Realizing Enveloping Algebras via Moduli Stacks

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Abstract

Let $\text{CF}(\text{Obj}\ A)$ denote the vector space of $\mathbb{Q}$-valued constructible functions on a given stack $\text{Obj}\ A$ for an exact category $A$. By using the Ringel–Hall algebra approach, Joyce proved that $\text{CF}(\text{Obj}\ A)$ is an associative $\mathbb{Q}$-algebra via the convolution multiplication and the subspace $\text{CF}^{\text{ind}}(\text{Obj}\ A)$ of constructible functions supported on indecomposables is a Lie subalgebra of $\text{CF}(\text{Obj}\ A)$ in [10]. In this paper, we show that there is a subalgebra $\text{CF}^{K\mathcal{S}}(\text{Obj}\ A)$ of $\text{CF}(\text{Obj}\ A)$ isomorphic to the universal enveloping algebra of $\text{CF}^{\text{ind}}(\text{Obj}\ A)$. Moreover we construct a comultiplication on $\text{CF}^{K\mathcal{S}}(\text{Obj}\ A)$ and a degenerate form of Green’s theorem. This generalizes Joyce’s work, as well as results of [3].

1 Introduction

Let $\Lambda$ be a finite dimensional $\mathbb{C}$-algebra such that it is a representative-finite algebra, i.e., there are finitely many finite dimensional indecomposable $\Lambda$-modules up to isomorphism. Let $\mathcal{I}(\Lambda) = \{X_1, \ldots, X_n\}$ be a set of representatives. Let $\mathcal{P}(\Lambda)$ be a set of representatives for the all isomorphism classes of $\Lambda$-modules. There is a free $\mathbb{Z}$-module $R(\Lambda)$ with a basis $\{u_X \mid X \in \mathcal{P}(\Lambda)\}$. Using the Euler characteristic, $\mathcal{P}(\Lambda)$ can be endowed with a multiplicative structure (see [22] and [13]). The multiplication is defined by

$$u_X \cdot u_Y = \sum_{A \in \mathcal{P}(\Lambda)} \chi(V(X,Y;A))u_A,$$

where $V(X,Y;A) = \{0 \subseteq A_1 \subseteq A \mid A_1 \cong X, A/A_1 \cong Y\}$ and $\chi(V(X,Y;A))$ is the Euler characteristic of $V(X,Y;A)$. Thus $(R(\Lambda), +, \cdot)$ is a $\mathbb{Z}$-algebra with identity $u_0$. Let $L(\Lambda)$ be a submodule of $R(\Lambda)$ which is spanned by $\{u_X \mid X \in \mathcal{I}(\Lambda)\}$. It follows that $L(\Lambda)$ is a Lie subalgebra of $R(\Lambda)$ with the Lie bracket $[u_X, u_Y] = u_X \cdot u_Y - u_Y \cdot u_X$. Riedtmann studied the universal enveloping algebra of $L(\Lambda)$. Let $R(\Lambda)'$ be the subalgebra of $R(\Lambda)$ generated by $\{u_X \mid X \in \mathcal{I}(\Lambda)\}$. Riedtmann showed that $R(\Lambda)'$ is isomorphic to the universal enveloping algebra of $L(\Lambda)$. These results have been generalized by two ways.

Joyce generalized Riedtmann’s work in the context of constructible functions (also stack functions) over moduli stacks. In [7], Joyce defined the Euler characteristics of constructible sets in $K$-stacks, pushforwards and pullbacks for constructible functions, where $K$ is an algebraically closed field. Let $A$ be
an abelian category and \( \text{CF}(\text{Obj}_A) \) the vector space of \( \mathbb{Q} \)-valued constructible functions on \( \text{Obj}_A(\mathbb{K}) \), where \( \text{Obj}_A \) is the moduli stack of objects in \( A \) and \( \text{Obj}_A(\mathbb{K}) \) the collection of isomorphism classes of objects in \( A \). Joyce proved that \( \text{CF}(\text{Obj}_A) \) is an associative \( \mathbb{Q} \)-algebra. The algebra \( \text{CF}(\text{Obj}_A) \) can be viewed as a variant of the Ringel-Hall algebra. Let \( \text{CF}^{\text{ind}}(\text{Obj}_A) \) be a subspace of \( \text{CF}(\text{Obj}_A) \) satisfying the condition that \( f([X]) \neq 0 \) implies \( X \) is an indecomposable object in \( A \) for every \( f \in \text{CF}^{\text{ind}}(\text{Obj}_A) \). Then \( \text{CF}^{\text{ind}}(\text{Obj}_A) \) is shown to be a Lie subalgebra of \( \text{CF}(\text{Obj}_A) \) (see [10] Theorem 4.9). Let \( \text{CF}^{\text{fin}}(\text{Obj}_A) \) be the subspace of \( \text{CF}(\text{Obj}_A) \) such that

\[
\text{supp}(f) = \{ [X] \in \text{Obj}_A(\mathbb{K}) \mid f([X]) \neq 0 \}
\]

is a finite set for every \( f \in \text{CF}^{\text{fin}}(\text{Obj}_A) \). Let

\[
\text{CF}^{\text{fin}}(\text{Obj}_A) = \text{CF}^{\text{ind}}(\text{Obj}_A) \cap \text{CF}^{\text{ind}}(\text{Obj}_A).
\]

Assume that a conflation \( X \to Y \to Z \) in \( A \) implies that the number of isomorphism classes of \( Y \) is finite for all \( X, Z \in \text{Obj}(A) \). With the assumption, Joyce proved that \( \text{CF}^{\text{fin}}(\text{Obj}_A) \) is an associative algebra and \( \text{CF}^{\text{ind}}(\text{Obj}_A) \) a Lie subalgebra of \( \text{CF}^{\text{fin}}(\text{Obj}_A) \). It follows that \( \text{CF}^{\text{fin}}(\text{Obj}_A) \) is isomorphic to the universal enveloping algebra of \( \text{CF}^{\text{ind}}(\text{Obj}_A) \). Joyce defined a comultiplication on \( \text{CF}^{\text{fin}}(\text{Obj}_A) \) and proved that \( \text{CF}^{\text{fin}}(\text{Obj}_A) \) is a bialgebra.

In [3], the authors extended Riedtmann’s results to algebras of representation-infinite type, i.e., the cardinality of isomorphism classes of indecomposable finite dimensional \( \Lambda \)-modules is infinite. Let \( R(\Lambda) \) be the \( \mathbb{Z} \)-module spanned by \( 1_{\mathcal{O}} \), where \( 1_{\mathcal{O}} \) is the characteristic function over a constructible set of stratified Krull-Schmidt \( \mathcal{O} \) (see [3] Section 3). The subspace \( L(\Lambda) \) of \( R(\Lambda) \) is spanned by \( 1_{\mathcal{O}} \), where \( \mathcal{O} \) are indecomposable constructible sets. The multiplication is defined by

\[
1_{\mathcal{O}_1} \cdot 1_{\mathcal{O}_2}(X) = \chi(\mathcal{O}_1; \mathcal{O}_2; X)
\]

where \( X \) is a \( \Lambda \)-module. Then \( R(\Lambda) \) is an associative algebra with identity \( 1_0 \) and \( L(\Lambda) \) a Lie subalgebra of \( R(\Lambda) \) with Lie bracket. The algebra \( R(\Lambda) \otimes \mathbb{Q} \) is the universal enveloping algebra of \( L(\Lambda) \otimes \mathbb{Q} \). The authors gave the degenerate form of Green’s formula and established the comultiplication of \( R(\Lambda) \) in [3].

The goal of this paper is to explicitly construct the enveloping algebra of \( \text{CF}^{\text{ind}}(\text{Obj}_A) \) by the methods in [3]. Let \( A \) be an exact category satisfying some properties. Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be a conflation in \( A \) and \( \text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \) the automorphism group of \( X \xrightarrow{f} Y \xrightarrow{g} Z \). The key idea in [3] is that \( V(X,Y;L) \) has the same Euler characteristic as its fixed point set under the action of \( \mathbb{C}^* \).

In this paper, we consider exact categories instead of categories of modules. Then as a substitute of the action of \( \mathbb{C}^* \), we analyze the action of a maximal torus of \( \text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \) on \( X \xrightarrow{f} Y \xrightarrow{g} Z \). The universal enveloping algebra of \( \text{CF}^{\text{ind}}(\text{Obj}_A) \) can be endowed with a comultiplication structure (Definition 4.1). It is compatible with multiplication (Theorem 4.6). The compatibility can be viewed as the degenerate form of Green’s theorem on Ringel-Hall algebras (see [4] or [20]).
The paper is organized as follows. In Section 2 we recall the basic concepts about stacks, constructible sets and constructible functions. In Section 3 we define the constructible sets of stratified Krull-Schmidt. We study the the subspace $\text{CF}^{KS}(\text{Obi}_{A})$ of $\text{CF}(\text{Obi}_{A})$ generated by characteristic functions supported on constructible sets of stratified Krull-Schmidt. Then $\text{CF}^{KS}(\text{Obi}_{A})$ provides a realization of the universal enveloping algebra of $\text{CF}^{\text{ind}}(\text{Obi}_{A})$. In Section 4 we give the comultiplication $\Delta$ in $\text{CF}^{KS}(\text{Obi}_{A})$ and prove that $\Delta$ is an algebra homomorphism.

2 Preliminaries

2.1 Constructible sets and constructible functions

From now on, let $K$ be an algebraically closed field with characteristic zero. We recall the definitions of constructible sets and constructible functions on $K$-stacks. These definitions are taken from Joyce [9].

**Definition 2.1.** Let $F$ be a $K$-stack. Let $F(K)$ denote the set of 2-isomorphism classes $[x]$ where $x: \text{Spec} K \to F$ are 1-morphisms. Every element of $F(K)$ is called a geometric point (or $K$-point) of $F$. For $K$-stacks $F$ and $G$, let $\phi : F \to G$ be a 1-morphism of $K$-stacks. Then $\phi$ induces a map $\phi_* : F(K) \to G(K)$ by $[x] \mapsto [\phi \circ x]$.

For any $[x] \in F(K)$, let $\text{Iso}_K(x)$ denote the group of 2-isomorphisms $x \to x$ which is called a stabilizer group. For ease of notations, $\text{Iso}_K(x)$ is used to denote the group instead of $\text{Iso}_K([x])$. If $\text{Iso}_K(x)$ is an affine algebraic $K$-group for each $[x] \in F(K)$, then we say $F$ with affine geometric stabilizers. A morphism of algebraic $K$-groups $\phi_x : \text{Iso}_K(x) \to \text{Iso}_K(\phi_*(x))$ is induced by $\phi : F \to G$ for each $[x] \in F(K)$.

A subset $\mathcal{O} \subseteq F(K)$ is called a constructible set if $\mathcal{O} = \bigcap_{i=1}^{n} F_i(K)$ for some $n \in \mathbb{N}^+$, where every $F_i$ is a finite type algebraic $K$-substack of $F$. A subset $S \subseteq F(K)$ is called a locally constructible set if $S \cap \mathcal{O}$ are constructible for all constructible subsets $\mathcal{O} \subseteq F(K)$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are constructible sets, then $\mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{O}_2$ and $\mathcal{O}_1 \setminus \mathcal{O}_2$ are constructible sets by [9, Lemma 2.4].

Let $\Phi : F(K) \to G(K)$ be a map. The set $\Gamma_\Phi = \{(x, \Phi(x)) \mid x \in F(K)\}$ is called the graph of $\Phi$. Recall that $\Phi$ is a pseudomorphism if $\Gamma_\Phi \cap (\mathcal{O} \times G(K))$ are constructible for all constructible subsets $\mathcal{O} \subseteq F(K)$. By [9, Proposition 4.6], if $\phi : F \to G$ is a 1-morphism then $\phi_*$ is a pseudomorphism, $\Phi(\mathcal{O})$ and $\Phi^{-1}(y) \cap \mathcal{O}$ are constructible sets for all constructible subset $\mathcal{O} \subseteq F(K)$ and $y \in G(K)$. If $\Phi$ is a bijection and $\Phi^{-1}$ is also a pseudomorphism, we call $\Phi$ a pseudoisomorphism.

Then we will recall the definition of the naïve Euler characteristic of a constructible subset of $F(K)$ in [9].

This is a useful result due to Rosenlicht [21].

**Theorem 2.2.** Let $G$ be an algebraic $K$-group acting on a $K$-variety $X$. There exist an open dense $G$-invariant subset $X_1 \subseteq X$ and a $K$-variety $Y$ such that
there is a morphism of varieties $\phi : X_1 \to Y$ which induces a bijection form $X_1(K)/G$ to $Y(K)$.

Let $X$ be a separated $K$-scheme of finite type, the Euler characteristic $\chi(X)$ of $X$ is defined by

$$\chi(X) = \sum_{i=0}^{2\dim X} (-1)^i \dim_{\mathbb{Q}_p} H^i_{\text{ cris}}(X, \mathbb{Q}_p),$$

where $p$ is a prime number, $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the ring of $p$-adic integers, $\mathbb{Q}_p$ is its field of fractions and $H^i_{\text{ cris}}(X, \mathbb{Q}_p)$ are the compactly-supported $p$-adic cohomology groups of $X$ for $i \geq 0$.

The following properties of Euler characteristic follow [3] and [9].

**Proposition 2.3.** Let $X$, $Y$ be separated, finite type $K$-schemes and $\varphi : X \to Y$ a morphism of schemes. Then:

1. If $Z$ is a closed subscheme of $X$, then $\chi(X) = \chi(X \setminus Z) + \chi(Z)$.
2. $\chi(X \times Y) = \chi(X) \times \chi(Y)$.
3. Let $X$ be a disjoint union of finitely many subschemes $X_1, \ldots, X_n$, we have

$$\chi(X) = \sum_{i=1}^{n} \chi(X_i).$$

4. If $\varphi$ is a locally trivial fibration with fibre $F$, then $\chi(X) = \chi(F) \cdot \chi(Y)$.
5. $\chi(K^n) = 1$, $\chi(K^n\mathbb{P}^n) = n + 1$ for all $n \geq 0$.

An algebraic $K$-stack $\mathcal{F}$ is said to be stratified by global quotient stacks if $\mathcal{F}(K) = \coprod_{i=1}^{s} \mathcal{F}_i(K)$ for finitely many locally closed substacks $\mathcal{F}_i$ where each $\mathcal{F}_i$ is 1-isomorphic to a quotient stack $[X_i/G_i]$, where $X_i$ is an algebraic $K$-variety and $G_i$ a smooth connected linear algebraic $K$-group acting on $X_i$. Then $\mathcal{F}$ is stratified by global quotient stacks.

Let $\mathcal{F} = \coprod_{i=1}^{s} \mathcal{F}_i$ where each $\mathcal{F}_i \cong [X_i/G_i]$ as above. By Theorem [2], there exists an open dense $G_i$-invariant subvariety $X_{i1}$ of $X_i$ for each $i$ such that there exists a morphism of varieties $\phi_{i1} : X_{i1} \to Y_{i1}$, which induces a bijection between $X_{i1}(K)/G_i$ and $Y_{i1}(K)$. Then $\phi_{i1}$ induces a 1-morphism $\theta_{i1} : G_{i1} \to Y_{i1}$, where $G_{i1}$ is 1-isomorphic to $[X_{i1}/G_i]$. Note that

$$\dim(X_{i(j-1)} \setminus X_{ij}) < \dim X_{i(j-1)}$$

for $j = 1, \ldots, k_i$. Using Theorem [2] again, we get a stratification

$$\mathcal{F}(K) = \coprod_{i=1}^{s} \prod_{j=1}^{k_i} G_{ij}(K)$$

for $s, k_i \in \mathbb{N}^+$, where $G_{ij} \cong [X_{ij}/G_i]$ such that $\phi_{ij} : X_{ij} \to Y_{ij}$ is a morphism of $K$-varieties and $\theta_{ij} : G_{ij} \to Y_{ij}$ a 1-morphism induced by $\phi_{ij}$. Let

$$Y = \prod_{i=1}^{s} \prod_{j=1}^{k_i} Y_{ij}$$

and $\Theta = \prod_{i=1}^{s} \prod_{j=1}^{k_i} (\theta_{ij})_* : \mathcal{F}(K) \to Y(K)$. Then $Y$ is a a separated $K$-scheme of finite type and $\Theta$ a pseudoisomorphism (see [9] Proposition 4.4 and Proposition 4.7).
Definition 2.4. Let $\mathcal{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\mathcal{C} \subseteq \mathcal{F}(\mathbb{K})$ a constructible set. Then $\mathcal{C}$ is pseudoÈEisomorphic to $Y(\mathbb{K})$, where $Y$ is a separated $\mathbb{K}$-scheme of finite type by [9, Proposition 4.7]. The naïve Euler characteristic of $\mathcal{C}$ is defined by $\chi^{na}(\mathcal{C}) = \chi(Y)$.

The following lemma is a generalization of Proposition 2.3 (4).

Lemma 2.5. Let $\mathcal{F}$ and $\mathcal{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. If $\mathcal{C} \subseteq \mathcal{F}(\mathbb{K})$, $\mathcal{D} \subseteq \mathcal{G}(\mathbb{K})$ are constructible sets, and $\Phi : \mathcal{C} \to \mathcal{D}$ is a surjective pseudomorphism such that all fibers have the same naïve Euler characteristic $\chi$, then $\chi^{na}(\mathcal{C}) = \chi \cdot \chi^{na}(\mathcal{D})$.

Proof. Because $\mathcal{C}$, $\mathcal{D}$ are constructible sets, there exist separated finite type $\mathbb{K}$-schemes $X$, $Y$ such that $\mathcal{C}$, $\mathcal{D}$ are pseudoisomorphic to $X(\mathbb{K})$, $Y(\mathbb{K})$ respectively. Therefore $\chi^{na}(\mathcal{C}) = \chi(X)$, $\chi^{na}(\mathcal{D}) = \chi(Y)$. Then $\Phi$ induces a surjective pseudomorphism between $X(\mathbb{K})$ and $Y(\mathbb{K})$, say $\phi : X(\mathbb{K}) \to Y(\mathbb{K})$. There exist two projective morphisms $\pi_1 : \Gamma_\phi \to X(\mathbb{K})$ and $\pi_2 : \Gamma_\phi \to Y(\mathbb{K})$. Note that $\pi_1$ is also a pseudoisomorphism, that is $\chi^{na}(\Gamma_\phi) = \chi(X)$, and all fibres of $\pi_2$ have the same naïve Euler characteristic $\chi$. Then $\chi^{na}(\Gamma_\phi) = \chi \cdot \chi(Y)$. Hence $\chi(X) = \chi \cdot \chi(Y)$. We finish the proof. \( \square \)

Definition 2.6. A function $f : \mathcal{F}(\mathbb{K}) \to \mathbb{Q}$ is called a constructible function on $\mathcal{F}(\mathbb{K})$ if the codomain of $f$ is a finite set and $f^{-1}(a)$ is a constructible subset of $\mathcal{F}(\mathbb{K})$ for each $a \in f(\mathcal{F}(\mathbb{K})) \setminus \{0\}$. Let $\text{CF}(\mathcal{F})$ denote the $\mathbb{Q}$-vector space of all $\mathbb{Q}$-valued constructible functions on $\mathcal{F}(\mathbb{K})$.

Let $S \subseteq \mathcal{F}(\mathbb{K})$ be a locally constructible set. The integral of $f$ on $S$ is

$$
\int_{x \in S} f(x) = \sum_{a \in f(S) \setminus \{0\}} a \chi^{na}(f^{-1}(a) \cap S)
$$

for each $f \in \text{CF}(\mathcal{F})$.

We recall the pushforwards and pullbacks of constructible functions due to Joyce [9].

Definition 2.7. Let $\mathcal{F}$ and $\mathcal{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers and $\phi : \mathcal{F} \to \mathcal{G}$ a 1-morphism. For each $f \in \text{CF}(\mathcal{F})$, the naïve pushforward $\phi^{na}_!(f) : \mathcal{F}(\mathbb{K}) \to \mathbb{Q}$ of $f$ is

$$
\phi^{na}_!(f)(t) = \sum_{a \in f(\phi^{-1}(t)) \setminus \{0\}} a \chi^{na}(f^{-1}(a) \cap \phi^{-1}(t))
$$

for each $t \in \mathcal{G}(\mathbb{K})$. Then $\phi^{na}_!(f)$ is a constructible function for each $f \in \text{CF}(\mathcal{F})$ by [9, Theorem 4.9].

Similarly, if $\Phi : \mathcal{F}(\mathbb{K}) \to \mathcal{G}(\mathbb{K})$ is a pseudomorphism, the naïve pushforward $\Phi^{na}_!(f) : \mathcal{F}(\mathbb{K}) \to \mathbb{Q}$ of $f \in \text{CF}(\mathcal{F})$ is defined by

$$
\Phi^{na}_!(f)(t) = \sum_{a \in f(\Phi^{-1}(t)) \setminus \{0\}} a \chi^{na}(f^{-1}(a) \cap \Phi^{-1}(t))
$$
for $t \in G(\mathbb{K})$. Recall that $\Phi^{na}(f) \in \text{CF}(G)$ by [9, Theorem 4.9].

If $\phi : F \to G$ is a 1-morphism such that $\chi(\text{Ker}(\phi_x)) = 1$ for all $x \in F(\mathbb{K})$, we can define a function $m_{\phi} : F(\mathbb{K}) \to \mathbb{Q}$ by

$$m_{\phi}(x) = \chi \left( \text{Iso}_K(\phi_*(x))/\phi_*(\text{Iso}_K(x)) \right)$$

for each $x \in F(\mathbb{K})$. For each $f \in \text{CF}(F)$, the pushforward $\phi_!(f) : G(\mathbb{K}) \to \mathbb{Q}$ of $f$ is defined by

$$\phi_!(f) = \phi^{na}_!(f \cdot m_{\phi}),$$

where $(f \cdot m_{\phi})(x) = f(x)m_{\phi}(x)$ for $x \in F(\mathbb{K})$. Note that $\phi_!(f) \in \text{CF}(G)$ (see [9]).

If $\phi$ is a 1-morphism of finite type, then $\phi^{-1}_!(D) \subset F(\mathbb{K})$ is a constructible set for each constructible subset $D$ of $G(\mathbb{K})$. Then $g \circ \phi_! \in \text{CF}(F)$ for $g \in \text{CF}(G)$. Recall that the pullback $\phi^* : \text{CF}(G) \to \text{CF}(F)$ of $\phi$ is defined by $\phi^*(g) = g \circ \phi_!$ and it is linear.

### 2.2 Stacks of objects and conflations in $A$

From now on, let $(A, S)$ be a Krull-Schmidt exact $\mathbb{K}$-category with idempotent complete (see A.1 and A.2). For simplicity, we write $A$ instead of $(A, S)$.

The isomorphism classes of $X \in \text{Obj}(A)$ and conflations $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $A$ are denoted by $[X]$ and $[X \xrightarrow{i} Y \xrightarrow{d} Z]$ (or $[(X, Y, Z, i, d)]$), respectively. Two conflations $X \xrightarrow{i} Y \xrightarrow{d} Z$ and $A \xrightarrow{f} B \xrightarrow{g} C$ are isomorphic if there exist isomorphisms $a : X \to A$, $b : Y \to B$ and $c : Z \to C$ in $A$ such that the following diagram is commutative

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y & \xrightarrow{d} & Z \\
& a \downarrow & & b \downarrow & c \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C
\end{array}$$

(1)

The morphism $(a, b, c)$ is called an isomorphism of conflations in $A$.

**Assumption 2.8.** Assume that $\text{dim}_K \text{Hom}_A(X, Y)$ and $\text{dim}_K \text{Ext}^1_A(X, Y)$ are finite for all $X, Y \in \text{Obj}(A)$. Let $K(A)$ denote the quotient group of the Grothendieck group $K_0(A)$ such that $[X] = 0$ in $K(A)$ implies that $X$ is a zero object in $A$, where $[X]$ denotes the image of $X$ in $K(A)$.

The following 2-categories are defined in [8].

Let $\text{Sch}_K$ be a 2-category of $K$-schemes such that objects are $K$-schemes, 1-morphisms morphisms of schemes and 2-morphisms only the natural transformations $\text{id}_f$ for all 1-morphisms $f$. Let $(\text{exactcat})$ denote the 2-category of all exact categories with 1-morphisms exact functors of exact categories and 2-morphisms natural transformations between the exact functors. If all morphisms of a category are isomorphisms, then the category is called a groupoid. Let $(\text{groupoids})$ be the 2-category with objects groupoids, 1-morphisms functors...
of groupoids and 2-morphisms natural transformations (also see [8 Definition 2.8]).

In [8 Section 7.1], Joyce defined a stack $\mathcal{F}_A : \text{Sch}_K \rightarrow (\text{exactcat})$ associated to the exact category $A$ (the original definition is for abelian category, it can be extended to exact categories directly), where $\mathcal{F}_A$ is a contravariant 2-functor and satisfies the condition $\mathcal{F}_A(\text{Spec}(K)) = A$. Applying $\mathcal{F}_A$, he defined two moduli stacks

$$\text{Obj}_A, \text{Exact}_A : \text{Sch}_K \rightarrow (\text{groupoids})$$

which are contravariant 2-functors ( [8 Definition 7.2]). The 2-functor

$$\text{Obj}_A = F \circ \mathcal{F}_A,$$

where $F : (\text{exactcat}) \rightarrow (\text{groupoids})$ is a forgetful 2-functor as follows. For an exact category $G$, $F(G)$ is a groupoid such that $\text{Obj}(F(G)) = \text{Obj}(G)$ and morphisms are isomorphisms in $G$. For $U \in \text{Sch}_K$, a category $\text{Exact}_A(U)$ is a groupoid whose objects are conflations in $\mathcal{F}_A(U)$ and morphisms isomorphisms of conflations in $\mathcal{F}_A(U)$.

Let $\eta : U \rightarrow V$ and $\theta : V \rightarrow W$ be morphisms of schemes in $\text{Sch}_K$. Obviously, the functors $\text{Obj}_A(\eta) : \text{Obj}_A(V) \rightarrow \text{Obj}_A(U)$ and $\text{Exact}_A(\eta) : \text{Exact}_A(V) \rightarrow \text{Exact}_A(U)$ are induced by $\mathcal{F}_A(\eta) : \mathcal{F}_A(V) \rightarrow \mathcal{F}_A(U)$. The natural transformations $\epsilon_{\theta, \eta} : \text{Obj}_A(\eta) \circ \text{Obj}_A(\theta) \rightarrow \text{Obj}_A(\theta \circ \eta)$ and $\epsilon_{\theta, \eta} : \text{Exact}_A(\eta) \circ \text{Exact}_A(\theta) \rightarrow \text{Exact}_A(\theta \circ \eta)$ are also induced by $\epsilon_{\theta, \eta} : \mathcal{F}_A(\eta) \circ \mathcal{F}_A(\theta) \rightarrow \mathcal{F}_A(\theta \circ \eta)$.

Let

$$K'(A) = \{ [X] \in K(A) | X \in \text{Obj}(A) \} \subset K(A).$$

For each $\alpha \in K'(A)$, Joyce defined $\text{Obj}^\alpha_A : \text{Sch}_K \rightarrow (\text{groupoids})$ which is a substack of $\text{Obj}_A$ in [8 Definition 7.4]. For each $U \in \text{Sch}_K$, $\text{Obj}^\alpha_A(U)$ is a full subcategory of $\text{Obj}_A(U)$. For each object $X$ in $\text{Obj}^\alpha_A(U)$, the image of $\text{Obj}_A(f)(X)$ in $K(A)$ is $\alpha$ for each morphism $f : \text{Spec}(K) \rightarrow U$.

Let $\eta : U \rightarrow V$ and $\theta : V \rightarrow W$ be morphisms in $\text{Sch}_K$. The functor $\text{Obj}^\alpha_A(\eta) : \text{Obj}^\alpha_A(V) \rightarrow \text{Obj}^\alpha_A(U)$ is defined by restriction from $\text{Obj}_A(\eta) : \text{Obj}_A(V) \rightarrow \text{Obj}_A(U)$. The natural transformation $\epsilon_{\theta, \eta} : \text{Obj}^\alpha_A(\eta) \circ \text{Obj}^\alpha_A(\theta) \rightarrow \text{Obj}^\alpha_A(\theta \circ \eta)$ is restricted from $\epsilon_{\theta, \eta} : \text{Obj}_A(\eta) \circ \text{Obj}_A(\theta) \rightarrow \text{Obj}_A(\theta \circ \eta)$.

For $\alpha, \beta, \gamma \in K'(A)$ and $\beta = \alpha + \gamma$, $\text{Exact}^{\alpha, \beta, \gamma}_A : \text{Sch}_K \rightarrow (\text{groupoids})$ is defined as follows. For $U \in \text{Sch}_K$, $\text{Exact}^{\alpha, \beta, \gamma}_A(U)$ is a full subcategory of $\text{Exact}_A(U)$. The objects of $\text{Exact}^{\alpha, \beta, \gamma}_A(U)$ are conflations

$$X \xrightarrow{d} Y \xrightarrow{d} Z \in \text{Obj}(\text{Exact}_A(U)),$$

where $X \in \text{Obj}(\text{Obj}^\alpha_A(U))$, $Y \in \text{Obj}(\text{Obj}^\beta_A(U))$ and $Z \in \text{Obj}(\text{Obj}^\gamma_A(U))$. Similarly, the morphism $\text{Exact}^{\alpha, \beta, \gamma}_A(\eta)$ and natural transformation $\epsilon_{\theta, \eta}$ are defined by restriction.

The following theorem is taking from [8 Theorem 7.5].

**Theorem 2.9.** The 2-functors $\text{Obj}_A$, $\text{Exact}_A$ are $\mathbb{K}$-stacks, and $\text{Obj}^\alpha_A$, $\text{Exact}^{\alpha, \beta, \gamma}_A$ are open and closed $\mathbb{K}$-substacks of them respectively. There are disjoint unions

$$\text{Obj}_A = \coprod_{\alpha \in K'(A)} \text{Obj}^\alpha_A, \text{Exact}_A = \coprod_{\alpha, \beta, \gamma \in K'(A)} \text{Exact}^{\alpha, \beta, \gamma}_A.$$
Assume that \( \text{Obj}_A \) and \( \text{Exact}_A \) are locally of finite type algebraic \( K \)-stacks with affine algebraic stabilizers. Recall that \( \text{Obj}_A(K) \) and \( \text{Exact}_A(K) \) are the collection of isomorphism classes of objects in \( A \) and the collection of isomorphism classes of conflations in \( A \), respectively. For each \( \alpha \in K^+(A) \), \( \text{Obj}_A(K) \) is the collection of isomorphism classes of \( X \in \text{Obj}(A) \) such that \( [X] = \alpha \) (see [10 Section 3.2]).

**Example 2.10.** Let \( Q = (Q_0, Q_1, s, t) \) be a finite connected quiver, where \( Q_0 = \{1, \ldots, n\} \) is the set of vertices, \( Q_1 \) is the set of arrows and \( s : Q_1 \to Q_0 \) (resp. \( t : Q_1 \to Q_0 \)) is a map such that \( s(\rho) \) (resp. \( t(\rho) \)) is the source (resp. target) of \( \rho \) for \( \rho \in Q_1 \). Let \( A = \mathbb{C}Q \) be the path algebra of \( Q \) and \( \text{mod-A} \) denote the category of all finite dimensional right \( A \)-modules.

Let \( d = (d_j)_{j \in Q_0} \) for all \( d_j \in \mathbb{N} \). There is an affine variety

\[
\text{Rep}(Q, d) = \bigoplus_{\rho \in Q_1} \text{Hom}(\mathbb{C}^{d_{\rho^1}}, \mathbb{C}^{d_{\rho^2}}).
\]

For each \( x = (x_\rho)_{\rho \in Q_1} \in \text{Rep}(Q, d) \), there is a \( \mathbb{C} \)-linear representation \( M(x) = (\mathbb{C}^{d_j}, x_\rho)_{j \in Q_0, \rho \in Q_1} \) of \( Q \). Let \( \text{rep}(Q) \) denote the category of finite dimensional \( \mathbb{C} \)-linear representations of \( Q \). Recall that \( \text{rep}(Q) \cong \text{mod-A} \). We identify \( \text{rep}(Q) \) with \( \text{mod-A} \).

The linear algebraic group

\[
\text{GL}(d) = \prod_{j \in Q_0} \text{GL}(d_j, \mathbb{C})
\]

acts on \( \text{Rep}(Q, d) \) by \( g.x = (g(t(\rho))x_\rho g(s(\rho))^{-1})_{\rho \in Q_1} \) for \( g = (g_j)_{j \in Q_0} \in \text{GL}(d) \).

A complex \( M^* = (M^{(i)}, \partial^i) \), where \( M^{(i)} \in \text{Obj}(\text{mod-A}) \) and \( \partial^{i+1}\partial^i = 0 \), is bounded if there exist some positive integers \( n_0 \) and \( n_1 \) such that \( M^{(i)} = 0 \) for \( i \leq -n_0 \) or \( i \geq n_1 \). Let \( \text{dim} M^{(i)} = d^{(i)} \) be the dimension vector of \( M^{(i)} \) for each \( i \in \mathbb{Z} \). The vector sequence \( (d^{(i)})_{i \in \mathbb{Z}} \) of \( M^* \) is denoted by \( \underline{d}(M^*) \).

Let \( \mathcal{C}(Q, d) \) denote the affine variety consisting of all complexes \( M^* \) with \( \underline{d}(M^*) = d \). The group \( G(d) = \prod_{i \in \mathbb{Z}} \text{GL}(d^{(i)}) \) is a linear algebraic group acting on \( \mathcal{C}(Q, d) \). The action is induced by the actions of \( \text{GL}(d^{(i)}) \) on \( \text{Rep}(Q, d^{(i)}) \) for all \( i \in \mathbb{Z} \), that is

\[
(g^{(i)}, (x^{(i)}, \partial^i)) \mapsto (g^{(i)}x^{(i)}, g^{(i+1)}\partial^i(g^{(i)})^{-1})^{(i)},
\]

Let \( \{P_1, \ldots, P_n\} \) be a set of representatives for all isomorphism classes of finite dimensional indecomposable projective \( A \)-modules. A complex \( P^* = \)

\[\ldots \to P^{(i-1)} \xrightarrow{\partial^{i-1}} P^{(i)} \xrightarrow{\partial^i} P^{(i+1)} \to \ldots \]

is projective if \( P^{(i)} \cong \bigoplus_{j=1}^n m^{(i)}_j P_j \) for \( m^{(i)}_j \in \mathbb{N} \) and \( i \in \mathbb{Z} \). Let

\[
\underline{e}(P^{(i)}) = m^{(i)} = (m^{(i)}_1, \ldots, m^{(i)}_n)
\]
be a vector corresponding to \( P^{(i)} \). By the Krull-Schmidt Theorem, \( \mathfrak{z}(P^{(i)}) \) is unique. The dimension vector of \( P^\bullet \) can be defined by

\[
\dim(P^\bullet) = (\ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots).
\]

A dimension vector \( \dim(P^\bullet) \) is bounded if \( P^\bullet \) is bounded.

Let \( \mathfrak{m} = (\mathfrak{m}_i)_{i \in \mathbb{Z}} \) be a bounded dimension vector and \( \mathfrak{d}(\mathfrak{m}) = (\mathfrak{d}_i)_{i \in \mathbb{Z}} \) be the vector sequence of a complex whose dimension vector is \( \mathfrak{m} \). Let \( \mathcal{P}^b(Q, \mathfrak{m}) \) be the set of all bounded projective complexes \( P^\bullet \) with \( \dim(P^\bullet) = \mathfrak{m} \) and \( \text{dis}(P^\bullet) = \mathfrak{d}(\mathfrak{m}) \). Note that \( \mathcal{P}^b(Q, \mathfrak{m}) \) is a locally closed subset of \( \mathcal{C}^b(Q, \mathfrak{d}(\mathfrak{m})) \). An action of \( G(\mathfrak{d}(\mathfrak{m})) \) on the variety \( \mathcal{P}^b(Q, \mathfrak{m}) \) is induced by the action of \( G(\mathfrak{d}(\mathfrak{m})) \) on \( \mathcal{C}^b(Q, \mathfrak{d}(\mathfrak{m})) \).

Let \( \mathcal{P}^b(Q) \) denote the exact category with objects bounded projective complexes and morphisms \( \phi : P^\bullet \to Q^\bullet \) morphisms between bounded projective complexes. The Grothendieck group

\[
K_0(\mathcal{P}^b(Q)) \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}^n_{(i)},
\]

where \( \mathbb{Z}^n_{(i)} = \mathbb{Z}^n \). Note that \( K(\mathcal{P}^b(Q)) = K_0(\mathcal{P}^b(Q)) \) and

\[
K'(\mathcal{P}^b(Q)) \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{N}^n_{(i)},
\]

where \( \mathbb{N}^n_{(i)} = \mathbb{N}^n \).

Joyce defined \( \mathcal{F}_{\text{mod-KQ}} \) in [8, Example 10.5]. Similarly, for each \( U \in \text{Sch}_K \), we define \( \mathcal{F}_{P^b(Q)}(U) \) to be the category as follows.

The objects of \( \mathcal{F}_{P^b(Q)}(U) \) are complexes of sheaves \( P^\bullet = (P^{(i)}, \partial^{(i)})_{i \in \mathbb{Z}}, \) where \( P^{(i)} = (\bigoplus_{j \in Q_0} X^{(i)}_j, x^i) \) and \( \partial^{(i)} : X^{(i)}_j \to X^{(i)}_{j+1} \) are exact sequences of free sheaves of finite rank on \( U \) and \( x^i = (x^i_{j, \rho}) \in Q_1, \) where \( X^{(i)}_j \) are sheaves of \( \mathcal{O}_U \)-modules for \( j \in Q_0 \). The morphisms of \( \mathcal{F}_{P^b(Q)}(U) \) are morphisms of complexes \( \phi^* : (P^{(i)}, \partial^{(i)}) \to (Q^{(i)}, \partial^{(i)}) \), where \( Q^{(i)} = (\bigoplus_{j \in Q_0} X^{(i)}_j, x^i) \) and \( \phi^* \) is a sequence of morphisms

\[
(\phi^i : P^{(i)} \to Q^{(i)})_{i \in \mathbb{Z}}
\]

with \( \phi^i = (\phi^i_j : X^{(i)}_j \to Y^{(i)}_j)_{j \in Q_0} \) such that \( \phi^{i+1} \partial^i = \partial^i \phi^i \) and \( \phi^{(i)}(\partial^i) = \phi^{(i)}(\partial^i) \) for all \( i \in \mathbb{Z} \) and \( \rho \in Q_1 \). It is easy to see that \( \mathcal{F}_{P^b(Q)}(U) \) is an exact category.

Let \( \eta : U \to V \) be a morphism in \( \text{Sch}_K \). A functor

\[
\mathcal{F}_{P^b(Q)}(\eta) : \mathcal{F}_{P^b(Q)}(V) \to \mathcal{F}_{P^b(Q)}(U)
\]

is defined as follows. If \( (P^{(i)}, \partial^{(i)})_{i \in \mathbb{Z}} \in \text{Obj}(\mathcal{F}_{P^b(Q)}(V)), \)

\[
\mathcal{F}_{P^b(Q)}(\eta)(P^{(i)}, \partial^{(i)})_{i \in \mathbb{Z}} = (\eta^*(P^{(i)}), \eta^*(\partial^{(i)}))_{i \in \mathbb{Z}}
\]
for $\eta^*(P^{(i)}) = (\bigoplus_{j \in Q_0} \eta^*(X_j^{(i)}), (\eta^*(x_j^i))_{j \in Q_0})$, where $\eta^*(X_j^{(i)})$ are the inverse images of $X_j^{(i)}$ by the morphism $\eta$, $\eta^*(D^i) : \eta^*(P^{(i)}) \to \eta^*(P^{(i+1)})$ with $\eta^*(D^{i+1})\eta^*(D^i) = 0$ for $i \in \mathbb{Z}$ and

$$\eta^*(x_j^i) : \eta^*(X_{s(\rho)}) \to \eta^*(X_{t(\rho)})$$

for $\rho \in Q_1$ are pullbacks of morphisms between inverse images. For a morphism $\phi^* : (P^{(i)}, D^i) \to (Q^{(i)}, D^i)$ in $\mathcal{F}_{\mathcal{P}^b(Q)}(V)$, the morphism

$$\mathcal{F}_{\mathcal{P}^b(Q)}(\eta)(\phi^*) : (\eta^*(P^*), \eta^*(D^*)) \to (\eta^*(Q^*), \eta^*(D^*))$$

is a sequence of morphisms

$$\left(\eta^*(\phi^j) : \left(\bigoplus_{j \in Q_0} \eta^*(X_j^{(i)}), (\eta^*(x_j^i))_{j \in Q_0}\right) \to \left(\bigoplus_{j \in Q_0} \eta^*(Y_j^{(i)}), (\eta^*(y_j^i))_{j \in Q_0}\right)\right)_{i \in \mathbb{Z}}$$

with $\eta^*(D^{i+1})\eta^*(D^i) = \eta^*(D^i)\eta^*(D^i)$, where $\eta^*(D^i)$ are pullbacks of morphisms between inverse images which satisfy $\eta^*(D^{i+1})\eta^*(D^i) = 0$, and

$$\eta^*(Q^*) = \left(\bigoplus_{j \in Q_0} \eta^*(Y_j^{(i)}), (\eta^*(y_j^i))_{j \in Q_0}\right)_{i \in \mathbb{Z}}$$

such that the pullbacks

$$\eta^*(\phi^j) : \eta^*(X_j^{(i)}) \to \eta^*(Y_j^{(i)})$$

satisfy $\eta^*(\phi^j)\eta^*(x_j^i) = \eta^*(y_j^i)\eta^*(\phi^j)$. Because locally free sheaves are flat, $\mathcal{F}_{\mathcal{P}^b(Q)}(\eta)(\phi^*)$ is an exact functor.

Let $\eta : U \to V$ and $\theta : V \to W$ be morphisms in $\text{Sch}_K$. As in [S Example 9.1], for each $P^* \in \text{Obj}(\mathcal{F}_{\mathcal{P}^b(Q)}(W))$, there is a canonical isomorphism $\epsilon_{\theta,\eta}(P^*) : \mathcal{F}_{\mathcal{P}^b(Q)}(\eta) \circ \mathcal{F}_{\mathcal{P}^b(Q)}(\theta)(P^*) \to \mathcal{F}_{\mathcal{P}^b(Q)}(\theta \circ \eta)(P^*)$. We get a 2-isomorphism of functors

$$\epsilon_{\theta,\eta} : \mathcal{F}_{\mathcal{P}^b(Q)}(\eta) \circ \mathcal{F}_{\mathcal{P}^b(Q)}(\theta) \to \mathcal{F}_{\mathcal{P}^b(Q)}(\theta \circ \eta)$$

by the canonical isomorphisms. Thus we have the 2-functor $\mathcal{F}_{\mathcal{P}^b(Q)}$.

The set $\text{Obj}_{\mathcal{P}^b(Q)}(\mathbb{C})$ consists of all isomorphism classes of complexes in $\mathcal{P}^b(Q)$.

As in [S Definition 7.7] and [10] Section 3.2, we have the following 1-morphisms

$$\pi_l : \text{Exact}_A \to \text{Obj}_A$$

which induces a map $(\pi_l)_* : \text{Exact}_A(K) \to \text{Obj}_A(K)$ defined by $[X \xrightarrow{i} Y \xrightarrow{d} Z] \mapsto [X]$;

$$\pi_m : \text{Exact}_A \to \text{Obj}_A$$

such that the induced map $(\pi_m)_* : \text{Exact}_A(K) \to \text{Obj}_A(K)$ maps $[X \xrightarrow{i} Y \xrightarrow{d} Z]$ to $[Y]$;

$$\pi_r : \text{Exact}_A \to \text{Obj}_A$$

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inducing the map \((\pi_r)_*: \text{Exact}_A(\mathbb{K}) \to \text{Obj}_A(\mathbb{K})\) by \([X \xrightarrow{i} Y \xrightarrow{d} Z] \mapsto [Z]\).

The map \(\pi_l \times \pi_r*: \text{Exact}_A(\mathbb{K}) \to \text{Obj}_A(\mathbb{K}) \times \text{Obj}_A(\mathbb{K})\) is defined by \((\pi_l \times \pi_r*)((X \xrightarrow{i} Y \xrightarrow{d} Z) = ([X], [Z])\). Note that \((\pi_l \times \pi_r*)_s = \pi_l* \times \pi_r*.

### 3 Hall Algebras

#### 3.1 Constructible sets of stratified Krull-Schmidt

These definitions are related to [3].

**Definition 3.1.** Let \(\mathcal{O}_1\) and \(\mathcal{O}_2\) be two constructible subsets of \(\text{Obj}_A(\mathbb{K})\), the direct sum of \(\mathcal{O}_1\) and \(\mathcal{O}_2\) is

\[
\mathcal{O}_1 \oplus \mathcal{O}_2 = \{[X_1 \oplus X_2] \mid [X_1] \in \mathcal{O}_1, [X_2] \in \mathcal{O}_2 \text{ and } X_1, X_2 \in \text{Obj}(A)\}.
\]

Let \(n\mathcal{O}\) denote the direct sum of \(n\) copies of \(\mathcal{O}\) for \(n \in \mathbb{N}^+\) and 0\(\mathcal{O} = \{[0]\}\). Similarly, let \(n\mathcal{X}\) denote the direct sum of \(n\) copies of \(\mathcal{X} \in \text{Obj}(A)\). A constructible subset \(\mathcal{O}\) of \(\text{Obj}_A(\mathbb{K})\) is called indecomposable if \(\mathcal{X} \in \text{Obj}(A)\) is indecomposable and \(\mathcal{X} \not\cong 0\) for every \([\mathcal{X}] \in \mathcal{O}\).

A constructible set \(\mathcal{O}\) is called to be of Krull-Schmidt if

\[
\mathcal{O} = n_1\mathcal{O}_1 \oplus n_2\mathcal{O}_2 \oplus \ldots \oplus n_k\mathcal{O}_k,
\]

where \(\mathcal{O}_i\) are indecomposable constructible sets and \(n_i \in \mathbb{N}\) for \(i = 1, \ldots, k\). If a constructible set \(\mathcal{Q} = \bigoplus_{i=1}^{n} \mathcal{Q}_i\), where \(\mathcal{Q}_i\) are constructible sets of Krull-Schmidt for \(1 \leq i \leq n\), namely \(\mathcal{Q}\) is a disjoint union of finitely many constructible sets of Krull-Schmidt, then \(\mathcal{Q}\) is said to be a constructible set of stratified Krull-Schmidt.

Let \(\mathcal{O}_1\) and \(\mathcal{O}_2\) be two indecomposable constructible sets. If \(\mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset\) and \(\mathcal{O}_1 \neq \mathcal{O}_2\), we have

\[
\mathcal{O}_1 \oplus \mathcal{O}_2 = 2(\mathcal{O}_1 \cap \mathcal{O}_2) \oplus \left(\bigoplus \left(\mathcal{O}_1 \setminus (\mathcal{O}_1 \cap \mathcal{O}_2)\right) \oplus \left(\mathcal{O}_2 \setminus (\mathcal{O}_1 \cap \mathcal{O}_2)\right)\right).
\]

If \(\mathcal{Q} = m_1\mathcal{O}_1 \oplus \ldots \oplus m_l\mathcal{O}_l\) is a constructible set of Krull-Schmidt, we can write \(\mathcal{Q} = \bigoplus_{i=1}^{l} \mathcal{Q}_i\) as a constructible set of stratified Krull-Schmidt, where

\[
\mathcal{Q}_i = n_{i1}\mathcal{O}_{i1} \oplus n_{i2}\mathcal{O}_{i2} \oplus \ldots \oplus n_{ik_i}\mathcal{O}_{ik_i}
\]

for indecomposable constructible sets \(\mathcal{O}_{ij}\) which are disjoint each other. Hence we can assume that \(\mathcal{O}_{1}, \ldots, \mathcal{O}_{i}\) are disjoint each other.

Let \(\text{CF}^{KS}(\text{Obj}_A)\) be the subspace of \(\text{CF}(\text{Obj}_A)\) which is spanned by characteristic functions \(1_{\mathcal{O}}\) for constructible sets of stratified Krull-Schmidt \(\mathcal{O}\), where each \(1_{\mathcal{O}}\) satisfies that \(1_{\mathcal{O}}([X]) = 1\) for \([X] \in \mathcal{O}\), and \(1_{\mathcal{O}}([X]) = 0\) otherwise.
Example 3.2. Let $\mathbb{P}^1$ be the projective line over $\mathbb{K}$ and $\text{coh}(\mathbb{P}^1)$ denote the category of coherent sheaves on $\mathbb{P}^1$.

Let $O(n)$ denote an indecomposable locally free coherent sheaf whose rank and degree are equal to 1 and $n$ respectively. Let $S_x^{[r]}$ be an indecomposable torsion sheaf such that $\text{rk}(S_x^{[r]}) = 0$, $\deg(S_x^{[r]}) = r$ and the support of $S_x^{[r]}$ is $\{x\}$ for $x \in \mathbb{P}^1$. The Grothendieck group $K_0(\text{coh}(\mathbb{P}^1)) \cong \mathbb{Z}^2$. The data $K(\text{coh}(\mathbb{P}^1))$ and $\mathcal{F}_{\text{coh}(\mathbb{P}^1)}$ are defined in [S] Example 9.1. The set of isomorphism classes of indecomposable objects in $\text{coh}(\mathbb{P}^1)$ is

$$\{[S_x^{[d]}] \mid x \in \mathbb{P}^1, d \in \mathbb{N}\} \cup \{[O(n)] \mid n \in \mathbb{Z}\}.$$ 

Recall that a non-trivial subset $U \subset \mathbb{P}^1$ is closed (resp. open) if $U$ is a finite (resp. cofinite) set. Let $O_d$ be a finite or cofinite subset of $\{[S_x^{[d]}] \mid x \in \mathbb{P}\}$ for each $d \in \mathbb{Z}^+$ and $O_0$ a finite subset of $\{[O(n)] \mid n \in \mathbb{Z}\}$. Then $O_d$ and $O_0$ are indecomposable constructible subsets of $\mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{K})$. Note that every indecomposable constructible subset of $\mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{K})$ is of the form

$$O_0 \sqcup O_{i_1} \sqcup \ldots \sqcup O_{i_n}$$

for $1 \leq i_1 < \ldots < i_n$. Then the finite direct sum $\bigoplus(O_0 \sqcup O_{i_1} \sqcup \ldots \sqcup O_{i_n})$ is a constructible set of Krull-Schmidt. Every constructible set of Krull-Schmidt in $\mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{K})$ is of the form. A constructible set of stratified Krull-Schmidt is a disjoint union of finitely many constructible sets of Krull-Schmidt.

Example 3.3. In Example 2.10 $\mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{C})$ is the set of all isomorphism classes of project complexes in $\mathcal{P}^b(Q, \underline{m})$. Note that

$$\mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{C}) = \bigoplus_{[m] \in K^b(\mathcal{P}^b(Q))} \mathcal{O}_{\text{coh}(\mathbb{P}^1)}([m]).$$

There is a canonical map

$$p_{\underline{m}} : \mathcal{P}^b(Q, \underline{m}) \to \mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{C})$$

which maps $P^\bullet$ to $[P^\bullet]$. A subset $U \subseteq \mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{C})$ is closed (resp. open) if $p_{\underline{m}^{-1}}(U)$ is closed (resp. open) in $\mathcal{P}^b(Q, \underline{m})$. A subset $V_{\underline{m}} \subseteq \mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{C})$ is locally closed if it is an intersection of a closed subset and an open subset of $\mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{C})$. A subset $O \subseteq \mathcal{O}_{\text{coh}(\mathbb{P}^1)}(\mathbb{C})$ is constructible if it is a finite disjoint union of locally closed sets $V_{\underline{m}}$. Every indecomposable constructible set $O$ is of the form $\bigcup_{[m] \in S} V_{[m]}$, where $S$ is a finite set and each complex in $p_{\underline{m}}^{-1}(V_{[m]})$ is an indecomposable complex.

3.2 Automorphism groups of conflations

For each $X \in \text{Obj}(A)$, suppose that $X = n_1X_1 \oplus n_2X_2 \oplus \ldots \oplus n_tX_t$, where $X_i$ are indecomposable for $i = 1, \ldots, t$ and $X_i \not\cong X_j$ for $i \neq j$. Then we have

$$\text{Aut}(X) \cong (1 + \text{rad End}(A)) \times \sum_{i=1}^t \text{GL}(n_i, \mathbb{K}).$$
The rank of maximal torus of $\text{Aut}(X)$ is denoted by $\text{rk} \, \text{Aut}(X)$. Let $n = n_1 + n_2 + \ldots + n_t$. Thus the number of indecomposable direct summands of $X$ is $n$, which is denoted by $\gamma(X)$. Note that $\gamma(X) = \text{rk} \, \text{Aut}(X)$. Let

$$\gamma(O) = \max \{ \gamma(X) \mid [X] \in O \}$$

for each constructible set $O$ in $\mathcal{Db}_A(\mathbb{K})$.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$ and $\text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$ denote the group of $(a_1, a_2, a_3)$ for $a_1 \in \text{Aut}(X), a_2 \in \text{Aut}(Y)$ and $a_3 \in \text{Aut}(Z)$ such that the following diagram is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \xrightarrow{g} Z \\
\downarrow{a_1} & & \downarrow{a_3} \\
X & \xrightarrow{f} & Y \xrightarrow{g} Z \\
\end{array}
$$

The homomorphism

$$p_1 : \text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \to \text{Aut}(Y)$$

is defined by $(a_1, a_2, a_3) \mapsto a_2$. If $p_1((a_1, a_2, a_3)) = p_1((a_1', a_2, a_3'))$ then $f(a_1 - a_1') = 0$ and $(a_3 - a_3')g = 0$. We have $a_1 = a_1'$ and $a_3 = a_3'$ since $f$ is an inflation and $g$ a deflation. Hence $p_1$ is an injective homomorphism of affine algebraic $\mathbb{K}$-groups and

$$\text{rk}(\text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)) = \text{rk} \, \text{Im} p_1 \leq \text{rk} \, \text{Aut}(Y) \quad (2)$$

Let

$$p_2 : \text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \to \text{Aut}(X) \times \text{Aut}(Z)$$

be a homomorphism given by $(a_1, a_2, a_3) \mapsto (a_1, a_3)$. If $p_2((a_1, a_2, a_3)) = p_2((a_1', a_2', a_3))$, then $(a_2 - a_2')f = 0$ and $g(a_2 - a_2') = 0$, we have

$$a_2 - a_2' \in (\text{Hom}(Z, Y)g) \cap (f \text{Hom}(Y, X)).$$

Observe that $\ker p_2$ is a linear space. It follows that $\chi(\ker p_2) = 1$ and

$$\text{rk} \, \text{Im} p_2 \leq \text{rk} \, \text{Aut}(X) + \text{rk} \, \text{Aut}(Z). \quad (3)$$

Two conflations $X \xrightarrow{i} Y \xrightarrow{d} Z$ and $X' \xrightarrow{i'} Y \xrightarrow{d'} Z'$ in $\mathcal{A}$ are said to be equivalent if there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y \xrightarrow{d} Z \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{i'} & Y \xrightarrow{d'} Z'
\end{array}
$$
where both \( f \) and \( g \) are isomorphisms. If the two conflations are equivalent, we write \( X \xrightarrow{i} Y \xrightarrow{d} Z \sim X' \xrightarrow{i'} Y' \xrightarrow{d'} Z' \). The equivalence class of \( X \xrightarrow{i} Y \xrightarrow{d} Z \) is denoted by \( (X \xrightarrow{i} Y \xrightarrow{d} Z) \). Define

\[
V(O_1, O_2; Y) = \{(X \xrightarrow{i} Y \xrightarrow{d} Z) \mid X \xrightarrow{i} Y \xrightarrow{d} Z \in \mathcal{S}, [X] \in O_1, [Z] \in O_2\},
\]

where \( \mathcal{S} \) is the collection of all conflations of \( \mathcal{A} \).

### 3.3 Associative algebras and Lie algebras

For \( f, g \in \text{CF}(\text{Obj}_\mathcal{A}) \), define \( f \cdot g \) by \( (f \cdot g)([X], [Y]) = f([X])g([Y]) \) for \( ([X], [Y]) \in \text{Obj}_\mathcal{A}(\mathbb{K}) \times \text{Obj}_\mathcal{A}(\mathbb{K}) \). Thus \( f \cdot g \in \text{CF}(\text{Obj}_\mathcal{A} \times \text{Obj}_\mathcal{A}) \). The pushforward of \( \pi_m \) is well-defined since \( p_1 \) is injective. The following definition of multiplication is taken from [10, Definition 4.1].

**Definition 3.4.** Using the following diagram

\[
\text{Obj}_\mathcal{A} \times \text{Obj}_\mathcal{A} \xleftarrow{\pi_l \times \pi_r} \text{Exfact}_\mathcal{A} \xrightarrow{\pi_m} \text{Obj}_\mathcal{A},
\]

we can define the convolution multiplication

\[
\text{CF}(\text{Obj}_\mathcal{A} \times \text{Obj}_\mathcal{A}) \xrightarrow{(\pi_l \times \pi_r)^*} \text{CF}(\text{Exfact}_\mathcal{A}) \xrightarrow{(\pi_m)_!} \text{CF}(\text{Obj}_\mathcal{A}).
\]

The multiplication \(* : \text{CF}(\text{Obj}_\mathcal{A}) \times \text{CF}(\text{Obj}_\mathcal{A}) \to \text{CF}(\text{Obj}_\mathcal{A})\) is a bilinear map defined by

\[
f * g = (\pi_m)_![(\pi_l \times \pi_r)^*(f \cdot g)] = (\pi_m)_![(\pi_l^* f) \cdot (\pi_r^* g)].
\]

Let \( O_1 \) and \( O_2 \) be constructible subsets of \( \text{Obj}_\mathcal{A}(\mathbb{K}) \), the meaning of \( 1_{O_1} * 1_{O_2} \) can be understood as follows. The function \( m_{\pi_m} : \text{Exfact}_\mathcal{A}(\mathbb{K}) \to \mathbb{Q} \), which is defined by

\[
m_{\pi_m}([X \xrightarrow{\ell} Y \xrightarrow{\ell_0} Z]) = \chi[\text{Aut}(Y)/p_1(\text{Aut}(X \xrightarrow{\ell} Y \xrightarrow{\ell_0} Z))],
\]

is a locally constructible function on \( \text{Exfact}_\mathcal{A}(\mathbb{K}) \) by [10, Proposition 4.16], namely \( m_{\pi_m} \mid \mathcal{O} \) is a constructible function on \( \mathcal{O} \) for every constructible subset \( \mathcal{O} \subseteq \text{Exfact}_\mathcal{A}(\mathbb{K}) \).

For each \([Y] \in \text{Obj}_\mathcal{A}(\mathbb{K})\),

\[
1_{O_1} * 1_{O_2}([Y]) = \sum_{c \in \Lambda^Y_{O_1, O_2}} c \chi^{na}(Q_c),
\]

where

\[
\Lambda^Y_{O_1, O_2} = \{ c = m_{\pi_m}([A \xrightarrow{\ell} Y \xrightarrow{\ell_0} B]) \mid [A] \in O_1, [B] \in O_2 \} \setminus \{0\}
\]

is a finite set, and

\[
Q_c = \{ [A \xrightarrow{\ell} Y \xrightarrow{\ell_0} B] \mid [A] \in O_1, [B] \in O_2, m_{\pi_m}([A \xrightarrow{\ell} Y \xrightarrow{\ell_0} B]) = c \}
\]

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are constructible sets for $c \in \Lambda^Y_{\mathcal{O}_1,\mathcal{O}_2}$. In fact, the 1-morphism $\pi_1 \times \pi_2$ is of finite type by [8 Theorem 8.4]. Hence $(\pi_2 \times \pi_1)^{-1}(\mathcal{O}_1 \times \mathcal{O}_2)$ is a constructible subset of $\mathcal{E}_{\text{fct}}_A$. Then

$$\Lambda^Y_{\mathcal{O}_1,\mathcal{O}_2} = m_{\pi_m} \left[ \left( (\pi_2 \times \pi_1)^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) \right) \cap \left( (\pi_2)^{-1}(\mathcal{O}_2) \right) \right] \setminus \{0\}$$

is a finite set by [9 Proposition 4.6]. Therefore

$$Q_c = m_{\pi_m}^{-1}(c) \cap \left[ \left( (\pi_2 \times \pi_1)^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) \right) \cap \left( (\pi_2)^{-1}(\mathcal{O}_2) \right) \right] \setminus \{0\}$$

are constructible for all $c \in \Lambda^Y_{\mathcal{O}_1,\mathcal{O}_2}$.

For each $([X],[Z]) \in \mathcal{O}_1 \times \mathcal{O}_2$, let

$$\Lambda^Y_{X,Z} = \{ c = m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) \mid [X \xrightarrow{f} Y \xrightarrow{g} Z] \in \mathcal{E}_{\text{fct}}_A(\mathbb{K}) \}$$

and

$$Q_{X,Z}^c = \{ [X \xrightarrow{f} Y \xrightarrow{g} Z] \mid m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) = c \},$$

where $\Lambda^Y_{X,Z}$ is a finite set and $Q_{X,Z}^c$ are constructible sets for all $c \in \Lambda^Y_{X,Z}$. Then

$$(1_{[X]} \ast 1_{[Z]})([Y]) = \sum_{c \in \Lambda^Y_{X,Z}} c\chi^{na}(Q_{X,Z}^c).$$

Let

$$\pi_1 : V(\mathcal{O}_1,\mathcal{O}_2; Y) \to \bigcup_{c \in \Lambda^Y_{X,Z}} Q_c$$

be a morphism given by $(X \xrightarrow{f} Y \xrightarrow{g} Z) \mapsto ([X \xrightarrow{f} Y \xrightarrow{g} Z])$. For each fibre of $\pi_1$, $\chi^{na}(\pi_1^{-1}([X \xrightarrow{f} Y \xrightarrow{g} Z])) = \chi \left( \text{Aut}(Y)/p_1 \left( \text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \right) \right)$. The set

$$\left\{ \chi \left( \text{Aut}(Y)/p_1 \left( \text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \right) \right) \mid [X \xrightarrow{f} Y \xrightarrow{g} Z] \in \bigcup_{c \in \Lambda^Y_{X,Z}} Q_c \right\}$$

is finite since $\chi(\text{Aut}(Y)/\text{Imp}_1) = m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z])$.

If $U \subseteq V(\mathcal{O}_1,\mathcal{O}_2; Y)$ is a constructible set, then

$$\chi^{na}(U) = \sum_{c} c\chi^{na}(P_c), \quad (4)$$

where $P_c = \{ [X \xrightarrow{f} Y \xrightarrow{g} Z] \mid [X \xrightarrow{f} Y \xrightarrow{g} Z] \in U, m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) = c \}$. Consequently, we have the naïve Euler characteristics of $V([X],[Z]; Y)$ and $V(\mathcal{O}_1,\mathcal{O}_2; Y)$.

**Lemma 3.5.** Let $X, Y, Z \in \text{Obj}(A)$ and $\mathcal{O}_1, \mathcal{O}_2$ be constructible sets. Then

$$\chi^{na}(V([X],[Z]; Y)) = \sum_{c \in \Lambda^Y_{X,Z}} c\chi^{na}(Q_{X,Z}^c) = 1_{[X]} \ast 1_{[Z]}([Y]),$$

$$\chi^{na}(V(\mathcal{O}_1,\mathcal{O}_2; Y)) = \sum_{c \in \Lambda^Y_{\mathcal{O}_1,\mathcal{O}_2}} c\chi^{na}(Q_c) = 1_{\mathcal{O}_1} \ast 1_{\mathcal{O}_2}([Y]).$$
The following result is due to [3, Proposition 6] and [10, Theorem 4.3].

**Theorem 3.6.** The $\mathbb{Q}$-space $\text{CF}(\text{Obj}_A)$ is an associative $\mathbb{Q}$-algebra, with convolution multiplication $*$ and identity $1_0$, where $1_0$ is the characteristic function of $[0] \in \text{Obj}_{\mathbb{A}}(\mathbb{K})$.

*Proof.* The proof of the theorem is quite similar to that in [10, Theorem 4.3] and so is omitted. \hfill $\square$

Joyce defined $\text{CF}^{\text{ind}}(\text{Obj}_A)$ to be the subspace of $\text{CF}(\text{Obj}_A)$ such that if $f([X]) \neq 0$ then $X$ is an indecomposable object in $\mathcal{A}$ for every $f \in \text{CF}^{\text{ind}}(\text{Obj}_A)$. There is a result of [3, Theorem 13] and [10, Theorem 4.9].

**Theorem 3.7.** The $\mathbb{Q}$-space $\text{CF}^{\text{ind}}(\text{Obj}_A)$ is a Lie algebra under the Lie bracket $[f,g] = f * g - g * f$ for $f, g \in \text{CF}^{\text{ind}}(\text{Obj}_A)$.

*Proof.* The proof is the same as the one used in [10, Theorem 4.9]. \hfill $\square$

### 3.4 The algebra $\text{CF}^{KS}(\text{Obj}_A)$

**Lemma 3.8.** Let $O_1$ and $O_2$ be two constructible subsets of $\text{Obj}_{\mathbb{A}}(\mathbb{K})$. For any $Y \in \text{Obj}(A)$, if $1_{O_1} \ast 1_{O_2}([Y]) \neq 0$, then there exist $X, Z \in \text{Obj}(A)$ such that $[X] \in O_1$, $[Z] \in O_2$ and $1_{[X]} \ast 1_{[Z]}([Y]) \neq 0$.

*Proof.* Let $Q_c$ and $\Lambda_{\mathcal{O}_1, \mathcal{O}_2}$ be as in Section 3.3. Let

$$\pi_2 : \bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c \to (\pi_{1c} \times \pi_{r*})(\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c)$$

be a map which maps $[X \to Y \xrightarrow{\Delta} Z]$ to $([X], [Z])$ and

$$m_m = m_{\pi_m}|_{\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c}.$$ 

It is easy to see that $m_m$ is a constructible function over $\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c$.

Because $\pi_l \times \pi_r$ is a 1-morphism, $\pi_2$ is a pseudomorphism by [9, Proposition 4.6]. Thus $\pi_2(\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c)$ is constructible and the naïve pushforward $(\pi_2)^{na}(m_m)$ of $m_m$ to $\pi_2(\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c)$ exists. Note that $(\pi_2)^{na}(m_m)$ is a constructible function on $\pi_2(\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c)$. In fact

$$(\pi_2)^{na}(m_m)([X],[Z]) = 1_{[X]} \ast 1_{[Z]}([Y])$$

for all $([X],[Z]) \in \pi_2(\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c)$. Therefore

$$\{1_{[X]} \ast 1_{[Z]}([Y]) \mid ([X],[Z]) \in \pi_2(\bigcup_{c \in \Lambda_{\mathcal{O}_1, \mathcal{O}_2}} Q_c)\}$$

is a finite set.
Lemma 3.9. Let \( \{([X_1], [Z_1]), \ldots, ([X_n], [Z_n])\} \) be a complete set of representatives for \( ([X], [Z]) \in \pi_2(\bigcup_{c \in A_{\gamma_1, \alpha_2}^c} Q_c) \) such that

\[
1_{[X_i]} \ast 1_{[Z_i]}([Y]) \neq 1_{[X_j]} \ast 1_{[Z_j]}([Y])
\]

for \( i \neq j \). Set

\[
p^{X_k, Z_k} = \left\{ ([A], [B]) \in \pi_2 \left( \bigcup_{c \in A_{\gamma_1, \alpha_2}^c} Q_c \right) \mid 1_{[A]} \ast 1_{[B]}([Y]) = 1_{[X_k]} \ast 1_{[Z_k]}([Y]) \right\}
\]

for all \( 1 \leq k \leq n \). Then \( p^{X_k, Z_k} \) are constructible sets for all \( 1 \leq k \leq n \) since

\[
p^{X_k, Z_k} = \left( (\pi_2)^n(m_m) \right)^{-1}(c_k),
\]

where \( c_k = (\pi_2)^n(m_m)([X_k], [Z_k]) \). Let \( \pi = \pi_2 \circ \pi_1 \) which maps \( \langle X \to Y \xrightarrow{\gamma} Z \rangle \) to \( ([X], [Z]) \). For each \( ([X], [Z]) \in \pi_2(\bigcup_{c \in A_{\gamma_1, \alpha_2}^c} Q_c) \),

\[
\chi^{(\pi^{-1}([X], [Z])))} = \chi^{(\pi([X], [Z])))} = \chi^{(V([X_k], [Z_k]; Y))}
\]

for some \( k \). According to Lemma 2.5 we have

\[
1_{O_1} \ast 1_{O_2}([Y]) = \chi^{(V(O_1, O_2; Y))} = \sum_{k=1}^n \chi^{(V([X_k], [Z_k]; Y))} \cdot \chi^{(p^{X_k, Z_k})}
\]

\[
= \sum_{k=1}^n (1_{[X_k]} \ast 1_{[Z_k]})([Y]) \cdot \chi^{(p^{X_k, Z_k})}.
\]

There exists \( ([X_k], [Z_k]) \) for some \( k \in \{1, \ldots, n\} \) such that \( 1_{[X_k]} \ast 1_{[Z_k]}([Y]) \neq 0 \) since \( 1_{O_1} \ast 1_{O_2}([Y]) \neq 0 \). \( \square \)

Let \( D_n(K) \) denote the group of invertible diagonal matrices in \( GL(n, K) \).

The following lemma is related to Riedtmann [18, Lemma 2.2].

**Lemma 3.9.** Let \( X, Y, Z \in \text{Obj}(\mathcal{A}) \) and \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be a conflation in \( \mathcal{A} \). If \( m_{\gamma_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) \neq 0 \), then \( \gamma(Y) \leq \gamma(X) + \gamma(Z) \). In particular, \( \gamma(Y) = \gamma(X) + \gamma(Z) \) if and only if \( Y \cong X \oplus Z \).

**Proof.** Recall that \( m_{\gamma_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) = \chi(\text{Aut} Y / \text{Im}(p_1)) \).

If \( \text{rk Aut}(Y) > \text{rk Im}(p_1) \), then the fibre of the action of a maximal torus of \( \text{Aut}(Y) \) on \( \text{Aut} Y / \text{Im}(p_1) \) is \( (K^*)^k \) for some \( k \geq 1 \), it forces \( \chi(\text{Aut} Y / \text{Im}(p_1)) = 0 \). Hence we have \( \text{rk Aut}(Y) = \text{rk Im}(p_1) \leq \text{rk Aut}(X) + \text{rk Aut}(Z) \).

We prove the second assertion by induction. First of all, suppose that \( X \not\cong 0 \) and \( Z \not\cong 0 \). If \( \text{rk Aut}(Y) = 2 \) and \( Y = Y_1 \oplus Y_2 \), then \( \text{rk Aut}(X) = \text{rk Aut}(Z) = 1 \) since \( X \) and \( Z \) are not isomorphic to \( 0 \). For \( t \in K^* \setminus \{1\} \), \( \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \in \text{Aut}(Y) \) and it is an element of a maximal torus \( D_2(K) \) of \( \text{Aut}(Y) \). A maximal torus of
Im(\(p_1\)) is also a maximal torus of Aut(\(Y\)) since \(\text{rk } \text{Aut}(Y) = \text{rk } \text{Im}(p_1)\). Because two maximal tori of a connected linear algebraic group are conjugate, there exists \(\alpha \in \text{Aut}(Y)\) such that \(\alpha \left( \begin{array}{cc} t & 0 \\ 0 & t^2 \end{array} \right) \alpha^{-1}\) lies in a maximal torus of \(\text{Im}(p_1)\). Hence there exist \(a \in \text{Aut}(X)\) and \(b \in \text{Aut}(Z)\) satisfying \((a, \alpha \left( \begin{array}{cc} t & 0 \\ 0 & t^2 \end{array} \right) \alpha^{-1}, b) \in\) Aut\((X \xrightarrow{f} Y \xrightarrow{g} Z)\), namely

\[
(a, \left( \begin{array}{cc} t & 0 \\ 0 & t^2 \end{array} \right), b) \in \text{Aut}(X \xrightarrow{\alpha^{-1} f} Y \xrightarrow{g} Z).
\]

Let \(f' = \alpha^{-1} f\) and \(g' = g \alpha\). Observe \((t, \left( \begin{array}{cc} t & 0 \\ 0 & t \end{array} \right), t) \in \text{Aut}(X \xrightarrow{f'} Y \xrightarrow{g'} Z)\).

Hence \(f'(a - t) = \left( \begin{array}{cc} 0 & 0 \\ 0 & t^2 - t \end{array} \right) f'\). Let \(s = \frac{1}{t^2 - t}(a - t) \in \text{End}(X)\) \((t \neq 0, 1)\).

Then \(f's = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) f'\). Because \(f'\) is an inflation and

\[s^2 = s\]. The category \(A\) is idempotent completion, consequently \(s\) has a kernel and an image such that \(X = \text{Ker}s \oplus \text{Im}s\). But \(X\) is indecomposable, without loss of generality we can assume \(X = \text{Ker}s\). Then \(s = 0\). Let \(f' = \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right)\) and \(g' = (g_1, g_2)\). It follows that

\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right) = f's = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = \left( \begin{array}{c} 0 \\ f_2 \end{array} \right).
\]

We have \(f_2 = 0\) and \(f' = \left( \begin{array}{c} f_1 \\ 0 \end{array} \right)\). The morphism \(Y_1 \oplus Y_2 \xrightarrow{(0,1)} Y_2\) is a deflation by \([2]\) Lemma 2.7. Because \((0,1)\left( \begin{array}{c} f_1 \\ 0 \end{array} \right) = 0\), there exists \(h \in \text{Hom}(Z, Y_1)\) such that \((0,1) = h(g_1, g_2)\). We have \(h g_1 = 0\) and \(h g_2 = 1_{Y_2}\). Observe \(g_2 h \in \text{End}(Z)\) and \((g_2 h)(g_2 h) = g_2 h\), so \(g_2 h\) has a kernel \(k : K \to Z\) and an image \(i : I \to Z\). Moreover \(Z \cong K \oplus I\). It follows that \(Z \cong K\) or \(Z \cong I\) since \(Z\) is indecomposable. If \(Z \cong K\) then \(g_2 h = 0\). But \(h g_2 h = h\), \(K = 0\). Thus \(h\) is an isomorphism and \(g_1 = 0\). We have \(Z \cong Y_2\). Similarly \(X \cong Y_1\). Hence \(X \oplus Z \cong Y_1 \oplus Y_2\).

Assume that the assertion is true for \(\text{rk } \text{Aut}(Y) = n < N\). When \(n = N\), we can assume \(\text{rk } \text{Aut}(X) = n_1\) where \(0 < n_1 < N\), then \(\text{rk } \text{Aut}(Z) = N - n_1 = n_2\).

Let \(Y = Y' \oplus Y_N\) and \(Y' = Y_1 \oplus \ldots \oplus Y_{N-1}\), where \(Y_i\) are indecomposable.

Observe that \(\left( \begin{array}{cc} tI_{N-1} & 0 \\ 0 & t^2 \end{array} \right)\) lies in a maximal torus of Aut\((Y)\) for \(t \in K^* \setminus \{1\}\).

There exists \((a, c, b) \in \text{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)\) such that \(c\) and \(\left( \begin{array}{cc} tI_{N-1} & 0 \\ 0 & t^2 \end{array} \right)\) are conjugate in Aut\((Y)\). For simplicity we assume \(c = \left( \begin{array}{cc} tI_{N-1} & 0 \\ 0 & t^2 \end{array} \right)\). So we
have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y' \oplus Y_N \\
& a & \downarrow c \\
X & \xrightarrow{f} & Y' \oplus Y_N \xrightarrow{g} Z \\
& b & \\
\end{array}
\]

where \( f = (f_1, f_2, \ldots, f_N)^t \) and \( g = (g_1, g_2, \ldots, g_N) \).

There is another commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(f', f_N)^t} & Y' \oplus Y_N \xrightarrow{(g^*, g_N)} Z \\
& tI_{n_1} & \downarrow tI_N \\
X & \xrightarrow{(f', f_N)^t} & Y' \oplus Y_N \xrightarrow{(g^*, g_N)} Z \\
& tI_{n_2} & \\
\end{array}
\]

where \( f^* = (f_1, f_2, \ldots, f_{N-1})^t \) and \( g^* = (g_1, g_2, \ldots, g_{N-1}) \). Then \( f = (f^*, f_N)^t, g = (g^*, g_N) \) and \( f(a - tI_{n_1}) = \begin{pmatrix} 0I_{N-1} & 0 \\ 0 & t^2 - t \end{pmatrix} f \). Let

\[
s_N = \frac{1}{t^2 - t}(a - tI_{n_1}).
\]

Then \( fs_N = \text{diag}(0, \ldots, 0, 1) f \). It follows \( f^*s_N = 0, f_Ns_N = f_N \) and \( g_Nf_N = g \begin{pmatrix} 0I_{N-1} & 0 \\ 0 & 1 \end{pmatrix} f = gfs_N = 0 \). Moreover \( s_N \) is an idempotent, we know that \( X = \ker s_N \oplus \im s_N \). If \( f_N \neq 0 \) then \( \im s_N \) is not isomorphic to \( 0 \). Similarly we can define \( s_1, s_2, \ldots, s_{N-1} \in \text{End}(X) \) with the property that \( fs_i = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0) f = (0, \ldots, 0, f_i, 0, \ldots, 0)^t \). Hence \( s_i \) is idempotent and if \( f_i \neq 0 \) then \( \im s_i \) is not isomorphic to \( 0 \) for each \( i \). Note that \( s_1 + s_2 + \ldots + s_N = 1_X \in \text{Aut}(X) \), it follows

\[
X = \im s_1 \oplus \ldots \oplus \im s_N.
\]

Hence \( f_i = 0 \) for some \( i \) since \( \text{rk} \text{Aut}(X) < N \). Without loss of generality, we assume \( f_N = 0 \). Let \( (0, \ldots, 0, 1) : Y_1 \oplus \ldots \oplus Y_N \to Y_N \), then

\[
(0, \ldots, 0, 1)(f_1, \ldots, f_N)^t = 0.
\]

Hence there exists \( h \in \text{Hom}(Z, Y_N) \) such that \( h(g_1, \ldots, g_N) = (0, \ldots, 0, 1) \), namely \( hg_1 = 0, \ldots, hg_{N-1} = 0 \) and \( hg_N = 1 \). Therefore \( Y_N \) is isomorphic to a direct summand of \( Z \). Assume that \( Z = Z' \oplus Y_N \) where \( \gamma(Z') = \gamma(Z) - 1 \). The morphism \((1, 0) : Z' \oplus Y_N \to Z' \) is a deflation, so \( g' = g^*(1, 0) : Y' \to Z' \) is a deflation by Definition A.1. Obviously, \((f_1, \ldots, f_{N-1})^t : X \to Y_1 \oplus \ldots \oplus Y_{N-1} \) is a kernel of \( g' \). Thus

\[
X \xrightarrow{(f_1, \ldots, f_{N-1})^t} Y_1 \oplus \ldots \oplus Y_{N-1} \xrightarrow{g'} Z'
\]

is a conflation. By hypothesis, \( Y_1 \oplus \ldots \oplus Y_{N-1} \cong X \oplus Z' \). Hence \( Y = Y_1 \oplus \ldots \oplus Y_N \cong X \oplus Z \). The proof is completed. \( \square \)
Remark 3.10. If \( 1_{[X]} \ast 1_{[Z]}([Y]) \neq 0 \), then \( \gamma(Y) \leq \gamma(X) + \gamma(Z) \), where the equality holds if and only if \( Y \cong \text{direct summand} \). 

Lemma 3.11. Let \( X, Y, Z \in \text{Obj}(\mathcal{A}) \) and \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be a conflation in \( \mathcal{A} \). If \( m_{\mathcal{A}}_\ast([X] \xrightarrow{f} Y \xrightarrow{g} Z) \neq 0 \), \( \gamma(Y) < \gamma(X) + \gamma(Z) \) and \( Y = \bigoplus_{i} Y_i \), then there exist two conflations \( X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \) such that \( f \cong \bigoplus_{i} f_i \) and \( g \cong \bigoplus_{i} g_i \).

Proof. Suppose that \( \text{rk} \text{Aut}(X) = n_1 \), \( \text{rk} \text{Aut}(X) = N \) and \( \text{rk} \text{Aut}(Z) = n_2 \). Then \( N < n_1 + n_2 \). For simplicity, we use the notation as above. Let \( Y = \bigoplus_{i=1}^n Y_i \), \( f = (f_1, f_2, \ldots, f_n) \), \( g = (g_1, g_2, \ldots, g_n) \) and the isomorphisms \( (a, b), (tI_{n_1}, tI_N, tI_{n_2}) \in \text{Aut}(X) \xrightarrow{f} Y \xrightarrow{g} Z) \), where \( c = \begin{pmatrix} tI_{n-1} & 0 \\ 0 & t^2 \end{pmatrix} \).

Recall that \( s_N = \frac{1}{t^2 - t} (a - tI_{n_1}) \in \text{End}(X) \) is an idempotent such that \( fs_N = (0, \ldots, 0, f_n)^t \) and \( X = \text{Ker} s_N \oplus \text{Im} s_N \). Similarly, there exists an idempotent \( r_N = \frac{1}{t - t^2} (b - tI_{n_2}) \) in \( \text{End}(Z) \) such that \( r_N g = (0, \ldots, 0, g_n) \) and \( Z = \text{Ker} r_N \oplus \text{Im} r_N \). Without loss of generality, we assume that \( f_N \neq 0 \) and \( g_N \neq 0 \). Because \( f_N s_N = f_N \) and \( r_N g_N = g_N \),

\[ g_N f_N = r_N g_N f_N s_N = r_N (g_1, \ldots, g_n) (f_1, \ldots, f_n)^t s_N = 0. \]

It is clear that \( i : \text{Ker} s_N \hookrightarrow X \) is a kernel of \( f_N : X \rightarrow Y_n \). There exists a morphism \( f'_N : \text{Im} s_N \rightarrow Y_n \) which is an image of \( f_N \) since \( X = \text{Ker} s_N \oplus \text{Im} s_N \). Similarly, we can find a morphism \( g'_N : Y_N \rightarrow \text{Im} r_N \) which is a coimage of \( g_N \) such that \( g_N = j g'_N \), where \( j : \text{Im}(r_N) \hookrightarrow Z \) is an image of \( g_N \). It is easy to check that \( f'_N \) is an inflation, \( g'_N \) a deflation and \( g'_N f'_N = 0 \). Let \( h : Y_N \rightarrow A \) be a morphism in \( \mathcal{A} \) such that \( h f'_N = 0 \). The morphism

\[ (0, \ldots, 0, h) : Y_1 \oplus \ldots \oplus Y_n \rightarrow A \]

satisfies \( (0, \ldots, 0, h) f = 0 \). There exists \( k \in \text{Hom}_\mathcal{A}(Z, A) \) such that

\[ (0, \ldots, 0, h) = k g \]

since \( g \) is a cokernel of \( f \). It follows that \( h = k g_N = k j g'_N \). Hence \( g'_N \) is a cokernel of \( f'_N \). Therefore \( \text{Im} s_N \xrightarrow{f'_N} Y_N \xrightarrow{g'_N} \text{Im} r_N \) is a conflation. By induction, every indecomposable direct summand of \( Y \) is extended by the direct summands of \( X \) and \( Z \). The proof is finished. \( \square \)
Lemma 3.12. Let $O_1$ and $O_2$ be two indecomposable constructible subsets of $\text{Obj}_A(\mathbb{K})$. Let $A \in \text{Obj}(A)$ and $\gamma(A) \geq 2$. If $[A] \notin O_1 \oplus O_2$, then $1_{O_1} \ast 1_{O_2}([A]) = 0$.

Proof. If $1_{O_1} \ast 1_{O_2}([A]) \neq 0$, then there exist $X, Y \in \text{Obj}(A)$ such that $[X] \in O_1$, $[Y] \in O_2$ and $1_{[X]} \ast 1_{[Y]}(A) \neq 0$ by Lemma 3.8. It follows that $\gamma(A) = 2$ and $A \cong X \oplus Y$ by Lemma 3.9 (also see [10, Theorem 4.9]). This leads to a contradiction. \hfill \Box

Corollary 3.13. Let $O_1$ and $O_2$ be indecomposable subsets of $\text{Obj}_A(\mathbb{K})$. If $O_1 \cap O_2 = \emptyset$, then

$$1_{O_1} \ast 1_{O_2} = 1_{O_1 \oplus O_2} + \sum_{i=1}^{m} a_i 1_{P_i},$$

where $P_i$ are indecomposable constructible subsets and $a_i = 1_{O_1} \ast 1_{O_2}([X])$ for $[X] \in P_i$.

Proof. Let $[M] \in O_1$ and $[N] \in O_2$. Then $M$ is not isomorphic to $N$ since $O_1 \cap O_2 = \emptyset$. Using the fact that $m_{\pi_m}([M \xrightarrow{(1,0)} M \oplus N \xrightarrow{(0,1)} N]) = 1$, we obtain

$$1_{O_1} \ast 1_{O_2}([M \oplus N]) = m_{\pi_m}([M \xrightarrow{(1,0)} M \oplus N \xrightarrow{(0,1)} N]) \cdot \nu_{\pi_m}([M \xrightarrow{(1,0)} M \oplus N \xrightarrow{(0,1)} N]) = 1.$$

By Lemma 3.12 we know that if $1_{O_1} \ast 1_{O_2}([X]) \neq 0$ and $[X] \notin O_1 \oplus O_2$, then $X$ is an indecomposable object. Note that

$$(1_{O_1} \ast 1_{O_2}(\text{Obj}_A(\mathbb{K}) \setminus (O_1 \oplus O_2))) \setminus \{0\} = \{a_1, a_2, \ldots, a_m\}.$$

Then $P_i = (1_{O_1} \ast 1_{O_2})^{-1}(a_i) \setminus (O_1 \oplus O_2)$ for $1 \leq i \leq m$. We complete the proof. \hfill \Box

Using Lemma 3.9 and Lemma 3.11 it is easy to see the following corollary:

Corollary 3.14. Let $O_1, O_2$ be two constructible sets. There exist finitely many constructible sets $Q_1, Q_2, \ldots, Q_n$ such that

$$1_{O_1} \ast 1_{O_2} = \sum_{i=1}^{n} a_i 1_{Q_i},$$

where $\gamma(Q_i) \leq \gamma(O_1) + \gamma(O_2)$ and $a_i = (1_{O_1} \ast 1_{O_2})([X])$ for any $[X] \in Q_i$.

For indecomposable constructible sets $O_1, \ldots, O_k$ and $X \in \text{Obj}(A)$, $1_{O_1} \ast 1_{O_2} \ast \ldots \ast 1_{O_k}([X]) \neq 0$ implies that $\gamma(X) \leq k$. In particular, $\gamma(X) = k$ means $X = X_1 \oplus \ldots \oplus X_k$ with $[X_i] \in O_i$ for $1 \leq i \leq k$.

Let $X_1, \ldots, X_m \in \text{Obj}(A)$ and there be $r$ isomorphic classes, we can assume that $X_1, \ldots, X_{m_1}$ are isomorphic, $X_{m_1+1}, \ldots, X_{m_2}$ are isomorphic, \ldots,
and \(X_{m-1}, \ldots, X_m\) are isomorphic, where \(m_1 + \ldots + m_r = m\). By [10], we have

\[
\text{Aut}(X_1 \oplus \ldots \oplus X_m)/\text{Aut}(X_1) \times \ldots \times \text{Aut}(X_m) \cong \mathbb{K}^l \times \prod_{i=1}^{r} (\text{GL}(m_i, \mathbb{K})/(\mathbb{K}^*)^{m_i}),
\]

(5)

\[
\chi(\text{Aut}(X_1 \oplus X_2 \oplus \ldots \oplus X_m)/\text{Aut}(X_1) \times \ldots \times \text{Aut}(X_m)) = \prod_{i=1}^{r} m_i!.
\]

(6)

**Proposition 3.15.** Let \(\mathcal{O}\) be an indecomposable constructible set. Then

\[
1^{*k}_{\mathcal{O}} = k!1_{k\mathcal{O}} + \sum_{i=1}^{t} m_i 1_{P_i},
\]

where \(\gamma(P_i) < k\) for each \(i\) and \(m_i = 1^{*k}_{\mathcal{O}}([X])\) for \([X] \in \mathcal{P}_i\).

**Proof.** We prove the proposition by induction. When \(k = 1\), it is easy to see that the formula is true. If \(k = 2\), then

\[
1^{*2}_{\mathcal{O}}([X \oplus Y]) = 1_{\mathcal{O}}([X]) \cdot 1_{\mathcal{O}}([X]) \cdot \chi(\text{Aut}(X \oplus X)/\text{Aut}(X) \times \text{Aut}(X)) = 2
\]

for \([X] \in \mathcal{O}\) and

\[
1^{*2}_{\mathcal{O}}([X \oplus Y]) = (1_{\mathcal{O}}([X]) \cdot 1_{\mathcal{O}}([Y]) + 1_{\mathcal{O}}([Y]) \cdot 1_{\mathcal{O}}([X])) \cdot \chi(\text{Aut}(X \oplus Y)/\text{Aut}(X) \times \text{Aut}(Y)) = 2,
\]

where \([X], [Y] \in \mathcal{O}\) and \(X \neq Y\). If \([X] \notin \mathcal{O} \oplus \mathcal{O}\) and \(\gamma(X) \geq 2\) then \(1^{*2}_{\mathcal{O}}([X]) = 0\) by Lemma 3.12. Hence \(1^{*2}_{\mathcal{O}} = 2 \cdot 1_{\mathcal{O} \oplus \mathcal{O}} + \sum_{i} m_i 1_{P_i}\) where \(P_i\) are indecomposable constructible sets by Corollary 3.14.

Now we suppose that the formula is true for \(k \leq n\). When \(k = n + 1\), we have

\[
1^{*(n+1)}_{\mathcal{O}} = 1^{*(n)}_{\mathcal{O}} \star 1_{\mathcal{O}} = (n!1_{n\mathcal{O}} + \sum c_{P'\mathcal{O}} 1_{P'}) \star 1_{\mathcal{O}},
\]

where \(\mathcal{P}'\) are constructible sets with \(\gamma(\mathcal{P}') < n\). If the formula is true for \(k = n + 1\), then

\[
n!1_{n\mathcal{O}} \star 1_{\mathcal{O}} = (n + 1)!1_{(n+1)\mathcal{O}} + \sum c_{\mathcal{Q}1_{\mathcal{Q}}},
\]

where \(\mathcal{Q}\) are constructible sets with \(\gamma(\mathcal{Q}) < n + 1\). Hence it suffices to show that the initial term of \(1_{n\mathcal{O}} \star 1_{\mathcal{O}}\) is \((n + 1)!1_{(n+1)\mathcal{O}}\), namely \((1_{n\mathcal{O}} \star 1_{\mathcal{O}})([X]) = n + 1\) for all \([X] \in (n + 1)\mathcal{O}\).

Assume that \(X = m_1X_1 \oplus m_2X_2 \oplus \ldots \oplus m_rX_r\), where \(X_1, \ldots, X_r \in \text{Obj}(\mathcal{A})\) which are not isomorphic to each other, \([X_i] \in \mathcal{O}\) for \(1 \leq i \leq r\), and \(m_1, \ldots, m_r\) are positive integers and \(m_1 + m_2 + \ldots + m_r = n + 1\).

\[
(1_{n\mathcal{O}} \star 1_{\mathcal{O}})([X]) = (1_{[(m_1-1)X_1 \oplus m_2X_2 \oplus \ldots \oplus m_rX_r]} \star 1_{[X_1]})([X])
\]
\[ + (1[m_1 X_1 \oplus (m_2 - 1)X_2 \oplus \cdots \oplus m_r X_r] \ast 1[X_2])([X]) \]
\[ + \cdots \]
\[ + (1[m_1 X_1 \oplus \cdots \oplus m_{r-1} X_{r-1} \oplus (m_r - 1)X_r] \ast 1[X_r])([X]) \]

Using Equation (6), it follows that
\[ 1^*{(m_1 - 1)}[X_1] \ast 1^*{m_2}[X_2] \ast \cdots \ast 1^*{m_r}[X_r] = (m_1 - 1)!m_2! \cdots m_r!1 \ast [(m_1 - 1)X_1 \oplus m_2 X_2 \oplus \cdots \oplus m_r X_r] + \cdots , \]
\[ 1^*{(m_1 - 1)}[X_1] \ast 1^*{m_2}[X_2] \ast \cdots \ast 1^*{m_r}[X_r] \ast 1[X_1] = (\prod_{i=1}^r m_i！)1 \ast [m_1 X_1 \oplus m_2 X_2 \oplus \cdots \oplus m_r X_r] + \cdots \]

Compare the initial monomials of the two equations, it follows that
\[ 1 \ast [(m_1 - 1)X_1 \oplus m_2 X_2 \oplus \cdots \oplus m_r X_r] \ast 1[X_1] = m_1 \cdot 1 \ast [m_1 X_1 \oplus m_2 X_2 \oplus \cdots \oplus m_r X_r] + \cdots \]

Thus 1 \ast [(m_1 - 1)X_1 \oplus m_2 X_2 \oplus \cdots \oplus m_r X_r] \ast 1[X_1]([X]) = m_1.

Similarly, we have 1 \ast [m_1 X_1 \oplus \cdots \oplus (m_i - 1)X_i \oplus \cdots \oplus m_r X_r] \ast 1[X_i]([X]) = m_i for \( i = 2, \ldots, r \). Hence \((1_{nO} \ast 1_O)([X]) = \sum_{i=1}^r m_i = n + 1 \) which completes the proof. \( \square \)

By induction, we have the following corollary.

**Corollary 3.16.** Let \( O_1, O_2, \ldots, O_k \) be indecomposable constructible sets which are pairwise disjoint. Then we have the following equations
\[ 1^*{m_1}[O_1] \ast 1^*{m_2}[O_2] \ast \cdots \ast 1^*{m_k}[O_k] = n_1!n_2! \cdots n_k!1 \ast [n_1 O_1 \oplus \cdots \oplus n_k O_k] + \cdots , \]
\[ 1 m_1 O_1 \oplus \cdots \oplus m_k O_k \ast 1[n_1 O_1 \oplus \cdots \oplus n_k O_k] = \prod_{i=1}^k \frac{(m_i + n_i)!}{m_i!n_i!}1 \ast [m_1 + n_1 O_1 \oplus \cdots \oplus (m_k + n_k) O_k] + \cdots , \]

where \( k \) is a positive integer and \( m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbb{N} \).

Let Ind(\( \alpha \)) be the subset of \( \text{Obj}^\alpha_{\mathcal{A}}(\mathbb{K}) \) such that \( X \) are indecomposable for all \([X] \in \text{Ind}(\alpha)\).

**Lemma 3.17.** For each \( \alpha \in \mathbb{K}^\prime(\mathcal{A}) \), Ind(\( \alpha \)) is a locally constructible set.

**Proof.** Assume \( \alpha, \beta, \gamma \in \mathbb{K}^\prime(\mathcal{A}) \setminus \{0\} \). The map
\[ f : \prod_{\beta, \gamma} \text{Obj}^\beta_{\mathcal{A}}(\mathbb{K}) \times \text{Obj}^\gamma_{\mathcal{A}}(\mathbb{K}) \to \text{Obj}^\alpha_{\mathcal{A}}(\mathbb{K}) \]
is defined by \( ([B], [C]) \mapsto [B \oplus C] \). It is clear that \( f \) is a pseudomorphism. Every \( \text{Obj}^\beta_{\mathcal{A}}(\mathbb{K}) \times \text{Obj}^\gamma_{\mathcal{A}}(\mathbb{K}) \) is a locally constructible set. For any constructible set \( \mathcal{C} \subseteq \text{Obj}^\beta_{\mathcal{A}}(\mathbb{K}) \times \text{Obj}^\gamma_{\mathcal{A}}(\mathbb{K}) \), there are finitely many \( \text{Obj}^\beta_{\mathcal{A}}(\mathbb{K}) \times \text{Obj}^\gamma_{\mathcal{A}}(\mathbb{K}) \) such that \( \mathcal{C} \cap (\text{Obj}^\beta_{\mathcal{A}}(\mathbb{K}) \times \text{Obj}^\gamma_{\mathcal{A}}(\mathbb{K})) \neq \emptyset \). Hence \( \Pi_{\beta, \gamma: \beta + \gamma = \alpha} \text{Obj}^\beta_{\mathcal{A}}(\mathbb{K}) \times \text{Obj}^\gamma_{\mathcal{A}}(\mathbb{K}) \) is locally constructible. Then Imf is a locally constructible set. It follows that \( \text{Ind}(\alpha) = \text{Obj}^\alpha_{\mathcal{A}}(\mathbb{K}) \setminus \text{Im}f \) is locally constructible. \( \square \)
The following proposition is due to [3] Proposition 11.

**Proposition 3.18.** Let $O_1, O_2$ be two constructible sets of Krull-Schmidt. It follows that

\[ 1_{O_1} * 1_{O_2} = \sum_{i=1}^{c} a_i 1_{Q_i} \]

for some $c \in \mathbb{N}^+$, where $a_i = 1_{O_1} * 1_{O_2}([X])$ for each $[X] \in Q_i$ and $Q_i$ are constructible sets of stratified Krull-Schmidt such that $\gamma(Q_i) \leq \gamma(O_1) + \gamma(O_2)$.

**Proof.** Because $O_1, O_2$ are constructible sets, the equation holds for some constructible sets $Q_i$ with $\gamma(Q_i) \leq \gamma(O_1) + \gamma(O_2)$ by Corollary 3.14.

For every $[Y] \in Q_i$, $1_{O_1} * 1_{O_2}([Y]) \neq 0$. By Lemma 3.3, there exist $X_1, Z_i \in \text{Obj}(\mathcal{A})$ such that $[X_1] \in O_1$, $[Z_i] \in O_2$ and $1_{[X_1]} * 1_{[Z_i]}([Y]) \neq 0$ since $1_{O_1} * 1_{O_2}([Y]) \neq 0$. Thanks to Lemma 3.11, we have that $\gamma(Y_i) \leq \gamma(X_i) + \gamma(Z_i)$. According to Lemma 3.11, all indecomposable direct summands of $Y_i$ are extended by the direct summands of $X_i$ and $Z_i$ since $1_{[X_i]} * 1_{[Z_i]}([Y_i]) \neq 0$.

By the discussion in Section 3.1, we can suppose that $O_1 = \bigoplus_{i=1}^t a_i C_i$ and $O_2 = \bigoplus_{j=1}^c b_j C_j$, where $a_i, b_j \in \{0, 1\}$ for all $i, j$ and $C_i$ are indecomposable constructible sets such that $C_i \cap C_j = \emptyset$ or $C_i = C_j$ for all $i \neq j$. Let $1 \leq r \leq t$, let $A_1, A_2, \ldots, A_r \mid \emptyset \neq A_i \subseteq \{1, \ldots, n\}$ for $i = 1, \ldots, r$ be an $r$-partition of $\{1, 2, \ldots, t\}$ if $A_1 \cup A_2 \cup \ldots \cup A_r = \{1, 2, \ldots, t\}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Obviously, the cardinal number of all partitions of $\{1, 2, \ldots, t\}$ is finite. Let $\{A_1, A_2, \ldots, A_r\}, \{B_1, B_2, \ldots, B_r\}$ be two $r$-partitions of $\{1, 2, \ldots, t\}$ and $c_k \in \mathbb{Q} \setminus \{0\}$ for $k = 1, 2, \ldots, r$. Set $O_{A_k} = \bigoplus_{i \in A_k} a_i C_i$ and $O_{B_k} = \bigoplus_{j \in B_k} b_j C_j$ for $1 \leq k \leq r$. Then we have

\[
\mathcal{R}_{A_k, B_k, c_k} = \{[X] \in O_{A_k} \oplus O_{B_k} \mid 1_{O_{A_k}} * 1_{O_{B_k}}([X]) = c_k\},
\]

\[
\mathcal{I}_{A_k, B_k, c_k} = \{[X] \mid X \text{ indecomposable, } 1_{O_{A_k}} * 1_{O_{B_k}}([X]) = c_k\}.
\]

This means that for each $[X] \in \mathcal{R}_{A_k, B_k, c_k}$, there exist $[A] \in O_{A_k}$ and $[B] \in O_{B_k}$ such that $X \cong A \oplus B$. For each $[Y] \in \mathcal{I}_{A_k, B_k, c_k}$, there exist $[C] \in O_{A_k}$ and $[D] \in O_{B_k}$ such that $C \rightarrow Y \rightarrow D$ is a non-split conflation in $\mathcal{A}$. Note that

\[
\mathcal{R}_{A_k, B_k, c_k} = ((1_{O_{A_k}} * 1_{O_{B_k}})^{-1}(c_k)) \cap (O_{A_k} \oplus O_{B_k}).
\]

By Corollary 3.14, $\mathcal{R}_{A_k, B_k, c_k} = \emptyset$ or $O_{A_k} \oplus O_{B_k}$. Hence $\mathcal{R}_{A_k, B_k, c_k}$ is a constructible set of Krull-Schmidt. There exist $\alpha_1, \ldots, \alpha_s \in K^*(\mathcal{A})$ such that $\mathcal{I}_{A_k, B_k, c_k} = (\Pi_{i=1}^s \text{Ind}(\alpha_i)) \cap ((1_{O_{A_k}} * 1_{O_{B_k}})^{-1}(c_k))$. By Lemma 3.17, $\mathcal{I}_{A_k, B_k, c_k}$ is an indecomposable constructible set.

Finally, $1_{O_1} * 1_{O_2}$ is a $\mathbb{Q}$-linear combination of finitely many $1_{\sum_{i=1}^c \alpha_i} 1_{\mathcal{R}_{A_k, B_k, c_k}}$, where $\mathcal{R}_{A_k, B_k, c_k}$ run through $\mathcal{R}_{A_k, B_k, c_k}$ and $\mathcal{I}_{A_k, B_k, c_k}$ for all $r$-partitions and $r = 1, 2, \ldots, t$. We finish the proof. $\square$
Thus we summarize what we have proved as the following theorem which is due to [3, Theorem 12].

**Theorem 3.19.** The \( \mathbb{Q} \)-space \( \text{CF}^{KS}(\mathcal{O}b_{j,A}) \) is an associative \( \mathbb{Q} \)-algebra with convolution multiplication \( \ast \) and identity \( 1_{[0]} \).

### 3.5 The universal enveloping algebra of \( \text{CF}^{\text{ind}}(\mathcal{O}b_{j,A}) \)

Let \( U(\text{CF}^{\text{ind}}(\mathcal{O}b_{j,A})) \) denote the universal enveloping algebra of \( \text{CF}^{\text{ind}}(\mathcal{O}b_{j,A}) \) over \( \mathbb{Q} \). The multiplication in \( U(\text{CF}^{\text{ind}}(\mathcal{O}b_{j,A})) \) will be written as \((x, y) \mapsto xy\).

There is a \( \mathbb{Q} \)-algebra homomorphism \( \Phi : U(\text{CF}^{\text{ind}}(\mathcal{O}b_{j,A})) \to \text{CF}^{KS}(\mathcal{O}b_{j,A}) \) defined by \( \Phi(1) = 1_{[0]} \) and \( \Phi(f_1 f_2 \ldots f_n) = f_1 \ast f_2 \ast \ldots \ast f_n \), where \( f_1, f_2, \ldots, f_n \) belong to \( \text{CF}^{\text{ind}}(\mathcal{O}b_{j,A}) \).

The following theorem is related to [3, Theorem 15].

**Theorem 3.20.** \( \Phi : U(\text{CF}^{\text{ind}}(\mathcal{O}b_{j,A})) \to \text{CF}^{KS}(\mathcal{O}b_{j,A}) \) is an isomorphism.

**Proof.** For simplicity of presentation, let

\[ U = U(\text{CF}^{\text{ind}}(\mathcal{O}b_{j,A})) \text{ and } \text{CF} = \text{CF}^{KS}(\mathcal{O}b_{j,A}). \]

Suppose that \( \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k \) are indecomposable constructible subsets of \( \mathcal{O}b_{j,A}(\mathbb{K}) \) which are pairwise disjoint. It follows that \( 1_{\mathcal{O}_1}, 1_{\mathcal{O}_2}, \ldots, 1_{\mathcal{O}_k} \) are linearly independent in \( \text{CF}^{\text{ind}}(\mathcal{O}b_{j,A}) \).

Let \( U_{\mathcal{O}_1 \ldots \mathcal{O}_k} \) denote the subspace of \( U \) which is spanned by all \( 1_{\mathcal{O}_1} \ast 1_{\mathcal{O}_2} \ast \ldots \ast 1_{\mathcal{O}_k} \) for \( n_i \in \mathbb{N} \) and \( i = 1, \ldots, k \).

Define \( \text{CF}_{\mathcal{O}_1 \ldots \mathcal{O}_k} \) to be the subalgebra of \( \text{CF} \) which is generated by the elements \( 1_{\mathcal{O}_1} \ast \ldots \ast 1_{\mathcal{O}_k} \) of \( \text{CF} \), where \( n_i \in \mathbb{N} \) for \( i = 1, 2, \ldots, k \).

The homomorphism \( \Phi \) induces a homomorphism

\[ \Phi_{\mathcal{O}_1 \ldots \mathcal{O}_k} : U_{\mathcal{O}_1 \ldots \mathcal{O}_k} \to \text{CF}_{\mathcal{O}_1 \ldots \mathcal{O}_k} \]

which maps \( 1_{\mathcal{O}_1} \ast 1_{\mathcal{O}_2} \ast \ldots \ast 1_{\mathcal{O}_k} \) to \( 1_{\mathcal{O}_1} \ast 1_{\mathcal{O}_2} \ast \ldots \ast 1_{\mathcal{O}_k} \).

First of all, we want to show that \( \Phi_{\mathcal{O}_1 \ldots \mathcal{O}_k} \) is injective.

For \( m \in \mathbb{N} \), let \( U^{(m)}_{\mathcal{O}_1 \ldots \mathcal{O}_k} \) be the subspace of \( U \) which is spanned by

\[ \left\{ 1_{\mathcal{O}_1}^{n_1} 1_{\mathcal{O}_2}^{n_2} \ldots 1_{\mathcal{O}_k}^{n_k} \mid \sum_{i=1}^{k} n_i \leq m, n_i \geq 0 \text{ for } i = 1, \ldots, k \right\} \]

Using the PBW Theorem, we obtain that

\[ \left\{ 1_{\mathcal{O}_1}^{n_1} 1_{\mathcal{O}_2}^{n_2} \ldots 1_{\mathcal{O}_k}^{n_k} \mid \sum_{i=1}^{k} n_i = m, n_i \geq 0 \text{ for } i = 1, \ldots, k \right\} \]

is a basis of the \( \mathbb{Q} \)-vector space \( U^{(m)}_{\mathcal{O}_1 \ldots \mathcal{O}_k} / U^{(m-1)}_{\mathcal{O}_1 \ldots \mathcal{O}_k} \) for \( m \geq 1 \).
Similarly, we define \( \text{CF}_{\mathcal{O}_1...\mathcal{O}_k}^{(m)} \) to be a subspace of \( \text{CF}_{\mathcal{O}_1...\mathcal{O}_k} \) such that each \( f \in \text{CF}_{\mathcal{O}_1...\mathcal{O}_k}^{(m)} \) is of the form \( \sum_{i=1}^{l} c_i \mathcal{O}_i \), where \( l \in \mathbb{N}^+ \), \( c_i \in \mathbb{Q} \), \( 1 \mathcal{O}_i \in \text{CF}_{\mathcal{O}_1...\mathcal{O}_k} \) and \( \mathcal{O}_i \) are constructible sets of Krull-Schmidt such that \( \gamma(\mathcal{O}_i) \leq m \).

In \( \text{CF}^{(m)} / \text{CF}^{(m-1)} \), the set

\[
\{1_{n_1} \mathcal{O}_1 \oplus 1_{n_2} \mathcal{O}_2 \oplus \cdots \oplus 1_{n_k} \mathcal{O}_k \mid \sum_{i=1}^{k} n_i = m, n_i \geq 0 \text{ for } i = 1, \ldots, k\}
\]

is linearly independent by the Krull-Schmidt Theorem.

For each \( m \geq 1 \), \( \Phi_{\mathcal{O}_1...\mathcal{O}_k} \) induce a map

\[
\Phi_{\mathcal{O}_1...\mathcal{O}_k}^{(m)} : \text{U}_{\mathcal{O}_1...\mathcal{O}_k}^{(m)} / \text{U}_{\mathcal{O}_1...\mathcal{O}_k}^{(m-1)} \rightarrow \text{CF}_{\mathcal{O}_1...\mathcal{O}_k}^{(m)} / \text{CF}_{\mathcal{O}_1...\mathcal{O}_k}^{(m-1)}
\]

which maps \( 1_{\mathcal{O}_1}^{n_1} \otimes 1_{\mathcal{O}_2}^{n_2} \otimes \cdots \otimes 1_{\mathcal{O}_k}^{n_k} \) to \( n_1!n_2!\cdots n_k!1_{\mathcal{O}_1}^{n_1} \oplus 1_{\mathcal{O}_2}^{n_2} \oplus \cdots \oplus 1_{\mathcal{O}_k}^{n_k} \) (also see Corollary 3.16), where \( \sum_{i=1}^{k} n_i = m \) and \( n_i \geq 0 \). From this we know that \( \Phi_{\mathcal{O}_1...\mathcal{O}_k}^{(m)} \) is injective for all \( m \in \mathbb{N} \). Obviously, both \( \text{U}_{\mathcal{O}_1\mathcal{O}_2...\mathcal{O}_n} \) and \( \text{CF}_{\mathcal{O}_1...\mathcal{O}_k} \) are filtered. From the properties of filtered algebra, we know that \( \Phi_{\mathcal{O}_1...\mathcal{O}_k} \) is injective. Hence \( \Phi : \text{U} \rightarrow \text{CF} \) is injective.

Finally, we show that \( \Phi \) is surjective by induction. When \( m = 1 \), the statement is trivial. Then we assume that every constructible function \( f = \sum_{i=1}^{l} a_i \mathcal{Q}_i \), lies in \( \text{Im}(\Phi) \), where \( a_i \in \mathbb{Q} \) and \( \mathcal{Q}_i \) are constructible sets of stratified Krull-Schmidt with \( \gamma(\mathcal{Q}_i) < m \).

Let \( n_1 + n_2 + \cdots + n_k = m \) and \( n_i \in \mathbb{N} \) for \( 1 \leq i \leq k \). Then

\[
\Phi(1_{\mathcal{O}_1}^{n_1} 1_{\mathcal{O}_2}^{n_2} \cdots 1_{\mathcal{O}_k}^{n_k}) = 1_{\mathcal{O}_1}^{n_1} \ast 1_{\mathcal{O}_2}^{n_2} \ast \cdots \ast 1_{\mathcal{O}_k}^{n_k}
\]

\[
= n_1!n_2!\ldots n_k!1_{\mathcal{O}_1}^{n_1} \oplus 1_{\mathcal{O}_2}^{n_2} \oplus \cdots \oplus 1_{\mathcal{O}_k}^{n_k} + \sum_{j=1}^{s} b_j P_j,
\]

where \( b_j \in \mathbb{Q} \) and \( P_j \) are constructible sets of stratified Krull-Schmidt with \( \gamma(P_j) < m \). By the hypothesis, \( \sum_{j=1}^{s} b_j P_j \in \text{Im}(\Phi) \). Hence \( 1_{\mathcal{O}_1\mathcal{O}_2...\mathcal{O}_k} \) lies in \( \text{Im}(\Phi) \). The algebra \( \text{CF} \) is generated by all \( 1_{\mathcal{O}_1\oplus \cdots \oplus \mathcal{O}_k} \), which proves that \( \Phi \) is surjective, the proof is finished. \( \square \)

4 Comultiplication and Green’s formula

4.1 Comultiplication

We now turn to define a comultiplication on the algebra \( \text{CF}^{KS}(\text{Obj}_A) \). For \( f, g \in \text{CF}(\text{Obj}_A) \), \( f \otimes g \) is defined by \( f \otimes g([X],[Y]) = f([X])g([Y]) \) for \( ([X],[Y]) \in (\text{Obj}_A \times \text{Obj}_A)(\mathbb{K}) = \text{Obj}_A(\mathbb{K}) \times \text{Obj}_A(\mathbb{K}) \) (see Definition 4.1). Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \).
some positive integers $n$, $p$.

$\text{rk} \, \text{Im}$ that $m$ of $\text{Aut}(\mathcal{A})$. Let Lemma 4.3.

$\chi$ If $\Delta(1_{\mathcal{A}})$.

$\Delta : \text{CF}^{\text{KS}}(\mathcal{O}_{\mathcal{A}}) \rightarrow \text{CF}^{\text{KS}}(\mathcal{O}_{\mathcal{A}} \times \mathcal{O}_{\mathcal{A}})$

is defined by $\Delta = (\pi_i \times \pi_r) \circ (\pi_m)^*$, where $\text{CF}^{\text{KS}}(\mathcal{O}_{\mathcal{A}} \times \mathcal{O}_{\mathcal{A}})$ is regarded as a topological completion of $\text{CF}^{\text{KS}}(\mathcal{O}_{\mathcal{A}}) \otimes \text{CF}^{\text{KS}}(\mathcal{O}_{\mathcal{A}})$.

The counit $\varepsilon : \text{CF}^{\text{KS}}(\mathcal{O}_{\mathcal{A}}) \rightarrow \mathbb{Q}$ maps $f$ to $f([0])$.

Note that $\Delta$ is a $\mathbb{Q}$-linear map since $(\pi_i \times \pi_r)$! and $(\pi_m)^*$ are $\mathbb{Q}$-linear map.

**Definition 4.2.** Let $\alpha = [A], \beta = [B] \in \mathcal{O}_{\mathcal{A}}(\mathbb{K})$ and $\mathcal{O} \subseteq \mathcal{O}_{\mathcal{A}}(\mathbb{K})$ be a constructible set of stratified Krull-Schmidt, define

$$h_{\alpha \beta} = \Delta(1_{\mathcal{O}})([A], [B]).$$

Let $\mathcal{O}_1$ and $\mathcal{O}_2 \subseteq \mathcal{O}_{\mathcal{A}}(\mathbb{K})$ be constructible sets, define

$$g_{\alpha \beta} = 1_{\mathcal{O}_1} \ast 1_{\mathcal{O}_2}(\alpha).$$

Because $\Delta(1_{\mathcal{O}})$ is a constructible function, $\Delta(1_{\mathcal{O}}) = \sum_{i=1}^{n} h_{\alpha \beta} \alpha_i 1_{\mathcal{O}_i}$ for some $\alpha_i, \beta_i \in \mathcal{O}_{\mathcal{A}}(\mathbb{K})$ and $n \in \mathbb{N}$, where $\mathcal{O}_i$ are constructible subsets of $\mathcal{O}_{\mathcal{A}}(\mathbb{K}) \times \mathcal{O}_{\mathcal{A}}(\mathbb{K})$.

**Lemma 4.3.** Let $X, Y, Z \in \text{Obj}(\mathcal{A})$. If $X \oplus Z$ is not isomorphic to $Y$, then $\Delta(1_{[Y]})([X], [Z]) = 0$.

**Proof.** If $\Delta(1_{[Y]})([X], [Z]) \neq 0$, there exists a conflation $X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z$ in $\mathcal{A}$ such that $m_{\pi_{1}, \pi_{r}}([X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z]) \neq 0$. Recall that

$$m_{\pi_{1}, \pi_{r}}([X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z]) = \chi([\text{Aut}(X) \times \text{Aut}(Z)]/\text{Imp}_g).$$

If $\text{rk} \, \text{Imp}_g < \text{rk} ([\text{Aut}(X) \times \text{Aut}(Z)])$, the fibre of the action of a maximal torus of $\text{Aut}(X) \times \text{Aut}(Z)$ on $([\text{Aut}(X) \times \text{Aut}(Z)])/\text{Imp}_g$ is $(\mathbb{K}^*)^l$ for $l > 0$. Then $\chi([\text{Aut}(X) \times \text{Aut}(Z)]/\text{Imp}_g) = 0$, which is a contradiction. Hence $\text{rk} ([\text{Aut}(X) \times \text{Aut}(Z))] = \text{rk} \, \text{Imp}_g$.

Assume that $\text{rk} \, \text{Aut}(X) = n_1$, $\text{rk} \, \text{Aut}(Z) = n_2$ and $\text{rk} \, \text{Aut}(Y) = n$ for some positive integers $n_1$, $n_2$ and $n$. Note that $D_{n_1} \times D_{n_2}$ is a maximal
torus of Aut(X) × Aut(Z). Because \( \text{rk}(\text{Aut}(X) \times \text{Aut}(Z)) = \text{rk} \ \text{Im}(p_2) \), each maximal torus of \( \text{Imp}_2 \) is also a maximal torus of \( \text{Aut}(X) \times \text{Aut}(Z) \). Therefore every maximal torus of \( \text{Imp}_2 \) and \( D_{n_1} \times D_{n_2} \) are conjugate. For simplicity, we can assume that \( D_{n_1} \times D_{n_2} \) is a maximal torus of \( \text{Imp}_2 \). For \( (t_1 I_{n_1}, t_2 I_{n_2}) \in D_{n_1} \times D_{n_2} \), where \( t_1 \neq t_2 \), there exists \( \tau \in \text{Aut}(Y) \) such that \( (t_1 I_{n_1}, \tau, t_2 I_{n_2}) \in \text{Aut}(X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z) \). Then we have the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{t_1 I_{n_1}} & \tau & \downarrow{t_2 I_{n_2}} \\
X & \xrightarrow{f} & Y \\
& \downarrow{t_2 I_{n_1}} & \downarrow{t_2 I_{n_2}} \\
& \downarrow{f} & \downarrow{g} \\
& Z & Z \\
\end{array}
\]

The morphism \( (t_2 I_{n_1}, t_2 I_{n_2}) \) is also in \( \text{Aut}(X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z) \). The following diagram is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{t_2 I_{n_1}} & \tau & \downarrow{t_2 I_{n_2}} \\
X & \xrightarrow{f} & Y \\
& \downarrow{t_2 I_{n_1}} & \downarrow{t_2 I_{n_2}} \\
& \downarrow{f} & \downarrow{g} \\
& Z & Z \\
\end{array}
\]

Consequently \( g(\tau - t_2 I_{n_2}) = 0 \). Because \( f \) is a kernel of \( g \), there exists \( h \in \text{Hom}(Y, X) \) such that \( \tau - t_2 I_{n_2} = fh \). Then \( \tau = fh + t_2 I_{n_2} \). We have

\[ f(t_1 I_{n_1}) = \tau f = (fh + t_2 I_{n_2})f, \]

it follows that

\[ fhf = f(t_1 I_{n_1}) - (t_2 I_{n_2})f = f(t_1 I_{n_1} - t_2 I_{n_2}). \]

Then \( h f = (t_1 - t_2) I_{n_1} \) since \( f \) is an inflation. Let \( f' = \frac{1}{t_1 - t_2} h \), then \( f' f = 1_X \). Hence \( X \) is isomorphic to a direct summand of \( Y \). The proof is completed. \( \square \)

For an indecomposable object \( X \in \text{Obj}(A) \), direct summands of \( X \) are only \( X \) and \( 0 \). Thus \( \Delta(1_{[X]}) = 1_{[X]} \otimes 1_{[0]} + 1_{[0]} \otimes 1_{[X]} \). It follows that \( \Delta(f) = f \otimes 1_{[0]} + 1_{[0]} \otimes f \) for \( f \in \text{CF}_{\text{ind}}(\text{Ob}b_A) \).

By Lemma 1.3, \( h_\alpha^{\beta} = 1 \) if \( \alpha \oplus \beta \in \mathcal{O} \), and \( h_\alpha^{\beta} = 0 \) otherwise. Let \( \mathcal{O} = n_1 \mathcal{O}_1 \oplus \ldots \oplus n_m \mathcal{O}_m \) be a constructible set of stratified Krull-Schmidt, where \( \mathcal{O}_i \) are indecomposable constructible sets for all \( 1 \leq i \leq m \). By Lemma 4.3, the formula \( \Delta(1_{\mathcal{O}}) = \sum_{i=1}^{n} h_\alpha^{\beta} \alpha_1 \mathcal{O}_i \) can be written as

\[ \Delta(1_{\mathcal{O}}) = \sum_{1 \leq i \leq m; 0 \leq k_i \leq n_i} 1_{k_1 \mathcal{O}_1 \oplus \ldots \oplus k_m \mathcal{O}_m} \otimes 1_{(n_1 - k_1) \mathcal{O}_1 \oplus \ldots \oplus (n_m - k_m) \mathcal{O}_m}. \]

Hence we have the following proposition.
**Proposition 4.4.** Let \( \mathcal{O} \) be a constructible set of stratified Krull-Schmidt, then \( \Delta(1_\mathcal{O}) \in \text{CF}^{KS}(\text{Obj}_\mathcal{A}) \otimes \text{CF}^{KS}(\text{Obj}_\mathcal{A}) \), i.e., the map

\[
\Delta : \text{CF}^{KS}(\text{Obj}_\mathcal{A}) \to \text{CF}^{KS}(\text{Obj}_\mathcal{A}) \otimes \text{CF}^{KS}(\text{Obj}_\mathcal{A})
\]

is well-defined.

### 4.2 Green’s formula on stacks

Recall that

\[
\int_{x \in S} f(x) \, d\mu = \sum_{a \in f(S) \setminus \{0\}} a \chi^\text{na}(f^{-1}(a) \cap S),
\]

where \( f \) is a constructible function and \( S \) a locally constructible set.

Let \( \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_\rho, \mathcal{O}_\sigma, \mathcal{O}_\tau, \mathcal{O}_\lambda \) be constructible sets and \( \alpha \in \mathcal{O}_1, \beta \in \mathcal{O}_2, \rho \in \mathcal{O}_\rho, \sigma \in \mathcal{O}_\sigma, \epsilon \in \mathcal{O}_\epsilon, \tau \in \mathcal{O}_\tau, \lambda \in \mathcal{O}_\lambda \) such that \( \mathcal{O}_\rho \oplus \mathcal{O}_\sigma = \mathcal{O}_1 \) and \( \mathcal{O}_\epsilon \oplus \mathcal{O}_\tau = \mathcal{O}_2 \).

The following theorem is the degenerate form of Green’s formula which is related to [3 Theorem 22].

**Theorem 4.5.** Let \( \mathcal{O}_1, \mathcal{O}_2 \) be constructible subsets of \( \text{Obj}_{2,\mathcal{A}(\mathbb{R})} \) and \( \alpha', \beta' \in \text{Obj}_{2,\mathcal{A}(\mathbb{R})} \), then we have

\[
g_{\mathcal{O}_2 \mathcal{O}_1}^{\alpha' \oplus \beta'} = \int_{\beta' \in \mathcal{O}_2} g_{\mathcal{O}_2}^{\beta'} g_{\mathcal{O}_1}^{\alpha'}. \]

**Proof.** By the proof of Lemma 3.3, \( g_{\mathcal{O}_2 \mathcal{O}_1}^{\alpha' \oplus \beta'} = \int_{\alpha' \in \mathcal{O}_1} g_{\mathcal{O}_2}^{\alpha'} g_{\mathcal{O}_1}^{\beta'} \). It suffices to prove the following formula

\[
g_{\beta' \alpha}^{\alpha' \oplus \beta'} = \int_{\beta' \alpha \in \mathcal{O}_2} g_{\mathcal{O}_2}^{\beta'} g_{\mathcal{O}_1}^{\alpha'}. \]

Suppose that \( [A] = \alpha, [B] = \beta, [A'] = \alpha', [B'] = \beta', [C] = \rho, [D] = \sigma, [E] = \epsilon \) and \( [F] = \tau \) for \( A, B, C, D, E, F \in \text{Obj}_{2,\mathcal{A}(\mathbb{R})} \). There are finitely many \((\rho, \sigma)\) and \((\epsilon, \tau)\) such that \( \rho \oplus \sigma = \alpha \) and \( \epsilon \oplus \tau = \beta \). The morphism

\[
i : \bigcup_{\substack{[C], [D], [E], [F] \in \mathcal{O}_1, [A] \in \mathcal{O}_2, [B] \in \mathcal{O}_2 \mathcal{O}_1}} V([C], [E]; A') \times V([D], [F]; B') \to V([A], [B]; A' \oplus B')
\]

is defined by

\[
(C \xrightarrow{f_1} A' \xrightarrow{g_1} E), (D \xrightarrow{f_2} B' \xrightarrow{g_2} F) \mapsto (C \oplus D \xrightarrow{f} A' \oplus B' \xrightarrow{g} E \oplus F),
\]

where \( f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \) and \( g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \). Because both \( C \xrightarrow{f_1} A' \xrightarrow{g_1} E \) and \( D \xrightarrow{f_2} B' \xrightarrow{g_2} F \) are conflations, \( C \oplus D \xrightarrow{f} A' \oplus B' \xrightarrow{g} E \oplus F \) is a conflation by [2 Proposition 2.9]. Hence the morphism is well-defined. Note that \( i \) is injective and \( g_{\rho \beta \sigma}^{\alpha' \oplus \beta'} = \chi^\text{na}(V([C], [E]; A') \times V([D], [F]; B')) \).
By \cite{9} Lemma 4.2, we have
\[ \chi^n(V([A],[B];A' \oplus B')) = \chi^n(\operatorname{Im}i) + \chi^n(V([A],[B];A' \oplus B') \setminus \operatorname{Im}i). \]

According to Lemma \ref{3.11}, if \( m_{\pi_m}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B]) \neq 0 \), then there exist two conflations \( C \xrightarrow{f_1} A' \xrightarrow{g_1} E \) and \( D \xrightarrow{f_2} B' \xrightarrow{g_2} F \) in \( A \) such that \( A \cong C \oplus D \), \( B \cong E \oplus F \), \( f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \) and \( g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \). Thus
\[ m_{\pi_m}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B]) = 0 \]
for any \( [A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B] \in V([A],[B];A' \oplus B') \setminus \operatorname{Im}i \). Using \cite{3}, it follows that \( \chi^n(V([A],[B];A' \oplus B') \setminus \operatorname{Im}i) = 0 \). Hence
\[
\Delta(x_{\alpha'} \oplus x_{\beta'}) = \chi^n(V([A],[B];A' \oplus B')) = \chi^n(\operatorname{Im}i) = \int_{\rho,\sigma,\tau} g_{\rho,\sigma,\tau} g_{\rho,\sigma,\tau}'.
\]
This completes the proof. \( \square \)

For all \( f_1, f_2, g_1, g_2 \in \mathrm{CF}^{\mathrm{KS}}(\mathcal{D}b_A) \), define \( (f_1 \otimes g_1) \ast (f_2 \otimes g_2) = (f_1 \ast f_2) \otimes (g_1 \ast g_2) \). Using Green’s formula, we have the following theorem due to \cite{3} Theorem 24.

**Theorem 4.6.** The map \( \Delta : \mathrm{CF}^{\mathrm{KS}}(\mathcal{D}b_A) \to \mathrm{CF}^{\mathrm{KS}}(\mathcal{D}b_A) \otimes \mathrm{CF}^{\mathrm{KS}}(\mathcal{D}b_A) \) is an algebra homomorphism.

**Proof.** The proof is similar to the one in \cite{3} Theorem 24. Let \( O_1, O_2 \in \mathcal{D}b_A(\mathbb{K}) \) be constructible sets of stratified Krull-Schmidt. Then
\[
\Delta(O_1 \ast O_2) = \Delta(\sum_{\lambda} \sum_{\alpha} g_{\alpha_1}^\lambda 1_{O_1} \otimes 1_{O_2}) = \sum_{\lambda} g_{\alpha_1}^\lambda \Delta(1_{O_1}) \]
\[
= \sum_{\lambda} g_{\alpha_1}^\lambda \sum_{\alpha',\beta'} h_{\alpha_1}^{\beta' \alpha'} 1_{O_{\alpha'}} \otimes 1_{O_{\beta'}} = \sum_{\alpha',\beta'} g_{\alpha_1}^{\alpha' \oplus \beta'} 1_{O_{\alpha'}} \otimes 1_{O_{\beta'}}.
\]

\[
\Delta(O_1) \ast \Delta(O_2) = \left( \sum_{\rho,\sigma} h_{\rho}^{\sigma} 1_{O_{\rho}} \otimes 1_{O_{\sigma}} \right) \ast \left( \sum_{\epsilon,\tau} h_{\epsilon}^{\tau} 1_{O_{\epsilon}} \otimes 1_{O_{\tau}} \right)
\]
\[
= \sum_{\rho,\sigma,\epsilon,\tau} h_{\rho}^{\sigma} h_{\epsilon}^{\tau} (1_{O_{\rho}} \ast 1_{O_{\sigma}}) \otimes (1_{O_{\epsilon}} \ast 1_{O_{\tau}})
\]
\[
= \sum_{\rho,\sigma,\epsilon,\tau} h_{\rho}^{\sigma} h_{\epsilon}^{\tau} (\sum_{\alpha',\beta'} g_{\alpha_1}^{\alpha'} g_{\beta'} 1_{O_{\alpha'}} \otimes 1_{O_{\beta'}})
\]
\[
= \sum_{\alpha',\beta'} (\sum_{\rho,\sigma,\epsilon,\tau} h_{\rho}^{\sigma} h_{\epsilon}^{\tau} g_{\alpha_1}^{\alpha'} g_{\beta'} 1_{O_{\alpha'}} \otimes 1_{O_{\beta'}}).
\]

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According to Theorem 4.5 it follows that
\[
\sum_{\rho,\sigma,\epsilon,\tau} h_{\rho\sigma}^0 h_{\sigma\epsilon}^0 g_{\epsilon\tau}^\epsilon g_{\tau\sigma}^\tau = g_{\rho\sigma}^\rho \oplus g_{\sigma\tau}^\tau.
\]
Therefore \(\Delta(1_{O_1} \ast 1_{O_2}) = \Delta(1_{O_1}) \ast \Delta(1_{O_2})\). We have thus proved the theorem. \(\Box\)

A Exact categories

We recall the definition of an exact category (see [11, Appendix A]).

Definition A.1. Let \(\mathcal{A}\) be an additive category. A sequence

\[
X \xrightarrow{f} Y \xrightarrow{g} Z
\]

in \(\mathcal{A}\) is called exact if \(f\) is a kernel of \(g\) and \(g\) is a cokernel of \(f\). The morphisms \(f\) and \(g\) are called inflation and deflation respectively. The short exact sequence is called a conflation. Let \(\mathcal{S}\) be the collection of conflations closed under isomorphism and satisfying the following axioms

A0 \(1_0 : 0 \to 0\) is a deflation.

A1 The composition of two deflations is a deflation.

A2 For every \(h \in \text{Hom}(X, X')\) and every inflation \(f \in \text{Hom}(X, Y)\) in \(\mathcal{A}\), there exists a pushout

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow h & & \downarrow h' \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

where \(f' \in \text{Hom}(X', Y')\) is an inflation.

A3 For every \(l \in \text{Hom}(Z', Z)\) and every deflation \(g \in \text{Hom}(Y, Z)\) in \(\mathcal{A}\), there exists a pullback

\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Z' \\
\downarrow l' & & \downarrow l \\
Y & \xrightarrow{g} & Z
\end{array}
\]

where \(g' \in \text{Hom}(Y', Z')\) is an deflation. Then \((\mathcal{A}, \mathcal{S})\) is called an exact category.

The definition of idempotent complete is taken from [2, Definition 6.1].

Definition A.2. Let \(\mathcal{A}\) be an additive category. The category \(\mathcal{A}\) is idempotent complete if for every idempotent morphism \(s : A \to A\) in \(\mathcal{A}\), \(s\) has a kernel \(k : K \to A\) and a image \(i : I \to A\) (a kernel of a cokernel of \(s\)) such that \(A \cong K \oplus I\). We write \(A \cong \text{Kers} \oplus \text{Im}s\), for simplicity.
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