ON NON-UNIFORM HYPERBOLICITY ASSUMPTIONS IN ONE-DIMENSIONAL DYNAMICS

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Abstract. We give an essentially equivalent formulation of the backward contracting property, defined by Juan Rivera-Letelier, in terms of expansion along the orbits of critical values, for complex polynomials of degree at least 2 which are at most finitely renormalizable and have only hyperbolic periodic points, as well as all $C^3$ interval maps with non-flat critical points.

1. Introduction

In the context of one-dimensional dynamics, widely adopted non-uniform hyperbolicity conditions involve expansion along the orbits of critical values, such as the Collet-Eckmann condition, the summability conditions and the large derivatives condition, see [3, 11, 4, 2, 1] among others. Recently, Rivera-Letelier [13] introduced a new notion called backward contraction which serves as a different type of non-uniform hyperbolicity condition. This condition is more convenient to use as it follows immediately that the first return maps to suitably chosen small neighborhoods of critical points have good combinatorial and geometric properties. For instance, this notion plays an important role in the work [1].

It has been realized that sufficient expansion along the orbits of critical points often implies backward contraction, see [13, Theorem A] and [1, Theorem 1]. In this paper, we study further the relation between these two types of non-uniformly hyperbolicity conditions. For notational definiteness, we shall mainly work on complex maps and leave the argument for interval maps in Appendix B.

Given a complex polynomial $f$, let $\text{Crit}(f)$ denote the set of critical points of $f$ in $\mathbb{C}$, let $J(f)$ denote the Julia set of $f$, and let

$$\text{Crit}'(f) = \text{Crit}(f) \cap J(f).$$

For every $z \in \text{Crit}(f)$ and $\delta > 0$ we denote by $B(f(z), \delta)$ the Euclidean ball of radius $\delta$ and centered at $f(z)$. Moreover, we denote by $\bar{B}(z, \delta)$ the connected component of $f^{-1}(B(f(z), \delta))$ that contains $z$.

**Definition 1.** Given a constant $r > 1$, we say that $f$ satisfies the backward contraction property with constant $r$ ($f \in BC(r)$ in short) if there exists...
\[ \delta_0 > 0 \] such that for every \( c \in \text{Crit}'(f) \), every \( 0 < \delta \leq \delta_0 \), every integer \( n \geq 1 \) and every component \( W \) of \( f^{-n}(B(c, r \delta)) \), we have that
\[ \text{dist}(W, CV(f)) \leq \delta \Rightarrow \text{diam}(W) < \delta \]
where \( CV(f) = f(\text{Crit}(f)) \). If \( f \in BC(r) \) for all \( r > 1 \), we will say that \( f \in BC(\infty) \).

In [13, Theorem A], the author showed that for a complex polynomial (or more generally a rational map), if
\[ \sum_{n=0}^{\infty} \frac{1}{|Df^n(f(c))|} < \infty \]
holds for all \( c \in \text{Crit}'(f) \), then \( f \) satisfies \( BC(\infty) \).

**Definition 2.** We say that a polynomial \( f \) satisfies the large derivative condition with constant \( K \) \( (f \in LD(K) \) in short) if there exists a neighborhood \( V \) of \( \text{Crit}'(f) \) such that for each \( c \in \text{Crit}'(f) \) and \( n \geq 1 \) with \( f^n(c) \in V \), we have
\[ |Df^n(f(c))| \geq K. \]
If \( f \in LD(K) \) for all \( K > 0 \), we will say that \( f \in LD(\infty) \).

Obviously, given a polynomial \( f \), if for every \( c \in \text{Crit}'(f) \), we have
\[ \lim_{n \to \infty} |Df^n(f(c))| = \infty \]
then \( f \) satisfies \( LD(\infty) \).

This definition was given first in [1] for \( C^3 \) interval maps with non-flat critical points and with all periodic points hyperbolic repelling, where it was proved that for such maps, \( f \in LD(\infty) \) implies \( f \in BC(\infty) \), where the properties \( LD(K) \) and \( BC(r) \) are defined as above except that we use the standard metric on the interval and use \( \text{Crit}'(f) = \text{Crit}(f) \). The proof uses a special tool in real one-dimensional dynamics, namely the one-sided Koebe principle, which has no complex analogy.

We shall prove the following in [2].

**Theorem A** (Large derivative implies backward contraction). For each integer \( d \geq 2 \), there exists \( K_0 = \tilde{K}_0(d) > 0 \) such that if \( f \) is a polynomial of degree \( d \) which is at most finitely renormalizable and has only hyperbolic periodic points and if \( f \) satisfies \( LD(\tilde{K}_0 r) \) for some \( r > 1 \), then \( f \) satisfies \( BC(r) \).

Recall that a renormalization of \( f \) is a map \( f^s : U \to V \), where \( s \) is a positive integer and \( V \supseteq U \) are Jordan disks, such that the following hold:
- \( f^s : U \to V \) is proper;
- \( U \) contains a critical point in \( J(f) \);
- the following set (called the filled Julia set of \( f^s : U \to V \)) is connected:
  \[ \{ z \in U : f^{sn}(z) \in U \text{ for all } n = 1, 2, \ldots \}; \]
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• for each \( c \in \text{Crit}(f) \), there exists at most one \( j \in \{0,1,\ldots,s-1\} \) with \( c \in f^j(U) \);
• \( U \not\supset J(f) \).

We say that \( f \) is infinitely renormalizable if there exists a sequence of renormalizations \( f^{s_k} : U_k \rightarrow V_k \) such that \( s_k \rightarrow \infty \).

As a consequence of the Schwarz lemma, we shall also prove the following converse statement in [3]:

**Theorem B** (Backward contraction implies large derivative). Let \( f \) be a polynomial of degree at least 2. There exists a constant \( r_0 \) depending on the maximal critical order of \( f \) such that if \( f \) satisfies \( \text{BC}(Kr_0) \) for some \( K > 1 \), then \( f \) satisfies \( \text{LD}(K) \).

Combining these two theorems, we obtain that

**Corollary 1.** Let \( f \) be a polynomial which is at most finitely renormalizable and has only hyperbolic periodic points. Then \( f \) satisfies \( \text{BC}(\infty) \) if and only if \( f \) satisfies \( \text{LD}(\infty) \).

Our proof of the Theorem [A] is based on the following complex bounds established in [6], which depends heavily on the recent analytic result [5] and the enhanced nest construction [7]. See also [12] for the case of Cantor Julia sets. Since the precise form is not stated explicitly in [9], we include a proof of the proposition in the Appendix [A] for completeness.

**Proposition 2** (Complex bounds). Assume that \( f \) is a complex polynomial of degree \( d \geq 2 \) which is at most finitely renormalizable and has only hyperbolic periodic points. Then there exists \( \rho_0 = \rho_0(d) > 0 \) such that for each \( c \in \text{Crit}'(f) \), there exists an arbitrarily small \( \rho_0 \)-nice topological disk which contains \( c \).

Recall that an open set \( V \subset \mathbb{C} \) is called nice if \( f^n(\partial V) \cap V = \emptyset \) for all \( n \geq 0 \). We say that \( V \) is \( \rho \)-nice if for each return domain \( U \) of \( V \), there is an annulus \( A \subset V \setminus \overline{U} \) such that \( U \) is contained in the bounded component of \( \mathbb{C} \setminus A \) and such that \( \text{mod}(A) \geq \rho \).

In appendix [B] we shall prove similar results for interval maps.

**Terminology and notation:**

A topological disk means a simply connected domain in \( \mathbb{C} \). An annulus \( A \) is a doubly connected domain in \( \mathbb{C} \), and the modulus \( \text{mod}(A) \) of \( A \) is defined to be \( \log R/r \), where \( A \) is conformal isomorphic to the round annulus \( \{ r < |z| < R \} \). Given an open set \( V \) and a set \( E \) with \( \overline{E} \subset V \), let

\[
\text{mod}(V; E) = \sup_A \text{mod}(A),
\]

where the supremum is taken over all annuli \( A \) with the property that \( A \subset V \setminus \overline{E} \) and \( E \) is contained in the bounded component of \( \mathbb{C} \setminus A \).

Given a nice set \( V \), let

\[
D(V) = \{ z \in \mathbb{C} : f^n(z) \in V \text{ for some } n \geq 1 \}.
\]
The first entry map $R_V : D(V) \to V$ is defined as $z \mapsto f^{k(z)}(z)$, where $k(z)$, called the entry time of $z$ into $V$, is the minimal positive integer such that $f^{k(z)}(z) \in V$. Since $f$ is continuous and $V$ is nice, $k(z)$ is constant in any component of $D(V)$. A component of $D(V)$ is called an entry domain. The map $R_V |_{D(V)} \to V$ is called the first return map of $V$, and a component of $D(V) \cap V$ is called a return domain. For any $x \in D(V)$, let $L_x(V)$ denote the entry domain which contains $x$. Moreover, for $x \in D(V) \cup V$, let $\hat{L}_x(V) = L_x(V)$ if $x \in D(V) \setminus V$, and let $\hat{L}_x(V) = V$ if $x \in V$.

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2. Large derivative implies backward contraction

The goal of this section is to prove Theorem A. So let us fix an integer $d \geq 2$ and let $f$ denote a polynomial of degree $d$. We assume throughout this section that $f$ is at most finitely renormalizable and has only hyperbolic periodic points. For each critical point $c$, let $\ell_c$ denote the order of $c$, and let $\ell_{\max}(f) = \max\{\ell_c : c \in \text{Crit}(f)\}$.

In the following we shall not state explicitly dependence of constants on the degree $d$. So a constant depending only on $d$ will be called universal.

2.1. Preparation. We shall use the following variation of the Koebe principle.

Lemma 3. For any $\rho > 0$ and $N \geq 1$, there exists $A_0 = A_0(\rho, N) > 0$ such that the following holds. Let $V \supset D, U \supset E$ be bounded topological disks and let $s$ be a positive integer with the following properties:

- $\text{mod}(V; D) \geq \rho$;
- $U$ is a component of $f^{-s}(V)$;
- the degree of $f^s : U \to V$ is at most $N$.
- $E$ is a connected component of $f^{-s}(D)$.

Then for any $x \in E$,

$$|Df^s(f(x))| \leq A_0 \frac{\text{diam}(f(D))}{\text{diam}(f(E))},$$

provided that $\text{diam}(D)$ is small enough.

Proof. Certainly we only need to prove the lemma in the case $\rho < 1$. By considering a suitable restriction of the map $f^s : U \to V$ we may assume that $\text{mod}(V; D) = \rho$.

Let $\varphi : f(U) \to \mathbb{D}$, $\psi : V \to \mathbb{D}$ be Riemann mappings with $\varphi(f(x)) = 0$ and $\psi(f^s(x)) = 0$, where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. Then $F := \psi \circ f^{-s} \circ \varphi^{-1}$ is a holomorphic map from $\mathbb{D}$ into itself and $F(0) = 0$. Thus by the Schwarz lemma, $|F'(0)| \leq 1$. Since $F$ is proper and $\deg(F) \leq N$, we have

$$\text{mod}(f(U); f(E)) \geq \frac{1}{\deg(F)} \text{mod}(V; D) \geq \frac{\rho}{N}.$$
By the Koebe distortion theorem, it follows that
\[ |\varphi'(f(x))| \leq C_1 \text{diam}(f(E))^{-1}, \]
where \( C_1 = C_1(\rho, N) \) is a constant. Since
\[ \text{mod}(\mathbb{D}; \psi(D)) = \text{mod}(V; D) = \rho < 1, \]
diam(\( \psi(D) \)) \( \geq e^{-1} \). Again by the Koebe distortion theorem, we obtain that
\[ |(\psi^{-1})'(0)| \leq C_2 \text{diam}(\mathbb{D}), \]
where \( C_2 = C_2(\rho) \). Finally, provided that diam(\( \mathbb{D} \)) is small enough, we have
\[ |Df(f^s(x))| \leq C_3 \text{diam}(\mathbb{D}) \]
where \( C_3 > 0 \) is a constant depending only on the degree of \( f \). Combining all these estimates, we obtain
\[ |Df(f^s(x))| = |Df(f^s(x))||F'(0)||\varphi'(f(x))||(|\psi^{-1})'(0)| \]
\[ \leq C_1 C_2 C_3 \frac{\text{diam}(f(D))}{\text{diam}(f(E))}. \]
Thus the lemma holds with \( A_0 = C_1 C_2 C_3. \) \( \square \)

Given an bounded open set \( \Omega \subset \mathbb{C} \) and \( z \in \Omega \), let
\[ IR(\Omega, z) = \inf_{w \in \partial \Omega} d(z, w), \]
\[ OR(\Omega, z) = \sup_{w \in \partial \Omega} d(z, w) \]
and
\[ \text{Shape}(\Omega, z) = \frac{OR(\Omega, z)}{IR(\Omega, z)}. \]

We shall use the following procedure to construct nice sets with bounded shape. Given a nice topological disk \( V \), a point \( z_0 \in V \) and a constant \( \lambda > 0 \), let \( B_V(z_0, \lambda) \) be the hyperbolic ball centered at \( z_0 \) and of radius \( \lambda \) (in the hyperbolic Riemann surface \( V \)), let \( V_s[z_0, \lambda] \) be the union of \( B_V(z_0, \lambda) \) and all the return domains of \( V \) that intersect this set, and let \( V[z_0, \lambda] \) be the filling of \( V_s[z_0, \lambda] \), i.e. the union of \( V_s[z_0, \lambda] \) and the bounded components of \( \mathbb{C} \setminus V_s[z_0, \lambda] \). Clearly, for each \( n \geq 1 \),
\[ f^n(\partial V[z_0, \lambda]) \cap V = \emptyset. \]

**Lemma 4.** For each \( \rho > 0 \) and \( \lambda > 0 \) there exist \( M > 1 \) and \( \rho' > 0 \) such that if \( V \) is a \( \rho \)-nice topological disk, then for any \( z_0 \in V \), \( V[z_0, \lambda] \) is \( \rho' \)-nice and \( \text{Shape}(V[z_0, \lambda], z_0) \leq M. \)

**Proof.** Write \( U = V[z_0, \lambda] \). Since \( V \) is \( \rho \)-nice, the hyperbolic diameters of return domains of \( V \) in \( V \) are uniformly bounded from above by a constant depending only on \( \rho \). It follows that the hyperbolic diameter of \( V_s[z_0, \lambda] \), hence that of \( V[z_0, \lambda] \) in \( V \) is bounded from above by a constant depending only on \( \rho \). Consequently, \( \text{mod}(V; U) \) is bounded away from zero. By the Riemann mapping theorem and the Koebe distortion theorem, we obtain that \( \text{Shape}(U, z_0) \) is bounded from above.
Let us prove that $U$ is $\rho'$-nice. Indeed, each return domain $W$ of $U$ is contained in a return domain $W'$ of $V$. The first return map $R_V$ to $V$ maps $W$ into either $U$ or a return domain of $V$. In both cases, we have that $\text{mod}(V; R_V(W))$ is bounded away from zero. Since $R_V|W'$ has bounded degree and since $W' \subset U$, we obtain that $\text{mod}(U; W)$ is bounded away from zero. \hfill $\square$

We say that $(\mathcal{V}', \mathcal{V})$ is an admissible pair of neighborhoods of a set $A \subset \text{Crit}'(f)$ if the following hold:

- $\mathcal{V} \subset \mathcal{V}'$;
- $\mathcal{V}$ (resp. $\mathcal{V}'$) is nice and each component of $\mathcal{V}$ (resp. $\mathcal{V}'$) is a topological disk containing exactly one point of $A$;
- for all $n \geq 1$ and $a \in A$,
  
  $$f^n(\partial \mathcal{V}_a) \cap \mathcal{V}_a' = \emptyset,$$
  
  where $\mathcal{V}_a$ (resp. $\mathcal{V}_a'$) denotes the component of $\mathcal{V}$ (resp. $\mathcal{V}'$) which contains $a$.

We say that $(\mathcal{V}', \mathcal{V})$ is called $\rho$-bounded if for every $a \in A$, we have

$$\text{mod}(\mathcal{V}_a'; \mathcal{V}_a) \geq \rho.$$

For each $c_0 \in \text{Crit}'(f)$, let

$$\text{Back}(c_0) = \{ c \in \text{Crit}'(f) : \{ f^n(c) : n \geq 0 \} \ni c_0 \} \ni c_0.$$ 

We shall need the following lemma which is essentially [7, Lemma 6.3].

**Lemma 5.** Let $c_0 \in \text{Crit}'(f)$ and let $(\mathcal{V}', \mathcal{V})$ be an admissible pair of neighborhoods of $\text{Back}(c_0)$. Assume that $\max_{c \in \text{Back}(c_0)} \mathcal{V}'_c$ is sufficiently small. Let $U$ be an entry domain of $\mathcal{V}$ with entry time $s$, let $c_1 \in \text{Back}(c_0)$ be such that $f^s(U) = \mathcal{V}_{c_1}$, and let $U'$ be the component of $f^{-s}(\mathcal{V}'_{c_1})$ that contains $U$, then the degree of $f^s : U' \to \mathcal{V}_{c_1}'$ does not exceed a universal constant $N_0$.

**Proof.** Assume that $\text{diam}(\mathcal{V}'_{c_1})$ is small enough, so that for any $c \in \text{Crit}'(f)$, if there exists $k \geq 1$ such that $f^k(c) \in \mathcal{V}_{c_1}$ then $c \in \text{Back}(c_1) \subset \text{Back}(c_0)$. For $c \in \text{Back}(c_0)$, and any $i \in \{ 1, 2, \ldots, s - 1 \}$, $f^i(U') \neq c$, for otherwise, we would have $f^i(U') \subset \mathcal{V}_c$, contradicting the hypothesis that $s$ is the entry time of $U$ into $\mathcal{V}$. \hfill $\square$

### 2.2. Nested nice sets

In this section, we shall prove the following proposition, which is a crucial step of the proof of Theorem A.

**Proposition 6** (Nested nice sets). *There exist universal constants $K_s > 0$, $\kappa_s > 0$ and $C_s > 0$ such that if $f$ satisfies $LD(K)$ with $K \geq K_s$, then for each $c_0 \in \text{Crit}'(f)$, there exists an infinite sequence of nice topological disks*

$$V_1 \supset V_2 \supset \cdots$$

*that contain $c_0$ such that the following hold:*

1. $\text{diam}(V_k) \to 0$ as $k \to \infty$;
2. for each $k \geq 1$, $\text{Shape}(f(V_k), f(c_0)) \leq C_s$;
3. for each \(k \geq 1\), \(\kappa_* \text{diam}(f(V_k)) \leq \text{diam}(f(V_{k+1}))\);
4. for each \(k \geq 1\), and any \(c \in \text{Back}(c_0)\), we have
\[
\text{diam}(f(\mathcal{L}_c(V_k))) \leq \frac{C_1}{K} \text{diam}(f(V_k)).
\]

To prove this proposition, we need a few lemmas.

Given a point \(c \in \text{Crit}(f)\), a topological disk \(W \ni c\) is called \((\rho,M)\)-bounded if \(W\) is \(\rho\)-nice and \(W\) satisfies \(\text{Shape}(W,c) \leq M\). Sometimes we say that \(W\) is uniformly bounded if it is \((\rho,M)\)-bounded for some universal constants \(\rho\) and \(M\).

**Lemma 7.** Given any \(\rho > 0\) and \(M > 0\), there exist \(K_1 = K_1(\rho) > 0\) and \(C = C(\rho,M) > 1\) such that if \(f\) satisfies \(LD(K)\) with \(K \geq K_1\), the following holds. Given \(c_0 \in \text{Crit}(f)\) and a \((\rho,M)\)-bounded puzzle piece \(\hat{V} \ni c_0\) with \(\text{diam}(\hat{V})\) sufficiently small, then \(V := \hat{V}[c_0,1]\) satisfies the following properties:

(i) for each \(c \in \text{Back}(c_0)\), we have
\[
\text{diam}(f(\mathcal{L}_c(V))) \leq \frac{C}{K} \text{diam}(f(V)).
\]

(ii) For each return domain \(U\) of \(V\), either
\[
U \subset \bar{B}(c_0, CK^{-1} \text{diam}(f(V)))
\]
or
\[
\text{diam}(U) \leq C \text{dist}(U,c_0).
\]

**Proof.** Let us first construct an admissible pair \((\hat{V},V)\) of neighborhoods of \(\text{Back}(c_0)\) as follows. Let \(\mathcal{D}\) be the collection of return domains of \(\hat{V}\) outside \(V\) and \(V\) itself. Let \(c_1, c_2, \ldots, c_m\) be the critical points in \(\text{Back}(c_0) \setminus \{c_0\}\). For each \(i = 0, 1, \ldots, m\), let \(\hat{V}^i = \mathcal{L}_{c_i}(\hat{V})\) and let \(t_i\) be the landing time of \(c_i\) into \(\hat{V}\). Let \(D_i\) be the element of \(\mathcal{D}\) that contains \(f^{t_i}(c_i)\) and let \(V^i = \text{Comp}_{c_i}(f^{-t_i}(D_i))\). (So \(t_0 = 0\), \(\hat{V}^0 = \hat{V}\), and \(V^0 = V\).) Moreover, let
\[
\hat{V} = \bigcup_{i=0}^m \hat{V}^i, \quad V = \bigcup_{i=0}^m V^i.
\]

For each \(i = 1, 2, \ldots, m\), the degree of the first entry map \(f^{t_i} : \hat{V}^i \to \hat{V}\) is bounded from above by \(N := \deg(f)^m\). Since \(\hat{V}\) is \(\rho\)-nice, by Lemma 4, there exists \(\rho' = \rho'(\rho) > 0\) such that \(\text{mod}(\hat{V};D) > \rho', D \in \mathcal{D}\). Thus
\[
\text{mod}(\hat{V}^i;V^i) \geq \rho'/N.
\]
Provided that \(\text{diam}(\hat{V})\) is small enough, by the assumption that \(f\) satisfies \(LD(K)\), we have
\[
|Df^{t_i}(f(c_i))| \geq K, i = 1, 2, \ldots, m.
\]
Thus, by Lemma 3 we have
\[ \text{diam}(f(V^i)) \leq \frac{A_0(\rho', N)}{|Df_t(f(c_i))|} \text{diam}(f(D_i)) \leq \frac{A_0(\rho', N)}{K} \text{diam}(f(\hat{V})). \]

Since \( \text{Shape}(\hat{V}, c_0) \leq M \), there exists \( M' \) depending only on \( M \) such that
\[ \text{diam}(f(\hat{V})) \leq M' \text{diam}(f(V)). \]

Since \( \mathcal{L}_c(V) \subset V^i \), we obtain that
\[ (1) \quad \text{diam}(f(\mathcal{L}_c(V))) \leq \text{diam}(f(V^i)) \leq C_1 K^{-1} \text{diam}(f(V)), \]
holds for all \( i = 1, 2, \ldots, m \), where \( C_1 = A_0(\rho', N) M' \) is a constant.

Assume now that \( f \) satisfies \( LD(K) \) with \( K \geq K_1 := A_0(\rho'/2, N) \).

Claim. There exists a constant \( C = C(\rho, M) \) such that if \( \text{diam}(\hat{V}) \) is small enough, then the following holds: for each return domain \( U \) of \( V \) with \( U \subset V \),
\[ \text{either } U \subset B(c_0, CK^{-1} \text{diam}(f(V))) \text{ or } \text{diam}(U) \leq C \text{ dist}(U, c_0). \]

To prove this claim, let \( \hat{D}_i \) be the topological disk with \( D_i \subset \hat{D}_i \subset \hat{V} \) and with \( \text{mod}(\hat{V}; \hat{D}_i) = \text{mod}(\hat{D}_i; D_i) = \text{mod}(\hat{V}; D_i)/2 \), and let \( \tilde{V}^i \) be the component of \( f^{-i}(D_i) \) that contains \( c_i \). Then
\[ \text{mod}(\tilde{V}^i; V^i) \geq \mu := \frac{\rho'}{2N}, \text{ mod}(\tilde{V}^i; V^i) \geq \mu \]
and
\[ \text{diam}(f(\tilde{V}^i)) \leq \frac{A_0(\rho'/2, N)}{K} \text{diam}(f(\hat{D}_i)) \leq \text{diam}(f(\hat{V})). \]

For each return domain \( U \) of \( V \) with return time \( s \) and with \( f^s(U) = V^i \), let \( \hat{U} \) and \( \tilde{U} \) be the components of \( f^{-s}(\hat{V}^i) \) and \( f^{-s}(\tilde{V}^i) \) that contain \( U \) respectively. By Lemma 5, the degree of \( f^s : \hat{U} \to \tilde{V}^i \) is bounded from above by \( N_0, \text{mod}(\hat{U}; U) \geq \rho'/2(2NN_0) \). If \( \hat{U} \not\supset c_0 \), then \( \text{diam}(U)/\text{dist}(U, c_0) \) is bounded from above by a constant depending on \( \rho \). If \( \hat{U} \supset c_0 \), then by Lemma 3 we obtain
\[ \text{diam}(f(\tilde{U})) \leq A_0(\mu, N) K^{-1} \text{diam}(f(\tilde{V}^i)) \leq A_0(\mu, N) K^{-1} \text{diam}(f(\hat{V})), \]
which implies that \( U \subset B(c_0, CK^{-1} \text{diam}(f(V))) \), where \( C = A_0(\mu, N) M' \) depends only on \( \rho \) and \( M \). This proves the claim.

Since each return domain of \( V \) is contained in a return domain of \( V \), the statement (ii) follows from the claim. Moreover, the claim implies that \( \text{diam}(f(\mathcal{L}_c(V))) \leq C K^{-1} \text{diam}(f(V)) \), which together with (1) implies (i).

\[ \square \]

**Lemma 8.** For each \( \rho > 0 \), there exists \( K_2 = K_2(\rho) > 0 \) such that if \( f \) satisfies \( LD(K_2) \), then the following holds. If \( W \) is a \( \rho \)-nice puzzle piece that contains a critical point \( c \in \text{Crit}(f) \) and if \( \text{diam}(W) \) is small enough, then either \( \mathcal{L}_c^3(W) = \emptyset \), or
\[ \text{mod}(W; \mathcal{L}_c^3(W)) \geq 1. \]
Proof. Let $\widehat{V} = W[c, 1], V = \widehat{V}[c, 1]$. By Lemma 4, there exists $\rho' > 0$ and $M > 1$ depending only on $\rho$ such that both $\widehat{V}$ and $V$ are $(\rho', M)$-bounded. By Lemma 7, if $f$ satisfies $LD(K)$ with $K \geq K_1(\rho')$, then
\[
\text{diam}(f(L_c(V))) \leq CK^{-1} \text{diam}(f(V)),
\]
where $C = C(\rho', M)$ (depending only on $\rho$). Thus $\text{mod}(V; L_c(V)) \geq 1$ provided that $K$ is large enough. Since $L_c^3(W) \subset L_c(V)$, the lemma follows. 

Lemma 9. For each $\rho > 0$, there exists $K_3 = K_3(\rho) > 0$ such that if $f$ satisfies $LD(K_3)$, then the following holds. Let $(V', V)$ be an admissible pair of neighborhoods of some $c_0 \in \text{Crit}(f)$ such that $V'$ is $\rho$-nice and $\text{mod}(V'; V) \geq 1$. Assume furthermore that $\text{diam}(V')$ is small enough. Then $V$ is $\rho_s$-nice, where $\rho_s > 0$ is a universal constant (independent of $\rho$).

Proof. Let $U$ be a return domain of $V$ with return time $s$. We need to find a universal bound for $\text{mod}(V; U)$. Let $U'_j$ be the component of $f^{-(s-j)}(V')$ which contains $f^j(U)$, $j = 0, 1, \ldots, s$. Then $U'_0 \subset V$ since $(V', V)$ is an admissible pair. If for each $c \in \text{Crit}'(f)$,
\[
\#\{0 \leq j < s : U'_j \ni c\} < 5,
\]
then $f^s : U'_0 \to V'$ has uniformly bounded degree, hence $\text{mod}(V; U)$ is bounded from below by a positive constant. So assume that (2) fails for some $c \in \text{Crit}'(f)$. Then clearly $c \in \text{Back}(c_0) \setminus \{c_0\}$ and there exists $1 \leq s_1 < s$ such that $U'_{s_1} \subset L_c^4(V')$. Therefore, there exists a minimal integer $s' \in \{1, 2, \ldots, s-1\}$ such that $U'_{s'} \subset L_c^3(V')$ for some $c' \in \text{Back}(c_0) \setminus \{c_0\}$. Let $W'_j$ be the component of $f^{-(s'-j)}(L_c(V'))$ which contains $f^j(U)$. By the minimality of $s'$, we have that for each $c \in \text{Crit}'(f)$,
\[
\#\{0 \leq j < s' : W'_j \ni c\} < 4.
\]
Thus $f^{s'} : W'_0 \to L_c(V')$ has uniformly bounded degree. Since $V'$ is $\rho$-nice, $L_c(V')$ is $\rho'$-nice, where $\rho' > 0$ is a constant depending only on $\rho$. By Lemma 8, if $f$ satisfies $LD(K_3)$ with $K_3 = K_3(\rho) = K_2(\rho')$, then
\[
\text{mod}(L_c(V'); f^{s'}(U)) \geq \text{mod}(L_c(V'); L_c^4(V')) \geq 1.
\]
It follows that $\text{mod}(V; U) \geq \text{mod}(W'_0; U)$ is bounded from below by a positive constant. 

Lemma 10. Given any $\rho > 0$ and $M > 0$ there exist constants $\hat{K} = \hat{K}(\rho, M) > 0$ and $\kappa \in (0, 1)$ such that if $f$ satisfies $LD(\hat{K})$, then the following holds. Given a $(\rho, M)$-bounded puzzle piece $\widehat{V}$ which contains $c_0 \in \text{Crit}(f)$ and such that $\text{diam}(\widehat{V})$ is small enough, there exists a $(\rho_1, M_1)$-bounded puzzle piece $\widehat{V}_1 \ni c_0$ such that
\[
\kappa \text{diam}(f(\widehat{V})) \leq \text{diam}(f(\widehat{V}_1)) \leq \frac{1}{e} \text{diam}(f(\widehat{V})),
\]
where $\rho_1 > 0, M_1 > 1$ are universal constants (independent of $\rho, M$).
we say that $f$ satisfies $LD(K)$ with a large constant $K$. By Lemma 7, each return domain $U$ of $V := \hat{V}[c_0, 1]$ satisfies one of the following: $U \subset B(c_0, CK^{-1} \text{diam}(f(V)))$ or $\text{diam}(U) \leq C \text{dist}(U, c_0)$, where $C = C(\rho, M)$ is a constant. By Lemma 4, Shape($V, c_0$), hence Shape($f(V), f(c_0)$) is bounded from above by a constant $M'$ depending on $\rho$ and $\ell_{c_0}$. Let $\varepsilon = (2M'(C + 2)e^{2\pi \ell_{c_0}})^{-1}$ and assume that $K > C/\varepsilon$. Let $\hat{V}$ be the filling of the union of $B(c_0, \varepsilon \text{diam}(f(V)))$ and all the return domain of $V$ that intersect $B(c_0, \varepsilon \text{diam}(f(V)))$. Then, $\hat{V} \subset B(c_0, (2M'e^{2\pi \ell_{c_0}})^{-1} \text{diam}(f(V)))$, hence $\text{mod}(V; \hat{V}) \geq 1$. By Lemma 9, we obtain that $\hat{V}$ is $\rho_\ast$-nice. Take $\hat{V}_1 = \hat{V}[c_0, 1]$. By Lemma 4, $\hat{V}_1$ is $(\rho_1, M_1)$-bounded for some universal constants $\rho_1$ and $M_1$. The estimate on the diameter of $\text{diam}(f(\hat{V}_1))$ follows from the construction. □

**Lemma 11.** There exist universal constants $\hat{K}_\ast > 0$ and $\hat{\kappa}_\ast > 0$ such that if $f$ satisfies $LD(\hat{K}_\ast)$, then for each $c_0 \in \text{Crit}(f)$, there exists an infinite sequence of $(\rho_1, M_1)$-bounded nice topological disks $\hat{V}_1 \supset \hat{V}_2 \supset \cdots$ that contain $c_0$ such that for each $k \geq 1$,

$$\hat{\kappa}_\ast \text{diam}(f(\hat{V}_k)) \leq \text{diam}(f(\hat{V}_{k+1})) \leq e^{-1} \text{diam}(f(\hat{V}_k))$$

**Proof.** It suffices to prove existence of an arbitrarily small $(\rho_1, M_1)$-bounded nice topological disk $\hat{V}_0 \ni c_0$ under the assumption that $f$ satisfies $LD(K)$ with a large $K$, since then we may apply Lemma 10 successively to obtain the desired sequence.

By Proposition 2 there exists an arbitrarily small $\rho_0$-nice topological disk $V \ni c_0$. Let $\hat{V} = V[c_0, 1]$. By Lemma 4, $\hat{V}$ is uniformly bounded, so by Lemma 10, we obtain existence of $\hat{V}_0$. □

**Proof of Proposition 6.** Let $K_\ast = \max(\hat{K}_\ast, K_1(\rho_1))$. Assume that $f$ satisfies $LD(K)$ with $K \geq K_\ast$. Let $\hat{V}_k$ be as in Lemma 11 and let $V_k = \hat{V}_k[c_0, 1]$. Then the first, second and third statements hold with suitable choices of $C_\ast$ and $\kappa_\ast$. By Lemma 4, Shape($V_k, c_0$), hence Shape($f(V_k), f(c_0)$), are uniformly bounded. By Lemma 7 the last statement holds. □

2.3. **Proof of Theorem A.** In [13, Section 6], the author introduced another notion called univalent pull back condition, which is closely related to backward contraction. (A similar notion, $\text{BC}^*(r)$, was used in [11].)

Given $\delta' > \delta > 0$, we say that $f$ satisfies the $(\delta, \delta')$-univalent pull back condition if for every $z \in \mathbb{C}$ and every integer $n \geq 1$ such that

- for each $j = 1, 2, \ldots, n - 1, f^j(z) \not\in \bigcup_{c \in \text{Crit}(f)} \overline{B}(c, \delta)$
- for some $c \in \text{Crit}(f)$, we have $f^n(z) \in \overline{B}(c, \delta')$,

then $f^n$ maps a neighborhood of $z$ conformally onto $\overline{B}(c, \delta')$. Given $r > 1$, we say that $f$ satisfies the univalent pull back condition with constant $r$
if for all $\delta > 0$ sufficiently small, $f$ satisfies the $(\delta, r\delta)$-univalent pull back condition.

The following is [13 Proposition 6.1 part 2].

**Lemma 12.** There exists a constant $r_0 > 1$ such that if $f$ satisfies the univalent pull back condition with constant $r r_0$, where $r > 1$ is a constant, then $f$ satisfies the backward contracting condition with constant $r$.

**Proof of Theorem 2.** Let $K_0 = \max(K_*, 2C_0^2\kappa_*^{-1}r_0)$, where $K_*, C_* > 0$ and $\kappa_* > 0$ are as in Proposition 6.

Fix $r > 1$ and assume that $f$ satisfies $LD(K)$ with $K = rK_0$. Let us prove that $f$ satisfies the univalent pull back condition with constant $r r_0$ which implies that $f$ satisfies the backward contraction condition with constant $r$ by Lemma 12. To this end, it suffices to prove that for each $c_0 \in \text{Crit}'(f)$ and each $\delta > 0$ small enough, if $U$ is a pull back of $\tilde{B}(c_0, \delta)$ that intersects $\text{Crit}'(f)$, then $\text{diam}(f(U)) \leq (rr_0)^{-1}\delta$. Let $V_k$ be as given in proposition 6. For each $\delta \in (0, (2C_0)^{-1}\text{diam}(V_1))$, there exists a maximal integer $k \geq 1$ such that $\tilde{B}(c_0, \delta) \subseteq V_k$. By part 2 and 3 of the proposition 6 we have

$$\text{diam}(f(V_k)) \leq 2C_0\kappa_*^{-1}\delta.$$ 

If $U$ is a pull back of $\tilde{B}(c_0, \delta)$ that contains a critical point $c$, then $U \subseteq \mathcal{L}_c(V_k)$. Thus by part 4 of the proposition 6 we obtain

$$\text{diam}(f(U)) \leq \text{diam}(f(\mathcal{L}_c(V_k))) \leq C_*K^{-1}\text{diam}(f(V_k)) \leq 2C_0^2\kappa_*^{-1}K^{-1}\delta \leq (rr_0)^{-1}\delta.$$

The proof is completed. \hfill $\Box$

### 3. Backward contraction implies large derivatives

In this section, we shall prove Theorem 13. Let $f$ be a polynomial which satisfies the backward contraction condition with constant $r > 4$. So there exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0]$ and each $c, c' \in \text{Crit}'(f)$, if $U \ni c$ is a component of $f^{-n}(\tilde{B}(c', r\delta))$ then $\text{diam}(f(U)) < \delta$. We continue to use $\ell_{\max}$ to denote the maximal order of critical points in the Julia set.

For each $n \geq 0$ and $c \in \text{Crit}'(f)$, let $\delta_n = 2^{-n}\delta_0$, $V_n^c = \tilde{B}(c, \delta_n)$ and let $V_n = \bigcup_{c \in \text{Crit}'(f)} V_n^c$.

By reducing $\delta_0$ if necessary, we may assume that for each $x \in V_n \setminus V_{n+2}$, and $n \geq 0$,

$$\kappa_0^{-1}\text{diam}(f(V_n)) \geq \text{diam}(V_n)|Df(x)| \geq \kappa_0 \text{diam}(f(V_n)),$$

and for each $c \in \text{Crit}'(f)$ and $0 < \delta < \delta' < r\delta_0$,

$$\kappa_0^{-1}\left(\frac{\delta'}{\delta}\right)^{1/\ell_c} \geq \frac{\text{diam}(\tilde{B}(c, \delta'))}{\text{diam}(\tilde{B}(c, \delta))} \geq \kappa_0 \left(\frac{\delta'}{\delta}\right)^{1/\ell_c}$$

where $\kappa_0$ is a constant depending only on $\ell_{\max}$. 


Lemma 13. For each \( n \geq 1 \) and \( c \in \text{Crit}'(f) \), if \( s \) is a positive integer with \( f^s(c) \in V_n \setminus V_{n+1} \) and with \( f^j(c) \notin V_{n+1} \) for all \( 1 \leq j < s \), then
\[
|Df^s(f(c))| \geq \kappa r,
\]
where \( \kappa > 0 \) is a constant depending only on \( \ell_{\max}(f) \).

Proof. Let \( c' \in \text{Crit}'(f) \) be such that \( f^s(c) \in V_n' \). For each \( j = 0, 1, \ldots, s \), let \( U_j \) be the connected component of \( f^{-(s-j)}(\tilde{B}(c', \delta_{n-1})) \) which contains \( f^j(c) \). Since \( f^j(c) \notin V_{n+1} \) for all \( 1 \leq j < s \), and since \( f \) is backward contracting with constant \( r > 4 \), we have \( \text{diam}(U_1) \leq \delta_{n-1}/r \), and \( U_j \cap \text{Crit}'(f) = \emptyset \) for all \( j = 1, 2, \ldots, s-1 \). Thus \( f^{s-1}: U_1 \rightarrow U_s \) is conformal. Since \( f^s(c) \in V_n' \), there is a constant \( \tau > 0 \) depending on the order of \( c' \) such that
\[
\tilde{B}(c', \delta_{n-1}) \supset B(f^s(c), \tau \text{diam}(\tilde{B}(c', \delta_{n-1}))).
\]
Applying the Schwarz lemma to the inverse of \( f^{s-1}: U_1 \rightarrow U_s \), we obtain
\[
|Df^s(f(c))| = |Df^{s-1}(f(c))||Df(f^s(c))| \geq \tau \frac{\text{diam}(\tilde{B}(c', \delta_{n-1}))}{\text{diam}(U_1)} |Df(f^s(c))|,
\]
which implies by (3) that \( |Df^s(f(c))| \geq 2\tau \kappa_0 r \). Defining \( \kappa = 2\tau \kappa_0 \) completes the proof. \( \square \)

Lemma 14. There exists \( r_* > 4 \) depending only on \( \ell_{\max}(f) \) such that if \( f \) satisfies \( BC(r_*) \) then the following holds. Let \( x \in V_{n+1} \) and let \( s \) be a positive integer with \( f^s(x) \in V_n \setminus V_{n+1} \) and with \( f^j(x) \notin V_{n+1} \) for all \( j = 1, 2, \ldots, s-1 \). Then \( |Df^s(f(x))| \geq 1 \).

Proof. Assume that \( f \) satisfies \( BC(r_*) \) with \( r_* \) sufficiently large so that
\[
4\kappa_0^2 \left( \frac{r_*}{2} \right)^\ell_{c'} \geq 1.
\]
Let \( c, c' \in \text{Crit}'(f) \) be such that \( x \in V_n' \) and \( f^s(x) \in V_n' \). For each \( j = 0, 1, \ldots, s \), let \( U_j \) be the connected component of \( f^{-(s-j)}(\tilde{B}(c', \kappa_0 \delta_{n-1})) \) which contains \( f^j(x) \). Since \( f^j(x) \notin \tilde{B}(\text{Crit}'(f), \delta_{n+1}) \) for all \( 1 \leq j < s \), \( f^{s-1}: U_1 \rightarrow U_s \) is conformal. Moreover, \( \text{diam}(U_1) \leq \delta_{n+1} \). Provided that \( r_* \) is large enough, \( U_s \) contains a ball centered at \( f^s(x) \) and of radius at least \( \text{diam}(U_1)/3 \). By the Schwarz lemma, we have
\[
|Df^s(f(x))| = |Df^{s-1}(f(x))||Df(f^s(x))| \\
\geq \frac{\text{diam}(U_s)}{3 \text{diam}(U_1)} |Df(f^s(x))| \\
= \frac{\text{diam}(\tilde{B}(c', r_* \delta_{n+1}))}{3 \text{diam}(U_1)} \frac{\text{diam}(V_n' \setminus V_{n+1})}{\text{diam}(V_n')} |Df(f^s(x))| \\
\geq \kappa_0^2 \left( \frac{r_*}{2} \right)^\ell_{c'} \frac{\text{diam}(fV_n')}{3 \text{diam}(U_1)} \\
\geq \frac{4\kappa_0^2}{3} \left( \frac{r_*}{2} \right)^\ell_{c'} \geq 1.
\]
Proof of Theorem B. Let \( r_0 = \max(r_*, \kappa^{-1}) \), where \( \kappa \) is as in Lemma 13 and \( r_* \) is as in Lemma 14.

Assume that \( f \) satisfies \( BC(Kr_0) \) with \( K \geq 1 \). We shall prove that for any \( S \geq 1 \) and \( c \in \text{Crit}'(f) \) with \( f^S(c) \in V_0, \) \( |Df^S(f(c))| \geq K \).

Given \( S \) and \( c \) as above, let us define inductively non-negative integers \( n_1 > n_2 > \cdots > n_m \) and \( S_1 < S_2 < \cdots < S_m = S \) as follows. First,
\[
n_1 = \max\{ n \geq 0 : f^j(c) \in V_n \text{ for some } 1 \leq j \leq S \},
\]
and
\[
S_1 = \max\{ s \leq S : f^s(c) \in V_{n_1} \}.
\]
(Observe that the backward contracting assumption on \( f \) implies that \( f \) has no critical relation, so \( n_1 \), and hence \( S_1 \) is well-defined.) If \( S_1 = S \) then we stop. Otherwise, let
\[
n_2 = \max\{ n \geq 0 : f^j(c) \in V_n \text{ for some } S_1 < j \leq S \},
\]
and
\[
S_2 = \max\{ s : S_1 < s \leq S, f^s(c) \in V_{n_2} \}.
\]
Repeating the argument, we must stop within finitely many steps.

By Lemma 13, \( |Df^S_1(f(c))| \geq \kappa Kr_0 \geq K \). By Lemma 14 for each \( i = 1, 2, \ldots, m - 1 \) we have
\[
|Df^{S_{i+1}-S_i}(f^{S_{i+1}}(c))| \geq 1.
\]
Thus
\[
|Df^s(f(c))| = |Df^{S_1}(f(c))| \prod_{i=1}^{m-1} |Df^{S_{i+1}-S_i}(f^{S_{i+1}}(c))| \geq K.
\]
\[ \square \]

Appendix A. A priori bounds

The goal of this section is to prove Proposition 2. Besides the results of [6], we shall also use some arguments in [7, Section 6]. Throughout this section, assume that \( f \) is a polynomial, at most finitely renormalizable and having only hyperbolic periodic points.

A.1. The puzzle construction. We shall now describe a puzzle partition which will provide nice topological disks as required. The construction given below is a modification of that in [6, Section 2] and makes use of equipotential curves for the Green function and all bounded periodic Fatou components, external rays and internal rays. The modification is necessary since we use the usual definition of renormalization rather than the one used in [6], and since we want to have an arbitrarily small puzzle piece for each \( c \in \text{Crit}'(f) \).

Let \( G \) denote the Green function of \( f \). A smooth external ray is a smooth gradient line of \( G \) which starts from infinity and tends to the Julia set of \( f \). A smooth external ray has a well-defined angle \( t \in \mathbb{R}/\mathbb{Z} \) at which the
ray goes to $\infty$. An external ray is either a smooth external ray, or else a limit of such rays. So for each $t \in \mathbb{R}/\mathbb{Z}$ there is exactly one smooth external rays $\mathcal{R}^t$, or two (non-smooth) external rays $\mathcal{R}^{t+}$ and $\mathcal{R}^{t-}$. Let $P_{\text{bad}}$ be the set of periodic points which are contained in the forward orbit of either a critical point or the landing point of a non-smooth external ray. Note that $P_{\text{bad}}$ is a finite set. Let $P_{\text{sep}}$ be the set of periodic points which are the common landing point of at least 2 external rays. (Here “sep” stands for “separable”.)

**Observation 1.** Let $K'$ be a periodic component of the filled Julia set of $f$ that contains a critical point but no attracting periodic points. Then $K' \cap P_{\text{sep}}$ is an infinite set.

In fact by [8], we only need to consider the case that $f$ has a connected Julia set. So $\partial K' = J(f)$ and $f$ has only repelling periodic points. Let $N_n$ (resp. $N'_n$) be the number of periodic points (resp. external rays) of $f$ which has period $n$. Then $N_1 = N'_1 + 1$ and for all $n > 1$, $N_n = N'_n$. Let us construct a sequence of integers $s_1 = 1 < s_2 < ...$, such that for each $j \geq 1$, $f$ has a periodic point $p_j$ of period $s_j$ which is not the landing point of an external ray of the same period. Since $N_1 > N'_1$, there exists a fixed point of $f$ which is not the landing point of an external ray fixed by $f$. Suppose now that $p_j$ and $s_j$ have been defined. Let $\gamma_j$ be an external ray landing at $p_j$ and let $s_{j+1}$ be the period of $\gamma_j$. Clearly, $s_{j+1} > s_j \geq 1$. Since $N_{s_{j+1}} = N'_{s_{j+1}}$, $f$ has a periodic point $p_{j+1}$ of period $s_{j+1}$ which is not the landing point of an external ray of period $s_{j+1}$. This proves the existence of $\{s_j\}_{j=1}^\infty$ and $\{p_j\}_{j=1}^\infty$. Since $p_j \in P_{\text{sep}}$ for all $j \geq 1$, the statement follows.

Let $\mathcal{K}$ be the collection of all periodic components of the filled Julia set of $f$ that contain a critical point. For each $K' \in \mathcal{K}$, choose $a_{K'} \in (K' \cap P_{\text{sep}}) \setminus P_{\text{bad}}$ and let $\Xi_{K'}$ be the union of the orbit of $a_{K'}$ and all external rays landing on this orbit. Moreover, let

$$\Theta_1 = \bigcup_{K' \in \mathcal{K}} \Xi_{K'}.$$

Now we shall define for each bounded Fatou component $B$, an equipotential curve $\Gamma_B$ and internal rays $\gamma^\theta_B$, $\theta \in \mathbb{R}/\mathbb{Z}$. We start by choosing bounded periodic Fatou components $B_1, B_2, \ldots, B_m$, such that the orbits of $B_i$'s are pairwise disjoint and such that the grand orbit of $\bigcup_{i=1}^m B_i$ covers the interior of the filled Julia set. By assumption, $f$ has an attracting periodic point $p_i \in B_i$ with period $s_i \geq 1$. We choose Jordan curves $\Gamma_{B_i} \subset B_i$ such that $B_i \setminus \Omega_i$ is disjoint from the orbit of all critical points and such that $\Gamma'_{B_i} := f^{-s_i}(\Gamma_{B_i}) \cap B_i$ lies in the interior of $B_i \setminus \Omega_i$, where $\Omega_i$ is the topological disk bounded by $\Gamma_{B_i}$. Then $\Gamma'_{B_i}$ is also a Jordan curve which bounds a topological disk $\Omega'_i$. Denote by $d_i$ the degree of the map $f^{s_i} : B_i \rightarrow B_i$. Choose a diffeomorphism

$$h_{B_i} : B_i \setminus \Omega_i \rightarrow \{1 < |z| \leq 2^{d_i}\}$$

such that
For $\theta \in \mathbb{R}/\mathbb{Z}$, define $\gamma_{\theta}^{B} := h_{B}^{-1}((r e^{2\pi i \theta} : 1 < r \leq 2^{d_{i}}))$. Given a bounded Fatou component $B$ there exists a unique $i \in \{1, 2, \ldots, m\}$ and a minimal non-negative integer $n_{B} \geq 0$ such that $f^{n_{B}}(B) = B_{i}$. Let $\Gamma_{B} = f^{-n_{B}}(\Gamma_{B_{i}}) \cap B$ which is also a Jordan curve, and define internal rays $\gamma_{\theta}^{B} = f^{-(n_{B} - i)}(\gamma_{\theta}^{B_{i}}) \cap B$.

**Observation 2.** There exists $\theta_{0} \in \mathbb{R}/\mathbb{Z}$ which is periodic under the map $\theta \to d_{i} \theta$, such that $\gamma_{\theta_{0}}^{B_{i}}$ converges to a periodic point in $\partial B_{i} \setminus P_{\text{bad}}$.

In fact, as in [4 Lemma 2.1], we can prove that for each $\theta \in \mathbb{R}/\mathbb{Z}$ which is periodic under the map $\theta \to d_{i} \theta$, $\gamma_{\theta_{i}}^{B_{i}}$ converges to a periodic point in $\partial B_{i}$. Moreover, for two distinct $\theta_{i}$, $\gamma_{\theta_{i}}^{B_{i}}$ converges to different points. Since $P_{\text{bad}}$ is finite, the statement follows.

For each $i = 1, 2, \ldots, m$, let us fix $\theta_{i} \in \mathbb{R}/\mathbb{Z}$ which is periodic under $t \to d_{i}t$ such that $\gamma_{\theta_{i}}^{B_{i}}$ converges to a periodic point $a_{i} \in \partial B_{i} \setminus P_{\text{bad}}$. Let

$$
\Xi_{i} = \bigcup_{k=0}^{s_{i}-1} \left( \Gamma_{f^{k}(B_{i})} \cup \bigcup_{j=0}^{\infty} \gamma_{f^{k}(B_{i})}^{d_{j} \theta_{i}} \right),
$$

let $\Xi_{i}'$ be the union of the orbit of $a_{i}$ and the external rays landing on this orbit, and let

$$
\Theta_{2} = \bigcup_{i=1}^{m} (\Xi_{i} \cup \Xi_{i}').
$$

Now we are ready to construct the puzzle. Let $\varepsilon > 0$ be such that the equipotential set $\Theta_{0} := \{G(z) = \varepsilon\}$ is disjoint from the grand orbit of all critical points. Let $\Theta = \Theta_{0} \cup \Theta_{1} \cup \Theta_{2}$. This is a finite union of smooth curves and periodic points, and each of these periodic points is the landing point of two or more smooth (external or internal) rays. Let $\mathcal{P}_{0}$ be the collection of all components of $\mathbb{C} \setminus \Theta$ which intersect the Julia set of $f$ and for $n \geq 1$, let $\mathcal{P}_{n}$ be the collection of components of $f^{-n}(P)$, where $P$ runs over all elements of $\mathcal{P}_{0}$. An element of $\mathcal{P}_{n}$ is called a **puzzle piece of depth** $n$.

Since $\Theta$ is disjoint from the orbit of critical points, for each $c \in \text{Crit}'(f)$ and each $n \geq 0$ there is a puzzle piece $P_{n}(c)$ of depth $n$ that contains $c$. If for each $c \in \text{Crit}'(f)$ and each integer $s \geq 1$, there exists $n \geq 0$ such that $f^{s}(c) \notin P_{n}(c)$, then the arguments in [4] show that $\text{diam}(P_{n}(c)) \to 0$ as $n \to \infty$ for each $c \in \text{Crit}'(f)$.

Otherwise, there exists $c \in \text{Crit}(f)$ and a minimal integer $s \geq 1$ such that $f^{s}(c) \in P_{n}(c)$ for all $n \geq 0$. The degree of $f^{s} : P_{n+s}(c) \to P_{n}(c)$ is non-increasing, hence eventually constant. So there exists $n_{0}$ such that all the critical points of $f^{s} : P_{n_{0}+s}(c) \to P_{n_{0}}(c)$ do not escape $P_{n_{0}+s}(c)$ under forward iteration of this map. Using the “thickening” technique [10], we can find topological disks $\widehat{P}_{n_{0}+s}(c) \supset P_{n_{0}+s}(c)$ and $\widehat{P}_{n_{0}}(c) \supset P_{n_{0}}(c)$ such
that \( f^s : P_{n_0+s}(c) \rightarrow P_{n_0}(c) \) is a renormalization of \( f \). Then arguing as in Observation 1, there exists a period point \( q \in P_{n_0+s}(c) \cap (P_{sep} \setminus P_{bad}) \). We add the orbit of \( q \) and the external rays landing on the orbit of \( q \) in the set \( \Theta \) and construct a new puzzle. Then either for each \( c \in \text{Crit}'(f) \) we obtain an arbitrarily small new puzzle pieces containing \( c \), or we obtain a new renormalization of \( f \). The new renormalization either has a larger period, or has a smaller degree. Repeating the argument, if \( f \) is not infinitely renormalizable, then we must stop within finitely steps.

In conclusion, if \( f \) is at most finitely renormalizable and has only hyperbolic periodic points, then we can construct a puzzle such that for each \( c \in \text{Crit}'(f) \) there exists an arbitrarily small puzzle pieces containing \( c \). In the following we fix such a puzzle.

A.2. The bounds. Given a recurrent critical point \( c \) of \( f \) and a nice topological disk \( V \ni c \), we say that a topological disk \( U \) is a child of \( V \) if there exist \( c' \in [c] \) and \( s \geq 1 \) such that \( U \ni c' \) and such that \( f^s : U \rightarrow V \) is a proper map with a unique critical point, where

\[
[c] = \{ \zeta \in \text{Crit}'(f) : \omega(c) = \omega(\zeta) \ni c, \zeta \}.
\]

We say that \( c \) is persistently recurrent if for each \( c' \in [c] \), each nice topological disk \( V \ni c' \) has only finitely many children. Otherwise, we say that the recurrent critical point \( c \) is reluctantly recurrent.

Let \( V \) be a nice topological disk which contains a critical point \( c \in \text{Crit}'(f) \). We say that \( V \) is essentially \( \rho \)-nice if for each return domain \( U \) of \( V \) which intersects \( \text{orb}(c) \), we have \( \text{mod}(V; U) \geq \rho \).

**Lemma 15.** For each \( c \in \text{Crit}'(f) \) which is persistently recurrent, there exists an arbitrarily small puzzle piece that contains \( c \) and is essentially \( \rho_1 \)-nice, where \( \rho_1 > 0 \) is a constant depending only on the degree of \( f \).

**Proof.** Given a small puzzle piece \( W \) that contains \( c \), we can construct a complex box mapping as in Lemma 2.2 of [6], but we take \( V \) to be the union of the components of the domain of the first entry map to \( W \) which intersect \( [c] \). Then the box mapping has only persistently recurrent critical points. Then by the proposition 10.1 of [9], the \( I_n \) constructed in §8 of [9] is essentially \( \rho_1 \)-nice for some universal \( \rho_1 > 0 \) when \( n \) is large enough. \( \square \)

Let us say that an open set \( V \) is a puzzle neighborhood of a set \( A \subset \text{Crit}'(f) \) if \( V \) is nice and each component of \( V \) is a puzzle piece containing exactly one point of \( A \). As before, If \( V' \supseteq V \) are two puzzle neighborhoods of \( A \) such that

\[
f^n(\partial V_a) \cap V'_a = \emptyset,
\]

holds for all \( n \geq 1 \) and \( a \in A \), where \( V_a \) (resp. \( V'_a \)) denotes the component of \( V \) (resp. \( V' \)) which contains \( a \), then we say that \( (V', V) \) is an admissible pair. We say that \( a \in A \) is special for the pair \( (V', V) \) if

\[
f^n(V_a) \cap V' = \emptyset \text{ for all } n \geq 1.
\]
Recall that \((V', V)\) is called \(\rho\)-bounded if for each \(a \in A\),
\[
\mod(V'_a; V_a) \geq \rho.
\]

**Lemma 16.** For any \(\rho > 0\) there exists \(\rho' > 0\) such that the following holds.
Let \(c \in \text{Crit}(f)\) and let \((V', V)\) be a \(\rho\)-bounded admissible pair of puzzle neighborhoods of \(\text{Back}(c)\). Assume that \(V'\) is sufficiently small. Then for each \(x \in D(V)\), we have
\[
\mod(L_x(V'); L_x(V)) \geq \rho'.
\]
Moreover, if \(c\) is special for \((V', V)\) then \(V_c\) is \(\rho'\)-nice.

**Proof.** Take \(x \in D(V)\). Let \(s\) be the entry time of \(x\) into \(V\), let \(U = L_x(V)\). Let \(a \in \text{Back}(c)\) be such that \(f^s(x) \in V_a\) and let \(U'\) be the component of \(f^{-s}(V'_a)\) which contains \(x\). By Lemma 5 the degree of \(f^s : U' \to V'_a\) is bounded from above by a constant \(N_0\). Thus \(\mod(U'; U) \geq N_0^{-1} \rho\). Since \(U' \subset L_x(V')\), \(\mod(L_x(V'), L_x(V)) \geq \mod(U'; U) \geq \rho/N_0\).

Now assume that \(c\) is special. Then for \(x \in D(V_c) \cap V_c\), we have \(L_x(V') \subset V_c\). Clearly, \(L_x(V_c) \subset L_x(V)\). Thus
\[
\mod(V_c; L_x(V_c)) \geq \mod(L_x(V'); L_x(V)) \geq \rho/N_0.
\]
This proves the last statement. \(\square\)

**Lemma 17.** For any \(\rho > 0\) there exists \(\rho' > 0\) such that the following holds.
Let \(c \in \text{Crit}(f)\) and let \((V', V)\) be a \(\rho\)-bounded admissible pair of puzzle neighborhoods of \(c\). Suppose that for each \(x \in \bigcup_{c' \in \text{Back}(c)} \text{orb}(c')\), either \(L_x(V) = \emptyset\) or \(\mod(V', L_x(V')) \geq \rho\).

If \(\text{diam}(V')\) is sufficiently small, then

(i) there exists a \(\rho'\)-bounded admissible pair \((V', V)\) of puzzle neighborhoods of \(\text{Back}(c)\) such that \(V_c' = V'\) and \(V_c = V\) and such that \(c\) is special for this admissible pair;
(ii) for any \(x \in D(V)\), we have \(\mod(L_x(V'); L_x(V)) \geq \rho'\);
(iii) \(V\) is \(\rho'\)-nice.

**Proof.** (i) Let \(c_0 = c, V_0' = V'\), \(V_0 = V\) and let \(c_1, c_2, \ldots, c_{b-1}\) be the critical points in \(\text{Back}(c) \setminus \{c\}\). For each \(i = 1, 2, \ldots, b - 1\), let \(V_i'\) be the entry domain of \(V'\) that contains \(c_i\) and let \(t_i\) be the entry time. If \(f^{t_i}(c_i) \in V\), let \(V_i\) be the component of \(f^{-t_i}(V)\) that contains \(c_i\). Otherwise, let \(V_i\) be the entry domain of \(V'\) that contains \(f^{t_i}(c_i)\) and let \(V_i\) be the component of \(f^{-t_i}(V_i)\) that contains \(c_i\). Put \(V' = \bigcup_{i=0}^{b-1} V_i'\) and \(V = \bigcup_{i=0}^{b-1} V_i\). Clearly, \((V', V)\) is an admissible pair of puzzle neighborhoods of \(\text{Back}(c)\), and \(c\) is a special critical point. Since the maps \(f^{t_i} : V_i' \to V'\) have uniformly bounded degree, the admissible pair \((V', V)\) is \(\rho'\)-bounded, where \(\rho' > 0\) is a constant.

(ii) By Lemma 16 and redefining \(\rho'\)-bounded, where \(\rho' > 0\) is a constant.
Since $\mathcal{V}'_i$ is an entry domain of $\mathcal{V}'_0$ for each $i = 1, 2, \ldots, b - 1$, we have

$$\mathcal{L}_x(\mathcal{V}') = \mathcal{L}_x(\mathcal{V}'_0) = \mathcal{L}_x(\mathcal{V}).$$

For $x \in D(V)$, $\mathcal{L}_x(\mathcal{V}) \supseteq \mathcal{L}_x(\mathcal{V})$. It follows that

$$\mod(\mathcal{L}_x(\mathcal{V}'; \mathcal{L}_x(\mathcal{V})) \geq \mod(\mathcal{L}_x(\mathcal{V}'), \mathcal{L}_x(\mathcal{V})) \geq \rho'.$$

(iii) Since $c$ is special for the pair $(\mathcal{V}', \mathcal{V})$, applying the last statement of Lemma 16 proves the result. \qed

**Lemma 18.** If $c \in \text{Crit}'(f)$ is reluctantly recurrent. Then for any $\rho > 0$, there exists an arbitrarily small puzzle neighborhood $V_k$ of $c$ which is $\rho$-nice. In particular, $V_k$ is essentially $\rho$-nice.

**Proof.** By [2, Lemma 6.5], there exists a puzzle piece $V \ni c$, a positive integer $N$ and a sequence of integers $s_k \to \infty$ with the following properties:

- $V$ is $\lambda$-nice for some $\lambda > 0$;
- $f^{s_k}(c) \in V$, and letting $V_k = \text{Comp}_c(f^{-s_k}V)$, we have $f^{s_k}: V_k \to V$ has degree at most $N$.

By replacing $V$ with some pull back of $V$, we may also assume

$$f^j(V_k) \not\ni c, \text{ for all } 1 \leq j < s_k,$$

which implies that

$$f^j(V_k) \cap V_k = \emptyset \text{ for all } 1 \leq j < s_k. \tag{5}$$

Let

$$\lambda_k = \inf_{y \in V \cap D(V_k)} \mod(V; \widehat{\mathcal{L}}_y(V_k)),$$

where $\widehat{\mathcal{L}}_y(V_k) = V_k$ if $y \in V_k$ and $\widehat{\mathcal{L}}_y(V_k) = \mathcal{L}_y(V_k)$ otherwise. Since $\widehat{\mathcal{L}}_y(V_k) \subset \mathcal{L}_y(V)$ for each $y \in V \cap D(V_k)$, we have $\lambda_k \geq \lambda$.

**Claim.** Each $V_k$ is $\lambda_k/N$-nice.

To prove this claim, take $x \in V_k \cap D(V_k)$. By [5], the return time of $x$ into $V_k$ is at least $s_k$ and hence $f^{s_k}(\mathcal{L}_x(V_k)) = \widehat{\mathcal{L}}_y(V_k)$, where $y = f^{s_k}(x)$. Therefore

$$\mod(V_k; \mathcal{L}_x(V_k)) \geq N^{-1} \mod(V; \widehat{\mathcal{L}}_y(V_k)) \geq \lambda_k/N.$$

Letting $U_k = \mathcal{L}_x(V_k)$, and applying Lemma 7, we obtain that for each $x \in D(U_k)$, $\mod(\mathcal{L}_x(V_k); \mathcal{L}_x(U_k)) \geq \rho_1$, where $\rho_1 > 0$ is a constant. It follows that $\lambda_k \to \infty$. The lemma follows. \qed

**Proof of Proposition 2.** By Lemmas 15 and 18, there exists a universal constant $\rho_1 > 0$ such that any $c \in \text{Crit}'(f)$, there exists an arbitrarily small puzzle piece that contains $c$ and is essentially $\rho_1$-nice (note that by the definition of essentially $\rho_1$-nice, the statement is trivial for non-recurrent $c \in \text{Crit}'(f)$). By Lemma 7 (ii), it follows that there exists an arbitrarily small $\rho_2$-bounded admissible pair $(\mathcal{V}', \mathcal{V})$ of $[c]$ for which $c$ is special, where $\rho_2 > 0$ is a constant.
Let us define a strictly increasing finite sequence \( \{\Omega_k\}_{k=0}^b \) of subsets of \( \text{Crit}'(f) \) as follows. Let \( \Omega_0 = \emptyset \). If \( \Omega_k \) is defined and \( \Omega_k \neq \text{Crit}'(f) \), then we proceed to define \( \Omega_{k+1} \) by taking \( c \in \text{Crit}'(f) \setminus \Omega_k \) with \( \text{Back}(c) \subset \Omega_k \cup [c] \) and letting \( \Omega_{k+1} = \Omega_k \cup [c] \). Clearly the procedure stops within \( \#\text{Crit}'(f) \) steps, so \( b \leq \#\text{Crit}'(f) \) and \( \Omega_b = \text{Crit}'(f) \). Note that for each \( k \), any \( c \in \Omega_k \), \( \text{Back}(c) \subset \Omega_k \).

We claim that for each \( k = 1, 2, \ldots, b \) and any \( c \in \Omega_k \setminus \Omega_{k-1} \), there exists an arbitrarily small \( \rho_2 \)-bounded admissible pair \((V', V)\) of puzzle neighborhoods of \( \Omega_k \) for which \( c \) is special. Let us prove this claim by induction. The case \( k = 1 \) has been proved above, so assume that the claim holds for \( k = k_0 - 1 \), \( 2 \leq k_0 \leq b \). Take \( c \in \Omega_{k+1} \setminus \Omega_k \). By induction hypothesis, for any \( n_0 \geq 1 \) there is a \( \rho_2 \)-bounded admissible pair of puzzle neighborhoods \((W', W)\) of \( \Omega_k \) such that the depth of \( W' \) is greater than \( n_0 \) for each \( c' \in \Omega_k \). Let us choose a \( \rho_2 \)-bounded admissible pair \((U', U)\) of puzzle neighborhoods of \([c] \) for which \( c \) is special, and such that the minimal depth of components of \( U' \) is greater than the maximal depth of components of \( W' \). Then \((W' \cup U', W \cup U)\) is a \( \rho_2 \)-bounded admissible puzzle neighborhood of \( \Omega_{k+1} \). Note that \( f^n(\partial U) \cap W' = \emptyset \) for all \( n \geq 0 \), for otherwise, we would obtain that \( c \in \text{Back}(c') \) for some \( c' \in \Omega_k \). Thus \( c \) is special for this admissible pair \((W' \cup U', W \cup U)\). This completes the induction step and thus the proof of the claim.

In particular, for any \( c \in \text{Crit}'(f) \), there exists an arbitrarily small \( \rho_2 \)-bounded admissible puzzle neighborhoods \((V', V)\) of \( \text{Back}(c) \) for which \( c \) is special. By Lemma 16 \( V_c \) is a \( \rho_0 \)-nice puzzle piece for some universal constant \( \rho_0 > 0 \).

\( \square \)

Appendix B. Interval maps

We shall prove corresponding results for a class of interval maps.

Recall that a map \( f : X \to X \) from a compact interval \( X \) of \( \mathbb{R} \) into itself is of class \( C^2 \) with non-flat critical points if \( f \) is of class \( C^1 \) on \( X \); of class \( C^3 \) outside \( \text{Crit}(f) := \{x \in X \mid D f(x) = 0\} \); and for each \( c \in \text{Crit}(f) \), there exists a number \( \ell_c > 1 \) (called the order of \( f \) at \( c \)) and diffeomorphisms \( \phi, \psi \) of \( \mathbb{R} \) of class \( C^3 \) with \( \phi(c) = \psi(f(c)) = 0 \) such that,

\[
|\psi \circ f(x)| = |\phi(x)|^{\ell_c}
\]

holds in a neighborhood of \( c \) in \( X \).

For such a map \( f \), we define \( J(f) \) to be the complement of the interior of the attracting basins of periodic attractors and \( \text{Crit}'(f) = \text{Crit}(f) \cap J(f) \). As in [11] we can define the properties \( BC(r) \) and \( LD(K) \) for \( C^3 \) interval maps with non-flat critical points. Let

\[
\ell_{\text{max}} = \max_{c \in \text{Crit}'(f)} \ell_c.
\]

For interval maps, we have
Theorem A’. For each $\ell > 1$, there exists $K_0 = K_0(\ell) > 0$ such that if $f$ is a $C^3$ interval maps with $\#\text{Crit}'(f) = N$ and with $\ell_{\text{max}} \leq \ell$, and if $f$ satisfies $LD(K_0 r)$ for some $r > 1$, then $f$ satisfies $BC(r)$.

This is proved in [1, Theorem 1], although the statement here is slightly more general. We observe that [1, Proposition 1]) remains true under the more general assumption here, if we require that $f^s(T)$ is contained a small neighborhood of $\text{Crit}'(f)$ (which is given by [15, Theorem C]). The dependence of constants follows from the proof.

We also have the following

Theorem B’. Let $f$ be a $C^3$ interval maps with non-flat critical points. There exists a constant $r_0 > 1$ depending only on $\ell_{\text{max}}$ such that if $f$ satisfies $BC(Kr_0)$ for some $K > 1$, then $f$ satisfies $LD(K)$.

The proof of this theorem follows the same outline as that of Theorem B, replacing the Schwarz lemma by the following real version.

Real Schwarz Lemma. Let $f$ be a $C^3$ interval maps with non-flat critical points. There exists $\eta = \eta(f) > 0$ and a universal constant $\theta \in (0, 1)$ such that if $f^n : U \to V$ is a diffeomorphism between intervals, $V \subset \tilde{B}(c, \eta)$ for some $c \in \text{Crit}(f)$ and $x \in U$ is such that $f^n(x)$ is the middle point of $V$, then

$$|Df^n(x)| \geq \theta \frac{|V|}{|U|}.$$ 

Proof. Let $\hat{V}$ be the open interval with $f^n(x)$ as middle point and with $|\hat{V}| = |V|/2$, and let $\hat{U} = f^{-n}(\hat{V}) \cap U$. By [15, Theorem C(2)], there exists a constant $K > 1$ such that

$$|Df^n(x)| \geq K^{-1} \frac{|\hat{V}|}{|\hat{U}|} \geq (2K)^{-1} \frac{|V|}{|U|},$$

provided that $\eta$ is sufficiently small. Thus the lemma holds with $\theta = 1/(2K)$.

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