Chaotic Quantization of Classical Gauge Fields

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Abstract

We argue that the quantized non-Abelian gauge theory can be obtained as the infrared limit of the corresponding classical gauge theory in a higher dimension. We show how the transformation from classical to quantum field theory emerges, and calculate Planck’s constant from quantities defined in the underlying classical gauge theory.

1 Introduction

Although much progress has been made in recent years, the question, how gravitation and quantum mechanics should be combined into one consistent unified theory of fundamental interactions, is still open. Superstring theory [1], which describes four-dimensional space-time as the low-energy limit of a ten- or eleven-dimensional theory (“M-theory” [2]), may provide the correct answer, but the precise form and content of the theory is not yet entirely clear. It is therefore legitimate to raise the question whether the fundamental description of nature at the Planck scale is really quantum mechanical, or whether the underlying theory could be a classical extension of general relativity. This questions was initially raised by ’t Hooft, who has argued that quantum mechanics can logically arise as low-energy limit of a microscopically deterministic, but macroscopically dissipative theory [3].

It is the goal of this manuscript to present an explicit example that shows how (Euclidean) quantum field theory can emerge in the infrared limit of a higher-dimensional,

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classical field theory. It is well known that relativistic quantum field theory in (3+1)-dimensional Minkowski space can be obtained by analytic continuation (“Wick rotation”) of the analogous statistical field theory defined on a four-dimensional Euclidean space. In fact, this concept provides the only known mathematically rigorous definition of interacting quantum field theories. Physical observables, such as vacuum expectation value of self-adjoint operators, can be reliably calculated in the Euclidean path integral formulation of the quantum field theory. This method has been extensively used to obtain numerical solutions of many relativistic quantum field theories.

We here show that in some cases, specifically for non-Abelian gauge fields, the functional integral of the three-dimensional Euclidean quantum field theory arises naturally as the long-distance limit of the corresponding classical gauge theory defined in (3+1)-dimensional Minkowski space. Because of the general nature of the mechanism underlying this transformation, for which we have coined the term chaotic quantization, it is expected to work equally well in other dimensions. For example, the four-dimensional Euclidean quantum gauge theory arises as the infrared limit of the (4+1)-dimensional classical gauge theory. We emphasize that the dimensional reduction is not caused by compactification; the classical field theory does not exhibit periodicity either in real or imaginary time.

The mechanism discussed below can be viewed as a physical analogue of the well-known technique of stochastic quantization [4]. In both cases, the quantum fluctuations arise from the stochastic noise in a higher-dimensional theory. However, chaotic quantization differs from stochastic quantization in two essential aspects: It only applies to certain field theories, including non-Abelian gauge theories, and it allows us to calculate Planck’s constant $\hbar$ in terms of fundamental physical quantities of the underlying higher-dimensional classical field theory. Accordingly, chaotic quantization provides a physical mechanism generating the quantum mechanics of fields and particles, while stochastic quantization is generally regarded as a convenient calculational technique, but not as a physical principle.

## 2 Chaoticity of Classical Yang-Mills Fields

The chaotic nature of classical non-Abelian gauge theories was first recognized twenty years ago [5, 6]. Over the past decade, extensive numerical solutions of spatially varying classical non-Abelian gauge fields on the lattice have revealed that the gauge field has positive Lyapunov exponents that grow linearly with the energy density of the field configuration and remain well-defined in the limit of small lattice spacing $a$ or weak-coupling [7, 8, 9]. More recently, numerical studies have shown that the (3+1)-dimensional classical non-Abelian lattice gauge theory exhibits global hyperbolicity. This conclusion is based on calculations of the complete spectrum of Lyapunov exponents [10] and on the long-time statistical properties of the Kolmogorov-Sinai (KS) entropy of the classical SU(2) gauge theory [11].

These results imply that correlation functions of physical observables decay rapidly,
and that long-time averages of observables for a single initial gauge field configuration are identical to their microcanonical phase-space average, up to Gaussian fluctuations which vanish in the long-time limit as $t_s^{-1/2}$, where $t_s$ is the observation time. Since the relative fluctuations of extensive quantities scale as $L^{-3/2}$, the microcanonical (fixed-energy) average can be safely replaced by the canonical average when the spatial volume probed by the observable becomes large. In the following we discuss the hierarchy of time and length scales on which this transformation occurs.

According to the cited results, the classical non-Abelian gauge field self-thermalizes on a finite time scale $\tau_{eq}$ given by the ratio of the equilibrium entropy and the KS-entropy, which determines the growth rate of the course-grained entropy:

$$\tau_{eq} = S_{eq}/h_{KS}.$$  \hfill (1)

At weak coupling, the KS-entropy for the $(3 + 1)$-dimensional SU(2) gauge theory scales as

$$h_{KS} \sim g^2 E \sim g^2 T (L/a)^3,$$  \hfill (2)

where $E$ is the total energy of the field configuration and $T$ is the related temperature defined by

$$E = T^2 \partial Z / \partial T.$$  \hfill (3)

Here $Z(T)$ is the partition function of the classical gauge field regularized by the lattice cut-off $a$. The equilibrium entropy of the lattice is independent of the energy and proportional to the number of degrees of freedom of the lattice: $S_{eq} \sim (L/a)^3$. The time scale for self-equilibration is thus given by

$$\tau_{eq} \sim (g^2 E a^3 / L^3)^{-1} \sim (g^2 T)^{-1}.$$  \hfill (4)

When one is interested only in long-term averages of observables, it is thus sufficient to consider the thermal classical gauge theory on a three-dimensional spatial lattice. Note that, although we are interested only in the long-distance properties of the quasi-thermalized field, we have to define the classical gauge field on a lattice rather than in the continuum. Due to the nonlinear interactions most of the energy contained in the initial field configuration ultimately cascades into modes with wavelengths near the ultraviolet cutoff $a$, and the limit of vanishing lattice spacing $a$ is not well defined. Without the lattice cutoff, we would be unable to replace the microcanonical average by a thermal average, because the cascade toward the ultraviolet would not end and no stationary limit would exist. We shall see later that the lattice cutoff $a$ also takes on an important physical role, as it enters into the definition of Planck's constant $\hbar$.

\footnote{In the convention adopted here, $g$ is the coupling strength of the classical Yang-Mills theory with dimension (energy $\times$ length)$^{-1/2}$. This choice ensures the proper dimensionality of the Yang-Mills action.}

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3 Chaoticity of Classical Yang-Mills Theory

The long-distance dynamics of non-Abelian gauge theories at finite temperature $T$ is known to reduce to the dynamics of the static chromomagnetic sector of the gauge field \[12\]. The scale beyond which this dimensional reduction is valid is given by the magnetic length scale $d_{\text{mag}} \sim (g^2 T)^{-1}$, where $g$ is the classical gauge coupling. The magnetic length scale is the same for the classical and the quantized gauge theories at finite temperature. $d_{\text{mag}}$ is independent of the lattice cutoff $a$. It is well recognized that the static magnetic sector in the thermal quantum field theory is essentially classical in nature and depends on $\hbar$ only via the scale of the thermal effective gauge coupling $g(T)$.

It is worth noting that in spite of many similarities between the thermal classical field theory and the thermal quantum field theory, there are major differences. The ultraviolet properties of the quantum field theory at finite temperature are controlled by the thermal length scale $d_{\text{th}} \sim \hbar / T$, which is a basically quantum mechanical concept. In the thermal classical field theory, the lattice spacing $a$ serves as ultraviolet regulator. Similarly, the electric screening length of the thermal quantum field theory is $d_{\text{el}} \sim \sqrt{\hbar / gT}$. In the thermal classical field theory electric fields are screened on the length scale $d_{\text{el}} \sim \sqrt{a / g^2 T}$. Only the magnetic length scales are equal for classical and quantal gauge fields. The inverse electric screening length is proportional to the plasma frequency $\omega_{\text{pl}}$ governing propagating long-wavelength modes. The damping of these plasma modes is of the order $\gamma_{\text{pl}} \sim d_{\text{mag}}^{-1}$, rendering the dynamics of the thermal gauge field purely dissipative and noisy on distances larger than $d_{\text{mag}}$.

The dynamic properties of thermal non-Abelian gauge fields at such long distances have been studied in much detail \[14, 15, 16\]. It is now understood that the real-time dynamics of the gauge field at such scales can be described, at leading order, by a Langevin equation

$$\sigma \frac{\partial A}{\partial t} = -D \times B + \xi,$$

where $D$ is the gauge covariant spatial derivative, $B = D \times A$ is the magnetic field strength, and $\xi$ denotes Gaussian distributed (white) noise with zero mean and variance

$$\langle \xi_i(x,t) \xi_j(x',t') \rangle = 2\sigma T \delta_{ij} \delta^3(x - x') \delta(t - t').$$

Here $\sigma$ denotes the color conductivity \[17\] of the thermal gauge field which is determined by the ratio $\omega_{\text{pl}}^2 / \gamma_{\text{th}}$ of the plasma frequency $\omega_{\text{pl}}$ and the damping rate $\gamma_{\text{th}}$ of a thermal gauge field excitation.

At leading logarithmic order in the quantum field theory, the color conductivity satisfies

$$\sigma^{-1} \sim \frac{\hbar}{T} \ln[d_{\text{mag}} / d_{\text{el}}].$$

\(^2\)The need for this length scale in the derivation of the Stefan-Boltzmann radiation law motivated Planck in 1900 to postulate the existence of the quantum of action $\hbar$. 

The derivation of the Langevin equation (5) for the classical thermal gauge theory proceeds completely in parallel to that for the quantum field theory, except that the plasma frequency \( \omega_{pl}^2 \sim g^2 T/a \), and the ratio between the magnetic and electric length scales depends on the combination \( g(Ta)^{1/2} \) instead of \( gh^{1/2} \). The separation of length scales requires \( g^2 Ta \ll 1 \). The color conductivity then scales as

\[
\sigma^{-1} \sim a \ln[d_{mag}/d_{el}].
\] (8)

This relation implies that the color conductivity is an ultraviolet sensitive quantity, which depends on the lattice cutoff. The various relations derived in this section are summarized in Table 1, where the results for the thermal quantum field theory are listed in parallel with those of the classical gauge theory.

4 Dimensional Reduction and Quantization

We will now show that an observer restricted to long distances in three-dimensional Euclidean space would interpret the dynamics of the classical gauge field as that of a quantum field in its vacuum state with the Planck constant

\[
h_3 = a T.
\] (9)

The starting point of our argument is the well-established fact that the random Gaussian process defined by the Langevin equation (5) generates three-dimensional field configurations with a probability distribution \( P[A] \) determined by the Fokker-Planck equation

\[
\sigma \frac{\partial}{\partial t} P[A] = \int d^3 x \frac{\delta}{\delta A} \left( T \frac{\delta P}{\delta A} + \frac{\delta W}{\delta A} P[A] \right).
\] (10)

Here \( W[A] \) denotes the magnetic energy functional

\[
W[A] = \int d^3 x \frac{1}{2} B(x)^2.
\] (11)

Any non-static excitations of the magnetic sector of the gauge field, i.e. magnetic fields \( B(k) \) not satisfying \( k \times B = 0 \), die away rapidly on a time scale of order \( \sigma/k^2 \), where \( k \) denotes the wave vector of the field excitation. For observers sensitive only to distances much larger than \( d_{mag} \) and times much longer than \( \sigma d_{mag}^2 \), measurements of the magnetic field yield averages with the equilibrium weight given by the stationary solution of the Fokker-Planck equation (10):

\[
P_0[A] = e^{-W[A]/T}.
\] (12)

The three-vector \( B_i \sim \epsilon_{ijk} F_{jk} \) incorporates all components of the field strength tensor \( f_{jk} \) in three dimensions. \( W/T \) is now identified as the three-dimensional action \( S_3 \) measured in units of Planck’s constant \( h_3 \):

\[
W/T = S_3/h_3,
\] (13)
where
\[ S_3[A] = -\frac{1}{4} \int dx_3 \int d^2x f^{ik} f_{ik}. \] (14)

Dimensional reasons require a rescaling of the gauge field strength with the fundamental length scale
\[ f^{ik} = \sqrt{a} F^{ik}. \] (15)

That the lattice spacing \( a \) is the proper rescaling parameter is seen by noting that the lattice versions of the two-dimensional and the three-dimensional integrals
\[ \int d^n x \rightarrow a^n \sum x \] (16)
differ by a factor \( a \). Together with the relations (11), (13), and (14) this reasoning demonstrates that \( \hbar_3 = aT \), as stated at the beginning of this section. The rescaling of the gauge field also fixes the three-dimensional coupling constant
\[ g_3^2 = \frac{g^2}{a} = \frac{g^2 T}{\hbar_3}, \] (17)
so that
\[ \sqrt{a}(\partial A + gA \times A) = (\partial A_{(3)} + g_3 A_{(3)} \times A_{(3)}). \] (18)

According to (10), an observer confined to the measurement of long-time and long-distance averages of microscopic observables associated with the classical gauge field measures the same values as would an observer “living” in the three-dimensional Euclidean world in the presence of a quantized gauge field in its vacuum state. It is important to note that this correspondence is not induced by a compactification of the Minkowskian time coordinate. There is no true thermal bath of gauge fields in the original Minkowski space theory, and the quasi-thermal solution of the (3+1)-dimensional classical field theory does not satisfy periodic boundary conditions in imaginary time.

The effective dimensional reduction found here is not caused by the discreteness of the excitations with respect to the time-like dimension, but by the dissipative nature of the (3+1)-dimensional dynamics. Magnetic field configurations satisfying \( D \times B = 0 \) can be thought of as low-dimensional attractors of the dissipative motion, and the chaotic dynamical fluctuations of the gauge field around the attractor can be consistently interpreted as quantum fluctuations of a vacuum gauge field in 3-dimensional Euclidean space.

We thus see that the mechanism of dimensional reduction discussed above is distinct from the mechanism that operates in thermal quantum field theories. In fact, the dimensional reduction by chaotic fluctuations and dissipation does not occur in scalar field theories, because – even in cases that exhibit chaos, such as two quarticly coupled scalar fields – there is no dynamical sector that survives after long-time averaging. Quasi-thermal fluctuations generate a dynamical “mass” for the scalar field(s)
and thus eliminate any arbitrarily slow field modes. In the case of gauge fields, the transverse magnetic sector is protected by the gauge symmetry, and it is this sector which survives the time average, without any need for fine-tuning of the microscopic theory.

5 General Considerations

It is worthwhile to review the essential ingredients of chaotic quantization. First, the underlying classical field theory must contain strongly coupled massless degrees of freedom. Such theories are generally chaotic at the classical level \[20\]. When observations are restricted to the infrared degrees of freedom, this corresponds to a coarse graining of the dynamical system, leading to strongly dissipative long-distance dynamics. The coupling to the short-distance modes generates uncorrelated noise, and the coarse-grained system obeys a dissipation-fluctuation theorem \[18\].

Second, the classical field theory at finite temperature must have degrees of freedom that remain unscreened. This condition generally requires the presence of a symmetry, such as gauge invariance. It is a reasonable expectation that such symmetries occur in any unified theory containing general relativity. The requirement also provides a simple and consistent explanation for the empirical fact that all fundamental interactions are described by gauge fields.

Our example for the chaotic quantization of a three-dimensional gauge theory in Euclidean space raises a number of questions:

1. Does the principle of chaotic quantization generalize to higher dimensions, in particular, to quantization in four dimensions?
2. Can the method be extended to describe field quantization in Minkowski space?
3. What type of deviations from the standard quantum field theory are caused by the existence of a microscopic classical dynamics?

The first question is most easily answered. As long as globally hyperbolic classical field theories can be identified in higher dimensions, our proposed mechanism should apply. Although we do not know of any systematic study of discretized field theories in higher dimensions, a plausibility argument can be made that Yang-Mills fields exhibit chaos in \((4+1)\) dimensions. For this purpose, we consider the infrared limit of a spatially constant gauge potential, as studied in refs. \[5, 6, 19\]. For the SU(\(N\)) gauge field in \((D+1)\) dimensions in the \(A_0 = 0\) gauge, there are \(3(N^2 - 1)\) interacting components of the vector potential and \(3(N^2 - 1)\) canonically conjugate momenta (the components of the electric field) that depend only on the time coordinate. The

\[3\] An exception may be the case where the excitation energy of the scalar field is just right to put the quasi-thermal field at the critical temperature of a second-order phase transition, where arbitrarily slow modes exist as fluctuations of the order parameter.
remaining gauge transformations and Gauss’ law allow to eliminate \((N^2 - 1)\) degrees of freedom from each. Next, rotational invariance in \(D\) dimensions permits to reduce the number of dynamical degrees of freedom by twice the number of generators of the group \(\text{SO}(D)\), i.e. by \(D(D-1)\). This leaves a \((D-1)(2N^2 - 2 - D)\)-dimensional phase space of the dynamical degrees of freedom and their conjugate momenta. For the dynamics to be chaotic, this number must be at least three. For the simplest gauge group \(\text{SU}(2)\), this condition permits infrared chaos in \(2 \leq D \leq 5\) dimensions, including the interesting case \(D = 4\). Higher gauge groups are needed to extend the chaotic quantization scheme to gauge fields in \(D > 5\) dimensions. Of course, this reasoning does not prove full chaoticity of the Yang-Mills field in these higher dimensions, it just indicates the possibility. Numerical studies will be required to establish the presence of strong chaos in these classical field theories.

The second question is more difficult to address. A formal answer would be that the Minkowski-space quantum field theory can (and even must) be obtained by analytic continuation from the Euclidean field theory. Any observable in the Minkowski-space theory that can be expressed as a vacuum expectation value of field operators can be obtained in this manner. If this argument appears somewhat unphysical, one might consider a completely different approach, beginning with a chaotic classical field theory defined in \((3+2)\) dimensions. Field theories defined in spaces with two time-like dimensions were first proposed by Dirac in the context of conformal field theory \([21]\) and have recently been considered as generalizations of superstring theory \([22]\). In that case, the reduction to one time dimension is achieved by gauge fixing. In the present case, the physical time dimension may be defined as the coordinate orthogonal to the total 5-momentum vector \(P^\mu\) of the initial field configuration.

In the presence of two time-like dimensions, “energy” becomes a two-component vector \(\vec{E}\) that is a part of the \((D + 2)\)-dimensional energy-momentum vector. If we select an initial field configuration with energy \(E_0\vec{n}\), where \(\vec{n}\) is a two-dimensional unit vector, this choice defines a preferred time-like direction \(\vec{n}\) in which the field thermalizes. Conservation of the energy-momentum vector ensures that the total energy component orthogonal to \(\vec{n}\) always remains zero. In this sense, the choice of an initial field configuration corresponds to a spontaneous breaking of the global \(\text{SO}(D,2)\) symmetry down to a global \(\text{SO}(D,1)\) symmetry. Whether this leads to an effective quantum field theory in \((D + 1)\) dimensional Minkowski space, remains to be investigated.

Finally, it is interesting to ask which deviations from the quantum field theory could be detected by a “slow” observer by means of very precise measurements. Clearly, an observer able to resolve the dynamics on the thermal or electric length scales of the underlying classical field theory would observe deviations from the di-
mensionally reduced vacuum field theory. For a space-time volume of linear dimension \( L \), the amplitude of the fluctuations is of the order \( (g^2 T L)^{-2} \). If, as one might suspect, the relevant microscopic length scale is of the order of the Planck scale, \( g^2 T \sim M_P \), quantities sensitive to the fluctuations around the infrared dynamics are suppressed by \( (M_P L)^2 \). For presently accessible length scales, the suppression factor is at least \( 10^{-34} \), and even smaller in low-energy precision tests of quantum mechanics. However, in principle, tests of Bell’s inequality in systems prepared with strong correlations on short time and distance scales can be used to establish at least an upper bound for the scale at which the transition from the classical dissipative dynamics to the quantum dynamics occurs.

It is a natural question to ask whether the mechanism of chaotic quantization outlined above corresponds to a hidden parameter theory of quantum mechanics. The answer is obviously positive, as the microscopic state of the system in a higher dimension is always precisely and deterministically defined. However, it is important to realize that the impossibility of hidden parameter descriptions of quantum mechanics is restricted to local theories, while our proposed mechanism operates in a higher dimensional space. A local dynamical theory in more dimensions generates fundamentally non-local effects in the lower-dimensional space. One can speculate that the time-scale associated with dimensional reduction, \( (g^2 T)^{-1} \), is the time for the collapse of the wave function. Our analysis predicts that this is the time required to average over the noise generated by the classical dynamics and to establish the stationary distribution of the Fokker-Planck equation (10).

6 Summary and Conclusions

Let us summarize. We have shown that a homogeneously excited, classical field theory in four dimensions can generate a three-dimensional Euclidean quantum field theory. Whereas the classical theory appears thermal for a four-dimensional observer, the three-dimensional observer experiences quantum fields at zero temperature. The essential feature facilitating this transformation is that the underlying deterministic theory contains a mechanism for information loss [3, 23, 24], here realized through its chaotic dynamics.

We can go further and speculate that the randomness caused by this intrinsic chaoticity of the underlying theory could generally lead to a reduction of the effective space-time dimensionality of the theory. In our example, the dimensional reduction is an effect of the quasi-thermal fluctuations. Another related, well-known phenomenon is the dimensional reduction of spin systems in arbitrarily weak, random magnetic fields [25], which finds its explanation in the hidden supersymmetry of the system [26].

We note that symmetries and physical laws may arise naturally from some essentially random dynamics, rather than being postulated from the beginning [27, 28, 29]. The goal of the program formulated in our example, is more restricted: microscopic
randomness is utilized as a foundation for “large-scale” physics that is described by quantum mechanics.

It is not clear whether the mechanism presented here for non-Abelian gauge fields is also at work in general relativity. Examples of chaotic behavior have been identified in the dynamics of classical gravitational fields [30, 31]. It has been found that the chaotic nature of the solutions may depend on the number of dimensions. A famous case is the evolution toward the singularity in the Bianchi type IX geometry, where the chaotic oscillatory approach changes to a monotonic approach in more than 10 dimensions [32, 33].

It has not been demonstrated that chaoticity is a general property of solutions of Einstein’s equations. This may not even be required, because an entirely different mechanism of information loss may be at work in general relativity than in Yang-Mills theory. Indeed, ’t Hooft has speculated that black hole formation may be the mechanism operating in the case of gravity [3]. Finally, our example does not contain fermion fields. It would be interesting to extend our study to supersymmetric theories, some of which have been shown to exhibit chaos in the infrared limit [34].

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**Appendix**

Here we present a qualitative analysis of the various length and time scales of a thermal Yang-Mills field in $D = (d + 1)$ space-time dimensions, for $d \geq 2$. The results are accurate up to logarithmic corrections that arise for various quantities in some dimensions. We decompose the field into Fourier components (suppressing vector and color indices):

$$A(x, t) = V^{1/2} \int d^d k a(k) e^{i k x - i \omega_k t}. \quad (19)$$

Equi-partition of the energy over the classical modes then implies that

$$|a(k)|^2 = T \omega_k^{-2}. \quad (20)$$

The presence of an “external” static color potential $A^0$ induces a polarization density

$$\rho_{pol} \sim g^2 V^{-1} \int d^d x A(x, t)^2 A^0 \sim g^2 A^0 \int d^d k |a(k)|^2 \sim g^2 A^0 \int d^d k \omega_k^{-2}. \quad (21)$$

With the ultraviolet lattice cut-off $k \leq a^{-1}$, one obtains:

$$d_{cl}^{-2} = \rho_{pol}/A_0 \sim g^2 T a^{2-d}. \quad (22)$$
The field theory can be considered to be in the weak coupling regime, when the electric screening length is much larger than the lattice constant $a$. This amounts to the condition
\[ \tilde{g}^2 \equiv g^2 Ta^{d-1} = (a/d_{el})^2 \ll 1, \tag{23} \]
which defines the effective weak coupling parameter $\tilde{g}$ of the $(d+1)$-dimensional classical Yang-Mills theory. With the help of this parameter, the electric screening length can be expressed simply as $d_{el} = a/\tilde{g}$.

The coupling constant of the dimensionally reduced quantum field theory in $D-1 = d$ dimensional Euclidean space is given by
\[ g^2_{D-1} = g^2 a^{-1} (Ta) = g^2 T, \tag{24} \]
independent of the number $d$ of space dimensions.

The transport coefficient describing color conductivity $\sigma$ is obtained by considering the polarization current induced by a constant electric field $E$. Schematically, it is given by
\[ j_{pol} = g^2 V^{-1} \int d^d x dt A(x, t)^2 E \sim g^2 E \int d^d k |a(k)|^2 \gamma(k)^{-1}, \tag{25} \]
where $\gamma(k)$ is the damping rate of a thermal field mode. This damping rate can be calculated in the standard way using the formula $\gamma = \sigma_{coll} n_{th}$, where $\sigma_{coll}$ denotes the “cross section” for a thermal excitation, and $n_{th}$ stands for the density of hard thermal excitations. In $d$ spatial dimensions one finds
\[ \sigma_{coll} \sim g^4 Ta \int d^{d-1} q (q^2 + d_{el}^{-2})^{-2} \sim g^4 Ta d_{el}^{5-d}. \tag{26} \]
The classical formula contains an additional factor $(Ta)$ describing the enhancement due to the classical occupation of thermal modes of the gauge field. The density of thermal excitations is
\[ n_{th} \sim \int d^d k |a(k)|^2 \omega_k \sim Ta^{1-d}. \tag{27} \]
Since $\gamma$ does not depend on $k$ in this approximation, we can pull it out of the integral over $k$ in (25) to obtain:
\[ j_{pol} \sim g^2 E \gamma^{-1} \int d^d k |a(k)|^2 \sim g^2 E \gamma^{-1} Ta^{2-d}. \tag{28} \]
Combining the expressions (23–28) we finally get the desired expression for the color conductivity:
\[ \sigma = j_{pol} / E \sim g^2 a / \sigma_{coll} \sim \alpha_{el}^{d-5} (g^2 T)^{-1}. \tag{29} \]
Specifically, in $d = 4$ spatial dimensions, the result is
\[ \sigma \sim a (g^2 T)^{-3/2}. \tag{30} \]

All results are summarized, and compared to the results obtained in the $(d+1)$-dimensional thermal quantum field theory, in Table 2.
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Table 1: Comparison of length scales in the classical and quantum Yang-Mills theories: hard thermal scale $d_{th}$, electric scale $d_{el}$, magnetic scale $d_{mag}$, plasma frequency $\omega_{pl}$, damping rate $\gamma_{th}$, and color conductivity $\sigma$.

| | Quantum field theory | Classical field theory |
|---|---------------------|------------------------|
| $d_{th}$ | $\hbar T^{-1}$ | $a$ |
| $d_{el}$ | $\hbar^{1/2}(gT)^{-1}$ | $(g^2 T/a)^{-1/2}$ |
| $d_{mag}$ | $(g^2 T)^{-1}$ | $(g^2 T)^{-1}$ |
| $d_{mag} \gg d_{el}$ | $g^2 h \ll 1$ | $g^2 T a \ll 1$ |
| $\omega_{pl}$ | $(gT)^2/\hbar$ | $g^2 T /a$ |
| $\gamma_{th}$ | $g^2 T \ln[d_{mag}/d_{el}]$ | $g^2 T \ln[d_{mag}/d_{el}]$ |
| $\sigma$ | $\omega_{pl}^2 /\gamma_{th}$ | $\omega_{pl}^2 /\gamma_{th}$ |

Table 2: Characteristic scales for classical and quantized thermal Yang-Mills theories in $D = (d + 1)$ space-time dimensions: weak coupling parameter $\tilde{g}$, electric screening length $d_{el}$, magnetic length scale $d_{mag}$, coupling constant of the dimensionally reduced quantum field theory $g_{D-1}$, and color conductivity $\sigma$.

| | Quantum field theory | Classical field theory |
|---|---------------------|------------------------|
| $\tilde{g}^2$ | $g^2 T^{d-3} \hbar^{4-d}$ | $g^2 T a^{4-d}$ |
| $d_{el}$ | $\hbar / (\tilde{g} T)$ | $a / \tilde{g}$ |
| $d_{mag}$ | $\hbar / (\tilde{g}^2 T)$ | $a / \tilde{g}^2$ |
| $g_{D-1}^2$ | $g^2 T / \hbar$ | $g^2 / a$ |
| $\sigma$ | $d_{el}^{d-5} (g^2 T)^{-1}$ | $d_{el}^{d-5} (g^2 T)^{-1}$ |