Exact Monodromy Group of N=2 Heterotic Superstring

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Abstract

We describe an \( N = 2 \) heterotic superstring model of rank-3 which is dual to the type-II string compactified on a Calabi-Yau manifold with Betti numbers \( b_{1,1} = 2 \) and \( b_{1,2} = 86 \). We show that the exact duality symmetry found from the type II realization contains the perturbative duality group of the heterotic model, as well as the exact quantum monodromies of the rigid \( SU(2) \) super-Yang-Mills theory. Moreover, it contains a non-perturbative monodromy which is stringy in origin and corresponds roughly to an exchange of the string coupling with the compactification radius.

September 1995
1. Introduction

During the last year, there has been considerable progress in understanding non-perturbative superstring theory based on the conjectured string-string duality [1]. $N = 2$ supersymmetric compactifications provide the first examples of non-trivial dynamics which are simple enough to be treated as laboratories for studying non-perturbative effects. The main observation is that the dilaton which plays the role of the string coupling constant belongs to a vector multiplet in the heterotic string and to a hypermultiplet in the type II string. Since abelian vector multiplets and neutral hypermultiplets do not interact in the low energy theory, tree-level type II theory provides the exact answer for the description of the moduli space of vector multiplets while tree-level heterotic theory is exact for hypermultiplets.

An interesting class of dual pairs consists in heterotic compactifications on $K_3 \times T^2$, on the one side, with type II compactifications on Calabi-Yau threefolds, on the other side [2, 3]. At generic points of their moduli space, the gauge group is abelian $U(1)^r$ and there are no charged massless hypermultiplets. In terms of the two Betti numbers of the Calabi-Yau manifold, $b_{1,1}$ and $b_{1,2}$, the rank is $r = b_{1,1} + 1$ and the number of hypermultiplets $b_{1,2} + 1$ where the +1 count the graviphoton and the type II dilaton, respectively. Moreover one considers only models with $b_{1,1} \geq 1$ to count for the heterotic dilaton. Among several concrete examples of dual pairs which have been proposed in ref. [2], there have been more quantitative tests only for the vector multiplet sector of two models of rank 3 and 4 which correspond in the heterotic side to models with 1 and 2 moduli, respectively, besides the dilaton [2, 4, 5, 6]. Their exact quantum moduli space is described on the type II side by the geometry of the Calabi-Yau hypersurfaces of degree 12 and 24, $X_{12}$ and $X_{24}$, correspondingly, which were studied extensively in refs. [7, 8] using mirror symmetry.

The main test consists in computing the perturbative correction to the heterotic prepotential [9, 10, 11] which, due to $N = 2$ non-renormalization theorems, occurs only at the
one loop level, and compare it with the exact expression obtained from type II \[2, 3, 4, 5\]. This procedure requires the correct identification of the heterotic dilaton as one of the \(b_{1,1}\) moduli of type II which allows to take the weak coupling limit corresponding to the large complex structure in the mirror Calabi-Yau. Besides this comparison of the low energy theory, further non-trivial tests have been performed for the rank 3 model by analyzing the structure of higher dimensional interactions \[6\].

Given these perturbative tests which strongly support the proposed equivalence between heterotic/type II dual pairs, it would be very interesting and instructive to study, using duality, the structure of non-perturbative effects in heterotic \(N = 2\) compactifications. A technical difficulty is that the exact results based on type II become simple only in certain variables parameterizing the complex structure of the mirror manifold, while the expressions for the periods and the mirror map to the special coordinates appropriate for the description of the heterotic vacua, are known only by their instanton expansions. A possible way to bypass this difficulty is to study the exact quantum symmetry and the effects of non-perturbative corrections to the perturbative \(T\)-duality transformations. In this work we address these issues in the context of a different rank-3 dual pair example based, on the type II side, on the Calabi-Yau hypersurface of degree 8, \(X_8\), for which the exact monodromy group was worked out in detail in ref. \[7\].

In section 2, we review the type II model \(X_8\) and its large complex structure limit which motivates its dual interpretation by identifying the heterotic dilaton and the classical \(T\)-duality group. The later is actually \(\Gamma_0(2)_{+}\), as was claimed in ref. \[5\], which is generated by the transformations \(T \rightarrow T + 1\) and \(T \rightarrow -1/2T\) and shares very similar properties with \(SL(2, \mathbb{Z})\).

In Section 3, we describe the heterotic dual realization and compute the one loop correction to the prepotential. At the particular point of the \(T\)-modulus space \(T = i/\sqrt{2}\) which is an order 2 fixed point, the \(U(1)\) of \(T\) is extended to \(SU(2)\) because of the ap-
pearance of two extra charged massless vector multiplets, and the prepotential develops a logarithmic singularity. The main difference with the other rank-3 dual pair based on $X_{12}$, besides the $T$-duality group, is that the $SU(2)$ enhanced gauge symmetry appears in this case with Kac-Moody level 2. Comparing the heterotic perturbative prepotential with the exact answer from the type II side in the weak coupling limit, we find a perfect agreement of the two expressions.

In Section 4, we derive the quantum monodromies of the one loop prepotential associated to its logarithmic singularity, following the method of ref. [10]. As a result the classical duality group gets modified to a subgroup of $Sp(6, \mathbb{Z})$ depending on 8 integer parameters. One of its two generators corresponding to the transformation $T \to -1/2T$ is identified with the Weyl reflection of the $SU(2)$ enhanced symmetry with its quantum monodromy determined in the rigid supersymmetric theory [12].

In Section 5, we present the exact symmetry group determined from type II using the results of ref. [7], and we identify the generators of the perturbative $T$-duality group and the quantized axionic shift which form a subgroup. Moreover among the remaining elements, we identify the non-perturbative quantum monodromy of the rigid $SU(2)$ theory associated to the appearance of massless monopoles or dyons [12]. This is a non-perturbative test of string-string duality which, in particular, exchanges world-sheet instantons of type II string with spacetime instantons of heterotic string. It also confirms the expectation that the enhanced $SU(2)$ symmetric point which is present in perturbation theory of $N = 2$ heterotic compactifications, disappears in the exact quantum theory, and the corresponding $SU(2)$ singularity splits into two branches where non-perturbative solitonic states become massless. It is remarkable that under string duality, this non-perturbative splitting is mapped on the type II side into the conifold locus of Calabi-Yau manifolds, while the dyonic hypermultiplets should be mapped into charged black holes which become massless at the conifold locus [13]. The exact symmetry group contains an additional non-perturbative
monodromy which is stringy in origin and is related to the dilaton. It corresponds to an $S \leftrightarrow T$ duality \footnote{After the completion of this work, we received the paper of ref. \cite{ref14} where similar results were obtained for the rank-3 dual pair based on $X_{12}$.} and implies the existence of new massless solitons with charges under the extra $U(1)$’s. All these properties seem to be much more generic than for this particular example.\footnote{After the completion of this work, we received the paper of ref. \cite{ref14} where similar results were obtained for the rank-3 dual pair based on $X_{12}$.}

2. The type II model

Let us consider the rank-3 type II superstring compactification on the manifold $X_8$ with Betti numbers $b_{1,1} = 2$ and $b_{1,2} = 86$, giving rise to 2 vector multiplets (besides the graviphoton) and 86 + 1 hypermultiplets including the dilaton. This manifold is defined as a hypersurface of degree 8 in the weighted projective space $WCP^4[1,1,2,2,2]$. The moduli space of the two vector multiplets can be studied using mirror symmetry \footnote{After the completion of this work, we received the paper of ref. \cite{ref14} where similar results were obtained for the rank-3 dual pair based on $X_{12}$.}. The mirror $X_8^*$ may be identified with the family of Calabi-Yau threefolds of the form $\{P = 0\}/\mathbb{Z}_4^3$, where

$$P = z_1^8 + z_2^8 + z_3^4 + z_4^4 - 8\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^4 z_2^4,$$  \hspace{1cm} (2.1)

depending on two complex structure deformation parameters $\psi$ and $\phi$. The three discrete group factors $\mathbb{Z}_4$ act on the coordinates as $(z_2, z_{2+m}) \rightarrow (-iz_2, iz_{2+m})$ for $m = 1, 2, 3$, respectively.

For a good description of the moduli space, it is convenient to enlarge $\mathbb{Z}_4^3$ to a group $\hat{G}$ acting also on $\psi$ and $\phi$,

$$(z_1, z_2, z_3, z_4, z_5; \psi, \phi) \rightarrow (\alpha^{a_1} z_1, \alpha^{a_2} z_2, \alpha^{2a_3} z_3, \alpha^{2a_4} z_4, \alpha^{2a_5} z_5; \alpha^{-a} \psi, \alpha^{-4a} \phi),$$  \hspace{1cm} (2.2)

where $\alpha = e^{2i\pi/8}$, $a_i$ are integers and $a = a_1 + a_2 + 2a_3 + 2a_4 + 2a_5$. $X_8^*$ is then obtained by considering the quotient $\{P = 0\}/\hat{G}$ and mod out the parameter space by a $\mathbb{Z}_8$ with
generator

\[(\psi, \phi) \rightarrow (\alpha \psi, -\phi).\]  

(2.3)

To describe the large complex structure limit, it is also convenient to introduce the variables

\[x = -\frac{2\phi}{(8\psi)^4} \quad \text{and} \quad y = \frac{1}{(2\phi)^2}.\]  

(2.4)

The Yukawa couplings are worked out in an explicit form in refs. [4, 8]. As follows from \(N = 2\) special geometry, there is a preferred coordinate system in which they are given as triple derivatives of an analytic prepotential [15, 16]. The corresponding special coordinates \(t_1\) and \(t_2\) are given by the inverted mirror map,

\[x(t_1, t_2) = \frac{1}{h(t_1)}(1 + \mathcal{O}(t_1(q_2)))\]  

(2.5)

\[y(t_1, t_2) = q_2 l(t_1)(1 + \mathcal{O}(t_1(q_2)))\]

where \(q_1 = e^{2\pi t_1}\), \(q_2 = e^{2\pi t_2}\) and \(\mathcal{O}(t_1(q_2))\) denotes terms which are at least of order one in \(q_2\) and order zero in \(q_1\). The function \(l(t_1) = 1 + \mathcal{O}(q_1)\), while \(h(t_1) = 1/q_1 + 104 + \mathcal{O}(q_1)\) will be given below. In these coordinates, the Yukawa couplings \(\mathcal{Y}_{ijk}\) are derived by a prepotential \(\mathcal{Y}\),

\[\mathcal{Y}_{ijk} \equiv -\partial_i \partial_j \partial_k \mathcal{Y},\]  

(2.6)

and they are given as power series in \(q_1\) and \(q_2\):

\[\mathcal{Y}_{111} = 8 + \sum_{n \geq 1} y_n q_1^n + \mathcal{O}(t_1(q_2))\]

\[\mathcal{Y}_{112} = 4 + \mathcal{O}(t_1(q_2))\]  

(2.7)

\[\mathcal{Y}_{122} = \mathcal{O}(t_1(q_2))\]

\[\mathcal{Y}_{222} = \mathcal{O}(t_1(q_2))\]

where \(y_n\) are known integer coefficients related to the world-sheet instanton numbers [4].
From eq. (2.7) one can extract the prepotential

\[ Y = -2t_1 t_2 - \frac{4}{3} t_1^3 + \frac{\alpha}{2} t_1^2 + (\delta - \frac{2}{3}) t_1 + \xi + g(t_1) \] (2.8)

\[ + \beta t_1 t_2 + \frac{\gamma}{2} t_2^2 + \epsilon t_2 + g^{NP}(t_1, t_2) \]

where \( \alpha, \beta, \gamma, \delta, \epsilon \) and \( \xi \) are arbitrary integration constants, and

\[ g(t_1) = \frac{1}{8t \pi^3} \sum_{n \geq 1} \frac{y_n}{n^3} q_1^n \] (2.9)

\[ g^{NP}(t_1, t_2) = \sum_{n \geq 1, m \geq 0} g^{NP}_{nm} q_2^n q_1^m \]

The above form of the prepotential suggests that this type II model has a heterotic dual realization by identifying \( t_2 \) with the heterotic dilaton \( S \) and \( t_1 \) with the single heterotic modulus \( T \). In fact, in this case, the general heterotic prepotential \( F \) takes the form

\[ F = -2ST^2 + f(T) + f^{NP}(T, S) , \] (2.10)

where the first two terms of the right hand side correspond to the classical and one-loop contributions, respectively. \( f^{NP} \) denotes the non-perturbative corrections

\[ f^{NP}(T, S) = \sum_{n \geq 1} f_a^{NP}(T) q_s^n \] (2.11)

where \( q_s = e^{2i\pi S} \) and \( \langle S \rangle = \frac{g_2}{\pi} + i \frac{8\pi^2}{g_s^2} \) in terms of the \( \theta \)-angle and the four-dimensional string coupling constant \( g_s \).

This identification is motivated by the fact that in the weak coupling limit \( q_s \to 0 \) (or equivalently \( y \to 0 \)), the discriminant locus of the mirror Calabi-Yau \( X_8^\star \)

\[ \Delta = \{(1 - 2^8 x^2 - 2^{18} x^2 y) \{1 - 4y\} = 0 \] (2.12)

becomes a perfect square, in complete analogy with the situation in the rigid \( SU(2) \) \( N = 2 \) supersymmetric Yang-Mills theory where the classically singular \( SU(2) \) point splits into two separate branches in the quantum theory \([12]\). The first factor in eq. (2.12) defines the
conifold singularity while the second factor gives rise to an isolated singularity $y = 1/4$ corresponding to the infinite strong coupling limit $S \to 0$ in the heterotic theory \cite{7, 2}.

The guide to construct the heterotic realization is to first find the classical $T$-duality group \cite{4,77}. This can be determined by the set of transformations which identify all solutions of the conifold singularity in the weak coupling limit $y = 0, x = 1/256$ in terms of the special coordinate $T$,

$$h(T) = 256 \quad \text{and} \quad S = i \infty ,$$

where $h$ is defined in eq. (2.3). In fact $h$ was found to be (up to an additive constant 104) the Haupmodul for $\Gamma_0(2)_+$ which should play the role of the heterotic $T$-duality group,

$$h(T) = \frac{[\theta_3^4(T) + \theta_4^4(T)]^4}{16[\eta(T)\eta(2T)]^8} ,$$

where $\theta_{3,4}$ are the Jacobi $\theta$-functions and $\eta$ the Dedekind function.

$\Gamma_0(2)_+$ is the group $\Gamma_0(2)$ of modular transformations

$$T \rightarrow \frac{aT + b}{cT + d} \equiv MT ; \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d$ integers, $c$ even and $\det M = 1$, together with an Atkin-Lenhner involution $W$ of determinant two \cite{8},

$$W : \quad T \rightarrow -\frac{1}{2T} .$$

The full group can be generated in terms of two elements which, for instance, can be chosen to be $W$ and the translation

$$U : \quad T \rightarrow T + 1 ,$$

or equivalently,

$$V : \quad T \rightarrow -\frac{2T + 1}{2T} .$$

$W$ and $V$ generate the little groups of the fixed points $i/\sqrt{2}$ and $(i - 1)/2$ which are of order 2 and 4, respectively. The fundamental domain is similar to the one of $SL(2, \mathbb{Z})$, 7
defined by
\[
\left\{ T_2 > 0, \frac{1}{2} \leq T_1 \leq 0, |T| \geq \frac{1}{\sqrt{2}} \right\} \cup \left\{ T_2 > 0, 0 < T_1 < 1/2, |T| > \frac{1}{\sqrt{2}} \right\}
\] (2.19)

with \( T = T_1 + iT_2 \). The \( SL(2, \mathbb{Z}) \) fixed points \( i \) and \( e^{2\pi i} \) of order 2 and 3 are now replaced by \( i/\sqrt{2} \) and \((i - 1)/2\) of order 2 and 4, while \( i\infty \) remains a cusp of width one. The function \( h \) of eq.\((2.14)\) is the analog of the \( j \)-invariant function of \( SL(2, \mathbb{Z}) \). The numerator and denominator in \((2.14)\) are respectively modular and cusp forms of weight 8 for \( \Gamma_0(2)_+ \), so that \( h \) is of weight 0 and it has a pole of order one at \( i\infty \) and a zero of order 4 at \((i - 1)/2\).

Moreover, \( h \) is a bijection between the fundamental domain \((2.19)\) and the Riemann sphere with three punctures such that \( h(i/\sqrt{2}) = 256 \), \( h((i - 1)/2) = 0 \) and \( h(i\infty) = \infty \). Finally the solutions of the equation \((2.13)\) for the conifold singularity in the perturbative limit consist in the images of \( i/\sqrt{2} \) under the action of \( \Gamma_0(2)_+ \), which shows also that this group is the modular one for \( T \).

3. The heterotic dual realization

The dual of the above type II model should be a rank-3 \( N = 2 \) heterotic vacuum with one vector multiplet modulus \( T \) besides the dilaton \( S \), and 87 hypermultiplets. The method of constructing such models is described in ref. \[2\]. The effective two derivative interactions of vector multiplets are described by a prepotential which has the general form of eq.\((2.10)\).

The Kähler potential of the moduli metric can then be written as

\[
K = -\ln(iY) \quad \text{with}
\]
\[
Y = 2F - 2\bar{F} - (T - \bar{T})(F_T + \bar{F}_T) - (S - \bar{S})(F_S + \bar{F}_S)
\]
\[
= 2(S - \bar{S})(T - \bar{T})^2 + 2(f + f^{NP} - \bar{f} - \bar{f}^{NP})
\]
\[
- (T - \bar{T})(f_T + f_T^{NP} + \bar{f}_T + \bar{f}_T^{NP}) - (S - \bar{S})(f_S^{NP} + \bar{f}_S^{NP})
\]

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where the subscripts denote differentiation with respect to the corresponding fields. Note that $F$ is defined up to the addition of any quadratic polynomial in $T$ and $S$ with real coefficients which leaves $Y$ invariant.

We now focus on the one loop contribution $f(T)$ which is actually the complete perturbative correction due to $N = 2$ non-renormalization theorems based on analyticity and the invariance under the axionic shift. The one loop correction to the Kähler metric is obtained by expanding eq. (3.1):

$$K_{TT} = K_{TT}^{(0)} \left[ 1 + \frac{2i}{S - S} \mathcal{I} + \cdots \right] ,$$

(3.2)

where the tree-level metric $K_{TT}^{(0)} = -2/(T - \bar{T})$ and $\mathcal{I}$ is given by

$$\mathcal{I}(T, \bar{T}) = \frac{i}{8} \left( \partial_T - \frac{2}{T - \bar{T}} \right) \left( \partial_T - \frac{4}{T - \bar{T}} \right) f(T) + c.c. \quad (3.3)$$

$\mathcal{I}$ can be computed by a one loop string calculation of an amplitude involving the antisymmetric tensor using the method of ref. [9]. From the expression (3.3) one can easily deduce the fifth derivative of the one loop prepotential, $f^{(5)}$, [11]:

$$f^{(5)}(T) = -\frac{8i}{(T - \bar{T})^2} \partial_T^5 \left( (T - \bar{T})^2 \mathcal{I} \right). \quad (3.4)$$

Since $T$ belongs to the coset $O(2,1)/O(2)$, the classical duality group is in general a subgroup of $GL^+(2, \mathbb{Z})$ which is the set of fractional linear transformations of the form (2.13) with $M$ being an integer matrix with positive determinant. As $\mathcal{I}$ is related to a physical amplitude, it is modular invariant. It is then straightforward to verify from eq. (3.4) that $f^{(5)}$ is a modular function of weight 6.

Integrating equation (3.4) one can determine $f$ up to a quartic polynomial,

$$f(T) = \int_{T_0}^{T} \frac{(T - T')^4}{4!} f^{(5)}(T')dT' , \quad (3.5)$$

where $T_0$ is an arbitrary point. The path of integration should not cross any singularity of $f^{(5)}$, while the result of the integral depends on the homology class of such paths. Different choices of homology classes of paths change $f$ by quartic polynomials. Moreover under a
modular transformation (2.13), \( f \) does not transform covariantly. Using its integral representation (3.3), we see that it has a weight \(-4\) up to an addition of a quartic polynomial, \( \mathcal{M}(T) \),

\[
f(T) \rightarrow f(MT) = (\det M)^2(cT + d)^{-4}[f(T) + \mathcal{M}(T)] .
\]

These transformations should leave the physical metric (3.3) invariant. Hence, one must have

\[
i \left( \partial_T - \frac{2}{T - T} \right) \left( \partial_T - \frac{4}{T - T} \right) \mathcal{M}(T) + c.c. = 0,
\]

which is satisfied only if \( \mathcal{M}(T) \) is a quartic polynomial with real coefficients. In fact, we will see below that this ambiguity is related to the non-trivial quantum monodromies [10].

Finally, in order to guarantee modular invariance of the full effective action, the dilaton should also transform. Imposing the general condition that modular transformations may be compensated by Kähler transformations in eq.(3.1), one finds:

\[
S \rightarrow S - \frac{c}{2} \frac{f_T + \mathcal{M}_T}{cT + d} + c^2 \frac{f + \mathcal{M}}{(cT + d)^2} + \frac{1}{12} \mathcal{M}_{TT} + \lambda_M ,
\]

up to an additive real constant \( \lambda_M \) which corresponds to the perturbative heterotic symmetry of the axion shift. It follows that in the presence of one loop corrections one can define a \( T \)-duality invariant dilaton \( S_{inv} \) [11],

\[
S_{inv} \equiv S - \frac{1}{12} f_{TT} ,
\]

which however is not a special coordinate of \( N = 2 \) Kähler geometry.

The above discussion applies to any rank-3 \( N = 2 \) heterotic string compactification. We now specialize to the dual candidate of the type II model described in Section 2, for which the classical duality group is \( \Gamma_0(2)_+ \). Furthermore the order 2 fixed point in the fundamental domain (2.19), \( T = i/\sqrt{2} \), which was found from the conifold singularity in the type II model, should correspond in the heterotic realization to the perturbative \( SU(2) \) enhanced symmetry point. Using the expressions for the left and right momenta, \( p_L \) and
\( p_R \), of the \( \Gamma^{(2, 1)} \) compactification lattice \[6\],

\[
\begin{align*}
  p_L &= \frac{i\sqrt{2}}{T - \bar{T}}(n_1 + n_2 T^2 + 2m\bar{T}) \\
  p_R &= \frac{i\sqrt{2}}{T - \bar{T}}(n_1 + n_2 T\bar{T} + m(T + \bar{T})) ,
\end{align*}
\]

(3.10)

one finds that the condition of additional massless states \( p_L = 0, p_R^2 \leq 2 \) at \( T = i/\sqrt{2} \) is satisfied for two solutions,

\[
SU(2) : \quad n_1 = \pm \frac{1}{2}, \quad n_2 = \pm 1, \quad m = 0 ,
\]

(3.11)

for which \( p_R = \pm 1 \). This implies that the charges for the vector multiplets sit in a lattice \( \Gamma_{1,0} \) defined by \( n_2 \) being odd integer, \( n_1 \in \mathbb{Z} + \frac{1}{2} \) and \( m \in \mathbb{Z} \), so that the spectrum is invariant under the modular group \( \Gamma_0(2)_+ \). Moreover, since \( \Gamma_{1,0} \) is odd, the enhanced \( SU(2) \) symmetry at \( T = i/\sqrt{2} \) has a Kac-Moody level 2 and a coupling constant \( g = g_s/\sqrt{2} \).

Note that there is no other point in the fundamental domain of \( T \) with additional charged massless states. However this lattice is not self-dual with respect to the inner product

\[
\frac{1}{2}(p_Lp'_L + \bar{p}_L\bar{p}'_L) - p_Rp'_R = (2mm' - n_1n'_2 - n_2n'_1).
\]

World-sheet modular invariance under \( \tau \rightarrow -1/\tau \) requires the existence of additional vector multiplets contained in the dual lattice which splits into 4 different sectors, with respect to their transformations under \( \tau \rightarrow \tau + 1 \), labeled by \( \Gamma_{1,\epsilon} \) and \( \Gamma_{2,\epsilon} \) for \( \epsilon = 0, \frac{1}{2} \). Applying once again the transformation \( \tau \rightarrow 1/\tau \), for instance in \( \Gamma_{2,0} \), one finds 2 more sectors labeled by \( \Gamma_{3,\epsilon} \):

\[
\begin{align*}
  \Gamma_{1,\epsilon} & : \quad n_1 \in \mathbb{Z} + \frac{1}{2} , \quad n_2 \text{ odd} \\
  \Gamma_{2,\epsilon} & : \quad n_1 \in \mathbb{Z} , \quad n_2 \text{ even} \quad \text{and} \quad m \in \mathbb{Z} + \epsilon \quad (3.12) \\
  \Gamma_{3,\epsilon} & : \quad n_1 \in \mathbb{Z} + \frac{n_2 + 1}{2} , \quad n_2 \in \mathbb{Z} \\
\end{align*}
\]

These 6 sectors couple to different blocks of the remaining conformal field theory in a way consistent with world-sheet modular invariance. At \( T = (i - 1)/2 \), one could get additional charged massless states in the sectors \( \Gamma_{1,\frac{1}{2}} \) and \( \Gamma_{2,0} \) corresponding to \( n_1 = \frac{n_2}{2} = m = \pm 1 \) and \( \pm \frac{1}{2} \), respectively. However, the corresponding coefficients associated to the remaining
conformal field theory should vanish since one knows that there are no extra charged massless states at this point. A similar argument applies for \( T = i e^{\frac{2\pi}{3}} \) in the sector \( \Gamma_{3,0} \) and \( T = e^{\frac{2\pi}{3}} \) in the sector \( \Gamma_{3,\frac{1}{2}} \) where one could have more charged massless states corresponding to \( n_1 = n_2 = \pm 1, m = 0 \) and \( n_1 = n_2 = 2m = \pm 1 \), respectively.

Using the results of ref. \[9\] it is straightforward to obtain an expression for \( I \) as an integral over the complex Teichmüller parameter \( \tau = \tau_1 + i\tau_2 \) of the world-sheet torus inside its fundamental domain,

\[
I = \sum_{\ell=1}^{6} \int \frac{d^2\tau}{\tau_2^{3/2}} \tilde{C}_\ell(\tau) \partial_\tau (\tau_2^{1/2} \sum_{\mathbb{F}_{L,R} \in \Gamma_\ell} e^{\pi i \tau |p_L|^2} e^{-\pi i \bar{\tau}_R}) ,
\]

where the sum extends over the six sectors denoted here by \( \Gamma_{\ell} \), and \( \tilde{C}_\ell \)'s are \( T \)-independent modular functions with well-defined transformation properties dictated by modular invariance of the integrand. Actually \( \tilde{C}_\ell \) is the trace of \((-1)^F q^{L_0-c/24} q^{-\tilde{L}_0-	ilde{c}/24}/\tilde{\eta} \) in the Ramond sectors of the corresponding remaining conformal blocks. We now use eq.(3.4) together with the form of the lattice momenta (3.10) to obtain:

\[
f^{(5)}(T) = \frac{512 \pi^2}{(T - \bar{T})^3} \sum_{\ell=1}^{6} \int \frac{d^2\tau}{\tau_2^{3/2}} \tilde{C}_\ell(\tau) \partial_\tau \left( \tau_2^2 \partial_\tau (\tau_2^{3/2} \sum_{\mathbb{F}_{L,R} \in \Gamma_\ell} p_{L,R}^2 e^{\pi i \tau |p_L|^2} e^{-\pi i \bar{\tau}_R}) \right) .
\]

One can easily show using the expression for the lattice momenta (3.10) that the integrand of \( \partial_T f^{(5)} \) reduces to a total derivative in \( \tau \) and the integral vanishes after integration by parts.

As \( \tau_2 \rightarrow \infty \), \( 2i\pi^2 \tilde{C}_\ell \) behaves as \( \bar{q}^{-1} \) and the one loop metric (3.13) has a logarithmic singularity at \( T = i/\sqrt{2} \) associated to the massless states (3.11),

\[
I \sim \frac{1}{\pi} \ln \left| T - \frac{i}{\sqrt{2}} \right|^2 \quad \text{for} \quad T \rightarrow i/\sqrt{2} .
\]

This singularity can also be found in the effective field theory by taking into account that \( \left| T - \frac{i}{\sqrt{2}} \right| \) is proportional to the mass of the two \( N = 2 \) vector multiplets near the \( SU(2) \) symmetric point, as it can be seen from eqs.(3.10) and (3.11). Note that the coefficient of the singularity is half of the corresponding coefficient in the other known rank-3 dual-pair
example \([3, 4, 8]\) because in our case \(SU(2)\) is realized with a Kac-Moody level 2. The singularity \((3.13)\) implies that the one loop prepotential behaves as

\[
f \sim \frac{4}{i\pi} \left( T - \frac{i}{\sqrt{2}} \right)^2 \ln \left( T - \frac{i}{\sqrt{2}} \right). \tag{3.16}\]

One can now use the duality symmetry \(\Gamma_0(2)_+\) to determine \(f\). As mentioned above, its fifth derivative \(f^{(5)}\) is a modular function of weight 6. Moreover it has a triple pole at \(T = i/\sqrt{2}\) with coefficient \(16/i\pi\) to reproduce the logarithmic singularity \((3.16)\), while it vanishes at \(T \to i\infty\), as seen from the integral representation \((3.14)\), implying that the metric does not grow at infinity. These requirements determine \(f^{(5)}\) up to a numerical constant which can also be fixed by the condition that the monodromy of \(f\) as \(T\) goes around \(i/\sqrt{2}\) is contained in the symplectic group \(Sp(6, \mathbb{Z})\) as dictated by \(N = 2\) supergravity. The result is:

\[
f^{(5)}(T) = \frac{64}{i\pi} \left( \frac{h_T}{h - h \left( \frac{i}{\sqrt{2}} \right)} \right)^3 \frac{5h + 3h \left( \frac{i}{\sqrt{2}} \right)}{h^2}, \tag{3.17}\]

where the modular invariant function \(h\) is given in eq.\((2.14)\).

A first non-trivial check of the proposed duality between this heterotic model and the type II compactification described in Section 2, is the comparison of the two corresponding prepotentials \(F\) and \(Y\) of eqs.\((2.10)\) and \((2.8)\). Indeed, the identification \(t_1 = T\) and \(t_2 = S\) implies that

\[
f^{(5)} = g^{(5)} = 4\pi^2 \sum_{n \geq 1} n^2 y_n q_i^n, \tag{3.18}\]

which we verified up to the fifth order in the \(q_i\) expansion using the numerical values for the coefficients \(y_n\)’s entering in the expression of the Yukawa couplings \((2.7)\) and given in ref. \([4]\).

4. Perturbative duality group

As we already mentioned, at the classical level the \(T\)-duality group of the heterotic model is \(\Gamma_0(2)_+\) generated by the transformations \(W\) and \(V\) defined in eqs.\((2.16)\) - \((2.18)\).
These generators obey the relations

\[ W^2 = V^4 = 1 \quad \quad UVW = 1 . \] (4.1)

Note the similarity of these relations with those of \( SL(2, \mathbb{Z}) \) obtained by replacing \( V \) with a generator of order 3.

The order 2 generator \( W \) corresponds to the Weyl reflection of the \( SU(2) \) gauge group at the enhanced symmetry point \( T = i/\sqrt{2} \) \(^{19} \). Away from this point, \( SU(2) \) is spontaneously broken by the vacuum expectation value, \( a \), of a Higgs field along the flat direction of the scalar potential, with the identification \( a \propto (T - i/\sqrt{2})/(T + i/\sqrt{2}) \) near the non-abelian point. Thus, the \( \mathbb{Z}_2 \) transformation \( T \rightarrow WT = -1/(2T) \) acts on \( a \) as the parity \( a \rightarrow -a \) which remains unbroken after the Higgs phenomenon.

At the quantum level, because of the singularity at \( T = i/\sqrt{2} \) the first relation of (4.1) is modified. \( W^2 \) instead of being the identity, is determined by the corresponding non-trivial monodromy \( M_{i/\sqrt{2}} \),

\[ W^2 = M_{i/\sqrt{2}} \quad V^4 = 1 \quad \quad UVW = 1 . \] (4.2)

In fact, when moving a point around the singularity the one loop prepotential \( f \) transforms as:

\[ M_{i/\sqrt{2}} : \quad f(T) \rightarrow f(T) + M_{i/\sqrt{2}}(T) \] (4.3)

with

\[ M_{i/\sqrt{2}}(T) = -4 \left( T - \frac{i}{\sqrt{2}} \right)^2 \left( T + \frac{i}{\sqrt{2}} \right)^2 \]

\[ = -4T^4 - 4T^2 - 1 . \]

The polynomial \( M_{i/\sqrt{2}} \) can be determined by expanding the expression (3.17) around \( T = i/\sqrt{2} \). It can also be easily obtained just from the knowledge of the leading behavior of \( f \) in eq.(3.16) together with the reality condition on the coefficients of \( M_{i/\sqrt{2}} \) which is required for modular invariance of the physical quantity \( I \) (see eq.(3.7)).
One can now use the modified group relations (4.2) to determine the transformation properties of \( f \) under the generators \( W \) and \( V \). This amounts to determine the corresponding quartic polynomials \( \mathcal{M} \) entering in eq.(3.6), which we will denote by \( \mathcal{M}_W \) and \( \mathcal{M}_V \) respectively. The first relation of eq.(4.2) gives:

\[
\mathcal{M}_W(T) + 4T^4 \mathcal{M}_W \left( \frac{-1}{2T} \right) = \mathcal{M}_{i/\sqrt{2}}(T) .
\]  

(4.4)

The general solution depends on two real parameters \( w_0 \) and \( w_1 \),

\[
\mathcal{M}_W(T) = -4(1 + w_0)T^4 + 2w_1T^3 - 2T^2 + w_1T + w_0 .
\]  

(4.5)

Imposing the second relation of eq.(4.2), one finds just one linear constraint among the five real coefficients \( v_n \) of the polynomial \( \mathcal{M}_V \). Finally, using the third relation of eq.(4.2), one can obtain the polynomial \( \mathcal{M}_U \) associated to the translation \( U \), as a function of the six independent parameters \( w_0, w_1 \) and \( v_n \),

\[
\mathcal{M}_U(T) = -\mathcal{M}_W(T+1) - 4(T+1)^4 \mathcal{M}_V \left( \frac{-1}{2T+2} \right) .
\]  

(4.6)

We now note that there is an additional constraint to \( \mathcal{M}_U \) coming from the behavior of \( f \) at infinity. Since \( f^{(5)} \) vanishes as \( T \to i\infty \), \( f \) behaves at most as \( T^4 \) which implies that the coefficient of \( T^4 \) in \( \mathcal{M}_U \) is zero. One then finds \( v_0 = 1 + w_0 \) and the polynomial \( \mathcal{M}_V \) becomes:

\[
\mathcal{M}_V(T) = \sum_{n=0}^{4} v_n T^n
\]  

(4.7)

with \( v_0 = 1 + w_0 \); \( v_4 = v_3 - \frac{4}{3}v_2 + 2v_1 - 4v_0 \).

We recall that the above transformations of \( f \) are accompanied by suitable \( T \)-dependent shifts of the dilaton, determined from eq.(3.8) up to additive \( T \)-independent constants.

The five independent parameters \( w_{0,1} \) and \( v_{1,2,3} \) entering in the polynomials \( \mathcal{M}_W \) and \( \mathcal{M}_V \) can be chosen arbitrarily using the freedom to add to \( f \) a quartic polynomial with real coefficients. This freedom is valid only if one neglects the non-perturbative corrections,
\( f^{NP} = 0 \) in eq.\((2.10)\), and it can be easily seen from eq.\((3.1)\). More precisely, adding to \( f \) a function \( P(T) \) one finds that \( Y \) remains invariant provided \( P \) satisfies eq.\((3.7)\) which implies that \( P \) is a quartic polynomial with real coefficients, and \( S \) is redefined as \( S \rightarrow S + P_{TT}/12 \).

In the presence of non-perturbative corrections, only the addition of a quadratic polynomial is allowed, which leaves \( S \) inert, corresponding to a change of basis.

Following the analysis of ref. \[10\], the perturbative \( T \)-duality group \( G \) contains a normal abelian subgroup \( H \) which is generated by elements obtained by conjugating the monodromy \( M_{i/\sqrt{2}} \) with elements \((2,13)\) of \( \Gamma_0(2)_+ \). A general element of \( H \) is then obtained by a sequence of such transformations, and shifts \( f \) by:

\[
f \rightarrow f + \sum_i N_i (\det M_i)^{-2} [2(a_i T + b_i)^2 + (c_i T + d_i)^2]^2 \equiv f + \sum_{n=0}^4 c_n T^n \quad N_i \in \mathbb{Z} . \tag{4.8}
\]

Since the generated coefficients \( c_n \)'s are 5 independent integer parameters, \( H \) is isomorphic to \( \mathbb{Z}^5 \). Moreover, the quotient \( G/H \) is isomorphic to \( \Gamma_0(2)_+ \) under which the \( c_n \)'s transform in the \( 5 \) representation (second rank traceless symmetric tensor) \[10\]. Thus, the perturbative \( T \)-duality group \( G \) involves 8 integer parameters. Note that \( G \) is not a semidirect product of \( \Gamma_0(2)_+ \) and \( H \) because \( \Gamma_0(2)_+ \) is not a subgroup of \( G \). Finally, the full perturbative symmetry group is the direct product of \( G \) with the constant dilaton shift,

\[
D : \quad S \rightarrow S + \lambda . \tag{4.9}
\]

The symplectic structure of \( N = 2 \) supergravity implies that all symmetry transformations of the effective low energy theory must be contained in the symplectic group \( Sp(6, \mathbb{R}) \) \[15, 20\] which is broken to \( Sp(6, \mathbb{Z}) \) by quantum effects. It is then convenient to introduce a field basis where all transformations act linearly. As usual, we define

\[
T = \frac{X^1}{X^0} \quad \text{and} \quad S = \frac{X^2}{X^0} , \tag{4.10}
\]

in terms of the homogeneous coordinates \( X^I \) with \( I = 0, 1, 2 \). The prepotential is a homogeneous function of degree 2,

\[
F(X^0, X^1, X^2) = (X^0)^2 F \left( \frac{X^1}{X^0}, \frac{X^2}{X^0} \right) , \tag{4.11}
\]
and the Kähler potential (3.1) is given in terms of
\[ Y = \bar{X}^I F_I - X^I \bar{F}_I \quad \text{with} \quad F_I = \frac{\partial F}{\partial X^I} . \] (4.12)

In this way all symmetries must act in the basis \((F_I, X^I)\) as symplectic transformations which leave the Kähler potential (4.12) manifestly invariant. Their symplectic action on the homogeneous basis is uniquely defined from the corresponding transformations of the fields \(T, S\) and the prepotential \(F^2\).

The invariance of the action under symplectic transformations implies that electric and magnetic charges form a symplectic vector. It is then useful to use a slightly different basis from (4.10), in which the perturbative transformations can be restricted consistently on the subspace of magnetically neutral states and therefore do not involve the inversion of the string coupling. They should therefore be represented by matrices of the form
\[
\begin{pmatrix}
a & ab \\
0 & a^{-1,t}
\end{pmatrix}
\] (4.13)

where \(b\) is a \(3 \times 3\) symmetric matrix, \(b = b^t\). For \(b \neq 0\) the effective action changes with \(F \bar{F}\) total derivative terms. Hence, classical duality transformations are block diagonal while the perturbative monodromies generate upper off-diagonal elements \(^2\). The new basis, obtained by a symplectic change of (4.10), is
\[
\bar{F}_I = F_I \quad \bar{X}^I = X^I \quad (I = 0, 1) \quad \bar{F}_2 = X^2 \quad \bar{X}^2 = -F_2 . \] (4.14)

It is now straightforward to obtain the matrix representation of the generators of the perturbative duality group, using eq. (3.6) with (4.5) - (4.7). We first fix the constant dilaton shift \(\lambda_M\) entering in the transformation of \(S\) in eq. (3.8) by imposing the group relations (4.2) \(V^4 = 1\), and \(UVW = 1\) \(i.e.\) that \(S\) is inert under the action of \(V^4\) and \(UVW\). The result is:
\[
\lambda_V = 0 ; \quad \lambda_U = w_1 - v_1 - \lambda_W . \] (4.15)

\(^2\)There is an overall sign ambiguity which is irrelevant for our purposes since we identify the element \(-1\) with the identity.
Note that $S$ is never inert under the action of the monodromy $W^2 = M_{i/\sqrt{2}}$ which corresponds to the first group relation. However, one can choose $\lambda_W$ in a way that this monodromy coincides with the one obtained in the rigid field theory as we will discuss below [21]. In our case, this requirement gives $\lambda_W = -2/3$. Furthermore, to simplify the expressions one has to make a choice for the five parameters $w_{0,1}$ and $v_{1,2,3}$ entering in the polynomials $M_W$ and $M_V$. We first choose $w_0 = -1/2$ and $w_1 = 0$, so that $M_W$ equals half of the monodromy polynomial $M_{i/\sqrt{2}}$. We then impose the requirement that $S$ is inert under the translation $U$ which yields $v_1 = w_1 = 0$ and $v_2 = 2 - 6\lambda_W = 6$. Finally we choose $v_3 = 16$, and the 3 polynomials (4.5) - (4.7) become:

$$M_W = -2T^4 - 2T^2 - \frac{1}{2}$$

$$M_V = 6T^4 + 16T^3 + 6T^2 + \frac{1}{2}$$

$$M_U = -4T^2 + 3.$$  \hspace{1cm} (4.16)

As a result, in the perturbative basis (4.14) the generators are:

$$\tilde{W} = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad \tilde{V} = \begin{pmatrix} -2 & 1 & -1 & -3 & 0 & 3 \\ -4 & 1 & 0 & -4 & 8 & 4 \\ -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -4 & -2 \end{pmatrix} \hspace{1cm} (4.17)$$

$$\tilde{U} = (\tilde{V}\tilde{W})^{-1} = \begin{pmatrix} 1 & -1 & 2 & 6 & 8 & 0 \\ 0 & 1 & -4 & 0 & -8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 4 & 1 \end{pmatrix} \quad \tilde{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
where in the second line we also give the matrix representation of the dilaton shift (4.9).
The upper off-diagonal $3 \times 3$ matrix is in fact $\lambda$ times the $O(2, 1)$ metric in the perturbative basis $(1, T, 2T^2)$ we are using [20, 10]. The parameter $\lambda$ should also be quantized at the non perturbative level, $\lambda = 1$, due to instanton effects.

The perturbative monodromy group is generated by the 3 generators $\tilde{W}$, $\tilde{V}$ and $\tilde{D}$. The generator $\tilde{W}$, whose classical part corresponds to the Weyl reflection of the $SU(2)$ enhanced gauge symmetry as we discussed in the beginning of this section, coincides with the perturbative monodromy $M_\infty$ of the rigid theory [12]. In fact by a linear change of our field variables which diagonalizes its classical part,

$$
\begin{align*}
\hat{F}_0 &= \frac{\tilde{F}_0 + \tilde{F}_2}{\sqrt{2}}, & \hat{F}_1 &= \tilde{F}_1, & \hat{F}_2 &= \frac{\tilde{F}_0 - \tilde{F}_2}{\sqrt{2}}, & \hat{X}^0 &= \frac{\tilde{X}^0 + \tilde{X}^2}{\sqrt{2}}, & \hat{X}^1 &= \tilde{X}^1, & \hat{X}^2 &= \frac{\tilde{X}^0 - \tilde{X}^2}{\sqrt{2}},
\end{align*}
$$

(4.18)

$\tilde{W}$ takes the form:

$$
\tilde{W} = \hat{M}_\infty = \begin{pmatrix}
-1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

(4.19)

Near the enhanced symmetry point, $\hat{X}^0 \sim 2i(T - i/\sqrt{2})$ and the subspace $(\hat{F}_0, \hat{X}^0)$ is associated to the rigid supersymmetric theory. The generator $\tilde{W}$ acts non-trivially only on this subspace and its action represented by the corresponding $2 \times 2$ submatrix can be identified with $M_\infty$ of ref. [12].

5. Exact symmetry group

The exact symmetry group of this model can be determined using duality from its type II realization based on the Calabi-Yau manifold $X_8$, discussed in Section 2. The full
The monodromy group was worked out in ref. [7] and was shown to be a subgroup of $Sp(6, \mathbb{Z})$, $G$, generated by 3 elements denoted by $A$, $T$ and $B$. The element $A$ generates an exact $\mathbb{Z}_8$ symmetry defined in eq.(2.3) and satisfies $A^8 = 1$. The other two generators $T$ and $B$ are associated to the monodromies around the conifold and the strong coupling loci, respectively, described by the discriminant (2.12), and they are subject to some group relations given in ref. [7]. Moreover, by considering the large complex structure limit, two independent (mutually commuting) translations were identified, acting on the special coordinates $t_{1,2}$ of eq.(2.5),

$$S_1 = (AT)^{-2} : t_1 \rightarrow t_1 + 1$$

$$S_2 = (ATB)^{-1} : t_2 \rightarrow t_2 + 1 .$$

(5.1)

Our first task is to identify the generators of the perturbative heterotic duality group derived in the previous section as elements of the type II monodromy group $G$. From the identification with the heterotic variables $t_1 = T$ and $t_2 = S$, one concludes that in a suitable basis one should have:

$$S_1 = U \quad S_2 = D|_{\lambda = 1} ,$$

(5.2)

where the matrix representation of $U$ and $D$ is given in eq.(4.17), in the perturbative basis (4.14). The one loop heterotic prepotential $f$ is shifted by the polynomial $\mathcal{M}_U$ of eq. (4.16) under the action of $U$, while it remains inert under $D$. These transformations allow us to determine the integration constants entering in the expression of the type II prepotential (2.8) for which the identification (5.2) should be valid,

$$\alpha = 4 \quad \delta = 3 \quad \beta = \gamma = \epsilon = 0 .$$

(5.3)

On the other hand, the 3 generators $A, T, B$ are given in ref. [7] in the symplectic basis $(F_i, X^I)$ of eq.(1.10) for a different choice of these parameters, namely $\alpha = \gamma = 0$, $\beta = -2$, $\epsilon = 0$. To keep the same notation as ref. [7], we are forced to use the symbol $T$ for the generator which should not be confused with the heterotic modulus.
$\delta = 3$ and $\epsilon = 1$. The values of the parameters can be changed by a conjugation with the symplectic matrix

$$
\begin{pmatrix}
1 & Q \\
0 & 1
\end{pmatrix}
$$

with

$$
Q =
\begin{pmatrix}
0 & \delta' & \epsilon' \\
\delta' & \alpha' & \beta' \\
\epsilon' & \beta' & \gamma'
\end{pmatrix}
$$

(5.4)

under which $\alpha, \beta, \cdots, \epsilon$ are shifted by $\alpha', \beta', \cdots, \epsilon'$. Performing such a conjugation, the two translations are identified according to eq.(5.2) in the basis (5.3), and the generators $A, T, B$ are:

$$
A =
\begin{pmatrix}
-2 & 0 & 1 & -2 & 0 & -1 \\
-4 & 1 & 0 & -4 & 4 & 0 \\
-3 & 1 & -1 & -4 & 0 & -1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
-1 & 0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

$$
B =
\begin{pmatrix}
1 & -1 & 2 & -2 & 0 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -2 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & -1
\end{pmatrix}
$$

(5.5)

$$
T =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

The fact that the generator $A$ is of order 8 suggests that the order 4 generator $V$ should be identified with a conjugation of $A^2$ (or its inverse). Indeed it is easy to verify that the matrix $\tilde{V}$ transformed in the original basis (4.10) is identical to $A^2$. Thus, we have shown that the perturbative dualities generated by $U$ and $V$, together with the quantized dilaton

\footnote{Note the sign mistakes in the first equation of page 521 of ref. \cite{ref7} where one reads “$\delta = -3$, $\epsilon = -1$” instead of their opposite values which lead to the correct conjugation matrix $m$.}
shift \( D|_{\lambda=1} \), form a subgroup of the type II symmetries. Using eqs. (5.1), (5.2) and the group relations (4.2) one has:

\[
U = (AT)^{-2}, \quad V = A^{2}, \quad W = A^{-1}TA, \quad D|_{\lambda=1} = (ATB)^{-1}.
\]  

\((5.6)\)

Our next task is to identify the quantum monodromy group \( \Gamma(2) \) of the \( SU(2) \) rigid field theory as a subgroup of \( G \). It is generated by two elements, \( M_{\infty} \) and \( M_{1} \), which satisfy the relation [12]:

\[
M_{\infty} = M_{1}M_{-1},
\]

\((5.7)\)

where \( M_{\infty} \) is the perturbative monodromy, while \( M_{1} \) and \( M_{-1} \) correspond to the monodromies around the points where dyonic hypermultiplets become massless and they are conjugate to each other. These properties can be used as a guide for the identification [21]. We have shown in the previous section that \( M_{\infty} \) coincides with the generator \( W \) of the heterotic duality group. A simple inspection of its form (5.6) suggests that \( M_{1} \) should be identified with \( T \) (or its conjugate \( A^{-1}TA \)). Indeed one can easily verify that in the basis (4.18) the generator \( T \) takes the form:

\[
\hat{T} = \hat{M}_{1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\((5.8)\)

One sees that in analogy with \( \hat{W} \), \( \hat{T} \) acts non-trivially only on the subspace of the rigid supersymmetric theory \( (\hat{F}_0, \hat{X}^0) \), and its action represented by the corresponding \( 2 \times 2 \) submatrix coincides with \( M_{1} \) of ref. [2]. Thus, we have shown that the non-perturbative monodromies of the \( SU(2) \) rigid field theory form a \( \Gamma(2) \) subgroup of the type II symmetry \( G \),

\[
M_{\infty} = W, \quad M_{1} = T, \quad M_{-1} = (AT)^{-1}T(ATA).
\]

\((5.9)\)
This result provides a non-perturbative test of the string duality and confirms that the $SU(2)$ enhanced symmetry which is present in perturbation theory of $N = 2$ heterotic compactifications is never restored when non-perturbative effects are taken into account (except in the trivial case of zero coupling). Instead, there are two singular (complex) lines in the moduli space of $S$ and $T$ where massless dyonic hypermultiplets appear with charges determined in the rigid theory. String duality maps these lines into the conifold locus of the Calabi-Yau manifold while the dyonic hypermultiplets are exchanged with the solitonic back hole hypermultiplets of type II string [13]. The position of the singular lines is determined by the equation $x = 2^{-8}/(1 \pm 2\sqrt{y})$ in terms of the large complex structure variables $x$ and $y$ of eq.(2.5).

The heterotic $T$-duality group extended with the quantized dilaton shift and the non-perturbative monodromy of the rigid field theory (5.6), (5.9) form a subgroup of the full exact symmetry $G$ generated, for instance, by the elements $A^2$, $A^{-1}TA$, $A^{-1}B$, and $T$. One needs just to introduce the “square root” of the perturbative generator $V = A^2$, or equivalently $B$, to recover the full $G$. The generator $B$ is related to the monodromy around the non-perturbative singularity $y = 1/4$ corresponding to infinite coupling, where new massless dyonic hypermultiplets appear with charges under the “dilaton” $U(1)$. This singular line which is not present in the rigid theory is a new stringy phenomenon related to the dilaton and seems to be a generic feature of string vacua. It would be interesting to find the corresponding solitonic solutions which become massless in the type II theory, in analogy with the black holes of the conifold locus.

From the matrix representation of the generator $B$ in eq.(5.5), one finds that it corresponds to the transformation,

\[
T \quad \rightarrow \quad T + S + 1
\]

\[
B : \quad S \quad \rightarrow \quad -S + 2
\]

\[
F(T, S) \quad \rightarrow \quad F(T, S) + S^2 - 1
\]

(5.10)
where $F$ is the full prepotential (2.10). The monodromy polynomial $S^2 - 1$ follows from the upper $3 \times 3$ off-diagonal block of the matrix. Note that redefining the dilaton as $S' \equiv S + T - 1$, $B$ corresponds to the exchange $T \leftrightarrow S'$ as suggested in ref. [3]. In fact $B^2$ corresponds to the monodromy one obtains when moving a point around the (ultra-)strong coupling singularity $S = 1$,

$$B^2 : \quad F \to F + 2(S - 1)^2 \quad (5.11)$$

which suggests that $F$ has a logarithmic singularity at $S = 1$ (modulo integer shifts by $S_2$).

One would like to use the exact symmetries discussed above in order to determine the low energy effective action of vector multiplets based on the prepotential $F$. Consider for instance $\partial^3 T F$ which is invariant under both transformations $B$ and $S_2$ (or equivalently $AT$). Unfortunately, to determine this function one needs an information about one more generator, $T$ or $A$ whose action on our variables is highly non-trivial. Similarly, the perturbative transformation $W$ (or $V$) is drastically modified.

**Acknowledgments** We would like to thank J. Lascoux and K.S. Narain for useful discussions.

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