Iterated period integrals and multiple Hecke $L$-functions

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Abstract

In this paper we express the multiple Hecke $L$-function in terms of a linear combination of iterated period integrals associated with elliptic cusp forms, which is introduced by Manin around 2004. This expression generalizes the classical formula of Hecke $L$-function obtained by the Mellin transformation of a cusp form. Also the expression gives a way of the analytic continuation of the multiple Hecke $L$-function.

1 Introduction

Recently, in [9, 10], Manin introduces a generalization of the period integrals of elliptic cusp forms by means of the iterated path integrals on the complex upper-half plane and discusses several related topics. For example, he studies the iterated version of Mellin transformation and its functional equation, the properties of non-commutative generating series of the iterated integrals, an analogy of the classical period theory and its interpretation in terms of non-abelian group cohomology, and so on. The result which we would like to focus in his paper is an expression of the iterated period integral in terms of special values of a multiple Dirichlet series in their convergent region (see §3.2 in [9]).

In this paper first we define the multiple Hecke $L$-function, which is essentially the same as the Dirichlet series which he introduced, and show the analytic continuation of the function. Next, as a main result, we give the expression of the iterated period integral in terms of a linear combination of $L$-functions, which holds in arbitrary region, and generalizes Manin’s expression. This expression also generalizes the classical formula (1) below given by Mellin transformation. As a consequence, one can write the iterated period integral as a $Q$-linear combination of special values of $L$-function.

Let $H = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$ be the complex upper-half plane. Assume that $f(z)$ is a holomorphic function on $H$ satisfying following two conditions:
(i) (Fourier expansion) $f(z)$ is periodic with period one and has a Fourier expansion of the form $f(z) = \sum_{m=1}^{\infty} c_m q^m$, where $q = e^{2\pi i z}$ whose coefficients $\{ c_m \}$ have at most polynomial growth in $m$ when $m \to \infty$: $c_m = O(m^M)$ for some $M > 0$.

*This work was partially supported by Priority Research Centers Program NRF 2009-0094069.
†This work was supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant 2009-0094069). Mathematics Subject Classification; Primary 11E45, Secondary 11M32.
(ii) (Cusp conditions) For any \( \gamma \in SL_2(\mathbb{Z}) \), there exists a constant \( C > 0 \) and an integer \( k \) such that
\[
(f|k\gamma)(z) := j(\gamma, z)^{-k} f(\gamma z) = O(e^{2\pi iCz}), \quad z \to i\infty,
\]
where \( j(\gamma, z) = cz + d \) and \( \gamma z = (az + b)/(cz + d) \) for \( \gamma = (a \ b \ c \ d) \).

For example, let \( \Gamma \) be a congruence subgroup of \( SL_2(\mathbb{Z}) \) containing the translation matrix \( (0 \ 1) \) and \( S_k(\Gamma) \) be the space of holomorphic \( \Gamma \)-cusp forms of weight \( k \). Then \( f(z) \in S_k(\Gamma) \) satisfies both (i) and (ii).

The Mellin transformation of \( f(z) \) gives the following expression:
\[
\int_{i\infty}^{0} f(z) \ z^{s-1} dz = -\Gamma(s)L(f, s) \tag{1}
\]
where \( \Gamma(s) = \int_{0}^{\infty} e^{-t s^{-1}} dt \) is the Gamma function and \( L(f, s) := (-2\pi i)^{-s} \sum_{m=1}^{\infty} c_m m^{-s} \) for \( \text{Re } s > 0 \) is the Hecke \( L \)-function attached to \( f \) which is normalized by \( (-2\pi i)^{-s} \) for the sake of convenience. (We take the non-positive imaginary axis as a branch cut in the \( z \)-plane, so that \( \text{arg}(z) \in [-\pi/2, 3\pi/2] \) to define the complex power throughout this paper.) (Following \[9\] [10], we chose \( i\infty \) as the base point of paths. The minus sign in RHS of \( (1) \) happens for this choice.) The condition (i) guarantees the convergence of the \( L \)-function in \( \text{Re } s > 0 \) and also (ii) implies the convergence of the integral in \( (1) \) in \( s \in \mathbb{C} \). For \( f \in S_k(\Gamma) \), the special values defined by the integral in \( (1) \) at critical points \( s = 1, \ldots, k-1 \) are called periods of \( f \) and play a fundamental role in the period theory.

Shokurov \[13\] constructs some varieties, so-called Kuga-Sato varieties, and interprets these numbers as periods of relative homology of the varieties. See also e.g. \[7\] [8] for the period theory. The equation \( (1) \) gives not only the interpretation of the periods as the special values of \( L \)-function but the way of the analytic continuation of \( L \)-functions.

Manin generalizes the integral in \( (1) \) as follows. For \( r = 1, \ldots, n \), let \( f_r \) be holomorphic functions on \( H \) satisfying the conditions (i) and (ii) above (for instance \( f_r \in S_{k_r}(\Gamma) \) for integers \( k_r \) and \( s_r \) be complex variables. For points \( a, z \in \overline{H} := H \cup P^1(\mathbb{Q}) \), we fix a path joining \( a \) to \( z \) on \( H \) and approaching \( a \) and \( z \) vertically if \( a \) or \( z \) is in \( P^1(\mathbb{Q}) \). Then we consider the iterated integral along the path:
\[
I_a^z \left( s_1, \ldots, s_n \right) := \int_a^{z_1} f_1(z_1) z_1^{s_1-1} dz_1 \int_a^{z_2} f_2(z_2) z_2^{s_2-1} dz_2 \cdots \int_a^{z_n} f_n(z_n) z_n^{s_n-1} dz_n. \tag{2}
\]
The integral converges again because of cusp conditions of \( f_r \)'s and defines a holomorphic function in \( z \in H \) and in \( (s_1, \ldots, s_n) \in \mathbb{C}^n \). For a connection between these integrals and the motif theory, see the recent paper \[15\] of Ichikawa. The purpose of this paper is to state a generalization of \( (1) \) in the case of the iterated integrals.

In \[2\] we introduce the multiple Hecke \( L \)-functions (Definition \( 1 \)) and generalize the identity \( (1) \) to an iterated case (Theorem \( 1 \)). The LHS in \( (1) \) will be replaced by Manin’s iterated integral. A linear combination of the multiple Hecke \( L \)-functions will appear in RHS. As a consequence, one can write the iterated integral as a \( \mathbb{Q} \)-linear combination of special values of \( L \)-function. Conversely the special values of \( L \)-function also can be expressed by the sum of iterated integrals (Corollary \( 1 \)). In \[3\] we give proofs of these
theorems. In last section, we discuss the further properties of some functions which will be introduced in §2 for the proofs.

In the last part of the Introduction, it is worth mentioning that the integral (2) can be seen as an analogy of the multiple polylogarithm (MPL):

\[
L_{\eta_1, \ldots, \eta_r}(z) := \sum_{m_1 > \cdots > m_r > 0} \frac{z^{m_1}}{m_1^{\eta_1} \cdots m_r^{\eta_r}} = \int_0^z \eta_1 \eta_2 \cdots \eta_r \eta(z)
\]

where \(|z| < 1\), \(n_1\) are positive integers with \(n = \sum n_1\) and \(\eta_j(z)\) are holomorphic 1-forms on \(P^1(C) \setminus \{0, 1, \infty\}\) defined by \(\eta_j(z) = dz/(1 - z)\) or \(dz/z\) if \(l = \sum_{m=1}^j n_m\) for some \(j = 1, \ldots, r\) or otherwise, respectively. If \(n_1 > 1\), the limits of (3) as \(z \to 1\) exist and are called multiple zeta values (MZVs). It is known that MZVs have a rich theory and a geometric origin associated to \(P^1(C) \setminus \{0, 1, \infty\}\) or the mixed Tate motifs over \(Z\). See [3, 4, 14].

As mentioned in [9, 10], the iterated integral (2) can be seen as an analogy of (3). The 1-forms on \(H\) replaced with those on \(P^1(C) \setminus \{0, 1, \infty\}\). In this reason, we expect a rich theory behind the values defined by (2) such as MZVs. As a consequence of results in this paper, those values will be able to interpreted as the special values of multiple Hecke \(L\)-function. For a related topic, see [6].

## 2 Periods and multiple Hecke \(L\)

In this section we state our main results. The proofs will be given in §3. Recall that \(f_r(\alpha, \ldots, \alpha)\) are holomorphic functions on \(H\) satisfying the conditions (i) and (ii) in §1. Suppose that these functions have the Fourier expansions of the forms \(f_r(z) = \sum_{m=1}^\infty c_m^{(r)} \eta_m^m\).

**Definition 1 (multiple Hecke \(L\)-function)** For \(s \in C\) with \(\text{Re } s > 0\) and for any integers \(\alpha_r \geq 1\), we define

\[
L(s) = L \left( f_1, f_2, \ldots, f_n \right) := (-2\pi i)^{-(s + \alpha_2 + \cdots + \alpha_n)} \sum_{m_1 > \cdots > m_n > 0} \frac{c_m^{(1)} \cdots c_m^{(n)}}{m_1^{\alpha_1} \cdots m_n^{\alpha_n}} \left( \frac{1}{m_1} \cdots \frac{n}{m_n} \right)
\]

Since Fourier coefficients \(c_m^{(r)}\) have polynomial order in \(m\) for any \(r\), there exists a constant \(M > 0\) such that \(|c_m^{(1)} \cdots c_m^{(n)}| = O(m_1^M)\). This implies the absolute convergence of \(L(s)\) in \(\text{Re}(s) > M + 1\).

In the following theorem, we claim that \(L(s)\) can be extended to an entire function on \(C\) for every fixed positive integers \(\alpha_2, \ldots, \alpha_n\). However the analytic continuation of this kind of Dirichlet series had been studied by Matsumoto-Tanigawa [11] in a general
context. They can regard other parameters $\alpha_2, \ldots, \alpha_n$ as complex variables and show the analytic continuation to $\mathbb{C}^n$ by using the Mellin-Barnes formula. Our method in the next section is very simple and different from theirs, but may not treat other parameters as variables.

**Theorem 1** For any fixed positive integers $\alpha_r$ for $2 \leq r \leq n$, the function $L(s)$ can be extended holomorphically to $\mathbb{C}$. Further, the following equations hold for any $s \in \mathbb{C}$:

(i) \( I_0^\infty \left( s, a_2, \ldots, a_n \right) = \Gamma(s, a_2, \ldots, a_n) \times \sum_{0 \leq j < a_r + j_r + 1} \left( \frac{s+j_r-1}{j_r} \right) \prod_{0 \leq j_r \leq n, j_l = j_{l+1} = 0} \left( \frac{a_l-1+j_l-1}{j_l} \right) L \left( s+j_r, a_2-j_r+j_2, \ldots, a_n-j_n+j_n+1 \right) \),

(ii) \( L \left( s, a_2, \ldots, a_n \right) = \frac{1}{\Gamma(s, a_2, \ldots, a_n)} \times \sum_{0 \leq j < a_r} \left( \frac{s+j_r-1}{j_r} \right) \prod_{0 \leq j_r \leq n, j_l = j_{l+1} = 0} \left( \frac{a_l-1+j_l-1}{j_l} \right) I_0^\infty \left( s+j_r, a_2-j_r+j_2, \ldots, a_n-j_n+j_n+1 \right) \).

where $\Gamma(s, a_2, \ldots, a_n) = (-1)^n \Gamma(s) \Gamma(a_2) \cdots \Gamma(a_n)$.

**Corollary 1** For any integers $\alpha_r \geq 1$, we have

(i) \( I_0^\infty \left( \alpha_1, \ldots, a_n \right) = \Gamma(\alpha_1, \ldots, a_n) \sum_{0 \leq j < a_r + j_r + 1} \left( \frac{\alpha_1-1+j_1-1}{j_1} \right) L \left( \alpha_1-j_1+j_2, \ldots, a_n-j_n+j_n+1 \right) \),

(ii) \( L \left( \alpha_1, \ldots, a_n \right) = \frac{1}{\Gamma(\alpha_1, \ldots, a_n)} \sum_{0 \leq j < a_r} \left( \frac{\alpha_1-1+j_1-1}{j_1} \right) I_0^\infty \left( \alpha_1-j_1+j_2, \ldots, a_n-j_n+j_n+1 \right) \).

3 Proof

Corollary 1 follows directly from Theorem 1 by substituting $s = \alpha_1$. Hence we will prove Theorem 1. The procedure of the proof of (i) is as follows: The LHS of (i) is the Mellin transformation of the function

$$F_\alpha \left( \frac{\alpha_1}{a_1}, a_2, \ldots, a_n \right) := f_1(z) I_\alpha \left( \frac{\alpha_2, \ldots, a_n}{f_2, \ldots, f_n} \right)$$

for $a = i \infty$. In the first step, we define an alternative family of functions $\tilde{F}_\alpha \left( a_1, a_2, \ldots, a_n \right)$ (Definition 2) then describe $F_\alpha$ by a linear combination of $\tilde{F}_\alpha$'s (Proposition 11). In the second step, we compute the Fourier expansion and Mellin transformation of $\tilde{F}_\alpha$ (Proposition 12, 13). In particular, the Mellin transformation of $\tilde{F}_\alpha$ gives the analytic continuation
of \(L(s)\). Last, we will compute the Mellin transformation of \(F^z_a\) by combining the results in the first and second steps. For the proof of (ii), we use an inversion description of \(\tilde{F}^z_a\) in terms of \(\tilde{F}^z_a\)'s in Proposition 1.

**Definition 2** Let \(a_r\) be positive integers for \(r = 1, \ldots, n\). For \(a \in \mathbb{H}\), we define holomorphic functions \(\tilde{I}^z_a\) and \(\tilde{F}^z_a\) in \(z \in \mathbb{H}\) by

\[
\tilde{I}^z_a \left( a_1^{\alpha}, \ldots, a_n^{\alpha} \right) := \int_a^z f_1(z_1)(z_1 - z)^{a_1 - 1}dz_1 \int_a^{z_1} \cdots \int_a^{z_{n-1}} f_n(z_n)(z_n - z_{n-1})^{a_n - 1}dz_n,
\]

\[
\tilde{F}^z_a \left( a_1^{\alpha}, \ldots, a_n^{\alpha} \right) := f_1(z)\tilde{I}^z_a \left( a_2^{\alpha}, \ldots, a_n^{\alpha} \right), \quad a \in \mathbb{H}.
\]

The integral converges again by virtue of the cusp conditions of \(f_i\)'s. In this definition, the parameters \(a_r\) should be positive integers, otherwise the integral depends on the choice of path because the factor \((z_r - z_{r-1})^{a_r - 1}\) has a singularity on \(H\) at \(z_r = z_{r-1}\). By definition \(\tilde{F}^z_a(f_1) = f_1(z)\) is independent on \(a\). The integral \(\tilde{I}^z_a(f_1) = \int_a^z f_1(z_1)(z_1 - z)^{a_1 - 1}dz_1\) is known as the Eichler integral of \(f\) when \(f \in S_k(\Gamma)\) and \(a = k - 1\). In this sense, \(\tilde{I}^z_a\) is an iterated version of the Eichler integral. (A modular property of \(\tilde{I}^z_a\) will be stated in the next section.)

**Proposition 1** For \(z, a \in \mathbb{H}\), we have

\[(i) \quad F^z_a \left( a_1^{\alpha}, \ldots, a_n^{\alpha} \right) = \sum_{0 \leq j_r < a_r + j_{r+1}} \prod_{l=2}^n (a_l + j_{l-1} - 1)_{z_l^2} \tilde{F}^z_a(f_1, f_2, \ldots, a_n - j_{n+1}) \]

\[(ii) \quad \tilde{F}^z_a \left( a_1^{\alpha}, \ldots, a_n^{\alpha} \right) = \sum_{0 \leq j_r < a_r} \prod_{l=2}^n (-1)^{j_l} (a_l - 1)_{z_l^2} \tilde{F}^z_a(f_1, f_2, \ldots, a_n - j_{n+1}) \]

**Proof.** Induction on \(n\). For (i), when \(n = 2\) it is easy. For \(n > 2\), by using the inductive hypothesis, we have

\[
F^z_a \left( a_1^{\alpha}, \ldots, a_n^{\alpha} \right) = f_1(z) \int_a^z z^{a_2 - 1} \tilde{F}^z_a(f_2, f_3, \ldots, a_n) \ dz_2
\]

\[
= f_1(z) \int_a^z z^{a_2 - 1} \prod_{0 \leq j_r < a_r + j_{r+1}} \prod_{l=3}^n (a_l + j_{l-1} - 1)_{z_l^2} \tilde{F}^z_a(f_2, f_3, \ldots, a_n - j_{n+1}) \ dz_2.
\]

By the following binomial expansion

\[
z_2^{a_2 + j_2 - 1} = (z_2 - z + z)^{a_2 + j_2 - 1} = \sum_{0 \leq j_2 < a_2 + j_3} \binom{a_2 + j_2 - 1}{j_2} (z_2 - z)^{a_2 - j_2 + j_3 - 1} z^{j_2}
\]

we get (i). (ii) follows directly from the binomial expansions. \(\square\)
Proposition 2 (Fourier expansion)  The Fourier expansions of $\overline{I}_{\infty}^z$ and $\overline{F}_{\infty}^z$ are given by

$$\overline{I}_{\infty}^z (a_1, \ldots, a_n) = \frac{\Gamma(a_1, \ldots, a_n)}{(-2\pi i)^{a_1+\cdots+a_n}} \sum_{m_1 > \cdots > m_n > 0} \frac{c_{m_1-m_2} \cdots c_{m_n-m_{n+1}}^{(n)}}{m_1^{a_1} \cdots m_n^{a_n}} q^{m_1},$$

$$\overline{F}_{\infty}^z (a_1, \ldots, a_n) = \frac{\Gamma(a_1, \ldots, a_n)}{(-2\pi i)^{a_2+\cdots+a_n}} \sum_{m_1 > \cdots > m_n > 0} \frac{c_{m_1-m_2} \cdots c_{m_n-m_{n+1}}^{(n)}}{m_2^{a_2} \cdots m_n^{a_n}} q^{m_1}.$$  

In particular $\overline{I}_{\infty}^z$ and $\overline{F}_{\infty}^z$ are periodic functions in $z$ of period one.

Proof. One can prove these by comparing the derivative of both-hand sides in $z$, and by an inductive argument.

Proposition 3 (Mellin transform)

$$\int_0^1 \overline{I}_{\infty}^z (a_1, \ldots, a_n) \, z^{s-1} \, dz = \Gamma(s,a_1, \ldots, a_n) L (s+a_1, a_2, \ldots, a_n),$$

$$\int_0^1 \overline{F}_{\infty}^z (a_1, \ldots, a_n) \, z^{s-1} \, dz = \Gamma(s,a_2, \ldots, a_n) L (s,a_2, \ldots, a_n).$$  

In particular, by eq. (8) we can extend $L(s)$ to an entire function.

Proof. By applying the Mellin transformation to the equations in Proposition 2 we easily get both expressions. Since the LHS of eq. (8) and $1/L(s)$ are entire, $L(s)$ can be extended entirely.

Proof of Theorem 1  In Proposition 3 we have already proved the analytic continuation of $L(s)$. For (i), by using eq. (5) and (8),

$$\int_0^1 \overline{I}_{\infty}^z (s, a_2, \ldots, a_n) \, dz = \int_0^1 \overline{F}_{\infty}^z (s, a_2, \ldots, a_n) \, dz \quad z^{s-1} \, dz$$

$$= \sum_{0 \leq j < a_r + a_{r+1}} \prod_{l=2}^n (a_{l+j_l+1})^{-1} \int_0^1 \overline{F}_{\infty}^z (s, a_{2+j_2}, \ldots, a_{r+j_r}) \, dz \quad z^{s+j_2-1} \, dz$$

$$= \sum_{0 \leq j < a_r + a_{r+1}} \prod_{l=2}^n (a_{l+j_l+1}) \Gamma(s+j_2, a_2, \ldots, a_{r+j_r}) \, L (s+a_2, \ldots, a_{r+j_r}).$$

The elementary equation below completes the proof of (i):

$$\prod_{l=2}^n (a_{l+j_l+1}) \Gamma(s+j_2, a_2, \ldots, a_{r+j_r}) = \Gamma(s, a_2, \ldots, a_n) \prod_{l=3}^n (a_{l-1+j_l-1}).$$

For (ii), apply the Mellin transformation to eq. (7), and use eq. (8).
4 Further properties of $\tilde{T}_a^z$ and $\tilde{F}_a^z$

In this section, we show several further properties of the functions $\tilde{T}_a^z$ and $\tilde{F}_a^z$ defined in the previous section.

4.1 Alternative expression

Since the function $\int_a^z f(w)(w-z)^{a-1} dw$ in $z$ gives the $a$-th anti-derivative of $f(z)$ up to a constant multiple in general, $\tilde{T}_a^z$ has an alternative simple expression in terms of an iterated integral:

Proposition 4 For any positive integers $a_r$ we have

$$\tilde{T}_a^z \left( a_1, ..., a_n \right) = (-1)^{a_1+\cdots+a_n} \Gamma(a_1,\ldots,a_n)$$

$$\times \left( \int_a^z dz_1 \int_a^{z_1} dz_2 \cdots \int_a^{z_{a_1-1}} f(z_1) dz_1 \cdots \int_a^{z_{a_1-1}} f(z_{a_1}) dz_1 \right) \int_a^{z_{a_1}} f(z_{a_1}) dz_{a_1}.$$

4.2 Modular properties and its applications

In this subsection we show modular properties of $\tilde{T}_a^z$ and $\tilde{F}_a^z$ and discuss its applications.

Let $a_r$ be positive integers for $r = 1, \ldots, n$. We fix the positive integers $k_r$ by $k_r = a_r + a_{r+1}$ for $r = 1, \ldots, n$ with $a_{n+1} := 1$ throughout this subsection. Conversely $a_r$ are expressed by $k_r$’s as $a_r = (-1)^{n-r+1} + \sum_{j=r}^{n} (-1)^{j-r} k_j$.

Proposition 5 For $a \in \overline{H}$ and $\gamma \in \text{GL}^+_2(\mathbb{Q})$ (+ means positive determinant), we have

$$(I_n) \quad \tilde{T}_a^z \left( a_1, ..., a_n \right) |_{(a_1+1)} \gamma = \tilde{T}_{\gamma^{-1}}^{-1} a^z \left( a_1, ..., a_n \right)$$

$$(F_n) \quad \tilde{F}_a^z \left( a_1, ..., a_n \right) |_{(a_1+1)} \gamma = \tilde{F}_{\gamma^{-1}}^{-1} a^z \left( a_1, ..., a_n \right)$$

where the slash action is defined in the usual manner: $(f|k \gamma)(z) := (\det \gamma)^{k/2} (\gamma z)^{-k} f(\gamma z)$ for any function $f$ on $H$.

Proof. First we check $(F_1)$ then show implications $(F_n) \Rightarrow (I_n)$ and $(I_{n-1}) \Rightarrow (F_n)$ for all $n$. $(F_1)$ is trivial:

$$f_1 |_{(a_1+1)} \gamma = f_1 |_{k_1 \gamma}$$

because of the choice of $k_1 = a_1 + 1$. For $(F_n) \Rightarrow (I_n)$, we have

$$\tilde{T}_a^z \left( a_1, ..., a_n \right) |_{(a_1+1)} \gamma = \left( \int_a^z \tilde{T}_a^z \left( a_1, a_2, ..., a_n \right) (z_1 - z)^{a_1-1} dz_1 \right) |_{(a_1+1)} \gamma$$

$$= (\det \gamma)^{-a_1-1/2} j(\gamma z)^{a_1-1} \int_a^z \tilde{T}_a^z \left( a_1, a_2, ..., a_n \right) (z_1 - z)^{a_1-1} dz_1$$

$$= (\det \gamma)^{-a_1-1/2} j(\gamma z)^{a_1-1} \int_a^z \tilde{T}_a^z \left( a_1, a_2, ..., a_n \right) (\gamma z_1 - z)^{a_1-1} d(\gamma z_1). \quad (9)$$
Substituting the formulas
\[ \gamma z_1 - \gamma z = (\det \gamma) f(\gamma, z_1)^{-1} j(\gamma, z)^{-1} (z_1 - z), \quad d(\gamma z_1) = (\det \gamma) f(\gamma, z_1)^{-2} dz_1 \]
and the induction hypothesis \((F_n)\):
\[ \tilde{F}_n^\gamma(z) \left( f_{1,1} \gamma, f_{2,1} \gamma, \ldots, f_{n,1} \gamma \right) = (\det \gamma)^{-a_1 + 1} \tilde{F}_n^\gamma(z) \left( f_{1,1}, f_{2,1}, \ldots, f_{n,1} \right) \]
into (9), we obtain the first implication:
\[ \tilde{F}_n^\gamma(z) \left( f_{1,1} \gamma, f_{2,1} \gamma, \ldots, f_{n,1} \gamma \right) (z_1 - z)^{a_1 - 1} dz_1 = \tilde{F}_n^\gamma(z) \left( f_{1,1}, f_{2,1}, \ldots, f_{n,1} \right) . \]
For \((I_{n-1}) \Rightarrow (F_n)\),
\[ \tilde{F}_n^z \left( f_{1,1} \gamma, f_{2,1} \gamma, \ldots, f_{n,1} \gamma \right) |_{(a_1+1) \gamma} = \left( f_1(z) \tilde{F}_n^z \left( f_{2,1} \gamma, \ldots, f_{n,1} \right) \right) |_{(a_1+1) \gamma} = (\det \gamma)^{a_1 + 1} j(\gamma, z)^{-(a_1+1) f_1(z) \tilde{F}_n^z \left( f_{2,1} \gamma, \ldots, f_{n,1} \right) .} (10) \]
By using the assumption \((I_{n-1})\):
\[ \tilde{F}_n^z \left( f_{2,1} \gamma, \ldots, f_{n,1} \right) = (\det \gamma)^{-a_2 + 1} j(\gamma, z)^{-a_2 + 1} \tilde{F}_n^z \left( f_{2,1} \gamma, \ldots, f_{n,1} \right) , \]
we have
\[ (10) = (\det \gamma)^{a_1 + a_2} j(\gamma, z)^{-a_1 + a_2} f_1(z) \tilde{F}_n^z \left( f_{2,1} \gamma, \ldots, f_{n,1} \right) = (f_1 |_{(a_1+a_2) \gamma}) (z) \tilde{F}_n^z \left( f_{2,1} \gamma, \ldots, f_{n,1} \right) \]
This completes the proof.

We show two applications of Proposition 5.

4.2.1 Functional equations

Let \( w_N = (0 \quad -1) \) (Fricke involution). Suppose that \( f_r \) are \( w_N \)-eigenfunctions of weight \( k_r \): \( f_{i,k} w_N = \epsilon_r f_r, \epsilon_r \in \{ \pm 1 \} \) for \( r = 1, \ldots, n \). Put \( g(z) := \frac{1}{\sqrt{N}} \int_0^z g(z) z^{s-1} dz \).

Theorem 2 It holds that
\[ \Lambda(s, g) = (-1)^{a_1 - 1 + s} N^{\frac{a_1 + 1}{2} - s} \epsilon \Lambda(-a_1 + 1 - s, g) \]
where \( \epsilon = \prod_{r=1}^n \epsilon_r \).

Proof. For any holomorphic function \( f(z) \) on \( H \) satisfying the condition (ii) in §1, one can show by the standard method that the Mellin transformation \( \Lambda(s, f) := \int_0^1 f(z) z^{s-1} dz \) satisfies
\[ \Lambda(s, \tilde{f}) = (-1)^s N^{-\frac{s}{2} - s} \Lambda(s, f) \]
where \( \alpha \) is any integer and \( \tilde{f}(z) = (f | w_N)(z) \). If we substitute \( g \) to \( f \) and \( -a_1 + 1 \) to \( \alpha \), then we obtain the theorem because \( \tilde{g} = \epsilon g \) is true by virtue of Proposition 5. \( \square \)
4.2.2 Twisted iterated integral

For any Dirichlet character $\chi \mod M$ we set $f^{\chi}(z) := \sum_{m=1}^{M} \chi(m)f\left(\frac{z+im}{M}\right)$. It is well-known \cite{12} that a holomorphic function $f$ on $H$ is $\Gamma_0(N)$-invariant of weight $k$: $f|_k \gamma = f$ for $\forall \gamma \in \Gamma_0(N)$ if and only if it is periodic with period one and satisfies

$$f^{\chi}|_k \sigma = \chi(-1)\overline{f^\chi}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for all Dirichlet characters $\chi \mod Nc$ for each $c \in \{1, \ldots, N\}$, where $\overline{\chi}$ is the complex conjugate of $\chi$.

Suppose $f_r \in S_k(\Gamma_0(N))$ and $\chi_r$ are Dirichlet characters of mod $Nc$ with $1 \leq c \leq N$ for $r = 1, \ldots, n$. Define two functions by

$$G_\chi(z) := \tilde{I}_{\tilde{T}}\left(\begin{smallmatrix} a_{11} & \cdots & a_{1n} \\ \tilde{T}_{11} & \cdots & \tilde{T}_{1n} \end{smallmatrix}\right), \quad G_{\overline{\chi}}(z) := \tilde{I}_{\tilde{T}}\left(\begin{smallmatrix} a_{11} & \cdots & a_{1n} \\ \tilde{T}_{11} & \cdots & \tilde{T}_{1n} \end{smallmatrix}\right)$$

and denote their Mellin transformations by $\Lambda(s, G_\chi)$ and $\Lambda(s, G_{\overline{\chi}})$ respectively. By Proposition \cite{5} and above fact, one can show that

$$G_\chi|_{(-\alpha_1+1)} \sigma = \chi_1(-1) \cdots \chi_n(-1)G_{\overline{\chi}}.$$

Hence we have the following identity:

**Theorem 3**

$$\Lambda(s, G_\chi) = (-1)^s \chi_1(-1) \cdots \chi_n(-1) \Lambda(1 - \alpha_1 - s, G_{\overline{\chi}}).$$

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