CLASSIFICATION OF FUNDAMENTAL GROUPS OF GALOIS COVERS
OF SURFACES OF SMALL DEGREE DEGENERATING
TO NICE PLANE ARRANGEMENTS

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ABSTRACT. Let $X$ be a surface of degree $n$, projected onto $\mathbb{CP}^2$. The surface has a natural Galois cover with Galois group $S_n$. It is possible to determine the fundamental group of a Galois cover from that of the complement of the branch curve of $X$. In this paper we survey the fundamental groups of Galois covers of all surfaces of small degree $n \leq 4$, that degenerate to a nice plane arrangement, namely a union of $n$ planes such that no three planes meet in a line. We include the already classical examples of the quadric, the Hirzebruch and the Veronese surfaces and the degree 4 embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$, and also add new computations for the remaining cases: the cubic embedding of the Hirzebruch surface $F_1$, the Cayley cubic (or a smooth surface in the same family), for a quartic surface that degenerates to the union of a triple point and a plane not through the triple point, and for a quartic 4-point. In an appendix, we also include the degree 8 surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ embedded by the $(2,2)$ embedding, and the degree $2n$ surface embedded by the $(1,n)$ embedding, in order to complete the classification of all embeddings of $\mathbb{CP}^1 \times \mathbb{CP}^1$, which was begun in [23].

1. Introduction

1.1. Background. Algebraic surfaces are classified by discrete and continuous invariants. Fixing the discrete invariants gives a family of algebraic surfaces parameterized by algebraic variety called the moduli space. All surfaces in the same moduli space have the same homotopy type and therefore the same fundamental group. So, fundamental groups are discrete invariants of the surfaces and form a central tool in their classification. We have no algorithm to compute the fundamental group of a given projective algebraic surface $X$, but

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we can cover \( X \) by a surface \( X_{\text{Gal}} \) with a computable fundamental group. The surface \( X_{\text{Gal}} \) is the Galois cover of \( X \), and its fundamental group is a crucial invariant of the surface \( X \). It is connected with the fundamental group of the complement of the branch curve of a generic projection of the surface \( X \).

One common solution to this problem is to degenerate \( X \) to a union of planes, whose branch curve is a line arrangement in the plane. The fundamental groups of line arrangements are easily computable and can be exploited to obtain information on the original branch curve and on the original surface. The idea of using degenerations for these purposes appears in [1], [14], [17], [23] and [28]. Degenerations of \( K3 \) surfaces were constructed in [14], [15] and [17], and are called pillow degenerations.

In recent years, braid groups have been becoming more and more popular in different branches of mathematics, such as the theory of Jones polynomials (which is a current interest of theoretical physicists). Former studies show that some nontrivial properties of braid groups are intimately connected with the existence of nontrivial geometrical objects of complex geometry and algebraic surfaces. Thus, our study of algebraic surfaces and their branch curves gives new insight in the field of braid groups and vice versa.

The applications of the braid group technique to the study of algebraic curves in general and to the topology of complements of curves in particular, started with Artin in [10] and [11]; see also Chisini [13] and Van Kampen [33]. The braid monodromy technique is due to ideas of Chisini, Enriques, Zariski and van Kampen in the 30’s (see [20], [33], [34], [35], [36]). It was revived by Moishezon in the late 1970’s, and was first presented in a complete form by Moishezon in [19] and [21], and Moishezon-Teicher in [20], [25] and [26]. Many examples of computations of braid monodromy have been executed, see for example in [1], [7], [26], [30]. Also, in [12] one can find a description of computations of braid monodromy and the fundamental group \( \pi_1(\mathbb{CP}^2 - S) \) of the complement of the branch curve of a surface.

In [12], the authors also considered the branch curve of a projection of a non-prime \( K3 \) surface. But they considered quotients of \( \pi_1(\mathbb{CP}^2 - S) \), where \( S \) denotes the branch curve, by a subgroup of commutators (commutators of geometric generators which are mapped to disjoint transpositions by the geometric monodromy representation, see Definition 2.2 in that paper). Their motivation came from the theory of symplectic manifolds and families of projections where the branch curves acquire and may lose pairs of transverse double points.
with opposite orientations; creating a pair of double points adds a commutation relation. The quotient there is the largest possible quotient of $\pi_1(\mathbb{CP}^2 - S)$.

Several other small cases have been considered individually: in [3], the authors compute the fundamental group of the complement of the branch curve of the Cayley cubic surface, and in [6], they compute it for the $(1, 1)$ embedding of the Hirzebruch surface $F_2$. ([31] also computes the fundamental group of the Galois cover of complete intersections.) In [4] and [5], the authors compute the fundamental group of the complement of the branch curve for the second Hirzebruch surface and for the product of a projective line and a torus, respectively.

Fundamental groups of Galois covers were first considered in [1], [2], [8] and [9]. In [2], we encounter new types of singularities, namely 3-points, local intersection points of three distinct lines, which had not been handled before, and whose analysis is necessary for the precise computations of the braid monodromy. Moreover, since the monodromies related to 6-points (first discussed in [26] and [29]) are quite hard to follow, [2] presents them in a precise way algebraically, accompanied by figures illustrating the computations. [8] deals with the Hirzebruch surface $F_1$, [22] and [16] deal with covers of Hirzebruch surfaces, while [9] deals with the product of two tori, whose branch curve was first investigated in [7]. Robb [31] computed the fundamental groups for complete intersection surfaces. The 4-point appears first in a local computation in [26]; we show it here in the context of an explicit algebraic surface.

The goals of this paper are twofold: first, to survey the fundamental groups of the complements of the branch curves and of the Galois covers of all degenerations of surfaces of low degrees (less than 5), and second, to complete the classification of Galois covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$, which was begun in [19], [21] and [27].

1.2. Method. In order to compute the possible fundamental group of the Galois cover of a surface of small degree with at worst isolated singularities and with a nice degeneration, we degenerate the surface to a union of planes, and then apply the regeneration process. The algebra is independent of the specific choice of surface, as it is determined entirely combinatorially. Therefore, to construct the set of all possible such fundamental groups of Galois covers of surfaces of degrees 2, 3 and 4, we construct combinatorial representations of each of the possible admissible arrangements of two, three or four planes, i.e., those in which no two planes meet in a line, and we verify that each of them in fact corresponds to a degeneration of
some algebraic surface. Note that there are many possible plane arrangements corresponding
to each diagram; however, the fundamental groups of the complement of the branch curve
and of the Galois cover are determined combinatorially, and are thus independent of the
specific arrangement. We then calculate the braid monodromy factorization and the funda-
mental groups using the method of Moishezon-Teicher. We present the computations in
some detail for the convenience of the reader, and so as to make all the examples completely
explicit. The computations proceed in three steps.

First, we compute the braid monodromy factorization of the branch curve, as defined in
[25, Prop. VI.2.1].

Consider the following setting (Figure 1). Let \( S \) be an algebraic curve in \( \mathbb{C}^2 \), with \( \pi : \mathbb{C}^2 \to \mathbb{C} \) be a generic projection onto the first coordinate. Define the
fiber \( K(x) = \{ y \mid (x, y) \in S \} \) in \( S \) over a fixed point \( x \), projected to the y-axis. Define
\( N = \{ x \mid \#K(x) < \deg(S) \} \) and \( M' = \{ s \in S \mid \pi|_s \text{ is not étale at } s \} \); note that \( \pi(M') = N \). Let
\( \{ A_j \}_{j=1}^q \) be the set of points of \( M' \) and \( N = \{ x_j \}_{j=1}^p \) their projections onto the x-axis. Recall
that \( \pi \) is generic, so we assume that \( \#(\pi^{-1}(x) \cap M') = 1 \) for every \( x \in N \). Let \( E \) (resp. \( D \))
be a closed disk on the x-axis (resp. the y-axis), such that \( M' \subset E \times D \) and \( N \subset \text{Int}(E) \).
We choose \( u \in \partial E \) a real point far enough from the set \( N \), so that \( x << u \) for every \( x \in N \).
Define \( C_u = \pi^{-1}(u) \) and number the points of \( K = C_u \cap S \) as \( \{ 1, \ldots, p \} \).

We now construct a g-base for the fundamental group \( \pi_1(E - N, u) \). Take a set of paths
\( \{ \gamma_j \}_{j=1}^q \) which connect \( u \) with the points \( \{ x_j \}_{j=1}^q \) of \( N \). Now encircle each \( x_j \) with a small
oriented counterclockwise circle \( c_j \). Denote the path segment from \( u \) to the boundary of this
circle by \( \gamma_j' \). We define an element (a loop) in the g-base as \( \delta_j = \gamma_j' c_j \gamma_j'^{-1} \). Let \( B_p[D, K] \) be
the braid group, and let \( H_1, \ldots, H_{p-1} \) be its frame (for complete definitions, see [25, Section
III.2]). The braid monodromy of \( S \) [11] is a map \( \varphi : \pi_1(E - N, u) \to B_p[D, K] \) defined as
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follows: every loop in \( E - N \) starting at \( u \) has liftings to a system of \( p \) paths in \( (E - N) \times D \) starting at each point of \( K = 1, \ldots, p \). Projecting them to \( D \) we obtain \( p \) paths in \( D \) defining a motion \( \{1(t), \ldots, p(t)\} \) (for \( 0 \leq t \leq 1 \)) of \( p \) points in \( D \) starting and ending at \( K \). This motion defines a braid in \( B_p[D,K] \). By the Artin Theorem [26], for \( j = 1, \ldots, q \), there exists a half-twist \( Z_j \in B_p[D,K] \) and \( \epsilon_j \in \mathbb{Z} \), such that \( \varphi(\delta_j) = Z_j^{\epsilon_j} \), where \( Z_j \) is a half-twist and \( \epsilon_j = 1, 2 \) or 3 (for an ordinary branch point, a node, or a cusp respectively). We now explain how to describe the \( Z_j \).

First, we recall the definition of an almost real curve from [26].

**Definition 1.** A curve \( S \) is called an almost real curve if

1. \( N \subseteq E \cap \mathbb{R} \),
2. \( N \subset E - \partial E \),
3. \( \forall x \in E \cap \mathbb{R} - N, \#(K \cap \mathbb{R})(x) \geq p - 1 \),
4. \( \forall x \in N, \#\pi^{-1}(x) \cap M' = 1 \),
5. The singularities can be
   
   (a) a branch point, topologically locally equivalent to \( y^2 + x = 0 \) or \( y^2 - x = 0 \),
   
   (b) a tacnode (locally a line tangent to a conic)
   
   (c) a local intersection of \( m \) smooth branches, transversal to each other.

We compute the braid monodromy around each singularity in \( S \). Let \( A_j \) be a singularity in \( S \) and denote by \( x_j \) its projection by \( \pi \) to the \( x \)-axis. We choose a point \( x'_j \) next to \( x_j \), such that \( \pi^{-1}(x'_j) \) is a typical fiber. If \( A_j \) is (b), (c) or (d), then \( x'_j \) is on the right side of \( x_j \). If \( A_j \) is (a), then \( x'_j \) is on the left side of \( x_j \) (the typical fiber in case (a), which is on the left side of this singularity, intersects the conic in two real points). We encircle \( A_j \) with a very small circle in such a way that the typical fiber \( \pi^{-1}(x'_j) \) intersects the circle in two points, say \( a, b \). We fix a skeleton \( \xi_{x'_j} \) which connects \( a \) and \( b \), and denote it as \( < a, b > \). The Lefschetz diffeomorphism \( \Psi \) [1, Subsection 1.9.5] allows us to get a resulting skeleton \( (\xi_{x'_j})\Psi \) in the typical fiber \( C_u \). This one defines a motion of its two endpoints. This motion induces a half-twist \( Z_j = \Delta < (\xi_{x'_j})\Psi > \). As above, \( \varphi(\delta_j) = \Delta < (\xi_{x'_j})\Psi >^{\epsilon_j} \). The braid monodromy factorization associated to \( S \) is \( \Delta_p^2 = \prod_{j=1}^{q} \varphi(\delta_j) \). The degree of the factorization (the sum of the \( \epsilon_j \)) is \( p \times (p - 1) \).
It is difficult to compute the braid monodromy factorization of a general branch curve directly, but it can be done indirectly by degeneration 1.2.

To do this, we can degenerate the surface to a union of planes, whose branch curve is a line arrangement in the plane, whose complement has a particularly simple fundamental group. The braid monodromy algorithm of [26] computes the braid monodromy factorization for the line arrangement. We can then regenerate the braid monodromy factorization via the regeneration rules of [29, p. 335-337] to obtain the braid monodromy factorization of the branch curve.

Note that geometrically, this regeneration of the braid monodromy factorization is equivalent to a local regeneration of the branch curve itself, via the regeneration lemmas in [26, p.38]:

1. regeneration of a branch point to two branch points;
2. regeneration of a node to two or four nodes;
3. regeneration of a tangency point to three cusps.

We note that a braid $Z^2_{ij}$ which is related to a node is regenerated to $Z^2_{ij,j'}$, which is a product of two braids $Z^2_{ij}, Z^2_{ij'}$. A braid $Z^3_{ij}$ related to a cusp is regenerated to a braid $Z^3_{ij,j'}$, which is a product of three braids, as follows: $Z^3_{ij}, (Z^3_{ij})Z^2_{ij,j'}, (Z^3_{ij})Z^2_{ij,j'}^{-1}$.

Throughout this paper we represent braids $B$ pictorially by paths $P$, such that $B$ is the half-twist with respect to the path $P$, that is, consider a map $f$ from a closed neighborhood $V$ of $P$ to a disc $U$, taking $P$ to the line segment $[-1,1]$; the braid $B$ is then defined by the conjugation of the 180 degree rotation in $U$ by $f$ on $V$, and the identity elsewhere. In the case of the products that arise from nodes and cusps, by combinations of the individual paths representing the half twists along those paths. (For simplicity, in the larger graphs, sometimes we draw one path representing all three in the portion where they overlap. Any dotted or dashed segment that connects to a path rather than a vertex is considered to continue along that path.)

\[X \in \mathbb{P}^n \xrightarrow{\text{degeneration}} X_0 \in \mathbb{P}^n \]
\[S \in \mathbb{P}^2 \xrightarrow{\text{regeneration}} S_0 \in \mathbb{P}^2 \]
After computing the braid monodromy factorization, the second step is to compute the fundamental group of the complement of the branch curve. By the Van Kampen theorem [33], there is a “good” geometric base \( \{ \Gamma_j \} \) of \( \pi_1(\mathbb{C}_{x_0} - S \cap \mathbb{C}_{x_0}, \ast) \), where \( \mathbb{C}_{x_0} \) is the fiber of the projection \( \pi|_{\text{aff}} \) above \( x_0 \), such that the group \( \pi_1(\mathbb{C}^2 - S, \ast) \) is generated by the images of \( \{ \Gamma_j \} \) in \( \pi_1(\mathbb{C}^2 - S, \ast) \) with the relations \( \varphi(\delta_i) \Gamma_j = \Gamma_j \) \( \forall i, j \). Recall that

\[
\pi_1(\mathbb{CP}^2 - \bar{S}) \simeq \pi_1(\mathbb{C}^2 - S) / \langle \prod_j \Gamma_j \rangle.
\]

Recall that the branch curve \( S \) of a smooth surface \( X \) contains branch points of the projection (where the curve is locally defined by the equation \( y = x^2 \)), nodes (locally \( y^2 = x^2 \)) and cusps (locally \( y^2 = x^3 \)). If the surface has isolated nodal/cuspidal singularities, then the branch curve is of the same form. Denote by \( a \) and \( b \) the two branches of the real part of \( S \) in a neighborhood of such a singular point, and let \( \Gamma_a, \Gamma_b \) be two non-intersecting loops in \( \pi_1(\mathbb{C}_{x_0} - S \cap \mathbb{C}_{x_0}, \ast) \) around the intersection points of the branches with the fiber \( \mathbb{C}_{x_0} \) (constructed by cutting each of the paths and creating two loops, which proceed along the two parts and encircle \( a \) and \( b \)); see [25, Proposition-Example VI.1.1]. Then by the van Kampen Theorem, we have the relations

\[
[\Gamma_a, \Gamma_b] = \Gamma_a \Gamma_b \Gamma_a^{-1} \Gamma_b^{-1} = 1 \text{ for a cusp,}
\]

\[
[\Gamma_a, \Gamma_b] = \Gamma_a \Gamma_b \Gamma_a^{-1} \Gamma_b^{-1} = 1 \text{ for a node, and}
\]

\[
\Gamma_a = \Gamma_b \text{ for a branch point.}
\]

These relations generate the group completely in the affine case. In the projective case, we have the additional ”projective” relation \( \prod \Gamma_j = 1 \).

The third step is to use the theorem of Moishezon-Teicher, [23], that there is an exact sequence

\[
0 \to K \to \pi_1(\mathbb{CP}^2 - \bar{S}) \to S_n \to 0,
\]

where the second map takes the generators \( \Gamma_i \) of the fundamental group to the transpositions in \( S_n \) according to the lexicographic order, and the fundamental group \( \pi_1(X_{\text{Gal}}) \) is the quotient of \( K \) by the relations \( < \Gamma_i^2 > \). We apply this exact sequence to obtain a presentation of the fundamental group of the Galois cover, and simplify the relations to produce a canonical presentation and identify the group, using various new group theoretic methods for each case.

1.3. Contents. The rest of this paper is organized as follows: Section 2 is a complete survey of the fundamental groups of the Galois covers of all the surfaces of degrees less than 5 that
degenerate to a "nice" planar arrangement, i.e., one in which no three planes meet in a line. In subsection 2.1, we consider the unique smooth quadric surface that degenerates to two planes. In subsection 2.2, we consider the two possible cubics: in 2.2.1, we analyze the (1,1) embedding of the Hirzebruch surface $F_1$ as a smooth cubic. In 2.2.2, we recall the degeneration of the Cayley cubic, which is singular, but note that there are also smooth surfaces that share that degeneration, and analyze the fundamental group of its Galois cover. In subsection 2.3, we consider the five possible quartics: in 2.3.1, we recall the (1,1) embedding of the Hirzebruch surface $F_2$. In 2.3.2, we recall the embedding of the Veronese surface $V_2$ and provide a more explicit analysis of its fundamental groups, and in 2.3.3 we analyze the (1,2) embedding of the Hirzebruch surface $F_0$, or $\mathbb{CP}^1 \times \mathbb{CP}^1$. In 2.3.4 we analyze the new case of a quartic surface degenerating to the union of a plane and the Cayley cubic degeneration, and in 2.3.5 we analyze the case of a quartic degenerating to a 4-point.

In the appendix, Section 3, we analyze for completeness those cases of $\mathbb{CP}^1 \times \mathbb{CP}^1$ not covered by the Moishezon-Teicher theorem [23, p.642], namely the (2,2) embedding and the (1,$n$) embedding for all positive integers $n$.

2. SMALL DEGREE CASES

Throughout this paper, we use the standard triangulation representation in which triangles denote planes, internal edges denote lines of intersection (we do not consider the boundary edges as they do not define well defined lines) and points of intersection of the internal edges denote points of intersection of the corresponding planes, and every arrangement is assumed to be embedded in the smallest dimensional projective space.

It is clear that every plane arrangement can be represented by a triangulation as long as no three planes meet in a line and no plane meets more than three other planes. In degrees less than 5, the second condition is vacuous. For the triangulation to represent a meaningful planar degeneration, it is necessary that if exactly two triangles meet at a point, they must meet along an edge, and if three or more triangles meet at a point, the dual graph of the triangulation must be connected. The reason is that the branch curve of an irreducible surface (and hence of a smooth surface or a surface with only isolated non-removable singularities) is connected. Hence the branch curve of the degeneration must also be connected. Moreover, the triangulation is planar since we can consider the edge graph of the triangulation, and
by Kuratowski’s theorem, every non-planar graph contains a subgraph homeomorphic to
the complete graph $K_5$ or the complete bipartite graph $K_{3,3}$, neither of which is possible
for triangulations of degree less than 10. Conversely, any triangulation determines a plane
arrangement in a projective space of sufficiently high degree.

We therefore consider planar triangulations such that the dual graph is connected.

The plane arrangement is not unique, but its combinatorial invariants, and therefore, in
particular, its braid monodromy factorization and the fundamental group of the complement
of the branch curve, are well defined. It is therefore enough to consider each possible plane
triangulation and to show that it actually corresponds to a degeneration of some smooth
surface.

2.1. **Quadric.** The smooth quadric surface degenerates to two planes, see Figure 3.

![Figure 3. Degeneration of the quadric](image)

The branch curve is a conic, which is smooth. There is only one braid relation, which is the
half twist given in Figure 4. The fundamental group of the complement has two generators

\[ \Gamma_1 \quad \text{and} \quad \Gamma'_1. \]

The projective relation is $\Gamma_1 \Gamma'_1 = e$. Hence $\Gamma_1^2 = e$. This group is thus $S_2$, so the
map $\rho$ in Exact Sequence (1) is the identity. Its kernel is therefore trivial. Note that every
degree 2 cover is Galois.

2.2. **Cubics.**

**Theorem 2.** There are two possible "nice" cubic degenerations (i.e., degenerations to line
arrangements such that no three planes meet in a line), those shown in Figures 5 and 7.

**Proof.** We construct the plane arrangements by gluing triangles together. The simplest
connected arrangement is that shown in Figure 5. By one extra gluing we obtain Figure 7.
It is not possible to glue the triangles any further without either gluing three triangles in
one edge or identifying the two endpoints of some edge, both of which are forbidden. \(\square\)
2.2.1. The Hirzebruch surface $F_1$. The Hirzebruch surface $F_1$, the ruled surface defined by $E = \mathcal{O} \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$, is embedded as a smooth cubic in $\mathbb{P}^4$ by the (1,1) embedding, i.e., by the linear system $|\ell_1 + \ell_2|$, where $\ell_1$ is the $(-1)$-curve and $\ell_2$ is the fibre. This surface degenerates to a union of three planes, as depicted in Figure 5. The branch curve $C_0$ is a line arrangement consisting of two intersecting lines. Regenerating it, we obtain the conic $(1, 1')$ and the tangent line $(2)$. When the line regenerates, the tangency regenerates into three cusps.

![Figure 5. Degeneration and Regeneration of the Hirzebruch surface $F_1$](image)

The braid monodromy factors are thus given in Figure 6. Here and throughout this paper, we use the group theoretic convention of denoting by $a^b$ the conjugation $b^{-1}ab$.

![Figure 6. $Z^3_{1',2',2', (Z_1')^2 Z_2'}$](image)

**Theorem 3.** The fundamental group of the Galois cover of the Hirzebruch surface $F_1$ is trivial.

**Proof.** By the Van Kampen theorem, the fundamental group $\pi_1(\mathbb{C}^2 \setminus S)$ is generated by the four generators $\Gamma_1, \Gamma_1', \Gamma_2$ and $\Gamma_2'$, subject to the relations

\begin{align*}
(2) \quad & \Gamma_1 = \Gamma_1', \\
(3) \quad & \Gamma_2 = \Gamma_2', \\
(4) \quad & \langle \Gamma_1, \Gamma_2 \rangle = e, \\
(5) \quad & \Gamma_1^{-2} = \Gamma_2^2.
\end{align*}

The map $\pi_1(\mathbb{C}^2 \setminus S) \to S_3$ is given by $\Gamma_1 \mapsto s_1$ and $\Gamma_2 \mapsto s_2$. Thus the kernel $K$ is generated by $\Gamma_1^2$ and $\Gamma_2^2$. Hence, the fundamental group $\pi_1(X_{\text{Gal}})$ is trivial. \[\square\]
2.2.2. **Triple point.** The union of three planes meeting at a triple point, as shown in Figure 7, was first studied as a degeneration of the Cayley cubic, a singular surface with four nodes. However, there are also smooth cubic surfaces that degenerate to this union: for example, the surface in \( \mathbb{CP}^3 \) defined by \( xyz + tw(x^2 + y^2 + z^2 + w^2) \), which degenerates when \( t = 0 \) to \( xyz \).

![Figure 7. The degeneration of the Cayley cubic](image)

In the paper [3], the initial braid monodromy factorization of the degenerate surface consisting of three planes meeting at a point is found to be \( \Delta_3^2 \), and by regenerating, the following braid monodromy factorization is obtained for the complement of the branch curve \( S \):

**Proposition 4.** [3] The braid monodromy factorization of \( S \) is given in (6), and its factors are represented by the paths in Figure 8.

\[
\Delta_3^2 = (Z_2^2)^{Z_2^2 y'} \cdot (Z_1^2)^{Z_2^2 y' Z_3^2 y'} \cdot (Z_1^2)^{Z_2^2 y' Z_3^2 y'} \cdot (Z_1^2)^{Z_2^2 y' Z_3^2 y'}
\]

\[
\cdot (Z_2^2)^{Z_1^2 y' Z_2^2 y'} \cdot (Z_2^2)^{Z_1^2 y' Z_2^2 y'} \cdot (Z_2^2)^{Z_1^2 y' Z_2^2 y'} \cdot (Z_2^2)^{Z_1^2 y' Z_2^2 y'}
\]

- The first, the fourth and the last two paths correspond to braids of branch points.
- The second and third paths correspond to braids of cusps, and the rest correspond to braids of nodes.

**Proposition 5.** [3] The fundamental group \( \pi_1(\mathbb{CP}^2 - S) \) is generated by \( \Gamma_1, \Gamma_2, \Gamma_3 \), subject to the relations

\[
\langle \Gamma_1, \Gamma_2 \rangle = e, \tag{7}
\]

\[
\langle \Gamma_2, \Gamma_3 \rangle = e, \tag{8}
\]

\[
\langle \Gamma_1, \Gamma_2^2 \Gamma_3^2 \rangle = e, \tag{9}
\]

\[
\langle \Gamma_1, \Gamma_2^{-1} \Gamma_3 \Gamma_2 \rangle = e, \tag{10}
\]

where we denote by \( \langle a, b \rangle = e \) the braid relation between \( a \) and \( b \), \( aba = bab \). The group \( G = \pi_1(\mathbb{CP}^2 - S) \) has the relations (7), (8), (10) and the additional projective relation

\[
\Gamma_2^2 \Gamma_3^2 = e. \tag{11}
\]
**Theorem 6.** The fundamental group of the Galois cover of the Cayley cubic (or its smoothing) is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** We relabel the generators of $G$ by replacing $\Gamma_3$ by $\Gamma'_3 = \Gamma_2^{-1} \Gamma_3 \Gamma_2$, to obtain the relations (7) and

\begin{align*}
(12) & \quad \langle \Gamma_2, \Gamma'_3 \rangle = e \\
(13) & \quad [\Gamma_1, \Gamma'_3] = e \\
(14) & \quad \Gamma_2 \Gamma'_1 \Gamma_2 \Gamma'_3 = e.
\end{align*}

Thus, this group is a quotient of the braid group $B_4$ by the relation (14). Since this relation is in the kernel of the map $B_4 \to S_4$, our fundamental group has a natural map to $S_4$, whose kernel is normally generated by $\Gamma_1^2$, $\Gamma_2^2$, and $\Gamma_3^2$. So $G/\langle \Gamma_1^2, \Gamma_2^2, \Gamma_3^2 \rangle \cong S_4$. Composing this map with a surjective map $S_4 \to S_3$, whose kernel is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and is normally generated by $s_1s_3$, we obtain the map $\pi_1(\mathbb{CP}^2 - \bar{S}) \to S_3$. The kernel $K/\langle \Gamma_1^2, \Gamma_2^2, \Gamma_3^2 \rangle$ is
isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by $\Gamma_1\Gamma_3$ and its conjugates. Hence we obtain $\pi_1(X_{\text{Gal}}) = K/(\Gamma_1^2, \Gamma_2^2, \Gamma_3^2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

2.3. Quartics.

**Theorem 7.** There are five possible quartic degenerations, corresponding to Figures 9, 12, 16, 19 and 25.

**Proof.** We construct the degenerations combinatorially by gluing triangles. Beginning with the arrangement of triangles found in Figure 5, we can add one more triangle in different places to obtain Figures 9, 12 and 16. Beginning with that found in Figure 7, we can add one more triangle to obtain the quartic arrangement in Figure 19, and then glue two more edges together to obtain that in Figure 25.

It is not possible to glue them any further (for instance, to obtain a non-simply-connected arrangement like the torus degenerations that appear in [5] in higher degrees), because in degree 4 this would force us to have either three planes meeting in a line, or two lines through the same pair of points, both of which are forbidden configurations.

2.3.1. The surface $F_2$. Consider the Hirzebruch surface $F_2$ defined by $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-2)$ on $\mathbb{P}^1$. It is a toric variety and can be embedded in $\mathbb{C}\mathbb{P}^5$ by the linear system $|\ell_1 + \ell_2|$, where $\ell_1$ is the $(-2)$-section and $\ell_2$ is the fibre. Its degeneration is a union of four planes in $\mathbb{C}\mathbb{P}^5$, as depicted in Figure 9, as shown in [6].

![Figure 9. Degeneration and Regeneration of the Hirzebruch surface $F_2$](image)

The branch curve $S_0$ in $\mathbb{C}\mathbb{P}^2$ is an arrangement of three lines. Regenerating it, the diagonal line regenerates to a conic, which is tangent to the lines 1 and 3. When the lines regenerate, each tangency regenerates into three cusps. We obtain the branch curve $S$, which is of degree 6 and which has six cusps.

As computed in [6], the braid monodromy factorization corresponding to $C$ gives an expression of $\Delta_6^2$ as the braids shown in Figure 10, and we also have the parasitic intersection braid given in Figure 11.
We apply the van Kampen Theorem to the above braids to get a presentation for \( \pi_1(\mathbb{C}P^2 \setminus S) \). After simplifying the relations, we find that the fundamental group is generated by \( \Gamma_1, \Gamma_2, \Gamma_3 \) subject to the relations

\[
\langle \Gamma_i, \Gamma_{i+1} \rangle = e, \text{ for } i=1, 2.
\]

\[
[\Gamma_1, \Gamma_3] = e,
\]

\[
\Gamma_1^{-2}\Gamma_2\Gamma_1^2 = \Gamma_3^{-2}\Gamma_2\Gamma_3^2.
\]

Clearly, the group \( \pi_1(\mathbb{C}P^2 \setminus S) \) is isomorphic to the braid group quotient \( B_4/\langle \Gamma_2\Gamma_3^2\Gamma_2\Gamma_3^2 \rangle \).

The map to \( S_4 \) is given by \( \Gamma_i \mapsto s_i \), and its kernel \( K \) is normally generated by the \( \Gamma_i^2 \). Hence, the quotient \( \pi_1(X_{\text{Gal}}) = K/\langle \Gamma_i^2 \rangle \) is trivial.

2.3.2. The Veronese surface \( V_2 \). We consider the Veronese surface of order 2, i.e., the embedding of \( \mathbb{C}P^2 \) into \( \mathbb{C}P^5 \) given by \( (x : y : z) \mapsto (x^2 : y^2 : z^2 : xy : yz : xz) \). This is a surface of degree 4, which was treated in a very abstract manner by [24]. The fundamental group of the Galois cover was found there to be \( \mathbb{Z}^4 \).

The degeneration is not necessary in this case, but for the sake of completeness in our survey of plane arrangements, we note that the surface degenerates to the union of four planes depicted in Figure 12, and we present the fundamental group explicitly in terms of generators and relations.
The branch curve consists of three lines meeting at three different vertices. We regenerate each vertex in turn, and use the van Kampen Theorem to obtain relations among the generators $\Gamma_i$ and $\Gamma'_i$ (for $i = 1, 2, 3$) of the fundamental group $\pi_1(\mathbb{CP}^2 \setminus S)$ of the complement of the branch curve in the projective plane.

Vertex 1 regenerates to a line (1) tangent to a conic (2,2'), as in Figure 13.

It gives rise to the braid monodromy factors $Z^3_{1 \Gamma'_1, \Gamma'}$ and $(Z^2_{2 \Gamma'_2})^{Z^2_{1 \Gamma'_1, \Gamma'}}$, which by the Van Kampen theorem yield the relations

\begin{align}
\langle \Gamma_1, \Gamma_2 \rangle &= e, \\
\langle \Gamma'_1, \Gamma_2 \rangle &= e, \\
\langle \Gamma^{-1}_1 \Gamma'_1 \Gamma_1, \Gamma_2 \rangle &= e, \\
\Gamma_2 \Gamma'_1 \Gamma_1 \Gamma^{-1}_1 \Gamma'_1 \Gamma^{-1}_2 &= \Gamma'_2.
\end{align}

Vertex 2 regenerates to a line (1) tangent to a conic (3,3'), and Vertex 3 to a line (3) tangent to a conic (2,2'). They give rise to the braid monodromy factors $Z^3_{1 \Gamma'_1, \Gamma}$ and $(Z^2_{3 \Gamma'_3})^{Z^2_{1 \Gamma'_1, \Gamma}}$, which yield the relations

\begin{align}
\langle \Gamma_1, \Gamma_3 \rangle &= e, \\
\langle \Gamma'_1, \Gamma_3 \rangle &= e, \\
\langle \Gamma^{-1}_1 \Gamma'_1 \Gamma_1, \Gamma_3 \rangle &= e, \\
\Gamma_3 \Gamma'_1 \Gamma_1 \Gamma_3 \Gamma^{-1}_1 \Gamma'_1 \Gamma^{-1}_2 &= \Gamma'_3.
\end{align}
and to the braid monodromy factors $Z^3_{2',3'}$ and $(Z_2')^2Z^3_{2',3'}$, which yield the relations

\begin{align}
\langle \Gamma'_2, \Gamma_3 \rangle &= e, \\
\langle \Gamma'_2, \Gamma_3' \rangle &= e, \\
\langle \Gamma'_2, \Gamma_3^{-1}\Gamma'_3\Gamma_3 \rangle &= e, \\
\Gamma'_3\Gamma_3\Gamma_2^{-1}\Gamma_3' &= \Gamma_2.
\end{align}

In addition to these relations, we have the projective relation

\begin{equation}
\langle \Gamma'_1, \Gamma_3 \rangle = \langle \Gamma_3, \Gamma'_1 \rangle = e.
\end{equation}

After simplifications, we find that the group $G$ is generated by the four generators $\Gamma_1, \Gamma_2, \Gamma_3$ and $\Gamma'_1$, under the relations

\begin{align}
\langle \Gamma_1, \Gamma_2 \rangle &= \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_3, \Gamma_1 \rangle = e \quad \text{and} \quad \langle \Gamma'_1, \Gamma_2 \rangle = \langle \Gamma_3, \Gamma'_1 \rangle = e,
\end{align}

and the map $\rho$ maps $\Gamma_1 \mapsto (1, 2), \Gamma_2 \mapsto (2, 3), \Gamma_3 \mapsto (2, 4)$ and $\Gamma'_1 \mapsto (1, 2)$.

**Lemma 8.** Ker $\rho$ is generated modulo the $\Gamma_i^2$ by

\begin{align}
(\Gamma_1\Gamma_2\Gamma_3\Gamma_2)^2, \quad (\Gamma_2\Gamma_3\Gamma_1\Gamma_3)^2, \quad (\Gamma'_1\Gamma_2\Gamma_3\Gamma_2)^2, \quad \text{and} \quad (\Gamma_2\Gamma_3\Gamma'_1\Gamma_3)^2.
\end{align}

**Proof of Lemma 8.** Consider the groups

\[ G_1 = \langle \Gamma_1, \Gamma_2, \Gamma_3 | \langle \Gamma_1, \Gamma_2 \rangle, \langle \Gamma_2, \Gamma_3 \rangle, \langle \Gamma_3, \Gamma_1 \rangle, \Gamma'_1_i \rangle \]

and \[ G_2 = \langle \Gamma'_1, \Gamma_2, \Gamma_3 | \langle \Gamma'_1, \Gamma_2 \rangle, \langle \Gamma_2, \Gamma_3 \rangle, \langle \Gamma_3, \Gamma'_1 \rangle, \Gamma'_1_i, \Gamma'_2_i \rangle \].

The maps $\rho$ from $G_1$ and $G_2$ to $S_4$ are given by the following two diagrams. As in [32], we represent maps from Coxeter groups to $S_n$ by diagrams with $n$ vertices, such that two vertices $i$ and $j$ are connected by an edge labelled $\Gamma$ if $\Gamma$ maps onto the transposition $(i, j)$. Note that if the edges $\Gamma_k$ and $\Gamma_l$ meet in a vertex, then $[\Gamma_k, \Gamma_l] = e$, and if $\Gamma_k$ and $\Gamma_l$ are disjoint, then $[\Gamma_k, \Gamma_l] = e$, since if $\Gamma_k \mapsto (a, b)$ and $\Gamma_l \mapsto (b, c)$, then $\langle (a, b), (b, c) \rangle = e$, and if $\Gamma_k \mapsto (a, b)$ and $\Gamma_l \mapsto (c, d)$, then $[(a, b), (c, d)] = e$. The lemma thus follows from [32, Theorem 2.3, p. 4].

Using the lemma, since the four generators commute modulo the $\Gamma_i^2$, it follows that Ker $\rho/\langle \Gamma_i^2 \rangle = Z^4$. 

\[ \boxed{} \]
Remark 9. As the branch curve is in fact a curve of degree 6 with nine cusps, the fundamental group of the complement of the curve can also be deduced from Zariski [35].

2.3.3. The embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a quartic. Consider the degeneration of the surface $\mathbb{CP}^1 \times \mathbb{CP}^1$, embedded in $\mathbb{P}^5$ by the linear system $|\ell_1 + 2\ell_2|$, where $\ell_1$ and $\ell_2$ are the pullbacks of the point classes from the two factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$. The degeneration is to a union of four planes depicted in Figure 16.

![Figure 16](image)

**Figure 16.** The $(1, 2)$-degeneration of $\mathbb{CP}^1 \times \mathbb{CP}^1$

**Theorem 10.** The fundamental group of the Galois cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in the $(1, 2)$-embedding is trivial.

**Proof.** The branch curve of the degeneration consists of three lines meeting at two different vertices.

Vertex 1 (respectively, 6) regenerates to a conic and gives rise to the trivial braid $Z_{1,1'}$ (resp. $Z_{6,6'}$), and hence to the relations

\begin{align}
\Gamma_1 &= \Gamma_1', \\
\Gamma_3 &= \Gamma_3'.
\end{align}
Vertex 5 (resp. 2) regenerates to a line 2 tangent to a conic (1,1') (resp., (3,3'), as in Figure 17 (resp., 18), giving rise to the braid monodromy factors $Z_{1',2}^3 Z_{2',2}^3$ and $(Z_{1'}^3 Z_{2'}^3)^{Z_{2,3}}$, and $(Z_{3'}^3 Z_{2,3})^{Z_{2,3}}$, and the relations

\begin{align}
(30) & \langle \Gamma_1', \Gamma_2 \rangle = \langle \Gamma_1', \Gamma_2 \rangle = \langle \Gamma_1', \Gamma_2^{-1} \Gamma_2 \Gamma_2 \rangle = e, \\
(31) & \Gamma_1 = \Gamma_2' \Gamma_2 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1}, \\
(32) & \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2', \Gamma_3 \rangle = \langle \Gamma_2^{-1} \Gamma_2 \Gamma_2, \Gamma_3 \rangle = e, \\
(33) & \Gamma_3' = \Gamma_2' \Gamma_2 \Gamma_3 \Gamma_2^{-1} \Gamma_2^{-1} \Gamma_3^{-1}.
\end{align}

We have also intersections that arise from lines that did not meet in the plane arrangement, but their images meet in $\mathbb{CP}^2$. We call these parasitic intersections. Explanation and configuration of how to construct the braids that correspond to these intersections, appear in [23, p. 616].

They give rise to the following relations:

\begin{align}
(34) & [\Gamma_2' \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1}, \Gamma_3] = e, \\
(35) & [\Gamma_2' \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma_3 \Gamma_3] = e, \\
(36) & [\Gamma_2' \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma_3 \Gamma_3] = e, \\
(37) & [\Gamma_2' \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma_3 \Gamma_3] = e.
\end{align}

From (28) and (29), we see that the group $\pi_1(\mathbb{CP}^2 \setminus S)$ is generated by $\Gamma_1$, $\Gamma_2$, $\Gamma_2'$ and $\Gamma_3$. Substituting into (31) and (33), we obtain

$$
\Gamma_1 = \Gamma_2' \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \text{ and } \Gamma_3 = \Gamma_2' \Gamma_2 \Gamma_3 \Gamma_2^{-1} \Gamma_2^{-1}.
$$
or in other words

\[(38) \quad [\Gamma_2', \Gamma_2, \Gamma_3] = [\Gamma_2', \Gamma_2, \Gamma_1] = e.\]

Hence, (34) reduces to \([\Gamma_1, \Gamma_3] = e\), and likewise (35), (36) and (37).

Applying (38), we get \(\Gamma_2'^{-1}\Gamma_1'\Gamma_2' = \Gamma_2'\Gamma_1^{-1}\Gamma_2^{-1}\), and hence \(\Gamma_1'\Gamma_2'\Gamma_3^{-1} = \Gamma_1^{-1}\Gamma_2\Gamma_1\), which implies that \(\Gamma_2' = \Gamma_1^{-1}\Gamma_2'\Gamma_1\). Likewise, \(\Gamma_2' = \Gamma_3^{-2}\Gamma_2'\Gamma_3\). Hence, the image of \(\Gamma_2'\) in the quotient \(G/\langle \Gamma_1', \Gamma_2', \Gamma_3' \rangle\) is equal to the image of \(\Gamma_2\). Thus \(G/\langle \Gamma_1', \Gamma_2', \Gamma_3' \rangle \cong \langle \Gamma_1, \Gamma_2, \Gamma_3' \rangle = \langle \Gamma_1, \Gamma_3 \rangle, [\Gamma_1, \Gamma_3], \Gamma_2', \Gamma_3' \rangle \cong S_4\). Hence, since the projection of the fundamental group of the branch curve complement to \(S_4\) is onto, it is an isomorphism; hence the fundamental group of \(X_{\text{Gal}}\), which is the kernel of this map, is trivial.

2.3.4. The union of a cubic degeneration and a plane. Consider a smooth quartic surface that degenerates to a union of three planes meeting at a point, and one plane not passing through that point, as shown in Figure 19. For example, if the ideal of the degenerated surface is \((x, y)(x, z)(w, y)\) or \((x^3w, x^3u, x^2w^2, x^2wu, xzw, xzw, xzw, xzw, xzw, xzw, xzw, xzw, xzw, xzw, xzw, xzw, xzw, xzw)\), then one can check by explicit computation that a generic deformation of this surface, such as

\[
\begin{align*}
(x^3w+t(x^4-y^4+z^4+w^4+u^4), x^3u+t(x^4+y^4-z^4-w^4-u^4), x^2w^2+t(x^4+y^4+z^4-w^4+u^4), x^2wu+t(x^4+y^4+z^4+w^4-u^4), x^2zw+t(2x^4+y^4+z^4+w^4+u^4), x^2zu+t(x^4+2y^4+z^4+w^4+u^4), xzw^2+t(x^4+y^4+2x^4+w^4+u^4), xzwu+t(x^4+y^4+z^4+2w^4+u^4), x^2yw+t(x^4+y^4+z^4+w^4+2u^4), x^2yu-t(x^4-3x^4+y^4+z^4+w^4+u^4), xyw^2+t(x^4+y^4-3z^4+w^4+u^4), xywu-t(x^4+y^4+z^4-3w^4+u^4), xzw-t(x^4+y^4+z^4+w^4-3u^4), xyzu-t(x^4+4y^4+z^4+w^4+u^4), yzw^2-t(x^4+y^4+4z^4+w^4+u^4), yzwu-t(x^4+y^4+4z^4+w^4+u^4),
\end{align*}
\]

is smooth.

![Figure 19. Degeneration to three planes with a triple point and one other plane](image)

**Theorem 11.** The fundamental group of the Galois cover of this surface is \(\mathbb{Z}^6 \times \mathbb{Z}_2^3\).

**Proof.** We break up the braid monodromy factorization into three components \(\Delta_1, \Delta_2\) and \(\Delta_4\). The factor \(\Delta_1\) corresponding to the triple point 1 in 19 is computed in [3] (where it is denoted \(\widetilde{\Delta}\)) as follows.
(39) \[ \Delta_1 = (Z_{13}'^2)^{Z_{23}',3} \cdot (Z_{13}^2)^{Z_{23}',3} \cdot (Z_{13}'^2)^{Z_{23}',3} \cdot (Z_{13}^2)^{Z_{23}',3} \]

\[ \cdot (Z_{12}'^2)^{Z_{23}',3} \cdot (Z_{13}'^2)^{Z_{23}',3} \cdot (Z_{12}'^2)^{Z_{23}',3} \cdot (Z_{13}^2)^{Z_{23}',3} \]

Note that the first, the fourth and the last two paths correspond to braids of branch points.
The second and third paths correspond to braids of cusps, and the rest correspond to braids of nodes.

\[ \text{Figure 20. } \Delta_1 \text{ braids from (39)} \]

\[ \Delta_1 \text{ thus gives rise to the following relations on the generators of the fundamental group:} \]

\[ \Gamma_1 = \Gamma_2' \]

\[ (\Gamma_1, \Gamma_2') = (\Gamma_1', \Gamma_2') = (\Gamma_2' \Gamma_1 \Gamma_2' \Gamma_1^{-1}, \Gamma_2') = e, \]

\[ (\Gamma_2' \Gamma_2 \Gamma_2^{-1}, \Gamma_3) = (\Gamma_2' \Gamma_2 \Gamma_2^{-1}, \Gamma_3') = (\Gamma_2' \Gamma_2 \Gamma_2^{-1}, \Gamma_3' \Gamma_3 \Gamma_2^{-1}) = e, \]

\[ \Gamma_1^{-1} \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_4^{-1} \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9 \Gamma_1^{-1} \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_4^{-1} \Gamma_2 \Gamma_1 \Gamma_2' = e \]

\[ [\Gamma_1', \Gamma_2'^{-1} \Gamma_3 \Gamma_2'] = [\Gamma_1', \Gamma_2'^{-1} \Gamma_3^{-1} \Gamma_2 \Gamma_2'^{-1} \Gamma_3 \Gamma_2'] = e, \]

\[ [\Gamma_1, \Gamma_2'^{-1} \Gamma_3 \Gamma_2'] = [\Gamma_1, \Gamma_2'^{-1} \Gamma_3^{-1} \Gamma_2 \Gamma_2'^{-1} \Gamma_3 \Gamma_2'] = e. \]

Vertices 2 and 4 are equivalent, and are both equivalent to Vertex 5 of the (1,2) degeneration of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), as depicted in Figure 17.

The factor \( \Delta_2 \) is thus as given in Figure 21, and gives rise to the relations
while $\Delta_4$ is given in Figure 22, and gives rise to the relations

$$\langle \Gamma'_3, \Gamma_4 \rangle = \langle \Gamma'_3, \Gamma'_4 \rangle = \langle \Gamma_3^{-1}, \Gamma'_4 \Gamma_4 \rangle = e,$$

$$\Gamma_3 = \Gamma'_3 \Gamma_4 \Gamma'_3 \Gamma_3^{-1} \Gamma'_4^{-1}.$$  

We also have the parasitic and projective relations.

The parasitic braids are explained in [23, p. 616], and are shown in Figure 23.

The relations are

$$[\Gamma_2, \Gamma_4] = e,$$
$$[\Gamma'_2, \Gamma_4] = e,$$
$$[\Gamma_2, \Gamma'_4] = e,$$
$$[\Gamma'_2, \Gamma'_4] = e.$$  

The projective relation is

$$\Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e.$$
By (47), (49) and (40), it is clear that the group is generated by the five generators $\Gamma'_1$, $\Gamma'_2$, $\Gamma'_3$, $\Gamma'_4$, and $\Gamma_4$, where the defining relations are

\[
\langle \Gamma'_1, \Gamma'_2 \rangle = e, \quad (41)
\]

\[
\langle \Gamma'_2, \Gamma'_3 \rangle = e, \quad \text{from (42) and (40)}
\]

\[
[\Gamma'_2, \Gamma'_4] = e, \quad (51)
\]

\[
\langle \Gamma'_2, \Gamma'_4 \rangle = e, \quad (46)
\]

\[
[\Gamma'_3, \Gamma'_4] = e, \quad (46)
\]

\[
\langle \Gamma'_3, \Gamma'_4 \rangle = e, \quad (48)
\]

\[
[\Gamma'_1, \Gamma'_2] = e.
\]

We map the generators

\[
\Gamma'_1 \mapsto (1, 3), \quad \Gamma'_2 \mapsto (1, 2), \quad \Gamma'_4 \mapsto (3, 4), \quad \Gamma_4 \mapsto (3, 4), \quad \Gamma'_3 \mapsto (2, 3),
\]

according to the configuration in Figure 19.

The diagrams (according to the convention of [32], given in 2.3.2) for the map to $S_4$ are thus as follows, and we have the additional relation $[\Gamma'_a, \Gamma'_2^{-1}\Gamma'_3\Gamma'_2] = e$, see Figure 24.

![Figure 24](image_url)

Hence, the by [32][p.3], the kernel of the map to $S_4$ is generated by $\Gamma'_1\Gamma'_2^{-1}\Gamma'_3\Gamma_2$, $\Gamma'_2\Gamma'_3^{-1}\Gamma'_1\Gamma'_3$, $\Gamma'_3\Gamma'_1^{-1}\Gamma'_2\Gamma_1$, arising from the cycle $\Gamma'_1$, $\Gamma'_2$, $\Gamma'_3$, and by the conjugates of $(\Gamma'_1\Gamma'_3\Gamma'_4\Gamma'_3)^2$ and $(\Gamma'_1\Gamma_3\Gamma_4\Gamma'_3)^2$, arising from the relation involving 3 edges meeting at a vertex (relation (4) in [32][p.4]).

Since the cycle only involves the vertices 1, 2, and 3, and since $S_3$ is generated by two transpositions, it is enough to take the preimages of two transpositions that arise from the cyclic relation. In particular, all elements of $K$ that arise from the cyclic relation are generated by $\Gamma'_1\Gamma'_2^{-1}\Gamma'_3\Gamma_2$ and $\Gamma'_2\Gamma'_3^{-1}\Gamma'_1\Gamma'_3$ and their conjugates. Since in $G/(\Gamma'_7)$ we have
\[ \Gamma_1^2 = (\Gamma_2^{-1}\Gamma_3^2\Gamma_2)^2 = (\Gamma_2\Gamma_3^2\Gamma_2)^2 = e, \] 

it follows from \([\Gamma'_1, \Gamma_2^{-1}\Gamma_3^2\Gamma_2] = e\) that \((\Gamma'_1\Gamma_2\Gamma_3^2\Gamma_2)^2 = e\).

Hence \(\Gamma'_1\Gamma_2\Gamma'_3\Gamma_2\), and its conjugate \(\Gamma'_2\Gamma'_3\Gamma'_2\), have order 2 in \(G/(\Gamma_1^2)\). Hence, since these two elements commute, the subgroup of \(K\) generated by these two elements is \(Z_2 \oplus Z_2\).

The generators of \(K\) that arise from three edges meeting in a vertex, and hence involve four vertices, are \((\Gamma'_1\Gamma_3\Gamma_4\Gamma_3)^2\), \((\Gamma'_1\Gamma_3\Gamma_4\Gamma_3)^2\) and their conjugates. Since \(S_4\) is generated by three transpositions, we need to take three conjugates of each of these elements. Hence, the group \(K\) is isomorphic to \(Z_3 \oplus Z_3 \rtimes Z_2^2\), or \(Z_6 \rtimes Z_2^2\).

\[ \square \]

2.3.5. **The 4-point.** The last possible degeneration of a quartic surface is to a plane arrangement with a 4-point, as in Figure 25.

![Figure 25. Surface degeneration diagram](image)

Again, this surface can be smooth or singular. For example, in \(\mathbb{CP}^4\), consider the surfaces defined by the ideal \((xz + tu(x + y + z + w), yw + tu(x + y + z - w))\). For generic \(t\) the surface is smooth, but when \(t = 0\) we get the four planes defined by the ideal \((xz, yw)\).

**Theorem 12.** The fundamental group of the Galois cover of the 4-point surface is \(Z_2^3\).

**Proof.** The degenerated surface has one 4-point and four 2-points (nodes). Then each node regenerates to two branch points. This gives rise to the braid monodromy factors \(\varphi_1 = Z_1 \cdot Z_1', ~ \varphi_2 = Z_2 \cdot Z_2', ~ \varphi_3 = Z_3 \cdot Z_3', ~ \varphi_4 = Z_4 \cdot Z_4',\) which are depicted in Figure 26.

The 4-point gives rise to the following braid monodromy factors, as computed in [6]. These braids appear in order in Figure 27.

\[
\varphi_5 = (Z^3_{1,2} \cdot Z^2_{3,4} \cdot h_1 \cdot h_2 \cdot (Z^2_{1,4})^Z_{1,2} \cdot Z^2_{1,4}) \cdot (Z^3_{1,2} \cdot (Z^2_{3,4})^Z_{1,4} \cdot h_3 \cdot h_4 \cdot (Z^2_{1,4})^Z_{1,2} \cdot (Z^2_{1,4})^Z_{1,2} \cdot (Z^2_{1,4})^Z_{1,2}).
\]
These braids give rise to the relations

\[ \Gamma_1 = \Gamma_1' \]
\[ \Gamma_2 = \Gamma_2' \]
\[ \Gamma_3 = \Gamma_3' \]
\[ \Gamma_4 = \Gamma_4' \]

\[ \langle \Gamma_1, \Gamma_2 \rangle = (\Gamma_1', \Gamma_2') = (\Gamma_1' \Gamma_2^{-1} \Gamma_2' \Gamma_2) = e \]
\[ \langle \Gamma_3, \Gamma_4 \rangle = (\Gamma_3', \Gamma_4') = (\Gamma_3' \Gamma_3^{-1} \Gamma_3' \Gamma_4) = e \]
\[ [\Gamma_2' \Gamma_2 \Gamma_2^{-1} \Gamma_2' \Gamma_2', \Gamma_4] = e \]
\[ [\Gamma_1, \Gamma_4] = e \]
\[ [\Gamma_1', \Gamma_2'] = (\Gamma_1', \Gamma_2') = (\Gamma_1^{-1} \Gamma_2' \Gamma_2' \Gamma_4) = e \]
\[ (\Gamma_3, \Gamma_4^{-1} \Gamma_4') = (\Gamma_3' \Gamma_4^{-1} \Gamma_4') = (\Gamma_3' \Gamma_3^{-1} \Gamma_3' \Gamma_4) = e \]
\[ [\Gamma_2' \Gamma_2 \Gamma_2^{-1} \Gamma_2' \Gamma_2', \Gamma_4^{-1} \Gamma_4'] = e \]
\[ [\Gamma_1^{-1} \Gamma_1^{-1} \Gamma_1, \Gamma_4^{-1} \Gamma_4'] = e \]

\[ \Gamma_2' \Gamma_2 \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = \Gamma_2 \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = e \]
\[ \Gamma_2' \Gamma_2 \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = \Gamma_2 \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = e \]
\[ \Gamma_2' \Gamma_2 \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = \Gamma_2 \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = e \]
\[ \Gamma_2' \Gamma_2 \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = \Gamma_2 \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2^{-1} \Gamma_2' \Gamma_2 = e \]
We also have the projective relation

\[(71) \Gamma_1'\Gamma_4'\Gamma_3'\Gamma_2'\Gamma_3\Gamma_1 = e.\]

Relations (55)-(58) simplify the relations as follows:

\[
\langle \Gamma_1, \Gamma_2 \rangle = e \\
\langle \Gamma_3, \Gamma_4 \rangle = e \\
[\Gamma_2^2\Gamma_1\Gamma_2^{-2}, \Gamma_4] = e \\
[\Gamma_1, \Gamma_4] = e \\
\Gamma_2^2\Gamma_1\Gamma_2\Gamma_1^{-1}\Gamma_2^{-2} = \Gamma_4\Gamma_3\Gamma_4^{-1} \\
\Gamma_4\Gamma_3^2\Gamma_2^2\Gamma_1 = e.
\]

In the group \(G/ \langle \Gamma_i^2 \rangle\), we get

\[
\langle \Gamma_1, \Gamma_2 \rangle = e \\
\langle \Gamma_3, \Gamma_4 \rangle = e \\
[\Gamma_1, \Gamma_4] = e \\
\Gamma_1\Gamma_2\Gamma_1^{-1} = \Gamma_4\Gamma_3\Gamma_4^{-1}.
\]

Using these relations, we have:

\[
e = \langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_4\Gamma_3\Gamma_4^{-1}, \Gamma_4 \rangle = \langle \Gamma_1\Gamma_2\Gamma_1^{-1}, \Gamma_1\Gamma_4\Gamma_1^{-1} \rangle = \langle \Gamma_2, \Gamma_4 \rangle.
\]

Since \(\Gamma_3 = \Gamma_4^{-1}\Gamma_1\Gamma_2\Gamma_1^{-1}\Gamma_4\), we get that \(G/ \langle \Gamma_i^2 \rangle\) is generated by \(\Gamma_1, \Gamma_2, \Gamma_4\), with the relations \(\Gamma_i^2 = e, \langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = [\Gamma_1, \Gamma_4] = e\). Therefore, \(G/ \langle \Gamma_i^2 \rangle \cong S_4\).

Since \(S_4\) is a finite group, the only map onto \(S_4\) is the identity map, and hence the kernel of the map is trivial. \(\square\)

### 3. Appendix: Calculations of Galois covers of \(\mathbb{CP}^1 \times \mathbb{CP}^1\)

#### 3.1. The case \((a, b) = (2, 2)\)

We now consider the degeneration of \(\mathbb{CP}^1 \times \mathbb{CP}^1\) of bidegree \((a = 2; b = 2)\), as shown in Figure 28. We expect \(\text{deg } \Delta_{16}^2 = 240\) conditions on the 16 Van Kampen generators \(\Gamma_1, \Gamma_1', \ldots, \Gamma_8, \Gamma_8'\).
We construct the braid relations from the degeneration. Vertex $a$ (resp., vertex $c$) is equivalent to vertex 1 (resp., vertex 3) of the $(1, 2)$-degeneration from 2.3.3, so giving rise to the single conditions

\[(72) \quad \Gamma_1 = \Gamma'_1\]
\[(73) \quad \Gamma_8 = \Gamma'_8.\]

Vertex $b$ and vertex $d$ are equivalent to vertex 2 of the $(1, 2)$-degeneration, giving rise to the sets of degree 10 conditions

\[(74) \quad \Gamma'_4 = \Gamma_4\Gamma'_2\Gamma_2\Gamma_4^{-1}\Gamma_2^{-1}\Gamma_4^{-1}\]
\[(75) \quad (a, \Gamma_4) = e, \quad \text{where } a = \Gamma_2, \Gamma'_2 \text{ or } \Gamma_2^{-1}\Gamma'_2\Gamma_2\]
\[(76) \quad \Gamma'_6 = \Gamma_4\Gamma'_2\Gamma_2\Gamma_6\Gamma_2^{-1}\Gamma_2^{-1}\Gamma_4^{-1}\]
\[(77) \quad [a, \Gamma_6] = e, \quad \text{where } a = \Gamma_2, \Gamma'_2 \text{ or } \Gamma_2^{-1}\Gamma'_2\Gamma_2.\]

The braids for the central vertex $e$ include those shown in Figures 29 and 30,
as well as those shown in Figures 31 and 32, which are all conjugated by $Z_{77',8}^{-2}Z_{1,22}^2$, those shown in Figures 33 and 34, which are all conjugated by $Z_{77',8}^{-1}Z_{22}^{-1}$, and those shown in Figures 35 and 36, which are all conjugated by $Z_{77',8}^{-1}Z_{22}^{-1}$. 

\[\text{Figure 30. } Z_{77',8}^{-2}Z_{1,33}, Z_{1,55}, Z_{1,77}, Z_{1,88}\]

\[\text{Figure 31. } (Z_{2,33})_{77',8}^{-2}Z_{22}^2, (Z_{3,55})_{77',8}^{-2}Z_{22}^2, (\tilde{Z}_{3,5})_{77',8}^{-2}Z_{22}^2\]

\[\text{Figure 32. } (\tilde{Z}_{3,5})_{77',8}^{-2}Z_{22}^2, (Z_{5,8})_{77',8}^{-2}Z_{22}^2, (Z_{2,7})_{77',8}^{-2}Z_{22}^2, (Z_{2,7})_{77',8}^{-1}Z_{22}^{-2}\]
The vertex thus gives rise to the following conditions:

\[(\Gamma_1, a) = e, \text{ where } a = \Gamma_2, \Gamma_2' \text{ or } \Gamma_2^{-1}\Gamma_2'\Gamma_2,\]

\[\Gamma_5' = \Gamma_5^{-1}\Gamma_5'\Gamma_7^{-1}\Gamma_7'\Gamma_8\Gamma_7\Gamma_7'\Gamma_8',\]

\[[b, \Gamma_8] = e, \text{ where } b = \Gamma_3 \text{ or } \Gamma_5',\]

\[(\Gamma_6^{-1}\Gamma_6'\Gamma_7^{-1}\Gamma_7'\Gamma_8\Gamma_7'\Gamma_8', c) = e, \text{ where } c = \Gamma_5, \Gamma_5', \text{ or } \Gamma_5^{-1}\Gamma_5'c,\]

\[[\Gamma_7^{-1}\Gamma_7'\Gamma_8\Gamma_7'\Gamma_8', b] = e, \text{ where } b = \Gamma_3 \text{ or } \Gamma_5',\]

\[(d, \Gamma_8) = e, \text{ where } d = \Gamma_7, \Gamma_7', \text{ or } \Gamma_7^{-1}\Gamma_7',\]

\[(\Gamma_2'\Gamma_2\Gamma_1^{-1}\Gamma_2'\Gamma_2^{-1}, b) = e, \text{ where } b = \Gamma_3, \Gamma_5, \text{ or } \Gamma_3^{-1}\Gamma_3'\Gamma_3,\]

\[[\Gamma_1, c] = e, \text{ where } c = \Gamma_5 \text{ or } \Gamma_5',\]

\[[\Gamma_1, d] = e, \text{ where } d = \Gamma_7 \text{ or } \Gamma_7',\]

\[[\Gamma_1, e] = e, \text{ where } e = \Gamma_8 \text{ or } \Gamma_5',\]
Figure 35. \((Z_2^{22} \cdot 8)^{Z_2^{22} \cdot 22}, (Z_2^{22} \cdot 8)^{Z_2^{22} \cdot 22}, (Z_1^{22} \cdot 22)^{22}\)
Vertex \( f \) and vertex \( h \) are equivalent to vertex 5 of the \((1, 2)\) degeneration, giving rise to the sets of degree 10 conditions

\[
\Gamma_4 = \Gamma_5^4 \Gamma_5 \Gamma_5^{-1} \Gamma_5^{-1},
\]
\[
(\Gamma_4', c) = e, \quad \text{where} \ c = \Gamma_5, \ \Gamma_5' \ \text{or} \ \Gamma_5^{-1} \Gamma_5' \Gamma_5,
\]
\[
\Gamma_6 = \Gamma_7 \Gamma_6 \Gamma_7^{-1} \Gamma_7^{-1},
\]
\[
(\Gamma_6', d) = 1, \quad \text{where} \ d = \Gamma_7, \ \Gamma_7' \ \text{or} \ \Gamma_7^{-1} \Gamma_7' \Gamma_7.
\]

Finally, for the parasitic intersections, we have the following braids:

![Figure 37](image1.png)

**Figure 37.** \(Z_{11',44'}^2, Z_{66',44'}^2, Z_{55',66'}^2, Z_{44',66'}^2, Z_{22',66'}^2\)

![Figure 38](image2.png)

**Figure 38.** \(Z_{11',66'}^2, (Z_{44',77'}^2)^{22'}, (Z_{66',66'}^2)^{22'}, s, (Z_{44',88'}^2)^{22'}, s\)
Thus the relations are

\[
[\Gamma'_5\Gamma_3\Gamma'_3\Gamma_2g^{-1}\Gamma_2^{2^{-1}}\Gamma_3^{-1}\Gamma_3^{1^{-1}}, h] = e,
\]

where \( g = \Gamma'_1\Gamma_1\Gamma'^{-1}_1 \) or \( \Gamma'_1 \), and \( h = \Gamma_4 \) or \( \Gamma'_4 \). \hfill (110)

\[
[b, h] = e, \text{ where } b = \Gamma_3 \text{ or } \Gamma'_3 \text{ and } h = \Gamma_4 \text{ or } \Gamma'_4. \hfill (111)
\]

\[
[d, i] = e, \text{ where } d = \Gamma_5 \text{ or } \Gamma'_5 \text{ and } i = \Gamma_6 \text{ or } \Gamma'_6. \hfill (112)
\]

\[
[\Gamma'_5\Gamma_5g\Gamma^{-1}_5\Gamma'^{-1}_5, g] = e, \text{ where } g = \Gamma'_4\Gamma_4\Gamma'^{-1}_4 \text{ or } \Gamma'_4 \text{ and } i = \Gamma_6 \text{ or } \Gamma'_6. \hfill (113)
\]

\[
[\Gamma'_2\Gamma_5\Gamma'_5\Gamma'_3\Gamma_3\alpha\Gamma^{-1}_3\Gamma_4^{-1}\Gamma'_4^{-1}\Gamma_5^{-1}, i] = e,
\]

where \( a = \Gamma'_2\Gamma_2\Gamma'^{-1}_2 \) or \( \Gamma'_2 \) and \( i = \Gamma_6 \) or \( \Gamma'_6. \hfill (114)
\]

\[
[\Gamma'_5\Gamma_5\Gamma'_4\Gamma'_4\Gamma'_3\Gamma_3\Gamma_2g\Gamma^{-1}_2\Gamma'_2\Gamma_3^{-1}\Gamma'_3^{-1}\Gamma_4^{-1}\Gamma'_4^{-1}\Gamma_5^{-1}, i] = e
\]

where \( g = \Gamma'_1 \), \( \Gamma'_1\Gamma_1\Gamma'^{-1}_1 \); \( i = \Gamma_6 \), \( \Gamma_6\Gamma^{-1}_6\Gamma_6. \hfill (115)
\]

\[
[\Gamma'_5\Gamma_5\Gamma_6\Gamma_6\Gamma_6\Gamma_6\Gamma_5\Gamma'_5\Gamma'_5\Gamma_5\Gamma_5\Gamma_2\Gamma_2\Gamma_1, i] = e.
\]

\hfill (116)

\[
[d, g] = e, \text{ where } h = \Gamma_4 \text{ or } \Gamma'_4; \text{ } d = \Gamma_7 \text{ or } \Gamma'_7, \hfill (117)
\]

\[
[\Gamma'_2\Gamma_7\Gamma'_7\Gamma_7, f] = e, \text{ where } i = \Gamma_6 \text{ or } \Gamma'_6; \text{ } f = \Gamma_8 \text{ or } \Gamma'_8, \hfill (118)
\]

\[
\text{where } h = \Gamma_4 \text{ or } \Gamma'_4; \text{ } f = \Gamma_8 \text{ or } \Gamma'_8.
\]

Finally, we also have the projective relation

\[
\Gamma'_8\Gamma_8\Gamma_7\Gamma'_7\Gamma'_6\Gamma_6\Gamma_6\Gamma_6\Gamma_5\Gamma'_5\Gamma'_4\Gamma'_4\Gamma_3\Gamma_3\Gamma_2\Gamma_2\Gamma_1 = e.
\hfill (119)
\]

**Theorem 13.** The fundamental group of the Galois cover is normally generated by \([\Gamma_2, \Gamma_4\Gamma_3\Gamma_4]\).

**Proof.** The proof will use the following key lemma repeatedly.

**Lemma 14.** Let \( G \) be a group containing \( x, y \) and \( y' \) such that \( \langle x, y \rangle = \langle x, y' \rangle = \langle x, y^{-1}y'y \rangle = e \), \( [x, y/y'] = e \) \( x^2 - y^2 = y^2 = e \). Then \( y = y' \).

**Proof of Lemma 14.** Since \( \langle x, y \rangle = e \), \( x^{-1}yx = yxy^{-1} \). Since \( \langle x, y' \rangle = e \), \( x^{-1}yx = yxy^{-1} \).

Since \( [x, y/y'] = e \), it follows that \( (x^{-1}y'x)(x^{-1}yx) = x^{-1}y'yx = y'y \). On the other hand, \( (x^{-1}y'x)(x^{-1}yx) = y'xy^{y^{-1}}yx = y'y \). On the other hand, \( (x^{-1}y'x)(x^{-1}yx) = y'xy^{-1}yx = y'y \). On the other hand, \( (x^{-1}y'x)(x^{-1}yx) = y'xy^{-1}yx = y'y \).

Since \( y^2 = 1 \), \( y^{-1} = y \). Hence \( xy^{-1}yx = e \), since \( x = x^{-1} \).

Since \( y^2 = 1 \), \( y^{-1} = y' \). Hence \( e = x^{-1}y^{-1}yx = x^{-1}y'yx = y'y \), so \( y' = y^{-1} = y \). \hfill \Box

Using the lemma, we continue the proof that the fundamental group of the Galois cover, the kernel of the map \( G \to S_8 \) is trivial modulo the subgroup normally generated by the \( \Gamma'_i \), by showing that each \( \Gamma_i \) is isomorphic to \( \Gamma'_i \) modulo the squares.
Starting from the fact that \( \Gamma_1' = \Gamma_1 \) and \( \Gamma_5' = \Gamma_8 \), we can substitute the relation (106) into (113) to obtain

\[
[\Gamma_4, \Gamma_6] = [\Gamma_4, \Gamma_6] = e.
\]

From the relations (76) and (77), we get that \( \Gamma_6' = \Gamma_4 \Gamma_6 \Gamma^{-1}_4 \). Since \( \Gamma_4 \) commutes with \( \Gamma_6 \), we get \( \Gamma_6' = \Gamma_6 \).

From the relation (108), using the fact that \( \Gamma_6' = \Gamma_6 \), it follows that \( [\Gamma_6, \Gamma_7' \Gamma_7] = e \), so by (109), we get that \( \Gamma_7' = \Gamma_7 \) using Lemma 14.

By (81), substituting \( \Gamma_6' \) into (106) and annihilating the squares, it follows that \( \Gamma_6' = \Gamma_6 \).

By (79), since \( \Gamma_8' = \Gamma_8 \), and \( \Gamma_7' = \Gamma_7 \), modulo \( \Gamma_8' \) and \( \Gamma_7' \) we get \( \Gamma_8 = \Gamma_5^{-1} \Gamma_5'^{-1} \Gamma_8' \Gamma_5' \)

Therefore, \( [\Gamma_8, \Gamma_5' \Gamma_5] = e \).

By (81), substituting \( \Gamma_6' \) and \( \Gamma_7' = \Gamma_7 \), we conclude that

\[
(\Gamma_8, c) = e,
\]

where \( c = \Gamma_5, \Gamma_5' \) or \( \Gamma_5^{-1} \Gamma_5 \). Thus, using Lemma 14 again, we conclude that \( \Gamma_5 = \Gamma_5' \).

By substituting \( \Gamma_5 = \Gamma_5' \) into (106) and annihilating the squares, it follows that \( \Gamma_4 = \Gamma_4' \).

Consider the relation (74), and substitute \( \Gamma_4 = \Gamma_4' \) to get \( [\Gamma_4, \Gamma_2' \Gamma_2] = e \). Thus, by (75) and Lemma 14, it follows that \( \Gamma_2 = \Gamma_2' \).

We now consider the projective relation (119). Since \( \Gamma_i = \Gamma_i' \) for all \( i \neq 3 \) and \( \Gamma_i^2 = e \) for all \( i \), we conclude that \( \Gamma_3 = \Gamma_3' \). Hence \( \Gamma_i = \Gamma_i' \) for all \( i \).

The defining relations (74)-(119) reduce to

\[
[\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma_5] = [\Gamma_1, \Gamma_6] = [\Gamma_1, \Gamma_7] = [\Gamma_1, \Gamma_8] = [\Gamma_2, \Gamma_6] = [\Gamma_2, \Gamma_6] = [\Gamma_2, \Gamma_8] = e,
\]

\[
[\Gamma_3, \Gamma_4] = [\Gamma_3, \Gamma_6] = [\Gamma_4, \Gamma_6] = [\Gamma_4, \Gamma_7] = [\Gamma_4, \Gamma_8] = [\Gamma_5, \Gamma_6] = [\Gamma_6, \Gamma_8] = e,
\]

\[
\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_1, \Gamma_3 \rangle = \langle \Gamma_1 \Gamma_2 \Gamma_1^{-1}, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_4, \Gamma_5 \rangle = \langle \Gamma_5, \Gamma_8 \rangle = \langle \Gamma_7, \Gamma_8 \rangle = \langle \Gamma_5, \Gamma_7 \Gamma_8 \Gamma_7^{-1} \rangle = e.
\]

From (90) and (92), (98) and (99), substituting \( \Gamma_i \) for \( \Gamma_i' \), we get

\[
\Gamma_1 \Gamma_2 \Gamma_1 \Gamma_3 \Gamma_1 \Gamma_2 \Gamma_1 = \Gamma_8 \Gamma_7 \Gamma_8 \Gamma_5 \Gamma_8 \Gamma_7 \Gamma_8.
\]

Hence, \( \Gamma_2 \Gamma_1 \Gamma_3 \Gamma_1 \Gamma_2 = \Gamma_7 \Gamma_8 \Gamma_5 \Gamma_8 \Gamma_7 \), since \( [\Gamma_1, \Gamma_7 \Gamma_8 \Gamma_5 \Gamma_8 \Gamma_7] = [\Gamma_8, \Gamma_1 \Gamma_2 \Gamma_1 \Gamma_3 \Gamma_1 \Gamma_2 \Gamma_1] = e. \)
Since $[\Gamma_i, \Gamma_8] = 1$ for $1 \leq i \leq 3$, we have
\[
e = [\Gamma_1\Gamma_2\Gamma_1\Gamma_3\Gamma_1\Gamma_2\Gamma_1, \Gamma_8] = [\Gamma_8\Gamma_7\Gamma_7\Gamma_8\Gamma_8\Gamma_7\Gamma_8, \Gamma_8]
\]
\[
= [\Gamma_7\Gamma_8\Gamma_8\Gamma_8\Gamma_7, \Gamma_8] = [\Gamma_8\Gamma_5\Gamma_8, \Gamma_7\Gamma_8\Gamma_7]
\]
\[
= [\Gamma_8\Gamma_5\Gamma_8, \Gamma_8\Gamma_7\Gamma_8] = [\Gamma_5, \Gamma_7].
\]

In the same way,
\[
e = [\Gamma_8\Gamma_7\Gamma_8\Gamma_8\Gamma_7\Gamma_8, \Gamma_1] = [\Gamma_1\Gamma_2\Gamma_1\Gamma_3\Gamma_1\Gamma_2\Gamma_1, \Gamma_1]
\]
\[
= [\Gamma_2\Gamma_1\Gamma_3\Gamma_2, \Gamma_1] = [\Gamma_1\Gamma_3\Gamma_1, \Gamma_2\Gamma_1\Gamma_2]
\]
\[
= [\Gamma_1\Gamma_3\Gamma_1, \Gamma_1\Gamma_2\Gamma_1] = [\Gamma_3, \Gamma_2].
\]

From $\Gamma_2\Gamma_1\Gamma_3\Gamma_1\Gamma_2 = \Gamma_7\Gamma_8\Gamma_5\Gamma_8\Gamma_7$, we get
\[
\Gamma_3 = \Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_5\Gamma_8\Gamma_2\Gamma_1 \quad \text{and} \quad \Gamma_5 = \Gamma_8\Gamma_7\Gamma_2\Gamma_1\Gamma_3\Gamma_1\Gamma_2\Gamma_7\Gamma_8.
\]

Thus,
\[
\langle \Gamma_3, \Gamma_6 \rangle = \langle \Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_5\Gamma_7\Gamma_2\Gamma_1, \Gamma_6 \rangle = \langle \Gamma_2, \Gamma_6 \rangle = e,
\]
\[
\langle \Gamma_2, \Gamma_5 \rangle = \langle \Gamma_2, \Gamma_8\Gamma_7\Gamma_2\Gamma_1\Gamma_3\Gamma_1\Gamma_2\Gamma_7\Gamma_8 \rangle = \langle \Gamma_2, \Gamma_1\Gamma_3\Gamma_1 \rangle = \langle \Gamma_2, \Gamma_3\Gamma_1\Gamma_3 \rangle = e,
\]
\[
\langle \Gamma_3, \Gamma_7 \rangle = \langle \Gamma_1\Gamma_2\Gamma_6\Gamma_8\Gamma_5\Gamma_8\Gamma_2\Gamma_2\Gamma_1, \Gamma_7 \rangle = \langle \Gamma_8, \Gamma_7 \rangle = e.
\]

Hence we obtain the Coxeter group with Dynkin diagram given in Figure 39, which, by [32, Theorem 2.3, p.4], has dual diagram given in Figure 40, and can be mapped to $S_8$, such that the kernel is normally generated by $[\Gamma_1, \Gamma_4\Gamma_5\Gamma_4]$. \hfill \Box

![Figure 39. The Coxeter diagram](image)

**Remark 15.** This result can also be deduced from [18].
3.2. The case \((a = 1, b = n)\). We now provide a more detailed proof of the following theorem (which appeared as an unproved corollary in [6]):

**Theorem 16.** The Galois cover of \((1, n)\)-embedding of \(\mathbb{CP}^1 \times \mathbb{CP}^1\) is simply connected, for all \(n\).

**Proof.** We first construct the braid relations.

All the vertices resemble those of the \((1, 2)\) degeneration. The two corners 1 and \(2n+2\) look like the corner vertices 1 and 6 of the \((1, 2)\) degeneration, and give rise to the following equations:

\[
\begin{align*}
\Gamma_1 &= \Gamma_1', \\
\Gamma_{2n-1} &= \Gamma_{2n-1}'.
\end{align*}
\]

The lower vertices look like vertex 2 of the \((1, 2)\) degeneration, and give rise to the following equations:

\[
\begin{align*}
\langle \Gamma_2, \Gamma_{2i+1} \rangle &= \langle \Gamma_2', \Gamma_{2i+1} \rangle = (\Gamma_2^{-1}\Gamma_2'\Gamma_2, \Gamma_{2i+1}) = e \\
\Gamma_{2i-1}' &= \Gamma_{2i+1}\Gamma_2\Gamma_2\Gamma_{2i+1}\Gamma_2^{-1}\Gamma_2'\Gamma_{2i+1}^{-1},
\end{align*}
\]

for \(i = 1, \cdots, n-1\). The upper vertices appear like vertex 5 of the \((1,2)\) degeneration, and give rise to the following equations:

\[
\begin{align*}
\langle \Gamma_5, \Gamma_{2j+1} \rangle &= \langle \Gamma_5', \Gamma_{2j+1} \rangle = (\Gamma_5^{-1}\Gamma_5'\Gamma_5, \Gamma_{2j+1}) = e \\
\Gamma_{2j-1}' &= \Gamma_{2j+1}\Gamma_5\Gamma_5\Gamma_{2j+1}\Gamma_2^{-1}\Gamma_5'\Gamma_{2j+1}^{-1},
\end{align*}
\]

for \(j = 1, \cdots, n-1\). The “parasitic” intersections give rise to the following equations:

\[
\begin{align*}
[a, b] &= e, \\
\text{where } a &= \Gamma_l \text{ or } \Gamma_l', \text{ and } b = \Gamma_{2k+1}, \Gamma_{2k+1}', \text{ or } \Gamma_{2k-1}^{-1}\Gamma_{2k-1}', \Gamma_{2k}^{-1}\Gamma_{2k}', \Gamma_{2k-1}\Gamma_{2k-1}^{-1}\Gamma_{2k-1}, \text{ for } k = 1, \cdots, n-1, \\
l &= 1, \cdots, 2k - 1. \text{ Finally, we also have the projective relation}
\end{align*}
\]

\[
\Gamma_{2n-1}'\Gamma_{2n-1}'\cdots\Gamma_1'\Gamma_1 = e.
\]
Now we simplify these relations.

From (125) \((j = 1)\), we get \(\Gamma_1 = \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2^{-1}\); hence \(\Gamma_2^{-1} \Gamma_1 \Gamma_2 = \Gamma_2 \Gamma_1 \Gamma_2^{-1}\). From (121), we can rewrite this as \(\Gamma_2^{-1} \Gamma_1 \Gamma_2 = \Gamma_2 \Gamma_1 \Gamma_2^{-1}\). From (126) \((j = 1)\), we have

\[
\langle \Gamma_1', \Gamma_2 \rangle = \langle \Gamma_1', \Gamma_2 \rangle = e \quad \text{and} \quad \Gamma_2' \Gamma_1 \Gamma_2 = \Gamma_1' \Gamma_2 \Gamma_1 \Gamma_2^{-1} = \Gamma_1' \Gamma_2 \Gamma_1'.
\]

Thus, \(\Gamma_2' \Gamma_1 = \Gamma_2 \Gamma_1\). Modulo the \(\Gamma_i^2\), this means that \(\Gamma_2 = \Gamma_2'\).

Likewise, from (123) and (124) \((i = 2)\), we get \(\Gamma_3' = \Gamma_3 \Gamma_2 \Gamma_2 \Gamma_3 \Gamma_2^{-1} \Gamma_2^{-1} \Gamma_3^{-1}\).

Now since \(\Gamma_2 = \Gamma_2'\) and \(\Gamma_2^2 = e\), we find that modulo the squares, \(\Gamma_3' = \Gamma_3 \Gamma_3 \Gamma_3^{-1} = \Gamma_3\).

By induction, we conclude that \(\Gamma_k' = \Gamma_k\) modulo the squares, for \(1 \leq k \leq 2n - 1\). Hence, \(G/\langle \Gamma_2^k \rangle \cong S_{2n}\). Hence, the kernel of the map \(G/\langle \Gamma_2^k \rangle \to S_{2n}\) is trivial. \(\square\)

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