Determination of lattice functions and 2nd-order transfer matrix for High Power Cyclotron.

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Abstract. This paper describes the development of beam dynamic simulation code for cyclotron. Starting from a description of beam dynamics in the cyclotron, lattice functions were determined and the solutions for the 2nd-order nonlinear Hamiltonian were revised. Based on the description of beam dynamics in the cyclotron, simulation code was also developed for cyclotron design.

1. Introduction
Cyclotrons have a wide range of medical, industrial, and research applications. High power cyclotrons (HPCs) are being developed to seek CP violation in neutrino sector [1] and transmute nuclear waste [2].

The main characteristics of HPCs are strong vertical focusing against the vertical space charge force, and high energy gain through the radio-frequency (RF) system against narrow turn separation. To extend the space-charge limit, vertical focusing should be strengthened to resist the vertical space charge force. The absence of longitudinal focusing leads to accumulated energy spread. Because of the non-zero dispersion, longitudinally-dependent radial motion reduces turn separation. Furthermore, intuitively, the effects of longitudinal space charge forces further reduce turn separation.

In a high-intensity cyclotron, space-charge effects on particle dynamics must be known in detail. Numerical simulations have been performed successfully in many high-intensity cyclotrons [3, 4], but to get deep insight on beam dynamics during magnet design for HPCs other types simulation code are required. Problems to be solved include determining the necessary increase in vertical focusing against space charge force, the degree by which nonlinear effects increase according to high vertical focusing, and how the dispersion effect limits beam intensity.

Equilibrium orbit (EO) code [5, 6] is one of the programs required for such purposes. During design of cyclotron magnets, this program provides useful information on tunes, isochronism, etc. Unfortunately, these programs do not provide information about lattice functions, which are essential parameters during cyclotron magnet design. These programs also do not provide information about nonlinear effects, which are important in design of high vertical focusing magnets for HPCs. In this paper, we extract the information for lattice functions and consider a 2nd-order transfer-matrix from the Hamiltonian for an HPC. Based on the theoretical exploration on cyclotron beam dynamics, we also introduce developed a program for designing cyclotron magnets. Section 2 describes the Hamiltonian in the Cyclotron. Section 3 describes determination of the lattice function. Section 4 gives results from 2nd-order transfer-matrix form Hamiltonian. Section 5 presents conclusions.
2. Hamiltonian in Cyclotron

In a cyclotron, the magnet is mirror-symmetric around a mid-plane. With this symmetry, the magnetic field in polar coordinates is obtained by calculating the Hamiltonian equation with the new Hamiltonian given by

\[ B_r(r, \theta, 0) = 0, \]
\[ B_\theta(r, \theta, 0) = 0, \]
\[ B_r(r, \theta, z) = -B_z(r, \theta, -z), \]
\[ B_\theta(r, \theta, z) = -B_\theta(r, \theta, -z), \]
\[ B_z(r, \theta, z) = B_z(r, \theta, -z). \]  

If \( z/r \ll 1 \), the magnetic field can be expanded in terms of \( z \), and terms higher than the 2nd-order can be ignored. With \( \vec{v} \cdot \vec{B} = \vec{v} \times \vec{B} = 0 \) conditions in a cyclotron, and Eqs. (1), the three-dimensional magnetic fields is expanded such that \[ B_r = \frac{z}{r} \frac{\partial B}{\partial r}, \]
\[ B_\theta = \frac{z}{a_\theta} \frac{\partial B}{\partial \theta}, \]
\[ B_z = B - \frac{1}{2} z^2 \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r} \frac{\partial B}{\partial r} + \frac{1}{r^2} \frac{\partial^2 B}{\partial \theta^2} \right) \]  

where \( B \) is the z-directional magnetic field in a mid-plane (i.e., \( B = B_z(r, \theta, z = 0) \)). Assume that a particle with momentum \( \vec{p} \) and charge \( q \) moves in a magnetic field that has vector potential \( \vec{A} \); then the canonical momentum of a particle \( \vec{p} \) is

\[ \vec{p} = \vec{p} + q \vec{A}. \]  

and the action integral in cylindrical coordinates is expressed as

\[ S = \int_{t_0}^t \left( P_r \frac{dx}{dt} + P_\theta \frac{dy}{dt} + P_z \frac{dz}{dt} - A_r \right) dt \]
\[ = \int_{t_0}^t \left( P_r \frac{dr}{dt} + P_\theta \frac{d\theta}{dt} + P_z \frac{dz}{dt} - A_r \frac{dr}{dt} - \frac{1}{2} z^2 \delta^2 B + \frac{1}{4} \frac{\partial B}{\partial r} \right) d\theta. \]  

From Eq. (4), if angle \( \theta \) is chosen instead of time \( t \) as independent variable, then the new Hamiltonian conjugate variables are \[ -E (= H) ; t, \]
\[ P_\theta = rp_\theta + q r A_\theta; \theta, \]
\[ P_r = p_r + q A_r; r, \]
\[ P_z = p_z + q A_z; z \]  

and the new Hamiltonian is

\[ H = -P_\theta - rp_\theta - q r A_\theta = -r \sqrt{(P^2 - (P_r - q A_r)^2 - (P_z - q A_z)^2)} - q r A_\theta. \]  

If \( \tau = \omega_0 t \) where \( \omega_0 \) is revolution frequency and \( t \) is the travel time of particles, then if we use a magnetic field with a value divided by charge \( q \) (i.e., \( \vec{B} \rightarrow \vec{B}/q \)), then the equation of motion can be obtained by calculating the Hamiltonian equation with the new Hamiltonian of Eq. (6) \[ \frac{dr}{d\theta} = \frac{\partial H}{\partial P_r} = \frac{r(P_r - A_r)}{(P^2 - (P_r - q A_r)^2 - (P_z - q A_z)^2)} = \frac{r P_r}{p_\theta^2}, \]
\[ \frac{dz}{d\theta} = \frac{\partial H}{\partial P_z} = \frac{z^2 \delta^2 B (B - 1 z^2 \delta^2 B + \frac{1}{2} \frac{\partial B}{r \partial \theta}) + z p_z \frac{\partial B}{p_\theta \partial \theta}}{\frac{r^2}{(P^2 - (P_r - q A_r)^2 - (P_z - q A_z)^2)^2}} = \frac{z \delta^2 B}{r \frac{\partial \delta^2 B}{d \theta}}. \]  

where \( E \) is total energy, \( m \) is total mass, \( \gamma \) is the relativistic factor, and \( c \) is the speed of light. In an EO, \( z = p_z = 0 \). From Eq. (7), equations of EO are expressed as

\[ \frac{dr}{d\theta} = \frac{r P_r}{p_\theta}, \]
\[
\frac{dp_r}{d\theta} = p_\theta - rB, \\
\frac{dr}{d\theta} = \frac{yr}{m_0 v_0}.
\] (8)

We expanded coordinates such that \( r \rightarrow r + x, \) \( p_r \rightarrow p_r + p_x \) and \( \tau \rightarrow \tau + \chi \) where \( r, p_r \) and \( \tau \) are values in the EO. To find an EO, initial values can be efficiently determined by a straightforward iteration process which is an extension of the familiar Newton method \([1, 10]\). Equations of displaced orbit (i.e., deviation of particle orbit from EO) can be revealed replacing \( r, p_r \) and \( \tau \) with their expanded terms \( r + x, p_r + p_x \) and \( \tau + \chi \) in equation of motion (Eqs. 7). By expanding equations of displaced orbit in terms of \( x, p_x, z, p_z, \tau \) up to 2\(^{nd}\) order, the 2\(^{nd}\)-order equations of motion are

\[
\frac{dx}{d\theta} = \frac{p_x}{p_\theta} x + \frac{r p_x^2}{p_\theta^2} + \frac{p^2}{2 p_\theta^2} + \frac{3 r p_r p_x^2}{2 p_\theta^3} - \frac{1}{2} \frac{r p_r p_x}{p_\theta} + \frac{1}{2} r \left( \frac{\phi}{p_\theta} \right)^2 p_x^2 + \frac{\partial B}{\partial \theta} \frac{1}{p_\theta} z p_x,
\]

\[
\frac{dz}{d\theta} = \frac{r}{p_\theta} p_z + \frac{1}{p_\theta} p_x x + \frac{r p_r}{p_\theta} p_z p_x,
\]

\[
\frac{dp_x}{d\theta} = \left( r \frac{\partial B}{\partial r} - p_r \frac{\partial B}{\partial \theta} \right) x + \left( \frac{\partial B}{\partial r} + r \frac{\partial^2 B}{\partial r^2} - p_r \frac{\partial^2 B}{\partial r \partial \theta} \right) x z + \frac{p_r^2}{p_\theta} \frac{\partial B}{\partial \theta} z p_x^2,
\]

By neglecting all nonlinear terms in Eqs. (9), a linear equation of motion can be expressed as

\[
\frac{dx}{d\theta} = \frac{p_x}{p_\theta} x + \frac{r p_x^2}{p_\theta^2},
\]

\[
\frac{dp_x}{d\theta} = \frac{p_r}{p_\theta} p_x - \left( B + r \frac{\partial B}{\partial r} \right) x,
\]

\[
\frac{dz}{d\theta} = \frac{r}{p_\theta} p_z,
\]

\[
\frac{dp_z}{d\theta} = \left( \frac{\partial B}{\partial r} + p_r \frac{\partial B}{\partial \theta} \right) z.
\] (10)

Solutions of equation Eq. (10) can be written in matrix form

\[
\begin{pmatrix}
  \frac{dx}{d\theta} \\
  \frac{dp_x}{d\theta} \\
  \frac{dz}{d\theta} \\
  \frac{dp_z}{d\theta}
\end{pmatrix} =
\begin{pmatrix}
  X & X_1 & X_2 & X_3 & X_4 \\
  Y & Y_1 & Y_2 & Y_3 & Y_4 \\
  Z & Z_1 & Z_2 & Z_3 & Z_4 \\
  W & W_1 & W_2 & W_3 & W_4
\end{pmatrix}
\begin{pmatrix}
  x \\
  p_x \\
  z \\
  p_z
\end{pmatrix},
\]

\[
\begin{pmatrix}
  x_0 \\
  p_x_0 \\
  z_0 \\
  p_z_0
\end{pmatrix} =
\begin{pmatrix}
  \phi_0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \\
  \psi_0 & \psi_1 & \psi_2 & \psi_3 & \psi_4 \\
  \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 \\
  \eta_0 & \eta_1 & \eta_2 & \eta_3 & \eta_4
\end{pmatrix}
\begin{pmatrix}
  x_0 \\
  p_x_0 \\
  z_0 \\
  p_z_0
\end{pmatrix}.
\] (11)

Matrices \( X \) and \( Z \) can be calculated using numerical methods such as the Runge-Kutta method.

3. Lattice functions in Cyclotron

Components in the transfer matrices can be calculated by numerical methods when the magnetic field distribution is known. The field distribution (Fig. 1) from reference \([11]\) was used. Two important parameters in beam optics are the radial and the vertical focusing frequencies, which are defined as the number of betatron oscillations per revolution of a reference particle. These frequencies measure the degree of focusing in the radial and the vertical planes, respectively. Once an EO is found, the linearized equations of motion are integrated for one sector (or one turn) to compute the transfer matrices. The focusing frequencies \( \nu_r \) and \( \nu_z \) are then computed using

\[
\cos(2\pi \nu_r) = \frac{1}{2} Tr(M_r),
\]
where $M_r$ and $M_z$ are the transfer matrices for one turn and the symbol $\text{Tr}$ signifies the trace of the matrix [12]. The magnetic fields must be carefully designed to avoid harmful resonances during the entire acceleration process. For the four-sector cyclotron that we consider here, the following structure resonances must be considered to affect the beam:

\[
\begin{align*}
4\nu_r &= 4 \\
3\nu_z &= 4 \\
2\nu_z &= 4 \\

\nu_r - 2\nu_z &= 0 \\
2\nu_r + 2\nu_z &= 4.
\end{align*}
\]

(13)

The particle’s focusing frequencies diverge between injection and extraction (Figure 2).

During a transformation along a circle in a cyclotron, the phase ellipse will continuously change its form and orientation, but not its area at a given energy. In matrix formulation, the ellipse parameters, which are also called Twiss parameters, the transform from initial Twiss parameters is given by [12]

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}_y =
\begin{pmatrix}
Y_{11}^2 & -2Y_{11}Y_{12}^2 & Y_{12}^2 \\
-Y_{11}Y_{21}^2 & Y_{12}^2 + Y_{21}Y_{22}^2 & -Y_{12}Y_{22}^2 \\
Y_{21}^2 & -2Y_{21}Y_{22}^2 & Y_{22}^2
\end{pmatrix}\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}_0
\]

(14)

where element $Y_{ij}$ of the $i$ th transfer matrix indicates $X_{ij}$ for the radial plane and $Z_{ij}$ for the vertical plane. Generally, the particle trajectory and its derivative in Twiss parameters can be expressed as [12]

\[
\begin{pmatrix}
y \\
\rho_r
\end{pmatrix}_y =
\begin{pmatrix}
\sqrt{\beta/\beta_0} (\cos \mu + \alpha_0 \sin \mu) \\
-\sqrt{\beta/\beta_0} (\alpha - \alpha_0) \cos \mu + (1 + \alpha \alpha_0) \sin \mu \\
\sqrt{\beta/\beta_0} \sin \mu
\end{pmatrix}\begin{pmatrix}
y \\
\rho_r
\end{pmatrix}_0.
\]

(15)

If the cyclotron is a circular accelerator, a periodic condition can be imposed with $(\alpha = \alpha_0, \beta = \beta_0)$ after one turn, and the transfer matrix is given by

\[
\begin{pmatrix}
Y_{11} & Y_{12} \\
Y_{12} & Y_{22}
\end{pmatrix} = \begin{pmatrix}
\cos \mu + \alpha \sin \mu & \beta \sin \mu \\
-\gamma \sin \mu & \cos \mu - \alpha \sin \mu
\end{pmatrix}
\]

(16)

Here phase term $\mu = 2\pi \nu$ at $\theta = 2\pi$. The initial Twiss function can be determined when an element of the transfer-matrix (Eq. 11) and focusing frequencies (Eq. 12) are calculated numerically. Beta-functions (Figure 3) determined from the transfer matrix in Eq. (14) have useful information during magnet design for cyclophon by implying beam size variation from initially assumed beam. In the figure, beta-function shows four-fold symmetry because the cyclotron has four sectors.

A cyclotron exerts no phase-restore force, so energy gain accumulates until extraction. Especially, a space-charge force can dramatically distort the beam distribution in longitudinal phase space in an HPC. Therefore, the dispersion function is an important parameter in descriptions of beam dynamics in cyclotrons, because it indicates longitudinally-dependent radial motion. This function in a cyclotron is

\[
\eta(E) = \frac{X_{co}(E) - X_{co}(E + \Delta E)}{\delta}
\]

(17)

where $X_{co}$ is a closed orbit at given energy, and $\delta$ is momentum deviation between a given energy and the next energy. This dispersion function (Figure 4) also shows four-fold symmetry due to the four-sector cyclotron.

### 4. 2nd order transfer-matrix in Cyclotron

In Section 2, the second order equations of motion (Eq. (9)) were derived and solutions of linear equation were given with a matrix form (Eq. 11). By putting $1^{\text{st}}$-order solutions (Eq. 11) to nonlinear terms of Eq. (9), they are transformed from nonlinear equations to linear inhomogeneous equations;
defining where \( f, g, \text{ and } h \) (Table 1), approximate solutions can be achieved by solving the following inhomogeneous equations:

\[
\begin{align*}
\frac{dx}{d\theta} &= \frac{p_r}{p_\theta} x + \frac{r p_r}{p_\theta} p_x + f_{11} x_0^2 + f_{12} x_0 p_{x0} + f_{13} p_{x0}^2 + f_{14} x_0^2 + f_{15} z_0 p_{z0} + f_{16} p_{z0}^2 \\
\frac{dp_x}{d\theta} &= -\frac{p_r}{p_\theta} p_x - \left( B + \frac{\partial B}{\partial r} \right) x + f_{21} x_0^2 + f_{22} x_0 p_{x0} + f_{23} p_{x0}^2 + f_{24} x_0^2 + f_{25} z_0 p_{z0} + f_{26} p_{z0}^2 \\
\frac{dz}{d\theta} &= \frac{r}{p_\theta} p_x + g_{11} x_0 z_0 + g_{12} x_0 p_{z0} + g_{13} p_{z0}^2 + g_{14} x_0 p_{x0}, \\
\frac{dp_z}{d\theta} &= \left( r \frac{\partial B}{\partial r} - \frac{p_r}{p_\theta} \frac{\partial B}{\partial \theta} \right) z + g_{21} x_0 z_0 + g_{22} x_0 p_{z0} + g_{23} p_{z0}^2 + g_{24} x_0 p_{x0} p_{z0} \\
\frac{dr}{d\theta} &= \frac{yr}{p_\theta} x + \frac{yp_r r}{p_\theta} p_x + h_{11} x_0^2 + h_{21} x_0 p_{x0} + h_{31} p_{x0}^2 + h_{41} x_0^2 + h_{51} z_0 p_{z0} + h_{61} p_{z0}^2
\end{align*}
\]  

(18)

If Green’s function is defined as

\[
\Lambda_x(\theta, \theta') = X(\theta)X(\theta') \\
\Lambda_z(\theta, \theta') = Z(\theta)Z(\theta'),
\]

where \( X \) and \( Z \) are matrices in Eq. (11) and vectors \( V_x \) and \( V_z \) are given as

\[
V_x = \begin{pmatrix} x_0^2 \\ x_0 p_{x0} \\ p_{x0}^2 \\ z_0 p_{z0} \\ p_{z0}^2 \end{pmatrix}, \quad V_z = \begin{pmatrix} x_0 z_0 \\ x_0 p_{z0} \\ p_{x0} z_0 \\ p_{x0} p_{z0} \end{pmatrix},
\]

(20)

then solutions of Eq. (18) are expressed as

\[
\begin{align*}
\begin{pmatrix} x \\ p_x \end{pmatrix} &= \begin{pmatrix} X_{11}(\theta) & X_{12}(\theta) \\ X_{21}(\theta) & X_{22}(\theta) \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix} + \sum_{n=1}^{6} \left( \int_{0}^{\theta} \Lambda_x(\theta, \theta') \begin{pmatrix} f_{11}(\theta') \\ f_{12}(\theta') \end{pmatrix} d\theta' \right) V_{x,n} \\
\begin{pmatrix} z \\ p_z \end{pmatrix} &= \begin{pmatrix} Z_{11}(\theta) & Z_{12}(\theta) \\ Z_{21}(\theta) & Z_{22}(\theta) \end{pmatrix} \begin{pmatrix} z_0 \\ p_{z0} \end{pmatrix} + \sum_{n=1}^{4} \left( \int_{0}^{\theta} \Lambda_z(\theta, \theta') \begin{pmatrix} g_{11}(\theta') \\ g_{21}(\theta') \end{pmatrix} d\theta' \right) V_{z,n}.
\end{align*}
\]

(21)

Similarly, solution \( \tau \) can be described as

\[
\tau = \int_{0}^{\theta} \left( \frac{yr}{p_\theta} (X_{11}(\theta) x_0 + X_{12}(\theta) p_{x0}) + \frac{yp_r r}{p_\theta} (X_{21}(\theta) x_0 + X_{22}(\theta) p_{x0}) \right) d\theta' + \sum_{n=1}^{6} \left[ \int_{0}^{\theta} \left( \frac{yr}{p_\theta} a_{1n} + \frac{yp_r r}{p_\theta} a_{2n} + h_n \right) d\theta' \right] V_{x,n}.
\]

(22)

Particle tracking in a 13-MeV cyclotron was performed using the linear transfer-matrix method (Eq. 11), the 2\(^{nd}\)-order transfer-matrix method (Eq. 21) and the full Runge-Kutta method (Eq. 9) to yield phase-space ellipses (Figure 5). Agreement is better between the 2\(^{nd}\)-order transfer-matrix method and the full Runge-Kutta method than between the linear transfer method and the full Runge-Kutta method. However, the Runge-Kutta method is rather slow.

5. Conclusion

This paper has described development of a beam dynamic simulation code for a cyclotron. Lattice functions, which are useful parameters to describe beam dynamics in circular accelerator, were calculated from a transfer matrix for the first time in cyclotron beam dynamics. The 2\(^{nd}\)-order transfer matrix was explored, and the phase-space beam motion from the 2\(^{nd}\)-order transfer matrix was compared with the result by Runge-Kutta method. The phase-space ellipses from the 2\(^{nd}\)-order transfer-matrix
method agreed well with the result obtained using the full Runge-Kutta method. Based on these beam dynamics results, cyclotron simulation code has been developed; it will be available soon.

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References
[1] J. M. Conrad and M. H. Shaevitz, Phys. Rev. Lett. 104, 141802 (2010).
[2] H. A. Abderrahim et al., Tech. Rep., DOE (2010), Accelerator and Target Technology for Accelerator Driven Transmutation and Energy Production, URL: http://www.science.doe.gov/hep/files/pdfs/ADSWhitePaperFinal.pdf.
[3] J.J. Yang, A. Adelmann, M. Humbel, M. Seidel, and T. J. Zhang, Phys. Rev. Accel. Beams 13, 064201 (2010).
[4] L. M. Onischenko, et al., Proc. of NUKLEONIKA, 48 p. 45-48 (2003).
[5] M. M. Gordon, Part. Acc. 16, 40 (1984).
[6] M. M. Gordon, Notes for 2nd order transfer matrix code, internal NSCL momo, 1986.
[7] T. L. Hart, D. J. Summers, and K. M. Paul, Technical Report, University of Mississippi 2010.
[8] B. F. Milton, Ph.D. thesis, Michigan University, 1986.
[9] H. A. Hagedoorn and N. F. Verster, Nucl. Instrum. Methods 18-19, 201 (1962).
[10] G-R. Hahn, private communication.
[11] S. H. Shin, et al., Journal of the Korean Physical Society, p. 1045-1051, v. 45 (2004).
[12] H. Wiedemann, Particle Accelerator Physics I (Springer, Berlin, 1993), p. 196
Figure 1. Field distribution for sector focused cyclotron [11]. Here sector number is 4.

Figure 2. Structure-resonance diagram and excursions of the focusing frequencies from injection to extraction; (a) $2\nu_r + 2\nu_z = 4$, (b) $\nu_r - 2\nu_z = 0$, and (c) $4\nu_r = 4$. 
Figure 3. Determined beta-function along theta at 1 MeV. Beta-function shows 4 fold symmetry due to 4 sector cyclotron.

Figure 4. Determined dispersion-function along theta at 1 MeV. Dispersion-function also shows 4 fold symmetry due to 4 sector cyclotron.
Figure 5. Horizontal phase-space. Red dots represent the result by Runge-Kutta method. Black dots represent phase space beam motion from (a) linear transfer matrix (b) the second order transfer matrix.
Table 1. Specification of fast count kicker power supply.

| Coefficient | Value |
|-------------|-------|
| $f_{11}$    | $\frac{\rho^2}{p_0}X_{11}X_{21} + \frac{3}{2} \left( \frac{\rho r_p}{p_0} \right)^2 X_{21}$ |
| $f_{12}$    | $\frac{\rho^2}{p_0} (X_{11}X_{22} + X_{12}X_{21}) + 2 \times \frac{3}{2} \left( \frac{\rho r_p}{p_0} \right)^2 X_{21}X_{22}$ |
| $f_{13}$    | $\frac{\rho^2}{p_0} X_{12}X_{22} + \frac{3}{2} \left( \frac{\rho r_p}{p_0} \right)^2$ |
| $f_{14}$    | $\frac{1 \times r_p}{2 \ p_0} Z_{21}^2$ |
| $f_{15}$    | $\frac{1 \times r_p}{p_0} Z_{21}Z_{22}$ |
| $f_{16}$    | $\frac{1 \times r_p}{2 \ p_0} Z_{22}^2$ |
| $f_{21}$    | $\left( - \frac{\partial^2 B}{\partial r^2} - \frac{r \partial B}{\partial r^2} \right) X_{11} - \frac{1}{2} \frac{p_0^2}{p_0} X_{21}^2$ |
| $f_{22}$    | $\left( -2 \frac{\partial B}{\partial r} - \frac{r \partial^2 B}{\partial r^2} \right) X_{11}X_{12} - \frac{p_0^2}{p_0} X_{21}X_{22}$ |
| $f_{23}$    | $\left( - \frac{\partial B}{\partial r} - \frac{r \partial^2 B}{\partial r^2} \right) X_{12} - \frac{1}{2} \frac{p_0^2}{p_0} X_{22}^2$ |
| $f_{24}$    | $\frac{1}{2} r \left( \frac{\partial^2 B}{\partial r^2} + \frac{1 \times 2 \partial B}{r \partial r} + \frac{1 \times 2 \partial^2 B}{r^2 \partial \theta^2} \right) Z_{11}^2 + \frac{1 \times r_p}{2 \ p_0} \frac{p_0}{p_0} Z_{11}Z_{21} - \frac{1 \times \frac{1 \times 2 \partial^2 B}{r^2 \partial \theta^2}}{2 \ p_0} Z_{21}^2$ |
| $f_{25}$    | $\frac{1}{2} \frac{r}{r} \left( \frac{\partial^2 B}{\partial r^2} + \frac{1 \times 2 \partial B}{r \partial r} + \frac{1 \times 2 \partial^2 B}{r^2 \partial \theta^2} \right) Z_{12}^2 + \frac{1 \times r_p}{p_0} \frac{p_0}{p_0} Z_{12}Z_{22} - \frac{1 \times \frac{1 \times 2 \partial^2 B}{r^2 \partial \theta^2}}{2 \ p_0} Z_{22}^2$ |
| $g_{11}$    | $\frac{1}{p_0} X_{11}Z_{21} + \frac{1 \times \rho r_p}{p_0} \frac{p_0}{p_0} X_{21}Z_{21}$ |
| $g_{12}$    | $\frac{1}{p_0} X_{11}Z_{22} + \frac{1 \times \rho r_p}{p_0} \frac{p_0}{p_0} X_{21}Z_{22}$ |
| $g_{13}$    | $\frac{1}{p_0} X_{12}Z_{21} + \frac{1 \times \rho r_p}{p_0} \frac{p_0}{p_0} X_{22}Z_{21}$ |
| $g_{14}$    | $\frac{1}{p_0} X_{12}Z_{22} + \frac{1 \times \rho r_p}{p_0} \frac{p_0}{p_0} X_{22}Z_{22}$ |
| $g_{21}$    | $\left( \frac{\partial B}{\partial r} + \frac{r \partial^2 B}{\partial r^2} \right) X_{11}Z_{11} - \frac{p_0^2}{p_0} \frac{p_0}{p_0} X_{21}Z_{11}$ |
| $g_{12}$    | $\left( \frac{\partial B}{\partial r} + \frac{r \partial^2 B}{\partial r^2} \right) X_{11}Z_{12} - \frac{p_0^2}{p_0} \frac{p_0}{p_0} X_{21}Z_{12}$ |
| $g_{13}$    | $\left( \frac{\partial B}{\partial r} + \frac{r \partial^2 B}{\partial r^2} \right) X_{12}Z_{11} - \frac{p_0^2}{p_0} \frac{p_0}{p_0} X_{22}Z_{11}$ |
| $g_{14}$    | $\left( \frac{\partial B}{\partial r} + \frac{r \partial^2 B}{\partial r^2} \right) X_{12}Z_{12} - \frac{p_0^2}{p_0} \frac{p_0}{p_0} X_{22}Z_{12}$ |
| $h_1$       | $\frac{\gamma_p \rho^2}{p_0} X_{11}X_{21} + \frac{1}{2} \left( \frac{\gamma r}{p_0} + \frac{3 \times \gamma p r^2}{p_0} \right) X_{21}$ |
| $h_2$       | $\frac{\gamma_p \rho^2}{p_0} (X_{11}X_{22} + X_{12}X_{21}) + 2 \times \frac{1}{2} \left( \frac{\gamma r}{p_0} + \frac{3 \times \gamma p r^2}{p_0} \right) X_{21}X_{22}$ |
| $h_3$       | $\frac{\gamma_p \rho^2}{p_0} X_{12}X_{22} + \frac{1}{2} \left( \frac{\gamma r}{p_0} + \frac{3 \times \gamma p r^2}{p_0} \right) X_{22}$ |
| $h_4$       | $\frac{1 \times \gamma r}{p_0}$ |
| $h_5$       | $\frac{1 \times \gamma r}{p_0}$ |
| $h_6$       | $\frac{1 \times \gamma r}{p_0}$ |