The Variance of Causal Effect Estimators for Binary V-structures

Jack Kuipers
D-BSSE, ETH Zurich, Mattenstrasse 26, 4058 Basel, Switzerland

Giussi Moffa
Division of Psychiatry, University College London, London, UK
Department of Mathematics and Computer Science, University of Basel, Basel, Switzerland

Abstract

Adjusting for covariates is a well established method to estimate the total causal effect of an exposure variable on an outcome of interest. Depending on the causal structure of the mechanism under study there may be different adjustment sets, equally valid from a theoretical perspective, leading to identical causal effects. However, in practice, with finite data, estimators built on different sets may display different precision. To investigate the extent of this variability we consider the simplest non-trivial non-linear model of a v-structure on three nodes for binary data. We explicitly compute and compare the variance of the two possible different causal estimators. Further, by going beyond leading order asymptotics we show that there are parameter regimes where the set with the asymptotically optimal variance does depend on the edge coefficients, a result which is not captured by the recent leading order developments for general causal models.

Keywords: Causality; Structure Learning; Bayesian Networks; Probability Theory.

1. Introduction

As a graphical representation of multivariate probability distributions, Bayesian networks are widely used statistical models with an underlying directed acyclic graph (DAG) structure. When taking DAGs to represent causal diagrams (Greenland et al. 1999; Pearl 2000; Hernán and Robins 2006; VanderWeele and Robins 2007) we may use a machinery based on the ‘do’ calculus of Pearl (1995) to estimate potential intervention effects of any variable on any other. Different graphical criteria exist to identify valid adjustment sets, among which the back-door criterion (Pearl 1993) is probably the most well known, and with alternative strategies developed more recently (Shpitser et al. 2010; Perković et al. 2017).

A valid adjustment set $Z$ for the effect of $X$ on $Y$ is such that for any probability distribution $p$ compatible with the underlying graphical structure $G$ the probability distribution of $Y$ after intervening on $X$ (setting it to some value) satisfies (Maathuis and Colombo 2015)

$$p(Y \mid do\ X) = \begin{cases} p(Y \mid X) & \text{if } Z = \emptyset \\ \int_Z p(Y \mid X, z)p(z)dz & \text{otherwise} \end{cases}$$

For linear Gaussian models, the marginalisation can be simply estimated by regressing $Y$ on $X$ and $Z$ and extracting the coefficient of $X$, hence the naming of ‘adjustment’ sets. This also holds for linear non-Gaussian causal models (Henckel et al. 2019).

The set of parents of $X$ always satisfies the back-door criterion and is therefore a valid adjustment set, but there may be many more depending on the graphical structure of the DAG (Perković et al. 2017). Although all valid adjustment sets provide consistent estimators of the causal effects,
for finite-sized data different adjustment sets can lead to different numerical estimates, and with different precisions.

In evaluating the variance of different estimators, Henckel et al. (2019) recently obtained the remarkable result that the asymptotically optimal adjustment set can be determined solely based on graphical criteria regardless of the edge coefficients. Even more recently, this has been extended to non-parametric estimators (Rotnitzky and Schuler, 2019) and the asymptotically optimal set has been further characterised (Witte et al., 2020).

To explore the precision of causal estimators for non-linear models we consider the simplest such case: a DAG with 3 nodes of binary variables organised in a v-structure with the outcome $Y$ of interest as a collider with parents $Z$ and $X$ (Figure 1), and with the latter being the exposure whose effect we wish to estimate.

For binary data and relatively small networks, one can explicitly marginalise over the remaining nodes in the DAG and its parameters (Moffa et al., 2017) to estimate causal effects. For the v-structure we compute the variance of the two different estimators corresponding to different adjustment sets, or to marginalisation with and without the edge from the third variable $Z$. In particular, we wish to compute the total causal effect of $X$ on $Y$:

$$F_R = p(Y | \text{do}(X = 1)) - p(Y | \text{do}(X = 0)) = p(Y | X = 1) - p(Y | X = 0)$$

(2)

where we can utilise the conditional probabilities since $X$ has no parents in the v-structure. We can therefore estimate the causal effect of $X$ on $Y$ from the raw conditionals in the data. For the v-structure we have the relationship

$$p(Y | X) = \sum_Z p(Y, Z | X) = \sum_Z p(Y | X, Z)p(Z)$$

(3)

so we can also estimate the causal effect of $X$ on $Y$ by marginalising over $Z$.

$$F_M = p(Y | X = 1, Z = 1)p(Z) + p(Y | X = 1, Z = 0)(1 - P(Z))$$

$$- p(Y | X = 0, Z = 1)p(Z) - p(Y | X = 0, Z = 0)(1 - P(Z))$$

(4)

The two choices of raw conditionals or marginalisation are analogous to adjusting for the empty set $\emptyset$ or $Z$ in linear models. It is instructive to also consider the DAG with the edge from $Z \rightarrow Y$ deleted. Since $Y$ would then be independent of $Z$, the marginalisation would reduce to the raw conditionals. The estimator using raw conditionals is therefore the same whether the edge from $Z \rightarrow Y$ is present or not, while the approach using marginalisation would give different estimates for the two cases.
2. Causal Estimates for a Binary V-structure

For both causal estimators we will use the maximum likelihood estimates of probabilities from the observed data. We consider the DAG in Figure 1 with the following probability tables:

\[
p(X = 1) = p_X, \quad p(Y = 1 \mid X = 0, Z = 0) = p_{Y|0}, \quad p(Y = 1 \mid X = 1, Z = 0) = p_{Y|2}
\]
\[
p(Z = 1) = p_Z, \quad p(Y = 1 \mid X = 0, Z = 1) = p_{Y|1}, \quad p(Y = 1 \mid X = 1, Z = 1) = p_{Y|3}
\]

When we generate data, as a collection of \( N \) binary vectors, from the DAG in Figure 1 instead of forward sampling along the topological order for this small example we can sample directly from a multinomial with probabilities

\[
\begin{array}{c|ccc|c|ccc|c}
X & Z & Y & p & X & Z & Y & p \\
\hline
0 & 0 & 0 & p_0 = (1 - p_X)(1 - p_Z)(1 - p_{Y|0}) & 1 & 0 & 0 & p_4 = p_X(1 - p_Z)(1 - p_{Y|2}) \\
0 & 0 & 1 & p_1 = (1 - p_X)(1 - p_Z)p_{Y|0} & 1 & 0 & 1 & p_5 = p_X(1 - p_Z)p_{Y|2} \\
0 & 1 & 0 & p_2 = (1 - p_X)p_Z(1 - p_{Y|1}) & 1 & 1 & 0 & p_6 = p_Xp_Z(1 - p_{Y|3}) \\
0 & 1 & 1 & p_3 = (1 - p_X)p_Zp_{Y|1} & 1 & 1 & 1 & p_7 = p_Xp_Zp_{Y|3}
\end{array}
\]

(6)

If we represent with \( N_i \) the number of sampled binary vectors indexed by \( i = 4X + 2Z + Y \), then the estimator of \( F \) from the raw conditionals is simply

\[
R = R_1 - R_0, \quad R_1 = \frac{N_5 + N_7}{N_4 + N_5 + N_6 + N_7}, \quad R_0 = \frac{N_1 + N_3}{N_0 + N_1 + N_2 + N_3}
\]

(7)

Using the marginalisation we would have the following estimator

\[
M = M_1 - M_0, \quad M_1 = M_{11} + M_{10}, \quad M_0 = M_{01} + M_{00}
\]

(8)

with the terms separated for later ease

\[
M_{11} = \frac{N_7}{(N_0 + N_7)} \frac{(N_2 + N_3 + N_6 + N_7)}{N}, \quad M_{01} = \frac{N_3}{(N_2 + N_3)} \frac{(N_2 + N_3 + N_6 + N_7)}{N}
\]
\[
M_{10} = \frac{N_5}{(N_4 + N_5)} \frac{(N_0 + N_1 + N_4 + N_5)}{N}, \quad M_{00} = \frac{N_1}{(N_0 + N_1)} \frac{(N_0 + N_1 + N_4 + N_5)}{N}
\]

(9)

2.1 Raw Conditionals

To compute \( E[R] \) we need to average over a multinomial sample

\[
E[R] = \sum_{N_0! \cdots N_7!} \frac{N!}{N_0! \cdots N_7!} p_0^{N_0} \cdots p_7^{N_7} R
\]

(10)

for which we use that fact that \( (p_0 + \cdots + p_7)^N \) generates the probability distribution when we perform a multinomial expansion. To obtain the terms needed for the expectation we define

\[
S_N = \{ [p_0 + p_2 + (p_1 + p_3)w]x + [p_4 + p_6 + (p_5 + p_7)v]z \}^N
\]

(11)

Then

\[
E[R_1] = \int \frac{v}{z} \frac{\partial}{\partial v} S_N \, dz \bigg|_{w=x=1, v=z=1} = \frac{v(p_5 + p_7)}{p_4 + p_6 + (p_5 + p_7)v} S_N \bigg|_{w=x=1, v=z=1}
\]

(12)
When we substitute for the generating variables, which sets \( S_N = 1 \), and perform the same steps for \( R_0 \) we obtain

\[
E[R] = \frac{p_5 + p_7}{p_4 + p_5 + p_6 + p_7} - \frac{p_1 + p_3}{p_0 + p_1 + p_2 + p_3} = \frac{p_5 + p_7}{p_X} - \frac{p_1 + p_3}{1 - p_X} \tag{13}
\]

The Variance. To compute the variance

\[
V[R] = V[R_1] - 2C[R_1, R_0] + V[R_0] \tag{14}
\]

we first show that the covariance is 0

\[
E[R_1 R_0] = \int \frac{w}{x} \frac{\partial}{\partial w} \int \frac{v}{z} \frac{\partial}{\partial v} S_N \, dz \, dx \bigg|_{w=x=1}^{w=x=1} = \int \frac{v(p_5 + p_7)}{p_4 + p_6 + (p_5 + p_7)v} S_N \, dx \bigg|_{w=x=1}^{w=x=1}
\]

\[
= \frac{v(p_5 + p_7)}{p_4 + p_6 + (p_5 + p_7)v} \cdot \frac{w(p_1 + p_3)}{p_0 + p_2 + (p_1 + p_3)w} S_N \bigg|_{w=x=1}^{w=x=1} = E[R_1]E[R_0] \tag{15}
\]

The more tricky terms are

\[
E[R_1^2] = \int \frac{v}{z} \frac{\partial}{\partial v} \int \frac{v}{z} \frac{\partial}{\partial v} S_N \, dz \, dx \bigg|_{w=x=1}^{w=x=1} = \int \frac{v(p_5 + p_7)}{p_4 + p_6 + (p_5 + p_7)v} S_N \, dz \bigg|_{w=x=1}^{w=x=1}
\]

\[
= \frac{v}{z} \frac{(p_5 + p_7)}{p_4 + p_6 + (p_5 + p_7)v} \int \left[ \frac{(p_4 + p_6)}{p_4 + p_6 + (p_5 + p_7)v} S_N + v \frac{\partial}{\partial v} S_N \right] \, dz \bigg|_{w=x=1}^{w=x=1}
\]

\[
= \frac{(p_5 + p_7)(p_4 + p_6)}{p_X^2} \int \left( 1 - p_X + p_X z \right)^N \, dz \bigg|_{z=1}^{z=1} + E[R_1]^2 \tag{16}
\]

The remaining integral can be expressed in terms of the hypergeometric function \( F \):

\[
\int \left( 1 - p_X + p_X z \right)^N \, dz = \sum_{k=1}^{N} \binom{N}{k} \frac{1}{k} p_X^k (1 - p_X)^{N-k}
\]

\[
= N p_X (1 - p_X)^{N-1} F \left( [1, 1 - N], [2, 2], - \frac{p_X}{1 - p_X} \right) \tag{17}
\]

Repeating the calculations for \( V[R_0] \) we obtain

\[
V[R] = \frac{(p_5 + p_7)(p_4 + p_6)}{p_X} N (1 - p_X)^{N-1} F \left( [1, 1 - N], [2, 2], - \frac{p_X}{1 - p_X} \right)
\]

\[
+ \frac{(p_1 + p_3)(p_0 + p_2)}{1 - p_X} N p_X^{N-1} F \left( [1, 1 - N], [2, 2], - \frac{1 - p_X}{p_X} \right) \tag{18}
\]

Bounds. This hypergeometric function in Equation (17) has a maximum value at around \( \frac{1}{N} \), and we note that if we divide by \( k + 1 \) instead of \( k \) in the sum we have the simple result

\[
\sum_{k=0}^{N} \binom{N}{k} \frac{1}{k+1} p_X^k (1 - p_X)^{N-k} = \frac{1}{p_X(N + 1)} - \frac{(1 - p_X)^{N+1}}{p_X(N + 1)} \tag{19}
\]
so that by considering the early terms in the sum we can bound

\[ \sum_{k=1}^{N} \left( \frac{N}{k} \right) \frac{1}{k} p_X^k (1 - p_X)^{N-k} > \frac{1}{p_X (N+1)}, \quad p_X > \frac{N - 1 + \sqrt{3N^2 + 4N + 1}}{N(N + 3)} \]  

(20)

which we can loosen to \( p_X > \frac{1 + \sqrt{3}}{N} \). This provides the following lower bound for the variance

\[ V[R] > \frac{(p_5 + p_7)(p_4 + p_6)}{p_X^3 (N + 1)} + \frac{(p_1 + p_3)(p_0 + p_2)}{(1 - p_X)^3 (N + 1)}, \quad \frac{1 + \sqrt{3}}{N} < p_X < \frac{N - 1 - \sqrt{3}}{N} \]  

(21)

To obtain a simple upper bound we can compute

\[ \sum_{k=1}^{N} \left( \frac{N}{k} \right) \frac{1}{k} p_X^k (1 - p_X)^{N-k} < 2 \sum_{k=1}^{N} \left( \frac{N}{k} \right) \frac{1}{k + 1} p_X^k (1 - p_X)^{N-k} < \frac{2}{p_X (N + 1)} \]  

(22)

so that the variance vanishes in the large \( N \) limit

\[ V[R] < \frac{2(p_5 + p_7)(p_4 + p_6)}{p_X^3 (N + 1)} + \frac{2(p_1 + p_3)(p_0 + p_2)}{(1 - p_X)^3 (N + 1)} \]  

(23)

### 2.2 Marginalisation

To compute the expected value \( E[M] \) we define

\[ T_N = \{ap_6s + ap_1st + bp_2u + bp_3uv + ap_4w + ap_5wx + bp_6y + bp_7yz\}^N \]  

(24)

where we include extra generating variables for all terms in our estimators. Then

\[ E[M_{11}] = \frac{1}{N} \left[ \int d_y \frac{bz}{y} \frac{\partial^2}{\partial b \partial z} \right] T_N \bigg|_{\begin{array}{c} a=b=1 \\ s=t=1 \\ u=v=1 \\ w=x=1 \\ y=z=1 \end{array}} = \left[ \frac{bz p_7 (p_2 u + p_3 u v + p_6 y + p_7 y z)}{(p_6 + p_7 z)} \right] T_{N-1} \bigg|_{\begin{array}{c} a=b=1 \\ s=t=1 \\ u=v=1 \\ w=x=1 \\ y=z=1 \end{array}} \]

(25)

and similarly for the other terms, leading to

\[ E[M] = \frac{p_7}{(p_6 + p_7)} p_Z + \frac{p_5}{(p_4 + p_5)} (1 - p_Z) - \frac{p_3}{(p_2 + p_3)} p_Z - \frac{p_1}{(p_0 + p_1)} (1 - p_Z) \]  

(26)

**A Variance.** For computing the variance we need to reapply the previous operators. If they act on different generating variables, they will simply recreate terms like the mean, so we focus on terms where they repeat. For example:

\[ E[M_{11}^2] \cdot N = \int d_y \frac{bz}{y} \frac{\partial^2}{\partial b \partial z} \left[ \frac{bz p_7 u (p_2 + p_3 v)}{(p_6 + p_7 z)} + bz p_7 y \right] T_{N-1} \bigg|_{\begin{array}{c} a=b=1 \\ s=t=1 \\ u=v=1 \\ w=x=1 \\ y=z=1 \end{array}} \]  

(27)
For the linear term in $y$, it is easiest if we rearrange and integrate first

$$b z \frac{\partial^2}{\partial z \partial b} \int dy bp7 z \ T_{N-1} \bigg|_{a=b=1, s=t=1, u=v=1, w=x=1} = \frac{p6p7}{(p6 + p7)^2} pZ + \frac{p7^2}{(p6 + p7)^2} + (N - 1) \frac{p7^2}{(p6 + p7)^2} pZ$$  \hspace{1cm} (28)$$

while for the rest of $E[M_{11}^2]$ we first differentiate wrt $b$

$$b \frac{\partial}{\partial b} \left[ b z p7 u(p2 + p3 v) (p6 + p7 z) \right] T_{N-1} \bigg|_{a=b=1, s=t=1, u=v=1, w=x=1, y=z=1} = (N - 1) \left[ \frac{z p7(p2 + p3)}{(p6 + p7 z)} \right] T_{N-2} + \left[ \frac{z p7(p2 + p3)}{(p6 + p7 z)} \right] T_{N-1}$$ \hspace{1cm} (29)$$

For the part with the factor of $y$, we again integrate first wrt $y$ and then differentiate to obtain

$$z \frac{\partial}{\partial z} \int dy (N - 1) z p7(p2 + p3) T_{N-2} \bigg|_{a=b=1, s=t=1, u=v=1, w=x=1, y=z=1} = (p2 + p3) \left[ \frac{p6p7}{(p6 + p7)^2} + (N - 1) \frac{p7^2}{(p6 + p7)^2} \right]$$ \hspace{1cm} (30)$$

on the rest we apply the operator for $z$

$$z \frac{\partial}{\partial z} \cdots \bigg|_{z=1} = \left[ \frac{p6p7(p2 + p3)}{(p6 + p7)^2} \right] T_{N-1} + (N - 1) \left[ \frac{p7^2(p2 + p3)}{(p6 + p7)^2} + \frac{p6p7(p2 + p3)}{(p6 + p7)^2} \right] T_{N-2} + (N - 1)(N - 2)p7^2(p2 + p3) \frac{p7^2}{(p6 + p7)} T_{N-3}$$ \hspace{1cm} (31)$$

The linear terms in $y$ give the following

$$\frac{p7^2}{(p6 + p7)^2} (p2 + p3) + (N - 1) \frac{p7^2}{(p6 + p7)^2} (p2 + p3)^2$$ \hspace{1cm} (32)$$

while the integrals lead to

$$\frac{p6p7(p2 + p3)}{(p6 + p7)} (N - 1)(1 - p6 - p7)^{N-2} F \left[ 1, 1, 2 - N, [2, 2], - \frac{p6 + p7}{1 - p6 - p7} \right]$$

$$+ \frac{p6p7(p2 + p3)^2}{(p6 + p7)^2} (N - 1)(N - 2)(1 - p6 - p7)^{N-3} F \left[ 1, 1, 3 - N, [2, 2], - \frac{p6 + p7}{1 - p6 - p7} \right]$$ \hspace{1cm} (33)$$

Combining all the terms, subtracting the mean part squared and simplifying slightly we obtain

$$V[M_{11}] \cdot N = \frac{p6p7(p2 + p3)}{(p6 + p7)} (N - 1)(1 - p6 - p7)^{N-2} F \left[ 1, 1, 2 - N, [2, 2], - \frac{p6 + p7}{1 - p6 - p7} \right]$$

$$+ \frac{p6p7(p2 + p3)^2}{(p6 + p7)^2} (N - 1)(N - 2)(1 - p6 - p7)^{N-3} F \left[ 1, 1, 3 - N, [2, 2], - \frac{p6 + p7}{1 - p6 - p7} \right]$$

$$+ \frac{p6p7}{(p6 + p7)^2} (p2 + p3 + pZ) + \frac{p7^2}{(p6 + p7)^2} pZ(1 - pZ)$$ \hspace{1cm} (34)$$
The Covariances. For the covariances where separate generating variables are used

\[ E[M_{11}M_{10}] = \frac{1}{N} \int dw \frac{ax}{w} \frac{\partial^2}{\partial a \partial w} \left[ \frac{bzp_7(p_2u + p_3uv + p_6y + p_7yz)}{(p_6 + p_7z)} \right] T_{N-1} \left| \begin{array}{c} a=b=1 \\ s=t=1 \\ u=v=1 \\ w=x=1 \\ y=z=1 \\ \end{array} \right. \]

it is easy to see that the operators act on \( T_{N-1} \) rather than the prefactor, so we repeat the calculation for the mean with \( N \) replaced by \( (N - 1) \) to obtain

\[ C[M_{11}, M_{10}] = -\frac{1}{N} E[M_{11}] E[M_{10}], \quad C[M_{01}, M_{10}] = -\frac{1}{N} E[M_{01}] E[M_{10}] \]

\[ C[M_{11}, M_{00}] = -\frac{1}{N} E[M_{11}] E[M_{00}], \quad C[M_{01}, M_{00}] = -\frac{1}{N} E[M_{01}] E[M_{00}] \] (36)

The more complicated cases are where the generating variables reoccur

\[ E[M_{11}M_{01}] = \frac{1}{N} \int du \frac{bv}{u} \frac{\partial^2}{\partial b \partial v} \left[ \frac{byp_7u(p_2 + p_3v)}{(p_6 + p_7z)} + bvp_7y \right] T_{N-1} \left| \begin{array}{c} a=b=1 \\ s=t=1 \\ u=v=1 \\ w=x=1 \\ y=z=1 \\ \end{array} \right. \]

For the term linear in \( u \) we first integrate then differentiate wrt \( v \) while for the other term we first differentiate then integrate to give

\[ E[M_{11}M_{01}] = \frac{1}{N} \frac{\partial}{\partial b} \left[ \frac{bp_3p_7}{(p_6 + p_7)} + \frac{bp_3p_7}{(p_2 + p_3)} \right] T_{N-1} \left| \begin{array}{c} a=b=1 \\ s=t=1 \\ u=v=1 \\ w=x=1 \\ y=z=1 \\ \end{array} \right. \]

\[ = E[M_{11}] E[M_{01}] + \frac{p_3p_7}{N(p_2 + p_3)(p_6 + p_7)} p_2 (1 - p_2) \] (38)
and

\[ C[M_{11}, M_{01}] = \frac{1}{N} \frac{p_3 p_7}{(p_2 + p_3)(p_6 + p_7)} p_Z (1 - p_Z) \]

\[ C[M_{01}, M_{00}] = \frac{1}{N} \frac{p_1 p_5}{(p_0 + p_1)(p_4 + p_5)} p_Z (1 - p_Z) \]

(39)

**The Variance.** Since the terms from the covariances simplify, the complete variance is

\[
V \cdot N = \frac{p_0 p_7 (p_2 + p_3)}{(p_6 + p_7)} (N - 1) (1 - p_6 - p_7)^{N-2} F \left( [1, 1, 2 - N], [2, 2], - \frac{p_6 + p_7}{1 - p_6 - p_7} \right) \\
+ \frac{p_0 p_7 (p_2 + p_3)^2}{(p_6 + p_7)} (N - 1) (N - 2) (1 - p_6 - p_7)^{N-3} F \left( [1, 1, 3 - N], [2, 2], - \frac{p_6 + p_7}{1 - p_6 - p_7} \right) \\
+ \frac{p_4 p_5 (p_0 + p_1)}{(p_4 + p_5)} (N - 1) (N - 2) (1 - p_4 - p_5)^{N-2} F \left( [1, 1, 2 - N], [2, 2], - \frac{p_4 + p_5}{1 - p_4 - p_5} \right) \\
+ \frac{p_4 p_5 (p_0 + p_1)^2}{(p_4 + p_5)} (N - 1) (N - 2) (1 - p_4 - p_5)^{N-3} F \left( [1, 1, 3 - N], [2, 2], - \frac{p_4 + p_5}{1 - p_4 - p_5} \right) \\
+ \frac{p_2 p_3 (p_6 + p_7)}{(p_2 + p_3)} (N - 1) (1 - p_2 - p_3)^{N-2} F \left( [1, 1, 2 - N], [2, 2], - \frac{p_2 + p_3}{1 - p_2 - p_3} \right) \\
+ \frac{p_2 p_3 (p_6 + p_7)^2}{(p_2 + p_3)} (N - 1) (N - 2) (1 - p_2 - p_3)^{N-3} F \left( [1, 1, 3 - N], [2, 2], - \frac{p_2 + p_3}{1 - p_2 - p_3} \right) \\
+ \frac{p_0 p_1 (p_4 + p_5)}{(p_0 + p_1)} (N - 1) (1 - p_0 - p_1)^{N-2} F \left( [1, 1, 2 - N], [2, 2], - \frac{p_0 + p_1}{1 - p_0 - p_1} \right) \\
+ \frac{p_0 p_1 (p_4 + p_5)^2}{(p_0 + p_1)} (N - 1) (N - 2) (1 - p_0 - p_1)^{N-3} F \left( [1, 1, 3 - N], [2, 2], - \frac{p_0 + p_1}{1 - p_0 - p_1} \right) \\
+ \frac{p_6 p_7}{(p_6 + p_7)^2} (p_2 + p_3 + p_Z) + \frac{p_4 p_5}{(p_4 + p_5)^2} (p_0 + p_1 + 1 - p_Z) \\
+ \frac{p_2 p_3}{(p_2 + p_3)^2} (p_6 + p_7 + p_Z) + \frac{p_0 p_1}{(p_0 + p_1)^2} (p_4 + p_5 + 1 - p_Z) \\
+ \left[ \frac{p_7}{(p_6 + p_7)} - \frac{p_5}{(p_4 + p_5)} - \frac{p_3}{(p_2 + p_3)} + \frac{p_1}{(p_0 + p_1)} \right]^2 p_Z (1 - p_Z) \]

(40)

We note that the hypergeometric functions can be written solely in terms of \( p_X \) and \( p_Z \) so that the variance is actually quadratic in \( p_{Y;i} \).

### 2.3 Relative Difference in Variances

To explore the difference in variances, we reparameterise the probabilities

\[ p_{Y;0} = q_0 - C , \quad p_{Y;1} = q_0 + C , \quad p_{Y;2} = q_1 - C , \quad p_{Y;3} = q_1 + C \]

(41)

so that for \( C = 0 \) the causal effect of \( X \) on \( Y \) is \( q_1 - q_0 \) and \( C \) is a measure of the effect of \( Z \) on \( Y \) (the same for each \( X \)).

We plot the difference in variances from the two estimators, \( \Delta = \frac{V[M] - V[R]}{V[R]} \). In Figure 2 we leave \( p_X \) free, set \( p_Z = \frac{2}{3} \) and set \( q_0 = \frac{1}{3}, q_1 = \frac{2}{3} \) and plot \( \Delta \) for \( N = 100 \) and \( N = 400 \). In the plot for \( N = 400 \) we also scaled \( C \) by 2. The behaviour and rescaled plots are very similar, suggesting a \( N^{-\frac{1}{2}} \) scaling.
3. Asymptotic Behaviour

To examine the asymptotic behaviour of the causal effect estimators in more detail, we expand the hypergeometric function

$$N^2 z^2 (1 - z)^{N-1} F \left( [1, 1, 1 - N], [2, 2], -\frac{z}{1 - z} \right) = 1 + \frac{(1 - z)}{Nz} + \ldots$$  \hspace{1cm} (42)

to obtain the following for the variance of $R$

$$V[R] \cdot N = \frac{(q_1 + (2p_z - 1)C)(1 - q_1 - (2p_z - 1)C)}{p_x} \left( 1 + \frac{(1 - p_x)}{Np_x} \right) + \frac{(q_0 + (2p_z - 1)C)(1 - q_0 - (2p_z - 1)C)}{(1 - p_x)} \left( 1 + \frac{p_x}{N(1 - p_x)} \right) + O(N^{-2})$$  \hspace{1cm} (43)

while

$$V[M] \cdot N = \frac{q_1(1 - q_1) - C^2}{p_x} \left( 1 + \frac{2(1 - p_x)}{Np_x} \right) - \frac{(2q_1 - 1)(2p_z - 1)C}{p_x} \left( 1 + \frac{2p_x}{N(1 - p_x)} \right) - \frac{(2q_0 - 1)(2p_z - 1)C}{(1 - p_x)} + O(N^{-2})$$  \hspace{1cm} (45)

lead to

$$(V[M] - V[R]) \cdot N = \frac{q_1(1 - q_1)(1 - p_x)}{Np_x^2} + \frac{q_0(1 - q_0)p_x}{N(1 - p_x)^2} - 4p_x(1 - p_z)C^2 + O(N^{-\frac{3}{2}})$$ \hspace{1cm} (46)

with root

$$C^* = \sqrt{\frac{1}{4Np_z(1 - p_z)}} \left[ \frac{q_1(1 - q_1)}{p_x}(1 - p_x)^2 + \frac{q_0(1 - q_0)}{(1 - p_x)p_x^2} \right]^{\frac{1}{2}}$$  \hspace{1cm} (47)

so that

$$\lim_{N \to \infty} V[M] - V[R] < 0, \quad C > C^*$$
$$\lim_{N \to \infty} V[M] - V[R] > 0, \quad C < C^*$$  \hspace{1cm} (48)

Note that although we used the scaling $C \sim N^{-\frac{1}{2}}$ to extract this result, it holds more generally. For example for fixed $C \neq 0$, it is trivial to see that $C > C^*$ for some $N$ and so that $V[M]$ will
become lower than $V[R]$. The asymptotically optimal adjustment set therefore uses marginalisation rather than raw conditioning, in line with previous results (Henckel et al., 2019; Rotnitzky and Smucler, 2019) from the leading order asymptotics.

For fixed $C = 0$ however, raw conditioning would be better. It is exactly by treating subleading terms, as we here do, that we can examine where the transition occurs and how it depends on the coefficients. For weaker effects of the edge from $Z \to Y$, with $C \lesssim N^{-\frac{1}{2}}$, the raw conditional can give a more precise estimate of the causal effect of $X$ on $Y$.

3.1 Detectable Edge Strengths

With a larger sample size, we may be able to detect and quantify smaller causal effects. Therefore we wish to get a feeling for the strength of the edge $Z \to Y$ we would detect from the data, or equivalently for which values of $C$ we would infer the presence of the edge. To do so, we calculate the expected difference in maximised log-likelihoods when including the edge compared to a DAG with the edge deleted:

$$E[\Delta l] = \frac{1}{2} + N \frac{7}{6} \ln \left( \frac{q_1 + C}{q_1} \right) + N \frac{6}{7} \ln \left( \frac{1 - q_1 - C}{1 - q_1} \right) + \ldots$$

$$= \frac{1}{2} + N p_x p_Z (q_1 + C) \ln \left( 1 + \frac{C}{q_1} \right) + N p_x p_Z (1 - q_1 - C) \ln \left( 1 - \frac{C}{1 - q_1} \right) + \ldots$$

$$= \frac{1}{2} + \frac{N}{2} \left[ \frac{p_x}{q_1 (1 - q_1)} + \frac{(1 - p_x)}{q_0 (1 - q_0)} \right] C^2 + O(C^3)$$

where the $\frac{1}{2}$ comes from Wilk’s theorem (Wilks, 1938) for the additional parameter when maximising all the probabilities relative to evaluating with the restriction $C = 0$.

The change is AIC is then

$$E[\Delta AIC] = 2 - 2E[\Delta l] = 1 - N \left[ \frac{p_x}{q_1 (1 - q_1)} + \frac{1 - p_x}{q_0 (1 - q_0)} \right] C^2 + O(C^3)$$

There is therefore an asymptotic regime where the edge is strong enough to detect on average using the AIC but the estimator from raw conditionals that does not use the edge has lower variance

$$N(C^*)^2 \geq NC^2 \geq \left[ \frac{p_x}{q_1 (1 - q_1)} + \frac{(1 - p_x)}{q_0 (1 - q_0)} \right]^{-1}$$

which follows from the Cauchy-Schwarz inequality. The regime only vanishes when $p_Z = \frac{1}{2}$ and $q_1 (1 - q_1) (1 - p_x)^2 = q_0 (1 - q_0) p_x^2$ and the two bounds become equal. Utilising the BIC instead ($E[\Delta BIC] = E[\Delta AIC] + \log(N)$) leads to a large regime where we would not detect the edge on average, but where the estimator using marginalisation that does rely on the edge has lower variance.

4. Discussion

To evaluate the precision of different estimators targeting the same causal effect in causal diagrams, we considered the simple case of a v-structure for binary data and explicitly computed the variance of the two different estimators for the effect of $X$ on the collider $Y$, with $Z$ as the other parent.

The results involve combinations of hypergeometric functions, suggesting that exact results for larger DAGs may be rather complex. Which estimator has the lower variance depends, among other
parameters, on the relative strength of the edge from $Z$ to $Y$. In general estimating the causal effect through marginalisation offers better performance in the presence of a stronger direct effect of $Z$ on $Y$. When the direct effect is weaker instead, ignoring the edge and estimating the causal effect through the raw conditionals provides higher precision.

By examining the asymptotic regime of large sample sizes, we could confirm the intuition that for edge strengths statistically detectable by the AIC, accounting for the edge in the estimation should generally lead to lower variance. Conversely, that the presence of statistically non-detectable edges should be ignored to achieve a lower variance.

Most importantly, we could also discover an asymptotic regime where raw conditional estimates, ignoring the edge, were more precise in the presence of statistically detectable edges. One way to appreciate the practical relevance of these findings is by observing that we can expect ranges of causal strengths which become statistically detectable from data before we can gain precision by accounting for them in the estimation. Our detailed asymptotic analysis for the v-structure goes beyond the leading-order asymptotic result where the optimal estimator does not depend on the edge coefficients [Henckel et al., 2019; Rotnitzky and Smucler, 2019].

Outside the asymptotic regime, for finite sample sizes the gain in precision when using marginalisation and thus explicitly accounting for the edge presence, appears to be linked to its strength. Although the example considered here is the simplest non-trivial DAG, this finding further supports the idea that learning the full structure of the graph, beyond simply identifying a valid adjustment set, may benefit the precision of causal inference. The practical limitation with observational data is that we can only learn structures up to an equivalence class, so that we need to consider the possible range of causal effects across the whole class [Maathuis et al., 2009], or implement Bayesian model averaging across DAGs [Moffa et al., 2017].

If we use a more stringent criterion to decide about the presence of edges, such as the BIC for example, which implements a stronger penalisation with respect to the AIC, we may end up missing edges too weak to detect on average, but whose presence would improve the precision of the causal estimation through marginalisation. In other words, for moderately weak direct effects, the selection of suitable adjustment sets may be relatively sensitive to the choice of the score. Analogously, we may expect that optimal causal estimation may also be sensitive to the choice of learning algorithm, whether constraint-based [Spirtes et al., 2000; Kalisch and Bühlmann, 2007], score-based search [Chickering, 2002], or Bayesian sampling [Friedman and Koller, 2003; Kuipers and Moffa, 2017; Kuipers et al., 2018]. Quantifying the extent by which the structure learning affects causal estimation constitutes an interesting line of further investigation.

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