SUPPORT POINTS AND THE BIEBERBACH CONJECTURE IN HIGHER DIMENSION

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ABSTRACT. We describe some open questions related to support points in the class \(S^0\) and introduce some useful techniques toward a higher dimensional Bieberbach conjecture.

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1. Introduction

Let \(\mathbb{B}^n \subset \mathbb{C}^n\) denote the unit ball for the standard Hermitian product in \(\mathbb{C}^n\), \(n \geq 1\). For the sake of simplicity, we consider only the case \(n = 1\) and \(n = 2\), but, in fact, all results which we are going to discuss for \(\mathbb{B}^2\) hold in any dimension. We denote \(\mathbb{D} := \mathbb{B}^1\).

Let \(S(\mathbb{B}^n)\) denote the class of univalent maps \(f : \mathbb{B}^n \to \mathbb{C}^n\) normalized so that \(f(0) = 0, df_0 = \text{id}\). We consider \(S(\mathbb{B}^n)\) as a subspace of the Frechét space of holomorphic maps from \(\mathbb{B}^n\) to \(\mathbb{C}^n\) with the topology of uniform convergence on compacta.

For \(n = 1\), the set \(S(\mathbb{D})\) is compact (see, e.g., [12]). For \(n > 1\), the set \(S(\mathbb{D})\) is not compact: a simple example is given by considering the sequence

\[\{(z_1, z_2) \mapsto (z_1 + mz_2, z_2)\}_{m \in \mathbb{N}},\]

which belongs to \(S(\mathbb{B}^2)\) but for \(m \to \infty\) does not converge.

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Since the set $S(\mathbb{D})$ is compact, for every continuous linear operator $L : \text{Hol}(\mathbb{D}, \mathbb{C}) \to \mathbb{C}$ there exists $f \in S(\mathbb{D})$ such that $\Re L(f) \geq \Re L(g)$ for all $g \in S(\mathbb{D})$. If $L$ is not constant, such an $f$ is called a support point for $L$.

It is known that every support point in $S(\mathbb{D})$ is unbounded and is, in fact, a slit map (see [14]). The most interesting linear functionals to be considered are perhaps those defined as $L_m(f) = b_m$, where $f(z) = \sum_{j \geq 0} b_j z^j \in S(\mathbb{D})$, $m \in \mathbb{N}$. The Bieberbach conjecture, proved in the 80’s by L. de Branges, states that $|b_m| \leq m$ for $f \in S(\mathbb{D})$ and that equality is reached only by rotations of the Koebe function.

In higher dimensions, since the class $S(\mathbb{B}^2)$ is not compact, one is forced to consider suitable compact subclasses. Convex maps and starlike maps form compact subclasses, but, for many purposes, these classes are too small. In [8] (see also [7]), it was introduced a compact subclass, denoted by $S^0(\mathbb{B}^2)$ (or simply $S^0$), for which the membership depends on the existence of a parametric representation, a condition that is always satisfied in dimension one thanks to the classical Loewner theory (see Section 2).

The class $S^0$ is strictly contained in $S := S(\mathbb{B}^2)$, but evidences are that every map in $S$ might be factorized as the composition of an element of $S^0$ and a normalized entire univalent map of $\mathbb{C}^2$ (this is known to be true for univalent maps $f$ on $\mathbb{B}^2$ which extend $C^\infty$ up to the boundary, $f(\mathbb{B}^2)$ is strongly pseudoconvex and $\overline{f(\mathbb{B}^2)}$ is polynomially convex; see [2]). Were this the case, one could somehow split the difficulties in understanding univalent maps on $\mathbb{B}^2$ into two pieces: understanding the compact class $S^0$ and automorphisms/Fatou-Bieberbach maps in $\mathbb{C}^2$. In light of this, the class $S^0$ seems to be a natural candidate to study in higher dimensions.

The present note focuses on support points on $S^0$. Our aim is, on the one hand, to state some natural open questions originating in the recent works [3, 4, 11, 13], and, on the other hand, to develop some new techniques to handle such problems (in particular, slice reduction and decoupling harmonic terms tricks). The paper [9] contains other open questions in this direction and an extensive bibliography on the subject, to which we refer the reader. Here we mainly focus our attention on those questions which relate the class $S^0$ to the (huge) group of automorphisms of $\mathbb{C}^2$, highlighting the deep differences between dimension 1 and dimension 2 (see Section 4).

We also develop the ideas in [4] (see Section 6), which allowed to construct an example of a bounded support point in $S^0$. With these tools in hand, we state a Bieberbach-type conjecture in $S^0$ for coefficients of pure terms in $z_1$ and $z_2$ in the expansion at the origin.

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1In [11], the result was extended to univalent maps on $\mathbb{B}^2$ which extend $C^1$ up to $\partial \mathbb{B}^2$ and whose image is Runge in $\mathbb{C}^2$, but, unfortunately, there is a gap in the proof.
2. The class $S^0$

In what follows we denote by $\mathbb{R}^+$ the semigroup of nonnegative real numbers and by $\mathbb{N}$ the semigroup of nonnegative integers.

Let

$$\mathcal{M} := \{ h \in \text{Hol}(\mathbb{B}^2, \mathbb{C}^2) : h(0) = 0, dh_0 = \text{id}, \text{Re} \langle h(z), z \rangle > 0, \forall z \in \mathbb{B}^2 \setminus \{0\} \},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{C}^2$.

The set $\mathcal{M}$ is compact in $\text{Hol}(\mathbb{B}^2, \mathbb{C}^2)$ endowed with the topology of uniform convergence on compacta (see [7]).

**Definition 2.1.** A Herglotz vector field associated with the class $\mathcal{M}$ on $\mathbb{B}^2$ is a mapping $G : \mathbb{B}^2 \times \mathbb{R}^+ \to \mathbb{C}^2$ with the following properties:

(i) the mapping $G(z, \cdot)$ is measurable on $\mathbb{R}^+$ for all $z \in \mathbb{B}^2$.

(ii) $-G(\cdot, t) \in \mathcal{M}$ for a.e. $t \in [0, +\infty)$.

**Remark 2.2.** Due to the estimates for the class $\mathcal{M}$ (see [7]), a Herglotz vector field associated with the class $\mathcal{M}$ on $\mathbb{B}^2$ is an $L^\infty$-Herglotz vector field on $\mathbb{B}^2$ in the sense of [5].

**Definition 2.3.** A family $(f_t)_{t \geq 0}$ of holomorphic mappings from $\mathbb{B}^2$ to $\mathbb{C}^2$ such that $f_t(0) = 0$ and $d(f_t)_0 = e^t \text{id}$ for all $t \geq 0$, is called a normalized regular family if

(i) the mapping $t \mapsto f_t$ is continuous with respect to the topology in $\text{Hol}(\mathbb{B}^2, \mathbb{C}^2)$ induced by the uniform convergence on compacta in $\mathbb{B}^2$,

(ii) there exists a set of zero measure $N \subset [0, +\infty)$ such that for all $t \in [0, +\infty) \setminus N$ and all $z \in \mathbb{B}^2$ the partial derivative $\frac{\partial f_t}{\partial t}(z)$ exists and is holomorphic.

For a given Herglotz vector field $G(z, t)$ associated with the class $\mathcal{M}$ on $\mathbb{B}^2$, a normalized solution to the Loewner-Kufarev PDE associated to $G(z, t)$ consists of a normalized regular family $(f_t)_{t \geq 0}$ such that the following equation is satisfied for a.e. $t \geq 0$ and for all $z \in \mathbb{B}^2$

$$\frac{\partial f_t}{\partial t}(z) = -d(f_t)_z \cdot G(z, t).$$

**Definition 2.4.** A normalized subordination chain $(f_t)_{t \geq 0}$ is a family of holomorphic mappings $f_t : \mathbb{B}^2 \to \mathbb{C}^2$, such that $f_t(0) = 0$, $d(f_t)_0 = e^t \text{id}$ for all $t \geq 0$, and for every $0 \leq s \leq t$ there exists $\varphi_{s,t} : \mathbb{B}^2 \to \mathbb{B}^2$ holomorphic such that $f_s = f_t \circ \varphi_{s,t}$. A normalized subordination chain $(f_t)_{t \geq 0}$ is called a normalized Loewner chain if for all $t \geq 0$ the mapping $f_t$ is univalent.

**Definition 2.5.** A normalized Loewner chain $(f_t)_{t \geq 0}$ on $\mathbb{B}^2$ is called a normal Loewner chain if the family $\{e^{-t}f_t(\cdot)\}_{t \geq 0}$ is normal.

From [7, Chapter 8], [11, Prop. 2.6] and [10], we have the following:
Theorem 2.6. (1) If \((f_t)_{t \geq 0}\) is a normalized Loewner chain on \(\mathbb{B}^2\), then it is a normalized solution to a Loewner-Kufarev PDE (2.1) for some Herglotz vector field \(G(z, t)\) associated with the class \(\mathcal{M}\) in \(\mathbb{B}^2\).

(2) Let \(G(z, t)\) be a Herglotz vector field associated with the class \(\mathcal{M}\) on \(\mathbb{B}^2\). Then there exists a unique normalized Loewner chain \((g_t)_{t \geq 0}\) — called the canonical solution — which is a normalized solution to (2.1). Moreover, \(\bigcup_{t \geq 0} g_t(\mathbb{B}^2) = \mathbb{C}^2\).

(3) If \((f_t)_{t \geq 0}\) is a normalized solution to (2.1), then \((f_t)_{t \geq 0}\) is a normalized subordination chain on \(\mathbb{B}^2\). Moreover, there exists a holomorphic mapping \(\Phi : \mathbb{C}^2 \to \bigcup_{t \geq 0} f_t(\mathbb{B}^2)\), with \(\Phi(0) = 0\) and \(d\Phi = \text{id}\) such that \(f_t = \Phi \circ g_t\), where \((g_t)_{t \geq 0}\) is the canonical solution to (2.1). In particular, \((f_t)_{t \geq 0}\) is a normalized Loewner chain if and only if \(\Phi\) is univalent.

Remark 2.7. Let \((f_t)_{t \geq 0}\) be a family of holomorphic mappings such that \(f_t(0) = 0, d(f_t)_0 = e^t\text{id}\) for all \(t \geq 0\). Assume that \((f_t)_{t \geq 0}\) satisfies (i) of Definition 2.3 and for all fixed \(z \in \mathbb{B}^2\) the mapping \(t \mapsto f_t(z)\) is absolutely continuous. If for all fixed \(z \in \mathbb{B}^2\) the family \((f_t)_{t \geq 0}\) satisfies (2.1) for a.e. \(t \geq 0\), then it a normalized solution to the Loewner-Kufarev PDE and thus is a regular family.

Definition 2.8. Let \(f \in \mathcal{S}\). We say that \(f\) admits parametric representation if
\[
f(z) = \lim_{t \to \infty} e^t \varphi(z, t)
\]
locally uniformly on \(\mathbb{B}^2\), where \(\varphi(z, 0) = z\) and
\[
\frac{\partial \varphi}{\partial t}(z, t) = G(\varphi(z, t), t), \quad \text{a.e. } t \geq 0, \quad \forall z \in \mathbb{B}^2,
\]
for some Herglotz vector field \(G\) associated with the class \(\mathcal{M}\) on \(\mathbb{B}^2\).

We denote by \(\mathcal{S}^0\) the set consisting of univalent mappings which admit parametric representation.

The following result is in [7, Chapter 8] (see also [8]):

Theorem 2.9. (1) A normalized univalent map \(f : \mathbb{B}^2 \to \mathbb{C}^2\) has parametric representation if and only if there exists a normal Loewner chain \((f_t)_{t \geq 0}\) on \(\mathbb{B}^2\) such that \(f_0 = f\).

(2) The class \(\mathcal{S}^0\) is compact in the topology of uniform convergence on compacta.

3. Support points and extreme points

Definition 3.1. (i) Let \(K\) be a compact subset of \(\text{Hol}(\mathbb{B}^2, \mathbb{C}^2)\) endowed with the topology of uniform convergence on compacta. A mapping \(f \in K\) is called a support point if there exists a continuous linear operator \(L : \text{Hol}(\mathbb{B}^2, \mathbb{C}^2) \to \mathbb{C}\) not constant on \(K\) such that \(\max_{g \in K} \text{Re} L(g) = \text{Re} L(f)\). We denote by \(\text{Supp}(K)\) the set of support points of \(K\).

(ii) A mapping \(f \in K\) is called an extreme point if \(f = tg + (1-t)h,\) where \(t \in (0, 1), g, h \in K,\) implies \(f = g = h\). We denote by \(\text{Ex}(K)\) the set of extreme points of \(K\).
Note that the notion of extreme points is not related to topology, but only on the geometry of the set. If \( K \) is a compact subset of \( \text{Hol}(\mathbb{B}^2, \mathbb{C}^2) \) and \( a \in K \) is a support point which maximizes the continuous linear operator \( L \), then \( \mathcal{L} := \{ b \in \text{Hol}(\mathbb{B}^2, \mathbb{C}^2) : \Re L(b) = \Re L(a) \} \) is a real hyperplane and \( \mathcal{L} \cap K \) contains extreme points. Therefore, for any continuous linear operator \( L \) which is not constant on \( K \) there exists a point \( a \in K \) which is both a support point (for \( L \)) and an extreme point for \( K \).

In dimension one it is known that all support points for \( S^0 \) are slit mappings (see [14]). In higher dimension, the situation is considerably more complicated.

**Proposition 3.2.** [13] Let \( f \in S^0 \) be a support point. Let \( G(z, t) \) be a Herglotz vector field associated with the class \( \mathcal{M} \) which generates a normal Loewner chain \((f_t)\) such that \( f_0 = f \). Then \( G(z, t) \) is a support point of \(-\mathcal{M}\) for a.e. \( t \geq 0 \).

**Question 3.3.** Let \( f \in S^0 \) be a support point.

1. Does there exist only one normal Loewner chain \((f_t)\) such that \( f_0 = f \)?
2. Does there exist a Herglotz vector field associated with the class \( \mathcal{M} \) which generates a normal Loewner chain \((f_t)\) with \( f_0 = f \) such that \( t \mapsto G(\cdot, t) \) is continuous and \( G(\cdot, t) \in \text{supp}(\mathcal{M}) \) for all \( t \geq 0 \)?

**Question 3.4.** Let \( G(z, t) \) be a Herglotz vector field associated with the class \( \mathcal{M} \) which generates a normal Loewner chain \((f_t)\).

1. If \( f_0 \) is extreme in \( S^0 \), is it true that \( G(z, t) \) is extreme in \(-\mathcal{M}\) for a.e. \( t \geq 0 \)?
2. If \( f_0 \) is extreme in \( S^0 \), is \( G(z, t) \) uniquely determined?

**Proposition 3.5.** [15] Let \((f_t)\) be a normal Loewner chain. Then for all \( t \geq 0 \), \( e^{-t}f_t \in S^0 \). Moreover, if \( f_0 \) is a support/extreme point for \( S^0 \), so is \( e^{-t}f_t \) for all \( t \geq 0 \).

**Definition 3.6.** Let \((f_t)_{t \geq 0}\) be a normalized Loewner chain in \( \mathbb{B}^2 \) and \( G(z, t) \) be the associated Herglotz vector field. We say that \((f_t)_{t \geq 0}\) is exponentially squeezing in \([T_1, T_2]\), for \( 0 \leq T_1 < T_2 \leq +\infty \) (with squeezing ratio \( a \in (0,1) \)) if for a.e. \( t \in [T_1, T_2] \) and for all \( z \in \mathbb{B}^2 \setminus \{0\} \),

\[
(3.1) \quad \Re \left( G(z, t), \frac{z}{\|z\|^2} \right) \leq -a.
\]

In [3] it is proved that \((3.1)\) is equivalent to: for all \( T_1 \leq s < t \leq T_2 \),

\[
(3.2) \quad \|f_t^{-1}(f_s(z))\| \leq e^{a(s-t)}\|z\|, \quad \text{for all } z \in \mathbb{B}^2.
\]

Hence, if \((f_t)\) is exponentially squeezing in \([T_1, T_2]\), then \( f_t \) is bounded for all \( t \in [0, T_2) \) and \( f_t(\mathbb{B}^2) \subset f_s(\mathbb{B}^2) \) for all \( T_1 \leq s < t < T_2 \).

Using the results of [3], or of [13] for support points, one can prove

**Proposition 3.7.** Let \((f_t)\) be a normal Loewner chain which is exponentially squeezing in \([T_1, T_2]\). Then \( f_0 \notin \text{Supp}(S^0) \cup \text{Ex}(S^0) \).
Example 3.8. Let \( f \in S^0 \). Let \((f_t)\) be one parametric representation of \( f \). Let \( r \in (0, 1) \). Consider \( f_{r,t}(z) := r^{-1}f_t(rz) \). Then \((f_{r,t})\) is an exponentially squeezing normal Loewner chain and in particular, \( f_r \in S^0 \setminus (\text{Supp}(S^0) \cup \text{Ex}(S^0)) \).

Question 3.9. Let \( f \in S^0 \setminus (\text{Supp}(S^0) \cup \text{Ex}(S^0)) \) be a bounded function. Is it true that \( f \) can be embedded into an exponentially squeezing Loewner chain?

Question 3.10. Let \( G(z, t) \) be a Herglotz vector field associated with the class \( \mathcal{M} \) which generates a normal Loewner chain \((f_t)\). Assume that
\[
\limsup_{z \to A} \text{Re} \left\langle G(z, t), \frac{z}{\|z\|^2} \right\rangle \leq -a,
\]
for some \( A \subset \partial \mathbb{B}^2 \) and a.e. \( t \geq 0 \). Is it true that \( f_0 \notin \text{Supp}(S^0) \cup \text{Ex}(S^0) \)?

4. AUTOMORPHISMS OF \( \mathbb{C}^2 \) AND SUPPORT POINTS

Let
\[
\text{Aut}_0(\mathbb{C}^2) := \{ f \in \text{Aut}(\mathbb{C}^2) : f(0) = 0, df_0 = \text{id} \}.
\]
Given \( f \in \text{Aut}_0(\mathbb{C}^2) \), for every \( r > 0 \), the map \( f^r : \mathbb{B}^2 \to \mathbb{C}^2 \) defined by
\[
f^r(z) = \frac{1}{r} f(rz),
\]
is normalized and univalent. For \( r << 1 \), the image \( f^r(\mathbb{B}^2) \) is convex, hence \( f^r \in S^0 \). For \( f \in \text{Aut}_0(\mathbb{C}^2) \), let
\[
r(f) := \sup \{ t > 0 : f^t \in S^0 \}.
\]
Since \( S^0 \) is compact and \( \{f^t\}_{t>0} \) is not normal except for \( f = \text{id} \), it follows that for \( f \in \text{Aut}_0(\mathbb{C}^2) \setminus \{\text{id}\} \)
\[
0 < r(f) < +\infty, \quad f^{r(f)} \in S^0.
\]

Question 4.1. Let \( f \in \text{Aut}_0(\mathbb{C}^2) \setminus \{\text{id}\} \). Is it true that \( f^{r(f)} \in \text{Supp}(S^0) \)?

The previous question has a positive solution in the case \( f(z_1, z_2) = (z_1 + az_2^2, z_2) \), as we discuss later (or see [4]).

Let
\[
\mathcal{A} := \{ f \in S^0 : \text{there exists } \Psi \in \text{Aut}(\mathbb{C}^2) : \Psi|_{\mathbb{B}^2} = f \}
\]
Note that in dimension one the analogue of \( \mathcal{A} \) contains only the identity mapping. In higher dimension we have

Theorem 4.2. \([11] \quad \overline{\mathcal{A}} = S^0\).

Take \( f \in S^0 \), and expand \( f \) as
\[
f(z_1, z_2) = (z_1 + \sum_{\alpha \in \mathbb{N}_2, |\alpha| \geq 2} b^1_{\alpha} z^\alpha, z_2 + \sum_{\alpha \in \mathbb{N}_2, |\alpha| \geq 2} b^2_{\alpha} z^\alpha).
\]
By a result of F. Forstnerič [6], for any \( M \in \mathbb{N} \) there exists \( g \in \text{Aut}_0(\mathbb{C}^2) \) such that \( f - g = O(\|z\|^{M+1}) \) (that is, \( f, g \) have the same jets up to order \( M \)). However, such a \( g \) does not belong to \( S^0 \) in general.

**Question 4.3.** For which \( \alpha \in \mathbb{N}^2 \) is it true that for any \( f \in S^0 \) there exists \( g \in A \) having the same coefficients \( b^i_\alpha \) as \( f \)?

## 5. Coefficient bounds in \( \mathbb{H}^2 \)

We use the following notation: \( f \in S^0 \),

\[
f(z_1, z_2) = (z_1 + \sum_{\alpha \in \mathbb{N}^2, |\alpha| \geq 2} b^1_\alpha z^\alpha, z_2 + \sum_{\alpha \in \mathbb{N}^2, |\alpha| \geq 2} b^2_\alpha z^\alpha).
\]

If \((f_t)\) is a normal Loewner chain, we denote by \( b^i_\alpha(t) \) the corresponding coefficients of \( f_t \).

For \( G(z, t) \) a Herglotz vector field associated with the class \( \mathcal{M} \)

\[
G(z, t) = (-z_1 + \sum_{\alpha \in \mathbb{N}^2, |\alpha| \geq 2} q^1_\alpha(t) z^\alpha, -z_2 + \sum_{\alpha \in \mathbb{N}^2, |\alpha| \geq 2} q^2_\alpha(t) z^\alpha).
\]

For an evolution family \( \varphi_{s,t} := f_t^{-1} \circ f_s \),

\[
\varphi_{s,t}(z_1, z_2) = (e^{s-t} z_1 + \sum_{\alpha \in \mathbb{N}^2, |\alpha| \geq 2} a^1_\alpha(s, t) z^\alpha, e^{s-t} z_2 + \sum_{\alpha \in \mathbb{N}^2, |\alpha| \geq 2} a^2_\alpha(s, t) z^\alpha).
\]

For \( f \in S^0, \alpha \in \mathbb{N}^2, |\alpha| \geq 2, j = 1, 2 \), let

\[
L^j_\alpha(f) := b^j_\alpha.
\]

The problem of finding the maximal possible sharp bound for coefficients of mappings in the class \( S^0 \), consists in fact in finding the support points in \( S^0 \) for the linear functionals \( L^j_\alpha \).

If \((f_1(z_1, z_2), f_2(z_1, z_2)) \in S^0 \), then \((f_2(z_2, z_1), f_1(z_2, z_1)) \in S^0 \). Therefore, it is enough to solve the problem for \( L^1_\alpha \). More generally, if \( U \) is a \( 2 \times 2 \) unitary matrix, given a normal Loewner chain \((f_t), (U^* f_t(U z))\) is again a normal Loewner chain. This enables us to assume that a given coefficient \( b^1_\alpha > 0 \), so that, in fact, \( \max_{g \in S^0} \text{Re} L^1_\alpha(g) = \max_{g \in S^0} |b^1_\alpha(g)| \).

Let \((f_t)\) be a normal Loewner chain, \( G(z, t) \) the associated Herglotz vector field and \((\varphi_{s,t})\) the associated evolution equation. Expanding the Loewner ODE one gets

\[
(5.1) \quad \frac{\partial a^1_\alpha(s, t)}{\partial t} = -a^1_\alpha(s, t) + q^1_\alpha(t) e^{\alpha|s-t|} + R_\alpha,
\]

where \( R_\alpha \) is the coefficient of \( z^\alpha \) in the expansion of

\[
\sum_{2 \leq |\beta| \leq |\alpha|-1} q^1_\beta(t) (e^{s-t} z_1 + \sum_{2 \leq |\beta| \leq |\alpha|-1} a^1_\beta(s, t) z^\beta)^{\gamma_1} (e^{s-t} z_1 + \sum_{2 \leq |\beta| \leq |\alpha|-1} a^2_\beta(s, t) z^\beta)^{\gamma_2}.
\]
Since \( f_0 = \lim_{t \to \infty} e^{t \varphi_{0, t}} \) uniformly on compacta, we have
\[
(5.2) \quad b^1_\alpha = \lim_{t \to \infty} e^{t a^1_\alpha (0, t)}. 
\]

Therefore, in order to get a sharp bound on the coefficients, one should try first to reduce the problem (if possible) to a simple problem involving the least possible number of coefficients of \( G \), then find a sharp bound for such coefficients and solve the associated ODE.

Below, we describe some methods which can be used to simplify the problem and then turn to some applications.

6. Operations in the class \( \mathcal{M} \)

6.1. Decoupling harmonic terms. Let \( G(z) \) be an autonomous Herglotz vector field associated with the class \( \mathcal{M} \). Then
\[
\Re \langle G(z), z \rangle \leq 0. 
\]

Such an inequality translates in terms of expansion as
\[
(6.1) \quad -|z_1|^2 - |z_2|^2 + \sum_{|\alpha| \geq 2} \Re q^1_\alpha z_1^{\alpha_1} \overline{z}_1^{\alpha_2} + \sum_{|\alpha| \geq 2} \Re q^2_\alpha z_1^{\alpha_1} \overline{z}_2^{\alpha_2} \leq 0. 
\]

Replacing \((z_1, z_2)\) by \((e^{i\theta_1} z_1, e^{i\theta_2} z_2)\), with \( \theta \in \mathbb{R} \) and \( k_1, k_2 \in \mathbb{Z} \), we obtain the expression
\[
(6.2) \quad -|z_1|^2 - |z_2|^2 + \sum_{|\alpha| \geq 2, (\alpha_1 - 1)k_1 + \alpha_2 k_2 = 0} \Re q^1_\alpha z_1^{\alpha_1} \overline{z}_1^{\alpha_2} + \sum_{|\alpha| \geq 2, \alpha_1 k_1 + (\alpha_2 - 1)k_2 = 0} \Re q^2_\alpha z_1^{\alpha_1} \overline{z}_2^{\alpha_2} + R(e^{i\theta}) \leq 0, 
\]

where \( R(e^{i\theta}) \) are harmonic terms with some common period. Integrating \((6.2)\) in \( \theta \) over such a period causes the term \( R(e^{i\theta}) \) to disappear, and we get a new expression
\[
(6.3) \quad -|z_1|^2 - |z_2|^2 + \sum_{|\alpha| \geq 2, (\alpha_1 - 1)k_1 + \alpha_2 k_2 = 0} \Re q^1_\alpha z_1^{\alpha_1} \overline{z}_1^{\alpha_2} + \sum_{|\alpha| \geq 2, \alpha_1 k_1 + (\alpha_2 - 1)k_2 = 0} \Re q^2_\alpha z_1^{\alpha_1} \overline{z}_2^{\alpha_2} \leq 0. 
\]

This means that the vector field
\[
G^{(k_1,k_2)}(z) = (-z_1 + \sum_{|\alpha| \geq 2, (\alpha_1 - 1)k_1 + \alpha_2 k_2 = 0} q^1_\alpha z_1^{\alpha_1} - z_2 + \sum_{|\alpha| \geq 2, \alpha_1 k_1 + (\alpha_2 - 1)k_2 = 0} q^2_\alpha z_2^{\alpha_2}), 
\]
is again a Herglotz vector field associated with the class \( \mathcal{M} \).
6.2. **Slice reduction.** Let \( \|v\| = 1 \). Let \( G(z) \) be an autonomous Herglotz vector field associated with the class \( \mathcal{M} \). For \( \zeta \in \mathbb{D} \), let
\[
-\zeta p_v(\zeta) = \langle G(\zeta v), v \rangle.
\]
It is easy to see that \( p_v(\zeta) = 1 + \tilde{p}_v(\zeta) \) belongs to the Carathéodory class; in particular, (see, e.g., [12]), its coefficients are bounded by 2. A direct computation gives
\[
p_v(\zeta) = 1 - \sum_{m=1}^{\infty} \left( \sum_{|\alpha|=m+1} \left( q_1^{1}v_1 + q_2^{2}v_2 \right) v^\alpha \right) \zeta^m.
\]
In particular, for all \( m \in \mathbb{N}, m \geq 1 \),
\[
\sup_{\|v\|=1} \left| \sum_{|\alpha|=m+1} (q_1^{1}v_1 + q_2^{2}v_2) v^\alpha \right| \leq 2.
\]
This condition is necessary but not sufficient for \( p_v \) to belong to the Carathéodory class. Observe that by [12, Corollary 2.3], if for some \( \|v\| = 1 \) and \( m \geq 1 \),
\[
\left| \sum_{|\alpha|=m+1} (q_1^{1}v_1 + q_2^{2}v_2) v^\alpha \right| = 2,
\]
then
\[
p_v(\zeta) = \sum_{l=1}^{m} t_m e^{i\theta + 2\pi i l/m} + z
\]
for some \( \theta \in \mathbb{R} \) and \( t_j \geq 0 \) with \( \sum t_j = 1 \).

A necessary and sufficient condition for \( p_v \) to belong to the Carathéodory class is the following ([12, Thm. 2.4]). For all \( m \geq 1 \),
\[
\sum_{k=0}^{m} \sum_{l=0}^{m} \left( \sum_{|\alpha|=k-l+1} (q_1^{1}v_1 + q_2^{2}v_2) v^\alpha \right) \lambda_k \lambda_l \geq 0,
\]
for all \( \lambda_0, \ldots, \lambda_m \in \mathbb{C} \), with the convention that for \( k - l + 1 \leq -2 \),
\[
\left( \sum_{|\alpha|=k-l+1} (q_1^{1}v_1 + q_2^{2}v_2) v^\alpha \right) = \left( \sum_{|\alpha|=l-k-1} (q_1^{1}v_1 + q_2^{2}v_2) v^\alpha \right)
\]
and \( \sum_{|\alpha|\leq1} (q_1^{1}v_1 + q_2^{2}v_2) v^\alpha = 2 \).
7. COEFFICIENT BOUNDS: $q_{m,0}^1$ AND $b_{0,m}^1$

Let $G(z)$ be an autonomous Herglotz vector field associated with the class $\mathcal{M}$ and fix $m \in \mathbb{N}$, $m \geq 2$. Using the trick of harmonic decoupling, consider $G^{(0,1)}(z)$. Then

$$G^{(0,1)}(z_1, z_2) = (-z_1 + \sum_{m \geq 2} q_{m,0}^1 z_1^m, -z_2 + \sum_{m \geq 2} q_{m,1}^2 z_1^m z_2).$$

From (6.4) we obtain for all $m \geq 2$,

$$(7.1) \quad \sup_{\|v\| = 1} |q_{m,0}^1 v_1|^2 + |q_{m-1,1}^2 v_2|^2 \leq 2 |v_1^{m-1}|$$

Taking $v_1 = 1$, $v_2 = 0$, we obtain

$$(7.2) \quad |q_{m,0}^1| \leq 2.$$  

This bound is sharp, as can be seen by considering the autonomous Herglotz vector field $G(z_1, z_2) = (-z_1 (1 + z_1) (1 - z_1)^{-1}, -z_2)$. Now, for $m = 2$, from (5.1), we obtain

$$a_{2,0}^1(t) = e^{-t} \int_0^t e^{-\tau} q_{2,0}^1(\tau) d\tau.$$  

By (7.2) and (5.2), we then have for all $f \in S^0$,

$$|b_{2,0}^1| \leq 2.$$  

The bound is sharp, as one sees by considering the map $(k(z_1), z_2)$, where $k(z_1)$ is the Koebe function in $\mathbb{D}$.

A similar bound for $|b_{m,0}^1|$ is not known.

**Question 7.1.** Is it true that $|b_{m,0}^1| \leq m$ for all $f \in S^0$ and $m \in \mathbb{N}$, $m \geq 3$?

Note that if the bound is correct, it is sharp as one sees from the function $(k(z_1), z_2)$, where $k(z_1)$ is the Koebe function in $\mathbb{D}$.

8. COEFFICIENT BOUNDS: $q_{0,m}^1$ AND $b_{0,m}^1$

Let $G(z)$ be an autonomous Herglotz vector field associated with the class $\mathcal{M}$. Fix $m \in \mathbb{N}$, $m \geq 2$. Using the decoupling harmonic terms trick, consider the vector field $G^{(m,1)}$, given by

$$G^{(m,1)}(z_1, z_2) = (-z_1 + q_{0,m}^1(t) z_1^m, -z_2).$$

Since $-G^{(m,1)} \in \mathcal{M}$, imposing the condition $\text{Re} \langle G^{(m,1)}(z), z \rangle \leq 0$, we get

$$-|z_1|^2 - |z_2|^2 + \text{Re} q_{0,m}^1 z_1^m z_2^m \leq 0.$$  

Setting $z_1 = xe^{i(\theta + \eta)}$, $z_2 = ye^{i\theta/m}$ with $x, y \geq 0$ and $q_{0,m}^1 e^{-i\eta} = |q_{0,m}^1|$, we obtain the equivalent equation

$$-x^2 - y^2 + |q_{0,m}^1| xy^m \leq 0, \quad x, y \geq 0, x^2 + y^2 \leq 1.$$
Using the method of Lagrange multipliers, one checks easily that the maximum for the function \((x, y) \mapsto -x^2 - y^2 + |q_{1, m}^1| z y^m\) under the constraint \(x, y \geq 0, x^2 + y^2 \leq 1\) is attained at the point \(x = \frac{1}{\sqrt{1+m}}, y = \sqrt{\frac{m}{2(1+m)}}\). Hence the previous inequality is satisfied if and only if

\[
|q_{1, m}^1| \leq \frac{(1 + m)^{m+1}}{m^2}.
\]

Note that (8.1) gives the sharp bound for the coefficients \(q_{1, m}^1\) in the class \(\mathcal{M}\).

As before, for \(m = 2\), from (5.1), (7.2) and (5.2) we then have for all \(f \in S^0\),

\[
|b_{0,2}^1| \leq \frac{3\sqrt{3}}{2}.
\]

In particular, the map

\[
\Phi : (z_1, z_2) \mapsto (z_1 + \frac{3\sqrt{3}}{2} z_2^2, z_2) \in S^0,
\]

is a bounded support point. Note also that \(\Phi\) is an automorphism of \(\mathbb{C}^2\) and provides an affirmative answer to Question 4.1 for this automorphism.

This result, together with the germinal idea of decoupling harmonic terms, was proved in [4].

**Question 8.1.** Is it true that

\[
|b_{0, m}^1| \leq u_m := \frac{(1 + m)^{m+1}}{m^2}
\]

for all \(f \in S^0\) and \(m \in \mathbb{N}, m \geq 3\)?

Note that if the bound is correct, it is sharp (consider the function \((z_1 + u_m z_2^m, z_2)\)).

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