A notion of symmetry witness related to Wigner’s theorem on symmetry transformations

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Abstract. A symmetry witness is a subset of the space of selfadjoint trace class operators that allows one to ascertain whether a linear map acting in that space is a symmetry transformation. This notion arises from a certain type of linear preserver problems. Precisely, a symmetry witness is a suitable set which is invariant with respect to an injective linear map in the Banach space of selfadjoint trace class operators where the quantum states live if and only if this map acts as a symmetry transformation. In particular, by a linear version of Wigner’s classical theorem, the set of pure states — the rank-one projections — is a symmetry witness. Linearity entails that the usual assumption of preservation of the transition probability between pure states becomes superfluous. This result extends to every set of projections of a fixed (finite) rank, with some suitable constraint on this rank. One then obtains a classification of the sets of projections of a fixed rank that are symmetry witnesses. These symmetry witnesses are projectable. Namely, formulating the mentioned result in terms of quantum states, the sets of ‘uniform’ density operators of a suitable fixed rank are symmetry witnesses as well.

1. Introduction
According to a celebrated classical theorem of Wigner, a symmetry transformation of a quantum system is implemented by means of a unitary or antiunitary operator, which is uniquely determined up to multiplication by a phase factor [1–9]. Otherwise stated, there is a canonical one-to-one correspondence between the symmetry transformations of a quantum system and the elements of the projective unitary-antiunitary group of the relevant Hilbert space. As a remarkable consequence — in general, projective — unitary group representations play a central role in quantum theory [10].

We stress that in Wigner’s seminal work symmetries were regarded as maps on pure states only and, as a defining property, the transition probability between pairs of states was assumed to be preserved by these maps. Linearity — or convex linearity — is simply not contemplated in this approach; it appears in subsequent formulations of Wigner’s theorem involving maps defined in different settings (e.g., the Kadison and Jordan-Segal automorphisms [4]). It is clear, however, that the linear structure is essentially already there, because the pure states, realized as rank-one projections, are embedded in the set of all states, which is a convex set in a linear space, the real Banach space of selfadjoint trace class operators. Moreover, the symmetry transformations of pure states extend in a straightforward way to linear maps acting in that space.

It is then natural to wonder whether assuming in advance that the maps one deals with be linear operators, and under suitable hypotheses, one can achieve theorems ‘of the Wigner
type' or, more generally, 'pseudo-Wigner' theorems, characterizing certain types of maps. By the latter expression we mean that one ends with maps of the general form $A \mapsto MAN$ or $A \mapsto MA^TN$, where $A$ belongs to some (real or complex) Banach space of bounded linear operators in a separable complex Hilbert space $H$, $M, N$ are suitable operators in $H$ and $A \mapsto A^T$ is the transposition map associated with a given orthonormal basis. A class of problems where this kind of maps typically arise are known in the literature under the name of linear preserver problems: one requires that a certain quantity, set or relation is preserved by a linear map and the solution consists in providing a suitable characterization of such a map; see [8,11] and references therein. They date back, at least, to the seminal work of Frobenius [12], who considered the matrix determinant as a preserved quantity and obtained a complete solution of the problem, in the form of a pseudo-Wigner theorem (in this case, $M, N$ are arbitrary nonsingular complex square matrices, with $\det(MN) = 1$).

In particular, a theorem of the Wigner type corresponds to the case where $M$ and $N$ are a unitary operator and its adjoint, respectively; i.e., a map that can be called — in general, with a slight abuse of terminology — a 'symmetry transformation' (note that such a result can be reformulated in terms of unitary and antiunitary operators, without using transposition).

Clearly, one can consider preserver problems that are not necessarily linear, and actually an exhaustive survey of such problems, especially in the context of quantum mechanics, can be found in the standard reference book [8]. E.g., considering bijective maps on the set of pure states and regarding the transition probability as a preserved quantity leads to Wigner’s theorem. On the other hand, one obtains a related linear preserver problem by considering linear maps in the real Banach space of selfadjoint trace class operators and requiring that the set of pure states be this time a preserved set, whereas the assumption of preservation of the transition probability between pure states is omitted. More generally, one may require that the set of (orthogonal) projections with a fixed rank be preserved. The problem can be reformulated in terms of certain families of density operators, thus it is relevant from the point of view of physics.

This intriguing problem has been considered, in the finite-dimensional case, in [13], where a partial solution has been found, and then reconsidered in full generality and completely solved in [9], giving rise to a theorem of the Wigner type. In the present contribution, we will discuss this theorem and various other related results. We will also argue that from this circle of ideas an interesting notion of symmetry witness emerges in a natural way. In short, a symmetry witness is suitable a set of selfadjoint trace class operators that allows one to detect the symmetry transformations among other linear maps. The set of all pure states of quantum system can be regarded as the prototype of this notion. More generally, as it will be clear later on, a symmetry witness arises whenever the solution of a linear preserver problem associated with a suitable symmetry witness candidate produces a theorem of the Wigner type.

The paper is organized as follows. In sect. 2, we review Wigner’s theorem and some related results. In sect. 3, we recall some pseudo-Wigner theorems in the linear setting. These results suggest that linearity may play a precise role in the derivation of theorems of the Wigner type, and in sect. 4 we show that this is indeed the case by discussing a linear version of Wigner’s theorem and other related facts. From this context then the notion of symmetry witness arises in a natural way; see sect. 5. In sect. 6, we provide a complete classification of the set of projections with a fixed rank that are symmetry witnesses; hence, a complete solution of the problem raised in [13]. Finally, in sect. 7, conclusions are drawn.

2. Wigner’s theorem and some related results

Let $\mathcal{H}$ be a separable complex Hilbert space, with $\dim(\mathcal{H}) \geq 2$, and let $\mathcal{P}_1(\mathcal{H})$ be the set of rank-one projections in $\mathcal{H}$, regarded as the set of pure states of a quantum system. For every pair $P, Q \in \mathcal{P}_1(\mathcal{H})$, one can define the transition probability

$$\text{tr}(PQ) \in [0,1].$$

(1)
According to Wigner [1], a symmetry transformation is a bijective map
\[ \Theta : \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H}) \] (2)
that preserves the transition probability; i.e.,
\[ \text{tr}(\Theta(P) \Theta(Q)) = \text{tr}(PQ), \quad \forall P, Q \in \mathcal{P}_1(\mathcal{H}). \] (3)

Symmetry transformations are characterized by the following celebrated theorem [1–9].

**Theorem 1** (Wigner). A bijection \( \Theta : \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H}) \) is a symmetry transformation if and only if it is of the form
\[ \Theta(P) = UP U^*, \quad \forall P \in \mathcal{P}_1(\mathcal{H}), \] (4)
where \( U \) is a unitary or antiunitary operator in \( \mathcal{H} \), uniquely determined up to multiplication by a phase factor.

**Remark 1.** Denoting by \( A \mapsto A^T \) (A bounded operator in \( \mathcal{H} \)) the transposition map associated with an orthonormal basis, we have:
\[ A^T = JA^*J, \] (5)
for some complex conjugation \( J \) in \( \mathcal{H} \) (an antiunitary operator such that \( J = J^* = J^{-1} \)). Hence, if \( U \) is an antiunitary operator in \( \mathcal{H} \), for every pure state \( P \) we can write:
\[ UP U^* = UP^*U^* = UP^T J U^* = WP^T W^*, \] (6)
for some unitary operator \( W = UJ \). Therefore, Wigner’s theorem can be expressed in terms of unitary operators and a (fixed) transposition.

There is a remarkable generalization of Wigner’s theorem, due to Uhlhorn [14], which is based on a somewhat weaker assumption: only the zero transition probability is preserved (in both directions). On the other hand, it can be shown by means of examples — see Example 1 below — that this result does not hold for \( \dim(\mathcal{H}) = 2 \).

Notice that, for \( P, Q \in \mathcal{P}_1(\mathcal{H}) \),
\[ \text{tr}(PQ) = 0 \iff P \perp Q, \quad \text{i.e. } PQ = 0; \] (7)
i.e., Uhlhorn’s result can be expressed in terms of maps that preserve the mutual orthogonality of rank-one projections.

**Theorem 2** (Uhlhorn). For \( \dim(\mathcal{H}) \geq 3 \), let \( \Theta : \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H}) \) be a bijective map preserving the zero transition probability — or, equivalently, the orthogonality — in both directions:
\[ \text{tr}(PQ) = 0 \iff \text{tr}(\Theta(P) \Theta(Q)) = 0, \quad P, Q \in \mathcal{P}_1(\mathcal{H}). \] (8)
Then, \( \Theta \) is of the form
\[ \Theta(P) = UP U^*, \quad \forall P \in \mathcal{P}_1(\mathcal{H}), \] (9)
where \( U \) is a unitary or antiunitary operator in \( \mathcal{H} \), uniquely determined up to multiplication by a phase factor.

**Example 1.** The fact that Uhlhorn’s theorem does not hold for \( \dim(\mathcal{H}) = 2 \) can be illustrated by a simple counterexample. Let us set \( \mathcal{H} = \mathbb{C}^2 \). In this case, every pure state \( P \) can be written (uniquely) in the form
\[ P = \frac{1}{2}(I_2 + a \cdot \sigma), \quad a \in \mathbb{R}^3, \quad \|a\| = 1, \] (10)
where $I_2$ is the $2 \times 2$ identity matrix and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices; i.e., there is a natural one-to-one correspondence between the elements of $\mathcal{P}_1(\mathcal{H})$ and the points of the unit sphere in $\mathbb{R}^3$ (the ‘Bloch sphere’). The transition probability can be computed in terms of this parametrization using the simple formula

$$\text{tr}((I_2 + a \cdot \sigma)(I_2 + b \cdot \sigma)) = 2(1 + a \cdot b).$$

(11)

Moreover, exploiting this correspondence every bijection $\Theta : \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H})$ induces a bijective map

$$\mathbb{R}^2 \ni X \equiv \{(0,0)\} \cup \{(\pi,0)\} \cup \left([0,\pi] \times [0,2\pi]\right) \ni (\theta,\phi) \mapsto \vartheta(\theta,\phi) \in X,$n

(12)

which is determined by the condition that $\vartheta(\theta,\phi) = (\theta',\phi')$, where

$$\frac{1}{2}(I_2 + \sin \theta' \cos \phi' \sigma_1 + \sin \theta' \sin \phi' \sigma_2 + \cos \theta' \sigma_3) = \Theta\left(\frac{1}{2}(I_2 + \sin \theta \cos \phi \sigma_1 + \sin \theta \sin \phi \sigma_2 + \cos \theta \sigma_3)\right):$$

conversely, a bijection $\vartheta$ of $X$ onto itself determines a bijection $\Theta$ of $\mathcal{P}_1(\mathcal{H})$ onto itself.

Now, let us set, in particular,

$$\vartheta(\theta,\phi) := (\theta,\phi), \text{ for } \theta \neq \pi/2, \quad \vartheta(\pi/2,\phi) := (\pi/2,\alpha(\phi)),$$

(13)

where

$$\alpha(\phi) := \phi^2/\pi, \quad \text{for } 0 \leq \phi < \pi, \quad \alpha(\phi) := (\phi - \pi)^2/\pi + \pi, \quad \text{for } \pi \leq \phi < 2\pi,$n

(14)

and let us denote by $\Theta_\vartheta$ the associated bijection of $\mathcal{P}_1(\mathcal{H})$ onto itself. By relation (11) it is clear that the antipodal points on the Bloch sphere parametrize mutually orthogonal projections; hence,

$$\text{tr}(PQ) = 0 \iff \text{tr}(\Theta_\vartheta(P) \Theta_\vartheta(Q)) = 0, \quad \forall P,Q \in \mathcal{P}_1(\mathcal{H}),$$

(15)

because

$$\alpha(\phi + \pi) = \phi^2/\pi + \pi = \alpha(\phi) + \pi, \quad \text{for } \phi \in [0,\pi).$$

(16)

On the other hand, by (11) it is also clear that the map $\Theta_\vartheta$ does not preserve, in general, the transition probability. Thus, it is not a symmetry transformation, and we have indeed a counterexample. Moreover, $\Theta_\vartheta$ cannot be extended to a linear map acting in the space of $2 \times 2$ hermitian matrices. The possibility of constructing a counterexample, relying on a bijection of $\mathcal{P}_1(\mathcal{H})$ onto itself that admits a linear extension, is actually excluded by Theorem 7 below.

Another remarkable formulation of Wigner’s theorem is expressed in terms of generic — pure or mixed — states. Then, let $\mathcal{S}(\mathcal{H})$ be the convex set of all quantum states (the normalized positive trace class operators in $\mathcal{H}$). A Kadison automorphism is a bijective map

$$\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$$

(17)

which preserves the convex structure of $\mathcal{S}(\mathcal{H})$; i.e., such that

$$\Phi(\epsilon \rho + (1-\epsilon)\sigma) = \epsilon \Phi(\rho) + (1-\epsilon)\Phi(\sigma), \quad \forall \epsilon \in [0,1], \quad \forall \rho,\sigma \in \mathcal{S}(\mathcal{H}).$$

(18)

The following classical result holds [3–6].

**Theorem 3** (Kadison). Let a bijection $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ be a Kadison automorphism. Then, $\Phi$ is of the form

$$\Phi(\rho) = U \rho U^*, \quad \forall \rho \in \mathcal{S}(\mathcal{H}),$$

(19)

where $U$ is a unitary or antiunitary operator in $\mathcal{H}$, uniquely determined up to multiplication by a phase factor.
It is worth noting that in this case there is no ‘preserved quantity’ in the hypotheses of the theorem, whereas a central role is played by the natural convex structure of $\mathcal{S}(\mathcal{H})$.

Another interesting result is a generalization of Uhlhorn’s theorem which is due to Šemrl [15]. Like the original result, it excludes the case where $\dim(\mathcal{H}) = 2$, and it is based on a preserved quantity — $\text{tr}(PQ) = 0$ — or, equivalently, on the preservation of the mutual orthogonality of projections:

$$\text{tr}(PQ) = 0 \iff PQ = 0;$$

but in this case $P, Q$ are not supposed, in general, to be pure states. They are assumed to belong to $\mathcal{P}(\mathcal{H})$, the set of (orthogonal) projections of rank $k \in \mathbb{N}$, with some constraint on $k$ in the finite-dimensional case.

**Theorem 6 (Arazy).** Let $\mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ — with $2 \leq 2k < \dim(\mathcal{H})$ — be a bijection that preserves orthogonality in both directions:

$$\forall P, Q \in \mathcal{P}(\mathcal{H}). \quad PQ = 0 \iff \Phi(P)\Phi(Q) = 0,$$

Then, $\Phi$ is of the canonical form

$$\Phi(P) = UPU^*, \quad \forall P \in \mathcal{P}(\mathcal{H}),$$

where $U$ is a unitary or antiunitary operator in $\mathcal{H}$.

Let us observe explicitly that, in principle, a plausible constraint in the hypotheses of the previous theorem would be that, for $\dim(\mathcal{H}) < \infty$, $2k \leq \dim(\mathcal{H}) \neq 2$; but the case where $2k = \dim(\mathcal{H}) \neq 2$ is actually not contemplated. We will further comment about this peculiar point in sect. 6. The physical significance of considering the set $\mathcal{P}(\mathcal{H})$, for $k \neq 1$, will also be discussed later on.

### 3. What about linearity?

In the literature, several pseudo-Wigner theorems arise from preserver problems concerning linear maps [8, 11]. Two classical examples, stemming from linear preserver problems where a Banach space norm is regarded as a preserved quantity, are due to Russo [16, 17] and Arazy [18], respectively. We will denote by $\mathcal{B}_1(\mathcal{H})$ the complex Banach space of trace class operators in $\mathcal{H}$ and, more generally, by $\mathcal{B}_p(\mathcal{H})$ the Schatten $p$-class, $1 \leq p \leq \infty$, with $\mathcal{B}_\infty(\mathcal{H})$ denoting the Banach space of compact operators (endowed with the operator norm).

**Theorem 5 (Russo).** Let $\Phi: \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a surjective linear isometry. Then, $\Phi$ is of the form

$$\Phi(A) = UAV \quad \text{or} \quad \Phi(A) = UA^TV, \quad \forall A \in \mathcal{B}_1(\mathcal{H}),$$

for some unitary operators $U, V$ in $\mathcal{H}$.

**Remark 2.** Reasoning as in Remark 1, for every transposition $A \mapsto A^\dagger$ there is a complex conjugation operator $J$ in $\mathcal{H}$ such that $A^\dagger = JA^*J$. Hence, given unitary operators $U, V$ in $\mathcal{H}$, we have that

$$UA^\dagger V = UJA^*JV = WA^*Z,$$

for some antiunitary operators $W = UJ$, $Z = JV$ in $\mathcal{H}$.

The previous result admits the following generalization concerning other Schatten classes of operators (with the exception of the Hilbert-Schmidt class $\mathcal{B}_2(\mathcal{H})$).

**Theorem 6 (Arazy).** Let $\Phi: \mathcal{B}_p(\mathcal{H}) \to \mathcal{B}_p(\mathcal{H})$ — with $1 \leq p \leq \infty$, $p \neq 2$ — be a linear isometry of the Banach space $\mathcal{B}_p(\mathcal{H})$ onto itself. Then, $\Phi$ is of the form

$$\Phi(A) = UAV \quad \text{or} \quad \Phi(A) = UA^TV, \quad \forall A \in \mathcal{B}_p(\mathcal{H}),$$

for all $A \in \mathcal{B}_p(\mathcal{H})$, where $U, V$ are unitary operators in $\mathcal{H}$. 

Remark 3. The fact that $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space entails that it admits further surjective linear isometries (unitary operators) wrt (25). On the other hand, if $\Phi: \mathcal{B}_2(\mathcal{H}) \to \mathcal{B}_2(\mathcal{H})$ is a surjective linear isometry such that $\Phi(\mathcal{P}_1(\mathcal{H})) = \mathcal{P}_1(\mathcal{H})$, then $\Phi(A) = UAU^*$ or $\Phi(A) = UA^*U^*$, for some unitary operator $U$ in $\mathcal{H}$ (by Wigner’s theorem, because the transition probability can be regarded as a Hilbert-Schmidt product, and by the fact that $\text{span}_c(\mathcal{P}_1(\mathcal{H}))$ is dense in $\mathcal{B}_2(\mathcal{H})$).

Remark 4. Results analogous to Arazy’s theorem hold for the complex Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators and for the real Banach space $\mathcal{B}(\mathcal{H})_s$ of selfadjoint bounded operators. Precisely, if $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective linear isometry, then it is of the form (25), for some unitary operators $U, V$ in $\mathcal{H}$; see Theorem A.9 of [8]. Moreover, if $\Phi: \mathcal{B}(\mathcal{H})_s \to \mathcal{B}(\mathcal{H})_s$ is a surjective linear isometry, then $\Phi(A) = s UAU^*$ or $\Phi(A) = s UA^*U^*$, for some unitary operator $U$ in $\mathcal{H}$ and $s \in \{1, -1\}$; see Theorem 2.6.1 of [8].

In quantum mechanics, one usually relates linearity with the Hilbert space associated with a quantum system — i.e., with the linear superposition of ‘state vectors’ — and with the observables (selfadjoint operators), rather than with the space of states $\mathcal{S}(\mathcal{H})$. Nevertheless, $\mathcal{S}(\mathcal{H})$ is a convex set in $\mathcal{B}_1(\mathcal{H})$, and in several contexts — e.g., the theory of open systems and quantum information — the linear structure of the whole space $\mathcal{B}_1(\mathcal{H})$ is actually exploited. Typical examples of topics where the linear structure of this space is relevant are [19–30]:

- quantum transmission channels;
- quantum dynamical semigroups and the associated master equations;
- entanglement detection.

Moreover, Kadison’s theorem and the results outlined in this section suggest that linearity may play a major role in the derivation of theorems of the Wigner type.

It seems then quite natural to reconsider Wigner’s theorem and some related results in the linear setting.

4. Symmetries in the linear setting: the role of linearity

From this point onwards, it will be convenient to work in the real Banach space $\mathcal{B}_1(\mathcal{H})_s$ of selfadjoint trace class operators in $\mathcal{H}$, where the quantum states (density operators) live. We will denote by $\mathcal{F}(\mathcal{H})_s \subset \mathcal{B}_1(\mathcal{H})_s$ the dense linear manifold of all selfadjoint operators in $\mathcal{H}$ of finite rank; i.e., $\mathcal{F}(\mathcal{H})_s = \text{span}_R(\mathcal{P}_1(\mathcal{H}))$. Clearly, in the finite-dimensional case $\mathcal{F}(\mathcal{H})_s, \mathcal{B}_1(\mathcal{H})_s$ and $\mathcal{B}(\mathcal{H})_s$ — regarded as linear spaces — actually coincide, and we prefer to adopt the ‘neutral’ notation $\mathcal{B}(\mathcal{H})_s$ whenever that case occurs.

In the linear setting, one can prove the following result [9], which is closely related to the classical form of Wigner’s theorem.

**Theorem 7** (Linear form of Wigner’s theorem). Let $\Phi$ be a densely defined linear operator in $\mathcal{B}_1(\mathcal{H})_s$ — with $\text{dom}(\Phi) = \mathcal{F}(\mathcal{H})_s$ — mapping the set $\mathcal{P}_1(\mathcal{H})$ of pure states onto itself. Then, $\Phi$ is closable, and its closure $\hat{\Phi}$ is a surjective isometry of the canonical form

$$\hat{\Phi}(A) = UAU^*, \quad \forall A \in \mathcal{B}_1(\mathcal{H})_s,$$

where $U$ is a unitary or antiunitary operator in $\mathcal{H}$, uniquely determined up to multiplication by a phase factor.

**Remark 5.** In this setting, the assumption of preservation of the transition probability between pure states is immaterial, and $\mathcal{P}_1(\mathcal{H})$ becomes a preserved set. A ‘symmetry transformation’ will be then simply a map of the form (26), without reference to Wigner’s original approach.

**Remark 6.** The linear manifold $\mathcal{F}(\mathcal{H})_s$ is dense in the Banach space $\mathcal{B}_p(\mathcal{H})_s := \mathcal{B}_p(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})_s$, the real Schatten $p$-class, for every $1 \leq p \leq \infty$. Hence, it is easily seen that the statement of Theorem 7 actually holds true with the space of selfadjoint trace class operators $\mathcal{B}_1(\mathcal{H})_s$ replaced with any of the Schatten classes $\mathcal{B}_p(\mathcal{H})_s$. 

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Also Kadison’s theorem, i.e. Theorem 3, can be revisited in the linear setting. In this case, the preserved set is formed by all density operators of finite rank. Precisely, one can prove the following fact [9].

**Theorem 8** (Linear form of Kadison’s theorem). Let $\Phi$ be a densely defined linear operator in $\mathcal{B}_1(\mathcal{H})_s$ — with $\text{dom}(\Phi) = \mathcal{F}(\mathcal{H})_s$ — mapping the convex set

$$\mathcal{F}\mathcal{F}(\mathcal{H}) := \mathcal{F}(\mathcal{H})_s \cap \mathcal{S}(\mathcal{H})$$

of finite rank density operators bijectively onto itself. Then, $\Phi$ is closable, and its closure $\hat{\Phi}$ is a surjective isometry of the canonical form (26); namely, a symmetry transformation.

Recently, in the finite-dimensional case Sarbicki et al have proved the following interesting result [13], which partially generalizes the linear version of Wigner’s theorem and is also related to Šemrl’s theorem (Theorem 4). Indeed, in this result one deals with linear maps preserving $\mathcal{P}_k(\mathcal{H})$, the set of projections of a fixed rank $k \geq 1$ (the assumption of preservation of the quantity $\text{tr}(PQ) = 0$ becomes superfluous).

**Theorem 9** (Sarbicki et al). Let $n = \text{dim}(\mathcal{H}) < \infty$ be a prime number and $\Phi: \mathcal{B}(\mathcal{H})_s \rightarrow \mathcal{B}(\mathcal{H})_s$ a linear map, mapping $\mathcal{P}_k(\mathcal{H})$ — with $k < n$ — bijectively onto itself. Then, $\Phi$ is a surjective isometry of the form

$$\Phi(A) = UAU^*, \quad \forall A \in \mathcal{B}(\mathcal{H})_s,$$

where $U$ is a unitary or antiunitary operator in $\mathcal{H}$.

Later on, Størmer [31] has proved another interesting theorem of the Wigner type, involving linear maps that act in the complex Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators in $\mathcal{H}$. These maps are assumed to be positive (hence, bounded; see, e.g., [32]) and unital (the identity is mapped to itself), and again to preserve the set of projections of a given rank $k \geq 1$.

We stress that, whereas the theorems of Wigner and Kadison — as well as their linear counterparts — are formulated in the ‘Schrödinger picture’ (symmetries act on states), Størmer’s result is worked out in the ‘Heisenberg picture’ (where, instead, observables are the physical objects undergoing symmetry transformations). Restricting to selfadjoint operators — i.e., to the real Banach space $\mathcal{B}(\mathcal{H})_s$ of (bounded) quantum observables — for a more direct comparison with Theorem 9 and with other results that will be discussed later on, we have the following equivalent version of Størmer’s theorem.

**Theorem 10** (Størmer). Let $\Phi: \mathcal{B}(\mathcal{H})_s \rightarrow \mathcal{B}(\mathcal{H})_s$ be a unital positive linear map, mapping $\mathcal{P}_k(\mathcal{H})$ — for some $k < \text{dim}(\mathcal{H})$ — onto itself. Then, $\Phi$ is of the form

$$\Phi(A) = UAU^*, \quad \forall A \in \mathcal{B}(\mathcal{H})_s,$$

where $U$ is a unitary or antiunitary operator in $\mathcal{H}$.

**Corollary 1.** Let $\Phi: \mathcal{B}(\mathcal{H})_s \rightarrow \mathcal{B}(\mathcal{H})_s$ be a surjective linear map. Then, the following facts are equivalent:

(i) $\Phi$ is isometric and positive;
(ii) $\Phi$ is isometric and unital;
(iii) $\Phi$ is positive and unital, and maps $\mathcal{P}_k(\mathcal{H})$, for some $k < \text{dim}(\mathcal{H})$, onto itself;
(iv) $\Phi$ is of the form (29), for some unitary or antiunitary operator $U$ in $\mathcal{H}$.

**Proof.** The statement follows from Theorem 10 and from the last assertion of Remark 4.

A comparison between the theorems of Sarbicki et al and Størmer suggests that — at least in the finite-dimensional case — the assumptions of positivity and unitality in the latter result may be relaxed. On the other hand, in Theorem 9 the assumption that $n = \text{dim}(\mathcal{H}) < \infty$ be a prime number appears rather singular. We will clarify both these points in sect. 6.
5. Symmetry witnesses

In the light of the results discussed in the previous section, it seems quite natural to replace, in the linear setting, preserved quantities — e.g., transition probability, fidelity, various metrics etc.; see [8] and references therein — with suitable preserved sets (e.g., $\mathcal{P}_k(\mathcal{H})$).

One can formalize this idea as follows [9].

**Definition 1** (Symmetry witnesses). We call a subset $\mathcal{W}$ of $\mathcal{B}_1(\mathcal{H})_k$, invariant with respect to every unitary or antiunitary transformation, and such that its linear span is dense in $\mathcal{B}_1(\mathcal{H})_k$ — i.e., such that

$$\text{span}_\mathbb{R}(\mathcal{W}) = \mathcal{B}_1(\mathcal{H})_k$$

(30)

— a **symmetry witness candidate**. A symmetry witness candidate $\mathcal{W}$ is called a **symmetry witness** if every injective linear operator $\Phi$ in $\mathcal{B}_1(\mathcal{H})_k$ — densely defined on $\text{dom}(\Phi) = \text{span}_\mathbb{R}(\mathcal{W})$, and preserving $\mathcal{W}$: $\Phi(\mathcal{W}) = \mathcal{W}$ — is of the form

$$\Phi(A) = UAU^*, \ \forall A \in \text{span}_\mathbb{R}(\mathcal{W}),$$

(31)

for some unitary or antiunitary operator $U$ in $\mathcal{H}$.

**Remark 7.** Note that every symmetry witness candidate $\mathcal{W}$ induces a linear preserver problem: to characterize the class of all injective linear operators in $\mathcal{B}_1(\mathcal{H})_k$, with domain $\text{span}_\mathbb{R}(\mathcal{W})$, that map $\mathcal{W}$ onto itself. Then, $\mathcal{W}$ is a symmetry witness if this class is formed precisely by all the unitary or antiunitary transformations.

**Remark 8.** Given a symmetry witness $\mathcal{W}$ and an injective linear operator $\Phi$ in $\mathcal{B}_1(\mathcal{H})_k$ — such that $\text{dom}(\Phi) = \text{span}_\mathbb{R}(\mathcal{W})$ and leaving $\mathcal{W}$ invariant — $\Phi$ is closable and its closure $\hat{\Phi}$ is a surjective isometry of the canonical form (26); i.e., a symmetry transformation. Hence, given a closed injective linear operator in $\mathcal{B}_1(\mathcal{H})_k$ whose domain contains $\text{span}_\mathbb{R}(\mathcal{W})$, one can check whether it is a symmetry transformation by verifying whether it preserves the witness $\mathcal{W}$.

**Example 2.** By the linear version of the theorems of Wigner and Kadison, the set of pure states $\mathcal{P}_1(\mathcal{H})$ and the convex set $\mathcal{F}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ of finite-rank density operators are symmetry witnesses.

We also want to introduce a notion of **projectable** symmetry witness. To this aim, let us consider the sets

$$\mathcal{B}_1(\mathcal{H})^+ := \{ A \in \mathcal{B}_1(\mathcal{H}) : A \geq 0, \ A \neq 0 \}, \ \mathcal{B}_1(\mathcal{H})^- := -\mathcal{B}_1(\mathcal{H})^+,$$

(32)

$$\mathcal{B}_1(\mathcal{H})^+_k := \mathcal{B}_1(\mathcal{H})^+_1 \cup \mathcal{B}_1(\mathcal{H})^-_k \subset \mathcal{B}_1(\mathcal{H})_k,$$

(33)

and let $\varpi$ be the natural projection map of $\mathcal{B}_1(\mathcal{H})^+_k$ onto $\mathcal{S}(\mathcal{H})$:

$$\varpi : \mathcal{B}_1(\mathcal{H})^+_k \ni A \mapsto \text{tr}(|A|)^{-1}|A| \in \mathcal{S}(\mathcal{H}).$$

(34)

We will say that a symmetry witness (candidate) $\mathcal{W}$ is positive, negative or signed if it is contained in $\mathcal{B}_1(\mathcal{H})^+_k$, $\mathcal{B}_1(\mathcal{H})^-_k$ or $\mathcal{B}_1(\mathcal{H})^+_k$, respectively.

If $\mathcal{W}$ is a signed symmetry witness candidate, it is easy to check that $\hat{\mathcal{W}} := \varpi(\mathcal{W}) \subset \mathcal{S}(\mathcal{H})$ is a symmetry witness candidate too, and this fact allows us to formulate the following definition.

**Definition 2** (Projectable symmetry witnesses). We call a signed symmetry witness **projectable** if its image through the map $\varpi$ is a symmetry witness too.

**Example 3.** The set $\mathcal{P}_k(\mathcal{H}) \subset \mathcal{B}_1(\mathcal{H})_k^+$, $k < \dim(\mathcal{H})$, and its projection $\mathcal{I}_k(\mathcal{H})_u := \varpi(\mathcal{P}_k(\mathcal{H}))$ are symmetry witness candidates. Indeed, they are invariant with respect to every unitary or antiunitary transformation; moreover, $\text{span}_\mathbb{R}(\mathcal{P}_k(\mathcal{H})) = \text{span}_\mathbb{R}(\mathcal{I}_k(\mathcal{H})_u) = \mathcal{S}(\mathcal{H})_k$, because every rank-one projection in $\mathcal{H}$ can be written as a linear combination $k + 1$ elements of $\mathcal{P}_k(\mathcal{H})$. The set $\mathcal{I}_k(\mathcal{H})_u$ consists of all uniform density operators of rank $k$; namely, of all mixed states whose eigenvalues form a uniform (constant) probability distribution. It is easy to see that if $\mathcal{P}_k(\mathcal{H})$ is a symmetry witness, then it is projectable; i.e., $\mathcal{I}_k(\mathcal{H})_u$ is a symmetry witness too. By Theorem 9, if $n = \dim(\mathcal{H}) < \infty$ is a prime number, then $\mathcal{P}_k(\mathcal{H})$, $k < n$, is a symmetry witness.
6. The sets of projections of a fixed rank as symmetry witnesses

As noted in the previous section, a natural linear preserver problem consists in determining the class of all injective linear operators — densely defined on \( \text{span}_\mathbb{R}(\mathcal{P}_k(\mathcal{H})) = \mathcal{F}(\mathcal{H})_s \), for some \( k < \dim(\mathcal{H}) \) — that leave the symmetry witness candidate \( \mathcal{P}_k(\mathcal{H}) \) invariant.

An almost complete solution of this problem is provided by the following result [9].

**Theorem 11.** Let \( \Phi \) be a densely defined linear operator in \( \mathcal{B}_1(\mathcal{H})_s \), with \( \text{dom}(\Phi) = \mathcal{F}(\mathcal{H})_s \).

Suppose that, given some \( k \in \mathbb{N} \) — with \( k = 1 \) if \( \dim(\mathcal{H}) = 2 \); with \( k < n/2 \) or \( n/2 < k < n \), if \( 3 \leq n = \dim(\mathcal{H}) < \infty \) — the following hypotheses are verified:

(i) \( \Phi(\mathcal{P}_k(\mathcal{H})) = \mathcal{P}_k(\mathcal{H}) \);
(ii) \( \Phi \) is injective.

(If \( \dim(\mathcal{H}) < \infty \) — or \( \dim(\mathcal{H}) = \infty \) and \( k = 1 \) — the second hypothesis is superfluous, because it follows from the first.)

Then, the operator \( \Phi \) is closable, and its closure \( \hat{\Phi} \) is a surjective isometry of the canonical form

\[
\hat{\Phi}(A) = UAU^*, \quad \forall A \in \mathcal{B}_1(\mathcal{H})_s, \tag{35}
\]

where \( U \) is a unitary or antiunitary operator in \( \mathcal{H} \), uniquely determined up to multiplication by a phase factor.

**Remark 9.** Theorem 11 can be reformulated in the complex Banach space \( \mathcal{B}_1(\mathcal{H}) \) of all trace class operators. Let \( \Phi_C \) be a linear operator in \( \mathcal{B}_1(\mathcal{H}) \), with \( \text{dom}(\Phi_C) = \text{span}_C(\mathcal{P}_k(\mathcal{H})) = \mathcal{F}(\mathcal{H}) \) (the linear manifold in \( \mathcal{B}_1(\mathcal{H}) \) of finite rank operators), and — for some \( k \in \mathbb{N} \) satisfying the constraints specified in the statement of Theorem 11 — let \( \Phi_C \) map \( \mathcal{P}_k(\mathcal{H}) \) onto itself. If \( \mathcal{H} \) is infinite-dimensional and \( k > 1 \), let us further assume that \( \Phi_C \) be injective. Then, \( \Phi_C \) is closable, and its closure \( \hat{\Phi}_C \) is a surjective isometry in \( \mathcal{B}_1(\mathcal{H}) \) which is either of the form

\[
\hat{\Phi}_C(A) = UAU^*, \quad \forall A \in \mathcal{B}_1(\mathcal{H}), \tag{36}
\]

for some unitary operator \( U \), or of the form

\[
\hat{\Phi}_C(A) = UAU^*, \quad \forall A \in \mathcal{B}_1(\mathcal{H}), \tag{37}
\]

for some antiunitary operator \( U \). As observed in Remark 1, we may consider, equivalently, the transformations generated by unitary operators only, possibly composed with a transposition associated with some orthonormal basis in \( \mathcal{H} \).

It is worth stressing that — in the case where \( 3 \leq n = \dim(\mathcal{H}) < \infty \) — the constraints set on \( k \) in Theorem 11 bring the following consequence: for \( n \) odd, every value of \( k < n \) gives rise to a symmetry witness, whereas, for \( n \geq 4 \) and \( n \) even, the value \( k = n/2 \) is not contemplated. As anticipated in sect. 3, this point is related to the exception \( 2k = \dim(\mathcal{H}) \neq 2 \) in Šemrl’s theorem (Theorem 4); see the proof [9] of Theorem 11 (which corresponds to Theorem 6 of [9]). The fact that the case where \( 1 < k = n/2 \) is indeed an exception is clarified by the following example.

**Example 4.** We will now provide a simple counterexample which shows that the statements of Šemrl’s theorem and of Theorem 11 actually do not hold in the aforementioned exceptional case. In fact, assuming that \( n = \dim(\mathcal{H}) = 2k \), \( k \in \mathbb{N} \), let us consider the linear map

\[
A^{(k)}: \mathcal{B}(\mathcal{H})_s \equiv \mathcal{B}_1(\mathcal{H})_s \to \mathcal{B}(\mathcal{H})_s, \quad A^{(k)}(A) := k^{-1} \text{tr}(A) I - A. \tag{38}
\]

Note that \( A^{(k)} \) maps \( \mathcal{P}_k(\mathcal{H}) \) bijectively onto itself: \( A^{(k)}(P) = I - P = P^\perp \), for every \( P \in \mathcal{P}_k(\mathcal{H}) \), where \( P^\perp \in \mathcal{P}_k(\mathcal{H}) \) is the projection onto the orthogonal complement of the subspace \( \text{ran}(P) \). Namely, \( A^{(k)} \) acts in \( \mathcal{P}_k(\mathcal{H}) \) as the orthocomplementation map; thus, restricted to this set, it
preserves orthogonality in both directions. Moreover, $\Lambda^{[i]}$ is injective, trace-preserving and unital; but, for $k > 1$, not positive, since it maps every pure state to a non-positive operator of rank $n - 1 = 2k - 1 \neq 1$. Therefore, if $k > 1$, $\Lambda^{[i]}$ cannot act as a symmetry transformation (not even on $\mathcal{P}_k(\mathcal{H})$, because span$_\mathbb{R}(\mathcal{P}_k(\mathcal{H})) = \mathcal{B}(\mathcal{H})_s$). For $k = 1$ ($\dim(\mathcal{H}) = 2$), we get the well known reduction map $\Lambda^{[i]}$, which, represented in terms of $2 \times 2$ hermitian matrices associated with an orthonormal basis, is the form

$$M \mapsto \sigma_2 M^T \sigma_2;$$

(39)

thus, a symmetry transformation ($\sigma_2$ is hermitian and unitary), coherently with Theorem 11.

It is now evident that the somewhat unnatural assumption — in Theorem 9 — that $n = \dim(\mathcal{H}) < \infty$ be a prime number is actually unnecessary; that result holds just because every prime number is odd. On the other hand, excluding all the even values of $n$ one cannot catch the the significant exception $2k = \dim(\mathcal{H}) \neq 2$ versus the case where $2k = \dim(\mathcal{H}) = 2$ (the linear version of Wigner’s theorem in dimension two, an ‘exception within the exception’).

Thus, the second of the two issues raised at the end of sect. 4 is clarified.

Theorem 11, complemented by the previous example, leads us to the following conclusion.

**Corollary 2.** The set $\mathcal{P}_k(\mathcal{H})$, of projections of a finite fixed rank $k$ — with $k = 1$, if $\dim(\mathcal{H}) = 2$; with $k < n/2$ or $n/2 < k < n$, if $3 \leq n = \dim(\mathcal{H}) < \infty$; arbitrary, if $\dim(\mathcal{H}) = \infty$ — is a symmetry witness. In the case where $4 \leq n = \dim(\mathcal{H}) < \infty$ and $n$ is even, a linear operator in $\mathcal{B}(\mathcal{H})_s$, mapping $\mathcal{P}_2(\mathcal{H})$ onto itself, is a trace-preserving and unital bijection, but, in general, not a symmetry transformation; thus, $\mathcal{P}_2(\mathcal{H})$ is not a symmetry witness.

**Proof.** The only assertion to be proven is that, for $4 \leq n = \dim(\mathcal{H}) < \infty$, with $n$ even, every linear operator in $\mathcal{B}(\mathcal{H})_s$ mapping $\mathcal{P}_2(\mathcal{H})$ onto itself is a trace-preserving and unital bijection. See the proof of Corollary 1 in [9].

A this point, we are able to show that Corollary 1 admits — in the finite-dimensional case — the following refinement, so clarifying also the first of the two issues raised at the end of sect. 4.

**Corollary 3.** Suppose that $\dim(\mathcal{H}) < \infty$, and let $\Phi: \mathcal{B}(\mathcal{H})_s \to \mathcal{B}(\mathcal{H})_s$ be a surjective linear map. Then, the following facts are equivalent:

(i) $\Phi$ is isometric (wrt the operator norm) and positive;

(ii) $\Phi$ is isometric and unital;

(iii) for some $k < \dim(\mathcal{H})$, $\Phi$ maps $\mathcal{P}_k(\mathcal{H})$ onto itself and, in the case where $2 < 2k = \dim(\mathcal{H})$, is also positive;

(iv) $\Phi$ is a symmetry transformation.

**Proof.** Taking into account Corollary 1 and Theorem 11, the only point that we still need to analyze is the case where $2k = \dim(\mathcal{H})$, $k > 1$, with $\Phi$ further assumed to be positive. By Corollary 2, we know that with the given assumptions $\Phi$ is unital, as well. Hence, by Corollary 1 the proof is complete.

Finally, we observe that the previous results can be rephrased in terms of uniform density operators, that are the physically relevant objects.

**Corollary 4.** Let $\Phi$ be a densely defined linear operator in $\mathcal{B}_1(\mathcal{H})_s$, with $\text{dom}(\Phi) = \mathcal{F}(\mathcal{H})_s$, mapping the set $\mathcal{A}_k(\mathcal{H})_u = \varpi(\mathcal{P}_k(\mathcal{H}))$, of uniform density operators of a fixed rank $k < \dim(\mathcal{H})$, onto itself. Assume, moreover, that

- in the case where $\dim(\mathcal{H}) = \infty$ and $k > 1$, $\Phi$ is injective;
- in the case where $2 < 2k = \dim(\mathcal{H}) < \infty$, $\Phi$ is positive.

Then, $\Phi$ is closable, and its closure $\hat{\Phi}$ is a symmetry transformation. If $k < \dim(\mathcal{H})$ and — for $\dim(\mathcal{H}) \geq 4 - 2k \neq \dim(\mathcal{H})$, $\mathcal{A}_k(\mathcal{H})_u$ is a symmetry witness; otherwise, it is not.
7. Conclusions

We can finally draw the following conclusions:

- A symmetry witness is a set \( \mathcal{W} \) in the Banach space \( \mathcal{B}(\mathcal{H})_s \) of selfadjoint trace class operators, that allows one to ascertain whether an injective linear operator \( \Phi \) acting in \( \mathcal{B}(\mathcal{H})_s \) is a symmetry transformation, by checking whether it leaves the witness invariant, i.e., whether \( \Phi(\mathcal{W}) = \mathcal{W} \); see Definition 1. Therefore, this notion is related to a suitable class of linear preserver problems; see Remark 7.

- By a linear version of Wigner’s theorem — Theorem 7 — one can regard the set of pure states \( \mathcal{P}_1(\mathcal{H}) \) as the prototype of a symmetry witness. Further remarkable examples are the convex set \( \mathcal{I}(\mathcal{H}) \) of all states (see Remark 1 of [9]) and the convex set \( \mathcal{F}\mathcal{I}(\mathcal{H}) \subset \mathcal{I}(\mathcal{H}) \) of finite-rank density operators (Theorem 8).

- By Theorem 11, the set \( \mathcal{P}_k(\mathcal{H}), k < \text{dim}(\mathcal{H}) \), is a symmetry witness too, with the peculiar exception — when \( 4 \leq n = \text{dim}(\mathcal{H}) < \infty \), \( n \) even — of the value \( k = n/2 \): a linear operator \( \Phi \) in \( \mathcal{B}(\mathcal{H})_s \equiv \mathcal{B}_k(\mathcal{H})_s \), mapping \( \mathcal{P}_2(\mathcal{H}) \) onto itself, is a trace-preserving and unital bijection, but, in general, not a symmetry transformation (Corollary 2). These symmetry witnesses are positive, and projectable (Definition 2) onto the symmetry witnesses \( \mathcal{I}_k(\mathcal{H})_u \) that have a clear physical interpretation: they consist of those density operators of a fixed rank whose eigenvalues form a uniform probability distribution.

- It is worth stressing that the set \( \mathcal{I}_k(\mathcal{H})_u \) has a precise role with regard to the majorization relation in \( \mathcal{I}(\mathcal{H}) \) [29,30]; i.e., it is formed by those density operators that are minimal, with respect to majorization, among all elements of \( \mathcal{I}(\mathcal{H}) \) with rank not larger than \( k \). Therefore, a (suitably defined) class of quantum entropies — e.g., the von Neumann, the Rényi or the Tsallis entropies — if restricted to the subset of \( \mathcal{I}(\mathcal{H}) \) of all density operators of rank not larger than \( k \), attain their maximum value precisely on \( \mathcal{I}_k(\mathcal{H})_u \) [29,30]. Moreover, considering in particular the finite-dimensional case, we observe that a quantum dynamical semigroup [19,21,24–30] that leaves the set \( \mathcal{I}_k(\mathcal{H})_u \) invariant, for some \( k < \text{dim}(\mathcal{H}) \), by Corollary 4 must be a one-parameter (semi-)group of unitary transformations; thus, it must actually describe the dynamics of a closed quantum system.

- The results outlined above solve a problem considered in [13]: to find a generalization of the linear version of Wigner’s theorem involving projections of a fixed rank. This problem has been first partially solved in [13], in the finite-dimensional case and with the (unnecessary) assumption that \( \text{dim}(\mathcal{H}) \) be a prime number (Theorem 9). A similar problem has been studied in [31] — see Theorem 10 — with no restriction on the Hilbert space dimension and working in a different setting: the sets of projections with a fixed rank are embedded in the Banach space of bounded operators (the dual of the Banach space of trace class operators); moreover, the linear maps acting on bounded operators are supposed to be positive and unital. A direct comparison with the results of [13], and with those discussed in sect. 6, is possible in the finite-dimensional case only, where the distinction between the Schrödinger and the Heisenberg pictures is not sharp. In this case, both positivity and unitality can be dispensed with: the single exception is the case of \( \mathcal{P}_2(\mathcal{H}) \), with \( n = \text{dim}(\mathcal{H}) \geq 4 \) and even. For these values of \( n \), unitality is automatically satisfied by a linear map mapping \( \mathcal{P}_2(\mathcal{H}) \) onto itself, whereas positivity implies that the map is a symmetry transformation.

- As explained in Remark 9, working in the real Banach space \( \mathcal{B}_k(\mathcal{H})_s \) (alternatively, in \( \mathcal{B}(\mathcal{H})_s \)) or in the complex Banach space \( \mathcal{B}_k(\mathcal{H}) \) (respectively, in \( \mathcal{B}(\mathcal{H}) \)) is unessential.

- At the beginning of sect. 6, we have observed that Theorem 11 provides an almost complete solution of the linear preserver problem associated with the symmetry witness candidate \( \mathcal{P}_k(\mathcal{H}), k < \text{dim}(\mathcal{H}) \). In the exceptional case where \( n = \text{dim}(\mathcal{H}) = 2k \), with \( k \geq 2 \) — according to an interesting conjecture [13] — every linear map in \( \mathcal{B}(\mathcal{H})_s \) mapping \( \mathcal{P}_k(\mathcal{H}) \)
onto itself would be either a symmetry transformation or a (trace-preserving, unital but non-positive map of the form $\Lambda^k: \mathcal{B}(\mathcal{H}) \ni A \mapsto \Lambda^k(UAU^*) \in \mathcal{B}(\mathcal{H})$, where $U$ is a unitary or antiunitary operator in $\mathcal{H}$ and $\Lambda^k$ is the map defined by (38). Proving this conjecture would amount to finding the single missing piece of the puzzle.

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