A TRIPLE-POINT WHITNEY TRICK

SERGEY A. MELIKHOV

ABSTRACT. We use a triple-point version of the Whitney trick to show that ornaments of three orientable $(2k-1)$-manifolds in $\mathbb{R}^{2k-1}$, $k > 2$, are classified by the $\mu$-invariant.

A very similar (but not identical) construction was found independently by I. Mabillard and U. Wagner, who also made it work in a much more general situation and obtained impressive applications. The present note is, by contrast, focused on a minimal working case of the construction.

1. Introduction

An ornament is a continuous map $f = \bigsqcup_{i=1}^n f_i$ from $X = \bigsqcup_{i=1}^n X_i$ to $Y$ that has no $i = j = k$ points, i.e. $f(X_i) \cap f(X_j) \cap f(X_k) = \emptyset$, whenever $i$, $j$ and $k$ are pairwise distinct. Note that $f$ is allowed to have triple points $f(x) = f(y) = f(z)$, where $x$, $y$, $z$ belong one or two of the $X_i$’s. We are interested in ornaments up to ornament homotopy, i.e. homotopy through ornaments.

Ornaments of circles in the plane were introduced by Vassiliev [14] as a generalization of doodles, previously studied by Fenn and Taylor [2]. Fenn and Taylor additionally required each circle to be embedded; however, Khovanov [4] redefined doodles as triple point free maps of circles in the plane, and Merkov proved that doodles in Khovanov’s sense are classified by their finite-type invariants [12]. Further references and examples can be found in [11], which is a more thorough companion paper to this brief note.

The problem of classification of ornaments of spheres in $\mathbb{R}^m$ is motivated, in particular, by geometric and algebraic constructions that go from link maps and their “quadratic” invariants to ornaments and their “linear” invariants; and conversely [11]. Link maps are, in turn, related to links by the Jin suspension and its variations, which likewise reduce some “quadratic” invariants of links to “linear” invariants of link maps [9], [13; §3].

2. $\mu$-invariant

We will consider only ornaments of the form $X_1 \sqcup X_2 \sqcup X_3 \to \mathbb{R}^m$. If $f = f_1 \sqcup f_2 \sqcup f_3$ is such an ornament, let $F$ be the composition

$$X_1 \times X_2 \times X_3 \xrightarrow{f_1 \times f_2 \times f_3} \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \setminus \Delta_{\mathbb{R}^m} \xrightarrow{\cong} S^{2m-1},$$

where $\Delta_{\mathbb{R}^m} = \{(x, x, x) \mid x \in \mathbb{R}^m\}$ and the homotopy equivalence is given, for instance, by $(x, y, z) \mapsto \frac{(2x-y-z, 2y-x-z)}{||(2x-y-z, 2y-x-z)||}$. Let $\mu(f) \in H^{2m-1}(X_1 \times X_2 \times X_3)$ be the image under $F^*$ of a fixed generator $\xi \in H^{2m-1}(S^{2m-1})$; to be precise, let us choose $\xi$ to correspond to the orientation of $S^{2m-1}$ given by its inwards co-orientation in the standardly oriented $\mathbb{R}^{2m}$. Clearly, $\mu(f)$ is invariant under ornament homotopy.
Let us now assume that each $X_i$ is a connected closed oriented $(2k-1)$-manifold and $m = 3k-1$. Then $F$ is a map between connected closed oriented $(6k-3)$-manifolds, and so $\mu(f)$ is an integer. In this simplest case, assuming additionally that each manifold $X_i$ is either PL or smooth, one can compute $\mu(f)$ as follows.

First let us note that since each $X_i$ is compact, for each ornament $f: X \to \mathbb{R}^m$ there exists an $\varepsilon > 0$ such that every map $f': X \to \mathbb{R}^m$, $\varepsilon$-close to $f$ (in the symmetric), is also an ornament, and moreover the rectilinear homotopy between $f$ and $f'$ is an ornament homotopy. Thus we are free to replace ornaments by their generic (PL or smooth) approximations. Similarly, ornament homotopies can be replaced by their generic approximations.

Now let us consider a homotopy between $f$ and the trivial ornament, which sends $X_1$, $X_2$ and $X_3$ to three distinct fixed points in $\mathbb{R}^m$. Its generic (PL or smooth) approximation $h_t$, if viewed as a map $X \times I \to \mathbb{R}^m \times I$, $(x, t) \mapsto (h_t(x), t)$, has only finitely many transverse $1 = 2 = 3$ points, which are naturally endowed with signs.\footnote{Every triple point of a generic map $F: N \to M$ from a 2$k$-manifold to a 3$k$-manifold corresponds to a transversal intersection point between the 3$k$-manifold $\Delta_M$ and the map $F^3: N^3 \to M^3$ from a 6$k$-manifold to a 9$k$-manifold.}

(See [1; II.4] concerning PL transversality.) The algebraic number of these $1 = 2 = 3$ points is easily seen to equal $\mu(f)$.\footnote{Each $1 = 2 = 3$ point of $h_t$ corresponds to a transversal intersection point between $\Delta_{\mathbb{R}^m} \times I$ and the map $X_1 \times X_2 \times X_3 \times I \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times I$, $(x, y, z, t) \mapsto (h_t(x, t), h_t(y, t), h_t(z, t), t)$. It is easily seen to be of the same sign.}

**Example 1.** The inclusions of the unit disks in the coordinate $2k$-planes $\mathbb{R}^k \times \mathbb{R}^k \times 0$, $\mathbb{R}^k \times 0 \times \mathbb{R}^k$ and $0 \times \mathbb{R}^k \times \mathbb{R}^k$ in $\mathbb{R}^{3k}$ yield a smooth map $B^{2k} \sqcup B^{2k} \sqcup B^{2k} \to B^{3k}$ with one transverse $1 = 2 = 3$ point. Restricting to the boundaries, we get the Borromean ornament $b: S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to S^{3k-1}$. By stereographically projecting $S^{3k-1}$ e.g. from $z = \frac{1}{\sqrt{3k}}(1, \ldots, 1)$ we also get an ornament $b_z: S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k-1}$.

On the other hand, the sphere of radius $\varepsilon \sqrt{k}$ centered at $(\varepsilon, \ldots, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ is tangent to each of the three unit $2k$-disks. By appropriately identifying the exterior of this sphere in the unit 3$k$-disk $B^{3k}$ with $S^{3k-1} \times I$, we get a smooth homotopy of $b$, and hence also of $b_z$, to the trivial ornament. It has one transverse $1 = 2 = 3$ point, which can be seen to be positive, and it follows that $\mu(b_z) = 1$.

In the case of doodles, the $\mu$-invariant was introduced in [2]. See [11] concerning relations between the $\mu$-invariant of ornaments and the triple $\mu$-invariant of link maps.

### 3. Classification

**Theorem 1.** Let $m = 3k-1$, $k > 2$ and let $X_1$, $X_2$, $X_3$ be connected closed oriented PL $(2k-1)$-manifolds. Then $\mu$ is a complete invariant of ornaments $X_1 \sqcup X_2 \sqcup X_3 \to \mathbb{R}^m$.

The proof is in the PL category. If the $X_i$ are smooth manifolds, the same construction with minimal (straightforward) amendments can be carried out in the smooth category.
Proof. Let $f$ and $g$ be generic PL ornaments of $X := X_1 \cup X_2 \cup X_3$ in $\mathbb{R}^m$ with $\mu(f) = \mu(g)$. Let $h : X \times I \to \mathbb{R}^m \times I$ be a generic PL homotopy between them. Since $\mu(f) = \mu(g)$, the $1 = 2 = 3$ points of $h$ can be paired up with opposite signs. Every such pair $(p^+, p^-)$ will now be canceled by a triple-point Whitney trick.

Let $p^\pm_i$ be the preimage of $p^\pm$ in $M_i := X_i \times I$. We first arrange that $(p^+_1, p^+_2)$ and $(p^-_1, p^-_2)$ be in the same component of the double point set $\Delta_1 := \{(x, y) \in M_1 \times M_2 \mid h(x) = h(y)\}$ (in case that initially they are not). To this end we pick points $(q^+_1, q^+_2)$ in the same components of $\Delta_1$ with $(p^+_1, p^+_2)$ and such that the double points $f(q^+_1) = f(q^+_2)$ and $f(q^-_1) = f(q^-_2)$ are not triple points.

Let us connect $q^+_1$ and $q^-_1$ by an arc $J_1$ in $M_1$, disjoint from the preimages of any double points (using that $k > 1$). Now we attach a thin 1-handle to $h(M_2)$ along the image of $J_1$. That is, we modify $h(M_2)$ into $h'(M_2)$, where $M'_2$ is obtained from $M_2$ by removing an oriented copy of $B^{2k} \times \partial I$ and pasting in $\partial B^{2k} \times I$. The embedded 1-handle $h'(\partial B^{2k} \times I)$ is constructed in a straightforward way. Namely, since $h$ is generic, $\Delta_1$ is an oriented $k$-manifold, immersed into the $2k$-manifold $M_1$ by the projection $\pi : M_1 \times M_2 \to M_1$. Let us take an oriented connected sum of its components along a ribbon $r(D^k \times I)$ in $M_1$ (going near $J_1$). Then $hr(D^k \times I)$ is naturally thickened to a solid rod $R(B^{2k} \times I)$ in $\mathbb{R}^m \times I$ whose lateral surface $R(\partial B^{2k} \times I)$ is the desired embedded 1-handle.

To restore the topology of $M_2$, we cancel the 1-handle geometrically by attaching a 2-handle along an embedded 2-disc $D$, which is disjoint from $h(M_1 \cup M_3)$ and meets $h'(M_2)$ only in $\partial D$ (such a disk exists since $k > 2$). That is, we modify $h'(M_2)$ into $h''(M_2)$, where $M''_2$ is obtained from $M'_2$ by removing an appropriately embedded copy of $B^{2k-1} \times \partial D^2$ and pasting in $\partial B^{2k-1} \times D^2$. As is well-known, this can be done so that $M''_2$ is homeomorphic to $M_2$. Since we do not care about self-intersections of individual components, we may define $h''$ on $\partial B^{2k-1} \times D^2$ to be an arbitrary generic map into a small neighborhood of $D \cup h'(B^{2k-1} \times \partial D^2)$.

Thus we may assume that $(p^+_1, p^+_2)$ and $(p^-_1, p^-_2)$ are in the same component of $\Delta_1$. To cancel the original $1 = 2 = 3$ points $p^+$ and $p^-$, let us connect $(p^+_1, p^+_2)$ and $(p^-_1, p^-_2)$ by an arc $J_{12}$ in $\Delta_1$ and attach a thin 1-handle to $h(M_3)$ along the image of $J_{12}$. (This 1-handle is the spherical block normal bundle of $h(M_1) \cap h(M_2)$ over the image of $J_{12}$. It

\begin{itemize}
  \item[3]Namely, $q^\pm_1$ has a regular neighborhood $N_{q_1}$ in $M_1$ that is homeomorphic to $[-1,1]^{2k}$ by an orientation preserving homeomorphism $\varphi_1$ such that $\varphi_1^{-1}(\Delta_1) = [-1,1]^{2k} \times \{0\}^{k-1}$ and $\varphi_1^{-1}(J_1) = \{0\}^{2k-1} \times [0,1]$. Let $Q = [-1,1]^{2k} \times \{0\}^{k-1}$ and let $N$ be a regular neighborhood of $J_1 \setminus (N_+ \cup N_-) \cup \varphi_1(Q \times 1) \cup \varphi_1(Q \times 1) \subseteq \partial M_1 \setminus (N_+ \cup N_-)$. Since a $k$-ball unknits in the interior of a $(2k - 1)$-ball, there is a homeomorphism $\psi : [-2,2]^{2k} \to N$ such that $\psi^{-1}(\partial N_+) = [-2,2]^{2k-1} \times \{\pm 2\}$ and $\psi(x, \pm 2) = \varphi_1(x, 1)$ for all $x \in Q$. Then $\varphi_1(Q \times 1) \cup \varphi_1(Q \times 1) \cup \psi(Q \times [-2,2])$ is the desired ribbon $r(D^k \times I)$.
  \item[4]If $N_1$ is a disk neighborhood of $J_1$ that is embedded by $h$, we may assume that $h(M_2)$ is transverse to a normal block bundle $\nu$ to $h(N_1)$, that is, $h(M_2)$ meets the total space $E(\nu)$ in $E(\nu)|_{h(M_2) \cap h(N_1)}$. Since $\nu$ is trivial, there is a homeomorphism $R : B^{2k} \times I \to E(\nu)|_{h(M_2) \cap h(N_1)}$ sending $B^{2k} \times \partial I$ onto $E(\nu)|_{h(\partial D^k \times J_1)}$.
  \item[5]In more detail, let us connect $q^+_2$ and $q^-_2$ by an arc $J_2$ in $M_2$, disjoint from the preimages of any double points. Let $H_1$ be a small regular neighborhood of $J_1 := J_2 \times \{0\} \cup J_2 \times [0,1]$ in $M_2 \times [0,2]$. Let $H_2$ be a small regular neighborhood of $D' := J_2 \times [0,1] \setminus H_1 \setminus H_2$. Then $M''_2$ can be identified with the frontier of $M_2 \times [-1,0] \cup H_1$ in $M_2 \times [-1,2]$ so that $h'(D')$ gets identified with $\partial D'$ and $M''_2$ with the frontier of $M_2 \times [-1,0] \cup H_1 \cup H_2$ in $M_2 \times [-1,2]$, which is homeomorphic to $M_2$.
is attached orientably since the two $1 = 2 = 3$ points have opposite signs.) The topology of $M_3$ can be restored using another 2-disk like before. In particular, this 2-disk is disjoint from $h(M_1 \sqcup M_2)$, so no new $1 = 2 = 3$ points arise.

Finally, we need to apply the “ornament concordance implies ornament homotopy in codimension three” theorem [7], [8]. (Alternatively, it should be possible to rework the above construction so as to keep the levels preserved at every step — but it would be a rather laborious exercise; compare [9; proofs of Lemmas 5.1, 5.4, 5.5].)

4. Discussion

Theorem 1 and its proof (in slightly less detailed form) were originally contained in the preprint [10], which I presented at conferences and seminars in 2006–07 and privately circulated at that time and in later years. For instance, the referee of the present paper (whose identity I know from his idiosyncratic remarks) does not deny that he received my preprint containing the proof of Theorem 1, exactly as it appears in [10], by email on May 23, 2006 and then again on July 7, 2006. I hesitated to publish [10] at that time as I hoped to get more progress on the conjectures stated in the introduction there; but other projects are still distracting me from this task.

In the meantime I. Mabillard and U. Wagner independently found and vastly generalized a version of the triple-point Whitney trick and also obtained nice applications leading to a disproof of the Topological Tverberg Conjecture [6]. (My only step in that direction was a feeble attempt to advertise the possibility of disproving the Topological Tverberg Conjecture by generalizing the construction of the present note — addressed, for instance, to P. Blagojević at the 2009 Oberwolfach Workshop on Topological Combinatorics.) Mabillard and Wagner call their construction the “triple Whitney trick”, but I prefer to reserve this title for a certain other device, extending Koschorke’s version of the Whitney–Haefliger construction [5; Proof of Theorem 1.15] and involving the triple-point Whitney trick as only one of several steps. It can be used to obtain a geometric proof of the Habegger–Kaiser classification of link maps in the 3/4 range [3], which will hopefully appear elsewhere (a sketch of this proof was presented in my talk at the Postnikov Memorial Conference in Będlewo, 2007).

References

[1] S. Buoncristiano, C. P. Rourke, and B. J. Sanderson, A Geometric Approach to Homology Theory, London Math. Soc. Lecture Note Series, vol. 18, Cambridge Univ. Press, Cambridge, 1976.
[2] R. Fenn and P. Taylor, Introducing doodles, Topology of Low-Dimensional Manifolds (Proc. Second Sussex Conf., 1977), Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, pp. 37–43.
[3] N. Habegger and U. Kaiser, Link homotopy in the 2-metastable range, Topology 37 (1998), 75–94.
[4] M. Khovanov, Doodle groups, Trans. Amer. Math. Soc. 349 (1997), no. 6, 2297–2315.
[5] U. Koschorke, On link maps and their homotopy classification, Math. Ann. 286 (1990), 753–782.
[6] I. Mabillard and U. Wagner, Eliminating higher-multiplicity intersections, I: A Whitney trick for Tverberg-type problems. arXiv:1508.02349.
[7] S. A. Melikhov, Pseudo-homotopy implies homotopy for singular links of codimension ≥ 3, Uspekhi Mat. Nauk 55 (2000), no. 3, 183–184; English transl., Russian Math. Surveys 55 (2000), 589–590.
[8] ______, *Singular link concordance implies link homotopy in codimension $\geq 3*$. Preprint, 1998; [arXiv:1810.08299v1].

[9] ______, *Self $C_2$-equivalence of two-component links and invariants of link maps*, J. Knot Theory Ram. **27** (2018), no. 13, 1842012. [arXiv:1711.03514v2].

[10] ______, *Gauss-type formulas for link map invariants*. Preprint, 2007; [arXiv:1711.03530v1].

[11] ______, *Gauss-type formulas for link map invariants*. [arXiv:1711.03530](https://arxiv.org/abs/1711.03530) (latest version).

[12] A. B. Merkov, *Vassiliev invariants classify plane curves and sets of curves without triple intersections*, Mat. Sb. **194** (2003), no. 9, 31–62; English transl., Sb. Math. **194** (2003), 1301–1330.

[13] M. Skopenkov, *Suspension theorems for links and link maps*, Proc. Amer. Math. Soc. **137** (2009), 359–369. [arXiv:math.GT/0610320].

[14] V. A. Vassiliev, *Invariants of ornaments*, Singularities and Bifurcations, Adv. Soviet Math., vol. 21, Amer. Math. Soc., Providence, RI, 1994, pp. 225–262.

**Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia**

*Email address: melikhov@mi-ras.ru*