EXISTENCE OF WEAK SOLUTIONS FOR $p(.)$-LAPLACIAN EQUATION VIA COMPACT EMBEDDINGS OF THE DOUBLE WEIGHTED VARIABLE EXponent SOBOLEV SPACES

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Abstract. In this study, we define double weighted variable exponent Sobolev spaces $W^{1,p(.)}(\Omega, \vartheta_0, \vartheta)$ with respect to two different weight functions. Also, we investigate the basic properties of this spaces. Moreover, we discuss the existence of weak solutions for weighted Dirichlet problem of $p(.)$-Laplacian equation

$$\begin{cases}
-\text{div} \left( \vartheta(x) |\nabla f|^{p(x)-2} \nabla f \right) = \vartheta_0(x) |f|^{q(x)-2} f & x \in \Omega \\
f = 0 & x \in \partial\Omega
\end{cases}$$

under some conditions of compact embedding involving the double weighted variable exponent Sobolev spaces.

1. Introduction

The history of potential theory begins in 17th century. Its development can be traced to such greats as Newton, Euler, Laplace, Lagrange, Fourier, Green, Gauss, Poisson, Dirichlet, Riemann, Weierstrass, Poincaré. We refer to the book by Kellogg [18] for references to some of the old works.

Kováčik and Rákosník [21] introduced the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^d)$ and the Sobolev space $W^{k,p(.)}(\mathbb{R}^d)$. They present some basic properties of the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^d)$ and the Sobolev space $W^{k,p(.)}(\mathbb{R}^d)$ such as reflexivity and Hölder inequalities were obtained. Also, Fan and Zhao [13] present important results for the variable exponent Lebesgue and Sobolev spaces. The study of electrorheological fluids is one of the important areas where these spaces have found applications, see [30]. As another area, we can say the study of variational integrals with non-standard growth, see [1], [36]. The boundedness of the maximal operator was an open problem in $L^{p(.)}(\mathbb{R}^d)$ for a long time. Diening [9] proved the first time this state over bounded domains if $p(.)$ satisfies locally log-Hölder continuous condition, that is,

$$|p(x) - p(y)| \leq \frac{C}{\ln|x-y|}, \quad x, y \in \Omega, \quad |x-y| \leq \frac{1}{2}$$

where $\Omega$ is a bounded domain. We denote by $P^{\text{log}}(\mathbb{R}^d)$ the class of variable exponents which satisfy the log-Hölder continuous condition. Diening later extended the result to unbounded domains by supposing, in addition, that the exponent $p(.) = p$ is a constant function outside a large ball. After this study, many absorbing and
cruicial papers appeared in non-weighted and weighted variable exponent spaces. For a historical journey, we refer [8], [11] and references therein.

The operator $-\Delta_{p(.)}f = - \text{div} \left( |\nabla f|^{p(x)-2} \nabla f \right)$ is called $p(.)$-Laplacian. The study of differential equations and variational problems with $p(.)$-growth conditions arouses much interest with the development of elastic mechanics, electrorheological fluid dynamics and image processing etc. We refer the readers [14], [19], [25], [28], [31], [34] and references therein. In general, the methods used in these works are base on continuous and compact embeddings between Lebesgue and Sobolev spaces.

In 2003, Fan and Zhang obtained a weak solution in $W^{1,p(.)}_0(\Omega)$ to the Dirichlet problem of $p(.)$-Laplacian
\[
\begin{cases}
- \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) = f(x, u) & x \in \Omega \\
u = 0 & x \in \partial \Omega
\end{cases}
\]
where $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function which satisfies the growth condition, see [12]. Moreover, in recent years, $p(.)$-Laplacian equations and variational problems with $p(.)$-growth conditions have been studied by several authors, see [5], [12], [17], [19], [24], [25].

In [14], the authors deal with two Dirichlet boundary value problems involving the weighted $p$-Laplacian. They give existence and multiplicity results under suitable conditions in constant exponent case. One of the our purpose is to extend to variable case of some of the results in [14].

In this study, we present and investigate double weighted variable exponent Sobolev spaces $W^{1,q(.)-p(.)}(\Omega, \vartheta_0, \vartheta)$ with respect to two different weight functions. The main purpose of this paper is to study the existence of weak solutions of $p(.)$-Laplacian problem
\[
(1.1) \quad \begin{cases}
- \text{div} \left( \vartheta(x) |\nabla f|^{p(x)-2} \nabla f \right) = \vartheta_0(x) |f|^{q(x)-2} f & x \in \Omega \\
f = 0 & x \in \partial \Omega
\end{cases}
\]
for $f \in W^{1,q(.)-p(.)}_0(\Omega, \vartheta_0, \vartheta)$ where $\Omega \subset \mathbb{R}^d$ is a bounded domain. Moreover, we discuss the necessary conditions for existence of weak solutions for (1.1) involving the Poincaré inequality in $W^{1,q(.)-p(.)}(\Omega, \vartheta_0, \vartheta)$, several continuous and compact embeddings.

2. Notation and Preliminaries

In this paper, we will work on $\Omega$ with Lebesgue measure $dx$. Also, the elements of the space $C_0^\infty(\Omega)$ are the infinitely differentiable functions with compact support. A normed space $(X, \|\cdot\|_X)$ is called a Banach function space (shortly BF-space), if Banach space $(X, \|\cdot\|_X)$ is continuously embedded into $L^1_{\text{loc}}(\Omega)$, briefly $X \hookrightarrow L^1_{\text{loc}}(\Omega)$, i.e. for any compact subset $K \subset \Omega$ there is some constant $c_K > 0$ such that $\|f|_K\|_{L^1(\Omega)} \leq c_K \|f\|_X$ for every $f \in X$. Moreover, a normed space $X$ is compactly embedded in a normed space $Y$, briefly $X \hookrightarrow Y$, if $X \hookrightarrow Y$ and the identity operator $I : X \rightarrow Y$ is compact, equivalently, $I$ maps every bounded sequence $(x_n)_{n \in \mathbb{N}}$ into a sequence $(I(x_n))_{n \in \mathbb{N}}$ that contains a subsequence converging in $Y$. Suppose that $X$ and $Y$ are two Banach spaces and $X$ is reflexive. Then $I : X \rightarrow Y$ is a compact operator if and only if $I$ maps weakly convergent sequences in $X$ onto convergent sequences in $Y$. More details can be found in [2].
Let $\Omega \subset \mathbb{R}^d$ is bounded and $\vartheta$ is a weight function. It is known that a function $f \in C_0^\infty (\Omega)$ satisfies Poincaré inequality in $L^1_0(\Omega)$ if and only if the inequality
\[
\int_\Omega |f(x)| \vartheta(x) \, dx \leq c(\text{diam} (\Omega)) \int_\Omega |\nabla f(x)| \vartheta(x) \, dx
\]
holds, see [10].

**Definition 1.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a domain with non-empty boundary $\partial \Omega$, denote
\[
L_{p^+}^\infty (\Omega) = \left\{ p(\cdot) \in L^\infty (\Omega) : \text{essinf}_{x \in \Omega} p(\cdot) > 1 \right\}
\]
for a measurable function $p(\cdot) : \Omega \rightarrow [1, \infty)$ (called a variable exponent on $\Omega$) by the symbol $P(\Omega)$. In this paper, the function $p(\cdot)$ always denotes a variable exponent. For $p(\cdot) \in P(\Omega)$, we put
\[
p^- = \text{essinf}_{x \in \Omega} p(x), \quad p^+ = \text{esssup}_{x \in \Omega} p(x).
\]
The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ consist of all measurable functions $f$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm
\[
\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{f}{\lambda} \right) \leq 1 \right\},
\]
where
\[
\varrho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} \, dx.
\]
Let $p^+ < \infty$. Then $f \in L^{p(\cdot)}(\Omega)$ if and only if $\varrho_{p(\cdot)}(f) < \infty$. The space $L^{p(\cdot)}(\Omega)$ is a Banach space with respect to $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$. If $p(\cdot) = p$ is a constant function, then the norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ coincides with the usual Lebesgue norm $\|\cdot\|_p$, see [21]. In this paper, we assume that all variable exponents are belong to $L_{p^+}^\infty (\Omega)$.

**Definition 2.** A measurable and locally integrable function $\vartheta : \Omega \rightarrow (0, \infty)$ is called a weight function. We say that $\vartheta_1 \prec \vartheta_2$ if only if there exists $c > 0$ such that $\vartheta_1(x) \leq c \vartheta_2(x)$ for all $x \in \Omega$. Now, we denote
\[
W(\Omega) = \left\{ \vartheta \in L^1_{\text{loc}} (\Omega) : \vartheta > 0 \text{ almost everywhere in } \Omega \right\}.
\]
For $p(\cdot) \in L^\infty (\Omega)$, we define
\[
W^{p(\cdot)}_{\vartheta} (\Omega) = \left\{ \vartheta \in W(\Omega) : \vartheta^{-\frac{1}{p^- - 1}} \in L^1_{\text{loc}} (\Omega) \right\}.
\]
Moreover, for $p(\cdot) \in L^\infty (\Omega)$ and $\vartheta \in W(\Omega)$, we consider the weighted variable exponent Lebesgue space
\[
L^{p(\cdot)}(\Omega, \vartheta) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_\Omega |f(x)|^{p(x)} \vartheta(x) \, dx < \infty \right\}
\]
with the Luxemburg norm
\[
\|f\|_{L^{p(\cdot)}(\Omega, \vartheta)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot), \vartheta} \left( \frac{f}{\lambda} \right) = \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} \vartheta(x) \, dx \leq 1 \right\}.
\]
The space $L^{p(\cdot)}(\Omega, \vartheta)$ is a Banach space with respect to $\| \cdot \|_{L^{p(\cdot)}(\Omega, \vartheta)}$. Moreover, $f \in L^{p(\cdot)}(\Omega, \vartheta)$ if and only if $\| f \|_{L^{p(\cdot)}(\Omega, \vartheta)} = \left\| f \vartheta^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot)}(\Omega)} < \infty$. It is known that we have the relationships between the modular $\varrho_{p(\cdot)}(\cdot)$ and the norm $\| \cdot \|_{L^{p(\cdot)}(\Omega, \vartheta)}$ are as follows

$$\min \left\{ \varrho_{p(\cdot)}(f)^{\frac{1}{p(\cdot)}}, \varrho_{p(\cdot)}(f)^{\frac{1}{q(\cdot)}} \right\} \leq \| f \|_{L^{p(\cdot)}(\Omega, \vartheta)} \leq \max \left\{ \varrho_{p(\cdot)}(f)^{\frac{1}{p(\cdot)}}, \varrho_{p(\cdot)}(f)^{\frac{1}{q(\cdot)}} \right\}$$

and

$$\min \left\{ \| f \|_{L^{p(\cdot)}(\Omega, \vartheta)}^{p(\cdot)}, \| f \|_{L^{q(\cdot)}(\Omega, \vartheta)}^{q(\cdot)} \right\} \leq \varrho_{p(\cdot)}(f) \leq \max \left\{ \| f \|_{L^{p(\cdot)}(\Omega, \vartheta)}^{p(\cdot)}, \| f \|_{L^{q(\cdot)}(\Omega, \vartheta)}^{q(\cdot)} \right\}$$

Also, if $0 < C \leq \vartheta$, then we have $L^{p(\cdot)}(\Omega, \vartheta) \hookrightarrow L^{p(\cdot)}(\Omega)$, since one easily sees that

$$C \int_{\Omega} |f(x)|^{p(x)} dx \leq \int_{\Omega} |f(x)|^{p(x)} \vartheta(x) dx$$

and $C \| f \|_{L^{p(\cdot)}(\Omega)} \leq \| f \|_{L^{p(\cdot)}(\Omega, \vartheta)}$. Moreover, the dual space of $L^{p(\cdot)}(\Omega, \vartheta)$ is $L^{p'(\cdot)}(\Omega, \vartheta^*)$, where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$ and $\vartheta^* = \vartheta^{1-p(\cdot)} = \vartheta^{\frac{1}{1-p(\cdot)}}$. For more details, we refer to [1], [4] and [20].

**Theorem 1.** (see [4]) If $\vartheta \in W_{p(\cdot)}(\Omega)$, then $L^{p(\cdot)}(\Omega, \vartheta) \hookrightarrow L^{1}_{\text{loc}}(\Omega) \hookrightarrow D'(\Omega)$, that is, every function in $L^{p(\cdot)}(\Omega, \vartheta)$ has distributional (weak) derivative, where $D'(\Omega)$ is distribution space.

**Remark 1.** (see [22]) If $\vartheta \notin W_{p(\cdot)}(\Omega)$, then the embedding $L^{p(\cdot)}(\Omega, \vartheta) \hookrightarrow L^{1}_{\text{loc}}(\Omega)$ need not hold.

Remark 1 says that the assumption $\vartheta \in W_{p(\cdot)}(\Omega)$ is necessary for distributional (weak) derivative techniques.

**Proposition 1.** Assume that $\vartheta \in W_{p(\cdot)}(\Omega)$, $\phi \in C_{0}^{\infty}(\Omega)$. Also, let a multi-index $\alpha \in \mathbb{N}^{n}_{0}$ be fixed. Then, the formula

$$L_{\alpha}(f) = \int_{\Omega} f D^{\alpha}\phi dx, \ f \in L^{p(\cdot)}(\Omega, \vartheta)$$

defines a continuous linear functional $L_{\alpha}$ on $L^{p(\cdot)}(\Omega, \vartheta)$ where $C_{0}^{\infty}(\Omega)$ is the space of $C^{\infty}(\Omega)$ functions with compact support in $\Omega$.

**Proof.** If we denote $Q = \text{supp}\phi$, then we have $Q = \overline{Q}$. By the Hölder inequality for $L^{p(\cdot)}(\Omega, \vartheta)$, we get

$$|L_{\alpha}(f)| \leq \int_{\Omega} |f| \vartheta^{\frac{1}{p(\cdot)}-\frac{1}{p'(\cdot)}} |D^{\alpha}\phi| dx$$

$$\leq c \left\| f \vartheta^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot)}(\Omega)} \left\| \vartheta^{-\frac{1}{q(\cdot)}} D^{\alpha}\phi \right\|_{L^{q(\cdot)}(\Omega)}$$

$$\leq C \left\| f \right\|_{L^{p(\cdot)}(\Omega, \vartheta)} < \infty.$$

where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. 

**Definition 3.** We set the weighted variable Sobolev spaces $W^{k,p(\cdot)}(\Omega, \vartheta)$ by

$$W^{k,p(\cdot)}(\Omega, \vartheta) = \left\{ f \in L^{p(\cdot)}(\Omega, \vartheta) : D^{\alpha} f \in L^{p(\cdot)}(\Omega, \vartheta), 0 \leq |\alpha| \leq k \right\}$$
Definition 4. The double weighted variable exponent Sobolev spaces $W^{k,p(\cdot)}(\Omega, \vartheta)$ is defined by

$$W^{k,p(\cdot)}(\Omega, \vartheta) = \left\{ f \in L^{p(\cdot)}(\Omega, \vartheta) : |\nabla f| \in L^{p(\cdot)}(\Omega, \vartheta) \right\}.$$ 

The function $\vartheta_{1,p(\cdot),\vartheta} : W^{1,p(\cdot)}(\Omega, \vartheta) \to [0, \infty)$ is shown as $\vartheta_{1,p(\cdot),\vartheta}(f) = \vartheta_{p(\cdot),\vartheta}(f) + \vartheta_{p(\cdot),\vartheta}(\nabla f)$. Also, the norm $\|f\|_{W^{1,p(\cdot)}(\Omega, \vartheta)} = \|f\|_{L^{p(\cdot)}(\Omega, \vartheta)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega, \vartheta)}$ makes the space $W^{1,p(\cdot)}(\Omega, \vartheta)$ a Banach space. If the exponent $p(\cdot)$ satisfies locally log-Hölder continuous condition, then a lot of regularities for variable exponent spaces holds. Because, the space $C_0^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega, \vartheta)$ under the circumstances, see [4]. More information on the classic theory of variable exponent spaces can be found in [11] and [21].

Throughout this paper, we assume that $1 < q^- \leq q(\cdot) \leq q^+ < p^- \leq p(\cdot) \leq p^+ < \lambda < \infty$, $\vartheta \in W_{p(\cdot)}(\Omega)$ and $\vartheta_0 \in W_{q(\cdot)}(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is a bounded domain.

3. Main Results

Before we consider the existence of weak solutions of (3.1), we present and investigate the double weighted variable exponent Sobolev spaces.

Definition 4. The double weighted variable exponent Sobolev spaces $W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ is defined by

$$W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta) = \left\{ f \in L^{q(\cdot)}(\Omega, \vartheta_0) : |\nabla f| \in L^{p(\cdot)}(\Omega, \vartheta) \right\}$$

or

$$W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta) = \left\{ f \in L^{q(\cdot)}(\Omega, \vartheta_0) : \frac{\partial f}{\partial x_i} \in L^{p(\cdot)}(\Omega, \vartheta) \right\}$$

equipped with the norm

$$\|f\|_{W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)} = \|f\|_{L^{q(\cdot)}(\Omega, \vartheta_0)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega, \vartheta)}.$$

Since $\vartheta \in W_{p(\cdot)}(\Omega)$ and $\vartheta_0 \in W_{q(\cdot)}(\Omega)$, it can be seen that $L^{p(\cdot)}(\Omega, \vartheta) \subset L^{q(\cdot)}_1(\Omega, \vartheta_0)$ and $L^{q(\cdot)}(\Omega, \vartheta_0) \subset L^{p(\cdot)}_1(\Omega)$. Therefore, the double weighted variable exponent Sobolev spaces $W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ is well-defined.

Now, we will give some basic properties of $W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$.

Proposition 2. The space $W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ is a Banach space with respect to the norm $\|\cdot\|_{W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)}$.

Proof. Let $(f_n)$ be a Cauchy sequence in $W^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$. Then, $(f_n)$ and $(\frac{\partial f_n}{\partial x_i})$ are Cauchy sequences in $L^{q(\cdot)}(\Omega, \vartheta_0)$ and $L^{p(\cdot)}(\Omega, \vartheta)$, respectively. Since the spaces $L^{q(\cdot)}(\Omega, \vartheta_0)$ and $L^{p(\cdot)}(\Omega, \vartheta)$ are Banach spaces, the sequence $(f_n)$ converges to some $f$ in $L^{q(\cdot)}(\Omega, \vartheta_0)$, and the sequence $(\frac{\partial f_n}{\partial x_i})$ converges to some $v_i$ in $L^{p(\cdot)}(\Omega, \vartheta)$ for $i = 1, 2, ..., d$. Hence, we have $f, v_i \in L^{1}_{loc}(\Omega)$, which are seen as distributions.
Now, we will show that each $v_i$ coincides with $\frac{\partial f}{\partial x_i}$ in the distributional sense. For every $\phi \in C_0^\infty(\Omega)$, by the Hölder inequality, we have

$$\left| \int_\Omega f_n \phi dx - \int_\Omega f \phi dx \right| \leq \int_\Omega |f_n - f| |\phi| dx$$

$$= \int_\Omega |f_n - f| p_{\phi_0}^{-\theta_0} q_{\phi_0}^{-\theta_0} |\phi| dx$$

$$\leq C \|f_n - f\|_{L^q(\Omega, \vartheta_0)} \left\| \phi \vartheta_0^{-\frac{1}{\theta_0}} \right\|_{L^r(\Omega)}$$

$$\leq C \|f_n - f\|_{L^q(\Omega, \vartheta_0)} \left\| \phi \right\|_{L^\infty(\Omega)} \left\| \vartheta_0^{-\frac{1}{\theta_0}} \right\|_{L^r(\text{supp}\phi)}$$

where $\text{supp}\phi \subset \Omega$ denotes the support of $\phi$ and $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. Since $\vartheta_0 \in W_q(\cdot)(\Omega)$, we get $\left\| \vartheta_0^{-\frac{1}{\theta_0}} \right\|_{L^r(\text{supp}\phi)} < \infty$. Moreover, if we consider the fact that $f_n \to f$ in $L^q(\Omega, \vartheta_0)$, we obtain

$$\int_\Omega f_n \phi dx \to \int_\Omega f \phi dx$$

as $n \to \infty$. In similar way, using $\frac{\partial f_n}{\partial x_i} \to v_i$ in $L^p(\Omega, \vartheta)$ and $\vartheta \in W_p(\cdot)(\Omega)$, we get

$$\int_\Omega \frac{\partial f_n}{\partial x_i} \phi dx \to \int_\Omega v_i \phi dx$$

as $n \to \infty$ for all $\phi \in C_0^\infty(\Omega)$ and $i = 1, 2, \ldots, d$. This yields

$$\int_\Omega v_i \phi dx = \lim_{n \to \infty} \int_\Omega \frac{\partial f_n}{\partial x_i} \phi dx$$

$$= - \lim_{n \to \infty} \int_\Omega f_n \frac{\partial \phi}{\partial x_i} dx$$

$$= - \int_\Omega f \frac{\partial \phi}{\partial x_i} dx$$

for all $\phi \in C_0^\infty(\Omega)$ and $i = 1, 2, \ldots, d$. It follows that $v_i = \frac{\partial f}{\partial x_i}$, hence $(f_n)$ converges to $f$ in $W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)$.

\[\blacksquare\]

**Remark 2.** The dual space of $W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ is $W^{-1, q'(\cdot), p'(\cdot)}(\Omega, \vartheta_0^*, \vartheta^*)$ where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$, $\frac{1}{q'(\cdot)} + \frac{1}{q'(\cdot)} = 1$, $\vartheta^* = \vartheta^{1-p'(\cdot)} = \vartheta^{-\frac{1}{p'(\cdot)}}$ and $\vartheta_0^* = \vartheta_0^{1-q'(\cdot)} = \vartheta_0^{-\frac{1}{q'(\cdot)-1}}$.

It is clear that $C_0^\infty(\Omega)$ is a subspace of $W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)$. Then, we define the space $W_0^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)$. Since $W_0^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ is a closed subset of $W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)$, then the space
Proposition 3. (see [14]) Let $|\mathbb{R}^d - \Omega| > 0$. If $f \in W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$, then the function

$$
\tilde{f}(x) = \begin{cases} 
    f(x) & \text{if } x \in \Omega \\
    0 & \text{if } x \in \mathbb{R}^d - \Omega
\end{cases}
$$

belongs to $W^{1,q(\cdot),p(\cdot)}(\mathbb{R}^d, \vartheta_0, \vartheta)$, and for each $i = 1, 2, ..., d$, one has

$$
\frac{\partial \tilde{f}}{\partial x_i}(x) = \begin{cases} 
    \frac{\partial f}{\partial x_i}(x) & \text{if } x \in \Omega \\
    0 & \text{if } x \in \mathbb{R}^d - \Omega
\end{cases}.
$$

The functions in $W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ can be extended by zero outside $\Omega$ into a function in $W^{1,q(\cdot),p(\cdot)}(\mathbb{R}^d, \vartheta_0, \vartheta)$. Also, it is easy to see that

$$
W_0^{1,q(\cdot),p(\cdot)}(\mathbb{R}^d, \vartheta_0, \vartheta) = W^{1,q(\cdot),p(\cdot)}(\mathbb{R}^d, \vartheta_0, \vartheta).
$$

Moreover, by Proposition 3, we can write that

$$
W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta) = \left\{ f \in L^{q(\cdot)}(\Omega, \vartheta_0) : |\nabla f|^{p(\cdot)} \in L^1(\Omega, \vartheta), \ f = 0 \text{ on } \partial \Omega \right\}
$$

equipped with the norm

$$
\|f\|_{W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)} = \|f\|_{L^{q(\cdot)}(\Omega, \vartheta_0)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega, \vartheta)}.
$$

The importance of $W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$ can be seen in various applications, such as Dirichlet problem for elliptic partial differential equations. That means the zero extension property above allows us to consider the space as a solution space for problems with Dirichlet type boundary conditions.

Now, we consider the Poincaré inequality in the space $W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta)$. Let $A \subset \mathbb{R}^d$. We define

$$
p_A^- = \text{essinf}_{x \in A \cap \Omega} p(x), \quad p_A^+ = \text{esssup}_{x \in A \cap \Omega} p(x).
$$

If $p_A^+ < \infty$ and if there exists $r > 0$ such that every $x \in \Omega$ either

$$
p_A^-(B(x, r)) \geq d
$$

or

$$
p_A^+(B(x, r)) \leq \frac{dp_A^-(B(x, r))}{d - p_B^-(r)}
$$
is valid, then the variable exponent $p(.)$ is said to satisfies the jump condition in $\Omega$ with constant $r$. Moreover we put

$$
p_B^-(r) = \begin{cases} 
    \frac{dp_A^-(B(x, r))}{d - p_B^-(B(x, r))} & \text{if } p_B^-(B(x, r)) < d \\
    p_B^+(B(x, r)) & \text{if } p_B^-(B(x, r)) \geq d
\end{cases}
$$

It is clear that if $\Omega$ is bounded and if $p(.)$ is continuous in $\overline{\Omega}$, then $p(.)$ satisfies the jump condition in $\Omega$ with some $r > 0$, see [15], [33].
Remark 3. Let $\Omega \subset \mathbb{R}^d$ be a bounded set. Then, the claim of Proposition 2.4 in [23] satisfies even if $p(\cdot) = 1$. This yields that the space $L^p(\Omega)$ is continuously embedded in $L^1(\Omega)$.

Now, we are ready to consider the Poincaré inequality for $W^{1,q(\cdot),p(\cdot)}_0(\Omega, \vartheta_0, \vartheta)$. 

**Theorem 2.** Let $\Omega$ be a bounded open domain in $\mathbb{R}^d$ with smooth boundary $\partial \Omega$. Moreover, assume that the exponent $q(\cdot)$ holds the jump condition in $\Omega$ with constant $r > 0$ and $\vartheta_0 < \vartheta$. Then, there is a constant $C > 0$ such that

$$
\|f\|_{L^q(\Omega, \vartheta_0)} \leq C \|\nabla f\|_{L^p(\Omega, \vartheta)}
$$

for every $f \in W^{1,q(\cdot),p(\cdot)}_0(\Omega, \vartheta_0, \vartheta)$.

**Proof.** Since $\Omega$ is a bounded set, $\overline{\Omega}$ is compact. Then, we can find $x_1, x_2, \ldots, x_n$ such that

$$
\Omega \subset \bigcup_{n=1}^t B(x_n, r).
$$

Because of the fact that $f \in W^{1,q(\cdot),p(\cdot)}_0(\Omega, \vartheta_0, \vartheta)$, the function $\tilde{f}$ can be taken as (3.1). Since the exponent $q(\cdot)$ holds the jump condition, we get by [23 Proposition 2.4] that

$$
\|f\|_{L^q(\Omega, \vartheta_0)} = \left\|f\right\|_{L^q(\mathbb{R}^d, \vartheta_0)} \leq \left\|\tilde{f} \chi_{B(x_1, r)} + \cdots + \chi_{B(x_n, r)} \right\|_{L^q(\mathbb{R}^d, \vartheta_0)} \\
\leq \sum_{n=1}^t \left\|f\right\|_{L^q(B(x_n, r), \vartheta_0)} \leq c \sum_{n=1}^t \left\|\tilde{f}\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} \\
\leq c \sum_{n=1}^t \left( \left\|\tilde{f} - f_{B_n}(x_n, r)\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} + \left\|f_{B_n}(x_n, r)\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} \right).
$$

It is note that the function $f_{B_n}(x_n, r)$ is average of $\tilde{f}$ over the balls $B(x_n, r)$ and defined as $f_{B_n}(x_n, r) = \frac{1}{|B(x_n, r)|} \int_{B(x_n, r)} \tilde{f}(x) \, \vartheta(x) \, dx$, see [10]. It is clear that $q_n(B(x_n, r)) \leq q(\cdot)$. Moreover, if we use the Poincaré inequality over the balls (see [16, Section 1]) and the embedding $L^{q_n}(B(x_n, r), \vartheta_0) \hookrightarrow L^{q_n}(B(x_n, r), \vartheta_0)$ (see [23 Proposition 2.4]), then we obtain

$$
\left\|\tilde{f} - f_{B_n}(x_n, r)\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} \leq r_c \left\|\nabla f\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} \\
\leq r_c \left\|\nabla f\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} \\
\leq r_c \left\|\nabla f\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} \\
\leq r_c \left\|\nabla f\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)}
$$

for all $n = 1, 2, \ldots, t$. Since $q(\cdot) < p(\cdot)$ and $\vartheta_0 < \vartheta$, we have $L^{p(\cdot)}(\Omega, \vartheta) \hookrightarrow L^{p(\cdot)}(\Omega, \vartheta_0) \hookrightarrow L^{p(\cdot)}(\Omega, \vartheta_0)$. This follows that

$$
\left\|\tilde{f} - f_{B_n}(x_n, r)\right\|_{L^{q_n}(B(x_n, r), \vartheta_0)} \leq r_c \left\|\nabla f\right\|_{L^{p(\cdot)}(\Omega, \vartheta)}.
$$
Moreover, if we use the Poincaré inequality in $L^1_\rho(\Omega)$ and Remark \[3\] then we get

\[ \left| \int_{B(x_n,r)} \vartheta \right| \leq \frac{C}{rd} \int_{\Omega} |f(x)| \vartheta(x) \, dx \leq \frac{C}{rd} \text{diam}(\Omega) \int_{\Omega} |\nabla f(x)| \vartheta(x) \, dx \]

\[ \leq \frac{C}{rd} \text{diam}(\Omega) c^{***} \|\nabla f\|_{L^p(\Omega, \vartheta_0)} \]

\[ \leq \frac{C}{rd} \text{diam}(\Omega) c^{***} \|\nabla f\|_{L^p(\Omega, \vartheta)} \]

for all $n = 1, 2, ..., t$. Since $\vartheta \in W^p_0(\Omega)$, we have

\[ \rho_{q_B(x_n,r), \vartheta_0} \left( \chi_{B(x_n,r)} \right) = \int_{B(x_n,r)} \vartheta(x) \, dx < \infty. \]

This yields that $\|1\|_{L^q_{\vartheta_B(x_n,r) \cap (B(x_n,r), \vartheta_0)}}$ depends only on $q_B(x_n,r)$. Hence the claim follows from the inequality (3.3). \qed

The inequality (3.2) is well known for the classical weighted Sobolev spaces $W^{1,p}_0(\Omega, \vartheta)$ under some conditions, see [26]. From now on, we assume that necessary conditions satisfy the inequality (3.2) in $W^{1,q}(\Omega, \vartheta_0, \vartheta)$.}

**Definition 5.** By the Theorem \[2\] we can present the norm on $W^{1,q}(\Omega, \vartheta_0, \vartheta)$ denoted by

\[ \|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)} = \|\nabla f\|_{L^p(\Omega, \vartheta)} \]

for every $f \in W^{1,q}(\Omega, \vartheta_0, \vartheta)$. It is noted that the norms $\|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)}$ and $\|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)}$ are equivalent norms on $W^{1,q}(\Omega, \vartheta_0, \vartheta)$. Then, the space $W^{1,q}(\Omega, \vartheta_0, \vartheta)$ is continuously embedded in $L^q(\Omega, \vartheta_0)$ if and only if the inequality (3.2) is satisfied for all $f \in W^{1,q}(\Omega, \vartheta_0, \vartheta)$.

4. APPLICATION

In this section, we discuss the $p(\cdot)$-Laplace operator $-\Delta_{p(\cdot), \vartheta} = -\text{div} \left( \vartheta(x) |\nabla f|^{p(x)-2} \nabla f \right)$. Let us consider the functional

\[ J(f) = \int_{\Omega} \left( \frac{1}{p(x)} |\nabla f|^{p(x)} \vartheta(x) - \frac{1}{q(x)} |f|^{q(x)} \vartheta_0(x) \right) \, dx \]

for all $f \in W^{1,q}(\Omega, \vartheta_0, \vartheta)$. Then $J \in C^1 \left( W^{1,q}(\Omega, \vartheta_0, \vartheta), \mathbb{R} \right)$, and the $p(\cdot)$-Laplace operator is the derivative operator of $J$ in the weak sense satisfies

\[ \langle J'(f), g \rangle = \int_{\Omega} \left( \vartheta(x) |\nabla f|^{p(x)-2} \nabla f \nabla g - \vartheta_0(x) |f|^{q(x)-2} f g \right) \, dx \]

for all $f, g \in W^{1,q}(\Omega, \vartheta_0, \vartheta)$.

**Definition 6.** We call that $f \in W^{1,q}(\Omega, \vartheta_0, \vartheta)$ is a weak solution of problem \[1\] if

\[ \int_{\Omega} \vartheta(x) |\nabla f|^{p(x)-2} \nabla f \nabla g \, dx = \int_{\Omega} \vartheta_0(x) |f|^{q(x)-2} f g \, dx \]

for all $g \in W^{1,q}(\Omega, \vartheta_0, \vartheta)$.\]
Let \( p(.) \in P(\Omega) \). Now, we define Sobolev conjugate of \( p(.) \) as

\[
p^*(.) = \begin{cases} 
\frac{d p(.)}{d-p(.)}, & p(.) < d \\
\infty, & p(.) \geq d 
\end{cases}
\]

**Theorem 3.** (see [10], [19]) Suppose that \( \Omega \subset \mathbb{R}^d \) is an open, bounded set with Lipschitz boundary and \( p(.) \in C^+ (\overline{\Omega}) \), \( p(.) \in P^{\log} (\Omega) \) with \( 1 < p^- \leq p^+ < d \). If \( r(.) \in L^\infty (\Omega) \) with \( r^- > 1 \) satisfies \( r(x) \leq p^+(x) \) for every \( x \in \Omega \), then we obtain the embedding \( W^{1,p(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega) \). Moreover, the compact embedding \( W^{1,p(.)}(\Omega) \hookrightarrow L^{r^*(.)}(\Omega) \) holds if \( \inf_{x \in \Omega} (p^*(x) - r(x)) > 0 \).

**Theorem 4.** Suppose that \( p(.) , q(.) \in C (\overline{\Omega}) \) and \( 1 < p(x), q(x) \) for all \( x \in \overline{\Omega} \) and moreover,

\[
(I) \quad 0 < \vartheta_1 \leq L^{\alpha(\cdot)}(\Omega) \quad \text{with} \quad 1 < \alpha(.) \in C (\overline{\Omega}),
\]

\[
(II) \quad \vartheta_0^{(\cdot)(\cdot)} \in L^1(\Omega) \text{ where } t(.) \in C (\overline{\Omega}) \text{ and } 1 < t(.) < q(.) < p(.) .
\]

\[
(III) \quad \vartheta_0 (x) \geq c > 0 \text{ for all } x \in \Omega.
\]

Then we get the compact embedding \( W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta) \hookrightarrow L^{r(\cdot)}(\Omega, \vartheta_1) \) for every \( r(.) \in C (\overline{\Omega}) \) and \( 1 < r(.) < \frac{q^*}{\alpha q} \) where \( \frac{1}{\alpha q} + \frac{1}{q} = 1 \).

**Proof.** Let \( f \in W_0^{1,q(\cdot),p(\cdot)}(\Omega, \vartheta_0, \vartheta) \). Then we write that \( f \in L^{q(\cdot)}(\Omega, \vartheta_0) \) and \( \nabla f \in L^{p(\cdot)}(\Omega, \vartheta) \). This follows (II) that \( \rho_{L^{\frac{p(.)}{p(x)}}(\Omega)} (|\nabla f|^{\frac{p(.)}{p(x)}} \vartheta^{\frac{p(x)}{p(.)}}) < \infty \) and \( \rho_{L^{\frac{p(.)}{p(x)-1}}(\Omega)} \left( \vartheta^{\frac{p(x)}{p(.)}} \right) < \infty \). By the Hölder inequality, we have

\[
\int_\Omega |\nabla f(x)|^{t(x)} \, dx \leq c_0 \left\| \nabla f \right\|^{\frac{t(.)}{p(.)}}_{L^{\frac{p(.)}{p(x)}}(\Omega)} \left( \vartheta^{\frac{p(x)}{p(.)}} \right)^{\frac{t(.)}{p(.)}}_{L^{\frac{p(.)}{p(x)-1}}(\Omega)}.
\]

If we consider the [23] Proposition 2.4 and (II), then we get \( \left\| \vartheta^{\frac{p(x)}{p(.)}} \right\|^{\frac{p(.)}{p(x)-1}}_{L^{\frac{p(.)}{p(x)-1}}(\Omega)} < \infty \). This follows that

\[
\left\| \vartheta^{\frac{p(x)}{p(.)}} \right\|^{\frac{p(.)}{p(x)-1}}_{L^{\frac{p(.)}{p(x)-1}}(\Omega)} \leq \left( \int_\Omega \left( \vartheta (x) \right)^{-\frac{t(.)}{p(x)-1}} dx + 1 \right)^{\frac{p(x)-1}{p(.)}} \leq c_1
\]

and

\[
(4.1) \quad \int_\Omega |\nabla f(x)|^{t(x)} \, dx \leq c_0 c_1 \left\| \nabla f \right\|^{\frac{t(.)}{p(.)}}_{L^{\frac{p(.)}{p(x)}}(\Omega)}.
\]

In general, we can suppose that \( \int_\Omega |\nabla f(x)|^{t(x)} \, dx > 1 \). By [23] Proposition 2.4 and

\[
(4.1) \quad \text{when } \int_\Omega |\nabla f(x)|^{p(x)} \vartheta (x) \, dx \leq 1,
\]

we have

\[
\left\| \nabla f \right\|_{L^{t(\cdot)}(\Omega)} \leq c_0 c_1 \left\| \nabla f \right\|^{\frac{t(.)}{p(.)}}_{L^{\frac{p(.)}{p(x)}}(\Omega)} \leq c_0 c_1 \left( \int_\Omega |\nabla f(x)|^{p(x)} \vartheta (x) \, dx \right)^{\frac{1}{p(x)}} \leq c_0 c_1 \left\| \nabla f \right\|_{L^{p(\cdot)}(\Omega, \vartheta)}.
\]
That means
\[ (4.2) \quad \| \nabla f \|_{L^t(\Omega)} \leq C \| \nabla f \|^{\frac{p^*}{p^* - 1}}_{L^{p^*}(\Omega, \vartheta)} \]
where \( C = (c_\alpha c_1)^{\frac{1}{\alpha}} > 0 \). By similar method, if \( \int_{\Omega} |\nabla f(x)|^{p(x)} \vartheta(x)\,dx > 1 \), we obtain
\[ (4.3) \quad \| \nabla f \|_{L^t(\Omega)} \leq C \| \nabla f \|^{\frac{p^* + q}{p^* + q - 1}}_{L^{p^*}(\Omega, \vartheta)} \]
where \( C = (c_\alpha c_1)^{\frac{1}{\alpha}} > 0 \). If we consider the inequalities (4.2) and (4.3), then we have \( \nabla f \in L^t(\Omega) \). In addition, by \( t(\cdot) < q(\cdot) \), \( |\Omega| < \infty \), (II) and (III), we get that \( L^{q(\cdot)}(\Omega, \vartheta_0) \hookrightarrow L^t(\Omega) \hookrightarrow L^{t(\cdot)}(\Omega) \). Therefore, we have \( f \in L^{t(\cdot)}(\Omega) \). This follows that \( f \in W^{1, t(\cdot)}(\Omega) \). Hence, the inclusion \( f \in W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \subset W^{1, t(\cdot)}(\Omega) \) is satisfied. Using the Banach Theorem in (7), we get
\[ (4.4) \quad W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \hookrightarrow W^{1, t(\cdot)}(\Omega) \]
By Theorem 3 we have compact embedding
\[ (4.5) \quad W^{1, t(\cdot)}(\Omega) \hookrightarrow L^s(\Omega) \]
for \( s(\cdot) < t^*(\cdot) \). Now, we define \( s(\cdot) = r(\cdot) \beta(\cdot) \). By the Hölder inequality for variable exponent Lebesgue space and (4), we have
\[ \int_{\Omega} |f(x)|^{r(\cdot)} \vartheta_1(x)\,dx \leq c_h \left\| |f|^{r(\cdot)} \right\|_{L^{r(\cdot)}(\Omega)} \| \vartheta_1 \|_{L^{\frac{1}{r(\cdot)}}(\Omega)} < \infty. \]
This follows that \( W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \subset L^r(\Omega, \vartheta_1) \). If we consider the Banach Theorem in (7), then we get \( W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \hookrightarrow L^r(\Omega, \vartheta_1) \). Now, we take a sequence \( (f_n)_{n \in \mathbb{N}} \subset W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \) such that \( f_n \rightharpoonup 0 \) in \( W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \) as \( n \to \infty \). This follows that \( f_n \rightharpoonup 0 \) in \( W^{1, t(\cdot)}(\Omega) \) by (4.4). Moreover, if we consider (4.3), then we get that \( f_n \to 0 \) in \( L^\ast(\Omega) \). Hence, we have
\[ \int_{\Omega} |f_n(x)|^{r(\cdot)} \vartheta_1(x)\,dx \leq c_h \left\| f_n \right\|_{L^{r(\cdot)}(\Omega)} \| \vartheta_1 \|_{L^{\frac{1}{r(\cdot)}}(\Omega)} \to 0 \]
that is, \( f_n \to 0 \) in \( L^{r(\cdot)}(\Omega, \vartheta_1) \). This completes the proof. \( \square \)

**Corollary 1.** Assume that all assumptions of Theorem 4 are satisfied. Then there exist \( C_1, C_2 > 0 \) such that
\[ \int_{\Omega} |f(x)|^{r(\cdot)} \vartheta_1(x)\,dx \leq \begin{cases} C_1 \left( \| f \|_{W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)} \right)^{r^+}, & \text{if } \| f \|_{W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)} > 1 \\ C_2 \left( \| f \|_{W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)} \right)^{r^-}, & \text{if } \| f \|_{W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)} < 1 \end{cases} \]
for all \( f \in W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \).

**Proof.** If we consider the Theorem 4 then we have \( W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \hookrightarrow L^{r^+}(\Omega, \vartheta_1) \) and \( W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta) \hookrightarrow L^{r^-}(\Omega, \vartheta_1) \) for \( 1 < r^- \leq r(\cdot) \leq r^+ < \frac{t(\cdot)}{p(\cdot)} \). Therefore, there are \( c_1, c_2 > 0 \) such that
\[ \| f \|_{L^{r^+}(\Omega, \vartheta_1)} = \left( \int_{\Omega} |f(x)|^{r^+} \vartheta_1(x)\,dx \right)^{\frac{1}{r^+}} \leq c_1 \| f \|_{W^{1, q(\cdot), p(\cdot)}(\Omega, \vartheta_0, \vartheta)} \]
and
\[
\|f\|_{L^r(\Omega, \vartheta_1)} = \left( \int_{\Omega} |f(x)|^r \vartheta_1(x) \, dx \right)^{\frac{1}{r}} \leq c_2 \|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)}^{-}
\]
for all \( f \in W^{1,q} \). This implies that
\[
\int_{\Omega} |f(x)|^{r(x)} \vartheta_1(x) \, dx \leq \int_{\Omega} \left( |f(x)|^{r^+} + |f(x)|^{r^-} \right) \vartheta_1(x) \, dx
\]
\[
\leq c_1^+ \left( \|f\|_{W^{1,q}_0(\Omega, \vartheta_1)}^{r^+} \right)^{\frac{1}{r^+}} + c_2^r \left( \|f\|_{W^{1,q}_0(\Omega, \vartheta_0, \vartheta)}^{-} \right)^{\frac{1}{r^-}}
\]
\[
\leq \begin{cases} 
C_1 \left( \|f\|_{W^{1,q}_0(\Omega, \vartheta_1)}^{r^+} \right)^{\frac{1}{r^+}} & \text{if } \|f\|_{W^{1,q}_0(\Omega, \vartheta_0, \vartheta)} > 1 \\
C_2 \left( \|f\|_{W^{1,q}_0(\Omega, \vartheta_0, \vartheta)}^{-} \right)^{\frac{1}{r^-}} & \text{if } \|f\|_{W^{1,q}_0(\Omega, \vartheta_0, \vartheta)} < 1
\end{cases}
\]

From now on, we assume that \( \vartheta_0 \) and \( \vartheta \) satisfy \( (I) \) and \( (II) \), \( (III) \), respectively. By the similar method in [12 Theorem 3.1], the following theorem is easy to see.

**Theorem 5.** Assume that \( p(\cdot) \) and \( q(\cdot) \) are the conjugate exponents of \( p(\cdot) \) and \( q(\cdot) \), respectively. Moreover, let \( \vartheta^* = \vartheta^{1-p(\cdot)} \) and \( \vartheta_0^* = \vartheta_0^{1-q(\cdot)} \). Then, we have
\[
(i) \quad J' : W^{1,q}(\Omega, \vartheta_0, \vartheta) \to W^{-1,q'}(\Omega, \vartheta_0, \vartheta) \quad \text{is continuous, bounded and strictly monotone operator.}
\]
\[
(ii) \quad J' \text{ is a mapping of type } (S_+) \text{, i.e., if } f_n \to f \text{ in } W^{1,q}(\Omega, \vartheta_0, \vartheta) \text{ and } \limsup_{n \to \infty} (J'(f_n) - J'(f), f_n - f) \leq 0, \text{ then } f_n \to f \text{ in } W^{1,q}(\Omega, \vartheta_0, \vartheta).
\]
\[
(iii) \quad J' : W^{1,q}(\Omega, \vartheta_0, \vartheta) \to W^{-1,q'}(\Omega, \vartheta_0^*, \vartheta^*) \quad \text{is a homeomorphism.}
\]

**Theorem 6.** The energy functional \( J \) is coercive and bounded below.

**Proof.** Let \( f \in W^{1,q}(\Omega, \vartheta_0, \vartheta) \) and \( \|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)} > 1 \). If we consider the Definition 5 and [23 Proposition 2.4] (or [11]), then we have
\[
J(f) = \int_{\Omega} \frac{1}{p(x)} |\nabla f|^{p(x)} \vartheta(x) \, dx - \int_{\Omega} \frac{1}{q(x)} |f|^{q(x)} \vartheta_0(x) \, dx
\]
\[
\geq \frac{1}{p^*} \int_{\Omega} |\nabla f|^{p(x)} \vartheta(x) \, dx - \frac{1}{q^-} \int_{\Omega} |f|^{q(x)} \vartheta_0(x) \, dx
\]
\[
\geq \frac{1}{p^*} \|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)}^{p^*} - \frac{1}{q^-} \max \left\{ \|f\|^q_{L^q(\Omega, \vartheta_0)} : \|f\|^p_{L^p(\Omega, \vartheta_0)} \leq 1 \right\}
\]
\[
\geq \frac{1}{p^*} \|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)}^{p^*} - \frac{1}{q^-} \max \left\{ \|f\|^q_{W^{1,q}(\Omega, \vartheta_0, \vartheta)} : \|f\|^p_{W^{1,q}(\Omega, \vartheta_0, \vartheta)} \leq 1 \right\}
\]
\[
= \frac{1}{p^*} \|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)}^{p^*} - \frac{1}{q^-} \|f\|^q_{W^{1,q}(\Omega, \vartheta_0, \vartheta)}.
\]
Since \( q^+ < p^- \), we have \( J(f) \to \infty \) as \( \|f\|_{W^{1,q}(\Omega, \vartheta_0, \vartheta)} \to \infty \). This completes the proof. \( \square \)

**Theorem 7.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set. Then the energy functional \( J \) is weakly lower semicontinuous.
Proof. Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of functions in \(W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)\) converging weakly to \(f \in W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)\). If we consider the Theorem 6 and Theorem 7, then we get  
\[
J(f) = \int_{\Omega} \frac{1}{p(x)} |\nabla f|^{p(x)} \vartheta(x) \, dx - \int_{\Omega} \frac{1}{q(x)} |f|^{q(x)} \vartheta_0(x) \, dx 
\leq \liminf_{n \to \infty} J(f_n).
\]
That is the desired result.

Corollary 2. If we consider the Theorem 6 and Theorem 7, then we get that \(J\) has a minimum point in \(W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)\), i.e., \(f\) is a weak solution of \((1.1)\), see [20, 35].

Theorem 8. The operator \(J\) satisfies the (PS) condition.

Proof. Let \((f_n)_{n \in \mathbb{N}}\) is a (PS) sequence, i.e.,
\[
|J(f_n)| \leq M
\]
and
\[
J'(f_n) \to 0 \text{ in } W_0^{-1,q(.)\cdot p(.)} (\Omega, \vartheta_0^*, \vartheta^*)
\]
where \(\frac{1}{p(.)} + \frac{1}{q(.)} = 1\), \(\frac{1}{p(.)} + \frac{1}{q(.)} = 1\), \(\vartheta_0^* = \vartheta_0^{1-q(.)}\) and \(\vartheta^* = \vartheta^{1-p(.)}\). Now, we want to prove that \((f_n)\) has a convergence subsequence. First, we will show that \((f_n)\) is bounded in \(W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)\). To see this, we assume that \((f_n)\) is not bounded. Hence, we can suppose that \(||f_n||_{W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)} > 1\) for all \(n \in \mathbb{N}\). This follows that
\[
M + ||f_n||_{W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)} 
\geq J(f_n) - \frac{1}{\lambda} \langle J'(f_n), f_n \rangle
\geq \int_{\Omega} \frac{1}{p(x)} |\nabla f_n|^{p(x)} \vartheta(x) \, dx - \int_{\Omega} \frac{1}{q(x)} |f_n|^{q(x)} \vartheta_0(x) \, dx 
\geq \left( \frac{1}{p^*} - \frac{1}{\lambda} \right) \int_{\Omega} |\nabla f_n|^{p(x)} \vartheta(x) \, dx + \left( \frac{1}{\lambda} - \frac{1}{q^*} \right) \int_{\Omega} |f_n|^{q(x)} \vartheta_0(x) \, dx
\geq \left( \frac{1}{p^*} - \frac{1}{\lambda} \right) ||f_n||_{W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)}^p + \left( \frac{1}{\lambda} - \frac{1}{q^*} \right) ||f_n||_{W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)}^q
\]
or equivalently
\[
M + ||f_n||_{W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)} + \left( \frac{1}{q^*} - \frac{1}{\lambda} \right) ||f_n||_{W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)}^q
\geq \left( \frac{1}{p^*} - \frac{1}{\lambda} \right) ||f_n||_{W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)}^p
\]
Therefore, we have \(\lambda \leq p^+\) which is a contradiction. This means that \((f_n)\) is bounded in \(W_0^{1,q(.)\cdot p(.)} (\Omega, \vartheta_0, \vartheta)\). By this boundedness, there exists a subsequence
Again, if we consider $W^1_{0,q(.),p(.)} (\Omega, \vartheta_0, \vartheta)$ that (4.6) and (4.7) hold, it is clear that $\Lambda$ is weakly lower semicontinuous. This follows by (32) that $\Lambda$ is weakly lower semicontinuous.

This follows by (4.6) that $\Lambda$ is a convex functional. Then there exist a $t \in [0,1]$ such that

$$
\langle \Lambda' (g), f - g \rangle = \lim_{t \to 0} \frac{\Lambda (g + t (f - g)) - \Lambda (g)}{t} = \Lambda (f) - \Lambda (g).
$$

This yields

$$
\langle \Lambda' (g), f - g \rangle = \lim_{t \to 0} \frac{\Lambda (g + t (f - g)) - \Lambda (g)}{t} \leq \Lambda (f) - \Lambda (g)
$$

or equivalently

$$
\langle \Lambda' (g), f - g \rangle \leq \Lambda (f) - \Lambda (g)
$$

This follows that

$$
0 = \lim_{n \to \infty} \langle \Lambda' (f_n), f - f_n \rangle \leq \Lambda (f) - \lim_{n \to \infty} \Lambda (f_n).
$$

Moreover, it is easy to see that $\Lambda$ is weakly lower semicontinuous. This follows by (4.7) that

$$
\lim_{n \to \infty} \Lambda (f_n) = \Lambda (f).
$$

Now, we are ready to prove that $f_n \to f$ in $W^1_{0,q(.),p(.)} (\Omega, \vartheta_0, \vartheta)$. Assume that the sequence $\{f_n\}$ is not convergent to $f$ in $W^1_{0,q(.),p(.)} (\Omega, \vartheta_0, \vartheta)$. Therefore, for $\varepsilon_1 > 0$,
there exists a subsequence \((f_{n_k})\) of \((f_n)\) such that \(\| f_{n_k} - f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)} \geq \varepsilon_1\). Since \(\Lambda\) is convex functional, we have

\[
\limsup_{n \to \infty} \Lambda \left( \frac{f_{n_k} + f}{2} \right) \leq \Lambda (f).
\]

Moreover, it is clear that \(\frac{f_{n_k} + f}{2} \rightharpoonup f\) in \(W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)\). Since \(\Lambda\) is weakly lower semicontinuous, we have

\[
\Lambda (f) \leq \liminf_{n \to \infty} \Lambda \left( \frac{f_{n_k} + f}{2} \right)
\]
which is a contradiction in sense to (4.8). This follows that \(f_n \rightharpoonup f\) in \(W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)\). This completes the proof. \(\square\)

**Theorem 9.** Let \(p^+ < q^- < \lambda\). Then, the Problem \((P1)\) has a nontrivial weak solution.

**Proof.** For this theorem, our motivation is based on Mountain Pass Theorem (see [35]). By Theorem 8 the energy functional \(J\) satisfies (PS) condition on \(W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)\).

If we consider the [23, Proposition 2.4] (or [4]), then we have

\[
J(f) = \int_{\Omega} \frac{1}{p(x)} |\nabla f|^{p(x)} \vartheta(x) \, dx - \int_{\Omega} \frac{1}{q(x)} |f|^{q(x)} \vartheta_0(x) \, dx
\]

\[
\geq \frac{1}{p^+} \int_{\Omega} |\nabla f|^{p^+} \vartheta(x) \, dx - \frac{1}{q^-} \int_{\Omega} |f|^{q^-} \vartheta_0(x) \, dx
\]

\[
\geq \frac{1}{p^+} \frac{\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)}^{p^+}}{\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)}^{q^-}} - \frac{1}{q^-} \frac{\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)}^{q^-}}{\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)}^{p^+}}
\]

\[
\geq \left( \frac{1}{p^+} - \frac{1}{q^-} \right) \frac{\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)}^{q^-}}{\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)}^{p^+}} > 0
\]

for \(\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)} \leq 1\). Thus, when \(\| f \|_{W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)} = \rho\) sufficiently small, we have \(J(f) > 0\). Moreover, since \(p^+ < q^-\), we get

\[
J(tg) = \int_{\Omega} \frac{1}{p(x)} |t \nabla g|^{p(x)} \vartheta(x) \, dx - \int_{\Omega} \frac{1}{q(x)} |tg|^{q(x)} \vartheta_0(x) \, dx
\]

\[
\leq t^{p^+} \int_{\Omega} |\nabla g|^{p^+} \vartheta(x) \, dx - t^{q^-} \int_{\Omega} |g|^{q^-} \vartheta_0(x) \, dx \longrightarrow -\infty
\]
as \(t \to \infty\) for \(g \in W^{1, q(.)}_{0}(\Omega, \vartheta_0, \vartheta)\setminus\{0\}\). It is note that \(J(0) = 0\). This follows that \(J\) satisfies the geometric conditions of the Mountain Pass Theorem (see [6], [27], [35]), and the operator \(J\) admits at least one nontrivial critical point. \(\square\)
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