Research Article

Approximate Cubic Lie Derivations on ρ-Complete Convex Modular Algebras

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In this article, we present generalized Hyers–Ulam stability results of a cubic functional equation associated with an approximate cubic Lie derivation on convex modular algebras $\rho$ with $\Delta_2$-condition on the convex modular functional $\rho$.

1. Introduction

In 1940, S. M. Ulam [1] raised the question concerning the stability of group homomorphisms. Let $G$ be a group and let $G'$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $f : G \rightarrow G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

(1)

for all $x, y \in G$, then there exists a homomorphism $F : G \rightarrow G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? D. H. Hyers [2] has solved the problem of Ulam for the case of additive mappings in 1941. The result was generalized by T. Aoki [3] in 1950, by Th.M. Rassias [4] in 1978, by J. M. Rassias [5] in 1992, and by P. Gavruta [6] in 1994. Over the past few decades, many mathematicians have investigated the generalized Hyers–Ulam stability theorems of various functional equations [7–12].

Now, we recall some basic definitions and remarks of modular spaces with modular functions, which are primitive notions corresponding to norms or metrics, as in the following [13–15].

Definition 1. Let $\chi$ be a linear space.

(a) A function $\rho : \chi \rightarrow [0, \infty]$ is called a modular if, for arbitrary $x, y \in \chi$,

(1) $\rho(x) = 0$ if and only if $x = 0$,

(2) $\rho(\alpha x) = \rho(x)$ for every scalar $\alpha$ with $|\alpha| = 1$,

(3) $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ for any scalars $\alpha, \beta$, where $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$;

(b) alternatively, if (3) is replaced by

$$(3') \rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$$

for every scalars $\alpha, \beta$, where $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

then we say that $\rho$ is a convex modular.

It is well known that a modular $\rho$ defines a corresponding modular space, i.e., the linear space $\chi_\rho$, given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}.$$  

(2)

Let $\rho$ be a convex modular. Then, we remark the modular space $\chi_\rho$ can be a Banach space equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf \{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\}.$$  

(3)

If $\rho$ is a modular on $\chi$, we note that $\rho(tx)$ is an increasing function in $t \geq 0$ for each fixed $x \in \chi$; that is, $\rho(\alpha x) \leq \rho(\beta x)$, whenever $0 \leq \alpha < \beta$. In addition, if $\rho$ is a convex modular on $\chi$, then $\rho(\alpha x) \leq \alpha \rho(x)$ for all $x \in \chi$ and $0 \leq \alpha \leq 1$. Moreover, we see that $\rho(\alpha x) \leq |\alpha| \rho(x)$ for all $x \in \chi$ and $|\alpha| \leq 1$.

Remark. (a) In general, we note that $\rho(\sum_{i=1}^{n} \alpha_i x_i) \leq \sum_{i=1}^{n} \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \geq 0$ $(i = 1, \ldots, n)$ whenever $0 < \sum_{i=1}^{n} \alpha_i = \alpha \leq 1$ [14].
(b) Consequently, we lead to \( \rho(\sum_{i=1}^{n} a_i x_i) \leq \sum_{i=1}^{n} |a_i| \rho(x_i) \) for all \( x_i \in \chi \) and \( 0 < \sum_{i=1}^{n} |a_i| = \alpha \leq 1 \).

**Definition 2.** Let \( \chi_\rho \) be a modular space and let \( \{x_n\} \) be a sequence in \( \chi_\rho \). Then,

1. \( \{x_n\} \) is \( \rho \)-convergent to \( x \in \chi_\rho \) and we write \( x_n \overset{\rho}{\to} x \) if \( \rho(x_n - x) \to 0 \) as \( n \to \infty \);

2. \( \{x_n\} \) is called \( \rho \)-Cauchy in \( \chi_\rho \) if \( \rho(x_n - x_m) \to 0 \) as \( n, m \to \infty \);

3. a subset \( K \) of \( \chi_\rho \) is called \( \rho \)-complete if and only if any \( \rho \)-Cauchy sequence in \( K \) is \( \rho \)-convergent to an element in \( K \).

They say that the modular \( \rho \) has the Fatou property if and only if \( \rho(x) \leq \liminf_{n \to \infty} \rho(x_n) \) whenever the sequence \( \{x_n\} \) is \( \rho \)-convergent to \( x \). A modular function \( \rho \) is said to satisfy the \( \Delta_2 \)-condition if there exists \( \kappa > 0 \) such that \( \rho(2x) \leq \kappa \rho(x) \) for all \( x \in \chi_\rho \).

In 2014, G. Sadeghi [16] has demonstrated generalized Hyers–Ulam stability via the fixed point method of a generalized Jensen functional equation

\[ f(rx + sy) = rf(x) + sh(y) \]

in convex modular spaces with the Fatou property and \( \Delta_2 \)-condition with \( 0 < \kappa \leq 2 \). In [15], the authors have proved the generalized Hyers–Ulam stability of quadratic functional equations via the extensive studies of fixed point theory in the framework of modular spaces whose modulars are convex and lower semicontinuous but do not satisfy any relatives of \( \Delta_2 \)-conditions (see also [17, 18]). Recently, the authors [14, 19, 20] have investigated stability theorems of functional equations in modular spaces without using the Fatou property and \( \Delta_2 \)-condition. In 2001, J. M. Rassias [21] has introduced to study Hyers–Ulam stability of the following cubic functional equation:

\[ f(2x + y) + f(x - y) + 3f(y) = 3f(x + y) + 6f(x), \] (4)

which is equivalent to

\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \] (5)

whose general solution is characterized as \( f(x) = B(x, x, x) \) where \( B \) is symmetric and additive for each fixed one variable [22]. For this reason, every solution of the cubic functional equation is said to be a cubic mapping.

Now, we say that \( \chi_\rho \) is called a (convex) modular algebra if the fundamental space \( \chi \) is an algebra over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) with (convex) modular \( \rho \) subject to \( \rho(ab) \leq \rho(a)\rho(b) \) for all \( a, b \in \chi \). A subset \( K \) of a convex modular algebra \( \chi_\rho \) is called \( \rho \)-complete if and only if any \( \rho \)-Cauchy sequence in \( K \) is \( \rho \)-convergent to an element in \( K \). Throughout the paper, \( \chi_\rho \) will be a \( \rho \)-complete convex modular algebra and the symbol \([a, b]\) will denote the commutator \( ab - ba \). We say that a mapping \( f \) is cubic homogeneous if \( f(\lambda x) = \lambda^3 f(x) \) for all vectors \( x \) and all scalars \( \lambda \), and a cubic homogeneous mapping \( f \) is called a cubic Lie derivation if \( f([x, y]) = [f(x), y^3] + [x^3, f(y)] \) for all vectors \( x, y \) [23, 24].

In this article, we first investigate generalized Hyers–Ulam stability of the equation

\[ f(3x - y) + f(x + y) = 2f(2x - y) + 12f(x) + 2f(y), \] (6)

in \( \rho \)-complete convex modular algebras without using the Fatou property and \( \Delta_2 \)-condition and then present alternatively generalized Hyers–Ulam stability of (6) using necessarily \( \Delta_2 \)-condition without the Fatou property in \( \rho \)-complete convex modular algebras.

### 2. Generalized Hyers–Ulam Stability of (6)

First of all, we remark that (6) is equivalent to the original cubic functional equation, and so every solution of (6) is a cubic mapping.

For notational convenience, we let the difference operators \( CE_f \) of cubic equation (6) and \( CD_f \) of cubic derivation be as follows:

\[ CE_f(\lambda x, \lambda y) = f(3\lambda x - \lambda y) + f(\lambda x + \lambda y) - 2\lambda^3 f(2x - y) - 12\lambda^3 f(x) - 2\lambda^3 f(y), \] (7)

\[ CD_f(x, y) = f((x, y)) - [f(x), y^3] - [x^3, f(y)] \]

for all \( x, y \) in a linear space \( \mathcal{X} \) and \( \lambda \in \Lambda = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \). In the following, we present a generalized Hyers–Ulam stability via direct method of the system \( CE_f = 0 \) and \( CD_f = 0 \) in \( \rho \)-complete convex modular algebras without using both the Fatou property and \( \Delta_2 \)-condition.

**Theorem 3.** Suppose that a mapping \( f : \chi_\rho \rightarrow \chi_\rho \) satisfies

\[ \rho(CE_f(\lambda x, \lambda y)) \leq \phi_1(x, y, z), \]

\[ \rho(CD_f(x, y)) \leq \phi_2(x, y) \] (8)

and \( \phi_1 : \chi_\rho^3 \rightarrow [0, \infty) \), \( \phi_2 : \chi_\rho^2 \rightarrow [0, \infty) \) are mappings such that

\[ \Phi(x, y, z) = \sum_{j=0}^{\infty} \phi_1(2^j x, 2^j y, 2^j z) < \infty, \]

\[ \lim_{n \to \infty} \phi_2(2^n x, 2^n y) = 0 \]

for all \( x, y, z \in \chi_\rho \) and \( \lambda \in \Lambda \). If for each \( x \in \chi_\rho \) the mapping \( r \rightarrow f(rx) \) from \( \mathbb{R} \) to \( \chi_\rho \) is continuous, then there exists a unique cubic Lie derivation \( F_1 : \chi_\rho \rightarrow \chi_\rho \) which satisfies equation (6) and

\[ \rho(f(x) - F_1(x)) \leq \frac{1}{16} \Phi(x, x, 0) \] (10)

for all \( x \in \chi_\rho \).
Proof. Putting \( y = x \) and \( \lambda = 1 \) in (8), we obtain
\[
\rho \left( 2f(2x) - 16f(x) \right) \leq \phi_1(x, x, 0),
\]
which yields
\[
\rho \left( f(2x) - 8f(x) \right) = \frac{1}{2} \rho \left( 2f(2x) - 16f(x) \right) \\
\leq \frac{1}{2} \phi_1(x, x, 0),
\]
\[
\rho \left( f(x) - \frac{f(2x)}{8} \right) \leq \frac{1}{8} \rho \left( f(2x) - 8f(x) \right) \\
\leq \frac{1}{16} \phi_1(x, x, 0)
\]
for all \( x \in \chi_\rho \). Since \( \sum_{j=0}^{n-1} (1/8^{j+1}) \leq 1 \), we prove the following functional inequality:
\[
\rho \left( f(x) - \frac{f(2^n x)}{2^{3n}} \right) \\
= \rho \left[ \sum_{j=0}^{n-1} \left( f \left( \frac{2^j x}{2^j} \right) - f \left( \frac{2^{j+1} x}{2^{j+1}} \right) \right) \right] \\
\leq \sum_{j=0}^{n-1} \frac{1}{2^{3(j+1)}} \rho \left( 8f \left( 2^j x \right) - f \left( 2^{j+1} x \right) \right) \\
\leq \frac{1}{16} \sum_{j=0}^{n-1} \phi_1 \left( 2^j x, 2^j x, 0 \right)
\]
for all \( x \in \chi_\rho \) by using the property of convex modular \( \rho \).
Now, replacing \( x \) by \( 2^m x \) in (13), we have
\[
\rho \left( \frac{f(2^m x)}{2^{3m}} - \frac{f(2^{m+n} x)}{2^{3(m+n)}} \right) \\
\leq \frac{1}{16} \sum_{j=m}^{m+n-1} \phi_1 \left( 2^j x, 2^j x, 0 \right)
\]
which converges to zero as \( m \to \infty \) by assumption (9). Thus the above inequality implies that the sequence \( \{ f(2^n x)/2^{3n} \} \) is \( \rho \)-Cauchy for all \( x \in \chi_\rho \) and so it is convergent in \( \chi_\rho \), since the space \( \chi_\rho \) is \( \rho \)-complete. Thus, we may define a mapping \( F_1 : \chi_\rho \to \chi_\rho \) as
\[
F_1(x) = \rho - \lim_{n \to \infty} \frac{f(2^n x)}{2^{3n}} \\
\lim_{n \to \infty} \rho \left( \frac{f(2^n x)}{2^{3n}} - F_1(x) \right) = 0,
\]
for all \( x \in \chi_\rho \).

Claim 1. \( F_1 \) is a cubic mapping satisfying approximation (10).
In fact, if we put \( (x, y, z) = (2^n x, 2^n y, 0) \) in (8) and then divide the resulting inequality by \( 2^{3n} \), one obtains
\[
\rho \left( \frac{CE \left( 2^n \lambda x, 2^n \lambda y \right)}{R \cdot 2^{3n}} \right) \leq \rho \left( \frac{CE \left( 2^n \lambda x, 2^n \lambda y \right)}{R \cdot 2^{3n}} \right) \\
\leq \frac{\phi_1(2^n x, 2^n y, 0)}{R \cdot 2^{3n}} \\
\to 0
\]
for all \( x, y \in \chi_\rho \), where \( R \geq 16|\lambda| + 3 \) is a fixed positive real. Thus we figure out by use of the first remark
\[
\rho \left( \frac{R}{16} CE_{F_1} \left( \lambda x, \lambda y \right) \right) = \rho \left( \frac{R}{16} CE \left( 2^n \lambda x, 2^n \lambda y \right) \right) \\
- \frac{CE \left( 2^n \lambda x, 2^n \lambda y \right)}{R \cdot 2^{3n}} \leq \frac{1}{R} \\
\cdot \rho \left( F_1 \left( 3 \lambda x - \lambda y \right) - \frac{f\left( 2^n (3 \lambda x - \lambda y) \right)}{2^{3n}} \right) + \frac{1}{R} \\
\cdot \rho \left( F_1 \left( \lambda x + \lambda y \right) - \frac{f\left( 2^n (\lambda x + \lambda y) \right)}{2^{3n}} \right) + \frac{2|\lambda|^3}{R} \\
\cdot \rho \left( F_1 \left( 2 x - y \right) - \frac{f\left( 2^n (2 x - y) \right)}{2^{3n}} \right) + \frac{12|\lambda|^3}{R} \\
\cdot \rho \left( F_1 \left( x \right) - \frac{f\left( 2^n x \right)}{2^{3n}} \right) + \frac{2|\lambda|^3}{R} \rho \left( F_1 \left( y \right) \right) \\
- \frac{f(2^n y)}{2^{3n}} \right) \to 0
\]
for all \( x, y \in \chi_\rho, \lambda \in \Lambda \) and all positive integers \( n \). Taking the limit as \( n \to \infty \), one obtains \( \rho(1/R)CE_{F_1}(\lambda x, \lambda y) = 0 \), and so \( CE_{F_1}(\lambda x, \lambda y) = 0 \) for all \( x, y \in \chi_\rho \). Hence, taking \( \lambda = 1 \) in \( CE_{F_1}(x, y) = 0 \), it follows that \( F_1 \) satisfies (6) and so it is
cubic. On the other hand, since $\sum_{i=0}^{n}(1/2^{3(i+1)} + 1/2^3) \leq 1$ for all $n \in \mathbb{N}$, it follows from (12) and the first remark that

$$\rho(f(x) - F_1(x)) = \rho \left( \frac{1}{2} \sum_{i=0}^{n} \frac{1}{2^{3(i+1)}} (2^3f(2^ix) - f(2^{i+1}x)) \right)$$

$$+ \frac{f(2^{n+1}x) - F_1(2x)}{2^{3n+1}} \leq \frac{1}{2} \sum_{i=0}^{n} \frac{1}{2^{3(i+1)}} \rho(C_{E_1}(2^ix, 2^ix)) + \frac{1}{2^3} \rho \left( \frac{f(2^{n+1}x) - F_1(2x)}{2^{3n+1}} \right) \leq \frac{1}{2} \sum_{i=0}^{n} \frac{1}{2^{3(i+1)}}$$

$$\cdot \rho(CE_1(2^ix, 2^ix)) + \frac{1}{2^3} \rho \left( \frac{f(2^{n+1}x) - F_1(2x)}{2^{3n+1}} \right) \leq \frac{1}{16} \sum_{i=0}^{n} \frac{1}{2^{3i}} \phi_1(2^ix, 2^ix, 0) \leq \frac{1}{2^3} \rho \left( \frac{f(2^n \cdot 2^ix) - F_1(2x)}{2^{3n+1}} \right)$$

for all $x \in \chi_\rho$ by taking $n \to \infty$ in the last inequality.

**Claim 2.** $F_1$ is cubic homogeneous. By (17), we have $CE_1(\lambda x, \lambda x) = 0$, which yields $F_1(2\lambda x) = 8\lambda^3F_1(x)$ for all $x \in \chi_\rho$ and $\lambda \in \Lambda$. From the assumption that for each $x \in \chi_\rho$ the mapping $r \mapsto f(rx)$ from $\mathbb{R}$ to $\chi_\rho$ is continuous, it follows that $F_1(\lambda x) = \lambda^3F_1(x)$ for all $x \in \chi_\rho$ and $\lambda \in \mathbb{R}$ by the same argument as in the paper [4, 25]. Thus, for any nonzero $\lambda \in \mathbb{C}$

$$F_1(\lambda x) = F_1 \left( 2 \frac{\lambda}{|\lambda|} \frac{|\lambda|}{2} x \right) = 8 \left( \frac{\lambda}{|\lambda|} \right)^3 F_1 \left( \frac{|\lambda|}{2} x \right) = 8 \left( \frac{\lambda}{|\lambda|} \right)^3 \left( \frac{|\lambda|}{2} \right)^3 F_1(x) = \lambda^3 F_1(x)$$

for all $x \in \chi_\rho$ and $\lambda \in \mathbb{C}$, which concludes that $F_1$ is cubic homogeneous.

**Claim 3.** $F_1$ is a cubic Lie derivation. From the second inequality in (9) and the second condition in (8), we arrive at

$$\rho \left( \frac{1}{4} CD_{F_1}(x, y) \right) \leq \rho \left( \frac{1}{4} CD_{F_1}(x, y) \right)$$

$$- \frac{CD_f(2^n x, 2^ny)}{4 \cdot 8^{2n}} + \frac{CD_f(2^n x, 2^ny)}{4 \cdot 8^{2n}} \leq \frac{1}{4}$$

for all $x, y \in \chi_\rho$, which tends to zero as $n$ tends to $\infty$. Therefore, one obtains $\rho((1/4)CD_{F_1}(x, y)) = 0$, and $F_1$ is a cubic Lie derivation.

**Claim 4.** $F_1$ is a unique cubic Lie derivation. To show the uniqueness of $F_1$, let us assume that there exists a cubic Lie derivation $G_1: \chi_\rho \to \chi_\rho$ which satisfies the inequality

$$\rho(f(x) - G_1(x)) \leq \frac{1}{16} \sum_{j=0}^{\infty} \phi_1 \left( 2^j x, 2^j y \right)$$

$$\sum_{j=0}^{\infty} \phi_1 \left( 2^j x, 2^j y \right) < \varepsilon, \quad \sum_{j=0}^{\infty} \phi_1 \left( 2^j x, 2^j y \right)$$

for all $x, y \in \chi_\rho$, but suppose $F_1(x_0) \neq G_1(x_0)$ for some $x_0 \in X$. Then there exists a positive constant $\varepsilon > 0$ such that $\varepsilon < \rho(F_1(x_0) - G_1(x_0))$. For such given $\varepsilon > 0$, it follows from (9) that there is a positive integer $n_0 \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} \phi_1(2^i x_0, 2^i x_0) < \varepsilon$. Since $F_1$ and $G_1$ are cubic mappings, we see from the equality $F_1(2^nx_0) = 2^{3n_0}F_1(x_0)$ and $G_1(2^nx_0) = 2^{3n_0}G_1(x_0)$ that

$$\rho \left( \frac{1}{4} CD_{F_1}(x, y) \right) \leq \rho \left( \frac{1}{4} CD_{F_1}(x, y) \right)$$

$$- \frac{CD_f(2^n x, 2^ny)}{4 \cdot 8^{2n}} + \frac{CD_f(2^n x, 2^ny)}{4 \cdot 8^{2n}} \leq \frac{1}{4}$$

for all $x, y \in \chi_\rho$, which tends to zero as $n$ tends to $\infty$. Therefore, one obtains $\rho((1/4)CD_{F_1}(x, y)) = 0$, and $F_1$ is a cubic Lie derivation.
which leads a contradiction. Hence the mapping \( F_1 \) is a
unique cubic Lie derivation near \( f \) satisfying approximation
(10) on the modular algebra \( \chi_\rho \).

As a corollary of Theorem 3, we obtain the following
result about cubic equation (6) associated with cubic
Lie derivation on the Banach algebra \( \chi_\rho \), which may be
considered as endowed with modular \( \rho = \| \cdot \| \).

**Corollary 4.** Suppose \( \chi_\rho \) is a Banach algebra
with norm \( \| \cdot \| \). For given real numbers \( \theta_1, \theta_2, \theta_3 \geq 0, r_1 < r_2 \) (\( i = 1, 2, 3 \)), and
\( a_1 + b_1 < 3, a_2 + b_2 < 6 \), suppose that a mapping \( f : \chi_\rho \rightarrow \chi_\rho \)
satisfies

\[
\left\| CE_f (\lambda x, \lambda y) \right\| \leq \theta_1 \| x \|^\alpha \| y \|^\beta
\]
\[+ \theta_2 \| x \|^\alpha \| y \|^\beta, \quad (24)
\]

\[
\left\| CD_f (x, y) \right\| \leq \theta_3 \| x \|^\alpha \| y \|^\beta
\]
\[(25)
\]

for all \( x, y \in \chi_\rho \), where \( x \neq 0 \) whenever \( r_i, a_i + b_i < 0 \)
and for each \( x \in \chi_\rho \), the mapping \( r \mapsto f (x) \) from \( \mathbb{R} \) to \( \chi_\rho \)
is continuous. Then there exists a unique cubic Lie derivation
\( F_1 : \chi_\rho \rightarrow \chi_\rho \) such that

\[
\rho \left( f (x) - F_1 (x) \right) \leq \frac{\theta_1 \| x \|^\alpha}{2 (2^{-2} - 2^{-1})} + \frac{\theta_2 \| x \|^\alpha}{2 (2^{-3} - 2^{-2})}
\]
\[+ \frac{\theta \| x \|^\alpha a_i}{2 (2^{-3} - 2^{-1} b_i)}
\]

for all \( x \in \chi_\rho \), where \( x \neq 0 \) whenever \( r_i, a_i + b_i < 0 \).

We observe that if the modular \( \rho \) satisfies the \( \Delta_2 \)-
condition, then \( \kappa \geq 1 \) for nontrivial modular \( \rho \), and \( \kappa \geq 2 \)
for nontrivial convex modular \( \rho \). See [13–16]. In the following
theorem, we prove generalized Hyers–Ulam stability of the
system \( CD_f = 0 \) and \( CE_f = 0 \) using necessarily \( \Delta_2 \)-
condition, which permits the existence of \( \rho \)-Cauchy sequence
in \( \chi_\rho \).

**Theorem 5.** Let \( \chi_\rho \) be a \( \rho \)-complete convex modular space
with \( \Delta_2 \)-condition. Suppose there exist two functions \( \varphi_1, \varphi_2 : \chi_\rho \rightarrow [0, \infty) \) for which a mapping \( f : \chi_\rho \rightarrow \chi_\rho \)
satisfies

\[
\rho \left( CE_f (\lambda x, \lambda y) \right) \leq \varphi_1 (x, y), \quad \Psi (x, y) = \sum_{j=1}^{\infty} k^{j/2} \varphi_1 \left( \frac{x}{2^j}, \frac{y}{2^j} \right) < \infty, \quad (26)
\]

\[
\rho \left( CD_f (x, y) \right) \leq \varphi_2 (x, y), \quad \lim_{n \rightarrow \infty} \kappa^{n \varphi_2} \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0 \quad (27)
\]

for all \( x, y \in \chi_\rho \), and \( \lambda \in \Lambda \). If for each \( x \in \chi_\rho \), the mapping
\( r \mapsto f (x) \) from \( \mathbb{R} \) to \( \chi_\rho \) is continuous, then there exists a
unique cubic Lie derivation \( F_2 : \chi_\rho \rightarrow \chi_\rho \) satisfying (6) and

\[
\rho \left( f (x) - F_2 (x) \right) \leq \frac{1}{4 \kappa^2} \Psi (x, x)
\]

for all \( x \in \chi_\rho \).

**Proof.** First, we remark that since \( \sum_{j=1}^{\infty} (k^{j/2}) \varphi_1 (0, 0) = \Psi (0, 0) < \infty \) and \( \rho (CE_f (0, 0)) \leq \varphi_1 (0, 0) \), we lead to
\( \varphi_1 (0, 0) = 0, CE_f (0, 0) = 0 \) and so \( f (0) = 0 \). Thus, it follows from (12) that

\[
\rho \left( f (x) - 8 f \left( \frac{x}{2} \right) \right) \leq \frac{1}{2} \varphi_1 \left( \frac{x}{2}, \frac{x}{2} \right)
\]

(29)

for all \( x \in \chi_\rho \). Thus, one obtains the following inequality by
the convexity of the modular \( \rho \) and \( \Delta_2 \)-condition:

\[
\rho \left( f (x) - 8^2 f \left( \frac{x}{2^2} \right) \right) \leq \frac{1}{2^2} \rho \left( 2 f (x) - 2 \cdot 8 f \left( \frac{x}{2} \right) \right)
\]
\[+ \frac{1}{2^3} \rho \left( 2^2 \cdot 8 f \left( \frac{x}{2} \right) - 2^2 \cdot 8^2 f \left( \frac{x}{2^2} \right) \right) \]
\[ \leq \frac{\kappa}{2^2} \varphi_1 \left( \frac{x}{2}, \frac{x}{2} \right) + \frac{\kappa^2}{2^2} \varphi_1 \left( \frac{x}{2^2}, \frac{x}{2^2} \right)
\]

(30)

for all \( x \in \chi_\rho \). Then using the repeated process for any \( n \geq 2 \),
we prove the following functional inequality:

\[
\rho \left( f (x) - 8^n f \left( \frac{x}{2^n} \right) \right) \leq \frac{1}{2^{n-2}} \sum_{j=1}^{\infty} \kappa^{j/2} \varphi_1 \left( \frac{x}{2^j}, \frac{x}{2^j} \right)
\]

(31)

for all \( x \in \chi_\rho \). In fact, it is true for \( n = 2 \). Assume that
inequality (31) holds true for \( n \). Thus, using the convexity of
the modular \( \rho \), we deduce

\[
\rho \left( f (x) - 8^{n+1} f \left( \frac{x}{2^{n+1}} \right) \right)
\]
\[= \rho \left( \frac{1}{2} \left( 2 f (x) - 2 \cdot 8 f \left( \frac{x}{2} \right) \right) \right)
\]
\[+ \frac{1}{2} \left( 2 \cdot 8 f \left( \frac{x}{2} \right) - 2 \cdot 8^{n+1} f \left( \frac{x}{2^{n+1}} \right) \right) \]
\[ \leq \frac{\kappa}{2} \rho \left( f (\frac{x}{2}) \right) + \frac{\kappa^4}{2^2} \rho \left( f (\frac{x}{2}) \right) \]
\[\leq 8^n f \left( \frac{x}{2^n} \right) \]
\[\leq \frac{1}{2} \varphi_1 \left( \frac{x}{2^n}, \frac{x}{2^n} \right) + \frac{\kappa^4}{2^2} \cdot \frac{1}{2} \kappa^2 \]
\[\sum_{j=1}^{\infty} \kappa^{j/2} \varphi_1 \left( \frac{x}{2^j}, \frac{x}{2^j} \right) \]

(32)

\[
\sum_{j=1}^{\infty} \kappa^{j/2} \varphi_1 \left( \frac{x}{2^j}, \frac{x}{2^j} \right) = \frac{1}{2 \kappa} \sum_{j=1}^{\infty} \kappa^{j/2} \varphi_1 \left( \frac{x}{2^j}, \frac{x}{2^j} \right)
\]
which proves (31) for \( n + 1 \). Now, replacing \( x \) by \( 2^{-m}x \) in (31), we have
\[
\rho \left( 2^{3m} f \left( \frac{X}{2^m} \right) - 2^{3(m+n)} f \left( \frac{X}{2^{m+n}} \right) \right) \\
\leq \kappa^{3m} \rho \left( f \left( \frac{X}{2^m} \right) - 2^{3n} f \left( \frac{X}{2^{m+n}} \right) \right) \\
\leq \frac{\kappa^{3m} n}{2^m} \phi_1 \left( \frac{X}{2^{m+1}} \right) - \frac{\kappa^{3m} n}{2^m} \phi_1 \left( \frac{X}{2^{m+n}} \right) \\
\leq \frac{\kappa^{3m} n}{2^m} \phi_1 \left( \frac{X}{2^{m+1}} \right) - \frac{\kappa^{3m} n}{2^m} \phi_1 \left( \frac{X}{2^{m+n}} \right) \\
= \frac{1}{2^m} \sum_{j=1}^{n} \frac{\kappa^{3m} n}{2^m} \phi_1 \left( \frac{X}{2^{m+1}} \right) \\
= \frac{1}{2^m} \sum_{j=1}^{n} \frac{\kappa^{3m} n}{2^m} \phi_1 \left( \frac{X}{2^{m+1}} \right),
\]
which converges to zero as \( m \to \infty \) by assumption (27). Thus, the sequence \( \{8^n f(x/2^n)\} \) is \( \rho \)-Cauchy for all \( x \in \chi_\rho \) and so it is \( \rho \)-convergent in \( \chi_\rho \) since the space \( \chi_\rho \) is \( \rho \)-complete. Thus, we may define a mapping \( F_2 : \chi_\rho \to \chi_\rho \) as
\[
F_2(x) = \rho - \lim_{n \to \infty} 8^n f \left( \frac{X}{2^n} \right) \\
\lim_{n \to \infty} \rho \left( 8^n f \left( \frac{X}{2^n} \right) - F_2(x) \right) = 0,
\]
for all \( x \in \chi_\rho \).

**Claim 1.** \( F_2 \) is a cubic mapping with estimation (28) near \( f \).

By \( \Delta_2 \)-condition without using the Fatou property, we can see the following inequality:
\[
\rho \left( f(x) - F_2(x) \right) \leq \frac{1}{2} \rho \left( f(x) - 2 \cdot 8^n f \left( \frac{X}{2^n} \right) + 2 \right) \\
+ \kappa \rho \left( 8^n f \left( \frac{X}{2^n} \right) - F_2(x) \right) + \frac{\kappa}{2} \rho \left( f(x) - 8^n f \left( \frac{X}{2^n} \right) \right) \\
+ n \sum_{j=1}^{n} \frac{\kappa^{3} n}{2^m} \phi_1 \left( \frac{X}{2^{m+1}} \right) - \frac{\kappa^{3} n}{2^m} \phi_1 \left( \frac{X}{2^{m+n}} \right) \\
\leq \frac{1}{4 \kappa^2} \sum_{j=1}^{n} \frac{\kappa^{3} n}{2^m} \phi_1 \left( \frac{X}{2^{m+1}} \right) = \frac{1}{4 \kappa^2} \Psi(x,x)
\]
by taking \( n \to \infty \), which yields approximation (28).

Now, setting \( (x,y) = (2^{-n}x,2^{-n}y) \) in (26) and multiplying the resulting inequality by \( 8^n \), we get
\[
\rho \left( 2^{3n} CE_f \left( 2^{-n} x, 2^{-n} y \right) \right) \leq \kappa^{3n} \phi_1 \left( 2^{-n} x, 2^{-n} y \right) \\
\leq \frac{\kappa^n}{2^n} \phi_1 \left( 2^{-n} x, 2^{-n} y \right) \\
\leq \frac{\kappa^n}{2^n} \phi_1 \left( 2^{-n} x, 2^{-n} y \right) \cdot \frac{\kappa^n}{2^n} \phi_1 \left( 2^{-n} x, 2^{-n} y \right),
\]
which tends to zero as \( n \to \infty \) for all \( x, y \in \chi_\rho \). Thus, it follows from the first remark that
\[
\rho \left( \frac{1}{R} CE_f \left( \frac{X, x, y}{} \right) \right) = \rho \left( \frac{1}{R} CE_f \left( \frac{X, x, y}{} \right) \right) \\
- \frac{2^n}{R} CE_f \left( \frac{X, x, y}{} \right) + \frac{2^n}{R} CE_f \left( \frac{X, x, y}{} \right) \\
\leq \frac{1}{R} \rho \left( F_2 \left( 3 \lambda x - \lambda y \right) - 2 \cdot 3^n f \left( \frac{3 \lambda x - \lambda y}{} \right) \right) \\
+ \frac{2^n}{R} \rho \left( F_2 \left( y \right) - 2 \cdot 3^n f \left( \frac{y}{} \right) \right) + \frac{1}{R} \\
\rho \left( F_2 \left( x + \lambda y \right) - 2 \cdot 3^n f \left( \frac{x + \lambda y}{} \right) \right) + \frac{2^n}{R} \\
\rho \left( F_2 \left( 2x - y \right) - 2 \cdot 3^n f \left( \frac{2x - y}{} \right) \right) + \frac{12^n}{R} \\
\rho \left( F_2 \left( x - 2 \cdot 3^n f \left( \frac{x}{} \right) \right) \right) + \frac{1}{R} \\
\rho \left( 2^n CE_f \left( \frac{x, \lambda y}{} \right) \right)
\]
for all \( x, y \in \chi_\rho \), \( \lambda \in \Lambda \), and all positive integers \( n \), where \( R \geq 16|\lambda|^3 + 3 \) is a fixed real number. Taking the limit as \( n \to \infty \), one obtains \( \rho(1/R CE_f \left( X, x, y \right)) = 0 \), and thus \( CE_f \left( x, y \right) = 0 \) for all \( x, y \in \chi_\rho \). Hence \( F_2 : \chi_\rho \to \chi_\rho \) satisfies (6), and so it is cubic.

**Claim 2.** \( F_2 \) is a cubic Lie derivation. By the same proof of Theorem 3, the mapping \( F_2 \) is a cubic homogeneous mapping. From the last inequality in (27) and the last condition in (26), one obtains that
\[
\rho \left( \frac{1}{4} CD_f \left( x, y \right) \right) = \rho \left( \frac{1}{4} CD_f \left( x, y \right) \right) \\
- 8^n \rho \left( CD_f \left( 2^{-n} x, 2^{-n} y \right) \right) + 8^n \rho \left( CD_f \left( 2^{-n} x, 2^{-n} y \right) \right) \\
\leq \frac{1}{4} \rho \left( F_2 \left( [x,y] \right) \right) + 8^n \rho \left( CD_f \left( 2^{-n} x, 2^{-n} y \right) \right) + \frac{1}{4} \\
\rho \left( 8^n \left( f \left( 2^{-n} x \right), y \right) - \left[ x, f \left( 2^{-n} x \right) \right] \right) + \frac{1}{4} \\
\rho \left( 8^n \left( f \left( 2^{-n} x \right), y \right) - \left[ x, f \left( 2^{-n} x \right) \right] \right) + \frac{1}{4} \\
\rho \left( 8^n \left( f \left( 2^{-n} x \right), y \right) - \left[ x, f \left( 2^{-n} x \right) \right] \right) + \frac{1}{4} \\
\rho \left( 8^n \left( f \left( 2^{-n} x \right), y \right) - \left[ x, f \left( 2^{-n} x \right) \right] \right) + \frac{1}{4} \\
\rho \left( 8^n \left( f \left( 2^{-n} x \right), y \right) - \left[ x, f \left( 2^{-n} x \right) \right] \right) + \frac{1}{4}
\]
(38)
for all $x, y \in \chi_\rho$, from which $CD_{F_1}(x, y) = 0$ by taking $n \to \infty$ and so $F_2$ is a cubic Lie derivation.

**Claim 3.** $F_2$ is unique. To show the uniqueness of $F_2$, let us assume that there exists a cubic Lie derivation $G_2 : \chi_\rho \to \chi_\rho$ which satisfies the approximation (28). Since $F_2$ and $G_2$ are cubic mappings, we see from the equalities $2 \rho F_2(2^{n} x) = F_2(x)$ and $2 \rho G_2 (2^{n} x) = G_2(x)$ that

$$
\rho (G_2(x) - F_2(x)) = \rho \left( \frac{2^{3(n+1)}}{2^3} (G_2 \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right)
+ \frac{2^{3(n+1)}}{2^3} \left( f \left( \frac{x}{2^n} \right) - F_2 \left( \frac{x}{2^n} \right) \right) \right) \leq \kappa \frac{3(n+1)}{2^3} \cdot \rho \left( f \left( \frac{x}{2^n} \right) - F_2 \left( \frac{x}{2^n} \right) \right)
+ \kappa \frac{3(n+1)}{2^3} \cdot \rho \left( G_2 \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right)
\leq \kappa \frac{3(n+1)}{2^3} \cdot \frac{1}{2^n} \sum_{j=1}^{\infty} \kappa \frac{4^j}{2^j} \phi_1 \left( \frac{x}{2^{j+n}} \right).
$$

which tends to zero as $n \to \infty$ for all $x \in \chi_\rho$. Hence the mapping $F_2$ is a unique cubic Lie derivation satisfying (28).

**Remark.** In Theorem 5 if $\chi_\rho$ is a Banach algebra with norm $\rho$, and so $\rho(2x) = 2\rho(x)$, $\kappa = 2$, then we see from (26) and (27) that there exists a unique cubic Lie derivation $F_2 : \chi_\rho \to \chi_\rho$, defined as $F_2(x) = \lim_{n \to \infty} 2^nf(x/2^n)$, $x \in \chi_\rho$, which satisfies (6) and

$$
\rho \left( f \left( x \right) - F_2 \left( x \right) \right) \leq \frac{1}{16} \sum_{j=1}^{\infty} 2^{3j} \phi_1 \left( \frac{x}{2^j} \right)
$$

for all $x \in \chi_\rho$.

As a corollary of Theorem 5, we obtain the following stability result of (6), which generalizes stability result on Banach algebras.

**Corollary 6.** Suppose $\chi_\rho$ is a Banach algebra with norm $\| \cdot \|$ and $\kappa = 2$. For given real numbers $\theta_1, \theta_2 \geq 0$, $r_1 > 3$ ($i = 1, 2$), $a_i + b_i > 3$, and $6 < a_2 + b_2$, if a mapping $f : \chi_\rho \to \chi_\rho$ satisfies

$$
\| CE_f (x, y) \| \leq \theta_1 \| x \|^a_1 + \theta_2 \| y \|^a_2 + \theta_1 \| x \|^b_1 \| y \|^b_1,
$$

$$
\| CD_f (x, y) \| \leq \theta_2 \| x \|^b_2 \| y \|^b_2,
$$

for all $x, y \in \chi_\rho$ and $\lambda \in \Lambda$, then there exists a unique cubic Lie derivation $F_2 : \chi_\rho \to \chi_\rho$ such that

$$
\| f \left( x \right) - F_2 \left( x \right) \| \leq \frac{\theta_1 \| x \|^a_1}{2 (2r_1 - 2^3)} + \frac{\theta_2 \| y \|^a_2}{2 (2r_2 - 8)}
+ \frac{\theta_1 \| x \|^b_1 + \theta_2 \| y \|^b_2}{2 (2^3 + b_1 - 8)}
$$

for all $x \in \chi_\rho$.

3. Conclusion

We introduce modular algebras with modular $\rho$ over $\mathbb{K}$ and obtain stability results of a cubic equation associated with cubic derivations on $\rho$-complete modular algebras, which generalizes stability results on Banach algebras.

**Data Availability**

Previously reported data were used to support this study and are available at [https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-017-1422-z](https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-017-1422-z) and [https://www.hindawi.com/journals/jfs/2015/461719/](https://www.hindawi.com/journals/jfs/2015/461719/). These prior studies (and datasets) are cited at relevant places within the text as [13–17].

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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