Quadratic Hyperboloids in Minkowski Geometries

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Abstract. A Minkowski plane is Euclidean if and only if at least one hyperbola is a quadric. We discuss the higher dimensional consequences too.

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1. Introduction

Let $I$ be an open, strictly convex, bounded domain in $\mathbb{R}^n$, (centrally) symmetric to the origin. Then function $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(x, y) = \inf\{\lambda > 0 : (y - x)/\lambda \in I\}$$

is a metric on $\mathbb{R}^n$ [1, IV.24], and is called Minkowski metric on $\mathbb{R}^n$. It satisfies the strict triangle inequality, i.e., $d(A, B) + d(B, C) = d(A, C)$ is valid if and only if $B \in AC$. A pair $(\mathbb{R}^n, d)$, where $d$ is a Minkowski metric, is called Minkowski geometry, and $I$ is called the indicatrix of it. In a Minkowski geometry $(\mathbb{R}^n, d)$,

$$D1 \ a \ set \ H_{d,F_1,F_2}^a := \{X : 2a = |d(F_1, X) - d(F_2, X)|\},$$

where $F_1, F_2 \in \mathbb{R}^n$ are called the focuses, and $a > 0$ is called the radius.

A hypersurface in $\mathbb{R}^n$ is called a quadric if it is the zero set of an irreducible polynomial of degree two in $n$ variables. We call a hypersurface quadratic if it is part of a quadric. Since every isometric mapping between two Minkowski geometries is a restriction of an affinity, and every affinity maps quadrics to quadrics, the quadraticity of a metrically defined hypersurface is a geometric property in each Minkowski geometry. Thus, the question arises

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whether the metrically defined hypersurfaces are quadrics. This question is answered for conics in [6].

We prove that (Theorem 4.3) a Minkowski plane is a model of the Euclidean plane, which means that the indicatrix is a bounded quadric [1, IV.25.4], if and only if at least one of the hyperbolas is a quadric, and that (Theorem 4.4) a Minkowski plane is analytic if and only if at least one of the hyperbolas is analytic.

As for higher dimensions, we prove (Theorem 5.1) that a Minkowski geometry is a model of the Euclidean geometry if and only if every central planar section of at least one quadric is either a hyperbola or an ellipse.

Similar problems for the ellipsoids were solved in [7].

2. Notations and Preliminaries

Points of \( \mathbb{R}^n \) are labeled as \( A, B, \ldots \), vectors are denoted by \( \overrightarrow{AB} \) or \( a, b, \ldots \), but we use these latter notations also for points if the origin is fixed. The open segment with end points \( A \) and \( B \) is denoted by \( \overrightarrow{AB} \), while \( \overrightarrow{AB} \) denotes the open ray starting from \( A \) passing through \( B \), finally, \( \overrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{AB} \).

On an affine plane, the affine ratio \((A, B; C)\) of the collinear points \( A, B \) and \( C \) satisfies \((A, B; C)\overrightarrow{BC} = \overrightarrow{AC}\) [1, III.15.10], and the cross ratio of the collinear points \( A, B \) and \( C, D \) is \((A, B; C, D) = (A, B; C)/(A, B; D)\) [1, VI.40.17].

It is easy to observe in D1 that a hyperboloid intersects line \( F_1F_2 \), the main axis, in exactly two points, whose distance is twice the radius. Further notions are the (linear) eccentricity \( c = d(F_1, F_2)/2 \), the numerical eccentricity \( \varepsilon = c/a \). The metric midpoint of the segment \( F_1F_2 \) is called the center.

Notations \( u_\varphi = (\cos \varphi, \sin \varphi) \) and \( u^\perp_\varphi := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2)) \) are frequently used. It is worth noting that, by these, we have \( \frac{d}{d\varphi}u_\varphi = u^\perp_\varphi \).

A quadric in the plane has the equation of the form

\[
Q^\sigma_\phi := \left\{ (x, y) : \begin{cases} 1 = x^2 + \sigma y^2 \text{ if } \sigma \in \{-1, 1\}, \\ x = y^2 \text{ if } \sigma = 0, \end{cases} \right\} \quad (D_q)
\]

in a suitable affine coordinate system \( \mathbf{s} \), and we call it elliptic, parabolic, or hyperbolic, if \( \sigma = 1, \sigma = 0, \) or \( \sigma = -1, \) respectively.

We usually polar parameterize the boundary \( \partial \mathcal{D} \) of a non-empty domain \( \mathcal{D} \) in \( \mathbb{R}^2 \) starlike with respect to a point \( P \in \mathcal{D} \) so that \( r : [-\pi, \pi) \to \mathbb{R}^2 \) is defined by \( r(\varphi) = r(\varphi)u_\varphi \), where \( r \) is the radial function of \( \mathcal{D} \) with base point \( P \).

We call a curve analytic if the coordinates of its points depend on its arc length analytically.

3. Utilities

In this section, the underlying plane is Euclidean.
Lemma 3.1. The border of a convex domain is an analytic curve if and only if any one of its radial functions is analytic.

Proof. Let $\mathcal{D}$ be an open convex domain containing the origin $O = (0, 0)$. Let $s \mapsto p(s)$ be an arc length parametrization of $\partial \mathcal{D}$, where $s \geq 0$, and let $\varphi \mapsto r(\varphi) = r(\varphi)u_\varphi$ be a polar parametrization of $\partial \mathcal{D}$ on $[-\pi, \pi)$ such that $p(0) = r(-\pi)$. Then,

$$s(\xi) = \int_{-\pi}^{\xi} |\dot{r}(\varphi)|d\varphi = \int_{-\pi}^{\xi} \sqrt{\dot{r}^2(\varphi) + r^2(\varphi)}d\varphi,$$

(3.1)

hence the function $s: \xi \mapsto s(\xi)$ is strictly monotonously increasing, and therefore its inverse function $\sigma: s(\xi) \mapsto \xi$ exists and is strictly monotonously increasing.

First, assume the analyticity of $r$. Then, as $r$ is bounded from below by a positive number, the integrand on the right-hand side of (3.1) is analytic, and therefore $s$ is analytic. As $s(\xi)$ is positive by (3.1), the analyticity of $\sigma$ follows from the analytic inverse function theorem [3, Theorem 4.2], and this implies the analyticity of $p(s) = r(\sigma(s)) = r(\sigma(s))u_{\sigma(s)}$.

Conversely, assume that $p$ is analytic. As the derivatives of the cosine and sine functions do not vanish simultaneously, $u_{\sigma(s)} = p(s)/|p(s)|$ proves that $\sigma$ is analytic. As the derivative $\sigma'(t) = 1/s(\sigma(t))$ vanishes nowhere, analyticity of $s$ follows again by the analytic inverse function theorem [3, Theorem 4.2]. Then the analyticity of $r(\xi) = \langle p(s(\xi)), u_\xi \rangle$ follows.

The lemma is proved. $\Box$

Notice that the differentiation of the last formula in the proof and then the substitution of the derivative of (3.1) give

$$\dot{r}(\xi) = \langle \dot{p}(s(\xi)), u_\xi \rangle \sqrt{\dot{r}^2(\xi) + r^2(\xi)}.$$  

(3.2)

Let $\mathcal{H}$ be a hyperbola with center $O$ and focuses $F_1$ and $F_2$. Let us label the intersection points of $F_1F_2$ and $\mathcal{H}$ so that $A \in \overline{F_1B}$. We clearly have $O \in AB \subset F_1F_2$, so we can choose a point $W$ on $F_1F_2$ such that $F_2 \in BW$.

There exists an angle $\Phi \in (0, \pi/2)$ such that a unique point $H$ exists on $\mathcal{H}$ for every $\varphi \in [0, \Phi) \cup (\pi - \Phi, \pi)$, such that $\angle WOH = \varphi$.

Given $\varphi_0 \in (0, \Phi)$, let $H_0 = H(\varphi_0)$, $\alpha_0 = \angle WF_1H_0$ and $\beta_0 = \angle WF_2H_0$. Assuming that $H_{2i}$ is defined for an $i \in \mathbb{N}$, we define sequences recursively as follows (see Fig. 1): $H_{2i+1} := \overline{F_1H_{2i}} \cap \mathcal{H}$, $\alpha_{2i+1} := \alpha_{2i}$, and $\beta_{2i+1} := \angle WF_2H_{2i+1}$; then $H_{2i+2} := F_2H_{2i+1} \cap \mathcal{H}$, $\alpha_{2i+2} := \angle WF_1H_{2i+2}$, and $\beta_{2i+2} := \beta_{2i+1}$. We clearly have $\varphi_{2i} \in (0, \Phi)$ and $\varphi_{2i+1} \in (\pi - \Phi, \pi)$ for every $i \in \mathbb{N}$.

Lemma 3.2. If $i \to \infty$, then $\alpha_{2i}$ and $\varphi_{2i}$ tend to zero, $\beta_{2i}$, $\beta_{2i+1}$, and $\varphi_{2i+1}$ tend to $\pi$, and $\alpha_{2i+2}/\alpha_{2i}$ tends to $(F_1, F_2; A, B)$.

Proof. We clearly have $\varphi_{2i} < \Phi < \pi/2$ and $\varphi_{2i+1} > \pi - \Phi > \pi/2$, and therefore

$$\alpha_{2i} < \pi - \beta_{2i}$$

and $\pi - \beta_{2i+1} < \alpha_{2i+1}$ (or $\pi - \beta_{2i+2} < \alpha_{2i}$),

$$\alpha_{2i+2} < \pi - \beta_{2i+2}$$

and $\pi - \beta_{2i+1} < \alpha_{2i}$,

hence $\beta_{2i+2} > \beta_{2i}$, $\alpha_{2i+2} < \alpha_{2i}$, and $\pi - \beta_{2i+2} < \alpha_{2i} < \pi - \beta_{2i}$. 

Thus, the sequences $\beta_{2i}$, $\beta_{2i+1}$ increase monotonously, while the sequences $\alpha_{2i}$, $\alpha_{2i+1}$ decrease monotonously. As these sequences are bounded, they are convergent.

Assuming $\lim_{i \to \infty} \beta_{2i} < \pi$, i.e., $\lim_{i \to \infty} (\pi - \beta_{2i}) > 0$, $\lim_{i \to \infty} \frac{\pi - \beta_{2i+2}}{\pi - \beta_{2i}} = 1$, and $\lim_{i \to \infty} \frac{\alpha_{2i}}{\pi - \beta_{2i}} = 1$ follow, hence the sinus law for triangle $\triangle F_1 F_2 H_{2i}$ implies

$$\lim_{i \to \infty} \frac{d(F_2, H_{2i})}{d(H_{2i}, F_1)} = \lim_{i \to \infty} \frac{\sin \alpha_{2i}}{\sin(\pi - \beta_{2i})} \cdot \lim_{i \to \infty} \frac{\pi - \beta_{2i}}{\alpha_{2i}} = 1,$$

which, by the continuity of $d$, gives $d(F_2, B) = d(B, F_1)$, a contradiction.

Thus, the sequences $\beta_{2i}$, $\beta_{2i+1}$, and $\varphi_{2i+1}$ also tend to $\pi$, and $\alpha_{2i}$, $\alpha_{2i+1}$, and $\varphi_{2i}$ tend to zero.

So, observing Fig. 1, we see that

$$h_1(\alpha_{2i}) := d(F_1, H_{2i}) \to d(F_1, B), \quad h_1(\alpha_{2i+1}) := d(F_1, H_{2i+1}) \to d(F_1, A),$$

$$h_2(\beta_{2i}) := d(F_2, H_{2i}) \to d(F_2, B), \quad h_2(\beta_{2i+1}) := d(F_2, H_{2i+1}) \to d(F_2, A).$$

The sine law for triangles $\triangle F_1 F_2 H_{2i}$ and $\triangle F_1 F_2 H_{2i+1}$ gives

$$\frac{h_2(\beta_{2i+1})}{h_1(\alpha_{2i+1})} = \frac{\sin \alpha_{2i+1}}{\sin(\pi - \beta_{2i+1})} \quad \text{and} \quad \frac{h_2(\beta_{2i+2})}{h_1(\alpha_{2i+2})} = \frac{\sin \alpha_{2i+2}}{\sin(\pi - \beta_{2i+2})},$$

respectively. Multiplying these by $\cos \beta_{2i+1}/\cos \alpha_{2i+1}$ and $\cos \beta_{2i+2}/\cos \alpha_{2i+2}$, respectively, and taking the ratio of the resulting fractions, we obtain

$$\tan \alpha_{2i+2} = \frac{h_2(\beta_{2i+2})}{h_1(\alpha_{2i+2})} \cos \beta_{2i+2} \frac{h_1(\alpha_{2i+1})}{h_2(\beta_{2i+1})} \cos \beta_{2i+1} \cos \alpha_{2i+1}.$$  

By (3.3), the right-hand side tends to $(F_1, F_2; A, B)$, so the proof is complete. \( \square \)

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be curves in the plane with analytic arc length parametrization on $[-1, 1]$ such that $\mathbf{r}_1(0) = \mathbf{r}_2(0)$ and $\dot{\mathbf{r}}_1(0) = \dot{\mathbf{r}}_2(0)$. Let $\ell$ be the line through $\mathbf{r}_1(0)$ that is orthogonal to $\dot{\mathbf{r}}_1(0)$, and let $F_1$, $F_2$, and $B$ be different points on $\ell$ such that $B \in F_1 F_2$ and $\mathbf{r}_1(0) \notin \{B, F_1, F_2\}$. Let $\mathbf{h}$ be an analytic arc length parameterization of a curve such that $B = \mathbf{h}(0)$ and $\mathbf{h}(1) = F_1$. Figure 1. Sequence of angles
Figure 2. Specially placed curves with different lines

\[ \dot{h}(0) = u_{\pi/2}. \]  
Every point \( H = h(s) \) determines two straight lines \( \ell_1 := F_1 H \) and \( \ell_2 := F_2 H \) closing small angles \( \alpha \) and \( \gamma = \pi - \beta \) with \( \ell \), respectively. Let the straight line \( \ell_j \) (\( j = 1, 2 \)) through the midpoint \( O \) of the segment \( F_1 F_2 \) be parallel to \( \ell_j \). See Fig. 2.

Denote the intersections of \( \ell_1 \) and \( \ell_2 \) with \( r_1 \) and \( r_2 \) by \( \bar{C}_1, \bar{D}_1 \) and \( \bar{C}_2, \bar{D}_2 \), respectively. Let \( s_i \) be the arc length parameter of \( r_i \) (\( i = 1, 2 \)), and define \( \delta(\alpha) = \langle C_1 - D_1, u_\alpha \rangle \) and \( \delta(\gamma) = \langle C_2 - D_2, u_\gamma \rangle \), where \( \gamma = \beta - \pi \).

Lemma 3.3. If the curves \( r_1 \) and \( r_2 \) are different in every neighborhood of the point \( K := r_1(0) \), and \( H \) tends to \( B \) on the curve \( h \), then

\[ \frac{\delta(\alpha)}{\delta(\gamma)} \to (F_2, F_1; B)^k, \quad \text{for an integer } k \geq 2. \]  

(3.4)

Proof. If \( r_1(i)(0) = r_2(i)(0) \) for every \( i \in \mathbb{N} \), then, by the analyticity of \( r_1 \) and \( r_2 \), \( r_1 = r_2 \) in a neighborhood of \( K \), so \( k := \min\{i \in \mathbb{N} : r_1(i)(0) \neq r_2(i)(0) \} \) is well defined and is at least two.

Letting \( H^\perp \) be the orthogonal projection of \( H \) onto \( \ell \), L'Hôpital’s rule gives

\[ \frac{|F_2 - B|}{|F_1 - B|} = \lim_{s \to 0} \frac{|F_2 - H^\perp|}{|F_1 - H^\perp|} = \lim_{s \to 0} \frac{\tan \alpha}{-\tan \gamma} = -\lim_{s \to 0} \frac{\dot{\alpha}}{\dot{\gamma}}. \]  

(3.5)

If \( \lim_{s \to 0} \frac{\delta(\alpha)}{\delta(\gamma)} \) exists, then L'Hôpital’s rule can be used, so we get

\[ \lim_{s \to 0} \frac{\delta(\alpha)}{\delta(\gamma)} = \lim_{s \to 0} \frac{\dot{\delta}(\alpha)\dot{\gamma}}{\dot{\delta}(\gamma)\dot{\alpha}} = \lim_{s \to 0} \frac{\dot{\delta}(\alpha)}{\dot{\delta}(\gamma)} \lim_{s \to 0} \frac{\dot{\alpha}}{\dot{\gamma}} = \cdots = \lim_{s \to 0} \frac{\delta^{(k)}(\alpha)}{\delta^{(k)}(\gamma)} \left( \lim_{s \to 0} \frac{\dot{\alpha}}{\dot{\gamma}} \right)^k \]

This proves the lemma.

\[ \square \]

4. One Hyperbola in a Minkowski Plane

We start by considering the Minkowski plane \((\mathbb{R}^2, d_I)\) with indicatrix \( I \).
By [4, (ii) of Theorem 3] every straight line parallel to the main axis intersects a hyperbola in exactly two points, hence if a hyperbola is a quadric, then it is a hyperbolic quadric.

Let $A, B$ be the intersections of line $F_1F_2$ with $\mathcal{H}_{d_x}^{a}$ such that $A \in F_1B$ and $B \in AF_2$. Let $\mathcal{I}_O$ be the translate of the indicatrix centered at the midpoint $O$ of $F_1F_2$, and let $I, J$ be the intersections of line $F_1F_2$ with $\partial \mathcal{I}_O$, so that $I \in OF_1$ and $J \in OF_2$. Furthermore, let $t_A, t_B$ and $t_I, t_J$, respectively, denote the tangents of the appropriate curve $\mathcal{H}_{d_x}^{a}$ or $\partial \mathcal{I}_O$ at $A, B$ and $I, J$, respectively. See Fig. 3.

Given the Euclidean metric $d_e$, we let $r$ be the radial function of $\mathcal{I}_O$ with respect to $O$, $\alpha = \angle (HF_1O), \gamma = \angle (HF_2B)$ $(\beta := \pi - \gamma)$ and $\varphi = \angle (HOB)$ for the points $H$ on the $B$-branch (that contains $B$) of $\mathcal{H}_{d_x}^{a}$. Finally, we define the lengths $h_1(\alpha) := d_e(F_1, H), h_2(\beta) := d_e(F_2, H)$, and $h(\varphi) := d_e(O, H)$. Then $d_T(F_1, H) = h_1(\alpha)/r(\alpha)$, and $d_T(F_2, H) = h_2(\beta)/r(\beta)$, so we have

$$2a = \frac{h_1(\alpha)}{r(\alpha)} - \frac{h_2(\beta)}{r(\beta)}.$$  \hspace{1cm} (4.1)

**Lemma 4.1.** If the hyperbola $\mathcal{H}_{d_x}^{a}$ is a quadric, then $t_A \parallel t_B \parallel t_I \parallel t_J$.

**Proof.** Since $\mathcal{H}_{d_x}^{a}$ is a quadric, $\varphi$ and $H$ are bijectively related, hence the functions $\alpha(\varphi), \beta(\varphi)$ are well defined.

The symmetry of $\mathcal{I}$ entails that $t_I \parallel t_J$, and it also follows that the affine center of the quadric $\mathcal{H}_{d_x}^{a}$ coincides with its metric center $O$, hence $t_A \parallel t_B$ too.

Choose a Euclidean metric $d_e$ so that $t_A \perp F_1F_2 \perp t_B$.

Differentiating (4.1) with respect to $\varphi$ leads to

$$0 = \frac{dh_1(\alpha)}{d\varphi} r(\alpha) - \frac{h_1(\alpha)}{r(\alpha)} \frac{dr(\alpha)}{d\alpha} d\alpha - \frac{dh_2(\beta)}{d\varphi} r(\beta) - \frac{h_2(\beta)}{r(\beta)} \frac{dr(\beta)}{d\beta} d\beta.$$  \hspace{1cm} (4.2)

As $\varphi = 0$ implies $\alpha = 0 = \pi - \beta$, and $\frac{dh_1}{d\alpha}(0) = \frac{dh_2}{d\beta}(\pi) = 0$ by $t_A \perp F_1F_2 \perp t_B$, (4.2) gives at $\varphi = 0$ that

$$r'(0) \left[ -h_1(0) \frac{d\alpha}{d\varphi}(0) + h_2(\pi) \frac{d\beta}{d\varphi}(0) \right] = 0.$$

![Figure 3. A hyperbola in a Minkowski plane](attachment:figure3.png)
Applying (3.5) for the present configuration, we obtain that the second factor in the left-hand side is negative, hence \( r'(0) = 0 \). Thus \( t_J \perp F_1F_2 \), so the lemma follows.

**Lemma 4.2.** If the hyperbola \( \mathcal{H}_{\alpha}^{a} \) is an analytic curve in a neighborhood of \( A \) and \( B \), then the curve \( \partial \mathcal{I}_O \) is analytic in a neighborhood of \( I \) and \( J \).

*Proof.* By Lemma 3.1 and its proof, the functions \( h_1, h_2, \) the angles \( \alpha(s), \beta(s) \), and the inverses of the angles, where \( s \) is the arc length parameter, are clearly analytic, hence we deduce that \( \beta(\alpha) \) and \( \alpha(\beta) \) are also analytic functions.

As \( x \mapsto 1/x \) is analytic in a neighborhood of 1, to prove that \( r(\alpha) \) is analytic in a neighborhood of 0, it is enough to prove that \( \bar{r}(\alpha) := 1/r(\alpha) \) is analytic in some neighborhood of 0. Bearing this in mind, we reformulate (4.1) as

\[
\bar{r}(\alpha) = \frac{h_2(\gamma(\alpha))}{h_1(\alpha)}\bar{r}(\gamma(\alpha)) + \frac{2a}{h_1(\alpha)}. \tag{4.3}
\]

Introduce the functions \( f(\alpha) := \gamma(\alpha), g(\alpha) := \frac{h_2(\gamma(\alpha))}{h_1(\alpha)} \), and \( e(\alpha) := \frac{2a}{h_1(\alpha)} \). Then \( f, g \) and \( e \) are analytic in a neighborhood of 0, \( f(0) = 0, \frac{df}{d\alpha}(0) = \frac{h_2(0)}{h_1(0)} < 1, g(0) = \frac{h_2(0)}{h_1(0)} < 1, \) and \( h(0) = \frac{2a}{h_1(0)} < 1. \) Furthermore, by (4.3), the function \( \phi(\alpha) := \bar{r}(\alpha) \) solves the functional equation \( \phi(\alpha) = g(\alpha)\phi(f(\alpha)) + h(\alpha) \). However, by [3, Theorem 4.6], such a functional equation has a unique solution, which additionally is analytic in a neighborhood of 0. Consequently, \( r(\alpha) \) is the reciprocal of that unique analytic solution, so \( \partial \mathcal{I}_O \) is analytic around \( J \), and, by its symmetry, around \( I \) too. \( \square \)

**Theorem 4.3.** A Minkowski plane is a model of the Euclidean plane if and only if at least one hyperbola is a quadric.

*Proof.* As every hyperbola is a quadric in the Euclidean plane, we only have to prove that a Minkowski plane is Euclidean if at least one hyperbola is a quadric.

Assume that \( \mathcal{H}_{\alpha}^{a} \) is a quadric.

We have \( t_A \parallel t_I \parallel t_J \parallel t_B \) by Lemma 4.1, and, as every (planar) quadric is an analytical curve, the border \( \partial \mathcal{I}_O \) is analytic in a neighborhood of \( I \) and \( J \) by Lemma 4.2, where \( O \) is the midpoint of \( F_1F_2 \). Furthermore, by the central symmetry of \( \mathcal{I}_O \) and the definition of \( \mathcal{H}_{\alpha}^{a} \), we have \( c = d_{\mathcal{I}}(F_1, O), \) \( AF_1 = F_2B \) and \( IA = BJ \), so \( O \) is the (affine) midpoint of both \( TJ \) and \( AB \). Additionally, we have \( a \cdot d_{\mathcal{I}}(O, J) = d_{\mathcal{I}}(O, B) \), because the definition of \( \mathcal{H}_{\alpha}^{a} \) implies

\[
2d_{\mathcal{I}}(O, B) = 2d_{\mathcal{I}}(O, F_2) - 2d_{\mathcal{I}}(F_2, B)
= d_{\mathcal{I}}(F_1, O) + d_{\mathcal{I}}(O, F_2) - d_{\mathcal{I}}(F_2, B) + 2a - d_{\mathcal{I}}(F_1, B) = 2a.
\]

Being a hyperbolic quadric, \( \mathcal{H}_{\alpha}^{a} \) has two asymptotes \( \ell_+ \) and \( \ell_- \) through \( O \). Let \( C_1 \) and \( C_2 \) be the points where they intersect the straight line \( t_A \).
Fix the affine coordinate system such as \( O = (0,0), J = (1,0), \) and \( C_1 = (c, \sqrt{c^2 - a^2}), \) and choose the Euclidean metric \( d_e \) so that \( \{(1,0),(0,1)\} \) is an orthonormal basis.

Let \( C \) denote the unit circle of \( d_e \). See Fig. 4.

Then both \( \mathcal{H}_{d_e;F_1,F_2}^a \) and \( \mathcal{H}_{d_e;F_1,F_2}^a \) are hyperbolic quadrics, and have two common tangents \( t_A \) and \( t_B \), two common asymptotes, and two common points \( A \) and \( B \), hence they coincide.

By the definition of \( \mathcal{H}_{d_e;F_1,F_2}^a \) we have \( h_1(\alpha) - h_2(\beta) = 2a \), which together with (4.1) implies

\[
\delta(\alpha) = \delta(\beta) \frac{h_2(\beta)}{h_1(\alpha) + 2a\delta(\beta)}, \tag{4.4}
\]

where \( \delta(\alpha) = 1 - r(\alpha) \) is the radial difference of \( C \) and \( \partial I_O \).

If in every neighborhood of \( I \) curves \( C \) and \( \partial I_O \) differ, then (4.4) implies

\[
\lim_{\varphi \to 0} \frac{\delta(\alpha)}{\delta(\beta)} = \frac{c - a}{c + a} = (F_2,F_1;B),
\]

which, by (3.4), implies \( (F_2,F_1;B) = 1 \). This contradicts \( a > 0 \), so the curves \( C \) and \( \partial I_O \) coincide in a neighborhood of \( I \).

However, if \( \delta(\beta_0) \neq 0 \) for a \( \beta_0 \), then no value of the 0-convergent sequence \( \beta_{2i} \) constructed in Lemma 3.2 can vanish by (4.4), therefore no \( \beta_0 \) can exist for which \( \delta(\beta_0) \neq 0 \). Similarly follows that no \( \alpha \) exists for which \( \delta(\alpha) \neq 0 \), hence \( C \) and \( \partial I_O \) coincide. \( \square \)

This kind of implication extends over to analyticity too.

**Theorem 4.4.** The indicatrix of a Minkowski plane is analytic if and only if one of the hyperbolas of the Minkowski plane is analytic.

**Proof.** First, assume that the Minkowski plane \( (\mathbb{R}^2,d_I) \) is analytic.

We use the notations introduced in the previous sections, and consider the hyperbola \( \mathcal{H}_{d^I;F_1,F_2}^a \).

Fix an arbitrary point \( H_0 \in \mathcal{H}_{d^I;F_1,F_2}^a \), and let the point \( R_i \in \mathcal{I} (i = 1,2) \) be such that \( OR_i \parallel F_iH_0 \). Let the straight line \( t_i \ (i = 1,2) \) be tangent to \( \mathcal{I} \)

![Figure 4. Coinciding hyperbolas \( \mathcal{H}_{d^I;F_1,F_2}^a = \mathcal{H}_{d_e;F_1,F_2}^a \)](image-url)
at $R_i$. Let $d_e$ be the Euclidean metric which satisfies $t_2 \perp OR_2$, $d_e(O, R_1) = d_e(O, R_2)$, and $d_e(O, J) = 1$. Then we have
\[
h_2^2(\beta) = h_1^2(\alpha) + 4c^2 - 4h_1(\alpha)c\cos \alpha, \ \text{and} \ \beta = \arcsin \frac{h_1(\alpha) \sin \alpha}{h_2(\beta)}.
\]
Substituting this into (4.1) results in the analytic equation
\[
F(\alpha, h_1(\alpha)) := \left(2a - \frac{h_1(\alpha)}{r(\alpha)}\right)^2 - \frac{h_1^2(\alpha) + 4c^2 - 4h_1(\alpha)c\cos \alpha}{r^2(\alpha)} \left(\arcsin \frac{h_1(\alpha) \sin \alpha}{\sqrt{h_1^2(\alpha) + 4c^2 - 4h_1(\alpha)c\cos \alpha}}\right) = 0.
\]
Since
\[
\frac{\partial}{\partial \alpha} F(\alpha, h_1(\alpha)) = 2\frac{-h_2(\beta)}{r(\alpha)} \frac{1}{r(\alpha)} - \frac{2h_1(\alpha) - 4c\cos \alpha}{r^2(\beta)} + \frac{2}{r^3(\beta)} \frac{h_2^2(\beta)}{\beta} \frac{\sin \alpha}{h_2(\beta)} = \frac{\sin \alpha}{h_2(\beta)} - \frac{1}{2} \frac{h_1(\alpha) \sin \alpha(2h_1(\alpha) - 4c\cos \alpha)}{h_2^3(\beta)},
\]
\[
\frac{\partial}{\partial \alpha} F(\alpha, h_1(\alpha)) \text{ vanishes if and only if}
\]
\[
- \frac{h_2(\beta)}{r(\alpha)} + \frac{h_1(\alpha) - 2c\cos \alpha}{r(\beta)} = \frac{h_2(\beta) \sin \alpha \frac{\dot{r}(\beta)}{r(\beta)} \cos \beta}{r(\beta)} \left(1 - \frac{h_1(\alpha)(h_1(\alpha) - 2c\cos \alpha)}{h_2^2(\beta)}\right).
\]
By (3.2), we have $\frac{\dot{r}(\beta)}{r(\beta)} = \cot \theta$, where $\theta$ is the angle between $F_1 \bar{H}$ and the tangent vector at $H$ of $\mathcal{H}_{d_2:F_1,F_2}$. Furthermore, it can be easily seen that $h_1(\alpha) - 2c\cos \alpha = h_2(\beta)\cos(\beta - \alpha)$. Thus, the above equation is equivalent to
\[
- \frac{h_2(\beta)}{r(\alpha)} + \frac{h_2(\beta) \cos(\beta - \alpha)}{r(\beta) = \frac{\cot \theta}{r(\beta)} \cos \beta (h_2(\beta) \sin \alpha - h_1(\alpha) \cos(\beta - \alpha) \sin \alpha).
\]
Since $h_2(\beta) \sin \beta = h_1(\alpha) \sin \alpha$, this equation simplifies to
\[
- \frac{1}{r(\alpha)} + \frac{\cos(\beta - \alpha)}{r(\beta)} \cos \beta = \frac{\cot \theta \sin \alpha - \sin(\beta - \alpha) \cot \theta}{\cos \beta} = - \sin(\beta - \alpha) \cot \theta.
\]
In sum, $\frac{\partial}{\partial \alpha} F(\alpha, h_1(\alpha))$ vanishes if and only if
\[
- \frac{\dot{r}(\beta)}{r(\alpha)} + \sin(\beta - \alpha)(\cot \theta + \cot(\beta - \alpha)) = 0. \tag{4.5}
\]
At $H_0$ we have $\theta = \pi/2$, and $r(\beta) = r(\alpha)$, therefore (4.5) reduces to $\cos(\beta - \alpha) = 1$, resulting in $\beta = \alpha$, a contradiction. Thus $\frac{\partial}{\partial \alpha} F(\alpha, h_1(\alpha)) \neq 0$ at $H_0$, hence the analytic implicit function theorem [3, Theorem 4.1] implies the analyticity of $h_1$ in a neighborhood of $\alpha$. As the point $H_0$ was chosen arbitrarily on $\mathcal{H}_{d_2:F_1,F_2}$, this proves that $\mathcal{H}_{d_2:F_1,F_2}$ is analytic.

Assuming now that the hyperbola is analytic, Lemma 4.2 proves the analyticity of the boundary of the indicatrix, where $F_1F_2$ intersects it. By (4.3) we have
\[
\frac{\dot{r}(\beta(\alpha))}{r(\alpha)} = \frac{h_1(\alpha)}{-h_2(\beta(\alpha))} \frac{\dot{r}(\alpha)}{r(\alpha)} + \frac{2a}{h_2(\beta(\alpha))}.
\]
This shows that if $\dot{r}$ is analytic in an interval $(-\epsilon, \epsilon)$, then it is also analytic in the interval $(-\beta(\epsilon), \beta(\epsilon))$. According to Lemma 3.2, this means that the boundary of the indicatrix is analytic at all the directions. \qed
5. Quadrics in a Minkowski Geometry

Before presenting the proof of Theorem 5.1, let us rephrase its statement for the planar case: if one hyperbola is a quadric, then the Minkowski plane is a model of the Euclidean geometry.

In a Minkowski geometry \((\mathbb{R}^n, d)\)

\[ (D2) \quad \mathcal{E}_{d,F_1,F_2}^a := \{ E : 2a = d(F_1, E) + d(E, F_2) \}, \]

where \(a > d(F_1, F_2)/2\), is called an **ellipse** if \(n = 2\), and an **ellipsoid** in higher dimensions, where \(F_1, F_2 \in \mathcal{M}\) are called the **focuses**, and \(a > 0\) is called the **radius**.

**Theorem 5.1.** A Minkowski geometry is a model of the Euclidean geometry if and only if through a point every planar section of at least one quadric is either a hyperbola or an ellipse.

**Proof.** As every planar section of each hyperbolic quadric is either a hyperbola or an ellipse in the Euclidean geometry, it is enough to prove that a Minkowski geometry is Euclidean if every planar section of at least one quadric is either a hyperbola or an ellipse.

Let the quadric \(Q\) be such that its every planar section is either a hyperbola or an ellipse. If the planar section is a hyperbola, then Theorem 4.3 implies that the parallel central planar section of the indicatrix is an ellipse. If the planar section is an ellipse, then \([7, \text{Theorem } 4.3]\) implies that the parallel central planar section of the indicatrix is an ellipse. Thus, the statement of the theorem follows immediately from \([2, \text{II.16.12}]\), which states for any integers \(1 < k < n\) that the border \(\partial K\) of a convex body \(K \subset \mathbb{R}^n\) is an ellipsoid if and only if every \(k\)-plane through an inner point of \(K\) intersects \(\partial K\) in a \(k\)-dimensional ellipsoid. \(\square\)

We omit the easy proof of the following result that closes this paper.

**Theorem 5.2.** A Minkowski geometry is a model of the Euclidean geometry if and only if there is a hyperplane and a point in it such that every line in the hyperplane through the point is parallel to main axis of some ellipsoid that is a quadric.

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References

[1] Busemann, H., Kelly, P.J.: Projective Geometries and Projective Metrics. Academic Press, New York (1953)
[2] Busemann, H.: The Geometry of Geodesics. Academic Press, New York (1955)
[3] Cheng, S.S., Li, W.: Analytic Solutions of Functional Equations. World Scientific, New Jersey (2008)
[4] Horváth, Á.G., Martini, H.: Conics in normed planes. Extracta Math. 26(1), 29–43. arXiv: 1102.3008 (2011)
[5] Hirschfeld, J.W.P.: Projective Geometries Over Finite Fields. Clarendon Press, Oxford (1979)
[6] Kurusa, A.: Conics in Minkowski geometry. Aequationes Math. 92, 949–961 (2018). https://doi.org/10.1007/s00010-018-0592-1
[7] Kurusa, Á.: Quadratic ellipses in Minkowski geometries (manuscript submitted)

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