On the equatorial motion of the charged test particles in Kerr-Newman-Taub-NUT spacetime and the existence of circular orbits

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Abstract
In this work, we perform a detailed analysis of the equatorial motion of the charged test particles in Kerr-Newman-Taub-NUT spacetime. By working out the orbit equation in the radial direction, we investigate possible orbit types. Concentrating particularly on the circular orbits, we discuss and determine the conditions for the existence of equatorial circular orbits. We study their stability as well. Next, we provide some sample plots of possible orbit configurations and give a detailed discussion of the orbit types.

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1 Introduction

A remarkable solution to Einstein-Maxwell field equations is known as the Kerr-Newman-Taub-NUT spacetime which describes a rotating electrically charged source equipped with a gravitomagnetic monopole moment (also identified as the NUT charge) [1, 2]. The spacetime includes four physical parameters; the gravitational mass, which is also called gravitoelectric charge; the gravitomagnetic mass (the NUT charge); the rotation parameter that is the angular speed per unit mass and the electric charge associated with the Maxwell field. In contrast to Kerr spacetime that is asymptotically flat, the rotating versions of spacetimes with gravitomagnetic monopole moment (Kerr-Taub-NUT and Kerr-Newman-Taub-NUT spacetimes) are asymptotically non-flat due to existence of the NUT charge. Although the Kerr-Taub-NUT and the Kerr-Newman-Taub-NUT spacetimes involve no curvature singularities, there exist conical singularities on the axis of symmetry. As discussed in [3], one can get rid of conical singularities by imposing a periodicity condition over the time coordinate. However, this inevitably leads to the emergence of closed timelike curves in the spacetime that makes it unphysical in the context of causality. To make the spacetime with the NUT charge physically relevant, one can investigate the global analysis of such spacetimes as in [4] and [5]. In this manner, an alternative physical interpretation of the spacetime with NUT charge has been given in [4] where the NUT metric is interpreted as a semi-infinite massless source of angular momentum (involving the singularity on the axis where $\theta = \pi$). Despite some unpleasing physical features of Taub-NUT spacetimes, the physical meaning of gravitomagnetic monopole moment and the physical interactions including NUT parameter have been comprehensively investigated. In [6] and [7], the physical meaning of NUT parameter has been exploited by examining the twisting effect of monopole moment on the orbit of the light rays. In [8] and [9], interaction of the massless scalar fields with gravitomagnetic monopole moment and gravitomagnetic effects regarding the NUT parameter have been investigated respectively. In [10], [11] and [12], some physical applications have been illustrated in the background of spacetimes involving the NUT parameter; namely in [10], gyromagnetic ratio related to Kerr-Newman Taub-NUT spacetime has been obtained, in [11], hidden symmetries of Kerr-Taub-NUT spacetime in Kaluza-Klein theory have been explored, while in [12], acceleration of particles on the background of Kerr-Taub-NUT spacetime has been studied.

In order to detect the existence of a NUT source in the universe, one can either investigate the effect of this parameter on the motion of light
or examine the effect on the motion of massive test particles. To accomplish and exploit such effect, one can study geodesics or the orbital motion on the background of spacetimes involving the NUT parameter. Such a study was initiated by [13] where the Schwarzschild type geodesics on the background of NUT spacetime has been examined by concluding that such spacetimes lie on the cones with the apex located at \( r = 0 \). Later in [14], a comprehensive analytic investigation of complete and incomplete geodesics in a (non-rotating) Taub-NUT spacetime has been realised. On the other hand, the motion of particles in a Kerr-Taub-NUT gravitational source immersed in a magnetic field has been investigated in [15]. In addition, in [16], energy extraction process (Penrose process) has been examined in rotating Kerr-Taub-NUT spacetime. In a recent work [17], an analytic expression for shadows (known as a special lensing property) of a Kerr-Newman-NUT spacetime has been obtained while examining the motion of photon in this background. Very recently in [18], timelike circular geodesics in (non-rotating) NUT spacetime has been studied while also discussing the Von Zeipel cylinders with respect to stationary observers and determining the relation of such cylinders to inertial forces.

This work is devoted to the study of equatorial orbits of charged massive test particle in Kerr-Newman-Taub-NUT spacetime. In previous works, the motion of charged massive test particles has been analytically investigated in Reissner-Nordström [19, 20], Reissner-Nordström-(Anti)-de-Sitter [21], Kerr-Newman [22] and Kerr-Newman-AdS (in \( f(R) \) gravity) [23] respectively. Our aim is to investigate the effect of the rotation and the NUT parameter on the equatorial motion of the charged test particles while concentrating on the existence of equatorial circular orbits and their stability in this background as well. In fact, we have recently examined the non-equatorial orbital motion of charged massive test particles in Kerr-Newman-Taub-NUT background in [24], where we have briefly mentioned the conditions for the existence of the equatorial orbits in rotating NUT spacetime without making a comprehensive analysis and investigation of such orbits. We should remark that, the study of equatorial orbits in rotating NUT spacetime deserves a separate care and investigation. As is also mentioned in [9], the equatorial orbits in rotating NUT spacetimes do not exist for arbitrary rotation and the NUT parameters. In [24], we have shown that, there exist equatorial orbits in such spacetimes provided that either the NUT parameter should vanish or a certain relation between the angular momentum and the energy of the test particle and the rotation parameter should exist (for arbitrary NUT parameter). In this work, we concentrate on the latter, i.e. we examine the equatorial orbits of the charged test particles in which
such a relation holds. We determine the possible orbit types and examine the possible orbit configurations depending on the value of the energy of the test particle while a special investigation is devoted to the study of circular orbits and their stabilities. To our knowledge, the study of circular orbits in rotating spacetimes has been initiated by Bardeen et al. [25] for Kerr spacetime. Later, the investigation of the existence of circular geodesics has been accomplished in Kerr-Newman spacetime [26, 27]. A detailed investigation of such orbits has also been realized in Kerr-de Sitter spacetimes including cosmological constant together with mass and rotation parameters [28]. In our study, we derive necessary conditions for the existence of equatorial circular orbits in Kerr-Newman-Taub-NUT spacetime and examine their stabilities while providing numerical calculations for inner-most stable circular orbits (ISCO) as well. Finally, we present the analytical solutions of the equations of motion over the equatorial plane by expressing them in terms of Weierstrass \( \wp \), \( \sigma \), and \( \zeta \) functions. We also provide plots of possible orbit types and calculate the perihelion shift for an equatorial bound orbit.

Organisation of the paper is as follows: In section 2, we provide an introduction to Kerr-Newman-Taub-NUT spacetime. In section 3, we obtain the governing equations of equatorial motion of the charged test particles. In section 4, we make a comprehensive analysis of possible orbit types. In section 5, we present a detailed investigation of equatorial circular orbits while discussing the stability of such orbits. In section 6, we provide a complete discussion of possible orbit configurations over the equatorial plane while also giving some sample plots for the possible orbit types. In the same section, we further examine the conditions for the existence of bound orbit in the region where the radial distance lies outside the outer singularity of the spacetime. Moreover, the Newtonian limit of the equatorial orbits are discussed while investigating the physical effect of the NUT parameter on the Newtonian orbits as well. In Appendix, we present the analytic solutions of the equatorial orbits while also calculating the perihelion shift for an equatorial bound orbit. We end up with some comments and conclusions.

2 Kerr-Newman-Taub-NUT spacetime

The Kerr-Newman-Taub-NUT spacetime is known as a stationary rotating solution of the Einstein-Maxwell field equations. The metric is asymptotically non-flat due to the existence of a NUT charge which is also identified as the gravitomagnetic monopole moment. In Boyer-Lindquist coordinates, Kerr-Newman-Taub-NUT spacetime can be written as (with asymptotically
non-flat structure),
\[ g = -\frac{\Delta}{\Sigma}(dt - \chi d\varphi)^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2 \theta}{\Sigma} (adt - (r^2 + \ell^2 + a^2)d\varphi)^2 \]
where
\[ \Sigma = r^2 + (\ell + a \cos \theta)^2, \]
\[ \Delta = r^2 - 2Mr + a^2 - \ell^2 + Q^2, \]
\[ \chi = a \sin^2 \theta - 2\ell \cos \theta. \]

Here, \( M \) can be identified as the parameter related to the physical mass of the gravitational source, \( a \) is associated with its angular momentum per unit mass while \( \ell \) specifies gravitomagnetic monopole moment of the source which is also identified as the NUT charge. Also, \( Q \) is specified as the electric charge. The electromagnetic field of the source can be expressed as the potential 1-form
\[ A = A_\mu dx^\mu = -\frac{Qr}{\Sigma}(dt - \chi d\varphi). \]

We use geometrized units such that \( c = 1 \) and \( G = 1 \). As is also remarked in [5], there exist two types of singularities related to that spacetime; one coming from the singularities of the metric components and the other resulting from vanishing of the determinant of the metric. The former results in the singularities at the spacetime coordinates where \( \Delta(r) = 0 \) and \( \Sigma(r, \theta) = 0 \) producing singularities at the horizons \( r = r_\pm \) and a conical singularity (at \( r = 0 \) and \( \cos \theta = -\frac{\ell}{a} \) with \( \ell^2 < a^2 \)) respectively. The latter occurs at \( \theta = 0 \) and \( \theta = \pi \). When the NUT parameter \( \ell = 0 \), the conical singularity obviously turns into the equatorial ring singularity at \( r = 0 \) and \( \theta = \frac{\pi}{2} \). We further remark that, the metric singularities related to radial coordinate exist at the locations
\[ r_\pm = M \pm \sqrt{M^2 - a^2 + \ell^2 - Q^2}, \]
where \( \Delta(r_\pm) = 0 \) provided that \( M^2 - a^2 + \ell^2 - Q^2 \geq 0 \). The singularity \( r_- \) can be identified as inner (or Cauchy) horizon while \( r_+ \) can be named as outer (or event) horizon. It can also be seen that, the spacetime allows a family of locally non-rotating observers which rotate with coordinate angular velocity given by
\[ \Omega = \frac{g_{t\phi}}{g_{\phi\phi}} = \frac{\Delta \chi - a \sin^2 \theta(r^2 + \ell^2 + a^2)}{\Delta \chi^2 - \sin^2 \theta(r^2 + \ell^2 + a^2)^2} \]
which is known as the frame dragging effect arising due to the presence of the off-diagonal component $g_{t\phi}$ of the metric. We also remark that at the outermost singularity $r_+$, the angular velocity can be calculated as

$$\Omega_+ = -\left. \left( \frac{g_{t\phi}}{g_{\phi\phi}} \right) \right|_{r=r_+} = \frac{a}{r_+^2 + \ell^2 + a^2}. \quad (2.6)$$

It can also be easily seen that the Killing vectors $\xi(t)$ and $\xi(\phi)$ generate two constants of motion namely the energy and the angular momentum of the test particle. Moreover, it is straightforward to show that the Killing vector $\xi = \xi(t) + \Omega_+ \xi(\phi)$ becomes null at the metric singularity where $r = r_+$.

### 3 The equations of motion

In this section, we examine the motion of charged test particle over the equatorial plane in Kerr-Newman-Taub-NUT spacetime. Traditionally, to obtain the equation of motions over the equatorial plane, one usually starts with the Lagrangian expression \[29\] and simply substitute $\theta = \frac{\pi}{2}$ in the resulting expressions. However, this standard technique cannot be directly applied for the Kerr-Newman-Taub-NUT spacetime since as is explicitly illustrated in \[9\] and \[24\], the existence of the equatorial orbits requires either the vanishing of the NUT parameter ($\ell = 0$) or a specific relation between the rotation parameter $a$, the energy $\bar{E}$, the angular momentum $\bar{L}$ of the test particle to hold. For that reason, one cannot directly substitute $\theta = \frac{\pi}{2}$ for the equatorial orbital motion. Instead, to get the field equations over the equatorial plane, one should start with the celebrated Hamilton-Jacobi equation, and then obtain the condition for the existence of equatorial orbits and substitute such a relation in the remaining governing equations. Therefore, to obtain the governing orbit equations over the equatorial plane, we find it useful to review our discussion raised in \[24\] starting with the Hamilton-Jacobi equation

$$2 \frac{\partial S}{\partial \tau} = g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} - qA_\mu \right) \left( \frac{\partial S}{\partial x^\nu} - qA_\nu \right). \quad (3.1)$$

Here, $\tau$ is an affine parameter and $q$ is the charge of the test particle. The existence of the timelike Killing vector $\xi(t)$ and spacelike Killing vector $\xi(\phi)$ for the Kerr-Newman-Taub-NUT spacetime \[27\] and the separability of the Hamilton-Jacobi equation in this spacetime \[30, 31, 32\] enforces the solution of that equation to be written as

$$S = -\frac{1}{2} m^2 \tau - Et + L\varphi + S_r(r) + S_\theta(\theta) \quad (3.2)$$
where the constants of motion $m$, $E$ and $L$ denote the mass, the energy and the angular momentum of the particle respectively. Then, Hamilton-Jacobi equation (3.1) results in two independent differential equations

$$\left(\frac{dS_r}{dr}\right)^2 = \frac{1}{\Delta} \left\{ -K - m^2 r^2 + \frac{1}{\Delta} \left[ (r^2 + a^2 + \ell^2) E - aL - qQr \right]^2 \right\} \quad (3.3)$$

and

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = K - m^2 (\ell + a \cos \theta)^2 - \left( \frac{\chi E - L}{\sin \theta} \right)^2, \quad (3.4)$$

where $K$ denotes the Carter separability constant. The use of the expression for the canonical momenta $P_\mu$ such that

$$P_\mu = \frac{\partial S}{\partial x^\mu} = mg_{\mu\nu} \frac{dx^\nu}{d\tau} + qA_\mu \quad (3.5)$$

and the identifications

$$P_t = -E, \quad P_\varphi = L, \quad (3.6)$$

result in the following equations of motion:

$$\Sigma^2 \left(\frac{d}{d\tau} \frac{dr}{\Sigma} \right)^2 = P_r(r), \quad (3.7)$$

$$\Sigma^2 \left(\frac{d}{d\tau} \frac{d\theta}{\Sigma} \right)^2 = P_\theta(\theta), \quad (3.8)$$

$$\frac{dt}{d\tau} = \frac{1}{\Sigma \Delta \sin^2 \theta} \left[ \bar{L} \left( \Delta \chi - a \sin^2 \theta (r^2 + \ell^2 + a^2) \right) + \bar{E} \left( -\Delta \chi + a \sin^2 \theta (r^2 + \ell^2 + a^2) \right) - \bar{q} Q r \sin^2 \theta (r^2 + \ell^2 + a^2) \right], \quad (3.9)$$

$$\frac{d\varphi}{d\tau} = \frac{1}{\Sigma \Delta \sin^2 \theta} \left[ \bar{E} \left( -\Delta \chi + a \sin^2 \theta (r^2 + \ell^2 + a^2) \right) + \bar{L} \left( \Delta - a^2 \sin^2 \theta \right) - \bar{q} Q a r \sin^2 \theta \right]. \quad (3.10)$$

Here,

$$\bar{E} := \frac{E}{m}, \quad \bar{L} := \frac{L}{m}, \quad \bar{q} := \frac{q}{m}, \quad (3.11)$$

$$P_r(r) = \left[ \bar{E} (r^2 + \ell^2 + a^2) - a\bar{L} - \bar{q} Q r \right]^2 - \Delta \left( \frac{K}{m^2} + r^2 \right) \quad (3.12)$$
and
\[ P_\theta(\theta) = \frac{K}{m^2} - (\ell + a \cos \theta)^2 - \left( \frac{\chi \tilde{E} - \tilde{L}}{\sin \theta} \right)^2. \] (3.13)

To this end, one can introduce a new time parameter \( \lambda \) (the so-called Mino time) as in [33] such that
\[ \frac{d\lambda}{d\tau} = \frac{1}{\Sigma}, \] (3.14)
and express the equations of motion in terms of Mino time:
\[ \left( \frac{dr}{d\lambda} \right)^2 = P_r(r), \] (3.15)
\[ \left( \frac{d\theta}{d\lambda} \right)^2 = P_\theta(\theta), \] (3.16)
\[ \frac{dt}{d\lambda} = \frac{\chi(\tilde{L} - \tilde{E}\chi)}{\sin^2 \theta} + \frac{(r^2 + a^2 + \ell^2)}{\Delta} \left[ \tilde{E}(r^2 + a^2 + \ell^2) - a\tilde{L} - \bar{q}Qr \right], \] (3.17)
\[ \frac{d\varphi}{d\lambda} = \frac{(\tilde{L} - \tilde{E}\chi)}{\sin^2 \theta} + \frac{a}{\Delta} \left[ \tilde{E}(r^2 + a^2 + \ell^2) - a\tilde{L} - \bar{q}Qr \right]. \] (3.18)

It is clear that when the conditions
\[ P_\theta|_{\theta = \frac{\pi}{2}} = 0, \quad \frac{dP_\theta}{d\theta} \bigg|_{\theta = \frac{\pi}{2}} = 0 \] (3.19)
are simultaneously met, the particle is confined to move over the equatorial plane. These two equations imply that
\[ \frac{K}{m^2} - \ell^2 - (a\tilde{E} - \tilde{L})^2 = 0 \] (3.20)
and
\[ \ell \left( a(1 - 2\tilde{E}^2) + 2\tilde{E}\tilde{L} \right) = 0. \] (3.21)
Then, the equation (3.21) further implies either \( \ell = 0 \) or the relation
\[ \tilde{L} = \frac{a(2\tilde{E}^2 - 1)}{2\tilde{E}}, \quad (\tilde{E} \neq 0) \] (3.22)
should be imposed between the rotation spacetime parameter \( a \), the energy \( \tilde{E} \) and the angular momentum \( \tilde{L} \) of the test particle. From expression (3.22) and the equations of motion (3.15)-(3.18), one can clearly see that, if \( a = 0 \) or \( \tilde{L} = 0 \) there exists no equatorial motion of the test particle in azimuthal \( \varphi \)-direction. Instead, there is only equatorial radial motion of the particle.
On the other hand, if \( a \neq 0 \), but \( 2E^2 = 1 \) (\( \bar{L} = 0 \)), the orbits of the charged test particle can be identified as the motion with vanishing orbital angular momentum. For this case, there exists equatorial motion in both radial and azimuthal directions. One more crucial remark is that, if \( 2E^2 > 1 \), \( \bar{L} > 0 \) implying that equatorial orbits are direct orbits (\( \bar{L} \) and \( a \) have the same sign also assuming that \( a > 0 \)). If on the other hand, \( 2E^2 < 1 \), \( \bar{L} < 0 \) which implies that the orbits are retrograde orbits (\( \bar{L} \) and \( a \) have opposite signs).

Now, we concentrate on the case where \( \ell \neq 0 \) and \( a \neq 0 \) with the Carter constant \( K \) becoming

\[
\frac{K}{m^2} = \ell^2 + \frac{a^2}{4E^2}. \quad (3.23)
\]

Then, if one substitutes (3.22) and (3.23) in the expressions (3.15), (3.17) and (3.18) (with \( \theta = \frac{\pi}{2} \)), one can obtain the governing equations that describe the motion of the test particle over the equatorial plane as

\[
\left( \frac{dr}{d\lambda} \right)^2 = \bar{P}_r(r), \quad (3.24)
\]

\[
\left( \frac{dr}{d\varphi} \right)^2 = \frac{\Delta^2(r)\bar{P}_r(r)}{a^2 \left( [\bar{E}(r^2 + \ell^2) + \frac{a^2}{2E} - \bar{q}Qr] - \frac{\Delta(r)}{2E} \right)^2}, \quad (3.25)
\]

\[
\left( \frac{dr}{dt} \right)^2 = \frac{\Delta^2(r)\bar{P}_r(r)}{(r^2 + a^2 + \ell^2) \left[ \bar{E}(r^2 + \ell^2) + \frac{a^2}{2E} - \bar{q}Qr \right] - \frac{a^2\Delta(r)}{2E} \right)^2, \quad (3.26)
\]

where we define

\[
\bar{P}_r(r) = \left[ \bar{E}(r^2 + \ell^2) + \frac{a^2}{2E} - \bar{q}Qr \right]^2 - \Delta(r) \left( r^2 + \ell^2 + \frac{a^2}{4E^2} \right), \quad (3.27)
\]

Finally, we should point out that when \( \ell = 0 \) (vanishing of the NUT parameter), these equations do not reduce to equations of motion that represent the orbits in Kerr-Newman spacetime since the existence of equatorial orbits in Kerr-Newman-Taub-NUT spacetime requires the condition (3.22) for \( \ell \neq 0 \).

4 Analysis of the orbits over the equatorial plane

One can easily see from (3.27) that, \( \bar{P}_r(r) \) is a fourth order polynomial in \( r \) with real coefficients. Furthermore, the radial motion is possible if \( \bar{P}_r(r) \geq 0 \). Then according to the roots of the polynomial \( \bar{P}_r(r) \), one can identify the
following orbit types in general [34]:

i. Bound orbit: When the particle moves in a region \( r_2 < r < r_1 \) (with \( r_1 \) and \( r_2 \) finite), the motion of the particle can be identified as bound. This is possible if \( \bar{P}_r(r) \) has four real roots or two real roots (with two complex roots) or two real double roots or one real triple root and one real root. In such cases, there may exist one or two bound regions. As a special case, the orbit is called circular if \( \bar{P}_r(r) \) has a real double root at \( r = r_c \) where \( r_c \) denotes the radius of the circular orbit.

ii. Flyby orbit: The orbit is called flyby if the particle starts from \( \mp \infty \) and comes to a point \( r = r_1 \) and goes back to infinity. Likewise, flyby orbits can arise when \( \bar{P}_r(r) \) has four real roots or two real roots (with two complex roots) or two real double roots or one real triple root and one real root. Similarly, for this case, there may exist one or two flyby orbits.

iii. Transit orbit: The orbit is said to be transit if the particle starts from \( \mp \infty \), crosses \( r = 0 \) and moves to \( \pm \infty \). This can be possible if \( \bar{P}_r(r) \) has no real roots.

One can further examine the possible orbit configurations depending on the value of the energy of the test particle:

1. The case for \( \bar{E} \neq 1 \):

If \( \bar{P}_r(r) \) has four different real roots, one can obtain two bound orbits for \( \bar{E} < 1 \), while for \( \bar{E} > 1 \) one can get one bound and two flyby orbits. If \( \bar{P}_r(r) \) has two different real zeros (and two complex conjugate roots), then one can get only one bound orbit for \( \bar{E} < 1 \), while for \( \bar{E} > 1 \), one can obtain two flyby orbits. If \( \bar{P}_r(r) \) has no real zeros, then the radial motion is not possible for \( \bar{E} < 1 \) since \( \bar{P}_r(r) < 0 \) for all \( r \), while one can get a transit orbit for \( \bar{E} > 1 \) since \( r \to \mp \infty, \bar{P}_r(r) \to \infty \). The orbit configurations can further be examined associated with the form of the radial potential. If \( \bar{P}_r(r) \) has the form

\[
\bar{P}_r(r) = (r - r_c)^2 \left( (\bar{E}^2 - 1)r^2 + \mu_1 r + \mu_2 \right) = (r - r_c)^2 b(r)
\]

where

\[
\mu_1 = 2 \left( (\bar{E}^2 - 1)r_c + M - \bar{q}Q\bar{E} \right), \tag{4.2}
\]

\[
\mu_2 = 3(\bar{E}^2 - 1)r_c^2 + 4\left( M - \bar{q}Q\bar{E} \right) r_c + 2\bar{E}^2 \ell^2 + Q^2(q^2 - 1) - \frac{a^2}{4\bar{E}^2} \tag{4.3}
\]
(i.e. $\bar{P}_r(r)$ has one double zero at $r = r_c$), it is obvious that a circular orbit exists at $r = r_c$. It is clear that such a form of $\bar{P}_r(r)$ is possible if the conditions

$$\left. \bar{P}_r(r) \right|_{r=r_c} = 0, \quad \left. \frac{d\bar{P}_r(r)}{dr} \right|_{r=r_c} = 0 \quad (4.4)$$

hold. We note that, these conditions imply the existence of circular orbits. Furthermore, provided that these two conditions hold, possible orbit configurations differ according to whether function $b(r)$ has two real zeros, one double zero or no real zeros. If $b(r)$ has two real zeros (i.e. $\mu_2^2 - 4(\bar{E}^2 - 1)\mu_2 > 0$), for $\bar{E} > 1$ one can obtain either one bound and two flyby or only two flyby orbits, while for $\bar{E} < 1$, one can get either one bound or two bound orbits.

In any case, there exists a circular orbit at $r = r_c$. If $b(r)$ has a double real zero (i.e. $\mu_2^2 - 4(\bar{E}^2 - 1)\mu_2 = 0$), for $\bar{E} > 1$ one bound and two flyby orbits can be observed while for $\bar{E} < 1$ there will be no radial motion. Meanwhile, in addition to circular orbit at $r = r_c$, there exists another circular orbit at a point with the condition $\mu_1^2 - 4(\bar{E}^2 - 1)\mu_2 = 0$ satisfied. If $b(r)$ has no real zeros (i.e. $\mu_2^2 - 4(\bar{E}^2 - 1)\mu_2 < 0$), for $\bar{E} > 1$ there exist two flyby orbits while for $\bar{E} < 1$ there will be no physical motion.

One can further analyse the orbit configurations for the case where $\bar{P}_r(r)$ has real triple zero i.e. $\bar{P}_r(r)$ has the form

$$\bar{P}_r(r) = (r - r_c)^3 ((\bar{E}^2 - 1)r + r_0) \quad (4.5)$$

where

$$r_0 = 3(\bar{E}^2 - 1)r_c + 2(M - \bar{q}Q\bar{E}) \quad (4.6)$$

(i.e. $\bar{P}_r(r)$ has a triple zero at $r = r_c$ where a circular orbit exists at that point). Orbit configurations can modify according to value of the energy $\bar{E}$ of the test particle. Then, for $\bar{E} > 1$, one can get two flyby orbits while for $\bar{E} < 1$, one can obtain only one bound orbit. We should also note that such a form of $\bar{P}_r(r)$ is possible if the conditions (4.4) are satisfied in addition to the condition

$$\left. \frac{d^2\bar{P}_r(r)}{dr^2} \right|_{r=r_c} = 0. \quad (4.7)$$

Moreover, the additional condition (4.7) can be considered as the condition that determines the transitions from stable to unstable (or vice-versa) circular orbits.

2. The case for $\bar{E} = 1$:
For $E = 1$, $\bar{P}_r(r)$ becomes a third order polynomial. Moreover, the orbit configurations can change according to the sign of the coefficient of the first term (i.e. the coefficient of $r^3$). For both of the cases $M > \bar{q}Q$ and $M < \bar{q}Q$, if $\bar{P}_r(r)$ has three distinct real roots, there exist one bound and one flyby orbits. If on the other hand $\bar{P}_r(r)$ has only one real root (together with two complex roots), then there exists only one flyby orbit. For this case, for $M > \bar{q}Q$, flyby orbit is observed in the interval $r_1 \leq r < +\infty$ while for $M < \bar{q}Q$, the flyby orbit can be realised in the interval $-\infty < r \leq r_1$ where $r_1$ is the real root of $\bar{P}_r(r)$. In addition, if $\bar{P}_r(r)$ is of the form

$$\bar{P}_r(r) = (r - r_c)^2(2(M - \bar{q}Q)r + \bar{r}_0) \quad (4.8)$$

where

$$\bar{r}_0 = 4(M - \bar{q}Q)r_c + 2\ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4}, \quad (4.9)$$

there certainly exists a circular orbit at $r = r_c$. Also, for such a form of $\bar{P}_r(r)$, one can observe either one bound and one flyby orbits or only one flyby orbit for both of the cases $M > \bar{q}Q$ and $M < \bar{q}Q$.

It is also of interest to examine the case where $E = 1$ and $M = \bar{q}Q$. For this special case, $\bar{P}_r(r)$ becomes a second order polynomial:

$$\left(2\ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4}\right)r^2 - \frac{a^2}{2}\bar{q}Qr + \ell^2 + (\ell^2 - Q^2)\left(\ell^2 + \frac{a^2}{4}\right) = 0. \quad (4.10)$$

In this case, the orbit configurations can modify according to the sign of the coefficient of $r^2$ term. Then, if

$$2\ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4} > 0, \quad (4.11)$$

one can obtain two flyby or a transit orbit according to whether $\bar{P}_r(r)$ has two different real zeros (or one double zero) or no zeros respectively. If on the other hand,

$$2\ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4} < 0, \quad (4.12)$$

one can get one bound orbit or no radial motion according to whether $\bar{P}_r(r)$ has two different real zeros or no zeros respectively. In any case, there exists a circular orbit at $r = r_c$ if $\bar{P}_r(r)$ has a double zero at that point. Some examples to orbit configurations analysed in this section are illustrated in discussion part while providing sample plots for possible orbit types as well.
5 Equatorial circular orbits and stability

In this section, we investigate the existence of circular orbits and their stabilities. It is clear that, when the conditions

$$\bar{P}_r(r_c) = 0, \quad \left. \frac{d\bar{P}_r}{dr} \right|_{r=r_c} = 0$$

are satisfied, one gets a circular orbital motion over the equatorial plane. These two conditions require that

$$3(\bar{E}^2 - 1)r_c^4 + 2(M - \bar{q}Q\bar{E})r_c^3 + \left(2\bar{E}^2\ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4E^2}\right)r_c^2$$

$$+ \left(2M\left(\ell^2 + \frac{a^2}{4E^2}\right) - \bar{q}Q\left(2\bar{E}\ell^2 + \frac{a^2}{E}\right)\right)r_c$$

$$+ \bar{E}^2\ell^4 + (\ell^2 - Q^2)\left(\ell^2 + \frac{a^2}{4E^2}\right) = 0,$$  \hspace{1cm} (5.2)

are satisfied, one gets a circular orbital motion over the equatorial plane. These two conditions require that

$$4(\bar{E}^2 - 1)r_c^4 + 6(M - \bar{q}Q\bar{E})r_c^3 + 2\left(2\bar{E}^2\ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4E^2}\right)r_c$$

$$+ 2M\left(\ell^2 + \frac{a^2}{4E^2}\right) - \bar{q}Q\left(2\bar{E}\ell^2 + \frac{a^2}{E}\right) = 0.$$  \hspace{1cm} (5.3)

We notice that the former (i.e. equation (5.2)) can also be written in the equivalent form

$$3(\bar{E}^2 - 1)r_c^4 + 4(M - \bar{q}Q\bar{E})r_c^3 + \left(2\bar{E}^2\ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4E^2}\right)r_c^2$$

$$- \bar{E}^2\ell^4 - (\ell^2 - Q^2)\left(\ell^2 + \frac{a^2}{4E^2}\right) = 0.$$  \hspace{1cm} (5.4)

There are many ways to solve equations (5.3) and (5.4) simultaneously. First, let us consider that the spacetime parameters are fixed. For fixed $M$, $Q$, $\ell$ and $a$, there remains $\bar{q}$, $\bar{E}$ and $r_c$ and one can hopefully try to solve these equations for $\bar{q}$ and $\bar{E}$ in terms of circular radius $r_c$. However, due to highly nonlinear structure of equations in $\bar{q}$ and $\bar{E}$, such an analytical solution does not seem to be possible. Instead, one can provide a numerical solution for $\bar{q}$ and $\bar{E}$ by fixing spacetime parameters and $r_c$. On the other hand, looking at the equations, one can see that equation (5.3) is a linear equation in $a^2$ and $\ell^2$ while (5.4) is a second order equation in $\ell^2$. Then, if
one eliminates \(a^2\) from (5.3) and substitute the resulting equation into (5.4), one obtains a second order equation in \(\ell^2\). It means that, equations (5.3) and (5.4) can simultaneously be solved analytically for \(\ell^2\) and \(a^2\) yielding

\[
a^2 = \frac{4\overline{E}^2}{r_c - M + 2\overline{E}\overline{q}Q} \left[ (2\overline{E}^2 r_c + M - \overline{E}\overline{q}Q) \ell_c^2 + 2(\overline{E}^2 - 1) r_c^3 + 3(M - \overline{q}Q\overline{E}) r_c^2 + Q^2(\overline{q}^2 - 1) r_c \right]
\]

(5.5)

where

\[
\ell_c^2 = \frac{1}{2 \left[ (2\overline{E}^2 + 1)(\overline{q}Q\overline{E} + r_c) + \overline{E}^2(r_c - M) \right]} \left[ \overline{q}Q(\overline{E}Q^2 - \overline{q}Qr_c) + 2r_c (\overline{E}^2 r_c + M) \right.
\]

\[
+ \left. (2\overline{E}r_c(r_c - M + 2\overline{q}Q\overline{E}) + \overline{q}r_c + \overline{E}Q^2) \sqrt{(\overline{q}Q - 2r_c\overline{E})^2 + 4r_c(M - r_c)} \right].
\]

(5.6)

Provided that the right hand sides of the expressions (5.5) and (5.6) are positive, these relations (i.e. (5.5) and (5.6)) determine the existence of circular orbits. In addition, there also exists a reality condition given by the inequality relation

\[
(\overline{q}Q - 2r_c\overline{E})^2 + 4r_c(M - r_c) \geq 0
\]

(5.7)

which makes the expression inside the square root in (5.6) positive.

Now, using the relations (5.5) and (5.6), it would also be physically illustrating to examine the behaviours of the NUT and rotation parameters graphically. We realise such a graphical analysis (2 and 3-dimensional plots) of these parameters with respect to change in circular radius \(r_c\) and the energy \(\overline{E}\) of the test particle. We choose the parameters such that the existence of circular orbits is guaranteed (i.e. we choose the parameters such that \(\ell_c^2\) and \(a^2\) will be positive). Then, the typical behaviours of \(\ell_c^2\) and \(a^2\) with respect to circular radius \(r_c\) and the energy \(\overline{E}\) of the test particle are illustrated in Figures 1-5.

Next, let us discuss the stability of circular orbits whose existence is determined by the expressions (5.5) and (5.6). It is clear that if the inequality

\[
\left. \frac{d^2 P_r}{d\ell^2} \right|_{r=r_c} < 0
\]

(5.8)

is satisfied, one can obtain stable circular orbits for which the stability condition reads

\[
12(\overline{E}^2 - 1)r_c^2 + 12(M - \overline{q}Q\overline{E})r_c + \left( 4\overline{E}^2 \ell_c^2 + 2Q^2(\overline{q}^2 - 1) - \frac{a^2}{2\overline{E}^2} \right) < 0.
\]

(5.9)
Figure 1: Plots of the NUT and rotation parameters with respect to radius of the circular orbit for different values of $\bar{q}$ with $M = 1$, $Q = 0.3$, $\bar{E} = 0.8$.

Figure 2: Plots of the NUT and rotation parameters with respect to radius of the circular orbit for different values of $\bar{E}$ with $M = 1$, $Q = 0.3$, $\bar{q} = 4$.

Figure 3: Plots of the NUT and rotation parameters with respect to $\bar{E}$. We take other parameters as $M = 1$, $Q = 0.3$, $\bar{q} = 4$ and $r_c = 0.1$ ($0 < r_c < M$).
Figure 4: Plots of the NUT and rotation parameters with respect to $\bar{E}$. We take other parameters as $M = 1$, $Q = 0.3$, $\bar{q} = 4$ and $r_c = 2$ ($r_c > M$).

Figure 5: 3d plots of the NUT and rotation parameter with respect to energy of the test particle and radius of the circular orbit. We take $M = 1$, $Q = 0.3$, $\bar{q} = 2$. 
Then, the expressions (5.5), (5.6) and (5.9) specify the conditions of a possible stable circular orbit. One cannot expect that all of the circular orbits are stable. Therefore, it is also of great interest to examine the transition from stable to unstable (or vice versa) circular orbits. For such an analysis, one has to solve

\[
\left. \frac{d^2 \bar{P}_r}{dr^2} \right|_{r=r_c} = \bar{P}''_r(r_c) = 0 \quad (5.10)
\]

whose physical solutions lead to existence of inner-most stable circular orbits (ISCO). Then, if one inserts (5.5) and (5.6) into (5.10), after some arrangements, one can obtain the equation which will determine the radius \( r_{ISCO} \) of innermost stable circular orbits which explicitly reads

\[
2 \left[ \bar{q} Q \bar{E} (4 \bar{E}^2 + 1) - M (2 \bar{E}^2 + 1) \right] \ell_c^2 \\
+8(\bar{E}^2 - 1) r_c^3 + 6[M - \bar{q} Q \bar{E} + 2(\bar{E}^2 - 1) (2\bar{q} Q \bar{E} - M)] r_c^2 \\
+12(M - \bar{q} Q \bar{E}^2 + 2Q^2 (\bar{q}^2 - 1) (2\bar{q} Q \bar{E} - M) = 0
\]

where one should use (5.6) for \( \ell_c^2 \). Due to highly nonlinear structure of this equation, one cannot solve it analytically. However, by fixing parameters \( \bar{E}, Q, \bar{q} \) and \( M \), this equation can be solved numerically to obtain the radius of ISCO of charged particles in Kerr-Newman-Taub-NUT spacetime (The uncharged versions of ISCO equation with NUT parameter have been investigated in [18] and [35]). We should also comment that even such a numerical calculation may not always give real (physical) solutions for \( r_{ISCO} \) which means that inner-most stable circular orbits \( r_{ISCO} \) may not exist for some particular values of charge \( \bar{q} \) and energy \( \bar{E} \) of the test particle.

On the other hand, let us suppose that the spacetime is fixed with parameters \( \ell, a, Q, M \). In this case, as we have already remarked at the beginning of this section, one has to solve equations (5.3) and (5.4) for \( \bar{q} \) and \( \bar{E} \) as a function of \( r_c \) and spacetime parameters and substitute them into \( \bar{P}_r''(r_c) \) and solve equation \( \bar{P}_r''(r_c) = 0 \) to determine \( r_{ISCO} \). However, one can see that the equations (5.3) and (5.4) cannot be solved analytically for \( \bar{q} \) and \( \bar{E} \) in terms of \( \ell, a, Q, M \) and \( r_c \). Then, for this case, one cannot obtain an explicit analytical expression similar to (5.11) to calculate \( r_{ISCO} \) in the usual manner. Now, we tabulate our results in Tables 1-2. Looking at the Tables 1 and 2 they are obtained by fixing \( \bar{E}, Q, \bar{q} \) and \( M \). Then, \( r_{ISCO} \) is determined from equation (5.11) (where we have used the expression with + sign for \( \ell_c^2 \)). It can be seen that circular orbits are stable for \( r_c > r_{ISCO} \) while they are unstable for \( r_c < r_{ISCO} \). Again, we should remark that our results cannot be directly compared to the results obtained for Kerr and
Table 1: Numerical calculation of $r_{ISCO}$ with respect to change in energy $\bar{E}$ of the test particle. $r_{ISCO}$ is calculated from (5.11) by fixing $\bar{q}$, $\bar{E}$, $Q$ and $M$. Numerically we take $\bar{q} = 100$, $Q = 0.1$ and $M = 1$.

| $E$  | $r_{ISCO}$ | $\bar{E}$ | $r_{ISCO}$ |
|------|------------|-----------|------------|
| 0.5  | 2.63875    | 1.3       | 1.91285    |
| 0.6  | 2.52452    | 1.4       | 1.84914    |
| 0.7  | 2.4159     | 1.5       | 1.78976    |
| 0.8  | 2.31476    | 2         | 1.54484    |
| 0.9  | 2.22125    | 2.5       | 1.36313    |
| 1    | 2.13494    | 3         | 1.22413    |
| 1.1  | 2.05519    | 3.5       | 1.11598    |
| 1.2  | 1.98136    | 4         | 1.03205    |

Kerr-Newman spacetimes (where the NUT parameter $\ell = 0$) since for Kerr-Newman-Taub-NUT spacetime, the existence of equatorial or bits requires the condition (3.22) for $\ell \neq 0$.

Before closing this section, one can also examine the special case where $\bar{q} = 0$ (the equatorial circular orbit for the uncharged test particle). For this special case, the energy of the test particle can be analytically calculated to yield

$$\bar{E}_c^2 = \frac{\alpha(r_c) + \sqrt{\alpha^2(r_c) + a^2(r_c^2 + \ell^2)(3r_c^2 - \ell^2)(r_c^2 + \ell^2 - Q^2)}}{2(r_c^2 + \ell^2)(3r_c^2 - \ell^2)}$$

(5.12)

where $r_c$ should obey the relation

$$2r_c^2\alpha(r_c) - 3(r_c^2 - \ell^2)\beta(r_c) - (3r_c^2 - \ell^2)\sqrt{\beta^2(r_c) + 2a^2r_c^2(r_c^2 + \ell^2)(r_c - M)} + 2r_c\sqrt{\alpha^2(r_c) + a^2(r_c^2 + \ell^2)(3r_c^2 - \ell^2)(r_c^2 + \ell^2 - Q^2)} = 0$$

(5.13)

as well. Here, we define

$$\alpha(r_c) = (r_c^2 + \ell^2)^2 + (r_c^2 - \ell^2)(2r_c^2 + \ell^2) - 4Mr_c^3$$

(5.14)

and

$$\beta(r_c) = r_c^2(2r_c^2 + Q^2) - (3r_c^2 + \ell^2)Mr_c.$$

(5.15)

Obviously, the existence of circular orbits for the case $\bar{q} = 0$, depends on the positivity of the right hand side of the equation (5.12).
Table 2: Numerical calculation of $r_{ISCO}$ with respect to change in the charge $\bar{q}$ of the test particle. $r_{ISCO}$ is calculated from (5.11) by fixing $\bar{q}$, $\bar{E}$, $Q$ and $M$. Numerically we take $Q = 0.1$ and $M = 1$ by making the calculations for $\bar{E} = 0.8$ and $\bar{E} = 1$.

| $\bar{E}$ | $\bar{q}$ | $r_{ISCO}$ | $\bar{E}$ | $\bar{q}$ | $r_{ISCO}$ |
|---------|---------|----------|---------|---------|----------|
| 0.8     | 50      | 1.30001  | 1       | 50      | 1.21042  |
| 0.8     | 100     | 2.31476  | 1       | 100     | 2.13494  |
| 0.8     | 150     | 3.34673  | 1       | 150     | 3.0805   |
| 0.8     | 200     | 4.38251  | 1       | 200     | 4.03048  |
| 0.8     | 250     | 5.41974  | 1       | 250     | 4.98211  |
| 0.8     | 300     | 6.45768  | 1       | 300     | 5.93454  |

6 Discussion of the orbits

In this section, we make a discussion of possible orbit types (bound, flyby and transit orbits) by providing sample plots over the equatorial plane. We also examine the conditions and examples of bound orbits emerging outside the metric singularity $r_+$ (i.e. the existence of bound orbits in the region where $r > r_+$). To further exploit the physical effect of the NUT parameter on the motion over equatorial plane, we make an analysis of the Newtonian orbits as well, where the radial variable $r$ for those orbits is assumed to be much larger than the Schwarzschild radius of the gravitational source ($r >> r_S = 2M$).

As is also outlined in section 4, the orbit types are determined by the radial potential $\bar{P}_r(r)$. Then, to illustrate the possible orbit types, it would be better to provide graphs of the radial potential $\bar{P}_r(r)$ with respect to radial distance $r$. These plots are realised in the graphs 6a-6g. The parameters can be chosen to obtain a physically acceptable radial motion (i.e $\bar{P}_r(r) \geq 0$). When looking at these graphs, one can see that 6a, 6b, 6c and 6d are plotted for $\bar{E} > 1$. In the graph 6a, one can observe that $\bar{P}_r(r)$ has four real zeros ordered as $0 < r_- < r_+ < r_4 < r_3 < r_2 < r_1$. Hence, one can conclude that there exist one bound and two flyby orbits. Bound orbit is seen in the region where $r_3 < r < r_2$ while flyby orbits are realized for $r \geq r_1$ and $r \leq r_4$. For $r \geq r_1$, if particle comes from infinity, it will turn from the turning point at $r = r_1$ and go back to infinity. For $r \leq r_4$, if particle comes from negative infinity, it will cross two singularities at $r = r_-$ and $r = r_+$ (i.e. Cauchy and event horizons) and turn back from the turning point at
In the graph 6b, as in 6a, $\bar{P}_r(r)$ has four real roots this time ordered as $r_4 < 0 < r_3 < r_− < r_+ < r_2 < r_1$. Therefore, it can be deduced that one observes one bound and two flyby orbits. The bound orbit is similarly observed in the region where $r_3 < r < r_2$ such that $r_3 < r_− < r_+ < r_2$. For this case, the bound orbit can be identified as two-world bound orbit since particle crosses two metric singularities at $r = r_−$ and $r = r_+$ and turns back at $r = r_2$. Meanwhile, flyby orbits can be observed for $-\infty < r < r_4$ and for $r_1 < r < \infty$. As for graph 6c, it is seen that the roots of $\bar{P}_r(r)$ are ordered as $0 < r_4 < r_3 < r_2 < r_1 < r_− < r_+$ which further implies that one bound and two flyby orbits are realised. Yet for this case, the bound orbit is observed in the region $r_3 < r < r_2 < r_− < r_+$ while one flyby orbit arises for $-\infty < r < r_4$ where the particle crosses the point $r = 0$ (We should note that since $\ell \neq 0$, $r = 0$ is not a metric singularity on the equatorial plane). In addition, the other flyby orbit is observed for $r_1 < r < \infty$. However, for this orbit, one can see that the particle crosses two metric singularities which further implies the existence of a two-world flyby orbit. Now for the graph 6d, one deduces that $\bar{P}_r(r)$ has no real zeros which requires the existence of a transit orbit. It means that if the particle starts into motion at $r \to \mp \infty$, it will cross $r = 0$ and move to $r \to \pm \infty$. Examining the graphs 6e and 6f, it is seen that they are plotted for $\bar{E} < 1$. From the plots, it is clear that the roots of $\bar{P}_r(r)$ for 6e are ordered as $r_4 < r_3 < 0 < r_2 < r_− < r_+ < r_1$ while the roots of $\bar{P}_r(r)$ for 6f are ordered as $r_4 < 0 < r_− < r_+ < r_3 < r_2 < r_1$. In both plots, one observes two bound orbits. In figure 6g, one bound orbit is observed for $r_4 < r < r_3 < 0$ while the second bound orbit is seen for $r_2 < r < r_1$ where $r_2 < r_−$ and $r_1 > r_+$ which implies that the second bound orbit can be identified as two-world bound orbit. On the other hand, in Figure 6h, one bound orbit is observed for $r_2 < r < r_1$ where $r_+ < r_2 < r_1$ while the second bound arises for $r_4 < r < r_3$ where in this case $r_4 < r_−$ and $r_3 > r_+$ which means that the bound orbit is two-world. Finally, for the graph 6g plotted for $\bar{E} = 1$, $\bar{P}_r(r)$ possesses three real zeros ordered as $0 < r_− < r_+ < r_3 < r_2 < r_1$ which further implies the existence of one flyby and one bound orbit. The flyby orbit occurs for $r < r_3$ and can be identified as two-world flyby since the metric singularities are crossed for the particle motion in the interval $-\infty < r < r_3$. On the other hand, the bound orbit occurs for $r_2 < r < r_1$ where $r_2 > r_+$. It is also of great interest to investigate the conditions for the (probable) existence of the bound orbits in the region where $r > r_+$ (in other words outside the outer singularity of the spacetime). These conditions will imply that a radial bound interval $r_2 \leq r \leq r_1$ may exist such that for that region, $\bar{P}_r(r) > 0$ and $r_2 > r_+$. To make such an analysis of bound motion, we follow
a similar procedure outlined in [36]. Now, we can affect a transformation 
\( r = R + r_+ \), where \( r_+ \) describes the metric singularity (i.e. \( \Delta(r_+) = 0 \)),
assuming that at least one region of binding exists where \( r > r_+ \). To this
end, we express \( \bar{P}_r(r) \) in terms of new variable \( R \). Then in terms of \( R \), we obtain
\[
\bar{P}_R(R) = A_4 R^4 + A_3 R^3 + A_2 R^2 + A_1 R + A_0 
\]
(6.1)
where
\[
A_0 = \frac{1}{4E^2} [a^2 + 2\bar{E} (\ell^2\bar{E} + r_+ (\bar{E}r_+ - \bar{q}Q))]^2, 
\]
(6.2)
\[
A_1 = \frac{\alpha^2}{2E^2} \left[ M - 2\bar{q}Q\bar{E} + r_+ (4\bar{E}^2 - 1) \right] + 2\ell^2 \left[ M - \bar{q}Q\bar{E} + r_+ (2\bar{E}^2 - 1) \right] 
+ 2r_+ [\bar{q}^2 Q^2 + r_+ (M - 3\bar{q}Q\bar{E}) + r_+^2 (2\bar{E}^2 - 1)], 
\]
(6.3)
\[
A_2 = \bar{q}^2 Q^2 + a^2 \left( 1 - \frac{1}{4E^2} \right) + \ell^2 (2\bar{E}^2 - 1) 
+ 2r_+ (2M - 3\bar{q}Q\bar{E}) + r_+^2 (6E^2 - 5), 
\]
(6.4)
\[
A_3 = 2 \left[ M - \bar{q}Q\bar{E} + 2r_+ (\bar{E}^2 - 1) \right] 
\]
(6.5)
and
\[
A_4 = \bar{E}^2 - 1. 
\]
(6.6)
At this stage, let us consider in what conditions this polynomial has positive
roots (which will lead to the existence of bound orbit for \( r > r_+ \)). We
remark that \( A_0 > 0 \). According to Descartes’ rule of sign, a polynomial
possessing real coefficients can not have more positive roots than the number
of variations of sign in its coefficients. Then if \( A_4 > 0 \) (\( \bar{E}^2 > 1 \)), a bound
region for \( r > r_+ \) may be realized under the conditions
\[
A_1 < 0, \quad A_2 > 0, \quad A_3 < 0 
\]
(6.7)
since we also have \( A_0 > 0 \). Then four variations of sign would be possible and
therefore there may exist four (distinct) real positive roots for \( \bar{P}_R(R) \). If the
above inequality conditions are simultaneously met, we have the possibility
of having a bound motion for \( \bar{E}^2 > 1 \) in the region where \( r > r_+ \). Figure
\[6a\] includes an example of the bound orbit for \( r > r_+ \). A straightforward
calculation shows that the inequality conditions for the existence of such a
bound orbit are satisfied for the given spacetime parameters. Furthermore,
as can be seen from the coefficients \( A_i \) (\( i = 1, 2, 3, 4 \)), we should remark that
for $\bar{E} > 1$ and $M > 3\bar{q}Q\bar{E}$, all $A_i$'s become positive such that there would be no sign change for the polynomial $\bar{P}_R(R)$. Therefore $\bar{P}_R(r)$ will not possess (real) roots for $r > r_+$ and as a result there would be no bound orbit for $r > r_+$ for the case where $\bar{E} > 1$ and $M > 3\bar{q}Q\bar{E}$.

On the other hand, if $A_4 < 0$ ($\bar{E}^2 < 1$), there exist at most three variations of sign since $A_0 > 0$. For this case, then either of the following inequalities should be simultaneously fulfilled for the existence of bound region(s):

\begin{align*}
A_1 &< 0, \quad A_2 < 0, \quad A_3 > 0, \\
A_1 &> 0, \quad A_2 < 0, \quad A_3 > 0, \\
A_1 &< 0, \quad A_2 > 0, \quad A_3 > 0, \\
A_1 &< 0, \quad A_2 > 0, \quad A_3 < 0.
\end{align*}

Similarly, this implies that if any one of the above conditions are simultaneously met, there may exist at most one region of binding outside the outer singularity where $r > r_+$. An example for the existence of the bound orbit for $r > r_+$ with $\bar{E} < 1$ is realised in figure 6f. It is seen that, for the spacetime parameters given in 6f, the conditions $A_1 < 0$, $A_2 > 0$ and $A_3 < 0$ (together with $A_0 > 0$ and $A_4 < 0$) are fulfilled.

On the other hand, for $\bar{E} = 1$ ($A_4 = 0$), the existence of such orbits for $r > r_+$ depends on the sign of $A_3$. For a third order polynomial, there should be at most three variations of sign in order to obtain a bound orbit for $r > r_+$. Then, one can conclude that, if $A_3 > 0$ ($M > \bar{q}Q$) there would be at most two variations of sign and therefore no bound orbit is seen for $r > r_+$. On the other hand, if $A_3 < 0$ ($M < \bar{q}Q$) and the conditions $A_2 > 0$, $A_1 < 0$ are simultaneously met, there would be three variations of sign and therefore a bound motion may be realised in the region where $r > r_+$. As a further comment, figure 6g includes an example of such a bound orbit with $\bar{E} = 1$.

Finally, to compare the orbital motion over the equatorial plane with Newtonian orbits, one can express the orbital equation (3.25) in terms of a new variable $u$ such that $r = \frac{1}{u}$. With this substitution, the orbital equation (3.25) turns into

\begin{equation}
\left(\frac{du}{d\varphi}\right)^2 = \frac{\Delta_u^2 \left(2(\ell^2 u^2 + 1)\bar{E}^2 + a^2 u^2 - 2\bar{q}Q\bar{E}u\right) - \Delta_u \left(4\bar{E}^2(\ell^2 u^2 + 1) + a^2 u^2\right)}{a^2 \left(2(\ell^2 u^2 + 1)\bar{E}^2 + a^2 u^2 - \Delta_u - 2\bar{q}Q\bar{E}u\right)}
\end{equation}

where we define

\begin{equation}
\Delta_u = 1 - 2Mu + (a^2 - \ell^2 + Q^2)u^2.
\end{equation}
(a) One bound and two flyby orbits for $M = 1, \bar{E} = 1.1, \bar{q} = 5, Q = 5, a = 3.4, \ell = 6, 0 < r_- < r_+ < r_3 < r_2 < r_1$.

(b) One bound and two flyby orbits for $M = 1, \bar{E} = 2, \bar{q} = 10, Q = 0.4, a = 0.9, \ell = 0.1, r_4 < 0 < r_3 < r_- < r_+ < r_2 < r_1$

(c) One bound and two flyby orbits for $M = 1, \bar{E} = 10, \bar{q} = 10, Q = 0.4, a = 0.9, \ell = 0.1, 0 < r_4 < r_3 < r_2 < r_1 < r_- < r_+$

(d) Transit orbit for $M = 1, \bar{E} = 10, \bar{q} = 0.8, Q = 0.4, a = 0.9, \ell = 0.1$

(e) Two bound orbits for $M = 1, \bar{E} = 0.073, \bar{q} = 18.74, Q = 0.4, a = 0.9, \ell = 0.1, r_4 < r_3 < 0 < r_2 < r_- < r_+ < r_1$

(f) Two bound orbits for $M = 1, \bar{E} = 0.9, \bar{q} = 5, Q = 5, a = 3.3, \ell = 5.94, r_4 < 0 < r_- < r_+ < r_3 < r_2 < r_1$

(g) One bound and one flyby orbits for $M = 1, \bar{E} = 1, \bar{q} = 5, Q = 5, a = 3.4, \ell = 6, 0 < r_- < r_+ < r_3 < r_2 < r_1$

Figure 6: Possible orbit types for different values of the energy $\bar{E}$. 
Now, to analyse the orbital equation for Newtonian orbits, we utilise the physically oriented approximations raised in [37]. Thus, if one assumes that the radius of a Newtonian orbit is much larger than the corresponding Schwarzschild radius of the gravitational source, the orbital equation (6.8) may be expanded around \( u = 0 \) up to third order in order to compare its solutions with those in a Schwarzschild background:

\[
\left( \frac{du}{d\varphi} \right)^2 \approx f(u) = D_0 + D_1 u + D_2 u^2 + D_3 u^3, \tag{6.10}
\]

where

\[
D_0 = \frac{4E^2(E^2 - 1)}{a^2(2E^2 - 1)^2}, \tag{6.11}
\]

\[
D_1 = \frac{16(2M\dot{E} - \dot{q}Q)(1 - \dot{E}^2)\dot{E}^3}{a^2(2E^2 - 1)^3} + \frac{8(M - \dot{q}Q\dot{E})\dot{E}^2}{a^2(2E^2 - 1)^2}, \tag{6.12}
\]

\[
D_2 = \frac{48(\dot{E}^2 - 1)(\dot{q}Q - 2M\dot{E})^2\dot{E}^4}{a^2(2E^2 - 1)^4} + \frac{16\dot{E}^4}{a^2(2E^2 - 1)^3} \left[ 2M\dot{E}(3\dot{q}Q - 2M\dot{E}) - 2\dot{q}^2Q^2 + (\dot{E}^2 - 1)(Q^2 - 2\ell^2) \right] + \frac{4\dot{E}^2}{a^2(2E^2 - 1)^2} \left[ 3a^2(\dot{E}^2 - 1) + Q^2(q^2 - 1) + 2\ell^2\dot{E}^2 \right] - 1, \tag{6.13}
\]

\[
D_3 = \frac{128\dot{E}^5}{a^2(2E^2 - 1)^5} (\dot{q}Q - 2M\dot{E})^3(\dot{E}^2 - 1) + \frac{96\dot{E}^4(\dot{q}Q - 2M\dot{E})}{a^2(2E^2 - 1)^4} \left[ (\dot{E}^2 - 1)(Q^2 - 2\ell^2)\dot{E} - (\dot{q}Q - 2M\dot{E})(\dot{E}\dot{q}Q + M(1 - 2\dot{E}^2)) \right] + \frac{16\dot{E}^3}{a^2(2E^2 - 1)^3} \left[ 3a^2(\dot{q}Q - 2M\dot{E})(\dot{E}^2 - 1) + 2\dot{E}^2(\dot{q}Q - M\dot{E})(2\ell^2 - Q^2) \right] + (\dot{q}Q - 2M\dot{E}) \left( 4M\dot{E}(M\dot{E} - \dot{q}Q) + \ell^2(3\dot{E}^2 - 1) + Q^2(q^2 - \dot{E}^2) \right) + \frac{8\dot{E}^2}{a^2(2E^2 - 1)^2} \left[ (M - \dot{q}Q\dot{E})(a^2(2\dot{E}^2 + 1) + \ell^2) + Ma^2(\dot{E}^2 - 1) \right] + 2M, \tag{6.14}
\]

provided that \( 2\dot{E}^2 - 1 \neq 0 \) and \( a \neq 0 \). We see that, all the terms in \( D_2 \) except \(-1\) and all the terms in \( D_3 \) describe corrections (or improvements) to the classical Newtonian orbits. It is remarkable that, unlike the other spacetime parameters \( (a, Q, \text{ and } M) \), the effect of the NUT parameter on the Newtonian orbits can be seen as the improvement where the NUT parameter contributes at least at the order \( u^2 \) (and the higher orders).
7 Conclusion

In this study, we have comprehensively examined the equatorial orbits of a charged test particle in the background of Kerr-Newman-Taub-NUT spacetime. Having obtained the governing orbit equations, we have made an analysis of possible orbit types that would come out via the analysis of radial potential $\bar{P}_r(r)$. We have accomplished a comprehensive investigation of equatorial orbit types with respect to the value of the energy $\bar{E}$ of the test particle and the form of the radial potential $\bar{P}_r(r)$. We have explicitly examined the cases for which $\bar{P}_r(r)$ has double and triple roots. Next, we have determined the conditions for the existence and stability of equatorial circular orbits. It is clear that the relations (5.5) and (5.6) determine the existence of equatorial circular orbits. In addition, the right hand sides of these expressions should be positive as well for the reality of NUT and rotation parameters. Next, we have discussed the stability of circular orbits while numerically solving ISCO equation for a charged particle in Kerr-Newman-Taub-NUT spacetime and presented our results in tabular form. In the forthcoming section, by providing plots for the radial potential $\bar{P}_r(r)$ with appropriately chosen physical parameters, we have presented a complete discussion of possible orbit types, namely we have investigated bound, flyby and transit orbits over the equatorial plane in detail. In addition, in discussion part, we have given the conditions for the (probable) existence of bound orbits outside the metric singularity $r_+$ (or event horizon) i.e. we have particularly investigated the existence of bound orbits in the region where $r > r_+$. Also, we have given explicit examples to a bound orbit outside the outer singularity that satisfies related existence conditions that we have found. As a further remark, we have made a comparison to Newtonian orbits where it is assumed that radial distance $r$ is much larger than Schwarzschild radius $r_S = 2M$ of corresponding gravitational source. We have explicitly seen that the effect of the NUT parameter on the Newtonian orbits can be interpreted as the improvement to such orbits unlike the other spacetime parameters ($a$, $Q$ and $M$). Finally, we have obtained the exact analytical solutions of the equatorial orbit equations in terms of Weierstrass $\wp$, $\sigma$ and $\zeta$ functions. Using these analytical solutions, we have also provided sample plots describing the bound and flyby orbital motions of the charged test particle in the regions where $r < r_-$ and $r > r_+$. In addition, as a physical observable, we have calculated the perihelion shift for a bound orbit over the equatorial plane where it obviously depends on the NUT and other physical parameters. We believe that, one can surely comment on the existence of the NUT parameter in the universe if the the-
oretical expression for the perihelion shift is compared with the numerical values provided through astronomical observations. For a future study, it would also be physically interesting to investigate the equatorial orbits of the charged test particles in rotating Taub-NUT spacetimes with cosmological constant (Kerr-Newman-Taub-NUT-(A)dS spacetimes). In particular, the investigation of the existence of equatorial circular orbits in such a space-time deserves further study to see the effect of the cosmological constant. These are devoted to future research.

Appendix A: Analytical Solutions

In this section, we provide analytical solutions to orbit equations (3.24)-(3.26) where $\bar{P}_r(r)$ is given by (3.27). First, the radial equation (3.24) can be expressed as

$$
\left(\frac{dr}{d\lambda}\right)^2 = \bar{P}_r(r) = B_0 + B_1 r + B_2 r^2 + B_3 r^3 + B_4 r^4
$$

(7.1)

where the coefficients read

$$
B_4 = E^2 - 1,
$$

(7.2)

$$
B_3 = 2(M - \bar{E}qQ),
$$

(7.3)

$$
B_2 = 2\bar{E}^2 \ell^2 + Q^2(\bar{q}^2 - 1) - \frac{a^2}{4E^2},
$$

(7.4)

$$
B_1 = 2M \left(\ell^2 + \frac{a^2}{4E^2}\right) - 2\bar{q}Q \bar{E} \left(\ell^2 + \frac{a^2}{2E^2}\right),
$$

(7.5)

and

$$
B_0 = \ell^4 E^2 - (Q^2 - \ell^2) \left(\ell^2 + \frac{a^2}{2E^2}\right).
$$

(7.6)

Performing the transformation (for $\bar{E}^2 \neq 1$)

$$
r = \frac{\alpha_3}{(4y - \frac{a^2}{E^2})} + r_1
$$

(7.7)

where $r_1$ is assumed to be one real root of $\bar{P}_r(r)$ and defining

$$
\alpha_1 = B_3 + 4B_4 r_1,
$$

(7.8)

$$
\alpha_2 = B_2 + 3B_3 r_1 + 6B_4 r_1^2,
$$

(7.9)
\[ \alpha_3 = B_1 + 2B_2r_1 + 3B_3r_1^2 + 4B_4r_1^3, \]  
(7.10)
equation (7.1) can be brought into the standard Weierstrass form
\[ \left( \frac{dy}{d\lambda} \right)^2 = \bar{P}_3(y) = 4y^3 - g_2y - g_3 \]  
(7.11)
whose solution can be written in terms of Weierstrass \( \wp \) function \[ y(\lambda) = \wp((\lambda - \lambda_0); g_2, g_3) \]  
(7.12)
with
\[ g_2 = \frac{1}{12} (\alpha_2^2 - 3\alpha_1\alpha_3), \quad g_3 = \frac{1}{8} \left( \frac{\alpha_1\alpha_2\alpha_3}{6} - \frac{B_4\alpha_3^2}{2} - \frac{\alpha_3^3}{27} \right). \]  
(7.13)
Then, the solution for radial coordinate \( r \) can be given by
\[ r = \frac{\alpha_3}{4\wp((\lambda - \lambda_0); g_2, g_3) - \alpha_3^3} + r_1. \]  
(7.14)
Next, from the integration of (3.25), one obtains
\[ \frac{1}{a}(\varphi - \varphi_0) = \int^r \frac{\left( \bar{E}(r^2 + \ell^2) + \frac{a^2}{2E} - \bar{q}Qr - \frac{\Delta(r)}{2E} \right)}{\Delta(r) \sqrt{P_r} \bar{r}} dr. \]  
(7.15)
Using the remark that \( \int^r \frac{dr}{\sqrt{P_r(r)}} = \int^y \frac{dy}{\sqrt{P_3(y)}} = \lambda - \lambda_0 \), one can accomplish the integration of the right hand side with respect to radial coordinate \( r \) resulting in
\[ \frac{1}{a}(\varphi - \varphi_0) = \left[ \left( \bar{E}\ell^2 + \frac{1}{2E} (\ell^2 - Q^2) \right) \omega_0 + \left( \frac{M}{E} - \bar{q}Q \right) \tilde{\omega}_0 + \left( \frac{2E^2 - 1}{2E} \right) \tilde{\omega}_0 \right] (\lambda - \lambda_0) \]
\[ + \sum_{i=1}^2 \sum_{j=1}^2 \left[ \left( \bar{E}\ell^2 + \frac{1}{2E} (\ell^2 - Q^2) \right) \omega_i + \left( \frac{M}{E} - \bar{q}Q \right) \tilde{\omega}_i + \left( \frac{2E^2 - 1}{2E} \right) \tilde{\omega}_i \right] \]
\[ \times \frac{1}{\varphi'(y_{ij})} \left[ \zeta(y_{ij})(\lambda - \lambda_0) + \ln \left( \frac{\sigma(s - y_{ij})}{\sigma(s_0 - y_{ij})} \right) \right]. \]  
(7.16)

Here, \( \varphi(y_{ij}) = y_i \) with \( \varphi(y_{11}) = \varphi(y_{12}) = y_1, \varphi(y_{21}) = \varphi(y_{22}) = y_2 \) where
\[ y_1 = \frac{1}{4\Delta(r_1)} \left( \frac{\alpha_2}{3} \Delta(r_1) - \alpha_3r_1 + \alpha_3r_1^3 \right), \]  
(7.17)
\[ y_2 = \frac{1}{4\Delta(r_1)} \left( \frac{\alpha_2}{3} \Delta(r_1) - \alpha_3 r_1 + \alpha_3 r_+ \right), \quad (7.18) \]

and the variables \( s \) and \( \lambda \) are related by \( s - s_0 = \lambda - \lambda_0 \), \( s_0 \) and \( \lambda_0 \) being integration constants. We further identify

\[ \omega_0 = \frac{1}{\Delta(r_1)}, \quad (7.19) \]

\[ \omega_1 = -\frac{\alpha_3 (r_1 - r_-)^2}{4\Delta^2(r_1)(r_+ - r_-)}, \quad (7.20) \]

\[ \omega_2 = \frac{\alpha_3 (r_1 - r_+)^2}{4\Delta^2(r_1)(r_+ - r_-)}, \quad (7.21) \]

\[ \bar{\omega}_0 = \frac{r_1}{\Delta(r_1)}, \quad (7.22) \]

\[ \bar{\omega}_1 = \frac{\alpha_3 (r_1 - r_-)}{4\Delta^2(r_1)(r_+ - r_-)} \left[ \Delta(r_1) + r_1(r_- - r_1) \right], \quad (7.23) \]

\[ \bar{\omega}_2 = \frac{\alpha_3 (r_+ - r_1)}{4\Delta^2(r_1)(r_+ - r_-)} \left[ \Delta(r_1) + r_1(r_+ - r_1) \right], \quad (7.24) \]

\[ \bar{\omega}_0 = \frac{r_1^2}{\Delta(r_1)}, \quad (7.25) \]

\[ \bar{\omega}_1 = -\frac{\alpha_3}{4\Delta^2(r_1)(r_+ - r_-)} [\Delta(r_1) + r_1(r_- - r_1)]^2, \quad (7.26) \]

\[ \bar{\omega}_2 = \frac{\alpha_3}{4\Delta^2(r_1)(r_+ - r_-)} [\Delta(r_1) + r_1(r_+ - r_1)]^2. \quad (7.27) \]

Finally, the integration of (3.26) yields

\[ t - t_0 = \int_r^\infty \frac{(r^2 + a^2 + \ell^2) \left( \bar{E}(r^2 + \ell^2) + \frac{a^2}{2E} - \bar{q}Qr \right)}{\Delta(r) \sqrt{P(r)}} dr \quad (7.28) \]
where upon integration, one can obtain the result

\[
\begin{align*}
t - t_0 &= \left[ \left( \bar{E} \ell^2 (a^2 + \ell^2) + \frac{a^2 \ell^2}{E} - \frac{a^2 Q^2}{2E} \right) \omega_0 + \left( \frac{Ma^2}{E} - (a^2 + \ell^2) \bar{q} Q \right) \bar{\omega}_0 \right. \\
&+ \left( \bar{E} (a^2 + 2\ell^2) \bar{\omega}_0 - Q \bar{\omega}_0 + \bar{E} \bar{\omega}_0 \right) (\lambda - \lambda_0) \\
&+ \frac{3}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{(E \omega_i - qQ \omega_i)}{\varphi'(y_{ij})} \left[ \zeta (y_{ij}) (\lambda - \lambda_0) + \ln \left( \frac{\sigma(s - y_{ij})}{\sigma(s_0 - y_{ij})} \right) \right] \\
&+ \frac{2}{E} \sum_{i=1}^{2} \sum_{j=1}^{2} \left[ \left( \bar{E} \ell^2 (a^2 + \ell^2) + \frac{a^2 \ell^2}{E} - \frac{a^2 Q^2}{2E} \right) \omega_i \right. \\
&+ \left( \frac{a^2 M}{E} - (a^2 + \ell^2) \bar{q} Q \right) \omega_i + \bar{E} (a^2 + 2\ell^2) \bar{\omega}_i \\
&\left. \times \frac{1}{\varphi'(y_{ij})} \left[ \zeta (y_{ij}) (\lambda - \lambda_0) + \ln \left( \frac{\sigma(s - y_{ij})}{\sigma(s_0 - y_{ij})} \right) \right] \right] \\
&- \bar{E} \bar{\omega}_4 \sum_{j=1}^{2} \frac{1}{\varphi'^2(y_{3j})} \left[ \left( \varphi(y_{3j}) + \frac{\varphi'(y_{3j})}{\varphi'(y_{3j})} \right)(\lambda - \lambda_0) \right. \\
&\left. + \left( \zeta(s - y_{3j}) + \frac{\varphi'(y_{3j})}{\varphi'(y_{3j})} \right) \ln \left( \frac{\sigma(s - y_{3j})}{\sigma(s - y_{3j})} \right) \right] \right]. & (7.29)
\end{align*}
\]

In addition, we identify \( \varphi(y_{3j}) = y_3 = \frac{a_3^2}{12} \) \((j = 1, 2) \) with \( \varphi(y_{31}) = \varphi(y_{32}) = y_3 = \frac{a_3^2}{12} \). We further calculate

\[
\hat{\omega}_0 = \frac{r_1^3}{\Delta(r_1)}, \quad (7.30)
\]

\[
\hat{\omega}_1 = \frac{a_3 [\Delta(r_1) + r_1 (r_+ - r_-)]^3}{4 \Delta^2(r_1)(r_+ - r_-)(r_1 - r_-)}, \quad (7.31)
\]

\[
\hat{\omega}_2 = \frac{a_3 [\Delta(r_1) + r_1 (r_+ - r_-)]^3}{4 \Delta^2(r_1)(r_+ - r_-)(r_+ - r_1)}, \quad (7.32)
\]

\[
\hat{\omega}_3 = \frac{a_3 \Delta(r_1)}{4(r_1 - r_-)(r_1 - r_+)}, \quad (7.33)
\]

\[
\hat{\omega}_0 = \frac{r_1^4}{\Delta(r_1)}, \quad (7.34)
\]

\[
\hat{\omega}_1 = \frac{\Delta^2(r_1) r_1^4 a_2^2 [\Delta(r_1) a_2 + 3(r_+ - r_1)a_3]^2 + 81 [\Delta(r_1) + (r_+ - r_1)r_1]^4 a_3^4}{324 \Delta^2(r_1)(r_+ - r_-)(r_- - r_1)^2 a_3^4}, \quad (7.35)
\]

28
\[
\omega_2 = \frac{\Delta^2(r_1)r_4^2\alpha_3^2 \left[ \Delta(r_1)\alpha_2 + 3(r_+ - r_1)\alpha_3 \right]^2 + 81 \left[ \Delta(r_1) + (r_+ - r_1) \right]^4 \alpha_3^4}{324\Delta^2(r_1)(r_+ - r_-)(r_+ - r_1)\alpha_3^3},
\]

(7.36)

\[
\omega_3 = \frac{\Delta(r_1) \left[ r_4^4\alpha_3^2 \left( \Delta(r_1)(M - r_1)\alpha_2 + 3(r_+ - r_1)(r_+ - r_1)\alpha_3 \right) + 81\Delta(r_1)(r_1 + M)\alpha_3^3 \right]}{162(r_1 - r_-)^2(r_1 - r_+)^2\alpha_3^3},
\]

(7.37)

\[
\omega_4 = \frac{\Delta(r_1) \left( r_4^4\alpha_3^2 + 81\alpha_3^3 \right)}{1296(r_1 - r_-)(r_1 - r_+)^2\alpha_3^3}.
\]

(7.38)

As a final remark, using these analytical solutions, we provide plots of the bound and flyby orbits over the equatorial plane for \( \bar{E} > 1 \), \( \bar{E} < 1 \) and \( \bar{E} = 1 \). These are illustrated in Figures 7-13.

**Appendix B: Calculation of the perihelion shift for a bound orbit**

Here, to get an expression for the perihelion shift for a bound orbit, we consider that the motion in the radial direction is bounded in the interval \( r_2 \leq r \leq r_1 \). Then, one can evaluate the fundamental period \( \Lambda_r \) for the radial motion as

\[
\Lambda_r = 2 \int_{r_2}^{r_1} \frac{dr}{\sqrt{P_r(r)}} = 2 \int_{y_0}^{\infty} \frac{dy}{\sqrt{P_3(y)}}
\]

(7.39)

where \( P_3(y) = 4y^3 - g_2y - g_3 \) with \( g_2 \) and \( g_3 \) introduced in (7.13). The integral can be calculated via the transformation

\[
x = \frac{1}{\kappa} \left( \frac{\bar{y}_2 - \bar{y}_3}{y - \bar{y}_3} \right)^{1/2}
\]

(7.40)

where \( \bar{y}_1, \bar{y}_2 \) and \( \bar{y}_3 \) correspond to the roots of the polynomial \( \bar{P}_3(y) = 0 \) (ordered as \( \bar{y}_3 < \bar{y}_2 < \bar{y}_1 \)) with \( \kappa^2 = \frac{\bar{y}_2 - \bar{y}_3}{\bar{y}_1 - \bar{y}_3} \). We also choose \( y_0 = \bar{y}_1 \). Then one gets the radial period as

\[
\Lambda_r = \frac{2}{\sqrt{\bar{y}_1 - \bar{y}_3}} K(\kappa)
\]

(7.41)
Figure 7: Bound and flyby orbits for $\bar{E} > 1$ with parameters $m = 1$, $M = 1$, $a = 0.9$, $Q = 0.4$, $\bar{q} = 10$, $\ell = 0.05$, $\bar{E} = 2$. Here, the roots of the radial potential $\bar{P}(r)$ satisfy $r_4 < 0 < r_3 < r_- < r_+ < r_2 < r_1$. Figure 8(b) illustrates the bound orbit in the region where $r_+ < r < r_2$. The dashed circles indicate the bounds of the radial motion. Figure 8(c) illustrates the flyby orbit in region where $r > r_1$. 
Figure 8: Bound and flyby orbits for $\bar{E} > 1$ with parameters $m = 1, M = 1, a = 0.9, Q = 0.4, \bar{q} = 15, \ell = 0.4, \bar{E} = 2$. Here, the roots of the radial potential $\bar{P}(r)$ satisfy $0 < r_4 < r_3 < r_- < r_+ < r_2 < r_1$. Figure 9(b) illustrates the bound orbit in the region where $r_+ < r < r_2$. The dashed circles indicate the bounds of the radial motion. Figure 9(c) illustrates the flyby orbit in region where $r > r_1$. 
Figure 9: Bound and flyby orbits for $\bar{E} > 1$ with parameters $m = 1, M = 1, a = 0.9, Q = 0.4, \bar{q} = 10, \ell = 0.4, \bar{E} = 2$. Here, the roots of the radial potential $\bar{P}(r)$ satisfy $0 < r_4 < r_3 < r_- < r_+ < r_2 < r_1$. Figure 10(b) illustrates the bound orbit in the region where $r_+ < r < r_2$. The dashed circles indicate the bounds of the radial motion. Figure 10(c) illustrates the flyby orbit in region where $r > r_1$. 
Figure 10: Bound orbit for $\bar{E} < 1$ with parameters $m = 1$, $M = 1$, $a = 3.3$, $Q = 5$, $\bar{q} = 5$, $\ell = 5.94$, $\bar{E} = 0.9$ ($\bar{L} > 0$). Here, the roots of the radial potential $\bar{P}(r)$ satisfy $r_4 < 0 < r_- < r_+ < r_3 < r_2 < r_1$. Figure 11(b) illustrates the bound orbit in the region where $r_2 < r < r_1$. The dashed circles indicate the bounds of the radial motion. On the $\bar{P}_r(r)$ graph the smallest root cannot be illustrated since $r_4 = -240.773$ is out of the scale range of the radial coordinate.
Figure 11: Bound orbit for $\bar{E} < 1$ with parameters $m = 1$, $M = 1$, $a = 0.9$, $Q = 0.4$, $\bar{q} = 18.74$, $\ell = 0.1$, $\bar{E} = 0.073$ ($\bar{L} < 0$). Here, the roots of the radial potential $\bar{P}(r)$ satisfy $r_4 < r_3 < 0 < r_2 < r_- < r_+ < r_1$. Figure 12(b) illustrates the bound orbit in the region where $r_4 < r < r_3$. The dashed circles indicate the bounds of the radial motion.
Figure 12: Bound and flyby orbits for $\bar{E} = 1$ with parameters $m = 1$, $M = 1$, $a = 3.4$, $Q = 5$, $q = 5$, $\ell = 6$, $(M - qQ < 0)$. Here, the roots of the radial potential $\bar{P}(r)$ satisfy $0 < r_- < r_+ < r_3 < r_2 < r_1$. Figure 13(b) illustrates the bound orbit in the region where $r_2 < r < r_1$. The dashed circles indicate the bounds of the radial motion. Figure 13(c) illustrates the flyby orbit in region where $-\infty < r < r_3$. 
Figure 13: Bound and flyby orbits for $\bar{E} = 1$ with parameters $m = 1$, $M = 1$, $a = 3.4$, $Q = 5$, $\bar{q} = -5$, $\ell = 6$, $(M - \bar{q}Q > 0)$. Here, the roots of the radial potential $\bar{P}(r)$ satisfy $r_3 < r_2 < r_1 < 0 < r_- < r_+$. Figure 14(b) illustrates the bound orbit in the region where $r_3 < r < r_2$. The dashed circles indicate the bounds of the radial motion. Figure 14(c) illustrates the flyby orbit in region where $r_+ < r < +\infty$. 
where \( K(\kappa) \) denotes the complete elliptic function with modulus \( \kappa \). Then, one can also evaluate the corresponding angular frequency

\[
\Upsilon_r = \frac{2\pi}{\Lambda_r} = \frac{\pi \sqrt{y_1 - y_3}}{K(\kappa)}
\]

(7.42)

for the radial motion. Furthermore, one can obtain the angular frequencies \( \Upsilon_\varphi \) and \( \Upsilon_t \) for the \( \varphi \)-motion and \( t \)-motion respectively from the solutions of \( \varphi(\lambda) \) and \( t(\lambda) \). By using the arguments exposed in [39], one can notice that the solutions \( \varphi(\lambda) \) and \( t(\lambda) \) can both be written in the forms

\[
\varphi(\lambda) = \Upsilon_\varphi (\lambda - \lambda_0) + \tilde{\varphi}(\lambda)
\]

(7.43)

and

\[
t(\lambda) = \Upsilon_t (\lambda - \lambda_0) + \tilde{t}(\lambda),
\]

(7.44)

where \( \Upsilon_\varphi \) and \( \Upsilon_t \) correspond to frequencies in Mino time for \( \varphi \)-motion and \( t \)-motion respectively. From these two solutions, one can get the corresponding angular frequencies as

\[
\Upsilon_\varphi = a \left[ \left( \bar{E} \ell^2 + \frac{1}{2E} (\ell^2 - Q^2) \right) \omega_0 + \left( \frac{M}{E} - \bar{q}Q \right) \tilde{\omega}_0 + \left( \frac{2\bar{E}^2 - 1}{2E} \right) \tilde{\omega}_0 \right]
\]

(7.45)

\[
+ a \sum_{i=1}^{2} \sum_{j=1}^{2} \left[ \left( \bar{E} \ell^2 + \frac{1}{2E} (\ell^2 - Q^2) \right) \omega_i + \left( \frac{M}{E} - \bar{q}Q \right) \tilde{\omega}_i + \left( \frac{2\bar{E}^2 - 1}{2E} \right) \tilde{\omega}_i \right] \zeta(y_{ij}) \frac{\wp'(y_{ij})}{\wp'(y_{ij})}
\]

and

\[
\Upsilon_t = \left[ \left( \bar{E} \ell^2 (a^2 + \ell^2) + \frac{a^2 \ell^2}{E} - \frac{a^2 Q^2}{2E} \right) \omega_0 + \left( \frac{Ma^2}{E} - (a^2 + \ell^2)\bar{q}Q \right) \tilde{\omega}_0 \right]
\]

\[
+ \bar{E} (a^2 + 2\ell^2) \tilde{\omega}_0 - \bar{q}Q \tilde{\omega}_0 + \bar{E} \tilde{\omega}_0
\]

\[
+ \sum_{i=1}^{3} \sum_{j=1}^{2} \left( \bar{E} \tilde{\omega}_i - \bar{q}Q \tilde{\omega}_i \right) \zeta(y_{ij}) - \bar{E} \tilde{\omega}_4 \sum_{j=1}^{2} \frac{1}{\wp'(y_{ij})} \left( \varphi(y_{3j}) + \frac{\varphi'(y_{3j})}{\wp'(y_{3j})} \right)
\]

\[
+ \sum_{i=1}^{2} \sum_{j=1}^{2} \left[ \left( \bar{E} \ell^2 (a^2 + \ell^2) + \frac{a^2 \ell^2}{E} - \frac{a^2 Q^2}{2E} \right) \omega_i + \left( \frac{a^2 M}{E} - (a^2 + \ell^2)\bar{q}Q \right) \tilde{\omega}_i \right]
\]

\[
+ \left( \frac{a^2 M}{E} - (a^2 + \ell^2)\bar{q}Q \right) \tilde{\omega}_i + \bar{E} (a^2 + 2\ell^2) \tilde{\omega}_i
\]

(7.46)

Finally, as is also outlined in [39], [40] and [41], the angular frequencies obtained using Mino time \( \lambda \) can be related to the angular frequencies \( \Omega_r \) and \( \Omega_\varphi \) obtained with respect to a distant observer time as

\[
\Omega_r = \frac{\Upsilon_r}{\Upsilon_t}, \quad \Omega_\varphi = \frac{\Upsilon_\varphi}{\Upsilon_t}.
\]

(7.47)
Obviously, these frequencies are not equal to each other. Therefore, it enables us to calculate the perihelion shift in the form

\[ \Omega_{\text{perihelion}} = \Omega_{\varphi} - \Omega_r. \quad (7.48) \]

It is clear that, the perihelion shift explicitly depends on the NUT parameter and other physical spacetime parameters as well as the charge and the energy of the test particle. If one makes a comparison of this theoretical expression with those provided in astronomical observations, one can surely argue the existence of the NUT parameter in the real physical world. Although we couldn’t provide a numerical value for the perihelion precision, one can comment that the NUT parameter and the charge of the test particle have a definite influence on the perihelion shift.

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