REMARKS ON FOURIER MULTIPLIERS AND APPLICATIONS TO THE WAVE EQUATION

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Abstract. Exploiting continuity properties of Fourier multipliers on modulation spaces and Wiener amalgam spaces, we study the Cauchy problem for the NLW equation. Local wellposedness for rough data in modulation spaces and Wiener amalgam spaces is shown. The results formulated in the framework of modulation spaces refine those in [3]. The same arguments may apply to obtain local wellposedness for the NLKG equation.

1. Introduction and results

In this short note we study the Cauchy problem for the nonlinear wave equation (NLW):

\begin{equation}
\begin{cases}
\partial_t^2 u - \Delta_x u = F(u) \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x),
\end{cases}
\end{equation}

with $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \geq 1$, $\Delta_x = \partial_{x_1}^2 + \ldots + \partial_{x_d}^2$, $F$ is a scalar function on $\mathbb{C}$, with $F(0) = 0$. The solution $u(t, x)$ is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. We will consider the case in which $F$ is an entire analytic function (in the real sense), and we shall highlight the special case $F(u) = \lambda |u|^{2k}u$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, where we have better results.

The integral version of the problem (1) has the form

\begin{equation}
u(t, \cdot) = K'(t)u_0 + K(t)u_1 + \mathcal{B}F(u),\end{equation}

where

\begin{equation}
K'(t) = \cos(t\sqrt{-\Delta}), \quad K(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}, \quad \mathcal{B} = \int_0^t K(t - \tau) \cdot d\tau.
\end{equation}

Here, for every fixed $t$, the operators $K'(t), K(t)$ in (3) are Fourier multipliers with symbols $\cos(2\pi t|\xi|)$, $\sin(2\pi t|\xi|)/(2\pi |\xi|)$, $\xi \in \mathbb{R}^d$. We recall that given a function

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σ on \( \mathbb{R}^d \) (the so-called symbol of the multiplier or, simply, multiplier), the corresponding Fourier multiplier operator \( H_\sigma \) is formally defined by

\[
H_\sigma f(x) = \int_{\mathbb{R}^d} e^{2\pi i \langle \xi \rangle} \sigma(\xi) \hat{f}(\xi) \, d\xi.
\]

So, continuity properties for multipliers in suitable spaces yield estimates for the linear part of the equation. These latter are then combined with abstract iteration (contraction) arguments to obtain local wellposedness of (1). We deal with this program in the framework of modulation spaces and also for Wiener amalgam spaces.

This was first considered, for the modulation spaces, in [1, 3, 14, 15], where the classical framework of \( L^p \), Sobolev or Besov spaces is abandoned in favour of such spaces, which permit to handle initial data which are not covered by the classical results. Precisely, a topic of great interest is the problem of the wellposedness of (1) (and other dispersive equations) in low regularity spaces. The classical results in this connections (see [10, 12] and the references therein) use the scale of Sobolev spaces \( H^s \) to measure the local regularity of the initial data. The modulation spaces (and also the Wiener amalgam spaces considered here) provide spaces where the local regularity is instead measured by the \( \mathcal{F}L^p \)-scale (that are the spaces of temperate distributions whose Fourier transform is in \( L^p \)). In order to state our results, we first introduce the spaces we deal with ([5, 7, 3]).

Let \( g \in S(\mathbb{R}^d) \) be a non-zero window function and consider the so-called short-time Fourier transform (STFT) \( V_g f \) of a function/tempered distribution \( f \) with respect to the window \( g \):

\[
V_g f(x, \xi) = \langle f, M_x T_{-x} g \rangle = \int e^{-2\pi i \xi y} f(y) g(x - y) \, dy,
\]

i.e., the Fourier transform \( \mathcal{F} \) applied to \( f T_{-x} g \).

For \( s \in \mathbb{R} \), we consider the weight function \( \langle x \rangle^s = (1 + |x|^2)^{s/2}, x \in \mathbb{R}^d \). If \( 1 \leq p, q \leq \infty, s \in \mathbb{R} \), the modulation space \( \mathcal{M}^{p,q}_s(\mathbb{R}^n) \) is defined as the closure of the Schwartz class with respect to the norm

\[
\|f\|_{\mathcal{M}^{p,q}_s} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p \, dx \right)^{q/p} \langle \xi \rangle^{sq} \, d\xi \right)^{1/q}
\]

(with obvious modifications when \( p = \infty \) or \( q = \infty \)).

Among the properties of modulation spaces, we record that they are Banach spaces whose definition is independent of the choice of the window \( g \in S(\mathbb{R}^d) \), \( \mathcal{M}^{2,2} = L^2, \mathcal{M}^{p_1,q_1}_s \hookrightarrow \mathcal{M}^{p_2,q_2}_s, \) if \( p_1 \leq p_2 \) and \( q_1 \leq q_2 \), \( \left( \mathcal{M}^{p,q}_s \right)' = \mathcal{M}^{p',q'}_{-s} \).

For a more general definition, involving different kinds of weight functions, both in the time and the frequency variables we refer to [7].
The Definition 1 has been accordingly extended to the quasi-Banach case $0 < p, q < 1$ in [3, 9]. Here the window function $g$ is to be restricted to the class

$$
\left\{ g \in \mathcal{S}(\mathbb{R}^d) : \text{supp} \hat{g} \subset \{ \xi : |\xi| \leq 1 \}, \text{ and } \sum_{k \in \mathbb{Z}^d} \hat{g}(\xi - \alpha k) \equiv 1, \forall \xi \in \mathbb{R}^d \right\},
$$

for a sufficiently small $\alpha > 0$, so that the set above is not empty. For other definitions of modulation spaces for all $0 < p, q \leq \infty$, we refer to [6, 11, 13].

We also recall the definition of the so-called Wiener amalgam spaces (3, 4, 11). First of all we denote by $L^p_s$, $s \in \mathbb{R}$, $0 < p \leq \infty$, the weighted $L^p$ space of function $f$ in $\mathbb{R}^d$ such that $\langle x \rangle^s f(x)$ is in $L^p$, with the obvious quasi-norm (norm if $p \geq 1$). Then, for $s, \gamma \in \mathbb{R}$, $1 \leq p, q < \infty$, a tempered distribution $f$ is in the Wiener amalgam space $W(\mathcal{F}L^q_s, L^q_\gamma)(\mathbb{R}^d)$ if $f$ is locally in $\mathcal{F}L^q_s(\mathbb{R}^d)$, that is, for every nonnull $g \in C_0^\infty(\mathbb{R}^d)$, $\mathcal{F}(fT_x g) \in L^q_s(\mathbb{R}^d)$ and

$$
\|f\|_{W(\mathcal{F}L^q_s, L^q_\gamma)} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\mathcal{F}(fT_x g)(y)|^q \langle y \rangle^{sq} dy \right)^{p/q} \langle x \rangle^{sp} dx \right)^{p} < \infty.
$$

When $p = \infty$ or $q = \infty$, we define $W(\mathcal{F}L^q_s, L^q_\gamma)(\mathbb{R}^d)$ as the closure of the Schwartz space with the norm in (3) (modified in the obvious way). This definition is independent of the test function $g \in \mathcal{S}(\mathbb{R}^d)$. For properties we refer to [4]. The definition above is extended to the cases $0 < p, q < 1$ in [3]; here the window function $g \in \mathcal{S}$ satisfies $\text{supp} \hat{g} \subset \{ |\xi| \leq 1 \}$, see also [11] for the quasi-Banach case in the global component $L^q$.

We now briefly present our wellposedness results (see the statements of Theorems 4.1, 4.3 and 4.4 for details). Consider first a nonlinearity $F(u)$, where $F$ is an entire real-analytic function. Then we will prove that (1) is wellposed in $\mathcal{M}^{p,1}_s(\mathbb{R}^d)$ for every $1 \leq p \leq \infty$, $s \geq 0$. In particular, for every $(u_0, u_1) \in \mathcal{M}^{p,1}_s(\mathbb{R}^d) \times \mathcal{M}^{p,1}_{s-1}(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution $u \in C([0, T]; \mathcal{M}^{p,1}_s(\mathbb{R}^d))$ to (2). Similarly, we shall show that (1) is also wellposed in $W(\mathcal{F}L^q_s, L^q_\gamma)$ for every $1 \leq p \leq \infty$, $s, \gamma \geq 0$.

We then consider the case of the power nonlinearity $F(u) = \lambda |u|^{2k} u$. We show that (1) is wellposed in $\mathcal{M}^{p,q}_s(\mathbb{R}^d)$ for every $1 \leq p \leq \infty$, $q' > 2kd$, $s \geq 0$. This refines a result in [3], where the authors assume $q' = \infty$ (and $u_1$ with the same regularity $s$ as $u_0$, rather than $s-1$). For the same nonlinearity we also prove wellposedness in $W(\mathcal{F}L^q_s, L^q_\gamma)$, $1 \leq p \leq \infty$, $q' > 2kd$, $s, \gamma \geq 0$.

The most interesting cases are of course when $s = \gamma = 0$, $p = \infty$, and $q$ is as large as possible. In this connection we notice that $\mathcal{M}^{\infty,q}_s \subset W(\mathcal{F}L^q_s, L^\infty)$, so Wiener amalgam spaces allow us to consider more general initial data than those in the above modulation spaces.
Let us highlight that the same arguments apply to the study of the Cauchy problem for the nonlinear Klein-Gordon equation (NLKG) (that is \([1]\) with the operator \(-\Delta_x\) replaced by \(I - \Delta_x\)). We omit details.

The basic tool of the proof is given by Fourier multiplier estimates on modulation spaces. Precisely, we will use (the first part of) the following refinement of results in \([1, 3]\), involving multipliers with symbols in the Wiener amalgam spaces \(W(\mathcal{F}L^p, L^q)\).

**Proposition 1.1.** Let \(s, t \in \mathbb{R},\ 0 < q \leq \infty\). Let \(\sigma\) be a function on \(\mathbb{R}^d\) and consider the Fourier multiplier operator defined in (4).

(i) If \(1 \leq p \leq \infty\), \(\sigma \in W(\mathcal{F}L^1, L^\infty)\), then the operator \(H_\sigma\) extends to a bounded operator from \(M^p,q_s\) into \(M^p,q_{s+\gamma}\), with

\[
\|H_\sigma f\|_{M^p,q_{s+\gamma}} \lesssim \|\sigma\|_{W(\mathcal{F}L^1, L^\infty)} \|f\|_{M^p,q_s}.
\]

(ii) If \(0 < p < 1\), \(\sigma \in W(\mathcal{F}L^p, L^\infty)\), then the operator \(H_\sigma\) is a bounded operator from \(M^p,q_s\) into \(M^p,q_{s+\gamma}\), with

\[
\|H_\sigma f\|_{M^p,q_{s+\gamma}} \lesssim \|\sigma\|_{W(\mathcal{F}L^p, L^\infty)} \|f\|_{M^p,q_s}.
\]

Similar estimates are proved for multipliers acting on Wiener amalgam spaces as well.

### 2. Preliminary results

In this section we collected some preliminary results.

This first lemma is proved in \([15, Corollary 4.2]\). The case \(p, p_i \geq 1\) was first proved by Feichtinger \([5]\). It is also proved in \([3]\) using the theory of multi-linear pseudodifferential operators.

**Lemma 2.1.** Let \(s \geq 0,\ 0 < p \leq p_i \leq \infty,\ 1 \leq r, q_i \leq \infty,\ N \in \mathbb{N}\), satisfy

\[
\sum_{i=1}^{N} \frac{1}{p_i} = \frac{1}{p}, \quad \sum_{i=1}^{N} \frac{1}{q_i} = N - 1 + \frac{1}{r},
\]

then we have

\[
\| \prod_{i=1}^{N} u_i \|_{M^{p,r}_s} \leq \prod_{i=1}^{N} \| u_i \|_{M^{p_i,q_i}_s}.
\]

In particular, for \(p_i = Np\), \(q_i = q\), \(i = 1, \ldots N\), we get

\[
\| \prod_{i=1}^{N} u_i \|_{M^{p,r}_s} \leq \prod_{i=1}^{N} \| u_i \|_{M^{p,q}_s}, \quad \frac{N}{q} = N - 1 + \frac{1}{r}.
\]
Lemma 2.2. Let $0 < p_i, p \leq \infty$, $1 \leq r, q_i \leq \infty$, $N \in \mathbb{N}$ satisfy (8), and $s \geq 0$, 
$\gamma = \sum_{i=1}^{N} \gamma_i$, $\gamma_i \in \mathbb{R}$. We have

$$
\| \prod_{i=1}^{N} u_i \|_{W(FL^r_i, L^p_i)} \leq \prod_{i=1}^{N} \| u_i \|_{W(FL^r_i, L^p_i)}.
$$

In particular, for $p_i = Np$, $q_i = q$, $\gamma_i = \gamma/N \geq 0$, $i = 1, \ldots, N$, we get

$$
(10) \quad \| \prod_{i=1}^{N} u_i \|_{W(FL^r_i, L^p_i)} \leq \prod_{i=1}^{N} \| u_i \|_{W(FL^r_i, L^p_i)}, \quad \frac{N}{q} = N - 1 + \frac{1}{r}.
$$

Proof. We choose a window function $g$ of the type $g(x) = \prod_{i=1}^{n} g_i(x)$, with $g_i \in \mathcal{C}^\infty_0(\mathbb{R}^d) \setminus \{0\}$, $i = 1, \ldots, N$. Then $M_{-x} \hat{g} = M_{-x} \hat{g}_1 * \ldots * M_{-x} \hat{g}_N$, so that

$$
\mathcal{F} \left( \prod_{i=1}^{N} u_i T_x g \right) = (\hat{u}_1 * M_{-x} \hat{g}_1) * \ldots * (\hat{u}_N * M_{-x} \hat{g}_N).
$$

Since $s \geq 0$, a repeated application of the inequality $\langle \xi \rangle^s \leq \langle \eta \rangle^s \langle \xi - \eta \rangle^s$ and Young inequality then give

$$
\| \prod_{i=1}^{N} u_i T_x g \|_{FL^r} \leq \prod_{i=1}^{N} \| u_i T_x g \|_{FL^r_i}.
$$

Now we multiply this inequality by $\langle x \rangle^\gamma$ and conclude by an application of H"older’s inequality.

Lemma 2.3. For $0 < p, q \leq \infty$, $s \in \mathbb{R}$ we have

$$
(11) \quad W(FL^r_s, L^p) \hookrightarrow W(FL^q_{s-1}, L^p), \quad \frac{d}{q} - \frac{d}{r} < 1.
$$

Proof. An application of H"older’s inequality shows that $L^r_s \hookrightarrow L^q_{s-1}$. Hence the desired embedding follows directly from the definition of Wiener amalgam spaces.

In the sequel the following convolution relations will be useful [5].

Lemma 2.4. For $i = 1, 2, 3$, let $B_i$ be one of the Banach spaces $FL^r_s$ $(1 \leq q \leq \infty$, $s \in \mathbb{R})$, $C_i$ be one of the Banach spaces $L^p_\gamma$ $(1 \leq p \leq \infty$, $\gamma \in \mathbb{R})$. If $B_1 \ast B_2 \hookrightarrow B_3$ and $C_1 \ast C_2 \hookrightarrow C_3$, we have

$$
(12) \quad W(B_1, C_1) \ast W(B_2, C_2) \hookrightarrow W(B_3, C_3).
$$
MULTIPLIER ESTIMATES

In this section we first prove estimates in $M_{p,q}^{s,\gamma}$ for the multipliers arising in the wave propagator, refining those in [1, 3]. Then we prove estimates in the Wiener amalgam spaces $W(F^{p,q}_{r,q}, L^{q,\gamma})$. We begin with the proof of Proposition 1.1.

Proof of Proposition 1.1. For the case $1 \leq p \leq \infty$, the arguments are a rearrangement of [1, Lemma 8], whereas the case $0 < p < 1$ one argues as in [2, Lemma 1].

For the sake of clarity, we shall detail the first case. Choose a window function $g_1 = g_0 \ast g_0 \in S(\mathbb{R}^d)$ and use $M_\xi g_1 = M_\xi g_0 \ast M_\xi g_0$, so that, by Young inequality,

\[
\|H_\sigma f\|_{M_{p,q}^{s,\gamma}}^q = \int_{\mathbb{R}^d} \|(H_\sigma f) \ast M_\xi g_1\|_p^q < (s+\gamma) d\xi \\
\leq \int_{\mathbb{R}^d} \|\mathcal{F}^{-1}(\sigma) \ast M_\xi g_0\|_{L^1}^q \|f \ast M_\xi g_0\|_{L^p}^q < (s+\gamma) d\xi \\
\leq \left(\sup_{\xi \in \mathbb{R}^d} \|\mathcal{F}^{-1}(\sigma) \ast M_\xi g_0\|_{L^1}^q < (s+\gamma) \right) \int_{\mathbb{R}^d} \|f \ast M_\xi g_0\|_{L^p}^q < (s+\gamma) d\xi \\
= \|\sigma\|_{W(F^{1,1}, L^{q,\gamma})}^q \|f\|_{M_{p,q}^{s,\gamma}},
\]

as desired.

We need also the following elementary result. We give the proof for the sake of completeness.

Lemma 3.1. Let $R > 0$ and $f \in C_0^\infty(\mathbb{R}^d)$ such that $\text{supp } f \subset B(y, R) := \{x \in \mathbb{R}^d, |x - y| \leq R\}$, with $y \in \mathbb{R}^d$. Then, for every $0 < p \leq \infty$, there exist an index $k = k(p) \in \mathbb{N}$ and a constant $C_{R,p} > 0$ (which depends only on $R$ and $p$) such that

\[
\|f\|_{F^{p,q}} \leq C_{R,p} \sup_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty}.
\]

Proof. We know $f \in C_0^\infty \subset F^{p,p}$. If we take $k \in \mathbb{N}$ such that $kp > d/2$, then

\[
\|f\|_{F^{p,p}} = \left(\int_{\mathbb{R}^d} \left(\frac{1}{\xi^{2k}} |f| \right)^p dx \right)^{1/p} \\
= \left(\int_{\mathbb{R}^d} \frac{1}{\xi^{2k}} \left|\int_{\mathbb{R}^d} e^{-2\pi ix\xi}(1 - \Delta)^k f(x)dx\right|^p d\xi \right)^{1/p} \\
\leq C_k \text{vol}(B(y, R)) \sup_{|\alpha| \leq 2k} \|\partial^\alpha f\|_{L^\infty} \leq C_{R,p} \sup_{|\alpha| \leq 2k} \|\partial^\alpha f\|_{L^\infty}.
\]

Now, we consider the family of multipliers $\sigma_{\alpha,\delta}$, and $\tau$ defined by

\[
\sigma_{\alpha,\delta} = \frac{\sin |\xi|^{\alpha}}{|\xi|^{\delta}}, \quad \delta \leq \alpha \leq 1, \quad \alpha > 0,
\]
Proposition 3.1. (i) The multipliers $\sigma_{\alpha,\delta}$ in (17) are in the space $W(F^{1}, L_{\infty})$.
(ii) For $\alpha = \delta = 1$, we have $\sigma_{1, 1} \in W(F^{p}, L_{\infty}^{p})$ for every $0 < p \leq \infty$.
(iii) The multiplier $\tau$ in (18) is in $W(F^{p}, L_{\infty}^{p})$ for every $0 < p \leq \infty$.

Proof. We first prove (i) and (ii). We consider a function $\chi \in C_{0}^{\infty}(\mathbb{R}^{d})$, $1 \leq \chi(\xi) \leq 2$, such that $\chi(\xi) = 1$ if $|\xi| \leq 1$, whereas $\chi(\xi) = 0$ if $|\xi| \geq 2$. Then, we split the multiplier into the sum of two functions $\sigma_{\text{sing}}$ and $\sigma_{\text{osc}}$, bearing the singularity at the origin and the oscillation at infinity, respectively:

\[
\sigma_{\alpha,\delta}(\xi) = \chi(\xi)\sigma_{\alpha,\delta}(\xi) + (1 - \chi(\xi))\sigma_{\alpha,\delta}(\xi) = \sigma_{\text{sing}}(\xi) + \sigma_{\text{osc}}(\xi).
\]

Singularity at the origin. First, we shall prove that $\sigma_{\text{sing}}$ is in $W(F^{1}, L_{\infty}^{s})$, for every $s \in \mathbb{R}$. Indeed, choose $g \in C_{0}^{\infty}(\mathbb{R}^{d})$ such that $\supp g \subset \{\xi \in \mathbb{R}^{d}, |\xi| \leq 1\}$, then $\sigma_{\text{sing}}(\xi)T_{x}g(\xi) = 0$ if $|x| > 3$, so that

\[
\|\sigma_{\text{sing}}\|_{W(F^{1}, L_{\infty}^{s})} = \text{ess sup}_{x \in \mathbb{R}^{d}}(|\sigma_{\text{sing}}T_{x}g|_{F^{1}}(x)) = \text{ess sup}_{|x| \leq 3}(|\sigma_{\text{sing}}T_{x}g|_{F^{1}}(x)) \leq (10)^{|s|/2}\text{ess sup}_{|x| \leq 3}(|\sigma_{\text{sing}}T_{x}g|_{F^{1}}(x)) \leq (10)^{|s|/2}\|\sigma_{\text{sing}}\|_{W(F^{1}, L_{\infty})} < \infty,
\]

for $\sigma_{\text{sing}} \in F^{1} \subset W(F^{1}, L_{\infty})$, (see [1], Theorem 9).

For $\alpha = \delta = 1$, the function $\sigma_{\text{sing}}$ is in $C_{0}^{\infty} \subset W(F^{p}, L_{\infty}^{p})$, for every $0 < p \leq \infty$.

Oscillation at infinity. At infinity the multipliers $\sigma_{\text{osc}}$ are in $W(F^{p}, L_{\infty}^{p})$, for every $0 < p \leq 1$. The proof uses Lemma 3.1, applied to the function $\sigma_{\text{osc}}T_{x}g \in C_{0}^{\infty}(\mathbb{R}^{d})$, with $g \in C_{0}^{\infty}(\mathbb{R}^{d})$. Precisely, we observe the decay properties of $\sigma_{\text{osc}}$,

\[
|\partial^{\alpha}\sigma_{\text{osc}}(\xi)| \lesssim \langle \xi \rangle^{-\delta}, \quad \forall \alpha \in \mathbb{Z}_{+}^{d}
\]

and those of the window $g$,

\[
|\partial^{\alpha}g(\xi - x)| \lesssim \langle x - \xi \rangle^{-N}, \quad \forall x, \xi \in \mathbb{R}^{d}, \ N \in \mathbb{N}.
\]

Combining the preceding estimates and the weight property

\[
\langle \xi \rangle^{-\delta}\langle x - \xi \rangle^{-|\delta|} \leq \langle x \rangle^{-\delta}
\]

yields

\[
\|\sigma_{\text{osc}}\|_{W(F^{p}, L_{\infty}^{p})} = \sup_{x \in \mathbb{R}^{d}}\|\sigma_{\text{osc}}T_{x}g\|_{F^{p}}(x) \delta < \infty.
\]

Finally, to prove (iii), observe that $\tau \in C_{0}^{\infty}(\mathbb{R}^{d})$ fulfills

\[
|\partial^{\alpha}\tau(\xi)| \lesssim 1, \quad \forall \xi \in \mathbb{R}^{d};
\]

so, arguing similarly to what done for $\sigma_{\text{osc}}$ before, we obtain the claim. \(\square\)
Corollary 3.2. Let $s \in \mathbb{R}$, $0 < q \leq \infty$.

(i) For every $1 \leq p \leq \infty$, the Fourier multiplier $H_{\sigma_{\alpha, \delta}}$, with symbol $\sigma_{\alpha, \delta}$ defined in (17), extends to a bounded operator from $\mathcal{M}^p_q(\mathbb{R}^d)$ into $\mathcal{M}^p_q(\mathbb{R}^d)$, with

$$\|H_{\sigma_{\alpha, \delta}}f\|_{\mathcal{M}^p_q} \lesssim \|\sigma_{\alpha, \delta}\|_{W(\mathcal{F}L^1, L_\infty^\infty)} \|f\|_{\mathcal{M}^p_q}. \quad (22)$$

(ii) For every $0 < p < 1$, $H_{\sigma_{1,1}}$ extends to a bounded operator from $\mathcal{M}^p_q(\mathbb{R}^d)$ into $\mathcal{M}^p_q(\mathbb{R}^d)$ with

$$\|H_{\sigma_{1,1}}f\|_{\mathcal{M}^p_{p+1}} \lesssim \|\sigma_{1,1}\|_{W(\mathcal{F}L^p, L_\infty^\infty)} \|f\|_{\mathcal{M}^p_q}. \quad (23)$$

(iii) For every $0 < p \leq \infty$, the Fourier multiplier $H_{\tau}$, with the symbol $\tau$ in (18) extends to a bounded operator from $\mathcal{M}^p_q(\mathbb{R}^d)$ into $\mathcal{M}^p_q(\mathbb{R}^d)$, with

$$\|H_{\tau}f\|_{\mathcal{M}^p_q} \lesssim \|\tau\|_{W(\mathcal{F}L^r, L_\infty^\infty)} \|f\|_{\mathcal{M}^p_q}, \quad (24)$$

where $r = \min\{1, p\}$.

Proof. The desired result follows from Propositions 3.1 and 1.1.

Proposition 3.3. (i) The functions $\sigma(\xi) = e^{\pm i|\xi|}$ belong to $M^{\infty,1}$, so that $\hat{\sigma} \in W(\mathcal{F}L^\infty, L^1)$.

(ii) The function $\sigma_{1,1}$ in (17) satisfies $\hat{\sigma}_{1,1} \in W(\mathcal{F}L^\infty_1, L^1_1)$ for every $\gamma \in \mathbb{R}$.

(iii) The function $\tau$ in (18) satisfies $\hat{\tau} \in W(\mathcal{F}L^\infty_1, L^1_1)$ for every $\gamma \in \mathbb{R}$.

Proof. Part (i) was proved in [1, Theorem 1]. In order to prove the second point, we split the symbol $\sigma_{1,1}$ as in (19) (with $\alpha = \delta = 1$). Then $\sigma_{\text{sing}} \in C^\infty_0(\mathbb{R}^d)$, so that $\hat{\sigma} \in \mathcal{S}(\mathbb{R}^d) \subset W(\mathcal{F}L^\infty_1, L^1_1)$ for every $\gamma \in \mathbb{R}$. We now treat the symbol $\sigma_{\text{osc}}$. We observe that $\hat{\sigma}_{\text{osc}}T_y = \hat{\sigma}_{\text{osc}} * M_{-y} \hat{g}$, where $\hat{\sigma}_{\text{osc}}(\xi) = \sigma_{\text{osc}}(-\xi)$. Hence we will obtain $\hat{\sigma}_{\text{osc}} \in W(\mathcal{F}L^\infty_1, L^1_1)$ if we prove that

$$\|\langle \xi \rangle \int e^{-2\pi i y \cdot x} \hat{g}(x) \sigma_{\text{osc}}(\xi - x) \, dx\|_{L^\infty} \leq C(1 + |y|^2)^{-N},$$

for an integer $N$ such that $N - \gamma > d/2$. We multiply this inequality by $(1 + |y|^2)^N$. By an integration by part and the Leibniz rule we see that it suffices to prove that

$$\langle \xi \rangle \int |\partial^\alpha \hat{g}(x) \partial^\beta \sigma_{\text{osc}}(\xi - x)| \, dx \leq C, \quad \forall \gamma \in \mathbb{R}, \ |\alpha| + |eta| \leq 2N.$$

Then one concludes by applying (20) (with $\delta = 1$), combined with the estimate $|\partial^\alpha \hat{g}(x)| \lesssim \langle x \rangle^{-N'}$, for all $x \in \mathbb{R}^d$, $N' \in \mathbb{N}$, and (21).

The proof of (iii) is completely similar.
Corollary 3.4. Let $s, \gamma \in \mathbb{R}, 1 \leq p, q \leq \infty$.
(i) The Fourier multiplier $H_{\sigma,1}$, with symbol $\sigma_{1,1}(\xi) = \frac{\sin|\xi|}{|\xi|}$ extends to a bounded operator from $W(\mathcal{F}L^p_s, L^q_{\gamma})$ into $W(\mathcal{F}L^p_{s+1}, L^q_{\gamma})$, with
\begin{equation}
\|H_{\sigma,1}f\|_{W(\mathcal{F}L^p_{s+1}, L^q_{\gamma})} \lesssim \|\sigma_{1,1}\|_{W(\mathcal{F}L^\infty_1, L^1_{1\gamma})} \|f\|_{W(\mathcal{F}L^p_s, L^q_{\gamma})}.
\end{equation}
(ii) The Fourier multiplier $H_{\tau}$, with symbol $\tau(\xi) = \cos|\xi|$ extends to a bounded operator from $W(\mathcal{F}L^p_s, L^q_{\gamma})$ into $W(\mathcal{F}L^p_s, L^q_{\gamma})$, with
\begin{equation}
\|H_{\tau}f\|_{W(\mathcal{F}L^p_s, L^q_{\gamma})} \lesssim \|\tau\|_{W(\mathcal{F}L^\infty_1, L^1_{1\gamma})} \|f\|_{W(\mathcal{F}L^p_s, L^q_{\gamma})}.
\end{equation}
Proof. Since $H_{\sigma}f = \hat{\sigma} \ast f$, when $\sigma$ is a temperate distribution and $f$ is a Schwartz function, the desired result follows at once from Proposition 3.3 and the convolution relations \cite{12} of Wiener amalgam spaces (recall, $L^p_{\gamma} \ast L^1_{1\gamma} \hookrightarrow L^p_{\gamma}$).

4. Local wellposedness of NLW

In this section we establish and prove the wellposedness result outlined in the Introduction.

Theorem 4.1. Assume $s \geq 0$, $1 \leq p \leq \infty$, $(u_0, u_1) \in \mathcal{M}^{p,1}_s(\mathbb{R}^d) \times \mathcal{M}^{p,1}_{s-1}(\mathbb{R}^d)$ and $F(z) = \sum_{j,k=0}^{\infty} c_{j,k} z^j \bar{z}^k$, an entire real-analytic function on $\mathbb{C}$ with $F(0) = 0$. For every $R > 0$, there exists $T > 0$ such that for every $(u_0, u_1)$ in the ball $B_R$ of center 0 and radius $R$ in $\mathcal{M}^{p,1}_s(\mathbb{R}^d) \times \mathcal{M}^{p,1}_{s-1}(\mathbb{R}^d)$ there exists a unique solution $u \in C^0([0,T]; \mathcal{M}^{p,1}_s(\mathbb{R}^d))$ to (2). Furthermore, the map $(u_0, u_1) \mapsto u$ from $B_R$ to $C^0([0,T]; \mathcal{M}^{p,1}_s(\mathbb{R}^d))$ is Lipschitz continuous.

The main features of the proof are given by Fourier multiplier estimates on modulation spaces, obtained in the previous section, together with the following classical iteration argument (see e.g. \cite{12}, Proposition 1.38).

Proposition 4.2. Let $\mathcal{N}$ and $\mathcal{T}$ be two Banach spaces. Suppose we are given a linear operator $\mathcal{B} : \mathcal{N} \to \mathcal{T}$ with the bound
\begin{equation}
\|\mathcal{B}f\|_\mathcal{T} \leq C_0 \|f\|_\mathcal{N}
\end{equation}
for all $f \in \mathcal{N}$ and some $C_0 > 0$, and suppose that we are given a nonlinear operator $F : \mathcal{T} \to \mathcal{N}$ with $F(0) = 0$, which obeys the Lipschitz bounds
\begin{equation}
\|F(u) - F(v)\|_\mathcal{N} \leq \frac{1}{2C_0} \|u - v\|_\mathcal{T}
\end{equation}
for all $u, v$ in the ball $B_\mu := \{u \in \mathcal{T} : \|u\|_\mathcal{T} \leq \mu\}$, for some $\mu > 0$. Then, for all $u_{lin} \in B_{\mu/2}$ there exists a unique solution $u \in B_\mu$ to the equation
\begin{equation}
u = u_{lin} + BF(u),
\end{equation}
with the map $u_{lin} \mapsto u$ Lipschitz with constant at most 2 (in particular, $\|u\|_\mathcal{T} \leq 2\|u_{lin}\|_\mathcal{T}$).
Proof of Theorem 4.1. We first observe that, by Corollary 3.2, for every $0 < p \leq \infty$ the multiplier $K'(t)$, with symbol $\cos(2\pi|\xi|)$, can be extended to a bounded operator on $\mathcal{M}^{p,1}_s$, with
\begin{equation}
\|K'(t)u_0\|_{\mathcal{M}^{p,1}_s} \leq C\|u_0\|_{\mathcal{M}^{p,1}_s}, \quad t \in [0, 1].
\end{equation}
The uniformity of the constant $C$, when $t$ varies in bounded subsets, follows from the proof of the boundedness property itself.

Similarly, the multiplier operator $K(t)$ with symbol $\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}$, satisfies the estimate
\begin{equation}
\|K(t)u_1\|_{\mathcal{M}^{p,1}_s} \leq C\|u_1\|_{\mathcal{M}^{p,1}_s}, \quad t \in [0, 1],
\end{equation}
for every $0 < p \leq \infty$.

Now we are going to apply Proposition 4.2 with $T = \mathcal{N} = C^0([0, T]; \mathcal{M}^{p,1}_s)$, where $T \leq 1$ will be chosen later on, with the nonlinear operator $\mathcal{B}$ given by the Duhamel operator in (3). Here $u_{\text{lin}} := K'(t)u_0 + K(t)u_1$ is in the ball $B_{\mu/2} \subset T$ by (29), (30), if $\mu$ is sufficiently large, depending on $R$. We see that (27) follows, with a constant $C_0 = O(T)$ from the Minkowski integral inequality (now $p \geq 1$) and (29).

In order to verify (28) observe that
\begin{equation}
F(z) - F(w) = \int_0^1 \frac{d}{dt}F(tz + (1-t)w) \, dt
= (z - w) \sum_{j,k,l,m \geq 0} c_{j,k,l,m} z^j \overline{w}^l w^m + (\overline{z} - \overline{w}) \sum_{j,k,l,m \geq 0} c'_{j,k,l,m} z^j \overline{w}^l w^m.
\end{equation}
Hence, applying the relation (9) for $q = r = 1$, we obtain, for $u, v \in \mathcal{M}^{p,1}_s$,
\begin{equation}
\|F(u) - F(v)\|_{\mathcal{M}^{p,1}_s} \leq \|u - v\|_{\mathcal{M}^{p,1}_s} \sum_{j,k,l,m \geq 0} (|c_{j,k,l,m}| + |c'_{j,k,l,m}|)\|u\|^{j+k}_{\mathcal{M}^{p,1}_s} \|v\|^{l+m}_{\mathcal{M}^{p,1}_s} < \infty.
\end{equation}
This expression is $\leq C_p\|u - v\|_{\mathcal{M}^{p,1}_s}$ if $u, v \in B_\mu$. Hence, by choosing $T$ sufficiently small we conclude the proof of existence, and also that of uniqueness among the solution in $T$ with norm $O(R)$. This last constraint can be eliminated by a standard continuity argument (cf. the proof of Proposition 3.8 in [12]).

Consider now the nonlinearity
\begin{equation}
F(u) = F_k(u) = \lambda |u|^{2k}u = \lambda u^{k+1}u^k, \quad \lambda \in \mathbb{C}, \ k \in \mathbb{N}.
\end{equation}
We have the following result.

Theorem 4.3. Let $F(u)$ as in (31), $1 \leq p \leq \infty$, $s \geq 0$, and
\begin{equation}
q' > 2kd.
\end{equation}
For every $R$ there exists $T > 0$ such that for every $(u_0, u_1)$ in the ball $B_R$ of center $0$ and radius $R$ in $\mathcal{M}^{p,q}_s(\mathbb{R}^d) \times \mathcal{M}^{p,q}_{s-1}(\mathbb{R}^d)$ there exists a unique solution
Proof. The proof goes as the one of Theorem 4.1, but with a different inequality for quasi-Banach spaces. Indeed, for any quasi-Banach space \(N\), it holds that \(\|F(z) - F(w)\| \leq C\|z - w\|\), where \(C\) is a constant. Then (28) is verified and this concludes the proof.

Now, the inclusion relations for modulation spaces \([5, 14]\) fulfill

\[\mathcal{M}_{s-1}^{p,r} \hookrightarrow \mathcal{M}_{s-1}^{p,q}\]

if \(\frac{d}{q} - \frac{d}{r} < 1\), that is (32). Then (28) is verified and this concludes the proof.

**Theorem 4.4.** Let \(F(u)\) be as in (31), \(1 \leq p \leq \infty\), \(s \geq 0\), \(\gamma \geq 0\), and \(q' > 2kd\). For every \(R\) there exists a unique solution \(u \in C^0([-\infty, T); W(\mathcal{F}L^q_s, L^r_\gamma)] \times W(\mathcal{F}L^q_{s-1}, L^r_\gamma))\) such that for every \((u_0, u_1)\) in the ball \(B_R\) of center \(0\) and radius \(R\), where \(\mathcal{F}\) is the Fourier transform, there exists a unique solution \(u \in C^0([-\infty, T); W(\mathcal{F}L^q_s, L^r_\gamma)] \times W(\mathcal{F}L^q_{s-1}, L^r_\gamma))\) to (2). Furthermore, the map \((u_0, u_1) \mapsto u\) from \(B_R\) to \(C^0([-\infty, T); W(\mathcal{F}L^q_s, L^r_\gamma)] \times W(\mathcal{F}L^q_{s-1}, L^r_\gamma))\) is Lipschitz continuous.

The same is true for an entire real-analytic nonlinearity \(F\) as in Theorem 4.1 if, in addition, \(q = 1\).

**Proof.** It follows from Corollary 3.4 that the following estimates hold:

\[\|K(t)u\|_{W(\mathcal{F}L^q_s, L^r_\gamma)} \lesssim \|u\|_{W(\mathcal{F}L^q_{s-1}, L^r_\gamma)}, \quad t \in [0, 1],\]

and

\[\|K(t)u\|_{W(\mathcal{F}L^q_s, L^r_\gamma)} \lesssim \|u\|_{W(\mathcal{F}L^q_{s-1}, L^r_\gamma)}, \quad t \in [0, 1].\]

We then argue as in the proof of the previous results. For example, for the nonlinearity (31) we choose \(\mathcal{T} = C^0([-\infty, T); W(\mathcal{F}L^q_s, L^r_\gamma)], \mathcal{N} = C^0([-\infty, T); W(\mathcal{F}L^q_{s-1}, L^r_\gamma))\). The estimate (28) here follows from the multilinear estimate (10) (with \(N = 2k + 1\)) combined with the inclusion in (11). The numerology is the same as that in the proof of Theorem 4.3. We omit the detail.

**Remark 4.5.** It would be interesting to know whether the previous results extend to the (smaller) spaces \(\mathcal{M}^{p,q}_{s-1}\) or \(W(\mathcal{F}L^q_s, L^r_\gamma)\), with \(p < 1\). The above method of proof does not cover this case because of the lack of the Minkowski integral inequality for quasi-Banach spaces. Indeed, for any quasi-Banach space \(Q\) there always exist an equivalent quasi-norm \(\|\cdot\|\) and a number \(0 < r < 1\) so that...
\[\| \cdot \|^r \text{ satisfies the triangle inequality. However, the corresponding integral version, namely}\]

\[\| \int_0^T u(t) \, dt \|^r \leq \int_0^T \| u(t) \|^r \, dt,\]

is false. One can see this by taking \( u(t) = av(t) \), where \( a \in \mathbb{Q} \), \( a \neq 0 \), is fixed, and \( v_\epsilon \in C([0, T]; \mathbb{R}) \) satisfies \( 0 \leq v_\epsilon \leq 1 \), \( v_\epsilon(t) = 1 \) per \( 0 \leq t \leq \epsilon \), \( v_\epsilon(t) = 0 \) per \( t \geq 2\epsilon \). Letting \( \epsilon \to 0^+ \) gives a contradiction.

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