New Fields on Super Riemann Surfaces

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Abstract

A new (1, 1)-dimensional super vector bundle which exists on any super Riemann surface is described. Cross-sections of this bundle provide a new class of fields on a super Riemann surface which closely resemble holomorphic functions on a super Riemann surface, but which (in contrast to the case with holomorphic functions) form spaces which have a well defined dimension which does not change as odd moduli become non-zero.

Super Riemann surfaces are (1, 1)-dimensional holomorphic complex supermanifolds which have interesting mathematical features and have been intensively studied because of their use in a very elegant and effective approach to the Polyakov quantization of the spinning string. In this approach, which has been developed by a number of authors, for instance in the works of Baranov, Manin, Frolov and Schwarz [1], Baranov and Schwarz [2], Belavin and Knizhnik [3], Rosly, Schwarz and Voronov [10] and Voronov [11], many

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classical techniques of algebraic geometry are generalised to the super setting. However these methods depend on certain spaces of fields on super Riemann surfaces having a super vector space structure (that is, a free module structure) which they do not in fact possess. Not only do these spaces lack this structure, but also the nature of these spaces may vary as one moves around the moduli space of super Riemann surfaces corresponding to fixed genus and spin structure on the underlying Riemann surface. This difficulty, which undermines the analysis used in the quantization method, has been recognised for some time - it seems to have been first mentioned in the literature by Giddings and Nelson [6], quoting comments of Witten, and then extensively analysed by Hodgkin [8].

The purpose of this paper is to describe a modification of the notion of function, which is very natural in the context of a super Riemann surface, and which allows one to construct a family of fields with the desired super vector space structure. As well as suggesting a means by which the Polyakov quantization procedure might be made valid, this new class of functions may make possible further developments in the general theory of super Riemann surfaces. The construction is based on a (1,1)-dimensional super vector bundle which is shown to be canonically defined on any super Riemann surface. It makes use of the superconformal structure of the super Riemann surface, in contrast to the functions usually considered which only use the superanalytic structure.

The structure of a super Riemann surface first appears in the work of Howe [9] on the superspace formulation of 2-dimensional superconformal gravity; here the construction is in terms of a real (2,2)-dimensional supermanifold. Subsequently this same structure was formulated very elegantly in terms of (1,1)-dimensional complex co-ordinates by Baranov and Schwarz [4] and by Friedan [5]. Much of the basic theory of super Riemann surfaces was developed in detail by Crane and Rabin [4], and their notation and terminology is largely used here.

Various equivalent definitions of the notion of super Riemann surface have been given; we use here an approach based on local co-ordinates because it is most appropriate for the explicit cohomological calculations which follow. Briefly, a super Riemann surface is a (1,1)-dimensional complex holomorphic supermanifold $M$ which satisfies the following additional condition: there must exist a covering of $M$ by co-ordinate neighbourhoods $(V_\alpha | \alpha \in \Lambda)$ with corresponding local co-ordinates $(z_\alpha, \zeta_\alpha)$ such that on overlapping co-ordinate...
neighbourhoods $V_\alpha, V_\beta$ the differential operators $D_\alpha = \partial/\partial \zeta_\alpha + \zeta_\alpha \partial/\partial z_\alpha$ and $D_\beta = \partial/\partial \zeta_\beta + \zeta_\beta \partial/\partial z_\beta$ are related by

$$D_\beta = M_{\alpha\beta}(\zeta_\beta, \zeta_\beta) D_\alpha. \quad (1)$$

It follows directly from this definition first that

$$M_{\alpha\beta}(\zeta_\beta, \zeta_\beta) = D_\beta \zeta_\alpha(\zeta_\beta) \quad (2)$$

and second that the expression for the co-ordinates $(z_\alpha, \zeta_\alpha)$ in terms of the co-ordinates $(z_\beta, \zeta_\beta)$ takes the form

$$z_\alpha = f_{\alpha\beta}(z_\beta) + \zeta_\beta \psi_{\alpha\beta}(z_\beta) \sqrt{f'_{\alpha\beta}(z_\beta)}$$

$$\zeta_\alpha = \psi_{\alpha\beta}(z_\beta) + \zeta_\beta \sqrt{f'_{\alpha\beta}(z_\beta)} + \psi_{\alpha\beta}(z_\beta) \psi'_{\alpha\beta}(z_\beta), \quad (3)$$

where the function $f_{\alpha\beta}$ is an even holomorphic function and the function $\psi_{\alpha\beta}$ is an odd holomorphic function. The Grassmann algebra $\mathbb{C}_S$ on which the super Riemann surface $M$ is modelled (and in which all functions take values) is here taken to be the infinite-dimensional complex Grassmann algebra with generators $1, \nu_{(1)}, \nu_{(2)}, \ldots$ so that a typical element $C$ of $\mathbb{C}_S$ may be expressed as

$$C = C_{\{\emptyset\}} + \sum_i C_{\{i\}} \nu_{\{i\}} + \sum_{i<j} C_{\{ij\}} \nu_{\{i\}} \nu_{\{j\}} + \ldots \quad (4)$$

where the coefficients $C_{\{\emptyset\}}, C_{\{i\}}, C_{\{ij\}}$ etc are complex numbers. The even and odd parts of $\mathbb{C}_S$ are denoted $\mathbb{C}_S^0$ and $\mathbb{C}_S^1$ respectively. In the proof of the main theorem of this paper we will use the $Z$-grading of $\mathbb{C}_S$,

$$\mathbb{C}_S = \mathbb{C}_S[[0]] \oplus \mathbb{C}_S[[1]] \oplus \mathbb{C}_S[[2]] \oplus \ldots, \quad (5)$$

where $\mathbb{C}_S[[r]]$ contains terms of level $r$ in the Grassmann generators, that is, linear combinations of exactly $r$ anticommuting generators. A useful mapping is the mapping $\epsilon : \mathbb{C}_S \to \mathbb{C}$ which is defined by

$$\epsilon C = C_{\{\emptyset\}}. \quad (6)$$

This map allows one to construct the underlying Riemann surface $M_{\{\emptyset\}}$ of $M$ as a Riemann surface with co-ordinate neighbourhoods $\{U_\alpha | \alpha \in \Lambda\}$, where $U_\alpha = \epsilon(V_\alpha)$, and local co-ordinates $z_{\alpha_{\{\emptyset\}}}$ which transform by

$$z_{\alpha_{\{\emptyset\}}} = f_{\alpha\beta_{\{\emptyset\}}}(z_{\beta_{\{\emptyset\}}}) \quad (7)$$
where the functions $f_{αβ}(∅)$ are holomorphic functions on $U_α \cap U_β$ such that $f_{αβ}(∅)(z_β(∅)) = \epsilon(f_{αβ}(z_β))$. The super Riemann surface $M$ also determines a choice of sign for the square roots $\sqrt{f_{αβ}'(z_β)}$ and hence a spin structure on $M_{∅}$. The corresponding line bundle of spinors is denoted $\sqrt{K}$.

Super Riemann surfaces on which the functions $ψ_{αβ}$ (which partly determine the super Riemann surface structure) all vanish are known as split super Riemann surfaces. On such surfaces the local co-ordinate representatives of a superholomorphic function take the form

$$g_α(z_α, ζ_α) = p + ζ_απ_α(z_α)$$  \hspace{1cm} (8)

where $p$ is a constant element of $\mathbb{C}S_0$ and $π_α$ are local representatives of a $\mathbb{C}S_1$-valued section of the spin bundle $\sqrt{K}$. Thus the space of all such functions has the super vector space structure $O(T) = (\mathbb{C}S_0 \otimes \mathbb{C}) \oplus (\mathbb{C}S_1 \otimes \Gamma(\sqrt{K}))$. However (although such structure is actually required in the superholomorphic quantization of the spinning string) the space of super holomorphic functions does not take this form on an arbitrary super Riemann surface. The simplest counterexample \[6, 8\] is the non-split super torus with odd modulus $ψ$. Here the most general even holomorphic function which can be defined has the local form

$$g(z, ζ) = c + \gamma ζ$$  \hspace{1cm} (9)

where $c$ is an arbitrary even constant and $γ$ is an odd constant such that $ψγ = 0$. The set $O(T)$ of such functions does not have the structure of a super vector space - that is, there do not exist complex vector spaces $V_0$ and $V_1$ such that

$$O(T) \cong (\mathbb{C}S_0 \otimes V_0) \oplus (\mathbb{C}S_1 \otimes V_1).$$

(In more technical terms, a super vector space is the even part of a free graded $\mathbb{C}_S$-module.) The super vector space structure is important because it guarantees the existence of bases and dimension, which do not always exist for a module over a Grassmann algebra.

The starting point of the constructions in this paper is the observation that, whenever $V_α \cap V_β \cap V_γ$ is not empty, the three functions $ψ_{αβ}, ψ_{βγ}$ and $ψ_{αγ}$ satisfy the relation

$$ψ_{αγ}(z_γ) = ψ_{αβ}(z_β) + M_{αβ}(z_β, ζ_β)ψ_{βγ}(z_γ)$$  \hspace{1cm} (10)
as can be shown using equations (3). Then, if for each $\alpha, \beta$ in $\Lambda$ such that $V_\alpha \cap V_\beta$ is not empty one defines the $(1, 1) \times (1, 1)$ supermatrix functions $\mathcal{N}_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ by

$$\mathcal{N}_{\alpha\beta}(z_\beta, \zeta_\beta) = \begin{pmatrix} 1 & M_{\alpha\beta}(z_\beta, \zeta_\beta)^{-1}\psi_{\alpha\beta}(z_\beta) \\ 0 & M_{\alpha\beta}(z_\beta, \zeta_\beta)^{-1} \end{pmatrix},$$

three crucial results follow. First, for all $\alpha$ in $\Lambda$,

$$\mathcal{N}_{\alpha\alpha} = I$$

(12)

where $I$ denotes the identity matrix, secondly for all $\alpha, \beta$ in $\Lambda$ such that $V_\alpha \cap V_\beta$ is not empty

$$\mathcal{N}_{\beta\alpha}(z_\alpha, \zeta_\alpha) = \mathcal{N}_{\alpha\beta}(z_\beta, \zeta_\beta)^{-1}$$

(13)

and thirdly for all $\alpha, \beta$ and $\gamma$ such that $V_\alpha \cap V_\beta \cap V_\gamma$ is not empty

$$\mathcal{N}_{\alpha\gamma}(z_\gamma, \zeta_\gamma) = \mathcal{N}_{\alpha\beta}(z_\beta, \zeta_\beta)\mathcal{N}_{\beta\gamma}(z_\gamma, \zeta_\gamma).$$

(14)

This shows that the collection of functions $\{\mathcal{N}_{\alpha\beta} : V_\alpha \cap V_\beta \to GL(1, 1|\mathbb{C})\}$ defines a $(1, 1)$-dimensional complex super vector bundle $E_M$ over the super Riemann surface $M$. Equipped with this bundle, the key definition of a superconformal function can now be given. Roughly speaking, the idea is that a superconformal function should be a cross-section of $E_M$ whose local representatives have the form

$$\begin{pmatrix} g_\alpha \\ D_\alpha g_\alpha \end{pmatrix}.$$  

(15)

(The work of Baranov and Schwarz [2] on zeroes and poles on super Riemann surfaces demonstrates the importance of the pair $(g, Dg)$.) In fact this simple definition of a superconformal function is slightly too restrictive, the odd component has to have some further freedom, (although it can be shown that co-ordinates can always be chosen in which a superconformal function does take the simple form (15)).

**Definition 1** A superconformal function $G$ on the super Riemann surface $M$ is a cross-section of $E_M$ whose local representatives $\begin{pmatrix} g_\alpha \\ \gamma_\alpha \end{pmatrix}$ are holomorphic and satisfy

$$\gamma_\alpha(z_\alpha, \zeta_\alpha) = D_\alpha(g_\alpha(z_\alpha, \zeta_\alpha) + r_\alpha(z_\alpha))$$

(16)

for some holomorphic function $r_\alpha$ on $V_\alpha$. 

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The main theorem of this paper will now be established. This theorem shows that the space of superconformal functions on a super Riemann surface has the desired super vector space structure.

**Theorem 1** The space $\mathcal{SC}(M)$ of superconformal functions on the super Riemann surface $M$ has the structure

$$\mathcal{SC}(M) \cong (\mathbb{C}_{S_0} \otimes \mathcal{O}(M_{\{\emptyset\}})) \oplus (\mathbb{C}_{S_1} \otimes \Gamma(\sqrt{K})).$$  \hspace{1cm} (17)

Proof

First note that if the vectors

$$\begin{pmatrix} g_\alpha \\ \gamma_\alpha \end{pmatrix} \quad \alpha \in \Lambda$$

are the local representatives of a superconformal function then the expansions of $g$ and $\gamma$ in terms of $z_\alpha$ and $\zeta_\alpha$ must (by equation (16)) take the form

$$\begin{pmatrix} g_\alpha \\ \gamma_\alpha \end{pmatrix} = \begin{pmatrix} p_\alpha(z_\alpha) + \zeta_\alpha \sigma_\alpha(z_\alpha) \\ \sigma_\alpha(z_\alpha) + \zeta_\alpha(p'_\alpha(z_\alpha) + r'_\alpha(z_\alpha)) \end{pmatrix},$$  \hspace{1cm} (18)

where $p_\alpha, \sigma_\alpha$ and $r_\alpha$ are holomorphic functions.

Substitution into the transformation law derived from (11) (inverted, because it leads to much simpler equations) gives

$$\begin{pmatrix} g_\beta \\ \gamma_\beta \end{pmatrix} = \begin{pmatrix} 1 & -\psi_{\alpha\beta} \\ 0 & M_{\alpha\beta} \end{pmatrix} \begin{pmatrix} g_\alpha \circ f_{\alpha\beta} \\ \gamma_\alpha \circ f_{\alpha\beta} \end{pmatrix} \hspace{1cm} (19)$$

and expansion in powers of $\zeta_\beta$ leads to four equations, the first of which shows that the functions $p_\alpha$ satisfy

$$p_\beta(z_\beta) = p_\alpha(f_{\alpha\beta}(z_\beta)).$$  \hspace{1cm} (20)

Using the fact that at level zero the functions $f_{\alpha\beta}$ are simply the transition functions of a Riemann surface, one finds, first at level zero in the Grassmann generators, and then at higher levels by induction, that

$$p_\alpha = p$$  \hspace{1cm} (21)
where \( p \) is a constant element of \( \mathbb{C} S_0 \) which is independent of \( \alpha \). This allows the other three equations derived from equation (19) to be simplified to give two independent conditions,

\[
\sigma_\beta = (\sigma_\alpha \circ f_{\alpha \beta}) \sqrt{f'_{\alpha \beta}} + \psi_{\alpha \beta} \psi'_{\alpha \beta} + \psi_{\alpha \beta}(r'_{\alpha} \circ f_{\alpha \beta}) \sqrt{f'_{\alpha \beta}} \\
(22)
\]

\[
r'_\beta = (r'_\alpha \circ f_{\alpha \beta}) f'_{\alpha \beta} + \psi'_{\alpha \beta}(\sigma_\alpha \circ f_{\alpha \beta}) + \psi_{\alpha \beta}(\sigma'_{\alpha} \circ f_{\alpha \beta}) f'_{\alpha \beta} \\
(23)
\]

which, with the condition (21), are necessary and sufficient for \( (g_\alpha \gamma_\alpha) \) to be the local representative a superconformal function.

The remainder of the proof involves constructing a linear map \( \Phi \) from the space \( (\mathbb{C} S_0 \otimes \mathcal{O}(M_{\emptyset})) \oplus (\mathbb{C} S_1 \otimes \Gamma(\sqrt{K})) \) into the space of superconformal functions on \( M \), and showing that it is invertible. A typical element of the domain of \( \Phi \) can be expressed as \( p \oplus \pi \) with \( p = p_{\emptyset} 1 + p_{\{1\}} \nu_{\{1\}} \nu_{\{2\}} + \ldots \) and \( \pi = \pi_{\{1\}} \nu_{\{1\}} + \pi_{\{2\}} \nu_{\{2\}} + \ldots \); however, since (22) and (23) are linear in \( r \) and \( \sigma \), it is sufficient to consider \( \pi = \pi_{\{i\}} \nu_{\{i\}} \) (with no summation), extending \( \pi \) to the full domain by linearity. The map \( \Phi \) will be constructed inductively, the induction being over the number \( n \) of generators \( \nu_{\{1\}}, \ldots, \nu_{\{n\}} \) of the Grassmann algebra on which the supermanifold is modelled. In order to establish that \( \Phi(p \oplus \pi) \) is superconformal as \( n \) increases by 1, it is necessary to use a further induction, over the Grassmann level of the terms in equations (22) and (23).

It may be verified by direct calculation that the even equation (23) is satisfied at level zero and the odd equation (22) at level 1 if we set

\[
\Phi(p \oplus \pi) = \left\{ \left( \frac{p + \zeta_\alpha \pi_{\alpha} \nu_{\{i\}}}{\pi_{\alpha} \nu_{\{i\}}} \right) \right\}. \\
(24)
\]

(so that \( r' = 0 \) at level zero).

To satisfy (22) at level 2 we set

\[
\Phi(p \oplus \pi) = \left\{ \left( \frac{p \oplus \zeta_\alpha \pi_{\alpha}(z_\alpha)}{\pi_{\alpha}(z_\alpha) + \zeta_\alpha r'_\alpha(z_\alpha)} \right) \right\} \\
(25)
\]

where the \( r_\alpha \) (which are zero at level zero) are determined by the following cohomological argument: we require

\[
\frac{d}{dz_\beta}(\psi_{\alpha \beta}(z_\beta) \pi_{\alpha}(f_{\alpha \beta}(z_\beta))) = r'_\beta(z_\beta) - r'_\alpha(f_{\alpha \beta}(z_\beta)) f'_{\alpha \beta}(z_\beta) \\
(26)
\]
at level 2. To see that there exist unique $r_\alpha$ (up to a constant) such that
this equation is satisfied the relation \((13)\) is used; the $\nu_{\{j\}}$ component of this
equation is
\[
\psi_{\alpha\gamma\{j\}} = \psi_{\alpha\beta\{j\}} \circ f_{\beta\gamma\{0\}} + \sqrt{f_{\alpha\beta\{0\}}' \circ f_{\beta\gamma\{0\}}} \psi_{\beta\gamma\{j\}},
\]
so that (using the fact that $\pi_{\{i\}}$ is a spinor on $M_{\{0\}}$)
\[
(\pi_{\alpha\{i\}} \circ f_{\alpha\gamma\{0\}})\psi_{\alpha\gamma\{j\}} = (\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{0\}} \circ f_{\beta\gamma\{0\}})(\psi_{\alpha\beta\{j\}} \circ f_{\beta\gamma\{0\}}) + (\pi_{\beta\{i\}} \circ f_{\beta\gamma\{0\}})\psi_{\beta\gamma\{j\}}.
\]
Thus the collection \(\{(\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{0\}})\psi_{\alpha\beta\{j\}}|\alpha, \beta \in \Lambda, V_\alpha \cap V_\beta \neq \emptyset\}\) defines an
element of $H^1(M_{\{0\}}, \mathcal{O})$, (or, in looser phrasing which will be used in the
remainder of the paper, \((\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{0\}})\psi_{\alpha\beta\{j\}}\) is an element of $H^1(M_{\{0\}}, \mathcal{O})$).
Now any class in $H^1(M_{\{0\}}, \mathcal{O})$ has a constant representative \((\mathfrak{f})\); thus there
exist $r_{\alpha\{ij\}}$ (unique up to a constant) in $C^0(M_{\{0\}}, \mathcal{O})$ which satisfy \((23)\) at
level 2, and thus \((23)\) is satisfied at level 2.

To satisfy \((22)\) at level 3 we set
\[
\Phi(p \oplus \pi) = \left( p + \zeta_i \sigma_\alpha(z_\alpha) \sigma_\alpha(z_\alpha) + \zeta_i \nu_{ij}(z_\alpha) \right)
\]
with each component of $r_\alpha = r_{\alpha\{ij\}}\nu_{\{i\}}\nu_{\{j\}}$ determined as before and $\sigma = \pi_{\{i\}}\nu_{\{j\}} + \sigma_{\{ijk\}}\nu_{\{i\}}\nu_{\{j\}}\nu_{\{k\}}$ where $\sigma_{\{ijk\}}$ is determined from \((22)\) by cohomological methods which will now be described.

It is sufficient to verify the \(\{ijk\}\) component of equation \((22)\), assuming
without loss of generality that $i < j < k$. Thus it must be shown that
\[
\sigma_{\beta\{ijk\}} = (\sigma_{\alpha\{ijk\}} \circ f_{\alpha\beta\{0\}})\sqrt{f_{\alpha\beta\{0\}}'} + (\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{0\}})(\pi_{\alpha\beta\{0\}}')f_{\alpha\beta\{jk\}}\sqrt{f_{\alpha\beta\{0\}}'} + \frac{1}{2}(\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{0\}})\nu_{\alpha\beta\{j\}} \nu_{\alpha\beta\{k\}} - [j \leftrightarrow k].
\]
To show that $\sigma_{\alpha\{ijk\}}$ can be chosen so that this level three equation is satisfied, first consider

$$T_{\alpha\beta ijk} = (\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{\emptyset\}})^2 f_{\alpha\beta\{jk\}}. \quad (31)$$

Using the consistency condition [1]

$$f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma} + \psi_{\beta\gamma}(\psi_{\alpha\beta} \circ f_{\beta\gamma\{\emptyset\}})\sqrt{f'_{\alpha\beta\{\emptyset\}} \circ f_{\beta\gamma\{\emptyset\}}}, \quad (32)$$

it then follows that

$$T_{\alpha\gamma ijk} = T_{\alpha\beta ijk} \circ f_{\beta\gamma\{\emptyset\}} + T_{\beta\gamma ijk} + U_{\alpha\beta\gamma ijk}. \quad (33)$$

where

$$U_{\alpha\beta\gamma ijk} = \left[ (\pi_{\alpha\{i\}} \circ f_{\alpha\gamma\{\emptyset\}})^2 \psi_{\beta\gamma}(\psi_{\alpha\beta} \circ f_{\beta\gamma\{\emptyset\}})\sqrt{f'_{\alpha\beta\{\emptyset\}} \circ f_{\beta\gamma\{\emptyset\}}} \right] - [j \leftrightarrow k]. \quad (34)$$

Using (10) it may be shown that $U_{\alpha\beta\gamma ijk}$ is in $H^2(M_{\emptyset}^{\{\emptyset\}}, O)$. Also, it may be verified by explicit calculation that

$$\frac{dU_{\alpha\beta\gamma ijk}}{dz_{\gamma}} = -2(A_{\alpha\gamma ijk} - (A_{\alpha\beta ijk} \circ f_{\beta\gamma\{\emptyset\}}) f'_{\beta\gamma\{\emptyset\}} - A_{\beta\gamma ijk}) \quad (35)$$

where

$$A_{\alpha\beta\{ijk\}} = \left[ (\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{\emptyset\}}) \psi_{\alpha\beta\{j\}}(r'_{\alpha\{ik\}} \circ f_{\alpha\beta\{\emptyset\}}) f'_{\alpha\beta\{\emptyset\}} \right]
+ \frac{1}{2}(\pi_{\alpha\{i\}} \circ f_{\alpha\beta\{\emptyset\}})^2 \psi_{\alpha\beta\{j\}} \psi'_{\alpha\beta\{k\}} - [j \leftrightarrow k]. \quad (36)$$

Thus, if functions $F_{\alpha\beta ijk}$ are chosen on $V_{\alpha} \cap V_{\beta}$ satisfying

$$\frac{dF_{\alpha\beta ijk}}{dz_{\beta}} = -2A_{\alpha\beta ijk}, \quad (37)$$

then

$$\frac{d}{dz_{\gamma}}(U_{\alpha\beta\gamma ijk} - F_{\alpha\gamma ijk} + F_{\alpha\beta ijk} \circ f_{\beta\gamma\{\emptyset\}} + F_{\beta\gamma ijk}) = 0 \quad (38)$$
so that \( U_{\alpha\beta\gamma \, ijk} - F_{\alpha\gamma \, ijk} + F_{\alpha\beta \, ijk} \circ f_{\beta\gamma \{\emptyset\}} + F_{\beta\gamma \, ijk} \) is an element of \( H^2(M_{\emptyset}, \mathbb{C}) \). Now \( H^2(M_{\emptyset}, \mathbb{C}) \) is trivial, and so there exist \( K_{\alpha\beta \, ijk} \) in \( C^1(M_{\emptyset}, \mathbb{C}) \) such that

\[
U_{\alpha\beta\gamma \, ijk} - F_{\alpha\gamma \, ijk} + F_{\alpha\beta \, ijk} \circ f_{\beta\gamma \{\emptyset\}} + F_{\beta\gamma \, ijk} = K_{\alpha\gamma \, ijk} - K_{\alpha\beta \, ijk} \circ f_{\beta\gamma \{\emptyset\}} - K_{\beta\gamma \, ijk}.
\]

(39)

Thus, if \( L_{\alpha\beta \, ijk} = F_{\alpha\beta \, ijk} + K_{\alpha\beta \, ijk} \),

\[
U_{\alpha\beta\gamma \, ijk} = L_{\alpha\gamma \, ijk} - L_{\alpha\beta \, ijk} \circ f_{\beta\gamma \{\emptyset\}} - L_{\beta\gamma \, ijk},
\]

(40)

and

\[
\frac{d}{dz_\beta} L_{\alpha\beta \, ijk} = -2A_{\alpha\beta \, ijk}.
\]

(41)

Thus \( T_{\alpha\beta \, ijk} - L_{\alpha\beta \, ijk} \) is in \( H^1(M_{\emptyset}, \mathcal{O}) \), and (again using the fact that every class in \( H^1(M_{\emptyset}, \mathcal{O}) \) has a constant representative), it can be deduced that there exist \( k_{\alpha \, ijk} \) in \( C^0(M_{\emptyset}, \mathcal{O}) \) such that

\[
\frac{d}{dz_\beta} (T_{\alpha\beta \, ijk} - L_{\alpha\beta \, ijk} + k_{\alpha \, ijk} \circ f_{\alpha\beta \{\emptyset\}} - k_{\beta \, ijk}) = 0.
\]

(42)

Using (41), we see that the level three equation (30) is then satisfied if

\[
\sigma_{\alpha \{ijk\}} = \frac{d}{dz_\alpha} k_{\alpha \, ijk} / 2\pi_{\alpha \{i\}},
\]

(43)

the division being possible because at any point where \( \pi_{\alpha \{i\}} \) has a zero, \( k_{\alpha \, ijk} \) must have one of at least one order higher.

The inductive construction of \( \Phi(p \oplus \pi) \) is then completed by similar arguments, although the details are too long to be included here. Simple induction over level however is difficult because of cross terms which arise; the double induction mentioned above, over number of generators and level, avoids this difficulty.

The proof of the theorem is completed by showing that \( \Phi \) is bijective. It is evident from its definition that \( \Phi \) is injective. To show that it is surjective, suppose that \( G \) is a superconformal function. From previous arguments we know that \( G \) has local representatives of the form

\[
\begin{pmatrix}
g_\alpha(z_\alpha, \zeta_\alpha) \\
g_\alpha'(z_\alpha, \zeta_\alpha)
\end{pmatrix} = \begin{pmatrix}
p + \zeta_\alpha \sigma_\alpha(z_\alpha) \\
\sigma_\alpha(z_\alpha) + \zeta_\alpha r'_\alpha(z_\alpha)
\end{pmatrix}.
\]

(44)
Suppose that $\sigma$ is decomposed level by level as

$$\sigma = \sigma[[1]] + \sigma[[3]] + \ldots \quad (45)$$

where $\sigma[[2k+1]]$ denotes the component of $\sigma$ of level $2k + 1$ in the generators $\nu_1, \nu_2, \ldots$ of $C_S$. Then, letting $\tilde{\sigma}_{2k+1}$ be defined inductively by

$$\tilde{\sigma}_1 = \sigma[[1]]$$

$$\tilde{\sigma}_{2k+1} = \sigma[[2k+1]] - \left[ \Phi(0 \oplus \left( \sum_{\ell=1}^{k} \tilde{\sigma}_{2\ell-1} \right)) \right] \quad (46)$$

it follows from the construction of $\Phi$ that each $\tilde{\sigma}_{2k+1}$ is an element of $C_{S1} \otimes \Gamma(\sqrt{K})$, and also that

$$G = \Phi\left(p \oplus \left( \sum_{k=0}^{\infty} \tilde{\sigma}_{2k+1} \right) \right) \quad (47)$$

so that $\Phi$ must be surjective.

Thus we have shown that the modified definition of a function embodied in the notion of a superconformal function leads to a function space with the desired properties. In the case of a split super Riemann surface a superconformal function is simply a holomorphic function, but the modification in the general case allows a constant structure as one moves away from the split part of super moduli space. It seems likely that this will be useful in applications. Further developments to be considered are the integration theory of such functions and the construction of analogous fields of higher spin.
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