L∞ STRUCTURES ON MAPPING CONES

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ABSTRACT. We show that the mapping cone of a morphism of differential graded Lie algebras $\chi: L \to M$ can be canonically endowed with an $L_\infty$-algebra structure which at the same time lifts the Lie algebra structure on $L$ and the usual differential on the mapping cone. Moreover, this structure is unique up to isomorphisms of $L_\infty$-algebras. The associated deformation functor coincides with the one introduced by the second author in [19].

INTRODUCTION

There are several cases where the tangent and obstruction spaces of a deformation theory are the cohomology groups of the mapping cone of a morphism $\chi: L \to M$ of differential graded Lie algebras. It is therefore natural to ask if there exists a canonical differential graded Lie algebra structure on the complex $(C_\chi, \delta)$, where

$$C_\chi = \bigoplus C^i_\chi, \quad C^i_\chi = L^i \oplus M^{i-1}, \quad \delta(l, m) = (dl, \chi(l) - dm),$$

such that the projection $C_\chi \to L$ is a morphism of differential graded Lie algebras.

In general we cannot expect the existence of a Lie structure: in fact the canonical bracket

$$l_1 \otimes l_2 \mapsto [l_1, l_2]; \quad m \otimes l \mapsto \frac{1}{2}[m, \chi(l)]; \quad m_1 \otimes m_2 \mapsto 0$$

satisfies the Leibniz rule with respect to the differential $\delta$ but not the Jacobi identity. However, the Jacobi identity for this bracket holds up to homotopy, and so we can look for the weaker request of a canonical $L_\infty$ structure on $C_\chi$.

More precisely, let $K$ be a fixed characteristic zero base field, denote by $DG$ the category of differential graded vector spaces, by $DGLA$ the category of differential graded Lie algebras, by $L_\infty$ the category of $L_\infty$ algebras and by $DGLA^2$ the category of morphisms in $DGLA$. The four functors

$$\begin{align*}
DGLA &\to L_\infty & \text{Quillen construction}, \\
L_\infty &\to DG & \text{forgetting higher brackets}, \\
DGLA^2 &\to DG & \{L \xrightarrow{\chi} M \} \mapsto C_\chi, \\
DGLA &\to DGLA^2 & L \mapsto \{L \to 0\},
\end{align*}$$

give a commutative diagram

$$\begin{array}{ccc}
DGLA &\to& L_\infty \\
\downarrow & & \downarrow \\
DGLA^2 &\to& DG
\end{array}$$

Our first result is
Theorem 1. There exists a functor $\tilde{C} : \text{DGLA}^2 \to L_\infty$ making the diagram

$$
\begin{array}{ccc}
\text{DGLA} & \to & L_\infty \\
\downarrow & & \downarrow \\
\text{DGLA}^2 & \to & DG
\end{array}
$$

commutative. If $\mathcal{F} : \text{DGLA}^2 \to L_\infty$ has the same properties, then for every morphism $\chi$ of differential graded Lie algebras, the $L_\infty$-algebra $\mathcal{F}(\chi)$ is isomorphic to $\tilde{C}(\chi)$.

In the above theorem, the functor $\tilde{C}$ is explicitly described. The linear term of the $L_\infty$-algebra $\tilde{C}(\chi)$ is by construction the differential $\delta$ on $C_\chi$, and the quadratic part turns out to coincide with the naive bracket described at the beginning of the introduction. An explicit expression for the higher brackets is given in Theorem 5.5.

The second main result of this paper is to prove that the deformation functor $\text{Def}_\tilde{C}(\chi)$ associated with the $L_\infty$ algebra $\tilde{C}(\chi)$ is isomorphic to the functor $\text{Def}_\chi$ defined in [19].

Given $\chi : L \rightarrow M$ it is defined a functor $\text{Def}_\chi : \text{Art} \rightarrow \text{Set}$, where $\text{Art}$ is the category of local Artinian $K$-algebras with residue field $K$:

$$
\text{Def}_\chi(A) = \{(x, e^a) \in (L^1 \otimes m_A) \times \exp(M^0 \otimes m_A) \mid dx + \frac{1}{2}[x,x] = 0, \ e^a \ast \chi(x) = 0\},
$$

where $\ast$ denotes the gauge action in $M$, and where $(l_0, e^{m_0})$ is defined to be gauge equivalent to $(l_1, e^{m_1})$ if there exists $(a,b) \in (L^0 \oplus M^{-1}) \otimes m_A$ such that

$$
l_1 = e^a \ast l_0, \quad e^{m_1} = e^b e^{m_0} e^{-\chi(a)}.
$$

Theorem 2. In the notation above, for every morphism of differential graded Lie algebras $\chi : L \rightarrow M$ we have

$$
\text{Def}_\tilde{C}(\chi) \simeq \text{Def}_\chi.
$$

The importance of Theorem 2 relies on the fact that it allows to study the functors $\text{Def}_\chi$, which are often naturally identified with geometrically defined functors, using the whole machinery of $L_\infty$-algebras. In particular this gives, under some finiteness assumption, the construction and the homotopy invariance of the Ku ranishi map [7, 8, 11], as well as the local description of corresponding extended moduli spaces.

As a final remark, we observe that $\text{DGLA}^2$ is a full subcategory of the category $\text{DGLA}^\Delta$ of cosimplicial differential graded Lie algebras and the generalization of Theorem 1 to $\text{DGLA}^\Delta$ (with $C_\chi$ replaced by the total complex) is essentially straightforward, using the ideas of this paper and Whitney-Dupont operators [4, 29, 3]. Naturally, it would be extremely interesting for applications to deformation theory to prove the analogue of Theorem 2 for cosimplicial DGLAs: at the moment we are unaware of simple descriptions of deformation functors associated to cosimplicial DGLAs.

Acknowledgment. Our thanks to Jim Stasheff for precious comments on the version v1 of this paper. The version v3 of this paper was written while the second author was at Mittag-Leffler Institute in Stockholm, during a special year on Moduli Spaces: the author is grateful for the support received and for the warm hospitality.

Keywords and general notation. We assume that the reader is familiar with the notion and main properties of differential graded Lie algebras and $L_\infty$-algebras (we refer to [7, 9, 11, 14, 15, 18] as introduction of such structures); however the basic definitions
are recalled in this paper in order to fix notation and terminology.

For the whole paper, \( \mathbb{K} \) is a fixed field of characteristic 0 and \( \textbf{Art} \) is the category of local Artinian \( \mathbb{K} \)-algebras with residue field \( \mathbb{K} \). For \( A \in \textbf{Art} \) we denote by \( m_A \) the maximal ideal of \( A \).

1. Conventions on graded vector spaces

In this paper we will work with \( \mathbb{Z} \)-graded vector spaces; we write a graded vector space as \( V = \oplus_{n \in \mathbb{Z}} V^n \), and call \( V^n \) the degree \( n \) component of \( V \); an element \( v \) of \( V^n \) is called a degree \( n \) homogeneous element of \( V \). We say that the graded vector space \( V \) is concentrated in degree \( k \) if \( V^i = \{0\} \) for \( i \neq k \). Morphisms between graded vector spaces are linear degree preserving maps, i.e. a map \( \varphi : V \to W \) is a collection of linear maps \( \varphi^n : V^n \to W^n \). The shift functor is defined as \( (V[k])^i := V^{i+k} \). We say that a linear map \( \varphi : V \to W \) is a degree \( k \) map if it is a morphism \( V \to W[k] \), i.e., if it is a collection of linear maps \( \varphi^n : V^n \to W^{n+k} \). The set of degree \( k \) linear maps from \( V \) to \( W \) will be denoted \( \text{Hom}^k(V, W) \).

Graded vector spaces are a symmetric tensor category with \( (V \otimes W)^k = \oplus_{i+j=k} V^i \otimes W^j \) and \( \sigma_{V,W} : V \otimes W \to W \otimes V \) given by \( \sigma(v \otimes w) := (-1)^{\deg(v) \cdot \deg(w)} w \otimes v \) on homogeneous elements. We adopt the convention according to which degrees are ‘shifted on the left’. By this we mean that we have a natural identification, called the suspension isomorphism, \( V[1] \simeq \mathbb{K}[1] \otimes V \) where \( \mathbb{K}[1] \) denotes the graded vector space consisting in the field \( \mathbb{K} \) concentrated in degree \(-1\). Note that, with this convention the canonical isomorphism \( V \otimes \mathbb{K}[1] \simeq V[1] \) is \( v \otimes 1 \mapsto (-1)^{\deg(v)} v[1] \). More in general we have the following decalage isomorphism

\[
V_1[1] \otimes \cdots \otimes V_n[1] \xrightarrow{\sim} (V_1 \otimes \cdots \otimes V_n)[n]
\]

\[
v_1[1] \otimes \cdots v_n[1] \mapsto (-1)^{\sum_{i=1}^n (n-i) \cdot \deg v_i} (v_1 \otimes \cdots \otimes v_n)[n].
\]

Since graded vector spaces are a symmetric category, for any graded vector space \( V \) and any positive integer \( n \) we have a canonical representation of the symmetric group \( S_n \) on \( \otimes^n V \). The space of coinvariants for this action is called the \( n \)-th symmetric power of \( V \) and is denoted by \( \text{Sym}^n V \). For instance

\[
V \otimes V = V \otimes V/(v_1 \otimes v_2 - (-1)^{\deg(v_1) \cdot \deg(v_2)} v_2 \otimes v_1).
\]

Twisting the canonical representation of \( S_n \) on \( \otimes^n V \) by the alternating character \( \sigma \mapsto (-1)^\sigma \) and taking the coinvariants one obtains the \( n \)-th antisymmetric (or exterior) power of \( V \), denoted by \( \wedge^n V \). For instance

\[
V \wedge V = V \otimes V/(v_1 \otimes v_2 + (-1)^{\deg(v_1) \cdot \deg(v_2)} v_2 \otimes v_1).
\]

By naturality of the decalage isomorphism, we have a commutative diagram

\[
\begin{array}{ccc}
\otimes^n(V[1]) & \xrightarrow{\text{decalage}} & (\otimes^n V)[n] \\
\sigma[1] \downarrow & & \downarrow (-1)^\sigma[n] \\
\otimes^n(V[1]) & \xrightarrow{\text{decalage}} & (\otimes^n V)[n]
\end{array}
\]

and so the decalage induces a canonical isomorphism

\[
\bigotimes^n (V[1]) \xrightarrow{\sim} \left( \bigwedge^n V \right)[n].
\]
As with ordinary vector spaces, one can identify \( \otimes^n V \) and \( \wedge^n V \) with suitable subspaces of \( \otimes^n V \), called the subspace of symmetric and antisymmetric tensors respectively.

**Remark 1.1.** Using the natural isomorphisms

\[
\text{Hom}^i(V, W[l]) \cong \text{Hom}^{i+l}(V, W)
\]

and the decalage isomorphism, we obtain natural identifications

\[
\text{dec}: \text{Hom}^i \left( \bigotimes^k V, W \right) \xrightarrow{\sim} \text{Hom}^{i+k-l} \left( \bigotimes^{k}(V[1]), W[l] \right),
\]

where

\[
\text{dec}(f)(v_1[1] \otimes \cdots \otimes v_k[1]) = (-1)^{ki+\sum_{j=1}^{k}(k-j) \cdot \deg(v_j)} f(v_1 \otimes \cdots \otimes v_k)[l].
\]

By the above considerations

\[
\text{dec}: \text{Hom}^i \left( \bigwedge^k V, V \right) \xrightarrow{\sim} \text{Hom}^{i+k-1} \left( \bigotimes^k(V[1]), V[1] \right).
\]

2. **Differential graded Lie algebras and \( L_\infty \)-algebras**

A differential graded Lie algebra (DGLA for short) is a Lie algebra in the category of graded vector spaces, endowed with a compatible degree 1 differential. More explicitly, it is the data \((V, d, [\cdot, \cdot])\), where \(V\) is a graded vector space, the Lie bracket

\[
[\cdot, \cdot]: V \wedge V \to V
\]

satisfies the graded Jacobi identity:

\[
[v_1, [v_2, v_3]] = [[v_1, v_2], v_3] + (-1)^{\deg(v_1) \cdot \deg(v_2)} [v_2, [v_1, v_3]],
\]

and \(d: V \to V\) is a degree 1 differential which is a degree 1 derivation of the Lie bracket, i.e.,

\[
d[v_1, v_2] = [dv_1, v_2] + (-1)^{\deg(v_1)} [v_1, dv_2].
\]

Morphisms of DGLAs are morphisms of graded vector spaces which are compatible with the differential and the bracket, namely

\[
\varphi(dv) = d\varphi(v) \quad (\varphi \text{ is a morphism of differential complexes})
\]

\[
\varphi[v_1, v_2] = [\varphi(v_1), \varphi(v_2)].
\]

Via the decalage isomorphisms one can look at the Lie bracket of a DGLA \(V\) as to a morphism

\[
q_2 \in \text{Hom}^1(V[1] \otimes V[1], V[1]), \quad q_2(v[1] \otimes w[1]) = (-1)^{\deg(v)} [v, w][1],
\]

Similarly, the suspended differential \(q_1 = d[1] = \text{id}_V \otimes d\) is a morphism of degree 1

\[
q_1: V[1] \to V[1], \quad q_1(v[1]) = -(dv)[1].
\]

Up to the canonical bijective linear map \(V \to V[1], v \mapsto v[1]\), the suspended differential \(q_1\) and the bilinear operation \(q_2\) are written simply as

\[
q_1(v) = -dv; \quad q_2(v \otimes w) = (-1)^{\deg(v)} [v, w],
\]

i.e., “the suspended differential is the opposite differential and \(q_2\) is the twisted Lie bracket”.

Define morphisms $q_k \in \text{Hom}^1(\odot^k(V[1]), V[1])$ by setting $q_k \equiv 0$, for $k \geq 3$. The map

$$Q^1 = \sum_{n \geq 1} q_n : \bigoplus_{n \geq 1} \odot^n V[1] \rightarrow V[1]$$

extends to a coderivation of degree 1

$$Q : \bigoplus_{n \geq 1} \odot^n V[1] \rightarrow \left( \bigoplus_{n \geq 1} \odot^n V[1] \right)$$

on the reduced symmetric coalgebra cogenerated by $V[1]$, by the formula

2.1.

$$Q(v_1 \odot \cdots \odot v_n) = \sum_{k=1}^n \sum_{\sigma \in S(k, n-k)} \varepsilon(\sigma) q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)})$$

where $S(k, n-k)$ is the set of unshuffles and $\varepsilon(\sigma) = \pm 1$ is the Koszul sign, determined by the relation in $\odot^n V[1]$

$$v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)} = \varepsilon(\sigma)v_1 \odot \cdots \odot v_n.$$ 

The axioms of differential graded Lie algebra are then equivalent to $Q$ being a codifferential, i.e., $QQ = 0$. This description of differential graded Lie algebras in terms of the codifferential $Q$ is called the Quillen construction [20]. By dropping the requirement that $q_k \equiv 0$ for $k \geq 3$ one obtains the notion of $L_\infty$-algebra (or strong homotopy Lie algebra), see e.g. [14, 15, 11]; namely, an $L_\infty$ structure on a graded vector space $V$ is a sequence of linear maps of degree 1

$$q_k : \odot^k V[1] \rightarrow V[1], \quad k \geq 1,$$

such that the induced coderivation $Q$ on the reduced symmetric coalgebra cogenerated by $V[1]$, given by the Formula 2.1 is a codifferential, i.e., $QQ = 0$. This condition implies $q_1 q_1 = 0$ and therefore an $L_\infty$-algebra is in particular a differential complex. Note that, by the above discussion, every DGLA can be naturally seen as an $L_\infty$-algebra; namely, a DGLA is an $L_\infty$-algebra with vanishing higher multiplications $q_k$, $k \geq 3$. Via the decalage isomorphisms of Remark 1.1, the multiplications $q_k$ of an $L_\infty$-algebra $V$ can be seen as morphisms

$$[\ldots, n]_n \in \text{Hom}^{2-n}(\bigwedge V, V).$$

The condition $QQ = 0$ then translates into a sequence of quadratic relations between the brackets $[\ldots, n]_n$, the first of which are $[[v_1]_1]_1 = 0$, i.e., $[\ ]_1$ is a degree 1 differential; $[[v_1, v_2]_2]_1 = [[v_1], v_2] + (-1)^{\deg_V(v_1)}[[v_1, v_2]]_2$, i.e., $[\ ]_1$ is a degree 1 derivation of the bracket $[\ ]_2$;

$$[v_1, [v_2, v_3]_2]_2 = [v_1, [v_2, v_3]_2] - (-1)^{\deg_V(v_1)\deg_V(v_2)}[[v_1], v_2, [v_1, v_3]_2]$$

i.e. the bracket $[\ ]_2$ satisfies the graded Jacobi identity up to the $[\ ]_1$-homotopy $[\ ]_3$, which explains the name ‘homotopy Lie algebras’. Note in particular that the $[\ ]_1$ cohomology of an $L_\infty$-algebra carries a natural structure of graded Lie algebra (or differential graded Lie algebra with trivial differential).
A morphism $f_\infty$ between two $L_\infty$-algebras $(V, q_1, q_2, q_3, \ldots)$ and $(W, \hat{q}_1, \hat{q}_2, \hat{q}_3, \ldots)$ is a sequence of linear maps of degree 0

$$f_n : \bigoplus_{n \geq 1} V[1] \rightarrow W[1], \quad n \geq 1,$$

such that the morphism of coalgebras

$$F : \bigoplus_{n \geq 1} V[1] \rightarrow \bigoplus_{n \geq 1} W[1]$$

induced by $F^1 = \sum_n f_n : \bigoplus_{n \geq 1} \bigodot^n V[1] \rightarrow W[1]$ commutes with the codifferentials induced by the two $L_\infty$ structures on $V$ and $W \text{[7,11,14,15,18]}$. An $L_\infty$-morphism $f_\infty$ is called linear (sometimes strict) if $f_n = 0$ for every $n \geq 2$. We note that a linear map

$$f_1 : V[1] \rightarrow W[1]$$

is a linear $L_\infty$-morphism if and only if

$$\hat{q}_n(f_1(v_1) \odot \cdots \odot f_1(v_n)) = f_1(q_n(v_1 \odot \cdots \odot v_n)), \quad \forall n \geq 1, v_1, \ldots, v_n \in V[1].$$

The category of $L_\infty$-algebras will be denoted by $L_\infty$ in this paper. Morphisms between DGLAs are linear morphisms between the corresponding $L_\infty$-algebras, so the category of differential graded Lie algebras is a (non full) subcategory of $L_\infty$.

If $f_\infty$ is an $L_\infty$ morphism between $(V, q_1, q_2, q_3, \ldots)$ and $(W, \hat{q}_1, \hat{q}_2, \hat{q}_3, \ldots)$, then its linear part

$$f_1 : V[1] \rightarrow W[1]$$

satisfies the equation $f_1 \circ q_1 = \hat{q}_1 \circ f_1$, i.e., $f_1$ is a map of differential complexes $(V[1], q_1) \rightarrow (W[1], \hat{q}_1)$. An $L_\infty$-morphism $f_\infty$ is called a quasiisomorphism of $L_\infty$-algebras if its linear part $f_1$ is a quasiisomorphism of differential complexes.

3. The suspended mapping cone of $\chi : L \rightarrow M$.

The suspended mapping cone of the DGLA morphism $\chi : L \rightarrow M$ is the graded vector space

$$C_\chi = \text{Cone}(\chi)[-1],$$

where $\text{Cone}(\chi) = L[1] \oplus M$ is the mapping cone of $\chi$. More explicitly,

$$C_\chi = \bigoplus_i C_\chi^i, \quad C_\chi^i = L^i \oplus M^{i-1}.$$

The suspended mapping cone has a natural differential $\delta \in \text{Hom}^1(C_\chi, C_\chi)$ given by

$$\delta(l, m) = (dl, \chi(l) - dm), \quad l \in L, m \in M.$$

Denote $M[t, dt] = M \otimes \mathbb{K}[t, dt]$ and define, for every $a \in \mathbb{K}$, the evaluation morphism

$$e_a : M[t, dt] \rightarrow M, \quad e_a(\sum m_i t^i + n_i t^i dt) = \sum m_i a^i.$$

It is easy to prove that every morphism $e_a$ is a surjective quasi-isomorphism of DGLA. Consider the DGLA

$$H_\chi = \{(l, m) \in L \times M[t, dt] \mid e_0(m) = 0, e_1(m) = \chi(l)\}$$

and the morphism

$$\iota : C_\chi \rightarrow H_\chi, \quad \iota(l, m) = (l, t\chi(l) + dt \cdot m).$$
Proposition 3.1. In the above notation, the morphism $i$ is an injective quasi-isomorphism of complexes. For every functor $F: \mathrm{DGLA}^2 \to L_\infty$ such that the diagram

$$
\begin{array}{ccc}
\mathrm{DGLA} & \longrightarrow & L_\infty \\
\downarrow & & \downarrow \\
\mathrm{DGLA}^2 & \longrightarrow & \mathrm{DG}
\end{array}
$$

commutes, there exists an $L_\infty$-morphism

$$i_\infty: F(\chi) \to H\chi$$

with linear term $i_1 = i$.

Proof. Defining

$$P = \{(l, m) \in L \times M[t, dt] \mid e_1(m) = \chi(l)\}.$$

We have a commutative diagram of morphisms of differential graded Lie algebras

$$
\begin{array}{ccc}
L & \xrightarrow{f} & P \\
\downarrow & & \downarrow \\
M & \xrightarrow{\text{Id}_M} & M
\end{array}
$$

and then two $L_\infty$-morphisms $F(\chi) \to F(\eta) \xleftarrow{h_\infty} H\chi$ whose linear parts are the two injective quasiisomorphisms

$$C_\chi \to C_\eta \xleftarrow{h} H\chi, \quad h(l, m) = ((l, m), 0).$$

A morphism of complexes $p: C_\eta \to H\chi$ such that $ph = \text{Id}_{H\chi}$ can be defined as

$$p((l, m), n) = (l, m + (t - 1)e_0(m) + dt \cdot n).$$

The composition of $p$ with the injective quasi-isomorphism $C_\chi \to C_\eta$ gives the map $i$. By general theory there exists a (non canonical) left inverse of $h_\infty$ with linear term equal to $p$ and then the morphism $i: C_\chi \to H\chi$ can be lifted to an $L_\infty$-quasiisomorphism. \qed

Denote by $\langle \cdot \rangle_1 \in \text{Hom}^1(C\chi[1], C\chi[1])$ and $q_1 \in \text{Hom}^1(H\chi[1], H\chi[1])$ the suspended differentials, namely

$$\langle(l, m)\rangle_1 = (-dl, -\chi(l) + dm), \quad l \in L, m \in M.$$  

$$q_1(l, m) = (-dl, -dm).$$

Notice that $i$ induce naturally an injective quasiisomorphism

$$i: C\chi[1] \to H\chi[1], \quad i(l, m) = (l, t\chi(l) + dt \cdot m).$$

The integral operator $\int_a^b: \mathbb{K}[t, dt] \to \mathbb{K}$ extends to a linear map of degree $-1$

$$\int_a^b: M[t, dt] \to M, \quad \int_a^b \left( \sum_i t^i m_i + t^i dt \cdot n_i \right) = \sum_i \left( \int_a^b t^i dt \right)n_i.$$

Lemma 3.2. In the above notation, consider the linear maps

$$\pi \in \text{Hom}^0(H\chi[1], C\chi[1]), \quad K \in \text{Hom}^{-1}(H\chi[1], H\chi[1])$$

defined as

$$\pi(l, m(t, dt)) = \left( l, \int_0^1 m(t, dt) \right), \quad K(l, m) = \left( 0, \int_0^t m - t \int_0^1 m \right).$$
Then $\pi$ is a morphism of complexes and

$$\pi_1 = \text{Id}_{C[1]}, \quad \iota \pi = \text{Id}_{H[1]} + Kq_1 + q_1K.$$ 

Proof. The proof of the Lemma is a simple exercise; we leave it to the reader. $\square$

4. Homotopy transfer of $L_\infty$ structures

A major result in the theory of $L_\infty$-algebras is the following homotopical transfer of structure theorem, which we learnt from [7, 13]. For the reader’s convenience, we give a sketch of the proof. Note that the version of the homotopy transfer we give here is slightly more general than the ones we are aware of in the existing literature.

**Theorem 4.1.** Let $(V, q_1, q_2, q_3, \ldots)$ be an $L_\infty$-algebra and $(C, \delta)$ be a differential complex. If there exist two morphisms of differential complexes

$$\iota: (C[1], \delta_{[1]}) \to (V[1], q_1) \quad \text{and} \quad \pi: (V[1], q_1) \to (C[1], \delta_{[1]})$$

such that the composition $\iota \pi$ is homotopic to the identity, then there exist an $L_\infty$-algebra structure $(C, \langle \rangle_1, \langle \rangle_2, \ldots)$ on $C$ extending its differential complex structure and an $L_\infty$-morphism $\iota_\infty$ extending $\iota$.

**Proof.** Let $K \in \text{Hom}^{-1}(V[1], V[1])$ be an homotopy between $\iota \pi$ and $\text{Id}_{V[1]}$, i.e., $q_1K + Kq_1 = \iota \pi - \text{Id}_{V[1]}$. Denote

$$q_+ = \sum_{n \geq 2} q_n: \bigoplus_{n \geq 2} \bigotimes^n V[1] \to V[1],$$

so that $Q^1 = q_1 + q_+$. Define a morphism of of graded coalgebras

$$\iota_\infty: \bigoplus_{n \geq 1} \bigotimes^n C[1] \to \bigoplus_{n \geq 1} \bigotimes^n V[1]$$

via the recursion

$$\iota_\infty^1 = \iota_1^1 + Kq_+ \iota_\infty$$

and a degree 1 coderivation

$$\hat{Q}: \bigoplus_{n \geq 1} \bigotimes^n C[1] \to \bigoplus_{n \geq 1} \bigotimes^n C[1]$$

by the formula

$$\hat{Q}^1 = \sum_{n \geq 1} \langle \rangle_n = \delta_{[1]} + \pi q_+ \iota_\infty.$$

Then we have

$$(Q \iota_\infty - \iota_\infty \hat{Q})^1 = Kq_+(Q \iota_\infty - \iota_\infty \hat{Q}).$$
Indeed,
\[
(Q_{t_\infty} - t_\infty \hat{Q})^1 = Q_{t_\infty}^1 - t_\infty \hat{Q} = q_{1t_\infty}^1 + q_{+t_\infty} - t_\infty \hat{Q}
\]
\[
= q_{1t_\infty} + q_1Kq_{+t_\infty} + q_{+t_\infty} - t_\infty \hat{Q}^1 - Kq_{+t_\infty} \hat{Q}
\]
\[
= q_{1t_\infty} + (\pi - \text{Id})q_{+t_\infty} + q_{+t_\infty} - t_\infty \hat{Q}^1 - Kq_{+t_\infty} \hat{Q}
\]
\[
= q_{1t_\infty} + \pi q_{+t_\infty} - Kq_{1q_{+t_\infty}} - t_\infty \hat{Q}^1 - Kq_{+t_\infty} \hat{Q}
\]
\[
= (q_{1t_\infty} - \delta_{[1]}) - Kq_{1q_{+t_\infty}} - Kq_{+t_\infty} \hat{Q}
\]
\[
= -Kq_{1q_{+t_\infty}} - Kq_{+t_\infty} \hat{Q}.
\]
Since \(0 = Q_{t_\infty}^1 = q_{1t_\infty}^1 + q_{+} = q_{+} + q_{+}Q\) we have \(q_{1t_\infty} = -q_{+}Q\) and therefore
\[
(Q_{t_\infty} - t_\infty \hat{Q})^1 = Kq_{+}(Q_{t_\infty} - t_\infty \hat{Q}).
\]
The map
\[
Q_{t_\infty} - t_\infty \hat{Q} : \bigoplus_{n \geq 1} \bigotimes_{1} \mathcal{C}[1] \to \bigoplus_{n \geq 1} \bigotimes_{1} \mathcal{V}[1]
\]
is a \(t_\infty\)-derivation and then, in order to prove that \(Q_{t_\infty} - t_\infty \hat{Q} = 0\), it is sufficient to show that \((Q_{t_\infty} - t_\infty \hat{Q})^1 = 0\). We shall prove by induction on \(n\) that \((Q_{t_\infty} - t_\infty \hat{Q})^1\) vanishes on \(\bigotimes^n \mathcal{C}[1]\); for \(n = 0\) there is nothing to prove. Let us assume \(n > 0\) and
\[
(Q_{t_\infty} - t_\infty \hat{Q})^1((\bigotimes^n \mathcal{C}[1])) = 0
\]
for every \(i < n\); then by coLeibniz rule, for every \(w \in \bigotimes^n \mathcal{C}[1]\) we have \((Q_{t_\infty} - t_\infty \hat{Q})(w) = (Q_{t_\infty} - t_\infty \hat{Q})^1(w) \in \mathcal{V}[1]\) and therefore
\[
(Q_{t_\infty} - t_\infty \hat{Q})^1(w) = Kq_{+}(Q_{t_\infty} - t_\infty \hat{Q})(w) = Kq_{+}(Q_{t_\infty} - t_\infty \hat{Q})^1(w) = 0.
\]
Therefore
\[
Q_{t_\infty} = t_\infty \hat{Q}.
\]
We also have
\[
(\hat{Q} \hat{Q})^1 = \hat{Q}^1 \hat{Q} = \hat{Q}_{1}^1 \hat{Q} + \pi q_{+t_\infty} \hat{Q} = \delta_{[1]} \hat{Q}^1 + \pi q_{+Q} t_\infty
\]
\[
= \delta_{[1]} \pi q_{+t_\infty} + \pi q_{+Q} t_\infty = \pi(q_{1t_\infty} + q_{+Q}) t_\infty.
\]
We have already noticed that \(q_{1t_\infty} = -q_{+}Q\) and then \((\hat{Q} \hat{Q})^1 = 0\). Since \(\hat{Q}\) is a coderivation, we find
\[
\hat{Q} \hat{Q} = 0.
\]
\[
\square
\]

The recursive definition of \(t_\infty^1\) can be explicitly solved in terms of a summation over rooted trees \([12\text{, Definition 6}]\); see also \([7\text{, 26}]\). Similarly, also the operator \(Q^1\) can be written as a sum over rooted trees. We sketch a proof of these facts following \([6]\). Let \(\mathcal{T}_K\) be the groupoid whose objects are directed rooted trees with internal vertices of valence at least two; trees in \(\mathcal{T}_K\) are decorated as follows: each tail edge of a tree in \(\mathcal{T}_K\) is decorated by the operator \(t\), each internal edge is decorated by the operator \(K\) and also the root edge is decorated by the operator \(K\); every internal vertex \(v\) carries the operation \(q_v\), where \(r\) is the number of edges having \(v\) as endpoint. Isomorphisms between objects in \(\mathcal{T}_K\) are isomorphisms of the underlying trees. Denote the set of isomorphism classes of objects of \(\mathcal{T}_K\) by the symbol \(T_K\). Similarly, let \(\mathcal{T}_\pi\) be the groupoid whose objects are directed rooted trees with the same decoration as \(\mathcal{T}_K\) except for the root edge, which is decorated by the operator \(\pi\) instead of \(K\). The set of isomorphism classes of objects of \(\mathcal{T}_\pi\) is denoted \(T_\pi\).
Via the usual operadic rules, each decorated tree $\Gamma \in T_K$ with $n$ tail vertices gives a linear map

$$Z_{\Gamma}(i, \pi, K, q_i) : C[1]^{\otimes n} \to V[1].$$

More precisely, let $\tilde{\Gamma}$ ba a planar embedding of $\Gamma$; the standard orientation of the plane induces a total ordering on tail vertices and then also a map

$$Z_{\tilde{\Gamma}}(i, \pi, K, q_i) : C[1]^{\otimes n} \to V[1]$$

evaluated according to the usual operadic rules. Then define $Z_{\Gamma}(i, \pi, K, q_i)$ as the composition of $Z_{\tilde{\Gamma}}(i, \pi, K, q_i)$ and the symmetrization map

$$C[1]^{\otimes n} \to C[1]^{\otimes n}, \quad v_1 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in S_n} \epsilon(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

It is straightforward to check that $Z_{\Gamma}(i, \pi, K, q_i)$ is well defined.

Similarly, each decorated tree in $T_\pi$ gives rise to a degree one multilinear operator on $C[1]$ with values in $C[1]$. Here is an example:

$$\Gamma = \overset{\pi}{\bullet} \quad q_2 \quad \overset{\pi}{\bullet} \quad q_2 \quad \overset{\pi}{\bullet} \quad q_2.$$

$$Z_{\Gamma}(i, \pi, K, q_i)(a \otimes b) = \pi q_2(i(a) \otimes i(b)) + (-1)^{\pi q_2(i(b) \otimes i(a))} = 2\pi q_2(i(a) \otimes i(b)).$$

**Proposition 4.2.** In the set-up of Theorem 4.1, if $K \in \text{Hom}^{-1}(V[1], V[1])$ satisfies the equation $q_1K + Kq_1 = i\pi - \text{Id}_{V[1]}$, then the operators $i_1^\infty$ and $\hat{Q}_1^\infty$ can be expressed as sums over decorated rooted trees via the formulas

$$i_1^\infty = i + \sum_{\Gamma \in T_K} \frac{Z_{\Gamma}(i, \pi, K, q_i)}{|\text{Aut}\Gamma|}; \quad \hat{Q}_1^\infty = \delta_1[I] + \sum_{\Gamma \in T_\pi} \frac{Z_{\Gamma}(i, \pi, K, q_i)}{|\text{Aut}\Gamma|}.$$

In particular, for $n \geq 2$, the $n$-th higher bracket defining the $L_\infty$ structure on $C[1]$ is

$$\langle \rangle_n = \sum_{\Gamma \in T_{\pi,n}} \frac{Z_{\Gamma}(i, \pi, K, q_i)}{|\text{Aut}\Gamma|},$$

where $T_{\pi,n}$ is the full subgroupoid of $T_\pi$ whose objects are rooted trees with exactly $n$ tail vertices.

**Proof.** Let $\mathcal{V}_K$ be the groupoid whose objects are rooted trees with a single internal vertex, of valence at least two. The root edge is decorated by the operator $K$, the tail edges are decorated by the operator $\text{Id}_{V[1]}$ and the vertex is decorated by the operator $q_r$, where $r$ is the number of tail edges, denote by $\mathcal{V}_{K,n}$ the full subgroupoid of $\mathcal{V}_K$ whose object have exactly $n$ tail edges; note that the set $V_{K,n}$ of isomorphism classes of objects of $\mathcal{V}_{K,n}$ consists of a single element and that if $\Gamma$ is an object in $\mathcal{V}_{K,n}$ with $n$-tail edges, then

$$\frac{Z_{\Gamma}(i, \pi, K, q_i)}{|\text{Aut}\Gamma|} = K q_n.$$

Also, let $\mathcal{I}$ be the groupoid whose objects are trees consisting of a single directed edge decorated by the operator $i$. Clearly, objects in $\mathcal{I}$ are all isomorphic to each other and have no nontrivial automorphisms; operadic evaluation gives, for an object $\Gamma$ in $\mathcal{I}$,

$$\frac{Z_{\Gamma}(i, \pi, K, q_i)}{|\text{Aut}\Gamma|} = i.$$
Each tree in $T_K$ can be seen as the composition of exactly one object of $V_K$ and a forest consisting of trees in $T_K$ and in $I$. Hence, the usual combinatorics of sums over graphs (see [6] for details) gives:

$$\mathfrak{1} + \sum_{\Gamma \in T_K} \frac{Z_{\Gamma}(1, \pi, K, q_i)}{|\text{Aut} \, \Gamma|} = \mathfrak{1} + \sum_{n \geq 2} \sum_{\Gamma_1 \in V_{K,n}} \frac{Z_{\Gamma_1}(1, \pi, K, q_i)}{|\text{Aut} \, \Gamma_1|} \left( \mathfrak{1} + \sum_{\Gamma_2 \in T_K} \frac{Z_{\Gamma_2}(1, \pi, K, q_i)}{|\text{Aut} \, \Gamma_2|} \right)^{\otimes n}$$

$$= \mathfrak{1} + \sum_{n \geq 2} K q_n \left( \mathfrak{1} + \sum_{\Gamma \in T_K} \frac{Z_{\Gamma}(1, \pi, K, q_i)}{|\text{Aut} \, \Gamma|} \right)^{\otimes n}.$$ 

This shows that

$$\mathfrak{1} + \sum_{\Gamma \in T_K} \frac{Z_{\Gamma}(1, \pi, K, q_i)}{|\text{Aut} \, \Gamma|}$$

satisfies the same recursion as $\mathfrak{1}_\infty$; since this recursion uniquely determines $\mathfrak{1}_\infty$, the formula for $\mathfrak{1}_\infty$ is proved. The formula for $\hat{Q}_1$ is proved by a completely similar argument. \[\square\]

5. The $L_\infty$ Structure on $C_\chi$

By Quillen construction [20], the $L_\infty$ structure on the differential graded Lie algebra $H_\chi$ is given by the brackets

$$q_k: \bigotimes_{k} (H_\chi[1]) \to H_\chi[1],$$

where $q_k = 0$ for every $k \geq 3$,

$$q_1(l, m(t, dt)) = (-dl, -dm(t, dt))$$

and

$$q_2((l_1, m_1(t, dt)) \odot (l_2, m_2(t, dt))) = (\mathfrak{1})^{\deg_{H_\chi}(l_1, m_1(t, dt))} (l_1, l_2), [m_1(t, dt), m_2(t, dt)]).$$

By Lemma [3,2] we can apply homotopy trasfer in order to costruc the explicit $L_\infty$ structure on $C_\chi$ and an explicit $L_\infty$-morphism $\mathfrak{1}_\infty: C_\chi \to H_\chi$ extending $\mathfrak{1}$.

According to Proposition [12] the linear maps of degree 1

$$\langle \rangle_n: \bigotimes^n C_\chi[1] \to C_\chi[1], \quad n \geq 2,$$

defining the induced $L_\infty$-algebra structure on $C_\chi$ are explicitly described in terms of summation over rooted trees. In our case, the properties

$$q_2(\text{Im} \, K \otimes \text{Im} \, K) \subseteq \ker \, \pi \cap \ker \, K, \quad q_k = 0 \ \forall \ k \geq 3,$$

imply that, fixing the number $n \geq 2$ of tails, there exists at most one isomorphism class of trees giving a nontrivial contribution.

- $n = 2$

This graph gives

$$\langle \gamma_1 \odot \gamma_2 \rangle_2 = \pi q_2(\mathfrak{1}(\gamma_1) \odot \mathfrak{1}(\gamma_2)).$$
\[ \langle \gamma_1 \odot \cdots \odot \gamma_n \rangle_n = \]
\[ = \frac{1}{2} \sum_{\sigma \in S_n} \sum_{\sigma(n-1) < \sigma(n)} \varepsilon(\sigma) \pi q_2(\iota(\gamma_{\sigma(1)}) \odot K q_2(\iota(\gamma_{\sigma(2)}) \odot \cdots \odot K q_2(\iota(\gamma_{\sigma(n-1)}) \odot \iota(\gamma_{\sigma(n)})) \cdots)) \]

\[ = \sum_{\sigma \in S_n} \varepsilon(\sigma) \pi q_2(\iota(\gamma_{\sigma(1)}) \odot K q_2(\iota(\gamma_{\sigma(2)}) \odot \cdots \odot K q_2(\iota(\gamma_{\sigma(n-1)}) \odot \iota(\gamma_{\sigma(n)})) \cdots)). \]

**Remark 5.1.** The above construction of the \(L_{\infty}\) structure on \(C_\chi\) commutes with tensor products of differential graded commutative algebras. This means that if \(R\) is a DGCA, then the \(L_{\infty}\)-algebra structure on the suspended mapping cone of \(\chi \otimes \text{id}_R\): \(L \otimes R \to M \otimes R\) is naturally isomorphic to the \(L_{\infty}\)-algebra \(C_\chi \otimes R\).

A more refined description involving the original brackets in the differential graded Lie algebras \(L\) and \(M\) is obtained decomposing the symmetric powers of \(C_\chi[1]\) into types:

\[ \bigodot^n (C_\chi[1]) = \bigodot^n \text{Cone}(\chi) = \bigoplus_{\lambda+\mu=n} \left( \bigodot^\mu M \right) \otimes \left( \bigodot^\lambda L[1] \right). \]

The operation \(\langle \rangle_2\) decomposes into

\[ l_1 \odot l_2 \mapsto (-1)^{\text{deg}_L(l_1)} [l_1, l_2] \in L; \quad m_1 \odot m_2 \mapsto 0; \]

\[ m \otimes l \mapsto \frac{(-1)^{\text{deg}_M(m)+1}}{2} [m, \chi(l)] \in M. \]

For every \(n \geq 2\) it is easy to see that \(\langle \gamma_1 \odot \cdots \odot \gamma_{n+1} \rangle_{n+1}\) can be nonzero only if the multivector \(\gamma_1 \odot \cdots \odot \gamma_{n+1}\) belongs to \(\bigodot^n M \otimes L[1]\). For \(n \geq 2\), \(m_1, \ldots, m_n \in M\) and \(l \in L[1]\) the formula for \(\langle \rangle_{n+1}\) described above becomes

\[ \langle m_1 \odot \cdots \odot m_n \otimes l \rangle_{n+1} = \]
\[ = \sum_{\sigma \in S_n} \varepsilon(\sigma) \pi q_2((dt)m_{\sigma(1)} \odot K q_2((dt)m_{\sigma(2)} \odot \cdots \odot K q_2((dt)m_{\sigma(n)} \odot t \chi(l)) \cdots)). \]

Define recursively a sequence of polynomials \(\phi_i(t) \in \mathbb{Q}[t] \subseteq \mathbb{K}[t]\) and rational numbers \(I_n\) by the rule

\[ \phi_1(t) = t, \quad I_n = \int_0^1 \phi_n(t) dt, \quad \phi_{n+1}(t) = \int_0^t \phi_n(s) ds - t I_n. \]
By the definition of the homotopy operator $K$ we have, for every $m \in M$

$$K((\phi_n(t)dt)m) = \phi_{n+1}(t)m.$$ 

Therefore, for every $m_1, m_2 \in M$ we have

$$Kq_2((dt \cdot m_1) \circ \phi_n(t)m_2) = -(-1)^{\deg_M(m_1)}\phi_{n+1}(t)[m_1, m_2].$$ 

Therefore, we find:

$$\langle m_1 \circ \cdots \circ m_n \otimes l \rangle_{n+1} =$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma)\pi q_2((dt)m_{\sigma(1)} \circ K q_2((dt)m_{\sigma(2)} \circ \cdots \circ K q_2((dt)m_{\sigma(n)} \circ t\chi(l)) \cdots))$$

$$= (-1)^{1+\deg_M(m_{\sigma(n)})} \sum_{\sigma \in S_n} \varepsilon(\sigma)\pi q_2((dt)m_{\sigma(1)} \circ K q_2((dt)m_{\sigma(2)} \circ \cdots \circ \phi_2(t)[m_{\sigma(n)}, \chi(l)] \cdots))$$

$$= (-1)^{n+1+\sum_{i=2}^n \deg_M(m_{\sigma(i)})} \sum_{\sigma \in S_n} \varepsilon(\sigma)\pi q_2((dt)m_{\sigma(1)} \circ \phi_n(t)[m_{\sigma(2)}, \cdots, [m_{\sigma(n)}, \chi(l)] \cdots])$$

$$= (-1)^{n+\sum_{i=1}^n \deg_M(m_{\sigma(i)})} I_n \sum_{\sigma \in S_n} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \cdots, [m_{\sigma(n)}, \chi(l)] \cdots)] \in M.$$ 

**Theorem 5.2.** For any morphism $\chi: L \to M$ of differential graded Lie algebras, let $\tilde{C}(\chi) = (C_\chi, \tilde{Q})$ be the $L_\infty$-algebra structure defined on $C_\chi$ by the above construction. Then

$$\tilde{C}: \text{DGLA}^2 \to L_\infty$$

is a functor making the diagram

$$\begin{array}{ccc}
\text{DGLA} & \longrightarrow & L_\infty \\
\downarrow \scriptstyle{\tilde{C}} & & \downarrow \scriptstyle{\tilde{C}} \\
\text{DGLA}^2 & \longrightarrow & \text{DG}
\end{array}$$

commutative. If $F: \text{DGLA}^2 \to L_\infty$ has the same properties, then for every morphism $\chi$ of differential graded Lie algebras, the $L_\infty$-algebra $F(\chi)$ is isomorphic to $\tilde{C}(\chi)$.

**Proof.** The functoriality of $\tilde{C}$ is clear; in fact, for every commutative diagram

$$\begin{array}{ccc}
L_1 & \xrightarrow{f_L} & L_2 \\
\scriptstyle{\chi_1} \downarrow & & \downarrow \scriptstyle{\chi_2} \\
M_1 & \xrightarrow{f_M} & M_2
\end{array}$$

of morphisms of differential graded Lie algebras, the natural map $(f_L, f_M): \tilde{C}(\chi_1) \to \tilde{C}(\chi_2)$ is a linear $L_\infty$-morphism.

If $F: \text{DGLA}^2 \to L_\infty$ has the same properties as $\tilde{C}$, according to Proposition 3.1, there exists two injective $L_\infty$ quasi-isomorphisms

$$\iota_\infty: \tilde{C}(\chi) \to H_\chi, \quad \hat{\iota}_\infty: F(\chi) \to H_\chi$$

with the same linear term $\iota$. The composition of $\iota_\infty$ with a left inverse of $\hat{\iota}_\infty$ is an isomorphism of $L_\infty$-algebras. \qed
Remark 5.3. As an instance of functoriality, note that the projection on the first factor $p_1: \bar{C}(\chi) \to L$ is a linear morphism of $L_\infty$-algebras. To see this, consider the morphism in $\text{DGLA}^2$

\[
\begin{array}{c}
L \\ \downarrow 1
\end{array} \longrightarrow \begin{array}{c}
\text{Id}_L \\ \downarrow 1
\end{array} \longrightarrow \begin{array}{c}
L \\ \downarrow 1
\end{array}
\]

We also have an explicit expression for the coefficients $I_n$ appearing in the formula for $\langle \rangle_{n+1}$; in the next lemma we show that they are, up to a sign, the Bernoulli numbers.

Lemma 5.4. For every $n \geq 1$ we have $I_n = -B_n/n!$, where the $B_n$ are the Bernoulli numbers, i.e., the rational numbers defined by the series expansion identity

\[
\sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{x^n}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots
\]

Proof. Keeping in mind the definition of $B_n$, we have to prove that

\[
1 - \sum_{n=1}^{\infty} I_n x^n = \frac{x}{e^x - 1}.
\]

Consider the polynomials $\psi_0(t) = 1$ and $\psi_n(t) = \phi_n(t) - I_n$ for $n \geq 1$. Then, for any $n \geq 1$,

\[
\frac{d}{dt} \psi_n(t) = \psi_{n-1}(t), \quad \int_0^1 \psi_n(t)dt = 0.
\]

Setting

\[
F(t, x) = \sum_{n=0}^{\infty} \psi_n(t)x^n,
\]

we have

\[
\frac{d}{dt} F(t, x) = \sum_{n=1}^{\infty} \psi_{n-1}(t)x^n = xF(t, x), \quad \int_0^1 F(t, x)dt = 1.
\]

Therefore,

\[
F(t, x) = F(0, x)e^{tx},
\]

\[
1 = \int_0^1 F(t, x)dt = F(0, x) \int_0^1 e^{tx} dt = F(0, x) \frac{e^x - 1}{x},
\]

and then

\[
F(0, x) = \frac{x}{e^x - 1}.
\]

Since $\psi_n(0) = -I_n$ for any $n \geq 1$ we get

\[
\frac{x}{e^x - 1} = F(0, x) = 1 - \sum_{n=1}^{\infty} I_n x^n.
\]

In fact an alternative proof of the equality $I_n = -B_n/n!$ can be done by observing that the polynomials $n!\psi_n(t)$ satisfy the recursive relations of the Bernoulli polynomials (see e.g. [22]).

Summing up the results of this Section, we have the following explicit description of the $L_\infty$ algebra $\bar{C}(\chi)$. 
Theorem 5.5. The $L_\infty$ algebra $\tilde{C}(\chi)$ is defined by the multilinear maps

$$\langle \rangle_n : \bigodot^n C_\chi[1] \to C_\chi[1],$$

given by

$$\langle (l, m) \rangle_1 = (-dl, -\chi(l) + dm);$$

$$\langle l_1 \circ l_2 \rangle_2 = (-1)^{\deg_L(l_1)}[l_1, l_2];$$

$$\langle m \circ l \rangle_2 = \frac{(-1)^{\deg_M(m)+1}}{2}[m, \chi(l)];$$

$$\langle m_1 \circ m_2 \rangle_2 = 0;$$

$$\langle m_1 \circ \cdots \circ m_n \circ l_1 \circ \cdots \circ l_k \rangle_{n+k} = 0, \quad n + k \geq 3, k \neq 1;$$

and

$$\langle m_1 \circ \cdots \circ m_n \circ l \rangle_{n+1} =$$

$$= -(-1)^{\sum_{i=1}^{n} \deg_M(m_i)} \frac{B_n}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \cdots, [m_{\sigma(n)}, \chi(l)]] \cdots]], \quad n \geq 2;$$

where the $B_n$ are the Bernoulli numbers, i.e., the rational numbers defined by the series expansion identity

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots$$

Remark 5.6. Via the decalage isomorphism $\otimes^n (C_\chi[1]) \cong \left(\wedge^n C_\chi\right)[n]$, the linear maps $\langle \rangle_n$ defining the $L_\infty$-algebra $\tilde{C}(\chi)$ correspond to multilinear operations $[\ ]_n : \wedge^n C_\chi \to C_\chi[2 - n]$ on $C_\chi$. In particular, the linear map $\langle \rangle_1$ corresponds to the differential $\delta$ on $C_\chi$

$$\delta : (l, m) \mapsto (dl, -\chi(l) + dm),$$

whereas the map $\langle \rangle_2$ corresponds to the following degree zero bracket

$$[\ ]_2 : C_\chi \wedge C_\chi \to C_\chi$$

$$[l_1, l_2]_2 = [l_1, l_2]; \quad [m, l]_2 = \frac{1}{2}[m, \chi(l)]; \quad [m_1, m_2]_2 = 0.$$

Note that this is precisely the naive bracket described in the introduction.

Remark 5.7. The occurrence of Bernoulli numbers is not surprising; it had already been noticed by K. T. Chen [2] how Bernoulli numbers are related to the coefficients of the Baker-Campbell-Hausdorff formula.

More recently, the relevance of Bernoulli numbers in deformation theory has been also remarked by Ziv Ran in versions v3-v6 of [21]. In particular, Ziv Ran’s “JacoBer” complex seems to be closely related with the coderivation $\hat{Q}$ defining the $L_\infty$ structure on $C_\chi$.

Bernoulli numbers also appear in some expressions of the gauge equivalence in a differential graded Lie algebra [27][3]. In fact the relation $x = e^a * y$ can be written as

$$x - y = \frac{e^{ad_a} - 1}{ad_a} ([a, y] - da).$$
Applying to both sides the inverse of the operator \( \frac{e^{ad_a} - 1}{ad_a} \) we get
\[
da = [a, y] - \sum_{n \geq 0} \frac{B_n}{n!} \text{ad}^n_a(x - y).
\]

The multilinear brackets \( \langle \rangle_n \) on \( \text{Cone}(\chi) = C_{\chi}[1] \) can be related to the Koszul (or ‘higher derived’) brackets \( \Phi_n \) of a differential graded Lie algebra as follows. Let \((M, \partial, [\cdot, \cdot])\) be a differential graded Lie algebra; the Koszul brackets
\[
\Phi_n: \bigotimes^n M \to M, \quad n \geq 1
\]
are the degree 1 linear maps defined as \( \Phi_1 = 0 \) and for \( n \geq 2 \)
\[
\Phi_n(m_1 \cdots m_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma) \cdot \langle \partial m_{\sigma(1)}, m_{\sigma(2)}, m_{\sigma(3)}, \cdots, m_{\sigma(n)} \rangle.
\]

Let \( L \) be the differential graded Lie subalgebra of \( M \) given by \( L := \partial M \) and let \( \chi: L \to M \) be the inclusion. We can identify \( M \) with the image of the injective linear map \( M \hookrightarrow \text{Cone}(\chi) \) given by \( m \mapsto (\partial m, m) \). Then we have \( \langle \partial m, m \rangle_1 = 0 \), \( \langle \partial m_1, m_1 \rangle \circ \cdots \circ \langle \partial m_{n+1}, m_{n+1} \rangle_{n+1} = 0, B_n(-1)^n(n+1) \Phi_{n+1}(m_1 \circ \cdots \circ m_{n+1}) \).

Since the multilinear operations \( \langle \rangle_n \) define an \( L_\infty \)-algebra structure on \( C_{\chi} = \text{Cone}(\chi)[-1] \), they satisfy a sequence of quadratic relations. Due to the above mentioned correspondence with the Koszul brackets, these relations are translated into a sequence of differential/quadratic relations between the odd Koszul brackets, defined as \( \{m\}_1 = 0 \) and
\[
\{m_1, \cdots, m_n\}_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma)(-1)^\sigma \cdot \langle \partial m_{\sigma(1)}, m_{\sigma(2)}, m_{\sigma(3)}, \cdots, m_{\sigma(n)} \rangle
\]
for \( n \geq 2 \). For instance, if \( m_1, m_2, m_3 \) are homogeneous elements of degree \( i_1, i_2, i_3 \) respectively, then
\[
\{m_1, m_2\}_2, m_3\} + (-1)^{i_1i_2 + i_1i_3} \{m_2, m_3\}_2, m_1\}_2 + (-1)^{i_2i_3 + i_1i_3} \{m_3, m_1\}_2, m_2\}_2 = 3/2 \cdot \partial\{m_1, m_2, m_3\}_3.
\]

The occurrence of Bernoulli numbers in the \( L_\infty \)-type structure defined by the higher Koszul brackets has been recently remarked by K. Bering in [4].

6. The Maurer-Cartan Functor

Having introduced an \( L_\infty \) structure on \( C_{\chi} \) in Section 5 we have a corresponding Maurer-Cartan functor \([7, 11]\) \( \text{MC}_{C_{\chi}}: \text{Art} \to \text{Set} \), defined as
\[
\text{MC}_{C_{\chi}}(A) = \left\{ \gamma \in C_{\chi}[1]^0 \otimes m_A \left| \sum_{n \geq 1} \frac{\langle \gamma \circ n \rangle_n}{n!} = 0 \right. \right\}, \quad A \in \text{Art}.
\]
Writing $\gamma = (l, m)$, with $l \in L^1 \otimes m_A$ and $m \in M^0 \otimes m_A$, the Maurer-Cartan equation becomes

$$0 = \sum_{n=1}^{\infty} \frac{\langle (l, m)^{\otimes n} \rangle_n}{n!}$$

$$= \langle (l, m) \rangle_1 + \frac{1}{2} \langle (l^\otimes 2) \rangle_2 + \langle (m \otimes l) \rangle_2 + \frac{1}{2} \langle (m^\otimes 2) \rangle_2 + \sum_{n \geq 2} \frac{n + 1}{(n + 1)!} \langle (m^{\otimes n} \otimes l) \rangle_{n+1}$$

$$= \left(-dl - \frac{1}{2}[l, l], -\chi(l) + dm - \frac{1}{2}[m, \chi(l)] + \sum_{n \geq 2} \frac{1}{n!} \langle (m^{\otimes n} \otimes l) \rangle_{n+1}\right) \in (L^2 \oplus M^1) \otimes m_A.$$

According to Theorem 5.5, since $\deg_M(m) = \deg_{C^\chi}[1](m) = 0$, we have

$$\langle (m^{\otimes n} \otimes l) \rangle_{n+1} = -\frac{B_n}{n!} \sum_{\sigma \in S_n} [m, [m, \ldots, [m, \chi(l)] \ldots]] = -B_n \operatorname{ad}_m^n(\chi(l)),$$

where for $a \in M^0 \otimes m_A$ we denote by $\operatorname{ad}_a : M \otimes m_A \to M \otimes m_A$ the operator $\operatorname{ad}_a(y) = [a, y]$.

The Maurer-Cartan equation on $C^\chi$ is therefore equivalent to

$$\begin{cases}
\left\{ 
\begin{array}{l}
dl + \frac{1}{2}[l, l] = 0 \\
\chi(l) - dm + \frac{1}{2}[m, \chi(l)] + \sum_{n=2}^{\infty} \frac{B_n}{n!} \operatorname{ad}_m^n(\chi(l)) = 0.
\end{array}
\right.
\end{cases}$$

Since $B_0 = 1$ and $B_1 = -\frac{1}{2}$, we can write the second equation as

$$0 = \chi(l) - dm + \frac{1}{2}[m, \chi(l)] + \sum_{n=2}^{\infty} \frac{B_n}{n!} \operatorname{ad}_m^n(\chi(l))$$

$$= [m, \chi(l)] - dm + \sum_{n=0}^{\infty} \frac{B_n}{n!} \operatorname{ad}_m^n(\chi(l)) = [m, \chi(l)] - dm + \frac{\operatorname{ad}_m}{e^{\operatorname{ad}_m} - 1}(\chi(l)).$$

Applying the invertible operator $\frac{e^{\operatorname{ad}_m} - 1}{\operatorname{ad}_m}$ we get

$$0 = \chi(l) + \frac{e^{\operatorname{ad}_m} - 1}{\operatorname{ad}_m}([m, \chi(l)] - dm).$$

In the right side of the last formula we recognize the explicit description of the gauge action

$$\exp(M^0 \otimes m_A) \times M^1 \otimes m_A \rightarrow M^1 \otimes m_A,$$

$$e^a * y = y + \sum_{n=0}^{+\infty} \frac{\operatorname{ad}_a}{(n + 1)!}([a, y] - da) = y + \frac{e^{\operatorname{ad}_a} - 1}{\operatorname{ad}_a}([a, y] - da).$$

Therefore, the Maurer-Cartan equation for the $L_\infty$-algebra structure on $C^\chi$ is equivalent to

$$\begin{cases}
dl + \frac{1}{2}[l, l] = 0 \\
e^m * \chi(l) = 0.
\end{cases}$$
7. Homotopy equivalence and the deformation functor

Recall that the deformation functor associated to an $L_\infty$-algebra $g$ is $\text{Def}_g = \text{MC}_g / \sim$, where $\sim$ denotes homotopy equivalence of solutions of the Maurer-Cartan equation: two elements $\gamma_0$ and $\gamma_1$ of $\text{MC}_g(A)$ are called homotopy equivalent if there exists an element $\gamma(t,dt) \in \text{MC}_g[t,dt](A)$ with $\gamma(0) = \gamma_0$ and $\gamma(1) = \gamma_1$. It is possible to prove that homotopy equivalence is an equivalence relation; in this paper we do not need this fact.

We have already described the functor $\text{MC}_{C_\chi}$ in terms of the Maurer-Cartan equation in $L$ and the gauge action in $M$. Now we want to prove a similar result for the homotopy equivalence on $\text{MC}_{C_\chi}$. We need some preliminary results.

**Proposition 7.1.** Let $(L,d,[\ ,\ ])$ be a differential graded Lie algebra such that:

1. $L = M \oplus C \oplus D$ as graded vector spaces.
2. $M$ is a differential graded subalgebra of $L$.
3. $d: C \to D[1]$ is an isomorphism of graded vector spaces.

Then, for every $A \in \text{Art}$ there exists a bijection

$$\alpha: \text{MC}_M(A) \times (C^0 \otimes m_A) \xrightarrow{\sim} \text{MC}_L(A), \quad (x,c) \mapsto e^x \cdot x.$$  

**Proof.** This is essentially proved in [24, Section 5] using induction on the length of $A$ and the Baker-Campbell-Hausdorff formula. Here we sketch a different proof based on formal theory of deformation functors [24, 23, 5, 16].

The map $\alpha$ is a natural transformation of homogeneous functors, so it is sufficient to show that $\alpha$ is bijective on tangent spaces and injective on obstruction spaces. Recall that the tangent space of $\text{MC}_L$ is $Z^1(L)$, while its obstruction space is $H^2(L)$. The functor $A \mapsto C^0 \otimes m_A$ is smooth with tangent space $C^0$ and therefore tangent and obstruction spaces of the functor

$$A \mapsto \text{MC}_M(A) \times (C^0 \otimes m_A)$$

are respectively $Z^1(M) \oplus C^0$ and $H^2(M)$. The tangent map is

$$Z^1(M) \oplus C^0 \ni (x,c) \mapsto e^x \cdot x = x - dc \in Z^1(M) \oplus d(C^0) = Z^1(M) \oplus D^1 = Z^1(L)$$

and it is an isomorphism. The inclusion $M \hookrightarrow L$ is a quasiisomorphism, therefore the obstruction to lifting $x$ in $M$ is equal to the obstruction to lifting $x = e^0 \cdot x$ in $L$. We conclude the proof by observing that, according to [5, Prop. 7.5], [16, Lemma 2.20], the obstruction maps of Maurer-Cartan functor are invariant under the gauge action. \hfill \square

**Corollary 7.2.** Let $M$ be a differential graded Lie algebra, $L = M[t,dt]$ and $C \subseteq M[t]$ the subspace consisting of polynomials $g(t)$ with $g(0) = 0$. Then for every $A \in \text{Art}$ the map $(x,g[t]) \mapsto e^{g(t)} \cdot x$ induces an isomorphism

$$\text{MC}_M(A) \times (C^0 \otimes m_A) \simeq \text{MC}_L(A).$$

**Proof.** The data $M$, $C$ and $D = d(C)$ satisfy the condition of Proposition 7.1. \hfill \square

**Corollary 7.3.** Let $M$ be a differential graded Lie algebra. Two elements $x_0, x_1 \in \text{MC}_M(A)$ are gauge equivalent if and only if they are homotopy equivalent.

**Proof.** If $x_0$ and $x_1$ are gauge equivalent, then there exists $g \in M^0 \otimes m_A$ such that $e^g \cdot x_0 = x_1$. Then, by Corollary 7.2, $x(t) = e^{tg} \cdot x_0$ is an element of $\text{MC}_{M[t,dt]}(A)$ with $x(0) = x_0$ and $x(1) = x_1$, i.e., $x_0$ and $x_1$ are homotopy equivalent.

Vice versa, if $x_0$ and $x_1$ are homotopy equivalent, there exists $x(t) \in \text{MC}_{M[t,dt]}(A)$ such that $x(0) = x_0$ and $x(1) = x_1$. By Corollary 7.2, there exists $g(t) \in M^0[t] \otimes m_A$ with $g(0) = 0$ such that $x(t) = e^{g(t)} \cdot x_0$. Then $x_1 = e^{g(1)} \cdot x_0$, i.e., $x_0$ and $x_1$ are gauge equivalent. \hfill \square
Theorem 7.4. Let $\chi : L \to M$ be a morphism of differential graded Lie algebras and let $(l_0, m_0)$ and $(l_1, m_1)$ be elements of $MC_{\chi}(A)$. Then $(l_0, m_0)$ is homotopically equivalent to $(l_1, m_1)$ if and only if there exists $(a, b) \in C^0_\chi \otimes m_A$ such that

$$l_1 = e^a * l_0, \quad e^{m_1} = e^{db} e^{m_0} e^{-\chi(a)}.$$

Remark 7.5. The condition $e^{m_1} = e^{db} e^{m_0} e^{-\chi(a)}$ can be also written as $m_1 \circ \chi(a) = db \circ m_0$, where $\circ$ is the Baker-Campbell-Hausdorff product in the nilpotent Lie algebra $M^0 \otimes m_A$.

As a consequence, we get that in this case the homotopy equivalence is induced by a group action; this is false for general $L_\infty$-algebras.

Proof. We shall say that two elements $(l_0, m_0)$, $(l_1, m_1)$ are gauge equivalent if and only if there exists $(a, b) \in C^0_\chi \otimes m_A$ such that

$$l_1 = e^a * l_0, \quad e^{m_1} = e^{db} e^{m_0} e^{-\chi(a)}.$$

We first show that homotopy implies gauge. Let $(l_0, m_0)$ and $(l_1, m_1)$ be homotopy equivalent elements of $MC_{\chi}(A)$. Then there exists an element $(\tilde{l}, \tilde{m})$ of $MC_{\chi[s, ds]}(A)$ with $(\tilde{l}(0), \tilde{m}(0)) = (l_0, m_0)$ and $(\tilde{l}(1), \tilde{m}(1)) = (l_1, m_1)$. According to Remark 7.1, the Maurer-Cartan equation for $(\tilde{l}, \tilde{m})$ is

$$\begin{cases} d\tilde{l} + \frac{1}{2}[\tilde{l}, \tilde{l}] = 0 \\ e^{\tilde{m}} \ast \chi(\tilde{l}) = 0 \end{cases}$$

The first of the two equations above tells us that $\tilde{l}$ is a solution of the Maurer-Cartan equation for $L[s, ds]$. So, by Corollary 7.2 there exists a degree zero element $\lambda(s)$ in $L[s] \otimes m_A$ with $\lambda(0) = 0$ such that $\tilde{l} = e^\lambda * l_0$. Evaluating at $s = 1$ we find $l_1 = e^{\lambda_1} * l_0$.

As a consequence of $\tilde{l} = e^\lambda * l_0$, we also have $\chi(\tilde{l}) = e^{\chi(\lambda)} * \chi(l_0)$. Set $\tilde{\mu} = \tilde{m} \circ \chi(\lambda) \circ m_0$, so that $\tilde{m} = \tilde{\mu} \circ m_0 \circ (-\chi(\lambda))$ and the second Maurer-Cartan equation is reduced to $e^{\tilde{\mu}} \ast (e^{m_0} \ast \chi(l_0)) = 0$, i.e., to $e^{\tilde{\mu}} \ast 0 = 0$, where we have used the fact that $(l_0, m_0)$ is a solution of the Maurer-Cartan equation in $C_\chi$. This last equation is equivalent to the equation $d\tilde{\mu} = 0$ in $C_\chi[s, ds] \otimes m_A$. If we write $\tilde{\mu}(s, ds) = \mu^0(s) + ds \mu^{-1}(s)$, then the equation $d\tilde{\mu} = 0$ becomes

$$\begin{cases} \mu^0 - d_M \mu^{-1} = 0 \\ d_M \mu^0 = 0, \end{cases}$$

where $d_M$ is the differential in the DGLA $M$. The solution is, for any fixed $\mu^{-1}$,

$$\mu^0(s) = \int_0^s d\sigma d_M \mu^{-1}(\sigma) = -d_M \int_0^s d\sigma \mu^{-1}(\sigma)$$

Set $\nu = -\int_0^1 ds \mu^{-1}(s)$. Then $m_1 = \tilde{m}(1) = (d_M \nu) \circ m_0 \circ (-\chi(\lambda_1))$. Summing up, if $(l_0, m_0)$ and $(m_1, l_1)$ are homotopy equivalent, then there exists $(d\nu, \lambda_1) \in (dM^{-1} \otimes m_A) \times (L^0 \otimes m_A)$ such that

$$\begin{cases} l_1 = e^{\lambda_1} * l_0 \\ m_1 = d\nu \circ m_0 \circ (-\chi(\lambda_1)) \end{cases},$$

i.e., $(l_0, m_0)$ and $(m_1, l_1)$ are gauge equivalent.

We now show that gauge implies homotopy. Assume $(l_0, m_0)$ and $(m_1, l_1)$ are gauge equivalent. Then there exists $(d\nu, \lambda_1) \in (dM^{-1} \otimes m) \times (L^0 \otimes m)$ such that

$$\begin{cases} l_1 = e^{\lambda_1} * l_0 \\ m_1 = d\nu \circ m_0 \circ (-\chi(\lambda_1)) \end{cases}.$$
Set \( \tilde{l}(s, ds) = e^{s\lambda_1} * l_0 \). By Corollary 7.2, \( \tilde{l} \) satisfies the equation \( d\tilde{l} + \frac{1}{2}[\tilde{l}, \tilde{l}] = 0 \). Set \( \tilde{m} = (d(s\nu)) \bullet m_0 \bullet (-\chi(s\lambda_1)) \). Reasoning as above, we find
\[
e^\tilde{m} * \chi(\tilde{l}) = e^{d(s\nu)} * 0 = 0.
\]
Therefore, \((\tilde{l}, \tilde{m})\) is a solution of the Maurer-Cartan equation in \( C_\chi[s, ds] \). Moreover \( \tilde{l}(0) = l_0, \tilde{l}(1) = l_1, \tilde{m}(0) = m_0 \) and \( \tilde{m}(1) = d\nu \bullet m_0 \bullet (-\chi(\lambda_1)) = m_1 \), i.e. \((l_0, m_0)\) and \((m_1, l_1)\) are homotopy equivalent. \( \square \)

8. Examples and applications

Let \( \chi: L \rightarrow M \) be a morphism of differential graded Lie algebras over a field \( \mathbb{K} \) of characteristic 0. In the paper [19] one of the authors has introduced, having in mind the example of embedded deformations, the notion of Maurer-Cartan equation and gauge action of \( L, M, \chi \); these notions reduce to the standard Maurer-Cartan equation and gauge action of \( L \) when \( M = 0 \). More precisely there are defined two functors of Artin rings \( \text{MC}_\chi \), \( \text{Def}_\chi: \text{Art} \rightarrow \text{Set} \), in the following way:

\[
\text{MC}_\chi(A) = \left\{ (x, e^a) \in (L^1 \otimes m_A) \times \exp(M^0 \otimes m_A) \mid dx + \frac{1}{2}[x, x] = 0, \ e^a * \chi(x) = 0 \right\},
\]

\[
\text{Def}_\chi(A) = \frac{\text{MC}_\chi(A)}{\text{gauge equivalence}},
\]

where two solutions of the Maurer-Cartan equation are gauge equivalent if they belong to the same orbit of the gauge action

\[
(\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)) \times \text{MC}_\chi(A) \rightarrow \text{MC}_\chi(A)
\]
given by the formula

\[
(e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm} e^a e^{-\chi(l)}) = (e^l * x, e^{dm} \bullet (-\chi(l))).
\]
The computations of Sections 6 and 7 show that \( \text{MC}_\chi \) and \( \text{Def}_\chi \) are canonically isomorphic to the functors \( \text{MC}_{\tilde{C}(\chi)} \) and \( \text{Def}_{\tilde{C}(\chi)} \) associated with the \( L_\infty \) structure on \( C_\chi \).

**Example 8.1.** Let \( X \) be a compact complex manifold and let \( Z \subset X \) be a smooth subvariety. Denote by \( \Theta_X \) the holomorphic tangent sheaf of \( X \) and by \( N_{Z|X} \) the normal sheaf of \( Z \) in \( X \).

Consider the short exact sequence of complexes

\[
0 \rightarrow \ker \pi \xrightarrow{\chi} A^0_X(\Theta_X) \xrightarrow{\pi} A^0_Z(N_{Z|X}) \rightarrow 0.
\]

It is proved in [19] that there exists a natural isomorphism between \( \text{Def}_\chi \) and the functor of embedded deformations of \( Z \) in \( X \). Therefore, the \( L_\infty \) algebra \( \tilde{C}(\chi) \) governs embedded deformations in this case.

Note that the DGLA \( A^0_Z(\Theta_Z) \) governs the deformations of \( Z \) and that the natural transformation

\[
\text{Def}_{\tilde{C}(\chi)} = \text{Def}_\chi \rightarrow \text{Def}_{A^0_Z(\Theta_Z)},
\]

\[\{\text{Embedded deformations of } Z\} \rightarrow \{\text{Deformations of } Z\}\]
is induced by the morphism in \( \text{DGLA}^2 \) given by the diagram

\[
\begin{array}{ccc}
\ker \pi & \rightarrow & A^0_Z(\Theta_Z) \\
\downarrow \chi & & \downarrow \\
A^0_X(\Theta_X) & \rightarrow & 0.
\end{array}
\]
The next result was proved in [19] using the theory of extended deformation functors; here we can prove it in a more standard way.

**Theorem 8.2.** Consider a commutative diagram of morphisms of differential graded Lie algebras

\[
\begin{array}{ccc}
L_1 & \xrightarrow{f_L} & L_2 \\
\downarrow{\chi_1} & & \downarrow{\chi_2} \\
M_1 & \xrightarrow{f_M} & M_2
\end{array}
\]

and assume that \((f_L, f_M): C_{\chi_1} \to C_{\chi_2}\) is a quasiisomorphism of complexes (e.g. if both \(f_L\) and \(f_M\) are quasiisomorphisms). Then the natural transformation \(\text{Def}_{\chi_1} \to \text{Def}_{\chi_2}\) is an isomorphism.

**Proof.** The map \((f_L, f_M): \tilde{C}(\chi_1) \to \tilde{C}(\chi_2)\) is a linear quasi-isomorphism of \(L_\infty\)-algebras and then induces an isomorphism of the associated deformation functors [11]. □

**Example 8.3.** Let \(\pi: A \to B\) be a surjective morphism of associative \(K\)-algebras and denote by \(I\) its kernel. The algebra \(B\) is an \(A\)-module via \(\pi\); this makes \(B\) a trivial \(I\)-module. Let \(K\) be the suspended Hochschild complex

\[
K = \text{Hoch}^\bullet(I, B)[-1],
\]

Note that the differential \(d\) of \(K\) is identically zero if and only if \(I \cdot I = 0\).

The natural map

\[
\alpha: \text{Hoch}^\bullet(A, A) \to K[1] = \text{Hoch}^\bullet(I, B)
\]

is a surjective morphism of complexes, and its kernel

\[
\ker \alpha = \{ f \mid f(I^\otimes) \subseteq I \}
\]

is a Lie subalgebra of \(\text{Hoch}^\bullet(A, A)\) endowed with the Hochschild bracket. Denote by \(\chi: \ker \alpha \hookrightarrow \text{Hoch}^\bullet(A, A)\) the inclusion. Since \(\chi\) is injective, the projection on the second factor induces a quasiisomorphism of differential complexes

\[
\text{pr}_2: C_{\chi} \to \text{Coker}(\chi)[-1] \simeq K,
\]

where the isomorphism on the right is induced by the map \(\alpha\). Therefore we have a canonical \(L_\infty\) structure (defined up to homotopy) on \(K\). This gives a Lie structure on the cohomology of \(K\), which is not trivial in general: consider for instance the exact sequence

\[
0 \to K\varepsilon \to \frac{\mathbb{K}[\varepsilon]}{(\varepsilon^2)} \xrightarrow{\pi} \mathbb{K} \to 0
\]

and

\[
f \in K^1 = H^1(K), \quad f(\varepsilon) = 1.
\]

Choose as a lifting the linear map \(g: \mathbb{K}[\varepsilon]/(\varepsilon^2) \to \mathbb{K}[\varepsilon]/(\varepsilon^2),\)

\[
g(1) = 0, \quad g(\varepsilon) = 1.
\]

Then

\[
dg(\varepsilon \otimes \varepsilon) = 2\varepsilon
\]

and so \(dg \in \ker \alpha\). Therefore, \((dg, g)\) is a closed element of \(C^1_{\chi}\) representing the cohomology class \(f \in H^1(K)\) and so

\[
[f, f] = \alpha(\text{pr}_2([(dg, g], (dg, g)]_2)) = \alpha([g, dg]).
\]
One computes
\[
[f, f](\varepsilon \otimes \varepsilon) = \pi([g, dg](\varepsilon \otimes \varepsilon))
\]
\[
= \pi(g(dg(\varepsilon \otimes \varepsilon)) - dg(g(\varepsilon) \otimes \varepsilon) + dg(\varepsilon \otimes g(\varepsilon)))
\]
\[
= \pi(g(2\varepsilon) - dg(1 \otimes \varepsilon) + dg(\varepsilon \otimes 1))
\]
\[
= 2,
\]

hence \([f, f] \neq 0\).

On the other hand, if \(A = B \oplus I\) as associative \(K\)-algebra, then the \(L_\infty\) structure on \(K\) is trivial. Indeed, considering \(K[1]\) as a DGLA with trivial bracket, the obvious map
\[
K[1] = \text{Hoch}^\bullet(I, B) \to \text{Hoch}^\bullet(A, A)
\]
gives a commutative diagram of morphisms of DGLAs

\[
\begin{array}{ccc}
0 & \to & \ker \alpha \\
\downarrow & & \downarrow \chi \\
K[1] & \to & \text{Hoch}^\bullet(A, A)
\end{array}
\]
such that the composition \(K \to C_\chi \to K\) is the identity. Therefore the \(L_\infty\)-algebra structure induced on \(K\) is isomorphic to \(\tilde{C}(0 \hookrightarrow K[1])\), which is a trivial \(L_\infty\)-algebra.

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