A generalization of the duality for finite multiple harmonic $q$-series

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Abstract

Recently, Bradley studied partial sums of multiple $q$-zeta values and proved a duality result. In this paper, we present a generalization of his result.

Keywords: finite multiple harmonic $q$-series

1 Introduction

Recently, finite multiple harmonic sums (MHSs for short) have been studied in connection with theoretical physics [1, 10]. In [5, 11], the $p$-divisibility of MHSs for primes $p$ have been investigated. MHSs have a remarkable property known as the duality and a generalization of this formula, which we call the difference formula for MHSs, was given in [6, Theorem 3.8]. On the other hand, in [2], Bradley proved a $q$-analog of the duality for MHSs. In the present paper, we shall consider a $q$-analog of the difference formula for MHSs. We note that the argument is parallel to that in [6].

Here, we explain the duality for finite multiple harmonic $q$-series due to Bradley. Let $0 < q < 1$. The $q$-analog of a non-negative integer $n$ is given by

$$[n]_q = \frac{1 - q^n}{1 - q}.$$  

For any multi-index (i.e. a finite sequence of positive integers) $\mu = (\mu_1, \ldots, \mu_p)$, we define

$$a_\mu(n) = \sum_{n_1 \geq \cdots \geq n_p \geq 0} \frac{q^{\mu_1 n_1 + \cdots + \mu_p n_p}}{[n_1 + 1]_q^{\mu_1} \cdots [n_p + 1]_q^{\mu_p}}, \quad 0 \leq n \in \mathbb{Z},$$

and

$$b_\mu(n) = \sum_{n_1 \geq \cdots \geq n_p \geq 0} \frac{q^{\mu_1 n_1 + \cdots + \mu_p n_p}}{[n_1 + 1]_q^{\mu_1} \cdots [n_p + 1]_q^{\mu_p}}, \quad 0 \leq n \in \mathbb{Z}.$$  

We note that the sum of the infinite series $\sum_{n=0}^\infty a_\mu(n)$ is the quantity known as the (non-strict) multiple $q$-zeta value, which has been investigated in recent
years [3, 4, 7, 8, 9]. The following is the duality for finite multiple harmonic $q$-series:

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \left[ \begin{array}{c} k \\ i \end{array} \right] q a_{\mu}(i) = b_{\mu^*}(k), \quad 0 \leq k \in \mathbb{Z},$$

where

$$\left[ \begin{array}{c} k \\ i \end{array} \right] q = \frac{[k]_q!}{[i]_q! [k-i]_q!}$$

is the $q$-binomial coefficient and $\mu^*$ is the dual multi-index of $\mu$. (The formula is slightly modified from Bradley’s one for the purpose of generalization.) The definition of $\mu^*$ will be given in Section 3. For example, we have

$$(2, 2)^* = (1, 2, 1), \quad (1, 1, 2)^* = (3, 1) \quad \text{and} \quad (4)^* = (1, 1, 1, 1)$$

by the diagrams

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

where the lower arrows are in the complementary slots to the upper arrows.

We next illustrate the main result of this paper. For a multi-index $\mu = (\mu_1, \ldots, \mu_p)$, the quantity $|\mu| = \mu_1 + \cdots + \mu_p$ is called the weight of $\mu$. We introduce nested sums

$$c_{\mu, \nu}(n, k) = \left[ \frac{n+k}{n} \right]^{-1}_q \sum_{n=n_1 \geq \cdots \geq n_p \geq 0 \atop k=k_1 \geq \cdots \geq k_r \geq 0} q^{(\mu_1-1)(n_1+1)+\cdots+(\mu_p-1)(n_p+1)+k_2+\cdots+k_r} \frac{[n_{i_1} + k_{j_1} + 1]_q \cdots [n_{i_m} + k_{j_m} + 1]_q}{[n_1 + k_1 + 1]_q \cdots [n_p + k_p + 1]_q},$$

where $0 \leq n, k \in \mathbb{Z}$,

for multi-indices $\mu = (\mu_1, \ldots, \mu_p)$ and $\nu = (\nu_1, \ldots, \nu_r)$ of the same weight $m$. The subscripts $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ are defined by

$$(i_1, \ldots, i_m) = (1, \ldots, 1, 2, \ldots, 2, \ldots, p, \ldots, p)$$

and

$$(j_1, \ldots, j_m) = (1, \ldots, 1, 2, \ldots, 2, \ldots, r, \ldots, r),$$

respectively. For example, for $\mu = (3, 1)$ and $\nu = (1, 1, 2)$, we have

$$c_{\mu, \nu}(n, k) = \left[ \frac{n+k}{n} \right]^{-1}_q \sum_{n=n_1 \geq n_2 \geq 0 \atop k=k_1 \geq k_2 \geq k_3 \geq 0} \frac{q^{2(n_1+1)+k_2+k_3}}{[n_1 + k_1 + 1]_q [n_1 + k_2 + 1]_q [n_1 + k_3 + 1]_q [n_2 + k_3 + 1]_q}.$$
The following is the main result of this paper: For any multi-index $\mu$, we have

$$\sum_{i=0}^{k} (-1)^i q^{i(i+1)/2} \left[\begin{array}{c} k \\ i \end{array}\right] \mu(n+i) = c_{\mu,\mu^*}(n,k), \quad 0 \leq n, k \in \mathbb{Z}. \tag{3}$$

As we see in Section 3, the equality $c_{\mu,\mu^*}(0,k) = b_{\mu^*}(k)$ holds. Hence the formula (3) is a generalization of the formula (1). In Section 2 we interpret the left-hand side of (3) as the $k$-th $q$-difference of the sequence $a_\mu \in \mathbb{C}^N$. The proof of (3) is given in Section 3.

### 2 $q$-differences of a sequence

In this section, we define the $k$-th $q$-difference of a sequence for a non-negative integer $k$ and give an explicit expression for it. Throughout this paper, we fix a complex number $q$ equal to neither 0 nor 1. (When dealing with multiple $q$-zeta values, we usually assume that $0 < q < 1$. But it is not necessary in finite expressions to restrict $q$ to the range $0 < q < 1$.) In the following, we denote by $N$ the set of non-negative integers.

**Definition 2.1.** For any $z \in \mathbb{C}$, we define the difference operator $\Delta_z : \mathbb{C}^N \to \mathbb{C}^N$ by putting

$$(\Delta_z a)(n) = a(n) - za(n + 1)$$

for any $a \in \mathbb{C}^N$ and any $n \in \mathbb{N}$.

**Definition 2.2.** For any $k \in \mathbb{N}$, we define the $k$-th $q$-difference operator by

$$\Delta_{q,k} = \Delta_{q,k} \circ \Delta_{q,k-1} \circ \cdots \circ \Delta_q,$$

where $\Delta_{q,0}$ is defined to be the identity on $\mathbb{C}^N$.

**Definition 2.3.** We define the operator $\nabla_q : \mathbb{C}^N \to \mathbb{C}^N$ by putting

$$(\nabla_q a)(n) = (\Delta_{q,n} a)(0)$$

for any $a \in \mathbb{C}^N$ and any $n \in \mathbb{N}$.

Let $\mathbb{C}[[X]]$ (resp. $\mathbb{C}[[X,Y]]$) be the ring of formal power series in one variable (resp. two variables) over $\mathbb{C}$. For a sequence $a \in \mathbb{C}^N$, we consider a formal power series

$$F_a(X,Y) = \sum_{n,k=0}^{\infty} (\Delta_{q,k} a)(n) \frac{X^n Y^k}{[n]_q! [k]_q!} \in \mathbb{C}[[X,Y]]. \tag{4}$$

The quantities

$$[n]_q = \frac{1-q^n}{1-q} \quad \text{and} \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q$$
are the $q$-integer and the $q$-factorial, respectively. As usual, we put $[0]_q! = 1$. The $q$-derivative of a formal power series $f(X) \in \mathbb{C}[[X]]$ is defined as
\[
D_q f(X) = \left( \frac{d}{dX} \right)_q f(X) = \frac{f(qX) - f(X)}{qX - X} \in \mathbb{C}[[X]].
\]
We have the $q$-Leibniz rule
\[
D^n_q (f(X)g(X)) = \sum_{k=0}^{n} \frac{n!}{k!} \left( D_k^q f(X) \right) \left( D_{n-k}^q g(q^k X) \right) \tag{5}
\]
for any $f(X), g(X) \in \mathbb{C}[[X]]$ and any $n \in \mathbb{N}$, where
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}
\]
is the $q$-binomial coefficient. We put $\partial_X = (\partial/\partial X)_q$ and $\partial_Y = (\partial/\partial Y)_q$. For any $f(X,Y) \in \mathbb{C}[[X,Y]]$, we define
\[
\Lambda_X f(X,Y) = f(qX,Y) \quad \text{and} \quad \Lambda_Y f(X,Y) = f(X,qY).
\]
By the definition of the $q$-derivative, we have
\[
(1 - q)X \partial_X = 1 - \Lambda_X \quad \text{and} \quad (1 - q)Y \partial_Y = 1 - \Lambda_Y. \tag{6}
\]
The $q$-commutator of operators $A$ and $B$ is defined as
\[
[A, B]_q = AB - qBA.
\]
We have the following $q$-commutation relations:
\[
[\partial_X, \Lambda_X]_q = [\partial_Y, \Lambda_Y]_q = 0,
[\Lambda_X, X]_q = [\Lambda_Y, Y]_q = 0,
[\partial_X, X]_q = [\partial_Y, Y]_q = 1. \tag{7}
\]
We note that for a formal power series
\[
f(X,Y) = \sum_{n,k=0}^{\infty} a(n,k) \frac{X^n Y^k}{[n]_q! \cdot [k]_q!} \in \mathbb{C}[[X,Y]]
\]
the equality
\[
(q\partial_X \Lambda_Y + \partial_Y - 1)f(X,Y) = \sum_{n,k=0}^{\infty} \left\{ q^{k+1} a(n+1,k) + a(n,k+1) - a(n,k) \right\} \frac{X^n Y^k}{[n]_q! \cdot [k]_q!} \tag{8}
\]
holds. From this, we easily see that
\[
(q\partial_X \Lambda_Y + \partial_Y - 1)F_a(X,Y) = 0. \tag{9}
\]
Lemma 2.4. If a formal power series \( f(X, Y) \in \mathbb{C}[[X, Y]] \) satisfies two conditions
\[
(q\partial_X \Lambda_Y + \partial_Y - 1)f(X, Y) = 0 \quad \text{and} \quad f(X, 0) = 0,
\]
then we have \( f(X, Y) = 0 \).

Proof. Let
\[
f(X, Y) = \sum_{n,k=0}^{\infty} a(n, k) \frac{X^n Y^k}{[n]_q! [k]_q!} \in \mathbb{C}[[X, Y]]
\]
satisfy the two conditions of the lemma. Then, by (8), we have
\[
q^{k+1}a(n+1, k) + a(n, k+1) - a(n, k) = 0 \quad \text{for any } n, k \in \mathbb{N}
\]
and
\[
a(n, 0) = 0 \quad \text{for any } n \in \mathbb{N}.
\]
Therefore we obtain the result by using induction on \( k \).

For any sequence \( a \in \mathbb{C}^\mathbb{N} \), we put
\[
f_a(X, Y) = \sum_{n=0}^{\infty} a(n) \frac{(X - q^n Y)(X - q^{2n} Y) \cdots (X - q^n Y)}{[n]_q!} \in \mathbb{C}[[X, Y]].
\]

We note that
\[
\partial_X \left\{ (X - q^m Y)(X - q^{m+1} Y) \cdots (X - q^n Y) \right\} = [n - m + 1]_q (X - q^m Y)(X - q^{m+1} Y) \cdots (X - q^{n-1} Y) \quad \text{(10)}
\]
and
\[
\partial_Y \left\{ (X - q^m Y)(X - q^{m+1} Y) \cdots (X - q^n Y) \right\} = -q^m [n - m + 1]_q (X - q^m Y)(X - q^{m+1} Y) \cdots (X - q^n Y) \quad \text{(11)}
\]
for any integers \( 1 \leq m \leq n \), which are immediate from the definition of the \( q \)-derivative. A \( q \)-analog of the exponential function is given by
\[
e(X) = \sum_{n=0}^{\infty} \frac{X^n}{[n]_q!} \in \mathbb{C}[[X]].
\]

Proposition 2.5. For any sequence \( a \in \mathbb{C}^\mathbb{N} \), we have
\[
F_a(X, Y) = f_a(X, Y)e(Y).
\]
Proof. It is easily seen that
\[ F_a(X, 0) = f_a(X, 0)e(0). \]
According to Lemma 2.4 and (9), we only have to prove the identity
\[ (q\partial_X \Lambda_Y + \partial_Y - 1) \{ f_a(X, Y)e(Y) \} = 0. \] (12)
By (10), (11) and the \( q \)-Leibniz rule (5), we have
\[ q\partial_X \Lambda_Y \{ f_a(X, Y)e(Y) \} = q \left\{ \sum_{n=1}^{\infty} a(n) \frac{(X - q^n Y)^i}{[n-1]_q!} \right\} e(qY) \]
and
\[ \partial_Y \{ f_a(X, Y)e(Y) \} = -q \left\{ \sum_{n=1}^{\infty} a(n) \frac{(X - q^n Y)^i}{[n-1]_q!} \right\} e(qY) + f_a(X, Y)e(Y). \]
From these, the identity (12) immediately follows.

Corollary 2.6. Let \( a \in \mathbb{C}_N \) be a sequence. Then, for any \( n, k \in \mathbb{N} \), we have
\[ (\Delta_{q,k} a)(n) = \sum_{i=0}^{k} (-1)^i q^\frac{i(i+1)}{2} \binom{k}{i}_q a(n + i). \]

Proof. We apply the operator \( \partial_X^k \partial_Y^i \) to both sides of the equation in Proposition 2.5
\[ \partial_X^k \partial_Y^i F_a(X, Y) = \partial_X^k \partial_Y^i \{ f_a(X, Y)e(Y) \}. \] (13)
The right-hand side is equal to
\[ \sum_{i=0}^{k} \binom{k}{i}_q (\partial_X^i \partial_Y f_a)(X, Y)e(q^i Y) \]
by the \( q \)-Leibniz rule (5). Since we have
\[ (\partial_X^i \partial_Y f_a)(0, 0) = (-1)^i q^\frac{i(i+1)}{2} a(n + i), \]
the desired equality follows from (13) on setting \( X = Y = 0 \).

Corollary 2.7. Let \( a \in \mathbb{C}_N \) be a sequence. Then, for any \( n \in \mathbb{N} \), we have
\[ (\nabla_q a)(n) = \sum_{k=0}^{n} (-1)^k q^\frac{k(k+1)}{2} \binom{n}{k}_q a(k). \]

Proof. It follows immediately from Corollary 2.6 on setting \( n = 0 \).
3 The difference formula for finite multiple harmonic \( q \)-series

We begin with the definition of the dual of a multi-index. A multi-index is a finite sequence of positive integers. For a multi-index \( \mu = (\mu_1, \ldots, \mu_p) \), the quantities \( |\mu| = \mu_1 + \cdots + \mu_p \) and \( l(\mu) = p \) are called the weight of \( \mu \) and the length of \( \mu \), respectively. The multi-indices of weight \( m \) are in one-to-one correspondence with the subsets of the set \( \{1, 2, \ldots, m-1\} \) by the mapping
\[
S_m: (\mu_1, \ldots, \mu_p) \mapsto \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_{p-1}\}.
\]
For example, in the case \( m = 3 \), we have
\[
(3) \mapsto \emptyset, \quad (1, 2) \mapsto \{1\}, \quad (2, 1) \mapsto \{2\} \quad \text{and} \quad (1, 1, 1) \mapsto \{1, 2\}
\]
from the diagrams

\[
\begin{array}{ccc}
1 & 2 & \circ \\
1 & 2 & \downarrow \\
1 & 2 & \circ \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & \circ \\
1 & 2 & \circ \\
1 & 2 & \circ \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & \circ \\
1 & 2 & \circ \\
1 & 2 & \circ \\
\end{array}
\]and \[
\begin{array}{ccc}
1 & 2 & \circ \\
1 & 2 & \circ \\
1 & 2 & \circ \\
\end{array}
\]

**Definition 3.1.** Let \( m \) be a positive integer and \( \mu \) a multi-index of weight \( m \). Then, we define the dual of \( \mu \) by
\[
\mu^* = S_{m-1}(S_m(\mu)^c),
\]
where \( S_m(\mu)^c \) denotes the complement of \( S_m(\mu) \) in the set \( \{1, 2, \ldots, m-1\} \).

Examples are given in [2]. We note that the equality
\[
(l(\mu) - 1) + (l(\mu^*) - 1) = |\mu| - 1
\]
holds for any multi-index \( \mu \). Now, we state the definition of the finite multiple harmonic \( q \)-series which are considered in this paper.

**Definition 3.2.** Let \( \mu = (\mu_1, \ldots, \mu_p) \) be a multi-index. Then, we put
\[
a_\mu(n) = \sum_{n = n_1 \geq \cdots \geq n_p \geq 0} \frac{q^{(\mu_1-1)(n_1+1)+\cdots+(\mu_p-1)(n_p+1)}}{[n_1+1]_q^{\mu_1} \cdots [n_p+1]_q^{\mu_p}}
\]
and
\[
b_\mu(n) = \sum_{n = n_1 \geq \cdots \geq n_p \geq 0} \frac{q^{(n_2+1)+\cdots+(n_p+1)}}{[n_1+1]_q^{\mu_1} \cdots [n_p+1]_q^{\mu_p}}
\]
for any non-negative integer \( n \).
Definition 3.3. Let \( \mu = (\mu_1, \ldots, \mu_p) \) and \( \nu = (\nu_1, \ldots, \nu_r) \) be multi-indices of the same weight \( m \). Then, we put

\[
c_{\mu, \nu}(n, k) = \left[ \frac{n+k}{n} \right]^{-1} \sum_{\substack{n_1 \geq \cdots \geq n_p \geq 0 \\ k_1 \geq \cdots \geq k_r \geq 0}} q^{(\mu_1-1)(n_1+1) + \cdots + (\mu_p-1)(n_p+1) + k_2 + \cdots + k_r} [n_{i_1} + k_{j_1} + 1]_q \cdots [n_{i_m} + k_{j_m} + 1]_q
\]

for any non-negative integers \( n \) and \( k \), where the subscripts \( i_1, \ldots, i_m, j_1, \ldots, j_m \) are defined by

\[
(i_1, \ldots, i_m) = (1, \ldots, 1, 2, \ldots, 2, \ldots, p, \ldots, p)_{\mu_1, \mu_2, \ldots, \mu_p}
\]

and

\[
(j_1, \ldots, j_m) = (1, \ldots, 1, 2, \ldots, 2, \ldots, r, \ldots, r)_{\nu_1, \nu_2, \ldots, \nu_r}.
\]

Let \( \mu \) and \( \nu \) be multi-indices of the same weight. Then, it is easily seen that

\[
c_{\mu, \nu}(n, 0) = a_{\mu}(n)
\]

for any \( n \in \mathbb{N} \). Moreover, by (14), we have

\[
c_{\mu, \mu^*}(0, k) = b_{\mu^*}(k)
\]

for any \( k \in \mathbb{N} \). For any multi-index \( \mu = (\mu_1, \ldots, \mu_p) \) with \( |\mu| \geq 2 \), we define a multi-index \( -\mu \) by

\[
-\mu = \begin{cases} 
(\mu_1 - 1, \mu_2, \ldots, \mu_p) & \text{if } \mu_1 \geq 2 \\
(\mu_2, \ldots, \mu_p) & \text{if } \mu_1 = 1.
\end{cases}
\]

We note that

\[
-(-\mu^*) = (\mu^*)^*.
\]

The following lemma states inductive relations of \( c_{\mu, \nu}(n, k) \).

Proposition 3.4. Let \( \mu = (\mu_1, \ldots, \mu_p) \), \( \nu = (\nu_1, \ldots, \nu_r) \) be multi-indices of the same weight greater than 1 and \( n, k \) non-negative integers.

(i) If \( \mu_1 \geq 2 \) and \( \nu_1 = 1 \), then we have

\[
q^{-n-k-1} \{ [n + k + 1]_q c_{\mu, \nu}(n, k) - [k]_q c_{\mu, \nu}(n, k - 1) \} = c_{-\mu, -\nu}(n, k).
\]

(ii) If \( \mu_1 = 1 \) and \( \nu_1 \geq 2 \), then we have

\[
[n + k + 1]_q c_{\mu, \nu}(n, k) - [n]_q c_{\mu, \nu}(n - 1, k) = c_{-\mu, -\nu}(n, k).
\]
Proof. Since the proof of (ii) is similar to that of (i), we prove only (i). We have

\[ [n+k+1]q^{c_{\mu,\nu}(n,k)} = \left[ \frac{n+k}{n} \right]^{-1} \sum_{n=n_1 \geq \ldots \geq n_p \geq 0} \frac{q^{(\mu_1-1)(n_1+1)+\ldots+(\mu_p-1)(n_p+1)+k_2+\ldots+k_r}}{[n_{i_2}+k_{j_2}+1]q \cdots [n_{i_m}+k_{j_m}+1]q} \]

and

\[ [k]q^{c_{\mu,\nu}(n,k-1)} = [k]q^{[n]q!\left[\frac{k-1}{n+k-1}\right]!} \sum_{n=n_1 \geq \ldots \geq n_p \geq 0} \frac{q^{(\mu_1-1)(n_1+1)+\ldots+(\mu_p-1)(n_p+1)+k_2+\ldots+k_r}}{[n_{i_2}+k_{j_2}+1]q \cdots [n_{i_m}+k_{j_m}+1]q} \]

\[ = \left[ \frac{n+k}{n} \right]^{-1} \sum_{n=n_1 \geq \ldots \geq n_p \geq 0} \frac{q^{(\mu_1-1)(n_1+1)+\ldots+(\mu_p-1)(n_p+1)+k_2+\ldots+k_r}}{[n_{i_2}+k_{j_2}+1]q \cdots [n_{i_m}+k_{j_m}+1]q} \]

Therefore we obtain

\[ [n+k+1]q^{c_{\mu,\nu}(n,k)} - [k]q^{c_{\mu,\nu}(n,k-1)} = \left[ \frac{n+k}{n} \right]^{-1} \sum_{n=n_1 \geq \ldots \geq n_p \geq 0} \frac{q^{(\mu_1-1)(n_1+1)+\ldots+(\mu_p-1)(n_p+1)+k_2+\ldots+k_r}}{[n_{i_2}+k_{j_2}+1]q \cdots [n_{i_m}+k_{j_m}+1]q} \]

from which the result follows immediately. \(\square\)

We restate Proposition 3.4 in terms of generating functions. For multi-indices \( \mu \) and \( \nu \) of the same weight, we define

\[ G_{\mu,\nu}(X,Y) = \sum_{n,k=0}^{\infty} c_{\mu,\nu}(n,k) \frac{X^n Y^k}{[n]q! [k]q!} \]

**Proposition 3.5.** Let \( \mu = (\mu_1, \ldots, \mu_p) \) and \( \nu = (\nu_1, \ldots, \nu_r) \) be multi-indices of the same weight greater than 1.

(i) If \( \mu_1 \geq 2 \) and \( \nu_1 = 1 \), then we have

\[ q^{-1} \Lambda_X^{-1} \Lambda_Y^{-1} \frac{1 - q^{\Delta_X \Delta_Y}}{1 - q} - Y \) \[ G_{\mu,\nu}(X,Y) = G_{-\mu,-\nu}(X,Y). \]

(ii) If \( \mu_1 = 1 \) and \( \nu_1 \geq 2 \), then we have

\[ \frac{1 - q^{\Delta_X \Delta_Y}}{1 - q} - X \) \[ G_{\mu,\nu}(X,Y) = G_{-\mu,-\nu}(X,Y). \]

**Proof.** These are immediate from Proposition 3.4. \(\square\)
We use Proposition 3.5 in order to prove Theorem 3.8 by induction, from which the main result follows easily. We need two lemmas.

Lemma 3.6. (i) We have

$$(q \partial_X \Lambda_Y + \partial_Y - 1)q^{-1} \Lambda^{-1}_X \Lambda^{-1}_Y \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y \right)$$

$$= q^{-2} \Lambda^{-1}_X \Lambda^{-1}_Y \left( \frac{1 - q^2 \Lambda_X \Lambda_Y}{1 - q} - qY \right) (q \partial_X \Lambda_Y + \partial_Y - 1).$$

(ii) We have

$$(q \partial_X \Lambda_Y + \partial_Y - 1) \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - X \right) = \left( \frac{1 - q^2 \Lambda_X \Lambda_Y}{1 - q} - X \right) (q \partial_X \Lambda_Y + \partial_Y - 1).$$

Proof. (i) By $q$-commutation relations (7), we have

$$[q \partial_X \Lambda_Y + \partial_Y - 1, \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y]_q$$

$$= [q \partial_X \Lambda_Y + \partial_Y, \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y]_q - (1 - q) \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y \right)$$

$$= (q \partial_X \Lambda_Y + \partial_Y - 1) - (1 - q) \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y \right).$$

We transpose the second term of the right-hand side to the left-hand side to obtain

$$(q \partial_X \Lambda_Y + \partial_Y - q) \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y \right) - q \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y \right) (q \partial_X \Lambda_Y + \partial_Y - 1)$$

$$= q \partial_X \Lambda_Y + \partial_Y - 1.$$

Multiplying by the operator $q^{-2} \Lambda^{-1}_X \Lambda^{-1}_Y$ from the left, we see that

$$(q \partial_X \Lambda_Y + \partial_Y - 1)q^{-1} \Lambda^{-1}_X \Lambda^{-1}_Y \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y \right)$$

$$- q^{-1} \Lambda^{-1}_X \Lambda^{-1}_Y \left( \frac{1 - q \Lambda_X \Lambda_Y}{1 - q} - Y \right) (q \partial_X \Lambda_Y + \partial_Y - 1)$$

$$= q^{-2} \Lambda^{-1}_X \Lambda^{-1}_Y (q \partial_X \Lambda_Y + \partial_Y - 1),$$

where we have used the identities

$$\Lambda^{-1}_X \partial_X = q \partial_X \Lambda^{-1}_X \quad \text{and} \quad \Lambda^{-1}_Y \partial_Y = q \partial_Y \Lambda^{-1}_Y.$$

If we transpose the second term of the left-hand side to the right-hand side, we obtain the result.
(ii) By a similar computation as in (i), we obtain
\[
[q\partial_X \Lambda_Y + \partial_Y - 1, \frac{1-q\Lambda_X \Lambda_Y}{1-q} - X]_q \\
= (q\partial_X \Lambda_Y + \partial_Y - 1) - q(1-\Lambda_X)\Lambda_Y - (1-q)X(\partial_Y - 1) \\
= (1-(1-q)X)(q\partial_X \Lambda_Y + \partial_Y - 1).
\]
The second equality is due to \[\Box\]. From this, the desired identity is easily derived.

**Lemma 3.7.** The mappings
\[
q^{-2}\Lambda_X^{-1}\Lambda_Y^{-1}\left(\frac{1-q^2\Lambda_X \Lambda_Y}{1-q} - qY\right) \quad \text{and} \quad \frac{1-q^2\Lambda_X \Lambda_Y}{1-q} - X
\]
from $\mathbb{C}[[X, Y]]$ to itself are injections.

**Proof.** We prove only the first one. The second is similar. Since the mapping $q^{-2}\Lambda_X^{-1}\Lambda_Y^{-1}$ is an injection, we only have to show that the mapping
\[
\frac{1-q^2\Lambda_X \Lambda_Y}{1-q} - qY
\]
is an injection. This is obviously a linear mapping. We suppose that the formal power series
\[
f(X, Y) = \sum_{n,k=0}^{\infty} a(n,k) \frac{X^n Y^k}{[n]q! [k]q!} \in \mathbb{C}[[X, Y]]
\]
is in the kernel of the above operator. Then we have
\[
[n+k+2]_q a(n,k) - q[k]_q a(n,k-1) = 0
\]
for any $n, k \in \mathbb{N}$. By induction on $k$, we see that $a(n,k) = 0$ for any $n, k \in \mathbb{N}$. This completes the proof.

**Theorem 3.8.** For any multi-index $\mu$, we have
\[
(q\partial_X \Lambda_Y + \partial_Y - 1)G_{\mu,\mu^*}(X, Y) = 0.
\]

**Proof.** The proof is by induction on $|\mu|$. In the case $|\mu| = 1$ (i.e. $\mu = (1)$), the theorem follows directly from
\[
G_{(1),(1)}(X, Y) = \sum_{n,k=0}^{\infty} \frac{X^n Y^k}{(n+k+1)q!}.
\]
Let $\mu = (\mu_1, \ldots, \mu_p)$ be a multi-index with $|\mu| \geq 2$. We put $\mu^* = (\mu_1^*, \ldots, \mu_p^*)$. If $\mu_1 \geq 2$, noting $\mu_1^* = 1$, we find that
\[
q^{-2}\Lambda_X^{-1}\Lambda_Y^{-1}\left(\frac{1-q^2\Lambda_X \Lambda_Y}{1-q} - qY\right) (q\partial_X \Lambda_Y + \partial_Y - 1)G_{\mu,\mu^*}(X, Y) = 0
\]
from Lemma 3.6 (i), Proposition 3.5 (i), (17) and the hypothesis of induction. According to Lemma 3.7, we have
\[(q\partial_X \Lambda_Y + \partial_Y - 1)G_{\mu, \mu^*}(X, Y) = 0.\]
Also in the case \(\mu_1 = 1\), we can argue in the same way. Therefore we have completed the proof.

The following is the main result of this paper.

**Corollary 3.9.** Let \(\mu\) be a multi-index. Then we have
\[(\Delta_{q,k} a_{\mu})(n) = c_{\mu, \mu^*}(n, k)\]
for any \(n, k \in \mathbb{N}\).

**Proof.** By (15), we have \(F_{a_\mu}(X, 0) = G_{\mu, \mu^*}(X, 0).\) (The formal power series \(F_a(X, Y)\) is defined in (4) for any sequence \(a \in \mathbb{C}^N.\)) Therefore we obtain
\[F_{a_\mu}(X, Y) = G_{\mu, \mu^*}(X, Y)\]
from Lemma 2.4, (9) and Theorem 3.8. This implies the corollary.

As a corollary of Corollary 3.9, we obtain the duality for finite multiple harmonic \(q\)-series due to Bradley.

**Corollary 3.10.** For any multi-index \(\mu\), we have
\[\nabla_q a_\mu = b_{\mu^*}.\]

**Proof.** Since we have (16), the corollary follows from Corollary 3.9 on setting \(n = 0.\)

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