Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops

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ABSTRACT: We present the results for three-loop gauge field anomalous dimensions in the SM calculated in the background field gauge within the unbroken phase of the model. The results are valid for the general background field gauge parameterized by three independent parameters. Both quantum and background fields are considered. The former are used to find three-loop anomalous dimensions for the gauge-fixing parameters, and the latter allow one to obtain the three-loop SM gauge beta-functions. Independence of beta-functions of gauge-fixing parameters serves as a validity check of our final results.

KEYWORDS: Renormalization Group, Standard Model

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1 Introduction

In spite of the fact that the Standard Model has many unsatisfactory aspects Nature still does not allow us to find some solid evidence for the existence of a more fundamental theory with new particles and/or interactions. Due to the joint efforts of both experimentalists and theoreticians we are about to enter the only unexplored part of the SM and unveil the mechanism of electroweak symmetry breaking. According to the recent experimental results, there is strong evidence for the existence of the Higgs boson, the last missing ingredient of the SM spectrum [1, 2].

The mass of the higgs seems to be located at the boundary of the so-called stability and instability regions [3–5] in the SM phase diagram (see Refs. [6–8] for recent studies). This fact implies that the SM can be potentially valid up to a very high scale (e.g., Plank scale).

In this situation, it is important to know how the running SM parameters evolve with energy scale. The analysis of high energy behavior is usually divided into two parts. The first one is the determination of running $\overline{\text{MS}}$-parameters from some (pseudo)observables. This procedure is usually referred to as “matching”. The second one utilizes renormalization group equations (RGEs) to find the corresponding values at some “New Physics” scale. In order to carry out such an analysis consistently one usually use $(L - 1)$-loop matching to find boundary conditions for $L$-loop RGEs (see, e.g., [9]). It is worth pointing that the advantage of the minimal-subtraction prescription lies in the fact that one needs to know only the ultraviolet (UV) divergent part of all the required diagrams. The latter has a simple polynomial structure in mass and momenta (once subdivergences are subtracted). Due to this, $\overline{\text{MS}}$ beta functions and anomalous dimensions can be relatively easily extracted from Green functions by solving a single scale problem with the help of the so-called infrared rearrangements (IRR) [10].

One- and two-loop results for SM beta functions have been known for quite a long time [11–21] and are summarized in [22]. Until recently, three-loop corrections were known only partially [23–28].
Having a well tested method for calculation of three-loop renormalization constants [29–31] and an experience in the calculations in the Standard Model and its minimal supersymmetric extension [32–34] we are planning to perform the calculation of all renormalization group coefficients in the third order of perturbation theory extending the results of Refs. [19, 35, 36] to one more loop.

In this paper, we present our first step in this direction: the results for three-loop anomalous dimensions of the SM gauge fields. Since we are only interested in UV-divergences for the fields and dimensionless parameters, we do not consider the effects related to spontaneous breaking of electroweak symmetry and, as a consequence, can neglect all dimensionful parameters of the model. Moreover, we made use of the background-field gauge (BFG) (see, e.g., Refs. [37, 38]) to carry out our calculation. In this gauge, due to the simple QED-like Ward identities involving background fields, one can easily obtain expressions for the beta-functions by considering the two-point functions with external background particles.

During the work on this project a few papers on the same topic appeared [39, 40] (gauge couplings) and [41] (Top Yukawa and higgs self-interactions). Since the authors of [39, 40] carried out a similar calculation, let us mention that our setup differs from that used in Ref. [39, 40] in several aspects.

Firstly, for the diagram generation we solely rely on FeynArts [42]. Since the diagrams are evaluated with the help of the MINCER package [43], a mapping to the MINCER notation for momenta is required. This problem was solved by hand with the help of the DIANA [44] topology files which were prepared during our previous calculations [29]. Based on these files a simple script was written which allows one to perform the mapping between the FeynArts and MINCER notation1.

Secondly, we do not consider the unbroken SM in a general Lorentz gauge, in which case we are forced to take into account vertex renormalization, but choose to work within the unbroken SM in a general background-field gauge. We keep the full dependence of the diagrams on the electroweak gauge-fixing parameters and take into account corresponding renormalization. Absence of these auxiliary parameters in the final expressions for beta-functions gives us an independent confirmation of the correctness of our calculation.

It is worth mentioning that in Refs. [39, 40] the SM in BFG was also considered. However, the corresponding calculation was carried out in the spontaneously broken phase and the model file distributed with FeynArts package was used. Since a consistent renormalization of the electroweak gauge-fixing parameters in the spontaneously broken phase requires a severe modification of corresponding part of the model file (see, e.g., Refs. [45–47]), the Landau gauge was chosen in [40] to avoid these kind of problems.

And lastly, since the unbroken SM in BFG is not implemented as a FeynArts model file, we are forced to use a package like FeynRules [48] or LanHEP [49]. Due to the fact that the authors are more accustomed to the latter, LanHEP was chosen to generate the required Feynman rules from the Lagrangian2.

1During the preparation of the final version of the paper a routine was written that automatically maps the FeynArts topologies onto that of MINCER.

2The authors of Refs. [39, 40] utilize FeynRules to obtain a model file for the unbroken SM.
The paper is organized as follows. In Section 2 we introduce our notation and present a brief description of the unbroken SM quantized in the background-field gauge. Section 3 describes the details of our calculation strategy. Finally, the results and conclusions can be found in Section 4. Appendix contains all the expressions for the considered renormalization constants.

2 The Standard Model in the unbroken phase. The background-field gauge

Let us briefly review the Lagrangian of the SM in the background-field gauge. We closely follow [50] albeit the fact that we introduce background fields only for gauge bosons. Moreover, as it was mentioned in Introduction, we neglect all the dimensionful couplings (i.e., mass parameters).

In our calculation we use the Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_H + \mathcal{L}_F + \mathcal{L}_{GF} + \mathcal{L}_{FP}. \tag{2.1}$$

Here $\mathcal{L}_G$ is the Yang-Mills part

$$\mathcal{L}_G = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{4} W_i^\mu W_i^\mu - \frac{1}{4} B_\mu B^\mu, \tag{2.2}$$

$$G_{\mu\nu}^a = \partial_{\mu} G_{\nu}^a - \partial_{\nu} G_{\mu}^a + g_s f^{abc} G_{\mu}^b G_{\nu}^c, \tag{2.3}$$

$$W_i^\mu = \partial_{\mu} W_i^\mu - \partial_{\nu} W_i^\nu + ig_2 \epsilon^{ijk} W_j^\mu W_k^\nu, \tag{2.4}$$

$$B_\mu = \partial_{\mu} B - \partial_{\nu} B^\nu, \tag{2.5}$$

where $G_{\mu}^a = \tilde{G}_{\mu}^a + \hat{G}_{\mu}^a (a = 1, \ldots, 8)$, $W_i^\mu = \tilde{W}_i^\mu + \hat{W}_i^\mu$, $(i = 1, 2, 3)$, and $B_\mu = \tilde{B}_\mu + \hat{B}_\mu$ are gauge fields for SU(3), SU(2) and U(1) groups. By $\mathcal{V} = (\hat{G}, \hat{W}, \hat{B})$ we denote quantum fields, and $\mathcal{V} = (\tilde{G}, \tilde{W}, \tilde{B})$ is used for their background counterpart. The corresponding gauge couplings are $g_s$, $g_2$, and $g_1$. The group structure constants enter into the commutation relations

$$[T^a, T^b] = if^{abc} T^c, \quad [\tau^i, \tau^j] = i\epsilon^{ijk} \tau^k, \tag{2.6}$$

with $T^a = \lambda^a/2$ and $\tau^i = \sigma^i/2$ being color and weak isospin generators.

The covariant derivative acting on a field which is charged under all the gauge groups looks like

$$D_\mu = \partial_\mu - ig_s T^a G_{\mu}^a - ig_2 \tau^i W_i^\mu + ig_1 \frac{Y_W}{2} B_\mu. \tag{2.7}$$

If a field is not charged under either group, the corresponding term is omitted. With the help of the covariant derivative one can write the following Higgs and fermionic parts of the Lagrangian:

$$\mathcal{L}_H = (D_\mu \Phi)^\dagger (D_\mu \Phi) - \lambda \left( \Phi^\dagger \Phi \right)^2, \tag{2.8}$$

$$\mathcal{L}_F = \sum_{i=1,2,3} \left( i \bar{Q}_i^L \tilde{D} Q_i^L + i \bar{L}_i^L \tilde{D} L_i^L + i \bar{u}_g^R \tilde{D} u_g^R + i \bar{d}_g^R \tilde{D} d_g^R + i \bar{t}_g^R \tilde{D} t_g^R \right) + \sum_{i,j=1,2,3} \left( Y_u^{ij} (Q_i^L \Phi^c) u_j^R + Y_d^{ij} (Q_i^L \Phi) d_j^R + Y_t^{ij} (L_i^L \Phi) t_j^R + \text{h.c.} \right). \tag{2.9}$$
where indices \( i, j = 1, 2, 3 \) count different fermion families, \( \lambda \) and \( Y_{u,d,l} \) are the higgs quartic and Yukawa matrices\(^3\), respectively. The left-handed quarks \( Q_L^g = (u_g, d_g)^L \) and leptons \( L_g^L = (\nu_g, l_g)^L \) form the SU(2) doublets while the right-handed quarks \( (u_R^g, d_R^g) \) and charged leptons \( l_R^g \) are the singlets with respect to SU(2). The Higgs doublet \( \Phi \) with \( Y_W = 1 \) has the following decomposition in terms of the component fields:

\[
\Phi = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}} (h + i\chi) \end{pmatrix}, \quad \Phi^c = i\sigma^2 \Phi^\dagger = \begin{pmatrix} -\phi^{-} \\ \frac{1}{\sqrt{2}} (h - i\chi) \end{pmatrix},
\]

(2.10)

Here a charge-conjugated Higgs doublet is introduced \( \Phi^c \) with \( Y_W = -1 \).

The gauge-fixing terms are introduced only for quantum fields

\[
\mathcal{L}_{GF} = -\frac{1}{2\xi_G} G_G^a G_G^a - \frac{1}{2\xi_W} C_W^i C_W^i - \frac{1}{2\xi_B} G_B^2,
\]

(2.11)

with

\[
G_G^a = \partial_\mu \tilde{G}_G^a + g_s f^{abc} \tilde{G}_G^b \tilde{G}_G^c, \\
C_W^i = \partial_\mu \tilde{W}_\mu^i + g_2 \epsilon^{ijk} \tilde{W}_\mu^j \tilde{W}_\mu^k, \\
G_B = \partial_\mu \tilde{B}_\mu.
\]

(2.12)

The ordinary derivatives are replaced by covariant ones containing the background fields. Due to this, the invariance of the effective action under background gauge transformations is not touched by introduction of (2.11).

The Fadeev-Popov part of the Lagrangian is given by

\[
\mathcal{L}_{FP} = -\epsilon_\alpha \delta G_\alpha \delta \theta^\beta c^\beta
\]

(2.13)

where \( \alpha, \beta = (G, W, B) \), and \( \delta G_\alpha / \delta \theta^\beta \) is the variation of gauge-fixing functions (2.12) under the following infinitesimal quantum gauge transformations

\[
\delta \tilde{G}_\mu^a = (D_\mu \theta_G)^a = \partial_\mu \theta_G^a + g_s f^{abc} G_\mu^b \theta_G^c, \\
\delta \tilde{W}_\mu^i = (D_\mu \theta_W)^i = \partial_\mu \theta_W^i + g_2 \epsilon^{ijk} W_\mu^j \theta_W^k, \\
\delta \tilde{B}_\mu = \partial_\mu \theta_B.
\]

(2.14)

It should be stressed that covariant derivatives in (2.14) involve the sum of quantum and background gauge fields \( V = \tilde{V} + \hat{V} \). The corresponding background transformations are obtained from (2.14) by the replacement \( V \rightarrow \hat{V} \).

The Feynman rules for the model described by the Lagrangian (2.1) were generated with the help of LanHEP \(^4\) [49].

It is worth mentioning here that our problem does not require the introduction of U(1) ghosts \( c_B, \bar{c}_B \) and background \( \hat{B} \) fields. This is due to the fact that the latter has the same interactions as its quantum counterpart \( B \) and the former decouples from other particles. Nevertheless, we keep them in our LanHEP model file to allow for possible generalizations to non-linear gauge-fixing as in Ref. [50].

\(^3\)In the actual calculation the diagonal Yukawa matrices were used. However, the result can be generalized with the help of additional tricks (see Sec.3 and Ref. [46]).

\(^4\)LanHEP 3.1.5, which was used by the authors, produces a wrong sign for the combination \( f^{abc} f^{dec} \) during export to the FeynArts model files. A new version with a fix is scheduled for November 2012.
3 Details of calculations

Due to the gauge invariance of the effective action for the background fields, QED-like Ward identities can be derived. The latter can be used to prove the following simple relations:

$$Z_{g_i} = Z_{\hat{V}_i}^{-1/2}, \quad i = 1, 2, 3$$  

(3.1)

with $Z_{\hat{V}_i}$ and $Z_{g_i}$ being renormalization constants for background fields $\hat{V}_i^\mu = (\hat{B}_\mu, \hat{W}_\mu, \hat{G}_\mu)$ and SM gauge couplings $g_i = (g_1, g_2, g_s)$, respectively.

Since we keep the full dependence on the gauge-fixing parameters $\xi_i$ during the whole calculation, we also need to know how $\xi_i = (\xi_B, \xi_W, \xi_G)$ are renormalized. Again, due to the Ward identities, the longitudinal part of the quantum gauge field propagators does not receive any loop corrections. As a consequence, the following identities hold:

$$Z_{\xi_i} = Z_{\tilde{V}_i}.$$  

(3.2)

Here $Z_{\xi_i}$ stands for the renormalization constants for the gauge-fixing parameters. The quantum gauge fields $\tilde{V}_i$ are renormalized in the $\overline{\text{MS}}$-scheme with the help of $Z_{\tilde{V}_i}$. It is clear from (3.1) and (3.2) that to carry out the calculation, one needs to consider gauge boson self-energies for both quantum $\tilde{V}$ and background $\hat{V}$ fields.

For calculation of the renormalization constants, following [26] (see also [10, 14, 51]), we use the multiplicative renormalizability of the corresponding Green functions. The renormalization constants $Z_V$ relate the dimensionally regularized one-particle-irreducible two-point functions $\Gamma_{V,\text{Bare}}$ with the renormalized one $\Gamma_{V,\text{Ren}}$ as:

$$\Gamma_{V,\text{Ren}} \left( Q^2, a_i \right) = \lim_{\epsilon \to 0} Z_V \left( \frac{1}{\epsilon}, a_i \right) \Gamma_{V,\text{Bare}} \left( Q^2, a_{i,\text{Bare}}, \epsilon \right),$$

(3.3)

where $a_{i,\text{Bare}}$ are the bare parameters of the model. For convenience, we introduce the following notation, which is closely related to that used in Ref. [40],

$$a_i = \left( \frac{5}{3} \frac{g_1^2}{16\pi^2}, \frac{g_2^2}{16\pi^2}, \frac{g_s^2}{16\pi^2}, \frac{Y_u^2}{16\pi^2}, \frac{Y_d^2}{16\pi^2}, \frac{Y_l^2}{16\pi^2}, \frac{\lambda}{16\pi^2}, \xi_G, \xi_W, \xi_G \right),$$

(3.4)

so we treat the gauge-fixing parameters along the same lines as couplings. Moreover, in the renormalization group analysis of the SM one usually employs the SU(5) normalization of the U(1) gauge coupling which leads to an additional factor $5/3$ in (3.4).

The bare parameters are related to the renormalized ones in the $\overline{\text{MS}}$-scheme by the following formula:

$$a_{k,\text{Bare}} \mu^{-2\rho_k\epsilon} = Z_{a_k} a_k(\mu) = a_k + \sum_{n=1}^{\infty} c^{(n)}_k \frac{1}{\epsilon^n},$$

(3.5)

where $\rho_k = 1/2$ for the gauge $(g_1, g_2, g_s)$ and Yukawa constants $(Y_u, Y_d, Y_l)$, $\rho_k = 1$ for the scalar quartic coupling constant $\lambda$, and $\rho_k = 0$ for the gauge fixing parameters. In order to extract a three-loop contribution to $Z_V$ from the corresponding self-energies, it is sufficient to know the two-loop renormalization constants for the gauge couplings and the one-loop results for the Yukawa couplings. This is due to the fact that the Yukawa vertices appear
for the first time only in the two-loop self-energies and the higgs self-coupling enters into
the result only at the third level of perturbation theory.

The four-dimensional beta-functions, denoted by \( \beta_i \), are defined via

\[
\beta_i(a_k) = \frac{d a_i(\mu, \epsilon)}{d \ln \mu^2} \bigg|_{\epsilon=0}.
\]

(3.6)

Here, again, \( a_i \) stands for both the gauge couplings and the gauge-fixing.

Given the fact that the bare parameters do not depend on the renormalization scale
the expressions for \( \beta_i \) can be obtained \[19\] by differentiation of (3.5) with respect to \( \ln \mu^2 \):

\[
- \rho_k \epsilon \left[ a_k + \sum_{n=1}^{\infty} \frac{c_k^{(n)}}{\epsilon^n} \right] = - \rho_k \epsilon a_k + \beta_k + \sum_{n=1}^{\infty} \sum_{l} (\beta_l - \rho_l \epsilon) \frac{\partial c_l^{(n)}}{\partial a_l} \frac{1}{\epsilon^n}.
\]

(3.7)

Taking in account only the leading order of the expansion in \( \epsilon \):

\[
\beta_k = \sum_l \rho_l a_l \frac{\partial c_l^{(1)}}{\partial a_l} - \rho_k \epsilon c_k^{(1)}.
\]

(3.8)

In MS-like schemes the renormalization constants for the Green functions may be
expanded as

\[
Z_\Gamma = 1 + \sum_{k=1}^{\infty} \frac{Z_k^{(k)}}{\epsilon^k}.
\]

(3.9)

Differentiating (3.9) with respect to \( \ln \mu^2 \) we simply get all-order expression for anomalous dimensions:

\[
\gamma_\Gamma \equiv - \mu^2 \frac{\partial \ln Z_\Gamma}{\partial \mu^2} = - \left[ \sum_j (\beta_j - \rho_j \epsilon) \frac{\partial Z_\Gamma}{\partial a_j} \right] Z_\Gamma^{-1}.
\]

(3.10)

It turns out that the above expression is finite as \( \epsilon \to 0 \) so

\[
\gamma_\Gamma = \sum_j a_j \rho_j \frac{\partial Z_\Gamma^{(1)}}{\partial a_j}.
\]

(3.11)

The advantage of (3.7) and (3.10) comes from the fact that it provides us with additional
confirmation of the correctness of the final result since beta functions and anomalous di-

mensions extracted directly from (3.7) and (3.10) are finite for \( \epsilon \to 0 \) only if \( c_k^{(n)} \) satisfy
the so-called pole equations \[52\], e.g.,

\[
\left[ \sum_l \rho_l a_l \frac{\partial}{\partial a_l} - \rho_k \right] c_k^{(n+1)} = \sum_l \beta_l \frac{\partial c_k^{(n)}}{\partial a_l}.
\]

(3.12)

In order to calculate the bare two-point functions for the quantum and background
fields, we generate the corresponding diagrams with the help of the FeynArts package \[42\].
It is worth pointing that we use the Classes level of diagram generation which allows us to
significantly reduce the number of generated diagrams since we do not distinguish fermion
generations. The complexity of the problem can be deduced from Table 1 that shows how
the number of the FeynArts generated diagrams increases with the loop level. Clearly, the presented numbers are an order of magnitude less than those given in Table I of Ref. [40], which somehow demonstrate the advantage of our approach.

The number of the SM fermion generations is introduced via counting fermion traces present in the generated expression for a diagram and multiplying it by \( n_G \). We separately count fermion traces involving the Yukawa interaction vertices and multiply them not by \( n_G \) but by \( n_Y \). This allows us to use the following substitution rules (c.f., [40]) to generalize the obtained expression to the case of the general Yukawa matrices

\[
\begin{align*}
  n_Y [a_u, a_d, a_l] & \to [\mathcal{Y}_u, \mathcal{Y}_d, \mathcal{Y}_l], \\
n_Y [a_u^2, a_d^2, a_l^2] & \to [\mathcal{Y}_{uu}, \mathcal{Y}_{dd}, \mathcal{Y}_{ll}], \\
n_Y^2 [a_u a_d, a_d a_l, a_u a_l] & \to [\mathcal{Y}_{ud}, \mathcal{Y}_{dl}, \mathcal{Y}_{ul}, \mathcal{Y}_{lu}], \\
n_Y a_u a_d & \to \mathcal{Y}_{ud} \quad (3.13)
\end{align*}
\]

where

\[
\begin{align*}
  \mathcal{Y}_u & = \frac{\text{tr} \, Y_u Y_u^\dagger}{16\pi^2}, & \mathcal{Y}_d & = \frac{\text{tr} \, Y_d Y_d^\dagger}{16\pi^2}, & \mathcal{Y}_l & = \frac{\text{tr} \, Y_l Y_l^\dagger}{16\pi^2}, \quad (3.14)
\end{align*}
\]

and

\[
\begin{align*}
  \mathcal{Y}_{uu} & = \frac{\text{tr} \, Y_u Y_u^\dagger Y_u Y_u^\dagger}{(16\pi^2)^2}, & \mathcal{Y}_{dd} & = \frac{\text{tr} \, Y_d Y_d^\dagger Y_d Y_d^\dagger}{(16\pi^2)^2}, \\
  \mathcal{Y}_{ud} & = \frac{\text{tr} \, Y_u Y_d Y_u Y_d^\dagger}{(16\pi^2)^2}, & \mathcal{Y}_{ll} & = \frac{\text{tr} \, Y_l Y_l^\dagger Y_l Y_l^\dagger}{(16\pi^2)^2}. \quad (3.15)
\end{align*}
\]

A comment is in order about the last substitution in (3.13). It turns out that \( \mathcal{Y}_{ud} \) is the only combination of up- and down-type Yukawa matrices, which can appear in the result for the three-loop gauge-boson self-energy within the SM. This can be traced to the following facts: 1) in the unbroken SM all the particles are massless so that chirality is conserved during fermion propagation; 2) only the Yukawa interactions flip the chirality of the incoming fermions; 3) there is no right-handed flavour changing current coupled to a SM gauge field. As a consequence, combinations like

\[
\text{tr} \, Y_u Y_d Y_u Y_d^\dagger \quad \text{and} \quad \text{tr} \, Y_d Y_u Y_d Y_u^\dagger
\]

which require at least two chirality-conserving transitions between right-handed up- and down-type quarks, do not show up in the result.

This type of counting is performed at the generation stage. A simple script converts the output of FeynArts to DIANA-like [44] notation and identifies MINCER topologies. This allows us to use the FORM [53] package COLOR [54] to do the SU(3) color algebra and MINCER [43] to obtain the \( \epsilon \)-expansion of diagrams. It is worth pointing that the expressions for all SM gauge couplings exhibit explicit dependence on number of colors \( N_c \), which stems from the fact that we have to sum over color when there is a (sub)loop with external color singlets coupled to quarks.
During our calculation we made use of naive anticommuting prescription for dealing with $\gamma_5$ (see, e.g., a nice review [60]). In this case, however, closed fermion loops with odd number of $\gamma_5$ ("odd traces") are not treated properly. In $D = 4$ such traces inevitably lead to the appearance of four-dimensional antisymmetric tensors $\epsilon_{\mu\nu\rho\sigma}$ that in the final result for a diagram should be contracted either between themselves or with external Lorentz indices and/or momenta. Since we are only interested in two-point functions the only non-zero combination that could potentially appear after loop integration is $\epsilon_{\mu\nu\rho\sigma}q^\mu q^\nu$ which originates at three loops from two odd traces. In this expression $q$ corresponds to external momentum, $\mu, \nu$ denote external Lorentz indices, and $\rho, \sigma$ are dummy indices representing the contractions due to internal vector boson propagators. A simple counting shows that both closed fermion lines are one-loop triangle (sub)graphs contributing to Adler-Bell-Jackiw gauge anomalies [55–57]. Having in mind the cancellation of such anomalies within the SM [58, 59], in our calculation we can safely put all the Dirac traces involving odd number of $\gamma_5$ to zero (see also the discussion in Ref. [40]). It is worth mentioning that the correct results for three-loop contribution to Yukawa coupling beta-functions [41] can not be obtained without special treatment of such traces.

### Table 1. Number of self-energy diagrams with external gauge fields, generated by FeynArts in the broken and unbroken SM, at one, two, and three loops.

|        | Broken $W^+/W^-$ | Unbroken $W^+$ |        | Broken $Z$ | Unbroken $Z$ |        | Broken $A$ | Unbroken $A$ |        | Broken $ZA$ | Unbroken $ZA$ |        | Broken $G$ | Unbroken $G$ |        | Total        | Total         |
|--------|-----------------|----------------|--------|-------------|---------------|--------|-------------|---------------|--------|-------------|---------------|--------|-------------|---------------|--------|-----------------|----------------|
| 1      | 10              | 11             | 4      | 9           | 11            | 4      | 7           | 4             | 7      | 4           | 4             | 4      | 4           | 4             | 4      | 37              | 36             |
| 2      | 339             | 389            | 6      | 281         | 371           | 6      | 218         | 66            | 218   | 67          | 67            | 66    | 67          | 67            | 66    | 1141            | 1113          |
| 3      | 21942           | 36647          |        | 19041       | 20144         |        | 14426       | 4060          |        | 16120       | 4183          |        | 3287        | 4060          |        | 74816           | 101137        |

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### 4 Results and conclusions

Here we present the results of our calculations in the form of the SM gauge beta-functions and anomalous dimension of the gauge-fixing parameters. From (3.1) and (3.2) it is clear that anomalous dimensions of the background fields are connected with the corresponding gauge coupling beta-functions

$$
\gamma_B = -\beta_1/a_1, \quad \gamma_W = -\beta_2/a_2, \quad \gamma_G = -\beta_s/a_s
$$

and for the quantum fields we have

$$
\gamma_B = \beta_{\xi_B}/\xi_B, \quad \gamma_W = \beta_{\xi_W}/\xi_W, \quad \gamma_G = \beta_{\xi_G}/\xi_G.
$$

The corresponding renormalization constants can be found in the Appendix.

At the end of the day, we have the following expressions for the beta-functions ($\lambda \equiv a_\lambda$):

$$
\beta_1 = a_1^2 \left( n_G \left( \frac{11}{45} N_c + \frac{3}{5} \right) + \frac{1}{10} \right)
$$
\[ \beta_2 = a_2^2 \left( n_G \left( \frac{N_c}{3} + \frac{1}{3} \right) - \frac{43}{6} \right) + a_2^2 \left( n_G \left( \frac{a_1 N_c}{60} + \frac{3 a_1}{20} + \frac{49 a_2 N_c}{12} + \frac{49 a_2}{12} + a_s C_F N_c \right) \right) + \frac{3}{10} a_1 \left( 259 a_2 - \frac{N_c Y_d}{2} - \frac{N_c Y_u}{2} + \frac{Y_l}{2} \right) + a_2^2 \left( n_G \left( \frac{287 a_1^2 N_c}{3600} - \frac{9 a_1^2}{400} + \frac{13 a_1 a_2 N_c}{240} + \frac{39 a_1 a_2}{80} - \frac{1}{2} a_1 a_s C_F N_c \right) \right) + \frac{1603 a_2^2 N_c}{27} + \frac{1603 a_2^2}{400} + \frac{13 a_2^2}{240} \left( a_s C_F N_c + \frac{133}{36} a_2^2 C_A C_F N_c - \frac{1}{2} a_2^2 C_F^2 N_c \right) + n_G^2 \left( \frac{121 a_1^2 N_c^2}{32400} - \frac{77 a_1^2 N_c}{1800} - \frac{33 a_1^2}{240} + \frac{415 a_2^2 N_c^2}{1200} - \frac{415 a_2^2 N_c}{240} - \frac{415 a_2^2}{432} \right) \]
\[\beta_s = a_s^2 \left( \frac{8TF_{NG}}{3} - \frac{11}{3} C_A \right) \]
\[+ a_s^2 \left( n_G - \frac{11 a_1 T_F}{15} + 3 a_2 T_F + \frac{40 a_s C_A T_F}{3} + 8 a_s C_F T_F \right) \]
\[+ \frac{34 a_s C_A^2}{3} - 4 T_F Y_d - 4 T_F Y_u \right) + a_s^2 \left( n_G - \frac{13 a_1^2 T_F}{60} - \frac{a_1 a_2 T_F}{20} \right) \]
\[+ \frac{22}{15} a_1 a_s C_A T_F - \frac{11}{15} a_1 a_s C_F T_F + \frac{7}{12} \right) \]
\[+ 6 a_2 a_s C_A T_F - 3 a_2 a_s C_F T_F + \frac{2830}{27} a_s^2 C_A^2 T_F + \frac{410}{9} a_s^2 C_A C_F T_F - 4 a_s^2 C_F T_F \right) \]
\[+ n_c^2 \left( - \frac{1331 a_s^2 T_F N_c}{8100} - \frac{121 a_s^2 T_F}{300} - \frac{11}{12} a_s^2 T_F N_c - \frac{11 a_s^2 T_F}{12} - \frac{632}{27} a_s^2 C_A T_F \right) \]
\[+ \frac{176}{9} a_s^2 C_F T_F^2 \right) \]
\[+ \frac{89 a_1 T_F Y_d}{20} - \frac{101 a_1 T_F Y_d}{20} - \frac{93 a_2 T_F Y_d}{4} \]
\[+ \frac{93 a_2 T_F Y_d}{4} \right) \]
\[+ \frac{2857}{54} a_s^2 C_A - 24 a_s C_A T_F Y_d - 24 a_s C_A T_F Y_d - 6 a_s C_F T_F Y_d - 6 a_s C_F T_F Y_u \]
\[+ 7 T_F N_c Y_d + 14 T_F N_c Y_d + 7 T_F N_c Y_u + 7 T_F Y_d Y_d + 9 T_F Y_d Y_u + 7 T_F Y_u Y_u \]
\[- 6 T_F Y_u Y_u \right), \quad (4.5) \]

\[\beta_{\xi_B} = \xi_B \left( - \frac{a_1}{10} + n_G \left( - \frac{3 a_1}{5} - \frac{11 a_1 N_c}{45} \right) \right) \]
\[+ \xi_B \left( - \frac{9 a_1^2}{50} - \frac{9 a_1 a_2}{10} + \frac{a_1 N_c Y_d}{6} + \frac{3 a_1 Y_d}{2} + \frac{17 a_1 N_c Y_u}{30} \right) \]
\[+ n_G \left( \frac{81 a_1^2}{100} - \frac{9 a_1 a_2}{20} - \frac{137 a_1^2 N_c}{300} - \frac{a_1 a_2 N_c}{20} - \frac{11 a_1 a_2 C_A N_c}{15} \right) \]
\[+ \xi_B \left( - \frac{489 a_2^2}{800} - \frac{783 a_2^2}{800} - \frac{3401 a_1 a_2}{800} - \frac{27 a_2^2 Y_d}{50} - \frac{9 a_1 a_2 Y_d}{10} + \frac{9 a_1 a_2 Y_d}{5} \right) \]
\[+ n_G \left( \frac{287 a_2^2}{2000} - \frac{27 a_2^2 a_1}{400} - \frac{1697 a_1^2 N_c}{8000} + \frac{a_1^2 a_2 N_c}{1200} + \frac{1}{45} a_1 a_2 Y_d \right) \]
\[+ \frac{137}{900} a_1 a_2 C_F N_c + \frac{1}{20} a_1 a_2 a_s C_F N_c - \frac{1463}{540} a_1 a_2 C_A C_F N_c + \frac{11}{30} a_1 a_2 C_F^2 N_c \right) \]
\[+ n_G^2 \left( \frac{981 a_2^2}{2000} + \frac{11 a_1 a_2}{80} + \frac{2837 a_1^2 N_c}{9000} + \frac{11 a_1 a_2 N_c}{72} + \frac{242}{135} a_1 a_2 C_F T_F N_c \right) \]
\[+ \frac{16577 a_2^2 N_c^2}{486000} + \frac{11 a_1 a_2 N_c^2}{720} + \frac{1267 a_2^2 N_c Y_d}{2400} + \frac{437}{160} a_1 a_2 N_c Y_d \]
\[+ \frac{17}{20} a_1 a_2 C_F N_c Y_d - \frac{17}{120} a_1 a_2 Y_d Y_d - \frac{61 a_1 N_c Y_d}{80} + \frac{2529 a_1^2 Y_d}{800} \]
\[+ \frac{1629 a_1 a_2 Y_d}{160} \right) \]
\[+ \frac{157}{60} a_1 N_c Y_d Y_d - \frac{99 a_1 Y_d}{40} - \frac{261 a_1 Y_d}{80} + \frac{2827 a_1^2 N_c Y_u}{2400} + \frac{157}{32} a_1 a_2 N_c Y_u \]
\[+ \frac{29}{20} a_1 a_2 C_F N_c Y_u - \frac{59}{60} a_1 N_c Y_d Y_u - \frac{199}{60} a_1 N_c Y_d Y_u - \frac{101}{120} a_1 N_c Y_u \]
\[+ 10 \]
\[
\beta_{\xi_W} = \xi_W \left( \frac{25 a_2}{6} - a_2 \xi_W + n_G \left( -\frac{a_2}{3} - \frac{a_2 N_c}{3} \right) \right) \\
+ \xi_W \left( -\frac{3 a_1 a_2}{10} + 113 a_2^2 + \frac{11 a_2^2 \xi_W}{2} - a_2 \xi_W + n_G \left( -3 a_1 a_2 \right) \right) \\
- \frac{13 a_2^2}{4} \left( \frac{a_1 a_2}{60} - \frac{a_2 N_c}{6} - \frac{a_2 a_s C_F N_c}{2} \right) + \frac{a_2 N_c \gamma_d}{4} + \frac{a_2 \gamma}{2} + \frac{a_2 N_c \gamma_u}{2} \\
+ \xi_W \left( -\frac{163 a_2^2}{1600} - \frac{33 a_1 a_2^2}{32} + \frac{143537 a_2^3}{576} - \frac{3 a_1 a_2^2}{10} - \frac{3 a_2 \lambda}{2} + 3 a_2 \lambda^2 \right) \\
- \frac{315 a_2^2 \xi_W}{8} - \frac{33 a_2^2 \xi_W}{4} - \frac{7 a_2^3 \xi_W}{4} + n_G \left( \frac{33 a_1^2 a_2}{400} + \frac{185 a_2^3}{144} + \frac{77 a_1^2 a_2}{1800} \right) \\
+ \frac{185 a_2^3 N_c}{72} + \frac{22}{9} a_2 a_2^2 C_F T_F N_c + \frac{121 a_1^2 a_2 N_c^2}{32400} + \frac{185 a_2^3 N_c^2}{144} + \frac{533}{480} a_2 a_2 N_c \gamma_d \\
+ \frac{79 a_2^3 \gamma_d}{32} - \frac{7 a_2 a_s C_F N_c \gamma_d}{8} - \frac{19 a_2 N_c \gamma_d}{16} + \frac{593}{480} a_2 a_2 N_c \gamma_u + \frac{79}{32} a_2^3 N_c \gamma_u \\
+ \frac{9 a_2 a_s C_F N_c \gamma_u}{8} - \frac{5 a_2 N_c \gamma_d \gamma_u}{8} - \frac{19 a_2 N_c \gamma_d \gamma_u}{16} + \frac{593}{480} a_2 a_2 N_c \gamma_u + \frac{79 a_2^3 N_c \gamma_u}{8} \\
- \frac{19 a_2 N_c \gamma_u}{16} - \frac{9 a_1 a_2^2 \xi(3)}{10} + \frac{3 a_2 \xi(3)}{2} - \frac{6 a_2 \xi_W \xi(3)}{2} - \frac{3 a_2^2 \xi_W \xi(3)}{2} \\
+ n_G \left( \frac{91 a_1 a_2^2}{400} + \frac{6 a_2^2 \xi_W}{144} + \frac{7025 a_2^3}{3600} + \frac{287 a_1 a_2 N_c}{300} \right) \\
- \frac{7025 a_2^3 N_c}{144} + \frac{1}{60} a_1 a_2 a_s C_F N_c + \frac{8 a_2^2 a_s C_F N_c}{36} + \frac{133 a_2^2 C_A C_F N_c}{36} \\
+ \frac{1}{2} a_2 a_s^2 C_F^2 N_c + 2 a_2 \xi_W N_c - \frac{9}{5} a_1 a_2^2 \xi(3) + 9 a_2^3 \xi(3) - \frac{1}{5} a_1 a_2^2 N_c \xi(3) \\
+ 9 a_2^3 N_c \xi(3) - 12 a_2^2 a_s C_F N_c \xi(3) \right), \tag{4.6}
\]

\[
\beta_{\xi_G} = \xi_G \left( \frac{13 a_2 C_A}{6} - \frac{a_2 C_A \xi_G}{2} - \frac{8 a_s T_F n_G}{3} \right) \\
+ \xi_G \left( \frac{59 a_2^2 C_A}{8} - \frac{11}{8} a_2^2 C_A^2 \xi_G - \frac{1}{4} a_2^2 C_A^2 \xi_G^2 + 4 a_s C_F \gamma_d + 4 a_s T_F \gamma_u \right) \\
+ \left( -\frac{11}{15} a_1 a_s T_F - 3 a_2 a_s T_F - 10 a_2^2 C_A T_F - 8 a_2^2 C_F T_F \right) n_G \\
+ \xi_G \left( \frac{9965 a_2^3 C_A^3}{288} - \frac{167}{32} a_2^3 C_A^3 \xi_G - \frac{33}{32} a_2^3 C_A^3 \xi_G^2 - \frac{7}{32} a_2^3 C_A^3 \xi_G^3 + n_G \left( \frac{121}{300} a_1^2 a_s T_F \right) \right) \\
+ \frac{11}{12} a_2^2 a_s T_F + \frac{304}{9} a_2^3 C_A T_F^2 + \frac{176}{9} a_2^3 C_F T_F^2 + \frac{1331 a_1^2 a_s T_F N_c}{8100} + \frac{11}{12} a_2^2 a_s T_F N_c \right) \tag{4.7}
\]
With the help of substitutions $C_A = N_c = 3$, $C_F = 4/3$, $T_F = 1/2$, $\gamma_a = \text{tr}\,\hat{T}$, $\gamma_d = \text{tr}\,\hat{B}$, $\gamma_i = \text{tr}\,\hat{L}$, $\gamma_{dd} = \text{tr}(\hat{B}^2)$, $\gamma_{uu} = \text{tr}(\hat{T}^2)$, $\gamma_{ll} = \text{tr}(\hat{L}^2)$, and $\gamma_{ud} = \text{tr}\,\hat{T}\hat{B}$ it is possible to show that the expressions presented above coincide with the results for the gauge beta functions obtained in Ref. [39].

As a consequence, one can be sure that the three-loop renormalization group equations obtained for the first time in Ref. [39] are correct and confirmed by an independent calculation. It is also worth mentioning that the obtained results can be used not only for the analysis of vacuum stability constraints within the SM (as in Refs. [6–8]) but also, e.g., for very precise matching of the SM with its supersymmetric extension since the corresponding three-loop renormalization group functions are already known from the literature [61–63]. Moreover, the leading two-loop decoupling corrections for the strongest SM couplings are also calculated within the MSSM in Refs. [32, 33, 64, 65].

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A Renormalization constants

Here we present the results for the renormalization constants from which the anomalous dimensions and beta-functions were extracted. It should be pointed out that the coefficients of the $\epsilon$-expansion satisfy the pole equations (3.12). The corresponding expressions together with the results for beta-functions can be found online\(^5\) in the form of Mathematica files.

\[
Z_{a_1} = 1 + a_1 \frac{1}{\epsilon} \left\{ n_G \left( \frac{11 N_c}{45} + \frac{3}{5} \right) + \frac{1}{10} \right\}
\]

\(^5\)As ancillary files of the arXiv version of the paper
\[ + a_1 \left\{ \frac{1}{\varepsilon} \left[ n_G^2 \left( \frac{121a_1N_c^2}{2025} + \frac{22a_1N_c}{75} + \frac{9a_1}{25} \right) + n_G \left( \frac{11a_1N_c}{225} + \frac{3a_1}{25} + \frac{a_1}{100} \right) \right] \right. \\
+ \frac{1}{\varepsilon} \left[ n_G \left( \frac{137a_1N_c}{1800} + \frac{81a_1}{200} + \frac{a_2N_c}{40} + \frac{9a_2}{40} + \frac{11a_sC_F N_c}{30} \right) + \frac{a_1}{100} + \frac{9a_2}{20} \\
- \frac{N_c Y_d}{12} - \frac{17N_c Y_u}{60} - \frac{3Y_u}{4} \right\} \right. \\
+ a_1 \left\{ \frac{1}{\varepsilon} \left[ n_G^3 \left( \frac{1331a_1^2N_c^3}{91125} + \frac{121a_1N_c^2}{1125} + \frac{33a_1^2N_c}{125} + \frac{27a_1}{125} \right) \right. \right. \\
+ n_G^2 \left( \frac{121a_1^2N_c^2}{6750} + \frac{11a_1N_c}{125} + \frac{27a_1}{125} \right) + n_G \left( \frac{11a_1N_c}{1500} + \frac{9a_2}{500} + \frac{a_1}{1000} \right) \right. \\
+ \frac{1}{\varepsilon} \left[ n_G \left( \frac{3731a_1N_c^2}{54000} + \frac{441a_1^2}{2000} + \frac{9a_1a_2N_c}{40} + \frac{117a_1a_2}{200} + \frac{a_1}{150}a_1a_sC_F N_c \right. \right. \\
- \frac{11}{270} a_1 N_c^2 Y_d - \frac{187}{15} a_1 N_c^2 Y_u - a_1 N_c Y_d - \frac{11}{2} a_1 N_c Y_d - \frac{17}{10} a_1 N_c Y_u - \frac{9a_1 Y_u}{10} \\
- \frac{7a_1^2 N_c}{720} - \frac{39a_1}{80} - \frac{121}{270} a_2^2 C_F C_F N_c \right] + n_G^2 \left( \frac{10549a_1^2N_c^2}{243000} + \frac{1519a_1^2N_c}{4500} \right) \right. \\
+ \frac{567}{1000} a_1^2 + \frac{11}{90} a_1 a_2 N_c^2 + \frac{7a_1a_2N_c}{50} + \frac{27a_1a_2}{100} + \frac{121}{675} a_1 a_s C_F N_c + \frac{11}{25} a_1 a_s C_F N_c \right. \\
+ \frac{a^2_2 N_c^2}{60} + \frac{a_2^2 N_c^2}{36} + \frac{a_2^2 N_c^2}{40} + \frac{44}{135} a_2^2 C_F T_F N_c \right. \\
+ \frac{21a_1^2}{1000} + \frac{9a_1a_2}{100} - \frac{11}{30} a_1 a_s C_F N_c \right. \\
+ \frac{2}{17} a_1 N_c Y_u + \frac{33a_1 Y_u}{80} - \frac{43a_1^2}{40} + \frac{a_2 N_c Y_d}{10} + \frac{17a_2 N_c Y_u}{10} + \frac{9a_2 Y_u}{16} \right. \\
+ \frac{1}{6} a_s C_F N_c Y_d + \frac{17}{30} a_s C_F N_c Y_u - \frac{N_c^2 Y_d^2}{36} - \frac{11}{90} N_c^2 Y_d Y_u - \frac{17N_c^2 Y_d^2}{180} \right. \\
- \frac{5N_c Y_d Y_u}{18} - \frac{31N_c Y_d Y_u}{24} + \frac{11N_c Y_u d}{60} - \frac{17N_c Y_u u}{120} - \frac{Y_d^2}{4} - \frac{3Y_u}{8} \\
+ \frac{1}{\varepsilon} \left[ n_G \left( \frac{1697a_1^2 N_c}{54000} + \frac{327a_1^2}{200} + \frac{a_1 a_2 N_c}{3600} - \frac{9a_1 a_2}{400} - \frac{137a_1 a_s C_F N_c}{2700} \right. \right. \\
+ \frac{a^2_2 N_c}{135} + \frac{9a^2_2}{10} - \frac{60}{60} a_2 a_s C_F N_c + \frac{1463a^2_2 C_F C_F N_c}{1620} - \frac{11}{90} a^2_2 C_F N_c \right. \\
+ n_G^2 \left( \frac{16577a^2_1N_c^2}{1458000} - \frac{2387a^2_1 N_c}{27000} - \frac{297a^2_1}{200} + \frac{11a^2_2 N_c}{2160} - \frac{11a^2_2 N_c}{240} - \frac{12a^2_2 N_c}{240} \right. \\
- \frac{242}{405} a^2_2 C_F T_F N_c \right. \\
+ \frac{163a^2_1}{800} + \frac{261 a_1 a_2}{800} + \frac{9a_1 \lambda}{50} - \frac{1267 a_1 N_c Y_d}{7200} \right. \\
- \frac{2827a_1 N_c Y_u}{7200} - \frac{843a_1 Y_u}{800} + \frac{340a_1^2}{960} + \frac{3a_2 \lambda}{10} - \frac{437a_2 N_c Y_d}{480} - \frac{157a_2 N_c Y_u}{96} \right. \\
- \frac{543a_1 Y_d}{160} - \frac{17}{60} a_s C_F N_c Y_d - \frac{29}{60} a_s C_F N_c Y_u - \frac{3 \lambda^2}{5} + \frac{17N_c^2 Y_d^2}{360} + \frac{59}{180} N_c^2 Y_d Y_u \right. \\
+ \frac{101 N_c^2 Y_u^2}{360} + \frac{157N_c Y_d Y_u}{180} + \frac{61N_c Y_u d}{240} + \frac{199N_c Y_u u}{180} + \frac{N_c Y_u d}{24} + \frac{113N_c Y_u u}{240} + \frac{33a_1^2}{40} + \frac{87Y_d}{80} \right) \right\}, \tag{A.1} \]
\[ Z_{a_2} = 1 + a_2 \frac{1}{c} \left\{ n_G \left( \frac{N_c}{3} + \frac{1}{3} \right) - \frac{43}{6} \right\} \\
+ a_2 \left\{ \frac{1}{\epsilon^2} \left[ n_G \left( \frac{a_2 N_c^2}{9} + \frac{2a_2 N_c}{9} + \frac{a_2}{9} \right) + n_G \left( -\frac{43a_2 N_c}{9} + \frac{43a_2}{9} \right) + \frac{1849a_2}{36} \right] + \left[ n_G \left( \frac{a_1 N_c}{120} + \frac{3a_1}{40} + \frac{49a_2 N_c}{24} + \frac{49a_2}{24} + \frac{a_s C_F N_c}{2} \right) \right] \right\} \\
+ \frac{3a_1}{20} \left\{ \frac{259a_2}{12} - \frac{N_c \gamma_d}{4} - \frac{N_c \gamma_u}{4} \right\} \right\} \\
+ a_2 \left( \frac{1}{\epsilon^3} \right) \left[ n_G \left( \frac{1849a_2 N_c}{36} + \frac{1849a_2^2}{36} \right) + n_G \left( -\frac{43}{18} a_2 N_c^2 - \frac{43a_2 N_c}{9} - \frac{43a_2^2}{18} \right) \right] \\
+ a_2 \left( \frac{1}{\epsilon} \right) \left[ \frac{7a_1 a_2 N_c}{360} - \frac{39a_1 a_2}{40} - \frac{22001a_2^2 N_c}{432} - \frac{22001a_2^2}{432} - \frac{43}{6} a_{a_s C_F N_c} - \frac{1}{6} a_2 N_c^2 \gamma_d \right] \\
- \frac{1}{6} a_2 N_c^2 \gamma_u - \frac{a_2 N_c \gamma_d}{6} - \frac{a_2 N_c \gamma_u}{6} - \frac{a_2 \gamma_d}{6} - \frac{a_2 \gamma_u}{6} - \frac{11}{18} a_{2s C_F N_c} \right\} \\
+ \frac{a_2^2}{18} \left( \frac{11a_1^2 N_c^2}{16200} + \frac{7a_1^2 N_c}{900} + \frac{3a_1^2}{200} + \frac{1}{18} a_{a_1 a_2 N_c^2} + \frac{a_{a_1 a_2 N_c}}{18} + \frac{a_{a_1 a_2}}{20} + \frac{343a_2 N_c^2}{216} \right) \\
+ \frac{343a_2 N_c}{108} + \frac{343a_2^2}{216} + \frac{1}{3} a_{a_2 C_F N_c^2} + \frac{1}{3} a_{a_2 C_F N_c} + \frac{4}{9} a_{2s C_F T_F N_c} \right\} + \frac{a_1^2}{200} \\
- \frac{3a_1 a_2}{20} + \frac{a_1 N_c \gamma_d}{48} + \frac{17a_1 N_c \gamma_u}{240} + \frac{3a_1 \gamma_l}{16} + \frac{77959a_2^2}{216} + \frac{181a_2 N_c \gamma_d}{48} \\
+ \frac{181a_2 N_c \gamma_u}{48} + \frac{181a_2 \gamma_l}{48} + \frac{1}{2} a_{a_2 C_F N_c \gamma_d} + \frac{1}{2} a_{a_2 C_F N_c \gamma_u} - \frac{N_c^2 \gamma_d^2}{12} - \frac{1}{6} a_{N_c^2 \gamma_d \gamma_u} \right\} \\
- \frac{N_c^2 \gamma_u^2}{12} - \frac{N_c \gamma_d \gamma_l}{8} - \frac{N_c \gamma_d \gamma_d}{8} - \frac{N_c \gamma_u \gamma_u}{8} - \frac{N_c \gamma_d \gamma_u}{8} - \frac{N_c \gamma_u \gamma_u}{8} - \frac{a_{1 \gamma l}}{8} \right\} \\
+ \frac{1}{\epsilon} \left[ n_G \left( \frac{287a_2^2 N_c}{10800} + \frac{91a_2^2}{1200} + \frac{13a_1 a_2 N_c}{720} + \frac{13a_1 a_2}{80} + \frac{1}{180} a_{a_1 a_2 C_F N_c} \right) \right] \\
+ \frac{a_1^2}{81} \left( \frac{1603a_2 N_c}{120} + \frac{1603a_2^2}{120} + \frac{13}{12} a_{a_2 C_F N_c} + \frac{133}{108} a_{2s C_A C_F N_c} - \frac{1}{6} a_{2s C_F T_F N_c} \right) \\
+ \frac{a_2^2}{27} \left( \frac{121a_2^2 N_c^2}{9720} - \frac{77a_2^2 N_c}{5400} - \frac{11a^2}{400} + \frac{415a_2 N_c^2}{1296} - \frac{415a_2 N_c}{648} - \frac{415a_2}{1296} \right) \\
- \frac{22}{27} a_{2s C_F T_F N_c} + \frac{163a_1^2}{4800} + \frac{187a_1 a_2}{160} + \frac{1}{10} \frac{a_{1 \gamma l}}{1440} - \frac{533a_1 N_c \gamma_d}{1440} - \frac{593a_1 N_c \gamma_u}{1440} \right\} \\
- \frac{17a_1 \gamma_l}{32} + \frac{66711a_2^2}{5184} + \frac{a_2 \gamma_l}{32} - \frac{81a_2 N_c \gamma_d}{32} - \frac{81a_2 N_c \gamma_u}{32} - \frac{81a_2 \gamma_l}{32} \right\} \\
- \frac{7}{12} a_{a_2 C_F N_c \gamma_d} - \frac{7}{12} a_{a_2 C_F N_c \gamma_u} - \frac{\lambda^2}{24} + \frac{5 N_c^2 \gamma_d^2}{24} + \frac{5 N_c^2 \gamma_d \gamma_u}{24} + \frac{5 N_c^2 \gamma_u^2}{24} \\
+ \frac{5 N_c \gamma_d \gamma_d}{48} + \frac{19 N_c \gamma_d \gamma_u}{12} + \frac{3 N_c \gamma_d \gamma_u}{8} + \frac{19 N_c \gamma_u \gamma_u}{48} \\
+ \frac{5 \gamma_d^2}{24} + \frac{19 \gamma_d \gamma_u}{48} \right\}, \quad (A.2)
\[ Z_{\alpha s} = 1 + a_s \frac{1}{\epsilon} \left\{ \frac{8 T_F n_G}{3} - \frac{11 C_A}{3} \right\} \\
+ a_s \left\{ \frac{1}{\epsilon^2} \left[ \frac{121 a_s C_A^2}{9} - \frac{176}{9} a_s C_A T_F n_G + \frac{64}{9} a_s T_F^2 n_G \right] \\
+ \frac{1}{\epsilon} \left[ n_G \left( \frac{11 a_1 T_F}{30} + \frac{3 a_2 T_F}{2} + \frac{20 a_s C_A T_F}{3} + 4 a_s C_F T_F \right) \\
- \frac{17 a_s C_A^2}{3} - 2 T_F \gamma_d - 2 T_F \gamma_u \right] \right\} \\
+ a_s \left\{ \frac{1}{\epsilon^2} \left[ - \frac{1331}{27} a_s^2 C_A^3 + \frac{968}{9} a_s^2 C_A^2 T_F n_G - \frac{704}{9} a_s^2 C_A T_F^2 n_G + \frac{512}{27} a_s^2 C_A T_F \right] \\
+ \frac{1}{\epsilon} \left[ n_G \left( \frac{11 a_1^2 T_F}{900} - \frac{121}{45} a_1 a_s C_A T_F - \frac{43 a_2^2 T_F}{12} - 11 a_2 a_s C_A T_F - \frac{2492}{27} a_s^2 C_A T_F \right) \\
- \frac{308}{9} a_s^2 C_A C_F T_F - \frac{32}{3} a_s T_F \gamma_d - \frac{32}{3} a_s T_F \gamma_u \right] \right\} \\
+ \frac{88}{45} a_1 a_s T_F^2 + \frac{1}{6} a_2^2 T_F N_c + \frac{a_2 T_F}{6} + 8 a_2 a_s T_F + \frac{1120}{27} a_s^2 C_A T_F^2 + \frac{224}{9} a_s^2 C_F T_F \right\} \\
+ a_1 T_F \gamma_d \gamma_u \left\{ \gamma_d \right\} + \frac{17 a_1 T_F^2}{30} + \frac{3 a_2 T_F}{2} + \frac{3 a_2 T_F}{2} + \frac{1309 a_s^2 C_A^2}{3} + \frac{44}{3} a_s C_A T_F \gamma_d \\
+ \frac{44}{3} a_s C_A T_F \gamma_u + 4 a_s C_F T_F \gamma_d + 4 a_s C_F T_F \gamma_u - \frac{2}{3} T_F N_c \gamma_d^2 - \frac{4}{3} T_F N_c \gamma_d \gamma_u \\
- \frac{2}{3} T_F N_c \gamma_u^2 - \frac{2 T_F \gamma_d \gamma_u}{3} - T_F \gamma_d \gamma_u - \frac{2 T_F \gamma_d \gamma_u}{3} + 2 T_F \gamma_u^2 \\
+ \frac{1}{\epsilon} \left[ n_G \left( - \frac{13 a_1^2 T_F}{180} - \frac{a_1 a_2 T_F}{60} + \frac{22}{45} a_1 a_s C_A T_F - \frac{11}{45} a_1 a_s C_F T_F + \frac{241 a_s^2 T_F}{36} \\
+ 2 a_2 a_s C_A T_F - a_2 a_s C_F T_F + \frac{2830}{81} a_s^2 C_A^2 T_F + \frac{410}{27} a_s^2 C_A C_F T_F - \frac{4}{3} a_s^2 C_F T_F \right) \\
+ n_G^2 \left( \frac{1331 a_1^2 T_F N_c}{24300} - \frac{121 a_1^2 T_F}{900} - \frac{11}{36} a_2^2 T_F N_c - \frac{11}{36} a_2 T_F^2 - \frac{632}{81} a_s^2 C_A T_F^2 \right) \\
- \frac{176}{27} a_s^2 C_A T_F^2 \right] \right\} + 2 T_F \gamma_d + 3 T_F \gamma_u \right\} , \tag{A.3} \]

\[ Z_{\xi s} = 1 + a_s \frac{1}{\epsilon} \left\{ n_G \left( - \frac{11 N_c}{45} - \frac{3}{5} - \frac{1}{10} \right) \right\} \\
+ a_s \left\{ \frac{1}{\epsilon} \left[ n_G \left( - \frac{137 a_1 N_c}{1800} - \frac{81 a_1}{200} - \frac{a_2 N_c}{40} - \frac{9 a_2}{40} - \frac{11 a_s C_F N_c}{30} \right) \right) \right\} \]
\[ Z_{\xi_W} = 1 + a_2 \left\{ \frac{1}{\epsilon} \left[ -\xi_W + n_G \left( -\frac{N_c}{3} - \frac{1}{3} \right) + \frac{25}{6} \right] \right\} \]

\[ + a_2 \left\{ \frac{1}{\epsilon^2} \left[ a_2 \xi_W^2 + n_G \left( \frac{a_2N_c}{3} + \frac{a_2\xi_W}{3} + \frac{a_2N_c}{2} + \frac{a_2}{2} \right) \right] - \frac{8a_2\xi_W}{3} - \frac{25a_2}{4} \right\} \]

\[ + \frac{1}{\epsilon} \left\{ n_G \left( -\frac{a_1}{120} - \frac{3a_1}{4} - \frac{13a_2N_c}{8} - \frac{13a_2}{8} - \frac{a_2C_F N_c}{2} \right) \right\} \]

\[ - \frac{3a_2}{20} - a_2 \xi_W^2 - \frac{11a_2\xi_W}{4} + \frac{113a_2}{8} + \frac{N_c}{4} + \frac{N_c\xi_d}{4} + \frac{3\xi_d}{4} \right\} \]

\[ + a_2 \left\{ \frac{1}{\epsilon} \left[ -\frac{a_2^2}{3} + n_G \left( -\frac{1}{3} \frac{a_2^2N_c}{3} - \frac{1}{3} \frac{a_2^2\xi_W}{3} - \frac{5}{6} \frac{a_2^2\xi_W}{2} N_c - \frac{5a_2^2\xi_W}{6} \right) \right. \right. \]

\[ - \frac{43a_2^2 N_c}{18} - \frac{43a_2^2}{18} + \frac{7a_2^2\xi_W}{6} + \frac{89a_2^2\xi_W}{12} + \frac{n_G^2 \left( \frac{a^2N_c}{18} + \frac{a_2^2 N_c}{9} + \frac{a_2^2}{18} \right) \right) \]

\[ + \frac{1525a_2^2}{72} \right\} \]
\[
Z_{\xi G} = 1 + a_s \left\{ \frac{1}{\epsilon} \left[ -\frac{C_A \xi G}{2} + \frac{13 C_A}{6} - \frac{8 T_F n G}{3} \right] \right\} + a_s \left[ \frac{1}{\epsilon^2} \left( \frac{1}{4} a_s C_A^2 \xi G^2 - \frac{17}{24} a_s C_A^2 - \frac{13 a_s C_A^2}{8} \right) \right] + n_G \left( \frac{4}{3} a_s C_A \xi G T_F + 2 a_s C_A T_F \right) + n_G \left( \frac{4}{3} a_s C_A \xi G T_F + 2 a_s C_A T_F \right) + n_G \left( \frac{4}{3} a_s C_A \xi G T_F + 2 a_s C_A T_F \right) + n_G \left( \frac{4}{3} a_s C_A \xi G T_F + 2 a_s C_A T_F \right)
\]
\[
\begin{align*}
&\frac{1}{\epsilon} \left[ n_G \left( -\frac{11a_1T_F}{30} - \frac{3a_2T_F}{2} - 5a_sC_A T_F - 4a_sC_T F \right) \right] \\
&\quad - \frac{1}{8}a_sC_{A\xi}\left( -\frac{11a_1C\xi}{16} - \frac{59a_sC}{16} + 2T_FY_d + 2T_FY_u \right) \right) \\
&\quad + \frac{1}{\epsilon^2} \left[ \frac{1}{3} \left( -\frac{1}{8}a_sC_{A\xi}\left( + \frac{1}{6}a_s^2C_{A\xi} + \frac{47}{48}a_s^2C_{A\xi} + \frac{403a_s^2C}{144} \right) \\
&\quad + \frac{1}{3} \left( -\frac{2}{3}a_s^2C_{A\xi} - \frac{5}{3}a_s^2C_{A\xi} - \frac{44}{9}a_s^2C_{A\xi} + \frac{16}{9}a_s^2C_{A\xi} \right) \right) \\
&\quad + \frac{1}{\epsilon^2} \left[ n_G \left( -\frac{11a_1^2T_F}{900} + \frac{11}{60}a_1a_sC_A\xi T_F + \frac{11}{60}a_sC_A T_F + \frac{43a_sT_F}{12} \right) \\
&\quad + \frac{3}{4}a_2a_sC_A\xi T_F + \frac{3}{4}a_2a_sC_A T_F + \frac{1}{3}a_s^2C_{A\xi} T_F + \frac{19}{6}a_s^2C_{A\xi} T_F \right) \right] \\
&\quad + \frac{27}{4}a_s^2C_A T_F + \frac{2sC_A C\xi T_F}{6} - \frac{62}{9}a_s^2C_A T_F \right) + \frac{7}{48a_s^2C_{A\xi}} + \frac{13}{24}a_s^2C_{A\xi} \right) \\
&\quad - \frac{143}{96}a_s^2C_{A\xi} - \frac{7957a_s^2C_A}{864} - a_s^2C_A T_F Y_d - a_s^2C_A T_F Y_u - a_s^2C_A T_F Y_d \right) \\
&\quad - a_s^2C_A T_F Y_u - 4a_s C_F T_F Y_d - 4a_s C_F T_F Y_u + \frac{2}{3}T_F N_c Y_d^2 + \frac{4}{3}T_F N_c Y_u \right) \\
&\quad + \frac{2}{3}T_F N_c Y_u^2 + \frac{2T_F T_Y Y_d^2 + \frac{2}{3}T_F T_Y Y_u^2 - 2T_F Y_d + T_F Y_d}{6} \right) \right] \\
&\quad + \frac{1}{\epsilon^2} \left[ n_G \left( \frac{13a_1^2T_F}{180} + \frac{a_1a_2T_F}{60} - \frac{22}{15}a_1a_sC_A T_F \right) \right] \right) \right] \\
&\quad + \frac{1}{45}a_1a_sC_A T_F + \frac{241a_1T_F}{36} - 6a_2a_sC_A T_F + \frac{9}{8}a_2a_sC_A T_F + a_2a_sC_F T_F \right) \\
&\quad + \frac{4}{3}a_s^2C_{A\xi} + \frac{12a_s^2C_A T_F}{3} + \frac{911}{27a_s^2C_A T_F} - \frac{16a_s^2C_A + 16a_s^2C_{A\xi} T_F}{3} \right) \\
&\quad - \frac{5}{27}a_s^2C_A C_F T_F + \frac{4}{3}a_s^2C_{A\xi} T_F + \frac{\xi^2}{1331a_1^2T_F N_c}{24300} + \frac{121a_1^2T_F}{24300} + \frac{11a_1a_sC_A T_F}{900} + \frac{11a_1a_sC_A T_F}{900} + \frac{31a_1a_sC_A T_F}{360} \right) \\
&\quad + \frac{11a_s^2C_A T_F}{36} + \frac{304}{27}a_s^2C_A + \frac{27}{6a_s^2C_A T_F} + \frac{176}{27}a_s^2C_{A\xi} T_F + \frac{89a_1T_F Y_d}{60} + \frac{101a_1T_F Y_u}{60} \right) \\
&\quad + \frac{31a_2T_F Y_d}{6} + \frac{31a_2T_F Y_u}{6} - \frac{7}{96}a_s^2C_{A\xi} - \frac{1}{16}a_s^2C_{A\xi} - \frac{1}{16}a_s^2C_{A\xi} - \frac{11}{32}a_s^2C_{A\xi} \right) \\
&\quad - \frac{1}{4}a_s^2C_{A\xi} T_F (3) - \frac{167}{96}a_s^2C_{A\xi} - \frac{3}{16}a_s^2C_{A\xi} - \frac{9965a_s^2C_A}{864} + \frac{250}{6}a_s^2C_{A\xi} T_F \right) \right] \\
&\quad + \frac{25}{6}a_1a_sC_A T_F Y_u + 2a_s C_F T_F Y_d + 2a_s C_F T_F Y_u - \frac{7}{3}T_F N_c Y_u^2 + \frac{14}{3}T_F N_c Y_d Y_u \right) \right] \\
&\quad - \frac{7}{3}T_F N_c Y_u^2 - \frac{77T_F Y_d Y_u}{3} - 3T_F Y_d - \frac{77T_F Y_d Y_u}{3} \right) + 2T_F Y_{uu} \right] \right) \right].
\end{align*}
\]
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