Asymptotic Optimality of Speed-Aware JSQ for Heterogeneous Systems

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The Join-the-Shortest-Queue (JSQ) load-balancing scheme is known to minimise the average delay of jobs in homogeneous systems consisting of identical servers. However, it performs poorly in heterogeneous systems where servers have different processing rates. Finding a delay optimal scheme remains an open problem for heterogeneous systems. In this paper, we consider a speed-aware version of the JSQ scheme for heterogeneous systems and show that it achieves delay optimality in the fluid limit. One of the major technical challenges in establishing this optimality result for heterogeneous systems is to show that the sequence of steady-state distributions indexed by the system size is tight in an appropriately defined space. We prove this through the use of exponential Lyapunov functions. Another difficulty is the characterisation of the fluid-limit which involves the stationary probabilities of a certain multi-dimensional Markov chain. By characterising these stationary probabilities and using the monotonicity of the system, we show that the fluid limit is unique and has a globally attractive fixed point.

CCS Concepts: • Networks → Network performance analysis; • Mathematics of computing → Markov processes.

Additional Key Words and Phrases: fluid limit, heterogeneous systems, Lyapunov functions, JSQ

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1 INTRODUCTION

Reducing the response times of jobs is a key challenge in modern large-scale service systems such as web server farms and cloud data centers. Efficient load balancing can significantly reduce the mean response time of jobs in such systems. Load balancing algorithms which achieve near optimal delay performance with minimum messaging overhead have been studied extensively for homogeneous systems consisting of identical servers [6, 12]. However, there exist relatively few works which study load balancing for heterogeneous systems [7, 18, 22].

Studying load balancing in heterogeneous systems is important due to two key reasons. First, by exploiting the diversity of server speeds, it may be possible to obtain lower average delay of jobs in heterogeneous systems than in homogeneous systems with the same total capacity. To illustrate this, we consider a simple example shown in Figure 1. Here, we compare two systems; one with identical servers and another with two different server speeds but having the same total capacity as the first system. In the first system, we apply the Join-the-Shortest-Queue (JSQ) scheme which is known to be delay-optimal in the homogeneous setting [26, 29]. In the second system, we apply...
Fig. 1. Comparison of mean response time of jobs between homogeneous and heterogeneous systems operating under JSQ and SA-JSQ schemes, respectively. For the homogeneous system, each server has service rate one; for the heterogeneous system a fraction $\gamma_i$ of servers have rate $\mu_i$. We choose $\gamma_1 = 1 - \gamma_2 = 1/5$, and $\mu_1 = 4\mu_2 = 20/8$.

a speed-aware version of the JSQ scheme, referred to as the Speed Aware JSQ (SA-JSQ) scheme in which each arrival is assigned to a server with the highest speed among the ones with the minimum queue length. Inter-arrival and service times have the same (exponential) distributions in both cases. As shown in Figure 1, we obtain a 60% reduction in the average delay in the heterogeneous system compared to that in the homogeneous system for loads less than 0.5. Thus, if the cost per unit of capacity is the same for both systems, then the utility obtained per unit of installed capacity is significantly higher for the heterogeneous system. This is one important advantage of designing heterogeneous systems over homogeneous systems.

Another important reason for studying heterogeneous systems is that most real systems consist of multiple generations of physical devices with different processing capabilities. In such systems, speed-unaware schemes, designed primarily for homogeneous systems, may perform very poorly [2, 7, 18]. For example, the SQ($d$) scheme, which assigns each incoming job to the shortest of $d$ randomly sampled queues, is known to result in a reduced stability region for heterogeneous systems [18]. This is because in the SQ($d$) scheme faster servers are not sampled as often as required. Other speed-unaware schemes such as JSQ and Join-the-Idle-Queue (JIQ) also have poor performance in the heterogeneous setting. Thus, to achieve efficient load balancing in heterogeneous systems, it is necessary to design schemes which take into account the service rates (speeds) of the servers.

There are various ways in which server speeds can be incorporated into job dispatching decisions; a natural scheme is to send each job to the server where the expected delay, i.e., the ratio of queue length to the server speed, is the minimum. However, even this scheme can be shown to be sub-optimal for finite system sizes. Finding the delay optimal scheme for heterogeneous systems is, in fact, an open problem to this date. In this paper, we aim to address this problem in the fluid limit where both the number of servers and the arrival rate of jobs approach to infinity while maintaining a constant ratio. We show that the SA-JSQ scheme, as described before, achieves the minimum possible average response time in this limiting regime. Our contributions are listed below:

1. **Lower bound**: Our first contribution is to obtain a lower bound on the mean response time of jobs that holds for any scheme in the heterogeneous setting. In obtaining this bound, we compare the heterogeneous system, which consists of separate queues, with a similar heterogeneous system where all the servers serve a single central queue. We show that the
system with separate queues stochastically dominates the system with the central queue. From this stochastic comparison result, we obtain the desired lower bound on the mean response time of jobs in the heterogeneous system. Note that studying the system with a single central queue and heterogeneous servers can be of independent interest.

(2) Stability and tightness: We next show that the heterogeneous system is stable under the SA-JSQ policy and obtain uniform bounds (not depending on the system size) on the tails of the stationary queue length distribution. The latter result is required to show the tightness of the sequence of stationary distributions indexed by the system size. The usual approach for proving tightness via coupling does not work in the heterogeneous setting as it is difficult to construct a coupling that maintains the desired dominance. Thus, we take a different approach and establish this result by analysing the drift of an exponential Lyapunov function. More specifically, we take the sum of moment generating functions of the queue lengths as the Lyapunov function and show that its drift is bounded for all possible states of the system. This result then enables us to analyze the steady-state drift of the exponential Lyapunov function.

(3) Fluid limit analysis: Our final contribution is the fluid limit analysis of the SA-JSQ scheme in the heterogeneous system. This analysis turns out to be considerably more challenging than conventional fluid analysis where the fluid limit equations are simple functions of the system’s state. Due to the separation of two time-scales, the fluid limit in the heterogeneous setting turns out to be a function of the stationary probabilities of a certain multi-dimensional Markov process which is difficult to characterise in general. Using some properties of this multi-dimensional Markov process, we uniquely characterise the fluid limit. Furthermore, using the motonicity of the finite system and the properties of the fluid limit equations we show that the limiting system has a unique and globally attractive fixed point that corresponds to the lower bound established before. This final result establishes the asymptotic optimality of SA-JSQ policy for heterogeneous systems.

1.1 Related works

There exists a vast literature on load-balancing policies for multi-server systems. Here, we only discuss the works that are most relevant to our paper. For a comprehensive review of existing works, we refer the reader to [24].

The JSQ policy was first shown to minimise the average delay of jobs for finite systems consisting of identical servers in [29] under the assumption of Poisson arrivals and exponential service times. This optimality result was later extended to general stochastic arrival processes and service-time distributions with non-decreasing hazard rates in [26]. Recent works [4, 15] have considered the fluid and diffusion limits for the JSQ scheme. In the fluid limit, it has been shown in [15] that the fraction of servers with two or more jobs converges to zero under the JSQ scheme. This implies that in the fluid limit all jobs find an idle server to join. In the Halfin-Whitt regime, where the normalized arrival \( \lambda \) varies with the system size \( N \) as \( \lambda = 1 - \beta / \sqrt{N} \) for some \( \beta > 0 \), it has been shown in [4] that the diffusion-scaled process approaches to a two-dimensional reflected Ornstein-Uhlenbeck (OU) process as \( N \to \infty \). The stationary distribution of this OU process has been studied in [3] which establishes that the steady-state fraction of servers with exactly two jobs scales as \( O(1/\sqrt{N}) \) and the fraction of servers with more than three jobs scales as \( O(1/N) \).

Another line of works explores load balancing schemes which require less communication between the servers and the job dispatcher. The SQ\( (d) \) scheme in which each arrival is assigned to the shortest among \( d \) randomly sampled queues was first analyzed independently in [25] and [14]. Using mean-field analysis, it was shown that for \( d \geq 2 \) the stationary queue length distribution has
a super-exponentially decaying tail for large system sizes. Thus, by querying only \( d \geq 2 \) servers at every arrival instant a significant reduction in the average delay is obtained in comparison to \( d = 1 \).
The Join-the-Idle-Queue (JIQ) further reduces communication overhead by keeping track of only the idle servers in the system. At each arrival instant it sends the incoming job to an idle server if one is available; otherwise, the job is sent to a randomly sampled server. This scheme was first proposed and analyzed by Lu et al. in [13] and it was shown that in the fluid limit the JIQ scheme achieves the same performance as the JSQ scheme. Thus, the JIQ scheme is asymptotically optimal for homogeneous systems in the fluid limit.

Relatively few works consider load balancing in heterogeneous systems. The SQ(d) scheme for heterogeneous systems has been analyzed in [10, 18, 30]. While [10] and [30] analyze the performance of the SQ(d) policy in light and heavy traffic regimes for finite system sizes, respectively, [18] considers its performance in the mean-field regime. It has been shown that the SQ(d) scheme suffers from a reduced stability region in heterogeneous systems due to infrequent sampling of faster servers. Subsequent works [7, 16] have studied variations of the SQ(d) scheme, aimed at improving its performance in heterogeneous systems while retaining the maximal stability region. The JIQ scheme has been analyzed in the heterogeneous setting by Stolyar [22]; it has shown that the average waiting time of jobs under the JIQ scheme approaches to zero in the fluid limit. This result, however, does not imply that the JIQ scheme is delay optimal in the fluid limit because in the JIQ scheme a job can be served at an idle slow server even if idle fast servers are available in the system.

Some recent works, e.g., [20, 27], study load balancing schemes for systems where jobs are constrained to be served only by specific subsets of servers. In these works, the focus is on finding conditions on the compatibility constraints such that the performance of classical load balancing algorithms such as JSQ, JIQ and SQ(d) remain asymptotically the same as in systems without compatibility constraints. In the work by Weng et al., a scheme similar to the SA-JSQ scheme has been considered for constrained heterogeneous systems. They prove the asymptotic optimality of this scheme using Lyapunov drifts methods. Although the scheme is technically the same as the SA-JSQ scheme for constrained heterogeneous systems, their analysis crucially relies on the assumption that the queues have finite buffer sizes. Hence, in that setting tightness and stability results follow immediately. In contrast, one of the main technical challenges in our setting is to prove the tightness of the stationary distributions as the queues in our model have infinite buffer sizes. Furthermore, the drift technique, applicable to finite-buffer systems, is difficult to generalise to our setting. Instead, to prove asymptotic optimality, we use martingale methods outlined in [28].

1.2 General Notations

We use the following notations throughout the paper. The sets \( \mathbb{R}, \mathbb{N}, \mathbb{Z} \) denote the set of real numbers, natural numbers, and non-negative integers, respectively. We also denote \( \mathbb{Z}_+ = \mathbb{Z} \cup \{\infty\} \).

For \( x, y \in \mathbb{R} \), we use \( x \wedge y \), \( x \vee y \), and \( (x)_+ \) to denote \( \min(x, y) \), \( \max(x, y) \), and \( \max(x, 0) \), respectively.

For any \( n \in \mathbb{N} \), \( [n] \) denotes the set \( \{1, 2, \ldots, n\} \). For any complete separable metric space \( E \), we denote \( D_E[0, \infty) \) to be the set of all \( cadlag \) functions from \([0, \infty) \) to \( E \) endowed with the Skorohod topology. Moreover, the notation \( \mathcal{B}(E) \) is used to denote the Borel sigma algebra generated by the set \( E \). The notation \( \Rightarrow \) is used for weak convergence. We use \( 1_{\{A \}} \) to denote indicator function for set \( A \).

1.3 Organisation

The rest of the paper is organized as follows. In Section 2, we introduce the system model and define the SA-JSQ policy. In Section 3, we obtain a lower bound on the mean response time of jobs that holds for any scheme in the heterogeneous system by comparing the system with a similar...
system having a central queue. In Section 4, we state our main results for the SA-JSQ policy. We prove the stability of the SA-JSQ scheme in Section 5 and the uniform bounds on the tails of the stationary queue lengths distribution in Section 6. The monotonicity property for SA-JSQ for a finite $N$ system is shown in Section 7. In Section 8, we prove the process convergence result for SA-JSQ and characterise its fixed point in Section 9. The proof of resource pooled optimality is given in Section 10. Numerical studies comparing different schemes under heterogeneous setting is given in Section 11. Finally, we conclude the paper in Section 12.

2 SYSTEM MODEL

We consider a system $\mathcal{M}_N$ consisting of $N$ parallel servers, each with its own queue of infinite buffer size. The servers are assumed to be heterogeneous, i.e., they have different service rates. Specifically, we assume that there are $M$ different server types or pools. Each type $j \in [M]$ server has a service rate of $\mu_j$. The proportion of type $j \in [M]$ servers in the system is assumed to be fixed at $\gamma_j \in [0, 1]$ with $\sum_{j \in [M]} \gamma_j = 1$. We further assume without loss of generality that $\mu_1 > \mu_2 > \ldots > \mu_M$ and $\sum_{j \in [M]} \gamma_j \mu_j = 1$ (normalised system capacity is unity). For simplicity of exposition we also assume that there exists $N \in \mathbb{N}$ such that $N\gamma_j \in \mathbb{N}$ for all $j \in [M]$.\(^1\)

Jobs are assumed to arrive at the system according to a Poisson process with a rate $N\lambda$. Each job requires a random amount of work, independent and exponentially distributed with mean 1. The inter-arrival and service times are assumed to be independent of each other. Upon arrival, each job is assigned to a server where it either immediately receives service (if the server is idle at that instant) or joins the corresponding queue to be served later. The queues are served according to the First-Come-First-Server (FCFS) scheduling discipline. For this system, the term queue length will refer to the total number of jobs in a queue including the current job in service.

Our main interest is to find a job assignment policy that minimises the steady state mean response time of jobs in the system. To this end, we shall analyse a modified version of the classical Join-the-Shortest-Queue (JSQ) policy which is known to be optimal for homogeneous systems [26]. We shall refer to the modified policy as the Speed-Aware JSQ or the SA-JSQ policy. It is defined as follows

**Definition 1.** Under the SA-JSQ policy, upon arrival of a job, it is sent to a server with the minimum queue length among all the servers in the system. Ties among servers of different types are broken by choosing the server type with the maximum speed and ties among servers of the same type are broken uniformly at random.

Note that unlike the classical JSQ policy, which breaks ties uniformly at random, the SA-JSQ policy breaks ties among servers of different types by choosing the server type with the maximum service rate. Furthermore, unlike the classical JSQ policy, the SA-JSQ policy is not optimal for the heterogeneous system for finite $N$: numerical simulations show that a scheme which assigns jobs to servers based on the shortest expected delay (i.e., queue length divided by the service rate) will perform better than the SA-JSQ policy. However, in subsequent sections of this paper, we shall show that not only the SA-JSQ policy results in a significant improvement in performance over classical JSQ policy, but also in the limit as $N \to \infty$ there is no better policy than SA-JSQ for the heterogeneous system.

3 LOWER BOUND ON THE MEAN RESPONSE TIME

A key ingredient in establishing the optimality of the SA-JSQ policy is finding a uniform lower bound on the steady state mean response time of jobs in system $\mathcal{M}_N$ for all $N$ under any stationary

\(^1\)Our asymptotic results do not depend on this assumption. However, the results for finite systems need to be modified slightly if this is not the case.
job assignment policy \( \Pi \). We later show that this lower bound is achieved by the SA-JSQ policy as 

\[ N \to \infty. \]

To obtain this lower bound, we show that the total number of jobs in system \( M_N \) under any stationary policy \( \Pi \) stochastically dominates the total number of jobs in a similar heterogeneous system \( M'_N \) working under a specific policy, referred to as the *Join-the-Fastest-Free-Server (JFFS)* policy.

**Definition 2.** Under the JFFS policy, if idle servers are available and there are jobs waiting in the central queue, then the head-of-the-line (HOL) job is assigned to the idle server with the highest service rate. Ties within the same server type are broken uniformly at random.

**JFFS policy:** If multiple servers are idle and the queue is non-empty, a decision needs to be made on which server the next job should be assigned to. To this end, we consider the following job assignment policy referred to as the *Join-the-Fastest-Free-Server (JFFS)* policy.

**Description of system \( M'_N \):** In system \( M'_N \), the servers remain the same as in system \( M_N \), i.e., there are \( N \) servers in total with \( N Y_j \) of them having service rate \( \mu_j \) for \( j \in [M] \). However, unlike system \( M_N \), we have a central queue for the whole system. Upon arrival, a job joins the central queue if all the servers are busy. Otherwise, the job is assigned to an idle server. Here, the central queue consists of only those jobs that are waiting for service and it excludes the jobs already in service.

**Definition 2.** Under the JFFS policy, if idle servers are available and there are jobs waiting in the central queue, then the head-of-the-line (HOL) job is assigned to the idle server with the highest service rate. Ties within the same server type are broken uniformly at random.

Hence, unlike system \( M_N \) where each server only serves its own queue, in system \( M'_N \), the central queue is served by all the servers. This system is, therefore, referred to as the *resource pooled system* \cite{21}.

The evolution of the resource pooled system \( M'_N \) under the JFFS policy can be described by the Markov chain \( Z^{(N)}(t) = (Z^{(N)}(t), t \geq 0) \), where \( Z^{(N)}(t) \in \mathbb{Z}_+ \) denotes the total number of jobs in system \( M'_N \) at time \( t \) under the JFFS policy. Clearly, under the JFFS policy, the number of busy type \( j \) servers at time \( t \geq 0 \) is given by \( (Z^{(N)}(t) - \sum_{i=1}^{t-1} N Y_i)_+ \land N Y_j \). Hence, the transition rate \( q_{Z^{(N)}}(k, l) \) of \( Z^{(N)} \) from state \( k \) to state \( l \) is given by

\[
q_{Z^{(N)}}(k, l) = \begin{cases} 
N \lambda, & \text{if } l = k + 1, \\
\sum_{j \in [M]} \mu_j ((k - \sum_{i=1}^{l-1} N Y_i)_+ \land N Y_j), & \text{if } l = k - 1, \\
0, & \text{otherwise}.
\end{cases} \tag{1}
\]

The following two propositions characterise the stationary behaviour of the chain \( Z^{(N)} \) for finite \( N \) and for \( N \to \infty \), respectively.

**Proposition 3.1.** For \( \lambda < 1 \), the Markov chain \( Z^{(N)} \) is positive recurrent. Furthermore, if \( Z^{(N)}(\infty) \) denotes the stationary number of jobs in the system, then

\[
\sup_N \frac{\mathbb{E}[Z^{(N)}(\infty)]}{N} \leq \lambda + \frac{\lambda}{1 - \lambda}. \tag{2}
\]

**Proposition 3.2.** If \( \lambda < 1 \), then for system \( M'_N \) operating under the JFFS policy we have as \( N \to \infty \)

\[
\frac{Z^{(N)}(\infty)}{N} \Rightarrow z^* \triangleq \max_{j \in [M]} \left( \frac{\lambda - \sum_{i=1}^{l-1} \mu_i Y_i}{\mu_j} \right). \tag{3}
\]

Proposition 3.1 implies that the resource pooled system is stable for \( \lambda < 1 \). It also provides a uniform upper bound on the steady-state expected (scaled) number of jobs in the system for all \( N \). These results are utilised to prove Proposition 3.2 which shows that the steady-state (scaled)
number of jobs in the resource pooled system converges weakly to \( z^* \) as \( N \to \infty \), where \( z^* \) is as defined in (3).

In (3), the expression within the brackets on the RHS is maximised for \( j = j^* \in [M] \) iff 
\[
\lambda \in \left[ \sum_{i=1}^{j^*-1} \frac{1}{\mu_i}, \sum_{i=1}^{j^*} \frac{1}{\mu_i} \right].
\]

Note that the existence of such a \( j^* \) is guaranteed because \( \lambda < 1 = \sum_{j \in [M]} \frac{1}{\mu_j} \). For \( j^* \) as defined above, it is easy to see that the fraction of busy type \( j \) servers is one for all \( j \in [j^* - 1] \) and zero for all \( j \geq j^* + 1 \). Furthermore, in pool \( j^* \), the fraction of busy servers is given by \( \frac{\lambda - \sum_{i=1}^{j^*-1} \frac{1}{\mu_i}}{\sum_{i=1}^{j^*} \frac{1}{\mu_i}} \). Since \( z^* < 1 \), the central queue in the limiting system is empty in the steady-state.

To show that the system \( \mathcal{M}_N \) operating under any stationary policy \( \Pi \) stochastically dominates the system \( \mathcal{M}_N' \) operating under the JFFS policy, we describe the state of the system \( \mathcal{M}_N \) under a stationary policy \( \Pi \) by the Markov chain \( X^{(N,\Pi)}(t) = (X_{i,j}^{(N,\Pi)}(t), i \geq 0) \), where \( X_{i,j}^{(N,\Pi)}(t) \) denotes the number of type \( j \) servers with at least \( i \) jobs. Let \( R^{(N,\Pi)}(t) = \sum_{i,j} X_{i,j}^{(N,\Pi)}(t) \) denote the total number of jobs in system \( \mathcal{M}_N \) at time \( t \geq 0 \) under policy \( \Pi \). We have the following stochastic comparison result.

**Theorem 1.** For any stationary policy \( \Pi \) if \( Z^{(N)}(0) \leq R^{(N,\Pi)}(0) \), then the processes \( Z^{(N)} \) and \( X^{(N,\Pi)} \) can be constructed on the same probability space such that \( Z^{(N)}(t) \leq R^{(N,\Pi)}(t) \) for all \( t \geq 0 \), i.e., \( Z^{(N)}(t) \leq_{st} R^{(N,\Pi)}(t) \), where \( \leq_{st} \) implies stochastic dominance.

The above theorem implies that if both \( Z^{(N)} \) and \( R^{(N,\Pi)} \) are ergodic chains, then \( Z^{(N)}(\infty) \leq_{st} R^{(N,\Pi)}(\infty), \) where \( Z^{(N)}(\infty) = \lim_{t \to \infty} Z^{(N)}(t) \) and \( R^{(N,\Pi)}(\infty) = \lim_{t \to \infty} R^{(N,\Pi)}(t) \) denote the stationary number of jobs in \( \mathcal{M}_N' \) and \( \mathcal{M}_N \), respectively. This further implies that \( \mathbb{E} \left[ Z^{(N)}(\infty) \right] \leq \mathbb{E} \left[ R^{(N,\Pi)}(\infty) \right] \) which by Little’s law gives the desired lower bound on the stationary mean response time of jobs in \( \mathcal{M}_N \) under any policy \( \Pi \).

By Little’s law, the steady-state mean response time, \( T'_{N,\Pi} \), of jobs in \( \mathcal{M}_N' \) under the JFFS policy is given by \( T'_{N,\Pi} = \frac{\mathbb{E} \left[ Z^{(N)}(\infty) \right]}{N \lambda} \). Hence, Proposition 3.2 implies \( \lim_{N \to \infty} T'_{N,\Pi} = \frac{z^*}{\lambda} \). Combining the above result with the stochastic comparison result in Theorem 1, we can conclude the following.

**Corollary 1.** The steady-state mean response time, \( T_{N,\Pi} \), of jobs in system \( \mathcal{M}_N \) under any stationary policy \( \Pi \) satisfies
\[
\liminf_{N \to \infty} T_{N,\Pi} \geq \frac{z^*}{\lambda} \tag{4}
\]

In subsequent sections, we establish that the above lower bound is achieved with equality when SA-JSQ is employed as the job assignment policy in \( \mathcal{M}_N \). This will imply the asymptotic optimality of the SA-JSQ policy for \( \mathcal{M}_N \).

## 4 ANALYSIS SETUP AND MAIN RESULTS

In this section, we introduce the key ingredients and notations for our analysis of the SA-JSQ policy. In particular, we discuss the different state-descriptors used in the analysis and discuss some key properties of the space they belong to. We also state our main results for the SA-JSQ policy and discuss their implications.

The state of the system \( \mathcal{M}_N \) at any time \( t \geq 0 \) under the SA-JSQ policy can be described in two different ways. These are as defined below:

1. **Queue-length descriptor:** We define the queue-length vector at time \( t \geq 0 \) as
   \[
   Q^{(N)}(t) = (Q_{k,j}^{(N)}(t), k \in [NY], j \in [M]),
   \]
   where \( Q_{k,j}^{(N)}(t) \) denotes the queue length of the \( k \)th server of type \( j \) at time \( t \).
We further define the space $S$ with a state in $S$ where the $\ell_1$-norm, denoted by $\|s\|_1$, is defined as $\|s\|_1 \triangleq \max_{j \in [M]} \sum_{i \geq 1} |s_{i,j}|$ for any $s \in S$. It is easy to verify that the space $S$ is complete and separable under the $\ell_1$-norm. Furthermore, if the system’s state $x^{(N)}$ belongs to $S^{(N)} \cap S$, then there are finitely many jobs in the system. Starting with a state in $S^{(N)} \cap S$ we can ensure that chain $x^{(N)}$ remains in $S \cap S^{(N)}$ for all $t \geq 0$ only if the process $x^{(N)}$, or equivalently the process $Q^{(N)}$, is positive recurrent. Our first main result states that this is the case when $\lambda < 1$.

**Theorem 2.** The process $Q^{(N)}$ is positive recurrent for each $\lambda < 1$ and each $N$.

The theorem above implies that for $\lambda < 1$ the stationary distributions of $Q^{(N)}$ and $x^{(N)}$ exist and they are unique. Let $Q^{(N)}(\infty) = \lim_{t \to \infty} Q^{(N)}(t)$ (resp. $x^{(N)}(\infty) = \lim_{t \to \infty} x^{(N)}(t)$) denote the state of the system distributed according to the stationary distribution of $Q^{(N)}$ (resp. $x^{(N)}$). The stability of the system guarantees that the steady-state expected number of jobs in the system is finite, i.e., $x^{(N)}(\infty)$ belongs to $S$ almost surely. However, to establish the asymptotic optimality of the SA-JSQ policy, we require the stronger result that the sequence $(x^{(N)}(\infty))_N$ of stationary states indexed by the system size is tight in $S$. Here, it is important to observe that $S$ is not a compact space. The relatively compact subsets of $S$ and the tightness criterion for sequences in $S$ are stated in Appendix E. The tightness criterion essentially requires the expected tail sums $E[\sum_{j \in [M]} \sum_{i \geq 1} x^{(N)}_{i,j}(\infty)]$ approach to zero as $t \to \infty$ uniformly in $N$. To show that $(x^{(N)}(\infty))_N$
satisfies this criterion, we need the following important result which states that the stationary tail probabilities of the queue lengths decay exponentially and the decay rate is uniform in $N$.

**Theorem 3.** If $\lambda < 1$, then under the SA-JSQ scheme, for each $N$, each $j \in [M]$, each $k \in [Ny_j]$, and each $l \geq 1$ the following bound holds for all $\theta \in [0, -\log \lambda)$

$$
P(Q_{k,j}^{(N)}(\infty) \geq l) \leq C_j(\lambda, \theta)e^{-l\theta},
$$

where $C_j(\lambda, \theta) = (1 - \lambda)/\mu_jNy_j(1 - \lambda e^\theta)) > 0$.

Our next main result establishes a key monotonicity property of the system for finite $N$. For two states $q$ and $\tilde{q}$ in $\mathbb{Z}_+^N$, we say $q \leq \tilde{q}$ if $q_{k,j} \leq \tilde{q}_{k,j}$ for each $k \in [Ny_j]$, $j \in [M]$. The inequality $s \leq \tilde{s}$ is similarly defined for $s, \tilde{s} \in S$. We have the following result.

**Theorem 4.** Consider two systems with initial states $Q^{(N)}(0)$ and $\tilde{Q}^{(N)}(0)$ satisfying $Q^{(N)}(0) \leq \tilde{Q}^{(N)}(0)$. Let $Q^{(N)}$ and $\tilde{Q}^{(N)}$ denote the corresponding processes describing the two systems under the SA-JSQ policy. Then, the processes $Q^{(N)}$ and $\tilde{Q}^{(N)}$ can be constructed on the same probability space such that $Q^{(N)}(t) \leq \tilde{Q}^{(N)}(t)$ for all $t \geq 0$.

The monotonicity property stated above implies that if two systems, both working under the SA-JSQ policy, start at two different initial states such that the queue lengths in the first system are dominated by the corresponding queue lengths in the second system, then this dominance is maintained for all $t \geq 0$. This property turns out to be essential in establishing the global asymptotic stability of the fluid limit process.

Our next set of results characterise the asymptotic properties of the process $x^{(N)}$ as $N \to \infty$. The first result states that the sequence $(x^{(N)}(t))_{N \geq 1}$ of processes indexed by $N$ converges weakly to a deterministic process $\xi = (x(t), t \geq 0)$ defined on $S$. To describe the limiting process, we define $l_j(s) = \min\{i : s_{i+1,j} < 1\}$ for any state $s \in S$ to be the minimum queue length in pool $j \in [M]$ in state $s$.

**Theorem 5.** (Process Convergence): Assume $x^{(N)}(0) \in S \cap S^{(N)}$ for each $N$ and $x^{(N)}(0) \Rightarrow \xi(0) \in S$ as $N \to \infty$. Then, the sequence $(x^{(N)}(t))_{N \geq 1}$ is relatively compact in $D_S[0, \infty)$ and any limit $x = (x(t) = (x_{i,j}(t)), i \geq 1, j \in [M]), t \geq 0$ of a convergent sub-sequence of $(x^{(N)}(N \geq 1)$ satisfies the following set of equations for all $t \geq 0, i \geq 1$ and $j \in [M]$

$$
x_{i,j}(t) = x_{i,j}(0) + \frac{\lambda}{Y_j} \int_0^t p_{i-1,j}(x(u))du - \int_0^t \mu_j(x_{i,j}(u) - x_{i+1,j}(u))du,
$$

where $p_{i-1,j}(s) \in [0, 1]$ is uniquely determined for each state $s \in S$. Furthermore, $p(s) = (p_{i-1,j}(s), i \geq 1, j \in [M])$ for each $s \in S$ satisfies the following properties

**P1.** $\sum_{j \in [M]} \sum_{i \geq 1} p_{i-1,j}(s) = 1$ for all $s \in S$.

**P2.** $p_{i-1,j}(s) = 0$ for all $i \geq l_j(s) + 2$ and for all $j \in [M]$.

**P3.** If $l_j(s) > 0$ for some $j \in [M]$, then $p_{i-1,j}(s) = 0$ for all $1 \leq i \leq l_j(s) - 1$.

**P4.** If $l_j(s) = 0$, then $p_{0,1}(s) = 1$.

**P5.** For some $j \in \{2, \ldots, M\}$. If $l_k(s) \geq 1$ for all $k \in [j - 1]$ and $l_j(s) = 0$, then $p_{0,1}(s) = 1$.

In the theorem above, $p_{i-1,j}(s)$ can be interpreted as the limiting probability of an incoming arrival being assigned to a type $j$ server with queue length $i - 1$ when the system is in state $s \in S$. With this interpretation, the properties of $p_{i-1,j}(s)$ listed above follow directly from the assignment rule under the SA-JSQ policy. Indeed, properties P4 and P5 state that if in state $s \in S$ pool $j \in [M]$ is the pool with the highest speed containing idle servers, then with probability 1, incoming jobs are assigned to idle servers in pool $j$ in state $s$. Similarly, properties P2 and P3 imply that for any
state $s \in S$ and any pool $j \in [M]$ with minimum queue-length $l_j(s)$, jobs can only be assigned to queues with lengths $l_j(s) - 1$ and $l_j(s)$ in the limiting system.

The exact expressions for $p_{i,j}(s)$ are complicated as they depend on the stationary probabilities of a multi-dimensional Markov chain (we explain this more in Section 8). In Theorem 5, we only list the properties essential to characterise the fixed point $x^* = (x^*_{i,j})$ of the fluid limit $x$. In our final result stated below, we show that the fixed point $x^*$ is unique and globally attractive.

**Theorem 6.** (i) If $\lambda < 1$, the fixed point $x^* = (x^*_{i,j}, i \geq 1, j \in [M])$ of the fluid limit $x$ described by (10) is unique in $S$ and is given by

$$x^*_{i,j} = \left(1 \land \frac{(\lambda - \sum_{k=1}^{i-1} \mu_k \gamma_k)}{\mu_j \gamma_j}\right) \forall j \in [M], \quad x^*_{i,j} = 0 \forall i \geq 2, j \in [M]. \quad (11)$$

(ii) (Global Stability): If $\lambda < 1$, then for any $x(0) \in S$ the fluid limit $x$ given by (10) satisfies $\lim_{t \to \infty} \|x(t) - x^*\|_1 = 0$.

(iii) (Interchange of Limits): Let $\lambda < 1$. Then, the sequence $(x^{(N)}(\infty))_N$ converges weakly to $x^*$, i.e., $x^{(N)}(\infty) \Rightarrow x^*$ as $N \to \infty$.

Note that the last statement of the theorem implies that the sequence of stationary distributions indexed by the system size concentrates on the point $x^*$. Furthermore, the first statement of the theorem implies that in state $x^* \in S$ the fraction of servers with two or more jobs is zero. This is similar to the classical JSQ result except that in this case if $\lambda \in [\sum_{k=1}^{i-1} \mu_k \gamma_k, \sum_{k=1}^{j-1} \mu_k \gamma_k)$, then all servers in pools $k \in [j-1]$ have exactly one job and in pool $j$ a fraction $(\lambda - \sum_{k=1}^{j-1} \mu_k \gamma_k)/\mu_j \gamma_j$ of servers have exactly one job; all remaining servers are idle. Thus, the total (scaled) number of jobs in state $x^*$ is equal to $z^*$ as defined by (3). Thus, by Little’s law, the mean response time of jobs $T_N$ under the SA-JSQ policy converges to $z^*/\lambda$ as $N \to \infty$, which by Corollary 1 implies the asymptotic optimality of the SA-JSQ policy.

### 5 STABILITY

To show that the process $Q^{(N)}$ is positive recurrent for all $\lambda < 1$, we use an appropriately defined Lyapunov function and show that its drift along any trajectory of $Q^{(N)}$ is negative outside a compact subset of the state space.

For any function $f : \mathbb{Z}_+^n \to \mathbb{R}$, defined on the state space of the process $Q^{(N)}$, the drift evaluated at a state $Q \in \mathbb{Z}_+^n$ is given by

$$G_{Q^{(N)}} f(Q) = \sum_{j \in [M]} \sum_{k \in [Ny_j]} \left[ r^{+\text{,}}_{k,j}(Q)(f(Q + e^{(N)}_{k,j}) - f(Q)) + r^{-\text{,}}_{k,j}(Q)(f(Q - e^{(N)}_{k,j}) - f(Q)) \right], \quad (12)$$

where $G_{Q^{(N)}}$ is the generator of $Q^{(N)}$; $e^{(N)}_{k,j}$ denotes the N-dimensional unit vector with one in the $(k,j)^{th}$ position; $r^{+\text{,}}_{k,j}(Q)$ are the transition rates from the state $Q$ to the states $Q \pm e^{(N)}_{k,j}$. Intuitively, the drift, defined above, represents the expected infinitesimal rate at which $Q^{(N)}(t)$ changes when $Q^{(N)}(t) = Q$. Under the SA-JSQ policy, for each state $Q \in \mathbb{Z}_+^n$ and each $k \in [Ny_j], j \in [M]$ we have

$$r^{+\text{,}}_{k,j}(Q) = \begin{cases} \frac{N\lambda}{|I_{\text{min,j}}(Q)|}, & \text{if } j = j^\dagger(Q) \text{ and } k \in I_{\text{min,j}}(Q) \\ 0, & \text{otherwise} \end{cases} \quad (13)$$

$$r^{-\text{,}}_{k,j}(Q) = \mu_j \mathbb{1}_{\{\delta_{k,j} > 0\}}, \quad (14)$$

where $I_{\text{min,j}}(Q)$ denotes the set of servers with the minimum queue length in pool $j$ in state $Q$ and $j^\dagger(Q)$ denotes the fastest pool that contains a server with the minimum queue length in state $Q$.
Q. The upward transition rate $r_{k,j}^N(Q)$ is obtained by multiplying the total arrival rate with the probability that the $k$th server in the $j$th pool receives an arrival. Similarly, the downward transition rate $r_{k,j}^{-N}(Q)$ is simply the service rate of the $k$th server in the $j$th pool if the server is busy.

**Proof of Theorem 2**: To prove Theorem 2, we compute the drift of the Lyapunov function $\Phi : \mathbb{Z}^N_+ \to [0, \infty)$ defined as follows

$$\Phi(Q) = \sum_{j \in [M]} \sum_{k \in [N_{Y_j}]} Q_{k,j}^2.$$  \hspace{1cm} (15)

From (12) we have that for any $Q \in \mathbb{Z}^N_+$

$$G(Q)^N \Phi(Q) = \sum_{j \in [M]} \sum_{k \in [N_{Y_j}]} r_{k,j}^N(Q)(2Q_{k,j} + 1) + r_{k,j}^{-N}(Q)(-2Q_{k,j} + 1)$$

$$= 2 \sum_{j \in [M]} \sum_{k \in [N_{Y_j}]} r_{k,j}^N(Q)Q_{k,j} + N\lambda - 2 \sum_{j \in [M]} \sum_{k \in [N_{Y_j}]} r_{k,j}^{-N}(Q)Q_{k,j} + \sum_{j \in [M]} \mu_j B_j(Q),$$ \hspace{1cm} (16)

where $B_j(Q)$ denotes the number of busy servers in pool $j$ in state $Q$. In the second equality, we have used the facts that $\sum_{j,k} r_{k,j}^N(Q) = N\lambda$ and $\sum_{k} r_{k,j}^{-N}(Q) = \mu_j B_j(Q)$ which follow easily from (13) and (14), respectively. We further note from (13) and (14) that $\sum_{j,k} r_{k,j}^N(Q)Q_{k,j} = N\lambda Q_{\text{min}}$ and $\sum_{j,k} r_{k,j}^{-N}(Q)Q_{k,j} = \mu_j \sum_{k} Q_{k,j}$, where $Q_{\text{min}}$ denotes the minimum queue length in state $Q$. Hence, from (16) we have

$$G(Q)^N \Phi(Q) = 2N\lambda Q_{\text{min}} - 2 \sum_{j} \mu_j \sum_{k} Q_{k,j} + N\lambda + \sum_{j} \mu_j B_j(Q)$$

$$\leq -2(1 - \lambda) \sum_{j} \mu_j \sum_{k} Q_{k,j} + N\lambda + N.$$ \hspace{1cm} (17)

In the second line, we have used the facts that $\sum_{j} \mu_j \sum_{k} Q_{k,j} \geq \sum_{j} \mu_j \sum_{k} Q_{\text{min}} = NQ_{\text{min}}$ and $B_j(Q) \leq N_{Y_j}$. Hence, it follows from the above that if $\lambda < 1$, then the drift is strictly negative whenever $\sum_{j} \mu_j \sum_{k} Q_{k,j} > \frac{N(1+\lambda)}{1-\lambda}$, and is bounded by $N(1+\lambda)$ otherwise. Thus, using the Foster-Lyapunov criterion for positive recurrence (see, e.g., Proposition D.3 of [11]) we conclude that $Q^{(N)}$ is positive recurrent.

### 6 UNIFORM BOUNDS AND TIGHTNESS

In this section, we first prove Theorem 3 which shows that the stationary queue length distribution has a uniformly decaying tail for all system sizes. We then use this uniform bound to establish the tightness of the sequence $(x^{(N)}(\infty))_N$ in $S$.

For homogeneous systems working under the classical JSQ policy, uniform bounds, similar to (9) can be obtained by coupling the system with another similar system working under a ‘worse performing’ policy such as the *random policy* in which the destination server for each job is chosen uniformly at random from the set of all servers. A coupling similar to the one described in [23] ensures that the total number of jobs in the first system is always smaller than that in the second system. However, in the heterogeneous setting, it is difficult to construct a similar coupling since the arrival of a job to a given pool in one system does not guarantee that the job will join the same pool in the other system.

To overcome this difficulty, we use a completely different approach based on analysing the drift of an exponential Lyapunov function. In particular, we analyse the drift of the Lyapunov function $\Psi_\theta : \mathbb{Z}^N_+ \to \mathbb{R}_+$ defined as

$$\Psi_\theta(Q) \triangleq \sum_{j \in [M]} \sum_{k \in [N_{Y_j}]} \exp(\theta Q_{k,j}).$$ \hspace{1cm} (19)
for some $\theta > 0$. They key idea we employ to prove Theorem 3 is the following: we show that for some positive values of $\theta$ the steady-state expected drift $\mathbb{E}[G_{Q}^{N}\Psi_{\theta}(Q^{N}(\infty))]$ of $\Psi_{\theta}$ is non-negative. From this, we obtain bounds on the weighted sum of moment generating functions of the stationary queue lengths of different pools, i.e., on $\mathbb{E}\left[\sum_{j \in [M]} \mu_{j} \exp(\theta Q_{k,j}^{N}(\infty))\right]$ for some positive $\theta$. Using Chernoff bounds, we then obtain the bounds on the tail probabilities as in Theorem 3.

Note that although we would normally expect the steady-state expected drift to satisfy

$$\mathbb{E}[G_{Q}^{N}\Psi_{\theta}(Q^{N}(\infty))] = 0,$$

proving this requires showing $\mathbb{E}[\Psi_{\theta}(Q^{N}(\infty))] < \infty$ which is only true for some $\theta > 0$ if all moments of the stationary queue lengths exist. However, the stability condition only guarantees the existence of the first moment, but it does not guarantee the existence of higher moments of the queue lengths. Thus, to prove Theorem 3, we use a weaker condition given by Proposition 1 of [8] which states that for any non-negative function $f : \mathbb{Z}^{N} \rightarrow \mathbb{R}_{+}$, if $\sup_{Q \in \mathbb{Z}^{N}} G_{Q}^{N} f(Q) < \infty$ then $\mathbb{E}[G_{Q}^{N} f(Q^{N}(\infty))] \geq 0$. However, to use this result we first need to show that the function $\Psi_{\theta}$ satisfies the above condition for some positive $\theta$. This is shown in the following lemma.

**Lemma 1.** For $\Psi_{\theta}$ as defined in (19) and $\theta \geq 0$, we have

$$G_{Q}^{N}\Psi_{\theta}(Q) \leq (1 - e^{-\theta}) \left( (\lambda e^{\theta} - 1) \sum_{j \in [M]} \sum_{k \in [N_{Y_{j}}]} \mu_{j} \exp(\theta Q_{k,j}) + \sum_{j \in [M]} \mu_{j} l_{j}(Q) \right),$$

where $l_{j}(Q)$ denotes the number of idle servers in pool $j \in [M]$ in state $Q$. In particular, for all $\theta \in [0, -\log \lambda)$ we have

$$\sup_{Q \in \mathbb{Z}^{N}_{+}} G_{Q}^{N}\Psi_{\theta}(Q) < \infty,$$

and the steady-state drift of $\Psi_{\theta}$ satisfies

$$\mathbb{E}[G_{Q}^{N}\Psi_{\theta}(Q^{N}(\infty))] \geq 0.$$  

**Proof.** From (12) and (19), we have that for any $Q \in \mathbb{Z}^{N}_{+}$

$$G_{Q}^{N}\Psi_{\theta}(Q) = \sum_{j \in [M]} \sum_{k \in [N_{Y_{j}}]} \left[ r_{k,j}^{+}(Q) \left( \exp(\theta Q_{k,j})(e^{\theta} - 1) \right) + r_{k,j}^{-}(Q) \left( \exp(\theta Q_{k,j})(e^{-\theta} - 1) \right) \right].$$

First, note that from (13), we can write

$$\sum_{j \in [M]} \sum_{k \in [N_{Y_{j}}]} r_{k,j}^{+}(Q) \exp(\theta Q_{k,j}) = \sum_{k \in [N_{Y_{j}}]} \frac{N \lambda I_{k \in I_{\text{min},j}(Q)}}{|I_{\text{min},j}(Q)|} \exp(\theta Q_{k,j}^{+}(Q)),$$

$$= N \lambda \sum_{k \in [N_{Y_{j}}]} \frac{I_{k \in I_{\text{min},j}(Q)}}{|I_{\text{min},j}(Q)|} \exp(\theta Q_{\text{min}}),$$

$$= N \lambda \exp(\theta Q_{\text{min}}),$$

where $Q_{\text{min}}$ denotes the minimum queue length in state $Q$ and the equality on the second line follows from the fact that for $k \in I_{\text{min},j}(Q)$ the queue length at the $k^{th}$ server of type $j^{t}(Q)$ is
We further note that for any $\theta \in \mathcal{Q}$ where $B$ we have
\[ \sum_{j \in [M]} \sum_{k \in [N_{Y_j}]} r_{k,j}^{-\mathcal{Q}}(\mathcal{Q}) \exp(\theta Q_{k,j}) = \sum_{j \in [M]} \sum_{k \in [N_{Y_j}]} \mu_j \mathbb{1}_{\{Q_{k,j} > 0\}} \exp(\theta Q_{k,j}), \]
\[ = \sum_{j \in [M]} \mu_j \sum_{k \in [N_{Y_j}]} \left(1 - \mathbb{1}_{\{Q_{k,j} = 0\}}\right) \exp(\theta Q_{k,j}), \]
\[ = \sum_{j \in [M]} \mu_j \sum_{k \in [N_{Y_j}]} \exp(\theta Q_{k,j}) - \sum_j \mu_j I_j(\mathcal{Q}). \] (25)

Hence, using (24) and (25), we can write
\[ G_{Q(\mathcal{N})} \Psi_\theta(\mathcal{Q}) = (e^\theta - 1) \left[ N \lambda \exp(\theta Q_{\text{min}}) - \frac{1}{e^\theta} \sum_{j \in [M]} \mu_j \sum_{k \in [N_{Y_j}]} \exp(\theta Q_{k,j}) + \frac{1}{e^\theta} \sum_j \mu_j I_j(\mathcal{Q}) \right]. \] (26)

We further note that for any $\theta \geq 0$ we have $\sum_j \mu_j \sum_k \exp(\theta Q_{k,j}) \geq \sum_j \mu_j \sum_k \exp(\theta Q_{\text{min}}) = N \exp(\theta Q_{\text{min}}) \sum_j \mu_j I_j = N \exp(\theta Q_{\text{min}})$. Using this fact in (26), we have that for any $\theta \geq 0$
\[ G_{Q(\mathcal{N})} \Psi_\theta(\mathcal{Q}) \leq (1 - e^{-\theta}) \left( \lambda e^\theta - 1 \right) \sum_{j \in [M]} \mu_j \sum_{k \in [N_{Y_j}]} \exp(\theta Q_{k,j}) + \sum_j \mu_j I_j(\mathcal{Q}). \]

This proves the first statement of the lemma. In order to prove the next statement, note that for all $\theta \in [0, -\log \lambda)$, we have $\lambda e^\theta - 1 < 0$. Therefore, from (20) it follows that for all $\theta \in [0, -\log \lambda)$ we have
\[ G_{Q(\mathcal{N})} \Psi_\theta(\mathcal{Q}) \leq (1 - e^{-\theta}) \sum_{j \in [M]} \mu_j I_j(\mathcal{Q}) \leq (1 - e^{-\theta}) N, \] (27)

where in the second inequality we have used the fact that $I_j(\mathcal{Q}) \leq N_{Y_j}$. Hence, for all $\theta \in [0, -\log \lambda)$ we have $\sup_{\mathcal{Q} \in \mathcal{Z}} G_{Q(\mathcal{N})} \Psi_\theta(\mathcal{Q}) < \infty$. The last statement of the lemma now follows from the application of Proposition 1 of [8].

**Proof of Theorem 3:** We are now equipped to prove Theorem 3 using the result of the lemma above. Taking expectation of (20) with respect to the stationary distribution and applying (22) we obtain
\[ \left(1 - \lambda e^\theta\right) \mathbb{E} \left[ \sum_j \mu_j \sum_k \exp(\theta Q_{k,j}^{(\mathcal{N})}(\infty)) \right] \leq \mathbb{E} \left[ \sum_j \mu_j (N_{Y_j} - B_j(\mathcal{Q}^{(\mathcal{N})}(\infty))) \right] = N(1 - \lambda), \] (28)

where $B_j(\mathcal{Q}) = N_{Y_j} - I_j(\mathcal{Q})$ denotes the number of busy servers in pool $j$ in state $\mathcal{Q}$. In the last inequality we have used the fact that due to ergodicity of the process $Q^{(\mathcal{N})}$, the steady state rate of departure from the system $\mathbb{E}[\sum_j \mu_j B_j(\mathcal{Q})]$ is equal to the arrival rate $N\lambda$. Hence, from (28) we have that for all $\theta \in [0, -\log \lambda)$
\[ \frac{N(1 - \lambda)}{1 - \lambda e^\theta} \geq \mathbb{E} \left[ \sum_{j \in [M]} \mu_j \sum_{k \in [N_{Y_j}]} \exp(\theta Q_{k,j}^{(\mathcal{N})}(\infty)) \right] = N \mathbb{E} \left[ \sum_{j \in [M]} \mu_j Y_j \exp(\theta Q_{k,j}^{(\mathcal{N})}(\infty)) \right] \]
\[ \geq N \mathbb{E} \left[ \mu_j Y_j \exp(\theta Q_{k,j}^{(\mathcal{N})}(\infty)) \right]. \]

where the second equality follows from the exchangeability of the stationary measure. Thus, for each $j \in [M]$ and $\theta \in [0, -\log \lambda)$ we have
\[ \mathbb{E} \left[ \exp(\theta Q_{k,j}^{(\mathcal{N})}(\infty)) \right] \leq \frac{1 - \lambda}{\mu_j Y_j} \frac{1 - \lambda}{1 - \lambda e^\theta}. \] (29)
Now, for each positive $\theta$ we have
\[
\mathbb{P}(Q^{(N)}_{k,j}(\infty) \geq l) = \mathbb{P}(\exp(\theta Q^{(N)}_{k,j}(\infty)) \geq \exp(\theta l)) \leq \frac{\mathbb{E}\left[\exp(\theta Q^{(N)}_{k,j}(\infty))\right]}{\exp(\theta l)}.
\]

The statement of the theorem now follows by using (29) on the above inequality. ■

Using Theorem 3, we now show that the sequence of stationary states $(x^{(N)}(\infty))_N$ is tight in the space $S$ under the $\ell_1$-norm. According to Prohorov’s theorem [1] the tightness of this sequence will imply that the sequence has convergent subsequences with limits in $S$. We shall show later that all convergent subsequences of $(x^{(N)}(\infty))_N$ have the same limit, thereby establishing the convergence of the original sequence $(x^{(N)}(\infty))_N$ to the same limit.

**Proposition 6.1.** The sequence $(x^{(N)}(\infty))_N$ of stationary states is tight in $S$ under the $\ell_1$-norm.

The necessary and sufficient criterion for tightness of the sequence $(x^{(N)}(\infty))_N$ in $S$ under the $\ell_1$-norm is derived in Lemma 10 of Appendix E and is given by
\[
\lim_{I \to \infty} \limsup_{N \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l} x^{(N)}_{i,j}(\infty) > \varepsilon \right) = 0, \forall \varepsilon > 0.
\]

(30)

The proof of Proposition 6.1 consists of verifying this condition using the uniform bounds derived in Theorem 3. The formal proof is given in Appendix B.

**7 MONOTONICITY**

In this section, we prove the monotonicity property stated in Theorem 4. The key idea here is to couple the arrivals and the departures of the two systems such that if the inequality $Q^{(N)}(t^-) \leq \tilde{Q}^{(N)}(t^-)$ is satisfied just before the arrival or departure event at time $t$ then it is also satisfied after the event has taken place.

**Proof of Theorem 4:** We refer to the systems corresponding to the processes $Q^{(N)}$ and $\tilde{Q}^{(N)}$ as the smaller and larger systems, respectively. Furthermore, in both systems, we call the $k^{th}$ server in the $j^{th}$ pool as the server with index $(k, j)$.

To couple the departures, we first generate a sequence of potential departure instants at the points of a Poisson process with rate $N$. At each potential departure instant, a server index $(k, j)$ is chosen as follows: First, a server type $j \in [M]$ is chosen with probability $\mu_j y_j$ (recall that $\sum_{j \in [M]} \mu_j y_j = 1$). Then, any server $k \in [N y_j]$ within the chosen type $j$ is selected uniformly at random. Once the server index $(k, j)$ is chosen as described above, the server with the index $(k, j)$ is selected for departure in both systems. In each system, an actual departure occurs from the selected server if the selected server is busy; otherwise, no departure occurs from the selected server (this is why the term potential departure is used to describe the event). Let $(k, j)$ denote the index of the chosen server and $D$ denote the potential departure instant. Assume that the inequality $Q^{(N)}_{k,j}(D^-) \leq \tilde{Q}^{(N)}_{k,j}(D^-)$ holds just before the departure. Then, we must have $Q^{(N)}_{k,j}(D) \leq \tilde{Q}^{(N)}_{k,j}(D)$. This is because $Q^{(N)}_{k,j}(D^-) > 0$ only if $\tilde{Q}^{(N)}_{k,j}(D^-) > 0$ in which case a departure occurs from both systems.

To couple the arrivals, we generate a common Poisson arrival stream with rate $N \lambda$ for both systems. At each arrival instant $A$, the job assignment decision is made following the SA-JSQ policy in each system independently of the other system, unless the pool containing the destination server is the same for both systems. If $j^d((Q^{(N)}(A^-)) = j^d((\tilde{Q}^{(N)}(A^-)) = j$, then we perform the following steps sequentially: (1) In the smaller system, the destination server is chosen uniformly at random from the servers having the minimum queue length in pool $j$. Let the index of this chosen server be $(k, j)$. (2) We check if the server having the same index $(k, j)$ in the larger system has the minimum queue length. If so, then this server is chosen as the destination server in the
larger system. (3) Otherwise, the destination server in the larger system is chosen uniformly at random (and independently of the smaller system) from the servers having the minimum queue length in pool \( j \) in the larger system. Let \( (k_s, j_s) \) and \( (k_l, j_l) \) be the indices of the destination servers in the smaller and the larger systems, respectively, at an arrival instant \( A \). Let \( S = Q^{(N)}_{k_s,j_s}(A^-) \) and \( L = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) \). Assume that \( Q^{(N)}(t) \leq \tilde{Q}^{(N)}(t) \) holds for all \( t < A \). Hence, \( S \leq L \). It is sufficient to show that the inequality

\[
Q^{(N)}(t) \leq \tilde{Q}^{(N)}(t) \text{ holds at } t = A. \tag{31}
\]

If \( (k_s, j_s) = (k_l, j_l) \), then the inequality trivially holds at \( t = A \). If \( (k_s, j_s) \neq (k_l, j_l) \), then we have the following possibilities:

1. **If** \( j_s < j_l \): In the larger system, the incoming arrival joins the server with index \( (k_l, j_l) \) and there is no arrival to the server with index \( (k_l, j_l) \) in smaller system. Hence, after the arrival we have \( Q^{(N)}_{k_l,j_l}(A^-) = Q^{(N)}_{k_l,j_l}(A^-) < \tilde{Q}^{(N)}_{k_l,j_l}(A^-) + 1 = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) \). Thus, for (31) to be violated we must have \( Q^{(N)}_{k_s,j_s}(A^-) = \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = S \). But since \( S \leq L \), we must have \( \tilde{Q}^{(N)}_{k_s,j_s}(A^-) \leq \tilde{Q}^{(N)}_{k_l,j_l}(A^-) \) which contradicts with the fact that the server with index \( (k_l, j_l) \) is the destination server in the larger system, i.e., it is the fastest server with the minimum queue length in the larger system, because a faster server of type \( j_s (< j_l) \) has smaller queue length. Hence, (31) must hold in this case.

2. **If** \( j_s = j_l = j, k_s \neq k_l \): Similarly as before, the inequality \( Q^{(N)}_{k_l,j_l}(A^-) < \tilde{Q}^{(N)}_{k_l,j_l}(A^-) \) holds after the arrival. For violation of (31), we therefore must have \( Q^{(N)}_{k_s,j_s}(A^-) = \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = S \) before the arrival. Since the arrival is assigned to the server with index \( (k_l, j) \) in the larger system, we must have \( L = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) \leq \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = S \). Hence, we must have \( S = L \). Also, since \( S = Q^{(N)}_{k_s,j_s}(A^-) \leq \tilde{Q}^{(N)}_{k_l,j_l}(A^-) = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) = L \), we must have \( Q^{(N)}_{k_s,j_s}(A^-) = \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) = L \). In this case, our coupling rule dictates that if we have chosen the index \( (k_s, j) \) as the destination server for the smaller system, then we must also choose the same indexed server in the larger system as the destination server. This leads to a contradiction because \( k_s \neq k_l \). Hence, (31) must hold in this case.

3. **If** \( j_s > j_l \): Similar to the first two cases, the inequality \( Q^{(N)}_{k_l,j_l}(A^-) < \tilde{Q}^{(N)}_{k_l,j_l}(A^-) \) holds. For violation of (31), we must have \( Q^{(N)}_{k_s,j_s}(A^-) = \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = S \) just before the arrival. Since \( L = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) \leq \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = S \), this implies \( S = L \). Furthermore, since \( S = Q^{(N)}_{k_s,j_s}(A^-) \leq \tilde{Q}^{(N)}_{k_l,j_l}(A^-) = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) = L \), we must have \( S = Q^{(N)}_{k_s,j_s}(A^-) = Q^{(N)}_{k_l,j_l}(A^-) = \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) = \tilde{Q}^{(N)}_{k_s,j_s}(A^-) = \tilde{Q}^{(N)}_{k_l,j_l}(A^-) = L \). Hence, there must be a tie between the servers of indices \( (k_s, j_s) \) and \( (k_l, j_l) \) in both systems. Therefore, according to the SA-JSQ policy the incoming arrival should be assigned to the server having index \( (i_s, j_s) \) in smaller system because \( j_l < j_s \). This leads to the contradiction because job has been assigned to the server having index \( (i_s, j_s) \) in the smaller system. Hence, (31) must hold in this case.

This completes the proof of the theorem.

**Remark 1.** It is easy to see that if two states \( Q \) and \( \tilde{Q} \) of the system \( M_N \) satisfy \( Q \leq \tilde{Q} \) then the corresponding empirical measure descriptors \( \bar{x} \) and \( \tilde{x} \) also satisfy \( \bar{x} \leq \tilde{x} \). Hence, the monotonicity property stated in terms of \( Q^{(N)} \) also holds if stated in terms of \( x^{(N)} \).
8 PROCESS CONVERGENCE

In this section, we outline the proof of Theorem 5 using the martingale approach of [28] and the time-scale separation technique of [9]. Here, we discuss the main steps of the proof and characterise the fluid limit process \( x \). The proof consists of the following three steps

**Step 1: Martingale representation:** The first step is to express the evolution of each component of the process \( x^{(N)} \) in terms of suitably defined martingales and a process \( V^{(N)} \) which evolves at a faster time scale than the components of \( x^{(N)} \). In particular, the process \( V^{(N)} = (V_{i,j}^{(N)}(t), i \geq 1, j \in [M]), t \geq 0 \) is defined as \( V_{i,j}^{(N)}(t) = N_{Y_j} - N_{Y_j}x_{i,j}^{(N)}(t) \). Thus, \( V_{i,j}^{(N)}(t) \) counts the number of type \( j \) servers with at most \( i - 1 \) jobs at time \( t \). It is easy to see that \( V^{(N)} \) is a Markov process defined on \( \mathbb{E} = (\mathbb{Z}_+^\infty)^M \) with transition rates

\[
V^{(N)} \rightarrow \begin{cases} 
V^{(N)} + e_{i,j}, & \text{at rate } N_{Y_j}(x_{i,j}^{(N)} - x_{i+1,j}^{(N)}) , \ i \geq 1, \ j \in [M], \\
V^{(N)} - e_{i,j}, & \text{at rate } N\lambda I \{V^{(N)} \in R_{i,j}\}
\end{cases}
\]

where \( R_{i,j} \) for all \( i \geq 1 \) and \( j \in [M] \) is defined as

\[
R_{i,j} = \{ v = (v_{i,k}) \in \mathbb{E} : \forall k \in [j - 1], v_{i-1,k} = 0 \forall l \in \{ j + 1, \ldots, M \}, 0 = v_{i-1,j} < v_{i,j}, \}
\]

The set \( R_{i,j} \) represents the set of states where the minimum queue length is at least \( i \) for the pools \( \{1, \ldots, j - 1\} \), exactly \( i - 1 \) for pool \( j \), and at least \( i - 1 \) for the pools in \( \{ j + 1, \ldots, M \} \). Thus, when \( V^{(N)}(t) \in R_{i,j} \), an incoming job under the SA-JSQ scheme will be assigned to a type \( j \) server with queue length \( i - 1 \). We can therefore express \( x_{i,j}^{(N)} \) for each \( i \geq 1 \) and \( j \in [M] \) as follows

\[
x_{i,j}^{(N)}(t) = x_{i,j}^{(N)}(0) + \frac{\lambda}{Y_j} \int_0^t \mathbb{1}_{\{V^{(N)}(s) \in R_{i,j}\}} ds - \mu_j \int_0^t (x_{i,j}^{(N)}(s) - x_{i+1,j}^{(N)}(s)) ds + \frac{1}{N_{Y_j}} (M_{i,j}^{(A,N)}(t) - M_{i,j}^{(D,N)}(t)),
\]

where \( M_{i,j}^{(A,N)} \) and \( M_{i,j}^{(D,N)} \) are martingales corresponding to the arrivals and departures at the component \( x_{i,j}^{(N)} \). The precise definitions of these martingales in terms of the counting processes are given in are given in Appendix C. It is important to note the difference in the time-scale for the processes \( x^{(N)} \) and \( V^{(N)} \). In a small interval \( [t, t + \delta] \), the process \( V^{(N)} \) experiences \( O(N\delta) \) transitions whereas the \( x^{(N)} \) changes only by \( O(\delta) \). Hence, for large \( N \) the process \( V^{(N)} \) reaches its steady-state while \( x^{(N)} \) remains almost constant in this interval. This separation of the two time-scales becomes crucial in characterising the limit of the indicator function \( \mathbb{1}_{\{V^{(N)}(s) \in R_{i,j}\}} \) appearing in (34).

Since the time-scales of \( V^{(N)} \) and \( x^{(N)} \) are different, they have different limits as \( N \to \infty \). To treat them as a single object and characterise its limit, we define the joint process \( (x^{(N)}, \beta^{(N)}) \) where \( \beta^{(N)} \) is a random measure defined on \( [0, \infty) \times \mathbb{E} \) as

\[
\beta^{(N)}(A_1 \times A_2) = \int_{A_1} \mathbb{1}_{\{V^{(N)}(s) \in A_2\}} ds.
\]

for any \( A_1 \in \mathcal{B}([0, \infty)) \) and \( A_2 \in \mathcal{B}(\mathbb{E}) \). Hence, (34) can be rewritten in terms of \( \beta^{(N)} \) for \( i \geq 1 \) and \( j \in [M] \) as follows

\[
x_{i,j}^{(N)}(t) = x_{i,j}^{(N)}(0) + \frac{\lambda}{Y_j} \beta^{(N)}([0, t] \times R_{i,j}) - \mu_j \int_0^t (x_{i,j}^{(N)}(s) - x_{i+1,j}^{(N)}(s)) ds + \frac{1}{N_{Y_j}} (M_{i,j}^{(A,N)}(t) - M_{i,j}^{(D,N)}(t)).
\]

**Step 2: Relative compactness:** The second step in proving Theorem 5 consists of showing that the sequence of processes \( (x^{(N)}, \beta^{(N)}) \) is relatively compact in \( D_2([0, \infty) \times \mathcal{L}_0) \) where \( \mathcal{L}_0 \) is defined as the space of measures on \( [0, \infty) \times \mathbb{E} \) satisfying \( \beta([0, t] \times \mathbb{E}) = t \) for each \( t \geq 0 \) and each \( \beta \in \mathcal{L}_0 \).
We equip \( \mathcal{L}_0 \) with the topology of weak convergence of measures restricted to \([0, t] \times E\) for each \( t \).

In the next lemma, we show that \( (x(N), \beta(N))_N \) is a relatively compact sequence in \( D_S[0, \infty) \times \mathcal{L}_0 \) and characterise the limit of any convergent subsequence.

**Lemma 2.** If \( x^N(0) \Rightarrow x(0) \in S \) as \( N \to \infty \), then the sequence \((x(N), \beta(N))_N\) is relatively compact in \( D_S[0, \infty) \times \mathcal{L}_0 \) and the limit \((x, \beta)\) of any convergent subsequence satisfies

\[
    x_{i,j}(t) = x_{i,j}(0) + \frac{\lambda}{Y_j} \beta([0, t] \times R_{i,j}) - \mu_j \int_0^t (x_{i,j}(s) - x_{i+1,j}(s))ds, \quad t \geq 0, \quad i \geq 1, \quad j \in [M].
\]

The proof of Lemma 2 consists of verifying standard conditions of relative compactness in \( D_S[0, \infty) \) given in Proposition 3.2.4 of [5]. This is achieved using an approach similar to [15]. The details are given in Appendix F.

**Step 3: Characterisation of the limit:** The final step in proving Theorem 5 is the characterisation the limit \( \beta([0, t] \times R_{i,j}) \) appearing in (36). To do so, we define for any \( x \in S \) a Markov process \( V_x \) on \( E \) with transition rates

\[
    V_x \rightarrow \begin{cases} 
    V_x + e_{i,j}, & \text{at rate } \mu_j(x_{i,j} - x_{i+1,j}), \\
    V_x - e_{i,j}, & \text{at rate } \frac{\lambda}{Y_j} \mathbb{I}_{\{x \in R_{i,j}\}},
    \end{cases} \quad i \geq 1, \quad j \in [M].
\]

From Lemma 2 and Theorem 3 of [9], it follows that the limit \( \beta([0, t] \times R_{i,j}) \) satisfies

\[
    \beta([0, t] \times R_{i,j}) = \int_0^t \pi_x(s) R_{i,j} ds, \quad i \geq 1, \quad j \in [M],
\]

where \( \pi_x \) is a stationary measure of the process \( V_x \) satisfying

\[
    \pi_x(\{V \in E : V_{i,j} = \infty\}) = 1, \quad \text{if } x_{i,j} < 1, \quad \text{for } j \in [M].
\]

Hence, we can write (36) in terms of \( \pi_x \) as

\[
    x_{i,j}(t) = x_{i,j}(0) + \frac{\lambda}{Y_j} \int_0^t \pi_x(s) (R_{i,j}) ds - \mu_j \int_0^t (x_{i,j}(s) - x_{i+1,j}(s)) ds, \quad i \geq 1, \quad j \in [M].
\]

We set \( p_{i-1,j}(x) = \pi_x(R_{i,j}) \). Hence, to complete the proof of Theorem 5, it remains to show that \( x \) uniquely determines the stationary measure \( \pi_x \) and it satisfies the properties P1-P5 listed in Theorem 5.

First note from (33) that the sets \( R_{i,j} \) for \( i \geq 1, \ j \in [M] \) form a partition of \( E \). Since \( \pi_x \) is a probability measure on \( E \), it follows that \( \sum_{i \in [1, j] \in [M]} p_{i-1,j}(x) = \sum_{i \in [1, j] \in [M]} \pi_x(R_{i,j}) = 1 \). This proves P1. Since \( \|x\|_1 < \infty \) for each \( x \in S \), there exists \( I_j(x) = \min \{i : x_{i+1,j} < 1\} \) for all \( j \in [M] \) and \( x \in S \). Observe that for each \( j \in [M] \) we have \( x_{i,j} = 1 \) for all \( 0 \leq i \leq I_j(x) \), and \( x_{i,j} < 1 \) for all \( i \geq I_j(x) + 1 \). Therefore, from (38), it follows that

\[
    \pi_x(\{V_j(x)+1 = V_j(x)+2, \cdots = \infty : \forall j \in [M]\}) = 1.
\]

Hence, from the definition of the set \( R_{i,j} \) in (33) it follows that \( p_{i-1,j}(x) = \pi_x(R_{i,j}) = 0 \) for all \( i \geq I_j(x) + 2 \) and \( j \in [M] \). This proves the property P2.

To prove P3, we note that if \( x_{i,j} = 1 \) for some \( i \geq 1, \ j \in [M] \), then \( dx_{i,j}/dt \leq 0 \). Using this fact in (39) we conclude that \( \lambda \pi_x(R_{i,j}) \leq \mu_j \int (x_{i,j} - x_{i+1,j}) \) for all \( i, j \) for which \( x_{i,j} = 1 \). The definition \( I_j(x) \) implies if \( I_j(x) > 0 \) for some \( x \) and some \( j \), then that \( x_{i,j} = 1 \) for all \( i \leq I_j(x) \). Hence, \( p_{i-1,j}(x) = \pi_x(R_{i,j}) = 0 \) for \( i \in [I_j(x) - 1] \). This shows P3. If \( I_j(x) > 0 \) for some \( j \in [M] \), then it can be verified using property P3 and the definition of \( R_{i,j} \) that

\[
    \pi_x(\{V_1,j = V_2,j = \cdots = V_{I_j(x)-1,j} = 0\}) = 1.
\]
Now suppose \( l_1(x) = 0 \). From (33), it follows that \( p_{0,1}(x) = \pi_x(R_{1,1}) = \pi_x(\{0 < V_{1,1}\}) = 1 \) since \( \pi_x(\{V_{1,1} = \infty\}) = 1 \) according to (38). This proves the P4. The proof of P5 follows similarly using (33), (38), and P2.

Hence, the only part left to prove Theorem 5 is to show that \( \pi_x \) is uniquely determined by \( x \) for all \( x \in S \). To show this, it is sufficient to prove that the stationary distribution of \( (V_{l_i(x),j} : j \in [M]) \) is uniquely determined by \( x \) because the stationary distribution of all other components of \( V_x \) has already been uniquely characterized by (40) and (41). The transition rates of the individual components of the chain \( (V_{l_1(x),1}, V_{l_2(x),2}, \ldots, V_{l_M(x),M}) \) are given by

\[
V_{l_k(x),k} \rightarrow \begin{cases} V_{l_k(x),k} + 1, & \text{at rate } \mu_k(x_{l_k(x),k} - x_{l_k(x),k-1}) \\ V_{l_k(x),k}, & \text{at rate } \frac{\lambda}{y_k} \mathbb{1}_{\{v \in R_{l_k(x),k}\}} \end{cases}, \forall k \in [M]. \tag{42}
\]

Note that the Markov chain given by (42) is defined on \( \mathbb{Z}^M_+ \) and has \( 2^M \) communicating classes. Among these communicating classes, there is only one restricted strictly to \( \mathbb{Z}^M_+ \); all other \( 2^M - 1 \) classes have at least one infinite component. To show the uniqueness \( \pi_x \) we need to show that it is concentrated only on a single communicating class among these \( 2^M \) classes. For this it is sufficient
to show \( \pi_x(V_{l_j(x),j} = \infty) = 0 \) or 1 for all \( j \in [M] \). To show this, we use the result of the next lemma which characterizes the stationary distribution of a finite dimensional Markov chain.

**Lemma 3.** For \( K \in \mathbb{N} \), let \( U = (U(t) = (U_j(t), j \in [K]) \in \mathbb{Z}^K_+ : t \geq 0 \) be a Markov chain with transition rates

\[
U \rightarrow \begin{cases} U + e_i, & \text{at rate } v_i \\ U - e_i, & \text{at rate } \frac{1}{\lambda_i} \mathbb{1}_{\{0 = U_1 = \cdots = U_{i-1} < U_i\}} \end{cases}, \forall i \in [K], \tag{43}
\]

where \( e_i \) denotes the \( K \)-dimensional \( i \)th unit vector. If \( \sum_{i \in [K]} \frac{v_i}{\lambda_i} < 1 \), then the Markov chain \( U \) is positive recurrent. Furthermore, if \( \pi \) denotes the stationary distribution of the chain, then we have

\[
\pi \{0 = U_1 = \cdots = U_{i-1} < U_i\} = \frac{v_i}{\lambda_i}, \forall i \in [K]. \tag{44}
\]

Using Lemma 3, we show that \( \pi_x(V_{l_j(x),j} = \infty) = 0 \) or 1 for all \( j \in [M] \) in Appendix I.

9 **FIXED POINT CHARACTERISATION**

In this section, we prove Theorem 6 which characterizes the fixed point \( x^* \) of the fluid limit \( x \) and shows that the fixed point globally attractive. For the proof of uniqueness of the fixed point, we explicitly use properties P1-P5 listed in Theorem 5. Moreover, to prove global stability we use the monotonicity of the process \( x^{(N)} \) proved in Theorem 4.

**Proof of Theorem 6.(i):** From (10) it follows that for \( x^* \in S \) to be a fixed point of the fluid limit \( x \), we must have

\[
\frac{\lambda}{y_j} p_{i-1,j}(x^*) = \mu_j(x^*_{l_i,j} - x^*_{l_{i-1},j}), \ i \geq 1, \ j \in [M]. \tag{45}
\]

Summing (45) over all \( i \geq 1 \) and for all \( j \in [M] \), we get \( \lambda \sum_{i \geq 1} \sum_{j \in [M]} p_{i-1,j}(x^*) = \sum_{j \in [M]} \mu_j y_j x^*_{1,j} \). Using P1, this implies that

\[
\lambda = \sum_{j \in [M]} \mu_j y_j x^*_{1,j}. \tag{46}
\]

Thus, \( x^*_{1,j} = 1 \) for all \( j \in [M] \) is not possible because the stability condition requires \( \lambda < 1 \). Hence, we must have \( x^*_{1,j} < 1 \) for at least one \( j \in [M] \). In the following, we consider different cases based on the interval in which \( \lambda \) belongs.
If $0 < \lambda < \mu_1 \gamma_1$: For $\lambda \in (0, \mu_1 \gamma_1)$, we show that $x^*_{i,1} = \lambda / \mu_1 \gamma_1$ and $x^*_{i,j} = 0$ for all $(i,j) \neq (1,1)$. Suppose $x^*_{i,1} < 1$, this means that $l_i(x^*) = 0$. Therefore, from property P4, we have $p_{0,1}(x^*) = 1$. Hence, summing (45) over all $i \geq 1$ and for $j = 1$, we get $x^*_{i,1} = \lambda / \mu_1 \gamma_1$. Similarly, summing (45) for all $i \geq m$ and for $j = 1$, we get $x^*_{m,j} = 0$ for any $m \geq 2$. By similar line of arguments as above, we can easily verify that $x^*_{i,j} = 0$ for all $i \geq 1$ and for all $j \in \{2, \ldots, M\}$. Now, suppose $x^*_{1,1} = 1$. Then from (46), with $x^*_{1,1} = 1$ implies that $\sum_{j=2}^{M} \mu_j y_j x^*_{1,j} = \lambda - \mu_1 \gamma_1 < 0$, which leads to a contradiction as $x^* \in S$.

If $\sum_{i=1}^{j-1} \mu_i y_i \leq \lambda < \sum_{i=1}^{j} \mu_i y_i$, for $j \in \{2, \ldots, M\}$: For this case, we show that $x^*_{1,k} = 1$ for all $k \in [j-1]$, $x^*_{1,j} = (\lambda - \sum_{i=1}^{j-1} \mu_i y_i) / \mu_j y_j$, $x_{1,k} = 0$ for all $k \geq j + 1$, and $x^*_{1,k} = 0$ for all $k \in [M]$ and for all $l \geq 2$. First, we use induction to prove that $x^*_{1,k} = 1$ for all $k \in [j-1]$. Suppose $x^*_{1,1} < 1$. This means that $l_i(x^*) = 0$. Therefore, using P4 and summing (45) for all $i \geq 1$ and $j = 1$, we get $x^*_{1,1} = \lambda / \mu_1 \gamma_1 \geq 1$, which contradicts the assumption that $x^*_{1,1} < 1$. Therefore, $x^*_{1,1} = 1$ and $l_i(x) \geq 1$. This proves base case for the induction. Now assume $x^*_{1,k} = 1$ for all $k \in [j-2]$, which implies that $l_k(x^*) \geq 1$ for all $k \in [j-2]$. Using the assumption that $x^*_{1,k} = 1$ for all $k \in [j-1]$, we show that $x^*_{1,j-1} = 1$. Suppose $x^*_{1,j-1} < 1$, which implies that $l_{j-1}(x^*) = 0$. Hence, using property P5 and using (45) we get $$$\sum_{i=1}^{j-1} \mu_i y_i x^*_{1,i} = \lambda - \sum_{i=1}^{j-1} \mu_i y_i \geq 1$$$, which contradicts the assumption that $x^*_{1,j-1} < 1$. Therefore, we must have $x^*_{1,j} = 1$ and $l_{j-1}(x^*) \geq 1$. Next, we prove that $x^*_{1,j} = (\lambda - \sum_{i=1}^{j-1} \mu_i y_i) / \mu_j y_j$. Suppose $l_j(x^*) \geq 1$. This implies that $x^*_{1,j} = 1$. Therefore, using (46), we have $\sum_{i=1}^{j} \mu_i x^*_{1,i} = \lambda - \sum_{i=1}^{j} \mu_i y_i < 0$, which is not possible as $x^* \in S$. Hence, we have $l_j(x^*) = 0$.

So far we have proved that $l_k(x^*) \geq 1$ for all $k \in [j-1]$, $l_j(x^*) = 0$ and $l_j(x^*) \geq 0$ for all $k \geq j + 1$. Therefore, using property P5 and equation (45), we can easily get $x^*_{1,k} = 0$ for all $k \geq j + 1$. Now using (46), we obtain $x^*_{1,j} = (\lambda - \sum_{i=1}^{j-1} \mu_i y_i) / \mu_j y_j$. Similarly, using property P5 and (45), we can easily verify that $x^*_{1,k} = 0$ for all $k \in [M]$ and for all $l \geq 2$.

The proofs of global stability and limit interchange are given in Appendix H.

10 RESOURCE POOLED OPTIMALITY

In this section, we prove Theorem 1. We use the following lemma which establishes that when both $M^*_N$ and $M_N$ have the same total number of jobs $i$, the rate of departure $q^{(N)}(i, i - 1)$ in $M^*_N$ is higher than the rate of departure in $M_N$. Note that the departure rate in $M_N$ at any time $t \geq 0$ under a policy $\Pi$ is $\sum_{i \in \{M\}} \mu_j x_{1,j}^{(N,\Pi)}$.

**Lemma 4.** For any stationary policy $\Pi$, if $Z^{(N)}(t) = R^{(N,\Pi)}(t) = i$ for some $t \geq 0$, then

$$
\sum_{j \in \{M\}} \mu_j x_{1,j}^{(N,\Pi)} \leq q^{(N)}(i, i - 1) = \sum_{j=1}^{M} \mu_j \left( i - \sum_{i=1}^{j-1} N_{Y_i} \right) \wedge N_{Y_j}.
$$

**Proof of Theorem 1:** We construct a coupling between the processes $Z^{(N)}$ and $X^{(N,\Pi)}$ such that if $Z^{(N)}(0) \leq R^{(N,\Pi)}(0)$ then $Z^{(N)}(t) \leq R^{(N,\Pi)}(t)$ for all $t \geq 0$. Let the current instant be $t$ and assume that $Z^{(N)}(t) \leq R^{(N,\Pi)}(t)$. We shall describe a way of generating the next event type (arrival or departure) and the time for the next event $s$ such that $Z^{(N)}(s) \leq R^{(N,\Pi)}(s)$ is maintained right after the event has taken place.

We generate the time until the next arrival for both systems as an exponentially distributed random variable with mean $N \lambda$. Hence, for both systems, arrivals occur at the same instants. If $Z^{(N)}(t) < R^{(N,\Pi)}(t)$ then the time until the next departure is generated independently for each system as exponential random variables with means $q^{(N)}(Z^{(N)}(t), Z^{(N)}(t) - 1)$ and $\sum_{j} \mu_j x_{1,j}^{(N,\Pi)}$ for systems $M^*_N$ and $M_N$, respectively. If $Z^{(N)}(t) = R^{(N,\Pi)}(t) = i$, then we generate the time $D$
(a) Mean response time as a function of normalized arrival rate $\lambda$ with $N = 1000$ servers. We set $\mu_1 = 4\mu_2 = 20/8$, $\gamma_1 = 1 - \gamma_2 = 1/5$, and $d_1 = d_2 = 2$.

(b) $d(x^{(N)}(\infty), x^*) = \sum_{i,j} |x_{i,j}^{(N)}(\infty) - x_{i,j}^*|$ as a function of system size $N$. We set $\mu_1 = 2\mu_2 = 4/3$, $\gamma_1 = 1 - \gamma_2 = 1/2$.

Fig. 2. Simulation plots

until the next departure for $M_N$ as an exponential random variable with mean $\mu_1 X_{1,i}^{(N)}(t)$.

We also generate another independent exponential random variable $C$ with mean $q_0 X_0(i, i - 1) - \sum_j \mu_j X_{i,j}^{(N)}(t) \geq 0$. Note that we can do so by Lemma 4. Now we generate the time until the next departure for $M_N$ as $D' = \min(D, C)$. Therefore, $D' \leq D$.

Once all the events have been generated as described above, the next event is taken to be the one with the earliest next event time. Due to the construction above, it is clear that $Z^{(N)}(s) \leq R^{(N)}(s)$ is maintained right after the next event. This completes the proof.

11 NUMERICAL STUDIES

In this section, we present simulation results for different load balancing schemes. For all simulations, we have assumed $M = 2$ and taken the number of arrivals to be $3 \times 10^5$. In Figure 2(a), we have plotted the mean response time of jobs for different schemes as a function of the normalized arrival rate $\lambda$. For performance comparison we also simulated a scheme proposed in [17] and referred to as the SQ$(d_1, d_2)$ scheme. For the SQ$(d_1, d_2)$ scheme, upon job arrival $d_j$ servers of type $j$ are sampled uniformly at random from the set of $N \gamma_j$ servers for $j \in [2]$. The job is then sent to the server with the minimum queue length among the sampled servers. Ties within different types servers are broken by selecting the type with the maximum rate. We see that with SA-JSQ we obtain upto 60% reduction in average response time of jobs compared to classical JSQ. As expected, the performance of SQ(2, 2) lies in between classical JSQ and SA-JSQ. To investigate the convergence rate to the fixed point of the fluid limit, in Figure 2(b), we have plotted the distance $d(x^{(N)}(\infty), x^*) = \sum_{i,j} |x_{i,j}^{(N)}(\infty) - x_{i,j}^*|$ as a function of $N$ for $\lambda \in \{0.5, 0.7, 0.9\}$. We note that for large values of $\lambda$ the distance is higher than that for smaller values of $\lambda$.

12 CONCLUSION AND FUTURE WORKS

In this paper, we have investigated speed-aware JSQ-type load balancing schemes for heterogeneous systems. We obtained a lower bound on the mean response time of jobs under any load balancing scheme by comparing the system with an appropriate resource pooled system. We showed that the lower bound is achieved by the SA-JSQ scheme in the fluid limit, thereby establishing the asymptotic optimality of SA-JSQ. Moreover, in establishing the fluid limit of SA-JSQ, we have proved uniform bounds on the stationary measures of queue lengths which are required to prove...
tightness. Using coupling, we have also shown that any stationary load balancing scheme in a system of parallel queues stochastically dominates the JFFS scheme in the resource pooled system. There are many interesting avenues for future work. Characterising the performance of the SA-JSQ scheme in the Halfin-Whitt regime remains as an open problem. It is also interesting to analytically characterise the distance between \( x^{(N)}(\infty) \) and the fixed point \( x^* \) as a function of the system size \( N \).

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To prove the proposition, we construct a coupling between the resource pooled system $M'_N$, and prove the results stated in Propositions 3.1 and 3.2.

\section*{A.1 Proof of Proposition 3.1}

To prove the proposition, we construct a coupling between the resource pooled system $M'_N$ and an M/M/1/N system with Poisson arrival rate $N\lambda$ and $N$ identical servers, each with rate 1. Let $Y^{(N)}(t)$ denote the number of jobs in the M/M/1/N system at time $t \geq 0$. The transition rates $q^{Y^{(N)}}(k, l)$, for $k, l \in \mathbb{Z}_+$, of the chain $Y^{(N)} = (Y^{(N)}(t), t \geq 0)$ are given by

\begin{equation}
q^{Y^{(N)}}(k, l) = \begin{cases} 
N\lambda, & \text{if } l = k + 1, \\
 k \land N, & \text{if } l = k - 1, \\
 0, & \text{otherwise}.
\end{cases}
\end{equation}

(48)

From the standard results on M/M/1/N queues, it follows that the process $Y^{(N)} = (Y^{(N)}(t), t \geq 0)$ is positive recurrent for $\lambda < 1$. Furthermore, if $Y^{(N)}(\infty)$ denotes the stationary number of jobs in the system, then from the standard results on M/M/1/N queues we have

\begin{equation}
\mathbb{E} \left[ \frac{Y^{(N)}(\infty)}{N} \right] = \lambda + \frac{\lambda}{1 - \lambda} \mathbb{P} \left[ Y^{(N)}(\infty) \geq N \right] \leq \lambda + \frac{\lambda}{1 - \lambda}.
\end{equation}

(49)

Hence, to prove the proposition it suffices to construct a coupling between $Z^{(N)}$ and $Y^{(N)}$ such that $Z^{(N)}(0) \leq Y^{(N)}(0)$ implies $Z^{(N)}(t) \leq Y^{(N)}(t)$ for all $t \geq 0$. First, note that for each Markov chain, the transition rate out of any state is bounded above by the constant $B = N(\lambda + 1)$. Hence, we can generate both $Z^{(N)}$ and $Y^{(N)}$ processes by constructing the corresponding uniformized discrete-time Markov chains $\tilde{Z}^{(N)} = (\tilde{Z}^{(N)}(m), m \in \mathbb{Z}_+)$ and $\tilde{Y}^{(N)} = (\tilde{Y}^{(N)}(m), m \in \mathbb{Z}_+)$. The one-step transition probabilities of these two chains from state $k$ to state $l$ are respectively given by

\begin{align*}
p^{Z^{(N)}}(k, l) &= \begin{cases} 
\frac{q^{Z^{(N)}}(k, l)}{B}, & \text{for } l \neq k, \\
1 - \sum_{l' \neq k} p^{Z^{(N)}}(k, l'), & \text{for } l = k,
\end{cases} \\
p^{\tilde{Z}^{(N)}}(k, l) &= \begin{cases} 
\frac{q^{\tilde{Z}^{(N)}}(k, l)}{B}, & \text{for } j \neq i, \\
1 - \sum_{l' \neq k} p^{\tilde{Z}^{(N)}}(k, l'), & \text{for } l = k,
\end{cases}
\end{align*}

where $q^{Z^{(N)}}$ and $q^{Y^{(N)}}$ denote the transition rates for $Z^{(N)}$ and $Y^{(N)}$, respectively. To construct the continuous-time sample paths of the original chains $Z^{(N)}$ and $Y^{(N)}$ on the same probability space,
we generate a common Poisson process with rate $\mathcal{B}$ and embed the time-steps of both $\tilde{Z}^{(N)}$ and $\tilde{Y}^{(N)}$ into the points of the Poisson process.

It is easy to see from the transition rates that $p^\tilde{Z}^{(N)}(k, k - 1) \geq p^\tilde{V}^{(N)}(k, k - 1)$ for all $k \in \mathbb{Z}_+$. We let the chains $\tilde{Z}^{(N)}$ and $\tilde{Y}^{(N)}$ evolve independently of each other except at instants when they become equal. If $\tilde{Z}^{(N)}(m) = \tilde{Y}^{(N)}(m) = k \in \mathbb{Z}_+$ for some time step $m \in \mathbb{Z}_+$, we first construct $\tilde{Y}^{(N)}(m + 1)$ according to the transition probabilities $p^\tilde{Y}^{(N)}$. Then we generate $\tilde{Z}^{(N)}(m + 1)$ as follows:

$$
\tilde{Z}^{(N)}(m + 1) = \begin{cases} 
\tilde{Y}^{(N)}(m + 1), & \text{if } \tilde{Y}^{(N)}(m + 1) \in \{i + 1, i - 1\} \\
\tilde{Y}^{(N)}(m) - \theta, & \text{otherwise},
\end{cases}
$$

where $\theta \in \{0, 1\}$ is a Bernoulli random variable with

$$
\mathbb{P}[^\theta = 1|\tilde{Z}^{(N)}(m) = \tilde{Y}^{(N)}(m) = k] = \frac{p^\tilde{Z}^{(N)}(k, k - 1) - p^\tilde{Y}^{(N)}(k, k - 1)}{p^\tilde{Y}^{(N)}(k, k)} \geq 0.
$$

Clearly, under the coupling described above $\tilde{Z}^{(N)}(m) \leq \tilde{Y}^{(N)}(m)$ for all $m \in \mathbb{Z}_+$ if $Z^{(N)}(0) = \tilde{Z}^{(N)}(0) \leq \tilde{Y}^{(N)}(0) = Y^{(N)}(0)$. Hence, we have $Z^{(N)}(t) \leq Y^{(N)}(t)$ for all $t \geq 0$.

### A.2 Proof of Proposition 3.2

To prove Proposition 3.2, we study the limit of the process $z^{(N)} = \left(z^{(N)}(t), t \geq 0\right)$ defined as

$$
z^{(N)}(t) = \frac{Z^{(N)}(t)}{N}, \quad t \geq 0.
$$

Thus, $z^{(N)}(t)$ denotes the scaled number of jobs in $\mathcal{M}^{(N)}_t$ under the JFFS scheme at time $t$. We first characterise the limit of the sequence of processes $(z^{(N)})_N$ in the lemma below.

**Lemma 5.** If $z^{(N)}(0) \Rightarrow z(0) \in \mathbb{R}$ as $N \to \infty$, then $z^{(N)} \Rightarrow z$ as $N \to \infty$, where $z = (z(t), t \geq 0)$ is the unique deterministic process satisfying the following integral equation

$$
z(t) = z(0) + \lambda t - \int_0^t \sum_{j=1}^M \mu_j \left( \left( z(s) - \sum_{i=1}^{j-1} Y_i \right) \wedge Y_j \right) \, ds. \quad (50)
$$

Furthermore, the process $z$ satisfying (50) has a unique fixed point $z^*$ given by

$$
z^* = \max_{j \in [M]} \left( \frac{\lambda - \sum_{i=1}^{j-1} \mu_i Y_i}{\mu_j} \right).
$$

**Proof.** Let $(A(t) : t \geq 0)$ and $(D(t) : t \geq 0)$ be independent unit-rate Poisson processes. We can express the evolution of the total number $Z^{(N)}(t)$ of jobs in the system $\mathcal{M}^{(N)}_t$ as follows:

$$
Z^{(N)}(t) = Z^{(N)}(0) + A(N\lambda t) - D(\int_0^t T(Z^{(N)}(s)) \, ds), \quad t \geq 0, \quad (52)
$$

where $Z^{(N)}(0) = Nz^{(N)}(0)$ and $T : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$
T(Z) = \sum_{j=1}^M \mu_j \left( Z - \sum_{i=1}^{j-1} N Y_i \right) \wedge N Y_j.
$$

Hence, for $Z \in \mathbb{Z}_+$, $T(Z)$ represents the total rate at which jobs depart the system in state $Z$. Dividing (52) by $N$, we have

$$
z^{(N)}(t) = z^{(N)}(0) + A^{(N)}(t) - D^{(N)}(t) + \lambda t - \int_0^t T'(z^{(N)}(s)) \, ds, \quad \forall t \geq 0, \quad (53)
$$

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where
\[
M_A^{(N)}(t) = \frac{A(N\lambda t) - N\lambda t}{N},
\]
\[
M_D^{(N)}(t) = \frac{D(\int_0^t T(Z^{(N)}(s)) \, ds) - \int_0^t T(Z^{(N)}(s)) \, ds}{N},
\]
\[
T'(z) = \frac{T(Z)}{N} = \sum_{j=1}^M \mu_j ((z - \sum_{i=1}^{j-1} y_i) \wedge y_j).
\]

Using Lemma 3.2 from [19], it can be easily verified that \(M_A^{(N)}\) and \(M_D^{(N)}\) are square-integrable martingales with respect to the filtration \(\mathcal{F}(N) = (\mathcal{F}_{N,t}, t \geq 0)\), where
\[
\mathcal{F}_{N,t} = \sigma\left[Z^{(N)}(0), A(N\lambda s), D\left(\int_0^s T(Z^{(N)}(u)) \, du\right), 0 \leq s \leq t\right].
\]
Moreover, quadratic variation process for \(M_A^{(N)}\) and \(M_D^{(N)}\) are given by
\[
[M_A^{(N)}](t) = \frac{A(N\lambda t)}{N^2}, \quad t \geq 0
\]
\[
[M_D^{(N)}](t) = \frac{1}{N^2} D(\int_0^t T(Z^{(N)}(s)) \, ds), \quad t \geq 0.
\]

Using the martingale functional central limit theorem [28], we have
\[
M_A^{(N)} \Rightarrow 0, \quad M_D^{(N)} \Rightarrow 0 \quad \text{as} \quad N \to \infty.
\]

From (53) and (50), we see that both \(z^{(N)}\) and \(z\) can be expressed as \(z^{(N)} = f(z^{(N)}(0), M_A^{(N)} - M_D^{(N)})\) and \(z = f(z(0), 0)\), where \(f : \mathbb{R} \times \mathcal{D}[0, \infty) \to \mathcal{D}[0, \infty)\) is defined as the mapping that takes \((b, y)\) to \(x\) determined by the following integral equation
\[
x(t) = b + y(t) + \lambda t - \int_0^t T'(x(s)) \, ds.
\]
Hence, if \(f\) is well-defined and continuous, then the continuous mapping theorem proves the first statement of the lemma. From Theorem 4.1 of [19]), it follows that to show that \(f\) is well-defined and continuous, it is sufficient to show that \(T'\) is Lipschitz continuous. Since, \(\mu_j ((z - \sum_{i=1}^{j-1} y_i) \wedge y_j)\) is Lipschitz with constant \(\mu_j\) for all \(j \in [M]\). Therefore, the sum of Lipschitz functions is again Lipschitz with constant \((\mu_1 \vee \mu_2 \vee \ldots \vee \mu_M)\). This establishes the first statement of the lemma.

To prove the second statement of the lemma, note that we can can express (50) in its differential form as follows
\[
\frac{d}{dt} z(t) = \lambda - T'(z(t)).
\]
Hence, any fixed point \(z^*\) of the process \(z\) must satisfy the equation \(\lambda - T'(z^*) = 0\). Since \(T'\) is a piecewise-linear map, it is easy to solve the above equation in closed form and find the unique solution to be (51).

\[\tag{51}\]

\textbf{Lemma 6.} Let \(z(u, t)\) denote the solution to (50) for \(z(0) = u\). Then for any \(u \in \mathbb{R}\), \(z(u, t) \to z^*\) as \(t \to \infty\). Furthermore, the sequence \((z^{(N)}(\infty))_N\) of stationary states converges weakly to \(z^*\) as \(N \to \infty\).

\[\tag{52}\]

\textbf{Proof.} We prove the first statement of the lemma by considering the following two cases (i) \(u \geq z^*\) and (ii) \(u < z^*\). We only provide the proof for the first case as the proof of the second case is similar.
If \( u \geq z^* \) then \( z(u, t) \geq z^* \) for all \( t \geq 0 \). To see this, assume on the contrary that \( z(u, t) < z^* \) for some \( t > 0 \). Since \( z(0, u) = u \geq z^* \) and \( z(u, \cdot) \) is continuous, there must exist a \( t_1 \in (0, t) \) such that \( z(u, t_1) = z^* \). But since \( z^* \) is a fixed point, this implies that \( z(t) = z^* \) for all \( t \geq t_1 \) which leads to a contradiction.

Now consider the Lyapunov function \( V : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) defined as \( V(z(u, t)) = z(u, t) - z^* \geq 0 \). Using (50) we have

\[
\frac{d}{dt} V(z(u, t)) = \lambda - T'(z(u, t)),
\]

where \( T'(z) = \sum_{j=1}^{M} \mu_j ((z - \sum_{i=1}^{j-1} y_i)_+ \land y_j) \). Since the fixed point \( z^* \) solves \( \lambda - T'(z^*) = 0 \), we have

\[
\frac{d}{dt} V(z(u, t)) = -(T'(z(u, t)) - T'(z^*)).
\]

We further note that \( T' \) is a strictly increasing function of its argument. Hence, \( V(z(u, t)) < 0 \) when \( z(u, t) > z^* \). This implies that \( V(z(u, t)) \rightarrow 0 \) as \( t \rightarrow \infty \), thereby proving the first part of the lemma.

To prove the second part of the lemma, we first show that the sequence \( (z^{(N)}(\infty))_N \) is tight. By the application of Markov inequality and the bound in (2), we have

\[
\sup_{N \geq 1} \mathbb{P}(z^{(N)}(\infty) > a) \leq \sup_{N \geq 1} \frac{\mathbb{E}[z^{(N)}(\infty)]}{a} \leq \frac{1}{a} \left( \frac{\lambda}{1 - \lambda} \right),
\]

This shows that to make \( \sup_{N \geq 1} \mathbb{P}(z^{(N)}(\infty) > a) < \epsilon \) for any \( \epsilon > 0 \), there exists appropriate choice \( a(\epsilon) \) not dependent on \( N \). This shows that the sequence \( (z^{(N)}(\infty))_N \) is tight. The rest of the lemma now follows from the same line of arguments as in the proof of the second statement of Theorem 6.

The proof of Proposition 3.2 follows directly from Lemma 6 by noting that \( z^{(N)}(\infty) = Z^{(N)}(\infty)/N \).

\section*{B \ PROOF OF PROPOSITION 6.1}

Fix any \( \epsilon > 0 \) and \( l \geq 1 \). Using Markov inequality, we obtain

\[
\mathbb{P} \left( \max_{j \in [M]} \sum_{i \geq l} x^{(N)}_{i,j}(\infty) > \epsilon \right) \leq \frac{\mathbb{E} \left[ \max_{j \in [M]} \sum_{i \geq l} x^{(N)}_{i,j}(\infty) \right]}{\epsilon} \leq \frac{1}{\epsilon} \mathbb{E} \left[ \sum_{j \in [M]} \sum_{i \geq l} x^{(N)}_{i,j}(\infty) \right].
\]

Since \( \left( x^{(N)}_{i,j}(\infty) \right)_i \) is a sequence non-negative random variables for each \( j \in [M] \), using monotone convergence theorem we can interchange the sum and the expectation on the RHS. Hence, we have

\[
\mathbb{P} \left( \max_{j \in [M]} \sum_{i \geq l} x^{(N)}_{i,j}(\infty) > \epsilon \right) \leq \frac{1}{\epsilon} \sum_{j \in [M]} \sum_{i \geq l} \mathbb{E} \left[ x^{(N)}_{i,j}(\infty) \right] = \frac{1}{\epsilon} \sum_{j \in [M]} \sum_{i \geq l} \mathbb{P} \left[ Q^{(N)}_{k,j}(\infty) \geq i \right],
\]

where the last equality follows from (5). Now, from Theorem 3 we know that for any \( \theta \in [0, -\log \lambda] \) we have

\[
\sum_{j \in [M]} \sum_{i \geq l} \mathbb{P} \left[ Q^{(N)}_{k,j}(\infty) \geq i \right] \leq \sum_{j \in [M]} \sum_{i \geq l} C_j(\lambda, \theta) e^{-i\theta} = C(\theta) e^{-l \theta},
\]

where \( C(\theta) = \frac{1}{(1 - \lambda e^\theta)(1 - e^{-\theta})} \sum_{j \in [M]} 1/\mu_j \). Since the RHS of the above inequality is not dependent on \( N \), using (55) we have

\[
\limsup_{N \rightarrow \infty} \mathbb{P} \left( \max_{j \in [M]} \sum_{i \geq l} x^{(N)}_{i,j}(\infty) > \epsilon \right) \leq C(\theta) e^{-l \theta}
\]
We first write the evolution of \( x^{(N)}_{i,j}(t) \) for all \( t \geq 0 \) in terms of number of arrivals and departures from system till time \( t \), for \( i \geq 1 \) and \( j \in [M] \) as

\[
x^{(N)}_{i,j}(t) = x^{(N)}_{i,j}(0) + \frac{1}{NY_j} A_{i,j} \left( N \lambda \int_0^t p_{i-1,j}^{(N)}(x^{(N)}(s))ds \right) - \frac{1}{NY_j} D_{i,j} \left( N \lambda \int_0^t (x^{(N)}_{i,j}(s) - x^{(N)}_{i+1,j}(s))ds \right),
\]

where \( A_{i,j} \) and \( D_{i,j} \) are mutually independent unit-rate Poisson processes and \( p_{i-1,j}^{(N)}(x^{(N)}(s)) = 1_{\{V^{(N)}(s) \in R_{i,j}\}} \). Define for all \( i \geq 1 \) and for all \( j \in [M] \)

\[
\begin{align*}
M^{(A,N)}_{i,j}(t) &= A_{i,j} \left( N \lambda \int_0^t p_{i-1,j}^{(N)}(x^{(N)}(s))ds \right) - N \lambda \int_0^t p_{i-1,j}^{(N)}(x^{(N)}(s))ds, \\
M^{(D,N)}_{i,j}(t) &= D_{i,j} \left( N \lambda \int_0^t (x^{(N)}_{i,j}(s) - x^{(N)}_{i+1,j}(s))ds \right) - N \lambda \mu_j \int_0^t (x^{(N)}_{i,j}(s) - x^{(N)}_{i+1,j}(s))ds.
\end{align*}
\]

We next show that the processes \( M^{(A,N)}_{i,j} = (M^{(A,N)}_{i,j}(t) : t \geq 0) \) and \( M^{(D,N)}_{i,j} = (M^{(D,N)}_{i,j}(t) : t \geq 0) \) are martingales with respect to the filtration \( F^{(N)} = \{F^N_t : t \geq 0\} \) augmented with all null sets, where

\[
F^N_t = \bigcup_{j \in [M]} G^N_{i,j} \quad \text{with}
\]

\[
G^N_{i,j} = \bigcup_{i \geq 1} \sigma \left( x^{(N)}_{i,j}(0), A_{i,j} \left( N \lambda \int_0^s p_{i-1,j}^{(N)}(x^{(N)}(u))du \right), D_{i,j} \left( N \lambda \mu_j \int_0^s (x^{(N)}_{i,j}(u) - x^{(N)}_{i+1,j}(u))du \right), 0 \leq s \leq t \right).
\]

**Lemma 7.** The processes \( M^{(A,N)}_{i,j} \) and \( M^{(D,N)}_{i,j} \), are square integrable \( F^{(N)} \)-martingales for all \( i \geq 1 \) and for all \( j \in [M] \). Moreover, the predictable quadratic variation processes are given by

\[
\begin{align*}
\langle M^{(A,N)}_{i,j} \rangle(t) &= N \lambda \int_0^t p_{i-1,j}^{(N)}(x^{(N)}(s))ds, \quad i \geq 1, \ j \in [M], \ t \geq 0, \\
\langle M^{(D,N)}_{i,j} \rangle(t) &= N \lambda \mu_j \int_0^t (x^{(N)}_{i,j}(s) - x^{(N)}_{i+1,j}(s))ds, \quad i \geq 1, \ j \in [M], \ t \geq 0.
\end{align*}
\]

**Proof.** Let \( I_{i,j}(t) = N \mu_j \int_0^t (x^{(N)}_{i,j}(s) - x^{(N)}_{i+1,j}(s))ds \) and \( L_{i,j}(t) = N \lambda \int_0^t p_{i-1,j}^{(N)}(x^{(N)}(s))ds \). Next, we prove that

\[
\begin{align*}
\mathbb{E}(I_{i,j}(t)) &< \infty, \quad \mathbb{E}(D_{i,j}(I_{i,j}(t))) < \infty, \quad i \geq 1, \ j \in [M], \\
\mathbb{E}(L_{i,j}(t)) &< \infty, \quad \mathbb{E}(A_{i,j}(L_{i,j}(t))) < \infty, \quad i \geq 1, \ j \in [M].
\end{align*}
\]

The proof of square-integrable martingales and its corresponding predictable quadratic variation processes then follows immediately using Lemma 3.2 from [19]. Note that using crude inequality we can write

\[
\begin{align*}
\mathbb{E}(I_{i,j}(t)) &\leq \mu_j t \left( NY_j \mathbb{E}(x^{(N)}_{i,j}(0)) + \mathbb{E}(A_{i,j}(L_{i,j}(t))) \right) \\
&\leq \mu_j \gamma_j N t + \mu_j N \lambda t^2 < \infty, \quad t \geq 0, \ i \geq 1, \ j \in [M],
\end{align*}
\]
We know that the maximum weight is $\mu$. To prove (47), we consider the following linear optimisation problem

\[
\begin{align*}
\mathbb{E} \left[ D_{i,j}(L_{i,j}(t)) \right] &\leq \mathbb{E} \left[ D_{i,j} \left( \mu_j t \left( N_{jt} x_{i,j}^{(N)}(0) + A_{i,j}(L_{i,j}(t)) \right) \right) \right] \\
&= \mathbb{E} \left[ \left\{ \mu_j t \left( N_{jt} x_{i,j}^{(N)}(0) + A_{i,j}(L_{i,j}(t)) \right) \right| A_{i,j}(L_{i,j}(t)) \right] \\
&\leq \mu_j t (N_{jt} + N\lambda t) < \infty, \quad t \geq 0, \quad i \geq 1, \quad j \in [M].
\end{align*}
\]

Similarly, we can show $\mathbb{E}(L_{i,j}(t)) < \infty$ and $\mathbb{E}(A_{i,j}(L_{i,j}(t))) < \infty$.

Now we can write the martingale representation of equation (57) for all $t \geq 0$, $i \geq 1$, and $j \in [M]$ as

\[
x_{i,j}^{(N)}(t) = x_{i,j}^{(N)}(0) + \frac{\lambda}{Y_j} \int_0^t \mathbb{1}_{(V^{(N)}(s) \in R_{i,j})} ds - \mu_j \int_0^t (x_{i,j}^{(N)}(s) - x_{i+1,j}^{(N)}(s)) ds + \frac{1}{N_{jt}} (M_{i,j}^{(A,N)}(t) - M_{i,j}^{(D,N)}(t)).
\]

(58)

In below lemma, we prove that the martingale part in (58) converges to 0 with respect to $t_1$.

**Lemma 8.** Following convergence holds as $N \to \infty$

\[
\left\{ \max_{j \in [M]} \frac{1}{N_{jt}} \sum_{i \geq 1} (|M_{i,j}^{(A,N)}(t)| + |M_{i,j}^{(D,N)}(t)|) \right\}_{t \geq 0} \Rightarrow 0.
\]

**Proof.** The proof is similar to the proof of Proposition 4.3 in [15] using Doob’s inequality.

\[\blacksquare\]

**D PROOF OF LEMMA 4**

To prove (47), we consider the following linear optimisation problem

\[
\begin{align*}
\max & \quad \sum_{j \in [M]} \mu_j x_{1,j}^{(N,\Pi)}(t), \\
\text{s.t.} & \quad \sum_{j \in [M]} x_{1,j}^{(N,\Pi)}(t) \leq i, \\
& \quad 0 \leq x_{1,j}^{(N,\Pi)}(t) \leq N_{jt}, \quad \forall j \in [M].
\end{align*}
\]

(59)

The first constraint in (59) is true as the total number of busy servers $\sum_{j \in [M]} x_{1,j}^{(N,\Pi)}(t)$ at time $t$ in $\mathcal{M}_N$ is always less than equal to the total number of customers in system that is $i$. The above optimisation problem has a unique maximum which is obtained in following way. First, note that the objective function is given as the weighted sum of $x_{1,j}^{(N,\Pi)}(t)$ with weight $\mu_j$ for $j \in [M]$. We know that the maximum weight is $\mu_1$. Therefore, the maximum value that $x_{1,j}^{(N,\Pi)}(t)$ takes is $(i \land N_{j1})$. Moreover, the second maximum weight is $\mu_2$, hence the maximum value that $x_{1,2}^{(N,\Pi)}(t)$ takes is $((i - N_{j1})_+ \land N_{j2})$. Proceeding in this way the maximum value that $x_{1,j}^{(N,\Pi)}(t)$ takes is

\[
\left( \left( i - \sum_{l=1}^{j-1} N_{jl} \right)_+ \land N_{jt} \right)
\]

for $j \in [M]$, which completes the proof.

\[\blacksquare\]

**E CHARACTERISATION OF COMPACT SETS AND TIGHTNESS CRITERIA**

In the lemma below, we characterise compact sets in the space $S$.

**Lemma 9.** A set $B \subseteq S$ is relative compact in $S$ if and only if

\[
\lim_{t \to \infty} \sup_{y \in B} \max_{j \in [M]} \sum_{l \geq t} y_{l,j} = 0.
\]

(60)
Proof. Suppose any $B \subseteq S$ satisfying (60). To show $B$ is relatively compact in $S$, we need to show that any sequence $(y^{(N)})_{N \geq 1}$ in $B$ has a Cauchy subsequence. Since $S$ is complete under $\ell_1$, therefore the sequence $(y^{(N)})_{N \geq 1}$ has a convergent subsequence whose limit lies in $B$ which will complete the proof.

Next we show that the sequence $(y^{(N)})_{N \geq 1}$ has a Cauchy subsequence. Fix any $\epsilon > 0$ and choose $l \geq 1$ such that

$$\max_{j \in [M]} \sum_{i \geq l} |y_{i,j}^{(N)}| < \frac{\epsilon}{4}, \forall N \geq 1. \quad (61)$$

Now consider the sequence of first coordinates $(y_{1,j}^{(N)})_{N \geq 1}$ for each $j \in [M]$. The sequence $(y_{1,j}^{(N)})_{N \geq 1}$ lies in $[0,1]$. Therefore, by Bolzano-Wiestrass theorem it has a convergent subsequence $(y_{1,j}^{(N)})_{k \geq 1}$. Moreover, along the indices $(N_k)_{k \geq 1}$, the sequence $(y_{2,j}^{(N_k)})_{k \geq 1}$ has a further convergent subsequence. Proceeding this way, we get a sequence of indices $(N_m)_{m \geq 1}$ along which all first $l-1$ coordinates converges. This implies that there exists a $N^e \in \mathbb{N}$ such that

$$\max_{j \in [M]} \sum_{i < l} |y_{i,j}^{(N)} - y_{i,j}^{(R)}| < \frac{\epsilon}{2}, \forall R, N \geq N^e. \quad (62)$$

Using (61), and (62), for all $R, N \geq N^e$ and for $y^{(N)}, y^{(R)} \in B$ we have

$$\left\|y^{(N)} - y^{(R)}\right\|_1 = \max_{j \in [M]} \sum_{i \geq l} |y_{i,j}^{(N)} - y_{i,j}^{(R)}| \leq \max_{j \in [M]} \sum_{i < l} |y_{i,j}^{(N)} - y_{i,j}^{(R)}| + \max_{j \in [M]} \sum_{i \geq l} |y_{i,j}^{(N)}| + \max_{j \in [M]} \sum_{i \geq l} |y_{i,j}^{(R)}| < \epsilon,$$

along the sequence of indices $(N_m)_{m \geq 1}$. This shows the existence of a Cauchy subsequence. Moreover, the limit point lies in $S$ follows from the completeness of $\ell_1$ space and the fact that $S$ is a closed subset of $\ell_1$.

Now for the only if part, let $B$ be a relatively compact set in $S$. Assume that there exists a $\epsilon > 0$ such that

$$\limsup_{l \to \infty} \max_{y \in B} \sum_{i \geq l} y_{i,j} > \epsilon. \quad (63)$$

This implies that for each $k \geq 1$, there exists a $y^{(k)} \in B$, such that $\max_{j \in [M]} \sum_{i \geq l} y_{i,j}^{(k)} \geq \frac{\epsilon}{2}$. Therefore, if $y^*$ be the limit of the sequence $(y^{(k)})_{k \geq 1}$, then (63) implies that $\max_{j \in [M]} \sum_{i \geq l} y_{i,j}^{*} \geq \frac{\epsilon}{2}$ for all $l \geq 1$, this leads to the contradiction that $y^* \in \ell_1$. 

Next, we prove the criteria for a sequence to be tight in $S$.

**Lemma 10.** A sequence $(y^{(N)})_{N \geq 1}$ of random elements in $S$ is tight iff for all $\epsilon > 0$ we have

$$\lim_{l \to \infty} \limsup_{N \to \infty} \mathbb{P}\left(\max_{j \in [M]} \sum_{i \geq l} y_{i,j}^{(N)} > \epsilon\right) = 0. \quad (64)$$

**Proof.** For if part, we construct a relatively compact set $B^\epsilon$ for any $\epsilon > 0$ such that

$$\mathbb{P}(y^{(N)} \notin B^\epsilon) < \epsilon, \forall N \in \mathbb{N}.\quad (65)$$
As \((y^{(N)})_{N \geq 1}\) satisfies (64), therefore there exists a \(l(e) \geq 1\) for all \(\epsilon > 0\) such that
\[
\limsup_{N \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l(e)} y_{i,j}^{(N)} > \epsilon \right) < \epsilon,
\]
and there exists a \(N^e \geq 1\) such that
\[
\mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l(e)} y_{i,j}^{(N)} > \epsilon \right) < \epsilon, \quad \forall N > N^e.
\]
Moreover, since \(y^{(1)}, \ldots, y^{(N^e)}\) are random elements of \(S\), there exists \(k(e) = \max \{l_1(e), \ldots, l_{N^e}(e)\}\) such that
\[
\mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq k(e)} y_{i,j}^{(N)} > \epsilon \right) < \epsilon, \quad \forall N \in \mathbb{N}.
\]
This implies that there exists an increasing sequence \((l(n))_{n \geq 1}\) such that
\[
\mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l(n)} y_{i,j}^{(N)} > \frac{\epsilon}{2^n} \right) < \frac{\epsilon}{2^n}, \quad \forall N \in \mathbb{N}.
\]
Define
\[
B^e = \left\{ y \in S : \max_{j \in [M]} \sum_{i \geq l(n)} y_{i,j} \leq \frac{\epsilon}{2^n}, \forall n \geq 1 \right\}.
\]
From Lemma 9, the set \(B^e\) is relatively compact in \(S\). Therefore, we have
\[
\mathbb{P}(y^{(N)} \notin B^e) = \mathbb{P}\left( \bigcup_{n \geq 1} \left\{ \max_{j \in [M]} \sum_{i \geq l(n)} y_{i,j}^{(N)} > \frac{\epsilon}{2^n} \right\} \right) \leq \sum_{n \geq 1} \mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l(n)} y_{i,j}^{(N)} > \frac{\epsilon}{2^n} \right) < \epsilon,
\]
where the first inequality follows from union bound. For only if part, assume that there exists an \(\epsilon > 0\) such that
\[
\lim_{l \to \infty} \limsup_{N \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l} y_{i,j}^{(N)} > \epsilon \right) > \epsilon.
\]
Since \((y^{(N)})_{N \geq 1}\) is tight in \(S\), therefore there exists a convergent subsequence \((y^{(N_k)})_{k \geq 1}\) with limit \(y^*\). From (65), we can write
\[
\epsilon < \lim_{l \to \infty} \limsup_{k \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l} y_{i,j}^{(N_k)} > \epsilon \right) \leq \lim_{l \to \infty} \limsup_{k \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l} y_{i,j}^{(N_k)} \geq \epsilon \right)
\]
\[
\leq \lim_{l \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{i \geq l} y_{i,j} \geq \epsilon \right).
\]
where the last inequality follows from Portmanteau’s theorem for closed set. This leads to the contradiction that \(y^* \in \ell_1\).

\[\blacksquare\]

**F PROOF OF LEMMA 2**

To prove relative compactness of the sequence \((x^{(N)}, \beta^{(N)})_{N \geq 1}\), we start with proving that for all finite time \(t\) the system occupancy state lies in some compact set.

**Lemma 11.** Assume \(x^{(N)}(0) \Rightarrow x(0) \in S\), as \(N \to \infty\). Then for any \(T \geq 0\), there exists a \(L(T, x(0)) > 2\) such that under the SA-JSQ policy, the probability that an arriving job joins a server with at-least \(L(T, x(0)) - 1\) active jobs upto time \(T\) tends to \(0\) as \(N \to \infty\).
Therefore, we have
\[ \mathbb{P}(X^{(N)}(t) \geq N(L(T, x(0)) - 1), t \in [0, T]) \to 0, \text{ as } N \to \infty. \]

Let \( m^{(N)}(t) = \min_{j \in [M]} m_j^{(N)}(t) \), where \( m_j^{(N)}(t) \) is the minimum queue length in the \( j^{th} \) pool at time \( t \) for \( N^{th} \) system. The probability at which an arrival joins a server with at-least \( L(T, x(0)) - 1 \) jobs during the time interval \([0, T]\) is given by
\[
\mathbb{P}\left( \{ m^{(N)}(t) \geq L(T, x(0)) - 1, t \in [0, T]\} \cap \{ \text{an arrival occur at } t \} \right),
\]
\[
\leq \mathbb{P}\left( m^{(N)}(t) \geq L(T, x(0)) - 1, t \in [0, T]\right),
\]
\[
= \mathbb{P}\left( X^{(N)}(t) \geq N(L(T, x(0)) - 1), t \in [0, T]\right), \tag{66}
\]
where \( X^{(N)}(t) \) is the total number of jobs in system at time \( t \). From conservation of flow, we can write \( X^{(N)}(t) = X^{(N)}(0) + A^{(N)}(t) - D^{(N)}(t) \), where \( D^{(N)}(t) \) is the total number of departures from system till time \( t \). Therefore, from (66), we can write
\[
\mathbb{P}\left( X^{(N)}(t) \geq N(L(T, x(0)) - 1), t \in [0, T]\right)
\]
\[
= \mathbb{P}\left( X^{(N)}(0) + A^{(N)}(t) - D^{(N)}(t) \geq N(L(T, x(0)) - 1), t \in [0, T]\right)
\]
\[
\leq \mathbb{P}\left( X^{(N)}(0) + A^{(N)}(t) \geq N(L(T, x(0)) - 1), t \in [0, T]\right), \tag{67}
\]
where \( M^{(N)}(t) = \frac{A^{(N)}(t) - NL(T, x(0))}{N} \), and \( x^{(N)}(0) = \frac{X^{(N)}(0)}{N} \). Now observer that for all \( \epsilon > 0 \) we have
\[
\mathbb{P}\left( \sup_{t \in [0, T]} |M^{(N)}(t)| > \epsilon \right) \to 0, \text{ as } N \to \infty.
\]

Therefore, we have
\[
\mathbb{P}\left( |M^{(N)}(t)| > 1, t \in [0, T] \right) \to 0, \text{ as } N \to \infty. \tag{68}
\]

From (66), (67), (68) and choosing \( L(T, x(0)) > 2 + x(0) + \lambda T \), where \( x(0) = \sum_{j \in [M]} \sum_{i \geq 1} \eta_j x_{ij}(0) \), we have
\[
\mathbb{P}\left( \{ m^{(N)}(t) \geq L(T, x(0)) - 1, t \in [0, T]\} \cap \{ \text{an arrival occur at } t \} \right) \to 0, \text{ as } N \to \infty.
\]

We now prove the relative compactness of the sequence \( (x^{(N)}(t), \beta^{(N)}))_{N \geq 1} \) by showing the relative compactness of individual component. Note that the space \( E \) is compact. Therefore, relative compactness of the sequence \( (\beta^{(N)})_{N \geq 1} \) follows from Prohorov’s theorem [1]. To prove relative compactness of the sequence \( (x^{(N)}(t))_{N \geq 1} \), we need to verify following conditions.

1. For every \( \eta > 0 \) and rational \( t \geq 0 \), there exists a compact set \( B_{\eta,t} \subset S \) such that
\[
\liminf_{N \to \infty} \mathbb{P}(x^{(N)}(t) \in B_{\eta,t}) \geq 1 - \eta \tag{69}
\]
We first prove condition (69). From Lemma 11, for any fix $t, s \in [t_{l-1}, t_l]$ with $\min_{l \in [n]}|t_l - t_{l-1}| > \delta$ such that
\[
\limsup_{N \to \infty} \mathbb{P}\left( \max_{l \in [n]} \sup_{s, t \in [t_{l-1}, t_l]} \left\| x^{(N)}(s) - x^{(N)}(t) \right\|_1 \geq \eta \right) < \eta. \tag{70}
\]

We first prove condition (69). From Lemma 11, for any fix $t \geq 0$ we have
\[
\lim_{N \to \infty} \mathbb{P}\left( x^{(N)}_{i,j}(t) \leq x^{(N)}_{i,j}(0), \forall i \geq L(t, x(0)), j \in [M] \right) = 1.
\]

Now it can be easily verified that the sequence $(x^{(N)}(0))_{N \geq 1}$ is tight in $S$. Therefore, using previous condition we can write for any $\epsilon > 0$
\[
\lim \limsup_{N \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{t \geq t_l} x^{(N)}_{i,j}(t) > \epsilon \right) \leq \lim \limsup_{N \to \infty} \mathbb{P}\left( \max_{j \in [M]} \sum_{t \geq t_l} x^{(N)}_{i,j}(0) > \epsilon \right) = 0. \tag{71}
\]

Hence, from Lemma 10 the sequence $(x^{(N)}(t))_{N \geq 1}$ is tight in $S$. This implies that the condition (69) is satisfied. Next for any $t_1 < t_2$, we consider
\[
\left| x^{(N)}_{i,j}(t_1) - x^{(N)}_{i,j}(t_2) \right| \leq \frac{\lambda}{Y_j} \beta^N([t_1, t_2] \times R_{i,j}) + \mu_j \int_{t_1}^{t_2} (x^{(N)}_{i,j}(s) - x^{(N)}_{i+1,j}(s))ds + \frac{1}{NY_j} |M^{(A,N)}_{i,j}(t_1) - M^{(D,N)}_{i,j}(t_1) - M^{(A,N)}_{i,j}(t_2) + M^{(D,N)}_{i,j}(t_2)| + o(1), i \geq 1, j \in [M].
\]

Using the above equation we can write $\ell_1$ distance between $x^{(N)}(t_1)$ and $x^{(N)}(t_2)$ as
\[
\left\| x^{(N)}(t_1) - x^{(N)}(t_2) \right\|_1 \\
\leq \max_{j \in [M]} \sum_{t \geq t_1} \frac{\lambda}{Y_j} \beta^N([t_1, t_2] \times R_{i,j}) + \max_{j \in [M]} \sum_{t \geq t_1} \int_{t_1}^{t_2} \mu_j (x^{(N)}_{i,j}(s) - x^{(N)}_{i+1,j}(s))ds + \frac{1}{NY_j} \sum_{t \geq t_1} |M^{(A,N)}_{i,j}(t_1) - M^{(D,N)}_{i,j}(t_1) - M^{(A,N)}_{i,j}(t_2) + M^{(D,N)}_{i,j}(t_2)| + o(1)
\]
\[
\leq \frac{\lambda}{Y_{\text{min}}} (t_2 - t_1) + \max_{j \in [M]} \int_{t_1}^{t_2} \mu_j x^{(N)}_{i,j}(s)ds + \frac{1}{NY_j} \sum_{t \geq t_1} |M^{(A,N)}_{i,j}(t_1) - M^{(D,N)}_{i,j}(t_1)(s)ds - M^{(A,N)}_{i,j}(t_2) + M^{(D,N)}_{i,j}(t_2)| + o(1)
\]
\[
\leq \frac{\lambda}{Y_{\text{min}}} + \mu_1 (t_2 - t_1) + \max_{j \in [M]} \frac{1}{NY_j} \sum_{t \geq t_1} |M^{(A,N)}_{i,j}(t_1) - M^{(D,N)}_{i,j}(t_1) - M^{(A,N)}_{i,j}(t_2) + M^{(D,N)}_{i,j}(t_2)| + o(1), \tag{72}
\]
where $Y_{\text{min}} = \min_{j \in [M]} Y_j$. From Lemma 8, the martingale part in (72) converges to 0 as $N \to \infty$. Moreover, from (72), it implies that for any finite partition $\{t_1, t_2, \ldots, t_n\}$ of $[0, T]$ with $\min_{i \in [n]}|t_i - t_{i-1}| > \delta$, we have
\[
\max_{l \in [n]} \sup_{s, t \in [t_{l-1}, t_l]} \left\| x^{(N)}(s) - x^{(N)}(t) \right\|_1 \leq \frac{\lambda}{Y_{\text{min}}} + \mu_1 \max_{l \in [n]}(t_l - t_{l-1}) + \zeta^{(N)},
\]
where $\mathbb{P}(\zeta^{(N)} > \frac{n}{2}) < \eta$ for all sufficiently large $N$. Take $\delta = \eta / (4(\frac{\lambda}{Y_{\text{min}}} + \mu_1))$ and any partition with $\max_{l \in [n]}(t_l - t_{l-1}) < \eta / (2(\frac{\lambda}{Y_{\text{min}}} + \mu_1))$ and $\min_{l \in [n]}|t_l - t_{l-1}| > \delta$, we have
\[
\max_{l \in [n]} \sup_{s, t \in [t_{l-1}, t_l]} \left\| x^{(N)}(s) - x^{(N)}(t) \right\|_1 \leq \eta,
\]

on the event \( \{ \xi^{(N)} \leq \frac{\eta}{2} \} \). Therefore, for sufficiently large \( N \) we obtain
\[
\mathbb{P} \left( \max_{t \in [n]} \sup_{s \in [t_1, t_1]} \| x^{(N)}(s) - x^{(N)}(t) \|_1 \geq \eta \right) \leq \mathbb{P}(\xi^{(N)} > \frac{\eta}{2}) < \eta.
\]
Hence the condition (70) is satisfied. Next, we show that the limit \((x, \beta)\) of any convergent subsequence of the sequence \((x^{(N)}, \beta^{(N)})_{N \geq 1}\) satisfies (36). We first show that the right side of (35) is a continuous map and then the result follows from an application of continuous mapping theorem. Consider,
\[
W_{i,j} \left( x^{(N)}(t), \beta^{(N)}, x^{(N)}(0), m^{(N)} \right)(t) = x_{i,j}^{(N)}(0) + \frac{\lambda}{y_j} \beta^{(N)}([0,t] \times R_{i,j}) - \mu_j \int_0^t (x_{i,j}^{(N)}(s) - x_{i+1,j}^{(N)}(s)) ds + m_{i,j}^{(N)}(t) \text{ for all } i \in [M],
\]
where \( m_{i,j}^{(N)}(t) = \frac{1}{N} \left( M_{i,j}^{(A,N)}(t) - M_{i,j}^{(D,N)}(t) \right) \). Now we next prove that the map \( W = (W_{i,j})_{i,j} \) is continuous. First, assume that the sequence \((x^{(N)}, m^{N})_{N \geq 1}\) converges to \((x, m)\), then there exists a \( N_1 \in \mathbb{N} \) such that \( \sup_{t \in [0,T]} \| x^{(N)}(t) - x(t) \|_1 < \epsilon/(4T) \) for all \( N \geq N_1 \). Therefore, we can write
\[
\sup_{t \in [0,T]} \int_0^T |x_{i,j}^{(N)}(t) - x_{i,j}(t)| ds \leq T \sup_{t \in [0,T]} \| x^{(N)}(t) - x(t) \|_1 < \epsilon/4.
\]
Also, there exists a \( N_2 \in \mathbb{N} \) such that \( \sup_{t \in [0,T]} \| m^{(N)}(t) - m(t) \|_1 < \epsilon/4 \). Second, assume that the sequence \((x^{(N)}(0))\) converges to \( x(0) \) with respect to \( \ell_1 \). Therefore, there exists a \( N_3 \in \mathbb{N} \) such that \( \| x^{(N)}(0) - x(0) \|_1 < \epsilon/4 \). Now we claim that there exists a \( N_4 \in \mathbb{N} \) such that
\[
\frac{\lambda}{y_j} \max_{j \in [M]} \sum_{i \geq 1} |\beta^{(N)}([0,T] \times R_{i,j}) - \beta([0,T] \times R_{i,j})| \leq \epsilon/4. \tag{73}
\]
The equation (73) implies that the convergence of the sequence \((\beta^{(N)}([0,T] \times R_{i,j}))_{i,j} \geq 1\) for any \( t \geq 0 \) is \( \ell_1 \) convergence. However, we know only the weak convergence of the sequence of measures \((\beta^{(N)})_{N \geq 1}\), which does not directly implies (73). Therefore, we show using weak convergence of the sequence \((\beta^{(N)})_{N \geq 1}\) that (73) is indeed true in our case. Since \( x(0) \in S \), there exists a \( m_j(x(0)) \) for all \( j \in [M] \) such that \( x_{i,j}(0) < 1 \) for all \( i \geq m_j(x(0)) \). Furthermore, from Lemma 11 we can write
\[
\lim_{N \to \infty} \mathbb{P} \left( \sup_{t \in [0,T]} x_{i,j}^{(N)}(t) \leq x_{i,j}^{(N)}(0), \forall i \geq L(T, x(0)), j \in [M] \right) = 1.
\]
Therefore, for \( N' = \max \left\{ m_j(x(0)), L(T, x(0)) \right\} \) we have
\[
\lim_{N \to \infty} \mathbb{P} \left( \sup_{t \in [0,T]} x_{i,j}^{(N)}(t) \leq 1, \forall i \geq N', j \in [M] \right) = 1. \tag{74}
\]
Using (74) and (33) we get
\[
\lim_{N \to \infty} \max_{j \in [M]} \sum_{i \geq N'} \beta^{(N)}([0,T] \times R_{i,j}) = \max_{j \in [M]} \sum_{i \geq N'} \beta([0,T] \times R_{i,j}) = 0. \tag{75}
\]
Also, weak convergence of \((\beta^{(N)})_{N}\) implies that
\[
\lim_{N \to \infty} \max_{j \in [M]} \sum_{i < N'} \beta^{(N)}([0,T] \times R_{i,j}) = \max_{j \in [M]} \sum_{i < N'} \beta([0,T] \times R_{i,j}). \tag{76}
\]
Hence, using (75) and (76) we get the desired result that is (73). Let \( \bar{N} = \max \{ N_1, N_2, N_3, N_4 \} \), then we have
\[
\sup_{t \in [0, T]} \left\| W\left( x^{(N)}(t), \beta^{(N)}, x^{(N)}(0), m^{(N)} \right) - W\left( x(t), \beta, x(0), m \right) \right\|_1 (t) < \epsilon.
\]
This shows that the map \( W \) is continuous, which completes the proof.

G PROOF OF LEMMA 3
To prove positive recurrent, we consider the Lyapunov function \( f : \mathbb{Z}^K_+ \rightarrow [0, \infty) \) as
\[
f(U) = \sum_{j \in [M]} U_j, \ U \in \mathbb{Z}^K_+.
\]
Now we can write the generator \( G_U \) acting on the function \( f \) for a state \( R \in \mathbb{Z}^K_+ \) as
\[
G_U f(R) = \sum_{j \in [K]} \nu_j - \sum_{j \in [K]} \lambda_j \mathbb{1}_{\{0=R_1=\cdots=R_{j-1}<R_j\}}.
\]
Assume that the event \( \{ R_1 > 0 \} \) happens. This implies that \( G_U f(R) < 0 \) when \( \sum_{j \in [K]} \nu_j < \min_{j \in [K]} \lambda_j \). Similarly, we have \( G_U f(R) < 0 \) if \( \sum_{j \in [K]} \nu_j < \min_{j \in [K]} \lambda_j \) and one of the event from the set
\[
\{ \{ R_1 > 0 \} \cup \{ R_2 > R_1 = 0 \} \cup \cdots \cup \{ 0 = R_1 = \cdots = R_{K-1} < R_K \} \},
\]
occurs. Otherwise \( G_U f(R) < \sum_{j \in [K]} \nu_j \). Therefore, using the Foster-Lyapunov criterion for positive recurrence from [11], we conclude that the chain \( U \) is positive recurrent.

Now observe that \( \pi(U_1 > 0) = v_1/\lambda_1 \) if the birth-death process for the component \( U_1 \) is stable. We use induction to prove (44). The base case is already proved above. Assume (44) is true for all \( j \in [K-1] \). Furthermore, we can set steady-state expected drift to 0 which gives
\[
\mathbb{E}[G_U f(U)] = \sum_{j \in [K]} \nu_j - \sum_{j \in [K]} \lambda_j \pi(0 = U_1 = \cdots = U_{j-1} < U_j) = 0.
\]
Hence, using (44) for all \( j \in [K-1] \) we get
\[
\pi(0 = U_1 = \cdots = U_{K-1} < U_k) = \frac{v_k}{\lambda_k}.
\]
Now we prove the necessary and sufficient condition for the chain \( U \) to be stable. For \( K = 2 \), we know that the chain \( (U_1, U_2) \) is stable if \( v_1 + v_2 < \min(\lambda_1, \lambda_2) \) and from (44) we have \( \pi(U_2 > U_1 = 0) = v_2/\lambda_2 \). Therefore, we can write
\[
\pi(U_1 = 0, U_2 = 0) = \pi(U_1 = 0) - \pi(U_2 > U_1 = 0) = 1 - \frac{v_1}{\lambda_1} - \frac{v_2}{\lambda_2} > 0,
\]
where the last inequality follows as the chain \( (U_1, U_2) \) is positive recurrent. Hence, the necessary and sufficient condition for the chain \( (U_1, U_2) \) to be positive recurrent is \( \frac{v_1}{\lambda_1} + \frac{v_2}{\lambda_2} < 1 \), which also implies that \( v_1 + v_2 < \min(\lambda_1, \lambda_2) \) holds. Similarly using induction we can easily prove this for general \( K \).

H PROOF OF THEOREM 6
Proof of Theorem 6.(ii): To prove this, we need the following lemma which extends the motonicity of the process \( x^{(N)} \) for finite \( N \) to the motonicity of the limiting process \( x \).

Lemma 12. Let \( x(u, \cdot) = (x(u, t), t \geq 0) \) denote a solution to (10) with \( x(0) = u \in S \). Then, for any \( u, v \in S \) satisfying \( u \leq v \) we have \( x(u, t) \leq x(v, t) \) for all \( t \geq 0 \).
Proof. First, note that for any \( u \in S \), there exists a sequence \((u^{(N)})_{N \geq 1}\) with \( u^{(N)} = (u_{i,j}^{(N)})_{i \geq 1, j \in [M]} \) such that \( \|u^{(N)} - u\|_1 \to 0 \) as \( N \to \infty \). We can simply construct such a sequence by setting \( u_{i,j}^{(N)} = \frac{[a_{i,j}^{(N)}]}{N^j} \) for each \( i \geq 1 \) and \( j \in [M] \). This construction also satisfies the property that if \( u, v \in S \) are such that \( u \leq v \) and if the sequences \((u^{(N)})_{N}\) and \((v^{(N)})_{N}\) are constructed from their corresponding limits \( u \) and \( v \) as described above, then \( u^{(N)} \leq v^{(N)} \) for all \( N \).

Let \( x^{(N)}(u^{(N)}, \cdot) = (x^{(N)}(u^{(N)}, t), t \geq 0) \) denote the process \( x^{(N)} \) started at \( x^{(N)}(0) = u^{(N)} \). Then by Theorem 4 we have that

\[
x^{(N)}(u^{(N)}, t) \leq x^{(N)}(v^{(N)}, t), \forall t \geq 0.
\]

Now letting \( N \to \infty \) and applying Theorem 5 gives the desired result. \( \square \)

From Lemma 12 it follows that for any \( x(0) \in S \) and any \( t \geq 0 \) we have

\[
x(t, \min(x(0), x^*)) \leq x(t, x(0)) \leq x(t, \max(x(0), x^*)),
\]

where \( \min(u, v) \) with \( u, v \in S \) is defined by taking the component-wise minimum. Hence, to prove global stability it is sufficient to show that \( \lim_{t \to \infty} \|x(t) - x^*\| = 0 \), holds for all initial states satisfying \( x(0) \geq x^* \) and \( x(0) \leq x^* \). We prove the convergence for the case \( x(0) \geq x^* \). The proof for the case \( x(0) \leq x^* \) is identical with obvious modifications. Let us define the Lyapunov function \( V : S \to \mathbb{R}_+ \) as follows:

\[
V(x) = \sum_{j \in [M]} y_j \sum_{i \geq 1} |x_{i,j} - x_{i,j}^*|.
\]

From Lemma 12 it follows that for \( x(0) \geq x^* \), we have \( x(t) \geq x^* \) for all \( t \geq 0 \). Thus, we have

\[
V(x(t)) = \sum_{j \in [M]} y_j \sum_{i \geq 1} \left(x_{i,j}(t) - x_{i,j}^*\right).
\]

Taking derivative of (80) with respect to \( t \) and using the fact that \( \frac{\lambda}{y_j} p_{i-1,j}(x^*) - \mu_j (x_{i,j}^* - x_{i+1,j}^*) = 0 \) for each \( i \geq 1 \) and \( j \in [M] \), we obtain

\[
\frac{dV(x(t))}{dt} = \sum_{j \in [M]} \sum_{i \geq 1} \lambda (p_{i-1,j}(x(t)) - p_{i-1,j}(x^*)) - \sum_{j \in [M]} \mu_j y_j (x_{i,j}(t) - x_{i,j}^*)
\]

\[
= - \sum_{j \in [M]} \mu_j y_j (x_{i,j}(t) - x_{i,j}^*),
\]

where the second equality follows from property P1. Hence, from the above derivative we see that \( \frac{dV(x(t))}{dt} < 0 \) whenever \( x(t) > x^* \). This implies that \( V(x(t)) \to 0 \) as \( t \to \infty \). Therefore, we have \( \|x(t) - x^*\|_1 \to 0 \) as \( t \to \infty \), which completes the proof.

Proof of Theorem 6.(iii): We first recall from Proposition 6.1 that the sequence \((x^{(N)}(\infty))_N\) is tight in \( S \) under the \( l_1 \)-norm. Hence, by Prohorov’s theorem, the sequence has convergent subsequences with limits in \( S \). Hence, it suffices to show that all convergent subsequences has the same limit point \( x^* \). Let \((x^{(N_k)}(\infty))_k\) be any such convergent subsequence of the sequence \((x^{(N)}(\infty))_N\) with limit point \( x^{**} \). Now, from Theorem 5, we have \( x^{**} \) as the fixed point of the map \( u \to x(u, t) \). Hence, by the global stability result proved earlier this implies that \( x^{**} = x^* \).

1 UNIQUE CHARACTERISATION OF STATIONARY DISTRIBUTION FROM GIVEN STATE

The proof of unique characterisation of \( \pi_x \) from \( x \) is as follows. Given \( x \in S \), we know \( l_j(x) \) for all \( j \in [M] \). Let \( m_i \) denote the \( i^{th} \) minimum of the set \( \{l_j(x) : j \in [M]\} \). Define \( A = \{m_1, \ldots, m_k\} \).
to be the set of non-decreasing distinct values of \( I_j(x) \), where \( k \) denote the number of distinct minimums of \( \{I_j(x) : j \in [M]\} \). For each \( m_i \), we define \( D_i = \{d_{i,1}, \ldots , d_{i,|D_i|}\} \) to be the sorted list of all pools with minimum queue length \( m_i \), i.e., \( d_{i,1} < \cdots < d_{i,|D_i|} \). Characterising the chain \((V_{i}(x), V_{i+1}(x), \ldots , V_{M}(x), M)\) is therefore equivalent to characterising the following chain

\[
C = \left( V_{m_1,d_{1,1}}, \ldots , V_{m_1,d_{1,|D_1|}}, V_{m_2,d_{2,1}}, \ldots , V_{m_2,d_{2,|D_2|}}, \ldots , V_{m_k,d_{k,1}}, \ldots , V_{m_k,d_{k,|D_k|}} \right).
\]

Now we claim that the rate of down transition of each component in the chain \( C \) is zero or it has the same form as in (43), i.e., the down rate of transition for the \( l \)th component of the vector \( C \) depends only on the components \( i \in [I-1] \) of \( C \).

The proof of above claim is as follows. To find the down rates of transition of each component, we visit the sets \( D_i \)'s in the order \( D_1, D_2, \ldots , D_{k} \) and within each \( D_i \) the pools are visited sequentially. Starting with set \( D_1 \), using the definition of \( R_{i,j} \) we can write the transition rates for the components \( V_{m_i,d_{i,1}}, \ldots , V_{m_i,d_{i,|D_i|}} \) as

\[
V_{m_i,d_{i,l}} \rightarrow \begin{cases} V_{m_i,d_{i,l}+1}, & \text{at rate } \mu_{d_{i,l}}(x_{m_i,d_{i,l}} - x_{m_i+1,d_{i,l}}) \\ V_{m_i,d_{i,l} - 1}, & \text{at rate } \frac{\lambda_{d_{i,l}}}{Y_{d_{i,l}}} \mathbf{1}_{\{0 = V_{m_i,d_{i,l-1}} - \cdots = V_{m_i,d_{i,l-1}} < V_{m_i,d_{i,l}}\}} \end{cases}, \forall i \in [|D_i|]. \tag{81}
\]

Now for the sets \( D_2, D_3, \ldots , D_{k} \), we have the following properties of the down transition rates of components. From the definition of \( R_{i,j} \), we can write \( \mathbf{1}_{\{V \in R_{m_i,d_{i,l}}\}} \) of the component \( V_{m_i,d_{i,l}} \) for \( i \in \{2, \ldots , k\}, l \in [|D_i|] \) as

\[
\{V_{m_i,r} = 0 \ \forall r \in [d_{i,l} - 1], V_{m_i-1,r} = 0 \ \forall r \geq d_{i,l} + 1, 0 = V_{m_i-1,d_{i,l}} < V_{m_i,d_{i,l}}\}. \tag{82}
\]

Using (82), we note that the down rate of transition is zero under following conditions (i) if any pool faster than \( d_{i,l} \) belongs to the set \( D_{l-1} \), (ii) if any pool slower than \( d_{i,l} \) belongs to the set \( D_{l-1} \) and \( m_l - 1 > m_{l-1} \). Otherwise, the down rate of transition has non-zero value. Moreover, from (82) it is clear that the non-zero down rate of \( V_{m_i,d_{i,l}} \) depends only on the components which are already visited. Furthermore, the definition of \( R_{i,j} \) only checks whether the previous components are zero or not. Therefore, the non-zero down rate has the form as given in (43).

Now for the components whose down transition rate is zero we have \( \pi(x(V_{i,j} = \infty)) = 1 \) because of non-zero up transition rates. Hence, we can exclude these components from the chain \( C \). The chain \( C \) will have components whose transition rates have the same form as in (43). Suppose \( Y \) is the chain obtained after excluding all infinite components from \( C \) and has dimension \( H \leq M \). Now we use Lemma 3 to verify that \( \pi(x(V_{i,j} = \infty)) = 1 \) or 0 for the components in \( Y \).

For the ease of notations, we are using the rates given in (43) as the rates of components of the chain \( Y \). Also we are assuming that \( Y = (Y_{m_i',p_i}, Y_{m_i',p_i}, \ldots , Y_{m_i',p_i}) \), where \( p_i \)'s are the leftover pools from chain \( C \) with minimum queue lengths \( m_i' \) for \( i \in [H] \).

Using Lemma 3, we know that the chain \( Y \) is stable if \( \sum_{i \in [H]} \frac{v_i}{v_i} < 1 \), where \( v_l = \mu_{p_i}(x_{m_i',p_i} - x_{m_i'+1,p_i}) \) and \( \lambda_i = \frac{\lambda_{p_i}}{Y_{p_i}} \) for all \( i \in [H] \). Now suppose \( \rho_1 < 1 \), then the component \( Y_{m_i',p_i} \) is stable. Now we show that \( \pi(x(Y_{m_i',p_i} = \infty)) = 0 \) with contradiction. Assume \( \pi(x(Y_{m_i',p_i} = \infty)) = \epsilon \in (0, 1] \). Also, assume \( \pi_x \) to be a stationary distribution of the Markov chain \( Y \). Therefore, we can write

\[
\pi_x(R_{m_i',p_i}) = (1 - \epsilon)\pi_x(Y_{m_i',p_i} > 0) + \epsilon = (1 - \epsilon)\frac{v_1}{\lambda_1} + \epsilon,
\]
where we get $\bar{\pi}_x(Y_{m'_i, p_1} > 0) = \frac{\nu}{\lambda_1}$ using (44). Now substitute this in differential form of (10) at time $t$, we get

$$\frac{dx_{m'_i, p_1}(t)}{dt} = \frac{\lambda}{y_{p_1}} \left( (1 - e) \frac{v'_1}{\lambda_1} + e \right) - \mu_{p_1} (x_{m'_i, p_1}(t) - x_{m'_1 + 1, p_1}(t))$$

$$= e \left( \frac{\lambda}{y_{p_1}} - \mu_{p_1} (x_{m'_i, p_1}(t) - x_{m'_1 + 1, p_1}(t)) \right) > 0,$$

where the last step follows as $\frac{\nu}{\lambda_1} < 1$. But if $x_{m'_i, p_1}(t) = 1$, we must have $\frac{dx_{m'_i, p_1}(t)}{dt} < 0$ which leads to a contradiction. Therefore, we have $\pi_x(Y_{m'_i, p_1} = \infty) = 0$. Suppose $\rho_1 \geq 1$, then the component $Y_{m'_i, p_1}$ is unstable and we have $\bar{\pi}_x(Y_{m'_i, p_1} \geq l) = 1$ for all $l \geq 0$. This shows that $\pi_x(Y_{m'_i, p_1} = \infty) = 1$.

For the component $Y_{m'_2, p_2}$, suppose $\rho_2 < 1 - \rho_1$, then the component $Y_{m'_2, p_2}$ is stable and we get $\pi_x(Y_{m'_2, p_2} = \infty) = 0$ using the same contradiction as above. If $\rho_2 \geq 1 - \rho_1$, then it is unstable and we have $\pi_x(Y_{m'_2, p_2} \geq l) = 1$ for all $l \geq 0$. This shows that $\pi_x(Y_{m'_2, p_2} = \infty) = 1$. Similarly, for general $i \in [H]$, if $\rho_i < 1 - \sum_{k=1}^{i-1} \rho_k$ then the component $Y_{m'_i, p_i}$ is stable and we get $\pi_x(Y_{m'_i, p_i} = \infty) = 0$ using the same contradiction as above, else we have $\pi_x(Y_{m'_i, p_i} = \infty) = 1$. 