Research Article

Some Inequalities of Generalized p-Convex Functions concerning Raina’s Fractional Integral Operators

Changyue Chen 1, Muhammad Shoaib Sallem 2, and Muhammad Sajid Zahoor 2

1School of Public Education, Shandong University of Finance and Economics, Taian, Shandong 271000, China
2Department of Mathematics, University of Okara, Okara, Pakistan

Correspondence should be addressed to Muhammad Shoaib Sallem; shaby455@yahoo.com

Received 23 April 2021; Revised 16 August 2021; Accepted 8 September 2021; Published 4 October 2021

Academic Editor: Ahmet Ocak Akdemir

Copyright © 2021 Changyue Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Convex functions play an important role in pure and applied mathematics specially in optimization theory. In this paper, we will deal with well-known class of convex functions named as generalized p-convex functions. We develop Hermite–Hadamard-type inequalities for this class of convex function via Raina’s fractional integral operator.

1. Introduction

The subject of fractional calculus got rapid development in the last few decades. As a matter of fact, fractional calculus give more accuracy to model applied problems in engineering and other sciences then classical calculus. In order to model recent complicated problems, scientists are using fractional inequalities and fractional equations. For more on this, we refer the books [1, 2]. The models with fractional calculus have been applied successfully in ecology, aerodynamics, physics, biochemistry, environmental science, and many other branches. For more about fractional calculus and models, we refer [3–5].

Fractional integral inequalities are considered one of the important tools to study the behavior and properties of solutions of various fractional problems [6–14]. There are many interesting generalization of fractional derivatives as per need of practical problems or some theoretical approach, for example, Raina’s fractional integral operator, Caputo-Fabrizio fractional integral, and extended Caputo-Fabrizio fractional integral. For recent work on it, we refer [15–20].

Convex functions also play an important role in pure and applied mathematics specially in optimization theory. Classical convexity does not fulfil needs of modern mathematics; therefore, several generalizations of convex functions are presented in literature. s-convex function [21], M-convex functions [22], and h-convex function [23] are some examples of generalized convex functions. It is always interesting to study properties of some generalized convex function in the setting of fractional integral operators. This paper is an effort in this direction. In this paper, we study the p-convex functions and present some of its properties in the setting of Raina’s fractional integral operators.

The paper is organized as follows. In Section 2, we present some basic definition and properties of Raina’s fractional integral operator. Section 3 is devoted for Hermite–Hadamard type inequalities for generalized p-convex functions in terms of Raina’s fractional integral operators.

2. Preliminaries

Here, we present some basic definitions and known results.

Definition 1 (convex function). A function $\phi: I \rightarrow \mathbb{R}$ is said to be convex function if the following inequality holds:

$$\phi((1-\theta)x + \theta y) \leq \theta \phi(x) + (1-\theta)\phi(y),$$  \hspace{1cm} (1)

for $\forall x, y \in I$ and $\theta \in [0, 1]$. 




One of the novel generalization of convexity is $\eta$-convexity introduced by M. R. Delavar and S. S. Dragomir in [24].

**Definition 2.** A function $\phi: I \rightarrow R$ is said to be generalized convex function with respect to $\eta: A \times A \rightarrow B$ for appropriate $A, B \subseteq R$ if

$$\phi(\beta x + (1 - \beta)y) \leq \phi(y) + \beta \eta(\phi(x), \phi(y)),$$

for $\forall x, y \in I$ and $\beta \in [0, 1]$.

In [25], Zhang and Wan gave definition of $p$-convex function as follows.

**Definition 3.** Let $I$ be a $p$-convex set. A function $f: I \rightarrow R$ is said to be $p$-convex function if

$$\phi\left(\left[8x^p + (1 - \beta)y^p\right]^{1/p}\right) \leq \beta \phi(x) + (1 - \beta)\phi(y),$$

holds, for all $x, y \in I$ and $\beta \in [0, 1]$.

In [26], the authors gave the definition of the generalized $p$-convex function as follows.

**Definition 4.** A function $\phi: I \rightarrow R$ is said to be generalized $p$-convex function with respect to $\eta: A \times A \rightarrow B$ for appropriate $A, B \subseteq R$ if

$$\phi(\beta x^p + (1 - \beta)y^p)^{1/p} \leq \phi(y) + \beta \eta(\phi(x), \phi(y)),$$

for $\forall x, y \in I$, $p > 0$ and $\beta \in [0, 1]$.

For some important properties and results about generalized $p$-convexity, see [26]. Moreover, in [26], the following Hermite–Hadamard type inequality for $p$-convex functions can be found.

**Theorem 1.** Let $\phi: I \rightarrow R$ be generalized $p$-convex function for $\xi_1, \xi_2 \in I$ with condition $\xi_1 < \xi_2$; then, we obtain the inequality

$$\phi\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{1/p} \leq \frac{p}{\xi_2^p - \xi_1^p} \int_{\xi_1}^{\xi_2} x^{p-1} \left[\phi(x) + \frac{\phi(\xi_1) + \phi(\xi_2)}{2} + \frac{1}{4} \eta(\phi(\xi_1), \phi(\xi_2)) + \eta(\phi(\xi_2), \phi(\xi_1))\right] dx,$$

where $\rho, \lambda > 0$, $|z| < R$ ($R$ is the set of real numbers), and $\sigma = (\sigma(1), \ldots, \sigma(k), \ldots)$ is a bounded sequence of positive real numbers.

Using (6), in [28], the authors defined the following left-sided and right-sided fractional integral operators, respectively:

$$\left(\mathcal{I}_{\rho, \lambda, \xi_1; w}^\alpha\phi\right)(z) = \int_{\xi_1}^{z} (z - \theta)^{(\lambda - 1)} \mathcal{I}_{\rho, \lambda, \xi_1; w}^\sigma z^w (\theta - \theta)^{\sigma(1)} \phi(\theta) d\theta, \quad (z > \xi_1),$$

$$\left(\mathcal{I}_{\rho, \lambda, \xi_1; w}^{\sigma} \phi\right)(z) = \int_{\xi_1}^{z} (\theta - z)^{(\lambda - 1)} \mathcal{I}_{\rho, \lambda, \xi_1; w}^\sigma z^w (\theta - z)^{\sigma(1)} \phi(\theta) d\theta, \quad (z < \xi_2),$$

where

$$\|\phi\|_p = \left(\int_{\xi_1}^{\xi_2} |\phi(z)|^p dz\right)^{1/p}. \quad (10)$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Let $\phi \in L[\xi_1, \xi_2]$. The right-hand side and left-hand side Riemann–Liouville fractional integral of order $\alpha > 0$ with $\xi_2 > \xi_1 > 0$ are defined by...
Journal of Mathematics 3

\[ f_{\xi_1,\phi}^a(z) = \frac{1}{\Gamma(a)} \int_{\xi_1}^z (z-k)^{a-1} \phi(k)dk, \quad x > \xi_1, \]
\[ f_{\xi_2,\phi}^a(z) = \frac{1}{\Gamma(a)} \int_{\xi_2}^z (k-z)^{a-1} \phi(k)dk, \quad z < \xi_2, \]

respectively, where \( \Gamma(a) \) is the Gamma function defined as \( \Gamma(a) \equiv \int_0^\infty e^{-k} k^{a-1}dk \).

\[ \phi(\xi_1) + \phi(\xi_2) \]
\[ = \frac{\xi_2 - \xi_1}{2(\xi_2 - \xi_1)^2 \mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_2 - \xi_1)^{\rho} \right]} \int_0^1 \left( 1 - \theta \right)^{\frac{1}{\rho}} \mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_2 - \xi_1)^{\rho} \phi(\theta \xi_1 + (1-\theta)\xi_1) \right] \theta d\theta \]
\[ - \int_0^1 \left( 1 - \theta \right)^{\frac{1}{\rho}} \mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_2 - \xi_1)^{\rho} \phi(\theta \xi_1 + (1-\theta)\xi_1) \right] \theta \theta d\theta. \]

### 3. Main Results

In this section, we establish new Hermite–Hadamard type inequalities for generalized \( p \)-convex functions in terms of Raina’s fractional integral operators.

\[ \phi \left( \left[ \frac{\xi_1 + \xi_2}{2} \right]^{1/p} \right) - N_{\eta} \leq \frac{\phi(\xi_1) + \phi(\xi_2)}{2} + \frac{\mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]}{2(\xi_2 - \xi_1)^2 \mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]} \]
\[ \leq \frac{\phi(\xi_1) + \phi(\xi_2)}{2} + \frac{\mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]}{2(\xi_2 - \xi_1)^2 \mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]} M_{\eta}. \]

where \( \sigma(k) = \sigma(k)(k\rho + \lambda) \), for all \( k = 0, 1, 2, \ldots \) and \( N_{\eta} \) and \( M_{\eta} \) are bounds of \( \phi \).

Proof. From inequality (6), we have

\[ 2\phi \left( \frac{\xi_1 + \xi_2}{2} \right)^{1/p} - N_{\eta} \leq \frac{\phi(\xi_1) + \phi(\xi_2)}{2} + \frac{\mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]}{2(\xi_2 - \xi_1)^2 \mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]} M_{\eta}. \]

Multiplying both sides by \( \phi(\xi_1) + \phi(\xi_2) \), we obtain

\[ \left[ 2\phi \left( \frac{\xi_1 + \xi_2}{2} \right)^{1/p} - N_{\eta} \right] \theta^{1-1} \mathcal{G}_{p,\lambda}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right] \leq \theta^{1-1} \mathcal{G}_{p,\lambda}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right] \phi(\xi_1) + (1-\theta)\xi_2^{\rho} + N_{\eta}. \]

### Lemma 1 (see [29, 30]). Let \( \lambda, \rho > 0, \omega \in \mathbb{R} \), and \( \sigma \) be a sequence of nonnegative real numbers. Let \( \phi: \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( (\xi_1, \xi_2) \) with \( \xi_1 < \xi_2 \) and \( \lambda > 0 \). If \( \phi' \in L[\xi_1, \xi_2] \), the following equality for the fractional integral operator holds:

\[ \int_{\xi_1}^{\xi_2} \phi^a(z) = \frac{1}{\Gamma(a)} \int_{\xi_1}^{\xi_2} (z-k)^{a-1} \phi(k)dk, \quad z > \xi_1, \]
\[ \int_{\xi_2}^{\xi_1} \phi^a(z) = \frac{1}{\Gamma(a)} \int_{\xi_2}^{\xi_1} (k-z)^{a-1} \phi(k)dk, \quad z < \xi_2, \]

Theorem 2. Let \( \phi: I \rightarrow \mathbb{R} \) be generalized \( p \)-convex function and provided \( \eta (\cdot, \cdot) \) is bounded from above on \( \phi(I) \times \phi(I) \) and \( \phi \in L[\xi_1, \xi_2] \) with \( \xi_1 < \xi_2 \) and \( p > 0 \). Then, following fractional integral inequality holds:

\[ \phi \left( \left[ \frac{x^p + y^p}{2} \right]^{1/p} \right) \leq \frac{\phi(\xi_1) + \phi(\xi_2)}{2} + \frac{\mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]}{2(\xi_2 - \xi_1)^2 \mathcal{G}_{p,\lambda+1}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right]} M_{\eta}. \]

where \( N_{\eta} \) are bounds of \( \phi \). Substitute \( x^p = \theta \xi_1^p + (1-\theta)\xi_2^p \)
and \( y^p = (1-\theta)\xi_1^p + \theta \xi_2^p \); then, (6) can be written as

\[ \left[ 2\phi \left( \frac{\xi_1 + \xi_2}{2} \right)^{1/p} - N_{\eta} \right] \theta^{1-1} \mathcal{G}_{p,\lambda}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right] \leq \theta^{1-1} \mathcal{G}_{p,\lambda}^{\sigma} \left[ w(\xi_1 - \xi_1)^{\rho} \phi(\xi_1) \right] \phi(\xi_1) + (1-\theta)\xi_2^{\rho} + N_{\eta}. \]
Integrate over $\theta \in [0, 1]$, we obtain

\[
\left[2\phi\left(\frac{\xi^p_2 + \xi^p_1}{2}\right)^{1/p} - N_\eta\right] \mathcal{F}_{\rho, \lambda + 1}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \leq \int_0^1 \theta \mathcal{F}_{\rho, \lambda}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \phi(\theta \xi^p_1 + (1 - \theta) \xi^p_2)^{1/p} d\theta
\]

\[
+ \int_0^1 \theta \mathcal{F}_{\rho, \lambda}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \phi(1 - \theta) \xi^p_1 + \theta \xi^p_2)^{1/p} d\theta
\]

\[
+ \int_0^1 \theta \mathcal{F}_{\rho, \lambda}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] N\eta d\theta.
\]

With the convenient change of the variable, we can observe that

\[
\int_0^1 \theta \mathcal{F}_{\rho, \lambda}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \phi(\theta \xi^p_1 + (1 - \theta) \xi^p_2)^{1/p} d\theta
\]

\[
= \frac{-p}{\xi^p_2 - \xi^p_1} \int_{\xi_1}^{\xi_2} \theta \mathcal{F}_{\rho, \lambda}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \left(\frac{\xi^p_2 - \xi^p_1}{\xi^p_2 - \xi^p_1}\right)^\rho \phi(\theta x + (1 - \theta) \xi^p_2)^{1/p} dx
\]

\[
= \frac{p}{\xi^p_2 - \xi^p_1} \int_{\xi_1}^{\xi_2} \theta \mathcal{F}_{\rho, \lambda}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \left(\frac{\xi^p_2 - \xi^p_1}{\xi^p_2 - \xi^p_1}\right)^\rho \phi(\theta x + (1 - \theta) \xi^p_2)^{1/p} dx
\]

\[
= \frac{p}{\xi^p_2 - \xi^p_1} \left(\mathcal{F}_{\rho, \lambda, \xi^p_1; \xi^p_2}^\sigma \phi(\xi^p_2)\right)\left(\xi_1\right).
\]

Similarly, the second integral can be written as

\[
\int_0^1 \theta \mathcal{F}_{\rho, \lambda}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \phi(1 - \theta) \xi^p_1 + \theta \xi^p_2)^{1/p} d\theta = \frac{p}{\xi^p_2 - \xi^p_1} \left(\mathcal{F}_{\rho, \lambda, \xi^p_2; w}^\sigma \phi(\xi^p_1)\right)\left(\xi_1\right).
\]

Now, equation (17) becomes

\[
\left[2\phi\left(\frac{\xi^p_2 + \xi^p_1}{2}\right)^{1/p} - N_\eta\right] \mathcal{F}_{\rho, \lambda + 1}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] \leq \frac{P}{\xi^p_2 - \xi^p_1} \left[\left(\mathcal{F}_{\rho, \lambda, \xi^p_1; \xi^p_2}^\sigma \phi(\xi^p_2)\right)\left(\xi_2\right) + \left(\mathcal{F}_{\rho, \lambda, \xi^p_2; \xi^p_1}^\sigma \phi(\xi^p_1)\right)\left(\xi_1\right)\right]
\]

\[
+ \frac{P}{\xi^p_2 - \xi^p_1} \mathcal{F}_{\rho, \lambda + 1}^\sigma \left[w(\xi^p_2 - \xi^p_1)^p\right] N\eta
\]

\[
\phi\left(\frac{\xi^p_2 + \xi^p_1}{2}\right)^{1/p} - N_\eta \leq \frac{P}{2(\xi^p_2 - \xi^p_1)} \left[\mathcal{F}_{\rho, \lambda, \xi^p_1; \xi^p_2}^\sigma \phi(\xi^p_2)\left(\xi_2\right) + \left(\mathcal{F}_{\rho, \lambda, \xi^p_2; \xi^p_1}^\sigma \phi(\xi^p_1)\right)\left(\xi_1\right)\right].
\]
which is the left-hand side of inequality (13). To prove right-hand side of (13), using the Definition 4 of generalized p-convex function,

\[
\phi(\theta \xi_1^p + (1 - \theta) \xi_2^p)^{1/p} \leq \phi(\xi_2) + \eta \phi(\xi_1), \phi(\xi_2), \phi(\xi_1)).
\]

(21)

Multiplying both inequalities by \(\theta^{k-1} \mathcal{J}_{\rho,\lambda}^\sigma [w(\xi_2^p - \xi_1^p)^p] \) and then adding, we obtain

\[
\theta^{k-1} \mathcal{J}_{\rho,\lambda}^\sigma [w(\xi_2^p - \xi_1^p)^p] \phi(\theta \xi_1^p + (1 - \theta) \xi_2^p)^{1/p} + \theta^{k-1} \mathcal{J}_{\rho,\lambda}^\sigma [w(\xi_2^p - \xi_1^p)^p] \phi(\theta \xi_1^p + (1 - \theta) \xi_2^p)^{1/p} \leq \theta^{k-1} \mathcal{J}_{\rho,\lambda}^\sigma [w(\xi_2^p - \xi_1^p)^p] [\phi(\xi_2) + \eta \phi(\xi_1), \phi(\xi_2)] + \theta^{k-1} \mathcal{J}_{\rho,\lambda}^\sigma [w(\xi_2^p - \xi_1^p)^p] [\phi(\xi_2) + \eta \phi(\xi_1, \phi(\xi_2))].
\]

(22)

Integrate over \(\theta \in [0, 1]\), we obtain

\[
\begin{align*}
\frac{1}{\sigma(k)} &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda + 1)} = \frac{1}{\Gamma(\alpha + 1)} \\
\mathcal{J}_{\rho,\lambda}^\sigma [w(\xi_2^p - \xi_1^p)^p] &= \frac{\alpha}{\Gamma(\alpha + 2)} \\
(\mathcal{J}_{\rho,\lambda}^\sigma |x|) &= I_1^\alpha |x|
\end{align*}
\]

(25)

Making the substitution in (13), we obtain (26).

**Remark 1.** In Corollary 1, if we take \(\eta(x, y) = x - y, \lambda = \alpha, \sigma = (1, 0, 0, \ldots), \omega = 0, \) and \(p = 1\), then we get Theorem 1.4 of [29, 31].

**Theorem 3.** Let \(\phi: [\xi_1, \xi_2] \rightarrow \mathbb{R}\) be a differentiable function on \((\xi_1, \xi_2)\) with \(\xi_1 < \xi_2\). If \(\phi'\) is a generalized p-convex function on \([\xi_1, \xi_2]\), then the following inequality for fractional integral operator holds:

\[
\frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{p[\mathcal{J}_{\rho,\lambda}^\sigma |x|] \phi(\xi_1) + \mathcal{J}_{\rho,\lambda}^\sigma |x| \phi(\xi_1) \phi(\xi_2)]}{2(\xi_2^p - \xi_1^p)\rho \mathcal{J}_{\rho,\lambda}^\sigma [w(\xi_2^p - \xi_1^p)^p]} \leq 2[\phi' (\xi_2) + \eta (|\phi' (\xi_1) |, \phi(\xi_2))].
\]

(26)
where $\sigma_1(k) = \sigma(k) = (1 - (1/2)^{k+1})$.

Proof. Using Lemma 1 and definition of generalized $p$-convexity of $|\phi'|$, we have

\[
\left| \frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{P\left( \mathcal{F}^\sigma_{\rho,\lambda,\xi_1,\omega} \phi(\xi_1) + \mathcal{F}^\sigma_{\rho,\lambda,\xi_1,\omega} \phi(\xi_1) \right)}{2(\xi_2^p - \xi_1^p)\mathcal{F}^\sigma_{\rho,\lambda,\xi_2,\omega} \left[ u(\xi_2^p - \xi_1^p) \right]^p} \right| \leq \frac{1}{2} \left| \frac{\xi_2^p - \xi_1^p}{\mathcal{F}^\sigma_{\rho,\lambda,\xi_2,\omega} \left[ u(\xi_2^p - \xi_1^p) \right]^p} \right| \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda + 1)} \left| u_k^k(\xi_2^p - \xi_1^p)^k \right|
\]

\[
= \frac{1}{2} \left| \frac{\xi_2^p - \xi_1^p}{\mathcal{F}^\sigma_{\rho,\lambda,\xi_2,\omega} \left[ u(\xi_2^p - \xi_1^p) \right]^p} \right| \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda + 1)} \left| u_k^k(\xi_2^p - \xi_1^p)^k \right| \times \left| \int_0^1 (1 - \theta)^{k+1} - \theta^{k+1} \right| \left( |\phi'(\xi_1)| + \eta(|\phi'(\xi_1)|, |\phi'(\xi_2)|) \right) \, d\theta
\]

It is easy to verify that

\[
\int_0^{1/2} (1 - \theta)^{k+1} - \theta^{k+1} \, d\theta = \frac{1}{k\rho + \lambda + 1},
\]

\[
\int_0^{1/2} (1 - \theta)^{k+1} - \theta^{k+1} \, d\theta = \frac{1}{k\rho + \lambda + 1},
\]

\[
\int_0^{1} (g^{k+1} - 8) (1 - g^{k+1}) \, d\theta = \frac{1}{k\rho + \lambda + 2} - \frac{1}{2(k\rho + \lambda + 1)}.
\]

Equation (27) becomes

\[
\left| \frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{P\left( \mathcal{F}^\sigma_{\rho,\lambda,\xi_1,\omega} \phi(\xi_1) + \mathcal{F}^\sigma_{\rho,\lambda,\xi_1,\omega} \phi(\xi_1) \right)}{2(\xi_2^p - \xi_1^p)\mathcal{F}^\sigma_{\rho,\lambda,\xi_2,\omega} \left[ u(\xi_2^p - \xi_1^p) \right]^p} \right| \leq \frac{1}{2} \left| \frac{\xi_2^p - \xi_1^p}{\mathcal{F}^\sigma_{\rho,\lambda,\xi_2,\omega} \left[ u(\xi_2^p - \xi_1^p) \right]^p} \right| \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda + 1)} \left| u_k^k(\xi_2^p - \xi_1^p)^k \right|
\]

\[
\times \left[ 2|\phi'(\xi_2)| \left( 1 - \frac{1}{k\rho + \lambda + 1} \right) + \eta(|\phi'(\xi_2)|, |\phi'(\xi_2)|) \frac{1}{k\rho + \lambda + 1} \right].
\]

Finally, we can write it as

\[
\left| \frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{P\left( \mathcal{F}^\sigma_{\rho,\lambda,\xi_1,\omega} \phi(\xi_1) + \mathcal{F}^\sigma_{\rho,\lambda,\xi_1,\omega} \phi(\xi_1) \right)}{2(\xi_2^p - \xi_1^p)\mathcal{F}^\sigma_{\rho,\lambda,\xi_2,\omega} \left[ u(\xi_2^p - \xi_1^p) \right]^p} \right| \leq \frac{1}{2} \left| \frac{\xi_2^p - \xi_1^p}{\mathcal{F}^\sigma_{\rho,\lambda,\xi_2,\omega} \left[ u(\xi_2^p - \xi_1^p) \right]^p} \right| \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda + 1)} \left| u_k^k(\xi_2^p - \xi_1^p)^k \right|
\]

\[
\times \left[ 2|\phi'(\xi_2)| + \eta(|\phi'(\xi_2)|, |\phi'(\xi_2)|) \frac{1}{k\rho + \lambda + 1} \right].
\]
where \( \sigma_1(k) = \sigma(k) = (1 - (1/2)^{k+1}) \), which is our required result.

\[
\left| \frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{p^\prime(\alpha + 1)}{2(\xi_2^\prime - \xi_1^\prime)^\alpha} \left[ (I_{\xi_2}^\prime \phi)(\xi_1) + (I_{\xi_1}^\prime \phi)(\xi_2) \right] \right| \leq \frac{\left( \xi_2^\prime - \xi_1^\prime \right)}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left[ 2|\phi'(\xi_2)| + \eta(|\phi'(\xi_1)|, |\phi'(\xi_2)|) \right]. \tag{31}
\]

**Proof.** By taking \( \lambda = \alpha, \sigma = (1, 0, 0, \ldots), w = 0, \) and \( p = 1, \) we obtain

\[
\mathcal{F}_{\rho,\lambda + 1}[w(\xi_2^\prime - \xi_1^\prime)^\alpha] = \frac{1}{\Gamma(\alpha + 1)},
\]

\[
\mathcal{F}_{\rho,\lambda + 2}[w(\xi_2^\prime - \xi_1^\prime)^\alpha] = \frac{1 - 1/2^\alpha}{\Gamma(\alpha + 2)}, \tag{32}
\]

\[
(\mathcal{G}_{\rho,\lambda, \xi_1, \xi_2, \omega}^\sigma \phi)(x) = I_{\xi_1}^\omega \phi x,
\]

\[
(\mathcal{G}_{\rho,\lambda, \xi_1, \xi_2, \omega}^\sigma \phi)(x) = I_{\xi_1}^\omega \phi x.
\]

Making the substitution in (35), we obtain (31).

**Remark 2.** In Corollary 2, if we take \( \eta(x, y) = x - y, \lambda = \alpha, \sigma = (1, 0, 0, \ldots), w = 0, \) and \( p = 1, \) then we obtain Theorem 1.5 in [29, 31].

**Theorem 4.** Let \( \phi: [\xi_1, \xi_2] \to \mathbb{R} \) be a differentiable function on \( (\xi_1, \xi_2) \) with \( \xi_1 < \xi_2. \) If \( |\phi'|^q \) is a generalized \( p \)-convex function on \([\xi_1, \xi_2]\), then the following inequality for fractional integral operator holds:

\[
\left| \frac{\phi(\xi_1) + \phi(\xi_2)}{2} - \frac{p^\prime(\alpha + 1)}{2(\xi_2^\prime - \xi_1^\prime)^\alpha} \left[ (I_{\xi_2}^\prime \phi)(\xi_1) + (I_{\xi_1}^\prime \phi)(\xi_2) \right] \right| \leq \frac{\left( \xi_2^\prime - \xi_1^\prime \right)}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left[ 2|\phi'(\xi_2)| + \eta(|\phi'(\xi_1)|, |\phi'(\xi_2)|) \right]. \tag{33}
\]

where \( \rho_1 = \rho p, \lambda_1 = \lambda p, \) and \( \sigma_1(k) = \sigma(k) = \left( \frac{1 - (1/2)^{(k+1)p}}{k(p + \lambda + 1)p_1 + 1} \right)^{1/p_1}, \tag{34} \)

**Proof.** Using Lemma 1, definition of generalized \( p \)-convexity of \( \phi \), and Hölder inequality, we have
\[
\left| \frac{\phi(\xi) + \phi(\xi)}{2} - p \left[ \left( F_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) + \left( F_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) \right] \right| \\
\leq \frac{\xi - \xi^2}{2} \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda + 1)} \left( F_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) k^p
\]
\[
\left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p} \left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p}
\]
\[
\left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p} \left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p}
\]
\[
\left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p} \left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p}
\]
\[
\left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p} \left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p}
\]
\[
\left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p} \left[ \left( 1 - (1 - \Theta) f_{p,\lambda,\xi}^{\sigma} \phi \right)(\xi) d\Theta \right]^{1/p}
\]

which completes the proof.

**Data Availability**

All data required for this research are included within the paper.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

Changyue Chen wrote the final version of this paper, verified the results, and arranged the funding for this paper, Muhammad Sheaab Saleem proposed the problem, proved the results, and supervised the work, and Muhammad Sajid Zahoor wrote the first version of the paper.

**Acknowledgments**

The authors are thankful to the University of Okara, Okara, Pakistan, for providing funds for this research. This work was funded by University of Okara, Okara, Pakistan.

**References**

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.

[2] V. Lakshmikantham and A. S. Vatsala, ”Basic theory of fractional differential equations,” *Nonlinear Analysis: Theory, Methods and Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.
[3] Y. Ouafik, “Modelling and simulation of a dynamic contact problem in thermo-piezoelectricity,” *Engineering and Applied Science Letters*, vol. 4, no. 2, pp. 43–52, 2021.

[4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, Netherlands, 2006.

[5] R. I. Gweryina, F. S. Kaduna, and M. Y. Kura, “Qualitative analysis of a mathematical model of divorce epidemic with anti-divorce therapy,” *Engineering and Applied Science Letters*, vol. 4, no. 2, pp. 1–11, 2021.

[6] S. Belarbi and Z. Dahmani, “On some new fractional integral inequalities,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 3, pp. 1–12, 2009.

[7] M. E. Omaba, L. O. Omenyi, and L. O. Omenyi, “Generalized fractional Hadamard type inequalities for \((Q_s)-class functions of the second kind,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 270–278, 2021.

[8] Z. Dahmani, “On Minkowski and Hermité-Hadamard integral inequalities via fractional integration,” *Annals of Functional Analysis*, vol. 1, no. 1, pp. 51–58, 2010.

[9] Z. Toghani and L. Gaggero–Sager, “Generalized fractional differential ring,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 279–287, 2021.

[10] M. A. Khan, T. Ali, S. S. Dragomir, and M. Z. Sarikaya, “Hermite-Hadamard type inequalities for conformable fractional integrals,” *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 112, no. 4, pp. 1033–1048, 2018.

[11] P. Fan, “New-type Hoeffding’s inequalities and application in tail bounds,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 248–261, 2021.

[12] M. Kunt and I. İçsan, “Hermite-Hadamard–Fejér type inequalities for p-convex functions,” *Arab Journal of Mathematical Sciences*, vol. 23, no. 2, pp. 215–230, 2017.

[13] M. Tariq, S. I. Butt, and S. I. Butt, “Some Ostrowski type integral inequalities via generalized harmonic convex functions,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 200–208, 2021.

[14] M. A. Noor, K. I. Noor, and S. Iftikhar, “Nonconvex functions and integral inequalities,” *Journal of Mathematics*, vol. 47, no. 2, p. 60, 2015.

[15] M. Gurbuz, A. O. Akdemir, S. Rashid, and E. Set, “Hermite–Hadamard inequality for fractional integrals of Caputo–Fabrizio type and related inequalities,” *Journal of Inequalities and Applications*, vol. 2020, no. 1, pp. 1–10, 2020.

[16] M. Caputo and M. Fabrizio, “A new definition of fractional derivative without singular kernel,” *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–13, 2015.

[17] M. A. Dokuyucu, E. Celik, H. Bulut, and H. M. Baskonus, “Cancer treatment model with the Caputo–Fabrizio fractional derivative,” *The European Physical Journal Plus*, vol. 133, no. 3, pp. 1–6, 2018.

[18] D. Baleanu, A. Jajarmi, H. Mohammadi, and S. Rezapour, “A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative,” *Chaos, Solitons and Fractals*, vol. 134, Article ID 109705, 2020.

[19] D. Baleanu, H. Mohammadi, and S. Rezapour, “A mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model,” *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–19, 2020.

[20] K. Shah, M. Sarwar, and D. Baleanu, “Study on Krasnoselskii’s fixed point theorem for Caputo–Fabrizio fractional differential equations,” *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–9, 2020.

[21] I. Iscan, “New estimates on generalization of some integral inequalities for s-convex functions and their applications,” 2012, http://arxiv.org/abs/1207.7114.

[22] K. Murota and A. Tamura, “New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities,” *Discrete Applied Mathematics*, vol. 131, no. 2, pp. 495–512, 2003.

[23] P. Burai and A. Hzy, “On approximately β-convex functions,” *Journal of Convex Analysis*, vol. 18, no. 2, pp. 447–454, 2011.

[24] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, “Generalized Riemann–Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of hadamard inequalities,” *IEEE access*, vol. 6, pp. 64946–64953, 2018.

[25] K. S. Zhang and J. P. Wan, “p-Convex functions and their properties,” *Pure Applied Mathematics*, vol. 23, no. 1, pp. 130–133, 2007.

[26] C. Y. Jung, M. S. Saleem, W. Nazeer, M. S. Zahoor, A. Latif, and S. M. Kang, “Unification of generalized and-convexity,” *Journal of Function Spaces*, vol. 2020, Article ID 4016386, 6 pages, 2020.

[27] R. K. Raina, “On generalized Wright’s hypergeometric functions and fractional calculus operators,” *East Asian Mathematical Journal*, vol. 21, no. 2, pp. 191–203, 2005.

[28] R. P. Agarwal, M. J. Luo, and R. K. Raina, “On Ostrowski type inequalities,” *Fasciculi Mathematici*, vol. 56, no. 1, pp. 5–27, 2016.

[29] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, “Hermite–Hadamard’s inequalities for fractional integrals and related fractional inequalities,” *Mathematical and Computer Modelling*, vol. 57, no. 9–10, pp. 2403–2407, 2013.

[30] E. Set, B. elik, and A. O. Akdemir, “Some new Hermite–Hadamard type inequalities for quasi-convex functions via fractional integral operator,” in *AIP Conference Proceedings*, vol. 1833, no. 1, AIP Publishing LLC, Article ID 020021, 2017.

[31] T. Ali, M. A. Khan, and Y. Khurshidi, “Hermite–Hadamard inequality for fractional integrals via eta-convex functions,” *Acta Mathematica Universitatis Comenianae*, vol. 86, no. 1, pp. 153–164, 2017.