Estimating the Variance of Measurement Errors in Running Variables of Sharp Regression Discontinuity Designs

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Abstract

Treatment effect estimation through regression discontinuity designs faces a severe challenge when the running variable is measured with errors, as the errors smooth out the discontinuity that the identification hinges on. Recent studies have shown that the variance of the measurement error term plays an important role on both bias correction and identification under such situations, but little is studied about how one can estimate the unknown variance from data. This paper proposes two estimators for the variance of measurement errors. The proposed estimators do not rely on any external source other than the data of the running variable and treatment assignment.

1 Introduction

Regression discontinuity design (RDD) is a powerful framework for estimating causal effect of a binary treatment variable on some outcome measurement. An RDD is built upon a key assumption that there exists a variable such that the treatment is assigned if and only if that variable exceeds a known threshold. A variable with this property is called a running variable. Intuitively, treatment effect estimation using an RDD compares the treated and untreated samples at around the threshold of the running variable. Assuming that other covariates are continuously distributed at the point, those slightly above the threshold and those slightly below are arguably quite similar except that only the former receives the treatment. Therefore differences in the outcome measurement between the two groups, if any, are attributable to the assignment of treatment.

Identification through an RDD is challenged when the running variable is measured with errors. Theoretically, even a small magnitude of measurement errors would nullify the estimation for the treatment effect leveraging the RDD assumption. This is because the measurement errors smooth out the discontinuity in the assignment at the threshold, hence the RDD assumption is broken. Note that an RDD with a mismeasured running variable does not form a fuzzy RDD; A fuzzy RDD assumes that the assignment probability is discontinuous at a threshold, while measurement errors wipe off the discontinuity.

Davezies and Le Barbanchon (2014) showed that the standard local polynomial regression yields a biased estimate for the treatment effect if the running variable is mismeasured. They then

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suggested an alternative estimator that is less susceptible to the measurement errors and examined the severity of the bias. Yanagi (2014) also studied a similar estimator, and proposed a method to alleviate the bias of the estimator. Finally, Pei and Shen (2017) proposed identification strategies that overcome the challenge from mismeasured running variables.

The series of studies agree that the variance of the measurement error term plays an important role on the bias correction as well as on the identification of the treatment effect. The analysis by Davezies and Le Barbanchon (2014) show that their estimator would be biased more severely when the running variable suffers from larger measurement errors. Yanagi (2014)’s bias correction approach requires that the variance is known from external sources. Similarly, one of the estimators proposed by Pei and Shen (2017) also utilizes external knowledge of the variance (See Approach 3 in §4.1). Despite its utility, only a handful of discussions have been devoted to how one can obtain or estimate the variance of measurement errors. Yanagi (2014) suggests that the variance can be estimated using an auxiliary data that provide the accurate distribution of the running variable. Given such data, the variance of measurement errors can be estimated by the difference between the true variance of the running variable and the variance of the mismeasured running variable. Such external sources, however, may not be available for all applications.

This paper proposes two estimators for the variance of the measurement error term that do not rely on any external sources; The estimation only requires data of running variables and the treatment assignment, which are available in virtually all RDD studies. The first estimator assumes that both the running variable and the measurement error follow the Gaussian distribution. Under this assumption, the conditional likelihood function can be written as a simple analytic formula, which can be maximized easily by standard numerical methods. The second estimator relaxes the Gaussian assumption and allows both the running variable and measurement error to follow arbitrary distributions characterized by finite numbers of parameters. Unlike the Gaussian case, the likelihood function in this case cannot be expressed by a simple form, and hence direct optimization would pose numerical stability issues. To overcome this challenge, a variant of the expectation maximization algorithm is suggested, which maximizes the likelihood function in a computationally efficient and stable manner.

After introducing the estimators, the results of simulation experiments are provided. All estimators successfully recover the true variance if the model assumption matches the data generation process. The estimators exhibit different degree of robustness towards misspecifications. The methods have been implemented as a package for R (R Core Team, 2017) and published on the author’s GitHub repository (https://github.com/kota7/rddsigma).

2 Model

Let $D \in \{0, 1\}$ denote the binary variable that indicates the assignment of treatment and a continuous variable $X$ the running variable for $D$. Suppose that $X$ and $D$ forms a sharp regression discontinuity design, i.e., $D = 1\{X > c\}$, with a known constant $c$.

Assume that $X$ is only observed with an additive error $U$:

$$W = X + U, \quad X \perp U$$

where $W$ is the mismeasured running variable, for which data are available. Throughout the paper $U$ is assumed to be continuous and has zero mean and a finite variance $\sigma^2$. The goal is to estimate
σ using a random sample of \( \{w_i, d_i\}_{i=1}^n \), where \( w_i \) and \( d_i \) represents the observations corresponding to \( W \) and \( D \) respectively.

### 2.1 Gaussian-Gaussian Case

Consider a case where both \( X \) and \( U \) follow the Gaussian distribution. The independence assumption of the two implies that they together follow the multivariate Gaussian distribution. This also implies that the sum, \( W \), is also Gaussian.

Let \( \mathbb{E}(X) = \mu_x \) and \( \text{Var}(X) = \sigma_x^2 \). Then, \( \mathbb{E}(W) = \mu_x \) and \( \text{Var}(W) = \sigma_x^2 + \sigma^2 \equiv \sigma_w^2 \). By the property of the multivariate Gaussian distribution (See e.g. Bishop, 2006, §2.3.1), the conditional distribution of \( U \) given \( W \) is also Gaussian and its parameters can be explicitly written as follows:

\[
\mathbb{E}(U|W) = \mu_{u|w} = \frac{\sigma^2}{\sigma_w^2} (W - \mu_x) \tag{1}
\]

and

\[
\text{Var}(U|W) = \sigma_{u|w}^2 = \left( 1 - \frac{\sigma^2}{\sigma_w^2} \right) \sigma^2. \tag{2}
\]

The conditional likelihood function can be constructed using (1) and (2). Consider \( p(D|W; \theta) \), that is, the conditional distribution of \( D \) given \( W \), where \( \theta = (\mu_x, \sigma_w, \sigma) \). Since \( D = 1|X > c = 1|U < W - c \), we have

\[
p(D|W; \theta) = \begin{cases} 
\Phi((W - c - \mu_{u|w})/\sigma_{u|w}) & \text{if } D = 1 \\
1 - \Phi((W - c - \mu_{u|w})/\sigma_{u|w}) & \text{if } D = 0 
\end{cases}
\]

where \( \Phi \) is the cumulative distribution function of the standard Gaussian distribution.

Although the likelihood function depends on three parameters, \((\mu_x, \sigma_w, \sigma)\), the first two can be estimated separately by the sample mean and standard deviation of \( W \). We can substitute these estimates into the likelihood function, and estimate \( \sigma \) by the maximum likelihood. Notice that this estimation process is a two-step maximum likelihood, and hence the variance of estimators needs to be adjusted appropriately (Murphy and Topel, 1985; Newey and McFadden, 1994).

### 2.2 Non-Gaussian Case

In this section, the Gaussian assumption in the previous section is relaxed. Assume instead that \( X \) and \( U \) follow some parametric distribution with a finite number of parameters. Under this assumption, the conditional likelihood is no longer be expressed by a simple analytic formula. Instead, estimation using the marginal likelihood function is considered.

Let \( p_x \) and \( p_u \) denote the probability density functions of \( X \) and \( U \) and suppose that they depend on parameters \( \theta_x \) and \( \theta_u \) respectively. The full likelihood function for a pair \((W, D)\) is given by

\[
\log p(W, D; \theta) = D \log \int_c^\infty p_x(x; \theta_x) p_u(W - x; \theta_u) dx \\
+ (1 - D) \log \int_{-\infty}^c p_x(x; \theta_x) p_u(W - x; \theta_u) dx
\]
The objective is to maximize the sum of log likelihood with respect to the parameters, that is,

\[ \hat{\theta} \equiv \arg\max_{\theta} \sum_{i=1}^{n} \log p(w_i, d_i; \theta) \]

Due to the complex expressions inside integrals, the direct maximization of this function by standard numerical routines tends to be computationally demanding and unstable. Alternative approach is to employ a variant of the expectation-maximization algorithm (EM algorithm) (Dempster et al., 1977), which turns out to be computationally more efficient and robust. Define the Q-function as below.

\[
Q(\theta, \theta' | W, D) = D \int_{c}^{\infty} h(\theta' | Z, D) (\log p(x; \theta_x) + \log p_u(W - x; \theta_u)) \, dx \\
+ (1 - D) \int_{c}^{\infty} h(\theta' | Z, D) (\log p(x; \theta_x) + \log p_u(W - x; \theta_u)) \, dx
\]

where the function \( h \) is defined as

\[
h(\theta, x | W, D = 1) = \frac{p_x(x; \theta_x)p_u(W - x; \theta_u)}{\int_{c}^{\infty} p_x(x; \theta_x)p_u(W - x; \theta_u) \, dx}
\]

\[
h(\theta, x | W, D = 0) = \frac{p_x(x; \theta_x)p_u(W - x; \theta_u)}{\int_{c}^{\infty} p_x(x; \theta_x)p_u(W - x; \theta_u) \, dx}
\]

The Q-function can be seen as the expectation of the likelihood function with respect to the weight \( h \). Its important property is that the Q-function provides a lower bound of the likelihood function holding \( \theta' \) fixed. Concretely, for any \((W, D)\) and \( \theta, \theta' \), the following inequality holds:

\[
\log p(W, D; \theta) - \log p(W, D; \theta') \geq Q(\theta, \theta' | W, D) - Q(\theta', \theta' | W, D).
\]

See the Appendix A for the proof of this inequality.

The inequality (5) motivates an algorithm such that the parameters are iteratively updated so as to maximize the sum of Q-functions:

\[
\theta^{(t+1)} \leftarrow \arg\max_{\theta} \sum_{i=1}^{n} Q(\theta, \theta^{(t)} | w_i, d_i)
\]

By the inequality (5), the objective function increases monotonically along the iterations, and hence it converges to a local maximum, provided that it is bounded.

Maximizing the Q-function tends to be computationally inexpensive and stable than maximizing the likelihood function directly. This is particularly the case for the distributions for which the maximum likelihood estimator has an explicit formula. An illustrative case is presented below where \( X \) is Gaussian and \( U \) is Laplace distribution.

Suppose \( X \) follows the Gaussian distribution and \( U \) the Laplace distribution, \textit{i.e.},
$p_x(x; \mu_x, \sigma_x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp \left( -\frac{(x - \mu_x)^2}{2\sigma_x^2} \right)$

$p_u(w; \sigma) = \frac{\sqrt{2}}{2\sigma} \exp \left( -\frac{\sqrt{2}|u|}{\sigma} \right)$. 

Note that $\text{Var}(U) = \sigma^2$.

Among the three parameters to estimate, $\theta = (\mu_x, \sigma_x, \sigma)$, $\mu_x$ can be estimated by the sample average of $W$. The two standard deviations are estimated by the EM algorithm. The Q-function is written as follows.

$Q(\theta, \theta'|w_i, d_i) = d_i \int_c^\infty h_i(\theta') \left[ \log p_x(x; \sigma_x) + \log p_u(w_i - x; \sigma) \right] dx$

$+ (1 - d_i) \int_{-\infty}^c h_i(\theta') \left[ \log p_x(x; \sigma_x) + \log p_u(w_i - x; \sigma) \right] dx,$

where $h_i(\theta) = h(\theta|w_i, d_i)$ is defined by substituting the $p_x$ and $p_u$ to (3) and (4). Setting $\sum_{i=1}^n \frac{\partial Q(\theta, \theta'|w_i, d_i)}{\partial \theta} = 0$ yields the first order conditions for the parameters.

$\sigma = \frac{\sqrt{2}}{n} \sum_{i=1}^n \left\{ d_i \int_c^\infty h_i(\theta')(w_i - x) dx + (1 - d_i) \int_{-\infty}^c h_i(\theta')(w_i - x) dx \right\}$ (6)

$\sigma_x^2 = \frac{1}{n} \sum_{i=1}^n \left\{ d_i \int_c^\infty h_i(\theta')(x - \mu_x)^2 dx + (1 - d_i) \int_{-\infty}^c h_i(\theta')(x - \mu_x)^2 dx \right\}$. (7)

The expressions within the integral can be seen as computing the weighted average of $|w_i - x|$ and $(x - \mu_x)^2$ respectively, as analogous to the variance estimator for the Laplace and the Gaussian distributions. Thanks to (6) and (7), the parameters can be updated at each iteration without relying on numerical optimization methods. This reduces the computation time and enhances the stability of the estimation. Analogous formulas can be obtained for many other parametric distributions, such as those belonging to the exponential family.

3 Simulation

This section provides the result of simulation experiments of the estimators introduced in the previous section. The methods have been implemented as an R package and published on the author’s GitHub repository (https://github.com/kota7/rddsigma).

The simulation data are generated from various combinations of distributions in order to examine the robustness towards misspecifications. $X$ has been generated from the Gaussian distribution and the exponential distribution, while $U$ has been generated from the Gaussian and Laplace distribution. For each pair of distributions, the variance of $X$ is set to one, and the variance of $U$, $\sigma$, to 0.2 and 1.2. The sample size is set to 500 and the cutoff point $c$ is set to one for all cases. As a result, eight sets of artificial data are generated as summarized in Table 1.
4 Concluding Remarks

This paper introduces two methods for estimating the variance of measurement errors in running variables of sharp regression discontinuity designs. In practice, estimates of the variance can be used as input to the bias correction process (Yanagi, 2014) or the robust estimation of treatment effect (Pei and Shen, 2017).

The first method is derived under the assumption that both the running variable and the measurement error follow the Gaussian distribution. Under this assumption the conditional likelihood function can be defined explicitly, and hence the parameters can be estimated easily by numerical optimization routines. Despite the strong assumptions on the variable distributions, the estimator
Fig. 1: Distribution of estimated $\sigma$. The numbers in the horizontal axis correspond to the IDs given in Table 1. Each panel corresponds to an estimation method: (A) Gaussian-Gaussian estimator, (B) non-Gaussian estimator with $X$ and $U$ following the Gaussian distribution, and (C) non-Gaussian estimator with $X$ following the Gaussian, and $U$ following the Laplace distribution.

exhibits robustness against misspecifications in the simulation exercises.

The second method relaxes the Gaussian assumption and allows both $X$ and $U$ to follow arbitrary distribution characterized by a finite number of parameters. A variant of EM algorithm is introduced, which maximizes the likelihood function efficiently and stably compared with direct optimization by standard numerical methods. This method performs as well as the first method when the model is correctly specified. However, the simulation experiment finds that the estimator tends to become unstable and even biased when the data are generated from different distribution from the model assumptions. The relatively low performance is observed when $X$ is generated from the exponential distribution. This can be because the Gaussian distribution cannot approximate the data well. This implies that appropriate choice of the distribution would enhance the accuracy of the model.

In practice, the first method will be useful in many cases for estimating the variance of measurement errors, as it is easy to implement and tends to be robust against misspecifications. The
second method can also be used as a robustness check for the estimation by the first method. It will also be useful in cases where the distribution can be approximated well by some known parametric distribution due to domain knowledge.

A Proof

This section provides a proof for the inequality (5):

$$\log p(W, D; \theta) - \log p(W, D; \theta') \geq Q(\theta, \theta'|W, D) - Q(\theta', \theta'|W, D).$$

To do so, the following lemma is useful (See Ishii and Ueda, 2014, Chapter 6 for more extensive discussion).

**Lemma.** Let

$$J(\theta) = \log \int_{x \in X} g(x; \theta)dx,$$

where $g$ is a positive-valued function and $X$ is a subset of the range of $g$. Define the corresponding $Q$-function by

$$Q(\theta, \theta') = \int_{x \in X} h(x; \theta') \log g(x; \theta) dx,$$

where

$$h(x; \theta) = \frac{g(x; \theta)}{\int_{X} g(x; \theta)dx}.$$

Then,

$$\log J(\theta) - \log J(\theta') \geq Q(\theta, \theta') - Q(\theta', \theta').$$

**Proof.**

$$\begin{align*}
\log J(\theta) - \log J(\theta') & = \int_{x \in X} g(x; \theta)dx - \int_{x \in X} h(x; \theta') \log g(x; \theta) dx \\
& = \int_{y \in X} h(y; \theta') \left( \log \int_{x \in X} g(x; \theta) dx - \log g(y; \theta) \right) dy \\
& = \int_{y \in X} h(y; \theta') \log \frac{\int_{x \in X} g(x; \theta) dx}{g(y; \theta)} dy \\
& = - \int_{y \in X} h(y; \theta') \log h(y; \theta) dy \\
& = - \int_{x \in X} h(x; \theta') \log h(x; \theta) dy.
\end{align*}$$

Construct the same equality with $\theta = \theta'$ and subtract from the both sides, then

$$\begin{align*}
\log J(\theta) - \log J(\theta') - Q(\theta, \theta') + Q(\theta', \theta') & = \int_{x \in X} h(x; \theta') \log \frac{h(x; \theta')}{h(x; \theta)} \\
& \geq 0.
\end{align*}$$
where the last line is due to the Gibb’s inequality. Hence,

\[
\log J(\theta) - \log J(\theta') \geq Q(\theta, \theta') - Q(\theta', \theta')
\]

To derive the inequality (5), apply the lemma with \( g(x; \theta) = p_x(x; \theta_x)p_a(W - x; \theta_a) \) and \( X = (c, \infty) \). Then, we obtain

\[
\log \int_c^\infty p_x(x; \theta_x)p_a(W - x; \theta_a) \geq \int_c^\infty h(\theta'|Z, D) \left( \log p_x(x; \theta_x) + \log p_a(W - x; \theta_a) \right) dx \quad (8)
\]

Similarly, applying the lemma with the same \( g \) function and \( X = (-\infty, c) \),

\[
\log \int_{-\infty}^c p_x(x; \theta_x)p_a(W - x; \theta_a) \geq \int_{-\infty}^c h(\theta'|Z, D) \left( \log p_x(x; \theta_x) + \log p_a(W - x; \theta_a) \right) dx \quad (9)
\]

(8) and (9) implies (5).

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