (3.11)

where $W = W_0^{(3)} = \sum_{n \in \mathbb{Z}} n^2 : \psi_n \psi_n :$ is a special element of $W_\infty$ algebra. The loop operators $O_k$ are converted to $J_k$ or $J_{-k}$ in the formula (3.10). Actually, $J_{\pm k}$ eventually become equivalent in the formula, since $g^{5dU(1)}_s$ satisfies the property [13]

$$J_k g^{5dU(1)}_s = g^{5dU(1)}_s J_{-k}, \text{ for } k \geq 0.$$  (3.12)

Viewing the coupling constants $t$ as a series of time variables, the right-hand side of (3.10) is the standard form of a tau function of 2-Toda hierarchy [27]. However, by virtue of (3.12), the two-sided time evolutions of 2-Toda hierarchy degenerate to one-sided time evolutions. This is precisely the reduction to 1-Toda hierarchy. Thus the generating function becomes a tau function of 1-Toda hierarchy. The Toda hierarchy also has a discrete variable $s$, which corresponds to the $U(1)$ charge of the Dirac sea. The formula (3.10) is easily generalized to the cases, where the charge $s$ is interpreted as the vev of the Higgs field.

4 Integral representation of energy functional

A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ gives rise to a series of decreasing integers $\lambda_i - i$ $(i = 1, 2, \cdots)$. Vice versa, a series of decreasing integers $x_1 > x_2 > \cdots$, in which the $j$-th integer becomes
\(x_j = -j\) for \(j > \exists k_0\), gives a partition. Such a series of decreasing integers is described by its density function as

\[
\rho_\lambda(x) = \sum_{i=1}^\infty \delta(x - (\lambda_i - i)), \quad x \in \mathbb{R}.
\] (4.1)

The statistical sum in the right-hand of (2.31) can be converted to a statistical sum over the density functions.

\[
\mathcal{Z}_{-i\hbar, i\hbar}^{U(1)}(t) = \sum_{\rho(\cdot)} e^{-\mathcal{E}[\rho(\cdot)]},
\] (4.2)

where \(\mathcal{E}[\rho(\cdot)]\) is the energy functional obtained by taking the logarithm of the statistical weight.

\[
\mathcal{E}[\rho_\lambda(\cdot)] = -\log \left\{ Q_{\lambda \lambda} [q_\lambda]^2 e^{\hbar^{-2} \sum_{k=1}^\infty \lambda_i c_k(\lambda)} \right\}.
\] (4.3)

The density function is rather implicit in the above expression of the energy functional. However, taking the consideration in [1], we can convert the expression into an integration of a quadratic form of the density function. Prior to giving such an integral representation of the energy functional, let us describe the basic formula as follows: Let \(f(x)\) be a function. For a partition \(\lambda\), we put

\[
F_\lambda = \sum_{s \in \lambda} f(h(s)),
\] (4.4)

where \(h(s)\) denotes the hook length of the box \(s \in \lambda\). The sum over the boxes in the right-hand side of (4.4) can be converted to an integration of a quadratic form of the density function. To see this, we take a function \(g(x)\) which satisfies

\[
g(x + 1) - 2g(x) + g(x - 1) = f(x),
\] (4.5)

\[
g(0) = 0.
\] (4.6)

By using such a function \(g(x)\), we can eventually express \(F_\lambda\) as

\[
F_\lambda = \frac{1}{2} \int_{x \neq y} dx dy g(|x - y|) \Delta \rho_\lambda(x) \Delta \rho_\lambda(y),
\] (4.7)

where \(\Delta\) denotes a difference operator of the form

\[
\Delta \rho(x) \equiv \rho(x) - \rho(x - 1).
\] (4.8)
We provide a proof of the formula \((4.7)\) in Appendix C.

As an application of the above formula, we describe an integral representation of the logarithm of the hook polynomial

\[
H_\lambda(q) = \prod_{s \in \lambda} (1 - q^{h(s)}). \tag{4.9}
\]

Note that, by taking \(f(x) = \log(1 - q^x)\) in \((4.4)\), we find \(F_\lambda = \log H_\lambda(q)\). Therefore, by choosing a function \(g(x; q)\) to satisfy the conditions \((4.5)\) and \((4.6)\), where \(f(x) = \log(1 - q^x)\), the formula \((4.7)\) gives the integral representation

\[
\log H_\lambda(q) = \frac{1}{2} \int_{x \neq y} dx \, dy \, g(|x - y|; q) \Delta \rho_\lambda(x) \Delta \rho_\lambda(y). \tag{4.10}
\]

In the above formula, we can take \(g(x; q)\) to be the logarithm of the \(q\)-analogue of the Barnes \(G\)-function as

\[
\log G_2(x + 1; q) = (1 - q)^{\frac{x(x-1)}{2}} \prod_{k=1}^{\infty} \left( \frac{1 - q^{x+k}}{1 - q^k} \right)^x (1 - q^k) \tag{4.11}
\]

To see that the above \(g(x; q)\) actually satisfies \((4.5)\) and \((4.6)\), we sufficiently note that \(G_2(x; q)\) is the second cousin in the hierarchy of the multiple \(q\)-gamma functions \(G_n(x; q)\) \((n = 0, 1, \cdots)\), which are defined by the following conditions [28]:

\[
\begin{align*}
(i) & \quad G_n(x + 1; q) = G_{n-1}(x; q)G_n(x; q), \\
(ii) & \quad G_n(1; q) = 1, \\
(iii) & \quad \frac{d^{n+1}}{dx^{n+1}} \log G_{n+1}(x + 1; q) \geq 0, \quad x \geq 0, \\
(iv) & \quad G_0(x; q) = [x]_q,
\end{align*} \tag{4.12}
\]

where \([x]_q = \frac{1-q^x}{1-q}\). The infinite product representation of \(G_2(x; q)\) follows from the above conditions as

\[
G_2(x + 1; q) = (1 - q)^{\frac{x(x-1)}{2}} \prod_{k=1}^{\infty} \left( \frac{1 - q^{x+k}}{1 - q^k} \right)^x (1 - q^k) \tag{4.13}
\]

The above infinite product representation leads to the expansion of \(g(x; q)\) in positive powers of \(q^x\) as

\[
g(x; q) = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{x}{1 - q^{-n}} - \frac{1}{(1 - q^n)(1 - q^{-n})} \right\} + \sum_{n=1}^{\infty} \frac{q^{nx}}{n(1 - q^n)(1 - q^{-n})}. \tag{4.14}
\]
Let us describe an integral representation of the energy functional (4.3). For this end, we first divide $E$ into two parts
\[ E[\rho(\cdot)] = E_1[\rho(\cdot)] + E_2[\rho(\cdot)], \quad (4.15) \]
where
\[ E_1[\rho(\cdot)] = -\log \left\{ Q|s_\lambda(q^\rho)|^2 \right\}, \quad (4.16) \]
\[ E_2[\rho(\cdot)] = -\hbar^{-2} \sum_{k=1}^{\infty} t_k \mathcal{O}_k(\lambda). \quad (4.17) \]
To obtain an integral representation of $E_1$, note that the $q$-hook formula (2.28) enables us to factorize $Q|s_\lambda(q^\rho)|^2$ into the product $q^{-\kappa(\lambda)/2} \times \left( \prod_{s \in \lambda} Qq^{h(s)} \right) H_\lambda(q)^{-2}$, where the logarithm of the first factor reads as
\[ -\log q^{-\frac{\kappa(\lambda)}{2}} = -\frac{\log q}{6} \int_{-\infty}^{+\infty} dx x^3 \Delta \rho(x), \quad (4.18) \]
while an integral representation of the logarithm of the second factor is easily obtained from the previous representation of $\log H_\lambda(q)$. By summing up these two, we eventually obtain the integral representation of $E_1$ as
\[ E_1[\rho(\cdot)] = -\frac{\log q}{6} \int_{-\infty}^{+\infty} dx x^3 \Delta \rho(x) + \int_{x \neq y} dx dy g(|x-y|; q, Q) \Delta \rho(x) \Delta \rho(y), \quad (4.19) \]
where $g(x; q, Q)$ is the logarithm of the following combination:
\[ e^{g(x; q, Q)} = Q^{-\frac{x(x+1)}{4}} q^{-\frac{x(x^2+1)}{12}} (1-q) \frac{x(x-1)}{4} G_2(x+1; q). \quad (4.20) \]
By using (4.13), the expansion of $g(x; q, Q)$ in positive powers of $q^x$ takes the form
\begin{align*}
g(x; q, Q) &= \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{x}{1-q^{-n}} - \frac{1}{(1-q^n)(1-q^{-n})} \right\} - \frac{x(x-1)}{4} \log Q - \frac{x(x^2-1)}{12} \log q \\
&+ \sum_{n=1}^{\infty} \frac{q^{nx}}{n(1-q^n)(1-q^{-n})}. \quad (4.21)\end{align*}
As regards the second part, we note
\[ -\int_{-\infty}^{+\infty} dx q^{kx} \Delta \rho_\lambda(x) = -\int_{-\infty}^{+\infty} dx q^{kx} \left( \rho_\lambda(x) - \rho_\lambda(x-1) \right) \]
The right-hand side of this formula becomes a finite sum by cancellation of terms between the two sums and gives rise to $O_k(\lambda)$. Thus we find

$$O_k(\lambda) = -\int_{-\infty}^{+\infty} dx q^{kx} \Delta \rho_\lambda(x).$$  \hspace{1cm} (4.23)$$

Therefore, the integral representation of $\mathcal{E}_2$ becomes

$$\mathcal{E}_2[\rho(\cdot)] = \hbar^{-2} \int_{-\infty}^{+\infty} dx \sum_{k=1}^{\infty} t_k q^{kx} \Delta \rho(x).$$  \hspace{1cm} (4.24)$$

5 Thermodynamic limit and Riemann-Hilbert problem

We consider the field theory limit of the $U(1)$ theory, which is achieved by letting $\hbar \to 0$ and amounts to the thermodynamic limit of the melting crystal model or the $q$-deformed random partition. Actually, the statistical average of the number of boxes of a partition, where the partition is identified with the Young diagram in the standard manner, becomes of order $\hbar^{-2}$. This implies that, as $\hbar$ goes to zero, partitions that dominate are those of order $\hbar^{-2}$. To realize the thermodynamic limit, we have to rescale the Young diagrams by changing the size of each side of a box $\hbar$ times the original one. Correspondingly, the rescaling of partitions can be organized in terms of the density function, by rescaling $x$ to $u = \hbar x$ as

$$\rho(x = \frac{u}{\hbar}) = \rho^{(0)}(u) + O(\hbar),$$  \hspace{1cm} (5.1)$$

where $\rho^{(0)}(u)$ denotes the scaled density function.

The statistical sum over the density functions in (4.2) can be replaced, as $\hbar$ goes to zero, with the sum over the scaled density functions

$$\mathcal{Z}_{-\hbar,\hbar}^{U(1)}(t) \simeq \sum_{\rho^{(0)}(\cdot)} e^{-\frac{1}{\hbar^2} \mathcal{E}^{(0)}[\rho^{(0)}(\cdot)]},$$  \hspace{1cm} (5.2)$$

where $\mathcal{E}^{(0)}$ denotes the classical energy functional obtained from $\mathcal{E}$ by

$$\mathcal{E}[\rho(x = \frac{u}{\hbar})] = \frac{1}{\hbar^2} \left\{ \mathcal{E}^{(0)}[\rho^{(0)}(u)] + O(\hbar) \right\}.$$  \hspace{1cm} (5.3)$$
Taking account of the expression (5.2), the field theory limit is the semi-classical limit realized by minimizing the classical energy functional.

The classical energy functional can be obtained from (4.19) and (4.24) by scaling the density function $\rho$ and the parameters as well. Noting that the difference operator $\Delta$ becomes the differential operator $\hbar d/du$ as $\hbar$ goes to zero, the classical energy functional takes the form

$$E(0)\left[\rho(0)\right] = \int_{u \neq v} dv d\rho^{(0)}(v) \rho^{(0)}(u) + \frac{R}{6} \int_{-\infty}^{+\infty} du u^2 d\rho^{(0)}(u),$$

(5.4)

where $g^{(0)}(u; R, \Lambda)$ denotes the classical limit of $g(x; q, Q)$ as

$$g(x = \frac{u}{\hbar}; q = e^{-R\hbar}, Q = (R\Lambda)^2) = \frac{1}{\hbar^2} \left\{ g^{(0)}(u; R, \Lambda) + O(\hbar) \right\}.$$

(5.5)

Explicitly, using the formula (4.21), it is given by

$$g^{(0)}(u; R, \Lambda) = \frac{R}{12} u^3 - \frac{\log R\Lambda}{2} u^2 - \frac{\zeta(2)}{R} u + \frac{\zeta(3)}{R^2} - \frac{1}{R^2} \sum_{n=1}^{\infty} \frac{e^{-nRu}}{n^3}.$$

(5.6)

The above function is subject to the following conditions:

$$\frac{d^2 g^{(0)}(u; R, \Lambda)}{du^2} = \log \left( \frac{\sinh \frac{Ru}{R\Lambda}}{\frac{Ru}{R\Lambda}} \right),$$

(5.7)

$$\frac{dg^{(0)}(0; R, \Lambda)}{du} = g^{(0)}(0; R, \Lambda) = 0.$$

(5.8)

Obtaining the thermodynamic limit reduces to solving the saddle point equation $\delta E^{(0)} / \delta \rho^{(0)}(u) = 0$, imposing the constraints [11] originally followed by $\Delta \rho_A$. The variational problem can be eventually summarized as the following integral equations:

$$\text{PP} \int_{-\infty}^{+\infty} dv 2\partial_v g^{(0)}(|u - v|; R, \Lambda) \frac{d\rho^{(0)}(v)}{dv} = -\frac{Ru^2}{2} - V'(u) \quad \text{on supp}(\frac{d\rho^{(0)}}{du}),$$

(5.9)

$$\int_{-\infty}^{+\infty} du \frac{d\rho^{(0)}(u)}{du} = -1,$$

(5.10)

$$\int_{-\infty}^{+\infty} du u \frac{d\rho^{(0)}(u)}{du} = 0,$$

(5.11)
where the integration symbol in the first equation means principle part, and we write the potential as
\[
V(u) = \sum_{k=1}^{\infty} t_k e^{-kR_u}. \tag{5.12}
\]

Solution of the variational problem gives the classical energy \(E^0(0) \equiv \mathcal{E}[\rho^0(\cdot)]\), where \(\rho^0\) denotes the solution. The vev of the loop operators \(\mathcal{O}_k\) can be realized by the partial differentiations of \(E^0(t)\) with respect to \(t_k\). Actually, it is the vev at non-vanishing coupling constants \(t\). We can express this quantity by using \(\rho^0\) in the form
\[
\frac{\partial E^0(t)}{\partial t_k} = \lim_{\hbar \to 0} \langle \mathcal{O}_k \rangle = \int_{-\infty}^{+\infty} du e^{-kR_u} \frac{d\rho^0(u)}{du}. \tag{5.13}
\]

To see the above formula, note that eq. (5.9) can be integrated to
\[
\text{PP} \int_{-\infty}^{+\infty} dv \, g^0(\{u - v\}; R, \Lambda) \frac{d\rho^0(v)}{dv} = -\frac{1}{2} \left( \frac{Ru^3}{6} + V(u) \right) + \frac{1}{2} C(t) \quad \text{on supp}(\frac{d\rho^0}{du}), \tag{5.14}
\]

where \(C(t)\) is the integration constant. By using the above equation and eq. (5.10), we can express \(E^0(0)\) as
\[
\mathcal{E}^0(t) = -\int_{u \neq v} du dv g^0(\{u - v\}; R, \Lambda) \frac{d\rho^0(u)}{du} \frac{d\rho^0(v)}{dv} - C(t). \tag{5.15}
\]
The partial differentiation of the right-hand side of the above equation with respect to \(t_k\) leads to the formula (5.13). Actually, making use of eqs. (5.14) and (5.10), it can be computed as
\[
\partial_{t_k} \mathcal{E}^0(t) = -2 \int_{-\infty}^{+\infty} du \frac{d\rho^0(u)}{du} \partial_{t_k} \left\{ \text{PP} \int_{-\infty}^{+\infty} dv g^0(\{u - v\}; R, \Lambda) \frac{d\rho^0(v)}{dv} \right\} - \partial_{t_k} C(t)
\]
\[
= -2 \int_{-\infty}^{+\infty} du \frac{d\rho^0(u)}{du} \partial_{t_k} \left\{ -\frac{1}{2} \left( \frac{Ru^3}{6} + V(u) \right) + \frac{1}{2} C(t) \right\} - \partial_{t_k} C(t)
\]
\[
= \int_{-\infty}^{+\infty} du \partial_{t_k} V(u) \frac{d\rho^0(u)}{du}. \tag{5.16}
\]
Thus, taking account of (5.12), we obtain the formula (5.13).
The variational problem can be reformulated as the Riemann-Hilbert problem to find out a certain analytic function. To see this, consider the integral transform
\[ \Phi(z) = \int_{-\infty}^{+\infty} du \coth \frac{R}{2}(z - u) \frac{d\rho^{(0)}(u)}{du}, \quad z \in \mathbb{C}. \] (5.17)

The above function is actually an analytic function on the cylinder \( \mathbb{C}^* = \mathbb{C}/\frac{2\pi i}{R} \mathbb{Z} \), since it is periodic with period \( 2\pi i/R \). The inverse-transform is realized by the imaginary part as
\[ \frac{d\rho^{(0)}(u)}{du} = \mp \frac{R}{2\pi} \Im \Phi(u \pm i0). \] (5.18)

Therefore, taking account of the integral transformation (5.17), the Riemann-Hilbert problem is stated to obtain an analytic function on \( \mathbb{C}^* - I \), where \( I \) is an interval in the real axis, and satisfies the following conditions:
\[ \Re \Phi(u \pm i0) = -1 - \frac{1}{R} V'''(u) \quad \text{when} \quad u \in I, \] (5.19)
\[ \Im \Phi(u \pm i0) = 0 \quad \text{when} \quad u \in \mathbb{R} - I, \] (5.20)
\[ \Phi(z) \to \mp 1 \quad \text{as} \quad \Re z \to \pm \infty, \] (5.21)
\[ \oint_C dz \, z\Phi(z) = 0 \quad (C: \text{a contour encircling} \ I \text{ anticlockwise}). \] (5.22)

Among the above conditions, the first two correspond to the saddle point equation (5.9). More precisely, they are a twice differentiated form of the saddle point equation. The last two equations respectively amount to (5.10) and (5.11).

### 6 Extended Seiberg-Witten geometry of $5d$ theory

The solution of the forgoing Riemann-Hilbert problem was obtained in [29], [8] in the case where all the coupling constants vanish. By generalizing the argument there, we can solve the Riemann-Hilbert problem (5.19)-(5.22), even in the case of non-vanishing coupling constants.

Let us employ the following curve which describes a fourth punctured \( \mathbb{C}P_1 \):
\[ C_\beta : y + y^{-1} = \frac{1}{R\Lambda}(e^{-Rz} - \beta), \quad z \in \mathbb{C}, \] (6.1)
where \( \beta \) is a real parameter satisfying the condition \( \beta > 2R\Lambda \). The above curve is an analogue of the so-called Seiberg-Witten hyperelliptic curve. Actually, it is a double covering of the
cylinder $\mathbb{C}^*$, and $y$ is uniformized on the curve. See Fig. 2. In particular, the branch points of $y$ are located at $z = -\frac{1}{R} \log(\beta \pm 2R\Lambda)$. Thereby, $y$ has a single cut along the interval $I$ in the real axis, where $I = \left[-\frac{1}{R} \log(\beta + 2R\Lambda), -\frac{1}{R} \log(\beta - 2R\Lambda)\right]$.

We make an ansatz on the solution of the Riemann-Hilbert problem (5.19)-(5.22) as

$$\Phi(z) = -1 + (2 + \varphi(z)) \frac{1}{R} \frac{d \log y}{dz},$$

where $\varphi(z)$ is a certain analytic function on $\mathcal{C}_\beta$. Through the above ansatz, conditions (5.19)-(5.22) impose constraints on $\varphi(z)$. Actually, the first three conditions are translated as

$$\Im \varphi(u \pm i0) = \begin{cases} \pm \frac{1}{iR} V'''(u) \sqrt{P(u - i0)} & u \in I, \\ 0 & u \in \mathbb{R} - I, \end{cases}$$

$$\varphi(z) = O(1) \quad \text{as} \quad \Re z \to +\infty,$$

$$\varphi(z) = O(e^{Rz}) \quad \text{as} \quad \Re z \to -\infty.$$  

In the above, $P(z)$ is a quadratic polynomial of $e^{Rz}$ given by

$$\frac{1}{R} d \log y = \frac{dz}{\sqrt{P(z)}},$$

In other word,

$$P(z) = (1 - \beta e^{Rz})^2 - (2R\Lambda e^{Rz})^2.$$
An analytic function satisfying the above three conditions can be realized by a contour integral of the form

$$\varphi(z) = -\frac{1}{4\pi i} \oint_C dw \left(1 + \coth \frac{R}{2} (z - w)\right) V'''(w) \sqrt{P(w)},$$  \hspace{1cm} (6.8)

where \(C\) is a closed curve that surrounds the interval \(I\) but leaves \(z\) outside as depicted in Fig. 3. To see that the above satisfies the conditions (6.3)-(6.5), note that the contour integration in the right-hand side of (6.8) can be written as an integration along \(I\) of the form

$$\varphi(z) = -\frac{1}{2\pi i} \int_I dv \left(1 + \coth \frac{R}{2} (z - v)\right) V'''(v) \sqrt{P(v - i0)}.$$  \hspace{1cm} (6.9)

When \(z = u \pm i0\), taking account of (6.7), the imaginary part of the right-hand side of this equation becomes \(\pm V'''(u) \sqrt{P(u - i0)/iR}\) if \(u \in I\), otherwise vanishes. Therefore, \(\varphi(z)\) given in (6.8) obeys the condition (6.3). The constant term of the kernel function in the contour integral (6.8) is required in order to satisfy the boundary conditions (6.4) and (6.5). To see this, note that the kernel function has the asymptotics

$$1 + \coth \frac{R}{2} z = 2 \sum_{n=0}^{\infty} e^{-nRz} \quad \text{as } \Re z \to \infty$$

$$= -2 \sum_{n=1}^{\infty} e^{nRz} \quad \text{as } \Re z \to -\infty.$$  \hspace{1cm} (6.10)

By plugging the above into the right-hand side of (6.9), we find that \(\varphi(z)\) satisfies the conditions (6.4) and (6.5).
6.1 Determination of \( \beta \)

In order to obtain the solution, taking \( \varphi(z) \) as in (6.8), we further need to impose the condition (5.22). This turns out to be an equation which determines the parameter \( \beta \) of the curve (6.1) in terms of \( t \).

We can calculate the right-hand side of (6.8) by residue calculus. To see this, we change the integration variable from \( w \) to \( W = e^{-Rw} \) as

\[
\varphi(z) = \frac{1}{2\pi i} \oint_{C'} \frac{dW}{W(W-e^{-Rz})} \frac{1}{R} V'''(W) \sqrt{(W-\beta)^2 - (2RA)^2},
\]

where the contour \( C' \), which is the image of \( C \) by the mapping \( w \mapsto \) \( W \), encircles the interval \([\beta-2RA, \beta+2RA]\) anticlockwise. In the outside of \( C' \), the integrand has poles at \( W = e^{-Rz}, \infty \) and is holomorphic elsewhere; the origin \( W = 0 \) is not a pole because \( V'''(W) = \sum_{k=1}^{\infty} t_k (kR)^3 W^k \) has a zero at \( W = 0 \). We can deform \( C' \) outward to small circles that surround the two poles clockwise. Thus the contour integral reduces to a sum of the residues (times \(-1\)) at those poles, so that we have

\[
\varphi(z) = -\frac{1}{R} V'''(z) \sqrt{P(z)} + M(z).
\]

The first and second terms in the right-hand side are the contribution of the residues at \( W = e^{-Rz} \) and \( W = \infty \), respectively. More explicitly,

\[
M(z) = \sum_{k=1}^{\infty} t_k M_k(z),
\]

\[
M_k(z) = -R^2 k^3 \sum_{n=0}^{k} d_{k-n} e^{-nRz}, \quad k = 1, 2, \ldots,
\]

where \( d_n = d_n(\beta) \) denote the coefficients of expansion of \( \sqrt{P(z)} \) in positive powers of \( e^{Rz} \):

\[
\sqrt{P(z)} = \sum_{n=0}^{\infty} d_n(\beta) e^{nRz} \quad \text{as } \Re z \to -\infty.
\]

The first few terms read \( d_0(\beta) = 1, d_1(\beta) = -\beta, \ldots \). Let us note that this expression of \( M_k(z) \) and \( M(z) \) readily implies that

\[
R^2 k^3 e^{-kRz} \sqrt{P(z)} + M_k(z) = O(e^{Rz}),
\]

\[
-\frac{1}{R} V'''(z) \sqrt{P(z)} + M(z) = O(e^{Rz}) \quad \text{as } \Re z \to -\infty.
\]
We can thus directly confirm that the right-hand side of (6.12) satisfies the boundary condition (6.5). Actually, to solve the foregoing Riemann-Hilbert problem, we could have started from (6.12) and proceeded to verifying the other conditions.

We now write down $Φ(z)$, plugging the expression (6.12) into (6.2), as follows:

$$Φ(z) = -\left(1 + \frac{1}{R}V''(z)\right) + (2 + M(z)) \frac{1}{R} \frac{d \log y}{dz}.$$  

(6.18)

By using this formula, the condition (5.22) becomes

$$\oint_C z(2 + M(z)) d \log y = 0.$$  

(6.19)

This equation can be understood as an equation that determines the parameter $β$ of the curve (6.1). To write down such an equation explicitly, let us introduce the quantities

$$J_m(β) = \oint_C z e^{-mRz} d \log y, \quad m = 0, 1, 2, \ldots.$$  

(6.20)

All the contour integrals that arise from components of $M(z)$ in $\oint_C z(2 + M(z)) d \log y$, taking account of (6.13) and (6.14), have the form (6.20) and are consequently expressed in terms of $J_m(β)$ and $d_n(β)$. Eventually, they are arranged to provide the following equation for $β$:

$$\left(2 - R^2 \sum_{k=1}^{∞} k^3 t_k d_k(β)\right) J_0(β) = R^2 \sum_{m=1}^{∞} \left(\sum_{k=m}^{∞} k^3 t_k d_{k-m}(β)\right) J_m(β).$$  

(6.21)

The solution of the Riemann-Hilbert problem (5.19)-(5.22) can be summarized as follows: Take the curve $C_{β=β(t)}$, where $β(t)$ is the solution of equation (6.21). Then, the analytic function in (6.18) gives the solution.

### 6.2 The case of single coupling constant

Let us examine the case of $t = (t_1, 0, 0, \ldots)$. In this case, eq. (6.21) is simplified to be

$$(2 + R^2 t_1 β) J_0(β) = R^2 t_1 J_1(β),$$  

(6.22)

where $J_{0,1}(β)$ are the integrals (6.20). Contour integrations that appear in (6.20) can be evaluated by computations utilizing the classical Jensen formula, and one eventually obtains

$$J_0(β) = \frac{2πi}{R} \log\left(\frac{β - \sqrt{β^2 - (2RA)^2}}{2(RA)^2}\right).$$  

(6.23)
Figure 4: The graph of $\beta = \beta(t_1, 0, 0, \cdots)$. Each corresponds respectively to the cases of $R = 0.1$, $R = 0.2$ and $R = 0.3$, where $\Lambda$ is fixed to one.

We provide a derivation of the above formulas in Appendix D. By plugging these values of the integrals into (6.22), eq.(6.21) in the case of $t = (t_1, 0, 0, \cdots)$ becomes

$$J_1(\beta) = \frac{2\pi i}{R} \left\{ \beta \log \left( \frac{\beta - \sqrt{\beta^2 - (2R\Lambda)^2}}{2(R\Lambda)^2} \right) - \left( \beta - \sqrt{\beta^2 - (2R\Lambda)^2} \right) \right\}. \quad (6.24)$$

When the coupling constant $t_1$ vanishes, the above equation is further simplified to be $\beta - \sqrt{\beta^2 - (2R\Lambda)^2} = 2(R\Lambda)^2$. Taking account of the constraint $\beta > 2R\Lambda$, which is required to have a single cut along the real axis, one has a unique solution $\beta = 1 + (R\Lambda)^2$ only when $0 < R\Lambda < 1$. This is precisely the solution obtained in [29,8]. Assuming that $0 < R\Lambda < 1$ as well, one also has a unique solution $\beta(t_1)$ of eq.(6.25) such that $\beta(0) = 1 + (R\Lambda)^2$. See Fig. 4.
6.3 Seiberg-Witten differential

The vev of the loop operator $O_k$ can be represented by using an analogue of the so-called Seiberg-Witten differential. Let $\Phi(z)$ be the foregoing solution described in the previous subsection. The inverse transform (5.18) gives rise to the minimizer $\rho_0(\star)$ as

$$
\frac{d\rho_0(\star)(u)}{du} = \begin{cases} 
\mp \frac{1}{2\pi i}(2 + M(u)) \frac{d\log y(u \pm i0)}{du} & u \in I, \\
0 & u \in \mathbb{R} - I.
\end{cases}
$$

The integral expression (5.13) of the vev of $O_k$ can be organized, by plugging the above formula into the expression and integrating by parts, to the following contour integral:

$$
\frac{\partial \mathcal{E}_{\star}(t)}{\partial t_k} = \lim_{\hbar \to 0} \langle O_k \rangle = -\frac{1}{2\pi i} \oint_C kRe^{-kRz}dS,
$$

where $dS = S'(z)dz$ is an analogue of the Seiberg-Witten differential. $S'(z)$ is given by the indefinite integral

$$
S'(z) = -\int^z (1 + \frac{1}{2}M(z))d\log y = -\log y + N(z)\sqrt{P(z)} + \text{const.},
$$

where $N(z)$ is a holomorphic function with an expansion similar to (6.13). Just like $\Phi(z)$, one can characterize $S'(z)$ by a Riemann-Hilbert problem, though we omit details.

The contour integral in the right-hand side of (6.27) can be converted to a residue integral. Actually, using coordinate $Z = e^{-Rz}$, we obtain

$$
\frac{\partial \mathcal{E}_{\star}(t)}{\partial t_k} = \lim_{\hbar \to 0} \langle O_k \rangle = \text{Res}_{Z=\infty} \left(kRZ^kdS\right).
$$

The above expression shows that $\mathcal{E}_{\star}(t)$ may be interpreted as a dispersionless tau function [30]. This is natural because the partition function is substantially a tau function of 1-Toda hierarchy and $\mathcal{E}_{\star}(t)$ gives the leading order part of the $\hbar$-expansion of $\log Z_{\star-i\hbar,i\hbar}(t)$.

7 Conclusion and discussion

We studied an extension of the Seiberg-Witten theory of 5d $\mathcal{N} = 1$ SYM on $\mathbb{R}^4 \times S^1$. We investigated correlation functions among the loop operators. These operators are analogues of the Wilson loop operators encircling the fifth-dimensional circle and give rise to the physical
observables of topological-twisted $5d\,\mathcal{N}=1$ SYM in the $\Omega$ background through the equivariant descent equation. The correlation functions were computed by using the localization technique. Generating function of the correlation functions of $U(1)$ theory equals a statistical sum over partitions and reproduces the partition function of the melting crystal model with external potentials, where the loop operators are converted to the external potentials of the melting crystal model. This eventually shows that, by regarding the coupling constants of the loop operators as a series of time variables, the generating function is a $\tau$ function of 1-Toda hierarchy. The 1-Toda hierarchy therefore describes a common integrable structure of $5d\,\mathcal{N}=1$ SYM in the $\Omega$ background and melting crystal model.

The thermodynamic limit of the partition function of this model was studied by applying an integral formula of the energy functional, and was reformulated as a Riemann-Hilbert problem to obtain an analytic function on $\mathbb{C}^* - I$, where $\mathbb{C}^*$ is a cylinder and $I$ is an interval in the real axis, and satisfies suitable conditions. We solved the Riemann-Hilbert problem that determines the limit shape of the main diagonal slice of random plane partitions in the presence of external potentials, and identified a relevant complex curve and the associated Seiberg-Witten differential, where the vev’s of the loop operators are expressed as residue integrals of the differential.

Lastly, let us briefly discuss the issue of $4d$ limit (see also the discussion on this issue in our previous work [13]). It seems quite difficult to achieve a reasonable $4d$ ($R \to 0$) limit in the present framework. The difficulty is rather obvious in the setup of the Riemann-Hilbert problem. As the definition (5.17) shows, it will not be $\Phi(z)$ but rather $R\Phi(z)$ that turns into the $4d$ counterpart

$$\Phi_{4d}(z) = \int_{-\infty}^{+\infty} \frac{du}{z-u} \frac{d\rho^{(0)}(u)}{du}$$

of the integral transform of $\frac{d\rho^{(0)}(u)}{du}$ in the limit as $R \to 0$. If we multiply both hand sides of the first equation (5.19) of the Riemann-Hilbert problem by $R$ and let $R \to 0$, the outcome is the equation

$$\Re \Phi_{4d}(u \pm i0) = \lim_{R \to 0} \frac{R}{2} \left(-1 - \frac{1}{R} V''(u)\right) = 0 \quad (u \in I)$$

that has no potential term. Thus, at least in this naive prescription, the external potentials simply decouple in the $4d$ limit. (As regards the constant term $-1$, it is natural that it disap-
pears, because this term originates in a 5d Chern-Simons term [7].) One might argue that this difficulty can be simply avoided by rescaling $t_k$’s as $t_k \rightarrow R^{-2}t_k$ before letting $R \rightarrow 0$. Actually, this does not resolve the problem, because the exponential functions $e^{-kR u}$ in the external potentials themselves become constant functions in this limit, so that the Riemann-Hilbert problem still takes a degenerate form:

$$\Re \Phi_{4d}(u \pm i0) = \frac{1}{2} \sum_{k=1}^{\infty} t_k k^3 \quad (u \in I). \quad (7.3)$$

Thus these naive procedures fail to generate the polynomial potentials $u^k$, $k = 1, 2, 3, \ldots$ on the right-hand side of the Riemann-Hilbert problem. A possible remedy for this difficulty will be to extend the set of loop operators $O_k$ to $k < 0$. One can thereby derive polynomial potentials from a linear combination of exponential potentials as, say,

$$\left( \frac{e^{-Ru} - e^{Ru}}{-2R} \right)^k \rightarrow u^k \quad (R \rightarrow 0). \quad (7.4)$$

This, however, causes some other problems. First of all, we have to point out that almost all part of our foregoing consideration assumes, implicitly or explicitly, that $q$ is in the range $0 < q < 1$. Flipping the signature of $k$ amounts to changing $q \rightarrow q^{-1}$. To introduce the loop operators $O_k$ for both $k > 0$ and $k < 0$ simultaneously, one will be forced to consider the case where $q = 1$ (and $|q| = 1$ more generally), which can only be reached as a kind of singular limit letting $q \rightarrow 1$. This is also the case for identification of the integrable structure; $q \rightarrow 1$ is a singular limit in which some properties of the quantum torus Lie algebra (crucial for identification of the integrable structure) break down [13]. Moreover, our method for solving the Riemann-Hilbert problem, too, has to be modified if the potential $V(z)$ contains both $e^{-kRz}$ and $e^{kRz}$. Thus the issue of the 4d limit is not straightforward in our present framework, raising a number of interesting questions.

**Acknowledgements**

K.T is supported in part by Grant-in-Aid for Scientific Research No. 18340061 and No. 19540179.
A The $T^2$-action on $A_E$

We treat integral curves of $V_{\epsilon_1, \epsilon_2}$ as curves on $\mathbb{R}^4$ in this appendix. Denote the integral curve passing through $x$ by $\gamma^x$. More precisely, it can be expressed as the solution of the differential equation

$$\frac{dx(s)}{ds} = V_{\epsilon_1, \epsilon_2}(x(s)), \quad x(0) = x. \tag{A.1}$$

Let $A$ be a gauge potential on the $SU(N)$-bundle $E$ on $\mathbb{R}^4$. We consider the parallel transports along the integral curves. In particular, along the above $\gamma^x$, the parallel transport of the fibre $E_x$ to $E_x$ is described by the holonomy operator $Pe^{-\int_{x(s)}^x A}$, where the path-ordered integration is achieved from $x(s)$ to $x$ along $\gamma^x$. Such parallel transports provide endomorphisms $\tau_{\epsilon_1, \epsilon_2}(s)$ of $E$ by the formula

$$(\tau_{\epsilon_1, \epsilon_2}(s)\phi)(x) = Pe^{-\int_{x(s)}^x A} \phi(x(s)), \tag{A.2}$$

where $\phi$ is a section of $E$. The above endomorphisms define a $T^2$-action on $\Omega^0(\mathbb{R}^4, E)$. In particular, they satisfy the relation $\tau_{\epsilon_1, \epsilon_2}(s) \cdot \tau_{\epsilon_1, \epsilon_2}(s') = \tau_{\epsilon_1, \epsilon_2}(s + s')$. The infinitesimal action is therefore given by

$$t_{\epsilon_1, \epsilon_2} \cdot \phi = \left. \frac{d}{ds} \tau_{\epsilon_1, \epsilon_2}(s)\phi \right|_{s=0}, \tag{A.3}$$

where $t_{\epsilon_1, \epsilon_2}$ denotes the generator of $T^2$ that gives $V_{\epsilon_1, \epsilon_2}$. The right-hand side of (A.3) can be computed by using (A.2). To see this, note that $x(s)$ takes a form $x(s) = x + sV_{\epsilon_1, \epsilon_2}(x) + O(s^2)$ for a very small $s$, thereby we have $Pe^{-\int_{x(s)}^x A} = 1 + sV_{\epsilon_1, \epsilon_2}A_\mu(x) + O(s^2)$ and $\phi(x(s)) = \phi(x) + sV_{\epsilon_1, \epsilon_2}^\mu \partial_\mu \phi(x) + O(s^2)$. By combining these two, we can write the formula (A.2) as

$$(\tau_{\epsilon_1, \epsilon_2}(s)\phi)(x) = \left( 1 + sV_{\epsilon_1, \epsilon_2}^\mu A_\mu(x) + O(s^2) \right) \left( \phi(x) + sV_{\epsilon_1, \epsilon_2}^\mu \partial_\mu \phi(x) + O(s^2) \right)$$

$$= \phi(x) + s \iota_{V_{\epsilon_1, \epsilon_2}} d_A \phi(x) + O(s^2), \tag{A.4}$$

where $\iota_{V_{\epsilon_1, \epsilon_2}}$ means an operation of contraction with the vector field $V_{\epsilon_1, \epsilon_2}$. Thus, by plugging the above into the right-hand side of (A.3), we obtain

$$t_{\epsilon_1, \epsilon_2} \cdot \phi = \iota_{V_{\epsilon_1, \epsilon_2}} d_A \phi. \tag{A.5}$$
Similarly, the endomorphisms induce a $T^2$-action on $A_E$, which is the space of all the gauge potentials on $E$, by the formula

$$d_{\tau_{\epsilon_1,\epsilon_2}(s)A} \tau_{\epsilon_1,\epsilon_2}(s) = \tau_{\epsilon_1,\epsilon_2}(s) d_A,$$  

(A.6)

where $d_A = d + A$ is a covariant differential on $E$. In the above formula, we use the same symbol to denote the $T^2$-action. By using the holonomy operator, the formula can be written in an explicit form as

$$d + (\tau_{\epsilon_1,\epsilon_2}(s) \cdot A)(x) = Pe^{-f_x^A}(d + A(x(s)))\left(Pe^{-f_x^A}A\right)^{-1},$$  

(A.7)

Let us describe the infinitesimal form of the above action. For a very small $s$, the right-hand side of (A.7) can be computed as

$$Pe^{-f_x^A}(d + A(x(s)))\left(Pe^{-f_x^A}A\right)^{-1} = \left(1 + s V_{\epsilon_1,\epsilon_2}^{\mu} \partial_\mu A(x) + O(s^2)\right) \left\{d + A(x) + s \left(V_{\epsilon_1,\epsilon_2}^{\nu} \partial_\nu A_\mu + A_\nu \partial_\mu V_{\epsilon_1,\epsilon_2}^{\nu}\right)(x) dx^\mu + O(s^2)\right\} \times \left(1 - s V_{\epsilon_1,\epsilon_2}^{\mu} A_\mu(x) + O(s^2)\right)$$

$$= d + A(x) + s \left(V_{\epsilon_1,\epsilon_2}^{\nu} \partial_\nu A_\mu - V_{\epsilon_1,\epsilon_2}^{\mu} \partial_\mu A_\nu + [V_{\epsilon_1,\epsilon_2}^{\nu}, A_\nu]\right)(x) dx^\mu + O(s^2)$$

$$= d + A(x) + s \tau_{V_{\epsilon_1,\epsilon_2} A}(x) + O(s^2),$$  

(A.8)

where $F_A = dA + A \wedge A$ is the curvature two form. Thus, the formula (A.7) reads as

$$\tau_{\epsilon_1,\epsilon_2}(s) \cdot A = A + s \tau_{V_{\epsilon_1,\epsilon_2} A} + O(s^2).$$  

(A.9)

Therefore the infinitesimal form of the $T^2$-action becomes

$$\tau_{\epsilon_1,\epsilon_2}(s) \cdot A = A + s \tau_{V_{\epsilon_1,\epsilon_2} A} + O(s^2).$$  

(A.10)

### B Proof of the identity (2.18)

We first rewrite the $Q$-transformations (2.6) and (2.7) in the forms

$$Q_{\epsilon_1,\epsilon_2} A(t) = \psi(t),$$

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\[ Q_{\epsilon_1,\epsilon_2} \psi(t) = dA(t) \phi(t) - \iota_{V_{\epsilon_1,\epsilon_2}} F_A(t) - \frac{dA(t)}{dt}, \]
\[ Q_{\epsilon_1,\epsilon_2} \phi(t) = -\iota_{V_{\epsilon_1,\epsilon_2}} \psi(t). \]  

(B.1)

By using the above expression, we can easily see that the combination \( F_A(t) - \psi(t) + \phi(t) \) satisfies the identity

\[ (dA(t) - \iota_{V_{\epsilon_1,\epsilon_2}} + Q_{\epsilon_1,\epsilon_2}) \left( F_A(t) - \psi(t) + \phi(t) \right) = \frac{dA(t)}{dt}. \]  

(B.2)

The above identity may be interpreted as a loop space analogue of the equivariant Bianchi identity [25]. Actually, when \( A(t), \psi(t) \) and \( \phi(t) \) are constant loops, that is, when they do not depend on \( t \), (B.2) reduces to

\[ (dA - \iota_{V_{\epsilon_1,\epsilon_2}} + Q_{\epsilon_1,\epsilon_2}) (F_A - \psi + \phi) = 0, \]  

(B.3)

where the combination \( F_A(x) - \psi(x) + \phi(x) \) is naturally identified with the \( T^2 \)-equivariant curvature of the universal connection [4], and (B.3) substantially describes the equivariant Bianchi identity.

Let us derive the formula (2.18) by using the identity (B.2). To do this, note that, since \( \psi(t) \) is a Grassmann-odd one-form on \( \mathbb{R}^4 \), while \( F_A(t) \) and \( \phi(t) \) are Grassmann-even two- and zero-forms, we have

\[ (d_{\epsilon_1,\epsilon_2} + Q_{\epsilon_1,\epsilon_2}) W(x; t_1, t_2) = -\int_{t_2}^{t_1} ds \tilde{W}(x; t_1, s) \wedge (d_{\epsilon_1,\epsilon_2} + Q_{\epsilon_1,\epsilon_2}) \left( F_A(s) - \psi(s) + \phi(s) \right)(x) \wedge W(x; s, t_2), \]  

(B.4)

where \( \tilde{W} \) is another generalization of the path-ordered integral (2.10), and is given by

\[ \tilde{W}(x; t_1, t_2) = P \exp \left\{ -\int_{t_2}^{t_1} dt \left( F_A(t) + \psi(t) + \phi(t) \right)(x) \right\}. \]  

(B.5)

Since \( O(x) \) is the trace of \( W(x; R, 0) \), using (B.4), we can compute \( (d_{\epsilon_1,\epsilon_2} + Q_{\epsilon_1,\epsilon_2}) O(x) \) as

\[ (d_{\epsilon_1,\epsilon_2} + Q_{\epsilon_1,\epsilon_2}) O(x) = -\int_0^R dt \text{Tr} \left\{ \tilde{W}(x; R, t) \wedge (d_{\epsilon_1,\epsilon_2} + Q_{\epsilon_1,\epsilon_2}) \left( F_A(t) - \psi(t) + \phi(t) \right)(x) \wedge W(x; t, 0) \right\}. \]  

(B.6)

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By using (B.2), the right-hand side of (B.6) can be further evaluated as

\[
(d_{e_1,e_2} + Q_{e_1,e_2})\mathcal{O}(x)
\]

\[
= - \int_0^R dt \text{Tr} \left\{ \frac{dA(x,t)}{dt} \wedge W(x; t, 0) \right\} \tag{B.7}
\]

\[
+ \int_0^R dt \text{Tr} \left\{ \bar{W}(x; R, t) \wedge A(x, t) \wedge \left( F_{A(t)} - \psi(t) + \phi(t) \right)(x) \wedge W(x; t, 0) \right\} \tag{B.8}
\]

\[
- \int_0^R dt \text{Tr} \left\{ \bar{W}(x; R, t) \wedge \left( F_{A(t)} + \psi(t) + \phi(t) \right)(x) \wedge A(x, t) \wedge W(x; t, 0) \right\}. \tag{B.9}
\]

In the above, we can arrange the first term of the right-hand side as

\[
(B.7) = - \int_0^R dt \text{Tr} \left\{ \frac{dA(x,t)}{dt} \wedge W(x; t, 0) \wedge W(x; R, t) \right\}
\]

\[
= - \int_0^R dt \text{Tr} \left\{ \frac{dA(x,t)}{dt} \wedge W(x; t, t - R) \right\}.
\]

Similarly, the second term and the third term can be arranged into

\[
(B.8) = \int_0^R dt \text{Tr} \left\{ A(x, t) \wedge \left( F_{A(t)} - \psi(t) + \phi(t) \right)(x) \wedge W(x; t, t - R) \right\},
\]

\[
(B.9) = - \int_0^R dt \text{Tr} \left\{ A(x, t) \wedge W(x; t, t - R) \wedge \left( F_{A(t-R)} - \psi(t - R) + \phi(t - R) \right)(x) \right\}.
\]

These two are particularly combined to give

\[
(B.8) + (B.9) = - \int_0^R dt \text{Tr} \left\{ A(x, t) \wedge \frac{dW(x; t, t - R)}{dt} \right\}.
\]

Therefore, by summing up (B.7), (B.8) and (B.9), we find

\[
(d_{e_1,e_2} + Q_{e_1,e_2})\mathcal{O}(x)
\]

\[
= - \int_0^R dt \text{Tr} \left\{ \frac{dA(x,t)}{dt} \wedge W(x; t, t - R) \right\} - \int_0^R dt \text{Tr} \left\{ A(x, t) \wedge \frac{dW(x; t, t - R)}{dt} \right\}
\]

\[
= - \int_0^R dt \frac{d}{dt} \left[ \text{Tr} \left\{ A(x, t) \wedge W(x; t, t - R) \right\} \right]. \tag{B.10}
\]

The last integral is integrated to be \( \text{Tr} A(x, 0) \wedge W(x; 0, -R) - \text{Tr} A(x, R) \wedge W(x; R, 0) \), and becomes, by virtue of the periodicity, zero. Thus we obtain the formula (2.18).
C Proof of the formula (4.7)

The formula (4.7) is stated as follows.

**Formula 1** Let \( f(x) \) be a function on \( \mathbb{R} \). Choose a function \( g(x) \) to satisfy the conditions (4.5) and (4.6). Then, for any partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), the following identity holds:

\[
\sum_{s \in \lambda} f(h(s)) = \int_{x>y} dx dy g(x - y) \Delta \rho_\lambda(x) \Delta \rho_\lambda(y), \tag{C.1}
\]

where \( h(s) \) denotes the hook length of the box \( s \in \lambda \), and \( \Delta \rho_\lambda(x) = \rho_\lambda(x) - \rho_\lambda(x - 1) \) is a difference of the density function (4.1) of \( \lambda \).

To prove the above formula, note that the density functions of a partition \( \lambda \) and its conjugate partition \( \tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots) \), where the Young diagram \( \tilde{\lambda} \) is obtained by flipping the Young diagram \( \lambda \) over its main diagonal, satisfy the relation \( \Delta \rho_{\tilde{\lambda}}(x) = \Delta \rho_\lambda(-x) \). We can thereby rewrite the right-hand side of (C.1) as

\[
\int_{x>y} dx dy g(x - y) \Delta \rho_\lambda(x) \Delta \rho_\lambda(y) = \int_{x>y} dx dy g(x - y) \Delta \rho_\lambda(x) \Delta \rho_{\tilde{\lambda}}(-y) \tag{C.2}
\]

We compute the right-hand side of (C.2). For this end, it is convenient to introduce the truth function \( \chi \) by

\[
\chi(n) = \begin{cases} 
1 & \text{iff } n \geq 1, \\
0 & \text{otherwise}.
\end{cases} \tag{C.3}
\]

Owing to the \( \delta \)-functions in the density functions, the right-hand side of (C.2) can be expressed by using the above \( \chi \) as

\[
\int_{x>y} dx dy g(x - y) \Delta \rho_\lambda(x) \Delta \rho_\lambda(y) = \sum_{i,j=1}^{\infty} g(\lambda_i + \tilde{\lambda}_j - i - j) \chi(\lambda_i + \tilde{\lambda}_j - i - j) + \sum_{i,j=1}^{\infty} g(\lambda_i + \tilde{\lambda}_j - i - j + 2) \chi(\lambda_i + \tilde{\lambda}_j - i - j + 2) - 2 \sum_{i,j=1}^{\infty} g(\lambda_i + \tilde{\lambda}_j - i - j + 1) \chi(\lambda_i + \tilde{\lambda}_j - i - j + 1). \tag{C.4}
\]

We can further simplify (C.4). To see this, note that the combination \( \lambda_i + \tilde{\lambda}_j - i - j \) becomes \( \leq -2 \) when the pair \((i, j)\) is not a box of the Young diagram \( \lambda \), while it becomes \( \geq 0 \) when the
pair is a box of $\lambda$. More precisely, when the pair $(i, j)$ is a corner of $\lambda$, which means that $(i, j)$ is a box of $\lambda$ but neither $(i + 1, j)$ nor $(i, j + 1)$ is, the combination $\lambda_i + \tilde{\lambda}_j - i - j$ is equal to 0, otherwise it takes $\geq 2$. Therefore, (C.4) is eventually simplified as

$$\int_{x>y} dxdy g(x - y) \Delta \rho_\lambda(x) \Delta \rho_\lambda(y) = \sum_{(i,j) \in \lambda} \left\{ g(\lambda_i + \tilde{\lambda}_j - i - j + 2) - 2g(\lambda_i + \tilde{\lambda}_j - i - j + 1) + g(\lambda_i + \tilde{\lambda}_j - i - j) \right\} + \sum_{\text{corners of } \lambda} g(0).$$ 

(C.5)

In the above, by virtue of the condition (4.5), the first term of the right-hand side becomes $\sum_{(i,j) \in \lambda} f(h(i,j))$, while the second term vanishes by the condition (4.6). Thus we obtain the formula (C.1).

### D Derivation of eqs. (6.23) and (6.24)

We derive eqs. (6.23) and (6.24) by using the classical Jensen formula and its variant. Let $a$ be a complex number such that $|a| \neq 0, 1$. The classical Jensen formula is an integration formula of the form

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \log|e^{i\theta} - a| = \log \max(1, |a|).$$ 

(D.1)

We also use the following variant of the above formula:

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta (e^{ik\theta} + e^{-ik\theta}) \log|e^{i\theta} - a| = \begin{cases} -\frac{1}{k} \Re a^{-k} & \text{iff } |a| > 1 \\ -\frac{1}{k} \Re a^k & \text{iff } 0 < |a| < 1 \end{cases}$$ 

(D.2)

where $k = 1, 2, \cdots$.

Let us first express $J_m(\beta)$ (6.20) in terms of a contour integral on the $y$-plane. To do this, note that, taking account of the relation $e^{-Rz} = R\Lambda(y + y^{-1}) + \beta$, the contour $C$ in the right-hand side of (6.20) can be chosen so that it maps to the unit circle $|y| = 1$ on the $y$-plane. We also note that, on the unit circle, by virtue of the condition $\beta > 2R\Lambda$, $R\Lambda(y + y^{-1}) + \beta$ takes positive real numbers. Thereby, the contour integral in the right-hand side of (6.20) can be
written as

\[ J_m(\beta) = -\frac{1}{R} \oint_{|y|=1} \frac{dy}{y} \left\{ R\Lambda(y + y^{-1}) + \beta \right\}^m \log |R\Lambda(y + y^{-1}) + \beta|. \]  

(D.3)

The above integral can be converted to a combination of the phase integrals that appear in the formulas (D.1) and (D.2). To see this, note that \( R\Lambda(y + y^{-1}) + \beta \) is factorized into

\[ R\Lambda(y + y^{-1}) + \beta = R\Lambda y^{-1}(y - \alpha)(y - \alpha^{-1}), \]  

(D.4)

where \( \alpha \equiv -\beta/2R\Lambda - \sqrt{(\beta/2R\Lambda)^2 - 1} \). By plugging the above into the right-hand side of (D.3), the integral becomes a sum of phase integrals of the form

\[ J_m(\beta) = -\frac{i}{R} \log R\Lambda \int_0^{2\pi} d\theta \left\{ R\Lambda(e^{i\theta} + e^{-i\theta}) + \beta \right\}^m \log |e^{i\theta} - \alpha| \]

\[-\frac{i}{R} \int_0^{2\pi} d\theta \left\{ R\Lambda(e^{i\theta} + e^{-i\theta}) + \beta \right\}^m \log |e^{i\theta} - \alpha^{-1}|. \]  

(D.5)

The above phase integrals can be computed by applying the formulas (D.1) and (D.2). For the cases of \( m = 0, 1 \), we particularly obtain eqs. (6.23) and (6.24).

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