Random matrix averages and the impenetrable Bose gas in Dirichlet and Neumann boundary conditions

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The density matrix for the impenetrable Bose gas in Dirichlet and Neumann boundary conditions can be written in terms of \( \prod_{l=1}^{n} | \cos \phi_1 - \cos \theta_l | | \cos \phi_2 - \cos \theta_l | \), where the average is with respect to the eigenvalue probability density function for random unitary matrices from the classical groups \( Sp(n) \) and \( O^+(2n) \) respectively. In the large \( n \) limit log-gas considerations imply that the average factorizes into the product of averages of the form \( \prod_{l=1}^{n} | t - x_l |^{2q} \) over the Jacobi unitary ensemble from random matrix theory. The latter task is accomplished by a duality formula from the theory of Selberg correlation integrals, and the large \( n \) asymptotic form is obtained. The corresponding large \( n \) asymptotic form of the density matrix is used, via the exact solution of a particular integral equation, to compute the asymptotic form of the low lying effective single particle states and their occupations, which are proportional to \( \sqrt{N} \).

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I. INTRODUCTION

The probability density functions (p.d.f.’s )

\[
\frac{1}{n!} \left( \frac{1}{2\pi} \right)^n \prod_{l=1}^{n} 4 \sin^2(\theta_l) \prod_{1 \leq j < k \leq n} 4(\cos \theta_k - \cos \theta_j)^2 \tag{1}
\]

\[
\frac{2}{n!} \left( \frac{1}{2\pi} \right)^n \prod_{1 \leq j < k \leq n} 4(\cos \theta_k - \cos \theta_j)^2 \tag{2}
\]

where \( 0 \leq \theta_j \leq \pi \) (\( j = 1, \ldots, n \)) occur in both random matrix theory and the quantum many body problem. In the former they are eigenvalue p.d.f.’s for classical groups with the Haar (uniform) measure – the group \( Sp(n) \) of \( n \times n \) unitary matrices with real quaternion elements (which are themselves \( 2 \times 2 \) matrices), and the group \( O^+(2n) \) of \( 2n \times 2n \) unitary matrices with real elements (real orthogonal matrices) and determinant equal to \(+1\), for \( 1 \) and \( 2 \) respectively. A self contained derivation of these facts can be found in [1, Chapter 2]. In the latter they are the absolute value squared of the ground state wave function for \( n \) free fermions on the interval \([0, \pi]\) with Dirichlet and Neumann boundary conditions respectively. As is similarly well known, and revised from first principles in our work [2], they are also the absolute value squared of the ground state wave function for \( n \) impenetrable bosons on the interval \([0, \pi]\) – in one-dimension the ground state wave function of the impenetrable Bose system is equal to the absolute value of the corresponding free Fermi system.

In studies relating to both these seemingly disparate interpretations of the p.d.f.’s \( 1 \) and \( 2 \) there is cause to investigate the function of \((\phi, m, n)\) defined by averaging

\[
\prod_{l=1}^{n} (\cos \phi - \cos \theta_l)^m \tag{3}
\]

with respect to these p.d.f.’s . In the random matrix interpretation this comes about in applications to \( L \)-function theory [3, 4, 5]. Briefly, there are families of \( L \)-functions with special symmetries which are known to have their
non-trivial zeros well described by eigenvalues of random matrices from the classical group corresponding to that symmetry. For \( \cos \varphi = 1 \) and small values of \( m \) the expected value of (3) can be computed with \( \{ \theta_l \} \) corresponding to the zeros of particular families of \( L \)-functions, and it can also be computed – with \( m \) a general non-negative integer – for the random matrix ensembles. This then allows for both a test of the original hypothesis relating \( L \)-functions to random matrices, and provides specific conjectures for the statistical properties of the zeros of the \( L \)-function families.

In the quantum many body interpretation the immediate interest is not in the average of (3), but rather the average of

\[
\prod_{l=1}^{n} |\cos \phi_1 - \cos \theta_l||\cos \phi_2 - \cos \theta_l|. \tag{4}
\]

This gives the ground state density matrix of the corresponding impenetrable Bose gas system (if the absolute value signs are removed, the average gives the ground state density matrix for the free Fermi system) \[2\]. However the study of (4) leads back to the computation of (3). Thus as noted in \[3\] and \[7\] for the problem of computing the asymptotic behavior of the ground state density matrix for the impenetrable bosons in the bulk and in an harmonic trap respectively, for large \( n \) and \( \phi_1 \) and \( \phi_2 \) fixed the average (3) is expected to factorize, and be proportional to the average of the product involving \( \phi_1 \) times the average of the product involving \( \phi_2 \). The latter are then the continuation in \( m \) of (3) from the even positive integers to the value \( m = 1 \) (this being a way to effectively study the average of the absolute value of (3)).

We remark that in the case of the density matrix for the impenetrable bosons in periodic boundary conditions the analogous task is to compute the average of

\[
\prod_{l=1}^{n} |e^{i\phi_1} - e^{i\theta_l}| |e^{i\phi_2} - e^{i\theta_l}|, \tag{5}
\]

with respect to the p.d.f.\[
\frac{1}{n!} \left( \frac{1}{2\pi} \right)^n \prod_{1 \leq j < k \leq n} |e^{i\theta_k} - e^{i\theta_j}|^2, \tag{6}
\]

where \(-\pi < \theta_j \leq \pi \) (\( j = 1, ..., n \)). In this case the asymptotic form for large \( n \) with \( \phi_1 \) and \( \phi_2 \) fixed is a special case of known asymptotic forms for Toeplitz determinants with singular generating functions of the so called Fisher-Hartwig type (for an extended discussion on this point and references to the relevant literature see \[2\]). The average of (4) with respect to (1) or (2) can readily be written as a Hankel determinant, but there is no known analogue of the Fisher-Hartwig asymptotic form.

In Section II of this work we show how, for \( m \) an even positive integer, the average of (3) with respect to the p.d.f.’s (1) and (2), which is by definition an \( n \)-dimensional integral, can be written as a \( m \)-dimensional integral. From the latter the large \( n \) asymptotic form of the average is deduced. In Section III the result of Section II is used to deduce the large \( n \) asymptotic form of the average (3) with respect to the p.d.f.’s (1) and (2) and thus of the ground state density matrix. We conclude in Section IV by applying our result for the asymptotic form of the density matrix to the computation of the occupations of the effective single particle states. We also make some remarks in relation to the wider setting of our asymptotic analysis, in which we use Coulomb gas arguments to formulate an analogue of the Fisher-Hartwig asymptotic form for a class of Jacobi unitary ensemble averages.

## II. DUALITY FORMULAS FOR MULTIPLE INTEGRALS AND ASYMPTOTIC ANALYSIS

### A. The duality formula

The change of variables

\[
x_j = \frac{1}{2} (\cos \theta_j + 1) \quad (0 \leq x_j \leq 1, \quad j = 1, ..., n) \tag{7}
\]

transforms (1) and (2) into the p.d.f.’s

\[
\frac{1}{n!} \left( \frac{1}{2\pi} \right)^n \prod_{l=1}^{n} (x_l(1 - x_l))^{1/2} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \tag{8}
\]
\[
\frac{2}{n!} \left( \frac{1}{2\pi} \right)^n \prod_{l=1}^{n} (x_l(1-x_l))^{-1/2} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2
\]

(9)

respectively. These p.d.f.’s in turn are special cases of the class of p.d.f.’s proportional to

\[
\prod_{l=1}^{n} x_l^{\lambda_1} (1-x_l)^{\lambda_2} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2
\]

(10)

known in random matrix theory as the Jacobi unitary ensemble. Also, under the change of variables (7) the task of computing the average of (3) becomes the task of computing the average of

\[
\prod_{l=1}^{n} (t - x_l)^m
\]

(11)

with respect to (8) and (9), or more generally with respect to (10).

In fact there is an advantage in further generalizing the setting of the computation of the average of (11) and considering the class of multiple integrals known as Selberg correlation integrals, defined by

\[
S_{n,m}(\lambda_1, \lambda_2, \lambda; t_1, \ldots, t_m) := \frac{1}{C} \int_{[0,1]^n} dx_1 \ldots dx_n \prod_{l=1}^{n} x_l^{\lambda_1} (1-x_l)^{\lambda_2} \prod_{i=1}^{m} (t_i - x_l) \times \prod_{1 \leq j < k \leq n} |x_k - x_j|^{2\lambda}
\]

(12)

\[
= \left( \prod_{l=1}^{n} \prod_{i=1}^{m} (t_i - x_l) \right) J(2\lambda)E_n
\]

Here

\[
C = S_n(\lambda_1, \lambda_2, \lambda) = \int_{[0,1]^n} dx_1 \ldots dx_n \prod_{l=1}^{n} x_l^{\lambda_1} (1-x_l)^{\lambda_2} \prod_{1 \leq j < k \leq n} |x_k - x_j|^{2\lambda},
\]

(13)

known as the Selberg integral, is the normalization chosen so that the coefficient of \(\prod_{i=1}^{m} t_i^n\) is unity, and the average over \(J(2\lambda)E_n\) refers to the p.d.f.

\[
\frac{1}{S_n(\lambda_1, \lambda_2, \lambda)} \prod_{l=1}^{n} x_l^{\lambda_1} (1-x_l)^{\lambda_2} \prod_{1 \leq j < k \leq n} |x_k - x_j|^{2\lambda}
\]

(14)

(the notation \(J(2\lambda)E_n\) denotes the Jacobi -(2\(\lambda\)) ensemble, which with \(\lambda = 1\) corresponds to the Jacobi unitary ensemble \(10\)). Setting \(t_1 = \ldots = t_m = t\) in (12) gives the average of (11) with respect to (14). The advantage in studying (12) is that we can put to use the discovery of 6, relating the Selberg correlation integrals to the theory of Jack polynomials (in the case \(\lambda = 1\) the Jack polynomials coincide with the Schur polynomials 7). In particular the Selberg correlation integrals were evaluated in terms of a generalization of the Gauss hypergeometric function \(_{2}F_{1}\) based on the Jack polynomials. It was realized by one of the present authors 10, 11, 12, 13, 14, 15 that theory initiated in 8 could be further developed and used to express the average of (9) with respect to (14) and its limiting forms as the Laguerre - (2\(\lambda\)) ensemble and the Gaussian - (2\(\lambda\)) ensemble, as \(m - \) dimensional integrals. Because the role of \(n\) and \(m\) is effectively interchanged, these integration identities have been referred to as duality formulas 16, 17, 18. One of their uses, as we will demonstrate in the case of the average of (11) with respect to (14), is in the computation of the large \(n\) asymptotics.

The particular duality formula of interest to us is given explicitly in 15. To state the result we must introduce the generalized circular ensemble, \(C^2E_N\), as the p.d.f. proportional to

\[
\prod_{1 \leq j < k \leq N} |z_k - z_j|^\beta, \quad (z_j = e^{i\theta_j}, -\pi < \theta_j < \pi, \ j = 1, \ldots, N)
\]

(15)
With this notation, we read off from eq. (3.41) that
\[
\left\langle \prod_{l=1}^{N} z_{l}^{(\eta_{1}-\eta_{2})/2}|1+z_{l}|^{\eta_{1}+\eta_{2}}(1+tz_{l})^{m} \right\rangle \propto \left\langle \prod_{l=1}^{m} |1-(1-t)x_{l}|^{N} \right\rangle_{C^{2}\beta N} \quad J(4/\beta)_{E_{m}} \quad \left| \begin{array}{l}
\lambda_{1}=2(\eta_{2}-m+1)/\beta-1 \\
\lambda_{2}=2(\eta_{1}+1)/\beta-1
\end{array} \right|
\]

But we want to make \( \prod_{l=1}^{n} (t-x_{l})^{m} \) the quantity being transformed, so \( \text{(16)} \) requires manipulation. For this we write
\[
t \mapsto 1 - \frac{1}{t}, \quad m \mapsto N, \quad \frac{2}{\beta} = \lambda, \quad N = n
\]

Noting that then
\[
\eta_{1} = \frac{1}{\lambda} (\lambda_{2} + 1) - 1, \quad \eta_{2} = \frac{1}{\lambda} (\lambda_{1} + 1) + n - 1,
\]
multiplying both sides of \( \text{(16)} \) by \( t^{mN} \) and taking the complex conjugate of the left hand side of \( \text{(16)} \) shows
\[
\left\langle \prod_{l=1}^{n} (t-x_{l})^{m} \right\rangle_{J(2\lambda)E_{n}} = A \left\langle \prod_{l=1}^{m} z_{l}^{(\lambda_{1}-\lambda_{2}/\lambda-n)/2}|1+z_{l}|^{(\lambda_{1}+\lambda_{2}/\lambda)2+n-2[t(1+z_{l})]-1} \right\rangle_{C(2/\lambda)E_{m}}
\]

where \( A \), the proportionality constant, is independent of \( t \). To specify \( A \) requires, in addition to the Selberg integral \( \text{(13)} \), the so called Morris integral
\[
M_{n}(a, b, \lambda) := \left( \frac{1}{2\pi} \right)^{n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{i=1}^{n} z_{i}^{(a-b)/2}|1+z_{i}|^{a+b} \prod_{1 \leq j < k \leq n} |z_{k}-z_{j}|^{2\lambda}.
\]

Then setting \( t = 1 \) in \( \text{(19)} \) shows that
\[
A = \frac{S_{n}(\lambda_{1}, \lambda_{2}+m, \lambda)}{S_{n}(\lambda_{1}, \lambda_{2}, \lambda)} M_{m}(0, 0, 1/\lambda)
\]
where \( \eta_{1} \) and \( \eta_{2} \) are given by \( \text{(18)} \). Both the Selberg integral and Morris integral have exact evaluations in terms of products of gamma functions (see e.g. \( \text{(1)} \)). In the case \( \lambda = 1 \) these read
\[
S_{n}(a, b, 1) = \sum_{j=0}^{n-1} \frac{\Gamma(a+1+j)\Gamma(b+1+j)\Gamma(2+j)}{\Gamma(a+b+1+n+j)} G(n+1+a) \frac{G(n+1+b)}{G(1+a)} G(n+1+a+b) G(n+2),
\]
\[
M_{n}(a, b, 1) = \prod_{j=0}^{n-1} \frac{\Gamma(a+b+1+j)\Gamma(2+j)}{\Gamma(a+1+j)\Gamma(b+1+j)} G(n+1+a+b) G(n+1+a) G(n+1+b) G(n+2),
\]

where \( G(z) \) denotes the Barnes G-function, related to the gamma function by the functional equation
\[
G(z+1) = \Gamma(z)G(z).
\]

**B. Asymptotics**

Our interest is in the asymptotic form of the \( J(2\lambda)E_{n} \) average in \( \text{(18)} \) in the case \( \lambda = 1 \) and \( m \) even. The experience of our previous study \( \text{(4)} \), in which we studied the same product averaged over the Gaussian unitary ensemble (eigenvalue p.d.f. of complex Hermitian matrices with Gaussian entries) using a result known in the literature \( \text{(19)} \) (see also \( \text{(20)} \)), tells us the related quantity
\[
\frac{Z_{n,\lambda_{1},\lambda_{2}}((X, q))}{Z_{n+q,\lambda_{1},\lambda_{2}}((\cdot, 0))}
\]
where
\[ Z_{n,\lambda_1,\lambda_2}((X,q)) = X^{\lambda_1 q}(1 - X)^{\lambda_2 q} \prod_{j=0}^1 dX_1 \cdot \prod_{j=1}^n dX_n \prod_{l=1}^n X_l^{\lambda_1}(1 - X_l)^{\lambda_2} |X - X_l|^{2q} \times \prod_{1 \leq j < k \leq n} |X_k - X_j|^2 \] (26)
is better suited for the purpose. In addition to multiplying the JUE\(n\) average in (19) by the \(t\)-dependent factor \(t^{\lambda_1 m/2(1 - t)^{\lambda_2 m/2}}\), a key feature of (25) is the normalization chosen so that in the interpretation of (26) as the configuration integral for a log-potential Coulomb gas the normalization has the same total charge.

It follows from (19), (21), and noting that we have the duality formula for (25) with configuration integral for a log-potential Coulomb gas the normalization has the same total charge.

That we have the duality formula for (25) with \(q = m/2, m\) even

\[ \frac{Z_{n,\lambda_1,\lambda_2}((t,m/2))}{Z_{n+m/2,\lambda_1,\lambda_2}((t',0))} = \frac{S_n(\lambda_1, \lambda_2 + m, 1)}{S_n(\lambda_1, \lambda_2 + m, 1) \cdot M_n(\lambda_1 + n, \lambda_2, 1)} \cdot \frac{t^{\lambda_1 m/2(1 - t)^{\lambda_2 m/2}}}{t^{\lambda_1 m/2(1 - t)^{\lambda_2 m/2}}} \]
\[ \times \prod_{j=1}^m \int_{-\pi}^{\pi} d\theta_1 \cdots \prod_{j=1}^m \int_{-\pi}^{\pi} d\theta_m \prod_{l=1}^m z_l^{(\lambda_1 - \lambda_2 - n)/2} |1 + z_l|^{\lambda_1 + \lambda_2 + n}|t(1 + z_l) - 1|^n \]
\[ \times \prod_{1 \leq j < k \leq m} |z_j - z_k|^2 \]
\[ = \frac{S_n(\lambda_1, \lambda_2 + m, 1)}{(2\pi i)^m S_n(\lambda_1, \lambda_2 + m, 1) \cdot M_n(\lambda_1 + n, \lambda_2, 1)} \cdot I_n(t), \] (28)

where
\[ I_n(t) := t^{\lambda_1 m/2(1 - t)^{\lambda_2 m/2}} \int_{\mathbb{C}^n} \prod_{l=1}^m (-x_l)^{\lambda_1 + \lambda_2 + n}(1 - x_l)^{-(\lambda_1 + \lambda_2 + n)}(1 - tx_l)^n \times \prod_{1 \leq j < k \leq m} (x_k - x_j)^2. \] (29)

In (29) \(C\) is any simple closed contour starting at \(x_j = 0\) in the complex plane and encircling \(x_j = 1\) anti-clockwise without crossing the interval \(x_j \in (0, 1)\). We obtain the second equality in (28) by writing the integrand in a form without absolute value signs, changing variables
\[ \frac{1}{2\pi i z_j} d\theta_j \] (30)
then noticing the integrand is analytic except at \(z_j = 0, 1\) and with a cut along the interval \(z_j \in (0, 1)\).

The large \(n\), fixed \(t \in (0, 1)\) asymptotic analysis of an integral very similar to (29) has been detailed in (14), and that analysis in turn follows the stationary phase analysis of a related multiple integral given in (11). Now, the \(n\)-dependent terms in the integrand of \(I_n(t)\) are
\[ x^n(1 - x)^{-n}(1 - tx)^n = \exp[n \log x_j - \log(1 - x_j) + \log(1 - tx_j)]]. \] (31)
As noted in (14), a simple calculation shows that the stationary point of the exponent occurs when
\[ x_j = 1 \pm \frac{1}{t} (1 - t) \]
\[ =: x_\pm. \] (32)

This suggests we deform the contours so that \(m/2\) integration variables, \((x_1, \ldots, x_{m/2})\) say, pass through \(x_+\), and the remaining pass through \(x_-\). We must then expand the integrand in the neighborhood of these stationary points. Because we have made a definite choice of the \(m/2\) variables, we must multiply by the combinatorial factor \(\left(\begin{array}{c} m \\ m/2 \end{array}\right)\).

From (14) we know that expanding the exponent in (31) to second order in \((x_+ - x_j)\) gives
\[ \exp[n \log x_j - \log(1 - x_j) + \log(1 - tx_j)] \sim \exp[-\frac{n}{2} \left( x_j - x_\pm \right)^2] \] (33)
where
\[ \alpha = \frac{t^2}{(1-tx)^2} + \frac{1}{x^2} - \frac{1}{(1-x)^2}. \]  

(34)

Regarding the leading order expansion of the other terms in the integrand we have
\[ \prod_{1 \leq j < k \leq m} (x_k - x_j)^2 \sim (x_+ - x_-)^{2(m/2)^2} \prod_{1 \leq j < k \leq m/2} (x_k - x_j)^2 \prod_{m/2+1 \leq j < k \leq m} (x_k - x_j)^2, \]

(35)

Hence
\[ I_n(t) \sim \left( \frac{m}{m/2} \right) [t^\lambda (1-t)^\lambda]^{m/2} (x_+ - x_-)^{2(m/2)^2} \prod_{1 \leq j < k \leq m/2} (x_k - x_j)^2 \]
\[ \times \left| \int_{m/2}^\infty dx_1 \ldots \int_{m/2}^\infty dx_{m/2} \prod_{i=1}^{m/2} \exp \left[ -\frac{n\alpha}{2x_i^2} \right] \prod_{1 \leq j < k \leq m/2} (x_k - x_j)^2 \right|^2 \]
\[ = \left( \frac{m}{m/2} \right) [t^\lambda (1-t)^\lambda]^{m/2} (x_+ - x_-)^{2(m/2)^2} \prod_{1 \leq j < k \leq m/2} (x_k - x_j)^2 \]
\[ \times |\alpha|^{-(m/2)^2} (V_{m/2})^2. \]

(37)

where
\[ V_{m/2} := \int_{m/2}^\infty dx_1 \ldots \int_{m/2}^\infty dx_{m/2} \prod_{i=1}^{m/2} \exp \left[ -\frac{1}{2x_i^2} \right] \prod_{1 \leq j < k \leq m/2} (x_k - x_j)^2. \]

(38)

From [14]
\[ |x_+| = \sqrt{\frac{1}{t}}, \quad |1-x_+| = \sqrt{\frac{1-t}{t}}, \quad |x_+ - x_-| = 2\sqrt{\frac{1-t}{t}}, \quad |\alpha| = \frac{2t^{3/2}}{(1-t)^{1/2}} \]

(39)

so [37] simplifies to read
\[ I_n(t) \sim (-1)^{m/2} \left( \frac{m}{m/2} \right) 2^{(m/2)^2} n^{-(m/2)^2} [t(1-t)]^{-m^2/8} (V_{m/2})^2. \]

(40)

Furthermore, we recognize \( V_{m/2} \) as a limiting case of the Selberg integral known as the Mehta integral [21], which has the evaluation
\[ V_{m/2} = (2\pi)^{m/4} \prod_{j=0}^{m/2-1} \Gamma(2+j) = (2\pi)^{m/4} G(m/2 + 2). \]

(41)

Recalling [28], our remaining task is to compute the asymptotic form of the combination of Selberg integrals and the Morris integral therein. According to [22] and [23], for this we require knowledge of the asymptotic expansion of the Barnes G-function. In fact Barnes himself showed [22]
\[ \log \left( \frac{G(n + a + 1)}{G(n + b + 1)} \right) \sim (b-a) n + \frac{a-b}{2} \log(2\pi) + \left( (a-b)n + \frac{a^2 - b^2}{2} \right) \log n + o(1). \]

(42)

Using this we find
\[ \frac{S_n(\lambda_1, \lambda_2 + m, 1)}{S_{n+m/2}(\lambda_1, \lambda_2, 1) M_m(\lambda_1 + n, \lambda_2, 1)} \sim \frac{n^{m^2/2-m^2}}{G(m+2)}. \]

(43)
Substituting (37) in (31), then substituting the result together with (18) in (28) we obtain the sought asymptotic formula

$$\frac{Z_{n, \lambda_1, \lambda_2}(t, q)}{Z_{n, \lambda_1, \lambda_2}(t, 0)} \sim \frac{1}{\pi^q} \frac{G^2(q + 1)}{G(2q + 1)} (2n)^{-q+q^2} [t(1 - t)]^{-q/2},$$

(44)

where we have set $m/2 = q$ and use has been made of the functional equation (24). We note that the right hand side of (44) is independent of the parameters $\lambda_1$ and $\lambda_2$.

A check on our workings to this stage is the special case $q = 1$. Then (25) coincides with the eigenvalue density in the JUE, $\rho^\text{JUE}(X)$, normalized so that its integral on $[0, 1]$ is unity. Setting $q = 1$ in (44) we read off that

$$\rho^\text{JUE}(X) \sim \frac{1}{\pi} (X(1 - X))^{-1/2},$$

(45)

which is indeed the known functional form (see e.g. (1)).

III. ASYMPTOTIC FORM OF THE DENSITY MATRICES

Consider the impenetrable Bose gas of $N + 1$ particles confined to the interval $[0, L]$ with Dirichlet boundary conditions. We know from [2] that the ground state density matrix $\rho_{N+1}^D(x, y)$ is given by

$$\rho_{N+1}^D(x, y) = \frac{(N + 1)}{C} \left( \sin \frac{\pi x}{L} \right) \left( \sin \frac{\pi y}{L} \right) \int_0^L dx_1 \ldots \int_0^L dx_N \prod_{l=1}^N \sin^2 \frac{\pi x_l}{L} \left| \cos \frac{\pi x_k}{L} - \cos \frac{\pi x_j}{L} \right|^2,$$

(46)

where

$$C = \int_0^L dx_1 \ldots \int_0^L dx_N \prod_{l=1}^{N+1} \sin^2 \frac{\pi x_j}{L} \prod_{1 \leq j < k \leq N+1} \left| \cos \frac{\pi x_k}{L} - \cos \frac{\pi x_j}{L} \right|^2.$$

(47)

Let us now change variables

$$\cos \frac{\pi x_j}{L} = 2X_j - 1$$

(48)

in both (46) and (47), and let us define

$$\rho_{N+1}^D(X, Y) := \rho_{N+1}^D(x, y) \bigg|_{\cos \frac{\pi x}{L} = 2X - 1, \cos \frac{\pi y}{L} = 2Y - 1}.$$  

(49)

Then in terms of the generalization of (26),

$$Z_{n, \lambda_1, \lambda_2}(X, q_1, Y, q_2) = |X - Y|^{2q_1} |X|^{\lambda_2} |Y|^{\lambda_1} |1 - Y|^{\lambda_2} |1 - Y|^{\lambda_1} \int_0^1 dx_1 \ldots \int_0^1 dx_n$$

$$\times \prod_{l=1}^n X_l^{\lambda_1} (1 - X_l)^{\lambda_2} |X - X_l|^{2q_1} |Y - X_l|^{2q_2} \prod_{1 \leq j < k \leq n} |X_k - X_j|^2$$

(50)

we have

$$\rho_{N+1}^D(X, Y) = \frac{\pi \rho}{|X - Y|^{1/2} |Y(1 - Y)|^{1/2} |X(1 - X)|^{1/2}} \frac{Z_{N, 1/2, 1/2}(X, 1/2, Y, 1/2)}{Z_{N+1, 1/2, 1/2}(X, 1/2, Y, 1/2)},$$

(51)

where $\rho := N/L$.

Similar considerations apply to the impenetrable Bose gas of $N + 1$ particles confined to the interval $[0, L]$ with Neumann boundary conditions. Again from [2] we know that the ground state density matrix $\rho_{N+1}^N(x, y)$ is given by

$$\rho_{N+1}^N(x, y) = \frac{(N + 1)}{C} \int_0^L dx_1 \ldots \int_0^L dx_N \prod_{l=1}^N \left| \cos \frac{\pi x_k}{L} - \cos \frac{\pi x_j}{L} \right| \left| \cos \frac{\pi y}{L} - \cos \frac{\pi x_j}{L} \right|^2,$$

(52)
where

$$C = \int_0^L dx_1 \ldots \int_0^L dx_{N+1} \prod_{1 \leq j < k \leq N+1} \left| \cos \frac{\pi x_k}{L} - \cos \frac{\pi x_j}{L} \right|^2.$$  

(53)

Defining

$$\rho^N_{N+1}(X, Y) := \rho^N_{N+1}(x, y) \left| \frac{\cos \pi x/L = 2X - 1}{\cos \pi y/L = 2Y - 1} \right.$$  

(54)

and changing variables according to (48) in (52) and (53) shows

$$\rho^N_{N+1}(X, Y) = \frac{\pi \rho}{|X - Y|^{1/2}} [X(1 - X)]^{1/4} [Y(1 - Y)]^{1/4} \frac{Z_{N+1/2, -1/2}((X, 1/2), (Y, 1/2))}{Z_{N+1/2, -1/2}((0, 0), (0, 0))}.  

(55)

As already noticed in [4], the log-gas interpretation of (50) allows us to predict that for large \( n \) it factorizes into a function of \( X \) and the same function of \( Y \), which are themselves of the form (20). Explicitly, we expect

$$\frac{Z_{n, \lambda_1, \lambda_2}((X, q_1), (Y, q_2))}{Z_{n+q_1, q_2, \lambda_1, \lambda_2}((0, 0), (0, 0))} \sim \frac{Z_{n, \lambda_1, \lambda_2}((X, q_1))}{Z_{n+q_1, \lambda_2}((0, 0))} \frac{Z_{n, \lambda_1, \lambda_2}((Y, q_2))}{Z_{n+q_2, \lambda_2}((0, 0))}  

(56)

As in [20], the key to choosing the correct normalizations is to balance the total charge in the log-gas interpretation. Setting \( q_1 = q_2 = q \) as required by (51) and (55) it follows from (56) that

$$\frac{Z_{n, \lambda_1, \lambda_2}((X, q), (Y, q))}{Z_{n+2q, \lambda_1, \lambda_2}((0, 0), (0, 0))} \sim \left( \frac{1}{\pi^2} \frac{G^2(4q + 1)}{G(2q + 1)} \right)^2 (2n)^{-q^2/2} \left[ X(1 - X) \right]^{-q^2/2} \left[ Y(1 - Y) \right]^{-q^2/2}.  

(57)

Substituting this asymptotic form with \( q = 1/2 \) in (51) and (55) we obtain that for large \( N \) and fixed \( X, Y \in (0, 1) \)

$$\rho^D_{N+1}(X, Y) \sim \rho^N_{N+1}(X, Y) \sim \rho \frac{G^4(3/2) [X(1 - X)]^{1/8} [Y(1 - Y)]^{1/8}}{\sqrt{2N} |X - Y|^{1/2}}.  

(58)

It is of interest to compare the asymptotic formula (58) against a numerical determination of say \( \rho^D_{N+1}(X, Y) \) or more conveniently \( \rho^D_{N+1}(X, 1 - X) \). To compute the latter we write it as a random matrix average. Thus it follows from the various definitions that

$$\rho^D_{N+1}(X, 1 - X) = \frac{8\rho}{N + 1} X(1 - X) \left\{ \prod_{l=1}^N (|4(1 - X) - 4X_l|) (|4X - 4X_l|) \right\}_{\text{JUE}_N} \bigg|_{\lambda_1 = \lambda_2 = 1/2}  

(59)

For each \( k = 1, 2, \ldots, M \) we sample from JUE$_N |_{\lambda_1 = \lambda_2 = 1/2}$ obtaining the \( N \)-tuple \((X^{(k)}_1, X^{(k)}_2, \ldots, X^{(k)}_N)\). Then the method of Monte Carlo integration tells us that

$$\rho^D_{N+1}(X, 1 - X) = \frac{8\rho}{N + 1} X(1 - X) \frac{1}{M} \prod_{k=1}^M \prod_{l=1}^N (|4(1 - X) - 4X_l^{(k)}|) (|4X - 4X_l^{(k)}|)  

+ O \left( \frac{1}{\sqrt{M}} \right).  

(60)

Fortuitously, we have available a recently discovered [23] random three term recurrence which generates a polynomial, the zeros of which have the p.d.f. \( J(2\lambda)E_n \). In the case of interest \((\lambda_1 = \lambda_2 = 1/2, \lambda = 1)\) the recurrence states

$$A_0(x) = 1  

A_1(x) = x - BD[n + 1/2, n + 1/2]  

A_j(x) = (w_2(x - 1) + w_0x)A_{j-1}(x) + w_1x(x - 1)A_{j-2}(x) \quad (j = 2, \ldots, n)  

(61)

where with

$$a \in \text{GD}[n + 1 - j + 1/2, 1], \quad b \in \text{GD}[j - 1, 1/2], \quad c \in \text{GD}[n + 1 - j + 1/2, 1]  

(62)$$
| $X$  | $\rho_{N+1}^{D,MC}(X,1-X)/\rho_{N+1}^{D}(X,1-X)$ |
|------|----------------------------------|
| 0.025 | 1.0958                           |
| 0.075 | 1.0039                           |
| 0.125 | 1.0363                           |
| 0.175 | 1.0098                           |
| 0.225 | 0.9439                           |
| 0.275 | 1.0080                           |
| 0.325 | 0.9692                           |
| 0.375 | 1.0338                           |
| 0.425 | 0.9706                           |
| 0.475 | 1.1309                           |

TABLE I: The ratio $\rho_{N+1}^{D,MC}(X,1-X)/\rho_{N+1}^{D}(X,1-X)$ where $\rho_{N+1}^{D,MC}(X,1-X)$ refers to the Monte Carlo expression (60), while $\rho_{N+1}^{D}(X,1-X)$ is the asymptotic form (58). We chose $N = 14$, and evaluated (60) with $M = 5000$.

and $d := a + b + c$ we have

$$w_0 = \frac{a}{d}, \quad w_1 = \frac{b}{d}, \quad w_2 = 1 - w_0 - w_1.$$  \tag{63}

Here BD[a, b] denotes the classical beta distribution, while GD[m, σ] denotes the classical gamma distribution. The theory of [23] tells us that $A_n(x)$ has its zeros distributed according to JUE|λ1 = λ2 = λ. Implementing (61) for fixed $n$ we thus computed the samples required in (60) for the Monte Carlo evaluation of $\rho_{N+1}^{D}(X,1-X)$. Forming the ratio then with the asymptotic form (58) gave the data in Table I.

**IV. PHYSICAL AND MATHEMATICAL IMPLICATIONS**

**A. Ground state occupation of effective single particle states**

The ground state density matrix is the theoretical quantity which quantifies the condensation of a Bose system. Thus if we decompose the density matrix

$$\rho_N(x, y) = \sum_{j=0}^{\infty} \lambda_j \phi_j(x) \phi_j(y),$$  \tag{64}

where the $\lambda_j, \phi_j$ are the eigenvalues and normalized eigenfunctions in the eigenvalue problem

$$\int \rho_N(x, y) \phi_j(y) \, dy = \lambda_j \phi_j(x),$$  \tag{65}

then by analogy with the free Fermi system in which (61) holds with $\lambda_j = 1$ ($j = 0, ..., N - 1$), $\lambda_j = 0$ ($j \geq N$), we see that the $\lambda_j$ have the physical interpretation as the occupation numbers of effective single particle states $\phi_j(x)$.

For Bose-Einstein condensation to occur we must have $\lambda_0$ proportional to $N$.

To study (65) in the case of the impenetrable Bose gas in Dirichlet or Neumann boundary conditions we restrict ourselves to large $N$ where use can be made of the asymptotic form of the density matrices (58). Before making the substitution, recalling the definitions (49) and (54) we must first change variables in (65) and redefine the eigenfunctions so that

$$\cos \pi y/L = 2Y - 1, \quad \phi_j(Y) = \phi_j(y)|_{\cos \pi y/L=2Y-1}, \quad \phi_j(X) = \phi_j(x)|_{\cos \pi x/L=2X-1}.$$  \tag{66}

Doing this we obtain, for large $N$, the integral equation

$$\sqrt{\frac{N}{\pi}} G^2(3/2) \int_0^1 \frac{[X(1-X)]^{1/8}[Y(1-Y)]^{1/8}}{|X-Y|^{1/2}} \phi_j(Y) \frac{dY}{\sqrt{Y(1-Y)}} = \lambda_j \phi_j(X).$$  \tag{67}

It follows immediately that

$$\lambda_j \propto \sqrt{N}.$$  \tag{68}
As noted in (7) this conclusion requires that \( j \) be fixed – for \( j \gg N \) we expect \( \lambda_j \propto (N/j)^4 \) in keeping with the corresponding result in periodic boundary conditions, since in this regime the boundary conditions are not expected to play a role.

Setting

\[
\lambda_j = \frac{G^4(3/2)}{\sqrt{2\pi}} \sqrt{N\lambda_j}
\]

and rearranging, (67) reads

\[
\int_0^1 \frac{\phi_j(Y)}{|X - Y|^{1/2}} \frac{dY}{[Y(1 - Y)]^{3/8}} = \bar{\lambda}_j \frac{\phi_j(X)}{[X(1 - X)]^{1/8}}.
\]

Remarkably the effective single particle ground state \( \phi_0(X) \), and the corresponding scaled occupation number \( \bar{\lambda}_0 \) can be computed exactly from (70). To see this requires knowledge of a piece of integral equation theory presented in Porter and Stirling [24]. The relevant theory tells us that the solution of the integral equation

\[
\int_0^1 \frac{\phi(t)}{|x - t|^\nu} dt = 1, \quad \nu < 1
\]

is

\[
\phi(x) = \frac{1}{\pi} \left( \cos \frac{\pi \nu}{2} \right) [x(1 - x)]^{(\nu - 1)/2}.
\]

Setting \( \nu = 1/2 \), it follows immediately that

\[
\phi_0(X) = \frac{1}{\sqrt{A}} [X(1 - X)]^{1/8}, \quad \bar{\lambda}_0 = \pi \sqrt{2}
\]

satisfies (71), where the normalization \( A \) is determined by the requirement that

\[
\frac{L}{\pi} \int_0^1 (\phi_0(X))^2 \frac{dX}{\sqrt{X(1 - X)}} = 1,
\]

and so

\[
A = \frac{L}{\pi} B(3/4, 3/4)
\]

where \( B(a,b) \) denotes the beta function. Substituting the exact evaluation of \( \bar{\lambda}_0 \) in (69) shows that in the large \( N \) limit

\[
\lambda_0 = G^4(3/2) \sqrt{N} = 1.3069 \sqrt{N}.
\]

To compute the higher order single particle states and their corresponding occupations we make the ansatz

\[
\phi_j(X) \propto \phi_0(X)p_j(X)
\]

where \( p_j(X) \) is a polynomial of degree \( j \). Now \( \{\phi_j(X)\} \) can always be chosen to be orthogonal (note that the measure is \( dX/\sqrt{X(1 - X)} \) on \([0,1]\)) so recalling (70) we require

\[
\int_0^1 \frac{p_j(X)p_k(X)}{(X(1 - X))^{1/4}} dX = 0 \quad j \neq k.
\]

Up to normalization, the unique polynomials with this property are the particular Gegenbauer polynomials

\[
p_j(X) = C_j^{1/4}(2X - 1),
\]

which we note are proportional to the particular Jacobi polynomials \( P_{j}^{-1/4,-1/4}(2X - 1) \). Normalizing (77) with the substitution (79) as in (71) shows

\[
\phi_j(X) = \sqrt{\frac{1}{L}} \sqrt{\frac{\Gamma(j + 1/4)\Gamma^2(1/4)}{\Gamma(j + 1/2)}} (X(1 - X))^{1/8} C_j^{1/4}(2X - 1).
\]
Now substituting (80) in (70) and setting \( X = 1 \) we obtain a definite integral for \( \tilde{\lambda}_j \) which can be found in (79), giving us the evaluation

\[
\tilde{\lambda}_j = \sqrt{2\pi} \frac{\Gamma(j + 1/2)}{j!}
\]  

(81)

and hence

\[
\lambda_j = G^4(3/2) \frac{\Gamma(j + 1/2)}{\sqrt{\pi}j!} \sqrt{N}.
\]  

(82)

To arrive at (80) we have made the ansatz (77). In fact a different approach can be taken to the problem, in which it is shown that an integral operator following from (70) commutes with the differential operator determining the polynomials \( \{C_j^{1/4}(2X - 1)\}_{j=0,1,2,...} \). This is done in Appendix A.

Finally, we note that substituting (58), (80) and (81) in (64) gives the following interesting identity

\[
\frac{1}{|X - Y|^{1/2}} = \sqrt{2 \pi} \Gamma^2(1/4) \sum_{j=0}^{\infty} (j + 1/4) C_j^{1/4}(2X - 1) C_j^{1/4}(2Y - 1).
\]  

(83)

### B. Generalized Fisher-Hartwig type asymptotics

One viewpoint of our asymptotic analysis of multiple integrals of the form (60) is that we are studying asymptotic problems of the Fisher-Hartwig class. Let us recall that the latter refers literally to Toeplitz determinants with both zeros and jump discontinuities is its generating function,

\[
D_N[e^{a(\theta)}] := \det[a_{i-j}]_{i,j=1,...,N}, \quad e^{a(\theta)} = \sum_{p=-\infty}^{\infty} a_p e^{ip\theta}
\]  

(84)

where

\[
a(\theta) = g(\theta) - i \sum_{r=1}^{R} b_r [\pi - (\theta - \phi_r) \mod 2\pi] + \sum_{r=1}^{R} a_r \log |2 - 2 \cos(\theta - \phi_r)|
\]  

(85)

with

\[
g(\theta) = \sum_{p=-\infty}^{\infty} g_p e^{ip\theta}, \quad \sum_{p=-\infty}^{\infty} |p||g_p|^2 < \infty.
\]  

(86)

Thus \( g(\theta) \) is a regular term, while at \( \phi_r \) \( (r = 1, ..., R) \) there is a jump discontinuity of strength \( b_r \) and a zero of order \( a_r \). To see the relationship with (60), we set \( b_r = 0 \) \( (r = 1, ..., R) \) thus eliminating the jump discontinuities, and recall the general formula relating a Toeplitz determinant to a multiple integral,

\[
D_N[e^{a(\theta)}] = \frac{1}{N!} \int_0^{2\pi} d\theta_1 ... \int_0^{2\pi} d\theta_N \prod_{l=1}^{N} e^{a(\theta_l)} \prod_{1 \leq j < k \leq N} |e^{i\theta_l} - e^{i\theta_j}|^2
\]

\[
= \frac{1}{N!} \int_0^{2\pi} d\theta_1 ... \int_0^{2\pi} d\theta_N \prod_{l=1}^{N} e^{g(\theta_l)} \left( \prod_{r=1}^{R} |e^{i\phi_r} - e^{i\phi_r}|^{2a_r} \right) \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^2.
\]  

(87)

Fisher and Hartwig [20] conjectured that in the case (80),

\[
D_N[e^{a(\theta)}] \sim e^{g_0 N} e^{\sum_{r=1}^{R} a_r^2 \log N} E
\]  

(88)

where \( E \) is independent of \( N \). This was subsequently proved, and it was furthermore shown

\[
E = e^{\sum_{k=1}^{\infty} k g_k g_k} \prod_{r=1}^{R} e^{-a_r (g(\phi_r) - g_0)} \prod_{1 \leq j < k \leq R} |e^{i\phi_k} - e^{i\phi_j}|^{-2a_k a_j} \prod_{r=1}^{R} \frac{G^2(1 + a_r)}{G(1 + 2a_r)}
\]  

(89)
which together tell us that

$$H_{n,\lambda_1,\lambda_2}[e^{h(x)} \prod_{r=1}^{R} [y_r - x]^{2q_r}] \over H_{n,\lambda_1,\lambda_2}[1],$$

(90)

where $h(x)$ is analytic on $(0,1)$. As our final issue, we would like to extend the log-gas argument used in the analysis of (94) to predict the large $n$ asymptotic form of (94).

From the log-gas perspective, the natural quantity to analyze is

$$H_n^m[h] := \int_0^1 dx_1 \cdots \int_0^1 dx_n \prod_{l=1}^m e^{h(x_l)} x_1^{\lambda_1}(1-x_1)^{\lambda_2} \prod_{1 \leq j<k \leq n} |x_k-x_j|^2.$$  

(92)

where for $m$ a non-negative integer

$$H_m^m[h] := \int_0^1 dx_1 \cdots \int_0^1 dx_m \prod_{l=1}^m e^{h(x_l)} x_1^{\lambda_1}(1-x_1)^{\lambda_2} \prod_{1 \leq j<k \leq m} |x_k-x_j|^2.$$  

(93)

Analogous to (96) we expect for large $n$ to factorize as

$$H_{n,\lambda_1,\lambda_2}[e^{h(x)} \prod_{r=1}^{R} [y_r - x]^{2q_r}] \over H_{n+\sum_{r=1}^{R} q_r,\lambda_1,\lambda_2}[e^{h(x)}] \sim \prod_{r=1}^{R} e^{-q_r h(y_r)} H_{n,\lambda_1,\lambda_2}[y_r-x]^{2q_r} \over H_{n+q_r,\lambda_1,\lambda_2}[1].$$  

(94)

where the second expression is motivated by inspection of the known results (88) and (89) for (87).

Thus we expect

$$H_{n,\lambda_1,\lambda_2}[e^{h(x)} \prod_{r=1}^{R} [y_r - x]^{2q_r}] \over H_{n,\lambda_1,\lambda_2}[1] \sim \prod_{1 \leq j<k \leq r} |y_k-y_j|^{-2q_j q_k} H_{n+\sum_{r=1}^{R} q_r,\lambda_1,\lambda_2}[e^{h(x)}] \over H_{n,\lambda_1,\lambda_2}[1] \times \prod_{r=1}^{R} e^{-q_r h(y_r)} H_{n,\lambda_1,\lambda_2}[y_r-x]^{2q_r} \over H_{n+q_r,\lambda_1,\lambda_2}[1].$$  

(95)

But according to (96) we have available both rigorous results (29, 30) as well as log-gas type arguments (31) which together tell us that

$$H_{n+Q,\lambda_1,\lambda_2}[y_r-x]^{2q_r} \over H_{n+\lambda_1,\lambda_2}[1] \sim \frac{1}{\pi^3} \frac{G^2(q_r+1)}{G(2q_r+1)} (2n)^{-q_r} G(2q_r+1)(y_r(1-y_r))^{-q_r^2/2}.$$  

(96)

Also, for the first ratio in (96) we have available both rigorous results (29, 30) as well as log-gas type arguments (31) which together tell us that

$$H_{n,\lambda_1,\lambda_2}[e^{h(x)} \prod_{r=1}^{R} [y_r - x]^{2q_r}] \over H_{n,\lambda_1,\lambda_2}[1] \sim \exp \left[ n + \frac{Q + (\lambda_1 + \lambda_2)/2}{\pi} \int_0^1 \frac{h(x)}{[x(1-x)]^{1/2}} dx \right] \times \exp \left[ -\frac{\lambda_1 + \lambda_2}{4}(h(0)+h(1)) \right] \times \exp \left[ \frac{1}{4 \pi^2} \int_0^1 dx \frac{h(x)}{[x(1-x)]^{1/2}} \int_0^1 dy \frac{h'(y)[y(1-y)]^{1/2}}{x-y} \right].$$  

(97)

Substituting (96) and (97) in (96) gives the analogue of (88),

$$H_{n,\lambda_1,\lambda_2}[e^{h(x)} \prod_{r=1}^{R} [y_r - x]^{2q_r}] \over H_{n,\lambda_1,\lambda_2}[1] \sim \exp \left[ n + \frac{\sum_{r=1}^{R} q_r + (\lambda_1 + \lambda_2)/2}{\pi} \int_0^1 \frac{h(x)}{[x(1-x)]^{1/2}} dx \right] \times \exp \left[ \frac{R}{2} (-q_r + q_r^2) \log 2n \right] K.$$  

(98)
where

\[ K = \prod_{1 \leq j < k \leq R} |y_k - y_j|^{-2q_j q_k} e^{-(\lambda_1 + \lambda_2)[h(0) + h(1)]/4} e^{-\sum_{r=1}^{R} q_r h(y_r)} \]

\[ \times \exp \left[ \frac{1}{4\pi^2} \int_0^1 dx \frac{h(x)}{x(1-x)^{1/2}} \int_0^1 dy \frac{h'(y)(y(1-y)^{1/2})}{x-y} \right] \prod_{r=1}^{R} (y_r(1-y_r))^{-q_r^2/2} \]

\[ \times \prod_{r=1}^{R} \frac{G^2(q_r + 1)}{\pi^q r^2} G^4(q_r + 1). \]  

(99)

\section*{C. Concluding remarks}

In our first paper on the impenetrable Bose gas \cite{2} we set ourselves the goal of providing the leading asymptotic form of the density matrix for the impenetrable Bose gas in a harmonic trap and in Dirichlet and Neumann boundary conditions. It was noted in \cite{2} that for the impenetrable Bose gas in periodic boundary conditions, the Fisher-Hartwig formula gave the asymptotic form

\[ \rho_{N+1}^C(x; 0) \sim \rho \frac{G^4(3/2)}{\sqrt{2\pi}} \left( \frac{\pi}{N \sin(\pi \rho x/N)} \right)^{1/2}. \]  

(100)

In \cite{7}, it was shown that for the harmonic well

\[ (2N)^{1/2} \rho_{N+1}^H(\sqrt{2NX}, \sqrt{2NY}) \sim N^{1/2} \frac{G^4(3/2)}{\pi} \frac{1 - X^2}{|X - Y|^{1/2}} (1 - Y^2)^{1/8}, \]  

(101)

while in the present paper, after changing variables according to (49), the asymptotic form of the density matrix is shown to have the leading asymptotic form (58). As a consequence of the scaling properties of these asymptotic forms, the occupation number of the low lying effective single particle states are all proportional to \( \sqrt{N} \), but with a proportionality constant dependent on the particular system.

To obtain the asymptotic forms we have used a combination of exact analysis, made possible by the theory of Selberg correlation integrals, and physical reasoning based on log-gas analogies. Taking this argument to its logical conclusion leads to a conjectured exact asymptotic formula, given by (98) and (99) for a Jacobi weight analogue of the Fisher-Hartwig formula.

\section*{APPENDIX A: PROOF THAT \{ \phi_0(X)C_j^{1/4}(2X - 1) \}_j=0,1,2,... ARE SOLUTIONS OF THE INTEGRAL EQUATION (70)}

The assertion that the \( \phi_j(X) \) given by (70) are solutions of (70) is equivalent to stating that the Gegenbauer polynomials are eigenfunctions of the integral operator

\[ K[f(\xi)] := \int_{-1}^{1} \frac{d\psi}{|\xi - \psi|^{1/2}(1 - \psi^2)^{1/4}} f(\psi). \]  

(A1)

where for convenience we are working on the interval \(-1 \leq \xi \leq 1\). In this Appendix we prove that \( K \) commutes with the differential operator, \( L \), which determines the Gegenbauer polynomials

\[ L := (1 - \xi^2) \frac{d^2}{d\xi^2} - \frac{3}{2} \xi \frac{d}{d\xi}, \]  

(A2)

with

\[ LC_j^{1/4}(\xi) = -j(j + 1/2)C_j^{1/4}(\xi). \]  

(A3)
We begin by identifying \( K[C_{j}^{1/4}(\xi)] \) with the following finite sum of hypergeometric functions

\[
K[C_{j}^{1/4}(\xi)] = \Omega_{j} \sum_{k=0}^{j} \frac{(-j)k(j+1/2)_{k}}{k!(3/4)_{k}} \left( \frac{1+\xi}{2} \right)^{1/4} 2F_{1} \left( \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}, \frac{1+\xi}{2} \right)
\]

\[
+ (-1)^{j} \Omega_{j} \sum_{k=0}^{j} \frac{(-j)k(j+1/2)_{k}}{k!(3/4)_{k}} \left( \frac{1-\xi}{2} \right)^{1/4} 2F_{1} \left( \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}, \frac{1-\xi}{2} \right),
\]

(A4)

where

\[
\Omega_{j} := \frac{\Gamma(3/4) \Gamma(j+1/2)}{\Gamma(5/4) j!}.
\]

To derive (A4) we break up the interval of integration in \( K[C_{j}^{1/4}(\xi)] \) into two regions, thus removing the need for the modulus, which gives

\[
K[C_{j}^{1/4}(\xi)] = \int_{\xi}^{1} (1 - \psi^{2})^{-1/4} C_{j}^{1/4}(\psi) \frac{d\psi}{\sqrt{\psi - \xi}} + (-1)^{j} \int_{-\xi}^{1} (1 - \psi^{2})^{-1/4} C_{j}^{1/4}(\psi) \frac{d\psi}{\sqrt{\psi + \xi}}
\]

\[
= z_{-}^{1/4} \int_{0}^{1} d\psi \psi^{-1/4} (1 - \psi)^{-1/2} (1 - z_{-} \psi)^{-1/4} C_{j}^{1/4}(1 - 2z_{-} \psi)
\]

\[
+ (-1)^{j} z_{+}^{1/4} \int_{0}^{1} d\psi \psi^{-1/4} (1 - \psi)^{-1/2} (1 - z_{+} \psi)^{-1/4} C_{j}^{1/4}(1 - 2z_{+} \psi)
\]

(A6)

where we have defined

\[
z_{\pm} := \frac{1 \pm \xi}{2}
\]

and in obtaining the second equality we have changed integration variables from \( \psi \) to \( (1 - \psi)/(1 - \xi) \) and \( (1 - \psi)/(1 + \xi) \) in the first and second integrals respectively. Substituting the following known expansion for the Gegenbauer polynomials

\[
C_{j}^{1/4}(\psi) = \frac{(-1)^{j} \Gamma(j+1/2)}{j! \sqrt{\pi}} \sum_{k=0}^{j} \frac{(-j)k(j+1/2)_{k}}{k!(3/4)_{k}} \left( \frac{1+\psi}{2} \right)^{k}
\]

(A8)

into (A6), and using the standard integral representation of the \( 2F_{1} \) function

\[
\int_{0}^{1} d\psi \psi^{-1/4} (1 - \psi)^{-1/2} = B(3/4, 1/2) 2F_{1} \left( \frac{1}{4} - k, \frac{3}{4}; 1/2; z \right),
\]

(A9)

then results in (A4).

We can now utilise known hypergeometric identities to facilitate the operation of the differential operator \( L \) on the expression for \( K[C_{j}^{1/4}(\xi)] \) given by (A4). We note that the structure of (A4) is of the form

\[
K[C_{j}^{1/4}(\xi)] = \Omega_{j} S_{j}(z_{+}) + (-1)^{j} \Omega_{j} S_{j}(z_{-}),
\]

(A10)

where

\[
S_{j}(z) := \sum_{k=0}^{j} \frac{(-j)k(j+1/2)_{k}}{k!(3/4)_{k}} z^{1/4} 2F_{1} \left( \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}; z \right).
\]

(A11)

In terms of both the variables \( z = z_{+} \) and \( z = z_{-} \) the differential operator \( L \) has the form

\[
L = z(1 - z) \frac{d^{2}}{dz^{2}} - \frac{3}{4} (2z - 1) \frac{d}{dz}
\]

(A12)

Utilising the identity

\[
\frac{d^{n}}{dz^{n}} \left[ z^{\delta} F_{p}^{q} \left( \alpha_{1}, ..., \alpha_{p} ; \rho_{1}, ..., \rho_{q} \left| z \right. \right) \right] = (\delta - n + 1)z^{\delta-n} p_{n+1} F_{q+1} \left( \delta + 1, \alpha_{1}, ..., \alpha_{p} ; \delta + 1 - n, \rho_{1}, ..., \rho_{q} \left| z \right. \right)
\]

(A13)
we find that
\[ L_2^{1/4} F_1 \left( \frac{1}{4} - k, 3; \frac{3}{4}; z \right) = \frac{z^{1/4}}{z} \left( \frac{-3}{16} \right) (1 - z) F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{3}{4}; z \right) \]
\[ + \frac{z^{1/4}}{z} \left( \frac{-3}{16} \right) (2z - 1) F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{1}{4}; z \right) \]  \hspace{1cm} (A14)
\[ L_2^{1/4} F_1 \left( \frac{1}{4} - k, 3; \frac{5}{4}; z \right) = -\frac{1}{2} z^{1/4} F_2 F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{7}{4}; z \right) \]
\[ - \frac{1}{4} z^{1/4} F_2 F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{1}{4}; z \right) \]  \hspace{1cm} (A15)
\[ L_2^{1/4} F_1 \left( \frac{1}{4} - k, 3; \frac{5}{4}; z \right) = -z^{1/4} F_1 \left( \frac{1}{4} + \frac{1}{2} \right) F_2 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}; z \right) \]
\[ - z^{1/4} F_1 \left( \frac{1}{4} - k \right) F_2 \left( \frac{1}{4} - (k - 1), \frac{3}{4}; \frac{5}{4}; z \right) \]  \hspace{1cm} (A16)

where the equalities in (A15) and (A16) respectively follow from the two particular contiguity relations \[32\]
\[ \left( -\frac{3}{16} \right) (1 - z) F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{3}{4}; z \right) = \frac{1}{16} [ (6 - 4k) z - 3 ] F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{1}{4}; z \right) \]
\[ - \frac{kz}{2} F_2 F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}; z \right), \]  \hspace{1cm} (A17)

and
\[ -\frac{1}{4} F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{1}{4}; z \right) = -k F_2 F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}; z \right) - \left( \frac{1}{4} - k \right) F_2 F_1 \left( \frac{1}{4} - (k - 1), \frac{3}{4}; \frac{5}{4}; z \right). \]  \hspace{1cm} (A18)

Hence from (A16) we obtain
\[ LS_j(z) = -z^{1/4} \sum_{k=0}^{j} \frac{(-1)^k (j + 1/2) k}{k! (3/4)_k} F_1 \left( \frac{1}{2} + k, \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}; z \right) \]
\[ - z^{1/4} \sum_{k=0}^{j} \frac{(-1)^k (j + 1/2) k}{k! (3/4)_k} k \left( \frac{1}{4} - k \right) F_2 F_1 \left( \frac{1}{4} - (k - 1), \frac{3}{4}; \frac{5}{4}; z \right). \]  \hspace{1cm} (A19)

Changing summation index in the second sum in (A19) from \( k \) to \( k - 1 \) and simplifying we deduce finally that
\[ LS_j(z) = -j (j + 1/2) \sum_{k=0}^{j} \frac{(-1)^k (j + 1/2) k}{k! (3/4)_k} z^{1/4} F_2 F_1 \left( \frac{1}{4} - k, \frac{3}{4}; \frac{5}{4}; z \right) \]
\[ = -j (j + 1/2) S_j(z), \]  \hspace{1cm} (A20)

which then implies that
\[ LKC_j^{1/4}(\xi) = -j (j + 1/2) KC_j^{3/4}(\xi) \]  \hspace{1cm} (A21)
\[ = KLC_j^{1/4}(\xi) \hspace{0.5cm} (j = 0, 1, 2, ...) \]  \hspace{1cm} (A22)

and so
\[ [K, L] = 0. \]  \hspace{1cm} (A23)

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