Analysis of the Game “2048” and Its Generalization in Higher Dimensions

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Abstract
We theoretically analyze the popular mobile app game ‘2048’ for the first time in $n$-dimensional space. We also present a strategy for computer and a winning strategy for player in certain conditions.

1 Introduction
‘2048’ [1] is a famous tile-mashing game created by Italian web developer Gabriele Cirulli and is extremely popular as a game app in mobile phones. In 2015’s computer olympiad, competitors were given to play this game. The game’s objective is to slide numbered tiles on a grid to combine them to create a tile with the number 2048; however, one can keep playing the game, creating tiles with larger numbers (such as a 32,768 tile) [2]. In this paper, we give a winning strategy and analyze the maximum value the player can reach for the standard two-dimensional board and also generalize for higher dimension. When the player has no legal moves (there are no empty spaces and no adjacent tiles with the same value), the game ends. Cirulli himself described 2048 as a clone of Veewo Studios’ app 1024, who has actually said in the description of the app to be a clone of Threes!. 2048 became a viral hit [3,4]. The game has been described by the Wall Street Journal as “almost like Candy Crush for math geeks” [5], and Business Insider called it “Threes on steroids” [7]. This game got second-place in a coding contest at Matlab Central Exchange was an AI system that would play 2048 on its own [6]. We can consider a computer is playing this game against a human player which is same as to think it as a two player game. We can add some new ingredient, such as we can construct an algorithm for the computer which will make the human player moves more difficult. In this paper, we give a winning strategy and analyze the maximum
value the player can reach for the standard two-dimensional board and also generalize for higher dimension.

When the human player (or the player) plays the game, initially the tiles of 2 and 4 appear randomly. We discuss the strategy that the computer uses in an attempt to defeat the player. After creating one tile, the player has many options to move the tile. We define move by the player and for which the player will reach the maximum value. After one move, the computer also has many options to create a tile for which the next move for the player would be most difficult. First the computer will create the tile by following it’s strategy. Then the player selects it’s move by winning strategy. By the winning strategy, the player can always reach the maximum value.

2 Analysis of the Game in two dimensions

In this section, we are going to construct the winning strategy for the player for $n_1 \times n_2$ board and also for the computer. To describe these mathematically, we need to define some terms for two-dimensional board. Every tile position in the two-dimensional board can be represented by a pair of co-ordinates $(i, j)$, where $0 \leq i \leq n_1 - 1$, and $0 \leq j \leq n_2 - 1$.

Definition 1 (Rectilinear neighbour): The rectilinear neighbour of a tile $(i, j)$, where $i \leq j$ or $i > j$, is said to be the row $i$ and column $j$ of that tile.

In Figure 1 we show the rectilinear neighbour for two dimensions.

![Rectilinear neighbour of a tile](image)

Figure 1: Rectilinear neighbour of a tile

Definition 2 (Diagonal neighbour): The diagonal neighbour of a tile is said to be any position that is not the rectilinear neighbour of that tile.
In Figure 2, we show the diagonal neighbour in 2-dimension.

![Diagram of diagonal neighbour of a tile]

**Figure 2: Diagonal neighbour of a tile**

**Note:** In this paper, we use the entire board as the $n$ neighbourhood region at a tile. In general, the neighbourhood region of a tile can be restricted to a certain radius.

**Remark 1** If we consider the rectilinear neighbour as a set then it will contain the tiles of the rows and the columns of that tile and the set diagonal neighbour is the complement set of rectilinear neighbour where the universal set is the whole board.

**Strategy for the computer:** Computer will move first by appearing a tile of value 4 at a random position $(i, j)$, where $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq n_2 - 1$. During the game, the computer will create a 4-valued tile at the rectilinear neighbour of $(i, j)$ if the value of the tile $(i, j)$ is greater than or equal to 4, otherwise create a tile of value 2 in the diagonal neighbour of the tile $(i, j)$.

The computer will create a tile of value 2 at the position $(i, j)$, when there is a tile of value 2 in the rectilinear neighbour of $(i, j)$. The computer will create a tile in the opposite direction of motion with respect to the tile moved by the player.

**Definition 3 (Previous tile):** For a tile position $(i, j)$, where $i \leq j$ or $i > j$, we define the previous tile as a function of $i$ and $j$ as follows,

$$f(i, j) = \begin{cases} 
(i - 1, j), & \text{if } (n_1 - i) \mod 2 = 1 \text{ and } j = n_2 - 1; \\
(i, j + 1), & \text{if } (n_1 - i) \mod 2 = 1 \text{ and } j < n_2 - 1; \\
(i, j - 1), & \text{if } (n_1 - i) \mod 2 = 0 \text{ and } j > 0; \\
(i + 1, j), & \text{OR, } (n_1 - i) \mod 2 = 0 \text{ and } j = 0; \\
\end{cases} \quad (1)$$

In Figure 3 we show the previous tile in two dimensions.
Definition 4 (Favourable situation): When the value of the tile position \((i,j)\) is \(2^m\) and the value of the previous tile of \((i,j)\) is \(2^{m+1}\) (where \(1 \leq m \leq n_1n_2\)) then it is called the favourable situation.

In Figure 4 we show the Favourable situation.

Definition 5 (Maximum value): Maximum value is the maximum value the player can achieve overall possible scenarios.

Definition 6 (Target cell): Consider the tile \(t_{\text{max}}\) currently at position \((i_{\text{max}}, j_{\text{max}})\) containing the maximum value say \(2^l\). In order to increase the value at \((i_{\text{max}}, j_{\text{max}})\) we need to create second tile \(t'\) of value \(2^l\) at position \((i', j')\), where \((i', j')\) is the previous element of \((i_{\text{max}}, j_{\text{max}})\) in the sequence specified by \(S_m\). We call \((i', j')\) as the current target cell. Given \(m\), the \(m\)-th term = \((i, j)\).

Let \(m = qn_2 + r\) then \(i = n_1 - 1 - q\) and \(j = n_2^{q \mod 2} - 1 + (-1)^q \mod 2r\)

For given \(m\), the sequence \(S_m = (n_1 - 1 - q, n_2^{q \mod 2} - 1 + (-1)^q \mod 2r)\), where \(m = qn_2 + r\).

Definition 7 (Merge): Add up of two same value tile is called merge.

**Winning strategy for the player:** In the beginning of the game, there is only one 4-valued tile \(t_1\) in the board, say, placed at a random position \((i,j)\), where \(0 \leq i \leq n_1 - 1, 0 \leq j \leq n_2 - 1\). The player will bring the tile \(t_1\) to the position \((n_1 - 1, 0)\), unless it is already in this position. Otherwise, the player can bring the tile by one move or will bring \(t_1\) to the rectilinear position of \((n_1 - 1, 0)\). For this move by the player, the computer will create another tile \(t_2\) against the player in the board. Then the player will move the tile \(t_1\) to the position \((n_1 - 1, 0)\). As per definition 10
$t_2$ will appear in the opposite direction of motion of $t_1$ and hence will not obstruct its movement to the position $(n_1 - 1, 0)$.

As the first step, the player will fix the target cell $t_{\text{max}}$ according to the sequence $S_m$, then the player will look at the value of the previous tile of the target cell. Next, the player will try to make the same valued tile in the target cell $t_{\text{max}}$. For each tile there are 4 possible moves and for each move there are 3 possible cases.

**Case 1:** If there is a 4-valued tile in the rectilinear position of the target cell $t_{\text{max}}$, then the player will bring that 4-valued tile in the target cell $t_{\text{max}}$ by one move.

**Case 2:** If there is a 2-valued tile in the rectilinear position of the target cell $t_{\text{max}}$, and there are two another 2-valued tiles in the board which can be merged, then the player will merge that 2-valued tile to get a 4-valued tile, otherwise, the player will bring the 2-valued tile to the position $t_{\text{max}}$ by one move.

**Case 3:** If there is no tile in the rectilinear position of the target cell $t_{\text{max}}$, then the player will bring a 4-valued tile to the rectilinear position of the target cell $t_{\text{max}}$ from the diagonal position of the target cell $t_{\text{max}}$. To bring this tile to the rectilinear position of $t_{\text{max}}$ the player will choose row wise move. If there is no 4-valued tile in the board then the player will merge two 2-valued tile to get a 4-valued tile.

When the player has made a 8-valued tile in the target cell then according to the sequence $S_m$, the player will choose another target cell and will continue this process. When the player has made a same valued tile with respect to the previous tile of the target cell $t_{\text{max}}$, then the player will merge those two tiles.

If the player will continue this game in this way, player can make the favourable situation and the maximum value.
Figure 5: Winning strategy for the player

Figure 6: Maximum value the player can reach
Whenever the player will be able to make the favorable situation it can reach the maximum value. To prove this result, we construct the next theorem.

**Theorem 1** In $n_1 \times n_2$ two-dimensional board, the player can reach the maximum value $2^{n_1 \times n_2 + 1}$ (where $n_1 \geq 1, n_2 \geq 1$).

**Proof:** We would like to use inductive argument on $n_1$ and $n_2$ respectively to prove this result. For the base case, take $n_1 = 1$ and $n_2 = 1$, the player can reach the maximum value $2^{(1 \times 1) + 1} = 2^2 = 4$, which is trivially true (by the favourable situation).

Let us assume that for $n_1 \times n_2$ board, the player can reach the maximum value $2^{(n_1 \times n_2 + 1)}$.

Now for $(n_1 + 1) \times n_2$ board, $(n_1 + 1)n_2$ boxes are there. After reaching $2^{(n_1 \times n_2 + 1)}$ there $n_2$ cells are left. When the player is playing using winning strategy, the player can make the favourable situation. The player fixes the target cell $t_{\text{max}}$ then it will try to make the same valued tile at position $t_{\text{max}}$ with respect to it's previous tile. If the value of $t_{\text{max}}$ is equal to it’s previous tile the player will merge them. To get the favourable situation, the player will make same value tile with respect to it’s previous tile and easily the player can reach $2^{(n_1 \times n_2 + 1)} + 1 = 2^{n_1 \times n_2 + 2}$. The player will continue this process for $n_2$ vacant cells to reach the maximum value

\[
= 2^{n_1 \times n_2 + 1} \times 2^{n_2} \\
= 2^{(n_1 + 1)n_2 + 1}.
\]

Then by the induction hypothesis the player can reach the maximum value $2^{(n_1 \times n_2 + 1)}$ for $n_1 \times n_2$ two dimensional board, as desired.

**Corollary 1** For the special case $n_1 = n_2 = n$, the maximum value the player can reach is $2^{(n^2 + 1)}$.

**Proof:** By Theorem 1 for $n_1 = n_2 = n$ the maximum value the player can reach is $2^{(n^2 + 1)}$ for $n \times n$ board.

**Algorithm for two-dimensional plane**
Algorithm 1: For Computer

1. Create a tile of 4 without at the position \((i, j)\) the board.
2. If there is a tile of 4 or bigger than 4 then create 4 in the nearest position else, appear 2 in the diagonal position.
3. Create a tile of value 2 if there a tile of value 2 in the nearest position
4. The computer will create a tile in the opposite direction of motion with respect to the tile moved by the player.

Algorithm 2: For player

1. Bring the tile \(t_1\) to the position \((n_1 - 1, 0)\)
2. Fix target cell \(t_{\text{max}}\) according to \(S_m\)
3. Look at the value of previous tile of \(t_{\text{max}}\)
4. Make the same valued tile in the target cell \(t_{\text{max}}\)

3 Generalization to ‘\(d\)’ dimensions

In this section, we are going to generalize the game in the most general form, i.e., a \(n_1 \times n_2 \times \ldots \times n_{d-1} \times n_d\) board in \(d\) dimensions. We count the maximum value the player can reach and discuss the winning strategy. Every \(d\)-dimensional board will thought as \(d\) tuples of coordinates \((i_1, i_2, \ldots, i_d)\).

We will define some terms in \(d\) dimensions.

Definition 8 (Rectilinear neighbour in \(d\) dimensions): Every tile position in \(d\)-dimensional board can be represented by the co-ordinates \((i_1, i_2, \ldots, i_d)\) where \(0 \leq i_j \leq n_j - 1\), where \(j = 1, 2, \ldots, d\). When the co-ordinates \(i_j\) (for \(j = 2, \ldots, d\)) are fixed and \(i_1\) varies then there is a row along \(n_{i_1}\). Similarly, when \(i_j\) varies for the different values of \(j = 1, 2, \ldots, d\) there are rows along \(i_j\). These rows are called the rectilinear neighbour of the tile \((i_1, i_2, \ldots, i_d)\) where \(0 \leq i_j \leq n_j - 1\).

Definition 9 (Diagonal neighbour): Diagonal neighbour of a tile is said to be any position that is not the rectilinear neighbour of that tile.

Definition 10 (Move of the player): The player chooses a tile and the direction \(r\) in which to move the chosen tile. There are \(2n\) options for directions; \(n\) mutually orthogonal axis-parallel directions and their opposite directions. Once the player chooses that direction then excluding the direction \(r\) we can identify the \((n - 1)\) dimensional hyperplane \(h\) containing the tile \(t\). Once the player initiates the move, each tile in the hyperplane \(h\) moves furthest possible distance along the direction \(d\).

Winning strategy for the player in \(d\) dimension:

As first step, the player fixes the direction \(a_2\). As we know, \(a_i \leq n_i - 1\) for all \(i = 1, 2, \ldots, d\) and \(n_i \geq 1\), when the player will reach \(n_2 - 1\) along the direction \(a_2\), it will fix the direction \(a_1\) for one step.

Next, the player goes through the direction \((-a_2)\) and after reaching \(((n_2 - 1) - (n_2 - 1)) = 0\), the player will fix the direction \(a_1\) again. After completing \(a_1\) and \(a_2\) in two dimensions the player...
will go to 3-dimension. For 3-dimensional board, we can think it as \( n_3 \) many layers of \( n_1 \times n_2 \) two-dimensional board. In three dimensions, the player will fix the direction \( a_3 \). After completing three dimensions the player will go to four dimensions and will fix the direction \( a_4 \) and so on, because we considered a \((d + 1)\)-dimensional \( n_1 \times n_2 \times \ldots \times n_d \times n_{d+1} \) board as \( n_{d+1} \) many \( d \)-dimensional \( n_1 \times n_2 \times \ldots \times n_d \) boards stacked up on one another. While choosing the direction the player will fix the target cell \( t_{\text{max}} \) according to the sequence \( S_m \).

The coordinates \((i_1, i_2, \ldots, i_d)\) becomes \( m \)-th term in the sequence where \( m = i_1 \times (n_2n_3 \ldots n_d) + i_2 \times (n_3 \ldots n_d) + \ldots i_{d-1} \times n_d + i_d \).

Let, \( m_1 = \left( \frac{m - i_d}{n_d} \right), \quad m_2 = \left( \frac{m_1 - i_{d-1}}{n_{d-1}} \right) \). Given \( m \), one can find the coordinates of that cell as follows, \( m_j = \left( \frac{m_{j-1} - i_{d-j+1}}{n_{d-j+1}} \right) \) for \( j = 1, 2, \ldots \) \( S_m = (m_d \mod n_1, m_{d-1} \mod n_2, \ldots, m_1 \mod n_{d-1} \mod n_d) \).

For each tile there are 2\( n \) possible moves and for each move there are 3 possible cases as we discussed in the winning strategy of two-dimensional board.

![Figure 7: Winning strategy for 3-dimensional board](image)

**Theorem 2** In \( n_1 \times n_2 \times \ldots \times n_d \) \( d \)-dimensional board, the player can reach the maximum value \( 2^{(\prod_{i=1}^{d} n_i) + 1} \).

**Proof:** We will prove it using inductive argument on \( d \).

For \( d = 2 \) (base case) the maximum value the player can reach for \( n_1 \times n_2 \) board is \( 2^{n_1n_2+1} \) from Theorem 1. Now let us assume that for \( d \)-dimensional \( n_1 \times n_2 \times \ldots \times n_d \) board, the player can reach the maximum value \( 2^{(\prod_{i=1}^{d} n_i) + 1} \).
A $(d+1)$-dimensional $n_1 \times n_2 \times \ldots \times n_d \times n_{d+1}$ board can be considered as $n_{d+1}$ many $d$-dimensional $n_1 \times n_2 \times \ldots \times n_d$ boards (called sub-boards here) stacked up on one another.

After reaching the maximum value in a $d$-dimensional sub-board, the player can continue the game in the adjacent $d$-dimensional sub-board.

By the winning strategy, the player can continue the game when at the $(0,0,\ldots,0)$-th position a $(d+1)$-dimensional hyper-cube of 8 is there and at the $(1,0,\ldots,0)$-th position a $(d+1)$-dimensional hyper-cube of 4 is there and at the $(1,0,\ldots,0)$-th position a $(d+1)$-dimensional hyper-cube of bigger value of 4 and 8 is there, then the game will be end. So, the maximum value is reached by the player $2(\prod_{i=1}^{d+1} n_i)+1$.

Now, we use inductive argument on $n_i$’s, each dimension size.

First we take $n_1 \times n_2 \times \ldots \times n_d$ board in $d$ dimensional plane. The player can reach the maximum value $2(\prod_{i=1}^{d} n_i)+1$.

Now, for $(n_1+1) \times n_2 \times \ldots \times n_d$ board in $d$-dimensional plane. After reaching $2(\prod_{i=1}^{d} n_i)+1$ there are $n_1n_2 \ldots n_d$ vacant cells. The player will fix the target cell $t_{max}$ and will try to make the same valued tile with respect to it’s previous tile. When the value of $t_{max}$ is equal to it’s previous tile then the player will merge them. For the favourable situation the player can reach $2(\prod_{i=1}^{d} n_i)+1$+ easily.

By continuing this process $n_2n_3 \ldots n_d$ times the player will reach $2(\prod_{i=1}^{d+1} n_i)+1+(\prod_{j=2}^{d} n_j)$. Then by Induction hypothesis the player can reach the maximum value $2(\prod_{i=1}^{d+1} n_i)+1$.

\textbf{Corollary 2} For the special case $n_1 = n_2 = \ldots = n_d = n$, the maximum value the player can reach is $2(n^{d+1})$.

\textbf{Proof:} By Theorem 2 for $n_1 = n_2 = \ldots = n_d = n$ the maximum value the player can reach is $2(n^{d+1})$ for $\underbrace{n \times n \times \ldots \times n}_{d \text{ times}}$ $d$-dimensional board.

\section{4 Conclusion}

In this paper, we have analyzed the popular mobile app game ‘2048’ for the first time and generalized it in $n$-dimensional space. Note that Huon Wilson \[8\], Cesar Kawakami \[9\] have developed this result for 4D and 5D respectively. We also presented a strategy for computer and a winning strategy for player in certain conditions.

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