The three- and four-Higgs couplings in the general two-Higgs-doublet model

D. Jurčiukonis\(^{(1)}\)\(^{‡}\) and L. Lavoura\(^{(2)}\)\(^{§}\)

\(^{(1)}\) University of Vilnius, Institute of Theoretical Physics and Astronomy, Saulètekio ave. 3, LT-10222 Vilnius, Lithuania

\(^{(2)}\) Universidade de Lisboa, Instituto Superior Técnico, CFTP, 1049-001 Lisboa, Portugal

July 23, 2018

Abstract

We apply the unitarity bounds and the bounded-from-below (BFB) bounds to the most general scalar potential of the two-Higgs-doublet model (2HDM). We do this in the Higgs basis, \textit{i.e.} in the basis for the scalar doublets where only one doublet has vacuum expectation value. In this way we obtain bounds on the scalar masses and couplings that are valid for all 2HDMs. We compare those bounds to the analogous bounds that we have obtained for other simple extensions of the Standard Model (SM), namely the 2HDM extended by one scalar singlet and the extension of the SM through two scalar singlets.
1 Introduction

In order to unveil the detailed mechanism of electroweak symmetry breaking it is crucial to measure the self-couplings of the boson with mass 125 GeV discovered in 2012 at the LHC [1]. In this paper we call that boson $h_1$. The Standard Model (SM) predicts $h_1$ to be a scalar and predicts its cubic and quartic couplings $g_3$ and $g_4$, which we define through

$$\mathcal{L} = \cdots - g_3 (h_1)^3 - g_4 (h_1)^4,$$

(1)

to be $g_3^{SM} \approx 32$ GeV and $g_4^{SM} \approx 0.032$, respectively. However, in Nature the scalar sector may be more complicated than in the SM and then $g_3$ and $g_4$ might have very different values. In this paper we survey the allowed values of $g_3$ and $g_4$ in three extensions of the SM:

- The SM plus two real, neutral scalar singlets and with a reflection symmetry on each of those singlets. Let SM2S denote this model, which we treat in section 2.
- The two-Higgs-doublet model (2HDM), which is the focus object of section 3.
- The 2HDM with the addition of one real, neutral scalar singlet and with a reflection symmetry of that singlet. This model, which we dub the 2HDM1S, is dealt with in section 4.

Our ingredients for bounding $g_3$ and $g_4$ in each of these models are:

- The bounded-from-below (BFB) and the unitarity conditions on the quartic part of the scalar potential of each model. We apply those conditions directly in the basis for the scalar doublets where only one of them has vacuum expectation value (VEV).
- The experimental bound on the oblique parameter $T$ [2].
- The (approximate) bound $\cos \vartheta > 0.9$ on the $h_1$ component $\cos \vartheta$ of the scalar doublet with nonzero VEV.

Other authors before us [3]–[7] have used the BFB and unitarity constraints in order to bound the scalar masses and couplings of the 2HDM. However, they have done it in the context of a constrained version of the model, viz. the 2HDM with a reflection symmetry acting on one of the scalar doublets, leading to $\lambda_6 = \lambda_7 = 0$ in the scalar potential of equation (33). In this paper we deal on the fully general 2HDM. We enforce the BFB and unitarity constraints in the so-called Higgs basis, i.e. the basis where only one of the doublets has VEV. Since that basis exists for every 2HDM, we thus obtain results that apply to every 2HDM.

At present there are only indirect, very rough bounds on $g_3$. Using the Standard Model Effective Theory developed in ref. [8] and experimental data [9], ref. [10] has found that $-8.4 < g_3 / g_3^{SM} < 13.4$. From the contribution of $g_3$ to the oblique parameters $S$ and $T$, ref. [11] derived $-14.0 < g_3 / g_3^{SM} < 17.4$. The authors of ref. [12] obtained firstly $-9.4 < g_3 / g_3^{SM} < 17.0$ and then [13] $-8.2 < g_3 / g_3^{SM} < 13.7$. The partial-wave
unitarity of $h_1 h_1 \rightarrow h_1 h_1$ scattering has been used \cite{14} to obtain $|g_3/g_3^{\text{SM}}| \lesssim 6.5$ and $|g_4/g_4^{\text{SM}}| \lesssim 65$. In an analysis of a specific three-Higgs-doublet model, ref. \cite{15} has found that in that model $-1.3 < g_3/g_3^{\text{SM}} < 20.0$ and $1.05 < g_4/g_4^{\text{SM}} < 1.6$.

The measurement of $g_3$ should be possible at future colliders. Reference \cite{16} concluded that one may be able to measure $g_3$ provided $-0.72 < g_3/g_3^{\text{SM}} < 20.0$ and $1.05 < g_4/g_4^{\text{SM}} < 1.6$. Unfortunately, measuring $g_4$ is probably more challenging \cite{17}.

1.1 $g_3$ and $g_4$ in the SM

The Standard Model has only one scalar doublet $\phi_1$. We write it

$$\phi_1 = \left( v + (H + iG^0)/\sqrt{2} \right), \tag{2}$$

where $v$ is the VEV, which is real and positive, and $G^+$ and $G^0$ are (unphysical) Goldstone bosons. In the SM $H$ coincides with the observed scalar $h_1$. The scalar potential is

$$V = \mu_1 \phi_1^\dagger \phi_1 + \frac{\lambda_1}{2} (\phi_1^\dagger \phi_1)^2. \tag{3}$$

The minimization condition of $V$ is $\mu_1 = -\frac{\lambda_1 v^2}{2}$. Therefore, in the unitary gauge where $G^+$ and $G^0$ do not exist,

$$V = -\frac{\lambda_1 v^4}{2} + \lambda_1 v^2 H^2 + \frac{\lambda_1 v^2}{\sqrt{2}} H^3 + \frac{\lambda_1}{8} H^4. \tag{4}$$

The second term in the right-hand side of equation (4) indicates that the squared mass $M_1$ of the observed scalar is given by $M_1 = 2\lambda_1 v^2$. Therefore,

$$V = -\frac{M_1 v^2}{4} + \frac{M_1}{2} (h_1)^2 + \frac{M_1}{2\sqrt{2} v} (h_1)^3 + \frac{M_1}{16 v^2} (h_1)^4 \tag{5a}$$

$$= \cdots + g_3^{\text{SM}} (h_1)^2 + g_4^{\text{SM}} (h_1)^4. \tag{5b}$$

Using the approximate experimental values

$$M_1 = (125 \text{ GeV})^2, \tag{6a}$$

$$v = 174 \text{ GeV}, \tag{6b}$$

one gathers from equation (5a) that

$$g_3^{\text{SM}} = \frac{M_1}{2\sqrt{2} v} = 31.7 \text{ GeV}, \tag{7a}$$

$$g_4^{\text{SM}} = \frac{M_1}{16 v^2} = 0.0323. \tag{7b}$$

It should be noted that the sign of $g_3$ implicitly depends on the sign of $h_1$. We fix that sign by noting that the covariant derivative of $\phi_1$ gives rise to a term

$$\mathcal{L} = \cdots + \frac{g^2}{2} W_{\mu}^+ W_{\mu}^- \left( v + \frac{H}{\sqrt{2}} \right)^2 \tag{8a}$$

$$= \cdots + \frac{g^2 v}{\sqrt{2}} W_{\mu}^+ W_{\mu}^- H. \tag{8b}$$

Thus, the coupling $W_{\mu}^+ W_{\mu}^- h_1$, viz. $g^2 v / \sqrt{2}$, is positive.
2 The Standard Model plus two singlets

We consider the Standard Model with the addition of two real $SU(2) \times U(1)$-invariant scalar fields $S_1$ and $S_2$. We assume two symmetries $S_1 \rightarrow -S_1$ and $S_2 \rightarrow -S_2$. We call this model the SM2S. The scalar potential is

$$V = V_2 + V_4,$$

$$V_2 = \mu_1 \phi_1^\dagger \phi_1 + m_1^2 S_1^2 + m_2^2 S_2^2,$$

$$V_4 = \frac{\lambda_1}{2} \left( \phi_1^\dagger \phi_1 \right)^2 + \frac{\psi_1}{2} S_1^4 + \frac{\psi_2}{2} S_2^4 + \psi_3 S_1^2 S_2^2 + \phi_1^\dagger \phi_1 \left( \xi_1 S_1^2 + \xi_2 S_2^2 \right).$$

2.1 Unitarity conditions

We derive the unitarity conditions on the parameters of $V_4$. We follow closely the method of ref. [20]. We write

$$\phi_1 = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \phi_1^\dagger = \begin{pmatrix} a^* & b^* \end{pmatrix}, \quad S_1^* = S_1, \quad S_2^* = S_2,$$

where $a$ and $b$ are complex fields. Then,

$$V_4 = \frac{\lambda_1}{2} (a^* a a + b^* b b + 2 a^* b^* a b) + \frac{\psi_1}{2} S_1^4 + \frac{\psi_2}{2} S_2^4 + \psi_3 S_1^2 S_2^2$$

$$+ (a^* a + b^* b) \left( \xi_1 S_1^2 + \xi_2 S_2^2 \right).$$

There are seven two-particle scattering channels ($Q$ is the electric charge, $T_3$ is the third component of weak isospin):

1. The channel $Q = 2$, $T_3 = 1$, with one state $aa$.
2. The channel $Q = 0$, $T_3 = -1$, with one state $bb$.
3. The channel $Q = 1$, $T_3 = 0$, with one state $ab$.
4. The channel $Q = 1$, $T_3 = 1$, with one state $ab^*$.
5. The channel $Q = 1$, $T_3 = 1/2$, with two states $aS_1$ and $aS_2$.
6. The channel $Q = 0$, $T_3 = -1/2$, with two states $bS_1$ and $bS_2$.
7. The channel $Q = 0$, $T_3 = 0$, with five states $S_1^2$, $S_2^2$, $S_1 S_2$, $a^* a$, and $b^* b$.

\footnote{The SM2S has already been mentioned in the literature as a model for Dark Matter, see ref. [18].}

\footnote{Strictly speaking, the unitarity conditions derived and utilized in this paper are the ones valid in the limit of infinite Mandelstam parameter $s$. For finite $s$ one must take into account the trilinear vertices that are induced from the quartic vertices when one substitutes one of the fields by its VEV. The unitarity conditions then become $s$-dependent and may be either more or less restrictive than the conditions in the limit of infinite $s$. See ref. [19].}
In order to derive the unitarity conditions one must write the scattering matrices for pairs of one incoming state and one outgoing state with the same $Q$ and $T_3$. Let the incoming state be $xy$ and let the outgoing state by $zw$, where $x$, $y$, $z$, and $w$ may be either $a$, $a^*$, $b$, $b^*$, $S_1$, or $S_2$. The corresponding entry in the scattering matrix is the coefficient of $xyz^*w^*$ in $V_4$, with the following additions:

For each $n$ identical operators in $xyz^*w^*$ there is an additional factor $n!$ in the entry.

If $x = y$ there is additional factor $2^{-1/2}$ in the entry.

If $z = w$ there is additional factor $2^{-1/2}$ in the entry.

One finds in this way that the scattering matrices for the channels 1, 2, 3, and 4 are

$$(\lambda_1).$$

The scattering matrices for the channels 5 and 6 are

$$
\begin{pmatrix}
2\xi_1 & 0 \\
0 & 2\xi_2
\end{pmatrix}.
$$

The scattering matrix for channel 7 is

$$
\begin{pmatrix}
6\psi_1 & 2\psi_3 & 0 & \sqrt{2}\xi_1 & \sqrt{2}\xi_1 \\
2\psi_3 & 6\psi_2 & 0 & \sqrt{2}\xi_1 & \sqrt{2}\xi_1 \\
0 & 0 & 4\psi_3 & 0 & 0 \\
\sqrt{2}\xi_1 & \sqrt{2}\xi_2 & 0 & 2\lambda_1 & \lambda_1 \\
\sqrt{2}\xi_1 & \sqrt{2}\xi_2 & 0 & \lambda_1 & 2\lambda_1
\end{pmatrix}.
$$

The matrix (14) is similar to the matrix

$$
\begin{pmatrix}
6\psi_1 & 2\psi_3 & 2\xi_1 & 0 & 0 \\
2\psi_3 & 6\psi_2 & 2\xi_2 & 0 & 0 \\
2\xi_1 & 2\xi_2 & 3\lambda_1 & 0 & 0 \\
0 & 0 & 0 & 4\psi_3 & 0 \\
0 & 0 & 0 & 0 & \lambda_1
\end{pmatrix}.
$$

The unitarity conditions are the following: the eigenvalues of all the scattering matrices should be smaller, in modulus, than $4\pi$. Thus, in our case,

$$
|\lambda_1| < 4\pi,
$$

$$
|\xi_1| < 2\pi,
$$

$$
|\xi_2| < 2\pi,
$$

$$
|\psi_3| < \pi,
$$

and the eigenvalues of

$$
\begin{pmatrix}
6\psi_1 & 2\psi_3 & 2\xi_1 \\
2\psi_3 & 6\psi_2 & 2\xi_2 \\
2\xi_1 & 2\xi_2 & 3\lambda_1
\end{pmatrix}
$$

should have moduli smaller than $4\pi$. 
2.2 Bounded-from-below conditions

One may write

\[ V_4 = \frac{1}{2} \begin{pmatrix} X & Y & Z \end{pmatrix} \begin{pmatrix} \lambda_1 & \xi_1 & \xi_2 \\ \xi_1 & \psi_1 & \psi_3 \\ \xi_2 & \psi_3 & \psi_2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \]  

(18)

where \( X = \phi^i \phi, Y = S_1^2 \), and \( Z = S_2^2 \) are positive definite quantities independent of each other. Therefore, the BFB conditions are \[ [21] \]

\[ \lambda_1 > 0, \quad (19a) \]
\[ \psi_1 > 0, \quad (19b) \]
\[ \psi_2 > 0, \quad (19c) \]
\[ a_1 \equiv \xi_1 + \sqrt{\lambda_1 \psi_1} > 0, \quad (19d) \]
\[ a_2 \equiv \xi_2 + \sqrt{\lambda_1 \psi_2} > 0, \quad (19e) \]
\[ a_3 \equiv \psi_3 + \sqrt{\psi_1 \psi_2} > 0, \quad (19f) \]
\[ \sqrt{\lambda_1 \psi_1 \psi_2} + \xi_1 \sqrt{\psi_1} + \xi_2 \sqrt{\psi_2} + \psi_3 \sqrt{\lambda_1 + \sqrt{2a_1 a_2 a_3}} > 0. \quad (19g) \]

2.3 Procedure

Let the VEV of \( S_1 \) be \( w_1 \) and let the VEV of \( S_2 \) be \( w_2 \). Then, the vacuum stability conditions are

\[ \mu_1 = -\lambda_1 v^2 - \xi_1 w_1^2 - \xi_2 w_2^2, \quad (20a) \]
\[ m_1^2 = -\psi_1 w_1^2 - \psi_3 w_2^2 - \xi_1 v^2, \quad (20b) \]
\[ m_2^2 = -\psi_2 w_2^2 - \psi_3 w_1^2 - \xi_2 v^2. \quad (20c) \]

Using equation \[ [2] \] with \( G^+ = 0 \) and \( G^0 = 0 \), i.e. in the unitary gauge, together with \( S_1 = w_1 + \sigma_1 \) and \( S_2 = w_2 + \sigma_2 \), one obtains

\[ V = -\frac{\lambda_1}{2} v^4 - \frac{\psi_1}{2} w_1^4 - \frac{\psi_2}{2} w_2^4 - \psi_3 w_1^2 w_2^2 - v^2 (\xi_1 w_1^2 + \xi_2 w_2^2) \]
\[ + \frac{1}{2} (H \quad \sigma_1 \quad \sigma_2) M (H \quad \sigma_1 \quad \sigma_2) \]  

(21b)
\[ + \frac{\lambda_1 v}{\sqrt{2}} H^3 + 2\psi_1 w_1 \sigma_1^3 + 2\psi_2 w_2 \sigma_2^3 \]  

(21c)
\[ + \xi_1 H \sigma_1 (\sqrt{2} \psi_1 + w_1 H) + \xi_2 H \sigma_2 (\sqrt{2} \psi_2 + w_2 H) \]  

(21d)
\[ + 2\psi_3 \sigma_1 \sigma_2 (w_1 \sigma_2 + w_2 \sigma_1) \]  

(21e)
\[ + \frac{\lambda_1}{8} H^4 + \frac{\psi_1}{2} \sigma_1^4 + \frac{\psi_2}{2} \sigma_2^4 + \frac{\xi_1}{2} H^2 \sigma_1^2 + \frac{\xi_2}{2} H^2 \sigma_2^2 + \psi_3 \sigma_1^2 \sigma_2^2, \]  

(21f)

where

\[ M = 2 \begin{pmatrix} \lambda_1 v^2 & \sqrt{2} \xi_1 w_1 & \sqrt{2} \xi_2 w_2 \\ \sqrt{2} \xi_1 w_1 & 2\psi_1 w_1^2 & 2\psi_3 w_1 w_2 \\ \sqrt{2} \xi_2 w_2 & 2\psi_3 w_1 w_2 & 2\psi_2 w_2^2 \end{pmatrix}. \]  

(22)
One diagonalizes the real symmetric matrix \( M \) as
\[
M = R^T \text{diag}(M_1, M_2, M_3) R,
\]
where \( R \) is a 3 \( \times \) 3 orthogonal matrix that may be parameterized as
\[
R = \begin{pmatrix}
c_1 & s_1 c_3 & s_1 s_3 \\
-s_1 c_2 & c_1 c_2 c_3 + s_2 s_3 & c_1 c_2 s_3 - s_2 c_3 \\
-s_1 s_2 & c_1 s_2 c_3 - c_2 s_3 & c_1 s_2 s_3 + c_2 c_3
\end{pmatrix}.
\]

Here, \( c_j = \cos \vartheta_j \) and \( s_j = \sin \vartheta_j \) for \( j = 1, 2, 3 \). One has
\[
\begin{pmatrix}
H \\
\sigma_1 \\
\sigma_2
\end{pmatrix} = R^T \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},
\]
where the \( h_j \) are the physical scalars, \( i.e. \) the eigenstates of mass; the scalar \( h_j \) has squared mass \( M_j \). We assume that \( h_1 \) is the already-observed scalar. The interactions of the scalars with \( W^+ W^- \) are given by equation \( (21) \), \( i.e. \)
\[
\mathcal{L} = \cdots + \frac{g^2 v}{\sqrt{2}} W^\mu W^{\mu*} (c_1 h_1 - s_1 c_2 h_2 - s_1 s_2 h_3). \tag{26}
\]

We define the sign of the field \( h_1 \) to be such that the coupling of \( h_1 \) to \( W^+ W^- \) has the same sign as in the Standard Model. Thus, we choose \( -\pi/2 < \vartheta_1 < \pi/2 \).

According to equation \( (24) \),
\[
g_3 = \frac{\lambda_1 v}{\sqrt{2}} c_1^3 + 2 \psi_1 w_1 s_1^3 c_3^3 + 2 \psi_2 w_2 s_1^3 s_3^3 \tag{27a}
+ \xi_1 c_1 s_1 c_3 \left( \sqrt{2} v s_1 c_3 + w_1 c_1 \right) + \xi_2 c_1 s_1 s_3 \left( \sqrt{2} v s_1 s_3 + w_2 c_1 \right) \tag{27b}
+ 2 \psi_3 s_1^3 c_3 s_3 \left( w_1 s_3 + w_2 c_3 \right) \tag{27c}
= \frac{M_1}{2 \sqrt{2} v} \left( c_1^3 + \frac{\sqrt{2} v}{w_1} s_1^3 c_3^3 + \frac{\sqrt{2} v}{w_2} s_1^3 s_3^3 \right) \tag{27d}
= g_3^{\text{SM}} \left( c_1^3 + \frac{\sqrt{2} v}{w_1} s_1^3 c_3^3 + \frac{\sqrt{2} v}{w_2} s_1^3 s_3^3 \right), \tag{27e}
\]
and
\[
g_4 = \frac{\lambda_1 v}{8} c_1^4 + \psi_1 \frac{v}{2} s_1^4 c_3^4 + \psi_2 \frac{v}{2} s_1^4 s_3^4 + \frac{\xi_1}{2} c_1^2 s_1^2 c_3^2 + \frac{\xi_2}{2} c_1^2 s_1^2 s_3^2 + \psi_3 s_1^4 c_3^4 s_3^4. \tag{28}
\]

The oblique parameter \( T \) is given by \( (25) \)
\[
T = T_{\text{singlets}} = \frac{3 s_1^2}{16 \pi v^2 m_W^2} \left\{ F \left( M_1, m_W^2 \right) - F \left( M_1, m_Z^2 \right) \right\} \tag{29a}
- c_2^2 \left[ F \left( M_2, m_W^2 \right) - F \left( M_2, m_Z^2 \right) \right] \tag{29b}
- s_2^2 \left[ F \left( M_3, m_W^2 \right) - F \left( M_3, m_Z^2 \right) \right], \tag{29c}
\]
where

\[ F(x, y) = \begin{cases} \frac{x + y}{2} - \frac{xy}{x - y} \ln \frac{x}{y} \quad & x \neq y, \\ 0 \quad & x = y. \end{cases} \] (30)

In our numerical work we use as input the nine quantities \( v, w_1, w_2, M_1, M_2, M_3, \vartheta_1, \vartheta_2, \) and \( \vartheta_3, \) which are equivalent to the nine parameters of the scalar potential \( \mu_1, m_1^2, m_2^2, \lambda_1, \psi_1, \psi_2, \psi_3, \xi_1, \) and \( \xi_2. \) We input equations (6) and choose arbitrary values for \( M_2 > 0 \) and \( M_3 > 0 \) such that \( M_2 \leq M_3 \) (this represents no lack of generality, it is just the naming convention for \( h_2 \) and \( h_3). \) The VEVs \( w_1 \) and \( w_2 \) are chosen positive; this corresponds to the freedom of choice of the signs of \( S_1 \) and \( S_2. \) The angle \( \vartheta_1 \) is in either the first or the fourth quadrant, with

\[ \cos \vartheta_1 > 0.9, \] (31)

so that the \( h_1W^+W^- \) coupling is within 10\% of its Standard Model value. The angle \( \vartheta_2 \) is in the first quadrant; this corresponds to a choice of the signs of the fields \( h_2 \) and \( h_3. \) The angle \( \vartheta_3 \) may be in any quadrant. We firstly compute \( T \) according to equation (29) and check that it is inside its experimentally allowed domain \([-0.04 < T < 0.20\]. We then compute

\[
\begin{align*}
\lambda_1 &= \frac{1}{2w^2} (M_1c_1^2 + M_2s_1^2c_2^2 + M_3s_1^2s_2^2), \quad (32a) \\
\psi_1 &= \frac{1}{4w_1^2} [M_1s_1^2c_2^2 + M_2(c_1c_2c_3 + s_2s_3)^2 + M_3(c_1s_2c_3 - c_2s_3)^2], \quad (32b) \\
\psi_2 &= \frac{1}{4w_2^2} [M_1s_1^2s_2^2 + M_2(c_1c_2s_3 - s_2c_3)^2 + M_3(c_1s_2s_3 + c_2c_3)^2], \quad (32c) \\
\xi_1 &= \frac{1}{2\sqrt{2vw_1}} [M_1c_1s_1c_3 - M_2c_1s_1c_2^2c_3 - M_3c_1s_1s_2^2c_3 \\
&\quad + (M_3 - M_2) s_1c_2s_2s_3], \quad (32d) \\
\xi_2 &= \frac{1}{2\sqrt{2vw_2}} [M_1c_1s_1s_3 - M_2c_1s_1c_2^2s_3 - M_3c_1s_1s_2^2s_3 \\
&\quad + (M_2 - M_3) s_1c_2s_2c_3], \quad (32e) \\
\psi_3 &= \frac{1}{4w_1w_2} [M_1s_1^2c_3s_3 + M_2(c_1^2c_2^2 - s_2^2) c_3s_3 + M_3(c_1^2s_2^2 - c_2^2) c_3s_3 \\
&\quad + (M_3 - M_2) c_1c_2s_2 (c_3^2 - s_3^2)]. \quad (32f)
\end{align*}
\]

We validate the input if the inequalities (16) and (19) hold and if the moduli of all three eigenvalues of the matrix (17) are smaller than \( 4\pi. \)

### 2.4 Results

A remarkable result of our numerical work is that there is an upper bound on the mass \( \sqrt{M_2}; \) even if the VEVs \( w_1 \) and \( w_2 \) are allowed to be as high as 100 TeV—and, correspondingly, the mass \( \sqrt{M_2} \) also grows to a value of that order—the mass \( \sqrt{M_2} \) remains much smaller. In figure 1 we depict the upper bound on \( \sqrt{M_2} \) as a function of \( c_1; \) when \( c_1 \to 1 \) the upper bound disappears, \( i.e. \) it tends to infinity.
Figure 1: The upper bound on the mass of the second-lightest scalar $\sqrt{M_2}$ versus $\cos \vartheta_1$ in the SM2S.

Figure 2: Scatter plots of the four-Higgs coupling $g_4$ versus the three-Higgs coupling $g_3$ in the SM2S. The dashed lines mark the values of both couplings in the SM. The left panel depicts all the points where both new-scalar masses are higher than 100 GeV; the right panel depicts the subset of those points where the two new scalars are both heavier than 1 TeV.
Figure 3: Scatter plots of the three-Higgs coupling $g_3$ (left panel) and of the four-Higgs coupling $g_4$ (right panel) versus $\cos \vartheta_1$ in the SM2S. The dashed lines mark the values of the couplings in the SM.

In figure 2 we display the predictions for $g_3$ and $g_4$. In order to produce that figure we have randomly generated $\sqrt{M_2}$ and $\sqrt{M_3}$ in the range 0 to 3 TeV and the VEVs $w_1$ and $w_2$ in the range 0 to 1.1 TeV. (Actually $g_4$ may become slightly larger, viz. $g_4 \lesssim 0.5$, if $w_1$ and $w_2$ are allowed to reach 10 TeV.) One sees that $g_3$ is always below its SM value and is always positive, while $g_4$ is almost always above its SM value. If the masses of the new scalars are higher, then $g_3$ takes values closer to the SM value. An important point is that $g_3$ remains of the same order of magnitude $\sim 30$ GeV as in the SM, but $g_4$ may easily be 15 times larger than in the SM.

In the left panel of figure 3 one sees that $g_3$ is correlated with $\cos \vartheta_1$: when $\cos \vartheta_1 \to 1$ the coupling $g_3$ necessarily approaches its SM value. This behaviour is because of equation (27d) and $c_1 > 0.9$, which implies $|s_1| \ll c_1$. On the other hand, $g_4$ is not correlated with $\cos \vartheta_1$, as one sees in the right panel of figure 3.

3 The two-Higgs-doublet model

We next consider the model with two scalar gauge-$SU(2)$ doublets $\phi_1$ and $\phi_2$ having the same weak hypercharge. This is usually known as 2HDM. The scalar potential is given by equation (9a), where

$$
V_2 = \mu_1 \phi_1^\dagger \phi_1 + \mu_2 \phi_2^\dagger \phi_2 + \left( \mu_3 \phi_1^\dagger \phi_2 + \text{H.c.} \right),
$$

$$
V_4 = \frac{\lambda_1}{2} \left( \phi_1^\dagger \phi_1 \right)^2 + \frac{\lambda_2}{2} \left( \phi_2^\dagger \phi_2 \right)^2 + \lambda_3 \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 + \lambda_4 \phi_1^\dagger \phi_2 \phi_2^\dagger \phi_1 + \lambda_5 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 + \lambda_6 \phi_1^\dagger \phi_1 \phi_1 \phi_2 + \lambda_7 \phi_1 \phi_2^\dagger \phi_1 \phi_2 + \text{H.c.}
$$

The two-Higgs-doublet model
where $\mu_{1,2}$ and $\lambda_{1,2,3,4}$ are real. The ten (real) coefficients in $V_4$ may be grouped as

$$\eta_{00} = \lambda_1 + \lambda_2 + 2\lambda_3,$$

(34a)

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 2 \Re(\lambda_6 + \lambda_7) \\ -2 \Im(\lambda_6 + \lambda_7) \\ \lambda_1 - \lambda_2 \end{pmatrix},$$

(34b)

$$E = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_{12} & \eta_{22} & \eta_{23} \\ \eta_{13} & \eta_{23} & \eta_{33} \end{pmatrix} = \begin{pmatrix} 2\lambda_4 + 2\Re\lambda_5 & -2\Im\lambda_5 & 2\Re(\lambda_6 - \lambda_7) \\ -2\Im\lambda_5 & 2\lambda_4 - 2\Re\lambda_5 & -2\Im(\lambda_6 - \lambda_7) \\ 2\Re(\lambda_6 - \lambda_7) & -2\Im(\lambda_6 - \lambda_7) & \lambda_1 + \lambda_2 - 2\lambda_3 \end{pmatrix}.$$  (34c)

Under a (unitary) change of basis of the scalar doublets, $\eta_{00}$ is invariant while

$$\eta \to O\eta, \quad E \to OEO^T,$$

(35)

where $O$ is an $SO(3)$ matrix. Only quantities and procedures that are invariant under the transformation (35) are meaningful.

### 3.1 Unitarity conditions

We write

$$\phi_1 = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} c \\ d \end{pmatrix}, \quad \phi_1^\dagger = (a^* \ b^*) \quad \phi_2^\dagger = (c^* \ d^*).$$

(36)

Then,

$$V_4 = \lambda_1 \frac{a^*a^*aa + b^*b^*bb}{2} + a^*b^*ab$$

(37a)

$$+ \lambda_2 \frac{c^*c^*cc + d^*d^*dd}{2} + c^*d^*cd$$

(37b)

$$+ (\lambda_3 + \lambda_4)(a^*c^*ac + b^*d^*bd)$$

(37c)

$$+ \lambda_3 (a^*d^*ad + b^*c^*bc)$$

(37d)

$$+ \lambda_4 (a^*d^*bc + b^*c^*ad)$$

(37e)

$$+ \lambda_5 \frac{a^*a^*cc + b^*b^*dd}{2} + a^*b^*cd$$

(37f)

$$+ \lambda_5^* \frac{c^*c^*aa + d^*d^*bb}{2} + c^*d^*ab$$

(37g)

$$+ \lambda_6 (a^*a^*ac + b^*b^*bd + a^*b^*ad + a^*b^*bc)$$

(37h)

$$+ \lambda_6^* (a^*c^*aa + b^*d^*bb + a^*d^*ab + b^*c^*ab)$$

(37i)

$$+ \lambda_7 (a^*c^*cc + b^*d^*dd + b^*c^*cd + a^*d^*cd)$$

(37j)

$$+ \lambda_7^* (c^*c^*ac + d^*d^*bd + c^*d^*bc + c^*d^*ad).$$

(37k)

The relevant scattering channels are [20].
1. The channel $Q = 2$, $T_3 = 1$, with three states $aa$, $cc$, and $ac$.

2. The channel $Q = 0$, $T_3 = -1$, with three states $bb$, $dd$, and $bd$.

3. The channel $Q = 1$, $T_3 = 0$, with four states $ab$, $cd$, $ad$, and $bc$.

4. The channel $Q = 1$, $T_3 = 1$, with four states $ab^*$, $cd^*$, $ad^*$, and $cb^*$.

5. The channel $Q = 0$, $T_3 = 0$, with eight states $aa^*$, $bb^*$, $cc^*$, $dd^*$, $ac^*$, $bd^*$, $ca^*$, and $db^*$.

Channel (5) produces the scattering matrix

\[
\begin{pmatrix}
2\lambda_1 & \lambda_1 & \lambda_3 + \lambda_4 & \lambda_3 & 2\lambda_6^* & \lambda_6^* & 2\lambda_6 & \lambda_6 \\
\lambda_1 & 2\lambda_1 & \lambda_3 & \lambda_3 + \lambda_4 & \lambda_6 & 2\lambda_6 & \lambda_6 & 2\lambda_6 \\
\lambda_3 + \lambda_4 & \lambda_3 & 2\lambda_2 & \lambda_2 & 2\lambda_7^* & \lambda_7^* & 2\lambda_7 & \lambda_7 \\
\lambda_3 & \lambda_3 + \lambda_4 & \lambda_2 & 2\lambda_2 & \lambda_7^* & 2\lambda_7^* & \lambda_7 & 2\lambda_7 \\
2\lambda_6 & \lambda_6 & 2\lambda_7 & \lambda_7 & \lambda_3 + \lambda_4 & \lambda_4 & 2\lambda_5 & \lambda_5 \\
\lambda_6 & 2\lambda_6 & \lambda_7 & 2\lambda_7 & \lambda_3 + \lambda_4 & \lambda_4 & 2\lambda_5 & \lambda_5 \\
2\lambda_6^* & \lambda_6^* & 2\lambda_7^* & \lambda_7^* & 2\lambda_5^* & \lambda_5^* & \lambda_3 + \lambda_4 & \lambda_4 \\
\lambda_6^* & 2\lambda_6^* & \lambda_7^* & 2\lambda_7^* & \lambda_5^* & \lambda_3 + \lambda_4 & \lambda_4 & \lambda_3 + \lambda_4
\end{pmatrix}
\]  

(38)

A similarity transformation transforms the matrix (38) into the direct sum of two $4 \times 4$ matrices

\[
\mathcal{M}_1 = \frac{1}{2} \left( \begin{pmatrix} \eta_{00} - 2I \\ \eta \end{pmatrix} E + 2I \times \mathbb{1}_{3 \times 3} \right),
\]

(39a)

\[
\mathcal{M}_2 = \frac{1}{2} \left( \begin{pmatrix} 3\eta_{00} - 2I \\ 3\eta \end{pmatrix} 3E + 2I \times \mathbb{1}_{3 \times 3} \right).
\]

(39b)

Here,

\[
I = \lambda_3 - \lambda_4 = \frac{\eta_{00} - \text{tr} \, E}{4}
\]

(40)

is invariant under a change of basis of the doublets. It is obvious that the eigenvalues of the matrices (39) are invariant under such a change too.

Channel (41) produces the scattering matrix

\[
\begin{pmatrix}
\lambda_1 & \lambda_4 & \lambda_6^* & \lambda_6 \\
\lambda_4 & \lambda_2 & \lambda_7^* & \lambda_7 \\
\lambda_6 & \lambda_7 & \lambda_3 & \lambda_5 \\
\lambda_6^* & \lambda_7^* & \lambda_5^* & \lambda_3
\end{pmatrix}
\]

(41)

which may readily be shown to be similar to $\mathcal{M}_1$. Channel (3) produces the scattering matrix

\[
\begin{pmatrix}
\lambda_1 & \lambda_5 & \lambda_6 & \lambda_6 \\
\lambda_5^* & \lambda_2 & \lambda_7^* & \lambda_7^* \\
\lambda_6^* & \lambda_7 & \lambda_3 & \lambda_4 \\
\lambda_6^* & \lambda_7 & \lambda_4 & \lambda_3
\end{pmatrix}
\]

(42)
The matrix (42) is similar to
\[
\begin{pmatrix}
0 & M_3 & 0 \\
M_3 & 0 & 0 \\
0 & 0 & I
\end{pmatrix},
\]
where
\[
M_3 = \begin{pmatrix}
\lambda_1 & \lambda_5 & \sqrt{2}\lambda_6 \\
\lambda_5^* & \lambda_2 & \sqrt{2}\lambda_7^* \\
\sqrt{2}\lambda_6^* & \sqrt{2}\lambda_7 & \lambda_3 + \lambda_4
\end{pmatrix}.
\]

Channels (1) and (2) also lead to the matrix \( M_3 \). Direct computation demonstrates that the eigenvalues of \( M_3 \) are invariant under the transformation (35).

Thus, the unitarity conditions for the scalar potential of the 2HDM are the following: the eigenvalues of the two \( 4 \times 4 \) matrices (39) and of the \( 3 \times 3 \) matrix (44), and \( I \) in equation (40), should have moduli smaller than \( 4\pi \). These conditions were first derived in ref. [24]. We emphasize that they are, as they should, invariant under a change of basis of the two doublets.

### 3.1.1 The case \( \lambda_6 = \lambda_7 = 0 \)

If \( \lambda_6 = \lambda_7 = 0 \), then \( \eta_1 = \eta_2 = \eta_{13} = \eta_{23} = 0 \) and this simplifies things considerably. The unitarity conditions are then

\[
\begin{align*}
|\lambda_3 + \lambda_4| &< 4\pi, \quad (45a) \\
|\lambda_3 - \lambda_4| &< 4\pi, \quad (45b) \\
|\lambda_3 + |\lambda_5|| &< 4\pi, \quad (45c) \\
|\lambda_3 - |\lambda_5|| &< 4\pi, \quad (45d) \\
a_+ &\equiv |\lambda_3 + 2\lambda_4 + 3|\lambda_5|| < 4\pi, \quad (45e) \\
a_- &\equiv |\lambda_3 + 2\lambda_4 - 3|\lambda_5|| < 4\pi, \quad (45f) \\
|\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 - \lambda_2)^2 + 4|\lambda_5|^2}| &< 8\pi, \quad (45g) \\
|\lambda_1 + \lambda_2 - \sqrt{(\lambda_1 - \lambda_2)^2 + 4|\lambda_5|^2}| &< 8\pi, \quad (45h) \\
|\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}| &< 8\pi, \quad (45i) \\
|\lambda_1 + \lambda_2 - \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_3^2}| &< 8\pi, \quad (45j) \\
b_+ &\equiv 3\lambda_1 + 3\lambda_2 + \sqrt{9(\lambda_1 - \lambda_2)^2 + 4(2\lambda_3 + \lambda_4)^2} < 8\pi, \quad (45k) \\
b_- &\equiv 3\lambda_1 + 3\lambda_2 - \sqrt{9(\lambda_1 - \lambda_2)^2 + 4(2\lambda_3 + \lambda_4)^2} < 8\pi. \quad (45l)
\end{align*}
\]
Figure 4: Scatter plots of $|\lambda_1|$ versus $|\lambda_5|$ and of $|\lambda_6|$ versus $|\lambda_7|$ with the unitarity conditions enforced. The dashed red lines indicate the bounds $|\lambda_{1,5}| < \frac{4\pi}{3}$ and $|\lambda_{6,7}| < 2\sqrt{2\pi}/3$, respectively.

3.1.2 The case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$

The case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ is not realistic because it produces a potential unbounded from below. Still, one may compute the unitarity conditions in that case and one obtains

$$\sqrt{|\lambda_6|^2 + |\lambda_7|^2} < 2\sqrt{2\pi}, \quad (46a)$$
$$\sqrt{|\lambda_6|^2 + |\lambda_7|^2 + |\lambda_6^2 + \lambda_7^2|} < \frac{4\pi}{3}. \quad (46b)$$

3.1.3 Consequences

We have numerically analyzed the unitarity conditions by giving random values to $\lambda_1, \lambda_2, \lambda_3, \lambda_4, |\lambda_5|, |\lambda_6|, |\lambda_7|$, arg $(\lambda_5^* \lambda_6 \lambda_7)$, and arg $(\lambda_6^* \lambda_7)$ and then checking whether all the unitarity conditions are met. We present in figures 4–6 scatter plots with more than 8,000 allowed points each. We have found that all the conditions (45) still hold even when $\lambda_6 = \lambda_7 = 0$ is not true; also, the conditions (46) still hold even when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ does not apply. In particular, the upper bounds (45b), (45c), (45f), (45k), and (45l) are sometimes attained, as illustrated in figures 6 and 5, respectively. For the individual parameters, the bounds

$$|\lambda_{1,2}| < \frac{4\pi}{3}, \quad (47a)$$
$$|\lambda_5| < \frac{4\pi}{3}. \quad (47b)$$
Figure 5: Scatter plots of $a_\pm$ and $b_\pm$—see equations (45e), (45f), (45k), and (45l)—with the unitarity conditions enforced. The red dashed lines indicate the bounds $a_\pm < 4\pi$ in the left plot and $b_\pm < 8\pi$ in the right plot.

Figure 6: Scatter plots of $\lambda_3$ versus $\lambda_4$ with the unitarity conditions enforced. The dashed red lines are given by the equations $|\lambda_3 - \lambda_4| = 4\pi$, $|2\lambda_3 + \lambda_4| = 4\pi$, and $|\lambda_3 + 2\lambda_4| = 4\pi$. 
\[ |\lambda_{6,7}| < \frac{2\sqrt{2\pi}}{3} \] (47c)

hold and are illustrated in figure [4]; the bound (47a) is suggested by inequality (45k) when \( \lambda_3, \lambda_4, \) and either \( \lambda_1 \) or \( \lambda_2 \) vanish; the bound (47b) is suggested by inequality (45e) when \( \lambda_3 = \lambda_4 = 0, \) and the bound (47c) is suggested by inequality (46b) when either \( \lambda_6 \) or \( \lambda_7 \) vanishes. Finally, \( (\lambda_3, \lambda_4) \) is always within the hexagon with sides \( |\lambda_3 - \lambda_4| = 4\pi, \) \( |2\lambda_3 + \lambda_4| = 4\pi, \) and \( |\lambda_3 + 2\lambda_4| = 4\pi, \) as illustrated in figure [6].

### 3.2 Bounded-from-below conditions

Necessary and sufficient conditions for the scalar potential of the 2HDM to be BFB were first derived in ref. [23]. Ivanov [25] and Silva [26] later produced other, equivalent conditions to the same effect. We have implemented numerically both the conditions of ref. [23] and those of ref. [26]. We have found that the Ivanov–Silva algorithm runs several times faster than the one of ref. [23]. We have also checked that all the points produced by either algorithm were validated by the other one.

The points in our scatter plots were produced by using the algorithm of ref. [26]. That algorithm runs as follows. One constructs the \( 4 \times 4 \) matrix

\[ \Lambda_E = \begin{pmatrix} \eta_0 & \eta^T \\ -\eta & -E \end{pmatrix} \] (48)

and one computes its four eigenvalues. Then the potential is BFB if all the following conditions apply:

- All four eigenvalues are real.
- All four eigenvalues are different from each other.
- Call \( \Lambda_0 \) the largest eigenvalue. Call the other three eigenvalues \( \Lambda_{1,2,3}. \) The eigenvalue \( \Lambda_0 \) is positive; thus,
  \[ \Lambda_0 > \Lambda_{1,2,3}, \quad \Lambda_0 > 0. \] (49)
  (Each of \( \Lambda_1, \Lambda_2, \) and \( \Lambda_3 \) may be either positive or negative.)
- \[ \frac{[(\Lambda_E - \Lambda_1 \times 1_{4\times4}) \times (\Lambda_E - \Lambda_2 \times 1_{4\times4}) \times (\Lambda_E - \Lambda_3 \times 1_{4\times4})]_{11}}{(\Lambda_0 - \Lambda_1)(\Lambda_0 - \Lambda_2)(\Lambda_0 - \Lambda_3)} > 0. \] (50)

It is possible to derive analytically some necessary conditions for boundedness-from-below. Let us parameterize

\[ \phi_1^\dagger \phi_1 = r^2 \sin^2 \theta, \quad \phi_2^\dagger \phi_2 = r^2 \cos^2 \theta, \quad \phi_1^\dagger \phi_2 = e^{i\alpha}r^2 h \sin \theta \cos \theta, \] (51)

where \( 0 \leq \theta \leq \pi/2 \) without loss of generality. Since, in the notation of equations (36),

\[ r^4 \left( 1 - h^2 \right) \sin^2 \theta \cos^2 \theta = \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 - \phi_1^\dagger \phi_2 \phi_2^\dagger \phi_1 = |ad - bc|^2 \geq 0, \] (52)
one concludes that $h^2 \leq 1$. Thus, without loss of generality $0 \leq h \leq 1$ while the phase $\alpha$ is arbitrary. Boundedness from below of $V_4$ means that

\[
\frac{\lambda_1}{2} \sin^4 \theta + \frac{\lambda_2}{2} \cos^4 \theta + \left[ \lambda_3 + \lambda_4 h^2 + \Re \left( \lambda_5 e^{2i\alpha} \right) h^2 \right] \sin^2 \theta \cos^2 \theta
\]

\[
+ 2h \Re \left( \lambda_6 e^{i\alpha} \right) \sin^2 \theta \cos \theta + 2h \Re \left( \lambda_7 e^{i\alpha} \right) \sin \theta \cos^3 \theta > 0
\]

(53a)

for any $\theta$, $h$, and $\alpha$. From the cases $\theta = 0$ and $\theta = \pi/2$ one derives

\[
\lambda_1 > 0, \quad \lambda_2 > 0.
\]

(54)

Making $\alpha \rightarrow \pi + \alpha$ in inequality (53), one concludes that

\[
2h \sin \theta \cos \theta \left| \Re \left[ (\lambda_6 \sin^2 \theta + \lambda_7 \cos^2 \theta) e^{i\alpha} \right] \right| < \frac{\lambda_1}{2} \sin^4 \theta + \frac{\lambda_2}{2} \cos^4 \theta
\]

\[
+ \left[ \lambda_3 + \lambda_4 h^2 \right] \sin^2 \theta \cos^2 \theta
\]

\[
+ \Re \left( \lambda_5 e^{2i\alpha} \right) h^2 \right] \sin^2 \theta \cos^2 \theta.
\]

(55c)

Therefore, the quantity in the right-hand side of inequality (55) must be positive for any $\theta$, $h$, and $\alpha$. It is easy to see that

\[
\rho \sin^4 \theta + \varsigma \cos^4 \theta + \varepsilon \sin^2 \theta \cos^2 \theta > 0 \quad \forall \theta \in \left[ 0, \frac{\pi}{2} \right] \quad \Leftrightarrow \quad \rho > 0, \quad \varsigma > 0, \quad \varepsilon > -2\sqrt{\rho}. \quad (56)
\]

Applying the statement (56) to the case $\rho = \lambda_1/2$, $\varsigma = \lambda_2/2$, $\varepsilon = \lambda_3 + \lambda_4 h^2 + \Re(\lambda_5 e^{2i\alpha}) h^2$ for any $h \in [0, 1]$ and $\alpha$, one concludes that

\[
\lambda_3 > -\sqrt{\lambda_1 \lambda_2}, \quad \lambda_3 + \lambda_4 - |\lambda_5| > -\sqrt{\lambda_1 \lambda_2}.
\]

(57)

Inequalities (54) and (57) are necessary and sufficient conditions for boundedness-from-below when $\lambda_6 = \lambda_7 = 0$ [27]; they are necessary conditions when $\lambda_6$ and $\lambda_7$ are nonzero.

We may now return to inequality (55), which implies, in principle, many more necessary conditions for boundedness-from-below. Setting for instance $\sin \theta = \cos \theta$ one concludes that

\[
2h \left| \Re \left[ (\lambda_6 + \lambda_7) e^{i\alpha} \right] \right| < \frac{\lambda_1 + \lambda_2}{2} + \lambda_3 + \lambda_4 h^2 + \Re \left( \lambda_5 e^{2i\alpha} \right) h^2,
\]

(58)

which must hold for any $h$ and $\alpha$. Therefore [28],

\[
2 |\lambda_6 + \lambda_7| < \frac{\lambda_1 + \lambda_2}{2} + \lambda_3 + \lambda_4 + |\lambda_5|.
\]

(59)

We have numerically analyzed the BFB conditions by giving random values to $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, $|\lambda_5|$, $|\lambda_6|$, $|\lambda_7|$, $\arg (\lambda_5^* \lambda_6 \lambda_7)$, and $\arg (\lambda_5^* \lambda_7)$ and then checking whether the BFB conditions are met. We have confirmed that the conditions (54), (57), and (59) always hold.\(^3\)

\(^3\)The BFB conditions worked out in this subsection are, clearly, the ones valid at tree level. At loop level the BFB conditions change, see ref. [29].
3.3 Procedure

We consider the most general 2HDM and purport to find out its ranges for $g_3$ and $g_4$. We use the Higgs basis for the scalar doublets; in that basis only $\phi_1^0$ has VEV and therefore $\phi_1$ has the expression (2), while

$$\phi_2 = \left( \frac{C^+}{(\sigma_1 + i\sigma_2) / \sqrt{2}} \right).$$

In equation (60), $\sigma_1$ and $\sigma_2$ are real fields and $C^+$ is the physical charged scalar of the 2HDM. We emphasize that using the Higgs basis represents no lack of generality, because both the unitarity and the BFB conditions are the same in any basis.

Since only $\phi_1$ has VEV, the vacuum stability conditions are $\mu_1 = -\lambda_1 v^2$ and $\mu_3 = -\lambda_6 v^2$ [30]. The coupling $\mu_2$ in equation (33a) is unrelated to the parameters of $V_4$; one may trade it for the charged-Higgs squared mass $M_C = \mu_2 + \lambda_3 v^2$. The mass terms of $H$, $\sigma_1$, and $\sigma_2$ are given by line (21b), with [30]

$$M = \begin{pmatrix}
2\lambda_1 v^2 & 2v^2 \Re \lambda_5 & -2v^2 \Im \lambda_5 \\
2v^2 \Re \lambda_5 & M_C + (\lambda_4 + \Re \lambda_5) v^2 & -v^2 \Im \lambda_5 \\
-2v^2 \Im \lambda_5 & -v^2 \Im \lambda_5 & M_C + (\lambda_4 - \Re \lambda_5) v^2
\end{pmatrix}.$$  

The matrix $M$ is diagonalized through equations (23) – (25).

The three invariants of $M$ are

\begin{align*}
I_1(M) &= 2M_C + 2 (\lambda_1 + \lambda_4) v^2, \\
I_2(M) &= M_C^2 + 2 (2\lambda_1 + \lambda_4) v^2 M_C + (4\lambda_1 \lambda_4 + \lambda_4^2 - |\lambda_5|^2 - 4 |\lambda_6|^2) v^4, \\
I_3(M) &= 2\lambda_1 v^2 M_C^2 + 4 (\lambda_1 \lambda_4 - |\lambda_6|^2) v^4 M_C \\
&\quad + 2 \left[ \lambda_1 \lambda_3^2 - \lambda_1 |\lambda_5|^2 - 2 \lambda_4 |\lambda_6|^2 + 2 \Re (\lambda_4^2 - \lambda_6^2) \right] v^6.
\end{align*}

We input parameters $\lambda_1, 2, \ldots, 7$ that satisfy both the unitarity conditions and the BFB conditions of subsections 3.1 and 3.2 respectively. We also use the values of $M_1$ and $v$ in equations (6). The two equations

\begin{align*}
M_1^3 - M_1^2 I_1(M) + M_1 I_2(M) - I_3(M) &= 0, \\
[M_1^2 I_1(M) - 2M_1 I_2(M) + 3I_3(M)] \cos^2 \vartheta_1 \\
+ M_1 I_1 [M_1 I_1(M) - M_1^2] - (M_1^2)_{11} I_1 - I_3(M) &= 0
\end{align*}

are quadratic in $M_C$. By affirming the fact that both quadratic equations (63) must hold for the same value of $M_C$, one is able to compute both $M_C$ and $\cos^2 \vartheta_1$. We thus get to know the full matrix $M$, hence its eigenvalues $M_2$ and $M_3$ and its diagonalizing matrix $R$.

We require $\cos \vartheta_1 > 0.9$. We also compute the oblique parameter

$$T = \frac{1}{16\pi s_{\vartheta_1}^2 m_W^2} \left[ s_{\vartheta_1}^2 F(M_C, M_1) + (1 - s_{\vartheta_1}^2 s_2^2) F(M_C, M_2) + (1 - s_{\vartheta_1}^2 s_2^2) F(M_C, M_3) \right]$$

\begin{align*}
&\quad - s_{\vartheta_1}^2 F(M_2, M_3) - s_{\vartheta_1}^2 c_2^2 F(M_1, M_3) - s_{\vartheta_1}^2 s_2^2 F(M_1, M_2) \right] + T_{\text{singlets}},
\end{align*}
where \( T_{\text{singlets}} \) is given by equation (22). We require \(-0.04 < T < 0.20\).

The four-Higgs vertex is given by

\[
g_4 = \frac{\lambda_1 c_1^2}{8} + \frac{\lambda_2 s_1^4}{8} + \frac{(\lambda_3 + \lambda_4) c_1^2 s_1^2}{4} + \frac{s_1^2 c_1^2 (c_3^2 - s_3^2) \Re \lambda_5}{4} - \frac{s_1^2 c_1^2 s_3 \Im \lambda_5}{2} \tag{65a}
\]

\[
+ \frac{s_1 c_1^3 (c_3 \Re \lambda_6 - s_3 \Im \lambda_6)}{2} + \frac{s_3^3 c_1 (c_3 \Re \lambda_7 - s_3 \Im \lambda_7)}{2}. \tag{65b}
\]

The three-Higgs vertex is given by

\[
g_3 = \frac{v}{\sqrt{2}} \left[ \lambda_1 c_1^3 + (\lambda_3 + \lambda_4) s_1^2 c_1 + s_1^2 c_1 (c_3^2 - s_3^2) \Re \lambda_5 - 2s_1^2 c_1 c_3 s_3 \Im \lambda_5 \right] \tag{66a}
\]

\[
+ 3s_1 c_1^2 (c_3 \Re \lambda_6 - s_3 \Im \lambda_6) + s_3^3 (c_3 \Re \lambda_7 - s_3 \Im \lambda_7) \right]. \tag{66b}
\]

We also want to consider the \( h_1 C^+ C^- \) vertex, which may be relevant in the discovery of the charged scalar. That vertex is given by

\[
V_4 = \cdots + h_1 C^+ C^- g_{1CC}, \tag{67}
\]

where, in the 2HDM,

\[
g_{1CC} = \sqrt{2}v (c_1 \lambda_3 + s_1 c_3 \Re \lambda_7 - s_1 s_3 \Im \lambda_7). \tag{68}
\]

### 3.4 Results

As we know from subsections 3.1 and 3.2, in general \( \lambda_1 \) can take any value in between 0 and \( 4\pi/3 \). Once the constraint \( \cos \vartheta_1 > 0.9 \) is imposed, however, \( \lambda_1 \) can be no larger than \( \sim 1 \); this is illustrated in figure 7. The closer \( \cos \vartheta_1 \) is to 1, the closer \( \lambda_1 \) must be to its SM value \( M_1/(2v^2) = 0.258 \); note that \( \lambda_1 \) is almost always larger than its SM value when \( \cos \vartheta_1 > 0.9 \); the minimum value that we have obtained for \( \lambda_1 \) is 0.217.

If \( \cos \vartheta_1 \lessgtr 0.99 \), then the masses of the new scalar particles of the 2HDM, namely \( \sqrt{M_C}, \sqrt{M_2}, \text{ and } \sqrt{M_3} \) can be no larger than \( \sim 700 \text{ GeV} \); if \( \cos \vartheta_1 \lessgtr 0.95 \), they can be no larger than \( \sim 550 \text{ GeV} \). When \( \cos \vartheta_1 \) becomes close to 1, the masses of the new scalar particles all grow in tandem to reach \( \text{O(TeV)} \); this is illustrated in figure 8. Moreover, when \( c_1 \rightarrow 1 \) the masses of all three new scalars become almost identical, as seen in figure 9. While \( \sqrt{M_C} - \sqrt{M_2} \) may be as large as 400 GeV if \( M_2 \) is close to zero (the minimum value that we have obtained for \( \sqrt{M_2} \) was 12 GeV), that mass difference becomes smaller than 100 GeV if both masses are larger than \( 1 \text{ TeV} \). (Note that \( M_C \) may be either larger or smaller than \( M_2 \); also remember that by convention \( M_2 \) is always smaller than \( M_3 \), but they may be smaller than \( M_1 \).)

We now come to the predictions for \( g_3 \) and \( g_4 \) in the 2HDM, which are depicted in figure 10.

One sees that \( g_3 \) and \( g_4 \) are broadly correlated with each other; \( g_3 \) may be up to three times larger than in the SM and \( g_4 \) may be up to six times larger than in the SM. An interesting feature is that \( g_3 \) may be zero or even negative, \textit{i.e.} it may have sign opposite to the one in the SM. (We recall that the sign of \( g_3 \) is measured relative to the sign of \( c_1 \); we arrange that \( c_1 \) is always positive.) On the other hand, \( g_4 \) is always positive because
Figure 7: Scatter plots of $\lambda_1$ versus $\cos \vartheta_1$ in the 2HDM. The dashed line marks the value of $\lambda_1$ in the SM.

Figure 8: Scatter plots of the masses of the extra scalars of the 2HDM versus $\cos \vartheta_1$. 

$\sqrt{M_i}$ [GeV]

- $i = C$
- $i = 2$
- $i = 3$
Figure 9: The difference between the mass of the charged scalar and the mass of the lightest non-SM neutral scalar \textit{versus} the mass of the charged scalar in the 2HDM.

Figure 10: Scatter plot of the four-Higgs coupling \( g_4 \) \textit{versus} the three-Higgs coupling \( g_3 \) in the 2HDM, for various values of \( c_1 \). The dashed lines mark the values of both couplings in the SM.
of the boundedness from below of the potential. Notice that, while $g_3$ has a much larger range in the 2HDM than in the SM2S, for $g_4$ the opposite thing happens—it may be as high as 0.45 in the SM2S, but no more than one half of that value in the 2HDM.

In figure 11 we depict the coupling $g_{1CC}$ of the 125 GeV neutral scalar to a pair of charged scalars in the 2HDM. One sees that that coupling may be as large as 1,500 GeV (with either sign) when the mass of the charged scalars is close to 500 GeV.

4 The two-Higgs-doublet model plus one singlet

We consider in this section the two-Higgs-doublet model with the addition of one real $SU(2)\times U(1)$-invariant scalar field $S$. We assume a symmetry $S \rightarrow -S$. As a shorthand, we shall dub this model the 2HDM1S. The quartic part of the scalar potential is

$$V_4 = \frac{\lambda_1}{2} (\phi_1^\dagger \phi_1)^2 + \frac{\lambda_2}{2} (\phi_2^\dagger \phi_2)^2 + \lambda_3 \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 + \lambda_4 \phi_1^\dagger \phi_2 \phi_2^\dagger \phi_1$$

(69a)

$$+ \left[ \frac{\lambda_5}{2} (\phi_1^\dagger \phi_2)^2 + \lambda_6 \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 + \lambda_7 \phi_2^\dagger \phi_2 \phi_1^\dagger \phi_1 + \text{H.c.} \right]$$

(69b)

$$+ \frac{\psi}{2} S^4$$

(69c)

$$+ S^2 \left( \xi_1 \phi_1^\dagger \phi_1 + \xi_2 \phi_2^\dagger \phi_2 + \xi_3 \phi_1^\dagger \phi_2 + \xi_3^* \phi_2^\dagger \phi_1 \right).$$

(69d)

4.1 Bounded-from-below conditions

Deriving necessary and sufficient BFB conditions for even a rather simple potential like the one in equation (69) is a notoriously difficult problem [31]. If $V_4$ were negative for
some possible values of $S^2$, $\phi_1^\dagger \phi_1$, $\phi_2^\dagger \phi_2$, and $\phi_1^\dagger \phi_2$, then $V_4$ would tend to $-\infty$ upon multiplication of those four values by an ever-larger positive constant. Therefore, we want $V_4$ to be positive for all possible values of $S^2$, $\phi_1^\dagger \phi_1$, $\phi_2^\dagger \phi_2$, and $\phi_1^\dagger \phi_2$. In order to guarantee this, we proceed in the following fashion.

**Necessary condition 1:** When $S^2 = 0$, equation (69) reduces to its first two lines, i.e. to the quartic potential of the 2HDM. Therefore, one must require the fulfilment of the conditions of subsection 3.2, viz. the four conditions in between equations (48) and (50).

**Necessary condition 2:** When $\phi_1^\dagger \phi_2 = 0$,

$$V_4 = \frac{1}{2} \begin{pmatrix} \phi_1^\dagger \phi_1 & \phi_2^\dagger \phi_2 & S^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_3 & \xi_1 \\ \lambda_3 & \lambda_2 & \xi_2 \\ \xi_1 & \xi_2 & \psi \end{pmatrix} \begin{pmatrix} \phi_1^\dagger \phi_1 \\ \phi_2^\dagger \phi_2 \\ S^2 \end{pmatrix}. \quad (70)$$

Since $\phi_1^\dagger \phi_1$, $\phi_2^\dagger \phi_2$, and $S^2$ are positive definite quantities, we must require [21]

$$\psi > 0, \quad (71a)$$
$$\lambda_1 > 0, \quad (71b)$$
$$\lambda_2 > 0, \quad (71c)$$

$$A_1 \equiv \xi_1 + \sqrt{\lambda_1 \psi} > 0, \quad (71d)$$
$$A_2 \equiv \xi_2 + \sqrt{\lambda_2 \psi} > 0, \quad (71e)$$
$$A_3 \equiv \lambda_3 + \sqrt{\lambda_1 \lambda_2} > 0, \quad (71f)$$

$$\sqrt{\lambda_1 \lambda_2 \psi} + \xi_2 \sqrt{\lambda_1} + \xi_1 \sqrt{\lambda_2} + \lambda_3 \sqrt{\psi} + \sqrt{2A_1 A_2 A_3} > 0. \quad (71g)$$

After enforcing these two necessary conditions, we know that $V_4 > 0$ either when only the first two lines of the potential (69) exist or when only the third line exists. If we guarantee that the fourth line is always positive too, then we will be sure that $V_4$ is always positive. We therefore have the following [4]

**Sufficient condition:** If, besides the two necessary conditions,

$$\xi_1 + \xi_2 > 0, \quad (72a)$$
$$\xi_1 \xi_2 - |\xi_3|^2 > 0, \quad (72b)$$

then $V_4$ is BFB.

We have numerically found the absolute minimum of $V_4$ for any set of parameters of the potential (69) that satisfies the two necessary conditions but does not meet the sufficient condition (72). We have done this by using $S^2 = 1$ together with equations (51) and by minimizing $V_4$ in the domain $r^2 > 0$, $0 \leq \theta \leq \pi/2$, $0 \leq h \leq 1$, and $0 \leq \alpha < 2\pi$. If the minimum of $V_4$ is positive, then the set of input parameters is good, else the set of input parameters is bad and one must discard it.

---

[4] We thank Igor Ivanov for pointing out this sufficient condition to us.
4.2 Unitarity conditions

There are the same five scattering channels as in the 2HDM, cf. subsection 3.1; but the channel \( Q = T_3 = 0 \) has an additional scattering state \( S^2 \). Additionally, there are two extra scattering channels:

- The channel \( Q = 1, \, T_3 = 1/2 \) with the two states \( aS \) and \( cS \).
- The channel \( Q = 0, \, T_3 = -1/2 \) with the two states \( bS \) and \( dS \).

Both these channels produce a scattering matrix

\[
M_4 = 2 \begin{pmatrix} \xi_1 & \xi_3 \\ \xi_3^* & \xi_2 \end{pmatrix}, \quad (73)
\]

Channels \([1,2]\) of subsection 3.1 again produce the scattering matrix \((44)\). Channel \((3)\) produces that matrix together with the additional eigenvalue \( I \) of equation \((40)\). Channel \((2)\) produces the scattering matrix \((39a)\). Finally, channel \((5)\) has the additional scattering state \( S^2 \) and therefore, instead of producing both the matrix \( M_1 \) of equation \((39a)\) and the matrix \( M_2 \) of equation \((39b)\), it produces \( M_1 \) together with

\[
M_2' = \begin{pmatrix} 6\psi & \sqrt{2} \xi^T \\ \sqrt{2} \xi & M_2 \end{pmatrix}, \quad \text{where} \quad \bar{\xi} = \begin{pmatrix} \xi_1 + \xi_2 \\ 2 \Re \xi_3 \\ -2 \Im \xi_3 \\ \xi_1 - \xi_2 \end{pmatrix}, \quad (74)
\]

Thus, the unitarity conditions for the 2HDM1S are the following: both \(|I|\) and the moduli of all the eigenvalues of the \( 2 \times 2 \) matrix \( M_4 \), of the \( 3 \times 3 \) matrix \( M_3 \), of the \( 4 \times 4 \) matrix \( M_1 \), and of the \( 5 \times 5 \) matrix \( M_2' \) must be smaller than \( 4\pi \).

4.3 Procedure

Just as in the previous section, we utilize the Higgs basis for the two doublets, i.e. equations \((2)\) and \((60)\). We also write \( S = w + \sigma \), where \( w \) is the VEV of the scalar \( S \) and \( \sigma \) is a field. The mass terms of the scalars are

\[
V = \cdots + M_C C^c C^+ + \frac{1}{2} \begin{pmatrix} H & \sigma_1 & \sigma_2 & \sigma \end{pmatrix} M \begin{pmatrix} H \\ \sigma_1 \\ \sigma_2 \\ \sigma \end{pmatrix}, \quad (75)
\]

with

\[
M = \begin{pmatrix} 2\lambda_1 v^2 & 2v^2 \Re \lambda_6 & -2v^2 \Im \lambda_6 & 2\sqrt{2}vw \xi_1 \\ 2v^2 \Re \lambda_6 & M_C + (\lambda_4 + \Re \lambda_5) v^2 & -v^2 \Im \lambda_5 & 2\sqrt{2}vw \Re \xi_3 \\ -2v^2 \Im \lambda_6 & -v^2 \Im \lambda_5 & M_C + (\lambda_4 - \Re \lambda_5) v^2 & -2\sqrt{2}vw \Im \xi_3 \\ 2\sqrt{2}vw \xi_1 & 2\sqrt{2}vw \Re \xi_3 & -2\sqrt{2}vw \Im \xi_3 & 4\psi^2 \end{pmatrix}, \quad (76)
\]
cf. equation (61). One diagonalizes \( M \) as

\[
M = R^T \text{diag} (M_1, M_2, M_3, M_4) R,
\]

\[
\begin{pmatrix}
H \\
\sigma_1 \\
\sigma_2 \\
\sigma
\end{pmatrix} = R^T \begin{pmatrix}
h_1 \\
h_2 \\
h_3 \\
h_4
\end{pmatrix},
\]

where \( R \) is a 4 \times 4 orthogonal matrix. The squared mass \( M_1 \) is given by equation (6a).

Without loss of generality, \( M_2 < M_3 < M_4 \). Just as in the previous sections, we require

\[
R_{11} \equiv c_1 > 0.9.
\]

The expression for the oblique parameter \( T \) is [22]

\[
T = \frac{1}{16\pi s_w^2 m_H^2} \left\{ \sum_{k=1}^{4} \left[ (R_{k2})^2 + (R_{k3})^2 \right] F(M_C, M_k) \right. \\
- \sum_{k=1}^{3} \sum_{k'=k+1}^{4} (R_{k2} R_{k'3} - R_{k'2} R_{k3})^2 F(M_k, M_{k'}) \\
+ 3 \sum_{k=2}^{4} (R_{k1})^2 \left[ F(M_k, m_Z^2) - F(M_k, m_W^2) \right] \\
+ 3 \left( c_1^2 - 1 \right) \left[ F(M_1, m_Z^2) + F(M_1, m_W^2) \right] \right\},
\]

and we demand \(-0.04 < T < 0.20\).

We input random values for the 15 real parameters \( M_C, \lambda_{1,2,3,4}, |\lambda_{5,6,7}|, \psi, \xi_{1,2}, |\xi_3|, \arg(\lambda_5^* \lambda_6 \lambda_7), \arg(\lambda_6^* \lambda_7), \) and \( \arg(\lambda_6^* \xi_3) \). We moreover input \( M_1 \) and \( v^2 \) given in equations (6). Then,

1. We require the input parameters to satisfy the BFB conditions of subsection 4.1—this may imply a numerical minimization of \( V_4 \) to check that \( V_4 > 0 \).

2. We require the input parameters to satisfy the unitarity conditions written after equation (74).

3. We compute the VEV \( w \) from the condition that \( M_1 \) should be an eigenvalue of the matrix \( M \).

4. We compute the full matrix \( M \), its eigenvalues \( M_{2,3,4} \), and its diagonalizing matrix \( R \); we choose the overall sign of \( R \) such that \( R_{11} \equiv c_1 > 0 \).

5. We impose both the condition (78) and the condition that the oblique parameter \( T \) is within its experimental bounds.

6. We compute the couplings

\[
g_3 = \frac{v}{\sqrt{2}} \left\{ \lambda_1 c_1^3 + (\lambda_3 + \lambda_4) c_1 \left[ (R_{12})^2 + (R_{13})^2 \right] \right\}
\]
The differences between the masses of the two lightest non-SM neutral scalars and the mass of the charged scalar versus the mass of the charged scalar in the 2HDM. Green points have all the scalars with mass larger than 500 GeV; magenta points have all the scalars with mass larger than 1 TeV.

\begin{align*}
+ c_1 \left[ (R_{12})^2 - (R_{13})^2 \right] \Re \lambda_5 - 2 c_1 R_{12} R_{13} \Im \lambda_5 \quad (80b) \\
+ 3 c_1^2 \left( R_{12} \Re \lambda_6 - R_{13} \Im \lambda_6 \right) + \left[ (R_{12})^2 + (R_{13})^2 \right] (R_{12} \Re \lambda_7 - R_{13} \Im \lambda_7) \right\} \\
+ 2 \psi w (R_{14})^3 + \xi_1 c_1 R_{14} \left( w c_1 + \sqrt{2} v R_{14} \right) + \xi_2 w R_{14} \left[ (R_{12})^2 + (R_{13})^2 \right] (80d) \\
+ \sqrt{2} R_{14} \left( v R_{14} + \sqrt{2} w c_1 \right) (R_{12} \Re \xi_3 - R_{13} \Im \xi_3), \quad (80e)
\end{align*}

\begin{align*}
g_1 &= \frac{\lambda_1 c_1^4}{8} + \frac{\lambda_2^2}{8} \left[ (R_{12})^2 + (R_{13})^2 \right] + \frac{\lambda_3 + \lambda_4}{4} c_1^2 \left[ (R_{12})^2 + (R_{13})^2 \right] \quad (81a) \\
+ \frac{\Re \lambda_5}{4} c_1^2 \left[ (R_{12})^2 - (R_{13})^2 \right] - \frac{3 \lambda_5}{2} c_1^2 R_{12} R_{13} \quad (81b) \\
+ \frac{c_1^3}{2} \left( R_{12} \Re \lambda_6 - R_{13} \Im \lambda_6 \right) + c_1 \left[ (R_{12})^2 + (R_{13})^2 \right] (R_{12} \Re \lambda_7 - R_{13} \Im \lambda_7) \quad (81c) \\
+ \frac{\psi}{2} (R_{14})^4 \quad (81d) \\
+ (R_{14})^2 \left\{ \frac{\xi_1 c_1^2}{2} + \frac{\xi_2}{2} \left[ (R_{12})^2 + (R_{13})^2 \right] + c_1 \left( R_{12} \Re \xi_3 - R_{13} \Im \xi_3 \right) \right\}, \quad (81e)
\end{align*}

\begin{align*}
g_{1CC} &= \sqrt{2} v \left( c_1 \lambda_3 + R_{12} \Re \lambda_7 - R_{13} \Im \lambda_7 \right) + 2 w \xi_2 R_{14}. \quad (82)
\end{align*}

### 4.4 Results

In figure 12, we have plotted the differences among the masses of the scalars against the mass of the charged scalar. One sees that $\sqrt{M_C}$ and $\sqrt{M_3}$ cannot be more than $\sim 300$ GeV from each other, but $\sqrt{M_2}$ may be as much as 2 TeV smaller than both of them.

In figure 13, we present a scatter plot of the mass of the lightest non-SM neutral scalar against $c_1$. One sees that, contrary to what happens in the 2HDM (cf. figure 8), $\sqrt{M_2}$
may reach 1 TeV even when $c_1$ is as low as 0.9.

We depict in figure 14 the three- and four-Higgs couplings $g_3$ and $g_4$. The ranges of the couplings are markedly different from the analogous ranges in the previous two models. In particular, $g_3$ in the 2HDM1S may be some eight times larger than in the 2HDM, and thirty times larger than in the SM; while $g_3 \in [-30 \text{ GeV}, 120 \text{ GeV}[$ in the 2HDM, $g_3 \in [-250 \text{ GeV}, 1 \text{ TeV}[$ in the 2HDM1S. Also, $g_4$ in the 2HDM1S may be twice as large as in the 2HDM (0.48 instead of 0.24). In the 2HDM1S there is no clear correlation between $g_3$ and $g_4$.

In figure 15 we have plotted the $h_1 C^+ C^-$ coupling $g_{1CC}$. Once again, that coupling in the 2HDM1S may be three times larger than in the 2HDM; very large values of $g_{1CC}$ occur even for $c_1$ very close to 1.

## 5 Conclusions

In this paper we have emphasized that both the bounded-from-below (BFB) conditions and the unitarity conditions for the two-Higgs-doublet model (2HDM) are invariant under a change of the basis used for the two doublets. Therefore, one may implement those conditions directly in the Higgs basis, viz. the basis where only one doublet has vacuum expectation value. This procedure allows one to extract bounds on the masses and couplings of the scalar particles of the most general 2HDM, disregarding any symmetry that a particular 2HDM may possess.

We have also utilized this procedure for two other models, namely the Standard Model with the addition of two real singlets (SM2S) and the 2HDM with the addition of one real singlet (2HDM1S), in both cases with reflection symmetries acting on each of the singlets. We have discovered that there are large variations among the couplings in these
Figure 14: The four-Higgs coupling $g_4$ versus the three-Higgs coupling $g_3$ in the 2HDM1S for various values of $c_1$. The dashed lines mark the values of the couplings in the SM.

Figure 15: Scatter plot of $g_{1CC}$ versus the mass of the charged scalars $C^\pm$ in the 2HDM1S.
three very simple extensions of the SM. We have focussed on the three couplings $g_3 (h_1)^3$, $g_4 (h_1)^4$, and $g_{1CC} h_1 C^+ C^-$, where $h_1$ is the observed neutral scalar with mass 125 GeV and $C^\pm$ are the charged scalars of the 2HDM. We have found, for instance, that:

- $0.6 < g_3 / g_3^{\text{SM}} \leq 1$ and $0.6 \leq g_4 / g_4^{\text{SM}} < 15$ is the SM2S. Thus, if one adds singlets to the SM, then $g_3$ becomes smaller but does not change order of magnitude, while $g_4$ may increase by one order of magnitude.

- $-0.5 < g_3 / g_3^{\text{SM}} < 3$ and $0 < g_4 / g_4^{\text{SM}} < 6$ is the 2HDM. In this case $g_3$ may have opposite sign and $g_4$ may be smaller than in the SM. Also, $g_{1CC}$ may be as high as 1.5 TeV (positive or negative) in the 2HDM.

- $-6.5 < g_3 / g_3^{\text{SM}} < 30$ and $0 < g_4 / g_4^{\text{SM}} < 15$ is the 2HDM1S. This model distinguishes itself by the possibility of very large—either positive or negative—values of $g_3$, and also large values of $g_4$ (uncorrelated with the ones of $g_3$). Once again, $g_{1CC}$ may be of order TeV in the 2HDM1S.

A comparison of the predictions of the three models for $g_3$ and $g_4$ is depicted in figure 16.

Figure 16: Scatter plot of $g_4 / g_4^{\text{SM}}$ versus $g_3 / g_3^{\text{SM}}$ in the three models that we have studied. The dashed lines mark the SM values $g_3 / g_3^{\text{SM}} = g_4 / g_4^{\text{SM}} = 1$.

We emphasize that our method may be used to obtain bounds and/or correlations among other parameters and/or observables of these models. Unfortunately, it may be difficult to generalize our work to more complicated models, both because they may contain too many parameters and because it is very difficult to derive full BFB conditions for even rather simple models.

Acknowledgements: L.L. thanks Pedro Miguel Ferreira and Igor Ivanov for useful discussions. D.J. thanks the Lithuanian Academy of Sciences for support through the
project DaFi2018. The work of L.L. is supported by the Portuguese Fundação para a Ciência e a Tecnologia through the projects CERN/FIS-NUC/0010/2015, CERN/FIS-PAR/0004/2017, and UID/FIS/00777/2013; those projects are partly funded by POCTI (FEDER), COMPETE, QREN, and the European Union.

References

[1] G. Aad et al. [ATLAS Collaboration], Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC, Phys. Lett. B 716 (2012) 1 [arXiv:1207.7214 [hep-ex]]; S. Chatrchyan et al. [CMS Collaboration], Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC, Phys. Lett. B 716 (2012) 30 [arXiv:1207.7235 [hep-ex]].

[2] C. Patrignani et al. [Particle Data Group], Review of particle physics, Chin. Phys. C 40 (2016) 100001.

[3] J. Baglio, O. Eberhardt, U. Nierste, and M. Wiebusch, Benchmarks for Higgs boson pair production and heavy Higgs boson searches in the two-Higgs-doublet model of type II, Phys. Rev. D 90 (2014) 015008 [arXiv:1403.1264 [hep-ph]].

[4] L. Wu, J. M. Yang, C. P. Yuan, and M. Zhang, Higgs self-coupling in the MSSM and NMSSM after the LHC Run I, Phys. Lett. B 747 (2015) 378 [arXiv:1504.06932 [hep-ph]].

[5] L. Bian and N. Chen, Higgs pair productions in the CP-violating two-Higgs-doublet model, JHEP 1609 (2016) 069 [arXiv:1607.02703 [hep-ph]].

[6] N. Chakrabarty and B. Mukhopadhyaya, High-scale validity of a two-Higgs-doublet scenario: Predicting collider signals, Phys. Rev. D 96 (2017) 035028 [arXiv:1702.08268 [hep-ph]].

[7] N. F. Bell, G. Busoni, and I. W. Sanderson, Self-consistent Dark Matter simplified models with an s-channel scalar mediator, JCAP 1703 (2017) 015 [arXiv:1612.03475 [hep-ph]]; N. F. Bell, G. Busoni, and I. W. Sanderson, Two Higgs doublet dark matter portal, JCAP 1801 (2018) 015 [arXiv:1710.10764 [hep-ph]]; M. Bauer, U. Haisch, and F. Kahlhoefer, Simplified dark matter models with two Higgs doublets: I. Pseudoscalar mediators, JHEP 1705 (2017) 138 [arXiv:1701.07427 [hep-ph]]; C.-F. Chang, X.-G. He, and J. Tandean, Two-Higgs-doublet-portal dark-matter models in light of direct search and LHC data, JHEP 1704 (2017) 107 [arXiv:1702.02924 [hep-ph]].

[8] M. Gorbahn and U. Haisch, Indirect probes of the trilinear Higgs coupling: $gg \to h$ and $h \to \gamma\gamma$, JHEP 1610 (2016) 094 [arXiv:1607.03773 [hep-ph]]; W. Bizoń, M. Gorbahn, U. Haisch, and G. Zanderighi, Constraints on the trilinear
Higgs coupling from vector boson fusion and associated Higgs production at the LHC, JHEP 1707 (2017) 083 [arXiv:1610.05771 [hep-ph]].

[9] ATLAS Collaboration, ATLAS-CONF-2016-049.

[10] S. D. Rindani and B. Singh, Indirect measurement of triple-Higgs coupling at an electron–positron collider with polarized beams, arXiv:1805.03417 [hep-ph].

[11] G. D. Kribs, A. Maier, H. Rzehak, M. Spannowsky, and P. Waite, Electroweak oblique parameters as a probe of the trilinear Higgs self-interaction, Phys. Rev. D 95 (2017) 093004 [arXiv:1702.07678 [hep-ph]].

[12] G. Degrassi, P. P. Giardino, F. Maltoni, and D. Pagani, Probing the Higgs self-coupling via single Higgs production at the LHC, JHEP 12 (2016) 080 [arXiv:1607.04251 [hep-ph]].

[13] G. Degrassi, M. Fedele, and P. P. Giardino, Constraints on the trilinear Higgs self-coupling from precision observables, JHEP 1704 (2017) 155 [arXiv:1702.01737 [hep-ph]].

[14] L. Di Luzio, R. Gröber, and M. Spannowsky, Maxi-sizing the trilinear Higgs self-coupling: how large could it be?, Eur. Phys. J. C 77 (2017) 788 [arXiv:1704.02311 [hep-ph]].

[15] D. Jurčiukonis and L. Lavoura, Lepton mixing and the charged-lepton mass ratios, JHEP 1803 (2018) 152 [arXiv:1712.04292 [hep-ph]].

[16] S. Di Vita, G. Durieux, C. Grojean, J. Gu, Z. Liu, G. Panico, M. Riembau, and T. Vantalon, A global view on the Higgs self-coupling at lepton colliders, JHEP 1802 (2018) 178 [arXiv:1711.03978 [hep-ph]].

[17] T. Plehn and M. Rauch, The quartic higgs coupling at hadron colliders, Phys. Rev. D 72 (2005) 053008 [hep-ph/0507321];
T. Liu, K. F. Lyu, J. Ren, and H. X. Zhu, Probing Quartic Higgs Self-Interaction, arXiv:1803.04359 [hep-ph].

[18] A. Abada, D. Ghaffor and S. Nasri, Two-singlet model for light cold dark matter, Phys. Rev. D 83 (2011) 095021 [arXiv:1101.0365 [hep-ph]]; A. Ahriche, A. Arhrib and S. Nasri, Higgs phenomenology in the two-singlet model, JHEP 1402 (2014) 042 [arXiv:1309.5615 [hep-ph]]; B. Grzadkowski and D. Huang, Spontaneous CP-violating electroweak baryogenesis and dark matter from a complex singlet scalar, arXiv:1807.06987 [hep-ph].

[19] M. D.Goodsell and F. Staub, Unitarity constraints on general scalar couplings with SARAH, arXiv:1805.07306 [hep-ph]; M. D. Goodsell and F. Staub, Improved unitarity constraints in two-Higgs-doublet models, arXiv:1805.07310 [hep-ph].
[20] M. P. Bento, H. E. Haber, J. C. Romão, and J. P. Silva, *Multi-Higgs doublet models: physical parametrization, sum rules and unitarity bounds*, JHEP **1711** (2017) 095 [arXiv:1708.09408 [hep-ph]].

[21] K. Kannike, *Vacuum stability conditions from copositivity criteria*, Eur. Phys. J. C **72** (2012) 2093 [arXiv:1205.3781 [hep-ph]].

[22] W. Grimus, L. Lavoura, O. M. Ogreid, and P. Osland, *A precision constraint on multi-Higgs-doublet models*, J. Phys. G **35** (2008) 075001 [arXiv:0711.4022 [hep-ph]].

[23] M. Maniatis, A. von Manteuffel, O. Nachtmann, and F. Nagel, *Stability and symmetry breaking in the general two-Higgs-doublet model*, Eur. Phys. J. C **48** (2006) 805 [hep-ph/0605184].

[24] S. Kanemura and K. Yagyu, *Unitarity bound in the most general two Higgs doublet model*, Phys. Lett. B **751** (2015) 289 [arXiv:1509.06060 [hep-ph]].

[25] I. P. Ivanov, *Minkowski space structure of the Higgs potential in the two-Higgs-doublet model*, Phys. Rev. D **75** (2007) 035001 [Erratum: *ibid.* 76 (2007) 039902] [hep-ph/0609018].

[26] I. P. Ivanov and J. P. Silva, *Tree-level metastability bounds for the most general two Higgs doublet model*, Phys. Rev. D **92** (2015) 055017 [arXiv:1507.05100 [hep-ph]].

[27] N. G. Deshpande and E. Ma, *Pattern of symmetry breaking with two Higgs doublets*, Phys. Rev. D **18** (1978) 2574;
K. G. Klimenko, *Conditions for certain Higgs potentials to be bounded below*, Theor. Math. Phys. **62** (1985) 58 [Teor. Mat. Fiz. **62** (1985) 87].

[28] P. M. Ferreira, R. Santos, and A. Barroso, *Stability of the tree-level vacuum in two Higgs doublet models against charge or CP spontaneous violation*, Phys. Lett. B **603** (2004) 219 [Erratum: *ibid.* **629** (2005) 114] [hep-ph/0406231].

[29] F. Staub, *Reopen parameter regions in two-Higgs doublet models*, Phys. Lett. B **776** (2018) 407 [arXiv:1705.03677 [hep-ph]].

[30] L. Lavoura and J. P. Silva, *Fundamental CP-violating quantities in a SU(2)⊗U(1) model with many Higgs doublets*, Phys. Rev. D **50** (1994) 4619 [hep-ph/9404276].

[31] I. P. Ivanov, M. Köpke, and M. Mühlleitner, *Algorithmic boundedness-from-below conditions for generic scalar potentials*, Eur. Phys. J. C **78** (2018) 413 [arXiv:1802.07976 [hep-ph]].