ABELIAN SANDPILE MODEL AND BIGGS-MERINO POLYNOMIAL FOR DIRECTED GRAPHS

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Abstract. We study the generating function of recurrent configurations of abelian sandpile model on a directed graph \( G \) called Biggs-Merino polynomial, named after Norman Biggs, who was the first to study this polynomial, and Criel Merino López, who proved that for an undirected graph the polynomial is equal to the Tutte polynomial of the graph evaluated at \( x=1 \). Implicit in the sandpile model is the choice of a vertex of \( G \) as a sink vertex, and it was conjectured by Perrot and Pham that Biggs-Merino polynomial is an invariant that is independent of the choice of the sink vertex. In this paper, we give a proof of the conjecture of Perrot and Pham by interpreting the Biggs-Merino polynomial as a product of two functions that does not depend on the sink. We also observe that Merino’s theorem can be generalized to the setting of Eulerian digraph, where the Biggs Merino polynomial is equal to a single variable generalization of Tutte polynomial called the greedoid Tutte polynomial. The proof of generalized Merino’s theorem uses the digraph version of Cori-Le Borgne algorithm to construct a bijection between arborescences of the digraph \( G \) and \( G^r \)-parking functions that preserves arborescence activities.

1. Introduction

Let \( G = (V, E) \) be a finite directed graph, which may have loops and multiples edges. The abelian sandpile model on \( G \) is a dynamical system in which we put a number of chips at each vertex of \( G \). If a vertex \( v \in V \) has at least as many chips as outgoing edges, then we can fire the vertex \( v \) by sending one chip along each outgoing edge to a neighbouring vertex.

The abelian sandpile model was introduced by Dhar [Dha90] as a model to study the concept of self-organized criticality introduced by Bak, Tang, and Wiesenfeld [BTW88]. Since then the abelian sandpile model has been studied in several different field of mathematics. In graph theory it was studied under the name of chip-firing game by Tardos [Tar88] and Björner, Lovász, and Shor [BLS91]; it appears in arithmetic geometry in the study of Jacobian of algebraic curves [Lor89, BN07]; in algebraic graph theory it relates to the study of potential theory on graphs [Big97b, BS13]; and several generalizations of abelian sandpile model has been studied in the existing literature, such as rotor-routing model [PDDK96], inverse toppling sandpile model [CPS12], and abelian network [BL13, BL14a, BL14b].

It is common to study abelian sandpile model by specifying a vertex of the digraph as the sink vertex. The sink vertex is not allowed to fire, and doing so on a strongly-connected digraph guarantees that after a finite sequence of firing moves the process will eventually stop, as eventually all vertices with the exception of the sink vertex will have less chips than the number of outgoing edges. Attached to the abelian sandpile model with a sink \( s \) is a generating function of \( s \)-recurrent
configurations of the model, which we name **Biggs-Merino polynomial** to honour Biggs [Big97a] and Merino López [ML97] who were the first to study this polynomial. Formally the Biggs-Merino polynomial is the sum:

$$\mathcal{B}(G, s; y) := \sum_{[\hat{c}] \in \text{Rec}(G, s)} y^{\text{dvl}([\hat{c}])},$$

where $\text{Rec}(G, s)$ is the quotient set of $s$-recurrent configurations by relation $\sim$ and $\text{dvl}$ is a function that measures the number of chips in an $s$-recurrent configuration. The $s$-recurrent configurations and the equivalence relation will be defined in Section 2, while the d-level will be defined in Section 4.

The study of Biggs-Merino polynomial for undirected graphs was started by Biggs [Big97a], who conjectured that the Biggs-Merino polynomial is equal to $y^{|E(G)|} \mathcal{T}(G; 1, y)$, where $\mathcal{T}(G; x, y)$ is the Tutte polynomial of the undirected graph $G$. The conjecture was proved to be true by Merino López [ML97] and is now known as Merino’s Theorem, and since then the Biggs-Merino polynomial has appeared in several different field of mathematics. In algebraic geometry the Biggs-Merino polynomial can be interpreted as the Hilbert series of a graded ring defined from a graph [PPW13]; in theoretical physics an analogue of Biggs-Merino polynomial in abelian avalanche model was studied by Gabrielov [Gab93]; and a two variable generalization of Biggs-Merino polynomial that is not equal to Tutte polynomial was studied by Lorenzini [Lor12].

The study of Biggs-Merino polynomial for directed graphs was started by Perrot and Pham [PV13], who extended the definition of Biggs-Merino polynomial to the setting of directed graphs. They proved that the polynomial $\mathcal{B}(G, s; y)$ does not depend on the choice of sink $s$ for a connected Eulerian digraph $G$, and they conjecture that statement holds true for all strongly-connected digraphs:

**Conjecture 1.1.** ([PV13, Conjecture 1]). For a strongly connected digraph $G$ the polynomial $\mathcal{B}(G, s; y)$ does not depend of the choice of the sink vertex $s$.

If the underlying graph $G$ is undirected, Conjecture 1.1 is a corollary of Merino’s Theorem, as the Biggs-Merino polynomial is equal to $y^{|E(G)|} \mathcal{T}(G; 1, y)$, which is an expression that is independent of the choice of sink. In this paper we give a proof of Conjecture 1.1, and the outline of the proof is presented below.

### 1.1. Outline of the proof of Conjecture 1.1

The idea of the proof is to relate two different types of abelian sandpile models, the **sinkless abelian sandpile model** $\text{Sand}(G)$ and the **abelian sandpile model with a sink** $\text{Sand}(G, s)$ at vertex $s$.

In the sinkless abelian sandpile model, a chip configuration $c$ is a non-negative vector indexed by $V(G)$. We have a special family of chip configurations called **recurrent configurations**, denoted by $\text{Rec}(G)$, and an equivalence relation $\sim$ on the elements of $\text{Rec}(G)$ (both notions will be defined in Section 2). We use $\text{Rec}(G, \sim) := \text{Rec}(G)/\sim$ to denote the quotient set of $\text{Rec}(G)$ by $\sim$.

The **level function** $\text{lvl} : \text{Rec}(G, \sim) \rightarrow \mathbb{N}_0$ is defined as

$$\text{lvl}(c) := \sum_{v \in V} c(v).$$

The level function is equal for any two configurations $c, d$ with $c \sim d$, and thus it is possible to define the level of an equivalence class $[c]$ as the level of a configuration $c$ contained in $[c]$. The generating
function $R(G; y)$ of $\text{Rec}(G, \sim)$ is a formal power series defined as:

$$R(G; y) := \sum_{[c] \in \text{Rec}(G, \sim)} y^{\text{slvl}(\hat{c})}.$$  

We remark that as its name implies, there is no notion of sink in the sinkless abelian sandpile model and the function $R(G; y)$ is independent of the choice of $s$.

In the abelian sandpile model with a sink at vertex $s$, an s-configuration $\hat{c}$ is a non-negative vector indexed by $V(G) \setminus \{s\}$. We have a special family of s-configurations called s-recurrent configurations, denoted by $\text{Rec}(G, s)$, and an equivalence relation $\sim$ on the elements of $\text{Rec}(G, s)$. We use $\text{Rec}(G, \sim) := \text{Rec}(G, s)/\sim$ to denote the quotient set of $\text{Rec}(G, s)$ by $\sim$.

The s-level function $\text{slvl} : \text{Rec}(G, s) \to \mathbb{N}_0$ is defined as:

$$\text{slvl}(\hat{c}) := \chi(\hat{c}) + \sum_{v \in V \setminus \{s\}} \hat{c}(v),$$

where $\chi(\hat{c}) := \min\{k \in \mathbb{N}_0 \mid c_k \text{ is a recurrent configuration}\}$ and $c_k$ is a chip configuration in $\text{Sand}(G)$ where $c_k(v) := \hat{c}(v)$ for $v \in V \setminus \{s\}$ and $c_k(s) := k$. The s-level of two s-recurrent configurations $\hat{c}, \hat{d}$ satisfying $\hat{c} \sim \hat{d}$ is equal (Lemma 3.2), and we define the s-level of an equivalence class $[\hat{c}] \in \text{Rec}(G, \sim)$ as the s-level of an element $\hat{c}$ contained in $[\hat{c}]$. The polynomial $P(G, s; y)$ is then defined as:

$$P(G, s; y) := \sum_{[\hat{c}] \in \text{Rec}(G, \sim)} y^{\text{slvl}([\hat{c}])}.$$  

There is another notion of level for s-recurrent configurations, called d-level function $\text{dvl}$, which is defined by:

$$\text{dvl}(\hat{c}) := \deg_{G, s}^+ + \sum_{v \in V \setminus \{s\}} \hat{c}(v),$$

for all $\hat{c} \in \text{Rec}(G, s)$, and we define the d-level of an element $[\hat{c}] \in \text{Rec}(G, \sim)$, as

$$\text{dvl}([\hat{c}]) := \max_{\hat{c} \in [\hat{c}]} \text{dvl}(\hat{c}).$$

The Biggs-Merino polynomial is defined as in Equation 1 using the notion of d-level defined above.

Let $\text{Rec}_n(G, \sim)$ be the subset of $\text{Rec}(G, \sim)$ with level equal to $n$, and $\text{Rec}_{\leq n}(G, \sim)$ be the subset of $\text{Rec}(G, \sim)$ with s-level less than or equal to $n$. The model $\text{Sand}(G)$ is related to the model $\text{Sand}(G, s)$ by a bijection $\Phi$ that sends $\text{Rec}_n(G, \sim)$ to $\text{Rec}_{\leq n}(G, \sim)$ for all $n$ (Lemma 3.5). The bijection between $\text{Rec}_n(G, \sim)$ and $\text{Rec}_{\leq n}(G, \sim)$ allows us to derive the following equality of formal power series:

$$R(G; y) = \frac{P(G, s; y)}{1 - y},$$

and since $R(G; y)$ is independent of the choice of the vertex $s$, we conclude that $P(G, s; y)$ is independent of the choice of $s$.

On the other hand, the value of s-level and d-level is equal for an arbitrary element of $\text{Rec}(G, \sim)$ (Lemma 4.3), which implies that $P(G, s; y) = B(G, s; y)$, and since $P(G, s; y)$ is independent of the choice of $s$ by previous argument, we conclude that $B(G, s; y)$ is also independent of the choice of $s$, proving Conjecture 1.1.
1.2. Merino’s Theorem for Eulerian digraphs. Let \( A(G, s) \) be the set of arborescences of \( G \) with a root at \( s \), the **greedoid Tutte polynomial** rooted at \( s \), first introduced by Björner, Korte, and Lovász [BKL85], is defined as:

\[
T(G, s; y) := \sum_{T \in A(G, s)} y^{e(T)},
\]

where \( e(T) \) denotes the external activity of \( T \) and is defined in Section 6. The greedoid Tutte polynomial is a single variable generalization of the ordinary Tutte polynomial \( T(G; x, y) \) for undirected graph \( G \) evaluated at \( x = 1 \).

When \( G \) is an Eulerian digraph, we observe that there is an analogue of Merino’s Theorem [ML97] for undirected graphs.

**Theorem 1.2.** (Generalized Merino’s theorem). If \( G \) is a connected Eulerian digraph, then the Biggs-Merino polynomial \( B(G, s; y) \) of \( G \) is equal to the greedoid Tutte polynomial \( T(G, s; y) \) of \( G \).

In particular, Merino’s Theorem for loopless undirected graphs is a consequence of Theorem 1.2. The proof of Theorem 1.2 uses a modification of Cori-Le Borgne algorithm [CLB03] to construct a bijection between arborescences of \( G \) and \( G^r \)-parking functions (to be defined in Section 6) that preserves the arborescence activities. When \( G \) is an Eulerian digraph, this bijection extends to a bijection between arborescences of \( G \) and s-recurrent configurations, as \( G^r \)-parking functions are known to be in bijection with s-recurrent configurations when \( G \) is Eulerian [HLM+08].

1.3. The outline of this paper. This paper is arranged as follows: In Section 2 we give a summary of basic theory of abelian sandpile model, including the definition of recurrent configurations and the equivalence relation defined on it. In Section 3 we discuss the connection between sinkless sandpile model and sandpile model with sink, and we prove that \( \text{Rec}_n(G, \sim) \) is in bijection with \( \text{Rec}_{\leq n}(G, \lessdot) \). In Section 4 we prove the equality between s-level and d-level for elements in \( \text{Rec}(G, \lessdot) \) to complete the proof of Conjecture 1.1. In Section 5 we give an example of the Biggs-Merino polynomial of a family of non-Eulerian digraphs. In Section 6 we give a proof of generalized Merino’s Theorem for Eulerian digraphs. Finally, in Section 7 we give a short discussion on possible future research directions.

2. Preliminaries

In this paper, we will mostly work with a finite directed graph \( G := (V(G), E(G)) \) with loops and multiple edges allowed, and we will use \( \mathcal{G} \) to indicate that the graph \( \mathcal{G} \) is undirected. We use \( V \) and \( E \) as a shorthand for \( V(G) \) and \( E(G) \) when the graph \( G \) is evident from the context. Each edge \( e \in E(G) \) is directed from its source vertex \( s(e) \) to its target vertex \( t(e) \). The outdegree of a vertex \( v \in V(G) \), denoted by \( \text{deg}^+(v) \), is the number of edges that has \( v \) as source vertex, while the indegree of a vertex \( v \in V(G) \), denoted by \( \text{deg}^-(v) \), is the number of edges that has \( v \) as target vertex.

A digraph \( G \) is Eulerian if \( \text{deg}^+_G(v) = \text{deg}^-_G(v) \) for all \( v \in V(G) \). A digraph is strongly connected if for any two vertices \( v, w \in V(G) \) there exists a directed path from \( v \) to \( w \). Throughout this paper, we always assume that our digraph \( G \) is strongly connected, and we remark that a connected Eulerian digraph is always strongly connected.
The Laplacian matrix $\Delta$ of a digraph $G$ is a square matrix $(\Delta_{i,j})_{|V|\times|V|}$ with the rows and columns indexed by $V$ and is defined by:

$$
\Delta_{i,j} := \begin{cases} 
\deg^{-}_G(i) - |\{e \in E \mid s(e) = t(e) = i\}| & \text{if } i = j; \vspace{1mm} \\
|\{e \in E \mid t(e) = i \text{ and } s(e) = j\}| & \text{if } i \neq j.
\end{cases}
$$

The reduced Laplacian matrix $\Delta_s$ of a digraph $G$ with respect to $s \in V$ is a square $(|V| - 1) \times (|V| - 1)$ matrix obtained by deleting the row and column of $\Delta$ that corresponds to $s$. Matrix Tree Theorem [Cha82] states that the determinant of $\Delta_s$ is equal to the number of reverse arborescences of $G$ rooted at $s$, where a reverse arborescence is a subgraph of $G$ such that for all $v \in V(G)$ there exists a unique directed path from $v$ to $s$ in the subgraph. Since $G$ is a strongly connected digraph, the determinant of $\Delta_s$ is a positive number for all $s \in V$.

The period constant $\alpha$ of $G$ is defined as $\alpha := \gcd_{v \in V}(\det \Delta_v)$, and the primitive period vector $r$ of $G$ is a positive vector indexed by $V$ with $r(v) := \frac{\det(\Delta_v)}{\alpha}$ for all $v \in V$. The primitive period vector $r$ is contained in the kernel of the Laplacian matrix $\Delta$ by Markov Chain Tree Theorem [AT89], and as we assume that $G$ is strongly connected, $\Delta$ has codimension 1 and $\ker \Delta$ is generated by $r$. In particular, since $\gcd_{v \in V}(r(v)) = 1$, any integer vector $q$ in $\ker \Delta$ is an integer multiple of $r$. We remark that when $G$ is an Eulerian digraph, the vector $(1, \ldots, 1)$ is contained in $\ker \Delta$, which implies that $r = (1, \ldots, 1)$ and $\alpha = \det(\Delta_v)$ for any $v \in V$.

2.1. Sinkless abelian sandpile model. The sinkless abelian sandpile model on a strongly connected digraph $G$, denoted by Sand($G$), starts with a number of chips at each vertex of $G$. If a vertex $v \in V$ has at least as many chips as outgoing edges, then $v$ is said to be unstable, and we can fire the vertex $v$ by sending one chip along each outgoing edge to a neighbouring vertex.

A chip configuration $c$ in Sand($G$) is a vector of non-negative integers indexed by the vertices of $G$ with $c(v)$ represents the number of chips in the vertex $v \in V$. For a chip configuration $c$, a firing move consists of reducing the number of chips of $c$ at a vertex $v \in V$ by $\deg^+_G(v)$, and then sending one chip along each outgoing edge of $c$ to a neighbouring vertex. Each finite sequence of firing moves is associated with an odometer $q \in \mathbb{N}^V$, where $q(v)$ is equal to the number of times the vertex $v$ is being fired in the sequence. Note that applying a finite sequence of firing moves with odometer $q$ to a chip-configuration $c$ gives us the chip configuration $c - \Delta q$.

Notice that after a firing move, some vertex $v$ may have negative chips, and so the resulting configuration is not a valid chip configuration as by definition a chip configuration is a non-negative integer vector. Hence, we say that a firing move on $c$ is legal if the vertex $v \in V$ that is fired is unstable and for any configuration $c$ such that $\deg^+_G(v) < c(v)$ for all $v \in V$.

For two configurations $c, d \in C(G)$, we say that $d$ is accessible from $c$, denoted by $c \rightarrow d$, if we can perform a finite sequence of legal firing moves on $c$ to arrive at $d$. Note that $\rightarrow$ is a transitive relation: if $c \rightarrow d$ and $d \rightarrow e$, then $c \rightarrow e$. A recurrent configuration, also known as a critical configuration, is a configuration $c$ such that $c$ is unstable and for any configuration $d$ such that $c \rightarrow d$, we also have $d \rightarrow c$. We use Rec($G$) to denote the set of all recurrent configurations of Sand($G$).

In the next lemma we present a test called burning test to check whether a given configuration is recurrent or not, which is a reformulation of [BZ92, Lemma 1.3] and [BZ92, Lemma 4.3].
Lemma 2.1. (Burning test for $\text{Sand}(G)$). A chip configuration $c$ is a recurrent configuration in $\text{Sand}(G)$ if and only if there exists a finite sequence of legal firing moves from $c$ back to $c$ such that each vertex $v \in V$ is fired exactly $r(v)$ times, where $r$ is the primitive period vector of $G$.

Proof. Please see Appendix A for details of the proof. \qed

We remark that the definition of recurrent configurations and burning test for sinkless abelian sandpile model are different from the standard definition because of the lack of sink. The standard definition of recurrent configurations and burning test will be presented in Section 2.2 for abelian sandpile model with sink, where the recurrent configuration is called $s$-recurrent configuration instead.

The next proposition gives two sufficient conditions for a chip configuration to be recurrent. We use $1_s \in \mathbb{N}_0^V$ to denote the chip configuration with $1_s(v) := 0$ for all $v \in V \setminus \{s\}$ and $1_s(s) := 1$.

Proposition 2.2. Let $c, d$ be chip configurations.

(i) If $c \rightarrow d$ and $c$ is a recurrent configuration, then $d$ is a recurrent configuration.

(ii) If $c$ is a recurrent configuration, then the configuration $c' := c + k1_s$ for some $k \in \mathbb{N}$ is also a recurrent configuration.

Proof. Please see Appendix A for details of the proof. \qed

For two recurrent configurations $c, d \in \text{Rec}(G)$, we construct an equivalence relation $\sim$ by setting $c \sim d$ if we have $c \rightarrow d$, or equivalently if $d$ is accessible from $c$ through a finite sequence of legal firing moves.

Note that the relation $\sim$ is reflexive as $c$ is accessible from $c$ through a finite sequence of legal firing moves of length zero. Transitivity of $\sim$ follows immediately from the transitivity of $\rightarrow$. The symmetry of $\sim$ follows from $c$ and $d$ being recurrent configurations, as if $c \rightarrow d$, then by definition of recurrence we also have $d \rightarrow c$. Thus $\sim$ is an equivalence relation on the set $\text{Rec}(G)$. We use $\text{Rec}(G, \sim)$ to denote the quotient set of $\text{Rec}(G)$ by $\sim$, and for any $c \in \text{Rec}(G)$ we use $[c]$ to denote the equivalence class in $\text{Rec}(G, \sim)$ that contains $c$.

2.2. Abelian sandpile model with sink. Let $s \in V$ be a fixed vertex, the abelian sandpile model with sink at $s$, denoted by $\text{Sand}(G, s)$, starts with a similar rule as for the sinkless sandpile model. We have a number of chips at each vertex of $G$, and when a vertex $v \in V$ has number of chips exceeding its outdegree, then $v$ can fire and send one chip along each outgoing edge to a neighbouring vertex. However, in this model we have a sink vertex $s$, with the rule that $s$ never fires and all chips that are sent to $s$ are removed from the game. For an extensive introduction to abelian sandpile model with sink, the reader is referred to [HLM+08]. For a non-technical overview of this model, the reader is referred to [LP10].

An $s$-chip configuration $\tilde{c}$ in $\text{Sand}(G, s)$, also called $s$-configuration, is a vector of non-negative integers indexed by $V \setminus \{s\}$, where $\tilde{c}(v)$ represents the number of chips in the vertex $v \in V \setminus \{s\}$. The reason in removing the entry that corresponds to $s$ is because in this model the sink does not possess any chips at any point of the game. An $s$-configuration $\tilde{c}$ is stable if $\tilde{c}(v) < \deg_G^+(v)$ for all $v \in V \setminus \{s\}$.

An $s$-firing move in $\text{Sand}(G, s)$ consists of reducing the number of chips of $\tilde{c}$ at an arbitrary vertex $v \in V \setminus \{s\}$ by $\deg_G^+(v)$, and then sending one chip along each outgoing edge of $\tilde{c}$ to a neighbouring vertex. An $s$-firing move is legal if the vertex $v \in V \setminus \{s\}$ being fired has at least as many chips as its outdegree. Given two $s$-configurations $\tilde{c}, \tilde{d}$, we use $\tilde{c} \xrightarrow{\text{leg}} \tilde{d}$ to denote that $\tilde{d}$ is obtained from $\tilde{c}$ by performing a legal $s$-firing move on the vertex $v \in V \setminus \{s\}$. It is convenient for us
to be able the fire vertex \(s\) as an s-firing move, and so we adopt the convention that firing a vertex \(s\) on an s-configuration \(\hat{c}\) will send \(\hat{c}\) back to \(\hat{c}\), and firing the sink vertex \(s\) is always legal. We say that a sequence of s-firing moves is reduced if the vertex \(s\) is never fired during the sequence, and note that for any sequence of s-firing moves, there exists a unique reduced sequence of s-firing moves obtained by removing all the moves that fire \(s\) in the sequence.

Each finite sequence of s-firing moves is associated with an s-odometer \(\hat{q}\) \(\in \mathbb{N}_0^{V\setminus\{s\}}\), where for any \(v \in V \setminus \{s\}\) we have \(\hat{q}(v)\) is equal to the number of times the vertex \(v\) is being fired in the sequence. Note that applying a finite sequence of s-firing moves with odometer \(\hat{q}\) to an s-configuration \(\hat{c}\) gives us the s-configuration \(\hat{c} - \Delta_s \hat{q}\).

Given two s-configurations \(\hat{c}, \hat{d}\), we say that \(\hat{d}\) is accessible from \(\hat{c}\) if there is a finite sequence of legal s-firing moves in \(\text{Sand}(G, s)\) from \(\hat{c}\) to \(\hat{d}\), denoted by \(\hat{c} \rightarrow_s \hat{d}\). Furthermore, if \(\hat{d}\) is a stable s-configuration, then we say that \(\hat{c}\) stabilizes to \(\hat{d}\). For a strongly connected digraph \(G\), any s-configurations \(\hat{c} \in \text{Sand}(G, s)\) stabilizes to a unique stable s-configuration \(\hat{d}\) \cite{HLM2008}, and we call \(\hat{d}\) the stabilization of \(\hat{c}\). The next lemma gives us a sufficient condition for two s-configurations to stabilize to the same s-configuration.

**Lemma 2.3.** \cite{HLM2008, Lemma 2.2}. If \(\hat{c}, \hat{d}\) are two s-configurations satisfying \(\hat{c} \rightarrow_s \hat{d}\), then \(\hat{d}\) is equal to \(\hat{d}\).

In the next lemma we present a property called the abelian property, which justifies the name “abelian sandpile model”. Described in words, this lemma tells us that the order of adding chips to an s-configuration followed by stabilizing does not change the final configuration.

**Lemma 2.4.** \cite{HLM2008, Corollary 2.6}. If \(\hat{c}_1, \hat{c}_2, \hat{c}_3\) are s-recurrent configurations, then \((\hat{c}_1 + \hat{c}_2 + \hat{c}_3)^{\circ} = (\hat{c}_1 + \hat{c}_2 + \hat{c}_3)^{\circ} = (\hat{c}_1 + \hat{c}_2 + \hat{c}_3)^{\circ}\).

We say that an s-configuration \(\hat{c}\) is s-recurrent if for any s-configurations \(\hat{c}_1\) there exists another s-configuration \(\hat{c}_2\) such that \((\hat{c}_1 + \hat{c}_2)^{\circ} = \hat{c}\). Note that this definition implies that an s-recurrent configuration is stable. We use \(\text{Rec}(G, s)\) to denote the set of s-recurrent configurations of \(\text{Sand}(G, s)\). We remark that while the definition of s-recurrence in \(\text{Sand}(G, s)\) appears to differ greatly from the definition of recurrence in \(\text{Sand}(G)\), in Section 3 we will see that the notion of s-recurrence and recurrence are strongly related.

In the next lemma we present the burning test for s-recurrent configurations in \(\text{Sand}(G, s)\), first discovered by Dhar \cite{Dhar1990} for undirected graphs and then discovered by Speer \cite{Speer1993} for directed graphs. The statement of the burning test below is due to Asadi and Backman \cite{Asadi2010} who discovered the test independently from Speer, and a stronger version of the test was discovered by Bond and Levine \cite[Theorem 5.5]{Bond2014} for a generalization of sandpile model called abelian network.

We denote the s-Laplacian vector \(\hat{u} \in \mathbb{N}_0^{V\setminus\{s\}}\) as the s-configuration with \(\hat{u}(v)\) is equal to the number of edges from \(s\) to \(v\) for all \(v \in V \setminus \{s\}\).

**Lemma 2.5.** \cite{HLM2008, Theorem 3}, \cite[Theorem 3.11]{Asadi2010}. For a strongly connected digraph \(G\), an s-configuration \(\hat{c}\) is s-recurrent in \(\text{Sand}(G, s)\) if and only if \((\hat{c} + r(s)\hat{u})^{\circ} = \hat{c}\), with \(r(s)\) is the entry of the primitive period vector \(r\) that corresponds to \(s\).

In the next proposition we give a sufficient condition for an s-configuration to be s-recurrent.

**Proposition 2.6.** \cite[Lemma 2.17]{HLM2008}. Suppose that \(\hat{d}\) is an s-configuration such that there exists an s-recurrent configuration \(\hat{c}\) and an s-configuration \(\hat{c}'\) satisfying \(\hat{d} = (\hat{c} + \hat{c}')^{\circ}\). Then \(\hat{d}\) is an s-recurrent configuration.
We now define an equivalence relation $\sim$ on $\text{Rec}(G, s)$. For any two $s$-recurrent configurations $\hat{c}, \hat{d} \in \text{Rec}(G, s)$, we say that $\hat{c} \sim \hat{d}$ if $\hat{d} = (\hat{c} + k\hat{u})^o$ for some non-negative integer $k \in \mathbb{N}_0$. Reflexivity of $\sim$ follows from the definition by taking $k = 0$ and transitivity follows from Lemma 2.4, as if $\hat{d} = (\hat{c} + k\hat{u})^o$ and $\hat{d}' = (\hat{d} + l\hat{u})^o$, then

$$(\hat{c} + (k + l)\hat{u})^o = ((\hat{c} + k\hat{u})^o + l\hat{u})^o = (\hat{d} + l\hat{u})^o = \hat{d}'.$$

To show symmetry, suppose that $\hat{d} = (\hat{c} + k\hat{u})^o$, then let $k'$ be the smallest positive multiple of $r(s)$ that is bigger than $k$, then

$$(\hat{d} + (k' - k)\hat{u})^o = ((\hat{c} + k\hat{u})^o + (k' - k)\hat{u})^o = (\hat{c} + k'\hat{u})^o = \hat{c},$$

where the second equality is due to Lemma 2.4 and the third equality is due to Lemma 2.5.

We remark that the relation $\sim$ is equivalent with the relation defined in [PV13, Section 6], and a proof of this equivalence is included in Appendix A (Proposition A.3).

We denote by $\text{Rec}(G, \sim)$ the quotient set of $\text{Rec}(G, s)$ by $\sim$, and for any $\hat{c} \in \text{Rec}(G, s)$ we use $[\hat{c}]$ to denote the equivalence class in $\text{Rec}(G, \sim)$ that contains $\hat{c}$. We remark that the size of $\text{Rec}(G, \sim)$ is equal to the period constant $\alpha$ (Proposition A.4) and thus is finite. When $G$ is an Eulerian digraph, we also have $\alpha$ is equal to $\det(\Delta_s) = |\text{Rec}(G, s)|$ as observed in the beginning of this section, so these two observations give us $|\text{Rec}(G, \sim)| = \alpha = |\text{Rec}(G, s)|$ and hence each equivalence class $[\hat{c}] \in \text{Rec}(G, \sim)$ contains exactly one $s$-recurrent configuration.

2.3. Comparison between two sandpile models. In Section 3, we need to interpret a chip configuration of sinkless sandpile model as an $s$-configuration of sandpile model with sink. In order to reduce the number of notations needed, we adopt the convention that $c \in \mathbb{N}_0^V$ is a chip configuration in $\text{Sand}(G)$, $\hat{c} \in \mathbb{N}_0^V \setminus \{s\}$ is an $s$-configuration with $\hat{c}(v) := c(v)$ for all $v \in V \setminus \{s\}$, and $\hat{c} \in \mathbb{N}_0^V$ is a chip configuration in $\text{Sand}(G)$ with $\hat{c}(v) := \hat{c}(v)$ for all $v \in V \setminus \{s\}$ and $\hat{c}(s) := 0$.

In the next proposition we highlight the connection between configurations $c, \hat{c}$, and $\hat{c}$.

**Proposition 2.7.** Let $c, d$ be chip configurations in $\text{Sand}(G)$.

(i) Suppose that $c \rightarrow^* d$ through a finite sequence of reduced legal $s$-firing moves. Then the associated sequence of firing moves is legal for $c$, and we have $c \rightarrow d + n 1_s$, where $n$ is the number of chips removed from the game during the legal $s$-firing moves from $c$ to $d$.

(ii) Suppose that $c \rightarrow^* d$ through a finite sequence of legal firing moves. Then the associated sequence of $s$-firing moves is legal for $\hat{c} + m\hat{u}$ and we have $\hat{c} + m\hat{u} \rightarrow^* \hat{d}$, where $m$ is the number of times the vertex $s$ is fired during the legal firing moves from $c$ to $d$.

**Proof.** Please see Appendix A for details of the proof. □

In the table below we give a list of comparisons between the sinkless sandpile model and sandpile model with sink, and the level function mentioned in the table will be defined in the next section:

|                        | $\text{Sand}(G)$ | $\text{Sand}(G,s)$ |
|------------------------|-------------------|---------------------|
| Set of chip configurations | $\mathbb{N}_0^V$  | $\mathbb{N}_0^V \setminus \{s\}$ |
| Set of recurrent configurations | $\text{Rec}(G)$ | $\text{Rec}(G,s)$ |
| Equivalence relation    | $\sim$            | $\sim$ |
| Set of equivalence classes for $\sim$ (\hat{c}) | $\text{Rec}(G,\sim)$ | $\text{Rec}(G,\sim)$ |
| Chip configurations     | $c, \hat{c}$     | $\hat{c}$ |
| Level function          | $\text{lvl}$     | $\text{slvl,dlvl}$ |
3. Connections between sinkless sandpile and sandpile with sink

3.1. Level function. For a recurrent configuration \( \mathbf{c} \in \text{Rec}(G) \), we define the level of \( \mathbf{c} \), denoted by \( \text{lvl}(\mathbf{c}) \), to be the number of chips contained in the configuration \( \mathbf{c} \), or formally

\[
\text{lvl}(\mathbf{c}) := \sum_{v \in V} c(v).
\]

Note that a legal firing move in \( \text{Sand}(G) \) does not change the number of chips in the game, so for any two recurrent configurations \( \mathbf{c}, \mathbf{d} \in \text{Rec}(G) \) such that \( \mathbf{c} \sim \mathbf{d} \), we have \( \text{lvl}(\mathbf{c}) = \text{lvl}(\mathbf{d}) \). We can thus extend the notion of level from \( \text{Rec}(G) \) to \( \text{Rec}(G, \sim) \) by

\[
\text{lvl}([\mathbf{c}]) := \text{lvl}(\mathbf{c}),
\]

where \([\mathbf{c}]\) is the equivalence class in \( \text{Rec}(G, \sim) \) containing \( \mathbf{c} \). For a natural number \( n \in \mathbb{N}_0 \), we use \( \text{Rec}_n(G, \sim) \) to denote the subset of \( \text{Rec}(G, \sim) \) containing all elements of \( \text{Rec}(G, \sim) \) with level \( n \).

The notion of level function for s-recurrent configurations is less straightforward to define, and we will need the following lemma.

Lemma 3.1. Suppose that \( \hat{\mathbf{c}} \) is an s-recurrent configuration in \( \text{Sand}(G, s) \) and \( \mathbf{r} \) is the primitive period vector of \( G \). Then \( \mathbf{r} + (\deg^+_G(s) \cdot \mathbf{r}(s)) \mathbf{1}_s \) is a recurrent configuration in \( \text{Sand}(G) \).

Proof. By Lemma 2.5, since \( \hat{\mathbf{c}} \) is s-recurrent, we have

\[
(\hat{\mathbf{c}} + \mathbf{r}(s) \hat{\mathbf{u}})^0 = \hat{\mathbf{c}},
\]

and in particular this means that \( \hat{\mathbf{c}} + \mathbf{r}(s) \hat{\mathbf{u}} \rightarrow^\ast \hat{\mathbf{c}} \) through a finite sequence of reduced legal s-firing moves. By Proposition 2.7 (i), this implies that \( \mathbf{r} + \mathbf{r}(s) \mathbf{1}_s \rightarrow^\ast \mathbf{r} + k \mathbf{1}_s \) for some \( k \in \mathbb{N}_0 \) through a sequence firing moves that do not fire \( s \). Note that a firing move in \( \text{Sand}(G) \) does not change the total amount of chips in the configuration, so the number of chips in the configuration \( \mathbf{r} + \mathbf{r}(s) \mathbf{1}_s \) is equal to the number of chips in \( \mathbf{r} + k \mathbf{1}_s \), and so we can conclude that \( k = \deg^+_G(s) \cdot \mathbf{r}(s) \).

On the other hand, there exists a finite sequence of legal firing moves from \( \mathbf{r} + (\deg^+_G(s) \cdot \mathbf{r}(s)) \mathbf{1}_s \) to \( \mathbf{r} + \mathbf{r}(s) \hat{\mathbf{u}} \), namely the sequence that fires \( s \) for \( \mathbf{r}(s) \) times. From previous paragraph we already know that there exists a finite sequence of legal firing moves sending \( \mathbf{r} + \mathbf{r}(s) \hat{\mathbf{u}} \) to \( \mathbf{r} + (\deg^+_G(s) \cdot \mathbf{r}(s)) \mathbf{1}_s \) where \( s \) is not fired during the sequence. Thus there is a finite sequence of legal firing moves from \( \mathbf{r} + (\deg^+_G(s) \cdot \mathbf{r}(s)) \mathbf{1}_s \) back to itself where the vertex \( s \) is fired \( \mathbf{r}(s) \) times. Let \( \mathbf{q} \) be the odometer of this sequence, it is easy to see that \( \Delta \mathbf{q} = 0 \) and \( \mathbf{q}(s) = \mathbf{r}(s) \). Since the primitive period vector \( \mathbf{r} \) generated the kernel of Laplacian matrix \( \Delta \), we conclude that \( \mathbf{q} = \mathbf{r} \). Thus \( \mathbf{r} + (\deg^+_G(s) \cdot \mathbf{r}(s)) \mathbf{1}_s \) passes the burning test in Lemma 2.1 and therefore is a recurrent configuration. \( \square \)

Let \( \mathbf{c} \in \text{Rec}(G, s) \) be an s-recurrent configuration, the s-index of \( \mathbf{c} \), denoted by \( \chi(\mathbf{c}) \), is the smallest integer \( k \) such that \( \mathbf{c} + k \mathbf{1}_s \) is a recurrent configuration in \( \text{Sand}(G) \). By Lemma 3.1 \( \chi(\hat{\mathbf{c}}) \) exists and \( \chi(\hat{\mathbf{c}}) \leq \deg^+_G(s) \cdot \mathbf{r}(s) \). Also note that \( \chi(\hat{\mathbf{c}}) \geq \deg^+_G(s) \), as \( \hat{\mathbf{c}} \) is stable and thus \( \mathbf{r} + (\deg^+_G(s) - 1) \mathbf{1}_s \) is stable configuration in \( \text{Sand}(G) \) and hence is not recurrent by definition. The function \( \varphi : \text{Rec}(G, s) \rightarrow \text{Rec}(G) \) is defined as

\[
\varphi(\hat{\mathbf{c}}) := \mathbf{r} + \chi(\mathbf{r}) \cdot \mathbf{1}_s.
\]
Note that by the definition of s-index, we have $\varphi(\bar{c})$ is contained in $\text{Rec}(G)$. The s-level of $\bar{c}$, denoted by $\text{slvl}(\bar{c})$, is defined as the number of chips contained in the configuration $\varphi(\bar{c})$, or formally

$$\text{slvl}(\bar{c}) := \text{lvl}(\varphi(\bar{c})) = \chi(\bar{c}) + \sum_{v \in V \setminus \{s\}} \hat{c}(v).$$

We would like to extend the notion of s-level from $\text{Rec}(G, s)$ to $\text{Rec}(G, \sim)$, and that can be done due to the next lemma.

**Lemma 3.2.** If $\bar{c}$ and $\hat{\bar{d}}$ are two s-recurrent configurations such that $\bar{c} \sim \hat{\bar{d}}$, then $\varphi(\bar{c}) \sim \varphi(\hat{\bar{d}})$.

**Proof.** We start by proving that if $\hat{\bar{d}} = (\bar{c} + \bar{u})^o$, then $\text{slvl}(\hat{\bar{d}}) \leq \text{slvl}(\bar{c})$. Let $\bar{e} := \bar{e} + (\chi(\bar{c}) - \deg_G(s)) \bar{1}_s + \bar{u}$ be a chip configuration in $\text{Sand}(G)$, note that we have $\hat{\bar{e}} = \bar{c} + \bar{u}$ and so $\hat{\bar{e}}^o = (\bar{c} + \bar{u})^o = \hat{\bar{d}}$. Since we have $\hat{\bar{e}}^o = \hat{\bar{d}}$, we can conclude that $\bar{e} \rightarrow_s \bar{d}$ and by Proposition 2.7(i), this implies that $\bar{e} \rightarrow_s \bar{d} + t\bar{1}_s$ for some $t \in \mathbb{N}_0$. On the other hand, let $\bar{c} := \varphi(\bar{c}) = \bar{e} + (\chi(\hat{\bar{c}}) - \deg^+_G(s)) \hat{\bar{1}}_s + \bar{u}$ which is equivalent with $\bar{c} \rightarrow_s \bar{e}$. Since we have previously concluded that $\bar{e} \rightarrow_s \bar{d} + t\bar{1}_s$, by transitivity of $\rightarrow_s$ we have $\bar{c} \rightarrow_s \hat{\bar{d}} + t\bar{1}_s$.

As $\bar{c} = \varphi(\bar{c})$ is recurrent by definition, by Proposition 2.2(i) we have $\bar{d} + t\bar{1}_s$ is also a recurrent configuration. By the minimality of $\chi$, this implies that $\chi(\hat{\bar{d}}) \leq t$, and thus we have

$$\text{slvl}(\hat{\bar{d}}) = \text{lvl}(\hat{\bar{d}} + \chi(\hat{\bar{d}}) \bar{1}_s) \leq \text{lvl}(\hat{\bar{d}} + t\bar{1}_s) = \text{lvl}(\bar{c}) = \text{slvl}(\bar{c}),$$

where the second equality is due to $\bar{c} \rightarrow_s \hat{\bar{d}} + t\bar{1}_s$. This shows that if $\hat{\bar{d}} = (\bar{c} + \bar{u})^o$, then $\text{slvl}(\hat{\bar{d}}) \leq \text{slvl}(\bar{c})$.

Now note that by definition if $\bar{c} \sim \hat{\bar{d}}$, then $\hat{\bar{d}} = (\bar{c} + k\bar{u})^o = ((\bar{c} + (k - 1)\bar{u})^o + \bar{u})^o$ for some $k \in \mathbb{N}_0$, so by previous argument we have $\text{slvl}(\bar{c} + (k - 1)\bar{u})^o) \leq \text{slvl}(\hat{\bar{d}})$ and by induction on $k$ we can conclude that $\text{slvl}(\bar{c}) \leq \text{slvl}(\hat{\bar{d}})$. Now note that the relation $\sim$ is symmetric, so if $\bar{c} \sim \hat{\bar{d}}$ then using the same argument we can derive $\text{slvl}(\hat{\bar{d}}) \geq \text{slvl}(\bar{c})$, and thus we conclude that $\text{slvl}(\bar{c}) = \text{slvl}(\hat{\bar{d}})$ if $\bar{c} \sim \hat{\bar{d}}$.

We again assume that $\hat{\bar{d}} = (\bar{c} + \bar{u})^o$, and the fact that $\text{slvl}(\bar{c}) = \text{slvl}(\hat{\bar{d}})$ implies that equality happens in Equation 2 and we conclude that then $\chi(\hat{\bar{d}}) = t$ and $\varphi(\hat{\bar{d}}) = \bar{d} + \chi(\hat{\bar{d}}) \bar{1}_s = \bar{d} + t\bar{1}_s$. Note that we have shown that $\bar{c} \rightarrow_s \bar{d} + t\bar{1}_s$, and since $\varphi(\bar{c}) = \bar{c}$ and $\varphi(\hat{\bar{d}}) = \bar{d} + t\bar{1}_s$, we conclude that $\varphi(\bar{c}) \rightarrow_s \varphi(\hat{\bar{d}})$. Note that both $\varphi(\bar{c})$ and $\varphi(\hat{\bar{d}})$ is recurrent by definition of $\varphi$, and hence by definition of $\sim$ we have $\varphi(\bar{c}) \sim \varphi(\hat{\bar{d}})$. The claim for the general case when $\bar{c} \sim \hat{\bar{d}}$ then follows by induction just as in the previous paragraph. \(\square\)

Let $[\bar{c}] \in \text{Rec}(G, \sim)$, we define the s-level of equivalence class $[\bar{c}] \in \text{Rec}(G, \sim)$ as:

$$\text{slvl}([\bar{c}]) := \text{slvl}(\bar{c}) = \text{lvl}(\varphi(\bar{c})).$$

and by Lemma 3.2 s-level is well defined. We use $\text{Rec}_n(G, \sim)$ to denote the subset of $\text{Rec}(G, \sim)$ containing elements of $\text{Rec}(G, \sim)$ with s-level equal to $n$, and $\text{Rec}_{\leq n}(G, \sim)$ to denote the subset of $\text{Rec}(G, \sim)$ containing all elements of $\text{Rec}(G, \sim)$ with s-level less than or equal to $n$.

### 3.2. Bijection between two set of recurrent configurations.

In this subsection we show that $\text{Rec}_n(G, \sim)$ is in bijection with $\text{Rec}_{\leq n}(G, \sim)$ via a function $\Phi$, and the following lemma will be useful in constructing $\Phi$. 
Lemma 3.3. If \( c \) is a recurrent configuration in \( \text{Sand}(G) \) then \( \hat{c}^\circ \) is an \( s \)-recurrent configuration in \( \text{Sand}(G,s) \). Furthermore, if \( d \) is a recurrent configuration satisfying \( c \sim d \), then \( \hat{c}^\circ \sim \hat{d}^\circ \).

Proof. Since \( c \) is recurrent, by burning test in Lemma 2.1 there exists a finite sequence of legal firing moves that sends \( c \) back to itself and each vertex \( v \in V \) is fired exactly \( r(v) \) times. By Proposition 2.7 (ii), this implies that there exists a finite sequence of legal s-firing moves sending \( \hat{c} + r(s)\hat{u} \) to \( \hat{c} \).

By Lemma 2.3, since we have \((\hat{c} + r(s)\hat{u}) \sim \hat{c} \), this implies that \((\hat{c} + r(s)\hat{u})^\circ = \hat{c}^\circ \). By Lemma 2.4 we can conclude that \((\hat{c}^\circ + r(s)\hat{u})^\circ = (\hat{c} + r(s)\hat{u})^\circ = \hat{c}^\circ \), and so \( \hat{c}^\circ \) passes the burning test in Lemma 2.5 and thus \( \hat{c}^\circ \) is an \( s \)-recurrent configuration.

For the second part of the lemma, Since \( c \sim d \), by the definition of \( \sim \) we have \( c \rightarrow d \). By Lemma 2.7 (ii) this implies \((\hat{c} + k\hat{u}) \rightarrow \hat{d} \) for some non-negative integer \( k \in \mathbb{N} \). By Lemma 2.3, we have \((\hat{c} + k\hat{u})^\circ = \hat{d}^\circ \). Therefore by Lemma 2.4 we can conclude that \((\hat{c}^\circ + k\hat{u})^\circ = (\hat{c} + k\hat{u})^\circ = \hat{d}^\circ \), and thus \( \hat{c}^\circ \sim \hat{d}^\circ \) by definition. \( \square \)

Let \( \Phi \) be a function from \( \text{Rec}(G) \) to \( \text{Rec}(G,s) \) defined as:

\[
\Phi(c) := \hat{c}^\circ,
\]

and note that the image of \( \Phi \) is contained in \( \text{Rec}(G,s) \) because of the first part of Lemma 3.3. The function \( \bar{\Phi} : \text{Rec}(G,\sim) \rightarrow \text{Rec}(G,\sim) \) is then defined as:

\[
\bar{\Phi}([c]) := [\Phi(c)],
\]

for all equivalence classes \([c]\) in \( \text{Rec}(G,\sim) \), and \( \bar{\Phi} \) is well defined due to the second part of Lemma 3.3.

We now take a slight detour to prove a proposition that may look mysterious but will be a key ingredient in the proof of the next lemma.

Proposition 3.4. Suppose that \( c \) is a recurrent configuration. Then \( c \sim (\varphi \circ \Phi)(c) + t\mathbf{1}_s \) for some non-negative number \( t \) and \( \text{lvl}(c) \geq \text{slvl}(\Phi(c)) \).

Proof. Let \( \bar{c} := \Phi(c) = \hat{c}^\circ \), then in particular we have \( \bar{c} \rightarrow_s \bar{c} \) and by Proposition 2.7 (i), this implies that \( c \rightarrow \bar{c} + k\mathbf{1}_s \) for some \( k \in \mathbb{N} \). Since \( c \) is recurrent, by Proposition 2.2(ii) \( \bar{c} + k\mathbf{1}_s \) is also recurrent and \( \bar{c} \sim \bar{c} + k\mathbf{1}_s \). Now note that by by Lemma 3.3, \( \bar{c} = \Phi(c) \) is an \( s \)-recurrent configuration and thus \( \chi(\bar{c}) \) is well defined, and since \( \bar{c} + k\mathbf{1}_s \) is recurrent, by the minimality of \( s \)-index we have \( \chi(\bar{c}) \leq k \). Set \( t := k - \chi(\bar{c}) \geq 0 \), and we have

\[
\bar{c} + k\mathbf{1}_s = \bar{c} + \chi(\bar{c})\mathbf{1}_s + (k - \chi(\bar{c}))\mathbf{1}_s = \varphi(\bar{c}) + t\mathbf{1}_s = (\varphi \circ \Phi)(c) + t\mathbf{1}_s,
\]

and since we have concluded that \( \bar{c} \sim \bar{c} + k\mathbf{1}_s \), this implies that \( c \sim (\varphi \circ \Phi)(c) + t\mathbf{1}_s \).

For the second part of the claim, it can be checked that

\[
\text{lvl}(c) = \text{lvl}((\varphi \circ \Phi)(c) + t\mathbf{1}_s) = \text{lvl}((\varphi \circ \Phi)(c)) + t = \text{slvl}(\Phi(c)) + t \geq \text{slvl}(\Phi(c)),
\]

as desired. \( \square \)

We now proceed to prove the claim made in the beginning of this subsection.

Lemma 3.5. The set \( \text{Rec}_n(G,\sim) \) is in bijection with \( \text{Rec}_n(G,\sim^\circ) \) via the function \( \bar{\Phi} \).

Proof. First, we need to show that the image of \( \text{Rec}_n(G,\sim) \) by \( \bar{\Phi} \) is contained in \( \text{Rec}_n(G,\sim^\circ) \). This follows immediately from Proposition 3.4, as given any element \([c] \in \text{Rec}_n(G,\sim) \), we have \( \text{slvl}(\bar{\Phi}([c])) = \text{slvl}(\Phi(c)) \leq \text{lvl}(c) = \text{lvl}([c]) = n \) as desired.
Next, we show that $\tilde{\Phi}$ is injective. Let $[c], [d] \in \text{Rec}_n(G, \sim)$, and suppose that $\tilde{\Phi}([c]) = \tilde{\Phi}([d])$. By Proposition 3.4 we have $c \sim (\varphi \circ \Phi)(c) + t_11_s$ for some $t_1 \in \mathbb{N}$ and $d \sim (\varphi \circ \Phi)(d) + t_21_s$ for some $t_2 \in \mathbb{N}$, so we can conclude that

$$t_1 = \text{lvl}(c) - \text{lvl}((\varphi \circ \Phi)(c)); \quad t_2 = \text{lvl}(d) - \text{lvl}((\varphi \circ \Phi)(d)).$$

Since $\tilde{\Phi}([c]) = \tilde{\Phi}([d])$, we have $\Phi(c) \sim \Phi(d)$ and by Lemma 3.2 this means that $(\varphi \circ \Phi)(c) \sim (\varphi \circ \Phi)(d)$ and in particular $\text{lvl}((\varphi \circ \Phi)(c)) = \text{lvl}((\varphi \circ \Phi)(d))$. Also note that $\text{lvl}(c) = \text{lvl}(d) = n$ since $[c]$ and $[d]$ is contained in $\text{Rec}_n(G, \sim)$. Together with Equation 3, we conclude that

$$t_1 = \text{lvl}(c) - \text{lvl}((\varphi \circ \Phi)(c)) = \text{lvl}(d) - \text{lvl}((\varphi \circ \Phi)(d)) = t_2.$$

Since $t_1 = t_2$ and $(\varphi \circ \Phi)(c) \sim (\varphi \circ \Phi)(d)$, we can then conclude that:

$$c \sim (\varphi \circ \Phi)(c) + t_11_s = (\varphi \circ \Phi)(c) + t_21_s \sim (\varphi \circ \Phi)(d) + t_21_s \sim d,$$

and thus $[c] = [d]$, which completes the proof for the injectivity.

Lastly, we show surjectivity by showing that for any $[\tilde{c}] \in \text{Rec}_\leq_n(G, \hat{\sim})$, there exists $[e] \in \text{Rec}_n(G, \sim)$ such that $\tilde{\Phi}([e]) = [\tilde{c}]$. Let $m \leq n$ be the s-level of $\tilde{c}$ and let $e := \varphi(\tilde{c}) + (n - m)1_s$. Note that $e$ has level equal to $n$ and by Proposition 2.2(ii) $e$ is a recurrent configuration since $\varphi(\tilde{c})$ is recurrent. So we have $[e] \in \text{Rec}_n(G, \sim)$, and note that we have:

$$\Phi(e) = \Phi(\varphi(\tilde{c}) + (n - m)1_s) = \Phi(\tau + (n + \chi(\tilde{c}) - m)1_s) = \tilde{c}^s = \tilde{c},$$

where last equality comes from the fact that $\tilde{c}$ is s-recurrent and thus stable. This shows that $\tilde{\Phi}([e]) = [\tilde{c}]$, which completes the proof of surjectivity.

\section{4. Proof of the conjecture of Perrot and Pham}

For a strongly connected digraph $G$, the generating function $\mathcal{R}(G; y)$ is the formal power series defined as

$$\mathcal{R}(G; y) := \sum_{n \geq 0} |\text{Rec}_n(G, \sim)| \cdot y^n,$$

we remark that the sum above is an infinite sum as $|\text{Rec}_n(G, \sim)| \geq |\text{Rec}_m(G, \sim)|$ for $n \geq m$ by Lemma 3.5. The generating function $\mathcal{P}(G, s; y)$ is defined as

$$\mathcal{P}(G, s; y) := \sum_{n \geq 0} |\text{Rec}_n(G, \hat{\sim})| \cdot y^n,$$

we remark that the sum for $\mathcal{P}(G, s; y)$ is a finite sum as $\text{Rec}(G, \hat{\sim})$ is a finite set as observed in Section 2, so $\mathcal{P}(G, s; y)$ is a polynomial.

\textbf{Theorem 4.1.} Let $G$ be a strongly connected digraph and $s$ be a vertex of $G$. We have the following equality of formal power series:

$$\mathcal{R}(G; y) = \frac{\mathcal{P}(G, s; y)}{(1 - y)}.$$

In particular, the left-hand side does not depend on the choice of $s$, and thus the polynomial $\mathcal{P}(G, s; y)$ is independent of the choice of $s$. 
Proof. By Lemma 3.5 we have
\[ |\text{Rec}_n(G, \sim)| = |\text{Rec}_{\leq n}(G, \sim)| = \sum_{m=0}^{n} |\text{Rec}_m(G, \sim)|. \]

Plugging in that relation into the definition of \( \mathcal{P}(G, s; y) \) and \( \mathcal{R}(y) \), we get
\[
\frac{\mathcal{P}(G, s; y)}{(1 - y)} = \frac{\sum_{m \geq 0} |\text{Rec}_m(G, \sim)| \cdot y^m}{(1 - y)}
\]
\[
= \left( \sum_{m \geq 0} |\text{Rec}_m(G, \sim)| \cdot y^m \right) \cdot \left( \sum_{k \geq 0} y^k \right)
\]
\[
= \sum_{n \geq 0} \left( \sum_{m=0}^{n} |\text{Rec}_m(G, \sim)| \right) y^n = \sum_{n \geq 0} |\text{Rec}_n(G, \sim)| \cdot y^n = \mathcal{R}(G; y),
\]
as desired. \( \square \)

In particular, Theorem 4.1 answers Conjecture 1.1, and the proof is outlined in the rest of this section.

For an element \( \hat{c} \in \text{Rec}(G, s) \), the \textbf{d-level} of \( \hat{c} \) is defined as
\[
dlvl(\hat{c}) := \deg_{\overline{G}}(s) + \sum_{v \in V \setminus \{s\}} \hat{c}(v).
\]

We remark that it is generally not true that \( dlv\bar{\text{v}}(\hat{c}) = dlv\bar{\text{v}}(\hat{d}) \) if \( \hat{c} \sim \hat{d} \). For example, let \( G \) be the digraph in Figure 1.

\[
\begin{tikzpicture}
  \node (v1) at (0,0) {1};
  \node (v2) at (1,0) {2};
  \node (v3) at (2,0) {3};
  \draw (v1) -- (v3);
  \draw (v3) -- (2,1.5);
  \draw (2,1.5) -- (v2);
  \draw (v2) -- (2,1.5);
\end{tikzpicture}
\]

\[ \Delta = \begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix} \]

\text{Figure 1. Example.}

Take \( v_1 \) as the sink, it is easy to check by Lemma 2.5 that the \( v_1 \)-configurator \( \hat{c} \) with \( \hat{c}(v_2) = 1 \) and the \( v_1 \)-configuraton \( \hat{d} \) with \( \hat{d}(v_2) = 2 \) are both \( v_1 \)-recurrent configurations in \( \text{Sand}(G, v_1) \). We also have \( \hat{c} \sim \hat{d} \), as we can get \( \hat{d} \) by adding one chip to \( \hat{c} \) at vertex \( v_2 \). Now notice that the d-level of \( \hat{c} \) is equal to \( \deg_{\overline{G}}(v_1) + \hat{c}(v_2) = 3 \) while the d-level of \( \hat{d} \) is equal to \( \deg_{\overline{G}}(v_1) + \hat{d}(v_2) = 4 \).

The notion of d-level of an equivalence class \([\hat{c}]\) in \( \text{Rec}(G, \sim) \) is thus defined by taking the maximum d-level among s-configurations in \([\hat{c}]\), or formally
\[
dlv\bar{\text{v}}([\hat{c}]) := \max_{\hat{c} \in [\hat{c}]} dlv\bar{\text{v}}(\hat{c}),
\]
for all \([\hat{c}] \in \text{Rec}(G, \sim)\). The \textbf{Biggs-Merino} polynomial \( B(G, s; y) \), which was due to Perrot and Pham [PV13] in the setting of strongly-connected digraphs, is defined by
\[
B(G, s; y) := \sum_{[\hat{c}] \in \text{Rec}(G, \sim)} y^{dlvl([\hat{c}])},
\]
we remark that the sum in Equation 5 is a finite sum as \( \text{Rec}(G, \sim) \) is a finite set.
Note that as observed in Section 2 for a connected Eulerian digraph $G$ each equivalence class $[c]$ in $\text{Rec}(G, \sim)$ contains exactly one $s$-recurrent configuration, so in this case the right hand side of Equation 5 becomes

$$\sum_{c \in \text{Rec}(G, s)} y^{\text{div}(\tilde{c})},$$

which was how the polynomial was originally defined by Biggs [Big97a] for undirected graphs. However, the polynomial in Equation 6 does depend on the choice of sink when $G$ is not Eulerian, so Perrot and Pham proposed to use the definition in Equation 5 instead as they conjecture that the expression in Equation 5 is independent of the choice of sink. Formally, their conjecture is as follows:

**Conjecture 1.1.** ([PV13, Conjecture 1]). For a strongly connected digraph $G$ the polynomial $B(G, s; y)$ is independent of the choice of vertex $s$.

In the rest of this section we prove Conjecture 1.1 by showing that $B(G, s; y)$ is equal to the polynomial $P(G, s; y)$, which we have shown to be independent of the choice of $s$ in Theorem 4.1. This is done by proving that for any $[c] \in \text{Rec}(G, \sim)$ the $s$-level and $d$-level of $[c]$ is equal, and the following proposition will be a key ingredient in the proof.

**Proposition 4.2.** Suppose that $\tilde{c}$ is an $s$-recurrent configuration and let $c := \varphi(\tilde{c})$. Then there exists a finite sequence of legal firing moves $c = c_0 \rightarrow c_1 \rightarrow \cdots \rightarrow c_k = c$ that sends $c$ back to itself which satisfies the following properties:

(i) Each vertex $v \in V$ is fired exactly $r(v)$ times.

(ii) If $v_i = s$ for some $i \leq k$, then $c_{i-1}(v) < \text{deg}_G^+(v)$ for all $v \in V \setminus \{s\}$.

(iii) There exists $i \leq k$ such that $v_i = s$ and $c_{i-1}(s) = \text{deg}_G^-(s)$.

**Proof.** Since $c = \varphi(\tilde{c})$, then by definition of $\varphi$ we have $c$ is a recurrent configuration and by Lemma 2.1 there exists a finite sequence of legal firing moves $c = c_0 \rightarrow c_1 \rightarrow \cdots \rightarrow c_k = c$ that sends $c$ back to itself and each vertex $v$ is fired exactly $r(v)$ times, so we have a sequence that satisfies property (i), and note that $k$ is equal to $\sum_{v \in V} r(v)$ and does not depend on the sequence.

For a given sequence of legal firing moves that satisfies property (i), let $i$ be the smallest natural number such that $v_i = s$ and there exists $v \in V \setminus \{s\}$ such that $c_{i-1}(v) \geq \text{deg}_G^+(v)$ and the vertex $v$ has not been fired for $r(v)$ times during the first $i$ moves. We refer to $i$ as the degeneracy index of the sequence, and if such an $i$ does not exist, then we say that the sequence is non-degenerate.

By observation above we have a sequence that satisfies property (i), and suppose that this sequence is degenerate. Let $i$ be the degeneracy index of this sequence, then there exists $v_j \in V \setminus \{s\}$ and $j > i$ such that $v_i = s$, $c_{i-1}(v_j) \geq \text{deg}_G^+(v)$, and $v_j$ has not been fired for $r(v)$ times during the first $i$ moves. Thus we can change the order of the sequence by firing $v_j$ as the $i$-th move instead while maintaining the rest of the sequence of the firing moves, i.e.

$$c = c_0 \rightarrow \cdots \rightarrow c_i \rightarrow c_i' \rightarrow c_{i+1}' \rightarrow \cdots \rightarrow c_j' \rightarrow \cdots \rightarrow c_k = c.$$

Note that the new sequence of firing moves is also a sequence of legal firing moves that satisfies property (i), and furthermore the degeneracy index of this sequence is now strictly bigger than the previous sequence.

Thus, we can keep doing this reordering of legal firing moves until we get a non-degenerate sequence that satisfies property (i). This non-degenerate finite sequence of legal firing moves has
the property that if \( v_i = s \) for some \( i \leq k \), then for all \( v \in V \setminus \{s\} \) either \( c_{i-1}(v) < \deg^+_G(v) \) or \( v \) has been fired for \( r(v) \) times.

Note that if \( v \in V \setminus \{s\} \) and \( v \) has been fired for \( r(v) \) times before the \( i \)-th firing move for some \( i \leq k \), then this means that the vertex \( v \) will not lose any chips for the rest of the firing moves as \( v \) will not be fired. This means that \( c_{i-1}(v) \leq c_k(v) = c(v) \), and note that since \( \hat{c} \) is s-recurrent and thus stable, we have \( c(v) = \hat{c}(v) < \deg^+_G(v) \) and thus this implies \( c_{i-1}(v) \leq c(v) < \deg^+_G(v) \). Combined with the observation that this sequence is non-degenerate, we then conclude that we have a finite sequence of firing moves such that if \( v_i = s \) for some \( i \leq k \), then for each \( v \in V \setminus \{s\} \) we have \( c_{i-1}(v) < \deg^+_G(v) \), and property (ii) is thus satisfied.

We now claim that for any finite sequence of legal firing moves for \( c = \varphi(\hat{c}) \) that satisfies (i), there exists \( i \) such that \( v_i = s \) and \( c_{i-1}(s) = \deg^+_G(s) \). Suppose that such an \( i \) does not exist, then this means that every time \( s \) is fired during the sequence of legal firing moves, the vertex \( s \) has at least \( \deg^+_G(s) + 1 \) chips. In that case then we can reduce one chip from vertex \( s \) of the configuration \( c \) and the given sequence of firing moves that we have is also legal for \( c - 1_s \). This means that there exists a sequence of legal firing moves from \( c - 1_s \) back to itself and such that each vertex \( v \in V \) is fired \( r(v) \) times. By Lemma 2.1 this implies that \( c - 1_s = \mathcal{G} + (\hat{\chi}(\hat{c}) - 1)1_s \) is recurrent, contradicting the minimality of \( \chi(\hat{c}) \). So there exists \( i \leq k \) such that \( v_i = s \) and \( c_{i-1}(s) = \deg^+_G(s) \). In particular, this shows that any finite sequence of legal firing moves that satisfies property (i) and (ii) will also satisfy property (iii), and the proof is complete. \( \square \)

We now proceed to show that s-level is equal to d-level for each equivalence class \([c] \in \text{Rec}(G, \sim)\).

**Lemma 4.3.** For each \([\hat{c}] \in \text{Rec}(G, \sim)\), we have \( \text{slvl}(\hat{c}) = \text{dlvl}(\hat{c}) \).

**Proof.** First we show that \( \text{slvl}(\hat{c}) \geq \text{dlvl}(\hat{c}) \) for all \([\hat{c}] \in \text{Rec}(G, \sim)\). We can without loss of generality assume that \( \hat{c} \) is an element in \([c] \) such that \( \text{dlvl}(\hat{c}) \) is maximum, and in particular this implies \( \text{dlvl}(\hat{c}) = \text{dlvl}(c) \). Now note that by definition \( \text{slvl}(\hat{c}) = \chi(\hat{c}) + \sum_{v \neq s} \hat{c}(v) \), and also note that \( \chi(\hat{c}) \geq \deg^+_G(s) \) as has been observed in the Section 3. Thus we can conclude that:

\[
\text{sln}(\hat{c}) = \chi(\hat{c}) + \sum_{v \neq s} \hat{c}(v) \geq \deg^+_G(s) + \sum_{v \neq s} \hat{c}(v) = \text{dlvl}(\hat{c}) = \text{dlvl}(c).
\]

We now show that \( \text{sln}(\hat{c}) \leq \text{dlvl}(\hat{c}) \) for all \([\hat{c}] \in \text{Rec}(G, \sim)\). It suffices to show that there exists an s-recurrent configuration \( \hat{d} \) contained in the equivalence class \([\hat{c}] \) such that \( \chi(\hat{d}) \leq \deg^+_G(s) \), as this will then imply:

\[
\text{sln}(\hat{c}) = \text{sln}(\hat{d}) = \chi(\hat{d}) + \sum_{v \neq s} d(v) \leq \deg^+_G(s) + \sum_{v \neq s} d(v) = \text{dlvl}(\hat{d}) \leq \text{dlvl}(\hat{c}).
\]

Let \( c := \varphi(\hat{c}) \), and let \( c = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} \ldots \xrightarrow{v_k} c_k = c \) be the sequence satisfying the three properties in Proposition 4.2. By Property (ii) and (iii) of the sequence, let \( i \) be a natural number such that \( c_{i-1}(v) < \deg^+_G(v) \) for all \( v \in V \setminus \{s\} \) and \( c_{i-1}(s) = \deg^+_G(s) \).

This implies \( c_{i-1}(v) = c_{i-1}(v) < \deg^+_G(v) \) for all \( v \in V \setminus \{s\} \), so \( \hat{c}_{i-1} \) is stable s-configuration and \( \hat{c}_{i-1} = \hat{c}_{i-1} \). On the other hand, since we have \( c \xrightarrow{v_i} c_{i-1} \), by Proposition 2.7(ii) there exists \( m \in \mathbb{N}_0 \) such that \( c + m \hat{u} \xrightarrow{v_i} \hat{c}_{i-1} \), and by Lemma 2.3 this implies \( (c + m \hat{u})^0 = \hat{c}_{i-1}^0 = \hat{c}_{i-1} \).

By Proposition 2.6 since \( (c + m \hat{u})^0 = \hat{c}_{i-1} \) and \( \hat{c} \) is s-recurrent, we have \( \hat{c}_{i-1} \) is s-recurrent, and by definition we also have \( \hat{c} \sim \hat{c}_{i-1} \). On the other hand, we have \( c_{i-1} = \hat{c}_{i-1} + \deg^+_G(s)1_s \) is a recurrent configuration, so we can conclude that \( \chi(\hat{c}_{i-1}) \leq \deg^+_G(s) \) by the minimality of \( \chi \).
Now take $\hat{d} := \hat{c}_{i-1}$, we have $\hat{d}$ is an $s$-recurrent configuration, $\hat{c} \sim \hat{d}$, and $\chi(\hat{d}) \leq \deg^+_G(s)$, as desired. \qed

Conjecture 1.1 now follows as a corollary of Theorem 4.1 and Lemma 4.3.

**Theorem 4.4.** Let $G$ be a strongly connected digraph and $s$ be a vertex of $G$. We have the following equality of formal power series:

$$B(G; s; y) = P(G; s; y) = (1 - y)R(G; y).$$

As the right hand side is independent of the choice of sink, the polynomial $B(G; s; y)$ is also independent of the choice of $s$.

**Proof.** The second equality follows immediately from Theorem 4.1, so it suffices to show that $P(G; s; y) = B(G; s; y)$. By Lemma 4.3, for any elements $[\hat{c}] \in \Rec(G, \hat{z})$ we have $\slvl([\hat{c}]) = \slvl([\hat{c}])$, and thus we can conclude that

$$B(G; s; y) = \sum_{[\hat{c}] \in \Rec(G, \hat{z})} y^{\slvl([\hat{c}])} = \sum_{[\hat{c}] \in \Rec(G, \hat{z})} y^{\slvl([\hat{c}])} = P(G; s; y),$$

as desired. \qed

5. **Examples**

In this section we give an example of Biggs-Merino polynomial for a family of non-Eulerian digraphs $G(n; a, b)$. Let $n, a, b$ be positive integers and let $G := G(n; a, b)$ be the digraph where $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $a$ edges from $v_i$ to $v_{i+1}$ and $b$ edges from $v_{i+1}$ to $v_i$ for $1 \leq i \leq n - 1$. Let $d := \gcd(a, b)$ and we can without loss of generality choose $v_1$ as the sink vertex by Theorem 4.4.

![Figure 2. The digraph $G(d; a, b)$.](image)

It is easy to check that the number of reverse arborescences of $G$ rooted at $v_i$, which is equal to the determinant of reduced Laplacian matrix $\Delta_{v_i}$, is equal to $b^{n-i}a^{i-1}$ for $1 \leq i \leq n$. Assume that $d = 1$, this implies that the period constant of $G$ is equal to $\alpha = \gcd_{1 \leq i \leq n} (\det \Delta_{v_i}) = 1$. Note that we have the size of $\Rec(G, \hat{z})$ is equal to period constant $\alpha$ (Proposition A.4), so we have $|\Rec(G, \hat{z})| = 1$. Note that by the definition of Biggs-Merino polynomial we have $B(G, v_1; 1) = |\Rec(G, \hat{z})| = 1$. So if $d = \gcd(a, b) = 1$ then we have $B(G, v_1; y)$ is a monomial.

Let $\hat{c}$ be the $v_1$-configuration with $\hat{c}(v_i) = \deg^+_G(v_i) - 1$ for $2 \leq i \leq n$. It is easy to check that $\hat{c}$ passes the burning test in Lemma 2.5 and $\hat{c}$ is $v_1$-recurrerent. Also note that any stable $v_1$-configuration $\hat{d}$ will have $\hat{d}(v_i) \leq \deg^+_G(v_i) - 1 = \hat{c}(v_i)$ for $2 \leq i \leq n$, so $\hat{c}$ is the stable $v_1$-configuration with the highest $d$-level. Thus we can conclude that for $d = 1$ we have Biggs-Merino polynomial to be equal to

$$B(G(n; a, b), v_1; y) = y^{\slvl([\hat{c}])} = y^{(n-1)(a+b-1)}.$$

To derive the Biggs-Merino polynomial of $G(n; a, b)$ in general case, we need the following lemma.
Lemma 5.1. Let $G$ be a strongly connected digraph, $d$ be a positive natural number, and $G^d$ be the digraph with $V(G^d) = V(G)$ and for each edge $e \in E(G)$ there exists $d$ copies of $e$ in $E(G^d)$. Then the Biggs-Merino polynomial of $G^d$ is equal to

$$B(G^d, s; y) = B(G, s; y^d) \cdot \left( \frac{1 - y^d}{1 - y} \right)^{|V(G)| - 1}.$$ 

Proof. We can observe that firing a vertex $v$ in $G$ is similar in firing a vertex $v$ in $G^d$, except that for each chips transferred in $G$ we have $d$ chips is transferred in $G^d$. For any chip configurations $c, d$ in $G^d$, we say that $c$ is of type $(t_v)_{v \in V}$, where $0 \leq t_v < d$ for all $v \in V$, if $c(v) \equiv t_v \mod d$ for all $v \in V$.

Define the mapping $\pi : \mathbb{N}_0^{V(G^d)} \to \mathbb{N}_0^{V(G)}$ by mapping a configuration $c$ in $\text{Sand}(G^d)$ to the configuration $\pi(c)$ in $\text{Sand}(G)$ by setting $\pi(c)(v) := \lfloor \frac{c(v)}{d} \rfloor$ for all $v \in V$. By the observation that transferring a chip in $G$ is equivalent to transferring $d$ chips in $G^d$, we conclude that a configuration $c$ in $\text{Sand}(G^d)$ passes the burning test in Lemma 2.1 if and only if the configuration $\pi(c)$ passes the same burning test in $\text{Sand}(G)$. Hence $c$ is recurrent in $\text{Sand}(G^d)$ if and only if $\pi(c)$ is recurrent in $\text{Sand}(G)$. By the same observation, a firing move in $\text{Sand}(G^d)$ does not change the type of the configuration, so we conclude that two configurations $c, d$ in $\text{Sand}(G^d)$ satisfies $c \sim d$ if and only if $c$ and $d$ is of the same type and $\pi(c) \sim \pi(d)$ in $\text{Sand}(G)$.

Note that for a configuration $e$ in $\text{Sand}(G)$, the pre-image of $e$ by $\pi$ is equal to

$$\pi^{-1}(e) = \{ c \in \mathbb{N}_0^{V(G^d)} \mid c(v) = d \cdot e(v) + t_v \text{ for all } v \in V(G^d), \ t_v = 0, 1, \ldots, d - 1 \}.$$ 

Together with the observation made above, we have for each equivalence class $[e] \in \text{Rec}(G, \sim)$ and a type $(t_v)_{v \in V}$ in $\text{Sand}(G^d)$, there exists a unique equivalence class $[c]$ in $\text{Rec}(G^d, \sim)$ containing configurations of the given type and such that $\pi$ maps $[c]$ to $[e]$, and note that $\text{lvl}([c]) = d \cdot \text{lvl}([e]) + \sum_{v \in V(G)} t_v$. Hence we can conclude that

$$\mathcal{R}(G^d; y) = \sum_{[e] \in \text{Rec}(G^d, \sim)} y^{\text{lvl}([e])} = \sum_{[e] \in \text{Rec}(G, \sim)} y^{d \cdot \text{lvl}([e])} \cdot \sum_{0 \leq t_v < d, \ v \in V} y^{\sum_{v \in V(G)} t_v} = \sum_{[e] \in \text{Rec}(G, \sim)} y^{d \cdot \text{lvl}([e])} \cdot (1 + y + \ldots + y^{d - 1})|V(G)| = \mathcal{R}(G; y^d) \cdot \left( \frac{1 - y^d}{1 - y} \right)^{|V(G)|},$$

Substituting $\mathcal{R}(G^d; y) = \frac{B(G^d, s; y)}{1 - y}$ and $\mathcal{R}(G; y^d) = \frac{B(G, s; y^d)}{1 - y^d}$, we can then conclude that

$$B(G^d, s; y) = B(G, s; y^d) \cdot \left( \frac{1 - y^d}{1 - y} \right)^{|V(G)| - 1},$$

as desired. \(\square\)
We can now compute the Biggs-Merino polynomial of $G(n; a, b)$ in general. By Lemma 5.1, the Biggs-Merino Polynomial of $G(n; a, b)$ is equal to

$$B(G(n; a, b), v_1; y) = B(G(n; a/b, b/d), v_1; y^d) \cdot \left(\frac{1 - y^d}{1 - y}\right)^{n-1}$$

$$= y^{(n-1)(a+b-d)} \cdot \left(\frac{1 - y^d}{1 - y}\right)^{n-1}.$$ 

A similar argument can be applied to a more general family of digraphs: Let $G$ be a strongly-connected digraph with period constant $\alpha = 1$ and $d$ a positive natural number, then the digraph $G^d$ has Biggs-Merino polynomial

$$B(G^d, s; y) = y^{d(|E(G)|-|V(G)|+1)} \left(\frac{1 - y^d}{1 - y}\right)^{|V(G)|-1}.$$ 

6. Connections to greedoid Tutte polynomial

We start this section by introducing greedoid Tutte polynomial and $G^r$-parking function, which will be the main objects of interest in this section.

6.1. Greedoid Tutte polynomial. Let $G = (V, E)$ be a directed graph with total order $<_e$ on the edges of $G$. A directed path $P$ of $G$ of length $k \geq 0$ is a sequence $v_0, e_0 v_1 e_1 \ldots e_{k-1} v_k$ where $v_0, \ldots, v_k \in V$, $e_0, \ldots, e_{k-1} \in E$, $e_i \neq v_j$ for distinct $i$ and $j$, and $e_i = (v_i, v_{i+1})$ for $0 \leq i \leq k - 1$. The total order $<_e$ can be extended to compare two distinct edge-disjoint directed paths $P_1$ and $P_2$, in which we have $P_1 <_e P_2$ if the smallest edge with respect to $<_e$ in $E(P_1) \cup E(P_2)$ is contained in $E(P_1)$.

An arborescence $T$ of $G$ rooted at $s \in V(G)$ is a subgraph $G$ such that for any $v \in V(G)$ there exists a unique directed path from $s$ to $v$. For each edge $e \in E(G) \setminus E(T)$, there are exactly two edge-disjoint directed paths $P_1 = v_0 e_0 \ldots e_{k-1} v_k$ and $P_2 = w_0 f_0 \ldots f_{l-1} w_l$ in $(V(G), E(T) \cup \{e\})$ that starts at the same vertex $v_0 = w_0$ and ends at the same vertex $v_k = w_l = t(e)$. We adopt the convention that $P_1$ is the path that contains $e$ (i.e. $e = e_{k-1}$), and we say that $e \in E(G) \setminus E(T)$ is externally active if $P_1 <_e P_2$. The external activity of the arborescence $T$ is the number of externally active edges in $T$ and is denoted by $e(T)$.

![Figure 3. P1 and P2 in T.](image-url)

Let $A(G, s)$ denote the set of arborescences of $G$ rooted at $s$. The single variable greedoid Tutte polynomial rooted at $s$, or simply greedoid Tutte polynomial, is the polynomial

$$T(G, s; y) := \sum_{T \in A(G, s)} y^{e(T)},$$

(7)
and the polynomial $T(G, s; y)$ does not depend on the choice of total order $<_r$ [BKL85, Theorem 6.1]. The greedoid Tutte polynomial was introduced by Björner, Korte, and Lovász [BKL85] for a more general family of combinatorial objects called greedoids, and the polynomial in Equation 7 is the greedoid Tutte polynomial of a specific class of greedoids called the branching greedoid of the digraph $G$. The reader is referred to [BZ92] for a general introduction to greedoids and related topics and [GM89, GT90, GM01] for a more detailed study on the greedoid Tutte polynomial.

The connection between greedoid Tutte polynomial and the Tutte polynomial for undirected graph is as follows: Let $G$ be a loopless connected undirected graph, and $G$ be the digraph with same vertex set as $G$ and with edge set obtained by replacing each edge $e := (v, w) \in E(G)$ of $G$ with a directed 2-cycle from $v$ to $w$ in $G$. The greedoid Tutte polynomial $T(G, s; y)$ of $G$ is equal to $y^{|E(G)|}T(G; 1, y)$, where $T(G, x, y)$ is the standard Tutte polynomial of $G$. A proof of this observation is included in Appendix B.

6.2. G-Parking function. For a digraph $G$, the G-parking function rooted at $s \in V$ is a function $f : V \setminus \{s\} \to \mathbb{N}_0$ with the property that for any non-empty subset $A \subseteq V \setminus \{s\}$ there exists $v \in A$ such that $f(v)$ is smaller than the number of edges from $v$ to $V \setminus A$. Parking function was originally defined by Konheim and Weiss [KW66] for complete graphs, and the notion of parking function for general directed graphs was introduced by Postnikov and Shapiro [PS04]. When $G$ is undirected, the $G$-parking function is also known as reduced divisors [BS13] and super-stable configurations [HLM+08].

Given a digraph $G$, let $G'$ be the graph obtained from $G$ by reversing the edge orientation of all edges of $G$. The $G'$-parking function rooted at $s \in V$ is then defined as the $G$-parking function for the digraph $G'$, or formally it is a function $f : V \setminus \{s\} \to \mathbb{N}_0$ such that for any non-empty subset $A \subseteq V \setminus \{s\}$ there exists $v \in A$ such that $f(v)$ is smaller than the number of edges from $V \setminus A$ to $v$. We use $\text{Park}(G, s)$ to denote the set of $G'$-parking functions rooted at $s$.

In this paper we will work with $G'$-parking function instead of $G$-parking function, for a reason that will be apparent soon. For an $s$-configuration $c$ in $\text{Sand}(G, s)$, the function associated to $c$ is the function $f : V \setminus \{s\} \to \mathbb{N}_0$ with $f(v) := \deg^+_G(v) - 1 - c(v)$ for all $v \in V \setminus \{s\}$, and vice versa.

**Lemma 6.1.** ([HLM+08, Theorem 4.4]). Let $G$ be a connected Eulerian digraph. Then an s-configuration $c$ is s-recurrent if and only if the function $f$ associated to $c$ is a $G'$-parking function.

We remark that for a general non-Eulerian digraph $G$, the duality between s-recurrent configurations and $G'$-parking functions in Lemma 6.1 fails to hold for both directions, as shown by the next two examples. For the first example, let $G$ be the digraph in Figure 4.

$$
\Delta = \begin{pmatrix}
1 & 0 & -4 \\
-1 & 2 & 0 \\
0 & -2 & 4
\end{pmatrix}
$$

**Figure 4.** Example 1.

Take $v_1$ as the sink vertex, it can be checked by Lemma 2.5 that the configuration $c \in C(G, v_1)$ where $c(v_2) = 1$ and $c(v_3) = 0$ is $v_1$-recurrent but the associated function $f$ with $f(v_2) = 0$ and $f(v_3) = 3$ is not a $G'$-parking function because for $A = \{v_3\}$ we have $f(v_3) = 3$ while there are only two edges from $V \setminus A$ to $v_3$. 

The following conditions:

We remark that those two conditions imply \( \sigma \) exactly once. We use \( \sigma \) and \( \Delta \) element \( \sigma \) of \( \Delta \) to denote the set of \( s \)-decreasing traversals of \( G \), \( s \)-recurrent configurations of \( G \) to denote the set of \( s \)-decreasing traversals of \( G \), and \( s \)-recurrent configurations being substituted by arborescences, and \( s \)-recurrent configurations being substituted by \( G^s \)-parking functions. In this subsection we give an implementation of Cori-Le Borgne algorithm in the setting \( G \) of \( G \) that preserves the spanning tree activities. Our observation is that their algorithm actually shows something more general: it applies to any directed graph \( G \), with spanning trees being substituted by arborescences, and \( s \)-recurrent configurations being substituted by \( G^s \)-parking functions. In this subsection we give a proof that when \( G \) is a connected Eulerian digraph, the Biggs-Merino polynomial \( B(G; s; y) \) is equal to the greedoid Tutte polynomial \( T(G, s; y) \) of \( G \). The proof of the claim uses a modification of Cori-Le Borgne algorithm [CLB03], which was originally used to construct a bijection between the set of spanning trees of \( G \) and \( s \)-recurrent configurations of \( G \) that preserves the spanning tree activities. Our observation is that their algorithm actually shows something more general: it applies to any directed graph \( G \), with spanning trees being substituted by arborescences, and \( s \)-recurrent configurations being substituted by \( G^s \)-parking functions. In this subsection we give an implementation of Cori-Le Borgne algorithm in the setting of directed graphs, with some modifications to the original approach in [CLB03] to account for the difference between the notion of external activities of arborescences and spanning trees.

Let \( G \) be a digraph with a total order \(<,\) on the edges of \( G \). An edge-vertex traversal \( \sigma := (\sigma_i)_{1 \leq i \leq |V| + |E|} \) of a digraph \( G \) is a sequence in which every vertex and edge of \( G \) appears exactly once. We use \( \sigma^{<i} := \{ \sigma_j, j < i \} \) to denote the set of elements that appears before the element \( \sigma_i \) in \( \sigma \).

For a fixed vertex \( s \in V \), an \( s \)-decreasing traversal \( \sigma \) is an edge vertex traversal that satisfies the following conditions:

(R1) If \( \sigma_i \) is contained in \( V \setminus \{s\} \), then \( \sigma_{i-1} \) is an edge with target vertex \( \sigma_i \).

(R2) If \( \sigma_i \) is an edge, then \( \sigma_i \) is the maximum edge with respect to \(<,\) among edges of \( G \) not contained in \( \sigma^{<i} \) and has source vertex contained in \( \sigma^{<i} \).

We remark that those two conditions imply \( \sigma_1 = s \) for all \( s \)-decreasing traversals \( \sigma \). We use \( \mathcal{D}(G, s) \) to denote the set of \( s \)-decreasing traversals of \( G \).

An edge \( e = \sigma_i \) in \( \sigma \) is strong if the target vertex \( t \) of \( e \) appears before \( e \) in \( \sigma \), or formally if \( t = \sigma_j \) for some \( j < i \). The strength of \( \sigma \), denoted by \( \text{str}(\sigma) \), is equal to the number of strong edges in \( \sigma \).

The following lemma will be helpful in the coming proofs.

**Lemma 6.2.** Let \( \sigma \) and \( \tau \) be two distinct \( s \)-decreasing traversals and \( k \) the minimal index such that \( \sigma_k \neq \tau_k \). Then one of \( \sigma_k \) and \( \tau_k \) is an edge and the other is a vertex.
Proof. Since \( k \) is minimal, we have \( \sigma <^k = \tau <^k \). If both \( \sigma_k \) and \( \tau_k \) are vertices, then by (R1) both \( \sigma_k \) and \( \tau_k \) are the target vertex of \( \sigma_{k-1} = \tau_{k-1} \), a contradiction. If both \( \sigma_k \) and \( \tau_k \) are edges, then by (R2) both \( \sigma_k \) and \( \tau_k \) are the maximum edge in \( E(G) \setminus \sigma <^k = E(G) \setminus \tau <^k \) and has source vertex contained in \( \sigma <^k = \tau <^k \), a contradiction. Hence one of \( \sigma_k \) and \( \tau_k \) is an edge while the other is a vertex. \( \square \)

Let \( \Upsilon \) be a function from \( D(G,s) \) to \( A(G,s) \) that maps an \( s \)-decreasing traversal \( \sigma \) to the arborescence \( \Upsilon(\sigma) \), where \( V(\Upsilon(\sigma)) := V(G) \) and \( E(\Upsilon(\sigma)) := \{ \sigma_{i-1} \mid \sigma_i \in V(G) \setminus \{s\} \} \).

Lemma 6.3. The mapping \( \Upsilon \) is a bijection between \( D(G,s) \) and \( A(G,s) \).

Proof. First, we show that \( \Upsilon \) maps an \( s \)-decreasing traversal \( \sigma \) to an arborescence of \( G \). It is easy to check that \( \Upsilon(\sigma) \) has \( |V(G)| - 1 \) edges, so it suffices to show that for any \( v \in V \) there exists a directed path from \( s \) to \( v \) in \( \Upsilon(\sigma) \). Note that by (R2) for any vertex \( v = \sigma_i \), there exists an edge in \( \Upsilon(\sigma) \) from a vertex in \( \sigma <^i \) to \( v \). By induction on \( i \), we conclude that there is a directed path from \( s = \sigma_1 \) to \( v \) in \( \Upsilon(\sigma) \).

Next, we show that \( \Upsilon \) is injective. Suppose that \( \sigma \) and \( \tau \) are two distinct \( s \)-decreasing traversals and \( k \) the minimal index such that \( \sigma_k \neq \tau_k \). Then by Lemma 6.2 we can assume that \( \sigma_k \) is a vertex while \( \tau_k \) is an edge. By definition of \( \Upsilon \), \( \sigma_{k-1} \) is an edge in \( \Upsilon(\sigma) \) but \( \tau_{k-1} \) is not an edge in \( \Upsilon(\tau) \), which shows that \( \Upsilon(\sigma) \) is not equal to \( \Upsilon(\tau) \).

Next, we show that \( \Upsilon \) is surjective. Let \( T \) be an arborescence of \( G \) rooted at \( s \), construct a decreasing traversal \( \sigma \) by setting \( \sigma_1 := s \), and then continue inductively on \( i \) as follows:

(i) If \( \sigma_i \) is an edge in \( E(T) \), then \( \sigma_{i+1} \) is the target vertex of \( \sigma_i \);

(ii) If \( \sigma_i \) is not an edge in \( E(T) \), then \( \sigma_{i+1} \) is the maximum edge with respect to \( <_e \) that is not contained in \( \sigma <^{i+1} \) and has source vertex contained in \( \sigma <^{i+1} \).

The construction gives us an edge-vertex traversal as \( T \) is an arborescence, and it is easy to check that \( \Upsilon(\sigma) = T \). Also note that \( \sigma \) is an \( s \)-decreasing traversal since \( \sigma \) obeys (R1) by (i) and obeys (R2) by (ii). This proves surjectivity. \( \square \)

Lemma 6.4. For any natural number \( n \), the mapping \( \Upsilon \) sends an \( s \)-decreasing traversal \( \sigma \) with strength equal to \( n \) to an arborescences \( T \) with external activities equal to \( n \).

Proof. It suffices to show that an edge \( e \) is strong in \( \sigma \) if and only if it is externally active in \( T \). First we made a useful observation about \( \sigma \) and \( T \). Suppose that \( \sigma_i \in V(T) \cup E(T) \), if \( \sigma_i \) is a vertex then by definition of \( \Upsilon \) the unique edge in \( E(T) \) with target vertex \( \sigma_i \) is contained in \( \sigma <^i \). On the other hand, if \( \sigma_i \) is an edge then by (R2) the source vertex of \( \sigma_i \) is contained in \( \sigma <^i \). Using those two observations, one can continue inductively to prove that if \( \sigma_i \in V(T) \cup E(T) \), then all vertices and edges contained in the unique directed path in \( T \) that starts at \( s \) and ends at \( \sigma_i \) is contained in \( \sigma <^i \).

Proof of \( \Rightarrow \) direction: Suppose that \( e = \sigma_i \) is strong in \( \sigma \) and \( e \) is not externally active in \( T \). Then since \( e \) is strong, by definition of \( \Upsilon \) we have \( e \) is not contained in \( E(T) \). Let \( P_1 = v_0 \sigma_0 \cdots e_{k-1} v_k \) and \( P_2 = w_0 f_0 \cdots f_{l-1} w_l \) be the two edge-disjoint directed paths in \( (V(T), E(T) \cup \{e\}) \) with \( v_0 = w_0, v_k = w_l = t \), and \( P_1 \) is the directed path that contains \( e \) i.e. \( e = e_{k-1} \).

Since \( \sigma_i \) is strong in \( \sigma \), we have \( t = v_k = w_l \) is contained in \( \sigma <^i \). Note that all vertices and edges in the path \( P_2 \) are contained in \( T \), so since \( w_l \) is contained in \( \sigma <^i \), by the observation in the beginning of the proof we have all edges and vertices in \( P_2 \) are contained in \( \sigma <^i \).

Since \( e \) is not externally active in \( T \), then the minimal edge \( f \) in \( E(P_1) \cup E(P_2) \) is contained in \( P_2 \). Then we have \( f = \sigma_j \) with \( j < i \). Note that the source vertex of \( e \) is not contained in \( \sigma <^i \), as if that is not the case then since \( f \) is smaller than \( e \) with respect to \( <_e \), by (R2) \( \sigma_j \neq f \) as \( e \) also...
satisfies the edge condition in (R2). So in particular there is a vertex in \( E(P_1) \) that is not contained in \( \sigma^{<j} \).

Let \( m \) be the minimal index such that \( v_m \in V(P_1) \) is not contained in \( \sigma^{<j} \), and note that by previous argument \( m < k - 1 \). Note that \( m \neq 0 \) since \( v_0 = w_0 \) is contained in \( P_2 \) and hence is in \( \sigma^{<j} \) by previous argument. Since \( 0 < m < k - 1 \), we conclude that \( e_{m-1} \in E(P_1) \) is contained in \( E(T) \). Since \( v_m \notin \sigma^{<j} \), by (R1) and definition of \( \Upsilon \) we have \( e_{m-1} \) is not contained in \( \sigma^{<j} \). On the other hand by the minimality of \( m \) we have \( v_{m-1} \) is in \( \sigma^{<j} \). Now, note that by the minimality of \( f \), we have \( f < e_{m-1} \), so by (R2) \( \sigma_j \neq f \) because \( e_{m-1} \) satisfies the edge condition in (R2) and is bigger than \( f \). Thus we get a contradiction.

Proof of \( \Leftarrow \) direction: suppose that \( e = \sigma_i \) is externally active in \( T \) and \( e \) is not strong in \( \sigma \). Let \( P_1 \) and \( P_2 \) be the two paths defined previously, then since \( e \) is externally active, the minimum edge \( e' \) with respect to \( <_{e} \) in \( E(P_1) \cup E(P_2) \) is contained in \( P_1 \).

Let \( e = \sigma_i \), then by the observation in the beginning of the proof we have all vertices and edges in \( P_1 \) except for \( v_k \) are contained in \( \sigma^{<i+1} \). In particular, \( e' \) is contained in \( \sigma^{<i+1} \) so \( e' = \sigma_j \) with \( j \leq i \). By the observation in the beginning we have \( v_0 = w_0 \) is contained in \( \sigma^{<j} \). So there is a vertex in \( E(P_2) \) that is contained in \( \sigma^{<j} \).

Let \( n \) be the maximal index such that \( w_n \in E(P_2) \) is contained in \( \sigma^{<j} \). Note that \( n \neq l \) since \( e \) is not strong in \( \sigma \) and hence \( t = w_l \) is not contained in \( \sigma^{<j} \) as \( t \) is not contained in \( \sigma^{<i+1} \). Since \( w_{n+1} \) is not contained in \( \sigma^{<j} \), by (R1) and definition of \( \Upsilon \), \( f_n \) is not contained in \( \sigma^{<j} \). Now, by the minimality of \( e' \) we have \( e' < e \), so by (R2) \( \sigma_j \neq e' \) because \( f_n \) satisfies the edge condition in (R2) and is bigger than \( e' \). Thus we get a contradiction. \( \square \)

We now define a function \( \Psi \) that maps an s-decreasing traversal \( \sigma \) to a \( G^r \)-parking function \( f \), where \( f(v) = |\{ e \in \sigma^{<i} | t(e) = \sigma_i = v \}| - 1 \) for all \( v \in V \setminus \{s\} \).

**Lemma 6.5.** The mapping \( \Psi \) is a bijection between \( D(G, s) \) and \( \text{Park}(G, s) \).

**Proof.** First, we show that \( \Psi \) maps an s-decreasing traversal \( \sigma \) to a \( G^r \)-parking function. Given any non-empty subset \( A \subseteq V \setminus \{s\} \), let \( i \) be the maximal index such that \( \sigma^{<i} \) does not intersect \( A \). Note that \( \sigma_i \) is a vertex in \( A \), and by (R2) each edge in \( \sigma^{<i} \) with target vertex \( \sigma_i \) has source vertex in \( \sigma^{<i} \). Hence we have

\[
(\Psi(\sigma))(\sigma_i) = |\{ e \in \sigma^{<i} | t(e) = \sigma_i = v \}| - 1 < \# \text{ edges from } V \setminus A \text{ to } \sigma_i,
\]

so \( \Psi(\sigma) \) is a \( G^r \)-parking function.

To show injectivity, let \( \sigma \) and \( \tau \) be two distinct s-decreasing traversals and \( k \) the minimal index such that \( \sigma_k \neq \tau_k \). By Lemma 6.2 we can assume that \( \sigma_k \) is a vertex and \( \tau_k \) is an edge. Let \( v = \sigma_k \), by definition of \( \Psi \) we can then conclude that \( (\Psi(\sigma))(v) < (\Psi(\tau))(v) \), which shows injectivity.

To show surjectivity, given a \( G^r \)-parking function \( f \), construct an s-decreasing traversal \( \sigma \) by setting \( \sigma_1 := s \), and then continue inductively on \( i \) as follows:

(i) If \( \sigma_i \) is an edge with target vertex \( v \notin \sigma^{<i+1} \) and \( \sigma^{<i+1} \) contains \( f(v) + 1 \) edges with target vertex \( v \), then \( \sigma_{i+1} = v \);

(ii) Else \( \sigma_{i+1} \) is the maximum edge with respect to \( <_e \) that is not contained in \( \sigma^{<i+1} \) and has source vertex contained in \( \sigma^{<i+1} \).

The algorithm does not terminate before we get an edge-vertex traversal if \( f \) is a \( G^r \)-parking function because for all \( v \in V \setminus \{s\} \), we have \( f(v) < \# \text{ edges from } V \setminus \{v\} \text{ to } v = \deg_G(v) \). It is easy to check \( \Psi(\sigma) = f \), and \( \sigma \) is an s-decreasing traversal since \( \sigma \) obeys (R1) by (i) and obeys (R2) by (ii). \( \square \)
Lemma 6.6. For any natural number $n$, the mapping $\Psi$ sends an $s$-decreasing traversal $\sigma$ with strength equal to $n$ to a $G^r$-parking function $f$ with $d$-level equal to $n$.

Proof. By definition of $\Psi$, strength, and $d$-level, we can conclude that

$$\text{str}(\sigma) = |\{e \in E(G) \mid e \text{ is strong in } \sigma\}| = |E(G)| - |\{e \in E(G) \mid e \text{ is not strong in } \sigma\}|$$

$$= |E(G)| - \sum_{v \in V \setminus \{s\}} |\{e \in \sigma^{-1} \mid t(e) = \sigma_i = v\}|$$

$$= |E(G)| - |V(G)| + 1 - \sum_{v \in V \setminus \{s\}} f(v) = \text{dlvl}(f),$$

which proves the lemma. \qed

Lemma 6.3, Lemma 6.4, Lemma 6.5, and Lemma 6.6 give us a bijection between the set of arborescences of $G$ and the set of $G^r$-parking functions that translates arborescences activities to $d$-level, which gives us the following theorem.

Theorem 6.7. Let $G$ be a strongly connected digraph and $s$ be a vertex of $G$. The greedoid Tutte polynomial $T(G,s;y)$ is equal to

$$T(G,s;y) = \sum_{n \geq 0} |\text{Park}_n(G,s)|y^n,$$

where $\text{Park}_n(G,s)$ is the set of $G^r$-parking functions rooted at $s$ with $d$-level equal to $n$. \qed

Theorem 6.7 generalizes a similar result of Kostić and Yan [KY08] for undirected graphs. We remark that there are several bijections in the existing literature between $s$-recurrent configurations and spanning trees of undirected graphs ([Big99, CRS02, Bac12, ABKS14]). In the setting of directed graphs there is a bijection between the set of arborescences of $G$ and the set of $G^r$-parking functions by Chebikin and Pylyavskyy [CP05], which is different from the bijection in Theorem 6.7 as their bijection does not preserve the notion of activities.

Theorem 1.2. (Generalized Merino’s Theorem) If $G$ is a connected Eulerian digraph, then the Biggs-Merino polynomial $B(G,s;y)$ is equal to the greedoid Tutte polynomial $T(G,s;y)$. In particular, the greedoid Tutte polynomial is independent of the choice of $s$.

Proof. When $G$ is a connected Eulerian digraph, the Biggs Merino polynomial is equal to the sum in Equation 6, and thus we have

$$B(G,s;y) = \sum_{n \geq 0} |\text{Rec}_n(G,s)|y^n,$$

and since $|\text{Rec}_n(G,s)|$ is equal to $|\text{Park}_n(G,s)|$ for all $n$ when $G$ is Eulerian by Lemma 6.1, the theorem follows immediately from Theorem 6.7. \qed

The Merino’s Theorem for loopless undirected graphs $G$ is a consequence of generalized Merino’s Theorem: the chip-firing game on a loopless undirected graph $G$ is equivalent with the chip-firing game on the directed graph $G$ obtained from replacing each edge of $G$ with a directed 2-cycle, so the Biggs-Merino polynomial $B(G,s;y)$ of $G$ is equal to the Biggs-Merino polynomial $B(G^r,s;y)$ of $G$. On the other hand, by Proposition 4.1 the greedoid Tutte polynomial $T(G,s;y)$ is equal to $y^{|E(G)|}T(G,1,y)$, where $T(G;x,y)$ is the Tutte polynomial of $G$. So by Theorem 1.2 we have $B(G,s;y) = y^{|E(G)|}T(G,1,y)$, as desired.
Theorem 1.2 answers another question raised in [PV13] on the combinatorial interpretation of $B(G, s; 2)$ when $G$ is a connected Eulerian digraph. It is shown in [GM89] that $T(G, s; 2)$ counts the number of spanning subgraphs of $G$ that is rooted at $s$ i.e. a subgraph of $G$ such that for all $v \in V(G)$, there exists a directed path in the subgraph that starts at $s$ and ends at $v$. Hence by Theorem 1.2 $B(G, s; 2)$ counts the number of said objects when $G$ is Eulerian.

We remark that for a general non-Eulerian digraph, the polynomial $B(G, s; y)$ is not equal to $T(G, s; y)$, as $B(G, s; y)$ does not depend on the choice of $s$ by Theorem 4.4, while $T(G, s; y)$ does depend on the choice of sink, as $T(G, s; 1)$ is equal to the number of arborescences of $G$ rooted at $s$, which depends on $s$. We also remark that the polynomial in Equation 6 (i.e. the original Biggs-Merino polynomial), is not equal to $T(G, s; y)$ when $G$ is not Eulerian, as when we substitute $y = 1$ to the polynomial in Equation 6 we get the number of reverse arborescences of $G$ rooted at $s$, which in general is not equal to the number of arborescences of $G$ rooted at $s$.

7. Concluding remarks

In this section we indicate a few possible directions for further research on Biggs Merino polynomials.

As remarked in Section 6, for non-Eulerian digraphs $G$ the Biggs-Merino polynomial of $G$ is not equal to the greedoid Tutte polynomial of $G$. A further research can be done on finding an expression of Biggs-Merino polynomial as a polynomial that counts a combinatorial object that is not related to sandpile model, possibly as the greedoid Tutte polynomial of a greedoid that arises naturally from the digraph $G$.

Another possible research direction is related to a method of computing the Biggs-Merino polynomial recursively. There is a deletion-contraction recursion formula [BKL85] for greedoid Tutte polynomial, and a Moebius inversion formula [PV13] for Biggs-Merino polynomial when $G$ is Eulerian. However, there are no known recurrence formulas for the Biggs-Merino polynomial for general non-Eulerian digraphs, and the existence of recursive formulas for polynomials closely related to the Biggs-Merino polynomial suggests that such a formula should exist.

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In this section we present proofs of several lemmas and propositions in Section 2.

**Lemma A.1.** (Lemma 2.1). A configuration $c$ in $\text{Sand}(G)$ is a recurrent configuration if and only if there exists a finite sequence of legal firing moves sending $c$ to $c$ such that each vertex $v \in V$ is fired exactly $r(v)$ times.

**Proof.** Proof for $\Rightarrow$ direction: Suppose that $c$ is a recurrent configuration, then by definition $c$ is unstable at some vertex $v$, so there exists a legal firing move $c \rightarrow d$ sending $c$ to a configuration $d$ in $\text{Sand}(G)$ that is distinct from $c$. Since $c$ is recurrent, we have $d \rightarrow c$ and thus there exists a finite sequence of legal firing moves of non-zero length that sends $c$ to $d$, then sends $d$ back to $c$, and let $q \in \mathbb{N}_0^V$ be the odometer of the sequence, so $q$ is a non-zero vector. Since $c$ is sent back to itself, we have $c - \Delta q = c$, which implies that $\Delta q = 0$. As the kernel of the Laplacian matrix $\Delta$ is spanned by the primitive period vector $r$, we have $q$ is a positive multiple of $r$.

It was shown in [BL92, Lemma 4.3] that if a finite sequence of firing moves with odometer $q$ being a positive multiple of $r$ is legal on a configuration $c$, then there exists a finite sequence of firing moves with odometer $r$ that is legal on $c$. This finishes the proof of $\Rightarrow$ direction.

Proof for $\Leftarrow$ direction: By assumption there exists a sequence of legal firing moves of non-zero length that can be performed on $c$, so $c$ is an unstable configuration. Let $d$ be a configuration in $\text{Sand}(G)$ that is accessible from $c$, we will prove that $d \rightarrow c$, and therefore showing that $c$ is a recurrent configuration.

Since $c \rightarrow d$ by assumption, then there exists a finite sequence of legal firing moves sending $c$ to $d$, and let $q$ be the odometer of the sequence. On the other hand, by assumption we have a finite sequence of firing moves with odometer $r$ that is legal on $c$ and sends $c$ back to itself. Thus, for any $n \in \mathbb{N}$ we can apply the same sequence of firing moves to $c$ for $n$ times to get another finite sequence of firing moves with odometer $nr$ that is legal on $c$ and sends $c$ back to itself. Since $r$ is a positive vector, we can assume that for sufficiently large $n$ we have $q(v) \leq nr(v)$ for all $v \in V$.

It was shown in [BL92, Lemma 1.3] that if a finite sequence of firing moves with odometer $q$ sends a chip configuration $c$ to $d$ legally and another sequence with odometer $q'$ sends $c$ to $d'$ legally, with $q(v) \leq q'(v)$ for all $v \in V$, then there exists a finite sequence of legal firing moves from $d$ to $d'$. Applying this property to our case, since there is a sequence of firing moves with odometer $q$ sending $c$ to $d$ legally and another sequence with odometer $nr$ sending $c$ to $c$ legally, with $q(v) \leq nr(v)$ for all $v \in V$, we conclude that there exists a finite sequence of legal firing moves from $d$ to $c$, or equivalently $d \rightarrow c$, proving $\Leftarrow$ direction.

In the next proposition, we use $1_s \in \mathbb{N}_0^V$ to denote the chip configuration with $1_s(v) := 0$ for all $v \in V \setminus \{s\}$ and $1_s(s) := 1$.

**Proposition A.2.** (Proposition 2.2). Let $c, d$ be chip configurations in $\text{Sand}(G)$.

(i) If $c \rightarrow d$ and $c$ is a recurrent configuration, then $d$ is a recurrent configuration.

(ii) If $c$ is a recurrent configuration, then the configuration $c' := c + k1_s$, for some $k \in \mathbb{N}$ is also a recurrent configuration.

\[\text{References}\]

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\[\text{Appendix A. Sandpile models}\]

In this section we present proofs of several lemmas and propositions in Section 2.

**Lemma A.1.** (Lemma 2.1). A configuration $c$ in $\text{Sand}(G)$ is a recurrent configuration if and only if there exists a finite sequence of legal firing moves sending $c$ to $c$ such that each vertex $v \in V$ is fired exactly $r(v)$ times.

**Proof.** Proof for $\Rightarrow$ direction: Suppose that $c$ is a recurrent configuration, then by definition $c$ is unstable at some vertex $v$, so there exists a legal firing move $c \rightarrow d$ sending $c$ to a configuration $d$ in $\text{Sand}(G)$ that is distinct from $c$. Since $c$ is recurrent, we have $d \rightarrow c$ and thus there exists a finite sequence of legal firing moves of non-zero length that sends $c$ to $d$, then sends $d$ back to $c$, and let $q \in \mathbb{N}_0^V$ be the odometer of the sequence, so $q$ is a non-zero vector. Since $c$ is sent back to itself, we have $c - \Delta q = c$, which implies that $\Delta q = 0$. As the kernel of the Laplacian matrix $\Delta$ is spanned by the primitive period vector $r$, we have $q$ is a positive multiple of $r$.

It was shown in [BL92, Lemma 4.3] that if a finite sequence of firing moves with odometer $q$ being a positive multiple of $r$ is legal on a configuration $c$, then there exists a finite sequence of firing moves with odometer $r$ that is legal on $c$. This finishes the proof of $\Rightarrow$ direction.

Proof for $\Leftarrow$ direction: By assumption there exists a sequence of legal firing moves of non-zero length that can be performed on $c$, so $c$ is an unstable configuration. Let $d$ be a configuration in $\text{Sand}(G)$ that is accessible from $c$, we will prove that $d \rightarrow c$, and therefore showing that $c$ is a recurrent configuration.

Since $c \rightarrow d$ by assumption, then there exists a finite sequence of legal firing moves sending $c$ to $d$, and let $q$ be the odometer of the sequence. On the other hand, by assumption we have a finite sequence of firing moves with odometer $r$ that is legal on $c$ and sends $c$ back to itself. Thus, for any $n \in \mathbb{N}$ we can apply the same sequence of firing moves to $c$ for $n$ times to get another finite sequence of firing moves with odometer $nr$ that is legal on $c$ and sends $c$ back to itself. Since $r$ is a positive vector, we can assume that for sufficiently large $n$ we have $q(v) \leq nr(v)$ for all $v \in V$.

It was shown in [BL92, Lemma 1.3] that if a finite sequence of firing moves with odometer $q$ sends a chip configuration $c$ to $d$ legally and another sequence with odometer $q'$ sends $c$ to $d'$ legally, with $q(v) \leq q'(v)$ for all $v \in V$, then there exists a finite sequence of legal firing moves from $d$ to $d'$. Applying this property to our case, since there is a sequence of firing moves with odometer $q$ sending $c$ to $d$ legally and another sequence with odometer $nr$ sending $c$ to $c$ legally, with $q(v) \leq nr(v)$ for all $v \in V$, we conclude that there exists a finite sequence of legal firing moves from $d$ to $c$, or equivalently $d \rightarrow c$, proving $\Leftarrow$ direction. \[\square\]

In the next proposition, we use $1_s \in \mathbb{N}_0^V$ to denote the chip configuration with $1_s(v) := 0$ for all $v \in V \setminus \{s\}$ and $1_s(s) := 1$.

**Proposition A.2.** (Proposition 2.2). Let $c, d$ be chip configurations in $\text{Sand}(G)$.

(i) If $c \rightarrow d$ and $c$ is a recurrent configuration, then $d$ is a recurrent configuration.

(ii) If $c$ is a recurrent configuration, then the configuration $c' := c + k1_s$, for some $k \in \mathbb{N}$ is also a recurrent configuration.
Proof. (i). First, we show that \( d \) is not a stable configuration. Since \( c \rightarrow d \) by assumption and \( c \) is recurrent, we conclude that \( d \rightarrow d \). If \( d \) is stable, then the only sequence of legal firing moves that can be performed on \( d \) is the sequence of length 0. So \( d = c \), which implies that \( c \) is stable, contradicting the assumption that \( c \) is unstable.

Next, let \( d_1 \) be a configuration in \( \text{Sand}(G) \) such that \( d \rightarrow d_1 \), we will show that \( d_1 \rightarrow d \). Since \( c \rightarrow d \) by assumption and \( d \rightarrow d_1 \), we conclude that \( c \rightarrow d_1 \) as \( - \rightarrow - \) is transitive, and by the recurrence of \( c \) we have \( d_1 \rightarrow c \). Since we have \( d_1 \rightarrow c \) and we have \( c \rightarrow d \) by assumption, by transitivity we conclude that \( d_1 \rightarrow d \). Therefore \( d \) is a recurrent configuration by definition of recurrence.

(ii). We use the burning test in Lemma 2.1. Since \( c \) is recurrent, then there exists a finite sequence of legal firing moves for \( c \) such that all vertices \( v \in V \) is fired exactly \( r(v) \) times. Since \( c' = c + k_1s \), any finite sequence of legal firing moves for \( c \) is also legal for \( c' \) and in particular there exists a finite sequence of legal firing moves for \( c' \) such that all vertices \( v \in V \) is fired exactly \( r(v) \) times. By Lemma 2.1, this implies that \( c' \) is a recurrent configuration.

In the next proposition we present an alternative definition of the relation \( \sim \) in Section 2, which is how the relation \( \sim \) was originally defined by Perrot and Pham in [PV13]. Note that \( \hat{u} \in N_0^{V \setminus \{s\}} \) is an integer vector with \( \hat{u}(v) \) is equal to the number of edges from \( s \) to \( v \) for each \( v \in V \setminus \{s\} \), and \( \Delta_s \) is the reduced Laplacian matrix at \( s \in V \) defined in Section 2.

**Proposition A.3.** Two \( s \)-recurrent configurations \( \hat{c} \) and \( \hat{d} \) satisfy \( \hat{c} \sim \hat{d} \) if and only if \( \hat{d} - \hat{c} \in (\hat{u}, \Delta_s Z^{V \setminus \{s\}})_Z \).

**Proof.** For the proof of \( \Rightarrow \) direction, note that if \( \hat{d} = (\hat{c} + k\hat{u})^0 \) for some \( k \in N_0 \), then in particular there exists a finite sequence of legal s-firing moves sending \( \hat{c} + k\hat{u} \) to \( \hat{d} \). This means that \( \hat{d} - \hat{c} = k\hat{u} - \Delta_s\hat{q}, \) where \( \hat{q} \) is the s-odometer of the sequence, and thus \( \hat{d} - \hat{c} \in (\hat{u}, \Delta_s Z^{V \setminus \{s\}})_Z \) as desired.

For the proof of \( \Leftarrow \) direction, suppose that \( \hat{d} - \hat{c} \in (\hat{u}, \Delta_s Z^{V \setminus \{s\}})_Z \), then without loss of generality we can assume that there exists \( k \in N_0 \) and \( \hat{q} \in N_0^{V \setminus \{s\}} \) such that \( \hat{d} - \hat{c} = k\hat{u} - \Delta_s\hat{q} \), so we have \( \hat{d} = \hat{c} + k\hat{u} - \Delta_s\hat{q} \). Let \( \hat{d}' := (\hat{c} + k\hat{u})^0 \), since \( \hat{c} \) is \( s \)-recurrent we have \( \hat{d}' \) is \( s \)-recurrent by Proposition 2.6, and note that \( \hat{d} - \hat{d}' \) is contained in \( \Delta_s Z^{V \setminus \{s\}} \). It was shown in [HLM+08, Corollary 2.16] that two \( s \)-recurrent configuration \( \hat{d}, \hat{d}' \) is equal if and only if \( \hat{d} - \hat{d}' \in \Delta_s Z^{V \setminus \{s\}} \). Thus, we have \( \hat{d} = \hat{d}' = (\hat{c} + k\hat{u})^0 \) where \( k \) is non-negative integer, and by definition we conclude that \( \hat{c} \sim \hat{d} \). □

Proposition A.3 allows us to compute the size of the set \( \text{Rec}(G, \sim) \), as can be seen in the next proposition.

**Proposition A.4.** The size of \( \text{Rec}(G, \sim) \) is equal to \( \alpha \), the period constant of \( G \).

**Proof.** It was shown in [HLM+08, Corollary 2.16] that the set \( \text{Rec}(G, s) \) is an abelian group isomorphic to the group quotient \( Z^{V \setminus \{s\}}/\Delta_s Z^{V \setminus \{s\}} \) via the inclusion map. Thus, by Proposition A.3, \( \text{Rec}(G, \sim) = \text{Rec}(G, s)/\sim \) is a group isomorphic with the group quotient \( Z^{V \setminus \{s\}}/(\hat{u}, \Delta_s Z^{V \setminus \{s\}})_Z \) via the inclusion map. By the third isomorphism theorem of groups, we have:

\[
|Z^{V \setminus \{s\}}/(\hat{u}, \Delta_s Z^{V \setminus \{s\}})_Z| = \frac{|Z^{V \setminus \{s\}}/\Delta_s Z^{V \setminus \{s\}}|}{|\langle \hat{u}, \Delta_s Z^{V \setminus \{s\}} \rangle_Z/\Delta_s Z^{V \setminus \{s\}}|}.
\]

Note that \( |Z^{V \setminus \{s\}}/\Delta_s Z^{V \setminus \{s\}}| \) is equal to \( \det \Delta_s \), and \( |\langle \hat{u}, \Delta_s Z^{V \setminus \{s\}} \rangle_Z/\Delta_s Z^{V \setminus \{s\}}| \) is equal to the order of \( \hat{u} \) in the group quotient \( \langle \hat{u}, \Delta_s Z^{V \setminus \{s\}} \rangle_Z/\Delta_s Z^{V \setminus \{s\}} \).
Since the Laplacian matrix $\Delta$ has co-dimension one and $\hat{\mathbf{u}}$ can be obtained by deleting the entry that corresponds to $s$ from the column of Laplacian matrix $\Delta$ that corresponds to $s$, we can conclude that there exists a one-to-one mapping between the set \{ $k \in \mathbb{Z}$ | $k\hat{\mathbf{u}} \in \Delta, s \} \setminus \{ s \}$ and $\ker \Delta \cap \mathbb{Z}^V$ by mapping $k$ to the unique vector $\mathbf{q} \in \mathbb{Z}^V$ with $\mathbf{q}(s) = k$ and $\Delta \mathbf{q} = \mathbf{0}$. Thus since any vector in $\ker \Delta \cap \mathbb{Z}^V$ is an integer multiple of the primitive period vector $\mathbf{r}$, we conclude that $\hat{\mathbf{u}}$ has order equal to $\mathbf{r}(s)$ in the group quotient $\langle \hat{\mathbf{u}}, \Delta, s \rangle = \ker \Delta \cap \mathbb{Z}^V \setminus \{ s \}$. Hence, we can conclude that

$$|\mathbb{Z}^V \setminus \{ s \}/\langle \hat{\mathbf{u}}, \Delta, s \mathbb{Z} \setminus \{ s \} \rangle| = \frac{\det \Delta_s}{\mathbf{r}(s)} = \frac{\det \Delta_s}{\det \Delta_s / \alpha} = \alpha,$$

as desired. \hfill \Box

In the next proposition, we prove the connection between chip configuration $\mathbf{c}$, $\hat{\mathbf{c}}$, and $\overline{\mathbf{c}}$ defined in Section 2.3.

**Proposition A.5.** (Proposition 2.7) Let $\mathbf{c}, \mathbf{d}$ be chip configurations in $\text{Sand}(G)$.

(i) Suppose that $\hat{\mathbf{c}} \rightarrow_s \mathbf{d}$ through a finite sequence of reduced legal $s$-firing moves. Then the associated sequence of firing moves is legal for $\mathbf{c}$, and we have $\mathbf{c} \rightarrow \mathbf{d} + n \mathbf{1}_s$, where $n$ is the number of chips removed from the game during the legal $s$-firing moves from $\mathbf{c}$ to $\mathbf{d}$.

(ii) Suppose that $\mathbf{c} \rightarrow \mathbf{d}$ through a finite sequence of legal firing moves. Then the associated sequence of $s$-firing moves is legal for $\hat{\mathbf{c}} + m \hat{\mathbf{u}}$ and we have $\hat{\mathbf{c}} + m \hat{\mathbf{u}} \rightarrow_s \mathbf{d}$, where $m$ is the number of times the vertex $s$ is fired during the legal firing moves from $\mathbf{c}$ to $\mathbf{d}$.

**Proof.** (i) Suppose that $\hat{\mathbf{c}} = \mathbf{c}_0 \rightarrow \mathbf{c}_1 \rightarrow \mathbf{c}_2 \rightarrow \ldots \rightarrow \mathbf{c}_k = \hat{\mathbf{d}}$, and $n_i$ denote the number of chips removed from the game after the $i$-th legal firing move from $\mathbf{c}$ to $\mathbf{d}$. We will prove that $\mathbf{c} = (\mathbf{c}_0 + n_0 \mathbf{1}_s) \rightarrow (\mathbf{c}_1 + n_1 \mathbf{1}_s) \rightarrow \ldots \rightarrow (\mathbf{c}_k + n_k \mathbf{1}_s) = \mathbf{d} + n \mathbf{1}_s$ and all the firing moves are legal, which proves the claim.

Let $0 < i < k$, since $\mathbf{c}_i \rightarrow \mathbf{c}_{i+1}$ is a reduced s-legal firing move by assumption, we have $v_i$ is not equal to $s$, and by definition $\mathbf{c}_i(v_i) = \mathbf{c}_{i+1}(v_i) \geq \deg^+_G(v_i+1)$. This implies that it is legal to fire vertex $v_{i+1}$ on $\mathbf{c}_i$, and hence it is legal to fire vertex $v_{i+1}$ on $\mathbf{c}_i + n_i \mathbf{1}_s$, and after performing the firing move on $\mathbf{c}_i$, we need to add $n_i - n_i$ chips to vertex $s$ to make up for chips lost when the same move is executed in $\text{Sand}(G, s)$. Thus we conclude that $(\mathbf{c}_i + n_i \mathbf{1}_s) \rightarrow (\mathbf{c}_{i+1} + n_i \mathbf{1}_s + (n_i - n_i) \mathbf{1}_s) = \mathbf{c}_{i+1} + n_i \mathbf{1}_s$, and the proof is complete.

(ii) Suppose that $\mathbf{c} = \mathbf{c}_0 \rightarrow \mathbf{c}_1 \rightarrow \mathbf{c}_2 \rightarrow \ldots \rightarrow \mathbf{c}_k = \mathbf{d}$ and $m_i$ denote the number of times the vertex $s$ is fired during the first $i$ legal firing moves from $\mathbf{c}$ to $\mathbf{d}$. We will prove that $(\hat{\mathbf{c}} + m \hat{\mathbf{u}}) = (\hat{\mathbf{c}} + (m_k - m_0) \hat{\mathbf{u}}) \rightarrow (\hat{\mathbf{c}}_1 + (m_k - m_1) \hat{\mathbf{u}}) \rightarrow (\hat{\mathbf{c}}_2 + (m_k - m_2) \hat{\mathbf{u}}) \rightarrow \ldots \rightarrow (\hat{\mathbf{c}}_k + (m_k - m_k) \hat{\mathbf{u}}) = \mathbf{d}$ and all the s-firing moves are legal, which proves the claim.

Let $0 \leq i < k$, by assumption we have $\mathbf{c}_i \rightarrow \mathbf{c}_{i+1}$, and we proceed by considering two possible cases. For the first case, if $v_{i+1}$ is contained in $V \setminus \{ s \}$, then since firing $v_{i+1}$ is legal for $\mathbf{c}_i$, this implies that $\hat{\mathbf{c}}_i(v_{i+1}) = \mathbf{c}_i(v_{i+1}) \geq \deg^+_G(v_{i+1})$. Thus firing vertex $v_{i+1}$ is a legal s-firing move that sends $\hat{\mathbf{c}}_i$ to $\mathbf{c}_{i+1}$, and thus is also a legal s-firing move from $\hat{\mathbf{c}}_i + (m_k - m_i) \hat{\mathbf{u}}$ to $\mathbf{c}_{i+1} + (m_k - m_i) \hat{\mathbf{u}}$. Since $v_{i+1} \neq s$, we have $m_i = m_i$, and we conclude that $(\hat{\mathbf{c}} + (m_k - m_i) \hat{\mathbf{u}}) \rightarrow (\hat{\mathbf{c}}_i + (m_k - m_i) \hat{\mathbf{u}}) = \hat{\mathbf{c}}_i + (m_k - m_i) \hat{\mathbf{u}}$. Hence we can conclude that $(\hat{\mathbf{c}}_i + (m_k - m_i) \hat{\mathbf{u}}) \rightarrow (\hat{\mathbf{c}}_i + (m_k - m_i) \hat{\mathbf{u}}) = \hat{\mathbf{c}}_{i+1} + (m_k - m_{i+1}) \hat{\mathbf{u}}$, and the proof is complete. \hfill \Box
APPENDIX B. CONNECTIONS BETWEEN TUTTE POLYNOMIAL AND GREEDOID TUTTE POLYNOMIAL

We start by recalling the definition of Tutte polynomial for undirected graphs. Consider an undirected connected graph $G$ with a total order $<_e$ on the edges of $G$. For a spanning tree $T$ of $G$, an edge $e \in E(T)$ is internally active in $T$ if it is the lowest edge, by the ordering of $<_e$, in the unique cocycle contained in $(E(G) \setminus E(T)) \cup \{e\}$. The internal activity of a tree $T$ is the number of internally active edges in $T$ and is denoted by $i(T)$. An edge $e \in E(G) \setminus E(T)$ is externally active if it is the lowest edge, by the ordering of $<_e$, in the unique cycle contained in $(V(G), E(T) \cup \{e\})$. The external activity of the spanning tree $T$ is the number of externally active edges in $T$ and is denoted by $e(T)$.

Let $\mathcal{A}(G)$ denote the set of the spanning trees of $G$. In [Tut54], Tutte introduced the following polynomial, called Tutte polynomial, as:

$$T(G; x, y) := \sum_{T \in \mathcal{A}(G)} x^{i(T)} y^{e(T)},$$

and it was shown that Tutte polynomial does not depend on the choice of $<_e$.

**Proposition B.1.** Let $G$ be a loopless connected undirected graph and $G$ be the directed graph obtained by replacing each edge $e$ of $G$ with a directed 2-cycle. For each vertex $s$, the greedoid Tutte polynomial $T(G; s; y)$ is equal to $y^{|E(G)|}T(G; 1, y)$.

**Proof.** Note that for each spanning tree $T$ of $G$, there is exactly one arborescence $T'$ of $G$ rooted at $s$ in which the underlying graph of $T$ is equal to $T$. Thus, the claim in the proposition will follow by proving that for any spanning tree $T$ of $G$, the arborescence $T'$ associated to $T$ has external activities $e(T) = |E(G)| + e(T)$.

We extend the ordering $<_e$ on the edges of $G$ to the edges of $G$ as follows, if $f$ and $f'$ are two edges in $E(G)$ such that the underlying edges $e \in E(G)$ of $f$ and the underlying edge $e' \in E(G)$ of $f'$ satisfies $e <_e e'$, then we set $f <_e f'$. If $f$ and $f'$ has the same underlying edge $e \in E(G)$, then we give an arbitrary order on the two edges. As both $T(G, s; y)$ and $T(G; x, y)$ does not depend on the total order $<_e$, it suffices to show that with respect to the total order $<_e$, for each edge $e \in E(G)$ exactly one of the two edges $e_1$ and $e_2$ in $E(G)$ associated to $e$ is externally active in $T$ if $e$ is not externally active in $T$, and both $e_1$ and $e_2$ are externally active in $T$ if $e$ is externally active in $T$.

We consider two possible cases for the edge $e \in E(G)$. First, suppose that $e \in E(T)$, then $e$ is not externally active in $T$, and without loss of generality we can assume that $e_2 \in E(T)$ while $e_1$ is not contained in $E(T)$. Since $e_2 \in E(T)$, we have $e_2$ is not externally active in $T$. Let $P_1$ and $P_2$ be the two edge-disjoint paths in $(V(G), E(T) \cup \{e_1\})$ with same starting vertex, same target vertex $t(e_1)$, and with $P_1$ containing $e_1$. It can be checked that $P_1$ is a directed cycle with two edges $e_1$ and $e_2$ while $P_2$ is a path of length zero, and thus by definition $e_1$ is externally active. So the claim follows as $e_1$ is externally active in $T$ while $e_2$ is not externally active in $T$.

For the second case, suppose that $e \in E(G) \setminus E(T)$. Let $P_1$ and $P_2$ be the two edge-disjoint paths in $(V(G), E(T) \cup \{e_2\})$ that has the same starting vertex and same target vertex $t(e_1)$, and $Q_1$ and $Q_2$ be defined similarly for $(V(G), E(T) \cup \{e_2\})$, and assume that $e_1$ is in $P_1$ and $e_2$ is in $Q_1$. It is easy to verify that $Q_2 = E(P_1) \setminus \{e_1\}$ and $P_2 = E(Q_1) \setminus \{e_2\}$, which implies $(E(P_1) \cup E(P_2)) \setminus \{e_1\} = (E(Q_1) \cup E(Q_2)) \setminus \{e_2\}$. We consider another two cases for the edge $e \in E(G) \setminus E(T)$.

For the first case, suppose that $e$ is externally active in $T$, then $e$ is the smallest edge in the unique cycle of $(V(G), E(T) \cup \{e\})$. This implies that $e_1$ is the smallest edge in $E(P_1) \cup E(P_2)$ and
similarly $e_2$ is the smallest edge in $E(Q_1) \sqcup E(Q_2)$. By definition, this implies that both $e_1$ and $e_2$ are externally active in $T$, as desired.

For the second case, suppose that $e$ is not externally active in $T$, then by definition the smallest edge in the unique cycle of $(V(G), E(T) \cup \{e\})$ is not $e$. This implies $e_1$ is not the smallest edge in $E(P_1) \sqcup E(P_2)$ and similarly $e_2$ is not the smallest edge in $E(Q_1) \sqcup E(Q_2)$. Let $f$ be the smallest edge in $E(P_1) \sqcup E(P_2)$, then $e_1 < e f$ and by the definition of $< e$, since $e_1$ and $e_2$ has the same underlying edge, we have $e_2 < e f$. Since we also have $(E(P_1) \sqcup E(P_2)) \setminus \{e_1\} = (E(Q_1) \sqcup E(Q_2)) \setminus \{e_2\}$, $f$ is not equal to $e_1$ or $e_2$, and $f$ is the smallest edge in $E(P_1) \sqcup E(P_2)$, we can conclude that $f$ is also the smallest edge in $E(Q_1) \sqcup E(Q_2)$. If $f$ is contained in $E(P_1)$, then $e_1$ is externally active in $T$ by definition, and since $E(Q_2) = E(P_1) \setminus \{e_1\}$, we have $f$ is contained in $E(Q_2)$, so $e_2$ is not externally active in $T$. On the other hand, if $f$ is contained in $E(P_2)$, then $e_1$ is not externally active in $T$ by definition, and since $E(P_2) = E(Q_1) \setminus \{e_2\}$, we have $f$ is contained in $E(Q_1)$, so $e_2$ is externally active in $T$. This shows that if $e$ is not externally active in $T$, then one of the edges $e_1$ and $e_2$ is externally active in $T$, while the other edge is not externally active in $T$, as desired. Now the proof is complete. 

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