Abstract

We show that (i) the standard fine structural properties for premice follow from normal iterability (whereas the classical proof relies on iterability for stacks of normal trees). We also show that (ii) every mouse which is finitely generated above its projectum, is an iterate of its core.

That is, let \( m < \omega \) and let \( M \) be a \( m \)-sound, \( (m, \omega_1 + 1) \)-iterable premouse. Then (i) \( M \) is \( (m + 1) \)-solid and \( (m + 1) \)-universal, \( (m + 1) \)-condensation holds for \( M \), and if \( m \geq 1 \) then \( M \) is super-Dodd-sound, a slight strengthening of Dodd-soundness. And (ii) if there is \( x \in M \) such that \( M \) is the \( r\Sigma^m_{m+1} \)-hull of parameters in \( \rho^M_{m+1} \cup \{x\} \), then \( M \) is a normal iterate of its \( (m + 1) \)th core \( C = \mathcal{E}_{m+1}(M) \); in fact, there is an \( m \)-maximal iteration tree \( T \) on \( C \), of finite length, such that \( M = M^*_T \), and \( i^*_T \) is just the core embedding.

Applying fact (ii), we prove that if \( M \models \text{ZFC} \) is a mouse and \( W \subseteq M \) is a ground of \( M \) via a forcing \( P \in W \) such that \( W \models "P \) is strategically \( \sigma \)-closed" and \( M|^{\aleph_1}_{\aleph_1} \in W \) (that is, the initial segment of \( M \) of height \( \aleph_1 \) is in \( W \)), then \( M \subseteq W \).

And if there is a measurable cardinal then there is a non-solid premouse.

The results hold for premice with Mitchell-Steel indexing, allowing extenders of superstrong type to appear on the extender sequence.

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1 Introduction

1.1 Background and goals

The large cardinal hierarchy constitutes a central focus in set theory. Our understanding of large cardinals has been greatly enriched through the discovery and study of fine structural inner models which exhibit them. These models are highly organized and admit a precise analysis, enabling a detailed understanding of their combinatorial properties, far beyond what can be achieved just working in ZFC plus large cardinals. They provide our key tool for establishing consistency strength lower bounds for various kinds of principles, particularly through \textit{core model} arguments. And they have been shown to arise naturally in other contexts, particularly in descriptive set theory, where they are intimately connected with models of determinacy.

The models come in two main varieties: \textit{pure extender mice} and \textit{strategy mice}. From now on, this article will deal exclusively with the pure extender variety, which we will refer to as \textit{mice}. Before we describe the aims of the paper, we give a brief outline of the basic features of mice.

Large cardinals at the level of measurability and beyond are typically exhibited by some kind of elementary (or partially elementary) embedding $j : P \to Q$ between structures $P, Q$ for the language of set theory (usually $P, Q$ will be transitive, and often $P = V$, the entire set-theoretic universe). Mice $M$ are set-theoretic structures having a transitive
universe of the form $\mathcal{J}_\alpha[\mathcal{E}]$ where $\mathcal{E}$ is a sequence of extenders with special properties.\footnote{A mouse has the form $M = (|M|, \in, \mathcal{E}, F)$ where $|M| = \mathcal{J}_\alpha[\mathcal{E}]$ is the universe of $M$, $\mathcal{E}$ is the aforementioned sequence of extenders, and $F \subseteq |M|$ is another extender.}

Extenders are essentially fragments of elementary embeddings. If $\beta \in \text{dom}(\mathcal{E})$ then $\beta < \alpha$ and $E = \mathcal{E}_\beta$ is an extender over $P = \mathcal{J}_\beta[|\beta|]$, and $E$ is essentially equivalent to the corresponding ultrapower embedding $i_E : P \to Q$ where $Q = \text{Ult}(P, E)$ is the ultrapower of $P$ by $E$. In some cases, $E$ will in fact be a total extender over the universe $\mathcal{J}_\alpha[\mathcal{E}]$ of $M$, and then $\text{Ult}(M, E)$ can be formed, and $E$ determines an ultrapower embedding $i_E^M : M \to \text{Ult}(M, E)$. But in general this can fail, and then we say that $E$ is only a partial extender (in the sense of $M$). We also write $E^M = \mathcal{E}$. A mouse also comes equipped with a further predicate $F \subseteq \mathcal{J}_\alpha[\mathcal{E}]$, and we write $F^M = F$ and $E^M = E^M \cap (F)$. The predicate $F$ is either $\emptyset$ or an extender over $\mathcal{J}_\alpha[\mathcal{E}]$. (So $F$ would be $\mathcal{E}_\alpha$, if we were to extend $M$ above height $\alpha$.)

The keys to our understanding of mice stem from their iterability and their fine structure. These keys are central to our analysis of the internal combinatorial properties of mice, the relationship between mice and the wider set-theoretic universe, and also their canonicity. A premouse is transitive and satisfies the same first order properties as does a mouse, but for a premouse there is no iterability requirement.

The iterability of $M$ requires, roughly, that we can transfinitely iterate the process of forming ultrapowers via extenders, always resulting in transitive models. It is defined precisely in terms of the iteration game, which is defined basically as follows, glossing over some details. The game is played between two players, I and II, and runs through some ordinal number $\lambda$ of stages. A run of the game produces a sequence $(M_\alpha)_{\alpha < \lambda}$ of premouse, amongst other things. We start with $M_0 = M$. Given $M_\alpha$ where $\alpha + 1 < \lambda$, player I selects some $E_\alpha$ from the extender sequence $E^M_\alpha$ of $M_\alpha$, and also some $\beta \leq \alpha$ and some $\gamma \leq \text{OR} \cap M_\beta$ such that $\text{Ult}(M_\beta|\gamma, E_\alpha)$ makes sense, and we define $M_{\alpha+1}$ to be this ultrapower. We simultaneously define an associated tree order on $\lambda$, setting here $\beta$ to be the tree-predecessor of $\alpha + 1$. At limit ordinal stages $\gamma < \lambda$, player II must select some set $b \subseteq \gamma$, such that $b$ is cofinal in $\gamma$ and linearly ordered by the tree ordering. We declare $b$ to be the set of tree-predecessors of $\gamma$, and set $M_\gamma$ to be the direct limit of the models $M_\alpha$ for $\alpha \in b$ (under ultrapower maps). Player II must ensure that all models produced are well-defined and transitive, and wins iff these conditions are maintained throughout. The entire array of information produced in a run of the game is called an iteration tree on $M$. The $\lambda$-iterability of $M$ requires the existence of a winning strategy for player II in this game (through $\lambda$ stages, as above). For example if $M$ is countable then a winning strategy exhibiting $\omega_1$-iterability is essentially a set of reals. Iterability is not in general simply a feature of the first order theory of $M$.

Fine structure comprises a collection of properties, of which Gödel’s condensation lemma for $L$ is a fairly prototypical instance.\footnote{The main properties we refer to here are condensation, solidity and universality of the standard parameter, Dodd-solidity, and the Initial Segment Condition (ISC). The definitions are recalled in §1.3 and elsewhere in the paper.} These properties are all first order: for each such property, there is a formula $\varphi$ (or sometimes a recursive theory $T$) such that given any structure $M$ with the first order signature of a mouse, the property holds of $M$ iff $M \models \varphi$ (respectively, $M \models T$).\footnote{For example, for $k < \omega$, $k$-soundness is expressible with a single formula $\varphi_k$, whereas $\omega$-soundness is expressed by the recursive theory $T = \{\varphi_k\}_{k < \omega}$.} So iterability is the more subtle notion. However, fine structure is very central to our understanding of mice, so central that it is even built
heavily into the very definition of mouse: a mouse is stratified in an increasing hierarchy of initial segments, each of which are also mice themselves, and part of the definition is that every proper segment must satisfy certain key fine structural requirements. In particular, they must be sound. Soundness is a strong and local form of the GCH. These properties are essential to the general development of the theory from the outset.

Typical constructions of mice are recursive in nature, building them, roughly, by recursion on their initial segments. Having produced a certain mouse $M$ at a stage of the construction, one must verify that it also has the right fine structural properties before proceeding. The proofs that $M$ has these properties rely heavily on the iterability of $M$. This paper gives new proofs of these properties, using an (at least superficially) weaker iterability hypothesis than do the classical proofs.

To explain the lesser iterability hypothesis, we need to refine our discussion of iteration trees that we began earlier. If $E = E_\alpha$ for some extender sequence $E$, then the length $lh(E)$ of $E$ is $\alpha$. The most fundamental kinds of iteration trees are normal trees, in which

(i) for all $\alpha \leq \beta$, we have $lh(E_\alpha) \leq lh(E_\beta)$,

(ii) given $E_\alpha$, the $T$-predecessor $\beta$ of $\alpha + 1$ is always chosen as small as possible that we can use $E_\alpha$ to form the next ultrapower, and

(iii) given $E_\alpha$ and $\beta$, $M_{\alpha+1} = \text{Ult}(P, E_\alpha)$ with $P$ the largest possible initial segment of $M_\beta$ over which $E_\beta$ is an extender.

The second main kind of iteration allows us to iterate linearly the process of forming normal trees. With this kind of iteration, we could, for example, first form a normal tree $T_0$ on $N_0 = M$, with last model $N_1 = M^{T_0}$, and then form a normal tree $T_1$ on $N_1$, with last model $N_2 = M^{T_1}$, etc, taking direct limits of the models $\langle N_\alpha \rangle_{\alpha < \eta}$ at limit stages $\eta$ of the process, and producing overall a (transfinite) sequence $\langle T_\alpha \rangle_{\alpha < \theta}$ of normal trees $T_\alpha$. This is called a stack of normal trees.$^5$

In the last few years, there has been significant work regarding and exploiting the relationship between normal trees and stacks of normal trees, particularly on the reduction of stacks to normal trees. This work is highly important in Steel’s recent progress in the analysis of HOD in models of determinacy $^{[32]}$. At essentially the same time as Steel’s work, the author showed $^{[25]}$ that normal iteration strategies satisfying inflation condensation can be extended to strategies for stacks of normal trees.$^6$ A significant part of this result was independently worked out by Steel $^{[32]}$. Steel and the author also worked out the more refined process of full normalization, establishing for example under large cardinals that $V_\delta \cap \text{HOD}^{L(\delta)}$ is the universe of a normal iterate of $M_\delta|\delta_0$, where $M_\delta$ is the canonical proper class mouse with infinitely many Woodin cardinals, and $\delta_0$ its least Woodin; see $^{[18]}$. Siskind and Steel have also developed this process in the context of comparison of strategy mice $^{[29]}$. Such techniques have been useful in work

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$^4$The description of normal we give here is a slight simplification of the precise one. By a normal iteration tree we will formally mean one which is $k$-maximal for some $k \leq \omega$.

$^5$This description is also slightly simplified. We will formally use the term stack in the sense of a $k$-maximal stack, for some $k \leq \omega$.

$^6$Versions of the main results in this paper were actually worked out prior to much of that for $^{[25]}$.

$^7$It was already known $^{[7]}$ that this model was the direct limit of iterates of $M_\omega|\delta_0$ given by stacks of trees; the new fact was the normality.
of Sargsyan, Schindler and the author [10], and the author [21], on Varsovian models, which explore self-iterability in mice and connections between inner models of mice and HOD in determinacy models.

Now the classical proof of the basic fine structural properties in mice relies heavily on the iterability of the mouse in question, and moreover, makes significant use of iterability for transfinite stacks of normal trees.\(^8\) In this paper we consider the problem of proving these properties from normal iterability alone.

The recent work mentioned above says a lot about the relationship between the two types of iterability, but whether normal iterability implies iterability for stacks in general, without the assumption of inflation condensation, seems to be open. So while the result of [25] mentioned above heads in the right direction, it does not suffice to establish fine structure from normal iterability. Doing so should provide a basic advance in our understanding of two of the fundamental building blocks of inner model theory.

On the other hand, an almost complete version of condensation (which generalizes Gödel’s theorem on condensation for levels of \(L\)) was proven in [24, Theorem 5.2], from normal iterability. The incompleteness of this version is due to the fact that in some cases it assumes a hypothesis on the solidity of the mouse \(M\) in question (see clause 1f of the cited theorem). (Solidity is another of the fine structural properties, and is related to condensation.)

These (partial) results suggest that normal iterability might indeed suffice to prove all of the fine structure properties, in full. The main goal of this paper is to show that it does. We show that normal iterability proves condensation (without the extra solidity assumption mentioned above), and proves solidity, universality, Dodd-solidity, and for pseudo-premice, the initial segment condition. To achieve this, we develop further the methods of [24]. Aside from providing a formal improvement of the classical results, the proof we give uses methods which do not feature in the classical proof, and these methods themselves might be of interest and provide new insight into the nature of mice and iteration trees.\(^9\)

We also prove some other facts of independent interest. In particular, we establish a simple criterion which ensures that a mouse is a normal iterate of its core. (The core of a mouse \(M\) is a natural hull of \(M\), corresponding to the “least” set \(A\) of ordinals which is definable from parameters over \(M\), but such that \(A \notin M\)). It has long been known that, roughly, every mouse which is below the level of a “cardinal which is strong to a measurable” is a normal iterate of its core, and that there are counterexamples to this phenomenon a little beyond that point (see §11.1 for discussion). However, the criterion we establish has no restriction on large cardinal complexity.

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\(^8\)This is through the use of a normal iteration strategy with the weak Dodd-Jensen property, the construction of which relies on iterability for stacks. For the classical proofs see [9], [34], [16], [12], [35], [36], [20].

\(^9\)The proof of solidity and universality we give here is modelled on that of condensation in [24], using structures related to the bicephali of [24] in place of the phalanxes of the classical proofs, and certain kinds of calculations from [24]. However, there are new fine structural difficulties to be handled here. For Dodd-solidity, there is an easy trick to reduce to the case where classical style arguments work. (This simplicity is probably an artifact of the assumption of 1-soundness; that is, one might formulate a generalization of Dodd-solidity which holds more generally, and the proof would then probably not be as direct.) Actually, the proof for Dodd-solidity we give here still involves significant work, but this is just due to the nature of the classical proof itself; the reduction to the (roughly) classical case itself is short.
Using an argument related to that for the previous one, we also show that if $M$ is a mouse, $\mu \in M$ and $M \models \text{"$\mu$ is a countably complete ultrafilter"}$ then $\mu$ is equivalent to a finite normal iteration tree on $M$, and hence has a simple description in terms of extenders in $E^M$. The result shows that no countably complete ultrafilters can appear in a mouse except for those which were essentially put directly in it. This generalizes Kunen’s classical result for $L[U]$, in which the only countably complete ultrafilters are those which are equivalent to finite iterates of $U$.

Finally, we establish a restriction on the possible kinds of set-forcing grounds of mice which model ZFC.

1.2 Results

We now list the main theorems we will prove. We work throughout with pure $L[\mathcal{E}]$ premice with Mitchell-Steel indexing, but allowing extenders of superstrong type in their extender sequence. The background theory is ZF (though it is straightforward to see that most of the results do not require much of ZF).

Theorem (Solidity and universality, 14.1). Let $m < \omega$. Then every $m$-sound, $(m, \omega_1 + 1)$-iterable premouse is $(m + 1)$-solid and $(m + 1)$-universal.\(^{11}\)

As an immediate corollary of the conjunction of this result with \(^{24}\), Theorem 5.2, we will also obtain the following result:

1.1 Theorem (Condensation). Let $k < \omega$. Let $M$ be a $k$-sound, $(k, \omega_1 + 1)$-iterable premouse. Let $H$ be a $k$-sound premouse and $\rho \in [\rho_{k+1}^H, \rho_k^H]$ be an $H$-cardinal such that $H$ is $\rho$-sound. Let $\delta = \text{card}^M(\rho)$. Let $\pi : H \to M$ be $k$-lifting with $\text{cr}(\pi) \geq \rho$. Then:

1. If $H \notin M$ then $\rho_{k+1}^H = \rho_{k+1}^M \leq \delta$, $H$ is the $\rho$-core of $M$, $\pi$ is the $\rho$-core map and $\pi(\bar{\rho}_{k+1}^H) = \bar{\rho}_{k+1}^M$.

2. If $H \in M$ then exactly one of the following holds:

   (a) $H \triangleleft M$,

   (b) $M|\rho$ is active and $H \triangleleft \text{Ult}(M|\rho, F^M|\rho)$,

   (c) $M|\rho$ is passive, $N = M|\rho^{+H}$ is active type 1 and $H = \text{Ult}_k(Q, F^N)$ where $Q \triangleleft M$ is such that $\rho = \delta^Q$ and $\rho_{k+1}^Q = \delta < \rho_k^Q$,

   (d) $k = 0$ and $H$ are active type 2 and $M|\rho$ is active type 2 and letting $R = \text{Ult}(M|\rho, F^M|\rho)$, then $N = R|\rho^{+H}$ is active type 1 and $H = \text{Ult}_0(M|\rho, F^N)$.

Note that in clauses 2(b)–2(d) above, $\delta < \rho = \delta^{+H}$, so $\rho$ is not an $M$-cardinal. The following fact is just an abridged version of \(^{24}\), Theorem 5.2. We state it here for ease of comparison with Theorem 1.1.

1.2 Fact (\(^{24}\)). Let $k, M, H, \rho, \delta, \pi$ be as in Theorem 1.1. Then:

1. If $H \notin M$ then:

\[^{10}\]This is essentially by a proof from the author’s dissertation \(^{22}\).

\[^{11}\]Note that we follow Zeman \(^{35}\) in our use of the terminology $(m + 1)$-solid, in that we do not incorporate $(m + 1)$-universality into it; see also \(^{27}, \S 1.2\) (Fine structure). This is in contrast to Mitchell-Steel \(^{9}\) and Steel \(^{34}\), where $(m + 1)$-solidity incorporates $(m + 1)$-universality by definition.
(i) If $M$ is $(k + 1)$-solid then $\rho^M_{k+1} = \rho^H_{k+1}$.

(ii) If $\rho^H_{k+1} = \rho^M_{k+1}$ then $H$ is the $\rho$-core of $M$, $\pi$ is the $\rho$-core map and $\pi(\rho^H_{k+1}) = \rho^M_{k+1}$.

(iii) If $\rho^H_{k+1} = \rho$ and $\rho^{+H} < \rho^{+M}$ then $M|\rho$ is active.

2. Part 2 of Theorem 1.1 holds.

Note that Theorem 1.1 follows immediately from Theorem 14.1 and Fact 1.2. We will also use Fact 1.2 in the proof of Theorem 14.1.

The next theorem is essentially due to Steel and Zeman, almost via their classical proofs. Note, however, that Steel’s proof is less general, as it is below superstrong extenders, and Zeman’s is with $\lambda$-indexing, not Mitchell-Steel. Super-Dodd-soundness is a strengthening of Dodd-soundness; see Definitions 7.1 and 7.4.

**Theorem** (Super-Dodd-soundness, 10.1). Let $M$ be an active, $(0, \omega_1 + 1)$-iterable pseudo-premouse, let $\kappa = \kappa^M$, and suppose that $M$ is either 1-sound or $\kappa^M$-sound. Then $M$ is super-Dodd-sound.

And the next is essentially due to Mitchell-Steel, by reducing to their classical proof from [9]:

**Theorem** (Initial Segment Condition, 10.6). Every $(0, \omega_1 + 1)$-iterable pseudo-premouse is a premouse.

$\mathbf{0}^\mathcal{T}$ is the least active mouse $M$ such that $M|\text{cr}(F^M) \models \text{“there is a strong cardinal”}$. Every mouse below $\mathbf{0}^\mathcal{T}$ is a normal iterate of its core (by the proof of [33, Theorem 8.13], or [4, §5]). But not too far beyond $\mathbf{0}^\mathcal{T}$, there are mice for which this fails. These things are discussed in detail in §11.1. The next result establishes a simple criterion which guarantees that a mouse is a normal iterate of its core, but without any limit on large cardinal complexity.

**1.3 Definition.** Let $\mathcal{T}$ be a successor length $m$-maximal iteration tree on an $m$-sound premouse $M$ such that $b^\mathcal{T}$ does not drop in model or degree. Let $\rho \in \text{OR}$. We say that $\mathcal{T}$ is almost-above $\rho$ iff for every $\alpha + 1 < \text{lh}(\mathcal{T})$, if $\text{cr}(E_\alpha^\mathcal{T}) < \rho$ then $M$ is active type 2, $m = 0$, $\alpha \in b^\mathcal{T}$, and $\text{cr}(i_\alpha^\mathcal{T})$ is the largest cardinal of $M_\alpha^\mathcal{T}$.

Let $\mathcal{T}$ be almost-above $\rho$. Note that if $\text{cr}(E_\alpha^\mathcal{T}) < \rho$ then $E_\alpha^\mathcal{T} = F(M_\alpha^\mathcal{T})$ and $(0, \alpha)^\mathcal{T}$ does not drop, so $E_\alpha^\mathcal{T}$ is a non-dropping image of $F^M$. Moreover, by taking $\alpha$ least such, we have $\text{cr}(i_\alpha^\mathcal{T}) \geq \rho$ and $\text{cr}(F^M) = \text{cr}(E_\alpha^\mathcal{T}) < \rho$. Since $\alpha \in b^\mathcal{T}$, it also follows that $\rho \leq \text{cr}(i_0^\mathcal{T})$. The definition of strongly finite below is given in 8.2, but it implies that $\mathcal{T}$ has finite length and every extender used along $b^\mathcal{T}$ is equivalent to a single measure (there is a finite set of generators which generates the extender).

**Theorem** (Projectum-finite generation, 11.5). Let $m < \omega$ and let $M$ be an $m$-sound, $(m, \omega_1 + 1)$-iterable premouse. Suppose that

$$M = \text{Hull}^M_{m+1}(\rho^M_{m+1} \cup \{x\})$$

for some $x \in M$. Then $M$ is an iterate of its $(m + 1)$st core $\mathfrak{c}_{m+1}(M)$. In fact, there is a successor length $m$-maximal tree $\mathcal{T}$ on $\mathfrak{c}_{m+1}(M)$, which is strongly finite and almost-above $\rho^M_{m+1}$, with $M = M_\infty^\mathcal{T}$. Moreover, $i_0^\mathcal{T}$ is the core map.
We will also deduce a related fact in Corollary 15.3. For now note that Theorem 11.5 has the following corollary:

**1.4 Corollary.** Let $M$ be any $m$-sound, $(m, \omega_1 + 1)$-iterable premouse. Let $A \subseteq \rho^{M}_{m+1}$ be $\bar{\Sigma}^{M}_{\omega_{m+1}}$-definable. Let $C = \mathcal{C}_{m+1}(M)$. Then $A$ is $\bar{\Sigma}^{C}_{\omega_{m+1}}$-definable.

**Proof.** We may assume that $M = \text{Hull}^{M}_{m+1}(\rho^{M}_{m+1} \cup \{x\})$ for some $x$, by taking a hull. So the theorem applies. But then standard calculations show that $A$ is $\bar{\Sigma}^{C}_{\omega_{m+1}}$.

The weaker version of the corollary in which we assume $(m, \omega_1, \omega_1 + 1)^*$-iterability\(^1\) is already clear via standard methods (take $M$ countable and a strategy with weak Dodd-Jensen, compare $M$ with the phalanx $((M, < \rho^{M}_{m+1}), C)$, and analyse the outcome). One can also use classical methods to prove the conclusion of Theorem 11.5 from $(m, \omega_1, \omega_1 + 1)^*$-iterability, and this classical style proof is easier.

We also incorporate a proof of the following theorem, which was established below the superstrong level in the author’s thesis [22, Theorem 4.8]. Its proof, modulo the validity of Dodd-soundness at the superstrong level, is essentially the same as in [22], but we will use an embellishment of the argument to prove both the super-Dodd-soundness and projectum-finite-generation theorems. So it serves as a good warm-up to those proofs.

**Theorem** (Measures in mice, 9.1). Let $M$ be a $(0, \omega_1 + 1)$-iterable premouse and $\mu \in M$ be such that $M \models \text{“} \mu \text{ is a countably complete ultrafilter}\text{”}$. Then there is a strongly finite $0$-maximal iteration tree on $M$ such that $b^T$ does not drop, $\text{Ult}_0(M, \mu) = M^T_{\infty}$ and the ultrapower map $\tilde{i}^{\mu, 0}_{\infty}$ is the iteration map $\tilde{i}^{T}_{\infty}$.

Using projectum-finite generation, we prove the following theorem, which relates to inner model theoretic geology, and some questions of Gabriel Goldberg and Stefan Miedzianowski; see [13]. We write $[M]$ for the universe of $M$.

**Theorem** (12.1). Let $M$ be a $(0, \omega_1 + 1)$-iterable premouse such that $[M] \models \text{ZFC}$ (here $M$ might be proper class). Let $W$ be a ground of $[M]$ via a forcing $P \in W$ such that $W \models \text{“} P \text{ is } \sigma\text{-strategically-closed}\text{”}$. Suppose $M[N]^W_M \in W$.\(^1\) Then $W = [M]$.

1.5 Remark. If $M$ is also tame then the assumption that $M[N]^M_M \in W$ is superfluous. This is shown in [19, Theorem 4.7]. It is not known to the author whether the same holds for non-tame, even for $\sigma$-closed forcing. Suppose $M$ is a mouse, $[M] \models \text{ZFC}$ and $W$ is a ground of $[M]$ via $\sigma$-closed forcing. Is $W = [M]$?

As an aside, we observe that solidity is non-trivial:

**Theorem** (§4). If there is a measurable cardinal then there is a non-solid premouse.

1.6 Remark. As mentioned above, the background theory for the paper is ZF. However, for the theorems listed above, we may assume ZFC. This is because the theorems only assert something about the first order theory of a premouse, assuming that it is $(k, \omega_1 +1)$-iterable, for some given $k < \omega$. But then fixing the premouse $M$ and a $(k, \omega_1 +1)$-iteration strategy $\Sigma$ for $M$, we can pass into $\text{HOD}_{\Sigma, M}$, or $L[\Sigma, M]$, appropriately defined, where we have ZFC and the original assumptions. Moreover, the only way that AC comes in is when proving that comparisons and related processes terminate; in those arguments

\(^{12}\)The distinction between $(k, \alpha, \beta)$- and $(k, \alpha, \beta)^*$-iterability is specified in [31, p. 1202].

\(^{13}\)Of course, as $P$ is $\sigma$-strategically-closed in $W$, automatically $R^M \subseteq W$. But note that $M[\omega^M_1]$ doesn’t just give $HC^M$, but also the restriction of the extender sequence to $\omega^M_1$.\(^8\)
we need that $\omega_1$ is regular and to be able to take an appropriate hull of $V_{\alpha}$. The rest of the paper goes through with ZF (we point out some details regarding this when proving $^{<\kappa}\mathcal{M}_k^{\text{iter}}$ is wellfounded, Lemma 13.7).

1.7 Remark. The author thanks the organizers for the opportunity to present key parts of an earlier version of the main arguments in this paper at the the 3rd Münster conference on inner model theory, the core model induction, and hod mice, at the University of Münster, Germany, in July 2015; cf. [14]. The overall argument is still the same as what was outlined there, but there are some significant differences in components. One of these is in the proof of of Lemma 11.6 (projectum-finite generation), where some methods from [24] and from [32] turned out to give a somewhat simpler approach to prove that lemma; this is explained in more detail at the beginning of §11. Second, there are some simplifications in the proof of solidity, one adopting a suggestion made by John Steel following the 2015 talk; this is explained in §14. Also, the idea for [25, Theorem 9.6] was only found during the conference; the author was only aware of a weaker version of this result prior to the conference, which was however enough to prove a weakened version of Lemma 13.7, which sufficed for the main theorems.

1.3 Notation and terminology

We assume general familiarity with [34] and/or [9], which develop the basic theory of Mitchell-Steel (MS) indexed premice. In particular, we work in MS-indexing, but allowing superstrong extenders on the extender sequence (see Remark 3.1). We work with MS fine structure, but modified as in [28, §5], meaning that we define the $n$th standard parameter $p_n$ to be what is denoted $q_n$ there, and the other fine structural notions are defined as there. We assume familiarity with generalized solidity witnesses (see [35, §1.12 + p. 326]), and also with [24, §1.1, §2]. Familiarity with other parts of [24], and also [27, §2], might help, but these can be referred to as necessary.

The reader should look through [24, §1.1] for a summary of the notation and terminology we use. We include some of the most central notions here, and some further terminology.

1.3.1 General

See [24, §1.1.1]. Given finite sequences $p, q$, we write $p \preceq q$ iff $p = q \upharpoonright \text{lh}(p)$. We write $q \prec p$ iff $q \preceq p$ and $q \neq p$.

We order $\text{OR}^{<\omega}$ in the “top-down” lexicographic ordering. That is, given $p, q \in \text{OR}^{<\omega}$, then

$$p < q \iff p \neq q \text{ and } \max(p \Delta q) \in q.$$ 

We sometimes identify $\text{OR}^{<\omega}$ with the strictly descending sequences of ordinals, so if $p = \{p_0, \ldots, p_{n-1}\}$ where $p_0 > \ldots > p_{n-1}$ then $p$ is identified with $(p_0, \ldots, p_{n-1})$, and $p \upharpoonright i$ denotes $\{p_0, \ldots, p_i-1\}$ for $i \leq n$, and $p \upharpoonright i$ denotes $p$ for $i \geq n$.

The author has not thought through adapting the arguments presented here to Jensen indexed premice or other forms of fine structure (such as in [35] or [5]).
We denote with $\widetilde{\text{OR}}_{15}$ the class of pairs $(z, \zeta) \in [\text{OR}]^{<\omega} \times \text{OR}$ such that $z = \emptyset$ or $\zeta \leq \min(z)$. We order $\text{OR}$ as follows. Let $(z, \zeta), (y, \upsilon) \in \text{OR}$. Write

$$z = \{z_0 > \ldots > z_{m-1}\} \text{ and } z_m = \zeta,$$

$$y = \{y_0 > \ldots > y_{n-1}\} \text{ and } y_n = \upsilon.$$

Then $(z, \zeta) < (y, \upsilon)$ iff either:

- $m < n$ and $z_i = y_i$ for all $i \leq m$, or
- there is $k \leq \min(m, n)$ such that $z|k = y|k$ and $z_k < y_k$.

Whenever we discuss the ordering of such tuples/pairs, it is with respect to these orderings. Note that they are wellfounded and set-like.

### 1.3.2 Premice and phalanxes

See [24, §1.1.2] for our use of the terminology premouse (with superstrongs) and associated notation. Here is a brief summary: For a premouse $M$, $F^M$ denotes its active extender, $E^M_\alpha$ is its internal extender sequence (without $F^M$), $E^M_\alpha = E^M \setminus \langle F^M \rangle$, and if $M$ is type 2, then $F^M_\kappa$ denotes the largest non-type Z proper segment of $F^M$. We write $M^{\text{pv}}$ for its passivization; that is, just like $M$ except with $F^{M^{\text{pv}}} = \emptyset$. And $M|\alpha$ is the initial segment of $M$ of ordinal height $\alpha$, including the extender active there, and $M||\alpha = (M|\alpha)^{\text{pv}}$. We abbreviate premouse with pm. Such notation is also employed with analogous meaning for other related structures $M$, such as pre-ISC-premice, which are like premice, but without any form of ISC (initial segment condition) required:

#### 1.8 Definition. A pre-ISC-premouse$^{17}$ is a structure $M = (J_\alpha[E], \in, E, G)$ such that there are $\delta, F, F, U, i, \kappa$ such that:

- $M^{\text{pv}} = (J_\alpha[E], \in, E, \emptyset)$ is a passive premouse with largest cardinal $\delta$,  
- $F$ is a weakly amenable short extender over $J_\alpha[E]$, 
- $U = \text{Ult}(J_\alpha[E], F)$ and $i : J_\alpha[E] \to U$ is the ultrapower map, 
- $\kappa = \text{cr}(F)$ and $J_\alpha[E] \models \text{“$\kappa^+$ exists”, so } \kappa^+ < \alpha$, 
- $M^{\text{pv}} = U|\delta^+U$, 
- either:
  - (a) $i(\kappa) > \alpha$ and $F$ is the $(\kappa, \alpha)$-extender derived from $i$ and there is $\nu < \alpha$ such that all generators of $F$ are $< \nu$, or
  - (b) $i(\kappa) = \delta$ and $F$ is the $(\kappa, \delta)$-extender derived from $i$.

---

$^{15}$In [27, Definition 2.6], the notation “$\mathcal{D}$” is used instead of “$\text{OR}$”.

$^{16}$Note that it is $\zeta \leq \min(z)$, not $\zeta < \min(z)$.

$^{17}$These are similar to the segmented-premice defined in [24, Definition 2.9]. (Segmented-premice are not defined in [27, §5], the implication to the contrary in [24, §1.1.2] notwithstanding.) But in [24, Definition 2.9], it was demanded that $\nu(F^M) \leq 1_{\text{gcd}}(M)$, which is not demanded for pre-ISC-premice.
and

- \( G = \bar{F} \) is the standard amenable predicate coding \( F \), as in \([34, 2.9-2.10]\), except that letting \( \nu = \nu(F) \) be the natural length of \( F \) (so \( \nu = \max(\kappa^+, \nu^-) \) where \( \nu^- \) is the strict sup of generators of \( F \)), we use \( \nu' = \max(\delta, \nu) \) in place of \( \nu \) in the definition of \( \bar{F} \). That is, \( \bar{F} \) is the set of pairs \((A, E)\) such that \( A \in M|\kappa^+ \) and \( E = F \upharpoonright (A \times [\nu']^{\leq \omega}) \).

Clearly every premouse is a pre-ISC-premouse.\(^{18}\)

When we write, for example, type 1/2, we mean “(an active premouse of) type 1 or type 2”. If \( M \) is an active pre-ISC-prem, \( \nu(M) = \nu(F^M) \) denotes the natural length of \( F^M \) (so \( \nu(F^M) = \max(\kappa^+, \nu^-) \) where \( \kappa = \text{cr}(F^M) \) and \( \nu^- \) is the strict sup of generators of \( F^M \)), and \( \lgcd(M) \) the largest cardinal of \( M \).

If \( N \) is an \( n \)-sound premouse, we write \((M, m) \leq (N, n)\) if \( M \leq N \) and if \( M = N \) then \( m = n \). If \( N \) is a pre-ISC-premous, we write \((M, m) \leq (N, 0)\) if \( M \leq N \) and if \( M = N \) then \( m = 0 \).

If \( M \) is a passive premouse and \( \xi < \text{OR}^M \), then \( M \upharpoonright \xi \) is the corresponding segment of \( M \) in the \( S \)-hierarchy. For example, if \( M = \mathcal{J}(N) \) (one step in the Jensen hierarchy above \( N \)), \( \lambda = \text{OR}^N \) and \( n < \omega \) then \( M \upharpoonright (\lambda + n) = \mathcal{S}_n(N) \). If \( M = (M^{pv}, F^M) \) is an active type 1/2 premouse, then \( M \upharpoonright \xi \) is the corresponding segment of \((M^{pv}, F^M) \), i.e. \( (M^{pv}|\xi, F^M \cap (M^{pv}|\xi)) \). And if \( M \) is active type 3 and \( \xi < \rho_0^M = \nu(F^M) \), then \( M \upharpoonright \xi = (M^{pv}|\xi, F^M \upharpoonright \xi) \).

Regarding phalanxes, we write for example \( \mathfrak{P} = ((P, < \kappa), (Q, \kappa), R, \lambda) \), where \( \kappa < \lambda \), for the phalanx consisting of 3 models \( P, Q, R \), where trees \( T \) on \( \delta \) have \( M^T_\beta = R \), first extender \( E^T_0 = (E^T_0, E^T_1, \ldots, E^T_n, \ldots) \) with \( \lambda \leq \text{lh}(E^T_0) \), extenders \( E^T_a \) with \( \text{cr}(E^T_a) < \kappa \) apply to \( P \), those with \( \text{cr}(E^T_a) = \kappa \) apply to \( Q \), and others apply to some segment of \( M^T_\beta \) for some \( \beta \geq 0 \). A \((p, q, r)\)-maximal tree \( T \) on \( \delta \) has \( \deg^T_0 = r \), and degrees \( p, q \) associated to \( P, Q \), so \( M^T_{p+1} = \text{Ult}_P(P, E^T_0) \) if \( \text{cr}(E^T_a) < \kappa \), etc, and \( T \) is otherwise formed according to the rules for \( n \)-maximal trees. We write \( M^T_0 = P \) and \( M^T_1 = Q \), and root\( T(a) \) is \(-2, -1, 0 \) denotes the root of \( a \) in the tree order. We also write, for example, \( \mathfrak{P}' = (((P, p), < \kappa), ((Q, q), \kappa), (R, r), \lambda) \) for the same phalanx, coupled with the specification of degrees \( p, q, r \) associated to the models \( P, Q, R \) of the phalanx. A degree-maximal tree on \( \mathfrak{P}' \) is just a \((p, q, r)\)-maximal tree on \( \mathfrak{P} \).

1.3.3 Fine structure

Let \( M \) be a \( k \)-sound premouse. See [24, §1.1.3] for notation such as \( \text{Hull}^M_{k+1}(X) \) (the \( r \Sigma_{k+1} \)-hull of \( M \)) from parameters in \( X \subseteq \mathcal{C}_0(M) \), \( \text{cHull}^M_{k+1}(X) \) (the transitive collapse variant) and the corresponding theory \( \text{Th}^M_{1 \Sigma_{k+1}}(X) \), and also \( \tilde{p}_{k+1} = (p_{k+1}, \ldots, p_1) \). Let \( q \in [\tilde{p}_k^M]^{< \omega} \). The solidity witnesses for \((M, q)\) are the hulls \( H_{q_i} = \text{cHull}^M_{k+1}(q_i \cup \{ q \}

\footnote{\textbf{Remark 2.10}. Every \( M \) is a pre-ISC-premous.}

\footnote{But a type 2 premouse is not a segmented-premous ([27, Definition 2.9]), because of the requirement of \( \nu(F^M) \leq \delta \) for segmented-premice. However, it is stated in [27, Remark 2.10] that every premouse is a segmented-premous, which was an oversight. It appears the author had two distinct notions in mind, which became blurred. The items [27, 2.11, 2.12, 2.13] are stated for segmented-premice \( P \), but they in fact also work fine for type 2 premice, which is a case of key interest, and apparently when the author wrote them, he had forgotten that a segregated-premous \( M \) was required to have \( \nu(F^M) \leq \lgcd(M) \). In any case, while the present paper depends on some of the proofs from [27, §2], it does not literally depend on those results themselves.}
both require that

part 1.1

28

(c)

part

28

2

1.9

Note here that we follow \[ \text{mice (Fact } x M \] See \[ \text{1.3.4 Extenders and ultrapowers} \]

generated

or is a

k

we mean a pair \((i, p^M_k)\), or essentially equivalently, the theories \(T_{q_i} = \text{Th}_{\Sigma_{k+1}}^M (q_i \cup \{ q \upharpoonright i, p^M_k \})\), where \(i < \text{lh}(q)\). We say that \((M, q)\) is \((k + 1)\)-solid iff \(H_{q_i} \in M\) (equivalently, \(T_{q_i} \in M\)) for each \(i < \text{lh}(q)\). And \(M\) is \((k + 1)\)-solid iff \((M, p^M_{k+1})\) is \((k + 1)\)-solid. We say that \(M\) is \((k + 1)\)-universal iff

\[
\mathcal{P}(\kappa) \cap M \subseteq \text{cHull}_{k+1}^M (\rho^M_{k+1} \cup \{ p^M_{k+1} \}).
\]

We say that \(k\) is the degree of this term. If \(k, \varphi, R, n, q\) are as in \([28, \text{Definition } 5.2]\), we define the partial function \(m_{\varphi, q} : \text{part } \mathcal{C}_0(R)^n \rightarrow \mathcal{C}_0(R)\) as there. We will, moreover, always use \(p^M_k\) for the parameter \(q\). Given an \(n\)-sound premouse \(R, x \in \mathcal{C}_0(R)\) and a term \(\tau = (k + 1, \varphi)\) of \(n > 0\) variables (therefore \(k, \varphi, R, n, q = p^M_k\) are as in \([28, \text{Definition } 5.2]\)), we write \(f^M_{\tau, x}\) for the partial function \(f^M_{\tau, x} : \text{part } \mathcal{C}_0(R)^n \rightarrow \mathcal{C}_0(R)\) where \(f^M_{\tau, x}(\bar{y}) = m_{\varphi, q, p^M_k}(x, \bar{y})\).

The following is just the variant of \([34, \text{Theorem } 5.1]\) with the iterability hypothesis being only normal iterability, and (as throughout the paper) allowing superstrong extenders in \(E^H\) and \(E^M\):

1.9 Fact (Condensation for \(\omega\)-sound mice). Let \(H, M\) be \((\omega, \omega_1 + 1)\)-iterable \(\omega\)-sound premice. Let \(\pi : H \rightarrow M\) be elementary with \(cR(\pi) = \rho\) where \(\rho = p^H_{\omega}\). Then either

(a) \(H \triangleleft M\), or

(b) \(M|\rho\) is active and \(H \triangleleft \text{Ult}(M|\rho, F^M|\rho)\).

Proof. Note that since \(\rho = p^M_{\omega} = cR(\pi)\) and by elementarity, we have \(\rho < p^M_{\omega}\), so \(H \in M\).

So Fact 1.2 part 2 applies, and gives the desired conclusion, since clauses (c) and (d) of Theorem 1.1 part 2 both require that \(H\) is non-sound. \(\square\)

1.3.4 Extenders and ultrapowers

See \([24, \text{§1.1.4}]\). Let \(E\) be a short extender over a pre-ISCP-premouse \(M, \kappa = cR(E)\) and \(U = \text{Ult}(M, E)\). We say \(E\) is weakly amenable to \(M\) if \(\mathcal{P}(\kappa) \cap U = \mathcal{P}(\kappa) \cap M\). Note that if all proper segments of \(M|\kappa + M\) satisfy the conclusions of condensation for \(\omega\)-sound mice (Fact 1.9), then weak amenability implies \(U|\kappa + U = M|\kappa + M\). We will have such condensation available when we form such ultrapowers, and we may use the stronger agreement implicitly at times. Let \(x \in U\). A set \(X\) of ordinals \(E\)-generates \(x\) (or just generates \(x\), if \(E\) is determined by context) if there is \(f \in M\) and \(a \in [x]^{\omega}\) such that \(x = [a, f]_E\). We say \(X\) generates \(E\) iff every \(x \in U\) is generated by \(X\). And \(E\) is finitely generated or is a measure if there is a finite set \(X\) generating \(E\).

We write \(p^M_{(i-1)} = OR^M\), and say \(M\) is \((-1)\)-sound. We write \(\text{Ult}_{(i-1)}(M, E) = \text{Ult}(M, E)\) (formed using functions in \(M\), and without squashing), and \(i^M_{E, (i-1)} = i^M_E\). Note that if \(M\) is a type 3 premouse, we have \(p^M_0 < p^M_{(i-1)}\).

For a sufficiently elementary embedding \(j, E_j\) denotes the extender derived from \(j\).
1.3.5 Embeddings

See [24, §1.1.5], in particular for the terminology c-preserving (cardinal-preserving), $\rho_n$-preserving, $\rho_n$-preserving, $\rho_n$-preserving, and the notation $i_{MN}: M \to N$ for a context determined map $i : M \to N$. One difference with [24, §1.1.5] is that, given a $\Sigma_\gamma$-elementary embedding $\pi : M \to N$ between premice $M, N$, we write $\text{Shift}(\pi)$ for the embedding denoted $\psi_\pi$ there. That is, if $M, N$ are active then

$$\text{Shift}(\pi) : \text{Ult}(M, F^M) \to \text{Ult}(N, F^N)$$

is the embedding induced by $\pi$ via the Shift Lemma, whereas if $M, N$ are passive then $\text{Shift}(\pi) = \pi$. We sometimes write $\hat{\pi} = \text{Shift}(\pi)$ to save space. So if $M, N$ are type 1/2 premice, then $M \subseteq \text{dom}(\pi)$ and $\pi = \text{Shift}(\pi) | M$, but if $M, N$ are type 3 then, as is the usual convention, fine structural maps $\pi : M \to N$ literally have domain $M^{\omega_\gamma} = \mathcal{C}_0(M)$, whereas $M \subseteq \text{dom}(\text{Shift}(\pi))$, so $\text{Shift}(\pi)$ acts directly on the elements of $M \setminus M^{\omega_\gamma}$ (such as $\nu^M_1 = \nu(F^M)$), whereas $\pi$ does not. In case $M, N$ are type 3, $\pi$ is $\nu$-low iff $\text{Shift}(\pi)(\nu^M) < \nu^N$, $\nu$-preserving iff $\text{Shift}(\pi)(\nu^M) = \nu^N$, and $\nu$-high iff $\text{Shift}(\pi)(\nu^M) > \nu^N$. In case $M, N$ are passive or type 1/2, we say $\pi$ is $\nu$-preserving.

The $n$-lifting embeddings (a weakening of the near $n$-embeddings of [34]), are introduced in [24, Definition 2.1] and their basic properties developed in [24, §2].

Let $M, N$ be active pre-ISC-premice and $\pi : M \to N$. We say $\pi$ is a $(−1)$-embedding iff $\pi$ is $\Sigma_1$-elementary and cofinal in $\rho_{−1}^N = \text{OR}^N$. We say $\pi$ is a near $(−1)$-embedding iff it is $\Sigma_1$-elementary, and $(−1)$-lifting iff $\pi$ is $\Sigma_0$-elementary. (Note that, for example, a near $m$-embedding between type 3 premice induces an $\text{r}\Sigma_{m+2}$-elementary embedding between the unsquashed structures. Likewise, a near $(−1)$-embedding is $\text{r}\Sigma_1$-elementary between the unsquashed structures. Degree $−(1)$ sits immediately below degree 0 in the squashed hierarchy.)

1.3.6 iteration trees and iterability

See [24, §1.1.6] and [25, §1.1.5]. Let $\mathcal{T}$ be an iteration tree. We write $\text{lh}(\mathcal{T})$ for the length of $\mathcal{T}$, $\text{lh}(\mathcal{T})^- = \{\alpha \mid \alpha + 1 < \text{lh}(\mathcal{T})\}$, and $<^\mathcal{T}$ for the tree order on $\text{lh}(\mathcal{T})$. We write $(M^\alpha, \text{deg}_\alpha^\mathcal{T})$ for the $\alpha$th model and degree of $\mathcal{T}$, for $\alpha < \text{lh}(\mathcal{T})$. Given $\alpha + 1 < \text{lh}(\mathcal{T})$, $\beta = \text{pred}^\mathcal{T}(\alpha + 1)$ denotes the $<^\mathcal{T}$-predecessor of $\alpha + 1$. $M^\alpha_{\beta+1}$ is the model $N \subseteq M^\alpha_\beta$ such that $M^\alpha_{\beta+1} = \text{Ult}_d(N, E^\mathcal{T}_\alpha)$, where $d = \text{deg}_{\alpha+1}^\mathcal{T}$, and $i^\mathcal{T}_{\alpha+1} : M^\alpha_{\beta+1} \to M^\mathcal{T}_\alpha$ is the ultrapower map. We write $\mathcal{D}^\mathcal{T}$ for the set of nodes at which $\mathcal{T}$ drops in model, so $\alpha + 1 \in \mathcal{D}^\mathcal{T}$ iff $M^\mathcal{T}_{\alpha+1} \subseteq M^\mathcal{T}_\alpha$; and $\mathcal{D}^\mathcal{T}_{\text{deg}}$ is likewise, but for drops in model or degree. If $\gamma \leq^\mathcal{T} \delta$ then $[\gamma, \delta]^\mathcal{T}$, $[\gamma, \delta]^\mathcal{T}$, $\gamma$, $\delta$, etc, are the corresponding $<^\mathcal{T}$-intervals. If $\gamma \leq^\mathcal{T} \delta$ and $[\gamma, \delta]^\mathcal{T} \cap \mathcal{D}^\mathcal{T} = \emptyset$ then $i^\mathcal{T}_\gamma : M^\mathcal{T}_\gamma \to M^\mathcal{T}_\delta$ is the iteration map, and if $\gamma = \alpha + 1$ then $i^\mathcal{T}_{\alpha+1, \delta} = i^\mathcal{T}_{\alpha+1} \circ i^\mathcal{T}_\alpha$. If $\gamma <^\mathcal{T} \delta$ then succ$^\mathcal{T}(\gamma, \delta)$ denotes the $<^\mathcal{T}$-successor of $\gamma$ in the interval $[\gamma, \delta]^\mathcal{T}$. A tree of length 1 is trivial in that it uses no extenders. In the context of an iteration tree $\mathcal{T}$ of successor length $\xi + 1$, $\xi$ denotes $\xi$. Given such a tree, $b^\mathcal{T}$ denotes $[0, \infty]^\mathcal{T}$, and if $(0, \infty]^\mathcal{T} \cap \mathcal{D}^\mathcal{T} = \emptyset$ then $i^\mathcal{T}$ denotes $i^\mathcal{T}_{b^\mathcal{T}}$. We say $\mathcal{T}$ is

\[\text{trivial}\] when we consider embeddings $\pi : M \to N$ of active pre-ISC-premice at “degree −1”, we have $\text{dom}(\pi) = M$. Note that this differs from the conventions when $M, N$ are active type 3 premice, where fine structural maps $\pi : M \to N$ are literally of the form $\pi : M^{\omega_\gamma} \to N^{\omega_\gamma}$.
terminally-non-dropping iff \( T \) has successor length and \( b^T \cap \mathcal{G}_{\text{deg}}^{\mathcal{E}} = \emptyset \); \( T \) is terminally-non-model-dropping iff \( T \) has successor length and \( b^T \cap \mathcal{T} = \emptyset \). We will also use much of the foregoing terminology and slight variants thereof for iteration trees on bicephali (see §5 and §14), without necessarily mentioning it explicitly.

Let \( M \) be an \( m \)-sound premouse. Recall that an iteration tree \( T \) on \( M \) is \( m \)-maximal if (i) \( \text{lh}(E^T_\alpha) \leq \text{lh}(E^T_\beta) \) for all \( \alpha < \beta \), (ii) \((M^T_\alpha, \text{deg}_\alpha^T) = (M, m)\), (iii) for all \( \alpha + 1 < \text{lh}(T) \), \( \text{pred}^T(\alpha + 1) \) is the least \( \beta \) such that \( \text{cr}(E^T_\alpha) < \nu(E^T_\beta) \), and (iv) for all such \( \alpha + 1, \beta \), \((M^T_{\alpha+1}, \text{deg}_{\alpha+1}^T)\) is the largest \((N, n) \leq (M^T_\beta, \text{deg}_\beta^T)\) such that \( \text{Ult}_\alpha(N,E^T_\alpha) \) is well-defined. An \((m, \alpha)\)-iteration strategy is one for \( m \)-maximal trees of length \( \leq \alpha \) (that is, player II wins if a tree of length \( \alpha \) is produced).

For the definitions of \( m \)-maximal stacks see [25, §1.1.5], and for the iteration game \( \mathcal{G}_{\text{fin}}(M, m, \omega_1 + 1) \) see [25, Definition 1.1].

1.10 Remark. In comparison arguments, in which we have two trees \( T, U \), we will make the conventional use of padding, setting \( E^T_\alpha = \emptyset \) or \( E^U_\alpha = \emptyset \) at stages \( \alpha \) of the comparison at which we do not use an extender in \( T \) or \( U \) respectively. In some comparison arguments we will also have stages \( \alpha \) at which \( E^T_\alpha = \emptyset = E^U_\alpha \). We will only explicitly define the rules for determining \( \text{pred}^T(\alpha + 1) \) for non-padded trees. Consider a padded tree \( T \). If \( E^T_\alpha = \emptyset \) then we set \( \text{pred}^T(\alpha + 1) = \alpha \) (and \( M^T_{\alpha+1} = M^T_\alpha \) etc). If \( E^T_\alpha \neq \emptyset \), we have an interval \([\beta_0, \beta_1]\) of ordinals \( \beta \) which might serve as \( \text{pred}^T(\alpha + 1) \). But here we will have \( E^T_\beta = \emptyset \) for all \( \beta \in [\beta_0, \beta_1] \), and \( M^T_\beta = M^T_{\beta_0} \) and \( \text{deg}_\beta = \text{deg}_{\beta_0} \) for all \( \beta \in [\beta_0, \beta_1] \). So it does not actually matter which \( \beta \in [\beta_0, \beta_1] \) is used as \( \text{pred}^T(\alpha + 1) \). But for specificity, we set \( \beta = \beta_1 \) (the unique ordinal \( \beta' \) such that \( E^T_{\beta'} \neq \emptyset \) and the rules for non-padded trees are satisfied at \( \beta' \)).

2 Proof outline

In the classical proof that sufficiently iterable countable premice \( M \) possess the various standard fine structural properties, iterability with respect to transfinite stacks of normal trees is used to show that there is a normal strategy for \( M \) with the weak Dodd-Jensen property (cf., for example, [34, §4.3]). Such a strategy is then used for various comparison arguments, and the weak Dodd-Jensen property is of central importance to the analysis of those comparisons.

Since we assume only normal iterability, we must prove the fine structural facts without relying on weak Dodd-Jensen. (In fact, by [25, Theorem 1.2], assuming \( \text{ZF} + \text{DC} \), a countable \( m \)-sound premouse is \((m, \omega_1, \omega_1 + 1)^*\)-iterable iff there is an \((m, \omega_1 + 1)^*\)-strategy for \( M \) with weak Dodd-Jensen.) In this section we will give an outline of the main arguments in the paper, highlighting the methods for getting around the lack of weak Dodd-Jensen.

As discussed in §1, condensation (Theorem 1.1) was almost already dealt with in [24, Theorem 5.2], and it will follow from that theorem and solidity (Theorem 14.1). The proofs of solidity (14.1) and projectum-finite generation (Theorem 11.5) which we give here are very related to the methods used in [24]. So we won’t say anything further directly regarding the details of the proof of condensation.
2.1 Solidity and universality

We begin with the most central results of the paper, solidity and universality, primarily focusing on solidity. Let $k < \omega$ and let $M$ be a $k$-sound, $(k, \omega_1 + 1)$-iterable premouse. Say we want to prove that $M$ is $(k + 1)$-solid. Let $\gamma \in p^M_{k+1}$ and let

$$H = cHull^M_{k+1}(\gamma \cup \{p^M_{k+1}\{\gamma\}, p^M_{k+1}\}).$$

We want to see $H \in M$. We may assume that $M$ is countable, and that for $\gamma' \in p^M_{k+1}$ with $\gamma' > \gamma$, the corresponding hull $H'$ is in $M$. For the purposes of this outline, let us also assume that $\gamma$ is an $M$-cardinal.

In the classical proof that $H \in M$ under these hypotheses, assuming also that there is an $(k, \omega_1 + 1)$-iteration strategy $\Sigma$ for $M$ with the weak Dodd-Jensen property, the phalanx $\mathcal{P} = ((M, < \gamma), H, \gamma)$ is compared with $M$, producing iteration trees $\mathcal{U}$ on $\mathcal{P}$ and $\mathcal{T}$ of $M$. The first extender $E$ used in $\mathcal{U}$ (if there is one) is taken from $\mathcal{P}^{H^k}$, with $\gamma < \text{lh}(E)$, and extenders $F$ with $\text{cr}(F) < \gamma$ apply to $M$, whereas if $\text{cr}(F) \geq \gamma$ then $F$ applies to $H$ or a later model of $\mathcal{U}$, with tree order determined by the rules of normality. Suppose that $\mathcal{U}$ is formed by lifting to a tree $\mathcal{U}'$ on $M$ via $\Sigma$, and also that $\mathcal{T}$ is via $\Sigma$. Then one shows that the comparison terminates with main branch $\mathcal{U}'$ of $\mathcal{U}$ above $H$; let $M^\mathcal{U}_{\infty}$ be the final iterate. One shows that there is no drop in model or degree along the main branch $\mathcal{U}'$ of $\mathcal{U}$, so we have an iteration map $\iota^\mathcal{U} : H \rightarrow M^\mathcal{U}_\infty$, which is a $k$-embedding, and $\text{cr}(\iota^\mathcal{U}) \geq \gamma$, because of the rule that extenders $E$ with $\text{cr}(E) < \gamma$ apply to $M$. (Possibly $\mathcal{U}$ is trivial, in which case $M^\mathcal{U}_\infty = H$ and $\iota^\mathcal{U}$ is the identity.) One also shows that either $M^\mathcal{U}_\infty \triangleleft M^\mathcal{P}_\infty$ or $\mathcal{U}'$ drops in model and $M^\mathcal{U}_\infty = M^\mathcal{P}_\infty$. These facts (together with fine structural analysis) lead to the conclusion that $H \in M$. But the arguments involved (showing that $\mathcal{U}'$ is above $H$, etc), make significant use of the weak Dodd-Jensen property.

The first key to working without weak Dodd-Jensen is that we prove solidity and universality by induction on a certain mouse order $<^p_k$. (The top-down induction on the elements of $p^M_{k+1}$ made implicit earlier is a sub-induction of this global one.) This order is related to Dodd-Jensen, and the induction provides a partial substitute for weak Dodd-Jensen, one which is provable from normal iterability alone. Given $k$-sound premice $P, Q$, we set $P <^p_k Q$ iff there is a finite $k$-maximal stack $\mathcal{T} = (T_i)_{i < n}$ on $Q$, consisting of finite trees $T_i$, non-dropping in model and degree on their main branches, and a $k$-embedding $\pi : P \rightarrow M^\mathcal{T}_\infty$ such that $\pi(p^P_{k+1}) < \iota^\mathcal{T}(p^Q_{k+1})$. This order is wellfounded within the $(k, \omega_1 + 1)$-iterable premice, so the induction makes sense (Lemma 13.7).

Since we are working by induction on $<^p_k$, for our particular $M$ introduced above, we may assume that all $P <^p_k M$ are $(k + 1)$-solid and $(k + 1)$-universal. This has useful consequences analogous to Dodd-Jensen, and is useful in showing that $M$ is $(k+1)$-solid and $(k+1)$-universal. In particular, we will see that $H <^p_k M$, and so $H$ is $(k+1)$-solid and $(k+1)$-universal. In certain cases, we will be able to use this to deduce that $H \in M$. This deduction, however, seems to require the application of Lemma 11.6 to premice $P <^p_k M$; this lemma is just like the projectum-finite generation theorem 11.5, except that it has extra solidity and universality hypotheses (which we know hold of $P <^p_k M$).

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20 The proof of Lemma 11.6 relies on Dodd-soundness, so this has a role in our proof of solidity and universality, albeit indirect.

21 Actually, it suffices to apply Corollary 11.7, which is a less fine consequence of Lemma 11.6.
The inductive hypothesis is also useful in a second manner: it has the consequence that degree \( k \) iteration maps on \( M \) preserve \( p_{k+1} \) (Lemma 13.8). This preservation fact is also used in establishing that \( M \) is \((k+1)\)-solid (whereas usually \( p_{k+1} \)-preservation is taken as a consequence of \((k+1)\)-solidity). It helps ensure that certain comparison arguments we perform (to be discussed below) end in a useful fashion, and also helps in their analysis.

The fact that \(<k\>_p\) is wellfounded within \((k, \omega_1 +1)\)-iterable premice follows easily from \cite[Theorem 9.6]{25}. The main point of \cite{25} was the construction of an iteration strategy for stacks of normal trees from a normal iteration strategy which has (a certain kind of) condensation. Here, we do not have any such strategy condensation assumption, but (as shown in \cite{25}) such an assumption is unnecessary here, because we can prove that \(<k\>_p\) is wellfounded by considering stacks of length \( \omega \) consisting of normal trees of finite length (and condensation is only relevant for stacks incorporating infinite trees, where the strategy’s branch choices come into play).\(^{22}\)

Using the inductive hypothesis (with \(<k\>_p\)) and some fine structural calculations, making use of condensation as in Facts 1.2 and 1.9, we will reduce to the case that

\[
M = \text{Hull}^M_{k+1}(\rho^M_{k+1} \cup \{p^M_{k+1}\}),
\]

along with some further easy restrictions which we will not detail here. And as mentioned above, the induction hypothesis leads to the fact that \( H \) is \((k+1)\)-solid and \((k+1)\)-universal.

After these initial steps, the second key to working without weak Dodd-Jensen enters the picture, which is to make use of (generalized) bicephalus comparisons in place of phalanx comparisons. Such comparison arguments are used by Woodin, the author \cite{24}, and Steel \cite{32}, for related purposes, but in various different contexts. They are motivated by the classical bicephalus comparisons used to establish, for example, the “uniqueness of the next extender” results in \cite[§9]{9}. In our present context, letting \( \pi : H \to M \) be the uncollapse map, we will consider \( B = (\gamma, \gamma, \pi, H, M) \) as a bicephalus, consisting of two structures \( H, M \), and some auxiliary information given by the “exchange ordinal” \( \gamma \) and the map \( \pi : H \to M \). (The repetition of \( \gamma \) is just for consistency with the more general case dealt with in §14.) We will form a comparison of \( B \) with \( M \), forming trees \( T \) on \( B \) and \( U \) on \( M \). The first extender \( E \) used in \( T \) will again have \( \gamma < \text{lh}(E) \). If we use an extender \( E^\alpha_T \) in \( T \) with \( \text{cr}(E^\alpha_T) < \gamma \) then we will have \( \text{pred}^T(\alpha + 1) = 0 \) and \( E^\alpha_T \) will apply to the entire bicephalus \( B \), not just \( M \) (nor just \( H \)). This will produce an ultrapower

\[
B^T_{\alpha+1} = \text{Ult}(B, E^\alpha_T) = (\gamma', \gamma', \pi', H', M'),
\]

with similar properties to those of \( B \). Here \( M' = \text{Ult}_k(M, E), \gamma' = i_E^{M,k}(\gamma) \), and \( H', \pi' \) come from the hull of \( M' \) at \( \gamma' \), analogous to the manner in which \( H, \pi \) come from the hull of \( M \) at \( \gamma \). See §14 for the precise definition. Similarly, if we have a node \( \beta < \text{lh}(T) \) and \( B^\beta_T = (\gamma^\beta_T, \gamma^\beta_T, \pi^\beta_T, H^\beta_T, M^\beta_T) \) is a bicephalus, \( \text{pred}^T(\xi + 1) = \beta \), \( \text{cr}(E^\xi_T) < \gamma^\beta_T \) and \( E^\xi_T \) is total over \( M^\beta_T \), then \( B^\beta_{\xi+1} \) will be the bicephalus \( \text{Ult}(B^\beta_T, E^\xi_T) \). Other extenders will apply to just a single model, forming an ultrapower in the usual fashion. For example, if \( \gamma \leq \text{cr}(E^\xi_T) \), then \( E^\xi_0 \) will apply to a segment of either \( H \) or \( M \), depending on the details.

\(^{22}\)A length \( \omega \) stack of finite trees lifts, via the methods of \cite{25}, into an infinite tree, of length some countable ordinal, but no strategy condensation is needed for that calculation.
of the situation. We will show that the comparison terminates, by basically the usual argument. Having a bicephalus $B^T_\beta$ indexed at $\beta$ helps to keep the comparison going at stage $\beta$, analogously to comparison arguments with classical bicephali. This helps to prevent the comparison from terminating for useless reasons, and substitutes partially for the fact, in a phalanx argument using weak Dodd-Jensen, that for the tree $T'$ on the phalanx $\mathcal{Q}$, $b^{T'}$ does not end up above $M$. So the comparison must terminate for “useful” reasons. But on the other hand, assuming that $H \notin M$, we will show that the iteration maps connecting $B$ to later bicephali $B^T_\beta$ will preserve fine structure nicely enough that the comparison cannot terminate for such “useful” reasons. This contradiction will give that $H \in M$. The material in §3 will help establish this preservation of fine structure.

The condensation result [24, Theorem 5.2] (of which Fact 1.2 is a simplification) was proved using related comparison arguments, involving bicephali (but of a different form).

This completes our discussion of the plan for proving solidity and universality.

2.2 Measures, Dodd-soundness, projectum-finite generation

We now give an outline of the kinds of arguments to be used in the main results in §9, §10 and §11. Each of these rely on a common kind of analysis of a comparison, but in somewhat different contexts.

As a representative case, which is also the simplest, let us focus on the setup for the argument for Theorem 9.1 in §9. So let $M$ be a passive mouse and $\mu \in M$ be such that $M \models \text{"}\mu$ is a countably complete ultrafilter". We want to see that there is a 0-maximal tree $T$ on $M$ with $M_\infty = \text{Ult}_0(M, \mu)$, $i^T$ exists and $i^T = i^M_\mu$, and moreover, $T$ has finite length and every extender used in $T$ is finitely generated, meaning that it is equivalent to its restriction to some finite set of generators.

Let $U = \text{Ult}_0(M, \mu)$. For the present outline, let us make some simplifying assumptions. Suppose that $U$ is wellfounded and iterable. Compare $M$ with $U$, with trees $U$ on $U$ and $T$ on $M$. Suppose that we get $M_U = Q$ where $Q = M_\infty^T$, the final branches $b^U, b^T$ are non-dropping and $k \circ i^M_\mu = j$, where $k = \sigma^U$ and $j = i^T$. See Figure 1 (some of the components of which are yet to be defined). We want to see then that $U$ is trivial; this will give that $U = Q$ and $k = \text{id}$ and $i^M_\mu = j$. We also want to see $T$ is finite and every extender used in $T$ is finitely generated.

Now since $k \circ i^M_\mu = j$, we have $\text{cr}(j) = \min(\text{cr}(i^M_\mu), \text{cr}(k)) \leq \text{cr}(k)$. Because $T, U$ arise from comparison and by considerations of compatibility of extenders, it follows that $\text{cr}(j) = \text{cr}(i^M_\mu) < \text{cr}(k)$. Let $x \in U$ be the seed of $\mu$, represented by the identity
function. The expectation is that $T$ is the minimal tree which “captures $x$”. Using a routine kind of finite support computation, to be discussed in §6.3, we can find a tree $T$ of finite length with last model $Q$, such that $b^T$ is non-dropping, and find a (suitably elementary) map $\zeta : Q \rightarrow Q$ such that $\zeta \circ i = j$ where $j = i^T$, and with $k(x) \in \operatorname{rg}(\zeta)$. Because of this and since $U = \{i^M_\mu(f(x) \mid f \in M\}$, we can define a map $\sigma : U \rightarrow Q$ with $\sigma \circ i^M_\mu = j$ and $\zeta \circ \sigma = k$. By the commutativity, we have $\operatorname{cr}(j) = \operatorname{cr}(k)$.

Suppose for simplicity that the first extender used along $b^T$ is a normal measure $E^T_\alpha$. By the commutativity, $E^T_\alpha$ must be the normal measure derived from the first extender $E^T_\alpha$ used along $b^T$. And since $k$ is an iteration map with $\operatorname{cr}(j) < \operatorname{cr}(k) = \min(\operatorname{cr}(\sigma), \operatorname{cr}(\zeta))$, we get that $U$, $Q$, and $Q$ agree below $\operatorname{cr}(k)$, and since $E^T_\alpha \notin Q$, therefore $E^T_\alpha \notin Q$. Considering the ISC, it follows that $E^T_\alpha = E^T_\alpha$ is a normal measure. So we have made progress toward our goals: the first extender used along $b^T$ is finitely generated, and its generator is contained in the range of $k$. (If we get that all generators of all extenders used along $b^T$ and in the range of $k$, then $U = Q$, so $U$ is trivial.)

The last paragraph is too much of a simplification, however, as will not be able to arrange that $E^T_\alpha$ is a normal measure in general (neither the later extenders used along $b^T$). But we will be able to arrange that all extenders used along $b^T$ (in fact in $T$ at all, in this case) are finitely generated, via a somewhat more careful finite support computation, detailed in §8. But this is not enough for the preceding argument to work, since we won’t yet be able to rule out the possibility that $k \neq \operatorname{id}$ and $\operatorname{cr}(k) \leq \gamma$ for some generator $\gamma$ of $E^T_\alpha$. In order to deal with this, we will need to be able to analyze the movement of certain key generators under the relevant kinds of embeddings (such as iteration embeddings). For this, we will need to know that all extenders in $E^M$ are Dodd-sound; the Dodd-solidity of the extenders will allow us to track images of Dodd parameters from Dodd-sound extenders (see §7). Using this, we will end up being able to show that $E^T_\alpha = E^T_\alpha$ and this extender is generated by a finite set which is in the range of $k$. (This can in general take multiple steps of analysis. It might be that $E^T_\alpha$ itself is non-Dodd-sound, but it will equivalent to the composition of a finite sequence of Dodd-sound extenders, and each step of analysis will correspond to one of those Dodd-sound extenders.) This argument can be iterated all the way along $b^T$, and this will eventually give that $U = Q = Q$ and $T$ is as desired.

Along with Theorem 9.1, we will use variants of the argument outlined above to prove Theorems 10.1 (super-Dodd soundness) and 11.5 (projectum-finite generation), and also Theorem 11.2. In §3 and §§5–8 we cover in detail the tools needed to complete the sketch outlined above, and the background for their adaptations to the other theorems just mentioned. (The results of the careful finite support computation in §8 will only be used directly in §9. But this key material will be adapted to and applied in the other contexts directly within §§10,11 themselves.) The proof of Theorem 11.5 in particular will involve a kind of bicephalus comparison (but the bicephali here will be different to those considered in the proof of solidity mentioned earlier). For this reason, much of the material in §§5–7 deals with these bicephali. In order to motivate the role of bicephali within that material, let us also say a little about the proof of 11.5.

Let $M$ satisfy the assumptions of Theorem 11.5 and $C = \mathcal{C}_{m+1}(M)$. Supposing also that $M$ is $(m+1)$-universal and $C$ is $(m+1)$-solid (hence $C$ is $(m+1)$-sound), we will use a comparison argument to prove the conclusions of the lemma, comparing a certain kind of bicephalus with itself. Letting $\rho = \rho^M_{m+1} = \rho^{C}_{m+1}$, the bicephalus will
be \( B = (\rho, C, M) \), consisting of the two models \( C \) and \( M \), and an “exchange ordinal” \( \rho \). In the comparison, we will form two iteration trees \( T, U \), both on \( B \). At nodes \( \alpha \) of \( T \), we will have either have a bicephalus \( B^T_\alpha = (\rho^T_\alpha, C^T_\alpha, M^T_\alpha) \), with fine structural properties much like those of \( B \), or a single premouse \( C^T_\alpha \), or a single premouse \( M^T_\alpha \). Much like in the solidity argument, if we have a bicephalus \( B^T_\alpha \) and \( \text{pred}^T(\beta + 1) = \alpha \) and \( \text{cr}(E^T_\beta) < \rho^T_\alpha \) and \( E^T_\beta \) is \( C^T_\alpha \)-total (equivalently, \( M^T_\alpha \)-total), then \( B^T_{\beta+1} \) will be a bicephalus \( \text{Ult}(B^T_\alpha, E^T_\beta) \). Other extenders will just apply to a single model. We will arrange the comparison such that it terminates in the following fashion: there is some node \( \alpha \) of \( U \) or of \( T \), but let us assume it is \( U \), such that:

(i) there is a bicephalus \( B^U_\alpha = (\rho^U_\alpha, C^U_\alpha, M^U_\alpha) \) indexed at \( \alpha \) in \( U \),

(ii) \( C^U_\alpha \) is an initial segment of one of the models of \( T \) indexed at stage \( \alpha \) (there might be one or two of them),

(iii) \( U \upharpoonright [\alpha, \infty) \) is trivial (meaning that it uses no extenders), and

(iv) \( T \upharpoonright [\alpha, \infty) \) is equivalent to an \( m \)-maximal “tree” \( T' \) on \( C^U_\alpha \) (recall point (ii) in this connection), has finite length (so \( \infty = \alpha + n \) for some \( n < \omega \)), uses only finitely generated extenders, and has last model \( M^U_\alpha \).

Point (iv) says that the theorem holds with respect to \( C^U_\alpha, M^U_\alpha \) in place of \( C, M \) (except for the wrinkle mentioned in Footnote 23). From here, we show that the picture reflects back to \( C, M \) themselves, proving the theorem (under the extra hypotheses of universality and solidity).\(^{24}\) The details on how the comparison is formed, and the reflection back to \( C, M \), will be discussed in \( \S \text{11.2} \), and we won’t say anything more about those things here. The analysis of the tail end \( (T, U) \upharpoonright [\alpha, \infty] \) of the comparison is very much like the analysis of the comparison in the proof of Theorem \( \text{9.1} \), which we outlined above. So in \( \S \text{6,7} \) we develop tools needed for that argument both for iteration trees on premice and on (this kind of) bicephali, and within \( \S \text{10,11} \), we develop variants of the more careful finite support argument from \( \S \text{8} \). The bicephali themselves and iteration trees on them are first described in \( \S \text{5} \).

2.3 Structure of main results

The logical structure of the proof of the central fine structural results is as follows:

- In \( \S \text{3} \) and \( \S \text{5–8} \) we lay out basic definitions and prove various lemmas. Parts of this material are quite standard or a small variant of standard theory, but its inclusion will hopefully make for a better reading experience. The material in \( \S \text{3} \) will be used throughout, the material in \( \S \text{5–8} \) used in \( \S \text{9–11} \).

\(^{23}\)Although \( T \) has wellfounded models, at this stage of the proof, we will have to allow the possibility that \( T' \) has illfounded models. (Note \( T' \) is on \( C^U_\alpha \), as opposed to being a normal continuation of \( T \upharpoonright (\alpha + 1) \), and we don’t seem to know here that \( C^U_\alpha \) is itself iterable.) But that illfoundedness will be strictly above the part of the model relevant to the comparison (that is, least disagreements will always occur in the wellfounded part, and the last model of \( T' \) will just be \( M^U_\alpha \), which is wellfounded). Actually, instead of working with \( T' \), we will work with a (real) tree on a certain phalanx, avoiding any possibility of illfoundedness.

\(^{24}\)The possibility of illfoundedness mentioned in Footnote 23 has no bearing on the tree on \( C \) which we get, since \( C \) is iterable, and this tree is only finite in length.
In §9, we prove Lemma 9.2, which is equivalent to Theorem 9.1 on measures in mice, except that it has the additional hypothesis that all proper segments of $M$ are Dodd-sound. This lemma is not needed for the main fine structural results; it is only used to deduce the actual Theorem 9.1 later in §10, and Theorem 9.1 is itself not needed. However, parts of the proof of Lemma 9.2 are referred to in §§10, 11.

In §10, we prove Lemma 10.2, which is equivalent to the super-Dodd-soundness Theorem 10.1, except that it has the additional hypothesis that the 1st core $C_1(M)$ is 1-sound (this is not immediate without 1-universality and 1-solidity).

In §11, we prove Lemma 11.6, which is equivalent to projectum-finite generation 11.5, except that it has the additional hypothesis that $M$ is $(m+1)$-universal and its $(m+1)$st core $C_{m+1}(M)$ is $(m+1)$-solid. The proof will rely on Lemma 10.2. We also deduce Corollary 11.7 from Lemma 11.6.

In §13, we introduce the mouse order $<_p$, and prove some facts about it.

In §14, we prove the solidity and universality Theorem 14.1, using Fact 1.2, the material in §3, §13, and Corollary 11.7. 25

In §15, we deduce that the full versions of Theorems 10.1, 11.5 and 1.1 hold.

If one takes Corollary 11.7 as a black box, the rest of the proof of solidity and universality is covered by just §§3, 13 and 14.

In §10 we also prove Theorem 10.6 (on pseudomice and the ISC) outright; it does not depend on the other results.

3 Fine structural preliminaries

We begin by laying out some general fine structural facts which we will need throughout. This extends somewhat the theory established in [9] and [34].

3.1 Remark (Superstrongs). Nearly all of the general MS theory, including that of [9] and [34], goes through at the superstrong level, with very little change to proofs. (Actually, we only need part of it, since we will be proving solidity etc here anyway.) Likewise for the results in [27], excluding [27, §5], which is appropriately generalized in [24]. We restate some of the theorems from those papers in this section, but without the superstrong restriction.

The main ubiquitous new feature is that if $T$ is an $m$-maximal tree on an $m$-sound premouse, we can have $\alpha + 1 < \text{lh}(T)$ with $\text{lh}(E^T_\alpha) = \text{lh}(E^T_{\alpha+1})$, but only under special circumstances (in particular that $E^T_\gamma$ is superstrong); see [24, §1.1.6]. This implies that when forming a comparison $(T, U)$ of premice, if $\gamma$ indexes the least disagreement

25The only proof that we know of for Corollary 11.7 is via Lemma 11.6, the proof of which relies on significant fine structure, including Dodd-solidity. This is the only way in which Dodd-solidity comes up in the solidity and universality. So a simpler proof of Corollary 11.7 might simplify the proof of solidity and universality overall.

The super aspect of super-Dodd-solidity is not used in the proof, however; only standard Dodd-solidity is relevant.
between \( M^T_\alpha, M^\mu_\alpha \), one should only set \( E^T_\alpha \neq \emptyset \) if \( F(M^T_\alpha|\gamma) \neq \emptyset \) and either \( F(M^\mu_\alpha|\gamma) = \emptyset \) or \( \nu(F(M^\mu_\alpha|\gamma)) \leq \nu(F(M^T_\alpha|\gamma)) \), and likewise symmetrically for \( E^\mu_\alpha \). In some variant situations, such as comparisons involving a tree \( U \) on a phalanx, we may have some models \( M^\mu_\alpha \) which fail to be pre-ISC-premice; but they will be pre-ISC-premice. For any active pre-ISC-premice \( S \), let \( \tilde{\nu}(S) = \max(\text{lgcd}(S), \nu(F^S)) \). Then we modify the decision procedure just mentioned (for determining whether \( E^T_\alpha \neq \emptyset \), etc) by using \( \tilde{\nu}(M^T_\alpha|\gamma) \) and \( \tilde{\nu}(M^\mu_\alpha|\gamma) \) in place of \( \nu(M^T_\alpha|\gamma) \) and \( \nu(M^\mu_\alpha|\gamma) \). We proceed in such a manner in all comparisons in this paper (whether mentioned explicitly or not).

One significant exception to the MS theory readily adapting is that Steel’s proof of Dodd-solidity does not immediately generalize to the superstrong level. Zeman proved the analogue for Jensen-indexed mice (at the superstrong level) in [36]. We establish the MS version in Theorem 10.1.

There is also one small tweak required in proofs of iterability of certain phalanxes and/or bicephali involved in the proofs of solidity, Dodd-solidity, etc. We discuss this in the proof of Claim 9 in the proof of Theorem 14.1, and (toward the end of) the proof of Claim 6 in the proof of Lemma 10.2.26

A premouse \( M \) is superstrong-small if there is no \( \alpha \leq \text{OR}^M \) with \( F^M|\alpha \) superstrong.

3.2 Lemma. The fine structure theory of [9, §4] goes through routinely (allowing for superstrong extenders). Likewise [9, 6.1.5]: If \( T \) is a \( k \)-maximal tree and \( \alpha + 1 < \text{lh}(T) \), then \( E \) is close to \( M^{\alpha+1}_T \).

Proof. We leave the verification to the reader; the presence of superstrong extenders has no substantial impact. (But one should bear Remark 3.1 in mind.) \( \square \)

This paper relies heavily on a result from [27], the proof of which is related to the calculations used in the proof of Strong Uniqueness in [9, §6]. We now explain this; the definition below follows [27, Definition 2.19], and is a variant of \((\rho^M_{m+1}, \rho^M_{m+1})\):

3.3 Definition. Let \( M \) be an \( m \)-sound premouse and \( x \in \mathcal{C}_0(M) \). Then \((z^M_{m+1}(x), \zeta^M_{m+1}(x))\) denotes the least \((z, \zeta) \in \text{OR} \) (see §1.3.1) such that \( \zeta \geq \omega \) and

\[ \text{Th}_{\Sigma^M_{m+1}}(\zeta \cup \{z, \tilde{p}^M_m, x\}) \notin M. \]

And \((z^M_{m+1}, \zeta^M_{m+1})\) denotes \((z^M_{m+1}(\emptyset), \zeta^M_{m+1}(\emptyset))\).

The relationship between \((z^M_{m+1}, \zeta^M_{m+1}), (p^M_{m+1}, \rho^M_{m+1})\) and \((m+1)\)-solidity is made clear in the following fact, which is an easy exercise (or see [27, Remark 2.20]):

3.4 Fact. Let \( M \) be as above and \((z, \zeta) = (z^M_{m+1}, \zeta^M_{m+1}) \) and \((p, \rho) = (p^M_{m+1}, \rho^M_{m+1})\). Then:

- \((z, \zeta) \leq (p, \rho) \) and \( \rho \leq \zeta \) and \( z \leq p \),
- \( M \) is \((m+1)\)-solid\(^{27} \) \( \iff \) \((z, \zeta) = (p, \rho) \iff z = p \iff \zeta = \rho \).
- \( M \) is non-(\(m+1\))-solid \( \iff \) \([\rho < \zeta \text{ and } z < p]) \iff \rho < \zeta \iff z < p.\)

\(^{26}\) The same issue arises in the proof of condensation. This was unfortunately overlooked by the author when writing the proof of [24, Theorem 5.2]. But it is easily corrected, as discussed in the last paragraph of the proof of Claim 9 of Theorem 14.1.

\(^{27}\) Recall here that, as in [32] but not [9], we do not incorporate \((m+1)\)-universality into \((m+1)\)-solidity.
The following definition slightly generalizes the picture of a sequence of extenders being applied along the branch of an iteration tree. In particular, when iterating bicephali in the proof of Lemma 11.6 (toward the proof of 11.5, projectum-finite generation), we will have such situations in which it seems we can’t assume that the extenders are close to the model to which they apply.

3.5 Definition. Let $N$ be a $k$-sound premouse. A degree $k$ abstract weakly amenable iteration of $N$ is a pair $\langle \langle N_\alpha \rangle_{\alpha \leq \lambda}, \langle E_\alpha \rangle_{\alpha < \lambda} \rangle$ such that $N_0 = N$, for all $\alpha < \lambda$, $N_\alpha$ is a $k$-sound premouse, $E_\alpha$ is a short extender weakly amenable to $N_\alpha$, $\text{cr}(E_\alpha) < \rho_k^{N_\alpha}$, $N_{\alpha + 1} = \text{Ult}_k(N_\alpha, E_\alpha)$, and for all limits $\eta \leq \lambda$, $N_\eta$ is the resulting direct limit. We say the iteration is wellfounded if $N_\lambda$ is also wellfounded (in which case $N_\lambda$ is a $k$-sound premouse).

Lemma 3.6 and Fact 3.7 below are proved essentially as is [27, Lemma 2.21] (the presence of superstrong extenders has no relevance to the proof). Together with Lemma 3.8, they will help us analyse the effect of iteration maps on critical objects (such as standard parameters and projecta) in comparison arguments such as the proof of solidity, Dodd-soldiety, and projectum-finite generation.

3.6 Lemma. Let $M$ be a $k$-sound premouse and $\eta < \rho_k^M$ be a limit ordinal. Let $S \subseteq \eta^{<\omega}$ be $M$-amenable; that is, for each $\alpha < \eta$, we have $S \cap \alpha^{<\omega} \in M$. Suppose $S \notin M$. Let $E$ be a short extender, weakly amenable to $M$. Let $U = \text{Ult}_k(M, E)$ and $i = i^{M,k}_E$. Suppose $U$ is wellfounded. Let $S^U = \bigcup_{\alpha < \eta} i(S \cap \alpha^{<\omega})$. Then $S^U \notin U$.

Likewise, if $\langle \langle M_\alpha \rangle_{\alpha \leq \lambda}, \langle E_\alpha \rangle_{\alpha < \lambda} \rangle$ is a wellfounded degree $k$ abstract weakly amenable iteration of $M_0 = M$, then $S^{M_\lambda} = \left( \bigcup_{\alpha < \eta} j_\alpha(S \cap \alpha^{<\omega}) \right) \notin M_\lambda$.

3.7 Fact (($z, \zeta$)-preservation). Let $N$ be $k$-sound and $\langle \langle N_\alpha \rangle_{\alpha \leq \lambda}, \langle E_\alpha \rangle_{\alpha < \lambda} \rangle$ a wellfounded degree $k$ abstract weakly amenable iteration of $N$. Let $j : N \to N_\lambda$ be the final iteration map and $x \in \mathfrak{C}_0(N)$. Then

$$z_{k+1}^{N'} = j(z_{k+1}^N) \quad \text{and} \quad \zeta_{k+1}^{N'} = \sup j^\omega z_{k+1}^N$$

and generally, $z_j^N(j(x)) = j(z_{j+1}^N(x))$ and $\zeta_j^N(j(x)) = \sup j^\omega \zeta_{j+1}^N(x)$.

3.8 Lemma (Preservation of fine structure). Let $N$ be a $k$-sound premouse$^{28}$, $\rho = \rho_k^N$ and $p = p_{k+1}^N$. Let $E$ be a short extender weakly amenable to $N$, with $\kappa = \text{cr}(E) < \rho$. Let $N' = \text{Ult}_k(N, E)$ and $j = i^{N,k}_E$. Suppose that $N'$ is wellfounded. Let $x \in \mathfrak{C}_0(N)$. Let $z_j^N(x) = j(z_{k+1}^N(x))$. Then $z_j^N(x) = j(z_{k+1}^N(x))$, $\zeta_j^N(x) = \sup j^\omega \zeta_{k+1}^N(x)$, $p' = j(p)$ and $p' = \sup j^\omega p_{k+1}^N$. Then:

1. $N'$ is $k$-sound and $j$ is a $k$-embedding.
2. $\text{Thr}_{\Sigma_{k+1}}(p' \cup p') \notin N'$.
3. $\rho_{k+1}^{N'} \leq p'$ and if $\rho_{k+1}^{N'} = \rho'$ then $p_{k+1}^N \leq p'$.
4. If $\kappa < \rho$ or $E$ is close to $N$ then $\rho_{k+1}^{N'} = \rho'$ (and so $p_{k+1}^N \leq p'$).

$^{28}$Note that we do not assume any iterability, nor that $N$ is $(k + 1)$-solid or $(k + 1)$-universal.
5. \( N' \) is \((k + 1)-\)solid iff \([N \text{ is } (k + 1)-\text{solid and } \rho_{k+1}^{N'} = \rho']\).

6. If \( N' \) is \((k + 1)-\)solid then \( p_{k+1}^{N'} = p' \).

7. \( N' \) is \((k + 1)-\)sound iff \([N \text{ is } (k + 1)-\text{sound and } \kappa < \rho]\).

Proof. Part 1 is standard. Part 2 follows from Fact 3.7 applied with \( x = p \); and part 3 is an immediate consequence of part 2.

Part 4: If \( \kappa \geq \rho \) and \( E \) is close to \( N \), the conclusion is standard. So suppose \( \kappa < \rho \). Let \( \alpha < \rho' \) and \( x \in N' \). Let \( \beta \in [\kappa, \rho) \) and \( y \in N \) be such that \( i_E(\beta) \geq \alpha \) and \( x = [a, f]_{E^k} \) for some function \( f \) which is \( r\Sigma^k \) (let \( y = f \) if \( k = 0 \)). Let \( t = \text{Th}_{N_{\Sigma_{k+1}}}(\beta \cup \{y\}) \). Then because \( a \in [i_E(\kappa)]^{<\omega} \) and using the proof that generalized witnesses compute solidity witnesses (see [35, §1.12 + p. 326]) one can show that \( i_E(t) \) computes \( t' = \text{Th}_{N_{\Sigma_{k+1}}}((\alpha \cup \{x\})) \), so \( t' \in N' \).

Part 5: Suppose first that \( N \) is \((k + 1)-\)solid and \( \rho_{k+1}^{N'} = \rho' \). So \( p_{k+1}^{N'} \leq p' \) by part 3. But \((N', p') \) is \((k + 1)-\)solid, which implies that \( p_{k+1}^{N'} \geq p' \), so \( p_{k+1}^{N'} = p' \), so \( N' \) is \((k + 1)-\)solid.

Now suppose that either \( N \) is non-\((k + 1)-\)solid or \( \rho_{k+1}^{N'} < \rho' \). Let \( (z, \zeta) = (z_{k+1}^{N'}, \zeta_{k+1}^{N'}) \) and \( (z', \zeta') = (j(z), \sup j^{<\omega} \zeta) = (z_{k+1}^{N'}, \zeta_{k+1}^{N'}) \) (the second equality by Fact 3.7). Then by Fact 3.4, if \( N \) is non-\((k + 1)-\)solid then \( \rho < \zeta \), so \( \rho_{k+1}^{N'} \leq \rho' < \zeta' \). On the other hand, if \( \rho_{k+1}^{N'} < \rho' \), then since \( \rho < \zeta \), we have \( p_{k+1}^{N'} < \rho' \), so \( \zeta_{k+1}^{N'} \leq \zeta' \). So in either case, \( \rho_{k+1}^{N'} < \zeta' = \zeta_{k+1}^{N'} \), so again by Fact 3.4, \( N' \) is non-\((k + 1)-\)solid.

Part 6: Suppose \( N' \) is \((k + 1)-\)solid. Then by part 5, \( N \) is also \((k + 1)-\)solid. So by Fact 3.4, \( p = z^N_{k+1} \) and \( p_{k+1}^{N'} = z^N_{k+1} \). But by Fact 3.7, \( j(z^N_{k+1}) = z^N_{k+1} \), so \( p' = j(p) = p_{k+1}^{N'} \), as desired.

Part 7: If \( N \) is \((k + 1)-\)sound and \( \kappa < \rho \), then by the previous parts, \( \rho' = \rho_{k+1}^{N'} \) and \( p' = p_{k+1}^{N'} \). But \( j(\kappa) < \rho' \), so

\[
N' = \text{Hull}_{k+1}^{N'}(j(\kappa) \cup \text{rg}(j)) = \text{Hull}_{k+1}^{N'}(\rho' \cup \{p'\}),
\]

so \( N' \) is sound.\(^{29}\)

Conversely, suppose first that \( N \) is non-\((k + 1)-\)sound but \( \kappa < \rho \). We may assume that \( N' \) is \((k + 1)-\)solid, so by the previous parts, \( N \) is \((k + 1)-\)solid, \( \rho' = \rho_{k+1}^{N'} \) and \( p' = p_{k+1}^{N'} \); moreover, there is some \( x \in \mathcal{C}_0(N) \) such that

\[
x \notin \text{Hull}_{k+1}^{N'}(\rho \cup \{p, p_{k+1}^{N'}\})
\]

But this statement is \( r\Pi_{k+1} \) in these parameters (note that \( \rho < \rho_{k+1}^{N} \), so \( \rho \in \text{dom}(j) \), as \( N \) is non-\((k + 1)-\)sound), so

\[
j(x) \notin \text{Hull}_{k+1}^{N'}(j(\rho) \cup \{p_{k+1}^{N'}, p_{k+1}^{N'}\}).
\]

Since \( \rho_{k+1}^{N'} = \rho' \leq j(\rho) \), it follows that \( N' \) is non-\((k + 1)-\)sound.

Finally suppose that \( \rho \leq \kappa \). We may assume \( N' \) is \((k + 1)-\)solid, so \( \rho' = \rho = \rho_{k+1}^{N'} \) and \( p' = p_{k+1}^{N'} \). But then \( \kappa \notin \text{Hull}_{k+1}^{N'}(\rho' \cup \{p', p_{k+1}^{N'}\}) \), so \( N' \) is non-\((k + 1)-\)sound.

\(^{29}\)This direction (that \( N \) being \((k + 1)-\)sound and \( \kappa < \rho \) implies \( N' \) is \((k + 1)-\)sound) can also be proved more directly, as opposed to going through the previous parts, and was observed earlier by Steve Jackson and the author.
The preceding lemma extends easily to abstract weakly amenable iterations:

**3.9 Corollary.** Let \( k < \omega \) and \( N \) be a \( k \)-sound premouse. Let \( \langle (N_\alpha)_{\alpha \leq \lambda}, (E_\alpha)_{\alpha < \lambda} \rangle \) be a wellfounded degree \( k \) abstract weakly amenable iteration on \( N \), with iteration maps \( j_{\alpha \beta} : N_\alpha \rightarrow N_\beta \). Then the obvious generalizations of the conclusions of Lemma 3.8 hold for the maps \( j_{\alpha \beta} : N_\alpha \rightarrow N_\beta \), where the hypothesis of part 4 is modified to say “if for each \( \alpha < \lambda \), either \( \text{cr}(E_\alpha) < \rho_{k+1}(N_\alpha) \) or \( E_\alpha \) is close to \( N_\alpha \),” and the characterization of soundness in part 7 modified to say “\( N \) is \((k+1)\)-sound and \( \text{cr}(E_\alpha) < \rho_{k+1}(N_\alpha) \) for each \( \alpha < \lambda \).”

**3.10 Lemma.** Let \( N \) be a \( k \)-sound premouse and \( T \) be a terminally non-dropping \( k \)-maximal tree on \( N \). Let \( N' = M_T^T \). Then \( \rho_{k+1}^N = \sup_{i \leq k+1} \rho_i^N \) and \( p_{k+1}^N \leq i^T(p_{k+1}^N) \). Moreover, let \( q \leq p_{k+1}^N \) be such that \( q \) is \((k+1)\)-solid for \( N \). Then \( i^T(q) \leq p_{k+1}^N \), and \( q = p_{k+1}^N \) iff \( i^T(q) = p_{k+1}^N \).

**Proof.** We just consider the case \( N' = \text{Ult}_k(N, E) \) where \( E \) is a short extender which is close to \( N \) with \( \kappa = \text{cr}(E) < \rho_k^N \); the generalization to the full lemma is then easy.

If \( \kappa < \rho_k^N \) then the fact that \( \rho_{k+1}^N = \sup_{i \leq k+1} \rho_i^N \) and \( p_{k+1}^N \leq i^T(p_{k+1}^N) \) follows from 3.8. If \( \rho_{k+1}^N \leq \kappa \), use the closeness of \( E \) to \( N \) and [9, §4] as usual. The “moreover” clause follows routinely, using generalized witnesses.

Finally, the following well-known notion will be relevant in §4, and also in the proof of solidity in §14:

**3.11 Definition.** For a \( n \)-sound premouse \( N \) and \( \rho_n^M \leq \rho < \rho_n^M \), we say that \( N \) has the \((n + 1, x)\)-hull property at \( \rho \) iff \( \mathcal{P}(\rho)^N \subseteq \text{hull}_{n+1}^N(\rho \cup \{x, p_n^N\}) \) (note the hull is collapsed).

## 4 Prologue: A non-solid premouse

Before really embarking on our endeavour of proving that iterable premice are (among other things) solid, it is maybe motivating to see that the goal cannot be overly trivial, by finding examples of non-solid premice. We do that in this section, assuming there is a measurable cardinal. The results here are not needed in later sections, however. The basic motivation for the construction is Lemma 3.8: If we can find a \( k \)-sound premouse \( M \) and an extender weakly amenable to \( M \) such that \( \rho^M_{k+1} \leq \text{cr}(E) < \rho^M_{k+1} \) and \( U = \text{Ult}_k(M, E) \) is wellfounded and \( \rho_{k+1}^U < \rho_{k+1}^M \), then \( U \) is non-\((k + 1)\)-solid. In order to achieve this, \( E \) must at least be non-close to \( M \) (though it seems this might not be enough by itself), and one might encode new information into some component measure of \( E \), and this information can then be present in \( U \). Actually we don’t need to work hard to encode new information; starting with some useful fine structural circumstances, things essentially take care of themselves.

**4.1 Theorem.** If there is a measurable cardinal then there is a non-solid premouse.

**Proof.** Suppose there is a measurable cardinal.

**Claim 1.** There is a \((0, \omega_1 + 1)\)-iterable 1-sound countable type 1 premouse \( M \) such that \( \rho_1^M = \omega_1^M, \rho_1^M = \emptyset, \) and \( \text{Hull}_1^M(\{x\}) \) is bounded in \( M \), for each \( x \in M \).
Proof. Let $L[D]$ be Kunen’s model for one measurable cardinal at $\kappa$. So $D \in L[D]$ and
$L[D] = \{D\}$ is the unique normal measure on $\kappa^+$. Let $P$ be the rearrangement of $L[D]$ as a
premouse. Let $E \in E^P$ be the extender given by $D$, and let $N = P \| lh(E)$. Then $N$ is
1-sound and $(0, \omega_1 + 1)$-iterable, with $\rho^N_1 = \kappa^+ = \kappa^+L[D]$.

Given $\alpha < \kappa^+L$, let $H_\alpha = \text{Hull}_1^N(\alpha)$ (recall that this denotes the uncollapsed hull).
Then $H_\alpha \cap \kappa^+L$ is bounded in $\kappa^+L$, because otherwise $\rho^N_1 \leq \kappa$. Since $N$ is type 1, we
have the usual $\Sigma^1_1$ cofinal function $f : \kappa^+L \to OR^N$. Therefore $H_\alpha \cap OR^N$ is bounded
in $OR^N$, so $H_\alpha \in N$ and $N_\alpha = cHull_1^N(\alpha) \in N$ and $\pi_\alpha \in N$ where $\pi_\alpha : N_\alpha \to H_\alpha$ is the
uncollapse map. Note $N_\alpha$ is a premouse and $\pi_\alpha$ is $\Sigma^1_1$-elementary, and $N_\alpha$ is $\alpha$-sound.

Now say $\alpha$ is good iff $\alpha < \omega_1^N$ and $\alpha = cr(\pi_\alpha)$, so $\alpha = \omega_1^{N_\alpha}$. Let $\alpha$ be good. Easily by
condensation, $N_\alpha \in N$, so $N_\alpha \subseteq Q$ where $Q$ is least with $\alpha = \omega_1^{N_\alpha} \leq OR^Q$ and $\rho^Q_1 = \omega$.
Therefore if $\alpha < \beta$ are both good, then $N_\alpha \subseteq N_\beta$ and $N_\alpha \in H_\beta$, and also
\[
\sup(H_\alpha \cap OR) < \sup(H_\beta \cap OR)
\]
(because $Th_{\Sigma^1_1}(\alpha) = Th_{\Sigma^1_1}(\alpha)$, so the parameter $N_\alpha \in H_\beta$ can be used to define the
object $H_\alpha$ in a $\Sigma^1_1$ fashion). Note that the good ordinals form a club $C \subseteq \omega_1^N$, and
$C \in N$. Let $\eta$ be the supremum of the first $\omega$-many good ordinals, so $\eta$ is also good.

Let $M = N_\eta$. We claim that $M$ is as desired. Let us verify that $\rho^N_1 = \eta = \omega_1^{N_\alpha}$.
Clearly $\rho^N_1 \leq \eta$, so we show $\omega_1^N \leq \rho^N_1$. Let $p \in |OR|^N$. Since $H_\eta = \bigcup_{\alpha < \eta} H_\alpha$, we can
fix $\alpha < \eta$ with $\pi_\eta(p) \in H_\alpha$. But then
\[
t = \text{def} \ Th_{\Sigma^1_1}(\eta) \cap \{\{p\}\} = Th_{\Sigma^1_1}(\pi_\eta(p)) = Th_{\Sigma^1_1}(\pi_\eta^{-1}(p)) = Th_{\Sigma^1_1}(\pi_\eta(p)),
\]
and since $N_\alpha \subseteq N|\eta$, therefore $t \in N_\eta$, which suffices. It easily follows that $\rho^N_1 = \emptyset$. And
Hull_1^N(\{x\}) is bounded in $N_\eta$ for each $x \in N_\eta$, by the properties of good $\alpha, \beta < \eta$, which
reflect into $M$. Note that $M = Hull^M_\eta(C \cap \eta)$, so $M$ is countable. The rest is clear. \hfill $\Box$

For the rest of the construction, we just need a mouse $M$ as in Claim 1. So fix such an $M$.\footnote{If $M$ is beyond $\mathcal{O}^\kappa$, then the proof to follow will depend on Theorem 11.5.} Let $U = \text{Ult}_0(M, F^M)$ and $i : M \to U$ the ultrapower map. Then by standard
preservation of fine structure, $\rho^U_1 = \rho^M_1 = \omega_1^M = \omega_1^U$. Recall the notation $M \upharpoonright \xi$ from
$\S 1.3$.

CLAIM 2. $\omega_1^M \subseteq Hull^M_1(X)$ for any unbounded $X \subseteq OR^M$, and $\omega_1^M \subseteq Hull^U_1(X)$ for any
unbounded $X \subseteq OR^U$.

Proof. For $\xi < OR^M$, let $\eta_\xi$ be the least $\eta > \xi$ such that
\[
Th_{\Sigma^1_1}(\omega^M_1) \neq Th_{\Sigma^1_1}(\omega^M_1),
\]
and let $\gamma_\xi$ be the least $\gamma < \omega^M_1$ such that
\[
Th_{\Sigma^1_1}(\gamma + 1) \neq Th_{\Sigma^1_1}(\gamma + 1).
\]
Now let $X \subseteq OR^M$ be unbounded. Let $\gamma < \omega^M_1$. Then there is $\xi \in X$ such that
Hull_1^M(\gamma) is bounded in $\xi$, so $Th_{\Sigma^1_1}(\gamma) = Th_{\Sigma^1_1}(\gamma)$. But then note that $\gamma_\xi \geq \gamma$, and
$\gamma_\xi \in Hull_1^M(X)$, but then since $\gamma_\xi < \omega^M_1$, we have $\gamma + 1 \subseteq Hull^M_1(X)$.

For $U$ it is likewise, as $i : M \to U$ is $\Sigma^1_1$-elementary and cofinal. \hfill $\Box$
Let \( \mu = \mathrm{cr}(F^M) \). Since \( U = \mathrm{Ult}_0(M,F^M) \), we have \( U = \mathrm{Hull}_1^\mu(\mathrm{rg}(i) \cup \{ \mu \}) \), and \( \mathrm{rg}(i) = \mathrm{Hull}_1^\mu(\omega_i^L) \). It follows that \( H' = \mathrm{Hull}_1^\mu(\{ \mu, x \}) \) is bounded in \( U \) for every \( x \in U \); for otherwise, Claim 2 implies that \( \omega \subseteq H' \), and hence \( \mathrm{rg}(i) \subseteq H' \), and hence \( U = H' \), but then \( \rho_1^U = \omega \), a contradiction.

Recall that \( M^{pv} \) was defined in §1.3.2. We will now find another embedding \( j : M^{pv} \to U^{pv} \) with \( \mathrm{cr}(j) = \mu \), use it to lift \( F^M \) to \( F' \) over \( U^{pv} \), defining a premouse \( U' = (U^{pv}, F') \), and deduce from this that \( K = \mathrm{Hull}^U(\{ \mu \}) \) is unbounded in \( U' \) and \( U' \) is non-solid. It seems to be the fusion of the (canonical) information in \( F^M \) together with the (non-canonical) information in \( j \) which leads to the failure of solidity.

**Claim 3.** There is \( j : M^{pv} \to U^{pv} \) such that \( j \) is cofinal and elementary (in the language of passive premouse), \( \mathrm{cr}(j) = \mu \), \( j \) is continuous at \( \mu^+ \), \( U \) has universe \( \{ j(f)(\mu) \mid f \in M \} \), and \( j \neq i \).

**Proof.** Note that \( i \) has all desired properties, except that \( i = i \). So let \( G \) be \( (L[U^{pv}], \mathbb{P}) \)-generic where \( \mathbb{P} = \mathrm{Col}(\omega, \mathrm{OR}^U) \). (We get such a \( G \in V \). For let \( M_1 = U \) and \( M_2 = \mathrm{Ult}_0(M_1,F^{M_1}) \). Then \( M_2 \) is countable and \( L[U^{pv}] \subseteq L[(M_2)^{pv}] \) and \( L[(M_2)^{pv}] \) has no segment projecting strictly across \( \mathrm{OR}^{M_2} \) (consider the length \( \mathrm{OR} \) iteration of \( M \) using \( F^M \) and its images). So \( \mathcal{P}(\mathrm{OR}^{M_2}) \cap L[U^{pv}] \subseteq M_2 \), and so is countable.) Fix a surjection \( g : \omega \to U^{pv} \) with \( g \in L[U^{pv},G] \). Let \( T = T_\theta \) be the natural tree of attempts to build a \( j \) with the desired properties, excluding that \( j \neq i \), with cofinality/surjectivity requirements arranged by using \( g \) for bookkeeping in the natural way. Clearly \( T \) is illfounded, since \( i \) is such an embedding. But we have \( T \in L[U^{pv},G] \), whereas \( i \notin L[U^{pv},G] \), so we get (many) such \( j \neq i \) with \( j \in L[U^{pv},G] \).

Fix any \( j \) as in Claim 3 (it need not be generic over \( L[U^{pv}] \)). Let \( E = E_j \). Note that \( U^{pv} = \mathrm{Ult}_0(M^{pv},E) \) and \( E \) is generated by \( \{ \mu \} \). Let \( U' = \mathrm{Ult}_0(M,E) \), i.e. including the active extender. Then \( j : M \to U' \) is cofinal \( \Sigma_1 \)-elementary, and \( U' \) is a premouse. We have (and let)

\[
H = \mathrm{rg}(j) = \mathrm{Hull}_1^{U'}(\omega_M^1) = \mathrm{Hull}_1^{U'}(\mu),
\]

and \( M \) is the transitive collapse of \( H \). (Note that if \( j \in L[U^{pv},G] \) where \( G \) is set-generic over \( L[U^{pv}] \), then \( U' \notin L[U^{pv},G] \), because from \( j \) and \( U' \) one can recover \( M \), but \( M \notin L[U^{pv},G] \).

**Claim 4.** \( \omega_1^M \subseteq \mathrm{Hull}_1^{U'}(X) \) for any unbounded \( X \subseteq \mathrm{OR}^{U'} \).

**Proof.** Like for \( U \) in the proof of Claim 2. \( \square \)

Let \( K = \mathrm{Hull}_1^{U'}(\{ \mu \}) \). The plan is to show that \( K \) is unbounded in \( U' \), and hence (by Claim 4, line (1) and choice of \( j \) (i.e., as in Claim 3)) \( K = U' \), so \( \rho_1^{U'} = \omega < \rho_1^M \), and so by Lemma 3.8, \( U' \) cannot be 1-solid (actually we will establish the failure of solidity without appeal to 3.8).

Let \( \bar{U} \) be the transitive collapse of \( K \) and \( \pi : \bar{U} \to U' \) the uncollapse. Write \( \pi(\bar{\mu}) = \mu \), \( \pi(\lambda) = \lambda = \mathrm{cr}(F^{U'}) \), etc. Let \( \bar{H} = \mathrm{Hull}_1^{\bar{U}}(\omega_1^{\bar{U}}) \) and \( \bar{M} \) be its transitive collapse and \( \bar{j} : M \to \bar{U} \) the uncollapse.

Recall the hull property from Definition 3.11.

**Claim 5.** We have:

1. \( \bar{U} \) is a premouse and \( \bar{U} = \mathrm{Hull}_1^{\bar{U}}(\{ \bar{\mu} \}) \), so \( \rho_1^{\bar{U}} = \omega \) and \( \rho_1^{\bar{U}} \leq \{ \bar{\mu} \} \).
2. $\tilde{H} = \text{Hull}_{V^1}(ω^1_V) = \text{Hull}_V(\bar{\mu})$

3. $\bar{\lambda}$ is the least ordinal in $\tilde{H}\setminus \bar{\mu}$, and therefore $\text{cr}(\bar{j}) = \bar{\mu}$ and $\bar{j}(\bar{\mu}) = \bar{\lambda}$.

4. $\bar{U}$ has the $(1, \emptyset)$-hull property at $\bar{\mu}$, so $\bar{M}|(\bar{\mu}^+)^{\bar{M}} = \bar{U}|(\bar{\mu}^+)\bar{U}$, so

$$\bar{M} = \text{cHull}_{V^1}(ω^1_V) = \text{cHull}_V(\bar{\mu}) \notin \bar{U}.$$ 

**Proof.** Part 1: This is routine.

Parts 2, 3: These follow easily from the analogous properties of $H, U'$ and $\Sigma_1$-elementarity.

Part 4: As $U'$ has the $(1, \emptyset)$-hull property at $\mu$, this also reflects. (Given $A \subseteq \mu$ with $\text{A} \in \text{rg}(\pi)$, $U' \models \text{"There is } \alpha < \omega_1 \text{ and } A' \subseteq \lambda = \text{cr}(\bar{F}) \text{ such that } A' \in \text{Hull}_V(\{\alpha\})$ and $A' \cap \mu = A''$. So $U'$ satisfies this regarding $\bar{A} = \pi^{-1}(A)$. This gives the $(1, \emptyset)$-hull property.)

**Claim 6.** The normal measure $D_j$ derived from $j$ is not that of any $M$-total $F \in E^+_M$.

**Proof.** We already know $D_j \neq F^M$. But if $F \in E^M$ is $M$-total type 1 then $F \neq D_j$ since

$$\text{Ult}_0(M^M, F) \neq \text{Ult}_0(M^M, F^M) = U^M = \text{Ult}_0(M^M, D_j).$$

**Claim 7.** The normal measure $D_j$ derived from $\bar{j}$ is not that of any $\bar{M}$-total $\bar{F} \in E^+_\bar{M}$.

**Proof.** Let $\bar{F} \in E^+_\bar{M}$ be $\bar{M}$-total type 1 with $\text{cr}(\bar{F}) = \bar{\mu}$. Let $F = j^{-1}(\pi(\bar{j}(\bar{F})))$, or $F = F^M$ if $\bar{F} = F^M$. Suppose for simplicity of notation that $\bar{F} \in E^\bar{M}$ (so $\bar{F} \neq F^M$); otherwise it is essentially the same. So $F \in E^M$ is $M$-total type 1 with $\text{cr}(F) = \mu$. By Claim 6, $E = E_j$ and $F$ have distinct derived normal measures. Fix $A \subseteq \mu$ with $A \in E(\mu) \setminus F(\mu)$. Then letting $D' = j(\bar{F}(\bar{\mu})) \in \text{Hull}_{V^1}(\bar{\mu})$, we have

$$U' \models \exists A' [A' \subseteq \lambda \text{ and } A' \in \text{Hull}_{V^1}(\mu) \text{ and } \mu \in A' \text{ but } A' \notin D']$$

(as witnessed by $A' = j(A)$). This is a $\Sigma_1$-assertion of the parameter $(\mu, D', \lambda)$, so reflects into $\bar{U}$ regarding $\bar{\mu}, \bar{D}, \bar{\lambda}$ where $\bar{\pi}(\bar{D}) = D'$. But $\bar{j}(\bar{F}(\bar{\mu})) = \bar{D}$, and this shows that $\bar{F}(\bar{\mu})$ and $D_j$ are distinct measures.

The following claim is the key step to prove:

**Claim 8.** $K$ is cofinal in $\text{OR}^{U'}$, and hence $\bar{U} = K = U'$.

**Proof.** Suppose $\xi = \sup K \cap \text{OR}^{U'} < \text{OR}^{U'}$. Let $\bar{U} = U' \upharpoonright \xi$. Then $\pi : \bar{U} \rightarrow \bar{U}$ is $\Sigma_1$-elementary. But since $\bar{U} \in U'$ and $\pi$ is determined by $(\pi(\bar{\mu}), \bar{U})$, therefore $\bar{U}, \pi \in U'$ also. Also $\rho^{\bar{U}}_\bar{U} = \omega$ and in fact $\bar{U} = \text{Hull}_V(\{\bar{\mu}\})$, so by Theorem 11.5 and the elementarity of $j$, it follows that $U' \models \text{"C = } \mathcal{C}_1(\bar{U}) \angle U'[ω^1_{U'}] \text{ and } \bar{U} \text{ is an iterate of } C, \text{ via a 0-maximal tree } \bar{T} \text{ of finite length"}$.

That is, whenever we have such a triple $(\bar{M}, \pi^*, M^*) \in M$ (in particular, with $\pi^* : M^* \rightarrow \tilde{M} = M \upharpoonright \zeta$ for some $\zeta$), then $M^*$ is an iterate of $C^* = \mathcal{C}_1(M^*) \angle M[ω^1_M]$ via a finite 0-maximal tree (by Theorem 11.5 and because $M^*$ is $(0, \omega + 1)$-iterable, since

31 If $M$ is below $0^*$, then instead of Theorem 11.5, we can just use the fact that every mouse below $0^*$ is an iterate of its core (see §11.1). Note that just because $\bar{U} = \text{Hull}_V(\{\bar{\mu}\})$, $\bar{T}$ must have finite length.
we have $\pi^*$. But the assertion that this happens for all such triples is $\text{rP}_1$, so lifts to $U'$. (Note that the assertion refers to $E^{U'}$, not just $E^{U'} = E^{U^\omega} = E^U$.)

Since $C$ is 1-sound and $\bar{U}$ is an iterate thereof, $\bar{U}$ is 1-solid and 1-universal. By Claim 5 parts 1 and 4, $p_{\bar{U}}^0 \leq \{\bar{\mu}\}$ but $\bar{M} = \text{cHull}_1^U(\omega_{\bar{U}}) \notin \bar{U}$, so since $\rho_{\bar{U}}^1 = \omega$ and by 1-solidity, either $p_{\bar{U}}^1 = \emptyset$ or $p_{\bar{U}}^1 = \{\alpha\}$ for some $\alpha < \omega_{\bar{U}}$. But in either case and by 1-universality,

$$C = \mathcal{C}_1(\bar{U}) = \text{cHull}_1^U(\omega_{\bar{U}}^1) = \bar{M},$$

and $\bar{j}$ is the core map, and $\bar{j} = i^\bar{T}$ (where $i^\bar{T}$ is the finite tree on $C = \bar{M}$ with last model $\bar{U}$ mentioned above). So $\text{cr}(i^\bar{T}) = \bar{\mu}$ and since $\bar{U} = \text{Hull}_1^U(\{\bar{\mu}\})$, therefore $\bar{U}$ uses only extender $\bar{E}$, which is type 1, $\bar{E} \in \mathcal{E}_+(\bar{M})$, and $\text{cr}(\bar{E}) = \bar{\mu}$. But we also have $\bar{j} = i^\bar{T}$, and this contradicts Claim 7.

So $K$ is cofinal in $U'$, so by Claim 4, $\omega_{U'} = \omega_1^M \subseteq K$, but $\mu \in K$, so $\bar{K} = U'$. □

Claim 9. $\rho_1^{U'} = \omega$ and $p_1^{U'} = \{\mu\}$ and $U'$ is non-1-solid.

Proof. By the previous claim, $\rho_1^{U'} = \omega < \omega_1^M = \rho_1^M$. Since $E$ is weakly amenable to $M$ and by Lemma 3.8 part 5, it follows that $U'$ is non-1-solid. But actually, we will argue directly, without using Lemma 3.8. It suffices to see $p_1^{U'} = \{\mu\}$, since $M = \text{cHull}_1^U(\mu)$ and $\bar{M} \notin U'$. For this, it suffices to see $p_1^{U'} \geq \{\mu\}$, by Claim 5 and since $U' = \bar{U}$. So let $p \in [\mu]^{<\omega}$. Then $A = \text{def} \text{cHull}_1^U(\{p\}) = \text{cHull}_1^M(\{p\})$, since $\text{cr}(j) = \mu$, and since $\rho_1^M = \omega_1^M$, $A \in M[\omega_1^M \subseteq U'$, so we are done. □

Since $U'$ is a non-solid premouse, this completes the proof.

We finish this section with an observation regarding uniqueness of active extenders in $\lambda$-indexed premice, which relates somewhat to the foregoing argument.

**4.2 Theorem.** Let $M$ be an active $\lambda$-indexed premouse with $\text{cof}(\text{OR}^M)$ uncountable. Then $E^{M[\alpha]}$ is the unique $E$ such that $(M \upharpoonright (\text{OR}^M), E)$ is a $\lambda$-indexed premouse.

Proof. Let $N = M^{P\omega}$. Let $M_0 = (N, E_0)$ and $M_1 = (N, E_1)$ be $\lambda$-indexed premice with $E_0 \neq E_1$. We will derive a contradiction. Let $i_n = i_{E_n}^N$ and $\kappa_n = \text{cr}(i_n)$ for $n = 0, 1$. We may assume that $\kappa_0 \leq \kappa_1$. Let $\lambda$ be the largest cardinal of $N$.

**Claim 1.** Let $\alpha \in [\lambda, \text{OR}^N) \cap \text{rg}(i_0) \cap \text{rg}(i_1)$. Let $i_n(\alpha_n) = \alpha$ for $n = 0, 1$. Then:

1. $i_0^\alpha(\alpha_0 + 1) \subseteq \text{rg}(i_1)$
2. if $\kappa_0 = \kappa_1$ then $\alpha_0 = \alpha_1$ and $i_0 \upharpoonright (\alpha_0 + 1) = i_1 \upharpoonright (\alpha_1 + 1)$.

Proof. Let $\beta \geq \alpha$ be least with $\rho_\alpha^{N[\beta]} = \lambda$. By elementarity, $\beta \in \text{rg}(i_0) \cap \text{rg}(i_1)$. Let $i_n(\beta_n) = \beta$. Then $\kappa_n \leq \alpha_n \leq \beta_n < (\kappa_n^+)^N$ and $\rho_n^{N[\beta_n]} = \kappa_n$. But then note that

$$\text{Th}^{N[\beta_0]}(\kappa_0) = \text{Th}^{N[\beta]}(\kappa_0) = \text{Th}^{N[\beta_1]}(\kappa_0).$$

If $\kappa_0 = \kappa = \kappa_1$, it follows that $\beta_0 = \beta_1$ and similarly, because $i_0 \upharpoonright \kappa = \text{id} = i_1 \upharpoonright \kappa$, it follows that $i_0 \upharpoonright (N[\beta_0]^N_\alpha) = i_1 \upharpoonright (N[\beta_1]^N_\alpha)$, which suffices in this case. And if $\kappa_0 < \kappa_1$, it is similar, but we get $i_0 \upharpoonright (N[\beta_0]) \subseteq i_1 \upharpoonright (N[\beta_1])$. □

**Claim 2.** $\text{rg}(i_0) \cap \text{rg}(i_1)$ is bounded in $N$. 28
Proof. Suppose not. As $E_0 \neq E_1$ and by the previous claim, it follows that $\kappa_0 < \kappa_1$ and so $\text{rg}(i_0) \subseteq \text{rg}(i_1)$. But then letting $E$ be the $(\kappa_0, \kappa_1)$-extender derived from $i_0$, note that $E$ is a whole segment of $E_0$ with $\lambda(E) = \kappa_1$. So by the initial segment condition for $\lambda$-indexed premouse, $E \in N$. But $E$ singularizes $(\kappa_1^+)^N$, a contradiction. \qed

Now fix $\gamma < OR^N$ above $\sup(\text{rg}(i_0) \cap \text{rg}(i_1))$. So let $\alpha_0 < (\kappa_1^+)^N$ be least with $i_0(\alpha_0) > \gamma$. Let $\beta_0 < (\kappa_1^+)^N$ be least with $i_1(\beta_0) > i_0(\alpha_0)$. Let $\alpha_1 < (\kappa_1^+)^N$ be least with $i_0(\alpha_1) > i_1(\beta_0)$. Etc. Let $\alpha = \sup_{n<\omega} \alpha_n$ and $\beta = \sup_{n<\omega} \beta_n$.

Since $\text{cof}(\text{OR}^N) = \text{cof}(\kappa_1^+)^N > \omega$, we have $\alpha < (\kappa_1^+)^N$ and $\beta < (\kappa_1^+)^N$. But then $i_0(\alpha + 1) \in N$ and $i_1(\beta + 1) \in N$, by weak amenability. So the construction of $\langle \alpha_n, \beta_n \rangle_{n<\omega}$ can be done inside $N$, so $\text{cof}^N(\alpha) = \omega = \text{cof}^N(\beta)$. So $i_0$ is continuous at $\alpha$ and $i_1$ continuous at $\beta$. But then $i_0(\alpha) = i_1(\beta) \in \text{rg}(i_0) \cap \text{rg}(i_1)$, a contradiction, completing the proof. \qed

Note the preceding argument is reminiscent of the Zipper Lemma. Given also the known methods of translating between extenders and iteration strategies, there is in fact a significant connection there. For the question of identifying the extender sequence of $L[\mathbb{E}]$ when $V = L[\mathbb{E}]$ for a Mitchell-Steel indexed premouse $L[\mathbb{E}]$ see [27] and [28].

5 Standard bicephali

As sketched in §2.2, the proof of projectum-finite-generation (Theorem 11.5, via Lemma 11.6) will use a comparison argument involving a (generalized) bicephalus of the form $B = (\rho, C, M)$, where $M$ is $k$-sound and $(k+1)$-universal, $C = \mathcal{E}_{k+1}(M)$ is $(k+1)$-solid and $\rho = \rho_{k+1}^M$. This bicephalus is therefore like the exact bicephali of $\mathcal{E}_{k+1}(M)$ when $\text{rg} \in (\omega_{k+1}^M, \omega_{k+1}^M)$, a contradiction.

For some self-containment, we give the basic definitions in this section, but omit various calculations which appear in [24]. In §§6–8, we will introduce and deal with with embeddings between iteration trees on premouse and on bicephali, finite support for such trees, and certain decompositions of iteration maps. These will be important in the main arguments in §§9,10,11, and for parts of this material (in §§6,7), it will use the basic definitions and notation relating to bicephali.

In the proof of solidity in §14, we will work with bicephali of a different kind, and also the trees on those bicephali will be formed somewhat differently to those we are about to describe. We will wait until §14 to introduce those bicephali formally, but the main idea and much of the setup and notation is the same.

\[\text{In fact, } M \text{ will be projectum-finitely-generated, as in the statement of Theorem } 11.5. \text{ So we could augment the premouse language with a new constant symbol } x, \text{ and interpret } x \text{ as the least } x \in [\rho^M_{k+1}]^\omega \text{ such that } M = \text{Hull}_{k+1}^M(\rho \cup \{x\}). \text{ Then } M \text{ is } \rho\text{-sound (in fact } (k+1)\text{-sound)} \text{ in the expanded language (with } p_{k+1} = \emptyset \text{ in the new language), and the iteration maps } i \text{ which act on bicephali will all easily preserve the interpretation of the symbol (when shifting } \rho \text{ to sup } i^* \rho = \rho_{k+1}^{M'}, \text{ where } M' \text{ is the iterate). Under this interpretation, the theory of } [24] \text{ goes through regarding the segments of iteration trees which shift bicephali. However, one obviously needs to adapt things for the parts which shift a non-dropping iterate } M' \text{ of } M \text{ above } \rho_{k+1}^{M'}.]\]
5.1 Definition. Let $M$ be an $m$-sound premouse and $\rho < \rho^M_m$. Say that $M$ is $(m+1,\rho)$-finitely generated iff there is some $g \in [\text{OR}^M]^\omega$ such that $\hat{M}_{m+1}^M(\rho \cup \{g, \rho^M_m\})$. Let $g^M(\rho)$ be the least such $g$.

The following is a variant of [24, Definition 3.1].

5.2 Definition. A 3-simple pre-exact bicephalus\(^{33}\) is a structure $B = (\rho, M, N)$ such that $M, N$ are premice, $\rho < \min(\text{OR}^M, \text{OR}^N)$ and $\rho$ is a cardinal of both $M, N, M||\rho^M = N||\rho^N$, for some $m, n \in \omega$, we have $\rho < \rho^M_m$ and $M$ is $m$-sound and $(m+1,\rho)$-finitely generated (hence $\rho^M_m + 1 \leq \rho$) and $\rho^N_n + 1 \leq \rho^N_n$ and $N$ is $n$-sound and $(n+1,\rho)$-finitely generated.

We say $B$ has degree $(m, n)$. We write $(\rho^B, M^0B, M^1B) = (\rho, M, N)$ and $B||\rho^M = M||\rho^N = N||\rho^M$. We say $B$ is trivial iff $M = N$.

Let $\tilde{B} = (\tilde{M}, \tilde{N}, \tilde{\rho})$ be a triple such that $\tilde{M}, \tilde{N}$ are structures for the premouse language satisfying the premouse axioms and $\tilde{\rho}$ is a linear order. We say that $\tilde{B}$ is wellfounded if $\tilde{M}, \tilde{N}$ are both wellfounded, and hence premice, and $\tilde{\rho}$ is a wellorder (and in this case we take $\tilde{M}, \tilde{N}, \tilde{\rho}$ to be transitive).

5.3 Remark. Note that in [24, Definition 3.1], it is allowed for example that $m = -1$, which means that $M$ is type 3 with $\rho = \rho^M_0 = \text{lgcd}(M)$; likewise for $N, n$. Although we could have developed the material in this section also for that case, doing so would have added complications (like those in [24, Definition 3.14], for example). We actually have no application for this case in this paper, so we have instead restricted our attention to $m, n \geq 0$ (hence the adjective “3-simple”). On the other hand, in [24, Definition 3.1], it is also assumed that $M, N$ are $\rho$-sound, whereas this assumption is weakened here. Actually we only need to use these bicephali under the added assumptions that $m = n, \rho = \rho^N_n + 1 = \rho^M_n, N$ is $(n+1)$-universal and $M = \mathcal{E}_{n+1}(N)$ is $(n+1)$-solid (hence $(n+1)$-sound), but $N$ is not $(n+1)$-sound.

From now on in this section, and until §14, the only bicephali we consider will be 3-simple pre-exact, so we drop these two adjectives and just write bicephalus.

5.4 Definition. Let $B$ be a bicephalus of degree $(m, n)$. A short extender $E$ is weakly amenable to $B$ iff $\text{cr}(E) < \rho^B$ and $E$ is weakly amenable to $M^B$ (equivalently, to $N^B$). Suppose $E$ is weakly amenable to $B$. Define

$$B' = \text{Ult}_{m,n}(B, E) = (M', N', \rho')$$

where $M' = \text{Ult}_m(M, E), N' = \text{Ult}_n(N, E)$ and $\rho' = \sup_{i \in E}\rho$.

We generalize abstract iterations of premice in the obvious manner.

5.5 Definition. Let $B$ be a standard bicephalus of degree $(m, n)$. A degree $(m, n)$ abstract weakly amenable iteration of $B$ is a pair $\langle B_\alpha \rangle_{\alpha \leq \lambda}, \langle E_\alpha \rangle_{\alpha < \lambda}$ such that $B_0 = B$, for all $\alpha < \lambda, B_\alpha$ is a degree $(m, n)$ standard bicephalus, $E_\alpha$ is a short extender weakly amenable to $B_\alpha$, $\text{cr}(E_\alpha) < \rho^{B_\alpha}$, $B_{\alpha+1} = \text{Ult}_{m,n}(B_\alpha, E_\alpha)$, and for all limits $\eta \leq \lambda, B_\eta$ is the resulting direct limit. We say the iteration is wellfounded if $B_\lambda$ is wellfounded. \(^{33}\)

\(^{33}\)But we normally just write bicephalus.
5.6 Lemma. With notation as in 5.4, suppose $B' = (\rho', M', N')$ is wellfounded. Let $i : M \to M'$ and $j : N \to N'$ be the ultrapower maps. Then:

1. $B'$ is a bicephalus of degree $(m, n)$.
2. $B'$ is trivial iff $B$ is trivial.
3. $i(g^M(\rho)) = g^{M'}(\rho')$ and $j(g^N(\rho)) = g^{N'}(\rho')$.
4. $i \restriction (B||\rho^B) \subseteq j$ and $i, j$ are continuous/cofinal at $\rho^B$.

Likewise for $B_\lambda$ etc if the abstract iteration in 5.5 is wellfounded.

Proof. This is mostly as in [24, Lemma 3.15], but the augmented bicephalus defined in [24, Definition 3.14] is not relevant, and there are other small differences. Let us first verify that $M'$ is $(m + 1, \rho')$-finitely generated and likewise for $N', n$, and part 3 holds. Let $g' = i(g^M(\rho))$. Then

$$M' = \text{Hul}_{m+1}^M(\rho' \cup \{g', \bar{p}_m^M\})$$

as usual. Moreover, $g'$ is least such, because otherwise there is $g'' \in [\text{OR}^M]^{<\omega}$ and $\beta < \rho$ with

$$g' \in \text{Hul}_{m+1}^M(i(\beta) \cup \{g'', \bar{p}_m^M\}),$$

but this is an $r\Sigma_{m+1}$ assertion over $M'$ about $(i(\beta), g', \bar{p}_m^M)$, so it pulls back to $M$ about $(\beta, g^M(\rho), \bar{p}^M_m)$, a contradiction. Likewise for $N', n$.

Parts 1 and 4 are quite routine consequences of what we have established above.

For part 2, suppose $B$ is non-trivial but $m = n$. Then we can fix an $r\Sigma_{m+1}$ formula $\varphi$ and $\bar{x} \in \rho^m$ such that $M, N \models \varphi(\bar{x}, g^M(\rho))$ iff $N \models \neg \varphi(\bar{x}, g^N(\rho))$. But then by part 3, $M' \models \varphi(i(\bar{x}), g^{M'}(\rho'))$ iff $N' \models \neg \varphi(j(\bar{x}), g^{N'}(\rho'))$, and since $i(\bar{x}) = j(\bar{x})$ by part 4, it follows that $M' \neq N'$.

We next define degree-maximal iteration trees $T$ on bicephali $B = (M, N, \rho)$, and associated notation and terminology. This follows [24, Definition 3.20]: as there, these are much like iteration trees on premice, but associated to each node $\alpha < \text{lh}(T)$ of $T$, will be a structure $B_\alpha$, which is either a bicephalus or a premouse. We write $\mathcal{B}$ for the set of nodes $\alpha$ such that $B_\alpha$ is a bicephalus. If $\alpha \in \mathcal{B}$ then we write $B_\alpha = (M^0_\alpha, M^1_\alpha, \rho_\alpha)$ for this bicephalus. Associated to each node $\alpha$ will also be a set $\text{side}_\alpha \subseteq \{0, 1\}$, with $\text{side}_0 \neq \emptyset$, indicating which “sides” of the base bicephalus $B$ are associated to $B_\alpha$. We will have $\text{side}_\alpha = \{0, 1\}$ iff $\alpha \in \mathcal{B}$.

Note that for $\alpha < \text{lh}(T)$, we have $\alpha \notin \mathcal{B}$ iff $\text{side}_\alpha = \{e\}$ for some $e$, and here $e \in \{0, 1\}$. If $\text{side}_\alpha = \{e\}$ then $B_\alpha$ will be a premouse, and we also write $B_\alpha = M^e_\alpha$. In general, $\mathcal{B} \cap [0, \alpha]^T$ will be a closed initial segment of $[0, \alpha]^T$. Let $\beta = \max(\mathcal{B} \cap [0, \alpha]^T)$, so $B_\beta = (M^0_\beta, M^1_\beta, \rho_\beta)$ is a bicephalus. Suppose $\beta < \alpha$ and $\text{side}_\alpha = \{e\}$. Then we can think of $B_\alpha = M^e_\alpha$ as being “above $M^e_\beta$”, analogous to how models of an iteration tree on a phalanx are “above” a model of the phalanx.

If $\alpha + 1 < \text{lh}(T)$ then we will have an integer exitside$_\alpha \in \text{side}_\alpha$ which indicates the “side” from which the exit extender $E^T_\alpha$ is taken, and in general this “side” is unrestricted. That is, we must just have $E^T_\alpha \in \mathcal{E}_+(M^e_\alpha)$ where $e = \text{exside}_\alpha$. So if $\text{side}_\alpha = \{e\}$ then
exitside, e. But if α ∈ B then in general, we are free to choose any e ∈ \{0, 1\} as exitside, and choose E^T_α ∈ E_+(M^T_α).

5.7 Definition. Let B = (M^0, M^1, ρ) be a bicephalus of degree (m^0, m^1). Let λ ∈ OR\{0\}. A degree-maximal iteration tree on B of length λ is a system

\[\mathcal{T} = \left(\langle \mathbf{\lessdot}^T, B, \mathcal{D}, \mathcal{D}_{\text{deg}}, \langle B, \rho, \text{sides}_\alpha \rangle_{\alpha < \lambda}, \langle M^\epsilon_\alpha, \deg^\epsilon_\alpha, \text{cr}^\epsilon_\alpha \rangle_{\alpha \leq \beta < \lambda} \land \epsilon < 2, \langle \text{exitside}_\alpha, \text{exit}_\alpha, B^*_\alpha \rangle_{\alpha + 1 < \lambda}, \langle M^\epsilon^*_{\alpha + 1}, \text{cr}^\epsilon^*_{\alpha + 1}, \text{cr}^\epsilon_{\alpha + 1, \beta} \rangle_{\alpha + 1 < \beta < \lambda} \land \epsilon < 2 \right),\]

with the following properties for all α < λ:

1. \langle \mathbf{\lessdot}^T \rangle is an iteration tree order on λ,
2. \mathcal{D} \subseteq \lambda is the set of dropping nodes,
3. \emptyset \neq \text{sides}_\alpha \subseteq \{0, 1\},
4. α ∈ B iff \text{sides}_\alpha = \{0, 1\},
5. 0 ∈ B \subseteq λ and B is closed in λ and closed downward under \langle \mathbf{\lessdot}^T \rangle,
6. B_0 = B and (\deg^0_\alpha, \deg^1_\alpha) = (m^0, m^1),
7. If α ∈ B then B_α = (M^0_\alpha, M^1_\alpha, ρ_\alpha) is a degree (m^0, m^1) bicephalus and (\deg^0_\alpha, \deg^1_\alpha) = (m^0, m^1).
8. If α /∈ B and \text{sides}_\alpha = \{e\} then B_\alpha = M^e_\alpha is a \deg^e_\alpha\text{-sound premouse, and } M^{1-e}_\alpha = \emptyset,
9. If α + 1 < λ then letting e = exitside_\alpha, we have e ∈ \text{sides}_\alpha and \text{exit}_\alpha \leq M^e_\alpha.
10. E_\alpha = F^{\text{exit}_\alpha} \neq \emptyset.
11. If α + 1 < β + 1 < \text{lht}(\mathcal{T}) then \text{lht}(E_\alpha) \leq \text{lht}(E_\beta).
12. If α + 1 < \text{lht}(\mathcal{T}) then β = \text{pred}^T(α + 1) is least with \text{cr}(E_\alpha) < \nu(E_\beta). Moreover,
(a) \text{sides}_\alpha+1 \subseteq \text{sides}_\beta
(b) α + 1 ∈ B iff [β ∈ B and \text{cr}(E_\alpha) < \rho_\beta \text{ and } E_\alpha \text{ is } B_\beta \text{-total}].
(c) If α + 1 ∈ B then \( B^*_\alpha = B_\beta, B_{α + 1} = \text{Ult}_{m^0, m^1}(B_\beta, E_\alpha) \) and \( i^0_{α + 1}, i^1_{α + 1} \) are the associated ultrapower maps (as in 5.4).
(d) If $\alpha + 1 \notin \mathcal{B}$ then $\text{side}_{\alpha+1} = \{ e \}$ where $e = \text{exitside}_\beta$, and moreover, $M_{\alpha+1}^e \subseteq M_\beta^e$ and $\deg^e_{\alpha+1}$ are determined as for degree-maximal trees on premice (with $\deg^e_{\alpha+1} \leq m_\alpha$ if $(0, \alpha + 1]^T \cap \mathcal{D} = \emptyset$),

$$M_\alpha^e = \text{Ult}_d(M_{\alpha+1}^e, E_\alpha)$$

where $d = \deg^e_{\alpha+1}$, and $i^\ast_{\alpha+1}$ is the ultrapower map.

(e) $\alpha + 1 \in \mathcal{D}$ iff $\text{side}_{\alpha+1} = \{ e \}$ and $M_{\alpha+1}^e \subsetneq M_\beta^e$.

(f) $\alpha + 1 \in \mathcal{D}_{\text{deg}}$ iff either $\alpha + 1 \in \mathcal{D}$ or $(\text{side}_{\alpha+1} = \{ e \}$ and $\deg^e_{\alpha+1} < \deg^e_\beta$).

13. The iteration map $i^e_{\alpha+1} : M_\alpha^e \to M_\beta^e$ is defined iff $[\alpha \leq T \beta \text{ and } \mathcal{D} \cap (\alpha, \beta]^T = \emptyset]$, and is then defined as usual; likewise for $i^e_{\alpha+1}$ and $i^e_{\alpha+1, \beta} = i^e_{\alpha+1, \beta} \circ i^e_{\alpha+1}$ if $\alpha + 1 < \text{lh}(T)$.

14. Suppose $\alpha$ is a limit. Then $\mathcal{D} \cap (0, \alpha)^T$ is finite and $\alpha \notin \mathcal{D} \cup \mathcal{D}_{\text{deg}}$, $\text{side}_\alpha = \lim_{\beta < \tau_\alpha} \text{side}_\beta$, and the models $M_\alpha^e$ are the resulting direct limits under the iteration maps $i^e_{\beta, \gamma}$ for $\beta \leq T \gamma < T \alpha$, etc., determining $B_\alpha$, etc. And $\deg^e_\alpha = \lim_{\beta < \tau_\alpha} \deg^e_\beta$. Here if $\alpha \in \mathcal{D}$ then $\rho_\alpha = \sup_{\beta < \tau_\alpha} (i^e_{\beta, \alpha} \rho_\beta)$.

Note that even if $E_\beta \in E_+ (M_\beta^e) \cap E_+ (M_\beta^f)$, we still specify $\text{exitside}_\beta$, and this is used to determine $\text{side}_{\alpha+1}$ when $\text{pred}^T (\alpha + 1) = \beta$.

Superscript $T$ denotes an object associated to $T$, so $B_\alpha^T = B_\alpha$, $M_\alpha^T = M_\alpha^e$, etc. We employ other notation associated to iteration trees (in particular as described in §1.3.6) in the obvious manner; for example, if $\text{lh}(T) = \xi + 1$ then $\infty$ may be used to denote $\xi$.

Also define $\mathcal{M}^T = \{ \alpha < \lambda \mid \text{side}_\alpha = \{ e \} \}$. Given for example a bicephalus $(P, Q, \rho)$, we may also write $\mathcal{P}^T = \mathcal{M}^{OT}$ and $P_\alpha = M_\alpha^{OT}$ and $i_{\alpha, \beta} = i^{OT}_{\alpha, \beta}$, and likewise $\mathcal{Q}^T = \mathcal{M}_\alpha^{OT}$ and $Q_\alpha = M_\alpha^{OT}$ and $j_{\alpha, \beta} = i^{OT}_{\alpha, \beta}$.

A putative degree-maximal iteration tree on $B$ is likewise, but if $\lambda = \alpha + 1$, then we do not demand that $B_\alpha$ be wellfounded or that it be a bicephalus/premouse. \(\blacksquare\)

The following lemma summarizes some basic facts. The proof is left to the reader, but it is much as for trees on premice, using Lemmas 5.6 and 3.9 (it is also like [24, Lemma 3.21]).

**5.8 Lemma.** Let $T$ be a putative degree-maximal tree on a degree $(m^0, m^1)$ bicephalus $(M^0, M^1, \rho)$ with wellfounded models. Then it is a degree-maximal tree, i.e. satisfies all conditions of Definition 5.7. Moreover, writing $\mathcal{B} = \mathcal{B}^T$, etc, we have:

1. Let $\alpha < \beta < \text{lh}(T)$. Then:
   - If $\beta \in \mathcal{B}$ then $\text{lh}(E_\alpha) \leq \rho_\beta$ and $\text{lh}(E_\alpha)$ is a cardinal of $B_\beta$.
   - exit$_\alpha(\text{lh}(E_\alpha) = \beta_\alpha || \text{lh}(E_\alpha)$.
   - If $e \in \{ 0, 1 \}$ and $\beta \in \mathcal{M}^e$ then either:
     - $\text{lh}(E_\alpha) < \text{OR}(M^e_\beta)$ and $\text{lh}(E_\alpha)$ is a cardinal of $M^e_\beta$, or
     - $\text{lh}(E_\alpha) = \text{OR}(M^e_\beta)$ and $\beta = \alpha + 1$ and $E_\alpha$ is superstrong and $M^e_\beta$ is active type 2.

2. If $\alpha + 1 < \text{lh}(T)$ then $E_\alpha$ is weakly amenable to $B^*_\alpha + 1$ and if $\alpha + 1 \notin \mathcal{B}$ then $E_\alpha$ is close to $B^*_\alpha + 1$. 

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3. Let $\alpha < \text{lh}(T)$ and $e \in \text{sides}_\alpha$ and $\delta \leq T \alpha$ with $(\delta, \alpha)^T \cap \mathcal{D}_{\text{deg}} = \emptyset$. Then $i^e_\delta$ is a $\deg^e_{\alpha}$-embedding, and if $\delta$ is a successor then $\bar{i}^e_\delta$ is a $\deg^e_{\alpha}$-embedding.

4. Suppose $b^T$ does not drop in model or degree and $e \in \text{sides}_\infty$. Let $m = m_e$ and $M = M_e$. Suppose $\rho_{m+1}^M = \rho$ and $M$ is $(m+1)$-sound. Let $\alpha = \max(\mathcal{B} \cap [0, \infty]^T)$. Then $M^e_\alpha$ is $(m+1)$-sound. If $\alpha < T \infty$ then $M^e_\infty$ is $m$-sound, $(m+1)$-solid and $(m+1)$-universal but not $(m+1)$-sound, $e_{m+1}(M^e_\infty) = M^e_\alpha$ and $i^e_{\alpha, \infty}$ is the core map, which is $p_{m+1}$-preserving.

5. Suppose $b^T$ drops in model or degree and let $\deg^e_e = k$ where $\text{sides}_\infty = \{e\}$. Let $\alpha + 1 \in b^T$ be least such that $(\alpha + 1, \infty)^T$ does not drop in model or degree. Then $M^e_\infty$ is $(k+1)$-universal and $(k+1)$-solid, but not $(k+1)$-sound, and $M^{e_+}_{\alpha+1} = e_{k+1}(M^e_\infty)$ and $i^{e_+}_{\alpha+1, \infty}$ is the core embedding, which is $p_{k+1}$-preserving.

5.9 Definition. The $(\omega_1 + 1)$-iteration game for a bicephalus $B$ is defined in the obvious manner, with players I and II building a putative degree-maximal tree on $B$, player I choosing the extenders $E^e_\alpha$, and exit sides $T^e_\alpha$, when it is ambiguous, and player II the branches, with player II having to ensure wellfoundedness. Iteration strategies and iterability associated to the game are then as usual.

6 Simple embeddings of iteration trees

In this section we discuss the construction of finite hulls $\bar{T}$ of iteration trees $T$, capturing a given element $x \in M^T_x$, in the sense that we have a natural copy map $\zeta : M^T_x \rightarrow M^\bar{T}_x$ with $x \in \text{rg}(\zeta)$. More generally, we will also consider hulls which are finite after some stage of $T$, allowing some control over the critical point of $\zeta$. The tree $T$ will be on a premouse or a bicephalus, and $\bar{T}$ on the same structure. The methods will be used in the main arguments in §§9–11, as outlined in §2.2. They mostly involve quite routine calculations like in the copying construction, but some of the details will be less routine, and in §6.1 and §6.2 below we make some preparations for the less routine aspects.

6.1 Projecta and definable cofinality

When considering a hull $\bar{T}$ of a tree $T$, we will keep track of the correspondence of fine-structural degrees $\deg^T_\alpha$ and $\deg^{\bar{T}}_\alpha$ between nodes $\alpha$ of $T$ and corresponding nodes of $\bar{T}$, by keeping track of the relevant projecta of $M^T_x$ and $M^{\bar{T}}_x$ and how they are shifted by iteration maps.\footnote{Around the time the author worked these things out, Steel worked out some related calculations.} \footnote{One might instead be able to use the methods of [11] to see that the copy maps associated to nodes at degree $n$ are near $n$-embeddings. But this would require some work also, particularly as we do have to allow the possibility that for trees on bicephali, some ultrapowers $\text{Ult}_n(M, E)$ will be formed without $E$ being close to $M$.} To keep track of the projecta, it is important to consider their definable cofinality, at the appropriate level of definability.

We are presently interested in $\rho^M_m$ where $M$ is an $m$-sound premouse, and how $\rho^M_m$ shifts under degree $m$ iteration maps $M \rightarrow \text{Ult}_m(M, E)$, and also, how these things relate to a copied ultrapower map $M' \rightarrow \text{Ult}_m(M', E')$. The $\text{v}_0^{\Sigma^M_0}$-cofinality of $\rho^M_m$ is relevant to this. There are 3 natural ways to define this notion of cofinality, and we will show that, in fact, they are all identical:
6.1 Definition. Let $M$ be an $m$-sound premouse and $\rho = \rho^M_m$.

The weak-$\Sigma^M_{\omega}$-cofinality of $\rho$, denoted $\text{wcof}^{\Sigma^M_{\omega}}(\rho)$, is the least $\kappa \in [\omega, \rho]$ such that either (i) $\kappa = \rho$, or (ii) $\rho \in M$ and $\kappa = \text{cof}^M(\rho)$, or (iii) $m > 0$ and there is $x \in \mathcal{C}_0(M)$ such that $\rho \cap \text{Hull}^M_\omega(\kappa \cup \{x\})$ is unbounded in $\rho$.

The $r^M_{\Sigma^M_{\omega}}$-cofinality of $\rho$, denoted $\text{cof}^{r^M_{\Sigma^M_{\omega}}}(\rho)$, is likewise, but with (iii) replaced by (iic) $m > 0$ and there is a cofinal normal function $f : \kappa \rightarrow \rho$ which is $r^M_{\Sigma^M_{\omega}}$.

The amenable-$r^M_{\Sigma^M_{\omega}}$-cofinality of $\rho$, denoted $\text{aco}^{r^M_{\Sigma^M_{\omega}}}(\rho)$, is likewise, but with (iii) replaced by (iiia) $m > 0$ and there is a cofinal normal function $f : \kappa \rightarrow \rho$ such that $f \upharpoonright \alpha \in M$ for each $\alpha < \kappa$, and the function $F : \kappa \rightarrow M$ is $r^M_{\Sigma^M_{\omega}}$, where $F(\alpha) = f(\alpha) \upharpoonright \alpha$.

We say $F$ is $m$-good for $M$ iff $F$ is as in (iiia), or letting $\kappa = \text{aco}^{r^M_{\Sigma^M_{\omega}}}(\rho)$, then $F \in M$, $F : \kappa \rightarrow M$, $F(\alpha) : \alpha \rightarrow \rho$ for each $\alpha < \kappa$, and $F : \kappa \rightarrow \rho$ is cofinal normal where $f = \bigcup_{\alpha < \kappa} F(\alpha)$. ⊣

6.2 Remark. If $m = 0$ and $M$ is non-type 3 then $\rho^0_M = \text{OR}^M = \text{cof}^{r^M_{\Sigma^M_{\omega}}}(\rho^0_M)$.

6.3 Lemma. Let $M$ be an $m$-sound premouse. Then there is an $F$ which is $m$-good for $M$ iff either $m > 0$ or $M$ is active type 3.

Proof. Let $\rho = \rho^M_m$ and $\kappa = \text{aco}^{r^M_{\Sigma^M_{\omega}}}(\rho)$. If $m = 0$ and $M$ is active type 3, then (iiia) fails trivially as $m = 0$, and $\rho = \nu(F^M) \in M$ (though $\rho \notin M^m$), so $\kappa = \text{cof}^M(\rho)$, and it follows there there is $F \in M$ as required. Now suppose $m > 0$, but there is no $F$ as in (iiia) for $\kappa$. Then note that also (i) does not attain for $\kappa$, so (ii) attains instead. Therefore there is $F \in M$ as required. Conversely, if $m = 0$ but $M$ is not active type 3, then (iiia) fails, and $\rho = \text{OR}^M \notin M$, so there is no $F \in M$ as above, so there is no $F$ which is $m$-good for $M$. □

6.4 Lemma. Let $M$ be an $m$-sound premouse and $\rho = \rho^M_m$. Then

$$\kappa = \text{def} \text{wcof}^{r^M_{\Sigma^M_{\omega}}}(\rho) = \text{cof}^{r^M_{\Sigma^M_{\omega}}}(\rho) = \text{aco}^{r^M_{\Sigma^M_{\omega}}}(\rho),$$

and if $\kappa \in M$ then $\kappa$ is regular in $M$.

Proof. If $m = 0$ then the three ordinals are the same by definition, and clearly if $\kappa \in M$ then $M \models \text{"\kappa is regular"}$.

So suppose $m > 0$. If $\rho \in M$ and $\rho$ is singular in $M$ then it is easy to see that all three values agree with $\text{cof}^M(\rho)$. So suppose that $\rho \notin M$, or $\rho \in M$ and $\rho$ is regular in $M$. Clearly

$$\text{wcof}^{r^M_{\Sigma^M_{\omega}}}(\rho) \leq \text{cof}^{r^M_{\Sigma^M_{\omega}}}(\rho) \leq \text{aco}^{r^M_{\Sigma^M_{\omega}}}(\rho) \leq \rho.$$ 

So suppose $\kappa = \text{wcof}^{r^M_{\Sigma^M_{\omega}}}(\rho) < \rho$; we must find a function $f : \kappa \rightarrow \rho$ witnessing that $\text{aco}^{r^M_{\Sigma^M_{\omega}}}(\rho) = \kappa$.

Since $\kappa < \rho^M_m = \rho$, we can easily find an $r^M_{\Sigma^M_{\omega}}$ function $g : \kappa \rightarrow \rho$ which is cofinal. (Take any $D \subseteq \rho$ and function $g' : D \rightarrow \rho$ which is cofinal and $r^M_{\Sigma^M_{\omega}}$, so in particular, $D$ is $r^M_{\Sigma^M_{\omega}}$. Since $\kappa < \rho^M_m$, therefore $D \in M$. By the minimality of $\kappa$, $D$ has ordertype $\kappa$, so $g'$ is easily modified to produce $g$ as desired.) Let $x \in \mathcal{C}_0(M)$ be such that $g$ is $r^M_{\Sigma^M_{\omega}}(\{x\})$. By the minimality of $\text{wcof}^{r^M_{\Sigma^M_{\omega}}}(\rho)$, note that $g^{\omega \alpha}$ is bounded in $\rho$ for each $\alpha < \kappa$. 35
We claim there is an \( r\Sigma^M_n \) function \( h : \rho^M_{\alpha<\kappa} \to \rho \) which is cofinal and monotone increasing. To see this we stratify \( r\Sigma^M_n \) truth as in \([9, \text{Appendix to } \S 2]\). If \( m = 1 \) and \( M \) is passive and has no largest proper segment, then we just look at the \( r\Sigma^M_n \) truths witnessed by proper segments \( t_n = M | \alpha \leq \kappa, \) for various \( \alpha \leq \text{OR}^M_n \). If \( m = 1 \) and \( M \) has a largest proper segment \( N, \) it is likewise, but letting \( \text{OR}^M_N = \eta, \) then \( t_{\eta+n} = M | (\eta + n) \) denotes \( \mathcal{S}_n(N), \) for \( n < \omega. \) And if \( m = 1 \) and \( M \) is active, one needs to augment proper segments of \( M \) with proper segments of the active extender of \( M. \) If \( m > 1, \) one considers theories of the form

\[
t_{\alpha} = \text{Th}_{r\Sigma^M_{\alpha<\kappa}}(\alpha \cup \{x, \rho^M_{\alpha<\kappa}\})
\]

for various \( \alpha < \rho^M_{m-1}, \) as coding witnesses to \( r\Sigma^M_n \) truths (here \( x \) is the parameter from which we defined \( g \) above). See \([9, \S 2]\) for more details. Now given \( \beta < \rho^M_{m-1}, \) let \( A_{\beta} \subseteq \kappa \) be the set of all \( \alpha < \kappa \) such that there is \( \xi \) such that \( t_{\beta} \) codes a witness to the \( (r\Sigma^m_n(\{x\})) \) statement \( \xi(g(\alpha)) = \xi^x. \) Let \( g_{\beta} : A_{\beta} \to \rho \) be the resulting function (so \( g_{\beta} \subseteq g \)). Then \( g_{\beta} \in M \) since \( t_{\beta} \in M. \) Since \( \rho \) is regular in \( M, \) therefore \( \text{rg}(g_{\beta}) \) is bounded in \( \rho. \)

Now let \( h : \rho^M_{m-1} \to \rho \) be \( h(\beta) = \text{sup}(\text{rg}(g_{\beta})). \) So \( h \) is \( r\Sigma^M_n, \) and clearly \( h \) is cofinal and monotone increasing.

We now claim that for each \( \alpha < \kappa, \) there is \( \beta < \rho^M_{m-1} \) such that \( g | \alpha \subseteq g_{\beta}, \) where \( g_{\beta} \) is as above. For otherwise, letting \( \alpha \) be the least counterexample, note that there is a cofinal \( r\Sigma^M_n \) function \( j : \alpha \to \rho^M_{m-1}, \) but then \( h \circ j : \alpha \to \rho \) is cofinal, contradicting the minimality of \( \kappa. \)

It follows that the function \( f' : \kappa \to \rho \) defined \( f'(\alpha) = \text{sup} g^\alpha \) is as desired, except that \( f' \) need not be strictly increasing. But from \( f' \) we can easily get an \( f \) as desired. \( \Box \)

**6.5 Remark.** Recall \( \hat{i} = \text{Shift}(i) \) was defined in \( \S 1.3.5. \) If \( M,F \) are as in the lemma below and \( M \) is type 3, it can be that \( F \in M \setminus M^m, \) in which case \( F \in \text{dom}(\hat{i}) \) but \( F \notin \text{dom}(i), \) which is why we need to deal with \( \hat{i} \) here.

**6.6 Lemma.** Let \( M \) be an \( m \)-sound premouse and \( E \) be an extender weakly amenable to \( M \) with \( \mu = cr(E) < \rho^M. \) Suppose \( U = \text{Ult}_m(M,E) \) is wellfounded. Let \( i = t^M_i, \) \( E \)

Write \( k^M = \text{cof}^\Sigma^M_n(\rho^M_m), \) and \( k^U = \text{cof}^\Sigma^U_n(\rho^U_m). \) Then:

1. \( k^U = \text{sup} \mu^k \) (recall also \( \rho^U_m = \text{sup} \mu^k \rho^M_m). \)

Moreover, suppose either \( m > 0 \) or \( M \) is active type 3, and let \( F : k^M \to M \) be \( m \)-good for \( M \) (see Lemma 6.3). Then:

2. Suppose \( m > 0 \) and \( \mu \neq k^M. \) Then:

   a. \( \mu^M = \rho^M_m, \) then \( \mu^U = \rho^U_m. \)

   b. \( \mu^M < \rho^M_m, \) then \( \mu^U = i(\mu^M). \)

   c. \( \mu^M = \rho^M_m, \) then \( \mu^U = \rho^U_m. \)

   d. \( \mu^M < \rho^M_m, \) then \( \mu^U = i(\mu^M). \)

   e. \( F \in M \) then \( \text{Shift}(i)(F) : k^U \to U \) is \( m \)-good for \( U. \)

   f. Given any \( r\Sigma^m_n \) term \( t \) and \( x \in \mathfrak{E}_0(M) \) such that \( F = t^M_i, \) we have that \( t^U_{(x)} : k^U \to U \) is \( m \)-good for \( U. \)
3. Suppose \( m > 0 \) and \( \mu = \kappa^M \) (so \( \kappa^M < \rho^M_{m} \)). Then:
   
   (a) If \( \rho^M_{m} = \rho^M_{0} \) then \( \rho^M_{m} < \rho^M_{m-1} = \rho^U_{0} \).  
   (b) If \( \rho^M_{m} < \rho^M_{0} \) then \( \rho^U_{m} < i(\rho^M_{m}) \).  
   (c) \( \kappa^U = \mu = \kappa^M < i(\kappa^N_{m}) \).  
   (d) If \( F \in M \) then \( \text{Shift}(i)(F) \upharpoonright \mu = i \circ F \) is \( m \)-good for \( U \).  
   (e) Given any \( r\Sigma_{m} \) term \( t \) and \( x \in \mathcal{C}_{0}(M) \) such that \( F = t^{M}_{x} \), we have that \( t^{U}_{i(x)} \upharpoonright \mu = i \circ F \) is \( m \)-good for \( U \).

4. Suppose \( m = 0 \) and \( M \) is type 3 and \( \mu \neq \kappa^M \). Then \( i \) is \( \nu \)-preserving and \( \text{Shift}(i)(\kappa^M) = \kappa^U \), and \( \text{Shift}(i)(F) : \kappa^U \to U \) is \( 0 \)-good for \( U \).

5. Suppose \( m = 0 \) and \( M \) is type 3 and \( \mu = \kappa^M \). Then \( i \) is \( \nu \)-high and \( \kappa^U = \mu = \kappa^M \), and \( \text{Shift}(i)(F) \upharpoonright \mu = i \circ F \) is \( 0 \)-good for \( U \).

**Proof.** Part 2: If \( M \) is type 3, then because \( m > 0 \), \( i \) is \( \nu \)-preserving.

Suppose first that \( \mu < \kappa^M \). Then since \( \kappa = \text{wcof}^{\Sigma_{m}^{M}}(\rho^M_{m}) \), all \( g : [\kappa]^{<\omega} \to \rho^M_{m} \) used in forming \( \text{Ult}_{m}(M, E) \) are all bounded in \( \rho^M_{m} \). All the parts now follow easily, as does that \( \kappa^U = \sup i^{\nu} \kappa^M \) in this case.

Now suppose instead that \( \kappa^M < \mu \), so \( \kappa^M < \mu < \rho^M_{m} \). Let \( g : [\mu]^{<\omega} \to \rho^M_{m} \) be \( r\Sigma_{m}^{M} \) and \( a \in [\nu_{E}]^{\omega} \). Let \( f = (\bigcup \text{rg}(F)) \). Note that \( h : [\mu]^{<\omega} \to \kappa^M \) is \( r\Sigma_{m}^{M} \), where \( h(u) \) is the least \( \alpha < \kappa^M \) such that \( h(u) < f(\alpha) \). Since \( \kappa^M < \mu < \rho^M_{m} \), therefore \( h \in M \). There is an \( E_{\kappa} \)-measure one set \( A \subseteq [\mu]^{<\omega} \) such that \( h \upharpoonright A \) is constant. Everything now follows easily, including again that \( \kappa^U = \sup i^{\nu} \kappa^M \).

Part 3: This is straightforward.

Parts 4.5: By calculations as in [9, §9] (one considers the correspondence between the simple ultrapower \( \text{Ult}(M, E) \) (formed without squashing) and \( \text{Ult}_{0}(M, E) \) (the unsquash of the ultrapower of the squash), and the respective ultrapower embeddings).

Part 1: If \( m = 0 \) and \( M \) is non-type 3, then \( \rho^M_{m} = \rho^N_{0} = \text{OR}^{M} = \kappa^M \) and \( \rho^U_{m} = \rho^U_{0} = \text{OR}^{U} = \kappa^U \), and also \( i \) is cofinal, which suffices. Otherwise see the discussion above. \( \square \)

**6.7 Definition.** We say that \( \pi : \mathcal{C}_{0}(M) \to \mathcal{C}_{0}(N) \) is \( m \)-preserving iff \( \pi \) is \( m \)-lifting and:

- \( \pi \) is \( \nu \)-preserving, and
- if \( m > 0 \) then for each \( i \in \{0, m\} \), we have:
  - \( \pi(\rho^M_{i}) = \rho^N_{i} \),
  - if \( \rho^M_{i} < \rho^N_{0} \) then \( \pi(\rho^M_{i}) = \rho^N_{i} \),
  - if \( \rho^M_{i} = \rho^N_{0} \) then \( \rho^N_{i} = \rho^N_{0} \),
  - if \( \text{cf}^{\Sigma_{m}^{M}}(\rho^M_{i}) = \rho^M_{0} \) then \( \text{cf}^{\Sigma_{N}^{N}}(\rho^N_{i}) = \rho^N_{0} \),
  - if \( \kappa = \text{cf}^{\Sigma_{m}^{M}}(\rho^M_{i}) < \rho^M_{0} \) then \( \pi(\kappa) = \text{cf}^{\Sigma_{N}^{N}}(\rho^N_{i}) \), and there is an \( r\Sigma_{m} \)-term \( t \) and \( x \in \mathcal{C}_{0}(M) \) such that \( f_{t,x}^{M} \) is \( i \)-good for \( M \) (where \( f_{t,x}^{M}(u) = t^{M}(x, u) \) and \( f_{t,x}^{N} \) is \( i \)-good for \( N \).
6.8 Lemma. Let $\pi : \mathcal{C}_0(M) \to \mathcal{C}_0(N)$ be $m$-preserving where $m > 0$. Suppose that $\kappa^M = \text{cof}^{\Sigma^m_1}(\rho^M_m) < \rho^M_m$. Let $t_1, x_1$ be such that $f^M_{t_1, x_1}$ is $m$-good for $M$. Then $f^N_{t_1, \pi(x_1)}$ is $m$-good for $N$.

Proof. Fix $t, x$ such that $f^M_{t, x}$ is $m$-good for $M$ and $f^N_{t, \pi(x)}$ is $m$-good for $N$. If $\rho^M_m < \rho^N_m$ then let $\rho = \rho^M_m$, and otherwise let $\rho = 0$. Let

$$T = \text{Th}_{\Sigma_m}(\kappa^M \cup \{x, x_1, \rho, \kappa^M\}).$$

Then $T \in \mathcal{C}_0(M)$, since $\kappa^M < \rho^M_m$. Note that the facts that

(i) $\forall \alpha < \kappa^M [t^M_1(x_1, \alpha) : \alpha \to \rho^M_m$ is a normal function],

(ii) $\forall \alpha < \beta < \kappa^M [t^M_1(x_1, \alpha) \subseteq t^M_1(x_1, \beta)],$

are simply expressed facts about $T$. Also, note that there is a fixed $\Sigma^m$ formula $\varphi$ such that

$$\forall \alpha, \beta < \kappa^M [t^M(x, \alpha + 1)(\alpha) < t^M_1(x_1, \beta + 1)(\beta) \iff \varphi(x, x_1, \alpha, \beta) \in T].$$

Therefore, that

(iii) $\sup \left( \text{rg} \left( \bigcup (\text{rg}(f^M_{t, x})) \right) \right) = \sup \left( \text{rg} \left( \bigcup (\text{rg}(f^M_{t_1, x_1})) \right) \right)$

is also a simple fact about $T$.

Now let $\kappa^N = \text{cof}^{\Sigma^m_1}(\rho^N_m)$. Since $\pi$ is $m$-preserving, if $\rho = 0$ then $\rho^N_m = \rho^N_0$, and if $\rho > 0$ then $\rho^N_m = \pi(\rho)$, and in any case,

$$\pi(T) = \text{Th}_{\Sigma_m}^{\Sigma^m}(\kappa^N \cup \{\pi(x), \pi(x_1), \pi(\rho), \kappa^N\}).$$

But then (i)–(iii) about $T$ lift to $\pi(T)$. Since $f^N_{t_1, \pi(x)}$ is $m$-good for $N$ by assumption, (i)–(iii) for $\pi(T)$ ensure that $f^N_{t_1, \pi(x_1)}$ is also $m$-good for $N$. \hfill \Box

6.2 Essentially degree-maximal trees

The kinds of hulls of iteration trees we consider will lead to the following slight generalization of degree-maximal trees:

6.9 Definition. Let $T$ be an iteration tree on a premouse or bicephalus, satisfying the requirements of degree-maximality, except for the monotone increasing length condition (i.e. that $\text{lh}(E^T_\alpha) \leq \text{lh}(E^T_\beta)$ for $\alpha < \beta$). We say that $T$ is essentially-degree-maximal iff $\nu(E^T_\alpha) \leq \nu(E^T_\beta)$ for all $\alpha + 1 < \beta + 1 < \text{lh}(T)$. Likewise essentially-m-maximal.

Let $T$ be essentially-degree-maximal and $\alpha + 1 < \text{lh}(T)$. We say that $E^T_\alpha$ is $T$-stable iff $\text{lh}(E^T_\alpha) \leq \text{lh}(E^T_\beta)$ for all $\beta \geq \alpha$.

6.10 Definition. Given an $m$-sound premouse $M$, an essentially-($m, \omega_1 + 1$)-iteration strategy for $M$ is just like an ($m, \omega_1 + 1$)-strategy for $M$, except that it works with essentially-$m$-maximal trees instead of $m$-maximal ones.
To calibrate expectations, the reader might consider the natural example. Suppose $T$ is essentially $m$-maximal on $M$, and uses just two extenders $E_0^T$ and $E_1^T$, with $\nu(E_0^T) \leq \nu(E_1^T) < \lh(E_0^T) < \lh(E_1^T)$. Let $\bar{T}$ be the $m$-maximal tree on $M$ that uses just one extender, $E_0^\bar{T} = E_1^T$. Then we claim $M_2^\bar{T} = M_1^T$. For if $\nu(E_1^T) < \nu(E_0^T)$ then $\mathrm{pred}^\bar{T}(2) = 0 = \mathrm{pred}^T(1)$, and clearly $M_2^\bar{T} = M_1^T \subseteq M$ and $\deg_2^\bar{T} = \deg_1^T$, so $M_2^\bar{T} = M_1^T$. On the other hand, suppose $\nu(E_0^T) \leq \nu(E_1^T)$. Then $\mathrm{pred}^\bar{T}(2) = 1$. Moreover, $2 \in \mathcal{D}^\bar{T}$, with $M_2^\bar{T} \prec M_1^T \lh(E_0^T)$. For $\lh(E_0^T) = \nu(E_0^T)^{+T}$ and

$$\nu(E_1^T) \leq \cr(E_1^T) < \cr(E_1^T)^{+\mathrm{exit}_T} < \lh(E_1^T) < \lh(E_0^T) = \nu(E_0^T)^{+\bar{T}},$$

and so $E_1^T$ is not $M_1^T$-total. In $\bar{T}$, we have $\mathrm{pred}^\bar{T}(1) = 0$, and since $M||\lh(E_0^T) = M_1^T||\lh(E_0^T)$, therefore $M_1^T = M_2^\bar{T}$ and $\deg_1^T = \deg_2^\bar{T}$, so $M_1^T = M_2^\bar{T}$.

The lemmas below generalize this example.

**6.11 Lemma.** Let $T$ be an essentially-degree-maximal tree and $\alpha + 1 < \lh(T)$. Then $E_\alpha^T$ is $T$-stable iff either:

- $\alpha + 2 = \lh(T)$,\(^{36}\) or
- $\lh(E_\alpha^T) < \lh(E_{\alpha+1}^T)$, or
- $\lh(E_\alpha^T) = \lh(E_{\alpha+1}^T)$ and either $\alpha + 3 = \lh(T)$ or $\lh(E_{\alpha+1}^T) < \lh(E_{\alpha+2}^T)$.

*Proof.* Suppose $\alpha + 2 < \lh(T)$, so $E_{\alpha+1}^T$ exists. If $\lh(E_\alpha^T) < \lh(E_{\alpha+1}^T)$ then $\lh(E_\alpha^T)$ is a cardinal of exit$^T_{\alpha+1}$, so $\lh(E_\alpha^T) \leq \nu(E_{\alpha+1}^T) \leq \nu(E_\beta^T) < \lh(E_\beta^T)$ for all $\beta > \alpha$, so $E_\alpha^T$ is $T$-stable. Now suppose $\lh(E_\alpha^T) = \lh(E_{\alpha+1}^T) < \lh(E_{\alpha+2}^T)$ (in particular, $E_{\alpha+2}^T$ exists). Then $E_{\alpha+1}^T$ is $T$-stable as above, and it follows that $E_\alpha^T$ is also $T$-stable. Conversely, these are the only options for $E_\alpha^T$ to be $T$-stable, since it is impossible to have $\lh(E_\alpha^T) = \lh(E_{\alpha+1}^T) = \lh(E_{\alpha+2}^T)$. If $\lh(E_\alpha^T) = \lh(E_{\alpha+1}^T)$ then $E_\alpha^T$ is superstrong and $E_{\alpha+1}^T$ is type 2, so $\lh(E_{\alpha+1}^T) < \lh(E_{\alpha+2}^T)$.\(\square\)

**6.12 Lemma.** Let $T$ be an essentially-$m$-maximal tree on a premouse $M$ of length $\alpha + 1$. Then there is an $m$-maximal tree $\bar{T}$ on $M$, with $\lh(\bar{T}) = \bar{\alpha} + 1 \leq \alpha + 1$, such that $M^\bar{T} = M^T$, and if $(0, \alpha)^T$ is non-(model, degree)-dropping then so is $(0, \bar{\alpha})^\bar{T}$ and $i^\bar{T} = i^T$.

Let $T$ be an essentially-degree-maximal tree on a bicephalus $B$. Then there is, analogously, a corresponding degree-maximal tree $\bar{T}$ on $B$.

*Proof.* This is a straightforward generalization of the example above; the extenders used in $\bar{T}$ are those $E$ which are $T$-stable, and the branches of $\bar{T}$ are those determined by $T$ in the obvious manner.\(\square\)

These observations also yield the following lemma:

**6.13 Lemma.** Let $M$ be an $m$-sound premouse and $\Sigma$ be an $(m, \omega_1 + 1)$-strategy for $M$. Then there is an essentially-$(m, \omega_1 + 1)$-strategy $\Sigma'$, with $\Sigma \subseteq \Sigma'$, and the trees $T$ on $M$ via $\Sigma'$ are exactly those which determine a tree $\bar{T}$ via $\Sigma$ as in Lemma 6.12 and its proof.

---

\(^{36}\)Note that $\alpha + 2 = \lh(T)$ iff $M_{\alpha+1}^T$ is the last model of $T$ iff $E_\alpha^T$ is the last extender of $T$. 

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6.3 Simple embeddings

In this subsection we describe and construct the first kind of hulls of iteration trees to be used in §§9–11 (cf. the discussion at the start of §6). In some of the definitions/lemmas, we state things literally only for iteration trees on bicephali; in those cases the version for trees on premice is just the obvious simplification thereof.

6.14 Definition. Let \( \varsigma : M \to N \) be a non-\( \nu \)-high embedding between premice. Let \( E \in \mathbb{E}_+^M \). Then \( \text{copy}(\varsigma, E) \) denotes \( F \), where either:

(i) \( E \in \mathbb{E}^M \) and \( F = \text{Shift}(\varsigma)(E) \) (see §1.3.5 and Remark 6.5), or

(ii) \( E = F^M \), \( \varsigma \) is non-\( \nu \)-low and \( F = F^N \), or

(iii) \( E = F^M \), \( \varsigma \) is \( \nu \)-low and \( F = F^N \mid \text{Shift}(\varsigma)(\nu(F^M)) \).

If \( \varsigma \) is a copy map arising in a typical copying construction, then \( F = \text{copy}(\varsigma, E) \) is the natural copy of \( E \) to an extender in \( F \in \mathbb{E}_+^N \). It is important that \( \varsigma \) is non-\( \nu \)-high here, since otherwise if \( F \in \mathbb{E}_+^N \setminus \mathbb{E}_0^N \) then \( \text{copy}(\varsigma, E) \notin \mathbb{E}_+^N \). Note that in general, we have \( \text{Shift}(\varsigma)(\nu(E)) \geq \nu(\text{copy}(\varsigma, E)) \). We define a variant copying process in 14.14, which handles \( \nu \)-high embeddings, assuming enough condensation.

6.15 Definition (\( \lambda \)-simple embedding). Let \( M \) be an \( m \)-sound premouse, \( T \) an \( m \)-maximal tree on \( M \) and \( T \) essentially-\( m \)-maximal on \( M \); or let \( B = (\rho, M, N) \) be a degree \((m, n)\) bicephalus, \( T \) be degree-maximal on \( B \), and \( T \) essentially-degree-maximal on \( B \). Let \( \lambda < \text{lh}(T) \). We say that

\[
\Phi = (\overline{T}, \varphi, \langle \sigma^0_{\alpha}, \sigma^1_{\alpha} \rangle_{\alpha < \text{lh}(\overline{T})})
\]

is a \( \lambda \)-simple embedding (of \( \overline{T} \)) into \( T \), written \( \overline{T} \rightleftharpoons_{\lambda \text{-sim}} T \), iff the following conditions hold, where we write \( \text{Shift}(\sigma) \) for maps \( \sigma \) in clauses 10–12:

1. \( \lambda + 1 \leq \text{lh}(\overline{T}) < \lambda + \omega \) and \( \text{lh}(\overline{T}) \leq \text{lh}(T) \).

2. \( \overline{T} \upharpoonright (\lambda + 1) = T \upharpoonright (\lambda + 1) \).

3. \( \varphi : \text{lh}(\overline{T}) \to \text{lh}(T) \) is order-preserving with \( \varphi \upharpoonright (\lambda + 1) = \text{id} \).

4. If \( \overline{T}, T \) are on \( B \), then \( \varphi \) preserves bicephalus/model structure; that is:

   (a) \( \text{sides}^\overline{T}_{\alpha} = \text{sides}^T_{\varphi(\alpha)} \).

   (b) if \( \text{sides}^\overline{T}_{\alpha} = \{n\} \) then \( \{\beta, \beta + 1, \text{pred}^\overline{T}(\beta + 1)\} \subseteq \text{rg}(\varphi) \) where \( \beta \) is least such that \( \beta + 1 \leq^\overline{T} \varphi(\alpha) \) and \( \text{sides}^{\overline{T}+1}_{\beta+1} = \{n\} \).

5. \( \varphi \) preserves tree, drop and degree structure. More precisely:

   (a) \( \alpha <^\overline{T} \beta \iff \varphi(\alpha) <^T \varphi(\beta) \).

   (b) if \( \beta + 1 \in (0, \varphi(\alpha)]^T \cap \mathcal{D}^T_{\text{deg}} \) then \( \{\beta, \beta + 1, \text{pred}^T(\beta + 1)\} \subseteq \text{rg}(\varphi) \).

   (c) \( \beta + 1 \in \mathcal{D}^T / \mathcal{D}^T_{\text{deg}} \) iff \( \varphi(\beta + 1) \in \mathcal{D}^T / \mathcal{D}^T_{\text{deg}} \).

   (d) \( \text{deg}^n_{\alpha} = \text{deg}^n_{\varphi(\alpha)} \).
6. If \( n \in \text{sides}^\top_{\alpha} \) then \( \sigma^n_{\alpha} : \mathcal{C}_0(M^n_{\alpha}) \to \mathcal{C}_0(M_{\varphi(\alpha)}^{n\top}) \) is \( \deg_{\alpha}^{n\top} \)-preserving embedding.

7. \( \sigma^n_{\alpha} = \text{id} \) for \( \alpha \leq \lambda \).

8. For \( \alpha + 1 < \text{lh}(\mathcal{T}) \), we have \( \text{exitside}_{\alpha}^\top = \text{exitside}_{\varphi(\alpha)}^\top \), and letting \( n = \text{exitside}_{\alpha}^\top \), we have \( E_{\varphi(\alpha)}^\top = \text{copy}(\sigma^n_{\alpha}, E_{\alpha}^\top) \).

9. Let \( \beta = \text{pred}^\top(\alpha + 1) \). Then:
   
   (a) \( \varphi(\beta) = \text{pred}^\top(\varphi(\alpha) + 1) \),
   
   (b) \( \varphi(\alpha) + 1 \leq^\top \varphi(\alpha + 1) \)
   
   (and note that also \( \text{sides}_{\alpha}^\top(\alpha + 1) = \text{sides}_{\varphi(\alpha) + 1}^\top = \text{sides}_{\alpha + 1}^\top \) and also \( (\varphi(\alpha) + 1, \varphi(\alpha + 1))^\top \cap \mathcal{D}_{\deg}^\top = \emptyset \)),
   
   (c) \( i_{\varphi(\alpha) + 1, \varphi(\alpha) + 1}^\top \) is \( i \)-preserving for each \( (n, i) \) with \( n \in \text{sides}_{\alpha + 1}^\top \) and \( i \leq \deg_{\alpha + 1}^{n\top} \).

10. Let \( \alpha + 1 \in \mathcal{D}^\top \) and \( \beta = \text{pred}^\top(\alpha + 1) \) and \( \{n\} = \text{sides}_{\alpha + 1}^\top \). Then \( \hat{\sigma}_{\beta}(M_{\alpha + 1}^{n\top}) = M_{\varphi(\alpha) + 1}^{n\top} \) (as mentioned above, \( \hat{\sigma} \) denotes \( \text{Shift}(\sigma) \) here and below).

11. Let \( \beta < \alpha < \text{lh}(\mathcal{T}) \) and \( n = \text{exitside}_{\beta} \) and \( \ell \in \text{sides}_{\alpha}^\top \). Then:
   
   - \( \hat{\sigma}_{\beta}^\top \mid \nu_{\beta}^\top \subseteq \sigma_{\ell} \), and
   
   - if \( E_{\beta}^\top \) is type 1 or 2 then \( \hat{\sigma}_{\beta}^\top \mid \text{lh}(E_{\beta}^\top) \subseteq \sigma_{\ell} \)
   
   (and recall \( \sigma \subseteq \hat{\sigma} \) in general).

12. Let \( \beta = \text{pred}^\top(\alpha + 1) \) and \( n \in \text{sides}_{\alpha + 1}^\top \). Then \( \sigma_{\alpha + 1}^n = i_{\varphi(\alpha) + 1, \varphi(\alpha) + 1}^\top \circ \sigma' \), where \( \sigma' \) is defined as in the Shift Lemma. That is, let \( d = \deg_{\alpha + 1}^{n\top} \), \( M^* = M_{\alpha + 1}^{n\top} \), \( M^* = M_{\varphi(\alpha) + 1}^{n\top} \), \( E = E^\top_{\alpha} \), \( E = E_{\varphi(\alpha)}^\top \), \( \ell = \text{exitside}_{\alpha}^\top \). Then for each \( a \in [\nu(E))^\omega \),
   
   if \( d = 0 \), then for each \( f \in \mathcal{C}_0(M^*) \) with \( f : [\text{cr}(E)]^{|a|} \to \mathcal{C}_0(M^*) \),
   
   \[
   \sigma'(\left[a, f\right]_{[\text{cr}(E)]^{|a|}}^{M^*, 0}) = \left[\hat{\sigma}_{\alpha}^\top(a), \hat{\sigma}_{\beta}^\top(f)\right]_{M^*, 0}^{M^*, 0},
   \]
   
   whereas if \( d > 0 \), then for each \( x \in \mathcal{C}_0(M^*) \) and degree \( d \) term \( t \) (see §1.3.3),
   
   \[
   \sigma'\left[\left[a, f_t\right]_{[\text{cr}(E)]^{|a|}}^{M^*, d}\right] = \left[\hat{\sigma}_{\alpha}^\top(a), f_t^{M^*, d}\right]_{M^*, d}^{M^*, d}.
   \]

Actually conditions 10 and 11 follow automatically from the others; we have stated them explicitly here as they help to make sense of the others.

Let \( \Phi : \mathcal{T} \to \lambda \sim \mathcal{T} \) with \( \Phi = (\mathcal{T}, \varphi, \sigma) \) where \( \sigma = (\sigma_{\alpha}^0, \sigma_{\alpha}^1)_{\alpha < \text{lh}(\mathcal{T})} \). We write \( \varphi \Phi = \varphi \) and \( \varphi_{\alpha}^\phi = \sigma_{\alpha}^\phi \). Given \( \beta = \alpha + 1 \in (0, \text{lh}(\mathcal{T})) \), write \( \Phi \mid \beta = (\mathcal{T} \mid \beta, \varphi \mid \beta, \sigma \mid \beta) \).

We now record a few more properties of \( \lambda \)-simple embeddings, writing \( \hat{\sigma} = \text{Shift}(\sigma) \):

6.16 Lemma. Let \( \Phi : \mathcal{T} \to \lambda \sim \mathcal{T} \), let \( \varphi = \varphi \Phi \) and \( \sigma^n_{\alpha} = \sigma_{\alpha}^\varphi \). Let \( \alpha, \beta, \xi < \text{lh}(\mathcal{T}) = \gamma + 1 \). Then:
1. If \( \alpha < ^T \beta \) and \( (\alpha, \beta) \uparrow \cap \mathcal{D}^T = \emptyset \) and \( e \in \text{sides}^T_\beta \) then \( \sigma^e_\beta \circ i^e_\alpha \uparrow = i^e_\alpha \circ \varphi(\beta) \circ \sigma^e_\alpha \).

2. If \( \alpha + 1 \in \mathcal{D}^T \) (so \( \varphi(\alpha + 1) = \varphi(\alpha) + 1 \)) and \( \xi = \text{pred}^T(\alpha + 1) \) and \( \text{sides}^T_{\alpha + 1} = \{e\} \) then
   \[
   \sigma^e_{\alpha + 1} \circ i^e_\alpha \uparrow = i^e_\alpha \circ \varphi(\alpha + 1) \circ \sigma^e_\xi | \mathcal{C}_0(\mathcal{M}^e \uparrow_{\alpha + 1}).
   \]

3. Suppose \( \bar{\nu} = \text{lh}(E^T_{\alpha + 1}) < \text{lh}(E^T_\alpha) \). Then \( \varphi(\alpha + 1) < ^T \varphi(\alpha) + 1 \) and \( E^T_\alpha \) is type 3.

   Let \( \hat{\nu} = \nu(E^T_{\alpha + 1}) \), \( \nu = \nu(E^T_{\varphi(\alpha)}) \), \( d = \text{exitside}^T_\alpha \), \( e \in \text{sides}^T_{\alpha + 1} \), \( \alpha^e_{\bar{\nu}} = i^e_{\varphi(\alpha + 1), \varphi(\alpha + 1)} \), \( \kappa = \text{cr}(j^e) \), \( \lambda = j^e(\kappa) \) and \( \iota = \text{lh}(E^T_{\varphi(\alpha + 1)}) \). Then
   \[
   \hat{\sigma}^e_{\bar{\nu}}(\hat{\nu}) = \nu = \kappa < \lambda = \sigma^e_{\alpha + 1}(\bar{\nu}) < \iota < \lambda^+ \cdot \mathcal{M}^e \uparrow_{\alpha + 1}.
   \]

4. Suppose \( \varphi(\alpha + 1) < ^T \varphi(\alpha + 1) \). Let \( \nu = \nu(E^T_{\varphi(\alpha)}) \) and \( \delta = \sup_{\xi < \varphi(\alpha) + 1} \nu(E^T_\xi) \). Let 
   \( \beta + 1 < \text{lh}(\mathcal{T}) \) and \( \kappa = \text{cr}(E^T_\beta) \) and \( d = \text{exitside}^T_\beta \). Then \( \sigma^e_\beta(\kappa) \notin [\nu, \delta] \).

5. Let \( \alpha < \beta \) and \( d = \text{exitside}^T_\alpha \), \( e \in \text{sides}^T_\beta \), \( \nu = \nu(E^T_{\alpha}), \iota = \text{lh}(E^T_{\alpha}) \). Then:
   \[
   \begin{align*}
   \hat{\sigma}^e_{\alpha} & = \nu = \sigma^e_\beta | \nu, \\
   \hat{\sigma}^d_{\alpha} & \leq \sigma^e_\beta(\nu) = \sigma^e_\beta(\nu), \\
   \hat{\sigma}^d_{\alpha} & (\nu) = \nu(E^T_{\varphi(\alpha)}), \\
   \text{if } E^T_\alpha \text{ is type 1/2 then } E^T_{\alpha + 1} \text{ is } \mathcal{T}-\text{stable, } \hat{\sigma}^d_{\alpha} | \iota = \sigma^e_\beta | \iota \text{ and } \sigma^e_\beta(\iota) = \text{lh}(E^T_{\varphi(\alpha)}).
   \end{align*}
   \]

Proof. We just discuss part 3; the rest is routine and left to the reader. Since \( \sigma^e_\alpha \) is \( \nu \)-preserving, letting \( \sigma' : \mathcal{C}_0(\mathcal{M}^e \uparrow_{\alpha + 1}) \to \mathcal{C}_0(\mathcal{M}^e \uparrow_{\varphi(\alpha) + 1}) \) be defined via the Shift Lemma (as in condition 12 of 6.15), \( \sigma'(\bar{\nu}) = \nu \) and either
   \[
   \begin{align*}
   (i) \ & \text{lh}(E^T_{\alpha}) < \rho_0(\mathcal{M}^e \uparrow_{\alpha + 1}) \text{ and } \sigma'(\text{lh}(E^T_{\alpha})) = \text{lh}(E^T_{\varphi(\alpha)}) \text{ (so } \text{lh}(E^T_{\varphi(\alpha)}) < \rho_0(\mathcal{M}^e \uparrow_{\varphi(\alpha) + 1})), \text{ or} \\
   (ii) \ & \bar{\nu} \text{ is the largest cardinal of } \mathcal{M}^e \uparrow_{\alpha + 1}, \text{ lh}(E^T_{\alpha}) = \rho_0(\mathcal{M}^e \uparrow_{\varphi(\alpha + 1)}), \nu \text{ is the largest cardinal of } \mathcal{M}^e \uparrow_{\varphi(\alpha) + 1} \text{ and } \text{lh}(E^T_{\varphi(\alpha)}) = \rho_0(\mathcal{M}^e \uparrow_{\varphi(\alpha) + 1})^37.
   \end{align*}
   \]

So letting \( \iota' = \sigma'(\bar{\nu}) \), we have \( \iota' < \text{lh}(E^T_{\varphi(\alpha)}) \leq \text{lh}(E^T_{\varphi(\alpha) + 1}) \) as \( \mathcal{T} \) is degree-maximal (not just essentially-degree-maximal). Note that this is independent of \( e \in \{0, 1\} \). So we may assume \( e = \text{exitside}^T_\alpha \). But then if \( \varphi(\alpha + 1) = \varphi(\alpha + 1) \) then \( \sigma^e_{\alpha + 1} = \sigma' \) and we must have that \( E^T_{\varphi(\alpha) + 1} = \text{copy}(\sigma', E^T_{\alpha + 1}) \), so \( \text{lh}(E^T_{\varphi(\alpha) + 1}) = \iota' \), contradicting that \( \iota' < \text{lh}(E^T_{\varphi(\alpha) + 1}) \). So \( \varphi(\alpha + 1) < \varphi(\alpha + 1) \) and \( \kappa < \iota' < j^e(\iota') = \iota = \text{lh}(E^T_{\varphi(\alpha + 1)}) \). So \( \nu = \kappa \), and the rest follows.

\[\square\]

6.17 Definition (reps). Let \( M \) be a premouse. We define \( \text{reps}^M : M \to \mathcal{P}(\mathcal{C}_0(M)) \). If \( M \) is non-type 3, then \( \text{reps}^M(x) = \{x\} \). If \( M \) is type 3, then \( \text{reps}^M(x) \) is the set of pairs \( (a, f) \) such that \( x = [a, f]^M_{P \cdot M} \). For an iteration tree \( \mathcal{T} \), \( \text{reps}^T_\alpha \) denotes \( \text{reps}^{M^\mathcal{T} T}_\alpha \).
A natural way to produce $\lambda$-simple embeddings is through is via finite support:

**6.18 Definition (Finite Support).** Let $\mathcal{T}$ be either an $m$-maximal tree on $m$-sound premouse $M$, or a degree-maximal tree on a bicephalus $B = (\rho, M, N)$.

A finite selection of $\mathcal{T}$ is a finite sequence $\mathcal{F} = (\mathcal{F}_\alpha^\beta)_{(\alpha, \beta) \in J}$ such that $J \subseteq \text{lh}(\mathcal{T}) \times \{0, 1\}$

and for all $(\theta, n) \in J$, we have $n \in \text{sides}_\theta^\mathcal{T}$, $\mathcal{F}_\theta^n \subseteq \mathcal{C}_0(M_\theta^n)$ and $\mathcal{F}_\theta^n$ is finite and non-empty. Write $J_\mathcal{F} = J$ and $I_\mathcal{F} = \{\alpha \mid \exists n \ [(\alpha, n) \in J]\}$.

A finite selection of $\mathcal{T}$ is a finite sequence $\mathcal{S} = (\mathcal{S}_\alpha^n)_{(\alpha, n) \in J}$ of $\mathcal{T}$ such that letting $J = J_\mathcal{F}$ and $I = I_\mathcal{F}$, we have:

1. $0 \in I$.
2. $J = \{(\alpha, n) \mid \alpha \in I \text{ and } n \in \text{sides}_\alpha^\mathcal{T}\}$
3. For each $(\alpha, n) \in J$ we have:
   - $\text{reps}_\alpha^\mathcal{T}(\nu(M_\alpha^nT)) \cap \mathcal{S}_\alpha^n \neq \emptyset$, and
   - letting $d = \text{deg}_\alpha^T$, if $\rho_d(M_\alpha^nT) < \rho_0(M_\alpha^nT)$ then $\rho_d(M_\alpha^nT) \in \mathcal{S}_\alpha^n$.
4. Let $(\beta + 1, e) \in J$ and $E = E_\beta^\mathcal{T}$ and $\ell = \text{exitside}_\beta^T$ and $\gamma = \text{pred}^T(\beta + 1)$ and $m = \text{deg}_{\beta + 1}^T$ and $M^* = M_{\beta + 1}^eT \leq M_\gamma^T$. Then:
   (a) $\beta, \gamma, \gamma + 1 \in I$.
   (b) For all $x \in \mathcal{S}_{\beta + 1}^e$ there is $(t, a, y)$ such that $t$ is an $r\Sigma_1^m$ term, $a \in [\nu_E]^{<\omega}$, reps$_\beta^T(a) \cap \mathcal{S}_{\beta}^e \neq \emptyset$, $y \in \mathcal{C}_0(M^*)$, reps$_\beta^T(y) \cap \mathcal{S}_{\gamma}^e \neq \emptyset$, and
      \[ x = [a, f_{t, y}^M]^m. \]
   (c) If $E \neq F(M_\beta^e)$ then reps$_\beta^T(E) \cap \mathcal{S}_{\beta}^e \neq \emptyset$.
5. Let $(\alpha, e) \in J$ with $\alpha$ a limit ordinal (so $0 \in I \cap \alpha$) and $\beta = \max(I \cap \alpha)$. Then $\beta < T\alpha$, $\beta$ is a successor, sides$^\mathcal{T}_\alpha = \text{sides}_\beta^\mathcal{T}$, $(\beta, T\alpha)T$ does not drop in model or degree, and $\mathcal{S}_\alpha^e \subseteq \mathcal{S}_\beta^e$.

Let $\mathcal{S}, J, I$ be as above. We write

\[ I'_\mathcal{S} = \{\alpha \in I \mid \alpha = \max(I) \text{ or } \alpha + 1 \in I\}. \]

Given a finite selection $\mathcal{F}$ of $\mathcal{T}$, we say that $\mathcal{S}$ is a finite support of $\mathcal{T}$ for $\mathcal{F}$ iff $I_\mathcal{F} \subseteq I'_\mathcal{S}$ and $F_\beta^e \subseteq \mathcal{S}_\beta^e$ for each $(\beta, e) \in J_\mathcal{F}$.

**6.19 Remark.** The above definition is mostly like that of [23, Definition 2.7], with the following small differences. First, we use reps$_\alpha^e$ in place of reps$_\alpha$. Second, we have added condition 3, in order to ensure that we get degree-preserving copy maps. Third, condition (g) of [23, 2.7] has been modified because our mice can have superstrong extenders. The point of (g)(iv) of [23, 2.7] was to ensure that finite support trees have the monotone length condition, which one cannot quite achieve here, and which is why we must allow essentially-degree-maximal, instead of degree-maximal trees.
We can now state the basic lemma on production of \( \lambda \)-simple embeddings mapping into a given tree \( T \):

**6.20 Lemma.** Let \( M, m, T \) or \( B = (\rho, M, N), T \) be as in Definition 6.18. Then:

1. For every finite selection \( \mathcal{F} \) of \( T \), there is a finite support of \( T \) for \( \mathcal{F} \).

2. Let \( S = (\mathcal{S}^n_\alpha)_{(\alpha, n) \in J} \) be a finite support of \( T \), and \( \lambda \in I^1_T \). Then there is a unique pair \((\mathcal{T}, \Phi)\) with

\[
\Phi : \mathcal{T} \rightarrow \lambda, \text{sim} \ T \quad \text{and} \quad \text{rg}(\varphi^\Phi) = (\lambda + 1) \cup I^1_T.
\]

Moreover, for each \( \alpha < \text{lh}(T) \) and \( e \in \text{side}_{\alpha}^T \), we have \( S_{\varphi^\Phi(\alpha)} \subseteq \text{rg}(\sigma^\Phi_e) \).

**Proof Sketch.** Part 1: This is a straightforward construction; a very similar argument is given in [23, Lemma 2.8]. (Here it is actually slightly easier, because of the changes mentioned in the remark above.)

Part 2: This is mostly a routine copying argument, maintaining the properties in 6.15 by induction, and using the properties in 6.16, and noting that every step is uniquely determined by the set \( I_T \), and that the closure of a finite support given by 6.18 is enough to keep things going. However, we will discuss the maintenance of the fact that \( \sigma^d_\alpha \) is a deg\( _{\varphi^\Phi} \)-preserving embedding, since this is not completely standard (though straightforward). (Note that this property ensures that the degrees in \( \mathcal{T} \) match those in \( T \).)

So let \( \alpha \geq \lambda \) and \( \beta = \text{pred}^T(\alpha + 1) \). Let \( e \in \text{side}_{\alpha + 1}^T \) and \( M^* = M^{e_T}_\alpha + 1 \) and \( M^* = M^{e_T}_{\varphi(\alpha + 1)} + 1 \) and \( \pi : e_M(M^*) \rightarrow e_M(M^*) \) be \( \pi = \text{Shift}(\sigma^\Phi_\alpha) \mid e_M(M^*) \). Let \( d = \text{deg}^T_{\alpha + 1} \).

Just because \( \pi \) is \( d \)-lifting, if \( \alpha + 1 \notin \mathcal{D}^T_{\text{deg}} \), then \( \varphi(\alpha + 1) \notin \mathcal{D}^T_{\text{deg}} \). If \( \alpha + 1 \notin \mathcal{D}_{\text{deg}}^T \), then by induction, \( \pi \) is \((d + 1)\)-preserving, which ensures \( \varphi(\alpha + 1) \in \mathcal{D}^T_{\text{deg}} \) and \( \text{deg} e_M \varphi^{\alpha + 1} = d \). Let \( M' = M^{e_T}_{\alpha + 1} + 1 \) and \( M' = M^{e_T}_{\varphi(\alpha + 1)} + 1 \). Let \( i_T = i_T^{e_T}_{\alpha + 1} \) and \( i_T = i_T^{e_T}_{\varphi(\alpha + 1)} + 1 \) and \( \sigma' : e_M(M') \rightarrow e_M(M') \) be defined via the Shift Lemma. So \( \sigma' \) is \( d \)-lifting and \( \pi \circ i_T = i_T \circ \pi \).

We claim that \( \sigma' \) is \( d \)-preserving. For let \( \mu = \text{cr}(E^T_{\alpha}) \). Then \( \sigma(\mu) = \text{cr}(E^T_{\varphi(\alpha)^{\alpha}}) \).

We have \( \mu < \rho^M_\alpha \) and \( \pi(\mu) < \rho^M_\alpha \). Let \( \kappa^M = \text{cof}^T \sum^\Delta_{\text{deg}} (\rho^M_\alpha) \) and \( \kappa^M = \text{cof}^T \sum^\Delta_{\text{deg}} (\rho^M_\alpha) \). Suppose first that \( \mu \neq \kappa^M \). Then since \( \pi \) is \( d \)-preserving, \( \pi(\mu) \neq \kappa^M \). By Lemma 6.6 and some easy calculations then, \( i_T \) and \( i_T \) are \( d \)-preserving. Since \( \pi \circ i_T = i_T \circ \pi \), together with Lemma 6.8, it follows that \( \sigma' \) is \( d \)-preserving, as desired. Now suppose instead that \( \mu = \kappa^M \). Then since \( \pi \) is \( d \)-preserving, \( \pi(\mu) = \kappa^M \) and moreover letting \( f^M_n \) be \( d \)-good for \( M^* \) (here \( t \) is an \( r \Sigma_4 \)-term and \( x \in e_M(M^*) \)), then \( f^M_n \) is also \( d \)-good for \( M^* \). So by Lemma 6.6, \( f^M_n \mid \kappa^M_\alpha \) is \( d \)-good for \( M^* \) and \( f^M_n \mid \kappa^M_\alpha \) is \( d \)-good for \( M^* \). But \( \kappa^{M^*} = \pi(\mu) = \kappa^{M^*} \), so \( \sigma'(\kappa^M) = \kappa^M \), so with the natural term \( t' \) and letting \( x' = \iota_t(x, \kappa^{M^*}) \) we get that \( f^M_n \) is \( m \)-good for \( M^* \) and

\[
\text{if } f^M_n(i_T, \sigma')(x') = f^M_n(i_T, \sigma'(\iota_t(x, \kappa^{M^*}))) \text{ then } \sigma'(\kappa^M) = f^M_n(i_T, \sigma'(\iota_t(x, \kappa^{M^*}))) \text{ is } \kappa^M_\alpha \text{ is } d \text{-good for } M^* \text{.}
\]

which is \( d \)-good for \( M^* \). The rest follows easily.

So \( \sigma' \) is \( d \)-preserving. If \( \varphi(\alpha + 1) = \varphi(\alpha + 1) + 1 \) then \( \sigma^e_{\alpha + 1} = \sigma' \), so we are done. Suppose instead that \( \varphi(\alpha + 1) < \varphi(\alpha + 1) \). Note then that by 6.18, \( \varphi(\alpha + 1) = \eta \) is a limit. So using properties 3 and 5 of 6.18, if \( \rho = \rho_d(M^T_{\eta}) < \rho_0(M^T_{\eta}) \) then \( \rho \in \text{rg}(i^T_{\varphi(\alpha + 1), \eta}) \).
which by \ref{6.6} easily implies that \( \vec{t}^{\mathcal{T}}_{\vec{\varphi}(\alpha)+1,\eta} \) is \( d \)-preserving. Since \( \sigma_{\alpha+1} = \vec{t}^{\mathcal{T}}_{\vec{\varphi}(\alpha)+1,\eta} \circ \sigma' \), this suffices.

7 Super-Dodd structure

As sketched in §2.2, the analysis of comparisons in §§9–11 will rely on keeping track of how embeddings such as iteration maps shift certain critical generators of extenders. The key to understanding this is the analysis of the Dodd structure of extenders, which is the topic of this section. We will actually describe a slight refinement, super-Dodd structure; this is relevant if the mice in question have extenders of superstrong type on their sequence.

The Dodd parameter and projectum of an active premouse \( M \) were introduced in [12, §3] and [16, §4]. The definitions we give for these objects below are stated differently, but they are equivalent. The super-Dodd parameter and projectum are refinements of these notions.

7.1 Definition. Let \( M \) be an active premouse, \( F = F^M \) and \( \mu = \text{cr}(F) \).

Recall that \((t^M, \tau^M)\), the Dodd parameter and projectum of \( M \), are the least \((t, \tau) \in \mathcal{OR} \) such that \( \mu^+ \leq \tau \) and \( F \) is generated by \( t \cup \tau \) (see §1.3.1 for the notation \( \mathcal{OR} \) and ordering thereof).

We define \((\tilde{t}^M, \tilde{\tau}^M)\), the super-Dodd parameter and projectum of \( M \), to be the least \((\tilde{t}, \tilde{\tau}) \in \mathcal{OR} \) such that \( F \) is generated by \( t \cup \tau \). \[ \]

7.2 Remark. Note that the difference between the definition of \((t^M, \tau^M)\) and that of \((\tilde{t}^M, \tilde{\tau}^M)\) is that the former includes the clause \( \mu^+ \leq \tau \), whereas the latter does not.

Let \((\tilde{t}, \tilde{\tau}) = (\tilde{t}^M, \tilde{\tau}^M)\). Then \( \tilde{t} \subseteq [\mu, \nu_F] \) and if \( t \neq \emptyset \) then \( \tilde{\tau} \leq \min(\tilde{t}) \). Either \( \tilde{t} = 0 \) or \( \tilde{\tau} > \mu^+ \). Let \((t, \tau) = (t^M, \tau^M)\). Then \( t = \tilde{t} \setminus [\mu] \) and \( \tau = \max(\tilde{\tau}, \mu^+) \). Steel observed that \( \tau \) is a cardinal of \( M \), so \( \tilde{\tau} \) is also (in fact, either 0 or an infinite cardinal).

We consider the super-Dodd parameter and projectum as, assuming the existence of mice with sufficient large cardinals, it is possible that \( [\mu] \not\subseteq \tilde{t}^M \). This information is recorded in \( \tilde{t}^M \), but not in \((t^M, \tau^M)\). So in this case, \((\tilde{t}^M, \tilde{\tau}^M)\) records more information, and also in this case, super-Dodd-soundness, to be defined below, is more demanding than Dodd-soundness. In case \( [\mu] \not\subseteq \tilde{t}^M \), \( F \) has a superstrong proper segment (see [27, Remark 2.5], in the context of which, \textit{premice} are superstrong-small). So \( \tilde{t}^M \) is just \( t^M \) in case \( M \) is superstrong-small (but even in this case, if \( \tilde{\tau}^M = 0 \) then \( \tilde{\tau}^M \neq \tau^M \)). In Zeman [36] (which uses \( \lambda \)-indexing), premice can have superstrong extenders, but the notion of Dodd parameter used there is analogous to [12]; it never includes the critical point.

7.3 Definition. Let \( \pi : M \to N \) be a \( \Sigma_0 \)-elementary embedding between premice,\(^{38}\) where \( M||\kappa^+ = N||\kappa^+ \) and \( \kappa = \text{cr}(\pi) \). Let \( \alpha \leq \beta \leq \pi(\kappa) \) and \( x \in [\beta]^{\omega} \) and let \( E \) be the \((\kappa, \beta)\)-extender derived from \( \pi \). Then \( E(\pi(\alpha)) \) denotes the set \( F \) of pairs \((A, y)\) such that \( A \in M||\kappa^+ \) and \( y \in [x \cup \alpha]^{\omega} \) and \( y \in \pi(A) \) (this differs from the notation in [12] when \( \alpha < \kappa^+ \)). So letting \( F = E(\pi(\alpha)) \), \( F \) is (or can be treated as) an extender over \( M \). Let \( U = \text{Ult}(M, F) \) and \( x' = [id, x]^{F}_M \) and:

\(^{38}\)This definition does not depend on \( F^M \), \( F^N \), so we may assume that \( M, N \) are passive. We may therefore also apply it to structures \( M', N' \) for a language extending that of passive premice, which are not themselves premice, but whose reducts \( M, N \) to the language of passive premice are premice.
– if \( \xi = \max(x' \cup \alpha) < i_F(\kappa) \) then let \( \gamma = \xi + U \), and
– otherwise let \( \gamma = i_F(\kappa) \) (note in this case \( x' \subseteq \alpha = i_F(\kappa) \)).

Then the trivial completion \( \text{trvcom}(F) \) of \( F \) is the \((\kappa, \gamma)\)-extender derived from \( i_F \) (this agrees with the standard notion when \( x \subseteq \alpha \)). The transitive collapse of \( F \) is \( \text{trvcom}(F) \upharpoonright (x' \cup \alpha) \).

We may identify \( F, \text{trvcom}(F) \), and the transitive collapse of \( F \).

7.4 Definition. Let \( M \) be an active pm, \( F = F^M, \alpha \in \text{OR}^M \cap \lambda^M \) and \( x \in [\text{OR}^M \cap \lambda^M]^{<\omega} \). The Dodd-witness for \((M, (x, \alpha))\) is the extender
\[
Dw^M(x, \alpha) = F \upharpoonright x \cup \alpha.
\]

We say that \( M \) is Dodd-solid iff for each \( \alpha \in i^M \) with \( \alpha > \kappa_F \), letting \( x = t^M(\alpha + 1) \), we have \( Dw^M(x, \alpha) \in M \). We say that \( M \) is super-Dodd-solid iff \( M \) is Dodd-solid and, if \( \kappa_F \in i^M \), then letting \( x = t^M(\kappa_F + 1) \), we have the component measure \( F_x \in M \) (or in notation as for the other witnesses, \( Dw^M(x, \kappa_F) \in M \)). We say that \( M \) is Dodd-amenable iff either \( \tau^M = \mu^+M \) or \( Dw^M(t^M, \alpha) \in M \) for every \( \alpha < \tau^M \).

For passive premice \( P \) we say that \( P \) is (trivially) Dodd-solid and super-Dodd-solid and Dodd-amenable. A premouse \( M \) is Dodd-sound iff \( M \) is Dodd-solid and Dodd-amenable, and \( M \) is super-Dodd-sound iff \( M \) is super-Dodd-solid and Dodd-amenable. ⊥

7.5 Remark. By Remark 7.2, for superstrong-small premice \( M \), super-Dodd-solidity (respectively, -soundness) is equivalent to Dodd-solidity (respectively, -soundness). But in case \( \{\mu\} \subseteq i^M \) where \( \mu = \alpha(F^M) \), super-Dodd-solidity demands that \( F^M \upharpoonright (i^M \setminus \{\mu\}) \in M \), which is not demanded by Dodd-solidity alone. We consider super-Dodd-solidity simply because it strikes the author as the more natural notion in the non-superstrong-small setting, and anyway, its proof (under appropriate hypotheses) is just slightly different from that for usual Dodd-solidity. In our application of Dodd structure, we need only Dodd-solidity, not super-Dodd-solidity.

Steel [16, Theorem 4.1]\(^{39}\) proved that every \((0, \omega_1, \omega_1 + 1)^\ast\)-iterable 1-sound superstrong-small premouse is Dodd-sound (hence all of its proper segments are also). In light of the previous paragraph, Steel’s result immediately implies super-Dodd-soundness for such premice. Zeman [36, Theorems 1.1, 1.2] then proved the corresponding fact for \( \lambda \)-indexed premice (with corresponding iterability hypothesis), but without the “superstrong-small” restriction. Zeman also proved some other related facts in that context.

We will establish in Theorem 10.1 super-Dodd-soundness for \((0, \omega_1 + 1)\)-iterable 1-sound premice (of course, here as elsewhere in the paper, this means with Mitchell-Steel indexing and without the superstrong-smallness restriction). Thus, in comparison with Steel’s result, we will assume only normal iterability, will not assume superstrong-smallness, and establish the stronger conclusion of super-Dodd-soundness. Actually, the main extra work required beyond Steel’s argument will be in handling superstrong extenders; the rest requires only minor modifications. It seems highly likely that the argument we give will be almost contained in the combination of Steel’s and Zeman’s. But the proof we give will follow more the lines of Steel’s.

\(^{39}\)The same result was claimed earlier in [12, Theorem 3.2], but the supposed proof there had a gap, which was filled in [16]. Recall that in [12] and [16], premice are by definition superstrong-small.
Toward the proof of super-Dodd-soundness, and also toward §§9, 11, we will develop various properties of Dodd and super-Dodd parameters and projecta. Some of the notions and facts come from or are slight variants of material from [12] and [27], and many properties of Dodd parameters and projecta established in those papers carry over to the present context; that is, they generalize to premice with superstrongs, and to super-Dodd parameters and projecta. We will also establish, in a certain context, an analogue (Lemma 7.11) of Zeman’s analysis in [36] of the relationship between the Dodd parameter $t^M$ and $p^M_1$. We begin by summarizing some generally useful notation and facts regarding weak hulls of type 2 premice, in Definition 7.6 and Lemma 7.7 below; this is as in [12, Claim 1, proof of Theorem 3.2(A), p. 176] and similar calculations in the proof of [27, Lemma 2.15].

7.6 Definition. Let $N$ be a type 2 premouse, $F = F^N$, and $X = \alpha \cup x$ where $\alpha \leq \nu = \nu_F$ and $x \in [\nu]^{<\omega}$. Then define

$$\text{Gen}'d^N(X) = \{x \in N \mid x \text{ is } F\text{-generated by some } t \in [X]^{<\omega}\}$$

$$= N \cap \{u^N_F(f)(t) \mid f \in N \land t \in [X]^{<\omega}\}$$

(cf. §1.3.4). Let $\kappa = \text{cr}(F)$, $G = F \upharpoonright X$ and $U_X = \text{Ult}(N|\kappa^+, G)$. Let

$$\pi : U_X \to U_\nu$$

be the natural factor map (here $U_\nu$ is just the special case of $U_X$ when $X = \nu$; equivalently, $U_\nu = \text{Ult}(N|\kappa^+, F)$). That is, let $\eta$ be the ordertype of $\text{Gen}'d^N(X) \cap \text{OR}$ and $\sigma : \eta \to \text{Gen}'d^N(X)$ be the isomorphism. Then

$$\pi([a, f]_G^{N|\kappa^+}) = [\sigma(a), f]_F^{N|\kappa^+}.$$ 

So $\text{Gen}'d^N(X) = \text{rg}(\pi) \cap N$.

Now suppose further that $\text{max}(x) + 1 = \nu$ and let $\pi(\nu_X) = \nu$. Then we will define a structure $N_X$ for the language of premice excluding the constant symbol $\dot{F}_X$, and an embedding $\pi_X : N_X \to N$ with $\pi(\nu_X) = \text{Gen}'d^N(X)$. Let $F_X$ be the trivial completion of (the transitive collapse of) $G$. So $\nu(F_X) = \nu_X$. We then define

$$N_X = \langle U_X \upharpoonright X^{+U_X}, \text{F}_X \rangle,$$

where $\text{F}_X$ is the amenable code of $F_X$ (as in [34, between 2.9 and 2.10]), and let $\pi_X : N_X \to N$ be the restriction $\pi_X = \pi | N_X$.

7.7 Lemma. Let $N, \nu, X, x, \kappa$, etc, be as in 7.6 with $\text{max}(x) + 1 = \nu$. Then $\text{cr}(\pi_X) \geq \kappa$.

Suppose $\text{cr}(\pi_X) > \kappa$. Then:

1. $N_X$ is a pre-ISC-premouse.
2. $\pi_X$ is cofinal in OR$^N$ and is $\Sigma_1$-elementary in the language of premice excluding the constant symbol $\dot{F}_X$.

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40It can be that $\text{cr}(\pi_X) = \kappa$ if there are superstrong extenders on $\mathbb{E}^N$. If $\text{cr}(\pi_X) = \kappa$ then $\text{rg}(\pi_X) \cap F^N = \emptyset$, and so $\pi_X$ is not $\Sigma_1$-elementary in the language with $\dot{E}$.

41Recall that this includes constant symbols for the sup of generators of the active extender, so $\pi_X(\nu(F_X^{+N})) = \nu(F_N)$. (It also includes a constant symbol for the critical point, but by hypothesis, $\text{cr}(\pi_X) > \kappa = \text{cr}(F_X^{+N}) = \text{cr}(F_N)$, so $\pi_X(\kappa) = \kappa$ already anyway.)

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3. $F_X$ has largest generator $\nu_X - 1 > \kappa^{+N_X}$, and is not type Z.

4. The following are equivalent:
   - $N_X$ can be expanded to a structure $N'_X$ for the full premouse language such that $\pi_X : N'_X \to N$ is $r\Sigma_1$-elementary (for this language).
   - $F_1^N \in \text{rg}(\pi_X)$.
   - $N_X$ satisfies the weak ISC.$^{42}$
   - $N_X$ satisfies the ISC (so $N_X$ is a premouse).

Proof. This is mostly as in [12, Claim 1, proof of Theorem 3.2(A), p. 176]. But part 4 was not covered there, so we just discuss that. It is straightforward to see that if $F_1^N \in \text{rg}(\pi_X)$ then $N_X$ satisfies the ISC and $\pi_X(F_1^N) = F_1^N$ (and so $\pi_X$ is $r\Sigma_1$-elementary). Now suppose that $N_X$ satisfies the weak ISC, as witnessed by $G \in N_X$. If $F^{N_X} | (\nu_X - 1)$ is non-type Z, so $G$ is the trivial completion of that extender, then it is straightforward see that $\pi_X(G) = F_1^N$ (and this is also non-type Z). Now suppose that $G' = F^{N_X} | (\nu_X - 1)$ is type Z, with largest generator $\gamma$, so $G$ is the trivial completion of $G' | \gamma \leq \gamma$ is a limit of generators of $F^{N_X}$, and $\gamma + U = \gamma + U'$ where $U = \text{Ult}(N_X, G)$ and $U' = \text{Ult}(N_X, G')$. The elementarity of $\pi_X$ gives that $\pi_X(\gamma)$ is a generator of $F$ and $\pi_X(G | \gamma) = F | \pi_X(\gamma)$, so $\pi_X(\gamma)$ is also a limit of generators of $F$. So it suffices to see that $\text{lh}(G) = \gamma^+ U = \nu_X - 1$, because then
   \[
   \text{lh}(\pi_X(G)) = \pi_X(\gamma)^+ \text{Ult}(N_X, \pi_X(G)) = \nu_X - 1,
   \]
   which easily implies $F | (\nu_X - 1)$ is type Z and $\pi_X(G) = F_1^N \in \text{rg}(\pi_X)$. But we have the factor map $\sigma : U' \to U''$ where $U'' = \text{Ult}(N_X, F^{N_X})$, and note $\text{cr}(\sigma) = \nu_X - 1$. So if $\gamma^+ U' \leq \nu_X - 1$ then
   \[
   \gamma^+ U = \gamma^+ U' = \gamma^+ U'' = \gamma^{+N_X},
   \]
   but as $G \in N_X$, in fact $\gamma^+ U < \gamma^{+N_X}$. So $\gamma^+ U = \gamma^+ U' = \nu_X - 1$, as desired. \hfill \Box

The following lemma is a slight generalization of [27, Corollary 2.17], which was an improvement of [12, Lemma 4.4].

7.8 Lemma. Let $M$ be a type 2 premouse, $\mu = \text{cr}(F^M)$, and $\gamma = \max(\rho_1^M, \mu^+ + M)$. Then $\tau^M \geq \gamma$. If $M$ is either Dodd-amenable or 1-sound then $\tau^M = \gamma$.

Proof. To see that $\rho_1^M \leq \tau^M$, observe $F^M = (\tau^M \cup t^M) \not\in M$. Therefore $\gamma \leq \tau^M$ by definition. If $M$ is Dodd-amenable, we get $\tau^M = \gamma$ by using Lemma 7.7 as in the proof of [27, 2.17]. If $M$ is 1-sound, note that by Lemma 7.7, $\rho_1^M \cup t$ generates $F^M$ for some $t \in [\text{OR}^M]^{<\omega}$, and deduce that $\tau^M \leq \gamma$. \hfill \Box

The simple lemma below is a condensation-based variant of the initial segment condition, which will be useful in §8, where we will develop some key technical setup for our proofs of super-Dodd-soundness and Projectum-finite-generation (Theorem 11.5).

7.9 Lemma. Let $M$ be a $(0, \omega + 1)$-iterable 1-sound type 2 premouse, $\mu = \text{cr}(F^M)$ and $\nu = \nu(F^M)$. Suppose $\mu^+ + M < \tau^M$. Then for every $r \in \nu^{<\omega}$ and $\theta < \tau^M$ there is $g \in \nu^{<\omega}$ and $M \cup \{r\}$ such that $r \cup \{\mu, \nu - 1\} \subseteq g$, $F_1^M$ is generated by $g$, $M$ is active type 2 and $F^M$ is the trivial completion of $F^M | g \cup \theta$.

$^{42}$That is, the largest non-type Z segment of $F^{N_X} | (\nu_X - 1)$ is in $N_X$. 48
Proof. By Lemma 7.8, $\rho^M_1 = \tau^M > \mu^+$. We may assume $\theta \geq \mu^+$ and $\theta$ is a cardinal in $M$. Applying [28, Lemma 2.3] with $\theta$ and $r$, we get an $M$, $g$ as required. 

7.10 Remark. Let $M, \theta, \kappa, \pi$ be as in 7.9, with $\mu^+ \leq \theta$. Then $\rho < OR^M < \rho^+$. and because $\kappa, \nu, F^M_\downarrow$ are generated by $g$ and by Lemma 7.7, the natural $\pi : M \to M$ is a 0-embedding with $cr(\pi) \geq \theta$.

Under mild large cardinal hypotheses, it can be that every extender in the sequence of a mouse $M$ is Dodd-sound, and yet iteration trees on $M$ use non-Dodd-sound extenders. We next analyse the nature of such “Dodd-unsoundness”, using the preservation of Dodd-solidity parameters and projecta described in Lemma 7.16.

We now analyse the relationship between $p^M_1$ and $t^M$, in a certain context; this is very similar to that in Jensen indexing, due to Zeman, in [36]. We will use the analysis in the proof of Dodd-solidity.

7.11 Lemma (Dodd parameter characterization). Let $M$ be a type 2 premouse with $F^M$ finitely generated; so $p^M_1 \leq \kappa^+M$ where $\kappa = cr(F^M)$. Suppose $M$ is $\kappa^+$-sound. Let $p = p^M_1 \setminus \kappa^+M$, $\bar{t} = t^M$ and $\xi \in OR$. Then:

1. $p \subseteq \bar{t}$,
2. $\bar{t}$ is the least tuple $t'$ such that $t'$ generates (that is, $F^M$-generates) $(p, F^M_\downarrow)$,
3. $u = \bar{t} \setminus p$ is the least tuple $u'$ such that $u' \cup p$ generates $F^M_\downarrow$ (therefore if $\xi \in u$ then $\xi \cup (\bar{t} \setminus (\xi + 1))$ does not generate $F^M_\downarrow$).
4. if $\xi \in p$ then:
   (a) $(\bar{t} \setminus p) \setminus (\xi + 1) = s \setminus (\xi + 1)$ where $s$ is least s.t. $s \cup (p \setminus (\xi + 1))$ generates $F^M_\downarrow$.
   (b) $\bar{t} \setminus (\xi + 1) \in \text{Hull}^M_1 (\{p \setminus (\xi + 1)\})$.

Proof. Part 2: Certainly $\bar{t}$ generates $(p, F^M_\downarrow)$, since $\bar{t}$ generates every element of $M$. Now let $t' \subseteq \bar{t}$ and suppose $t'$ generates $(p, F^M_\downarrow)$. Let $X = t'$ and $\pi_X : M_X \to M$ be as in Definition 7.6, so $rg(\pi_X) = \text{Gen'}^M(X)$. Then $p_1, F^M_\downarrow \in rg(\pi_X)$. Since $\kappa = cr(F^M_\downarrow)$, therefore $\kappa \in rg(\pi_X)$, which easily gives that $\kappa^+ \subseteq \text{rg}(\pi_X)$. So by Lemma 7.7, $M_X$ is a premouse and $\pi_X$ is $r\Sigma_1$-elementary (in the full premouse language) with $\kappa^+ \cup \{p\} \subseteq \text{Gen'}^M(X) = \text{rg}(\pi_X)$.

So $p^M_1 \in \text{rg}(\pi_X)$, and since $M$ is 1-sound with $p^M_1 \leq \kappa^+$, therefore $M \subseteq \text{rg}(\pi_X)$, so $t'$ generates $F^M$. Therefore $t' = \bar{t}$.

We now prove the remaining parts together. Let

$$p = p^M_1 \setminus \kappa^+M = \{\eta_0 > \eta_1 > \ldots > \eta_{n-1}\}$$

and $\eta_n = 0$ (here $n = 0$ if $p = \emptyset$).

CLAIM. For each $i \leq n$, we have:

1. $p \mid i = \{\eta_0, \ldots, \eta_{i-1}\} \subseteq \bar{t}$,
2. \((\bar{t}(p \upharpoonright i))\backslash (\eta_i + 1) = s\backslash (\eta_i + 1)\), where \(s\) is the least tuple such that \(s \cup (p \upharpoonright i)\) generates \(F^M_i\).

**Proof.** The proof is by induction on \(i \leq n\). Trivially \(p \upharpoonright 0 \subseteq \bar{t}\).

Now fix \(i \leq n\) and suppose \(p \upharpoonright i \subseteq \bar{t}\), and let \(s\) be the least tuple as in clause 2. Since \(\bar{t}\) generates \(F^M_i\), clearly \(s' = \bar{t}(p \upharpoonright i)\) is such that \(s' \cup (p \upharpoonright i)\) generates \(F^M_i\). So suppose \(s\backslash (\eta_i + 1) < (\bar{t}(p \upharpoonright i))\backslash (\eta_i + 1)\). Then note that \(s \cup p\) generates \((F^M_i, p_1^M)\) (here \(s \cup p\) generates \(\kappa\) because it generates \(F^M_i\), and hence generates all points \(\leq \kappa^M\)). Therefore \(s \cup p\) generates \(F^M_i\), but note \(s \cup p \not\subseteq \bar{t}\), a contradiction.

Now suppose \(i < n\), i.e. \(\eta_i > \kappa^M\). We show \(\eta_i \in \bar{t}\). Suppose not. Then

\[\eta_i \in \text{Hull}^M_i(\eta_i \cup \{\bar{t}(\eta_i + 1)\}).\]

But by part 2 of the claim, \(\bar{t}(\eta_i + 1) \in \text{Hull}^M_i(\{p \upharpoonright i\})\). (Use the standard trick for minimization. That is, we clearly get some \(s_0 \in \text{Hull}^M_i(\{p \upharpoonright i\})\) such that \(s_0 \cup (p \upharpoonright i)\) generates \(F^M_i\). But if \(s_0\) is not the least, then we also get some \(s_1 < s_0\) with this property, and so on. Eventually some \(s_m\) is the least.) Therefore

\[\eta_i \in \text{Hull}^M_i(\eta_i \cup \{p \upharpoonright i\}),\]

contradicting the minimality of \(p_1^M\).

The remaining parts of the lemma follow easily from the claim and its proof. \(\square\)

### 7.12 Remark.

In §§9–11, we will need to understand the action of iteration maps on Dodd parameters and projecta, and variants thereof. Suppose \(M\) is a Dodd-sound type 2 premouse and \(E\) is a weakly amenable extender over \(M\), with \(\tau^M \leq \text{cr}(E)\). Suppose that \(N = \text{Ult}_0(M, E)\) is wellfounded. Then \((t^N, \tau^N)\) relates to \((t^M, \tau^M)\), but the relationship depends heavily on \(E\), as, for example, all generators of \(E\) are generators of \(F^N\). (So, for example, letting \(j = \bar{j}^M_0 : M \rightarrow N\), if \(E\) is a normal measure and \(\kappa = \text{cr}(E)\) then \(t^N = j(t^M) \cup \{\kappa\}\) and \(\tau^M = \tau^N\); if \(E\) is type 3 then \(t^N = j(t^M) \setminus \nu(E)\) and \(\tau^N = \nu(E)\).) A certain variant of the Dodd parameter and projectum, considered in [27], is preserved in a fashion more analogous to the standard parameter and projectum. We recall that next, and define the “super-” variant thereof.

Let \(M\) be a premouse and \(N\) a structure for the premouse language.\(^{43}\) Recall from [27, Definition 2.7] that an embedding \(j : M \rightarrow N\) is *Dodd-appropriate* if \(j\) is \(\Sigma_0\)-elementary, cardinal preserving, \(M||\mu^{+M} = N||\mu^{+N}\) where \(\mu = \text{cr}(j)\) (so \(\mu\) is inaccessible in \(M\)), and there is \(\lambda \in \text{OR} \cap \text{wfp}(N)\) such that \(\lambda \leq j(\mu)\) and \(E_j \upharpoonright \lambda \notin N\). Note that this does not require that \(N\) be wellfounded.

Suppose \(j : M \rightarrow N\) is Dodd-appropriate and let \(E = E_j \upharpoonright \xi\) where \(\xi \leq j(\mu)\) and \(E_j\) has no generators in \([\xi, j(\mu)]\), so \(N[j(\mu) = \text{Ult}(M, E)|j_0^M(\mu)\). Also essentially from [27, Definition 2.7], the *Dodd-solidity parameter and projectum of \(j\) (or of \(E\)), denoted \((s_j, \sigma_j)\), are the least \((s, \sigma) \in \text{OR}\) such that \(\sigma \geq \mu^{+M}\) and \(E|s \cup \sigma \notin N\). We also write \((s_E, \sigma_E) = (s_j, \sigma_j)\).

\(^{43}\)Like in 7.3, \(F^M, F^N\) are not relevant here, so we may assume \(M, N\) are passive, and we may also apply the definition to structures \(M', N'\) for a larger language, as long as the reduct \(M\) to the language of passive preimage is a premouse.
7.13 Definition. Let $M$ be a premouse and $j : M \to N$ be Dodd-appropriate.\footnote{Remarks analogous to those in Footnote 38 hold here also.} Let $\mu = \text{cr}(j)$ and let $E = E_j | \xi$ where $\xi \leq j(\mu)$ and $E$ has no generators in $[\xi, j(\mu))$.

Almost as in \cite[Definition 2.29]{Dodd-core}, the Dodd-similarity core of $E$, denoted $\mathfrak{C}_D(E)$, is the extender $\text{trv}(E | s_E \cup s_E)$. And if $E = F^p$ for some active premouse $P$, then $(s^p, \sigma^p)$ denotes $(s_E, \sigma_E)$ and $\mathfrak{C}_D(P)$ denotes the pre-ISC-pm (Definition 1.8) $\tilde{P}$ such that $\tilde{P}|_{\mu^+} = P|_{\mu^+}$, $F^p = \mathfrak{C}_D(E)$ and $\nu(F^p) \geq \delta$, where $\delta$ is the largest cardinal of $\tilde{P}$ (this works via calculations as in Definition 7.6 and Lemma 7.7).

The super-Dodd-similarity parameter and projectum of $j$ (or of $E$), denoted $(\tilde{s}_j, \tilde{\sigma}_j) = (\tilde{s}_E, \tilde{\sigma}_E)$, are the least $(\tilde{s}, \tilde{\sigma}) \in \widetilde{\text{OR}}$ such that $E | \tilde{s} \cup \tilde{\sigma} \notin N$. And if $E = F^p$ for some active premouse $P$, then we write $(\tilde{s}^p, \tilde{\sigma}^p) = (\tilde{s}_E, \tilde{\sigma}_E)$.

Note that if $j : M \to N$ is Dodd-appropriate and $E, \mu$ as in 7.13 then either $\tilde{s}_E > \mu^+M$ or $\tilde{s}_E = 0$. Various facts regarding Dodd-similarity parameters and projecta established in \cite{Dodd-core} generalize to mice with superstrongs, and to super-Dodd-similarity parameters and projecta. Note also that the (super-)Dodd-similarity parameter and projectum is an analogue of $(\tilde{z}_{k+1}, \tilde{\zeta}_{k+1})$ from Definition 3.3. The following facts are clear:

7.14 Fact. Let $M$ be a type 2 premouse. Then:

$$(\tilde{s}_M, \tilde{\sigma}_M) \leq (t_M, \tau_M),$$

$M$ is Dodd-sound iff $(\tilde{s}_M, \tilde{\sigma}_M) = (t_M, \tau_M),$

$$(\tilde{s}_M, \tilde{\sigma}_M) \leq (\tilde{t}_M, \tilde{\tau}_M),$$

$M$ is super-Dodd-sound iff $(\tilde{s}_M, \tilde{\sigma}_M) = (\tilde{t}_M, \tilde{\tau}_M).$

We now give a couple of lemmas which describe how the (super-)Dodd-similarity parameter and projectum are shifted by iteration maps.

7.15 Lemma. Let $M, U$ be premice, $i : M \to U$ be Dodd-appropriate and $\kappa = \text{cr}(i)$. Let $u < \omega$ be such that $U$ is $u$-sound. Let $F$ be weakly amenable to $U$ with $\kappa < \text{cr}(F) < \rho_u$. Suppose $W = \text{Ult}_u(U, F)$ is wellfounded. Let $j = i^U_u : U \to W$. Then

$$s_{joi} = j(s_i) \text{ and } \sigma_{joi} = \sup j^n \sigma_i,$$

$$(3)$$

$$\tilde{s}_{joi} = j(\tilde{s}_i) \text{ and } \tilde{\sigma}_{joi} = \sup j^n \tilde{\sigma}_i.$$  

$(4)$

Moreover, if $j' : U \to W'$ is the iteration map of some wellfounded abstract degree $u$ weakly amenable iteration on $U$ (Definition 3.5), via extenders $F'$ each with $\text{cr}(F') > \kappa$, then $j'$ preserves $(s, \sigma)$ and $(\tilde{s}, \tilde{\sigma})$ analogously; that is, lines (3) and (4) hold after replacing $j$ with $j'$ throughout.

Proof. We literally just discuss $(\tilde{s}, \tilde{\sigma})$-preservation for $j : U \to W$. For $(s, \sigma)$ it is almost the same, and the last paragraph of the lemma is an easy corollary.

\footnote{In \cite{Dodd-core}, the Dodd-core was formally defined to be the transitive collapse of $E | s_\cup s$ (as opposed to its trivial completion, but these are essentially equivalent). That terminology was not so good, as the Dodd-core of extender $E$ should, by analogy with the standard projectum and parameter, be defined by restricting the extender to $\tau' \cup t'$, where $\tau'$ is least such that for some $t''$, $E | (\tau' \cup t'') \notin \text{Ult}(M, E)$, and $t'$ is then the minimal witness for $\tau'$. So we opted here for Dodd-similarity core instead.}
Let $E = E_i \upharpoonright i(\kappa)$, so $E \notin U$. If $\text{cr}(j) > i(\kappa)$ then (by weak amenability) the conclusion is immediate, so suppose $\text{cr}(j) \leq i(\kappa)$. Let $G = E_{j\circ i} \upharpoonright j(i(\kappa))$. If $X \subseteq i(\kappa)$ and $E \upharpoonright X \in U$, then $j(E \upharpoonright X) = G \upharpoonright j(X)$, as $\kappa^+ \in G < \text{cr}(j)$. So we just need to see that $W$ does not contain any fragments of $G$ which are too large.

Suppose that $\overline{\sigma}_i > \kappa^+$. It suffices to see that

$$G \upharpoonright (\overline{\sigma}_i) \cup \sup j(\overline{\sigma}_i) \notin W.$$  

But this follows from Lemma 3.6, applied to the set $E \upharpoonright (\overline{\sigma}_i \cup \overline{\sigma}_i)$, coded as a subset of $\overline{\sigma}_i$ (which is appropriately amenable to $U$, but not in $U$).

If instead $\overline{\sigma}_i = 0$, then the measure $E \upharpoonright \overline{s}_i \notin U$, but this is a subset of $\kappa^+$, and $\mathcal{P}(\kappa^+) \cap U = \mathcal{P}(\kappa^+) \cap W$, so we are done. \hfill \Box

We also need the following slight variant of the preceding lemma:

**7.16 Lemma.** Let $M$ be a type 2 premouse. Let $d \in \{0, 1\}$ be such that $M$ is $d$-sound, and if $d = 1$, suppose $M$ is Dodd-sound. Let $E$ be weakly amenable to $M$ with $\text{cr}(E) < \rho^M_d$. Suppose that $M' = \text{Ult}_d(M, E)$ is wellfounded. Let $i = i^M_d$. Let $F = F^M$ and $F' = F^{M'}$. Then:

$$s_{F'} = i(s_F) \text{ and } \sigma_{F'} = \sup i^*\sigma_F,$$

$$\overline{s}_{F'} = i(\overline{s}_F) \text{ and } \overline{\sigma}_{F'} = \sup i^*\overline{\sigma}_F.$$

**Proof sketch.** A complete argument is given in [27, §2] (formally below superstrong), but we give a sketch here. We literally just discuss $\text{cr}(\mathcal{S}, \sigma)$-

 preservation.

If $d = 0$, the proof is basically as in the proof of Lemma 7.15. (Maybe $\text{cr}(E) \leq \text{cr}(F)$, but that is fine, considering the definition of $F$. If $\sigma_F = 0$ then the measure $F \upharpoonright \overline{s}_F \notin M$, and this is coded amenably as a subset of $\mu^+M$ where $\mu = \text{cr}(F)$.)

Suppose $d = 1$. So $\text{cr}(E) < \rho^M_1$. Suppose $\mu^+M < \rho^M_1$, so by Lemma 7.8 and Fact 7.14, $\rho^M_1 = \tau^M = \sigma^M = \overline{\tau}^M = \overline{\sigma}^M$ and $\tau^M = s^M = \overline{\tau}^M = \overline{s}^M$. We have $p^M_{1'} = i(p^M_1)$ and $\rho^M_1 = \sigma'$ where $\sigma' = \sup i^*\overline{\sigma}^M$, so

$$M' = \text{Hull}^M_{1'}(i(\overline{\sigma}^M_{1'}) \cup \sigma').$$

But because $M$ is Dodd-sound, $i(F^M, p^M_1) = (M^M_{1'}, p^M_{1'})$ is generated by $F' \upharpoonright s' \cup \sigma'$ where $s' = i(\overline{s}^M)$. But then by Lemma 7.7, $F' \upharpoonright (s' \cup \sigma')$ generates all of $M'$, so $F' \upharpoonright (s' \cup \sigma') \notin M'$.

Now suppose $\rho^M_1 \leq \mu^+M$. Since $M$ is Dodd-sound, $\tau^M = \sigma^M = \overline{\tau}^M = \overline{s}^M$ and $\tau^M = s^M$ and $\overline{\sigma}^M = 0$ and either $\overline{s}^M = s^M$ or $\overline{s}^M = s^M \cup \{\mu\}$. Either way, $F^M \upharpoonright \overline{s}^M \notin M$, which is an r$\Pi^2_M$ statement about the parameter $\overline{\tau}^M$, and since $i$ is $r\Sigma^2$-elementary, therefore $F^M \upharpoonright s' \notin M$ where $s' = i(\overline{s}^M)$. It follows that $\overline{s}^M = s'$ and $\overline{s}^M = 0$. \hfill \Box

If $T$ is a $k$-maximal iteration tree on a $k$-sound premouse $M$, the extenders $E^T_{\alpha}$ need not be Dodd-sound, even if every extender in $E^M_{\alpha}$ is Dodd-sound. But often in cases of interest (always, if every extender in $E^M_{\alpha}$ is Dodd-sound) $E^T_{\alpha}$ has a natural decomposition into a sequence of Dodd-sound extenders. That sequence can be derived in a fairly simple manner from extenders used in $T$. We discuss this next.

**7.17 Definition.** Let $M$ be an $m$-sound premouse all of whose proper segments are Dodd-sound, and $T$ be $m$-maximal on $M$. Let $\beta + 1 < \ellh(T)$. We say $\beta$ is Dodd-nice (for $T$) iff either $(0, \beta]^T$ drops in model, or $E^T_{\beta} \in \mathcal{E}(M^T_{\beta})$, or $F^M$ is Dodd-sound.
Similarly, let $T$ a degree-maximal tree on a bicephalus $B = (\rho, M^0, M^1)$ where all proper segments of $M^0, M^1$ are Dodd-sound. Let $\beta + 1 < \lh(T)$. We say $\beta$ is Dodd-nice (for $T$) iff either $(0, \beta]^T$ drops in model, or letting $e = \text{exitside}_{\beta}^T$, we have $E_{\beta}^T \in \mathbb{E}(M_{\beta}^T)$ or $M^e$ is Dodd-sound.

Let $T$ be a tree as in one of the cases above. Let $\lambda < \lh(T)$. We say that $T$ is $\lambda$-Dodd-nice iff $\beta$ is Dodd-nice for every $\beta \geq \lambda$ with $\beta + 1 < \lh(T)$. If $b^T$ exists, we say that $b^T$ is $\lambda$-Dodd-nice iff $\beta$ is Dodd-nice for every $\beta \geq \lambda$ with $\beta + 1 \in b^T$.

7.18 Remark. The Dodd-niceness of $\beta$ does not imply that $E_{\beta}^T$ is Dodd-sound (assuming some large cardinals). For example, suppose that $M$ is active type 2, $(0, \omega_1 + 1)$-iterable, all initial segments of $M$ are Dodd-sound, and there is $\kappa$ such that $\tau^M \leq \kappa < \kappa + M < \text{OR} M$, and $\kappa$ is $M$-measurable via some $E \in \mathbb{E} M$. Let $T$ be the $0$-maximal tree on $M$ with $E_0^T = E$ and $E_1^T = F(M^T_1)$. Then 1 is Dodd-nice but $E_1^T$ is not Dodd-sound, by Fact 7.14 and Lemma 7.16. Indeed, letting $F = F^M$ and $F' = F(M^T_1) = E^T_1$, by 7.14 and 7.16, $\tau_F = \sigma_F = \sigma_{F'} \leq \kappa = \text{cr}(i_{01}^T)$ and $s_{F'} = i_{01}^T(s_F) = i_{01}^T(t_F)$. But then it easily follows that the elements of $M^T_1$ which are $F'$-generated by $\sigma_{F'} \cup \{s_{F'}\}$ are precisely those in $\text{rg}(i_{01}^T)$, and so $E_1^T$ is not Dodd-sound (since Dodd-soundness would require that all elements of $M^T_1$ were $F'$-generated by $\sigma_{F'} \cup \{s_{F'}\}$).

The next lemma analyzes failures of Dodd-soundness in general; is a routine variant of results in [27, §2] (particularly [27, Remark 2.30]), so we leave the direct adaptation to the reader.

7.19 Lemma. Let either

- $m < \omega$, $M$ be an $m$-sound premouse all of whose proper segments are Dodd-sound, and $T$ be $m$-maximal on $M$, or

- $B = (\rho, M^0, M^1)$ be a bicephalus such that all proper segments of $M^0, M^1$ are Dodd-sound, and $T$ degree-maximal on $B$.

Let $\beta + 1 < \lh(T)$ be such that $\beta$ is Dodd-nice for $T$. Let $e = \text{exitside}_{\beta}^T$ and $E = E_{\beta}^T$. Then $E$ fails to be Dodd-sound iff we have:

(i) $M^{e+\lambda}T$ is active type $2$ and $E = F(M^{e+\lambda T})$, and

(ii) there is $\varepsilon + 1 \leq T \beta$ such that $(\varepsilon + 1, \beta]^T \cap \emptyset = \emptyset$ (so $M^{e+\lambda+1}T$ is active type $2$) and $\text{cr}(E_{\varepsilon}^T) \geq \sigma_E$ where $\bar{E} = F(M^{e+1, T})$.

Moreover, suppose $E$ is non-Dodd-sound and $\varepsilon$ is least as above. Let $P = M^{e+1, T}$, $\bar{P} = M^{e+T}_{\varepsilon+1}$, $\bar{E} = F^{\bar{P}}$ and $\delta = \text{pred}^T(\varepsilon + 1)$. Then:

1. $\bar{E} = \mathbf{c}_{\text{Ds}}(E)$ and $\bar{P} = \mathbf{c}_{\text{Ds}}(P)$.

2. $\bar{E} = F^{\bar{P}}$ is Dodd-sound.

3. $\delta$ is the unique $\delta' < \lh(T)$ such that $\bar{P} \leq M^{\delta'}_\beta$.

4. $\text{deg}^T(\varepsilon + 1) = 0$.

5. $\rho_1^P = \tau_E = \sigma_E \leq \text{cr}(E_{\varepsilon}^T) = \text{cr}(i_{\varepsilon+1, \beta}^{e+T})$.

6. $s_E = i_{\varepsilon+1, \beta}(t_{\bar{E}}) = i_{\varepsilon+1, \beta}(t_E)$. 

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7. \( \sigma_E < \nu(E^T) < \text{lh}(E^T) \leq \text{OR}^\beta \).

8. \( \delta \) is the least \( \delta' \) such that \( \sigma_E < \text{lh}(E^T_{\delta'}) \).

Note that if \( E = E^T_\alpha \in E_+ (M^T_\beta) \) is Dodd-sound, then there is also a unique \( \xi \) such that \( C_{\text{Dop}}(E) \in E_+ (M^T_\xi) \), because \( C_{\text{Dop}}(E) = E \); in fact, \( \xi = \beta \).

Proof. The characterization of Dodd-soundness and parts 1–6 in the “Moreover” clause follow by a straightforward induction, using Lemma 7.16. Let us just discuss parts 7 and 8, assuming the rest. Part 7: We have \( \sigma_E < \text{cr}(E^T_\delta) < \nu(E^T_\delta) \) by hypothesis and normality, and \( \sigma_E = \sigma_E \) by part 5. And \( \nu(E^T_\delta) < \text{lh}(E^T_\delta) \leq \text{OR}^\beta \) since \( \tilde{P} = M^T_{\alpha+1} \), so \( E^T_\delta \) measures exactly \( \mathcal{P}(\kappa) \cap \tilde{P} \) where \( \kappa = \text{cr}(E^T_\delta) \), but \( \rho^\beta \leq \sigma_E \leq \kappa \), so \( \text{lh}(E^T_\delta) \leq \text{OR}^\beta \).

Part 8: Let \( \delta' \) be least with \( \sigma_E < \text{lh}(E^T_\delta) \). So \( \delta' \leq \delta \) by part 7; we need to see \( \delta' = \delta \).

This is like part of the proof of Closeness [9, Lemma 6.1.5]. Suppose that \( \delta' < \delta \) and let \( \gamma \) be least such that \( \gamma \geq \delta' \) and \( \gamma + 1 \leq \delta \). So \( \sigma_E < \text{lh}(E^T_{\gamma'}) \leq \text{lh}(E^T_{\gamma'}) \). We have \( \tilde{P} \leq M^T_{\gamma'} \) and \( \rho^\beta \leq \sigma_E = \sigma_E < \text{lh}(E^T_{\gamma'}) \).

It follows from the rules of degree-maximal trees that \( \tilde{P} = M^T_{\gamma'} \) (that is, otherwise \( \tilde{P} = M^T_{\alpha+1} \) and \( \text{lh}(E^T_{\delta'}) \leq \text{OR}^\beta \), but then \( \text{lh}(E^T_{\delta'}) \) is a cardinal in \( M^T_{\gamma'} \), contradicting the fact that \( \rho^\beta < \text{lh}(E^T_{\gamma'}) \leq \text{OR}^\beta \)). So by the minimality of \( \varepsilon \), \( M^T_{\gamma+1} \) is Dodd-sound (and active type 2) and \( \text{cr}(E^T_{\gamma+1}) < \text{cr}(F(M^T_{\gamma+1})) \). But then by Lemma 7.16 and straightforward elementarity considerations in case \( \text{deg} E^T_{\gamma+1} > 1 \), it follows that

\[
\sigma_{F(M^T_{\gamma+1})} = \sup \{ \text{lh}(E^T_{\gamma'}) \} \geq \text{lh}(E^T_{\gamma'}) > \sigma_E.
\]

Likewise for all \( \gamma' \) with \( \gamma + 1 < \gamma' \leq \gamma + 1 \leq \delta \), giving that

\[
\sigma_E = \sigma_{F(M^T_{\gamma+1})} = \sup \{ \text{lh}(E^T_{\gamma'}) \} > \sigma_E,
\]

contradicting part 5.

We now adapt Dodd ancestry and Dodd decomposition, described in [27, 2.31–2.36], to our context. They provide a decomposition of non-Dodd-sound extenders used in degree-maximal iteration trees into linear sequences of Dodd-sound extenders, and also, corresponding decompositions of iteration maps. The Dodd ancestry relation \( \alpha <^T_{\text{Da}} \beta \), introduced first, indicates when an extender \( E^T_{\alpha} \) of \( \mathcal{T} \) “hereditarily contributes generators” to an extender \( E^T_{\beta} \) which are not absorbed into the Dodd-solidity-core of \( E^T_{\beta} \).

7.20 Definition. Let \( (m, M, \mathcal{T}) \) or \( (B, \mathcal{T}) \) be as in 7.19. We define the Dodd ancestry relation \( \leq^T_{\text{Da}} \) on \( \text{lh}(\mathcal{T})^- \). Let \( \alpha + 1, \beta + 1 < \text{lh}(\mathcal{T}) \). Say \( \alpha <^* \beta \) iff \( \beta \) is Dodd-nice for \( \mathcal{T} \), \( E^T_\beta \) is non-Dodd-sound and \( \varepsilon + 1 \leq \delta \leq \beta \) where \( \varepsilon \) is least as in 7.19. Then \( \leq^T_{\text{Da}} \beta \) denotes the transitive closure of \( <^* \), and \( \leq_{\text{Da}} \) the reflexive closure of \( <^T_{\text{Da}} \).

Given \( \alpha \leq^T_{\text{Da}} \beta \), let the trace of \( (\beta, \alpha) \) be the sequence \( (\beta_0, \beta_1, \ldots, \beta_n) \) where \( \beta_0 = \beta \), \( \beta_n = \alpha \) and \( \beta_{i+1} = ^* \beta_i \) for \( i < n \) (so \( \alpha = \beta \) iff \( n = 0 \)).

7.21 Remark. It is easy to see that if \( \alpha <^T_{\text{Da}} \beta \) (so \( \beta \) is Dodd-nice by definition) then \( \alpha \) is also Dodd-nice. In the example discussed in Remark 7.18, \( 0 <^* 1 \), so \( 0 <^T_{\text{Da}} 1 \). And as described in the following lemma (and cf. [27, 2.27, 2.28]), \( E^T_1 \) is equivalent to the composition \( E^0_0 \circ F^M \), and so

\[
M^T_1 = \text{Ult}_0(M, E^T_1) = \text{Ult}_0(\text{Ult}_0(M, F^M), E^T_0),
\]

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and the ultrapower maps agree; this decomposes the Dodd-unsound extender $E_T^r$ into composition of two Dodd-sound extenders. Let us consider a slightly more complex example. With the hypotheses as in the same example, suppose further that $E$ is type 2 and there is $\mu$ which is $M\lhd(E)$-measurable via some $D \in \mathbb{R}^M$, and $\tau_E \leq \mu < \mu^M[|E|] < \lhd(E)$. Let $U$ be the 0-maximal tree with $\lhd(U) = 4$, $E_0^M = D$, $E_1^M = F(M_1^U)$ and $E_2^M = F(M_2^U)$. (Note that $M_1^U = M\lhd(E)$ and $\deg_1^U = 0$, since $\rho_1^M[|E|] \leq \tau_E \leq \mu$, so $M_1^U = \Ult_0(M\lhd(E), D)$ is indeed active with the image of $E$, $\kappa = \cr(E) = \cr(E_1^U) < \mu$ and $E_1^M = F(M_1^U)$ is $M$-total and $\pred_1^U(2) = 0$ and $M_2^U = M$, so $M_2^U = \Ult_0(M, E_1^M)$, and then likewise, $M_3^U = \Ult_0(M, E_2^M)$ is active with the image of $F^M$, and $\cr(E_2^M) = \cr(F^M)$, so $M_3^U = \Ult_0(M, E_2^M)$. Then $0 <^\ast 1 <^\ast 2$, and so $0 <^\ast_{D_a} 1 <^\ast_{D_a} 2$ and $0 <^\ast_{D_a} 2$. And we have

$$M_3^T = \Ult_0(M, E_2^T) = \Ult_0(\Ult_0(\Ult_0(M, F^M), E), D)$$

and the ultrapower maps agree, and again, $E_2^T$ is Dodd-unsound, but $F^M$, $E$, $D$ are each Dodd-sound.

The following lemma generalizes this kind of analysis, decomposing any extender $E_\xi^T$ such that $\xi$ is Dodd-nice for $T$ into a sequence $\langle G_\gamma \rangle_{\gamma < \lambda}$ of Dodd-sound extenders, whose composition is equivalent to $E_\xi^T$. The sequence will have strictly critical points, but need not be “normal”, in that we can have $\gamma < \delta < \lambda$ with $\cr(G_\delta) < \nu(G_\gamma)$. By composing such sequences, we in fact analyze a full iteration map $i_{\beta, \gamma}^T$ (assuming all the relevant ordinals are Dodd-nice).

7.22 Lemma (Dodd decomposition). Let $(\alpha, M, T)$ or $(B, T)$ be as in 7.19. Let $\alpha <^T \beta$ and $e \in \text{sides}_T^\beta$ be such that $(\alpha, \beta]^T$ does not drop in model or degree, $k = \deg^{E^\alpha} = \deg^{E^\beta}_T$, and suppose $\xi$ is Dodd-nice for $T$ for every $\xi + 1 \in (\alpha, \beta]^T$. Let $\vec{G} = \langle G_\gamma \rangle_{\gamma < \lambda}$ enumerate, in order of increasing critical point, the set of all extenders of the form $\mathfrak{E}_{D_a}(E^\xi)$ where $\delta \leq\!_T D_a \xi$ for some $\xi + 1 \in (\alpha, \beta]^T$. Then:

1. Let $\xi + 1, \zeta + 1 \in (\alpha, \beta]^T$ and $\theta \leq\!_T D_a \xi$ and $\delta \leq\!_T D_a \zeta$ with $\theta \neq \delta$. Then $\cr(E^\theta) \neq \cr(E^\delta)$, so the enumeration of $\vec{G}$ mentioned in the lemma statement is well-defined. In fact, let $\vec{\theta} = (\theta_0, \ldots, \theta_m)$ be the trace of $(\xi, \theta)$ and $\vec{\delta} = (\delta_0, \ldots, \delta_n)$ the trace of $(\zeta, \delta)$. Then:

   (a) If $m < n$ and $\vec{\theta} = \vec{\delta} \upharpoonright m$ then $\theta <\!^T D_a \theta$, so $\delta < \theta$ and

   $$\cr(E^\theta) < \sigma E^\delta \leq \cr(E^\delta) < \nu(E^\delta) < \nu(E^\theta).$$

   (b) If $n < m$ and $\vec{\delta} = \vec{\theta} \upharpoonright n$ then likewise (with $\delta, \theta$ exchanged, so $\theta <\!^T D_a \delta$, etc).

   (c) Suppose there is $k < \min(m, n)$ such that $\vec{\theta} \upharpoonright k = \vec{\delta} \upharpoonright k$ but $\theta_k \neq \delta_k$. If $\theta_k < \delta_k$ then $\theta < \delta$ and

   $$\cr(E^\theta) < \nu(E^\delta) \leq \cr(E^\delta) < \nu(E^\delta);$$

   if $\delta_k < \theta_k$ then likewise.

2. There is an abstract degree $k$ weakly amenable iteration of $M^\alpha_{\mathfrak{E}}$ of the form

$$\langle (N_\gamma)_{\gamma \leq \lambda}, (G_\gamma)_{\gamma < \lambda} \rangle,$$

with abstract iteration maps $j_{\gamma_1, \gamma_2} : N_{\gamma_1} \rightarrow N_{\gamma_2}$. 55
3. \( N_\lambda = M_\beta^T \) and \( j_\lambda = i_{\alpha,\beta}^T \).

4. For \( \gamma \in [0, \lambda) \) and \( \delta \) with \( G_\gamma = E_{D}\alpha(E_\delta^T) \) and \( \mu = \alpha(E_\delta^T) = \alpha(G_\gamma) \), we have:
   - \( E_\delta^T / \nu(E_\delta^T) \) is the \((\mu, \nu(E_\delta^T))\)-extender derived from \( j_\gamma \lambda \).
   - For any \( \beta' < \text{lh}(T) \) such that \( \beta < T \beta' \) and \( (\beta, \beta')^T \cap \mathcal{D}^T = \emptyset \), \( E_\delta^T / \nu(E_\delta^T) \) is the \((\mu, \nu(E_\delta^T))\)-extender derived from \( i_{\beta,\beta'} T \).
   - Let \( j = j_{\gamma,\lambda} \) and \( N' = \text{cHull}_{k+1}(\text{rg}(j) \cup \sigma_j \cup \{s_j\}) \) and \( j' : N' \rightarrow N_\lambda \) be the uncollapse map. Then \( N' = N_{\gamma+1} \) and \( j' = j_{\gamma+1,\lambda} \). Moreover, if also, \( \alpha \) has form \( \varepsilon + 1 \) where \( \varepsilon \) is Dodd-nice for \( T \), and \( \vec{G}^* \) enumerates, in order of increasing critical point, all those extenders either in \( \vec{G} \) or of form \( \mathcal{E}_{D}\alpha(E_\delta^T) \) where \( \delta \leq \text{rg}(\varepsilon) \), then the analogous facts hold of \( \vec{G}^* \), \( M_{\varepsilon+1}^T \), \( i_{\varepsilon,\alpha}^T \) (replacing \( \vec{G}, M^T, i_{\varepsilon+1,\alpha} \) respectively).

Proof. This is as in [27, 2.32, 2.35, 2.36]. Note that the proof uses the associativity of extenders described in [27, 2.27, 2.28]; these facts continue to hold, with the same proofs, without the superstrong-smallness assumption that was formally present in [27].

7.23 Definition (Dodd decomposition). In the context of the lemma above, the sequence \( \vec{G} \) is called the Dodd decomposition of \( i_{\alpha,\beta}^T \), and in the context of the “Moreover” clause, \( \vec{G}^* \) is that of \( i_{\varepsilon,\alpha}^T \).

8 Capturing with strongly finite trees

In this section we discuss constructions of finite (of finite after some stage) support trees \( \mathcal{T} \) of iteration trees \( T \), capturing a given \( x \in M_\infty^T \) with a copy map \( \varsigma : M_\infty^T \rightarrow M_\infty^T \) having \( x \in \text{rg}(\varsigma) \) like in §6, but with more specialized properties than those considered there. In particular, under some Dodd-soundness assumptions, we will show how to arrange that every extender used in \( \mathcal{T} \) is finitely generated (and also slight variations thereof). Recall from §2.2 that we want such methods for the argument in §9, so the results in this section will be used there. Although we won’t literally use the results after §9, their proofs will be used. This is because will use slight variants of the results in §§10.11. These variants will be described in those sections, but the proofs, being only embellishments on those presented here, will only be sketched there.

8.1 Definition. Let \( m < \omega \), let \( M \) be \( m \)-sound and let \( \mathcal{T}, T \) be terminally non-dropping \( m \)-maximal trees on \( M \). Suppose \( \mathcal{T}, T \) have successor lengths \( \xi + 1, \xi + 1 \) respectively and let \( x \in \mathcal{E}_0(M_\xi^T) \) and \( \alpha < \rho_0(M_\xi^T) \). Then we say that \( \mathcal{T} \) captures \((T, x, \alpha)\) iff there is \( \lambda \in b^T \cap b^\infty \) and \( \sigma \) such that:
   - \( \mathcal{T} \upharpoonright (\lambda + 1) = T \upharpoonright (\lambda + 1) \),
   - \( \lambda \leq^T \xi \) and \( \lambda \leq^T \xi \), and
   - \( \sigma : M_\xi^T \rightarrow M_\xi^T \) is an \( m \)-embedding with \( \{x\} \cup \alpha \subseteq \text{rg}(\sigma) \) and \( \sigma \circ i_{\lambda,\xi}^T = i_{\lambda,\xi}^T \).

We say that \( \mathcal{T} \) captures \((T, x, 0)\).
The following definition and lemma will be used in §9:

**8.2 Definition** (Strongly finite). Let \( M \) be an \( m \)-sound premouse all of whose initial segments are Dodd-sound (including \( M \) itself), and \( T \) be a terminally-non-dropping successor length \( m \)-maximal tree on \( M \). We say that \( T \) is strongly finite iff \( \text{lh}(T) < \omega \) and for each \( \alpha + 1 \in b^T \) and each \( \gamma \leq T \alpha \), \( \xi_{D\alpha}(E^T_\gamma) \) is finitely generated (so \( \sigma_{E^T_\gamma} = \text{cr}(E^T_\gamma)^{\text{exit}_T} \)).

It is not completely obvious from the definition, but in strongly finite trees \( T \), all extenders \( E^T_\chi \) used either feed generators into the eventual iteration map, in that \( \chi \leq T \text{Da} \varepsilon \) for some \( \varepsilon + 1 \leq T \infty \), or \( E^T_\chi \) is used for a rather trivial reason to do with indexing. In this trivial case, basically \( E^T_\chi \) is type 2 and just has to be used in order to reveal the next measure which does contribute generators into the eventual iteration map, and the critical point of that measure is \( \text{lgcd}(\text{exit}_T^T) \). (If we were working with \( \lambda \)-indexing instead of Mitchell-Steel indexing, and \( T' \) were the natural version of \( T \) in that hierarchy, then the \( \lambda \)-indexed version \( E \) of \( E^T_\chi \) would not be used in \( T' \), as we could produce the next measure without using \( E \).) We make this precise with the following definition and lemma:

**8.3 Definition.** Let \( M \) be \( m \)-sound and \( T \) be \( m \)-maximal on \( M \). Let \( \chi < \chi' < \text{lh}(T) \). We say that \( \chi \) is \( \chi' \)-transient (with respect to \( T \)) iff for some \( \eta \), we have:

- \( M^{T \chi} \) is active type 2,
- \( \chi = \text{pred}^T(\eta + 1) < T \eta + 1 \leq T \chi' \),
- \( (\eta + 1, \chi']^T \) does not drop, and
- \( \text{cr}(i_{\eta+1}^{T \chi}) = \text{lgcd}(M^{T \chi+1}_{\eta+1}) \).

\( \dashv \)

**8.4 Lemma.** Let \( m, M \) be as in Definition 8.2. Let \( T \) be a strongly finite terminally-non-dropping \( m \)-maximal tree on \( M \). Then for each \( \chi + 1 < \text{lh}(T) \), one of the following options holds, where \( \xi + 1 = \text{lh}(T) \):

(i) there are \( \varepsilon, \chi' \) such that \( \varepsilon + 1 \leq T \xi \) and \( \chi' \leq T \varepsilon \), and either:

(a) \( \chi = \chi' \), or
(b) \( \chi < T \chi' \) and \( \chi \) is \( \chi' \)-transient,

or

(ii) \( \chi < T \xi \) and \( \chi \) is \( \xi \)-transient.

Moreover, the three options (i)a, (i)b, (ii) are mutually exclusive.

Proof sketch. The proof is like that for Subclaim 1.1 in the proof of Lemma 8.6 below, and so we leave it to the reader to extract it from there. (Below superstrong, the argument appeared within the proof of [22, Theorem 4.8].)

Although we do not need it, it is natural to observe that if \( M \) is as above and non-type 2 then strongly finite trees on \( M \) produce models which can also be produced via linear (but possibly non-normal) iterations, via Dodd-sound measures. It is similar if \( M \) is type 2, but there is a slight wrinkle. Cf. [22, Theorem 4.8(d)]:

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8.5 Proposition. Let $M,m$ be as in 8.2 and let $\mathcal{T}$ be a strongly finite terminally non-dropping $m$-maximal tree on $M$. Then there is a (possibly non-normal, possibly non-$m$-maximal) tree $\mathcal{L}$ on $M$ which has no drops in model or degree (anywhere), $\deg^*_\alpha = m$ for all $\alpha < \lh(\mathcal{L})$, with $M^\mathcal{L}_\alpha = M^\mathcal{T}_\alpha$ and $i^\mathcal{L} = i^\mathcal{T}$ and every extender used along the main branch $b^\mathcal{L}$ of $\mathcal{L}$ is finitely generated and Dodd-sound, $\cr(E^\mathcal{L}_\alpha) < \cr(E^\mathcal{T}_\beta)$ for $\alpha + 1, \beta + 1 \in b^\mathcal{L}$ with $\alpha < \beta$, and moreover:

- if $M$ is non-type 2 then $\mathcal{L}$ is linear, so $\alpha + 1 \in b^\mathcal{L}$ for every $\alpha + 1 < \lh(\mathcal{L})$,
- if $M$ is type 2, then for each $\alpha + 1 < \lh(\mathcal{L})$, if $\alpha + 1 \notin b^\mathcal{L}$ then $\alpha \in b^\mathcal{L}$ and $\cr(i^\mathcal{L}_\alpha) = \lgcd(M^\mathcal{L}_\alpha)$ and $E^\mathcal{L}_\alpha = F(M^\mathcal{L}_\alpha)$.

Proof. Let $\tilde{G} = \langle G_\alpha \rangle_{\alpha < \lambda}$ be the Dodd decomposition of $i^\mathcal{T}$ and $M_\alpha = \Ult_m(M,\tilde{G}^\mathcal{L} | \alpha)$. Then for each $\alpha < \lambda$, either:

1. $\cr(G_\alpha) + M_\alpha < \OR^{M_\alpha}$ and $G_\alpha \in \mathbb{E}^{M_\alpha}_{+}$, or
2. $\cr(G_\alpha) = \lgcd(M_\alpha)$, $M$ is active type 2, and $G_\alpha \in \mathbb{E}_+(\Ult(M_\alpha, F^{M_\alpha}))$.

This is similar to the proof of Subclaim 1.1 in the proof of Lemma 8.6 below, so we omit further discussion. The proposition follows easily from this observation; for example, if $M$ is non-type 2 then $\mathcal{L}$ is just the degree $m$ linear iteration with $E^\mathcal{L}_\alpha = G_\alpha$.

\[\square\]

8.6 Lemma. Let $m,M$ be as in Definition 8.2. Let $\mathcal{U}$ be a terminally-non-dropping $m$-maximal tree on $M$ and $x \in M^\mathcal{U}_\omega$. Then there is a strongly finite terminally-non-dropping $m$-maximal tree $\mathcal{T}$ on $M$ capturing $(\mathcal{U},x)$.

In particular, if $M$ is not type 2, then $M^\mathcal{T}_\omega$ is a linear iterate of $M$ (but possibly via a non-normal tree).

The proof below the superstrong level, which is essentially identical to the one allowing superstrongs below,\footnote{As mentioned earlier, the proof that iterable sound mice with Mitchell-Steel indexing are Dodd-sound is not quite the direct translation of the proof below the superstrong level. But the present lemma simply assumes the relevant Dodd-soundness, so this is not an issue here.} appeared as part of the proof of the author’s thesis [22, Theorem 4.8].

Proof. By Lemma 6.20, we already know there is a terminally non-dropping $m$-maximal tree $\mathcal{T}$ on $M$ capturing $(\mathcal{U},x)$ with $\lh(\mathcal{T}) < \omega$ (but we don’t claim that $\mathcal{T}$ is strongly finite). Call such a tree $\mathcal{T}$ a candidate. We will show that the “least” candidate is strongly finite, for a certain natural notion of “leastness”, which we now formulate. Let $\mathcal{T}$ be a candidate. Note that $\mathcal{T}$ is $\geq 0$-Dodd-nice (Definition 7.17), since every initial segment of $M$ is Dodd-sound. Let $\xi + 1 = \lh(\mathcal{T}) < \omega$. Let $A$ be the set of ordinals $\beta$ such that $\beta \leq_T \alpha$ for some $\alpha + 1 \in b^\mathcal{T}$. Let $\langle \kappa_\ell \rangle_{i<\ell} \enumerate \{ \cr(E^\mathcal{T}_\beta) \} \beta \in A$ in decreasing order. So $\cr(i^\mathcal{T}) = \kappa_{\ell-1}$. Let $\beta_\ell \in A$ be such that $\kappa_\ell = \cr(E^\mathcal{T}_{\beta_\ell})$, and $\gamma_\ell = \lh(\mathcal{E}_D^\alpha(E^\mathcal{T}_{\beta_\ell}))$. Then the index of $\mathcal{T}$ is $\langle \gamma_\ell \rangle_{i<\ell}$. Let $\mathcal{T}$ be the candidate of lexicographically minimal index (noting that there is one), and adopt the notation just described for this $\mathcal{T}$ (that is, $\xi, \gamma_\ell$, etc). The following claim completes the proof:

Claim 1. $\mathcal{T}$ is strongly finite.
Proof. Suppose not. We will construct a candidate $\bar{T}$ with smaller index than $T$, a contradiction, basically by replacing the core $C_{\mathcal{D}_\theta}(E_T^\bar{T})$ of a certain extender $E_T^\bar{T}$ used in $T$, whose core $C_{\mathcal{D}_\theta}(E_T^\bar{T})$ is not finitely generated, with a sub-extender whose core is finitely generated. So let $a < \ell$ be least (so $\kappa_a$ largest) such that $C_{\mathcal{D}_\theta}(E_T^\bar{T})$ is not finitely generated. Let $\kappa = \kappa_a$, $\beta = \beta_a$, $Q = C_{\mathcal{D}_\theta}(E_T^\bar{T})$ and $F = F^Q$. So $\kappa + Q < \sigma = \sigma_F = \tau_F$.

By Lemma 7.19 there is a unique $T < \theta$ with $Q \leq M_{\theta}^T$; also by 7.19, $\theta \leq T \beta$ and $\theta$ is the least ordinal with $\sigma < \ellh(E_T^\bar{T})$.

Let $\bar{\theta} \leq \theta$ be least with $\ellh(E_T^\bar{T}) > \kappa + Q$. Let $\bar{\theta} + k = \theta$ (so $k < \omega$) and $\bar{\xi} + k = \xi$.

Let $\chi$ be least with $\chi \geq \theta$ and $\chi + 1 \leq T \xi$. Let $\bar{\chi} + k = \chi$. Our plan is to select some $g \in \nu^\omega$ and $R \triangleleft M_{\theta}^\bar{T}$ with $F^R \approx F | g$, and such that we can define an $m$-maximal tree $\bar{T}$ on $M$ such that:

- $\bar{T} | \bar{\theta} + 1 = T | \bar{\theta} + 1$ and $\ellh(\bar{T}) = \bar{\xi} + 1$ and $(0, \bar{\xi}] \bar{T} \cap \ellh_{\deg} = 0$.

- Replacing the role of $Q$ in $T$ with $R$ in $\bar{T}$, we perform a kind of reverse copy construction, much like in the proof of Lemma 6.20, so that $T \upharpoonright [\theta, \xi]$ will be a "copy" of $\bar{T} \upharpoonright [\bar{\theta}, \bar{\xi}]$. Moreover, $\bar{\chi}$ is least such that $\bar{\chi} \geq \theta$ and $\bar{\chi} + 1 \leq T \xi$, and for $\alpha \in [\bar{\chi} + 1, \bar{\xi}]^\bar{T}$, the copying process will yield a copy map

$$\pi_\alpha : M^{\bar{T}}_{\alpha} \to M^{T}_{\alpha + m}$$

with $\ellh(\pi_\alpha) = \kappa + Q$.

- The final copy map $\pi_{\bar{\xi}} : M^{\bar{T}}_{\bar{\xi}} \to M^{T}_{\bar{\xi}}$ is an $m$-embedding with $\pi_{\bar{\xi}} \circ i^{\bar{T}} = i^T$.

- $\tau^{-1}(x) \in \text{rg}(\pi_{\bar{\xi}})$, where $\tau : M^{\bar{T}}_{\bar{\xi}} \to M^{\bar{T}}_{\bar{\xi}} |_{\ellh}$ witnesses that $\bar{T}$ captures $(\bar{\mathcal{U}}, \bar{x})$.

It will follow that $\tau \circ \pi_{\bar{\xi}} : M^{\bar{T}}_{\bar{\xi}} \to M^T_{\bar{\xi}}$ witnesses that $\bar{T}$ captures $(\bar{\mathcal{U}}, \bar{x})$, and therefore $\bar{T}$ will be a candidate. From the construction it will also be clear that $\bar{T}$ has index strictly less than does $T$, which will be a contradiction.

So we need to select $g$ and $R$ and build $\bar{T}$. Now $F = F^Q$ is either type 2 or 3. In either case, we will choose a type 2 premouse $Q' \triangleleft Q$ with $F^{Q'} \subseteq F$, a type 2 premouse $R \triangleleft Q'$, and a 0-embedding $\pi_{RQ'} : R \to Q'$ with

$$\ellh(\pi_{RQ'}) = \kappa + Q = \kappa + Q.$$


**Subclaim 1.1.** Let $\chi \in [\theta, \xi)$. Then $\text{cr}(E_{\chi}^T) \notin (\kappa, \sigma)$, so $\text{pred}^T(\chi + 1) \notin (\bar{\theta}, \theta)$. In fact, one of the following options holds:

(i) there are $\varepsilon, \chi'$ such that $\varepsilon + 1 \leq^T \xi$ and $\chi' \leq^T_{\text{Da}} \varepsilon$, and either:

(a) $\chi = \chi'$, or

(b) $\chi <^T \chi'$ and $\chi$ is $\chi'$-transient and $E_{\chi'}^T = F(M_{\chi'})$,

or

(ii) $\chi <^T \xi$ and $\chi$ is $\xi$-transient.

Moreover, the three options (i)a, (i)b and (ii) are mutually exclusive.

Recall here that if (i)b holds and $\eta + 1 = \text{succ}^T(\chi, \chi')$, then $E_{\chi}^T = F(M_{\eta + 1}^T)$ and $\text{cr}(i_{\eta + 1, \chi}) = \text{lgcd}(M_{\eta + 1}^T)$, so $\text{cr}(E_{\chi}^T) = \text{cr}(E_{\chi'}^T)$. Likewise, if (ii) holds and $\eta + 1 = \text{succ}^T(\chi, \xi)$, then $E_{\chi}^T = F(M_{\eta + 1}^T)$, so $\text{cr}(E_{\chi}^T) = \text{cr}(F(M_{\xi}^T))$.

**Proof.** The mutual exclusivity is easy to see. We prove that one of the options holds at each $\chi \in [\theta, \xi]$ by induction on $\chi$; it is straightforward to see that this implies $\text{cr}(E_{\chi}^T) \notin (\kappa, \sigma)$ (use Lemma 7.19).

**Case 1.** $\chi = \theta$.

If $\theta = \beta$ then $E_{\theta}^T = E_{\beta}^T$ is Dodd-sound and (i)a holds. So suppose $\theta <^T \beta$, so $E_{\beta}^T$ is non-Dodd-sound. We have $\kappa_\alpha = \kappa = \text{cr}(F) < \text{lh}(E_{\theta}^T)$, and since $E_{\beta}^T$ is non-Dodd-sound, in fact $\alpha > 0$ and $\kappa_{\alpha - 1}$ is the critical point of the iteration map $Q \to \text{exit}_{\beta}^T$, so $\kappa_{\alpha - 1} < \text{lh}(E_{\theta}^T)$. Let $i < \ell$ be least (so $\kappa_i$ largest) such that $\kappa_i < \text{lh}(E_{\theta}^T)$ (so $i \leq a - 1$ and $\kappa_i \geq \kappa_{a - 1}$). Then $\text{exit}_{\theta}^T||\kappa_i^{+\text{exit}_{\theta}^T} = \text{C}_{\text{Da}}(\text{exit}_{\theta}^T)||\kappa_i^{+\text{exit}_{\theta}^T_{\text{Da}}}$. By Lemma 7.22, $\beta_i <^T_{\text{Da}} \beta$ and $\sigma \leq \kappa_i$. (We have $\kappa_i < \nu(E_{\theta}^T)$, since $E_{\beta}^T$ is non-Dodd-sound, hence type 2, and $\kappa_i < \text{lh}(E_{\theta}^T) \leq \text{lh}(E_{\beta}^T)$.)

**Subcase 1.1.** $\kappa_i < \text{lgcd}(\text{exit}_{\theta}^T)$, so $\kappa_i^{+\text{exit}_{\theta}^T} < \text{lh}(E_{\theta}^T)$.

Then $\theta$ is the unique $\theta'$ such that $\text{C}_{\text{Da}}(\text{exit}_{\theta}^T) \leq M_{\theta'}$. To see this, by Lemma 7.19, it suffices to see that $\theta$ is the least $\theta'$ such that $\sigma(E_{\theta'}^T) < \text{lh}(E_{\theta}^T)$. But this holds because (i) $\theta$ is the least $\theta'$ such that $\sigma < \text{lh}(E_{\theta'}^T)$, (ii) $\sigma \leq \kappa_i$, and (iii) by choice of $a$ and the subcase hypothesis,

$$\sigma(E_{\theta}^T) = \tau(\text{C}_{\text{Da}}(E_{\theta}^T)) = \kappa_i^{+\text{exit}_{\theta}^T} = \kappa_i^{+\text{exit}_{\theta}^T} < \text{lh}(E_{\theta}^T).$$

Now $E_{\beta_i}^T$ is Dodd-sound, for otherwise, $i > 0$ and $\kappa_{i - 1}$ is the critical point of the iteration map $\text{C}_{\text{Da}}(\text{exit}_{\beta_i}^T) \to \text{exit}_{\beta_i}^T$, but then by the previous paragraph, $\kappa_{i - 1} < \nu(E_{\theta}^T) < \text{lh}(E_{\theta}^T)$, contradicting the choice of $i$. It follows that $\beta_i = \theta$ and (i)a holds, completing this subcase.

**Subcase 1.2.** $\kappa_i = \text{lgcd}(\text{exit}_{\theta}^T)$.

Since $\text{lgcd}(\text{exit}_{\theta}^T) = \kappa_i = \text{cr}(E_{\theta}^T)$ is a limit cardinal of $\text{exit}_{\beta_i}^T$, therefore $\text{exit}_{\theta}^T$ is type 2 or 3.

Suppose $\text{exit}_{\theta}^T$ is type 2. Since $\kappa_i = \text{lgcd}(\text{exit}_{\theta}^T), \theta = \text{pred}^T(\beta_i + 1)$. So if $\beta_i + 1 <^T \xi$ then (ii) holds. Otherwise, there is $\varepsilon$ such that $\varepsilon + 1 <^T \xi$ and $\beta_i <^T_{\text{Da}} \varepsilon$. But then (i)b holds, with some $\chi' \leq^T_{\text{Da}} \varepsilon$ and $\beta_i + 1 = \text{succ}^T(\theta, \chi')$.
Now suppose \( \text{exit}_{\beta}^T \) is type 3, so \( \kappa_i = \nu(E_{\beta}^T) \). We will reach a contradiction (recall we are presently assuming \( \theta <^T \beta \)). Note that \( \text{pred}^T(\beta_i + 1) = \theta + 1 \).

We claim there is \( \delta + 1 \leq \xi_T \) such that \( \theta \leq \text{Da}_\delta \). For fix \( \epsilon + 1 \leq \xi_T \) such that \( \beta_i \leq \text{Da}_\delta \). If \( \beta_i = \epsilon \) then

\[
\theta + 1 = \text{pred}^T(\beta_i + 1) <^T \beta_i + 1 \leq \xi_T,
\]

so \( \delta = \theta \) works. So suppose \( \beta_i < \text{Da}_\epsilon \). We claim that \( \delta = \epsilon \) works, and that \( \theta < \text{Da}_\epsilon \). In fact, let \( \eta \) be such that \( \beta_i < \text{Da}_\eta \leq \text{Da}_\epsilon \). Then

\[
\theta + 1 = \text{pred}^T(\beta_i + 1) <^T \beta_i + 1 \leq \eta,
\]

and note that \( E_{\beta_i}^T \) is total over \( M_{\beta_i+1}^T = M_{\beta_i+1}^* \). We claim that also \( \theta < \text{Da}_\eta \). For otherwise, \( F(M_{\beta_i+1}^T) = \mathcal{C}_{\text{Da}}(E_{\eta}^T) \) is Dodd-sound. But in the latter case, \( \text{cr}(E_{\eta}^T) < \tau(F(M_{\beta_i+1}^T)) \), so

\[
\text{cr}(E_{\beta_i}^T) = \kappa_i = \nu(E_{\beta_i}^T) \leq i_{E_{\eta}^T(\text{cr}(E_{\eta}^T))) < \tau(F(M_{\beta_i+1}^T)) = \sigma(F(M_{\beta_i+1}^T)),
\]

contradicting the fact that \( \beta_i < \text{Da}_\eta \).

So there is \( \delta + 1 \leq \xi_T \) with \( \theta \leq \text{Da}_\delta \). Since \( E_{\beta_i}^T \) is type 3, and by the choice of \( a \), therefore \( \text{cr}(E_{\beta_i}^T) \leq \kappa = \nu(E_{\beta_i}^T) \). There is also \( \epsilon + 1 \leq \xi_T \) with \( \beta < \text{Da}_\epsilon \), and recall that (we are presently assuming) \( \theta \neq \beta \). So by Lemma 7.22, it follows that \( \sigma(E_{\beta_i}^T) \leq \kappa = \text{cr}(E_{\beta_i}^T) \) (otherwise, by that lemma, we must have \( \sigma(E_{\beta_i}^T) \leq \text{cr}(E_{\beta_i}^T) \), contradicting that \( \text{cr}(E_{\beta_i}^T) \leq \kappa \)). But since \( E_{\beta_i}^T \) is type 3, \( \sigma(E_{\beta_i}^T) = \nu(E_{\beta_i}^T) \geq \sigma > \kappa \), a contradiction.

**Case 2.** \( \chi > \theta \), (i)a holds at stage \( \chi - 1 \), and \( \chi \neq ^T \xi \).

Then there is \( \iota \) such that \( \chi - 1 < \text{Da}_\iota \) and \( \iota < \iota \) for some \( \iota + 1 \leq \xi_T \); let \( \iota \) be least such. Then \( \chi < \iota \). If \( \chi = \iota \) then (i)a holds (at stage \( \chi \)), suppose \( \chi < \iota \). Let \( b < \ell \) with \( \kappa_b = \text{cr}(E_{\iota}^T) \), so \( \beta_b + 1 = \text{succ}^T(\chi, \iota) \). Then \( \kappa_b < \text{lh}(E_{\chi}^T) \). Let \( i \) be least such that \( \kappa_i < \text{lh}(E_{\chi}^T) \).

**Subcase 2.1.** \( \kappa_i < \text{lgcd}(\text{exit}_{\chi}^T) \), so \( \kappa_{i+\text{exit}_{\chi}^T} < \text{lh}(E_{\chi}^T) \).

Then \( \mathcal{C}_{\text{Da}}(\text{exit}_{\beta_i}^T) \subseteq M_{\chi}^T \), \( E_{\beta_i}^T \) is Dodd-sound and \( \beta_i = \chi \), all as in Subcase 1.1 of Case 1, so (i)a holds.

**Subcase 2.2.** \( \kappa_i = \text{lgcd}(\text{exit}_{\chi}^T) \).

Then \( E_{\chi}^T \) is type 2 and (i)b holds (since (ii) is ruled out by case hypothesis). This is established as in Subcase 1.2 of Case 1.

**Case 3.** \( \chi > \theta \) and (i)a holds at stage \( \chi - 1 \), and \( \chi <^T \xi \).

Let \( b < \ell \) with \( \kappa_b = \text{cr}(E_{\chi}^T) \), so \( \beta_b + 1 = \text{succ}^T(\chi, \xi) \). Then \( \kappa_b < \text{lh}(E_{\chi}^T) \). From here we proceed just as in Case 2.

**Case 4.** \( \chi > \theta \) and (ii) holds at stage \( \chi - 1 \).

This is basically like Case 3, but letting \( \kappa_b = \text{lgcd}(\text{exit}_{\chi-1}^T) \), then \( \chi - 1 <^T \xi \) and \( \beta_b + 1 = \text{succ}^T(\chi - 1, \xi) \) (instead of \( \chi <^T \xi \) and \( \beta_b + 1 = \text{succ}^T(\chi, \xi) \)). But as before, consider the least \( i \) such that \( \kappa_i < \text{lh}(E_{\chi}^T) \), so \( i \leq b \) and \( \kappa_b \leq \kappa_i \), etc.

**Case 5.** \( \chi > \theta \) and (i)b holds at stage \( \chi - 1 \).

This is a slight variant of Case 4, completing all cases and hence the proof of the subclaim. □
By Subclaim 1.1, $\mathcal{T} \cap [\theta, \xi]$ can be viewed as a tree $T'$ on the phalanx
$$\Phi(T \mid (\bar{\theta} + 1)) \supset ((Q, 0)),$$
where in $T'$, if $cr(E^T_\alpha) \leq \kappa$ then $pred^{T'}(\alpha + 1) \leq \bar{\theta}$ (so $E^T_\alpha$ applies to a model of $\mathcal{T} \mid (\bar{\theta} + 1)$), whereas if $cr(E^T_\alpha) > \kappa$ then $pred^{T'}(\alpha + 1) = \bar{\theta} + 1$, with $(M^T_{\alpha + 1}, \text{deg}_{\alpha + 1}) \leq (Q, 0)$. In fact, by the subclaim, if $cr(E^T_\alpha) > \kappa$ then $cr(E^T_\alpha) \geq \sigma$, and recall that exit$_\theta^T \leq Q$ and $\rho^Q_1 \leq \sigma$, so requiring $(M^T_{\alpha + 1}, \text{deg}_{\alpha + 1}) \leq (Q, 0)$ corresponds with what happens in $\mathcal{T}$. In particular, if $\alpha \geq \bar{\theta}$ and $cr(E^T_\alpha) > \kappa$ and $pred^{T}(\alpha + 1) = \bar{\theta}$ then $(M^T_{\alpha + 1}, \text{deg}_{\alpha + 1}) \leq (Q, 0)$.

Let $Q', g, R$ and $\pi_{R\ell Q'} : R \rightarrow Q$ be as discussed earlier. Then $R < M^T_{\bar{\theta}}$ and $\rho^Q_\theta = \kappa + Q$. We attempt to form $\mathcal{T}$, extending $\mathcal{T} \cap (\bar{\theta} + 1) = \mathcal{T} \cap (\bar{\theta} + 1)$, viewing $\mathcal{T} \cap [\theta, \xi]$ as a tree $T'$ on the corresponding phalanx $\Phi(\mathcal{T} \mid (\bar{\theta} + 1)) \supset ((R, 0))$, requiring that it copies in the usual manner to $T'$, with initial copy maps given by identity maps and $\pi_{R\ell Q'}$. As long as $Q', g$ are chosen large enough, this succeeds. We discuss a little further below why this makes sense, particularly in the case that $Q$ is type 3, and hence $Q' \neq Q$.

Suppose first that $Q$ is type 2, so $Q' = Q$. Then reverse copying proceeds in the usual fashion, as long as the copy maps $\pi_\delta : M^T_{\bar{\theta}} \rightarrow M^T_{\bar{\theta} + m}$ (for $\delta < \xi$) are $\nu$-preserving, and $E^T_{\bar{\theta} + m} \in \pi_{\ell \delta}$. Just as in the proof of 6.20, by choosing $g$ large enough, we can ensure that these conditions are indeed maintained.

Now suppose instead that $Q$ is type 3. Here $Q'$ is type 2 with $F^{Q'} \subseteq F^Q$ and $Q' \not\approx Q$, and (we may arrange that) $\pi_{R\ell Q'} : R \rightarrow Q'$ is a 0-embedding. In this case we may not be able to “reverse copy” ultrapowers of $Q$ to ultrapowers of $R$, for example, since $Q' \neq Q$. So we want to point out why this does not cause a problem. Because $Q$ is type 3, we have $\beta = \theta$ and $\nu = \nu(F^Q)$, so $E^Q_\delta = F^Q$. At stage $\theta$ in $\mathcal{T}$, we “reverse copy” in the sense that we set $E^T_{\bar{\theta}} = F^R$. This then yields the copy map(s) at stage $\bar{\theta} + 1$ via essentially the Shift Lemma: Given an $n$-sound $N$ with $N \upharpoonright [\kappa^+ N] = Q \upharpoonright [\kappa^+ Q]$ and $\kappa < \rho^N_n$, we get a map
$$\pi : \text{Ult}_n(N, F^R) \rightarrow \text{Ult}_n(N, F^Q)$$
by setting $\pi([a, f]_{E^R_{\bar{\theta}}}) = [\pi_{R\ell Q'}(a), f]_{E^Q_{\bar{\theta}}}$; By taking $Q', g$ large enough, we get $\sigma \in \text{rg}(\pi)$ here (and $\nu(F^Q) \in \text{rg}(\pi)$ as $\pi_{R\ell Q'}$ is a 0-embedding); let $\pi(\bar{\sigma}) = \sigma$. Note that then $\nu(F^R) < \bar{\sigma}$, as $\pi_{R\ell Q'}(\nu(F^R)) = \nu(F^{Q'}) < \nu(F^Q) = \sigma$. Letting $\alpha \in [\theta, \xi]$, since $cr(E^T_\alpha) \notin (\kappa, \sigma)$ and $\sigma = \nu(F^Q) = \nu(E^Q_\delta)$, if $cr(E^T_\alpha) > \kappa$ then $pred^{T'\bar{\delta}}(\alpha + 1) > \theta$. It will follow that in $\mathcal{T}$, for all stages $\alpha \geq \bar{\theta}$, $cr(E^T_\alpha) \notin (\kappa, \sigma)$, and if $\alpha > \bar{\theta}$ then root$_T^\mathcal{T}(\alpha) \leq \bar{\theta}$ (that is, $\alpha$ is above a node of $\Phi(\mathcal{T} \mid (\bar{\theta} + 1))$, not above $R$). So in $\mathcal{T}$, we never need to take an ultrapower of any $R' \subseteq R$, nor in fact of any $R' \subseteq M^T_{\bar{\theta}}$. So assuming that we have taken $Q', g$ large enough, the copying process succeeds, as desired. We leave the remaining (straightforward) details to the reader.

Now $\mathcal{T}$ is a candidate with index less than that of $\mathcal{T}$, a contradiction which completes the proof that $\mathcal{T}$ is strongly finite.

This completes the proof of the theorem.

9 Measures in mice

We pause briefly to use the machinery of the previous sections to analyse measures in mice. The argument is mostly that given in [22, Theorem 4.8], though that was formally
proven only below superstrong. The version in [22] is also more restrictive in that it assumes that $N \models \text{KP}^* + \kappa^{++}$ exists, where the measure is over $\kappa$. For the version here, this assumption is reduced to that of $N \models \kappa^+$ exists, where the measure is over $\kappa$ (in particular, without $\text{KP}^*$). Although we do not literally need the theorem itself later, we will adapt its proof in the proofs of super-Dodd-solidity 10.1 and projectum-finite generation 11.5, so it serves as a good warm-up. We will also omit some of the details of those proofs when they are similar enough to those in this section. So like §8, the proofs of the main theorems of the paper do depend on the arguments in this section. However, it should be noted that the details relating to the coding of finite iteration trees definably over mice, from [27], which come up at the beginning of the proof, are not relevant to later proofs in the paper. If one weakens the theorem below by adding the assumption that $N$ has no largest cardinal, then one can avoid the use of any coding apparatus, as we indicate during the proof.

9.1 Theorem. Let $N$ be a $(0, \omega_1+1)$-iterable premouse and $D \in N$ such that $N \models \text{"D is a countably complete ultrafilter". Then there is a strongly finite, terminally-non-dropping, 0-maximal tree $T$ on $N$ with $\text{Ult}_0(N,D) = M^T_\infty$ and $i^N_D = i^T$.47

We cannot prove the theorem above until we know that all proper segments of $N$ are Dodd-sound, which will be established in §10. For now we reduce it to this issue:

9.2 Lemma. Adopt the hypotheses of Theorem 9.1, and assume that all proper segments of $N$ are Dodd-sound. Then the conclusion of Theorem 9.1 holds.

Proof. We sketched the main plan for the proof in the first part of §2.2. That plan should be read before starting the details below. In the first part of the proof, we will get into the kind of situation assumed for the sketch in §2.2. We do this by taking some $M \triangleleft N$ over which there is an $N$-ultrafilter $D$ definable which is a counterexample to the lemma, taking a nice enough countable hull $M$ of $M$, over which we can define a preimage $D$ of some such $D$, and showing that the relevant fine structural ultrapower $\bar{U} = \text{Ult}_m(M, D)$ is iterable. At that point, we will basically have arranged the assumptions for the plan, and so we then execute that plan by comparing $M$ with $\bar{U}$ and analysing the outcome.

So suppose the lemma fails. Let $\theta \geq \omega$ be such that $D$ is an ultrafilter over $\theta$. Then $\mathcal{P}(\theta)^N \in N$ (as $D \in N$), so $\theta^{++} N \prec \text{OR}^N$. We have $D \subseteq N|\theta^{++} N$, so can let $M \triangleleft N$ with $\theta^M = \theta^{++}$ and $D \in \mathcal{J}(M)$.

Let $0 < n < \omega$ with $\rho^M_n = \theta^{++} = \rho^M_\omega$. We claim that there is a strongly finite, terminally-non-dropping $n$-maximal tree $T$ on $M$ such that $M^T_n = \text{Ult}_n(M, D)$ and $i^T = i^{M,n}_D$. This yields a tree $T''$ on $N$ witnessing the lemma. For using the regularity of $\theta^{++} N$ in $N$, one gets that $n$-maximal trees $U$ on $M$ are equivalent to $\omega$-maximal trees $U'$ on $M$. That is, every function $f: \alpha \to \mathcal{C}_0(M)$ with $\alpha \leq \theta$ such that $f$ is definable from parameters over $M$, is actually $\Sigma^M_\omega$-definable, and this fact is preserved for non-degree-dropping $n$-maximal, hence $\omega$-maximal, iterates. Similarly, $\omega$-maximal trees $U'$ on $M$ correspond to 0-maximal trees $U''$ on $N$ such that $U''$ is based on $M$, as long as $U''$ has wellfounded models, and moreover, if $\beta^U$ exists and is non-dropping, then $\beta^{U''}$ exists and is non-dropping, and $\beta^{U''} = \beta^{U'} \restriction M$. (For some ordinals $\alpha$ we have $M^{T'}_\alpha = M^{T''}_\alpha$, in which case $(0, \alpha)^T$ and $(0, \alpha)^{T''}$ do not drop in model or degree; for some $\alpha$ we have $M^{T'}_\alpha = M^{T''}_\alpha$.) So given a tree $T$ on $M$ as claimed, the corresponding tree $T''$ on $N$ witnesses the lemma.

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47 Of course, since $T$ is 0-maximal, we must have $E^T_n \in E_+ (M^T_n)$, so the theorem is non-trivial.
So suppose the claim fails. The failure is a first order statement about $M$ (without parameters). For if $D' \subseteq M^{\theta+N}$ is definable (from parameters) over $M$, and $M \models "D'"$ is a countably complete ultrafilter over $\theta'$, then $D' \in N$ and $N \models "D'"$ is a countably complete ultrafilter over $\theta'$, since $P(\theta)^N \subseteq M$. And [27, 3.15–3.20] gives a coding of finite $n$-maximal iteration trees on $\theta$-sound $(n, \omega)$-iterable premice, which is uniformly definable over such premice. Note that the definability depends on the restriction to $n$-maximal for a particular $n$. As usual, the superstrong version of this is almost identical with that literally in [27].48 However, for our present purposes, it suffices to consider only a coding of strongly finite trees $T$, and for such $T$, every extender $E^T_\alpha$ is type 1 or type 2 and hence not superstrong. So the coding apparatus in [27] is essentially sufficient for our purposes. As remarked earlier, if $N$ has no largest cardinal, then we can avoid appealing to any coding of iteration trees at all. For in this case, instead of using $M$ as above, we can take $M = N|\theta++N$, and then the relevant iteration trees can be defined directly over $M$.49 We continue literally with the general case (using a coding apparatus), but the remaining details are essentially the same in the simplified version just mentioned.

Using these two things, we can fix $k \in [n, \omega)$ and $m \gg k$ and an $r\Sigma_m$ statement

$$\psi$$

asserting over $M$ (and also over similar premice) “I am $k$-sound and there is a countably complete ultrafilter $D'$ over $\theta$ which is $r\Sigma_k$-definable from parameters over me, such that for every strongly finite non-terminal $n$-maximal tree $T$ on me and every $\alpha < i_0^T(\theta)$, if $\alpha$ generates $M^T_{\infty\infty}$, i.e. if

$$M^T_{0\infty} = \text{Hull}^{M^T_{\infty\infty}}_{r\Sigma_m}(\text{rg}(i^T_{0\infty}) \cup \{ \alpha \}),$$

then $D'$ is not the ultrafilter derived from $i^T_{0\infty}$ with seed $\alpha$”.

Working in $N$, let $\bar{M} = e\text{Hull}_{m+1}(\bar{p}_n^M)$ and $\pi : \bar{M} \rightarrow M$ be the uncollapse map. Let $\pi(\bar{\theta}) = \theta$ etc. Then $\bar{M}$ is $(m+1)$-sound with

$$\rho_{m+1}^M = \omega < \bar{\theta} = \rho_m^M = \bar{\rho}_n^M$$

and $\pi(p_i^M) = p_i^M$ for $i \leq m+1$ (note $p_i^M = \emptyset$ for $i > n$) and

$$\bar{M} = \text{Hull}^{\bar{M}_{m+1}}_{r\Sigma_m}(\bar{\rho}_n^M) = \text{Hull}^{\bar{M}_{m+1}}_{\infty\infty}(\emptyset),$$

as $\bar{\rho}_n^M$ is simply enough definable over $\bar{M}$ anyway. As $m$ is large enough, $\bar{M} \models \psi$, so we can fix an analogous ultrafilter $\bar{D}$ over $\bar{M}$, and may assume that its definition lifts to give $D$ over $M$. Note that $n$-maximal trees on $\bar{M}$ are equivalent to $m$-maximal. And because all initial segments of $M$ are Dodd-sound, so are all initial segments of $\bar{M}$.

48 In the notation of [27, 3.15–3.20], the ordinal $\lambda_{\alpha}^M$ is somewhat inconveniently defined in the case that $E^{T}_\alpha$ is superstrong and $E^T_\alpha \neq F(M^T_\alpha)$ (in this case it is the index of $E^T_\alpha$). Here, if $\mu = \text{cr}(E^T_\alpha)$, then instead of considering functions $f : [\mu]^{<\omega} \rightarrow \mu$ when representing ordinals $< \lambda_{\alpha}^M$, one must of course instead consider functions $f : [\mu^{+\alpha}]^{<\omega} \rightarrow (\mu^{+\alpha})^N$. Alternatively, one might redefine $\lambda_{\alpha}^M$ in this case as $\lambda(E^T_\alpha)$, and allow the possibility that $\lambda_{\alpha}^M = \lambda^{\alpha+1}_{\alpha+1}$.

49 In [22], an argument more along these lines was given, though even there there is not quite that much space available.
Let $X = \bigcap (\text{rg} (\pi) \cap D)$. Then $X \neq \emptyset$ by countable completeness in $N$. Let $x \in X$. Let $\bar{U} = \text{Ult}_n (\bar{M}, \bar{D}) = \text{Ult}_m (\bar{M}, \bar{D})$ (and note $i_{\bar{M}, \bar{D}}^M = i_{\bar{M}, \bar{D}}^M$ is an $m$-embedding and is continuous at $\bar{\theta}^+ \bar{M}$). Like for $\bar{M}$, $n$-maximal trees on $\bar{U}$ are equivalent to $m$-maximal.

The realization map $\sigma : \bar{U} \to M$, defined

$$\sigma ([f]_{\bar{M}, \bar{D}}^n) = \pi (f)(x),$$

is an $n$-lifting embedding. So $\bar{U}$ is $(n, \omega_1 + 1)$-iterable (in $V$), hence $(m, \omega_1 + 1)$-iterable, as is $\bar{M}$. Note $\bar{U} = \text{Hull}_{m+1}^{\bar{U}} (\bar{x})$ where $\sigma (\bar{x}) = x$.

We are now basically in the situation we assumed when describing the plan for the proof in §2.2, which we will now execute in detail. So consider the $m$-maximal comparison $(\bar{U}, \bar{T})$ of $(\bar{U}, \bar{M})$. We will show that $\bar{U}$ is trivial, $\bar{T}$ is strongly finite, $\bar{M} = M^\infty$ and $i_{\bar{M}, \bar{D}}^M = i^T$, which will be a contradiction.

Well, because $\bar{M}$ is $(m+1)$-sound with $\rho_{m+1}^\bar{M} = \omega$, etc, we get that $b^U, b^T$ do not drop, $M^\infty_U = Q = M^\infty_D$, $j, k$ are $m$-embeddings where $j = i^T$ and $k = \bar{i}^T$, as is $i = i_{\bar{M}, \bar{D}}^M$, and $k \circ i = j$ (see Figure 2).

Now since all initial segments of $M$ are Dodd-sound, by Lemma 8.6, we can fix a strongly finite $m$-maximal tree $\bar{T}$ capturing $(\bar{T}, k(\bar{x}))$. We will eventually show that $\bar{T} = \bar{T}$. Let $Q = M^\infty_D$ and $\zeta : Q \to Q$ be the capturing map. So $\zeta$ is an $m$-embedding and $\zeta \circ i^T = i^T = j$. Let $j = i^T$. Let $\tilde{k} : \bar{U} \to \bar{Q}$ be $\tilde{k} = \zeta^{-1} \circ k$; this makes sense as $k(\bar{x}) \in \text{rg} (\zeta)$. Then all maps are $m$-embeddings and the full resulting diagram commutes (see Figure 3).

We will analyze $T$ and its relationship to $\bar{T}$, by analyzing the Dodd decompositions of $j = i^T$ and $\tilde{j} = i^T$, eventually showing that these two Dodd decompositions are identical.

**Claim 1.** Let $\kappa = \text{cr} (j)$. Then:
1. $\kappa = \text{cr}(i) = \text{cr}(j) = \kappa^+ \leq \text{cr}(k) = \min(\text{cr}(k), \text{cr}(\varsigma))$.

2. $\bar{M}||\kappa^+\bar{M} = \bar{U}||\kappa^+\bar{U} = Q||\kappa^+Q = Q||\kappa^+Q$.

\textit{Proof.} By commutativity and since $k$ is an iteration map, it suffices to see that $\text{cr}(k) > \kappa$.
Also by commutativity, $\text{cr}(k) \geq \kappa$, so suppose $\text{cr}(k) = \kappa$. We have $\text{cr}(i) \geq \kappa$. So for each $A \in \mathcal{P}(\kappa) \cap \bar{M}$, we have $i(A) \cap \kappa = A$, so

$$k(A) = k(i(A) \cap \kappa) = k(i(A)) \cap k(\kappa) = j(A) \cap k(\kappa).$$

Therefore $j, k$ are compatible through $k(\kappa) \leq j(\kappa)$. But because $j, k$ are iteration maps arising from comparison, this is impossible. \qed

\textbf{Claim 2.} We have:

1. $s_{\kappa i} = s_j = k(s_i)$ and $\sigma_{\kappa i} = \sigma_j = \text{sup} k^\varsigma \sigma_i$.
2. $s_{\bar{k} \kappa i} = s_j = \bar{k}(s_i)$ and $\sigma_{\bar{k} \kappa i} = \sigma_j = \text{sup} \bar{k}^\varsigma \sigma_i$.
3. $s_{\varsigma j} = s_j = \varsigma(s_j)$ and $\sigma_{\varsigma j} = \sigma_j = \text{sup} \varsigma^\varsigma \sigma_j$.
4. $\kappa^+\bar{M} = \sigma_i = \sigma_j = \kappa$.

\textit{Proof.} Part 1: This holds because $k$ is an iteration map and by Claim 1 and Lemma 7.15. Lemma 7.15 applies because $E_i \upharpoonright i(\kappa) \not\subseteq \bar{U}$. For if $E_i \upharpoonright i(\kappa) \subseteq \bar{U}$ then note that by commutativity and Claim 1, we have

$$k(E_i \upharpoonright i(\kappa)) = E_j \upharpoonright j(\kappa) \in Q,$$

which is impossible, as $j$ is an iteration map.

Part 2: We have $\bar{k} \circ i = \bar{j}$, so $s_{\bar{k} \kappa i} = s_j$ and $\sigma_{\bar{k} \kappa i} = \sigma_j$. We also have $\kappa^+\bar{M} < \text{cr}(k) = \min(\text{cr}(\bar{k}), \text{cr}(\varsigma))$. Therefore by commutativity, $\bar{k}$ maps fragments of $E_i$ to fragments of $E_j = E_{\kappa i}$. So we just need to see we don’t get too large a fragment of $E_j$ appearing in $\bar{Q}$. (Note that we don’t know (yet) that $\bar{k}$ is an iteration map, so we can’t just use Lemma 7.15 for this.) But if

$$E_j \upharpoonright (\bar{k}(s_i) \cup \text{sup} \bar{k}^\varsigma \sigma_i) \in \bar{Q},$$

then applying $\varsigma$, we would have

$$E_j \upharpoonright (\varsigma(\bar{k}(s_i)) \cup \varsigma(\text{sup} \bar{k}^\varsigma \sigma_i)) \in \bar{Q},$$

since, much as for $\bar{k}$, $\varsigma$ maps fragments of $E_j$ to fragments of $E_j = E_{\kappa i}$. But as $\varsigma(\bar{k}(s_i)) = k(s_i)$ and $\varsigma(\text{sup} k^\varsigma \sigma_i) \geq \text{sup} k^\varsigma \sigma_i$, this contradicts part 1.

Part 3: $\circ \bar{j} = j$, so $s_{\circ \bar{j}} = s_j$ and $\sigma_{\circ \bar{j}} = \sigma_j$. But by part 1, commutativity and 2,

$$s_j = k(s_i) = \varsigma(\bar{k}(s_i)) = \varsigma(s_j),$$

$$\sigma_j = \text{sup} k^\varsigma \sigma_i = \text{sup} \varsigma \circ \bar{k}^\varsigma \sigma_i = \text{sup} \varsigma^\varsigma \sigma_j.$$  

Part 4: This follows immediately from the preceding parts together with the strong finiteness of $\bar{T}$ (in particular using that the Dodd core of the first extender used along $b^\bar{T}$ is finitely generated). \qed
The following claim shows that the first extenders in the Dodd decompositions of \( \bar{j} \) and \( j \) are identical. Let \( \bar{\alpha} \) be least such that \( \bar{\alpha} + 1 \in b^\bar{T} \), and \( \alpha \) likewise for \( T \).

**Claim 3.** \( E_{D_\alpha}(E_{D_\alpha}^T) = E_{D_\alpha}(E_{D_\alpha}^{T_0}) \).

**Proof.** By the previous claim, we have \( \sigma_j = \sigma_{\bar{j}} = \kappa^+ M \), and \( \varsigma(s_j) = s_j \). But then \( E_j \upharpoonright (\sigma_j \cup s_j) \) is equivalent to \( E_{\bar{j}} \upharpoonright (\sigma_{\bar{j}} \cup s_j) \), and these extenders are equivalent to the Dodd cores mentioned in the claim. \( \square \)

So we have shown that \( \bar{j} = i^T \) and \( j = i^T \) yield the same first extenders in their Dodd decompositions. We now want to show that they have the same second extenders, etc, proceeding all the way through.

Let \( G_0 = \mathcal{C}_{D_\alpha}(E_{D_\alpha}^T) = \mathcal{C}_{D_\alpha}(E_{D_\alpha}^{T_0}) \), and let \( M_1 = \text{Ult}_m(M, G_0) \). Let \( j_{01} : M \to M_1 \) and \( j_{1\infty} : M_1 \to Q \) and \( j_{1\infty} : M_1 \to Q \) be the Dodd decomposition maps (Lemma 7.22); in particular, \( j_{01} \) is the ultrapower map. So

\[
M_1 = \text{cHull}_{m+1}(\kappa^+ M \cup \{s_j\}),
\]

\( j_{1\infty} \) is the uncollapse map and \( j_{1\infty}(t_{G_0}) = s_j \), and likewise regarding \( j_{1\infty} : M_1 \to Q \).

Since \( \varsigma(s_j) = s_j \) and \( \sigma_j = \sigma_{\bar{j}} = \kappa^+ M \), we have \( \varsigma \circ j_{1\infty} = j_{1\infty} \).

By the claims, \( \kappa^+ M \cup \{s_j\} \subseteq \text{rg}(\tilde{k}) \), so we can define \( i_1 : M_1 \to \bar{U} \) by \( i_1 = \tilde{k}^{-1} \circ j_{1\infty} \).

We get an extended commuting diagram (Figure 4).

**Claim 4.** \( \bar{M} \neq \bar{Q} \).

**Proof.** Suppose \( \bar{M} = \bar{Q} \), so \( \langle G_0 \rangle \) is the full Dodd decomposition of \( \bar{j} \), so \( j_{1\infty} = \text{id} \).

Then \( i_1 : \bar{Q} \to \bar{U} \) and \( \bar{k} \circ i_1 = \text{id} \) (because all the relevant generators are in \( \text{rg}(\tilde{k}) \)). Therefore \( \bar{Q} = \bar{U} \) and \( \bar{k} = i_1 = \text{id} \) and \( \bar{U} \) is a normal iterate of \( M \), via \( \bar{T} \), and \( i = i^T = j_{0\infty} \).

(Therefore the comparison of \( \bar{M} \) with \( U \) actually yields \( \bar{T} = \bar{T} \) and \( U \) is trivial.) Moreover, \( \bar{T} \) is strongly finite. But we arranged that \( M \models \psi \) (from line (5)), which ensured that no such \( \bar{T} \) exists, a contradiction. \( \square \)

Since \( \bar{M} \neq \bar{Q} \) and \( \bar{M} \) appears along the Dodd decomposition of \( \bar{j} = i^T \), \( \text{cr}(j_{1\infty}) \) exists. By commutativity, \( \text{cr}(\bar{j}_{1\infty}) \) also exists (so \( \bar{M} \neq Q \) also). We now get the following slight variant of Claim 1:

**Claim 5.** Let \( \kappa_1 = \text{cr}(j_{1\infty}) \). Then:

\[
\]
1. \( \kappa_1 = \text{cr}(i_1) = \text{cr}(j_{1\infty}) = \text{cr}(j_{1\infty}) < \kappa_1^{+U} < \text{cr}(k) = \min(\text{cr}(\bar{k}), \text{cr}(\varsigma)) \).

2. \( \bar{M}_1||\kappa_1^{+\bar{M}_1} = \bar{U}||\kappa_1^{+U} = \bar{Q}||\kappa_1^{+\bar{Q}} = Q||\kappa_1^{+Q} \).

**Proof.** Let \( \bar{\alpha}, \alpha \) be as before; that is, \( \bar{\alpha} \) is least such that \( \bar{\alpha} + 1 \in b^\top \), and \( \alpha \) likewise for \( T \). If \( E_\alpha^T \) is Dodd-sound, or equivalently, \( G_0 = E_\alpha^T \), then \( \bar{M}_1 = M_{\alpha + 1}^T \), and and \( j_{1\infty} = i_{\bar{\alpha} + 1, \infty} \), and we can argue as in the proof of Claim 1 to show that \( \text{cr}(k) > \text{cr}(j_{1\infty}) \), and hence \( \text{cr}(i_1) = \text{cr}(j_{1\infty}) = \text{cr}(j_{1\infty}) \), etc. Suppose instead that \( E_\alpha^T \) is non-Dodd-sound. Let \( \gamma \) be such that \( \gamma < T_{\bar{\alpha}} \alpha \) with \( \text{cr}(E_\gamma^T) \) minimal. Then \( E_\gamma^T \upharpoonright \nu(E_\gamma^T) \), which is used in the comparison, is derived from \( j_{1\infty} \). So we can still argue as before to show that \( \text{cr}(k) > \text{cr}(j_{1\infty}) \), etc. \( \Box \)

We next adapt Claim 2; the proof is essentially identical, so we leave it to the reader:

**Claim 6.** We have:

1. \( s_{k\bar{i}_1} = s_{j_{1\infty}} = k(s_{i_1}) \) and \( \sigma_{k\bar{i}_1} = \sigma_{j_{1\infty}} = \sup k^n \sigma_{i_1} \).
2. \( s_{k\bar{i}_1} = s_{j_{1\infty}} = k(s_{i_1}) \) and \( \sigma_{k\bar{i}_1} = \sigma_{j_{1\infty}} = \sup k^n \sigma_{i_1} \).
3. \( s_{\varsigma\bar{j}_{1\infty}} = s_{j_{1\infty}} = \varsigma(s_{i_1}) \) and \( \sigma_{\varsigma\bar{j}_{1\infty}} = \sigma_{j_{1\infty}} = \sup \varsigma^n \sigma_{j_{1\infty}} \).
4. \( \kappa^{+\bar{M}_1} = \sigma_{i_1} = \sigma_{j_{1\infty}} = \sigma_{j_{1\infty}} \).

Let \( \bar{\alpha}_1 \) be least such that \( \bar{\alpha}_1 + 1 \in b^\top \setminus \{\bar{\alpha} + 1\} \), and \( \alpha_1 \) likewise for \( T \) (with respect to \( \alpha \)). We now adapt Claim 3; again the proof is just like before:

**Claim 7.** \( \mathcal{C}_{Ds}(E_{\bar{\alpha}_1}^T) = \mathcal{C}_{Ds}(E_{\alpha_1}^T) \).

So \( \bar{j} \) and \( j \) also have the same second extenders in their respective Dodd decompositions.

We are now in a position to define \( G_1 = \mathcal{C}_{Ds}(E_{\bar{\alpha}_1}^T) = \mathcal{C}_{Ds}(E_{\alpha_1}^T), \bar{M}_2, i_2 : \bar{M}_2 \to \bar{U} \), etc, and we get another commuting diagram, with \( \bar{M}_2 \) situated between \( \bar{M}_1 \) and \( \bar{Q} \).

This process extends directly to all finite stages of the Dodd decomposition of \( j \). But \( \bar{T} \) is finite, so we reach some stage \( \ell < \omega \) with \( \bar{M}_\ell = \bar{Q} \). However, the proof of Claim 4 adapts directly (with “\( \ell \)” replacing “1” and \( "\langle G_0, \ldots, G_{\ell - 1}\rangle" \) replacing “\( \langle G_0 \rangle \)”) to show that in fact, \( \bar{M}_\ell \neq \bar{Q} \). This contradiction completes the proof. \( \Box \)

## 10 Super-Dodd-soundness

The following theorem is basically due to the combination of work of Steel [12, Theorem 3.2], [16, Theorem 4.1] and Zeman [36, Theorems 1.1, 1.2]. The proof we give here is somewhat different, however. In its proof, we will use the methods of the proof of Lemma 9.2, with which the reader should probably be familiar.

**10.1 Theorem** (Super-Dodd-soundness). Let \( M \) be an active, \( (0, \omega_1 + 1) \)-iterable premouse, let \( \kappa = \kappa^M \), and suppose that either:

(a) \( M \) is 1-sound, or

(b) \( M \) is \( \kappa^{+M} \)-sound.

Then \( M \) is super-Dodd-sound.
In this section we prove the following lemma, which is the central argument for the proof of super-Dodd-soundness:

**10.2 Lemma (Super-Dodd-soundness).** Assume the hypotheses of Theorem 10.1. Suppose further that \( \mathcal{C}_1(M) \) is 1-sound. Then the conclusion of Theorem 10.1 holds.

**10.3 Remark.** The hypothesis that \( \mathcal{C}_1(M) \) is 1-sound is actually superfluous, by Theorem 14.1, but we will have to use the lemma in order to prove this fact.

The proof of super-Dodd-soundness to follow proceeds by first reducing to the case in which normal iterability yields an iteration strategy with the weak Dodd-Jensen property, and then approximately follows Steel’s proof. Actually this reduction is very easy, and just takes a few lines of argument. However, Steel’s proof makes significant use of the assumption that \( M \) has no extenders of superstrong type on its sequence, so does not fully suffice for our purposes. There is some more work involved in adapting things to the superstrong level. There are also some minor modifications for the super- aspect of super-Dodd-soundness. But none of this adaptation work relates directly to proving fine structure from normal iterability. Moreover, Zeman [36] already proved (standard) Dodd-soundness for 1-sound mice with \( \lambda \)-indexing, at the superstrong level (from \( (0, \omega_1, \omega_1 + 1)^\ast \)-iterability). So after the reduction to the case that we have a strategy with weak Dodd-Jensen, one could presumably follow Zeman’s proof closely, translating it to Mitchell-Steel indexing, at least for standard Dodd-soundness. The proof we give presumably does have significant overlap with Zeman’s, but is instead based on Steel’s and an elaboration of the proof of Theorem 9.1.  

Before we start, note the following, which will be useful, since \( M \) is active:

**10.4 Definition.** Let \( T \) be a 0-maximal iteration tree. For \( \alpha + 1 < \text{lh}(T) \), say \( \alpha \) is \( T \)-special iff \( (0, \alpha]^T \cap \mathcal{D}^T = \emptyset \) and \( E^T_\alpha = F(M^T_\alpha) \).

**10.5 Lemma.** Let \( T \) be 0-maximal, \( \alpha \) be \( T \)-special and \( \beta = \text{pred}^T(\alpha + 1) \). Then \( \beta \leq^T \alpha \).
If \( \beta < \alpha \) then \( (\beta, \alpha]^T \) is non-dropping, \( \text{cr}(i^T_{\beta \alpha}) > \text{cr}(E^T_\alpha) \), and \( \gamma \) is non-\( T \)-special for every \( \gamma + 1 \in (\beta, \alpha]^T \). Hence, if all proper segments of \( M^T_0 \) are Dodd-sound then \( \gamma \) is Dodd-nice for each \( \gamma + 1 \in (\beta, \alpha]^T \).

**Proof of Lemma 10.2.** The plan is as follows. We first reduce to case (a), that \( M \) is 1-sound, and then reduce further to the case in which normal iterability is sufficient to provide an iteration strategy with the weak Dodd-Jensen property. From there we proceed with a comparison argument similar to Steel’s original proof, augmented with further analysis which is necessary to handle the presence of superstrong extenders. We will give the rough idea of this comparison argument after we perform the reductions just mentioned.

So let us first reduce to case (a). Suppose that (b) holds. Let \( C = \mathcal{C}_1(M) \) and let \( \pi : C \to M \) be the core map. So (by our added assumption for the lemma) \( C \) is 1-sound, so assuming the result in case (a), \( C \) is super-Dodd-sound. So if \( \kappa = \kappa^M < \rho^M_1 \) then \( M = C \)

\[\text{69}\]
is super-Dodd-sound. And if $\kappa = \rho_1^M$ then note that we still have $M = C$, since $\pi$ is a 0-embedding (and $M$ is type 2), so $\text{rg}(\pi)$ is cofinal in $\kappa^+$. So we may assume $\rho_1^M < \kappa$.

Since $\pi$ is $\Sigma_1$-elementary, it suffices to see $\pi(\bar{t}^C) = \bar{t}^M$ and $\bar{t}^M = 0$. Let $t' \in [\nu(M)]^{<\omega}$ be least generating $(F_{\bar{t}}^M, \rho_1^M)$ (with respect to $F^M$). Since $F_{\bar{t}}^M, \rho_1^M \in \text{rg}(\pi)$, we have $t' \in \text{rg}(\pi)$. And $\kappa$ is generated by $t'$, because $F_{\bar{t}}^M$ is.

We claim $t' = \bar{t}^M$ and $\bar{t}^M = 0$. In fact, because $M$ is $\kappa^+$-sound and $\rho_1^M, F_{\bar{t}}^M$ are generated by $t'$, by hypothesis (b) and Lemma 7.7, $t'$ generates $F^M$. It follows that $\bar{t}^M = 0$ and $\bar{t}^M \leq t'$. But then the minimality of $t'$ implies that $\bar{t}^M = t'$.

So $\bar{t}^M \in \text{rg}(\pi)$ and $\bar{t}^M = 0$. But then it is easy to see that $\pi(\bar{t}^C) = \bar{t}^M$, completing the proof.

Now assume that (a) holds; that is, $M$ is 1-sound. We will show $M$ is super-Dodd-sound. We may and do assume that $M$ is type 2 and all its proper segments are Dodd-sound. Moreover, $M$ is Dodd-amenable. For let $\kappa = \kappa^M$ and suppose that $\kappa^+ < \tau^M$ (otherwise Dodd-amenability is immediate). Then by Lemma 7.8, $\kappa^+ < \tau^M = \rho_1^M$. But then for every $\alpha < \tau^M$, we have $F \upharpoonright \alpha \cup \{t^M\} \in M$, since this is coded as an $\Sigma_1^M$ subset of $(\kappa^+ \times \alpha)^{<\omega}$, and $\kappa^+, \alpha < \rho_1^M$. So it suffices to prove that $M$ is super-Dodd-solid.

We now take the $\Sigma_1$-hull over $M$ of a singleton $\{q\}$, with $q$ selected capturing the relevant objects and such that the resulting transitive collapse is sound. For this, we apply [28, Definition 2.2, Lemma 2.3]: Let $(q, \omega) \in D$ be 1-self-solid for $M$ with $\bar{t}^M, \bar{t}^M \in \text{Hull}_1^M(\{q\})$, let $\bar{M} = \text{Hull}_1^M(\{q\})$ and $\pi : \bar{M} \rightarrow M$ be the uncollapse (such a pair $(q, \omega)$ exists by [28, Lemma 2.3]; in the notation there, the “$q$” and “$\omega$” are written in the other order). So $\rho_1^{\bar{M}} = \omega$ and $\bar{M}$ is sound.

Claim 1. If $M$ is super-Dodd-solid then so is $\bar{M}$.

Proof. Note $\pi(\bar{t}^M \upharpoonright \text{lh}(t^M)) = \bar{t}^M$, so if $\bar{E} \in M$ witnesses super-Dodd-solidity for $\bar{M}$, then $\pi(\bar{E} \upharpoonright \text{lh}(t^M))$ witnesses super-Dodd-solidity for $M$. \hfill \Box

Also, all proper segments of $\bar{M}$ are Dodd-solid. So resetting notation, we may assume:

Assumption 1. $\rho_1^M = \omega$.

Recall $M$ is also 1-sound. We can therefore use these fine structural circumstances to substitute for any appeals to weak Dodd-Jensen in the proofs. (Moreover, the unique $(0, \omega_1 + 1)$-strategy for $M$ has weak Dodd-Jensen.) Thus, we have completed the portion of the proof of super-Dodd-solidity which is directly relevant to the main theme of the paper (that is, proving fine structural facts from normal iterability). From here on we can just approximate Steel’s proof of Dodd-solidity (the conjunction of [12], [16], [20]), modified to deal with superstrong extenders and to prove super-Dodd-solidity. We will, however, give a full account.

In the main argument (which we will only need to use under some further assumptions, specified in Assumption 2), we will consider a comparison of $M$ with a phalanx $\mathfrak{U}$. This phalanx could, for example, be of form $\mathfrak{U} = ((M, < \zeta), U, \zeta)$ where $U = \text{Ult}_0(M, G)$ for some Dodd-solidity witness $G$ corresponding to $\zeta \in t^M$, where, for example, $\zeta$ is an $M$-cardinal and $\kappa < \zeta < \text{lgcd}(M)$.

This is the direct analogue of the proof of ISC in [9, §10]. In Steel’s proof of Dodd-solidity, say in case $\zeta$ is an $M$-cardinal and $\zeta < \text{lgcd}(M)$, instead of comparing with $\mathfrak{U}$, $M$ is actually compared with the
nor $b^T$ drops, $b^T$ is above $U$, and $i^M \circ i^M_G = i^T$. This will give that $G$ is a sub-extender of that derived from $i^T$, using a set of generators of form $\zeta \cup x$, for some finite $x$. By analyzing the extenders used along $b^T$, we will see that this sub-extender can in fact be computed working inside $M$. The main difficulty will be in handling the possibility that some $T$-special extender $E^T_\alpha$ is used along $b^T$ (that is, $E^T_\alpha$ is a non-dropping image of $F^M$). The special case of this in which the first extender used along $b^T$ is $T$-special, is somewhat easier to handle. The other case (the first extender used along $b^T$ is non-$T$-special, but some later one is $T$-special) is readily ruled out in Steel’s proof, because it implies that there are superstrong extenders in $E^M$. But that argument does not suffice for us here, so we need to handle it directly, which takes some more work (see especially Claim 21 of Case 2). The measure analysis argument from the proof of Theorem 9.1 will be useful in this regard. (Another variant of the measure analysis argument will also be needed later in the proof of Theorem 11.5.)

So let $\zeta \in \tilde{M}$, let $w = \tilde{M} \setminus (\zeta + 1)$, and let $G = Dw^M(w, \zeta)$. We must see $G \in M$.

We first dispense (in Claims 3 and 4 below) with some easy cases.

Claim 2. $F^M$ is generated by $\tilde{M}$, so $F^M \upharpoonright \tilde{M} \notin M$.

Proof. We have $\tau^M = \kappa^+M$, by Lemma 7.8, Assumption 1 and since $M$ is 1-sound. So $\tilde{M} = 0$, which suffices.

Claim 3. If $w = 0$, or equivalently, $\nu(F^M) = \zeta + 1$, then $G \in M$.

Proof. If $G$ is non-type $Z$, this is by the ISC. Suppose $G$ is type $Z$. We argue like in [16, 2.7]; here is a sketch. Since $G$ is type $Z$, it has a largest generator $\xi$, which is a limit of generators (of both $G$ and $F^M$). So $G = G \upharpoonright \xi$ is type 3, and by the ISC, there is $R \triangleleft M$ with $F^R = G$. Let

$$\tilde{U} = \text{Ult}_0(M, \tilde{G}) \text{ and } U = \text{Ult}_0(M, G) \text{ and } W = \text{Ult}_0(M, F^M),$$

and let $\tilde{\pi} : \tilde{U} \to U$ and $\pi : U \to W$ be the factor maps. So $\text{cr}(\tilde{\pi}) = \xi$ and $\text{cr}^{(\pi)} = \zeta > \xi$.

In fact, $\text{cr}(\pi) = \zeta = \xi^+U$. For since $\text{cr}^{(\pi)} > \xi$, we have $\text{cr}^{(\pi)} \geq \xi^+U$. But $\xi^+U$ is not a $W$-cardinal, since $\tilde{G} \in W$ (by coherence), and therefore $\text{cr}^{(\pi)} \leq \xi^+U$.

Subclaim 3.1. $\xi$ is the largest cardinal of $M$, and $\xi$ is inaccessible in $M$.

Proof. Since $R \triangleleft M$, we have $\tilde{G} \in M$. It follows that $\text{card}^{(M)}(\zeta) \leq \xi$, and since $\zeta \geq \text{lgcd}(M)$, therefore $\xi \geq \text{lgcd}(M)$. So we just need to see that $\xi$ is inaccessible in $M$. As $G$ is type $Z$, $\text{lh}(F^R) = \xi^+U = \xi^+U$. By condensation for $\omega$-sound mice (Fact 1.9) with $\tilde{\pi}, \tilde{U} \upharpoonright \xi^+U = U \upharpoonright \xi^+U$, and $\xi$ is inaccessible in both $\tilde{U}$ and $U$. But $\pi(\xi) = \xi$, so $\xi$ is inaccessible in $W$, and so by coherence, $\xi$ is also inaccessible in $M$, as desired.

Now consider the phalanx

$$\mathcal{S} = ((M, < \zeta), (R, \xi), U, \zeta = \xi^+U),$$

phalanx $((M, < \zeta), H, \zeta)$, where $H$ is as described just before Claim 5 below. But these two comparisons are just slight translations of one another. We find it a little more convenient to work with $\mathcal{S}$ instead.
Note $\mathcal{H}$ is $((0, -1, 0), \omega_1 + 1)$-iterable\textsuperscript{52}, because we can lift $(0, -1, 0)$-maximal trees on it to trees on the $(0, 0, 0)$-iterable phalanx

$$\mathcal{H}' = \left( (M, < \xi), (M, \xi), W, \text{OR}^M = \xi^+W \right),$$

using the identity map $M \rightarrow M$, the inclusion map $R \rightarrow R \lhd M$, and $\pi : U \rightarrow W$ as lifting maps. The execution of the lifting process is much like (but somewhat simpler than) that in the proof of Claim 6 below, so we omit further details here. We compare $\mathcal{H}$ with $M$, resulting in trees $U, T$ respectively.

**Claim 4.** $\zeta \in p_1^M \setminus \kappa^+M$ then $G \in M$.

**Proof.** Let $\rho = p_1^M \setminus (\zeta + 1)$ and $N = \text{cHull}_1^M (\zeta \cup \{p\})$ and $\pi : N \rightarrow M$ the uncollapse. By 1-solidity, $N \in M$. By Lemma 7.11, $w = F^M \setminus (\zeta + 1) \in \text{rg}(\pi)$. But then $G = Dw^M (w, \zeta)$ is equivalent to $F^N \in M$.

By Claims 3 and 4, Lemma 7.11 and induction, and noting that $\text{cr}(F^M) \notin p_1^M$, we may assume:

**Assumption 2.**

- $w \neq \emptyset$,
- $Dw^M (w \setminus (\alpha + 1), \alpha) \in M$ for all $\alpha \in w$, and
- $\zeta \notin p_1^M$, so $F^M_\zeta$ is not generated by $\zeta \cup w$.

Under these assumptions we will deal with the main case, which will involve a comparison similar to that in the proof of Claim 3. This will be a comparison between a phalanx $\mathcal{H}$, defined below, and $M$; we introduced an example case of this comparison in the outline given just after Assumption 1. Let $U = \text{Ult}_0 (M, G), W = \text{Ult}_0 (M, F^M)$, and $\pi : U \rightarrow W$ be the factor map. So $\text{cr}(\pi) = \zeta$. Let $X = w \cup \zeta, H = M_X$ and $\pi^- = \pi_X : H \rightarrow M$ (as

\textsuperscript{52}The degree $-1$ is used because $R$ is active type 3 with $\rho_0^R = \nu(F^R) = \xi$, and extenders $E$ only apply to $R$ with $\text{cr}(E) = \xi$. The ultrapower $\text{Ult}_{-1}(R, E) = \text{Ult}(R, E)$ is just the usual ultrapower, formed without squashing. Iterability just requires that all models produced are wellfounded, not that they be premice. If $\text{cr}(E^R) = \xi$ then $M^M_{\alpha+1}$ is not a premouse, even if it is wellfounded, because it fails the ISC.}
in Definition 7.6). Note that $F^H = \text{trvcom}(G)$, $H^{\nu\nu} = U|\max(\bar{w}) + U$ where $\pi(\bar{w}) = w$, $\pi^- = \pi|H$, and $\text{cr}(\pi^-) = \text{cr}(\pi) = \zeta$. So $\zeta$ is an $H$-cardinal and a $U$-cardinal.

Claim 5. If $\zeta > \kappa$ then $H$ fails the ISC.

Proof. By Assumption 2 and Lemma 7.7.

If $zeta$ is an $M$-cardinal, define the phalanx

$$\mathcal{M} = ((M, < \zeta), U, \zeta).$$

And if $\text{card}^M_\zeta = \delta < \zeta$ (in which case $\zeta = \delta^U$), define the phalanx

$$\mathcal{M} = ((M, < \delta), (R, \delta), U, \zeta),$$

where $R < M$, $\zeta = \delta^R, r \in \{-1\} \cup \omega$ and $\rho_{r+1}^R = \delta < \rho_1^R$, where $\rho_{r+1}^R = \text{OR}^R$ (which is only relevant in case $R$ is type 3).

Claim 6. $\mathcal{M}$ is $((0, 0), \omega_1 + 1)$-iterable, or $((0, r, 0), \omega_1 + 1)$-iterable, as appropriate.\footnote{For the meaning in case $r = -1$, see Footnote 52.}

The iterability proof is mostly a routine copying process, and we won’t need the details of the process later. For these reasons we postpone it to later, beginning on page 82. (However, there is a new wrinkle which appears at the superstrong level, as we explain here.)

We now compare $\mathcal{M}$ with $M$. Recall that we use the slight tweak of the usual comparison process described in Remark 3.1, and note that in the present comparison, there can be $\alpha < \text{lh}(U)$ such that $M^\mathcal{M}_\alpha$ fails to be a premouse. This happens just if $R$ exists and is active type 3 with $\delta = \text{lgcd}(R)$, $\alpha$ is above $R$, and $(R, \alpha)^R \cap G^R = \emptyset$; here the iteration map $\iota^R_{M^\mathcal{M}, \alpha} : R \rightarrow M^\mathcal{M}_\alpha$ has critical point $\delta$ and $F^{M^\mathcal{M}_\alpha}$ fails the ISC with respect to $F^R \notin M^\mathcal{M}_\alpha$.

Claim 7. There is a successful comparison $(\mathcal{U}, T)$ of $(\mathcal{M}, M)$.

Proof. This is like in the classical fine structure proofs. (Because $U$ is a premouse, the usual arguments with the ISC work to show that an attempted comparison cannot last through length $\omega_1 + 1$. Note that the failures of the ISC in models $M^\mathcal{M}_\alpha$ mentioned above do not interfere with comparison termination, because $\text{cr}(F^M) = \text{cr}(F^R)$, so $F^{M^\mathcal{M}_\alpha}$ cannot be used as the “typical” extender along a branch of length $\omega_1$.)

We will now analyze the comparison.

Claim 8. Let $\alpha + 1 < \text{lh}(U)$. Then:

1. $E^M_\alpha$ is close to $M^{\mathcal{U}}_{\alpha+1}$.

2. If $0 < \iota^M \alpha + 1$ and $(0, (\alpha + 1)^M \cap G^M = \emptyset$ and $\text{cr}(E^M_\alpha) < \iota^M_0(\text{cr}(F^U))$ where $\beta = \text{pred}^M(\alpha + 1)$ then $(E^M_\alpha)_a \in M^\mathcal{M}_\beta$ for every $a \in [\nu(E^M_\beta)]^{<\omega}$.\footnote{For the meaning in case $r = -1$, see Footnote 52.}
with \( \text{cr}(\pi) = \zeta \), so by condensation for \( \omega \)-sound mice (Fact 1.9), either (a) \( U|\zeta^+U = W|\zeta^+U = M|\zeta^+U = R|\zeta^+U \), or (b) \( R = M|\zeta = W|\zeta \) is active with extender \( F = FR \), and \( U|\zeta^+U = \text{Ult}(R, FR)|\zeta^+U \). In either case, \( \mu_a \) is close to \( R \).

Part 2: This follows easily from an inspection of the proof of [9, 6.1.5].

Recall that we are working under hypothesis (a) (that \( M \) is 1-sound) and by Assumption 1, \( \rho^M_1 = \omega \). So with Claim 8, it follows that fine structure is preserved by the iteration maps in the usual way, \( \mathcal{U}^i \) is above \( U \), neither \( b^T \) nor \( b^M \) drops, \( M^\mathcal{T}_\infty = Q = M^\mathcal{T}_\infty \), and letting \( j = i^T, k = i^M \) and \( i_G = i^M_G \), we have \( k \circ i_G = j \), and \( \text{cr}(k) \geq \zeta \).

Claim 9. We have:

1. \( j \) is short; that is, for each \( \beta \in (0, \infty)^T \), we have \( \text{cr}(i^T_\beta) \leq i^T_\beta(\kappa) \).
2. \( k \circ i_G \) is short; that is, for each \( \beta \in [0, \infty)^M \), we have \( \text{cr}(i^M_\beta) \leq i^M_\beta(i_G(\kappa)) \).

Proof. Part 1: Suppose otherwise and let \( \beta \) be the least counterexample. Let \( \kappa' = j(\kappa) \); so \( \kappa' = i^T_\beta(\kappa) \) and since \( Q = M^\mathcal{T}_\infty \),

\[
M^T_\beta = \text{chull}_1^Q(\kappa' \cup \text{rg}(j)) = \text{chull}_1^Q(\kappa' \cup \{p^Q_1\}).
\]

Let \( \gamma \in [0, \infty)^M \) be least such that either \( \gamma + 1 = \text{lh}(U) \) or \( i^M_\gamma(i_G(\kappa)) < i^M_\beta(i_G(\kappa)) \). Then \( \kappa' = k(i_G(\kappa)) = i^M_\gamma(i_G(\kappa)) \), and since \( Q = M^\mathcal{T}_\infty \) and \( k^\omega(\zeta \cup \{w^U\}) \subseteq \kappa' \),

\[
M^\mathcal{T}_\gamma = \text{chull}_1^Q(\kappa' \cup \text{rg}(k)) = \text{chull}_1^Q(\kappa' \cup \{p^Q_1\}),
\]

so \( M^\mathcal{T}_\gamma = M^T_\beta \). But this implies that \( M^\mathcal{T}_\gamma = Q = M^T_\beta \), so the comparison terminates there, a contradiction.

Part 2 is similar.

Claim 10. \( \text{cr}(i_G) = \text{cr}(j) = \kappa < \kappa^+M = \kappa^+U = \kappa + Q < \text{cr}(k) \).

Proof. If \( \zeta > \kappa \), this is already clear. Otherwise use the proof of Claim 1 in the proof of Lemma 9.2.

We have \( \pi : U \rightarrow W \). Let \( \pi(w^U) = w \), so \( U = \text{Hull}_1^U(\zeta \cup \{p^U_1, w^U\}) \).

Let \( \xi + 1 \in b^T \) with \( \text{pred}^T(\xi + 1) = 0 \). For \( \mathcal{T} \)-special see Definition 10.4.

54In fact, arguments similar to those given for part 1 give strict inequality here; that is, for each \( \beta \in (0, \infty)^T \), we have \( \text{cr}(i^T_\beta) < i^T_\beta(i_G(\kappa)) \). Therefore \( \hat{\beta}^i \) is continuous at \( i_G(\kappa) \).

If \( M \) is below superstrong, this has the consequence that the same holds for \( \beta \) in \( (0, \infty)^T \); that is, for each such \( \beta \), we have \( \text{cr}(i^T_\beta) < i^T_\beta(\kappa) \). For suppose \( \beta \in (0, \infty)^T \) is such that \( \text{cr}(i^T_\beta) = i^T_\beta(\kappa) \).

Then the short extender derived from \( i^T \) has a whole segment of length \( i^T_\beta(\kappa) \), so \( \{i^T(f)(\kappa) \mid f \in M\} \) is bounded by \( i^T_\beta(\kappa) \). But if \( M \) is below superstrong then \( \{i_G(f)(\kappa) \mid f \in M\} \) is unbounded in \( i_G(\kappa) \) (see the proof of [16, Lemma 4.4]). And \( \text{cr}(\hat{\beta}^i) > \kappa \); this is clear as \( M \) is below superstrong, which implies \( \text{cr}(\hat{\beta}^i) \geq \zeta > \kappa \), but actually holds more generally by Claim 10 below. But then since \( \hat{\beta}^i \) is continuous at \( i_G(\kappa) \), it follows that \( \{\hat{\beta}^i(i_G(f))(\kappa) \mid f \in M\} \) is unbounded in \( \hat{\beta}^i(i_G(\kappa)) \). But \( \hat{\beta}^i \circ i_G = i^T \), so this just says that \( \{i^T(f)(\kappa) \mid f \in M\} \) is unbounded in \( i^T(\kappa) \), a contradiction.

It follows that if \( M \) is below superstrong and \( \varepsilon + 1 \in b^T \) and \( \varepsilon \) is \( \mathcal{T} \)-special, then \( \text{pred}^T(\varepsilon + 1) = 0 \).

This gives a much simpler proof of Claim 21 under these hypotheses, and this Steel’s argument basically uses this kind of simpler argument at that point.
CASE 1. $\xi$ is $T$-special.

Let $M' = M_\xi^T$, $s' = s_{FM'}$ and $\sigma' = \sigma_{FM'}$. We have

$$k(s_{iG}) = s_{k\circ iG} = s_j = s' = i_{0G}^T(s_{FM}),$$

$$\sup k^\sigma s_{iG} = \sigma_{k\circ iG} = \sigma_j = \sigma' = \sup i_{0G}^T \sigma_{FM},$$

using Lemma 7.15 for the first equality in each line, the fact that $k \circ i_G = j$ for the second equalities, that $FM'$ is derived from $j$ for the third equalities, and Lemma 7.16 for the final equalities. By induction, $w \leq s_{FM}$, so $w' = i_{0G}^T(w) \leq s'$, so $w' \leq k(s_{iG})$ and $lh(s_{iG}) \geq lh(w)$.

**Claim 11.** $s_{iG} \mid lh(w) < w^U$.

**Proof.** If $s_{iG} \mid lh(w) > w^U$ then the extender derived from $i_G$ with support $w^U \cup \zeta$ belongs to $U$, which is impossible as this is equivalent to $G$ itself.

So suppose $w^U \leq s_{iG}$. Then $\max(s_{iG}) = \max(w^U)$. So if $\zeta > \kappa$ then $H$ satisfies weak ISC and hence ISC (by Lemma 7.7), contradicting Claim 5. So $\zeta = \kappa$, so $U = \text{Hull}_1^\xi(\{U \cup p_1^1\})$ and $i^\theta(w^U) = w'$. But then $\text{cr}(k) = \kappa$, since the fact that $\zeta = \kappa$ implies that $k$ is not $FM'$-generated by $w$, hence not $FM'$-generated by $w'$, and hence $\kappa \notin \text{Hull}_1^\xi(\{p_1^1, w'\})$. This contradicts Claim 10. \hfill \Box

**Subcase 1.1.** $\kappa = \zeta$.

In this case we just need to see that the measure $(FM)_w \in M$. But since $j = k \circ i_G$ and $\text{cr}(k) > \kappa$ and $k(s_{iG} \mid lh(w)) = w'$, we have

$$(FM)_w = (FM')_w' = G_{s_{iG} \mid lh(w)} = (FM)_{\pi(s_{iG} \mid lh(w))},$$

and since $s_{iG} \mid lh(w) < w^U$, we have $\pi(s_{iG} \mid lh(w)) < \pi(w^U) = w$. By induction, the Dodd-solidity witnesses at each $\alpha \in w$ belong to $M$, and note now that $(FM)_w$ is a component measure of one of these, and hence in $M$.

**Subcase 1.2.** $\kappa < \zeta$.

In this case we will show $FM \triangleright FM' \in M$, contradicting Claim 2. By Claim 5, $H$ fails the ISC, so $\max(s_{iG}) < \max(w^U)$. Note that $k(\max(w^U))$ is a generator of some extender $E$ used along $b^\theta$, and since $s' = k(s_{iG})$ and $\nu(FM') = \max(s') + 1$, we have $k(\max(s_{iG})) < \text{cr}(E)$. It follows that $U$ has an inaccessible cardinal $\chi$ with $\max(s_{iG}) < \chi < \max(w^U)$; take $\chi$ least such. Let $\theta \in b^\theta$ be least such that either $\theta + 1 = lh(U)$ or $i_{0G}^\theta(\chi) \leq \text{cr}(i_{0G}^\theta)$, so in fact by the minimality of $\chi$, $i_{0G}^\theta(\chi) < \text{cr}(i_{0G}^\theta)$ and $i_{0G}^\theta(\chi) = k(\chi)$ and $i_{0G}^\theta(s_{iG}) = k(s_{iG}) = s'$. Let $t' = i_{0G}^\theta(FM')$. We have $\max(t') = \max(s_{FM'})$, so $\max(t') = \max(s') < k(\chi)$. By a finite support argument like those in the proof of Lemma 6.20, there is $(\mathcal{V}, \zeta)$ such that $\zeta$ is a $(0, 0)$- or $(0, r, 0)$-maximal tree of finite length on $\mathcal{V}$, $b^\zeta$ is above $U$ and does not drop, $\zeta : M_\zeta^\mathcal{V} \to M_\theta^\mathcal{V}$ is an $0$-embedding, $\zeta \circ i_0^\mathcal{V} = i_0^\theta$ and $t' \in \text{rg}(\zeta)$. By Claim 8 part 2, for every $\alpha + 1 \in b^\theta$, every component measure of $E_\alpha$ is in $M_\beta^\mathcal{V}$, where $\beta = \text{pred}^\mathcal{V}(\alpha + 1)$.

Let $(\alpha_i)_{i < \ell}$ enumerate those $\alpha_i$ such that $\alpha_i + 1 \in b^\mathcal{V}$. Then with some more finite support calculations, we can find $(\bar{V}, \bar{\zeta})$ such that $\bar{V} = (\bar{U}_\alpha)_{\alpha < \ell} \cdot \bar{E} = (E_\alpha)_{\alpha < \ell}$ is an abstract $0$-maximal iteration of $U_0 = U$ (so $U_{\alpha + 1} = U_{\alpha + 1}(U_\alpha, E_\alpha)$ for $\alpha < \ell$) such that

$$E_\alpha \in U_\alpha |_{E_\alpha}^{U_0} \bar{E}(\chi)$$

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for each \( \alpha < \ell \), each \( E_\alpha \) is a finitely generated extender, and \( \xi : U_\ell \to M^{\aleph_\alpha}_\ell \) is a 0-embedding such that \( \xi \circ L_{\alpha,0}^{U_\ell} \) is the present case hypothesis that \( \nu \). Note that \( F^M | \bar{\ell}^M \) is equivalent to the measure derived from \( 1^{U_{\ell,0}}_E \circ i_G \) with seed \( \xi^{-1}(\xi^{-1}(t')) \), so it suffices to see that \( \mu \in M \).

Note that \( \bar{E} \in U|\chi \). So since \( \chi < \max(w^U) \), we have \( \bar{E} \in U' = \text{Ult}_0(M, G') \) where \( G' = E_{i_G} \mid \max(w^U) \) is the extender derived from \( i_G \) of length \( \max(w^U) \). And \( \mu \) is just the measure derived from \( 1^{U_{\ell,0}}_E \circ i_G \) with the same seed \( \xi^{-1}(\xi^{-1}(t')) \) (note this seed is below \( \chi \)).

Let \( F' = F^M \mid \max(w) \) and \( U'' = \text{Ult}_0(M, F') \). Let \( \varpi : U' \to U'' \) be the factor map induced by \( \pi \mid \max(w^U) \). Then \( \varpi \circ i_{\varpi(E)} = i_{\varpi(E)} \), so \( \mu \) is the measure derived from \( i_{\varpi(E)} \circ i_{\varpi(E)}^{M,0} \) with seed \( \varpi(\xi^{-1}(\xi^{-1}(t'))) \). But because \( F' \in M \) and \( \varpi(\bar{E}) \in \text{Ult}_0(M | \kappa^{+M}, F') \), it follows that \( \mu \in M \), as desired.

**Case 2.** \( \xi \) is non-\( T \)-special.

In this case we will use an argument like that in the proof of Lemma 9.2. For this, we must see that we can capture the relevant fragment of the extender derived from \( i_T \) with a tree \( T \) which is, modulo \( \zeta \), strongly finite. We make this precise as follows. Let us say that a 0-maximal tree \( S \) on \( M \) is **nicely-\( \zeta \)-strongly finite** if

1. \( \text{lh}(S) < \omega \),
2. \( S \) is non-trivial, \( b^S \) does not drop and \( \text{cr}(i^S) = \kappa = \text{cr}(F^M) \) (so \( \kappa^{+M} < \text{lh}(E^S_0) \)),
3. \( \zeta < \text{lh}(E^S_0) \),
4. either \( \zeta = \delta \) or \( \zeta = \delta^{+\text{exit}^S} \), where \( \delta = \text{card}^M(\zeta) \),
5. \( \zeta' \) is non-\( S \)-special, where \( \zeta' \) is least such that \( \zeta' + 1 \in b^S \), and
6. for each \( \alpha + 1 \in b^S \) and \( \beta = \text{pred}^S(\alpha + 1) \), we have:
   
   (a) if \( \beta = 0 \) (so \( \text{cr}(E^S_\alpha) = \kappa \) and \( \alpha \) is non-\( S \)-special) then:
      
      i. if \( \kappa = \zeta \) then \( \mathcal{C}_{D_s}(E^S_\alpha) \) is finitely generated (so \( \sigma(E^S_\alpha) = \kappa^{+M} \)),
      ii. if \( \kappa < \zeta \) then \( \sigma(E^S_\alpha) \leq \zeta \) (so note \( \sigma(E^S_\alpha) \in \{ \delta, \zeta \} \)),
      iii. \( \mathcal{C}_{D_s}(E^S_\alpha) \) is finitely generated for each \( \gamma < \alpha \) (and note \( \delta \leq \text{cr}(E^S_\alpha) \)),
   
   (b) if \( \beta > 0 \) and \( \alpha \) is non-\( S \)-special then \( \mathcal{C}_{D_s}(E^S_\gamma) \) is finitely generated for each \( \gamma < \alpha \),
   
   (c) if \( \beta > 0 \) and \( \alpha \) is \( S \)-special (so Lemma 10.5 applies) then \( \mathcal{C}_{D_s}(E^S_\alpha) \) is finitely generated for each \( \theta < \alpha \), for each \( \gamma + 1 \in (\beta, \alpha] \).

**Claim 12.** Let \( x \in \mathcal{E}_0(M)^{\aleph_0}_{\ell} \). Then there is a nicely-\( \zeta \)-strongly finite 0-maximal tree \( S \) on \( M \) which captures \( (T, x, \zeta) \) (see Definition 8.1).

**Proof.** This is a straightforward variant of the proof of Lemma 8.6. Say a 0-maximal tree \( S \) on \( M \) is a **candidate** if it satisfies conditions 1–5 of nice-\( \zeta \)-strong-finiteness and \( S \) captures \( (T, x, \zeta) \). By the properties of \( T \), we get candidates from straightforward finite support (condition 5 uses the present case hypothesis that \( \xi \) is non-\( T \)-special). Define the **index** of a candidate like before, but excluding the lengths of \( S \)-special extenders.
That is, let $A$ be the set of all ordinals $\beta$ such that there is $\alpha + 1 \in b^S$ such that either $\beta < S_\alpha$ or $\beta = \alpha$ is non-$S$-special. Now proceed as before: let $\langle \kappa_i \rangle_{i < \ell}$ enumerate $\{ \text{cr}(E^S_{\beta_i}) \mid \beta \in A \}$ in decreasing order, let $\beta_i \in A$ be such that $\kappa_i = \text{cr}(E^S_{\beta_i})$, and let $\gamma_i = \text{lh}(E^S_{\beta_i})$. Then the index of $S$ is $\langle \gamma_i \rangle_{i < \ell}$. Let $S$ be the candidate of lexicographically least index.

It suffices to see that $S$ is nicely-$\zeta$-strongly finite, so suppose not. So condition 6 fails. As in the proof of 8.6, we will use this to construct a candidate $\tilde{S}$ with smaller index than $S$, a contradiction.

Adopt the notation introduced in the definition of index (for this $S$). As condition 6 fails, we can fix the least $a < \ell$ such that $\mathcal{C}_{\mathcal{D}_a}(E^S_{\beta_a})$ is not of the desired form. A perusal of the clauses in condition 6 shows that this just means that $\sigma(E^S_{\beta_a}) > \zeta$ and $\mathcal{C}_{\mathcal{D}_a}(E^S_{\beta_a})$ is not finitely generated. Let $\kappa = \kappa_a$, $\beta = \beta_a$, $Q = \mathcal{C}_{\mathcal{D}_a}(\text{exit}^S_{\beta_a})$ and $F = F^Q$. So $\text{max}(\kappa^{+Q}, \zeta) < \sigma = \sigma_F = \tau_\mathcal{F} = \sigma(E^S_{\beta_a})$. And like before, by 7.19 there is a unique $\theta \leq \beta$ such that $Q \leq M^S_{\theta}$; moreover, $\theta \leq ^S \beta$ and $\theta$ is the least ordinal such that $\sigma < \text{lh}(E^S_{\theta})$.

Let $\theta$ be least such that $\text{lh}(E^S_{\theta}) > \text{max}(\kappa^{+Q}, \zeta)$. (Recall that $\text{lh}(E^S_{\theta}) \geq \zeta$, and $\text{lh}(E^S_{\theta}) = \zeta$ iff $M[\zeta]$ is active. It easily follows that the same holds for $S$. So if $\kappa^{+Q} < \zeta$ then $\theta \in \{0, 1\}$. Let $\theta + m = \theta$ (so $m < \omega$) and $\bar{\theta} + m + 1 = \bar{\theta} + 1 = \text{lh}(S)$. Much as in the proof of 8.6, we will find $g \in \nu^\omega_\bar{\theta}$ and $R \triangleleft M^S_{\bar{\theta}}$ with $F^R \approx F \vDash \zeta \cup g$, such that we can define a candidate $\tilde{S}$ such that $\tilde{S} \restriction \bar{\theta} + 1 = S \restriction \bar{\theta} + 1$, $\text{lh}(\tilde{S}) = \bar{\theta} + 1$, $(0, \bar{\theta})^S \cap \mathcal{G}^S = \emptyset$, and $\tilde{S} \restriction [\bar{\theta}, \bar{\theta}]$ is built as a reverse copy of $S \restriction [\theta, \bar{\theta}]$. This will result in a final copy map $\pi_{\bar{\theta}} : M^S_{\bar{\theta}} \rightarrow M^S_{\tilde{\theta}}$, which will be a $0$-embedding with $\pi_{\bar{\theta}} \circ i^S_{\theta} = i^S_{\bar{\theta}}$ and $\bar{\zeta} \cup \{ \tau^{-1}(x) \} \subseteq \text{rg}(\pi_{\bar{\theta}})$, where $\tau : M^S_{\bar{\theta}} \rightarrow M^S_{\tilde{\theta}}$ witnesses that $\tilde{S}$ captures $(U, x, \bar{\zeta})$. It will follow that $\tau \circ \pi_{\bar{\theta}} : M^S_{\bar{\theta}} \rightarrow M^S_{\tilde{\theta}}$ witnesses that $\tilde{S}$ captures $(U, x, \bar{\zeta})$.

If $\kappa^{+Q} \geq \zeta$, this will be done essentially like before. Suppose instead that $\kappa^{+Q} < \zeta$. Then $\text{pred}^S(\beta + 1) = 0$. (For $\zeta \leq \text{lh}(E^S_{\theta})$ and either $\delta = \zeta$ is an $M$-cardinal, or $\zeta = \delta^{+\text{exit}^S_{\beta_a}}$. Note then that $\nu(E^S_{\theta}) \geq \delta$ for each $\gamma$, and so if $\text{pred}^S(\beta + 1) \neq 0$ then $\delta \leq \kappa$, so $\kappa^{+Q} \geq \zeta$). So $\beta$ is non-$S$-special and $\kappa = \text{cr}(F^M)$. Suppose $Q$ is type 2. Then $\text{max}(\kappa^{+Q}, \zeta) < \tau_\mathcal{F} = \rho^Q_1$, $\zeta$ is a $Q$-cardinal, and $Q$ is 1-sound. So for cofinally many $g \in [\nu(F)]^{<\omega}$, there is a type 2 segment $R \triangleleft Q$ with $F^R$ equivalent to $F \vDash \zeta \cup \{ g \}$ and $\rho^R_1 = \tau(F^R) = \zeta$, so we choose an appropriate $R$ of this form, much as before. If instead $Q$ is type 3, then we again choose some sufficiently large type 2 segment $Q' \triangleleft Q$ with $F^{Q'} \subseteq F$, and then if still $\rho^Q_1 > \zeta$, proceed as we did in the type 2 case with $Q'$ replacing $Q$.

To see that we can arrange the reverse copying, we establish the analogue of Subclaim 1.1 from the proof of 8.6, classifying the various extenders used in $S \restriction [\theta, \bar{\theta}]$: 

**SUBCLAIM 12.1.** Let $\chi \in [\theta, \bar{\theta}]$. Then $\text{cr}(E^S_{\chi}) \notin (\kappa, \sigma)$, so $\text{pred}^S(\chi + 1) \notin (\bar{\theta}, \theta)$. In fact, exactly one of the following holds:

(i) there are $\varepsilon, \chi'$ such that $\varepsilon$ is non-$S$-special, $\varepsilon + 1 \leq S \varepsilon$ and $\chi' \leq S_{\beta_a} \varepsilon$, and either:

(a) $\chi = \chi'$, or

(b) $\chi < S \chi'$ and $\chi$ is $\chi'$-transient and $E^S_{\chi'} = F(M^S_{\chi'})$,

or

(ii) there are $\varepsilon, \varepsilon, \chi'$ such that $\varepsilon$ is $S$-special, $\varepsilon + 1 \leq S \varepsilon$ and either:

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- $\chi' = \varepsilon' = \varepsilon$, or
- letting $\iota = \text{pred}^S(\varepsilon + 1)$, then $\iota <^S \varepsilon' + 1 \leq^S \varepsilon$ and $\chi' \leq^\text{Da} \varepsilon'$,

and either:

(a) $\chi = \chi'$, or
(b) $\chi <^S \chi'$ and $\chi$ is $\chi'$-transient and $E^S_{\chi'} = F(M^S_{\chi'})$,

or

(iii) $\chi <^S \vartheta$ and $\chi$ is $\vartheta$-transient.

Moreover, the 5 options (i)a, (i)b, (ii)a, (ii)b, (iii) are mutually exclusive.

**Proof.** This is just a slight variant of the proof of Subclaim 1.1 of Claim 1 in the proof of 8.6. We leave the details to the reader. \hfill $\Box$

Like in the proof of 8.6, Subclaim 12.1 ensures that we can carry out the reverse copying. This contradicts the minimality of the index of $S$, as desired. \hfill $\Box$

By the proof of Subclaim 12.1, we have:

**CLAIM 13.** Let $S$ be nicely-\(\varsigma\)-strongly finite and $\chi + 1 < \vartheta + 1 = \text{lh}(S)$. Then exactly one of the 5 options (i)a, (i)b, (ii)a, (ii)b, (iii) of Subclaim 12.1 of the proof of Claim 12 holds with respect to $S$.

**GOAL 1.** We will prove $(*_{1})$, which is the conjunction of the following statements:
- $\mathcal{U}$ is trivial, $U = M^T_{\infty}$ and $i_G = i^T$,
- $\mathcal{T}$ is nicely-\(\varsigma\)-strongly finite, and
- $\alpha$ is non-$\mathcal{T}$-special for every $\alpha + 1 \in b^T$.

**CLAIM 14.** If ($*$) holds then $G \in M$.

**Proof.** By Claim 9, $j = i^T$ is short. So if $\kappa = \delta = \varsigma$ then $(\kappa^{+\omega})^M < \text{OR}^M$ and $\mathcal{T}$ is based on $M||\gcd(M)$, so easily $G \in M$. Now suppose $\kappa^{+M} \leq \delta < \varsigma$. Because $w \neq \emptyset$, we have $\varsigma < \text{lh}(\mathcal{T})$.

We claim that for each $\chi + 1 < \text{lh}(\mathcal{T})$, option (i) of Claim 13 holds. For option (iii) is ruled out by the shortness of $i^T$, and option (ii) is ruled out since by ($*$), there is no $\mathcal{T}$-special $\varepsilon$ with $\varepsilon + 1 \in b^T$. It follows that we can consider $\mathcal{T}$ as a tree $\mathcal{T}'$ on $M||\gcd(M)$. In fact, letting $\alpha + 1 = \text{succ}^T(0, \infty)$, then $\mathcal{T} = \mathcal{T}_0 \mathcal{T}_1$ where $\mathcal{T}_0 = \mathcal{T} \upharpoonright (\alpha + 1)$ and $\mathcal{T}_1$ is based on $M^\mu_{\varsigma} \mu^{+\omega}$, where $\mu = i^{T_0}_{\alpha+1}(\kappa)$. Since $M^\mu \models \text{ZFC}^-$, $\mathcal{T}' \in M$, so $G \in M$. \hfill $\Box$

So it suffices to prove ($*$). Fix a nicely-\(\varsigma\)-strongly finite tree $\mathcal{T}$ capturing $(\mathcal{T}, i^U(w^U), \varsigma)$ (exists by Claim 12). We will proceed like in the proof of Lemma 9.2 to show that $\mathcal{T} = \mathcal{T}$ and $\mathcal{U}$ is trivial, with further embellishments to rule out $\mathcal{T}$-special extenders along the main branches of $\mathcal{T}$, $\mathcal{U}$.

Let $\tilde{Q} = M^T_{\infty}$ and $\varsigma : \tilde{Q} \to Q$ be the capturing map. Let $\tilde{j} = i^T$. So $\varsigma \circ \tilde{j} = j$. Let $k : U \to \tilde{Q}$ be $k = \varsigma^{-1} \circ k$. Then the diagram in Figure 5 commutes. Because $\mathcal{T}$ is short, and by Claim 10, we have:

\[\text{The requirement that } E^S_{\chi'} = F(M^S_{\chi'}) \text{ is redundant in the case that } \chi' = \varepsilon, \text{since } \varepsilon \text{ is } S\text{-special.}\]
Lemma 9.2 and 9.2 (and since 7.15 recall $\bar{\kappa}$ of Lemma are as in the proof of Claim 15.

So $\bar{\kappa}$ is non-$\bar{\kappa}$-special and non-$\bar{\kappa}$-special respectively. Let $\bar{\xi} = \bar{\xi}$ and $\bar{T} \upharpoonright \langle \xi + 2 \rangle = \bar{T} \upharpoonright \langle \xi + 2 \rangle$ (and $\xi \in b^{\bar{\kappa}} \cap b^{\bar{\kappa}}$). Let $\bar{\epsilon} \in b^{\bar{\kappa}} \cap b^{\bar{\kappa}}$ be largest such that:

$- \quad \bar{T} \upharpoonright (\beta + 1) = \bar{T} \upharpoonright (\beta + 1)$, and

$- \quad \bar{\epsilon}$ is non-$\bar{\epsilon}$-special for every $\bar{\epsilon} + 1 \leq \bar{\beta}$. 

Figure 5: The diagram commutes.

Claim 15. $\bar{j} = i^{\bar{T}}$ is short, the models $M, U, Q, Q$ agree through their common value for $\kappa^+$, and

$$\text{cr}(i_G) = \text{cr}(\bar{j}) = \text{cr}(\bar{\epsilon}) = \kappa < \kappa^+ \leq \text{cr}(\bar{k}) = \min(\text{cr}(\bar{k}), \text{cr}(\bar{\xi})).$$

Let $i = i_G$ and $s = s_i$ and $\sigma = \sigma_i$. Let $s' = s_j$ and $\sigma' = \sigma_j$.

Claim 16. We have:

1. $s' = s_j = s_{k_{oi}} = k(s_i)$ and $\sigma' = \sigma_j = \sigma_{k_{oi}} = \sup k^u \sigma_i$.
2. $s_{k_{oi}} = s_j = k(s_i)$ and $\sigma_{k_{oi}} = \sigma_j = \sup k^u \sigma_i$.
3. $s_{coj} = s_j = c(s_j)$ and $\sigma_{coj} = \sigma_j = \sup c^u \sigma_i$.
4. $\max(\delta, \kappa^+ M) \leq \sigma_i = \sigma_j = \sigma_j \in \{\kappa^+ M, \delta, \bar{\xi}\}$.

Proof. Part 1 is by Lemma 7.15. Parts 2 and 3 are as in the proof of Claim 2 of Lemma 9.2. For part 4, the fact that $\max(\delta, \kappa^+ M) \leq \sigma_j \in \{\kappa^+ M, \delta, \bar{\xi}\}$ follows from the nice-$\zeta$-smallness of $\bar{T}$. Since $\max(\kappa, \kappa^+ M) \leq \text{cr}(\bar{k}) \leq \text{cr}(\bar{k})$, the preceding parts then give that $\sigma_i = \sigma_j = \sigma_j$. \hfill \Box

Let $\bar{\xi}$ be least such that $\bar{\xi} + 1 \in b^{\bar{T}}$ (recall $\bar{\xi}$ is likewise for $\bar{T}$).

Claim 17. $E_{\text{Dh}}(E_{\bar{\xi}}) = E_{\text{Dh}}(E_{\bar{\xi}}^T)$.

Proof. As in the proof of Claim 3 of Lemma 9.2 (recall $\bar{\xi}$ is non-$\bar{T}$-special and $\bar{\xi}$ is non-$\bar{T}$-special). \hfill \Box

Now going on with the analysis of the Dodd-decompositions of $E_{\bar{\xi}}^T$ and $E_{\bar{\xi}}^T$ as in the proof of Lemma 9.2 (and since $\xi, \bar{\xi}$ are non-$\bar{T}$-special and not-$\bar{T}$-special respectively), we get that $\xi = \bar{\xi}$ and $\bar{T} \upharpoonright \langle \xi + 2 \rangle = \bar{T} \upharpoonright \langle \xi + 2 \rangle$ (and $\xi \in b^{\bar{T}} \cap b^{\bar{T}}$). Let $\beta \in b^{\bar{T}} \cap b^{\bar{T}}$ be largest such that:

$- \quad \bar{T} \upharpoonright (\beta + 1) = \bar{T} \upharpoonright (\beta + 1)$, and

$- \quad \bar{\epsilon}$ is non-$\bar{\epsilon}$-special for every $\bar{\epsilon} + 1 \leq \bar{\beta}$. 

So $\xi + 1 \leq \beta$. Let $\bar{j}_{\beta \infty} : M^T_{\beta} \rightarrow \bar{Q}$ and $j_{\beta \infty} : M^T_{\beta} \rightarrow Q$ be the iteration maps. Then $c \circ j_{\beta \infty} = j_{\beta \infty}$ and as in the proof of Lemma 9.2, we can define $i_{\beta} : M^T_{\beta} \rightarrow U$ such that $\bar{k} \circ i_{\beta} = j_{\beta \infty}$. Let $M^* = M^T_{\beta} = M^T_{\beta}$. 

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Claim 18. If \( M^* = \bar{Q} \) then \((*)\) holds (see Goal 1), so \( G \in M \).

Proof. If \( M^* = \bar{Q} \) then \( U = \bar{Q}, i^\bar{T} = i_G \) and \( \bar{T} = T \), all as in the proof of Lemma 9.2, so \((*)\) holds.

So suppose \( M^* \neq \bar{Q} \) (and hence \( M^* \neq Q \)). Let \( \bar{\varepsilon} + 1 = \text{succ}(\beta, \infty) \) and \( \varepsilon + 1 = \text{succ}^T(\beta, \infty) \).

Claim 19. If \( M^* \neq \bar{Q} \) (hence \( M^* \neq Q \)) then either \( \varepsilon \) is \( T \)-special or \( \bar{\varepsilon} \) is \( \bar{T} \)-special.

Proof. Otherwise we can carry on with the analysis of Dodd decomposition, and reach a contradiction to the maximality of \( \beta \).

So we may assume, for a contradiction:

Assumption 3. \( \bar{Q} \neq M^* \neq Q \) and either \( \varepsilon \) is \( T \)-special or \( \bar{\varepsilon} \) is \( \bar{T} \)-special.

Let
\[
i^* : M^* \to U \quad \text{and} \quad j^* : M^* \to \bar{Q} \quad \text{and} \quad j^* : M^* \to Q
\]
be the maps resulting from the analysis above. So \( k \circ i^* = j^* \) and \( \zeta \circ j^* = j^* \) and \( k \circ i^* = j^* \). We have \( \text{cr}(j^*) = \text{cr}(E^\zeta_\delta) \geq \delta \), since \( \nu(E^\delta_\zeta) \geq \delta \). (Note that \( E^\delta_\zeta \) might be superstrong with \( \nu(E^\delta_\zeta) = \delta \), in which case \( \text{cr}(E^\delta_\zeta) = \delta \), and then if \( \delta < \zeta \), we have \( \text{cr}(j^*) < \zeta \).) Let \( \mu^* = \text{cr}(j^*) = \text{cr}(F^{M^*}) \). As usual we have:

Claim 20. \( \text{cr}(i^*) = \text{cr}(j^*) = \mu^* < \text{cr}(k) = \min(\text{cr}(\bar{k}), \text{cr}(\zeta)) \).

Most of the rest of the proof is devoted to the following claim:

Claim 21. \( \varepsilon \) is non-\( T \)-special.

Proof. Suppose otherwise. We will show that the measure \( \mu = (F^{M^*})_{i^*M^*} \in M \), contradicting Claim 2. This will be similar to the argument used in Subcase 1.2 of Case 1, by deriving \( \mu \) from measures already known to be in \( M \).

Let \( s^* = s_{i^*} \) and \( \sigma^* = \sigma_{i^*} \). Let \( M' = M^{i^*} \). Let \( s' = s_{F^{M'}} \) and \( \sigma' = \sigma_{F^{M'}} \). By Claim 20 and the usual calculation, \( s' = k(s^*) \) and \( \sigma' = \sup k^{< \sigma^*} \). Let \( w' = i^*_b(w) \), so \( w' \leq s' \).

Let \( \zeta' = i^*_b(\zeta) \).

Subclaim 21.1. \( s^* | \text{lh}(w) < w'U \).

Proof. Much as before, we must have \( s^* | \text{lh}(w) \leq w'U \). Suppose \( s^* | \text{lh}(w) = w'U \). Note that
\[
\zeta' \notin \text{Hull}^Q(\zeta' \cup \{ p_1^Q, w' \})
\]
(as \( Q = M^T \) and \( F^{M^*} \) is used along \( b^T \)). Since \( k(w', p_1^Q) = (w', p_1^Q) \) and \( \text{cr}(k) \geq \zeta \), it follows that \( U \neq Q \), so \( U \) is non-trivial, and in fact, there is \( i + 1 \in b^T \) such that \( \text{cr}(E^U_i) \leq \zeta' < \nu(E^U_i) \).

Now \( F^i_{M'} \in \text{rg}(k) \) (basically because \( k(s^*) = s' \)). Therefore
\[
F^i_{M'} \notin \text{Hull}^Q(\text{cr}(E^U_i) \cup \{ p_1^Q, w' \})
\]
(see §1.3.2 for \( F^i_1 \)). But by Assumption 2, \( F^i_{M'} \) is not generated by \( \zeta' \cup w' \), and therefore
\[
F^i_{M'} \notin \text{Hull}^Q(\zeta' \cup \{ p_1^Q, w' \}),
\]
whereas \( \text{cr}(E^U_i) \leq \zeta' \), a contradiction, proving the subclaim.

\( \square \)
So let $m < \text{lh}(w)$ be such that $s^* | m = w^U | m$ but $(s^*)_m < (w^U)_m$.

**Subclaim 21.2.** $U \models \text{"There is an inaccessible } \chi \text{ with } (s^*)_m < \chi < (w^U)_m".$

**Proof.** We have $w' \preceq s' = k(s^*)$, so

(i) if $m > 0$ then $k((s^*)_m) = (w')_m < k((w^U)_m) < k((s^*)_{m-1}) = (w')_{m-1}$ and

(ii) if $m = 0$ then $k((s^*)_0) = (w')_0 = \nu(F^{M'}) - 1 < k((w^U)_0)$.

But

$$(w^U)_m \notin \text{Hull}^U((w^U)_m \cup \{p_1^U, w^U \upharpoonright m\}),$$

and $k$ lifts this fact, and therefore in case (i), $k((w^U)_m)$ is a $(w' | m)$-generator of $F^{M'}$, and in case (ii), $T$ uses an extender $E$ along $b_T$, after $F^{M'}$, for which $k((w^U)_0)$ is a generator. In case (ii), we get

$$k((s^*)_m) < \chi' = \text{cc}(E) \leq k((w^U)_m)$$

and $\chi'$ is a limit of inaccessibles in $Q$, which easily suffices. In case (i), $T$ uses an extender $E$ on the branch leading to $M'$, for which $k((w^U)_m)$ is a generator, such that line (6) also holds, and again this easily suffices. \[\square\]

Let $\chi$ be least as in Subclaim 21.2. Let $t = \vec{t}^M$ and $t' = i_{00}^\chi(t)$. Recall $k(s^* | \text{lh}(w)) = w' \preceq t'$. Let $\theta \in b^t$ be least such that either $\theta + 1 = \text{lh}(U) \text{ or } i_{00}^\chi(\chi) \leq \text{cc}(i_{00}^\chi)$, so in fact by the minimality of $\chi$, $i_{00}^\chi(\chi) < \text{cc}(i_{00}^\chi)$ and $i_{00}^\chi(\chi) = k(\chi)$ and $i_{00}^\chi((s^*)_m) = k((s^*)_m) = (t')_m < k(\chi)$. Note that $t' \in \text{rg}(i_{00}^\chi)$.

As in Subcase 1.2 of Case 1, there is $(\mathcal{V}, \varsigma)$ such that $\mathcal{V}$ is a $(0, 0)$- or $(0, r, 0)$-maximal tree of finite length on $\mathcal{U}$, $b^\mathcal{U}$ is above $U$ and does not drop, $\varsigma : M^\mathcal{U}_\infty \rightarrow M^\mathcal{U}$ is a 0-embedding, $\varsigma \circ i_{00}^\mathcal{V} = i_{00}^\mathcal{V} \circ \varsigma$ and $\hat{t} \in \text{rg}(\varsigma)$. Also as there, we can find $\ell < \omega$ and $(\mathcal{V}, \vec{t}, E, \varsigma)$ such that $U = \langle U_\alpha \rangle_{\alpha \in \ell}$, $\vec{E} = \langle E_m \rangle_{m \in \ell}$, and $\hat{\mathcal{V}} = (\vec{U}, \vec{E})$ is an abstract 0-maximal iteration of $U_0 = U$, $E_m \in U_\alpha[\mathcal{V}]$ and $E_m$ is a finitely generated extender, and $\varsigma : U_\ell \rightarrow M^\mathcal{U}_\ell$ is a 0-embedding with $\varsigma \circ i^U_{\mathcal{V}} = i^\mathcal{V}_0$ and $\varsigma^{-1}(\hat{t}) \in \text{rg}(\varsigma)$. Note that $\mu$ is the measure derived from $i^U_{\mathcal{V}} \circ i_{\mathcal{V}}$ with seed $\vec{\varsigma}^{-1}(\varsigma^{-1}(\hat{t}))$.

We have $\vec{E} \in U[\mathcal{V}]$. So since $\chi < (w^U)_m$, we have $\vec{E} \in U' = \text{Ult}_0(M, G')$ where

$$G' = E_{\vec{E}} \upharpoonright \{(w^U \upharpoonright m) \cup (w^U)_m\}.$$

So letting $\vec{\varsigma} : \text{Ult}_0(U', \vec{E}) \rightarrow \text{Ult}_0(U, \vec{E}) = U_\ell$ be the map given by the Shift Lemma applied to the factor map $U' \rightarrow U$, $\mu$ is the measure derived from $i^U_{\vec{E}} \circ i^{M_0}_{G'}$ with seed $\vec{\varsigma}^{-1}(\varsigma^{-1}(\hat{t}))$.

Let $F' = F^M \upharpoonright \{(w \upharpoonright m) \cup w_m\}$ and $U'' = \text{Ult}_0(M, F')$. Let $\varpi : U' \rightarrow U''$ be the factor map induced by $\pi \upharpoonright \{(w^U \upharpoonright m) \cup (w^U)_m\}$. Then $\varpi \circ i^{M_0}_{G'} = i^{F'}_{G'}$, so $\mu$ is the measure derived from $i^U_{\varpi(\mathcal{V})} \circ i^{F'}_{G'}$ with seed $\varpi(\vec{\varsigma}^{-1}(\varsigma^{-1}(\hat{t})))$. Since $F' \in M$ and $\varpi(\vec{E}) \in \text{Ult}_0(M[\kappa^+M], F')$, it follows that $\mu \in M$, as desired.

This completes the proof that $\varepsilon$ is non-$T$-special (Claim 21). \[\square\]

The following claim will now give the desired contradiction:
Claim 22. $\bar{e}$ is non-$T$-special.\(^{56}\)

Proof. Because $\varepsilon$ is non-$T$-special and as in the proof of Closeness \([9, 6.1.5]\), for every $a \in [\nu(E^T_\alpha)]_\omega$, we have $(E^T_\alpha)_a \in M^*$. Suppose $E^T_\alpha$ is $T$-special and let $i = i(E^T_\alpha)$. Then the measure $D = (E^T_\alpha)_i \notin M^*$, and $\text{cr}(i) > \mu^*$ (recall $M^* = M^T_\beta = M^T_\gamma$), so $D \notin M^*$. But note that by Claim 20 and $\alpha$-preservation, we have $\zeta(\max(i)) + 1 = \nu(E^T_\alpha)$, so $D = (E^T_\alpha)_i \in M^*$, a contradiction. \(\square\)

So we have shown that $\varepsilon$ is non-$T$-special and $\bar{e}$ is non-$\bar{T}$-special, contradicting Assumption 3, completing the analysis of Case 2.

This completes everything other than the postponed verification of the iterability of $\bar{\Psi}$, which we finally tend to:

Proof of Claim 6. We will show that $\bar{\Psi}$ is $((0,0), \omega_1 + 1)$-iterable or $((0,r), \omega_1 + 1)$-iterable respectively.\(^{57}\) Toward this iterability, let $\gamma$ be any $M$-cardinal with $\kappa < \gamma$ and let

$$\bar{\Psi}_\gamma = ((M, < \gamma), W, \gamma).$$

We will show that $\bar{\Psi}_\gamma$ is $((0,0), \omega_1 + 1)$-iterable, and that we can reduce trees on $\bar{\Psi}$ to trees on $\bar{\Psi}_{\gamma(\zeta)}$.

Subclaim 22.1. $\bar{\Psi}_\gamma$ is $((0,0), \omega_1 + 1)$-iterable.

Proof. The proof is just a simple case of normalization calculations. Write $\bar{\Psi} = \bar{\Psi}_\gamma$. First, instead of considering (unpadded) $(0,0)$-maximal trees $W$ on $\bar{\Psi}$ themselves, we will consider padded $(0,0)$-maximal trees $W$ of a special form. We write $-1, 0$ for the two roots of $W$ (so $M^W_\mathcal{R} = M$ and $M^W_\emptyset = W$). Say that a padded $(0,0)$-maximal tree $W$ on $\bar{\Psi}$ is nicely padded iff:

1. For each $\alpha + 1 < \text{lh}(W)$, if $0 \leq W \alpha$ and $(0, \alpha)^W \cap \mathcal{D}^W = \emptyset$ and $E^W_\alpha \neq \emptyset$ then $\text{lh}(E^W_\alpha) < i_0^W(\text{OR}^M)$.

2. For each $\alpha + 1 < \text{lh}(W)$, if $E^W_\alpha = \emptyset$ then:

   (a) $0 \leq W \alpha$ and $(0, \alpha)^W \cap \mathcal{D}^W = \emptyset$,

   (b) $\text{pred}^W(\alpha + 1) = -1$, $M^W_\alpha = M^W_\alpha$, $M^W_{\alpha + 1} = M$, and $i^W_\alpha = i^W_\alpha \circ i^W_{\text{OR}^M}$,

   (c) $\alpha + 2 < \text{lh}(W)$,

   (d) $i^W_0(\text{OR}^M) < \text{lh}(E^W_{\alpha+1})$.

3. For each $\alpha + 1 < \text{lh}(W)$, if $E^W_\alpha \neq \emptyset$ then $\text{pred}^W(\alpha + 1)$ is the least $\beta$ such that $\text{cr}(E^W_\alpha) < \nu^W_\beta$, where $\nu^W_\beta = \nu(E^W_\beta)$ if $E^W_\beta \neq \emptyset$, and $\nu^W_\beta = i^W_0(\nu(F^M))$ if $E^W_\beta = \emptyset$.

4. Everything else is as for $(0,0)$-maximal trees.

\(^{56}\)Note that this does not seem to follow immediately from the construction of $\bar{T}$, as we have not yet shown that $\bar{T}$ has no $T$-special extender along $b^T$, but only that $E^T_\alpha$ is non-$T$-special. And we have to allow the possibility that $b^T$ has ordertype $> \omega$ and infinitely many extenders are used along $b^T$ prior to the first $T$-special extender there. So $E^T_\alpha$ might ostensibly be the reflection of such a $T$-special extender in $\bar{T}$.

\(^{57}\)See Footnote 52 for the case $r = -1$. 

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Clearly nicely padded \((0,0)\)-maximal trees on \(\mathfrak{M}\) are equivalent to unpadded \((0,0)\)-maximal trees on \(\mathfrak{M}\).

Now let \(\mathfrak{M}\) be the phalanx \(((M,<\gamma),M,\gamma)\). Note that \((0,0)\)-maximal trees \(T\) on \(\mathfrak{M}\) are equivalent to 0-maximal trees on \(M\) with \(\gamma < \text{lh}(E_T^\gamma)\). So let \(\Sigma_{\mathfrak{M}}\) be the iteration strategy for \(\mathfrak{M}\) equivalent to some \((0,\omega_1+1)\)-strategy for \(M\).

Let \(\mathcal{W}\) be a nicely padded \((0,0)\)-maximal tree on \(\mathfrak{M}\) and \(T\) be \((0,0)\)-maximal on \(\mathfrak{M}\). We say that \(((\mathcal{W},T)\) is a nice pair\) iff \(\text{lh}(T) = \text{lh}(\mathcal{W})\), \(\langle T = \langle\mathcal{W}\) and for each \(\alpha + 1 < \text{lh}(\mathcal{W})\), we have:

- if \(E^\mathcal{W}_\alpha \neq \emptyset\) then \(E^T_\alpha = E^\mathcal{W}_\alpha\), and
- if \(E^\mathcal{W}_\alpha = \emptyset\) then \(E^T_\alpha = F(M^T_\alpha)\).

We get a \((0,0)\)-maximal iteration strategy \(\Sigma_{\mathfrak{M}}\) for \(\mathfrak{M}\) producing nicely padded trees \(\mathcal{W}\) such that for some \((0,0)\)-maximal tree \(T\) on \(\mathfrak{M}\) via \(\Sigma_{\mathfrak{M}}\), \(((\mathcal{W},T)\) is a nice pair. To see this, observe the following points regarding nice pairs \(\mathcal{W},T)\):

A. \(\mathcal{D}T = \mathcal{D}^\mathcal{W}\) and \(\deg T = \deg^\mathcal{W}\).

B. For each \(\alpha < \text{lh}(\mathcal{W})\), if either \(-1 \leq^\mathcal{W} \alpha\) or \([0 \leq^\mathcal{W} \alpha\) and \((0,\alpha)^W \cap \mathcal{D}^\mathcal{W} \neq \emptyset\) then:

(i) \(M^T_\alpha = M^W_\alpha\),
(ii) for each \(\beta \leq^T \alpha\) with \((\beta,\alpha)^T \cap \mathcal{D}^T = \emptyset\), we have \(i^T_\beta = i^\mathcal{W}_\beta\) and if \(\beta\) is a successor then \(M^T_\beta = M^\mathcal{W}_\beta\) and \(i^T_\beta = i^\mathcal{W}_\beta\).

C. For each \(\alpha < \text{lh}(\mathcal{W})\), if \(0 \leq^\mathcal{W} \alpha\) and \((0,\alpha)^W \cap \mathcal{D}^\mathcal{W} = \emptyset\) then:

(a) \(M^W_\alpha = \text{Ult}_0(M,F^{M^T_\alpha})\),
(b) \(i^T_0 = i^\mathcal{W}_0 \upharpoonright M\),
(c) \(F^{M^T_\alpha}\) is just the \((\kappa, \text{OR}^{M^T_\alpha})\)-extender derived from \(i^\mathcal{W}_0 \circ i^{TM}_\alpha\).
(d) Suppose \(\alpha\) is a successor \(\beta + 1 < \text{lh}(\mathcal{W})\) and let \(\varepsilon = \text{pred}^\mathcal{W}(\beta + 1)\). Then \(\kappa = \text{cr}(F^{M^\varepsilon}) < \gamma \leq \text{cr}(E_\gamma^\mathcal{W}) < \text{OR}(M^T_\varepsilon)\), and because \(\text{OR}^{M^T_\varepsilon}\) is a successor cardinal of \(M^\varepsilon\) = \(\text{Ult}_0(M,F^{M^T_\varepsilon})\), the same functions are used in forming \(\text{Ult}_0(M^T_\varepsilon,E_\gamma^\mathcal{W})\) as in forming \(i^\mathcal{W}_\varepsilon,\beta + 1(M^\varepsilon||\text{OR}^{M^T_\varepsilon})\).

With these points in mind, it is easy to see that \(\Sigma_{\mathfrak{M}}\) is indeed a \(((0,0),\omega_1+1)\)-strategy for \(\mathfrak{M}\).

We now reduce trees on \(\mathfrak{U}\) to trees on \(\mathfrak{M}_{\pi(\zeta)}\), giving the desired iterability of \(\mathfrak{U}\). Recall (from directly after Assumption 2) that \(\pi : U \rightarrow W\) is the factor map. We have \(\text{cr}(\pi) = \zeta \geq \kappa\), so \(\pi(\zeta)\) is a \(W\)-cardinal such that \(\kappa < \pi(\zeta) < \text{OR}^M\), and hence an \(M\)-cardinal. So by the previous claim, \(\mathfrak{M}_{\pi(\zeta)}\) is \(((0,0),\omega_1+1)\)-iterable. But we can reduce trees \(\mathfrak{U}\) on \(\mathfrak{U}\) to trees \(\mathcal{W}\) on \(\mathfrak{M}_{\pi(\zeta)}\) via standard copying, except for one possibly non-standard detail which occurs at the superstrong level. We describe this below, along with some more detail in general.

If \(\zeta\) is an \(M\)-cardinal then \(\mathfrak{U} = ((M,<\zeta),U,\zeta)\), so we can lift directly to \(\mathfrak{M}_{\pi(\zeta)} = ((M,<\pi(\zeta)),W,\pi(\zeta))\) using id : \(M \rightarrow M\) and \(\pi : U \rightarrow W\) as initial copy maps. This case is routine, so we move on.
Suppose instead that \( \text{card}^M(\zeta) = \delta < \zeta = \delta^+ U \), so \( R \triangleleft M \) is defined, \( \delta^+ U = \delta^+ R \) and 

\[
\mathcal{U} = ((M, < \delta), (R, \delta), U, \delta^+ U).
\]

Then \( \pi(\zeta) = \delta^+ W = \delta^+ M \) and we lift \( (0, r, 0) \)-maximal trees on \( \mathcal{U} \) to essentially-\( (0, 0, 0) \)-maximal trees on the phalanx 

\[
\Phi^\pi(\zeta) = ((M, < \delta), (M, \delta), W, \delta^+ M),
\]

using \( \text{id}: M \rightarrow M \), inclusion \( R \rightarrow R \triangleleft M \) and \( \pi: U \rightarrow W \) as initial copy maps. But this suffices, since \( \Phi^\pi(\zeta) \) is clearly equivalent to \( \Phi^\pi(\zeta) \), so is \( (0, 0, 0) \)-maximally iterable, and just like in the proofs of Lemmas 6.12 and 6.13, this implies it is essentially-\( (0, 0, 0) \)-maximally iterable. (Note that \( \Phi^\pi(\zeta) \) has three models, where \( \Phi^\pi(\zeta) \) has only two.)

The reason we can only expect essentially-\( (0, 0, 0) \)-maximality instead of \( (0, 0, 0) \)-maximality for the tree on \( \Phi^\pi(\zeta) \) is as follows. Let \( \mathcal{U} \) be \( (0, r, 0) \)-maximal, formed by lifting to \( W \) on \( \Phi^\pi(\zeta) \). Suppose \( \text{cr}(E^\mathcal{U}_\alpha) = \delta \) and \( E^\mathcal{U}_\alpha, E^W_\alpha \) have superstrong type. Then \( M^\mathcal{U}_{\alpha + 1} = \text{Ult}(R, E^\mathcal{U}_\alpha) \) and \( \text{cr}(E^W_\alpha) = \delta \) and \( M^W_{\alpha + 1} = \text{Ult}(M, E^W_\alpha) \). We get the copy map

\[
\pi_{\alpha + 1}: M^\mathcal{U}_{\alpha + 1} \rightarrow S_{\alpha + 1} \triangleleft M^W_{\alpha + 1},
\]

where \( S_{\alpha + 1} = i^W_{\alpha + 1}(R) \), defined via the Shift Lemma (from earlier copy maps) in a straightforward manner. But \( i^W_{\alpha + 1}(\delta) = \lambda(E^W_\alpha) = \nu(E^W_\alpha) \), since \( E^W_\alpha \) is superstrong, and since \( \rho^W_\alpha = \delta \), we have \( \rho^W_{\alpha + 1} = \nu(E^W_\alpha) \). Therefore \( S_{\alpha + 1} \triangleleft M^W_{\alpha + 1} \triangleleft M^W_{\alpha + 1} \), so \( \text{OR}(S^W_{\alpha + 1}) < \nu(E^W_\alpha) + M^W_{\alpha + 1} = \text{lh}(E^W_\alpha) \). But we will have \( E^\mathcal{U}_{\alpha + 1} \in E_\alpha(M^\mathcal{U}_{\alpha + 1}) \), so \( E^W_{\alpha + 1} \in E_\alpha(S^W_{\alpha + 1}) \) is the copy of \( E^\mathcal{U}_{\alpha + 1} \) under \( \pi_{\alpha + 1} \). Therefore \( \text{lh}(E^W_{\alpha + 1}) < \text{lh}(E^W_\alpha) \), so \( W \) fails to be \( (0, 0, 0) \)-maximal. On the other hand, we have \( \text{lh}(E^\mathcal{U}_\alpha) \leq \text{lh}(E^\mathcal{U}_{\alpha + 1}) \), and because of this, it is straightforward to see that \( \nu(E^W_\alpha) = \lambda(E^W_\alpha) \leq \nu(E^W_{\alpha + 1}) \). Moreover, it is only in this situation that we can have \( \text{lh}(E^W_{\alpha + 1}) < \text{lh}(E^W_\alpha) \). So \( W \) will be essentially-\( (0, 0, 0) \)-maximal.

The remaining details are quite standard, so we leave them to the reader. (There are also more details in the related proof of Claim 9 of the proof of Theorem 14.1.)

This completes the proof of the super-Dodd-soundness lemma 10.2.

We can now complete the proof of the theorem on measures in mice:

**Proof of Theorem 9.1.** By Lemma 10.2, all proper segments of \( M \) are Dodd-sound, so Lemma 9.2 applies, so we are done.

And we finish this section with the proof of the ISC for normally iterable pseudopreme (cf. [9, §10]):

**10.6 Theorem** (Initial segment condition). Let \( M \) be a \((0, \omega_1 + 1)\)- iterable pseudopreme. Then \( M \) is a premouse.

**Proof.** We use a trick similar to that used at the beginning of the proof of the super-Dodd-soundness lemma 10.2, to reduce to the classical case. Let \( \gamma \) be the largest generator of \( F^M \), let \( \delta = \text{lgcd}(M) \) and let \( G = F^M \upharpoonright \delta \); so \( G \in M \). Let

\[
\tilde{M} = \text{cHull}_1^M(\{\gamma, G\}),
\]

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let \( \pi : \bar{M} \to M \) be the uncollapse and let \( \pi(\bar{\gamma}, \bar{\delta}, \bar{G}) = (\gamma, \delta, G) \). It is straightforward to see that \( \bar{\gamma} \) is the largest generator of \( F^{\bar{M}} \), \( \bar{\delta} = \gcd(M) \), \( \bar{G} = F^{\bar{M}} \upharpoonright \bar{\delta} \), and if \( M \) is a premouse then so is \( \bar{M} \). So, resetting notation, we may assume that \( \bar{M} = M \) and \( \pi = \text{id} \). In particular, \( \rho_1^M = \omega \). It is then easy to see that if \( M' \) is a pseudo-premouse and \( \sigma : M \to M' \) and \( \tau : M \to M' \) are cofinal and \( \Sigma_1 \)-elementary, then \( \sigma(\gamma, G) \) is \( \tau(\gamma, G) \), and therefore that \( \sigma = \tau \). Using these facts in place of weak Dodd-Jensen, the standard proof of the ISC for \((0, \omega_1, \omega_1 + 1)^*\)-iterable pseudo-premice goes through. \( \square \)

11 Projectum-finitely generated mice

11.1 When mice are (or are not) iterates of their cores

In this section we will give the main argument for the projectum-finite generation theorem 11.5, which gives a simple criterion guaranteeing that a mouse \( M \) is an iterate of its core \( C \). If we have an \( m \)-sound, \((m, \omega_1 + 1)\)-iterable premouse \( M \), the criterion, projectum-finite generation, is just that \( M = \text{Hull}_{m+1}(\rho_{m+1} M \cup \{x\}) \) for some \( x \in C_0(M) \).

But before we begin, we discuss some already known results which are related, and slight variants thereof. These results will demonstrate that one cannot simply drop the criterion completely: we can have \( m < \omega \) and an \((m, \omega_1 + 1)\)-iterable, \( m \)-sound premouse \( M \) which is not an \( m \)-maximal iterate of \( C_{m+1}(M) \). This phenomenon first occurs roughly at the point of a cardinal which is “strong past a measurable”. We will make this more precise in what follows; its interpretation is the deciding factor here.

11.1.1 Mice which are iterates of their cores

Recall \( 0^\# \) (0-pistol) is the least active mouse \( M \) such that \( M|\text{cr}(FM) \models \text{"There is a strong cardinal".} \) If \( 0^\# \) does not exist, then by [33, Theorem 8.13] or [4, §5], (cf. also [23, Theorem 6.1]), every universal weasel \( W \) is a normal iterate of the core model \( K \). It is pointed out in [15, between 8.20 and 8.21] that if \( 0^\# \) (0-hand-grenade, [15, Definition 2.3]) does not exist, then for the core model \( K \) as developed in [15], if \( K \models \text{"there are no } \mu, \kappa \text{ such that } \kappa \text{ is measurable, } \mu < \kappa \text{ and } \mu \text{ is } < \kappa \)-strong”, then the same conclusion holds. In fact, an elaboration of the proof yields a little more, as shown in Theorem 11.2 below.

11.1 Definition. Let \( M \) be an active premouse and let \( \mu = \text{cr}(F^M) \). Say that \( F^M \) is of \textit{limit space type} if there is \( \kappa > \mu \) such that \( \kappa \) is measurable in \( \text{Ult}(M, F^M) \) and \( \nu(F^M) = \kappa^+M \). So \( \text{OR}^M = \kappa^+ \upharpoonright \text{Ult}(M, F^M) \) and by coherence, the order 0 measure on \( \kappa \) in \( \text{Ult}(M, F^M) \) is also in \( \text{E}^M \).

11.2 Theorem. Let \( k < \omega \) and let \( K \) be a \((k + 1)\)-sound \((k, \omega_1 + 1)\)-iterable premouse such that \( \rho_{k+1}^K = \omega \), and let \( \Sigma_K \) be the unique \((k, \omega_1 + 1)\)-iteration strategy for \( K \). Let \( U \) be a \((k, \omega_1 + 1)\)-iterable premouse such that \( \mathcal{C}_{k+1}(U) = K \). Suppose that \( U \) is not a \( k \)-maximal \( \Sigma_K \)-iterate of \( K \). Then there is a \( K \)-total extender \( E \in E^K \) which is of \textit{limit space type}.
Proof. Consider the $k$-maximal comparison $(U, T)$ of $(U, K)$, using some $(k, \omega_1 + 1)$-strategy for $U$ and $\Sigma_K$ for $T$. Then $M^{U, k}_T = M^T_k$ and $b^U, b^T$ do not drop in model or degree. Let $Q = M^{T, k}_K$. Since $U$ is not a normal iterate of $K$ via $\Sigma_K$, $U$ is non-trivial. Note that by Lemma 3.8, since $Q$ is $(k + 1)$-solid, so is $U$, and $i^U(\rho^U_{k+1}) = \rho^Q_{k+1}$. Letting $\pi : K \to U$ be the core map, we have $i^U \circ \pi = i^T$.

Standard arguments with the hull and definability properties give that $\text{cr}(i^T) < \text{cr}(i^U)$, and moreover, that there is $\alpha + 1 \in b^T$ such that $\text{cr}(E^T_\alpha) < \text{cr}(i^T) < \nu(E^T_\alpha)$. This is by calculations like in [33, Example 4.3 and Remark, p. 29]. Let $\kappa = \text{cr}(i^U)$ and $\mu = \text{cr}(E^T_\alpha)$.

Claim. $E^T_\alpha$ has an initial segment which is of limit space type.

Proof. Suppose otherwise. Let $\beta = \text{pred}^T(\alpha + 1)$, $j = i^T_{\beta, \infty}$ and $k = i^U$. Noting that $\text{rg}(j) \subseteq \text{rg}(k)$, let $\pi' : M^T_\beta \to U$ be such that $k \circ \pi' = j$.

By the ISC, $E^T_\alpha | \kappa \in E^{M^T_{k+1}}$, and so $E^T_\alpha | \kappa \in U$. Since $k \circ \pi' = j$ and $\text{cr}(k) = \kappa$, $k(E^T_\alpha | \kappa) = E_j | k(\kappa)$, where $E_j$ is the short extender derived from $j$. But then $E_j | k(\kappa) \in \mathbb{E}$, so $k(\kappa) < \nu(E^T_\alpha)$. Since $\kappa$ is measurable in $U$, $k(\kappa)$ is measurable in $Q$, and hence in $M^{T, k+1}_T$. But since $E^T_\alpha$ is not of limit space type, it follows that $k(\kappa) < \nu(E^T_\alpha) < k(\kappa) + Q = k(\kappa) + M^{T, k+1}_T = \text{lh}(E^T_\alpha)$, and $E^T_\alpha$ is type 2. Also since $E^T_\alpha$ has no segment of limit space type, exit$_I^{E^T_\alpha}$ has no measurable cardinal $\theta$ such that $\text{cr}(E^T_\alpha) < \theta < k(\kappa)$, and there is a unique total measure on $k(\kappa)$ in $E(M^{T, k+1}_T)$.

Like in §9 and §10, we have $s_{E^T_\alpha} = s_j = k(s_{\pi'})$ and $\sigma_{E^T_\alpha} = \sigma_j = \sup k \sigma_{\pi'}$. Since $E^T_\alpha$ is type 2, $s_{E^T_\alpha} \neq \emptyset$ and $k(\kappa) \leq \max(s_{E^T_\alpha}) < \text{lh}(E^T_\alpha)$. So $\kappa \leq \max(s_{\pi'}) < k + U$. Therefore $s_{\pi'} \subseteq \kappa$, and therefore $\sigma_{\pi'} = \sigma_j = \sigma_{E^T_\alpha}$.

Let $F = \mathcal{C}_\alpha(M^T_\alpha)$ and $\tau : \text{Ult}(M^T_\alpha, F) \to Q$ be the standard factor map. If $E^T_\alpha$ is Dodd-sound then $F = E^T_\alpha$ and $\tau = i^T_{\alpha+1, \infty}$ and $\nu(E^T_\alpha) < \text{cr}(i^T_{\alpha+1, \infty})$. But by the previous paragraph, we can define $\pi'' : \text{Ult}(M^T_\alpha, F) \to U$ such that $k \circ \pi'' = \tau$. Since $\text{cr}(k) = \kappa$, $\text{cr}(\tau) \leq \kappa$. So $E^T_\alpha$ is non-Dodd-sound.

Let $\eta < T \alpha$ be such that $\mathcal{C}_{\eta}(M^T_\alpha) \subseteq M^T_{\eta}$, let $\gamma + 1 = \text{succ}^T(\eta, \alpha)$, and let $\xi : \mathcal{C}_{\eta}(M^T_{\alpha}) \to M^T_{\alpha}$ be the iteration map $\xi = i^T_{\eta+1, \alpha}$. By the remarks above, the extenders used along the branch $(\eta, \alpha)$ are just the order 0 measures on the largest cardinal of the current model (starting with $\mathcal{C}_{\eta}(M^T_{\alpha})$). If $\text{cr}(\xi) \geq \kappa$ then let $F' = F_{\mathcal{C}_{\eta}(M^T_{\alpha})}$. If $\text{cr}(\xi) < \kappa$ then let $\xi \equiv T \alpha$ be such that $\xi = \alpha$ or $\text{lgcd}(M^T_{\xi}) \geq \kappa$, and let $F' = F^{M^T_{\xi}}$. Now let $\pi' : \text{Ult}(M^T_{\alpha}, F') \to Q$ be the factor map (this is given by the tail end of the iteration of the order 0 measures just mentioned, followed by $i^T_{\alpha+1, \infty}$). Let $\pi''' : \text{Ult}(M^T_{\alpha}, F') \to U$ be such that $k \circ \pi''' = \tau'$. We have $\text{cr}(\pi''') \geq \kappa$. But $\kappa \in \text{rg}(\pi''')$, since $\kappa$ is a successor measurable of $U$. So $\pi'''(\kappa) = \kappa$ and $U|_{\kappa + U} = \text{Ult}(M^T_{\alpha}, F')|_{\kappa + U} = \text{Ult}(M^T_{\alpha}, F')|_{\kappa + \text{lh}(M^T_{\alpha}, F')}$. Let $k(\kappa) = \tau'(\kappa)$ and by commutativity, the short $(\kappa, k(\kappa))$ extenders derived from $k$ and $\tau'$ are identical. But these arise from comparison, a contradiction.

Now note that if $\text{cr}(E^T_\alpha) = \text{lgcd}(M^T_{\alpha})$, so $M^T_{\alpha}$ is active type 2, then since $\text{cr}(E^T_\alpha)$ is a limit of measurables, $F(M^T_{\alpha})$ has a proper segment which is of limit space type, and since $(0, \beta^T)$ does not drop, the same holds of $K$, as desired. So suppose $\text{cr}(E^T_\alpha)^{+M^T_{\alpha}} < \text{OR}(M^T_{\alpha})$. Therefore $\text{cr}(E^T_\alpha)^{+M^T_{\alpha}} < \text{lh}(E^T_\alpha)$. Now arguing like in the proof of closedness [9, 6.1.5], it follows
again that there is an $M^+_D$-total extender $E \in \mathbb{E}_+(M^+_D)$ which is of limit space type, and it easily follows that the same holds of $\mathbb{E}^K_N$.

The result above is close to optimal. One cannot obtain from the hypotheses of the theorem an active mouse $N$ and $\kappa \in (\text{cr}(F^N), \nu(F^N))$ such that $\kappa$ is measurable in $\text{Ult}(N, F^N)$ and $\kappa^{++ \text{Ult}(N, F^N)} < \nu(F^N)$. For letting $\gamma$ be the least generator of $F^N$ which is $> \xi = \kappa^{++ \text{Ult}(N, F^N)}$, then $\gamma = \kappa^{++ \text{Ult}(N, F^N)}$ and $F^N \upharpoonright \xi = F^N \upharpoonright \gamma$. Since $\kappa$ is measurable in $\text{Ult}(N, F^N)$, letting $D$ be the order 0 measure on $\kappa$ there, we have $\text{lh}(D) < \text{OR}^N$, and since we have the factor embedding $\pi : \text{Ult}(N, F^N \upharpoonright \xi) \rightarrow \text{Ult}(N, F^N)$, and $\pi \upharpoonright (\xi + 1) = \text{id}$, in fact $\text{lh}(D) < \gamma$. This is significantly beyond the least mouse $P$ which models $\text{ZFC}$ and such that there are $D, E \in \mathbb{E}^P$ which are $P$-total and such that $\text{cr}(E) < \text{cr}(D) < \text{lh}(D) < \text{lh}(E)$. But such a mouse $P$ is enough to arrange a mouse which is not an iterate of its core, as we will see.

### 11.1.2 A mouse which is not an iterate of its core

In [33, pp. 85–87], Steel outlines a situation in which the core model $K$ exists and there is a universal weasel which is not an iterate of $K$ (and $K$ is “below two strong cardinals”). In Schindler [15, Lemma 8.21] (this result is credited there to Steel), more details of such a construction are given, with a more precise large cardinal restriction. But the hypothesis of [15, 8.21] was not actually quite enough for its stated purpose. For the hypothesis of [15, 8.21] is that $0^* \text{ does not exist (} 0^*, 0\text{-hand-grenade, is defined in [15, Definition 2.3]), and letting } K \text{ be the core model (as constructed in [15] under this hypothesis) } 58$, there are $\mu < \kappa \in \text{OR such that } \kappa \text{ is measurable in } K \text{ and } K|\kappa \models "\mu \text{ is strong}"$, and the conclusion is that there is a generic extension of $K$ in which there is a universal weasel $U$ which is not an iterate of $K$. Because $0^*$ does not exist, it follows that there is a (possibly proper class) successful comparison of $K$ and $U$. But a slight modification of the proof of Theorem 11.2 and standard core model arguments show that if there is such a universal weasel $U$, then there is a $K$-total $E \in \mathbb{E}^K$ such that $E$ is of limit space type. But this is strictly beyond the hypothesis of [15, 8.21].

We next give a slight modification of that of [15, 8.21], which achieves its goal from a slightly stronger hypothesis, and derive from this an example of a mouse which is not an iterate of its core.

### 11.3 Theorem

Let $K$ be a fully iterable premouse which models $\text{ZFC + "there is no proper class premouse with a Woodin cardinal"}$, and suppose there is no $K' \subseteq K$ such that $K' \models \text{ZFC + "OR is Woodin"}$. Suppose there are $\mu, \kappa, D, E$ such that:

- $D \in \mathbb{E}^K$ is a $K$-total order 0 measure on $\kappa$,
- $E \in \mathbb{E}^K$ such that $\text{cr}(E) = \mu < \kappa$ and $\text{lh}(D) < \text{lh}(E)$.

Then there are $\theta < \xi < \kappa$ such that, letting $G$ be $(K, \text{Col}(\omega, \xi^{+\kappa}))$-generic, $K[G] \models "\text{There is a proper class premouse } W \text{ and a (possibly proper class) comparison } (T, U)"

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58 The precise details of the core model under this hypothesis are not particularly important. But if $M$ is a premouse which is below $0^*$ and $T$ is a limit length normal tree on $M$, then $M(T)$ is the $Q$-structure for itself, and in fact the class of ordinals $\mu < \delta(T)$ such that $M(T) \models "\mu \text{ is a strong cardinal}"$ gives a counterexample to Woodinness. This ensures that if a comparison of proper class models runs through OR stages, then there are (proper class sized) cofinal branches at the end.
of \((K,W)\) which terminates in a common proper class premouse \(Q = M^T = M^\mu\), with \(b^\sigma, b^\mu\) non-dropping, and \(U\) is non-trivial, and in fact, \(W|\theta^{++}K = K|\theta^{++}K\), but \(W|\theta^{++}K\) is active", and therefore, \(W\) is not an iterate of \(K\).

**Proof.** We follow in outline the argument for [15, 8.21], but making use of the stronger hypothesis and giving more details.

Let \(E,D \in \mathcal{E}^K\) witness the hypothesis of the theorem, with \((\text{lh}(E),\text{lh}(D))\) lexicographically least possible. Let \(\mu = \text{cr}(E)\) and \(\kappa = \text{cr}(D)\). Let \(U = \text{Ult}(K,E)\). Let \(g_0\) be \((K,\text{Col}(\omega,\mu^{++}))\)-generic. Of course, \(\mu^{++}U = \mu^{++}K\) and \(g_0\) is also \((U,\text{Col}(\omega,\mu^{++}U))\)-generic.

**Claim 1.** There is a tree \(T \in K[g_0]\) such that \(T \subseteq \omega \times (\omega \times \text{OR}^K)^{<\omega}\) and in \(K[g_0]\),

1. for all \(\alpha < \kappa\), \(\text{Col}(\omega,\alpha)\) forces that for all \(\Pi_3^1\) formulas \(\varphi\) and all reals \(x\), if \((\varphi,x) \in p[T]\) then \(\varphi(x)\), and

2. \(\text{Col}(\omega,\kappa^{++})\) forces that, letting \(g_1\) be the name for the generic filter, for all reals \(x \in U[g_0][g_1]\), if \(\varphi(x)\) then \((\varphi,x) \in p[T]\).

**Proof.** This will be via Woodin’s arguments as in [30, §3] and as in [8, Theorem 1.5.12]. Fix first the standard tree \(S \in K\) such that \(S \subseteq \omega \times (\langle \omega \times \omega \rangle \times <\omega\mu)\) and in \(K\), for each \(\alpha < \mu\), \(\text{Col}(\omega,\alpha)\) forces that \(p[S]\) is a universal \(\Pi_3^1\) set. We get this as usual from the closure of \(K|\mu\) under sharps. Let \(S'\) be the Shoenfield tree for \(\Sigma_3^1\) on \(\omega \times \mu\). So in \(K\), \(S, S'\) are \(< \mu\)-complementing. Let \(R\) be the tree for a universal \(\Sigma_3^1\) set derived from \(S\) in the usual manner.

For \(a \in i_E^K(\langle \mu\rangle)^{<\omega}\), let \(\nu_a\) be the measure over \(\mu^{\text{lh}(a)}\) derived from \(i_E^K\) with seed \(a\); so for \(X \subseteq \mu^{\text{lh}(a)}\), we have

\[X \in \nu_a \iff a \in i_E^K(X)\]

So for each \(a, \nu_a \in K|\mu^{++}\). Let \(\sigma_a = i_E^K(\nu_a)\). So \(U \models \sigma_a\) is \(\mu'\)-complete measure over \((\mu')^{\text{lh}(a)}\), where \(\mu' = i_E^K(\mu)\). Write \(\dim(\sigma_a) = \text{lh}(a)\).

Now work in \(K[g_0]\), where \(\mu^{++}\) is countable. Let \(\langle \tau_n, \theta_n \rangle_{n<\omega}\) enumerate all pairs \((\tau, \theta)\) such that \(\tau \in \omega \times \omega, \theta = \nu_a\) for some \(a\), \(\text{lh}(\tau) = \dim(\nu_a)\), and such that for each \(n < \omega\), \(\text{lh}(\tau_n) = \dim(\theta_n)\), there is \(m < n\) such that \((\tau_n, \theta_m) = (\tau_n, \theta_n) \upharpoonright m\), where the latter restriction denotes the pair \((\tau_n \upharpoonright m, \theta_n \upharpoonright m)\), where \(\theta_n \upharpoonright m\) denotes the projection of \(\theta_n\) to the first \(m\) coordinates. For \(m < \omega\), let \(U_m = \text{Ult}(U, \theta_m)\). For \(\ell < m < \omega\) such that \((\tau_\ell, \theta_\ell) = (\tau_m, \theta_m) \upharpoonright \ell\), let \(i_{\ell m} : U_\ell \to U_m\) be the canonical factor map. Let \(T\) be the tree consisting of tuples

\[(\varphi, \vec{x}, \vec{\alpha}) \in \omega \times \omega \times \omega \text{ with } \varphi(u) \text{ is a } \Pi_3^1 \text{ formula of form } \forall x \psi(x,u), \text{ where } \psi \text{ is } \Sigma_2^1, \text{ and for some } n < \omega, \text{ we have } n = \text{lh}(\vec{x}) = \text{lh}(\vec{\alpha}), \text{ and for all } \ell, m \text{ with } \ell < m < n \text{ and } (\tau_\ell, \theta_\ell) = (\tau_m, \theta_m) \upharpoonright \ell, \text{ if } i_E^K(S_{\neg \psi}, \text{lh}(\tau_m), \tau_m) \in \theta_m \text{ then } i_{\ell m}(\alpha_\ell) > \alpha_m,\]

where \(\vec{\alpha} = (\alpha_0, \ldots, \alpha_{n-1})\). (Note here that if \(\ell < m < n\) and \((\tau_\ell, \theta_\ell) = (\tau_m, \theta_m) \upharpoonright \ell\) and \(i_E^K(S_{\neg \psi}, \text{lh}(\tau_m), \tau_m) \in \theta_m\) then also \(i_E^K(S_{\neg \psi}, \text{lh}(\tau_\ell), \tau_\ell) \in \theta_\ell\).)

**Subclaim 1.1.** \(T\) is as desired.

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Proof. First let us verify clause 1. So let $\alpha < \kappa$ and $g_1$ be \((K[g_0], \Col(\omega, \alpha))\)-generic. Let $x \in \mathbb{R}^{K[g_0][g_1]}$ and $\varphi$ be $\Pi^1_1$, and suppose that \((\varphi, x) \in p[T]\). Let $\bar{\alpha} \in \omega$ OR be such that \((\varphi, x, \bar{\alpha}) \in [T]\). We have to see that $K[g_0][g_1] \models \varphi(x)$, so suppose otherwise. We have $V^K_{\kappa} = V^K_{\kappa}$, so $K[g_0][g_1] = \mathcal{U}^{U[g_0][g_1]}$, so $U[g_0][g_1] \models \neg \varphi(x)$. But $i^K_\mathcal{E}(R)$ projects to the universal $\Sigma^1_3$ set in $U[g_0][g_1]$, since $\alpha < \kappa < i^K_\mathcal{E}(\mu)$. So letting $\varphi(x)$ be the formula $\forall y \psi(x, y)$ where $\psi$ is $\Sigma^3_2$, we can fix $y, f \in U[g_0][g_1]$ such that $y \in \omega$ and $((\neg \psi), x, y, f) \in \{i^K_\mathcal{E}(S)\}$. But then letting $a_n = f | n$ $\langle \nu_n \rangle_{n \in \omega} < K[g_0][g_1]$ is a tower of measures over $K$, and $\langle \sigma_\nu \rangle_{n < \omega}$ is a tower of measures over $U$, and each $\sigma_\nu$ canonically extends to a measure $\sigma^*_\nu$ over $U[g_0][g_1]$, and the fact that $(\varphi, x) \in p[T]$ establishes that $\text{Ult}(U[g_0][g_1], \langle \sigma^*_\nu \rangle_{n < \omega})$ is illfounded, but note that this ultrapower factors into the ultrapower $\text{Ult}(U, i^K_\mathcal{E}(E))$, which is wellfounded, a contradiction.

Now let us verify clause 2: Let $g_1$ be \((K[g_0], \Col(\omega, \kappa^+K))\)-generic, let $x \in \mathbb{R} \cap U[g_0][g_1]$, let $\varphi$ be $\Pi^1_1$, and suppose $K[g_0][g_1] \models \varphi(x)$. We want to see that $(\varphi, x) \in p[T]$. Since $x \in U[g_0][g_1] \subseteq K[g_0][g_1]$, we have $U[g_0][g_1] \models \varphi(x)$ also. Since $\kappa^+K < i^K_\mathcal{E}(\mu)$, $i^K_\mathcal{E}(R)$ projects to a universal $\Sigma^3_3$ set in $K[g_0][g_1]$. So $((\neg \varphi), x) \notin p[i^K_\mathcal{E}(R)]$. So $i^K_\mathcal{E}(R)(\neg \varphi), x$ is wellfounded, so has a rank function in $U[g_0][g_1]$. By using restrictions of this rank function to various sub-trees (which are all still in $U[g_0][g_1]$ and hence relevant to forming the ultrapowers $U_\nu = \text{Ult}(U[g_0][g_1], \theta^*_\mu)$, where $\theta^*_\mu$ is the extension of $\theta_\mu$ to $U[g_0][g_1]$), we get a sequence $\bar{\alpha} \in \omega \cap U[g_0][g_1]$ such that $(\varphi, x, \bar{\alpha}) \in [T]$, which suffices.

Since $T$ works, we have established the claim. \hfill $\square$

We will apply the claim to the $\Pi^1_1$ notion of a good premouse, which is as follows. Working in any set-generic extension $K[G]$ of $K$, we say that a countable premouse $N$ is good if it is sound, and for every countable $N$-premouse $M$, if $M$ is $\Pi^1_2$-iterable (above $N$, this means), then $M$ is equivalent to a premouse $M'$ (so all of its proper segments are sound, as a premouse, as opposed to an $N$-premouse), and $M'$ is $\Pi^1_2$-iterable (not just above $N$).

Let $g_1$ be $\Col(\omega, \kappa^+K)$-generic and $g = (g_0, g_1)$.

Claim 2. $K[g] \models \text{"}K[\text{lh}(D) \text{ is good}"}$.

Proof. Let $N = K[\text{lh}(D)]$. We have $\kappa^+K = \kappa^+U = \omega^1_1[g]$, and in $K[g]$, $U[\kappa^+K]$ is equivalent to the stack of the collection $\mathcal{E}$ of all $N$-premece $Q$ which are sound, project to $N$, and are $\Pi^1_2$-iterable (as $N$-premice). For $\text{lh}(D) = \text{OR}^N$ is a strong cutpoint of $U$, by the minimality of $\text{lh}(E)$. So every $P \subset U[\kappa^+K]$ with $N \not\subset P$ and $\rho_\omega^P \leq \text{OR}^N$, is equivalent to an $N$-premece $P \in \mathcal{E}$. Conversely, if $Q \in \mathcal{E}$ but $Q$ is not equivalent to any $Q' \not\subset U$, then we could compare $Q$ with $U$, and $Q$ would out-iterate $U$, which is impossible, as $U$ computes too many successor cardinals correctly, as we are working in $K[g]$. So $N$ satisfies the requirements of goodness with respect to every countable sound $\Pi^1_2$-iterable $N$-premice $P$ in $K[g]$ which projects to $N$. But if $P \in K[g]$ is a countable $N$-premice which is $\Pi^1_2$-iterable, then since $\kappa^+K = \omega^1_1[g]$, we can fix $Q \in \mathcal{E}$ such that $Q \not\subset P$, and then comparing $P$ with $Q$, we get that $Q$ out-iterates $P$, and so since $Q$ is a fully iterable premice (not just $N$-premice), so is $P$.

The following claim will yield the ordinal $\xi$ witnessing the theorem:

Claim 3. There are $\xi, \theta$ such that $\mu^+K < \theta < \xi < \kappa$ and $K[g_0] \models \text{"}it is forced by } \Col(\omega, \xi^+)\text{"} that there is an active good premice $N$ such that $N^{PV} = K[\theta^+K]$.

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Proof. Let $M = \text{Ult}(K, D)$, so $g_0$ is also $(M, \text{Col}(\omega, \mu^+))^\text{-generic}$, and $D$ extends canonically to measure $D^*$ over $K[g_0]$, and $M[g_0] = \text{Ult}(K[g_0], D^*)$. Let $\gamma = \delta_D(\kappa+K)$, so $\gamma$ is where $D$ would be indexed in Jensen indexing. Let $g_i$ be $(K[g_0], \text{Col}(\omega, \gamma))$-generic (of course, Col$(\omega, \gamma)$ is equivalent to Col$(\omega, K^\gamma)$ in $K[g_0]$, as $	ext{card}^{K[g_0]}(\gamma) = \kappa+K$). Then $g_i$ is also $(M[g_0], \text{Col}(\omega, \gamma))$-generic (but $\gamma$ is an $M[g_0]$-cardinal). Let $g' = (g_0, g_1')$. Let $z \in \mathbb{R} \cap M[g_1']$ be the canonical code for $M[\gamma]$ determined by $(M[\gamma], g_1')$.

Working in $M[g_1']$, let $T'_z$ be the tree of attempts to build a real $y$ and a sequence $\vec{\alpha}$ such that $(\varphi_0, z \uplus y, \vec{\alpha}) \in [i_D^{K[\varphi_0]}(T)]$, where $\varphi_0$ asserts that $y$ codes a passive premouse $N$ and $z$ codes an extender $F$ such that for some other active type 1 premouse $(N', F')$ with $N' \subset N$, letting $\theta = \text{cr}(F')$, then $N = \text{Ult}(N'[\theta^+\mathbb{N}', F'])$ and $F$ is the extender derived from $i^{N'[\theta^+\mathbb{N}'}, (N', F')$.

We claim that $T'_z$ is illfounded. For working in $K[g']$, similarly define the tree $T_z$ of attempts to build a real $y$ and sequence $\vec{\alpha}$ such that $(\varphi_0, z \uplus y, \vec{\alpha}) \in [T]$ (where $\varphi_0$ is as above). Then $T_z$ is illfounded, since $z \uplus y \in U[g']$ where $y$ codes the extender of $i_D^{K[\kappa+K]}$, and by clause 2. But noting that $i_D^{K[\varphi_0]}, T_z \subseteq T'_z$, it follows that $T'_z$ is illfounded.

So it is forced over $M[g_0]$ by $\text{Col}(\omega, \gamma)$ that letting $z$ be the canonical code for $M[\gamma]$, $T'_z$ is illfounded (which is defined from $i_D^{K[\varphi_0]}(T)$). This fact pulls back under $i_D^{K[g_0]}$ to $K[g_0]$ and $K[\kappa+K]$ and $T$.

To establish the claim, we need to reflect this to some $\xi < \kappa$ replacing $\kappa$, so that we can apply clause 1 of Claim 1. Well, instead of $g_1'$ being $(M[g_0], \text{Col}(\omega, \gamma))$-generic, let $g''_1$ be $(K[g_0], \text{Col}(\omega, K^\gamma))$-generic, let $g''_0 = (g_0, g'_1)$, and let $z \in M[g''_0]$ be the canonical code for $K[\kappa+K]$ determined by $(K[\kappa+K], g''_0)$. Then working in $M[g''_0]$, define $T'_z$ be as before (hence, still using $i_D^{K[\varphi_0]}(T)$). Then since $g''_1$ is also $(K[g_0], \text{Col}(\omega, K^\gamma))$-generic, by what was established above, we know that $T_z$ is illfounded. But note that $i_D^{K[g_0]}, T_z \subseteq T'_z$, so $T'_z$ is illfounded. Since $\kappa < i_D^{K[g_0]}(\kappa)$, it follows by elementarity that there is $\xi < \kappa$ such that in $K[g_0]$, Col$(\omega, K^\gamma)$ forces that if $z'$ is the canonical real coding $K[\xi+K]$, then $T'_z$ (defined over $K[g_0][h]$ from $T$, where $h = (K[g_0], \text{Col}(\omega, K^\gamma))$) is illfounded. But since $\xi < \kappa$, the claim therefore follows from clause 1 of Claim 1. \hfill \Box

Fix $\theta < \xi < \kappa$ and $N$ witnessing Claim 3, with $N \in K[g_0][g_1]$ where $g_1$ is generic over $K[g_0]$ for Col$(\omega, K^\gamma)$. So $N^{PV} = K[\theta^+\mathbb{N}]$ and $N$ is good and active. Let $g = (g_0, g_1)$.

Working in $K[g]$, where there is no proper class inner model with a Woodin cardinal, using [6], we can build $K(N)$ (the core model over $N$), which is a proper class $N$-premouse which is iterable for set-sized trees. Since $N$ is good, $K(N)$ is equivalent to a proper class premouse $W$ such that $N \subset W$ and $W$ is iterable for set-sized trees (not just above $N$). (The latter is a straightforward reflection and consequence of goodness; just take an countable elementary substructure of a putative failure of iterability, containing the relevant $Q$-structures.)

CLAIM 4. Work in $K[g]$. Then $W$ has the properties claimed by the theorem.

Proof. Because $K,W$ are both fully iterable in $K[g]$, we can compare them through length OR, or until the comparison $(\mathcal{T}, \mathcal{U})$ terminates, if earlier. If $\text{lh}(\mathcal{T}, \mathcal{U}) < \text{OR}$ then clearly $M^g_{\mathcal{T}} = M^g_{\mathcal{U}}$ and there are no drops on $b^T, b^U$. Suppose $\text{lh}(\mathcal{T}, \mathcal{U}) = \text{OR}$. Since $K$ is externally iterable and $\mathcal{T}$ is correct, we can externally fix a $\mathcal{T}$-cofinal wellfounded branch $b$ (or $T$ uses only set-many extenders in $K[g]$). By the smallness hypothesis,
$M(T, U) \models \text{“OR is not Woodin”}$; in other words, $M(T, U)$ is the Q-structure for itself. Now working in $\mathcal{J}(K[g])$, where OR$^K$ is inaccessible, we can continue the comparison, using this Q-structure to determine the $T$- and $U$-cofinal branches $b, c$ as usual (see the proof of [19, Lemma 2.1] for such a calculation). We claim that $i_b^U = \text{OR}^K \subseteq \text{OR}^K$ and $i_c^U = \text{OR}^K \subseteq \text{OR}^K$. For by the usual weasel argument in $\mathcal{J}(K[g])$, either $i_b^U = \text{OR}^K \subseteq \text{OR}^K$ or $i_c^U = \text{OR}^K \subseteq \text{OR}^K$. But then by weak covering for $K(N)$ in $K[g]$, and since we are working in $K[g]$ (or $\mathcal{J}(K[g])$), it follows that there is $C \in \mathcal{J}(K[g])$ such that $C \subseteq \text{OR}^K$ is club, and for every $\eta \in C$ which is a singular strong limit cardinal of $K$, we have $\eta + M(T, U) = \eta + K$. So the usual calculations therefore give what was claimed.

As a corollary, we get a mouse which is not an iterate of its core:

11.4 Corollary. Let $K$ be as in Theorem 11.3 and countable, and suppose also that $K$ is pointwise definable. Let $W$ witness the theorem. Then $\mathcal{J}(K), \mathcal{J}(W)$ are premice, $\mathcal{J}(K)$ is sound with $\rho_1^{\mathcal{J}(K)} = \omega$, $\mathcal{J}(K), \mathcal{J}(W)$ are both $(0, \omega_1 + 1)$-iterable, $\mathcal{C}_1(\mathcal{J}(W)) = \mathcal{J}(K)$ but $\mathcal{J}(W) \neq \mathcal{J}(K)$, and letting $\gamma$ be the index of least disagreement between $\mathcal{J}(W), \mathcal{J}(K)$, then $\mathcal{J}(W)|_\gamma$ is active and $\gamma = \theta + \mathcal{J}(K)$ for some $\theta$. Therefore $\mathcal{J}(W)$ is not a 0-maximal iterate of $\mathcal{J}(K)$.

Proof. $\mathcal{J}(K)$ is a sound premouse with $\rho_1^K = \omega$ and $\rho_1^K = \{\text{OR}^K\}$ because $K \models \text{ZFC}$ and is pointwise definable. And $\mathcal{J}(W)$ is a premouse because $W \models \text{ZFC}$. Moreover, 0-maximal trees on $\mathcal{J}(K)$ are equivalent to those on $K$, and hence $\mathcal{J}(K)$ is $(0, \omega_1 + 1)$-iterable. So by [25], $\mathcal{J}(K)$ is in fact $(0, \omega_1, \omega_1 + 1)^*$-iterable. And 0-maximal trees on $\mathcal{J}(W)$ are also equivalent to those on $W$. So by the theorem, there is a successful comparison $(T', U')$ of $\mathcal{J}(K), \mathcal{J}(W)$, and we get $i_{T'} : \mathcal{J}(W) \rightarrow M_{\infty}^{T'} = M_\infty^{T'}$. So $\mathcal{J}(W)$ is $(0, \omega_1 + 1)$-iterable, and $\mathcal{C}_1(\mathcal{J}(W)) = \mathcal{C}_1(M_{\infty}^{T'}) = \mathcal{J}(K)$. But since there is $\theta$ as in Theorem 11.3, we are done.

11.2 Projectum-finite generation

In this section we prove a key lemma toward the proof of the following theorem; recall that almost-above was defined in Definition 1.3. We will rely on [24, §2], and also the methods used in §9 of the present paper, with both of which the reader should be familiar.

11.5 Theorem (Projectum-finite-generation). Let $m < \omega$ and $M$ be an $m$-sound, $(m, \omega_1 + 1)$-iterable premouse. Suppose that

$$M = \text{Hull}_{m+1}^M(\rho_{m+1}^M \cup \{x\})$$

for some $x \in M$. Then there is a successor length $m$-maximal tree $T$ on $\mathcal{C}_{m+1}(M)$, such that $b^T$ does not drop in model or degree, $T$ is strongly finite and almost-above $\rho_{m+1}^M$, and $M = M_{\infty}^T$. Moreover, $i_{T_{\infty}}$ is just the core map.

The lemma just says that the theorem holds assuming also that $M$ is $(m+1)$-universal and $\mathcal{C}_{m+1}(M)$ is $(m + 1)$-solid. So together with Theorem 14.1, it immediately yields the theorem.
11.6 Lemma. Adopt the hypotheses of Theorem 11.5. Suppose further that $M$ is $(m+1)$-universal and $\mathcal{C}_{m+1}(M)$ is $(m+1)$-solid. Then the conclusion of Theorem 11.5 holds.

The lemma has the following immediate corollary, using standard calculations:

11.7 Corollary. Adopt the hypotheses of Corollary 11.4. Suppose further that $M$ is $(m+1)$-universal and $\mathcal{C}_{m+1}(M)$ is $(m+1)$-solid. Then $A$ is $\sum_{m+1}(M)$.

11.8 Remark. Our proof of solidity (Theorem 14.1) will use Corollary 11.7, which follows immediately from Lemma 11.6. Given iterability for stacks, one can directly prove the conclusion of Corollary 11.7 (in fact even Corollary 1.4, which is just like Corollary 11.7 but without the extra $(m+1)$-universality and $(m+1)$-solidity hypotheses) by very standard arguments. It seems tempting to try to prove Corollary 11.7 (from only normal iterability) via a bicephalus argument, but otherwise using standard calculations, and hence avoiding the extra effort involved in our proof of Lemma 11.6. However, the author has not been successful in this, as a difficulty arises in one special case. We will describe this more carefully later, in Remark 11.15.

The plan for the proof of Lemma 11.6 is as follows: it is motivated by the classical use of bicephali such as in [9, §9], and also the use of bicephali in [24]. We may assume $M$ is countable. Let $C = \mathcal{C}_{m+1}(M)$ and $\rho = \rho_{m+1}^M$. Then $B = (C, M, \rho)$ is a degree $(m, m)$ bicephalus. We will establish that $B$ is iterable, and compare it with itself, producing trees $T, U$ on $B$. The comparison will proceed roughly via selecting extenders for least disagreement, but at some stages of comparison, for example when $B^M_\alpha$ is a bicephalus, this can be ambiguous, since we might consider either $M^\mathsf{Ord}_\alpha$ (the model above $C$) or $M^{\mathsf{Ord}_\alpha}$ (that above $M$) for least disagreement with the model(s) $C^T = M^\mathsf{Ord}_\alpha$ and/or $M^{\mathsf{Ord}_\alpha} = M^\mathsf{Ord}_\alpha$ from $T$ (whichever are defined). Likewise with $T, U$ exchanged. We will make the rules for deciding from which models we choose the least disagreeing extenders precise, and arrange them so that the comparison can only terminate in a useful fashion. We want to see that one of the comparison trees, say $T$, uses no extenders, and $U$ can be translated to an essentially equivalent tree $U'$ on $C$ witnessing the statement of the lemma (so $U'$ is strongly finite, $M^{\mathsf{Ord}_\alpha}_\infty = M$, etc.). Now the comparison will terminate for essentially the usual reasons. By the material in §5, if $B^M_\alpha$ is a bicephalus then it has fine structural properties analogous to those of $B$. Using the rules of the comparison and an variant of the proof of Lemma 9.2, will show that the comparison reaches a stage $\alpha$ at which in one of the trees, say $U$, we have a bicephalus $B^M_\alpha = (C^M_\alpha, M^\mathsf{Ord}_\alpha, \rho^M_\alpha)$, with $C^M_\alpha \leq M^\mathsf{Ord}_\alpha$ for some $e \in \{0, 1\}$, the exit extenders $E^M_\alpha$ and $\hat{E}^M_\alpha$ are chosen by least disagreement between $M^\mathsf{Ord}_\alpha$ and $C^M_\alpha \leq M^\mathsf{Ord}_\alpha$, and the tail end $(T, U) | [\alpha, \infty)$ of $(T, U)$ after stage $\alpha$ has essentially the form with respect to $B^M_\alpha$ that we want to show that $(T, U)$ has with respect to $B$. (In particular, $U | [\alpha, \infty)$ is trivial, and $T | [\alpha, \infty)$ translates to a "tree" $T'$ on $C^M_\alpha$ (recall $C^M_\alpha \leq M^\mathsf{Ord}_\alpha$) which is strongly finite, $M^\mathsf{Ord}_\alpha = M^\mathsf{Ord}_\alpha$, etc.)\(^{59}\) In particular, $\text{lh}(T, U) = \alpha + n + 1$ for some $n < \omega$. So if $\alpha = 0$, we would be done. But if instead $\alpha > 0$, we will show that we can pull back the form of $(T, U) | [\alpha, \alpha + n]$ to $(T, U) | [0, n]$.

\(^{59}\)The reason for the qualifier "essentially" and the quotation marks around "tree" is that it seems we might not know the resulting "tree" actually has fully wellfounded models, at least in a certain special case, in which that "tree" might use extenders overlapping $\rho^M_\alpha$. However, this illfoundedness could only occur up strictly above where the least disagreement occurs, so it wouldn’t cause a problem. In the end, we will instead describe this "tree" as a (real) tree on a certain phalanx, avoiding any such possibility of illfoundedness.
It proceeds roughly as follows. Let $G$ be the $(\rho, \rho'^M)$-extender derived from $1^{\omega_3M}_{\omega_0}$. Let $\gamma$ be least such that $M|\gamma \neq C|\gamma$. The calculation will show that
\[ Ult_0(M|\gamma, G) \subseteq M^M_\alpha \]
and
\[ Ult_0(C|\gamma, G) \subseteq C^M_\alpha, \]
and that these two structures have the same ordinal height and are distinct, and therefore they form the least disagreement between $M^M_\alpha$ and $C^M_\alpha$. But as mentioned earlier, we had $C^M_\alpha \leq M^T_\alpha$, and $(E^M, E^T_\alpha)$ constitute the least disagreement between $(M^M_\alpha, M^T_\alpha)$, and so exit$^M_\alpha = Ult_0(M|\gamma, G)$ and exit$^T_\alpha = Ult_0(C|\gamma, G)$. But since we already know $U|\alpha, \infty$ uses no extenders and $E^T_\alpha \neq \emptyset$, it follows that $F^M|\gamma = \emptyset$ and $F^C|\gamma \neq \emptyset$, and that the fine structural properties of $C|\gamma$ are closely related to those of exit$^T_\alpha$. This all generalizes to a similar situation at stages $i \in [0, n)$ (with slight wrinkles foreshadowed in Footnote 59). From this correspondence between $(T, U)|[0, n)$ and $(T, U)|[\alpha, \alpha + n]$, we will conclude that actually $(T, U)$ is as desired, and $\alpha = 0$, a contradiction.

Before beginning the proof of the lemma, we will describe the methods we will use to establish lines (7) and (8) above. The calculation is a generalization of the methods used in the analysis of comparisons in [24, §4], and in particular of how stages of comparison lift under ultrapower maps; this kind of analysis of preservation of comparison (under ultrapower maps) is also developed further in [18, §8]. A key component of the calculation (Lemma 11.11 below) involves arguments with condensation much like those in [24, §4], and which were worked out more generally by Steel, and then sharpened independently by Steel and the author (see [18]). These calculations are integral within full normalization [18]. We first recall a basic definition from [9], though slightly modified:

11.9 Definition. Let $M$ be an $m$-sound premouse and $\gamma \leq OR^M$. The extended drop-down sequence of $((M, m), \gamma)$ is the sequence $((M_i, m_i))_{i<n}$ where $(M_0, m_0) = (M|\gamma, 0)$ and for $i < n$, $(M_{i+1}, m_{i+1})$ is the lexicographically least $(M', m')$ such that $(M_i, m_i) \leq (M', m') \leq (M, m)$ and either $(M', m') = (M, m)$ or $\rho^M_{m'+1} < \rho^M_{m_i+1}$, and moreover, $n < \omega$ is as large as possible under these conditions.

11.10 Remark. Note $(M, m) = (M_n, m_n)$ by definition. Write $\rho^M_{m+1} = 0$, and $\rho^K_k = \rho^K_k$ for other $K, k$. Let $i < n$. Note that if $M_i, m_i + 1$ then $\theta = \rho^M_{m_i + 1} = \rho^M_{m_i}$ is an $M_{i+1}$-cardinal and $\rho^M_{m_{i+1} + 1} < \theta \leq \rho^M_{m_{i+2}}$. And if $M_i = M_{i+1}$ then again $\rho^M_{m_{i+1} + 1} < \rho^M_{m_{i+1}} = \rho^M_{m_{i+2}}$. It can however be that $\rho^M_{m_i + 1} = \rho^M_{m_i}$, but by the preceding remarks, this implies $i = 0 < n$ (so $m_i = m_0 = 0$) and $\rho^M_0 = \rho^M_1 = \rho^M_0$ (so $M_0$ is passive or type 3).

For $m < \omega$ and $m$-sound premouse $M$, suitable condensation below $(M, m)$ is defined in [24, Definition 2.10]. By [24, Lemma 2.11], this notion is preserved by iteration maps at the degrees we need it, and by [24, Lemma 2.14] (or alternatively, the finer [24, Theorem 5.2]), it is a consequence of $(m, \omega_1 + 1)$-iterability (in fact $(m - 1, \omega_1 + 1)$-iterability if $m > 0$).

Some form of the following lemma is due to Steel.\footnote{Components of the argument were shown in [24], in particular [24, Lemma 3.17(11)]. Steel showed that $U_0 \leq N$ under the circumstances of the lemma. The full lemma as stated here was then observed by the author and Steel independently, but it comes out quite readily from Steel’s proof that $U_0 \leq N$.} The proof uses an iterated version of the proof of [24, Lemma 3.17(11)]:
11.11 Lemma (Dropdown lifting). Let $M$ be an $m$-sound premouse. Suppose suitable condensation holds below $(M, m)$. Let $\rho < \text{OR}^M$ be an $M$-cardinal with $\rho \leq \rho^M_m$. Let $(M_\alpha)_{\alpha \leq \lambda}, (E_\alpha)_{\alpha < \lambda}$ be a wellfounded degree $m$ abstract iteration of $M$ with abstract iteration maps $j_{\alpha \beta} : M_\alpha \to M_\beta$, where $\text{cr}(E_\alpha) < \sup j_{0\alpha} \rho$ for each $\alpha < \lambda$. Let $\pi = j_{0\lambda}$ and $N = M_\lambda$. Let $G$ be the $(\rho, \sup \pi^\rho)$-extender over $M$ derived from $\pi$.

Let $\rho \leq \gamma \leq \text{OR}^M$ and $((M_i, m_i))_{i \leq n}$ be the extended dropdown sequence of $((M, m), \gamma)$. Let $U_i = \text{Ult}_{m_i}(M_i, G)$. Then $U_i \trianglelefteq N$ for each $i \leq n$, and $\langle (U_i, m_i) \rangle_{i \leq n}$ is the extended dropdown sequence of $((N, m), \text{OR}^U)$.

Proof. By definition, $(M_i, m_i) = (M, m)$, and $U_n = \text{Ult}_m(M, G)$, so $(U_n, m)$ is the terminal element of the extended dropdown sequence of $((N, m), \text{OR}^U)$.

Now suppose $U_{i+1} \trianglelefteq N$ where $i < n$, and we will show the following, writing $\bar{\rho}^M_{m+1} = \bar{\rho}^N_{m+1} = 0$ and $\bar{\rho}^K_{k+1} = \rho^K_{k+1}$ for other $K, k$:

CLAIM. We have:

1. $U_i \trianglelefteq U_{i+1}$,
2. if $U_i = U_{i+1}$ then $\bar{\rho}^{U_{i+1}+1}_{m+1} < \bar{\rho}_{m+1} = \bar{\rho}_{m+1}^U = \bar{\rho}_{m+1}^U$,
3. if $U_i \neq U_{i+1}$ then $\bar{\rho}^{U_{i+1}}_{m+1} < \bar{\rho}^{U_{i}}_{m+1} = \bar{\rho}^U_{m+1}$ and $\theta$ is a cardinal of $U_{i+1}$,
4. if $\bar{\rho}_{m+1}^M < \bar{\rho}_{m+1}^M$ then $\bar{\rho}_{m+1}^U < \bar{\rho}_{m+1}^U$,
5. if $\bar{\rho}_{m+1}^M = \bar{\rho}_{m+1}^M$ then $\bar{\rho}_{m+1}^U = \bar{\rho}_{m+1}^U$ (see Remark 11.10).

Proof.

CASE 1. $A_{i+1} = M_i$.

Let $A = M_{i+1} = M_i$. We have then $m_{i+1} > m_i$ and $\bar{\rho}^A_{m_{i+1}+1} < \bar{\rho}^A_{m_i+1}$.

For $m_i \leq k < m_{i+1}$, let $B_k = \text{Ult}_k(A, G)$ and $i_k = i_{A, k} : A \to B_k$. For $m_i \leq k < m_{i+1}$ let $\sigma_{k, i+1} : B_k \to B_{k+1}$ be the natural factor map, so $\sigma_{k, i+1} \circ i_k = i_{k+1}$ and $\sigma_{k, i+1}$ is $\bar{\rho}_{k+1}$-preserving $k$-lifting. Almost as in the proof of [24, Lemma 3.17(11)], we have $B_k \trianglelefteq B_{k+1}$. For self-containment, here is the reason: Using Corollary 3.9, both $B_k, B_{k+1}$ are $(k + 1)$-sound with $\rho^A = \sup i_k \circ \bar{\rho}^A_{k+1}$ and $\rho^B_{k+1} = \sup i_{k+1} \circ \bar{\rho}^A_{k+1}$.

and $\rho^B_{k+1} = \sup i_{k+1} \circ \bar{\rho}^A_{k+1}$. Let $\sigma = \sigma_{k, i+1}$. Then part 1 holds in this case as:

1. If $\sigma^B_{k+1}$ is unbounded in $\bar{\rho}^{B_{k+1}}$, then by [24, Lemma 2.4], $\sigma$ is a $k$-embedding, which implies $\sigma = \text{id}$ and $B_k = B_{k+1}$.

2. If $\sigma^B_{k+1}$ is bounded in $\bar{\rho}^{B_{k+1}}$, then by [24, Lemma 2.4], suitable condensation applies, and as $\rho'$ is a cardinal in $B_{k+1}$, we get $B_k \trianglelefteq B_{k+1}$.

Now $\bar{\rho}^A_{m_{i+1}+1} < \bar{\rho}^A_{m_i+1}$, so

$$\bar{\rho}^{U_{i+1}}_{m+1} = \sup i_{m+1} \circ \bar{\rho}^A_{m_{i+1}+1} < \sup i_{m+1} \circ \bar{\rho}^A_{m_i+1} = \bar{\rho}^{U_{i+1}}_{m_i+1}.$$
and since $\theta = \rho_{m+1}^A = \ldots = \rho_{m+1}^A$, $i_k(\theta)$ is independent of $k \in \{m,m+1\}$, so

$$U_{i+1}^I / \rho_{m+1}^A < U_{m+1}^I = B_{m+1}^I = \rho_{m+1}^A = \ldots = \rho_{m+1}^A = \rho_{m+1}^A = \rho_{m+1}^A = U_{i+1}^I$$

and

$$\rho_{m+1}^A < \rho_{m+1}^A \iff \rho_{m+1}^A < \rho_{m+1}^A.$$  

If $\rho_{m+1}^A = \rho_{m+1}^A$ then also $i = 0 = m_0$, by Remark 11.10.

So if $U_{i+1} = U_{i+1}$ then we get part 2 as needed, and if $U_{i+1} < U_{i+1}$ then note that $\rho_{m+1}^A = \rho_{m+1}^A$ is a cardinal of $U_{i+1}$, so $\rho_{i+1}^A = \rho_{m+1}^A$, giving part 3. Parts 4 and 5 also follow easily.

**Case 2.** $M_i < M_{i+1}$.

So $\theta = \rho_{m+1}^M$ is an $M_{i+1}$-cardinal, $\rho_{m+1}^A < \theta \leq \rho_{m+1}^M$, and either:

- $\rho_{m+1}^M > \theta < \rho_{m+1}^M$ or
- $i = 0 = m_0$, $\rho_0^M = \rho_1^M = \rho_0^M = \theta$ and $M_0$ is passive or type 3.

If $M_{i+1}$ is active let $\psi : U_{i+1}(M_{i+1}, F M_{i+1}) \rightarrow U_{i+1}(U_{i+1}, F U_{i+1})$ be given by the Shift Lemma with $j = i_{G}^{M_{i+1},M_{i+1}}$; otherwise let $\psi = j$. Note $M_i \in \text{dom}(\psi)$ and $\psi \downarrow \theta = j \downarrow \theta$.

Let

$$\sigma : U_i = \text{Ult}_{m_i}(M_i,G) \rightarrow \psi(M_i)$$

be the natural factor map. Note that $\rho_{m+1}^U = \sup \psi\theta$ is a cardinal in $U_{i+1}$, because if $\mu^{M_{i+1}} \leq \theta < \rho_{m+1}^M$ then $\rho_{m+1}^M$ is $\sigma^{M_{i+1}}$-regular, and since and $\rho \leq \mu^{M_{i+1}}$ (recall $G$ is the $(\rho, \sup \pi^\gamma)$-extender derived from $\pi$), therefore $j$ is continuous at $\mu^{M_{i+1}}$. Also $\sigma \downarrow \rho_{m+1}^U = \text{id}$, and much as before, by elementarity, suitable condensation applies to $\sigma$, giving $U_i < U_{i+1}$. (In the case that $M_{i+1}$ is type 3 and $\psi(M_i) \notin U_{i+1}$, we have $\psi(M_i)|\text{OR}^{U_{i+1}} = U_{i+1}|\text{OR}^{U_{i+1}}$ by coherence, and we get $U_i \triangleleft \psi(M_i)$ by condensation, but $\rho_{m+1}^U < \text{OR}^{U_{i+1}}$, so $U_i < U_{i+1}$.) The desired properties now easily follow. □

This completes the proof of the claim, and the theorem is an easy consequence. □

We can now begin the main argument of this section.

**Proof of Lemma 11.6.** We may assume that $M$ is not $(m+1)$-sound. Let $\rho = \rho_{m+1}^C = \rho_{m+1}^M$ and $\pi : C \rightarrow M$ the core map. So $C \neq M$ but $\rho^C \neq \rho^M$. Note $\rho < \rho_0^C \leq \rho_0^M$.

Define the bicephalus $B = (\rho, C, M)$, which is degree-maximally iterable, as there is an easy copying argument to lift trees on $B$ to trees on $M$ (there are such iterability proofs in [24]). For a degree-maximal tree $\mathcal{T}$ on $B$, recalling notation from §5, we write $C_\alpha^\mathcal{T} = M_\alpha^\mathcal{T}$ and $M_\alpha^\mathcal{T} = M_\alpha^T$ and $i_{\alpha\beta}^T = i_{\alpha\beta}^T$ and $j_{\alpha\beta}^T = j_{\alpha\beta}^T$ etc, and $\rho_\alpha^\mathcal{T}$ for $(\rho_\alpha^T) + C_\alpha^\mathcal{T} = (\rho_\alpha^T) + M_\alpha^\mathcal{T}$.

We compare $B$ with itself, producing padded degree-maximal trees $\mathcal{T}, \mathcal{U}$ respectively, much as in the bicephalus and cephalanx comparisons in [24]. The comparison proceeds

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One might also consider comparing $B$ with $C$, or $B$ with $M$. Say we produce trees $\mathcal{T}$ on $B$ and $\mathcal{U}$ on $C$ or on $M$. If we are comparing $B$ with $C$, it seems we might reach a stage $\alpha$ such that $(0, \alpha]^T$ does not drop and $C_\alpha^\mathcal{T} \triangleleft B_\alpha^\mathcal{T}$, and so the comparison terminates at that point, without a useful outcome. If instead we are comparing $B$ with $M$, then we might reach a stage $\alpha$ with $\alpha \in \mathcal{B}^T$ and $(0, \alpha]^T$ does not drop. $\rho_\alpha^T = \rho_0^M(M_\alpha^T)$ and $M_\alpha^T \triangleleft (\rho_\alpha^T)^+M_\alpha^T = M_\alpha^T \triangleleft (\rho_\alpha^T)^+M_\alpha^T$, but $M_\alpha^T \neq M_\alpha^T$. If we then move into $C_\alpha^\mathcal{T}$ in $\mathcal{T}$, the comparison might terminate above $C_\alpha^\mathcal{T}$ and above $M_\alpha^T$, but this does not seem to yield relevant information about $B$. If instead we move into $M_\alpha^T$ in $\mathcal{T}$, the comparison might terminate above $M_\alpha^T$ and $M_\alpha^T$, but this also does not seem to yield useful information, at least not directly.
basically via least disagreement. However, there are two caveats, just like in [24]. The first is that when selecting extenders $E$ by least disagreement, we must also minimize on $\nu(E)$ before proceeding, as described in Remark 3.1. Secondly, when there are multiple models available on one (or both) side(s), we have to take some care in determining exactly how to proceed, also as in [24]. There can be stages $\alpha$ at which we have a bicephalous $B^T_\alpha$ in $T$ say, and we decide that we want to make sure that the next extender used in $T$ comes from some particular model of $B^T_\alpha$, say $C^T_\alpha$. In this case, we pad in both trees at this stage, setting $E^T_\alpha = \emptyset = E^U_\alpha$, but restrict the available models in $T$ to include only $C^T_{\alpha + 1} = C^T_\alpha$ at stage $\alpha + 1$, and we say that in $T$ we move into $C^T_\alpha$ at stage $\alpha$. We can similarly move into $M^T_\alpha$ in $T$. Likewise in $U$. To facilitate this notationally, we will keep track of sets $S^T_\alpha$, $S^U_\alpha$, $A^T_\alpha$, $A^U_\alpha$, specifying the models available at stage $\alpha$. Here $\emptyset \neq S^T_\alpha \subseteq \text{sides}^T_\alpha$ and $A^T_\alpha = \{ M^T_\alpha \mid e \in S^T_\alpha \}$, and likewise for $S^U_\alpha$, $A^U_\alpha$.

We will only ever take exit extenders from available models. Formally, this means that if $S^T_\alpha = \{ n \}$ and $E^T_\alpha \neq \emptyset$ then exitside$^T_\alpha = n$. Moreover, once we have restricted the set of available models, we may not remove this restriction, except by using an exit extender from that model; formally, this means that if $E^T_\beta = \emptyset$ for all $\beta \in [\alpha, \gamma)$ then $S^T_\gamma \subseteq S^T_\alpha$. If $E^T_\alpha \neq \emptyset$ and $\alpha + 1 \in \text{trees}^T$, however, then both models are automatically available at stage $\alpha + 1$ in $T$, i.e. $S^T_{\alpha + 1} = \{ 0, 1 \}$. Likewise for a limit $\eta \in \text{trees}^T$ such that $T$ uses extenders cofinally $< \eta$. (As with standard comparison, there can also be stages $\alpha$ where $E^T_\alpha \neq \emptyset = E^U_\alpha$, or vice versa.) If $E^T_\alpha \neq \emptyset$ and $S^T_\alpha = \{ 0, 1 \}$, then it will actually be that $E^T_\alpha \in E(C^T_\alpha) \cap E(M^T_\alpha)$, and in fact $\text{lh}(E^T_\alpha) < \rho^T_\alpha$; in this case we always set exitside$^T_\alpha = 0$. Once we have specified how to select exitside$^T_\alpha$ and $E^T_\alpha$, the remaining features of $T \cup (\alpha + 2)$ are determined by the rules for degree-maximality (Definition 5.7) and Remark 1.10. Likewise for $U$.

We now explain the rules for the comparison. The basic point is that, at stage $\alpha$, if we have $S^T_\alpha = \{ 0, 1 \}$, and we decide whether to move into $C^T_\alpha$ or $M^T_\alpha$ at stage $\alpha$, we choose $C^T_\alpha$ unless this would prematurely end the comparison. Similarly in $U$, we prefer $C^U_\alpha$ over $M^U_\alpha$. If $S^T_\alpha = \{ 0, 1 \} = S^U_\alpha$, we will only move into a model on both sides at stage $\alpha$ if $B^T_\alpha \parallel \rho^T_\alpha = B^U_\alpha \parallel \rho^U_\alpha$. In this case, if $C^T_\alpha = C^U_\alpha$, then we either (i) move into $C^T_\alpha = C^U_\alpha$ in $T$ and $M^T_\alpha$ in $U$, or (ii) move into $C^T_\alpha = C^U_\alpha$ in $U$ and $M^T_\alpha$ in $T$ (randomly choosing one option, for symmetry). Formally, this is executed as follows. We will construct degree-maximal trees $T$, $U$. We start with $B^T_0 = B = B^U_0$ and $S^T_0 = \{ 0, 1 \} = S^U_0$. Suppose we have $(T, U) \cup (\alpha + 1)$ and $S^T_\alpha$, $S^U_\alpha$.

**Case 1.** $S^T_\alpha = \{ 0, 1 \} = S^U_\alpha$ and $B^T_\alpha \parallel \rho^T_\alpha = B^U_\alpha \parallel \rho^U_\alpha$ where $\rho_\alpha = \min(\rho^T_\alpha, \rho^U_\alpha)$.

We set $E^T_\alpha = \emptyset = E^U_\alpha$, and will move into some model(s).

**Subcase 1.1.** $\rho_\alpha = \rho^T_\alpha = \rho^U_\alpha = \rho^T_\alpha + \rho^U_\alpha$ and $C^T_\alpha = C^U_\alpha$.

Either: In $T$ move into $C^T_\alpha$, and in $U$ move into $M^U_\alpha$; or, in $T$ move into $M^T_\alpha$, and in $U$ move into $C^T_\alpha$ (irrespective of whether $M^T_\alpha = M^U_\alpha$).

**Subcase 1.2.** $\rho_\alpha = \rho^T_\alpha + \rho^U_\alpha$, and $C^T_\alpha \neq C^U_\alpha$.

In $T$ move into $C^T_\alpha$, and in $U$ move into $C^U_\alpha$.

(Note here that $C^T_\alpha \not\leq C^U_\alpha \not\leq C^T_\alpha$.)

**Subcase 1.3.** $\rho_\alpha = \rho^T_\alpha + \rho^U_\alpha$ and $C^T_\alpha \neq C^U_\alpha$.

In $T$ move into $M^T_\alpha$. There is no change in $U$.

**Subcase 1.4.** $\rho_\alpha = \rho^T_\alpha + \rho^U_\alpha$, and $C^T_\alpha \not\leq C^U_\alpha \not\leq \rho^T_\alpha + \rho^U_\alpha$.

In $T$ move into $C^T_\alpha$. There is no change in $U$.
**Subcase 1.5.** $\rho_+ = \rho_+^{\alpha T} < \rho_+^{\alpha T}$.

By symmetry with Subcases 1.3 and 1.4.

**Case 2.** $S_0^T \not\{0, 1\} \not\mathcal{S}$.

Select extenders by least disagreement, if possible. (That is, letting $S_0^T = \{d\}$ and $S_0^\alpha = \{e\}$, if $M_0^\alpha \leq M_0^\alpha$ or vice versa, then the comparison terminates.

Otherwise proceed by least disagreement between these models, in the manner described in Remark 3.1. If $E_0^\alpha \not\emptyset$ set $S_0^\alpha = d$, and likewise for $\mathcal{U}$, e.)

**Case 3.** $S_0^T \not\{0, 1\} \not\mathcal{S}$.

Select extenders by least disagreement, if possible. (But maybe $B_0^\alpha \not\emptyset B_0^\alpha || \rho_+^{\alpha T}$, in which case the comparison terminates.)

**Case 4.** $S_0^T \not\{0, 1\} \not\mathcal{S}$ and $B_0^\alpha || \rho_+^{\alpha T} = B_0^\alpha || \rho_+^{\alpha T}$.

We set $E_0^\alpha = E_0^\alpha = \emptyset$ and move into some model, according to subcases.

**Subcase 4.1.** $C_0^T \not\subset B_0^\alpha$.

In $\mathcal{T}$, move into $M_0^\alpha$ (so set $S_0^\alpha = \{1\}$).

**Subcase 4.2.** $C_0^T \not\subset B_0^\alpha$.

In $\mathcal{T}$, move into $C_0^T$ (so set $S_0^\alpha = \{0\}$).

**Case 5.** $S_0^T \not\{0, 1\} \not\mathcal{S}$.

By symmetry with Cases 3 and 4.

**Case 6.** $S_0^T = \{0, 1\} \not\mathcal{S}$ and $B_0^\alpha || \rho_+ \not\emptyset B_0^\alpha || \rho_+^{\alpha T}$ where $\rho_+ = \min(\rho_+^{\alpha T}, \rho_+^{\alpha T})$.

Select extenders by least disagreement.

This completes all cases. The rules for degree-maximality (Definition 5.7), along with Remark 1.10 regarding padding, now determine $(\mathcal{T}, \mathcal{U}) \not\{0, 1\} \not\mathcal{S}$. (In particular, if $E_0^\alpha \not\emptyset$ then $\text{pred}^\alpha (\alpha + 1)$ is the least $\beta$ such that $E_0^\beta \not\emptyset$ and $\text{cr}(E_0^\alpha) < \nu(E_0^\beta)$.) If $E_0^\alpha = \emptyset$ and we do not move into a model at stage $\alpha$ in $\mathcal{T}$, then $S_{0+1}^\alpha = S_0^\alpha$. If $E_0^\alpha \not\emptyset$ then $S_{0+1}^\alpha = \text{side}_0^\alpha$. Given $(\mathcal{T}, \mathcal{U}) \not\eta$ where $\eta$ is a limit, we extend to $(\mathcal{T}, \mathcal{U}) \not\eta$ using our iteration strategy for $B$. If $\mathcal{T} \not\eta$ is eventually only padding then $S_0^\alpha = \lim_{\alpha < \eta} S_0^\alpha$.

If $T \not\eta$ not eventually only padding then $S_0^\alpha = \text{side}_0^\alpha$. Likewise for $\mathcal{U}$.

If $\alpha \in \mathcal{B}^\alpha$ but $S_0^\alpha = \{\alpha\}$, let movin$^\alpha (\alpha)$ be the largest $\beta < \alpha$ such that $S_0^\beta = \{0, 1\}$ (so in $\mathcal{T}$ we move into $M_0^\beta T = M_0^\beta$ at stage $\beta$). Likewise for $\mathcal{U}$.

**Claim 1.** The following properties hold of $\mathcal{T}$, and likewise for $\mathcal{U}$:

1. If $\alpha + 1 \in \mathcal{B}^\alpha$ then $E_0^\alpha$ is weakly amenable to $B_0^\alpha$.

2. Let $\alpha \in \mathcal{B}^\alpha$. Then:
   (a) $\rho_0^\alpha = \rho_0^\alpha (C_0^\alpha) = \rho_0^\alpha (M_0^\alpha)$
   (b) $i_0^\alpha, j_0^\alpha$ are $\rho_0^\alpha$-preserving $m$-embeddings,
   (c) $M_0^\alpha$ is $(m + 1)$-solid and $(m + 1)$-universal but not $(m + 1)$-sound,
   (d) $C_0^\alpha = \mathcal{C}_m ^\alpha (M_0^\alpha)$ is $(m + 1)$-sound,
   (e) letting $\pi_\alpha : C_0^\alpha \rightarrow M_0^\alpha$ be the core map (so $\pi_0 = \pi$) and $\beta \leq \alpha$,
      $$\pi_\alpha \circ i_\beta^\alpha = j_\beta^\alpha \circ \pi_\beta,$$

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3. If $\alpha + 1 \notin \mathcal{R}^T$ then $E^T_\alpha$ is close to $B^*_\alpha+1$. Therefore (even if extenders applied to bicephali $B^*_\alpha$ are not close to $C^T_\alpha$) all iteration maps preserve fine structure as usual.

**Proof.** Part 2 follows from part 1 and Corollary 3.9. Part 3 uses a simple variant of the argument of [9, 6.1.5]. □

**Claim 2.** We have:

- $E^T_\alpha = \emptyset = E^d_\alpha$ iff we move into a model at stage $\alpha$ in some tree.
- If $S^T_\alpha = \{0, 1\}$ and $E = E^T_\alpha \neq \emptyset$ or $E = E^d_\alpha \neq \emptyset$, then $\text{lh}(E) < \rho^T_\alpha$. Likewise for $\mathcal{U}$.

Let $\alpha + 1, \beta + 1 < \text{lh}(T, \mathcal{U})$ with $E^T_\alpha \neq \emptyset \neq E^d_\beta$. Let $\nu = \min(\nu^T_\alpha, \nu^d_\beta)$. Then:

- If $\alpha = \beta$ then $\text{lh}(E^T_\alpha) = \text{lh}(E^d_\beta)$ and $\nu(E^T_\alpha) = \nu(E^d_\beta)$.
- In general, $E^T_\alpha \upharpoonright \nu \neq E^d_\beta \upharpoonright \nu$.

Hence, the comparison terminates.

**Proof.** This easily follows from the rules of comparison and as usual. □

**Claim 3.** We have:

1. If we move into $M^T_\alpha$ at stage $\alpha$ then $C^T_\alpha \preceq B^{M^T}_\alpha$, so at stage $\alpha + 1$ we select the least disagreement between $M^T_\alpha$ (in $T$) and $C^T_\alpha$ (in $\mathcal{U}$) (which exists).

2. If we move into $C^T_\alpha$ at stage $\alpha$ then for each $\beta > \alpha$ and $n \in S^d_\beta$, we have $C^T_\alpha \not\preceq M^{n,d}_\beta$. Likewise symmetrically.

**Proof.** Part 1 is just a matter of checking the definitions. Regarding part 2, consider, for example, the case that we move into $C^T_\alpha$ in $T$ in Subcase 1.4. Then because $\rho^T_\alpha < \rho^{n,d}_\alpha$ and either $\rho^{n,d}_\alpha$ is a cardinal of $C^d_\alpha$ or $\rho^{n,d}_\alpha = \text{OR}(C^d_d_\alpha)$, and $\rho^{n,d}_\alpha$ is a cardinal of $M^{d}_\alpha$, we have $C^T_\alpha \not\preceq C^d_\alpha$ and $C^T_\alpha \not\preceq M^{d}_\alpha$. And $C^T_\alpha \not\preceq M^{d}_\alpha$ as $C^T_\alpha$ is $(m+1)$-sound but $M^{d}_\alpha$ is not, and $C^T_\alpha \not\preceq C^{\beta}_\alpha$ because $\rho^{C^T_\alpha}_{m+1} \neq \rho^{C^T_\alpha}_{m+1}$. So suppose $\beta > \alpha$ is least such that $C^T_\alpha \preceq C^d_\beta$ or $C^T_\alpha \preceq M^{d}_\beta$. Then there is $\gamma \in (\alpha, \beta)$ with $E^d_\gamma \neq \emptyset$, and note then that $\text{lh}(E^d_\gamma) \geq \rho^T_\alpha$. But then since $\rho_{m+1}(C^T_\alpha) = \rho^{C^T_\alpha}_{m+1} < \text{lh}(E^d_\gamma)$, we have $C^T_\alpha \not\preceq C^d_\beta$ and $C^T_\alpha \not\preceq M^{d}_\beta$. So $C^T_\alpha = C^d_\beta$ or $C^T_\alpha = M^{d}_\beta$. But if $\beta \in \mathcal{R}^d$ then

$$\rho_{m+1}(M^{d}_\beta) = \rho_{m+1}(C^{d}_\beta) \geq \text{lh}(E^d_\gamma) > \rho_{m+1}(C^T_\alpha),$$

so $C^T_\alpha \not\in \{C^d_\beta, M^{d}_\beta\}$. So $\beta \not\in \mathcal{R}^d$; let sides$^d_\beta = \{n\}$, so $C^T_\alpha = M^{n,d}_\beta$. As $C^T_\alpha$ is $(m+1)$-sound, $\text{deg}_{n,d} \geq m + 1$ and $\text{lh}(E^d_\gamma) \leq \rho_{m+1}(M^{n,d}_\beta)$, a contradiction. The remaining cases are similar. □

\[^{62}\text{If } M \text{ is type 2 and } k = 0 \text{ and } \kappa = \text{cr}(F^M) \text{ and } \kappa^+ = \rho = \rho^M, \text{ then we do not yet know that } F^M \text{ is close to } C; \text{ in particular this case causes some obstacles to be dealt with.}\]
Let \( \xi + 1 = \lh(T, U) \). We say that the comparison terminates early if \( \xi = \beta + 1 \) and \( E^T = D^U \).

Claim 4. \( S^T_\xi \neq \{0, 1\} \neq S^U_\xi \).

Proof. By the comparison rules, \( S^T_\xi \neq \{0, 1\} \) or \( S^U_\xi \neq \{0, 1\} \). Suppose \( S^T_\xi = \{0, 1\} \) and \( S^U_\xi = \{n\} \), so Case 3 attains at stage \( \xi \) and \( M_\xi^U \triangleleft B^T_\xi \| \rho^+_\xi, \) so \( M^U_\xi \) is fully sound. Therefore \( \xi \in B^U \) and \( n = 0 \), so \( C_\xi^U \triangleleft B^T_\xi \). But \( C_\xi^U = C^U_\beta \) where \( \beta = \movin^U(\xi) \). This contradicts Claim 3. \( \square \)

So let \( \mathcal{A}^T_\xi = \{N^T_\xi\} \) and \( \mathcal{A}^U_\xi = \{N^U_\xi\} \), so \( N^T_\xi \subseteq N^U_\xi \) or vice versa. By Claim 3, it follows that \( N^T_\xi, N^U_\xi \) are non-sound and (let) \( N = N^T_\xi = N^U_\xi \).

Claim 5. Either:

- \( S^T_\xi = \{1\} \) and \( b^T \) does not drop in model or degree, or
- \( S^U_\xi = \{1\} \) and \( b^U \) does not drop in model or degree.

Proof. Standard fine structure shows that either \( b^T \) or \( b^U \) doesn’t drop in model or degree. We can assume \( b^T \) doesn’t. So suppose \( S^T_\xi = \{0\} \). Then by Claim 3, \( \xi \notin B^T \), and hence \( N = C^T_\xi \) is m-sound, but not \( (m+1) \)-sound. But then if \( S^U_\xi = \{0\} \) or \( b^U \) drops in model or degree (hence in model), then the core map \( \mathcal{E}_{m+1}(N) \rightarrow N \) is an iteration map of both \( T \) and \( U \), which contradicts comparison. \( \square \)

So we can assume that \( S^U_\xi = \{1\} \) and \( b^U \) does not drop in model or degree, so in \( U \) we moved into \( M^U_\lambda \) at stage \( \lambda = \movin^U(\xi) \), and \( \lambda \leq \xi \), and (let)

\[ k = i^U_\lambda : M_\lambda^U \rightarrow M_\xi^U = N. \]

By Claim 3 then, \( \mathcal{E}_{m+1}(N) = C^U_\lambda \subseteq B^T_{\lambda+1} \). But \( C^U_\lambda \neq N \), so letting \( \text{sides}_\xi^T = \{e\} \) and \( \eta + 1 = \succ^T(\lambda + 1, \xi) \), then

\[ j = i^{e^U}_\eta : M^{e^U}_\eta \rightarrow C^U_\lambda \rightarrow N = M^{e^U}_\xi \]

is the core map, and \( j \neq \text{id} \). Note \( k \circ \pi^U_\lambda = j \) (recall \( \pi^U_\lambda \) is the core map). We will show that \( U \upharpoonright [\lambda, \infty) \) is trivial and \( T \upharpoonright [\lambda, \infty) \) is essentially a strongly finite tree \( T' \) on \( C^U_\lambda \) which is almost-above \( B^T_\lambda \), with \( M^T_\infty = N = M^U_\lambda \) (so \( k = \text{id} \) and \( j = \pi^U_\lambda \)). This will take an argument like the proof of Lemma 9.2 (on measures in mice). For this we first need to establish the analogue of Lemma 8.6 (on strongly finite trees).

11.12 Definition. Let \( S \) be degree-maximal on \( B \), of successor length \( \zeta + 1 \). Let \( x \in \mathcal{C}_0(M^T_\zeta) \) and \( \rho' = \rho_{m+1}(M^T_\zeta) = \rho^M_\lambda \). Then we say that \( S \) captures \( (T, x, \rho' + 1) \) iff there is \( \sigma \) such that:

1. \( \lambda \in b^S \cap b^T \),
2. \( S \upharpoonright (\lambda + 1) = T \upharpoonright (\lambda + 1) \),
3. either \( \lambda = \zeta = \xi \) (so \( S = T \)) or:
- \( \lambda < \xi \) (note then that since \( \lambda < ^T \xi \), letting \( \eta + 1 = \text{succ}^T(\lambda, \xi) \), we have 
\( (\eta + 1, \xi] \cap \mathcal{G}_{\text{deg}} = \emptyset \),
- \( \lambda < \xi \) and letting \( \bar{\eta} + 1 = \text{succ}^S(\lambda, \xi) \), then \( \text{side}_{\bar{\eta} + 1}^S = \{ e \} \) and \( (\bar{\eta} + 1, \xi]^S \cap \mathcal{G}_{\text{deg}} = \emptyset \),
- \( M^{eS}_{\bar{\eta} + 1} = M^{eT}_{\bar{\eta} + 1} \) and \( \text{deg}_{\bar{\eta} + 1}^S = m = \text{deg}_{\bar{\eta} + 1}^T \).

4. \( \sigma : M^S_\xi \rightarrow M^T_\xi \) is an \( m \)-embedding with \( \{ x \} \cup (\rho' + 1) \subseteq \text{rg}(\sigma) \) and \( \sigma \circ \iota_{\bar{\eta} + 1, \xi}^{eS} = \iota_{\bar{\eta} + 1, \xi}^{eT} \) (so \( \text{cr}(\sigma) > \rho' \)).

11.13 Definition (Core-strongly-finite). Let \( S \) be degree-maximal on \( B \), of successor length \( \zeta + 1 \), with \( \zeta \notin \mathcal{B}^S \). Let \( \text{side}^S = \{ e' \} \) and \( m' = \text{deg}^S_e \). We say \( S \) is core-strongly finite iff for some \( \lambda', \zeta \in \text{OR} \) and \( n < \omega \), we have:
- \( \text{lh}(T) = \zeta + 1 \) and \( \zeta = \lambda' + n \),
- \( \lambda' < ^S \zeta \),
- either \( \text{side}^S \) drops in model or \( e' = 0 \),
- \( \mathcal{C}_{m' + 1}(M^e_{\xi'}^S) \subseteq M_{\lambda'}^{e'} \),
- \( \mathcal{C}_{d_\alpha}(E^S_\gamma) \) is finitely generated for all \( \gamma \leq^S d_\alpha \), for all \( \alpha \) with \( \lambda' < ^S \alpha + 1 \leq ^S \zeta \).

11.14 Definition. Let \( S \) be degree-maximal on \( B \). Let \( \chi < \chi' < \text{lh}(S) \) and \( d \in \text{side}^S_{\chi'} \). We say that \( \chi \) is \( (d, \chi') \)-transient (with respect to \( S \)) iff for some \( \eta \), we have:
- \( M_{\chi'}^d \) is active type 2,
- \( \chi = \text{pred}^S(\eta + 1) < ^S \eta + 1 \leq ^S \chi' \),
- \( (\eta + 1, \chi'] \cap ^S \) does not drop (so \( M_{\bar{\eta} + 1}^{dS} \) is active type 2 and \( \iota_{\bar{\eta} + 1, \chi'}^{dS} : M_{\bar{\eta} + 1}^{dS} \rightarrow M_{\chi'}^{dS} \)) and
- \( \text{cr}(\iota_{\bar{\eta} + 1, \chi'}^{dS}) = \text{lcm}(M_{\bar{\eta} + 1}^{dS}) \).

Note that if \( \chi \) is \( (d, \chi') \)-transient then, with notation as above, \( \text{side}^S_{\chi'} = \text{side}^S_{\bar{\eta} + 1} = \{ d \} \) and \( E^S_\chi = \mathcal{F}(M_{\bar{\eta} + 1}^{dS}) \). In particular, \( d \) is uniquely determined by \( \chi' \). We say \( \chi \) is \( \chi' \)-transient (for \( S \)) iff there is \( d \) such that \( \chi \) is \( (d, \chi') \)-transient.

We now state and prove the existence of core-strongly-finite trees on bicephali, capturing the kind of information which we need for the present argument:

CLAIM 6. Let \( \rho' = \rho^S_\lambda \) and \( x \in \mathcal{C}_0(M^S_\xi) \). Then there is a degree-maximal tree \( S \) on \( B \) such that:

(i) \( S \) captures \((T, x, \rho' + 1)\),
(ii) \( \text{side}^S \) is \( \geq \lambda \)-Dodd-nice, and
(iii) \( S \) is core-strongly finite.

\(^{63}\)Note that by the other conditions, \( \text{side}^S \) is \( \geq \lambda' \)-Dodd-nice, so \( \alpha \) is Dodd-nice (see Definition 7.17).
Proof. We will first observe that there is $S$ satisfying conditions (i) and (ii). By then minimizing as in the proof of Lemma 8.6, we will see that condition (iii) is also satisfied.

**Subclaim 6.1.** There is $S$ satisfying requirements (i) and (ii).

**Proof.** By Lemma 6.20, we can fix $S, \zeta, n, \Phi$ such that $n < \omega$, $\text{lh}(S) = \zeta + 1 = \lambda + n + 1$, $\Phi : S \rightarrow_{\lambda, \text{sim}} T$ (see 6.15), and $S$ captures $(T, x, \rho' + 1)$.

So it suffices to see that $b^S$ is $\geq \lambda$-Dodd-nice, and since $\Phi : S \rightarrow_{\lambda, \text{sim}} T$, for this it is easily enough to see that $b^T$ is $\geq \lambda$-Dodd-nice. But since $C$ is Dodd-sound, the only extenders $E^T_{\alpha}$ which might violate this are those which are an image of $F^M$ (that is, $\text{extside}^T_{\alpha} = 1, (0, \alpha)^T \cap \mathcal{G}^T = \emptyset$ and $E^T_{\alpha} = F(M^T_{\alpha})$). So suppose $\alpha + 1 \in (\lambda, \infty)^T$ and $\alpha$ has this form. Letting $\beta \leq T \alpha$ be least such that either $\beta = \alpha$ or $\text{cr}(i^T_{\beta, \alpha}) > \text{cr}(F(M^T_{\beta}))$, note that $\beta = \text{pred}^T(\alpha + 1)$ and that $(0, \alpha + 1)^T \cap \mathcal{G}^T = \emptyset$. But since $\lambda < T \alpha + 1$ and recalling properties of $T$, it follows that $b^T \cap \mathcal{G}^T = \emptyset$ and $\text{sides}^T_{\infty} = \emptyset$, so $e = 0$. But $1 \in \text{sides}^T_{\alpha} \subseteq \text{sides}^T_{\beta}$, and since $\text{sides}^T_{\alpha + 1} = \emptyset$ and $\text{pred}^T(\alpha + 1) = \beta$, therefore $\beta \in \mathcal{B}^T$ and $\text{extside}^T_{\alpha} = 0$ and $\rho(B^T_{\alpha}) \leq \text{cr}(E^T_{\alpha}) < \nu(E^T_{\alpha})$. Since $\text{extside}^T_{\alpha} = 1$, therefore $\beta < \alpha$. Let $\varepsilon$ be least such that $\varepsilon + 1 \leq T \alpha$ and $\varepsilon \geq \beta$. Then $\text{cr}(E^T_{\varepsilon}) > \text{cr}(E^T_{\varepsilon}) = \text{cr}(F(M^T_{\beta}))$, by choice of $\beta$. So $\text{pred}^T(\varepsilon + 1) \geq \text{pred}^T(\alpha + 1) = \beta$. But by choice of $\varepsilon$, $\text{pred}^T(\varepsilon + 1) \leq \beta$. So $\text{pred}^T(\varepsilon + 1) = \beta$. But since $\text{cr}(E^T_{\varepsilon}) \geq \text{cr}(E^T_{\alpha}) \geq \rho(B^T_{\alpha})$ and $\text{extside}^T_{\beta} = 0$, therefore $\text{sides}^T_{\alpha + 1} = \emptyset$, so since $1 \in \text{sides}^T_{\alpha}$, we can’t have $\varepsilon + 1 \leq T \alpha$, a contradiction. \hfill $\square$

Call a tree $S$ as in the statement of Subclaim 6.1 a candidate. Given a candidate $S$, adopting the notation above, define the index of $S$ as in the proof of Lemma 8.6, except that we use the set $A$ of ordinals $\beta \leq \beta_0$ for some $\alpha + 1 \in (\lambda, \zeta)^S$ (hence $\beta \geq \lambda$; note $\text{cr}(E^S_{\alpha}) \geq \rho'$).

**Subclaim 6.2.** $S$ is core-strongly finite.

**Proof.** Suppose not. We construct a candidate $\bar{S}$ with smaller index than $S$, a contradiction. Let $\langle \gamma_i \rangle_{i < \ell}$ be the index of $S$ and $\kappa_i, \beta_i$ be as usual, so $A = \{ \beta_i \mid i < \ell \}$ and $\kappa_i = \text{cr}(E^S_{\beta_i})$ with $\kappa_i > \kappa_{i+1}$ for $i + 1 < \ell$. Let $a < \ell$ be least such that $\mathcal{C}_{\mathcal{D}_a}(E^S_{\beta_a})$ is not finitely generated. Let $\kappa = \kappa_a, \beta = \beta_a, Q = \mathcal{C}_{\mathcal{D}_a}(\text{exit}^S_\beta)$ and $F = F^Q$. So $\kappa^+Q < \sigma < \sigma_F = \tau_F$. Let $\theta$ be as in the proof of Claim 1 of Lemma 8.6, so $\theta$ is least such that $\sigma < \text{lh}(E^S_\theta)$; also $Q \leq M^S_\theta$. Now $\lambda \leq \theta$. For since $\sigma$ is a cardinal of $\text{exit}^S$, we have $\sigma \leq \nu(E^S_\theta) < \text{lh}(E^S_\theta)$. And $\kappa_{\ell-1} = \text{cr}(E^S_{\beta_{\ell-1}})$ and $\text{pred}^S(\beta_{\ell-1}) = \lambda$. But $\kappa_{\ell-1} \leq \kappa < \sigma \leq \nu(E^S_\theta)$, so by normality, $\lambda = \text{pred}^S(\beta_{\ell-1} + 1) \leq \theta$.

Let $\theta$ be least such that $\text{lh}(E^S_\theta) > \kappa^+Q$; much as above, $\lambda \leq \theta < \theta$. Let $\tilde{\theta} + h \equiv (\text{so} \ h < \omega)$ and $\tilde{\zeta} + h = \zeta$. Let $\chi$ be least such that $\chi \geq \theta + \chi + 1 \leq T \zeta$. Let $\tilde{\chi} + h = \chi$. Our plan is to select some $g \in \nu_{\tilde{\theta}^F}$ and $R \cap M^S_\theta$ with $F^R \approx F \mid g$, and such that we can define a degree-maximal tree $\bar{S}$ on $B$ such that:

- $\bar{S} \restriction \theta + 1 = S \restriction \tilde{\theta} + 1$ and $\text{lh}(\bar{S}) = \tilde{\zeta} + 1$.
- $\lambda < \bar{S} \tilde{\zeta}$ and letting $\bar{\gamma} + 1 = \text{suc} \bar{S}(\lambda, \bar{\zeta})$, we have:
  - $\text{sides}^S_{\bar{S} + 1} = \text{sides}^S_{\bar{S}} = \{ \varepsilon \}$,
  - $\langle \bar{\gamma} + 1, \bar{\zeta} \rangle^S \cap \mathcal{G}^S_{\text{deg}^S} = \emptyset$ and $\text{deg}^S_{\bar{\zeta} + 1} = \text{deg}^S_{\bar{\zeta}} = m$, and
Let $\bar{\mathcal{S}}$ be the reverse copy of $\mathcal{S}$.

- $M^c_{\gamma+1} = \mathfrak{c}_{m+1}(M^c_{\zeta}) = \mathfrak{c}_{m+1}(M^c_{\zeta}) = M^c_{\gamma+1}$.

- $b^\mathcal{S}$ is $\lambda$-Dodd-nice.

- Replacing the role of $Q$ in $\mathcal{S}$ with $R$ in $\bar{\mathcal{S}}$, we perform a kind of reverse copy construction, much like in the proof of Lemma 8.6, so that $\mathcal{S} \upharpoonright [\theta, \zeta]$ will be a “copy” of $\bar{\mathcal{S}} \upharpoonright [\bar{\theta}, \bar{\zeta}]$. Moreover, $\chi$ is least such that $\bar{\chi} \geq \bar{\theta}$ and $\bar{\chi} + 1 \leq \bar{\zeta}$, and for $\alpha \in [\chi + 1, \bar{\zeta}]$, the copying process will yield a copy map

$$\pi^\mathcal{S}_\alpha : M^c_{\alpha} \rightarrow M^c_{\alpha}$$

with $\text{cr}(\pi^\mathcal{S}_\alpha) > \kappa^+R = \kappa^+Q$.

- The final copy map $\pi^\mathcal{S}_\zeta : M^c_{\zeta} \rightarrow M^c_{\zeta}$ is an $m$-embedding with $\pi^\mathcal{S}_\zeta \circ j^\mathcal{S} = j^\mathcal{S}$ where $j^\mathcal{S} : \mathfrak{c}_{m+1}(M^c_{\zeta}) \rightarrow M^c_{\zeta}$ is the iteration map and likewise $j^\mathcal{S}$ (so $\text{dom}(j^\mathcal{S}) = \text{dom}(j^\mathcal{S})$, by earlier points).

- $\tau^{-1}(x) \in \text{rg}(\pi^\mathcal{S}_\xi)$, where $\tau : M^c_{\xi} \rightarrow M^c_{\xi}$ witnesses that $\mathcal{S}$ captures $(T, x, \rho' + 1)$; it will follow that $\tau \circ \pi^\mathcal{S}_\xi : M^c_{\xi} \rightarrow M^c_{\xi}$ witnesses that $\bar{\mathcal{S}}$ captures $(\bar{T}, x, \rho' + 1)$.

Therefore $\bar{\mathcal{S}}$ will be a candidate.

We will select $g, R, Q'$ and build $\bar{\mathcal{S}}$ as in the proof of Claim 1 of Lemma 8.6. The following claim helps us see that the copying can be executed without problems, and is the analogue of Subclaim 1.1 from there:

**Subsubclaim 6.2.1.** Let $\chi \in [\theta, \zeta]$. Then $\text{cr}(E^\mathcal{S}_\chi) \notin (\kappa, \sigma)$, so $\text{pred}^\mathcal{S}(\chi + 1) \notin (\bar{\theta}, \bar{\theta})$. In fact, one of the following options holds:

(i) there are $\varepsilon, \chi'$ such that $\lambda \leq^\mathcal{S} \varepsilon + 1 \leq^\mathcal{S} \zeta$ and $\chi' \leq^\mathcal{S} \varepsilon$, and either:

(a) $\chi = \chi'$, or

(b) $\chi \leq^\mathcal{S} \chi'$ and $\chi$ is $\chi'$-transient and $E^\mathcal{S}_{\chi'} = F(M^c_{\chi'})$, where sides$^\mathcal{S}_{\chi'} = \{d\},$

or

(ii) $\lambda \leq^\mathcal{S} \chi \leq^\mathcal{S} \xi$ and $\chi$ is $\xi$-transient.

Moreover, the three options (i)a, (i)b and (ii) are mutually exclusive.

**Proof.** This is just a slight variant of the proof of Subclaim 1.1 within the proof of Lemma 8.6.

Recall here that if (i)b holds and $\eta + 1 = \text{succ}^\mathcal{S}(\chi, \chi')$, then $E^\mathcal{S}_\chi = F(M^c_{\eta+1})$, so $\text{cr}(E^\mathcal{S}_\chi) = \text{cr}(E^\mathcal{S}_{\chi'})$. Likewise, if (ii) holds and $\eta + 1 = \text{succ}^\mathcal{S}(\chi, \xi)$, then $E^\mathcal{S}_\chi = F(M^c_{\eta+1})$, so $\text{cr}(E^\mathcal{S}_\chi) = \text{cr}(F(M^c_{\xi})).$

By Subsubclaim 6.2.1, $\mathcal{S} \upharpoonright [\theta, \zeta]$ can be viewed as a tree $\mathcal{S}'$ on the “phalanx” $\Phi(\mathcal{S} \upharpoonright (\bar{\theta} + 1)) \sim ((Q, 0))$, where the nodes of $\Phi(\mathcal{S} \upharpoonright (\bar{\theta} + 1))$ index either a single model or a bicephalus, corresponding to $\mathcal{S} \upharpoonright (\bar{\theta} + 1)$, but $Q$ is just a single model, irrespective of what is indexed at $\theta$ in $\mathcal{S}$, and in $\mathcal{S}'$, if $\text{cr}(E^\mathcal{S}_\alpha) \leq \kappa$ then $\text{pred}^\mathcal{S}(\alpha + 1) \leq \bar{\theta}$ (so $E^\mathcal{S}_{\alpha}$ applies to a model or bicephalus of $\mathcal{S} \upharpoonright (\bar{\theta} + 1)$), whereas if $\text{cr}(E^\mathcal{S}_\alpha) > \kappa$ then $\text{pred}^\mathcal{S}(\alpha + 1) \geq \bar{\theta} + 1$, for $\alpha \in \mathfrak{c}$.\hfill $\square$
and if \( \text{pred}^{S'}(\alpha + 1) = \tilde{\theta} + 1 \) then we take \( (M_{\alpha+1}^{S'}, \deg^{S'}_{\alpha+1}) \leq (Q, 0) \) as large as possible. In fact, by the subclaim, if \( \text{cr}(E^{S'}_{\alpha}) > \kappa \) then \( \text{cr}(E^{S'}_{\alpha}) \geq \sigma \), and recall that \( \text{exit}^{S}_{\theta} \leq Q \) and \( \rho^{Q}_{1} \leq \sigma \), so having \( (M^{S'}_{\alpha+1}, \deg^{S'}_{\alpha+1}) \leq (Q, 0) \) in case \( \text{pred}^{S'}(\alpha + 1) = \tilde{\theta} \) corresponds with what happens in \( S \). In particular, if \( \alpha \geq \theta \) and \( \text{cr}(E^{S}_{\alpha}) > \kappa \) and \( \text{pred}^{S}(\alpha + 1) = \theta \) then \( (M^{S}_{\alpha+1}, \deg^{S}_{\alpha+1}) \leq (Q, 0) \) where \( \epsilon_{\theta} = \text{exit}^{S}_{\theta} \).

With these observations, the construction of \( Q', g, R, S \) is just a slight variant of that in the proof of Claim 1 of Lemma 8.6.

The index of \( S \) is less than that of \( S \), unreasonable, establishing the subclaim. \( \square \)

And hence that of the claim.

**Claim 7.** Let \( S \) be core-strongly finite, \( \text{lh}(S) = \zeta + 1 \), sides\( ^{S}_{\zeta} = \{ e' \} \) and \( m' = \deg^{S}_{\zeta} \). Let \( \lambda' <^{S} \chi' \) be such that \( \text{c}\_\text{c}(M^{e'_{\zeta}^{S}}_{\chi'}) \leq M^{e_{\zeta}^{S}}_{\chi} \). Then for every \( \chi \in [\lambda', \zeta) \), one of the following options holds:

(i) there are \( \varepsilon, \chi' \) such that \( \lambda <^{S} \varepsilon + 1 \leq^{S} \chi' \) and \( \chi' \leq^{S} \text{Da} \varepsilon \), and either:
   
   (a) \( \chi = \chi' \), or
   
   (b) \( \chi <^{S} \chi' \) and \( \chi \) is \( \chi' \)-transient and \( E^{S}_{\chi'} = F(M_{\chi'}^{dS}) \), where \( \text{side}^{S}_{\chi} = \{ d \} \),
   
   or

(ii) \( \lambda' \leq^{S} \chi <^{S} \zeta \) and \( \chi \) is \( \zeta \)-transient.

Moreover, the three options (i)a, (i)b and (ii) are mutually exclusive.

**Proof.** See the proof of Subsubclaim 6.2.1 of Subclaim 6.2 of Claim 6. \( \square \)

**Claim 8.** \( T \) is core-strongly-finite and \( E_{\alpha}^{\mu} = \emptyset \) for all \( \gamma \geq \lambda \), so \( M_{\alpha}^{\mu} = M_{\lambda}^{\mu} \) and \( k = \text{id} \).

**Proof.** The proof is like that of the analysis of measures in mice, Lemma 9.2. Fix \( x \in M \) with \( M = \text{Hull}_{m+1}(\rho^{M}_{m+1} \cup \{ x \}) \). By Claim 6, we can fix a degree-maximal tree \( \tilde{T} \) on \( B \) such that \( \tilde{T} \) captures \( (T, i_{\text{HDD}}^{T}(x), \rho^{T}_{1}) \), \( b^{T} \) is \( \lambda \)-Dodd-nice, and \( \tilde{T} \) is core-strongly-finite. Then \( \tilde{T} \vert (\lambda + 1) = \tilde{T} \vert (\lambda + 1), \) sides\( ^{\tilde{T}}_{\infty} = \text{side}^{T}_{\infty} = \{ e \} \), and by the proof of Subclaim 6.1 of Claim 6, \( b^{T} \) is also \( \lambda \)-Dodd-nice.

Let \( \tilde{j} = C_{\lambda}^{T} \rightarrow \tilde{N} = M^{\infty}_{\lambda} \) and \( j : C_{\lambda}^{T} \rightarrow N = M^{\infty}_{\lambda} \) be the \( (m+1) \)-core maps, which are also the tail iteration maps given by \( \tilde{T} \) and \( T \) respectively. Let \( \varsigma : \tilde{N} \rightarrow N \) witness the capturing, so \( \varsigma \) is an \( m \)-embedding with \( \varsigma \circ \tilde{j} = j, i_{\text{HDD}}^{T}(x) \in \text{rg}(\varsigma) \) and \( \text{cr}(\varsigma) > \rho^{T}_{1} \).

Define \( \tilde{k} = \varsigma^{-1} \circ k \). Then the usual diagram commutes (Figure 6).

It is now routine to execute the argument from the proof of Theorem 9.1 (which also appeared in the proof of super-Dodd-solidity). The only formal difference is that here the trees are only being analyzed above \( \lambda \), and also that the core of interest, \( C_{\lambda}^{T} \), might be a proper segment of \( M^{T}_{\lambda} = M^{T}_{\infty} \), in which case the first extenders used along \( (\lambda, \infty)^{T} \) and \( (\lambda, \infty)^{T} \) cause a drop in model to \( C_{\lambda}^{T} \); similarly, if \( (0, \lambda)^{T} \cap \theta^{T} \neq \emptyset \) and \( \deg^{T}_{\lambda} > m \), then they could cause a drop in degree to \( m \), without a drop in model. But these details have no significant impact. We leave the execution to the reader. \( \square \)

The remainder of the proof involves a slight split into two cases, as follows:
Figure 6: The diagram commutes.

(a) $C, M$ are active type 2, $m = 0$ and $\kappa = \text{cr}(F^C) = \text{cr}(F^M) < \rho$; and
(b) otherwise.

Let $B' = (C', M', \rho') = B^U_\lambda$. Suppose case (a) holds. Then we define $\bar{C} = C|\kappa + C$ and the phalanx

\[
\mathcal{S} = ((\bar{C}, \kappa), C, \rho),
\]

and likewise, letting $\kappa' = \text{cr}(F^{C'}) = \text{cr}(F^{M'}) < \rho'$, let $\bar{C}' = C'|(\kappa') + C'$ and $\mathcal{S}' = ((\bar{C}', \kappa'), C', \rho')$. Say a $(0, 0)$-maximal tree $V$ on $\mathcal{S}$ is relevant if for all $\alpha + 1 < \text{lh}(V)$, either $\text{cr}(E^V_\alpha) \geq \rho$ or $\text{cr}(E^V_\alpha) = \kappa$ (and as usual, $\rho \leq \text{lh}(E^V_0)$, so in fact $\rho < \text{lh}(E^V_0)$).

Likewise for $V', \mathcal{S}', \kappa', \rho'$. Note that $\mathcal{S}$ is $(0, \omega_1 + 1)$-iterable with respect to relevant trees, as is $\mathcal{S}'$; for the case of $\mathcal{S}'$, this is because relevant trees on $\mathcal{S}'$ can be lifted to normal continuations of $U | (\lambda + 1)$. In case (b), we don’t need to define $\mathcal{S}, \mathcal{S}'$.

Let $(\mathcal{V}', \mathcal{W}')$ be the initial segment of the $m$-maximal comparison of

(i) $(\mathcal{S}', M')$, in case (a) above holds,

(ii) $(C', M')$ otherwise,

through length $\theta'$, where $\theta' \in [1, \omega]$ is largest such that $E^\mathcal{V}'_\gamma \neq \emptyset = E^\mathcal{W}'_\gamma$ for all $\gamma + 1 < \theta'$(so iterability of $M'$ is irrelevant, by choice of $\theta'$).

Claim 9. $(\mathcal{V}', \mathcal{W}')$ is finite, $\mathcal{W}'$ (on $M'$) is trivial, $\mathcal{V}'$ (on $\mathcal{S}'$ or $C'$) is finite and uses the same extenders as does $T | [\lambda, \xi]$, and $M_{\theta' - 1} = M^{\mathcal{V}'} = M'$.

Proof. The tree $V'$ is just a simple reorganization of $T | [\lambda, \xi]$, and the claim follows easily from Claims 7 and 8. That is, recall that $E^T_\alpha = \emptyset$ (as in $U$, we moved into $M^U_\lambda$ at stage $\lambda$), $\text{lh}(T) = \xi + 1 = \lambda + n + 1$ and $n > 1$ (as $E^T_\lambda = \emptyset$ but $E^T_{\lambda + 1} \neq \emptyset$), and $E^T_{\lambda + 1 + i} \neq \emptyset$ for all $i + 1 < n$. So we want to see that $E^\mathcal{V}'_i = E^T_{\lambda + 1 + i}$ for all $i + 1 < n$, and $M^\mathcal{V}'_{n - 1} = M^{\mathcal{V}'}_{\lambda + n} = M'$. Clearly $E^\mathcal{V}'_0 = E^T_{\lambda + 1}$, since $M^\mathcal{V}'_0 = C^U_{\lambda} \leq M^T_{\lambda + 1}$ where exitside$^T_{\lambda + 1} = d$. So the point is that given $i$ with $1 < i + 1 \leq n$, if it is not the case that

\[
sides^T_{\lambda + i + 1} = \{e\} \quad \text{and} \quad M^\mathcal{V}'_{i + 1} = M^T_{\lambda + i + 1}
\]

then $\text{cr}(E^\mathcal{V}'_{\lambda + i}) < \rho'$, so by Claim 7, $\lambda + i$ is $\xi$-transient for $T$. Since $M' = M^T_{\xi}$, it follows that case (a) holds, $\lambda + 1 \leq T \lambda + i < T \xi$ and $E = E^\mathcal{V}'_{i + 1} = E^T_{\lambda + i + 1}$ is an image of $F^{C'}$ with
\[ \kappa' = \text{cr}(E) = \text{cr}(F^{C'}) < \rho'. \] If \( i > 1 \) then let \( \mu \) be

\[ \mu = \text{cr}(i_{\lambda+1}^{\gamma}) = \text{lglcd}(M^T_{\lambda+1}), \]

and if \( i = 1 \) then let \( \mu \) be

\[ \mu = \text{cr}(j) = \text{lglcd}(C'). \]

Since (a) holds and \( \kappa' = \text{cr}(E) \), we have \( M^\gamma_i = \text{Ult}(\bar{C}', E) \). So note for each \( d \in \text{side}_{\lambda+1} \), we have

\[ M^\gamma_{\lambda+1} = M^\gamma_i = \text{Ult}(\bar{C}', E) = M^{\gamma'}_i, \]

But letting \( \gamma + 1 = \text{suc}^T(\lambda + i, \xi) \), \( E_\gamma^\nu \) (which is finitely generated and Dodd-nice, with critical point \( \mu \)) is produced by a finite iteration of the model in line (9), and either \( E_i^\nu = E_{\lambda+1}^T \) is the first extender in that iteration, or \( E_\gamma^\nu \) is Dodd-sound and \( E_i^\nu = E_{\lambda+1}^T \). The claim easily follows from these considerations.

Let \((\nu, W)\) be the initial segment of the maximal comparison of

(i) \((\delta, M)\), in case (a) above holds,

(ii) \((C, M)\) otherwise,

through length \( \theta \), where \( \theta \in [1, \omega] \) is largest such that \( E_\gamma^W \neq \emptyset = E_\gamma^W \) for all \( \gamma + 1 < \theta \).

Let \( G \) be the \((\rho, \rho')\)-extender over \( C \) derived from \( i_{CC'} = i_{0\lambda}^{\mu} \), or equivalently, from \( i_{MM'} = i_{0\lambda}^{\mu} \). Let \( \bar{G} \) be the restriction of \( G \) to a \((\kappa, \kappa')\)-extender. So \( M^\gamma_{\lambda+1} = \text{Ult}(\bar{C}, \bar{G}) \), \( M^\nu_0 = C' = \text{Ult}(\rho, G) \), and \( M^\nu_0 = M' = \text{Ult}(\rho, G) \), and the iteration maps \( i_{CC'} : C \to C' \) and \( i_{MM'} : M \to M' \) are just the associated ultrapower maps.

The following claim describes the relationship between \((\nu, W)\) and \((\nu', W')\):

**Claim 10.** For all \( \gamma < \min(\theta, \theta') \), we have:

1. \( \text{deg}_\gamma^{\nu'} = \text{deg}_\gamma^{\nu'} \); let \( d = \text{deg}_\gamma^{\nu'} \),

2. \( M^\nu_i = \text{Ult}_d(M^\nu_i, G) \),

3. if \( \gamma + 1 < \theta' \) then exit\(^{\nu'}_{\gamma} = \text{Ult}_d(\text{exit}_\gamma^{\nu'}, G) \neq \emptyset \) is active,

4. if \( \gamma + 1 < \theta' \) then \( E_{\gamma}^{W'} = E_{\gamma}^{W'} = \emptyset \),

5. if \( \gamma + 1 = \theta' \) then \( M = M^\nu_i \) and \( M' = M^\nu_i \).

**Proof.** For this, we use calculations with commutativity of ultrapowers, and with condensation, like those used in the analyses of comparisons in [24, §4]. However, we need a more general form, like that used in full normalization. A key component of this is Lemma 11.11, and the main idea for that was due to Steel. The combination of methods also relates to the analysis of the interaction between comparison and full normalization, which is due to the author; see [18, §8].

The proof being presented for Lemma 11.6 differs in approach with what I had in mind during the presentation of (most of) the results of this paper at the 2015 Münster conference in inner model theory; that approach did not depend on Lemma 11.11 or any full normalization calculations. Eventually I noticed that the present argument is simpler and more direct than the earlier approach, so I adopted it.
First consider $\gamma = 0$. In this case, part 1 is by definition (and $d = m$), and part 2 just says that $C' = \text{Ult}_d(C, G)$, which is also by definition. We also know that $M' = \text{Ult}_d(M, G)$ and that, in case (a), $C' = \text{Ult}_0(C, G)$.

Now fix $\gamma < \min(\theta, \theta')$ and suppose that all parts hold at all $\gamma' < \gamma$, and parts 1 and 2 hold at $\gamma$.

Since $E^W_{\gamma'} = \emptyset$ for all $\gamma' < \gamma$, we have $M^W_{\gamma} = M$. Let $\eta$ be least such that either $M^Y_{\gamma}|\eta \neq M|\eta$ or $\eta = \min(\text{OR}^M, \text{OR}^M)$. Note here that if $M^Y_{\gamma}|\eta = M|\eta$ then in fact $M^Y_{\gamma} \subseteq M$, since $M$ is not sound.

Note that Lemma 11.11 applies to $(M, M', G, m)$ and also to $(M^Y_{\gamma}, M^Y_{\gamma'}, G, d)$, where $d = \deg^Y_{\gamma} = \deg^Y_{\gamma'}$. Therefore,

$$U_0 = \text{Ult}_0(M^Y_{\gamma}|\eta, G) \leq M^Y_{\gamma'},$$

$$U_1 = \text{Ult}_0(M|\eta, G) \leq M'.$$

Since $M^Y_{\gamma}|\eta = M|\eta$, we get $U^W_{\gamma'} = U^W_1$, and the ultrapower maps agree.

Suppose $M^Y_{\gamma}|\eta \neq M|\eta$. Since the ultrapower maps agree, the distinction between the active extenders of these models lifts to give $F_{U_0} \neq F_{U_1}$. So $U_0, U_1$ constitute the least disagreement between $M^Y_{\gamma'}$ and $M^W_{\gamma'}$. Therefore $\gamma + 1 < \theta'$. But $W'$ is trivial, so either

$- F_{U_0} \neq \emptyset = F_{U_1}$, so $F^M_{\gamma}|\eta \neq \emptyset = F^M|\eta$, or

$- F_{U_0} \neq \emptyset \neq F_{U_1}$ and $\nu(F_{U_0}) < \nu(F_{U_1})$, so $F^M_{\gamma}|\eta \neq \emptyset \neq F^M|\eta$ and $\nu(F^M_{\gamma}|\eta) < \nu(F^M|\eta)$.

It follows that $\gamma + 1 < \theta$, with $E^Y_{\gamma} = F(M^Y_{\gamma}|\eta)$ and $E^W_{\gamma} = \emptyset$.

Now suppose instead that $M^Y_{\gamma} = M|\eta$, so $M^Y_{\gamma} \subseteq M$. Let $d = \deg^Y_{\gamma}$. If $M^Y_{\gamma} \subseteq M^W_{\gamma}$ then again by 11.11, we have $M^V_{\gamma} = \text{Ult}_d(M^Y_{\gamma}, G) \subseteq M^W_{\gamma} = M'$, so $\gamma + 1 = \theta'$ and $M^\theta_{\gamma+1} \subseteq M'$, contradicting Claim 9. So $M^Y_{\gamma} = M$. It follows that $d = m$ and root$^V(\gamma) = 0$. For if root$^V(\gamma) = -1$ then $M^Y_{\gamma} \models \text{ZFC}^-$, whereas $M \not\models \text{ZFC}^-$. So root$^V(\gamma) = 0$, but then $M^Y_{\gamma}$ is $d$-sound but not $(d + 1)$-sound, so $d = m$. So again by 11.11, $M^Y_{\gamma} = \text{Ult}_m(M^Y_{\gamma}, G) = \text{Ult}_d(M, G) = M'$, so $\gamma + 1 = \theta' = \theta$.

This establishes parts 3–5 for $\gamma$. We next consider parts 1, 2 for $\gamma + 1$, assuming that $\gamma + 1 < \min(\theta, \theta')$, and so $M^Y_{\gamma}|\eta \neq M|\eta$.

Suppose first that $\kappa = \text{cr}(E^Y_{\gamma'}) \geq \rho$. Then $\kappa' = \text{cr}(E^Y_{\gamma'}) \geq \rho'$. Let $\delta = \text{pred}^V(\gamma + 1)$. Using part 3, it easily follows that $\delta = \text{pred}^V(\gamma + 1)$. Let $M^* = M^\gamma_{\gamma+1}$ and $d = \deg^Y(\gamma + 1)$. Then $(M^*, d)$ is an element of the extended dropdown sequence of $((M^Y_{\delta}, \deg^Y_{\delta}), \text{OR}(\text{exit}^Y_{\delta}))$, and so by dropdown lifting 11.11, $(\text{Ult}_d(M^*, G), d)$ is likewise an element of the extended dropdown sequence of $((M^Y_{\delta}, \deg^Y_{\delta}), \text{OR}(\text{exit}^Y_{\delta}))$. Considering the positions of the various projecta and how they are shifted up by the ultrapower maps, we moreover get

$$M^\gamma_{\gamma+1} = \text{Ult}_d(M^*, G)$$

and $d = \deg^Y_{\gamma+1}$. It remains to see that

$$\text{Ult}_d(M^Y_{\gamma+1}, G) = M^\gamma_{\gamma+1}.$$
This is a calculation very much like [24, Lemma 3.9]. Note that \( \text{Ult}_d(M_{\gamma+1}^\gamma, E_\gamma^\gamma) \) results from the two-step abstract \( d \)-maximal iteration of \( M_{\gamma+1}^\gamma \) given by first using \( E_\gamma^\gamma \), then using \( G \). On the other hand, \( M_{\gamma+1}^{\gamma'} \) results from first using \( G \) (producing \( M_{\gamma+1}^{\gamma'} \)), and then \( E_\gamma^{\gamma'} \). We want to see that these result in the same model; that is,

\[
\text{Ult}_d(\text{Ult}_d(M_{\gamma+1}^{\gamma'}, E_\gamma^{\gamma'}), G) = \text{Ult}_d(\text{Ult}_d(M_{\gamma+1}^{\gamma'}, G), E_\gamma^{\gamma'}).
\]

So letting \( j, k \) be the two resulting ultrapower maps (with domain \( M_{\gamma+1}^{\gamma'} \)), it suffices to see that the models are the \( \text{r}\Sigma_d \)-hull of \( \text{rg}(j) \) (or \( \text{rg}(k) \), respectively) and ordinals below \( \text{lh}(E_\gamma^{\gamma'}) \), and that the \((\kappa, \text{lh}(E_\gamma^{\gamma'}))\)-extenders derived from \( j, k \) are identical (as then both models are just the degree \( d \) ultrapower of \( M_{\gamma+1}^{\gamma'} \) by that common extender, and hence equal).

But note that \((\text{exit}_\gamma^{\gamma'})^\text{pr} \) is a cardinal proper segment of \( M_{\gamma+1}^{\gamma'} \) of ordinal height \( \text{lh}(E_\gamma^{\gamma'}) \leq \rho_d(M_{\gamma+1}^{\gamma'}) \), so

\[
i^{\text{exit}_\gamma^{\gamma'}}_{G, \gamma, d} \mid \text{exit}_\gamma^{\gamma'}, \quad i^{\text{exit}_\gamma^{\gamma'}, 0}_{G, \gamma, d},
\]

and these maps \( \text{exit}_\gamma^{\gamma'} \) cofinally into \( \text{exit}_\gamma^{\gamma'} \). But then since \( i^{\text{exit}_\gamma^{\gamma'}, 0}_{G, \gamma, d} \) maps fragments of \( E_\gamma^{\gamma'} \) to fragments of \( E_\gamma^{\gamma'} \), it follows that the two derived extenders are identical, as desired.

The second case is that \( \text{cr}(E_\gamma^{\gamma'}) = \kappa \), so \( \text{cr}(E_\gamma^{\gamma'}) = \kappa' \), and \( M_{\gamma+1}^{\gamma'} = \text{Ult}_0(\mathcal{C}, E_\gamma^{\gamma'}) \), and \( M_{\gamma+1}^{\gamma+1} = \text{Ult}_0(\mathcal{C}', E_\gamma^{\gamma'}) \) This case works almost the same, except that \( \mathcal{C}' = \text{Ult}_0(\mathcal{C}, G) \) instead of \( \text{Ult}_0(\mathcal{C}, G) \). However, we still have \( \text{exit}_\gamma^{\gamma'} = \text{exit}_\gamma^{\gamma'}, G \) and \( M_{\gamma+1}^{\gamma'} = \text{Ult}_0(\mathcal{M}_{\gamma+1}^{\gamma'}, G) \), both using the full \( G \). We leave the details to the reader. This completes the proof the claim. \( \square \)

Now by Claim 9, \( \theta' < \omega \), and so using Claim 10, \( \theta = \theta' \) and \( (\mathcal{V}, \mathcal{W}) \) is a successful comparison, of length \( \theta \), with \( \mathcal{W} \) trivial and \( M_\theta^\theta = M \). It can’t be that (a) holds and \( \text{root}^\mathcal{V}(\theta - 1) = -1 \), because in that case, \( M_{\theta - 1}^\theta \models \text{ZFC}^- \), although \( M \not\models \text{ZFC}^- \). It follows that \( T \) is equivalent to \( \mathcal{V} \), and \( T \) is trivial, so \( \lambda = 0 \), \( B = B' \) and \( M_\xi^T = M \). But \( T \) is core-strongly-finite by Claim 8. Note then that at stage 0, in \( T \) we moved into \( C \) (and in \( U \) we moved into \( M \) and \( b^T \cap D_{\text{deg}}^\mathcal{U} = \emptyset \). If (b) holds, it now follows that \( \mathcal{V} \) is a tree on \( C \) of the right form (that is, it witnesses the lemma with respect to \( C, M \)); it is almost-above \( \rho \) by Claim 7). Otherwise, letting \( \mathcal{V}' \) be the \( m \)-maximal tree on \( C \) which is equivalent to \( \mathcal{V} \) (using the same extenders), then \( \mathcal{V}' \) has the right form. This completes the proof.

\( \square \)

11.15 Remark. As mentioned in Remark 11.8, it seems one might try to prove just Corollary 11.7 by comparing bicephali, but using more standard calculations instead of proving Claim 10 above (which also requires Lemma 11.11). But there seems to be a difficulty, which we now describe.

Start by forming the same bicephalus comparison as before. Say it reaches a bicephalus \( B' = (C', M', \rho') = B'_{\xi^T} \) in \( T \), and that \( C' \subseteq M_\xi^{\xi^T} \), and \( (T, \mathcal{U}) \) has length \( \alpha + 1 \), where \( \lambda < T \alpha \), and \( (0, \alpha]^T \) does not drop in model or degree, but sides \( i_{\alpha}^T = \{1\} \), so...
$N' = M^T_1$ is defined, but $M' \neq N'$. Suppose that $\lambda \in \theta'$ and the core map $k : C' \rightarrow N'$ is in fact an iteration map along the last part of the final branch $\theta'$. Then standard arguments already show that

$$t' = Th(M^C_1) \cup \{i^T(x)\}$$

is $\Sigma_{m+1}$, where $x \in M$. Fix $p' \in C'$ such that $t'$ is $\Sigma_{m+1}(\{p'\})$.

One would like to be able to pull this back to $C, M$. But we can construct a finite tree $\tilde{T}$ on $T$, whose last model is a bicepsal $\tilde{B} = (\tilde{C}, \tilde{M}, \tilde{\rho})$, which captures the parameter $p'$. Let $\sigma^0 : \tilde{C} \rightarrow \tilde{C}$ and $\sigma^1 : \tilde{M} \rightarrow \tilde{M}'$ be the capturing maps, which are $m$-embeddings, and $\sigma^0 \circ i^{0T} = i^{0T}$ and $\sigma^1 \circ i^{1T} = i^{1T}$ and $\sigma(\tilde{\rho}) = p'$, and $\sigma^0 \upharpoonright \tilde{\rho} = \sigma^1 \upharpoonright \tilde{\rho}$. By the elementarity etc, it follows that

$$i = ThM_{\Sigma_\infty}(\tilde{\rho} \cup \{i^{1T}(\tilde{x})\})$$

is $\Sigma_{m+1}(\{\tilde{p}\})$. And $\tilde{T}$ is finite, and the iteration maps $i^{0T}$ and $i^{1T}$ agree over $\rho$. So the desired theory $t$ is just $(i^{0T})^{-1} \circ i^{1T}$. Since $\tilde{T}$ is finite, one might expect that $t$ should therefore be definable from parameters over $C$. But the problem is, that $C, M$ might be active and there might be a use of $F^M$ (or some iterate of $F^M$) along $b^T$, hence with $\kappa = cr(F^M) < \rho$. Note then that $\kappa + M \leq \rho = P_{m+1}^M$, and so if $m > 0$ or $M$ is type 3 or $\kappa + M < \rho$, then the component measures of $F^M$ would be in $M$, and in $C$, and so we could replace the use of $F^M$ with some measure in $C$, which would suffice. So the real problem case is that $C, M$ are type 2, $m = 0$, and $\kappa + M = (\kappa^+)^C = \rho^C = \rho^M$. In this case, the author does not see how to avoid the proof given above. Moreover, in this case, it does seem that using $F^M$ along $b^T$ might ostensibly provide the parameter $\tilde{p}$.

### 12 Interlude: Grounds of mice via strategically $\sigma$-closed forcings

In this brief section we use Theorem 11.5 (finitely generated hulls) to prove Theorem 12.1. However, the rest of the paper does not depend on this section. Recall that if $N \models \text{ZFC}$ then a ground of $N$ is a $W \subseteq N$ such that there is $\mathbb{P} \in W$ and $G \in N$ such that $G$ is $(W, \mathbb{P})$-generic and $N = W[G]$.

#### 12.1 Theorem

Let $M$ be a $(0, \omega_1 + 1)$-iterable mouse whose universe $|M|$ models ZFC. Let $W$ be a ground of $|M|$ via a forcing $\mathbb{P} \in W$ such that $W \models \text{"$\mathbb{P}$ is $\sigma$-strategically-closed".}$ Suppose $M \models_{\mathbb{P}} W$. Then the forcing is trivial; that is, $W = |M|$.

For the proof we use the following two facts. The first is well-known; see [3]:

#### 12.2 Fact (Ground definability, Laver/Woodin)

There is a formula $\varphi_{gd}$ in the language of set theory, such that the following holds: Let $N$ be a model of ZFC and $W \subseteq N$ be a ground of $N$. Then there is $p \in W$ such that $W = \{x \in N \mid N \models \varphi_{gd}(p, x)\}$.

The second is by [28]:

---

\*\*\*If $M$ is also tame, then the hypothesis that $M \models_{\mathbb{P}} W$ is superfluous, as shown in [19, Theorem 4.7]. Of course, as $\mathbb{P}$ is $\sigma$-strategically-closed in $W$, automatically $M \subseteq W$. But note that $M \models_{\mathbb{P}} W$ gives the extender sequence of length $\omega_1^M$, not just HC.$^M$.\*\*\*
12.3 Fact. There is a formula \( \varphi_{pm} \) in the language of set theory such that for any mouse \( M \) with no largest cardinal, given any \( N \in M \), we have \( N \prec M \iff M \models \varphi_{pm}(M|\omega_1^M, N) \).

12.4 Definition. Let \( H \) be a transitive class which satisfies “there is no largest cardinal”. Let \( X \in H \). If there is a premouse \( P \) such that \( \vert P \vert = H \) and for all \( N \in H \), we have

\[
N \prec P \iff H \models \varphi_{pm}(X, N),
\]

then let \( S^H(X) \) denote \( P \); otherwise \( S^H(X) \) is undefined. Say that \( X \) is good for \( H \) if \( S^H(X) \) is defined. Note that \( \{ X \in H \mid X \text{ is good for } H \} \) is a definable class of \( H \), uniformly in such \( H \).

Proof of Theorem 12.1. Suppose the theorem is false, as witnessed by \( M, W, \mathbb{P} \). Let \( m = M|\omega_1^M \). Then we may assume that for all sufficiently large limit cardinals \( \lambda \) of \( W \), \( \mathbb{P} \) forces “\( \bar{m} \) is good for \( H_\lambda \) and \( \bar{m} = S(\bar{m})|\omega_1^{S(\bar{m})} \)”. Later we will also add some further statements which we assume are forced by \( \mathbb{P} \), as they become relevant. We force with \( \mathbb{P} \times \mathbb{P} \). Write \( \check{S}_0^\lambda \) and \( \check{S}_1^\lambda \) for the names for \( S_{H_\lambda[W|G_i]}(\bar{m}) \) and \( S_{H_\lambda[W|G_0]}(\bar{m}) \) respectively, where \( G_0 \times G_1 \) is the \( \mathbb{P} \times \mathbb{P} \)-generic.

Fix some sufficiently large limit cardinal \( \lambda \) of \( W \), with \( \mathbb{P} \in H_\lambda^W \), and such that \((*)\) there is \( \eta < \lambda \) such that there is \( (p, q) \in \mathbb{P} \times \mathbb{P} \) forcing “\( \check{S}_0^\lambda|\eta \neq \check{S}_1^\lambda|\eta \)”. Let \( \eta \) be the least witness to \((*)\) (relative to \( \lambda \)). Note then that \( M|\eta \in W \) and \( \mathbb{P} \times \mathbb{P} \) forces “\( \check{S}_0^\lambda|\eta = \check{S}_1^\lambda|\eta = M|\eta \)”. There are two possibilities:

(i) There is \( (p, q) \in \mathbb{P} \times \mathbb{P} \) forcing “\( \check{S}_0^\lambda|\eta \) and \( \check{S}_1^\lambda|\eta \) are both active, and \( \check{S}_0^\lambda|\eta \neq \check{S}_1^\lambda|\eta \),

or

(ii) otherwise.

Claim 1. If (ii) holds, then \( \eta \) is not a cardinal of \( W \).

Proof. In case (ii), any condition \((p, q)\) which forces that \( \check{S}_0^\lambda|\eta \) is active decides all elements of the active extender \( E = F^{\check{S}_0^\lambda|\eta} \), because otherwise we get case (i). So \( E \in W \), and the active premouse \((M|\eta, E) \in W \). But the active extender of a premouse definably collapses its index, so \( \eta \) is not a cardinal in \( W \).

\( \square \)

Fix \( (p_0, q_0) \in \mathbb{P} \) witnessing \((*)\) and the choice of \( \eta \), and witnessing (i) if it holds.

Using a local version of Fact 12.2 (for example in [26]); or by choosing \( \lambda \) closed enough, one could use more well known versions), we can fix a formula \( \psi \) in the language of set theory and \( a_0, a_1 \in H_\lambda^W \) such that \( \mathbb{P} \times \mathbb{P} \) forces “for all \( x \in H_\lambda^W[G_i] \), we have \( x \in H_\lambda^W \) iff \( H_\lambda^W[G_i] \models \psi(x, a_i) \)”, for each \( i \in \{0, 1\} \).

Fix a strategy \( \Sigma \) witnessing the strategic-\( \sigma \)-closure of \( \mathbb{P} \) in \( W \), and let \( ^* \in W \) be a wellorder of \( \mathbb{P} \). Let

\[
z = (M|\omega_1^M, a_0, a_1, \mathbb{P}, ^*, \Sigma, p_0, q_0).
\]

Fix a recursive enumeration \( \langle s_n \rangle_{n < \omega} \) of all formulas in the language of passive preimage.

We now define a run of the “\( \mathbb{P} \)-game” \( \langle p_n, p'_n \rangle_{n < \omega} \), according to \( \Sigma \), with conditions \( p_n \) played by Player I, and \( p'_n \) by player II. We chose \( p_0 \) earlier. Let \( p'_0 = \Sigma(p_0) \). Given \( p'_n \), let \( p_{n+1} \) be the \( ^* \)-least \( p < p'_n \) such that \( (p, q_0) \) either forces

\[
\check{S}_0^\lambda \models \text{“there is a unique ordinal } \alpha \text{ such that } c_n(z, \alpha) \text{”} \tag{11}
\]

\(^{67}\text{Actually one can take } a_0 = a_1 = H_\lambda^W \text{ for a sufficiently large } \alpha < \lambda.\)

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or forces its negation, and such that if \((p, q_0)\) forces (11) then there is some \(\alpha < \lambda\) such that \((p, q_0)\) forces \(s^\mathcal{A}_\lambda \models \varsigma_n(\bar{z}, \bar{\alpha})^\mathcal{A}\). Set \(p'_n = \Sigma(p_0, p_1, \ldots, p_{n+1})\). This determines the
sequence. Choose \(p_\omega \leq p_n, p'_n\) for all \(n < \omega\).

Define \(\langle q_n, q'_n \rangle_{n<\omega}\) completely symmetrically. Let \(q_\omega \leq q_n, q'_n\) for all \(n < \omega\).

For those \(n < \omega\) such that \(p_{n+1}\) forces (11), let \(\alpha_n\) be the witnessing ordinal, and otherwise let \(\alpha_n = 0\). Likewise define \(\beta_n\) for \(q_n+1\).

Note that the forcing statements used in determining the sequences above are all definable over \(\mathcal{H}_\lambda^W\) from the parameter \(z\), uniformly in each level \(\Sigma_n\) (regarding the complexity of \(\varsigma_i\)); this uses Fact 12.3. So letting

\[
H^W = \text{Hull}^\mathcal{H}_\lambda^W(\{z\})
\]

(the uncollapsed hull of all elements of \(\mathcal{H}_\lambda^W\) definable over \(\mathcal{H}_\lambda^W\) in the language of set theory) from the parameter \(z\), then for each \(n < \omega\),

\[
\{z, p_n, p'_n, q_n, q'_n, \alpha_n, \beta_n\} \subseteq H^W.
\]

(But we do not claim that the sequence \(\langle p_n, q_n \rangle_{n<\omega} \in H^W\), for example.)

Now let \(G_0 \times G_1\) be \((W, \mathbb{P})\)-generic with \((p_\omega, q_\omega) \in G_0 \times G_1\). Let \(M_i = (\hat{S}_i)^{G_0 \times G_1}\). Let

\[
H_i = \text{Hull}_{\Sigma_i}^M(\{z\})
\]

(note this hull is computed using the language of premise).

Let \(A = \{\alpha_n \mid n < \omega\}\) and \(B = \{\beta_n \mid n < \omega\}\).

Claim 2. \(A = H_0 \cap \mathsf{OR} = H^W \cap \mathsf{OR} = H_1 \cap \mathsf{OR} = B.\)

Proof. We have \(A \subseteq H_0 \cap \mathsf{OR}\) by choice of \(p_\omega\). To see \(H_0 \cap \mathsf{OR} \subseteq A\), let \(\alpha \in H_0 \cap \mathsf{OR}\).

Then there is some formula \(\varsigma\) such that \(M_0 \models \"\alpha\ is the unique ordinal such that \(\varsigma(z, \alpha)\"\). But then \(\varsigma = \varsigma_n\) for some \(n\), so \(p_{n+1}\) forces this, and \(\alpha = \alpha_n \in A\).

So \(A = H_0 \cap \mathsf{OR}\), and by symmetry, \(B = H_1 \cap \mathsf{OR}\).

By line (12), \(A \cup B \subseteq H^W\).

It just remains to see that \(H^W \cap \mathsf{OR} \subseteq A \cap B\). Let \(\alpha \in H^W \cap \mathsf{OR}\). Note that \(\mathcal{H}_\lambda^W \subseteq M_0\) and by the (local) ground definability fact and our choice of \(z\), \(\mathcal{H}_\lambda^W\) is a class of \(M_0\) definable from the parameter \(z\). Therefore \(\alpha \in H_0\), so \(\alpha \in A\). By symmetry, \(\alpha \in H_1\), so \(\alpha \in B\), completing the proof. \(\square\)

Now let \(H'_i = \text{Hull}_{\Sigma_i}^{|M_i|}(\{z, \lambda\})\) (here \(J(M_i)\) denotes one step in the \(J\)-hierarchy above \(M_i\)). A standard computation shows that \(H'_i \cap \lambda = H_i \cap \mathsf{OR}\).

Let \(K_i\) be the transitive collapse of \(H'_i\). Let \(\pi_i : K_i \to H'_i\) be the uncollapse map.

Let \(\pi_i(\bar{z}_i, \bar{\lambda}_i) = (z, \lambda)\). Then \(K_i = \text{Hull}_{\Sigma_i}^{|K_i|}(\{\bar{z}_i, \bar{\lambda}_i\})\). So \(\rho_i^{K_i} = \omega\) and \(K_i\) is finitely \(\Sigma_i\)-generated. Since \(\omega^W_1 = \omega^M_i\), we have \(\text{cr}(\pi_i) = \omega^K_i = \omega^W_1 \cap H'_i\), so by Claim 2, \(\text{cr}(\pi_i)\) is independent of \(i \in \{0, 1\}\).

Let \(C_i = \mathcal{C}_1(K_i)\). We may assume here that \(K_i\) is 1-solid and 1-universal, and \(C_i\) is likewise and 1-sound, and that \(C_i \trianglelefteq M_i|\omega^M_i\), since these are first-order facts about \(M\). But \(M_0|\omega^M_0 = m = M_1|\omega^M_1\), by assumption. Therefore \(C_0, C_1 \trianglelefteq m\). But also, \(\omega^C_0 = \omega^K_0 = \omega^K_1 = \omega^C_1\) and \(C_0, C_1\) project to \(\omega\). It follows that \(C_0 = C_1\).

We can also assume that Theorem 11.5 holds with respect to \(K_0, K_1\), since this also holds for \(M\). Therefore \(K_0, K_1\) are both finite normal iterates of \(C_0 = C_1\).
Now note that $\text{OR}^{K_i}$ is the ordertype of $H_i \cap \text{OR}$, which is independent of $i$. We have $\pi_i : K_i \to H_i'$ and $\text{rg}(\pi_i) = H_i'$, and therefore $\pi_0 \restriction \text{OR} = \pi_1 \restriction \text{OR}$. By our choice of $(p_0, q_0)$ and $(p_\omega, q_\omega)$, we have $\eta \in \text{rg}(\pi_0), \text{rg}(\pi_1)$, $M_0||\eta = M_1||\eta$ and $M_0||\eta \neq M_1||\eta$. Let $\pi_i(\bar{\eta}) = \eta$. Clearly then $K_0||\bar{\eta} = K_1||\bar{\eta}$. If $M_0||\eta$ is passive and $M_1||\eta$ active, this reflects and $K_0||\eta$ is passive and $K_1||\eta$ is active. If both $M_0||\eta$ and $M_1||\eta$ are active, then the active extenders differ, and because this was forced by $(p_0, q_0)$ over $H_0^\omega$, and it was also forced that $M_0||\eta = M_1||\eta$, it easily follows that there are ordinals in $H_0^\omega, H_1'$ witnessing the distinction between the extenders. (That is, let $F_i = F^{M_1||\eta}$.) Clearly $\kappa_i = \text{cr}(F_i) \in H_i'$ (so $\kappa_0, \kappa_1 \in H_i'$). If $\kappa_0 \neq \kappa_1$ then we are done. If $\kappa_0 = \kappa_1$, then there is some $\alpha < \kappa_i^{\text{acc}(M_1||\eta)} \cap H_i'$ and some $\bar{\beta} \in [\eta]^{<\omega} \cap H_i'$ such that $\bar{\beta} \in F_0(A_\alpha)$ iff $\bar{\beta} \notin F_1(A_\alpha)$, where $A_\alpha$ is the $\alpha$th subset of $[\kappa_i]^{<\omega}$ in the order of construction of $M_\alpha||\eta$.) Note then that $K_0||\bar{\eta} \neq K_1||\bar{\eta}$, and the kind of disagreement corresponds to the case above (that is, whether one is passive and the other active, or both are active but distinct). So $\bar{\eta}$ indexes the least disagreement between $K_0, K_1$.

Now let $T_i$ be the finite normal tree on $C_0 = C_1$ whose last model is $K_i$, given by Theorem 11.5. Note that $T_i$ is determined by $K_i$ (it is the result of comparing $C_i$ with $K_i$, but no extenders get used on the $K_i$ side, and also the trees are finite length, so there are no branches to choose). Note also that if $K_i|\xi$ is active then $T_i$ uses no extender with index $\xi$, so $K_i|\xi = M_0^\xi|\xi$ where $n$ is least such that either $\text{lh}(T_i) = n + 1$ or $\text{lh}(E_n^T) \geq \xi$. (This holds even if there is $m + 1 < \text{lh}(T_i)$ such that $M_n^T||\text{lh}(E_n^T)$ is active. For in this case, $E_n^T$ is superstrong and $M_n^T||\eta$ is active type 2 with $\text{OR}(M_n^T) = \text{lh}(E_n^T)$.) But then $0, m + 1 \restriction T_i$ drops in model, since $C_i$ is passive. Therefore $m + 2 < \text{lh}(T_i)$ and $E_{m+1}^T = F^{M_{m+1}^T}$, so $\text{OR}(M_{m+1}^T)$ is passive in $K_i$.) Since $K_0||\bar{\eta} = K_1||\bar{\eta}$, we get $n < \omega$ with $T_0 \restriction (n + 1) = T_1 \restriction (n + 1)$, and $M_n^T||\bar{\eta} = K_0||\bar{\eta} = K_1||\bar{\eta}$, and since at least one side is active at $\bar{\eta}$, say $K_0||\bar{\eta}$ is active, we must have $M_n^T||\bar{\eta} = K_0||\bar{\eta}$ (as $K_0$ doesn’t move in the comparison). But likewise for $K_1$, and as $K_0||\bar{\eta} \neq K_1||\bar{\eta}$, and $K_1$ doesn’t move in comparison, $K_1||\bar{\eta}$ is passive. So $T_i$ uses an extender indexed at $\bar{\eta}$, so $\bar{\eta}$ is a cardinal of $K_1$. Therefore, as $\lambda$ was a limit cardinal of all models and $\pi_i$ is sufficiently elementary, $\eta$ is a cardinal in $M_1$, hence in $W[G_1]$, hence in $W$. But as $K_1||\bar{\eta}$ is passive, so is $M_1||\eta$, so we are in case (ii) above, so by Claim 1, $\eta$ is not a cardinal in $W$, contradiction. \qed

13 The mouse order $<_p^k$

Our proof of solidity and universality will be by induction on a certain mouse (or mouse-parameter) order. We introduce this now.

13.1 Definition. $M_k$ denotes the class of all $k$-sound preemtice, and $M_k^\text{iter}$ denotes class of all $(k, \omega_1 + 1)$-iterable preemtice in $M_k$.

13.2 Remark. Let $M \in M_k$. A $k$-maximal stack on $M$ was defined in §1.3. The iteration game $G_\text{fin}(M, k, \omega_1 + 1)$ was introduced in [25, Definition 1.1], and proceeds as follows: Player I first plays a $k$-maximal stack $\bar{T} = \langle T_i \rangle_{i \leq m^1}$, where $m < \omega$ and each $T_i$ has finite length, and then the players proceed to play a round of the $(n, \omega_1 + 1)$-iteration game on $M_{\omega_1}^T$, where $n = \text{deg}_{\omega_1}$. The following fact is one case of [25, Theorem 9.6]:

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Let $M \in \mathcal{M}^\text{iter}_k$. Then there is a winning strategy $\Sigma^*_M$ for player II in $\mathcal{G}_{\text{fin}}(M, k, \omega_1 + 1)$. Moreover, given any $(k, \omega_1 + 1)$-strategy $\Sigma$ for $M$, there is a canonical winning strategy $\Sigma^*$ for II in this game determined by $\Sigma$.

Let $\vec{T} = \langle T_n \rangle_{n < \omega}$ be any $k$-maximal stack on $M$ of length $\omega$, consisting of finite length trees $T_n$. Then $(0, \infty]^T_n$ drops for only finitely many $n < \omega$ and $M_{\vec{T}}^\infty$ is wellfounded.

13.4 Definition (Parameter order). Let $\mathcal{M}^*_k$ denote the class of pairs $(M, p)$ such that $M \in \mathcal{M}_k$ and $p \in [\rho^M]^{<\omega}$. We order $\mathcal{M}^*_k$ as follows. For $(M, p), (N, q) \in \mathcal{M}^*_k$, let $(M, p) <^p_k (N, q)$ if there is a $k$-maximal stack $\vec{T} = \langle T_i \rangle_{i < n}$ on $N$ where $n < \omega$ and each $T_i$ is terminally-non-dropping with $\text{lh}(T_i) < \omega$, and there is a $k$-embedding $\pi : M \rightarrow M_{\vec{T}}^\infty$ such that $\pi(p) < i\vec{T}^\infty(q)$.

We are actually only interested in the following sub-order of $<^p_k$:

13.5 Definition (Standard parameter order). We order $\mathcal{M}^*_k$ as follows. Given $M, N \in \mathcal{M}_k$, let $M <^p_k N$ iff $(M, p^M_k) <^p_k (N, p^N_k)$.

13.6 Lemma. If $(M, p) <^p_k (N, q)$ and $N \in \mathcal{M}^\text{iter}_k$ then $M \in \mathcal{M}^\text{iter}_k$.

Proof. By Fact 13.3 and using the witnessing map $\pi$ to copy trees. □

By the following lemma, we can arrange our proof of solidity and universality by induction on $<^p_k$:

13.7 Lemma. $<^p_k \upharpoonright \mathcal{M}^\text{iter}_k$ is transitive and wellfounded. So is $<^p_k \upharpoonright \mathcal{M}^\text{iter}_k$.

Proof. The transitivity of $<^p_k \upharpoonright \mathcal{M}^\text{iter}_k$ follows from Fact 13.3 and the following copying argument (cf. Figure 7). Let $(M_2, q_2) <^p_k (M_1, q_1) <^p_k (M_0, q_0)$, as witnessed by trees $\vec{T}_{12}$ on $M_1$ and $\vec{T}_{01}$ on $M_0$, and maps

$$\pi_{21} : M_2 \rightarrow M_{\vec{T}}_{\infty 12} \text{ and } \pi_{10} : M_1 \rightarrow M_{\vec{T}}_{\infty 01}.$$ 

Let $\vec{T}'$ be the $\pi_{10}$-copy of $\vec{T}_{12}$ to a tree on $M_{\vec{T}}_{\infty 01}$, let $\vec{T}_{02} = \vec{T}_{01} \upharpoonright \vec{T}'$ and

$$\sigma : M_{\vec{T}}_{\infty 12} \rightarrow M_{\vec{T}}_{\infty 02}.$$
be the final copy map and \( \pi_{20} = \sigma \circ \pi_{21} \). Then \( \vec{T}_{02} \) and \( \pi_{20} \) witness the fact that \((M_2, q_2) \leq p^* \) \((M_0, q_0) \). \(^{68}\)

We now show that \( <p^*_k \) is wellfounded below every \((M, p) \) with \( M \in \mathcal{M}^{\text{iter}}_k \). So fix such an \((M, p) \). The argument is essentially the usual one for Dodd-Jensen, but recall that we do not assume AC; one needs a little care to deal with this. First define a more restrictive relation \(<' \) on \( \mathcal{M}_k \) by setting \((N_1, q_1) <' (N_0, q_0) \) iff \((N_1, q_1) < p^*_k (N_0, q_0) \) as witnessed by a stack \( \vec{T} \) (on \( N_0 \)) with \( N_1 = M^\vec{T}_k \) and the map \( \text{id} : N_1 \to N_1 \) (so \( q_1 < i^\vec{T}(q_0) \)). Note that \(<' \) is also transitive. The set of all \((N, q) \) such that \((N, q) <' (M, p) \) is wellordered, as is the set of all witnessing trees \( \vec{T} \), since those trees are finite. So if \(<' \) is illfounded then there is a \(<'\)-descending sequence \( \langle (N_m, q_m) \rangle_{m<\omega} \), and a sequence \( \langle \vec{T}_m \rangle_{m<\omega} \) of witnessing stacks. But letting \( \vec{T} \) be the concatenation of all \( \vec{T}_m \), then \( \vec{T} \) is a length \( \omega \), \( k \)-maximal stack on \( M \) such that \( M^\vec{T}_k \) is illfounded, contradicting Fact 13.3.

Now suppose that \( <p^*_k \) is illfounded below \((M, p) \). Say that \((N, q) \) is bad if \((N, q) \) is in the illfounded part of \( <p^*_k \) and either \((N, q) = (M, p) \) or \((N, q) <' (M, p) \). Since \(<' \) is wellfounded below \((M, p) \), we can fix a bad \((N, q) \) such that there is no bad \((N', q') \) \(<' (N, q) \). Let \((P', r') \leq p^*_k (N, q) \) with \((P', r') \) also in the illfounded part of \( <p^*_k \), as witnessed by \( \vec{T} \) and \( \pi : P' \to M^\vec{T}_k \). Let \( P = M^\vec{T}_k \) and \( r = \pi(r') \). So \((P, r) \) \(<' (N, q) \), so by choice of \((N, q) \), \((P, r) \) is in the wellfounded part of \( <p^*_k \). But by copying using \( \pi \),

\[
\text{for every } (P'', r'') \leq p^*_k (P', r') \text{, we have } (P'', r'') < p^*_k (P, r),
\]

so \( <p^*_k \) is wellfounded below \((P, r) \), a contradiction. \( \square \)

**13.8 Lemma.** Let \( M \in \mathcal{M}^{\text{iter}}_k \) and suppose \( H \) is \((k+1)\)-solid for all \( H <^p_k M \). Let \( \vec{T} \) be \( k \)-maximal on \( M \) of successor length, such that \( b^\vec{T} \) does not drop in model or degree. Then \( \rho^N_{k+1} = i^\vec{T}(p^M_{k+1}) \) and \( \rho^N_{k+1} = \sup i^\vec{T} \rho^M_{k+1} \).

**Proof.** Write \( N = M^\vec{T}_k \). By preservation of fine structure (Lemma 3.8 and Corollary 3.9), and without considering \( <^p_k \), we have:

(a) \( \rho^N_{k+1} = \sup i^\vec{T} \rho^M_{k+1} \) and \( \rho^N_{k+1} \leq i^\vec{T}(p^M_{k+1}) \),

(b) \( N \) is \((k+1)\)-solid if \( M \) is \((k+1)\)-solid,

(c) if \( M \) is \((k+1)\)-solid then \( \rho^N_{k+1} = i^\vec{T}(p^M_{k+1}) \).

So suppose \( \rho^N_{k+1} < i^\vec{T}(p^M_{k+1}) \); we will reach a contradiction.

Suppose \( \text{lh}(\vec{T}) < \omega \). Then \( N <^p_k M \), as witnessed by \( \vec{T} \) and \( \text{id} : N \to N \). Therefore by the lemma’s hypothesis, \( N \) is \((k+1)\)-solid. So by (b), \( M \) is \((k+1)\)-solid, so by (c), \( \rho^N_{k+1} = i^\vec{T}(p^M_{k+1}) \), a contradiction.

So \( \text{lh}(\vec{T}) \geq \omega \). Let \( \vec{T} \) be a \( k \)-maximal tree of finite length which captures \((\vec{T}, p^N_{k+1}, 0) \) (exists by 6.20). Write \( \bar{N} = M^\vec{T}_k \). So \( b^\vec{T} \) does not drop in model or degree, and there is a \( k \)-embedding \( \sigma : \bar{N} \to N \) with \( \sigma \circ i^\vec{T} = i^\vec{T} \) and \( \bar{p} \in \bar{N} \) with \( p^N_{k+1} = \sigma(\bar{p}) \). Using (a) and its version for \( \vec{T} \) and commutativity, we have \( \rho^N_{k+1} = \sup \sigma^* \rho^N_{k+1} \), so \( \sigma(\rho^N_{k+1}) \geq \rho^N_{k+1} \). Note \(^{68}\) \( <^p_k \) might not be transitive on all of \( \mathcal{M}_k \) (including non-iterable premice), because the copying construction might produce an illfounded model. This is why Fact 13.3 is relevant to transitivity.
that $\bar{p} < i^T(p_{k+1}^M)$, since $\sigma(\bar{p}) = p_{N+1}^N < i^T(p_{k+1}^M) = \sigma(i^T(p_{k+1}^M))$. As in the previous paragraph, $i^T$ is $p_{k+1}$-preserving, so $\bar{p} < p_{N+1}^N = i^T(p_{k+1}^M)$. So
\[ t = Th_{\Sigma_{k+1}}(\rho_{k+1}^N \cup \{\bar{p}, \bar{p}_k^N\}) \in N. \]
So $\sigma(t) \in N$, and note that from $\sigma(t)$, we get
\[ Th_{\Sigma_{k+1}}(\sigma(\rho_{k+1}^N) \cup \{\sigma(\bar{p}), \bar{p}_k^N\}) \in N. \]
(This is as in the computation of solidity witnesses from generalized solidity witnesses, using that $\sigma$ is $r_{\Sigma_{k+1}}$-elementary.) But as $\sigma(\bar{p}) = p_{N+1}^N$ and $\sigma(\rho_{k+1}^N) \geq \rho_{k+1}^N$, this is a contradiction. □

14 Solidity and universality

In this section we prove that normally iterable mice are solid and universal. For the entire section, we will fix some $k < \omega$, and deal with $k$-sound premice $M$, and proving fine structure at the $(k + 1)$th level, like $(k + 1)$-solidity, etc. To reduce notation, we will usually drop the subscript "$(k + 1)$" from the notation $p_{k+1}^M$, $\rho_{k+1}^M$, $C_{k+1}(M)$, Hull$_{k+1}^M$, $\text{cHull}_{k+1}^M$, writing instead $p_{k+1}$, etc. (But we still write $p_k^M$, etc, for the objects at the $k$th level.)

In proving the $(k + 1)$-solidity of a premouse $\tilde{M}$, we will want to show that for certain $\tilde{H}$ and near $k$-embeddings
\[ \tilde{\pi} : \tilde{H} \to \tilde{M}, \]
we have $\tilde{H} \in \tilde{M}$. In some cases we will deduce this from Lemma 11.6 (on projectum-finitely generated mice), in others, we will quote facts on iterable bicephali from [24], and in the remaining cases, where the main work is, we will form and analyze a comparison of a certain kind of bicephalus $B = (\delta, \gamma, \tilde{\pi}, \tilde{H}, M)$. The details of the relevant kind of bicephali are given in Definition 14.2, but its two models are $\tilde{H}$ and $\tilde{M}$, and $\gamma = \text{cr}(\tilde{\pi})$ is an element of $p_{k+1}^M$, and $\delta = \text{card}^M(\gamma)$.

14.1 The main argument

14.1 Theorem (Solidity and universality). Let $k < \omega$ and let $M$ be a $k$-sound, $(k, \omega_1+1)$-iterable premouse. Then $M$ is $(k + 1)$-solid and $(k + 1)$-universal.

We gave an outline for the plan of the proof in §2.1. The reader might want to review that prior to beginning the proof below to get the general idea in mind (though we won’t actually rely on it).

Proof. It suffices to consider only premice in $\mathcal{M}_{k}^\text{iter} \cap \text{HC}$, since given any $M \in \mathcal{M}_{k}^\text{iter}$, working in $L[M]$, we can find a countable elementary substructure of $M$.

The proof is by induction on $<^P_k \upharpoonright (\mathcal{M}_{k}^\text{iter} \cap \text{HC})$ (see §13), and this order is in large part a substitute for Dodd-Jensen. So fix $M \in \mathcal{M}_{k}^\text{iter} \cap \text{HC}$ and suppose by induction that
\[ \forall H \left((H <^P_k M) \implies H \text{ is } (k + 1)\text{-solid and } (k + 1)\text{-universal}\right); \quad (13) \]
note here that every \( H <^p_k M \) is in HC, since the trees \( T \) witnessing that \( H <^p_k M \) are finite. We must prove that \( M \) is \((k + 1)\)-solid and \((k + 1)\)-universal.

Before we begin the main argument, sketched in §2.1, we want to observe that we may assume that \( M = \text{Hull}^k(M) \rho^M \cup \bar{p}^M_{k+1} \), and also reduce the problem to \((k + 1)\)-solidity (dispensing with \((k + 1)\)-universality). Let \( \rho = \rho^M \). Let \( \bar{M} = \mathcal{C}(M) \) and let \( \pi : \bar{M} \rightarrow M \) be the uncollapse. Then \( \pi \) is a \( k \)-embedding and \( \bar{M} \) is \((k, \omega_1 + 1)\)-iterable. Let \( \pi(q) = p^M \). We have

\[
\bar{M} = \text{Hull}^k(M \cup \{q, p^M \}).
\]  

(14)

**Claim 1.** \( \rho^\bar{M} = \rho \) and \( p^\bar{M} = q \).

**Proof.** That \( \rho^\bar{M} = \rho \) is as usual: By line (14), \( \rho^M \leq \rho \). Conversely, given \( \delta < \rho \) and \( A \subseteq \delta \) which is \( r\Sigma^k_{k+1} \)-definable, then since \( \pi \) is \( r\Sigma^k_{k+1} \)-elementary and \( \text{cr}(\pi) \geq \rho \), \( A \) is also \( r\Sigma^M_{k+1} \)-definable, so \( A \in M[\delta^+ \subseteq M[\rho \subseteq \bar{M}] \). So \( \rho^\bar{M} = \rho \).

Since \( \rho^\bar{M} = \rho \) and by line (14), \( p^\bar{M} \leq q \). Suppose \( p^\bar{M} < q \). Then \( \bar{M} <^p_k M \), as witnessed by \( T, \pi \), where \( T \) is trivial. So by line (13), \( \bar{M} \) is \((k + 1)\)-solid and \((k + 1)\)-universal. So letting \( C = \mathcal{C}(\bar{M}) \) and \( \sigma : C \rightarrow \bar{M} \) the core map, we have \( \rho^C = \rho \) and \( \pi(p^C) = p^\bar{M} \), so \( \pi(\sigma(p^C)) < p^M \). Therefore \( C \in M \) (by the minimality of \( p^M \)) and \( C <^p_k M \), so \( C \) is also \((k + 1)\)-solid, so \( C \) is \((k + 1)\)-sound. So by line (14) and Corollary 11.7 applied to \( \bar{M} \), the theory

\[
t = \text{Th}_{r\Sigma^k_{k+1}}(\rho \cup \{q, p^M \}),
\]
coded as a subset of \( \rho \), is definable from parameters over \( C \). Since \( C \in M \), therefore \( t \in M \). But \( t \) is equivalent to

\[
t' = \text{Th}_{r\Sigma^k_{k+1}}(\rho \cup \{p^M, p^M \}),
\]
so \( t' \in M \), a contradiction. \( \square \)

**Claim 2.** If \( \bar{M} \) is \((k + 1)\)-solid then \( M \) is \((k + 1)\)-solid and \((k + 1)\)-universal.

**Proof.** Suppose \( \bar{M} \) is \((k + 1)\)-solid. Then by Claim 1 and line (14), \( \bar{M} \) is \((k + 1)\)-sound, and \( \pi(p^\bar{M}) = p^M \), so \( M \) is \((k + 1)\)-solid. Consider \((k + 1)\)-universality. We have \( \bar{M} = \mathcal{C}(M) \not\subseteq M \), and the core map \( \pi : \bar{M} \rightarrow M \) satisfies the requirements of condensation Fact 1.2, with the parameter “\( \rho' \)" there being \( \rho = \rho^\bar{M} = \rho^M \). If \( \rho^\bar{M} < \rho^M \), then by Fact 1.2 part 1(iii), \( M[\rho \text{ is active}, \) but \( \rho \) is an \( M \)-cardinal. So \( \rho^M = \rho^M \). But then \( \bar{M}[\rho^\bar{M}] = M[\rho^M] \), since \( \text{cr}(\pi) \geq \rho \) and if \( \text{cr}(\pi) = \rho \) then we can apply condensation for \( \omega \)-sound mice (Fact 1.9) to \( \pi \mid N : N \rightarrow \pi(N) \) for each \( N \in M \) with \( \rho_N = \rho \), to see that \( N \in M \). This gives in particular that \( \mathcal{P}(\rho) \cap \bar{M} = \mathcal{P}(\rho) \cap M \), so \( M \) is \((k + 1)\)-universal, as desired. \( \square \)

So it suffices to prove that \( \bar{M} \) is \((k + 1)\)-solid.

**Claim 3.** For every \( H \), if \( H <^p_k \bar{M} \) then \( H <^p_k M \).

**Proof.** By Claim 1, and via copying as in the proof of transitivity of \( <^p_k \).

So the induction hypothesis, line (13), still applies after replacing \( \bar{M} \) with \( M \). We reset notation, writing “\( M \)” instead of “\( \bar{M} \)” ; this amounts to proving \( M \) is \((k + 1)\)-solid under the added assumption, which we now make, that
Assumption 1. $M = \text{Hull}^M(p^M \cup \bar{p}^M)$.  

For $q \leq p^M$, we will prove that $(M, q)$ is $(k + 1)$-solid, by induction on $\text{lh}(q)$. So let $q < p^M$ be such that $(M, q)$ is $(k + 1)$-solid and let $n = \text{lh}(q)$. For $k$-sound premice $N$, write $q^N$ for $p^N|n$. Let $\gamma = \max(p^M \setminus q^M)$. Let 

$$H = \text{cHull}^M\{(q^M, \bar{p}^M_k) \cup \gamma\}$$

and let $\pi : H \to M$ be the uncollapse and $\pi(\gamma) = q^M$ (we have since discarded the core map $\pi$ defined earlier). We have to see that $H \in M$. So we now assume otherwise, and will draw out a contradiction:

Assumption 2. $H \notin M$. \(^{69}\)

We now prepare for the main argument. Let $B = (\delta, \gamma, \pi, H, M)$. We will consider $B$ as a kind of bicephalus, for which $\delta$ is the primary exchange ordinal; so extenders $E$ with $\text{cr}(E) < \delta$ will lift the entire bicephalus, whereas if $\text{cr}(E) \geq \delta$ then $E$ will apply to just one model. The plan is to form and analyse a comparison of $B$ with $M$. This bicephalus $B$ will be iterable because $M$ is, and as we will be able to lift trees on $B$ to trees on $M$. The comparison between $B$ and $M$ will terminate via essentially the usual argument for comparison of mice. The fine structural properties of $B$ will be preserved nicely by iterations, so that iterates $B'$ will have similar properties. But using these properties, and the precise rules for forming the comparison, we will be able to argue that the comparison cannot terminate, giving the desired contradiction.

We will specify precisely how we form iteration trees on $B$ in Definition 14.7. But before that, it will be useful to establish some more of the fine structural properties of $B$, some of which will be abstracted, in Definition 14.2, into the kind of bicephali $B'$ which will arise as iterates of $B$. We will then establish the basic properties of iteration trees on $B$, adapting the picture for trees on premice and phalanxes. After that, immediately following Claim 13, we will specify the precise rules for the comparison, and then proceed to the actual analysis of the comparison, which constitutes Claims 14–24.

Claim 4. $\pi$ is a $k$-embedding.

Proof. Suppose not. Then $\pi^*\bar{p}^H_k$ is bounded in $\rho^M_k$, which implies $H \in M$; see [24, Lemma 2.4]. This contradicts Assumption 2. \(\square\)

Therefore $\text{cr}(\pi) = \gamma < \rho^H_k$ and $\pi(\gamma) < \rho^M_k$. Let $\delta = \text{card}^M(\gamma)$. As usual, there is a significant break into two cases, and using condensation for $\omega$-sound mice (Fact 1.9) in the usual manner like in the proof of Claim 2, we have either:

1. $\delta = \gamma$ is a limit cardinal of $M$ and inaccessible in $H$, $\gamma + H \leq \gamma^+M$ and $H||\gamma^+H = M||\gamma^+H$, or

2. $\gamma = \delta + H$ and $\pi(\gamma) = \delta^+M$ and $H||\gamma = M||\gamma$, and either:
   - $M||\gamma$ is passive and $H||\gamma^+H = M||\gamma^+H$, or
   - $M||\gamma$ is active with an $M$-total extender $F$ and $H||\gamma^+H = \text{Ult}(M, F)||\gamma^+H$.

\(^{69}\)Once we have proven that in fact, $H \in M$, we will anyway be able to deduce a precise description of $H$ in terms of proper segments of $M$ via Theorem 11.5, and thus recover at that point information that might otherwise have been obtained by arguing directly, without contradiction.
So in any case, we have $H||\gamma^+H \in M$.

**Claim 5.** $H \not\in M$, and if $M|\gamma$ is active then $H \not\in \Ult(M|\gamma,F^M|M)$.

**Proof.** We have $H \not\in M$, so if $H \subseteq M$ then $H \land M$ and $H \in M$. And if $M|\gamma$ is active then $U$ (as above) is in $M$, so if $H \subseteq U$ then $H \in M$. \hfill\Box

**Claim 6.**

1. If $\gamma = \text{lgcd}(H)$ then $H,M$ are active type 2 and $k = 0$.

2. If $H,M$ are active type 1 or 2 and $k = 0$ then $\gamma \notin [k,\kappa^{+M}]$, where $\kappa = \text{cr}(F^M)$.

**Proof.** Suppose $\gamma = \text{lgcd}(H)$. Then if $H$ is passive then $H = H||\gamma^+H \in M$. If $H$ is active type 3 then $\rho_0^H = \gamma$, contradicting that $\gamma < \rho_0^H$. If $H$ is type 1 then $\gamma > \delta$ and $\gamma = \nu(F^H)$, so $k = 0$, but then $\pi^\gamma$ is bounded in $\pi(\gamma) = \nu(F^M)$, so by standard calculations, $\pi$ is bounded in $\text{OR}^M = \rho_0^M = \rho_k^M$, a contradiction. So $H$ is type 2, so $M$ is also, and also $\rho_1^H \leq \gamma$, so $k = 0$.

Now suppose $H,M$ are active type 1/2, $k = 0$ and $\gamma \in [k,\kappa^{+M}]$. Note that $\kappa < \gamma = \kappa^+H < \kappa^{+M}$ and $\pi^\kappa\kappa^+H = \kappa^+H$ is bounded in $\kappa^{+M}$. So again $\pi$ is bounded in $\rho_k^M$. \hfill\Box

**Claim 7.** We have:

1. $\pi(p^H) < p^M$, so $H <_k^p M$,

2. $H$ is $(k+1)$-solid and $(k+1)$-universal with $\rho^H \leq \gamma$,

3. $q^H = r$, and either $p^H = q^H$ or $p^H = q^H \upharpoonright s$ for some $s \subseteq \gamma$,

4. $p^M < \rho^H$.

**Proof.** Parts 1–3: Recall $\pi(r) = q^M$. So $H = \text{Hull}^H(\gamma \cup \{r,p_k^M\})$. So $\rho^H \leq \gamma$ and either:

(i) $p^H \leq r$, so $\pi(p^H) \leq q^M < p^M$, or

(ii) $p^H = r \upharpoonright s$ where $s \subseteq \gamma$, so $\pi(p^H) = q^M \upharpoonright \pi(s) = q^M \upharpoonright s < p^M$.

So in either case $\pi(p^H) < p^M$, so by line (13), $H$ is $(k+1)$-solid and $(k+1)$-universal.

We have verified parts 1 and 2. Now consider part 3. It is enough to see that either $p^H = r$ or $p^H = r \upharpoonright s$ for some $s \subseteq \gamma$, since this implies $q^H = r$, as we already know that $H$ is $(k+1)$-solid. And since (i) or (ii) above holds, it therefore suffices to see that $p^H \not= r$. So suppose $p^H < r$. Let $C = \mathcal{C}(H)$. First note that since $H <_k^p M$, also $C \not<_k^p M$, $C$ is $(k+1)$-sound and $\rho^C = \rho^H$.

Now we will show that $C \in M$ and that $H$ is finitely generated above its projectum; that is,

$$H = \text{Hull}^H(p^H \cup \{x\}) \text{ for some } x \in \mathcal{C}_0(H). \quad (15)$$

Given this, as in the proof of Claim 1, we can use Lemma 11.6 to deduce that $H \in M$.

So, since $p^H < r$, we have $\pi(p^H) < q^M$, and because $(M,q^M)$ is solid and $\rho^H \leq \gamma = \text{cr}(\pi)$, therefore $C \in M$.

Now let us establish line (15). Since $C \in M$ and $M|\delta \subseteq C$, we have $\rho^C = \rho^H \geq \delta$. Also $M|\gamma \subseteq M$ and clearly $\rho^H \leq \gamma$. So $\rho^C = \rho^H \in [\delta,\gamma)$ and if $\gamma$ is an $M$-cardinal then $\rho^H = \gamma = \delta$. But if $\rho^H = \gamma$ then line (15) holds, as witnessed by $x = (r,p_k^M)$, so suppose
\[ \rho^H = \delta < \delta^+ = \gamma. \] Then since \( H \) is \((k+1)\)-universal, we have \( \gamma \subseteq \text{Hull}^H((\delta+1) \cup \{p^M_k\}) \) (note the hull is uncollapsed), and line (15) follows, as desired.

Part 4: Since \( \text{cr}(\pi) \geq \rho^M \), easily \( \rho^H \geq \rho^M \). So suppose \( \rho = \rho^H = \rho^M \). Let

\[ t = \text{Th}_{\Sigma_k+1}(\rho \cup \{p^H, \tilde{p}^H_k\}) = \text{Th}_{\Sigma_k+1}(\rho \cup \{\pi(p^H), \tilde{p}^M_k\}). \]

Since \( \pi(p^H) < p^M \) (by Claim 7), \( t \in M \), so \( C = \mathcal{C}(H) \in M \). Since \( H \) is \((k+1)\)-universal, \( C \) codes a surjection \( \rho \rightarrow H \cap \mathcal{P}(\rho) \), so it follows that \( \rho^H < \rho^M \). Since \( H|\delta = M|\delta \) and \( \delta \) is an \( M \)-cardinal, therefore \( \delta \leq \rho \), so \( \gamma \in \{\rho, \rho^H\} \). So if \( \gamma = \rho \) then since \( \pi(\gamma) < p^M \), we have \( H \in M \) by the minimality of \( p^M \). And if \( \gamma = \rho^H \) then since \( H \) is \((k+1)\)-universal, \( H = \text{Hull}^H(\rho \cup \{r, \alpha, \tilde{p}^M_k\}) \) for some \( \alpha < \gamma \), and since \( \pi(\gamma) < p^M \), the minimality of \( p^M \) again yields that \( H \in M \).

\[ \square \]

In the following definition we abstract out the key properties of \( B \) that we have established so far, and which will also hold for the kinds of iterates \( B' \) to be considered:

14.2 Definition. A structure \( B' = (\delta', \gamma', \pi', H', M') \) is a pre-relevant bicephalus of degree \( k \) and length \( n \) iff:

- \( M' \) is a \( k \)-sound pm, \( \text{lh}(p^{M'}) \geq n \), \( (M', q^{M'}) \) is solid, \( \delta' = \text{card}^{M'}(\gamma') \) and

\[ M' = \text{Hull}^{M'}((\gamma' + 1) \cup \{q^{M'}, \tilde{p}^{M'}_k\}), \]

- \( H' \) is a \( k \)-sound pm, \( \text{lh}(p^{H'}) \geq n \), \( (H', q^{H'}) \) is solid, \( \gamma' \) is an \( H' \)-cardinal, \( \gamma' \leq \min(q^{H'}) \) and

\[ H' = \text{Hull}^{H'}(\gamma' \cup \{q^{H'}, \tilde{p}^{H'}_k\}), \]

- \( \pi' : \mathcal{C}_0(H') \rightarrow \mathcal{C}_0(M') \) is a \( k \)-embedding, \( \text{cr}(\pi') = \gamma' \) and \( \pi(q^{H'}) = q^{M'} \).

Let \( B' \) be a pre-relevant bicephalus with notation as above. We write \( \delta^{B'} = \delta' \) and \( \gamma^{B'} = \gamma' \) etc. Note that \( H' = \text{cHull}^{M'}(\gamma' \cup \{q^{M'}, \tilde{p}^{M'}_k\}) \) and \( \pi' \) is the uncollapse map. Note that \( q^{M'} \leq p^{M'} \) and either

(i) \( q^{M'} \sim (\gamma) \leq p^{M'} \), or

(ii) \( p^{M'} < q^{M'} \sim (\gamma') \).

We say that \( B' \) is relevant iff (i) holds.

14.3 Definition. Let \( B' \) be a pre-relevant bicephalus and \( E \) be a short extender. We say that \( E \) is weakly amenable to \( B' \) iff \( \text{cr}(E) < \delta' \) and \( E \) is weakly amenable to \( B' |\delta^{B'} = H'|\delta^{B'} = M'|\delta^{B'} \). We (attempt to) define

\[ \text{Ult}(B', E) = (\tilde{\delta}, \tilde{\gamma}, \tilde{\pi}, \tilde{H}, \tilde{M}) \]

as follows. We set \( \tilde{M} = \text{Ult}_k(M', E) \); suppose this is wellfounded. Letting \( j = i_{E,k}^{M'} \), then \( (\tilde{\delta}, \tilde{\gamma}) = j(\delta', \gamma') \),

\[ \tilde{H} = \text{cHull}^{\tilde{M}}(\tilde{\gamma} \cup \{q^{\tilde{M}}, \tilde{p}^{\tilde{M}}_k\}), \]

and \( \tilde{\pi} : \tilde{H} \rightarrow \tilde{M} \) is the uncollapse. Define \( i_E^{B'} : H' \rightarrow \tilde{H} \) as \( i_E^{B'} = \tilde{\pi}^{-1} \circ j \circ \pi' \).

\[ \square \]

\[ ^70 \text{An alternative to these last two sentences would be to argue via Lemma 11.6 as in the proof of parts 1–3, but the argument provided is simpler.} \]
14.4 Definition. Let \( B' \) be a relevant bicephalus of degree \( k \). An abstract degree \( k \) weakly amenable iteration of \( B' \) is the obvious analogue of Definition 3.5: a pair \( (\langle E_\alpha, \alpha < \lambda \rangle, \langle B_\alpha, \alpha < \lambda \rangle) \) where \( B_0 = B' \), \( B_\alpha \) is a relevant bicephalus for each \( \alpha < \lambda \), each \( E_\alpha \) is a short extender weakly amenable to \( B_\alpha \) (so \( \text{cr}(E_\alpha) < \delta B_\alpha \)), \( B_{\alpha+1} = \text{Ult}(B_\alpha, E_\alpha) \), and \( B_\eta \) is the resulting direct limit when \( \eta \leq \lambda \) is a limit. Wellfoundedness of the iteration requires that \( B_\lambda \) is wellfounded.

14.5 Lemma. Continuing as in Definition 14.3, if \( \tilde{M} \) is wellfounded then:

1. \( j \) is a \( k \)-embedding with \( j(q^{M'}) = q^{\tilde{M}} \).
2. \( H' \in M' \) iff \( \tilde{H} \in \tilde{M} \).
3. \( \tilde{B} \) is a pre-relevant bicephalus.
4. \( \tilde{\pi}, i_E^{B'} \) are \( k \)-embeddings, \( i_E^{B'} \mid (\gamma')^{+H'} \subseteq j \) and \( \tilde{\pi} \circ i_E^{B'} = j \circ \pi' \).
5. If \( j \) is \( p_{k+1} \)-preserving and \( B' \) is relevant then \( \tilde{B} \) is relevant.

Likewise for abstract degree \( k \) weakly amenable iterations of \( B' \), with \( B_\lambda \) replacing \( \tilde{B} \), \( j \circ \pi \) replacing \( j \), etc.

Proof. Part 1 is completely routine. Part 2 is a consequence of this and \((z, \zeta)\)-preservation (Fact 3.7). That is, for the more subtle direction, suppose that \( H' \notin M' \). Then since \( (M', q^{M'}) \) is \((k+1)\)-solid, we have \( q^{M'} \leq z^{M'} \) and \( (z^{M'}, \zeta^{M'}) \leq (q^{M'}, \gamma') \). So \( (z^{\tilde{M}}, \zeta^{\tilde{M}}) = (j(z^{M'}), \sup j^{+} \zeta^{M'}) \leq (j(q^{M'}), j(\gamma')) \), but \( j(q^{M'}) = q^{\tilde{M}} \) and \( (\tilde{M}, q^{\tilde{M}}) \) is \((k+1)\)-solid, so

\[ \text{Th}_{\tilde{M}, \Sigma_{k+1}, \{j(\gamma') \cup \{q^{\tilde{M}}, \tilde{p}^{\tilde{M}}\}\}} \notin \tilde{M}, \]

so \( \tilde{H} \notin \tilde{M} \).

For the remaining parts, one should first prove everything other than the fact that \( i \mid (\gamma')^{+H'} \subseteq j \), where \( i = i_E^{B'} \), and we leave those first parts to the reader and assume them. Let us now deduce that \( i \mid (\gamma')^{+H'} \subseteq j \). We first show \( i(\gamma') = j(\gamma') \). Note that

\[ \pi'(\gamma') = \text{the least } \gamma^* \in \text{Hull}^M(\gamma' \cup \{q^{M'}, \tilde{p}^{M'}\}) \setminus \gamma'. \]

We just need to see that

\[ j(\pi'(\gamma')) = \text{the least } \gamma^* \in \text{Hull}^{\tilde{M}}(j(\gamma') \cup \{q^{\tilde{M}}, \tilde{p}^{\tilde{M}}\}) \setminus j(\gamma'). \]

But supposing that \( \gamma^* \in [j(\gamma'), j(\pi'(\gamma'))] \) is also in that hull, then the existence of such a \( \gamma^* \) is an \( \Sigma_{k+1} \) assertion of the parameter \( j(q^{M'}, \gamma', \tilde{p}^{M'}) \), hence pulls back to \( M' \), a contradiction.

So \( i(\gamma') = j(\gamma') \). But then it is easy to deduce that \( i \mid \mathcal{P}(\gamma') \cap H' \subseteq j \), using that \( \tilde{\pi} \circ i = j \circ \pi' \) and \( \text{cr}(\pi') = \gamma' \) and \( \text{cr}(\tilde{\pi}) = j(\gamma') \). It follows that \( i \mid (\gamma')^{+H'} \subseteq j \). \( \square \)

14.6 Remark. Continuing as in Definition 14.4, suppose that \( i_{E_{\alpha}, k} \) is \( p_{k+1} \)-preserving for each \( \alpha < \lambda \), and \( M_{\lambda} \) is wellfounded. Then note that \( B_\lambda \) is also a relevant bicephalus and the maps \( j_{\alpha \lambda} : M_\alpha \rightarrow M_\lambda \) are \( p_{k+1} \)-preserving \( k \)-embeddings, and the analogue of Lemma 14.5 holds.
We now define the kind of iteration tree on $B$ which we will use for comparison: a degree-maximal iteration tree. These are analogous to those in §5, but there are some key differences: when forming an ultrapower $\bar{B} = \text{Ult}(B', E)$ of a bicephalus and the associated maps $i : H' \to \bar{H}$ and $j : M' \to \bar{M}$, we follow 14.3, and when $\delta^{B'} < \gamma^{B'}$ and $\text{cr}(E) = \delta^{B'}$, then we do not form a bicephalus, but we need to be careful about how to proceed: in condition in 14.7(12e)(i) below we apply the extender $E_{\alpha}$ to some $Q \triangleleft M_{\beta}$, although one might have considered applying it to $H_{\beta}$. One further difference, the analogue of the anomalous case in phalanx iterations (see Footnote 52), is that we can have non-premice appearing in the tree (they arise in the situation just mentioned, if $Q$ is type 3 with $\nu^Q_0 = \delta_\beta$); thus, we only say pre-ISC-premouse in condition 14.7(8).

**14.7 Definition.** Let $B'$ be a degree $k$ relevant bicephalus. A degree-maximal iteration tree $\mathcal{T}$ on $B'$ of length $\lambda$ is a system

$$\mathcal{T} = \left(\langle T, \mathcal{R}, \mathcal{D}, \mathcal{D}_{\text{deg}}, (B_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}, \pi_{\alpha}, \text{sides}_{\alpha})_{\alpha < \lambda}, \langle M_{\alpha}^e, \text{deg}_{\alpha}^e, i_{\alpha}^e \rangle_{\alpha \leq \beta < \lambda \text{ and } e < 2}, \right.$$  

\[\text{exit}_{\alpha}, \text{exit}_{\alpha}, E_{\alpha}, B_{\alpha+1}^e, i_{\alpha+1}^e, \rangle_{\alpha+1 < \lambda}, \langle M_{\alpha+1}^e, i_{\alpha+1}^e \rangle_{\alpha+1 < \lambda \text{ and } e < 2}\right),$$

with the following properties for all $\alpha < \lambda$, where we write $H_{\alpha} = M_{\alpha}^0$, $M_{\alpha} = M_{\alpha}^1$, $i_{\alpha\beta} = i_{\alpha,\beta}^0$, $j_{\alpha\beta} = i_{\alpha,\beta}^1$, etc:

1. $\langle T, \mathcal{R} \rangle$ is an iteration tree order on $\lambda$,
2. $\mathcal{D} \subseteq \lambda$ is the set of dropping nodes,
3. $\emptyset \neq \text{sides}_{\alpha} \subseteq \{0, 1\},$
4. $\alpha \in \mathcal{D}$ iff sides$_{\alpha} = \{0, 1\},$
5. $0 \in \mathcal{D} \subseteq \lambda$ and $\mathcal{D} \cap [0, \alpha]^T$ is a closed initial segment of $[0, \alpha]^T$,
6. $B_0 = B'$ and $(\text{deg}_{0}^0, \text{deg}_{0}^1) = (k, k),$
7. If $\alpha \in \mathcal{D}$ then $B_{\alpha} = \langle \gamma_{\alpha}, \delta_{\alpha}, \pi_{\alpha}, H_{\alpha}, M_{\alpha} \rangle$ is a degree $k$ relevant bicephalus and $(\text{deg}_{\alpha}^0, \text{deg}_{\alpha}^1) = (k, k)$.
8. If $\alpha \notin \mathcal{D}$ and sides$_{\alpha} = \{e\}$ then $B_{\alpha} = M_{\alpha}^e$ is a deg$^e_{\alpha}$-sound pre-ISC-premouse, and $M_{\alpha}^{e-1} = \emptyset$.
9. If sides$_{\alpha} = \{0\}$ then $B_{\alpha} = H_{\alpha}$ is a premouse,
10. If $\alpha + 1 < \lambda$ then $e = \text{exit}_{\alpha} \in \text{sides}_{\alpha}$, exit$_{\alpha} \subseteq M_{\alpha}^e$, and $E_{\alpha} = F^{\text{exit}_{\alpha}} \neq \emptyset$.
11. If $\alpha + 1 < \beta + 1 < \text{lh}(\mathcal{T})$ then $\text{lh}(E_{\alpha}) \leq \text{lh}(E_{\beta})$.

12. Suppose $\alpha + 1 < \text{lh}(\mathcal{T})$ and let $\beta = \text{pred}^T(\alpha + 1)$. Then:

   (a) $\beta$ is the least $\beta'$ such that $\text{cr}(E_{\alpha}) < \nu(\text{exit}_{\beta'})$.
   (b) sides$_{\alpha+1} \subseteq \text{sides}_{\beta}$

\footnote{Recall from Remark 3.1 that for an active pre-ISC-pm $S$, $\nu(S) = \max(\nu(S), \text{lgcd}(S))$.}
(c) \( \alpha + 1 \in \mathcal{B} \) iff \( \beta \in \mathcal{B} \) and \( \text{cr}(E_\alpha) < \delta_\beta \) and \( E_\alpha \) is \( B_\beta \text{-total} \).

(d) If \( \alpha + 1 \in \mathcal{B} \) then \( B^{*}_{\alpha+1} = B_\beta \) and \( B^{\ast}_{\alpha + 1} = \text{Ult}(B_\beta, E_\alpha) \) and \( i^{t}_{\alpha+1}, j^{*}_{\alpha+1} \) are the associated maps (all defined as in 14.3).

(e) If \( \beta \in \mathcal{B}^T \) but \( \alpha + 1 \notin \mathcal{B}^T \) then:

(i) if \( \delta^B < \gamma^B \) and \( \text{exitside}_\beta = 0 \) and \( \gamma_\beta \leq \text{lh}(E_\beta) \) and \( \delta_\beta = \text{cr}(E_\alpha) \), then \( \text{sides}_{\alpha + 1} = \{1\} \), and

(ii) otherwise, \( \text{sides}_{\alpha + 1} = \{\text{exitside}_\beta\} \).

(f) If \( \text{sides}_{\alpha + 1} = \{c\} \) then \( M^{\ast}_{\alpha + 1} \leq M^t_\beta \) and \( d = \text{deg}^{\ast}_{\alpha + 1} \) are determined as usual for degree-maximality (with \( d \leq k \) if \( (0, \alpha + 1)^T \cap \mathcal{D} = \emptyset \)),

\[
M^{\ast}_{\alpha + 1} = \text{Ult}_d(M^{\ast}_{\alpha + 1}, E_\alpha),
\]

and \( i^{\ast}_{\alpha + 1} \) is the ultrapower map. Here if \( M^{\ast}_{\alpha + 1} \) is type 3 with largest cardinal \( \text{cr}(E_\alpha) \), then \( d = -1 \), so the ultrapower is just that formed using functions in \( M^{\ast}_{\alpha + 1} \), without squashing; see §13.4 and Definition 14.8 below. Also if \( \text{deg}^{\ast} = -1 \) and \( \alpha + 1 \notin \mathcal{D} \) then \( d = -1 \).

13. The remaining objects are determined as usual, with direct limits at limit \( \eta \), so \( H_\eta, M_\eta \) are the direct limits under the iteration maps, and for \( \alpha < T \eta \), set \( \gamma_\eta = i_{\alpha\eta}(\gamma_\alpha) = j_{\alpha\eta}(\gamma_\alpha) \) and likewise for \( \delta_\eta \), and \( \pi_\eta \circ i_{\alpha\eta} = j_{\alpha\eta} \circ \pi_\alpha \).

Note that part of the definition is that for each \( \alpha \in \mathcal{B} \), \( B_\alpha \) is a degree \( k \) relevant bicephalus. Also define \( \mathcal{B}^T = \{ \alpha < \lambda \mid \text{sides}_\alpha = \{0\} \} \) and \( \mathcal{T}^T \) likewise but with \( 1 \) instead of \( 0 \). And for \( \alpha < \text{lh}(T) \), define \( \mathcal{B}^T_+ (\alpha) = \max(\mathcal{B}^T \cap [0, \alpha]^T) \). \( \dashv \)

14.8 Definition. Continue with the notation from 14.7. As mentioned above, if \( \delta < \gamma \) and \( \text{sides}_\beta = \{0, 1\} \) and \( \text{exitside}_\beta = \{0\} \) and \( \gamma_\beta \leq \text{lh}(E_\beta) \) and \( \beta = \text{pred}(\alpha + 1) \) and \( \text{cr}(E_\alpha) = \delta_\beta \), then we set \( \text{sides}_{\alpha + 1} = \{1\} \) (which is important as \( H_\beta \) need not be \( \delta_\beta \)-sound), and \( M^{*}_{\alpha + 1} = J_\beta \) is the least \( J^* \circ M_\beta \) such that \( \rho^{\ast}_{\eta_{0}} = \delta_\beta \) and \( \gamma_\beta \leq \text{OR}^\beta \).

Note \( J^* = J_\beta(J_0) \). We say that \( \alpha + 1 \) is a mismatched dropping node of \( \mathcal{T} \), and all nodes \( \xi \) such that \( \alpha + 1 \leq T^\mathcal{T} \xi \) and \( (\alpha + 1, \xi)^T \cap \mathcal{D} = \emptyset \) we call weakly anomalous nodes. If \( M^\gamma | \gamma^T \) is also type 3, we call such nodes \( \xi \) anomalous nodes. In case \( \alpha + 1 \) is an anomalous mismatched dropping node, \( M^{*}_{\alpha + 1} = M^{\ast}_{\beta} \gamma_\beta \) and \( \text{deg}_{\alpha+1} = -1 \) and \( \nu(M^{\ast}_{\alpha+1}) = \delta_\beta = \text{cr}(E_\alpha) \), so we form \( M_{\alpha+1} = \text{Ult}_{-1}(M^{\ast}_{\alpha+1}, E_\alpha) = \text{Ult}(M^{\ast}_{\alpha+1}, E_\alpha) \) (see §13.4). And in case \( \zeta = \text{pred}^T(\alpha' + 1) \) is anomalous and \( \alpha' + 1 \notin \mathcal{D} \), we have \( \text{deg}_{\alpha+1} = \text{deg}_\xi = -1 \) and \( M_{\alpha'+1} = \text{Ult}_{-1}(M^\zeta_\xi, E_{\alpha'}) \). If \( \xi \) is anomalous then \( M_\xi \) is not a premouse, as it fails the ISC.

14.9 Remark. For anomalous \( \xi \), if \( E_\xi = F(M_\xi) \), then \( \nu(E_\xi) = \sup_{\zeta < \xi} \nu(E_\zeta) \), so it is possible that \( \nu(E_\xi) < \text{lgcd}(M_\xi) \), in which case \( \nu(E_\xi) < \nu(\text{exit}_\xi) = \text{lgcd}(M_\xi) \). We use \( \text{lgcd}(\text{exit}_\xi) \) as the exchange ordinal in this situation mainly because it is more convenient in the iterability proof later.

14.10 Definition. Let \( B' \) be a degree \( k \) relevant bicephalus. A putative degree-maximal tree \( \mathcal{T} \) on \( B' \) is a system satisfying all of the requirements of an iteration tree on \( B' \), with all models formed as in 14.7, except that if \( \text{lh}(\mathcal{T}) = \alpha + 1 > 1 \) then we make no demands on the wellfoundedness of \( B'^T_k \), nor its first order properties. And \( \mathcal{T} \) is relevance-putative iff it is putative degree-maximal, and if \( \text{lh}(\mathcal{T}) = \alpha + 1 \) then \( B_\alpha \) is wellfounded, and if \( \alpha \in \mathcal{B}^T \) then \( B_\alpha \) is pre-relevant. \( \dashv \)
Using Lemma 14.5, it is straightforward to verify:

**Claim 8.** If $T$ is a putative degree-maximal tree on $B$ with wellfounded models, then $T$ is relevance-putative.

**14.11 Definition.** A $((k,k),\omega_1 + 1)$-iteration strategy for a degree $k$ relevant bicephalus $B'$ is defined using the iteration game defined with (putative) degree-maximal trees $T$ on $B'$. If a putative tree is reached which is not a true degree-maximal tree, then player I wins.

An almost $((k,k),\omega_1 + 1)$-iteration strategy for a degree $k$ relevant bicephalus $B'$ is as above, except that if tree is reached which is relevance-putative but not a true tree, then player II wins immediately.

**14.12 Remark.** We have now completed our digression which began with Definition 14.2 above, and return to the context of the proof of solidity. Let $B = (\delta, \gamma, \pi, H, M)$ be from there. Note that $B$ is a relevant bicephalus (see 14.2). As stated earlier, the plan is to form and analyse a comparison of $B$ with $M$, forming a degree-maximal tree on $B$ (see 14.7). Before we begin this, we discuss in Claims 9–13 below the iterability of $B$ and various preservation facts which are essential to the analysis of the comparison.

**Claim 9 (Almost iterability).** $B$ is almost $((k,k),\omega_1 + 1)$-iterable.

**14.13 Remark.** The proof is a copying process much like that used in the proof of condensation from normal iterability, [24, Theorem 5.2]. In order to first focus the more novel aspects of the solidity proof, we postpone the proof of Claim 9 to §14.2. In the proof of Claim 11 below, we will need the following details from the copying process: Let $T$ be a putative degree-maximal tree on $B$, of finite length $\alpha + 1 < \omega$, with $\alpha \in \mathcal{B}^T \cup \mathcal{M}^T$. Then we will have a $k$-maximal tree $U$ on $M$ with $\text{lh}(U) = \alpha + 1$, such that $(0, \alpha)^U \cap \mathcal{S}^U = \emptyset$, and a $k$-embedding $\sigma : M^U_\alpha \to M^U_\beta$ such that $\sigma \circ j_0^T = i_0^U$. The almost iterability proof is self-contained and can be read directly at this stage, so the reader who prefers to proceed more linearly through the logic should make a detour there now.

We next discuss closeness of extenders to their target models in trees $T$ on $B$. We restrict to our particular $B$, as opposed to dealing with an arbitrary relevant bicephalus, because we will use Claim 6 to rule out certain cases in which closeness would otherwise not obviously hold.

**Claim 10 (Closeness).** Let $T$ be any putative degree-maximal tree on $B$. Let $\xi + 1 < \text{lh}(T)$ and $\beta = \text{pred}^T(\xi + 1)$. Then:

1. If $1 \in \text{sides}_T^\xi \xi + 1$ then $E_T^\xi$ is close to $M^T_\xi$.  
2. If $\text{sides}_T^{\xi + 1} = \{0\}$ then $E_T^\xi$ is close to $H_T^{\xi + 1}$.

---

72 In the original version of the argument presented at the Münster conference 2015, all comparison arguments were formed between bicephali and themselves (see [14]). Afterward, John Steel suggested that a comparison between a bicephalus and a premouse might suffice, and said that he had also been working on related arguments toward [32]. For the present proof, such a simplification did indeed work out. The author did not see how to simplify the proof of projectum-finite generation in this way, though in that case, the bicephali and rules for comparison are simpler anyway.

73 However, if $\text{sides}_T^{\xi + 1} = \{0, 1\}$, then it is not relevant whether $E_T^\xi$ is close to $H_T^\xi$, as recall that $H_T^{\xi + 1}$ is not formed in general as $\text{Ult}_k(H_T^{\xi + 1}, E_T^\xi)$, but as a certain hull of $M^T_\xi$.  

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The proof of closeness is very close to the usual one (see [9, 6.1.5]), so it is also
to postponement to §14.2.

Claim 11. Let \( T \) be any relevance-putative degree-maximal tree on \( B \). Let \( \alpha \in B^T \cup M^T \)
be such that \((0, \alpha] \cap D^T_{\deg} = \emptyset \). Then:

1. \( T \) is degree-maximal.
2. \( \rho^M_T = \sup \iota^T_0 \rho^T \leq \delta^T_\alpha \) and \( j^T_0 \) is \( p_{k+1} \)-preserving.
3. If \( \alpha \in B^T \) then:
   
   (a) \( \rho^{H^T_\alpha} \leq \gamma^T_\alpha \), \( H^T_\alpha \) is \( \gamma^T_\alpha \)-sound and \( \rho^{H^T_\alpha} = \rho^{H^T_\alpha} \gamma^T_\alpha = \iota^T_0 (\rho^H \gamma) = \iota^T_0 (\rho^H) \).

   (b) \( \rho^{M^T_\alpha} < \rho^{H^T_\alpha} \).
123(6,11),(992,992)

(c) \( H^T_\alpha \notin M^T_\alpha \) and \( M^T_\alpha \) is non-solid.

Proof. Part 2: The proof is similar to that of Lemma 13.8, but now using the Closeness
Claim 10 for part of the argument. Let \( N = M^T_\alpha \) and \( j = j^T_0 : M \rightarrow N \). By Claim 10,
all extenders applied along the branch from \( M \) to \( N \) are close to their target model, so
together with Corollary 3.9 (and without considering \( <_k^p \)) we have:

(a) \( \rho^N = \sup \rho^M \) and \( \rho^N \leq j(\rho^M) \).

(b) \( N \) is non-(\( k+1 \))-solid (as \( M \) is non-(\( k+1 \))-solid as \( H \notin M \)).

So it suffices to see that \( j(\rho^M) = p^N \), so suppose \( p^N < j(\rho^M) \).

Suppose that \( \text{lh}(T) < \omega \). Then by Remark 14.13 and as \( \alpha \in B^T \cup M^T \), we have a
\( k \)-maximal tree \( U \) on \( M \) such that \( \text{lh}(U) < \omega \) and \( \theta^U \) does not drop in model or
degree, and we have a \( k \)-embedding \( \sigma : N \rightarrow M^U_\alpha \) such that \( \sigma \circ j = \iota^U_\alpha \). Therefore
\( \sigma(j(\rho^M)) = \iota^U_\delta (\rho^M) \).

Since \( p^N < j(\rho^M) \), we have \( \sigma(p^N) < \sigma(j(\rho^M)) = \iota^U_\delta (\rho^N) \) so \( N <_k^p M \), as witnessed
by \( U \) and \( \sigma : N \rightarrow M^U_\alpha \). By our global inductive hypothesis (line (13)), \( N \) is \( (k+1) \)-
solid, contradicting (b).

So \( \text{lh}(T) \geq \omega \). But like done in the proof of Lemma 13.8, we can build a finite
length tree \( T \) capturing \((T, \rho^N)\), and this leads to a contradiction just as there. (Define
\( \text{capturing} \) for such trees like in 8.1, and construct \( T \) via a finite support argument like
in 6.20. We leave the details to the reader.)

Part 1: Since \( \gamma \in \rho^M \), this is an immediate consequence of part 2.

Part 3a: This follows easily from the \( \gamma \)-soundness of \( H \) (use preservation of generalized
solidity witnesses under near \( k \)-embeddings).

Part 3b: Write \( B' = (\delta', \gamma', \pi', H', M') = B^T_\alpha \), and \( i = i^T_0 \) and \( j = j^T_0 \). Using
properties of \( \pi' \) and that \( j \) is \( p_{k+1} \)-preserving, it is easy to reduce to the case that

\[ \rho^{H'} = \rho^{M'} = \delta' < \gamma' < (\rho^{M'})^M', \]

so assume this. We will show that \( H \in M \), a contradiction.

\[ ^{74}\text{If } T \text{ is finite, then we also get that } H^T_\alpha \text{ is } (k+1) \text{-solid and } (k+1) \text{-universal, like in the proof}
\text{ of the earlier parts. But the author is not sure whether one can show that } H^T_\alpha \text{ is } (k+1) \text{-solid and}
\text{ (k+1)-universal for infinite } T. \]
Note that either \( p^{H'} = q^{H'} \) or \( p^{H'} = q^{H'} \land (\beta) \) for some \( \beta \in [\delta', \gamma'] \). And line (16) and part 2 imply together that \( \rho^M = \delta < \gamma \) and \( i, j \) are continuous at \( \delta \).

Let \( T \) be a finite tree on \( B \) capturing \( \langle T, \{ \delta', \gamma' \}, p^T \rangle \), meaning here in particular that \( \text{lh}(T) = \bar{\alpha} + 1 \) and \( \bar{\alpha} \in B^T \) and letting \( B = B^T_{\bar{\alpha}} = (\delta, \gamma, \bar{\pi}, H, M) \), we have \( k \)-embedding capturing maps \( \sigma : \bar{H} \to H' \) and \( \tau : M \to M' \) with \( \delta', \gamma', p^{H'} \in \text{rg}(\sigma) \) and \( \delta', \gamma', p^M \in \text{rg}(\tau) \) and \( \pi' \circ \sigma = \tau \circ \bar{\pi} \), and the capturing maps commute with the iteration maps. Note \( \sigma(\bar{\delta}) = \tau(\bar{\delta}) = \delta' = \rho^{H'} \). Let \( \bar{\rho} = \delta \). Let \( \sigma(\bar{\rho}) = \rho^{H'} \). Let \( \bar{i} = \bar{i}^T_{\bar{\rho}} \) and \( \bar{j} = \bar{j}^T_{\bar{\rho}} \).

Now \( H <^p_k M \). For \( \pi(p^H) < p^M \), since \( \bar{j} \) is \( p_{k+1} \)-preserving and considering the hull of \( M \) that forms \( \bar{H} \). So we get \( H <^p_k M \) by lifting \( T \) to a tree \( U \) on \( M \) as in part 2 (again using Remark 14.13).

Since \( H <^p_k M \), \( \bar{H} \) is \((k+1)\)-solid and \((k+1)\)-universal.

Recall \( \sigma(\bar{\rho}) = p^{H'} \) and \( \sigma(\bar{\rho}) = \delta' \). Let

\[
\bar{i} = \text{Th}_{\delta_{2k+1}} \left( \bar{\rho} \cup \{ \bar{\rho}, \bar{p}^H \} \right).
\]

Then \( \bar{i} \notin \bar{H} \), because otherwise \( \sigma(\bar{i}) \in H' \), and then from \( \sigma(\bar{i}) \) we can recover the theory

\[
t' = \text{Th}_{\delta_{2k+1}} \left( \rho' \cup \{ p^{H'}, \bar{p}^H \} \right),
\]

but \( t' \notin H' \). So \( \bar{p}^H \leq \bar{\rho} \), but by part 2 and since \( \bar{j}, \bar{j} \) are continuous at \( \delta \), we have \( \rho^M = \bar{\rho} = \delta \), which implies \( \rho^H \geq \bar{\rho} \). So \( \rho^H = \bar{\rho} \), and \( p^H \leq \bar{\rho} \).

By \((k+1)\)-universality for \( H \), it follows that there is \( \beta < \bar{\gamma} \) such that

\[
\bar{H} = \text{Hull}^H(\bar{\rho} \cup \{ \beta, q^{H'}, \bar{p}^H \} \).
\]

But \( \pi(q^{H'} \land (\beta)) < p^M \), and it follows that \( \bar{H} \in M \), so \( H \in M \) by Lemma 14.5, as desired.

Part 3c follows from Lemma 14.5, since \( H \notin M \). \( \square \)

Claim 12. There is a \(((k, k), \omega_1 + 1)\)-iteration strategy for \( B \). Moreover, every almost \(((k, k), \omega_1 + 1)\)-strategy for \( B \) is an (actual) \(((k, k), \omega_1 + 1)\)-strategy.

Proof. This is an immediate consequence of Claims 9 and 11. \( \square \)

We now summarize the fine structural properties of iterates \( H^{\bar{T}}_\alpha \) and \( M^{\bar{T}}_\alpha \) when \( \alpha \notin B^T \). This complements Claim 11:

Claim 13. Let \( T \) be any degree-maximal tree on \( B \). Then:

1. Suppose \( \alpha \notin B^T \) and \( (0, \alpha]^T \cap D^T_{\text{deg}} = \emptyset \) and let \( \beta = \beta^T_\alpha \). Then:

   a. Suppose \( \alpha \in H^T \). Then:

      - \( H^{\bar{T}}_\alpha \) is a \( k \)-sound premouse,
      - \( \rho^{H^{\bar{T}}_\alpha} = \rho^{H^{\bar{T}}_\alpha} \leq \gamma^{\bar{T}_\alpha} \leq cr(i^{\bar{T}_\alpha}) \),
      - \( H^{\bar{T}}_\beta \) is the \( \gamma^{\bar{T}_\alpha} \)-core of \( H^{\bar{T}}_\alpha \) and \( i^{\bar{T}_\alpha} \) is the \( \gamma^{\bar{T}_\alpha} \)-core map,
      - \( i^{\bar{T}_\alpha} \) is \( p_{k+1} \)-preserving. \( ^{75} \)

\( ^{75} \) But the author does not know whether \( i^{\bar{T}_\alpha}, i^{\bar{T}_\alpha} \) are \( p_{k+1} \)-preserving.
(b) Suppose $\alpha \in \mathcal{M}^T$. Then:
- $M^T_\alpha$ is a $k$-sound premouse,
- $\rho^{M^T_\alpha} = \rho^{M^T_\beta} \leq \delta^T_\beta \leq \cr(j^T_\beta, \alpha)$,
- $M^T_\beta$ is the $\delta^T_\beta$-core of $M^T_\alpha$ and $j^T_\beta, \alpha$ is the $\delta^T_\beta$-core map,\(^{76}\)
- $j^T_\beta, \alpha$ are $p_{k+1}$-preserving, and
- $M^T_\alpha$ is non-solid.

2. If $\alpha \in \mathcal{M}^T$ and $(0, \alpha]^T \cap \mathcal{D}^T_{\deg} \neq \emptyset$ and $\alpha$ is non-anomalous then letting $d = \deg^T(\alpha)$ and $\xi + 1 \leq T \alpha$ be largest such that $\xi + 1 \in \mathcal{D}^T_{\deg}$, and letting $\beta = \text{pred}^T(\xi + 1)$, we have:
- $M^T_\beta$ is premouse, and is $d$-sound, $(d + 1)$-solid, $(d + 1)$-universal, but fails to be $(d + 1)$-sound,
- $M^T_{\xi + 1} = \mathcal{C}_{d+1}(M^T_\alpha)$ is $(d + 1)$-sound and $\mathcal{C}_{d+1}(M^T_\alpha) \subseteq M^T_\beta$,
- $j^T_{\xi + 1, \alpha}$ is the core map,
- $\rho_{d+1}(M^T_\xi) = \rho_{d+1}(\mathcal{C}_{d+1}(M^T_\alpha)) \leq \cr(j^T_{\xi + 1, \alpha}),$
- $j^T_{\xi + 1, \alpha}$ is $p_{d+1}$-preserving.

3. If $\alpha \in \mathcal{H}^T$ and $(0, \alpha]^T \cap \mathcal{D}^T_{\deg} \neq \emptyset$, it is like in part 2 (with $H^T_\alpha$, not $M^T_\alpha$).

4. Suppose $\alpha$ is anomalous. Then it is like in part 2, except that $M^T_\alpha$ is not a premouse.

Proof. Let $\mathcal{T}$ be any relevance-putative tree of length $\alpha + 1$.

Part 1: Suppose $\alpha \in \mathcal{H}^T$. We have $\gamma^T_\beta \leq \cr(i^T_\beta)$ because otherwise letting $\xi + 1 = \text{succ}^T(\beta, \alpha)$, we would have $\cr(E^T_\xi) = \delta^T_\beta$, but then $\xi + 1$ should in fact be a mismatched dropping node, so $\xi + 1 \notin \mathcal{H}^T$. And by the closeness (Claim 10), all the extenders used along the branch $(\beta, \alpha]^T$ are close to the models to which they apply; since $H^T_\beta$ is $\gamma^T_\beta$-sound and $\rho(H^T_\beta) \leq \gamma^T_\beta$, this suffices. If instead $\alpha \in \mathcal{M}^T$, use Claims 10 and 11 (in particular for $p_{k+1}$-preservation), together with Corollary 3.9.

The remaining parts follow in the usual manner from closeness (Claim 10).

We are now ready to proceed with the comparison. We compare $B$ with $M$, defining padded trees $U, T$ respectively, with $U$ being degree-maximal on $B$ and $T$ being $k$-maximal on $M$. We will also define $S^U, \mathcal{S}^U, \text{movin}^U$, with $\emptyset \neq S^U_\alpha \subseteq \text{side}^U_\alpha$; these bookkeeping devices are defined completely analogously to those in the proof of Lemma 11.6 (see the paragraph immediately following the proof of Claim 1 of that lemma’s proof). As before, if $S^U_\alpha = \{0, 1\}$, we may move into a model of $U$ at stage $\alpha$, setting $S^U_{\alpha + 1} = \{0\}$ or $S^U_{\alpha + 1} = \{1\}$ and $E^U_\alpha = \emptyset = E^T_\alpha$. And as usual, after selecting models for potential exit extenders, we minimize on $\bar{\nu}(E)$ before actually selecting extenders $E$ (see Remark 3.1).

Let us describe the rules for forming the comparison (how we move into models and select extenders). We start with $S^U_0 = \{0, 1\}$ at stage 0. At stage $\alpha$, if $S^U_\alpha \neq \{0, 1\}$,\(^{76}\) Here the $\delta^T_\beta$-core of $M^T_\alpha$ just means cHull$^{M^T_\alpha} (\delta^T_\beta \cup \{p^{M^T_\alpha} \setminus \delta^T_\beta \})$; the terminology does not presuppose any solidity.

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we select extenders as usual (or terminate the comparison). Suppose \( S^u_\delta = \{0,1\} \). If \( B^u_\delta (\delta^u_\delta \neq M^u_\delta | \delta^u_\delta \) we select extenders as usual (and we do not move into any model in this case). Suppose \( B^u_\delta (\delta^u_\delta = M^u_\delta | \delta^u_\delta \). Then we move into a model in \( U \), setting \( E^u_\alpha = \emptyset = E^T_\gamma \). If \( M^u_\delta = M^u_\gamma \) then we move into \( H^u_\alpha \) (set \( S^u_{\alpha+1} = \{0\} \)), and otherwise we move into \( M^u_\alpha \) (set \( S^u_{\alpha+1} = \{1\} \)). Of course, we use some \((k,k), \omega_1 + 1\)-strategy to form \( U \), and some \((k,\omega_1 + 1)\)-strategy to form \( T \). This completes the description of the comparison.

We now proceed to the analysis of the comparison. We want to see that we reach a stage \( \beta \) with \( \beta \in B^u_\delta \), and at that stage, we move into \( H^u_\beta \) in \( U \), \( U \) does not use any extenders after that point, and then the comparison terminates at a stage \( \eta \) shortly after \( \beta \), with \( H^u_\beta = H^u_\alpha = M^u_\gamma \). We will then analyze the fine structure of \( H^u_\alpha, M^u_\alpha \), and using the fact that \( H^u_\alpha \notin M^u_\gamma \), reach a contradiction.

The first thing to verify is that the trees \( U, T \) are normal.

**Claim 14.** Let \( \xi < \zeta \leq \beta < \text{lh}(U, T) \). Then:

1. Suppose \( \xi < \beta \) and \( S^u_\beta = \{0,1\} \). Then \( \text{lh}(E) \leq \delta^u_\beta \) whenever \( E = E^u_\xi \neq \emptyset \) or \( E = E^T_\xi \neq \emptyset \).

2. Suppose \( \beta + 1 < \text{lh}(U, T) \) and \( P, Q, R \neq \emptyset \) with \( P \in \{\text{exit}^u_\beta, \text{exit}^T_\gamma\}, Q \in \{\text{exit}^u_\zeta, \text{exit}^T_\xi\} \) and \( R \in \{\text{exit}^u_\beta, \text{exit}^u_\zeta\} \). Then:
   
   (a) \( \text{OR}^P \leq \text{OR}^Q \) and \( \bar{\nu}^P \leq \bar{\nu}^Q \).
   
   (b) If \( \zeta = \xi \) then \( \text{OR}^P = \text{OR}^Q \) and \( \bar{\nu}^P = \bar{\nu}^Q \).
   
   (c) Suppose \( \zeta < \xi \) and \( \bar{\nu}^P = \bar{\nu}^Q \). Then \( \xi < \zeta + 1 \) is an anomalous mismatched dropping node, \( J = M|\gamma \) is active type 3, \( E^u_\xi \neq \emptyset \) is superstrong, \( Q = \text{exit}^u_\zeta \), \( E^\gamma_\zeta = \emptyset, E^\gamma_{\zeta+1} = \emptyset \), and if \( \zeta + 1 < \beta \) then \( \bar{\nu}^Q < \bar{\nu}^P \).

**Proof.** The proof is by induction on \( \beta \). Limits are easy. Suppose \( \beta = \eta + 1 \) for some \( \eta \).

**Part 1:** Since \( S^u_{\eta+1} = \{0,1\} \), either \( E^u_\eta \neq \emptyset \) or \( E^\gamma_\eta \neq \emptyset \). Let \( \lambda = \text{lh}(E^u_\eta) \) or \( \lambda = \text{lh}(E^\gamma_\eta) \), whichever is defined. If \( E^u_\eta \neq \emptyset \) then setting \( \chi = \text{pred}^u(\eta + 1) \), we have \( \text{cr}(E^u_\eta) < \delta^u_\eta \), so \( \lambda \leq \delta^u_{\eta+1} \). Suppose \( E^u_\eta \neq \emptyset \neq E^\gamma_\eta \) and \( \delta^u_{\eta+1} < \lambda \). Then \( S^u_{\eta} = \{0,1\} \) and \( B^u_\eta = B^u_{\eta+1} \), and \( B^u_\eta \upharpoonright \delta^u_\eta = M^u_\gamma \upharpoonright \delta^u_\eta \), so by the rules of comparison, we move into either \( H^u_\eta \) or \( M^u_\eta \) at stage \( \eta \), so \( E^u_\eta = \emptyset = E^\gamma_\eta \) (and \( S^u_{\eta+1} \neq \{0,1\} \)), a contradiction.

We leave the rest to the reader. \( \square \)

**Claim 15.** There is no pair \((\zeta, \xi)\) such that \( \zeta < \text{lh}(U) \) and \( \xi < \text{lh}(T) \) and \( E^u_\zeta \upharpoonright \nu(E^u_\zeta) = E^T_\xi \upharpoonright \nu(E^T_\xi) \neq \emptyset \).

**Proof.** Suppose \((\zeta, \xi)\) is so. As \( E^T_\xi \upharpoonright \nu(E^T_\xi) \) satisfies the ISC, so does \( E^u_\zeta \upharpoonright \nu(E^u_\zeta) \). So \( \zeta \) is non-anomalous, so \( \bar{\nu}(\text{exit}^u_\zeta) = \nu(E^u_\zeta) = \nu(E^T_\xi) = \bar{\nu}(\text{exit}^T_\xi) \). So \( \zeta \neq \xi \) by the comparison rules, and using Claim 14, it follows that \( \zeta = \xi + 1 \) and \( \zeta \) is anomalous, contradiction. \( \square \)

**Claim 16.** The comparison terminates at some stage \( \alpha < \omega_1 \).

The proof is just a slight variant of the usual one, and is relegated to §14.2.

Now that we know the comparison terminates, we will analyze the manner in which it does, and use the properties of \( H, M \) (in particular that \( H \notin M \)) and the preservation
properties of the iteration maps, to arrive at a contradiction. Let $\alpha + 1$ be the length of the full comparison $(U, T)$. So the comparison terminates at stage $\alpha$, and in fact, in the following fashion:

Claim 17. $S^U_\alpha = \{0\}$ and $H^U_\alpha \subseteq M^T_\alpha$.

In the proof and later, given $\beta \in B^U$, write $\delta^U_\beta = (\delta^U_{\beta})^{+U}_\alpha$ (so $(\delta^U_{\beta})^{+U}_\alpha \leq \delta^U_\beta$).

Proof. Suppose $S^U_\alpha = \{0, 1\}$. Then $M^T_\alpha \leq B^U_\alpha \delta^U_\alpha$. But then $b^T \cap \mathcal{P}^T_{deg} = \emptyset$ and $M^T_\alpha$ is solid, so by Corollary 3.9, $M$ is solid, a contradiction. If $S^U_\alpha = \{0\}$ then we can similarly rule out $M^T_\alpha \leq H^U_\alpha$, giving the claim. So suppose $S^U_\alpha = \{1\}$; we will reach a contradiction.

Subclaim 17.1. $M^U_\alpha = M^T_\alpha$, $b^T, b^U$ do not drop in model or degree, and $j^U, i^T$ are $p_{k+1}$-preserving.

Proof. If $M^T_\alpha < M^U_\alpha$ it is again as before. If $M^U_\alpha < M^T_\alpha$ then $M^U_\alpha$ is a sound premouse, and by Claim 13, $b^T \cap \mathcal{P}^T_{deg} = \emptyset$ and $M$ is solid. So $M^U_\alpha = M^T_\alpha$.

Suppose both $b^T, b^U$ drop in model or degree. Then the usual arguments combined with Claim 13 yield a contradiction. (Anomalous extenders are not “partial”, so they do not interfere with the incompatibility of extenders relevant to this argument. That is, let $\theta + 1 \in B^U_{deg}$ and $\theta + 1 \leq U \xi + 1 \leq U \alpha$. Then exit $\xi$ is a premouse; that is, it is not the case that $\xi$ is anomalous and $E^U_\xi = F(M^U_\xi)$. For suppose otherwise. Then $\xi + 1 \in B^U$. For letting $\chi = B^U(\xi)$ and $J_\chi = M^U_\chi \rightarrow U \chi$ and $j : J_\chi \rightarrow M^U_\chi$ be the iteration map. $J_\chi$ is active type 3 with $\nu(F^J_\chi) = \delta^U_\chi = \chi(j)$, so $\chi(E^U_\xi) = \chi(F^J_\xi) < \delta^U_\xi$ and $E^U_\xi$ is $B^U$-total, and this implies that pred$^U(\xi + 1) \leq U \chi$ and $\xi + 1 \in B^U$.)

If one side drops but the other does not, then $M^U_\alpha = M^T_\alpha$ is a premouse and is solid, which again implies $M$ is solid. So neither side drops. The $p_{k+1}$-preservation now follows Lemma 13.8 and Claims 11 and 13.

Subclaim 17.2. $\alpha \in B^U$.

Proof. Suppose not, so sides$^U_\alpha = S^U_\alpha = \{1\}$. Let $\beta = \text{min}^U(\alpha)$. So $\beta \leq U \alpha$ and (by Claim 13) $j^U_\beta$ is $p_{k+1}$-preserving, $M^U_{\beta}$ is the $\delta^U_{\beta}$-core of $M^U_\alpha$, $j^U_{\beta}$ is the $\delta^U_{\beta}$-core map and $\delta^U_{\beta} \leq \text{cr}(j^U_{\beta})$. So $M^U_{\beta} \| \delta^U_{\beta} = M^U_{\alpha} \| \delta^U_{\alpha}$ and

$$p = \text{def} j^U_{\beta}(q^{M^U_{\beta}} \setminus \langle \gamma^U_{\beta} \rangle) = p^M \setminus \delta^U_{\beta} = i^T_{M\alpha}(p^M \setminus \delta).$$

Subsubclaim. There is $\beta' \leq T \alpha$ with $M^T_{\beta'} = M^U_{\beta}$.

Proof. We first show that there is no $\xi + 1 \leq T \alpha$ with $E_{\xi}^T \neq \emptyset$ and $\text{cr}(E_{\xi}^T) < \delta^T_{\beta} < \nu(E_{\xi}^T)$. This is just via the (relevant version of) the argument with the ISC and hull property (in the sense of 3.11) from [33, Example 4.3], combined with the $p_{k+1}$-preservation that we have. That is, suppose $\xi$ is a counterexample. Then $M^T_\alpha$ does not have the $(k+1, p)$-hull property at $\delta^T_{\beta}$, as $p \in \text{rg}(i^T_{M\alpha})$ (in fact, the ISC gives $E = E_{\xi}^T \setminus \delta^U_{\alpha} \subseteq M^T_\alpha$, but

$$E \notin \text{Ult}_k(M^T_{\xi+1}, E) = \text{cHull}^T_\alpha(\delta^T_{\beta} \cup \{\hat{p}^M_{\alpha}, p\});$$

cf. [33] for more details). But $M^U_{\beta} = \text{cHull}^U_\alpha(\delta^U_{\beta} \cup \{\hat{p}^M_{\alpha}, p\})$, so $M^U_{\alpha}$ does have the $(k+1, p)$-hull property at $\delta^U_{\beta}$, contradicting that $M^U_\alpha = M^T_\alpha$. 127
So let \( \beta' \leq^T \alpha \) be least such that either \( \beta' = \alpha \) or \( \delta^u_{\beta'}(i^u_{\beta'}(p^T \setminus \delta)) = p \), it now easily follows that \( M^T_{\beta'} = M^T_{\beta} \), as desired. \( \square \)

Now at stage \( \beta \), in \( \mathcal{U} \), we moved into \( M^T_{\beta} \), so by the rules of comparison, \( M^T_{\beta} \neq M^T_{\beta'} \), so \( \beta \neq \beta' \). But if \( \beta' < \beta \), then note that \( E^\xi_{\beta} = \emptyset \) for all \( \xi \in [\beta', \beta) \), but then \( M^T_{\beta} = M^T_{\beta'} \), a contradiction. So \( \beta < \beta' \), and note that \( E^\xi_{\beta'} = \emptyset \) for all \( \xi \in [\beta', \beta') \). But then \( M^T_{\beta'} = M^T_{\beta} \), and \( S^u_{\delta^m_{\beta'}} = \{1\} \) although \( \text{sides}^u_{\delta^m_{\beta'}} = \{0, 1\} \) (so \( \beta' \in \mathcal{B}^U \)). Hence, the comparison terminates at stage \( \beta' \), so \( \beta' = \alpha \), contradicting the assumption that \( \text{sides}^u_{\alpha} = \{1\} \), proving the subclaim. \( \square \)

So \( S^u_{\delta^m_{\beta}} = \{1\} \) but \( \alpha \in \mathcal{B}^U \). Let \( \beta = \text{movin}^U(\alpha) \). So \( \beta < \alpha \) and at stage \( \beta \), in \( \mathcal{U} \) we move into \( M^T_{\beta} = M^T_{\beta} \) in \( \mathcal{U} \); also, \( E^\xi_{\beta} = \emptyset \) for all \( \xi \in [\beta, \alpha) \), and \( E^\xi_{\alpha} = \emptyset \), but \( E^\xi_{\alpha} \neq \emptyset \) for all \( \xi \in [\beta, \alpha) \). Since we moved into \( M^T_{\beta} \), we have \( M^T_{\beta} \neq M^T \), so \( E^T_{\beta+1} \neq \emptyset \). Also, \( M^T_{\beta} [\delta^m_{\beta} = M^T_{\beta} [\delta^m_{\beta} \), which is passive, so \( \delta^u_{\beta} < \text{lh}(E^T_{\beta+1}) \). As \( M^T_{\beta} = M^T_{\beta} \) and \( M^T_{\beta} = M^T_{\beta+1} \), therefore 
\[
M^T_{\beta} [\delta^u_{\beta} = M^T_{\beta} [\delta^u_{\beta} = M^T_{\beta+1} [\delta^u_{\beta}
\]
and \( \delta^u_{\beta} = (\delta^u_{\beta})^{\text{exit}_{\beta+1}} \). Let \( \zeta \geq \beta + 1 \) be least such that \( \zeta + 1 \leq^T \alpha \). Let \( \kappa = \text{cr}(E^\zeta_{\beta}) \).
Since \( M^T_{\beta} = M^T_{\beta} \) is \( \delta^m_{\beta} \)-sound and by \( p_{k+1} \)-preservation, we have \( \kappa \leq \gamma^T_{\beta} \), hence \( \kappa \leq \delta^m_{\beta} \). But \( \delta^m_{\beta} \in \text{rg}(i^T_{\gamma^T}) \), so \( i^T_{\gamma^T+1}(\kappa) = \delta^m_{\beta} \). Therefore \( E^\zeta_{\beta} \) is superstrong and \( \zeta = \beta + 1 \) and \( \text{lh}(E^\zeta_{\beta}) = \delta^u_{\beta} \). So \( \beta + 2 = \alpha \). Let \( \varepsilon = \text{pred}^T(\beta + 2) \). Then computing with the hull property like before, we get that there is \( \chi <^U \beta \) such that \( M^T_{\chi} = M^T_{\beta} \), so \( \chi \in \mathcal{B}^U \), and also get that \( E^\chi_{\beta+1} \) was also used in \( \mathcal{U} \), a contradiction. \( \square \)

By Claim 17, \( S^u_{\alpha} = \{0\} \). However, by the next claim, sides^u_{\alpha} = \{0, 1\}:

**Claim 18.** \( \alpha \in \mathcal{B}^U \).

**Proof.** Suppose not. By Claim 17, then \( \alpha \in \mathcal{H}^U \) and \( H^u_{\delta^m_{\alpha}} \leq M^T_{\alpha} \), but \( H^u_{\delta^m_{\alpha}} \) is not sound. So \( H^u_{\delta^m_{\alpha}} = M^T_{\alpha} \). So \( (0, \alpha \delta m) \) does not drop in model or degree, as otherwise \( M \) is solid. Let \( \beta = \text{movin}^U(\alpha) \). By Claim 13, \( i^u_{\beta} \colon H^u_{\beta} \to H^u_{\alpha} \) is the \( \gamma^U_{\beta} \)-core map, \( \rho(H^u_{\alpha}) = \rho(H^u_{\beta}) \leq \gamma^T_{\beta} \leq \text{cr}(i^u_{\beta}) \), etc. But then we get a pair of identical extenders used in \( \mathcal{U} \) and \( T \), much like before; this is a contradiction. \( \square \)

So \( \alpha \in \mathcal{B}^U \) but \( S^u_{\alpha} = \{0\} \). Let \( \beta = \text{movin}^U(\alpha) \). Let \( B^u_{\beta} = B' = (\delta', \gamma', \pi', H', M') \). So \( H' = H^u_{\beta} \leq M' \), and in \( \mathcal{U} \) we move into \( H' \) at stage \( \beta \), and therefore 
\[
M' = M^T_{\beta} \tag{17}
\]
\( E^T_{\beta} = \emptyset \neq E^T_{\beta'} \), and note that \( E^\xi_{\beta} = \emptyset \neq E^T_{\beta} \) for all \( \xi > \beta \) with \( \xi + 1 < \text{lh}(U, T) \) (and by the next claim and since \( H' \neq M' = M^T_{\beta} \), such a \( \xi \) exists).

**Claim 19.** We have:

(i) \( (0, \beta \delta m) \) does not drop in model or degree,

(ii) \( \gamma' = \delta^u_{\beta'}(\gamma) = i^T_{\beta}(\gamma) \),

(iii) \( H' = M^T_{\alpha} \).

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(iv) $E^T_{\beta+1} \neq \emptyset$ exists, so $\alpha \geq \beta + 2$.

**Proof.** Part (i): By line (17), $M^T_{\beta} = M^T_{\beta}$. So if $(0, \beta|^T$ drops then $M^T_{\beta}$ is $(k + 1)$-solid, so $M$ is too, contradiction.

Part (ii): Using the previous part, $j_{0|\beta}$ and $i_{0|\beta}$ are $p_{k+1}$-preserving, and since $\gamma \in p^M$ and $M^T_{\beta} = M^T_{\beta}$, therefore $\gamma' = j^M_{0|\beta}(\gamma) = i^M_{0|\beta}(\gamma)$.

Part (iii): By Claim 17, $H' \subseteq M^T_{\alpha}$. So suppose $H' \not< M^T_{\alpha}$. We claim that either

- $H' \not< M'$, or

- $\gamma > \delta$ and $M'|\gamma'$ is active with $F' = E^T_{\beta+1}$, and $H' \not< U = \text{def \ Ult}(M'|\gamma', F')$,

which suffices, since then $H' \in M'$, so $H \in M$, contradicting Assumption 2.

So suppose $H' \not< M'$. Then $E^T_{\beta+1} \neq \emptyset$. But as $\rho^H \leq \gamma'$, it follows that $\text{lh}(E^T_{\beta+1}) = \gamma' > \delta'$, and so $E^T_{\beta+1} = F' = F^M|\gamma'$. If $F'$ is non-superstrong, then immediately $H' \not< U$. Suppose $F'$ is superstrong. Let $\zeta = \text{def} \ U(\beta + 2)$. Since $i^T_{0|\beta}(\gamma) = \gamma'$, we have $F' \in \text{rg}(i^T_{0|\beta})$, which gives $\zeta \leq T \beta$. If $\zeta = \beta$ then we are done, so suppose $\zeta < T \beta$. Let $\mu = \text{cr}(F')$ and note $\mu^{+ + M^T_{\alpha}} < \text{cr}(i^T_{\zeta})$, so

$$M^T_{\zeta}|\mu^{+ + M^T_{\alpha}} = M^T_{\beta}|\mu^{+ + \delta'_{\alpha}},$$

$$U|((\delta')^{+ + U} = M^T_{\zeta+1}|((\delta')^{+ + \delta'_{\alpha+1}}),$$

so $H' \not< U$, as desired.

Part (iv): This follows immediately from the previous parts. \qed

Note at this point that it might not be obvious that $H'$ itself is solid, so we cannot immediately rule out having $H' = M^T_{\alpha}$ with $(0, \alpha]^T$ non-dropping.

**Claim 20.** $\gamma^{+H} < \gamma^{+M}$.

**Proof.** Suppose not. Then note $\delta = \gamma$ and $(\delta\alpha)^{+H^\alpha} = \delta^{+H}$. We have $E^T_{\beta+1} \neq \emptyset$ by Claim 19. But $M^T_{\beta} = M^H_{\beta}$, so $\delta^{+H} < \text{lh}(E^T_{\beta+1})$, which contradicts the fact that $H^\alpha$ is $\delta^\alpha$-sound and projects $\leq \delta^\alpha_{\alpha}$. \qed

Recall that $E^T_{\beta} = \emptyset$, and after moving into $H'$ in $U$, the only extenders used in the comparison are the $E^T_{\beta+n}$ with $n > 0$, and there is at least one such extender. Call these extenders *terminal*. There is either 1 or 2 of these:

**Claim 21.** Either:

1. $E^T_{\beta+1}$ is the only terminal extender and either:
   
   (i) $\delta' < \gamma' = \text{lh}(E^T_{\beta+1})$, or
   
   (ii) $\gamma' < (\gamma')^{+H} = \text{lh}(E^T_{\beta+1})$ and $E^T_{\beta+1}$ is type 1/3,

   or

2. $E^T_{\beta+1}, E^T_{\beta+2}$ are the only terminal extenders, and:

   \[\text{As usual, if } \gamma = \text{lcd}(H) \text{ then } \gamma^{+H} \text{ denotes OR}^H.\]
\[ \delta' < \gamma' = \text{lh}(E_{\beta+1}^T), \] and
\[ \gamma' < (\gamma')^{+H'} = \text{lh}(E_{\beta+2}^T), \] and \( E_{\beta+2}^T \) is type 1/3.

**Proof.** This is because \( T \) cannot use any extenders with generators \( \geq \gamma' \), which follows from routine fine structure together with the facts that \( H' \) is \( \gamma' \)-sound with \( p^{H'} \setminus \gamma' = q^{H'} \), and if \( b^T \cap \mathcal{D}_{\mathrm{deg}}^T = \emptyset \) then \( t^T \) is \( p_{k+1} \)-preserving. \( \square \)

**Claim 22.** We have:

1. \( \hat{i}^T_{0,\beta} \downarrow (H||\gamma^{+H}) \subseteq \check{j}_{0,\beta} \), and
2. \( (\gamma')^{+H'} = \sup \hat{i}^T_{0,\beta} \uparrow \gamma^{+H} = \sup j_{0,\beta} \uparrow \gamma^{+H} = \sup i^T_{0,\beta} \uparrow \gamma^{+H} \).

**Proof.** Part 1: By Lemma 14.5.

Part 2: We first prove that \( (\gamma')^{+H'} = \sup j_{0,\beta} \uparrow \gamma^{+H} \). This holds because \( j_{0,\beta} \) is a \( k \)-embedding, \( H = \text{Hull}^H(\gamma \cup \{q^H, p_k^H\}) \) and likewise for \( H' \). That is, we can write \( \mathcal{E}_0(H) \) as an increasing union of hulls
\[ \mathcal{E}_0(H) = \bigcup_{\xi < \rho^H} \tilde{H}_\xi, \]
where for \( \xi < \rho^H \), \( \tilde{H}_\xi \) is the set of all \( x \in \mathcal{E}_0(H) \) with \( x \in \text{Hull}^H(\gamma \cup \{q^H, p_k^H\}) \) as witnessed by some segment/theory below “level \( \xi \)” (using the stratification of \( r\Sigma_{k+1} \) as in [9, §2]). Let \( \eta_\xi = \tilde{H}_\xi \cap \gamma^{+H} \). Then each \( \eta_\xi < \gamma^{+H} \), and \( \sup_{\xi < \rho^H} \eta_\xi = \gamma^{+H} \). But \( \tilde{H}_\xi \) is determined by the corresponding \( r\Sigma_k \) theory (or “initial segment”) \( t_\xi \in H \), and \( \hat{i}^T_{0,\beta}(t_\xi) = t^T_{0,\beta} \), where \( t^T_{0,\beta} \) is defined analogously with respect to \( H' \). Thus, \( \hat{i}^T_{0,\beta}(\eta_\xi) = \eta_{\hat{i}^T_{0,\beta}(\xi)} \), where \( \eta_{\hat{i}^T_{0,\beta}(\xi)} \) is defined analogously to \( \eta_\xi \). We also have \( \sup_{\xi < \rho^H} \eta_{\hat{i}^T_{0,\beta}(\xi)} = (\gamma')^{+H'} \), and \( \hat{i}^T_{0,\beta} \) is a \( k \)-embedding, so \( \rho^H_k = \sup \hat{i}^T_{0,\beta} \uparrow \rho^H_k \), so we are done.

So \( (\gamma')^{+H'} = \sup \hat{i}^T_{0,\beta} \uparrow \gamma^{+H} = \sup j_{0,\beta} \uparrow \gamma^{+H} \), using part 1. But now note that although \( T \) is a tree on \( M \), not \( B \), we can make all the analogous definitions with \( T \), defining \( H^T_{\beta} \) as the relevant collapsed hull of \( M^T_\beta \), and defining \( \pi^T_{0,\beta} : H^T_{\beta} \rightarrow M^T_\beta \) etc. Everything we have just done also applies there, but since \( M^T_{\beta} = M^T_{\beta}, j^T_{0,\beta}, i^T_{0,\beta} \) are \( p_{k+1} \)-preserving, and these parameters together with the model determine the hull \( H' \), it follows that \( \sup i^T_{0,\beta} \uparrow \gamma^{+H} = (\gamma')^{+H'} \) also. (But note we don’t claim \( j^T_{0,\beta} (\gamma^{+H} \subseteq i^T_{0,\beta}) \) \( \square \)

The remaining two claims rule out all possibilities, giving the desired contradiction:

**Claim 23.** \( M|\gamma \) is not passive.

**Proof.** Suppose \( M|\gamma \) is passive. By condensation for \( \omega \)-sound mice (Fact 1.9), \( M||\gamma^{+H} = H||\gamma^{+H} \).

**Case 1.** \( M|\gamma^{+H} \) is active with extender \( G \) with \( \kappa = \text{cr}(G) < \delta \).

We have \( j_{0,\beta}^T(\gamma) = \gamma' \), so \( M'|\gamma' \) is passive. And because \( \kappa < \delta < \gamma^{+H} = \text{lh}(G), G \) is \( M \)-total and \( \text{cof}^M(\text{lh}(G)) = \kappa^{+H} \). It follows that \( j_{0,\beta}^T \) is continuous at \( \gamma^{+H} \), so by Claim 22, \( j_{0,\beta}^T(\gamma^{+H}) = (\gamma')^{+H'} \). Therefore \( M'|(\gamma')^{+H'} \) is active with \( G' = j_{0,\beta}^T(G) \). So \( E^T_{\beta+1} = G' \), and by Claim 21, this is the only terminal extender. Also \( i_{0,\beta}^T(G) = G' = j_{0,\beta}^T(G) \), by 130
Claim 22 and for the same continuity reasons as for $j_0^M$. (Note that we don’t know that $j_0^M = i_{0^*}$ though; in fact if the embeddings are non-trivial, then they must be distinct.)

Let $\zeta = \text{pred}^M(\beta + 2)$. Let $\kappa' = \text{cr}(G') = i_{0^*}(\kappa) < \delta'$. Note that because $G' = i_0^T(G)$, we have $\zeta \leq \beta.78$ So $0, \zeta^T$ does not drop in model or degree. Moreover, $G'$ is $M_\zeta^T$-total and $\kappa' < i_{0^*}(\delta) < \rho_k(M_\zeta^T).79$ so $(0, \beta + 2)^T$ also does not drop in model or degree. And either $M_\zeta^T = M_\zeta^T$ or $\kappa' < \text{cr}(i_{0^*})$.

Now if $\kappa' < \text{cr}(M_\zeta^T)$ then by Lemma 3.8 and as $H' = M_{\beta + 2}^T$, we have

$$\rho^{H'} = \rho(M_{\beta + 2}^T) = \sup \text{cof}(M_\zeta^T) = \rho(M_\zeta^T) \geq \text{lh}(G') = (\gamma')^{+H'},$$

contradicting the fact that $\rho^{H'} \leq \gamma'$. So $\rho(M_{\beta + 2}^T) \leq \kappa' < \text{cr}(i_{0^*})$. Therefore, recalling line (17), we have

$$\rho^{M_{\beta + 2}} = \rho(M_{\beta + 2}^T),$$

but by Claim 11(3), we have $\rho^{M_{\beta + 2}} < \rho^{H'}$, contradiction.

Case 2. Otherwise.

So either $M_{\gamma + H}$ is passive, or active with an extender $G$ with $\text{cr}(G) \geq \delta$. Note that if $G$ exists then in fact $G$ is type 1, $\text{cr}(G) = \delta < \gamma < \gamma^+H$, and $G$ is $M$-partial.

Suppose $j_0^M(\gamma + H) = (\gamma')^{+H'}$. Then $M_{\beta + 2}^T(\gamma')^{+H'}$ is passive or active type 1. Passivity contradicts Claim 21, so it is active type 1, but then note

$$H' = M_{\beta + 2}^T = \text{Ult}(M_{\beta + 2}, G')$$

with $M_{\beta + 2}^T \triangleright M'$, and therefore $H' \in M'$, contradiction.

So $j_{0^*}^M(\gamma + H) > (\gamma')^{+H'}$. Let $\theta = \text{cof}^M(\gamma + H)$. Since $\gamma + H < \rho_\zeta^M$, it follows that $\theta$ is measurable in $M$ and there is some $\zeta < M$ such that $\text{cr}(j_{0^*}^M(\theta)) = j_{0^*}^M(\theta)$. Letting $\zeta$ be least such, then $j_{0^*}^M$ is continuous at $\theta$ and hence at $\gamma + H$. Letting $f \in M$ be such that $f : \theta \to \gamma + H$ is normal, it follows that

$$j_{0^*}^M(f) \upharpoonright j_{0^*}^M(\theta) = j_{0^*}^M(\theta) \iff (j_{0^*}^M(f) : j_{0^*}^M(\theta) \to \sup j_{0^*}^M)^{\gamma + H}$$

is also a normal function, which is in $M'$. So (together with Claim 22), we have $\text{cof}^M((\gamma')^{+H'}) = j_{0^*}^M(\theta)$, which is inaccessible in $M'$. Therefore $M'((\gamma')^{+H'})$ is not active with an extender $F$ having $\text{cr}(F) < \delta'$ (such an $F$ would be $M'$-total, which contradicts the cofinality just computed). But it is active by Claim 21, hence with a non-$M'$-total extender $F$, which implies that $F$ is type 1 with $\text{cr}(F) = \delta' < \gamma' < (\gamma')^{+H'}$, which gives $H' \in M'$ as before. $\square$

Claim 24. $M|\gamma$ is not active.

Proof. Suppose $M|\gamma$ is active. So $\gamma > \delta$ and $\text{cr}(F) < \delta$ and $F$ is $M$-total. Let $U = \text{Ult}(M, F)$. By condensation for $\omega$-sound mice (Fact 1.9), $H||\gamma + H = U||\gamma + H$. By Claim 19, $\gamma' = j_{0^*}^M(\gamma) = i_{0^*}^T(\gamma)$, so $E^T_{\beta + 1} = F' = j_{0^*}^M(F) = i_{0^*}^T(F)$.

\footnote{In fact, $\zeta$ is the least $\zeta^* \leq \beta$ such that $\zeta^* = \beta$ or $i_{0^*}^T(\kappa) < \text{cr}(i_{0^*}^T(\kappa))$, which is the least $\zeta^* \leq \beta$ (as opposed to $\leq T$) such that $\zeta^* = \beta$ or $\kappa' < \nu(E^T_T)$.}

\footnote{Note we refer here to $\rho_k$, not $\rho_{k + 1}$.}
Case 1. $U|\gamma^+H$ is active with an extender $G$.

Then $j^{\mathcal{M}}_{\gamma^+H}$ and $i^T_{\gamma^+H}$ are continuous at $\gamma^+H$. For if $\text{cr}(G) < \delta$, this is as before, and if $\text{cr}(G) = \delta$ then $\text{cof} M(\gamma^+H) = \text{cof} M(\delta^+H)$, definably over $M|\text{lh}(G) = M|\gamma^+H$, but since $\delta^+H = \gamma$ and $M|\gamma$ is active with $F$, the embeddings are continuous at $\gamma$, and hence at $\gamma^+H$. So in any case, $E^T_{\gamma^+H}$ exists and $E^T_{\gamma^+H} = G' = i^{\mathcal{M}}_{\gamma^+H}(G) = i^T_{\gamma^+H}(G)$. And since $H'$ is $\gamma'$-sound with $\rho'H \leq \gamma'$ and by $p_{k+1}$-preservation, $G'$ is type $1/3$.

We claim that $G$ is type 1, so $\text{cr}(G) = \delta$. For otherwise, $G$ is type 3, so $\text{cr}(G') < \delta$ and $\text{cr}(G') < \delta'$ and $G' \in \text{rg}(j^T_{\gamma^+H})$, so we can reach a contradiction as in Subcase 1.

And we claim that $F$ is type 3. For otherwise $F$ is type 2, so note that $H' = \text{Ult}_0(M'|\gamma', G') \in M'$, a contradiction.

So $F$ is type 3 and $G$ type 1, so $\nu(F) = \delta = \text{cr}(G)$. So $E^T_{\gamma^+H} = F' = j^{\mathcal{M}}_{\gamma^+H}(F)$ and $j^T_{\gamma^+H} = G' = j^{\mathcal{M}}_{\gamma^+H}(G)$. But then we again reach a contradiction like in Subcase 1 (but just with $G' \circ F'$ replacing the single extender $G'$ considered there).

Case 2. $U|\gamma^+H$ is passive.

We already know $E^T_{\gamma^+H} = F'$. Suppose $E^T_{\gamma^+H}$ exists. Then as before and since $U|\gamma^+H$ is passive, $j^T_{\gamma^+H}$ is discontinuous at $\gamma^+H$, which, as in Case 2, implies that $\text{cr}(E^T_{\gamma^+H}) = \delta'$. But then if $F'$ is type 2, this gives $H' \in M'$, and if $F'$ is type 3, gives a contradiction like in Case 1.

So $E^T_{\gamma^+H}$ does not exist, so $H' = M^T_{\gamma^+H}$. Since $F' \in \text{rg}(j^T_{\gamma^+H})$, as usual $\zeta = \text{pred}^T(\beta + 2) \leq^T \beta$, so $(0, \beta + 2]^T$ does not drop in model or degree, and $H' = \text{Ult}_k(M^T_{\zeta}, F')$. Let $\kappa' = \text{cr}(F')$.

If $\kappa' < \rho(M^T_{\zeta})$ then much as in Case 1 and since $\rho'H' \leq \gamma'$, we have

$$\rho'H' = \rho(M^T_{\beta+2}) = \text{sup} i^T_{\zeta, \beta+2} \rho(M^T_{\zeta}) = \text{lh}(F') = \gamma',$$

which implies that $F'$ is superstrong and $\rho(M^T_{\zeta}) = (\kappa')^{+M^T_{\zeta}}$. But $\text{lh}(p^{M^T_{\beta+2}}) > n = \text{lh}(q^M) = \text{lh}(\rho'H \setminus \gamma)$, and since $i^T_{\zeta, \beta+2}$ is $p_{k+1}$-preserving, therefore $\text{lh}(p^{M^T_{\beta+2}}) > n$, so $\text{lh}(\rho'H') > n$, which contradicts the fact that $H'$ is $\gamma'$-sound with $\rho'H' = \gamma'$ and $\text{lh}(\rho'H' \setminus \gamma') = n$.

So $\rho^{M^T_{\zeta}} \leq \kappa'$, which leads to a contradiction like at the end of Case 1. \qed

The last two claims are incompatible, yielding a contradiction which completes the proof of solidity and universality. \qed

14.2 Iterability, closeness and comparison termination

In this section we give the proofs of Claims 9, 10 and 16 from the proof of Theorem 14.1.

Proof of Claim 9. The proof is similar to that in the proof of condensation from normal iterability, [24, Theorem 5.2 proof, Claim 5]. However, the details of the bi-cephali-cephalanxes there and trees formed on them are somewhat different to the bi-cephali and trees currently under consideration. Moreover, there is an issue which arises in connection with superstrong extenders, which was unfortunately ignored in [24], leading to a minor mistake in the proof of [24, Theorem 5.2 proof, Claim 5]. So we will handle it correctly and in some detail here. The issue is just the analogue of that discussed at

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the end of the proof of Claim 6, within the proof of Lemma 10.2, which led to the consideration of essentially-$(0,0,0)$-maximal trees on the phalanx $2^\omega \mathcal{W}_1$, instead of standard $(0,0,0)$-maximal trees. (If one is not familiar with the setup in [24, Theorem 5.2], nor the proof of Claim 6 of 10.2, then one should simply read ahead for the present, as the details regarding the issue are reasonably self-contained. If more details are desired after reading here, then [24] might be useful, though the setup there is somewhat different.)

First let us describe the basic mechanism we will use for copying extenders. Recall the extender copying function $\text{copy}(\varsigma, E)$ from Definition 6.14, which is defined under the assumption that $\varsigma$ is a non-$\nu$-high embedding between premice. This will not suffice for our present purposes, as we will have to work with copy maps which might be $\nu$-high. So we now define a variant $\text{co} \text{copy}(\varsigma, E)$, which also works when $\varsigma$ is $\nu$-high, assuming enough condensation (which we will have). The desire is that when copying an iteration $\rho$ is the set of $(q, F)$ and by commutativity. We now set

\[ \varsigma, E \]

So we now define a variant concopy($\varsigma, \mathcal{E}$), which also works when $\varsigma$ is $\nu$-high, assuming enough condensation (which we will have). The desire is that when copying an iteration tree $\tilde{T}$ to a tree $\mathcal{U}$, we can copy $E_\alpha^T$ to $E_\alpha^U = \text{copy}(\varsigma, E)$, and moreover, in such a manner that $\tilde{T}, \mathcal{U}$ have the same tree order (and more).\footnote{The improved method was noticed by the author in about 2015. Prior to that point, the standard method for copying in this situation was to use an extra extender in the upper tree, so as to “reveal” the extender $\text{Shift}(\varsigma)(E)$ (see §1.3.5), and then to copy $E$ to that extender. But this creates notational hassles, in particular because the trees are indexed differently; the method we describe here takes a little thought at the outset, but works out simpler in the end.}

14.14 Definition. Let $P, N$ be $k$-sound premice such that all proper segments of $N$ satisfy standard condensation facts (which facts become clear below). Let $\varsigma : P \to N$ be $k$-lifting.

Let $E \in \mathcal{E}_P^P$. Set $\text{copy}(\varsigma, E) = \text{copy}(\varsigma, E)$ unless $\varsigma$ is $\nu$-high and $E \in \mathcal{E}_P^{\nu}(P)$, so assume this is the case. In particular, $P, N$ are active type 3, $\nu(F^{\mathcal{W}}) < \text{lh}(E) < OR^P$, and so $\tilde{\varsigma}(E) \notin \mathcal{E}_N^N$ where $\tilde{\varsigma} = \text{Shift}(\varsigma)$ (see §1.3.5). We want to replace $\tilde{\varsigma}(E)$ with a certain hull in $\mathbb{E}_N^N$. We will also define some other associated objects, $Q \triangleleft P, Q' \triangleleft N$ and $\varsigma' : Q \to Q'$.

Let $Q \triangleleft P$ be least such that $\rho_{Q}^{Q} = \nu$ and $E \in \mathcal{E}_P^{\nu}$. Let $Q^{\uparrow} = \tilde{\varsigma}(Q)$, so $Q^{\uparrow} \triangleleft \text{Ult}(N, F^N)$ and $\rho_{Q}^{Q^{\uparrow}} = \tilde{\varsigma}(\nu) > \nu(F^N)$. Let $q < \omega$ be least such that $\rho_{q}^{Q} = \nu$ (so if $\rho_{q}^{Q} = \nu$ then $q = 0$). Let

\[ H' = \text{Hull}_{\theta + 1}(\nu(F^N) \cup \{\rho_{q+1}^{Q}\}) \]

$Q'$ be the transitive collapse of $H'$, and $\sigma : Q' \to H'$ the uncollapse.

Note that $Q'$ is $(q+1)$-sound, because (i) $\nu(F^N)$ is a cardinal in $N$ and in $\text{Ult}(N, F^N)$, so $\rho_{q+1}^{Q} = \nu(F^N)$; and (ii) letting $w \in Q$ be the set of $(q + 1)$-solidity witnesses for $Q$, then there is $\gamma < \nu$ such that $w \in \text{Hull}_{\gamma+1}(\{\gamma, \rho_{q}^{Q}\})$ which implies $\tilde{\varsigma}(w) \in H'$, but $\tilde{\varsigma}(w)$ is the set of $(q + 1)$-solidity witnesses for $Q'$.

So by condensation with $\sigma$ (in $\text{Ult}(N, F^N)$), we get $Q' \triangleleft \text{Ult}(N, F^N)$, so by coherence, $Q' \triangleleft N$. Let $\varsigma' : Q \to Q'$ be $\varsigma' = \sigma^{-1} \circ \tilde{\varsigma} \restriction Q$. Then $\varsigma'$ is a $\rho_{q+1}$-preserving near-$q$-embedding and $\varsigma \subseteq \varsigma'$. And $\varsigma'$ is $\nu$-preserving, because $\tilde{\varsigma} \restriction Q : Q \to Q^{\uparrow}$ is $\nu$-preserving and by commutativity. We now set $F = \text{copy}(\varsigma', E)$, so $F \in \mathcal{E}_P^{Q'} \subseteq \mathcal{E}_N^N$. Moreover, because $\varsigma \subseteq \varsigma'$, $\varsigma'$ has appropriate properties to use as a copy map at this stage. \(\square\)

14.15 Definition. Let $P, N$ be active pre-ISC-premice. Let $\varsigma : P \to N$ be $(-1)$-lifting (that is, $\Sigma_0$-elementary).\footnote{See §1.3.5. Recall the $\Sigma_0$-elementarity is with respect to $\mathcal{E}_P^P, F^P$, where $F^P$ is encoded as specified in Definition 1.8. Also recall that dom($\varsigma$) = $P$; there is no squashing being considered here.} Let $E \in \mathcal{E}_P^{P^P}$. Then concopy($\varsigma, E$) denotes $F^N$ if $E = F^P$, \(\square\)
and denotes $\varsigma(E)$ otherwise.

Note that in all cases above, we have $\text{Shift}(\varsigma)(\nu(E)) \geq \nu(\text{concopy}(\varsigma, E))$.

Now fix a $(k, \omega_1 + 1)$-strategy $\Sigma$ for $M$. By Lemma 6.13, $\Sigma$ induces a canonical strategy $\Sigma'$ for essentially-$k$-maximal trees. (We will only need to make use of the “essentially" aspect of $\Sigma'$ when $\alpha + 1$ in a special circumstance, explained below.)

We define an almost $((k, k), \omega_1 + 1)$-strategy $\Gamma$ for $B$, such that trees $T$ on $B$ via $\Gamma$ lift to (essentially-$k$-maximal) trees $\mathcal{U}$ on $M$ via $\Sigma'$. This will suffice, since by Claim 8, we just need to ensure wellfoundedness of the models of $\mathcal{U}$. Much as in Definition 14.8, for $\alpha \in \mathcal{B}^T$, let $J^T_\alpha = J^T_{\alpha}(J)$ where $J \alpha M$ is least such that $\gamma \leq \text{OR}^T$ and $\rho^T_\omega = \delta$. We define copy maps $\sigma_\alpha$, $\varsigma_\alpha$, and structures $K^T_\alpha \leq M^T_\alpha$ (for some values of $\alpha$, some of these are irrelevant), with the following properties:

1. $\text{lh}(\mathcal{U}) = \text{lh}(T)$ and $<^\mathcal{U} = <^T$.

2. Suppose $\alpha \in \mathcal{B}^T$, so we have $B^T_\alpha = (\delta_\alpha, \gamma_\alpha, \pi_\alpha, H^T_\alpha, M^T_\alpha)$. Then:
   - $[0, \alpha]^T \cap \mathcal{D}^T_{\text{deg}} = \emptyset$, so $\deg^T_\alpha = k$,
   - $\sigma_\alpha : M^T_\alpha \rightarrow M^T_\alpha$ is a $\nu$-preserving $k$-embedding and $\sigma_\alpha \circ J^T_{\alpha} = i^T_0$.
   - $\varsigma_\alpha = \sigma_\alpha \circ \pi_\alpha : H^T_\alpha \rightarrow M^T_\alpha$ is a $k$-embedding, and note $\varsigma_\alpha \upharpoonright \gamma_\alpha = \sigma_\alpha \upharpoonright \gamma_\alpha$.

(However, $\pi_\alpha$, and therefore also $\varsigma_\alpha$, might be $\nu$-high. In fact $\pi_0$ might be $\nu$-high. But $\pi_\alpha, \varsigma_\alpha$ are non-$\nu$-low, since they are $k$-embeddings.)

3. Suppose $\alpha \in \mathcal{H}^T$. Then:
   - $\varsigma_\alpha : H^T_\alpha \rightarrow M^T_\alpha$ is c-preserving $\deg^T_0$-lifting.
   - If $[0, \alpha]^T \cap \mathcal{D}^T_{\text{deg}} = \emptyset$ then $[0, \alpha]^T \cap \mathcal{D}^T_{\text{deg}} = \emptyset$ and $\varsigma_\alpha \circ i^T_0 = i^T_{\alpha} \circ \pi_0$ and $\varsigma_\alpha$ is a $k$-embedding (note $\pi_0 = \varsigma_0 = \pi$).

4. Suppose $\alpha \in \mathcal{M}^T$ and $\alpha$ is not weakly anomalous. Then:
   - $\sigma_\alpha : M^T_\alpha \rightarrow M^T_\alpha$ is c-preserving $\deg^T_0$-lifting.
   - If $[0, \alpha]^T \cap \mathcal{D}^T_{\text{deg}} = \emptyset$ then $[0, \alpha]^T \cap \mathcal{D}^T_{\text{deg}} = \emptyset$ and $\sigma_\alpha \circ i^T_0 = i^T_{\alpha} \circ \pi_0$ and $\sigma_\alpha$ is a $k$-embedding.

5. Suppose $\alpha \in \mathcal{M}^T$ and $\alpha$ is weakly anomalous. Then:
   - $K^T_\alpha \leq M^T_\alpha$.
   - If $\alpha$ is non-anomalous then $\sigma_\alpha : M^T_\alpha \rightarrow K^T_\alpha$ is c-preserving $\deg^T_0$-lifting.
   - If $\alpha$ is anomalous (so $\deg^T_0 = -1$) then $K^T_\alpha$ is active type 3 and $\sigma_\alpha : M^T_\alpha \rightarrow K^T_\alpha$ is $(-1)$-lifting. Note that in this case, $M^T_\alpha$ is a pre-ISC-premouse which fails the ISC, with $\nu(F(M^T_\alpha)) \leq \text{lgcd}(M^T_\alpha)$, and $\sigma_\alpha$ is also c-preserving. There is no squashing involved, so $\text{dom}(\sigma_\alpha) = M^T_\alpha$ (even though $K^T_\alpha$ is actually a premouse). See §1.3.5.

Moreover, let $\beta = \max(\mathcal{B}^T \cap [0, \alpha]^T)$ and $\varepsilon + 1 = \text{succ}^T(\beta, \alpha)$. Then:
   - $J^T_\beta = M^T_{\varepsilon + 1} \upharpoonright M^T_\beta$ and $[\varepsilon + 1, \alpha]^T \cap \mathcal{D}^T = \emptyset$ and $J^T_{\varepsilon + 1, \alpha} : J^T_\beta \rightarrow M^T_\alpha$ and $\text{cr}(j^T_{\varepsilon + 1, \alpha}) = \delta_\beta$.
Suppose there is no $\chi \in [\varepsilon + 1, \alpha)^T$ such that $\text{cr}(j^{\alpha}_{j^{\alpha}_{\chi}}) \geq j^{\alpha}_{j^{\alpha}_{\chi}} T_{\delta}(\beta)$. Then $[0, \alpha]^T \cap \mathcal{D}^{\alpha}_{\mathrm{deg}} = \emptyset$ and $K^{\alpha}_{\alpha} = \mathcal{i}^{\alpha}_{0\alpha}(J) \leq M^{\alpha}_{\alpha}$ and

$$\sigma_\alpha \circ j^{\alpha}_{j^{\alpha}_{\chi},\alpha} \circ j^{\alpha}_{0\alpha} \mid J = j^{\alpha}_{0\alpha} \mid J.$$ 

Suppose there is $\chi$ as above. Then $\alpha$ is non-anomalous and $K^{\alpha}_{\alpha} = M^{\alpha}_{\alpha}$. Moreover, taking $\chi$ least such, and $\xi + 1 = \text{succ}^T(\chi, \alpha)$, then $\mathcal{D}^{\alpha}_{\mathrm{deg}} \cap [0, \alpha]^T = \{\xi + 1\}$ and $M^{\alpha}_{\xi+1} = K^{\alpha}_{\chi} = \mathcal{i}^{\alpha}_{0\alpha}(J)$ and $\mathcal{i}^{\alpha}_{\xi+1} : K^{\alpha}_{\chi} \rightarrow K^{\alpha}_{\alpha} = M^{\alpha}_{\alpha}$, and

$$\sigma_\alpha \circ j^{\alpha}_{j^{\alpha}_{\chi},\alpha} \circ j^{\alpha}_{0\alpha} \mid J = \mathcal{i}^{\alpha}_{\xi+1,\alpha} \circ \mathcal{i}^{\alpha}_{0\alpha} \mid J.$$ 

6. Let $\alpha + 1 < \text{lh}(T)$. If $\text{exitside}^{T}_{\alpha} = 0$ then $E^{\alpha}_{\alpha} = \text{concopy}(\sigma_\alpha, E^{\alpha}_{\alpha})$, whereas if $\text{exitside}^{T}_{\alpha} = 1$ then $E^{\alpha}_{\alpha} = \text{concopy}(\sigma_\alpha, E^{\alpha}_{\alpha})$.

7. Let $\alpha + 1 < \text{lh}(T)$. If $\alpha + 1 \in \mathcal{D}^T \cup \mathcal{H}^T$ then $\sigma_{\alpha+1}$ is just the map given by the natural application of the Shift Lemma. (If $\alpha + 1$ is a mismatched dropping node and $\beta = \text{pred}^T(\alpha + 1)$, then apply the Shift Lemma to $\sigma_\beta \mid J^T_\beta : J^T_\beta \rightarrow \mathcal{i}^{\beta}_{0\beta}(J)$ and the map $\psi : \text{exit}_\alpha \rightarrow \text{exit}_\alpha$, extracted from the definition of concopy and either $\sigma_\alpha$ (if $\text{exitside}^{T}_{\alpha} = 0$) or $\sigma_\alpha$ (if $\text{exitside}^{T}_{\alpha} = 1$).) If $\alpha + 1 \in \mathcal{H}^T$ then $\sigma_\alpha$ is likewise produced by the Shift Lemma.

8. If $\eta < \text{lh}(T)$ is a limit (so $[0, \eta]^T = [0, \eta]^{\beta}$ and $[0, \eta]^{\beta} = \Sigma'(T \mid \eta)$), then $\sigma_\eta, \sigma_\eta$ are just formed via commutativity as usual.

These conditions determine how $T, U$ are formed, and also $\sigma_\alpha, \sigma_\alpha, K^{\alpha}_{\alpha}$ in the cases that they are relevant. We now make a couple of remarks on the less standard considerations here.

First, in the context of clause 6, suppose $\sigma_\alpha$ is $\nu$-high and $\nu(H^{\alpha}_{\alpha}) < \text{lh}(E^{\alpha}_{\alpha}) < \text{OR}(H^{\alpha}_{\alpha})$. Then defining $\sigma'_{\alpha}, Q, Q'$ as in 14.14, note that if $\beta + 1 < \text{lh}(T)$ and $\text{pred}^T(\beta + 1) = \alpha$ and $\nu(H^{\alpha}_{\alpha}) \leq \text{cr}(E^{\alpha}_{\alpha})$, then $H^{\alpha}_{\alpha} \cup J \leq Q$ and $M^{\alpha}_{\alpha+1} \leq Q'$, and $\mathcal{i}^{\alpha}_{\beta} \mid H^{\alpha}_{\alpha+1} \rightarrow M^{\alpha}_{\alpha+1}$ is the appropriate map with which to apply the Shift Lemma when defining $\sigma_{\alpha+1}$.

Second, let us consider the manner in which $U$ can fail to be $k$-maximal, and what happens in that situation. Suppose we have determined $T \mid (\alpha + 1)$ and $U \mid (\alpha + 1)$ and $\langle \sigma_\beta, \sigma_\beta, J^T_\beta \rangle_{\beta \leq \alpha}$, and exitside$^T_{\alpha}$ and $E^{\alpha}_{\alpha}$ and are chosen appropriately. In particular, $E^{\alpha}_{\alpha}$ is $T \mid (\alpha + 1)$-normal; that is, $\text{lh}(E^{\alpha}_{\beta}) \leq \text{lh}(E^{\alpha}_{\alpha})$ for all $\beta < \alpha$. Then $E^{\alpha}_{\alpha}$ is determined by clause 6. We get $\text{lh}(E^{\alpha}_{\beta}) \leq \text{lh}(E^{\alpha}_{\alpha})$ for all $\alpha < \beta$ much as usual, except in the case that $\alpha = \varepsilon + 1$ for some $\varepsilon$, $\varepsilon + 1$ is a mismatched dropping node and $E^{\alpha}_{\varepsilon}$ is of superstrong type (hence $E^{\alpha}_{\varepsilon}$ is also of superstrong type). Suppose this is the case. Let $\beta = \text{pred}^T(\varepsilon + 1) = \max(\mathcal{D}^T \cap [0, \varepsilon + 1)^T)$. So $\beta \leq T^{\varepsilon}$ and $\beta = \max(\mathcal{D}^T \cap [0, \varepsilon + 1)^T)$ and exitside$^T_{\beta} = 0$ and $\gamma_{\beta} = \beta + H^{\alpha}_{\beta} \leq \text{lh}(E^{\alpha}_{\beta})$.

Now $K^{\alpha}_{\varepsilon+1} = \mathcal{i}^{\beta}_{0,\varepsilon+1}(J) = \mathcal{i}^{\beta}_{\varepsilon+1}(J)$ where $J' = \mathcal{i}^{\beta}_{0\beta}(J)$, and $J' < M^{\alpha}_{\beta}$ with $\rho^J_{\beta} = \mathcal{i}^{\beta}_{0\beta}(\delta) = \mathcal{c}_{\beta}(\mathcal{c}(E^{\alpha}_{\beta})) = \mathcal{cr}(E^{\alpha}_{\beta})$, so $K^{\alpha}_{\varepsilon+1} < M^{\alpha}_{\varepsilon+1}$ with $\rho^J_{\beta} = \mathcal{i}^{\beta}_{0\beta}(\delta) = \mathcal{c}_{\beta}(\mathcal{c}(E^{\alpha}_{\beta})) = \mathcal{cr}(E^{\alpha}_{\beta})$ (since $E^{\alpha}_{\beta}$ is superstrong). But $\text{lh}(E^{\alpha}_{\beta}) = \nu(E^{\alpha}_{\beta})$ $+$ $M^{\alpha}_{\varepsilon+1}$, so $\text{OR}(K^{\alpha}_{\varepsilon+1}) < \text{lh}(E^{\alpha}_{\beta})$. Since $\sigma_{\varepsilon+1} : M^{\alpha}_{\varepsilon+1} \rightarrow K^{\alpha}_{\varepsilon+1}$ and sides$^{\alpha}_{\varepsilon+1} = \{1\}$, we must have exitside$^{\alpha}_{\varepsilon+1} = 1$ and $E^{\alpha}_{\varepsilon+1} \in \mathcal{E}_+(M^{\alpha}_{\varepsilon+1})$, and as $E^{\alpha}_{\varepsilon+1} = \mathcal{concopy}(\sigma_{\varepsilon+1}, E^{\alpha}_{\varepsilon+1})$ in $\mathcal{E}_+(K^{\alpha}_{\varepsilon+1})$, so $\text{lh}(E^{\alpha}_{\beta}) < \text{lh}(E^{\alpha}_{\beta})$. However, $\text{lh}(E^{\alpha}_{\beta}) \leq \text{lh}(E^{\alpha}_{\varepsilon+1})$ (with equality iff $M|\varepsilon$ is active iff $J = M|\varepsilon$), and
as $E^T_\zeta$ also has superstrong type, we have $\sigma_{\zeta+1}(\nu(E^T_\zeta)) = \nu(E^T_\zeta)$, so either $\lh(E^T_\zeta) = \lh(E^T_{\zeta+1})$ and $\exit^T_{\zeta+1} = K^U_{\zeta+1}$, or $\lh(E^T_\zeta) < \lh(E^T_{\zeta+1})$ and

$$\text{Shift}(\sigma_{\zeta+1})(\lh(E^T_\zeta)) = \nu(E^T_\zeta)^{+\nu_{\zeta+1}} < \lh(E^T_{\zeta+1}).$$

In either case, $\nu(E^T_\zeta) \leq \nu(E^T_{\zeta+1})$, as required for the essential-$\varpi$-maximality of $U$.

Note also that if $\zeta + 1$ is anomalous then $J = M|\gamma$ and $J^T_\beta = M^T_{\zeta+1} | \gamma| = M^*_T$ are type 3 and $\nu(F(J^T_\beta)) = \delta_\beta = \cr(E^T_\zeta)$, and

$$M^T_{\zeta+1} = \text{Ult}_-1(J^T_\beta, E^T_\zeta) = \text{Ult}(J^T_\beta, E^T_\zeta),$$

which is a pre-ISC-premouse with largest cardinal

$$i_{E^T_\zeta}(\delta_\beta) = \lambda(E^T_\zeta) = \nu(E^T_\zeta) = \nu(F(M^T_{\zeta+1})), \tag{2}$$

$K^U_{\zeta+1}$ is a type 3 premouse with largest cardinal $\nu(E^T_\zeta) = \lambda(E^T_\zeta)$, and $\sigma_{\zeta+1} : M^T_{\zeta+1} \rightarrow K^U_{\zeta+1}$ is $\Sigma_0$-elementary. In this case, $E^T_{\zeta+1} = F(M^T_{\zeta+1})$, since $\lh(E^T_\zeta) = \OR(M^T_{\zeta+1})$, and since $\lh(E^T_\zeta) \leq \lh(E^T_{\zeta+1})$, $F(M^T_{\zeta+1})$ is the only valid option for $E^T_{\zeta+1}$. So $\nu(E^T_\zeta) = \nu(F(M^T_{\zeta+1}))$, so the rules of normality ensure that there is no $\alpha$ with $\pred^T_\alpha(\alpha + 1) = \zeta + 1$, so $\zeta + 1$ is an end-node of $\mathcal{T}$, and also of $\mathcal{U}$. On the other hand, if $\zeta + 1$ is non-anomalous (but it is weakly anomalous in the present discussion) then there is in general nothing preventing there being $\alpha < \lh(T)$ with $\zeta + 1 < T \alpha$, hence with $\nu(E^T_\zeta) \leq \cr(i_{\zeta+1,\alpha}^T)$, even with $(\zeta + 1, \alpha)^T \cap H^T = \emptyset$; in this case we get $(0, \alpha)^T \cap H^T = \emptyset$ and $K^U_\alpha = M^T_\alpha$. (It can be that $\deg^T_\alpha < \deg^T_{\zeta+1}$ here; it seems it might also be that $\deg^T_\alpha < \deg^T_{\zeta+1}$.)

The remaining details of the copying process are left to the reader; they are basically routine. The overall picture is also similar to that in the proof of [24, Theorem 5.2], though there, the possibility that $\mathcal{U}$ fails to be $k$-maximal was overlooked. That proof should be corrected by allowing $\mathcal{U}$ (as there) to be essentially-$k$-maximal. The proof there then adapts much as above.

**Proof of Claim 10.** This is basically as in the proof of Closeness [9, Lemma 6.1.5], proceeding by induction on $\xi$. However, there are some small, but important, differences. They are mostly because we are iterating bicephali, as opposed to premice. Similar issues arise in the classical proof of solidity etc., when one iterates phalanxes, but some of these things were not covered in detail in [9], for example. So we discuss enough details to describe the differences. One should also keep in mind that we allow superstrong extenders on $E$, whereas this was not the case in [9], however, this does not have a significant impact here.

Let $\kappa = \cr(E^T_\xi)$. So $\kappa < \nubar(\exit^T_\beta)$ and $\kappa^{+\exit^T_\beta} \leq \OR(\exit^T_\beta)$. Let $e = \exit^T_\beta$ and $E = E^T_\xi$, so $E \in \mathcal{E}_+(M^T_{\beta})$.

**Case 1.** $\xi + 1 \in B^T$.

So we must see that $E$ is close to $M^T_\beta$. We have $\beta \in B^T$ and $\kappa < \delta^T_\beta$ and $\kappa^{+\exit^T_\beta} = \kappa^{+B^T_\beta} \leq \delta^T_\beta$, so $\kappa^{+B^T_\beta} < \lh(E^T_\beta)$.

**Subcase 1.1.** $\beta = \xi$.

We may assume $E \in \mathcal{E}_+(H^T_\beta)$. We have $H^T_\beta ||^{\kappa^{+H^T_\beta}} \subseteq M^T_\beta$, so if $E \in H^T_\beta$ then $E$ is close to $M^T_\beta$. Otherwise $E = F^T_{H^T_\beta}$, which is close to $M^T_\beta$ because $\pi_{\beta} : H^T_\beta \rightarrow M^T_\beta$ is $r\Sigma_1$-elementary with $\kappa^{+H^T_\beta} < \cr(\pi_{\beta})$, so $F^T_{H^T_\beta}$ is a sub-extender of $F^T_{M^T_\beta}$. 136
Subcase 1.2. \( \beta < \xi \).

By the proof of [9, 6.1.5], we may assume \( E = F(M_{\xi}^T) \). Likewise, we may assume either \( \xi \) is anomalous, or \( M_{\xi}^T \) is active type 2 with \( \rho_1(M_{\xi}^T) \leq \kappa^+M_{\xi}^T \).

Let \( \zeta \) be least with \( \zeta \geq \beta \) and \( \zeta + 1 \leq T \). It follows that \( (\zeta + 1, \xi)^T \cap D_{\text{deg}} = \emptyset \) and either (i) \( \zeta + 1 \in \mathcal{B}^T \) and \( k = 0 \), or (ii) \( \zeta + 1, \xi \notin \mathcal{B}^T \) and \( \deg_{\zeta+1}^T = \deg_{\xi}^T \in \{-1, 0\} \) and \( \rho_1(M_{\zeta+1}^T) = \rho_1(M_{\xi}^T) \leq \kappa^+M_{\xi}^T \leq \kappa^+M_{\zeta}^T < \text{cr}(E_{\xi}^T) \), and therefore \( \text{pred}_{\xi}^T(\zeta + 1) = \beta \) (as in the proof of [9, 6.1.5]).

Subsubcase 1.2.1. \( \zeta + 1 \in \mathcal{B}^T \) and \( k = 0 \).

First suppose \( e = 1 \). By induction, the extenders used along \( (\beta, \xi)^T \) are close to the \( M_{\alpha}^T \)'s. As \( \kappa < \text{cr}(E_{\xi}^T) \), it follows as usual that \( E \) is close to \( M_{\beta}^T \), as desired.

Now suppose \( e = 0 \). Let \( \beta' = \mathcal{B}_{\xi}(\xi) \). Then the extenders used along \( (\beta', \xi)^T \) are close to the \( H_{\beta}^T \)'s, and so \( E \) is close to \( H_{\beta}^T \). But \( \pi_{\beta'} : H_{\beta}^T \rightarrow M_{\beta}^T \) is r\( \Sigma_1 \)-elementary and \( \text{cr}(E) < \text{cr}(E_{\xi}^T) < \delta_{\beta'} \), so \( E \) is close to \( M_{\beta}^T \). But the extenders used along \( (\beta, \beta')^T \) are close to the \( M_{\alpha}^T \)'s, and it follows that \( E \) is close to \( M_{\beta}^T \), as desired.

Subsubcase 1.2.2. \( \zeta + 1 \notin \mathcal{B}^T \) and \( \deg_{\zeta+1}^T = \deg_{\xi}^T \in \{-1, 0\} \).

By induction, the extenders used along \( (\beta, \xi)^T \) are close to their target models, so essentially as usual, \( E \) is close to \( M_{\zeta+1}^T \), and since \( M_{\zeta+1}^T \leq H_{\beta}^T \) or \( M_{\zeta+1}^T \leq M_{\beta}^T \), this suffices as before, using \( \pi_{\beta} \) to lift the r\( \Sigma_1 \) definitions of measures if \( M_{\zeta+1}^T \leq H_{\beta}^T \). (Maybe \( \zeta + 1, \xi \) are anomalous, in which case \( e = 1 \) and \( M_{\zeta+1}^T, M_{\beta}^T \) are not premice; in this case \( \text{close} \) is defined as usual, but using (unsquashed) \( \Sigma_1 \)-definability, and the usual calculations work.)

Case 2. \( \xi + 1 \notin \mathcal{B}^T \).

If \( \xi + 1 \) is not a mismatched dropping node, this is just like in [9]. So suppose \( \xi + 1 \) is mismatched dropping. So \( \beta \in \mathcal{B}^T \), \( \xi + 1 \in \mathcal{M}^T \), \( M_{\xi+1}^T = J_{\beta}^T = \iota_{0, \beta}(J) \), \text{exitside}_{\beta} = 0 \) and \( \kappa = \delta_{\beta} < \kappa^+H_{\beta} = \gamma_{\beta}^T < \text{lh}(E_{\beta}^T) \) (recall \( \gamma < \text{OR}^H \)).

Subclaim 9.1. The component measures of \( E \) are all in \( H_{\beta}^T \).

**Proof.** Suppose not. Suppose \( \xi = \beta \). Since \text{exitside}_{\beta} = 0 \), we must have \( E_{\beta}^T = F_{\beta}^H \) and \( H_{\beta}^T \) is type 1/2 and \( \rho_1^{H_{\beta}^T} \leq \kappa^+H_{\beta} = \gamma_{\beta}^T \). But this contradicts Claim 6 part 2.

So \( \xi > \beta \). Note that \( E_a \notin M_{\xi}^T \) for some \( a \), so \( M_{\xi}^T \) is active type 2, \( E = F(M_{\xi}^T) \) and \( \rho_1(M_{\xi}^T) \leq \kappa^+H_{\xi} = \gamma_{\beta}^T \). Let \( \zeta \geq \beta \) be least such that \( \zeta + 1 \leq T \). Arguing much as usual, we get that \( \text{pred}_{\xi}(\zeta + 1) = \beta \) and \( (\zeta + 1, \xi)^T \cap D_{\text{deg}} = \emptyset \), \( \kappa < \text{cr}(E_{\xi}^T) \), etc, so \( \zeta + 1 \in \mathcal{M}^T \) and \( H_{\zeta+1}^T \leq H_{\beta}^T \). So by induction, \( F_{\beta}^H \) is close to \( H_{\zeta+1}^T \). So if \( H_{\zeta+1}^T \prec H_{\beta}^T \) we are done. Otherwise \( H_{\zeta+1}^T = H_{\beta}^T \) is active type 2 and \( \rho_1(H_{\beta}^T) \leq \gamma_{\beta}^T \) and \( k = 0 \), again contradicting Claim 6. \( \square \)

So the component measures are all in \( H_{\beta}^T \)|\( \chi \), where \( \chi = \kappa^+H_{\beta}^T \). But by condensation using \( \pi_{\beta} \)\(^{82}\) either

(i) \( H_{\beta}^T \)|\( \chi = M_{\beta}^T \)|\( \chi \), or

\(^{82}\)Should point out that this is is internal to \( M_{\beta}^T \), preserved by \( j_{0, \beta} \) from \( M \).
(ii) $M|\gamma \uparrow_\beta$ is active with extender $F$ and $H^T_\beta||\chi = U||\chi$ where $U = \text{Ult}(M^T_\beta, F)$.

If (i) holds, note that since $J \triangleleft M$ and $\rho^J_\alpha = \kappa$, we have $\chi \leq \text{OR}^J$, so $H^T_\beta||\chi = J||\chi$, so the measures are in $J$, and so $E$ is close to $J$. And if (ii) holds, then $J = M|\gamma \uparrow_\beta$, and then since the measures are in $U||\chi$, it again follows that $E$ is close to $J$, as desired.

This completes the proof of closeness (or its approximation).

Proof of Claim 16. This is by the usual argument unless the comparison produces a tree $\mathcal{U}$ uses extenders $E^T_\alpha$ which are not premouse extenders (that is, not the active extender of a premouse), so assume this is the case. So then $M|\gamma$ is active type 3, $\delta < \gamma$, and $\delta = \text{cr}(E)$ for some $H$-total extenders on $\mathcal{E}^{\mathcal{U}}_\delta$.

Running the usual argument, we take an elementary $\varphi : A \rightarrow V_\Omega$ with some large enough $\Omega$ and $A$ countable transitive, and $\varphi(\omega_1^A) = \omega_1$, and everything relevant in $\text{rg}(\varphi)$. Let $\kappa = \omega_1^A = \text{cr}(\varphi)$. As usual, $\mathcal{U}|\omega_1, T|\omega_1$ are both cofinally non-padded and $\kappa <^T \omega_1$ and $\kappa <^T \omega_1$. We may assume that $\kappa \in B^\mathcal{U}$ (so $\omega_1 \in B^\mathcal{U}$ also) as otherwise the extenders used along $(\kappa, \omega_1)^{\mathcal{U}T}$ satisfy the ISC. We have that $M^\mathcal{U}_\kappa, M^\mathcal{U}_{\omega_1}, M^T_\kappa, M^T_{\omega_1}$ all have the same $\mathcal{P}(\kappa)$, $\mathcal{P}(\kappa) \cap H^{\mathcal{E}}_{\kappa} \subseteq \mathcal{P}(\kappa) \cap M^T_\kappa$, and

$$ ((\kappa, \omega_1)^{\mathcal{U}}) \subseteq\mathcal{P}(\kappa) \cap H^{\mathcal{E}}_{\kappa} \subseteq \mathcal{P}(\kappa) \cap M^T_\kappa. $$

Let $\alpha + 1 = \text{succ}^\mathcal{U}((\kappa, \omega_1)^{\mathcal{U}T})$ and $\beta + 1 = \text{succ}^\mathcal{T}((\kappa, \omega_1)^{\mathcal{T}T})$. Then $E^\mathcal{U}_\alpha$ and $E^\mathcal{T}_\beta$ have critical point $\kappa$, measure the same $\mathcal{P}(\kappa)$, and letting $\iota = \min(\text{exit}^\mathcal{U}_\alpha, \text{exit}^\mathcal{T}_\beta)$, we have $E^\mathcal{U}_\alpha|\iota = E^\mathcal{T}_\beta|\iota$.

By the rules of comparison, $\alpha \neq \beta$. If $\alpha < \beta$ then we would have $\text{exit}^\mathcal{U}_\alpha < \text{exit}^\mathcal{T}_\beta$, but this contradicts the ISC as usual. So $\beta < \alpha$. So if exit$^\mathcal{U}_\alpha$ is a premouse, then by Claim 14, we get $\nu(E^\mathcal{U}_\alpha) = \nu(E^\mathcal{T}_\beta) < \nu(E^\mathcal{E}_\alpha) = \nu(E^\mathcal{E}_\beta)$, but then exit$^\mathcal{U}_\alpha$ fails ISC, a contradiction. So $\alpha$ is anomalous. Let $\zeta$ be least such that $\zeta + 1 <^\mathcal{T} \alpha$ and $\zeta + 1$ is anomalous. Let $Q^\ast = M^\mathcal{U}_{\zeta + 1}$, so $Q^\ast$ is a type 3 premouse and $\text{cr}(E^\mathcal{U}_\alpha) = \nu(Q^\ast)$. We have $F^{Q^\ast}|\nu(Q^\ast) \subseteq E^\mathcal{U}_\alpha$, and $F^{Q^\ast} \notin B^{\mathcal{U}T}_{\alpha + 1}$. The ISC and compatibility then gives $F^{Q^\ast} = E^\mathcal{T}_\beta$.

But $\zeta$ is non-anomalous (anomalous extenders apply to bicephali), so $E^\mathcal{T}_\zeta$ is a premouse extender. Let $\xi + 1 = \text{succ}^\mathcal{T}((\beta + 1, \omega_1)^{\mathcal{T}T})$. Then standard extender factoring calculations give that $E^\mathcal{T}_\xi = E^\mathcal{U}_\xi$. This contradicts Claim 15. □

15 Conclusion

Now that we know that solidity and universality holds, we can deduce that the desired theorems on condensation, super-Dodd-soundness and projectum-finite-generation hold, as discussed in §2:

Proof of Theorems 1.1, 10.1 and 11.5.

Theorem 1.1: Given a premouse $M$ which is $m$-sound and $(m, \omega_1 + 1)$-iterable, by Theorem 14.1, $M$ is $(m + 1)$-solid, which by Fact 1.2 parts 1(i) and 1(ii) immediately yields the result.

The other results hold because by Theorem 14.1, we have the necessary solidity and universality to apply and Lemmas 10.2 and 11.6. □

\(^{83}\)See e.g. [27, §5].
Let us deduce a few simple corollaries to the main theorems. The main one is naturally:

**15.1 Corollary.** Let $R$ be countable and transitive, and suppose $R \models \text{ZFC} + \text{"$\delta$ is Woodin"}$. Suppose there is an $(\omega_1 + 1)$-iteration strategy $\Sigma$ for (coarse) normal iteration trees on $R$. Let $C^R$ be the maximal $L[\mathbb{E}]$-construction of $R$ (using extenders of $R$ to background the construction in an appropriate manner). Then $C^R$ does not break down; it converges on a class $M$ of $R$ such that $M \models \text{"$\delta$ is Woodin"}$. 

**Proof.** One can run the standard proof from [9, §11], using the results of the paper, since normal trees on creatures of $C^R$ can be lifted to normal trees on $R$. 

We can deduce a canonical form for all solidity witnesses:

**15.2 Corollary.** Let $M$ be a $k$-sound, $(k, \omega_1 + 1)$-iterable premouse. Let $\gamma \in \rho_{k+1}$ and $H = H_\gamma$ the solidity witness at $\gamma$ (the collapsed hull form). Let $\rho = \rho_{k+1}(H)$ and $C = \mathcal{C}_{k+1}(H)$. Then either $C \triangleleft M$ or $M/\rho$ is active with extender $F$ and $C \triangleleft U = \text{Ult}(M, F)$. Moreover, $H$ is an iterate of $C$ via a $k$-maximal tree $T$ which is strongly finite, and almost-above $\rho$.

**Proof.** This follows easily from the theorems, together with Fact 1.2 part 2, to get $C \triangleleft M$ or $C \triangleleft U$, and using Theorem 11.5 to get the rest. (Since $H \in M$, note that either $\rho = \gamma$ or $\rho = \text{card}^M(\gamma)$ and $\gamma = \rho^{1+H}$. Using this and $(k+1)$-universality, observe that $H$ is projectum-finitely generated.)

**15.3 Corollary.** Let $M$ be an $(m + 1)$-sound, $(m, \omega_1 + 1)$-iterable premouse, where $m < \omega$. Let $E$ be a finitely generated short extender which is weakly amenable to $M$ with $\rho_{m+1}^M \leq \text{cr}(E) < \rho_{m+1}^M$. Let $U = \text{Ult}_m(M, E)$. Then $U$ is $(m, \omega_1 + 1)$-iterable iff there is an $m$-maximal tree $T$ on $M$ of finite length $n+1$ with $U = M_{T, n+1}$ (moreover, if $T$ exists, then it is unique, $[0, n]^T$ does not drop in model or degree, and $i_{E, m}^M$ is the iteration map).

**Proof.** If $U = M_{T, n}^T$ for such a tree $T$, then $U$ is $(m, \omega_1 + 1)$-iterable by Fact 13.3.

Now suppose $U$ is $(m, \omega_1 + 1)$-iterable. By Theorem 14.1, $U$ is $(m+1)$-solid. Since $E$ is weakly amenable to $M$ and $M$ is $(m + 1)$-solid, by Lemma 3.8, we have $\rho_{m+1}^U = \rho_{m+1}^M$ and $\rho_{m+1}^U = i_{E, m}^M(\rho_{m+1}^M)$. Letting $E$ be generated by $x$, it follows that $U = \text{Hull}_{m+1}^M(\rho_{m+1}^M \cup \{x\})$, so Theorem 11.5 applies and proves the corollary.

**15.4 Remark.** An anonymous referee noticed that we get an alternate proof of Theorem 9.1, by arguing first as there, until the point at which $\tilde{M}, \tilde{U}, \tilde{x}$ have been defined and $M, \tilde{U}$ shown iterable. Then, since $\tilde{M} = \mathcal{C}_{m+1}(\tilde{U})$ and $\tilde{U} = \text{Hull}_{m+1}^\tilde{U}(\{\tilde{x}\})$, Theorem 11.5 gives that $\tilde{U}$ is an iterate of $\tilde{M}$ via a strongly finite terminally non-dropping $m$-maximal tree, which is a contradiction as in the proof of Theorem 9.1.

The theorems in this paper leave the following questions open:
15.5 **Question.** Does $\omega_1$-iterability suffice to prove fine structure (either for normal trees, or for stacks)? That is, let $m < \omega$. Is it true that every $(m, \omega_1, \omega_1)$-iterable premouse is $(m + 1)$-solid? Does this follow from $(m, \omega_1)$-iterability? Likewise for the other fine structural properties.

It is not only in comparison termination that $(\omega_1 + 1)$-iterability is used; it also used in the proof in [25, Theorem 9.6], which was used to verify that $<^P$ is wellfounded.

15.6 **Question.** Does normal iterability imply iterability for stacks (without assuming any condensation for the normal strategy)?

15.7 **Question.** Does normal iterability suffice to prove the basic fine structural results for mice with long extenders (in the $\kappa^{+n}$-supercompactness region)?

15.8 **Question.** Let $M$ be the least active sound mouse such that $F^M$ is of limit space type. Is there a mouse $U$ whose core is $M$, but such that $U$ is not an iterate of $M$?

15.9 **Question.** Let $M$ be a non-tame mouse modelling ZFC, and $W \subseteq M$ be a ground of the universe $[M]$ of $M$, via a forcing $\mathbb{P} \in W$ such that $W \models \text{"$\mathbb{P}$ is $\sigma$-closed"}$. Is $W = [M]$?

In the last question, if $M$ is instead tame, the answer is “yes”, by [19, Theorem 4.7].

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