Numerical results for gauge theories near the conformal window

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Abstract. A novel strong interaction beyond the standard model could provide a dynamical explanation of electroweak symmetry breaking. Experimental results strongly constrain properties of models that realise this mechanism. Whether these constraints are obeyed by any strongly interacting quantum field theory is a non-perturbative problem that needs to be addressed by first-principle calculations. Monte Carlo simulations of lattice regularised gauge theories is a powerful tool that enables us to address this question. Recently various lattice investigations have appeared that have studied candidate models of strongly interacting dynamics beyond the standard model. After a brief review of the main methods and of some recent results, we focus on the analysis of SU(2) gauge theory with one adjoint Dirac fermion flavour, which is shown to have a near-conformal behaviour with an anomalous dimension of order one. The implications of our findings are also discussed.

1. Introduction and motivations
The recent experimental discovery of the Higgs boson [1, 2] is one of the most remarkable confirmations of the validity of the standard model (SM) of particle physics interactions, which embodies electroweak symmetry breaking through the presence of a condensate for the Higgs field. However, despite the spectacular success of this theory (leading for instance to the most accurate predictions for the fine constant structure $\alpha$ [3]), from the theoretical point of view there are clear indications that the SM (or more precisely the Higgs sector of the SM) is not a fundamental theory. One of the problems with the SM Higgs mechanism is related to fine tuning: due to loop contributions, the natural order of magnitude for the mass of the Higgs boson is the mass of the cut-off of the theory (which, ultimately, is the Planck scale $m_P \simeq 10^{19}$ GeV), while experimentally it is found that the Higgs mass $m_H$ is around 125 GeV. If the SM were to be a fundamental theory, such a low value for $m_H$ could only result from very precise cancellations that span seventeen orders of magnitude.

Although the scenario of the SM as a fundamental theory could be related to asymptotic safety in gravity [4], a huge cancellation is generally regarded as the manifestation of the fact that the theory is either incomplete or not fundamental. For instance, the mass of the scalar is protected from acquiring values of the order of the cut-off scale if the theory is supersymmetric, with contributions of bosonic and fermionic particles cancelling each others in the self energy of the Higgs. Another possibility is that the Higgs boson be a composite particle resulting from a new strong interaction. This scenario is referred to as strongly interacting dynamics beyond the standard model or more commonly as technicolor. The basic idea of this framework is that the...
Electroweak symmetry is broken spontaneously by a chiral condensate of a new strong dynamics involving particles that are not in the SM [5–7]. This mechanism of electroweak symmetry breaking is lifted from Quantum Chromodynamics (QCD), in which the quark chiral condensate does break electroweak symmetry, although the phenomenology of this breaking is three orders of magnitude weaker than required in the SM. Although a simple copy of QCD with a larger \( \Lambda \) parameter is incompatible with electroweak precision measurements [8], it is possible to reconcile dynamical electroweak symmetry breaking with experiments if one assumes that the running of the gauge coupling of the novel interaction is slower than in QCD [9–11]. This happens if the theory is close to the conformal window. In this case, approximate scale invariance in the infrared (known as near-conformality) determines the appearance of a light scalar particle in the spectrum [11] that can play the role of the SM Higgs boson. The basic ideas of technicolor are reviewed for instance in [12].

Determining the dynamical features of technicolor-like theories is a non-perturbative problem. As such, it requires approaches that are controlled at strong coupling. In this respect, much progress has been recently achieved using string-gauge duality techniques, starting from first-principle top-down approaches [13–20], as well as in the probe brane approximation [21–30] and utilising bottom-up techniques [31–40]. A valid alternative to analytic treatments is a numerical approach based on the framework of Lattice Gauge Theories. Originally devised for studying low energy phenomena in QCD, the lattice can be proficiently used also for studying strongly interacting dynamics beyond the SM. Since the physics in this case is inherently different from that of QCD, a novel set of numerical tools has been developed to investigate gauge theories near the conformal window and to answer the different type of questions that arise in this context.

In only few years after the original lattice studies of dynamical electroweak symmetry breaking due to a novel interaction [41–45], the field of lattice investigations of near-conformal gauge theories has made an enormous progress in answering crucial questions related to the theoretical viability of strongly interacting dynamics beyond the SM. The purpose of this work is to provide an introduction to the framework of dynamical electroweak symmetry breaking and justify the need for lattice calculations. An overview of lattice techniques is provided in Sect. 3. Sect. 4 and Sect. 5 report on numerical results for two gauge theories (respectively SU(2) with two adjoint Dirac fermions and SU(2) with one adjoint Dirac fermion) that can be used to advance our understanding of dynamical electroweak symmetry breaking due to a new strong force. Finally, Sect. 6 presents concluding remarks and gives an overview of future directions.

2. Novel strong dynamics and electroweak symmetry breaking
The SM is a \( SU(2)_L \otimes U(1)_Y \) gauge theory coupling doublets of left-handed fermions to four gauge bosons. In addition to fermionic matter and gauge bosons, the SM contains a doublet of scalars with a quartic self-interaction potential having minima at a non-perturbative vacuum. The scalar field gets a non-trivial vacuum expectation value (vev), reducing the gauge group to \( U(1)_{EM} \) and providing mass to three gauge bosons. Besides, fermions get mass from the Higgs vev via a Yukawa interaction. This picture has been confirmed to an extraordinary degree of accuracy by electroweak precision measurements performed at LEP and at Tevatron and by the recent discovery of the Higgs boson at the LHC.

However, from a more fundamental point of view, the picture is still unsatisfactory. One of the key questions is the following. Due to quantum fluctuations, the Higgs mass is expected to get corrections of the order of the natural cut-off (Planck scale); what does keep it at around 125 GeV? The answer to this question is not unique. Elegant and appealing scenarios can be
conjectured if one considers the SM as an effective theory that is only valid at scales below the TeV. Following this route, the Higgs field is complemented with or replaced by a whole new sector. This sector should provide mass to the weak interaction gauge bosons and to the SM fermions while at the same time being compatible with the very stringent phenomenological constraints. The recent breakthrough at the LHC imposes the presence of a light scalar, the state that in the SM is identified with the Higgs boson.

Among possible scenarios, one of the proposed possibilities is a new interaction that couples new fermions to new gauge bosons. In QCD, chiral symmetry is spontaneously broken by the formation of a chiral condensate, which is not invariant under $SU(2)_L \otimes U(1)_Y$. Hence, the strong force by itself provides electroweak symmetry breaking. However, the SM strong force is not able to account for the phenomenologically observed electroweak symmetry breaking of the SM, its typical infrared (IR) scale being $\Lambda_{QCD} \simeq 200$ MeV, since in the SM, electroweak symmetry breaking is characterised by an intrinsic scale of about 250 GeV. Hence, going down the strong interaction route to explain electroweak symmetry breaking, we need a new force. Mass to the weak gauge bosons is provided through this strong dynamics as follows. As a consequence of spontaneous symmetry breaking, a set of Goldstone bosons emerge. Among those bosons, three provide the longitudinal component to the $Z$ and $W^\pm$ bosons and the others acquire mass of the order of the strong scale of the theory (around one TeV). In order to provide mass to the SM fermions, a further non-Abelian interaction is required that connects these fermions with the fermions of the new force.

Despite the appeal of this framework and the comfort that we have a deep understanding of chiral symmetry breaking, assuming a QCD-like new strong force generates tension between the need to provide mass to fermions and suppression of flavour changing neutral currents. The tension can be mitigated and hopefully removed if the theory does not have a standard QCD-like dynamics, but is instead characterised by a dynamics with two scales. At the larger scale, $\Lambda_{UV}$, the physics would be perturbative and dominated by the Gaussian fixed point. At scales below $\Lambda_{IR}$, the smaller scale, the dynamics will be chiral symmetry breaking. In an intermediate regime, the coupling might in principle evolve slowly (i.e. it could walk instead of running) [9–11]. Yet another requirement is the anomalous dimension $\gamma_*$ of the chiral condensate being very close to one [49].

Walking can be achieved by adjusting the parameters of the theory (number of flavours and of colours) so that the model is still confining, but in the vicinity of the onset of the conformal window. It is worth at this point to remind the possible IR behaviours (phases) of a non-Abelian gauge theory. In the perturbative region, the evolution of the gauge coupling $g$ with the renormalisation scale $\mu$ is given by

$$\beta(g^2) = \mu \frac{dg}{d\mu} = -b_0 g^3 - b_1 g^5 + \ldots ,$$

and the two universal (i.e. renormalisation scheme independent) coefficients $b_0$ and $b_1$ can be computed in perturbation theory. For a $SU(N)$ gauge theory with $N_f$ flavours of massless fermions in the representation $R$ of the gauge group, one obtains

$$b_0 = \frac{1}{(4\pi)^2} \left( \frac{11}{3} N - \frac{4}{3} T_R N_f \right), \quad b_1 = \frac{1}{(4\pi)^4} \left[ \frac{34}{3} N^2 - \frac{20}{3} N T_R N_f - 4 \frac{N^2 - 1}{d_R} N_f \right],$$

where $T_R$ and $d_R$ are respectively the normalisation of the trace and the dimension of the representation $R$. Physically interesting theories are those that display asymptotic freedom. This property implies $b_0 > 0$. In QCD, also $b_1$ is bigger than zero, and this gives rise to confinement and chiral symmetry breaking. However, for particular choices of $N_f$ and of $R$, it is possible to obtain $b_1 < 0$ while retaining asymptotic freedom. This gives rise to a zero
in the beta function $\beta(g^2)$, which determines the existence of a fixed point in the IR. For the scenario to be consistent, one needs the fixed point coupling $g_*$ to be small. A similar IR fixed point is known as the Caswell-Banks-Zaks fixed point [50, 51]. It is believed (and confirmed in supersymmetry, where analytic calculations are possible) that IR conformality survives also for values of $N_f$ at which the coupling is stronger, up to a lower value $N_{c,l}$ at which the dynamics becomes confining and chiral symmetry breaking like in QCD. At fixed $N$, the values of $N_f$ such that $N_{c,l} \leq N_f \leq N_{c,u}$, where the upper value corresponds to the loss of asymptotic freedom, identifies theories with an IR fixed point. This region is called the conformal window.

Since walking can happen below the onset of the conformal window, the problem is characterised by a strong coupling and requires an investigation method that is controlled in the non-perturbative regime. Below, we shall use lattice regularisation followed by Monte Carlo simulations. In order for numerical methods to work, a non-zero fermion mass is needed. Results for the massless limit can then be obtained by extrapolating data for non-zero fermion mass to the chiral point. In order to understand lattice results, in the remainder of this section we review what happens to IR conformal gauge theories when they are deformed with a small mass term.

We start by reviewing the behaviour of the QCD spectrum as the fermion mass $m$ is varied. At $m = \infty$, mesons decouple and we have a Yang-Mills spectrum. If the mass $m$ is lowered, we are in a heavy quark regime, in which mesons are more massive than glueballs and scalar and the pseudoscalar mesons are nearly degenerate. If we lower the mass, when we hit the chiral scale, the pseudoscalars will start to emerge as the pseudo-Goldstone bosons of chiral symmetry breaking and get lighter, while the rest of the spectrum will settle at the QCD scale. Finally, in the chiral limit the pseudoscalars become massless. This is sketched in Fig. 1a.

In the massless case, for a theory with an IR fixed point, if we assume a regular behaviour for the renormalisation group (RG) functions, we find

$$g \to g_* : \begin{cases} \beta(g) \approx \beta_*(g - g_*) \\ \gamma(g) \approx \gamma_* \end{cases},$$

where $\gamma_*$ is the fixed point value of the chiral condensate anomalous dimension $\gamma(g)$. In order to understand what happens when we deform the theory with a mass term, we remark that the mass is a relevant direction in the RG sense. As a consequence, any non-zero mass term determines the loss of IR conformality and the appearance of a confining and chiral symmetry breaking spectrum. The running of the mass with the renormalisation scale $\mu$ from a reference
scale $\mu_0$ is given by

$$m(\mu) = m(\mu_0) \exp \left\{ - \int g(\mu) \frac{\gamma(z)}{g(\mu_0)} \beta(z) \, dz \right\} \equiv Z(\mu, \mu_0, \Lambda) m(\mu_0) .$$

We can define a renormalised mass $M$ from the condition $m(M) = M$. A large value of $M$ destroys conformality and the theory looks like Yang-Mills with heavy matter. In this case, the mesons will have mass $m_{\text{mes}} \simeq 2M$ and the glueballs $m_{\text{glue}} \simeq B_{\text{glue}} \Lambda$, with $\Lambda$ the Yang-Mills scale and $B_{\text{glue}}$ some numeric coefficient not necessarily small (in QCD, for instance, the mass of the lightest glueball is about eight times $\Lambda_{\text{QCD}}$). We are interested in the opposite regime, in which $M \ll \Lambda$. An analytic calculation can be performed near the Caswell-Banks-Zaks fixed point [52]. The central result is the emergence of an IR scale $\Lambda_{\text{IR}}$ related to $M$ but exponentially suppressed with respect to it:

$$\Lambda_{\text{IR}} = M e^{-\frac{1}{2} \frac{b_{\text{YM}}}{g^2} \mu_0} \ll M \ll \Lambda .$$

(4)

$\Lambda_{\text{IR}}$ controls an effective large-distance Yang-Mills spectrum, with the fermions being in a heavy quark regime. At energies much lower than $M$, the spectrum is that of a pure Yang-Mills theory with scale $\Lambda_{\text{IR}}$. Mesons are bound states of the quark-antiquark pairs interacting via the YM static potential, the bound energy is small with respect to the mass of the fermions, and the correction to the potential due to quark-antiquark pair creation are negligible. As a result, mesons are effectively quenched and have a mass that is much larger than the mass of the lowest-lying glueballs. As the mass $M$ is reduced, the IR physics is always the same, provided that all the masses are rescaled with $M$.

One can show that spectrum of a particle $X$ varies with the constituent mass $m$ and with the anomalous dimension of the condensate at fixed point as follows:

$$m_X = A_X \mu \frac{\gamma}{1+\gamma} m(\mu) \frac{1}{1+\gamma} .$$

(5)

On a lattice of spacing $a$, choosing $\mu = a^{-1}$ gives

$$a m_X = A_X (am)^{\frac{1}{1+\gamma}} ,$$

(6)

with $A_X$ independent of $m$ (at the leading order, in the chiral region). As a consequence, ratios of masses are constant as a function of $m$. Note that this is in stark contrast with QCD, for which the ratio of the mass of any confining state over the pseudoscalar mass diverges in the chiral limit.

In summary, for a Caswell-Banks-Zaks fixed point, one expects that for large constituent masses the scenario is still that of Yang-Mills plus heavy quarks. As the mass deformation regime is reached, the spectrum will scale uniformly towards the chiral limit, keeping constant mass ratios. This locking of the spectrum arises in the heavy quark regime (Fig. 1b), which will be a feature of the mass-deformed theory regardless of how small the mass deformation is.

If we leave the perturbative fixed point, analytic predictions are not possible. However, we still expect that the general feature of the perturbative description (i.e. constant spectral ratios) characterises the mass deformed regime, but the locking will happen when the spectrum of the large distance theory has left the heavy quark regime. In particular, we might expect that for this IR theory the pseudoscalar mesons emerge as pseudo-Goldstone bosons related to chiral symmetry breaking, as in QCD, or, more interestingly, they can have a mass that is comparable to that of the lightest glueball (Fig. 1c).

In the context of our discussion of confining and chiral symmetry breaking gauge theories versus IR conformal gauge theories, walking can be seen as a cross-over phenomenon: a theory
is walking if for an intermediate regime of energies it presents features that one would attribute to a IR conformal gauge theory, but as the energy is reduced, eventually becomes confining and chiral symmetry breaking. Because of the behaviour at intermediate scales, walking is also referred to as near-conformality.

Since the analysis we have performed so far is only semi-quantitative, our discussion leaves unanswered important fundamental questions, starting from whether this scenario is realised at all in a concrete gauge theory. Those issues will be addressed via Monte Carlo calculations on the lattice.

3. Lattice setup

Before discussing our results, we briefly review how gauge theories are formulated on a lattice. The advantage of the lattice formulation over the continuum one is that the former is amenable to Monte Carlo numerical simulations, which can be used to study non-perturbative features of the theory.

Consider a gauge theory with gauge group $G = SU(N)$ and $N_f$ fermionic fields in the representation $R$ of $G$. A formulation of the theory that preserves gauge invariance on a Euclidean four-dimensional grid with an isotropic spacing $a$ can be obtained as follows. If $A_\mu(x)$ is the vector potential, we define the link variable $U_\mu(i)$ on the grid point $i = x/a \equiv (i_0, \ldots, i_3)$ as

$$U_\mu(i) = \text{Pexp} \left( ig \int_{ai}^{a(i+\hat{\mu})} A_\mu(x) \, dx^\mu \right),$$

(7)

with $\hat{\mu}$ the versor in direction $\mu$ and $g$ the gauge coupling. $U_\mu(i)$ is naturally associated with the link $(i; \hat{\mu})$ that joins the point $i$ with the point $i + \hat{\mu}$. The variable corresponding to the negative direction $-\mu$ is given by

$$U_{-\mu}(i) = (U_\mu(i))^\dagger \equiv U^\dagger_\mu(i).$$

(8)

Under a gauge transformation $G \in G$, from the transformation law of the field $A_\mu$ and the definition of $U_\mu(i)$ given in (7), one immediately obtains the transformation

$$U_\mu(i) \rightarrow G(i)U_\mu(i)G(i)\dagger(i + \hat{\mu}).$$

(9)

As a consequence, the parallel transport of the link variable along an elementary square of the lattice (known as the plaquette)

$$U_{\mu\nu}(i) = U_\mu(i)U_\nu(i + \hat{\mu})U^\dagger_\mu(i + \hat{\nu})U^\dagger_\nu(i)$$

(10)

transforms in the adjoint representation of $G$:

$$U_{\mu\nu}(i) \rightarrow G\dagger(i)U_{\mu\nu}(i)G(i).$$

(11)

On the lattice, $U_{\mu\nu}(x)$ plays the same role as the field tensor $G_{\mu\nu}(x)$ in the continuum.

The simplest form for the lattice action of gauge fields is given by the Wilson action

$$S_W = \beta \sum_{i, \nu < \mu} \text{Tr} \left[ 1 - \text{Re} \left( U_{\mu\nu}(i) \right) \right],$$

(12)

where $\beta = 2N/g^2$, $\text{Tr}$ indicates the trace and $\text{Re}$ is the real part. Note that the Wilson action is written only in terms of the trace of (the real part of) the plaquette, and, as a consequence of
Eq. (11), is gauge-invariant, as it should be. It is possible to show that in the naive continuum limit $a \to 0$

$$S_W = -\frac{1}{2g^2} \sum_i \left[ \text{Tr} \left( G_{\mu \nu} G^{\mu \nu} \right) a^4 + \mathcal{O}(a^6) \right] \simeq S_{\text{cont}} + \mathcal{O}(a^2) \ ,$$

(13)
i.e. the lattice action differs from the continuum action by terms that are of order $a^2$. A weak coupling calculation shows that the subleading terms are irrelevant in the RG sense as the cut-off $a$ is removed. Hence, the Wilson action is in the same universality class as the continuum action, i.e. the two actions describe the same continuum physics.

With the gauge part of the action formulated in terms of variables defined on links, gauge invariance of bilinears involving fermion fields is easy to implement. The naive discretisation of the derivative term in the Dirac action is done in terms of finite differences. In the lattice action, when the natural choice of defining fermion fields on lattice sites is performed, this determines the appearance of bilinear terms involving spinors evaluated on nearest-neighbour lattice points, i.e. terms of the form $\bar{\psi}(i)\psi(i + \hat{\mu})$. Here $\psi$ is the lattice spinor, which corresponds to the continuum spinor multiplied by $a^{3/2}$ to make it dimensionless. Moreover, for simplicity we will start by considering fermions in the fundamental representation. In the presence of a gauge field, the fermion bilinear terms introduced above are modified with the insertion of the link variable living on the lattice link joining the relevant sites:

$$\bar{\psi}(i)\psi(i + \hat{\mu}) \to \bar{\psi}(i)U_\mu(i)\psi(i + \hat{\mu}) \ ,$$

(14)

with the resulting term being gauge invariant if $\psi(i) \to G(i)\psi(i)$ and $\bar{\psi}(i) \to \bar{\psi}(i)G^\dagger(i)$, as a straightforward discretisation of gauge transformations would suggest.

The final building block of our lattice gauge theory is the action of a free fermion field. On the continuum, this can be expressed as

$$S_f = \int \bar{\psi}(x)D(x, y)\psi(y)d^4xd^4y \ ,$$

(15)

where the Dirac operator $D$ is the (diagonal) bilinear form

$$D(x, y) = \delta^4(x - y) \left( i\partial_y - m \right) \ ,$$

(16)

with $m$ being the Lagrangian fermion mass. With the discretisation of the derivative, this takes the form of a band-diagonal matrix, in which only diagonal terms and terms connecting nearest-neighbours are different from zero. However, a straightforward discretisation of the Dirac operator leads to the notorious fermion doubling problem: for each lattice flavour, sixteen degenerate flavours are generated in the continuum limit. The doubling problem is a consequence of the Nielsen-Ninomya no-go theorem, which states the impossibility of having a lattice action that respects chiral symmetry, has only nearest-neighbour interactions and is free from doublers, showing that the preservation of these three properties is intimately related to the realisation of the Lorentz group, explicitly broken by the lattice discretisation. Hence, in order to have a lattice action with a continuum limit free of doublers, one needs to give up either ultralocality (i.e. the fact that interactions have a finite radius) or break explicitly chiral symmetry at $m = 0$. At this stage, it is worth stressing that regardless of the discretisation strategy chosen, in the continuum limit one must recover the original theory. Hence, different discretisations will be equivalent near the continuum limit. From the computational point of view, however, there could be advantages in choosing a formulation over the other. Below, we focus on the Wilson formulation, which explicitly breaks chiral symmetry at the Lagrangian level. Chiral symmetry is then recovered by tuning the unrenormalised fermion mass to a critical value that is itself.
an output of the calculation. The advantage of this approach over others relies on smaller computational costs while keeping the physics close to that of the continuum formulation.

In the Wilson formulation, the lattice Dirac operator in the presence of gauge fields is given by

$$ D(i,j) = \delta_{i,j} - \kappa \left( (1 - \gamma_\mu) R[U_\mu(i)] \delta_{j,i+\bar{\mu}} (1 + \gamma_\mu) R[U_\mu^\dagger(i-\bar{\mu})] \delta_{j,i-\bar{\mu}} \right), $$

where $k = 1/(8 + 2am)$ is called the hopping parameter and we have allowed for the fermion field to be in a generic representation $R$ of $G$ by indicating by $R[U]$ the element $U \in G$ expressed in the representation $R$. Note that when $m = 0$, $\kappa = 1/8$, which leads to explicit breaking of the chiral symmetry. Because of this breaking, the mass gets additively renormalised and chiral symmetry is recovered at a value $k_c$ of the hopping parameter that needs to be determined in the simulation. With this definition of the Dirac operator, we can write the lattice fermion action for $N_f$ degenerate fermion flavours described by the spinors $\psi_1, \ldots, \psi_{N_f}$ as

$$ S_f = \sum_{l=1}^{N_f} \sum_{i,j} \bar{\psi}_l(i) D(i,j) \psi_l(j) $$

and the full action as

$$ S = S_f + S_W, $$

with $S_W$ given in Eq. (13).

After performing the Grassmann integrals over the fermion fields, the path integral of the theory reads

$$ Z = \int \langle DU_\mu(i) \rangle (\det(D))^{N_f} e^{-S_W}, $$

where $\det(D)$ is the determinant of the Dirac operator and $\langle DU_\mu(i) \rangle$ the path integral measure of the link variables. In the path integral formulation, the vacuum expectation value of an operator $O \equiv O(\psi_1, \ldots, \psi_{N_f}; U_\mu)$ is given by

$$ O = \frac{1}{Z} \int \langle D\psi(i) \rangle \langle D\bar{\psi}(i) \rangle \langle DU_\mu(i) \rangle O(\psi_1, \ldots, \psi_{N_f}; U_\mu) e^{-S}, $$

where we have introduced the fermion path integral measure $\langle D\psi(i) \rangle \langle D\bar{\psi}(i) \rangle$.

If the theory is formulated on a finite lattice, expressions like (21) give rise to well-defined integrals that can be evaluated efficiently using Monte Carlo techniques: after integrating out the fermion fields and mapping the gauge fields into a Markovian process, the latter generates configurations distributed according to the path integral measure weighted with the Boltzmann term $e^{-S+N_f \log \det(D)}$, which is then interpreted as a probability measure. Since these configurations carry already the correct information about the probability measure, observables are determined as simple averages over the Markovian process. With $C^{(i)} \equiv \{ U_\mu^{(i)} \}$ the realisation of the fields at step $i$ of the Markov process, we define $O_N$ as the estimate of $\langle O \rangle$ for a Markov chain of length $N$:

$$ O_N = \frac{1}{N} \sum_{i=1}^{N} O(C^{(i)}). $$
It can be proved that

$$\langle O \rangle - O_N = O(N^{-1/2}) ,$$

(23)
i.e. $O_N$ converges to $O$ in the limit $N \to \infty$, with the difference being of order $1/\sqrt{N}$ for finite $N$. Hence, this approximation is controlled. Moreover, a statistical error (computed as the standard deviation of $O$ over the probability measure) gives the confidence interval of the measurement.

Following the ideas outlined above, in a numerical study of a lattice gauge theory one first investigates the value of an observable $O$ at fixed parameters $k$ and $a$ on a fixed-size grid. Then, the calculation is repeated, extrapolating first to infinite volume, then to the chiral limit and finally removing the cut-off $a$ (which in an asymptotically free gauge theory amounts to sending $\beta \to \infty$).

A prerequisite for the lattice formulation is the Wick rotation, to move from Minkowki to Euclidean space. In fact, all the considerations presented so far have been performed on a lattice with Euclidean metric. While in principle it is possible to reconstruct Minkowskian correlation functions from the Euclidean ones, a wide class of observables (e.g. spectral masses) can be accessed directly in Euclidean space. Among the observables we are interested in, a central role will be played by masses of bound states, which are extracted from correlation functions of operators transforming with the quantum numbers of interest. If $C(\tau)$ is a correlation function of operators of quantum numbers $J^{PC}$, from general principles we know that

$$C(\tau) \approx e^{-\tau M_{J^{PC}}} ,$$

(24)

where $M_{J^{PC}}$ is the lowest mass in the sector with quantum numbers $J^{PC}$. Hence, in order to extract masses, one measures expectation values of correlators in lattice simulations and fit the expected asymptotic behaviour.

Since in the Wilson formulation of lattice fermions the Lagrangian mass is additively renormalised, it is useful to define a mass that is only multiplicatively renormalised. This can be done through the axial Ward identity. The corresponding mass, $m_{PCAC}$, is zero in the chiral limit, and can hence be used to understand how the massive case approaches the massless limit. For this reason, we shall use $m_{PCAC}$ instead of the Lagrangian fermion mass in our analysis in the following section. As a function of $m_{PCAC}$, we can determine the anomalous dimension from the spectrum of the theory using the relation provided in Eq. (6). When dealing with a lattice of finite size, it is important to keep into account the finite extension $L$ in the calculation of scaling behaviours. As a result [53], Eq. (6) is replaced by

$$Lam_X = F_X \left((aLm_{PCAC})^{1/\gamma} \right) ,$$

(25)

where $F_X$ is a universal function of the scaling variable $x = L(aLm_{PCAC})^{1/\gamma}$. The practical meaning of this expression is that values of $m_X$ at different lattice sizes $L$ are described by a universal curve of $x$. Due to striking similarities with Monte Carlo investigations of second order phase transitions (which IR conformality is closely related to), this method is called Finite Size Scaling (FSS).

Having introduced the generalities of Lattice Gauge Theories, Monte Carlo simulations and the analysis techniques we shall be using to understand the IR conformal regime, below we examine some non-perturbative results for gauge theories that are relevant for our understanding of strongly interacting dynamics beyond the SM, using as a guide the discussion of Sect. 2.
4. SU(2) Gauge Theory with two adjoint Dirac flavours

One of the first theories potentially relevant for dynamical electroweak symmetry breaking [54] that was studied numerically is the SU(2) gauge theory with two adjoint fermions. After the first explorations [41, 44, 55, 56], systematic investigations of the spectrum were performed [57–60], which pointed out how the general behaviour of bound state masses as a function of the fermion mass is qualitatively different from the one observed in QCD. In particular, it was noticed in [57] and confirmed with a more detailed analysis in [59, 60], that the behaviour of the spectrum in the chiral limit is compatible with the scenario presented in [52], where in the vicinity of a weakly coupled IR fixed point a dynamical Yang-Mills scale is generated that is exponentially suppressed with respect to the renormalised fermion mass.

The results of an investigation of the spectrum of the theory are plotted in Fig. 2. This is reminiscent of the scenario of locking when the locking scale is much larger than the Yang-Mills scale (see Fig. 1b). Once again, this is compatible with the IR fixed point being of the Caswell-Banks-Zaks type, which determines the emergence of a spectrum that shares the qualitative features of that of a QCD-like theory with heavy quark. Note that this spectrum arises in the IR independently of the Lagrangian fermion mass, which, although in a non-trivial way, in this case plays the sole role of setting the scale of the effective IR Yang-Mills theory. Another indication pointing in this direction is the degeneracy of the pseudoscalar meson and the vector meson [57, 60].

From a more quantitative perspective, precise numerical data allow us to extract the value of the anomalous dimension of the chiral condensate. The first measurement for this theory was provided in [53] using a FSS technique. Subsequently, other analyses using this and alternative methods have been performed. A summary of results obtained using FSS, the Schrödinger Functional (SF) technique, Monte Carlo Renormalisation Group (MCRG) methods and the Dirac Mode Number Scaling (MNS) is provided in Tab. 1, together with reference to the original works that describe in details those methods and the corresponding results. Within the variability of the results (for which the most recent ones have to be considered as more reliable, since they better reflect the evolution of our understanding of the underlying tools and techniques), the clear feature that emerges is that the anomalous dimension appears to be well below one, in stark contrast with the phenomenological requirements. Hence, the conclusion that can be drawn is that, while displaying an interesting IR behaviour that is clearly distinguished from that of QCD-like theories and in principle promising as a mechanism

Figure 2. The spectrum of SU(2) gauge theory with two Dirac adjoint flavours as a function of the PCAC mass at $\beta = 2.2$ (from [57]).
Table 1. Summary of numerical results for the anomalous dimension $\gamma_*$ in SU(2) gauge theory with two adjoint Dirac fermions. For comparison, some analytical estimates are reported in the last two lines. For details about methods and analyses, we refer to the quoted literature.

| Method    | $0.05 < \gamma_* < 0.25$ |
|-----------|---------------------------|
| FSS [53]  | $0.05 < \gamma_* < 0.56$ |
| SF [61]   | $0.05 < \gamma_* < 0.20$ |
| FSS [60]  | $0.22 \pm 0.06$          |
| MCRG [62] | $0.26 < \gamma_* < 0.60$ |
| SF [63]   | $0.31 \pm 0.06$          |
| FSS [64]  | $0.51 \pm 0.16$          |
| MNS [65]  | $0.37 \pm 0.02$          |
| MNS [66]  | $0.38 \pm 0.02$          |
| Perturbative 4-loop [67] | 0.500 |
| All-orders hypothesis [68] | 0.46 |

of dynamical electroweak symmetry breaking, from a quantitative point of view the model is not phenomenologically viable. The small anomalous dimension is compatible with the qualitative observation that the emerging Yang-Mills spectrum has a constituent quark mass well above the Yang-Mills scale, a feature that characterises IR conformal theories with a fixed point of the Caswell-Banks-Zaks type [52]. Despite the small anomalous dimension, the model has proven to be very interesting from the conceptual point of view, as it shows at work some of the key ideas of technicolor.

To our knowledge, SU(2) with two adjoint Dirac fermions is the first gauge theory that has been shown to display an IR conformal or near-conformal behaviour, the two scenarios being very hard to disentangle in a lattice calculation. Over the years, the model has been object of wide attention and more refined studies [69–74]. Currently, there seems to be no doubt in the field that the IR behaviour of the theory is definitely different from that of QCD. Despite the little relevance for phenomenology, the theory is still the subject of intensive numerical study, since it can serve as a solid toy model to understand (near-)conformal gauge theories. Ongoing investigations [66] are focussing on a better understanding of finite size effects, which play a key role in numerical studies of IR conformal systems. The first indications seem to confirm the broad picture that emerged on smaller lattices. The next step will be to perform a systematic exploration towards the continuum limit. This could help to understand whether the different extracted values for the anomalous dimension of the condensate could be due to scaling violations related to the fact that the gauge coupling is marginal in the RG sense [75].

5. SU(2) Gauge Theory with one adjoint Dirac flavour

The model discussed in the previous section shows at work some of the ideas that motivate us to study strongly interacting dynamics beyond the SM as a mechanism of electroweak symmetry breaking. However, it fails the crucial test of phenomenological viability, due to the small anomalous dimension of the condensate. From a theoretical point of view, we can ask ourselves whether models with an anomalous dimension that is compatible with phenomenological requests do exist. The general agreement is that anomalous dimensions of order one can arise near the onset of the conformal window. In the case of SU(2) with adjoint Dirac fermions, given that

\footnote{For this reason, exceeding on the side of caution, we prefer to talk about (near-)conformal behaviour, indicating with this expression that both conformality and near-conformality are in fact possible.}
the case $N_f = 2$ is (near-)conformal, we can ask what happens when we take one single flavour. A possibility is that the theory be confining in a QCD-like fashion. However, it is also possible that it be walking or stay conformal. If the IR dynamics is different from that of QCD, we expect to observe an anomalous dimension that is larger than that of the two flavour case and, if the theory is near the onset of conformality, the anomalous dimension should be of order one.

We can test these expectations by performing numerical simulations of the one-flavour model [76, 77]. We focus our attention on the mass spectrum of the theory at finite fermion mass, aiming at extrapolating to the chiral limit to probe the IR behaviour of the theory for massless constituent fermions. We first note that the theory has a non-trivial chiral symmetry breaking pattern that reduces the global symmetry from SU(2) to a residual SO(2). This chiral symmetry breaking pattern can be exposed by rewriting the Dirac flavour in terms of two Majorana or two Weyl components. We refer to [76,77] for details. The classification of fermion bilinears in terms of their transformation law under the Lorentz group, parity and the residual SO(2) group (which leads to the conservation of a quantum number that can be taken as the baryonic charge) shows that the spectrum contains mesons (i.e. bound states of two fermions with zero baryon number), baryons (or diquarks), which are bound states of two fermions with baryon number $B = \pm 2$ and spin-$\frac{1}{2}$ states (single fermions dressed with gluons) in addition to the usual Yang-Mills glueball spectrum. The Goldstone bosons emerging from the spontaneous breaking of the chiral symmetry are scalar baryons. For a full classification of the spectrum, we refer to [76].

The spectrum of the theory is reported in Figs. 3a and 3b in units of the square root of the string tension $\sigma$. These figures show that mass ratios are constant towards the massless limit. As previously seen, this is a typical signature of IR (near-)conformality. We also note that the scalar baryon (shown in Fig. 3b) is heavier than the scalar meson, which in turn has the same mass as the scalar glueball (Fig. 3a). The would-be Goldstone boson being not the lightest state of the theory when the mass goes to zero is incompatible with the scenario of chiral symmetry breaking, while it is possible in an IR (near-)conformal gauge theory. As already noticed, the scalar glueball and the scalar meson are degenerate. This indicates that those states (which are notionally distinct in QCD) are in fact the same state, which emerges as the lightest scalar state either by computing scalar correlators in the mesonic sector or by computing correlators of gluonic operators that are invariant under spatial rotations and parity.
Figure 4. (a) The mass of the $B = 2$ baryon after rescaling of the data assuming $\gamma_* = 1$; (b) anomalous dimension $\gamma_*$ determined by studying the Dirac operator mode number scaling in a calculation performed on a $48 \times 24^3$ lattice at $am = 1.523$ and $\beta = 2.05$. From [76].

A FSS analysis of the spectrum can be performed to get a handle on the anomalous dimension of the condensate. In Fig. 4a we plot the combination $a L m_{2+}$ (with $m_{2+}$ the mass of the $B = 2$ baryon measured on a lattice of extension $2L \times L^3$, where $2L$ is the temporal size and $L$ the spatial size) as a function of $L (am_{\text{PCAC}})^{1/(1+\gamma_*)}$ assuming $\gamma_* = 1$. The collapse of the data onto a single curve shows that the anomalous dimension takes values near one. Inspection of similar plots obtained by varying $\gamma_*$ suggests that $0.9 \leq \gamma_* \leq 1$. The anomalous dimension can be studied also using the MNS technique described in [65]. Fig. 4b shows a typical results of fits of the anomalous dimension when varying the fit interval (whose upper value is the abscissa of the plot and various lower values are represented by the different points at fixed abscissa). For reasonable values of the upper value (between 0.35 and 0.45) the fit gives stable results that can be summarised as $\gamma_* = 0.925(25)$, a value that is compatible with that extracted using FSS techniques.

The lattice results reported in this section (which should still be regarded as exploratory, as simulations have been performed for a single $\beta$ on a limited set of volumes and fermion masses) suggest that (a) SU(2) gauge theory with one adjoint Dirac flavour is (near-)conformal; (b) the theory has an anomalous dimension of the order of that required by phenomenology; (c) the lightest particle is the scalar particle. From the conceptual point of view, these three properties are very important, because they show from first principles that near the conformal window anomalous dimensions of order one do arise and the scalar (i.e. the would be Higgs particle of the SM) is naturally light. Although this theory is still of no phenomenological interest, as the symmetry breaking pattern can account for the mass of only two SM gauge bosons, our results provide a numerical proof that the key ideas of strongly interacting dynamics beyond the SM are realised in concrete systems.

6. Conclusions
In this work, we have presented lattice results aimed at answering some of the most crucial conceptual questions still open in the framework of strongly interacting dynamics beyond the SM. We have provided numerical evidence of the existence of IR (near-)conformal gauge theories by anticipating their spectral signature and finding confirmation for it in candidate theories using first-principle Monte Carlo calculations. The investigation of SU(2) gauge theory with one adjoint Dirac fermion suggests that there exist strongly interacting theories that are (near-)conformal and have an anomalous dimension of order one and a light scalar field. The
large anomalous dimension is needed to fulfil the phenomenological constraints provided by electroweak precision measurements, while the light scalar is identified with the Higgs boson. The fact that LHC experiments have not seen other new particles in addition to the Higgs is compatible with the scalar being the lightest particle in the spectrum of the novel strong dynamics. With the LHC restarting its operations, the quest for a dynamical explanation of electroweak symmetry breaking might soon find an answer. While none of the models studied so far can provide a realistic phenomenological description of electroweak symmetry breaking due to a new strong force, lattice investigations (which include studies of other theories that we have been unable to cover here, see e.g. [42, 43, 45] and more recently [78–82] for other approaches and results in different models) have greatly contributed to a deeper understanding of crucial non-perturbative features of the framework. Together with other conceptual advances, they have confirmed strongly interacting dynamics beyond the SM as one of the most robust sets of ideas that could explain electroweak symmetry breaking [83].

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