Hyperfine Mass Splittings of Baryons
Containing a Heavy Quark in Large N QCD

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Abstract

The hyperfine mass splittings of baryons containing a heavy quark are derived at leading order in large $N$ QCD. Hyperfine splittings either preserve or violate heavy quark spin symmetry. Previous work proves that the splittings which preserve heavy quark spin symmetry are proportional to $J^2$ at order $1/N$, where $J$ is the angular momentum of the light degrees of freedom of the baryon. This work proves that the splittings which violate heavy quark spin symmetry are proportional to $J \cdot S_Q$ at order $1/(N m_Q)$ in the $1/N$ and $1/m_Q$ expansions.
The large $N$ expansion provides a quantitative and systematic method for the calculation of baryonic properties in QCD. Recent work [1][2][3] proves that baryon-pion couplings are determined by a single uncalculable coupling constant in large $N$. The couplings derived in large $N$ QCD satisfy light quark spin-flavor symmetry relations. These symmetry relations are robust up to corrections of order $1/N^2$ since the leading order $1/N$ correction vanishes. Large $N$ predictions for other properties of baryons also can be considered. This work focuses on the hyperfine mass splittings of baryons containing a single heavy quark. The splittings fall into two categories: splittings which preserve heavy quark spin symmetry and splittings which violate heavy quark spin symmetry. Previous work [4] proves that the spin symmetry-preserving splittings are proportional to $J^2$ at leading order in the $1/N$ expansion, where $J$ is the angular momentum of the light degrees of freedom of the baryon. This letter proves that the leading order spin symmetry-violating splittings are proportional to $J \cdot S_Q$. Ref. [1] proves that baryon mass splittings are first allowed at order $1/N$ in the $1/N$ expansion. The spin-symmetry-violating splittings are suppressed by an additional factor of $1/m_Q$ since heavy quark spin symmetry is a good symmetry in the infinite heavy quark mass $m_Q \to \infty$ limit [5]. These large $N$ results are identical to the predictions of the large $N$ Skyrme and non-relativistic quark models [6].

The proof of the above result begins with a discussion of the spectrum of baryon states in large $N$. In the $m_Q \to \infty$ limit, heavy baryon states are constructed from the spin states of the $S_Q = \frac{1}{2}$ heavy quark and the states for the light degrees of freedom (which, by definition, include everything except the heavy quark) of the baryon. In large $N$ with $N$ odd and $N_f = 2$ light flavors, the light degrees of freedom consist of the degenerate tower of $(I, J)$ states $(0, 0), (1, 1), (2, 2), \ldots, ((N-1)/2, (N-1)/2)$, where $I$ and $J$ are the isospin and the angular momentum of the light degrees of freedom, respectively. The $(0, 0)$ state in the tower corresponds to a single baryon, the spin-$\frac{1}{2}$ $\Lambda_Q$ baryon, with the spin of the baryon determined by the spin of the heavy quark. All other $(I, J)$ states correspond to a degenerate doublet of heavy baryon multiplets with isospin $I$ and total spin equal to $I \pm \frac{1}{2}$, since $J = I$ for the given tower of states. For instance, the pair of baryon states with light degrees of freedom given by the $(1, 1)$ state corresponds to the spin-$\frac{1}{2}$ $\Sigma_Q$ and the spin-$\frac{3}{2}$ $\Sigma_Q^*$. All the heavy quark baryon states are degenerate up to mass splittings of order $1/N$ [4]. The $1/N$ mass splittings can be divided into two categories. Hyperfine splittings amongst states in the tower produce baryon mass splittings which do not depend on the heavy quark spin. An example of this type of splitting is the $(\Sigma_Q)_{\text{ave}} - \Lambda_Q$ mass difference, where $(\Sigma_Q)_{\text{ave}}$ is the spin-averaged mass of the $\Sigma_Q$ and $\Sigma_Q^*$ baryons. The heavy
quark spin-independent mass splittings are identical to the hyperfine mass splittings of
the tower of states for the light degrees of freedom. Ref. [4] proves that these splittings
are proportional to $J^2$ at leading order in large $N$. This work concentrates on hyperfine
mass splittings which depend on the heavy quark spin. For the remainder of this paper,
these splittings will be referred to as heavy quark splittings in order to distinguish them
from the heavy quark spin-independent splittings. An example of this type of splitting is
the $\Sigma^*_Q - \Sigma_Q$ mass difference. An analogous heavy quark splitting exists for each state
in the tower. These heavy quark spin-dependent splittings arise at order $1/m_Q$ because
they violate heavy quark spin symmetry. This paper proves that the heavy quark spin-
dependent mass splittings are proportional to $J \cdot S_Q$ at leading order in large $N$ QCD. The
leading order splitting occurs at order $1/(N m_Q)$ in the $1/N$ and $1/m_Q$ expansions.

The derivation of this result for the heavy quark splittings is similar to the derivation
for the hyperfine splittings given in Ref. [4]. For this reason, many of the details of this
calculation are not repeated here. The starting point of the proof is the consideration of
chiral logarithmic corrections to baryon masses due to pion loops. For the present calculation,
the discussion given in Ref. [4] must be extended to include separate renormalizations
of the two baryon masses corresponding to each $(I, J)$ state of the tower, such as the $\Sigma_Q$
and $\Sigma^*_Q$ for the $(1,1)$ state. The chiral logarithmic correction to a baryon mass difference
is equal to the difference of the chiral logarithmic corrections to each of the masses. The
crucial point of the proof is the realization that these loop corrections grow with one power
of $N$ more than a baryon mass splitting. Consistency of the large $N$ limit requires that
the loop correction to a baryon mass difference be the same order or higher order in the
$1/N$ expansion as the mass difference. Thus, consistency of the large $N$ limit requires an
exact cancellation amongst the chiral logarithms at leading order. The condition of exact
cancellation results in equations relating baryon mass differences. These equations have a
unique solution, and they determine all ratios of baryon mass differences.

The derivation of the large $N$ consistency conditions for the heavy quark splittings is
completely analogous to the derivation of Ref. [4]. The diagrams contributing to the chiral
logarithmic corrections are proportional to a single Feynman diagram, so all kinematic
factors factor out of the consistency conditions. Evaluation of Clebsch-Gordan factors
arising from each diagram is considerably more involved in the case at hand, however,
since the pion couplings to heavy quark baryon states must be used rather than pion
couplings to the light degrees of freedom, which are simpler. These baryon-pion couplings are derived in Ref. [2]. The couplings are of the form
\[ \overline{B} G^{ai} B \frac{\partial^a \pi^i}{f_\pi}, \] (1)
where \( a = 1, 2, 3 \) labels the angular momentum channel of the \( p \)-wave pion, \( i = 1, 2, 3 \) labels the isospin of the pion, and \( G^{ai} \) is an operator with unit spin and isospin. The matrix elements of \( G^{ai} \) for heavy baryon states \( |II_z, J J_z\rangle \) with isospin \( I \), third component of isospin \( I_z \), spin \( J \), and third component of spin \( J_z \) are parametrized in terms of a single unknown coupling constant in large \( N \),
\[ \langle I' I_z', J' J_z' | G^{ai} | II_z, J J_z \rangle = N g' (-1)^{I + I + S_Q + J' - I' - 1 - 1} \frac{1}{\sqrt{(2I + 1)(2J + 1)}} \left\{ \begin{array}{c c c}
I & I' & I' \\
J Q & J' & J
\end{array} \right\} \left( \begin{array}{c c c}
I & 1 & I' \\
I_z & i & I_z'
\end{array} \right) \left( \begin{array}{c c c}
J & 1 & J' \\
J_z & a & J_z'
\end{array} \right), \] (2)
where \( g' \) is the arbitrary coupling constant of \( O(1) \), and the quantity in curly braces is the 6\( j \) symbol. The spin of the initial heavy quark baryon is either \( J = I + \frac{1}{2} \) or \( J = I - \frac{1}{2} \), whereas the spin of the final baryon is either \( J' = I' + \frac{1}{2} \) or \( J' = I' - \frac{1}{2} \), since \( S_Q = \frac{1}{2} \) and the assumed tower for the light degrees of freedom implies that the angular momentum of the light degrees of freedom is equal to its isospin. Since the heavy quark carries zero isospin, the isospin of the light degrees of freedom is also the isospin of the heavy quark baryon. The Clebsch-Gordan factors for the pion loop diagrams can be evaluated using Eq. (2), the definition of the 6\( j \) symbol, and the identity
\[ \sum_{I_{2z}, J_{2z}, i, a} (-1)^i (-1)^a \left( \begin{array}{c c c}
I_1 & 1 & I_2 \\
I_{1z} & i & I_{2z}
\end{array} \right) \left( \begin{array}{c c c}
J_1 & 1 & J_2 \\
J_{1z} & a & J_{2z}
\end{array} \right) \left( \begin{array}{c c c}
I_2 & 1 & I_1 \\
I_{2z} & -i & I_{1z}
\end{array} \right) \left( \begin{array}{c c c}
J_2 & 1 & J_1 \\
J_{2z} & -a & J_{1z}
\end{array} \right) = (-1)^{I_1 - I_2} (-1)^{J_1 - J_2} \sqrt{\frac{(2I_2 + 1)(2J_2 + 1)}{(2I_1 + 1)(2J_1 + 1)}}, \] (3)
which sums over the \( z \)-components of isospin and spin of intermediate baryon and pion states in a loop diagram. A special case of Eq. (3) is given in Ref. [4]. After considerable manipulation, a very simple consistency condition relating the heavy quark splittings is obtained.
The consistency condition for the heavy quark splitting $\Delta M_i \equiv B_i^* - B_i$ amongst baryons with light degrees of freedom corresponding to the $i^{\text{th}}$ state in the tower is given by

$$\Delta M_{i-1} \frac{J_{i-1}}{J_i} - \Delta M_i \left[ \frac{J_i}{J_{i+1}} + \frac{J_{i+1}}{J_i} \right] + \Delta M_{i+1} \frac{J_{i+2}}{J_{i+1}} = 0,$$  

(4)

where the angular momenta of the light degrees of freedom satisfy $J_{i+1} = J_i + 1$ and $J_i = I_i$ for the assumed tower. Thus, the $i = 1$ state corresponds to the $\Lambda_Q$ baryon with $J_1 = 0$, whereas the $i = 2$ state corresponds to the $\Sigma_Q$ and $\Sigma_Q^*$ baryons with $J_2 = 1$. Note that the consistency condition for the heavy quark splitting $\Delta M_i$ depends only on the splittings $\Delta M_{i-1}, \Delta M_i,$ and $\Delta M_{i+1}$ of the $(i - 1), i,$ and $(i + 1)$ states in the tower because single pion exchange can only change the spin or isospin of the initial baryon by one unit. Hence, the allowed intermediate baryon states in one-loop diagrams for an external state with light degrees of freedom $i$ are the states with light degrees of freedom $(i - 1), i,$ and $(i + 1)$. The consistency condition Eq. (4) for the heavy quark splitting of state $i$ assumes that the splitting $\Delta M_{i-1}$ exists. This is not the case for the first non-vanishing heavy quark splitting $\Delta M_2 = (\Sigma_Q^* - \Sigma_Q)$ since the first state in the tower, $(0, 0)$, corresponds to a single baryon state and hence there is no analogue of $\Delta M$ for this state. The $\Delta M_2$ heavy quark splitting satisfies a truncated version of Eq. (4), given by

$$-\Delta M_2 \left[ \frac{J_2}{J_3} + \frac{J_3}{J_2} \right] + \Delta M_3 \frac{J_4}{J_3} = 0,$$  

(5)

where $J_2 = 1$. The solution to Eqs. (4) and (5) is unique. Given a hyperfine mass splitting $\Delta M_2$, Eq. (5) determines $\Delta M_3$. Given $\Delta M_2$ and $\Delta M_3$, Eq. (4) then determines the mass splitting $\Delta M_4$. All remaining mass splittings are determined recursively using Eq. (4).

The unique solution to the recursion relations (4) and (5) produces heavy quark splittings with the same ratios as those produced by the operator $\mathbf{J} \cdot \mathbf{S}_Q$. The proof of this assertion is as follows. Consider the initial recursion relation Eq. (4). This equation fixes the ratio $\Delta M_3/\Delta M_2$,

$$\frac{\Delta M_3}{\Delta M_2} = \frac{J_2^2 + J_3^2}{J_2 J_4},$$  

(6)

which must be compared with the ratio produced by the operator $\mathbf{J} \cdot \mathbf{S}_Q$. The difference of $\mathbf{J} \cdot \mathbf{S}_Q$ values for states with total spin $J + \frac{1}{2}$ and $J - \frac{1}{2}$ is given by

$$\frac{1}{2} ((J + \frac{1}{2})(J + \frac{1}{2}) - (J - \frac{1}{2})(J + \frac{1}{2})) = \frac{1}{2}(2J + 1),$$  

(7)
since $\mathbf{J} \cdot \mathbf{S}_Q = \frac{1}{2}(\mathbf{J}^2 - \mathbf{J}_2^2 - \mathbf{S}_Q^2)$, and $\mathbf{J}^2$ and $\mathbf{S}_Q^2$ are the same for the $\mathcal{J} = J + \frac{1}{2}$ and $\mathcal{J} = J - \frac{1}{2}$ states. Thus, the ratio obtained from $\mathbf{J} \cdot \mathbf{S}_Q$ is

$$\frac{\Delta M_3}{\Delta M_2} = \frac{(2J_3 + 1)}{(2J_2 + 1)}. \quad (8)$$

Eqs. (6) and (8) are not equivalent for arbitrary $J_2$. However, for $J_2 = 1$, both expressions yield

$$\Delta M_3 = \frac{5}{3} \Delta M_2. \quad (9)$$

It remains to prove that given an initial splitting ratio of $(2J_3 + 1)/(2J_2 + 1)$, all subsequent ratios satisfy

$$\frac{\Delta M_{i+1}}{\Delta M_i} = \frac{(2J_{i+1} + 1)}{(2J_i + 1)}. \quad (10)$$

The recursion relation Eq. (4) yields

$$\frac{\Delta M_{i+1}}{\Delta M_i} = \frac{J_{i+1}}{J_{i+2}} \left\{ \left[ \frac{J_{i+1}}{J_i} + \frac{J_i}{J_{i+1}} \right] - \frac{\Delta M_{i-1}}{\Delta M_i} \frac{J_{i-1}}{J_i} \right\}. \quad (11)$$

Assuming that the ratio

$$\frac{\Delta M_i}{\Delta M_{i-1}} = \frac{(2J_i + 1)}{(2J_{i-1} + 1)}, \quad (12)$$

Eq. (11) then implies that Eq. (10) is satisfied. This result is most easily seen by making the substitution $J_i = \frac{n}{2}$. Then the sought result is equivalent to the identity

$$\frac{(n+2)}{(n+4)} \left\{ \left[ \frac{(n+2)}{n} + \frac{n}{(n+2)} \right] \frac{n-1}{(n+1)} \frac{(n-2)}{n} \right\} = \frac{(n+3)}{(n+1)}, \quad (13)$$

which is true for arbitrary $n$. This completes the proof that the heavy quark splittings are proportional to $\mathbf{J} \cdot \mathbf{S}_Q$.

It is worth noting that the heavy quark splittings $\Delta M_i$ are in the same ratios as the nucleon hyperfine mass splittings derived in Ref. [4]. There is a simple reason for this unanticipated result. The recursion relation derived for the $\mathbf{J} \cdot \mathbf{S}_Q$ heavy quark splittings is very similar to that for the $\mathbf{J}_2$ hyperfine splittings. The recursion relation derived in Ref. [4] is of the form

$$\frac{1}{X_{i-1} Y_{i-1}} - \left[ \frac{1}{X_i} + X_i \right] + X_{i+1} Y_{i+1} = 0, \quad (14)$$

where

$$X_i = \frac{2J_{i+1} + 1}{2J_i + 1} = \frac{J_{i+1} + \frac{1}{2}}{J_i + \frac{1}{2}}. \quad (15)$$
are the coefficients of the recursion equation and
\[ Y_i = \frac{J_{i+1}}{J_i} \] (16)
are equal to the ratios of hyperfine mass splittings,
\[ Y_i = \frac{(M_{i+1} - M_i)}{(M_i - M_{i-1})} . \] (17)
For the nucleon hyperfine mass splittings, such as the splitting \( \Delta - N \), the \( J_i \) take half-integral values. Note that the \( X_i \) for half-integral values of \( J_i \) are equal to the \( Y_i \) for integral values of \( J_i \), with \( X_1(J_1 = 1/2) = Y_2(J_2 = 1) \). Thus, Eq. (14) for half-integral \( J_i \) is identical to the recursion relation of the heavy quark splittings for integral \( J_i \),
\[ \frac{1}{X_{i-1}} \frac{1}{Y_{i-1}} - \left[ \frac{1}{Y_i} + Y_i \right] + X_{i+1}Y_{i+1} = 0 , \] (18)
where the \( Y_i \) are the coefficients of the recursion equation and the \( X_i \) are the ratios of heavy quark splittings,
\[ X_i = \frac{\Delta M_{i+1}}{\Delta M_i} . \] (19)

The derivation of this paper assumed that the number of colors \( N \) was odd. The results obtained for \( N \) odd are also true for \( N \) even. For an even number of colors, the light degrees of freedom of the heavy quark baryons form the tower of half-integral \( (I,J) \) states \((1/2, 1/2), (3/2, 3/2), (5/2, 5/2), ..., ((N - 1)/2, (N - 1)/2)\). In this case, the first non-vanishing heavy quark splitting is the splitting \( \Delta M_1 \) corresponding to the first state in the tower with \( J_1 = \frac{1}{2} \). The consistency condition for \( \Delta M_1 \) implies
\[ \Delta M_2 = 2 \Delta M_1 . \] (20)
The recursion relation for the heavy quark splitting of state \( i \) is identical to Eq. (14). The unique solution of Eqs. (14) and (20) yields heavy quark splittings proportional to \( \mathbf{J} \cdot \mathbf{S_Q} \). When the number of colors \( N \) is even, the spectrum of baryons which contain no heavy quarks is given by the integral \( (I,J) \) tower. Consideration of Eqs. (14) and (18) shows that the ratios of heavy quark splittings for half-integral angular momentum of the light degrees of freedom are identical to the ratios of hyperfine splittings for the integral \( (I,J) \) tower of baryon states. In this case \( X_1(J_1 = 0) = Y_1(J_1 = 1/2) \).

In conclusion, this work proves that the heavy quark splittings of baryons containing a single heavy quark are proportional to \( \mathbf{J} \cdot \mathbf{S_Q} \). The splittings also are proportional to \( 1/(N m_Q) \), since consistency of the large \( N \) expansion requires degeneracy of the baryon states up to order \( 1/N \) [1], and heavy quark spin symmetry violation first occurs at order \( 1/m_Q \).
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References

[1] R. Dashen and A.V. Manohar, UCSD/PTH 93-16
[2] E. Jenkins, UCSD/PTH 93-17
[3] R. Dashen and A.V. Manohar, UCSD/PTH 93-18
[4] E. Jenkins, UCSD/PTH 93-19
[5] N. Isgur and M.B. Wise, Phys. Lett. B232 (1989) 113, Phys. Lett. B237 (1990) 527
[6] E. Jenkins and A.V. Manohar, Phys. Lett. B294 (1992) 273