ON DETERMINATION OF ZERO-SUM ℓ-GENERALIZED SCHUR NUMBERS FOR SOME LINEAR EQUATIONS

BIDISHA ROY AND SUBHA SARKAR

Abstract. Let \( r, m \) and \( k \geq 2 \) be positive integers such that \( r \mid k \) and let \( v \in \left[0, \left\lfloor \frac{k}{r} \right\rfloor \right] \) be any integer. For any integer \( \ell \in [1, k] \) and \( \epsilon \in \{0, 1\} \), we let \( E^{(\ell, \epsilon)}_v \) be the linear homogeneous equation defined by \( E^{(\ell, \epsilon)}_v : x_1 + \cdots + x_{k-(rv+\epsilon)} = x_{k-(rv+\epsilon-1)} + \cdots + \ell x_k \). We denote the number \( S^{(\ell, \epsilon)}_k(k; r; v) \), which is defined to be the least positive integer \( t \) such that for any \( m \)-coloring \( \chi : [1, t] \rightarrow \{0, 1, \ldots, m-1\} \), there exists a solution \((\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)\) to the equation \( E^{(\ell, \epsilon)}_v \) that satisfies the \( r \)-zero-sum condition, namely, \( \sum_{i=1}^k \chi(\hat{x}_i) \equiv 0 \pmod{r} \). In this article, we completely determine the constant \( S^{(2,1)}_{1,2}(k; r; 0), S^{(1,1)}_{1,2}(k; r; 0), S^{(1,1)}_{k,2}(k; 2; 1) \) and \( S^{(1,0)}_{k,2}(k; r; v) \). Also, we prove upper bound for the constants \( S^{(2,1)}_{1,2}(k; 2; 0) \) and \( S^{(1,1)}_{k,2}(k; 2; v) \).

1. INTRODUCTION

For a given positive integer \( m \), the classical Schur number, denoted by \( S(m) \) is defined as follows. For a given positive integer \( m \), the number \( S(m) \) is defined to be the least positive integer \( t \) such that every \( m \)-coloring of the interval \([1, t]\) admits a monochromatic solution to the linear equation \( x_1 + x_2 = x_3 \).

In 1933, Rado \cite{7} introduced the generalized Schur number and defined it as follows. For given positive integers \( k \geq 2 \) and \( m \geq 1 \), the generalized Schur number \( S(k; m) \) is the least positive integer \( t \) such that every \( m \)-coloring of the interval \([1, t]\) admits a monochromatic solution to the equation \( x_1 + \cdots + x_{k-1} = x_k \). He also proved that \( S(k; m) \) is finite. Moreover, he characterized any linear homogeneous equation with non-zero coefficients that admits a monochromatic solutions as follows.

Theorem 1. \cite{7} Let \( E : a_1 x_1 + \cdots + a_k x_k = 0 \) be any linear homogeneous equation with integer coefficients and at least one of the coefficients is non-zero. Then \( E \) admits monochromatic solution to any \( m \)-coloring of natural numbers if and only if there exist \( 1 \leq i_1 < i_2 < \ldots < i_d \leq k \) such that \( a_{i_1} + a_{i_2} + \cdots + a_{i_d} = 0 \).

Before we proceed further, we need the following definition.

Definition 1. For a given positive integer \( r \geq 2 \), we call a sequence \((a_1, a_2, \ldots, a_n)\) of integers to be an \( r \)-zero-sum sequence if \( \sum_{i=1}^n a_i \equiv 0 \pmod{r} \).

The zero-sum theory has received a lots of attention and has vast literature, see for instance \cite{1}, \cite{3}, \cite{4} and \cite{5}.

In \cite{8}, Robertson replaced the “monochromatic property” in the definition of the generalized Schur number by “zero-sum property” and introduced zero-sum generalized Schur number. A similar kind of topic has been addressed in \cite{2}.

Let \( r \) and \( k \) be given positive integers such that \( r \mid k \). For given integers \( \ell \in [1, k], v \in \left[0, \left\lfloor \frac{k}{r} \right\rfloor \right] \) and \( \epsilon \in \{0, 1\} \) we set the linear homogeneous equation as follows.

\[
E^{(\ell, \epsilon)}_v : x_1 + \cdots + x_{k-(rv+\epsilon)} = x_{k-(rv+\epsilon-1)} + \cdots + \ell x_k
\]
1. Hence, by Theorem 1, we cannot conclude the finiteness of the constant $m \geq \frac{k-1}{2r}$. Theorem 5. Let $r, m$ and $k \geq 2$ be given positive integers with $r|k$ and let $v \in \left[0, \left\lfloor \frac{k-1}{2r} \right\rfloor \right]$ be any integer and $\epsilon \in \{0, 1\}$. For any integer $\ell \in [1, k]$, the zero-sum $\ell$-generalized Schur number $S_{k,m}^{(\ell,\epsilon)}(k; r; v)$ is defined to be the least positive integer $t$ such that for every $m$-coloring $\chi : [1, t] \to \{0, 1, \ldots, m-1\}$, there is a solution $(x_1, x_2, \ldots, x_k)$ to the equation $E_{\ell,\epsilon}^{(\ell,\epsilon)}$ satisfying $r$-zero-sum condition, namely, $\sum_{i=1}^{k} \chi(x_i) \equiv 0 \pmod{r}$.

For any integers $\ell \in [1, k]$, $v \in \left[1, \left\lfloor \frac{k-1}{2r} \right\rfloor \right]$ and $\epsilon \in \{0, 1\}$ or $\ell \in [1, k-1], v = 0$ with $\epsilon = 1$, the linear equation $E_{\ell,\epsilon}^{(\ell,\epsilon)}$ defined in (1) satisfies the condition of Theorem 1 and hence $S_{k,m}^{(\ell,\epsilon)}(k; r; v)$ are finite numbers for any $k, r$ and $m$.

When $\ell = \epsilon = 1$, $v = 0$ and $m = r$, the constant $S_{4,r}^{(1,1)}(k; r; 0)$ is denoted by $S_{4}(k; r)$ in the literature (see [8]) and is called the zero-sum generalized Schur number. In [8], Robertson proved a lower bound for $S_{4}(k, r)$ when $r = 3$ and 4.

Recently in [6], E. Metz showed the exact values of this constant for $r = 3, 4$. Moreover, he proved the following two results.

**Theorem 2.** [6] Let $r$ and $k$ be positive integers with $r|k$ and $k \geq 2r$. Then,

$$S_{4}(k; r) = S_{4}^{(1,1)}(k; r; 0) \leq \begin{cases} kr - r & \text{if } r \text{ is an odd prime} \\ 4k - 5 & \text{if } r = 4 \\ kr - \sum_{i=1}^{t}(p_i - 1) - 1 & \text{if } r \geq 6 \text{ and } r = p_1 \ldots p_t \text{ be the prime decomposition of } r \\ kr - 1 & \text{if } r \text{ is an even integer} \end{cases}$$

**Theorem 3.** [6] Let $r$ and $k$ be positive integers with $r|k$. Then,

$$S_{4}(k; r) = S_{4}^{(1,1)}(k; r; 0) \geq \begin{cases} kr - r & \text{if } r \text{ is an odd integer} \\ kr - 1 & \text{if } r \text{ is an even integer} \end{cases}$$

Also, when $\ell = \epsilon = 1$, $v = 0$ and $m = 2$ in the Definition 2, the constant $S_{4,2}^{(1,1)}(k; r; 0)$ is denoted by $S_{4,2}(k; r)$ in the literature. In [8], Robertson calculated the value of this constant for some cases and was completely determined in [9]. More precisely,

**Theorem 4.** [8] [9] Let $r$ and $k$ be positive integers with $r|k$ and $k > r$. Then,

$$S_{4,2}(k; r) = S_{4}^{(1,1)}(k; r; 0) = rk - 2r + 1.$$ 

In this article, we shall consider the case $\ell > 1$. First note that, when $\ell = k - 1, v = 0$ and $\epsilon = 1$, the condition of Theorem 1 is satisfied by $E_{0}^{(k-1,1)}$. Also $(1, 1, \ldots, 1)$ satisfies the equation $E_{0}^{(k-1,1)}$ together with the $r$-zero-sum condition. Thus we see that $S_{4,m}^{(k-1,1)}(k; r; 0) = 1$ for each $m \geq 2$.

However, the equation $E_{0}^{(k,1)} : x_1 + \cdots + x_{k-1} = kx_k$ does not satisfy the condition of Theorem 1. Hence, by Theorem 1 we cannot conclude the finiteness of the constant $S_{4,2}^{(k,1)}(k; r; 0)$. We indeed prove the finiteness of this constant by calculating the exact value of it as follows.

**Theorem 5.** Let $r$ and $k$ be positive integers with $r|k$ with $k \geq 2$. Then,

$$S_{4,2}^{(k,1)}(k; r; 0) = \begin{cases} 3 & \text{if } r = 2 \text{ and } k \geq 4 \\ 4 & \text{if } r \geq 3 \text{ and } k = r \text{ or } k = 2r \\ 3 & \text{if } r \geq 3 \text{ and } k \geq 3r. \end{cases}$$
When \( \ell = m = r = 2, \epsilon = 1 \) and \( v = 0 \), we have the following upper bound.

**Theorem 6.** Let \( k \geq 4 \) be an even positive integer. Then,

\[
S^{(2,1)}_{3,2}(k; 2; 0) \leq \left\lfloor \frac{k}{4} \right\rfloor + \frac{k}{2} - 1.
\]

Now we move for the case when \( v \) is not zero in the equation \( E^{(\ell, \epsilon)}_v \).

**Theorem 7.** Let \( k \) be an even positive integer and \( v \in [1, \left\lfloor \frac{k-1}{2} \right\rfloor] \). Then

\[
S^{(1,1)}_{3,2}(k; 2; v) \leq \left( \frac{k}{2} - 2v \right).
\]

When \( v = 1 \) in Theorem 7, we prove the exact value in the following theorem.

**Theorem 8.** Let \( k \) be an even positive integer with \( k \geq 6 \). Then,

\[
S^{(1,1)}_{3,2}(k; 2; 1) = \left( \frac{k}{2} - u - 2 \right), \text{ where } u = \begin{cases} t & : \text{if } k = 10t + s \text{ and } s \in \{6, 8\} \\ t - 1 & : \text{if } k = 10t + s \text{ and } s \in \{2, 4\} \end{cases}.
\]

In the following theorem, we compute the exact value when \( \epsilon = 0, \ell = 1 \) and \( m = r \).

**Theorem 9.** Let \( r \) and \( k \) are positive integers with \( r | k \) and \( v \in [1, \left\lfloor \frac{k-1}{2r} \right\rfloor] \) also an integer. Then,

\[
S^{(1,0)}_{3,r}(k; r; v) = \frac{k}{r} - \left\lfloor \frac{(v-1)k}{vr} \right\rfloor - 1.
\]

## 2. Proof of the Theorem

Here we consider the equation \( E^{(k,1)}_0 : x_1 + \cdots + x_{k-1} = kx_k \).

**Case I:** \( (r = 2 \text{ and } k \geq 4) \)

Let \( \chi : [1, 3] \to \{0, 1\} \) be any 2-coloring. We may assume that \( \chi(1) = 0 \), since \( \chi \) admits a 2-zero sum solution if and only if \( \chi \) defined by \( \hat{\chi}(i) = 1 - \chi(i) \) does. Considering the solution \( (1, 1, 2, 1) \) and using the color of 1, we can conclude \( \chi(2) = 1 \) (for otherwise, we are done).

Since \( 2\chi(3) + (k - 3)\chi(2) + \chi(2) \equiv 0 \pmod{2} \), we see that \( (3, 3, 2, \ldots, 2, 2) \) is a 2-zero sum solution to the equation \( E^{(k,1)}_0 \). This proves that, in this case, we get \( S^{(k,1)}_{3,2}(k; 2; 0) \leq 3 \).

For proving the lower bound, we consider the coloring \( \chi : [1, 2] \to \{0, 1\} \) with \( \chi(1) = 0 \) and \( \chi(2) = 1 \). If \( (x_1, \ldots, x_k) \) is a solution to \( E^{(k,1)}_0 \), then \( x_k \neq 2 \) because \( x_1 + \cdots + x_{k-1} \leq 2(k-1) \) where as \( kx_k = 2k \). If \( x_k = 1 \), then \( kx_k = k \) and hence we can get only one solution to the equation \( E^{(k,1)}_0 \) which is \( (1, 1, 2, 1) \) but it does not satisfy 2-zero-sum condition. Hence, we can conclude \( S^{(k,1)}_{3,2}(k; 2; 0) = 3 \).

**Case II:** \( (r \geq 3 \text{ and } k = r) \) or \( (r \geq 3 \text{ and } k = 2r) \)

Let \( \chi : [1, 4] \to \{0, 1\} \) be any 2-coloring. We may assume that \( \chi(1) = 0 \). Looking at the solution \( (1, 1, 2, 1) \) and using the color of 1, we can conclude \( \chi(2) = 1 \) (for otherwise, we are done). Now, considering the solution \( (2, 2, 3, 3, 2) \) and using the color of 2, we can
conclude $\chi(3) = 0$ (for otherwise, we are done). Again, consider the solution $(2, \ldots, 2, 4, 2)$.

Then, if $\chi(4) = 1$, then we are done. If $\chi(4) = 0$, then by observing that $(3, 3, 4, 4, 4, 3)$ is an $r$-zero-sum solution to the equation $E_0^{(k, 1)}$. Hence, since $r \geq 3$ and $k$ is any integer such that $r|k$, we get $S_{3/2}^{(k, 1)}(k; r; 0) \leq 4$.

Now it remains to prove the lower bound. We consider $\chi : [1, 3] \to \{0, 1\}$ such that $\chi(1) = 0 = \chi(3)$ and $\chi(2) = 1$.

**Subcase 1.** ($r \geq 3$ and $k = r$)

Since $k = r$, first one can observe that an $r$-zero sum solution to the equation $E_0^{(k, 1)}$ is also a monochromatic solution and vice versa.

Second, note that by taking $x_i = 2$ for all $i = 1, 2, \ldots, k$, we cannot get any solution to the equation $E_0^{(k, 1)}$. Also, since $kx_k \geq r$, by taking only $x_i = 3$ or $x_i = 1$, one can not get any solution to the equation $E_0^{(k, 1)}$. Thus, under this coloring, it is impossible to get a monochromatic solution to the equation $E_0^{(k, 1)}$. Hence, we can conclude $S_{3/2}^{(k, 1)}(k; r; 0) \geq 4$.

**Subcase 2.** ($r \geq 3$ and $k = 2r$)

By taking $x_k = 1$, the only possible solution is $(1, \ldots, 1, 2, 1)$, which is not an $r$-zero sum solution. Again, by taking $x_k = 2$, the only possible solution to the equation $E_0^{(k, 1)}$ is $(3, 3, 2, \ldots, 2, 2)$, which is not an $r$-zero sum solution under the coloring $\chi$. Finally, if we take $x_k = 3$, then, $kx_k = 3k$ and $x_1 + \cdots + x_{k-1} \leq 3k - 3$. Hence, there is no solution with $x_k = 3$ also. Thus we get, $S_{3/2}^{(k, 1)}(k; r; 0) \geq 4$.

**Case III:** ($r \geq 3$ and $k \geq 3r$)

Suppose for a contradiction that there exists a 2-coloring $\chi : [1, 3] \to \{0, 1\}$ for which $E_0^{(k, 1)}$ does not have any $r$-zero-sum solution for some $r \geq 3$. We may assume that $\chi(1) = 0$. Looking at the solution $(1, \ldots, 1, 2, 1)$ and using the color of 1, we can conclude $\chi(2) = 1$. Now, considering the solution $(2, \ldots, 2, 3, 3, 2)$ and using the color of 2, we can conclude $\chi(3) = 0$.

If $k$ is an odd multiple of $r$, then we first observe that $\frac{k-3}{2}$ is a positive integer for any parity of $r$. Hence, in this case, we see that $(\underbrace{2, \ldots, 2}_{(r-1)-\text{times}}, \underbrace{1, \ldots, 1}_{(2r-1)-\text{times}}, \underbrace{3, \ldots, 3}_{2 \text{-times}}, 2)$ is an $r$-zero sum solution to the equation $E_0^{(k, 1)}$.

If $k$ is an even multiple of $r$, then $\frac{k-4}{2}$ is a positive integer as $k \geq 3r$. Thus, in this case, we see that $(\underbrace{2, \ldots, 2}_{(2r-1)-\text{times}}, \underbrace{1, \ldots, 1}_{(k-5r-1)-\text{times}}, \underbrace{3, \ldots, 3}_{(k-5r-1)-\text{times}}, 2)$ is an $r$-zero sum solution to the equation $E_0^{(k, 1)}$, which is a contradiction. Hence, we have $S_{3/2}^{(k, 1)}(k; r; 0) \leq 3$.

For the lower bound, we consider the coloring $\chi : [1, 2] \to \{0, 1\}$ such that $\chi(1) = 0$ and $\chi(2) = 1$. Note that $x_k \neq 2$ as $kx_k = 2k$ and $x_1 + \cdots + x_{k-1} \leq 2k - 2$. Therefore, $x_k = 1$ and
ON DETERMINATION OF ZERO-SUM $\ell$-GENERALIZED SCHUR NUMBERS

hence $kx_k = k$. Thus, there is only one solution, namely, $(1, \ldots, 1, 2, 1)$, to the equation $E_0^{(k,1)}$, which does not satisfy $r$-zero-sum condition. This proves the lower bound and the theorem. \qed

3. PROOF OF THE THEOREM

Here we consider the equation $E_0^{(2,1)} : x_1 + \cdots + x_{k-1} = 2x_k$.

Let us denote $S = \left\lceil \frac{k}{4} \right\rceil + \frac{k}{2} - 1$. Let $\chi : [1, \left\lceil \frac{k}{4} \right\rceil + \frac{k}{2} - 1] \to \{0, 1\}$ be a 2-coloring for which $E_0^{(2,1)}$ does not have any 2-zero-sum solution. We may assume that $\chi(1) = 0$, since $\chi$ admits a 2-zero sum solution if and only if $\hat{\chi}$ defined by $\hat{\chi}(i) = 1 - \chi(i)$ does.

Note that, $\chi(\frac{k}{2})$ is either 0 or 1. In the following table, we determine the color of $2, S, \frac{k}{2} + 1$ and 3 using some solutions of $E_0^{(2,1)}$.

| Solution to the equation $E_0^{(2,1)}$ | $\chi(\frac{k}{2}) = 0$ | $\chi(\frac{k}{2}) = 1$ |
|----------------------------------------|-------------------------|-------------------------|
| $(1, \ldots, 1, 2, \frac{k}{2})$ (k-2)-times | $\chi(2) = 1$ | $\chi(2) = 0$ |
| $(1, \ldots, 1, \frac{k}{2}, S)$ (k-2)-times | $\chi(S) = 1$ | $\chi(S) = 0$ |
| $(1, \ldots, 1, 2, 2, 2, \frac{k}{2} + 1)$ (k-4)-times | $\chi(\frac{k}{2} + 1) = 0$ | $\chi(\frac{k}{2} + 1) = 1$ |
| $(1, \ldots, 1, 2, 3, \frac{k}{2} + 1)$ (k-3)-times | $\chi(3) = 0$ | $\chi(3) = 0$ |

Case I: $(4|k)$

When $k = 4$, we see that $(1, 1, 2, 2)$ is a 2-zero sum solution to the equation $x_1 + x_2 + x_3 = 2x_4$. Thus $S_{3,2}^{(2,1)}(4; 2; 0) \leq 2$ which is equal to $S = \left\lceil \frac{4}{4} \right\rceil + \frac{4}{2} - 1$.

Let $k = 4t$ for some integer $t \geq 2$. In this case, $S = \left\lceil \frac{4t}{4} \right\rceil + \frac{4t}{2} - 1 = 3t - 1$. Thus, using above table and considering some more solution of the equation $E_0^{(2,1)}$, we have the following table.

| Solution to the equation $E_0^{(2,1)}$ | $\chi(\frac{k}{2}) = 0$ | $\chi(\frac{k}{2}) = 1$ |
|----------------------------------------|-------------------------|-------------------------|
| $(1, \ldots, 1, 2, \ldots, 2, S - 1)$ (2t-2)-times | $\chi(S - 1) = 1$ | $\chi(S - 1) = 1$ |
| $(1, \ldots, 1, 2t - 2, S - 1)$ (4t-3)-times | $\chi(2t - 2) = 0$ | $\chi(2t - 2) = 0$ |

Thus, we get a solution $(1, \ldots, 1, 3, 2t - 2, S)$ to the equation $E_0^{(2,1)}$ which satisfies 2-zero-sum condition in both cases $\chi(\frac{k}{2}) = 0$ or 1.

Case II: $(4$ does not divide $k)$

In this case, we can write $k = 4t + 2$ for some integer $t \geq 1$. Also we observe that $S = \left\lceil \frac{4t + 2}{4} \right\rceil + \frac{4t + 2}{2} - 1 = 3t + 1$. Thus, using the first table and considering some more solutions to
the equation $E_0^{(2,1)}$, we get the following table;

| Solution to the equation $E_0^{(2,1)}$ | $\chi(\frac{k}{2}) = 0$ | $\chi(\frac{k}{2}) = 1$ |
|----------------------------------------|------------------------|------------------------|
| $\left(1, \ldots, 1, \frac{2}{2}, \ldots, \frac{2}{2}, S - 1\right)$ | $\chi(S - 1) = 0$ | $\chi(S - 1) = 1$ |
| $4t$-times \ (2t-1)-times | $\chi(2t) = 1$ | $\chi(2t) = 0$ |

Thus, we get a solution $(1, \ldots, 1, 3, 2t, S)$ to the equation $E_0^{(2,1)}$ which satisfies 2-zero-sum condition in both cases $\chi(\frac{k}{2}) = 0$ or 1. This proves the theorem. □

4. PROOF OF THE THEOREM 7

Here we consider the equation $E_1^{(1,1)} : x_1 + \cdots + x_{k-(2v+1)} = x_{k-2v} + \cdots + x_k$.

Let us denote $t := (\frac{k}{2} - 2v)$. Let $\chi : [1, t] \to \{0, 1\}$ be any 2-coloring. Since $v \leq \frac{k-1}{2}$ and $k$ is even, we get $k \geq 4v + 2$.

Now, observe that

$$\left(1, \ldots, 1, \frac{k}{2} - 2v, \ldots, \frac{k}{2} - 2v\right)$$

is a solution to the equation

(2) \hspace{1cm} \begin{align*}
  x_1 + \cdots + x_{k-(2v+1)} &= x_{k-2v} + \cdots + x_k, \\
  \text{because } \ x_{k-2v} + \cdots + x_k &= (2v + 1)(\frac{k}{2} - 2v) = vk + \frac{k}{2} - 4v^2 - 2v \\
  \therefore \ x_1 + \cdots + x_{k-(2v+1)} &= \underbrace{1 + \cdots + 1}_{(k-4v)-\text{times}} + \underbrace{(\frac{k}{2} - 2v) + \cdots + (\frac{k}{2} - 2v)}_{(2v-1)-\text{times}} \\
  &= k - 4v + (2v - 1)(\frac{k}{2} - 2v) \\
  &= vk + \frac{k}{2} - 4v^2 - 2v.
\end{align*}

Since $2|k$, we get

$$\sum_{i=1}^{k} \chi(x_i) = (k - 4v)\chi(1) + 4v\chi\left(\frac{k}{2} - 2v\right) \equiv 0 \pmod{2}.$$

Thus, we get $S^{(1,1)}_{3,2}(k; 2; v) \leq (\frac{k}{2} - 2v)$. □

5. PROOF OF THE THEOREM 8

Here we consider the equation $E_1^{(1,1)} : x_1 + \cdots + x_{k-3} = x_{k-2} + x_{k-1} + x_k$ and denote it by $E(k)$ for simplicity.

Case 1: $(u = 0)$

In this case, $k \in [6, 14]$ and we show that $S^{(1,1)}_{3,2}(k; 2; 1) = (\frac{k}{2} - 2)$. 

In this case, we first prove the lower bound.

Subcase 1. \((k = 6)\).

In this case, clearly, \((1, 1, 1, 1, 1, 1)\) is the only 2-zero sum solution and we are done.

Subcase 2. \((k = 8)\)

The lower bound follows \(S^{(1,1)}(8; 2; 1) \geq 2\) as \((1, \ldots, 1)\) is not a solution of \(E(8)\).

Subcase 3. \((k = 10)\)

\(S^{(1,1)}(10; 2; 1) \geq 3\) follows because \(E(10)\) does not have any solution in \([1, 2]\).

Subcase 4. \((k = 12)\)

We observe that \(E(12)\) does not have any solution in \([1, 2]\) and the only solution of \(E(12)\) in \([1, 3]\) is \((1, \ldots, 1, 3, 3, 3)\). Thus, considering the 2-coloring \(\chi(1) = \chi(2) = 0, \chi(3) = 1\) of \([1, 3]\), we see that \(E(12)\) does not have a 2-zero sum solution in \([1, 3]\). Hence, we get \(S^{(1,1)}(12; 2; 1) \geq 4\).

Subcase 5. \((k = 14)\)

The only solutions of \(E(14)\) in \([1, 4]\) are \((1, \ldots, 1, 4, 4, 3)\) and \((1, \ldots, 1, 2, 4, 4, 4)\). Thus, if we consider the 2-coloring \(\chi(1) = \chi(2) = 0, \chi(3) = \chi(4) = 1\) of \([1, 4]\), then \(E(14)\) does not have a 2-zero sum solution in \([1, 4]\). Hence, \(S^{(1,1)}(14; 2; 1) \geq 5\).

By putting \(v = 1\) in Theorem 7, the upper bound follows.

Case II: \((u \text{ is odd})\)

For proving the upper bound, on the contrary, let \(\chi : [1, \frac{k}{2} - u - 2] \rightarrow \{0, 1\}\) be a 2-coloring such that \(\chi\) doesn’t admit a 2-zero sum solution to \(E(k)\). We may assume that \(\chi(1) = 0\), since \(\chi\) admits a 2-zero sum solution if and only if the induced coloring \(\hat{\chi}\) defined by \(\hat{\chi}(i) = 1 - \chi(i)\) does.

Subcase 1. \((4|k)\)

Since \(u \text{ is odd}, u - 1\) and \(3u - 1\) both are even. Using \(\chi(1) = 0\) and considering the solution \((1, \ldots, 1, \frac{k}{4} + \frac{u - 1}{2}, \frac{k}{4} + \frac{u - 1}{2}, \frac{k}{2} - u - 2)\), we get \(\chi(\frac{k}{2} - u - 2) = 1\) (otherwise, we are done). Since \(\frac{k}{2} - u - 2 \geq \frac{k}{4} + \frac{3u - 1}{2} \Leftrightarrow k \geq 10u + 6\), we can consider the solution \((1, \ldots, 1, \frac{k}{4} + \frac{3u - 1}{2}, \frac{k}{4} + \frac{3u - 1}{2}, \frac{k}{2} - 3u - 2)\) which produces \(\chi(\frac{k}{2} - 3u - 2) = 1\). But the solution \((1, \ldots, 1, \frac{k}{2} - 3u - 2, \frac{k}{2} - u - 2, \frac{k}{2} - u - 2, \frac{k}{2} - u - 2)\) satisfies 2-zero sum condition and proves this subcase.

Subcase 2. \((4 \not| k)\)

In this subcase, we write \(k = 4t + 2\) for some integer \(t\), as \(k\) is even and hence \(\frac{k}{2} - u - 2 = 2t - u - 1\). Note that since \(u \text{ is odd}, 10u + 6\) is divisible by 4. Since \(4 \not| k\), in this subcase we see that \(k \geq 10u + 8\).

Since \(u + 1\) and \(3u + 1\) both are even, we consider the solution \((1, \ldots, 1, 2t - u - 2, \frac{2t + u + 1}{2}, \frac{2t + u + 1}{2})\) and using the color of 1, we get \(\chi(2t - u - 2) = 1\). Since \(\frac{k}{2} - u - 2 = 2t - u - 1 \geq \frac{2t + 3u + 1}{2} \Leftrightarrow k \geq 10u + 8\), by taking the solution \((1, \ldots, 1, 2t - 3u - 2, \frac{2t + 3u + 1}{2}, \frac{2t + 3u + 1}{2})\), we conclude that \(\chi(2t - 3u - 2) = 1\). Therefore the solution \((1, \ldots, 1, 2t - 3u - 2, 2t - u - 2, 2t - u - 2)\) is a 2-zero-sum solution to the equation \(E(k)\) irrespective of the color of \(2t - u - 1\).
Case III: \((u \text{ is even})\)

Let \(\chi : [1, \frac{k}{2} - u - 2] \to \{0, 1\}\) be a 2-coloring such that \(\chi\) doesn’t admit a 2-zero sum solution to \(E(k)\). We may assume that \(\chi(1) = 0\).

Subcase 1. \((4|k)\)

Since \(u\) is even, \(4\) divides \(10u\) and hence \(k \geq 10u + 8\). Now, using \(\chi(1) = 0\) and by considering the solution \((1, \ldots, 1, \frac{k}{4} + \frac{u}{2}, \frac{k}{4} + \frac{u}{2}, \frac{k}{2} - u - 3)\), we conclude that \(\chi(\frac{k}{2} - u - 3) = 1\). Also, since \(\frac{k}{2} - u - 2 \geq \frac{k}{4} + \frac{3u}{2} \iff k \geq 10u + 8\), from the solution \((1, \ldots, 1, \frac{k}{4} + \frac{3u}{2}, \frac{k}{4} + \frac{3u}{2}, \frac{k}{2} - 3u - 3)\), we get \(\chi(\frac{k}{2} - 3u - 3) = 1\). Note that whatever color of \(\frac{k}{2} - u - 2\) may be, the solution \((1, \ldots, 1, \frac{k}{2} - 3u - 3, \frac{k}{2} - u - 2, \frac{k}{2} - u - 2)\) satisfies 2-zero sum condition to the equation \(E(k)\) and finishes this subcase.

Subcase 2. \((4 / k)\)

In this case, since \(k\) is even, we can write \(k = 4t + 2\) for some integer \(t\) and \(\frac{k}{2} - u - 2 = 2t - u - 1\). Now, consider the solution \((1, \ldots, 1, 2t - u - 1, \frac{2t+u}{2}, \frac{2t+u}{2})\) and using the color of 1, we conclude \(\chi(2t - u - 1) = 1\). Since \(\frac{k}{2} - u - 2 = 2t - u - 1 \geq \frac{2t+3u}{2} \iff k \geq 10u + 6\), the solution \((1, \ldots, 1, 2t - 3u - 1, \frac{2t+3u}{2}, \frac{2t+3u}{2})\) implies that \(\chi(2t - 3u - 1) = 1\). Then, the solution \((1, \ldots, 1, 2t - 3u - 1, 2t - u - 1, 2t - u - 1, 2t - u - 1)\) is a 2-zero-sum solution to the equation \(E(k)\). Thus this proves the subcase and the upper bound.

Now we prove the lower bound for all positive integer \(u\) with \(k \geq 16\).

Case I: \((4|k)\)

Subcase 1. \((u \text{ is odd})\)

Let \(\chi : [1, \frac{k}{2} - u - 3] \to \{0, 1\}\) be a 2-coloring with \(\chi(1) = \chi(2) = \cdots = \chi(\frac{k}{4} - \frac{u+3}{2}) = 0 \) and \(\chi(\frac{k}{4} - \frac{u+3}{2} + 1) = \cdots = \chi(\frac{k}{2} - u - 3) = 1\).

If there exists a solution and the color of all the \(x_i\)'s are 0, then \(x_1 + \cdots + x_{k-3} \geq k - 3\) and \(x_{k-2} + x_{k-1} + x_k \leq 3(\frac{k}{4} - \frac{u+3}{2})\). Since \(k - 3 > 3(\frac{k}{4} - \frac{u+3}{2})\), this is not possible. Therefore, any 2-zero sum solution, at least two of \(x_{k-2}, x_{k-1}\) and \(x_k\) must have color 1 and the other has color 0 or all three have color 1. In the first case, we get

\[
x_{k-2} + x_{k-1} + x_k \leq \frac{k}{4} - \frac{u+3}{2} + 2 \left(\frac{k}{2} - u - 3\right) = \frac{5k - 10u - 30}{4}
\]

whereas \(x_1 + \cdots + x_{k-3} \geq k - 3\). Note that \(\frac{5k-10u-30}{4} < k - 3 \iff k < 10u + 18\). Since \(4|k\), we get, \(x_1 + \cdots + x_{k-3} > x_{k-2} + x_{k-1} + x_k\) if and only if \(k \leq 10u + 14\). Therefore, in this case, we conclude that there is no 2-zero-sum solution.

For the second case, we assume \(\chi(x_i) = 1\) for all \(i = k - 2, k - 1, k\) and \(\chi(x_j) = 1\) for some \(j \in \{1, 2, \ldots, k - 3\}\). Hence, we get, \(x_{k-2} + x_{k-1} + x_k \leq 3(\frac{k}{2} - u - 3)\) whereas \(x_1 + \cdots + x_{k-3} \geq k - 4 + \left(\frac{k}{4} - \frac{u+3}{2} + 1\right) = \frac{5k-2u-18}{4}\). Note that \(3(\frac{k}{2} - u - 3) < \frac{5k-2u-18}{4} \iff k < 10u + 18 \iff k \leq 10u + 14\). This condition implies that the chosen color has no 2-zero sum solution.

Subcase 2. \((u \text{ is even})\)
In this case, consider the 2-coloring $\chi : [1, \frac{k}{2} - u - 3] \to \{0, 1\}$ with $\chi(1) = \cdots = \chi(\frac{k}{4} - \frac{u+2}{2}) = 0$ and $\chi(\frac{k}{2} - \frac{u+2}{2} + 1) = \cdots = \chi(\frac{k}{2} - u - 3) = 1$.

If there exists a solution and the color of all the $x_i$’s are 0, then $x_1 + \cdots + x_{k-3} \geq k - 3$ and $x_{k-2} + x_{k-1} + x_k \leq 3(\frac{k}{4} - \frac{u+2}{2})$. Since $k - 3 > 3(\frac{k}{4} - \frac{u+2}{2})$, this type of solution is not possible. Therefore, any 2-zero sum solution, at least two of $x_{k-2}, x_{k-1}$ and $x_k$ must have color 1. Hence, in the first case, we get

$$x_{k-2} + x_{k-1} + x_k \leq \frac{k}{4} - \frac{u+2}{2} + \frac{2}{2} (\frac{k}{2} - u - 3) = \frac{5k - 10u - 28}{4}$$

whereas $x_1 + \cdots + x_{k-3} \geq k - 3$. Note that $\frac{5k - 10u - 28}{4} < k - 3 \iff k < 10u + 16$. Since $4 | k$, we get, $x_1 + \cdots + x_{k-3} > x_{k-2} + x_{k-1} + x_k$ if and only if $k \leq 10u + 14$. Therefore, in this case, we conclude that there is no 2-zero-sum solution.

For the second case, we assume $\chi(x_i) = 1$ for all $i = k - 2, k - 1, k$ and $\chi(x_j) = 1$ for some $j \in \{1, 2, \ldots, k - 3\}$. Hence, we get, $x_{k-2} + x_{k-1} + x_k \leq 3(\frac{k}{4} - u - 3)$ whereas $x_1 + \cdots + x_{k-3} \geq k - 4 + (\frac{k}{4} - \frac{u+2}{2} + 1) = \frac{5k - 2u - 16}{4}$. Note that $3(\frac{k}{4} - u - 3) < \frac{5k - 2u - 16}{4} \iff k < 10u + 20 \iff k \leq 10u + 18$. Thus, for $k \leq 10u + 14$, we conclude that the chosen color has no 2-zero sum solution. This finishes the proof of the lower bound.

Case II: ($4 \not| k$)

In this case, we can write $k = 4t + 2$ for some integer $t$.

Subcase 1. ($u$ is even)

For proving the lower bound, consider the 2-coloring $\chi : [1, \frac{4t+2}{2} - u - 3] \to \{0, 1\}$ with $\chi(1) = \cdots = \chi(\frac{4t+2}{4} - \frac{u+2}{2}) = 0$ and $\chi(\frac{4t+2}{4} + 1) = \cdots = \chi(\frac{4t+2}{2} - u - 3) = 1$.

If there exists a solution and the color of all the $x_i$’s are 0, then $x_1 + \cdots + x_{k-3} \geq k - 3 = 4t - 1$ and $x_{k-2} + x_{k-1} + x_k \leq 3(2t - u - 2)$ whereas $x_1 + \cdots + x_{k-3} \geq k - 4 + (2t - \frac{u+2}{2} + 1) = \frac{10t - 5u - 10}{2}$. Note that $3(2t - u - 2) < \frac{10t - 5u - 10}{2} \iff k < 10u + 18 \iff k \leq 10u + 16$. Hence, for $k \leq 10u + 14$, we conclude that there is no 2-zero-sum solution.

Subcase 2. ($u$ is odd)

For proving the lower bound, consider the 2-coloring $\chi : [1, \frac{4t+2}{2} - u - 3] \to \{0, 1\}$ with $\chi(1) = \cdots = \chi(t - \frac{u+2}{2}) = 0$ and $\chi(t - \frac{u+2}{2} + 1) = \cdots = \chi(2t - u - 2) = 1$.

If there exists a solution and the color of all the $x_i$’s are 0, then $x_1 + \cdots + x_{k-3} \geq k - 3 = 4t - 1$ and $x_{k-2} + x_{k-1} + x_k \leq 3(t - \frac{u+3}{2})$. Since $4t - 1 > 3(t - \frac{u+3}{2})$, this type of solution is not possible. Therefore, in any 2-zero sum solution, at least two of $x_{k-2}, x_{k-1}$ and $x_k$ must have color 1 and the other has color 0 or all three have color 1. Hence, in the first case, we get

$$x_{k-2} + x_{k-1} + x_k \leq t - \frac{u+3}{2} + 2 (2t - u - 2) = \frac{10t - 5u - 11}{2}$$
whereas \( x_1 + \cdots + x_{k-3} \geq k - 3 = 4t - 1 \). Note that \( \frac{10t-5u-11}{2} < 4t - 1 \iff k \leq 10u + 18 \). Since \( 4 \not| k \), we get, \( x_1 + \cdots + x_{k-3} > x_{k-2} + x_{k-1} + x_k \) if and only if \( k \leq 10u + 16 \). Therefore, in this case, we conclude that there is no 2-zero-sum solution.

For the second case, we assume \( \chi(x_i) = 1 \) for all \( i = k - 2, k - 1, k \) and \( \chi(x_j) = 1 \) for some \( j \in \{ 1, 2, \ldots, k-3 \} \). Hence, we get, \( x_{k-2} + x_{k-1} + x_k \leq 3(2t - u - 2) \) whereas \( x_1 + \cdots + x_{k-3} \geq k - 4 + (t - \frac{u+5}{2} + 1) = \frac{10t-u-5}{2} \). Note that \( 3(2t - u - 2) < \frac{10t-u-5}{2} \iff k < 10u + 16 \). Thus, for \( k \leq 10u + 14 \), we conclude that the chosen color has no 2-zero sum solution. This proves the lower bound and the theorem. \( \square \)

6. PROOF OF THE THEOREM

Here we consider the equation

\[
E_v^{(1,0)} : x_1 + \cdots + x_{k-vr} = x_{k-vr+1} + \cdots + x_k.
\]

Let us denote \( s := \frac{k}{r} - \left\lfloor \frac{(u-1)k}{vr} \right\rfloor - 1 \) for simplicity.

Case I: \( (k = 2vr) \)

In this case, the number of variables in both sides of the (3) are equal and hence \( (1, \ldots, 1) \) is an \( r \)-zero sum solution. Thus we get, \( S_{k,v}^{(1,0)}(k; r; v) = 1 \), as desired.

Case II: \( (k > 2vr) \)

Since \( r | k \), by division algorithm we write \( k - 2vr = vrt + ir \) for some non-negative integers \( t \) and \( i \) with \( i \in [1, v] \). Therefore, we get

\[
k = vrt + (2v+i)r \iff (v-1)k = v(v-1)rt + (2v+i)(v-1)r \]
\[
\iff \frac{(v-1)k}{vr} = (v-1)t + \frac{(v-1)(2v+i)}{v} \]
\[
\iff \left\lfloor \frac{(v-1)k}{vr} \right\rfloor = (v-1)t + (2v+i - 3).
\]

Hence, we have \( s = \frac{k}{r} - \left\lfloor \frac{(v-1)k}{vr} \right\rfloor - 1 = \frac{k}{r} - (v-1)t - (2v+i-3) - 1 \).

For proving the lower bound we show that in the interval \([1, s-1]\), equation (3) does not have any solution. First, we observe that

\[
x_{k-vr+1} + \cdots + x_k \leq vr(s-1) = vr \left( \frac{k}{r} - \left\lfloor \frac{(v-1)k}{vr} \right\rfloor - 1 - 1 \right)
\]
\[
= vr \left( \frac{k}{r} - (v-1)t - (2v+i-3) - 1 \right) = vk - v(v-1)rt - v(2v+i-3)r - vr - vr
\]
\[
= vk - (v-1)k + (2v+i)(v-1)r - v(2v+i-3)r - 2vr
\]
\[
= k - (2v+i)r + 3vr - 2vr = k - vr - ir
\]

but \( x_1 + \cdots + x_{k-vr} \geq k - vr \). Thus, we get \( S_{k,v}^{(1,0)}(k; r; v) \geq \frac{k}{r} - \left\lfloor \frac{(v-1)k}{vr} \right\rfloor - 1 \).

Now, we proceed to prove the upper bound and consider an arbitrary \( m \)-coloring \( \chi : [1, s] \to \{0, 1, \ldots, m - 1\} \). Next, we consider

\[
(4) \quad (x_1, x_2, \ldots, x_{k-vr}, x_{k-vr+1}, \ldots, x_k) = (\underbrace{1, \ldots, 1}_{(k-vr)-times}, \underbrace{s, \ldots, s}_{vr-times}, \underbrace{s-1, \ldots, s-1}_{(vr-ir)-times})
\]
and show that it is a solution to the equation \([4]\). Note that the value of \(x_1 + \cdots + x_{k-\nu r}\) is \(k - \nu r\) and the value of \(x_{k-\nu r+1} + \cdots + x_k\) is \(ir s + (v-i)r(s-1)\) which is exactly equal to \(k - \nu r\) because,

\[
ir s + (v-i)r(s-1) = ir \left( \frac{k}{r} - \left\lfloor \frac{(v-1)k}{\nu r} \right\rfloor - 1 \right) + (v-i)r \left( \frac{k}{r} - \left\lfloor \frac{(v-1)k}{\nu r} \right\rfloor - 1 - 1 \right)
\]

\[
= ir \left( \frac{k}{r} - (v-1)l - (2v+i-3) - 1 \right) + (v-i)r \left( \frac{k}{r} - (v-1)l - (2v+i-3) - 1 - 1 \right)
\]

\[
= vk - v(v-1)r - (2v+i-3)vr - 2vr + ir
\]

\[
= vk - (v-1)k + (2v+i)(v-1)r - (2v+i-3)vr - 2vr + ir = k - \nu r.
\]

Thus, it remains to show that the tuple defined in \([\text{4}]\) satisfies the \(r\)-zero sum condition. Since \(r \mid k\), we see that

\[
\sum_{i=1}^{k} \chi(x_i) = (k - \nu r)\chi(1) + ir(\chi(s)) + (\nu r - ir)\chi(s-1) \equiv 0 \pmod{r}.
\]

Therefore, we get \(S^{(1,0)}_{k,r}(k;r;v) \leq \frac{k}{r} - \left\lfloor \frac{(v-1)k}{\nu r} \right\rfloor - 1\) and hence the theorem. \(\square\)

Acknowledgement. We would like to sincerely thank Prof. S. D. Adhikari and Prof. R. Thangadurai for their insightful remarks. This work was done while the authors were visiting the Department of Mathematics, Ramakrishna Mission Vivekananda Educational and Research Institute and they wish to thank this institute for the excellent environment and for their hospitality.

References

[1] A. Bialostocki and P. Dierker, On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings, Discrete Math. 110 (1992), 1-8.
[2] N. Brown, On Zero-Sum Rado Numbers for the Equation \(ax_1 + x_2 = x_3\), Masters thesis, South Dakota State University, (2017).
[3] Y. Caro, Zero-sum problems, a survey, Discrete Math. 152 (1996), 93-113.
[4] P. Erdős, A. Ginzburg, and A. Ziv, Theorem in additive number theory, Bulletin Research Council Israel 10F (1961), 41-43.
[5] W. D. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: a survey, Expo. Math 24 (2006), 337-369.
[6] E. Metz, Upper and Lower Bounds on Zero-Sum Generalized Schur Numbers, \[\text{arXiv:1803.03851v1}\].
[7] R. Rado, Studien zur Kombinatorik (German), Math. Z. 36 (1933), no. 1, 424-470.
[8] A. Robertson, Zero-sum generalized Schur numbers, \[\text{arXiv:1802.03382v1}\].
[9] A. Robertson, Bidisha Roy and Subha Sarkar, The Determination of 2-color zero-sum generalized Schur Numbers, \[\text{arXiv:1803.00861v2}\].

(Bidisha Roy and Subha Sarkar) Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad 211019, India

E-mail address, Subha Sarkar: subhasarkar@hri.res.in
E-mail address, Bidisha Roy: bidisharoy@hri.res.in