On the Interplay of Regularity and Decay in Case of Radial Functions II. Homogeneous Spaces

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Abstract We deal with decay and boundedness properties of elements of radial subspaces of homogeneous Besov and Triebel-Lizorkin spaces. For the region of parameters which are of interest for us these homogeneous spaces are larger than the inhomogeneous counterparts. By switching from the inhomogeneous spaces to the homogeneous classes the properties of the radial elements change. Our investigations are based on the atomic decompositions for radial subspaces in the sense of Epperson and Frazier (J. Fourier Anal Appl. 1:311–353, 1995). Finally, we apply these results for deriving some assertions on compact embeddings on unbounded domains.

Keywords Radial functions · Homogeneous Besov and Triebel-Lizorkin spaces · Compact embeddings

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1 Introduction

Starting at the end of the seventies several authors have investigated the interplay of regularity and decay properties of radial functions, see e.g. Strauss [27] and Lions [16]. A detailed study of this interplay for radial functions belonging to inho-
Homogeneous spaces of Besov and Lizorkin-Triebel type has been made in our recent paper [24].

Elementary calculations with smooth radial function $g$ vanishing at infinity prove that

$$|x|^{d-1}|g(x)| \leq c_d \| \nabla g(x) \|_{L^1(\mathbb{R}^d)}.$$

see, e.g., Lions [16] and [24]. This inequality shows that it makes sense to switch to homogeneous function spaces, since only the norm of the homogeneous Sobolev space $W^1_1(\mathbb{R}^d)$ occurs on the right-hand side of the last inequality. The first serious step in this direction has been done by Cho and Ozawa [6]. These authors proved the inequality

$$|x|^{\frac{d}{2} - s}|g(x)| \leq c \| g \|_{\dot{H}^s(\mathbb{R}^d)}$$

for radial functions $g$ belonging to $\dot{H}^s(\mathbb{R}^d) \cap L^t(\mathbb{R}^d)$, $t = d/(\frac{d}{2} - s)$, $\frac{1}{2} < s < \frac{d}{2}$.

Here we are going to prove the following generalization.

**Main Theorem** Let $d \geq 2$, $0 < p < \infty$, $\max(\sigma_p, 1/p) < s < d/p$. Then there exists a constant $c$ such that

$$|x|^{\frac{d}{p} - s}|g(x)| \leq c \| g \|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)}$$

holds for all radial functions $g$ belonging to the intersection of the homogeneous Besov space $\dot{B}^s_{p,\infty}(\mathbb{R}^d)$ with the radial subspace of the Lorentz space $L^t,\infty(\mathbb{R}^d)$, $t = d/(\frac{d}{p} - s)$.

In view of the elementary embedding $\dot{H}^s(\mathbb{R}^d) \hookrightarrow \dot{B}^s_{2,\infty}(\mathbb{R}^d)$ the inequality (2) really generalizes (1). The restrictions for $s$ are essentially optimal. We will comment on this in Sect. 4. We would like to point out that in contrast to homogeneous spaces the decay properties at infinity of radial functions belonging to inhomogeneous Besov or Triebel-Lizorkin spaces do not depend on the smoothness parameter $s$, cf. [23, 24].

The literature on inhomogeneous function spaces with different symmetry constraints is quite rich, cf. e.g. [7, 14–16, 23, 26, 27]. The authors study such problems as decay near infinity, controlled unboundedness around the origin, higher regularity out of the origin and compactness of Sobolev embeddings. Much less seems to be known about the properties of radial functions belonging to homogeneous spaces. It is our aim in this paper to investigate decay properties of radial functions in the general framework of homogeneous Besov and Triebel-Lizorkin spaces on $\mathbb{R}^d$. We deal also with the regularity out of the origin and compactness of some embeddings. In particular, we extend the above-mentioned results of Cho and Ozawa to $p \neq 2$. Our treatment heavily relies on atomic decompositions for radial subspaces as developed by Epperson and Frazier [8].

The paper is organized as follows. In Sect. 2 our main effort consists in defining what is meant by a radial element in homogeneous function spaces. Section 3 is devoted to the study of regularity properties of radial functions out of the origin. Next, in Sect. 4, we shall prove our Main Theorem. A further part of this section is taken for the investigation of some limiting cases. Finally, we investigate the compactness
of embeddings in the framework of radial subspaces of the homogeneous function spaces. Let us mention that a naive extension of the known compact embeddings from the non-homogeneous case to the homogeneous one does not work. Finally, in two appendices, we collect the definitions and some properties of various types of homogeneous function spaces.

**Notation** As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. If $X$ and $Y$ are two quasi-Banach spaces, then the symbol $Y \hookrightarrow X$ indicates that the embedding is continuous. As usual, the symbol $c$ denotes positive constants which depend only on the fixed parameters $s, p, q$ and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols “$\lesssim$” and “$\gtrsim$” instead of “$\leq$” and “$\geq$”, respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \leq cB$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. For two quasi-Banach spaces $X$ and $Y$ the set of all linear and continuous operators, which map $X$ into $Y$, is denoted by $L(X, Y)$.

We shall use the following conventions throughout the paper:

- If $E$ denotes a space of functions on $\mathbb{R}^d$ then by $RE$ we mean the subset of radial functions in $E$ and we endow this subset with the same quasi-norm as the original space.
- Homogeneous Besov and Lizorkin-Triebel spaces are denoted by $\dot{B}^s_{p,q}(\mathbb{R}^d)$ and $\dot{F}^s_{p,q}(\mathbb{R}^d)$, respectively. Inhomogeneous Besov and Lizorkin-Triebel spaces are denoted by $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$.
- If there is no reason to distinguish between these two scales we will use the notation $\dot{A}^s_{p,q}(\mathbb{R}^d)$. If, in one assertion, several spaces of type $\dot{A}^s_{p,q}(\mathbb{R}^d)$ occur, then one has to replace $A$ always by the same letter (if not otherwise stated). Similarly for the radial subspaces.
- The symbol $[f]$ or $([f])_m$ always refers to a class of functions, see (3) and (38). In case, we formulate an assertion for the class, then it means the assertion holds for all elements of the class.
- The set $C^\infty_0(\mathbb{R}^d)$ is the collection of all infinitely differentiable and compactly supported functions on $\mathbb{R}^d$. Throughout the paper $\varphi \in C^\infty_0(\mathbb{R}^d)$ denotes a radial cut-off function, i.e., $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 3/2$.

## 2 Radial Elements of Homogeneous Besov and Triebel-Lizorkin Spaces

Following an idea of Cho and Ozawa [6] we shall describe the classes of functions we are interested in as intersections of homogeneous spaces and Lebesgue spaces (Lorentz spaces), see Sect. 2.3.

### 2.1 Radial Distributions

In homogeneous function spaces of the above type one considers classes of functions

$$[f] := \{ f + p : p \text{ polynomial over } \mathbb{R}^d \}, \quad f \in S'(\mathbb{R}^d), \quad (3)$$
instead of functions itself. Sometimes this causes some difficulties, many times assertions require a particular interpretation as, e.g., the inequality (2). For definitions and basic properties of homogeneous spaces of Besov and Triebel-Lizorkin type we refer to Appendix A below and to the references given there.

Following [8] we use the following definition of radiality.

**Definition 1** Let \( f \in \mathcal{S}'(\mathbb{R}^d) \).

(i) The distribution \( f \) is called radial if it is invariant under rotations around the origin, i.e.

\[
f(\varphi \circ \Phi) = f(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^d),
\]

for all such rotations \( \Phi \).

(ii) We call the class \([f]\) radial if \([f]\) contains a radial distribution \(g\).

The following simple observation will be of some use. Let \([f] \in \dot{A}^s_{p,q}(\mathbb{R}^d)\) be radial and let \(g\) be one of the radial elements in \([f]\). Let \((\varphi_j)_{j}\) be the smooth, homogeneous, dyadic and radial decomposition of unity, defined in Appendix A in formula (36) below. Then the distribution

\[
\sigma(g) := \sum_{j = -\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F}g]
\]

is radial as well, since the Fourier transform of a radial function is radial. However, the right-hand side does not depend on the particular element \(g\) in \([f]\). Hence, we may write

\[
\sigma([f]) = \sum_{j = -\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F}h], \quad (4)
\]

where \(h \in [f]\) is arbitrary. The mapping \([f] \mapsto \sigma([f])\) has some further nice properties which we are going to recall now. By \(\sigma(\dot{B}^s_{p,q}(\mathbb{R}^d))\) we denote the set of all images under the mapping \(\sigma\) equipped with the quasi-norm

\[
\|\sigma([f])|\sigma(\dot{B}^s_{p,q}(\mathbb{R}^d))\| := \|[f]|\dot{B}^s_{p,q}(\mathbb{R}^d)\|.
\]

Later on we shall need the following formula with respect to the real method of interpolation, see [1, 2, 28] for the basics.

**Lemma 1** Let \(0 < p, q, q_1, s \leq \infty, s_0, s_1 \in \mathbb{R}, s_0 \neq s_1\), and \(0 < \Theta < 1\). Then, with \(s := (1 - \Theta)s_0 + \Theta s_1\), we have

\[
\left(\sigma(\dot{B}^s_{p,q_0}(\mathbb{R}^d)), \sigma(\dot{B}^s_{p,q_1}(\mathbb{R}^d))\right)_{\Theta,q} = \sigma(\dot{B}^s_{p,q}(\mathbb{R}^d)).
\]

(5)

**Remark 1** The formula

\[
(\dot{B}^s_{p,q_0}(\mathbb{R}^d), \dot{B}^s_{p,q_1}(\mathbb{R}^d))_{\Theta,q} = \dot{B}^s_{p,q}(\mathbb{R}^d),
\]

(6)
also in the above generality, can be found in [13] with a reference to [19, Chap. 11]. This result is stated also in [30, 5.2.5] but without proof.

Proof The mapping \( \sigma : \dot{B}^{s_0}_{p,q_0}(\mathbb{R}^d) \rightarrow \sigma(\dot{B}^{s_0}_{p,q_0}(\mathbb{R}^d)) \) is an isometry. Hence, (5) follows immediately from (6). \( \square \)

2.2 Lorentz Spaces

Some properties of Besov spaces are partly easier described in terms of Lorentz spaces than in terms of Lebesgue spaces. For a measurable function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) its non-increasing rearrangement is denoted by \( f^* \), i.e.,

\[
\sup_{0 < t < \infty} t^{1/p} f^*(t)
\]

is finite. We refer to [1, Chap. 4], [2, 1.3] or [28, 1.18.6] for the basic properties of these spaces. They represent natural refinements of the Lebesgue spaces \( L^p(\mathbb{R}^d) \) in view of the identity \( L^p(\mathbb{R}^d) = L^p,p(\mathbb{R}^d) \). Below we shall use the formula

\[
(L_{t_0}(\mathbb{R}^d), L_{t_1}(\mathbb{R}^d))_{\Theta,q} = L_{t,q}(\mathbb{R}^d), \quad 0 < \Theta < 1,
\]

where

\[
\frac{1}{t} = \frac{1}{t_0} + \frac{\Theta}{t_1}
\]

and \( 0 < t_0 < t_1 \leq \infty, t_0 \leq q \), see [2, Theorem 5.2.1] or [28, Theorem 1.18.7/2].

Let \( C_0(\mathbb{R}^d) \) be the space of all uniformly continuous functions vanishing at infinity. Finally, we shall use the following abbreviation

\[
\sigma_p := d \max\left(0, \frac{1}{p} - 1\right).
\]

There is no connection between this number \( \sigma_p \) and the mapping \( \sigma \). These are just standard notations in this field and we hope there will be no misunderstanding.

Lemma 2 Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \).

(i) Let \( \sigma_p < s < d/p \). Then

\[
\sigma \in \mathcal{L}(\dot{B}^{s}_{p,q}(\mathbb{R}^d), L_{t,q}(\mathbb{R}^d)), \quad t := \frac{d}{p - s},
\]
and

$$\sigma \in \mathcal{L}(\dot{F}^s_{p,q} (\mathbb{R}^d), L_t(\mathbb{R}^d)), \quad t := \frac{d}{p - s}.$$ 

(ii) Let \( s = d/p \). Then

$$\sigma \in \mathcal{L}(\dot{B}^{d/p}_{p,1} (\mathbb{R}^d), C_0(\mathbb{R}^d)).$$

(iii) Let \( 0 < p \leq 1 \) and \( s = d/p \). Then

$$\sigma \in \mathcal{L}(\dot{F}^{d/p}_{p,q} (\mathbb{R}^d), C_0(\mathbb{R}^d)).$$

**Proof Step 1.** Proof of (i). Observe, that \( s > \sigma p \) implies \( 1 < t < \infty \). Now we turn to Besov spaces. We have

$$\dot{B}^s_{p,\min(1,p)}(\mathbb{R}^d) \hookrightarrow \dot{B}^s_{p,p}(\mathbb{R}^d) = \dot{F}^s_{p,p}(\mathbb{R}^d) \hookrightarrow \dot{F}^0_{t,2}(\mathbb{R}^d),$$

see [13] or Lemma 10. Next we use the Littlewood-Paley characterization of \( L_t(\mathbb{R}^d) \), i.e.

$$\| f \|_{\dot{F}^0_{t,2}(\mathbb{R}^d)} = \left\| \left( \sum_{j=\infty}^{\infty} |\mathcal{F}^{-1} [\varphi_j \mathcal{F} f]|^2 \right)^{1/2} \right\|_{L_t(\mathbb{R}^d)}$$

$$\times \left\| \sum_{j=\infty}^{\infty} \mathcal{F}^{-1} [\varphi_j \mathcal{F} f] \right\|_{L_t(\mathbb{R}^d)}$$

$$= \| \sigma([f]) \|_{L_t(\mathbb{R}^d)}.$$ 

For the latter we refer to, e.g., [17]. Hence we got

$$\sigma \in \mathcal{L}(\dot{B}^s_{p,\min(1,p)}(\mathbb{R}^d), L_t(\mathbb{R}^d)), \quad t := \frac{d}{d - s}.$$ 

Next we use Lemma 1. We choose \( s_1 > s_0 > \sigma p \) and \( \Theta \in (0, 1) \) such that \( s := (1 - \Theta)s_0 + \Theta s_1 \). Furthermore, we put

$$t_i := \frac{d}{d - s_i}, \quad i = 1, 2.$$ 

Obviously this implies

$$\frac{1}{t} = \frac{1 - \Theta}{t_0} + \frac{\Theta}{t_1}.$$ 

Simple monotonicity properties of the real method yield

$$\sigma(\dot{B}^s_{p,q}(\mathbb{R}^d)) = (\sigma(\dot{B}^{s_0}_{p,\min(1,p)}(\mathbb{R}^d)), \sigma(\dot{B}^{s_1}_{p,\min(1,p)}(\mathbb{R}^d)))_{\Theta, q}$$

$$\hookrightarrow (L_{t_0}(\mathbb{R}^d), L_{t_1}(\mathbb{R}^d))_{\Theta, q} = L_{t, q}(\mathbb{R}^d).$$
This proves (i) in case of Besov spaces. The assertion concerning the Triebel-Lizorkin spaces follows from Lemma 10 and the Littlewood-Paley characterization of $L_t(\mathbb{R}^d)$.

**Step 2.** Proof of (ii). We shall use the Nikol’skij inequality

$$
\sup_{x \in \mathbb{R}^d} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](x)| \leq c 2^{jd/p} \|\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\|_{L^p(\mathbb{R}^d)},
$$

where $c$ is independent of $f \in \mathcal{S}'(\mathbb{R}^d)$ and $j \in \mathbb{Z}$, see [30, 1.3.2]. By Paley-Wiener theorem $\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$ is a smooth function. Using the Nikol’skij inequality we obtain

$$
\sup_{x \in \mathbb{R}^d} \left| \sum_{j=-M}^{N} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](x) \right| \leq c \|f\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)},
$$

with the same constant as in (7), independent of $M, N \in \mathbb{N}$. Convergence of the sum $\sum_{j=-M}^{N} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$ in $C(\mathbb{R}^d)$ follows by the same type of argument. This proves uniform continuity and boundedness of the limit. Since each summand $\mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$ is an uniformly continuous function in $L^p(\mathbb{R}^d), p < \infty$ we obtain

$$
\lim_{|x| \to \infty} \left| \sum_{j=-M}^{N} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](x) \right| = 0
$$

for each fixed pair $(M, N) \in \mathbb{N}^2$. By the uniform convergence of $\sum_{j=-M}^{N} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]$ to $\sigma(f)$ this yields

$$
\lim_{|x| \to \infty} \sigma(f)(x) = 0
$$

as well.

**Step 3.** Observe, that the third part follows from the second in combination with Jawerth’s embedding theorem

$$
\dot{F}^{d/p}_{p,q}(\mathbb{R}^d) \hookrightarrow \dot{B}^{d/u}_{u,p}(\mathbb{R}^d), \quad 1 < u < \infty,
$$

see [13] or Lemma 10. □

**Remark 2** (i) Essentially the Lemma is well-known. We refer to Peetre [19, Theorem 7, Chap. 11, p. 242] for the first part (inhomogeneous case) and to Bourdaud [3] for the second.

(ii) The limiting situation $s = \sigma_p$ is investigated in Peetre [19, Theorem 7, Chap. 11, p. 242] and in Vybiral [33].

(iii) Let $\sigma_p < s < d/p$. For later use we mention that

$$
\sigma \in \mathcal{L}(\dot{B}_{p,q}^s(\mathbb{R}^d), L_t(\mathbb{R}^d)), \quad t := \frac{d}{d/p - s}, \quad q \leq t,
$$

and

$$
\sigma \in \mathcal{L}(\dot{F}_{p,q}^s(\mathbb{R}^d), L_t(\mathbb{R}^d)), \quad t := \frac{d}{d/p - s}, \quad 0 < q \leq \infty.
$$
2.3 Radial Subspaces of Homogeneous Besov-Triebel-Lizorkin Spaces

Lemma 2 gives us the possibility to describe the subspaces of homogeneous Besov-Triebel-Lizorkin spaces we are interested in, in a much more transparent way.

Lemma 3 Let $0 < p < \infty$ and $0 < q \leq \infty$.

(i) Let $\sigma_p < s < d/p$. Then
\[
\sigma(R \dot{A}^s_{p,q}(\mathbb{R}^d)) = \dot{A}^s_{p,q}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d), \quad t := \frac{d}{d/p - s}.
\]

(ii) Let $s = d/p$. Then
\[
\sigma(R \dot{B}^{d/p}_{p,1}(\mathbb{R}^d)) = \dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d).
\]

(iii) Let $0 < p \leq 1$ and $s = d/p$. Then
\[
\sigma(R \dot{F}^{d/p}_{p,q}(\mathbb{R}^d)) = \dot{F}^{d/p}_{p,q}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d).
\]

Proof If $[f]$ is radial, then $\sigma([f])$ is a radial distribution. In case of (i) Lemma 2 yields $\sigma([f]) \in RL_{t,\infty}(\mathbb{R}^d)$ and, of course, within $[f]$ the function $\sigma([f])$ is the only element in $[f]$ with this property. Similarly we argue in case (ii) and (iii), respectively. □

Remark 3 (i) Using Remark 2 we obtain
\[
\sigma(R \dot{B}^s_{p,q}(\mathbb{R}^d)) = \dot{B}^s_{p,q}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d), \quad t := \frac{d}{d/p - s}, \quad q \leq t,
\]
and
\[
\sigma(R \dot{F}^s_{p,q}(\mathbb{R}^d)) = \dot{F}^s_{p,q}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d), \quad t := \frac{d}{d/p - s}, \quad 0 < q \leq \infty.
\]

(ii) Cho and Ozawa [6] have used the above identity to introduce radial subspaces of $\dot{H}^s(\mathbb{R}^d) = \dot{B}^s_{2,2}(\mathbb{R}^d) = \dot{F}^s_{2,2}(\mathbb{R}^d)$.

2.4 Homogeneous Versus Inhomogeneous Spaces

Let us start with recalling the relations between the homogeneous spaces $\dot{A}^s_{p,q}(\mathbb{R}^d)$ and their inhomogeneous counterparts $A^s_{p,q}(\mathbb{R}^d)$. If $0 < p < \infty$, $0 < q \leq \infty$ and $s > \sigma_p$, then
\[
A^s_{p,q}(\mathbb{R}^d) = L_p(\mathbb{R}^d) \cap \dot{A}^s_{p,q}(\mathbb{R}^d)
\]
and
\[
\|g|A^s_{p,q}(\mathbb{R}^d)\| \asymp \|g|L_p(\mathbb{R}^d)\| + \|[g]|\dot{A}^s_{p,q}(\mathbb{R}^d)\|.
\]
cf. [22, Proposition 2.6.2/3] or [2, Theorem 6.3.2]. Formula (8) has to be interpreted in the following way:

• if inside the class \([f] \in \dot{A}^s_{p,q}(\mathbb{R}^d)\) is one representative \(g\) belonging to \(L_p(\mathbb{R}^d)\), then this function \(g\) belongs to the inhomogeneous space \(A^s_{p,q}(\mathbb{R}^d)\) and the quasi-norm equivalence (9) holds.

• On the other hand, if \(g \in A^s_{p,q}(\mathbb{R}^d)\) then the associated class \([g] \in \dot{A}^s_{p,q}(\mathbb{R}^d)\) and the quasi-norm equivalence (9) holds.

By means of such an interpretation it is clear that under the given restrictions the homogeneous spaces are larger than its inhomogeneous counterparts. Hence, decay and boundedness properties of elements of radial subspaces of homogeneous spaces can be quite different from those of radial subspaces of inhomogeneous spaces. It is instructive to look at the following family of test functions. For \(\alpha > 0\) and \(\delta \geq 0\) we define

\[
g_{\alpha,\delta}(x) := (1 + |x|^2)^{-\alpha/2}(\log(e + |x|^2))^{-\delta}. \tag{10}\]

Elementary calculations yield the following.

**Lemma 4** Let \(1 \leq p < \infty\) and \(0 < q < \infty\).

(i) The function \(g_{\alpha,\delta}\) belongs to \(L_{p,q}(\mathbb{R}^d)\) if, and only if either \(\alpha > d/p\) or \(\alpha = d/p\) and \(\delta q > 1\).

(ii) Let \(m \in \mathbb{N}\). Then \(g_{\alpha,\delta} \in \dot{W}^m_p(\mathbb{R}^d)\) if, and only if either \(\alpha + m > d/p\) or \(\alpha + m = d/p\) and \(\delta > 1/p\).

**Remark 4** Comparing (i) and (ii) it becomes obvious that the conditions for belonging to the space \(\dot{W}^m_p(\mathbb{R}^d)\) are weaker than those for belonging to \(L_{p,q}(\mathbb{R}^d)\).

These assertions extend in a natural way to fractional order of smoothness. For a proof we refer to Appendix A below.

**Proposition 1** Let \(\delta > 0\).

(i) We suppose

\[
\sigma_{p,q} := d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right) < s < \frac{d}{p}. \tag{11}\]

Then \(g_{\alpha,\delta}\) belongs to \(\dot{F}^s_{p,q}(\mathbb{R}^d)\) if either \(\alpha > \frac{d}{p} - s\) \((\delta \geq 0\) arbitrary) or \(\alpha = \frac{d}{p} - s\) and \(\delta > 1/p\).

(ii) We suppose \(\sigma_p < s < d/p\). Then \(g_{\alpha,\delta}\) belongs to \(\dot{B}^s_{p,q}(\mathbb{R}^d)\) if either \(\alpha > \frac{d}{p} - s\) or \(\alpha = \frac{d}{p} - s\) and \(\delta > 1/q\).

**Remark 5** (i) In a certain sense locally there is not much difference between homogeneous spaces and inhomogeneous spaces. The main difference concerns the behaviour near infinity.
(ii) Let either $\sigma_{p,q} < s < d/p$ (if $A = F$) or $\sigma_p < s < d/p$ (if $A = B$). Then the embeddings

$$A^s_{p,q} (\mathbb{R}^d) \hookrightarrow \dot{A}^s_{p,q} (\mathbb{R}^d) \cap L_t,\infty (\mathbb{R}^d), \quad t := \frac{d}{d - s},$$

are strict.

### 3 Local Regularity of Radial Functions Outside the Origin

Radial functions have some extra regularity outside the origin. This has been observed for the first time by Lions [16]. We have dealt with this in the framework of inhomogeneous spaces in [24].

The extension to homogeneous spaces is more or less obvious. Let $[f] \in RA^s_{p,q} (\mathbb{R}^d)$. We use (4). For any element $g$ of $[f]$ we have

$$g = p + \sigma ([f]) = p + f_0 + f_1,$$

where $p$ denotes an appropriate polynomial and

$$f_0 := \sum_{j=-\infty}^{-1} \mathcal{F}^{-1} (\varphi_j \mathcal{F} f) \quad \text{and} \quad f_1 := \sum_{j=0}^{\infty} \mathcal{F}^{-1} (\varphi_j \mathcal{F} f).$$

By classical properties of the Fourier transform (the Fourier transform of a radial function is radial) and because of the fact, that also the functions $\varphi_j$ are radial, both, $f_0$ and $f_1$ are radial. The first sum $f_0$ is an entire analytic function of exponential type, whereas the second sum $f_1$ belongs to inhomogeneous space $RA^s_{p,q} (\mathbb{R}^d)$, see, e.g., [24]. Thus, the local smoothness depends on the second sum and therefore it is the same as in case of radial inhomogeneous spaces.

**Lemma 5** Let $[f] \in RA^{q_0}_{p_0,q_0} (\mathbb{R}^d)$. Then, for any $g \in [f]$, we have $g \in A^{s_1,\elloc}_{p_1,q_1} (\mathbb{R}^d)$ if $f_1 \in RA^{s_1}_{p_1,q_1} (\mathbb{R}^d)$.

Now we simply refer to [24], where we have dealt with this question in certain detail. The outcome are the following corollaries. To describe this in more detail we need Hölder-Zygmund spaces. Recall, that $C^s (\mathbb{R}^d) = B^s_{\infty,\infty} (\mathbb{R}^d)$ in the sense of equivalent norms if $s \notin \mathbb{N}_0$. Of course, also the spaces $B^s_{\infty,\infty} (\mathbb{R}^d)$ with $s \in \mathbb{N}$ allow a characterization by differences. We refer to [30, 2.2.2, 2.5.7] and [31, 3.5.3]. We shall use the abbreviation

$$Z^s (\mathbb{R}^d) = B^s_{\infty,\infty} (\mathbb{R}^d), \quad s > 0.$$

**Corollary 1** Let $d \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$, and $s > 1/p$. Let $\varphi \in C^\infty_0 (\mathbb{R}^d)$ be a radial function such that $0 \notin \text{supp} \varphi$. If $[f] \in \dot{RA}^s_{p,q} (\mathbb{R}^d)$, then for all $g \in [f]$ we have $\varphi g \in Z^{s-1/p} (\mathbb{R}^d)$. 
Remark 6 Here and in what follows we use another (standard) interpretation. The elements of homogeneous spaces are classes of functions and the elements of such a class are functions defined almost everywhere. Let \( \{ g \} \) be such an equivalence class with respect to coincidence almost everywhere. If it contains a continuous representative then we shall call \( \{ g \} \) continuous and speak of values of \( \{ g \} \) at any point. In the latter case we call also the class \( \{ g \} \) continuous (since all its elements are continuous in the above sense).

Some limiting cases are collected in the next theorem.

Corollary 2 Let \( d \geq 2 \) and \( 0 < p < \infty \).

(i) If \( \{ f \} \in R\dot{B}^{1/p}_{p,1}(\mathbb{R}^d) \), then all \( g \in \{ f \} \) are continuous outside the origin.

(ii) Let \( 0 < p \leq 1 \). If \( \{ f \} \in R\dot{F}^{1/p}_{p,\infty}(\mathbb{R}^d) \), then all \( g \in \{ f \} \) are continuous outside the origin.

Remark 7 Let \( s = d/p \). Then it follows from Lemma 2 that \( \{ f \} \in R\dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \) implies that \( f \) is continuous on \( \mathbb{R}^d \). Similarly, if \( \{ f \} \in R\dot{F}^{d/p}_{p,\infty}(\mathbb{R}^d) \), \( 0 < p \leq 1 \), then \( f \) is continuous on \( \mathbb{R}^d \).

4 Decay Properties of Radial Functions

In this section we shall deal with the elements of the spaces \( \sigma(\dot{A}^{s}_{p,q}(\mathbb{R}^d)) \), hence with true functions. However, below we shall use the reformulation of this condition given in Remark 17 in Appendix A.

Theorem 1 Let \( d \geq 2, 0 < p < \infty, s > \sigma_p \) and in addition

\[
\frac{1}{p} < s < \frac{d}{p}.
\]

(13)

(i) There exists a constant \( c > 0 \) such that

\[
|x|^{\frac{d}{p} - s} |g(x)| \leq c \|[g]\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)}
\]

holds for all \( g \in \dot{B}^s_{p,\infty}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d) \), \( t = d/(\frac{d}{p} - s) \), and all \( x \neq 0 \).
(ii) There exist a positive constant $c > 0$ and a function $g \in RL_{t, \infty}(\mathbb{R}^d)$, $t = d/(d - p - s)$, such that $[g] \in \dot{B}^{s}_{p, \infty}(\mathbb{R}^d)$ and

$$|x|^{d - s} |g(x)| \geq c \|[g]\| \dot{B}^{s}_{p, \infty}(\mathbb{R}^d)$$

holds for all $x \neq 0$.

Proof Step 1. We shall use the smooth radial atomic decomposition of Epperson and Frazier, cf. Theorem 6. in Appendix B below.

Let $g \in \dot{B}^{s}_{p, \infty}(\mathbb{R}^d) \cap RL_{t, \infty}(\mathbb{R}^d)$. Then $g = \sigma([g])$. Let $x \neq 0$. We choose $r \in \mathbb{Z}$ such that $2^{-r-1} \leq |x| \leq 2^{-r+1}$. Furthermore, there exists an atomic decomposition of $[g]$ such that

$$\sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k} \in [g]$$

and

$$\|[g]\| \dot{B}^{s}_{p, \infty}(\mathbb{R}^d) \asymp \|(s_{j,k})_{j,k} \| \dot{b}_{p, \infty} \asymp \sup_{j \in \mathbb{Z}} \left( \sum_{k=0}^{\infty} |s_{j,k}|^p \max(1, k)^{d-1} \right)^{1/p}.$$

Recall, $\text{supp} a_{j,k} \subset \tilde{A}_{j,k}$. We claim that one can find natural numbers $n_1, n_2$ and $n_3$ independent of $x$ and $r$ such that the following holds:

- If $j \leq r + n_1$ and $x \in \tilde{A}_{j,k}$, then $k = 0, 1, \ldots, n_2$.
- If $j > r + n_1$ and $x \in \tilde{A}_{j,k}$, then $2^{j-r+n_1} < k \leq 2^{j-r+n_1} + n_3$.

This is elementary and left to the reader. By using this and (48) we find

$$\left| \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k}(x) \right|$$

$$\lesssim \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} |s_{j,k}| 2^{-j(s - d/p)} \chi_{\tilde{A}_{j,k}}(x)$$

$$\lesssim \sum_{j < r + n_1} \sum_{k=0}^{n_2} |s_{j,k}| 2^{-j(s - d/p)} + \sum_{j \geq r + n_1} \sum_{k=2^{j-r-n_1}}^{2^{j-r-n_1} + n_3} |s_{j,k}| 2^{-j(s - d/p)}$$

$$\lesssim \sum_{j < r + n_1} 2^{-j(s - d/p)} \left( \sum_{k=0}^{n_2} |s_{j,k}|^p k^{d-1} \right)^{1/p}$$

$$+ \sum_{j \geq r + n_1} 2^{-j(s - d/p)} 2^{(j-r-n_1) \frac{1-d}{d} \left( \sum_{k=2^{j-r-n_1}}^{2^{j-r-n_1} + n_3} |s_{j,k}|^p k^{d-1} \right)^{1/p}}.$$
\[ \lesssim \| (s,j,k)_{j,k} \|_{\dot{B}^p_{p,\infty}} \left( \sum_{j < r + n_1} 2^{-j(s - \frac{d}{p})} + \sum_{j \geq r + n_1} 2^{-j(s - \frac{d}{p})} 2^{(j-r-n_1)\frac{1-d}{p}} \right) . \]

But
\[ \sum_{j < r + n_1} 2^{-j(s - \frac{d}{p})} \lesssim 2^{-r(s - \frac{d}{p})} \quad \text{if } s < \frac{d}{p}, \]
and
\[ \sum_{j \geq r + n_1} 2^{-j(s - \frac{d}{p})} 2^{(j-r-n_1)\frac{1-d}{p}} \lesssim 2^{-r(s - \frac{d}{p})} \quad \text{if } s > \frac{1}{p}. \]

Thus
\[ \left| \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k}(x) \right| \lesssim \| (s,j,k)_{j,k} \|_{\dot{B}^p_{p,\infty}} \| x \|^{s - \frac{d}{p}}. \quad (16) \]

This proves (i) with \( \sigma([g]) \) replaced by the specific atomic decomposition used in (16). Obviously, the atomic decomposition satisfies
\[ \lim_{|x| \to \infty} \left| \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k}(x) \right| = 0. \]

On the other hand we know \( \sigma([g]) \in L_{t,\infty}(\mathbb{R}^d) \), \( t = d/(d - s) \). Hence, \( \sigma([g]) = \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k} \).

Step 2. Let \( \varphi \) be a smooth radial function supported in \( \tilde{A}_{1,0} \) such that
\[ 0 \leq \varphi(x) \leq 1, \quad x \in \mathbb{R}^d, \quad \text{and} \quad \varphi(x) = 1 \quad \text{if } |x| \leq 2^{-1} \mu_{\nu,0}. \]

We put
\[ a_j(x) := 2^{-j(s - \frac{d}{p})} \varphi(2^j x), \quad x \in \mathbb{R}^d. \]

Then \( a_j \) is a smooth radial molecule in the sense of [8], associated to \( \tilde{A}_{j,0} \), such that
\[ a_j(x) = 2^{-j(s - \frac{d}{p})} \quad \text{if } x \in \tilde{A}_{j,0}. \]

Furthermore, the functions \( a_j \) satisfy (47) and (48) (\( |\gamma| \leq s + 1 \)) up to a general constant. We define
\[ f := \sum_{j \in \mathbb{Z}} a_j, \]
then \( [f] \in R\dot{B}^s_{p,\infty}(\mathbb{R}^d) \) and \( \| [f] \| R\dot{B}^s_{p,\infty}(\mathbb{R}^d) \| \lesssim 1 \), see Lemma 13. For \( 2^{-r-1} < |x| \leq 2^{-r}, \), \( r \in \mathbb{Z} \), we derive from the nonnegativity of \( \varphi \)
\[ \sum_{j \in \mathbb{Z}} a_j(x) \geq \sum_{j \leq r - 1} a_j(x) = \sum_{j \leq r - 1} 2^{-j(s - \frac{d}{p})} > 2^{(r-1)(\frac{d}{p} - s)}. \]

This implies (15). \( \square \)
The class \([g]\), associated to the function \(g\) constructed in the proof of (ii), does not belong to any of the spaces \(\dot{B}^s_{p,q}(\mathbb{R}^d)\), \(q < \infty\). In addition it does not belong to \(\dot{F}^s_{p,\infty}(\mathbb{R}^d)\). However, if we concentrate for given \(x\) to the function \(a_f\), where \(|x| \asymp 2^{-r}\), then we immediately obtain the following weaker version of a negative result.

**Lemma 6** Let \(d, p, q\) and \(s\) be as in Theorem 1. Then there exists a positive constant \(c > 0\) such that for all \(x \neq 0\) there exists a nontrivial function \(g \in C^\infty_0(\mathbb{R}^d)\), \([g] \in \dot{B}^s_{p,q}(\mathbb{R}^d)\), and

\[
|x|^\frac{d}{p} - s |g(x)| \geq c \|\[g]\| \dot{B}^s_{p,q}(\mathbb{R}^d).
\]

If in addition \(s > \sigma_{p,q}\), then the assertion remains true if we replace \(B\) by \(F\).

**Remark 8** There are more explicit functions which realize the extremal behaviour up to logarithmic terms. For example, let

\[
g(x) := \psi(x)|x|^{s-d}\frac{p}{d}(-\log |x|)^{-\delta} + (1 + |x|^2)^{-\frac{d}{p}}(\log(e + |x|^2))^{-\delta}, \quad x \in \mathbb{R}^d.
\]

Under the restrictions of Theorem 1 the class \([g]\) belongs to \(\dot{B}^s_{p,\infty}(\mathbb{R}^d)\) if \(\delta > 0\), see [22, Lemma 2.3.1] and Proposition 1.

There are some more results explaining the sharpness of (14). The first one is a consequence of the homogeneity of the norms.

**Lemma 7** Let \(0 < p < \infty\), \(0 < q \leq \infty\), and \(s \leq d/p\). Let \(\varrho : (0, \infty) \to (0, \infty)\) be continuous. Let us assume that there exists a constant \(c\) such that

\[
\varrho(|x|)|x|^{d}\frac{p}{d} - s\left|\sigma([f])(x)\right| \leq c\|[f]\| \dot{A}^s_{p,q}(\mathbb{R}^d)\|
\]

holds for all \([f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d)\) and all \(x \neq 0\). Then \(\varrho\) must be bounded.

**Proof** Let \(\lambda > 0\). With a function \(f\) also \(f(\lambda \cdot)\) belongs to \(R\dot{A}^s_{p,q}(\mathbb{R}^d)\) or more exactly, the corresponding classes. Furthermore, we have

\[
\|[f(\lambda \cdot)]\| \dot{A}^s_{p,q}(\mathbb{R}^d)\| \asymp \lambda^{s-d/p}\|[f]\| \dot{A}^s_{p,q}(\mathbb{R}^d)\|,
\]

see, e.g., [19] or [30, Remark 5.1.3/4]. Clearly \(\sigma(f(\lambda \cdot)) = \sigma(f)(\lambda \cdot)\). We apply the inequality (17) with \(f(\lambda \cdot)\) and obtain for the particular choice \(\lambda = |x|^{-1}\)

\[
\varrho(|x|)|\sigma(f)(1, 0, \ldots, 0)| \leq c\|[f]\| \dot{A}^s_{p,q}(\mathbb{R}^d)\|.
\]

Choosing \(f\) such that \(\sigma(f)(1, 0, \ldots, 0) \neq 0\) we obtain the boundedness of \(\varrho\). \(\square\)

**Remark 9** Comparison with the inhomogeneous situation. Let \(d, p, q\) and \(s\) as in Theorem 1. In [24] we proved...
\[ |x|^{d-s} |f(x)| \lesssim \| f \|_{B^s_{p,\infty}(\mathbb{R}^d)} \quad \text{if } |x| < 1, \]
\[ |x|^{d-1} |f(x)| \lesssim \| f \|_{B^s_{p,\infty}(\mathbb{R}^d)} \quad \text{if } |x| > 1, \]
(19)
valid for all \( f \in RB^s_{p,\infty}(\mathbb{R}^d) \). Thus the behaviour near zero is the same as in the case of homogeneous spaces, but the decay at infinity is improved by switching to the smaller inhomogeneous spaces.

4.2 Decay of Radial Functions—Limiting Cases

In this subsection we deal with the limiting situations, i.e., \( s = \frac{1}{p} \) and \( s = \frac{d}{p} \). To begin with we state positive results, first for Besov spaces, second for Triebel-Lizorkin spaces.

**Theorem 2** Let \( d \geq 2 \) and \( 0 < p < \infty \).

(i) Let in addition \( p > 1 - \frac{1}{d} \). Then there exists a constant \( c \) such that
\[ |x|^{d-1} |g(x)| \leq c \| [g] \|_{\dot{B}^{1/p}_{p,1}(\mathbb{R}^d)} \]
holds for all \( g \in \dot{B}^{1/p}_{p,1}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d), \ t = dp/(d-1) \).

(ii) Let \( s = d/p \). There exists a constant \( c \) such that
\[ |g(x)| \leq c \| [g] \|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)} \]
holds for all \( g \in \dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d) \).

**Proof**

**Step 1.** Proof of (ii). It is enough to mention that this is an immediate consequence of Lemma 2.

**Step 2.** Proof of (i). Observe that \( \sigma_p < 1/p \) if, and only if, \( p > 1 - 1/d \), and
\[ \dot{B}^{1/p}_{p,1}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d) = \dot{B}^{1/p}_{p,1}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d), \ t = dp/(d-1), \]
see Remark 3. The proof of the inequality (20) can be done similar to the proof of (14). We are going to use the notation from there. Again our point of departure is an atomic decomposition of \([g]\) such that
\[ \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k} \in [g] \]
and
\[ \| [g] \|_{\dot{B}^{1/p}_{p,1}(\mathbb{R}^d)} \asymp \| (s_{j,k})_{j,k} \|_{\dot{B}^{1/p}_{p,1}} \asymp \sum_{j \in \mathbb{Z}} \left( \sum_{k=0}^{\infty} |s_{j,k}|^p \max (1, k)^{d-1} \right)^{1/p}. \]
Let $|x| \asymp 2^{-r}$. Then, by using (48), we conclude

$$\left| \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k}(x) \right| \lesssim 2^{-r(1-d)/p} \sum_{j<r+n_1} 2^{-j-r}\left(\frac{1-d}{p}\right)^{1/p} \left( \sum_{k=0}^{n_2} |s_{j,k}|^p k^{d-1} \right)^{1/p}$$

$$+ 2^{-r(1-d)/p} \sum_{j\geq r+n_1} 2^{-j-r}\left(\frac{1-d}{p}\right)^{1/p} \left( \sum_{k=2^{j-r-n_1}}^{2^{j-r-n_1}+n_3} |s_{j,k}|^p k^{d-1} \right)^{1/p}$$

$$\lesssim 2^{-r(1-d)/p} \| (s_{j,k})_{j,k} \| \hat{b}_{p,1}.$$

This proves (20). \hfill \Box

**Theorem 3** Let $d \geq 2$ and $0 < p \leq 1$.

(i) Let in addition $p > 1 - \frac{1}{d}$. There exists a constant $c$ such that

$$|x|^{d-1} \| g(x) \| \leq c \| [g] \| \hat{F}_{p,\infty}^{1/p}(\mathbb{R}^d) \|$$

(21)

holds for all $g \in \hat{F}_{p,\infty}^{1/p}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d)$, $t = dp/(d - 1)$.

(ii) There exists a constant $c$ such that

$$|g(x)| \leq c \| [g] \| \hat{F}_{p,\infty}^{d/p}(\mathbb{R}^d) \|$$

(22)

holds for all $g \in \hat{F}_{p,\infty}^{d/p}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d)$.

**Proof** The inequality (22) follows from Lemma 2. Hence, it remains to deal with (21). Again we make use of the notation used in the proof of Theorem 1. This time we use an atomic decomposition of $[g]$ such that

$$\sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k} \in [g]$$

and

$$\| [g] \| \hat{F}_{p,\infty}^{1/p}(\mathbb{R}^d) \| \asymp \| (s_{j,k})_{j,k} \| \hat{f}_{p,\infty} \| \leq \sup_{j \in \mathbb{Z}} \sup_{k=0,1,...} |s_{j,k}| \hat{\chi}_{j,k}(p) \| L_p(\mathbb{R}^d) \|. $$

\hfill \Box
Let $|x| \asymp 2^{-r}$, $r \in \mathbb{Z}$. We divide this atomic decomposition of $[g]$ into three sums

$$
\begin{align*}
g_1 &:= \sum_{j<r+n_1} \sum_{k=0}^{\infty} s_{j,k} a_{j,k}(x), \\
g_2 &:= \sum_{r+n_1 \leq j < 0} \sum_{k=0}^{\infty} s_{j,k} a_{j,k}(x), \\
g_3 &:= \sum_{j \geq 0} \sum_{k=0}^{\infty} s_{j,k} a_{j,k}(x).
\end{align*}
$$

(23)

Here $g_2 \equiv 0$ if $r + n_1 \geq 0$. Since the atoms are radial also $g_1$, $g_2$ and $g_3$ are radial. Concerning $g_1$ we find (see (48) and use $p \leq 1$)

$$
|x| \frac{d-1}{p} |g_1(x)| \lesssim \sum_{j<r+n_1} 2^{(j-r)(d-1)/p} \sum_{k=0}^{n_2} |s_{j,k}| \lesssim \sup_j \left( \sum_{k=0}^{n_2} |s_{j,k}|^p \right)^{1/p}
$$

$$
\lesssim \sup_j \left( \sum_{k=0}^{n_2} |s_{j,k}|^p \int_{\mathbb{R}^d} |\tilde{\chi}_{j,k}^{(p)}(y)|^p dy \right)^{1/p}
$$

$$
\lesssim \left( \int_{\mathbb{R}^d} \sup_{j,k} |\tilde{\chi}_{j,k}^{(p)}(y)|^p dy \right)^{1/p} = \| (s_{j,k})_{j,k} | \hat{f}_p, \infty \|
$$

On the other hand, $g_3$ belongs to the inhomogeneous space $RF_{p,\infty}^1(\mathbb{R}^d)$ and

$$
\| g_3 | F_{p,\infty}^{1/p}(\mathbb{R}^d) \| \lesssim \| (s_{j,k})_{j,k} | \hat{f}_p, \infty \|,
$$

(24)

cf. [8, Sect. 6]. Now it follows from Theorems 10, 13 in [24] and (24) that

$$
|x| \frac{d-1}{p} |g_3(x)| \lesssim \| (s_{j,k})_{j,k} | \hat{f}_p, \infty \|.
$$

It remains to estimate the second sum. If $a_{j,k}$ is an atom centered at $\mathbb{A}_{j,k}$ then $\tilde{a}_{j,k}(x) := 2^{r(1-d)} a_{j,k}(2^{-r}x)$ is an atom centered at $\mathbb{A}_{j-r,k}$. In consequence the function

$$
\tilde{g}_2(x) := 2^{r(1-d)/p} g_2(2^{-r}x) = \sum_{r+n_1 \leq j < 0} \sum_{k=0}^{\infty} s_{j,k} \tilde{a}_{j,k}(x)
$$

belongs to $RF_{p,\infty}^{1/p}(\mathbb{R}^d)$ (by the same reasonings as in case of $g_3$) and

$$
\| \tilde{g}_2 | F_{p,\infty}^{1/p}(\mathbb{R}^d) \| \lesssim \| (s_{j,k})_{j,k} | \hat{f}_p, \infty \|,
$$

cf. [8, Sect. 6]. Applying once more Theorem 10, 13 in [24] we get

$$
|x| \frac{d-1}{p} |g_2(x)| \lesssim |2^{r(1-d)/p} g_2(x)| \lesssim |\tilde{g}_2(2^{-r}x)| \lesssim \| \tilde{g}_2 | F_{p,\infty}^{1/p}(\mathbb{R}^d) \|,
$$

(25)

since $|2^{-r}x| \asymp 1$. Now (23)–(25) imply (21).
Since the inhomogeneous spaces $RA^{s}_{p,q}(\mathbb{R}^d)$ are subspaces of $\sigma(R\dot{A}^s_{p,q}(\mathbb{R}^d))$ we can use some negative results obtained in [24] in the context of the inhomogeneous spaces.

**Corollary 3** Let $0 < p < \infty$ and $0 < q \leq \infty$.

(i) If either $\sigma_p < s < 1/p$ (if $A = B$) or $\sigma_{p,q} < s < 1/p$ (if $A = F$) then the following holds: For all sequences $(x^j)_{j=1}^{\infty}$ such that $\lim_{j \to \infty} |x^j| = \infty$, there exists a radial function $f \in RA^s_{p,q}(\mathbb{R}^d), \|f|\dot{A}^s_{p,q}(\mathbb{R}^d)\| = 1$, such that $f$ is unbounded in any neighborhood of $x^j$, $j \in \mathbb{N}$.

(ii) Let $s = 1/p > \sigma_p$ and $1 < q \leq \infty$. Then the assertion in (i) remains true for the spaces $RB^{1/p}_{p,q}(\mathbb{R}^d)$.

(iii) Let $\sigma_{p,q} < s = 1/p < 1$. Then the assertion in (i) remains true for the spaces $RF^{1/p}_{p,q}(\mathbb{R}^d)$.

**Remark 10** We comment on the case $s = d/p$. If $1 < q \leq \infty$, then the inhomogeneous space $RB^{d/p}_{d,q}(\mathbb{R}^d)$ contains functions which are unbounded in a neighborhood of the origin, see [4]. Hence, if $g$ is such a function, the class $[g]$ contains elements which are all unbounded in a neighborhood of the origin. Similar arguments apply to the cases $RF^{d/p}_{p,q}(\mathbb{R}^d), 1 < p < \infty, 0 < q \leq \infty$.

For convenience of the reader we formulate the consequences for potential and Sobolev spaces, see Appendix A for a definition.

**Corollary 4** Let $1 < p < \infty$. Then the following assertions are equivalent.

(i) There exists a constant $c$ such that

$$|x|^{d/p-s}|g(x)| \leq c\|[g]\|_{\dot{H}^s_p(\mathbb{R}^d)}\|$$

holds for all $g \in \dot{H}^s_p(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d), t = d/(d/p - s)$, and all $x \neq 0$.

(ii) We have $1/p < s < d/p$.

**Proof** The equivalence follows from the identity $\dot{H}^s_p(\mathbb{R}^d) = \dot{F}^s_{p,2}(\mathbb{R}^d)$, see Remark 16(iii), Theorem 1 and Corollary 4.

**Remark 11** For $p = 2$ the inequality (26) has been proved in a simpler way by Cho and Ozawa [6]. They used tools from Fourier analysis. By explicit counterexamples they disproved (26) in case $\sigma(\dot{H}^{d/2}_{2}(\mathbb{R}^d))$.

**Corollary 5** Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then the following assertions are equivalent.

(i) There exists a constant $c$ such that

$$|x|^{d/m}|g(x)| \leq c\|[g]_m\|_{\dot{W}^m_p(\mathbb{R}^d)}\|$$

holds for all $g \in \dot{W}^m_p(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d), t = d/(d/p - m)$, and all $x \neq 0$. 

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We have \( 1 \leq m < d/p \).

**Proof** The assertions follows immediately from the existence of an isomorphism which maps \( \dot{W}_p^m (\mathbb{R}^d) \) onto \( \dot{F}^m_{p,2} (\mathbb{R}^d) \), see the remarks in Sect. A.2. \( \square \)

**Remark 12** As a consequence of Theorem 1 and the embeddings \( \dot{B}^m_{1,1} (\mathbb{R}^d) \hookrightarrow \dot{W}^m_1 (\mathbb{R}^d) \hookrightarrow \dot{B}^m_{1,\infty} (\mathbb{R}^d) \), we also obtain inequality (27) in case \( p = 1 \) and \( 1 \leq m < d \), i.e.

\[
|x|^{d-m} |g(x)| \leq c \| [g]_m \| \dot{W}^m_1 (\mathbb{R}^d)
\]

holds for all \( g \in \dot{W}^m_1 (\mathbb{R}^d) \cap RL_{t,\infty} (\mathbb{R}^d), t = d/(d-m) \), and all \( x \neq 0 \).

5 **Compact Embeddings**

Many authors applied the decay properties of radial functions to prove compactness of related embeddings. For inhomogeneous spaces of Besov and Triebel-Lizorkin type we have done this in [23]. For convenience of the reader we recall a part of these results.

**Proposition 2** Let \( 0 < p_0, q_0 \leq \infty, 1 \leq p_1 \leq \infty \) and \( s_0 \in \mathbb{R} \). Let \( A \in \{ B, F \} \).

(i) Let \( d \geq 2 \). Then the embedding \( RA^s_{p_0,q_0} (\mathbb{R}^d) \hookrightarrow L_{p_1} (\mathbb{R}^d) \) is compact if, and only if, \( p_0 < p_1 \) and

\[
s_0 > d, \left( \frac{1}{p_0} - \frac{1}{p_1} \right).
\]

(ii) Let \( d = 1 \). For all triples \( (s_0, p_0, q_0) \) the space \( RA^s_{p_0,q_0} (\mathbb{R}) \) is not compactly embedded into \( L_{p_1} (\mathbb{R}) \).

**Remark 13** Here we tacitly allow \( p = \infty \) in case of Triebel-Lizorkin spaces. Originally Triebel had introduced \( F^s_{\infty,q} (\mathbb{R}^d), 1 < q \leq \infty \), essentially by duality, see [30]. However, the nowadays used definition of \( F^s_{\infty,q} (\mathbb{R}^d), 0 < q \leq \infty \), has been found by Frazier and Jawerth in [11]. We refer to this article as well as to the lecture notes [34] for the definition and for a collection of properties. At this moment we only recall that

\[
B^s_{\infty,q} (\mathbb{R}^d) \hookrightarrow F^s_{\infty,q} (\mathbb{R}^d) \hookrightarrow F^s_{\infty,\infty} (\mathbb{R}^d) = B^s_{\infty,\infty} (\mathbb{R}^d),
\]

cf. [22, Remark 2.2.3/3] or [34, Propositions 2.2, 2.4].

The naive extension of Proposition 2, i.e., replacing \( RA^s_{p_0,q_0} (\mathbb{R}^d) \) by \( \sigma (RA^s_{p_0,q_0} (\mathbb{R}^d)) \), does not lead to compact embeddings.

**Lemma 8** Let \( d \geq 2 \). Let \( 0 < p_0, q_0 \leq \infty, 1 < p_1 < \infty \) and \( s_0 \in \mathbb{R} \). Let \( A \in \{ B, F \} \). Then \( \sigma (RA^s_{p_0,q_0} (\mathbb{R}^d)) \) is never compactly embedded into \( L_{p_1} (\mathbb{R}^d) \).
Proof Using (18) it is easily seen that
\[ s_0 - \frac{d}{p_0} = 0 - \frac{d}{p_1} \]
is a necessary condition for the validity of an embedding \( \sigma(R\dot{A}_{p_0,q_0}^0(\mathbb{R}^d)) \hookrightarrow L_{p_1}(\mathbb{R}^d) \). Let us assume \( p_0 < p_1 \) for a moment. Then
\[ s_0 = d \left( \frac{1}{p_0} - \frac{1}{p_1} \right) > \sigma_{p_0} \]
follows and by applying Lemma 3 we find
\[ \sigma(R\dot{F}_{p_0,q_0}^0(\mathbb{R}^d)) = \dot{F}_{p_0,q_0}^0(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d), \quad t = \frac{d}{p_0 - s_0} = p_1. \]

Recall \( R\dot{F}_{p_0,q_0}^0(\mathbb{R}^d) \hookrightarrow \sigma(R\dot{F}_{p_0,q_0}^0(\mathbb{R}^d)) \), see Sect. 2.4, and the Littlewood-Paley assertion \( \sigma(R\dot{F}_{p_1,2}^0(\mathbb{R}^d)) = L_{p_1}(\mathbb{R}^d) \), then compactness of the embedding \( \sigma(R\dot{F}_{p_0,q_0}^0(\mathbb{R}^d)) \hookrightarrow L_{p_1}(\mathbb{R}^d) \) would imply compactness of the embedding \( RF_{p_0,q_0}^0(\mathbb{R}^d) \hookrightarrow L_{p_1}(\mathbb{R}^d) \) but this is in contradiction with Proposition 2.

If \( p_0 = p_1 \), then \( s_0 = 0 \). Again Proposition 2 yields the non-compactness of the embedding. If \( p_0 > p_1 \) then there exists no embedding. In such a case we have \( s_0 < 0 \) and there are examples of singular distributions, see, e.g., [25], belonging to \( \sigma(R\dot{F}_{p_0,q_0}^0(\mathbb{R}^d)) \).

In what follows we shall deal with two less obvious situations where we still have compact embeddings on unbounded domains. For both cases we follow the strategy that the local behaviour of elements of \( \sigma(R\dot{A}_{p,q}^s(\mathbb{R}^d)) \) is as that of \( RA_{p,q}^s(\mathbb{R}^d) \) but globally they are different. This suggests a splitting: we shall treat their behaviour in the unit ball and independently the behaviour outside of the unit ball.

5.1 Compactness of Embeddings Into Sums of Lebesgue Spaces

Let \( X \) and \( Y \) be Banach spaces. For simplicity we assume that \( X, Y \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \). By \( X + Y \) we denote the space of all tempered distributions \( f \), which can be represented as a sum \( f = f_1 + f_2 \), where \( f_1 \in X \) and \( f_2 \in Y \). \( X + Y \) becomes a Banach space if equipped with the norm
\[ \| f \|_{X + Y} := \inf \left\{ \| f_1 \|_{X} + \| f_2 \|_{Y} : f = f_1 + f_2 \right\}. \]

Before we turn to compactness of embeddings we shall investigate continuity of embeddings.

Lemma 9 Let \( d \geq 2, 0 < p < \infty, 0 < q \leq \infty, 1 \leq p_1 \leq p_2 \leq \infty \) and
\[ \sigma_p < s < \frac{d}{p}. \quad (28) \]
(i) Then $\sigma(\mathring{F}^s_{p,q}(\mathbb{R}^d))$ is continuously embedded into $L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)$ if

$$p \leq p_1 \leq \frac{d}{d - s} \leq p_2 \leq \infty.$$ 

(ii) Then $\sigma(\mathring{B}^s_{p,q}(\mathbb{R}^d))$ is continuously embedded into $L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)$ if

$$p \leq p_1 < \frac{d}{d - s} < p_2 \leq \infty.$$ 

If in addition $q \leq \frac{d}{d - s}$, then $p_1 = \frac{d}{d - s}$ and $p_2 = \frac{d}{d - s}$ becomes admissible.

**Proof Step 1.** Proof of (i). As in (12) we work with the decomposition $\sigma([g]) = g_0 + g_1$, where

$$g_0 := \sum_{j=-\infty}^{-1} \mathcal{F}^{-1}(\varphi_j \mathcal{F}g) \quad \text{and} \quad g_1 := \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\varphi_j \mathcal{F}g).$$

From $[g] \in \dot{F}^s_{p,q}(\mathbb{R}^d)$ we derive $[g_0] \in \dot{F}^s_{p,q}(\mathbb{R}^d)$ and $[g_1] \in \dot{F}^s_{p,q}(\mathbb{R}^d)$ by using standard Fourier multiplier assertions (similar to the proof of the independence (in the sense of equivalent quasi-norms) of $\|\cdot\|_{\dot{F}^s_{p,q}(\mathbb{R}^d)}$ from the chosen smooth dyadic decomposition of unity). Concerning $g_1$ we get in addition that $g_1 \in F^s_{p,q}(\mathbb{R}^d)$. By the known embedding relations for inhomogeneous spaces we obtain

$$g_1 \in L_{p_1}(\mathbb{R}^d) \quad \text{if} \quad p \leq p_1 \leq \frac{d}{d - s}. \quad (29)$$

For $g_0$ we have $[g_0] \in \dot{F}^{s_0}_{p,q}(\mathbb{R}^d)$ for any $s_0 \geq s$ by using obvious monotonicity properties. Lemma 2 yields

$$\sigma([g_0]) \in L_{p_2}(\mathbb{R}^d) \quad \text{if} \quad \frac{d}{d - s} \leq p_2 < \infty. \quad (30)$$

Clearly, $\sigma([g_i]) = g_i, \ i = 0, 1$. Next we use Theorem 1 with respect to $g_0$ and obtain boundedness of this function on $|x| \geq 1$. Finally, since $g_0$ is entire function it is bounded on $|x| \leq 1$. Hence, $p_2 = \infty$ is admissible in (30). Now the claim follows from (29) and (30).

**Step 2.** Let $[g] \in R\mathring{B}^s_{p,q}(\mathbb{R}^d)$. Similarly as in Step 1 we derive

$$g_1 \in L_{p_1}(\mathbb{R}^d) \quad \text{if} \quad \begin{cases} p \leq p_1 \leq \frac{d}{d - s} \quad \text{and} \quad 0 < q \leq \frac{d}{d - s}, \\ p \leq p_1 < \frac{d}{d - s} \quad \text{and} \quad \frac{d}{d - s} < q \leq \infty, \end{cases}$$
and
\[ g_0 \in L_{p_2}(\mathbb{R}^d) \quad \text{if} \quad \begin{cases} \frac{d}{p-s} \leq p_2 \leq \infty \text{ and } 0 < q \leq \frac{d}{p-s}, \\ \frac{d}{p-s} < p_2 \leq \infty \text{ and } \frac{d}{p-s} < q \leq \infty. \end{cases} \]

Here one has to use \([g_0] \in \dot{B}^{s_{q_0}}_{p,q}(\mathbb{R}^d)\) for any \(s_0 > s\) with \(q_0\) arbitrary. \(\square\)

**Remark 14** For the existence of the embedding \(\sigma(R\dot{A}^s_{p,q}(\mathbb{R}^d)) \hookrightarrow L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)\) under the condition (28) the relation
\[ p_1 \leq \frac{d}{p-s} \leq p_2 \]
is necessary. This can be seen as follows. First, let us assume \(p_2 < d/(d_p - s)\). Consider our test function \(g_{\alpha,\delta}\) with \(\alpha = \frac{d}{p-s}\) and \(\delta > 1/q\). Then \(g_{\alpha,\delta} \in \sigma(\dot{B}^s_{p,q}(\mathbb{R}^d))\), see Proposition 1. Obviously \(g_{\alpha,\delta} \not\in L_{p_2}(\mathbb{R}^d)\). Because of \(|g_{\alpha,\delta}(x)| \leq 1\) for all \(x\) this implies \(g_{\alpha,\delta} \not\in (L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d))\). Second, we assume \(p_1 > d/(d_p - s)\). This time we consider the test function \(f_{\alpha,\delta}\) given by
\[ f_{\alpha,\delta}(x) := \psi(x)|x|^{-\alpha}(\log(e + |x|^2))^{-\delta}, \]
with \(\alpha = \frac{d}{p-s}\) and \(\delta > 1/q\). Then
\[ f_{\alpha,\delta} \in B^s_{p,q}(\mathbb{R}^d) \hookrightarrow \sigma(\dot{B}^s_{p,q}(\mathbb{R}^d)), \]
see [22, Lemma 2.3.1/1]. Obviously \(f_{\alpha,\delta} \not\in L_{p_1}(\mathbb{R}^d)\). Because of \(|f_{\alpha,\delta}(x)| \geq 1\), \(|x| \leq 1\), this implies \(f_{\alpha,\delta} \not\in (L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d))\). This proves the claim for \(A = B\). In case \(A = F\) we apply the elementary embedding \(\sigma(\dot{B}^s_{p,\min(p,q)}(\mathbb{R}^d)) \hookrightarrow \sigma(R\dot{F}^s_{p,q}(\mathbb{R}^d))\) and argue as above.

**Theorem 4** Let \(d \geq 2\), \(0 < p < \infty\), \(1 \leq p_1 \leq p_2 \leq \infty\) and \(s\) as in (28). Then \(\sigma(R\dot{A}^s_{p,q}(\mathbb{R}^d))\) is compactly embedded into \(L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)\) if
\[ p < p_1 < \frac{d}{p-s} < p_2 \leq \infty. \]

**Proof** Step 1. Let \(A = B\). Let \((g_\ell)\) be a bounded sequence in \(\sigma(R\dot{B}^s_{p,q}(\mathbb{R}^d))\). Again we shall work with the decomposition \(g_\ell = g_{0,\ell} + g_{1,\ell}\), where
\[ g_{0,\ell} := \sum_{j=-\infty}^{-1} \mathcal{F}^{-1}(\varphi_j \mathcal{F} g_\ell) \quad \text{and} \quad g_{1,\ell} := \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\varphi_j \mathcal{F} g_\ell), \]
see (12). Both series converge in \(S'(\mathbb{R}^d)\) and the limits are radial. Furthermore, we have \(g_{1,\ell} \in R\dot{B}^s_{p,q}(\mathbb{R}^d)\) and
\[ \|g_{1,\ell}\|_{R\dot{B}^s_{p,q}(\mathbb{R}^d)} \lesssim \|g_\ell\|_{R\dot{B}^s_{p,q}(\mathbb{R}^d)} \]
(31)
by using standard Fourier multiplier techniques as in case of proving independence of $\hat{B}_{p,q}^{s}(\mathbb{R}^d)$ from the generating function $\varphi$). Here the constant in the inequality (31) does not depend on $\ell$. From Proposition 2 we derive the existence of a convergent subsequence $(g_{1,\ell_k})_k$ and a radial function $g_1 \in L_{p_1}(\mathbb{R}^d)$ such that

$$\lim_{k \to \infty} \| g_1 - g_{1,\ell_k} |_{L_{p_1}(\mathbb{R}^d)} \| = 0.$$ 

Observe

$$\frac{d}{p} - \frac{d}{p_2} > s \iff \frac{d}{p} - \frac{d}{p_2} < p_2.$$ 

In view of this and by applying Lemma 10 we find

$$\| g_{0,\ell_k} |_{L_{p_2}(\mathbb{R}^d)} \| \lesssim \| g_{0,\ell_k} |_{\hat{B}_{p,q}^{d} - \frac{d}{p_2} (\mathbb{R}^d)} \| \lesssim \| g_{0,\ell_k} |_{\hat{B}_{p,q}^{s}(\mathbb{R}^d)} \|$$

(compare with the last argument in proof of Lemma 9). The parameter $q_0$ can be chosen as small as we want. Since our functions $g_{0,\ell_k}$ have all a Fourier transform supported in a ball with radius $\lesssim 1$ the Bernstein inequality yields

$$\| g_{0,\ell_k} |_{W^{m}_{p_2}(\mathbb{R}^d)} \| \lesssim \| g_{0,\ell_k} |_{L_{p_2}(\mathbb{R}^d)} \| \lesssim 1$$

for a fixed but arbitrary $m \in \mathbb{N}$. Let $p_2 < w \leq \infty$. From Proposition 2 we derive the existence of a subsequence of $(g_{0,\ell_k})_k$ (for simplicity also denoted by $(g_{0,\ell_k})_k$) and a radial function $g_0 \in L_{w}(\mathbb{R}^d)$ such that

$$\lim_{k \to \infty} \| g_0 - g_{0,\ell_k} |_{L_{w}(\mathbb{R}^d)} \| = 0.$$ 

Hence, the sequence $(g_{\ell_k})_k$ is convergent in $L_{p_1}(\mathbb{R}^d) + L_{w}(\mathbb{R}^d)$ but the conditions for $p_2$ and $w$ coincide.

Step 2. In case $A = F$ we apply the elementary embeddings stated in Lemma 10(iv). $\square$

5.2 Compactness of Embeddings—Exterior Domains

We consider spaces defined on the complement of a ball with center in the origin. For simplicity we choose $\Omega := \mathbb{R}^d \setminus \{ x : |x| < 1 \}$. Let $[f] \in R\hat{A}_{p,q}^{s}(\mathbb{R}^d)$. Under the restrictions of Lemma 2 $\sigma(f)$ is a radial function. By $\tau(f)$ we denote the restriction of this function to $\Omega$. We define

$$R\hat{A}_{p,q}^{s}(\Omega) := \left\{ \tau(f) : [f] \in R\hat{A}_{p,q}^{s}(\mathbb{R}^d) \right\},$$

$$\| \tau(f) |_{R\hat{A}_{p,q}^{s}(\Omega)} \| := \inf \left\{ \| g \|_{R\hat{A}_{p,q}^{s}(\mathbb{R}^d)} : \tau(f) = \tau(g) \right\}.$$

These restrictions can be understood in the pointwise sense, since the elements in $R\hat{A}_{p,q}^{s}(\mathbb{R}^d)$ are continuous outside the origin, see Theorem 1. Of course, the restrictions of radial functions in $A_{p,q}^{s}(\mathbb{R}^d)$ to $\Omega$ belong to $R\hat{A}_{p,q}^{s}(\Omega)$, but this embedding is proper, see Proposition 1. By using a similar notation for the inhomogeneous spaces a direct consequence of Proposition 2 is the following: the embedding
RA_{p_0,q_0}(\Omega) \hookrightarrow L_{p_1}(\Omega) is compact if \( p_0 < p_1 \) and \( s_0 > d(\frac{1}{p_0} - \frac{1}{p_1}) \). For those exterior domains this can be partly improved. Let \( C(\mathbb{R}^d) \) be the space of all uniformly continuous functions equipped with the supremum norm.

**Theorem 5** Let \( d \geq 2, 0 < p < \infty \),

\[
\max\left(\sigma_p, \frac{1}{p}\right) < s < \frac{d}{p} \quad \text{and} \quad \frac{d}{p - s} < p_1 < \infty.
\]

(32)

Then \( R\dot{B}^s_{p,\infty}(\Omega) \) is compactly embedded into \( L_{p_1}(\Omega) \cap RC(\Omega) \).

**Proof** The proof relies on Theorem 1 and the Arzela-Ascoli Theorem.

Let \((g_\ell)\ell\) be a bounded sequence in \( R\dot{B}^s_{p,\infty}(\Omega) \). From Corollary 1 we already know that \( \tau(g_\ell) \in Z^{s-1/p,loc}(\Omega) \) since \( s > 1/p \), hence, in particular it is continuous.

**Step 1.** We shall prove the compactness of the embedding \( R\dot{B}^s_{p,\infty}(\Omega) \hookrightarrow RC(\Omega) \).

Theorem 1 yields

\[
|x|^\frac{d}{p - s}|g_\ell(x)| \leq c\|g_\ell\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)} \lesssim 1
\]

for all \( \ell \) and all \( |x| \geq 1 \). Hence, \((g_\ell)\ell\) is a bounded sequence in \( RC(\Omega) \). Next we use the decompositions \( g_\ell = g_{0,\ell} + g_{1,\ell} \), where

\[
g_{0,\ell} := \sum_{j=-\infty}^{-1} \mathcal{F}^{-1}(\varphi_j \mathcal{F}g_\ell) \quad \text{and} \quad g_{1,\ell} := \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\varphi_j \mathcal{F}g_\ell),
\]

see (12). As above, see the proof of Theorem 4, we conclude \((g_{1,\ell})\ell\) is a bounded sequence in \( RB^s_{p,\infty}(\mathbb{R}^d) \). Let \( \varrho \in C^\infty(\mathbb{R}^d) \) be a radial function such that \( \varrho(x) = 0 \) if \( |x| \leq 1/2 \) and \( \varrho(x) = 1 \) if \( |x| \geq 3/4 \). Such a function is a pointwise multiplier for all inhomogeneous spaces, see e.g. [30, 2.8.2] or [22, 4.7]. By making use of a trace theorem proved in [24] this implies that the traces to the real axis of \((\varrho g_{1,\ell})\ell\) form a bounded sequence in \( RB^s_{p,\infty}(\mathbb{R}) \). Standard embedding relations yield that the traces form a bounded sequence in \( Z^{s-1/p}(\mathbb{R}) \). Applying a result on radial extensions from [24] we finally obtain that \((\varrho g_{1,\ell})\ell\) forms a bounded sequence in \( Z^{s-1/p}(\mathbb{R}^d) \). Obviously this implies the equicontinuity of the family \((g_{1,\ell})\ell\) on \( \Omega \). By Arzela-Ascoli Theorem we conclude the existence of a subsequence \((g_{1,\ell_k})\ell_k\) and a limit function \( g_1 \in RC(\Omega) \) such that

\[
\lim_{k \to \infty} \|g_1 - g_{1,\ell_k}|C(\Omega)\| = 0.
\]

Furthermore, as in the proof of Theorem 4, we conclude that \((g_{0,\ell_k})\ell_k\) is a bounded sequence in \( RC(\mathbb{R}^d) \) as well as in \( RC^1(\mathbb{R}^d) \). Again by Arzela-Ascoli Theorem we conclude the existence of a subsequence of \((g_{0,\ell_k})\ell_k\) (denoted again by \((g_{0,\ell_k})\ell_k\)) and a limit function \( g_0 \in RC(\Omega) \) such that

\[
\lim_{k \to \infty} \|g_0 - g_{0,\ell_k}|C(\Omega)\| = 0.
\]
Hence, \((g_\ell k)_k\) is a convergent sequence with limit \(g := g_0 + g_1\). This proves the claim, i.e., \(R \hat{B}_p^s(\Omega) \hookrightarrow RC(\Omega)\) is compact.

**Step 2.** Let \(M > 1\) be a fixed number. For a continuous function \(h\) we denote by \(\tau_M(h)\) its restriction to the set \(\Omega_M := \{x \in \mathbb{R}^d : 1 \leq |x| \leq M\}\). By using Step 1 we find
\[
\lim_{k \to \infty} \|g - g_\ell k\|_{L_{p_1}(\Omega_M)} = 0.
\]
Without loss of generality we may assume
\[
\|g - g_\ell M\|_{L_{p_1}(\Omega_M)} < M^{-1}, \quad M = 2, 3, \ldots
\]
Next observe, that by an application of Theorem 1 we get
\[
|g_\ell(x)| \lesssim |x|^{s - \frac{d}{p}}
\]
and hence
\[
\int_{|x| > M} |g_\ell(x)|^{p_1} \, dx \lesssim M^{d - p_1 \left(\frac{d}{p} - s\right)} = M^{-\varepsilon},
\]
where \(\varepsilon := p_1 \left(\frac{d}{p} - s\right) - d > 0\), see (32). From (33) and (34) we also derive inequality (35) with \(g_\ell M\) replaced by \(g\). Because of
\[
\|g - g_\ell M\|_{L_{p_1}(\Omega)} \lesssim \|g - g_\ell M\|_{L_{p_1}(\Omega_M)} + M^{-\varepsilon/p_1} \lesssim M^{-1} + M^{-\varepsilon/p_1}
\]
we obtain convergence for \(M\) tending to infinity. This proves the theorem. \(\square\)

**Remark 15** Let \(0 < q \leq \infty\) and \(A \in \{B, F\}\). From the elementary embeddings for Besov-Triebel-Lizorkin spaces it follows that under the conditions of Theorem 5 the embedding \(R \hat{A}^s_{p,q}(\Omega) \hookrightarrow L_{p_1}(\Omega) \cap RC(\Omega)\) is compact.

**Appendix A: Homogeneous Besov and Triebel-Lizorkin Spaces**

**A.1 Distribution Spaces Modulo Polynomials**

General references for homogeneous Besov and Triebel-Lizorkin spaces are \([10–12, 19, 30]\). For convenience of the reader we recall the definition and a few properties of these spaces.

Let \(\varphi \in C_0^\infty(\mathbb{R}^d)\) be a radial function such that \(\text{supp} \varphi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 3/2\}\) and \(\varphi(\xi) = 1\) if \(|\xi| \leq 1\). Then we define
\[
\varphi_j(\xi) := \varphi(2^{-j} \xi) - \varphi(2^{-j+1} \xi), \quad \xi \in \mathbb{R}^d, \quad j \in \mathbb{Z}.
\]
This leads to a specific homogeneous smooth dyadic decomposition of unity since
\[
\sum_{j=-\infty}^{\infty} \varphi_j(\xi) = 1, \quad \xi \neq 0.
\]
We shall identify tempered distributions modulo polynomials. In fact, we consider the classes

$$[f] := \{ f + p : p \text{ polynomial over } \mathbb{R}^d \}, \quad f \in S'(\mathbb{R}^d).$$

**Definition 2** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$.

(i) Let $0 < p \leq \infty$. Then the homogeneous Besov space $\dot{B}_p^{s,q}(\mathbb{R}^d)$ is the collection of all classes $[f]$ such that

$$\| [f] \|_{\dot{B}_p^{s,q}(\mathbb{R}^d)} := \left( \sum_{j=-\infty}^{\infty} 2^{jsq} \| \mathcal{F}^{-1} [\varphi_j(\xi) \mathcal{F} f(\xi)](\cdot) \|_{L_p(\mathbb{R}^d)} \right)^{1/q} < \infty.$$

(ii) Let $0 < p < \infty$. Then the homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^d)$ is the collection of all classes $[f]$ such that

$$\| [f] \|_{\dot{F}_p^{s,q}(\mathbb{R}^d)} := \left( \sum_{j=-\infty}^{\infty} 2^{jsq} | \mathcal{F}^{-1} [\varphi_j(\xi) \mathcal{F} f(\xi)](\cdot) |^{q} \right)^{1/q} \| L_p(\mathbb{R}^d) \| < \infty.$$

**Remark 16** (i) The definition makes sense since

$$\mathcal{F}^{-1} [\varphi_j F(f + p)] = \mathcal{F}^{-1} [\varphi_j F f]$$

for all polynomials $p$, all $f \in S'(\mathbb{R}^d)$, and all $j \in \mathbb{Z}$. Moreover, the spaces $\dot{B}_p^{s,q}(\mathbb{R}^d)$ and $\dot{F}_p^{s,q}(\mathbb{R}^d)$ are independent of the resolution of unity up to equivalence of quasi-norms. Furthermore, we always have

$$\sum_{j=-\infty}^{\infty} \mathcal{F}^{-1} (\varphi_j F g) \in [f], \quad \forall g \in [f].$$

(ii) The spaces $\dot{B}_p^{s,q}(\mathbb{R}^d)$ and $\dot{F}_p^{s,q}(\mathbb{R}^d)$ are quasi-Banach spaces.

(iii) Let $1 < p < \infty$. Define $\dot{H}_p^{s}(\mathbb{R}^d)$ as the collection of all classes $[f]$ such that

$$\mathcal{F}^{-1} [\xi^s \mathcal{F} f(\xi)](\cdot) \in L_p(\mathbb{R}^d)$$

equipped with the induced norm. Usually $\dot{H}_p^{s}(\mathbb{R}^d)$ are called homogeneous potential spaces. Then $\dot{H}_p^{s}(\mathbb{R}^d)$ coincides with $\dot{F}_p^{s,2}(\mathbb{R}^d)$ in the sense of equivalent norms.

The following well-known continuous embeddings are of some use for us.

**Lemma 10** Let $s, s_0, s_1 \in \mathbb{R}$ and $0 < q, q_0, q_1 \leq \infty$.

(i) Let $0 < p_0 \leq p_1 < \infty$. We have $\dot{F}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \dot{F}_{p_1,q_1}(\mathbb{R}^d)$ if

$$s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1} \quad (37)$$

and either $p_0 < p_1$ or $p_0 = p_1$ and $q_0 \leq q_1$. 

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(ii) Let $0 < p_0, p_1 \leq \infty$. We have
\[ \dot{B}^{s_0}_{p_0, q_0}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_1}_{p_1, q_1}(\mathbb{R}^d) \] if (37), $p_0 \leq p_1$, and $q_0 \leq q_1$ hold.

(iii) Let $0 < p_0 < p_1 \leq \infty$. We have
\[ \dot{B}^{s_0}_{p_0, q_0}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s}_{p, q}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_1}_{p_1, q_1}(\mathbb{R}^d) \]
if $q_0 \leq p \leq q_1$.

(iv) Let $0 < p < \infty$. We have
\[ \dot{B}^{s}_{p, q}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s}_{p, q}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s}_{p, q}(\mathbb{R}^d) \]
if $q_0 \leq \min(p, q)$ and $\max(p, q) \leq q_1$.

Remark 17
(i) Observe, for fixed $s$ and $p$ the Besov space $\dot{B}^{s}_{p, \infty}(\mathbb{R}^d)$ is the largest in the both scales $\dot{B}^{s}_{p, q}(\mathbb{R}^d)$ and $\dot{F}^{s}_{p, q}(\mathbb{R}^d)$.
(ii) For proofs we refer, e.g., to [13]. This reference does not cover the second embedding in part (iii). For this part we refer to Franke [9], but see also [32].

A.2 Function Spaces Modulo Polynomials of a Certain Degree

First of all we wish to mention that the mapping
\[ [f] \rightarrow \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f] \]
is an isomorphism which maps $\dot{F}^0_{p, 2}(\mathbb{R}^d)$ onto $L_p(\mathbb{R}^d)$ if $1 < p < \infty$, see [18] or [29, Theorem 3.2.1]. Next we turn to homogeneous Sobolev spaces $\dot{W}^m_p(\mathbb{R}^d)$. This time we consider classes of functions modulo polynomials of degree $m - 1$, i.e. we put
\[ [f]_m := \{ g \in S'(\mathbb{R}^d) : g = f + \sum_{|\alpha| < m} a_\alpha x^\alpha \}. \] (38)

Then $\dot{W}^m_p(\mathbb{R}^d)$ is the collection of all classes $[f]_m$ such that $D^\alpha f \in L_p(\mathbb{R}^d)$, $|\alpha| = m$. For $m \in \mathbb{N}$ and $1 < p < \infty$ there exists an isomorphism of $F^m_{p, 2}(\mathbb{R}^d)$ onto $\dot{W}^m_p(\mathbb{R}^d)$. Proofs can be found in [18] or in [29, Theorem 3.2.1]. A fractional order version is given by the following definition.

Definition 3
Let $0 < q \leq \infty$.

(i) Let $0 < p \leq \infty$. Assume $\sigma_p < s < m$, for some natural number $m$. Then the class $[f]_m$ of regular distributions belongs to $\dot{B}^{s, m}_{p, q}(\mathbb{R}^d)$ if
\[ N^{s}_{p, q}(f) := \left( \int_0^{\infty} t^{-s q} \sup_{|h| < t} \| \Delta^m_h f(x) \|_{L_p(\mathbb{R}^d)} \frac{dt}{t} \right)^{1/q} < \infty. \]
(ii) Let $0 < p < \infty$. Assume $\sigma_{p,q}(d) < s < m$, for some natural number $m$. Then the class $[f]_m$ of regular distributions belongs to $\dot{F}^{s,m}_{p,q}(\mathbb{R}^d)$ if
\[
M^s_{p,q}(f) := \left( \int_{\mathbb{R}^d} \left( \int_0^\infty t^{-sq} \left( \int_{|h| \leq t} \frac{d\tau}{\tau} \int_0^t \frac{|\Delta^m_h f(x)| dh}{\tau} \right)^{p/q} dx \right)^{1/p} \right) < \infty.
\]
(39)

**Proposition 3** Under the restrictions from Definition 3 we have the following: there exists an isomorphism of $\dot{A}^{s,m}_{p,q}(\mathbb{R}^d)$ onto $A^{s}_{p,q}(\mathbb{R}^d)$.

**Remark 18** With $A = B$ Proposition 3 can be found in [30, Theorem 5.2.3/2]. With $A = F$ it is proved in [5]. Both references cover the case of Banach spaces only. However, the methods from [5] extend to the quasi-Banach space case.

**The Regularity of Some Test Functions** Now we turn to the regularity of $g_{\alpha, \delta}$, see (10), with respect to the fractional order spaces. There are several possibilities to attack this problem. We decided for using differences, see Proposition 3. The following two lemmas will cover Proposition 1.

**Lemma 11** Let $0 < p < \infty$, $0 < q \leq \infty$ and $\sigma_{p,q} < s < d/p$.

(i) The function $g_{\alpha,0}$ belongs to $\dot{F}^s_{p,q}(\mathbb{R}^d)$ if, and only if $\alpha > \frac{d}{p} - s$.

(ii) Let $\delta > 0$. Then $g_{\alpha, \delta}$ belongs to $\dot{F}^s_{p,q}(\mathbb{R}^d)$ if either $\alpha > \frac{d}{p} - s$ (\( \delta \geq 0 \) arbitrary) or $\alpha = \frac{d}{p} - s$ and $\delta > 1/p$.

**Proof.** Step 1. Proof of (i) in case $q = \infty$. For given $s > 0$ there exists an integer $M \in \mathbb{N}_0$ and a real positive number $\tau \in (0, 1]$ such that $s = M + \tau$. The function $g_{\alpha,0}$ is smooth. Hence we may apply the Mean Value Theorem. This implies
\[
\Delta^M_h g_{\alpha,0}(x) = |h|^M D^M_e (g_{\alpha,0})(\xi),
\]
where $e$ is the direction from $x$ to $x + Mh$ and $\xi = x + te$ for some $t \in (0, |M||h|)$, $D^M_e$ denotes the $M$-th order derivative of $g_{\alpha,0}$ (restricted to this line) in direction $e$. We obtain
\[
|\Delta^{M+1}_h g_{\alpha,0}(x)| = |h|^M |\Delta^M_h g_{\alpha,0}(x + h) - \Delta^M_h g_{\alpha,0}(x)|
\]
\[
= |h|^M |D^M_e (g_{\alpha,0})(\eta) - D^M_e (g_{\alpha,0})(\xi)|
\]
\[
\lesssim |h|^{M+\tau} \sup_{|\xi - x| \leq M|h|} \sup_{|\eta - x| \leq (M+1)|h|} \max_{|\beta| = M} \frac{|D^\beta g_{\alpha,0}(\xi) - D^\beta g_{\alpha,0}(\eta)|}{|\xi - \eta|^\tau}.
\]
(40)

Mathematical induction yields
\[
D^\beta g_{\alpha,0}(\xi) = (1 + |\xi|^2)^{-\frac{\beta}{2} - |\beta|} p_{\beta}(\xi), \quad \xi \in \mathbb{R}^d,
\]
where $p_{\beta}$ is a polynomial of degree at most $|\beta|$. Let $|x| = r > 2 \max((M + 1)|h|, 1)$ and $|\beta| = M$. Then

$$
\frac{|(1 + |\xi|^2)^{-\frac{q}{2} - |\beta|}p_{\beta}(\xi) - (1 + |\eta|^2)^{-\frac{q}{2} - |\beta|}p_{\beta}(\eta)|}{|\xi - \eta|^\tau}
\leq (1 + (r - (M + 1)|h|)^2)^{-\frac{q}{2} - |\beta|} \frac{|p_{\beta}(\xi) - p_{\beta}(\eta)|}{|\xi - \eta|^\tau}
+ c(r + (M + 1)|h|)^{|\beta|} \frac{(1 + |\xi|^2)^{-\frac{q}{2} - |\beta|} - (1 + |\eta|^2)^{-\frac{q}{2} - |\beta|}}{|\xi - \eta|^\tau}
\lesssim r^{-|\beta| - \tau} = r^{-\alpha - s},
$$

(41)
as long as $|\xi - x|, |\eta - x| \leq (M + 1)|h|$. Hence, with (40) and (41) we get

$$
\int_{|x| > 2} \left( \sup_{0 < t < |x|/(2M + 2)} t^{-s} t^{-d} \int_{|h| < t} |\Delta_h^{M+1} g_{\alpha,0}(x)| \, dh \right)^p \, dx
\lesssim \int_{|x| > 2} |x|^{-(\alpha + s)p} \, dx
\lesssim \int_2^\infty r^{d-1} r^{-(\alpha + s)p} \, dr < \infty \quad \text{if } d/p < \alpha + s.
$$

(42)

This can be complemented by the obvious inequality

$$
\int_{|x| < 2} \left( \sup_{t > 0} t^{-s} t^{-d} \int_{|h| < t} |\Delta_h^{M+1} g_{\alpha,0}(x)| \, dh \right)^p \, dx
\lesssim \sup_{|x| < 2} \left( \sup_{0 < t < 1} t^{-s} \sup_{|h| < t} |\Delta_h^{M+1} g_{\alpha,0}(x)| \right)^p
+ \max_{\ell = 0, 1, \ldots, M+1} \sup_{|x| < 2} \left( \sup_{t > 1} \sup_{|h| < t} |g_{\alpha,0}(x + \ell h)| \right)^p < \infty,
$$

(43)
since $g_{\alpha,0}$ is a bounded $C^\infty$ function. According to (39) it remains to check

$$
\int_{|x| > 2} \left( \sup_{|x|/(2M + 2) < t} t^{-s} t^{-d} \int_{|h| < t} |\Delta_h^{M+1} g_{\alpha,0}(x)| \, dh \right)^p \, dx
\lesssim \max_{\ell = 0, 1, \ldots, M+1} \int_{|x| > 2} \left( \sup_{|x|/(2M + 2) < t} t^{-s} t^{-d} \int_{|h| < t} |g_{\alpha,0}(x + \ell h)| \, dh \right)^p \, dx
\lesssim \int_{|x| > 2} \left( \sup_{|x|/(2M + 2) < t} t^{-s} \int_{|y| < (3M + 3)t} |g_{\alpha,0}(y)| \, dy \right)^p \, dx
\lesssim \int_{|x| > 2} \left( \sup_{|x|/(2M + 2) < t} t^{-s - \alpha} \right)^p \, dx
\lesssim \int_2^\infty r^{d-1} r^{-(\alpha + s)p} \, dr < \infty \quad \text{if } d/p < \alpha + s.
$$

(44)
Combining (42)–(44) we have proved sufficiency in (i) with \( q = \infty \). Now, let \( q < \infty \).

We choose \( 0 < p_0 < p \) and define

\[
s_0 := s - \frac{d}{p} + \frac{d}{p_0}.
\]

From the arguments used above we conclude that \( g_{\alpha,0} \in \dot{F}^{s_0}_{p_0,\infty}(\mathbb{R}^d) \), but \( \dot{F}^{s_0}_{p_0,\infty}(\mathbb{R}^d) \hookrightarrow \dot{F}^s_{p,q}(\mathbb{R}^d) \), see Lemma 10(i). Observe, that also in case \( s = M + 1 \) the above estimate is sufficient to guarantee \( g_{\alpha,0} \in \dot{F}^s_{p,\infty}(\mathbb{R}^d) \).

Step 2. Sketch of the proof of (ii). We only describe the needed modifications for the limiting situation \( s + \alpha = d/p \).

Substep 2.1. Let \( q = \infty \). First observe, that Leibniz formula for derivatives of products yields

\[
D^\beta g_{\alpha,\delta}(\xi) = (1 + |\xi|^2)^{-\frac{\gamma}{2} - |\beta|} p_\beta(\xi)(\log(e + |\xi|^2))^{-\delta} + \sum_{e + \vartheta = \beta, \vartheta \neq 0} (1 + |\xi|^2)^{-\frac{\gamma}{2} - |\vartheta|} p_\vartheta(\xi) D^\vartheta(\log(e + |\xi|^2))^{-\delta},
\]

where \( p_\vartheta \) are polynomials of degree at most \( |\vartheta| \). The chain rule yields

\[
D^\vartheta(\log(e + |\xi|^2))^{-\delta} = \sum_{\ell=1}^{\frac{\vartheta}{|\vartheta|}} (\log(e + |\xi|^2))^{-\delta - \ell} \times \sum_{\gamma^1 + \cdots + \gamma^\ell = \vartheta} c_{\ell,\gamma^1,\ldots,\gamma^\ell} D^\gamma^1(\log(e + |\xi|^2)) \cdots D^\gamma^\ell(\log(e + |\xi|^2))
\]

for appropriate constants \( c_{\ell,\gamma^1,\ldots,\gamma^\ell} \). Obviously

\[
|D^\gamma(\log(e + |\xi|^2))| \lesssim |\xi|^{-|\gamma|}, \quad |\xi| \geq 1, \ |\gamma| > 0,
\]

and consequently

\[
|D^\vartheta(\log(e + |\xi|^2))| \lesssim |\xi|^{-|\vartheta|} (\log(e + |\xi|^2))^{-\delta - 1}, \quad |\xi| \geq 1, \ |\vartheta| > 0. \quad (45)
\]

As in (41) we derive

\[
\frac{|(1 + |\xi|^2)^{-\frac{\gamma}{2} - |\beta|} p_\beta(\xi)(\log(e + |\xi|^2))^{-\delta} - (1 + |\eta|^2)^{-\frac{\gamma}{2} - |\beta|} p_\beta(\eta)(\log(e + |\eta|^2))^{-\delta}|}{|\xi - \eta|^\tau} 
\lesssim r^{-\alpha - s} \left( (\log(e + r^2))^{-\delta} + \frac{|(\log(e + |\xi|^2))^{-\delta} - (\log(e + |\eta|^2))^{-\delta}|}{|\xi - \eta|^\tau} \right)
\lesssim r^{-\alpha - s} (\log(e + r^2))^{-\delta}, \quad (46)
\]
where $|\xi - x|, |\eta - x| \leq (M + 1)|h|, |x| = r > 2 \max((M + 1)|h|, 1)$ and $|\beta| = M$. Now, let $P_{\alpha, \varrho}(\xi) := (1 + |\xi|^2)^{-\frac{\alpha}{2}} - |\rho|^p \varrho(\xi)$. Similarly as in (46), see also (41), and using (45) we find

$$|P_{\alpha, \varrho}(\xi)D^\vartheta(\log(e + |\xi|^2)) - P_{\alpha, \varrho}(\eta)D^\vartheta(\log(e + |\eta|^2)) - \delta| \leq r^{-\alpha-|\varrho|-s} \tau \leq r^{-\alpha-|\varrho|-\tau}$$

under the same restrictions as in (46). This leads to the following modification of (42)

$$\int_{|x| > 2} \left( \sup_{0 < t < 1/|x|/(2M+2)} t^{-s} t^{-d} \int_{|h| < t} |\Delta_{h}^{M+1} g_{\alpha, \delta}(x)|^p dh \right)^{1/p} dx \lesssim \int_2^{\infty} r^{d-1} r^{-(\alpha+s)p} (\log(e + r^2))^{-\delta p} dr < \infty \text{ if } \delta p > 1.$$}

The term

$$\int_{|x| < 2} \left( \sup_{t > 0} t^{-s} t^{-d} \int_{|h| < t} |\Delta_{h}^{M+1} g_{\alpha, \delta}(x)|^p dh \right)^{1/p} dx$$

can be estimated as in (43). For the modification of (44) observe

$$\int_0^t (1 + r^2)^{-\alpha/2} (|\log(e + r^2)|)^{-\delta} dr \lesssim r^{d-\alpha} (\log(e + t))^{-\delta}$$

uniformly in $t > 1/M$, since $\alpha < d$. This proves $g_{\alpha, \delta} \in \dot{F}_{\varrho, \infty}^{d-\alpha} (\mathbb{R}^d)$ if $\delta > 1/p$.

**Substep 2.2.** Let $0 < q \leq \infty$, $0 < p < \infty$ and $0 < s := \frac{d}{p} - \alpha < \frac{d}{p}$. We choose $0 < p_0 < p$ and define

$$s_0 := s - \frac{d}{p} + \frac{d}{p_0} = \frac{d}{p_0} - \alpha.$$}

From Substep 2.1 and Lemma 10(i) we conclude that

$$g_{\alpha, \delta} \in \dot{F}_{p_0, \infty}^{s_0} (\mathbb{R}^d) \hookrightarrow \dot{F}_{p, q}^{s} (\mathbb{R}^d)$$

if $\delta > 1/p_0$. For $p_0 \uparrow p$ the claim follows.

**Step 3.** Necessity in (i). Let us assume $g_{\alpha, 0} \in \dot{F}_{p, \infty}^{s} (\mathbb{R}^d)$ with $\alpha = \frac{d}{p} - s$. Then, by Lemma 10(i), it follows that the class $[g_{\alpha, 0}]$ contains at least one element which belongs to $L_t(\mathbb{R}^d)$, $t = \frac{d}{p - s}$. Since there is at most one element in such a class which decays near infinity we get $g_{\alpha, 0} \in L_t(\mathbb{R}^d)$. By Lemma 4(i) we conclude $\delta > 1/t > 0$. This is a contradiction. □
Lemma 12  Let $0 < p < \infty$, $0 < q \leq \infty$ and $\sigma_p < s < d/p$.

(i) The function $g_{\alpha,0}$ belongs to $\dot{B}^s_{p,q}(\mathbb{R}^d)$ if $\alpha > \frac{d}{p} - s$.

(ii) Let $\delta > 0$. Then $g_{\alpha,\delta}$ belongs to $\dot{B}^s_{p,q}(\mathbb{R}^d)$ if either $\alpha > \frac{d}{p} - s$ or $\alpha = \frac{d}{p} - s$ and $\delta > 1/q$.

Proof  Both assertions follow from Lemma 11 by using Lemma 10(iii). □

Appendix B: Radial Distributions and Atomic Decompositions

We recall a construction of Epperson and Frazier [8]. We will do that with certain detail because we are going to use it with a different normalization.

Let $J_\nu$ denote the Bessel function of order $\nu$, $\nu \geq -\frac{1}{2}$, defined by

$$J_\nu(t) := \begin{cases} \frac{(t/2)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_1^1 (1 - y^2)^{\nu - 1/2} e^{ity} \, dy & \text{if } \nu > -\frac{1}{2}, \\ \left(\frac{2}{\pi t}\right)^{1/2} \cos t & \text{if } \nu = -\frac{1}{2}, \end{cases} \quad t \in \mathbb{R}. $$

Let $\mu_{\nu,0} < \mu_{\nu,1} < \cdots$ be the positive zeros of $J_\nu$. We put $\mu_{\nu,-1} := 0$. Then

$$\mu_{\nu,k} = \pi \left(k + \frac{\nu}{2}\right) + O\left(\frac{1}{k+1}\right)$$

and

$$\mu_{\nu,k} - \mu_{\nu,k-1} = \pi + O\left(\frac{1}{k+1}\right).$$

For $k = 0, 1, 2, \ldots$ we introduce associated annuli (balls, if $k = 0$)

$$A_{j,k} := \{x \in \mathbb{R}^d : 2^{-j} \mu_{\nu,k-1} \leq |x| \leq 2^{-j} \mu_{\nu,k}\}, \quad j \in \mathbb{Z},$$

$$\tilde{A}_{j,k} := \{x \in \mathbb{R}^d : 2^{-j}(\mu_{\nu,k-1} - 1) \leq |x| \leq 2^{-j}(\mu_{\nu,k} + 1)\}, \quad j \in \mathbb{Z}.$$

From now on we fix $\nu = \frac{d-2}{2}$ and drop it in notation.

Next we recall the definition of smooth radial atoms from [8].

Definition 4  Let $s \in \mathbb{R}$ and $0 < p < \infty$. A radial function $a$ is called a smooth radial atom associated to $A_{j,k}$ if it satisfies the following conditions:

$$\text{supp } a \subset \tilde{A}_{j,k}, \quad (47)$$

$$\int a(x)\,dx = 0,$$

$$\sup_{x \in \mathbb{R}^d} |D^\gamma a(x)| \leq c_\gamma 2^{-j(s - |\gamma| - \frac{d}{p})}, \quad \forall \gamma \in \mathbb{N}_0^d, \quad (48)$$

Here $c_\gamma := 1$ if $|\gamma| \leq s + 1$ and $c_\gamma$ must be independent of $j$ and $k$ if $|\gamma| > s + 1$. 

\[ \text{Birkhäuser} \]
As usual one has to introduce associated sequence spaces as well. Let \( \chi_{A_{j,k}} \) denote the characteristic function of the set \( A_{j,k} \). Then we define \( \tilde{\chi}(p)_{j,k} := 2^{jd/p} \chi_{A_{j,k}} \). The announced sequence spaces are then given by

\[
\dot{b}_{p,q} := \left\{ (s_{j,k})_{j,k} : s_{j,k} \in \mathbb{C}, \right\},
\]

\[
\| (s_{j,k})_{j,k} \|_{\dot{b}_{p,q}} := \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k=0}^{\infty} |s_{j,k}| L_p(\mathbb{R}^d) \right)^q \right)^{1/q} < \infty
\]

and

\[
\dot{f}_{p,q} := \left\{ (s_{j,k})_{j,k} : s_{j,k} \in \mathbb{C}, \right\},
\]

\[
\| (s_{j,k})_{j,k} \|_{\dot{f}_{p,q}} := \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k=0}^{\infty} |s_{j,k}| \tilde{\chi}(p)_{j,k} L_p(\mathbb{R}^d) \right)^q \right)^{1/q} < \infty
\].

Again we will use these notation with \( \dot{a}_{p,q} \) in place of \( \dot{b}_{p,q} \) or \( \dot{f}_{p,q} \) if there is no need to distinguish these cases. Now we are in position to formulate the result of Epperson and Frazier [8], see Theorem 4.1 and the comments in Sect. 5.

**Theorem 6** Suppose \( 0 < p < \infty \), \( 0 < q \leq \infty \) and either \( s > \sigma_{p,q} - 1 \) if \( A = F \) or \( s > \sigma_p - 1 \) if \( A = B \). For \( [f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d) \) there exist smooth radial atoms \( a_{j,k} \) associated to \( A_{j,k}, j \in \mathbb{Z}, k \in \mathbb{N}_0 \), and a sequence \( (s_{j,k})_{j,k} \in \dot{a}_{p,q} \), such that

\[
\sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k} a_{j,k} \in [f]
\]

and

\[
\|[f]||R\dot{A}^s_{p,q}(\mathbb{R}^d) \| \asymp \| (s_{j,k})_{j,k} \|_{\dot{a}_{p,q}}.
\]

**Remark 19** The identity (49) should be interpreted in the following way. The sequence \( (f_n)_n \), where

\[
f_n = \sum_{j=-n}^{n} \sum_{k=0}^{\infty} s_{j,k} a_{j,k},
\]

converges to some \( g \in [f] \) with respect to the quasi-norm in \( R\dot{A}^s_{p,q}(\mathbb{R}^d) \) as \( n \) tends to infinity, if \( q < \infty \), and in \( S'(\mathbb{R}^d)/\mathcal{P} \) if \( q = \infty \).

We need another result of Epperson and Frazier, see Theorem 3.1 and the comments in Sect. 5 in [8].
Lemma 13 Suppose $0 < p < \infty$, $0 < q \leq \infty$ and either $s > \sigma_{p,q}$ if $A = F$ or $s > \sigma_p$ if $A = B$. There exists a positive constant $c$ such that for any sequence $(a_{j,k})_{j \in \mathbb{Z}, k \in \mathbb{N}_0}$ of radial functions satisfying the conditions (47), (48) (restricted to values of $\gamma$ such that $|\gamma| \leq s + 1$) and any sequence $(s_{j,k})_{j,k} \in \dot{A}_{p,q}$ the inequality

$$\left\| \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} s_{j,k} a_{j,k} \right\|_{\dot{A}_{p,q}^s} \leq c \| (s_{j,k})_{j,k} \|_{\dot{A}_{p,q}}$$

holds.

Remark 20 Radial subspaces of homogeneous Besov spaces have been characterized in a wavelet-style by Rauhut [20] and Rauhut and Rösler [21]. These methods could be used here as well.

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