ON THE COMPUTATION OF RATIONAL POINTS OF A HYPERSURFACE OVER A FINITE FIELD

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\textbf{Abstract.} We analyze a family of algorithms for computing rational points of hypersurfaces defined over a finite field based on searches on “vertical strips”, namely searches on parallel lines in a given direction. We consider two basic models of generation of vertical strips: an “independent” model, where repetitions are allowed, and a “nonindependent one”, where repetitions are avoided. We determine the asymptotic probability distribution of the number of searches and show that it decays with an exponential ratio in both models. We also analyze the probability distribution of outputs, using the notion of Shannon entropy, and prove that both models are somewhat close to any “ideal” equidistributed algorithm.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of $q$ elements, let $X_1, \ldots, X_r$ be indeterminates over $\mathbb{F}_q$ and let $\mathbb{F}_q[X_1, \ldots, X_r]$ denote the ring of $r$-variate polynomials with coefficients in $\mathbb{F}_q$ for any positive integer $r$. Let $\mathcal{F}_{r,d} := \{F \in \mathbb{F}_q[X_1, \ldots, X_r] : \deg(F) \leq d\}$ for integers $r \geq 2$ and $d \geq 2$ and let $F$ be an arbitrary element of $\mathcal{F}_{r,d}$. In this paper we address the problem of finding an $\mathbb{F}_q$-rational zero of $F$, namely a point $x \in \mathbb{F}_q^r$ with $F(x) = 0$.

It is well-known that the elements of $\mathcal{F}_{r,d}$ have $q^{r-1}$ zeros in $\mathbb{F}_q^r$ on average. More precisely, we have the following result (see, e.g., [15, Theorem 6.16]):

\begin{equation}
\frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} N(F) = q^{r-1},
\end{equation}

where $N(F) := |\{x \in \mathbb{F}_q^r : F(x) = 0\}|$. The average deviation from the expected value $q^{r-1}$ has also been analyzed (see, e.g., [15, Theorem 6.17]). If the polynomial $F$ under consideration is absolutely irreducible, that is, it is irreducible as an element of $\mathbb{F}_q[X_1, \ldots, X_r]$, where $\mathbb{F}_q$ denotes the algebraic closure of $\mathbb{F}_q$, then explicit upper bounds on the deviation $|N(F) - q^{r-1}|$ are known (see, e.g., [5]). We remark that “most” elements of $\mathcal{F}_{r,d}$ are absolutely irreducible (see [22] for a precise estimate on the number of absolutely irreducible elements of $\mathcal{F}_{r,d}$).

This suggests a strategy to find an $\mathbb{F}_q$-rational zero of a given $F \in \mathcal{F}_{r,d}$. Since the expected number of zeros of $F$ is equal to the number of elements of $\mathbb{F}_q^{r-1}$, given $a_1 \in \mathbb{F}_q^{r-1}$, one may try to find a zero of $F$ having $a_1$ in its first $r - 1$ coordinates. This amounts to compute a zero in $\mathbb{F}_q$ of the univariate polynomial $F(a_1, X_r)$, which can be done from the vector of coefficients of $F(a_1, X_r)$ with $O(d \log q)$ arithmetic operations in $\mathbb{F}_q$, up to logarithmic factors (see, e.g., [20, Corollary 14.16]). As an element of $\mathcal{F}_{r,d}$ has $D := \binom{d+r}{r}$ coefficients, the number of arithmetic operations in $\mathbb{F}_q$ required to compute the vector of coefficients of $F(a_1, X_r)$ is $O(D)$, up to logarithmic factors. If the polynomial $F(a_1, X_r)$ has no zeros in $\mathbb{F}_q$, then a further element $a_2 \in \mathbb{F}_q^{r-1}$ can be picked up to see whether $F(a_2, X_r)$ has a zero in $\mathbb{F}_q$. The algorithm proceeds in this way until a zero of $F$ in $\mathbb{F}_q^r$ is obtained. As a
consequence, the whole procedure requires $O(NS(F) \cdot D)$ arithmetic operations in $\mathbb{F}_q$, where $NS(F)$ is the number of elements $a \in \mathbb{F}_q^{\ast}$ which has to be picked until a polynomial $F(a, X_r)$ with a zero in $\mathbb{F}_q$ is obtained.

In the case $r = 2$, this idea was proposed and analyzed in [21]. The corresponding strategy was called a “Search on Vertical Strips” (SVS for short). This paper will be devoted to analyze the SVS strategy for elements of $\mathcal{F}_{r,d}$ from a probabilistic point of view. For this purpose, we shall concentrate on the critical point of this algorithm, namely the number of vertical strips which must be generated.

There are two possible approaches to generate vertical strips. On one hand, one may generate vertical strips, that is, elements of $\mathbb{F}_q^{\ast}$, randomly and independently. This will be called the independent model in what follows. On the other hand, as repeated vertical strips lead to useless searches, one may require that each new vertical strip is distinct from all the previous ones. More precisely, given an input polynomial $F \in \mathcal{F}_{r,d}$, let $1 \leq s \leq q^{-1}$ and assume that the SVS algorithm has performed $s - 1$ unsuccessful searches on the vertical strips determined by $a_1, \ldots, a_{s-1} \in \mathbb{F}_q^{\ast}$. Then the $s$th step of this variant of the SVS algorithm generates at random an element $a_s \in \mathbb{F}_q^{\ast} \setminus \{a_1, \ldots, a_{s-1}\}$ and searches for an $\mathbb{F}_q$-rational zero of $F(a_s, X_r)$. This is what we call the non-independent model.

Each choice of vertical strips determines a concrete “version” of the SVS algorithm. Therefore, our analysis takes all these versions into account. In Sections 2 and 3 we analyze, for a given $s \geq 1$ the probability distribution of the number of searches performed by the algorithm. For this purpose, we consider sets $\mathcal{F}_{r,d}^{\text{ind}}$ and $\mathcal{F}_{r,d}^{\text{nonind}}$ which represent all possible choices of vertical strips, and the random variables $C_{r,d}^{\text{ind}} : \mathcal{F}_{r,d}^{\text{ind}} \times \mathcal{F}_{r,d} \to \mathbb{N} \cup \{\infty\}$ and $C_{r,d}^{\text{nonind}} : \mathcal{F}_{r,d}^{\text{nonind}} \times \mathcal{F}_{r,d} \to \mathbb{N} \cup \{\infty\}$ which count the number of vertical strips that are searched in both models. Our main result asserts that the probability that $s$ vertical strips are searched is

$$(1.2) \quad P_{r,d}[C_{r,d}^{\text{nonind}} = s] = (1 - \mu_d)^{s-1} \mu_d + O(q^{-1/2}),$$

where $\mu_d := \sum_{j=1}^{d} (-1)^{j-1}/j!$. We remark that the quantity $\mu_d$ arises also in connection with another classical combinatorial problem over finite fields, that of the value set of univariate polynomials (cf. [13], [17]). More precisely, for a polynomial $f \in \mathbb{F}_q[T]$ of degree $d$, the cardinality $\mathcal{V}(f)$ of the value set of $f$ is defined as $\mathcal{V}(f) := |\{f(c) : c \in \mathbb{F}_q\}|$. In [1], Birch and Swinnerton–Dyer established the following classical result: if $f \in \mathbb{F}_q[T]$ is a generic polynomial of degree $d$, then $\mathcal{V}(f) = \mu_d q + O(1)$.

The estimate (1.2) relies on the analysis of Section 3, where the behavior of the SVS algorithm for a fixed choice of the first $s$ vertical strips $a_1, \ldots, a_s \in \mathbb{F}_q^{\ast}$ is determined. More precisely, we consider the behavior of the variant of the SVS algorithm which proceeds to evaluate the input polynomial $F$ in each $a_i$, successively with $i := 1, 2, \ldots$, until a root in $\mathbb{F}_q$ of a polynomial $F(a_i, X_r)$ is obtained. It turns out that the probability that the $s$ vertical strips are searched is essentially that of the right–hand side of (1.2). This may be considered as a “realistic” version of the SVS algorithm in the sense of [1]. As the author states, “when a randomized algorithm is implemented, one always uses a sequence whose later values come from earlier ones in a deterministic fashion. This invalidates the assumption of independence and might cause one to regard results about probabilistic algorithms with suspicion.” Our results show that the probabilistic behavior of the SVS algorithm is not essentially altered when a fixed choice of vertical strips is considered.

Another critical aspect in the analysis of the SVS strategy is the distribution of outputs. Given $F \in \mathcal{F}_{r,d}$, any of the variants of the SVS algorithm outputs an $\mathbb{F}_q$–rational zero of $F$. This zero is determined by certain random choices which are made during the execution of the algorithm. As a consequence, a relevant point in the analysis of the SVS algorithm is the probability distribution of the outputs. For an ideal algorithm, outputs should be equiprobably distributed. For this reason, in [21] the basic SVS strategy for bivariate polynomials over $\mathbb{F}_q$ is modified so that all possible $\mathbb{F}_q$–rational zeros of the input polynomial have the same probability of being the
output. Such a modification can be also be applied to both variants of the SVS algorithm considered here.

Nevertheless, we shall pursue here a different course of action, analyzing the average distribution of outputs by means of the concept of Shannon entropy. If the output of the SVS algorithm on a given input polynomial $F$ tends to be concentrated on a few $\mathbb{F}_q$-rational zeros of $F$, then the “amount of information” that we obtain might be said to be “small”. On the other hand, if all the $\mathbb{F}_q$-rational zeros of $F$ are equally possible outputs, then the amount of information provided by the algorithm may be considered to be larger. Following [3] (see also [2]), where homotopy algorithms for polynomial systems over the complex numbers are analyzed, we define a Shannon entropy $H^\text{var}_F$ associated to an input $F \in \mathbb{F}_{r,d}$ and any of the variants $\text{var} \in \{\text{ind}, \text{nind}\}$ of the SVS algorithm, which measures how “concentrated” are the outputs of the SVS algorithm on input $F$. Then we analyze the average entropy of both variants of the SVS algorithm when $F$ runs through all the elements of $\mathbb{F}_{r,d}$, namely

$$H^\text{var}_F := \frac{1}{|\mathbb{F}_{r,d}|} \sum_{F \in \mathbb{F}_{r,d}} H^\text{var}_F.$$

For an “ideal” algorithm for the search of $\mathbb{F}_q$-rational zeros of elements of $\mathbb{F}_{r,d}$, from the point of view of the probability distribution of outputs, and $F \in \mathbb{F}_{r,d}$, it is easy to see that $H^\text{ideal}_F = \log |\mathbb{F}_{r,d}|$. It follows that

$$H^\text{ideal} \leq \log(q^r-1).$$

The main results of Section 5 assert that, for $\text{var} \in \{\text{ind}, \text{nind}\}$,

$$H^\text{var}_F \geq \frac{1}{2\mu_d} \log(q^r-1)(1 + O(q^{-1})).$$

Since $1/2\mu_d \approx 0.79$ for large $d$, in view of (1.3) we may paraphrase (1.4) as saying that both variants of the SVS algorithm are at least 79 per cent as good as any “ideal” algorithm, from the point of view of the distribution of the outputs.

The proof of (1.4) relies on an analysis of the expected number of vertical strips of the elements of $\mathbb{F}_{r,d}$ which may be of independent interested. Denote by $NS(r,d)$ be the average number of vertical strips with $\mathbb{F}_q$-rational zeros of $F$, when $F$ runs through all the elements of $\mathbb{F}_{r,d}$. We prove that

$$NS(r,d) = \mu_d q^{r-1} + O(q^{-2}).$$

We also estimate the variance of the number of vertical strips with $\mathbb{F}_q$-rational zeros.

The paper is organized as follows. Section 2 is devoted to the analysis of the probability that one or two vertical strips are searched. In Section 3 we estimate the expected number of vertical strips to be searched for a given choice of $s \geq 3$ (distinct) vertical strips. We express the probability that $s$ vertical strips are searched in terms of average cardinalities of value sets and apply estimates for the latter in order to establish an explicit estimate of the former. In Section 4 we apply the results of Section 3 to establish (1.2). Section 5 is concerned with the probability distribution of outputs in both models. In Section 5.1 we establish (1.5) and an estimate of the corresponding variance. In Section 5.2 we apply these estimates to prove (1.4). Finally, in Section 6 we exhibit a few simulations aimed at confirming the asymptotic result (1.2).

2. Analysis of the Probability of $C = 1$ and $C = 2$

We start the probabilistic analysis of the SVS algorithm discussing how frequently one or two vertical strips are searched. As it will become evident, this will happen in most cases. Therefore, accurate estimates on the probability of these two cases is critical for an accurate description of the behavior of the algorithm.
2.1. Estimates for the probability of $C = 1$. For positive integers $r \geq 2$ and $d \geq 2$, we shall estimate the probability that the SVS algorithm, on input an element of $\mathcal{F}_{r,d} := \{ F \in \mathbb{F}_q[X_1, \ldots, X_r] : \deg(F) \leq d \}$, finds a root in the first vertical strip, under any of the two models of generation of vertical strips mentioned in the introduction. Observe that both models proceed in the same way at the first step. Therefore we shall not distinguish them and shall drop the superscript $\text{ind}$ from the notations of this section. As $r$ and $d$ are fixed, we shall also drop the indices $r$ and $d$ from the notations.

Each possible choice for the first vertical strip is represented by an element of $\mathbb{F}_q^{-1}$. As a consequence, we may represent the situation by means of the random variable $C_1 := C_{1,r,d}^{\text{var}} : \mathbb{F}_q^{-1} \times \mathcal{F}_{r,d} \to \{1, \infty\}$ defined in the following way:

$$
C_1(a, F) := \begin{cases} 
1 & \text{if } F(a, X_r) \text{ has an } \mathbb{F}_q\text{-rational zero}, \\
\infty & \text{otherwise}.
\end{cases}
$$

We consider the set $\mathbb{F}_q^{-1} \times \mathcal{F}_{r,d}$ endowed with the uniform probability $P_1 := P_{1,r,d}$ and study the probability of the set $\{C_1 = 1\}$. In the next result we provide an exact formula for this probability.

**Theorem 2.1.** For $q > d$, we have the identity

$$
P_1[C_1 = 1] = \sum_{j=1}^{d} (-1)^{j-1} \left(\frac{q}{j}\right) q^{-j} + (-1)^d \left(\frac{q-1}{d}\right) q^{-d}.
$$

**Proof.** For any $F \in \mathcal{F}_{r,d}$, we denote by $VS(F)$ the set of vertical strips where $F$ has an $\mathbb{F}_q$-rational zero and by $NS(F)$ the number of such vertical strips, that is,

$$
VS(F) := \{ a \in \mathbb{F}_q^{-1} : (\exists x_r \in \mathbb{F}_q) F(a, x_r) = 0 \}, \quad NS(F) := |VS(F)|.
$$

It is easy to see that $\{C_1 = 1\} = \bigcup_{F \in \mathcal{F}_{r,d}} VS(F) \times \{F\}$. Since this is a union of disjoint subsets of $\mathbb{F}_q^{-1} \times \mathcal{F}_{r,d}$, it follows that

$$
P_1[C_1 = 1] = \frac{1}{q^{-1} |\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} NS(F).
$$

Fix $F \in \mathcal{F}_{r,d}$. Observe that

$$
VS(F) = \bigcup_{x \in \mathbb{F}_q} \{ a \in \mathbb{F}_q^{-1} : F(a, x) = 0 \}.
$$

As a consequence, by the inclusion–exclusion principle we obtain

$$
NS(F) = \left| \bigcup_{x \in \mathbb{F}_q} \{ a \in \mathbb{F}_q^{-1} : F(a, x) = 0 \} \right|
$$

$$
= \sum_{j=1}^{q} (-1)^{j-1} \sum_{X_j \subseteq \mathbb{F}_q} \left| \{ a \in \mathbb{F}_q^{-1} : (\forall x \in X_j) F(a, x) = 0 \} \right|,
$$

where $X_j$ runs through all the subsets of $\mathbb{F}_q$ of cardinality $j$. We conclude that

$$
\sum_{F \in \mathcal{F}_{r,d}} NS(F) = \sum_{F \in \mathcal{F}_{r,d}} \sum_{j=1}^{q} (-1)^{j-1} \sum_{X_j \subseteq \mathbb{F}_q} \left| \{ a \in \mathbb{F}_q^{-1} : (\forall x \in X_j) F(a, x) = 0 \} \right|.
$$

For any $j$ with $1 \leq j \leq q$, we denote

$$
\mathcal{N}^j := \frac{1}{q^{-1} |\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} \sum_{X_j \subseteq \mathbb{F}_q} \left| \{ a \in \mathbb{F}_q^{-1} : (\forall x \in X_j) F(a, x) = 0 \} \right|,
$$

where $X_j$ runs through all the subsets of $\mathbb{F}_q$ of cardinality $j$. If $j \leq d$ and $X_j \subseteq \mathbb{F}_q$ is a set of cardinality $j$, then the equalities $F(a, x) = 0 (x \in X_j)$ are $j$ linearly–independent conditions on the coefficients of $F$ in the $\mathbb{F}_q$–vector space $\mathcal{F}_{r,d}$. It
follows that

\[ N_j = \frac{1}{q^{r-1} |F_{r,d}|} \sum_{X_j \subseteq \mathbb{F}_q} \sum_{a \in \mathbb{F}_q^{r}} \left| \{ F \in F_{r,d} : (\forall x \in X_j) F(a,x) = 0 \} \right| \]

(2.2) 

\[ = \frac{1}{q^{r-1+\dim F_{r,d}}} \sum_{X_j \subseteq \mathbb{F}_q} \sum_{a \in \mathbb{F}_q^{r}} q^{\dim F_{r,d}-j} = \binom{q}{j} q^{-j} \]

On the other hand, if \( j > d \) and \( X_j \subseteq \mathbb{P}_q \) is subset of cardinality \( j \), then the condition \( F(a,x) = 0 \) is satisfied for every \( x \in X_j \) if and only if \( F(a, X_r) = 0 \). The condition \( F(a, X_r) = 0 \) is expressed by means of \( d + 1 \) linearly–independent linear equations on the coefficients of \( F \) in \( F_{r,d} \). We conclude that

\[ N_j = \frac{1}{q^{r-1} |F_{r,d}|} \sum_{X_j \subseteq \mathbb{F}_q} \sum_{a \in \mathbb{F}_q^{r}} \left| \{ F \in F_{r,d} : (\forall x \in X_j) F(a,x) = 0 \} \right| \]

(2.3) 

\[ = \frac{1}{q^{r-1+\dim F_{r,d}}} \sum_{X_j \subseteq \mathbb{F}_q} \sum_{a \in \mathbb{F}_q^{r}} q^{\dim F_{r,d}-(d+1)} = \binom{q}{j} q^{-d-1} \]

Combining (2.2) and (2.3) we obtain

\[ P_1[C_1 = 1] = \frac{\sum_{j=1}^{d} N_j}{\sum_{j=1}^{d+1} N_j} \sum_{j=1}^{d} (-1)^{j-1} \binom{q}{j} q^{-j} + \sum_{j=d+1}^{q} (-1)^{j-1} \binom{q}{j} q^{-d-1} \]

Finally, since

(2.4) 

\[ \sum_{j=d+1}^{q} (-1)^{j-1} \binom{q}{j} = \sum_{j=0}^{d} (-1)^{j} \binom{q}{j} = (-1)^{d} \binom{q-1}{d} \]

we readily deduce the statement of the theorem. \( \square \)

Next we discuss the asymptotic behavior of the probability \( P_1[C_1 = 1] \). Fix \( d \geq 2 \). From Theorem 2 we can see that

\[ P_1[C_1 = 1] = \mu_d + O(q^{-1}), \quad \mu_d := \sum_{j=1}^{d} (-1)^{j-1} \binom{q}{j} j! \]

To show this, given positive integers \( k,j \) with \( k \leq j \), we shall denote by \( \lceil k \rceil \) the unsigned Stirling number of the first kind, namely the number of permutations of \( n \) elements with \( k \) disjoint cycles. The following properties of the Stirling numbers are well–known (see, e.g., [12 §A.8]):

\[ \lceil j \rceil = 1, \quad \lceil j \rceil^{j-1} = \frac{j}{2}, \quad \sum_{k=0}^{j} \binom{j}{k} = j! \]

We shall also use the following well–known identity:

(2.5) 

\[ \binom{q}{j} = \sum_{k=0}^{j} \frac{(-1)^{j-k}}{j!} \binom{j}{k} q^k \]

According to Theorem 2.4 and (2.5), we have

\[ P_1[C_1 = 1] = \sum_{j=1}^{d} (-1)^{j-1} \sum_{k=0}^{j} \frac{(-1)^{j-k}}{j!} \binom{j}{k} q^{k-j} + (-1)^{d} \binom{q-1}{d} q^{-d-1} \]

\[ = \sum_{j=1}^{d} \frac{(-1)^{j-1}}{j!} \binom{j}{j} + \sum_{j=1}^{d} \frac{(-1)^{j}}{j!} \binom{j}{j-1} q^{-1} \]

\[ + \sum_{j=1}^{d} \sum_{k=0}^{j-2} \frac{(-1)^{k-1}}{j!} \binom{j}{k} q^{k-j} + (-1)^{d} \binom{q-1}{d} q^{-d-1} \]
It follows that

\[ P_1(C_1 = 1) = \mu_d + \frac{1}{q} \sum_{j=1}^{d} \frac{(-1)^j}{j!} \left( \frac{j}{2} \right) - \frac{1}{q} \sum_{j=1}^{d} \sum_{k=0}^{j-2} \frac{(-1)^k}{j!} \left[ \frac{j}{k!} \right] q^{j-k} + \frac{(-1)^d}{q^{d+1}} \left( \frac{q - 1}{d} \right). \]

As a consequence, we obtain

\[ |P_1(C_1 = 1) - \mu_d| \leq \frac{1}{4q} + \frac{d}{q^2} + \frac{1}{2q}. \]

We have therefore the following result.

**Corollary 2.2.** For \( q > d \),

\[ |P_1(C_1 = 1) - \mu_d| \leq \frac{2}{q}. \]

As \( d \) tends to infinity, the number \( P_1(C_1 = 1) \) tends to \( 1 - e^{-1} = 0.6321 \ldots \), where \( e \) denotes the basis of the natural logarithm. This explains the numerical results in the first row of the tables of the simulations of Section 6.

It is worth remarking that the quantity \( P_1(C_1 = 1) \) is closely connected with the probability that a univariate polynomial of degree at most \( d \) has \( \mathbb{F}_q \)-rational roots. More precisely, consider the set \( \mathcal{F}_{1,d} \) of univariate polynomials of degree at most \( d \) with coefficients in \( \mathbb{F}_q \), endowed with the uniform probability \( p_{1,d} \), and the random variable \( N_{1,d} : \mathcal{F}_{1,d} \to \mathbb{Z}_{\geq 0} \) which counts the number of \( \mathbb{F}_q \)-rational zeros, namely

\[ N_{1,d}(f) := |\{ x \in \mathbb{F}_q : f(x) = 0 \}|. \]

The random variable \( N_{1,d} \) has been implicitly studied in the literature (see, e.g., [9 §2] or [14 Theorem 3]). It can be proved that, for \( d < q \),

\[ p_{1,d}[N_{1,d} > 0] = P_1(C_1 = 1). \]

### 2.2. Estimates on the probability that \( C = 2 \)

Next we analyze the probability that the SVS algorithm performs exactly two searches in both models of generation of vertical strips.

We start with the independent model. In this model, each possible choice for the first two vertical strips is an arbitrary element \( \mathbf{a} := (a_1, a_2) \in \mathbb{F}_q^2 : \mathbb{F}_q \neq \mathbb{F}_q \). Therefore, we introduce the random variable \( C_2^{\text{ind}} := C_{2,r,d}^{\text{ind}} : \mathbb{F}_q^2 \times \mathcal{F}_{r,d} \to \{1, 2, \infty\} \)

defined as follows:

\[ C_2^{\text{ind}}(\mathbf{a}, F) := \begin{cases} 1 & \text{if } N_{1,d}(F(a_1, X_r)) > 0, \\ 2 & \text{if } N_{1,d}(F(a_1, X_r)) = 0 \text{ and } N_{1,d}(F(a_2, X_r)) > 0, \\ \infty & \text{otherwise.} \end{cases} \]

We consider the set \( \mathbb{F}_q^2 \times \mathcal{F}_{r,d} \) endowed with the uniform probability \( P_2^{\text{ind}} := P_{2,r,d}^{\text{ind}} \)

and analyze the probability \( P_2^{\text{ind}}[C_2^{\text{ind}} = 2] \).

Now we consider the nonindependent model. Here we appreciate for the first time a difference among both models: since repetitions are only allowed in the independent model, elements \( \mathbf{a} \in \mathbb{F}_q^{2(r-1)} \) of the form \( \mathbf{a} := (a_1, a_1) \) cannot be chosen for the first two vertical strips. Therefore, we denote by \( \mathbb{F}_q^{\text{ind}} \) the set of all possible choices for the first two vertical strips and by \( N_2^{\text{ind}} \) its cardinality, that is,

\[ \mathbb{F}_q^{\text{ind}} := \{ \mathbf{a} : (a_1, a_2) \in \mathbb{F}_q^2 : a_1 \neq a_2 \}, \quad \mathbb{F}_q^{\text{ind}} = |\mathbb{F}_q^{\text{ind}}| = q^{r-1}(q^{r-1} - 1). \]

We shall study the random variable \( C_2^{\text{nc}} := C_{2,r,d}^{\text{ind}} : \mathbb{F}_q^{\text{ind}} \times \mathcal{F}_{r,d} \to \{1, 2, \infty\} \)

defined as in the independent model, where the set \( \mathbb{F}_q^{\text{ind}} \times \mathcal{F}_{r,d} \) is endowed with the uniform probability \( P_2^{\text{ind}} := P_{2,r,d}^{\text{ind}} \). We aim to determine the probability \( P_2^{\text{ind}}[C_2^{\text{ind}} = 2] \).

The probability \( P_2^{\text{nc}}[C_2^{\text{nc}} = 2] \) in both models will be expressed in terms of probabilities concerning the random variables \( C_{2,r,d} : \mathcal{F}_{r,d} \to \{1, 2, \infty\} \) which count the number of searches that are performed on the vertical strips defined by \( \mathbf{a} :=... \)
Lemma 2.3. Denote $N_2^\text{ind} := q^{2(r-1)}$. We have
\[
P_2^\text{ind}(C_2^\text{ind} = 2) = \frac{1}{N_2^\text{ind}} \sum_{a \in F_2^\text{ind}} p_{r,d}(C_{a,r,d} = 2),
\]
\[
P_2^\text{ind}(C_2^\text{ind} = 2) = \frac{1}{N_2^\text{ind}} \sum_{a \in F_2^\text{ind}} p_{r,d}(C_{a,r,d} = 2).
\]

Proof. We prove the first assertion. Observe that
\[
\{C_2^\text{ind} = 2\} = \bigcup_{F \in F_{r,d}} \{a \in F_2^\text{ind} : C_{a,r,d}(F) = 2\} \times \{F\}.
\]
Since this is union of disjoint sets, we conclude that
\[
P_2^\text{ind}(C_2^\text{ind} = 2) = \frac{1}{N_2^\text{ind}} \sum_{F \in F_{r,d}} \left| \left\{a \in F_2^\text{ind} : C_{a,r,d}(F) = 2\right\} \right|.
\]
Observe that
\[
\frac{1}{|F_{r,d}|} \sum_{F \in F_{r,d}} \left| \left\{a \in F_2^\text{ind} : C_{a,r,d}(F) = 2\right\} \right| = \frac{1}{|F_{r,d}|} \sum_{F \in F_{r,d}} \sum_{a \in F_2^\text{ind}, C_{a,r,d}(F) = 2} 1
\]
\[
= \frac{1}{|F_{r,d}|} \sum_{a \in F_2^\text{ind}} \sum_{F \in F_{r,d}, C_{a,r,d}(F) = 2} 1
\]
\[
= \frac{1}{|F_{r,d}|} \sum_{a \in F_2^\text{ind}} \left| \{F \in F_{r,d} : C_{a,r,d}(F) = 2\} \right|
\]
\[
= \sum_{a \in F_2^\text{ind}} p_{r,d}(C_{a,r,d} = 2),
\]
which readily implies the first assertion. The second assertion follows with a very similar argument. \qed

Next we estimate the probability $p_{r,d}(C_{a,r,d} = 2)$ for a given $a \in F_2^\text{ind}$.

Proposition 2.4. For $q > d$ and $a := (a_1, a_2)$, we have
\[
\left| p_{r,d}(C_{a,r,d} = 2) - \mu_d(1 - \mu_d) \right| \leq \frac{2}{q}.
\]

Proof. Observe that
\[
\{C_{a,r,d} = 2\} = \{F \in F_{r,d} : N_{1,d}(F(a_2, T)) > 0\} \setminus \{F \in F_{r,d} : N_{1,d}(F(a_1, T)) > 0\}.
\]
The number of elements of $F_{r,d}$ having $\mathbb{F}_q$-rational zeros in the vertical strip defined by $a_2$ is determined in Theorem 2.1. Therefore, it remains to find the number $N_{a,2}$ of elements of $F_{r,d}$ having $\mathbb{F}_q$-rational zeros both in the vertical strips defined by $a_1$ and $a_2$. We have
\[
N_{a,2} = \left| \bigcup_{x \in \mathbb{F}_q} \bigcup_{y \in \mathbb{F}_q} \{F \in F_{r,d} : F(a_1, x) = F(a_2, y) = 0\} \right|.
\]
Given sets $X \subset \mathbb{F}_q$ and $Y \subset \mathbb{F}_q$, we denote
\[
S_{a}(X, Y) := \{F \in F_{r,d} : F(a_1, x) = F(a_2, y) = 0 \text{ for all } x \in X \text{ and } y \in Y\}.
\]
Then the inclusion–exclusion principle implies
\begin{equation}
N_{a,2} = \sum_{j=1}^{q} \sum_{k=1}^{q} (-1)^{j+k} \sum_{X_j \subset \mathbb{F}_q} \sum_{Y_k \subset \mathbb{F}_q} |S_{a}(X_j, Y_k)|.
\end{equation}
where the sum runs over all subsets $\mathcal{X}_j \subset \mathbb{F}_q$ and $\mathcal{Y}_k \subset \mathbb{F}_q$ of $j$ and $k$ elements respectively.

**Claim.** $\frac{N_{a,2}}{|\mathcal{F}_{r,d}|} = (P_1[C_1 = 1])^2$.

**Proof of Claim.** For $1 \leq j, k \leq q$, let

$$N_{j,k} := \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} |\mathcal{S}_a(\mathcal{X}_j, \mathcal{Y}_k)|.$$

We determine $N_{j,k}$ according to whether one of the following four cases occurs.

First suppose that $j, k \leq d$. As $a_1 \neq a_2$, the equalities $F(a_1, x) = 0, F(a_2, y) = 0$ for all $x \in \mathcal{X}_j$ and $y \in \mathcal{Y}_k$ impose $j + k$ linearly–independent conditions on the coefficients of $F \in \mathcal{F}_{r,d}$. Therefore, $|\mathcal{S}_a(\mathcal{X}_j, \mathcal{Y}_k)| = q^{\dim \mathcal{F}_{r,d} - j - k}$, which implies

$$N_{j,k} = \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} q^{\dim \mathcal{F}_{r,d} - j - k} = \binom{q}{j} \binom{q}{k} q^{\dim \mathcal{F}_{r,d} - j - k}.$$ 

The second case is determined by the conditions $j > d$ and $k \leq d$. If $j > d$ and $\mathcal{X}_j \subset \mathbb{F}_q$ is subset of cardinality $j$, then the condition $F(a_1, x) = 0$ is satisfied for every $x \in \mathcal{X}_j$ if and only if $F(a_1, X_r) = 0$. The condition $F(a_1, X_r) = 0$ is expressed by $d + 1$ linearly–independent linear equations on the coefficients of $F \in \mathcal{F}_{r,d}$. On the other hand, the equalities $F(a_2, y) = 0$ for every $y \in \mathcal{Y}_k$ impose $k$ additional linearly–independent conditions on the coefficients of $F$. We conclude that

$$N_{j,k} = \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} q^{\dim \mathcal{F}_{r,d} - (d+1) - k} = \binom{q}{j} \binom{q}{k} q^{\dim \mathcal{F}_{r,d} - (d+1) - k}.$$ 

The third case, namely $j \leq d$ and $k > d$, is completely analogous to the second one. Finally, when $j > d$ and $k > d$, the conditions under consideration imply $F(a, X_r) = F(b, X_r) = 0$. We readily deduce that

$$N_{j,k} = \binom{q}{j} \binom{q}{k} q^{2(d+1)}.$$ 

From the expression for $N_{j,k}$ of the four cases under consideration, we infer that

$$\frac{N_{a,2}}{|\mathcal{F}_{r,d}|} = \frac{1}{|\mathcal{F}_{r,d}|} \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} N_{j,k}$$

$$= \sum_{j=1}^d \sum_{k=1}^d (-1)^{j+k} \binom{q}{j} \binom{q}{k} q^{j-k} + 2 \sum_{j=1}^d \sum_{k=d+1}^q (-1)^{j+k} \binom{q}{j} \binom{q}{k} q^{j-(d+1)}$$

$$+ \sum_{j=d+1}^q \sum_{k=d+1}^q (-1)^{j+k} \binom{q}{j} \binom{q}{k} q^{j-(d+1)}.$$ 

By (2.3) and elementary calculations we obtain

$$\frac{N_{a,2}}{|\mathcal{F}_{r,d}|} = \left( \sum_{j=1}^d (-1)^j \binom{q}{j} q^{-j} \right)^2 - 2 \left( \sum_{j=1}^d (-1)^j \binom{q}{j} q^{-j} \right) (-1)^d \binom{q-1}{d} q^{-d-1}$$

$$+ \left( \binom{q-1}{d} q^{-d-1} \right)^2.$$ 

This and Theorem 2.1 readily imply the claim. \hfill \Box

Combining the previous claim and Theorem 2.1 we deduce that

$$p_{r,d}[C_{a,r,d} = 2] = P_1[C_1 = 1] - \frac{N_{a,2}}{|\mathcal{F}_{r,d}|} = (1 - P_1[C_1 = 1]) P_1[C_1 = 1].$$

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) := (1 - x)x$. The Mean Value theorem shows that there exists $\xi \in (0, 1)$ such that

$$(1 - P_1[C_1 = 1]) P_1[C_1 = 1] - (1 - \mu_d) \mu_d = f'(\xi) (P_1[C_1 = 1] - \mu_d).$$
As the function $x \mapsto f'(x)$ maps the real interval $[0, 1]$ to $[-1, 1]$, we conclude that $|f'(\xi)| \leq 1$. Therefore, from Corollary 2.2 it follows that

$$|(1 - P_1[C_1 = 1])P_2[C_1 = 1] - (1 - \mu_d)\mu_d| \leq |P_1[C_1 = 1] - \mu_d| \leq \frac{2}{q}.$$  

This immediately implies the statement of the proposition. □

Proposition 2.4 is the critical step in the analysis of the behavior of the probability $P_2^{\text{var}}[C_2^{\text{var}} = 2]$ in both models, which is estimated in the next result.

**Theorem 2.5.** For any $q > d$ and $\var \in \{\text{ind}, \text{mnd}\}$,

$$|P_2^{\text{var}}[C_2^{\text{var}} = 2] - (1 - \mu_d)\mu_d| \leq \frac{2}{q}.$$  

**Proof.** We first consider the nonindependent model. By Lemma 2.3 and Proposition 2.4 we obtain

$$|P_2^{\text{ind}}[C_2^{\text{ind}} = 2] - (1 - \mu_d)\mu_d| \leq \frac{1}{N_2^{\text{ind}}} \sum_{a \in F_2^{\text{ind}}} |p_{r,d}[C_{a,r,d} = 2] - (1 - \mu_d)\mu_d| \leq \frac{2}{q}.$$  

On the other hand, concerning the independent model, we observe that, if an element $a := (a_1, a_2) \in F_2^{\text{ind}}$ with $a_1 = a_2$ is generated, then the SVS algorithm will never stop at the second search, that is, $p_{r,d}(C_{a_1,a_2}, r,d = 2) = 0$. As a consequence,

$$|P_2^{\text{ind}}[C_2^{\text{ind}} = 2] - (1 - \mu_d)\mu_d| \leq \frac{1}{N_2^{\text{ind}}} \sum_{a \in F_2^{\text{ind}}} |p_{r,d}[C_{a,r,d} = 2] - (1 - \mu_d)\mu_d|$$

$$\leq \frac{N_2^{\text{ind}}}{N_2^{\text{ind}}} \frac{2}{q} = \frac{2}{q}.$$  

This finishes the proof of the theorem. □

We finish the section with a remark concerning the spaces we have considered so far to discuss the probability that the SVS algorithm searches only the first or the first two vertical strips. For the analysis of the probability of one search, we have considered $F_1^{\text{var}} := \mathbb{F}_q^{r-1}$ and the random variable $C_1^{\text{var}} : F_1^{\text{var}} \times F_{r,d} \to \{1, \infty\}$, while the analysis of the probability of two searches has been done considering the random variable $C_2^{\text{var}} : F_2^{\text{var}} \times F_{r,d} \to \{1, 2, \infty\}$. To link both analysis, in Lemma 4.1 below we shall prove that

$$P_2^{\text{var}}[C_2^{\text{var}} = 1] = P_1^{\text{var}}[C_1^{\text{var}} = 1],$$

which shows the consistency of the models underlying Theorems 2.4 and 2.5. In the Section 3 we shall show that the analysis of the probability that $s$ vertical strips are searched can be done in a unified framework for any $s \geq 1$.

### 3. The number of searches for given vertical strips

As can be inferred from Section 2 a critical step in the probabilistic analysis of SVS algorithm for both models is the determination of the probability that $s$ vertical strips are searched, for a given choice of $s$ pairwise–distinct vertical strips. The cases $s = 1$ and $s = 2$ were discussed in Section 2. In this section the analysis of the general case is carried out.

Fix $2 \leq s \leq \min\{\binom{d+r-1}{r-1}, q^{-1}\}$ and $a_1, \ldots, a_s \in \mathbb{F}_q^{r-1}$. Suppose that $a_i \neq a_j$ for $i \neq j$ and denote $a := (a_1, \ldots, a_s)$. In this section we analyze the probability that the SVS algorithm performs $s$ trials until it reaches a vertical strip with an $\mathbb{F}_q$–rational zero of the polynomial under consideration, assuming that $a_1, \ldots, a_s$ are the choices for the first $s$ vertical strips to be considered.

Let $T$ be an indeterminate over $\mathbb{F}_q$ and $\Phi := \Phi_a : F_{r,d} \to F_{1,d}^s$ the $\mathbb{F}_q$–linear mapping defined as

$$\Phi(F) := (F(a_1, T), \ldots, F(a_s, T)).$$

We first obtain a characterization of the image of $\Phi$. This characterization will allow us to express the probability that the SVS algorithm performs $s$ trials in terms of
the average cardinality of the value set of certain families of univariate polynomials with prescribed coefficients.

As we explain below, there exists a unique positive integer \( \kappa_s \leq d \) such that

\[
\frac{(\kappa_s + r - 2)}{r - 1} \leq \frac{(\kappa_s + r - 1)}{r - 1}.
\]

In the sequel we shall assume that the points \( a_1, \ldots, a_s \) under consideration satisfy the condition we now state. For \( 1 \leq j \leq \kappa_s \), let \( D_j := \binom{\Omega}{r-1} \) and let \( \Omega_j := \{\omega_1, \ldots, \omega_{D_j}\} \subset (\mathbb{Z}_{\geq 0})^{r-1} \) denote the set of \((r-1)\)-tuples \( \omega_k := (\omega_{k,1}, \ldots, \omega_{k,r-1}) \) with \( |\omega_k| := |\omega_{k,1} + \cdots + \omega_{k,r-1}| \leq j \). Let \( a_{\omega_k}^s := a_{\omega_{k,1}}^{r-1}a_{\omega_{k,r-1}}^s \) for \( 1 \leq i \leq s \) and \( 1 \leq k \leq D_j \). Then we require that the multivariate Vandermonde matrix

\[
(3.2)
\begin{pmatrix}
\mathcal{A}_{\omega_1} & \cdots & \mathcal{A}_{\omega_{D_j}} \\
\vdots & \ddots & \vdots \\
\mathcal{A}_{\omega_1} & \cdots & \mathcal{A}_{\omega_{D_j}}
\end{pmatrix} 
\in \mathbb{F}_q^{s \times D_j}
\]

has maximal rank \( \min\{D_j, s\} \) for \( 1 \leq j \leq \kappa_s \).

We briefly argue that this is a mild requirement which is likely to be satisfied by any “reasonable” choice of the elements \( a_1, \ldots, a_s \in \mathbb{F}_q^{r-1} \).

Let \( \mathcal{M}_j := \mathcal{A}^{\omega_1}_1 \cdots \mathcal{A}^{\omega_{D_j}}_1 \) be \((r-1)\)-tuples of indeterminates over \( \mathbb{F}_q \), that is, \( \mathcal{A}_i := (a_{\omega_{i,1}}, \ldots, a_{\omega_{i,r-1}}) \) for \( 1 \leq i \leq s \), and denote by \( \mathcal{V}_j \) the following \( \min\{D_j, s\} \times \min\{D_j, s\} \) Vandermonde matrix with entries in \( \mathbb{F}_q[\mathcal{A}_1, \ldots, \mathcal{A}_s] \):

\[
\mathcal{V}_j := \begin{pmatrix}
\mathcal{A}^{\omega_1}_1 & \cdots & \mathcal{A}^{\omega_{\min\{D_j, s\}}}_1 \\
\vdots & \ddots & \vdots \\
\mathcal{A}^{\omega_1}_1 & \cdots & \mathcal{A}^{\omega_{\min\{D_j, s\}}}_1
\end{pmatrix}.
\]

Assume that the numbering of \( \Omega_j := \{\omega_1, \ldots, \omega_{D_j}\} \subset (\mathbb{Z}_{\geq 0})^{r-1} \) is made according to degrees, i.e., \(|\omega_k| \leq |\omega_l| \) whenever \( k \leq l \). In particular, \( \omega_1 = (0, \ldots, 0) \).

By \cite{10} Theorem 1.5 it follows that det \( \mathcal{V}_j \) is absolutely irreducible, namely it is an irreducible element of \( \mathbb{F}_q[\mathcal{A}_1, \ldots, \mathcal{A}_s] \), for \( 1 \leq j \leq \kappa_s \). Let \( \delta_j \) denote the degree of det \( \mathcal{V}_j \). We have the upper bound \( \delta_j \leq jD_j \). Then \cite{5} Theorem 5.2 proves that the number \( \mathcal{N}_j \) of \((r-1)\)-tuples \( a_1, \ldots, a_s \in \mathbb{F}_q^{r-1} \) annihilating det \( \mathcal{V}_j \) satisfies the following estimate:

\[
(3.3)
|\mathcal{N}_j - q^{s(r-1)-1}| \leq (\delta_j - 1)(\delta_j - 2)q^{s(r-1)-\frac{s}{2}} + 5\delta_j^{3/2}q^{s(r-1)-2}.
\]

Any choice of \( a_1, \ldots, a_s \) avoiding these \( \mathcal{N}_j = \mathcal{O}(q^{s(r-1)-1}) \) tuples for \( 1 \leq j \leq \kappa_s \) will satisfy our requirements. Furthermore, many “bad” choices \( a_1, \ldots, a_s \) annihilating the polynomial det \( \mathcal{V}_j \) for a given \( j \) will also work, as other minors of the Vandermonde matrix \( \mathcal{M}_j \) of \( \mathcal{V}_j \) may be nonsingular. In particular, for \( s \leq r \) and \( a_1, \ldots, a_s \) affinely independent, our requirement is satisfied.

Summarizing, denote \( \mathcal{V}_s := \prod_{j=1}^{\kappa_s} \det \mathcal{V}_j \in \mathbb{F}_q[\mathcal{A}_1, \ldots, \mathcal{A}_s] \) and let

\[
(3.4)
\mathcal{B}_s := \{a := (a_1, \ldots, a_s) \in \mathbb{F}_q^{s(r-1)} : \mathcal{V}_s(a) = 0\}.
\]

Then \( |\mathcal{B}_s| = \mathcal{O}(q^{s(r-1)-1}) \) and all the results of this section will be valid for any \( a \in \mathbb{F}_q^{s(r-1)} \setminus \mathcal{B}_s \).

3.1. A characterization of the image of \( \Phi \). In order to characterize the image \( \text{Im}(\Phi) \) of \( \Phi \), we shall express each element of the \( \mathbb{F}_q \)-linear space \( \mathcal{F}_{r,d} \) by its coordinates in the standard monomial basis \( \mathcal{B} \) of \( \mathcal{F}_{r,d} \), considering the monomial order we now define. Denote by \( \mathcal{B}_i \) the set of monomials of \( \mathbb{F}_q[X_1, \ldots, X_{r-1}] \) of degree at most \( i \) for \( 0 \leq i \leq d \), with the standard lexicographical order defined by setting \( X_1 < X_2 < \cdots < X_{r-1} \). Then the basis \( \mathcal{B} \) is considered with the order \( \mathcal{B} = \{X_1, \ldots, X_{r-1}^{-1} \mathcal{B}_1, \ldots, X_{r-1}^{-1} \mathcal{B}_{d-1}, \mathcal{B}_d \} \), where each set \( X_1^{-i} \mathcal{B}_i \) is ordered following the order induced by the one of \( \mathcal{B}_i \). In other words, any \( F \in \mathcal{F}_{r,d} \) can be uniquely
expressed as
\[ F = \sum_{i=0}^{d} F_i(X_1, \ldots, X_{r-1})X_i, \]

where each \( F_i \) has degree at most \( d - i \) for \( 0 \leq i \leq d \). Then the vector of coefficients \((F)_B\) of \( F \) in the basis \( \mathcal{B} \) is given by \((F)_B = ((F_0)_{\mathcal{B}_0}, \ldots, (F_0)_{\mathcal{B}_d})\). On the other hand, we shall express the elements of \( \mathcal{F}_{r,d}^i \) in the basis \( \mathcal{B}' := \{T^d, \ldots, T, 1\}^s \).

Let
\[ D_i := \binom{i + r - 1}{r - 1} = |\mathcal{B}_i| \quad (0 \leq i \leq d), \quad D := \binom{d + r}{r} = |\mathcal{B}| = \sum_{i=0}^{d} |\mathcal{B}_i|. \]

We also set \( D_{-1} := 0 \). Observe that the sequence \((D_i)_{i \geq -1}\) is strictly increasing. Therefore, there exists a unique \( \kappa_s \in \mathbb{N} \) such that
\[ D_{\kappa_s - 1} < s \leq D_{\kappa_s}. \]

By definition it follows that \( \kappa_s \leq d \).

The matrix \( \mathbb{M}_\Phi \in \mathbb{F}_q^{(d+1) \times D} \) of \( \Phi \) with respect to the bases defined above can be written as the following block matrix:
\[ \mathbb{M}_\Phi = \begin{pmatrix} M_1 \\ \vdots \\ M_s \end{pmatrix}, \]

where \( M_i \in \mathbb{F}_q^{(d+1) \times D} \) is the diagonal block matrix
\[ M_i := \begin{pmatrix} M_{i,0} \\ M_{i,1} \\ \vdots \\ M_{i,d} \end{pmatrix}, \quad M_{i,j} := (\alpha_i^\alpha : |\alpha| \leq j) \in \mathbb{F}_q^{1 \times D_j}. \]

Our first result concerns the dimension of \( \text{Im}(\Phi) \).

**Lemma 3.1.** For \( s \leq \min\{D_d, q^{r-1}\} \), we have \( \dim \text{Im}(\Phi) = (\kappa_s - 1 + r) + s(d - \kappa_s + 1) \).

**Proof.** Let \( \mathbf{h} := (h_1, \ldots, h_s) \) be an element of \( \text{Im}(\Phi) \). Then there exists \( F \in \mathcal{F}_{r,d} \) with \( \mathbf{h} = \Phi(F) \). Denote by \((F)_B = ((F_0)_{\mathcal{B}_0}, \ldots, (F_0)_{\mathcal{B}_d})\) the coordinates of \( F \) in the basis \( \mathcal{B} \). Then the block structure of the matrix \( \mathbb{M}_\Phi \) implies
\[ \Phi(F) = \sum_{j=0}^{d} \begin{pmatrix} M_{1,j} \\ \vdots \\ M_{s,j} \end{pmatrix} (F_{d-j})_{\mathcal{B}_j} T^{d-j}. \]

As \( \mathbf{a} \in \mathcal{B}_s \), we have
\[ \text{rank} \begin{pmatrix} M_{1,j} \\ \vdots \\ M_{s,j} \end{pmatrix} = \min\{D_j, s\} = \begin{cases} D_j & \text{for } 0 \leq j \leq \kappa_s - 1, \\ s & \text{for } \kappa_s \leq j \leq d. \end{cases} \]

As a consequence,
\[ \dim \text{Im}(\Phi) = \sum_{j=0}^{\kappa_s - 1} D_j + s(d - \kappa_s + 1) = \left(\kappa_s - 1 + r\right) + s(d - \kappa_s + 1). \]

This proves the lemma. \( \square \)

We shall rewrite the expression for the dimension of \( \text{Im}(\Phi) \) in a suitable form for our needs. For this purpose, we shall use the following simple combinatorial identity.
Remark 3.2. For positive integers $R, K$, we have

\begin{equation}
\sum_{j=0}^{\infty} \binom{j + R}{R} = (R + 1) \left( \binom{R + 1 + K}{R + 2} \right).
\end{equation}

Indeed,

\[
\sum_{j=0}^{\infty} \binom{j + R}{R} = \sum_{j=1}^{\infty} \frac{(j + R)!}{R!(j - 1)!} = (R + 1) \sum_{j=0}^{\infty} \binom{j + R + 1}{R + 1} = (R + 1) \left( \binom{R + 1 + K}{R + 2} \right).
\]

This shows (3.6). \hfill \square

Now we rewrite the expression for the dimension of $\text{Im}(\Phi)$.

Remark 3.3. Under the above notations,

\[
\dim \text{Im}(\Phi) = D_{d,r,s} := \sum_{j=0}^{\kappa_s - 1} (d + 1 - j)(D_j - D_{j-1}) + (d + 1 - \kappa_s)(s - D_{\kappa_s - 1}).
\]

Indeed, since $\sum_{j=0}^{k}(D_j - D_{j-1}) = D_k$, we conclude that $D_{d,r,s}$ may be expressed in the following way:

\[
D_{d,r,s} = -\sum_{j=0}^{\kappa_s - 1} j(D_j - D_{j-1}) + (d + 1 - \kappa_s)s + \kappa_s D_{\kappa_s - 1}.
\]

Taking into account (3.6), we obtain

\[
D_{d,r,s} = -(r - 1) \left( \frac{\kappa_s + r - 2}{r} \right) + (d + 1 - \kappa_s)s + \kappa_s D_{\kappa_s - 1}.
\]

A simple calculation finishes the proof of the remark. \hfill \square

Next we determine a suitable parameterization of $\text{Im}(\Phi)$. To this end, let $\Phi^*: \text{Im}(\Phi) \to \mathbb{E}^{D_{d,r,s}}$ be the $\mathbb{E}$–linear mapping defined by

\[
\Phi^*(h) := h^*,
\]

where $h := (h_1, \ldots, h_s)$, $h_i := (h_{d,i}, \ldots, h_{0,i}) \in \mathbb{E}^{d+1}$ for $1 \leq i \leq s$ and $h^* := (h^*_1, \ldots, h^*_s)$ is defined in the following way:

\begin{equation}
\begin{cases}
(h_{d-j,i}, \ldots, h_{0,i}) & \text{for } D_{j-1} < i \leq D_j, \ 0 \leq j \leq \kappa_s - 1, \\
(h_{d-\kappa_s,i}, \ldots, h_{0,i}) & \text{for } D_{\kappa_s - 1} < i \leq s.
\end{cases}
\end{equation}

Remark 3.3 shows that $\Phi^*$ is well–defined.

Lemma 3.4. $\Phi^*$ is an isomorphism.

Proof. Since $\Phi^*$ is a linear mapping between $\mathbb{E}$–vector spaces of the same dimension, it suffices to show that, if $h \in \text{Im}(\Phi)$ and $\Phi^*(h) := h^* = \mathbf{0}$, then $h = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector of both vector spaces.

Fix $h := \Phi(F) \in \text{Im}(\Phi)$ with $h^* = \mathbf{0}$. From (3.5) we deduce that

\begin{equation}
(M_{1,j}) (F_{d-j}) B_j = \begin{pmatrix} h_{d-j,1} \\
\vdots \\
M_{s,j} h_{d-j,s} \end{pmatrix}.
\end{equation}

Fix $j$ with $0 \leq j \leq \kappa_s - 1$. Then the element $h_{d-j,i}$ is included in the definition of $h^*_i$ if and only if $i \leq D_j$. As $h^* = \mathbf{0}$ by hypothesis, it follows that $h_{d-j,i} = 0$ for $1 \leq i \leq D_j$ and we have the identity:

\[
\begin{pmatrix} M_{1,j} \\
\vdots \\
M_{D_j,j} \\
M_{D_{j+1},j} \\
\vdots \\
M_{s,j} \end{pmatrix} (F_{d-j}) B_j = \begin{pmatrix} 0 \\
\vdots \\
0 \\
0 \\
\vdots \\
h_{d-j,D_{j+1}} \\
\vdots \\
h_{d-j,s} \end{pmatrix}.
\]
Since the upper \((D_k \times D_k)\)-submatrix of the matrix in the left–hand side is invertible, we conclude that \((F_{d-k})_{S_k} = 0\). This implies \(h_{d-j,k+1} = \cdots = h_{d-j,s} = 0\). On the other hand, for \(j \geq \kappa_s\) the element \(h_{d-j,i}\) is included in the definition of \(h^*_i\) for \(1 \leq i \leq s\) and therefore \(h_{d-j,i} = 0\) for \(1 \leq i \leq s\). This shows that \(h = 0\). \(\square\)

Denote by \(\Psi := (\psi_1, \ldots, \psi_s) : \mathbb{F}_q^{D_{d,r,s}} \to \text{Im}(\Phi)\) the inverse mapping of \(\Phi^*\). We need further information concerning the mappings \(\psi_i\).

**Lemma 3.5.** Let be given \(h^*_i := (h_{d-j,i}, \ldots, h_{0,i}) \in \mathbb{F}_q^{d-j+i+1}\) for \(1 \leq i \leq s\), where \(j_i\) is the unique nonnegative integer with \(0 \leq j_i \leq \kappa_s\) and \(D_{j_i-1} < i \leq D_{j_i}\). Let \(h^* := (h^*_1, \ldots, h^*_s) \in \mathbb{F}_q^{D_{d,r,s}}\) and \(h := \Psi(h^*)\). Denote

\[
    h_i := \psi_i(h^*) := h_{d,i} T^d + \cdots + h_{d-j_i+1,i} T^{d-j_i+1} + h_{d-j_i,i} T^{d-j_i} + \cdots + h_{0,i}.
\]

Then \(h_{d,j_i}, \ldots, h_{d-j_i+1,i}\) are uniquely determined by \(h^*_1, \ldots, h^*_s\).

**Proof.** Fix \(k\) with \(0 \leq k \leq j_i - 1\). Write \(h := \Phi(F)\). In the proof of Lemma 3.1 we prove that

\[
    \begin{pmatrix}
        M_{1,k} \\
        \vdots \\
        M_{D_k,k}
    \end{pmatrix}
    (F_{d-k})_{S_k}
    =
    \begin{pmatrix}
        h_{d-k,1} \\
        \vdots \\
        h_{d-k,D_k}
    \end{pmatrix},
\]

where the \((D_k \times D_k)\)-matrix in the left–hand side is invertible. The element \(h_{d-k,l}\) is included in the definition of \(h^*_i\) if and only if \(l \leq D_k\). Furthermore, we have \(k \leq j_i - 1 \leq j_i - 1\). We conclude that the vector in the right–hand side is uniquely determined by \(h^*_1, \ldots, h^*_s\), and thus so is \((F_{d-k})_{S_k}\). Therefore, the identity

\[
    \begin{pmatrix}
        M_{1,k} \\
        \vdots \\
        M_{i,k}
    \end{pmatrix}
    (F_{d-k})_{S_k}
    =
    \begin{pmatrix}
        h_{d-k,1} \\
        \vdots \\
        h_{d-k,i}
    \end{pmatrix},
\]

shows that the element \(h_{d-k,i}\) is uniquely determined by \(h^*_1, \ldots, h^*_s\). \(\square\)

### 3.2. The probability of \(s\) searches in terms of cardinalities of value sets.

For \(a := (a_1, \ldots, a_s) \in \mathbb{F}_q^{s(r-1)}\) as before, we are interested in estimating the probability of the set of polynomials of \(F_{r,d}\) for which the SVS algorithm performs \(s\) trials on the vertical strips determined by \(a_1, \ldots, a_s\). For this purpose, we consider the set \(F_{r,d}\) endowed with the uniform probability \(p_{r,d}\) and the random variable \(C_a := C_{a,r,d} : F_{r,d} \to \{1, 2, \ldots, s, \infty\}\) which counts the number of searches that the SVS algorithm performs for a given input on the vertical strips determined by \(a_1, \ldots, a_s\), \(C_a(F) = \infty\) meaning that \(F\) has no \(\mathbb{F}_q\)-rational zeros on these \(s\) vertical strips.

We start with the following elementary result.

**Lemma 3.6.** Let \(\mathcal{V}\) and \(\mathcal{W}\) be \(\mathbb{F}_q\)-linear spaces of finite dimension and \(\Phi : \mathcal{V} \to \mathcal{W}\) any \(\mathbb{F}_q\)-linear mapping. Consider \(\mathcal{V}\) and \(\mathcal{W}\) endowed with the uniform probabilities \(P_{\mathcal{V}}\) and \(P_{\mathcal{W}}\) respectively. Then for any \(A \subset \mathcal{W}\) we have

\[
    P_{\mathcal{V}}(\Phi^{-1}(A)) = \frac{|A \cap \text{Im}(\Phi)|}{|\text{Im}(\Phi)|} = \frac{P_{\mathcal{W}}(A \cap \text{Im}(\Phi))}{P_{\mathcal{W}}(\text{Im}(\Phi))} =: P_{\mathcal{W}}(A) =: P_{\text{Im}(\Phi)}(A).
\]

**Proof.** We have

\[
    \frac{1}{|\mathcal{V}|} |\Phi^{-1}(A)| = \frac{1}{|\mathcal{V}|} \sum_{w \in A} |\Phi^{-1}(w)| = \frac{1}{|\mathcal{V}|} |\text{Ker}(\Phi)| |A \cap \text{Im}(\Phi)|.
\]

By the Dimension theorem and the equality \(|\mathcal{S}| = q^\dim \mathcal{S}\), which holds for any \(\mathbb{F}_q\)-linear space \(\mathcal{S}\), we obtain

\[
    \frac{1}{|\mathcal{V}|} |\Phi^{-1}(A)| = \frac{|A \cap \text{Im}(\Phi)|}{|\text{Im}(\Phi)|} = \frac{P_{\mathcal{W}}(A \cap \text{Im}(\Phi))}{P_{\mathcal{W}}(\text{Im}(\Phi))}.
\]

This finishes the proof of the lemma. \(\square\)
Consider the $\mathbb{F}_q$–linear mapping $\Phi$ of (3.1). Since $\text{Im}(\Phi)$ is an $\mathbb{F}_q$–linear subspace, by Lemma 3.6 it follows that

$$p_{r,d}[C_{\mathbb{F}_q}] = s = \frac{|(\{N = 0\}^{s-1} \times \{N > 0\}) \cap \text{Im}(\Phi)|}{|\text{Im}(\Phi)|},$$

where $N := N_{1,d}$ denotes the random variable which counts the number zeros in $\mathbb{F}_q$ of the elements of $F_{1,d}$. As a consequence, we need an estimate of the quantity $R_s := |(\{N = 0\}^{s-1} \times \{N > 0\}) \cap \text{Im}(\Phi)|$.

According to Lemma 3.7 each element $h \in \text{Im}(\Phi)$ can be uniquely expressed in the form $h = \Psi(h^*)$, where $h^*$ is defined as in (3.7). Hence,

$$(3.10) \quad R_s = \sum_{h^* \in \mathbb{F}_q^{d-r,s}} 1_{\{N=0\}^{s-1} \times \{N > 0\}}(\Psi(h^*)),$$

where $1_{\{N=0\}^{s-1} \times \{N > 0\}} : F_{1,d} \rightarrow \{0,1\}$ denotes the characteristic function of the set $\{N = 0\}^{s-1} \times \{N > 0\}$. By Lemma 3.3 the coordinate $\psi_i(h^*)$ depends only on $h_i^* := (h_i^*, \ldots, h_s^*)$ for $1 \leq i \leq s$. We shall therefore write $\psi_i(h^*)$ as $\psi_i(h_i^*)$ for $1 \leq i \leq s$, with a slight abuse of notation.

First, we rewrite the expression (3.10) for $R_s$ in a suitable form for our purposes.

**Lemma 3.7.** Let $h := (\sum_{j=0}^d h_{j,1} T_j, \ldots, \sum_{j=0}^d h_{j,s} T_j)$ be an arbitrary element of $\text{Im}(\Phi)$ and let $h^* := \Psi(h^*) \in \mathbb{F}_q^{D_{d-r,s}}$ be defined as in (3.7). For $s \leq \min\{D_{d,r}, q^{-1}\}$, the following identity holds:

$$(3.11) \quad R_s = \sum_{h_i^* \in \mathbb{F}_q^{d-r,s}} \cdots \sum_{h_{1,s}^* \in \mathbb{F}_q^{d-r,s}} 1_{\{N=0\}^{s-1} \times \{N > 0\}}(\Psi(h^*)).$$

**Proof.** Recall that $h^* := (h_1^*, \ldots, h_s^*)$ is defined as follows:

$$h_i^* := \begin{cases} (h_{d-j,i}, \ldots, h_{0,i}) & \text{for } D_{j-1} < i \leq D_j, \ 0 \leq j \leq \kappa_s - 1, \\ (h_{d-\kappa_s,i}, \ldots, h_{0,i}) & \text{for } D_{\kappa_s-1} < i \leq s. \end{cases}$$

We may rewrite (3.10) in the following way:

$$R_s = \sum_{h_1^* \in \mathbb{F}_q^{d-r+1}} \cdots \sum_{h_s^* \in \mathbb{F}_q^{d-r,s+1}} 1_{\{N=0\}^{s-1} \times \{N > 0\}}(\Psi(h^*)).$$

As a consequence of the remarks before Lemma 3.7 it follows that

$$1_{\{N=0\}^{s-1} \times \{N > 0\}}(\Psi(h^*)) = \prod_{i=1}^{s-1} 1_{\{N=0\}}(\psi_i(h_i^*)) \cdot 1_{\{N > 0\}}(\psi_s(h_s^*)) = \prod_{i=1}^{s-1} 1_{\{N=0\}}(\psi_i(h_i^*)) \cdot 1_{\{N > 0\}}(\psi_s(h_s^*)).$$

Then the previous expression for $R_s$ can be rewritten as follows:

$$R_s = \sum_{h_1^* \in \mathbb{F}_q^{d-r+1}} \cdots \sum_{h_{s-1}^* \in \mathbb{F}_q^{d-r,s-1}} 1_{\{N=0\}}(\psi_1(h_1^*)) \cdots 1_{\{N=0\}}(\psi_{s-1}(h_{s-1}^*)) \sum_{h_s^* \in \mathbb{F}_q^{d-r,s+1}} 1_{\{N > 0\}}(\psi_s(h_s^*)), \tag{3.11}$$

which readily implies the statement of the lemma. \hfill \Box

For $1 \leq i \leq s - 1$, fix $h_i^* := (h_{d-j_i,i}, \ldots, h_{0,i}) \in \mathbb{F}_q^{d-j_i+1}$, where $j_i$ is the unique integer with $0 \leq j_i \leq \kappa_s$ and $D_{j_i-1} < i \leq D_{j_i}$. For each $h_s^* := (h_{d-\kappa_s,s}, \ldots, h_{0,s}) \in \mathbb{F}_q^{d-\kappa_s+1}$, denote by $f_{h_s^*}$ the polynomial

$$f_{h_s^*} := \psi_s(h_1^*, \ldots, h_s^*) := h_{d,s} T^{d+1} + h_{d-\kappa_s,s} T^{d-\kappa_s+1} + h_{d-\kappa_s,s+1} T^{d-\kappa_s+2} + \cdots + h_{0,s}.$$

According to Lemma 3.7 we are interested in estimating the sum

$$\sum_{h_s^* \in \mathbb{F}_q^{d-\kappa_s+1}} 1_{\{N > 0\}}(f_{h_s^*}).$$
We observe that
\[
\sum_{h_\ast \in \mathbb{F}_q^{d-k_s+1}} 1_{\{N>0\}}(f_{h_\ast}) = \sum_{(h_{d-k_s}, \ldots, h_1) \in \mathbb{F}_q^{d-k_s}} \sum_{h_0, c \in \mathbb{F}_q} 1_{\{N>0\}}(f_{h_\ast}) = \mathcal{V}(f_{h_\ast}),
\]
(3.12)
where \(f_{h_\ast} := h_{d,s}T^d + \cdots + h_{d-k_s+1,s}T^{d-k_s+1} + h_{d-k_s}T^{d-k_s} + \cdots + h_{1,s}T\) is the polynomial obtained from \(f_{h_\ast}\) by setting its constant coefficient to zero and \(\mathcal{V}(f)\) denotes the cardinality of the value set of \(f \in \mathbb{F}_q[T]\), namely \(\mathcal{V}(f) := |\{f(c) : c \in \mathbb{F}_q\}|\). As a consequence, the quantity (3.11) can also be described as the sum of the cardinalities of the value sets of all the elements of \(\mathcal{F}_{1,d}\), Lemma 3.3 proves that \(h_{d,s}, \ldots, h_{d-k_s+1,s}\) are uniquely determined by \(h_{s-1} := (h_1, \ldots, h_{s-1})\). Thus, the sum in the right-hand side of (3.12) takes as argument the cardinalities of the value sets of all the elements of \(\mathcal{F}_{1,d}\) having its first \(k_s\) coefficients \((h_{d,s}, \ldots, h_{d-k_s+1,s})\) prescribed (and the constant coefficient set to zero). Set \(\psi_s^{fix}(h_{s-1}) := (h_{d,s}, \ldots, h_{d-k_s+1,s})\) and denote by \(\mathcal{V}(d, k_s, \psi_s^{fix}(h_{s-1}))\) the average cardinality of the value set of the family \(\{f_{h_\ast} : h_{s} \in \mathbb{F}_q^{d-k_s}\}\), namely
\[
\mathcal{V}(d, k_s, \psi_s^{fix}(h_{s-1})) := \frac{1}{q^{d-k_s}} \sum_{h_\ast \in \mathbb{F}_q^{d-k_s}} \mathcal{V}(f_{h_\ast}).
\]
(3.13)
Now we express the probability that \(C_\ast = s\) in terms of \(\mathcal{V}(d, k_s, \psi_s^{fix}(h_{s-1}))\).

**Proposition 3.8.** For \(s \leq \min\{D_d, q^{-1}\}\), the following identity holds:
\[
p_{r,d}[C_\ast = s] = \frac{1}{q^{d+1}} \sum_{h_\ast \in \mathbb{F}_q^{d+1}} \cdots \frac{1}{q^{d-j_s-1+1}} \sum_{h_{s-1} \in \mathbb{F}_q^{d-j_s-1+1}} \frac{\mathcal{V}(d, k_s, \psi_s^{fix}(h_{s-1}))}{q},
\]
(3.14)

**Proof.** By Remark 3.3 we deduce that
\[
\dim \text{Im} (\Phi) = \sum_{j=0}^{k_s-1} (d+1-j)(D_j - D_{j-1}) + (d+1-k_s)(s-D_{k_s-1})
\]
\[
= \sum_{j=0}^{k_s} \min(D_j, s) (d-j) + s = \sum_{i=1}^{s} (d-j_i) + s.
\]
Combining this with (3.9) and Lemma 3.7 we obtain
\[
p_{r,d}[C_\ast = s] = \frac{1}{q^{d+1}} \sum_{h_\ast \in \mathbb{F}_q^{d+1}} \cdots \frac{1}{q^{d-j_s-1+1}} \sum_{h_{s-1} \in \mathbb{F}_q^{d-j_s-1+1}} \frac{1_{\{N>0\}}(\psi_s(h_\ast))}{q^{d-k_s+1}} 
\]
(3.12) and (3.13) complete the proof of the proposition.

Suppose further that \(s \leq \min\{D_{d-2}, q^{-1}\}\). As we explain in the next section, for any \(h_{s-1}^*\) such that \(f_{h_\ast}^*\) is of degree \(d\), the average cardinality in (3.13) can be estimated in the following way:
\[
|\mathcal{V}(d, k_s, \psi_s^{fix}(h_{s-1}))| - \mu_d q | \leq \varepsilon_{d,s} q^{1/2} + \eta_{d,s},
\]
where \(\varepsilon_{d,s} > 0\) and \(\eta_{d,s} > 0\) are constants which admit a universal upper bound independent of \(q\). More generally, for \(1 \leq i \leq s-1\) and \(1 \leq k \leq i-1\), fix \(h_k := (h_{d-j_k}, \ldots, h_{0,k}) \in \mathbb{F}_q^{d-j_k+1}\), where \(j_k\) is the unique nonnegative integer with \(0 \leq j_k \leq k_s\) and \(D_{j_k-1} < k \leq D_{j_k}\). For each \(h_k^* := (h_{d-j_k}, \ldots, h_{0,i}) \in \mathbb{F}_q^{d-j_k+1}\), denote by \(f_{h_k^*}\) the polynomial
\[
f_{h_k^*} := \psi_i(h_1^*, \ldots, h_s^*) := h_{d,s}T^d + \cdots + h_{d-j_k+1}T^{d-j_k+1} + h_{d-j_k}T^{d-j_k} + \cdots + h_{0,i}.
\]
According to Lemma 3.5, the coefficients \( h^*_r := (h^*_1,\ldots,h^*_s) \) are uniquely determined by \( h^*_r := (h^*_1,\ldots,h^*_r,\ldots,h^*_s) \). Consequently, we set \( \psi^\text{fix}_i(h^*_r) := (h^*_1,\ldots,h^*_r,\ldots,h^*_s) \) and consider the average cardinality \( \mathcal{V}(d,j_i,\psi^\text{fix}_i(h^*_r)) \) of the value set of the family \( \{ F_{h^*_r} : h^*_r \in \mathbb{F}_q^{d-j_i} \} \), namely

\[
\mathcal{V}(d,j_i,\psi^\text{fix}_i(h^*_r)) := \frac{1}{q^{d-j_i}} \sum_{\tilde{h}^*_r \in \mathbb{F}_q^{d-j_i}} \mathcal{V}(F_{\tilde{h}^*_r}).
\]

where \( F_{h^*_r} := h_1T^d + \cdots + h_{d-j_i+1}T^{d-j_i+1} + h_{d-j_i}T^{d-j_i} + \cdots + h_1T \) is the polynomial obtained from \( f_{h^*_r} \) by setting its constant coefficient to zero. In the next section we shall exhibit quantities \( \varepsilon_{d,i} > 0 \) and \( \eta_{d,i} > 0 \), which admit a universal upper bound independent of \( q \), such that

\[
|\mathcal{V}(d,j_i,\psi^\text{fix}_i(h^*_r)) - \mu_d q| \leq \varepsilon_{d,i} q^{1/2} + \eta_{d,i}
\]

for any \( h^*_r \) for which \( f_{h^*_r} \) is of degree \( d \). We need the following remark.

**Remark 3.9.** For each \( h := (h_1,\ldots,h_s) \in \text{Im}(\Phi) \), we have \( h_{d,1} = \cdots = h_{d,s} \). Indeed, from (3.5) we deduce that

\[
\begin{pmatrix}
M_{1,0} \\
\vdots \\
M_{s,0}
\end{pmatrix}
(F_d)_{B_0} = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
(F_d)_{B_0} = \begin{pmatrix}
h_{d,1} \\
\vdots \\
h_{d,s}
\end{pmatrix}.
\]

This implies \( h_{d,1} = \cdots = h_{d,s} = (F_d)_{B_0} \). In particular, the coefficient \( h_{d,1} \) of the monomial \( T^d \) in the polynomial \( h_1 \) uniquely determines the coefficient \( h_{d,j} \) of the monomial \( T^j \) in \( h_j \) for \( 2 \leq j \leq s \).

Our next result expresses the probability of \( s \) searches in terms of the quantities \( \varepsilon_{d,i} \) and \( \eta_{d,i} \) (1 \( i \leq s \)).

**Theorem 3.10.** For \( s \leq \min\{D_d-2,q^{r-1}\} \), we have

\[
|p_{r,d}[C = s] - (1 - \mu_d)^{s-1}\mu_d| \leq \sum_{i=1}^{s} \varepsilon_{d,i} q^{-1/2} + \left(1 + \sum_{i=1}^{s} \eta_{d,i}\right) q^{-1}.
\]

**Proof.** Denote \( C := C_{d,s} \). We split the expression for \( p_{r,d}[C = s] \) of Proposition 3.8 into two sums, depending on whether \( h_{d,1} = 0 \) or not. More precisely, we write

\[
p_{r,d}[C = s] = p_{r,d}[C = s, F_d = 0] + p_{r,d}[C = s, F_d \neq 0],
\]

where

\[
p_{r,d}[C = s, F_d = 0] = \frac{1}{q^{d+1}} \sum_{\tilde{h}^*_r \in \mathbb{F}_q^{d+1}} \cdots \frac{1}{q^{d-j_{s-1}+1}} \sum_{\tilde{h}^*_r \in \mathbb{F}_q^{d-j_{s-1}+1}} \mathcal{V}(d,\kappa_s,\psi^\text{fix}_i(h^*_r)) = \frac{1}{q^{d+1}} \sum_{h_{d,1} = 0} \cdots \sum_{h_{d,s} = 0} \mathcal{V}(d,\kappa_s,\psi^\text{fix}_i(h^*_r)) = \frac{1}{q^{d+1}} \sum_{h_{d,1} = 1} \cdots \sum_{h_{d,s} = 1} \mathcal{V}(d,\kappa_s,\psi^\text{fix}_i(h^*_r)).
\]

Concerning the first term, we are considering the intersection of the \( \mathbb{F}_q \)-linear space \( \text{Im}(\Phi) \) with the linear subspace \( \mathcal{F}_{1,d-1}^r \). As the former is not contained in the latter, the dimension of the intersection falls at least by one, and Lemma 3.6 implies

\[
p_{r,d}[C = s, F_d = 0] \leq \frac{|\text{Im}(\Phi) \cap \mathcal{F}_{1,d-1}^r|}{|\text{Im}(\Phi)|} \leq \frac{q^{\dim \text{Im}(\Phi)-1}}{q^{\dim \text{Im}(\Phi)}} = \frac{1}{q}.
\]
On the other hand, it is easy to see that the expression for \( p_{r,d}[C = s, F_d \neq 0] \) may be rewritten in the following way:

\[
p_{r,d}[C = s, F_d \neq 0] = \mu_d \frac{q - 1}{q^{d+1}} \sum_{h_1^r \in \mathbb{Z}^{d+1}} \cdots \sum_{h_{d-1}^r \in \mathbb{Z}^{d-1+1}} \sum_{\substack{h_s^r \in \mathbb{Z}^d \ni \psi_1(h_s^r) = 0 \\ h_{d,1}^r = 1}} 1 + T_s,
\]

where

\[
T_s := \frac{q - 1}{q^{d+1}} \sum_{h_1^s \in \mathbb{Z}^{d+1}} \cdots \sum_{h_{d-1}^s \in \mathbb{Z}^{d-1+1}} \sum_{\substack{h_s^s \in \mathbb{Z}^d \ni \psi_1(h_s^s) = 0 \\ h_{d,1}^s = 1}} \left( \frac{\mathcal{V}(d, \kappa_s, \psi_{\text{fix}}(h_{s-1}^s))}{q} - \mu_d \right).
\]

In particular, from (3.14) and Remark 3.9 we conclude that

\[
|T_s| \leq \frac{q - 1}{q} (\varepsilon_{d,s} q^{-1/2} + \eta_{d,s} q^{-1}).
\]

Observe that

\[
\frac{1}{q^{d-j-1+1}} \sum_{h_{j-1}^s \in \mathbb{Z}^{d-j-1+1}} 1 = 1 - \frac{1}{q^{d-j-1+1}} \sum_{h_{j-1}^s \in \mathbb{Z}^{d-j-1+1}} \sum_{\substack{h_s^s \in \mathbb{Z}^d \ni \psi_1(h_s^s) = 0 \\ h_{d,1}^s = 1}} \frac{\mathcal{V}(d, j_s, \psi_{\text{fix}}(h_{s-1}^s))}{q}.
\]

Therefore, we may rewrite \( p_{r,d}[C = s, F_d \neq 0] \) in the following way:

\[
p_{r,d}[C = s, F_d \neq 0] = (1 - \mu_d) \frac{1}{q^{d+1}} \sum_{h_1^r \in \mathbb{Z}^{d+1}} \cdots \sum_{h_{d-1}^r \in \mathbb{Z}^{d-1+1}} \sum_{\substack{h_s^s \in \mathbb{Z}^d \ni \psi_1(h_s^s) = 0 \\ h_{d,1}^r = 1}} 1 + T_{s-1} + T_s,
\]

where

\[
T_{s-1} := \mu_d \frac{q - 1}{q^{d+1}} \sum_{h_1^r \in \mathbb{Z}^{d+1}} \cdots \sum_{h_{d-1}^r \in \mathbb{Z}^{d-1+1}} \sum_{\substack{h_s^s \in \mathbb{Z}^d \ni \psi_1(h_s^s) = 0 \\ h_{d,1}^r = 1}} \left( \mu_d - \frac{\mathcal{V}(d, j_s-1, \psi_{\text{fix}}(h_{s-1}^s))}{q} \right).
\]

Furthermore, from (3.15) and Remark 3.9 it follows that

\[
|T_{s-1}| \leq \mu_d \frac{q - 1}{q} (\varepsilon_{d,s-1} q^{-1/2} + \eta_{d,s-1} q^{-1}).
\]

Arguing as above, we see that

\[
p_{r,d}[C = s, F_d \neq 0] = (1 - \mu_d)^{s-1} \mu_d \frac{q - 1}{q} + \sum_{i=1}^{s} T_i,
\]

where \( T_i \) is defined as in (3.17) and

\[
T_i := \mu_d (1 - \mu_d)^{s-i+1} \frac{q - 1}{q} \sum_{h_1^r \in \mathbb{Z}^{d+1}} \cdots \sum_{h_{d-1}^r \in \mathbb{Z}^{d-1+1}} \sum_{\substack{h_s^s \in \mathbb{Z}^d \ni \psi_1(h_s^s) = 0 \\ h_{d,1}^r = 1}} \left( \mu_d - \frac{\mathcal{V}(d, j_i, \psi_{\text{fix}}(h_{s-1}^s))}{q} \right)
\]

for \( 1 \leq i \leq s - 1 \). From (3.16) and Remark 3.9 we easily deduce the statement of the theorem. \( \square \)
3.3. Estimates for average cardinalities of value sets and the probability of $C_n = s$. Theorem 3.10 shows that the probability that the SVS algorithm stops after $s \leq \min \{D_{d-2}, q^{-1}\}$ attempts can be expressed in terms of the average cardinality of the value set of certain families of univariate polynomials.

Value sets are a classical subject of combinatorics over finite fields (cf. [15], [17]). Let $V(d, 0)$ denote the average cardinality of the value set of all monic polynomials in $F_q[T]$ of degree $d$ with $f(0) = 0$. It is well-known that

\[
V(d, 0) = \sum_{j=1}^{d} (-1)^{j-1} \binom{q}{j} q^{1-j} = \mu_d q + O(1),
\]

where $\mu_d := \sum_{j=1}^{d} (-1)^{j-1}/j!$ and the constant underlying the $O$-notation depends only on $d$ (see [18], [19]).

On the other hand, if we consider the average cardinality $V(f)$ for all the monic polynomial $f$ of degree $d$ with some coefficients prescribed, the results are less precise. Indeed, let be given $j$ with $1 \leq j \leq d - 2$ and $a := (a_{d-1}, \ldots, a_{d-j}) \in F_q^s$. For every $b := (b_{d-j-1}, \ldots, b_1)$, let

\[
f_b^i := T^d + \sum_{i=1}^{s} a_{d-i} T^{d-i} + \sum_{i=s+1}^{d-1} b_{d-i} T^{d-i}.
\]

In [5] it is shown that, for $p := \text{char}(F_q) > d$,

\[
V(d, j, a) := \frac{1}{q^{d-j-1}} \sum_{b \in F_q^{d-j-1}} V(f_b) = \mu_d q + O(q^{1/2}),
\]

where the constant underlying the $O$-notation depends only on $d$ and $j$ (see also [19]). Suppose that $q > d$. In [7], the following explicit estimate for $1 \leq j \leq d/2 - 1$ is obtained:

\[
|V(d, j, a) - \mu_d q| \leq \frac{e^{-1}}{2} + \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} + \frac{7}{q}.
\]

This result holds without any restriction on the characteristic $p$ of $F_q$. On the other hand, in [10] it is proved that, if $p > 2$ and $1 \leq j \leq d - 3$, then

\[
|V(d, j, a) - \mu_d q| \leq d^2 2^{d-1} q^{-\frac{1}{2}} + 133 d^5 e^{2\sqrt{d}} - d.
\]

Estimates (3.19) and (3.20) provide admissible expressions for the quantities $\epsilon_{d,i}$ and $\eta_{d,i}$ (1 \leq i \leq s) of the statement of Theorem 3.10. As a consequence, we have the following result.

**Theorem 3.11.** Let be given $a := (a_1, \ldots, a_s) \in \mathbb{F}_q^{s(r-1)} \setminus B_s$, where the set $B_s$ is defined in 3.3. For $s \leq \min \{\left(\frac{d}{d/2 + r-1}\right), q^{r-1}\}$, we have

\[
|pr_d[C_{a,r,d} = s] - (1 - \mu_d)^{s-1} \mu_d| \leq \left(\frac{e^{-1}}{2} + \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} + 1\right) q^{-1} + 7sq^{-2}.
\]

On the other hand, if $p > 2$ and $s \leq \min \{\left(\frac{d+r-3}{r-1}\right), q^{r-1}\}$, then

\[
|pr_d[C_{a,r,d} = s] - (1 - \mu_d)^{s-1} \mu_d| \leq sd^2 2^{d-1} q^{-\frac{1}{2}} + (133sd^5 e^{2\sqrt{d}} + 1) q^{-1}.
\]

**Proof.** With notations as in Section 3.2 fix $i$ with $1 \leq i \leq s$ and fix $h_k^i := (h_{d-k}, k, \ldots, h_0, k) \in \mathbb{F}_q^{d-k+1}$ for $1 \leq k \leq i - 1$, where $j_k$ is the unique nonnegative integer with $0 \leq j_k \leq \kappa_s$ and $D_{j_k - 1} < k \leq D_{j_k}$. For each $h_k^* := (h_{d-j_k}, \ldots, h_0, k) \in \mathbb{F}_q^{d-j_k+1}$, denote

\[
f_{h_k^*} := \psi_i(h_{d-j_k}, \ldots, h_0, k) = h_{d-j_k} T^d + \ldots + h_{d-j_k+1} T^{d-j_k+1} + h_{d-j_k+1} T^{d-j_k+1} + \ldots + h_0.
\]

Lemma 3.20 shows that the coefficients $h_{d,i}, \ldots, h_{d,j_i+1,i}$ are uniquely determined by $h_{i-1}^* := (h_{1}, \ldots, h_{1}^*)$. Consequently, set $\psi_i^{\text{fix}}(h_{i-1}^*) := (h_{d,i}, \ldots, h_{d-j_i+1,i})$ and...
consider the average cardinality
\[ V(d, j, \psi^\text{fix}(h_{i-1}^*)) := \frac{1}{q^{d-j_i}} \sum_{h_i \in C_i} V(f_{h_i}). \]
where \( f_{h_i} := h_{d,i}T^d + \cdots + h_{d-j_i+1,i}T^{d-j_i+1} + h_{d-j_i,i}T^{d-j_i} + \cdots + h_{1,i}T \) is the polynomial obtained from \( f_{h_i} \) by setting its constant coefficient to zero. According to (3.10), for \( 1 \leq j_i - 1 \leq \frac{d}{2} - 1 \) and any \( h_{i-1}^* \) such that \( f_{h_i} \) is of degree \( d \) we have
\[ (3.21) \quad \left| V(d, j, \psi^\text{fix}(h_{i-1}^*)) - \mu_d \right| \leq \frac{e^{-1}}{2} + \frac{(d - 2)^5e^{2\sqrt{d}}}{2^d - 2} q^{-1} + 7q^{-2}. \]
On the other hand, from (3.20) it follows that, if \( p > 2 \) and \( j_i - 1 \leq d - 3 \), then
\[ (3.22) \quad \left| V(d, j, \psi^\text{fix}(h_{i-1}^*)) - \mu_d \right| \leq d^2 2^{d-1} q^{-\frac{d}{2}} + 133d^{d+5}e^{2\sqrt{d} - d} q^{-1}. \]
Suppose that \( s \leq \min \{ \{(d/2 + r - 1), q^{-r-1} \} \} \). Therefore \( j_s := \kappa_s \leq d/2 \) and (3.21) holds for \( 1 \leq i \leq s \). This implies that
\[ \varepsilon_{d,i} := 0, \quad \eta_{d,i} := \left( \frac{e^{-1}}{2} + \frac{(d - 2)^5e^{2\sqrt{d}}}{2^d - 2} \right) q^{-1} + 7q^{-2} \]
are admissible definitions for \( \varepsilon_{d,i} \) and \( \eta_{d,i} \) (1 \( \leq i \leq s \)) in the statement of Theorem 3.10, which shows the first assertion of the theorem.

On the other hand, for \( p > 2 \) and \( j_i - 1 \leq d - 2 \), it follows that (3.21) holds for \( 1 \leq i \leq s \) and hence
\[ \varepsilon_{d,i} := d^2 2^{d-1}, \quad \eta_{d,i} := 133d^{d+5}e^{2\sqrt{d} - d} \]
are admissible definitions for \( \varepsilon_{d,i} \) and \( \eta_{d,i} \) (1 \( \leq i \leq s \)) in the statement of Theorem 3.10. This readily implies the second assertion of the theorem. \( \square \)

We remark that the approach of the proof of Theorem 3.11 cannot be applied to estimate the probability that \( s > s^* = (d/2 + r - 1) \) vertical strips are searched, since the behavior of the mapping \( \Phi := \Phi_{\alpha} : F_{r,d} \to F_{r,d} \) of (3.1) may change significantly in this case. In what concerns "large" values of \( s \), from Theorem 3.11 one easily deduces that
\[ pr_{r,d}[C_{r,d} \geq s^\ast] = (1 - \mu_d) s^\ast + \mathcal{O}(q^{-1/2}). \]
As \( |1 - \mu_d| \leq 1/2 \), from the expression of \( s^\ast \) it follows that the main term of this probability decreases exponentially with \( r \) and \( d \).

4. Probabilistic analysis of the SVS algorithm

In this section we determine the probability distribution of the number searches that the SVS algorithm performs in both models of generation of vertical strips. Similarly to Section 2 for \( s \geq 3 \) we denote
\[ F_{q} := F_{q}^{-1} \times \cdots \times F_{q}^{-1} \text{ (s times)}, \quad N_{\text{ind}} := |F_{\text{ind}}|, \]
\[ F_{\text{ind}} := \{(a_1, \ldots, a_s) \in F_{q}^s : a_i \neq a_j \text{ for } i \neq j \}, \quad N_{\text{ind}} := |F_{\text{ind}}|, \]
and consider the random variable \( C_{\text{var}} := C_{\text{var}_{r,d}} : F_{\text{var}} \times F_{r,d} \to \{1, \ldots, s, \infty\} \) defined for \( \alpha := (a_1, \ldots, a_s) \in F_{\text{var}} \) and \( F \in F_{r,d} \) in the following way:
\[ C_{\text{var}}(\alpha, F) := \left\{ \begin{array}{ll}
\min \{j : N_{1,d}(F(a_j, X_r)) > 0\} & \text{if } \exists j \text{ with } N_{1,d}(F(a_j, X_r)) > 0,
\infty & \text{otherwise}.
\end{array} \right. \]
We consider the set \( F_{\text{var}} \times F_{r,d} \) as before endowed with the uniform probability \( P_{\text{var}} := P_{\text{var}_{r,d}} \) and analyze the probability \( P_{\text{var}}(C_{\text{var}} = s) \). To link the probability spaces determined by \( F_{\text{var}} \times F_{r,d} \) and \( P_{\text{var}} \) for all \( s \geq 1 \), we have the following result.

Lemma 4.1. Let \( s > 1 \), \( \varphi \in \{\text{ind, nind}\} \) and let \( \pi_{\varphi} : F_{\text{var}} \times F_{r,d} \to F_{\text{var}_{s-1}} \times F_{r,d} \) be the mapping induced by the projection \( F_{\text{var}} \to F_{\text{var}_{s-1}} \) on the first \( s - 1 \) coordinates. If \( S \subseteq F_{\text{var}_{s-1}} \times F_{r,d} \), then \( P_{\text{var}}[\pi_{\varphi}^{-1}(S)] = P_{\text{var}}[S] \).
Proof. We first consider the case $\text{var} = \text{ind}$. Note that
\[
\pi_s^{-1}(S) = \bigcup_{F \in \mathcal{F}_r,d} \{ \{a_1, \ldots, a_s\} \in F_s^{\text{ind}} : (a_1, \ldots, a_{s-1}, F) \in S \} \times \{F\}
\]
\[
= \bigcup_{F \in \mathcal{F}_r,d} \bigcup_{(a_1, \ldots, a_{s-1}) \in F_s^{\text{ind}} : (a_1, \ldots, a_{s-1}, F) \in S} \{ \{a_1, \ldots, a_{s-1}\} \times (F_q^{-1})_s \setminus \{a_1, \ldots, a_{s-1}\} \} \times \{F\}.
\]
It follows that
\[
P^{\text{ind}}[\pi_s^{-1}(S)] = \frac{1}{N_s^{\text{ind}}[F_r,d]} \sum_{F \in \mathcal{F}_r,d} \sum_{\{a\} \in F_s^{\text{ind}} : (\{a\}, F) \in S} (q^{r-1} - s + 1)
\]
\[
= \frac{1}{N_s^{\text{ind}}[F_r,d]} \sum_{F \in \mathcal{F}_r,d} \sum_{\{a\} \in F_s^{\text{ind}} : (\{a\}, F) \in S} \left| \left\{ \{a\} \in F_s^{\text{ind}} : (\{a\}, F) \in S \right\} \right| = P^{\text{ind}}[S].
\]

The identity $P^{\text{ind}}[\pi_s^{-1}(S)] = P^{\text{ind}}[S]$ is shown by a similar argument. □

According to the Kolmogorov extension theorem (see, e.g., [11, Chapter IV, Section 5, Extension Theorem]), the conditions of “consistency” of Lemma 4.1 imply that the probabilities $P_s$ ($s \geq 1$) can be put in a unified framework. More precisely, in the independent model we define $F^{\text{ind}} := (F_q^{r-1})^N$. For every $s \geq 1$, denote by $\pi_s^{\text{ind}} : F^{\text{ind}} \times \mathcal{F}_r,d \to F_s^{\text{ind}} \times \mathcal{F}_r,d$ the mapping induced by the projection $F^{\text{ind}} \to F_s^{\text{ind}}$ on the first $s$ coordinates. Then there exists a unique probability measure $P^{\text{ind}}$ defined on $F^{\text{ind}} \times \mathcal{F}_r,d$ such that
\[
P^{\text{ind}}[\pi_s^{-1}(S)] = P^{\text{ind}}[S]
\]
for every $S \subset F^{\text{ind}} \times \mathcal{F}_r,d$. On the other hand, in the nonindependent model we simply define $F^{\text{ind}} := F_q^{r-1}$ and $P^{\text{ind}} := P^{\text{ind}}$. In other words, for $\text{var} \in \{\text{ind}, \text{ind}\}$ there exists a probability measure $P^{\text{ind}}$ defined on $F^{\text{ind}}$, which allows us to interpret consistently all the results of this paper. In the same vein, the sequence of variables $(C^{\text{var}}_{a,r,d})_{a \in \mathcal{F}_s}$ is naturally extended to a random variable $C^{\text{var}} : F^{\text{var}} \times \mathcal{F}_r,d \to \mathbb{N} \cup \{\infty\}$. Consequently, we shall drop the subscript $s$ from the notations $P^{\text{var}}$ and $C^{\text{var}}$ in what follows.

For the analysis of the probability distribution of the number of searches, we express the probability $P^{\text{var}}[C^{\text{var}} = s]$ in terms of probabilities concerning the random variables $C_{a,r,d} : \mathcal{F}_r,d \to \mathbb{N}$, which count the number of vertical strips that are searched when the choice for the first $s$ vertical strips is $a$. As the result is a slight generalization of Lemma 2.3, we shall only sketch its proof.

Lemma 4.2. For $\text{var} \in \{\text{ind}, \text{ind}\}$, we have
\[
P^{\text{var}}[C^{\text{var}} = s] = \frac{1}{N_s^{\text{var}}[\mathcal{F}_r,d]} \sum_{a \in \mathcal{F}_s} \sum_{F \in \mathcal{F}_r,d} p_{r,d}[C_{a,r,d} = s].
\]

Proof. The set $\{C^{\text{var}} = s\}$ can be expressed as the following union of disjoint sets:
\[
\{C^{\text{var}} = s\} = \bigcup_{F \in \mathcal{F}_r,d} \{a \in \mathcal{F}_s : C_{a,r,d}(F) = s\} \times \{F\}.
\]
Therefore,
\[
P^{\text{var}}[C^{\text{var}} = s] = \frac{1}{N_s^{\text{var}}[\mathcal{F}_r,d]} \sum_{F \in \mathcal{F}_r,d} \sum_{a \in \mathcal{F}_s} \left| \left\{ a \in \mathcal{F}_s : C_{a,r,d}(F) = s \right\} \right| \times \{F\}
\]
\[
= \frac{1}{N_s^{\text{var}}[\mathcal{F}_r,d]} \sum_{a \in \mathcal{F}_s} \sum_{F \in \mathcal{F}_r,d} \left| \left\{ F \in \mathcal{F}_r,d : C_{a,r,d}(F) = s \right\} \right| \times \{F\}
\]
\[
= \frac{1}{N_s^{\text{var}}[\mathcal{F}_r,d]} \sum_{a \in \mathcal{F}_s} \sum_{F \in \mathcal{F}_r,d} p_{r,d}[C_{a,r,d} = s],
\]
which shows the lemma. □
In Theorem 3.3 we determine the asymptotic behavior of the probability \( p_{r,d}[C_{\mathbf{a},r,d} = s] \) for “general” \( \mathbf{a} \). More precisely, let \( \mathcal{B}_s \subset \mathcal{F}_s^{\text{ind}} \) be the set of \( \mathbf{a} \). For each \( \mathbf{a} \in \mathcal{F}_s^{\text{ind}} \setminus \mathcal{B}_s \), the estimate of Theorem 3.3 holds. By (3.3) it follows that \( |\mathcal{B}_s| = \mathcal{O}(q^{(r-1)} \log(1/q)) \), where the \( \mathcal{O} \)-constant depends on \( s \), \( d \) and \( r \), but is independent of \( q \).

Now we are ready to estimate the probability \( P^\text{var}[C_{\text{var}} = s] \). First we consider the nonindependent model. By the definition of \( \mathcal{B}_s \) we easily conclude that \( \mathcal{F}_s^{\text{ind}} \setminus \mathcal{B}_s \subset \mathcal{F}_s^{\text{ind}} \). Lemma 4.2 implies

\[
P^\text{ind}[C_{\text{ind}} = s] = \frac{1}{N_s^{\text{ind}}} \sum_{\mathbf{a} \in \mathcal{F}_s^{\text{ind}}} p_{r,d}[C_{\mathbf{a},r,d} = s] = \frac{1}{N_s^{\text{ind}}} \sum_{\mathbf{a} \in \mathcal{F}_s^{\text{ind}} \setminus \mathcal{B}_s} p_{r,d}[C_{\mathbf{a},r,d} = s] = \frac{1}{N_s^{\text{ind}}} \sum_{\mathbf{a} \in \mathcal{F}_s^{\text{ind}} \setminus \mathcal{B}_s} p_{r,d}[C_{\mathbf{a},r,d} = s] + \mathcal{O}(q^{-1}).
\]

Let \( \mathbf{a} \in \mathcal{F}_s^{\text{ind}} \setminus \mathcal{B}_s \). According to Theorem 3.3 for \( s \leq \binom{d/2}{r-1} \) we have \( p_{r,d}[C_{\mathbf{a},r,d} = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1}) \). On the other hand, for \( p > 2 \) and \( s \leq \binom{d+r-3}{r-1} \), it holds that \( p_{r,d}[C_{\mathbf{a},r,d} = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1/2}) \). As a consequence, we have the following result.

**Theorem 4.3.** For \( s \leq \binom{d/2}{r-1} \), we have

\[
P^\text{ind}[C_{\text{ind}} = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1}).
\]

On the other hand, if \( p > 2 \) and \( s \leq \binom{d+r-3}{r-1} \), then

\[
P^\text{ind}[C_{\text{ind}} = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1/2}).
\]

Next we analyze the probability that \( s \) vertical strips are searched in the independent model. Given \( \mathbf{a} := (\mathbf{a}_1, \ldots, \mathbf{a}_s) \in \mathcal{F}_s^{\text{ind}} \), we denote by \( n(\mathbf{a}) \) the maximum number of \( \mathbf{a}_i \) which are pairwise distinct. In particular, \( \mathcal{F}_s^{\text{ind}} \) consists of the \( \mathbf{a} \in \mathcal{F}_s^{\text{ind}} \) with \( n(\mathbf{a}) = s \). Assume that \( s > 1 \) and let be given \( \mathbf{a} \in \mathcal{F}_s^{\text{ind}} \) with \( n(\mathbf{a}) = 1 \). Then \( p_{r,d}[C_{\mathbf{a},r,d} = s] = 0 \), since all the elements of \( \mathcal{F}_{r,d} \) having \( \mathbb{F}_q \)-rational zeros on the vertical strip defined by \( \mathbf{a}_s \) have also \( \mathbb{F}_q \)-rational zeros on the vertical strip defined by \( \mathbf{a}_1 \). As a consequence, we may write

\[
P^\text{ind}[C_{\text{ind}} = s] = \frac{1}{N_s^{\text{ind}}} \sum_{j=2}^{s-1} \sum_{\mathbf{a} : n(\mathbf{a}) = j} p_{r,d}[C_{\mathbf{a},r,d} = s]
\]

\[
= \frac{N_s^{\text{ind}}}{N_s^{\text{ind}}} P^\text{ind}[C_{\text{ind}} = s] = \frac{1}{N_s^{\text{ind}}} \sum_{j=2}^{s-1} \sum_{\mathbf{a} : n(\mathbf{a}) = j} p_{r,d}[C_{\mathbf{a},r,d} = s].
\]

Observe that \( N_s^{\text{ind}} \{ \mathbf{a} \in \mathcal{F}_s^{\text{ind}} : n(\mathbf{a}) = j \} = \mathcal{O}(q^{1-r}) \) for \( 2 \leq j \leq s - 1 \). Therefore, from Theorem 4.3 we deduce the following result.

**Theorem 4.4.** For \( s \leq \binom{d/2}{r-1} \), we have

\[
P^\text{ind}[C_{\text{ind}} = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1}).
\]

On the other hand, if \( p > 2 \) and \( s \leq \binom{d+r-3}{r-1} \), then

\[
P^\text{ind}[C_{\text{ind}} = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1/2}).
\]

5. On the probability distribution of the outputs

This section is devoted to the analysis of the probability distribution of the outputs of both variants of the SVS algorithm. For this purpose, following [3] (see also [2]), we use the concept of Shannon entropy. For \( F \in \mathcal{F}_{r,d} \), denote \( Z(F) := \{ \mathbf{x} \in \mathbb{F}_q^r : F(\mathbf{x}) = 0 \} \) and \( N(F) := |Z(F)| \). We define a Shannon entropy \( H^\text{var}_F \)
associated with $F$ and a variant $\mathcal{A}^{\text{var}}$ of the SVS algorithm, with \( \var \in \{ \text{ind}, \text{nind} \} \), as
\[
H_{\var}^{\text{var}} = \sum_{x \in Z(F)} -P_{x,F}^{\var} \log(P_{x,F}^{\var}),
\]
where \( P_{x,F}^{\var} \) is the probability that the algorithm \( \mathcal{A}^{\text{var}} \) outputs \( x \) on input \( F \) and \( \log \) denotes the natural logarithm. It is well–known that \( H_{\var}^{\text{var}} \leq \log |Z(F)| \), and equality holds if and only if \( P_{x,F}^{\var} = 1/|Z(F)| \) for every \( x \in Z(F) \). We shall consider the average entropy of both variants of the SVS algorithm when \( F \) runs through all the elements of \( \mathcal{F}_{r,d} \), namely
\[
H^{\text{var}} := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} H_{\var}^{\text{var}}.
\]

For an “ideal” algorithm for the search of \( \mathbb{F}_q \)-rational zeros of elements of \( \mathcal{F}_{r,d} \), from the point of view of the probability distribution of outputs, and \( F \in \mathcal{F}_{r,d} \), the probability \( P_{x,F}^{\text{ideal}} \) that a given \( x \in Z(F) \) occurs as output is equal to \( 1/N(F) \). As a consequence, according to the definition (5.1), the corresponding entropy is
\[
H_{F}^{\text{ideal}} := \sum_{x \in Z(F)} -P_{x,F}^{\text{ideal}} \log(P_{x,F}^{\text{ideal}}) = \sum_{x \in Z(F)} \log(1/N(F)) = \log(N(F)).
\]

By the concavity of the function \( x \mapsto \log x \) and (1.1), we conclude that
\[
H^{\text{ideal}} := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} H_{F}^{\text{ideal}} \leq \log \left( \frac{\sum_{F \in \mathcal{F}_{r,d}} N(F)}{|\mathcal{F}_{r,d}|} \right) = \log(q^{-1}),
\]
where the last identity is due to (1.1). In our analysis below, we shall exhibit lower bounds on the average entropy \( H^{\text{var}} \) with \( \var \in \{ \text{ind}, \text{nind} \} \) which nearly match this upper bound.

5.1. On the number of vertical strips. As it will become apparent, a critical point in the study of the behavior of \( H^{\text{var}} \) for \( \var \in \{ \text{ind}, \text{nind} \} \) is the analysis of the probability distribution of the random variable \( NS : \mathcal{F}_{r,d} \to \mathbb{Z}_{\geq 0} \) which counts the number of vertical strips of the elements of \( \mathcal{F}_{r,d} \).

Recall that, for an element \( F \in \mathcal{F}_{r,d} \), we denote by \( VS(F) \) the set of vertical strips where \( F \) has an \( \mathbb{F}_q \)-rational zero and by \( NS(F) \) its cardinality, that is,
\[
VS(F) := \{ a \in \mathbb{F}_q^{-1} : (\exists x_r \in \mathbb{F}_q) F(a, x_r) = 0 \}, \quad NS(F) := |VS(F)|.
\]

We start with the following result, which concerns the average number \( NS(r,d) \) of vertical strips in \( \mathcal{F}_{r,d} \), namely
\[
NS(r,d) := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} NS(F).
\]

Lemma 5.1. The number \( NS(r,d) \) satisfies
\[
NS(r,d) = \sum_{k=1}^{d} (-1)^{k-1} \binom{q}{k} q^{r-1-k} + (-1)^d \binom{q-1}{d} q^{r-d-2} = \mu_d q^{r-1} + O(q^{r-2}).
\]

Proof. According to (2.1), \( NS(r,d) = q^{r-1}P[C = 1] \). Then the first assertion of the lemma follows immediately from Theorem (2.1). The second assertion follows from Corollary (2.2). \( \square \)

Next we determine the variance \( NS_2(r,d) \) of the random variable which counts the number of vertical strips of the elements of \( \mathcal{F}_{r,d} \), that is,
\[
NS_2(r,d) := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} (NS(F) - NS(r,d))^2 = \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} NS(F)^2 - NS(r,d)^2.
\]
Proposition 5.2. The variance $NS_2(r, d)$ satisfies
$$NS_2(r, d) = -q^{1-r} (NS(r, d))^2 + NS(r, d) = \mu_d(1 - \mu_d) q^{r-1} + O(q^{r-2}).$$

Proof. Recall the notations $F_2^\mathrm{ind} := F_{r-1}^q \times F_{r-1}^q$, $F_2^\mathrm{ind} := F_2^q \setminus \{(a, a) : a \in F_{r-1}^q\}$ and $N_2^\mathrm{ind} := |F_2^\mathrm{ind}|$. Fix $F \in F_{r,d}$. We have
$$NS(F)^2 = \left| \bigcup_{x,y \in \mathbb{F}_q} \{(a_1, a_2) \in F_2^\mathrm{ind} : F(a_1, x) = F(a_2, y) = 0\} \right|.$$ Then the inclusion–exclusion principle implies
$$\sum_{F \in F_{r, d}} NS(F)^2 = \sum_{F \in F_{r, d}} \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{X_j \subset \mathbb{F}_q} \sum_{Y_k \subset \mathbb{F}_q} S(X_j, Y_k)$$
$$= \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{X_j \subset \mathbb{F}_q} \sum_{Y_k \subset \mathbb{F}_q} S(X_j, Y_k),$$
where $X_j$ and $Y_k$ run through all the subsets of $\mathbb{F}_q$ of cardinality $j$ and $k$, respectively, and, for arbitrary subsets $X \subset \mathbb{F}_q$ and $Y \subset \mathbb{F}_q$,
$$S(X, Y) := |\{(a, a) \in F_2^\mathrm{ind} : (\forall x \in X)(\forall y \in Y) F(a, x) = F(a, y) = 0\}|.$$ For $a := (a_1, a_2) \in F_2^\mathrm{ind}$ and subsets $X \subset \mathbb{F}_q$ and $Y \subset \mathbb{F}_q$, denote
$$S_a(X, Y) := \{F \in F_{r,d} : (\forall x \in X)(\forall y \in Y) F(a, x) = 0, F(a, y) = 0\}.$$ It follows that
$$\sum_{F \in F_{r, d}} NS(F)^2 = \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{X_j \subset \mathbb{F}_q} \sum_{Y_k \subset \mathbb{F}_q} \sum_{a \in F_2^\mathrm{ind}} |S_a(X_j, Y_k)|$$
$$= \sum_{a \in F_2^\mathrm{ind}} \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{X_j \subset \mathbb{F}_q} \sum_{Y_k \subset \mathbb{F}_q} |S_a(X_j, Y_k)| =: \sum_{a \in F_2^\mathrm{ind}} N_{a, 2},$$
where $N_{a, 2}$ is defined as in (2.4). If $a \in F_2^\mathrm{ind}$, then the claim in the proof of Proposition 2.4 asserts that
$$\frac{N_{a, 2}}{|F_{r, d}|} = (P[C = 1])^2.$$ On the other hand, for $(a, a) \in F_2^\mathrm{ind} \setminus F_2^\mathrm{ind}$, by elementary calculations it can be seen that
$$N(a, a) := \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{X_j \subset \mathbb{F}_q} \sum_{Y_k \subset \mathbb{F}_q} |S(a, a)(X_j, Y_k)| = \sum_{j=1}^q (-1)^{j-1} \sum_{X_j \subset \mathbb{F}_q} |S_a(X_j)|,$$ where $S_a(Z) := \{F \in F_{r,d} : (\forall z \in Z) F(a, z) = 0\}$ for any subset $Z \subset \mathbb{F}_q$. As a consequence,
$$\frac{1}{|F_{r,d}|} \sum_{F \in F_{r,d}} NS(F)^2 = \sum_{a \in F_2^\mathrm{ind}} \frac{N_{a, 2}}{|F_{r,d}|} + \frac{1}{|F_{r,d}|} \sum_{a \in F_{r-1}^q} \sum_{j=1}^q (-1)^{j-1} \sum_{X_j \subset \mathbb{F}_q} |S_a(X_j)|$$
$$= N_2^\mathrm{ind} (q^{1-r} NS(r, d))^2 + \frac{1}{|F_{r,d}|} \sum_{F \in F_{r,d}} NS(F).$$ The statement of the proposition follows easily from Lemma 2.1.

As a consequence of the Chebyshev inequality we obtain a lower bound on the number of $F \in F_{r,d}$ whose number $NS(F)$ of vertical strips differs roughly a half from the expected value $NS(r, d)$.

Corollary 5.3. The number of $F \in F_{r,d}$ for which $NS(F) \leq NS(r, d)/2$ is at most $O(q^{-r} q^{r-1})$. 


Proof. By Lemma 5.1 and Proposition 5.2, the Chebyshev inequality implies
\[ p_{r,d} \left( |NS(F) - NS(r, d)| \geq \frac{NS(r, d)}{2} \right) \leq 4 \left( 1 - q^{-r} NS(r, d) \right). \]
Taking into account that
\[ \frac{4(1 - q^{-r} NS(r, d))}{NS(r, d)} = \frac{4(1 - \mu_d)}{\mu_d} q^{-r} + O(q^{-r}), \]
the corollary readily follows. \( \square \)

5.2. A lower bound for the entropy in both models. In order to analyze the Shannon entropy, it is necessary to determine the probability \( P_{var}^{F_{rd}} \) that an element \( x := (a, x) \in \mathbb{F}_q^r \) occurs as output of the SVS algorithm on input \( F \in \mathbb{F}_{r,d} \).

Given an input polynomial \( F \in \mathbb{F}_{r,d} \), and the vertical strip defined by an element \( a \in \mathbb{F}_q^{r-1} \), the SVS algorithm proceeds to search for \( \mathbb{F}_q \)-rational zeros of the univariate polynomial \( f := F(a, T) \). If this search is done using the randomized algorithm of Cantor and Zassenhaus (see [6]), then all the \( \mathbb{F}_q \)-rational zeros of \( f \) are equiprobable. Indeed, the algorithm splits \( f \) recursively into two factors, one of which is \( \gcd(T(q-1)/2 - 1, f(T + b)) \) for a random \( b \in \mathbb{F}_q \), and continues with the smaller factor. In the sequel we shall assume that the search of roots in \( \mathbb{F}_q \) of elements of \( \mathbb{F}_{r,d} \) is performed using a randomized algorithm of the Cantor–Zassenhaus type for which all outputs are equiprobable.

For the analysis of the distribution of outputs, we slightly modify the probabilistic model considered so far. For a suitable \( rd \in \mathbb{N} \), denote by \( \Omega_d \subset \mathbb{F}_q^r \) the set of all possible random choices of elements of \( \mathbb{F}_q^r \) made by the routine of the SVS algorithm which searches for \( \mathbb{F}_q \)-rational zeros of the elements of \( \mathbb{F}_{r,d} \). We consider \( \Omega_d \) with the uniform probability, \( F_{var} \times \mathbb{F}_{r,d} \times \Omega_d \) with the probability measure \( P_{var} \) defined in Section 4 for \( var \in \{ \text{ind}, \text{nind} \} \), and the space \( F_{var} \times \mathbb{F}_{r,d} \times \Omega_d \) with the product probability \( P_{var} \). Finally, we shall consider probabilities related to the random variable \( C_{out}^{var} \) : \( F_{var} \times \mathbb{F}_{r,d} \times \Omega_d \rightarrow \mathbb{F}_q \times \{ \emptyset \} \) defined in the following way: for a triple \( (a, F, \gamma) \in F_{var} \times \mathbb{F}_{r,d} \times \Omega_d \), if \( F \) has an \( \mathbb{F}_q \)-rational zero on any of the vertical strips defined by \( a \), and \( a_j \) is the first vertical strip with this property, then \( C_{out}^{var}(a, F, \gamma) := (a_j, x) \), where \( x \in \mathbb{F}_q \) is the zero of \( F(a_j, T) \) computed by the corresponding routine in the SVS algorithm determined by the random choice \( \gamma \). Otherwise, we define \( C_{out}^{var}(a, F, \gamma) := \emptyset \). In these terms, the probability \( P_{var}^{F_{rd}} \) that an element \( x := (a, x) \in \mathbb{F}_q^r \) occurs as output of the SVS algorithm on input \( F \in \mathbb{F}_{r,d} \) may be expressed as the conditional probability \( P_{var}^{F_{rd}}[C_{out}^{var} = x | F] \), namely
\[ P_{var}^{F_{rd}}[C_{out}^{var} = x | F] := \frac{P_{var}^{F_{rd}}[\{ C_{out}^{var} = x \} \cap (F_{var} \times \{ F \} \times \Omega_d)]}{P_{var}^{F_{rd}}[F_{var} \times \{ F \} \times \Omega_d]}. \]

Now we are ready to determine the probability \( P_{var}^{F_{rd}} \) in both models. For this purpose, we shall denote by \( N_a(F) \) the number of \( \mathbb{F}_q \)-rational zeros of \( F \) in the vertical strip defined by \( a \), i.e.,
\[ N_a(F) := |\{ x \in \mathbb{F}_q : F(a, x) = 0 \}|. \]
We have the following result.

Lemma 5.4. Let \( F \in \mathbb{F}_{r,d} \), \( x := (a, x) \in Z(F) \) and \( var \in \{ \text{ind}, \text{nind} \} \). Then
\[ P_{var}^{F_{rd}} = \frac{1}{NS(F) N_a(F)}. \]
Proof. In both models, if \( x \) occurs as output at the \( j \)-th step, then the SVS algorithm must have chosen elements \( a_1, \ldots, a_{j-1} \) for the first \( j-1 \) searches such that \( N_{a_k}(F) = 0 \) for \( 1 \leq k \leq j-1 \), and the element \( a \) for the \( j \)-th search. Finally, the routine searching for roots in \( \mathbb{F}_q \) of \( F(a, T) \) must output \( x \), which occurs with probability \( 1/N_a(F) \).
Now we consider the independent model. Given $k \in \mathbb{N}$, the probability of choosing $a_k$ with $N_a(F) = 0$ is equal to $1/(q^{r-1} - NS(F))$. As a consequence,

$$P_{\text{ind}}(N_a(F) = 0, \ldots, N_{a_{j-1}}(F) = 0, a_j = a|F) = \left(\frac{q^{r-1} - NS(F)}{q^{r-1}}\right)^{j-1} \cdot \frac{1}{q^{r-1}}.$$ 

Since the number of steps may be arbitrarily large, it follows that

$$P_{\text{ind}}(N_a(F) = 0, \ldots, N_{a_{j-1}}(F) = 0, a_j = a|F) = \frac{1}{q^{r-1}N_a(F)} \sum_{j=1}^{\infty} \left(\frac{q^{r-1} - NS(F)}{q^{r-1}}\right)^{j-1} = \frac{1}{NS(F) N_a(F)}.$$ 

This shows the assertion of the lemma in the independent model.

Next we consider the nonindependent model. Recall that, in this model, the element $a_j \in \mathbb{F}_q^{r-1}$ for the $j$th search is randomly chosen among the elements of $\mathbb{F}_q^{r-1} \setminus \{a_1, \ldots, a_{j-1}\}$ with equiprobability. Therefore, if $a$ arises as the choice for the $j$th step, then the SVS algorithm must have chosen pairwise–distinct elements $a_1, \ldots, a_{j-1} \in \mathbb{F}_q^{r-1} \setminus NS(F)$ for the first $j - 1$ searches. The probability of these choices is

$$P_{\text{nind}}(N_a(F) = 0, \ldots, N_{a_{j-1}}(F) = 0, a_j = a|F) = \frac{1}{q^{r-1}N_a(F)} \sum_{j=1}^{\infty} \left(\frac{q^{r-1} - NS(F)}{q^{r-1}}\right)^{j-1} = \frac{1}{q^{r-1}N_a(F)}.$$ 

As there are $q^{r-1} - NS(F)$ elements $b \in \mathbb{F}_q^{r-1}$ with $N_b(F) = 0$, the algorithm performs at most $q^{r-1} - NS(F) + 1$ searches. Finally, when the element $a$ is chosen, the probability to find $x$ as the root in $F$ of $F(a, T)$ is equal to $1/N_a(F)$.

It follows that

$$P_{\text{nind}}(N_a(F) = 0, \ldots, N_{a_{j-1}}(F) = 0, a_j = a|F) \frac{1}{N_a(F)} = \frac{1}{q^{r-1}N_a(F)} \sum_{j=0}^{q^{r-1} - NS(F) - 1} \frac{q^{r-1} - NS(F)}{j!}.$$ 

According to, e.g., [13] §5.2, Problem 1,

$$\sum_{j=0}^{q^{r-1} - NS(F) - 1} \frac{q^{r-1} - NS(F)}{j!} = \frac{q^{r-1} - NS(F)}{q^{r-1} - NS(F)}.$$ 

We conclude that

$$P_{\text{nind}}(N_a(F) = 0, \ldots, N_{a_{j-1}}(F) = 0, a_j = a|F) = \frac{1}{q^{r-1}N_a(F)} \frac{q^{r-1} - NS(F)}{NS(F) N_a(F)}.$$ 

This completes the proof of the lemma. □

As a consequence of Lemma [5.4], we see that the probability $P_{\text{var}}(x|F)$ that an arbitrary element $x \in \mathbb{F}_q^r$ occurs as output of the SVS algorithm is the same in both models. Therefore, the entropy $H_{\text{var}}^{\text{log}}$ will be the same in both models. For this reason, we shall drop the superscript var ∈ \{ind, nind\} and consider the entropy

$$H_F = \sum_{(a,x) \in Z(F)} \frac{\log (NS(F) N_a(F))}{NS(F) N_a(F)}.$$ 

We shall determine the asymptotic behavior of the average entropy

$$H := \frac{1}{|F_{r,d}|} \sum_{F \in F_{r,d}} H_F = \frac{1}{|F_{r,d}|} \sum_{F \in F_{r,d}} \sum_{(a,x) \in Z(F)} \frac{\log (NS(F) N_a(F))}{NS(F) N_a(F)}.$$
Observe that
\[(5.5) \quad \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in Z(F)} 1 = \sum_{(a,x) \in \mathbb{E}^r_q} |\{F \in \mathcal{F}_{r,d} : F(a, x) = 0\}| = q^{\dim \mathcal{F}_{r,d} + r - 1}.
\]

Further, the function \( f : (0, +\infty) \to \mathbb{R}, f(x) := \log x/x \) is increasing in the interval \([e, +\infty)\) and convex in the interval \([e^{1/2}, +\infty)\). By Corollary 5.3 the probability of the set of \( F \in \mathcal{F}_{r,d} \) having up to \( e^{3/2} \approx 4.843 \) vertical strips is \( \mathcal{O}(q^{-r}) \). Therefore, we have
\[
H = \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in Z(F)} 1 \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in Z(F)} \frac{\log(\text{NS}(F)N(a)(F))}{\text{NS}(F)N(a)(F)} 1
\]
\[
\ge q^{-1} \left( \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in Z(F)} \text{NS}(F)N(a)(F) \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in Z(F)} 1 \right) (1 + \mathcal{O}(q^{-r})).
\]

Next we analyze the numerator
\[
\mathcal{N} := \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in Z(F)} \text{NS}(F)N(a)(F)
\]
in the argument of \( f \) in the last expression.

**Lemma 5.5.** We have \( \mathcal{N} = 2 \mu_d q^{2r-2+\dim \mathcal{F}_{r,d}}(1 + \mathcal{O}(q^{-1})) \).

**Proof.** For \( F \in \mathcal{F}_{r,d} \) and \( a \in V S(F) \), we have
\[
\text{NS}(F) = \left| \bigcup_{x \in \mathbb{E}^r_q} \{a \in \mathbb{E}^{r-1}_q : F(a, x) = 0\} \right|, \quad \mathcal{N}(F) = \left| \{x \in \mathbb{E}^r_q : F(a, x) = 0\} \right|.
\]
As a consequence,
\[
\mathcal{N} = \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in \mathbb{E}^r_q} \sum_{y \in \mathbb{E}^r_q \cap \mathbb{E}^r_q} \left| \bigcup_{x \in \mathbb{E}^r_q} \{b \in \mathbb{E}^{r-1}_q : F(b, z) = 0\} \right|
\]
\[
= \sum_{F \in \mathcal{F}_{r,d}} \sum_{(a,x) \in \mathbb{E}^r_q} \sum_{y \in \mathbb{E}^r_q \cap \mathbb{E}^r_q} \sum_{k=1}^{q} (-1)^{k-1} \sum_{Z_k \subseteq \mathbb{E}^r_q} \left| \{b \in \mathbb{E}^{r-1}_q : F(b, T)|_{Z_k} = 0\} \right|
\]
\[
= \sum_{k=1}^{q} (-1)^{k-1} \sum_{a \in \mathbb{E}^{r-1}_q} \sum_{x \in \mathbb{E}^r_q} \sum_{y \in \mathbb{E}^r_q} \sum_{Z_k \subseteq \mathbb{E}^r_q} \mathcal{N}_{a,x,y,Z_k},
\]
where
\[
\mathcal{N}_{a,x,y,Z_k} := \sum_{F \in \mathcal{F}_{r,d}} \left| \{b \in \mathbb{E}^{r-1}_q : F(b, T)|_{Z_k} = 0\} \right|
\]
\[
= \sum_{b \in \mathbb{E}^{r-1}_q} \left| \{F \in \mathcal{F}_{r,d} : F(a, x) = 0, F(a, y) = 0, F(b, T)|_{Z_k} = 0\} \right|.
\]

Suppose that \( k \leq d \). For \( b \neq a \) and \( x \neq y \), the equalities \( F(a, x) = 0, F(a, y) = 0, F(b, T)|_{Z_k} = 0 \) are linearly–independent conditions on the coefficients of \( F \). If \( b \neq a \) and \( x = y \), then we have \( k+1 \) linearly–independent conditions. Finally, for \( b = a \), the number of linearly–independent conditions depends on the size of the intersection \( \{x, y\} \cap Z_k \). It follows that
\[
\mathcal{N}_{a,x,y,Z_k} = (q^{r-1} - 1) q^{\dim \mathcal{F}_{r,d} + k - 1} |\{x, y\} \cup Z_k| + q^{\dim \mathcal{F}_{r,d} - \min(d+1,|\{x, y\} \cup Z_k|)}.
\]
Therefore, by elementary calculations we obtain
\[ \sum_{x \in \mathbb{F}} \sum_{y \in \mathbb{F}} \sum_{Z_k \subseteq F_{r,d}} N_{a,x,y,z_k} = (q^{-1} - 1) \left( \frac{q}{k} \right)^{\dim F_{r,d} - k} q^{\dim F_{r,d} - k} \left( \frac{q^2 - q}{q^2} + \frac{q}{q^2} \right) (1 + O(q^{-r})) \]
\[ = \frac{2q - 1}{q} \left( \frac{r-1}{r} - 1 \right) \left( \frac{q}{k} \right)^{\dim F_{r,d} - k} (1 + O(q^{-r})). \]

Now assume that \( k > d \). Then the condition \( F(b,T)|z_k \equiv 0 \) is equivalent to \( F(b,T) = 0 \). Arguing as above, we deduce that
\[ \sum_{x \in \mathbb{F}} \sum_{y \in \mathbb{F}} \sum_{Z_k \subseteq F_{r,d}} N_{a,x,y,z_k} = \frac{2q - 1}{q} \left( \frac{r-1}{r} - 1 \right) \left( \frac{q}{k} \right)^{\dim F_{r,d} - (d+1)} (1 + O(q^{-r})). \]

Putting these equalities together and using (2.4), we obtain
\[ N = 2q^{2r-2+\dim F_{r,d}} \frac{2q - 1}{2q} (1 - q^{-r}) \]
\[ = \left( \frac{d}{k} \right) \sum_{k=1}^{q} \left( -1 \right)^{k-1} \left( \frac{q}{k} \right)^{q-k} \left( \frac{q}{k} \right)^{-k} \left( \frac{q}{k} \right)^{q-d-1} (1 + O(q^{-r})) \]
\[ = 2 \mu_d q^{2r-2+\dim F_{r,d}} (1 + O(q^{-1})). \]

This finishes the proof of the lemma.

Combining (5.6) with (5.5) and Lemma 5.5 it follows that
\[ H \geq q^{-1} f \left( \frac{2 \mu_d q^{2r-2+\dim F_{r,d}} (1 + O(q^{-1}))}{q^{\dim F_{r,d} - (d+1)}} \right) (1 + O(q^{-r})). \]

In other words, we have the following result.

**Theorem 5.6.** If \( H \) denotes the average entropy for any of the models of generation of vertical strips, then
\[ H \geq \frac{1}{2 \mu_d} \log(q^{-1})(1 + O(q^{-1})). \]

Recall that, according to (5.5), for an algorithm for which the outputs are equidistributed we have the upper bound \( H \leq \log(q^{-1}) \). For large \( d \) we have
\[ \frac{1}{2 \mu_d} \approx \frac{1}{2(1 - e^{-1})} \approx 0.79. \]

We may therefore paraphrase Theorem 5.6 as saying that any of the variants of the SVS algorithm under consideration is at least 79 per cent as good as any “ideal” algorithm.

### 6. Simulations on Test Examples

Now we describe the results on the distribution of the number of searches that were obtained by executing the SVS algorithm on random samples of elements \( F_{r,d} \), for given values of \( q, r \) and \( d \). Recall that \( C^{\text{var}} : F^{\text{var}} \times F_{r,d} \to \mathbb{N} \cup \{ \infty \} \) denotes the random variable which counts the number of searches that are performed for all possible choices of vertical strips, with or without repetitions according to the model \( \text{var} \in \{ \text{ind}, \text{nind} \} \). Theorems 4.3 and 4.4 shows that
\[ P^{\text{var}}[C^{\text{var}} = s] \approx (1 - \mu_d)^{s-1} \mu_d. \]

The simulations we exhibit were aimed to test whether the right–hand side of the previous expression approximates the left–hand side on the examples that were considered. For this purpose, given a random sample \( S \subset F_{r,d} \) and \( a \in F^{\text{var}}_a \), we shall use the following notations:
\[ p_{a, S} := p_{r,d}[S \cap C^{\text{var}}_{a,r,d} = s], \quad \tilde{p}_a := (1 - \mu_d)^{s-1} \mu_d. \]
We shall take \( N := 30 \) choices of \( \alpha \in \mathbb{F}^{\text{var}} \), and compute the sample means
\[
\mu_{s}^{\text{ind}} := \frac{1}{N} \sum_{i=1}^{N} p_{s,i}^{\text{ind}}, \quad \mu_{s}^{\text{nind}} := \frac{1}{N} \sum_{i=1}^{N} p_{s,i}^{\text{nind}}.
\]
Furthermore, we shall consider the corresponding relative errors:
\[
\epsilon_{s}^{\text{ind}} := \frac{|\mu_{s}^{\text{ind}} - \hat{\mu}_{s}|}{\mu_{s}^{\text{ind}}}, \quad \epsilon_{s}^{\text{nind}} := \frac{|\mu_{s}^{\text{nind}} - \hat{\mu}_{s}|}{\mu_{s}^{\text{nind}}}.
\]
We shall only consider relatively moderate values of \( s \), since for higher values of \( s \) the probabilities \( p_{s}^{\text{ind}} \) and \( p_{s}^{\text{nind}} \) are so small that the corresponding information becomes uninteresting.

6.1. \textbf{Examples with} \( q := 67 \) \textbf{and} \( r := 2 \). In this section we consider random samples of bivariate polynomials with coefficients in the finite field \( \mathbb{F}_{67} \). In Table 1 we consider a random sample \( S \) of 1000000 polynomials of \( \mathbb{F}_{67}[X_1, X_2] \) of degree at most \( d := 30 \) and analyze how many vertical strips are searched on this sample. Therefore, we have \( \hat{\mu}_{s} := (1 - \mu_{30})^{s-1} \mu_{30} \), where \( \mu_{30} := 0.6321205588 \ldots \).

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( s \) & \( \mu_{s}^{\text{ind}} \) & \( \mu_{s}^{\text{nind}} \) & \( \hat{\mu}_{s} \) & \( \epsilon_{s}^{\text{ind}} \) & \( \epsilon_{s}^{\text{nind}} \) \\
\hline
1 & 0.634698 & 0.635031 & 0.632121 & 0.004061 & 0.004583 \\
2 & 0.185658 & 0.231664 & 0.232544 & 0.004101 & 0.004259 \\
3 & 0.114003 & 0.084627 & 0.085548 & 0.004101 & 0.004259 \\
4 & 0.04165 & 0.030921 & 0.031471 & 0.004101 & 0.004259 \\
5 & 0.015201 & 0.011279 & 0.011578 & 0.004101 & 0.004259 \\
6 & 0.005168 & 0.004101 & 0.004259 & 0.004101 & 0.004259 \\
7 & 0.001574 & 0.001509 & 0.001567 & 0.004101 & 0.004259 \\
8 & 0.001259 & 0.000553 & 0.000576 & 0.004101 & 0.004259 \\
9 & 0.000501 & 0.000199 & 0.000212 & 0.004101 & 0.004259 \\
10 & 0.000178 & 0.000076 & 0.000078 & 0.004101 & 0.004259 \\
11 & 0.000058 & 0.000025 & 0.000029 & 0.004101 & 0.004259 \\
12 & 0.000025 & 0.000010 & 0.000011 & 0.004101 & 0.004259 \\
13 & 0.000017 & 0.000038 & 0.000039 & 0.004101 & 0.004259 \\
14 & 0.000003 & 0.000011 & 0.000011 & 0.004101 & 0.004259 \\
15 & 0.000005 & 0.000001 & 0.000001 & 0.004101 & 0.004259 \\
\hline
\end{tabular}
\caption{Random sample with \( q = 67 \), \( r = 2 \) and \( d = 30 \).}
\end{table}

Our second example concerns a sample 1000000 polynomials of \( \mathbb{F}_{67}[X_1, X_2] \) of degree at most \( d := 5 \). Therefore, we have \( \hat{\mu}_{s} := (1 - \mu_{5})^{s-1} \mu_{5} \), where \( \mu_{5} := 0.6333333 \ldots \). The corresponding results are summarized in Table 2.
6.2. Examples with \( q := 11 \) and \( r := 2 \). Next we consider random samples of polynomials of \( \mathbb{F}_{11}[X_1, X_2] \). Our first example is a sample of 1000000 polynomials of degree at most \( d := 3 \). On the other hand, we consider a further sample of 1000000 polynomials of degree at most \( d := 8 \). Results are given in Tables 3 and 4 respectively.

Table 3. Random sample with \( q = 11 \), \( r = 2 \) and \( d = 3 \).

| \( s \) | \( \tilde{\mu}_s^{\text{ind}} \) | \( \tilde{\mu}_q^{\text{ind}} \) | \( \tilde{p}_s \) | \( e_s^{\text{ind}} \) | \( e_q^{\text{ind}} \) |
|-------|-----------------|-----------------|-------|----------|----------|
| 1     | 0.6635736       | 0.635885       | 0.633333 | 0.003778 | 0.004012 |
| 2     | 0.231624        | 0.231459       | 0.232222 | 0.002582 | 0.003298 |
| 3     | 0.081522        | 0.084318       | 0.085148 | 0.044487 | 0.009844 |
| 4     | 0.030461        | 0.030727       | 0.031221 | 0.024966 | 0.016085 |
| 5     | 0.012368        | 0.011188       | 0.011448 | 0.074405 | 0.023224 |
| 6     | 0.004872        | 0.004091       | 0.004197 | 0.138482 | 0.025996 |
| 7     | 0.001932        | 0.001481       | 0.001539 | 0.203540 | 0.039029 |
| 8     | 0.000003        | 0.000543       | 0.000564 | 0.375166 | 0.040109 |
| 9     | 0.000349        | 0.000195       | 0.000207 | 0.408009 | 0.056976 |
| 10    | 0.000131        | 0.000069       | 0.000076 | 0.421127 | 0.085938 |
| 11    | 0.000049        | 0.000029       | 0.000028 | 0.432258 | 0.030685 |
| 12    | 0.000025        | 0.000009       | 0.000010 | 0.593068 | 0.129198 |
| 13    | 0.000015        | 0.000003       | 0.000003 | 0.743825 | 0.133380 |
| 14    | 0.000007        | 0.000002       | 0.000001 | 0.805016 | 0.085740 |
| 15    | 0.000003        | 0.000001       | 0.000001 | 0.842862 | 0.057169 |

Table 4. Random sample with \( q = 11 \), \( r = 2 \) and \( d = 8 \).

| \( s \) | \( \tilde{\mu}_s^{\text{ind}} \) | \( \tilde{\mu}_q^{\text{ind}} \) | \( \tilde{p}_s \) | \( e_s^{\text{ind}} \) | \( e_q^{\text{ind}} \) |
|-------|-----------------|-----------------|-------|----------|----------|
| 1     | 0.641352        | 0.661205       | 0.666666 | 0.008036 | 0.008260 |
| 2     | 0.215778        | 0.233556       | 0.222222 | 0.029865 | 0.050571 |
| 3     | 0.073214        | 0.075859       | 0.074074 | 0.011752 | 0.023530 |
| 4     | 0.019652        | 0.025931       | 0.024691 | 0.256421 | 0.047821 |
| 5     | 0.015875        | 0.008932       | 0.008230 | 0.481548 | 0.078591 |
| 6     | 0.007855        | 0.003059       | 0.002743 | 0.650716 | 0.103016 |

6.3. Examples with \( r = 3 \) and \( q := 11 \), \( q := 67 \) and \( q := 8 \). Finally, we consider two samples of 1000000 polynomials of \( \mathbb{F}_5[X_1, X_2, X_3] \). The first sample contains polynomials of degree at most \( d := 5 \) with coefficients in \( \mathbb{F}_{11} \), while the second one contains polynomials of degree at most \( d := 5 \) with coefficients in \( \mathbb{F}_{67} \). Results are exhibited in Tables 5 and 6 respectively.
Table 5. Random sample with \( q = 11, r = 3 \) and \( d = 5 \).

| \( s \) | \( \mu_{s}^{\text{ind}} \) | \( \mu_{s}^{\text{ind}} \) | \( \hat{p}_{s} \) | \( \epsilon_{s}^{\text{ind}} \) | \( \sigma_{s}^{\text{ind}} \) |
|---|---|---|---|---|---|
| 1 | 0.649683 | 0.649494 | 0.633333 | 0.025166 | 0.024881 |
| 2 | 0.227548 | 0.227637 | 0.232222 | 0.020543 | 0.020145 |
| 3 | 0.077141 | 0.079769 | 0.085148 | 0.103804 | 0.067430 |
| 4 | 0.029618 | 0.027999 | 0.031221 | 0.054119 | 0.119075 |
| 5 | 0.010695 | 0.098222 | 0.011448 | 0.133993 | 0.165519 |
| 6 | 0.003725 | 0.003419 | 0.004198 | 0.126953 | 0.227683 |
| 7 | 0.001347 | 0.001213 | 0.001539 | 0.142230 | 0.269344 |
| 8 | 0.000545 | 0.000421 | 0.000564 | 0.035909 | 0.340555 |
| 9 | 0.000192 | 0.000149 | 0.000207 | 0.078273 | 0.382851 |
| 10 | 0.000068 | 0.000050 | 0.000076 | 0.119500 | 0.504379 |
| 11 | 0.000022 | 0.000017 | 0.000028 | 0.281997 | 0.662509 |
| 12 | 0.000011 | 0.000002 | 0.000010 | 0.052593 | 0.500062 |
| 13 | 0.000004 | 0.000002 | 0.000004 | 0.072689 | 0.726225 |
| 14 | 0.000001 | 0.000001 | 0.000001 | 0.210049 | 0.523767 |
| 15 | 0.000001 | 0.000000 | 0.000001 | 0.206637 | 0.207058 |

Table 6. Random sample with \( q = 67, r = 3 \) and \( d = 5 \).

| \( s \) | \( \mu_{s}^{\text{ind}} \) | \( \mu_{s}^{\text{ind}} \) | \( \hat{p}_{s} \) | \( \epsilon_{s}^{\text{ind}} \) | \( \sigma_{s}^{\text{ind}} \) |
|---|---|---|---|---|---|
| 1 | 0.635842 | 0.635802 | 0.633333 | 0.003945 | 0.003883 |
| 2 | 0.231469 | 0.231571 | 0.232222 | 0.003255 | 0.002810 |
| 3 | 0.084408 | 0.084286 | 0.085148 | 0.008764 | 0.010237 |
| 4 | 0.030993 | 0.030732 | 0.031221 | 0.017209 | 0.015898 |
| 5 | 0.011192 | 0.011192 | 0.011447 | 0.022803 | 0.022809 |
| 6 | 0.004061 | 0.004081 | 0.004197 | 0.033550 | 0.028645 |
| 7 | 0.001485 | 0.001482 | 0.001539 | 0.036417 | 0.038865 |
| 8 | 0.000526 | 0.000541 | 0.000564 | 0.073753 | 0.042865 |
| 9 | 0.000205 | 0.000200 | 0.000207 | 0.007567 | 0.039628 |
| 10 | 0.000076 | 0.000074 | 0.000076 | 0.002699 | 0.002618 |
| 11 | 0.000027 | 0.000027 | 0.000028 | 0.026543 | 0.017780 |
| 12 | 0.000009 | 0.000010 | 0.000010 | 0.066246 | 0.033320 |
| 13 | 0.000004 | 0.000005 | 0.000004 | 0.064961 | 0.078891 |
| 14 | 0.000002 | 0.000001 | 0.000001 | 0.124645 | 0.111938 |
| 15 | 0.000000 | 0.000000 | 0.000001 | 0.508529 | 0.257107 |

We end this section by considering random samples of polynomials with coefficients in a non–prime field, namely \( E_{8}[X_{1}, X_{2}, X_{3}] \). In Table 7 the results for a sample of 100000 polynomials of degree at most \( d := 3 \) are exhibited.

Table 7. Random sample with \( q = 8, r = 3 \) and \( d = 3 \).

| \( s \) | \( \mu_{s}^{\text{ind}} \) | \( \mu_{s}^{\text{ind}} \) | \( \hat{p}_{s} \) | \( \epsilon_{s}^{\text{ind}} \) | \( \sigma_{s}^{\text{ind}} \) |
|---|---|---|---|---|---|
| 1 | 0.663445 | 0.662970 | 0.666666 | 0.004855 | 0.005566 |
| 2 | 0.206555 | 0.223378 | 0.222222 | 0.108038 | 0.005176 |
| 3 | 0.075090 | 0.075137 | 0.074074 | 0.013008 | 0.014511 |
| 4 | 0.031267 | 0.025553 | 0.024691 | 0.210306 | 0.033719 |
| 5 | 0.016340 | 0.008046 | 0.008230 | 0.496310 | 0.048099 |
| 6 | 0.005855 | 0.002822 | 0.002743 | 0.531402 | 0.027708 |

Summarizing, the results of Tables 4 and 5 show that the behavior predicted by the asymptotic estimates of Theorems 4.3 and 4.4 can also be appreciated in the
numerical experiments we performed. In general, it seems that experiments using the non-independent model fit better the main term of our asymptotic estimates. Nevertheless, as the cost of the SVS algorithm grows exponentially with the number \( r \) of variables under consideration, our experiments only considered the cases \( r = 2 \) and \( r = 3 \).

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