On generalized Fisher informations and Cremér-Rao type inequalities

Shigeru Furuichi
Department of Computer Science and System Analysis, College of Humanities and Sciences, Nihon University, 3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan
E-mail: furuichi@chs.nihon-u.ac.jp

Abstract. We define $q$-Fisher informations and show $q$-Cramér-Rao type inequalities that the $q$-Gaussian distribution with special $q$-variances attains the minimum value of the $q$-Fisher informations.

1. Introduction
The maximum entropy principle is fundamental in statistical physics. In nonextensive statistical physics, the maximum Tsallis entropy principle is also important and has been studied by many literatures [17, 18, 10, 19, 3, 1, 7]. The maximum Tsallis entropy principle states that Tsallis distribution ($q$-Gaussian distribution) attains the maximum value of the Tsallis entropy, subject to the constraint on the $q$-expectation value and the $q$-variance. On the other hand, it is well known that the Gaussian distribution maximize the entropy and minimize the Fisher information (Cramér-Rao inequality) [4]. Therefore it is quite natural that we analogically have the following question: $q$-Gaussian minimizes $q$-Fisher information? We here consider it for two cases in nonextensive statistical physics. This brief report is based on our previous paper [7] so that we omit the proofs.

2. The $q$-Fisher information based on the $q$-expectation value
We denote the $q$-logarithmic function $\ln_q$ by

$$\ln_q(x) \equiv \frac{x^{1-q} - 1}{1 - q}, \quad (q \in \mathbb{R}, q \neq 1, x > 0).$$

In this section, we give a definition of the $q$-Fisher information, based on the $q$-expectation value $E_q$.

Definition 1 For random variable $X$ with probability density function $f(x)$, we define the $q$-score function $s_q(X)$ and the $q$-Fisher information $J_q(X)$ by

$$s_q(x) \equiv \frac{d\ln_q f(x)}{dx},$$
$$J_q(X) \equiv E_q \left[ s_q(X)^2 \right],$$

where the $q$-expectation value $E_q$ is defined by $E_q[g(X)] \equiv \int_{-\infty}^{\infty} f(x)^q g(x)dx$ for random variables $g(X)$ for any continuous function $g(x)$ and the probability density function $f(x)$.
Then we have the following theorem.

**Theorem 1** (*q*-Cramér-Rao type inequality [7]) For given the *q*-expectation value $\mu_q \equiv E_q[X]$, the *q*-variance $\sigma_q^2 \equiv E_q[(X - \mu_q)^2]$ and the random variables $X$ with the probability density function $p(x)$ we have the following *q*-Cramér-Rao type inequality:

$$J_q(X) \geq \frac{1}{\sigma_q^2} \quad \text{for} \quad q \in [0, 1) \cup (1, 3).$$

In addition, the equality holds if the probability density function is given by the *q*-Gaussian density function:

$$\phi_q(x) \equiv \frac{1}{Z_q} \exp_q \left\{ -\frac{\beta_q(x - \mu_q)^2}{\sigma_q^2} \right\}, \quad Z_q = \int_{-\infty}^{\infty} \exp_q \left\{ -\frac{\beta_q(x - \mu_q)^2}{\sigma_q^2} \right\}, \quad \beta_q = \frac{1}{3 - q},$$

with the *q*-variance:

$$\sigma_q = \frac{2}{\Gamma_q/(3 - q)} \frac{q^{3-q} \Gamma_q(1-q)^{1/2}}{\Gamma_q(1/2, 1/q)}, \quad (0 \leq q < 1)$$

or

$$\sigma_q = \frac{2}{\Gamma_q/(3 - q)} \frac{q^{3-q} \Gamma_q(q - 1)^{1/2}}{\Gamma_q(1/2, 1/q, 1/2)}, \quad (1 < q < 3).$$

### 3. The *q*-Fisher information based on the normalized *q*-expectation value

In this section, we give another definition of *q*-Fisher information, based on the normalized *q*-expectation value $E_q^{(nor)}$.

**Definition 2** For the random variable $X$ with the probability density function $f(x)$, we define the *q*-score function $s_q(x)$ and the *q*-Fisher information $J_q(X)$ by

$$s_q(x) \equiv \frac{d \ln f(x)}{dx}, \quad (6)$$

$$J_q^{(nor)}(X) \equiv E_q^{(nor)} [s_q(X)^2], \quad (7)$$

where the normalized *q*-expectation value $E_q^{(nor)}$ is defined by $E_q^{(nor)}[g(X)] \equiv \frac{\int g(x)f(x)^{qdx}}{\int f(x)^{qdx}}$ for random variables $g(X)$ for any continuous function $g(x)$ and the probability density function $f(x)$.

Note that our *q*-Fisher informations defined in Definition 1 and Definition 2 are different from those in several literature [13, 12, 2, 11, 15, 14, 8]. Then we have the following theorem.

**Theorem 2** (*q*-Cramér-Rao type inequality [7]) Given the random variable $X$ with the probability density function $p(x)$, the *q*-expectation value $\mu_q \equiv E_q^{(nor)}[X]$ and the *q*-variance $\sigma_q^2 \equiv E_q^{(nor)}[(X - \mu_q)^2]$, we have the following *q*-Cramér-Rao type inequality:

$$J_q^{(nor)}(X) \geq \frac{1}{\sigma_q^2} \left( \int p(x)^qdx - 1 \right) \quad \text{for} \quad q \in [0, 1) \cup (1, 3).$$
Immediately we have
\[ J_q^{(nor)}(X) \geq \frac{1}{\sigma_q^2} \text{ for } q \in (1,3). \]  \hspace{1cm} (9)

In addition, the equality holds if the probability density function is given by the \( q \)-Gaussian density function:
\[ \phi_q(x) = \frac{1}{Z_q} \exp_q \left\{ -\frac{\beta_q(x - \mu_q)^2}{\sigma_q^2} \right\}, \quad Z_q = \int_{-\infty}^{\infty} \exp_q \left\{ -\frac{\beta_q(x - \mu_q)^2}{\sigma_q^2} \right\}, \quad \beta_q = \frac{1}{3 - q}, \]

with the \( q \)-variance:
\[ \sigma_q = \frac{2^{\frac{1}{q-1}} (3-q)^{\frac{q+1}{q(q-1)}} (1-q)^{\frac{1}{2}}}{B \left( \frac{1}{2}, \frac{1}{1-q} \right)}, \quad (0 \leq q < 1) \]  \hspace{1cm} (10)

or
\[ \sigma_q = \frac{2^{\frac{1}{q-1}} (3-q)^{\frac{3+q}{q(q-1)}} (q-1)^{\frac{1}{2}}}{B \left( \frac{1}{q-1} - \frac{1}{2}, \frac{1}{2} \right)}, \quad (1 < q < 3). \]  \hspace{1cm} (11)

The \( q \)-variances \( \sigma_q \) given in the above, Eq.(10) and Eq.(11) are same to those in Eq.(4) and Eq.(5). The proof is given in [7] in an analogical way to that in [9]. Both theorems in section 2 and section 3 recover the usual Cramér-Rao inequality \( J_1(X) \geq \frac{1}{\sigma_1^2} \) \( (J_1^{(nor)}(X) \geq \frac{1}{\sigma_1^2}) \) in the limit \( q \to 1 \).

4. On an example of the \( q \)-Fisher informations \( J_q \) and \( J_q^{(nor)} \) and the feature of the \( q \)-variance \( \sigma_q \) appeared in the previous sections

Here, we give the typical example for two \( q \)-Fisher informations. For the random variable \( G \) obeying the \( q \)-Gaussian distribution
\[ \phi_q(x) = \frac{1}{Z_q} \exp_q \left\{ -\frac{\beta_q(x - \mu_q)^2}{\sigma_q^2} \right\}, \]

where \( \beta_q \equiv \frac{1}{3-q} \) and the \( q \)-partition function \( Z_q \equiv \int_{-\infty}^{\infty} \exp_q \left\{ -\frac{\beta_q(x - \mu_q)^2}{\sigma_q^2} \right\} dx \), the \( q \)-score function is calculated as
\[ s_q(x) = -\frac{2\beta_q Z_q^{q-1}}{\sigma_q^2} (x - \mu_q), \]

which is linear with respect to \( x \). Thus we can calculate the \( q \)-Fisher informations as
\[ J_q(G) = J_q^{(nor)}(G) = \frac{4\beta_q^2 Z_q^{2q-2}}{\sigma_q^2}. \]  \hspace{1cm} (12)

Note that we have
\[ \lim_{q \to 1} J_q(G) = \lim_{q \to 1} J_q^{(nor)}(G) = \frac{1}{\sigma_1^2}. \]  \hspace{1cm} (13)

Finally we give comments on \( \sigma_q \). We firstly note on the limit \( q \to 1 \) for the \( q \)-variances \( \sigma_q \) given in Eq.(4) and Eq.(5) (or Eq.(10) and Eq.(11)). The following results were checked by the computer software:
\[ \lim_{q \to 1-0} \sigma_q = \lim_{q \to 1-0} \frac{2^{\frac{1}{q}} (3-q)^{\frac{q+1}{2(q-1)}} (1-q)^{\frac{1}{2}}}{B \left( \frac{1}{2}, \frac{1}{1-q} \right)} = \lim_{r \to +0} \frac{2^{1-r} (2+r)^{r-2}}{B \left( \frac{1}{2}, \frac{1}{1} \right)} = \frac{1}{\sqrt{2e\pi}}. \]
and
\[
\lim_{q \rightarrow 1+0} \sigma_q = \lim_{q \rightarrow 1+0} \frac{2^{\frac{1}{1-q}} (3-q)^{\frac{1}{2-q}} U(q - 1)^{\frac{1}{2}}}{B\left(\frac{1}{q-1} - \frac{1}{2}, \frac{1}{2}\right)} = \frac{1}{\sqrt{2e\pi}}.
\]

Furthermore, the following figure represents \(\sigma_q\) for \(q \in [0, 3)\).

![Figure 1](image_url)

**Figure 1.** The \(q\)-variances \(\sigma_q\) given in Eq.(4) and Eq.(5) (or Eq.(10) and Eq.(11))

5. Conclusion

In our previous paper [6], we gave the rough meaning of the parameter \(q\) from the information-theoretical viewpoint. In [6], we showed that the Tsallis entropies for \(q \geq 1\) had the subadditivity and therefore we had several information-theoretical properties in the case of \(q \geq 1\). However, the Tsallis entropies for \(q < 1\) did not have such properties. As similar as the case of the Tsallis entropies, in Theorem 2 we have found that the \(q\)-Fisher information have the quite same situation such that we have \(J_q^{\text{nor}}(X) \geq \frac{1}{\sigma_q}\) for \(q \geq 1\), however for the case of \(q < 1\), we do not have any relation between \(J_q^{\text{nor}}(X)\) and \(\frac{1}{\sigma_q}\) other than the inequality (8). Therefore these results give us the difference of the \(q\)-Fisher information \(J_q^{\text{nor}}(X)\) for \(q \in [0, 1)\) and \(J_q^{\text{nor}}(X)\) for \(q \in (1, 3)\), as similar as the Tsallis entropies did in [6]. Summarizing these results, we may conclude that the Tsallis entropies and the \(q\)-Fisher information \(J_q^{\text{nor}}(X)\) based on the normalized \(q\)-expectation value, make a sense for the case of \(q \geq 1\) in our setting. As for the \(q\)-Fisher information \(J_q(X)\) based on the \(q\)-expectation value, we do not find such difference on Cramér-Rao type inequality.

We close this section giving a comment on a possible application of our \(q\)-Fisher informations. The famous central limit theorem in probability theory, shows the distribution function of the standardized sum of an independent sequence of random variables convergences to Gaussian distribution under a certain assumption. The classical central limit theorem is usually proved by the characteristic function. However we have another method of its proof. Namely, the Fisher information can be applied to prove the classical central limit theorem [5, 16, 9]. Recently, the \(q\)-central limit theorem for \(q \geq 1\) was proved in [20] by introducing new notions such as the \(q\)-independence, the \(q\)-Fourier transformation and the \(q\)-characteristic function and so on. Therefore we may expect that a new proof of the \(q\)-central limit theorem may be given by applying the \(q\)-Fisher information in the future.
Acknowledgement
The author was partially supported by the Japanese Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (B), 18300003 and Grant-in-Aid for Encouragement of Young Scientists (B), 20740067.

References
[1] S. Abe, Heat and entropy in nonextensive thermodynamics: transmutation from Tsallis theory to Rényi-entropy-based theory, Physica A, Vol.300 (2001), pp.417-423.
[2] S. Abe, Geometry of escort distributions, Phys. Rev. E, Vol.68 (2003), 031101.
[3] S. Abe, S. Martínez, F. Pennini, and A. Plastino, Nonextensive thermodynamic relations, Phys. Lett. A, Vol.281 (2001), pp.126-130.
[4] S. Amari and H. Nagaoka, Methods of Information Geometry, OXFORD UNIV. PRESS, 2000.
[5] L.D. Brown, A proof of the central limit theorem motivated by the Cramér-Rao inequality, G. Kallianpur, P.R. Krishnaiah and J.K. Ghosh eds., Statistics and Probability: Essays in Honor of C.R. Rao, North-Holland Publishing Campany (1982), pp.141-148.
[6] S. Furuichi, Information theoretical properties of Tsallis entropies, J. Math. Phys., Vol.47 (2006), 023302.
[7] S. Furuichi, On the maximum entropy principle and the minimization of the Fisher information in Tsallis statistics, J. Math. Phys., Vol.50 (2009), 013303.
[8] H. Hasegawa, Stationary and dynamical properties of information entropies in nonextensive systems, Phys. Rev. E, Vol.77 (2008), 031133.
[9] O. Johnson, Information theory and the central limit theorem, Imperial College Press, 2004.
[10] S. Martínez, F. Nicolás, F. Pennini and A. Plastino, Tsallis’ entropy maximization procedure revisited, Physica A, Vol.286 (2000), pp.489-502.
[11] F. Pennini and A. Plastino, Power-law distributions and Fisher’s information measure, Physica A, Vol.334 (2004), pp.132-138.
[12] F. Pennini, A.R. Plastino and A. Plastino, Rényi entropies and Fisher informations as measures of nonextensivity in a Tsallis setting, Physica A, Vol.258 (1998), pp.446-457.
[13] A. Plastino, A. R. Plastino and H. G. Miller, Tsallis nonextensive thermostatistics and Fisher’s information measure, Physica A, Vol.235 (1997), pp.557-588.
[14] M. Portesi, F. Pennini and A. Plastino, Geometrical aspects of a generalized statistical mechanics, Physica A, Vol.307 (2002), pp.273-282.
[15] M. Portesi, A. Plastino, and F. Pennini, Information measures based on Tsallis’ entropy and geometric considerations for thermodynamic systems, Physica A, Vol.365 (2006), pp.173-176.
[16] R. Shimizu, On Fisher’s amount of information for location family, G.P. Patil et al. (eds.), Statistical Distributions in Scientific Work, Vol.3, D. Reidel Publishing Company, Dordrecht-Holland (1975), pp.305-312.
[17] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys., Vol.52 (1988), pp.479-487.
[18] C. Tsallis et al., Nonextensive Statistical Mechanics and Its Applications, edited by S. Abe and Y. Okamoto (Springer-Verlag, Heidelberg, 2001); see also the comprehensive list of references at http://tsallis.cat.cbpf.br/biblio.htm.
[19] C. Tsallis, R.S. Mendes and A.R. Plastino, The role of constraints within generalized nonextensive statistics, Physica A, Vol.261 (1998), pp.534-554.
[20] S. Umarov, C. Tsallis and S. Steinberg, On a $q$-central limit theorem consistent with nonextensive statistical mechanics, Milan Journal of Mathematics, Vol.76 (2008), pp.307-328.