The Wahlquist metric cannot describe an isolated rotating body

Michael Bradley¹, Gyula Fodor², Mattias Marklund³ and Zoltán Perjés²
1 Department of Plasma Physics, Umeå University, S–901 87
Umeå, Sweden
2 KFKI Research Institute for Particle and Nuclear Physics,
H–1525, Budapest 114, P.O.B. 49, Hungary
3 Department of Mathematics and Applied Mathematics,
University of Cape Town, 7701 Rondebosch, Cape Town, South Africa

November 2, 2021

Abstract

It is proven that the Wahlquist perfect fluid space-time cannot be smoothly joined to an exterior asymptotically flat vacuum region. The proof uses a power series expansion in the angular velocity, to a precision of the second order. In this approximation, the Wahlquist metric is a special case of the rotating Whittaker space-time. The exterior vacuum domain is treated in a like manner. We compute the conditions of matching at the possible boundary surface in both the interior and the vacuum domain. The conditions for matching the induced metrics and the extrinsic curvatures are mutually contradictory.

1 Introduction

There is an embarrassing hiatus in the theory of rotating fluid bodies in general relativity. Whereas a large number of spherically symmetric static equilibrium states are known with their matching to the ambient vacuum [1], there is currently no global space-time model available for a rotating fluid
body in an asymptotically flat vacuum. None of the few existing rotating fluid metrics could thus far been matched to a vacuum domain. An ominous case is that of the Wahlquist metric [2] which has been around and much studied for three decades. There is a parameter range within which the zero-pressure surface of the Wahlquist metric is prolate along the axis of rotation [3], and this indicates the action of an external force. Yet, no exact proof is known of the nonexistence of an asymptotically flat vacuum exterior.

A convenient way of learning about the properties of fluids is to look at the slow-rotation limit. After all, if a property (for example, that of being possible to join a vacuum exterior) holds in an exact sense, then it is expected that the property holds in the slow-motion limit as well. In the present work we develop an approximative scheme to the rotating Whittaker space-time. There are at least two reasons why this space-time is of interest for the research of the problem of relativistic fluid states: first, it is the static limit of the Wahlquist space-time; secondly, the equation of the rotation potential \( \omega \) can be solved in terms of elementary functions.

In this paper, the problem of matching of the rotating Whittaker metric to the vacuum exterior is investigated to a precision of quadratic order in the rotation parameter. It will be assumed that the fluid undergoes a circular rotation, that is, the velocity lies in the plane of the stationary \( \partial / \partial t \) and axial \( \partial / \partial \varphi \) Killing vectors. For rigid rotation, \( u^\varphi = \Omega u^t \) where \( \Omega \) is constant. We are using the approximation scheme for slow rotation, developed by Hartle [6], based on a power series expansion in the angular velocity \( \Omega \) of the fluid. We keep at most second order terms. The metric has the form

\[
\begin{align*}
ds^2 &= [1 + 2h(r, \vartheta)]A(r)^2dt^2 - [1 + 2m(r, \vartheta)]B(r)^2dr^2 \\
&
- \left[1 + 2k(r, \vartheta)\right]C(r)^2\left\{d\vartheta^2 + \sin^2 \vartheta [d\varphi + (\Omega - \omega(r)) dt]\right\} \quad (1)
\end{align*}
\]

where \( A, B \) and \( C \) are the metric functions of the static configuration. The rotation potential \( \omega \) is of order \( \Omega \), and the functions \( h, m \) and \( k \) are of order \( \Omega^2 \).

The \((t, \varphi)\) component of Einstein equation gives a second order partial differential equation for \( \omega \). The solution can be sought in the form

\[
\omega(r, \vartheta) = \sum_{l=1}^{\infty} \omega_l(r) \left[-\frac{1}{\sin \vartheta} \frac{dP_l(\cos \vartheta)}{d\vartheta}\right] \quad (2)
\]

The equations for the coefficients \( \omega_l \) with different values of \( l \) decouple. The functions \( \omega_l(r) \) for \( l > 1 \) cannot be regular at infinity. Hence, by the matching
conditions it follows that the rotation potential is a function of the radial coordinate alone, $\omega(r, \vartheta) = \omega(r)$ even in the fluid region. The potential $\omega$ satisfies a second-order ordinary linear differential equation.

We expand the second-order metric functions in Legendre polynomials,

$$h(r, \vartheta) = \sum_{l=0}^{\infty} h_l(r) P_l(\cos \vartheta)$$

and similarly for $m(r, \vartheta)$ and $k(r, \vartheta)$. The quantities with different $l$ decouple once again. The equations for $l > 2$ do not include inhomogeneous terms with $\omega$, and it is asserted that the solution for $l > 2$ does vanish as is the case in the static limit. Hence the Legendre expansion takes the form

$$h(r, \vartheta) = h_0(r) + h_2(r) P_2(\cos \vartheta)$$

$$m(r, \vartheta) = m_0(r) + m_2(r) P_2(\cos \vartheta)$$

$$k(r, \vartheta) = k_0(r) + k_2(r) P_2(\cos \vartheta)$$

Hartle uses the freedom in the choice of radial coordinate to set $k_0 = 0$. The second-order perturbations satisfy a set of inhomogeneous linear differential equations, with a driving term quadratic in the rotation function $\omega$.

In Sec. 2, we review the form of the vacuum metric to the required accuracy. In Sec. 3, we transform the slowly rotating Wahlquist metric to the Hartle form. What we obtain is a particular case of the rotating Whittaker fluid in this approximation. The matching surface $S$ at $p = 0$ is an ellipsoidal cylinder of rotation. We compute the normal $n$ and extrinsic curvature $K_{ab}$ of the embeddings of the surface $S$ in both domains, in Sec. 4. These results are employed in Sec. 5, for setting up the junction conditions at $S$. Here we conclude that the matching equations have no solution for the free constants of integration: they form an inconsistent set. In the light of earlier investigations on the nature of the matching equations of perfect fluids, where it was found that the equations form an overdetermined set, our result is perhaps not that surprising. However, we argue as follows. If the Wahlquist metric can be matched in general, then the matching would exist in the slow-rotation limit. Hence we conclude that the Wahlquist solution cannot be matched to an asymptotically flat region.
2 The Vacuum Exterior

The metric of the ambient empty region to quadratic order has been computed in [6], albeit in a slightly different notation. The unperturbed metric is described by the Schwarzschild solution,

$$A^2 = \frac{1}{B^2} = 1 - \frac{2M}{r}, \quad C = r.$$  \hfill (7)

The perturbed metric is of the form \([\text{[I]}\], with \(\Omega = 0\) and

$$\omega = \frac{2aM}{r^3},$$  \hfill (8)

$$h_0 = \frac{1}{r - 2M} \left( \frac{a^2M^2}{r^3} + \frac{r}{2M}c_2 \right)$$  \hfill (9)

$$m_0 = \frac{1}{2M - r} \left( \frac{a^2M^2}{r^3} + c_2 \right)$$  \hfill (10)

$$h_2 = 3c_1r(2M - r) \log \left(1 - \frac{2M}{r}\right) + a^2 \frac{M}{r^4}(M + r) + 2c_1 \frac{M}{r} \left(3r^2 - 6Mr - 2M^2\right) \frac{r - M}{2M - r}$$  \hfill (11)

$$k_2 = 3c_1(r^2 - 2M^2) \log \left(1 - \frac{2M}{r}\right) - a^2 \frac{M}{r^3}(2M + r) - 2c_1 \frac{M}{r}(2M^2 - 3Mr - 3r^2)$$  \hfill (12)

$$m_2 = 6a^2 \frac{M^2}{r^4} - h_2$$  \hfill (13)

To second order in the rotational parameter, the general slowly rotating solution is characterized by the mass parameter \(M\), the first order small rotation parameter \(a\), and the second order small constants \(c_1\) and \(c_2\).

A particular solution (the slowly rotating Kerr space-time) is given by \(c_1 = c_2 = 0\) and metric functions

$$h_0 = \frac{-a^2M^2}{r^3(2M - r)}$$  \hfill (14)

$$h_2 = \frac{a^2M(M + r)}{r^4}$$  \hfill (15)

4
\[ m_0 = \frac{a^2 M^2}{r^3(2M - r)} \quad (16) \]
\[ m_2 = \frac{a^2 M(5M - r)}{r^4} \quad (17) \]
\[ k_2 = -\frac{a^2 M(2M + r)}{r^4}. \quad (18) \]

### 3 The Wahlquist solution

The Wahlquist solution is given by the metric \(^2\)

\[
 ds^2 = f \left( dt - \tilde{A} d\phi \right)^2 
 - r_0^2 \left( \zeta^2 + \xi^2 \right) \left[ \frac{d\zeta^2}{(1 - \tilde{k}^2\zeta^2) \tilde{h}_1} + \frac{d\xi^2}{(1 + \tilde{k}^2\xi^2) \tilde{h}_2} + \frac{\tilde{h}_1 \tilde{h}_2}{\tilde{h}_1 - \tilde{h}_2} d\phi^2 \right] \quad (19)
\]

where

\[
 f = \frac{\tilde{h}_1 - \tilde{h}_2}{\zeta^2 + \xi^2}, \quad \tilde{A} = r_0 \left( \frac{\xi^2 \tilde{h}_1 + \zeta^2 \tilde{h}_2}{\tilde{h}_1 - \tilde{h}_2} - \xi_A^2 \right) \quad (20)
\]
\[
 \tilde{h}_1(\zeta) = 1 + \zeta^2 + \frac{\zeta}{\kappa^2} \left[ \zeta - \frac{1}{\tilde{k}} \sqrt{1 - \tilde{k}^2\zeta^2} \arcsin \left( \tilde{k} \zeta \right) \right] \quad (21)
\]
\[
 \tilde{h}_2(\xi) = 1 - \xi^2 - \frac{\xi}{\kappa^2} \left[ \xi - \frac{1}{\tilde{k}} \sqrt{1 + \tilde{k}^2\xi^2} \arcsinh \left( \tilde{k} \xi \right) \right] \quad (22)
\]

The constant \( \xi_A \) is defined by \( \tilde{h}_2(\xi_A) = 0 \). The fluid velocity is proportional to \( \partial / \partial t \) and the pressure and density are

\[
 p = \frac{1}{2} \mu_0 \left( 1 - \kappa^2 f \right), \quad \mu = \frac{1}{2} \mu_0 \left( 3\kappa^2 f - 1 \right) \quad (23)
\]

Since the constants are related by the equation of state

\[
 \mu_0 = \mu + 3p = \frac{2\tilde{k}^2}{\kappa^2 r_0^2} \quad (24)
\]

we substitute

\[
 \tilde{k} = \kappa r_0 \sqrt{\frac{\mu_0}{2}}. \quad (25)
\]
Making the coordinate transformation

\[ \zeta = \frac{r}{r_0}, \quad \xi = \cos \Theta \left(1 + \frac{1}{12} r_0^2 \mu_0 \right) \]  

(26)

in the limit \( r_0 \to 0 \) we get Whittaker’s spherically symmetric static solution\[4\].

Making a further coordinate transformation

\[ \sin X = \kappa r \sqrt{\frac{\mu_0}{2}}, \]  

(27)

we get a somewhat simpler form for the metric in the \( r_0 \to 0 \) limit

\[ ds^2 = f_0 dt^2 - \frac{2}{\mu_0 \kappa^2} \left[ \frac{dX^2}{f_0} + \sin^2 X \left( d\Theta^2 + \sin^2 \Theta d\varphi^2 \right) \right] \]  

(28)

where

\[ f_0 = 1 + \frac{1}{\kappa^2} (1 - X \cot X) \].  

(29)

Since the vanishing pressure surface is at \( \kappa^2 f_0 = 1 \), from the positivity of the pressure follows that the constant \( \kappa \) satisfies \( 0 < \kappa < 1 \), and the \( X \) coordinate has the range \( 0 < X < X_s < \frac{\pi}{2} \).

In the slowly rotating limit, if we keep only linear terms in \( r_0 \), the only change will be that we write \( (d\varphi - \omega dt)^2 \) in place of \( d\varphi^2 \) in the metric. To this accuracy we have \( \xi_A = 1 \). Thus the non-diagonal term in the metric is proportional to \( \sin^2 \Theta \), we get

\[ \omega = \frac{\mu_0 r_0}{2 \sin^2 X} (1 - X \cot X) \]  

(30)

Keeping second-order terms in \( r_0 \), the metric has the form

\[ ds^2 = f_0(1 + 2h) dt^2 - \frac{2}{\mu_0 \kappa^2} \frac{1 + 2m}{f_0} dX^2 \]  

\[ - \frac{2}{\mu_0 \kappa^2} \sin^2 X \left[ (1 + 2k) d\Theta^2 + \sin^2 \Theta (1 + 2n) (d\varphi - \omega dt)^2 \right] \]  

(31)

where

\[ h = \frac{\cos^2 \Theta}{2 \sin^2 X} (X \cot X - 1) \mu_0 r_0^2 \]  

(32)

\[ m = \frac{\kappa^2 \sin X - \cos^2 \Theta (\kappa^2 \sin X + \sin X - X \cos X)}{\sin X (\kappa^2 \sin X + \sin X - X \cos X)^2} \kappa^2 r_0^2 \]  

(33)
\[ k = -\frac{1}{3\kappa^2} \left[ 3\kappa^2 \cos^2 X \cos^2 \Theta + \sin^2 X \left( 1 + \cos^2 \Theta \right) \right] r_0^2 \]  
(34)

\[ n = \frac{\sin^2 \Theta}{3\kappa^2 \sin X} \left[ 3 \sin^2 \Theta (\sin X - \cos X) \right. \\
\left. + \cos^2 \Theta \sin^3 X - 3\kappa^2 \sin X \right] r_0^2 \]  
(35)

To get into Hartle’s coordinates, \( n = k \) and \( k_0 = 0 \), we make an infinitesimal coordinate transformation

\[ X = x + \alpha(x, \vartheta) r_0^2 \quad , \quad \Theta = \vartheta + \beta(x, \vartheta) \sin \vartheta r_0^2 \]  
(36)

where

\[ \alpha(x, \vartheta) = \alpha_0 + \alpha_2 \sin^2 \vartheta \]  
(37)

\[ \beta(x, \vartheta) = \beta_1 \cos \vartheta \]  
(38)

and

\[ \alpha_0 = -\frac{\mu_0}{12\kappa^2 \cos x \sin^3 x} \left\{ \left[ 3x^2 - (2x^2 - 3) \sin^2 x \right] \cos^2 x \\
- x \cos x \sin x (6 - 5 \sin^2 x) + \kappa^2 \sin^4 x \right\} \]  
(39)

\[ \alpha_2 = -\frac{\mu_0}{8\kappa^2 \cos x \sin^4 x} \left\{ x \cos x \sin x (\kappa^2 + 6 - 5 \sin^2 x) \\
+ (2\kappa^4 - \kappa^2 + 2x^2 - 3) \sin^2 x \cos^2 x - 3x^2 \cos^2 x \right\} \]  
(40)

\[ \beta_1 = -\frac{\mu_0}{12 \sin^3 x} \left[ 3(x \cos x + \kappa^2 \sin x - \sin x) - (3\kappa^2 - 2) \sin^3 x \right] \]  
(43)

To avoid the appearance of logarithm terms in \( \vartheta \) we rescale the angular coordinate by a small constant factor

\[ \varphi \longrightarrow \left[ 1 + \frac{1}{4} r_0^2 \mu_0 \left( 1 - \kappa^2 \right) \right] \varphi \]  
(44)

This will also ensure that there will be no conical singularity at the axis. We also rescale the time coordinate \( t \) to cancel a second order small constant term in \( h \).

The metric (31) takes the form (11)

\[ ds^2 = f_0 (1 + 2h) dt^2 - \frac{1 + 2m}{\mu_0 \kappa^2 f_0} dx^2 \]

\[ - \frac{2}{\mu_0 \kappa^2} \sin^2 x (1 + 2k) \left[ d\vartheta^2 + \sin^2 \vartheta (d\varphi - \omega dt)^2 \right] \]  
(45)
with

\[ f_0 = 1 + \frac{1}{\kappa^2} (1 - x \cot x) \]  

(46)

and

\[ \omega = \frac{\mu_0 r_0}{2 \sin^2 x} (1 - x \cot x) \]  

(47)

For the second order small quantities we obtain

\[ h_0 = \frac{5 \sin^2 x - 6 - 2 \cos^2 x \sin^2 x}{24 [(\kappa^2 + 1) \sin x - x \cos x] \sin^2 x \cos x} \mu_0 r_0^2 \]

\[ -\frac{4 x^2 \sin^2 x - 3 \sin^2 x - 3 x^2 - 4 \sin^4 x}{24 [(\kappa^2 + 1) \sin x - x \cos x] \sin^3 x} \mu_0 r_0^2 \]  

(48)

\[ h_2 = \left( x \cos x \frac{\sin^2 x + 3 \kappa^2 - 3}{6 \sin^3 x} + \frac{\kappa^2}{6} \right) \mu_0 r_0^2 \]

\[ -\frac{1}{4} + \frac{3 - 6 \kappa^2 - 2 x^2}{12 \sin^2 x} + \frac{x^2}{4 \sin^4 x} \left( \frac{\mu_0 r_0^2}{2 \kappa^2} \right) \]  

(49)

\[ m_0 = -\left[ \frac{\sin^2 x}{12 \cos^2 x} + x \frac{\cos^3 x - 2 \cos^2 x \sin x - 3 \sin^3 x}{8 [(\kappa^2 + 1) \sin x - x \cos x] \sin^3 x} \cos x \right] \mu_0 r_0^2 \]

\[ + \frac{x \sin^3 x + (x^2 - 7) \cos x \sin^2 x - (2 x^2 + 3) \cos^3 x}{24 [(\kappa^2 + 1) \sin x - x \cos x] \cos x \sin x} \mu_0 r_0^2 \]  

(50)

\[ m_2 = \left( \frac{15 x^2}{\sin^4 x} - 6 x \cos x \frac{\kappa^2 + 5 - \sin^2 x}{\sin^4 x} \right) \mu_0 r_0^2 \]

\[ -2 \kappa^2 - 7 - \frac{14 x^2 - 15 - 6 \kappa^2}{\sin^4 x} \left( \frac{\mu_0 r_0^2}{2 \kappa^2} \right) \]  

(51)

\[ k_2 = \left( \frac{5 x^2 - 3 + 3 \kappa^2}{\sin^2 x} - 2 x^2 + 3 - \kappa^2 \right) \mu_0 r_0^2 \]

\[ -x \cos x \frac{3 \kappa^2 - 6 + 5 \sin^2 x}{\sin^3 x} - \frac{3 x^2}{\sin^4 x} \left( \frac{\mu_0 r_0^2}{12 \kappa^2} \right) \]  

(51)

All these second order quantities go to zero as \( x \to 0 \), which shows that the center is regular to the required order.

The pressure is \( p = p_0 + r_0^2 [p_{20} + p_{22} P_2 (\cos \vartheta)] \) where

\[ p_0 = \frac{\mu_0}{2} \left( x \cot x - \kappa^2 \right) \]  

(52)

\[ p_{20} = \frac{\mu_0^2}{24 \sin^4 x \cos x} \left[ x \sin x (3 \sin^2 x - 2) \right] \]
\[ p_{22} = -\frac{2\kappa^4 \mu_0^2}{12 \sin^2 x} + \frac{\mu_0^2}{24 \kappa^2 \sin^5 x} \left[ x \cos x - (\kappa^2 + 1) \sin x \right] \]
\[ \left[ (2\kappa^2 - 3) \sin^4 x + 2x \sin x \cos x (\kappa^2 - 3 + \sin^2 x) \right. \]
\[ - \left. \left( 2x^2 - 3 + 4\kappa^4 + 2\kappa^2 \right) \sin^2 x + 3x^2 \right] \] (54)

4 The matching surface

In the fluid region, the matching surface \( S \) is defined by the condition of vanishing pressure, \( p = 0 \). In the limit of no rotation, the matching surface is the history of the sphere \( x = x_1 \) satisfying
\[ x_1 \cot x_1 = \kappa^2 \]. (55)

For slow rotation the equation of the matching surface \( S \) is
\[ x = x_1 + r_2^2 \xi \] (56)

with
\[ \xi = -\left[ \frac{p_{20} + p_{22} P_2(\cos \vartheta)}{p'_0} \right] \bigg|_{x=x_1} \] (57)

where we denote \( d/dx \) by a prime. Substituting (52) - (54) and (55), we get that \( \xi = \xi_0 + \xi_2 P_2(\cos \vartheta) \), where the constants \( \xi_0 \) and \( \xi_2 \) are defined by
\[ \xi_0 = \frac{\kappa^{10} - 2\kappa^8 + 2x_1^2 \kappa^6 + \kappa^6 + x_1^2 \kappa^4 + x_1^4 \kappa^2 - x_1^2 \kappa^2 + x_1^4}{12x_1 \kappa^2 (\kappa^4 - \kappa^2 + x_1^2)} \] (58)
\[ \xi_2 = -\frac{4\kappa^{10} + \kappa^8 + 4x_1^2 \kappa^6 - 8\kappa^6 + 2x_1^2 \kappa^4 + 3\kappa^4 - 4x_1^2 \kappa^2 + x_1^4}{12x_1 \kappa^2 (\kappa^4 - \kappa^2 + x_1^2)} \] (59)

In the vacuum exterior region, suitable hypersurfaces for matching are determined by the condition
\[ \tilde{\Omega}^2 g_{\varphi \varphi} + 2\tilde{\Omega} g_{\varphi t} + g_{tt} = 1 - \tilde{C} \] (60)

where \( \tilde{\Omega} \) and \( \tilde{C} \) are constants. In the limit of no rotation, the matching surface is the history of the sphere \( r = r_1 \). For slow rotation the deformation of the surface is described by
\[ r = r_1 + a^2 \chi \] (61)
where $\chi$ is a function of $\vartheta$ and $a$ is the small rotational parameter. Substituting into (60) and keeping only second order small terms in the rotational parameter, we get

$$\chi = \chi_0 + \chi_2 P_2(\cos \vartheta)$$

where $\chi_0$ and $\chi_2$ are constants to be determined by the matching conditions.

The function $y = x - r_o^2 \xi$ characterizes the constant pressure hypersurfaces, with $y = x_1$ on the matching surface. Denoting the coordinates as $x_{(W)}^a = (t, x, \vartheta, \varphi)$, the normal one-form has the components

$$n_a^{(W)} = \left(0, 1, 3r_o^2 \xi_2 \sin \vartheta \cos \vartheta, 0\right) \sqrt{-g_{11}^{(W)}}$$

where $g_{ab}^{(W)}$ denotes the metric components in the Wahlquist region. Using the coordinates $x_{(V)}^a = (t, r, \vartheta, \varphi)$ and introducing the notation $g_{ab}^{(V)}$ for the metric in the vacuum region, the normal form of the possible matching surfaces is

$$n_a^{(V)} = \left(0, 1, 3a^2 \xi_2 \sin \vartheta \cos \vartheta, 0\right) \sqrt{-g_{11}^{(V)}}$$

The extrinsic curvature $K = K_{ab} dx^a dx^b \rvert_S$ of the surface $S$ is defined in terms of the normal $n_a$ and the projector $h_{ab} = g_{ab} + n_a n_b$ where $g_{ab}$ are the components of (1) and the coordinates are restricted to $S$. The quantities $K_{ab}$ are given by $K_{ab} = h_{ac} h_{bd} n^{(c,d)}$. From the expression

$$K_{ab} = n_{(a,b)} - n^r \left( g_{(a|r,b)} - \frac{1}{2} g_{ab,r} \right) + n_{(a,r)} n^r \left( n_{(b)}, - n_{(|r,b)} \right)$$

for the extrinsic curvature we obtain

$$K_{00} = \frac{1}{2} g_{00,1} n^1,$$

$$K_{03} = \frac{1}{2} g_{03,1} n^1,$$

$$K_{12} = -\frac{1}{2} g_{22,1} n^2,$$

$$K_{22} = \frac{1}{2} g_{22,1} n^1 + n_{2,2},$$

$$K_{33} = \frac{1}{2} g_{33,1} n^1 + n_2 \sin \vartheta \cos \vartheta.$$

On the matching surface $S$, given by Eqs. (56) and (61) respectively, the parts $K_{11} dr^2$ and $K_{12} d\vartheta d\varphi$ of the extrinsic curvature will both be of fourth order in the expansion parameters, hence dropped.
5 Junction conditions

In this section, we search for isometric embeddings of the matching surface $S$ in the vacuum and Wahlquist domains, respectively. We equate with each other the respective induced extrinsic curvatures $K(V)$ and $K(W)$ of $S$, in the vacuum and in the Wahlquist region. Hence, in terms of the induced metric $ds^2|_S$ the equations of matching are

$$ds^2(V)|_S = ds^2(W)|_S \quad K(V)|_S = K(W)|_S .$$

The matching to zero-order in the rotation parameter takes place on the cylinder which is the product, $S^2 \times R$, of the metric two-sphere and the time. We take advantage of the freedom in taking constant linear combinations of the time and azimuthal coordinates. We apply a rigid rotation in the fluid region by setting $\varphi \to \varphi + \Omega t$ where $\Omega$ is a constant. Then we re-scale the interior time coordinate $t \in R$ by $t \to c_4(1 + r_0^2c_3)t$ with further constants $c_3$ and $c_4$ to be determined from the matching conditions.

From the zero-order matching conditions we get the following relations:

$$M = \frac{r_1}{2\kappa^2}(\kappa^2 - \cos^2 x_1)$$

$$r_1 = \frac{2^{1/2}}{\kappa \mu_0^{1/2}} \sin x_1$$

$$c_4 = \cos x_1$$

We next perform the matching to first order, with the matching surface still being the product $S^2 \times R$. The matching equations are the $(t, \varphi)$ components of Eqs. (71) from which we get the parameter values

$$\Omega = \frac{\mu_0 x_1 r_0}{6 \sin x_1 \cos x_1}$$

$$a = \frac{r_0}{3 \cos x_1} \frac{2x_1 \cos^2 x_1 - 3 \sin x_1 \cos x_1 + x_1}{\sin x_1 \cos x_1 - x_1} .$$

To second order in the rotation parameter, the matching surface $S$ is an ellipsoidal cylinder characterized by the embedding conditions (56) and (61). The values of the metric coefficients and their derivatives on $S$ are given by a power series expansion in $r_0$ in the fluid and in $a$ in the vacuum regions, respectively. In the vacuum, the nonvanishing components of the normal are

$$n_1(V) = -\frac{\cos x_1}{\kappa} (1 - m(V)) \quad n_2(V) = 3 \cos \theta \sin \theta \frac{\kappa a^2 x_2}{\cos x_1}$$
In the interior, we have

\[ n_{1(W)}^1 = -\left(\frac{\mu_0}{2}\right)^{1/2} (1 - m_{(W)}) \quad n_{2(W)}^2 = 3 \cos \theta \sin \theta \left(\frac{2}{\mu_0}\right)^{1/2} r_0^2 \xi_2. \]  

(78)

Substituting in the matching conditions (71), and Taylor expanding in the powers of the rotation parameter, we get a set of lengthy linear equations. We need to solve these for the parameters \( \chi_0, \chi_2, c_1, c_2 \), and \( c_3 \) in terms of \( r_0, x_1 \), and \( \mu_0 \). The system of equations turns out to have no solution at all, which proves that the slowly rotating Wahlquist solution cannot be matched to an asymptotically flat vacuum exterior.

6 Conclusions

The physical reason that the Wahlquist metric cannot describe an isolated rotating fluid body in equilibrium is apparently that this fluid medium would not withhold the hydrostatic and gravitational forces acting in the neighborhood of the junction surface, without any deformation. To achieve equilibrium, additional, external forces are needed. There may exist a rotating fluid geometry with more degrees of freedom where all the conditions of matching can be met. In fact, in our approximation where we keep quadratic terms in the angular velocity, the perturbations \( h_0 \) and \( m_2 - k_2 \) are governed by uncoupled linear ordinary differential equations, of which the Wahlquist metric represents a particular solution. It is not difficult to obtain the general solution for the function \( h_0 \). We find, however, that this ‘generalized’ Wahlquist metric [with \( m_2 - k_2 \) as given in Eqs. (48)] still fails to satisfy the matching conditions. There is no a priori reason that an initially spherically symmetric perfect fluid body could not be set in a rotational stationary equilibrium state. It would be most surprising if the general rigidly rotating Whittaker metric could be shown impossible to match with the asymptotically flat vacuum exterior. Therefore, it will be desirable to carry out a check that this is not indeed the case.

Even though we have shown the impossibility of the Wahlquist metric describing an isolated rotating body in equilibrium, conceivably there may exist configurations where a Wahlquist fluid ball in an ambient vacuum domain is kept in equilibrium by an external force. A simple picture of this sort is when a fluid ball is surrounded by nested shells of vacuum and matter. It is our
intention to investigate the stability of such more elaborate configurations in a perturbative treatment.

7 Acknowledgments

This work has been partially supported by the OTKA grant T022533. G.F. wishes to acknowledge the hospitality of the Department of Plasma Physics of Umeå University. Z.P. thanks the Wenner-Gren foundation for financial support. M.B. was partially supported by the NFR.

References

[1] M. S. R. Delgaty and K. Lake, Comput. Phys. Commun. 115 395 (1998), gr-qc/9809013

[2] H. D. Wahlquist, Phys. Rev. 172, 1291 (1968). We use a tilde notation for Wahlquist’s functions

[3] C. Hoenselaers, in Proc. Fourth Monash General Relativity Workshop, Eds. A. Lun, L. Brewin and E. Chow, Monash Univ., (1993)

[4] J. M. Whittaker, Proc. Roy. Soc. A 306, 1 (1968)

[5] D. R. Brill and J. M. Cohen, Phys. Rev. 143, 1011 (1966)

[6] J. B. Hartle, Astrophys. J. 150, 1005 (1967)

[7] M. Mars and J. M. M. Senovilla, Mod.Phys.Lett.A, 13,1509 (1998)

[8] W. Roos, Gen. Rel. Grav. 7, 431 (1976)