DERIVATION OF 3D ENERGY-CRITICAL NONLINEAR SCHröDINGER EQUATION AND BOGOLIUBOV EXCITATIONS FOR BOSE GASES

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Abstract. We derive the 3D quintic NLS as the mean field limit of a Bose gas with three-body interactions. The quintic NLS is energy-critical, leading to several new difficulties in comparison with the cubic NLS which emerges from Bose gases with pair-interactions. Our method is based on Bogoliubov’s approximation, which also provides the information on the fluctuations around the condensate in terms of a norm approximation for the N-body wave function.

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1. Introduction

The 3D energy-critical nonlinear Schrödinger equation (NLS) reads

\[
\begin{aligned}
  i\partial_t \varphi(t, x) &= -\Delta \varphi(t, x) + b_0 |\varphi(t, x)|^4 \varphi(t, x), & x \in \mathbb{R}^3, \ t > 0 \\
  \varphi(0, x) &= \varphi_0(x).
\end{aligned}
\]

(1)

The well-posedness of (1) in the defocusing case \(b_0 > 0\) was first proved by Bourgain [9] and Grillakis [24] for radial data and then by Colliander, Keel, Staffilani, Takaoka, and Tao [17] for general data. In this paper, we will derive (1) as a macroscopic description for the microscopic many-body Schrödinger equation of bosons in a mean-field limit.

From first principles of quantum mechanics, the dynamics of a Bose gas in 3D with \(N\) particles is described by the \(N\)-body Schrödinger equation

\[
\begin{aligned}
  i\partial_t \Psi_N(t) &= H_N \Psi_N(t), \\
  \Psi_N(t = 0) &= \Psi_{N,0}.
\end{aligned}
\]

(2)

Here \(\Psi_N(t)\) is a wave function in the symmetric space \(L^2_s((\mathbb{R}^3)^N)\) and \(H_N\) is the Hamiltonian of the system. In this paper, we consider the case of non-relativistic bosons interacting via a three-body interaction potential,

\[
H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N^2} \sum_{1 \leq i < j < k \leq N} N^{6\beta} V(N^{\beta} (x - y), N^{\beta} (x - z)).
\]

(3)

Here \(V : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\) has the following symmetry condition

\[
V(x, y) = V(y, x), \quad V(x - y, x - z) = V(y - x, y - z) = V(z - y, z - x).
\]

(4)

Note that for any fixed parameter \(\beta > 0\), in the limit \(N \to \infty\) the re-scaled potential

\[
V_N(x - y, x - z) = N^{6\beta} V(N^{\beta} (x - y), N^{\beta} (x - z))
\]

(5)

converges weakly to the delta interaction

\[
b_0 \delta_{x=y=z}, \quad b_0 = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} V(x, y) \, dx \, dy.
\]

(6)

The bigger \(\beta\) is, the more singular the potential is. The coupling constant \(N^{-2}\) in front of the interaction terms in (3) places us in the mean-field regime, when the kinetic energy and the interaction energy are comparable in the large \(N\) limit.

We are interested in the macroscopic behavior of the system when \(N \to \infty\). To the leading order we expect the system to exhibit the Bose-Einstein condensation. This is the phenomenon when most of particles occupy a common single quantum state, namely in terms of the wave function

\[
\Psi_N(t) \approx \varphi(t)^\otimes N
\]

(7)

in an appropriate sense, for a function \(\varphi(t)\) in \(L^2(\mathbb{R}^3)\). A formal computation using the limiting interaction potential (6) suggests that \(\varphi(t)\) solves the quintic NLS (1). Making this computation rigorous, however, is a nontrivial problem.

In the present paper, we will justify the approximation (7) (with \(\varphi(t)\) solving the quintic NLS (1)) for all \(0 < \beta < 1/6\), leading to an extension of the recent important result of X. Chen and Holmer [16] (see also T. Chen and Pavlović [14] for related results in lower dimensions). Moreover, we will go beyond the leading order and obtain information on the fluctuations around the condensate, in terms of a norm approximation for the wave...
function. In particular, we will also extend the norm approximation obtained by X. Chen [15] in the mean-field case \( \beta = 0 \).

When the particles interact only via pair interactions, the condensate should be effectively described by the cubic NLS (instead of the quintic NLS (1)). The well-posedness of the defocusing cubic NLS has been proved by Bourgain [8] and Dodson [19]. The rigorous derivation of the cubic NLS from many-body Schrödinger equation is the subject of a vast literature; see [49, 3, 1, 20, 21, 22, 33, 17, 31, 45, 41, 13, 10]. In particular, we refer to the seminal work of Erdös, Schlein and Yau [21] on the critical case \( \beta = 1 \), where the cubic NLS is replaced by the Gross-Pitaevskii equation with the true scattering length of the pair interaction (see also [33, 4, 45, 13, 10] for later developments). The norm approximation with pair interactions has also attracted many studies [30, 23, 27, 25, 37, 46, 40, 5, 35, 41, 11]; in particular, we refer to [11] for the last development which covers all \( 0 < \beta < 1 \) (the case \( \beta = 1 \) remains open).

We will benefit from the methods developed to handle the pair-interaction case, in particular the justification of Bogoliubov’s argument [6] in [37, 40, 41, 11]. However, it turns out that the analysis in the case of three-body interactions is significantly more complicated and several new ideas are needed. Our main results are presented in the next section.

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2. Main results

2.1. Convergence of reduced density matrices. The proper meaning of the Bose-Einstein condensation [7],

\[
\Psi_N(t) \approx \varphi(t) \otimes N,
\]

should be given in terms of the reduced density matrices. Recall that the one-body density matrix \( \gamma_{\Psi_N}^{(1)} \) of a \( N \)-body wave function \( \Psi_N \) is a non-negative trace class operator on \( L^2(\mathbb{R}^3) \) with kernel

\[
\gamma_{\Psi_N}(x; y) = N \int \Psi_N(x, x_2, \ldots, x_N) \overline{\Psi_N(y, x_2, \ldots, x_N)} \, dx_2 \cdots dx_N.
\]

(8)

(We use the convention that inner products in Hilbert spaces are linear in the first argument and anti-linear in the second.)

The precise meaning of (7) reads, in the limit \( N \to \infty \),

\[
\frac{\langle \varphi(t), \gamma_{\Psi_N}^{(1)} \varphi(t) \rangle}{N} \to 1,
\]

(9)

namely the expectation of the number of particles in mode \( \varphi(t) \) is mostly equal to \( N \). Equivalently, we can rewrite (9) as

\[
\frac{\text{Tr}(Q(t) \gamma_{\Psi_N}^{(1)} Q(t))}{N} \to 0, \quad Q(t) = 1 - |\varphi(t)\rangle\langle \varphi(t)|.
\]

(10)

Moreover, since \( |\varphi(t)\rangle\langle \varphi(t)| \) is a rank-one projection, (9)–(10) is equivalent to the trace class convergence

\[
\text{Tr} \left| N^{-1} \gamma_{\Psi_N}^{(1)} - |\varphi(t)\rangle\langle \varphi(t)| \right| \to 0.
\]

(11)

As explained in the introduction, it is natural to expect that \( \varphi(t) \) solves the quintic NLS [1]. Our first result is a rigorous justification of this fact.
Theorem 1 (Convergence of reduced density matrices). Assume that \( 0 \leq V \in C_c(\mathbb{R}^6) \). Let \( \varphi(t) \) be the solution to the quintic NLS \( (1) \) with initial function \( \varphi(0) \in H^1(\mathbb{R}^3), \| \varphi(0) \|_{L^2} = 1 \). Let \( \Psi_N(t) \) be the solution to the Schrödinger equation \( (2) \) with the initial state \( \Psi_{N,0} \) in \( L^2_2((\mathbb{R}^3)^N) \) satisfying

\[
\text{Tr} \left[ (1 - \Delta) Q(0) \gamma_{\Psi_{N,0}}^{(1)} Q(0) \right] \leq C. \tag{12}
\]

Assume that \( 0 < \beta < 1/6 \). Then for all \( t > 0 \) and for all \( \alpha < \min \{ \beta/2, (1 - 6\beta)/4 \} \) fixed, we have

\[
\text{Tr} \left| N^{-1} \gamma_{\Psi_N(t)}^{(1)} - |\varphi(t)\rangle \langle \varphi(t)| \right| \leq C t N^{-\alpha}. \tag{13}
\]

Here \( C_t \) is continuous in \( t \) and independent of \( N \).

This kind of results was obtained recently by X. Chen and Holmer \[16\] for bosons on the torus \( T^3 \) and for \( 0 < \beta < 1/9 \). In fact, our result can be extended to the torus case as well (see Appendix for further explanation). Unlike the BBGKY approach in \[16\], our method gives explicit error estimate and can be adapted easily when an external potential or a magnetic field appears. On the other hand, here we do not cover the case \( \varphi(0) \in H^1(\mathbb{R}^3) \) as in \[16\] which is of certain mathematical interest. We rather think of the physical situation when the initial state \( \Psi_N(0) \) is the ground state of a trapped system, and in this case the condensate \( \varphi(0) \) is expected to be sufficiently regular.

The assumption \( 0 < \beta < 1/6 \) in Theorem 1 is a technical condition which allows us to control the interaction potential by the kinetic operator. Note that the total interaction potential felt by the \( i \)-th particle is

\[
\frac{1}{N^2} \sum_{j,k: j \neq i, k \neq i} N^{6\beta} V(N^\beta (x_i - x_j), N^\beta (x_i - x_k)).
\]

Thus the total interaction potential felt by a single particle may be as large as \( N^{6\beta} \) in the worst case (when all particles collapse to a singular point). In our method, we need to control the potential energy per particle by the total kinetic energy of all particles which is normally of order \( N \) (since the system occupies a volume of order 1). This requires \( N^{6\beta} \ll N \), namely \( \beta < 1/6 \). This condition is reminiscent of the well-known threshold \( \beta < 1/3 \) in the pair-interaction case \[20, 45, 25, 40\] where the total interaction potential felt by the \( i \)-th particle is

\[
\frac{1}{N} \sum_{j \neq i} N^{3\beta} \tilde{V}(N^\beta (x_i - x_j))
\]

which is as large as \( N^{3\beta} \) in the worst case. We will come back to the explanation for the smallness condition on \( \beta \) with more details later.

It is natural to expect that the result in Theorem 1 holds true for larger \( \beta \)'s up to a critical value (possibly \( \beta = 1 \)) where some subtle correction emerges to the leading order due to few-particle scattering processes. However, achieving larger \( \beta \)'s will need a substantial improvement of the current method.

2.2. Norm approximation. To describe the fluctuations around the condensate, it is convenient to switch to a Fock space representation where the number of particles is not fixed. We define the bosonic Fock space

\[
\mathcal{F} = \bigoplus_{n \geq 0} L^2_2((\mathbb{R}^{3n})).
\]
The creation operator $a^*(f)$ and the annihilation operator $a(f)$, for some $f \in L^2(\mathbb{R}^3)$, are defined as

$$(a^*(f)\Psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\Psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}),$$

$$(a(f)\Psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int f(x_n)\Psi^{(n+1)}(x_1, \ldots, x_n, x_{n+1}) \, dx_{n+1}$$

for all $\Psi \in \mathcal{F}$. They satisfy the canonical commutation relations (CCR)

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0, \quad [a(f), a^*(g)] = \langle f, g \rangle, \quad \forall f, g \in L^2(\mathbb{R}^3). \quad (14)$$

We will also use the operator-valued distributions $a_x^*$ and $a_x$ defined via

$$a^*(f) = \int_{\mathbb{R}^3} f(x)a_x^* \, dx, \quad a(f) = \int_{\mathbb{R}^3} f(x)a_x \, dx, \quad \forall f \in L^2(\mathbb{R}^3), \quad (15)$$

which satisfy the CCR

$$[a_x^*, a_y] = [a_x, a_y] = 0, \quad [a_x, a_y^*] = \delta(x - y), \quad \forall x, y \in \mathbb{R}^3.$$

The creation and annihilation operators provide a convenient way to express many operators on Fock space in compact forms. For example, if $A$ is a self-adjoint operator on the one-particle space $L^2(\mathbb{R}^3)$ with kernel $A(x, y)$, then we can define its quantization on Fock space by

$$d\Gamma(A) = \bigoplus_{n=0}^{\infty} \sum_{j=0}^{n} A_j = \int \int A(x; y)a_x^*a_y \, dx \, dy.$$

In particular, the number operator is

$$\mathcal{N} = d\Gamma(1) = \int a_x^*a_x \, dx$$

and the $N$-body Hamiltonian $H_N$ can be extended to an operator on Fock space as

$$H_N = d\Gamma(-\Delta) + \frac{1}{6N^2} \int \int \int V_N(x, y, z)a_x^*a_y^*a_xa_ya_z \, dx \, dy \, dz. \quad (16)$$

Now we want to go further to analyze the fluctuations around the condensate. We are interested in the norm approximation of the form

$$\Psi_N(t) \approx \sum_{n=0}^{N} u(t)^{\otimes(N-n)} \otimes_s \psi_n(t); \quad \psi_n(t) := \sum_{n=0}^{N} \frac{(a^*(u(t)))^{N-n}}{(N-n)!} \psi_n(t)$$

where

$$\Phi(t) = (\psi_n(t))_{n=0}^{\infty} \in \mathcal{F}_+(t) = \bigoplus_{n=0}^{\infty} \left(\{u(t)\perp\right)^{\otimes n}$$

describes excited particles, with

$$\{u(t)\perp = Q(t)L^2(\mathbb{R}^3), \quad Q(t) = 1 - |u(t)\rangle\langle u(t)|.$$

The natural candidate for the condensate state $u(t)$ can be obtained by formally inserting the purely uncorrelated ansatz $u(t)^{\otimes N}$ into the Schrödinger equation, leading to the quintic Hartree equation

$$\begin{cases} 
  i\partial_t u(t, x) = \left(-\Delta + \frac{1}{2} \int \int |u(t, y)|^2 V_N(x - y, x - z)|u(t, z)|^2 \, dy \, dz\right) u(t, x) \\
  u(0, x) = u_0(x).
\end{cases} \quad (18)$$
We have ignored the $N$-dependence in the notation of $u(t)$ for simplicity. The weak convergence $[11]$ implies that $u(t)$ converges to the solution $\varphi(t)$ of the quintic NLS $[11]$ when $N \to \infty$. The difference between $u(t)$ and $\varphi(t)$ is not visible in the leading order of the condensate (Theorem $1$). However, the choice of $u(t)$ is better for the refined estimate $[17]$. The behavior of $u(t)$ can be controlled in a uniform way in $N$ (see Section $3$ for details).

Bogoliubov’s approximation $[6]$ suggests that the excited state $\Phi(t) = (\psi_n(t))_{n=0}^\infty$ is determined by the Bogoliubov equation

\[
\begin{cases}
  i\partial_t \Phi(t) = \mathbb{H}(t) \Phi(t), \\
  \Phi(0) = \Phi_0
\end{cases}
\]  

with the quadratic generator

\[\mathbb{H}(t) = d\Gamma(h + K_1) + \frac{1}{2} \iint \left( K_2(t, x, y) a_x^* a_y^* + K_2(t, x, y) a_x a_y \right).\]

Here

\[h(t) := -\Delta + \frac{1}{2} \iint V_N(x - y, x - z)|u(t, y)|^2|u(t, z)|^2 \, dy \, dz\]

is the one-body Hartree operator appearing in $[13]$, and

\[K_1 := Q(t) \tilde{K}_1 Q(t), \quad \tilde{K}_1[f](x) = \iint V_N(x - y, x - z)|u(t, y)|^2u(t, x)f(y) \, dy \, dz,\]

\[K_2 := \left( Q(t) \otimes Q(t) \right) \tilde{K}_2, \quad \tilde{K}_2(x, y) = \left( \int V_N(x - y, x - z)|u(t, z)|^2 \, dz \right) u(t, x)u(t, y).\]

The existence and uniqueness of the solution to the Bogoliubov equation $[19]$ is well-known $[37]$ (see Section $4$ for further discussions).

Our second result is a rigorous derivation for the Bogoliubov equation $[19]$.

**Theorem 2** (Norm approximation). Assume that $0 \leq V \in C_0(\mathbb{R}^6)$. Let $u(t)$ be the Hartree dynamics $[13]$ with an initial state $u(0) \in H^1(\mathbb{R}^3)$, $\|u(0)\|_{L^2} = 1$. Let $\Phi(t) = (\psi_n(t))_{n=0}^\infty$ be the Bogoliubov dynamics $[19]$ such that the initial state $\Phi(0)$ satisfies

\[\Phi_0 \in \bigoplus_{n=0}^\infty \left( \{u(0)\}^\perp \right)^\otimes_n, \quad \|\Phi(0)\| = 1, \quad \langle \Phi_0, d\Gamma(1 - \Delta)\Phi_0 \rangle \leq C.\]

Let $\Psi_N(t)$ be the Schrödinger dynamics $[2]$ with the initial state

\[\Psi_{N,0} = \sum_{n=0}^N u(0)^\otimes(N-n) \otimes_s \psi_n(0).\]

Assume that $0 < \beta < 1/6$. Then for all $t > 0$ and for all $\alpha < (1 - 6\beta)/4$ fixed, we have

\[\left\| \Psi_N(t) - e^{-i\int_0^t \chi(s) \, ds} \sum_{n=0}^N u(t)^\otimes(N-n) \otimes_s \psi_n(t) \right\|_{L^2((\mathbb{R}^3)^N)}^2 \leq C_{t,\alpha} N^{-\alpha}\]

where

\[\chi(t) = \frac{2N + 3}{6} \iint V_N(x - y, x - z)|u(t, x)|^2|u(t, y)|^2|u(t, z)|^2 \, dx \, dy \, dz.\]

As explained in $[10]$, the one-particle density matrix $(\gamma(t), \alpha(t))$ of the Bogoliubov dynamics $\Phi(t)$, defined by the kernels

\[\gamma(t, x, y) = \langle \Phi(t), a_x^* a_y \Phi(t) \rangle, \quad \alpha(t, x, y) = \langle \Phi(t), a_x a_y \Phi(t) \rangle,\]
is the unique solution to the
\[
\begin{align*}
i\partial_t \gamma &= h\gamma - \gamma h + K_2\alpha - \alpha^*_h K_2^* \\
i\partial_t \alpha &= h\alpha + \alpha h^T + K_2 + K_2^T \gamma + \gamma K_2,
\end{align*}
\]
(24)

Thus our result in Theorem 2 also gives a rigorous derivation for (24) as an effective description for the density of the excited particles. Moreover, note that if \( \Phi_0 \) is a quasi-free state, then the solution \( \Phi(t) \) to the Bogoliubov equation (19) is a quasi-free state for all \( t > 0 \) and (24) is indeed equivalent to the Bogoliubov equation (19). Nevertheless, our Theorem 2 works in a general situation and does not require the quasi-free restriction.

Our result in Theorem 2 extends the norm approximation obtained by X. Chen [15] on the mean-field case \( \beta = 0 \) (to be precise, the work in [15] deals with the setting of the fluctuations around coherent states in Fock space rather than the fluctuations around factorized states in \( N \)-particle space, but our method applies for both cases). Our analysis is completely different from [15] and will be explained below.

2.3. Ideas of the proofs. Now let us quickly explain the main ingredients of the proofs of Theorem 1 and Theorem 2.

First at all, as a preliminary step, we need to prove the well-posedness of the quintic Hartree equation (18). In particular, it is important to derive the uniform (i.e. \( N \)-independent) bound in \( H^4(\mathbb{R}^3) \),
\[
\|u(t, \cdot)\|_{H^4(\mathbb{R}^3)} \leq C_t \|u(0, \cdot)\|_{H^4(\mathbb{R}^3)}
\]
(which in turn provides uniform bound on \( \|u(t, \cdot)\|_{L^\infty} \) and \( \|\partial_t u(t, \cdot)\|_{L^\infty} \) by Sobolev’s embedding). The proof requires nontrivial modifications from the analysis for the quintic NLS [11] in [17]. More precisely, we will treat the quintic Hartree equation (18) as a perturbation of the quintic NLS [11] and use the method developed in [17] to extend the Strichartz’s estimate in \( L^{10}_{t,x} \) for the quintic Hartree solution.

Next, we start the many-body analysis with the general approach as in the pair-interaction case [37, 40, 41, 11]. This approach is based on a unitary transformation \( U_N(t) \) introduced in [36] which maps the original \( N \)-particle space \( L^2((\mathbb{R}^3)^N) \) to the truncated Fock space built up on the orthogonal complement \( \{u(t)\}^\perp \) of quintic Hartree equation:
\[
U_N(t) : \sum_{n=0}^N u(t) \otimes (N-n) \otimes \otimes_\sigma \psi_n(t) \mapsto \bigoplus_{n=0}^N \psi_n(t)
\]
Heuristically, this operator \( U_N(t) \) factors out the condensate and implements the c-number substitution in Bogoliubov’s idea [6]. Thus it remains to analyze the transformed dynamics
\[
\Phi_N(t) = U_N(t)\Psi_N(t)
\]
in the excited Fock space. The assertion in Theorem 1 is essentially equivalent to
\[
\langle \Phi_N(t), N\Phi_N(t) \rangle \ll N.
\]

To propagate the latter bound in time, we need to show that the generator of \( \Phi_N \) can be controlled by its kinetic part. The main difficulty lies on the fact that the interaction part of the generator of \( \Phi_N \) depends heavily on \( N \) and behaves badly when there are too many particles. To overcome this difficulty, we will use the localization technique in Fock space and focus on low particle sectors. This idea was used also in the pair-interaction case [37, 40, 41, 11]. However, while in the pair-interaction case it is sufficient to restrict
to $\mathcal{N} \ll N$, in our three-body interaction case we need to restrict further to $\mathcal{N} \ll N^{1-2\beta}$. This is due to the cubic term

$$\frac{1}{\sqrt{N}} \iiint V_N(x - y, x - z) a_x^* a_y^* a_z^* \, dx \, dy \, dz$$

(25)

which does not appear in the pair-interaction case. By using the diagonalization result on the quadratic Hamiltonian [43], we can bound the above cubic term by

$$\eta d \Gamma(1 - \Delta) + \eta^{-1} C_1 MN^{4\beta - 1}, \quad \forall \eta \geq C_1 \sqrt{M} N^{2\beta - 1}$$

on the truncated Fock space of $\mathcal{N} \leq M$. The condition $M \ll N^{1-2\beta}$ is necessary to take $\eta$ of order 1. This leads to a good kinetic bound for the truncated dynamics $\Phi_{N,M}$, which has a generator similar to that of $\Phi_N$ but restricted to the truncated space $\mathcal{N} \leq M$.

To complete the proof of Theorem [1], we need to show that the truncated dynamics $\Phi_{N,M}$ is sufficiently close to $\Phi_N$. Heuristically, this step is doable if $M$ is sufficiently large, namely, the effect of the cut-off $\mathcal{N} \leq M$ is negligible. Technically, this step will be done by comparing the two generators and using the kinetic estimate of $\Phi_{N,M}$ (plus the round kinetic bound $O(N)$ for $\Phi_N$), resulting the condition $M \gg N^{4\beta}$. Putting the latter condition together with the previous one $M \ll N^{1-2\beta}$, we obtain the net condition $\beta < 1/6$ at the end.

The norm approximation in Theorem [2] requires to compare $\Phi_N(t)$ with the Bogoliubov dynamics $\Phi(t)$. From the proof of Theorem [1], we know that $\Phi_N$ is close to $\Phi_{N,M}$ if $M \gg N^{4\beta}$. Therefore, it is natural to compare the truncated dynamics $\Phi_{N,M}$ with the Bogoliubov dynamics $\Phi(t)$. It turns out that this step can be done if $M \ll N^{1-5\beta}$. Thus if $\beta < 1/9$ then the norm approximation follows. To improve the range of $\beta$, we will use an iteration technique: we will compare $\Phi_{N,M}$ with a further truncated dynamics $\Phi_{N,M}$ with $M \ll M$ (where we can use the improved kinetic bounds for both, instead of using the round kinetic bound $O(N)$ for $\Phi_N$). This technique allows us to bring down the cut-off parameter $\tilde{M}$ to $N^\beta$ (after many but finite iteration steps). And finally, we compare $\Phi_{N,M}$ with $\Phi(t)$, which requires $\tilde{M} \ll N^{1-5\beta}$. All this leads to the condition $\beta < 1/6$ for the norm approximation in Theorem [2] which is fortunately the same as the condition in Theorem [1].

In order to extend our results to more singular potentials (i.e. larger $\beta$'s), it is crucial to have a better bound for the cubic term (25). This issue goes beyond the current knowledge of Bogoliubov theory and seems very interesting. We refer to [7] for a recent important contribution to the analysis of the cubic term in the pair-interaction case. The problem in the three-body interaction case, however, is completely open.

**Organization of the paper.** We will discuss the quintic Hartree equation (18) in Section 3 and Bogoliubov equation (19) in Section 4. Then in Section 5 we explain the general strategy to derive these effective equations from the many-body Schrödinger equation (2). Then we settle in Section 6 key operator estimates on Fock space. Our main Theorems 1 and 2 are proved in Sections 7 and 8 respectively. In Appendix, we explain the extension of our results with $\mathbb{R}^3$ replaced by the torus $T^3$.

### 3. Quintic Hartree Equation

In this section we study the well-posedness of the quintic Hartree equation (18),

$$\begin{aligned}
   &\left\{ \begin{array}{l}
   i\partial_t u(t,x) = \left( -\Delta + \frac{1}{2} \iint V_N(x - y, x - z)|u(t,y)|^2|u(t,z)|^2 \, dy \, dz \right) u(t,x) \\
   u(0,x) = u_0(x).
   \end{array} \right.
\end{aligned}$$

We will prove
Theorem 3 (Uniform estimates for quintic Hartree equation). Let $u_0 \in H^4(\mathbb{R}^3)$ be an initial state. Then for every time $T > 0$, there exists a unique solution to equation (18) on $[0,T]$ and it satisfies

$$\|u(t,\cdot)\|_{H^4(\mathbb{R}^3)} \leq C_t, \quad \|\partial_t u(t,\cdot)\|_{H^2(\mathbb{R}^3)} \leq C_t.$$  

(26)

Here the constant $C_t$ is dependent on $\|u_0\|_{H^4}$ and $t$ (it is continuous and increasing in $t$), but independent of $N$.

Moreover, when $N \to \infty$, $u(t)$ converges to the solution $\varphi(t)$ to the quintic NLS (1) (with the same initial condition $\varphi_0 = u_0$):

$$\|u(t,\cdot) - \varphi(t,\cdot)\|_{L^2(\mathbb{R}^3)}^2 \leq C_t N^{-\beta}.$$  

(27)

Remark 4. Note that from (26) and Sobolev’s inequality we obtain

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_t, \quad \|\partial_t u(t,\cdot)\|_{L^\infty} \leq C_t.$$  

These bounds will be used repeatedly in the paper.

The well-posedness result of the energy-critical NLS (1),

$$\left\{\begin{array}{l}
    i\partial_t \varphi(t, x) = -\Delta \varphi(t, x) + b_0|\varphi(t, x)|^4\varphi(t, x) \\
    \varphi(0, x) = \varphi_0(x)
  \end{array}\right.$$  

was proved in 2008 by Colliander, Keel, Staffilani, Takaoka, and Tao [17]. The adaptation from the local equation (11) to the nonlocal one (18) is not obvious and we will explain the details below. Note that a similar result to Theorem 3 for the cubic Hartree equation (which involves only a two-body interaction potential) was proved in 2013 by Grillakis and Machedon [25]. It turns out that the analysis for the 3-body case is significantly more complicated and we could not simply follow the analysis in [25].

Our proof of Theorem 3 is organized as follows. First, in Section 3.1 we will prove the existence of uniqueness of a local solution to (18). This step is standard; we will follow [12, 50, 39] and the references therein. Next, in Section 3.2 we extend the local solution to a global one and derive the $N$-independent estimates. This is the crucial step where we need to interpret (3) as a perturbation of (1). By employing the time-extension technique in [17], we can go from local estimate to global estimate. This will be explained in Sections 3.2 and 3.3. Finally, we conclude the proof of Theorem 3 in Section 3.4.

3.1. Local well-posedness. First we prove the existence and uniqueness of a solution to equation (18) in a short time interval.

Lemma 5 (Local existence of quintic Hartree equation). For every $u_0 \in H^1(\mathbb{R}^3)$, there exists a constant $\varepsilon > 0$ depending on $\|u_0\|_{H^1(\mathbb{R}^3)}$ such that for any time interval $I$ containing 0 with

$$\|e^{it\Delta}u_0\|_{L^6_t L^{18}_x(I \times \mathbb{R}^3)} \leq \varepsilon,$$  

(28)

there exists a unique solution $u \in C(I, H^1(\mathbb{R}^3)) \cap L^6_t W^{1,\infty}_x(I \times \mathbb{R}^3)$ of (18).

Proof. We will use a fixed-point argument similarly to the the energy-critical NLS case [39, Theorem 5.5] (see also [12, 50] and the references therein). Let

$$E(I, a) = \left\{ v \in C(I, H^1(\mathbb{R}^3)) \cap L^6_t W^{1,\infty}_x(I \times \mathbb{R}^3) : \|v\|_{L^6_t W^{1,\infty}_x(I \times \mathbb{R}^3)} \leq a \right\}.$$  

(29)
equipped with the norm
\[ \|v\|_{E(I,a)} = \sup_{t \in I} \|v(t)\|_{H^1(\mathbb{R}^3)} + \|v\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)}. \]

This makes \(E(I,a)\) a complete metric space. Define for \(u \in E(I,a)\)
\[ \Phi(u)(t) = e^{it\Delta}u_0 - \frac{i}{2} \int_0^t e^{i(t-s)\Delta} \int \|u(s,y)\|_{H^1(\mathbb{R}^3)} \|u(z)\|_{H^1(\mathbb{R}^3)} \|u(s,z)\|_{H^1(\mathbb{R}^3)} dy dz u(s) ds. \] (30)

We want to prove that \(\Phi\) is a contraction map on \(E(I,a)\) and then apply the contraction mapping principle.

In the following let us ignore the time dependence in the notation of \(u\) for simplicity. By the product rule for gradient and Hölder’s, Young’s, Sobolev’s inequalities we have
\[ \left\| \nabla \left( \int \|u(y)\|^2 V_N \|u(z)\|^2 dy dz \right) \right\|_{L^3_t L^{18}_x(I \times \mathbb{R}^3)} \lesssim \left\| \int \|u\|^2 V_N \|u\|^2 dy dz \right\|_{L^3_t L^{18}_x(I \times \mathbb{R}^3)} \lesssim \left\| \nabla u \right\|_{L^6_t L^{18}_x(I \times \mathbb{R}^3)} \left\| \int |u|^2 V_N |u|^2 dy dz \right\|_{L^3_t L^{18}_x(I \times \mathbb{R}^3)} \lesssim \left\| \nabla u \right\|_{L^6_t L^{18}_x(I \times \mathbb{R}^3)} \left\| V \right\|_{L^1(I \times \mathbb{R}^3)} \left\| u \right\|_{L^6(I \times \mathbb{R}^3)} \left\| u \right\|_{L^9(I \times \mathbb{R}^3)} \right\|_{L^3_t L^{18}_x(I \times \mathbb{R}^3)} \lesssim \left\| \nabla u \right\|_{L^6_t L^{18}_x(I \times \mathbb{R}^3)}^5. \] (31)

Hence using (28) and Strichartz estimate [48] (also consider [50]) we obtain
\[ \left\| \Phi(u) \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)} \lesssim \left\| e^{it\Delta}u_0 \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)} + \left\| \frac{i}{2} \int_0^t e^{i(t-s)\Delta} \int |u(s,y)|^2 V_N |u(s,z)|^2 dy dz u(s) ds \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)} \lesssim \varepsilon + \left\| u \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)}^5. \]

Therefore, for \(u \in E(I,a)\) we have \(\left\| \Phi(u) \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)} \lesssim \varepsilon + a^5\) and hence choosing \(a > 0\) small enough and then \(\varepsilon > 0\) small we have
\[ \left\| \Phi(u) \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)} \lesssim a. \]

Moreover, again using Strichartz estimate we have
\[ \sup_{t \in [0,T]} \left\| \Phi(u)(t) \right\|_{H^1(\mathbb{R}^3)} \lesssim \left\| u_0 \right\|_{H^1(\mathbb{R}^3)} + \left\| \int \|u|^2 V_N |u|^2 dy dz u \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)} \lesssim \left\| u_0 \right\|_{H^1(\mathbb{R}^3)} + \left\| u \right\|_{L^6_t W^{1,18}_x(I \times \mathbb{R}^3)}^5. \]

Putting altogether we obtain \(\Phi(E(I,a)) \subseteq E(I,a)\).
To show that $\Phi$ is a contraction map we see for $u, v \in E(I, a)$
\[
\|\nabla (\Phi(u) - \Phi(v))\|_{L_I^6 L_x^{18}} \lesssim \left\| \iint |u|^2 V_N |u|^2 \, dy \, dz \, \nabla u - \iint |v|^2 V_N |v|^2 \, dy \, dz \, \nabla v \right\|_{L_I^4 L_x^{18}}^\frac{6}{7} + \left\| \left( \iint |u|^2 V_N |u|^2 \, dy \, dz - \iint |v|^2 V_N |v|^2 \, dy \, dz \right) \nabla v \right\|_{L_I^4 L_x^{18}}^\frac{6}{7}.
\]

The first term can be estimated similarly to \((31)\), which gives
\[
\left\| \iint |u|^2 V_N |u|^2 \, dy \, dz \, \nabla (u - v) \right\|_{L_I^6 L_x^{18}}^\frac{6}{7} \lesssim \|\nabla u\|_{L_I^6 L_x^{18}(I \times \mathbb{R}^3)} \|\nabla (u - v)\|_{L_I^4 L_x^{18}}^\frac{6}{7}.
\]

The second one follows by
\[
\left\| \left( \iint |u|^2 V_N |u|^2 \, dy \, dz - \iint |v|^2 V_N |v|^2 \, dy \, dz \right) \nabla v \right\|_{L_I^6 L_x^{18}(I \times \mathbb{R}^3)} \lesssim \|\nabla u\|_{L_I^6 L_x^{18}(I \times \mathbb{R}^3)} \|\nabla (u - v)\|_{L_I^4 L_x^{18}(I \times \mathbb{R}^3)}.
\]

Putting all together we obtain
\[
\|\nabla (\Phi(u) - \Phi(v))\|_{L_I^6 L_x^{18}} \lesssim a^4 \|\nabla (u - v)\|_{L_I^6 L_x^{18}(I \times \mathbb{R}^3)}.
\]

Similarly we obtain
\[
\|\Phi(u) - \Phi(v)\|_{L_I^6 L_x^{18}} \lesssim a^4 \|u - v\|_{L_I^6 L_x^{18}}
\]

and together
\[
\|\Phi(u) - \Phi(v)\|_{L_I^6 W_x^{1,18}} \lesssim a^4 \|u - v\|_{L_I^6 W_x^{1,18}}.
\]

Furthermore, using the above and Strichartz estimate we obtain
\[
\sup_{t \in I} \|\Phi(u) - \Phi(v)\|_{H^1(I \times \mathbb{R}^3)} \lesssim \left\| \iint |u|^2 V_N |u|^2 \, dy \, dz \, u - \iint |v|^2 V_N |v|^2 \, dy \, dz \, v \right\|_{L_I^6 W_x^{1,18}}^\frac{6}{7} \lesssim a^4 \|u - v\|_{L_I^6 W_x^{1,18}}.
\]

\[
\tag{33}
\]
By this we have
\[ \| \Phi(u) - \Phi(v) \|_{E(I, a)} \lesssim a^4 \| u - v \|_{E(I, a)} \]
which gives for \( a > 0 \) small enough that \( \Phi \) is a contraction map on \( E(I, a) \). Using the contracting mapping principle we have that there exists a unique \( u \in C(I : H^1(\mathbb{R}^3)) \cap L^6_t W^{1, 16}_x (I \times \mathbb{R}^3) \) which solves (18) with initial data \( u_0 \in H^1(\mathbb{R}^3) \). \( \square \)

In the following we will derive the global theory of equation (18). We will use the well-known global well-posedness theory of the energy-critical NLS
\[ i \partial_t \tilde{u} = -\Delta \tilde{u} + b_0 |\tilde{u}|^4 \tilde{u} \] (34)
(consider for example [17] and [50]) and consider equation (18) as a perturbation. For that we will prove an adapted version of [17, Lemma 3.9 and Lemma 3.10] for the quintic Hartree equation, which will give global spacetime bounds for solutions of equation (18).

3.2. Short time perturbation.

Lemma 6 (Short time Perturbation). Let \( I(t \geq 0) \) be a compact interval and \( \tilde{u} \) be a solution of
\[ i \partial_t \tilde{u} = -\Delta \tilde{u} + \frac{1}{2} \int |\tilde{u}(y)|^2 V_N(x - y, x - z) |\tilde{u}(z)|^2 \, dy \, dz + e \] (35)
on \( I \times \mathbb{R}^3 \) for some function \( e \). Assume that we have
\[ \| \nabla \tilde{u} \|_{L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3)} \leq \varepsilon_0, \quad \| \nabla e \|_{L^6_t L^6_x (I \times \mathbb{R}^3)} \leq \varepsilon, \] (36)\( (37) \)
for some constants \( \varepsilon_0, \varepsilon > 0 \) small enough. For \( t_0 \in I \) let \( u(t_0) \) be close to \( \tilde{u}(t_0) \). More precisely:
\[ \| \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0)) \|_{L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3)} \leq \varepsilon. \] (38)
We then conclude that there exists a solution \( u \) of (18) on \( I \times \mathbb{R}^3 \) with initial state \( u(t_0) \) at \( t_0 \), which fulfills the following spacetime bounds:
\[ \| u - \tilde{u} \|_{L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3)} \lesssim \| \nabla (u - \tilde{u}) \|_{L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3)} \lesssim \varepsilon, \] (39)
\[ \| \nabla (i \partial_t + \Delta) (u - \tilde{u}) \|_{L^6_t L^6_x (I \times \mathbb{R}^3)} \lesssim \varepsilon. \] (40)

Proof. By the local theory given in Lemma 5 we can prove (39)-(40) as a priori estimates, meaning that we assume that \( u \) already exists on \( I \).

Let \( v = u - \tilde{u} \). Define
\[ S(t) = \| \nabla (i \partial_t + \Delta) v \|_{L^6_t L^6_x (I \times \mathbb{R}^3)}. \]
Using [38], Strichartz estimate and Duhamel’s formula we can estimate the \( L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3) \) Norm of \( v \) by \( S(t) \):
\[ \| \nabla v \|_{L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3)} \lesssim \| \nabla (v - e^{i(t-t_0)\Delta} v(t_0)) \|_{L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3)} + \| \nabla e^{i(t-t_0)\Delta} v(t_0) \|_{L^6_t L^{\frac{20}{3}}_x (I \times \mathbb{R}^3)} \lesssim S(t) + \varepsilon. \] (41)
On the other hand we know that \( v \) solves the following equation:

\[
i \partial_t v = -\Delta v + \frac{1}{2} \iint |v(y) + \tilde{u}(y)|^2 V_N(x - y, x - z) |v(z) + \tilde{u}(z)|^2 \, dy \, dz (v + \tilde{u}) - \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z) |\tilde{u}(z)|^2 \, dy \, dz \tilde{u} - e
\]

(42)

In order to estimate \( S(t) \) we have to estimate several terms of \( H_2 \) which contain different powers of \( v \) and \( \tilde{u} \). Since all of the terms can be estimated in the same strategy, we just give a few examples. We start with the term \( \iint |v(y)|^2 V_N(x - y, x - z) |v(z)|^2 \, dy \, dz \nabla v \). By Hölder’s inequality we have

\[
\left\| \iint |v(y)|^2 V_N(x - y, x - z) |v(z)|^2 \, dy \, dz \nabla v \right\|_{L^2_t L^\frac{5}{2} (I \times \mathbb{R}^3)} \leq \left\| \iint |v(y)|^2 V_N(x - y, x - z) |v(z)|^2 \, dy \, dz \right\|_{L^\frac{5}{2}_y L^\frac{5}{2} (I \times \mathbb{R}^3)} \left\| \nabla v \right\|_{L^{10}_t L^{30}_y (I \times \mathbb{R}^3)}.
\]

The first term can be estimated using first Young’s inequality and then Sobolev’s inequality and \( H_1 \):

\[
\left\| \iint |v(y)|^2 V_N(x - y, x - z) |v(z)|^2 \, dy \, dz \right\|_{L^\frac{5}{2}_y L^\frac{5}{2} (I \times \mathbb{R}^3)} \lesssim \left\| \iint |v(y)|^4 V_N(x - y, x - z) \, dy \, dz \right\|_{L^\frac{5}{2}_y L^\frac{5}{2} (I \times \mathbb{R}^3)} \leq \left\| \|V\|_{L^1(\mathbb{R}^6)} \|v\|_{L^{10}_y (\mathbb{R}^3)} \right\|_{L^\frac{5}{2}_y (I)} \lesssim \|v\|_{L^{10}_t L^3_y (\mathbb{R}^3)} \lesssim \|\nabla v\|_{L^4_t L^\frac{30}{7} (I \times \mathbb{R}^3)} \lesssim (S(t) + \varepsilon)^4.
\]

Hence, using the above and \( H_1 \) again we obtain

\[
\left\| \iint |v(y)|^2 V_N(x - y, x - z) |v(z)|^2 \, dy \, dz \nabla v \right\|_{L^2_t L^\frac{5}{2} (I \times \mathbb{R}^3)} \lesssim (S(t) + \varepsilon)^5.
\]

For the readers convenience we also consider the mixed term \( \iint |\tilde{u}|^2 V_N \nabla |v|^2 \, dy \, dz \) as a second example. First we use again Hölder’s inequality and get

\[
\left\| \iint |\tilde{u}|^2 V_N \nabla |v|^2 \, dy \, dz \right\|_{L^\frac{5}{2}_y L^\frac{5}{2} (I \times \mathbb{R}^3)} \lesssim \left\| \iint |\tilde{u}|^2 V_N \nabla |v|^2 \, dy \, dz \right\|_{L^\frac{5}{2}_y L^{15}_y (I \times \mathbb{R}^3)} \|v\|_{L^{10}_t L^3_y (\mathbb{R}^3)}.
\]

The first term can then be estimated by Minkowski’s inequality

\[
\left\| \iint |\tilde{u}|^2 V_N \nabla |v|^2 \, dy \, dz \right\|_{L^\frac{5}{2}_y L^\frac{5}{2} (I \times \mathbb{R}^3)} \lesssim \left\| \iint |\tilde{u}|^2 V_N |\nabla v| |v| \, dy \, dz \right\|_{L^\frac{5}{2}_y L^{15}_y (I \times \mathbb{R}^3)} \lesssim \left\| \iint V_N (y, z) \left( \int |\tilde{u}(x - y)|^{20} |v(x - z)|^{15} \nabla v(x - z) \right)^{\frac{15}{15}} \, dy \, dz \right\|_{L^\frac{5}{2}_y (I)} \lesssim \left\| \|V\|_{L^1(\mathbb{R}^6)} \|\tilde{u}\|_{L^{10}_y (\mathbb{R}^3)} |v|_{L^{10}_y (\mathbb{R}^3)} \right\|_{L^{30}_y (\mathbb{R}^3)} \lesssim \|\nabla v\|_{L^4_t L^\frac{30}{7} (I \times \mathbb{R}^3)} \lesssim (S(t) + \varepsilon)^4.
\]
\[ \lesssim \| \tilde{u} \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)}^2 \| v \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \| \nabla v \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \]
\[ \lesssim \| \nabla \tilde{u} \|_{L^5_{t,x}L^\infty(I \times \mathbb{R}^3)}^2 \| \nabla v \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)}^2 \lesssim \varepsilon_0^2 (S(t) + \varepsilon)^2. \]

Since by Sobolev’s inequality and (11) we have that \( \| v \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \lesssim \| \nabla v \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \lesssim S(t) + \varepsilon \), we conclude
\[ \left\| \int \int |\tilde{u}|^2 V_N \nabla |v|^2 \, dy \, dz \, v \right\|_{L^6_t L^{5/2}_x(I \times \mathbb{R}^3)} \lesssim \varepsilon_0^2 (S(t) + \varepsilon)^3. \]

These and similar estimates yield
\[ S(t) \lesssim \varepsilon + \sum_{j=1}^5 (S(t) + \varepsilon)^j \varepsilon_0^{5-j}. \]

If we choose \( \varepsilon_0 \) small enough, a standard continuity argument gives \( S(t) \lesssim \varepsilon \) for all \( t \in I \). This proves (40). Using (41) we also conclude (39). This finishes the proof. \( \square \)

### 3.3. Long time perturbation

We now prove a version of Lemma 6 without the smallness condition (36) by using Lemma 6 iteratively.

**Lemma 7 (Long time Perturbation).** Let \( I \) be a compact interval and \( \tilde{u} \) be a function on \( I \times \mathbb{R}^3 \) which solves
\[
i\partial_t \tilde{u} = -\Delta \tilde{u} + \frac{1}{2} \int \int |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz \, \tilde{u} + \varepsilon, \tag{43} \]
for some function \( \varepsilon \) on \( I \times \mathbb{R}^3 \). Moreover assume that \( \tilde{u} \) fulfills the following spacetime bounds:
\[
\| \tilde{u} \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \leq M, \tag{44}
\]
\[
\| \nabla \tilde{u} \|_{L^5_{t,x}L^\infty(I \times \mathbb{R}^3)} \leq E, \tag{45}
\]
\[
\| \nabla \varepsilon \|_{L^5_{t,x}L^{6/5}_x(I \times \mathbb{R}^3)} \leq \varepsilon. \tag{46}
\]

for some constants \( M, E \geq 0 \) and some small enough \( \varepsilon > 0 \). For \( t_0 \in I \) let \( u(t_0) \) be close to \( \tilde{u}(t_0) \) in the sense that
\[
\| \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0)) \|_{L^{10}_{t,x}L^\infty(I \times \mathbb{R}^3)} \leq \varepsilon. \tag{47}
\]

We then conclude that there exists a solution \( u \) of (18) on \( I \times \mathbb{R}^3 \) with initial state \( u(t_0) \) at \( t_0 \), which fulfills the following spacetime bounds:
\[
\| u - \tilde{u} \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \lesssim \| \nabla (u - \tilde{u}) \|_{L^{10}_{t,x}L^\infty(I \times \mathbb{R}^3)} \lesssim C(M, E)\varepsilon, \tag{48}
\]
\[
\| \nabla (i\partial_t + \Delta) (u - \tilde{u}) \|_{L^{5/2}_{t,x}L^{\infty}(I \times \mathbb{R}^3)} \lesssim C(M, E)\varepsilon. \tag{49}
\]

**Proof.** Let \( \delta > 0 \). Using (41) we can split \( I \) into finetly many subintervals \( I_1, ..., I_{C(M)} \) such that
\[
\| \tilde{u} \|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \lesssim \delta
\]
for each \( j \). Using Duhamel’s formula, Strichartz’s estimate and estimates similar to the ones in Lemma 6 and Lemma 7 we obtain
\[
\| \nabla \tilde{u} \|_{L^{10}_{t,x}L^\infty(I \times \mathbb{R}^3)} \]
the solution of the perturbed equation given by the quintic NLS

Proof of Theorem 3.

3.4 Conclusion of Theorem 3.

We want to apply Lemma 7 with \( \tilde{u} \) being the solution of the perturbed equation given by the quintic NLS

\[ i\partial_t \tilde{u} = -\Delta \tilde{u} + b_0 |\tilde{u}|^4 \tilde{u} \]

\[ = -\Delta \tilde{u} + \frac{1}{2} \int |\tilde{u}(y)|^2 V_N(x - y, x - z) |\tilde{u}(z)|^2 \, dy \, dz \, \tilde{u} + e_N, \]

We can now apply Lemma 6 inductively. Using this for the first subinterval gives

\[ \| \nabla \tilde{u} \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_j \times \mathbb{R}^3)} \leq E + \tilde{u}. \]

Summing over all finite intervals gives

\[ \| \nabla \tilde{u} \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_j \times \mathbb{R}^3)} \leq C(M, E). \] (50)

If we now choose \( \varepsilon_0 \) as in Lemma \(^{13}\) and use \(^{15}\) we can split \( I \) again into finitely many subintervals \( I_1, \ldots, I_{C(M, E, \varepsilon_0)} \) with \( I_j = [t_j, t_{j+1}] \) and

\[ \| \nabla \tilde{u} \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_j \times \mathbb{R}^3)} \leq \varepsilon_0. \] (51)

We can now apply Lemma 6 inductively. Using this for the first subinterval \( I_0 \) gives

\[ \| u - \tilde{u} \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_0 \times \mathbb{R}^3)} \leq \| \nabla (u - \tilde{u}) \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_0 \times \mathbb{R}^3)} \leq \varepsilon, \]

\[ \| \nabla (i\partial_t + \Delta) (u - \tilde{u}) \|_{L^{2}_t L^{\infty}_{x}} \|_{(I_0 \times \mathbb{R}^3)} \leq \varepsilon. \]

Now proceeding iteratively using Duhamel’s formula we see that

\[ \| \nabla e^{i\Delta(t-t_1)} (u(t_1) - \tilde{u}(t_1)) \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_1 \times \mathbb{R}^3)} \]

\[ \leq \| \nabla e^{i\Delta(t-t_0)} (u(t_0) - \tilde{u}(t_0)) \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_1 \times \mathbb{R}^3)} \]

\[ + \left\| e^{i\Delta(t-t_1)} \int_{t_0}^t e^{i\Delta(t-s)} \nabla (i\partial_t + \Delta) (u - \tilde{u})(s) \, ds \right\|_{L^{10}_t L^{\infty}_{x}} \|_{(I_1 \times \mathbb{R}^3)} \]

\[ \leq \varepsilon + \left\| \int_{t_0}^t e^{i\Delta(t-s)} \nabla (i\partial_t + \Delta) (u - \tilde{u})(s) \, ds \right\|_{L^{2}} \|_{(I_1 \times \mathbb{R}^3)} \]

\[ \leq \varepsilon + \| \nabla (i\partial_t + \Delta) (u - \tilde{u}) \|_{L^{2}_t L^{\infty}_{x}} \|_{(I_0 \times \mathbb{R}^3)} \leq \varepsilon. \]

For \( \varepsilon > 0 \) small enough we can iterate this procedure and obtain

\[ \| u - \tilde{u} \|_{L^{10}_t L^{\infty}_{x}} \|_{(I_1 \times \mathbb{R}^3)} \leq C(j) \varepsilon, \]

\[ \| \nabla (i\partial_t + \Delta) (u - \tilde{u}) \|_{L^{2}_t L^{\infty}_{x}} \|_{(I_1 \times \mathbb{R}^3)} \leq C(j) \varepsilon, \]

for all \( j \). Summing over all finite intervals we obtain \(^{13}\) and \(^{19}\). \qed

3.4 Conclusion of Theorem 3. We now apply the two previous Lemmas to prove Theorem 3.

Proof of Theorem 3. Global well-posedness. We want to apply Lemma 7 with \( \tilde{u} \) being the solution of the perturbed equation given by the quintic NLS
with initial state $\tilde{u}(0, x) = u_0$. Here we have defined the perturbation

$$e_N = b_0|\tilde{u}|^4 \tilde{u} - \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz \, \tilde{u}.$$ 

In order to use Lemma 7 we want to show that $\|\nabla e_N\|_{L^2_\alpha L^6_\beta(I \times \mathbb{R}^3)}$ is arbitrarily small for $N$ large. We see that

$$\begin{align*}
\|\nabla e_N\|_{L^2_\alpha L^6_\beta(I \times \mathbb{R}^3)} & \lesssim \left\| \left( \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz - b_0|\tilde{u}|^4 \right) \nabla \tilde{u} \right\|_{L^2_\alpha L^6_\beta(I \times \mathbb{R}^3)} \\
& \quad + \left\| \nabla \left( \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz - b_0|\tilde{u}|^4 \right) \tilde{u} \right\|_{L^2_\alpha L^6_\beta(I \times \mathbb{R}^3)} \\
& \lesssim \left\| \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz - b_0|\tilde{u}|^4 \right\|_{L^2_\alpha L^6_\beta(I \times \mathbb{R}^3)} \\
& \quad + \left\| \nabla \left( \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz - b_0|\tilde{u}|^4 \right) \tilde{u} \right\|_{L^2_\alpha L^6_\beta(I \times \mathbb{R}^3)}.
\end{align*}$$

We now show that the first term

$$\left\| \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz - b_0|\tilde{u}|^4 \right\|_{L^2_\alpha L^6_\beta}$$

will be arbitrarily small for $N$ big enough. The second term will follow similarly. We see by Minkowski’s inequality

$$\begin{align*}
\left\| \frac{1}{2} \iint V_N(y, z) \left( |\tilde{u}(x - y)|^2 |\tilde{u}(x - z)|^2 - |\tilde{u}(x)|^4 \right) \, dy \, dz \right\|_{L^2_\alpha L^6_\beta} \\
& \leq \frac{1}{2} \iint V_N(y, z) \left( |\tilde{u}(x - y)|^2 |\tilde{u}(x - z)|^2 - |\tilde{u}(x)|^4 \right) \, dy \, dz \\
& \leq \frac{1}{2} \iint V_N(y, z) \left( |\tilde{u}(x - y)|^2 |\tilde{u}(x - z)|^2 - |\tilde{u}(x)|^4 \right) \|\tilde{u}\|_{L^2_\alpha L^6_\beta} \, dy \, dz \\
& \leq \frac{1}{2} \iint V_N(y, z) \left( |\tilde{u}(x - y)|^2 |\tilde{u}(x - z)|^2 - |\tilde{u}(x)|^4 \right) \|\tilde{u}\|_{L^2_\alpha L^6_\beta} \, dy \, dz \\
& = 2 \iint_{|y| \leq CN^{-\beta}} V_N(y, z) \left( |\tilde{u}(x - y) - \tilde{u}(x)|^2 \right) \|\tilde{u}\|_{L^2_\alpha L^6_\beta} \, dy \, dz \\
& \leq C \|\tilde{u}\|_{L^2_\alpha L^6_\beta} \sup_{|y| \leq CN^{-\beta}} \|\tilde{u}(x - y) - \tilde{u}(x)\|_{L^2_\alpha L^6_\beta}.
\end{align*}$$

Here we have used the fact that $V_N(y, \cdot)$ is zero for $|y| > CN^{-\beta}$ for some $C > 0$. Note that $\tilde{u}$ is independent of $N$ and $\|\tilde{u}\|_{L^2_\alpha L^6_\beta} \leq C$ by [17] Theorem 1.1, we have the continuity by translation

$$\lim_{|y| \to 0} \|\tilde{u}(x - y) - \tilde{u}(x)\|_{L^2_\alpha L^6_\beta} = 0.$$ 

Therefore, the right side of (52) is arbitrarily small for large $N$.

We are now able to apply Lemma 7 which gives the existence and uniqueness of a solution $u$ to (18) (we omit the $N$-dependence of $u$ in the notation).
Finally we come to the proof of (26). By Lemma 7 we also know that the solution \( u \) to (13) obeys the following spacetime bound
\[
\int_0^T \int_{\mathbb{R}^3} |u(t, x)|^2 \, dx \, dt < \infty.
\]
(53)
Using this we can split up \([0, T]\) into finitely many subintervals \( I_0, \ldots, I_K \) such that on each \( I_j \)
\[
\|u\|_{L_{t,x}^{10}(I_j \times \mathbb{R}^3)} \leq \delta
\]
for some small \( \delta > 0 \). Now for any multi-index \( \alpha \) with \( |\alpha| \leq 4 \) we obtain by using Strichartz estimate on the first interval \( I_0 \)
\[
\|D^\alpha u\|_{L_t^{10}L_x^{\frac{30}{7}}(I_0 \times \mathbb{R}^3)} \lesssim \|D^\alpha u_0\|_{L_t^2L_x^2} + \left\| \frac{1}{2} \int \int |u|^2 V_N |u|^2 \, dy \, dz \, D^\alpha u \right\|_{L_t^2L_x^2}
\lesssim \|u_0\|_{H^4(\mathbb{R}^3)} + \|u\|_{L_{t,x}^{10}(I_0 \times \mathbb{R}^3)} \|D^\alpha u\|_{L_t^{10}L_x^{\frac{30}{7}}(I_0 \times \mathbb{R}^3)}
\lesssim \|u_0\|_{H^4(\mathbb{R}^3)} + \delta^4 \|D^\alpha u\|_{L_t^{10}L_x^{\frac{30}{7}}(I_0 \times \mathbb{R}^3)}.
\]
(54)
For \( \delta > 0 \) small enough we obtain
\[
\|D^\alpha u\|_{L_t^{10}L_x^{\frac{30}{7}}(I_0 \times \mathbb{R}^3)} \lesssim \|u_0\|_{H^4(\mathbb{R}^3)}.
\]
Using this and Strichartz estimate again, we get
\[
\|D^\alpha u(t, \cdot)\|_{L_t^2(\mathbb{R}^3)} \lesssim \|u_0\|_{H^4(\mathbb{R}^3)} + \left\| \frac{1}{2} \int \int |u|^2 V_N |u|^2 \, dy \, dz \, D^\alpha u \right\|_{L_t^2L_x^2} \lesssim \|u_0\|_{H^4(\mathbb{R}^3)}
\]
for any \( t \in I_0 \) and from this \( \|u(t, \cdot)\|_{H^4(\mathbb{R}^3)} \lesssim \|u_0\|_{H^4(\mathbb{R}^3)} \) for any \( t \in I_0 \). This procedure can now be iterated and we obtain
\[
\|u(t, \cdot)\|_{H^4(\mathbb{R}^3)} \lesssim \|u_0\|_{H^4(\mathbb{R}^3)}
\]
for all \( t \geq 0 \).

Thus we have proved the first bound in (26). The second bound follows from the first and the Hartree equation (13). Indeed, we have
\[
\|\partial_t u(t, \cdot)\|_{L_t^2(\mathbb{R}^3)} \leq \|\Delta u(t, \cdot)\|_{L_t^2(\mathbb{R}^3)} + \|u(t, \cdot)\|_{L_t^4(\mathbb{R}^3)} \|u(t, \cdot)\|_{L_t^\infty(\mathbb{R}^3)} \lesssim C_t
\]
and a similar estimate with \( -\Delta (\partial_t u(t, \cdot)) \). This finishes the proof of (26).

**Convergence to the quintic NLS solution.** Now we turn to the proof of (27). We compute the derivative of the norm distance using equation (13) and (11). This gives
\[
\frac{d}{dt} \|u(t) - \varphi(t)\|_{L_t^2(\mathbb{R}^3)}^2 = 2\Re \langle u(t), \left( \frac{1}{2} \int \int |u(t)|^2 V_N |u(t)|^2 \, dy \, dz - b_0 |\varphi(t)|^4 \right) \varphi(t) \rangle
= 2\Re \langle u(t), \left( \frac{1}{2} \int \int |u(t)|^2 V_N |u(t)|^2 \, dy \, dz - b_0 |u(t)|^4 \right) \varphi(t) \rangle + 2\Re \langle u(t), (b_0 |u|^4 - b_0 |\varphi|^4) \varphi(t) \rangle.
\]
(56)
To estimate the first term in (56) we see that
\[
\left| \frac{1}{2} \int \int |u(t, y)|^2 V_N(x - y, x - z)|u(t, z)|^2 \, dy \, dz - b_0 |u(t, x)|^4 \right|
= \frac{1}{2} \int \int V_N(x - y, x - z) |u(t, y)|^2 |u(t, z)|^2 - |u(t, x)|^4 \, dy \, dz
\]
From this we obtain

\[ \|V\| \leq \left| \iint V_N(x - y, x - z)(|u(t, y)|^2 - |u(t, x)|^2)|u(t, z)|^2 \, dy \, dz \right| + \left| \iint V_N(x - y, x - z)(|u(t, z)|^2 - |u(t, x)|^2)|u(t, x)|^2 \, dy \, dz \right|. \]

We now proceed with the first term, since both terms can be estimated similarly. Using that \( V \) has compact support and hence \( V(y, \cdot) = 0 \) for \( |y| > C \) for some \( C > 0 \), this gives

\[
\left| \iint V_N(x - y, x - z)(|u(t, y)|^2 - |u(t, x)|^2)|u(t, z)|^2 \, dy \, dz \right|
\leq \iint_{|y| \leq CN^{-\beta}} V_N(y, x - z) \, |u(t, x - y)|^2 \, |u(t, z)|^2 \, dy \, dz
\leq C N^{-\beta} \int_0^1 \int_{|y| \leq CN^{-\beta}} V_N(y, x - z) \, |\nabla u(t, x - sy)|^2 \, y \, ds \, |u(t, z)|^2 \, dy \, dz \, ds
\leq C N^{-\beta}.
\]

In the last inequality we have used Theorem 3 to bound all factors containing \( u \) by \( C_t \). From this we obtain

\[
|\langle u(t), \left( \frac{1}{2} \iint |u(t)|^2 V_N|u(t)|^2 \, dy \, dz - b_0|u(t)|^4 \right) \varphi(t) \rangle| \leq C N^{-\beta}.
\]

Now for the second term in (35), we use the elementary inequality

\[ ||a|^4 - |b|^4| \leq C||a - b||(|a|^3 + |b|^3) \]

and obtain

\[
|\langle u(t), (|u(t)|^4 - |\varphi(t)|^4) \varphi(t) \rangle| \leq \int |u(t, x)| |\varphi(t, x)| |u(t, x)|^4 - |\varphi(t, x)|^4 | \, dx
\leq C \int |u(t, x) - \varphi(t, x)||u(t, x)|^5 + |\varphi(t, x)|^5 | \, dx
\leq C \|u(t) - \varphi(t)\|_{L^2(\mathbb{R}^3)} (\|u(t)\|_{L^{10}(\mathbb{R}^3)}^5 + \|\varphi(t)\|_{L^{10}(\mathbb{R}^3)}^5)
= C_t \|u(t) - \varphi(t)\|_{L^2(\mathbb{R}^3)}^2.
\]

Here we have used \( \|\varphi(t)\|_{L^{10}(\mathbb{R}^3)}^5 \leq C \|\varphi(t)\|_{H^2(\mathbb{R}^3)}^5 \leq C \), which was proven in [17, Corollary 1.2], and \( \|u(t)\|_{L^{10}(\mathbb{R}^3)} \leq C \|u(t)\|_{H^2(\mathbb{R}^3)}^5 \leq C_t \), which holds true by Theorem 3.

Putting now both estimates together, we obtain

\[
\frac{d}{dt} \|u(t) - \varphi(t)\|_{L^2(\mathbb{R}^3)}^2 \leq C_t \left( N^{-\beta} + \|u(t) - \varphi(t)\|_{L^2(\mathbb{R}^3)}^2 \right),
\]

which completes the proof of (27). \( \square \)

4. Bogoliubov equation

In this section we discuss the Bogoliubov dynamics [19],

\[
\begin{cases}
i \partial_t \Phi(t) = \mathbb{H}(t) \Phi(t), \\
\Phi(0) = \Phi_0.
\end{cases}
\]

Recall that the quadratic generator \( \mathbb{H}(t) \) in (20) is built up on the Hartree dynamics \( u(t) \) in (18) with \( u_0 \in H^4(\mathbb{R}^3) \). All useful properties of (19) are collected in the following
Lemma 9 (Bounds on Bogoliubov Hamiltonian). For every $\varepsilon > 0$ and $\eta > 0$ we have
\begin{equation}
\pm \left( \|\mathbb{H}(t)\| + \|d\Gamma(\Delta)\| \right) \leq \eta \|\mathbb{H}(1 - \Delta)\| + C_t \mathcal{N} + C_{t,\varepsilon} \eta^{-1} N^{\beta + \varepsilon}, \tag{58}
\end{equation}
\begin{equation}
\pm \partial_t \|\mathbb{H}(t)\| \leq \eta \|\mathbb{H}(1 - \Delta)\| + C_t \mathcal{N} + C_{t,\varepsilon} \eta^{-1} N^{\beta + \varepsilon}, \tag{59}
\end{equation}
\begin{equation}
\pm [i(\mathbb{H}(t), \mathcal{N})] \leq \eta \|\mathbb{H}(1 - \Delta)\| + C_t \mathcal{N} + C_{t,\varepsilon} \eta^{-1} N^{\beta + \varepsilon}. \tag{60}
\end{equation}

To prove Lemma 9 we will use a well-known property on the ground state energy of quadratic Hamiltonians, see e.g. [13, 18]. The following result is taken from [13] Lemma 9.

Lemma 10 (Pairing term estimate). Let $H > 0$ be a self-adjoint operator on $L^2(\mathbb{R}^3)$ and let $K$ be a Hilbert-Schmidt operator on $L^2(\mathbb{R}^3)$ with symmetric kernel $K(x,y) = K(y,x)$ and satisfying $KH^{-1}K^* \leq H$. Then
\begin{equation}
\pm \frac{1}{2} \int \left( K(x,y)a_x^* a_y^* + K(x,y)a_x a_y \right) \, dx \, dy \leq \|\mathbb{H}(H)\| + \frac{1}{2} \|H^{-1/2}K\|^2_{L^2(\mathbb{R}^6)}
\end{equation}
as quadratic forms on Fock space.

In application of Lemma 10 the following kernel estimates will be useful.

Lemma 11 (Kernel estimate). Let $K_2$ be the operator on $L^2(\mathbb{R}^3)$ with kernel $K_2(t,x,y)$ as in (20). Then we have $\|K_2\|_{op} \leq C_t$ and for all $\varepsilon > 0$,
\begin{equation}
\| (1 - \Delta)^{-\frac{1}{2}} K_2(t, \cdot, \cdot) \|^2_{L^2(\mathbb{R}^6)} \leq C_{t,\varepsilon} N^{\beta + \varepsilon}, \tag{61}
\end{equation}
\begin{equation}
\| (1 - \Delta)^{-\frac{1}{2}} \partial_t K_2(t, \cdot, \cdot) \|^2_{L^2(\mathbb{R}^6)} \leq C_{t,\varepsilon} N^{\beta + \varepsilon}. \tag{62}
\end{equation}

Proof of Lemma 11. First, we consider the operator bound. For every $f \in L^2(\mathbb{R}^3)$, we denote $\tilde{f} = Q(t)f$ and use the Cauchy-Schwarz inequality to estimate
\begin{align*}
\|\langle f, K_2 f \rangle \| & = \left| \int \int \int f(x) \tilde{f}(y) |u(t,z)|^2 u(t,x) u(t,y) V_N(x-y,x-z) \, dx \, dy \, dz \right| \\
& \leq \|u(t, \cdot)\|^4_{L^\infty(\mathbb{R}^3)} \int \int \int \frac{|f(x)|^2 + |\tilde{f}(y)|^2}{2} V_N(x-y,x-z) \, dx \, dy \, dz \\
& = \|u(t, \cdot)\|^4_{L^\infty(\mathbb{R}^3)} \|f\|^2_{L^2(\mathbb{R}^3)} \|\tilde{f}\|^2_{L^2(\mathbb{R}^3)} \|V_N\|_{L^1(\mathbb{R}^3)} \leq C_t \|f\|^2_{L^2(\mathbb{R}^3)}.
\end{align*}

Therefore,
\begin{equation}
\|K_2\|_{op} \leq C_t.
\end{equation}

Next, to prove (61) we use an interpolation argument as in [25, 40, 41]. By definition we know that
\begin{equation}
K_2 = \left( Q(t) \otimes Q(t) \right) \tilde{K}_2
\end{equation}
and hence
\begin{equation}
K_2 - \tilde{K}_2 = \left( (Q(t) - 1) \otimes (Q(t) - 1) \right) \tilde{K}_2 + \left( Q(t) \otimes (Q(t) - 1) \right) \tilde{K}_2.
\end{equation}
We now want to prove a $L^2$ bound on the first term, the second term will follow similarly. We see that

$$\left\| \left( Q(t) - 1 \right) \otimes 1 \tilde{K}_2(t, \cdot, \cdot) \right\|^2_{L^2(\mathbb{R}^6)} = \left\| u(t) \langle u(t) \rangle \otimes 1 \tilde{K}_2(t, \cdot, \cdot) \right\|^2_{L^2(\mathbb{R}^6)}$$

$$= \iint \left| u(t, \bar{x}) \tilde{K}_2(t, \bar{x}, y) \, d\bar{x} \right|^2 \left| u(t, x) \right|^2 \, dx \, dy$$

$$= \iint \left| \int u(t, \bar{x})u(t, z) \, u_N(\bar{x} - y, \bar{x} - z) \, u(t, y) \, dz \right|^2 \left| u(t, x) \right|^2 \, dx \, dy$$

$$\leq \| u(t, \cdot) \|^5_{L^\infty(\mathbb{R}^3)} \| V \|_{L^2(\mathbb{R}^6)}^2 \| u(t, \cdot) \|_{L^2(\mathbb{R}^6)} \| u(t, \cdot) \|_{L^2(\mathbb{R}^6)} \leq C_t,$$

where we have used Theorem 3 in the last inequality. This gives

$$\| K_2(t, \cdot, \cdot) - \tilde{K}_2(t, \cdot, \cdot) \|^2_{L^2(\mathbb{R}^6)} \leq C_t.$$ 

Since $(1 - \Delta_x)^{-\frac{1}{2}} \leq 1$ on $L^2$ we see that

$$\| (1 - \Delta_x)^{-\frac{1}{2}} K_2(t, \cdot, \cdot) - (1 - \Delta_x)^{-\frac{1}{2}} \tilde{K}_2(t, \cdot, \cdot) \|^2_{L^2(\mathbb{R}^6)} \leq C_t. \quad (63)$$

Hence, we only need to prove (61) with $K_2$ replace by $\tilde{K}_2$ to get the desired result. We will prove

$$\| (1 - \Delta)^{-\frac{1}{2} - \varepsilon} \tilde{K}_2(t, \cdot, \cdot) \|^2_{L^2(\mathbb{R}^6)} \leq C_t \varepsilon \quad (64)$$

$$\| \tilde{K}_2(t, \cdot, \cdot) \|^2_{L^2(\mathbb{R}^6)} \leq C_t N^\beta \quad (65)$$

for any $\varepsilon > 0$ and then use interpolation. To prove (64) we first calculate the Fourier transform of $\tilde{K}_2$:

$$\tilde{K}_2(t, p, q) = \iint e^{-i2\pi(p \cdot x + q \cdot y)} \int |u(t, z)|^2 V_N(x - y, x - z) \, dz \, u(t, x)u(t, y) \, dx \, dy$$

$$= \iint e^{-i2\pi(p \cdot x + q \cdot y)}|u(t, x - z)|^2 V_N(x - y, z)u(t, x)u(t, y) \, dx \, dy \, dz$$

$$= \iint e^{-i2\pi(p \cdot \bar{x} + (p \cdot q) \cdot y)}|u(t, \bar{x} + y - z)|^2 V_N(\bar{x}, z)u(t, \bar{x} + y)u(t, y) \, d\bar{x} \, dy \, d\bar{z}$$

$$= \iint e^{-i2\pi p \cdot \bar{x} V_N(\bar{x}, z)(|u_{\bar{z} \cdot -}|^2 u_{\bar{z} \cdot}) (p + q) \, d\bar{x} \, d\bar{z}.$$ 

Here we have defined the short-hand notation for the translation $u_{\bar{z} \cdot} = u(\bar{x} + \cdot)$. Therefore, by the Cauchy-Schwarz inequality

$$|\tilde{K}_2(t, p, q)|^2 \leq \left( \int \int |V_N(\bar{x}, z)|(|u_{\bar{z} \cdot}|^2 u_{\bar{z} \cdot} u)(p + q) \, d\bar{x} \, d\bar{z} \right)^2 \leq \| V \|^4_{L^1(\mathbb{R}^6)} \int \int |V_N(\bar{x}, z)|(|u_{\bar{z} \cdot}|^2 u_{\bar{z} \cdot} u)(p + q)^2 \, d\bar{x} \, d\bar{z}.$$

Moreover, using Theorem 3 and Plancherel’s we have that

$$\int |(|u_{\bar{z} \cdot}|^2 u_{\bar{z} \cdot} u)(p + q)|^2 \, dq \leq \| u(t, \cdot) \|^6_{L^\infty(\mathbb{R}^3)} \int |u(t, x)|^2 \, dx = C_t.$$ 

Hence, we see that for all $\varepsilon > 0$

$$\| (1 - \Delta)^{-\frac{1}{2} - \varepsilon} \tilde{K}_2 \|^2_{L^2(\mathbb{R}^6)} = \iint (1 + |p|^2)^{-\frac{1}{2} - 2\varepsilon} |\tilde{K}_2(t, p, q)|^2 \, dp \, dq.$$
\[
\leq \|V\|_{L^1(\mathbb{R}^6)} \iint (1 + |p|^2)^{-\frac{3}{2} - 2\epsilon} \mathcal{V}_N(x, z)|(|u_{x-z}|^2u_{x} + p + q)|^2 \, dp \, dq \, d\tilde{x} \, dz
\leq C_t \|V\|_{L^1(\mathbb{R}^6)}^2 \iint (1 + |p|^2)^{-\frac{3}{2} - 2\epsilon} \, dp = C_{t, \epsilon},
\]
where we have used \( \int (1 + |p|^2)^{-\frac{3}{2} - 2\epsilon} \, dp \leq C_\epsilon \) in three dimensions. To prove (63) we calculate
\[
\|\tilde{K}_2\|_{L^2(\mathbb{R}^6)}^2 = \int \int |u(t, x)|^2 |u(t, y)|^2 \int |u(t, z)|^2 \mathcal{V}_N(x - y, x - z) \, dz \, dx \, dy
\leq \|u(t, \cdot)\|^6_{L^\infty(\mathbb{R}^3)} \int \int |u(t, x)|^2 \int \mathcal{N}^{\beta} \mathcal{V}(\mathcal{N}^\beta(x - y), \mathcal{N}^\beta(x - z)) \, dz \, dx \, dy
= \|u(t, \cdot)\|^6_{L^\infty(\mathbb{R}^3)} \|u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \int \int \mathcal{N}^{\beta} \mathcal{V}(\mathcal{N}^\beta y, z) \, dz \, dy
= C_t \mathcal{N}^{\beta} \int \int \mathcal{V}(y, z) \, dz \, dy = C_t \mathcal{N}^{\beta},
\]
where we have used that \( \mathcal{V} \) has compact support. Therefore, by interpolation
\[
\|(1 - \Delta)^{-\frac{1}{2}} \tilde{K}_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^6)}^2 \leq C_{t, \epsilon} \mathcal{N}^{\beta + \epsilon},
\]
which proves (61).

For the proof of (62) we see that
\[
\partial_t K_2(t) = \partial_t Q(t) \otimes Q(t) \tilde{K}_2(t) + Q(t) \otimes \partial_t Q(t) \tilde{K}_2(t) + Q(t) \otimes Q(t) \partial_t \tilde{K}_2(t).
\]
Similarly to the derivation of (63) one can prove bounds for each term and obtain
\[
\|(1 - \Delta)^{-\frac{1}{2}} \partial_t K_2(t, \cdot, \cdot) - (1 - \Delta)^{-\frac{1}{2}} \partial_t \tilde{K}_2(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^6)} \leq C_t.
\]
Therefore, we again only need to prove (62) with \( \partial_t K_2 \) replaced by \( \partial_t \tilde{K}_2 \). This works similar to the derivation of (61) and we omit the details. This finishes the proof. \( \square \)

Now we are able give

**Proof of Lemma 3** Consider
\[
\mathbb{D}(t) + d\Gamma(\Delta) = d\Gamma(h(t) + \Delta + K_1) + \frac{1}{2} \iint \left( K_2(x, y) a_{x} a_{y}^* + K_2(x, y) \right) a_{x} a_{y} \, dx \, dy.
\]

By definition of \( h(t) \) we have that
\[
h(t) + \Delta = \iint |u(t, y)|^2 \mathcal{V}_N(x - y, x - z)|u(t, z)|^2 \, dy \, dz
\]
which is a multiplication operator. By Theorem 3 the corresponding function can be bounded by
\[
\left| \iint |u(t, y)|^2 \mathcal{V}_N(x - y, x - z)|u(t, z)|^2 \, dy \, dz \right| \leq \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}^4 \mathcal{V}_N \|L^1(\mathbb{R}^6) \leq C_t. \quad (66)
\]

Moreover, the operator \( K_1 \) in (20) satisfies
\[
\|K_1\|_{op} \leq C_t \quad (67)
\]
(this can be proved similarly to the bound \( \|K_2\|_{op} \leq C_t \) in Lemma 11). Thus we have proved that \( \pm(h(t) + \Delta + K_1) \leq C_t \), and hence
\[
\pm d\Gamma(h + \Delta + K_1) \leq C_d \mathcal{N}. \quad (68)
\]
Next, by applying the paring term estimate in Lemma 10 with 
\[ H = \eta (1 - \Delta) + \|K_2\|_{op}, \quad \eta > 0, \]
and the kernel estimate in Lemma 11 we find that
\[
\pm \frac{1}{2} \int \int \left( K_2(t, x, y)a_x^*a_y^* + \overline{K_2(t, x, y)a_xa_y} \right) 
\leq \eta d\Gamma(1 - \Delta) + \|K_2\|_{op}N + \eta^{-1}\|1 - \Delta\|_2^{-1/2}K_2\|_{L^2(\mathbb{R}^6)}^2 
\leq \eta d\Gamma(1 - \Delta) + C_\eta N + C_{t,\varepsilon}^{-1}N^{1/2}. \tag{69}
\]
Combining this with (68) we conclude
\[
\pm \left( H(t) + d\Gamma(\Delta) \right) \leq \eta d\Gamma(1 - \Delta) + C_\eta N + C_{t,\varepsilon}^{-1}N^{3/2 + \varepsilon}
\]
which finishes the proof of (68). The bound on \[ \partial_t H(t) \] can be proven similarly. Moreover, we see that
\[
i[H(t), N] = - \int \int \left( iK_2(t, x, y)a_x^*a_y^* + i\overline{K_2(t, x, y)a_xa_y} \right) \, dx \, dy
\]
and (69) also follows from the same argument. This ends the proof. \[ \square \]

Finally we conclude

**Proof of Theorem 8.** Using the bound in Lemma 9, the existence and uniqueness of the solution \( \Phi(t) \in F(u(t)^{-1}) \) to the Bogoliubov equation [19] follow from the abstract results in [37, Theorems 7, 8].

It remains to prove the kinetic bound (57). By Lemma 9 we have
\[
A(t) = H(t) + C_{t,\varepsilon}(N + N^{\beta + \varepsilon}) \geq \frac{1}{2}d\Gamma(1 - \Delta),
\]
if we choose \( C_{t,\varepsilon} \) large enough. Using the equation for \( \Phi(t) \), we see that
\[
\frac{d}{dt} \langle \Phi(t), A(t)\Phi(t) \rangle = \langle \Phi(t), \partial_t A(t)\Phi(t) \rangle + \langle \Phi(t), i[H(t), A(t)]\Phi(t) \rangle 
\leq \langle \Phi(t), \partial_t (H(t) + \partial_t C_{t,\varepsilon}(N + N^{\beta + \varepsilon}))\Phi(t) \rangle + \langle \Phi(t), i[H(t), N]\Phi(t) \rangle 
\leq C_{t,\varepsilon}\langle \Phi(t), A(t)\Phi(t) \rangle.
\]
Thus, using Gronwall’s inequality we get
\[
\langle \Phi(t), A(t)\Phi(t) \rangle \leq e^{C_{t,\varepsilon}}\langle \Phi(0), A(0)\Phi(0) \rangle.
\]
Since
\[
A(0) \leq C_\varepsilon(d\Gamma(1 - \Delta) + N^{\beta + \varepsilon}),
\]
we obtain that
\[
\langle \Phi(t), d\Gamma(1 - \Delta)\Phi(t) \rangle \leq 2\langle \Phi(t), A(t)\Phi(t) \rangle \leq C_\varepsilon e^{C_{t,\varepsilon}}\left( \langle \Phi(0), d\Gamma(1 - \Delta)\Phi(0) \rangle + N^{\beta + \varepsilon} \right).
\]
This finishes the proof of the kinetic estimate (57). \[ \square \]
5. Transformation of the many-body dynamics

Our general strategy to derive effective equations from the many-body Schrödinger equation (2) is similar to that in the pair-interaction case \cite{37, 40, 41, 11}. Let \( \{u(t)\} \) be the Hartree dynamics and recall from \cite{36} Section 2.3 the following operator

\[
U_N(t) = \bigoplus_{k=0}^{N} Q(t)^{\otimes k} \frac{(a(u(t)))^{N-k}}{\sqrt{(N-k)!}}, \quad Q(t) = 1 - |u(t)\rangle\langle u(t)|.
\]

(70)

It is a unitary operator from \( L^2_2(\mathbb{R}^3)^N \) to the truncated Fock space

\[
\mathcal{F}_{+}^{\leq N}(t) = \mathbb{1}_{\leq N} \mathcal{F}_{+}(t) = \bigoplus_{n=0}^{N} \left( \{u(t)\} \right)^{\otimes n}, \quad \mathbb{1}_{\leq N} = \mathbb{1}(N \leq N)
\]

with the inverse

\[
U_N(t)^* = \bigoplus_{k=0}^{N} (a^*(u(t)))^{N-k} \frac{\sqrt{(N-k)!}}{\sqrt{(N-k)!}}.
\]

(71)

Of course we can extend \( U_N(t)^* \) to the whole Fock space \( \mathcal{F}_{+}(t) \) by setting value 0 outside the truncated space \( \mathcal{F}_{+}^{\leq N}(t) \) (in this way \( U_N(t) \) is a partial unitary operator from \( L^2_2(\mathbb{R}^3)^N \) to \( \mathcal{F}_{+}(t) \)).

As explained in \cite{36}, \( U_N(t) \) provides a rigorous implementation of the c-number substitution in Bogoliubov’s heuristic argument \cite{36}, via the actions

\[
\begin{align*}
U_N(t)a^*(u(t))a(u(t))U_N(t)^* & = N - \mathcal{N}, \\
U_N(t)a^*(f)a(u(t))U_N(t)^* & = a^*(f)\sqrt{N - \mathcal{N}}, \\
U_N(t)a^*(u(t))a(f)U_N(t)^* & = \sqrt{N - \mathcal{N}}a(f), \\
U_N(t)a^*(f)a(g)U_N(t)^* & = a^*(f)a(g)
\end{align*}
\]

(72)

where \( f, g \in \{u(t)\} \). When \( \mathcal{N} \ll N \) with \( \mathcal{N} \) the particle number operator, the quantity \( \sqrt{N - \mathcal{N}} \) is close to the scalar value \( \sqrt{N} \), leading to a quantitative justification of Bogoliubov’s approximation in the sector of few particles.

The unitary operator \( U_N(t) \) allows us to transform the Schrödinger equation (2) to an equation in Fock space. Recall the phase factor in \cite{23}:

\[
\chi(t) = \frac{2N + 3}{6} \iiint V_N(x - y, x - z)|u(t, x)|^2|u(t, y)|^2|u(t, z)|^2 \, dx \, dy \, dz.
\]

Lemma 12 (Transformed many-body dynamics). Let \( \Psi_N(t) \) be the solution of (2). Then

\[
\Phi_N(t) := e^{-i\int_0^t \chi(s) \, ds} U_N(t) \Psi_N(t) \in \mathcal{F}_{+}^{\leq N}(t)
\]

solves

\[
i\partial_t \Phi_N(t) = \hat{H}_N(t) \Phi_N(t) = \left( \hat{H}(t) + \frac{1}{2} \sum_{j=0}^{6} (R_j + R_j^*) \right) \Phi_N(t)
\]

(74)

where

\[
R_0 = \frac{1}{6} \left\langle u(t)^{\otimes 3}, V_N u(t)^{\otimes 3} \right\rangle \left( \frac{3\mathcal{N}^2 + 6\mathcal{N} + 2}{N} - \frac{\mathcal{N}(\mathcal{N} + 1)(\mathcal{N} + 2)}{N^2} \right)
\]

\[
+ \, d\Gamma \left( Q(t) \left( \frac{1}{2} \iint |u(t, y)|^2 V_N |u(t, z)|^2 \, dy \, dz + K_1 \right) Q(t) \right) \left( \frac{(N - \mathcal{N})(N - \mathcal{N} - 1)}{N^2} - 1 \right)
\]
\[ R_1 = \left( \frac{(N - N)(N - N - 1)}{N^2} - 1 \right) \sqrt{N - N} a \left( Q(t) \int |u(t, y)|^2 V_N \right), \]

\[ R_2 = \int \int K_2(t, x, y) a_x^* a_y^* \left( \sqrt{N - N - 1} V_N (N - N - 2) - 1 \right), \]

\[ R_3 = \frac{1}{3N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t)) V_N 1 \otimes 1 \otimes 1 (x, y, z; x', y', z') u(t, x') u(t, y') u(t, z') \]

\[ \times a_x^* a_y^* a_z^* \ dx \ dy \ dz \ dx' \ dy' \ dz' \sqrt{N - N - 2} \sqrt{N - N - 1} \]

\[ + \frac{2}{N^2} \int \int \int (Q(t) \otimes Q(t) \otimes Q(t)) \int |u(t, z)|^2 V_N \ dx Q(t) \otimes 1 (x, y; x', y') u(t, y') \times \]

\[ \times a_x^* a_y^* a_x' \ dx \ dy \ dz \ dx' \ dy' \ dz' \sqrt{N - N - N - 1} \]

\[ + \frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t))(x, y, z; x', y', z') u(t, z) u(t, x') u(t, z') \times \]

\[ \times a_x^* a_y^* a_x' \ dx \ dy \ dz \ dx' \ dy' \ dz' \sqrt{N - N - N - 1} \]

\[ R_4 = \frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t)) V_N Q(t) \otimes 1 \otimes 1 (x, y, z; x', y', z') u(t, y') u(t, z') \times \]

\[ \times a_x^* a_y^* a_z^* \sqrt{N - N - 1} \int \int \ dx \ dy \ dz \ dx' \ dy' \ dz' \]

\[ + \frac{1}{2N^2} \int \int \int (Q(t) \otimes Q(t) \otimes Q(t)) \int |u(t, z)|^2 V_N \ dx Q(t) \otimes Q(t)(x, y; x', y') \times \]

\[ a_x^* a_y^* a_x' \ dx \ dy \ dz \ dx' \ dy' \sqrt{N - N} \]

\[ + \frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t))(x, y, z; x' y', z') u(t, z) u(t, y') \times \]

\[ \times a_x^* a_y^* a_x' \ dx \ dy \ dz \ dx' \ dy' \ dz' \sqrt{N - N - 1} \]

\[ R_5 = \frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t)) V_N Q(t) \otimes Q(t) \otimes 1 (x, y, z; x', y', z') u(t, z') \times \]

\[ a_x^* a_y^* a_z^* \sqrt{N - N - 2} \int \int \ dx \ dy \ dz \ dx' \ dy' \ dz' \]

\[ R_6 = \frac{1}{6N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t)) V_N Q(t) \otimes Q(t) \otimes Q(t)(x, y, z; x', y', z') \times \]

\[ \times a_x^* a_y^* a_x' a_y' \ dx \ dy \ dz \ dx' \ dy' \ dz'. \]

**Proof.** From the definition (33) and the Schrödinger equation (12) we find that

\[ i \partial_t \Phi_N(t) = \left( \chi(t) + i \dot{U}_N(t)U_N^* t + U_N(t)H_NU_N^* t \right) \Phi_N(t). \]  

(75)

The time-derivative of \( U_N(t) \) has been computed in [37, Lemma 6]

\[ i \dot{U}_N(t)U_N^* t = a^*(u(t))a(Q(t)i\dot{u}(t)) - \sqrt{N - N} a(Q(t)i\dot{u}(t)) \]

\[ - a^*(Q(t)i\dot{u}(t))\sqrt{N - N} - (i\dot{u}(t), u(t))(N - N) \]

\[ = a^*(u(t))a(Q(t)h(t)u(t)) - \sqrt{N - N} a(Q(t)h(t)u(t)) \]

\[ - a^*(Q(t)h(t)u(t))\sqrt{N - N} - (u(t), h(t)u(t))(N - N). \]  

(76)

Here in the last identity we have used the Hartree equation \( i\dot{u}(t) = h(t)u(t) \).

Then we compute the conjugation \( U_N(t)H_NU_N^* t \) using (13) and (72):

\[ U_N(t)H_NU_N^* t = \frac{1}{6N^2} \langle u(t) \rangle^{\otimes 3}, V_N u(t)^{\otimes 3}(N - N - 1)(N - N - 2)(N - N - 3) \]
It remains to sum (76) and (77), then use the identity (Lemma 13) exactly implies the equation (74) from (75).

Equation (74) can be approximated by the Bogliubov equation (19). The task of estimating \( F \) for all \( L \) with kernel \( F \). Then we have

\[ \| \nabla u(t, \cdot) \|_{L^2}^2 (N - N) + \sqrt{N - N} a(Q(t)h(t)u(t)) + a^*(Q(t)h(t)u(t)) \sqrt{N - N} \]

\[ + d \Gamma(Q(t)h(t)Q(t)) + K_1 + \frac{1}{2} \iint \left( K_2(t, x, y) a^*_x a_y + \overline{K_2(t, x, y)} a_x a_y \right) dx \, dy \]

\[ + \frac{1}{2} \sum_{j=1}^6 (R_j + R_j^*). \]  

(77)

The choice of the phase factor \( \chi(t) \) in (23),

\[ \chi(t) = \frac{2N + 3}{6} \langle u(t)^{\otimes 3}, V_N u(t)^{\otimes 3} \rangle, \]

exactly implies the equation (74) from (75).

Formally, in the limit \( N \to \infty \) all the error terms \( R_j' \)'s are negligible and the transformed equation (74) can be approximated by the Bogliubov equation (19). The task of estimating all \( R_j' \)'s will be carried out in the next section.

6. Operator bounds on Fock space

Now we collect various useful bounds on the the error terms \( R_j' \)'s in (74). We will prove

**Lemma 13 (Error bounds on truncated Fock spaces).** For \( 0 \leq m \leq N \), we have the following quadratic form bounds on the truncated Fock space \( F_+^{\leq m}(t) \):

\[ \pm (R_j + R_j^*) \leq \eta \Gamma(1 - \Delta) + \eta^{-1} C \mu N^{4\beta - 1}, \]  

(78)

\[ \pm i[R_j + R_j^*, N] \leq \eta \Gamma(1 - \Delta) + \eta^{-1} C \mu N^{4\beta - 1}, \]  

(79)

\[ \pm \partial_t (R_j + R_j^*) \leq \eta \Gamma(1 - \Delta) + \eta^{-1} C \mu N^{4\beta - 1}, \]  

(80)

for all \( j = 0, 1, 2, 3, 4, 5, 6 \) and for all

\[ \eta \geq C \mu \max \{ \sqrt{m} N^{2\beta - 1}, \sqrt{m^2} N^{5\beta - 3}, m^2 N^{4\beta - 2} \}. \]

We will use the following well-known Sobolev type estimate (see [44, Lemma 3.2]).

**Lemma 14 (Kinetic bound for translation-invariant potentials).** For \( 0 \leq W \in L^{3/2}(\mathbb{R}^3) \), the multiplication operator \( W(x - y) \) satisfies

\[ 0 \leq W(x - y) \leq \| W \|_{L^{3/2}(\mathbb{R}^3)} (\Delta_x) \]  

(81)

as quadratic forms on \( L^2(\mathbb{R}^3 \times \mathbb{R}^3) \).

We will also need the following kernel estimate (c.f. Lemma 11).

**Lemma 15 (Kernel estimate).** For every \( z \in \mathbb{R}^3 \) fixed, let \( k_z \) be the operator on \( L^2(\mathbb{R}^3) \) with kernel

\[ k_z(x, y) = u(t, x) u(t, y) V_N(x - y, x - z). \]

Then we have

\[ \| (1 - \Delta_x)^{-1/2} k_z \|_{L^2(\mathbb{R}^3)}^2 \leq C \mu N^{4\beta}. \]  

(82)
Proof of Lemma 13. First we calculate the Fourier transform
\[
\hat{k}_z(p, q) = \int e^{-i2\pi(p \cdot x + q \cdot y)} u(t, x) u(t, y) V_N(x - y, x - z) \, dx \, dy
\]

\[
e^{-i2\pi p \cdot z} \int e^{-i2\pi N^{-\beta}(p \cdot x + q \cdot y)} u(t, N^{-\beta} x + z) u(t, N^{-\beta} y) V(x - y, x) \, dx \, dy
\]

\[
e^{-i2\pi p \cdot z} \hat{f}(N^{-\beta} p, N^{-\beta} q),
\]

where we have defined \( f(x, y) = u(t, N^{-\beta} x + z) u(t, N^{-\beta} y) V(x - y, x) \). From this and Hardy’s inequality we get

\[
\| (1 - \Delta_x)^{-1/2} k_z \|_{L^2}^2 = \iint (1 + |p|^2)^{-1} |\hat{f}(N^{-\beta} p, N^{-\beta} q)|^2 \, dp \, dq
\]

\[
= N^{4\beta} \iint (N^{-2\beta} + |p|^2)^{-1} |\hat{f}(p, q)|^2 \, dp \, dq
\]

\[
\leq N^{4\beta} \iint |\hat{f}(p, q)|^2 \, dp \, dq \leq 4N^{4\beta} \iint |\nabla_p \hat{f}(p, q)|^2 \, dp \, dq
\]

\[
= 4N^{4\beta} \iint |x|^2 |u(t, N^{-\beta} x)|^2 |u(t, N^{-\beta} y)|^2 |V(x - y, x)| \, dx \, dy
\]

\[
\leq C_t N^{4\beta} \iint |x|^2 V(x - y, x) \, dx \, dy \leq C_t N^{4\beta}.
\]

In the last step we have used that \( V \) has compact support. This finishes the proof. \( \square \)

Proof of Lemma 15. Proof of (78). We proceed term by term.

\[ j = 0 \] Let us consider

\[
R_0 = \frac{1}{6} \left\langle u(t)^{\otimes 3}, V_N u(t)^{\otimes 3} \right\rangle \left( \frac{3N^2 + 6N + 2}{N} - \frac{N(N + 1)(N + 2)}{N^2} \right),
\]

\[
+ \, d\Gamma \left( Q(t) \left( \frac{1}{2} \iint |u(t, y)|^2 V_N|u(t, z)|^2 \, dy \, dz + K_1 \right) Q(t) \right) \left( \frac{(N - N)(N - N - 1)}{N^2} - 1 \right)
\]

\[
=: R_{0,1} + R_{0,2}.
\]

For the first term \( R_{0,1} \), using Theorem 13 we know that

\[
\iint |u(t, x)|^2 |u(t, y)|^2 |u(t, z)|^2 V_N(x - y, x - z) \, dx \, dy \, dz \leq C_t.
\]

Moreover,

\[
\left| \frac{3N^2 + 6N + 2}{N} - \frac{N(N + 1)(N + 2)}{N^2} \right| \leq C \frac{(N + 1)^2}{N}
\]

With that we obtain

\[
\pm R_{0,1} \leq C_t \frac{(N + 1)^2}{N}.
\]

For the second term \( R_{0,2} \), from the one particle operator bounds (66) and (67) we get

\[
\pm d\Gamma \left( Q(t) \left( \frac{1}{2} \iint |u(t)|^2 V_N|u(t)|^2 \, dy \, dz + K_1 \right) Q(t) \right) \leq C_t N.
\]
Moreover, we have the simple bound on $\mathcal{F}_+^{\leq N}(t)$
\[
\left| \frac{(N - N)(N - N - 1)}{N^2} - 1 \right| \leq C\frac{(N + 1)}{N}.
\]
Putting both together (the relevant operators commute) we obtain
\[
\pm R_{0,2} \leq C_t \frac{(N + 1)^2}{N}.
\]
Thus in summary, as quadratic forms on $\mathcal{F}_+^{\leq m}(t)$,
\[
\pm R_0 \leq C_t \frac{(N + 1)^2}{N} \leq C_t \frac{m}{N} (N + 1).
\]
Note that $N \leq d \Gamma (1 - \Delta)$.

\section{Derivation of 3D Energy-Critical NLS and Excitations}

\subsection*{Derivation}

We consider
\[
R_1 = \left( \frac{(N - N)(N - N - 1)}{N^2} - 1 \right) \sqrt{N - N} \ a \left( Q(t) \int \int |u(t)|^2 V_N |u(t)|^2 \ dy \ dz \ u(t) \right).
\]

For any $\Phi \in \mathcal{F}_+^{\leq m}(t)$, by the Cauchy-Schwarz inequality we see that
\[
|\langle \Phi, R_1 \Phi \rangle| \leq 2 \left\| \left( \frac{(N - N)(N - N - 1)}{N^2} - 1 \right) \sqrt{N - N} \Phi \right\| \times
\]
\[
\times \left\| a \left( Q(t) \frac{1}{2} \int \int |u(t)|^2 V_N |u(t)|^2 \ dy \ dz \ u(t) \right) \right\|.
\]

For the second term we use the obvious inequality $a^*(v)a(v) \leq \|v\|^2 \mathcal{N}$ combined with
\[
\left\| Q(t) \frac{1}{2} \int \int |u(t)|^2 V_N |u(t)|^2 \ dy \ dz \ u(t) \right\|^2_{L^2(\mathbb{R}^3)}
\]
\[
\leq \left\| \frac{1}{2} \int \int |u(t)|^2 V_N |u(t)|^2 \ dy \ dz \ u(t) \right\|^2_{L^2(\mathbb{R}^3)}
\]
\[
\leq \|u(t, \cdot)\|^2_{L^8(\mathbb{R}^3)} \|V\|_{L^1(\mathbb{R}^3)} \|u(t, \cdot)\|^2_{L^2(\mathbb{R}^3)} = C_t.
\]

For the first term, we use the simple bound
\[
\left| \left( \frac{(N - N)(N - N - 1)}{N^2} - 1 \right) \sqrt{N - N} \right| \leq C\frac{N + 1}{\sqrt{N}}.
\]

Putting together, and using $\Phi \in \mathcal{F}_+^{\leq m}(t)$ we obtain
\[
|\langle \Phi, R_1 \Phi \rangle| \leq C_t \left\langle \Phi, \mathcal{N} \Phi \right\rangle \frac{1}{2} \left\langle \Phi, \frac{(N + 1)^2}{N} \Phi \right\rangle \frac{1}{2} \leq C_t \sqrt{\frac{m}{N}} (\Phi, (N + 1) \Phi).
\]

Thus we have the quadratic form estimate on $\mathcal{F}_+^{\leq m}(t)$:
\[
\pm (R_1 + R_1) \leq C_t \sqrt{\frac{m}{N}} (N + 1).
\]

\section{Derivation of 3D Energy-Critical NLS and Excitations}

\subsection*{Derivation}

We consider
\[
R_2 = \int \int K_2(t, x, y) a_x a_y \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right).
\]
For any \( \Phi \in F_+^{\leq m}(t) \), we have
\[
|\langle \Phi, R_2 \Phi \rangle| = \left| \iint K_2(t, x, y) \langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right) \Phi \rangle \, dx \, dy \right|
\]
\[
= \iiint V_N(x - y, x - z)|u(t, z)|^2 u(t, x) u(t, y) \times \langle a_x a_y \Phi, \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right) \Phi \rangle \, dx \, dy \, dz
\]
\[
\leq \|u(t, \cdot)\|_{L^\infty}^3 \iint V_N(x - y, x - z)|u(t, z)| \times \|a_x a_y \Phi\| \left\| \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right) \Phi \right\| \, dx \, dy \, dz. \tag{83}
\]

In the above we could replace \( K_2(t, x, y) = Q(t) \otimes Q(t) \tilde{K}_2(t, x, y) \) by \( \tilde{K}_2(t, x, y) \), namely could ignore the projection \( Q(t) \), since \( \Phi \) belongs to the excited Fock space \( F_+^{\leq m}(t) \).

In \( 83 \) we can use again \( \|u(t, \cdot)\|_{L^\infty} \leq C_{t} \). Moreover, using \( \sqrt{1 - s} = 1 + O(s) \) with \( s > 0 \) small, it is straightforward to see that
\[
\left| \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right| \leq C \frac{N + 1}{N},
\]
and hence
\[
\left\| \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right) \Phi \right\| \leq C \left( \Phi, \frac{(N + 1)^2}{N^2} \Phi \right)^{\frac{1}{2}} \leq C \frac{\sqrt{m}}{N} \langle \Phi, (N + 1) \Phi \rangle^{1/2}
\]
because \( \Phi \in F_+^{\leq m}(t) \).

Next, by the Cauchy-Schwarz inequality we can estimate
\[
\iint V_N(x - y, x - z)|u(t, z)||a_x a_y \Phi| \, dx \, dy \, dz
\]
\[
\leq \left( \iint V_N(x - y, x - z)|u(t, z)|^2 \, dx \, dy \, dz \right)^{1/2} \times
\]
\[
\times \left( \iint V_N(x - y, x - z)||a_x a_y \Phi|^2 \, dx \, dy \, dz \right)^{1/2}
\]
\[
\leq C \left( \iint V_N(x - y, x - z)||a_x a_y \Phi|^2 \, dx \, dy \, dz \right)^{1/2}.
\]

By Lemma \( 14 \) we can estimate
\[
\iint V_N(x - y, x - z) \, dz = \int N^{3\beta} V(N^\beta(x - y), z) \, dz \leq C N^{\beta} \Delta_x.
\]
Therefore,
\[
\iint V_N(x - y, x - z)a_x^* a_y^* a_x a_y \, dz \leq C N^{\beta} d\Gamma(-\Delta) N, \tag{84}
\]
and hence, since \( \Phi \in F_+^{\leq m}(t) \),
\[
\iint V_N(x - y, x - z)||a_x a_y \Phi|^2 \, dx \, dy \, dz \leq C N^{\beta} m \langle \Phi, d\Gamma(-\Delta) \Phi \rangle. \tag{85}
\]
Thus we can conclude from \(^{33}\) that
\[
|\langle \Phi, R_2 \Phi \rangle| \leq C_t mN^{\beta/2-1}(\Phi, d\Gamma(1-\Delta)\Phi).
\]
Therefore, we have the quadratic form bound on \(F_+^{\leq m}(t)\):
\[
\pm (R_2 + R_3^2) \leq C_t mN^{\beta/2-1}d\Gamma(1-\Delta).
\] (86)

Note that \(mN^{\beta/2-1} \leq \sqrt{mN^{2\beta-1}}\) when \(m \ll N\).

\[j = 3\] Now we consider
\[
R_3 = \frac{1}{3N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t)V_N 1 \otimes 1 \otimes 1) (x, y, z; x', y', z') u(t, x') u(t, y') u(t, z')
\]
\[
\times a_x a_y a_z \frac{dx \, dy \, dz \, dx' \, dy' \, dz'}{\sqrt{N-N-2N-N-1}}
\]
\[
+ \frac{2}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \int |u(t, z)|^2 V_N \, dz Q(t) \otimes 1)(x, y; x', y') u(t, y') \times
\]
\[
\times a_x a_y a_z \frac{dxdy \, dx' \, dy'}{\sqrt{N-N}(N-N-1)}
\]
\[
+ \frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes 1 V_N 1 \otimes 1 \otimes Q(t))(x, y, z; x', y', z') u(t, z) u(x, x') u(t, z')
\]
\[
\times a_x a_y a_z \frac{dxdy \, dx' \, dy' \, dz'}{\sqrt{N-N}(N-N-1)}
\]
\[
=: R_{3.1} + R_{3.2} + R_{3.3}.
\]

As before, we will estimate the expectation of \(R_3\) against an arbitrary vector \(\Phi \in F_+^{\leq m}(t)\).
Again, we can ignore the projection \(Q(t)\), since \(\Phi\) belongs to the excited Fock space \(F_+(t)\).

\[R_{3.1}\] This is the most complicated term (which does not appear in the pair-interaction case). We start with
\[
|\langle \Phi, R_{3.1} \Phi \rangle| = \frac{1}{3N^2} \int \cdots \int V_N (x-y, x-z) u(t, x) u(t, y) u(t, z) \langle \Phi, a_x a_y a_z \Phi_1 \rangle \, dx \, dy \, dz
\]
\[
\leq \frac{1}{3N^2} \int |u(t, z)| \left| \left\langle a_x \Phi, \left( \int k_z(x, y) a_y a_z \, dx \, dy \right) \Phi_1 \right| \right| \, dz
\]
with
\[
\Phi_1 = \sqrt{N-N-2N-N-1} \sqrt{N-N} \Phi,
\]
\[
k_z(x, y) = V_N (x-y, x-z) u(t, x) u(t, y).
\] (87)

For every \(z \in \mathbb{R}^3\) fixed, we can think of \(k_z(x, y)\) is the kernel of an operator \(k_z\) on \(L^2(\mathbb{R}^3)\).
Let us prove that
\[
\pm (k_z + k_z^*) \leq C_t N^\beta (-\Delta).
\] (88)

Indeed, for every \(f \in L^2(\mathbb{R}^3)\) by the Cauchy-Schwarz inequality and Lemma 14 we have
\[
|\langle f, k_z f \rangle| \leq \int \int |f(x)||f(y)||V_N (x-y, x-z)| |u(t, x)||u(t, y)| \, dx \, dy
\]
\[
\leq \|u(t, \cdot)\|_{L^\infty}^2 \frac{|f(x)|^2 + |f(y)|^2}{2} V_N (x-y, x-z) \, dx \, dy
\]
\[
\leq C_t \int |f(x)|^2 N^{3\beta} \left( \int V(y, N^\beta (x-z)) \, dy \right) \, dx
\]
Combining with the kernel estimate in Lemma 15 we find that

\[ z_0 \forall z \in \mathbb{R}^3 \]  

Thus for every \( z \in \mathbb{R}^3 \) fixed, we can use the pairing term estimate in Lemma 10 with \( K(x, y) = k_z(x, y), \quad H = \eta C_t N^3(1 - \Delta), \quad \eta \geq 1. \)

Combining with the kernel estimate in Lemma 15, we find that

\[ \pm \frac{1}{2} \left( \int \int k_z(x, y) a_x^* a_y^* \, dx \, dy + h.c. \right) \leq C_t \left( \eta N^3 \, d\Gamma(1 - \Delta) + \eta^{-1} N^3 \right), \quad \forall \eta \geq 1. \quad (89) \]

Using (89) and the Cauchy-Schwarz inequality we also have

\[ \left| \langle a_z \Phi, \left( \int \int k_z(x, y) a_x^* a_y^* \, dx \, dy \right) \Phi_1 \rangle \right| \leq C_t \left( \eta N^3 \, d\Gamma(1 - \Delta) + \eta^{-1} N^3 \right) \Phi_1^{1/2} \left( \langle a_z \Phi, \left( \eta N^3 \, d\Gamma(1 - \Delta) + \eta^{-1} N^3 \right) a_z \Phi \rangle \right)^{1/2}. \]

Integrating the above estimate against \( |u(t, z)| \, dz \) and using the Cauchy-Schwarz inequality we get

\[ \left| \langle \Phi, R_{3,1} \Phi \rangle \right| \leq \frac{1}{3N^2} \int |u(t, z)| \left| \int \int k_z(x, y) a_x^* a_y^* \langle a_z \Phi, \Phi_1 \rangle \, dx \, dy \right| \, dz \]

\[ \leq \frac{C_t}{N^2} \left( \eta N^3 \, d\Gamma(1 - \Delta) + \eta^{-1} N^3 \right) \Phi_1^{1/2} \times \]

\[ \int |u(t, z)| \left( \langle a_z \Phi, \left( \eta N^3 \, d\Gamma(1 - \Delta) + \eta^{-1} N^3 \right) a_z \Phi \rangle \right)^{1/2} \, dz \]

\[ \leq \frac{C_t}{N^2} \left( \eta N^3 \, d\Gamma(1 - \Delta) + \eta^{-1} N^3 \right) \Phi_1^{1/2} \times \]

\[ \|u(t, \cdot)\|_{L^2} \left( \int \left( \langle a_z \Phi, \left( \eta N^3 \, d\Gamma(1 - \Delta) + \eta^{-1} N^3 \right) a_z \Phi \rangle \right) \, dz \right)^{1/2}. \]

The term involving \( \Phi_1 \) can be estimated using

\[ 0 \leq \sqrt{N - \mathcal{N}} - 2 \sqrt{N - \mathcal{N}} - 1 \sqrt{N - \mathcal{N}} \leq N^{3/2} \]

on \( F_{\pm}^{\leq m} (t) \) (and that \( \mathcal{N} \) commutes with \( d\Gamma(1 - \Delta) \)). For the other term, we use

\[ \int a_x^* a_z \, dz = \mathcal{N} \]

and

\[ \int a_z^* d\Gamma(1 - \Delta) a_z \, dz = \int \int a_z^* a_x^* (1 - \Delta_x) a_x a_z \, dx \, dz \]

\[ = \int \int a_x^* (1 - \Delta_x) (a_x^* a_x) a_z \, dx \, dz \]

\[ = \int \int a_x^* (1 - \Delta_x) (a_x a_x^* - \delta_{x = z}) a_z \, dx \, dz \]

\[ = d\Gamma(1 - \Delta)(\mathcal{N} - 1). \]
Therefore, when $\Phi \in F_+^m(t)$, we have
\[
|\langle \Phi, R_{3,1} \Phi \rangle| \leq C_t \sqrt{\frac{m}{N}} \left( \Phi, \left( \eta N^\beta \, d\Gamma(1 - \Delta) + \eta^{-1} N^{3\beta} \right) \Phi \right), \quad \forall \eta \geq 1.
\]
Thus
\[
\pm (R_{3,1} + R_{3,1}^*) \leq C_t \left( \eta \sqrt{m N^{2\beta-1}} d\Gamma(1 - \Delta) + \eta^{-1} \sqrt{m N^{6\beta-1}} \right), \quad \forall \eta \geq 1.
\]
This is equivalent to
\[
\pm (R_{3,1} + R_{3,1}^*) \leq \eta d\Gamma(1 - \Delta) + \eta^{-1} C_t m N^{4\beta-1}, \quad \forall \eta \geq 1.
\]

[R3.2] By the Cauchy-Schwarz inequality we have that
\[
|\langle \Phi, R_{3,2} \Phi \rangle| = \frac{2}{N^2} \left| \iiint V_N(x - y, x - z) |u(t, z)|^2 u(t, y) \times \Phi, a_x^*_y a_x \sqrt{N - N(N - N - 1)} \Phi \right| dx \, dy \, dz
\]
\[
\leq \frac{2}{N^2} \left( \iiint V_N(x - y, x - z) |a_x a_y \Phi|^2 dx \, dy \, dz \right)^{\frac{1}{2}} \times \left( \iiint V_N(x - y, x - z) \left| a_x \sqrt{N - N(N - N - 1)} \Phi \right|^2 dx \, dy \, dz \right)^{\frac{1}{2}}.
\]
The first term has been estimated as in [35],
\[
\iiint V_N(x - y, x - z) |a_x a_y \Phi|^2 dx \, dy \, dz \leq CN^\beta m(\Phi, d\Gamma(-\Delta)\Phi).
\]
The second term can be computed explicitly (by doing the integration over $y, z$ first)
\[
\iiint V_N(x - y, x - z) |a_x \sqrt{N - N(N - N - 1)} \Phi|^2 dx \, dy \, dz
\]
\[
= \left( \iiint V(x, y) dx \, dy \right) \int \left| a_x \sqrt{N - N(N - N - 1)} \Phi \right|^2 dx
\]
\[
= \left( \iiint V(x, y) dx \, dy \right) \left( \Phi, C \sqrt{N - N(N - N - 1)} \sqrt{N - N(N - N - 1)} \Phi \right)
\]
\[
\leq CN^3 \left( \Phi, (N + 1) \Phi \right).
\]
Putting all together, we obtain
\[
|\langle \Phi, R_{3,2} \Phi \rangle| \leq \frac{C_t}{N^2} \sqrt{N^\beta m(\Phi, d\Gamma(-\Delta)\Phi)} \sqrt{N^3 \left( \Phi, (1 + N) \Phi \right)} \leq C_t \sqrt{m N^{\beta-1}} (\Phi, d\Gamma(1 - \Delta)\Phi).
\]
Thus
\[
\pm (R_{3,2} + R_{3,2}^*) \leq C_t \sqrt{m N^{\beta-1}} (\Phi, d\Gamma(1 - \Delta)\Phi).
\]

[R3.3] Using the Cauchy-Schwarz inequality we get for $\Phi \in F_+^m(t)$
\[
|\langle \Phi, R_{3,3} \Phi \rangle| = \frac{1}{N^2} \left| \iiint V_N(x - y, x - z) u(t, z) u(t, x) u(t, y) \times \right.
\]
\[
\left. \Phi, \beta_m \sqrt{N - N(N - N - 1)} \Phi \right| dx \, dy \, dz
\]
\[
\leq \frac{1}{N^2} \left( \iiint V_N(x - y, x - z) |u(t, z) u(t, x) u(t, y)|^2 dx \, dy \, dz \right)^{\frac{1}{2}} \times \left( \iiint V_N(x - y, x - z) \left| \beta_m \sqrt{N - N(N - N - 1)} \Phi \right|^2 dx \, dy \, dz \right)^{\frac{1}{2}}.
\]

Therefore, when $\Phi \in F_+^m(t)$, we have
\[
|\langle \Phi, R_{3,3} \Phi \rangle| \leq C_t \sqrt{m N^{\beta-1}} (\Phi, d\Gamma(1 - \Delta)\Phi).
\]
Thus we can proceed exactly as for the previous term $R_{3,2}$ and obtain

$$\langle \Phi, R_{3,3} \Phi \rangle \leq C_t \sqrt{mN^{\beta-1}} \langle \Phi, d\Gamma(1-\Delta) \Phi \rangle.$$ 

Thus

$$\pm (R_{3,3} + R_{3,3}) \leq C_t \sqrt{mN^{\beta-1}} d\Gamma(1-\Delta).$$

\[
R_4 = \frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes 1 \otimes 1)(x, y, z; x', y', z') u(t, y') u(t, z') \times \nonumber \]
\[
a_x^* a_y^* a_z^* a_{x'} \sqrt{N-N-1} \sqrt{N-N} \|x,y,z \rangle \langle y,z \|
onumber \]
\[
+ \frac{1}{2N^2} \int \int \int \int \int (Q(t) \otimes Q(t)) \int |u(t, z)|^2 V_N d\gamma z \langle Q(t) \otimes Q(t) \rangle (x, y; x', y') \times \nonumber \]
\[
a_x^* a_y^* a_{x'} a_{y'} d\gamma x d\gamma y d\gamma x' d\gamma y' (N-N) \nonumber \]
\[
+ \frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes 1 V_N Q(t) \otimes 1 \otimes Q(t)) (x, y, z; x', y', z') u(t, z) u(t, y') \times \nonumber \]
\[
a_x^* a_y^* a_{x'} a_{y'} (N-N) d\gamma x d\gamma y d\gamma x' d\gamma y' d\gamma z' \nonumber \]
\[
=: R_{4,1} + R_{4,2} + R_{4,3}. \nonumber \]

Again, we will estimate the expectation of $R_4$ against $\Phi \in F_x^{\leq m}(t)$, and for this purpose we ignore the projection $Q(t)$ in the computation because $\Phi$ belongs to the excited Fock space.

\[R_{4,1}\] By the Cauchy-Schwarz inequality we can estimate

\[
\langle \Phi, R_{4,1} \Phi \rangle = \frac{1}{N^2} \int \int \int \int V_N (x-y, x-z) u(t, y) u(t, z) \times \nonumber \]
\[
\times \langle \Phi, a_x^* a_y^* a_z^* a_{x'} \sqrt{N-N-1} \sqrt{N-N} \Phi \rangle \|x,y,z \rangle \langle y,z \| \nonumber \]
\[
\leq \frac{1}{N^2} \int \int \int \int V_N (x-y, x-z) |u(t, y) u(t, z)| \times \nonumber \]
\[
\times ||a_x a_y a_z \Phi|| ||a_x \sqrt{N-N-1} \sqrt{N-N} \Phi|| \|x,y,z \rangle \langle y,z \| \nonumber \]
\[
\leq \frac{C_t}{N^2} \left( \int \int \int V_N (x-y, x-z) ||a_x a_y a_z \Phi||^2 \|x,y,z \rangle \langle y,z \| \right)^{1/2} \times \nonumber \]
\[
\times \left( \int \int |u(t, y)|^2 |u(t, z)|^2 ||a_x \sqrt{N-N-1} \sqrt{N-N} \Phi||^2 \|x,y,z \rangle \langle y,z \| \right)^{1/2} \nonumber \]
\[
\leq C_t \langle \Phi, R_6 \Phi \rangle^{1/2} \langle \Phi, N \Phi \rangle^{1/2}. \nonumber \]
Then using the bound $R_6 \leq Cm^2N^{4β-2}dΓ(−Δ)$ (see (91) below) we find that

$$\langle Φ, R_{4,1}Φ \rangle \leq C_mN^{2β-1}dΓ(1 − Δ).$$

Thus

$$±(R_{4,1} + R_{4,1}^{*}) \leq C_mN^{2β-1}dΓ(1 − Δ).$$

**[R_{4,2}]** By Lemma [14] and the facts that $\|u(t,·)\|_{L^∞} \leq C_t$ and

$$\left\| \int V_N(·,z)dz \right\|_{L^{3/2}(R^3)} = \left( \int \left\| N^{β} \int V(N^βx,N^βz)dz \right\|^2 dx \right)^{\frac{1}{2}} \leq N^β \left( \int \left\| \int V(x,z)dz \right\|^2 dx \right)^{\frac{1}{2}} \leq CN^β$$

we get

$$0 \leq R_{4,2} \leq C_tN^{β-1}dΓ(−Δ)N.$$  

On the truncated Fock space $\mathcal{F}^{≤m}(t)$, we obtain

$$0 \leq R_{4,2} \leq C_tN^{β-1}dΓ(−Δ). \quad (90)$$

**[R_{4,3}]** Using the Cauchy-Schwarz inequality we have for $Φ \in \mathcal{F}^{≤N}(t)$

$$\|Φ, R_{4,1}Φ \| = \frac{1}{N^2} \left| \int \int \int V_N(x-y,x-z)u(t,z)u(t,y)Φ,a_x^*a_y^*a_z(N−N)Φ dx dy dz \right| \leq \frac{1}{N^2} \left( \int \int \int V_N(x-y,x-z)\|u(t,z)\|^2\|a_xa_y√N−NΦ\|^2 dx dy dz \right)^{\frac{1}{2}} \times \left( \int \int \int V_N(x-y,x-z)\|u(t,y)\|^2\|a_xa_z√N−NΦ\|^2 dx dy dz \right)^{\frac{1}{2}} = \langle Φ, R_{4,2}Φ \rangle.$$  

Therefore, from the above bound of $R_{4,2}$ we have

$$±(R_{4,3} + R_{4,3}) \leq 2R_{4,2} \leq C_tN^{β-1}dΓ(−Δ)$$

as quadratic forms on $\mathcal{F}^{≤m}(t)$.

**$j = 5$** We consider

$$R_5 = \frac{1}{N^2} \int \cdots \int (Q(t)⊗Q(t)⊗Q(t))V_NQ(t)⊗Q(t)⊗1(x,y,z;x′,y′,z′)u(t,z′) \times a_x^*a_y^*a_z^*a_{x′}a_{y′}√N−N−2 dx dy dz dx′ dy′ dz′.$$  

For $Φ \in \mathcal{F}^{≤m}(t)$, by the Cauchy-Schwarz inequality and the previous bound on $R_{4,2}$, we have

$$\|Φ, R_{5}Φ \| = \frac{1}{N^2} \left| \int \int \int V_n(x−y,x−z)u(t,z)\langle Φ,a_x^*a_y^*a_z^*a_xa_y√N−N+2Φ \rangle dx dy dz \right| \leq \frac{1}{N^2} \left| \int \int \int |V_n(x−y,x−z)|\|u(t,z)\|\|a_xa_ya_zΦ\|\|a_xa_y√N−N+2Φ\|dy dy dz \right|$$
We consider the proof of the commutator bounds in (78). This finishes the proof of (78).

Therefore, in particular, on the truncated Fock space \( \mathcal{F}_0 \) we have

\[
\langle \Phi, R_{4,2} \Phi \rangle^{1/2} \langle \Phi, R_6 \Phi \rangle^{1/2} \leq C t \sqrt{m^3 N^{5\beta - 3}} \langle \Phi, d\Gamma (1 - \Delta) \Phi \rangle.
\]

Here in the last estimate we have used the bound on \( R_{4,2} \) in (90) above and the bound \( 0 \leq R_6 \leq C m^2 N^{4\beta - 2} d\Gamma (-\Delta) \) in (91) below. Thus

\[
\pm (R_5 + R_6^\circ) \leq C t \sqrt{m^3 N^{5\beta - 3}} d\Gamma (1 - \Delta).
\]

\( j = 6 \) We consider

\[
R_6 = \frac{1}{6N^2} \int \cdots (Q(t) \otimes Q(t) \otimes Q(t)V_N Q(t) \otimes Q(t) \otimes Q(t)(x, y, z; x', y', z') \times a_x a_y a_x a_x a_x a_z \ dx \ dy \ dz \ dx' \ dy' \ dz'.
\]

By Lemma 14 we have

\[ 0 \leq \int V_N (x - y, x - z) \leq \sup_z N^{6\beta} V_N (x - y, z) \leq C N^{3\beta} \sup_z N^{3\beta} V_N (x - y, z) \| L^{3/2} (-\Delta x) \| \leq C N^{4\beta} (-\Delta x).\]

Therefore,

\[ 0 \leq R_6 \leq C N^{4\beta - 2} d\Gamma (-\Delta) N^2.\]

In particular, on the truncated Fock space \( \mathcal{F}_0 \),

\[ 0 \leq R_6 \leq C m^2 N^{4\beta - 2} d\Gamma (-\Delta).\]

This finishes the proof of (78).

**Proof of the commutator bounds in (79).** These bounds follows from (78) with \( \eta = 1 \) and the fact that

\[
[R_0, \mathcal{N}] = [R_{4,2}, \mathcal{N}] = [R_{4,3}, \mathcal{N}] = [R_6, \mathcal{N}] = 0,
\]

\[
[R_1, \mathcal{N}] = R_1, \quad [R_2, \mathcal{N}] = -R_2
\]

\[
[R_{3,1}, \mathcal{N}] = -3R_{3,1}, \quad [R_{3,2}, \mathcal{N}] = R_{3,2}, \quad [R_{3,3}, \mathcal{N}] = R_{3,3},
\]

\[
[R_{4,1}, \mathcal{N}] = -2R_{4,1}, \quad [R_5, \mathcal{N}] = -R_5.
\]

**Proof of the derivative bounds (80).** Heuristically these bounds are similar to (78).

For the reader’s convenience we will again go term by term.

\( j = 0 \) Since

\[
\left| \partial_t \int \cdots \left| u(t, x) \right|^2 \left| u(t, y) \right|^2 \left| u(t, z) \right|^2 V_N (x - y, x - z) \ dx \ dy \ dz \right|
\]

\[
\leq 3 \int \cdots \left| \partial_t u(t, x) \right| \left| u(t, x) \right| \left| u(t, y) \right|^2 \left| u(t, z) \right|^2 V_N (x - y, x - z) \ dx \ dy \ dz \leq C_t,
\]
we obtain
\[ \pm \partial_t R_{0,1} = \pm \frac{1}{6} \partial_t \iint \int |u(x)|^2 |u(y)|^2 |u(z)|^2 V_N(x - y, x - z) \, dx \, dy \, dz \times \left( \frac{3N^2 + 6N + 2}{N} - \frac{N^3 + 3N^2 + 2N}{N^2} \right) \leq C_t \left( \frac{N + 1}{N} \right)^2. \]

Moreover, using \( \|Q(t)\|_{op}, \|\partial_t Q(t)\|_{op} \leq C_t \), we have
\[ \left\| \partial_t \left( Q(t) \left( \frac{1}{2} \iint |u(t)|^2 V_N |u(t)|^2 \, dy \, dz + K_1 \right) Q(t) \right\|_{op} \leq C_t. \]

Hence, as above
\[ \pm \partial_t R_{0,2} = \pm d \Gamma \left( \partial_t \left( Q(t) \left( \frac{1}{2} \iint |u(t)|^2 V_N |u(t)|^2 \, dy \, dz + K_1 \right) Q(t) \right) \right) \times \left( \frac{(N - N)^2 - (N - N)}{N} - 1 \right) \leq C_t \left( \frac{N + 1}{N} \right)^2. \]

In summary,
\[ \pm \partial_t R_0 \leq C_t \left( \frac{N + 1}{N} \right)^2 \leq C_t \frac{m}{N}(N + 1). \]

\[ j = 1 \] The term \( \partial_t R_1 \) can be estimated similarly to \( R_1 \) and we get
\[ \pm \partial_t (R_1 + R_1^*) \leq C_t \sqrt{ \frac{m}{N} (N + 1) }. \]

\[ j = 2 \] Let \( \Phi \in \mathcal{F}_+^{\leq m}(t) \). Then we have
\[ \langle \Phi, \partial_t R_2 \Phi \rangle = \iint \left( (\partial_t Q(t) \otimes 1 + 1 \otimes \partial_t Q(t)) \tilde{K}_2(t, x, y) + \partial_t \tilde{K}_2(t, x, y) \right) \times \left\langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right) \Phi \right\rangle \, dx \, dy, \]
where we have used
\[ \partial_t K_2(t, x, y) = \partial_t Q(t) \otimes Q(t) \tilde{K}_2(t, x, y) + Q(t) \otimes \partial_t Q(t) \tilde{K}_2(t, x, y) \]
and then left out all \( Q(t) \) since \( \Phi \in \mathcal{F}_+^{\leq m}(t) \).

As in [50], we have
\[
\left| \iint \partial_t \tilde{K}_2(t, x, y) \langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right) \Phi \rangle \, dx \, dy \right| \\
\leq \iint \left| \partial_t |u(t, z)|^2 u(t, x) u(t, y) + 2 |u(t, z)|^2 \partial_t u(t, x) u(t, y) \right| V_N(x - y, x - z) \times \\
\times \left| \left\langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{N - N - 1} \sqrt{N - N} (N - N - 2)}{N^2} - 1 \right) \Phi \right\rangle \right| \, dx \, dy \, dz \\
\leq C_t \left( \iint \iint |u(t, z)|^2 |V_N(x - y, x - z)| |a_x a_y| \, dx \, dy \, dz \right)^{\frac{1}{2}} \times 
\]
Thus \( \frac{\sqrt{N - N - 1}}{N^2} \sqrt{N - N - 2} - 1 \) \( \Phi \)

\[ \leq C_t(\Phi, R_{4,2}^2) \frac{1}{\sqrt{N + 1}}(\Phi, (N + 1)\Phi)^{1/2} \]

\[ \leq C_t \sqrt{mN^{\beta - 1}}(\Phi, d\Gamma(1 - \Delta)\Phi). \]

The other terms follow as

\[ \|\langle \partial_t u \rangle (u \otimes 1)K_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^6)}^2 = \int \int \|u(t, \tilde{x})K_2(t, \tilde{x}, y)\| \|\partial_t u(t, x)\|^2 dx \, dy \]

\[ = \int \int \int \int \|u(t, \tilde{x})u(t, z)\|^2 u(t, \tilde{x})V_N(\tilde{x} - y, \tilde{x} - z) u(t, y \) \| \|\partial_t u(t, x)\|^2 dx \, dz \, d\tilde{x} \, dy \]

\[ \leq \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}^8 \|V\|_{L^1(\mathbb{R}^6)}^2 \|\partial_t u(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \leq C_t. \]

Using this and similar estimates we obtain

\[ \int \int \left( \langle \partial_t Q(t) \otimes 1 + 1 \otimes \partial_t Q(t) \rangle \right) K_2(t, x, y) \times \]

\[ \times \langle \Phi, a^2 a^y \left( \frac{\sqrt{N - N - 1}}{N^2} + \sqrt{N - N - 2} - 1 \right) \Phi \rangle dx \, dy \]

\[ \leq \left\| \langle \partial_t Q(t) \otimes 1 + 1 \otimes \partial_t Q(t) \rangle \right\|_L^2(\mathbb{R}^6) \left( \int \int \|a^2 a^y \Phi\|^2 dx \, dy \right)^{1/2} \times \]

\[ \left\| \left( \frac{\sqrt{N - N - 1}}{N^2} + \sqrt{N - N - 2} - 1 \right) \Phi \right\| \]

\[ \leq C_t \frac{1}{\sqrt{N}} \left( \Phi, N^2 \Phi \right)^{1/2} \left( \Phi, (N + 1)\Phi \right)^{1/2} \leq C_t \sqrt{mN^{\beta - 1}}(\Phi, (N + 1)\Phi). \]

Putting everything together we obtain

\[ \|\Phi, \partial_t R_2 \Phi\| \leq C_t \sqrt{mN^{\beta - 1}}(\Phi, d\Gamma(1 - \Delta)\Phi). \]

Thus

\[ \pm \partial_t (R_2 + R_2') \leq C_t \sqrt{mN^{\beta - 1}} d\Gamma(1 - \Delta). \]

**Let \( j = 3 \)**

\[ \Phi \in \mathcal{F}^{<\infty}_+(t). \]

Then we have

\[ \langle \Phi, \partial_t R_{3,1} \Phi \rangle \]

\[ = \frac{1}{3N^2} \int \cdots \int \left[ Q(t) \otimes Q(t) \right] V_N 1 \otimes 1 \otimes 1 (x, y, z; x', y', z') \partial_t (u(t, x')u(t, y')u(t, z')) + \]

\[ + \partial_t (Q(t) \otimes Q(t) \otimes Q(t) V_N 1 \otimes 1 \otimes 1 (x, y, z; x', y', z') u(t, x')u(t, y')u(t, z')) \times \langle \Phi, a^2 a^y a^x \sqrt{N - N - 2} \sqrt{N - N - 1} \Phi \rangle dx \, dy \, dz \, dx' \, dy' \, dz'. \]

The first term containing \( \partial_t (u(t, x')u(t, y')u(t, z')) \) can be estimated exactly as for \( R_{3,1}. \)

The second term containing \( \partial_t Q(t) \) can be evaluated as

\[ \frac{1}{3N^2} \left| \int \cdots \int \left( \partial_t Q(t) \otimes Q(t) \otimes Q(t) V_N 1 \otimes 1 \otimes 1 (x, y, z; x', y', z') u(t, x')u(t, y')u(t, z') \right) \times \langle \Phi, a^2 a^y a^x \sqrt{N - N - 2} \sqrt{N - N - 1} \Phi \rangle dx \, dy \, dz \, dx' \, dy' \, dz' \right| \]
\[
= \frac{1}{3N^2} \left| \cdots \int (\partial_t Q(t))(x; x') V_N(x' - y, x' - z) \delta(y - y') \delta(z - z') u(t, x') u(t, y') u(t, z') \right.
\times \langle \Phi, a_x^* a_y^* a_z^* \sqrt{N - N - 2\sqrt{N - N - 1\sqrt{N - N}}\Phi} \rangle \, dx \, dy \, dz \, dy' \, dz' \\
\left. \right| \right.
= \frac{1}{3N^2} \left| \iiint (\partial_t Q(t))(x; x') V_N(x' - y, x' - z) u(t, x') u(t, y) u(t, z) \right.
\times \langle \Phi, a_x^* a_y^* a_z^* \sqrt{N - N - 2\sqrt{N - N - 1\sqrt{N - N}}\Phi} \rangle \, dx \, dy \, dz \, dy' \, dz' \\
\leq C_t \left( \eta \, d\Gamma(1 - \Delta) + \eta^{-1} C_t m N^{4\beta - 1} \right) \Phi, \quad \forall \eta \geq C_t \sqrt{m N^{2\beta - 1}}
\]

where the last estimate follows similarly as for \( R_{3,1} \) again. Hence, we have
\[
\pm \partial_t (R_{3,1} + R_{3,1}^*) \leq C_t \left( \eta \, d\Gamma(1 - \Delta) + \eta^{-1} C_t m N^{4\beta - 1} \right), \quad \forall \eta \geq C_t \sqrt{m N^{2\beta - 1}}.
\]

**R_{3,2}** Let \( \Phi \in \mathcal{F}^{\leq m}_+(t) \). Then we have
\[
\langle \Phi, \partial_t R_{3,2} \Phi \rangle = \frac{2}{N^2} \int \cdots \int \left[ (Q(t) \otimes Q(t) V_N Q(t) \otimes 1)(x, y; x', y') \partial_t (|u(t, z)|^2 u(t, y')) + \right.
\left. + \partial_t (Q(t) \otimes Q(t) V_N Q(t) \otimes 1)(x, y; x', y') |u(t, z)|^2 u(t, y') \right] \times \\
\times \langle \Phi, a_x^* a_y^* a_z^* \sqrt{N - N - 1\sqrt{N - N - 1}}\Phi \rangle \, dx \, dy \, dz
\]
The first term containing \( \partial_t (|u(t, z)|^2 u(t, y')) \) can be estimated similarly to \( R_{3,2} \), which is
\[
\frac{2}{N^2} \left| \iiint V_N(x - y, x - z) \partial_t (|u(t, z)|^2 u(t, y')) \langle \Phi, a_x^* a_y^* a_z^* \sqrt{N - N - 1\sqrt{N - N - 1}}\Phi \rangle \, dx \, dy \, dz \right|
\leq \frac{4}{N^2} \left( \iiint |V_N(x - y, x - z)| \partial_t u(t, y')^2 |u(t, z)|^2 \|a_x(N - N - 1)\Phi\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \times \\
\times \left( \iiint |V_N(x - y, x - z)| |u(t, z)|^2 \|a_x a_y \sqrt{N - N + 1}\Phi\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \leq \frac{C_t}{N^2} \left( \int \|a_x(N - N - 1)\Phi\|^2 \, dx \right)^{\frac{3}{2}} \times \\
\times \left( \iiint |V_N(x - y, x - z)| |u(t, z)|^2 \|a_x a_y \sqrt{N - N + 1}\Phi\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \leq C_t \sqrt{m N^{2\beta - 1}} \Phi, \, d\Gamma(-\Delta) \Phi.
\]

Here we have omitted all the \( Q(t) \) since \( \Phi \in \mathcal{F}^{\leq m}_+(t) \). For the other terms containing \( \partial_t Q(t) \) we use the following kernel estimate
\[
\left| (\partial_t Q(t))(x; x') \right| = |\partial_t u(t, x) \overline{u(t, x')} + u(t, x) \partial_t \overline{u(t, x')} | \leq q(t, x) q(t, x')
\]
with \( q(t, x) = |u(t, x)|^2 + |\partial_t u(t, x)| \). Using Theorem (3) we get
\[
\|q(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C_t, \quad \|q(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_t
\]
We now decompose \( \partial_t (Q(t) \otimes Q(t) V_N Q(t) \otimes 1) \) into three terms. The first one containing \( (\partial_t Q(t) \otimes Q(t) V_N Q(t) \otimes 1) \) can be estimated as
\[
\frac{2}{N^2} \left| \iiint (\partial_t Q(t) \otimes Q(t) \int |u(t, z)|^2 V_N \, dz Q(t) \otimes 1)(x, y; x', y') u(t, y') \times \\
\right|
\]
\[
\times \langle \Phi, a_x^* a_y^* a_{x'} \sqrt{N - N} (N - N - 1) \Phi \rangle dxdy dz'
\]

\[
= \frac{2}{N^2} \left| \iiint (\partial_t Q(t)(x; x')) \int |u(t, z)|^2 V_N(x' - y, x' - z) \, dz \delta(y - y') u(t, y') \times \\
\times \langle \Phi, a_x^* a_y^* a_{x'} \sqrt{N - N} (N - N - 1) \Phi \rangle dxdy dz' \right|
\]

\[
\leq \frac{1}{N^2} \left( \iiint q(t, x) q(t, x') \int |u(t, z)|^2 V_N(x' - y, x' - z) \, dz |u(t, y)| \times \\
\times \|a_x a_y \sqrt{N - N - 1} \Phi \| \|a_{x'} (N - N - 1) \Phi \|^2 dxdy dz' \right) \frac{1}{2} \times \\
\times \left( \iiint |q(t, x')|^2 |u(t, z)|^2 V_N(x' - y, x' - z) \|a_x a_y \sqrt{N - N - 1} \Phi \|^2 dxdy dz' \right) \frac{1}{2}
\]

\[
\leq \frac{2}{N^2} \left( \int \int |q(t, x)|^2 |u(t, z)|^2 V_N(x' - y, x' - z) \, dz |u(t, y)| \|a_x a_y \sqrt{N - N - 1} \Phi \|^2 dxdy dz' \right) \frac{1}{2}
\]

The second one \((Q(t) \otimes \partial_t Q(t) V_N Q(t) \otimes 1)\) follows as

\[
\frac{2}{N^2} \left| \iiint (Q(t) \otimes \partial_t Q(t) \int |u(t, z)|^2 V_N dz Q(t) \otimes 1)(x, y; x', y') u(t, y') \times \\
\times \langle \Phi, a_x^* a_y^* a_{x'} \sqrt{N - N} (N - N - 1) \Phi \rangle dxdy dz' \right|
\]

\[
= \frac{2}{N^2} \left| \iiint (\partial_t Q(t))(y; y') \int |u(t, z)|^2 V_N(x - y', x - z) \, dz \delta(x - x') u(t, y') \times \\
\times \langle \Phi, a_x^* a_y^* a_{x'} \sqrt{N - N} (N - N - 1) \Phi \rangle dxdy dz' \right|
\]

\[
\leq \frac{2}{N^2} \left( \int \int q(t, y) q(t, y') \int |u(t, z)|^2 V_N(x - y', x - z) \, dz |u(t, y')| \times \\
\times \|a_x a_y \sqrt{N - N - 1} \Phi \| \|a_{x'} (N - N - 1) \Phi \|^2 dxdy dz' \right) \frac{1}{2}
\]

\[
\leq \frac{2}{N^2} \left| q(t, \cdot) \right|_{L^\infty(\mathbb{R}^3)} \left| u(t, \cdot) \right|_{L^\infty(\mathbb{R}^3)} \left| V \right|_{L^1(\mathbb{R}^6)} \times \\
\times \left( \int \int |a_x a_y \sqrt{N - N - 1} \Phi \|^2 dxdy \right) \frac{1}{2} \left( \int \int |q(t, y)|^2 \|a_x (N - N - 1) \Phi \|^2 dxdy \right) \frac{1}{2}
\]

\[
\leq \frac{C_t}{N} \langle \Phi, N^2 (N - N - 1) \Phi \rangle \frac{1}{2} \langle \Phi, N \Phi \rangle \frac{1}{2} \leq C_t \sqrt{mN^{-1}} \langle \Phi, N \Phi \rangle.
\]

For the third one \((Q(t) \otimes V_N \partial_t Q(t) Q(t) \otimes 1)\) we can estimate

\[
\frac{2}{N^2} \left| \iiint (Q(t) \otimes \partial_t Q(t) \int |u(t, z)|^2 V_N dz \partial_t Q(t) \otimes 1)(x, y; x', y') u(t, y') \times \\
\times \langle \Phi, a_x^* a_y^* a_{x'} \sqrt{N - N} (N - N - 1) \Phi \rangle dxdy dz' \right|
\]

\[
= \frac{2}{N^2} \left| \iiint \int |u(t, z)|^2 V_N(x - y, x - z) \, dz (\partial_t Q(t))(x; x') \delta(y - y') u(t, y') \times \\
\times \langle \Phi, a_x^* a_y^* a_{x'} \sqrt{N - N} (N - N - 1) \Phi \rangle dxdy dz' \right|
\]

\[
\leq \frac{2}{N^2} \left( \int \int |u(t, z)|^2 V_N(x - y, x - z) \, dz q(t, x) q(t, x') |u(t, y)| \times \\
\times \|a_x a_y \sqrt{N - N - 1} \Phi \| \|a_{x'} (N - N - 1) \Phi \|^2 dxdy dz' \right) \frac{1}{2}
\]
\[
\times \|a_x a_y \sqrt{N - N + 1}\Phi\| \|a_x' (N - N' - 1)\Phi\| \, dx \, dy \, dz
\]
\[
\leq \frac{2}{N^2} \left( \iiint |q(t, x)|^2 |u(t, z)|^2 |V_N(x - y, x - z)| |u(t, y)|^2 \|a_x' (N - N - 1)\Phi\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \times
\]
\[
\times \left( \iiint |q(t, x')|^2 |u(t, z)|^2 |V_N(x - y, x - z)| |a_x a_y \sqrt{N - N + 1}\Phi\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \leq C_t \sqrt{mN^3 - 1}(\Phi, d\Gamma(1 - \Delta)\Phi).
\]
Here we have used the bound on \(R_{4,2}\) in (90). Collecting all above estimates, we obtain
\[
|\langle \Phi, \partial_t R_{3,2}\Phi \rangle| \leq C_t \sqrt{mN^3 - 1}(\Phi, d\Gamma(1 - \Delta)\Phi).
\]
Thus
\[
\pm \partial_t (R_{3,2} + R_{4,2}^*) \leq C_t \sqrt{mN^3 - 1} \, d\Gamma(1 - \Delta).
\]

\[R_{3,3}\] Let \(\Phi \in \mathcal{F}^{\leq m}_+(t)\). We have that
\[
\langle \Phi, \partial_t R_{3,3}\Phi \rangle = \frac{1}{N^2} \int \cdots \int \left( \frac{Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes Q(t)}{Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes Q(t)}(x, y, z; x', y', z') \partial_t \left( \frac{u(t, z)u(t, x')u(t, y')}{u(t, z)u(t, x')u(t, y')} \right) \right.
\]
\[
+ \left( \left( \partial_t (Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes Q(t)) \right) (x, y, z; x', y', z') \frac{u(t, z)u(t, x')u(t, y')}{u(t, z)u(t, x')u(t, y')} \right) \times
\]
\[
\times \left( \Phi, a_x^* a_y^* a_z^* \sqrt{N - N(N - N - 1)\Phi} \right) \, dx \, dy \, dz \right|^{\frac{1}{2}} \times
\]
\[
\left( \int \int V_N(x - y, x - z) |a_x (N - N - 1)\Phi| |a_y (N - N - 1)\Phi| \, dx \, dy \, dz \right)^{\frac{1}{2}} \leq C_t \sqrt{mN^3 - 1}(\Phi, d\Gamma(1 - \Delta)\Phi).
\]
Let us now decompose \(\partial_t (Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes Q(t))\) into three terms. For the first one containing \((\partial_t Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes Q(t))\) we get
\[
\frac{1}{N^2} \int \cdots \int \left( \partial_t Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes Q(t) \right) (x, y, z; x', y', z') \frac{u(t, z)u(t, x')u(t, y')}{u(t, z)u(t, x')u(t, y')} \times
\]
\[
\times \left( \Phi, a_x^* a_y^* a_z^* \sqrt{N - N(N - N - 1)\Phi} \right) \, dx \, dy \, dz \right|^{\frac{1}{2}} \times
\]
\[
\left( \int \int V_N(x - y, x - z) |a_x a_y \sqrt{N - N + 1}\Phi\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \leq C_t \sqrt{mN^3 - 1}(\Phi, d\Gamma(1 - \Delta)\Phi).
\]
\[ \times \langle \Phi, a_x^* a_y^* a_z^* \sqrt{N-N}(N-N-1)\Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \]

\[ \leq \frac{1}{N^2} \iint \int \int q(t, x)q(t, x')|V_N(x' - y, x' - z)|u(t, z)|u(t, x')|u(t, y)| \times \]

\[ \times |a_x a_y \sqrt{N-N+1}\Phi| |a_z(N-N-1)\Phi| \, dx \, dy \, dz \, dx' \]

\[ \leq \frac{C_t}{N^2} \left( \iint \int |V_N(x' - y, x' - z)| a_x a_y \sqrt{N-N+1}\Phi^2 \, dx \, dy \, dz \, dx' \right)^{\frac{1}{2}} \times \]

\[ \left( \iint \int |q(t, x)|^2 |V_N(x' - y, x' - z)| a_z(N-N-1)\Phi^2 \, dx \, dy \, dz \, dx' \right)^{\frac{1}{2}} \]

\[ \leq \frac{C_t}{N^2} \left( \iint |a_x a_y \sqrt{N-N+1}\Phi^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \left( \iint |a_z(N-N-1)\Phi|^2 \, dz \right)^{\frac{1}{2}} \]

\[ \leq C_t \sqrt{mN^{3}} \langle \Phi, (N+1)\Phi \rangle. \]

The term involving \((Q(t) \otimes \partial_t Q(t) \otimes 1V_N 1 \otimes 1 \otimes Q(t))\) can be treated similarly. The third term \((Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes \partial_t Q(t))\) goes as follows:

\[ \frac{1}{N^2} \left| \cdots \int (Q(t) \otimes Q(t) \otimes 1V_N 1 \otimes 1 \otimes \partial_t Q(t))(x, y, z; x', y', z')u(t, z)u(t, x')u(t, y') \times \right. \]

\[ \times \langle \Phi, a_x^* a_y^* a_z^* \sqrt{N-N}(N-N-1)\Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \]

\[ = \frac{1}{N^2} \left| \cdots \int V_N(x - y, x - z)(\partial_t Q(t))(z; z')\delta(x - x')\delta(y - y')u(t, z)u(t, x')u(t, y') \times \right. \]

\[ \times \langle \Phi, a_x^* a_y^* a_z^* \sqrt{N-N}(N-N-1)\Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \]

\[ \leq \frac{1}{N^2} \iint \int \int |V_N(x - y, x - z)|q(t, z)q(t, z')|u(t, z)|u(t, x')|u(t, y')| \times \]

\[ \times |a_x a_y \sqrt{N-N+1}| |a_z(N-N-1)\Phi| \, dx \, dy \, dz \, dx' \]

\[ \leq \frac{C_t}{N^2} \left( \iint \int |q(t, z')|^2 |V_N(x - y, x - z)|u(t, z)|^2 |a_x a_y \sqrt{N-N+1}|^2 \, dx \, dy \, dz \, dx' \right)^{\frac{1}{2}} \times \]

\[ \left( \iint \int |V_N(x - y, x - z)||u(t, x')|^2 |a_z(N-N-1)\Phi|^2 \, dx \, dy \, dz \, dx' \right)^{\frac{1}{2}} \]

\[ \leq \frac{C_t}{N^2} \left( \iint \int |V_N(x - y, x - z)|u(t, z)|^2 |a_x a_y \sqrt{N-N+1}|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \times \]

\[ \times \langle \Phi, \mathcal{N}(N-N-1)^2\Phi \rangle^{\frac{1}{2}} \]

\[ \leq C_t \sqrt{mN^3} \langle \Phi, d\Gamma(1 - \Delta)\Phi \rangle. \]

Putting everything together we obtain:

\[ |\langle \Phi, \partial_t R_{3,3} \Phi \rangle| \leq C_t \sqrt{mN^3} \langle \Phi, d\Gamma(1 - \Delta)\Phi \rangle. \]

Thus

\[ \pm \partial_t (R_{3,3} + R_{3,3}^*) \leq C_t \sqrt{mN^3} d\Gamma(1 - \Delta). \]

\[ j = 4 \quad [R_{4,1}] \] Let \( \Phi \in \mathcal{F}_{+}^{\leq m}(t) \). Then we have:

\[ \langle \Phi, \partial_t R_{4,1} \Phi \rangle \]
The second one containing \( \partial_t (u(t,y'))u(t,z') \) can be estimated similarly to \( R_{4,1} \). For the other terms we decompose \( \partial_t (Q(t) \otimes Q(t))V_NQ(t) \otimes 1 \otimes 1 \) into four terms. For the first term \( \partial_t (Q(t) \otimes Q(t))V_NQ(t) \otimes 1 \otimes 1 \) we have

\[
\frac{1}{N^2} \left| \int \cdots \int (\partial_t Q(t) \otimes Q(t))V_NQ(t) \otimes 1 \otimes 1 (x, y, z; x', y', z')u(t, y')u(t, z') \times \right.
\]

\[
\left. \langle \Phi, a_x^* a_y^* a_z^* a_x^* a_y^* a_z^* \sqrt{N - N' - 1} \sqrt{N - N' \Phi} \rangle dx dy dz dx' dy' dz' \right| \]

\[
\leq \frac{1}{N^2} \| u(t, \cdot ) \|_{L^\infty}^2 \left( \int \int \int |q(t,x')|^2 V_N(x' - y, x' - z) \| a_x a_y a_z \|_2^2 dx dy dz dx' \right)^\frac{1}{2} \times \]

\[
\left. \left( \int \int \int \int |q(t,x')|^2 |V_N(x' - y, x' - z)\| a_x \sqrt{N - N' - 1} \sqrt{N - N' \Phi} \|_2^2 dx dy dz dx' \right)^\frac{1}{2} \right| \]

\[
\leq C_{mN^{\gamma - 1}} \langle \Phi, d\Gamma(\Delta) \Phi \rangle^\frac{1}{2} \langle \Phi, N \Phi \rangle^\frac{1}{2} \]

\[
\leq C_{mN^{\gamma - 1}} \langle \Phi, d\Gamma(1 - \Delta) \Phi \rangle.
\]

The second one containing \( (Q(t) \otimes \partial_t Q(t) \otimes Q(t))V_NQ(t) \otimes 1 \otimes 1 \) can be bounded as

\[
\frac{1}{N^2} \left| \int \cdots \int (Q(t) \otimes \partial_t Q(t) \otimes Q(t))V_NQ(t) \otimes 1 \otimes 1 (x, y, z; x', y', z')u(t, y')u(t, z') \times \right.
\]

\[
\left. \langle \Phi, a_x^* a_y^* a_z^* a_x^* a_y^* a_z^* \sqrt{N - N' - 1} \sqrt{N - N' \Phi} \rangle dx dy dz dx' dy' dz' \right| \]

\[
\leq \frac{1}{N^2} \left| \int \int \int |q(t,y')|V_N(x' - y', x - z) \delta(x - x') \delta(z - z')u(t, y')u(t, z') \times \right.
\]

\[
\left. \langle \Phi, a_x^* a_y^* a_z^* a_x^* a_y^* a_z^* \sqrt{N - N' - 1} \sqrt{N - N' \Phi} \rangle dx dy dz dx' dy' dz' \right| \]

\[
\leq \frac{1}{N^2} \left| \int \int \int |u(t, y')|^2 V_N(x' - y', x - z) \| a_x a_y a_z \|_2^2 dx dy dz dy' \right|^\frac{1}{2} \times \]

\[
\left. \left( \int \int \int \int |q(t,y')|^2 |V_N(x' - y', x - z)\| a_x \sqrt{N - N' - 1} \sqrt{N - N' \Phi} \|_2^2 dx dy dz dy' \right)^\frac{1}{2} \right| \]

\[
\leq C_{tN^{\gamma - 1}} \langle \Phi, d\Gamma(-\Phi) \rangle^\frac{1}{2} \langle \Phi, \Phi \rangle^\frac{1}{2}.
\]
Let $\Phi\leq C_t m N^{\beta-1}(\Phi, d\Gamma(1-\Delta)\Phi)$.

The third term $\langle Q(t) \otimes Q(t) \otimes \partial_t Q(t) V_N Q(t) \otimes 1 \otimes 1 \rangle$ can be treated similarly. For the fourth term $\langle Q(t) \otimes Q(t) \otimes Q(t) V_N \partial_t Q(t) \otimes 1 \otimes 1 \rangle$ we estimate

$$\frac{1}{N^2} \left| \int \cdots \int (Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes 1 \otimes 1)(x, y, z; x', y', z') u(t, y') u(t, z') \times \right.$$

$$\times \langle \Phi, a_x a_y a_z a_{x'} \sqrt{N-N-1} \sqrt{N-N} \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \left| \right.

$$

$$\frac{1}{N^2} \left| \int \cdots \int V_N(x-y, x-z)(\partial_t Q(t))(x; x')\delta(y-y')\delta(z-z') u(t, y') u(t, z') \times \right.$$

$$\times \langle \Phi, a_x a_y a_z a_{x'} \sqrt{N-N-1} \sqrt{N-N} \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \left| \right.

$$

$$\leq \frac{1}{N^2} \left| \int \cdots \int V_N(x-y, x-z)|q(t, x)q(t, x') u(t, y) u(t, z)| \times \right.$$

$$\times \|a_x a_y a_z \Phi\|\|a_x \sqrt{N-N-1} \sqrt{N-N} \Phi\| \, dx \, dy \, dz \, dx' \left| \right.

$$

$$\leq \frac{1}{N^2} \|u(t, \cdot)\|_{L^\infty}^2|q(t, \cdot)| \|q(t, \cdot)|_{L^\infty} \left( \int \left| q(t, x)^2 V_N(x-y, x-z)\|a_x a_y a_z \Phi\|^2 \right| dx \, dy \, dz \right)^{1/2} \times \right.$$

$$\times \left( \int \left| q(t, x)^2 V_N(x-y, x-z)\|a_x \sqrt{N-N-1} \sqrt{N-N} \Phi\| \right| dx \, dy \, dz \right)^{1/2}$$

$$\leq C_t \left( \int \int \int V_N(x-y, x-z)\|a_x a_y a_z \Phi\|^2 \right)^{1/2} \langle \Phi, N(1-\Delta) \Phi \rangle$$

$$\leq C_t (\Phi, R_0 \Phi)^{1/2} (\Phi, N^\beta \Phi)^{1/2} \leq C_t m N^{2\beta-1}(\Phi, d\Gamma(1-\Delta)\Phi).$$

In the last estimate, we have used the bound on $R_6$ in (91). Hence, we conclude that

$$\left| \langle \Phi, \partial_t R_{4,1} \Phi \rangle \right| \leq C_t m N^{2\beta-1}(\Phi, d\Gamma(1-\Delta)\Phi).$$

This means

$$\pm \partial_t (R_{4,1} + R_{4,1}^*) \leq C_t m N^{2\beta-1} d\Gamma(1-\Delta).$$

**R4.2** Let $\Phi \in \mathcal{F}_{<m}(t)$. Then we have

$$\langle \Phi, \partial_t R_{4,2} \Phi \rangle = \frac{1}{2N^2} \int \cdots \int \left[ (Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t))(x, y; x', y') \partial_t |u(t, z)|^2 + \right.$$

$$\left. + \partial_t (Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t))(x, y; x', y') |u(t, z)|^2 \right] \times \langle \Phi, a_x a_y a_x a_y (N-\Delta) \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz.$$
\[
\begin{align*}
\times \langle \Phi, a'_x a'_y a'_z(N - N)\Phi \rangle \, dxdy' dy' \\
= \frac{1}{2N^2} \int \int \int \int (\partial_t Q(t))(x, x') |u(t, z)|^2 V_N(x' - y, x' - z) \, dz \delta(y - y') \times \\
\times \langle \Phi, a'_x a'_y a'_z(N - N)\Phi \rangle \, dxdy' dy' \\
\leq \frac{1}{2N^2} \int \int \int q(t, x)q(t, x')|u(t, z)|^2 V_N(x' - y, x' - z) \times \\
\times a_x a_y \sqrt{N - N} a_x a_y \sqrt{N - N} \, dxdy' dz' \\
\leq \frac{C_t}{N^2} \left( \int \int \int V_N(x' - y, x' - z) \|a_x a_y \sqrt{N - N}\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \times \\
\times \left( \int \int \int |q(t, x)|^2 |u(t, z)|^2 V_N(x' - y, x' - z) \|a_x a_y \sqrt{N - N}\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \\
\leq \frac{C_t}{N} \langle \Phi, N^2(N - N)\Phi \rangle \frac{1}{2} \langle \Phi, R_{4,2}\Phi \rangle \\
\leq C_t \sqrt{mN^2-1} \langle \Phi, \partial_t(1 - \Delta)\Phi \rangle.
\end{align*}
\]

Here we have used the previous bound on \(R_{4,2}\). The other three terms can be estimated in the same way. With this we can conclude

\[
|\langle \Phi, \partial_t R_{4,2}\Phi \rangle| \leq C_t \sqrt{mN^2-1} \langle \Phi, \partial_t(1 - \Delta)\Phi \rangle.
\]

Thus

\[
\pm \partial_t(R_{4,2} + R'_{4,2}) \leq C_t \sqrt{mN^2-1} \, d\Gamma(1 - \Delta).
\]

**[R4.3]** Let \( \Phi \in F^m_{+}(t) \). Then we have

\[
\begin{align*}
(\Phi, \partial_t R_{4,3}\Phi) = \frac{1}{N^2} \int \cdots \int \left[ (Q(t) \otimes Q(t) \otimes 1V_NQ(t) \otimes 1 \otimes Q(t))(x, y, z; x', y', z') \partial_t \left( \overline{u(t, z)}u(t, y') \right) + \\
+ \partial_t(Q(t) \otimes Q(t) \otimes 1V_NQ(t) \otimes 1 \otimes Q(t))(x, y, z; x', y', z') \overline{u(t, z)}u(t, y') \right] \times \\
\times \langle \Phi, a'_x a'_y a'_z(N - N)\Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz'.
\end{align*}
\]

For the first term, which contains \( \partial_t \left( \overline{u(t, z)}u(t, y') \right) \), we get

\[
\begin{align*}
\frac{1}{N^2} \int \cdots \int (Q(t) \otimes Q(t) \otimes 1V_NQ(t) \otimes 1 \otimes Q(t))(x, y, z; x', y', z') \partial_t \left( \overline{u(t, z)}u(t, y') \right) \times \\
\times \langle \Phi, a'_x a'_y a'_z(N - N)\Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \\
\leq \frac{2}{N^2} \left( \int \int |\partial_t u(t, z)|^2 V_N(x - y, x - z) \|a_x a_y \sqrt{N - N}\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \times \\
\times \left( \int \int |u(t, y)|^2 V_N(x - y, x - z) \|a_x a_y \sqrt{N - N}\|^2 \, dx \, dy \, dz \right)^{\frac{1}{2}} \\
\leq C_t mN^{\beta-1} \langle \Phi, d\Gamma(-\Delta)\Phi \rangle.
\end{align*}
\]
Let us now decompose $\partial_t(Q(t) \otimes Q(t) \otimes 1 \otimes Q(t))$ into four terms. For the first term $(\partial_t Q(t) \otimes Q(t) \otimes 1 \otimes Q(t))$ we get

$$\frac{1}{N^2} \left| \cdots \right| \int \langle \partial_t Q(t) \otimes Q(t) \otimes 1 \otimes Q(t) \rangle (x, y, z; x', y', z') u(t, z) u(t, y') \times \Phi, a_x^* a_y a_x a_z (N - N) \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$= \frac{1}{N^2} \left| \cdots \right| \int \langle \partial_t Q(t)(x; x') V_N(x' - y, x' - z) \delta(y - y') \delta(z - z') u(t, z) u(t, y') \times \Phi, a_x^* a_y a_x a_z (N - N) \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$\leq \frac{1}{N^2} \left| \cdots \right| \int \int \int q(t, x) q(t, y') V_N(x' - y, x' - z) |u(t, z)| |u(t, y')| \times \times \| a_x a_y \sqrt{N - N} \Phi \| \| a_x a_z \sqrt{N - N} \Phi \| \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$\leq \frac{C_1}{N^2} \left( N, N \Phi \right) \frac{1}{2} \left( \int \int \int V_N(x' - y, x' - z) |u(t, y')| |u(t, y')| \times \times \| a_x a_y \sqrt{N - N} \Phi \| \| a_x a_z \sqrt{N - N} \Phi \| \, dx \, dy \, dz \, dx' \, dy' \, dz' \right)^{1/2}$$

$$\leq C_1 \sqrt{m N^2} \langle \Phi, \sqrt{N} \Phi \rangle^{1/2} \langle \Phi, d\Gamma(\Delta) \Phi \rangle^{1/2} \leq C_1 \sqrt{m N^2} \langle \Phi, d\Gamma(\Delta) \Phi \rangle.$$

The second term $(Q(t) \otimes \partial_t Q(t) \otimes 1 \otimes Q(t))$ goes as

$$\frac{1}{N^2} \left| \cdots \right| \int \langle Q(t) \otimes \partial_t Q(t) \otimes 1 \otimes Q(t) \rangle (x, y, z; x', y', z') u(t, z) u(t, y') \times \Phi, a_x^* a_y a_x a_z (N - N) \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$= \frac{1}{N^2} \left| \cdots \right| \int \langle \partial_t Q(t)(y; y') V_N(x' - y, x' - z) \delta(x - x') \delta(z - z') u(t, z) u(t, y') \times \Phi, a_x^* a_y a_x a_z (N - N) \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$\leq \frac{1}{N^2} \left| \cdots \right| \int \int \int q(t, y) q(t, y') V_N(x' - y, x' - z) |u(t, z)| |u(t, y')| \times \times \| a_x a_y \sqrt{N - N} \Phi \| \| a_x a_z \sqrt{N - N} \Phi \| \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$\leq \frac{C_1}{N^2} \left( N, N \Phi \right) \frac{1}{2} \left( \int \int \int V_N(x' - y, x' - z) |u(t, y')| |u(t, y')| \times \times \| a_x a_y \sqrt{N - N} \Phi \| \| a_x a_z \sqrt{N - N} \Phi \| \, dx \, dy \, dz \, dx' \, dy' \, dz' \right)^{1/2}$$

$$\leq C_1 \sqrt{m N^2} \langle \Phi, \sqrt{N} \Phi \rangle^{1/2} \langle \Phi, d\Gamma(\Delta) \Phi \rangle.$$

For the third term $(Q(t) \otimes Q(t) \otimes 1 \otimes \partial_t Q(t) \otimes 1 \otimes Q(t))$ we see that

$$\frac{1}{N^2} \left| \cdots \right| \int \langle Q(t) \otimes Q(t) \otimes 1 \otimes \partial_t Q(t) \otimes 1 \otimes Q(t) \rangle (x, y, z; x', y', z') u(t, z) u(t, y') \times \Phi, a_x^* a_y a_x a_z (N - N) \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$= \frac{1}{N^2} \left| \cdots \right| \int \int \int \int V_N(x - y, x - z) \delta(x - x') \delta(y - y') \delta(z - z') u(t, z) u(t, y') \times \times \Phi, a_x^* a_y a_x a_z (N - N) \Phi \rangle \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$

$$\leq \frac{1}{N^2} \left| \cdots \right| \int \int \int \int \int \int V_N(x - y, x - z) q(t, x) q(t, x') \times \times \| a_x a_y \sqrt{N - N} \Phi \| \| a_x a_z \sqrt{N - N} \Phi \| \, dx \, dy \, dz \, dx' \, dy' \, dz' \right|$$
\[ \leq C_t \sqrt{N^3 - 1} \langle \Phi, d\Gamma(1 - \Delta) \rangle. \]

Thus
\[ \pm \partial_t (R_{4,3} + R'_{4,3}) \leq C_t \sqrt{mN^3 - 1} d\Gamma(1 - \Delta). \]

For \( \Phi \in \mathcal{F}_{+}^{m}(t) \) we have that
\[ \langle \Phi, \partial_t R_5 \Phi \rangle = \frac{1}{N^2} \int \cdots \int \left[ \langle Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t) \otimes 1 \rangle(x, y, z; x', y', z') \partial_t u(t, z') + \right. \]
\[ + \left. \langle \partial_t (Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t) \otimes 1) \rangle(x, y, z; x', y', z') u(t, z') \right] \times \]
\[ \times \langle \Phi, a_x^* a_y^* a_z^* a'_{x'} a'_y \sqrt{N - N - 2\Phi} \rangle dxdydzdz'. \]

The term containing \( \partial_t u(t, z') \) can be estimated as
\[ \frac{1}{N^2} \int \cdots \int \langle Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t) \otimes 1 \rangle(x, y, z; x', y', z') \partial_t u(t, z') \times \]
\[ \times \langle \Phi, a_x^* a_y^* a_z^* a'_{x'} a'_y \sqrt{N - N - 2\Phi} \rangle dxdydzdz' \]
\[ \leq \frac{1}{N^2} \left( \iint V_N(x - y, x - z)\|a_x a_y a_z \Phi\|^2 dxdydz \right)^{\frac{1}{2}} \times \]
\[ \times \left( \iint V_N(x - y, x - z)\|\partial_t u(t, z)^2\|a_x a_y a_z \sqrt{N - N - 2\Phi} dx dydz \right)^{\frac{1}{2}} \]
\[ \leq C_t \sqrt{mN^3 - 1} \langle \Phi, d\Gamma(1 - \Delta) \rangle \]
\[ \leq C_t \sqrt{mN^3 - 1} \langle \Phi, d\Gamma(1 - \Delta) \rangle \]
\[ \leq C_t \sqrt{mN^3 - 1} \langle \Phi, d\Gamma(1 - \Delta) \rangle \]
\[ \leq C_t \langle \Phi, d\Gamma(1 - \Delta) \rangle. \]

Here we have used the bound on \( R_6 \) in [111] in the last estimate.

For the other terms we decompose \( \partial_t(Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t) \otimes 1) \) into five terms. The first one containing \( \langle \partial_t Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t) \otimes 1 \rangle \) can be bounded as
\[ \frac{1}{N^2} \int \cdots \int \langle \partial_t Q(t) \otimes Q(t) \otimes Q(t) V_N Q(t) \otimes Q(t) \otimes 1 \rangle(x, y, z; x', y', z') u(t, z') \times \]
\[ \times \langle \Phi, a_x^* a_y^* a_z^* a'_{x'} a'_y \sqrt{N - N - 2\Phi} \rangle dxdydzdz' \]
\[ = \frac{1}{N^2} \int \cdots \int \langle \partial_t Q(t) \rangle(x; x') V_N(x' - y, x' - z) \delta(y - y') \delta(z - z') u(t, z') \times \]
\[ \times \langle \Phi, a_x^* a_y^* a_z^* a'_{x'} a'_y \sqrt{N - N - 2\Phi} \rangle dxdydzdz' \]
\[ \leq \frac{1}{N^2} \iint q(t, x) q(t, x') |V_N(x' - y, x' - z)||u(t, z)|| \times \]
\[ \times \|a_x a_y a_z \Phi\| a_x a_y \sqrt{N - N - 2\Phi} dx dydzdz' \]
\[ = C_t \langle \Phi, d\Gamma(1 - \Delta) \rangle. \]
\[
\left\| Q(t) \otimes Q(t) \otimes Q(t) \right\|_{V^N(t) \otimes Q(t) \otimes Q(t) \otimes 1} \leq C_4 \left| \Phi(t) \right| \leq C \gamma \left( \Phi, \left. \delta \gamma \right(1 + \Delta) \right).
\]

The term with \((Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes 1)\) and \((Q(t) \otimes Q(t) \otimes Q(t)) \otimes Q(t) \otimes Q(t) \otimes 1)\) can be treated similarly. The fourth term \((Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes 1)\) can be bounded as

\[
\frac{1}{N^2} \left\| \left( Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \right) \right\|_{V^N(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes 1} \leq C \gamma \left( \Phi, \left. \delta \gamma \right(1 + \Delta) \right).
\]

The other term containing \((Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes 1)\) can be bounded in the same way. Putting everything together we get

\[
\left| \left< \Phi, \partial_t R_3 \Phi \right> \right| \leq C_5 \gamma \left( \Phi, \left. \delta \gamma \right(1 + \Delta) \right).
\]

This means

\[
\partial_t (R_3 + R_5^2) \leq C_5 \gamma \left( \Phi, \left. \delta \gamma \right(1 + \Delta) \right).
\]

**Case 6** Let \( \Phi \in \mathcal{F}^{c,m}_+(t) \). Then we have

\[
\left< \Phi, \partial_t R_6 \Phi \right> = \frac{1}{6N^2} \int \left( \partial_t (Q(t) \otimes Q(t) \otimes Q(t)) \right) \left( x, y, z, x', y', z' \right) \times \left< \Phi, a_x^* a_y^* a_z^* a_{x'} a_{y'} a_{z'} \Phi \right> dx \, dy \, dz \, dx' \, dy' \, dz'
\]

We can decompose \( \partial_t (Q(t) \otimes Q(t) \otimes Q(t)) \otimes Q(t) \otimes Q(t) \otimes Q(t) \) into six terms which can be estimated all the same. Therefore, we only need to prove the bound for \( \partial_t (Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t)) \). We have

\[
\frac{1}{6N^2} \int \left( \partial_t (Q(t) \otimes Q(t) \otimes Q(t)) \right) \left( x, y, z, x', y', z' \right) \times \left< \Phi, a_x^* a_y^* a_z^* a_{x'} a_{y'} a_{z'} \Phi \right> dx \, dy \, dz \, dx' \, dy' \, dz'
\]

\[
= \frac{1}{6N^2} \int \left( \partial_t Q(t) \right) (x', x') V_N(x' - y, x' - z) \delta(y - y') \delta(z - z') \times \left< \Phi, a_x^* a_y^* a_z^* a_{x'} a_{y'} a_{z'} \Phi \right> dx \, dy \, dz \, dx' \, dy' \, dz'
\]

\[
\leq \frac{1}{6N^2} \int \left( q(t, x) q(t, x') \right) V_N(x' - y, x' - z) ||a_x a_y a_z|| ||a_x a_y a_z|| dx \, dy \, dz \, dx'
\]
Moreover, from the explicit computation (77) we have the simple estimate

\[ \langle \Phi_N(t), \partial_t \Phi_N(t) \rangle = \langle \Phi_N(t), 6N \Phi_N(t) \rangle + O(C_t N). \]  

7.1. Step 1: Round kinetic bound for full dynamics.

**Lemma 16** (Round kinetic bound for full dynamics). *Let \( \beta < 1/4 \) and let \( \Phi_N(0) \) satisfy (93). Then*

\[ \langle \Phi_N(t), d\Gamma(1 - \Delta) \Phi_N(t) \rangle \leq C_t N. \]  

**Proof.** First of all by the energy conservation of the Schrödinger dynamics, we have

\[ \langle \Phi_N(t), \partial_t \Phi_N(t) \rangle = \langle \Phi_N(t), 6N \Phi_N(t) \rangle. \]  

Moreover, from the explicit computation (94) we have the simple estimate

\[ \langle \Psi_N(t), H_N \Phi_N(t) \rangle = \langle \Phi_N(t), U_N H_N U_N^* \Phi_N(t) \rangle = \langle \Phi_N(t), \tilde{H}_N(t) \Phi_N(t) \rangle + O(C_t N). \]  

Here again we used the bound on \( R_6 \) in (91). Thus

\[ \pm \partial_t R_6 \leq C_t \sqrt{m^3 N^{5\beta - 3}} \langle \Phi, d\Gamma(1 - \Delta) \Phi \rangle. \]  

This finishes the proof of (90) and by that the proof of Lemma 13. □
Next, using (94) with $m = N$ and $\eta = C_1N^{4\beta}$ we find that
\[ \pm \tilde{H}_N(t) \leq C_1N^{4\beta}d\Gamma(1 - \Delta) + C_1N. \]
Therefore, the assumption (93) ensures that
\[ \pm \langle \Phi_N(0), \tilde{H}_N(0)\Phi_N(0) \rangle \leq CN. \]

Combining the latter estimate with (98) and (99) we deduce that
\[ \pm \langle \Psi_N(t), H_N\Psi_N(t) \rangle \leq C_tN \tag{100} \]
Since $V_N \geq 0$, we then obtain
\[ \langle \Psi_N(t), d\Gamma(-\Delta)\Psi_N(t) \rangle \leq C_tN. \tag{101} \]

Next, by decomposing $1 = P(t) + Q(t)$, with $P(t) = |u(t)\rangle\langle u(t)|$ we have the Cauchy-Schwarz inequality
\[ -\Delta = P(t)(-\Delta)P(t) + Q(t)(-\Delta)Q(t) + P(t)(-\Delta)Q(t) + Q(t)(-\Delta)P(t) \]
\[ \geq (1 - \eta^{-1})P(t)(-\Delta)P(t) + (1 - \eta)Q(t)(-\Delta)Q(t), \quad \forall \eta > 0. \]
Taking $\eta = 1/2$ and using $P(t)(-\Delta)P(t) = \|\nabla u(t, \cdot)\|_{L^2}^2P(t)$ we obtain
\[ Q(t)(1 - \Delta)Q(t) \leq C_t(1 - \Delta). \]
Consequently,
\[ d\Gamma(Q(t)(1 - \Delta)Q(t)) \leq 2d\Gamma(-\Delta) + C_tN. \]
Thus from (101) we deduce (97):
\[ \langle \Phi_N, d\Gamma(1 - \Delta)\Phi_N \rangle = \langle \Psi_N, d\Gamma(Q(t)(1 - \Delta)Q(t))\Psi_N \rangle \leq C_t \langle \Psi_N, d\Gamma(1 - \Delta)\Psi_N \rangle \leq C_tN. \]

\section*{7.2. Step 2: Improved kinetic bound for truncated dynamics.}
For every $M \geq 1$, we introduce an intermediate dynamics on truncated Fock space $\mathcal{F}_+^{\leq M}(t)$.
\[ i\partial_t \Phi_{N,M}(t) = 1_{\leq M}\tilde{H}_N(t)1_{\leq M}\Phi_{N,M}(t), \quad \Phi_{N,M}(0) = 1_{\leq M}\Phi_N(0). \tag{102} \]

Our idea is that if $M$ is significantly smaller than $N$, then we will have a better control on the kinetic energy of the truncated dynamics $\Phi_{N,M}$.

\textbf{Lemma 17} (Refined kinetic bound for truncated dynamics). Let $\Phi_N(0)$ satisfy (93). Then for all $1 \ll M \ll N^{1-2\beta}$, and for all $t > 0$, $\varepsilon > 0$ we have
\[ \langle \Phi_{N,M}(t), d\Gamma(1 - \Delta)\Phi_{N,M}(t) \rangle \leq C_{t,\varepsilon} \left( MN^{4\beta - 1} + N^{3+\varepsilon} \right). \tag{103} \]

\textbf{Proof}. When $M \ll N^{1-2\beta}$, we have
\[ \max\{\sqrt{MN^{2\beta - 1}}, \sqrt{M^3N^{5\beta - 3}}, M^2N^{4\beta - 2} \} = \sqrt{MN^{2\beta - 1}} \ll 1. \]
Therefore, we can apply (94)-(96) with $\eta = 1/2$ and obtain
\[ \pm 1_{\leq M}\left( \tilde{H}_N(t) - d\Gamma(-\Delta) \right)1_{\leq M} \leq \frac{1}{2}d\Gamma(1 - \Delta) + C_{t,\varepsilon} \left( N + MN^{4\beta - 1} + N^{3+\varepsilon} \right), \]
\[ \pm 1_{\leq M}[i\tilde{H}_N(t),\mathcal{N}]1_{\leq M} \leq \frac{1}{2}d\Gamma(1 - \Delta) + C_{t,\varepsilon} \left( N + MN^{4\beta - 1} + N^{3+\varepsilon} \right). \]
we have sectors with large and small particle numbers. Take To bound this expression we will use a method from [42, 11] where we will separate the inequality and the kinetic bound (103), which gives

The term containing large particle numbers can be bounded using the Cauchy-Schwarz proof.

Using the mass conservation

Therefore, Gronwall’s inequality implies that

Here recall that for large enough, we have the quadratic form estimates

Step 3: Truncated vs. full dynamics. Now we show that if is sufficiently large, then the truncated dynamics in (102) is close to the full dynamics .

Lemma 18 (Norm approximation to full dynamics). Let and let satisfy (93). Then for all and for all we have

Proof. Using the mass conservation

we have

To bound this expression we will use a method from [42, 11] where we will separate the sectors with large and small particle numbers. Take and write

The term containing large particle numbers can be bounded using the Cauchy-Schwarz inequality and the kinetic bound (103), which gives
Here we have used the fact that
\[ \langle \Phi_N(t), \mathbb{1}^{\leq m} \Phi_{N,M}(t) \rangle = \mathbb{1}^{\leq m} \left( \mathbb{1}^{\leq m} - \mathbb{1}^{\leq m} \mathbb{1}^{\leq M} \tilde{H}_N(t) \mathbb{1}^{\leq M} \right) \Phi_{N,M}(t) \]
\[ = \left\langle \Phi_N(t), i \tilde{H}_N(t), \mathbb{1}^{\leq m} \Phi_{N,M}(t) \right\rangle. \tag{111} \]

Here we have used the fact that
\[ \mathbb{1}^{\leq m} \mathbb{1}^{\leq M} \tilde{H}_N(t) \mathbb{1}^{\leq M} = \mathbb{1}^{m} \tilde{H}_N(t) \]
which is true because \( m \leq M - 3 \) and \( \tilde{H}_N(t) \) contains at most 3 creation and at most 3 annihilation operators. To bound this expression we will average over \( m \in [M/2, M - 3] \).

The key point of this argument is summarized in the following lemma.

**Lemma 19.** For \( 1 \leq M \ll N^{1-2\beta} \) we have the operator bound
\[ \pm \frac{1}{M/2 - 2} \sum_{m=M/2}^{M-3} i \tilde{H}_N(t), \mathbb{1}^{\leq m} \leq C_{t,\varepsilon} \left( d\Gamma(1 - \Delta) + M N^{4\beta - 1} + N^{\beta + \varepsilon} \right). \tag{113} \]

Let us postpone the proof of Lemma 19 and proceed to conclude Lemma 17. Using (111), Lemma 19 and the kinetic estimates in Lemmas 16, 17, we obtain

\[ \left| \frac{1}{M/2 - 1} \sum_{m=M/2}^{M-3} \frac{d}{dt} \left\langle \Phi_N(t), \mathbb{1}^{\leq m} \Phi_{N,M}(t) \right\rangle \right| \]
\[ = \left| \frac{1}{M/2 - 1} \sum_{m=M/2}^{M-3} \left\langle \Phi_N(t), i \tilde{H}_N(t), \mathbb{1}^{\leq m} \Phi_{N,M}(t) \right\rangle \right| \]
\[ \leq \frac{C_{t,\varepsilon}}{M} \left( d\Gamma(1 - \Delta) + M N^{4\beta - 1} + N^{\beta + \varepsilon} \right) \left\langle \Phi_N(t) \right\rangle^{1/2} \times \left( \left\langle \Phi_{N,M}(t) \right\rangle \left( d\Gamma(1 - \Delta) + M N^{4\beta - 1} + N^{\beta + \varepsilon} \right) \Phi_{N,M}(t) \right)^{1/2} \]
\[ \leq \frac{C_{t,\varepsilon}}{M} \sqrt{N + M N^{4\beta - 1} + N^{\beta + \varepsilon}} \sqrt{M N^{4\beta - 1} + N^{\beta + \varepsilon}} \]
\[ \leq C_{t,\varepsilon} \left( N^{4\beta - 1} + \frac{N^{2\beta}}{\sqrt{M}} + \frac{N^{(1+\beta+\varepsilon)/2}}{M} \right). \]

Furthermore, the assumption (93) ensures that
\[ \left\langle \Phi_N(0), \mathbb{1}^{\leq m} \Phi_{N,M}(0) \right\rangle = \left\langle \Phi_N(0), \mathbb{1}^{\leq m} \Phi_N(0) \right\rangle = 1 - \left\langle \Phi(0), \mathbb{1}^{\gtrsim m} \Phi(0) \right\rangle \]
\[ \geq 1 - \left( \frac{N}{m} \Phi(0) \right) \geq 1 - \frac{C}{M}. \]

Hence, we get
\[ \Re \frac{1}{M/2 - 1} \sum_{m=M/2}^{M-3} \left\langle \Phi_N(t), \mathbb{1}^{\leq m} \Phi_{N,M}(t) \right\rangle \]
We write Proof of Lemma 19.

This finishes the proof of Lemma 17. □

Moreover, from (110) we immediately see that

\[ \mathfrak{R} \frac{1}{M/2 - 1} \sum_{m=M/2}^{M-3} \langle \Phi_N(t), \mathbb{1}^{\leq m} \Phi_{N,M}(t) \rangle \geq -C_{t,\varepsilon} \left( \frac{N^{(4\beta - 1)/2}}{\sqrt{M}} + \frac{N^{(1+\beta+\varepsilon)/2}}{M} \right). \] (114)

Summing these two estimates and using the decomposition (109), we get

\[ \mathfrak{R} \langle \Phi_N(t), \Phi_{N,M}(t) \rangle \geq 1 - C_{t,\varepsilon} \left( \frac{N^{(4\beta - 1)/2}}{\sqrt{M}} + \frac{N^{(1+\beta+\varepsilon)/2}}{M} \right). \]

Thus in conclusion, (108) implies that

\[ ||\Phi_N(t) - \Phi_{N,M}(t)||^2 \leq C_{t,\varepsilon} \left( \frac{N^{(4\beta - 1)/2}}{\sqrt{M}} + \frac{N^{(1+\beta+\varepsilon)/2}}{M} \right). \]

This finishes the proof of Lemma 17. □

It remains to give

**Proof of Lemma 19**. We write

\[ [\tilde{H}_N(t), \mathbb{1}^{\leq m}] = \mathbb{1}^{> m} \tilde{H}_N(t) \mathbb{1}^{\leq m} - \mathbb{1}^{\leq m} \tilde{H}_N(t)^{> m}. \]

Since both terms can be treated similarly we will only handle the first one. Let

\[ \mathcal{E}_1 = \alpha^+ \left( \langle Q(t) \frac{1}{2} \int |u(t)|^2 V_N |u(t)|^2 \, dy \, dz \, u(t) \rangle \sqrt{N-N} \left( N - N' - 1 \right)^2 - (N-N' - 1)^2 - N^2 \right. \]

\[ + \frac{1}{N^2} \int \int \int \int (Q(t) \otimes Q(t) \int |u(t,z)|^2 V_N \, dz \, Q(t) \otimes 1) (x, y; x', y') u(t, y') \times \]

\[ a_x^* a_y^* a_x' a_y' \, dy \, dy' \sqrt{N-N} \left( N-N' - 1 \right) \]

\[ + \frac{1}{2N^2} \int \cdots \int \left( Q(t) \otimes Q(t) \otimes 1 V_N \otimes 1 \otimes Q(t) \right) (x, y, z; x', y', z') u(t, z) u(t, x') u(t, z') \times \]

\[ a_x^* a_y^* a_x' a_y' \, dy \, dy \, dy' \sqrt{N-N} \left( N-N' - 1 \right) + h.c. \]

\[ + \frac{1}{2N^2} \int \cdots \int \left( Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes 1 \otimes 1 \right) (x, y, z; x', y', z') u(t, y') u(t, y') u(t, z') \times \]

\[ a_x^* a_y^* a_x' a_y' \sqrt{N-N} - 1 \sqrt{N-N} \, dx \, dy \, dz \, dy' \, dz' + h.c. + \]

\[ \mathcal{E}_2 = \frac{1}{2} \int K(x, y) a_x^* a_y^* \, dx \, dy \sqrt{N-N-1} \sqrt{N-N-1} \left( N-N-2 \right) \]

\[ + \frac{1}{2N^2} \int \cdots \int \left( Q(t) \otimes Q(t) \otimes Q(t) V_N \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes 1 \right) (x, y; x', y', z') u(t, z') \times \]

\[ a_x^* a_y^* a_x' a_y' \sqrt{N-N-2} \, dx \, dy \, dz \, dy' \, dz' + h.c. \]

\[ \mathcal{E}_3 = \frac{1}{6N^2} \int \cdots \int \left( Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) V_N \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \otimes Q(t) \right) (x, y, z; x', y', z') u(t, z') \times \]

\[ a_x^* a_y^* a_z^* \, dx \, dy \, dz \, dy' \, dz' \sqrt{N-N-2} \sqrt{N-N-2} \sqrt{N-N-1} \sqrt{N-N-1} \]
With this we can write
\[ 1^m \tilde{H}_N(t) 1^m = E_1 1(N = m) + E_2 1(m - 1 \leq N \leq m) + E_3 1(m - 2 \leq N \leq m). \]
From this we get
\[
\sum_{m=M/2}^{M-3} 1^m \tilde{H}_N(t) 1^m = E_1 1(M/2 \leq N \leq M - 3)
+ E_2 \left[ 1(M/2 - 1 \leq N \leq M - 4) + 1(M/2 \leq N \leq M - 3) \right]
+ E_3 \left[ 1(M/2 - 2 \leq N \leq M - 5) + 1(M/2 - 1 \leq N \leq M - 4) + 1(M/2 \leq N \leq M - 3) \right]
\]
Thus we have the quadratic form estimate
\[
\left( X, E_3 1(M/2 \leq N \leq M - 3) X \right) \leq \left( X, d\Gamma(1 - \Delta) + MN^{4\beta - 1} X \right)^{1/2} 
\times \left( 1(M/2 \leq N \leq M - 3) X, d\Gamma(1 - \Delta) + MN^{4\beta - 1} 1(M/2 \leq N \leq M - 3) X \right)^{1/2}
\leq \left( X, d\Gamma(1 - \Delta) + MN^{4\beta - 1} X \right).
\]
Thus we have the quadratic form estimate
\[
\pm \left( E_3 1(M/2 \leq N \leq M - 3) + h.c. \right) \leq d\Gamma(1 - \Delta) + MN^{4\beta - 1}.
\]
All the other terms can be bound similarly using the Lemmas \[9\] and \[13\]. For example, we have
\[
\pm \left( E_2 1(M/2 \leq N \leq M - 3) + h.c. \right) \leq d\Gamma(1 - \Delta) + N^{\beta + \varepsilon},
\]
and similarly,
\[
\frac{1}{M/2 - 2} \sum_{m=M/2}^{M-3} \frac{i[H(t), 1^m]}{M} \leq \frac{C_{t, \varepsilon}}{M} \left( d\Gamma(1 - \Delta) + N^{\beta + \varepsilon} \right).
\]
This ends the proof of Lemma \[19\] \(\Box\)

7.4. Step 3: Conclusion of Theorem \[1\]

Proof of Theorem \[1\]. Let \( 0 < \beta < 1/6 \). Let \( u(t) \) and \( \varphi(t) \) be the solution to the quintic Hartree equation \[18\] and the quintic NLS \[1\] with the same initial condition \( \varphi_0 = u_0 \in H^4(\mathbb{R}^3) \). We define
\[
\Phi_N(t) = e^{-i\int_0^t \varphi(s) ds} U_N(t) \Psi_N(t) = e^{-i\int_0^t \varphi(s) ds} U_N(t) e^{-iH_N} U_N^*(0) \Phi_N(0).
\]
The actions \[12\] lead to the key identity
\[
Q(t) \gamma^{(1)}_{\Phi_N(t)} Q(t) = \gamma^{(1)}_{\Phi_N(t)}.
\]
In particular, recall that the assumption \[12\] in Theorem \[1\] is equivalent to \[33\],
\[
\langle \Phi_N(0), d\Gamma(1 - \Delta) \Phi_N(0) \rangle = \text{Tr} \left( (1 - \Delta) \gamma^{(1)}_{\Phi_N(0)} \right) = \text{Tr} \left( (1 - \Delta) Q(0) \gamma^{(1)}_{\Psi_N(0)} Q(0) \right) \leq C.
\]
By applying Lemma \[18\] we find that for all \( 1 \leq M \ll N^{1-2\beta} \)}
the truncated dynamics $\Phi_{N,M}(t) \in \mathcal{F}_+^{\leq M}(t)$ defined by

$$i\partial_t \Phi_{N,M}(t) = \mathbb{1}^{\leq M} \hat{H}_N(t) \mathbb{1}^{\leq M} \Phi_{N,M}(t), \quad \Phi_{N,M}(0) = \mathbb{1}^{\leq M} \Phi_N(0)$$

satisfies the norm approximation

$$\|\Phi_N - \Phi_{N,M}\|^2 \leq C_{t,\epsilon} \left( N^{(4\beta-1)/2} + \frac{N^{2\beta}}{\sqrt{M}} + \frac{N^{(1+\beta+\epsilon)/2}}{M} \right). \quad (117)$$

Now we transfer (117) to an estimate on the one-body density matrix. Since $\Phi_{N,M}(t) \in \mathcal{F}_+^{\leq M}(t)$, we have the obvious estimate

$$\|\Phi_N(t) - \Phi_{N,M}(t)\|^2 = \|\mathbb{1}^{>M} \Phi_N(t)\|^2 + \|\mathbb{1}^{\leq M} \Phi_N(t) - \Phi_{N,M}(t)\|^2 \geq \|\mathbb{1}^{>M} \Phi_N(t)\|^2.$$

Thus (117) gives

$$\|\mathbb{1}^{>M} \Phi_N(t)\|^2 \leq C_{t,\epsilon} \left( N^{(4\beta-1)/2} + \frac{N^{2\beta}}{\sqrt{M}} + \frac{N^{(1+\beta+\epsilon)/2}}{M} \right). \quad (118)$$

Then by the Cauchy-Schwarz inequality and the fact that $1 - Q(t) = |u(t)\rangle\langle u(t)|$ is a rank-one projection, we obtain

$$\text{Tr} \left[ N^{-1} \gamma_{\Psi_N}(t) - |u(t)\rangle\langle u(t)| \right] \leq \sqrt{N^{-1} \text{Tr} \left( Q(t) \gamma_{\Psi_N}(t) Q(t) \right)}$$

$$\leq \sqrt{MN^{-1} + C_{t,\epsilon} \left( \frac{N^{2\beta}}{\sqrt{M}} + \frac{N^{(1+\beta+\epsilon)/2}}{M} \right).} \quad (119)$$

On the other hand, from (27) in Theorem 3 we find that

$$\text{Tr} \left[ |u(t)\rangle\langle u(t)| - |\varphi(t)\rangle\langle \varphi(t)| \right] \leq 2 \|u(t) - \varphi(t)\|_{L^2(\mathbb{R}^3)} \leq C t N^{-\beta/2}. \quad (120)$$

Thus in summary, from (119) and (120) we conclude by the triangle inequality that

$$\text{Tr} \left[ N^{-1} \gamma_{\Psi_N}(t) - |\varphi(t)\rangle\langle \varphi(t)| \right]$$

$$\leq C t N^{-\beta/2} + \sqrt{MN^{-1} + C_{t,\epsilon} \left( N^{(4\beta-1)/2} + \frac{N^{2\beta}}{\sqrt{M}} + \frac{N^{(1+\beta+\epsilon)/2}}{M} \right).} \quad (121)$$

It remains to optimize the right side of (121) over $1 \ll M \ll N^{1-2\beta}$. Choosing

$$M = N^{1-2\beta-\epsilon}$$

for $\epsilon > 0$ arbitrarily small (but independent of $N$), we deduce from (121) that

$$\text{Tr} \left[ N^{-1} \gamma_{\Psi_N}(t) - |\varphi(t)\rangle\langle \varphi(t)| \right] \leq C_{t,\epsilon} N^{-\alpha} \quad (122)$$
for any constant
\[ \alpha < \min \left\{ \frac{\beta}{2}, \frac{1-6\beta}{4} \right\}. \]

This completes the proof of Theorem \(1\). \(\square\)

\section{8. Proof of Theorem \(2\)}

In this section we prove the norm approximation in Theorem \(2\). As explained in Section \(7\) it boils down to prove that the transformed dynamics \(\Phi_N(t) = e^{-i \int_0^t \chi(s) \, ds} U_N(t) \Psi_N(t)\) in \((74)\) converges to the Bogoliubov dynamics \(\Phi(t)\) in \((19)\).

We will follow the strategy of the previous section to consider the truncated dynamics \(\Phi_{N,M}\) in \((102)\),
\[ i \partial_t \Phi_{N,M}(t) = 1^{\leq M} \tilde{H}(t) 1^{\leq M} \Phi_{N,M}(t), \quad \Phi_{N,M}(0) = 1^{\leq M} \Phi_N(0). \]

The norm approximation between \(\Phi_{N,M}\) and \(\Phi_N\) was already provided in Lemma \(18\). Therefore, it is natural to compare \(\Phi_{N,M}\) with the Bogoliubov dynamics \(\Phi(t)\).

\subsection*{8.1. Step 1: Truncated vs. Bogoliubov dynamics.}

\textbf{Lemma 20} (Truncated vs. Bogoliubov dynamics). Let \(1 \ll M \ll N^{1-2\beta}\). Then
\[ \|\Phi_{N,M}(t) - \Phi(t)\|^2 \leq C_{t,\epsilon} \left( \frac{N^{\beta+\epsilon}}{M} + \sqrt{MN N^{5\beta-1+2\epsilon}} \right), \quad (123) \]

\textbf{Proof}. Similarly to \((108)\), we have
\[ \|\Phi_{N,M}(t) - \Phi(t)\|^2 \leq 2 \left( 1 - \Re \langle \Phi_{N,M}(t) , \Phi(t) \rangle \right). \]

Now picking \(M/2 \leq m \leq M - 3\) we write
\[ \langle \Phi_{N,M}(t) , \Phi(t) \rangle = \langle \Phi_{N,M}(t) , 1^{>m} \Phi(t) \rangle + \langle \Phi_{N,M}(t) , 1^{\leq m} \Phi(t) \rangle. \quad (124) \]

The first term containing the many particle sector can be bounded using \((57)\):
\[ \|\langle \Phi_{N,M}(t) , 1^{>m} \Phi(t) \rangle\| \leq \|1^{>m} \Phi(t)\| \leq \langle \Phi(t) , (N/m) \Phi(t) \rangle^{1/2} \leq C_{t,\epsilon} \frac{N^{(\beta+\epsilon)/2}}{\sqrt{M}}. \quad (125) \]

For the second term in \((124)\) we calculate the derivative
\[ \frac{d}{dt} \langle \Phi_{N,M}(t) , 1^{\leq m} \Phi(t) \rangle = i \langle \Phi_{N,M}(t) , \left( 1^{\leq M} \tilde{H}(t) 1^{\leq M} 1^{\leq m} - 1^{\leq m} \tilde{H}(t) 1^{\leq M} 1^{\leq m} \right) \Phi(t) \rangle \]
\[ = i \langle \Phi_{N,M}(t) , [\tilde{H}(t) , 1^{\leq m}] + \left( (\tilde{H}(t) - \tilde{H}(t)) 1^{\leq m} \right) \Phi(t) \rangle, \quad (126) \]

where we have used, as in \((122)\),
\[ 1^{\leq M} \tilde{H}(t) 1^{\leq M} 1^{\leq m} = \tilde{H}(t) 1^{\leq m}. \]

To bound the first term in \((126)\) containing the commutator \([\tilde{H}(t) , 1^{\leq m}]\), we average over \(m \in [M/2 , M-3]\), then use \((115)\) and the kinetic bounds in Theorem \(8\) and Lemma \(17\).

We obtain
\[ \left| \frac{1}{M/2 - 1} \sum_{m=M/2}^{M-3} \langle \Phi_{N,M}(t) , i[\tilde{H}(t) , 1^{\leq m}] \Phi(t) \rangle \right| \]
\[ \leq \frac{C_{t,\epsilon}}{M} \left\langle \Phi_{N,M}(t) , \left( \text{d} \Gamma(1 - \Delta) + N^{\beta+\epsilon} \right) \Phi_{N,M}(t) \right\rangle^{\frac{1}{2}} \left\langle \Phi(t) , \left( \text{d} \Gamma(1 - \Delta) + N^{\beta+\epsilon} \right) \Phi(t) \right\rangle^{\frac{1}{2}} \]
\[ \leq \frac{C_{t,\epsilon}}{M} \sqrt{MN} N^{4\beta-1} + N^{\beta+\epsilon} \cdot \sqrt{N^{\beta+\epsilon}} \leq \frac{C_{t,\epsilon}}{M} \left( \sqrt{MN} N^{5\beta-1+\epsilon} + N^{\beta+\epsilon} \right). \quad (127) \]
In order to estimate the second term in (125), we use (78) in Lemma 13 when \( m \ll N^{1-2\beta} \),

\[
1 \leq (\tilde{H}_N(t)-\mathbb{H}(t))1^{m} \leq \eta d\Gamma(1-\Delta) + \eta^{-1}C_{t}mN^{4\beta-1}, \quad \forall \eta \geq C_{t}\sqrt{mN^{2\beta-1}}.
\]

Then for all \( M/2 \leq m \leq M-3 \), by the Cauchy-Schwarz inequality and the kinetic bounds in Theorem 8 and Lemma 17, we can estimate

\[
\left| \left\langle \Phi_{N,M}(t), (\tilde{H}_N(t)-\mathbb{H}(t))1^{m}\Phi(t) \right\rangle \right| = \left| \left\langle \Phi_{N,M}(t), 1^{m+3}(\tilde{H}_N(t)-\mathbb{H}(t))1^{m+3}1^{m}\Phi(t) \right\rangle \right| \\
\leq C_{t} \left\langle \Phi_{N,M}(t), \left( \eta d\Gamma(1-\Delta) + \eta^{-1}M^{4\beta-1} \right) \Phi_{N,M}(t) \right\rangle^{\frac{1}{2}} \times \\
\times \left\langle \Phi(t), \left( \eta d\Gamma(1-\Delta) + \eta^{-1}M^{4\beta-1} \right) \Phi(t) \right\rangle^{\frac{1}{2}} \\
\leq C_{t}\sqrt{\eta(MN^{4\beta-1}+N^{\beta+\varepsilon}) + \eta^{-1}M^{4\beta-1} \cdot \sqrt{\eta N^{\beta+\varepsilon} + \eta^{-1}M^{4\beta-1}}} \\
\leq C_{t}\sqrt{\eta^{2}N^{2(\beta+\varepsilon)} + (1+\eta^{2})MN^{5\beta-1} + (1+\eta^{-2})(MN^{4\beta-1})^{2}}
\]

for all \( \eta \geq C_{t}\sqrt{MN^{2\beta-1}} \). By choosing

\[
\eta = C_{t}\sqrt{MN^{3\beta-1}}
\]

we obtain

\[
\left| \left\langle \Phi_{N,M}(t), (\tilde{H}_N(t)-\mathbb{H}(t))1^{m}\Phi(t) \right\rangle \right| \leq C_{t,\varepsilon}\sqrt{MN^{5\beta-1+2\varepsilon}}. \tag{128}
\]

Averaging the latter bound over \( M/2 \leq m \leq M-3 \), and combining with (127), we deduce from (126) that

\[
\left| \frac{1}{M/2-1} \sum_{m=M/2}^{M-3} \frac{d}{dt} \left( \Phi_{N,M}(t), 1^{m}\Phi(t) \right) \right| \leq C_{t,\varepsilon} \left( \frac{N^{\beta+\varepsilon}}{M} + \sqrt{MN^{5\beta-1+2\varepsilon}} \right).
\]

Then using the the bound

\[
\left\langle \Phi_{N,M}(0), 1^{m}\Phi(0) \right\rangle = 1 - \left\langle \Phi_{N}(0), 1^{m}\Phi_{N}(0) \right\rangle \geq 1 - \frac{C}{M},
\]

which follows from assumption \( \left\langle \Phi_{N}(0), \mathcal{N}\Phi_{N}(0) \right\rangle \leq C \), we obtain as in (114),

\[
\Re \frac{1}{M/2-1} \sum_{m=M/2}^{M-3} \left\langle \Phi_{N,M}(t), 1^{m}\Phi(t) \right\rangle \geq 1 - C_{t,\varepsilon} \left( \frac{N^{\beta+\varepsilon}}{M} + \sqrt{MN^{5\beta-1+2\varepsilon}} \right).
\]

Putting the latter estimate together with (125) (after averaging over \( M/2 \leq m \leq M-3 \)), we conclude that from (124) that

\[
\left\| \Phi_{N,M}(t) - \Phi(t) \right\|^{2} \leq C_{t,\varepsilon} \left( \frac{N^{\beta+\varepsilon}}{M} + \sqrt{MN^{5\beta-1+2\varepsilon}} \right).
\]

This is the desired estimate. \( \square \)
8.2. **Step 2: A further truncated dynamics.** Recall that from Lemma 18 and Lemma 20 we have proved that for all \(1 \ll M \ll N^{1-2\beta}\),

\[
\|\Phi_N(t) - \Phi_{N,M}(t)\|^2 \leq C_{t,\varepsilon} \left( N^{(4\beta - 1)/2} + \frac{N^{2\beta}}{\sqrt{M}} + \frac{N^{(1+\beta+\varepsilon)/2}}{M} \right),
\]

\[
\|\Phi_{N,M}(t) - \Phi(t)\|^2 \leq C_{t,\varepsilon} \left( \frac{N^{\beta+\varepsilon}}{M} + \sqrt{MN^{5\beta-1+2\varepsilon}} \right).
\]

Obviously, we can deduce \(\|\Phi_N(t) - \Phi(t)\| \to 0\) if we can control all the error terms in the above estimates. However, to make both error terms small simultaneously, we would need

\[
\frac{N^{2\beta}}{\sqrt{M}} \cdot \sqrt{MN^{5\beta-1+2\varepsilon}} = \sqrt{N^{9\beta-1+2\varepsilon}} \to 0
\]

which requires \(\beta < 1/9\).

To extend the norm approximation to all \(\beta < 1/6\), we need a further step. We will introduce another truncated dynamics \(\Phi_{N,\bar{M}}\) with \(\bar{M} \ll M\) and apply Lemma 20 to the new one. Of course, to make this strategy work we need to show that \(\Phi_{N,\bar{M}}\) is sufficiently close to \(\Phi_{N,M}\). This is the content of the following

**Lemma 21** (Intermediate norm approximation). Let \(1 \ll \bar{M} \leq M \ll N^{1-2\beta}\). Then

\[
\|\Phi_{N,M}(t) - \Phi_{N,\bar{M}}(t)\|^2 \leq C_{t,\varepsilon} \left( N^{(4\beta - 1)/2} + \frac{N^{(\beta+\varepsilon)/2}}{\sqrt{M}} + \frac{M}{\bar{M}}N^{4\beta-1} + \frac{\sqrt{MN^{5\beta+\varepsilon-1}}}{M} \right).
\]

**Proof.** The proof strategy follows similarly as in Lemma 18. We use again

\[
\|\Phi_{N,M}(t) - \Phi_{N,\bar{M}}(t)\|^2 \leq 2 \left( 1 - \Re \langle \Phi_{N,M}(t), \Phi_{N,\bar{M}}(t) \rangle \right).
\]

(129)

For any \(\bar{M}/2 \leq m \leq \bar{M} - 3\) we split the right side as

\[
\langle \Phi_{N,M}(t), \Phi_{N,\bar{M}}(t) \rangle = \langle \Phi_{N,M}(t), 1_{\leq m} \Phi_{N,\bar{M}}(t) \rangle + \langle \Phi_{N,M}(t), 1_{> m} \Phi_{N,\bar{M}}(t) \rangle.
\]

The second term can be bounded similarly to (110):

\[
|\langle \Phi_{N,M}(t), 1_{> m} \Phi_{N,\bar{M}}(t) \rangle| \leq \|\Phi_{N,M}(t)\| \cdot \|1_{> m} \Phi_{N,\bar{M}}(t)\| \leq C_{t,\varepsilon} \left( N^{(4\beta - 1)/2} + \frac{N^{(\beta+\varepsilon)/2}}{\sqrt{M}} \right).
\]

(130)

The bound for the few particle sectors follows again by averaging over \(m \in [\bar{M}/2, \bar{M} - 3]\), then using Lemma 19 and the kinetic bound in Lemma 17 (for both \(\Phi_{N,M}(t)\) and \(\Phi_{N,\bar{M}}(t)\)). This gives

\[
\frac{1}{\bar{M}/2 - 1} \sum_{m=\bar{M}/2}^{\bar{M}-3} \frac{1}{\bar{M}/2 - 1} \sum_{m=\bar{M}/2}^{\bar{M}-3} \langle \Phi_{N,M}(t), i[H_N(t), 1_{\leq m}] \Phi_{N,\bar{M}}(t) \rangle \mid
\]

\[
= \frac{1}{\bar{M}/2 - 1} \sum_{m=\bar{M}/2}^{\bar{M}-3} \langle \Phi_{N,M}(t), i[H_N(t), 1_{\leq m}] \Phi_{N,\bar{M}}(t) \rangle \mid
\]

\[
= \frac{1}{\bar{M}/2 - 1} \sum_{m=\bar{M}/2}^{\bar{M}-3} \langle \Phi_{N,M}(t), i[H_N(t), 1_{\leq m}] \Phi_{N,\bar{M}}(t) \rangle \mid
\]
Proof of Theorem 2. Let $\Phi_{N,M}(t)$.

Step 3: Conclusion of Theorem 2. Therefore, (129) gives the desired inequality.

Combining with (130) we obtain which follows from assumption $\langle \Phi(0), N^\beta \Phi(0) \rangle \leq C$, we obtain as in (114),

Putting this together with (130) we obtain

Putting this together with (130) we obtain

Therefore, (129) gives the desired inequality.

8.3. Step 3: Conclusion of Theorem 2

Proof of Theorem 2. Let $\Phi_N(t) = U_N(t) \Psi_N(t)$ as in (74). For every $1 \leq M \leq N$, let $\Phi_{N,M}(t)$ be the truncated dynamics defined as in (102).

For every $0 < \beta < 1/6$ and $0 < \alpha < (1 - 6\beta)/4$, we can find a finite number $K > 0$ and a decreasing sequence $\{M_k\}_{k=1}^K$ such that

$$N^{1-2\beta} \gg M_1 \geq \max\{N^{4\beta + 2\alpha}, N^{(1+\beta)/2+\alpha}\}, \quad (131)$$

$$M_k \geq M_{k+1} > \max\{\sqrt{M_k}, M_k N^{-\beta}, N^{3+2\alpha}\}, \quad \forall k = 1, 2, ..., K - 1, \quad (132)$$

$$M_K = N^{3+2\alpha}. \quad (133)$$

From Lemmas 18, 21 and 20 and the above choice of $\{M_k\}_{k=1}^K$, we have the norm approximations

$$\|\Phi_N(t) - \Phi_{N,M_1}(t)\|^2 \leq C_{t,\varepsilon} \left( N^{(4\beta-1)/2} + \frac{N^{2\beta}}{M_1} + \frac{N^{(1+\beta+\epsilon)/2}}{M_1} \right) \leq C_{t,\varepsilon} N^{-\alpha + \epsilon}, \quad (134)$$

$$\|\Phi_{N,M_k}(t) - \Phi_{N,M_{k+1}}(t)\|^2 \leq C_{t,\varepsilon} \left( N^{(4\beta-1)/2} + \frac{N^{\beta+\epsilon}}{M_{k+1}} + \sqrt{\frac{M_k}{M_{k+1}} N^{4\beta-1} + \frac{M_k N^{5\beta+\epsilon-1}}{M_{k+1}}} \right), \quad (135)$$

$$\leq C_{t,\varepsilon} N^{-\alpha + \epsilon}, \quad \forall k = 1, 2, ..., K - 1, \quad (136)$$

$$\|\Phi_{N,M_K}(t) - \Phi(t)\|^2 \leq C_{t,\varepsilon} \left( \frac{N^{\beta+\epsilon}}{M_K} + \sqrt{M_K N^{5\beta+1+2\epsilon}} \right) \leq C_{t,\varepsilon} N^{-\alpha + \epsilon}. \quad (137)$$
The Littlewood-Paley projectors will use a decomposition into dyadic integers. The following strong functional spaces are introduced by Herrer-Tataru-Tzvetkov \[29\]:

\[ \psi \] in Section 3 are needed.

Finally, since \( U_N(t) \) is a (partial) unitary operator we have that

\[ \| \Psi_N(t) - U_N(t) \mathbb{1} \leq \| \Phi_N(t) - \mathbb{1} \leq \| \Phi_N(t) - \Phi(t) \| . \]

and this finishes the proof of Theorem 2. 

\[ \square \]

Appendix A. Extension to systems on torus

Our main results in Theorem 1 and 2 remain valid when the configuration space \( \mathbb{R}^3 \) is replaced by the torus \( T^3 = (\mathbb{R}/(2\pi \mathbb{Z}))^3 \). In our proofs, the domain issue only emerges in the level of the effective theory. Therefore, the main concern is the well-posedness of the quintic Hartree equation on \( T^3 \).

Let us start by briefly recalling the definition of the Littlewood-Paley projections. Take \( \Phi \rightarrow \mathbb{R} \rightarrow [0, 1] \) a smooth even function with \( \psi(y) = 1 \) if \( |y| \leq 1 \) and \( \psi(y) = 0 \) if \( |y| \geq 2 \). We will use a decomposition into dyadic integers

\[ M = 2^j, \quad j \in \mathbb{Z}. \]

The Littlewood-Paley projectors \( P_{\leq M} \) and \( P_M \) are then defined by

\[ P_{\leq M} f(\xi) := \psi(|\xi|/M)f(\xi), \quad \xi \in \mathbb{Z}^3, \]

\[ P_M f := P_{\leq 1} f, \quad P_M f := P_{\leq M} f - P_{\leq M/2} f \quad \text{if} \quad M \geq 2. \]

The following strong functional spaces are introduced by Herr-Tataru-Tzvetkov \[29\]

\[ \| u \|_{X^s(\mathbb{R})}^2 = \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2)^s \| e^{it\xi^2} \hat{u}(t, \xi) \|_{L^2_t}^2, \]

\[ \| u \|_{Y^s(\mathbb{R})}^2 = \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2)^s \| e^{it\xi^2} \hat{u}(t, \xi) \|_{L^2_t}^2, \]

Theorem 22. Let \( u_0 \in H^4(T^3) \) be an initial state. Then for every time \( T > 0 \), there exists a solution to equation (134) on \( [0, T] \) which satisfies

\[ \| u(t, \cdot) \|_{H^s(T^3)} \leq C_t, \quad \| \partial_t u(t, \cdot) \|_{H^2(T^3)} \leq C_t. \]  

Here the constant \( C_t \) is dependent on \( \| u_0 \|_{H^4} \) and \( t \), but independent of \( N \).

Again the result is proven by considering the Hartree equation as a perturbation of the quintic NLS. Even though the basic idea is very similar to the \( \mathbb{R}^3 \) case, the details in the torus case are more involved because some Strichartz’s estimates are no longer available. In the following we will follow the analysis for quintic NLS on torus by Ionescu and Pausader \[31\] [32] and will mainly focus on the places when nontrivial modifications from the \( \mathbb{R}^3 \) case in Section 3 are needed.

Let us start by briefly recalling the definition of the Littlewood-Paley projections. Take \( \psi : \mathbb{R} \rightarrow [0, 1] \) a smooth even function with \( \psi(y) = 1 \) if \( |y| \leq 1 \) and \( \psi(y) = 0 \) if \( |y| \geq 2 \). We will use a decomposition into dyadic integers

\[ M = 2^j, \quad j \in \mathbb{Z}. \]

The Littlewood-Paley projectors \( P_{\leq M} \) and \( P_M \) are then defined by

\[ P_{\leq M} f(\xi) := \psi(|\xi|/M)f(\xi), \quad \xi \in \mathbb{Z}^3, \]

\[ P_M f := P_{\leq 1} f, \quad P_M f := P_{\leq M} f - P_{\leq M/2} f \quad \text{if} \quad M \geq 2. \]

The following strong functional spaces are introduced by Herr-Tataru-Tzvetkov \[29\]

\[ \| u \|_{X^s(\mathbb{R})}^2 = \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2)^s \| e^{it\xi^2} \hat{u}(t, \xi) \|_{L^2_t}^2, \]

\[ \| u \|_{Y^s(\mathbb{R})}^2 = \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2)^s \| e^{it\xi^2} \hat{u}(t, \xi) \|_{L^2_t}^2, \]
with $U^p$ and $V^p$ are defined by Hadac-Herr-Koch \[31\]. For any bounded time interval $I \subset [0, \infty)$, we denote $X^s(I)$ and $Y^s(I)$ in the usual way as restriction norms. As suggested by Ionescu-Pausader \[31\], we will also use the following spacetime norm
\[
\|u\|_{Z(I)} := \sum_{p \in \{p_0, p_1\}} \sup_{J \subseteq I, |J| \leq 1} \left( \sum_M M^{5-p/2} \|P_M u(t)\|_{L^p_t(L^2_x)}^p \right)^{1/p},
\]
(136)

which satisfies $\|u\|_{Z(I)} \lesssim \|u\|_{X^1(I)}$. Finally we will also use a norm interpolating between the $Z(I)$ and $X^1(I)$ norm
\[
\|u\|_{Z'(I)} = \|u\|_{Z(I)}^{1/2} \|u\|_{X^1(I)}^{1/2}.
\]

A.1. Estimate of the nonlinear term. In order to prove Theorem \[22\] it is important to have good control on the nonlinear term in (134). For that we will use the following Lemma. This will be proved similarly to \[31, Lemma 3.2\] (see also \[29, Proposition 4.1\] and \[32, Lemma 3.2\]).

**Lemma 23.** Let $s \geq 1$, and $u_j \in X^s(I)$, $j = 1 \ldots 5$. Then we have on a time interval $I = (a, b)$, $|I| \leq 1$ the following estimate
\[
\left\| \int_a^t e^{i(t-s)\Delta} \int \tilde{u}_1(s, y) \tilde{u}_2(s, y) V_N(x - y, x - z) \tilde{u}_3(s, z) \tilde{u}_4(s, z) \, dy \, dz \, \tilde{u}_5(s) \, ds \right\|_{X^s(I)} \lesssim \|u_{\sigma(1)}\|_{X^s(I)} \prod_{j=2}^5 \|u_{\sigma(j)}\|_{Z'(I)},
\]
(137)

for any permutation $\sigma$ on $\{1, \ldots, 5\}$. Here $\tilde{u}_j$ can either be $u_j$ or $\overline{u}_j$.

**Proof.** We will just proof the case where $\sigma$ is the identity since the case for all other permutations follows in the same way. By \[29, Proposition 2.11\] we know that
\[
\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq M} \left( \int \tilde{u}_1 \tilde{u}_2 V_N \tilde{u}_3 \tilde{u}_4 \, dy \, dz \, \tilde{u}_5 \right) \, ds \right\|_{X^s(I)}
\]
with
\[
\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq M} \left( \int \tilde{u}_1 \tilde{u}_2 V_N \tilde{u}_3 \tilde{u}_4 \, dy \, dz \, \tilde{u}_5 \right) \, ds \right\|_{X^s(I)} \lesssim \sup \left\| \int_{I \times T^3} P_{\leq M} \left( \int \tilde{u}_1(t, y) \tilde{u}_2(t, y) V_N(x - y, x - z) \tilde{u}_3(t, z) \tilde{u}_4(t, z) \, dy \, dz \, \tilde{u}_5(t, x) \right) \, v(t, x) \, dx \, dt \right\|,
\]
where the supremum is taken over all $v \in Y^{-s}(I)$ with $\|v\|_{Y^{-s}} = 1$. Define $u_0 = P_{\leq N} v$. With this notation we need to prove
\[
\left\| \int_{I \times T^3} \int \tilde{u}_1(t, y) \tilde{u}_2(t, y) V_N(x - y, x - z) \tilde{u}_3(t, z) \tilde{u}_4(t, z) \, dy \, dz \, \tilde{u}_5(t, x) u_0(t, x) \, dx \, dt \right\| \lesssim \|u_0\|_{Y^{-s}(I)} \|u_1\|_{X^s(I)} \prod_{j=2}^5 \|u_j\|_{Z'(I)}.
\]
(138)

The desired result will then follow by taking the limit $M \to \infty$. 

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We now use the dyadic decompositon $u_k = \sum_{M_k} P_{M_k} u_k$. With that and the Cauchy-Schwarz inequality we see that in order to bound the left hand side of (38) it suffices to bound
\[
S = \sum_{M} \int \int V_N(y, z) \|P_{M_0} \tilde{u}_0 P_{M} \tilde{u}_2 (\cdot - y) P_{M_4} \tilde{u}_4 (\cdot - z)\|_{L^2_{\nu} (I \times T^2)} \times \|P_{M_1} \tilde{u}_1 (\cdot - y) P_{M_3} \tilde{u}_3 (\cdot - z) P_{M_5} \tilde{u}_5\|_{L^2_{\nu} (I \times T^2)} \, dy \, dz,
\]
where $M$ is the set of tuples $(M_0, M_1, M_2, M_3, M_4, M_5)$ of dyadic numbers satisfying $M_i \geq 1$ and $M_5 \leq M_4 \leq \cdots \leq M_1$. We first consider the subset $M_1 \subset M$ with $M_2 \leq M_0 \sim M_1$: Using Lemma 3.1 and translation invariance of the $Y^0$ and $Z'$ norms we get
\[
S_1 \lesssim \int \int V_N(y, z) \, dy \, dz \sum_{M_1} \left( \frac{M_5}{M_1} + \frac{1}{M_3} \right) \frac{\delta}{\delta} \left( \frac{M_4}{M_0} + \frac{1}{M_2} \right) \times 
\]
\[
\times \|P_{M_1} u_1\|_{Y^0} \|P_{M_2} u_3\|_{Z'} \|P_{M_4} u_4\|_{Z'} \|P_{M_5} u_5\|_{Z'} \|P_{M_0} u_0\|_{Y^0} \|P_{M_2} u_2\|_{Z'} \|P_{M_4} u_4\|_{Z'}.
\]
for some $\delta > 0$. By using the Cauchy-Schwarz inequality we can sum with respect to $M_2, M_3, M_4$, and $M_5$ which gives
\[
S_1 \lesssim \prod_{j=2}^{5} \|u_j\|_{Z'} \sum_{M_0 \sim M_1} \|P_{M_0} u_0\|_{Y^0} \|P_{M_1} u_1\|_{Y^0} \sim \prod_{j=2}^{5} \|u_j\|_{Z'} \sum_{M_0 \sim M_1} M_0^{-\delta} \|P_{M_0} u_0\|_{Y^0} M_1^{\delta} \|P_{M_1} u_1\|_{Y^0}.
\]
Using again Cauchy-Schwarz in $M_1$ we get
\[
S_1 \lesssim \|u_0\|_{Y^0-z} \|u_1\|_{Y^0} \prod_{j=2}^{5} \|u_j\|_{Z'} \lesssim \|u_0\|_{Y^0-z} \|u_1\|_{X^s} \prod_{j=2}^{5} \|u_j\|_{Z'}.
\]
For the subset $M_2 \subset M$ with $M_0 \leq M_2 \sim M_1$ we use both estimates in Lemma 3.1 and obtain
\[
S_2 \lesssim \sum_{M_2} \int \int V_N(y, z) \, dy \, dz \left( \frac{M_5}{M_1} + \frac{1}{M_3} \right) \delta \left( M_0^{1/2-5/p_0} M_2^{1/2-5/p_0} M_4^{10/p_0-2} \times \right.
\]
\[
\times \|P_{M_1} u_1\|_{Y^0} \|P_{M_2} u_3\|_{Z'} \|P_{M_4} u_4\|_{Z'} \|P_{M_5} u_5\|_{Z'} \|P_{M_0} u_0\|_{Z} \|P_{M_2} u_2\|_{Z} \|P_{M_4} u_4\|_{Z}.
\]
Now using Strichartz’s estimate Corollary 2.2 (and the embedding $Y^0(I) \hookrightarrow U^p_{\Delta} (I, L^2) \hookrightarrow U^p_{\Delta} (I, L^2)$, with notation $U^p_{\Delta}$ in we have
\[
\|P_{M_0} u_0\|_{Z(I)} \lesssim M_0 \left( \|u_0\|_{U^p_{\Delta} (I, L^2)} + \|u_0\|_{U^p_{\Delta} (I, L^2)} \right) \lesssim M_0 \|P_{M_0} u_0\|_{Y^0(I)}.
\]
Putting this in the above we obtain
\[
S_2 \lesssim \sum_{M_2} \left( \frac{M_5}{M_1} + \frac{1}{M_3} \right) \delta \left( M_0^{3/2-5/p_0} M_2^{1/2-5/p_0} M_4^{10/p_0-2} \times \right.
\]
\[
\times \|P_{M_1} u_1\|_{Y^0} \|P_{M_2} u_3\|_{Z'} \|P_{M_4} u_4\|_{Z'} \|P_{M_5} u_5\|_{Z'} \|P_{M_0} u_0\|_{Y^0} \|P_{M_2} u_2\|_{Z} \|P_{M_4} u_4\|_{Z}.
\]

\[ \leq \sum_{M_2} \left( \frac{M_5}{M_1} + \frac{1}{M_3} \right) \delta \| P_{M_1} u_1 \|_{Y^0} \| P_{M_1} u_3 \|_{Z} \| P_{M_5} u_5 \|_{Z^\prime} \| P_{M_6} u_0 \|_{Y^0} \| P_{M_2} u_2 \|_{Z} \| P_{M_4} u_4 \|_{Z}. \]

We can now use the Cauchy-Schwarz to sum over \( M_1, M_4, M_3 \) to get

\[ S_2 \lesssim \prod_{j=3}^5 \| u_j \|_{Z^\prime} \sum_{M_1 \sim M_2 \gg M_0} \| P_{M_1} u_1 \|_{Y^0} \| P_{M_6} u_0 \|_{Y^0} \| P_{M_2} u_2 \|_{Z}. \]

\[ \lesssim \prod_{j=2}^5 \| u_j \|_{Z^\prime} \sum_{M_1 \gg M_0} \| P_{M_1} u_1 \|_{Y^0} \| P_{M_0} u_0 \|_{Y^0}. \]

\[ \lesssim \prod_{j=2}^5 \| u_j \|_{Z^\prime} \sum_{M_1 \gg M_0} \left( \frac{M_0}{M_1} \right)^s \| P_{M_1} u_1 \|_{Y^0} \| P_{M_0} u_0 \|_{Y^0}. \]

Now using Schur’s Lemma we obtain

\[ S_2 \lesssim \| u_0 \|_{Y^{-s}} \| u_1 \|_{Y^s} \prod_{j=2}^5 \| u_j \|_{Z^\prime} \lesssim \| u_0 \|_{Y^{-s}} \| u_1 \|_{X^s} \prod_{j=2}^5 \| u_j \|_{Z^\prime}. \]

This finishes the proof. \( \square \)

### A.2. Local well-posedness

In this section we will prove the local theory for the quintic Hartree equation on \( \mathbb{T}^3 \). We will proceed very similarly to \[31\] Proposition 3.3 (see also \[32\] Propositon 3.3). In fact the only real difference is the treatment of the nonlinear term which is provided by Lemma \[24\].

**Lemma 24** (Local well-posedness). For every \( u_0 \in H^1(\mathbb{T}^3) \) there exists an \( \varepsilon > 0 \) depending on \( \| u_0 \|_{H^1(\mathbb{T}^3)} \) such that on any time interval \( I \ni 0, |I| \leq 1 \) with

\[ \| e^{it\Delta} u_0 \|_{Z(I)} \leq \varepsilon \]  

there exists a unique solution \( u \in X^1(I) \) of \[134\].

If furthermore \( u \in X^1(I) \) is a solution of \[134\] on some open interval \( I \) satisfying

\[ \| u \|_{Z(I)} < \infty \]  

then \( u \) can be extended to some neighborhood of \( I \) and

\[ \| u \|_{X^1(I)} \leq C(\| u_0 \|_{H^1(\mathbb{T}^3)}, \| u \|_{Z(I)}) \]  

**Proof.** We will proceed in the standard way by using a fixed point argument. Let \( E := \| u_0 \|_{H^1(\mathbb{T}^3)} \) and define the set

\[ E(I, a) = \{ v \in X^1(I) : \| v \|_{X^1(I)} \leq 2E, \| v \|_{Z(I)} \leq a \}, \]

which is closed in \( X^1(I) \). We will consider the following map

\[ \Phi(u)(t) = e^{it\Delta} u_0 - \frac{i}{2} \int_0^t e^{i(t-s)\Delta} \int |u(s, y)|^2 V_N(x - y, x - z) |u(s, z)|^2 \, dy \, dz \, u(s) \, ds. \]

Using \[24\] Proposition 2.10 for the linear term and Lemma \[23\] for the nonlinear term, we see that

\[ \| \Phi(u)(t) \|_{X^1(I)} \leq \| u_0 \|_{H^1(\mathbb{T}^3)} + C \| u \|_{Z(I)}^4 \| u \|_{X^1(I)} \leq E + C a^4 E \]

\[ \| \Phi(u)(t) \|_{Z(I)} \leq \varepsilon + C \| u \|_{Z(I)}^4 \| u \|_{X^1(I)} \leq \varepsilon + C a^4 E. \]
In the second line we have used assumption (139) and that the $Z'(I)$ can be bounded by the $X^1(I)$ norm for the nonlinear term. Hence by choosing $a = 2\varepsilon$ and then $\varepsilon > 0$ small enough we obtain that $\Phi(E(I, a)) \subseteq E(I, a)$.

Moreover, using again Lemma 23 we obtain that

$$\|\Phi(u) - \Phi(v)\|_{X^1(I)} \lesssim (\|u\|_{Z(I)}^2 + \|v\|_{Z(I)}^2) \|u - v\|_{X^1(I)} \lesssim a^4 \|u - v\|_{X^1(I)}.$$  

For $a > 0$ small enough this gives that $\Phi$ is a contraction map on $E(I, a)$. Using now the contraction mapping principle we see that there exists a unique $u \in E(I, a)$ which solves (134).

To see the uniqueness of the solution in the whole space $X^1(I)$ we let us assume that there exist $u, v \in X^1(I)$ solution of (134). If we choose an open subinterval $J \subseteq I$ containing 0 we have that for $J$ small enough $u, v \in E(J, a)$. By uniqueness in $E(J, a)$ we know that $u|_J = v|_J$. Hence, the set \{u = v\} is open and closed in $I$ and therefore equal to $I$.

The extension result for finite $Z(I)$ norm (140) follows in the same way as in [31, Proposition 3.3] (see also [32, Lemma 3.4]). □

A.3. Proof of Theorem 22. In this section we will prove the global existence and regularity stated in Theorem 22. We will use the global well-posedness theory of the quintic nonlinear Schrödinger equation on $\mathbb{T}^3$

$$i\partial_t \tilde{u} = -\Delta \tilde{u} + b_0 |\tilde{u}|^4 \tilde{u}$$  

provided in [31]. The quintic Hartree equation (134) will then be considered as a perturbation of (142). Using a stability result similar to [31, Proposition 3.4] in the case of the quintic NLS, this will give the global theory of (134).

Lemma 25 (Stability). Assume $I$ is an open bounded interval and $\tilde{u}$ in $X^1(I)$ a solution of the perturbed equation

$$i\partial_t \tilde{u} = -\Delta \tilde{u} + \frac{1}{2} \int |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz + e$$  

for some function $e$. Moreover, assume that

$$\|\tilde{u}\|_{Z(I)} \leq M,$$  

$$\|\tilde{u}\|_{L^\infty_t L^{\frac{4}{3}}_x(I \times \mathbb{T}^3)} \leq E,$$  

$$\|u_0 - \tilde{u}(0)\|_{H^1(\mathbb{T}^3)} + \left\| \int_{I} e^{i(t-s)\Delta} e(s, \cdot) \, ds \right\|_{X^1(I)} \leq \varepsilon$$  

for some $M, E \geq 0$ and some small enough $\varepsilon > 0$.

Then there exists a solution $u \in X^1(I)$ of (134) with

$$\|u\|_{X^1(I)} \leq C(M, E).$$  

Using Lemma 23 for the nonlinear term, the proof of Lemma 25 follows similarly to [31, Proposition 3.4] (consider also [32, Proposition 3.5]) and is therefore omitted.

Now we come to the proof of the main result Theorem 22.

Proof Theorem 22. In order to prove Theorem 22 we want to apply Lemma 25 with $\tilde{u}$ being the solution of the perturbed equation given by the quintic NLS on $\mathbb{T}^3$

$$i\partial_t \tilde{u} = -\Delta \tilde{u} + b_0 |\tilde{u}|^4 \tilde{u}$$  

$$= -\Delta \tilde{u} + \frac{1}{2} \int |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz \tilde{u} + e_N,$$
with initial state $\tilde{u}(0, x) = u_0$. Here we have defined the perturbation

$$e_N = b_0 |\tilde{u}|^4 \tilde{u} - \frac{1}{2} \iint |\tilde{u}(y)|^2 V_N(x - y, x - z)|\tilde{u}(z)|^2 \, dy \, dz \, \tilde{u}.$$ 

In order to use Lemma 23 we want to show that

$$\left\| \int_0^t e^{(t-s)\Delta} e_N(s, \cdot) \, ds \right\|_{X^1(I)}$$

is arbitrarily small for $N$ large. Without loss of generality we assume $|I| \leq 1$ in order to apply Lemma 23. By the triangle inequality we have

$$\left\| \int_0^t e^{(t-s)\Delta} e_N(s, \cdot) \, ds \right\|_{X^1(I)} \leq \left\| \int_0^t e^{(t-s)\Delta} \iint V_N(y, z) \left( |\tilde{u}(x - y)|^2 - |\tilde{u}(x)|^2 \right) |\tilde{u}(x)|^2 \, dy \, dz \, \tilde{u}(s, \cdot) \, ds \right\|_{X^1(I)} + \left\| \int_0^t e^{(t-s)\Delta} \iint V_N(y, z) \left( |\tilde{u}(x - z)|^2 - |\tilde{u}(x)|^2 \right) |\tilde{u}(x - y)|^2 \, dy \, dz \, \tilde{u}(s, \cdot) \, ds \right\|_{X^1(I)}.$$ 

We will only consider the first term since the second term will follow the same way. From Lemma 23 we have that

$$\left\| \int_0^t e^{(t-s)\Delta} \iint V_N(y, z) \left( |\tilde{u}(x - y)|^2 - |\tilde{u}(x)|^2 \right) |\tilde{u}(x)|^2 \, dy \, dz \, \tilde{u}(s, \cdot) \, ds \right\|_{X^1(I)} \leq \left\| \tilde{u} \right\|_{X^4(I)}^4 \sup_{|y| \leq CN^{-\beta}} \left\| \tilde{u}(\cdot - y) - \tilde{u} \right\|_{X^1(I)}.$$ 

where we have used that $V_N(y, \cdot) = 0$ for $|y| > CN^{-\beta}$ since $V$ has compact support. Using the global well-posedness of the quintic NLS on $\mathbb{T}^3$ proven in 31 we see that this expression is arbitrarily small for $N$ large. By Lemma 23 we now obtain that (134) has a solution $u \in X^1(I)$.

To conclude the regularity result (134) we use that $\|u\|_{X^1(I)}$ is finite and the nonlinear estimate in Lemma 23. Here it is important that the right side of (137) includes the weaker norm of $Z'$ which can be made arbitrarily small by localizing in time (unlike the $X^1$-norm). To be precise, since $\|u\|_{Z'(I)}$ is finite, for every $\delta > 0$ the time interval $I$ can be split up into finitely many subintervals $I_0, \cdots, I_K$ such that

$$\|u\|_{Z'(I_j)} \leq \delta$$

for each $j = 1, \cdots, K$. We also assume $|I_j| \leq 1$ for each $j = 1, \cdots, K$. Now using Duhamel’s formula and Lemma 23 we obtain

$$\|u\|_{X^4(I_0)} \leq \|u_0\|_{H^4(\mathbb{T}^3)} + \|u\|_{Z'(I_0)}^2 \|u\|_{X^4(I_0)} \leq \|u_0\|_{H^4(\mathbb{T}^3)} + \delta^4 \|u\|_{X^4(I_0)}.$$ 

If we choose $\delta > 0$ small enough, this gives

$$\|u\|_{X^4(I_0)} \lesssim \|u_0\|_{H^4(\mathbb{T}^3)}.$$ 

The embedding $X^4(I_0) \hookrightarrow L^\infty(I_0, H^4(\mathbb{T}^3))$ now gives

$$\|u(t, \cdot)\|_{H^4(\mathbb{T}^3)} \lesssim \|u_0\|_{H^4(\mathbb{T}^3)}.$$
for each $t \in I_0$. Using this we can iterate the procedure and obtain $\|u(t, \cdot)\|_{H^1(T^3)} \lesssim \|u_0\|_{H^1(T^3)}$ for all $t \in I$. □

References

[1] R. Adami, F. Golse, and A. Teta, Rigorous derivation of the cubic NLS in dimension one, J. Stat. Phys. 127 (6) (2007), 1193–1220.
[2] Z. Ammari and F. Nier, Mean field limit for bosons and infinite dimensional phase-space analysis, Ann. Henri Poincaré 9 (2008), 1503–1574.
[3] C. Bardos, F. Golse, and N.J. Mauser, Weak coupling limit of the N-particle Schrödinger equation, Methods Appl. Anal. 7 (2000), no. 2, 275–293.
[4] N. Benedikter, G. de Oliveira, and B. Schlein, Quantitative derivation of the Gross-Pitaevskii equation, Comm. Pure Appl. Math. 68 (2015), no. 8, 1399–1482.
[5] C. Boccato, S. Cenatiempo, and B. Schlein, Quantum many-body fluctuations around nonlinear Schrödinger dynamics, Ann. Henri Poincaré 18 (2017), no. 1, 113–191.
[6] N. N. Bogoliubov, On the theory of superfluidity, J. Phys. (USSR), 11 (1947), 23.
[7] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, Bogoliubov Theory in the Gross-Pitaevskii Limit, Preprint 2018, arXiv:1801.01389.
[8] J. Bourgain, Scattering in the energy space and below for 3d NLS, Journal d’analyse Mathématiques 75 (1998), 267–297.
[9] J. Bourgain, Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case, J. Amer. Math. Soc. 12 (1999), no. 1, pp. 145–171.
[10] C. Brennecke and B. Schlein, Gross-Pitaevskii dynamics for Bose-Einstein condensates. Preprint (2017) arxiv:1702.05625.
[11] C. Brennecke, P. T. Nam, M. Napiórkowski, and B. Schlein, Fluctuations of N-particle quantum dynamics around the nonlinear Schrödinger equation, Preprint (2017) arXiv:1710.09743.
[12] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes Vol. 10, American Mathematical Society 2003.
[13] T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer, Unconditional uniqueness for the cubic Gross-Pitaevskii hierarchy via quantum de Finetti, Commun. Pure Appl. Math. 68 (2015), no. 10, 1845–1884.
[14] T. Chen and N. Pavlović, The quintic NLS as the mean field limit of a Boson gas with three-body interactions, J. Funct. Anal., 260 (2011), pp. 959–997.
[15] X. Chen, Second Order Corrections to Mean Field Evolution for Weakly Interacting Bosons in the Case of Three-body Interactions, Archive for Rational Mechanics and Analysis 203 (2012), pp. 455–497.
[16] X. Chen and J. Holmer, The Derivation of the Energy-critical NLS from Quantum Many-body Dynamics, Preprint (2018) arXiv:1803.08082.
[17] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Ann. of Math. 167 (2), (2008), pp. 767–865.
[18] J. Dereziński, Bosonic quadratic Hamiltonians, J. Math. Phys. 58 (2017), 121101.
[19] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical nonlinear Schrödinger equation when $d \geq 3$, J. Amer. Math. Soc. 25 (2012), pp. 429–463.
[20] L. Erdős, B. Schlein, and H.-T. Yau, Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems, Invent. Math. 167 (2007), 515–614.
[21] L. Erdős, B. Schlein, and H.-T. Yau, Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate, Ann. of Math. (2) 172 (2010), 291–370.
[22] J. Fröhlich, A.Knowles, and S. Schwarz, On the mean-field limit of bosons with Coulomb two-body interaction, Commun. Math. Phys. 288 (2009), 1023–1059.
[23] J. Ginibre and G. Velo, The classical field limit of scattering theory for nonrelativistic many-boson systems, I, Commun. Math. Phys. 66 (1979), 37–76.
[24] M. Grillakis, On nonlinear Schrödinger equations, Comm. Partial Differential Equations 25 (2000), pp. 1827–1844.
[25] M. Grillakis and M. Machedon, Pair excitations and the mean field approximation of interacting Bosons I, Comm. Math. Phys. 324 (2003), pp. 601–636.
[26] M. Grillakis and M. Machedon, Pair excitations and the mean field approximation of interacting Bosons, II, Comm. PDE. 42 (2017), no. 1, 24–67.
[27] M. G. Grillakis, M. Machedon, and D. Margetis, Second-order corrections to mean field evolution of weakly interacting bosons. I, Commun. Math. Phys. 294 (2010), 273–301.
[28] M. Hadac, S. Herr, and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, Ann. Inst. H. Poincaré, Anal. Non Linéaire 26 (2009), no. 3, 917–941.

[29] S. Herr, D. Tataru, and N. Tzvetkov, Global well-posedness of the energy critical nonlinear Schrödinger equation with small initial data in $H^1(T^d)$, Duke Math. J. (2011) Vol. 159, no. 2, p. 329–349.

[30] K. Hepp, The classical limit for quantum mechanical correlation functions, Comm. Math. Phys. 35 (1974), 265–277.

[31] A. D. Ionescu and B. Pausader, The energy-critical defocusing NLS on $\mathbb{T}^3$, Duke Math. J. 161 (2012), 1581–1612.

[32] A. D. Ionescu and B. Pausader, Global well-posedness of the energy-critical defocusing NLS on $\mathbb{R} \times T^3$, Commun. Math. Phys. 312 (2012), 781–831.

[33] S. Klainerman and M. Machedon, On the uniqueness of solutions to the gross-pitaevskii hierarchy, Commun. Math. Phys. 279 (2008), 169–185.

[34] A. Knowles and P. Pickl, Mean-field dynamics: singular potentials and rate of convergence, Commun. Math. Phys. 298 (2010), 101–138.

[35] E. Kuz, Exact Evolution versus Mean Field with Second-order correction for Bosons Interacting via Short-range Two-body Potential, Differential Integral Equations 30 (2017), no. 7/8, 587–630.

[36] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej, Bogoliubov spectrum of interacting Bose gases, Comm. Pure Appl. Math., 68 (2015), pp. 413–471.

[37] M. Lewin, P. T. Nam, and B. Schlein, Fluctuations around Hartree states in the mean-field regime, Amer. J. Math., 137 (2015), pp. 1613–1650.

[38] E. H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics, American Mathematical Society, 2001.

[39] F. Linaris and G. Ponce, Introduction to Nonlinear Dispersive Equations Second equation, Universitext, Springer, 2015.

[40] P. T. Nam and M. Napiórkowski, Bogoliubov correction to the mean-field dynamics of interacting bosons, Adv. Theor. Math. Phys. 21 (2017), 683–738.

[41] P. T. Nam and M. Napiórkowski, A note on the validity of Bogoliubov correction to mean-field dynamics, J. Math. Pure. Appl. 108 (2017), 662–688.

[42] P.T. Nam and M. Napiórkowski, Norm approximation for many-body quantum dynamics: focusing cases in low dimensions, Preprint (2017).

[43] P. T. Nam, M. Napiórkowski, and J. P. Solovej, Diagonalization of bosonic quadratic Hamiltonians by Bogoliubov transformations, J. Funct. Anal., 270 (11) (2016), pp. 4340–4368.

[44] P. T. Nam, N. Rougerie, and R. Seiringer, Ground states of large bosonic systems: The Gross-Pitaevskii limit revisited, Analysis & PDE 9 (2016), 459-485.

[45] P. Pickl, Derivation of the time dependent Gross Pitaevskii equation with external fields, Rev. Math. Phys. 27 (2015), 1550003.

[46] A. Pizzol. Bose particles in a box I. A convergent expansion of the ground state of a three-modes Bogoliubov Hamiltonian, Preprint (2015) arXiv:1511.07022.

[47] I. Rodnianski and B. Schlein, Quantum fluctuations and rate of convergence towards mean field dynamics, Commun. Math. Phys. 291 (2009), 31–61.

[48] R. S. Strichartz, Restriction of Fourier Transform to Quadratic Surfaces and Decay of Solutions of Wave Equations, Duke Math. J., 44 (1977), 70 5774.

[49] H. Spohn, Kinetic equations from Hamiltonian dynamics: Markovian limits, Rev. Modern Phys. 52 (1980), pp. 569–615.

[50] T. Tao, Introduction to Nonlinear Dispersive Equations Second equation, CBMS 106, AMS, 2006.

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