ROI constrained Auctions

Benjamin Heymann

Criteo, b.heymann@criteo.com

Abstract. A standard result from auction theory is that bidding truthfully in a second
price auction is a weakly dominant strategy, or, in the language of digital advertising, 'the
cost per mille (eCPM) is equal to the click through rate (CTR) times the cost per clicks
(CPC)'. However, such assertion is incorrect if the buyers are subject to ROI constraints.
More generally, several predictions derived by the traditional auction theory literature fail to
apply. This is what I propose to discuss in this article.
I formalize the notion of ROI constrained auctions and derive a Nash equilibrium for second
price auctions. I then extend this result to any combination of first and second price payment
rules and do an asymptotic analysis of the equilibrium as the number of bidders increases.
Further I expose a revenue equivalence property, and finish with a proposal for a dynamic
extension of the bidder ROI constrained optimization problem.

Keywords: Auction theory · Ad auctions · Bidding strategy · Equilibrium · Control · HJB.

1 Introduction

What should you bid in a second price ad auction for a display with a known click-through rate
(CTR), for a given cost-per-click (CPC)? The most probable (and possibly incorrect) answer is
"CPC times CTR". However, the right answer is "What do you mean by cost-per-click?". In this
article, we challenge the current modeling approach for ad auctions. We argue that when some
advertisers' business constraints apply, the expected outcome of the auction may depart from the
traditional literature.

Advertising is a major source of revenue for Internet publishers, and as such, is financing a large
portion of the Internet. About 200 billions USD were spend in 2017 on digital advertising. Banners
for display advertising are usually sold through a high frequency one unit auction mechanism called
RTB (Real Time Bidding) by or on behalf of advertisers.

When a user reaches a publisher page, it triggers a call to an RTB platform. The RTB platform
then calls potential buyers, who have a few millisecond to answer with a bid request. This results in
an allocation of the display to a bidder, in exchange of a payment. The allocation and the payment
are defined by the rules of the auction, which are not uniformly defined among the RTB platforms.
The industry standard is currently shifting from Second Price Auctions to First Price Auctions.

The literature on display advertising auctions has been growing over the last decade, pushed by
the emergence of Internet giants whose business models are centered on digital advertising. One
track of research takes the viewpoint of the seller and focuses on how to build "good" auctions, and
relies mostly on mechanism design theory [25], while the dual track brings the buyer view point
under scrutiny, and focuses on the design of bidding strategies [13][16][7][5]. The reader can refer to
[11] and [9] for an introduction to auction theory.

Usually, in performance markets, the advertiser has a target Return on Investment (ROI) and/or
a budget in mind. We will assume that the budget constraint is absent or not binding. This topic of
budget constrains has already been addressed in the literature [2][17]. We will focus the analysis on
the ROI constraint. The ROI is measured in term of average action per unit of money spent, the
action being a click, a visit, a conversion... In the literature and in the industry we usually use the
inverse of the ROI, expressed in Cost Per Click (CPC), Cost per Sale (COS) or more generally, Cost
Per Action (CPA). For instance, if (1) I am facing a competition uniformly distributed between 0 and 1, (2) the CTR of every display is 0.5, (3) the auction is second price and (4) I am ready to pay 1 USD on average for a click. Then if I bid 0.5, I will win half of the time. For every display won, I get on average half a click, and I pay \[\int_0^{0.5} t \, dt = 1/4,\] thus my expected CPC is \(1/(4*0.5) = 0.5 < 1.\) I should increase my bid to raise my empirical CPC.

We pinpoint the following paradox: on the one hand, a lot of research effort is being invested in the design of good, sophisticated auctions and bidding strategies, but on the other hand, one key aspect of the end buyer (the brand) objective program is being neglected more often than not.

This work brings several contributions to the table: we introduce the buyer’s ROI constrained optimization problem, and exhibit a solution, we find the symmetric equilibrium in this setting and compare its properties with standard results from the literature, in particular, no reserve price should be used in the symmetric setting. Since in practice the ROI is computed in average over time, it create a dependency between auction, we thus also discuss extensions to dynamic settings and derive some properties of the solutions.

## 2 Modeling Assumptions and Notations

We start by introducing the simplest ROI constrained optimization problem for a buyer. There is one item (a display opportunity in our context) to be sold through an auction. In all but Section 6, we assume that there is only one auction to be held. Unless otherwise stated, we also assume the auction to follow a second price rule, with \(n\) bidders competing for the item.

Each item brings a value \(v\) to the bidder of interest. This value is independently distributed among bidders, with the same distribution of cumulative \(F\) and density \(f\) of support \([0, 1]\). At bid time, each bidder knows his value for the item to be sold. For example, in the context of ad auction, \(v\) can be thought of as the CTR or the conversion probability.

We will denote by \(b\) the bid of a given bidder of interest, and by \(b^-\) the maximum of the bids of the other bidders (the price to beat). In the proofs, we will use the letter \(g\) and \(G\) for the density and the cumulative distributions of the competitors best bid (or best value, depending on the context). Without loss of generality, we assume \(v \in [0, 1]\).

For a random variable \(X\), we denote by \([X] \in \{0, 1\}\) the binary random variable that takes the value 1 when \(X \geq 0\), and 0 otherwise.

We assume the bidders do not want to have their ROI below a given threshold. Throughout the paper, we will not refer to the ROI directly, but to its inverse (homogeneous to a cost per click, a cost per sale, a cost per action...), that we will refer to with the letter \(T\), because it is our "targeted cost per something". We assume \(T\) to be the same for all buyers.

The bidder wants to maximize his expected value, subject to an \(ex \ ante\) constraint in expectancy representing the targeted ROI. The constraint is \(ex \ ante\) because in practice, the same bidding strategy is going to be applied repeatedly (or even simultaneously) on several similar auctions. More formally, the bidder is looking for a bid function \(b : [0, 1] \to \mathbb{R}^+\) solution of

\[
\max_b \mathbb{E}_{v, b^-}[b^- < b]v, \tag{1}
\]

subject to \(\mathbb{E}_{v, b^-}[b^- < b]b^- \leq T\mathbb{E}_{v, b^-}[b^- < b]v\) (ROI). The average is computed over the distribution of values and prices to beat. The RHS of the constraint corresponds to the expected value earned by the buyer times the target, while the LHS is the expected spend. Observe that if we remove the constraint, the buyer would buy all the opportunities no matter the cost.

The decisional interactions among buyers can be modeled as a game with constraints on the strategies profile. For a given set of opponent bidding strategies, the solution of (1) is the best reply of the bidder. If we find a strategies profile such that all components are best replies against the others, we say that this strategy profile is a (constrained) Nash equilibrium.
3 Bidding Behavior and Symmetric Equilibrium

The main result of this section is the derivation of a symmetric Nash equilibrium for an ROI constrained Second Price Auction. We then generalize this result to linear combinations of First Price and Second Price Auctions.

3.1 Second Price

Theorem 1 (Best Reply). For any distribution of the price to beat \( b^- \), there exists \( \lambda^* \in \mathbb{R}^+ \) such that a solution of (1) writes:

\[
\begin{align*}
\lambda^* & = \frac{1}{\lambda^* + T} \gamma
\end{align*}
\]

Proof. See Appendix

Lemma 1 (ROI monotony). If the bidder bid proportionally to the value i.e. there exists \( \alpha \) such that \( b(v) = \alpha v \), then the ROI is non-increasing in \( \alpha \).

Proof. Fix \( u \in [0,1] \), \( \alpha > 0 \), then the expected spend knowing that the value is \( v \) and the auction is won is \( \int_0^u tg(t)dt \), where \( g \) is the probability density of the price to beat. This quantity is increasing in \( \alpha \). Meanwhile the value is \( v \). We get the result by integration on \( v \).

If \( v_1, v_2, \ldots v_n \) are \( n \) independent draws from \( F \), we denote by \( v_i^{(n)} \) the average of the \( i^{th} \) greater draw.

Theorem 2 (Equilibrium Bid). The unique symmetric constrained Nash equilibrium strategy is

\[
\begin{align*}
b^*(v) & = \frac{T\hat{b}(v)}{\gamma} \gamma(F,n) = n \left( \frac{v_i^{(n-1)}}{v_i^{(n)}} - \frac{n-1}{n} \right)
\end{align*}
\]

Proof. See Appendix

Discussion Theorem 1 is the answer to the introductory question. Despite its simplicity, it shows something that is probably overlooked: it may happen in a second price auction that the bidder’s optimal bid depends on the competition. Another useful insight is that the bid is linear in the value, which implies that simple bid multipliers could be used optimally. Observe that the result holds also in non-symmetric settings.

The proof relies on the strong incentive compatibility of the Second Price Auction in standard setting reinterpreted on the Lagrangian of the optimization problem. The parameter \( \lambda^* \) should be interpreted as the Lagrangian multiplier associated with the ROI constraint.

Informally speaking, the harder the constraint, the bigger \( \lambda \), the smaller the bid. When \( \lambda \) is set to \(+\infty\), one bids only for displays that do not consume the constraint. When \( \lambda \) goes to zero, the bid diverges.

Lemma 1 is a useful observation to understand what is happening. Observe that since the bid is linear in the value and the objective increasing in the bid multiplier, the optimal bid multiplier is the maximal admissible one.

Theorem 2 is one of our main result. Observe that \( \gamma \) is only a function of the number of opponents and \( F \). It is smaller than one (see Appendix). In particular, the bid is proportional to the target \( Tv \) with a factor greater than one. Compared with the standard setting, the seller’s expected revenue is multiplied by \( T/\gamma \). The limit of \( \gamma \) as the number of bidders \( n \) goes to infinity, as well as other natural questions concerning the equilibrium will be studied in in Section 5. Note that Theorem 2 is a necessary and sufficient condition for a symmetric Nash equilibrium, therefore it is unique, but we cannot claim that we have identified the only NE (see Section 5).
3.2 Generalization

We now generalize the reasoning to auctions which are convex combinations of First Price and Second Price. We motivate this extension by the use in the industry of auctions which mix together first and second pricing rules, such as soft-floor.

If we denote by $S(x,y)$ the payment rule of the auction when the bidder bid $b$ and the best competing bid is $b^-$, then we can make the following observations: (1) for any $s \geq 0$, $S(sb, sb^-) = sS(b, b^-)$, (2) $S(b, b) = b$, (3) $S(b, b^-)$ is the sum of a linear function of $b$ and a linear function of $b^-$. We denote by $(\hat{b}, \hat{b}^-)$ symmetric equilibrium bidding strategy profile of the standard auction with payment rule $S$ and i.i.d. values distributed according to $F$ (and no ROI constraints).

**Theorem 3.** The bid $b^*(v) = \frac{T}{\gamma}v$, is a symmetric equilibrium strategy.

**Proof.** See Appendix

**Example :** We can check this formula on a FPA with 2 buyers and uniform distribution to recover $\gamma = 1$.

4 Examples

4.1 Power Law Case: derivation of $\gamma$

We take $F(v) = v^a$ over $[0,1]$ with $a > 0$. We get that $v^{(n)}_1 = an/an+1$. Therefore we can write

$$\frac{v^{(n-1)}_1}{v^{(n)}_1} = \frac{n-1}{n} \frac{an+1}{a(n-1)+1},$$

which leads to

$$\gamma(a, n) = (n - 1) \left( \frac{an + 1}{a(n - 1) + 1} - 1 \right). \quad (5)$$

We observe that $\gamma$ is increasing in $n$ and $a$. It converges to 1 as $n$ (or $a$) goes to infinity. A plot of $\gamma$ is displayed in Figure 1(a).

**Remark:** If we consider the 2 bidders case, and denote by $\alpha$ and $\beta$ their respective bid multipliers, then the payment $\pi$ and $V$ of the first bidder are respectively equal to

$$\pi = \beta(\frac{\alpha}{\beta})^{a+2} \frac{1}{(a+2)(2a+3)}$$

and

$$V = (\frac{\alpha}{\beta})^{a+1} \frac{1}{(a+1)(2a+3)}. \quad (6)$$

We see that the ROI of the first bidder does not depends on the second bidder strategy. Therefore, we have in this case something similar to a strategy dominance.

4.2 Numerical estimation of $\gamma$

The trick is to remark that $\gamma$ is equal to the ratio of spend vs value when $T = 1$. We derive the following algorithm to estimate $\gamma$ in with different $F$ and $n$:

1. Choose $n$, $F$ and a number of samples $N$
2. $\text{Spend} = 0; \; \text{Value} = 0$
3. Generate $v \sim F$ and $v^- \sim F^n$
4. If $v > v^-$ then : $\text{Spend} = v^-$ and $\text{Value} = v^-$
5. Repeat from 3. (N-1) times
6. $\gamma \approx \frac{\text{Spend}}{\text{Value}}$
The results are displayed in Figure 1. For the lognormal and the Poisson distributions, the results are similar to those of the power distribution: monotony and convergence to 1. It seems that the bigger the variance, the bigger $\gamma$. Without providing a proof, we see that we can get an insightful intuition from Equation 8 of the next section.

The exponential distribution provides more exotic results: $\gamma$ seems to be only a function of $n$ (i.e. independent of the scale parameter). This can be easily proved analytically using the formula $E(\max(v_1, v_2)) = 1/\lambda_1 + 1/\lambda_1 - 1/\lambda_1 + \lambda_2$ and a recursion on $v_1^{(n)}$. It is however still increasing in $n$, and seems to converge to 1 as $n$ goes to infinity.

![Graphs showing scaling factor as a function of distribution parameter for different numbers of bidders](image)

(a) Power Distribution  
(b) Exponential Distribution  
(c) LogNormal Distribution  
(d) Poisson Distribution

Fig. 1: The scaling factor as a function of the distribution parameter, for different numbers of bidders

5 Properties of the Equilibrium

We recover a standard result ([12] from auction theory, adapted to ROI constrained auctions.

**Theorem 4 (Revenue equivalence).** All the auctions described in Section 8.3 bring the same expected revenue to the seller.
Proof. The expected payment of a buyer is equal to his expected value time $T$. Since the total welfare is fixed, the expected payment of every buyer is the same no matter the auction.

On the other hand, the following observation is quite unusual.

**Theorem 5 (Optimal Reserve Price).** With ROI constraints, the optimal reserve price in a symmetric setting is zero.

**Proof.** Same argument as before: a reserve price would decrease the social welfare, and thus decrease the expected payment.

This completely departs from the idea that reserve prices should be used to increase the seller’s revenue. Observe that in practice in the case of display advertising, the buyer may take some time to react to a change of ROI. Consequently, measuring the seller’s long term expected revenue uplift is a technical and business challenge. Moreover, the reserve price, by decreasing the welfare, may on the long run trigger a decrease of the target ROI, since buyers have to do an arbitrage between volume and ROI.

**Theorem 6 (Convergence).**

For any number of bidders $n$, $\gamma(F,n) \leq 1$. Moreover if the value $v$ is bounded, then

$$
\lim_{n \to \infty} \gamma(F,n) = 1
$$

**Proof.** Let us denote by $Y^{(k)}$ the maximum of $k$ random variables drawn with the distribution $F$. One just need to observe that

$$
\gamma(F,n) = 1 - \frac{\mathbb{E}(v - Y^{n-1}|v > Y^{n-1})}{v^{(n)}_1}.
$$

This can be interpreted in the following way: when there is a great number of bidders, the competition tends to become first price.

**Remark 1.** There may be a non symmetric equilibrium.

The basic idea is that one bidder can bid higher than necessary to force another bidder to leave the auction because his (ROI) constraint cannot be satisfied.

**Proof.** We concentrate on exhibiting a counter-example. Take $n = 2$, $T = 1$, $F$ is the uniform distribution over $[2, 3]$, and denote by $(\alpha_i)_{i=1,2}$ the bid multipliers of the two bidders. Set $\alpha_2 = 6$. Then if $\alpha_1 \leq 4$, bidder 1 do not win any auction, but the (ROI) constraint is satisfied.

On the other hand, if $\alpha_1 > 4$, then the spend vs value ratio is bigger than 4, and the (ROI) constraint is violated. We can check that if $\alpha_1 = 0$ the (ROI) constraint is satisfied for 2. Therefore $(\alpha_1, \alpha_2) = (0, 6)$ is an asymmetric Nash equilibrium.

In addition to this, observe that the bidders may be tempted to bid other strategies than the linear best reply. For example consider this example: Take $n = 2$, an exponential distribution with parameter 1, $T = 1$. One agent may increase its profit by bidding with an affine function. Compare $(b_1(v_1), b_2(v_2)) = (2v_1 + 1, 2v_2)$ with $(b_1(v_1), b_2(v_2)) = (3v_1, 3v_2)$.

On a simulation with $10^8$ auctions, we get in the first case an effective inverse ROI of 0.92, and a revenue of 0.84 for bidder 1 (resp. 1.0 and 0.61 for bidder 2), while in the second case, we get an effective inverse ROI of 1, and a revenue of 0.75 for bidder 1 (resp. 1.0 and 0.75 for bidder 2). Those simple, informal examples indicate that the bidders may be tempted to bid aggressively to weaken the other bidders ROI.
6 Dynamic Bidder Problem

In practice, the buyer behaviors may differ from the static case: (1) the dynamics allows the buyer
to take decisions based on the past events, (2) the linear constraint in expectation does not reflect
the buyer risk aversion, (3) the benefit the buyer gets from a won auction is stochastic (for example,
if he is only interested in clicks or conversions), (4) from a business perspective, the constraint is
"smooth", and there is a trade-off between ROI and volumes that can be expressed in different
ways.

We propose in this section a continuous time optimal control framework to express and study
the buyer’s problem in a dynamic fashion. The approach combines the advantages of a powerful
and mathematically clean expressiveness with theoretical insights and numerical tools. We only
model one individual buyer. Yet note that the optimal control formulation is a first building block
for the study of the full market dynamics (with mean field games or differential games).

6.1 Deterministic Dynamic Formulation

When we neglect the stochastic aspect of the reward, the problem is very similar to the static one.
In this section, $G$ is the cumulative distribution of the price to beat. We denote by $\tau$ the time
horizon. We denote by $R$ the instantaneous revenue and by $C$ the instantaneous cost:

$$
R(b) = \int_0^{+\infty} dv G(b)vf(v), \quad C(\alpha) = \int_0^{+\infty} dv f(v) \int_0^b dt g(t). \quad (9)
$$

The bidder is now maximizing with respect to $b$

$$
\int_0^\tau R(b_t) dt,
$$

subject to $\dot{X}(t) = T R(b_t) - C(b_t), \quad X_\tau \geq 0 \quad \text{and} \quad X_0 = x_0.$$

We use the Pontryagin Maximum Principle to argue that a necessary condition for an optimal
control is that there exists a multiplier $p(t)$ so that $b(t)$ maximizes

$$
H = p(T R - C) + R. \quad \text{Moreover} \quad \dot{p} = -\frac{\partial H}{\partial x} = 0.
$$

We thus recover the main result of the static case: the bid is linear in $v$. Moreover in
the absence of stochasticity, the multiplier is constant. Observe that in practice, one could use the
solution of problem (10) to implement a Markov Decision Process (MDP), using $x_0$ as the current
state and thus get an online bidding engine.

6.2 Stochastic Formulation

The number of displays is much higher than the number of clicks/sales, therefore I neglect the
stochasticity of the competition/price over the stochasticity that comes from the actions. Since we
see a lot of displays, we apply the law of large number to model the uncertainty with a white noise.
Therefore there is an additional, stochastic term in the revenue, $T \sigma_i dW_t$, where $W_t$ is a Brownian
motion. We get that an optimal, stochastic bid $b$ should maximize (see [10] for a reference)

$$
E_{b^{-},v}[b > b^{-}]((Tv - b^{-})p + v + T^2 \sigma_i^2 M). \quad (11)
$$

Since the realization of the action follows a binomial law, $\sigma = v(1 - v)$. Assuming $v \ll 1$,
we can approximate it as $v$. Therefore every infinitesimal terms of the Hamiltonian becomes

$$
((Tv - b^{-})p + v + vMT^2)[b > b^{-}] \quad \text{reproducing} \quad \text{the previous reasoning we get}
$$

$$
b^* = v(T + \frac{MT^2 + 1}{p}). \quad (12)
$$
Conclusion: Once again, the bid factor approach is justified! Observe that this conclusion comes from an approximation, (therefore the bid is not strictly speaking optimal for the initial problem). Last but not least, we have reduced the control set from $\mathbb{R}^+ \to \mathbb{R}^+$ to $\mathbb{R}^+$. This reduction gives access to a well understood approach to solve this kind of problem: the continuous time equivalent of Dynamic Programming.

Observe that our problem definition is incomplete: we need to take into account the constraint on the final state. We restate this constraint with a penalty $K(x)$. Using dynamic programing, the buyer problem is equivalent to the following Hamilton-Jacobi-Bellman equation:

$$
\partial_t V + \sup_{\alpha} \left( (T \mathcal{R}(\alpha) - C(\alpha)) \partial_x V + \frac{T^2}{2} \mathcal{R}(\alpha) \partial_{xx} V + \mathcal{R}(\alpha) \right) = 0 \quad \text{and} \quad V_T = K,
$$

with

$$
\mathcal{R}(\alpha) = \int_0^{+\infty} dv G(\alpha v) v f(v), \quad \text{and} \quad C(\alpha) = \int_0^{+\infty} dv f(v) \int_0^{\alpha v} dt g(t).
$$

6.3 Numerical Resolution

We solve an instance of the problem with the numerical optimal stochastic control solver BocopHJB \cite{3}. BocopHJB compute the Bellman function by solving the non-linear partial differential equation (HJB). Observe that the state is one dimensional, which make this kind of solver very efficient.

We take $G(b^{-}) = (b^{-})^a$ on $[0, 1]$ and $v$ uniformly distributed on $[0, 1]$. We get that $\mathcal{R}(\alpha) = \frac{\alpha^2}{2\alpha^2}$ if $\alpha < 1$ and $\mathcal{R}(\alpha) = \frac{\alpha^a}{(\alpha+2)(\alpha^2+1)} + \frac{1}{2}(1 - \frac{1}{\alpha^2})$ otherwise, similarly, $C(\alpha) = \frac{a \alpha^{a+1}}{(\alpha+1)(\alpha+2)}$ for $\alpha \leq 1$ and $C(\alpha) = \frac{a a^{a+1}}{(\alpha+1)(\alpha+2)}((\alpha+2)\alpha - a - 1)$ for $\alpha \geq 1$ (see Figure 2). We take $T = 0.5$ and a linear penalty for negative $X_T$. The result of simulated trajectories are displayed on Figure 3.

![Fig. 2: $\mathcal{R}$ and $C$ and empirical $T$ for two values of $a$](image)

7 Conclusion

We have formalized the existence of ROI constraints in the buyers bidding problem. These constraints are of paramount importance in performance markets, yet they tend to be neglected in the current
literature. We have seen how standard results translate into this context, sometimes with surprising insights. We have given a proposal to study the dynamic, stochastic setting by combining a first order approximation with the Hamilton-Jacobi-Bellman approach. This allowed us in particular to propose a numerical method.

This work raises questions that would probably deserve more extended study. In particular, can we generalize the first three theorems to other constraints, auction rules and settings? Can the existence of the aggressive behaviors pointed out at the end of Section 5 poses a problem to the market? We may also want to put under scrutiny the harder, but more realistic case of correlated values. The dynamic case would also probably deserve a specific study.

Acknowledgement

I would like to thank Jérémie Mary for his helpful comments.

References

1. Agarwal, D., Ghosh, S., Wei, K., You, S.: Budget pacing for targeted online advertisements at linkedin. In: Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining. pp. 1613–1619. ACM (2014)
2. Balseiro, S.R., Besbes, O., Weintraub, G.Y.: Repeated auctions with budgets in ad exchanges: Approximations and design. Management Science 61(4), 864–884 (2015)
3. Bonnans, F., Giorgi, D., Heymann, B., Martinon, P., Tissot, O.: BocopHJB 1.0.1 – User Guide. Technical Report RT-0467, INRIA (Sep 2015), https://hal.inria.fr/hal-01192610
4. Bonnans, J.: Stochastic Optimization, Lecture Notes
5. Cai, H., Ren, K., Zhang, W., Malialis, K., Wang, J., Yu, Y., Guo, D.: Real-time bidding by reinforcement learning in display advertising. In: Proceedings of the Tenth ACM International Conference on Web Search and Data Mining. pp. 661–670. ACM (2017)
6. Diemert, E., Meynet, J., Galland, P., Lefortier, D.: Attribution modeling increases efficiency of bidding in display advertising. In: Proceedings of the ADKDD’17. p. 2. ACM (2017)
7. Fernandez-Tapia, J., Guéant, O., Lasry, J.M.: Optimal real-time bidding strategies. Applied Mathematics Research eXpress 2017(1), 142–183 (2016)
8. Golrezaei, N., Lin, M., Mirrokni, V., Nazerzadeh, H.: Boosted second-price auctions for heterogeneous bidders (2017)
9. Krishna, V.: Auction theory. Academic press (2009)
The Lagrangian of the optimization problem writes

\[ L^b(\lambda, b) = E_{b^-, v}[b^- < b](v - \lambda(b^- - Tv)) \]  

(15)

By recognizing the payoff of a Second Price auction with value \( v(1/\lambda + T) \), we educe that for each \( \lambda \) and \( v \), a \( b \) maximizing \( L \) is \( v(1/\lambda + T) \). The existence of a stationary \( \lambda^* \) is a consequence of the minmax Theorem (see Theorem 1.1129 in [4]).

B Proof of Theorem 2

To derive the first order condition, we follow the same path as in the proof of Theorem 3. We thus need to argue on the uniqueness.

Let us go back to the Lagrangian expression, and consider a value \( v \). We start with a small technical consideration: observe that the bids can be modified on a zero measure set without impacting the profit. To simplify the technical analysis, we assume without loss of generality that the bids are maximizers of the integrand. We denote by \( b \) a symmetric equilibrium bidding strategy.

Remember that bidding truthfully is a weakly dominant strategy for a second price auction. Similarly, we can say that for a given \( \lambda \), \( b(v, \lambda) \) is not strictly dominated, no matter the competition. Still, it is possible that some other bid provide as much in the Lagrangian expression. Such thing may only happen if \( b \) has a discontinuity around \( v \). Since \( b \) is linear where it is not discontinuous, and increasing, the only possibility is that \( b \) has positive jumps on a set of positive measures. Since \( b \) is bounded this is not possible (by Baire’s theorem).

C Proof of Theorem 3

First observe that, the Lagrangian of the bidder optimization problems writes:

\[ \lambda E_{v, b^-}((1/\lambda + T)v - S(b(v), b^-)) \]  

\[ [b > b^-], \]

thus we can look for a point-wise maximizer. Denote by \((\hat{b}, \hat{b}^-)\) the equilibrium bid of the same auction without any ROI constraints. Observe that if the price to beat is distributed like \((1/\lambda + T)\hat{b}^-(v^-)\) (for a given \( \lambda > 0 \)), then:

\[ b^*(v) = \arg\max_b E_{b^-}((1/\lambda + T)v - S(b, b^-)(v^-)) \]

\[ = (1/\lambda + T)\arg\max_b E_{b^-}((1/\lambda + T)v - S((1/\lambda + T)b, (1/\lambda + T)\hat{b}^-)(v^-)) \]

\[ = (1/\lambda + T)\arg\max_b E_{b^-}v - S(b, \hat{b}^-)(v^-)) \]

\[ = (1/\lambda + T)\hat{b}(v). \]

Therefore, for a given \( \lambda \), if the competition uses the proposed linear bid, one should also bid linearly to maximize the Lagrangian. We denote by \( G \) the price to beat distribution and by \( \kappa \) the degree of
"second priceness": $S(b, b^-) = ((1 - \kappa)b + \kappa b^-)[b > b^-]$. Observe that the first term of the integrand in the Lagrangian definition is equal to $E_v b\lambda G(b\lambda)$. So, let us deal with the second term. Since

$$E_v, v^- S(b\lambda(v), b^- (v^-))[v > v^-] = E_v \int_0^{b\lambda(v)} S(b\lambda(v), t)g(t)dt$$

$$= E_v S(b\lambda(v), b\lambda(v))G(b) - \int_0^{b\lambda(v)} \kappa G(t)dt$$

$$= E_v b\lambda G(b\lambda) - \int_0^{b\lambda(v)} \kappa G(t)dt,$$

we get that

$$L = \lambda \kappa E_v \int_0^{b\lambda(v)} G(t)dt.$$

The first order condition on $\lambda$ writes

$$0 = E_v, b^- \left( (1/\lambda + T)v - S(b(v), b^-) [b > b^-] - \kappa/\lambda E_v G(b\lambda(v))vdv. \right)$$

Then using successively the homogeneity of $S$, a simplification by $\kappa$, and the de facto definition of $\gamma$ as $\lambda T/1+\lambda T$, we get

$$\gamma = \frac{E_v^-[v > v^-]}{E_v[v > v^-]},$$

which we simplify using the following formula on the order statistics (see [9]): $EY_2^{(n)} = E n Y_1^{(n-1)} - (n - 1)Y_1^{(n)}$. 