Axion electrodynamics in $p + is$ superconductors

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We perform a systematic study of axion electrodynamics in $p + is$ superconductors. Unlike the superconducting Dirac/Weyl systems, the induced electric field does not enter into the axion action. Furthermore, in addition to the usual axion angle which is defined as the phase difference between the superconducting phases on the two Fermi surfaces of different helicities, the axion field contains an additional sinusoidal term. Our work reveals the differences for axion electrodynamics between the relativistic cases and the $p + is$ superconductors.

I. INTRODUCTION

Spin triplet superconductivity and paired superfluidity have a complex spin-orbit entangled structure in the Cooper pair wavefunctions,\textsuperscript{11,12} leading to exotic behaviors like topological properties.\textsuperscript{12,13} Another interesting type of Cooper pairing is the one with spontaneous time reversal symmetry breaking which arises when two or more channels of pairing instabilities compete and coexist.\textsuperscript{12,14} A mixture of triplet and singlet Cooper pairings which breaks time reversal symmetry has been theoretically studied in different contexts, showing exotic properties including nontrivial bulk electromagnetic and gravitational response.\textsuperscript{12,13} Quantized surface thermal Hall effects,\textsuperscript{12,15} chiral Majorana fermions propagating along the magnetic domain wall on the surface,\textsuperscript{16} and high order topology.\textsuperscript{17} Recently, there has been experimental evidence of triplet pairing gap functions with spontaneous time reversal symmetry breaking in real material.\textsuperscript{18,19}

Axion as an elementary particle was proposed in the high energy context more than four decades ago,\textsuperscript{20,21} which has been considered as a candidate for dark matter and dark energy, thought its existence still remains inconclusive. On the other hand, the dynamical axion field has been proposed to exist in topological systems as a condensed matter realization of axions.\textsuperscript{22,23,24} The coupling between the axion angle $\theta$ and the electromagnetic field is of the form $\int d^4x \partial\bar{\theta} \cdot \vec{B}$ (up to an overall numerical constant factor), resulting in various magnetoelectric effects, where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively. In particular, the three-dimensional (3D) $p + is$ superconductor has been considered as a superconducting platform which hosts axion field,\textsuperscript{23,25} where the triplet pairing component is invariant under spin-orbit coupled rotations analogous to the pairing of the $^3$He-B superfluid.

In this work, we perform a systematic derivation of the coupling between the axion angle and the electromagnetic field in $p + is$ superconductors, including the contributions from both the orbital and spin channels. The axion electrodynamics has been already studied in superconducting 3D Dirac and Weyl systems with mixed-parity pairing which breaks time reversal symmetry.\textsuperscript{26,27} On the other hand, we find that there are several differences between the $p + is$ superconductors and the superconducting Dirac/Weyl case. Firstly, the electromagnetic part of the action is not of the $\vec{E} \cdot \vec{B}$ form, but is $\nabla(\phi - \hbar \partial_t \hat{\Phi}/e) \cdot \vec{B}$, where $\phi$ and $\hat{\Phi}$ are the electric potential and the superfluid phase, respectively. In particular, the induced electric field $\partial_t \hat{A}/c$ does not appear in the action. The second difference is that the axion angle $\Theta_{ax}$ in the $p + is$ superconductors is not just $\theta_{ax}$, defined as the phase difference between the superconducting phases on the two Fermi surfaces of different helicities. In addition to $\theta_{ax}$, $\Theta_{ax}$ also acquires a sinusoidal term $\sin(\theta_{ax})$. These differences arise from a lack of Lorentz symmetry in $p + is$ superconductors.

II. MODEL HAMILTONIAN

We consider a superconducting 3D centrosymmetric electronic system which exhibits a mixture of singlet and triplet pairing symmetries. The band dispersion is

$$\xi_{\alpha}(\vec{k}) = \frac{\hbar^2}{2m} k^2 - \epsilon_F, \quad (1)$$

in which $\epsilon_F = \frac{\hbar^2}{2m} k_f^2$ is the Fermi energy where $k_f$ is the Fermi wavevector, and $\alpha = \uparrow, \downarrow$ is the spin index. The pairing Hamiltonians $\hat{P}_s(\vec{k})$ and $\hat{P}_p(\vec{k})$ for the s-wave and $^3$He-B like p-wave pairing gap functions are defined as

$$\hat{P}_s(\vec{k}) = (i \sigma_2)_{\alpha \beta} c_{\alpha}^\dagger(\vec{k}) c_{\beta}(\vec{k}),$$
$$\hat{P}_p(\vec{k}) = \frac{1}{k_f} k_{j} (i \sigma_j \sigma_2)_{\alpha \beta} c_{\alpha}^\dagger(\vec{k}) c_{\beta}(\vec{k}), \quad (2)$$

respectively, in which: $\alpha, \beta$ are spin indices; $c_{\alpha}^\dagger(\vec{k})$ is the electron creation operator with momentum $\vec{k}$ and spin $\alpha$; $\sigma_j$’s ($j = 1, 2, 3$) are the three Pauli matrices in spin space; and repeated indices imply summations. We note that $\hat{P}_p(\vec{k})$ is invariant under spin-orbit coupled SO(3) rotations, which has the same form as the pairing in the $^3$He-B superfluid.\textsuperscript{16}
The pattern of the mixed-parity pairing gap function can be determined by a Ginzburg-Landau free energy analysis. Keeping up to quartic terms and neglecting terms involving temporal and spatial derivatives, the most general form of the free energy invariant under both time reversal \((T)\) and inversion \((P)\) symmetries is given by

\[
F = -\alpha_s \Delta_3 \lambda s - \alpha_p \Delta_3 \lambda p + \beta_s |\Delta s|^4 + \beta_p |\Delta p|^4 + \gamma_1 |\Delta p|^2 |\Delta s|^2 + \gamma_2 (\Delta_3 \lambda s \Delta_3 \lambda p + c.c.),
\]

where \(\Delta_3 (\lambda = s, p)\) are the pairing gap functions in the \(\lambda\)-channel. When the instabilities in the \(s\) and \(p\)-wave channels coexist, both \(\alpha_s\) and \(\alpha_p\) are negative. At tree level, the coefficients \(\beta_s, \gamma_j (\lambda = s, p, j = 1, 2)\) are all determined by the electronic band structure, independent of the interactions. In particular, close to the superconducting transition point, \(\gamma_2 = 5\frac{\zeta(3)}{\pi^2 T_F} N_F\) is generically positive \(^{23}\) where \(\zeta(z)\) \((z \in \mathbb{C})\) is the Riemann zeta function; \(T_F\) is the superconducting transition temperature (where for simplicity, a degenerate transition temperature for both \(s\) and \(p\)-wave channels is assumed); and \(N_F = \frac{m k_F}{\pi^2 \hbar^2}\) is the density of states at the Fermi level for a single spin component. An important implication of a positive \(\gamma_2\) is that a relative \(\pi/2\) phase difference between \(\Delta_s\) and \(\Delta_p\) is energetically favorable as can be readily seen from Eq. \(3\), leading to a superconducting pairing of the \(p \pm is\) form. Notice that the \(p \pm is\) pairing spontaneously breaks both time reversal and inversion symmetries, but remains invariant under the combined \(PT\)-operation up to an overall gauge transformation \(^{24}\)

We note that in addition to the intrinsic \(p \pm is\) superconductors, the \(p \pm is\) pairing symmetry can also be realized via proximity effects, as discussed in Ref. \(^{21}\).

In the remaining parts of this article, \(|\Delta_3|\) is denoted as \(\Delta_3\), for short, and the \(\pi/2\) superconducting phase difference will be explicitly displayed by considering the \(p \pm is\) pairing Hamiltonian \(\Delta_p \Psi_p(\vec{k}) + i \Delta_s \Psi_s(\vec{k})\). In the Bogoliubov-de-Gennes (BdG) formalism, the mean field Hamiltonian acquires the form

\[
H_{\text{BdG}} = \frac{1}{2} \sum_{\vec{k}} \Psi^\dagger(\vec{k}) H(\vec{k}) \Psi(\vec{k}),
\]

in which

\[
\Psi^\dagger(\vec{k}) = (c_{\uparrow}^\dagger(\vec{k}) c_{\downarrow}^\dagger(\vec{k}) c_{\uparrow}(-\vec{k}) c_{\downarrow}(-\vec{k})),
\]

and the matrix kernel \(H(\vec{k})\) is

\[
H(\vec{k}) = \left( \frac{\hbar^2}{2m} k^2 - \epsilon_F \right) \gamma^0 + \Delta_3 \lambda \vec{\gamma} \cdot \vec{\gamma} + \Delta_s \gamma^4,
\]

where \(\gamma_i = (\gamma^1, \gamma^2, \gamma^3)\), and the matrices \(\gamma^\mu\) \((\mu = 0, 1, 2, 3, 4)\) are defined as

\[
\gamma^0 = \sigma_0 \tau_3, \quad \gamma^1 = -\sigma_3 \tau_1, \quad \gamma^2 = -\sigma_0 \tau_2, \quad \gamma^3 = \sigma_1 \tau_1, \quad \gamma^4 = \sigma_2 \tau_1,
\]

where \(\tau^i (i = 1, 2, 3)\) are the Pauli matrices in the Nambu space, and \(\sigma_0, \sigma_1\) and \(\tau_0\) denote the \(2 \times 2\) identity matrices in the spin and Nambu spaces, respectively. It can be straightforwardly verified that the five gamma-matrices \(\gamma^\mu (\mu = 0, 1, 2, 3, 4)\) satisfy the anticommutation relations

\[
\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu}.
\]

Later we will also consider spatially and temporally varying pairing gap functions \(\Delta_\lambda (\vec{r}, t) (\lambda = s, p)\). Including the minimal coupling to electromagnetic potentials \(A^\mu(\tau)\), the Hamiltonian in real space becomes

\[
H_{\text{BdG}} = \frac{1}{2} \int d^3r \sum_{\vec{r}} \Psi^\dagger(\vec{r}) \hat{T}^\rho \gamma^\rho \Psi(\vec{r}),
\]

where the summation is over \(\rho = 0, 1, 2, 3, 4\); \(\Psi^\dagger(\vec{r})\) is the Fourier transform of \(\Psi^\dagger(\vec{k})\); and

\[
\hat{T}^0 = \frac{\hbar^2}{2m} (\vec{r} - \vec{A})^2 - \mu(\vec{r}),
\]

\[
\hat{T}^j = \frac{1}{2 k_f} (\Delta_\lambda (\vec{r}, t), -i \partial_j) \gamma^j, \quad (j = 1, 2, 3),
\]

\[
\hat{T}^4 = \Delta_s (\vec{r}, t).
\]

In addition to Eq. \(9\), there is also a Zeeman term in the presence of a magnetic field, i.e.,

\[
H_{\text{ZM}} = \frac{1}{2} g \mu_B \int d^3r \sum_{\vec{r}} \Psi^\dagger(\vec{r}) \left[-i \frac{1}{2} \epsilon_{ijk} B_i \gamma^j \gamma^k \right] \Psi(\vec{r}),
\]

in which \(g\) is the Landé factor, \(\mu_B\) is the Bohr magneton, \(\vec{B} = \frac{1}{2} \nabla \times \vec{A}\) is the magnetic field, and the \(1/2\) factor in the front comes from the spin-1/2 nature of the electrons.

### III. Transverse Supercurrent in the Presence of Spatial Inhomogeneities

In this section, we perform a quick real space calculation of the transverse supercurrent induced by static electric field and spatial inhomogeneity of \(\Delta_s\). The corresponding term in the axion action \(S_{\text{ax}}\) can then be determined from the relation \(j_i = -e \frac{\partial A_i}{\partial x} (i = 1, 2, 3)\) where \(e\) is the light velocity. Later in Sec. \(\text{IV}\) we will carry out a systematic path integral calculation. In addition to the supercurrent discussed in this section, there is also bound current originating from the spin magnetic moment which is not related to electron transport, as will be discussed in detail in Sec. \(\text{VII}\). In BdG form, the matrix kernel of the operator of the electric current density \(j_i(\vec{x}) (i = 1, 2, 3)\) at position \(\vec{x}\) is

\[
\hat{j}_i(\vec{x}) = -\frac{e \hbar}{4m} \left[ \delta(\vec{\hat{r}} - \vec{x}) (-i \nabla_{\hat{x}}) + (-i \nabla_{\vec{x}}) \delta(\vec{\hat{r}} - \vec{x}) \right] \sigma_0 \tau_0(\vec{r}) \sigma_i(\vec{r}),
\]

in which \(\vec{r}\) is the coordinate operator. The expectation value of \(\hat{j}_i(\vec{x})\) is

\[
\left< \hat{j}_i(\vec{x}) \right> = \text{Tr}(\hat{j}_i(\vec{x}) \hat{G}),
\]
in which the Green’s function $\hat{G}$ is

$$\hat{G} = \frac{1}{-\partial_\tau - H},$$

(14)

where $\tau$ is the imaginary time, and $H$ is the matrix kernel of the Hamiltonian $H_{BGK}$ in the presence of a spatially varying electric potential $\phi(\vec{r})$ and s-wave pairing gap function $\Delta_s(\vec{r})$. We emphasize that the symbol “$\text{Tr}$” denotes the trace operation of an operator, which, in addition to the trace of the matrix structure in the spin and Nambu spaces, also involves the integral over spatial coordinates. In what follows, we use “$\text{tr}$” to indicate the trace which is only taken over the $4 \times 4$ matrix structure.

The Green’s function $\hat{G}$ can be rewritten as

$$\hat{G} = (-) \frac{1}{-\partial_\tau^2 + H^2} (-\partial_\tau + H).$$

(15)

The Hamiltonian squared $H^2$ can be separated as

$$H^2 = H_0^2 + Q,$$

(16)

in which

$$H_0^2 = \hat{T}^p \hat{T}^p,$$

$$Q = \sum_{0 \leq p < s \leq 4} [\hat{T}^p, \hat{T}^s] \gamma^p \gamma^s,$$

(17)

where $\hat{T}^p$ is defined in Eq. (10) and the anticommutation relations Eq. (5) is used. Straightforward calculations show that

$$\hat{T}^p \hat{T}^p = \left[ -\frac{\hbar^2}{2m} \nabla^2 - \mu(\vec{r}) \right] ^2 + \frac{\Delta^2}{k_f^2} (-\nabla^2) + [\Delta_s(\vec{r})]^2,$$

$$[\hat{T}^0, \hat{T}^4] = -i \frac{\Delta^2}{k_f^2} \partial_\tau \mu,$$

$$[\hat{T}^0, \hat{T}^4] = -\frac{\hbar^2}{2m} (\nabla^2 \Delta_s + 2 \nabla \Delta_s \cdot \nabla),$$

$$[\hat{T}^0, \hat{T}^4] = -\frac{\hbar^2}{2m} \partial_\tau \Delta_s,$$

(18)

where $\mu(\vec{r}) = \epsilon_F + e \phi(\vec{r})$, where $\phi(\vec{r})$ is the electric potential.

Expanding $\langle \hat{j}_i(\vec{x}) \rangle$ in powers of $Q$, we obtain

$$\langle \hat{j}_i(\vec{x}) \rangle = \sum_{n=0}^{\infty} (-)^n \frac{1}{n!} \text{Tr} \left[ \hat{j}_i(\vec{x}) \left( \frac{1}{-\partial_\tau^2 + H_0^2} Q \right)^n \right] \times \frac{1}{-\partial_\tau^2 + H_0^2} (-\partial_\tau + H).$$

(19)

In what follows, we will only keep terms up to $n = 2$.

The $n = 0$ term in Eq. (19) vanishes: The odd-in-$\partial_\tau$ term vanishes after Matsubara frequency summation; and the other terms contain a trace of a single $\gamma$-matrix, hence are also zero.

The $n = 1$ term also vanishes. Removing the odd-in-$\partial_\tau$ term, the $n = 1$ term contains one $\Delta_H$ and one $H$ under the trace operation. However, $\Delta_H$ and $H$ are products of two and one $\gamma$-matrices, respectively. Hence the result vanishes since the trace of a product of three $\gamma$-matrices is zero.

Finally we consider the $n = 2$ term

$$\langle \hat{j}_i^{(2)}(\vec{x}) \rangle = - \text{Tr} \left[ \hat{j}_i(\vec{x}) \left( \frac{1}{-\partial_\tau^2 + H_0^2} Q \right)^2 \left( \frac{1}{-\partial_\tau^2 + H_0^2} \right) H \right].$$

(20)

To lowest order in the gradient expansion $\mu(\vec{r})$ and $\Delta_s(\vec{r})$ can be taken as constants in $H_0$. The only way to have a nonzero trace is a multiplication of all the five $\gamma$-matrices. Since the electric current operator $\hat{j}_i(\vec{x})$ contains a $-i\partial_\tau$, the $H$ term must contribute $\frac{\Delta^2}{k_f^2} (-i\partial_\tau) \gamma^i$ so that the trace is nonzero. Therefore,

$$\langle \hat{j}_i^{(2)}(\vec{x}) \rangle = 2 \frac{\Delta^2}{k_f^2} \text{Tr} \left[ \hat{j}_i(\vec{x}) \frac{1}{-\partial_\tau^2 + H_0^2} \partial_\tau \mu \frac{1}{-\partial_\tau^2 + H_0^2} \partial_\tau \Delta_s \right]$$

$$\times \frac{1}{-\partial_\tau^2 + H_0^2} (-i\partial_\tau) \text{tr} (\gamma^0 \gamma^i \gamma^j \gamma^k),$$

(21)

in which the trace of the product of five $\gamma$-matrices can be easily evaluated using

$$\text{tr} (\gamma^0 \gamma^i \gamma^j \gamma^k \gamma^l) = -4 \epsilon_{ijk}.$$

(22)

Using Eq. (12), $\langle \hat{j}_i^{(2)}(\vec{x}) \rangle$ can be evaluated as

$$\langle \hat{j}_i^{(2)}(\vec{x}) \rangle = -\epsilon_{ijk} \frac{2 \hbar}{m} \frac{\Delta^2}{k_f^2} \int d\vec{y} d\vec{z}$$

$$\{ (\vec{x})(-i\partial_\tau) \frac{1}{-\partial_\tau^2 + H_0^2} |\vec{y}\rangle \partial_\tau \mu(\vec{y}) \langle \vec{y}|(-i\partial_\tau) \frac{1}{-\partial_\tau^2 + H_0^2} |\vec{z}\rangle \partial_\tau \Delta_s(\vec{z}) \}$$

$$\times \frac{1}{-\partial_\tau^2 + H_0^2} (-i\partial_\tau) \delta^2(\vec{x} - \vec{z}).$$

(23)

In momentum space, Eq. (23) becomes

$$\langle \hat{j}_i^{(2)}(\vec{k}) \rangle = -\epsilon_{ijk} \frac{2 \hbar}{m} \frac{\Delta^2}{k_f^2}$$

$$\times \int \frac{d^3k}{(2\pi)^3} \left\{ \int \frac{d\vec{k}_3}{(2\pi)^3} \right\}$$

$$\times \frac{1}{\omega_n} \left[ \prod_{\alpha=1}^{3} \frac{1}{\omega_n^2 + |H_0(k_{\alpha})|^2} \right]$$

$$\times \frac{1}{\omega_n} \left[ \prod_{\alpha=1}^{3} \frac{1}{\omega_n^2 + |H_0(k_{\alpha})|^2} \right] e^{i k_{\alpha} \cdot (\vec{r}_n - \vec{r}_{n+1})},$$

(24)

in which $\beta$ is the inverse temperature; $\omega_n = (2n + 1) \pi / \beta$ is the fermionic Matsubara frequency; $\vec{r}_{1,2,3} = \vec{x}, \vec{y}, \vec{z}$; and $\vec{r}_4 = \vec{r}_1$.

To lowest order in the gradient expansion, we can set $\vec{y} = \vec{z}$ in $\partial_\tau \mu(\vec{y}) \partial_\tau \Delta_s(\vec{z})$ within Eq. (24), then integrating over $\vec{z}$ gives a momentum delta function $\delta^{(3)}(\vec{k}_2 - \vec{k}_3)$. 

Furthermore, if the higher order terms in the gradient expansion are neglected, then \( \vec{y} \) can be set as \( \vec{x} \) in \( \partial_{j} \mu(\vec{y}) \partial_{k} \Delta_{s}(\vec{y}) \), and the integration over \( \vec{y} \) gives \( \delta^{(3)}(\vec{k}_{1} - \vec{k}_{2}) \). As a result, we obtain

\[
\tilde{j}_{i}(\vec{x}) = \sum_{n=0}^{2} \left( \tilde{\gamma}^{(n)}(\vec{x}) \right)
\]

in which \( \partial_{j} \mu = -\epsilon \partial_{j} \phi \) is used, and the coefficient \( D \) is

\[
D(\Delta_{s}, \Delta_{p}) = \frac{4\hbar}{m} \left( \frac{\Delta_{p}}{k_{f}} \right)^{3} \times \frac{1}{\beta} \int \frac{d^{3}k}{(2\pi)^{3}} \frac{k^{2}/3}{\left[ \omega_{n}^{2} + \xi^{2}(\vec{k}) + \left( \frac{2\pi}{k_{f}} \right)^{2}k^{2} + \Delta_{s}^{2} \right]^{3}},
\]

where the replacement \( k_{f}^{2} \rightarrow k^{2}/3 \) is used which holds under integration. For simplicity, we consider zero temperature such that \( \frac{1}{\beta} \sum_{n} \) can be replaced by \( \int \frac{d\theta}{2\pi} \). In the weak pairing limit \( \Delta_{s}, \Delta_{p} \ll \epsilon_F, k \) can be simply taken as \( k_{f} \) in \( \left( \frac{2\pi}{k_{f}} \right)^{2} \) in the denominator, and \( \int dk = \int d\xi/(\hbar v_{f}) \) where \( v_{f} = \hbar k_{f}/m \) is the Fermi velocity. Eq. (26) can then be evaluated, yielding

\[
D(\Delta_{s}, \Delta_{p}) = \frac{\Delta_{p}^{3}}{6\pi^{2}\hbar \left( \Delta_{p}^{2} + \Delta_{s}^{2} \right)^{2}}.
\]

The expression of \( j_{i} \) can be obtained from \( \tilde{j}_{i} = -e\delta S_{ax}/\delta A_{i} \), where

\[
S_{ax} = -\frac{e^{2}}{c} \int d^{4}x D(\Delta_{s}, \Delta_{p}) \epsilon_{ijk} A_{i} \partial_{j} \phi \partial_{k} \Delta_{s}.
\]

Writing \( D(\Delta_{s}, \Delta_{p}) \) as a derivative of \( \Theta_{ax} \), i.e.,

\[
D(\Delta_{s}, \Delta_{p}) = -\frac{1}{24\pi^{2}\hbar} \frac{\partial \Theta_{ax}}{\partial \Delta_{s}},
\]

and performing an integration by part, we arrive at

\[
S_{ax} = -\frac{\alpha}{24\pi^{2}} \int d^{4}x \Theta_{ax} \epsilon_{ijk} \partial_{i} \phi \partial_{j} A_{k},
\]

where \( \alpha = \frac{e^{2}}{c} \) is the fine structure constant, and \( \int d^{4}x = \int d^{3}rdt \). Imposing the boundary condition \( \Theta_{ax} = 0 \) for a pure s-wave superconductor (i.e., \( \Delta_{s} \gg \Delta_{p} \)), \( \Theta_{ax} \) is determined to be

\[
\Theta_{ax}(\Delta_{s}, \Delta_{p}) = 24\pi^{2} \int_{\Delta_{s}}^{\infty} dx D(x, \Delta_{p}) = \pi - \frac{2\Delta_{s}\Delta_{p}}{\Delta_{s}^{2} + \Delta_{p}^{2}} - 2 \arctan(\frac{\Delta_{s}}{\Delta_{p}}).
\]

Therefore, defining

\[
\theta_{ax} = \pi - 2 \arctan(\frac{\Delta_{s}}{\Delta_{p}}),
\]

\( \Theta_{ax} \) can be written as

\[
\Theta_{ax} = \theta_{ax} - \sin(\theta_{ax}).
\]

Here we have two comments regarding the axion angle in Eq. (33) and the action in Eq. (30). First note that the pairings on the two degenerate Fermi surfaces of different helicities (i.e., \( \vec{k} \cdot \vec{\sigma} = \pm 1 \)) are \( \left| \Delta \right| e^{\pm i\theta_{ax}} \). However, the axion angle \( \Theta_{ax} \) is not just \( \theta_{ax} \), which is the difference between the superconducting phases on the two Fermi surfaces, but acquires an additional \( \sin(\theta_{ax}) \) term. Secondly, the conventional axion term in a relativistic system is of the form \( ~ \vec{E} \cdot \vec{B} \) where \( \vec{E} = -\nabla \phi - \partial_{i} A_{i} \). On the other hand, Eq. (30) only contains the \( \nabla \phi \vec{B} \) term, which is reasonable since in this section we did not include a time-dependent vector potential. However, we will see in Sec. IV that in fact, the \( \partial_{i} \vec{A} \vec{B} \) term is missing even in the general situation. Therefore, replacing \( \phi \) by \( \phi - \frac{\gamma}{2} \partial_{i} \Phi \) to keep gauge invariance, Eq. (30) is already the full axion action, where \( \Phi \) is the U(1)-breaking superconducting phase mode. As will be discussed in Sec. IV, the difference between the \( p + i s \) and the relativistic cases is the result of a lack of Lorentz invariance in \( p + i s \) superconductors.

IV. PATH INTEGRAL FORMULATION

In this section, we formulate the systematic path integral approach to the axion action in \( p + i s \) superconductors. We assume the four-fermion interaction to be of the form

\[
H_{int} = -g_{s} \frac{1}{L^{3}} \sum_{k} P_{s}^{\dagger}(\vec{k}) P_{s}(\vec{k}) - g_{p} \frac{1}{L^{3}} \sum_{k} P_{p}^{\dagger}(\vec{k}) P_{p}(\vec{k}),
\]

in which \( L \) is the linear size of the system in space and \( g_{s} > 0 \) (\( \lambda = s, p \)) are coupling constants, and \( P_{s}^{\dagger}(\vec{k})'s \) (\( \lambda = s, p \)) are defined in Eq. (2). After performing a Hubbard-Stratonovich transformation, the partition function in the imaginary time formalism can be written as

\[
Z = \int D[c^{\dagger}, c] D[\Delta_{s}^{\ast}, \Delta_{s}] D[\Delta_{p}^{\ast}, \Delta_{p}] e^{-S}
\]

in which

\[
S = S_{\Delta} + S_{f},
\]

where

\[
S_{\Delta} = \sum_{\lambda=s,p} \frac{1}{g_{\lambda}} \int d\tau d^{3}r |\Delta_{\lambda}|^{2},
\]

and the fermionic part \( S_{f} \) is

\[
S_{f} = \frac{1}{2} \int d\tau d^{3}r \Psi^{\dagger}(\tau, \vec{r}) (\partial_{\tau} + H_{f}) \Psi(\tau, \vec{r}),
\]

in which \( \Psi^{\dagger}(\tau, \vec{r}) \) is a set of Grassmann numbers defined through the Fourier transform of \( \Psi^{\dagger}(\vec{k}) \) in Eq. (5), and
We note that in momentum space, \( \Delta \) through the following replacements:

\[
H \to H_f, \\
\Phi \to \Phi, \\
\Phi' \to \Phi', \\
\Phi_2 \to \Phi_2, \\
\epsilon_c^{\cdot} \mathcal{A} \to \mathcal{A}^{\cdot}, \\
\epsilon_c \mathcal{A} \to \mathcal{A}.
\]

in which \( H_f \) \((i, j) = (1, 2)\) is the \((i, j)\)-block of \( H_f \); \( i \Phi \) is the electric potential in imaginary time; \( \{ A, B \} = AB + BA \) denotes the anticommutator of the operators \( A \) and \( B \); and \( \Phi \) and \( \Phi_t \) are the superconducting phase mode and the Leggett mode, respectively.

The phase mode \( \Phi \) can be absorbed into electromagnetic potentials by performing a gauge transformation through the following replacements:

\[
-ic \Phi \to \Phi' = -ic \Phi + \partial_t \Phi, \\
e_{c} \mathcal{A} \to \mathcal{A}^{\cdot} = \mathcal{A} + h \nabla \Phi.
\]

Assuming a background \( \Delta_{\lambda} \) \((\lambda = s, p)\) and including small fluctuations of the different modes, \( H_f \) becomes

\[
H_f = H_{f0} + \Delta H_f,
\]

in which \( H_{f0} \) is simply Eq. 6 and

\[
\Delta H_f = \Delta H_A^{(1)} + \Delta H_A^{(2)} + \Delta H_\phi + \Delta H_Z \\
+ \Delta H_p + \Delta H_s + \Delta H_t,
\]

where

\[
\Delta H_A^{(1)} = \frac{\hbar}{2m} \{ \mathcal{A}^{\cdot}(\tau, \vec{r}), -i\nabla \} \sigma_0 \tau_0, \\
\Delta H_A^{(2)} = \frac{\hbar}{2m} [ \mathcal{A}^{\cdot}(\tau, \vec{r}) ]^2 \gamma^0, \\
\Delta H_\phi = \phi' (\tau, \vec{r}) \gamma^0, \\
\Delta H_Z = -\frac{i}{4} \gamma_\mu \epsilon_{ijk} B_i \gamma_j \gamma_k, \\
\Delta H_p = \frac{1}{2k_f} \{ \delta \Delta_p(\tau, \vec{r}), -i\nabla \} \cdot \gamma, \\
\Delta H_s = \delta \Delta s (\tau, \vec{r}) \gamma^4, \\
\Delta H_t = \Delta \delta \Phi_1 (\tau, \vec{r}) \sigma_2 \tau_2.
\]

We note that in momentum space, \( \Delta H_A^{(1)}(\vec{q}) = \frac{\hbar}{2m} (2\vec{k} + \vec{q}) \cdot \mathcal{A}(\vec{q}) \) and \( \Delta H_p(\vec{q}) = \frac{1}{k_f} (\vec{k} + \vec{q}/2) \cdot \delta \Delta_p(\vec{q}) \).

Since the fermionic quasiparticles are fully gapped, the action for the collective bosonic degrees of freedom can be obtained by integrating over the fermions, resulting in \( \frac{1}{4} Tr n (-\partial_\tau - H_f) \). In what follows, we will only consider the axion terms. They arise in the third order terms in the Trn-expansion, i.e.,

\[
S_f^{(3)} = -\frac{1}{6} Tr [(\mathcal{G}_0 \Delta H_f)^3],
\]

in which \( \mathcal{G}_0 = (\partial_\tau + H_0)^{-1} \). Here we note that as discussed in Sec. V and Sec. VI, in addition to the axion terms, there are other nonvanishing terms in \( S_f^{(3)} \) which involve two spacetime derivatives, as a consequence of a lack of Lorentz symmetry. We do not explicitly calculate these additional terms since the calculations are very cumbersome. A list of such terms based on a symmetry analysis is included in Appendix A.

V. ORBITAL CONTRIBUTION TO AXION ELECTRODYNAMICS

In this section, we calculate the orbital contribution to the axion action based on the path integral approach. There are three diagrams which contain two \( A^\mu \)'s \((\mu = 0, 1, 2, 3)\) where \( A^0 = \phi \) and one \( \delta \Delta_\lambda \) \((\lambda = s, p)\), as shown in Fig. 1 (I, II, III). Diagram III – though not zero – does not contribute to the axion action, hence we neglect. We will only calculate the terms involving two spacetime derivatives in diagrams I, II in Fig. 1.

A. Diagram I

This diagram potentially can contribute to \( \int d^4x \partial_\mu \mathcal{A}^\mu \cdot \vec{B} \) in the axion action. However, we demonstrate that in fact, this contribution vanishes.

I. \( \lambda = s \)

One term contributing to the \( \lambda = s \) case is

\[
\mathcal{D}_s^{(1)} = \frac{1}{12} \sum \omega_n \int \frac{d^3 \vec{k}}{(2\pi)^3} Tr [ \mathcal{G}_0(\omega_n, \vec{k}) \frac{\hbar}{2m} (2\vec{k} + \vec{q}) \cdot \mathcal{A}(\vec{q}, \vec{k}) \mathcal{G}_0(\omega_n + \Omega_2, \vec{k} + \vec{q}) \\
\times \frac{\hbar}{2m} (2\vec{k} - \vec{q}_1) \cdot \mathcal{A}(-\Omega_1 - \Omega_2, -\vec{q}_1 - \vec{q}_2) \\
\times \mathcal{G}_0(\omega_n - \Omega_1, \vec{k} - \vec{q}_1) \delta \Delta s (\Omega_1, \vec{q}_1) \gamma^4 ],
\]

in which the minus sign coming from the fermion loop cancels with the sign in Eq. 43, and

\[
\mathcal{G}_0(\omega_n, \vec{k}) = \frac{a_{\chi}(\omega_n, \vec{k}) \gamma^\chi}{a_{\chi}(\omega_n, \vec{k}) a_{\chi}(\omega_n, \vec{k})},
\]

where the summation of \( \chi \) is over \( \chi = 0, 1, 2, 3, 4, 5; \gamma^5 = \sigma_0 \tau_0; \) and

\[
a_0 = \xi \kappa, \ a_i = \frac{\Delta \kappa}{k_f} \ (i = 1, 2, 3), \ a_4 = \Delta s, \ a_5 = i \omega_n(46)
\]
Plugging Eq. (45) into Eq. (44), the trace of the numerators of $G_0$ gives
\[
\text{tr} \left[ a_{x_1}(\omega_n, \vec{k}) \gamma^x a_{x_2}(\omega_n + \Omega_2, \vec{k} + \vec{q}_2) \gamma^{x_2} \right]
\]
\[\quad a_{x_3}(\omega_n - \Omega_1, \vec{k} - \vec{q}_1) \gamma^{x_3} \gamma^4 \] (47)

We note that the $O(\Omega_j)$ ($j = 1, 2$) term in Eq. (47) is $8\omega_n \Delta_s (\Omega_1 - \Omega_2)$.

To generate $\int d^4x \partial_\nu \vec{A} \cdot \nabla \times \vec{A}$, we must consider terms in Eq. (44) which involve one $\Omega_j$ and one $\vec{q}_j$ ($j = 1, 2$). It is straightforward to see that the $O(\Omega_j)$ term in $1/\sum_{\omega_n} a_n(\omega_n + \Omega, \vec{k} + \vec{q})^2 (\Omega = -\Omega_1, \Omega_2$ and $\vec{q} = -\vec{q}_1, \vec{q}_2$) is proportional to $\omega_n, \Omega$, similar as the $O(\Omega_j)$ term in the trace in Eq. (47). Since the Matsubara summation of terms involving odd powers of $\omega_n$ vanishes, we conclude that there is no contribution to $\int d^4x \partial_\nu \vec{A} \cdot \nabla \times \vec{A}$ from Diagram I for $\lambda = s$.

2. $\lambda = p$

The analysis for $\lambda = p$ in Diagram I is exactly similarly and the contribution again vanishes.

B. Diagram II

This diagram potentially can contribute to $\int d^4x \nabla \phi' \cdot \vec{B}$ in the axion action. We show that it gives exactly the axion action derived in Sec. III.

One term contributing to the $\lambda = s$ case is $-\frac{1}{\beta} \text{Tr}[G_0 \Delta H_A (1) G_0 \Delta H_s G_0 \Delta H_s]$. Including the combinatoric factor of 6 and using Eq. (45), we obtain
\[
D_s^{II} = \frac{1}{\beta} \sum_{\omega_n} \int d^4k (\frac{1}{(2\pi)^3})^2 m (2\vec{k} + \vec{q}) \cdot \vec{A}(\Omega_2, \vec{q}_2)
\]
\[\times \phi'(\Omega_1 - \Omega_2, -\vec{q}_1, \vec{q}_2) \delta \Delta_s (\Omega_1, \vec{q}_1)
\times \text{tr} \left[ a_{x_1}(\omega_n, \vec{k}) \gamma^x a_{x_2}(\omega_n + \Omega_2, \vec{k} + \vec{q}_2) \gamma^{x_2} \gamma^0 \right]
\[\quad \cdot a_{x_3}(\omega_n - \Omega_1, \vec{k} - \vec{q}_1) \gamma^{x_3} \gamma^4 \right],
\]
\[\times \Pi_{\nu}^{\lambda_1 = 1} \frac{1}{a_{x}(\omega_n + \Omega_1', \vec{k} + \vec{q}_1') a_{x}(\omega_n + \Omega_1', \vec{k} + \vec{q}_1')} \] (48)
in which $\Omega_1' = 0, \Omega_2' = \Omega_2, \Omega_3' = -\Omega_1$, and $\vec{q}_1' = \vec{q}_2 = \vec{q}_1$. Recall that we want a term $\sim \epsilon_{i j k} \partial_i \phi \partial_j A_k$ in the action. Such term can be generated from the trace of a multiplication of the product of five $\gamma$-matrices as shown in Eq. (22). Then the $\epsilon_{i j k}$ term within the trace in Eq. (48) can be evaluated as
\[
\text{tr}[...] = -4 \left( \frac{\Delta_p}{k_f} \right)^3 \epsilon_{i j k} k_i (q_{j k} - q_{1 k})
\]
\[= 4 \left( \frac{\Delta_p}{k_f} \right)^3 \epsilon_{i j k} q_{j k} q_{1 k}. \] (49)

Since Eq. (49) already contains a product of two wavevector $q$'s, we can set $\vec{q}_j, \Omega_j$ to be zero in the re-
mainling parts of Eq. (48). This gives

$$D_s^H = D(\Delta_s, \Delta_p)\epsilon_{ijk}q_{2i}q_{1k} \times A'_G(\Omega_2, \tilde{q}_2)\delta \Delta_s(\omega_1, \tilde{q}_1)\phi'(-\Omega_1 - \Omega_2, \tilde{q}_1 - \tilde{q}_2).$$  

(50)

in which $D(\Delta_s, \Delta_p)$ is given exactly by Eq. (26). Integration by part and transforming to the real space, the corresponding term in the action is

$$\int d^4x D(\Delta_s, \Delta_p)\vec{A}' \cdot (\nabla \phi' \times \nabla \delta \Delta_s).$$  

(51)

2. $\lambda = p$

When $\lambda = p$, $D_p^H$ can be obtained from $D_s^H$ by replacing $\delta \Delta_s$ in Eq. (48) with $\delta \Delta_p$, and changing the trace to

$$\text{tr}[a_{\chi_1}(\omega_n, \tilde{k})\gamma^1 a_{\chi_2}(\omega_n + \Omega_2, \tilde{k} + \tilde{q}_2)\gamma^2 \cdot a_{\chi_3}(\omega_n - \Omega_1, \tilde{k} - \tilde{q}_1)\gamma^3 \frac{1}{k_f}(\tilde{k} - \frac{\tilde{q}_1}{2}) \cdot \gamma_i].$$  

(52)

The $\epsilon_{ijk}$ term in Eq. (52) can be straightforwardly evaluated as

$$-4\frac{\Delta_s^2}{k_f^2}\epsilon_{ijk}q_{2i}q_{1k}q_{k}.$$  

(53)

As a result, the corresponding term in the axion action in real space is

$$-D'(\Delta_s, \Delta_p)\vec{A}' \cdot (\nabla \phi' \times \nabla \delta \Delta_p),$$  

(54)

in which

$$D'(\Delta_s, \Delta_p) = \frac{\Delta_s}{\Delta_p} D(\Delta_s, \Delta_p).$$  

(55)

C. Diagram III

This diagram does not contribute to the axion action as explained at the beginning of this section, though it does contribute to non-axion terms as discussed in Appendix A.

D. Orbital contribution to the axion action

Combining Eq. (51) together and using $\nabla \delta \Delta_s = \nabla \Delta_s$, we obtain

$$S_{ax}^o = \int d^4x \vec{A}' \cdot [\nabla \phi' \times (D\nabla \Delta_s - D'\nabla \Delta_p)],$$  

(56)

in which $D'(D')$ is $D(\Delta_s, \Delta_p)$ ($D(\Delta_s, \Delta_p)$) for short. Plugging in the expression of $D$ given in Eq. (27), we have

$$D\nabla \Delta_s - D'\nabla \Delta_p = \frac{1}{6\pi^2\hbar} \left[\frac{1}{1 + (\frac{\Delta_s}{\Delta_p})^2}\right] \nabla \frac{\Delta_s}{\Delta_p}. $$  

(57)

Using the integral

$$\int_{\frac{\Delta_s}{\Delta_p}}^{\infty} \frac{dx}{(1 + x^2)^2} = \frac{1}{4\pi} - 2\arctan(\frac{\Delta_s}{\Delta_p}) - 2\frac{\Delta_s/\Delta_p}{1 + (\Delta_s/\Delta_p)^2},$$  

(58)

$S_{ax}^o$ becomes

$$S_{ax}^o = -\frac{\alpha}{24\pi^2} \int d^4x \vec{A}' \cdot [\nabla \phi' \times \nabla \Theta_{ax}^o],$$  

(59)

where $\Theta_{ax}^o$ coincides exactly with the expression in Eq. (35). Integrating by parts, employing Eq. (40), and transforming to the real time (i.e., $i\phi \rightarrow \phi$), we obtain

$$S_{ax}^o = -\frac{\alpha}{24\pi^2} \int d^4x \Theta_{ax}^o \nabla(\phi - \hbar/e \partial_t \Phi) \cdot \vec{B},$$  

(60)

where $\nabla \times \nabla \Phi = 0$ is used. This is exactly what we have obtained in Eq. (30).

The differences between Eq. (60) and the axion action in the relativistic case have been discussed by the end of Sec. III. The essential reason is a lack of Lorentz symmetry in the $p + is$ case. Here we note that on a technical level, the band dispersion in the normal metal phase of $p + is$ superconductor does not contain negative energy states. Therefore, unlike the relativistic case, there is no inter-band transition (from negative to positive energy bands) in the $p + is$ superconducting case, which leads to the different behaviors between the two situations.

VI. Zeeman contribution to axion electrodynamics

In this section, we calculate the Zeeman contribution to the axion action based on the path integral approach.

A. Diagram IV

This diagram potentially can contribute to $\int d^4x \nabla \phi' \cdot \vec{B}$ in the axion action. We will derive its explicit expression.

1. $\lambda = s$

One term contributing to the $\lambda = s$ case is $-\frac{1}{3} \text{Tr}[G_0(\Delta H_2G_0(\Delta H_2G_0(\Delta H_s))$. Including the combinatorial factor of 6, we obtain

$$D_s^IV = \frac{1}{3} \sum_{\omega_n} \int \frac{d^4k}{(2\pi)^3} \text{Tr} \left[ G_0(\omega_n, \tilde{k})\phi'(\Omega_2, \tilde{q}_2)\gamma^0 G_0(\omega_n + \Omega_2, \tilde{k} + \tilde{q}_2) \times (-)^{\frac{1}{4}iq\mu B}\epsilon_{ijk}\gamma^k B_i(-\Omega_1 - \Omega_2, -\tilde{q}_1 - \tilde{q}_2) \times G_0(\omega_n - \Omega_1, \tilde{k} - \tilde{q}_1)\delta \Delta_s(\Omega_1, \tilde{q}_1)\gamma^4 \right].$$  

(61)
Only considering the axion term \( \int d^4x \nabla \phi' \cdot \vec{B} \), we can set \( \Omega_1 = \Omega_2 = 0 \), and it is enough to expand \( D_s^{IV} \) up to linear order in \( \tilde{q}_x \) (\( \alpha = 1, 2 \)). Calculations show that

\[
D_s^{IV} = E_{1s_i}q_{1i}B_i + E_{2s_i}q_{2i}B_i + O(q_{\alpha j}q_{\beta k}),
\]

in which

\[
E_{asi} = -2g_\mu B \frac{\Delta \frac{1}{k_f}}{\beta} \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} M_{asi},
\]

where

\[
M_{asi} = M_{ai}, E_{asi} = E_{ai},
\]

\[
\text{where } i = x, y, z.
\]

We will only keep the leading order terms in an expansion over \( \Delta_k / \varepsilon_f (\lambda = s, p) \). Then in \( E_{1s} \), all \( k_f^2 \) can be replaced by \( k_f^2 / 3 \) in the integrand, and \( \int d^3k/(2\pi)^3 \) can be set as \( N_f \int d\xi \), where \( N_f = \frac{mk_f}{2\pi^2 k_f^3} \). On the other hand, \( M_{2s} \) contains a term \(-\frac{k_f^2}{2m}(k_f^2 + \frac{2}{3}k_f^2)\varepsilon_f^2\frac{1}{k_f^2}\), which is one order less than the other terms in the numerator. Therefore, in this case, \( \int d^3k/(2\pi)^3 \) should be set as \( N_f \int (1 + \frac{\xi}{2\pi} \cdot \kappa) d\xi \), and

\[
M_{2s} \rightarrow -\frac{\frac{3}{2} \xi \varepsilon_f}{[\omega_n^2 + \xi_k^2 + \frac{1}{3} \Delta_k^2 k_f^2 + \Delta_k^2]^3} + \frac{\omega_n^2 - \frac{1}{3} \xi_k^2 + \frac{1}{2} \Delta_k^2 - \Delta_k^2}{[\omega_n^2 + \xi_k^2 + \frac{1}{3} \Delta_k^2 k_f^2 + \Delta_k^2]^3} + \frac{4 \Delta_k^2 \xi_k^2}{[\omega_n^2 + \xi_k^2 + \frac{1}{3} \Delta_k^2 k_f^2 + \Delta_k^2]^3},
\]

where the first term can combine with \( N_f \int \frac{\xi}{2\pi} d\xi \) within \( \int d^3k/(2\pi)^3 \) giving a non-zero contribution, and the third term in Eq. (66) comes from expanding the denominator using \( \Delta_k^2 k_f^2 / k_f^2 = \Delta_k^2 (1 + \xi_k^2 / \varepsilon_f) \). Again at zero temperature, \( \frac{1}{\beta} \sum_{\omega_n} = \int \frac{d\omega}{2\pi} \). Performing the integrations \( \int d\omega \int d\xi \), we obtain

\[
E_{1s}(\Delta_s, \Delta_p) = \frac{g \mu_B m}{12\pi^2 h^2} \frac{\Delta_p (3\Delta_s^2 + \Delta_p^2)}{(\Delta_p^2 + \Delta_s^2)^2},
\]

\[
E_{2s}(\Delta_s, \Delta_p) = \frac{g \mu_B m}{4\pi^2 h^2} \frac{\Delta_p (\Delta_s^2 - \Delta_p^2)}{(\Delta_p^2 + \Delta_s^2)^2}.
\]

Transforming back to the real space, the action becomes

\[
S_{ax}^Z = \int d^4x [E_{1s} \nabla \delta \Delta_s \cdot \vec{B} \phi' + E_{2s} \delta \Delta_s \vec{B} \cdot \nabla \phi'].
\]

Integrating by parts, dropping total derivative terms, and using \( \nabla \cdot \vec{B} = 0 \), we obtain the axion action

\[
\int d^4x (E_{1s} - E_{2s}) \phi' \nabla \delta \Delta_s \cdot \vec{B}.
\]

2. \( \lambda = p \)

Similar as the \( \lambda = s \) case, the expression with a spatially varying \( \delta \Delta_p \) is

\[
D_p^{IV} = \frac{1}{\beta} \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \text{tr}[G_0(\omega_n - \tilde{k}) \phi'(\Omega_2, \tilde{q}_2)G_0(\omega_n + \Omega_2, \tilde{k} + \tilde{q}_2) \times (-\frac{1}{4}ig \mu_B \epsilon_{ijk} \gamma^j \gamma^k B_i (-\Omega_1 - \Omega_2 - \tilde{q}_1 - \tilde{q}_2) \times G_0(\omega_n - \Omega_1 - \tilde{q}_1) \frac{1}{k_f} \delta \Delta_p (\Omega_1, \tilde{q}_1) (\tilde{k} - \frac{\tilde{q}_1}{2}) \cdot \gamma^j].
\]

Again setting \( \Omega_1 = \Omega_2 = 0 \), and keeping the linear in \( \tilde{q}_x^2 (\alpha = 1, 2) \), we obtain

\[
D_p^{IV} = E_{1p}^{i}q_{1i}B_i + E_{2p}^{i}q_{2i}B_i + O(q_{\alpha j}q_{\beta k}),
\]

in which

\[
E_{op} = -g \mu_B \frac{\Delta \frac{1}{k_f}}{\beta} \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} M_{op},
\]

where \( (k_1 \neq k_j \neq k_k) \)

\[
M_{1p} = \frac{\omega_n^2 + \xi_k^2 + \frac{1}{3} \Delta_k^2 k_f^2 + \Delta_k^2}{[\omega_n^2 + \xi_k^2 + \frac{1}{3} \Delta_k^2 k_f^2 + \Delta_k^2]^3},
\]

\[
M_{2p} = 4\frac{\Delta^2 \xi_k^2}{[\omega_n^2 + \xi_k^2 + \frac{1}{3} \Delta_k^2 k_f^2 + \Delta_k^2]^3}.
\]

We note that again, the values of Eq. (73) do not depend on \( i = x, y, z \). Calculations show that (up to leading order in \( \Delta_k / \varepsilon_f \), where \( \lambda = s, p \))

\[
E_{op} = -\frac{\Delta_p}{\Delta_p} E_{as}.
\]

Correspondingly, the contribution to the axion action is

\[
\int d^4x (E_{1p} - E_{2p}) \phi' \nabla \delta \Delta_p \cdot \vec{B}.
\]
B. Diagram V

This diagram potentially can contribute to \( \int d^4 x \partial_t \vec{A}'. \vec{B} \) in the axion action. However, in a way similar with the discussions in Sec. III, it can be seen that the axion contribution from this diagram vanishes, since the integrand is odd with respect to the fermionic Matsubara frequency.

C. Zeeman contribution to the axion action

Combining Eqs. (69, 75) and using Eq. (74), we obtain

\[
S_{ax}^Z = \int d^4 x \phi' E_s^Z (\Delta_s, \Delta_p) \nabla(\frac{\Delta_s}{\Delta_p}) \cdot \vec{B},
\]

in which

\[
E_s^Z = \frac{g \mu_B m}{3 \pi^2 \hbar^2} \frac{1}{[1 + (\Delta_s/\Delta_p)^2]^2}.
\]

Using

\[
\int_0^{\Delta_p} \frac{dx}{(1 + x^2)^2} = \frac{1}{2} \left[ \arctan(\Delta_s/\Delta_p) + \frac{\Delta_s/\Delta_p}{1 + (\Delta_s/\Delta_p)^2} \right],
\]

and plugging in \( \mu_B = \frac{e \hbar}{2m_c} \), \( S_{ax}^Z \) becomes

\[
S_{ax}^Z = \frac{g e}{24 \pi^2 \hbar c} \int d^4 x \phi' \nabla \Theta_{ax}^Z \cdot \vec{B},
\]

in which \( \alpha \) is again the fine structure constant, and

\[
\Theta_{ax}^Z = 2 \arctan(\frac{\Delta_s}{\Delta_p}) + \frac{2 \Delta_s \Delta_p}{\Delta_s^2 + \Delta_p^2}.
\]

Further performing an integration by part and transforming to real time, we obtain

\[
S_{ax}^Z = \frac{g \alpha}{24 \pi^2} \int d^4 x \Theta_{ax}^Z \nabla(\phi - \frac{\hbar}{e} \partial_t \Phi) \cdot \vec{B},
\]

in which \( g = 2 \) in vacuum, but can be somewhat arbitrary in solid state materials. The action \( S_{ax}^Z \) in Eq. (81) can also contribute to transverse supercurrent, similar to its orbital counterpart as discussed in Sec. III. However, we note that unlike the orbital case, the current arising from Eq. (81) are bound currents, which are not related to wavepacket transport of electrons.

Here we note that \( \Theta_{ax}^Z \) is again the fine structure constant, and

\[
\Theta_{ax}^Z = \frac{\pi}{2} - \Theta_{ax}^Z,
\]

where the expression of \( \Theta_{ax}^Z \) is given in Eq. (83). This difference comes from the integral \( \int_{\lambda_0}^{\lambda} d\lambda' \) in Eq. (85), unlike the one in Eq. (58) which is \( \int_{\Delta_s/\Delta_p}^{\infty} dx \). The choice of the lower and upper bounds of the integration in Eq. (85) is consistent with Ref. [24] where the free energy describing the magnetoelectric effect in the spin channel is calculated in the vicinity of \( T_c \). It is shown in Ref. [24] that for a pure \( p \)-wave superconductor, the spin magnetoelectric effect vanishes. Therefore, in the current situation, \( \Theta_{ax}^Z \) should be zero for a pure \( p \)-wave superconductor in the absence of \( \Delta_s \).

VII. CONCLUSION

In conclusion, we have studied the coupling between the axion angle and the electromagnetic field in \( p + is \) superconductors. Including both the orbital and Zeeman contributions, the axion action is derived as

\[
S_{ax} = -\frac{\alpha}{24 \pi^2} \int d^4 x \Theta_{ax} \nabla(\delta - \frac{\hbar}{e} \partial_t \Phi) \cdot \vec{B},
\]

in which \( \alpha \) is the fine structure constant, and the axion angle is

\[
\Theta_{ax} = \Theta_{ax}^o - \Theta_{ax}^Z = -g \pi + (1 + g)(\theta_{ax} - \theta_{ax}^Z),
\]

where \( g \) is the Landé factor and \( \theta_{ax} = \pi - 2 \arctan(\Delta_s/\Delta_p) \). The axion action in Eq. (82) has two differences compared with the relativistic case. The induced electric field does not appear, therefore \( S_{ax} \) does not have Lorentz symmetry. In addition, the axion angle \( \Theta_{ax} \) also contains a sinusoidal term \( \sin(\theta_{ax}) \), not just \( \theta_{ax} \).

Our work reveals the crucial difference for axion electrodynamics between the \( p + is \) superconducting and the superconducting Dirac/Weyl systems.

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Appendix A: Symmetry allowed non-axion terms

In this appendix we examine all the symmetry allowed terms which contain one \( \Delta_\lambda \) (\( \lambda = s, p \), two \( A^\mu 's \) \( (\mu = t, x, y, z) \), and two spacetime derivatives. Up to an overall factor \( F(\Delta_s^2, \Delta_p)\Delta_\lambda \) (where \( F \) is a function which can be determined by calculating the corresponding diagram), the terms invariant under 3D rotations and \( \mathcal{PT} \)-operation are

\[
\begin{align*}
&\partial_t \vec{A}' \cdot \partial_t \vec{A}' \delta \Delta_\lambda \\
&\partial_t^2 \vec{A}' \cdot \vec{A}' \delta \Delta_\lambda \\
&\partial_t \phi' \partial_t \phi' \delta \Delta_\lambda \\
&\partial_t^2 \phi' \phi' \delta \Delta_\lambda \\
&(\nabla \cdot \vec{A}') (\nabla \cdot \vec{A}') \delta \Delta_\lambda \\
&\nabla^2 \vec{A}' \cdot \vec{A}' \delta \Delta_\lambda \\
&\nabla \phi' \cdot \nabla \phi' \delta \Delta_\lambda \\
&\nabla^2 \phi' \phi' \delta \Delta_\lambda
\end{align*}
\]
\[
\partial_t \vec{A} \cdot (\nabla \times \vec{A}) \delta \Delta \lambda \\
\vec{A} \cdot (\nabla \times \partial_t \vec{A}) \delta \Delta \lambda \tag{A3}
\]

\[
(\nabla \cdot \vec{A}) \partial_t \phi' \delta \Delta \lambda \\
(\nabla \times \vec{A}) \cdot (\nabla \phi') \delta \Delta \lambda \tag{A4}
\]

\[
\partial_t \vec{A} \cdot (\nabla \times \vec{A}) \delta \Delta \lambda \\
\vec{A} \cdot (\nabla \times \partial_t \vec{A}) \delta \Delta \lambda \tag{A5}
\]

in which \( \lambda = s, p \). The terms in Eq. (A3) vanish as discussed in Sec. V A, and Eq. (A5) is the axion term in which

\[
\text{with } f \text{ and } f' \text{ compared with the axion terms. However, we note that none of the non-axion terms can contribute to the effects like transverse supercurrent as discussed in Sec. III and in fact, they do not exhibit magnetoelectric effects.}
\]

\[
\int d^4x [N_f (\hbar \partial_t \Phi - e \phi)^2 + \frac{n_s}{2m} (\hbar \nabla \Phi + \frac{e \vec{A}}{c})^2], \tag{A6}
\]

in which \( n_s \) is the superfluid density. Notice that in the long wavelength limit, additional spacetime gradient terms are suppressed by factor of \((q \xi_f)^n \sim (h v_f q/\Delta)^{1/2}\) where \( \xi_f \) is the coherence length, and \( q \) can be either \(|\vec{q}|\) or \(|\Omega|/v_f \). For simplicity, consider the term \( \partial_t \vec{A} \cdot \nabla \times \vec{A} \) (although this term vanishes as discussed in Eq. (A3), it works as an illustration for the other non-axion terms, regarding the order of the coefficients). The coefficient \( C \) of this term should be on order of \( \frac{1}{\mathcal{N}^q} \). Using \( n_s \sim k_f^3 \), it is straightforward to obtain \( C \sim (\epsilon_f/\Delta)^2 \). Other hand, recall that the coefficients of the axion terms are of \( O[(\Delta/\epsilon_f)^q] \). Therefore, generically, the non-axion terms can be much larger, i.e., enhanced by a factor of \((\epsilon_f/\Delta)^2 \) compared with the axion terms. However, we note that none of the non-axion terms can contribute to the effects like transverse supercurrent as discussed in Sec. III and in fact, they do not exhibit magnetoelectric effects.

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