ON A STOCHASTIC WAVE EQUATION
DRIVEN BY A NON-GAUSSIAN LÉVY PROCESS *

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Abstract

This paper investigates a damped stochastic wave equation driven by a non-Gaussian Lévy noise. The weak solution is proved to exist and be unique. Moreover we show the existence of a unique invariant measure associated with the transition semigroup under mild conditions.

Key words: Damped wave equation, Lévy noise, invariant measure
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1 Introduction

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space, and on which, \(\tilde{N}(dz, dt) := N(dz, dt) - \pi(dz)dt\) defines a compensated Poisson random measure of a Poisson random measure \(N : \mathcal{B}(Z \times [0, \infty)) \times \Omega \rightarrow \mathbb{N} \cup \{0\}\) with the characteristic measure \(\pi(\cdot)\) on \((Z, \mathcal{B}(Z))\) with \(Z = \mathbb{R}^m (m \in \mathbb{N})\). The characteristic measure \(\pi(\cdot)\) satisfies that
\[
\pi(\{0\}) = 0, \quad \int_Z 1 \wedge |z|^2 \pi(dz) < \infty. \tag{1.1}
\]
According to (1.1), for \(Z_1 = \{z \in Z; |z| \leq 1\}\), we can define
\[
\bar{\theta} = \int_{Z_1} |z|^2 \pi(dz), \quad \theta = \pi(Z \setminus Z_1). \tag{1.2}
\]

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In the current paper, we are concerned with the following hyperbolic equation with a non-Gaussian Lévy noise perturbation:

\[
\begin{aligned}
\frac{\partial^2 u(t,\xi)}{\partial t^2} + \kappa \frac{\partial u(t,\xi)}{\partial t} - \Delta u(t,\xi) \\
= \int_{Z_1} a(u(t-,\xi),z) \tilde{N}(dz,t) \\
+ \int_{Z \setminus Z_1} b(u(t-,\xi),z) \tilde{N}(dz,t), \\
(t,\xi) \in [0,\infty) \times D,
\end{aligned}
\]  

\hspace{1cm} (1.3)

where the domain \( D \subset \mathbb{R}^d \) is a bounded open set with sufficiently regular boundary \( \partial D \) and \( \kappa > 0 \) denotes the damped coefficient. The random measure \( \tilde{N}(dz,dt) = N(dz,dt) - \pi(dz)dt \) denotes the compensated Poisson random measure through the compensator of \( N(dz,dt) \). In addition, the functions \( a : \mathbb{R} \times Z_1 \to \mathbb{R} \) and \( b : \mathbb{R} \times Z \setminus Z_1 \to \mathbb{R} \) are some regular functions with the exact conditions in Section 2 below.

White noise perturbed stochastic wave equations have been investigated in the literature (see e.g. [1, 3, 4, 5, 6, 7, 12] and the references therein). In Chow [6], the global (weak) solutions of stochastic wave equations with polynomial nonlinearity were explored by constructing appropriate Lyapunov functionals. In a successive paper, Chow [7] discussed the asymptotic behavior of the global (weak) solution to a semilinear stochastic wave equation by using the energy approach. Brzniak et al. [5] studied an abstract stochastic wave equation: stochastic beam equation and Lyapunov functions techniques were used to prove the existence of global mild solutions and asymptotic stability of the zero solution. Barbu et al. [1] demonstrated the existence of an invariant measure for the transition semigroup associated with a stochastic wave equation with the nonlinear dissipative damping and further established the uniqueness in some special case. In Bo et al. [4], the authors used appropriate energy inequalities to give sufficient conditions such that the local solutions of a class of (strong) damped stochastic wave equations are blowup with a positive probability or explosive in \( L^2 \)-sense.

A recent work in Peszat and Zabczyk [11, 12] was to consider the following wave equation driven by an impulsive noise,

\[
\frac{\partial^2 u(t)}{\partial t^2} = [\Delta u(t) + f(u(t))]dt + b(u(t))PdZ(t),
\]  

\hspace{1cm} (1.4)

where \( f, b : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous, \( P \) is a regularizing linear operator and the impulsive noise \( Z = (Z_t)_{t \geq 0} \) is formulated as a Poisson random measure. By estimating the stochastic convolution w.r.t. Poisson random
measure, the authors proved that (1.4) admits a unique mild solution, provided the intensity measure of $Z$ and eigenvectors of the Laplace operator jointly satisfy a finite infinite series condition.

Compared with the above mentioned literature, we discuss several other aspects of the differences in this article. First, the objective equation we considered is the damped wave equation (with the damped term $\kappa \frac{\partial u(t)}{\partial t}$) which is used to model nonlinear phenomena in relativistic quantum mechanics with local interaction (see e.g. [15, 16]). Second, this paper focuses on the notion of the weak solution which is a stronger form than the mild notion. Third, the perturbation can include a general non-Gaussian Lévy noise which is much wider than the one considered in [11, 12]. Specially, we don’t make any assumptions for the Lévy measure in the process of proving the existence and uniqueness of the weak solution. Finally, we also explore the invariant measure associated with the weak solution, which was not considered in [11, 12].

The paper is organized as follows: In the coming section, some preliminaries and hypothesis are given. In Section 3, the existence of a unique weak solution to (1.3) is established. Section 4 is devoted to proving the existence of a unique invariant measure corresponding to the weak solution under mild conditions.

## 2 Preliminaries and hypothesis

We begin with some basic notation, functional spaces and inequalities, which will be used frequently in the following sections.

Define a linear operator $A$ by

$$Au = -\Delta u, \quad u \in D(A) = H^2(D) \cap H^1_0(D).$$

where $H^p(D)$ is the set of all functions $u \in L^2(D)$ which have generalized derivatives up to order $p$ such that $D^\alpha u \in L^2(D)$ for all $\alpha : |\alpha| \leq p$, and $H^p_0(D)$ denotes the closure of $C^\infty_0(D)$ in $H^p(D)$. Set $H = L^2(D)$ and $V = H^1_0(D)$. Then $A$ is a positive self-adjoint unbounded operator on $H$. On the other hand, both $H$ and $V$ are Hilbert spaces if we endow them with usual inner products $\langle \cdot, \cdot \rangle$ and $\ll \cdot , \cdot \rangle$, respectively. Furthermore,

$$D(A) \subset V \subset H \subset V^*,$$

where $V^*$ denotes the dual space of $V$, and the embedding $V \subset H$ is compact. Thus there exists an orthonormal basis of $H$, $(e_k)_{k=1,2,\ldots}$ which
consists of eigenvectors of $A$ such that $Ae_k = \lambda_k e_k$ for $k = 1, 2, \ldots$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, with $\lim_{k \to \infty} \lambda_k = +\infty$. According to the spectral theory, for each $s \in \mathbf{R}$, we can define Hilbert space $V_{2s} = D(A^s)$, under the following inner product and the norm:

\[
\langle u, v \rangle_{2s} := \sum_{k=1}^{\infty} \lambda_k^{2s} \langle u, e_k \rangle \langle v, e_k \rangle, \quad (2.3)
\]

\[
|u|_{2s} := \left[ \sum_{k=1}^{\infty} \lambda_k^{2s} |\langle u, e_k \rangle|^2 \right]^{1/2}. \quad (2.4)
\]

Obviously $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0$ and $\ll \cdot, \cdot \gg = \langle \cdot, \cdot \rangle_1$. For parsimony, we set $| \cdot | = | \cdot |_0$ and $\| \cdot \| = | \cdot |_1$. The following Poincare-type inequality are well known (see e.g. Temam [17] and Zeidler [19]):

\[
|u|_{\alpha_1} \leq \lambda_1^{\alpha_1 - \beta_1} |u|_{\beta_1}, \quad \text{for } \alpha_1 \leq \beta_1, \text{ and } u \in D(A^{\beta_1/2}). \quad (2.5)
\]

At the end of this section, we make the following basic assumptions:

(H1) $a, b : \mathbf{R} \times Z \to \mathbf{R}$ are measurable and there exists a constant $\ell_a > 0$ such that

\[
a(0, z) \equiv 0, \quad |a(x, z) - a(y, z)|^2 \leq \ell_a |x - y|^2 |z|^2.
\]

Remark 2.1 An example of the function pair $(a, b)$ is $a(x, z) = b(x, z) = \sigma(x)z$ in (H1), where $\sigma : \mathbf{R} \to \mathbf{R}$ is a Lipschitzian map with Lip-coefficient $\sqrt{\ell_a}$ and $\sigma(0) = 0$. In the case, the perturbation in (1.3) can be rewritten as

\[
\sigma(u(t))dL_t,
\]

where $(L_t)_{t \geq 0}$ is a Lévy process (with Lévy measure $\pi(\cdot)$) given by

\[
L_t = \int_0^t \int_{Z_1} z\tilde{N}(dz, ds) + \int_0^t \int_{Z \setminus Z_1} zN(dz, ds),
\]

by employing the Lévy-Khintchine Theorem (see e.g. Sato [14]).

In the coming section, we shall prove existence and uniqueness of the weak solutions to (1.3). A $V \times H$-valued $(\mathcal{F}_t)_{t \geq 0}$-adapted process $X = (X(t))_{t \geq 0} = ((u(t), v(t)))_{t \geq 0}$ is called a weak solution of (1.3) with an initial
value \( X(0) = (\varphi, \psi) \in V \times H \), if it fulfills the following two conditions:

1. \( X \in C([0, T]; V) \times \mathbb{D}([0, T]; H) \) for each \( T > 0 \), \( \mathbb{P} \)-a.s. and
2. For all test pairs \( \phi = (\phi_1, \phi_2)^T \in D(A^*) \), it holds that

\[
\langle X^T(t), \phi \rangle = \langle X^T(0), \phi \rangle + \int_0^t \langle X^T(s), A^* \phi \rangle \, ds + \int_0^t \langle G(X^T(s)), \phi \rangle \, ds
\]

almost surely for \( t \geq 0 \), where \( X^T(t) = (u(t), v(t))^T \) and \( A^* \) denotes the adjoint operator of \( A \) and \( D(A^*) \) is its domain of the definition. In addition,

\[
A = \begin{bmatrix} 0 & I \\ -A & -\kappa I \end{bmatrix},
\]

\[
G(X^T(t)) = \begin{bmatrix} 0 \\ \int_{Z_1} a(u(t-), z) \tilde{N}(dz, t) + \int_{Z_1 \setminus Z_1} b(u(t-), z) \tilde{N}(dz, t) \end{bmatrix}.
\]

### 3 Existence and uniqueness

The aim of this section is to establish the existence of a unique weak solution for (1.3) under the condition \((H1)\).

The following result concentrates on the counterpart with small jumps.

**Lemma 3.1** Suppose that \( h \in L^2([0, T] \times Z_1; V) \) and \( Y(0) = (\varphi, \psi) \in V \times H \). Then for any \( T > 0 \), there exists a unique weak solution \( (Y(t))_{t \geq 0} = ((u(t), v(t)))_{t \geq 0} \in C([0, T]; V) \times \mathbb{D}([0, T]; H) \) for the system:

\[
\begin{cases}
du(t) = v(t)dt \\
dv(t) = -[\kappa v(t) + Au(t)] dt + \int_{Z_1} h(t-; z) \tilde{N}(dz, dt), \\
u(0) = \varphi, \quad v(0) = \psi.
\end{cases}
\]

**Proof.** We are first to define,

\[
g(t) = \int_0^t \int_{Z_1} h(s, z) \tilde{N}(dz, ds), \quad t \geq 0.
\]

Since \( h \in L^2([0, T] \times Z_1; V) \), \( g \in L^2([0, T]; V) \). Let’s consider the system,

\[
\begin{cases}
\bar{u}(t) = [\bar{v}(t) + g(t)] dt \\
\bar{v}(t) = -[\kappa (\bar{v}(t) + g(t)) + Au(t)] dt
\end{cases}
\]

\[1\]For \( T > 0 \), \( \mathbb{D}([0, T]; H) \) denotes the space of all RCLL \((\mathcal{F}_t)_{t \geq 0}\)-adapted random processes.
In light of Lions [10], (3.2) admits a unique weak solution \( Z(t) = (u(t), \bar{v}(t)) \) such that \( Z \in C([0, T]; V) \times C([0, T]; H) \). Let \( v(t) = \bar{v}(t) + g(t) \). Then \( Y(t) = (u(t), v(t)) \) solves (3.1) and furthermore \( Y \in C([0, T]; V) \times \mathbb{D}([0, T]; H) \). Thus we complete the proof of the lemma.

**Proposition 3.1** Let the condition \((H1)\) hold. Then for \( X(0) = (\varphi, \psi) \in V \times H \), there exists a unique weak solution \( X = (X(t))_{t \geq 0} = ((u(t), v(t)))_{t \geq 0} \) for the system:

\[
\begin{aligned}
    &du(t) = v(t)dt \\
    &dv(t) = -[\kappa v(t) + Au(t)] dt + \int_{Z_1} a(u(t-), z) \tilde{N}(dz, dt), \\
    &u(0) = \varphi, \quad v(0) = \psi.
\end{aligned}
\]

**Proof.** Let’s construct a sequence of \((\tilde{F}_t)_{t \geq 0}\)-adapted random processes \((X^n)_n \geq 0\) by \( X^n(t) = X(0) \) for all \( t \geq 0 \), and for \( n \geq 0 \), \( X^{n+1} = (X^{n+1}(t))_{t \geq 0} = ((u^{n+1}(t), v^{n+1}(t))_{t \geq 0} \in C([0, T]; V) \times \mathbb{D}([0, T]; H) \) being the unique weak solution for the following system:

\[
\begin{aligned}
    &d\text{u}^{n+1}(t) = v^{n+1}(t)dt \\
    &d\text{v}^{n+1}(t) = -[\kappa v^{n+1}(t) + Au^{n+1}(t)] dt + \int_{Z_1} a(u^{n}(t-), z) \tilde{N}(dz, dt), \\
    &u^{n+1}(0) = \varphi, \quad v^{n+1}(0) = \psi.
\end{aligned}
\]

By virtue of Lemma 3.1, it follows that \( X^{n+1} \) exists. In what follows, we show that the sequence \((X^n)_{n \geq 1}\) is cauchy in \( C([0, T]; V) \times \mathbb{D}([0, T]; H) \) compatible with the uniform topology. The Itô rule (see e.g. Ikeda and Watanabe [9]) for \( |v^{n+1}(t) - v^n(t)|^2 \) yields that,

\[
\begin{aligned}
    &|X^{n+1}(t) - X^n(t)|^2_{V \times H} \\
    &= \|u^{n+1}(t) - u^n(t)\|^2 + |v^{n+1}(t) - v^n(t)|^2 \\
    &= \|u^{n+1}(t) - u^n(t)\|^2 - 2\kappa \int_0^t |v^{n+1}(s) - v^n(s)|^2 ds - \|u^{n+1}(t) - u^n(t)\|^2 \\
    &\quad + 2 \int_0^t \int_{Z_1} |a(u^n(s), z) - a(u^{n-1}(s), z)|^2 \pi(dz) ds \\
    &\quad + \int_0^t \int_{Z_1} [[v^{n+1}(s) - v^n(s)] + (a(u^n(s), z) - a(u^{n-1}(s), z))]^2 \\
    &\quad \quad - |v^{n+1}(s) - v^n(s)|^2 \tilde{N}(dz, ds) \\
    &= -2\kappa \int_0^t |v^{n+1}(s) - v^n(s)|^2 ds + 2 \int_0^t \int_{Z_1} |a(u^n(s), z) - a(u^{n-1}(s), z)|^2 \pi(dz) ds.
\end{aligned}
\]
\[ + \int_0^t \int_{Z_1} \left[ \left| (v^{n+1}(s) - v^n(s)) + (a(u^n(s), z) - a(u^{n-1}(s), z)) \right|^2 \right. \\
- \left. \left| v^{n+1}(s) - v^n(s) \right|^2 \right] \tilde{N}(dz, ds). \] (3.5)

In light of the condition \((H1)\) and Poincare-type inequality (2.5), one gets,

\[
2 \int_0^t \int_{Z_1} \left| a(u^n(s), z) - a(u^{n-1}(s), z) \right|^2 \pi(dz)ds \\
\leq 2 \ell_0 \int_0^t \int_{Z_1} \left| u^n(s) - u^{n-1}(s) \right|^2 |z|^2 \pi(dz)ds \\
= \frac{2 \ell_0}{\lambda_1} \int_0^t \left\| u^n(s) - u^{n-1}(s) \right\|^2 ds. \quad (3.6)
\]

Now we turn to the last term of the r.h.s. of (3.5). For \(t \geq 0\), define

\[
II(t) = 2 \int_0^t \int_{Z_1} \left( v^{n+1}(s) - v^n(s), a(u^n(s), z) - a(u^{n-1}(s), z) \right) \tilde{N}(dz, ds) \\
+ \int_0^t \int_{Z_1} \left| a(u^n(s), z) - a(u^{n-1}(s), z) \right|^2 \tilde{N}(dz, ds) \\
:= II_1(t) + II_2(t). \quad (3.7)
\]

Then for the term \(II_1\),

\[
[II_1, II_1]_{t}^{1/2} \\
= \left[ \int_0^t \int_{Z_1} \left( v^{n+1}(s) - v^n(s), a(u^n(s), z) - a(u^{n-1}(s), z) \right)^2 N(dz, ds) \right]^{1/2} \\
\leq 2 \left[ \int_0^t \int_{Z_1} \left| v^{n+1}(s) - v^n(s) \right|^2 \left| a(u^n(s), z) - a(u^{n-1}(s), z) \right|^2 N(dz, ds) \right]^{1/2} \\
\leq 2 \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right| \\
\times \left[ \int_0^t \int_{Z_1} \left| a(u^n(s), z) - a(u^{n-1}(s), z) \right|^2 N(dz, ds) \right]^{1/2} \\
\leq \frac{1}{4 \sqrt{6}} \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right|^2 \\
+ 4 \sqrt{6} \int_0^t \int_{Z_1} \left| a(u^n(s), z) - a(u^{n-1}(s), z) \right|^2 N(dz, ds). \quad (3.8)
\]

As a consequence, the Davis inequality and Poincare-type inequality (2.5)
jointly imply that,

\[
E \left[ \sup_{0 \leq s \leq t} |II_1(s)| \right] \\
\leq 2\sqrt{6} E \left[ [II_1, II_1]_t^{1/2} \right] \\
\leq \frac{1}{2} E \left[ \sup_{0 \leq s \leq t} |v^{n+1}(s) - v^n(s)|^2 \right] + \frac{48\tilde{\theta}\ell_a}{\lambda_1} \int_0^t E \left\| u^n(s) - u^{n-1}(s) \right\|^2 ds. 
\]

(3.9)

As for the term \(II_2\), analogously we have,

\[
[II_2, II_2]_t^{1/2} \\
= \left[ \int_0^t \int_{Z_1} |a(u^n(s), z) - a(u^{n-1}(s), z)|^4 N(dz, ds) \right]^{1/2} \\
\leq \ell_a \left[ \int_0^t \int_{Z_1} |u^n(s) - u^{n-1}(s)|^4 z^4 N(dz, ds) \right]^{1/2} \\
\leq \frac{1}{16\sqrt{6}} \sup_{0 \leq s \leq t} \left\| u^n(s) - u^{n-1}(s) \right\|^2 \\
+ \frac{4\sqrt{6}\ell_a^2}{\lambda_1^2} \int_0^t \int_{Z_1} \left\| u^n(s) - u^{n-1}(s) \right\|^2 z^4 N(dz, ds),
\]

(3.10)

and so,

\[
E \left[ \sup_{0 \leq s \leq t} |II_2(s)| \right] \\
\leq \frac{1}{8} E \left[ \sup_{0 \leq s \leq t} \left\| u^n(s) - u^{n-1}(s) \right\|^2 \right] + \frac{48\tilde{\theta}\ell_a^2}{\lambda_1^2} \int_0^t E \left\| u^n(s) - u^{n-1}(s) \right\|^2 ds,
\]

(3.11)

where we used the fact \( \int_{Z_1} |z|^4 \pi(dz) \leq \tilde{\theta} \).

In the following, we divide (3.5) into two respective parts \( \|u^{n+1}(t) - u^n(t)\|^2 \) and \( |v^{n+1}(t) - v^n(t)|^2 \) and estimate them respectively. According to (3.5) and (3.6), we can conclude that for all \( t > 0 \),

\[
E \left[ \sup_{0 \leq s \leq t} \left\| u^{n+1}(s) - u^n(s) \right\|^2 \right] + E \left[ \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right|^2 \right] \\
\leq \frac{2\tilde{\theta}\ell_a^2}{\lambda_1} E \int_0^t \left\| u^n(s) - u^{n-1}(s) \right\|^2 ds + E \left[ \sup_{0 \leq s \leq t} II_1(s) \right] + E \left[ \sup_{0 \leq s \leq t} II_2(s) \right].
\]
From (3.9) and (3.11), it follows that,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| u^{n+1}(s) - u^n(s) \right\|^2 \right] + \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right| \right] \\
\leq \frac{1}{8} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| u^n(s) - u^{n-1}(s) \right\|^2 \right] + C_1 \mathbb{E} \int_0^t \left\| u^n(s) - u^{n-1}(s) \right\|^2 \, ds \\
+ \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right| \right],
\]
where \( C_1 = \frac{5\delta \ell \alpha + 48\delta \ell^2}{\lambda_1} \). This implies that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| u^{n+1}(s) - u^n(s) \right\|^2 \right] \leq \frac{1}{8} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| u^n(s) - u^{n-1}(s) \right\|^2 \right] \\
+ C_1 \mathbb{E} \int_0^t \left\| u^n(s) - u^{n-1}(s) \right\|^2 \, ds.
\]
Analogously, using (3.5) and (3.6), one gets,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| v^{n+1}(s) - v^n(s) \right\|^2 \right] \\
\leq -2\kappa \mathbb{E} \int_0^t \left| v^{n+1}(s) - v^n(s) \right|^2 \, ds + \frac{2\delta \ell a}{\lambda_1} \mathbb{E} \int_0^t \left\| u^n(s) - u^{n-1}(s) \right\|^2 \, ds \\
+ \mathbb{E} \left[ \sup_{0 \leq s \leq t} II_1(s) \right] + \mathbb{E} \left[ \sup_{0 \leq s \leq t} II_2(s) \right].
\]
We also apply (3.9) and (3.11) to conclude that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right|^2 \right] \\
\leq -2\kappa \mathbb{E} \int_0^t \left| v^{n+1}(s) - v^n(s) \right|^2 \, ds + C_1 \mathbb{E} \int_0^t \left\| u^n(s) - u^{n-1}(s) \right\|^2 \, ds \\
+ \frac{1}{8} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| u^n(s) - u^{n-1}(s) \right\|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right|^2 \right].
\]
As a consequence,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| v^{n+1}(s) - v^n(s) \right|^2 \right] \\
\leq -4\kappa \mathbb{E} \int_0^t \left| v^{n+1}(s) - v^n(s) \right|^2 \, ds + \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| u^n(s) - u^{n-1}(s) \right\|^2 \right] \\
+ 2C_1 \mathbb{E} \int_0^t \left\| u^n(s) - u^{n-1}(s) \right\|^2 \, ds.
\]
Consequently, for all $t > 0$,

$$
\begin{align*}
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X^{n+1}(s) - X^n(s)|^2_{V \times H} \right] \\
\leq -4\kappa \mathbb{E} \int_0^t |v^{n+1}(s) - v^n(s)|^2 \, ds + \frac{3}{8} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X^n(s) - X^{n-1}(s)|^2_{V \times H} \right] \\
+ 3C_1 \mathbb{E} \int_0^t |X^n(s) - X^{n-1}(s)|^2_{V \times H} \, ds.
\end{align*}
$$

(3.12)

For each $0 < t \leq T$, let $V^n(t) = \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X^{n+1}(s) - X^n(s)|^2_{V \times H} \right]$ with $n \geq 0$. Then (3.12) can be rewritten as

$$
V^n(t) \leq \frac{3}{8} V^{n-1}(t) + 3C_1 \int_0^t V^{n-1}(s) ds, \quad n \geq 1,
$$

A recursive scheme for the above relation between $V^n$ and $V^{n-1}$ shows that for each $T > 0$, there exists a constant $C_T > 0$ such that

$$
V^n(t) \leq C_T \sum_{i=0}^n C^n_i \left( \frac{3}{8} \right)^{n-i} \frac{C^n_i}{i!} = C_T \left( \frac{3}{8} \right)^n \sum_{i=0}^n C^n_i \frac{(8C_T/3)^i}{i!}
$$

$$
\leq C_T \left( \frac{3}{4} \right)^n \exp \left( \frac{8C_T}{3} \right),
$$

where we used the fact $\sum_{i=0}^n C^n_i = 2^n$ and hence $C^n_i \leq 2^n$ for each $i = 0, 1, \ldots, n$. This recursive result further yields that there exists a random process $X \in C([0, T]; V) \times \mathbb{D}([0, T]; H)$ such that

$$
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^n(t) - X(t)|^2_{V \times H} \right] = 0.
$$

(3.13)

Letting $n \to +\infty$ in (3.4) to conclude that $(X(t))_{t \geq 0}$ is a weak solution of (3.3). The uniqueness of $(X(t))_{t \geq 0}$ follows from the Itô rule and Gronwall Lemma. We omit its proof.

\textbf{Theorem 3.1} Suppose that the condition (H1) holds. Then for $X(0) = (\varphi, \psi) \in V \times H$, (1.3) admits a unique weak solution $X = (X(t))_{t \geq 0} = (u(t), v(t))_{t \geq 0}$.

\textbf{Proof.} It follows from (1.1) that, $\pi(Z \setminus Z_1) < \infty$. Then the process $(N(Z \setminus Z_1 \times [0, t]))_{t \geq 0}$ has only finite jumps in each finite interval of $\mathbb{R}_+$.
i.e., there exist increasing jump times $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$. Moreover, $(N(A \times [0, t]))_{(A, t) \in \mathcal{B}(Z \setminus Z_1) \times \mathbb{R}_+}$ can be represented by a $Z$-valued point process $(p(t))_{t \geq 0}$ with the domain $D_p$ as a countable subset of $\mathbb{R}_+$. That is,

$$N(A \times [0, t]) = \sum_{s \in D_p, s \leq t} 1_A(p(s)), \quad \text{for } t > 0 \text{ and } A \in \mathcal{B}(Z \setminus Z_1). \quad (3.14)$$

Therefore for $k = 1, 2, \ldots$, $\tau_k \in \{t \in D_p; p(t) \in Z \setminus Z_1\}$. For each $n \in \mathbb{N}$, we easily see that $\tau_k$ is an $(\bar{\mathcal{F}}_t)_{t \geq 0}$-stopping time and $\tau_k \to \infty$, as $k \to \infty$. For each $T \in (0, \tau_1)$, By virtue of Proposition 3.2, there exists a unique weak solution $X^0 \in C([0, T]; V) \times D([0, T]; H)$ on $[0, \tau_1)$. Construct the following

$$X^1(t) = \begin{cases} X^0(t), & t \in [0, \tau_1), \\
X^0(\tau_1) + \begin{bmatrix} 0 \\ b(u(\tau_1), p(\tau_1)) \end{bmatrix}^T, & t = \tau_1. \end{cases}$$

Therefore $(X^1(t))_{0 \leq t \leq \tau_1}$ uniquely solves (3.1) in the time interval $[0, \tau_1]$. Furthermore we define

$$\begin{cases} \tilde{X}_0^1 = X^1(\tau_1), \\
\tilde{p}(t) = p(t + \tau_1), \\
D_{\tilde{p}} = \{t \geq 0; t + \tau_1 \in D_p\}, \\
\tilde{\mathcal{F}}_t = \mathcal{F}_{\tau_1+t}. \end{cases}$$

Note that $\tau_2 - \tau_1 \in \{t \in D_p; \tilde{p}(t) \in Z \setminus Z_1\}$. Then we can construct a process $(\tilde{X}^1(t))_{0 \leq t \leq \tau_2 - \tau_1}$ by a same way as for $(X^1(t))_{0 \leq t \leq \tau_1}$. Thus we let

$$X^2(t) = \begin{cases} X^1(t), & 0 \leq t \leq \tau_1, \\
\tilde{X}^1(t - \tau_1), & \tau_1 \leq t \leq \tau_2. \end{cases}$$

Then $X^2(t)$ is a unique weak solution of (1.3) in the time interval $[0, \tau_2]$. Hence the existence of the unique global weak solution follows from the above successive procedure, and the theorem is proved. \qed

4 \hspace{2em} Invariant measure

In the section, we shall study the existence of a unique invariant measure associated with the transient semigroup $(\mathcal{P}_t)_{t \geq 0}$ defined by

$$\mathcal{P}_t \Phi((\varphi, \psi)) = \mathbb{E} \left[ \Phi(X^0_t((\varphi, \psi))) \right], \quad (\varphi, \psi) \in V \times H, \quad \Phi \in C_b(V \times H), \quad (4.1)$$
where $X^0_t((\varphi, \psi)) = (u^0_t(\varphi), v^0_t(\psi))$ denotes the weak solution of (1.3) with the initial value $(\varphi, \psi) \in V \times H$ at time-zero. As for the Markov property of $X^0_t((\varphi, \psi))$, we refer to Bo et al. \[3\].

To establish the invariant measure for $(\mathcal{P}_t)_{t \geq 0}$, set

$$\delta_0 = \frac{\lambda_1}{2\kappa} \land \frac{\kappa}{4}, \quad (4.2)$$

and $\rho_\delta(t) = \delta u(t) + v(t)$ with the weak solution $(X(t))_{t \geq 0} = (u(t), v(t))_{t \geq 0}$ to (1.3). Then we claim that,

**Lemma 4.1** For all positive $\delta \leq \delta_0$ and $t \geq 0$, it holds that

$$|\rho_\delta(t)|^2 + \|u(t)\|^2 \leq |\delta \varphi + \psi|^2 + \|\varphi\|^2 - \int_0^t \left[ \delta \|u(s)\|^2 + \kappa |\rho_\delta(s)|^2 \right] ds$$

$$+ \int_0^t \int_{Z_1} |a(u(s), z)|^2 \pi(dz) ds + M_t$$

$$+ \int_0^t \int_{Z \setminus Z_1} \left[ |b(u(s), z)|^2 + 2 \langle \rho_\delta(t), b(u(s), z) \rangle \right] \pi(dz) ds,$$

\[4.3\]

where $(M_t)_{t \geq 0}$ is a RCLL $(\tilde{\mathcal{F}}_t)_{t \geq 0}$-martingale with mean zero and which is given by

$$M_t = \int_0^t \int_{Z_1} \left[ |\rho_\delta(s) + a(u(s), z)|^2 - |\rho_\delta(s)|^2 \right] \tilde{N}(dz, ds)$$

$$+ \int_0^t \int_{Z \setminus Z_1} \left[ |\rho_\delta(s) + b(u(s), z)|^2 - |\rho_\delta(s)|^2 \right] \tilde{N}(dz, ds), \quad t \geq 0.$$

**Proof.** By virtue of (1.3), the process $(\rho_\delta(t))_{t \geq 0}$ is a RCLL $(\tilde{\mathcal{F}}_t)_{t \geq 0}$-semimartingale which satisfies the following dynamics,

$$d\rho_\delta(t) = (\delta - \kappa) \rho_\delta(t) dt - [\delta(\delta - \kappa) + A] u(t) dt + \int_{Z_1} a(u(t), z) \tilde{N}(dz, dt)$$

$$+ \int_{Z \setminus Z_1} b(u(t), z) N(dz, dt), \quad (4.4)$$

$$\rho_\delta(0) = \delta \varphi + \psi.$$

On the other hand, we remark that for $\delta \leq \delta_0$ and $t \geq 0$,

$$\delta(\kappa - \delta) \langle u(t), \rho_\delta(t) \rangle - (\kappa - \delta) |\rho_\delta(t)|^2 - \delta \|u(t)\|^2 \leq -\frac{\delta}{2} \|u(t)\|^2 - \frac{\kappa}{2} |\rho_\delta(t)|^2.$$

\[4.5\]
Then apply the Itô rule w.r.t. Poisson random measures (see Ikeda and Watanabe [9]) to \( \frac{1}{2} |\rho(t)|^2 \), the desired result follows from (4.4) and (4.5) immediately. \( \square \)

Hereafter, we define an energy functional \( E^\delta \) on \( V \times H \) by
\[
E^\delta(u, v) = |\delta u + v|^2 + \|u\|^2, \quad (u, v) \in V \times H.
\]

In order to explore the invariant measure, we impose the following condition on the function \( b : \mathbb{R} \times Z \setminus Z_1 \to \mathbb{R} \), \( (H2) \) There exists \( \ell_b > 0 \) such that
\[
b(0, z) \equiv 0, \quad |b(x, z) - b(y, z)|^2 \leq \ell_b |x - y|^2.
\]

**Remark 4.1** Note that the condition \( (H2) \) rules out the case of \( b(x, z) = \sigma(x)z \) in Remark 2.1. To incorporate the case into the section, we impose the condition,
\( (H2)' \) There exists \( \ell_b > 0 \) such that
\[
b(0, z) \equiv 0, \quad |b(x, z) - b(y, z)|^2 \leq \ell_b |x - y|^2 \left| z \right|^p, \quad \text{with the integer } p \geq 2,
\]
\[
\theta_p = \int_{Z \setminus Z_1} |z|^p \pi(\text{d}z) < \infty.
\]

The last condition in \( (H2)' \) is equivalent to that the Lévy process \( (L_t)_{t \geq 0} \) admits the finite \( p \)-order moment. Compared with \( (H2) \) and \( (H2)' \), we also note that if \( (H2) \) holds, then Lévy measure \( \pi(\cdot) \) is unrestrictive. However it rules out the case in Remark 2.1. If \( (H2)' \) is assumed to be true, then the case in Remark 2.1 is included, but an additional condition on \( \pi(\cdot) : \theta_2 < \infty \) has to be imposed. However the essential proofs in the section by employing \( (H2) \) and \( (H2)' \) are indistinctive.

Consequently,

**Lemma 4.2** Suppose the triple \( (\ell_a, \ell_b, \kappa) \) satisfies that,
\[
\frac{\theta \ell_a + 2\ell_b}{\lambda_1} < \delta_0, \quad \text{and} \quad \kappa > \theta,
\] (4.6)
where $\hat{\theta}$, $\underline{\theta}$ are defined in (1.2). Then under the conditions (H1)–(H2), or under the conditions (H1)–(H2)' for the triple $(\ell_a, \ell_b, \kappa)$ satisfying (4.6) with $\underline{\theta}$ replaced by $\theta_p$, there exist positive constants $\delta \leq \delta_0$ and $\lambda = \lambda(\delta)$ such that

$$E^\delta(u(t), v(t)) \leq E^\delta(\varphi, \psi) - \lambda \int_0^t E^\delta(u(s), v(s))ds + M_t, \quad t \geq 0,$$

where the RCLL $(\bar{\mathcal{F}}_t)_{t \geq 0}$-martingale $(M_t)_{t \geq 0}$ is defined in Lemma 4.1.

**Remark 4.2** 1. Note that the parameter $\delta_0$ depends on $\kappa$ (see (4.2)). However, we can choose a pair $(\ell_a^*, \ell_b^*) \in (0, \infty)^2$ (at least when they are small enough) such that

$$\underline{\theta} \vee \sqrt{2\lambda_1} < \frac{\lambda_1^2}{2\theta \ell_a + 4\theta \ell_b^*}.$$

Taking any $\kappa^* \in (\underline{\theta} \vee \sqrt{2\lambda_1}, \lambda_1^2/[2\theta \ell_a + 4\theta \ell_b^*])$. Then the triple $(\ell_a^*, \ell_b^*, \kappa^*)$ fulfills (4.6).

2. If the condition (H2) is placed by (H2)', then the constant $\underline{\theta}$ should be placed by $\theta_p$ in (4.6). In the case, we choose a pair $(\ell_a^*, \ell_b^*) \in (0, \infty)^2$ (at least when they are small enough) such that

$$\theta_p \vee \sqrt{2\lambda_1} < \frac{\lambda_1^2}{2\theta \ell_a^* + 4\theta_p \ell_b^*}.$$

We are now in a position to prove Lemma 4.3.

**Proof of Lemma 4.3.** Using the conditions (H1)–(H2) and Poincare-type inequality (2.5), it follows that

$$\int_{Z_1} |a(u(t), z)|^2 \pi(dz) \leq \hat{\theta} \ell_a |u(t)|^2 \leq \frac{\hat{\theta} \ell_a}{\lambda_1} \|u(t)\|^2, \quad t \geq 0, \quad (4.7)$$

and

$$\left| \int_{Z_1} \left[ |b(u(t), z)|^2 + 2 \langle \rho_\delta(t), b(u(t), z) \rangle \right] \pi(dz) \right| \leq \frac{2\theta \ell_b}{\lambda_1} \|u(t)\|^2 + \Theta |\rho_\delta(t)|^2, \quad t \geq 0. \quad (4.8)$$

Thanks to (4.6), we can choose a positive $\delta \in (\hat{\theta} \ell_a/\lambda_1 + 2\hat{\theta} \ell_b/\lambda_1, \delta_0]$, and then Lemma 4.1 yields that,

$$|\rho_\delta(t)|^2 + \|u(t)\|^2 \leq |\rho_\delta(0)|^2 + \|\varphi\|^2 - \int_0^t \left[ \delta - \hat{\theta} \ell_a/\lambda_1 - 2\hat{\theta} \ell_b/\lambda_1 \right] \|u(s)\|^2 ds$$
For, from the Gronwall Lemma, it follows that for some positive constants $C$ a unique invariant measure $\nu$.

Proof. By virtue of Theorem 3.3, there exists a unique solution $(X_t)_{t\geq 0}$ for Theorem 4.1. Under the same conditions as in Lemma 4.3, there exists a unique solution $(X_t)_{t\geq 0}$ for the transient semigroup $(P_t)_{t\geq 0}$ defined by (4.1).

In what follows, we state the main result of the section.

Theorem 4.1 Under the same conditions as in Lemma 4.3, there exists a unique invariant measure $\nu(\cdot)$ on $(V \times H, B(V \times H))$ for the transient semigroup $(P_t)_{t\geq 0}$ defined by (4.1).

Proof. We adopt the method used in Chow [7]. Let $(\tilde{N}(A \times [0,t]))_{A \in B(Z)}$ be an independent copy of the Poisson random measure $(N(A \times [0,t]))_{A \in B(Z)}$ for $t \geq 0$. For any $A \in B(Z)$ and $t \in \mathbb{R}$, define

$$
\tilde{N}(A \times [0,t]) = N(A \times [0,t]), \quad \text{if } t \geq 0, \quad \text{and}
$$

$$
\tilde{N}(A \times [t,0]) = \tilde{N}(A \times [0,-t]), \quad \text{if } t < 0.
$$

Let $\tilde{N}$ be the compensated Poisson random measure of $\tilde{N}$. For each $s \in \mathbb{R}$, consider the system:

$$
\begin{cases}
du(t, \xi) = v(t, \xi)dt, \\
dv(t, \xi) = -[kv(t, \xi) + Au(t, \xi)]dt + \int_{Z_1} a(u(t, \xi), z)\tilde{N}(dz, dt) \\
+ \int_{Z_2} b(u(t, \xi), z)\tilde{N}(dz, dt), \\
u(s, \xi) = \varphi(\xi), \quad v(s, \xi) = \psi(\xi), \quad \xi \in D.
\end{cases}
$$

By virtue of Theorem 3.3, there exists a unique solution $(X^s_t((\varphi, \psi)))_{t,s} \in C([s,T]; V) \times D([s,T]; H)$ for each $T > 0$, provided $(\varphi, \psi) \in V \times H$. Therefore, from the Gronwall Lemma, it follows that for some positive constants $\delta \leq \delta_0$ and $\lambda = \lambda(\delta)$,

$$
\mathbb{E}\left[\mathbb{E}^\delta(X^s_t((\varphi, \psi)))\right] \leq e^{-\lambda(t-s)}\mathbb{E}\left[\mathbb{E}^\delta((\varphi, \psi))\right], \quad t > s. \quad (4.10)
$$

For $s_1 > s_2 > 0$, define

$$
\hat{X}^{1,2}_t((\varphi, \psi)) = (\hat{u}(t), \hat{v}(t)) = (u^{-s_1}_t(\varphi) - u^{-s_2}_t(\varphi), v^{-s_1}_t(\psi) - v^{-s_2}_t(\psi)).
$$
Then $\tilde{X}_t^{1,2}((\varphi, \psi))$ fulfills that

$$
\begin{align*}
\begin{cases}
d\tilde{u}(t, \xi) = \tilde{v}(t, \xi)dt, \\
\tilde{v}(t, \xi) = -[\kappa \tilde{v}(t, \xi) + A\tilde{u}(t, \xi)]dt + \int_{Z_1} \tilde{a}(u_t^{s_1}(\xi), u_t^{s_2}(\xi), z)\tilde{N}(dz, dt), \\
\tilde{u}(-s_2, \xi) = u_{s_2}^{s_1}(\xi) - \varphi(\xi), \\
\tilde{v}(-s_1, \xi) = v_{s_2}^{s_1}(\xi) - \psi(\xi),
\end{cases}
\end{align*}
$$

where for $(\xi, z) \in D \times Z$,

$$
\begin{align*}
\tilde{a}(u_t^{s_1}(\xi), u_t^{s_2}(\xi), z) &:= a(u_t^{s_1}(\xi), z) - a(u_t^{s_2}(\xi), z), \\
\tilde{b}(u_t^{s_1}(\xi), u_t^{s_2}(\xi), z) &:= b(u_t^{s_1}(\xi), z) - b(u_t^{s_2}(\xi), z).
\end{align*}
$$

Let $\tilde{\rho}(t) = \delta \tilde{u}(t) + \tilde{v}(t)$ with $t \geq 0$. Then from Lemma 4.3, it follows that there exist positive $\delta \leq \delta_0$ and $\lambda = \lambda(\delta)$ such that

$$\mathbb{E}\left[\mathbb{E}^\delta(\tilde{X}_t^{1,2}((\varphi, \psi)))\right] \leq e^{-\lambda(t+s_2)}\mathbb{E}\left[\mathbb{E}^\delta(\tilde{u}(-s_2), \tilde{v}(-s_2))\right], \quad t > -s_2. \tag{4.12}\label{4.12}$$

Thanks to (4.10), there exists a positive constant $C > 0$ such that

$$\mathbb{E}\left[\mathbb{E}^\delta(\tilde{X}_t^{1,2}((\varphi, \psi)))\right] \leq Ce^{-\lambda(t+s_2)}[1 + \mathbb{E}^\delta(\varphi, \psi)], \quad t > -s_2. \tag{4.13}\label{4.13}$$

Then by virtue of (4.13), one gets,

$$\mathbb{E}\left[\mathbb{E}^\delta(X_0^{s_1}((\varphi, \psi)) - X_0^{s_2}((\varphi, \psi)))\right] \leq Ce^{-\lambda s_2}[1 + \mathbb{E}^\delta(\varphi, \psi)]. \tag{4.14}\label{4.14}$$

This implies that $(X_0^s)_{s \geq 0}$ is Cauchy in $L^2(\Omega; V \times H)$. As a consequence, there exists a unique random vector $X_0^{-\infty}((\varphi, \psi)) \in L^2(\Omega; V \times H)$ such that $X_0^{-s}((\varphi, \psi)) \rightarrow X_0^{-\infty}((\varphi, \psi))$, as $s \rightarrow \infty$ in $L^2(\Omega; V \times H)$ sense. We remark that the vector processes

$$X_0^s((\varphi, \psi)) = (u_0^s(\varphi), v_0^s(\psi)) \quad \text{and} \quad X_s^0((\varphi, \psi)) = (u_s^0(\varphi), v_s^0(\psi))$$

admit the same distribution on the same probability space for each $s \geq 0$. Let $\nu(\cdot)$ be the induced probability measure of $X_0^{-\infty}((\varphi, \psi))$ on $(V \times H, \mathcal{B}(V \times H))$. Then $\nu(\cdot)$ is the unique invariant measure for the transient semigroup $(\mathcal{P}_t)_{t \geq 0}$. Thus the proof of the theorem is finished. \hfill \square

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