WEAK SOLUTIONS FOR ONE-DIMENSIONAL NON-CONVEX ELASTODYNAMICS

SEONGHAK KIM AND YOUNGWOO KOH

ABSTRACT. We explore local existence and properties of classical weak solutions to the initial-boundary value problem of a one-dimensional quasilinear equation of elastodynamics with non-convex stored-energy function, a model of phase transitions in elastic bars proposed by Ericksen [19]. The instantaneous formation of microstructures of local weak solutions is observed for all smooth initial data with initial strain having its range overlapping with the phase transition zone of the Piola-Kirchhoff stress. As byproducts, it is shown that such a problem admits a local weak solution for all smooth initial data and local weak solutions that are smooth for a short period of time and exhibit microstructures thereafter for some smooth initial data. In a parallel exposition, we also include some results concerning one-dimensional quasilinear hyperbolic-elliptic equations.

1. INTRODUCTION

The evolution process of a one-dimensional continuous medium with elastic response can be modeled by quasilinear wave equations of the form

\[ u_{tt} = \sigma(u_x)_x, \]

where \( u = u(x,t) \) denotes the displacement of a reference point \( x \) at time \( t \) and \( \sigma = \sigma(s) \) the Piola-Kirchhoff stress, which is the derivative of a stored-energy function \( W = W(s) \geq 0 \). With \( v = u_x \) and \( w = u_t \), one may study equation (1.1) as the system of conservation laws:

\[
\begin{cases}
  v_t = w_x, \\
  w_t = \sigma(v)_x.
\end{cases}
\]

In case of a strictly convex stored-energy function, the existence of weak or classical solutions to equation (1.1) and its vectorial case has been studied extensively: Global weak solutions to system (1.2) and hence equation (1.1) are established in a classical work of DiPerna [18] via vanishing viscosity method in the framework of compensated compactness of Tartar [43] for \( L^\infty \) data and later by Lin [29] and Shearer [39] in an \( L^p \) setup. This framework is also used to construct global weak solutions to (1.1) via relaxation methods.
by Serre [38] and Tzavaras [45]. An alternative variational scheme is studied by Demoulini et al. [15] via time discretization. However global existence of weak solutions to the vectorial case of (1.1) is still open. In regard to classical solutions to (1.1) and its vectorial case, one can refer to Dafermos and Hrusa [11] for local existence of smooth solutions, to Klainerman and Sideris [28] for global existence of smooth solutions for small initial data in dimension 3, and to Dafermos [13] for uniqueness of smooth solutions in the class of BV weak solutions whose shock intensity is not too strong.

Convexity assumption on the stored-energy function is often regarded as a severe restriction in view of the actual behavior of elastic materials (see, e.g., [22, Section 2] and [7, Section 8]). However there have not been many analytic works dealing with the lack of convexity on the energy function. For the vectorial case of equation (1.1) in dimension 3, measure-valued solutions are constructed for polyconvex energy functions by Demoulini et al. [16]. Also by the same authors [17], in the identical situation, it is shown that a dissipative measure-valued solution coincides with a strong one provided the latter exists. Assuming convexity on the energy function at infinity but not allowing polyconvexity, measure-valued solutions are obtained by Rieger [37] for the vectorial case of (1.1) in any dimension. Despite of all these existence results, there has been no example of a non-convex energy function with which (1.1) admits classical weak solutions in general, not to mention its vectorial case. Among some optimistic and pessimistic opinions, Rieger [37] expects such solutions even showing microstructures of phase transitions. Moreover, such expected phenomenology is successfully implemented in some numerical simulations [5, 36].

In this paper, we study weak solutions to the one-dimensional initial-boundary value problem of non-convex elastodynamics

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \sigma(u_x)_x & \text{in } \Omega_T = \Omega \times (0, T), \\
u(0, t) &= u(1, t) = 0 & \text{for } t \in (0, T), \\
u = g, \ u_t = h & \text{on } \Omega \times \{t = 0\},
\end{align*}
\]

(1.3)

where \(\Omega = (0, 1) \subset \mathbb{R}\) is the domain occupied by a reference configuration of an elastic bar, \(T > 0\) is a fixed number, \(g\) is the initial displacement of the bar, \(h\) is the initial rate of change of the displacement, and the stress \(\sigma : (-1, \infty) \to \mathbb{R}\) is given as in Figure 1. The zero boundary condition here amounts to the physical situation of fixing the end-points of the bar. In this case, the energy function \(W : (-1, \infty) \to [0, \infty)\) may satisfy \(W(s) \to \infty\) as \(s \to -1^+\); but this is not required to obtain our result. On the other hand, we consider (1.3) as a prototype of the hyperbolic-elliptic problem with a non-monotone constitutive function \(\sigma : \mathbb{R} \to \mathbb{R}\) as in Figure 2.

Problem (1.3) with a non-monotone stress \(\sigma(s)\) as in Figure 2 is proposed by Ericksen [19] as a model of the phenomena of phase transitions in elastic bars. There have been many studies on such a problem that usually fall into two types. One direction of study is to consider the Riemann problem of the
system of conservation laws of mixed type \((1.2)\) initiated by James \[23\] and followed by numerous works (see, e.g., Shearer \[40\], Pego and Serre \[34\] and Hattori \[21\]). Another path is to study the viscoelastic version of equation \((1.1)\): To name a few among initiative works, Dafermos \[12\] considers the equation \(u_{ttt} = \sigma(u_x, u_{xt})_x + f(x, t)\) under certain parabolicity and growth conditions and establishes global existence and uniqueness of smooth solutions together with some asymptotic behaviors as \(t \to \infty\). Following the work of Andrews \[2\], Andrews and Ball \[3\] prove global existence of weak
solutions to the equation \(u_{tt} = u_{xxt} + \sigma(u_x)_x\) for non-smooth initial data and study their long-time behaviors. For the same equation, Pego [33] characterizes long-time convergence of weak solutions to several different types of stationary states in a strong sense. Nonetheless, up to our best knowledge, the main theorem below may be the first general existence result on weak solutions to (1.3), not in the stream of the Riemann problem nor in that of non-convex viscoelastodynamics.

Let \(\sigma(s)\) be given as in Figure 1 or 2 (see section 2). For an initial datum \((g, h) \in W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)\), we say that a function \(u \in W_1^{1,\infty}(\Omega_T)\) is a weak solution to problem (1.3) provided \(u_x > -1\) a.e. in \(\Omega_T\) in case of Figure 1, for all \(\varphi \in C_\infty_c(\Omega \times [0, T])\), we have

\[
\int_{\Omega_T} (u_t \varphi_t - \sigma(u_x) \varphi_x) \, dx \, dt = - \int_0^1 h(x) \varphi(x, 0) \, dx,
\]

and

\[
\begin{cases}
   u(0, t) = u(1, t) = 0 & \text{for } t \in (0, T), \\
   u(x, 0) = g(x) & \text{for } x \in \Omega.
\end{cases}
\]

The main result of the paper is the following theorem that will be separated into two detailed statements in section 2 along with some corollaries.

**Theorem 1.1.** Let \(\sigma(s)\) be as in Figure 1 or 2 and let \((g, h) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega)\) with \(s_1^* < g'(x_0) < s_2^*\) at some \(x_0 \in \Omega\). In case of Figure 1, assume also \(g'(x) > -1\) for all \(x \in \Omega\). Then there exists a finite number \(T > 0\) for which problem (1.3) admits infinitely many weak solutions.

Existence and non-uniqueness of weak solutions to problem (1.3) have been generally accepted (especially, in the context of the Riemann problem) and actively studied in the community of solid mechanics. Such non-uniqueness is usually understood to be arising from a constitutive deficiency in the theory of elastodynamics, reflecting the need to incorporate some additional relations (see, e.g., Slemrod [42], Abeyaratne and Knowles [1] and Truskinovsky and Zanzotto [44]).

Global existence of Lipschitz continuous weak solutions to problem (1.3) is not directly obtained in the course of proving Theorem 1.1 as it would require a global classical solution to some modified hyperbolic problem in our method of proof and such a global one might not exist due to a possible shock formation at a finite time. However, we still expect the existence of global \(W_1^{1,p}\)-solutions \((p < \infty)\) to (1.3).

The rest of the paper is organized as follows. Section 2 describes precise structural assumptions on the functions \(\sigma(s)\) corresponding to Figures 1 and 2 respectively. Then detailed statements of the main result, Theorem 1.1 with respect to Figures 1 and 2 are introduced separately as Theorems 2.1 and 2.4 with relevant corollaries in each case. Section 3 begins with a motivational approach to solve problem (1.3) as a homogeneous partial differential inclusion with a linear constraint. Then the main results in
precise form, Theorems 2.1 and 2.4 are proved at the same time under a pivotal density fact, Theorem 3.1. The proofs of the corollaries to the main results are also included in section 3. In section 4, a major tool for proving the density fact is established in a general form. Lastly, section 5 carries out the proof of the density fact.

In closing this section, we introduce some notations. Let \( m, n \) be positive integers. We denote by \( \mathbb{M}^{m \times n} \) the space of \( m \times n \) real matrices and by \( \mathbb{M}^{m \times n}_{sym} \) that of symmetric \( n \times n \) real matrices. We use \( O(n) \) to denote the space of \( n \times n \) orthogonal matrices. For a given matrix \( M \in \mathbb{M}^{m \times n} \), we write \( M_{ij} \) for the component of \( M \) in the \( i \)th row and \( j \)th column and \( M^T \) for the transpose of \( M \). For a bounded domain \( U \subseteq \mathbb{R}^n \) and a function \( w^* \in W^{m,p}(U) \) \((1 \leq p \leq \infty)\), we use \( W^{m,p}_w(U) \) to denote the space of functions \( w \in W^{m,p}(U) \) with boundary trace \( w^* \).

2. Precise statements of main theorems

In this section, we present structural assumptions on the functions \( \sigma(s) \) for Case I: non-convex elastodynamics and Case II: hyperbolic-elliptic problem corresponding to Figures 1 and 2 respectively. Then we give the detailed statement of the main result, Theorem 1.1, in each case, followed by some relevant byproducts.

(Case I): For the problem of non-convex elastodynamics, we impose the following conditions on the stress \( \sigma : (-1, \infty) \rightarrow \mathbb{R} \) (see Figure 1).

Hypothesis (NC): There exist two numbers \( s_2 > s_1 > -1 \) with the following properties:

(a) \( \sigma \in C^3((-1, s_1) \cup (s_2, \infty)) \cap C((-1, s_1] \cup [s_2, \infty)) \).
(b) \( \lim_{s \downarrow -1^+} \sigma(s) = -\infty \).
(c) \( \sigma : (s_1, s_2) \rightarrow \mathbb{R} \) is bounded and measurable.
(d) \( \sigma(s_1) > \sigma(s_2) \), and \( \sigma'(s) > 0 \) for all \( s \in (-1, s_1) \cup (s_2, \infty) \).
(e) There exist two numbers \( c > 0 \) and \( s_1 + 1 > \rho > 0 \) such that \( \sigma'(s) \geq c \) for all \( s \in (-1, s_1 - \rho] \cup [s_1 + 1, \infty) \).
(f) Let \( s_1^* \in (-1, s_1) \) and \( s_2^* \in (s_2, \infty) \) denote the unique numbers with \( \sigma(s_1^*) = \sigma(s_2) \) and \( \sigma(s_2^*) = \sigma(s_1) \), respectively.

(Case II): For the hyperbolic-elliptic problem, we assume the following for the constitutive function \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) (see Figure 2).

Hypothesis (HE): There exist two numbers \( s_2 > s_1 \) satisfying the following:

(a) \( \sigma \in C^3((-\infty, s_1) \cup (s_2, \infty)) \cap C((-\infty, s_1] \cup [s_2, \infty)) \).
(b) \( \sigma : (s_1, s_2) \rightarrow \mathbb{R} \) is bounded and measurable.
(c) \( \sigma(s_1) > \sigma(s_2) \), and \( \sigma'(s) > 0 \) for all \( s \in (-\infty, s_1) \cup (s_2, \infty) \).
(d) There exists a number \( c > 0 \) such that \( \sigma'(s) \geq c \) for all \( s \in (-\infty, s_1 - 1] \cup [s_2 + 1, \infty) \).
(e) Let \( s_1^* \in (-\infty, s_1) \) and \( s_2^* \in (s_2, \infty) \) denote the unique numbers with \( \sigma(s_1^*) = \sigma(s_2) \) and \( \sigma(s_2^*) = \sigma(s_1) \), respectively.
In both cases, for each \( r \in (\sigma(s_2), \sigma(s_1)) \), let \( s_-(r) \in (s_1^*, s_1) \) and \( s_+(r) \in (s_2, s_2^*) \) denote the unique numbers with \( \sigma(s_+(r)) = r \). We may call the interval \((s_1^*, s_2^*)\) as the phase transition zone of problem (1.3), since the formation of microstructures and breakdown of uniqueness of weak solutions to (1.3) begin to occur whenever the range of the initial strain \( g' \) overlaps with the interval \((s_1^*, s_2^*)\) as we can see below from Theorems 2.1 and 2.4 and their corollaries.

We now state the main result on Case I: weak solutions for non-convex elastodynamics under Hypothesis (NC).

**Theorem 2.1.** Let \((g, h) \in W^{3,2}_0(\Omega) \times W^{2,2}_0(\Omega)\) satisfy \( g'(x) > -1 \) for all \( x \in \Omega \) and \( s_1^* < g'(x_0) < s_2^* \) for some \( x_0 \in \Omega \). Let \( \sigma(s_2) < r_1 < r_2 < \sigma(s_1) \) be any two numbers with \( s_-(r_1) < g'(x_0) < s_+(r_2) \). Then there exist a finite number \( T > 0 \), a function \( u^* \in \cap_{k=0}^3 C^k([0, T]; W_0^{3-k,2}(\Omega)) \) with \( u^* > -1 \) on \( \Omega_T \), where \( W_0^{0,2}(\Omega) = L^2(\Omega) \), and three disjoint open sets \( \Omega^1_T, \Omega^2_T, \Omega^3_T \subset \Omega_T \) with \( \Omega^2_T \neq \emptyset \), \( \partial \Omega^1_T \cap \partial \Omega^2_T = \emptyset \), and

\[
\begin{align*}
\partial \Omega^1_T \cap \Omega_0 &= \{(x, 0) \mid x \in \Omega, g'(x) < s_-(r_1)\}, \\
\partial \Omega^2_T \cap \Omega_0 &= \{(x, 0) \mid x \in \Omega, s_-(r_1) < g'(x) < s_+(r_2)\}, \\
\partial \Omega^3_T \cap \Omega_0 &= \{(x, 0) \mid x \in \Omega, g'(x) > s_+(r_2)\},
\end{align*}
\]

where \( \Omega_0 = \Omega \times \{t = 0\} \), such that for each \( \epsilon > 0 \), there exist a number \( T_\epsilon \in (0, T) \) and infinitely many weak solutions \( u \in W_0^{1,\infty}(\Omega_T) \) to problem (1.3) with the following properties:

(a) Approximate initial rate of change:

\[
\|u_t - h\|_{L^\infty(\Omega_{T_\epsilon})} < \epsilon,
\]

where \( \Omega_{T_\epsilon} = \Omega \times (0, T_\epsilon) \).

(b) Classical part of \( u \):

(i) \( u \equiv u^* \) on \( \Omega^1_T \cup \Omega^2_T \),

(ii) \( u(x, 0) = h(x) \) \( \forall (x, 0) \in (\partial \Omega^1_T \cup \partial \Omega^3_T) \cap \Omega_0 \),

(iii) \( u(x, t) \in \begin{cases} (s_-(r_1)) & \forall (x, t) \in \Omega^1_T, \\
& > s_+(r_2) & \forall (x, t) \in \Omega^3_T.\end{cases}\)

(c) Microstructure of \( u \):

\( u_x(x, t) \in [s_-(r_1), s_-(r_2)] \cup [s_+(r_1), s_+(r_2)] \), a.e. \((x, t) \in \Omega^2_T\).

(d) Interface of \( u \):

\( u_x(x, t) \in \{s_-(r_1), s_+(r_2)\} \), a.e. \((x, t) \in \Omega_T \setminus \bigcup_{i=1}^3 \Omega^i_T\).

As a remark, note that corresponding deformations of the elastic bar, \( d(x, t) = u(x, t) + x \), should satisfy

\[
d_x(x, t) = u_x(x, t) + 1 > -1 + 1 = 0, \text{ a.e. } (x, t) \in \Omega_T;
\]

this guarantees that for a.e. \( t \in (0, T) \), such deformations \( d : [0, 1] \times \{t\} \rightarrow [0, 1] \) are strictly increasing with \( d(0, t) = 0 \) and \( d(1, t) = 1 \). Moreover, for such a \( t \in (0, T) \), the deformations \( d(x, t) \) are smooth (as much as the initial displacement \( g \)) for the values of \( x \in [0, 1] \) for which slope \( d_x(x, t) \in (0, s_-(r_1) + 1) \cup (s_+(r_2) + 1, \infty) \) and are Lipschitz elsewhere with \( d_x(x, t) \in \cdots \).
\([s_-(r_1) + 1, s_-(r_2) + 1] \cup [s_+(r_1) + 1, s_+(r_2) + 1]\) a.e. Therefore, these dynamic deformations fulfill a natural physical requirement of invertibility for the motion of an elastic bar not allowing interpenetration.

As byproducts of Theorem 2.1, we obtain the following two results for non-convex elastodynamics. The first is on local existence of weak solutions to problem (1.3) for all smooth initial data. The second gives local weak solutions to (1.3) that are all identical and smooth for a short period of time and then evolve microstructures along with the breakdown of uniqueness for some smooth initial data.

**Corollary 2.2.** For any initial datum \((g, h) \in W^{3,2}_0(\Omega) \times W^{2,2}_0(\Omega)\) with \(g' > -1\) on \(\Omega\), there exists a finite number \(T > 0\) for which problem (1.3) has a weak solution.

**Corollary 2.3.** Let \((g, h) \in W^{3,2}_0(\Omega) \times W^{2,2}_0(\Omega)\) satisfy \(g' > -1\) on \(\Omega\). Assume \(\max_\Omega g' \in (s_1^*, s_1)\) or \(\min_\Omega g' \in (s_2, s_2^*)\). Then there exist finite numbers \(T > T' > 0\) such that problem (1.3) admits infinitely many weak solutions that are all equal to some \(u^* \in \cap_{k=0}^3 C^k([0, T']; W^{3-k,2}_0(\Omega))\) in \(\Omega_{T'}\) and evolve microstructures from problem (1.3) for all smooth initial data.

The following is the main result on **Case II:** hyperbolic-elliptic equations under Hypothesis (HE).

**Theorem 2.4.** Let \((g, h) \in W^{3,2}_0(\Omega) \times W^{2,2}_0(\Omega)\) with \(s_1^* < g'(x_0) < s_2^*\) for some \(x_0 \in \Omega\). Let \(\sigma(s_2) < r_1 < r_2 < \sigma(s_1)\) be any two numbers with \(s_-(r_1) < g'(x_0) < s_+(r_2)\). Then there exist a finite number \(T > 0\), a function \(u^* \in \cap_{k=0}^3 C^k([0, T]; W^{3-k,2}_0(\Omega))\), and three disjoint open sets \(\Omega_1^T, \Omega_2^T, \Omega_3^T \subset \Omega_T\) with \(\Omega_1^T \neq \emptyset\) and \(\partial \Omega_1^T \cap \partial \Omega_2^T = \emptyset\), and

\[
\begin{align*}
\partial \Omega_1^T \cap \Omega_0 &= \{ (x, 0) \mid x \in \Omega, g'(x) < s_-(r_1) \}, \\
\partial \Omega_2^T \cap \Omega_0 &= \{ (x, 0) \mid x \in \Omega, s_-(r_1) < g'(x) < s_+(r_2) \}, \\
\partial \Omega_3^T \cap \Omega_0 &= \{ (x, 0) \mid x \in \Omega, g'(x) > s_+(r_2) \},
\end{align*}
\]

such that for each \(\epsilon > 0\), there exist a number \(T_\epsilon \in (0, T)\) and infinitely many weak solutions \(u \in W^{1,\infty}_u(\Omega_T)\) to problem (1.3) satisfying the following properties:

(a) Approximate initial rate of change:
\[\|u_t - h\|_{L^\infty(\Omega_\epsilon)} < \epsilon.\]

(b) Classical part of \(u\):
(i) \(u \equiv u^*\) on \(\Omega_1^T \cup \Omega_2^T\),
(ii) \(u_t(x, 0) = h(x)\) \(\forall (x, 0) \in (\partial \Omega_1^T \cup \partial \Omega_3^T) \cap \Omega_0\),
(iii) \(u_x(x, t) \begin{cases} < s_-(r_1) & \forall (x, t) \in \Omega_1^T \\ > s_+(r_2) & \forall (x, t) \in \Omega_3^T \end{cases}\)

(c) Microstructure of \(u\):
\[u_\epsilon(x, t) \in [s_-(r_1), s_-(r_2)] \cup [s_+(r_1), s_+(r_2)], \text{ a.e. } (x, t) \in \Omega_T.\]

(d) Interface of \(u\):
\[u(x, t) \in \{s_-(r_1), s_+(r_2)\}, \text{ a.e. } (x, t) \in \Omega_T \setminus (\cup_{i=1}^3 \Omega_i^T).\]
Corollary 2.5. For any initial datum \((g, h) \in W^{3,2}_0(\Omega) \times W^{2,2}_0(\Omega)\), there exists a finite number \(T > 0\) for which problem (1.3) has a weak solution.

Corollary 2.6. Let \((g, h) \in W^{3,2}_0(\Omega) \times W^{2,2}_0(\Omega)\) satisfy \(\max_{\Omega} g' \in (s_1^*, s_1)\) or \(\min_{\Omega} g' \in (s_2^*, s_2)\). Then there exist finite numbers \(T > T' > 0\) such that problem (1.3) admits infinitely many weak solutions that are all equal to some \(u^* \in \cap_{k=0}^{3-k} C^k([0, T']; W^{3-k,2}_0(\Omega))\) in \(\Omega_{T'}\) and evolve microstructures from \(t = T'\) as in Theorem 2.4.

3. Proof of main theorems

In this section, we prove the main results, Theorems 2.1 and 2.4, with some essential ingredient, Theorem 3.1, to be verified in sections 4 and 5. The proofs of related corollaries are also included.

Our exposition hereafter will be parallelwise for Cases I and II.

3.1. An approach by differential inclusion. We begin with a motivational approach to attack problem (1.3) for both Cases I and II. To solve equation (1.1) in the sense of distributions in \(\Omega_{T'}\), suppose there exists a vector function \(w = (u, v) \in W^{1,\infty}(\Omega_{T'}; \mathbb{R}^2)\) such that

\[
\begin{align*}
v_x &= u_t \\
v_t &= \sigma(u_x)
\end{align*}
\]

a.e. in \(\Omega_{T'}\).

We remark that this formulation is motivated by the approach in [47] and different from the usual setup of conservation laws (1.2). For all \(\varphi \in C_c^\infty(\Omega_T)\), we now have

\[
\int_{\Omega_T} u_t \varphi_t \, dx \, dt = \int_{\Omega_T} v_x \varphi_t \, dx \, dt = \int_{\Omega_T} v_t \varphi_x \, dx \, dt = \int_{\Omega_T} \sigma(u_x) \varphi_x \, dx \, dt;
\]

hence having (3.1) is sufficient to solve (1.1) in the sense of distributions in \(\Omega_{T'}\). Equivalently, we can rewrite (3.1) as

\[
\nabla w = \begin{pmatrix} u_x & u_t \\
v_x & v_t \end{pmatrix} = \begin{pmatrix} u_x & v_x \\
v_x & \sigma(u_x) \end{pmatrix}
a.e. \text{ in } \Omega_T,
\]

where \(\nabla\) denotes the space-time gradient. Set

\[
\Sigma_{\sigma} = \left\{ \begin{pmatrix} s & b \\
b & \sigma(s) \end{pmatrix} \in M^{2 \times 2}_{sym} \mid s, b \in \mathbb{R} \right\}.
\]

We can now recast (3.1) as a homogeneous partial differential inclusion with a linear constraint:

\[
\nabla w(x, t) \in \Sigma_{\sigma}, \quad \text{a.e. } (x, t) \in \Omega_T.
\]

We will solve this inclusion for a suitable subset \(K\) of \(\Sigma_{\sigma}\) to incorporate some detailed properties of weak solutions to (1.3).

Homogeneous differential inclusions of the form \(\nabla w \in K \subset M^{m \times n}\) are first encountered and successfully understood in the study of crystal microstructure by Ball and James [4], Chipot and Kinderlehrer [6] and with
3.2. **Proof of Theorems 2.1 and 2.4.** Due to the similarity between Theorems 2.1 and 2.4, we can combine their proofs into a single one.

To start the proof, we assume functions \( g, h \) and numbers \( r_1, r_2 \) are given as in Theorem 2.1 (Case I) and in Theorem 2.4 (Case II). For clarity, we divide the proof into several steps.

**Modified hyperbolic problem:** Using elementary calculus, from Hypothesis (NC) (Case I), Hypothesis (HE) (Case II), we can find a function \( \sigma^* \in C^3(\mathbb{R}) \) such that

\[
\sigma^*(s) = \sigma(s) \quad \text{for all} \quad s \in (-1, \infty) \quad \text{(Case I)},
\]

\[
\sigma^*(s) < \sigma(s) \quad \text{for all} \quad s_-(r_1) < s \leq s_-(r_2),
\]

\[
\sigma^*(s) > \sigma(s) \quad \text{for all} \quad s_+(r_1) \leq s < s_+(r_2) \quad \text{(see Figure 3 for both cases)}.
\]
value problem
\[
\begin{align*}
\Omega_T^1 &= \{(x, t) \in \Omega_T \mid u^*_x(x, t) < s_-(r_1)\}, \\
\Omega_T^2 &= \{(x, t) \in \Omega_T \mid s_-(r_1) < u^*_x(x, t) < s_+(r_2)\}, \\
\Omega_T^3 &= \{(x, t) \in \Omega_T \mid u^*_x(x, t) > s_+(r_2)\}, \\
F_T &= \Omega_T \setminus (\bigcup_{i=1}^3 \Omega_T^i).
\end{align*}
\]
then (2.21) holds \textbf{(Case I)}, (2.22) holds \textbf{(Case II)}, and \( \partial \Omega_T^1 \cap \partial \Omega_T^3 = \emptyset \). As \( s_-(r_1) < u^*_x(x_0, 0) < s_+(r_2) \), we also have \( \Omega_T^2 \neq \emptyset \); so \( |\Omega_T^2| > 0 \).

We define
\[
v^*(x, t) = \int_0^x h(z) \, dz + \int_0^t \sigma^*(u^*_x(x, \tau)) \, d\tau \forall (x, t) \in \Omega_T.
\]
Then \( w^* := (u^*, v^*) \) satisfies
\[
v^*_x = u^*_t \quad \text{and} \quad v^*_t = \sigma^*(u^*_x) \quad \text{in} \ \Omega_T.
\]
Note that this implies \( v^* \in C^2(\Omega_T) \); hence \( w^* \in C^2(\Omega_T; \mathbb{R}^2) \).

**Related matrix sets:** Define the sets (see Figure 3)
\[
\begin{align*}
\hat{K}_+ &= \{(s, \sigma(s)) \in \mathbb{R}^2 \mid s_-(r_1) \leq s \leq s_+(r_2)\}, \\
\hat{K} &= \hat{K}_+ \cup \hat{K}_-, \\
\hat{U} &= \{(s, r) \in \mathbb{R}^2 \mid r_1 < r < r_2, 0 < \lambda < 1, s = \lambda s_-(r) + (1 - \lambda)s_+(r)\}, \\
K &= \left\{ \begin{pmatrix} s & b \\ b & r \end{pmatrix} \in M_{2 \times 2}^{\text{sym}} \mid (s, r) \in \hat{K}, |b| \leq \gamma \right\}, \\
U &= \left\{ \begin{pmatrix} s & b \\ b & r \end{pmatrix} \in M_{2 \times 2}^{\text{sym}} \mid (s, r) \in \hat{U}, |b| < \gamma \right\},
\end{align*}
\]
where \( \gamma := \|u_0^*\|_{L^\infty(\Omega_T)} + 1 \).

**Admissible class:** Let \( \epsilon > 0 \) be given. Choose a number \( T_\epsilon \in (0, T] \) so that \( \|u^*_t - h\|_{L^\infty(\Omega_{T_\epsilon})} < \epsilon/2 =: \epsilon' \). We then define the \textit{admissible class} \( \mathcal{A} \) by
\[
\mathcal{A} = \left\{ \begin{array}{l}
\begin{array}{l}
w = (u, v) \in W^{1, \infty}_w(\Omega_T; \mathbb{R}^2) \\
\quad \text{in} \ \Omega_T \setminus \Omega_T^3
\end{array}
\end{array}
\right\}_{w \equiv w^* \in C^2(\Omega_T; \mathbb{R}^2)}
\]
It is easy to see from (3.2) and (3.4) that \( w^* \in A \neq \emptyset \). For each \( \delta > 0 \), we also define the \( \delta \)-approximating class \( A_\delta \) by

\[
A_\delta = \left\{ w \in A \mid \int_{\Omega_T^2} \text{dist}(\nabla w(x, t), K) \, dx \, dt \leq \delta|\Omega_T^2| \right\}.
\]

**Density result:** One crucial step for the proof of Theorem 2.1 (Case I), Theorem 2.1 (Case II) is the following density fact whose proof appears in section 5 that is common for both cases.

**Theorem 3.1.** For each \( \delta > 0 \),

\( A_\delta \) is dense in \( A \) with respect to the \( L^\infty(\Omega_T; [0, 2]) \)-norm.

**Baire’s category method:** Let \( X \) denote the closure of \( A \) in the space \( L^\infty(\Omega_T; [0, 2]) \), so that \( (X, L^\infty) \) is a nonempty complete metric space. As \( U \) is bounded in \( M^{2 \times 2} \), so is \( A \) in \( W^{1,\infty}(\Omega_T; [0, 2]) \); thus it is easily checked that

\[ X \subset W^{1,\infty}(\Omega_T; [0, 2]). \]

Note that the space-time gradient operator \( \nabla \) is a Baire-one function (see, e.g., [9, Proposition 10.17]). So by the Baire Category Theorem (see, e.g., [9, Theorem 10.15]), the set of points of discontinuity of the operator \( \nabla \), say \( D_\nabla \), is a set of the first category; thus the set of points at which \( \nabla \) is continuous, that is, \( C_\nabla := X \setminus D_\nabla \), is dense in \( X \).

**Completion of proof:** Let us confirm that for any function \( w = (u, v) \in C_\nabla \), its first component \( u \) is a weak solution to (1.3) satisfying (a)–(d). Towards this, fix any \( w = (u, v) \in C_\nabla \).

(1.4) & (1.5): To verify (1.4), let \( \varphi \in C_c^\infty(\Omega \times [0, T]) \). From Theorem 3.1 and the density of \( A \) in \( X \), we can choose a sequence \( w_k = (u_k, v_k) \in A_{1/k} \) such that \( w_k \to w \) in \( X \) as \( k \to \infty \). As \( w \in C_\nabla \), we have \( \nabla w_k \to \nabla w \) in \( L^1(\Omega_T; M^{2 \times 2}) \) and so pointwise a.e. in \( \Omega_T \) after passing to a subsequence if necessary. By (3.1) and the definition of \( A \), we have \( (u_k)_x = (u_k)_t \) in \( \Omega_T \) and \( (v_k)_x(x, 0) = v_k^+ - u_k^+(x, 0) = u_k^+(x, 0) = h(x) \) \( (x \in \Omega) \); so

\[
\int_{\Omega_T} (u_k)_t \varphi_t \, dx \, dt = \int_{\Omega_T} (v_k)_x \varphi_t \, dx \, dt
\]

\[
= -\int_{\Omega_T} (v_k)_t \varphi \, dx \, dt - \int_0^1 (v_k)_x(x, 0) \varphi(x, 0) \, dx
\]

\[
= \int_{\Omega_T} (v_k)_t \varphi_x \, dx \, dt - \int_0^1 h(x) \varphi(x, 0) \, dx,
\]

that is,

\[
\int_{\Omega_T} ((u_k)_t \varphi_t - (v_k)_t \varphi_x) \, dx \, dt = -\int_0^1 h(x) \varphi(x, 0) \, dx.
\]

On the other hand, by the Dominated Convergence Theorem, we have

\[
\int_{\Omega_T} ((u_k)_t \varphi_t - (v_k)_t \varphi_x) \, dx \, dt \to \int_{\Omega_T} (u_t \varphi_t - v_t \varphi_x) \, dx \, dt;
\]
thus
\begin{equation}
\int_{\Omega_T} (u_t \varphi_t - v_t \varphi_x) \, dx \, dt = - \int_0^1 h(x)\varphi(x, 0) \, dx.
\end{equation}

Also, by the Dominated Convergence Theorem,
\[ \int_{\Omega_T^2} \operatorname{dist}(\nabla w_k(x, t), K) \, dx \, dt \to \int_{\Omega_T^2} \operatorname{dist}(\nabla w(x, t), K) \, dx \, dt. \]

From the choice \( w_k \in \mathcal{A}_{1/k} \), we have
\[ \int_{\Omega_T^2} \operatorname{dist}(\nabla w_k(x, t), K) \, dx \, dt \leq \frac{|\Omega_T^2|}{k} \to 0; \]
so
\[ \int_{\Omega_T^2} \operatorname{dist}(\nabla w(x, t), K) \, dx \, dt = 0. \]

Since \( K \) is closed, we must have
\begin{equation}
\nabla w(x, t) \in K \subset \Sigma_{\sigma}, \quad \text{a.e. } (x, t) \in \Omega_T^2.
\end{equation}

For each \( k \), we have \( w_k \equiv w^* \) in \( \Omega_T \setminus \tilde{\Omega}_T^{w_k} \) for some open set \( \Omega_T^{w_k} \subset \Omega_T^2 \) with \( |\partial \Omega_T^{w_k}| = 0 \), and so \( \nabla w_k \equiv \nabla w^* \) in \( \Omega_T \setminus \tilde{\Omega}_T^{w_k} \); thus \( w = w^* \) and \( \nabla w = \nabla w^* \) a.e. in \( \Omega_T \setminus \tilde{\Omega}_T^{w} \). By \((3.2)\) and \((3.4)\), we have
\[ v_x = u_t \quad \text{and} \quad v_t = \sigma^*(u_x^*) = \sigma(u_x) \quad \text{a.e. in } \Omega_T \setminus \tilde{\Omega}_T^{2}. \]

This together with \((3.6)\) implies that \( \nabla w \in \Sigma_{\sigma} \) a.e. in \( \Omega_T \). In particular, \( v_t = \sigma(u_x) \) a.e. in \( \Omega_T \). Reflecting this to \((3.5)\), we have \((1.4)\). As \( w = w^* \) on \( \partial \Omega_T \), we also have \((1.5)\).

(a), (b), (c) & (d): As \( w = w^* \) a.e. in \( \Omega_T \setminus \tilde{\Omega}_T^{2} \), it follows from the continuity that \( u \equiv u^* \) in \( \Omega_T^1 \cup \tilde{\Omega}_T^{2} \); so (b) is guaranteed by the definition of \( \Omega_T^1 \) and \( \tilde{\Omega}_T^{2} \), with \( u_x^* > -1 \) on \( \Omega_T \) for Case I. Since \( \nabla w = \nabla w^* \) a.e. in \( \Omega_T \setminus \tilde{\Omega}_T^{2} \), we have \( u_x = u_x^* \in \{s_-(r_1), s_+(r_2)\} \) a.e. in \( F_T \); so (d) holds. From \( w_k \in \mathcal{A}_{1/k} \subset \mathcal{A} \), we have \( |(u_k)_t(x, t) - h(x)| < \epsilon' \) for a.e. \( (x, t) \in \Omega_T \). Taking the limit as \( k \to \infty \), we obtain that \( |u_t(x, t) - h(x)| \leq \epsilon' \) \( \leq \epsilon < \epsilon' \) for a.e. \( (x, t) \in \Omega_T \); hence \( \|u_t - h\|_{L^\infty(\Omega_T)} \leq \epsilon' = \epsilon < \epsilon \). Thus (a) is proved. From \((3.6)\) and the definition of \( K \), (c) follows.

**Infinitely many weak solutions:** Having shown that the first component \( u \) of each pair \( w = (u, v) \) in \( \mathcal{C}_\Sigma \) is a weak solution to \((1.3)\) satisfying (a)–(d), it remains to verify that \( \mathcal{C}_\Sigma \) has infinitely many elements and that no two different pairs in \( \mathcal{C}_\Sigma \) have the first components that are equal. Suppose on the contrary that \( \mathcal{C}_\Sigma \) has finitely many elements. Then \( w^* \in \mathcal{A} \subset X = \mathcal{C}_\Sigma = \mathcal{C}_\Gamma \), and so \( u^* \) itself is a weak solution to \((1.3)\) satisfying (a)–(d); this is a contradiction. Thus \( \mathcal{C}_\Sigma \) has infinitely many elements. Next, we check that for any two \( w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in \mathcal{C}_\Sigma \),
\[ u_1 = u_2 \iff v_1 = v_2. \]
Suppose \( u_1 \equiv u_2 \) in \( \Omega_T \). As \( \nabla w_1, \nabla w_2 \in \Sigma_\sigma \) a.e. in \( \Omega_T \), we have, in particular, that
\[
(v_1)_x = (u_1)_t = (u_2)_t = (v_2)_x \text{ a.e. in } \Omega_T.
\]
Since both \( v_1 \) and \( v_2 \) share the same trace \( u^* \) on \( \partial \Omega_T \), it follows that \( v_1 \equiv v_2 \) in \( \Omega_T \). The converse can be shown similarly. We can now conclude that there are infinitely many weak solutions to (1.3) satisfying (a)–(d).

The proof of Theorem 2.1 (Case I), Theorem 2.4 (Case II) is now complete under the density fact, Theorem 3.1, to be justified in sections 4 and 5.

3.3. Proofs of Corollaries 2.2, 2.3, 2.5 and 2.6. We proceed the proofs of the companion versions of Corollaries 2.2 and 2.5 and Corollaries 2.3 and 2.6, respectively.

Proof of Corollaries 2.2 and 2.5. Let \((g, h) \in W^{3,2}(\Omega) \times W_0^{2,2}(\Omega)\) be any given initial datum, with \(g' > -1\) on \(\Omega\) for Case I. If \(g'(x_0) \in (s_1^*, s_2^*)\) for some \(x_0 \in \Omega\), then the result follows immediately from Theorem 2.1 (Case I), Theorem 2.4 (Case II).

Next, let us assume \(g'(x) \notin (s_1^*, s_2^*)\) for all \(x \in \bar{\Omega}\). We may only consider the case that \(g'(x) \geq s_2^*\) for all \(x \in \bar{\Omega}\) as the other case can be shown similarly. Fix any two \(\sigma(s_2) < r_1 < r_2 < \sigma(s_1)\), and choose a function \(\sigma^* \in C^3(-1, \infty)\) (Case I), \(\sigma^* \in C^3(\mathbb{R})\) (Case II) in such a way that (3.2) is fulfilled. By [11, Theorem 5.2] (Case I), [11, Theorem 5.1] (Case II), there exists a finite number \(\hat{T} > 0\) such that the modified initial-boundary value problem (3.3), with \(T\) replaced by \(\hat{T}\), admits a unique solution \(u^* \in \cap_{k=0}^3 C^k([0, \hat{T}]; W^{3-k,2}(\Omega))\), with \(u^*_{x'} > -1\) on \(\Omega_{\hat{T}}\) for Case I. Now, choose a number \(0 < T \leq \hat{T}\) so that \(u^*_{x'} \geq s_+ (r_2)\) on \(\hat{T}_T\). Then \(u^*\) itself is a classical and thus weak solution to problem (1.3).

Proof of Corollaries 2.3 and 2.6. Let \((g, h) \in W^{3,2}(\Omega) \times W_0^{2,2}(\Omega)\) satisfy \(\max_{\Omega} g' \in (s_1^*, s_1)\) or \(\min_{\Omega} g' \in (s_2, s_2^*)\). In Case I, assume also \(g' > -1\) on \(\bar{\Omega}\). We may only consider the case that \(M := \max_{\Omega} g' \in (s_1^*, s_1)\) as the other case can be handled in a similar way. Choose two numbers \(\sigma(s_2) < r_1 < r_2 < \sigma(s_1)\) so that \(s_-(r_1) > M\). Then take a \(C^3\) function \(\sigma^*(s)\) satisfying (3.2). Using [11, Theorem 5.2] (Case I), [11, Theorem 5.1] (Case II), we can find a finite number \(\hat{T} > 0\) such that modified problem (3.3), with \(T\) replaced by \(\hat{T}\), has a unique solution \(u^* \in \cap_{k=0}^3 C^k([0, \hat{T}]; W^{3-k,2}(\Omega))\), with \(u^*_x > -1\) on \(\Omega_{\hat{T}}\) for Case I. Then choose a number \(0 < T' \leq \hat{T}\) so small that \(u^*_{x'} \leq s_-(r_1)\) on \(\hat{T}_T\) and that \(s^*_1 < u^*_x(x_0, T')\) for some \(x_0 \in \Omega\). With the initial datum \((u^*(, T'), u^*_x(, T')) \in W^{3,2}(\Omega) \times W_0^{2,2}(\Omega)\) at \(t = T'\), with \(u^*_x(, T') > -1\) on \(\bar{\Omega}\) for Case I, we can apply Theorem 2.1 (Case I), Theorem 2.4 (Case II) to obtain, for some finite number \(T > T'\), infinitely many
weak solutions \( \tilde{u} \in W^{1,\infty}(\Omega \times (T', T)) \) to the initial-boundary value problem

\[
\begin{aligned}
\tilde{u}_{tt} &= \sigma(\tilde{u}_x)_x & \text{in } \Omega \times (T', T), \\
\tilde{u}(0, t) &= \tilde{u}(1, t) = 0 & \text{for } t \in (T', T), \\
\tilde{u} &= \tilde{u}^*, \quad \tilde{u}_t = \tilde{u}_t^* & \text{on } \Omega \times \{t = T'\}
\end{aligned}
\]

satisfying the stated properties in the theorem. Then the glued functions

\( u = u^* \chi_{\Omega \times (0, T')} + \tilde{u} \chi_{\Omega \times [T', T)} \)

are weak solutions to problem (1.3) fulfilling the required properties.

\[ \square \]

4. Rank-one smooth approximation under linear constraint

In this section, we prepare the main tool, Theorem 4.1, for proving the density result, Theorem 3.1. Instead of presenting a special case that would be enough for our purpose, we exhibit the following result in a generalized and refined form of [35, Lemma 2.1] that may be of independent interest (cf. [30, Lemma 6.2]).

**Theorem 4.1.** Let \( m, n \geq 2 \) be integers, and let \( A, B \in M^{m \times n} \) be such that \( \text{rank}(A - B) = 1 \); hence

\( A - B = a \otimes b = (a_i b_j) \)

for some non-zero vectors \( a \in \mathbb{R}^m \) and \( b \in \mathbb{R}^n \) with \( |b| = 1 \). Let \( L \in M^{m \times n} \) satisfy

\[ Lb \neq 0 \quad \text{in} \quad \mathbb{R}^m, \]

and let \( \mathcal{L} : M^{m \times n} \to \mathbb{R} \) be the linear map defined by

\[ \mathcal{L}(\xi) = \sum_{1 \leq i \leq m, 1 \leq j \leq n} L_{ij} \xi_{ij} \quad \forall \xi \in M^{m \times n}. \]

Assume \( \mathcal{L}(A) = \mathcal{L}(B) \) and \( 0 < \lambda < 1 \) is any fixed number. Then there exists a linear partial differential operator \( \Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m) \) satisfying the following properties:

(1) For any open set \( \Omega \subset \mathbb{R}^n \),

\[ \Phi v \in C^{k-1}(\Omega; \mathbb{R}^m) \quad \text{whenever } k \in \mathbb{N} \text{ and } v \in C^k(\Omega; \mathbb{R}^m) \]

and

\[ \mathcal{L}(\nabla \Phi v) = 0 \quad \text{in} \quad \Omega \quad \forall v \in C^2(\Omega; \mathbb{R}^m). \]

(2) Let \( \Omega \subset \mathbb{R}^n \) be any bounded domain. For each \( \tau > 0 \), there exist a function \( g = g_\tau \in C^\infty_c(\Omega; \mathbb{R}^m) \) and two disjoint open sets \( \Omega_A, \Omega_B \subset \subset \Omega \) such that

(a) \( \Phi g \in C^\infty(\Omega; \mathbb{R}^m) \),

(b) \( \text{dist}(\nabla \Phi g, [-\lambda(A - B), (1 - \lambda)(A - B)]) < \tau \text{ in } \Omega \),

(c) \( \nabla \Phi g(x) = \begin{cases} 
(1 - \lambda)(A - B) & \forall x \in \Omega_A, \\
-\lambda(A - B) & \forall x \in \Omega_B,
\end{cases} \)

(d) \( ||\Omega_A| - \lambda|\Omega|| < \tau, \quad ||\Omega_B| - (1 - \lambda)|\Omega|| < \tau \),
where \([-\lambda(A-B), (1-\lambda)(A-B)\] \) is the closed line segment in \( \ker L \subset \mathbb{M}^{m \times n} \) joining \(-\lambda(A-B)\) and \((1-\lambda)(A-B)\).

**Proof.** We mainly follow and modify the proof of [35, Lemma 2.1] which is divided into three cases.

Set \(r = \text{rank}(L)\). By (4.1), we have \(1 \leq r \leq m \wedge n =: \min\{m, n\}\).

**Case 1:** Assume that the matrix \(L\) satisfies \(L_{ij} = 0\) for all \(1 \leq i \leq m, 1 \leq j \leq n\) but possibly the pairs \((1, 1), (1, 2), \ldots, (1, n), (2, 2), \ldots, (r, r)\); hence \(L\) is of the form

\[
L = \begin{pmatrix}
L_{11} & L_{12} & \cdots & L_{1r} & \cdots & L_{1n} \\
L_{22} & & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
L_{rr} & & & & & \\
\end{pmatrix}
\in \mathbb{M}^{m \times n}
\]

and that

\[A - B = a \otimes e_1\] for some nonzero vector \(a = (a_1, \ldots, a_m) \in \mathbb{R}^m\),

where each blank component in (4.2) is zero. From (4.1) and \(\text{rank}(L) = r\), it follows that the product \(L_{11} \cdots L_{rr} \neq 0\). Since \(0 = L(A - B) = L(a \otimes e_1) = L_{11}a_1\), we have \(a_1 = 0\).

In this case, the linear map \(L : \mathbb{M}^{m \times n} \to \mathbb{R}\) is given by

\[L(\xi) = \sum_{j=1}^{n} L_{1j}\xi_{1j} + \sum_{i=2}^{r} L_{ii}\xi_{ii}, \quad \xi \in \mathbb{M}^{m \times n}.
\]

We will find a linear differential operator \(\Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m)\) such that

\[
L(\nabla \Phi v) \equiv 0 \quad \forall v \in C^2(\mathbb{R}^n; \mathbb{R}^m).
\]

So our candidate for such a \(\Phi = (\Phi^1, \ldots, \Phi^m)\) is of the form

\[
\Phi^i v = \sum_{1 \leq k \leq m, 1 \leq l \leq n} a_{kl}^i v_{x_k x_l},
\]

where \(1 \leq i \leq m, v \in C^1(\mathbb{R}^n; \mathbb{R}^m)\), and \(a_{kl}^i\)'s are real constants to be determined; then for \(v \in C^2(\mathbb{R}^n; \mathbb{R}^m)\), \(1 \leq i \leq m\), and \(1 \leq j \leq n\),

\[
\partial_{x_j} \Phi^i v = \sum_{1 \leq k \leq m, 1 \leq l \leq n} a_{kl}^i v_{x_k x_l x_j}.
\]
Rewriting (4.13) with this form of $\nabla \Phi v$ for $v \in C^2(\mathbb{R}^n; \mathbb{R}^m)$, we have

\[
0 = \sum_{1 \leq k \leq m, 1 \leq j, l \leq n} L_{1j}a_{kl}^1v_{xix_j}^k + \sum_{i=2}^r \sum_{1 \leq k \leq m, 1 \leq l \leq n} L_{ii}a_{kl}^1v_{xix_i}^k
\]

\[
= \sum_{k=1}^m \left( \sum_{j=2}^r L_{1j}a_{kl}^1v_{xix_j}^k + \sum_{l=2}^r (L_{1l}a_{kl}^1 + L_{ll}a_{kl}^1)v_{xix_l}^k + \sum_{j=r+1}^n L_{1j}a_{kl}^1v_{xix_j}^k \right)
\]

\[
+ \sum_{2 \leq j < l \leq r} (L_{1j}a_{kl}^1 + L_{1l}a_{kl}^1 + L_{ll}a_{kl}^1)v_{xix_j}^k + \sum_{2 \leq j \leq r, r+1 \leq l \leq n} (L_{1j}a_{kl}^1 + L_{1l}a_{kl}^1)v_{xix_j}^k
\]

\[
+ \sum_{r+1 \leq j \leq l \leq n} (L_{1j}a_{kl}^1 + L_{1l}a_{kl}^1)v_{xix_j}^k.
\]

Should (4.13) hold, it is thus sufficient to solve the following algebraic system for each $k = 1, \cdots, m$ (after adjusting the letters for some indices):

\[
0 = L_{11}a_{1k}^1, \quad \forall j = 1, \cdots, m
\]

\[
L_{1j}a_{kl}^1 + L_{lj}a_{kl}^1 = 0, \quad \forall j = 2, \cdots, r
\]

\[
L_{1j}a_{kl}^1 + L_{lj}a_{kl}^1 + L_{ll}a_{kl}^1 = 0, \quad \forall j = 2, \cdots, r
\]

\[
L_{1l}a_{kl}^1 + L_{ll}a_{kl}^1 = 0, \quad \forall j = 1, \cdots, r
\]

\[
L_{1j}a_{kl}^1 + L_{lj}a_{kl}^1 = 0, \quad \forall j = 1, \cdots, r
\]

\[
L_{1l}a_{kl}^1 = 0, \quad \forall j = 1, \cdots, r
\]

Although these systems have infinitely many solutions, we will solve those in a way for a later purpose that the matrix $(a_{kl}^1)_{2 \leq j,k \leq m} \in M^{(m-1) \times (m-1)}$ fulfills

\[
a_{21}^j = a_j, \quad \forall j = 2, \cdots, m, \quad \text{and} \quad a_{1k}^1 = 0 \quad \text{otherwise.}
\]

Firstly, we let the coefficients $a_{kl}^1$ $(1 \leq i, k \leq m, 1 \leq l \leq n)$ that do not appear in systems (4.5)–(4.12) $(k = 1, \cdots, m)$ be zero with an exception that we set $a_{21}^j = a_j$ for $j = r+1, \cdots, m$ to reflect (4.13). Secondly, for $1 \leq k \leq m, k \neq 2$, let us take the trivial (i.e., zero) solution of system (4.5)–(4.12). Finally, we take $k = 2$ and solve system (4.5)–(4.12) as follows.
with (4.13) satisfied. Since $L_{11} \neq 0$, we set $a_{21}^1 = 0$; then (4.5) is satisfied. So we set
\[
a_{21}^j = -\frac{L_{11}}{L_{jj}} a_{2j}^1, \quad a_{2j}^j = -\frac{L_{jj}}{L_{11}} a_j \quad \forall j = 2, \ldots, r;
\]
then (4.7) and (4.13) hold. Next, set
\[
a_{2j}^j = \frac{L_{1j}}{L_{jj}} a_{2j}^1 = \frac{L_{1j}}{L_{11}} a_j \quad \forall j = 2, \ldots, r;
\]
then (4.6) is fulfilled. Set
\[
a_{2j}^l = -\frac{L_{1j} a_{2j}^1 + L_{1j} a_{2j}^1}{L_{ll}} = \frac{L_{1j} l_{1j} a_j + L_{1j} L_{ll} a_i}{L_{ll} L_{11}}, \quad a_{2j}^j = 0
\]
for $j = 3, \ldots, r$ and $l = 2, \ldots, j - 1$; then (4.8) holds. Set
\[
a_{2j}^1 = 0 \quad \forall j = r + 1, \ldots, n;
\]
then (4.9) and (4.10) are satisfied. Lastly, set
\[
a_{2j}^1 = 0, \quad a_{2j}^j = -\frac{L_{1j} a_{2j}^1}{L_{ll}} = \frac{L_{1j}}{L_{11}} a_j \quad \forall j = r + 1, \ldots, n, \forall l = 2, \ldots, r;
\]
then (4.11) and (4.12) hold. In summary, we have determined the coefficients $a_{kl}^i$ ($1 \leq i, k \leq m, 1 \leq l \leq n$) in such a way that system (4.3)–(4.12) holds for each $k = 1, \ldots, m$ and that (4.13) is also satisfied. Therefore, (1) follows from (4.3) and (4.4).

To prove (2), without loss of generality, we can assume $\Omega = (0, 1)^n \subset \mathbb{R}^n$. Let $\tau > 0$ be given. Let $u = (u^1, \ldots, u^m) \in C^\infty(\Omega; \mathbb{R}^m)$ be a function to be determined. Suppose $u$ depends only on the first variable $x_1 \in (0, 1)$. We wish to have
\[
\nabla \Phi u(x) \in \{ -\lambda a \otimes e_1, (1 - \lambda) a \otimes e_1 \}
\]
for all $x \in \Omega$ except in a set of small measure. Since $u(x) = u(x_1)$, it follows from (4.4) that for $1 \leq i \leq m$ and $1 \leq j \leq n$,
\[
\Phi^i u = \sum_{k=1}^m a_{k1}^i u_{x_1}^k; \quad \text{thus} \quad \partial_{x_1} \Phi^i u = \sum_{k=1}^m a_{k1}^i u_{x_1 x_1}^k.
\]
As $a_{k1}^1 = 0$ for $k = 1, \ldots, m$, we have $\partial_{x_1} \Phi^1 u = \sum_{k=1}^m a_{k1}^1 u_{x_1 x_1}^k = 0$ for $j = 1, \ldots, n$. We first set $u^1 \equiv 0$ in $\Omega$. Then from (4.13), it follows that for $i = 2, \ldots, m$,
\[
\partial_{x} \Phi^i u = \sum_{k=2}^m a_{k1}^i u_{x_1 x_1}^k = a_{21}^i u_{x_1 x_1}^2 = a_{1}^{2} u_{x_1 x_1}^2 = \left\{ \begin{array}{ll} a_{1}^{2} u_{x_1 x_1} & \text{if } j = 1, \\ 0 & \text{if } j = 2, \ldots, n. \end{array} \right.
\]
As $a_1 = 0$, we thus have that for $x \in \Omega$,
\[
\nabla \Phi u(x) = (u^2)^2 u(x_1) a \otimes e_1.
\]
For irrelevant components of $u$, we simply take $u^3 = \cdots = u^m \equiv 0$ in $\Omega$. Lastly, for a number $\delta > 0$ to be chosen later, we choose a function $u^2(x_1) \in C^\infty(0, 1)$ such that there exist two disjoint open sets $I_1, I_2 \subset \mathbb{R}$
so $|I_1| - \lambda| < \tau/2$, $|I_2| - (1 - \lambda)| < \tau/2$, $\|u^2\|_{L^{\infty}(0,1)} < \delta$, $\|u^2\|_{L^{\infty}(0,1)} < \delta$, $-\lambda \leq (u^2)^\nu(x_1) \leq 1 - \lambda$ for $x_1 \in (0,1)$, and

$$(u^2)^\nu(x_1) = \begin{cases} 1 - \lambda & \text{if } x_1 \in I_1, \\ -\lambda & \text{if } x_1 \in I_2. \end{cases}$$

In particular,

$$\nabla \Phi u(x) \in [-\lambda a \otimes e_1, (1 - \lambda) a \otimes e_1] \quad \forall x \in \Omega.$$  

We now choose an open set $\Omega_\epsilon' \subset \subset \Omega' := (0,1)^{n-1}$ with $|\Omega' \setminus \Omega_\epsilon'| < \tau/2$ and a function $\eta \in C_\infty(\Omega')$ so that

$$0 \leq \eta \leq 1 \text{ in } \Omega', \quad \eta \equiv 1 \text{ in } \Omega', \quad \text{and } |\nabla \eta^i| < \frac{C}{\tau^i} \quad (i = 1,2) \text{ in } \Omega',$$

where $x' = (x_2, \cdots, x_n) \in \Omega'$ and the constant $C > 0$ is independent of $\tau$. Now, we define $g(x) = \eta(x')u(x_1) \in C_\infty(\Omega; \mathbb{R}^m)$. Set $\Omega_A = I_1 \times \Omega_\epsilon'$ and $\Omega_B = I_2 \times \Omega_\epsilon'$. Clearly, (a) follows from (1). As $g(x) = u(x_1) = u(x)$ for $x \in \Omega_A \cup \Omega_B$, we have

$$\nabla \Phi g(x) = \begin{cases} (1 - \lambda)a \otimes e_1 & \text{if } x \in \Omega_A, \\ -\lambda a \otimes e_1 & \text{if } x \in \Omega_B; \end{cases}$$

hence (c) holds. Also,

$$||\Omega_A| - \lambda|\Omega| = ||\Omega_A| - \lambda| = ||I_1||\Omega'| - \lambda| = ||I_1| - |\Omega'| \setminus \Omega_\epsilon'| - \lambda| < \tau,$$

and likewise

$$||\Omega_B| - (1 - \lambda)|\Omega| < \tau;$$

so (d) is satisfied. Note that for $i = 1, \cdots, m$,

$$\Phi^i g = \Phi^i(\eta u) = \sum_{1 \leq k \leq m, 1 \leq l \leq n} a^i_{kl}(\eta u^k)_{x_1} = \eta \Phi^i u + \sum_{1 \leq k \leq m, 1 \leq l \leq n} a^i_{kl} \eta_{x_1} u^k$$

$$= \eta \Phi^i u + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_1} = \eta a^i_{21} u^2_{x_1} + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_1}.$$ 

So

$$||\Phi g||_{L^{\infty}(\Omega)} \leq C \max\{\delta, \delta^{-1}\} < \tau$$

if $\delta > 0$ is chosen small enough; so (e) holds. Next, for $i = 1, \cdots, m$ and $j = 1, \cdots, n$,

$$\partial_{x_j} \Phi^i g = \eta_{x_j} a^i_{21} u^2_{x_1} + \eta \partial_{x_j} \Phi^i u + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_1} + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_1 x_j};$$

hence from (4.14),

$$\text{dist}(\nabla \Phi g, [-\lambda a \otimes e_1, (1 - \lambda) a \otimes e_1]) \leq C \max\{\delta^{-1}, \delta^{-2}\} < \tau \text{ in } \Omega$$

if $\delta$ is sufficiently small. Thus (b) is fulfilled.
(Case 2): Assume that $L_{i1} = 0$ for all $i = 2, \cdots, m$, that is,

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ 0 & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & L_{m2} & \cdots & L_{mn} \end{pmatrix} \in \mathbb{M}^{m \times n}$$

and that

$$A - B = a \otimes e_1$$

for some nonzero vector $a \in \mathbb{R}^m$; then by (4.1), we have $L_{11} \neq 0$.

Set

$$\hat{L} = \begin{pmatrix} L_{22} & \cdots & L_{2n} \\ \vdots & \ddots & \vdots \\ L_{m2} & \cdots & L_{mn} \end{pmatrix} \in \mathbb{M}^{(m-1) \times (n-1)}.$$

As $L_{11} \neq 0$ and $\text{rank}(L) = r$, we must have $\text{rank}(\hat{L}) = r - 1$. Using the singular value decomposition theorem, there exist two matrices $\hat{U} \in O(m-1)$ and $\hat{V} \in O(n-1)$ such that

$$\hat{U}^T \hat{L} \hat{V} = \text{diag}(\sigma_2, \cdots, \sigma_r, 0, \cdots, 0) \in \mathbb{M}^{(m-1) \times (n-1)},$$

where $\sigma_2, \cdots, \sigma_r$ are the positive singular values of $\hat{L}$. Define

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in O(m), \quad V = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in O(n).$$

Let $L' = U^T L V$, $A' = U^T A V$, and $B' = U^T B V$. Let $L' : \mathbb{M}^{m \times n} \to \mathbb{R}$ be the linear map given by

$$L'(\xi') = \sum_{1 \leq i \leq m, 1 \leq j \leq n} L'_{ij} \xi'_{ij} \quad \forall \xi' \in \mathbb{M}^{m \times n}.$$ 

Then, from (4.15), (4.16) and (4.17), it is straightforward to check the following:

1. $A' - B' = a' \otimes e_1$ for some nonzero vector $a' \in \mathbb{R}^m$, $L'e_1 \neq 0$, $L'(A) = L'(B)$, and $L'$ is of the form (4.2) in Case 1 with $\text{rank}(L') = r$.

Thus we can apply the result of Case 1 to find a linear operator $\Phi' : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m)$ satisfying the following:

1. For any open set $\Omega' \subset \mathbb{R}^n$,

$$\Phi' \nu' \in C^{k-1}(\Omega'; \mathbb{R}^m)$$

whenever $k \in \mathbb{N}$ and $\nu' \in C^k(\Omega'; \mathbb{R}^m)$ and

$$L'(\nabla \Phi' \nu') = 0 \quad \text{in} \quad \Omega' \quad \text{for all} \quad \nu' \in C^2(\Omega'; \mathbb{R}^m).$$

2. Let $\Omega' \subset \mathbb{R}^n$ be any bounded domain. For each $\tau > 0$, there exist a function $g' = g'_\tau \in C^\infty_c(\Omega'; \mathbb{R}^m)$ and two disjoint open sets $\Omega'_{A'}, \Omega'_{B'} \subset \subset \Omega'$ such that

(a) $\Phi' g' \in C^\infty_c(\Omega'; \mathbb{R}^m)$,
(b') $\text{dist}(\nabla \Phi g', [-\lambda(A' - B'), (1 - \lambda)(A' - B')] < \tau$ in $\Omega'$,

(c') $\nabla \Phi g'(x) = \begin{cases} (1 - \lambda)(A' - B') & \forall x \in \Omega'_A, \\
-\lambda(A' - B') & \forall x \in \Omega'_B, \end{cases}$

(d') $\|\Omega'_A| - \lambda|\Omega'| < \tau$, $\|\Omega'_B| - (1 - \lambda)|\Omega'| < \tau$,

(e') $\|(\Phi')\|_{L^\infty(\Omega')} < \tau$.

For $v \in C^1(\mathbb{R}^n; \mathbb{R}^m)$, let $v' \in C^1(\mathbb{R}^n; \mathbb{R}^m)$ be defined by $v'(y) = U^T v(V y)$ for $y \in \mathbb{R}^n$. We define $\Phi v(x) = U \Phi'(V^T x)$ for $x \in \mathbb{R}^n$, so that $\Phi v \in C(\mathbb{R}^n; \mathbb{R}^m)$. Then it is straightforward to check that properties (1') and (2') of $\Phi'$ imply respective properties (1) and (2) of the linear operator $\Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C(\mathbb{R}^n; \mathbb{R}^m)$.

(Case 3): Finally, we consider the general case that $A$, $B$ and $L$ are as in the statement of the theorem. As $|b| = 1$, there exists an $R \in O(n)$ such that $R^T b = e_1 \in \mathbb{R}^n$. Also there exists a symmetric (Householder) matrix $P \in O(m)$ such that the matrix $L' := P L R$ has the first column parallel to $e_1 \in \mathbb{R}^m$. Let

$$A' = PAR \quad \text{and} \quad B' = PBR.$$ Then $A' - B' = a' \otimes e_1$, where $a' = Pa \neq 0$. Note also that $L' e_1 = PL R b = PL b \neq 0$. Define $L'(\xi') = \sum_{i,j} L'_{ij} \xi'_i \xi'_j \in M^{m \times n}$; then $L'(A') = L(A) = L(B) = L'(B')$. Thus by the result of Case 2, there exists a linear operator $\Phi' : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C(\mathbb{R}^n; \mathbb{R}^m)$ satisfying (1') and (2') above.

For $v \in C^1(\mathbb{R}^n; \mathbb{R}^m)$, let $v' \in C^1(\mathbb{R}^n; \mathbb{R}^m)$ be defined by $v'(y) = P v(R y)$ for $y \in \mathbb{R}^n$, and define $\Phi v(x) = P \Phi v'(R^T x) \in C(\mathbb{R}^n; \mathbb{R}^m)$. Then it is easy to check that the linear operator $\Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C(\mathbb{R}^n; \mathbb{R}^m)$ satisfies (1) and (2) by (1') and (2') similarly as in Case 2.

\[\square\]

5. Proof of density result

In this final section, we prove Theorem 3.1, which plays a pivotal role in the proof of the main results, Theorems 2.1 and 2.4.

To start the proof, fix a $\delta > 0$ and choose any $w = (u, v) \in A$ so that $w \in W_{w^*}^1(\Omega_T; \mathbb{R}^2) \cap C^2(\Omega_T; \mathbb{R}^2)$ satisfies the following:

$$(5.1) \quad \begin{cases} w \equiv w^* \text{ in } \Omega_T \setminus \overline{\Omega_T'} \text{ for some open set } \Omega_T' \subset \subset \Omega_T \text{ with } |\partial \Omega_T'| = 0, \\
\nabla w(x, t) \in U \text{ for all } (x, t) \in \Omega_T^2, \text{ and } \|u_t - h\|_{L^\infty(\Omega_T)} < \varepsilon'. \end{cases}$$

Let $\eta > 0$. Our goal is to construct a function $w_\eta = (u_\eta, v_\eta) \in A_\delta$ with $\|w - w_\eta\|_{L^\infty(\Omega_T)} < \eta$; that is, a function $w_\eta \in W_{w^*}^1(\Omega_T; \mathbb{R}^2) \cap C^2(\Omega_T; \mathbb{R}^2)$ with the following properties:

$$(5.2) \quad \begin{cases} w_\eta \equiv w^* \text{ in } \Omega_T \setminus \overline{\Omega_T''} \text{ for some open set } \Omega_T'' \subset \subset \Omega_T^2 \text{ with } |\partial \Omega_T''| = 0, \\
\nabla w_\eta(x, t) \in U \text{ for all } (x, t) \in \Omega_T^2, \|u_\eta\|_{L^\infty(\Omega_T)} < \varepsilon', \\
\int_{\Omega_T^2} \text{dist}(\nabla w_\eta(x, t), K) \text{ d}x\text{d}t \leq \delta |\Omega_T^2|, \text{ and } \|w - w_\eta\|_{L^\infty(\Omega_T)} < \eta. \end{cases}$$

For clarity, we divide the proof into several steps.
(Step 1): Choose a nonempty open set $G_1 \subset \Omega^T_{\nu} \setminus \partial \Omega^T_{\nu}$ with $|\partial G_1| = 0$ so that
\begin{equation}
\int_{(\Omega^T_{\nu} \setminus \partial \Omega^T_{\nu}) \setminus G_1} \text{dist}(\nabla w(x,t), K) \, dx \, dt \leq \frac{\delta}{5}|\Omega^T_{\nu}|.
\end{equation}
Since $\nabla w \in U$ on $G_1$, we have $\|u_t - h\|_{L^\infty(G_1), \gamma - \|u_t\|_{L^\infty(G_1)}} < \gamma$; then fix a number $\theta$ with
\begin{equation}
0 < \theta < \min\{\epsilon' - \|u_t - h\|_{L^\infty(\Omega^T_{\nu})}, \gamma - \|u_t\|_{L^\infty(G_1)}\}.
\end{equation}
For each $\mu > 0$, let
\begin{align*}
G^\mu_2 &= \{(x,t) \in G_1 \mid \text{dist}((u_x(x,t), v_t(x,t)), \partial \tilde{U}) > \mu\}, \\
H^\mu_2 &= \{(x,t) \in G_1 \mid \text{dist}((u_x(x,t), v_t(x,t)), \partial \tilde{U}) < \mu\}, \\
F^\mu_2 &= \{(x,t) \in G_1 \mid \text{dist}((u_x(x,t), v_t(x,t)), \partial \tilde{U}) = \mu\}.
\end{align*}
Since $\lim_{\mu \to 0^+} |H^\mu_2| = 0$, we can find a $\nu \in (0, \min\{\frac{\delta}{5}, \theta\})$ such that
\begin{equation}
\int_{H^\nu_2} \text{dist}(\nabla w(x,t), K) \, dx \, dt \leq \frac{\delta}{5}|\Omega^T_{\nu}|, \quad G^\nu_2 \neq \emptyset, \quad \text{and} |F^\nu_2| = 0.
\end{equation}
Let us write $G_2 = G^\nu_2$ and $H_2 = H^\nu_2$. Choose finitely many disjoint open squares $B_1, \ldots, B_N \subset G_2$ parallel to the axes such that
\begin{equation}
\int_{G_2 \setminus \bigcup_{i=1}^{N} B_i} \text{dist}(\nabla w(x,t), K) \, dx \, dt \leq \frac{\delta}{5}|\Omega^T_{\nu}|.
\end{equation}
(Step 2): Dividing the squares $B_1, \ldots, B_N$ into smaller disjoint sub-squares if necessary, we can assume that
\begin{equation}
|\nabla w(x,t) - \nabla w(\bar{x}, \bar{t})| < \frac{\nu}{8}
\end{equation}
whenever $(x,t), (\bar{x}, \bar{t}) \in B_i$ and $i = 1, \ldots, N$. Now, fix any $i \in \{1, \ldots, N\}$. Let $(x_i, t_i)$ denote the center of the square $B_i$, and write
\begin{equation}
(s_i, r_i) = (u_x(x_i, t_i), v_t(x_i, t_i)) \in \tilde{U};
\end{equation}
then $\text{dist}((s_i, r_i), \partial \tilde{U}) > \nu$. Let $\alpha_i > 0$, $\beta_i > 0$ be chosen so that
\begin{equation}
(s_i - \alpha_i, r_i), (s_i + \beta_i, r_i) \in \tilde{U}, \quad \text{dist}((s_i - \alpha_i, r_i), \tilde{K}_-) = \frac{\nu}{2}, \quad \text{and}
\end{equation}
\begin{equation}
\text{dist}((s_i + \beta_i, r_i), \tilde{K}_+) = \frac{\nu}{2}.
\end{equation}
To apply Theorem 4.1 with $m = n = 2$ to the square $B_i$, let
\begin{equation}
A_i = \begin{pmatrix} s_i - \alpha_i & b_i \\ b_i & r_i \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} s_i + \beta_i & b_i \\ b_i & r_i \end{pmatrix},
\end{equation}
where $b_i = u_t(x_i, t_i)$; then
\begin{equation}
A_i - B_i = \begin{pmatrix} -\alpha_i - \beta_i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\alpha_i - \beta_i \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{equation}
Let $L : \mathbb{M}^{2 \times 2} \to \mathbb{R}$ be the linear map defined by
\begin{equation}
L(\xi) = -\xi_{21} + \xi_{12} \quad \forall \xi \in \mathbb{M}^{2 \times 2},
\end{equation}
with its associated matrix $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; then
\[
\mathcal{L}(A_i) = \mathcal{L}(B_i)(=0) \quad \text{and} \quad C_i = \lambda_i A_i + (1 - \lambda_i) B_i,
\]
where $C_i = \nabla w(x, t)$ and $\lambda_i = \frac{\beta}{\alpha_i + \beta_i} \in (0, 1)$. By Theorem 4.1, there exists a linear operator $\Phi_i : C^1(\mathbb{R}^2; \mathbb{R}^2) \to C^0(\mathbb{R}^2; \mathbb{R}^2)$ satisfying properties (1) and (2) in the statement of the theorem with $A = A_i$, $B = B_i$ and $\lambda = \lambda_i$. In particular, for the square $B_i \subset \mathbb{R}^2$ and a number $0 < \tau < \min\{\frac{\pi}{2}, \theta, \eta, \frac{\delta(\Omega^2 \setminus \Omega)}{SN}\}$ with $S := \max_{r_1 \leq r \leq r_2} (s_+(r) - s_-(r)) > 0$, we can find a function $g_i \in C^\infty_c(B_i; \mathbb{R}^2)$ such that
\[
\Phi_i g_i \in C^\infty(B_i; \mathbb{R}^2), \quad \mathcal{L}(\nabla \Phi_i g_i) = 0 \quad \text{in} \quad B_i,
\]
\[
\text{dist}(\nabla \Phi_i g_i, [-\lambda_i(A_i - B_i), (1 - \lambda_i)(A_i - B_i)]) < \tau \quad \text{in} \quad B_i,
\]
\[
|B_i^1| < \tau, \quad \|\Phi_i g_i\|_{L^\infty} < \tau,
\]
where
\[
B_i^1 = \{(x, t) \in B_i \mid \text{dist}(\nabla \Phi_i g_i(x, t), [-\lambda_i(A_i - B_i), (1 - \lambda_i)(A_i - B_i)]) > 0\}.
\]
Let $B_i^2 = B_i \setminus B_i^1$. We finally define
\[
w_\eta = w + \sum_{i=1}^N \chi_{B_i^2} \Phi_i g_i \quad \text{in} \quad \Omega_T.
\]

(Step 3): Let us check that $w_\eta = (u_\eta, v_\eta)$ is indeed a desired function satisfying (5.2). It is clear from (5.1) and the construction above that $w_\eta \in W^{1,\infty}_w(\Omega_T \setminus \bar{\Omega}_T; \mathbb{R}^2) \cap C^2(\Omega_T \setminus \bar{\Omega}_T; \mathbb{R}^2)$. Set $\Omega_T^{w_\eta} = \Omega_T \cup (\cup_{i=1}^N B_i)$, then $\Omega_T^{w_\eta} \subset \subset \Omega_T^2$, $|\partial \Omega_T^{w_\eta}| = 0$, and $w_\eta = w = w^* \text{ in } \Omega_T \setminus \Omega_T^{w_\eta}$. From (5.7), (5.8), $\nu < \theta$ and $\tau < \nu/8$, it follows that for $i = 1, \ldots, N$,
\[
\nabla w_\eta = \nabla w + \nabla \Phi_i g_i \in [A_i, B_i]_{\nu/4} \subset U \quad \text{in} \quad B_i,
\]
where $[A_i, B_i]_{\nu/4}$ is the $\frac{\nu}{4}$-neighborhood of the closed line segment $[A_i, B_i]$ in the space $M_{sym}^{2 \times 2}$; thus $\nabla w_\eta \in U$ in $\Omega_T^2$. By (5.4) and (5.8) with zero antidiagonal of $A_i - B_i$, we have
\[
\|u_\eta - h\|_{L^\infty(\Omega_T)} \leq \|u_t - h\|_{L^\infty(\Omega_T)} + \tau < \|u_t - h\|_{L^\infty(\Omega_T)} + \theta < \epsilon',
\]
\[
\|w - w_\eta\|_{L^\infty(\Omega_T)} = \|\sum_{i=1}^N \chi_{B_i^2} \Phi_i g_i\|_{L^\infty(\Omega_T)} < \tau < \eta.
\]
Lastly, note that
\[
\int_{\Omega_T^2} \text{dist}(\nabla w_\eta(x, t), K) \, dx dt = \int_{(\Omega_T^2 \setminus \partial \Omega_T^2) \setminus G_1} \text{dist}(\nabla w(x, t), K) \, dx dt
\]
\[
+ \int_{H_2} \text{dist}(\nabla w(x, t), K) \, dx dt + \int_{G_2 \cup (\cup_{i=1}^N B_i)} \text{dist}(\nabla w(x, t), K) \, dx dt
\]
\[
+ \sum_{i=1}^N \int_{B_i} \text{dist}(\nabla w(x, t) + \nabla \Phi_i g_i(x, t), K) \, dx dt =: I_1 + I_2 + I_3 + I_4.
\]
Observe here that for \( i = 1, \cdots, N \),
\[
\int_{B_i} \text{dist}(\nabla w + \nabla \Phi_i g_i, K) \, dx \, dt = \int_{B_i} \text{dist}(\nabla w + \nabla \Phi_i g_i, K) \, dx \, dt \\
+ \int_{B_i} \text{dist}(\nabla w(x, t) + \nabla \Phi_i g_i, K) \, dx \, dt \\
\leq S |B_i^1| + \nu |B_i^2| \leq S \tau + \frac{\delta}{5} |B_i^2| \leq \frac{\delta |\Omega_2^1|}{5N} + \frac{\delta}{5} |B_i^2|.
\]
Thus \( I_4 \leq \frac{2\delta}{5} |\Omega_2^1| \); whence with (5.3), (5.5) and (5.6), we have \( I_1 + I_2 + I_3 + I_4 \leq \frac{\delta}{5} + \frac{\delta}{5} + \frac{\delta}{5} + \frac{\delta}{5} |\Omega_2^1| = \delta |\Omega_2^1| \).
Therefore, (5.2) is proved, and the proof is complete.

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Institute for Mathematical Sciences, Renmin University of China, Beijing 100872, PRC

E-mail address: kimseo14@ruc.edu.cn

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, ROK

E-mail address: ywkoh@kias.re.kr