Geometric imprint of CP violation in two flavor neutrino oscillations

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February 5, 2010

Abstract

In vacuum or constant density matter, the two flavor neutrino oscillation formulae are insensitive to the presence of CP violating phases owing to the fact that the CP phase can be gauged away. In sharp contrast to the above case, we show that the CP violating phases can not be gauged away in presence of adiabatically changing background density accompanied by varying CP phases. We present a pure geometric visualization of this fact by exploiting Pancharatnam’s prescription of cyclic quantum projections. Consequently the topological phase obtained in Phys. Rev. D 79, 096013 (2009) can become geometric if CP violation occurs in a varying density medium.

Pacs numbers: 03.65.Vf,14.60.Pq

Keywords: geometric phases, neutrino oscillations, CP violation

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1 Introduction

Pontecorvo’s insightful idea of neutrino oscillations \[1\] has met phenomenal success over the past several decades in explaining the observed deficit of neutrinos from a wide variety of astrophysical and terrestrial sources. It is now well established that the different neutrino flavors oscillate among themselves while conserving the lepton number. The coherent phase picked up by the evolving states that shows up in transition amplitudes leads to the phenomena of oscillations. The fact that there is a geometric interpretation of the standard neutrino oscillation formalism was recently shown \[2\] for the specific case of CP conservation (CPC) and two flavors.

The topological\(^1\) phase obtained in Ref. \[2\] was robust and did not depend upon the nature of the evolution. We pointed out that as long as the singular point in ray space was encircled, the phase of \(\pi\) remained irrespective of the evolution being in vacuum or in medium. The \(\pi\) anholonomy was first noticed in Refs. \[3\] and \[4\] as a phenomenon associated with real and symmetric Hamiltonians. The presence of this topological phase was ascribed to the structure of leptonic mixing matrix. This implied that there was no unexpected surprise due to this topological phase and that the standard quantum mechanical treatment takes into account Pancharatnam’s phases correctly (in fact is a realization of the same).

The purpose of the present work is to extend our analysis \[2\] to the CP violating (CPV) case while restricting ourselves to two neutrino flavors. We will explicitly find the conditions under which it is possible to destroy the robustness of the topological phase by the inclusion of CPV terms. We consider the general CPV form of neutrino Hamiltonian for the case of two neutrino flavors and derive an expression for the oscillation probability for an arbitrary medium with in the adiabatic approximation. Then we show that the cross terms in the probability have a purely geometric interpretation by employing Pancharatnam’s prescription of cyclic quantum collapses. By doing so we establish a crucial geometric difference between the CPC and the CPV case. Having established our result, we briefly discuss our result in two possible physical scenarios where CPV can appear in the Hamiltonian and leave its imprint at the level of detection probability.

The information on neutrino masses and mixings gleaned from most oscillation experiments can be conveniently analysed considering two neutrino situation or quasi-two-neutrino-situation (for example, the one mass scale dominant approximation) \[5\]. So, we focus only on two flavor case in the present work which provides a clean visualization of physical effects via rotations and reflections on the Poincaré sphere.

This paper is organized as follows. In Sec. \[2\], we describe the general Hamiltonian for two flavor neutrinos, including the CPV term. We also discuss possible sources of such a CPV term. Then in Sec. \[3\], we obtain the neutrino oscillation formulae in presence of

\(^{1}\)By our definition, the topological phase refers to phase factors that are insensitive to small changes in the circuit (and are invariant under deformations of circuit), while geometric phases are sensitive to such changes. We use this distinction between the two terms throughout this article (unless otherwise specified).
CPV term and establish connection of the cross terms appearing in the probability with the Pancharatnam’s geometric phase. We explicitly demonstrate (pictorially) the geometric differences between the case of constant density and adiabatically changing matter background, both for CPC and CPV situations which is the central result of this article. In this context, we also comment on the action of CP and its connection with Wigner’s theorem in Sec. 4. We end with concluding remarks in Sec. 5.

2 Hamiltonian for two flavor neutrinos

It is well-known that the two flavor neutrino system is equivalent to a two state quantum system \[4, 6\] in the ultra-relativistic limit (for equal fixed momenta of the two neutrinos). Upon linearizing the Dirac equation in the ultra-relativistic limit, the propagation of neutrinos through vacuum or matter can be described by an effective Schrödinger-like equation.

Let us examine the general hermitian\(^2\) form of Hamiltonian for any two state system \[7\],

\[
\mathcal{H} = \hat{\mathbf{r}} \cdot \mathbf{\sigma} + r_0 \mathbb{I}_2
\]

\[
= \begin{pmatrix}
    z & x - iy \\
    x + iy & -z
\end{pmatrix} + r_0 \mathbb{I}_2,
\]

\[
\simeq \begin{pmatrix}
    -\cos \vartheta & \sin \vartheta \ e^{-i \varphi} \\
    \sin \vartheta \ e^{i \varphi} & \cos \vartheta
\end{pmatrix},
\]

(1)

where, \(x, y\) and \(z\) are the three independent parameters appearing in the Hamiltonian. The term proportional to Identity \(r_0\) leads to non-zero trace and adds an overall phase factor, \(e^{-i \int r_0 dt}\) to the evolving state. But the states have the freedom of redefinition up to a phase so this extra phase does not affect oscillations and we can safely omit this term and deal with traceless Hamiltonian (last row of Eq. (1)) spanned by the Pauli matrices. \(\vartheta\) and \(\varphi\) are the polar angles and \(\hat{\mathbf{r}}\) is a unit vector. The eigenvalues of Eq. (1) are \(\lambda_{\pm} = \pm \int_0^t \sqrt{x^2 + y^2 + z^2} \, dt'\) = \(\pm 1\) and the normalized eigenstates are:

\[
| \vartheta, \varphi, + \rangle = \begin{pmatrix}
    \cos(\vartheta/2) e^{-i \varphi} \\
    \sin(\vartheta/2)
\end{pmatrix}, \quad \text{and} \quad | \vartheta, \varphi, - \rangle = \begin{pmatrix}
    -\sin(\vartheta/2) e^{-i \varphi} \\
    \cos(\vartheta/2)
\end{pmatrix},
\]

(2)

The ray space in this case is a two-dimensional sphere \(S^2\) which is known as Poincaré sphere (also referred to as the Bloch sphere in quantum mechanics). Pictorially, oscillations are unitary rotations about an axis on the Poincaré sphere while CP transformation can be represented as reflection in the \(x - z\) plane on the Poincaré sphere. The terms appearing in the Hamiltonian due to vacuum, normal matter (with standard interactions (SI) and non-standard interactions (NSI)) and due to neutrino backgrounds with SI encountered by the neutrinos as

\[^2\text{neglecting absorption}\]
they propagate between the source and the detector are listed in Table 1. In general, oscillations in vacuum or any medium can be viewed as the phenomenon of elliptic birefringence in optics as explained in [2].

Let us comment upon the CP properties of different media mentioned in Table 1 and appearance of CPV term in the Hamiltonian. The action of CP can be understood in terms of $H ightarrow H^\dagger$. Since only $\sigma_y$ changes sign under CP, CPV implies that $y \neq 0$. Vacuum is CP symmetric which implies $y = 0$. The other two terms ($x$ and $z$) depend on mixing angle in vacuum $\vartheta/2$ and the vacuum oscillation frequency $\omega = \delta m^2/2p$ where $\delta m^2$ is the mass-squared difference and $p \simeq E$ is the neutrino momentum or energy in the ultra-relativistic limit. Normal matter is CP asymmetric since neutrinos and antineutrinos interact with matter in different ways (Mikheyev-Smirnov-Wolfenstein [8, 9] effect can enhance oscillation in the neutrino channel and suppress in the antineutrino channel) but due to the absence of flavor changing neutral currents in the Standard Model (SM) there are no off diagonal terms and the Hamiltonian still preserves CP ($y = 0$) in the absence of intrinsic CPV phases.

An important consequence of this is that if vacuum mixing and neutrino masses are zero, then neutrinos can not change flavor. In Table 1, the second row corresponds to this case. $V_C = \sqrt{2}G_F n_e$ is the charged current potential due to coherent forward scattering of $\nu_e$ with electrons in matter. $G_F$ is the Fermi coupling constant and $n_e$ is the electron number density.

The minimal theoretical scenario needed to describe neutrino flavor oscillations is the requirement of neutrino masses, for which there is concrete evidence from various oscillation experiments. It is well-known that the SM by itself predicts massless neutrinos. The simplest way is to add dimension five non-renormalizable terms consistent with symmetries and particle content of the SM which leads to desired Majorana masses for the left-handed neutrinos. But, the neutrino interactions involving the light fields are still assumed to be described by weak interactions within the SM and this is minimal SM extension for accommodating neutrino mass. However, once we invoke new physics in order to explain the

\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Medium & $x$ & $y$ & $z$ \\
\hline
Vacuum & $\omega/2 \sin \vartheta$ & 0 & $-\omega/2 \cos \vartheta$ \\
Normal matter + SI & $\omega/2 \sin \vartheta$ & 0 & $-\omega/2 \cos \vartheta + V_C/2$ \\
Normal matter + NSI & Re[$\omega/2 \sin \vartheta + \epsilon_{ey}/2$] & Im[$\omega/2 \sin \vartheta + \epsilon_{ey}/2$] & $-\omega/2 \cos \vartheta + V_C/2 + (\epsilon_{ee} - \epsilon_{yy})/2$ \\
Neutrino backgrounds + SI & Re[$\omega/2 \sin \vartheta + B_{ey}/2$] & Im[$\omega/2 \sin \vartheta + B_{ey}/2$] & $-\omega/2 \cos \vartheta + V_C/2 + B/2$ \\
\hline
\end{tabular}
\end{center}
\end{table}

Table 1: The three independent elements of $H$ in different kinds of media - vacuum, normal matter with SI, ordinary matter with NSI and mixed state neutrino backgrounds (see text).

\footnote{The angle $\vartheta$ is connected to the mixing angle $\Theta$ used in the standard neutrino oscillation literature by $\vartheta = 2\Theta$.}
non-zero neutrino masses, it seems rather unnatural to leave out the NSI which allow for flavor changing interactions as well as are new sources of CP violation which can affect production, detection and propagation of neutrinos [11]. Some of the early attempts discussing new sources of lepton flavor violation (for instance, R-parity violating supersymmetry) were geared towards providing an alternate explanation for the observed deficit of neutrinos coming from the Sun in the limiting case of zero neutrino mass and absence of vacuum mixing [12, 13, 14]. In recent years, the emphasis has shifted towards understanding the interplay between SI and NSI and whether future oscillation experiments can test these NSI apart from determining the oscillation parameters precisely. This has led to an upsurge in research activity in this direction [15, 16, 17, 18, 19, 20, 21, 22, 23]. Non-standard physics which allows for flavor changing interactions may also lead to additional CPV phases which appear as $y \neq 0$ in Eq. (1).

The first possibility to have $y \neq 0$ is the case of NSI in propagation (which affects the coherent forward scattering of neutrinos with background matter)\(^4\) which induces new CPV phases. Such extrinsic\(^5\) CPV phases can mimic the signature of intrinsic CPV phases thereby affecting the determination of intrinsic phase appearing in the leptonic mixing matrix. In the three flavor case, the problem becomes very complicated and one needs to carefully disentangle the extrinsic CPV from the intrinsic one by looking at the CPV observables [19]. Here in the two flavor case, there are no intrinsic phases [24] but extrinsic CPV phases can actually appear at the level of Hamiltonian. If we consider only those operators that arise at a scale much lower than lepton number violating scale then the relevant interaction is

\[
\mathcal{L} = \sum_{f; \alpha, \beta} \frac{4}{\sqrt{2}} G_F \bar{\nu}_\alpha L \gamma^\mu \nu^\mu_{BL} \left( \epsilon f_{\alpha \beta} f_{L}^\gamma f_{L}^\gamma f_{L}^{\gamma} + \epsilon f_{\alpha \beta} f_{R}^{\gamma} f_{L}^{\gamma} f_{R} \right),
\]

where $f = e, p, n$ and $\alpha, \beta = e, \mu, \tau$. Since the scale at which this interaction arises is supposed to be not too far from the electroweak scale $\Lambda_{ew} \sim G_F^{-1/2}$, its coupling may be parameterized by $\epsilon_{\alpha \beta} G_F$, where $\epsilon \sim (\Lambda_{ew}/\Lambda_{NP})^2$. The terms appearing in the flavor Hamiltonian in presence of ordinary matter and NSI are given in third row of Table [1]. At the level of underlying Lagrangian describing NSI, the NSI coupling of the neutrino can be to $e, u, d$ or $e, p, n$. But from phenomenological point of view, only the sum (incoherent) of all these individual contributions is relevant. The effect of coherent forward scattering induced by such interaction terms in Lagrangian on the propagation of neutrinos in ordinary neutral unpolarized medium is governed by the parameters,

\[
\epsilon_{\alpha \beta} = \sum_{f=e, u, d} \frac{n_f}{n_e} \epsilon_{\alpha \beta}^f,
\]

where $n_f$ is the density of fermion $f$ in medium crossed by the neutrino. Also, $\epsilon^f = \epsilon^f_{L} + \epsilon^f_{R}$.

\(^4\)We focus only on NSI during propagation of neutrinos through matter. However it should be noted that NSI can also affect production and detection processes as discussed in Refs. [13, 16, 17, 23].

\(^5\)The term extrinsic refers to matter-induced CPV phases.
Note that a CPV term in the Hamiltonian can have its origin in different sources, for example by inclusion of non-standard neutrino-matter interactions during propagation as described above or via CPV neutrino background which is encountered by neutrinos emanating from a dense supernova core. In the latter case, if the background neutrinos are in a mixed state given by $|\nu_b\rangle = \gamma_e|\nu_e\rangle + \gamma_y|\nu_y\rangle$, CPV can arise in the two flavor neutrino Hamiltonian without the need to invoke any new physics \[25, 26\]. This is due to neutrino self interactions \[27, 28, 29\]. The pure state flavor density matrix for a given momentum mode $\mathbf{p}$ is

$$\rho_\mathbf{p} = |\nu_b\rangle\langle \nu_b| = \begin{pmatrix} |\gamma_e|^2 & \gamma_e \gamma_y^* \\ \gamma_e^* \gamma_y & |\gamma_y|^2 \end{pmatrix}. \tag{5}$$

Background neutrinos will be in mixed state naturally if they also undergo oscillations. The terms of fourth row in Table II are given by

$$B = \sqrt{2} GF \int d^3\mathbf{q} (1 - \cos \theta_{\mathbf{p},\mathbf{q}}) \left[(\bar{\rho}_\mathbf{q} - \bar{\rho}_\mathbf{q})_{ee} - (\bar{\rho}_\mathbf{q} - \bar{\rho}_\mathbf{q})_{yy}\right],$$

$$B_{ey} = \sqrt{2} GF \int d^3\mathbf{q} (1 - \cos \theta_{\mathbf{p},\mathbf{q}}) \left[(\bar{\rho}_\mathbf{q} - \bar{\rho}_\mathbf{q})_{ey}\right],$$

$$B_{ye} = \sqrt{2} GF \int d^3\mathbf{q} (1 - \cos \theta_{\mathbf{p},\mathbf{q}}) \left[(\bar{\rho}_\mathbf{q} - \bar{\rho}_\mathbf{q})_{ye}\right], \tag{6}$$

where $\theta_{\mathbf{p},\mathbf{q}}$ is the angle between the direction of propagating neutrino with momentum $\mathbf{p}$ and the direction of other neutrinos in the ensemble with momentum label $\mathbf{q}$. The quantity $\rho_{\mathbf{q},\gamma\gamma}(\bar{\rho}_{\mathbf{q},\gamma\gamma})$ stands for the matrix element of the density matrix operator, $\langle \nu_\gamma | \rho_{\mathbf{q}} | \nu_\gamma \rangle (\langle \bar{\nu}_\gamma | \bar{\rho}_{\mathbf{q}} | \bar{\nu}_\gamma \rangle).$

Hence for mixed state background neutrinos, all the three terms of the Hamiltonian $x$, $y$ and $z$ can be non-zero in contrast to the case of vacuum or ordinary matter with SI. Thus we note that the Hamiltonian with CPV neutrino background requires all the three parameters for its complete description. The background potential matrix is no longer diagonal and therefore even massless neutrinos can oscillate.

We should emphasize that within the SM, the only situation where one can get flavor off-diagonal terms in the Hamiltonian without any new physics is the case of dense neutrino backgrounds. Here off-diagonal terms arise not due to any new flavor changing interactions but due to background neutrinos being in a mixed flavor state. If the background neutrinos were in pure flavor state then the off-diagonal terms in the potential matrix would disappear and one ends up with a CPC Hamiltonian as in the case of ordinary matter when only SI are taken into account. Hence such a mixed state neutrino background provides us with a minimal scenario for CPV effects to play a role. In addition, invoking NSI in the case of dense neutrino backgrounds leads to extra source of CPV phases in the Hamiltonian \[20\] over and above those induced by mixed state neutrino background.

The question then arises if there is any visible consequence of extrinsic CPV phase or $y \neq 0$ at the level of probability in the two flavor case whatever its source might be. It

\[6\] Not an impure state in the conventional sense.
is well-known that in the case of CPV medium with constant density, this term is of no consequence. Below we will first establish this fact by showing that the extra phases in the mixing matrix can be gauged away \[24\]. By using Pancharatnam’s prescription of cyclic quantum collapses we show that the geometric phase is always restricted to be topological (two pairs of orthogonal states can always be made to lie on a great circle) \[2\]. Further we will discuss the case of varying density CPV medium where it is possible to have an observable effect of CPV phases at the probability level. This is connected intimately to the fact that the Pancharatnam’s phase can become geometric (different from \(\pi, 0\)) in such a situation. We give a pure geometric view of this effect by using Pancharatnam’s prescription \[30\].

3 CPV phases and the two flavor oscillation probability

Solving the full neutrino evolution equation in matter exactly is formidable analytically because of the fact that density in general is not constant but varies in space (or time in natural units). Usually one assumes adiabaticity: the variation of density is small over one oscillation length. This is the quasi-static approximation \[31\].

If we start with the most general form of mixing matrix which includes the CPV phases, we can argue that only one angle is sufficient to parameterize this matrix and the extra phase can always be gauged away by appropriate redefinition of the flavor states \[24, 32, 33\]. This can be shown explicitly as follows

\[
U = U_{\nu L}U_{l_L}^\dagger = \begin{pmatrix} \cos(\vartheta/2)e^{i\alpha} & \sin(\vartheta/2)e^{i\beta} \\ -\sin(\vartheta/2)e^{i\gamma} & \cos(\vartheta/2)e^{i(-\alpha+\beta+\gamma)} \end{pmatrix},
\]

If we define

\[
P_\nu = \begin{pmatrix} e^{-i\alpha} \\ e^{-i\gamma} \end{pmatrix}, \quad P_l = \begin{pmatrix} 1 \\ e^{i(-\alpha+\beta)} \end{pmatrix},
\]

then under a transformation \(U \rightarrow P_\nu U P_l^\dagger\), we can eliminate the three phases from the mixing matrix. The mass eigenstates are defined up to a phase and therefore we can redefine them \(\nu_{L,R} \rightarrow P_\nu \nu_{L,R}\) (Note that \(\nu_L\) is denoted by |\(\vartheta_i, \varphi_i, \pm\rangle\) where \(i = 1, 2\) runs over the number of generations) and similarly for \(l_{L,R} \rightarrow P_l l_{L,R}\). With the above transformation, the mass matrices remain real and unchanged. This leads to no observable CPV (Dirac type) in the two flavor case. But, a Majorana type CPV phase can survive even in the two flavor case \[32, 33, 34, 35\]. Note that in certain extended models, such as left right symmetric models and supersymmetric models, it is possible to have CPV only with two flavors. The CPV characteristic of Majorana phases can be probed in effects connected with the neutrino mass term such as neutrinoless double beta decay \[34\]. But the main point is that neutrino
oscillation experiments are insensitive to Dirac or Majorana nature of neutrinos \[32, 35\]. So, it suffices to take the mixing matrix to be real,

\[
\mathcal{U} = \begin{pmatrix}
\cos(\varphi/2) & \sin(\varphi/2) \\
-\sin(\varphi/2) & \cos(\varphi/2)
\end{pmatrix},
\]

without any loss of generality. This leads to no observable significance of intrinsic CPV phases as far as two flavor neutrino oscillation formalism is concerned \[35\]. Note that at any instant, we can always get rid of the intrinsic CPV phases in the mixing matrix.

However in case of an adiabatic evolution, is it still possible to define a gauge transformation which can gauge away the dependence of CPV phases. Let us ask if the transition probability for the two flavor neutrinos is at all sensitive to the CPV phases in presence of adiabatic evolution. In what follows, we consider neutrinos propagating through an arbitrary medium and obtain the expression for transition probability and analyse the effect of CPV phase in the Hamiltonian in Eq. (1).

To keep the discussion general, we start with a neutrino created as a flavor state \(|\nu_\alpha\rangle\) and detected as \(|\nu_\beta\rangle\). The state \(|\nu_\alpha\rangle\) is

\[
|\nu_\alpha\rangle = |\vartheta_1, \varphi_1, +\rangle + |\vartheta_1, \varphi_1, -\rangle,
\]

where \(|\vartheta_1, \varphi_1, \pm\rangle\) denote orthogonal pair of energy eigenstates of the Hamiltonian \(H(\vartheta_1, \varphi_1)\) (Eq. (1)). The two orthogonal mass states \(|\vartheta_1, \varphi_1, \pm\rangle\) evolve into \(|\vartheta_2, \varphi_2, \pm\rangle\) as a result of adiabatic evolution \[31\]

\[
|\vartheta_1, \varphi_1, \pm\rangle \rightarrow e^{-iD_{\pm}} |\vartheta_2, \varphi_2, \pm\rangle,
\]

with \(D_{\pm} = \pm \int_{t_0}^{t} \sqrt{x^2 + y^2 + z^2} \, dt' + \int_{t_0}^{t} r_0 \, dt'\) as the dynamical phases obtained from Eq. (1) where \(x, y, z\) for specific medium can be read off from Table 1.

This is the zeroth order adiabatic approximation. However the two states can also pick up a geometric contribution as they traverse the region between the source and the detector \[4\]. In general the geometric component picked up by the individual eigenstates under adiabatic evolution vanishes on account of evolution being along the geodesic curve\[7\]. But, a net geometric phase can appear if we can perform interference experiment between these energy eigenstates in the energy space. Neutrinos are produced and detected as flavor states which are superpositions of mass states, so this condition of realising interference in energy space is automatically achieved \[2\]. The appearance of geometric phase can be understood from the description of vertical lift \[36\] which accounts for the net non-zero geometric phase during any evolution (closed or open). Note that open loops can also be closed by connecting them by the shorter geodesic. The geometric phase is determined by the path of the state vector on the Poincaré sphere.

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\[7\] Except when the evolution along the geodesic curve happens to cross the antipodal point and the geometric phase then becomes \(\pi\).
The states $|\vartheta_1, \varphi_1, \pm\rangle$ and $|\vartheta_2, \varphi_2, \pm\rangle$ are connected by parallel transport rule on the Poincaré sphere. The two time evolved states $e^{-iD\pm}|\vartheta_2, \varphi_2, \pm\rangle$ form the final flavor state $|\nu_\beta\rangle$ at the detector

$$|\nu_\beta\rangle = \nu_{\beta,+}|\vartheta_2, \varphi_2, +\rangle + \nu_{\beta,-}|\vartheta_2, \varphi_2, -\rangle.$$  

Note that the initial and final flavor states are shown as red bullets in Figs. 1 and 2.

$|\vartheta_i, \varphi_i, \pm\rangle$ ($i = 1, 2$) correspond to the orthogonal pairs of mass (energy) eigenstates lying on distinct elliptic axes depending upon the values of $\vartheta_i, \varphi_i$ (shown as blue bullets in Figs. 1 and 2).

To illustrate this point of view, let us carefully examine the expression for transition probability along the lines of Ref. [2]. The probability for flavor transition $\nu_\alpha \rightarrow \nu_\beta$ is given by squaring the amplitude,

$$P(\nu_\alpha \rightarrow \nu_\beta) = |A(\nu_\alpha \rightarrow \nu_\beta)|^2$$

$$= \langle \nu_\alpha | \vartheta_1, \varphi_1, + \rangle \langle \vartheta_2, \varphi_2, + | \nu_\beta \rangle \langle \vartheta_2, \varphi_2, + | \nu_\beta \rangle \langle \vartheta_1, \varphi_1, + | \nu_\alpha \rangle$$

$$+ \langle \nu_\alpha | \vartheta_1, \varphi_1, - \rangle \langle \vartheta_2, \varphi_2, - | \nu_\beta \rangle \langle \vartheta_2, \varphi_2, - | \nu_\beta \rangle \langle \vartheta_1, \varphi_1, - | \nu_\alpha \rangle$$

$$+ \left[ \langle \nu_\alpha | \vartheta_1, \varphi_1, - \rangle e^{iD_--iD_+} \langle \vartheta_2, \varphi_2, - | \nu_\beta \rangle \langle \vartheta_2, \varphi_2, + | \nu_\beta \rangle e^{-iD_+} \langle \vartheta_1, \varphi_1, + | \nu_\alpha \rangle + c.c. \right].$$

The two direct terms correspond to classical probability and two cross terms contain the effect of geometric and dynamical phases. In the limit when coherence is lost (for instance when the path length is much larger than the oscillation length), the total probability reduces to the classical probability. The quantum mechanical effect of oscillation essentially lies in the cross terms.

At this juncture, we would like to introduce Pancharatnam’s idea of closed loop quantum collapses and the related geometric interpretation of the cross terms appearing in the expression of probability (Eq. (13)). Given any three rays $|\mathcal{A}\rangle$, $|\mathcal{B}\rangle$ and $|\mathcal{C}\rangle$ in the projective Hilbert space for a two state system (such that the neighbouring rays are non-orthogonal) the phase of the complex number $\langle \mathcal{A} | \mathcal{C} \rangle \langle \mathcal{C} | \mathcal{B} \rangle \langle \mathcal{B} | \mathcal{A} \rangle$ is given by $\Omega/2$ where $\Omega$ is the solid angle subtended by the geodesic triangle at the center of the Poincaré sphere. This excess phase is known as the Pancharatnam phase. Pancharatnam’s phase reflects the curvature of ray space and is independent of any parameterization or slow variation. Thus it can also appear in situations where the Hamiltonian is constant in time i.e. for the case of vacuum or constant density matter in the case of neutrino oscillations. Furthermore, note the fact that Schrödinger evolution (possibly) interrupted by measurements can lead to Pancharatnam’s phase. The crucial requirement for Pancharatnam’s phase to be well-defined is to have cyclic projection of (at least three) states and the consecutive collapses should be between nonorthogonal rays.

In Eq. (13) the cross term is related to the two path interferometer in energy space shown in Ref. [4]. The structure of the cross term (without the dynamical evolution) is a
series of cyclic quantum collapses with intermediate adiabatic evolutions given by $|\nu_\alpha\rangle \rightarrow |\vartheta_1, \varphi_1, +\rangle \rightarrow |\vartheta_2, \varphi_2, +\rangle \rightarrow |\nu_\beta\rangle \rightarrow |\vartheta_2, \varphi_2, -\rangle \rightarrow |\vartheta_1, \varphi_1, -\rangle \rightarrow |\nu_\beta\rangle$ which essentially maps a trajectory (as illustrated in Figs. 1 and 2) and subtends a solid angle of $\Omega$ at the center of the Poincaré sphere. The center of the Poincaré sphere is a singular point which is the same as the degeneracy point corresponding to the null Hamiltonian. So, the appearance of geometric phase is intimately connected to the existence of this singular point.

The interference term (without the dynamical phase) can be expressed as $re^{i\beta}$ and therefore picks up a phase which will be $\beta = \Omega/2$ (half the solid angle) due to Pancharatnam’s prescription. Note that even though the collapse process leads to a loss in probability ($r \neq 1$) the phase of the cross term remains unaltered whether we perform continuous evolution (unitary) or collapses (non-unitary) as long as it is cyclic.

The Pancharatnam geometric phase is given by

$$\beta = \text{Arg} \left[ \langle \nu_\alpha | \vartheta_1, \varphi_1, - \rangle \langle \vartheta_2, \varphi_2, - | \nu_\beta \rangle \langle \nu_\beta | \vartheta_2, \varphi_2, + \rangle \langle \vartheta_1, \varphi_1, + | \nu_\alpha \rangle \right]$$

A few remarks concerning the states $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ are in order. The states $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ can be any general flavor states. Pure flavor states ($|\nu_e\rangle$ and $|\nu_\mu\rangle$) are very special (real) linear superposition of mass eigenstates and correspond to those produced and detected via weak interactions. If either $|\nu_\alpha\rangle$ or $|\nu_\beta\rangle$ are identical to any one of the four eigenstates involved before and after the adiabatic evolution, the interference term vanishes and then it is meaningless to talk about the phase of such a term, whether dynamical or geometric. It is easy to understand this from Pancharatnam’s definition of parallelism between any two nonorthogonal states.

For the case of appearance probability $P(\nu_\alpha \rightarrow \nu_\beta)$ one gets a geometric phase $\beta^{\text{app}} = \Omega/2$. For $\nu_\beta = \nu_\alpha$, i.e. survival probability $P(\nu_\alpha \rightarrow \nu_\alpha)$ of the same flavor, it is easy to see that the path formed by cyclic collapses will again form a closed trajectory and subtend a non-zero solid angle at the origin. Therefore the interference term in survival probability will pick up a geometric phase which can be computed once we know the geometric phase in the case of appearance probability to be $\Omega/2$ by imposing unitarity which means that the interference terms should have a relative sign. Thus the cross term in survival probability will pick up a phase given by $e^{i\beta} = e^{i(\pi - \Omega/2)}$. Hence both appearance and survival probabilities will pick up different geometric phases

$$\beta^{\text{app}} = \Omega/2 \quad \text{and} \quad \beta^{\text{surv}} = \pi - \Omega/2.$$ 

Next we discuss the distinct physical situations that can arise. In Figs. 1 and 2, we pictorially depict the geometric differences in cross terms between the CPC and CPV cases both for constant and varying density situations. The CPC case was discussed in Ref. [2]. In the present study with CPV phases, the density and CPV phase are two parameters which can either be constant or varying with distance, we list three possible cases:

(A). Constant density case :- The pictorial depiction of the direction of collapse processes is shown in Fig. 4. In this case, we have four states (the two pair of mass and
Figure 1: Pictorial depiction of the cross terms (Eq. 13) in case of appearance and survival probability for the CPC case. The direction of collapse processes connecting the four (six) neutrino states appearing in constant (varying) density case in the cross term are represented on the Poincaré sphere by straight lines with arrows (see text). Note that the CPC mass states are labelled by $|\vartheta_i, \pm\rangle$ ($i = 1, 2$). Cases (a) and (b) correspond to constant density situation while (c) and (d) stand for varying density situation. The states $|\vartheta_2, \pm\rangle$ are obtained after adiabatically evolving $|\vartheta_1, \pm\rangle$. Adiabatic evolution between the mass states is shown by a dotted line while dashed lines refer to collapse processes.
Figure 2: Direction of collapse processes connecting the four (six) neutrino states in absence (presence) of variation of density (and CPV phases) corresponding to the cross terms in probability (Eq. (13)) are represented on the Poincaré sphere by straight lines with arrows (see text). Note that the in presence of CPV, $|\theta_i, \phi_i, \pm \rangle (i = 1, 2)$ are used to denote the mass states. Cases (a) and (b) correspond to appearance and survival probability in a CPV ($\varphi \neq 0$) but constant density situation. Cases (c) and (d) stand for appearance and survival probability in presence of CPV and varying density case. Adiabatic evolution between the mass states is shown by a dotted line while dashed lines refer to collapse processes.
flavor states being orthogonal) that will lie on a great circle as shown in Fig. 3 (a) and (b). By appropriate gauge transformations the great circle can be made to coincide with the equatorial great circle. The bottomline is that we can use real form of the leptonic mixing matrix (Eq. (9)) and the phase will remain topological as in the CPC case [2]. The Pancharatnam geometric phase is given by

$$\beta = \text{Arg}[\pm \cos^2(\vartheta/2)\sin^2(\vartheta/2)] = \pi, 0.$$  \hfill (16)

Using Pancharatnam’s prescription of quantum collapses, we have shown that in the two flavor context, CPV phase leaves no imprint of its existence in oscillation probability for the case of constant density matter, hence providing a purely geometric view of this result.

It is then natural to ask if CPV plays any relevant role in presence of adiabatic evolution or alternatively one can ask if it is possible to gauge away the CPV phase which itself is evolving in time (or distance). We again use Pancharatnam’s prescription to analyze this situation.

(B). Density and CPV phase varying with distance :- In this case, we have the complex Hamiltonian (Eq. (11)) and the eigenstates will not be real in general so the ray space is the full Poincaré sphere $S^2$ (instead of a great circle $S^1$) in contrast to the case (A) listed above (see Fig. 3 (c) and (d)). There are four inner products appearing in the cross term, $\langle \nu_\alpha \parallel \vartheta_1, \varphi_1, - \rangle$, $\langle \vartheta_2, \varphi_2, - \parallel \nu_\beta \rangle$, $\langle \nu_\beta \parallel \vartheta_2, \varphi_2, + \rangle$ and $\langle \vartheta_1, \varphi_1, + \parallel \nu_\alpha \rangle$ which are connected to the elements of the mixing matrix [2]. We saw that by a suitable gauge transformation, the mixing matrix can be made real for two flavor case (Eq. (9)). This implies that $|\nu_\alpha\rangle$ and $|\vartheta_1, \varphi_1, \pm\rangle$ can be chosen to be real and lying on the equatorial great circle ($x$-$z$ plane). However once this transformation is carried out, we do not have the freedom to choose the phases of the mass and flavor states any further. In general, the adiabatically evolved mass states $|\vartheta_2, \varphi_2, \pm\rangle$ will be complex. Also, the final flavor state $|\nu_\beta\rangle$ which will be a linear superposition of states $|\vartheta_2, \varphi_2, \pm\rangle$ with complex coefficients given by Eq. (11). The three states $|\nu_\beta\rangle$ and $|\vartheta_2, \varphi_2, \pm\rangle$ will also lie on a different great circle and this is clearly illustrated in Fig. 3 (c) and (d). The fact that the two great circles (one containing the initial flavor and mass states and the other one containing the final flavor and mass states) do not coincide will lead to a net geometric phase (which differs from topological phase obtained for the CPC case [2]).

Let us compute explicitly the cross term for this case. Two of the inner products are real and positive, i.e., $\langle \nu_\alpha \parallel \vartheta_1, \varphi_1, - \rangle = \sin \vartheta_1$ and $\langle \vartheta_1, \varphi_1, + \parallel \nu_\alpha \rangle = \cos \vartheta_1$. But the other two are complex, given by $\langle \vartheta_2, \varphi_2, - \parallel \nu_\beta \rangle = \cos \vartheta_2 e^{i(\alpha-\beta+\gamma)}$ and $\langle \nu_\beta \parallel \vartheta_2, \varphi_2, + \rangle = -\sin \vartheta_2 e^{-i\gamma}$. This implies that in general the geometric phase picked up will be given by

$$B_{\text{app}} = \pi - (\alpha - \beta) \quad \text{and} \quad B_{\text{surv}} = \pi - B_{\text{app}} = (\alpha - \beta).$$ \hfill (17)
In the limit, \((\alpha - \beta) \to 0\), the phase becomes topological and that would actually correspond to a constant density (CPC or CPV) situation.

Geometrically, we see that during the course of adiabatic evolution the mass state \(|\vartheta_2, \varphi_2, \pm \rangle\) gets lifted from one plane containing \(|\vartheta_1, \varphi_1, \pm \rangle\) and thus explore the full ray space instead of a great circle (with \langle \vartheta_2, \varphi_2, \pm | \vartheta_1, \varphi_1, \pm \rangle\) being complex in general). We have shown above that when presence of CPV induced by background is coupled with an adiabatic evolution with varying background density, the cross term of the probability (see Eq. (13)) will depend upon the extent of CPV or variation of \(\varphi_i\) under adiabatic evolution (\(\varphi_2 \neq \varphi_1\)). This leads to a geometric phase. Turning this argument around, we can conclude that CPV in the Hamiltonian can cause a deviation of \(e^{i\beta}\) from \(\pm 1\) in presence of an adiabatic evolution.

\(\text{(C). Density varying and CPV phase constant :-}\) In this case, all the six states involved in the collapse process with intermediate adiabatic evolutions will be confined to a great circle and the phase will be topological in nature. This will be identical to the CPC situation.

The key result of this article can be stated as follows. The topological phase obtained in Ref. [2] for CPC situation was robust and did not depend on any details of time evolution. We have shown that the presence of CPV in the Hamiltonian leads to a deviation of \(e^{i\beta}\) from \(\pm 1\) in presence of an adiabatic evolution iff the CPV phase also varies in space or in time.

4 The cross terms and the action of CP

Let us examine the form of cross terms appearing in general expression (Eq. (13)) for probability

\[
\mathcal{T} = \langle \nu_\alpha | \vartheta_1, \varphi_1, - \rangle e^{iD_-} \langle \vartheta_2, \varphi_2, - | \nu_\beta \rangle \langle \nu_\beta | \vartheta_2, \varphi_2, + \rangle e^{-iD_+} \langle \vartheta_1, \varphi_1, + | \nu_\alpha \rangle + \text{c.c.}
\]

\[
= [r e^{i\beta} e^{i(D_- - D_+)} + r e^{-i\beta} e^{-i(D_- - D_+)}
\]

\[
= 2r \cos[\beta + (D_- - D_+)]
\]

\[
= 2r [\cos \beta \cos(D_- - D_+) - \sin \beta \sin(D_- - D_+)]
\]

(18)

where \(r\) is the (real) amplitude of the cross term and \(r \leq 1\). Geometrically we can see that the amplitude \(r\) will not change under CP. Inner product of two rays is a measure of distance between the two rays and this distance will not change under CP (reflection) operation.

From the above expression (Eq. (18)), it is clear that for the special case when \(\beta = \pi\) or \(0\) i.e. when the phase is topological in character, \(\sin \beta \to 0\) and we have only the first term (cos \(\beta\) term) contributing to the probability. However, when the phase becomes geometric, we will have both the terms contributing to the cross term.
Now we discuss the effect of CP transformation on the expression for transition probability (Eq. (13)). CP transformation is an anti-unitary operation like the time-reversal operation and is given by

\[ \phi_i \rightarrow -\phi_i \quad (i = 1, 2) \]

\[ \Sigma^{CP} = \left[ (\nu_\alpha | \vartheta_1, -\varphi_1, -) e^{iD_-} (\vartheta_2, -\varphi_2, - | \nu_\beta) (\nu_\beta | \vartheta_2, -\varphi_2, +) e^{-iD_+} (\vartheta_1, -\varphi_1, + | \nu_\alpha) + \text{c.c.} \right] \]

\[ = \left[ re^{-i\beta} e^{i(D_+ - D_-)} + re^{i\beta} e^{-i(D_+ - D_-)} \right] \]

\[ = 2r \cos[(\beta) + (D_+ - D_-)] \]

\[ = 2r [\cos \beta \cos(D_- - D_+) + \sin \beta \sin(D_- - D_+)] \] \quad (19)

CP transformation connects two different Hamiltonians or two different backgrounds that are CP partners, connected by \( \phi_i \rightarrow -\phi_i \quad (i = 1, 2) \). If we look at the form of cross term (Eq. (18)) and CP-transformed cross term (Eq. (19)), we note that the cross term and hence the transition probability will in general be different for the two CP partners. For the special case, when the phase is topological (\( \sin \beta \rightarrow 0 \)) irrespective of the CP properties of the medium, the probability is invariant under CP. The cross term will no longer remain invariant if the presence of CPV is coupled with an adiabatic evolution as the phase \( \beta \) becomes geometric. Pairwise the two Hamiltonians (connected by CP transformation) are distinguishable at the level of probability and CPV coupled with adiabatic evolution plays a crucial role for this to happen.

We now discuss the implications of Wigner’s theorem and ray space isometries in the present context. We can draw a connection between the above results and those rigorously obtained in Ref. [37] for any quantum mechanical system. It was pointed out that Wigner’s theorem and the isometries of ray space can be most easily understood by using geometric ideas. The Pancharatnam phase \( \beta \) itself is not an invariant under isometries of the ray space, but \( \cos \beta \) is. So under isometries of the ray space, \( \beta \) can be preserved (unitary rotations) or reversed (anti-unitary reflections). Wigner\(^8\) proved that any ray space isometry can be realized on the Hilbert space of quantum mechanics by either a unitary or anti-unitary transformation. The only measurable quantity is the transition probability which is defined as an overlap between rays and measures the distance between them. This result is captured in Eqs. (18) and (19). In two flavor case, when the geometric phase is restricted to be topological (CPC case and CPV in constant density situation), the cross terms in the probability depend only on \( \cos \beta \) and we can not distinguish between unitary (\( \beta \rightarrow \beta \)) and anti-unitary (\( \beta \rightarrow -\beta \)) operations. But, in case the phase is geometric as in case of adiabatic evolution coupled with CPV, we have an extra term \( \propto \sin \beta \) which will change sign under CP transformation if CP is violated. In our present work, we have given a geometric visualization of the operation of CP in case of two flavor neutrino oscillations, which is anti-unitary. Thus, for the two state system, the Poincaré sphere description is quite rich and it encompasses the full set of possible operations, including unitary as well as anti-unitary operations.

\(^8\)A complete algebraic proof of Wigner’s theorem is given in Ref. [38].
5 Conclusion

We have shown that inclusion of CPV in the two flavor neutrino Hamiltonian can make the Pancharatnam phase geometric (and path-dependent) from topological, in case of adiabatic evolution of the mass eigenstates only when the CPV phases also vary with distance. The standard quantum mechanical treatment of two flavor neutrino oscillation contains Pancharatnam's phase which no longer remains topological when varying CPV phases and varying background density are taken into account. We have given a neat geometric illustration of the fact that the two flavor oscillation probability is sensitive to the presence of CPV phase in the Hamiltonian for the case adiabatically varying density profile.

Concerning the observability of the geometric phase, we should note that the cross term in the probability should be accessible to measurements. We mentioned two physical settings in Sec. 2 where the CPV phase could appear in the Hamiltonian.

In the first case of ordinary matter and CPV induced by NSI in propagation, we should note that the extrinsic CPV phases are due to the $\epsilon_{\alpha\beta}^{\delta}$ terms, which do not vary with distance. This would correspond to case (A) (if density is constant) or case (C) (if density is varying) described in Sec. 3. In both the cases, we have shown that the phase remains topological. A special case (R-parity violating supersymmetry) of non-standard interactions in ordinary matter was pointed out in Ref. [39] to show that in two flavor case, it is possible to have more than one essential parameters (electron, proton and neutron number densities) leading to non-zero Berry's [40] phase if the cyclic condition was also met in certain long baseline neutrino oscillation experiments. Note that there are other more exotic non-standard ways to get flavor off-diagonal and CPV terms leading to a modification of the dispersion relation in the neutrino Hamiltonian for example, violation of Lorentz invariance or CPT violation [41].

The second case discussed in Sec. 2 was that of dense neutrino backgrounds where CPV can appear even when SI are taken into consideration and this was solely due to the fact that background neutrinos in mixed state can induce CPV terms in the Hamiltonian. Neutrinos traversing dense supernova cores in a region outside the neutrinosphere pass through varying electron and neutrino number densities [3]. This could actually correspond to case B of Sec. 3 if the CPV phase was also varying with distance where the Pancharatnam prescription predicts a geometric phase. In case of neutrinos coming from Supernovae it is impossible to detect the effects due to CPV induced geometric phases since the oscillations would get averaged out by the time the neutrinos reach the detectors on the Earth and the cross term will not be accessible to measurements.

If in a hypothetical situation, it was possible to realise both varying CPV phases and varying density, then Pancharatnam's prescription predicts a geometric phase in the transition probability different from $\pi, 0$. The problem then reduces to disentangling the geometric component from the total phase in the transition probability. The dynamical phase depends on the energy while the geometric part is independent of energy. So, by carrying out measurements at different energy values, we can extract the geometric part from the total phase.
measured in oscillation experiments.

To conclude it turns out that with only two flavors the robustness of the topological phase is very hard to destroy even in the presence of CPV as long as the density and/or the CPV phases are constant. For the two physical settings discussed above, the phase remains topological whether CP is violated or not. Thus, it calls for a study of the full three flavor situation where the physics of geometric phase can play a role [12] due to the presence of intrinsic CPV phase in the leptonic mixing matrix.

Acknowledgments :- The author is deeply indebted to Joseph Samuel and Supurna Sinha for numerous useful discussions leading to the present work and critical comments on the manuscript. We acknowledge Michael V. Berry for helpful email correspondence. It is a pleasure to thank Sudhir K. Vempati for many valuable discussions related to neutrinos and Walter Winter for helpful discussions and comments. We thank Basudeb Dasgupta and Amol Dighe for discussion concerning supernova neutrinos.

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