QCD Corrected $1/m_b$ Contributions to $B\bar{B}$–Mixing*

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Abstract

We calculate the QCD corrected effective Hamiltonian for $B\bar{B}$–Mixing in heavy quark effective theory including corrections of the order $\Lambda_{QCD}/m_b$. The matrix elements of the subleading operators are estimated using the vacuum insertion assumption. We show that the major part of the subleading corrections may be absorbed into the heavy meson decay constant $f_B$; the remaining corrections are only due to QCD effects and give an enhancement of $\Delta M$ of 5%.

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1 Introduction

Oscillations between particle and antiparticle have been observed first in the neutral Kaon system [1] and later also in the neutral B meson system [2, 3]. Such oscillations are predicted by the standard model and proceed through the so called box diagrams depicted in fig. 1. Since the top quark as well as the W boson are heavy compared to the B meson it is convenient to describe the oscillations in an effective theory, where the top quark and the W boson are integrated out.

We shall concentrate in this paper on the oscillations in the neutral B meson system. The effective Hamiltonian, which is obtained after integrating out the top quark and the W boson, consists of only one operator

\[ H_{\text{eff}} = \frac{G_F^2}{\pi^2} |V_{tb}^* V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) O_0 \]  

where

\[ O_0 = (\bar{d}_L \gamma_\mu b)(\bar{d}_L \gamma^\mu b) \]  

and \( \Phi \) is a function which arises from integrating out the top quark and the W boson at the same scale [4]

\[ \Phi(x) = \frac{1}{4} + \frac{9}{4(1-x)} - \frac{3}{2(1-x)^2} - \frac{3}{2} \frac{x^2 \ln x}{(1-x)^3} \]  

To evaluate the mass shift \( \Delta M \) relevant for the oscillations, one needs to calculate the matrix element of this effective interaction between a neutral B meson and its antiparticle; it is given by

\[ \Delta M = \frac{1}{2m_B} \langle B^0 | H_{\text{eff}} | \bar{B}^0 \rangle \]

\[ = \frac{G_F^2}{2\pi^2} |V_{tb}^* V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) \frac{1}{m_B} \langle B^0 | O_0 | \bar{B}^0 \rangle \]  

\[ = \frac{G_F^2}{6\pi^2} |V_{tb}^* V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) \eta_{\text{QCD}}(\mu) f_B^2 B_B(\mu) m_B \]  

where \( f_B \) is the B meson decay constant defined by

\[ \langle 0 | \bar{d}_L \gamma_\mu \gamma_5 b | B^0 (p) \rangle = i f_B p_\mu. \]
The coefficient $\eta_{\text{QCD}}$ contains the short distance contribution of the QCD corrections which is calculable in perturbation theory. It depends on the factorization scale $\mu$ at which the short distance contributions become separated from the long distance ones; the non-perturbative long distance contributions are contained in the “bag parameter” $B_B$, which depends in such a way on $\mu$ that the matrix element of the Hamiltonian is scale independent.

As an estimate for the order of magnitude of the effect the so called “vacuum insertion” assumption has been frequently used in the past, which corresponds to the case $B_B = 1$. However, since $B_B$ is scale dependent this assumption has to be supplemented by a statement about the scale where it is assumed to be valid.

The short distance contribution from scaling down to a scale $m_b \leq \mu \leq M_W$ has been calculated in leading logarithmic approximation some time ago \[5\] and yields

$$\eta_{\text{QCD}}(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{6/23} m_b \leq \mu \leq M_W, \quad (6)$$

which gives a correction factor of $\eta_{\text{QCD}} \sim 0.85$ at the scale of the $b$ quark. The corrections induced by subleading logarithms which are to be considered in conjunction with the $O(\alpha_s)$ matching conditions at the $W$ scale have also been calculated \[6\].

If one could get some information on the parameter $B_B$ at a scale of the order of the $B$ quark mass, one could in fact predict the amplitude of the oscillations as a function of the CKM matrix elements and the top quark mass. However, at the present time the only non-perturbative information available is from lattice methods \[7\], which gives some information on the parameter $f_B^2 B_B$ at low scales\[1\]. Thus it is desirable to scale further down to these very low scales. This is possible by switching at the point $\mu = m_b$ to heavy quark effective theory \[8\], in which the $b$ quark is integrated out and replaced by a static color source. The leading logarithmic result for the scaling $\mu \leq m_b$, i.e. the resummation of logarithms $(\alpha_s \ln(m_b/\mu))^n$, has been calculated in \[9\]. The result is

$$\eta_{\text{QCD}}(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(m_b)} \right)^{12/25} \eta_{\text{QCD}}(m_b) \quad m_c \leq \mu \leq m_b \quad (7)$$

\[1\]In fact, the lattice results indicate that the “bag parameter” $B_B$ at these low scales is indeed close to the value $B_B = 1$ used in the vacuum insertion assumption \[1\].
\[
\eta_{\text{QCD}}(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right)^{12/27} \eta_{\text{QCD}}(m_c) \quad \mu \leq m_c.
\] (8)

This result is indeed very peculiar, since the anomalous dimension of the four fermion operator \( \mathcal{O}_0 \) is the sum of the anomalous dimensions of the two left handed currents involved. In fact, if factorization like e.g. vacuum insertion assumption takes place, one would expect this behaviour of the anomalous dimension to all orders. However, the QCD scaling below \( m_b \) has been calculated even to subleading order [10] and it seems that this factorization does not hold for the subleading result.

Heavy quark effective theory is a systematic expansion in inverse powers of the heavy quark mass. The purpose of the present paper is to extend the existing calculations of the scaling below \( m_b \) by including terms of the order \( 1/m_b \) into the effective Hamiltonian. The motivation is twofold. Firstly, there are indications from the lattice that the \( 1/m_b \) corrections could be large, even for the \( b \) quark. Secondly, this is the first attempt to deal with the \( 1/m_b \) corrections in a non-leptonic process, of which the Hamiltonian for \( B\bar{B} \) mixing is certainly the easiest to deal with.

In section 2 we shall discuss an appropriate operator basis for the \( 1/m_b \) corrections to \( B\bar{B} \) mixing and calculate the anomalous dimension matrix and the Wilson coefficients at some low scale \( \mu \). In section 3 we shall estimate the \( 1/m_b \) effects by using the vacuum insertion assumption and the results on the \( 1/m \) corrections for the heavy meson decay constants obtained from QCD sum rules [11, 12].

2 The Operator Basis in the Order \( 1/m_b \)

To leading and subleading order in \( 1/m \), the effective Lagrangian for a heavy quark moving with velocity \( v \) is given by [13] (we use \( D_\mu = \partial_\mu - igA_\mu T^a \))

\[
\mathcal{L} = \bar{h}_v^+ i v \cdot D h_v^+ + \bar{\rho}_v^+ h_v^+ + \bar{h}_v^+ \rho_v^+ \\
+ \frac{1}{2m} \left( \bar{h}_v^+ i D^\perp + \bar{\rho}_v^+ \right) \left( i D^\perp h_v^+ + R_v^+ \right)
\] (9)

where the transverse derivative \( D - \dot{x}v \cdot D \) is denoted as \( D^\perp \), the heavy quark field is defined by

\[
h_v^+(x) = e^{imv\cdot x} \frac{1 + \gamma_5}{2} b(x),
\] (10)
and the sources $\rho_i^+ + v$ and $R_i^+ + v$ for the heavy quark and antiquark fields with velocity $v$ have been retained. Since there is also a heavy antiquark involved, one has to use also the effective Lagrangian for an antiquark which can be obtained from (9) just by replacing $v$ with $-v$ and $h_i^+$ with
\begin{equation}
  h_i^-(x) = e^{-imv \cdot x} \frac{1 - \gamma^0}{2} b(x),
\end{equation}
along with the proper replacements of the sources: $\rho_i^+ \rightarrow \rho_i^-$, $R_i^+ \rightarrow R_i^-$. Using the quark-type parameterization for one factor in the effective Hamiltonian (2) and the antiquark-type parameterization in the other factor (using the nomenclature of [13]) one obtains the effective Hamiltonian density at the matching scale $\mu = m_b$ in terms of heavy quark fields
\begin{equation}
  H_{\text{eff}} = \frac{G_F^2}{\pi^2} |V_{tb}V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right)
  \times \left[ \eta_{\text{QCD}}(m_b) \mathcal{O}_0'(m_b) + \frac{1}{m_b} \left( a_1(m_b) X_1^+(m_b) + \sum_{i=1}^{3} c_i(m_b) \mathcal{O}_i^+(m_b) \right) \right].
\end{equation}
At the matching scale there are two local operators
\begin{align}
  \mathcal{O}'_0 &= (\bar{d}_L \gamma_\mu h_i^+) (\bar{d}_L \gamma^\mu h_i^-) \\
  X_1^+ &= (\bar{d}_L \gamma_\mu i D^\perp h_i^+) (\bar{d}_L \gamma^\mu h_i^-) + (\bar{d}_L \gamma_\mu h_i^-) (\bar{d}_L \gamma^\mu i D^\perp h_i^-),
\end{align}
and three nonlocal operators $\mathcal{O}_i^+$, which are time-ordered products originating from the $1/m$ piece of the Lagrangian:
\begin{align}
  \mathcal{O}_1^+ &= i \int d^4 x \, T \left[ \mathcal{O}_0'(0) (\bar{h}_v^+(i v \cdot D)^2 h_v^+) + \bar{h}_v^-(i v \cdot D)^2 h_v^- (x) \right] \\
  \mathcal{O}_2^+ &= i \int d^4 x \, T \left[ \mathcal{O}_0'(0) (\bar{h}_v^+(i D)^2 h_v^+) + \bar{h}_v^- (i D)^2 h_v^- (x) \right] \\
  \mathcal{O}_3^+ &= i \int d^4 x \, T \left[ \mathcal{O}_0'(0) (\bar{h}_v^+ \frac{g_2}{2} \sigma^{\mu\nu} G_{\mu\nu} h_v^+ + \bar{h}_v^- \frac{g_2}{2} \sigma^{\mu\nu} G_{\mu\nu} h_v^-) (x) \right].
\end{align}
The coefficients of the $1/m_b$ terms are
\begin{align}
  a_1(m_b) &= \frac{1}{2} \eta_{\text{QCD}}(m_b), \\
  c_2(m_b) &= c_3(m_b) = -c_1(m_b) = \frac{1}{2} \eta_{\text{QCD}}(m_b),
\end{align
In order to calculate the leading logarithmic QCD corrections to this effective Hamiltonian, one first has to set up a basis of operators bearing the correct dimension and quantum numbers. Since there are three independent momenta in the transition, and five independent gamma matrix structures possible, one arrives at a basis of 15 local operators:

\begin{align*}
P_1 & = (\bar{d}_L iv \cdot D h_v^+)(\bar{d}_L h_v^-) \\
P_2 & = (\bar{d}_L iD h_v^+)(\bar{d}_L h_v^-) \\
P_3 & = (\bar{d}_L iD \gamma^\mu h_v^+)(\bar{d}_L \gamma^\mu h_v^-) \\
P_4 & = (\bar{d}_L \gamma^\mu iv \cdot D h_v^+)(\bar{d}_L \gamma^\mu h_v^-) \\
P_5 & = i\epsilon_{\lambda\mu\nu\rho} v^\lambda (\bar{d}_L iD^\mu \gamma^\nu h_v^+)(\bar{d}_L \gamma^\rho h_v^-) \\
Q_1 & = (\bar{d}_L h_v^+)(\bar{d}_L iv \cdot D h_v^-) \\
Q_2 & = (\bar{d}_L \gamma^\mu h_v^+)(\bar{d}_L iD^\mu h_v^-) \\
Q_3 & = (\bar{d}_L h_v^+)(\bar{d}_L iD h_v^-) \\
Q_4 & = (\bar{d}_L \gamma^\mu h_v^+)(\bar{d}_L \gamma^\mu iv \cdot D h_v^-) \\
Q_5 & = i\epsilon_{\lambda\mu\nu\rho} v^\lambda (\bar{d}_L \gamma^\nu h_v^+)(\bar{d}_L iD^\mu \gamma^\rho h_v^-) \\
R_1 & = \left[i v \cdot \partial (\bar{d}_L h_v^+)\right](\bar{d}_L h_v^-) \\
R_2 & = \left[i\partial_\mu (\bar{d}_L \gamma^\mu h_v^+)\right](\bar{d}_L h_v^-) \\
R_3 & = \left[i\partial_\mu (\bar{d}_L h_v^+)\right](\bar{d}_L \gamma^\mu h_v^-) \\
R_4 & = \left[i v \cdot \partial (\bar{d}_L \gamma^\mu h_v^+)\right](\bar{d}_L \gamma^\mu h_v^-) \\
R_5 & = i\epsilon_{\lambda\mu\nu\rho} v^\lambda \left[i\partial^\mu (\bar{d}_L \gamma^\nu h_v^+)\right](\bar{d}_L \gamma^\rho h_v^-)
\end{align*}

Operators containing \(\gamma_5\) or \(\sigma\) matrices can be eliminated by the projection operators implicit in the \(d_L\) and \(h_v^\pm\) fields.

These operators can mix with each other through the diagrams of fig.2. In addition, there are also the diagrams of fig. 3 which introduce mixing of the nonlocal operators \(O_i^+\) with the local operators. Note that there is no mixing in the other direction since the local operators do not require the nonlocal ones as counterterms.

One can employ the symmetries of the effective Hamiltonian to simplify the mixing matrix. Firstly, the lowest order operator as well as the \(1/m_b\) cor-
rections in \((12)\) have the property of being symmetric under Fierz transformations, i.e. after exchanging the two light quarks one recovers the original operator by rearranging the indices of the gamma matrices\(^2\). Therefore it is useful to switch to an operator basis \(\{X_i, Y_j\}\) of definite Fierz parity:

\[
\mathcal{F}X_i = X_i \quad \text{and} \quad \mathcal{F}Y_j = -Y_j.
\] (17)

In the case at hand, there are seven linear combinations of \(P, Q,\) and \(R\) with positive and eight combinations with negative Fierz parity.

Secondly, the operators can be divided into a class of operators \(\{X_i^+, Y_j^+\}\) which are symmetric under the exchange \(v \rightarrow -v\), and another class of antisymmetric operators \(\{X_i^-, Y_j^-\}\). It turns out that there is no mixing between these two sets. Since the operators introduced by tree-level matching \((12)\) have positive parity both under Fierz and \(v \rightarrow -v\) transformations, the \(X^-\) and \(Y^-\) operators can be neglected altogether.

Thirdly, one can eliminate all operators which vanish because of the equations of motion

\[
v \cdot Dh_v^i = 0, \quad Dd_L = 0
\] (18)

since they do not contribute to physical matrix elements and do not mix with operators that do contribute.

The anomalous dimensions of the nonlocal operators is obtained simply by adding the anomalous dimension of the lowest order effective Hamiltonian \(\mathcal{O}_0^\prime\) and the anomalous dimensions of the \(1/m\) terms in the effective Lagrangian \([14]\). Since the former is just twice the anomalous dimension of the involved current, this pattern of factorization is found again for the nonlocal part of the recoil corrections to the four fermion matrix element.

Evaluating the one loop diagrams depicted in fig. 2 and 3 we obtain the result summarized in tab. (tab. 4): The mixing matrix of the relevant operators separates into two blocks. The chromomagnetic moment operator \(\mathcal{O}_3^+\) mixes with the Fierz symmetric operators

\[
X_1^+ = P_2 + P_3 - P_5 + Q_2 + Q_3 - Q_5 \\
X_2^+ = 4R_1 + R_4
\] (19)

the first one being the operator that already occurred in the matching conditions \([13]\). The kinetic energy operator \(\mathcal{O}_2^+\) requires as counterterm a Fierz

\(^2\) For the \(R_i\) operators which arise in the mixing, in order to obtain the Fierz transform a partial integration is also necessary.
Table 1: The anomalous dimension matrix of the relevant operators. A factor $\alpha_s/12\pi$ has been omitted in all elements.

antisymmetric operator which is renormalized multiplicatively

$$Y_1^+ = 2R_4.$$  \hfill (20)

With these definitions, the effective Hamiltonian at scales $\mu < m_b$ is given including the $1/m_b$ terms by

$$H_{\text{eff}} = \frac{G_F^2}{\pi^2} |V_{tb}V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) \left[ \eta_{\text{QCD}}(\mu) \mathcal{O}_0'(\mu) + \frac{1}{m_b} \left\{ c_2(\mu) \mathcal{O}_2^+ (\mu) + c_3(\mu) \mathcal{O}_3^+ (\mu) + a_1(\mu) X_1^+ (\mu) + a_2(\mu) X_2^+ (\mu) + b_1(\mu) Y_1^+ (\mu) \right\} \right].$$

The array of Wilson coefficients $\{c_k, a_i, b_j\}$ is the solution of the renormalization group equation

$$\mu \frac{d}{d\mu} \begin{pmatrix} c \\ a \\ b \end{pmatrix} + \frac{\alpha_s(\mu)}{12\pi} \gamma^T \begin{pmatrix} c \\ a \\ b \end{pmatrix} = 0 \hfill (22)$$

where $\gamma$ is the anomalous dimension matrix as listed in tab. 1.

The entries in the anomalous dimension matrix have been calculated in a general covariant gauge and have been found to be gauge independent, as it should be the case. However, there is a gauge dependence in the counterterms of the $\mathcal{O}_i$ operators corresponding to operators which vanish according to the equations of motion (18), indicating that these terms are unphysical.

The solution of (22) can be written in the form

$$\begin{pmatrix} c(\mu) \\ a(\mu) \\ b(\mu) \end{pmatrix} = \exp \left[ \frac{1}{2(33 - 2n_f)} \gamma^T \ln \frac{\alpha_s(\mu)}{\alpha_s(m_b)} \right] \begin{pmatrix} c(m_b) \\ a(m_b) \\ b(m_b) \end{pmatrix} \hfill (23)$$
where \( n_f = 4 \) for \( \mu > m_c \), and \( n_f = 3 \) for \( \mu < m_c \). With the initial conditions at \( \mu = m_b \) taken from (15) the solution (23) reads

\[
\begin{align*}
  c_2(\mu) &= \zeta(\mu)^{12} \\
  c_3(\mu) &= \zeta(\mu)^3 \\
  a_1(\mu) &= -\frac{9}{11} \zeta(\mu)^3 - \frac{4(413 + \sqrt{649})}{649(-27 + \sqrt{649})} \zeta(\mu)^{(45 - \sqrt{649})/4} \\
  &\quad + \frac{4(413 - \sqrt{649})}{649(27 + \sqrt{649})} \zeta(\mu)^{(45 + \sqrt{649})/4} \\
  a_2(\mu) &= \frac{14}{11} \zeta(\mu)^3 - \frac{40(295 + 11\sqrt{649})}{649(27 + \sqrt{649})} \zeta(\mu)^{(45 - \sqrt{649})/4} \\
  &\quad + \frac{40(295 - 11\sqrt{649})}{649(-27 + \sqrt{649})} \zeta(\mu)^{(45 + \sqrt{649})/4} \\
  b_1(\mu) &= -\zeta(\mu)^3 + \zeta(\mu)^{20}
\end{align*}
\]

(24)

with

\[
\zeta(\mu) = \left( \frac{\alpha_s(m_c)}{\alpha_s(m_b)} \right)^{1/25} \left( \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right)^{1/27}
\]

(25)

for \( \mu < m_c \). Inserting the values

\[
m_b = 5 \text{ GeV}, \quad m_c = 1.5 \text{ GeV}, \quad \Lambda_{\text{QCD},4} = 0.2 \text{ GeV}
\]

(26)

and the one loop approximation for \( \alpha_s \), we obtain at \( \mu = 1 \text{ GeV} \)

\[
(c_2, c_3, a_1, a_2, b_1) = (0.60, 0.47, 0.52, -0.16, -0.15)
\]

(27)

together with the value \( \eta_{\text{QCD}}(1 \text{ GeV}) = 1.20 \).

### 3 Estimate of the Matrix Elements

In order to state numerical results, one has to calculate the matrix elements of the operators as given above, evaluated between eigenstates of the lowest-order HQET Lagrangian since the mass dependence resides now in the coefficients. As long as a non-perturbative calculation is not available, one has to rely on model assumptions. We shall estimate all matrix elements
using the vacuum insertion assumption. As pointed out in the introduction
we expect this assumption to at least give the correct order of magnitude
since the parameter $B_B$ calculated from the lattice is indeed of order unity.

Using the vacuum insertion assumption for the matrix elements of the
effective Hamiltonian we shall encounter the same matrix elements as in the
discussion of the heavy meson decay constant $f_B$ which has been analyzed
to order $1/m_B$ using QCD sum rules [11, 12]. The meson decay constant
$f_B$ and its leading recoil corrections may be parameterized in terms of four
constants, which have been estimated in [11]. Both results for $f_B$ [11, 12] are
consistent and we shall use the values given in [11] for the four parameters.

The parameters describing the heavy meson decay constant are defined
by the following matrix elements
\begin{align}
\langle 0 | \overline{d} \Gamma h^+ | B^0 (v) \rangle &= \frac{1}{2} F(\mu) \text{Tr} \left\{ \Gamma M^+(v) \right\} \quad (28) \\
\langle 0 | i \partial_\mu (\overline{d} \Gamma h^+ ) | B^0 (v) \rangle &= \frac{1}{2} F(\mu) \bar{\Lambda} v_\mu \text{Tr} \left\{ \Gamma M^+(v) \right\} \quad (29) \\
i \int d^4x \langle 0 | T \left[ (d \Gamma h^+)(0) (\bar{h}^+_v (iD)^2 h^+_v)(x) \right] | B^0 (v) \rangle &= F(\mu) G_1(\mu) \text{Tr} \left\{ \Gamma M^+(v) \right\} \quad (30) \\
\int d^4x \langle 0 | T \left[ (d \Gamma h^+)(0) g \frac{2}{3} (\bar{h}^+_v \sigma_{\mu\nu} G^{\mu\nu} h^+_v)(x) \right] | B^0 (v) \rangle &= 6F(\mu) G_2(\mu) \text{Tr} \left\{ \Gamma M^+(v) \right\} \quad (31)
\end{align}

The constant $\bar{\Lambda}$ is the difference between the heavy quark and the heavy
meson mass; the other three constants $F(\mu), G_1(\mu)$ and $G_2(\mu)$ depend on the
renormalization point $\mu$. In [11] these constants have been estimated at the
low scale $\mu \sim 2\bar{\Lambda}$. The values obtained from QCD sum rules for the four
parameters are [11]
\begin{align}
\bar{\Lambda} &= 500 \text{ MeV}, \quad F(2\bar{\Lambda}) = 0.36 \text{ GeV}^{3/2} \quad (32) \\
G_1(2\bar{\Lambda}) &= -0.5 \text{ GeV}, \quad G_2(2\bar{\Lambda}) = -55 \text{ MeV}
\end{align}

where the effective value for $G_1$ as discussed in [11] has been used. Fur-
thermore, the matrix $M^+(v)$ appearing in eqs.(28-31) is the representation
matrix for a heavy pseudoscalar meson containing a heavy quark
\begin{align}
M^+(v) &= \frac{(-i)}{2} \sqrt{m_B (1 + \gamma_5)}. \quad (33)
\end{align}
Vacuum insertion for the lowest order operator $O'_0$ corresponds to the replacement

$$\langle B^0|O'_0|\bar{B}^0 \rangle \rightarrow \frac{4}{3}\langle B^0|\bar{d}_L\gamma_\mu h^-|0\rangle \langle 0|\bar{d}_L\gamma^\mu h^+|\bar{B}^0 \rangle,$$  \hspace{1cm} (34)

where the factor $4/3$ is a colour factor: The light quark operators may be contracted in two ways with the light quarks in the mesons which corresponds to taking the Fierz transform. One of the possibilities is colour singlet and contributes with a factor 1, the other one is a combination of colour singlet and octet, of which only the singlet contributes. This yields in total a factor of $4/3$ for a Fierz symmetric operator like $O'_0$. Similarly, one would obtain a factor of $2/3$ for a Fierz antisymmetric operator.

To evaluate this the matrix elements involving mesons containing a heavy anti-quark are needed. They are given by the same constants $\bar{\Lambda}, F(\mu), G_1(\mu)$ and $G_2(\mu)$ due to charge conjugation and are evaluated by replacing the matrix $M^+(v)$ by

$$M^-(v) = \frac{(-i)}{2} \sqrt{m_B}(1 - \not{v})\gamma_5.$$  \hspace{1cm} (35)

Thus we obtain for the matrix element of $O'_0$

$$\langle B^0|O'_0|\bar{B}^0 \rangle = \frac{1}{3} F^2(\mu) m_B.$$  \hspace{1cm} (36)

After vacuum insertion all the matrix elements of the local operators in the order $1/m_B P_i, Q_i$ and $R_i$ may be expressed in terms of $F(\mu)$ and $\bar{\Lambda}$. The non-vanishing matrix elements of the local operators are

$$\langle B^0|P_2|\bar{B}^0 \rangle = \langle B^0|Q_3|\bar{B}^0 \rangle = -\frac{5}{24}\bar{\Lambda}m_B F^2(\mu)$$

$$\langle B^0|P_3|\bar{B}^0 \rangle = \langle B^0|Q_2|\bar{B}^0 \rangle = -\frac{1}{24}\bar{\Lambda}m_B F^2(\mu)$$

$$\langle B^0|P_5|\bar{B}^0 \rangle = \langle B^0|Q_5|\bar{B}^0 \rangle = \frac{1}{12}\bar{\Lambda}m_B F^2(\mu)$$

$$\langle B^0|R_1|\bar{B}^0 \rangle = -\frac{7}{24}\bar{\Lambda}m_B F^2(\mu)$$

$$\langle B^0|R_2|\bar{B}^0 \rangle = -\langle B^0|R_3|\bar{B}^0 \rangle = -\frac{5}{24}\bar{\Lambda}m_B F^2(\mu)$$

$$\langle B^0|R_4|\bar{B}^0 \rangle = \frac{1}{6}\bar{\Lambda}m_B F^2(\mu)$$
where the two possible contractions of the light quark in the four fermion operators have been taken into account. For the local operators with a definite Fierz parity, the result is

$$\langle B_0^0 | X_1^+ | B_0^0 \rangle = -\frac{2}{3} \bar{\Lambda} m_B F^2(\mu)$$
$$\langle B_0^0 | X_2^+ | B_0^0 \rangle = -\Lambda m_B F^2(\mu)$$
$$\langle B_0^0 | Y_1^+ | B_0^0 \rangle = \frac{1}{3} \Lambda m_B F^2(\mu)$$

(38)

We shall now define the vacuum insertion for the non-local operators $O_i^+$. For the piece of $O_2$ containing the quark operators $h_v^+$ we define vacuum insertion by

$$i \int d^4x \langle B_0^0 | T \left[ O_0(0) (\bar{h}_v^+ (iD)^2 h_v^+) (x) \right] [\overline{B}^0(v)]$$

(39)

$$\rightarrow \frac{4}{3} \langle B_0^0 | \bar{d}_L \gamma_\mu h_v^- | 0 \rangle i \int d^4x \langle 0 | T \left[ (\bar{d}_L \gamma^\mu h_v^+)(0) (\bar{h}_v^+ (iD)^2 h_v^+) (x) \right] [\overline{B}^0(v)]$$

where the colour factor of $4/3$ is the same as in the lowest order piece $O_0'$. Thus this matrix element may be expressed in terms of $F(\mu)$ and $G_1(\mu)$. Analogously we define vacuum insertion for the matrix element of the non-local operator $O_3^+$ and obtain

$$\langle B_0^0 | O_2^+ | B_0^0 \rangle = \frac{4}{3} m_B F^2(\mu) G_1(\mu)$$

(40)

$$\langle B_0^0 | O_3^+ | B_0^0 \rangle = 8 m_B F^2(\mu) G_2(\mu)$$

while the matrix elements of $O_1$ vanish due to the equations of motion of the heavy quark.

Within the framework of the vacuum insertion assumption, we now can insert the expressions for the matrix elements (38) and (40) into the effective Hamiltonian (21):

$$\Delta M = \frac{G_F^2}{6\pi^2} |V_{tb}^* V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) \times F^2(\mu) \left[ \eta_{\text{QCD}}(\mu) + \frac{\Lambda}{m_b} \left\{ 4c_2(\mu) \frac{G_1(\mu)}{\Lambda} + 24c_3(\mu) \frac{G_2(\mu)}{\Lambda} \right. \right.
\left. - 2a_1(\mu) - 3a_2(\mu) + b_1(\mu) \right\}$$

(41)
With the numerical values taken from (27) and (32), appropriate for the scale \( \mu = 1 \) GeV, we obtain

\[
\Delta M = \frac{G_F^2}{6\pi^2} |V_{tb}^\ast V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) F^2(1 \text{ GeV}) \left( 1.20 - 4.4 \frac{\bar{\Lambda}}{m_b} \right)
\]

\[= \frac{G_F^2}{6\pi^2 m_B} |V_{tb}^\ast V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) 0.51 \text{ GeV}^4 \] (42)

Thus, with the parameters taken from QCD sum rule analysis, the 1/m\(_b\) terms amount to a large correction of about -40% in this case. The most important 1/m\(_b\) corrections arise from the nonlocal operators; furthermore, the fact that \( G_1 \) is not well known [11] introduces a large uncertainty in the final result.

However, the matrix elements of the nonlocal operators factorize in leading log order in the same way as the leading operator \( \mathcal{O}_0' \) does, because the diagrams of fig. 3 which could lead to non-factorizable contributions only introduce local operators as counterterms, and the local operators in turn do not mix with the nonlocal ones. Therefore one can easily absorb these contributions into the pseudoscalar decay constant \( f_B \), if the contributions of subleading local operators to \( f_B \) are taken into account properly.

Explicitly, the square of the pseudoscalar decay constant is given in leading log approximation up to order 1/m\(_b\) by [11, 15]

\[f_B^2 = \frac{F^2(\mu)}{m_B} \zeta^{12}(\mu) \left( 1 + 2 \frac{G_1(\mu)}{m_b} + 12 \zeta^{-9}(\mu) \frac{G_2(\mu)}{m_b} - \frac{\bar{\Lambda}}{m_b} [1 + d(\mu)] \right), \] (43)

where for \( \mu < m_c \)

\[\zeta(\mu) = \left( \frac{\alpha_s(m_c)}{\alpha_s(m_b)} \right)^{1/25} \left( \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right)^{1/27}. \] (44)

The radiative corrections in the local term

\[d(\mu) = \frac{16}{9} (\zeta^{-9}(\mu) - 1 - \ln \zeta^{-9}(\mu)) \] (45)

remain very small for reasonable values of \( \mu \). In particular, \( d(1 \text{ GeV}) = 0.05 \).

Comparing the coefficients of the lowest order operator and the nonlocal 1/m\(_b\) terms in (11) and (13) shows that these terms factorize and we have

\[\eta_{QCD}(\mu) = c_2(\mu) = \zeta^{12}(\mu) \eta_{QCD}(m_b) \]

\[c_3(\mu) = \zeta^3(\mu) \eta_{QCD}(m_b). \] (46)
This may be used to rewrite \((41)\) by absorbing the large nonlocal \(1/m_b\) contributions into \(f_B^2\):

\[
\Delta M = \frac{G_F^2}{6\pi^2} |V_{tb}^* V_{td}|^2 m_t^2 \Phi \left( \frac{m_t^2}{M_W^2} \right) \eta_{\text{QCD}}(m_b) f_B^2 m_B \times \left( 1 + \frac{\Lambda}{m_b} \left[ 1 + d(\mu) \right. \right.
\]
\[
\left. \left. + \frac{\zeta_{-12}^{12}(\mu)}{\eta_{\text{QCD}}(m_b)} \left( -2a_1(\mu) - 3a_2(\mu) - a_3(\mu) - b_1(\mu) + b_2(\mu) \right) \right] \right). \tag{47}
\]

This expression has the form of eq. \((4)\) from which we may read off the bag factor

\[B_B(m_b) = 1 + 0.45 \frac{\Lambda}{m_b} = 1.05 \tag{48}\]

where \(\mu = 1\) GeV has been taken as the scale where vacuum insertion is assumed to be valid for the subleading matrix elements.

In fact, the leading log result only implies that in the static limit \(B_B(\mu)\) is scale independent for \(\Lambda < \mu < m_b\), where \(\Lambda\) is the scale where perturbation theory breaks down. However, the static value \(B_{B_{\text{stat}}}^B\) of \(B_B\) is not fixed and we have simply set its value to unity. For an arbitrary value of \(B_{B_{\text{stat}}}^B\) our results for \(\Delta M\) would be multiplied by this factor. In particular, eq. \((48)\) also is multiplied by \(B_{B_{\text{stat}}}^B\) and thus we have calculated the \(1/m_b\) contributions to \(B_B\) as defined in \((4)\). We note that these corrections can be calculated perturbatively.

### 4 Discussion of the results

We presented a complete calculation of the leading log QCD corrections to order \(1/m_b\) for the effective Hamiltonian relevant for \(B\bar{B}\) mixing. The matrix elements have been estimated using the vacuum insertion assumption which reduces matrix elements of four fermion operators to a product of current matrix elements between the meson and the vacuum. The latter have recently been estimated using the QCD sum rule approach \((1)\). As far as \(f_B\) is concerned the \(1/m_b\) corrections according to \((1)\) are indeed large; however, in our analysis we have absorbed these large corrections into \(f_B\).

The remaining corrections are calculable subleading corrections to the bag.

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parameter which are of the order \((\alpha_s/\pi)(\bar{\Lambda}/m_b) \ln(m_b/\mu)\) with \(\mu = 2\bar{\Lambda}\). They arise only due to QCD effects and hence are small. We find an enhancement factor for \(B_B\) of 1.05.

We may also compare the results to experiment [16]. Using a value for \(x\)

\[
x = \Delta M \cdot \tau_b = 0.67 \pm 0.10
\]  

we may study the possible values of \(|V_{td}|\) as a function of the top quark mass. Keeping in mind the above assumptions and using a value of \(f_B\) obtained from the lattice [7]

\[
f_B = 205 \text{ MeV}
\]

we find

\[
0.008 \leq |V_{td}| \leq 0.015.
\]  

The range given is due to a variation of the top quark mass between 110 and 170 GeV; however, we have not taken into account the uncertainties in the \(f_B\) determination from the lattice (\(\pm 40 \text{ MeV}\)) and the experimental error in \(x\).

Finally, we want to stress that the major part of our calculation as given in section 2 does not rely on vacuum insertion assumption. Eventually, the matrix elements of the four fermion operators should be evaluated non-perturbatively, e.g. by lattice calculations. Since lattice calculations work best at low scales, our formulae are needed to connect the low scales with the \(b\) mass scale where the matching to the high energy theory is done.

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Figure 1: Feynman diagrams inducing transitions between a neutral $B$ meson and its antiparticle.

Figure 2: Diagrams with insertions of the local operators
Figure 3: Diagrams with insertions of the nonlocal operators