ORBIT SPACES AND LEAF SPACES OF FOLIATIONS AS GENERALIZED MANIFOLDS

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ABSTRACT. The paper gives a categorical approach to generalized manifolds such as orbit spaces and leaf spaces of foliations. It is suggested to consider these spaces as sets equipped with some additional structure which generalizes the notion of atlas. The approach is compared with the known ones that use the Grothendieck topos, the Haefliger classifying space and the Connes non-commutative geometry. The main aim of this paper is to indicate that the suggested approach permits to simplify essentially the modern theory of the leaf space of foliations and the orbit space of diffeomorphism groups and to obtain some new results on the characteristic classes of foliations. As an application, it is shown that the first Chern class is non-trivial for the leaf space of the Reeb foliation and its geometrical meaning is indicated.

1. Introduction

Such objects as leaf spaces of foliations and orbit spaces of diffeomorphism groups are subjects of intensive study during last several decades. There is the vast literature devoted to the investigation of these spaces. They have a very complicated structure, in particular, often their topology is very bad. By this reason usually such a space is substituted by some model which 'encodes' the information about this space. For example, for the leaf spaces of foliations there are three approaches to construct such models, namely, by means of the Grothendieck topos, the Haefliger classifying space and the Connes non-commutative geometry. Central to all these approaches is the construction of a smooth étale groupoid $G$. The three approaches above then become special instances of the general procedure of associating to a smooth groupoid a classifying topos $\mathcal{Sh}(G)$, a classifying space $BG$, and a convolution algebra $C^\infty_c(G)$. Thus the study of the leaf space of a foliation is included in the more general theory of smooth groupoids. In all of these examples the leaf space of a foliation or the orbit space of a diffeomorphism group are present only virtually.

In contrast to these approaches we consider the objects above as sets equipped with some additional structure which generalizes the notion of atlas. Moreover, we construct a category containing among its objects both manifolds and all objects above. A typical example of such an object is an orbifold. Thus, our construction is based on the idea of the atlas consisting of charts in contrast, for example, to the non-commutative geometry which is based on the idea of non-commutative, in general, algebra of functions. Although the topology of basic sets of objects for the category above is often bad or trivial but the smooth structure is very rich. First such an approach was proposed in the paper [21]. The more general categorical construction which contains all categories we need next is proposed in the paper [22]. Note that the objects of these categories possess the natural structure of Grothendieck's topology and, in many cases, they are defined uniquely by some

1 The paper contains unpublished results by Mark V. Losik (27.01.1935–12.05.2013). The paper is made up of 3 tex-files found on the computer of M.V. Losik by his student A. Galaev. The text was not changed, only several misprints were corrected and the abstract was added.
smooth étale groupoid. This allows us to compare our approach with the known ones.

The main aim of this paper is to indicate that our approach permits to simplify essentially the modern theory of the leaf space of foliations and the orbit space of diffeomorphism groups and to obtain some new results on the characteristic classes of foliations.

The characteristic classes of a foliation \( \mathfrak{F} \) of codimension \( n \) on a smooth manifold \( M \) are well known. Moreover, it is known that they can be defined as cohomology classes on the spaces of leaves of foliations. Usually, the characteristic classes above are derived from the relative cohomologies \( H^*(W_n, O(n)) \) and \( H^*(W_n, GL(n, \mathbb{R})) \) of the Lie algebra of formal vector fields with respect to the orthogonal group \( O(n) \) or from the Pontrjagin classes of the principal \( GL(n, \mathbb{R}) \)-bundle associated to the space of leaves of foliation. All these characteristic classes can be obtained as the cohomology classes of the Čech-De Rham cohomology theory constructed in [5].

Although there are explicit formulas for all these characteristic classes, there is no information about non-triviality of the characteristic classes above. In the paper [21] it was shown that the formal first Chern class of the cohomology \( H^*(W_1, O(1)) \) is in general non-trivial in some specific cohomology theory. One of the aims of this paper is to prove that the first Chern class is non-trivial for the leaf space of the Reeb foliation and to indicate its geometrical meaning.

Unless specified, all manifolds next are finite dimensional Hausdorff manifolds of \( C^\infty \)-class. Throughout the paper all categories are proposed to be small.

2. Preliminaries

Let \( \mathcal{C} \) be a category. Next we denote the space of objects of \( \mathcal{C} \) by \( \text{Ob}\mathcal{C} \) and the set of morphisms of \( \mathcal{C} \) by \( \mathcal{C}_1 \). Let \( a \in \text{Ob}\mathcal{C} \). A set \( S \) of \( \mathcal{C} \)-morphisms \( f : b \to a \) is an \( a \)-sieve, if the composition of an arbitrary \( \mathcal{C} \)-morphism \( c \to b \) and \( f \) belongs to \( S \). The minimal \( a \)-sieve containing a given set \( T \) of \( \mathcal{C} \)-morphisms with the target \( a \) is called an \( a \)-sieve generated by \( T \). The set of \( \mathcal{C} \)-morphisms \( c \to b \) whose compositions with \( f \) belong to \( S \), is called a restriction of \( S \) to \( b \) (with respect to \( f \)) and denoted by \( f^*S \).

Recall that a Grothendick topology on a category \( \mathcal{C} \) is constituted by assigning to each \( a \in \text{Ob}\mathcal{C} \) the set \( \text{Cov}(a) \) of \( a \)-sieves, called covers, such that the following axioms hold:

(1) For any \( a \in \text{Ob}\mathcal{C} \), the \( a \)-sieve generated by identity morphism \( 1_a \) is a cover;
(2) For any \( \mathcal{C} \)-morphism \( f : b \to a \) and \( S \in \text{Cov}(a) \), we have \( f^*S \in \text{Cov}(b) \);
(3) Let \( S \in \text{Cov}(a) \) and let \( R \) be an \( a \)-sieve. Then \( R \in \text{Cov}(a) \) if, for any \( \mathcal{C} \)-morphism \( f : b \to a \) from \( S \), we have \( f^*R \in \text{Cov}(b) \).

A category \( \mathcal{C} \) with a Grothendieck topology on \( \mathcal{C} \) is called a site.

Referring to the paper [22] for the general exposition we consider the following six categories.

1. The category \( \mathcal{M}_n \) with objects smooth manifolds of dimension \( n \) and morphisms local diffeomorphisms (i.e. étale morphisms) of one manifold into another one.
2. The full subcategory \( \mathcal{D}_n \) of the category \( \mathcal{M}_n \) with objects open submanifolds of \( \mathbb{R}^n \).
3. The category \( \mathcal{M}_\infty \) with objects smooth manifolds with model space \( \mathbb{R}^\infty \) and morphisms local diffeomorphisms (i.e. étale morphisms) of one manifold into another one. In [1] it is shown that the usual smooth technique can be extended naturally to such manifolds.
4. The category \( \mathcal{M} \) of smooth manifolds (of arbitrary finite dimension) and morphisms smooth maps of one manifold into another one.
5. The full subcategory $D$ of the category $\mathcal{M}$ with objects open submanifolds of $\mathbb{R}^n$ ($n = 0, 1, \ldots$).

6. Let $G$ be a Lie group. Consider a category $\mathcal{P}_n(G)$ with objects smooth principal $G$-bundles with $n$-dimensional bases whose morphisms are morphisms of such principal $G$-bundles which are projected to the étale maps of bases.

Let $\mathcal{C}$ be any of the categories above. For any $a \in \mathcal{C}_0$, an $a$-sieve $S$ is called a cover, if $S$ contains the set of inclusions $a_i \rightarrow a$, where $a_i$ is an open cover of $a$. It is easy to see that these data define a Grothendieck topology on $\mathcal{C}$.

Next we apply some general constructions of [22] to the category $\mathcal{C}$.

Denote by $J$ the forgetful functor from $\mathcal{C}$ to the category $\text{Sets}$ of sets. Let $X$ be a set. A $\mathcal{C}$-chart on $X$ is a map $k : c \rightarrow X$, where $c \in \mathcal{C}_0$. Put $D(k) = c$. Given two $\mathcal{C}$-charts $k_1$ and $k_2$, a morphism from $k_1$ to $k_2$ is a $\mathcal{C}$-morphism $f : D(k_1) \rightarrow D(k_2)$ such that $k_1 = k_2 \circ f$. Let $\mathcal{C}(X)$ be a category of $\mathcal{C}$-charts on $X$. By definition, $D : \mathcal{C}(X) \rightarrow \mathcal{C}$ is a covariant functor. For a set $\Phi(X)$ of $\mathcal{C}$-charts on $X$, denote by $\mathcal{C}_{\Phi(X)}$ the full subcategory of $\mathcal{C}(X)$ with the object set $\Phi(X)$.

2.1. Definition. A set $\Phi(X)$ of $\mathcal{C}$-charts on $X$ is called a $\mathcal{C}$-atlas on $X$, if the set $X$ with the set of maps $k : J \circ D(k) \rightarrow X$ ($k \in \Phi(X)$) is an inductive limit $\lim_j J \circ D$ of the functor $J \circ D : \mathcal{C}_{\Phi(X)} \rightarrow \text{Sets}$.

Let $\Phi(X)$ be an $\mathcal{C}$-atlas on $X$. Evidently the functor $D : \mathcal{C}_{\Phi(X)} \rightarrow \mathcal{C}$ maps $\mathcal{C}_{\Phi(X)}$ onto some subcategory $\mathcal{C}(\Phi(X))$ of $\mathcal{C}$ and the pair $(X, \Phi(X))$ can be considered as an inductive limit of the restriction of $J$ to $\mathcal{C}(\Phi(X))$. Conversely, for any subcategory $\mathcal{B}$ of $\mathcal{C}$, the inductive limit of the restriction of $J$ to $\mathcal{B}$ is an $\mathcal{C}$-atlas on some set $X$. Thus, one can define an $\mathcal{C}$-atlas as an inductive limit of the restriction of $J$ to some subcategory of $\mathcal{C}$. The set $X$ for this $\mathcal{C}$-atlas is determined by this inductive limit up to bijective map. Note that, for every set $\Phi(X)$ of $\mathcal{C}$-charts on $X$, by the definition of an inductive limit, $\lim_j J \circ D$ is a set $\tilde{X}$ given with a set of maps $\tilde{k} : J \circ D(k) \rightarrow \tilde{X}$ ($k \in \Phi(X)$) such that the set of maps $\tilde{k} : D(k) \rightarrow \tilde{X}$ is a $\mathcal{C}$-atlas on $\tilde{X}$ and there is a unique map $h : \tilde{X} \rightarrow X$ such that, for every $k \in \Phi(X)$, $k = h \circ \tilde{k}$.

Let $\Phi$ be a set of $\mathcal{C}$-charts on $X$. We shall say that, for a $b \in \mathcal{C}_0$ a $\mathcal{C}$-chart on $X$ of the type $k \circ D(f)$, where $k \in \Phi$, $c = D(k)$, and $f : b \rightarrow c$ is a morphism of $\mathcal{C}$, is obtained by restriction of $k$ to $b$ with respect to $f$. Denote by $R(\Phi)$ the set of $\mathcal{C}$-charts on $X$ obtained by restriction of the charts from $\Phi$ with respect to all appropriate morphisms of $\mathcal{C}$. Clearly, we have $\Phi \subset R(\Phi)$ and $R^2(\Phi) = R(\Phi)$.

We shall say that a $\mathcal{C}$-chart $k$ on $X$ is obtained by gluing from $\Phi$, if there are a family $\{k_i\}$ of $\mathcal{C}$-charts from $\Phi$ and the set of morphisms $\{f_i : D(k_i) \rightarrow D(k)\}$ generating a cover of $D(k)$, such that $k_i = k \circ f_i$. Denote by $G(\Phi)$ the set of $\mathcal{C}$-charts on $X$ obtained by gluing from $\Phi$. Clearly, we have $\Phi \subset G(\Phi)$ and $G^2(\Phi) = G(\Phi)$.

Put $\Phi = G \circ R(\Phi)$.

2.2. Proposition. ([22], Proposition 2.1.2) The correspondence $\Phi \rightarrow \Phi$ is a closure on the set of $\mathcal{M}_n$-atlases on $X$, i.e. the following conditions hold:

1. $\Phi \subset \Phi$;
2. If $\Phi_1 \subset \Phi_2$, then $\Phi_1 \subset \Phi_2$;
3. $\Phi = \Phi$.

Evidently, for a $\mathcal{C}$-atlas on $X$, $\Phi$ is a $\mathcal{C}$-atlas on $X$. A $\mathcal{C}$-atlas $\Phi$ on $X$ is called closed if $\Phi = \Phi$.

A pair $(X, \Phi)$, where $\Phi$ is a closed $\mathcal{C}$-atlas on $X$, is called $\mathcal{C}$-space. Two $\mathcal{C}$-atlases $\Phi_1$ and $\Phi_2$ on $X$ are called equivalent if $\Phi_1 = \Phi_2$. By definition, equivalent $\mathcal{C}$-atlases on $X$ determine the same structure of $\mathcal{C}$-space on $X$. It is evident that two $\mathcal{C}$-atlases $\Phi_1$ and $\Phi_2$ on $X$ are equivalent iff $R(\Phi_1) = R(\Phi_2)$. 
2.3. **Definition.** Let \((X_1, \Phi_1)\) and \((X_2, \Phi_2)\) be two \(C\)-spaces. A morphism 
\((X_1, \Phi_1) \to (X_2, \Phi_2)\) is a map \(f : X_1 \to X_2\), if, for each \(k \in \Phi_1\), we have \(f \circ k \in \Phi_2\).

Thus, we have a category \(C_{sp}\) of \(C\)-spaces, in particular, we have the categories 
\(M_{\infty, sp}, M_{sp}, D_{n, sp}, D_{sp}\) and \(P(G)_{n, sp}\). Note that the category of \(D\)-spaces coincides with the category of diffeological spaces introduced by Souriau [33].

For \(c \in C_0\), the identity map \(c : c \to J(c)\) is an \(C\)-chart on \(c\) and \(\Phi(c) = \{\text{id}\}\) is an \(C\)-atlas on \(c\). Therefore, one can consider \(c\) as an object of the category \(C_{sp}\) and the corresponding map \(C \to C_{sp}\) is an isomorphism of the category \(C\) onto a full subcategory of the category \(C_{sp}\).

It is evident that each object of the category \(M_n\) is an object of the category \(D_{n, sp}\). Since the category \(D_n\) is a full subcategory of \(M_n\), the corresponding natural map from \(D_{n, sp}\) into \(M_{n, sp}\) is a category isomorphism (see [22], Theorem 2.3.4). Nevertheless, later it is useful to consider both of these categories. Moreover, each \(M_n\)-atlas \(\Phi\) on \(X\) is a \(M\)-atlas on \(X\) (with the larger set of morphisms of charts) and equivalent \(M_{n, sp}\)-atlasses on \(X\) are equivalent as \(M\)-atlases on \(X\).

3. **Étale structures and étale groupoids**

Consider the following generalizations of an \(M_n\)-space.

3.1. **Definition.** Let \(\Phi\) be a set of \(M_n\)-charts on a set \(X\) and let \(B_{\Phi}\) be a subcategory of the category \(C_{\Phi}\) with the same set of objects \(\Phi\). An étale structure on \(X\) is a category \(\mathcal{E}_{\Phi}\) such that \((\mathcal{E}_{\Phi})_0 = \Phi\) given with a full covariant functor \(\Pi : \mathcal{E}_{\Phi} \to B_{\Phi}\) which is identical on the set of objects satisfying the following condition: the set of maps \(k : J \circ D(k) \to X (k \in \Phi)\) is an inductive limit \(\varinjlim J \circ D \circ \Pi\) of the functor \(J \circ D \circ \Pi : \mathcal{E}_{\Phi} \to \text{Sets}\). An étale structure \(\mathcal{E}_{\Phi}\) on \(X\) is called effective if \(\mathcal{E}_{\Phi} = B_{\Phi}\).

An étale structure \(\mathcal{E}_{\Phi}\) on \(X\) is called locally invertible if for any \(k_1, k_2 \in \Phi\), each \(\mathcal{E}_{\Phi}\)-morphism \(f : k_1 \to k_2\), and each \(x \in D(k_1)\), there are \(k_3 \in \Phi\) and \(\mathcal{E}_{\Phi}\)-morphism \(g : k_3 \to k_1\) such that \(y = f(x) \in D(k_3)\), \(g(y) = x\) and \(fg = 1_{D(k_3)}\).

It is clear that each étale structure \(\mathcal{E}_{\Phi}\) on \(X\) defines uniquely the corresponding structure of \(M_n\)-space on \(X\) and, therefore, the corresponding structure of diffeological space.

For example, let \(\Gamma\) be a pseudogroup of local diffeomorphisms of an \(n\)-dimensional manifold \(M\), \(\Phi(\Gamma)\) consists of the restrictions of the projection \(M \to M/\Gamma = X\) to the domains of the local diffeomorphisms from \(\Gamma\), and morphisms of \(\mathcal{E}_{\Phi(\Gamma)}\) are local diffeomorphisms from \(\Gamma\). By definition, \(\mathcal{E}_{\Phi(\Gamma)}\) is an effective locally invertible étale structure on \(X\).

Let \(\mathcal{E}_{\Phi(\Gamma)}\) be an effective étale structure on \(X\) with the set of objects \(\Phi\). For each \(M \in M_0\), \(k \in \Phi\), and a smooth map \(f : M \to D(k)\), the composition \(k \circ f\) is an \(M\)-chart on \(X\).

3.2. **Definition.** A diffeological structure on the set \(X\) over an effective invertible étale structure \(\mathcal{E}_{\Phi}\) on \(X\) is a category \(\mathcal{E}_{\Phi, d}\) whose objects are the pairs \((k, f)\), where \(k \in \Phi\) and \(f : M \to D(k)\) is a smooth map for some \(M \in M_0\), and, for any objects \((k_1, f_1), (k_2, f_2)\), a morphism \((k_1, f_1) \to (k_2, f_2)\) is defined whenever \(f_1\) and \(f_2\) have the same domain \(M\) and in this case is a morphism \(g : k_1 \to k_2\) belonging to \((\mathcal{E}_{\Phi})_1\) such that \(f_2 = g \circ f_1\).

Consider the covariant functor \(\mathcal{E}_{\Phi, d} \to \text{Sets}\).

Denote by \(\Phi_d\) the structure of diffeological space on \(X\) induced naturally by \(\mathcal{E}_{\Phi}\) on \(X\).

Now we show that an \(M_n\)-space has a rich smooth structure.

Let \(F : M_n \to \text{Sets}\) be a covariant functor. Define an extension of \(F\) to the category \(M_{n, sp}\) as follows. Let \(X\) be a \(M_n\)-space and let \(\Phi\) be a \(M_n\)-atlas on
$X$. By definition of inductive limit, $\lim \overset{\rightarrow}{F \circ D}$ is a set $\tilde{X}$ with the set of maps $\tilde{\Phi}$: 
\[ \tilde{k} : F \circ D(k) \to \tilde{X} \ (k \in \Phi) \] compatible with the morphisms of the category $\mathcal{C}_k$. 
Put $F(\Phi) = \tilde{X}$. It is easy to check that, if two $\mathcal{M}_n$-atlases $\Phi_1$ and $\Phi_2$ on $X$ are equivalent, there is a natural bijective map $F(\Phi_1) \to F(\Phi_2)$. We extend thus the functor $F$ to the category $\mathcal{M}_n$. It is easy to see that this extension is compatible with functor morphisms. 
For example, let $S(M)$ be the set of smooth singular simplexes on a manifold $M$. Evidently $S(M)$ is a covariant functor $\mathcal{M}_n \to \mathcal{S}_{\text{sets}}$ which has a natural extension to the category $\mathcal{M}_n$. Then we have a natural notion of the complex of finite smooth singular chains on each $\mathcal{M}_n$-space.

Let $F : \mathcal{M}_n \to \mathcal{M}_N \ (F : \mathcal{M}_n \to \mathcal{M}_\infty)$ be a covariant functor and let $\Phi$ be a $\mathcal{M}_n$-atlas on $X$. Put $\tilde{F} = J \circ F$ and $\tilde{X} = \lim \overset{\rightarrow}{\tilde{F} \circ D}$. Then the set of maps $\tilde{\Phi}$: 
\[ \tilde{k} : \tilde{F} \circ D(k) \to \tilde{X} \ (k \in \Phi) \] is an $\mathcal{M}_N$-atlas ($\mathcal{M}_\infty$-atlas) on the set $\tilde{X} = F(\Phi)$. We obtain thus the covariant functor $F : \mathcal{M}_n \to \mathcal{M}_N \ (F : \mathcal{M}_n \to \mathcal{M}_\infty)$ which is an extension of the initial functor and is compatible with functor morphisms. We have thus the notions of the tangent bundle, the frame bundle, and so on for any $\mathcal{M}_n$-space.

Assume that a category $\mathcal{C}$ has products and the kernel of each pair of morphisms exists. For example, $\mathcal{C}$ is a category of sets, groups, rings or algebras. Let $F : \mathcal{M}_n \to \mathcal{C}$ be a contravariant functor. Define an extension of $F$ to the category of $\mathcal{M}_n$-spaces as follows. Let $\Phi$ be a $\mathcal{M}_n$-atlas on $X$. By the definition of projective limit, is an object $\tilde{X}$ of the category $\mathcal{C}$ with the set of morphisms $\Phi$:
\[ k : \tilde{x} \to F \circ D(k) \ (k \in \Phi) \] compatible with the morphisms of the category $\mathcal{C}_k$. It is known that the projective limit $\lim \overset{\leftarrow}{\pi}$ exists. It is easy to see that, if two $\mathcal{M}_n$-atlases $\Phi_1$ and $\Phi_2$ on $X$ are equivalent, there is a natural isomorphism $\tilde{\Phi}_1 \to \tilde{\Phi}_2$. Put $F(\Phi) = \tilde{\Phi}$. It is evident that $F$ is an extension of the initial functor to the category $\mathcal{M}_n$, which is compatible with functor morphisms. If $F(M)$ is a sheaf on each $n$-dimensional manifold $M$, $F(X, \Phi)$ is called a sheaf on a $\mathcal{M}_n$-space $(X, \Phi)$. The examples of sheaves are the tensor algebra $\Theta(M)$ of tensor fields, the de Rham complex $\Omega^*(M)$ on a manifold $M$, and so on.

It is easy to see that the notion of integral of $p$-differential form over a $p$-dimensional finite cochain on a manifold $M$ has a natural extension to any $\mathcal{M}_n$-space and the corresponding Stokes theorem is true.

Recall the definition of smooth and étale groupoids.

A groupoid is a category $\mathcal{G}$ all whose morphisms are isomorphisms. Thus, we have a set $\mathcal{G}_0$ of objects $x, y, \ldots$ and the set $\mathcal{G}_1$ of morphisms $f, g, \ldots$. Each morphism $f : x \to y$ has a source $s(f) = x$ and a target $t(f) = y$. Two morphisms $f$ and $g$ with $s(f) = t(g)$ can be composed as $fg : s(g) \to t(f)$. This composition is associative, has a unit $1_x : x \to x$ for each $x \in \mathcal{G}_0$, and has an inverse $i(f) = f^{-1} : t(f) \to s(f)$ for each $f \in \mathcal{G}_1$. Put $\mathcal{G}_2 = \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 = \{(f, g) \in \mathcal{G}_1 \times \mathcal{G}_1 : s(f) = t(g)\}$. All the structure is contained in a diagram
\[
\begin{array}{ccc}
\mathcal{G}_2 & \overset{m}{\longrightarrow} & \mathcal{G}_1 \\
\downarrow{\iota} & & \downarrow{\iota} \\
\mathcal{G}_0 & \overset{u}{\longrightarrow} & \mathcal{G}_1.
\end{array}
\]
Here $s$ and $t$ are the source and the target, $m$ denotes composition, $i$ is the inverse, and $u(x) = 1_x$. The groupoid is smooth if $\mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds ($\mathcal{G}_1$ is non-Hausdorff, in general), all the structure maps in (3.2.1) are smooth, and $s$ and $t$ are submersions, so that $\mathcal{G}_2$ is a smooth manifold as well. A smooth groupoid is étale if all the structure maps in (3.2.1) are étale maps (it suffices to require this for $s$).

Let $\mathcal{C}$ be a category. Define an equivalence relation $x \sim y$ on $\mathcal{C}_0$ generated by the relations of the following type: there is a morphism $f$ of the category $\mathcal{C}$ such that either $f : x \to y$, or $f : y \to x$. Thus, the quotient set $\mathcal{C}_0/\sim$ equals $\pi_0(\mathcal{C})$, the
set of connected components of $C$. It is evident that, if $C = \mathcal{G}$ is a groupoid, $x \sim y$ iff there is a morphism $f : x \to y$ of the groupoid $\mathcal{G}$.

Let $\mathcal{E}_\Phi$ be an étale structure on a set $X$. Define a category $\mathcal{G}(\mathcal{E}_\Phi)$ as follows. The set of objects $\mathcal{G}(\mathcal{E}_\Phi)_0$ equals the disjoint union of domains of $\mathcal{M}_n$-charts from $\Phi$. For $x, y \in \mathcal{G}(\mathcal{E}_\Phi)_0$ and $k_1, k_2 \in \Phi$, a morphism $x \to y$ is a germ at $x$ of a $\mathcal{E}_\Phi$-morphism $g : k_1 \to k_2$ such that $x \in D(k_1)$, $y \in D(k_2)$, and $D(g)(x) = y$. If an étale structure $\mathcal{E}_\Phi$ on $X$ is locally invertible, the category $\mathcal{G}(\mathcal{E}_\Phi)$ is a groupoid.

By definition, the set of objects $\mathcal{G}(\mathcal{E}_\Phi)_0$ of $\mathcal{G}(\mathcal{E}_\Phi)$ is an $n$-dimensional manifold. Endow the set of morphisms $\mathcal{G}(\mathcal{E}_\Phi)_1$ of $\mathcal{G}(\mathcal{E}_\Phi)$ with a topology generated by the sets of germs of morphisms $f : k_1 \to k_2$ of the category $\mathcal{E}_\Phi$ at points $x \in D(k_1)$. Since such sets are $n$-dimensional manifolds, $\mathcal{G}(\mathcal{E}_\Phi)_1$ is an $n$-dimensional manifold as well.

By construction, for a locally invertible étale structure $\mathcal{E}_\Phi$ the category $\mathcal{G}(\mathcal{E}_\Phi)$ is an étale groupoid.

3.3. Remark. Note that, in the construction above, the category $\mathcal{G}(\mathcal{E}_\Phi)$ very often can be reduced essentially. Assume that we have a surjective étale map $\mathcal{G}(\mathcal{E}_\Phi)_0 \to M$, where $M$ is an $n$-dimensional manifold. Then one can take for $\mathcal{G}(\mathcal{E}_\Phi)_0$ the manifold $M$ and, for $x, y \in M$, take for a $\mathcal{E}$-morphism $x \to y$ the germ of local diffeomorphism $(M, x) \to (M, y)$ which can be lifted to a germ of some $\mathcal{G}(\mathcal{E}_\Phi)$-morphism $k_1 \to k_2$. By definition, the reduced category have the same quotient space $X = \mathcal{G}(\Phi)/\sim$. For example, let $\Gamma$ be a pseudogroup of local diffeomorphisms of an $n$-dimensional manifold $M$ and let $\mathcal{E}_{\Phi(\Gamma)}$ be the corresponding locally invertible étale structure on $X = M/\Gamma$. By the standard procedure, the set of objects $\mathcal{G}(\mathcal{E}_{\Phi(\Gamma)})_0$ is the disjoint union of domains of local diffeomorphisms from $\Gamma$. But there is the natural étale projection $\mathcal{G}(\mathcal{E}_{\Phi(\Gamma)})_0 \to M$. Therefore, one can take for the object set of the reduced category $\mathcal{G}(\mathcal{E}_{\Phi(\Gamma)})$ the manifold $M$.

Let $\mathcal{G}$ be a smooth groupoid and $X = \mathcal{G}_0/\sim$. Since $s$ is a submersion, for each morphism $g : x \to y$ from $\mathcal{G}_1$, there is an open neighborhood $U$ of $x \in \mathcal{G}_0$ and a smooth section $\sigma : U \to \mathcal{G}_1$ of the map $s$ such that $\sigma(x) = g$. Put $f_\sigma = t \circ \sigma : U \to \mathcal{G}_0$.

Define a category $\mathcal{E}(\mathcal{G})$ as follows. The objects of this category are open subsets of $\mathcal{G}_0$. For $U, V \in \mathcal{E}(\mathcal{G})_0$, a section $\sigma : U \to \mathcal{G}_1$ of the source map $s$ with the property that $f_\sigma(U) \subset V$ is a morphism of the category $\mathcal{E}(\mathcal{G})$. The composition of morphisms $\sigma : U \to \mathcal{G}_1$ and $\tau : V \to \mathcal{G}_1$, where $f_\sigma(U) \subset V$, is defined by $(\tau \sigma)(x) = \tau(f_\sigma(x)) \cdot \sigma(x)$ ($x \in U$), where the multiplication is taken in $\mathcal{G}$. Note that the category $\mathcal{E}(\mathcal{G})$ is similar to the category $\text{Emb}(\mathcal{G})$ defined in [25].

We claim that, for each section $\sigma : U \to \mathcal{G}_1$, the map $f_\sigma$ is étale. Let $x \in U$ and $\sigma(x) = g : x \to y$. Consider a section $\sigma' : U' \to \mathcal{G}_1$ such that $\sigma'(y) = g^{-1}$. Since the problem is a local one, one may suppose that $f_{\sigma'}(U) \subset U'$. Then the composition $\tau = \sigma' \circ \sigma$ is a section over $U$. If the neighborhood $U' \times x$ is sufficiently small, there is a diffeomorphism $d : \sigma'(U) \to \tau(U)$ (as submanifolds of $\mathcal{G}_1$) such that $\sigma' \circ f_\sigma = d \circ \sigma$. This implies our claim.

Let $p : \mathcal{G}_0 \to X$ be the projection. For each open subset $U$ of $\mathcal{G}_0$, denote by $k_U$ the restriction of $p$ to $U$. By definition, $k_U$ is a $\mathcal{M}_n$-chart on $X$. For two open subsets $U, V$ of $\mathcal{G}_0$ and the section $\sigma$ of the source map $s$ such that $f_\sigma(U) \subset V$, one can consider $f_\sigma$ as a morphism $k_U \to k_V$ of $\mathcal{M}_n$-charts on $X$. Denote by $\Phi = \Phi(\mathcal{G})$ the set of $\mathcal{M}_n$-charts on $X$ of type $k_U$. Consider also the category $\mathcal{B}(\mathcal{G})$ with $\Phi$ as the set of objects whose morphisms are the étale maps $f_\sigma$ as above. We have the natural covariant functor $\Pi : \mathcal{E}(\mathcal{G}) \to \mathcal{B}(\mathcal{G})$ such that, for an open subset $U \subset \mathcal{G}_0$, we have $\Pi(U) = U$ and, for a section $\sigma$ as above, we have $\Pi(\sigma) = f_\sigma : k_U \to k_V$. By definition, the functor $\Pi$ is full.

Consider the covariant functor $J \circ D \circ \Pi : \mathcal{E}(\mathcal{G}) \to \text{Sets}$. Prove that the set of maps $k_U : J \circ D(k_U) : U \to X$ is an inductive limit $\text{lim}_\leftarrow J \circ D$.
It is obvious that the codomains of the \( M_\sigma \)-charts \( k_\sigma \) cover \( X \). Let \( U, V \in \mathcal{E}(G)_0 \), \( x \in U \), \( y \in V \), and \( k_U(x) = k_V(y) \). Since \( X = G_0/\sim \), there is a \( G \)-morphism \( g : x \to y \). First assume that \( x = y \) and \( U \subset V \). Consider the section \( \varepsilon : U' \to G_1 \) defined by \( \varepsilon(x') = 1_U \) for any \( x' \in U \). Then \( \varepsilon \) is a morphism \( k_U \to k_V \) such that \( f_{\varepsilon}(x) = y \) and we are done. In the general case consider a smooth section \( \sigma : U \to G_1 \) of the source map \( s \) such that \( \sigma(x) = g \). By the argument above one can assume that \( f_{\sigma}(U) \subset V \). Then for \( f_{\sigma} \) as a morphism \( k_U \to k_V \), we have \( f_{\sigma}(x) = y \) and we are done again. Thus, \( \mathcal{E}(G) \) is a locally invertible étale structure on \( X \).

Let \( G \) be a smooth groupoid and let \( \mathcal{E}(G) \) be the corresponding locally invertible étale structure on \( X = G_0/\sim \). Consider the groupoid \( \mathcal{G}(\mathcal{E}(G)) \). By definition we have an obvious étale map \( \mathcal{G}(\mathcal{E}(G))_0 \to G_0 \). Then, by remark \( \mathcal{E} \) one can consider the corresponding reduced groupoid \( \mathcal{G}(\mathcal{E}(G)) \) with \( G_0 \) as the set of objects.

By construction, for any étale groupoid \( G \), we have \( \mathcal{G}(\mathcal{E}(G)) = G \). Thus, to study an étale groupoid \( G \) one can equivalently use the corresponding étale structure \( \mathcal{E}_G \).

Consider the category \( \mathcal{E}(G) \) for which \( \mathcal{E}(G)_0 = S \), \( \mathcal{E}(G)_1 \) coincides with the set of morphisms defined above, and the composition of morphisms defined naturally.

3.4. Proposition. \( \mathcal{E}(G) \) is an étale structure on \( X = G_0/\sim \).

Proof. For each \( \sigma \in S \) corresponding to a representation \( W = U \times V \), put \( D(\sigma) = \sigma(U) \) and \( k_\sigma = p \circ s|_{D(\sigma)} : D(\sigma) \to X \), where \( p : G_0 \to X \) is the projection. By definition, \( k_\sigma \) is an \( M_\sigma \)-chart on \( X \). It is clear that each point of \( X \) lies in the codomain of one of the maps \( k_\sigma \).

Let, for \( \sigma, \sigma' \in S \) and \( g \in D(\sigma) \) and \( g' \in D(\sigma') \), we have \( k_\sigma(g) = k_{\sigma'}(g') \). Let \( \sigma \in S \) be a section corresponding to a representation \( W = U \times V \) of a neighborhood of \( g \in G_1 \) such that \( g \in D(\sigma) \). Since \( i : G_1 \to G_1 \) is a diffeomorphism, \( j(W)' = U' \times V' \) is a representation Consider, for the sections \( \sigma \) and \( \sigma' \), the corresponding representations \( W = U \times V \) and \( W' = U' \times V' \). First assume that we have \( g = g' \) and \( U \subset U' \). Then the inclusion \( U \subset U' \) induces a morphism \( f : \sigma \to \sigma' \) such that \( f(g) = g' \). Then, in the general case, one may suppose that \( f_{\sigma}(U) \subset U' \). Under this assumption, the map \( f_{\sigma} \) induces a morphism \( f : \sigma \to \sigma' \) such that \( k_\sigma(g) = k_{\sigma'}(g') \). Thus, \( k_\sigma(g) \) does not depend on the choices of the representation \( W \) and the section \( \sigma \) through \( g \).

Let \( t(g) = s(g') \). Consider the representation \( W' = U' \times V' \) of a neighborhood \( g' \) and the section \( \sigma' \) such that \( \sigma'(t(g)) = g' \). One may assume that \( f_{\sigma'}(U) \subset U' \). Then \( f_{\sigma'} \) induces a morphism \( f : \sigma \to \sigma' \). Then, for the morphism \( f \), we have \( f(g) = g' \).

Consider the general case. By the initial assumption, there are morphisms \( g_1 : t(g) \to s(g') \), \( g_2 : t(g_1) = s(g') \), and sections \( \sigma_1, \sigma_2 \in S \) such that \( \sigma_1(t(g)) = g_1 \) and \( \sigma_2(t(g_1)) = g' \) respectively. We may suppose that defined for sufficiently small neighborhoods \( U \) and \( U_1 \) of \( s(g) \) and \( s(g_1) \) there are morphisms \( F_1 : \sigma \to \sigma_1 \) and \( F_2 : \sigma_1 \to \sigma' \).

By assumption, there is a morphism \( g'' : t(g) \to t(g') \) from \( G_1 \). Consider the representation \( W'' = U'' \times V'' \) of a neighborhood of \( g'' \) and the section \( \sigma'' \) such that \( \sigma''(t(g)) = g'' \). One may assume that \( f_{\sigma''}(U) \subset U'' \) and \( f_{\sigma''} \subset U' \). Then \( f_{\sigma''} \) induces a morphism \( F : \sigma \to \sigma'' \) and \( f_{\sigma''} \) induces a morphism \( G : \sigma'' \to \sigma' \). Then, for the morphism the morphism \( G \circ F \), we have \( G \circ F(g) = g' \).

First assume that \( s(g) = s(g') \). By assumption, there is a morphism \( g'' \in G_1 \) with \( s(g'') = t(g') \) and \( t(g'') = t(g') \). Let \( W'' = U'' \times V'' \) be the corresponding representation of a neighborhood of \( g'' \) and let \( \sigma'' \) be the corresponding section such that \( \sigma''(s(g'')) = g'' \).
Put \( f_\sigma = t \circ \sigma \). We claim that \( f_\sigma \) is an étale map of \( U \) into \( G_0 \). Consider the corresponding representation \( W' = U' \times V' \) for the morphism \( g' = g^{-1} \). Since the problem is a local one, we may suppose that \( f(U) \subseteq U' \) and choose a section \( \sigma' : U' \to W' \). It suffices to consider the case when \( x = y \). Then \( \sigma' \circ f \) is a local section of \( s \) over \( U \) and there is a diffeomorphism \( d \) of \( \sigma(U) \) to \( \sigma' \circ f(U) \) (as submanifolds of \( G_1 \)) such that \( \sigma' \circ f = d \circ \sigma \). This implies our statement.

Denote by \( S \) the set of sections \( \sigma \) of the type above. Let \( k_\sigma \) be the restriction of the projection \( G_0 \to (G_0/\sim) = X \) to \( U_g \). Then we have an indexed family \( \{ k_\sigma \}_{\sigma \in S} \) of \( \mathcal{M}_n \)-charts on \( X \). For \( \sigma_1, \sigma_2 \in S \), a morphism \( \sigma_1 \to \sigma_2 \) is the étale map \( f_{\sigma_1} \) whenever \( f_{\sigma_1}(D(k_{\sigma_1})) \subseteq D(k_{\sigma_2}) \). Thus one obtains a locally invertible étale structure \( \mathcal{E}(\mathcal{G}) \) on \( X \).

In general \( \mathcal{E}(\mathcal{G}) \) has a more fine structure. Actually, for each local representation \( W = U \times V \) as above, we have a one-to-one correspondence between \( V \) and the set \( K_V \) of sections \( \sigma_z \) \((z \in V)\) of the type \( s_z(x) = (x, z) \). This correspondence defines a structure of a manifold on \( K_V \). Define a topology on \( S \) generated by all the sets \( K_V \). Moreover, one obtains on \( S \) a structure of a manifold of dimension \( \dim G_1 - \dim G_0 \).

Then, for each \( v \in V_g \), the map \( z \mapsto (z, v_0) \) defines a local section \( \sigma_v \) of \( s \) over \( U_g \).

Put \( fr_g, v = t \circ \sigma_v \). We claim that \( f = fr_g, v \) is an étale map of \( U_g \to G_0 \). Let, for some morphism \( g' : y \to z \) from \( G_1, U_{g'} \) be a neighborhood of \( y \in G_0 \) with the corresponding representation \( W_{g'} = U_{g'} \times V_{g'} \). Since the problem is a local one, we may suppose that \( f(U_g) \subseteq U_{g'} \). It suffices to consider the case when \( x = y \). Let \( \sigma_v' \) be the corresponding section \( U_{g'} \to G_1 \) over \( U_{g'} \) for \( g' = g^{-1} \). Then \( \sigma_v' \circ f \) is a local section of \( s \) over \( U_g \) and there is a diffeomorphism \( d \) of \( \sigma_v'(U_{g'}) \) to \( \sigma_v' \circ f(U_{g'}) \) (as submanifolds of \( G_1 \)) such that \( \sigma_v' \circ f = d \circ \sigma_v \). This implies our statement.

For each representation \( r_g : W_g = U_g \times V_g \) as above, we denote by \( k_{r_g} \) the restriction of the projection \( G_0 \to X = G_0/\sim \) to \( U_g \). Then we have an indexed family \( \Phi = \{ k_{r_g} \} \) of \( \mathcal{M}_n \)-charts on \( X \) with the set of indexes \( I = \{ r_g \} \). For any two \( \mathcal{M}_n \)-charts \( k_{r_g} \) and \( k_{r_g'} \) on \( X \), such that for some \( v \in V_g \) we have \( fr_g, v(U_g) \subseteq U_{g'} \), the corresponding map \( fr_g, v : U_g \to U_{g'} \) is a morphism of \( \mathcal{M}_n \)-charts on \( X \). Thus one obtains an indexed locally invertible structure \( \mathcal{E}(\Phi) \) on \( X \).

Let \( \mathcal{G} \) be an étale groupoid and let \( X = \mathcal{G}_0/\sim \). Assume that \( \sigma \) is a section of \( s : G_1 \to G_0 \) on a neighborhood \( U \) of \( x \in G_0 \) with \( s(x) = y \) and \( U \) is so small that \( t \circ \sigma \) is a diffeomorphism from \( U \) onto its image. Consider a \( \mathcal{M}_n \)-chart \( k_{\sigma} \) on \( X \) which is the restriction of the projection \( G_0 \to X \) to \( U \). Then the \( \{ k_{\sigma} \}_{\sigma \in S} \) is an indexed family \( \Phi_S \) of \( \mathcal{M}_n \)-charts on \( X \). Consider the locally invertible indexed étale structure \( \mathcal{E}(\Phi_S) \) on \( X \) with the set of objects \( S \) whose morphisms \( \sigma_1 \to \sigma_2 \) are local diffeomorphisms \( t \circ \sigma_1 : U_1 \to U_2 \), where \( U_1 = D(k_{\sigma_1}) \) and \( U_2 = D(k_{\sigma_2}) = t \circ \sigma_1(U_1) \).

We generalize definition 2.3 as follows.

3.5. Definition. Let \( \mathcal{E}_{\Phi_1} \) and \( \mathcal{E}_{\Phi_2} \) be two étale structures on the sets \( X_1 \) and \( X_2 \) respectively. A map \( f : X_1 \to X_2 \) is called a morphism of \( \mathcal{E}_{\Phi_1} \to \mathcal{E}_{\Phi_2} \) if the following conditions hold:

1. For each \( k \in \Phi_1 \), we have \( f \circ k \in \Phi_2 \);
2. For each morphism \( m : k_1 \to k_2 \) from \( \mathcal{E}_{\Phi_1} \), the map \( m : D(f \circ k_1) = D(k_1) \to D(k_2) = D(f \circ k_2) \) is a morphism \( f \circ k_1 \to f \circ k_2 \) from \( \mathcal{E}_{\Phi_2} \).

Now we give examples of invertible \( \mathcal{M}_n \)-atlas.

1. Let \( N \) be an \((m + n)\)-dimensional manifold and let \( \mathcal{F} \) be a foliation of codimension \( n \) on \( N \). Denote by \( L(\mathcal{F}) \) the set of leaves of \( \mathcal{F} \) and by \( \Phi(\mathcal{F}) \) the set of smooth maps \( M \to N \) transverse to the leaves of \( \mathcal{F} \), where \( M \in (\mathcal{M}_n)_0 \).
3.6. Theorem. ([21], [22]) The set $\Phi(F)$ of $M_n$-charts on $L(F)$ is a locally invertible $M_n$-atlas on $L(F)$.

Let $T$ be a complete transversal of the foliation $F$, i.e. an embedded $n$-submanifold $M \subset N$ which transverse to the leaves of $F$ hitting each leaf at least once. Let $p : T \to L(F)$ be a projection. It is easy to check that the set $\Phi(T)$ of the restrictions of $p$ to open submanifolds of $T$ is a locally invertible $M_n$-atlas on $L(F)$ and $\Phi(T)$ is equivalent to the $M_n$-atlas $\Phi(F)$. By remark 3.3 the set of objects for the reduced groupoid $G(\Phi(T))$ equals $T$. Evidently, the groupoid $G_p(T)$ coincides with the so-called étale model $\text{Hol}_T(N, F)$ of the holonomy groupoid $\text{Hol}(N, F)$ of the foliated manifold $(N, F)$ (see, for example, [24], [35]).

Remark. For study the leaf space of a foliated manifold $(N, F)$ one usually uses a holonomy groupoid of $F$ with $N$ as the set of objects. From the point of view accepted in this paper this is not natural since different foliated manifolds may have isomorphic leaf spaces.

2. Let $X$ be a one point set $\{pt\}$ and let the set of $M_n$-charts be the set $\Phi(\{pt\})$ of all maps $M \to \{pt\}$, where $M \in \text{Ob} M_n$. It is evident that the pair $(\{pt\}, \Phi(\{pt\}))$ is a final object of the category $M_n$.

3. Consider the space $\mathbb{R}^n$ and the pseudogroup $\Gamma(\mathbb{R}^n)$ of all local diffeomorphisms of $\mathbb{R}^n$. It is clear that $\mathbb{R}^n/\Gamma(\mathbb{R}^n)$ is an one point set $\{pt\}$ and the set $\Phi(\Gamma(\mathbb{R}^n))$ of $M_n$-charts on $\{pt\}$ is a groupoid $M_n$-atlas on $\{pt\}$. Then the corresponding $M_n$-space is a final object of the category of $M_n$ spaces. By definition, the groupoid corresponding to $\Phi(\Gamma(\mathbb{R}^n))$ is the classifying Haefliger groupoid $\Gamma_n$ [15].

4. Morita equivalence

Let $\mathcal{G}$ and $\mathcal{H}$ be two smooth groupoids. A covariant functor $F : \mathcal{G} \to \mathcal{H}$ is smooth if the corresponding maps $\mathcal{G}_0 \to \mathcal{H}_0$ and $\mathcal{G}_1 \to \mathcal{H}_1$ are smooth. It is clear that the morphism $E_{\mathcal{G}_1} \to E_{\mathcal{H}_1}$ of locally invertible étale structures on the sets $X_1$ and $X_2$ respectively induces a smooth covariant functor $\mathcal{G}(E_{\mathcal{G}_1}) \to \mathcal{G}(E_{\mathcal{H}_1})$ of the corresponding étale groupoids.

Recall some definitions.

Let $\mathcal{G}$ be a smooth groupoid and let $M$ be a smooth manifold. A right action of $\mathcal{G}$ on $M$ consists of two smooth maps $\pi : M \to \mathcal{G}_0$ (the moment map) and $m : M \times_{\mathcal{G}_0} \mathcal{G}_1 = \{(x, g) : \pi(x) = t(g)\} \to M$ (action) such that, denoting $m(x, g) = xg$, we have

$$(xg)h = x(gh), \quad x1 = x, \quad \pi(xg) = s(g).$$

We will call $M$ a right $\mathcal{G}$-space with the moment map $\pi$. A left $\mathcal{G}$-space is defined similarly.

A (right) $\mathcal{G}$-bundle over the manifold $B$ consists of a smooth right $\mathcal{G}$-space $E$ and a smooth map $p : E \to B$ which is $\mathcal{G}$-invariant (i.e. $p(xg) = p(x)$). It is called principal, if $p$ is a surjective submersion and the map $E \times_{\mathcal{G}_0} \mathcal{G}_1 \to E \times_B E$, defined by $(e, g) \to (e, eg)$, is a diffeomorphism.

Let $\mathcal{G}$ and $\mathcal{H}$ be two smooth groupoids.

4.1. Definition. A homomorphism of groupoids (or the Hilsum-Skandalis map) $\mathcal{G} \to \mathcal{H}$ ([14], [23], [26]) consists of a manifold $P$, smooth maps (source and target): $s_{\mathcal{P}} : P \to \mathcal{G}_0$, $t_{\mathcal{P}} : P \to \mathcal{H}_0$, a left action of $\mathcal{G}$ on $P$ with the moment map $s_{\mathcal{P}}$, a right action of $\mathcal{H}$ on $P$ with the moment map $t_{\mathcal{P}}$, such that

1. $s_{\mathcal{P}}$ is $\mathcal{H}$-invariant, $t_{\mathcal{P}}$ is $\mathcal{G}$-invariant;
2. the actions of $\mathcal{G}$ and $\mathcal{H}$ commute: $(gp)h = g(ph)$;
3. $s_{\mathcal{P}} : P \to \mathcal{G}_0$, as an $\mathcal{H}$-bundle with the moment map $t_{\mathcal{P}}$ is principal.
The composition of two homomorphisms $P : G \to H$ and $Q : H \to K$ is defined by dividing $P \times_{H_0} Q$ by the action of $H$: $(p, q)h = (ph, h^{-1}q)$, and taking the natural actions of $G$ and $K$.

Thus, we have the category of smooth groupoids. Two smooth groupoids $G$ and $H$ are called Morita equivalent they are isomorphic in the category of smooth groupoids and an isomorphism $G \to H$ is called a Morita equivalence.

Now we compare definition 3.35 with the definition of a morphism of smooth groupoids.

4.2. Theorem. Let $E_{\Phi_1}$ and $E_{\Phi_2}$ be locally invertible étale structures on the sets $X$ and $Y$ respectively and let $G = E_{\Phi_1}$ and $H = E_{\Phi_2}$ be the corresponding étale groupoids. Then each homomorphism $G \to H$ induces a morphism $f : X \to Y$ of the corresponding diffeological spaces defined by $E_{\Phi_1}$ and $E_{\Phi_2}$. Conversely, each morphism $f : X \to Y$ of the diffeological spaces defines a homomorphism $G \to H$.

Proof. Let $P : G \to H$ be a homomorphism of smooth groupoids. By definition of the actions of $G$ and $H$ on $P$, for each $p \in P$, we have $G_{sp}(p) = G_{sp}(gp)$ and $tP(p)H = tP(pH)$. Moreover, by (3) of 4.1, $sp$ induces a surjective map of orbit spaces $P/G \to (G_0/G)$.

Let, for $p_1, p_2 \in P$, we have $G_{sp}(p_1) = G_{sp}(p_2)$. Then there is a $g \in (G)$ such that $sp(p_1) = sp(gp_1)$ and by (3) of 4.1 there is a unique $h \in (H)$ such that $gp_1 = ph$. Therefore, by (1) of 4.1, we have $tP(p_1H) = tP(gp_1H) = tP(p_1H)$. This implies that the correspondence $sp(\hat{G}) \to tP(pH)$ defines a map of orbit spaces $fP : G_0/G \to H_0/H_1$.

Put $X = G_0/G$ and $Y = H_0/H$ and consider the natural structures of diffeological spaces on $X$ and $Y$ defined by the groupoids $G$ and $H$.

Since, by (3) of 4.1, the $sp$ is a surjective submersion, for each $x \in G_0$, there are local sections of $sp \to G_0$ defined in sufficiently small neighborhoods of $x$. Let $\sigma : U \to P$ be such a section. Then the composition $tP \circ \sigma : U \to H_0$ is a smooth map and its composition with $H_0 \to H_0/H$ is a $M$-chart on $Y$ as a diffeological space. Let $\sigma_1, \sigma_2 : U \to P$ be two such sections.

By (3) of 4.1, there is a smooth map $h : U \to H_1$ such that for each $y \in U$ we have $\sigma_2(y) = \sigma_1(y)h(y)$. Then the $M$-charts on $Y$ induced by $\sigma_1$ and $\sigma_2$ as above coincide. Since the restrictions of the projection $G_0 \to G_0/G$ to the open subsets above form a $M$-atlas on $X$ of the structure of a diffeological space on $X$, the construction above induces a morphism of diffeological spaces $X \to Y$.

Conversely, let $G, H$ be two étale groupoids determined by the locally invertible $M_n$-atlases $\Phi_1$ on $X = G_0/G$ and $\Phi_2$ on $Y = H_0/H$ and let $f : X \to Y$ be a morphism of the corresponding diffeological spaces.

Put $P = \{(c, d, h) : c \in G_0, d \in H_0, f \circ pG(c) = pH(d), h \in H_1, t(h) = d\}$. Consider the following equivalence relation on $P$: $(c, d, h) \sim (c', d', h')$, where $h' \in H_1$ and $h' : d \to d'$, and the corresponding quotient set $P = P/\sim$. Define a topology on $P$ as follows. For each $c \in G_0$, consider a restriction $pG : G_0 \to X$ to some neighborhood $U$ of $c$ as a $M$-chart on $X$. Then, if $U$ is sufficiently small, the composition $f \circ pG : D$-chart on $Y$ of the type $pH \circ \varphi_U$, where $\varphi_U$ is a smooth map $U \to H_0$. Let $V$ be a neighborhood of $d = \varphi_U(c) \in H_0$ such that there is a smooth section $\sigma : V \to H_1$. We may assume that $\varphi_U(U) \subset V$. Consider a subset $P(U, \varphi, \sigma) = P$ defined by the points of $P$ of the type $(c', \varphi(c'), \sigma \circ \varphi(c'))$, where $c' \in U$. Consider the topology on $P$ generated by the subsets $P(U, \varphi, \sigma)$ for all $c \in G_0$, the maps $\varphi$, and the sections $\sigma$. Since each subset $P(U, \varphi, \sigma)$ is a smooth manifold, it is easy to check that there is a unique structure of a smooth manifold on $P$ such that each $P(U, \varphi, \sigma)$ is an open submanifold of $P$. 
Define the maps $\tilde{s}_P : \tilde{P} \to \tilde{G}_0$ and $\tilde{t}_P : \tilde{P} \to \tilde{H}_0$ as follows: $\tilde{s}_P(c, d, h) = c$ and $\tilde{t}_P(c, d, h) = s(h)$. It is easy to check that the maps $\tilde{s}_P$ and $\tilde{t}_P$ induce the smooth maps $s_P : P \to \tilde{G}_0$ and $t_P : P \to \tilde{H}_0$. Consider the left action of $G$ on $\tilde{P}$ and the right action of $H$ on $\tilde{P}$: $g(c, d, h) = (t(g), d, h)$, where $g \in G$ and $s(g) = c$, and $(c, d, h)h' = (c, d, hh')$, where $h' \in H_1$ and $t(h') = s(h)$. It is easy to check that these actions induce the smooth actions on $P$. It is clear that the map $s_P$ is $H$-invariant, the map $t_P$ is $G$-invariant, the actions of $G$ and $H$ commute, and $s_P : P \to \tilde{G}_0$, as an $H$-bundle with the moment map $t_P$, is principal. Thus, we have a homomorphism of groupoids $G \to H$.

**Remark.** Let $\varphi : G \to H$ be a smooth covariant functor. By definition, $\varphi$ induces a map $f : G_0 \to H_0$ which is the morphism of the corresponding diffeological spaces. It is easy to check that the smooth morphism of groupoids $G \to H$, defined by $P_{\varphi} = G_0 \times_{H_0} H_1 = \{(x, h) : \varphi(x) = t(h)\}$, $s_P(c, h) = c$, $t_P(c, h) = s(h)$ and the natural actions of $G$ and $H$ on $P_{\varphi}$, is a partial case of the construction of Theorem 4.2.

The following corollaries are evident.

4.3. **Corollary.** Let $\Phi_1$ and $\Phi_2$ be locally invertible $M_n$-atlases on the sets $X$ and $Y$ respectively and let $G = \mathcal{G}(\Phi_1)$ and $H = \mathcal{G}(\Phi_2)$ be the corresponding etale groupoids. Then the categories $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent iff the diffeological spaces $X$ and $Y$ defined by the $M_n$-atlases $\Phi_1$ and $\Phi_2$ are isomorphic as objects of the category $\mathcal{M}_n$.

4.4. **Corollary.** Let $\Phi_1$ and $\Phi_2$ be equivalent locally invertible $M_n$-atlases on the set $X$. Then the corresponding etale groupoids $\mathcal{G}(\Phi_1)$ and $\mathcal{G}(\Phi_2)$ are Morita equivalent.

5. **Classifying toposes**

Let $\Phi$ be a $M_n$-atlas or $M$-atlas on $X$ such that, for each $(M, k) \in \Phi$ and each open subset $U \subset M$ the restriction of $(M, k)$ with respect to the inclusion $U \subset M$ belongs to $\Phi$. Let $S$ be a contravariant functor on the category $\mathcal{C}_n$. For $(M, k) \in \Phi$, consider the standard category $\mathcal{C}(M)$ of $M$ as topological space, i.e. the category with objects open subsets $U$ of $M$ and morphisms the inclusions $U_1 \subset U_2$ of such open subsets. By assumption $\mathcal{C}(M)$ is a subcategory of $\mathcal{C}_n$. The functor $S$ is called a sheaf on $\Phi$ if, for each $(M, k) \in \Phi$, the restriction of $S$ to $\mathcal{C}(M)$ is a sheaf on $M$. It is easy to see that, if $\Phi$ is an invertible $M_n$-atlas on $X$, a sheaf on $\Phi$ defines a $\mathcal{C}(\Phi)$-sheaf in sense of [24] and, conversely, each $\mathcal{C}(\Phi)$-sheaf induces a sheaf on $\Phi$. Therefore, the classifying topos $\mathcal{B}(\mathcal{C}(\Phi))$ could be considered as the space of sheaves on $\Phi$.

Let $\Phi$ be a $M_n$-atlas on $X$ as above and let $(X, \Psi)$ be the corresponding diffeological space. Since, by definition, $\Psi = \Phi$, using the inverse image functor one can extend uniquely each sheaf on $\Phi$ to the sheaf on $\Psi$. This remark implies the following theorem.

5.1. **Theorem.** Let $\Phi_1$ and $\Phi_2$ be locally invertible $M_n$-atlases on the sets $X$ and $Y$ respectively, $\mathcal{G} = \mathcal{G}(\Phi_1)$ and $\mathcal{H} = \mathcal{G}(\Phi_2)$ the corresponding etale groupoids, $(X, \Psi_1)$ and $(Y, \Psi_2)$ the corresponding diffeological spaces. Then each morphism $f : X \to Y$ of diffeological spaces induces a morphism $\mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ of toposes. If $f$ is an isomorphism, it induces an isomorphism of the corresponding toposes.

Any sheaf of sets, groups, rings or algebras on the category $M_n$ ($\mathcal{D}_n$) induces a sheaf of sets, groups, rings or algebras on any $M_n$-space. For a foliated manifold $(N, F)$, such a sheaf on the $M_n$-space $N/F$ coincides with the known notion of transversal sheaf of a foliation. In general, the construction above shows how to extend the notion of the transversal structure of foliations to arbitrary $M_n$-spaces.
6. Nerve and classifying space for a smooth groupoid

For a groupoid $\mathcal{G}$, denote by $\mathcal{G}_n$ the space of composable strings of morphisms in $\mathcal{G}_n$:

\[(6.0.1) \quad x_0 \xleftarrow{g_1} x_1 \xleftarrow{g_2} \ldots \xleftarrow{g_n} x_n.\]

The spaces $\mathcal{G}_n$ ($n \geq 0$) with the face maps $d_i : \mathcal{G}_n \to \mathcal{G}_{n-1}$ defined in the usual way:

\[d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) & \text{if } 1 \leq i \leq n-1 \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}\]

form a simplicial space, the nerve of $\mathcal{G}$, denoted by $\mathcal{N}(\mathcal{G})$. Its geometric realization is the classifying space $B\mathcal{G}$.

For a smooth groupoid $\mathcal{G}$, the notion above agree with the earlier notation for $n = 0, 1, 2$. Thus, the space $\mathcal{G}_n$ is a fibered product $\mathcal{G}_1 \times \mathcal{G}_0 \times \cdots \times \mathcal{G}_0 \mathcal{G}_1$ and, hence, has a structure of a smooth manifold.

A Morita equivalence $\varphi : \mathcal{G} \to \mathcal{H}$ induces a weak homotopy equivalence $B\mathcal{G} \to B\mathcal{H}$ ([15]).

Lemma [13] and [24] imply the following

6.1. Theorem. Let $\Phi_1(X)$ and $\Phi_2(X)$ be two full $\mathcal{M}_n$-atlases on $X$ such that $\Phi_1(X) \subset \Phi_2(X)$ and let $\mathcal{G}_{\Phi_1}(X)$ and $\mathcal{G}_{\Phi_2}(X)$ be the corresponding étale groupoids. Then the classifying spaces of $\mathcal{G}_{\Phi_1}(X)$ and $\mathcal{G}_{\Phi_2}(X)$ are homotopy equivalent.

Let $\mathcal{G}$ be a étale groupoid such that its set of objects $\mathcal{G}_0$ is an $n$-dimensional manifold $M$. For example, for the Haefliger groupoid $\Gamma_n$ we have $M = \mathbb{R}^n$. Let $\mathcal{F}$ be a foliation on a manifold $N$ of codimension $n$ and let $T$ be a complete transversal of $\mathcal{F}$. Consider the full $\mathcal{M}_n$-atlas $\Phi$ on the set of leaves $N/\mathcal{F}$ determined by $T$ and the corresponding étale category $\mathcal{G}_{\Phi}$. For this category we have $M = T$.

In this case one can define the so-called embedding category $\text{Emb}(\mathcal{G})$. Objects of the category $\text{Emb}(\mathcal{G})$ are the elements of a fixed base $U$ of the topology of $M = G_0$ consisting of contractible open subsets. For two such open subsets $U$ and $V$, each section $\sigma : U \to G_1$ of the source map $s$ with the property that $t \circ \sigma : U \to G_0$ defines an embedding $\tilde{\sigma} : U \to V$, is a morphism of the category $\text{Emb}(\mathcal{G})$. The composition in $\text{Emb}(\mathcal{G})$ is defined by $\tilde{\tau} \circ \tilde{\sigma}(x) = \tau(t(\sigma(x))) \cdot \sigma(x)$ (multiplication in $G$).

The nerve of the category $\text{Emb}(\mathcal{G})$ is the simplicial set, whose geometric realization is the classifying space of $\text{Emb}(\mathcal{G})$ and is denoted by $B\text{Emb}(\mathcal{G})$. There is the following

6.2. Theorem. ([25]) For any étale groupoid $\mathcal{G}$ the classifying spaces $B(\mathcal{G})$ and $B\text{Emb}(\mathcal{G})$ are weakly homotopy equivalent.

In contrast to $B\mathcal{G}$, the classifying space $B\text{Emb}(\mathcal{G})$ is a CW-complex.

6.3. Corollary. The classifying spaces $B\mathcal{G}_{\Phi}$ and $B\text{Emb}_{\Phi}$ are weakly homotopy equivalent.

7. The Čech-De Rham cohomology of $\mathcal{M}_n$-spaces

Consider the double complex

\[C^{k,l} = \check{C}^k(U, \Omega^l) = \prod_{U_0 \xrightarrow{h_1} \ldots \xrightarrow{h_k} U_k} \Omega^l(U_0).\]

Here the product ranges over all $k$-tuple composable morphisms, and $\Omega^l(U_0)$ is the space of differential forms on $U_0$. For $\omega \in C^{k,l}$, we denote its component by $\omega(h_1, \ldots, h_k) \in \Omega^l(U_0)$. Let $X$ be a $\mathcal{M}_n$-space defined by a full $\mathcal{M}_n$-atlas $\Phi$ and
let $\mathcal{G}_\Phi$ be the corresponding étale groupoid. Assume that the space of objects of $\mathcal{G}_\Phi$ is an $n$-dimensional manifold $M$.

Let $X$ be a $\mathcal{M}_n$-space defined by a full $\mathcal{M}_n$-atlas $\Phi$ formed by the charts of the type $(U, k)$, where $U \in \mathcal{U}$. Let $\mathcal{D}_\Phi$ be the subcategory of the category $\mathcal{M}_n$ associated with $\Phi$ and let $\text{Emb}(\mathcal{D}_\Phi)$ be a subcategory of $\mathcal{D}_\Phi$ with the same set of objects and morphisms of the type $g : U \to V$, where $g$ is an embedding of $U$ into $V$. By definition, $\text{Emb}(\mathcal{D}_\Phi)$ is the embedding category for the étale groupoid $\mathcal{G}_\Phi$.

Consider the Čech complexes for $\mathcal{G}_\Phi(X)$ as follows:

$$
\hat{\mathcal{C}}_\mathcal{U}(\Phi(X), \Omega^p): \prod_{U_0} \Omega^p(U_0) \xrightarrow{\delta} \prod_{U_0 \supseteq U_1} \Omega^p(U_0) \xrightarrow{\delta} \prod_{U_0 \supseteq U_1 \supseteq U_2} \Omega^p(U_0) \xrightarrow{\delta} \ldots,
$$

where the product is taken over the strings of composable arrows of $\mathcal{G}_\Phi$, with the boundary

$$(7.0.1) \quad (\delta \omega)(g_1, \ldots, g_{p+1}) = g_1^* \omega(g_2, \ldots, g_{p+1}) + \sum_{i=1}^p (-1)^i \omega(g_1, \ldots, g_{i+1}g_i, \ldots, g_{p+1}) + (-1)^{p+1} \omega(g_1, \ldots, g_p).$$

The Čech-De Rham complex $(\hat{\mathcal{C}}_\mathcal{U}(\Phi(X), \Omega^*), D)$ of a $\mathcal{M}_n$-space $X$ is the total complex for the double complex $(\hat{\mathcal{C}}_\mathcal{U}(\Phi(X), \Omega^*), \delta, d)$, where $d$ is the De Rham differential and $D = \delta \pm d$ with the standard sign convention.

Denote by $\tilde{H}_q^\delta(X, \mathbb{R})$ the cohomology of the total complex $(\hat{\mathcal{C}}_\mathcal{U}(\Phi(X), \Omega^*), D)$ and call it the Čech-De Rham cohomology of the $\mathcal{M}_n$-space $X$ (relative to $\Phi(X)$ and $\mathcal{U}$).

Consider the first spectral sequence of the double complex $(\hat{\mathcal{C}}_\mathcal{U}(X, \Omega^*), \delta, d)$. It is clear that, for the spectral sequence, we have $E_2^{pq} = E_\infty^{pq} = 0$ for $q > 0$ and $E_2^{0q} = E_\infty^{0q} = H^p(B\mathcal{G}_\Phi, \mathbb{R})$. Therefore, we have an isomorphism $\tilde{H}_q^\delta(X, \mathbb{R}) = H^*(B\mathcal{G}_\Phi, \mathbb{R})$.

Thus, we have the following

7.1. Theorem. For a $\mathcal{M}_n$-space $X$ defined by a full $\mathcal{M}_n$-atlas $\Phi$, the Čech-De Rham cohomology $\tilde{H}_q^\delta(\Phi(X), \mathbb{R})$ does not depend on the choice neither of a full $\mathcal{M}_n$-atlas $\Phi$, nor of a base $\mathcal{U}$ and is denoted by $H^*(X, \mathbb{R})$.

The cohomology $H^*(X, \mathbb{R})$ is called the real cohomology of $\mathcal{M}_n$-space $X$ and denoted by $H^*(X, \mathbb{R})$.

8. Categories $\mathcal{P}_n(G)$ and $\mathcal{P}_n(G)$-spaces

Let $G$ be a Lie group. Consider a category $\mathcal{P}_n(G)$ with objects smooth principal $G$-bundles with $n$-dimensional bases and morphisms the morphisms of such principal $G$-bundles which project to the étale maps of bases. Applying the general procedure of [22] to this category one obtains the category of $\mathcal{P}_n(G)$-spaces. By definition, a $\mathcal{P}_n(G)$-chart on a set $Y$ is a pair $(k, P)$, where $k : P \to Y$, where $P \in \text{Ob} \mathcal{P}_n(G)$. Morphisms of such charts are defined by means of morphisms of the category $\mathcal{P}_n(G)$. A $\mathcal{P}_n(G)$-atlas on a set $Y$ is the set $\Psi$ of $\mathcal{P}_n(G)$-charts on $Y$ such that the set of maps $k : J \circ I_{\Phi(Y)}(P, k) = J(P) \to Y ((P, k) \in \Psi(Y)))$ is an inductive limit $\varinjlim J \circ I_{\Phi(Y)}$ of the functor $J \circ I_{\Phi}$, where the functors $J$ and $I_{\Phi(Y)}$ are similar to those of definition [2.1]. Therefore, one defines a $\mathcal{P}_n(G)$-space as a set $Y$ with a a maximal $\mathcal{P}_n(G)$-atlas $\Psi_m(Y)$ on $Y$. For a principal smooth $G$-bundle $P$ with a base $B$ the projection $p : P \to B$ is a covariant functor from $\mathcal{P}_n(G)$ to $\mathcal{M}_n$. By the definition of the category $\mathcal{P}_n$, for any $\mathcal{P}_n$-space $Y$ the $\mathcal{P}_n$-atlas on $Y$ induces a $\mathcal{M}_n$-atlas on some set $B_Y$ so that the extension of the functor $P \to B$ gives the covariant functor $Y \to B_Y$ from the category of $\mathcal{P}_n$-spaces to the category $\mathcal{M}_n_{\text{sp}}$. The $\mathcal{M}_n$-set $B_Y$ is called the base of the $\mathcal{P}_n$-space $Y$. 
A \( \mathcal{P}_n(G) \)-atlas \( \Psi \) on a set \( Y \) is called full if the corresponding \( \mathcal{M}_n \)-atlas on the base \( By \) is full.

**Remark.** 1. For a \( \mathcal{M}_n \)-space \( X \), consider the corresponding frame bundle \( Fr(X) \) (\( Fr \) is the extension of the functor putting in correspondence to a smooth manifold its frame bundle). By definition, \( Fr(X) \) is a \( \mathcal{P}_n(\text{GL}(n, \mathbb{R})) \)-space.

2. Consider a transversal principal \( G \)-bundle \( P \) over a foliation \((N, \mathcal{F})\) of codimension \( n \). One can consider \( P \) as a principal \( G \)-bundle over \( N \) with a smooth action of the holonomy groupoid \( \text{Hol}(\mathcal{F}) \) of the foliation \( \mathcal{F} \) on \( P \) which commutes with the action of \( G \) on \( P \) (see [13]). It is evident that the action of \( \text{Hol}(\mathcal{F}) \) on \( P \) induces an action on \( N \). It is easy to construct a full \( \mathcal{P}_n(G) \)-atlas on the set \( Y \) of orbits of the groupoid \( \text{Hol}(\mathcal{F}) \) on \( N \).

Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \) and let \( p : P \to B \) be a smooth principal \( G \)-bundle. For \( \xi \in \mathfrak{g} \), denote by \( X_\xi \) the corresponding vector field on \( P \) induced by the action of \( G \) on \( P \). Let \( T_p \) be a tangent space at \( p \in P \) and let \( l_p : T_p \to \mathfrak{g} \) be a linear map such that, for any \( \xi \in \mathfrak{g} \), we have \( l_p(X_\xi(p)) = \xi \). The space \( \hat{P} \) of all such linear maps \( l_p \) (\( p \in P \)) is a smooth fiber bundle over \( P \) with the projection \( l_p \to p \).

Consider a right action of \( G \) on \( \hat{P} \) defined by: \( g \to \alpha(g) = \text{ad}^{-1}(g) \circ l_p \circ (R_g)_{\text{adj}}^{-1} \), where \( g \to R_g \) is the action of \( G \) on \( P \).

The following lemma is evident.

**8.1. Lemma.** The fiber bundle \( \hat{P} \to P \) possesses the following properties:

1. the fibers of the fiber bundles \( \hat{P} \to P \) and \( \hat{P}/G \to B \) are affine spaces and, then, the spaces \( \hat{P}/G \) and \( B \) are homotopy equivalent;
2. the transformation \( \alpha_g \) is an automorphism of the affine fiber bundle \( \hat{P} \to P \);
3. the correspondence \( g \to \alpha_g \) defines a free action of \( G \) on \( \hat{P} \) and \( P \to \hat{P}/G \) is a smooth principal \( G \)-bundle;
4. \( \hat{P}/G \) is a smooth fiber bundle over \( B \) with affine fibers;
5. each connection form \( \omega \) on \( P \) is a smooth \( G \)-invariant section of the fiber bundle \( \hat{P} \to P \) or a smooth section of the fiber bundle \( \hat{P}/G \to B \).

Define a differential 1-form \( \varkappa \) on \( \hat{P} \) with values in \( \mathfrak{g} \) as follows. For a tangent vector \( w \) at \( l_p \in \hat{P} \), put \( \varkappa(w) = l_p(\hat{p}, w) \in \mathfrak{g} \). The following lemma follows from the definitions directly.

**8.2. Lemma.** The form \( \varkappa \) have the following properties:

1. \( \varkappa \) is a connection form on the principal \( G \)-bundle \( \hat{P} \);
2. \( \varkappa \) is invariant under the natural action of \( G \) of the group of automorphisms of the principal \( G \)-bundle \( P \) on \( \hat{P} \);
3. Let \( \omega \) be a connection form on \( P \) and \( s : P \to \hat{P} \) is the corresponding section. Then we have \( \omega = s^* \varkappa \).

The connection \( \varkappa \) is called a canonical connection. We denote by \( K \) the curvature form of \( \varkappa \).

Now we apply the Chern-Weil homomorphism to the canonical connection \( \varkappa \). Assume that the Lie group \( G \) is reductive. It is known that the space of invariant polynomials \( I(\mathfrak{g}) \) on the Lie algebra \( \mathfrak{g} \) is a free algebra generated by a finite set of homogeneous polynomials.

Let \( F \in I(\mathfrak{g}) \) is a homogeneous polynomial of degree \( p \). Denote by \( F(K) \) the differential \( 2p \)-form on \( \hat{P}/G \) obtained by the substitution of \( K \) instead of a variable of \( F \) with the following alternation. The form \( F(K) \) is a closed form on the space \( \hat{P}/G \) and its cohomology class depends only on the structure of the principal \( G \)-bundle \( P \). This cohomology class is called the **characteristic class** of \( P \) corresponding to the polynomial \( F \). Since, by (1) of lemma 8.1, the spaces \( \hat{P}/G \) and \( B \)
are homotopy equivalent, we obtain the homomorphism $I(\mathfrak{g}) \to H^*(B, \mathbb{R})$ which is called the characteristic Chern-Weil homomorphism.

Let $\omega$ be a connection form on the principal $G$-bundle $P \to B$ and $s : B \to \tilde{P}/G$ the corresponding section of the bundle $\tilde{P}/G \to B$. Then $s^*F(K) = F(R)$, where $R$ is the curvature form of $\omega$, and the map $F \to F(K)$ defines the standard characteristic Chern-Weil homomorphism associated with the connection $\omega$. It is clear that it does not depend on the choice of $\omega$.

Consider the covariant functors $P \to \tilde{P}$, $P \to B$, $P \to \tilde{P}/G$, and their extensions to the category of $\mathcal{P}_n(G)$-spaces.

Moreover, consider the contravariant functors $P \to \Omega^*(\tilde{P})$, $P \to \Omega^*(\tilde{P}/G)$, and the functor morphisms $\tilde{P} \to \tilde{P}/G$ and $\tilde{P}/G \to B$, and their extensions to the category of $\mathcal{P}_n(G)$-spaces.

Thus, we get the de Rham complexes $\Omega(\tilde{P})$, $\Omega(\tilde{P}/G)$, and $\Omega(BY)$ for any $\mathcal{P}_n(G)$-space $Y$ and their cohomologies $H^*(\tilde{P})$, $H^*(\tilde{P}/G)$, and $H^*(BY)$. It is clear that the characteristic homomorphism $I(\mathfrak{g}) \to \Omega^*(\tilde{P}/G)$ is extended to the characteristic homomorphisms $I(\mathfrak{g}) \to \Omega^*(\tilde{P}/G)$ and $I(\mathfrak{g}) \to H^*(\tilde{P}/G)$ for any $\mathcal{P}_n(G)$-space $Y$.

**Remark.** The construction of the characteristic homomorphism $I(\mathfrak{g}) \to \Omega^*(\tilde{P}/G)$ is based on the statement (2) of lemma 8.2. For a $\mathcal{P}_n(G)$-space $Y$, consider a morphism $\tilde{Y}/G \to BY$ and the corresponding homomorphism $H^*(BY) \to H^*(\tilde{Y}/G)$. In contrast to the isomorphism $H^*(B) \to H^*(\tilde{Y}/G)$ (see (1) of lemma 8.1), the homomorphism $H^*(BY) = H^*(\tilde{Y}/G)$ is not an isomorphism in general. Therefore, the image of the characteristic homomorphism $I(\mathfrak{g}) \to H^*(\tilde{Y}/G)$ cannot be projected to $H^*(BY)$.

Now we show how to extend the characteristic homomorphism to $\mathcal{P}_n(G)$-spaces.

Let $\Psi$ be a full $\mathcal{P}_n(G)$-atlas on a $\mathcal{P}_n(G)$-space $Y$ and let $\Phi(B_Y)$ be the corresponding $\mathcal{M}_n$-atlas on the base $B_Y$. Let us take an $n$-dimensional manifold $M$ and a base $\mathcal{U}$ of the topology of $M$ consisting of contractible open subsets of $M$. Assume that the $\mathcal{M}_n$-atlas $\Phi(B_Y)$ is formed by the charts of the type $(U, k)$, where $U \in \mathcal{U}$. For each $U \in \mathcal{U}$, let us choose a $\mathcal{P}_n(G)$-chart $(P_U, k) \in \Psi$ which induces a $\mathcal{M}_n$-chart $(U, k)$ on $B_Y$. Since $U$ is contractible, for any $U, V \in \mathcal{U}$ and each embedding $\sigma : U \to V$ there is a unique morphism of principal G-bundles $P_U \to P_V$ which projects to $\sigma$.

Consider the Čech-De Rham complex $(\tilde{\mathcal{C}}_{\mathcal{U}}(\Phi(X), \Omega^*), D)$ for $\Phi(B_Y)$. By the arguments above, each string of composable morphisms of the $\mathcal{M}_n$-atlas $\Phi(B_Y)$

$$\tilde{\mathcal{C}}_{\mathcal{U}}(\Phi(B_Y), \Omega^p) : \prod_{U_0} \Omega^p(U_0) \xrightarrow{\delta} \prod_{U_0 \supseteq \cup_{U_1}} \Omega^p(U_0) \xrightarrow{\delta} \prod_{U_0 \supseteq \cup_{U_1} \cup_{U_2}} \Omega^p(U_0) \xrightarrow{\delta} \ldots,$$

could be covered by the string of composable morphisms of $\mathcal{P}_n(G)$-atlas on $Y$. Applying the extension of the functor $\tilde{P} \to \tilde{P}/G$ to the latter string we get the string

$$\prod_{U_0} \Omega^p(P_{U_0}) \xrightarrow{\delta} \prod_{U_0 \supseteq \cup_{U_1}} \Omega^p(P_{U_0}) \xrightarrow{\delta} \prod_{U_0 \supseteq \cup_{U_1} \cup_{U_2}} \Omega^p(P_{U_0}) \xrightarrow{\delta} \ldots$$

and then we have the corresponding Čech-De Rham complex $(\tilde{\mathcal{C}}_{\mathcal{U}}(\Phi(X), \Omega^*), D)$ for $\Phi(B_Y)$.

9. The first Chern class for the Reeb foliation

9.1. The cohomologies $H^*(W_n)$ and $H^*(W_n, \mathfrak{gl}(n, \mathbb{R}))$. Let $W_n$ be the algebra of formal vector fields in $n$ variables, i.e. the topological vector space of $\infty$-jets at
0 of smooth vector fields on \(\mathbb{R}^n\) with the bracket induced by the Lie bracket of vector fields on \(\mathbb{R}^n\). Consider \(\mathbb{R}\) as a trivial \(W_n\)-module. The complex \(C^*(W_n) = \{C^q(W_n), d^q\}\) of standard continuous cochains of \(W_n\) with values in \(\mathbb{R}\) is defined as follows: \(C^q(W_n)\) is the space of continuous skew-symmetric \(q\)-forms on \(W_n\) with values in \(\mathbb{R}\) and the differential \(d^q : C^q(W_n) \to C^{q+1}(W_n)\) is defined by the following formula:

\[
(d^q c)(\xi_1, \ldots, \xi_{q+1}) = \sum_{i,j} (-1)^{i+j} c([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{q+1}),
\]

where \(c \in C^q(W_n), \xi_1, \ldots, \xi_{q+1} \in W_n\), and, as usual, \(\hat{\xi}\) means that the term \(\xi\) is omitted. We denote the cohomology of this complex by \(H^*(W_n) = H^p(W_n)\).

The natural action of \(GL(n, \mathbb{R})\) on \(\mathbb{R}^n\) induces an action of \(GL(n, \mathbb{R})\) on \(C^*(W_n)\) by automorphisms of this complex. Then we have the subcomplex of relative cochains \(C^*(W_n, GL(n, \mathbb{R}))\) of \(W_n\) with respect to \(GL(n, \mathbb{R})\) consisting of \(GL(n, \mathbb{R})\)-invariant cochains from \(C^*(W_n)\). We denote the cohomology of this complex by \(H^*(W_n, GL(n, \mathbb{R})) = \{H^p(W_n, GL(n, \mathbb{R}))\}\).

Recall some facts about the cohomology \(H^*(W_n, GL(n, \mathbb{R}))\) ([1], [6], [8]). By definition, \(C^*(W_n)\) is a graded differential algebra and the differential \(d\) is an antiderivation of degree 1. For \(\xi^i \in \mathbb{R}[\mathbb{R}^n]\) and \(\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x^i} \in W_n\), put

\[
c^i_{j_1 \ldots j_r}(\xi) = \frac{\partial^r \xi^i}{\partial x^{j_1} \ldots \partial x^{j_r}} (0),
\]

where \(x^i (i = 1, \ldots, n)\) are the standard coordinates in \(\mathbb{R}^n\). By definition, we have \(c^i_{j_1 \ldots j_r} \in C^1(W_n)\). Moreover, \(c^i_{j_1 \ldots j_r}\) for \(r = 0, 1, \ldots, n\) and \(i, j_1 \ldots j_r = 1, \ldots, n\) are generators of the DG-algebra \(C^*(W_n)\). Since \(d = \{d^q\}\) is an antiderivation of degree 1 of \(C^*(W_n)\), it is uniquely determined by the following conditions:

\[
d c^i_{j_1 \ldots j_r} = \sum_{0 \leq k \leq r} \sum_{s_1 < \ldots < s_k} \sum_{l=1}^n c^i_{j_1 \ldots j_{s_l} \ldots j_{s_k} \ldots j_{s_k} \ldots j_{s_k}} c^j_{s_{l-1}} c^k_{s_k} \wedge c^l_{j_{s_k} \ldots j_{s_k}}.
\]

Put

\[
\gamma = (c^i_j), \quad \Psi^j_k = \sum_{k=1}^n c^i_{j_k} \wedge c^k_j, \quad \text{and} \quad \Psi = (\Psi^j_k).
\]

It is known that

\[
\Psi_p = \text{tr}(\Psi \wedge \ldots \wedge \Psi) \quad (p = 1, \ldots, n)
\]

are cocycles of \(C^*(W_n, GL(n, \mathbb{R}))\) and the cohomology classes of these cocycles generate \(H^*(W_n, GL(n, \mathbb{R}))\). The cohomology class of \(\Psi_p\) is called \(p\)th formal Chern class.

### 9.2. The space of frames of infinite order and the Gelfand-Kazhdan form.

Let \(M\) be a connected orientable \(n\)-dimensional smooth manifold. Denote by \(S(M)\) the space of frames of infinite order of \(M\), i.e. \(\infty\)-jets at 0 of germs at 0 of smooth regular at 0 \(\in \mathbb{R}^n\) maps from \(\mathbb{R}^n\) into \(M\). It is known that \(S(M)\) is a manifold with model space \(\mathbb{R}^\infty ([11])\).

Define the canonical Gelfand-Kazhdan 1-form \(\omega\) with values in \(W_n\) on \(S(M)\) ([7] and [11]). Let \(\tau\) be a tangent vector at \(s \in S(M)\) and let \(s(u)\) be a curve on \(S(M)\) such that \(\tau = \frac{ds}{du}(0)\). One can represent \(s(u)\) by a smooth family \(k_u\) of germs at 0 of regular at 0 \(\in \mathbb{R}^n\) maps \(\mathbb{R}^n \to M\), i.e. \(s(u) = j_0^\infty k_u\). Then put

\[
\omega(\tau) = -j_0^\infty \frac{ds}{du} (k_0^{-1} \circ k_u)(0).
\]

Let \(c \in C^q(W_n)\). For each \(s \in S(M)\) and \(X_1, \ldots, X_q \in T_s\), put

\[
\omega_c(X_1, \ldots, X_q) = c(\omega(X_1), \ldots, \omega(X_q)).
\]
It is known that \( c \mapsto \omega_c \) is a homomorphism of the complexes \( \alpha : C^*(W_n) \to \Omega^*(S(M)) \). Moreover, we have \( \alpha(C^*(W_n, \text{GL}(n, \mathbb{R}))) = \Omega^*(S(M)/\text{GL}(n, \mathbb{R})) \).

It is easy to check that \( \beta = (\beta_j) = -(\alpha(c_i)) \) is a connection form on a principal \( \text{GL}_n(\mathbb{R}) \)-bundle \( S(M) \to S(M)/\text{GL}_n(\mathbb{R}) \), \( R = \Omega^2 + \beta \) is the curvature form of this connection and the image of the formal Chern class \( \Psi_p \) under \( \alpha \) equals \( \alpha(\Psi_p) = \text{Tr}(R \wedge \cdots \wedge R) \), \( p \) times.

**Case** \( n = 1 \). Now apply the constructions above to the case \( n = 1 \). Let \( x \) be a coordinate on \( M_1 \) near \( x_0 \in M_1 \) and \( k = k(u) \) be a smooth map \( \mathbb{R} \to M_1 \) regular at \( u = 0 \). By definition, \( s = j_0^\infty k \in S(M_1) \). Thus, one can consider \( x_0 = k(0), x_p = \frac{dk}{du}(0) \) \((p = 1, \ldots)\) as coordinates on \( S(M_1) \), where \( x_1 \neq 0 \). Let \( s(u) \) be a curve in \( S(M_1) \). It can be obtained as follows: \( s(u) = j_0^\infty k(u)(t) \), where for each \( u \), \( k(u)(t) \) is a smooth map \( \mathbb{R} \to M_1 \) regular at \( t = 0 \). Then \( \tau = \frac{d}{du}j_0^\infty k(u)(t)|_{u=0} \) is a tangent vector to the curve \( s(u) \) at \( s(0) \). By definition, we have \( \omega(\tau) = -j_0^\infty \frac{d}{du}(k^{-1}(0) \circ k(u))|_{u=0} \).

Let \( \omega = (\omega_0, \omega_1, \omega_2, \ldots) \) and \( x_p \) \((p = 0, \ldots)\) be the coordinates on \( S(M_1) \) above. By construction, we have

\[
\begin{align*}
\omega_0 &= -\frac{dx_0}{x_1}, & \omega_1 &= -\frac{x_2 dx_0}{x_1} - \frac{dx_1}{x_1}, \\
\omega_2 &= \left(\frac{x_3}{x_1} - \frac{2x_2^2}{x_1^2}\right) dx_0 + \frac{x_2^2 dx_1}{x_1} - \frac{dx_2}{x_1}.
\end{align*}
\]

Consider the action of \( \text{GL}(1, \mathbb{R}) = \mathbb{R}^* \) on \( S(M_1) \). By definition, we have for \( \lambda \in \mathbb{R}^* \) and \( s = (x_p) \): \( \lambda s = (\lambda x_p) \). Then one can take \( y_0 = x_0, y_p = (x_1^p)^{-1} x_p \) \((p = 2, \ldots)\) for the coordinates on \( S(M_1)/\text{GL}(1, \mathbb{R}) \). By definition, the first Chern class \( c_1 \) in these coordinates is defined by the 2-form \( c_{1,y} = dy_2 \wedge dy_0 \) on \( S(M_1)/\text{GL}(1, \mathbb{R}) \). It is clear that this form is exact on \( M_1 \). But we will show that, in general, this form is not exact on an \( \mathcal{D}_1 \)-space.

**9.3. Reeb’s foliation.** Describe Reeb’s foliation on the sphere \( S^3 \). Consider the sphere \( S^3 \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \) given by the equation \( |x|^2 + |y|^2 = 2 \), where \( |x| \) is the standard norm in \( \mathbb{R}^2 \). Take two subsets \( S^3_i \)(\(i = 1, 2\)) of \( S^3 \) defined by the equations \( |x|^2 \leq |y|^2 \) and \( |y|^2 \leq |x|^2 \) respectively. The projection \( (x, y) \to x \) induces the structure of fiber bundle on \( S^3_i \) with base \( D_2 = \{|x|^2 \leq 1\} \) and, for each \( x \in D_2 \), the fiber is the circle \( S^1 \) given by the equation \( |y|^2 = 2 - |x|^2 \). Therefore, \( S^3 \) is diffeomorphic to \( D_2 \times S^1 \). Similarly, \( S^3_2 \) is diffeomorphic to \( D_2 \times S^1 \), where \( D_2 = \{|y|^2 \leq 1\} \) and the fiber over \( y \in D_2 \) is given by the equation \( |x|^2 = 2 - |y|^2 \).

By definition, we have \( S^3 = S^3_1 \cup S^3_2 \).

Consider a smooth function \( f(t) \) on the interval \(|t| < 1\) satisfying the following conditions:

\[
\begin{align*}
(9.3.1) & \quad f(0) = 0, \quad f(t) \geq 0, \quad f(-t) = f(t), \\
(9.3.2) & \quad \lim_{t \to \pm 1} \frac{df}{dt} = \infty, \quad \lim_{t \to \pm 1} \frac{1}{|t|} \frac{df}{dt} = 0, \quad \text{for } p = 0, 1, \ldots
\end{align*}
\]

Consider the foliation of codimension 1 on \( S^3_1 \) with the leaf \( L = S^1 \times S^1 \) and the leaves \( L_\alpha \) for each \( \alpha \in \mathbb{R} \), consisting of the points of the type \((x, |y|e^{2\pi i (\alpha + f(t))})\), where \( x \) is an interior point of \( D_2 \) and we regard \( \mathbb{R}^4 \) as \( \mathbb{R}^2 \times \mathbb{C} \). It is clear that the leaf \( L \) is compact and the leaf \( L_\alpha \) is diffeomorphic to \( \mathbb{R}^2 \).

Similarly, we define a foliation of codimension 1 on \( S^3_2 \) using, in general, another function \( f \). Combining these two foliations we get a foliation on \( S^3 \) called the Reeb foliation and is denoted by \( \mathfrak{F}_R \).
Consider the curves \( \gamma_1: \alpha \to (0,0,\sqrt{2}e^{2\pi i t}) \) \((\alpha \in \mathbb{R})\) and \( \gamma_2: t \to \left(\frac{\alpha}{t^2}, \sqrt{\frac{\alpha}{2t}}, \sqrt{\frac{\alpha}{2t}}\right) \) \(|t| < \sqrt{2}\) on \( S^3 \). By construction, the curves \( \gamma_1 \) and \( \gamma_2 \) are transverse to Reeb’s foliation on \( S^3 \) and, then, can be considered as \( D_1 \)-charts on the space of leaves \( S^3/\mathbb{R} \). Since \( \gamma_1(\alpha) \in L_\alpha \) and \( \gamma_2(t) \in L_\alpha \) for \( f(|t|) + \alpha = 0 \), the map \( \varphi: \gamma_2 \to \gamma_1 \) is a morphism \( F \) of \( D_1 \)-charts \( \gamma_2 \to \gamma_1 \).

Denote by \( \alpha_i \) and \( t_i \) \((i \neq 1)\) the extension of the coordinates \( \alpha \) and \( t \) to the \( D_1 \)-space \( S(S^3/\mathbb{R})/\text{GL}(1, \mathbb{R}) \). By definition, the 2-form \( c_1 \), defining the first Chern class, is given in these coordinates by the forms \( c_{1,0} = d\alpha_0 \land d\alpha_2 \) and \( c_{1,1} = dt_2 \land dt_0 \), respectively. Evidently we have \( \varphi^*c_{1,0} = c_{1,1} \).

9.4. Theorem. The first Chern class \( c_1 \) is non-trivial in the complex \( \Omega^*(S(S^3/\mathbb{R})/\text{GL}(1, \mathbb{R})) \).

Proof. First we write the extension of the morphism \( \varphi \) to \( S(S^3/\mathbb{R}) \). Let \( f(x) \) and \( g(u) \) be two functions and let \( h = g \circ f \). Recall the classical Faà di Bruno formula for the \( n \)-th derivative of \( h^{(n)} \):

\[
(9.4.1) \quad h^{(n)} = n! \sum_{k=1}^{n} \frac{g^{(k)}}{k!} f^{(i_1)} \cdots f^{(i_k)} \frac{t^{i_1}}{i_1!} \cdots \frac{t^{i_k}}{i_k!}
\]

Applying (9.4.1) for \( g = -f(t) \) and \( f = t(u) \) we get

\[
\alpha_0 = -f(t_0), \quad \alpha_n = -n! \sum_{k=1}^{n} \frac{f^{(k)}}{k!} t^{i_1} \cdots t^{i_k} \frac{1}{i_1!} \cdots \frac{1}{i_k!}, \quad n \geq 1
\]

Then the extension of the morphism \( \varphi \) to \( S(S^3/\mathbb{R})/\text{GL}(1, \mathbb{R}) \) is defined by the formulas

\[
(9.4.2) \quad \beta_0 = -f(y_0), \quad \beta_n = (-1)^{n-1} \left( n! \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(y_0) t^{i_1} \cdots t^{i_k} \frac{1}{i_1!} \cdots \frac{1}{i_k!} + f^{(n)}(y_0) \right)
\]

where \( y_1 = \ldots = y_n = 1 \), \( n \geq 2 \), and \( \beta_0 \) are the coordinates on \( S(S^3/\mathbb{R})/\text{GL}(1, \mathbb{R}) \) corresponding to \( t_i \) and \( \alpha_i \).

By induction with respect to \( n \) from (9.4.1) for \( h(t) = \frac{1}{f(t)} \) and (9.3.1) we get for \( n > 1 \)

\[
(9.4.3) \quad \lim_{t \to \pm 1} \frac{f^{(n)}(t)}{(f(t))^n} = 0 \quad \text{and} \quad \lim_{t \to \pm 1} \frac{d}{dt} \left( \frac{f^{(n)}(t)}{(f(t))^n} \right) = 0.
\]

In particular, (9.4.3) implies \( \lim_{y_0 \to \pm 1} \beta_n = 0 \) for \( n > 1 \).

Assume that the class \( c_1 \) is trivial. Then there is a form \( \gamma \in \Omega^1(S(S^3/\mathbb{R})/\text{GL}(1, \mathbb{R})) \) such that \( f = d\gamma \). This means that, if \( \gamma_{\beta} \) and \( \gamma_{y} \) are the expressions of the form \( \gamma \) in the coordinates \( \beta_n \) and \( y_n \), respectively, we have \( d\gamma_{\beta} = c_{1,\beta}, d\gamma_{y} = c_{1,y} \), and \( \varphi^*\gamma_{\beta} = \gamma_{y} \).

Let \( \gamma_{\beta} = y_0 d\beta_0 + \sum_{n \geq 2} y_n d\beta_n \), where \( y_i \) are smooth functions of a finite number of \( \beta_k \) \((k \neq 1)\) and the sum have a finite set of summands. So we have \( \gamma_{\beta} = \beta_2 d\beta_1 + \lambda \), where \( \lambda = \sum_{n \neq 1} \lambda_n d\alpha_n \) is a closed 1-form in the finite set of the variables \( \beta_n \) \((n \neq 1)\) and the sum has a finite set of summands. Moreover, we have

\[
\varphi^*\gamma_{\beta} = A_0 d\gamma_0 + \sum_{n \geq 2} A_n d\gamma_n,
\]

where \( A_0 \) and \( A_n \) are smooth functions of a finite number of \( y_k \) \((k \neq 1)\) and the sum have a finite set of summands. By definition, the 1-form \( \varphi^*\gamma \) has a smooth extension to a neighborhood of each point \((\pm 1, y_2, \ldots)\).
On the other hand, we have

\[ (9.4.4) \quad \varphi^* \gamma_\beta = (y_2 + \frac{f''}{f} - \lambda_0 \circ \varphi f')dy_0 + \sum_{n \geq 2} \lambda_n \circ \varphi d((\beta_n \circ \varphi), \]

where \( f = f(y_0) \). From (9.4.4) we get

\[ (9.4.5) \quad A_0 = y_2 + \frac{f''}{f} - \lambda_0 \circ \varphi f' + \sum_{n \geq 2} (-1)^{n-1} \lambda_n \circ \varphi \left( n! \sum_{k=1}^{n-1} \frac{1}{k!} \left( \frac{f(k)}{f'} \right)^n \right) \sum_{i_1 + \cdots + i_k = n} \frac{y_{i_1}}{i_1!} \cdots \frac{y_{i_k}}{i_k!} + \left( \frac{f(n)}{f'} \right)^n. \]

Fix a \( \beta_0 \). There exists a sequence \( \{y_{0,n}\} \) such that \( \lim_{n \to \infty} y_{0,n} = \pm 1 \) and \( -f(y_{0,n}) \equiv \beta_0 \mod 1 \). Consider \( \lim_{n \to \infty} \frac{1}{f(y_{0,n})} A_0(y_{0,n}, y_2, \ldots) \). By (9.4.5), (9.3.1), and (9.4.3), we get \( \lambda_0(\beta_0, 0, \ldots, 0) = 0 \). Therefore, we have

\[ \lambda_0(\beta_0, \beta_2, \ldots) = \sum_{i \geq 2} \mu_i(\beta_0, \beta_2, \ldots) \beta_i, \]

where \( \mu_i \) are smooth functions of \( \beta_0, \beta_2, \ldots \). In particular, we have

\[ (9.4.6) \quad \lambda_0(\beta_0, \beta_2, 0, \ldots, 0) = \mu(\beta_0, \beta_2) \beta_2, \]

where \( \mu(\beta_0, \beta_2) = \mu_2(\beta_0, \beta_2, 0, \ldots, 0) \) is a smooth function of \( \beta_0, \beta_2 \) and \( \mu(\beta_0, 0) = \frac{\partial A_0}{\partial y_2}(\beta_0, 0, \ldots, 0) \).

Consider \( \lim_{n \to \infty} \frac{\partial A_0}{\partial y_2}(y_{0,n}) \). Since (9.3.2) and (9.4.3) imply \( \lim_{n \to \pm 1} f' \frac{\partial \beta_n}{\partial y_2} = 0 \) for \( n > 2 \) and \( \lim_{y_0 \to \pm 1} f \frac{\partial \beta_n}{\partial y_2} = -1 \), we get

\[ \frac{\partial A_0}{\partial y_2}(\pm 1, y_2, \ldots) = 1 + \frac{\partial \lambda_0}{\partial \beta_2}(\beta_0, 0, \ldots, 0). \]

Since the form \( \lambda_0 \) is closed, we have

\[ (9.4.7) \quad \frac{\partial \lambda_0}{\partial \beta_0}(\beta_0, 0, \ldots, 0) = \frac{\partial \lambda_0}{\partial \beta_2}(\beta_0, 0, \ldots, 0) = c = \text{const}. \]

By definition, the form \( \lambda_0 \) is periodic with respect to \( \beta_0 \) with period 1. This implies that \( \lambda_0 \) and \( \frac{\partial \lambda_0}{\partial \beta_2} \) are periodic with respect to \( \beta_0 \) with the same period as well. It is easy to see that, if the derivative of a smooth periodic function is constant, this constant equals 0. Then (9.4.7) implies \( \frac{\partial \lambda_0}{\partial \beta_2}(\beta_0, 0, \ldots, 0) = c = 0 \). Thus, \( \mu(\beta_0, \beta_2) = \nu(\beta_0, \beta_2) \beta_2 \), where \( \nu(\beta_0, \beta_2) \) is a smooth function of \( \beta_0 \) and \( \beta_2 \), and, by (9.4.6), we have \( \lambda_0(\beta_0, \beta_2, 0, \ldots, 0) = \nu \beta_2^2 \).

Put now in (9.4.5) \( y_0 = 0 \) for \( n \geq 2 \). We get

\[ A_0(y_0, 0, \ldots, 0) = \frac{f''}{f} - \nu(-f(y_0), -\frac{f''}{f'}) \left( \frac{f''}{f'} \right)^2 + \sum_{n \geq 2} (-1)^{n-1} \lambda_n \left( -f(y_0), -\frac{f''}{f'} \right)^n, \]

where \( f = f(y_0) \). This equation can be rewritten in the following form

\[ \frac{f''}{f'} = \frac{A_0(y_0, 0, \ldots, 0) - \sum_{n \geq 2} (-1)^{n-1} \lambda_n \left( -f(y_0), -\frac{f''}{f'} \right)^n \left( \frac{f''}{f'} \right)^n}{1 - \nu(-f(y_0), -\frac{f''}{f'}) \left( \frac{f''}{f'} \right)} \]

Therefore, by (9.4.3) \( \frac{f''}{f'} \) is bounded near \( y_0 = \pm 1 \). But this contradicts to the properties (9.3.2) of the function \( f \). This contradiction proves the theorem.
Now we study the geometrical meaning of the first Chern class of Reeb’s foliation. First we recall that Reeb’s foliation is determined up to isomorphism by the holonomy group of the compact leaf $S^1 \times S^1$ (see [32]) and by some considerations on orientation studied in details in [23]. The generator of this holonomy group is defined by the conjugation class of the germ of a diffeomorphism $g(t)$ of the interval $(-\epsilon, \epsilon)$ which is infinitely tangent to the identity, i.e. we have $g'(1) = 1$ and $g^{(n)}(1) = 0$ for $n > 1$. Moreover, $g = g_1 \circ g_2$, where $g_1$ and $g_2$ are germs of diffeomorphisms which are infinitely tangent to the identity, $g_1$ equals the identity for $t > 1$, and $g_2$ equals the identity for $t > 1$. Moreover, we may suppose that $g_1$ is monotone function.

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On April 19 of 2013 M.V. Losik informed A. Galaev in an email that the first Chern class defines the Reeb foliation up to the orientation. Unfortunately, the proof of this statement is missing. It is a challenge to recover the proof.
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