The oscillations of oblate drop under the influence of a
alternating electric field

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Abstract. The forced oscillations of incompressible fluid drop under the alternating electric field are considered. In equilibrium, the drop has the form of a rotational figure with arbitrary contact angle bounded axially parallel solid planes. The drop is surrounded by an incompressible fluid with another density. The external uniform electric field acts as an external force that causes motion of the contact line. In order to describe this contact line motion the modified Hocking boundary condition is applied: the velocity of the contact line is proportional to the deviation of the contact angle and the speed of the fast relaxation processes, which frequency is proportional to twice the frequency of the electric field. Using of this equation can qualitatively describe the experimental dependence of the contact angle vs voltage in contrast to the Young-Lippmann equation.

1. Introduction
Recent years have shown keen interest in the dynamics of droplets and bubbles on dielectric substrate in electric field (electrowetting-on-dielectric, EWOD) [1-4]. In the study of electrowetting one of the problems [5,6] is the lack of coordination of the experimental results with the predictions of the theoretical model using the Young-Lippmann equation [1,7,8] (figure 1):

\[
\cos \vartheta = \cos \vartheta_0 + \frac{\varepsilon \varepsilon_0 s}{2d\sigma} V^2,
\]

where \( V \) is the value of the applied voltage, \( \vartheta_0 \) is the contact angle without applied voltage – equilibrium contact angle, which is defined by well-known Young’s equation, \( \sigma \) is the interfacial tension between the droplet and the surrounding fluid, \( d \) is the thickness of the dielectric film, \( \varepsilon_0 \) and \( \varepsilon \) are the permittivity of vacuum and the dielectric layer, respectively. The last term in the equation (1) is the modified term of Lippmann equation [9].

Figure 1. Contact angle \( \theta \) as function of voltage \( V \). The difference between the experimental data and the theoretical solution using the Young-Lippmann equation

In article [10] authors sought to develop a theory describing droplet oscillations induced by electrowetting and proposed another condition – modified Hocking boundary condition. Original

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effective boundary condition has been proposed by Hocking for small oscillations of the contact line [11]. The velocity of the contact line is assumed to be proportional to the deviation of the contact angle from its equilibrium value (for simplicity, the equilibrium contact angle is considered to be \( \pi/2 \)):

\[
\frac{\partial \zeta^*}{\partial t} = \Lambda \cdot \nabla \zeta^*,
\]

(2)

where \( \zeta^* \) is the deviation of the interface from the equilibrium position, \( \tilde{k} \) is the external normal to the solid surface, \( \Lambda \) is a phenomenological constant (the so-called wetting parameter or Hocking parameter), having the dimension of velocity. The equation (2) is well proven in the study of natural oscillations and fast relaxation processes [12-16].

In [10] we believed that the external electric field plays the role of source of motion and forces the contact angle to change in time. Hereafter, we assume that the external force is time periodic \( \cos \omega t \). For very fast relaxation processes at the triple line, the contact angle evolves according to the law \( \cos \omega t \). Taking into account the more general case when relaxation effects occur at timescales comparable with that of the external force, we can formulate an effective boundary condition to be fulfilled at the contact line

\[
\frac{\partial \zeta^*}{\partial t} = \mp \Lambda \left( \frac{\partial \zeta^*}{\partial t} + A^* \cos(2\omega t) \right),
\]

(3)
i.e. the velocity of the contact line is proportional to the deviation of the contact angle and the speed of the fast relaxation processes, which frequency is proportional to twice the frequency of the electric field.

In the present paper, we consider the behavior of an oblate drop sandwiched between two flat plates under AC. The drop has equilibrium form of rotational figure with arbitrary contact angle whereas cylindrical drop with right contact angle was considered in [10]. The natural oscillations of oblate drop were researched in [15].

2. Problem formulation

Following [10], we consider the dynamic behavior of an incompressible liquid drop of density \( \rho_i^* \) surrounded by other liquid of density \( \rho_e^* \). The system is bounded by two parallel solid surfaces (see figure 2) separated by a distance \( h^* \). The equilibrium contact angle \( \theta_0 \) between the side surface of the drop and the solid surface is arbitrary. The external uniform alternating electric field acts as an external force that causes the contact line motion.

Let the surface of the drop be described by the equation \( r^* = R^*(z) + \zeta^* (\alpha^*, z^*, t^*) \), where \( R^*(z) \) is the equilibrium radius of the drop. We use \( \sigma^{-1/2} \sqrt{\left( \rho_i^* + \rho_e^* \right) R_0^{-3}} \), \( R_0 = R(0) \), \( h^* \), \( \rho_i^* + \rho_e^* \), \( A^* \), \( A^* \sigma \left( R_0^{-2} \right) \), \( A^* \sqrt{\sigma \left( \left( \rho_i^* + \rho_e^* \right) R_0^{-3} \right)} \) as the scales for the time, length, height, density, deviation of drop surface from its equilibrium position, pressure, and velocity potential, respectively. The system is described by the following equations [10,15]:

\[
\rho_j = -\rho \phi_{ji}, \quad \Delta \phi_j = 0, \quad j = i, e,
\]

(4)

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} + b^2 \frac{\partial^2}{\partial z^2},
\]

\[
r = R: \left[ \phi_r \right] = 0, \quad \zeta_r = \phi_r, \quad \left[ p \right] = \zeta + \zeta_{uu} + b^2 \zeta_{zz},
\]

(5)

\[
\zeta = \pm \frac{1}{2} : \phi_z = 0,
\]

(6)

\[
r = R^* : \zeta = \mp \lambda \left( \zeta - \zeta_{0z} + a \cos(2\omega t) \right), \quad \zeta_{0z} = \cot \theta_0
\]

(7)

where \( p \) is the fluid pressure, the square brackets denote the jump in the quantity at the interface between the external liquid and the drop. The boundary-value problem (5)-(8) involves six
parameters: the aspect ratio, the dimensionless density, the wetting parameter, the AC frequency and amplitude
\[ b = R_0 h^{-1}, \quad \rho_i = \rho_i^* \left( \rho^*_e + \rho^*_i \right)^{-1}, \quad \rho_e = \rho_e^* \left( \rho^*_e + \rho^*_i \right)^{-1}, \quad \lambda = \Lambda^* \sigma^{-3/2} \sqrt{\left( \rho^*_e + \rho^*_i \right) R_0^3}, \]
\[ \omega = \omega^* \sigma^{-3/2} \sqrt{\left( \rho^*_e + \rho^*_i \right) R_0^3}, \quad a = A^* C \sigma^{-3/2} \sqrt{\left( \rho^*_e + \rho^*_i \right) R_0^3} / 2. \]

**Figure 2.** Problem geometry (1 – electrode, 2 – dielectric layer).

### 3. Uniform electric field

We consider the case of small distances between the solid planes. Assuming that \( b \gg 1 \), i.e. \( \beta = b^{-1} \ll 1 \), we expand the function \( R(z) \) (equilibrium shape) in a series in the small parameter \( \beta \):
\[ R(z) = 1 + \beta R_1(z) + \ldots \] In order to investigate the problem it is convenient to begin with a consideration of the uniform electric field. Our interest deals with the axisymmetric oscillation modes, therefore system of the equations and boundary conditions (5)–(8) are axially symmetric.

The flow domain in the drop can be divided into a region having the shape of a cylinder with unit radius and a narrow transition region between the cylinder and the surrounding liquid. In the transition region, the velocity potential and the pressure of the liquids can vary significantly in the radial direction, so that it is necessary to introduce the expansion of radial coordinate \( r = (r - R(z)) \beta^{-1} \). In view of axial symmetry, the solution of the Laplace equation (6) is written as
\[ \phi_1(r,z,t) = \Re \left( a_0 r + \sum_{k=1}^{\infty} a_k \exp \left( (2k + 1) \pi b r \right) \sin \left( (2k + 1) \pi z \right) e^{i \omega t} \right), \quad (8) \]
\[ \phi_2(r,z,t) = \Re \left( a_0 r^{-1} + \sum_{k=1}^{\infty} b_k \exp \left( -(2k + 1) \pi b r \right) \sin \left( (2k + 1) \pi z \right) e^{i \omega t} \right), \quad (9) \]
The kinematic condition on the free surface (the second condition in (6)) gives the expression for the surface deviation
\[ \zeta(z,t) = \Re \left( dz + \sum_{k=0}^{\infty} c_k \cos \left( (2k + 1) \pi z \right) e^{i \omega t} \right). \quad (10) \]

Substituting solutions (9)–(11) into (5)–(8) we obtain expressions for the unknown amplitudes \( a_0, b_0, c_k \) and \( d \) by Galerkin method.

The contact angle at the upper plate versus square root of amplitude \( a \) (i.e. proportional to AC potential \( V \)) is given in figure 3 for different values of the Hocking parameter \( \lambda \) and AC frequency (equilibrium contact angle \( \theta_p = \pi / 2 \)). The dependences obtained in qualitative agreement with the experimental data. However the maximum deviation of contact angle tends to \( \pi / 2 \), i.e. \( \theta \to 0 \) or \( \theta \to \pi \) whereas contact angle is finite in experiments.

Different wetting situations are addressed by changing the values of the parameter \( \lambda \) and range between two limiting cases: i) for \( \lambda = 0 \), the contact line motion is completely uncoupled from the external field – the contact line remains fixed and the droplet overall is motionless; ii) the opposite
limit, \( \lambda \gg 1 \), corresponds to infinitely fast relaxation of the contact angle, when the latter quickly adjusts to the slowly changing force. In both these situations, there is no dissipation at the contact line. Apparently the fluid viscosity must be taken into account. It is giving a finite amplitude for both limit cases of Hocking boundary especially in limiting case \( \lambda \gg 1 \).

![Figure 3](image)

**Figure 3.** The contact angle \( \vartheta \) vs square root of amplitude \( \alpha \) for three different values of the Hocking parameter \( \lambda \) and of the oscillation frequency \( \omega \) (\( b = 1, \rho_i = 0.7 \)).

(a) \( \omega = 5 \), (b) \( \omega = 10 \), (a) \( \omega = 100 \),
\( \lambda = 0.1 \) (solid line), \( \lambda = 1 \) (dashed line), \( \lambda = 10 \) (dotted line).

Figure 4 shows the contact angle of highly-condensed drops as function of the square root of the amplitude for two values of the equilibrium contact angle \( \vartheta_0 \). From these curves shows that for any values \( \vartheta_0 \) curves behave the same way.

![Figure 4](image)

**Figure 4.** The contact angle \( \vartheta \) vs square root of amplitude \( \alpha \) for \( \vartheta_0 = 75^\circ, 120^\circ \).

(a) \( \omega = 5 \), (b) \( \omega = 10 \), (a) \( \omega = 10 \),
\( \lambda = 0.1 \) (solid line), \( \lambda = 1 \) (dashed line), \( \lambda = 10 \) (dotted line).

**4. Conclusions**
The dependences of surface amplitude and contact angle from oscillation amplitude and frequency are obtained for Hocking parameter, geometric parameters and equilibrium contact angle. It is shown that The curve of contact angle \( \vartheta \) vs square root of amplitude obtained in qualitative agreement with the experimental data. However the maximum deviation of contact angle tends to \( \pi/2 \), i.e. \( \vartheta \rightarrow 0 \) or \( \vartheta \rightarrow \pi \) whereas contact angle is finite in experiments. Similar dependences are obtained for cylindrical drop (\( \vartheta_0 = 90^\circ \)) under nonuniform electric field [10]. Consequently, there are two ways of further development of problem solution: 1. fluid viscosity (see for example [13]), 2. contact angle hysteresis (see for example [14]). Both of these cases limit the amplitude of oscillation and may limit the change of the contact angle.

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5. References
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