SO(10) Operator Analysis For $\nu_\mu \nu_\tau$ Oscillations.

H.-C. Cheng
M.S. Gill
and
L.J. Hall

Department of Physics
University of California
and
Theoretical Physics Group
Physics Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
Berkeley, California 94720

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY90-21139.
Abstract

In grand unified theories the flavor mixing angles of leptons are related to those of quarks. In this paper we study SO(10) theories with a precise group theoretic relation at the GUT scale for mixing between the heaviest two generations: $\theta_{\mu\tau}^0 = \kappa |V_{cb}^0|$. A comprehensive operator search yields all possible cases where $\kappa$ is a group theory Clebsch. The resulting predictions for $\nu_\mu\nu_\tau$ oscillations are scaled from the grand to the weak scales. We find that all but one of the models which have such a relationship between $\theta_{\mu\tau}^0$ and $V_{cb}^0$, and are not already excluded, will be probed by the CHORUS and NOMAD experiments. A more precise mixing measurement, for example by the proposed P803 experiment, could distinguish between the models.
Disclaimer

This document was prepared as an account of work sponsored by the United States Government. Neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial products process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof of The Regents of the University of California and shall not be used for advertising or product endorsement purposes.

Lawrence Berkeley Laboratory is an equal opportunity employer.
1 Introduction

The standard model, while extremely successful, has 18 free parameters, 13 of which are in the flavor sector. In seeking a more fundamental theory we can be guided by the requirement that at least some of these parameters should be predicted. Symmetries provide essentially the only tool which is sufficiently developed to yield such predictions: imposing extra symmetries on a theory leads to a reduction in the number of free parameters. Such symmetries have been studied in the flavor sector for over 20 years. Experiment has recently provided a hint as to which symmetries should be imposed: the only one of the 18 parameters of the standard model which has been successfully predicted to a high level of significance is the weak mixing angle [1]. This suggests that we should pursue theories which have both supersymmetry and grand unified symmetry.

Mass relations from grand unified theories can typically only be checked at the 20-30% level. This is because many of the masses and mixing angles are not precisely known, and because the predictions depend on the strong gauge coupling constant. Hence if predictions from the flavor sector are to be highly significant, there should be as many of them as possible. This has been a guiding principle behind several recent works on the charged fermion mass sector [2,3,4]. The first of these was based on the Georgi-Jarlskog texture [5], which is a highly successful ansatz for the form of the Yukawa matrices at the GUT scale involving six independent operators. Including one physical phase and the ratio of Higgs vacuum expectation values, tan $\beta$, this scheme describes the 13 observable flavor parameters in terms of 8 free parameters, thereby predicting 5 of the standard model flavor parameters.

A more recent analysis [4] is much more ambitious. A general operator analysis is done for the flavor sector of $SO(10)$ theories [6], and several models are found where just the minimum possible number of operators, 4, successfully describe flavor physics. This search for a maximally predictive flavor sector is based on the observation that a set of spontaneously broken family symmetries generally leads to a hierarchy of operators such that the dominant contribution to fermion masses arises from just a few such operators[3,5,7]. This reduces the number of free parameters by two compared with the Georgi-Jarlskog case, and
hence predicts 7 of the 13 standard model flavor parameters. While there is no guarantee that these minimal $SO(10)$ flavor models are correct, they do merit attention: it is surprising that such economic descriptions of fermion masses exist, and they have already reached a level of some significance by virtue of having 7 predictions in agreement with present measurements. They will be further tested as the flavor parameters become better measured.

How should these theories be extended to include predictions for neutrino masses and mixings? It might be thought that no further assumptions are necessary: since the families fall into 16-dimensional representations of $SO(10)$, $16_i \ i = 1, 2, 3$, and since $16_i \supset u_i, d_i, e_i, \nu_i$ it could be expected that the operators which generate the Yukawa matrices $U, D$, and $E$ for charged fermions will automatically generate the Yukawa matrices for neutrinos. Unfortunately this is not the case. The neutrino masses involve both the Dirac Yukawa matrix, $\bar{\psi}$, which couples doublet to singlet neutrinos, and also the Majorana Yukawa matrix, $\Lambda$, which involves only singlet neutrinos. Given a grand unified model which specifies $U, D$, and $E$ there are always further assumptions which must be made for $\bar{\psi}$, and especially $\Lambda$, to be specified[8]. Indeed, the more neutrino mass predictions one wants to make the more assumptions must typically be made.

For example, the Georgi-Jarkskog ansatz can be studied within particular $SO(10)$ schemes for neutrino mass predictions. If a broad class of models is studied (as was done by Harvey, Ramond, and Reiss in [5]) few if any precise numerical predictions can be made. However, it is possible to narrow one’s focus to the case that the necessary 6 $SO(10)$ invariant operators have a very special form, in which case the maximal number of predictions can be made: both mass ratios and all mixing angles and phases of the leptonic Kobayashi-Maskawa matrix [9]. In the case of charged fermion masses it is worth seeking maximally predictive theories, because they can immediately be tested by their predictions. This is not the case for models for neutrino masses. While we have an existence proof of a maximally predictive model [9] no aspect of the neutrino sector has yet been tested and it may be that maximally predictive theories of this sort will fall victim to the large number of assumptions inherent in their construction. However, some principle is required in guiding a search for grand unified models for neutrino mass prediction. What should replace the idea of
maximal predictivity based on spontaneously broken family symmetries that was used in the charged fermion mass sector [4]?

In this paper we enumerate a set of minimal assumptions which allows the prediction of just one quantity, the \( \nu_\mu \) to \( \nu_\tau \) mixing angle \( \theta_{\mu\tau} \equiv |V_{\nu_{\mu\tau}}| \) and give the possible predictions for this angle that result from \( SO(10) \) grand unification. The reason for concentrating on this angle is that we believe that if the grand unified framework is correct, then this is the parameter of the neutrino sector with the best prospect of being measured over the coming years. A prediction for \( \theta_{\mu\tau} \) takes the form:

\[
\theta_{\mu\tau} = \kappa |V_{cb}| \eta \tag{1.1}
\]

where \( \kappa \) is a GUT generalized Clebsch-Gordan group theory factor expected to have a simple numerical value (eg. 1, 3, 1/3, 2/3, ...) and \( \eta \) is a renormalization group correction which can be computed and is expected to be close to unity. The present oscillation limits correspond to \( \theta_{\mu\tau} < 0.032 \), and the CHORUS [10] and NOMAD [11] experiments will reduce this to 0.01. Hence current experiments are precisely at the right sensitivity to check relations of the form (1.1), assuming \( \Delta m^2 \) is large enough.

In the grand unified framework the heaviest neutrino is \( \nu_\tau \) and hence the largest \( \Delta m^2 \) will be for \( \nu_e \nu_\tau \) and \( \nu_\mu \nu_\tau \) oscillations. Since one typically finds \( \theta_{e\tau} \) to be of order \( |V_{ub}| \), the amplitude for these oscillations are prohibitively small. For CHORUS and NOMAD to see a positive signal, \( m_{\nu_e} \) would have to be at least 1 eV. Is this likely to be true? The simplest estimate for this mass is \( m_{\nu_\tau} \approx \nu^2 / V \) where \( \nu \) is the electroweak vev, and \( V \) is the scale of \( B-L \) breaking, presumably the GUT scale. This produces the very disappointing expectation that \( m_{\nu_\tau} \approx 10^{-2} eV \). In the models we discuss below there is a reason why this estimate should be enhanced by about a factor of 30. Furthermore, the overall magnitude of the neutrino masses cannot be accurately predicted, the above is simply an order of magnitude guess. This is to be contrasted with relations of the form of (1.1) which can give precise predictions at the 20% level of accuracy. While it is hard to argue theoretically that \( \Delta m^2 \) must be large enough for laboratory observations of \( \nu_\mu \nu_\tau \) oscillations, the situation for \( \nu_e \nu_\mu \) oscillations is only worse. Furthermore, there are hints from both the solar neutrino
and from dark matter that $\Delta m^2 \approx 100 \text{eV}^2$.

In this paper we do not consider predictions for other quantities in the neutrino sector. The reason for this restricted view is the hope that predictions of the form of (1.1) can be obtained with relatively mild assumptions. We will find that this hope is only partly borne out. Predictions can be made based on studies of just two $SO(10)$ invariant operators: the ones contributing to the 23 and 33 entries of the Yukawa matrices. One need not consider in any detailed way the operators for the lighter generations. Furthermore, the operators for the heavier generations are the ones with the simpler structure. Nevertheless, the list of assumptions which we are forced to make is still uncomfortably long.

The light neutrino mass matrix is given by

$$ m_\nu = \frac{v^2}{V} \Lambda^{-1} \Lambda^T $$

(1.2)

where $v = 247 \text{GeV}$ is the electroweak vev, and $V$ is the $B-L$ breaking vev. It is because $V$ is unknown that the scale of the neutrino masses cannot be predicted. Once an $SO(10)$ invariant model for $U$, $D$, and $E$ has been written down, the matrix $\Lambda$ will follow immediately. At the GUT scale the contributions to $\lambda_{ij}$ will be Clebschs times the $U_{ij}$ contributions. Hence the difficulty in making predictions for neutrino mass ratios and mixing angles lies not with $\Lambda$, but with $\Lambda$. Suppose the two Higgs doublets of the low energy theory, taken to be the minimal low energy supersymmetric model, originate from some GUT representations $\{\phi\}$. The crucial question is whether the singlet field which acquires vev $V$ lies wholly within this same set $\{\phi\}$. If not then there will be entries of $\Lambda$ which are completely unrelated to those of $U$, $D$, $E$. To avoid this difficulty we assume that the form of $\Lambda$ is such that $\theta_{\mu\tau}$ depends only on the entries of $\Lambda$ and not on those of $\Lambda$. While this is a strong assumption, it nevertheless includes a very large class of models. It has the very obvious advantage that provided $\Lambda$ possesses certain features which puts it in this class of models, we do not need to know anything else about $\Lambda$ in order to make our prediction.

To convince the reader that there are many models where $\theta_{\mu\tau}$ does not depend on the $\Lambda_{ij}$ entries, consider the following. It is phenomenologically well motivated to have $U_{11} = U_{13} = U_{31} = D_{11} = D_{13} = D_{31} = 0$ [12]. Essentially all
predictive ansätze have this form, because it results, when \( U_{12} = U_{21} \) and \( D_{12} = D_{21} \), in the successful relations \( V_{ub}/V_{cb} = \sqrt{m_u/m_c} \) and \( V_{td}/V_{ts} = \sqrt{m_d/m_s} \). The \( SO(10) \) theory will then give \( \lambda_{11} = \lambda_{13} = \lambda_{31} = 0 \).

Hence we take

\[
\sim = \begin{pmatrix}
0 & d & 0 \\
d' & c & b \\
0 & b' & a
\end{pmatrix}
\] (1.3)

and a completely general \( \Lambda \):

\[
\Lambda = \begin{pmatrix}
F & D & E \\
D & C & B \\
E & B & A
\end{pmatrix}
\] (1.4)

with

\[
\Lambda^{-1} = \frac{1}{\det \Lambda} \begin{pmatrix}
\bar{F} & \bar{D} & \bar{E} \\
\bar{D} & \bar{C} & \bar{B} \\
\bar{E} & \bar{B} & \bar{A}
\end{pmatrix}
\] (1.5)

where \( \bar{A} = CF - D^2, \bar{B} = ED - BF, \bar{C} = AF - E^2, \bar{D} = BE - AD, \bar{E} = BD - CE \) and \( \bar{F} = AC - B^2 \).

One finds the contributions to \( \theta_{\mu\tau} \) from diagonalization of \( m_\nu \) of equation (2) to be

\[
\theta_{\mu\tau}^{(\nu)} = \frac{b'd'\bar{D} + ad'\bar{E} + b'c\bar{C} + (ac + bb')\bar{B} + ab\bar{A}}{b'^2C + 2ab'\bar{B} + a^2\bar{A}}
\] (1.6)

If all terms in numerator and denominator are comparable, there is little hope for a prediction for \( \theta_{\mu\tau} \). However, if the \( \bar{A} \) terms dominate then \( \theta_{\mu\tau} = b/a \) depends only on \( \sim \) not on \( \Lambda \). If \( \bar{B} \) dominates the \( \theta_{\mu\tau} = \frac{b}{2a} + \frac{c}{2b} \), while if \( \bar{C} \) dominates \( \theta_{\mu\tau} = c/b' \). In this paper we will assume that either \( \bar{A} \) or \( \bar{B} \) dominates, and that the resulting prediction is either \( b/a \) or \( b/2a \). (i.e. in the latter case we assume \( c/b' \ll b/a \)). The resulting predictions depend only on the 23 and 33 entries of \( \sim \), which come from simple operators chosen to yield correct heavy generation masses and \( V_{cb} \). These are the cases which lead to predictions of the form of equation (1.1).

There are several ways that \( \bar{A} \) or \( \bar{B} \) dominance can occur. For example, suppose that \( \Lambda \) has the same pattern of zeros as the other matrices so that \( \Lambda_{11} = F = 0 \) and \( \Lambda_{13} = E = 0 \). In this case \( \bar{B} = \bar{C} = 0 \) so that (1.6) reduces to
$$\theta_{\mu\tau}^{(\nu)} = \frac{b'd\tilde{D} + a\tilde{d}'\tilde{E} + ab\tilde{A}}{a^2\tilde{A}}$$  \hspace{1cm} (1.7)$$

with \( \tilde{A} = -D^2, \tilde{D} = -AD \) and \( \tilde{E} = BD \). We will argue below that the simplest operator for the 33 entry which gives masses to the heaviest generation leads to \( A = 0 \) so \( \tilde{D} \) can be dropped. In the numerator of (1.7) there is a simple competition between two terms and presumably in a reasonable fraction of theories the \( \tilde{A} \) term dominates.

We are now ready to list our assumptions

(1) We study a supersymmetric grand unified \( SO(10) \) theory, broken at the scale of grand unification to the MSSM.

(2) We assume \( \theta_{\mu\tau} \) does not involve the elements of the Majorana Yukawa matrix \( \Lambda \), and that

$$\theta_{\mu\tau}^{(\nu)} = \frac{\lambda_{23}}{\lambda_{33}} \quad \text{or} \quad \frac{1}{2} \frac{\lambda_{23}}{\lambda_{33}} \hspace{1cm} (1.8)$$

(3) The 33 entries of \( U, D, E \), and \( \tilde{\cdot} \) matrices arise from a single operator. The unique possibility which gives a successful top mass prediction is \( O_1 = 16_3 10 16_3 \) \([13]\).

(4) The dominant contribution to the 23 entries of the \( U, D, E \), and \( \tilde{\cdot} \) matrices arises from a single operator \( O_2 \).

(5) We also demand that either \( D_{23} = E_{23} = 0 \), or that \( O_2 \) contains the same 10 as \( O_1 \). The reason for this assumption will emerge below.

### 2 GUT Scale Predictions

Let us now use these assumptions to make predictions for \( \theta_{\mu\tau} \). The \( U, D, E \), and \( \tilde{\cdot} \) matrices of the two heaviest generations take the following forms:

$$U = \begin{pmatrix} U_{22} & B_U \\ U_{32} & A_U \end{pmatrix}, \quad D = \begin{pmatrix} D_{22} & B_D \\ D_{32} & A_D \end{pmatrix}, \quad \tilde{\cdot} = \begin{pmatrix} \lambda_{22} & \chi B_U \\ \lambda_{32} & A_U \end{pmatrix}, \quad E = \begin{pmatrix} E_{22} & \epsilon B_D \\ E_{32} & A_D \end{pmatrix} \hspace{1cm} (2.1)$$
where $\epsilon$ and $\chi$ are group Clebschs and where all matrix elements are a priori complex before phase redefinitions on the fields. $A_U$ and $A_D$ are not equal in general because the two light $SU(2)$ doublets do not necessarily lie entirely in the 10 which generates the 33 entries. The mass hierarchy in different generations requires that the 33 entries be much larger than the other matrix elements, in which case $|V_{cb}|$ and $\theta_{\mu\tau}$ at GUT scale are given by

$$\left|V_{cb}^0\right| = \left|\frac{B_U}{A_U} - \frac{B_D}{A_D}\right|$$

(2.2)

$$\theta^0_{\mu\tau} = \left|\frac{\chi B_U}{2A_U} - \frac{\epsilon B_D}{A_D}\right| \quad \text{or} \quad \left|\frac{\chi B_U}{2A_U} - \frac{\epsilon B_D}{A_D}\right|$$

(2.3)

where the superscript "0" represents the quantity at the GUT scale. In general $\frac{B_U}{A_U}$ and $\frac{B_D}{A_D}$ are unrelated and cannot be made real simultaneously by redefining the relative phases of the quark fields. The unknown relative phase and magnitude between $\frac{B_U}{A_U}$ and $\frac{B_D}{A_D}$ prevent us from making any definite predictions. Therefore, to get the simple relation (1.1) requires that either $B_D = 0$ or that $O_2$ contains the same 10 as $O_1$ in which case $\frac{B_D}{A_D}$ and $\frac{B_U}{A_U}$ have the same phase and are related by a group Clebsch factor $\xi$ (this is the explanation for assumption #5 above). For either case we can write:

$$B_D = \frac{\xi B_U}{A_U} \quad \text{(where } \xi = 0 \text{ if } B_D = 0)$$

(2.4)

Then

$$\left|V_{cb}^0\right| = \left|(1 - \xi) \frac{B_U}{A_U}\right|$$

(2.5)

$$\theta^0_{\mu\tau} = \left|\left(\chi - \epsilon\xi\right) \frac{B_U}{A_U}\right| \quad \text{or} \quad \left|\left(\frac{\chi}{2} - \epsilon\xi\right) \frac{B_U}{A_U}\right|$$

(2.6)

We then find the following $\kappa$ factors in eq. (1.1):

$$\Rightarrow \kappa = \left|\frac{\chi - \epsilon\xi}{1 - \xi}\right| (\equiv \kappa_1) \quad \text{or} \quad \left|\frac{\chi}{1 - \xi} - \epsilon\xi\right| (\equiv \kappa_2)$$

(2.7)

In what follows we study extensively the possible $O_2$’s and then calculate the various $\kappa$ factors that result from them.
2.1 Operators Contributing to the Two-Three Yukawa Matrix Elements

For $O_2$ being a dimension-4 operator, there are only 3 possibilities, $16_2 \times 10 \times 16_3$, $16_2 \times 120 \times 16_3$, and $16_2 \times \overline{126} \times 16_3$. However, $V_{cb} \approx \frac{1}{20}$ hints that $O_2$ may be a higher-dimension operator which is suppressed by some powers of masses of order $M_P$ (for examples, see Table 3). For simplicity we will restrict ourselves to dimension-5 operators. Also, if the 23 entry is generated by a dimension-6 or higher operator, then the expansion in powers of $\frac{v}{M}$ is $\sim \frac{1}{4} \to \frac{1}{5}$ which is becoming uncomfortably large.

A dimension-5 operator has the form

$$\frac{1}{M_P} \phi_1 \phi_2 16_3$$

(2.8)

To make an $SO(10)$ invariant the product of the $\phi_1 \times \phi_2$ must contain a 10, 120, or $\overline{126}$ of $SO(10)$ since $16 \times 16 = 10 + 120 + 126$ (the 10 and 120 of $SO(10)$ are self-conjugate). For representations of dimension $\leq 126$, the possible combinations of $\phi_1 \phi_2$ are:

$$45 \times 10 \supset 10, 120,$$

$$45 \times 120 \supset 10, 120, \overline{126},$$

(2.9a)

$$45 \times \overline{126} \supset 120, \overline{126},$$

$$54 \times 10 \supset 10,$$

$$54 \times 120 \supset 120,$$

(2.9b)

$$54 \times 126 \supset \overline{126},$$

$$16 \times 16 \supset 10, 120,$$

$$\overline{16} \times \overline{16} \supset 10, 120, \overline{126}.$$  

(2.9c)
45 $\times$ 126 also contains 120, but this does not couple to 16 $\times$ 16.

For operators (2.9a), we consider the case in which the 45 vev lies in a definite direction in the $SO(10)$ group space: in one of the hypercharge $Y$, $B - L$, $T_{3R}$, or $X$ directions, where $X$ preserves the $SU(5)$ subgroup. In fact, $SO(10)$ may be broken to $SU(5)$ by a vev of $45_X$ at a larger scale than $SU(5)$ is broken. This means that the mass scale appearing in the denominator in (2.8) can be $\langle 45_X \rangle$ (on the order of the GUT scale) as well as masses of order $M_P$.

When a 45 vev acts on a fermion in a 16, it gives a numerical Clebsch which is the charge of the fermion under the particular group generator corresponding to the direction of this vev. Therefore the 23 entries of the fermion mass matrices are just related by these Clebschs. These Clebschs are listed in Table 1.

|   | $X$ | $Y$ | $B - L$ | $T_{3R}$ |
|---|-----|-----|---------|----------|
| $q$ | 1   | 1   | 1       | 0        |
| $u^c$ | 1   | -4  | -1      | -1       |
| $d^c$ | -3  | 2   | -1      | 1        |
| $l$ | -3  | -3  | -3      | 0        |
| $e^c$ | 1   | 6   | 3       | 1        |
| $N^c$ | 5   | 0   | 3       | -1       |

Table 1: The charges (normalized to integers) of fermion fields under the group generators $X$, $Y$, $B - L$, and $T_{3R}$.

For operators (2.9b), the product of a 54 and another representation contains only one of the 10, 120 and $\overline{126}$, so the predictions are the same as those of the dimension-4 operators and need not be discussed separately.

The $16^2$ and $\overline{16}^2$ of (2.9c) are more complicated. Under the $SU(5)$ decomposition, $16 = 1 + \overline{5} + 10$ and $\overline{16} = 1 + 5 + 10$. We can see that $16^2$ only contributes to the masses of down quarks and charged leptons and $\overline{16}^2$ only contributes to the masses of up quarks and neutrinos (because the former arises from the $SU(5)$ operator $10 \times \overline{5} \times \overline{5}$ and the latter from $10 \times 5 \times 10$). Therefore, if $O_2 = \frac{B}{M} 16_2 \overline{16}^2 16_3$, it is natural to have $D_{23} = E_{23} = 0$. Now the vev of the 5 of $SU(5)$ breaks $SU(2)_L$ so it must be light. Only the singlet of $SU(5)$ could develop a vev at the order of the GUT scale. This operator is more complicated because in addition to the Dirac masses, it also gives Majorana masses...
to both left handed and right handed neutrinos. In this case, the light neutrino mass matrix receives an additional contribution beyond that of eq. (1.2) and we discuss this below.

We are now ready to list the possible operators and the resulting Clebsch coefficients.

### 2.2 The Clebsch Coefficients $\kappa_1$ and $\kappa_2$

1. Dimension-4 operators:

   (1) $O_2 = 16_2 \ 10 \ 16_3$, where the 10 is the same 10 as in $O_1$. In this case we have $\frac{B_U}{A_U} = \frac{B_D}{A_D} \Rightarrow V_{cb} = 0$, which conflicts with experiment, so it’s excluded.

   (2) $O_2 = 16_2 \ 10' \ 16_3$, where the 10’ is different from the 10 in $O_1$ and has a vev contributing to $U$ and $\tilde{U}$ only. In this case $\xi = 0, \ \chi = 1$ and therefore $\kappa_1 = 1, \ \kappa_2 = \frac{1}{2}$.

   (3) $O_2 = 16_2 \ \overline{126} \ 16_3$. The $\overline{126}$ contributes to the up quark and the Dirac neutrino mass matrix elements with the ratio $U_{23} : \lambda_{23} = 1 : -3$. One obtains $\xi = 0, \ \chi = -3$ and $\kappa_1 = 3, \ \kappa_2 = \frac{3}{2}$.

   (4) $O_2 = 16_2 \ 120 \ 16_3$. 120 contains 2 independent vevs which contribute to the masses of up quarks and neutrinos. To be able to get predictions we have to assume that the vev lies in some particular direction. In the $SU(5) \times U(1)$ decomposition the 2 vevs lie in the 5 and 45 directions of $SU(5)$ and contribute to the neutrino and up quark masses separately, so $\lambda_{23}$ and $U_{23}$ are not related. In the $SU(4) \times SU(2) \times SU(2)$ decomposition, the 2 vevs lie in the $(2,2,1)$ and $(2,2,15)$ directions which give the ratios $U_{23} : \lambda_{23} = 1 : 1$ and $U_{23} : \lambda_{23} = 1 : -3$ respectively. The former is identical to the case of 10’ and the latter is identical to the case of $\overline{126}$.

   The Clebschs for dimension-4 operators are summarized in Table 2.

2. Dimension-5 operators, the 2 scalars can be:

   (1) $10 \times 45$, where 10 is the same 10 in $O_1$. The Clebschs for different operators are listed in Table 3.

   The following models all have $\xi = 0$.

   (2) $10' \times 45$ (different 10 than in $O_1$). The results are listed in Table 3.

   (3) $\overline{126} \times 45$. The results are listed in Table 4.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{Operator} & $\xi$ & $\chi$ & $\kappa_1$ & $\kappa_2$ \\
\hline
$16_2 \, 10 \, 16_3$ & $V_{cb} = 0$ & & & \\
\hline
$16_2 \, 10' \, 16_3$ & 0 & 1 & 1 & $\frac{1}{2}$ \\
\hline
$16_2 \, 126 \, 16_3$ & 0 & $-3$ & 3 & $\frac{1}{2}$ \\
\hline
$16_2 \, 120 \, 16_3$ & Same as $10'$ and $126$ & & & \\
\hline
\end{tabular}
\caption{The Clebschs for all dimension-4 operators.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{Operator} & $\xi$ & $\chi$ & $\epsilon$ & $\kappa_1$ & $\kappa_2$ \\
\hline
$16_2 \, 10 \, \frac{45_X}{\mathcal{M}} \, 16_3$ & $-3$ & 5 & $-\frac{1}{3}$ & 1 & $\frac{3}{4}$ \\
\hline
$16_2 \, 10 \, \frac{45_Y}{\mathcal{M}} \, 16_3$ & $-\frac{1}{2}$ & 0 & 3 & 1 & 1 \\
\hline
$16_2 \, 10 \, \frac{45_B - L}{\mathcal{M}} \, 16_3$ & 1 & $-3$ & $-3$ & $V_{cb} = 0$ \\
\hline
$16_2 \, 10 \, \frac{45_{T_R}}{\mathcal{M}} \, 16_3$ & $-1$ & 1 & 1 & 1 & $\frac{3}{4}$ \\
\hline
$16_2 \, \frac{45_X, Y, B - L, T_R}{\mathcal{M}} \, 10 \, 16_3$ & $V_{cb} = 0$ & & & \\
\hline
$16_2 \, 10 \, \frac{45_Y}{(45_X)} \, 16_3$ & $\frac{1}{6}$ & 0 & $-9$ & $\frac{9}{5}$ & $\frac{9}{3}$ \\
\hline
$16_2 \, 10 \, \frac{45_B - L}{(45_X)} \, 16_3$ & $-\frac{1}{3}$ & $-\frac{3}{5}$ & 9 & $\frac{9}{5}$ & $\frac{81}{40}$ \\
\hline
$16_2 \, 10 \, \frac{45_{T_R}}{(45_X)} \, 16_3$ & $\frac{1}{3}$ & $\frac{1}{5}$ & $-3$ & $\frac{9}{5}$ & $\frac{33}{20}$ \\
\hline
$16_2 \, \frac{45_X, Y, B - L, T_R}{(45_X)} \, 10 \, 16_3$ & $V_{cb} = 0$ & & & \\
\hline
\end{tabular}
\caption{The Clebschs for the operators containing scalars $10 \times 45$.}
\end{table}
| Operator                  | $\chi$ | $\kappa_1$ | $\kappa_2$ |
|--------------------------|--------|------------|------------|
| $16_2 10' \frac{45X}{M} 16_3$ | 5      | 5          | $\frac{5}{2}$ |
| $16_2 10' \frac{45Y}{M} 16_3$ | 0      | 0          | 0          |
| $16_2 10' \frac{45_B-L}{M} 16_3$ | −3     | 3          | $\frac{3}{2}$ |
| $16_2 10' \frac{45_T_R}{M} 16_3$ | 1      | 1          | $\frac{1}{2}$ |
| $16_2 \frac{45X}{M} 10' 16_3$ | −3     | 3          | $\frac{3}{2}$ |
| $16_2 \frac{45Y}{M} 10' 16_3$ | −3     | 3          | $\frac{3}{2}$ |
| $16_2 \frac{45_B-L}{M} 10' 16_3$ | −3     | 3          | $\frac{3}{2}$ |
| $16_2 \frac{45_T_R}{M} 10' 16_3$ | $V_{cb} = 0$ |

Table 4: The Clebschs for the operators containing scalars $10' \times 45$. 

12
| Operator | $\chi$ | $\kappa_1$ | $\kappa_2$ |
|----------|--------|------------|------------|
| $16_2 \ 126 \ \chi \ 45 \ \frac{X}{M} \ 16_3$ | $-15$ | $15$ | $\frac{15}{2}$ |
| $16_2 \ 126 \ \chi \ 45 \ \frac{M}{X} \ 16_3$ | $0$ | $0$ | $0$ |
| $16_2 \ 126 \ 45 \ \frac{Y - T}{M} \ 16_3$ | $9$ | $9$ | $\frac{9}{2}$ |
| $16_2 \ 126 \ \chi \ 45 \ \frac{T_{1R}}{M} \ 16_3$ | $-3$ | $3$ | $\frac{3}{2}$ |
| $16_2 \ 126 \ \chi \ 45 \ \frac{Y}{M} \ 126 \ 16_3$ | $9$ | $9$ | $\frac{9}{2}$ |
| $16_2 \ \chi \ 45 \ \frac{Y}{M} \ 126 \ 16_3$ | $9$ | $9$ | $\frac{9}{2}$ |
| $16_2 \ \chi \ 45 \ \frac{Y}{M} \ 126 \ 16_3 \ V_{cb} = 0$ | $0$ | $0$ | $0$ |
| $16_2 \ 126 \ \chi \ 45 \ \frac{Y}{(45_X)} \ 16_3$ | $9$ | $9$ | $\frac{9}{10}$ |
| $16_2 \ 126 \ \chi \ 45 \ \frac{Y - T}{(45_X)} \ 16_3$ | $-3$ | $3$ | $\frac{3}{2}$ |
| $16_2 \ 126 \ \chi \ 45 \ \frac{T_{1R}}{(45_X)} \ 16_3$ | $-3$ | $3$ | $\frac{3}{2}$ |
| $16_2 \ 126 \ \chi \ 45 \ \frac{Y}{(45_X)} \ 126 \ 16_3 \ V_{cb} = 0$ | $0$ | $0$ | $0$ |

Table 5: The Clebschs for the operators containing scalars $\overline{126} \times 45$. 

13
(4) $120 \times 45$. Similar to the $16_2 120 16_3$, either no prediction can be made or it gives the same results as those of the $10' \times 45$ and $126 \times 45$ cases.

(5) $\overline{16}^2$. Predictions can be obtained if the 2 $\overline{16}$'s are the same otherwise there will be too many arbitrary parameters to be fixed. Under $SU(5)$, $\overline{16} = 1 + 5 + \overline{10}$. The 1 gets a vev $\langle 1 \rangle$ on the order of the GUT scale and the 5 gets a vev $\langle 5 \rangle$ on the order of the weak scale. While the product $\langle 1 \rangle \langle 5 \rangle$ contributes to the Dirac masses of up quarks and neutrinos, the $\langle 1 \rangle \langle 1 \rangle$ and $\langle 5 \rangle \langle 5 \rangle$ contribute only to the Majorana masses of the right handed and the left handed neutrinos respectively. When there is a direct contribution to the light neutrino mass matrix, equation (2) must be modified to

$$m_\nu = \tilde{v}^2 - \frac{v^2}{V} \Lambda^T$$

(16)

where $\overset{\circ}\nu$ is the Yukawa matrix of the left handed neutrinos and $\tilde{v}$ has mass dimension unity. Here we choose $\tilde{v} = \frac{v^2}{V}$ so that $\overset{\circ}\nu$ can be compared with $\overset{\circ}\Lambda$ and $\Lambda$. If this operator is the dominant source of the Majorana masses of the neutrinos ($\tilde{B}$ dominates in equation 6), $(m_\nu)_{23}$ receives comparable contributions from both terms as we can see the first term $\sim (\langle 5 \rangle)^2$ and the second term $\sim \frac{(\langle 1 \rangle)^2}{(\langle 1 \rangle)^2}$.

From the group theory we know $16 \times \overline{16} \supset 1, 45, 210, \overline{16}^2 \supset 10, \overline{126}$ (no 120 because the 2 $\overline{16}$'s are the same and 120 is antisymmetric). There are several ways to combine these fields into $SO(10)$ invariants.

(i) $(16_2 \overline{16})_1 (16_3 16_3)_1$: This operator does not contribute to $U_{23}$ and is thus not interesting.

(ii) $(16_2 \overline{16})_{45} (16_3 16_3)_{45}$: This gives the ratio $U_{23} : \lambda_{23} : \Lambda_{23} : \nu_{23} = 8 : 3 : -5 : -5 \Rightarrow \chi = \frac{3}{8}$. The ratio of the two terms in $(m_\nu)_{23}$ is $-5 : \frac{3^2}{5} = \frac{-5}{3}$ so $\kappa_2$ has to be modified by a factor $\left| \frac{5^2}{3} - 1 \right| = \frac{16}{9}$. In this case $\kappa_2 = \frac{16}{9} \times \frac{3}{2} = \frac{1}{3}$.

(iii) $(16_2 \overline{16})_{210} (16_3 16_3)_{210}$: This gives the ratio $U_{23} : \lambda_{23} : \Lambda_{23} : \nu_{23} = 4 : 9 : 5 : 5 \Rightarrow \chi = \frac{9}{7}$. Similarly $\kappa_2$ is modified by a factor $\left| \frac{9^2}{7} - 1 \right| = \frac{56}{81}$. In this case $\kappa_2 = \frac{56}{81} \times \frac{3}{2} = \frac{7}{6}$.

(iv) $(16_2 16_3)_{10} (\overline{16} \overline{16})_{10}$: This operator does not contribute to the Majorana masses so the predictions are the same as those of $16_2 10' 16_3$, i.e., $\kappa_2 = \frac{1}{2}$.

(v) $(16_2 16_3)_{126} (\overline{16} \overline{16})_{126}$: This gives the ratio $U_{23} : \lambda_{23} : \Lambda_{23} : \nu_{23} = 1 : -3 : -8 : -8 \Rightarrow \chi = -3$. $\kappa_2$ is modified by a factor $\left| \frac{8^2}{3} - 1 \right| = \frac{55}{9}$. Therefore we
have \( \kappa_2 = \left| \frac{55}{9} \times \frac{3}{2} \right| = \frac{55}{6} \).

The \( \kappa_1 \) predictions result from the condition that \( \tilde{A} \) dominates \( AD^2 \gg B^2F, \Lambda^{-1} \) is dominated by \( \Lambda_{33} \) in equation (1.6), so there is no simple relation between the two terms in \( (m_\nu)_{23} \). Predictions can be obtained only when the first term is negligible, or in terms of \( \Lambda \) matrix elements, \( A \ll B \). In this case, \( \kappa_1 = \frac{3}{8}, \frac{9}{4}, 1, 3 \) for the operators (ii), (iii), (iv), and (v) respectively. The results are summarized in Table 6.

| Operator | \( \chi \) | \( \kappa_1 \) | \( \kappa_2 \) |
|----------|------|------|------|
| \((16_2 \overline{16})_1 (\overline{16} 16_3)_1\) | - | - | - |
| \((16_2 \overline{16})_{45} (\overline{16} 16_3)_{45}\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) | \(\frac{1}{3}\) |
| \((16_2 \overline{16})_{210} (\overline{16} 16_3)_{210}\) | \(\frac{9}{4}\) | \(\frac{9}{4}\) | \(\frac{7}{4}\) |
| \((16_2 16_3)_{10} (\overline{16} 16)_{10}\) | 1 | 1 | \(\frac{1}{2}\) |
| \((16_2 16_3)_{126} (\overline{16} 16)_{126}\) | -3 | 3 | \(\frac{55}{6}\) |

Table 6: The Clebschs for the \(16^2 \overline{16}^2\) operators

3 The Renormalization Factor \( \eta \)

In this section we shall discuss the renormalization group (RG) dependence of the predictions on the mass scale, \( t = \ln(\mu) \). As discussed, the predictions we have found all take the form of equation (1.1) with the \( \kappa \) predictions as shown in the last section. The \( \kappa \)'s are pure group theory coefficients and are completely determined by the matrix texture and mass-producing operators one chooses, and the relations involving these generalized Clebschs are assumed to hold strictly at the GUT scale. These coefficients involve no dynamic information about the RG running down to the weak scale or the actual experimental inputs, and we have chosen to include all of this dependence in the \( \eta \) factor. In fact, both \( \theta_{\mu\tau} \) and \( V_{cb} \) renormalize (differently) and the original form of the prediction at the GUT scale, \( \theta_{\mu\tau}^0 = \kappa |V_{cb}^0| \), when run down to the electroweak scale at approximately \( m_t \) becomes equation (1.1), where the unsuperscripted \( \theta_{\mu\tau} \) and \( V_{cb} \) are taken to be the values at \( m_t \) (taken to be on the order of the SUSY-breaking scale so that our MSSM RGE's are valid down to this scale).
3.1 Heavy Yukawa Coupling RGE’s

The procedure for obtaining these η factors is as follows: we must first solve the RGE’s for the running of the three heaviest generation Yukawa couplings[14]:

\[
\begin{align*}
\frac{d(h_t)}{dt} &= - \left( \frac{h_t}{16\pi^2} \right) \left( 6h_t^2 + h_b^2 - G_U \right) \\
\frac{d(h_b)}{dt} &= - \left( \frac{h_b}{16\pi^2} \right) \left( 6h_b^2 + h_t^2 - G_D \right) \\
\frac{d(h_\tau)}{dt} &= - \left( \frac{h_\tau}{16\pi^2} \right) \left( 3h_b^2 + 4h_\tau^2 - G_E \right)
\end{align*}
\]

(3.1)

where the \(G_{(U,D,E)}\) are the parts of the equation that come from gauge boson renormalization:

\[
\begin{align*}
G_U &= \frac{13}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \\
G_D &= \frac{7}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \\
G_E &= \frac{9}{5} g_1^2 + 3g_2^2
\end{align*}
\]

(3.2)

These equations must be solved numerically, subject to the boundary conditions \(h_b(M_{GUT}) = h_\tau(M_{GUT})\) (which is not in general equal to \(h_t(M_{GUT})\) since we have chosen to solve the more general case of \(A_U \neq A_D\)), and \(\eta_b(m_t) = \frac{m_b(m_t)}{m_t(m_t)}\) (where \(\eta_b\) is the boundary factor for the running of \(h_b\)). Here, we must run the extracted experimental values of \(m_b(m_b)\) and \(m_\tau(m_\tau)\) up to the values at \(m_t\), which involves only a gauge boson renormalization over this range: for \(m_b\) we divide through by the factor \(\eta_b\) [2], and we can actually take \(m_\tau(m_t) \approx m_\tau(m_\tau)\) since lepton masses renormalize only by QED and weak interactions, and thus change very little over this range. The factor \(\eta_b\) depends on the value of \(\alpha_S(M_Z)\). Since we have not specified the value of \(\frac{h_t(M_{GUT})}{h_b(M_{GUT})}\), we must fix a value for this in order to obtain solutions for \(h_t, h_b\) and \(h_\tau\). This is equivalent to varying the value of \(\tan \beta\) we choose, since fixing \(\tan \beta\) will fix the value of \(h_\tau(m_\tau)\) (through \(m_\tau(m_\tau) = \sqrt{2} h_\tau(m_\tau) \cos \beta\)). This then fully determines the solutions for \(h_t(\mu), h_b(\mu)\) and \(h_\tau(\mu)\). Using these solutions we can...
then determine the definite integrals of the these heavy Yukawas squared from $m_t$ to $M_{GUT}$, which are factors in the definitions of the $\eta$’s as we shall now see.

### 3.2 RGE’s for $V_{cb}$ and $\theta_{\mu\tau}$

The RGE’s for $V_{cb}$ and $\theta_{\mu\tau}$ are [15]:

\[
\frac{dV_{cb}}{dt} = -\left(\frac{V_{cb}}{16\pi^2}\right)(h_t^2 + h_b^2)
\]
\[
\frac{d\theta_{\mu\tau}}{dt} = -\left(\frac{\theta_{\mu\tau}}{16\pi^2}\right)(h_{\tau}^2)
\]  

(3.3)

These equations have the analytic solutions:

\[
\theta_{\mu\tau} = \theta_{\mu\tau}^0 f_{\tau}
\]
\[
V_{cb} = V_{cb}^0 f_t f_b
\]  

(3.4)

with

\[
f_i = \exp\left(\int_{m_Z}^{M_{GUT}} h_i^2 dt\right)
\]  

(3.5)

i=t,b, $\tau$.

Thus $\theta_{\mu\tau}^0 = \kappa |V_{cb}^0|$ becomes $\theta_{\mu\tau} = \kappa |V_{cb}|\eta$ with

\[
\eta = \frac{f_{\tau}}{f_t f_b}
\]  

(3.6)

The renormalization correction $\eta$ is shown in Figure 1 as a function of $\tan \beta$ for different $\alpha_S$ values.
4 Discussion of Results.

From Figure 1 we can see that the renormalization correction $\eta$ has strong dependence on $\alpha_S(M_Z)$ and a somewhat weaker dependence on $m_b$ and the unknown $\tan \beta$. This makes our predictions less definite. With all uncertainties included, $\eta = 0.74 \pm 0.11$ and $V_{cb}\eta = 0.033 \pm 0.007$. Therefore the cases of $\kappa = 1$ just lie on the edge of the current experimental limit set by FNAL E531 (as shown in Fig. 2). The CHORUS and NOMAD experiments will test $\theta_{\mu\tau}$ within the range 0.01-0.032 which corresponds to $\kappa = 0.3-1$. Most of the models we have discussed have $\kappa > 1$ and therefore large $\Delta m^2$ is already excluded in these models (even taking uncertainties into account). However, there are still many models which give $\kappa$ values which are allowed. These models are listed in Table 7 (except the uninteresting case $\kappa = 0$, which would mimic no $\nu_\mu\nu_\tau$ oscillations).

| Operator                      | $\kappa_1$ | $\kappa_2$ |
|-------------------------------|------------|------------|
| $16_2 10' 16_3$               | 1          | $\frac{1}{2}$ |
| $16_2 10'$ $\frac{45_1 R}{M}$ 16_3 | 1          | $\frac{1}{2}$ |
| $16_2 \frac{45^y}{(45X)} 10'$ 16_3 | 1          | $\frac{1}{2}$ |
| $16_2 \frac{45^B-L}{(45X)} 10'$ 16_3 | 1          | $\frac{1}{2}$ |
| $(16_2 16_3)_{10} (16 10')_{10}$ | 1          | $\frac{1}{2}$ |
| $16_2 10' \frac{45^y}{M}$ 16_3 | 1          | $\frac{3}{2}$ |
| $16_2 10' \frac{45^y_{10R}}{M}$ 16_3 | 1          | $\frac{3}{2}$ |
| $16_2 10' \frac{45^y_{10R}}{M}$ 16_3 | 1          | $\frac{3}{2}$ |
| $16_2 10' \frac{45^y_{10R}}{M}$ 16_3 | 1          | $\frac{3}{2}$ |
| $16_2 10' \frac{45^y_{10R}}{M}$ 16_3 | 1          | $\frac{3}{2}$ |
| $16_2 \frac{45^y_{10R}}{(45X)} 16_3$ | $\frac{3}{2}$ | $\frac{3}{10}$ |
| $16_2 \frac{45^y_{10R}}{(45X)} 16_3$ | $\frac{3}{2}$ | $\frac{3}{10}$ |
| $16_2 \frac{45^y_{10R}}{(45X)} 16_3$ | $\frac{3}{2}$ | $\frac{3}{10}$ |
| $16_2 \frac{45^y_{10R}}{(45X)} 16_3$ | $\frac{3}{2}$ | $\frac{3}{10}$ |
| $16_2 \frac{45^y_{10R}}{(45X)} 16_3$ | $\frac{3}{2}$ | $\frac{3}{10}$ |
| $(16_2 16)_{45} (16 16_3)_{45}$ | $\frac{3}{2}$ | 1          |
| $(16_2 16)_{210} (16 16_3)_{210}$ | $\frac{3}{10}$ | $\frac{1}{3}$ |

Table 7: The $\kappa$ values for which large $\Delta m^2$ is not excluded. (Note: The 120 can be substituted for the 10' or 126. See Sec. 2.2, Dimension-4 operators, #4.)

The preferred $\kappa$ values are 1 and $\frac{1}{2}$ as they occur most in these allowed models. The corresponding $\sin^2 2\theta_{\mu\tau}$ values are $4 \times 10^{-3}$ and $1 \times 10^{-3}$, respec-
tively. We can see from Figure 2 that they are well within the region which will be probed by the CHORUS and NOMAD experiments (the NOMAD and CHORUS limits are comparable). In fact, all $\kappa$ values except those given by $16_2 \times 10^{45 \frac{\alpha_s}{V_{cb}}} 16_3$ in Table 7 can be probed by these experiments.

## 5 Conclusion

There are two ideas which underlie much of the thinking about neutrino masses and mixings:

- Neutrinos are light because of the see-saw mechanism (eq. 1.2), with a large mass responsible for the Majorana masses.

- Neutrino mixing is expected to be broadly similar to quark mixing, because, at some fundamental level, leptons are similar to quarks.

These ideas become concrete in the framework of grand unified theories, which provide both a unification of quarks and leptons and a very large mass scale for the right handed neutrinos. It is well-known that the idea of grand unification suggests a hierarchy of neutrino masses: $m_{\nu_e} : m_{\nu_\mu} : m_{\nu_\tau} \approx m_\mu : m_\tau : m_e$ (or perhaps $m_\mu^2 : m_\tau^2 : m_e^2$) and of neutrino mixing angles: $\theta_{ij} \approx |V_{ij}|$. To what extent can these suggestions be sharpened into precise numerical predictions for parameters of the neutrino sector?

In this paper we have shown that a sequence of five assumptions about the grand unified theory allows the mixing angle $\theta_{\mu\tau}$ to be precisely predicted via eq. (1.1). However there are many Clebschs, $\kappa$, which can appear in such a prediction, and we have no reason to expect that any one is preferred. Hence we have searched for all such possible Clebschs. With the exception of just one case, CHORUS and NOMAD will discover $\nu_\mu - \nu_\tau$ oscillations, if $\Delta m^2$ is large enough. However, they will not be able to provide a significant numerical test of any of our predictions. There are two reasons for this. The first is that given the smearing of our predictions due to the experimental uncertainties of $\alpha_S$ and $V_{cb}$, and given the large number of possible $\kappa$ values, our predictions span the entire region which CHORUS and NOMAD will probe. A considerable improvement in this situation can be expected in the future as the uncertainties, especially on $V_{cb}$,
will be reduced. The second difficulty is that the statistics of the CHORUS and NOMAD experiments will not be high enough to distinguish between models with close $\kappa$ values (for example $\frac{3}{8}$ and $\frac{1}{2}$). Hence, if these experiments do discover $\nu_\mu - \nu_\tau$ oscillations, it will be very important to do further experiments with higher statistics, such as the Fermilab proposal P803[16]. For example, if $\kappa = \frac{1}{2}$, P803 will be able to determine the value of $\sin^2 2\theta_{\mu\tau}$ to an accuracy of about 10%, assuming that $\Delta m^2$ is large enough.

Finally, even if the assumptions made in this paper are incorrect, grand unified theories are still expected to yield relations of the form of eq. (1.1): $\theta_{\mu\tau} = \kappa |V_{cb}| \eta$. The difference is that, in this more general case, $\kappa$ need not take a special value corresponding to a group theory Clebsch. For example it might be a linear combination of several Clebschs. Nevertheless, the expectation is that $\kappa \approx 1$. In principle $\eta$ could differ from the values shown in figure 1. In practice, theories with perturbative couplings do not give results very different from those of the MSSM shown in figure 1. Hence we conclude that CHORUS and NOMAD will probe precisely the range of $\theta_{\mu\tau}$ of interest to grand unified theories. A null result by P803 would imply that, within the context of grand unified theories, $m_{\nu_\tau} \leq 3$ eV, too small to be of much interest for dark matter.

References

[1] H. Georgi and S. Glashow, Phys. Rev. Lett. 32, 438 (1974); H. Georgi, H. Quinn and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974); S. Dimopoulos, S. Raby and F. Wilczek, Phys. Rev. D24, 1681 (1981); S. Dimopoulos and H. Georgi, Nucl. Phys. B193, 150 (1981); L. Ibanez and G. G. Ross, Phys. Lett. 105B, 439 (1981); M. B. Einhorn and D. R. T. Jones, Nucl. Phys. B196, 475 (1982); W. J. Marciano and G. Senjanovic, Phys. Rev. D25, 3092 (1982).

[2] S. Dimopoulos, L. J. Hall and S. Raby, Phys. Rev. Lett. 68, 752 (1992); Phys. Rev. D45, 4192 (1992); P. Ramond, UFIFT-92-4 (1992); H. Arason, D. Castano, E. J. Piard and P. Ramond, Phys. Rev. D47, 232 (1993); G. Anderson, S. Raby, S. Dimopoulos and L. J. Hall, Phys. Rev. D47, 3702 (1993); V. Barger, M. S. Berger, T. Han and M. Zralek, Phys. Rev. Lett.
68, 3394 (1992); V. Barger, M. S. Berger and P. Ohmann, *Phys. Rev. D47*, 1093 (1993).

[3] P. Ramond, R. G. Roberts and G. G. Ross, preprint RAL-93-010 (1993).

[4] G. Anderson, S. Dimopoulos, L. J. Hall, S. Raby and G. Starkman, LBL preprint 33531 (1993).

[5] H. Georgi and C. Jarlskog, *Phys. Lett. B86*, 297 (1979); H. Georgi and D. V. Nanopoulos, *Nucl. Phys. B159*, 16 (1979); J. Harvey, P. Ramond and D. B. Reiss, *Phys. Lett. B92*, 309 (1980), *Nucl. Phys. B199*, 223 (1992).

[6] H. Georgi, AIP Conference, Williamsburg (1974); H. Fritzsch and P. Minkovsky, *Ann. Phys. (NY) 93*, 193 (1975).

[7] C. D. Froggatt and H. B. Nielsen, *Nucl. Phys. B147*, 277 (1979); S. Dimopoulos, *Phys. Lett. B29B*, 417 (1983).

[8] One idea is to reduce the number of couplings by having few $SO(10)$ Higgs representations and by imposing CP conservation, K. S. Babu and R. N. Mohapatra, Bartol preprint BA-92-54 (1992).

[9] S. Dimopoulos, L. J. Hall and S. Raby, *Phys. Rev. D47*, R3697 (1993).

[10] CHORUS collaboration, N. Armenise et al., CERN-SPSC/90-42 (1990).

[11] NOMAD collaboration, P. Astier et al., CERN-SPSC/91-21 (1991).

[12] L. J. Hall and A. Rašin, LBL preprint 33668 (1993).

[13] B. Ananthanarayan, G. Lazarides and Q. Shafi, *Phys. Rev. D44*, 1613 (1991); L. J. Hall, R. Rattazzi and U. Sarid, LBL preprint 33997 (1993).

[14] T.P. Cheng, E. Eichten and L-F. Li *Phys. Rev. D9*, 225 (1974); M. Machacek and M. Vaughn, *Nucl. Phys. B236*, 221 (1984).

[15] K. S. Babu and Q. Shafi, *Phys. Rev. D47*, 5004 (1993).

[16] Fermilab P803 Proposal, K. Kodama et al. (1990), updated (June 1993).
Figure Captions

Figure 1: A plot of the renormalization correction factor $\eta$ versus $\tan \beta$ for $\alpha_s(m_Z) = .110, .118, \text{ and } .126$. On the solid (dashed) [dotted] curve the $\overline{\text{MS}}$ values of the running $b$ quark mass is $m_b(m_b) = 4.25 (4.35) [4.15]$ GeV.

Figure 2: The $\Delta m^2$ vs. $\sin^2 2\theta_{\mu\tau}$ plot including the current (FNAL 531) and future (NOMAD and CHORUS) expected limits, and the primary predictions of this paper (for $\kappa = \frac{1}{2}, 1$), with error bars.