Invariants, Projection Operators and
SU(N)×SU(N) Irreducible Schwinger Bosons

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Abstract
We exploit SU(N) Schwinger bosons to construct and analyze the coupled irreducible representations of SU(N)×SU(N) in terms of the invariant group. The corresponding projection operators are constructed in terms of the invariant group generators. We also construct SU(N)×SU(N) irreducible Schwinger bosons which directly create these coupled irreducible states. The SU(N) Clebsch Gordan coefficients are computed as the matrix elements of the projection operators.

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1 Introduction

It is well known that the Schwinger representation of the SU(2) Lie algebra [1] has played important roles in widely different branches of physics such as nuclear physics [2], condensed matter physics [3], quantum optics [4], gauge theories [5], quantum gravity [6] etc. They have also played equally significant role in the study of Lie groups [7]. In particular, in the context of representation theory of SU(N) group [8] the Schwinger bosons enable us to construct their unitary irreducible representations with enormous ease and simplicity [9, 10, 11, 12, 13, 14]. The SU(2) case, studied extensively by Schwinger himself, provides the easiest example of this simplification. The Hilbert space \( \mathcal{H}_{sb} \) associated with two Schwinger bosons is isomorphic to the representation space of SU(2) and is the simplest possible representation or model space [15] of SU(2). However, in the case of higher \( N \geq 3 \), this simple isomorphism is lost due to the existence of certain SU(N) invariant operators. These invariant operators follow U(N−1) algebra [9, 11, 13, 14] and lead to SU(N) invariant directions in the Hilbert space \( \mathcal{H}_{sb} \) associated with SU(N) group. Any two states which differ by an overall presence of such invariant operators will transform in the same way under SU(N). This leads to the problem of multiplicity which in turn makes the representation theory of SU(N) \( (N \geq 3) \) much more
involved compared to SU(2) [9, 10, 11, 13, 14]. The standard way to handle this problem is by demanding that the Schwinger boson states follow the symmetries of SU(N) Young tableaux. In [13, 14] we showed that these SU(N) Young tableau symmetries can be easily realized by imposing certain SU(N) invariant constraints on $\mathcal{H}_{ab}$. We further defined SU(N) irreducible Schwinger bosons which weakly commute with the above constraints and hence directly create states which are invariant under all SU(N) Young tableau symmetries (see section 2.2). These SU(N) irreducible states created by the monomials of SU(N) irreducible Schwinger bosons are also multiplicity free. This makes the construction of all SU(N) irreducible representations exactly analogous to the simple SU(2) case. Thus the SU(N) invariant constraint formulation provides a novel approach to study the SU(N) representation theory. The purpose and motivation of the present work is to show that the above ideas can also be naturally extended to the study of the coupled representations of the direct product group SU(N) × SU(N). We discuss the simplest and well studied SU(2) × SU(2) case first and then go to higher SU(N) × SU(N) groups. In fact, to our knowledge even the $N = 2$ results (section 2) are new and have many novel features. In particular, we show that all coupled angular momentum irreducible representations can be projected out directly from the decoupled angular momentum states by certain projection operators. These projection operators are built from the invariant $Sp(2, R) \times SU(2)$ generators which commute with the total angular momentum generators. The SU(2) Clebsch Gordan coefficients are simply the matrix elements of the above projection operators in the decoupled basis (see eqn. (15)) and can be easily computed (see section 2.3). Further, using the invariant algebra we also construct SU(2) × SU(2) irreducible Schwinger bosons which directly create all possible coupled SU(2) × SU(2) irreducible states. As expected, these simple SU(2) techniques based on the invariant groups have natural extension to all higher SU(N) groups. This is significant and important as, inspite of vast amount of literature on SU(2) group and very specific techniques valid for $N = 3, 4$ etc., general computational methods to handle SU(N) group for arbitrary N are hard to find [8, 10].

The plan of the paper is as follows. The section 2 deals with the simplest SU(2) case. In this section we briefly construct the total angular momentum group invariant $Sp(2, R) \times SU(2)$ algebras [1]. Using this invariant algebra we construct projection operators which directly project the direct product Hilbert space to various irreducible Hilbert spaces characterized by the net angular momentum and net magnetic quantum numbers. Further, we construct SU(2) × SU(2) irreducible Schwinger bosons which trivially satisfy the above constraints and directly create the direct product irreducible representations. We then show that the Clebsch Gordan coefficients can be very easily computed as the matrix elements of the projection operators using the invariant algebra. In section 3 we show that the above SU(2) techniques have very natural extension to all higher SU(N) groups.

**Direct product representations and invariant algebras**

In the following sections we show that the symmetries of the direct product Young tableaux (see Figure 1 and Figure 2) can also be realized through certain group invariant constraints (see eqn. (16)). Further, these group invariant constraints lead to projection operators which project out the coupled irreducible representations from the direct product of two irreducible
representations. As mentioned earlier, we discuss the simple $SU(2)$ group (section 2) first and then generalize these ideas and techniques to $SU(N)$ group with arbitrary $N$ (section 3).

2 Representations of $SU(2) \times SU(2)$ and invariants

The Schwinger boson representations of $SU(2) \times SU(2)$ Lie algebra is:

$$J^a_1 \equiv \frac{1}{2} a^+ \alpha (\sigma^\alpha)_{\alpha\beta} a_\beta, \quad J^a_2 \equiv \frac{1}{2} b^+ \alpha (\sigma^\alpha)_{\alpha\beta} b_\beta,$$

where $\sigma^\alpha$ denote the Pauli matrices, $(a_\alpha, a^+_\alpha)$ and $(b_\alpha, b^+_\alpha)$ with $\alpha = 1, 2$ are the two Schwinger boson doublets. It is easy to check that the operators in (1) satisfy the $SU(2)$ Casimirs:

$$\hat{J}_1 \cdot \hat{J}_1 \equiv \frac{n_a}{2} \left(\frac{n_a}{2} + 1\right), \quad \hat{J}_2 \cdot \hat{J}_2 \equiv \frac{n_b}{2} \left(\frac{n_b}{2} + 1\right).$$

In (2), $\hat{n}_a = \vec{a} \cdot \vec{a} = (a_1^+ a_1 + a_2^+ a_2)$ and $\hat{n}_b = \vec{b} \cdot \vec{b} = (b_1^+ b_1 + b_2^+ b_2)$ are the number operators with eigenvalues $n_a = n_a^1 + n_a^2$ and $n_b = n_b^1 + n_b^2$ respectively. The decoupled angular momentum states are:

$$|j_1, m_1\rangle = \frac{(a_1^+)^{j_1 + m_1} (a_2^+)^{j_1 - m_1}}{\sqrt{(j_1 + m_1)! (j_1 - m_1)!}} |0\rangle, \quad |j_2, m_2\rangle = \frac{(b_1^+)^{j_2 + m_2} (b_2^+)^{j_2 - m_2}}{\sqrt{(j_2 + m_2)! (j_2 - m_2)!}} |0\rangle.$$

The representations of $SU(2)$ can also be characterized by the eigenvalues of the total occupation number operator as,

$$|n_a^1, n_a^2\rangle = \frac{(a_1^+)^{n_a^1} (a_2^+)^{n_a^2}}{\sqrt{n_a^1! n_a^2!}} |0\rangle, \quad |n_b^1, n_b^2\rangle = \frac{(b_1^+)^{n_b^1} (b_2^+)^{n_b^2}}{\sqrt{n_b^1! n_b^2!}} |0\rangle.$$

In (4) $n_a^1 = j_1 + m_1, n_a^2 = j_1 - m_1, n_b^1 = j_2 + m_2, n_b^2 = j_2 - m_2$. The direct product states $|n_a^1, n_a^2\rangle \otimes |n_b^1, n_b^2\rangle$ will often be denoted by $|n_a^1, n_a^2, n_b^1, n_b^2\rangle$. The total angular momentum generators are:

$$J^a_T = J^a_1 + J^a_2.$$

The corresponding group will be denoted by $SU(2)_T$. We now construct all possible $SU(2)_T$ invariants out of the two Schwinger boson doublets in (1). The first set of invariant operators is:

$$k_+ \equiv \vec{a} \cdot \vec{b}^+, \quad k_- \equiv \vec{a} \cdot \vec{b}, \quad k_0 = \frac{1}{2} (\hat{n}_a + \hat{n}_b + 2).$$

(6)
In (6) the invariants \( k_\pm \) are the antisymmetric combination of the two doublets: 
\[
a^\dagger \cdot \tilde{b}^\dagger \equiv \epsilon_{\alpha\beta}a_\alpha^\dagger b_\beta^\dagger = (a_1^\dagger b_2^\dagger - a_2^\dagger b_1^\dagger) \quad \text{and} \quad a \cdot \tilde{b} \equiv \epsilon_{\alpha\beta}a_\alpha b_\beta = (a_1 b_2 - a_2 b_1). \]
It is easy to check that \( k_+, k_- \) and \( k_0 \) commute with \( SU(2)_T \) generators \( J^a \) in (5) and satisfy \( Sp(2,\mathbb{R}) \) algebra:
\[
[k_-, k_+] = 2k_0, \quad [k_0, k_\pm] = \pm k_\pm. \tag{7} \]
The discrete unitary irreducible representations \( |k, q\rangle \) of \( Sp(2,\mathbb{R}) \) relevant for us in this work are characterized by the eigenvalues of \( k^2 \equiv k_1^2 + k_2^2 - k_0^2 = \frac{1}{2}(k_+ k_- + k_- k_+) - k_0^2 \) and \( q \) satisfying:
\[
k_0 |k, q\rangle = q |k, q\rangle, \quad k^2 |k, q\rangle = k(1 - k) |k, q\rangle. \tag{8} \]
In (8), \( q = k, k+1, k+2, \cdots \). The \( Sp(2,\mathbb{R}) \) raising and lowering operators satisfy: 
\[
k_\pm |k, q\rangle = [(q \pm k)(q \mp k \pm 1)]^{\frac{1}{2}} |k, q \pm 1\rangle \quad \text{and} \quad k_-|k, q = k\rangle = 0.
\]
Similarly, another \( SU(2)_T \) invariant algebra is obtained by defining \( [1] \):
\[
\kappa_+ \equiv a^\dagger \cdot b, \quad \kappa_- \equiv b^\dagger \cdot a, \quad \kappa_0 \equiv \frac{1}{2}(\hat{n}_a - \hat{n}_b). \tag{9} \]
These generators satisfy the standard \( SU(2) \) algebra:
\[
[k_+, k_-] = 2k_0, \quad [k_0, k_\pm] = \pm k_\pm. \tag{10} \]
It is easy to check that the \( Sp(2, R) \) and \( SU(2) \) generators in (6) and (9) respectively commute with each other as well as with \( SU(2)_T \) in (5). Therefore, the coupled irreducible representations of \( SU(2) \times SU(2) \) can also be labeled by the quantum numbers of the \( Sp(2, R) \times SU(2) \) group (see (26) and (27)).

### 2.1 The Projection operators, invariants and symmetries of Young tableaux

In this section we consider the coupled angular momentum states obtained by taking the direct product of two arbitrary angular momentum states \( |j_1, m_1\rangle \) and \( |j_2, m_2\rangle \) as shown in Figure [1].

The Young tableaue decomposition in Figure [1] corresponds to the standard expansion of the decoupled basis in terms of the coupled basis:
\[
|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=j_1+j_2}^{|j_1-j_2|} C_{j_1,m_1,j_2,m_2}^{j,m} |j_1; j_2; j, m\rangle, \tag{11} \]
In (11) \( m = m_1 + m_2 \). The same series can also be obtained by defining projection operators \( \mathcal{P}_j \) which directly project the decoupled state to a particular coupled state \( |j_1; j_2; j, m\rangle \). In terms of projection operators, the expansion (11) takes the form:
\[
|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=j_1+j_2}^{|j_1-j_2|} \mathcal{P}_j |j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv \sum_{r=0}^{\min(2j_1,2j_2)} \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle. \tag{12} \]
Figure 1: Graphical or Young tableau representation of the identity (12). The $SU(2) \otimes SU(2)$ coupled states on the right hand side can be directly obtained from the decoupled state by the corresponding projection operators (see (14)). The coupled states on the right hand sides also carry $Sp(2,R) \times SU(2)$ quantum numbers (see (26) and (27)).

In (12) $r$ is the number of two boxes (invariants) on the right hand side of Figure 1, i.e.,

$$r = j_1 + j_2 - j.$$  

(13)

Comparing the series (12) with the standard expansion in (11) we get:

$$\mathcal{P}_r |j_1, m_1 \rangle \otimes |j_2, m_2 \rangle = C_{j_1,m_1;j_2,m_2}^{j,m} |j_1, j_2; j, m \rangle.$$  

(14)

In (14) $j = j_1 + j_2 - r$ and $m = m_1 + m_2$. Taking the norms of each side of (14) and using $\mathcal{P}_r^2 = \mathcal{P}_r$ we get:

$$C_{j_1,m_1;j_2,m_2}^{j,m} = \sqrt{\langle j_1,j_2,m_1,m_2|\mathcal{P}_r|j_1,j_2,m_1,m_2 \rangle}.$$  

(15)

In (15) we have used the notation $|j_1,j_2,m_1,m_2 \rangle \equiv |j_1,m_1 \rangle \otimes |j_2,m_2 \rangle$. The Clebsch Gordan coefficients in (15) will be explicitly computed in section 2.3.

We now construct the projection operators defined in (12). The Figure 1 and (12) imply that the projection operators can only depend on the $SU(2)_T$ invariant operators ($\hat{n}_a, \hat{n}_b, k_+, \kappa_+$) discussed in section 2. We first consider $r = 0$ ($j = j_1 + j_2$) case. The Figure 1 implies that $\mathcal{P}_0(\equiv \mathcal{P})$ should completely symmetrize the $SU(2)$ indices so that $j = j_1 + j_2$. Therefore, we demand:

$$k_- (\mathcal{P} |j_1, m_1 \rangle \otimes |j_2, m_2 \rangle) = 0.$$  

(16)

As $\mathcal{P}$ depends only on the $SU(2)_T$ invariant operators, we can write the most general form as:

$$\mathcal{P} = \sum l_{\{q\}}(\hat{n}_a, \hat{n}_b) (k_+)^{q_1}(k_-)^{q_2}(\kappa_+)^{q_3}(\kappa_-)^{q_4}$$  

(17)

In (17) $l_{\{q\}} \equiv l_{q_1,q_2,q_3,q_4}(\hat{n}_a, \hat{n}_b)$ are the number operator dependent operators. Further, as the projection operator should not change the number of either a or b type oscillators, we get $q_1 = q_2, q_3 = q_4$. On the other hand, the identity $\epsilon_{\alpha \beta \epsilon \gamma \delta} = \delta_{\alpha \gamma} \delta_{\beta \delta} - \delta_{\alpha \delta} \delta_{\beta \gamma}$ implies:

$$\kappa_+ \kappa_- = \hat{n}_a \hat{n}_b - k_+ k_-.$$  

(18)
Thus all $SU(2)$ operators $\kappa_+ \kappa_-$ in (17) can be removed in terms of $Sp(2,\mathbb{R})$ operators $k_+ k_-$. Therefore, the most general form of the projection operator is:

$$P = \sum_{q=0}^{\infty} l_q(\hat{n}_a, \hat{n}_b) (k_+)^q (k_-)^q.$$  (19)

The constants $l_q$ can be computed (see appendix A.1) by using the constraint (16) and lead to:

$$l_q(\hat{n}_a, \hat{n}_b) = \frac{(-1)^q (\hat{n}_a + \hat{n}_b - q)!}{q! (\hat{n}_a + \hat{n}_b)!}$$  (20)

Note that the constant term in (20) is chosen to be unity (i.e $l_0 = 1$) so that:

$$P^2 = P \delta_{rs} = P = P$$

as $k_- P = 0$. The Figure 1 now immediately implies that all other projection operators are of the form:

$$P_r = N_r (k_+)^r P (k_-)^r = N_r (k_+)^r P (k_-)^r$$  (21)

The constant coefficients $N_r$ are fixed by demanding that the operators $P_r$ satisfy $P_r^2 = P_r$ (see appendix A.1). We thus get:

$$N_r = \frac{(n_a + n_b - 2r + 1)!}{r! (n_a + n_b - r + 1)!}$$  (22)

Note that these coefficient can also be computed by demanding completeness property:

$$\sum_{r=0}^{\min(2j_1,2j_2)} P_r = I.$$  (23)

The completeness property (23) which is manifest in the defining expansion (12) is proved in appendix A.1. It is also easy to check that the Hilbert spaces projected by different projection operators in (21) are orthogonal:

$$P_r P_s = \delta_{rs} P_r, \quad r, s = 0, 1, 2, \cdots \min(2j_1, 2j_2).$$  (24)

In (24) we have used the $Sp(2,\mathbb{R})$ commutation relation (6) and the constraints $k_- P = 0$ ($r > s$), $P k_+ = 0$ ($r < s$). We note that the coupled angular momentum states $P (k_-)^r |j_1, m_1\rangle \otimes |j_2, m_2\rangle$ in the expansion (12) also belong to the $Sp(2,\mathbb{R})$ irreducible representations with lowest $Sp(2,\mathbb{R})$ magnetic quantum number $q = q_0 = (j_1 + j_2 - r + 1)$ as:

$$k_0 P (k_-)^r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = q_0 P (k_-)^r |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$  (25)

$$k^2 P (k_-)^r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = q_0(1 - q_0) P (k_-)^r |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$  (26)
To get the second eigenvalue equation we have used $k_- \mathcal{P} = 0$ to replace $k^2(\equiv \frac{1}{2}(k_+ k_- + k_- k_+)) - k_0^2$ by $\frac{1}{2}[k_-, k_+] - k_0^2 = k_0(1 - k_0)$. The equations (25) immediately imply:

$$k_0 \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = (j_1 + j_2 + 1) \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$  

(26)

$$k^2 \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = q_0(1 - q_0) \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$  

Similarly, it is easy to check that the quantum numbers of $SU(2)_T$ invariant $SU(2)$ group in (9) are:

$$\kappa_0 \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = (j_1 - j_2) \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle,$$  

(27)

$$\kappa^2 \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = j(j + 1) \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$  

Note that $j = J_1 + J_2 = r$ in (27).

### 2.2 $SU(2) \times SU(2)$ irreducible Schwinger bosons

It is known that all possible $SU(N)$ irreducible representations can be written as monomials of $SU(N)$ irreducible Schwinger bosons [13 [14]. This construction is the $SU(N)$ extension of the Schwinger $SU(2)$ construction [11]. In this section we apply these ideas to construct the coupled states $|J_1, J_2, J, M\rangle$ in [14] as monomials of $SU(2) \times SU(2)$ irreducible Schwinger bosons (see equation (37)). The $SU(2) \times SU(2)$ irreducible Schwinger boson creation operators create states which satisfy $k_- = 0$ and therefore correspond to maximally symmetric (or states with highest angular momentum) states. All other states can be constructed by applying the invariant operators on such maximally symmetric states. Note that this procedure is also illustrated by Figure 1. The first coupled state on the right hand side with $n_a + n_b = 2j_1 + 2j_2$ is the maximally symmetric state. All other coupled states on the right hand side are obtained by multiplications of the invariant $k_+$ (i.e., two boxes arranged vertically in Figure 1) on such maximally symmetric states. As in [13 [14], we define:

$$A^\dagger_\alpha \equiv a^\dagger_\alpha + f(\hat{n}_a, \hat{n}_b) k_+ b_\alpha, \quad B^\dagger_\alpha \equiv b^\dagger_\alpha + g(\hat{n}_a, \hat{n}_b) k_+ a_\alpha.$$  

(28)

Note that by construction (28) the $SU(2) \times SU(2)$ transformation properties of $A^\dagger_\alpha$ and $B^\dagger_\alpha$ are exactly same as those of $a^\dagger_\alpha$ and $b^\dagger_\alpha$ respectively. We now demand:

$$k_- A^\dagger_\alpha \mathcal{P} |j_1, m_1\rangle \otimes |j_2, m_2\rangle = 0, \quad k_- B^\dagger_\alpha \mathcal{P} |j_1, m_1\rangle \otimes |j_2, m_2\rangle = 0.$$  

(29)

The above constraints can be solved in terms of the unknown operator valued functions $f(\hat{n}_a, \hat{n}_b)$ and $g(\hat{n}_a, \hat{n}_b)$:

$$f(\hat{n}_a, \hat{n}_b) = -\frac{1}{(n_a + n_b)}, \quad g(\hat{n}_a, \hat{n}_b) = \frac{1}{(n_a + n_b)}.$$  

(30)
Note that the $f(\hat{n}^a, \hat{n}^b)$ and $g(\hat{n}^a, \hat{n}^b)$ in (30) are well defined as they always follow a creation operator in (28). As an example of the states created by the $SU(2) \times SU(2)$ irreducible Schwinger bosons we consider the state: $A^\dagger_\alpha B^\dagger_\beta |0\rangle = A^\dagger_\alpha B^\dagger_\beta |0\rangle = \frac{1}{2} \left( a^\dagger_\alpha b^\dagger_\beta + a^\dagger_\beta b^\dagger_\alpha \right) |0\rangle$. We note that it is already symmetric in the $SU(2)$ indices $\alpha$ and $\beta$ and no explicit symmetrization is needed. In fact, $SU(2)$ implies weak equality. In other words the equations (32) are true only on the projected section of the Hilbert space which satisfies the constraint $k_- = 0$. The equivalence of (32) and (28) can be easily established by substituting $\mathcal{P}$ from (19) in (32) and noting that $l_1(\hat{n}_a, \hat{n}_b) = f(\hat{n}_a, \hat{n}_b) = -g(\hat{n}_a, \hat{n}_b)$. The completely symmetric states of $SU(2)_T$ can be easily defined through the irreducible Schwinger bosons:

$$|j_1, j_2; j = j_1 + j_2, m\rangle \equiv N^{-1}_{j_1m_1, j_2m_2} \left( A^\dagger_{j_1+m_1} \right) \left( B^\dagger_{j_2+m_2} \right) |0\rangle. \quad \text{(33)}$$

To compute the normalization constant $N^{-1}_{j_1m_1, j_2m_2}$ in (33) we note that the right hand side of the above equation can also be written in terms of decoupled states as:

$$(N^{-1}_{j_1m_1, j_2m_2})^{-1} |j_1, j_2; j = j_1 + j_2, m\rangle = \mathcal{P} |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$

In the first step above we have introduced identity as $\mathcal{P}$. We then replace the irreducible Schwinger bosons by their defining equations (28) and used $\mathcal{P}k_+ = 0$ in the second step to get the decoupled states at the end. To compute the normalization $N^{-1}_{j_1m_1, j_2m_2}$ in (33) we notice that the completely symmetric states are given by (14) at $r = 0$:

$$\mathcal{P} |j_1, m_1\rangle \otimes |j_2, m_2\rangle = C^{j_1+j_2, m}_{j_1, m_1, j_2, m_2} |j_1, j_2; j = j_1 + j_2, m\rangle.$$

8
Comparing this with (34) we get: \( N_{j_1, m_1}^{j_2, m_2} C_{j_1, m_1; j_2, m_2}^{j = j_1 + j_2, m} = 1 \). Therefore,

\[
N_{j_2, m_2}^{j_1, m_1} = \frac{1}{C_{j_1, m_1; j_2, m_2}^{j = j_1 + j_2, m}} = \left[ \frac{(2j_1 + 2j_2)!}{(2j_1)!(2j_2)!} \frac{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!}{(j_1 + j_2 + m_1 + m_2)!(j_1 + j_2 - m_1 - m_2)!} \right]^{1/2} \tag{35}
\]

For example, we put \( j_1 = 2, m_1 = 0, j_2 = 1, m_2 = 0 \) in (33) and replace the irreducible Schwinger bosons by their defining equations (28) and (30) to get:

\[
|j_1 = 2, j_2 = 1, j = 3, m = 0\rangle = N_{10}^{20} \left[ \frac{3}{5} |2, 0\rangle |1, 0\rangle + \frac{\sqrt{3}}{5} |2, -1\rangle |1, 1\rangle + \frac{\sqrt{3}}{5} |2, 1\rangle |1, -1\rangle \right]. \tag{36}
\]

Therefore, explicit normalization of the above state gives: \( N_{10}^{20} = \sqrt{\frac{3}{5}} \) which is also the value obtained by (35) with \( C_{j_1 = 2, m_1 = 0; j_2 = 1, m_2 = 0} = \sqrt{\frac{3}{5}} \). With this value of normalization, the expansion (36) further gives:

\[
C_{j_1 = 2, m_1 = 0; j_2 = 1, m_2 = 1} = C_{j_1 = 2, m_1 = 1; j_2 = 1, m_2 = -1} = \sqrt{\frac{1}{5}}.
\]

The same values are also obtained from the Clebsch Gordan series (41) obtained using the invariants in the next section. The above example provides a self consistency check on the procedure. The discussions in the previous section imply that an arbitrary consistent coupled state can be written as:

\[
|j_1, j_2; j, m\rangle = N_{j_1, j_2}^{j} (k_+)^{j_1 + j_2 - j} |(j_1 - j_2 + j)/2, (j_2 - j_1 + j)/2; j, m\rangle. \tag{37}
\]

Note that the state \( |(j_1 - j_2 + j)/2, (j_2 - j_1 + j)/2; j, m\rangle \) is maximally symmetric and is of the form (33). The normalization constants \( N_{j_1, j_2}^{j} \) can be easily computed using the commutation relations (7) as \( k_- |(j_1 - j_2 + j)/2, (j_2 - j_1 + j)/2; j, m\rangle = 0 \). They are given by:

\[
N_{j_1, j_2}^{j} = \sqrt{\frac{(2j_1 + 2j_2 + 1)!}{(j_1 + j_2 - j)!(3j_1 + 3j_2 - j + 1)!}}.
\]

We again emphasize that all possible \( SU(2) \times SU(2) \) coupled states in (37) are monomials of the irreducible Schwinger bosons. All the symmetries of the coupled Young tableaux on the right hand side of Figure 1 are already present in (37) and there is no need for explicit symmetrization or anti-symmetrization by hand. Thus the the irreducible Schwinger bosons (37) can be thought of as the generalization of \( SU(2) \) Schwinger bosons (3) which directly lead to coupled angular momentum states.

The \( SU(2) \times SU(2) \) irreducible Schwinger bosons satisfy the following algebra:

\[
\begin{align*}
\left[ A^\dagger_\alpha, A^\dagger_\beta \right] &= 0, \quad \left[ B^\dagger_\alpha, B^\dagger_\beta \right] = 0, \quad \left[ A^\dagger_\alpha, B^\dagger_\beta \right] = 0, \\
\left[ A_\alpha, A^\dagger_\beta \right] &= \delta_{\alpha\beta} - \frac{1}{\hat{n}_a + \hat{n}_b + 1} a_\alpha^\dagger a_\beta + \frac{1}{(\hat{n}_a + \hat{n}_b)(\hat{n}_a + \hat{n}_b + 1)} k_+ \tilde{b}_\beta a_\alpha \\
\left[ B_\alpha, B^\dagger_\beta \right] &= \delta_{\alpha\beta} + \frac{1}{\hat{n}_a + \hat{n}_b + 1} a_\alpha^\dagger a_\beta - \frac{1}{(\hat{n}_a + \hat{n}_b)(\hat{n}_a + \hat{n}_b + 1)} k_+ \tilde{a}_\beta b_\alpha.
\end{align*}
\tag{38}
\]
2.3 The projection operators and $SU(2)$ Clebsch Gordan Coefficients

The Clebsch Gordan coefficients are given by the defining equation (41):

$$C_{j_1, m_1 j_2, m_2}^{j_1 + j_2 - r, m} = \langle j_1, j_2; j = j_1 + j_2 - r, m | \mathcal{P}_r | j_1, m_1, j_2, m_2 \rangle$$

(39)

This can be rewritten as:

$$C_{j_1, m_1 j_2, m_2}^{j_1 + j_2 - r, m} = \frac{\langle j_1, m_1' = j_1, j_2, m_2' = m - j_1 | \mathcal{P}_r \mathcal{P}_r | j_1, m_1, j_2, m_2 \rangle}{C_{j_1, m_1' j_2, m_2'}^{j_1 + j_2 - r, m = m - j_1}}$$

(40)

$$= \frac{\langle j_1, j_2, m - j_1 | \mathcal{P}_r | j_1, m_1, j_2, m_2 \rangle}{[\langle j_1, j_2, m - j_1 | \mathcal{P}_r | j_1, j_2, m - j_1 \rangle]^2}$$

We have used $\mathcal{P}_r^2 = \mathcal{P}_r$ in (40). As shown in appendix B.1, the above matrix elements of $\mathcal{P}_r$ can be easily computed to give:

$$C_{j, m}^{j_1, m_1 j_2, m_2} = \delta_{m, m_1 + m_2} \sqrt{\frac{(j_1 - j_2 + j)! (j_2 - m_2)! (j_2 + m_2)! (j_1 + m_1)! (2j + 1)(j - m)!}{(j_1 + j_2 + j + 1)! (j_2 - j_1 + j)! (j_1 + j_2 - j)! (j_1 - m_1)! (j + m)!}}$$

$$\min\{j_1 - j_2 + j, j_2 - j_1 + j\} \sum_{q=0}^{\min\{j_1 - j_2 + j, j_2 - j_1 + j\}} (-1)^{j_1 - j_2 + m_1} (2j - q)! (j_1 + j_2 - j + q)! q! (j_1 - j_2 + j - q)! (j - m - q)! (j_2 - j + m_1 + q)!$$

(41)

The series representing Clebsch Gordon coefficient in (41) matches with the expansion given in (39). In section 3.2 this $SU(2)$ computation will be extended to $SU(N)$.

3 Invariants and representations of $SU(N) \times SU(N)$

We now generalize the previous $SU(2)$ ideas and techniques to direct product of two conjugate representations of $SU(N)$. For simplicity we choose these to be $N$ and $N^*$ representations of $SU(N)$. We write the corresponding generators as:

$$Q_1^a \equiv \frac{1}{2} a^+ \alpha (\Lambda^a)_{\alpha}^\beta a_{\beta}, \quad Q_2^a \equiv -\frac{1}{2} b^\dagger \alpha (\bar{\Lambda}^a)_{\alpha}^\beta b_{\beta}.$$  

(42)

In (42), $a = 1, 2, \ldots (N^2 - 1)$ and $\alpha, \beta = 1, 2 \ldots N$. $\Lambda^a$’s are the generalized Gell-Mann matrices for $N$-plets of $SU(N)$ and $-\bar{\Lambda}^a$ are the dual matrices corresponding to the $N^*$-plets of $SU(N)$. From (42) it is clear that $a^\dagger$’s transform as $N$ under one $SU(N)$ and $b^\dagger$’s transform as $N^*$ under another $SU(N)$.

Like in $SU(2)$ case (4) the decoupled $N$ and $N^*$ irreducible representations

---

Footnote:\footnote{4}{For $N \geq 3$ the $N$ and $N^*$ representations are not equivalent. Therefore, we now use upper $a^+\alpha$ and lower $b^\dagger_\alpha$ indices to differentiate between the two conjugate representations.}
Figure 2: Graphical or Young tableau representation of the identity (49). The coupled $SU(N) \times SU(N)$ states on the right hand side can be directly obtained by projection operators (see (50)) and also carry Sp(2,R) quantum numbers (see (57)).

\[
\begin{align*}
|\{n\}_a\rangle &\equiv |n_1^a, n_2^a, \ldots n_N^a\rangle = \frac{(a_1^a)^{n_1^a}(a_2^a)^{n_2^a} \cdots (a_N^a)^{n_N^a}}{\sqrt{n_1^a!n_2^a! \cdots n_N^a!}} |0\rangle, \\
|\{n\}_b\rangle &\equiv |n_1^b, n_2^b, \ldots n_N^b\rangle = \frac{(b_1^b)^{n_1^b}(b_2^b)^{n_2^b} \cdots (b_N^b)^{n_N^b}}{\sqrt{n_1^b!n_2^b! \cdots n_N^b!}} |0\rangle
\end{align*}
\]

(43)

In (43) \{n\} represents N partitions of n. The two Casimirs are the two total number operators $\hat{n}_a$ and $\hat{n}_b$ with eigenvalues $n_a$ and $n_b$ respectively:

\[
n_1^a + n_2^a + \cdots + n_N^a = n_a, \quad n_1^b + n_2^b + \cdots + n_N^b = n_b.
\]

(44)

We will often denote the $SU(N)$ direct product state $|\{n\}_a\rangle \otimes |\{n\}_b\rangle$ by $|n_1^a n_2^a \cdots n_N^a n_1^b n_2^b \cdots n_N^b\rangle$.

As in the case of $SU(2)$ (5), we define the total $SU(N)$ flux operators:

\[
Q_T^a = Q_1^a + Q_2^a.
\]

(45)

The corresponding group will be denoted by $SU(N)_T$. At this stage we can also define the coupled $SU(N) \times SU(N)$ states through the Clebsch Gordan decomposition (see Figure 2) as:

\[
|\{n\}_a\rangle \otimes |\{n\}_b\rangle = \sum_{r=0}^{\text{min}(n_a,n_b)} C_{\{n\}_a,\{n\}_b}^r |\{n_a - r\}_a\{n_b - r\}_b; r\rangle
\]

(46)

As in Figure 2, the coupled states denoted by $|\{n_a - r\}_a\{n_b - r\}_b; r\rangle$ in (46) represents the invariant operator $(k_+)^r$ acting on the completely traceless tensor state of rank $(n_a - r, n_b - r)$. 

11
We now define the following $SU(N) \times SU(N)$ invariant operators:

\[ k_+ \equiv a^\dagger \cdot b^\dagger, \quad k_- \equiv a \cdot b, \quad k_0 = \frac{1}{2}(\hat{n}_a + \hat{n}_b + N). \]  

(47)

In (47) the invariants are the scalar products of $N$ and $N^*$ representations: $a^\dagger \cdot b^\dagger = (a_1^1 b_1^1 + a_1^2 b_2^1 + \cdots + a_1^N b_N^1)$ and $a \cdot b = (a_1 b_1 + a_2 b_2 + \cdots + a_N b_N)$. It is easy to check that they again satisfy $Sp(2,\mathbb{R})$ algebra (7):

\[ [k_-, k_+] = 2k_0, \quad [k_0, k_\pm] = \pm k_\pm. \]  

(48)

Like in $SU(2)$ case, the $SU(N) \times SU(N)$ projection operators are defined as:

\[ \{n_a\} \otimes \{n_b\} = \sum_{r=0}^{\min(n_a, n_b)} P_r \{n_a\} \otimes \{n_b\}. \]  

(49)

Comparing (46) with (49) we get:

\[ P_r \{n_a\} \otimes \{n_b\} \equiv C_{\{n_a\}, \{n_b\}} \{n_a - r\}, \{n_b - r\}; r. \]  

(50)

In (50) $C_{\{n_a\}, \{n_b\}}$ are the Clebsch Gordan coefficients. Taking the norms on both the sides of (50) we get a simple expression for the $SU(N)$ Clebsch Gordan coefficients:

\[ C_{\{n_a\}, \{n_b\}} = \sqrt{\langle \{n_a\} | \otimes \langle \{n_b\} | P_r \{n_a\} \otimes \{n_b\} \rangle} \]  

(51)

We will explicitly compute these coefficients in section 3.2.

We now construct the projection operators defined in (50). Note that the condition of tracelessness is exactly same as demanding the constraint $k_- = 0$. Using this fact and the invariant algebra (48) the $SU(N)$ projection operator $P_0$ can be easily constructed like in $SU(2)$ case (see appendix A.2):

\[ P \equiv P_0 = \sum_{q=0}^{\infty} L_q \left( \hat{n}_a, \hat{n}_b \right) (k_+)^q (k_-)^q \]  

(52)

where

\[ L_q \left( \hat{n}_a, \hat{n}_b \right) = \frac{(-1)^q (\hat{n}_a + \hat{n}_b + N - 2 - q)!}{q! (\hat{n}_a + \hat{n}_b + N - 2)!}. \]  

(53)

Again, the projection operator in (52) satisfies

\[ P^2 = PP = \left( 1 - \frac{1}{\hat{n}_a + \hat{n}_b + N - 2} k_+ k_- + \cdots \right) P = P, \]  

as $k_- P = 0$: Note that the $SU(N)$ projection operator (52) reduces to the $SU(2)$ projection operator (19) at $N = 2$. Like in $SU(2)$ case, all other projection operators in (49) or equivalently in Figure 2 are of the form:

\[ P_r = N_r \ (k_+)^r \ P_0 \ (k_-)^r = N_r \ (k_+)^r \ P \ (k_-)^r \]  

(54)
The constant coefficients \( N_r \) are fixed by demanding that the operators \( \mathcal{P}_r \) satisfy: \( \mathcal{P}_r^2 = \mathcal{P}_r \) and are given by (see appendix A.2):

\[
N_r = \frac{(n_a + n_b + N - 2r - 1)!}{r!(n_a + n_b + N - r - 1)!}
\]  

(55)

As expected, (55) reduces to (22) at \( N = 2 \). Like in SU(2) case the projection operators satisfy the orthogonality and completeness properties:

\[
\sum_{r=0}^{\min(n_a,n_b)} \mathcal{P}_r = I, \quad \mathcal{P}_r \mathcal{P}_s = \delta_{rs} \mathcal{P}_r, \quad r, s = 0, 1, 2, \ldots \min(n_a, n_b).
\]  

(56)

The orthogonality relation can be proven exactly like in the SU(2) case (see (24)) and the completeness relation, manifest in (49), is proved in appendix A.2.

3.1 SU(N) × SU(N) irreducible Schwinger bosons

Like in the section 2.2 (also see [13, 14]), we define:

\[
A_\alpha^\dagger \equiv a_\alpha^\dagger + F(\hat{n}_a, \hat{n}_b)k_+ b_\alpha^\dagger, \quad B_\alpha^\dagger \equiv b_\alpha^\dagger + G(\hat{n}_a, \hat{n}_b)k_+ a_\alpha.
\]  

(58)

Note that by construction (58) the SU(N)_T transformation properties of \( A_\alpha^\dagger \) and \( B_\alpha^\dagger \) are exactly same as those of \( a_\alpha^\dagger \) and \( b_\alpha^\dagger \) respectively. We now demand:

\[
k_- A_\alpha^\dagger \mathcal{P}\{n_a^i\} \otimes \{n_b^i\} = 0, \quad k_- B_\alpha^\dagger \mathcal{P}\{n_a^i\} \otimes \{n_b^i\} = 0.
\]  

(59)

The above constraints can be solved in terms of the unknown functions \( F(\hat{n}_a, \hat{n}_b) \) and \( G(\hat{n}_a, \hat{n}_b) \):

\[
F(\hat{n}_a, \hat{n}_b) = G(\hat{n}_a, \hat{n}_b) = -\frac{1}{(\hat{n}_a + \hat{n}_b + N - 2)}.
\]  

(60)
Therefore, explicit normalization of the above state gives:

\[
\left| \{ n_a \}, \{ n_b \}; r = 0 \right> = N_{\{ n_a \}, \{ n_b \}}^{\{ n_a \}} \frac{(A^1_{11})^{n_a^1} \cdots (A^{1N})^{n_a^N} (B^1_1)^{n_b^1} \cdots (B^1_N)^{n_b^N}}{\sqrt{(n_a^1)!(n_a^2)\cdots(n_a^N)!(n_b^1)!(n_b^2)\cdots(n_b^N)!}} |0\rangle
\]

\[
= N_{\{ n_a \}, \{ n_b \}}^{\{ n_a \}} \mathcal{P} \frac{(a^{11})^{n_a^1} \cdots (a^{1N})^{n_a^N} (b^1_1)^{n_b^1} \cdots (b^1_N)^{n_b^N}}{\sqrt{(n_a^1)!(n_a^2)\cdots(n_a^N)!(n_b^1)!(n_b^2)\cdots(n_b^N)!}} |0\rangle
\]

\[
= N_{\{ n_a \}, \{ n_b \}}^{\{ n_a \}} \mathcal{P} \{ n_a^i \} \otimes \{ n_b^i \}
\]

(61)

The first state of the Clebsch Gordon series for \( SU(N) \times SU(N) \) as given in Figure 2 can be easily defined through the irreducible Schwinger bosons:

\[
\left| \{ n_a \}, \{ n_b \}; r = 0 \right> = N_{\{ n_a \}, \{ n_b \}}^{\{ n_a \}} (A^1_{11})^{n_a^1} \cdots (A^{1N})^{n_a^N} (B^1_1)^{n_b^1} \cdots (B^1_N)^{n_b^N} |0\rangle
\]

The first state of the Clebsch Gordon series for \( SU(3) \) states (43) with partitions: \( n_a^1 = 1, n_a^2 = 0; n_b^1 = 1, n_b^2 = 1, n_b^3 = 0 \). In (61) we replace the irreducible Schwinger bosons by their defining equation (58) and (30) to get,

\[
\left| \{ n_a^1 = 1, n_a^2 = 0, n_a^3 = 0 \}, \{ n_b^1 = 1, n_b^2 = 1, n_b^3 = 0 \}; r = 0 \right> = N_{1,1,0,0}^{1,0,0,0} A^1_B B^{1\dagger} |0\rangle
\]

\[
= N_{1,1,0,0}^{1,0,0,0} \left[ \begin{array}{c} 3 \\ 4 \\ 1 \\ 1 \\ 0 \\ 0 \end{array} \right] = \sqrt{\frac{2!}{4!3!3!}} \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right] = \sqrt{\frac{3}{4}} \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right]
\]

Therefore, explicit normalization of the above state gives: \( N_{1,1,0,0}^{1,0,0,0} = \sqrt{\frac{3}{4}} \). On the other hand, this normalization can also be computed by using (62) and the SU(3) Clebsch Gordan expression (68) obtained in the next section. Putting the above values of occupation numbers and \( r = 0 \) in (68) we get:

\[
(N_{1,1,0,0}^{1,0,0,0})^{-1} = C_{\{ n_a^1 = 1, n_a^2 = 0, n_a^3 = 0 \}, \{ n_b^1 = 1, n_b^2 = 1, n_b^3 = 0 \}}^{0,0,0,0} = \sqrt{\frac{2!}{4!3!3!}} (4! - 3!) = \sqrt{\frac{3}{4}}.
\]
Infact, at this stage we can cross check the other values of the $SU(N)$ Clebsch Gordan coefficients present in (63) with their values computed from the $SU(N)$ Clebsch Gordan expression (68) in the next section. The decomposition (63) implies

$$C_{\{00\},\{00\}}^{r=0} = -\frac{1}{\sqrt{6}} \text{ and } C_{\{01\},\{01\}}^{r=0} = -\frac{1}{\sqrt{12}}.$$  

As can be checked, these are also the values obtained from (68) after putting $N = 3$, various occupation numbers and $r = 0$. Thus the above simple state provides three self consistency checks on our procedure.

The discussions in the previous section and Figure 2 imply that an arbitrary coupled state can be written as:

$$|\{n_a\}, \{n_b\}; r\rangle = \mathcal{N}_{n_a,n_b}^{r} (k_{+})^{r} |\{n_a - r\}, \{n_b - r\}; r = 0\rangle$$  

The normalization constants $\mathcal{N}_{n_a,n_b}^{r}$ can be easily computed as $k_{-}|\{n_a\}, \{n_b\}; r = 0\rangle = 0$ and are given by:

$$\mathcal{N}_{n_a,n_b}^{r} = \frac{(n_a + n_b + N - 1)!}{r!(n_a + n_b + N + r - 1)!}.$$  

We again emphasize that except the invariant term all the $SU(N) \times SU(N)$ coupled states in (37) are monomials of the irreducible Schwinger bosons. The present construction of coupled states is a straightforward generalization of the original construction to the decoupled $SU(2)$ angular momentum states [3].

We note that the $SU(N) \times SU(N)$ irreducible Schwinger bosons satisfy:

$$[A^\dagger_\alpha, A^\beta_\beta] = 0, \quad [B^\dagger_\alpha, B^\beta_\beta] = 0, \quad [A^\dagger_\alpha, B^\dagger_\beta] = 0$$

$$[A_\alpha, A^{\dagger_\beta}] = \delta^\beta_\alpha - \frac{1}{n_a + n_b + N - 1} b^\dagger_\alpha b^\beta + \frac{1}{(n_a + n_b + N - 1)(n_a + n_b + N - 2)} k_{+} a_\alpha b^\beta$$

$$[B^\alpha, B^{\dagger_\beta}] = \delta^\alpha_\beta - \frac{1}{n_a + n_b + N - 1} a^{\dagger_\alpha} a_\beta + \frac{1}{(n_a + n_b + N - 1)(n_a + n_b + N - 2)} k_{+} a_\beta b^\alpha.$$  

These relations reduce to (38) for $N = 2$.

### 3.2 The Projection operators and $SU(N)$ Clebsch Gordon Coefficients

We write [49] and [50] as

$$|\{n_a\} \otimes |\{n_b\}\rangle = \sum_{r=0}^{n} P_r |\{n_a\}\rangle \otimes |\{n_b\}\rangle \equiv \sum_{r=0}^{n} C_{\{n_a\},\{n_b\}}^{r} |\{n_a - r\}, \{n_b - r\}; r\rangle$$
where, \( n = \min(n_a, n_b) \). Hence the Clebsch Gordon Coefficients can be computed as in the \( SU(2) \) case:

\[
C^{r}_{\{n_a\},\{n_b\}} = \left\langle n_a, 0, \ldots, 0 \right| \otimes \left\langle \bar{n}_b, \bar{n}_b^2, \ldots, \bar{n}_b^N \right| \mathcal{P}_r \left| n_a^n, n_a^2, \ldots, n_a^N \right\rangle \otimes \left| n_b^1, n_b^2, \ldots, n_b^N \right\rangle \left[ \left\langle n_a, 0, \ldots, 0 \right| \otimes \left\langle \bar{n}_b, \bar{n}_b^2, \ldots, \bar{n}_b^N \right| \mathcal{P}_r \left| n_a, 0, \ldots, 0 \right\rangle \otimes \left| \bar{n}_b^1, \bar{n}_b^2, \ldots, \bar{n}_b^N \right\rangle \right]^\frac{1}{2} \tag{67}
\]

In the above equation \( \{\bar{n}_b^1, \cdots \bar{n}_b^N\} \) are the values of the occupation numbers corresponding to the special choice \( \{n_a^1 = n^a, 0, 0, \ldots, 0\} \) so that the total magnetic quantum numbers on both sides of the projection operator remain unchanged\(^5\). They are given by:

\[
\bar{n}_b = n_a - n_a^1 + n_b^1 \quad \text{and} \quad \bar{n}_b^i = n_b^1 - n_a^i \quad i = 2, 3, \ldots, N
\]

As shown in appendix B.2, the matrix elements of \( \mathcal{P}_r \) in (67) can be easily computed to give,

\[
C^{r}_{\{n_a\},\{n_b\}} = \sum_q (-1)^q \frac{q + r)!}{q!} \frac{(n_a + n_b + N - 2r - 1)!n_a^1!n_b^2! \cdots n_b^N!}{r!(n_a + n_b + N - r - 1)!n_a^1!n_b^2! \cdots n_a^N!n_b^1! \cdots n_b^N!} \tag{68}
\]

Note that this \( SU(N) \) Clebsch Gordon series reduces to the \( SU(2) \) Clebsch Gordon series \( (41) \) for \( N = 2 \). This can be checked by identifying \( (b_2^\dagger, b_1^\dagger) \) of \( SU(N) \) with \((b_1^\dagger, -b_2^\dagger) \) of \( SU(2) \) respectively so that \( a^1 \cdot b^1 \ (SU(N) \text{ invariant}) \rightarrow a^1 \cdot b^1 \ (SU(2) \text{ invariant}) \) and putting:

\[
\begin{align*}
n_a^1 &= j_1 + m_1 \quad n_b^1 = j_2 - m_2 \quad \bar{n}_b^1 = j_2 - (m - j_1) \\
n_a^2 &= j_1 - m_1 \quad n_b^2 = j_2 + m_2 \quad \bar{n}_b^2 = j_2 + (m - j_1).
\end{align*}
\]

The \( SU(2) \) Casimirs in (68) are: \( n_a = n_a^1 + n_a^2 = 2j_1 \), \( n_b = n_b^1 + n_b^2 = \bar{n}_b^1 + \bar{n}_b^2 = 2j_2 \) and \( r = j_1 + j_2 - j \).

\(^5\)Note that the \( SU(N) \) states \( |n^1, n^2, \ldots, n^N\rangle \) in \( (43) \) can also be characterized by \( SU(N) \) Casimir \( n = n^1 + n^2 + \cdots + n^N \) along with the \( SU(N) \) magnetic quantum numbers \( \{h_i\} (i = 1, 2, \ldots, (N - 1)) \) as:

\[
\begin{align*}
h_a^1 &= n_a^1 - n_a^2 \\
h_a^2 &= n_a^1 + n_a^2 - 2n_a^3 \\
\vdots \\
h_a^{N-1} &= n_a^1 + n_a^2 + \cdots + n_a^{N-1} - (N - 1)n_a^N \\
h_b^1 &= n_b^1 - n_b^2 \\
h_b^2 &= 2n_b^3 - n_b^1 - n_b^2 \\
\vdots \\
h_b^{N-1} &= (N - 1)n_b^N - n_b^1 - n_b^2 - \cdots - n_b^{N-1}.
\end{align*}
\]
4 Summary and discussions

In this work, we have investigated the role of $SU(N) \times SU(N)$ invariant groups in the Clebsch Gordan decomposition of direct product of two $SU(N)$ irreducible representations. The techniques were completely based on the invariant groups and their algebras enabling us to handle all $SU(N)$ within a single framework. It was crucial to use Schwinger construction to get all possible invariants. The invariant group generators were used to construct projection operators to get all possible coupled irreducible representations. The $SU(N)$ Clebsch Gordan coefficients were computed as matrix elements of these projection operators. Using the invariant algebra we also constructed $SU(N) \times SU(N)$ irreducible Schwinger bosons which directly creates the coupled irreducible states. Note that in the case of $SU(N)$ ($N \geq 3$) we only considered direct product of $N$ and $N^*$ representations leading to $Sp(2,R)$ invariant algebras. In fact the analysis of section 3 is also valid for any two $SU(N)$ fundamental conjugate representations of dimensions $N_C^r$ each. For simplicity we had chosen $r = 1$. It will be interesting to extend these techniques to direct product of two arbitrary $SU(N)$ irreducible representations. The invariant group involved will then be much larger. All possible projection operators and the irreducible Schwinger bosons will again depend on the invariant operators or generators of the invariant group. The work in this direction is in progress and will be reported elsewhere.
A The projection operators

In this appendix we construct and prove the completeness property of $SU(N) \times SU(N)$ projection operators.

A.1 $SU(2) \times SU(2)$

We start with the construction of projection operator associated with symmetrization:

$$\mathcal{P}(|j_1, m_1 \rangle \otimes |j_2, m_2 \rangle) = C_{j_1, j_2, j_1, j_2, j_1, j_2, j, j} (j_1, j_2, j = j_1 + j_2, m = m_1 + m_2)$$

(69)

We note that the $SU(2) \times SU(2)$ transformation property as well as the total number of $a^\dagger$'s ($= 2j_1 = n_a$) and $b^\dagger$'s ($= 2j_2 = n_b$) are same for the decoupled and coupled states on the left and right hand side of (69). Therefore the projection operator is of the form:

$$\mathcal{P} = \sum_{q=0}^{\min\{n_a, n_b\}} l_q(\hat{n}_a, \hat{n}_b) k_+^q k_-^q$$

(70)

where, $k_\pm$ are the $SU(2)$ invariant $Sp(2, \mathbb{R})$ operators defined in \[6\]. The unknown coefficients $l_q(\hat{n}_a, \hat{n}_b)$ can be easily fixed by demanding that the projected state is completely symmetric in all the $SU(2)$ indices and therefore should be annihilated by $k_-, \ i.e.$

$$k_- \mathcal{P} |j_1, m_1 \rangle \otimes |j_2, m_2 \rangle = 0.$$ \hspace{1cm} (71)

After using $k_\pm k_q^\pm = [k_-, k_q^+] = q(\hat{n}_a + \hat{n}_b - q + 3)k_+^{q-1}$ we get the recurrence relation:

$$l_{q+1}(\hat{n}_a, \hat{n}_b) = -\frac{1}{(q + 1)(\hat{n}_a + \hat{n}_b - q)} l_q(\hat{n}_a, \hat{n}_b),$$

leading to:

$$l_q(\hat{n}_a, \hat{n}_b) = \frac{(-1)^q (\hat{n}_a + \hat{n}_b - q)!}{q! (\hat{n}_a + \hat{n}_b)!}$$ \hspace{1cm} (72)

Note that $l_0(\hat{n}_a, \hat{n}_b) = 1$ implying the projection operator $\mathcal{P}^2 = \mathcal{P}$.

To compute the $\mathcal{P}_r$ normalization coefficients in (22) we use the required $\mathcal{P}_r^2 = \mathcal{P}_r$ property:

$$\mathcal{P}_r \mathcal{P}_r = N_r^2 \{(k_+)^r \mathcal{P} (k_-)^r\} \{ (k_+)^r \mathcal{P} (k_-)^r\} = N_r^2 (k_+)^r \mathcal{P} [(k_-)^r, (k_+)^r] \mathcal{P} (k_-)^r = N_r^2 (k_+)^r \mathcal{P} (k_-)^r = \mathcal{P}_r.$$

(73)

In (73) we have used $k_- \mathcal{P} = 0$ to replace $k_\pm k_q^\pm$ by the commutator $[k_-^r, k_+^r] = r! (\hat{n}_a + \hat{n}_b + r + 1)! (\hat{n}_a + \hat{n}_b - r)!$ and replaced the number operators $\hat{n}_a, \hat{n}_b$ by their eigenvalues $n_a = 2j_1, n_b = 2j_2$ at the end. The above eqn. gives:

$$N_r = \frac{(n_a + n_b - 2r + 1)!}{r!(n_a + n_b - r + 1)!}$$ \hspace{1cm} (74)
One can easily check that \( P_r P_{s \neq r} = 0 \) as \( k_- \mathcal{P} = 0(r > s) \) and \( \mathcal{P} k_+ = 0(r < s) \) leading to orthonormal irreducible Hilbert spaces characterized by the net angular momentum quantum numbers.

We now prove the completeness property of the projection operators \( P_r \). We start with:

\[
\sum_r P_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_r P_r \sum_{j=|j_1-j_2|}^{j_1+j_2} C'_{j_1,m_1;j_2,m_2} |j_1,j_2;j,m\rangle
\]

\[
= \sum_r P_r \sum_{j=|j_1-j_2|}^{j_1+j_2} C'_{j_1,m_1;j_2,m_2} N_{j,m} (J^-)^{j-m} |j_1,j_2;j,m = j\rangle
\]

\[
= \sum_{j=|j_1-j_2|}^{j_1+j_2} C'_{j_1,m_1;j_2,m_2} N_{j,m} (J^-)^{j-m} \sum_r |j_1,j_2;j,m = j\rangle = P_0 |j_1,j_2;j,m = j\rangle.
\]

(75)

In (75) the total lowering operator is defined as \( J^- \equiv J^-_a + J^-_b \equiv a_1^\dagger a_2 + b_1^\dagger b_2 \) and \( N_{j,m} = \sqrt{(j+m)!/(2j! (j-m)!)} \) are the corresponding constants. We have also used the fact that the projection operators \( P_r \) commute with \( J^- \) and satisfy orthonormality condition (24).

### A.2 SU(N) × SU(N)

We can exactly follow the \( SU(2) \) techniques of the previous section and use the relation \( [k_-, k^q_+] = q(\hat{n}_a + \hat{n}_b + N + 1 - q)k^q_+ \) to obtain the \( SU(N) \times SU(N) \) projection operator [53]:

\[
L_q(n_a, n_b) = \frac{(-1)^q (n_a + n_b + N - 2 - q)!}{q! (n_a + n_b + N - 2)!}
\]

Similarly, as in \( SU(2) \) case [24]:

\[
P_r \mathcal{P} P_r = N_r^2 \{k_+^r \mathcal{P} k_+^r \} \{k^-_+ \mathcal{P} k^-_+ \} = N_r^2 k_+^r \mathcal{P} [k_+^r, k^-_+] \mathcal{P} k_+^r
\]

\[
= N_r r! \frac{(n_a + n_b + N - r - 1)!}{(n_a + n_b + N - r - 1)!} P_r \equiv P_r.
\]

(76)

Above we have used the relation \( [k_+^r, k_+^r] = \frac{r!(\hat{n}_a + \hat{n}_b + N + r - 1)!}{(n_a + n_b + N - 1)!} \). We thus get:

\[
N_r = \frac{(n_a + n_b + N - 2r - 1)!}{r!(n_a + n_b + N - r - 1)!}
\]

(77)

As in \( SU(2) \) case the different projected or irreducible spaces are orthonormal: \( P_r P_{s \neq r} = 0 \) as \( k_- \mathcal{P} = 0(r > s) \) and \( \mathcal{P} k_+ = 0(r < s) \). The completeness property of the \( SU(N) \) projection operators also follows exactly as in the \( SU(2) \) case.
B Matrix elements of Projection operators

In this appendix we compute the matrix elements of projection operators in (40) and (67) to get the $SU(2)$ and $SU(N)$ Clebsch Gordan coefficients.

B.1 $SU(2) \times SU(2)$

The numerator in (40) is:

$$\langle j_1, j_1, j_2, m - j_1 | \mathcal{P}_r | j_1, m_1, j_2, m_2 \rangle = N_r \begin{vmatrix} j_1 & 2j_1 & 0 \\ j_2 + m - j_1 & j_1 + j_2 - m & k^+ \mathcal{P} k^- \\ j_2 + m_1 & j_1 - m_1 \end{vmatrix} = K(j_1, m_1, j_2, m_2, q, r)$$

In the first step we have written the decoupled angular momentum states in terms of the occupation number basis. In the second step we have substituted the expansion (70) of $\mathcal{P}$ with $n_a = 2j_1 - r$, $n_b = 2j_2 - r$ for the coefficient $l_q$ in (72). Note that the matrix elements $K$ can be easily computed as both $k^+ k^-$ and $k^+ k^-$ in (78) can be replaced by monomials of harmonic oscillator creation and annihilation operators respectively:

$$k^+ \rightarrow (a_1 b_2)^{q+r}, \quad k^+ \rightarrow (-1)^{q+r-s} C_s (a_1 b_2)^s (a_2 b_1)^{q+r-s}.$$

Above $s = q + r + m_1 - j_1$. Substituting these monomials in (78) leads to:

$$K = (-1)^{q+r-s}(q+r)! \frac{s!(q+r-s)!(j_1 + m_1 - s)!(j_2 - m_2 - s)!(j_1 - m_1 - q - r + s)!(j_2 + m_2 - q - r + s)!}{(2j_1)!(j_2 + m - j_1)!(j_2 - m + j_1)!(j_1 + m_1)!(j_2 + m_2)!(j_2 - m_2)!}. \quad (79)$$

Substituting $N_r$ from (74), $l_q(2j_1 - r, 2j_2 - r)$ from (72) and $K$ from above with $s = q + r + m_1 - j_1$, the matrix element (78) takes the form:

$$\langle j_1, j_2, m - j_1 | \mathcal{P}_r | j_1, m_1, j_2, m_2 \rangle = \frac{(2j_1 + 2j_2 - 2r + 1)!}{r!(2j_1 + 2j_2 - r + 1)!(2j_1 + 2j_2 - 2r)!} \times \sqrt{\frac{(2j_1)!(j_2 - m + j_1)!(j_1 + m_1)!(j_2 + m_2)!(j_2 - m_2)!}{(j_2 + m - j_1)!(j_1 - m_1)!(j_2 - m_2)!}} \sum_{q=0}^{\min(2j_1-r,2j_2-r)} \frac{(-1)^q}{(q + r + m_1 - j_1)!(j_1 + j_2 - m - q - r)!(2j_1 - q - r)!} \quad (80)$$
Putting $r = j_1 + j_2 - j$ in the above equation we get:

$$\langle j_1, j_2, m - j_1 | P_r | j_1, m_1, j_2, m_2 \rangle = \left[ \frac{(2j + 1)!}{(2j)! (j_1 + j_2 - j)!(j_1 + j_2 + j + 1)!} \right] \sqrt{\frac{(2j_1)! (j_1 + j_2 - m)! (j_1 + m_1)! (j_2 + m_2)! (j_2 - m_2)!}{(j_1 - m_1)! (j_2 - j_1 + m)!}}$$

$$\sum_{q=0}^{\min(j_1-j_2+j_2-j_1+j)} (-1)^{q+j_1-m_1} \frac{(j_1 + j_2 - j + q)!(2j - q)!}{(q)! (j_2 - j + q + m_1)!(j_1 - j_2 + j - q)!(j - m - q)!}.$$  \hspace{1cm} (81)

For the denominator of (40), we substitute $m_1 = j_1$ and $m_2 = m - j_1$ in (81) to obtain,

$$\langle j_1, j_1, j_2, m - j_1 | P_r | j_1, j_1, j_2, m - j_1 \rangle$$

$$= \frac{(2j + 1)! (2j_1)! (j_2 + j_1 - m)!}{(2j)! (j_1 + j_2 - j)!(j_1 + j_2 + j + 1)!} \sum_{q=0}^{q_{\text{max}}} (-1)^q \frac{(2j - q)!}{q! (j_1 - j_2 + j - q)!(j - m - q)!}. \hspace{1cm} (82)$$

In (82) the upper limit on the sum over q is $q_{\text{max}} \equiv \min(2j_1 - r, 2j_2 - r) = \min(j_1 - j_2 + j, j_2 - j_1 + j)$. This above series in q is summed using the formula:

$$\sum_{q=0}^{q_{\text{max}}} (-1)^q \frac{(C - q)!}{q! (A - q)!(B - q)!} = C! A!B! (C - A)! (C - B)! \times \frac{(C - A)!(C - B)!}{C!(C - A - B)!}.$$  \hspace{1cm} (83)

Finally, the denominator in (40) is:

$$\sqrt{\langle j_1, j_1, j_2, m - j_1 | P_r | j_1, j_1, j_2, m - j_1 \rangle}$$

$$= \sqrt{(j + m)! (j_2 - j_1 + j)!(2j + 1)!(2j_1)! (j_2 + j_1 - m)!}$$

$$\sqrt{(j_1 + j_2 - j)!(j_1 + j_2 + j + 1)!(j - m)! (j_2 - j_1 + m)! (j_1 - j_2 + j)!}.$$  \hspace{1cm} (84)

The final expression of the Clebsch Gordon coefficient in (41) is now obtained by dividing (81) by (84).

### B.2 SU(N) × SU(N)

Similarly the matrix element in the numerator of $SU(N)$ Clebsch Gordon coefficient expression \[67\] is:

$$\langle \{ n_a^1 = n_a \}, \{ \bar{n}_b \} | P_r P_r | \{ n_a \}, \{ n_b \} \rangle$$

$$= N_r \sum_{q} l_q (n_a - r, n_b - r) \left[ \begin{array}{c} n_a \ 0 \ \cdots \ 0 \\ \bar{n}_b \ n_b^1 \ \cdots \ n_b^N \end{array} \right]_{K((n_a), (n_b), q, r)}$$

$$= N_r \sum_{q} l_q (n_a - r, n_b - r) K((n_a), \{ n_b \}, q, r)$$  \hspace{1cm} (85)
The matrix element $K(\{n_a\}, \{n_b\}, q, r)$ are calculated in the same way as in the $SU(2)$ case. In the computation of $K(\{n_a\}, \{n_b\}, q, r)$ in (85) $k^{q+r}_+$ and $k^{q+r}_-$ can be replaced by the following monomials of Schwinger bosons:

$$k^{q+r}_+ \rightarrow (a_1^1 b_1^{11})^q r_+^r, \quad k^{q+r}_- \rightarrow \frac{(q + r)!}{\beta_1! \cdots \beta_N!} (a_1^1 b_1^{11})^{\beta_1} (a_2^2 b_2^{2})^{\beta_2} \cdots (a_N^N b_N^{N})^{\beta_N}$$ (86)

Equating the occupation numbers in the matrix element in (85) we get:

$$\beta_1 = q + r + n_a^+ - n_a^-, \quad \beta_2 = n_a^2, \quad \beta_3 = n_a^3, \quad \ldots, \beta_N = n_a^N,$$

leading to:

$$K(\{n_a\}, \{n_b\}, q, r) = \sqrt{n_a! n_b^1! n_b^{2}! \cdots n_b^N!} n_a^1! n_a^2! \cdots n_a^N! n_b^1! n_b^{2}! \cdots n_b^N!$$

$$\times \frac{(q + r)!}{(n_a^+ - n_a^- + q + r)! n_a^2! \cdots n_a^N!}$$

$$\times \frac{1}{(n_a - q - r)! (\bar{n}_b^1 - q - r)! n_b^2! \cdots n_b^N!}$$ (87)

Now substituting the values of $N_a$ and $l_q(n_a - r, n_b - r)$ from A.2 and the matrix element $K$ from above we finally get the numerator of (67) as:

$$\langle \{n_a^1 = n_a\}, \{\bar{n}_b\} | \mathcal{P}_r | \{n_a\}, \{n_b\} \rangle = \frac{(n_a + n_b + N - 2r - 1)!}{r!(n_a + n_b + N - r - 1)!} \sqrt{n_a! n_b^1! n_b^{2}! \cdots n_b^N!}$$

$$\times \frac{n_a^1! n_a^2! \cdots n_a^N! n_b^1! n_b^{2}! \cdots n_b^N!}{q!(n_a - q - r)! (\bar{n}_b^1 - q - r)! (n_a^+ - n_a^- + q + r)!}$$ (88)

Like in $SU(2)$ case, the denominator of (67) is the square-root of the numerator with $n_a^1 = n_a$, $n_a^2 = n_a^3 = \cdots = n_a^N = 0$ and $n_b^i = \bar{n}_b^i$, $\forall i$. The final expression for the denominator in (67) is:

$$\langle \{n_a^1 = n_a\}, \{\bar{n}_b\} | \mathcal{P}_r | \{n_a\}, \{n_b\} \rangle = \frac{(n_a + n_b + N - 2r - 1)!}{r!(n_a + n_b + N - r - 1)!} \sqrt{n_a! n_b^1! n_b^{2}! \cdots n_b^N!}$$

$$\times \sum_q (-1)^q(n_a + n_b + N - 2 - 2r - q)! q!(n_a - q - r)! (\bar{n}_b^1 - q - r)! (n_a^+ - n_a^- + q + r)!$$ (89)

In (89) the last sum has been performed using (83) again. Finally, the $SU(N)$ Clebsch Gordon coefficient expansion (68) is obtained by dividing (88) with square root of (89).
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[17] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonsky, Singapore, Singapore: World Scientific (1988) 514p (The Clebsch Gordan coefficient expansion (41) matches exactly with the expansion (6) of section 8.2 (page 238) after using the identity $C_{j_1, m_1; j_2, m_2, m}$).