Maps that preserve left (right) $K$-Cauchy sequences

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Abstract

It is well-known that on quasi-pseudometric space $(X,q)$, every $q$*-Cauchy sequence is left (or right) $K$-Cauchy sequence but the converse does not hold in general. In this article, we study a class of maps that preserve left (right) $K$-Cauchy sequences that we call left (right) $K$-Cauchy sequentially-regular maps. Moreover, we characterize totally bounded sets on a quasi-pseudometric space in terms of maps that preserve left $K$-Cauchy and right $K$-Cauchy sequences and uniformly locally semi-Lipschitz maps.

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1. Introduction

Let $f : (X,d) \to (Y,d')$ be a map between two metric spaces $(X,d)$ and $(Y,d')$. Then $f$ is called Cauchy sequentially-regular (or Cauchy continuous) if $(f(x_n))$ is a Cauchy sequence on $(Y,d')$ whenever $(x_n)$ is a Cauchy sequence on $(X,d)$. In [12], Snipes investigated the class of Cauchy sequentially-regular maps. He proved that a map $f : (X,d) \to (Y,d')$ is uniformly continuous if and only if $f$ preserves parallel sequences. Moreover, He characterized Cauchy sequentially-regular maps in terms of maps that preserve equivalent sequences.

In addition, Snipes observed that the class of Cauchy sequentially-regular maps from the metric space $(X,d)$ into the metric space $(Y,d')$ sits between the class of uniformly continuous maps from the metric space $(X,d)$ into the metric space $(Y,d')$ and the class of continuous maps from the metric space $(X,d)$ into the metric space $(Y,d')$. He also proved that a map $f : (X,d) \to (Y,d')$ is Cauchy sequentially regular whenever $f : (X,d) \to (Y,d')$ is continuous and $(X,d)$ is complete. Finally Snipes in [12] investigated extension maps of Cauchy sequentially-regular maps into a complete metric space.

After Snipes paper [12], the concept of Cauchy sequential regularity got an interest in the community of mathematicians. For instance in [8], Jain and Kundu proved that each Cauchy sequentially-regular map is uniformly continuous if and only if the completion of a
metric space \((X, d)\) is a UC-space (space on which each continuous function is uniformly continuous).

Furthermore, in [6], Di Maio et al. proved that a metric space is complete if and only if each continuous function defined on it is a Cauchy sequentially-regular map.

Moreover, in [2, 3], Beer, characterized Cauchy sequentially-regular maps in terms of a family on which they must be strongly uniformly continuous on totally bounded sets and uniformly continuous on totally bounded sets. Recently, Beer proved that on a metric space \((X, d)\), a subset \(B\) of \(X\) is totally bounded if and only its image \(f(B)\) is \(d'\)-bounded whenever \(f : (X, d) \to (Y, d')\) is Cauchy sequentially-regular map and \((Y, d')\) is a metric space. In addition, he showed that a subset \(B\) of a metric space \((X, d)\) is totally bounded if and only if its image \(f(B)\) is bounded subset of \(\mathbb{R}\) whenever \(f : (X, d) \to (\mathbb{R}, |.|)\) is uniformly locally Lipschitz.

It is no doubt that uniform continuity and continuity plays an important role in the study of Cauchy sequentially-regular maps on metric spaces. Observe that for any two quasi-metric spaces \((X, q)\) and \((Y, p)\). If a map \(f : (X, q) \to (Y, p)\) is quasi-uniformly continuous (or uniformly continuous), then \(f : (X, q^*) \to (Y, p^*)\) is also uniformly continuous but the converse is not true, in general (see Example 3.3). It is well-known that on a quasi-pseudometric space, if a sequence is \(q^\ast\)-Cauchy, then it is left (right) \(K\)-Cauchy sequence but the converse is not true (see for instance [4, 13]). As might be expected these have led to the conjecture that the maps from a quasi-pseudometric space into another quasi-pseudometric space that preserve left (right) \(K\)-Cauchy sequences that we call left (right) \(K\)-Cauchy sequentially regular maps (see Definition 4.8) need to be studied carefully.

The aim of this paper is a careful study of the above-mentioned conjecture. Moreover, for map \(f : (X, q) \to (Y, p)\), where \((X, q)\) and \((Y, p)\) are quasi-pseudometric spaces, we study connections between left (right) \(K\)-Cauchy sequentially regular maps and \(q^\ast\)-Cauchy sequentially maps. We also show that a continuous map from a left (right) Smyth complete quasi-metric space into a quasi-pseudometric space is left (right) \(K\)-Cauchy sequentially-regular. Finally, we characterize totally bounded sets on a quasi-pseudometric space in terms of left and right \(K\)-Cauchy sequentially-regular maps and uniformly locally semi-Lipschitz maps which extend an important result due to Beer and Garrido ([1, Theorem 3.2]) in our settings.

2. Preliminaries

In this section we summarize some basic results on quasi-pseudometric spaces. For more details about quasi-pseudometric spaces we recommend the following articles [4, 7, 9, 13].

Let \(X\) be a set and \(q : X \times X \to [0, \infty)\) be a function. Then \(q\) is an quasi-pseudometric on \(X\) if

(a) \(q(x, x) = 0\) for all \(x \in X\),
(b) \(q(x, y) \leq q(x, z) + q(z, y)\) for all \(x, y, z \in X\).

If \(q\) is a quasi-pseudometric on \(X\), then the pair \((X, q)\) is called an quasi-pseudometric space.

If the function \(q\) satisfies the condition

(c) for any \(x, y \in X, q(x, y) = 0 = q(y, x)\) implies \(x = y\) instead of condition (a), then \(q\) is called a \(T_0\)-quasi-metric on \(X\) and the pair \((X, q)\) is called \(T_0\)-quasi-metric space (see for instance [9]).

Furthermore, if \(q\) is a quasi-pseudometric on \(X\), then the function \(q^\prime : X \times X \to [0, \infty)\) defined by \(q^\prime(x, y) = q(y, x)\), for all \(x, y \in X\) is also a quasi-pseudometric on \(X\) and it is called the conjugate quasi-pseudometric of \(q\).
Note that for any $q$ quasi-pseudometric on $X$, the function $q^s$ defined by $q^s(x, y) := \max\{q(x, y), q'(x, y)\}$ for all $x, y \in X$ is a pseudometric on $X$, usually called the symmetrised quasi-pseudometric of $q$.

If $(X, q)$ is a quasi-pseudometric space. Then we associate the topology $\tau(q)$ on $X$ with respect to $q$ where its base is given by the family $\{D_q(x, \epsilon) : x \in X$ and $\epsilon > 0\}$ with $D_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}$.

If $A \subseteq X$ and $\epsilon > 0$, then the set $D_q(A, \epsilon)$ is defined by

$$D_q(A, \epsilon) := \bigcup_{a \in A} D_q(a, \epsilon).$$

We say that a subset $A$ of $X$ is $q$-totally bounded provided that $A$ is $q^s$-totally bounded, i.e. for any $\epsilon > 0$, there exists $F \in \mathcal{F}$ with $F \subseteq A \subseteq D_q(F, \epsilon)$, where $\mathcal{F}$ is the collection of all finite subsets of $X$ (see [14, Definition 5]).

In the sequel we denote by $\mathcal{T}_q^s(X)$ the collection of all $q$-totally bounded subsets of $X$.

It is well-known that $\mathcal{T}_q^s(X)$ forms a bornology on $X$, i.e.

(a) $\{x\} \in \mathcal{T}_q^s(X)$ whenever $x \in X$,

(b) if $A \subseteq B$ with $B \in \mathcal{T}_q^s(X))$, then $A \in \mathcal{T}_q^s(X)$,

(c) if $A, B \in \mathcal{T}_q^s(X)$, then $A \cup B \in \mathcal{T}_q^s(X)$.

It is easy to see that $\mathcal{T}_q^s(X) = \mathcal{T}_q^s(X) = \mathcal{T}_q^s(X)$.

**Definition 2.1.** Let $(X, q)$ be a quasi-pseudometric space. A sequence $(x_n)$ in $X$ is called:

(a) left $K$-Cauchy if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$q(x_k, x_m) < \epsilon \text{ whenever } n, k \in \mathbb{N} \text{ with } N \leq k \leq n.$$  

(b) right $K$-Cauchy if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$q(x_k, x_m) < \epsilon \text{ whenever } n, k \in \mathbb{N} \text{ with } N \leq k \leq n.$$ 

(c) $q^s$-Cauchy if it is a Cauchy sequence in the symmetrised quasi-pseudometric $q^s$.

Note that if we want to emphasise the quasi-pseudometric $q$ on $X$, we shall say that a sequence is right $q$-$K$-Cauchy and left $q$-$K$-Cauchy.

For a quasi-pseudometric space $(X, q)$. It well-known that these above three concepts are associated as follows:

$q^s$-Cauchy $\implies$ left $K$-Cauchy and $q^s$-Cauchy $\implies$ right $K$-Cauchy.

**Remark 2.2.** Let $(X, q)$ be quasi-pseudometric space.

(a) A sequence $(x_n)$ in $X$ is left $q$-$K$-Cauchy if and only if $(x_n)$ is right $q^l$-$K$-Cauchy.

(b) A sequence $(x_n)$ in $X$ is $q^s$-Cauchy if and only if the sequence $(x_n)$ is both left $q$-$K$-Cauchy and right $q$-$K$-Cauchy.

The following definition can be found for instance on [4].

**Definition 2.3.** Let $(X, q)$ be quasi-pseudometric space. We say that $(X, q)$ is:

(a) bicomplete if its associated pseudometric space $(X, q^s)$ is complete, that is, every $q^s$-Cauchy sequence is $q^s$-convergent;

(b) sequentially left (right) $K$-complete if every left (right) $K$-Cauchy sequence is $q$-convergent;

(c) sequentially left (right) Smyth complete if every left (right) $K$-Cauchy sequence is $q^s$-convergent;
3. Uniformly continuous and semi-Lipschitz maps

An asymmetric norm on a real vector space $X$ is a function $\|\cdot\| : X \to [0, \infty)$ satisfying the conditions

1. $\|x\| = \|-x\| = 0$ then $x = 0$;
2. $\|ax\| = a\|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$,

for all $x, y \in X$ and $a \geq 0$. Then the pair $(X, \|\|)$ is called an asymmetric normed space.

The conjugate asymmetric norm $\|\|_*$ of $\|\|$ and the symmetrisation norm $\|\|_s$ of $\|\|$ are defined respectively by

$\|x\| := \| - x\|$ and $\|x\|_* := \max\{|x|, \|x\|\}$ for any $x \in X$.

An asymmetric norm $\|\|$ on $X$ induces a quasi-metric $q_{\|\|}$ on $X$ defined by

$q_{\|\|}(x, y) = \|x - y\|$ for any $x, y \in X$.

If $(X, \|\|, \|\|)$ is normed lattice space, then the function $\|\|_u$ defined by $\|x\|_u := \|x^+\|$, where $x^+ = \max\{x, 0\}$ is an asymmetric norm on $X$.

A basic but interesting example we point out the asymmetric norm $u$ on $\mathbb{R}$ (considered as a real vector space) defined for any $y \in \mathbb{R}$ by $u(y) = y^+$, where $y^+ = \max\{y, 0\}$, it follows that $u^-(y) = \max\{-x, 0\} = y^-$ and $u^+(y) = \max\{y^+, y^-\} = |x|$. In addition, the asymmetric norm $u$ induces the quasi-metric $q_u$ on $\mathbb{R}$ defined by $q_u(x, y) = (x - y)^+ = \max\{x - y, 0\}$ whenever $x, y \in \mathbb{R}$.

The following is a well-known definition.

**Definition 3.1.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces. A map $f : (X, q) \to (Y, p)$ is called quasi-uniformly continuous (or uniformly continuous) if for any $\epsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$, then $p(\varphi(x), \varphi(y)) < \epsilon$ for all $x, y \in X$.

**Lemma 3.2.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces. If the map $f : (X, q) \to (Y, p)$ is uniformly continuous, then the function $f : (X, q^*) \to (Y, p^*)$ is uniformly continuous.

**Example 3.3.** We equip $X = \mathbb{R}_+ = [0, \infty)$ with the quasi-metric $q$ defined by $q(x, y) = (y - x)^+$ for any $x, y \in [0, \infty)$ and $Y = \mathbb{R}$ is equipped with the $T_0$-quasi-metric $p$ defined by $p(x, y) = (y - x)^+$ for any $x, y \in \mathbb{R}$. Then

(i) the function $f(x) = -\sqrt{x}$ whenever $x \in \mathbb{R}_+$ is uniformly continuous from $(\mathbb{R}_+, \|\|)$ into $(\mathbb{R}, \|\|)$.

(ii) the function $f(x) = -\sqrt{x}$ whenever $x \in \mathbb{R}_+$ is not uniformly continuous from $(\mathbb{R}_+, q)$ into $(\mathbb{R}, p)$.

Let $(X, q)$ be a quasi-metric space and $(Y, \|\|)$ be an asymmetric normed space. Then a map $f : (X, q) \to (Y, \|\|)$ is called semi-Lipschitz if there exists $k \geq 0$ such that

$\|f(x) - f(y)\| \leq kq(x, y)$ for all $x, y \in X$. (3.1)

The number $k$ satisfying (3.1) is called semi-Lipschitz constant for $f$ and the map $f$ is called $k$-semi-Lipschitz. For more details about semi-Lipschitz maps we recommend the reader to see [5].

**Definition 3.4.** Let $(X, q)$ be a quasi-metric space and $(Y, \|\|)$ be an asymmetric normed space. Then:

(a) A map $f : (X, q) \to (Y, \|\|)$ is called locally semi-Lipschitz provided that for all $x \in X$, then there exists $\delta(x) > 0$ such that $f|_{D_q(x, \delta(x))}$ is semi-Lipschitz.
(b) A function \( f : (X, q) \rightarrow (Y, \|\cdot\|) \) is called \textit{uniformly locally semi-Lipschitz} provided that for all \( x \in X \), there exists \( \delta > 0 \) (\( \delta \) does not depend to \( x \)) such that \( f|_{D_q(x, \delta)} \) is semi-Lipschitz.

**Lemma 3.5.** Let \((X, q)\) be a quasi-metric space and \((Y, \|\cdot\|)\) be an asymmetric normed space. If function \( f : (X, q) \rightarrow (Y, \|\cdot\|) \) is locally semi-Lipschitz, then \( f : (X, q^s) \rightarrow (Y, \|\cdot\|) \) is locally semi-Lipschitz.

**Proof.** Suppose that \( f : (X, q) \rightarrow (Y, \|\cdot\|) \) is locally semi-Lipschitz. Let \( x \in X \), there exists \( \delta(x) > 0 \) and \( k \geq 0 \) such that for any

\[
y, z \in D_{q^s}(x, \delta(x)) \subseteq D_q(x, \delta(x))
\]

we have

\[
\|f(y) - f(z)\| \leq kq(y, z) \leq kq^s(y, z) \tag{3.2}
\]

and

\[
\|f(z) - f(y)\| \leq kq(z, y) \leq kq^s(y, z). \tag{3.3}
\]

Combining (3.2) and (3.3) for some \( k \geq 0 \) we have

\[
\|f(y) - f(z)\| \leq kq(y, z) \leq kq^s(y, z)
\]

whenever \( y, z \in D_{q^s}(x, \delta(x)) \). Thus the function \( f : (X, q^s) \rightarrow (Y, \|\cdot\|) \) is locally semi-Lipschitz. \( \Box \)

**Remark 3.6.** Let \((X, q)\) be a quasi-metric space and \((Y, \|\cdot\|)\) be an asymmetric normed space. If a function \( fi : (X, q) \rightarrow (Y, \|\cdot\|) \) is locally semi-Lipschitz, then \( \varphi|_{D_q(x, \delta_x)} \) is continuous whenever \( x \in X \) and for some \( \delta_x > 0 \).

4. Left (right) \( K \)-Cauchy sequentially regular maps

**Definition 4.1.** \((\text{compare [10, Definition 3.1 and Definition 3.4]})\). Let \((X, q)\) be a quasi-pseudometric space. Let \((x_n)\) and \((y_n)\) be sequences in \(X\).

(a) We say that the sequences \((x_n)\) and \((y_n)\) are \textit{parallel} with respect to \( q \) (noted by \((x_n)\equiv_q(y_n)\)) if for any \( \epsilon > 0 \), there exists \( n_\epsilon \in \mathbb{N} \) such that \( q(x_n, y_n) < \epsilon \) whenever \( n \geq n_\epsilon \).

(b) We say that the sequences \((x_n)\) and \((y_n)\) are \textit{equivalent} with respect to \( q \) (noted by \((x_n)\equiv q(y_n)\)) if for any \( \epsilon > 0 \), there exists \( n_\epsilon \in \mathbb{N} \) such that \( q^s(x_k, y_n) < \epsilon \) whenever \( n, k \geq n_\epsilon \).

Note that the concept of parallel sequences in quasi-pseudometric spaces is not new. For instance, in [10], Moshoko introduced concepts of parallel sequences and equivalent sequences in order to study extensions of maps that preserve \( q^s \)-Cauchy sequences in a quasi-pseudometric space \((X, q)\). But it is well-known that on a quasi-pseudometric space \((X, q)\), any \( q^s \)-Cauchy sequence in \(X\) is left \( K \)-Cauchy (right \( K \)-Cauchy), still the converse is not true in general. Our definitions of parallel and equivalent sequences are motivated from metric point of view of parallel and equivalent sequences (see [12]) and by the fact that parallel sequences are preserved by uniformly continuous maps and equivalent sequences are preserved by Cauchy-sequentially-regular maps. However, we are studying maps that preserve left \( K \)-Cauchy (right \( K \)-Cauchy) sequences. This explains why our Definition 4.1(2) is more general than [10, Definition 3.4]. We point out that in [7], Doitchinov introduced the concept of cosequence sequences which is similar to the concept of parallel sequences with connections to Cauchy sequences in a quasi-pseudometric space. From cosequence sequences, he defined equivalent sequences for a quasi-metric space satisfying some properties (that he called balanced quasi-metric space).

**Lemma 4.2.** Let \((X, q)\) be a quasi-pseudometric space. Let \((x_n)\) and \((y_n)\) be sequences in \(X\) and \( a \in X \). If \((x_n)\) is \( q^s \)-convergent to \( a \) and \((y_n)\) is \( q^s \)-convergent to \( a \), then \((x_n)\equiv q(y_n)\).
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Let $12$. Then the following statements are equivalent.

**Proof.** Let $\epsilon > 0$. Suppose that $(x_n)$ is $q^a$-convergent to $a$ and $(y_n)$ is $q^a$-convergent to $a$. We show that $(x_n) \equiv q^r(y_n)$. Then there exists $n_\epsilon \in \mathbb{N}$ and $n'_\epsilon \in \mathbb{N}$ such that

$$q^a(a, x_n) < \frac{\epsilon}{2} \quad \text{if} \quad n \geq n_\epsilon \quad (4.1)$$

and

$$q^a(y_n, a) < \frac{\epsilon}{2} \quad \text{if} \quad n \geq n'_\epsilon. \quad (4.2)$$

Let $N = \max\{n_\epsilon, n'_\epsilon\}$. If $N \leq k, n$, then

$$q^a(x_k, y_n) = q^a(y_n, x_k) \leq q^a(y_n, a) + q^a(a, x_k) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $(x_n) \equiv_q(y_n)$. \hfill \Box

The following lemma is a consequence of the definition and Remark 2.2.

**Lemma 4.3.** Let $(X, q)$ be a quasi-pseudometric space. Let $(x_n)$ and $(y_n)$ be sequences in $X$. If $(x_n) \equiv q(y_n)$, then the sequence $(x_n)$ is left $K$-Cauchy and right $K$-Cauchy.

We leave the proof of the following lemma.

**Lemma 4.4.** Let $(X, q)$ be a quasi-pseudometric space and $(x_n)$ and $(y_n)$ be any two sequences in $X$ and $a \in X$. If $(x_n)$ is $q$-convergent to $a$ and $(y_n)||_q(x_n)$, then $(y_n)$ is $q$-convergent to $a$.

**Lemma 4.5.** Let $(X, q)$ be a quasi-pseudometric space and $(x_n)$ and $(y_n)$ be any two sequences in $X$. It is true that $(x_n) \equiv q(y_n)$ if and only if the sequence $(z_n)\equiv (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ is left $K$-Cauchy and right $K$-Cauchy, where $z_n := (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$.

**Proof.** $(\Rightarrow)$ Let $\epsilon > 0$. Suppose that $(x_n) \equiv_q(y_n)$. Then there exists $n_\epsilon \in \mathbb{N}$ such that

$$q^a(x_k, y_m) < \epsilon \quad \text{whenever} \quad k, m \geq n_\epsilon.$$

It follows that the sequence $z_n = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ is $q^a$-Cauchy sequence. Hence the sequence $z_n$ is left $K$-Cauchy and right $K$-Cauchy.

$(\Leftarrow)$ Suppose that the sequence $z_n = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ is left $K$-Cauchy and right $K$-Cauchy. Then the sequence $z_n = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ is $q^a$-Cauchy. Therefore, we have that $(x_n) \equiv_q(y_n)$ by [12, Theorem 1 (4)]. \hfill \Box

The following proposition is obvious. Therefore, we omit the proof.

**Proposition 4.6** (compare [10, Theorem 3.2]). Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces. Then the following statements are equivalent.

1. The map $f : (X, q) \rightarrow (Y, p)$ is uniformly continuous.
2. Whenever $(x_n)||_q(y_n)$ in $X$ and $f : (X, q) \rightarrow (Y, p)$ is a map, then $(f(x_n))||_p(f(y_n))$ in $Y$.

**Remark 4.7.** We point out that it is easy to find an example of two sequences which are parallel with respect to $q$ but they are not parallel with respect to $q^r$.

**Definition 4.8.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces. A map $f : (X, q) \rightarrow (Y, p)$ is called:

(a) A left $K$-Cauchy sequentially-regular if for any left $K$-Cauchy sequence $(x_n)$ in $X$, then the sequence $(f(x_n))$ is left $K$-Cauchy in $Y$.

(b) A right $K$-Cauchy sequentially-regular if for any right $K$-Cauchy sequence $(x_n)$ in $X$, then the sequence $(f(x_n))$ is left $K$-Cauchy in $Y$. 
Proposition 4.9. Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces and $f : (X, q) \to (Y, p)$ be a map. Then we have that the map $f$ is left $K$-Cauchy sequentially-regular in $X$ if and only if whenever $(x_n)_{n \in \mathbb{N}} \equiv_q (y_n)_{n \in \mathbb{N}}$ in $X$, then $(f(x_n))_{n \in \mathbb{N}} \equiv_p (f(y_n))_{n \in \mathbb{N}}$ in $Y$.

Proof. (⇒) Suppose that $f$ is left $K$-Cauchy and right $K$-Cauchy sequentially-regular. If $(x_n)_{n \in \mathbb{N}} \equiv_q (y_n)_{n \in \mathbb{N}}$ in $X$, then it follows that the sequence $(x_1, y_1, x_2, y_2, \cdots)$ is left $K$-Cauchy and right $K$-Cauchy sequence in $X$ by Lemma 4.5. Thus the sequence $(f(x_1), f(y_1), f(x_2), f(y_2), \cdots)$ is left $K$-Cauchy and right $K$-Cauchy sequence in $Y$ from the assumption on the map $f$. Hence $(f(x_n))_{n \in \mathbb{N}} \equiv_q (f(y_n))_{n \in \mathbb{N}}$ in $Y$ by Lemma 4.5.

(⇐) Assume that $f$ preserves equivalent sequences. Let $(x_n)$ be a left $K$-Cauchy sequence and right $K$-Cauchy in $X$. Since $(x_n)_{n \in \mathbb{N}} \equiv_q (x_n)_{n \in \mathbb{N}}$, then we have that $(f(x_n))_{n \in \mathbb{N}} \equiv_q (f(x_n))_{n \in \mathbb{N}}$ in $Y$. Therefore, the sequence $(f(x_n))_{n \in \mathbb{N}}$ is left $K$-Cauchy and right $K$-Cauchy sequence in $Y$. □

Theorem 4.10. Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces and $f : (X, q) \to (Y, p)$ be a map. Then the following hold.

1. If the map $f$ is uniformly continuous, then $f$ is left $K$-Cauchy (right $K$-Cauchy) sequentially-regular.

2. If the map $f$ is right $K$-Cauchy and right $K$-Cauchy sequentially-regular, then $f$ is continuous with respect to $\tau(q^*)$ and $\tau(p^*)$.

Proof. (1) Let $\epsilon > 0$. Suppose that $f$ is uniformly continuous. We only show that $f$ is left $K$-Cauchy sequentially regular and for $f$ right $K$-Cauchy will follow by symmetry. Let $(x_n)$ be any left $q$-$K$-Cauchy sequence in $X$.

Then there exists $\delta > 0$ because $f$ is uniformly continuous such that

$$q(x_k, x_n) < \delta \quad \text{whenever} \quad N \leq k \leq n$$

for some $N \in \mathbb{N}$ since $(x_n)$ is left $K$-Cauchy sequence in $X$. It follows that

$$p(f(x_k), f(x_n)) < \epsilon \quad \text{whenever} \quad N \leq k \leq n$$

for some $N \in \mathbb{N}$. Hence $(f(x_n))$ is left $K$-Cauchy in $Y$.

(2) Suppose that $f$ is right $K$-Cauchy and right-$K$-Cauchy sequentially-regular. If $(x_n)$ be sequence in $X$ such that $(x_n)$ is $q^*$-convergent to $a \in X$. We show that the sequence $(f(x_n))$ is $p^*$-convergent to $f(a)$.

We consider the constant sequence $(a)$ which is $q^*$-convergent to $a$. Then we have that $(x_n) \equiv_q (a)$ by Lemma 4.2. It follows that the sequence $(x_1, a, x_2, a, \cdots)$ is left $K$-Cauchy and right $K$-Cauchy ($q^*$-Cauchy) in $X$.

From our assumption we have $(f(x_1), f(a), f(x_2), f(a), \cdots)$ is left $K$-Cauchy and right $K$-Cauchy ($p^*$-Cauchy) in $Y$ with a convergent subsequence $(f(a))$ which $p^*$-convergent to $f(a)$. Thus the sequence $(f(x_n))_{n \in \mathbb{N}}$ is $p^*$-convergent to $f(a)$. □

Example 4.11 (compare [15, Example 1]). Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\} \cup \{n : n \in \mathbb{N} \setminus \{1\}\}$. We equip $X$ with the quasi-metric $q$ defined by $q(x, x) = 0$ for any $x \in X$, $q(0, 1/n) = 1/n$ for any $n \in \mathbb{N}$, $q(1/n, 1/m) = 1/n$ whenever $n < m$, $q(0, n) = 2^{-n}$ whenever $n \in \mathbb{N} \setminus \{1\}$, $q(n, m) = |2^{-n} - 2^{-m}|$ whenever $n, m \in \mathbb{N} \setminus \{1\}$ and $q(x, y) = 1$ otherwise.

It is easy to see that sequences $(1/n)$ and $(n)$ are left $q$-$K$-Cauchy in $X$ and both are $q$-convergent to 0. If we consider the function $g : (X, q) \to (X, q)$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/n & \text{if } x = n \in \mathbb{N} \\ n & \text{if } x = 1/n \quad \text{and} \quad n \in \mathbb{N} \setminus \{1\}. \end{cases}$$
Then the function $g$ preserves left $q$-$K$-Cauchy sequences since $g((1/n)) = (n)$ and $g((n)) = (1/n)$ and $g$ is continuous.

**Theorem 4.12.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces and $f : (X, q) \to (Y, p)$ be a uniformly continuous map. Then $f : (X, q) \to (Y, p)$ is left $q$-$K$-Cauchy sequentially-regular if and only if $f : (X, q^t) \to (Y, p^t)$ is right $q^t$-$K$-Cauchy sequentially-regular.

**Proof.** We only prove the necessary condition and the sufficient condition follows by similar arguments. It is obvious that $f : (X, q) \to (Y, p)$ is uniformly continuous if and only if $f : (X, q^t) \to (Y, p^t)$ is uniformly continuous.

Suppose that $f : (X, q) \to (Y, p)$ is left $q$-$K$-Cauchy sequentially-regular and let $(x_n)$ be a right $q^t$-$K$-Cauchy sequence. Then the sequence $(x_n)$ is left $q$-$K$-Cauchy in $X$ by Remark 2.2 (1).

Moreover, the sequence $(f(x_n))$ is left $p$-$K$-Cauchy in $Y$ from the assumption. But the sequence $(f(x_n))$ is right $p^t$-$K$-Cauchy in $Y$ again by Remark 2.2 (1). Hence $f : (X, q^t) \to (Y, p^t)$ is right $q^t$-$K$-Cauchy sequentially-regular.

**Theorem 4.13.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces. If the uniformly continuous map $f : (X, q) \to (Y, p)$ is left $K$-Cauchy and right $K$-Cauchy sequentially-regular, then $f : (X, q^t) \to (Y, p^t)$ is $q^t$-$K$-Cauchy sequentially regular.

**Proof.** Let $(x_n)$ be a $q^t$-Cauchy sequence in $X$. Then $(x_n)$ is both left and right $q$-$K$-Cauchy in $X$ by Remark 2.2 (2).

Furthemore, $(f(x_n))$ is both left and right $p$-$K$-Cauchy in $Y$ since $f : (X, q) \to (Y, p)$ is both left $K$-Cauchy and right $K$-Cauchy sequentially-regular.

Moreover, the sequence $(f(x_n))$ is $p^t$-$K$-Cauchy by Remark 2.2 (2). Therefore, the uniformly continuous map $f : (X, q^t) \to (Y, p^t)$ is $q^t$-$K$-Cauchy in $X$.

**Theorem 4.14.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces and $f : (X, q) \to (Y, p)$ be a map. Then whenever $(X, q)$ is left Smyth complete and the map $f$ is continuous, then $f$ is left $K$-Cauchy sequentially-regular.

**Proof.** Suppose that $(X, q)$ is left Smyth complete and the map $f$ is continuous. If the sequence $(x_n)$ is left $K$-Cauchy, then there exists $x \in X$ such that $(x_n)$ is $q^t$-convergent to $x$ by the left Smyth completeness of $(X, q)$.

Then, the sequence $(f(x_n))$ is $p^t$-convergent to $f(x)$ since the map $f : (X, q^t) \to (Y, p^t)$ is continuous. Hence the sequence $(f(x_n))$ is $q^t$-Cauchy. Therefore, the sequence $(f(x_n))$ is left $K$-Cauchy by Remark 2.2(2).

**Corollary 4.15.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces and $f : (X, q) \to (Y, p)$ be a map. Then whenever $(X, q)$ is right Smyth complete and the map $f$ is continuous, then $f$ is right $K$-Cauchy sequentially-regular.

**Remark 4.16.** In Theorem 4.14, if we replace the left Smyth completeness by the sequentially left $K$-completeness, the theorem does not hold because for a sequence being left $K$-Cauchy does not guarantee the existence of the limit (see [13, Example 2]).

5. Total boundedness and left $K$-Cauchy sequential regularity

Let $(X, q)$ be a quasi-pseudometric space. An arbitrary subset $A$ of $X$ is called $q$-bounded if and only if there exists $x \in X$, $r > 0$ and $s > 0$ such that $A \subseteq D_q(x, r) \cap D_{q^t}(x, s)$. Note that one can replace $D_q(x, r) \cap D_{q^t}(x, s)$ by $D_q[x, r] \cap D_{q^t}[x, s]$.

Note that the above definition is slightly different from [16]. In the sense of [16] a subset $A$ of $X$ can be $q$-bounded and not necessary $q'$-bounded. Obviously in our context a subset $A$ is $q$-bounded if and only if it is $q'$-bounded. But $q$-boundedness (or $q'$-boundedness) does not imply $q^*$-boundedness. Moreover, if $q$ is an extended quasi-pseudometric on $X$ (i.e. the distance between two point can be $\infty$), then a subset $B$ of $X$ can be included in $D_q(x, \varepsilon)$ for some $x \in X$ but its diameter $\text{diam}(B) = \{q(y, z) : y, z \in B\} = \infty$ (see [16, p. 2022]).

Let $\mathcal{B}_q(X)$ be the collection of all $q$-bounded subsets of $X$ whenever $(X, q)$ is quasi-pseudometric space. It is easy to see that

(a) whenever $x \in X$, then $\{x\} \in \mathcal{B}_q(X)$,
(b) whenever $A \subseteq B \subseteq X$ and $B \in \mathcal{B}_q(X)$, then $A \in \mathcal{B}_q(X)$,
(c) whenever $A, B \in \mathcal{B}_q(X)$, then $A \cup B \in \mathcal{B}_q(X)$.

It follows that $\mathcal{B}_q(X)$ forms a bornology on $X$ and this bornology is called the quasi-pseudometric bornology determined by $q$. Furthermore, We have the following observations instead of the one observed in [11]

$$\mathcal{B}_q(X) = \mathcal{B}_q(X)$$

and

$$\mathcal{B}_q(X) = \mathcal{B}_q(X).$$

**Remark 5.1.** Let $(X, q)$ be a quasi-pseudometric space and $A \subseteq X$. It is easy to see that:

(i) If $A \in \mathcal{T}\mathcal{B}_q(X)$, then $A \in \mathcal{B}_q(X)$.

(ii) Whenever $F$ is finite subset of $X$, $F \in \mathcal{T}\mathcal{B}_q(X)$.

**Proposition 5.2.** Let $(X, q)$ and $(Y, p)$ be quasi-pseudometric spaces and $f : (X, q) \rightarrow (Y, p)$ be a map. Then $f|_T$ is uniformly continuous, whenever $T \in \mathcal{T}\mathcal{B}_q(X)$ if and only if $f$ is Cauchy sequentially regular.

**Proof.** ($\Rightarrow$) Assume that $f : (T, q) \rightarrow (Y, p)$ is uniformly continuous with $T$ is $q^*$-totally bounded. Let $(x_n)$ be a $q^*$-Cauchy sequence. Then $\{x_n : n \in \mathbb{N}\}$ is $q^*$-totally bounded and $f : (T, q^*) \rightarrow (Y, p^*)$ is uniformly continuous. It follows that $f$ is Cauchy sequentially regular from [2, Proposition 5.7(2)].

($\Leftarrow$) Without loss of generality we suppose that $f : (T, q^*) \rightarrow (Y, p^*)$ is not uniform continuous and $T$ is $q^*$-totally bounded. Then for any $n \in \mathbb{N}$, there exists two sequences $(x_n), (t_n)$ in $T$ such that

$$q^*(x_n, t_n) < \frac{1}{n} \text{ and } p^*(f(x_n), f(t_n)) \geq \varepsilon \text{ for some } \varepsilon > 0. \quad (5.3)$$

From the $q^*$-totally boundedness of $T$, suppose that the sequence $(t_n)$ is $q^*$-Cauchy, then the sequence $(t_1, x_1, t_2, x_2, \cdots)$ is $q^*$-Cauchy but its image $(f(t_1), f(x_1), f(t_2), f(x_2), \cdots)$ under $f$ is not $p^*$-Cauchy from (5.3).

**Theorem 5.3.** Let $(X, q)$ be a quasi-pseudometric space and $F$ be a nonempty subset of $X$. Then following conditions are equivalent:

(1) $F$ is $q$-totally bounded;

(2) Whenever $(Y, ||.||)$ is an asymmetric normed space and the map $f : (X, q) \rightarrow (Y, q||.||)$ is left and right $K$-Cauchy sequentially regular, then $f(F) \in \mathcal{B}_{q||.||}(Y)$;

(3) Whenever $(Y, ||.||)$ is an asymmetric normed space and the map $f : (X, q) \rightarrow (Y, q||.||)$ is uniformly locally semi-Lipschitz, then $f(F) \in \mathcal{B}_{q||.||}(Y)$;

(4) Whenever the function $f : (X, q) \rightarrow (\mathbb{R}, q_a)$ is uniformly locally semi-Lipschitz, then $f(F)$ is $q_a$-bounded set $\mathbb{R}$. 


Proof. (1) $\implies$ (2) Suppose $f : (X, q) \to (Y, q_{\|\cdot\|})$ is left and right $K$-Cauchy sequentially regular and $F$ is $q$-totally bounded. We have that $f : (X, q^s) \to (Y, q_{\|\cdot\|})$ is $q^s$-Cauchy sequentially regular by Theorem 4.13. Since $F$ is $q^s$-totally bounded as a $q$-totally bounded. Then $f(F) \in \mathcal{B}_{q_{\|\cdot\|}}(Y)$ by [1, Theorem 3.2]. Thus from inclusion (5.1) we have $f(F) \in \mathcal{B}_{q_{\|\cdot\|}}(Y)$.

(2) $\implies$ (3) and (3) $\implies$ (4) Follows from Lemma 3.5 and [1, Theorem 3.2].

(4) $\implies$ (1) Suppose that $F$ is not $q$-totally bounded. We show that there exists a semi-Lipschitz function $g : (D_q(x, \delta), q) \to (\mathbb{R}, q_u)$ for any $x \in X$ and some $\delta > 0$.

Since $F$ is not $q$-totally bounded, then we have that

$$F \notin \bigcup_{k=1}^{n} D_q(f_k, \epsilon), \text{ where } f_k \in F \text{ whenever } k \in \{1, \ldots, n\}$$

for some $\epsilon > 0$. By induction, we construct a sequence $(f_n)$ in $F$ such that whenever $n \in \mathbb{N}$ we have $f_{n+1} \notin \bigcup_{k=1}^{n} D_q(f_k, \epsilon)$.

It follows that the family $\{D_q(f_n, \epsilon) : n \in \mathbb{N}\}$ is uniformly discrete. Furthermore, we have for any $x \in X$, there exists $n' \in \mathbb{N}$ such that

$$\emptyset \neq D_q(x, \epsilon/4) \cap D_q(f_{n'}, \epsilon/4) \subseteq D_q(x, \epsilon/4) \cap D_q(f_{n'}, \epsilon/4)$$

from [2, Proposition 3.8].

Let $g$ be a function defined by

$$g(x) = \begin{cases} n - \frac{4n}{\epsilon} q(f_n, x) & \text{if } x \in D_q(f_n, \epsilon/4) \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the function $g$ is unbounded with respect to $u$. We now show that $g$ is a semi-Lipschitz. Consider $x, y \in D_q(f_n, \epsilon/4)$, then

$$q_u(g(x), g(y)) = (g(x) - g(y))^+ = \left[\left(n - \frac{4n}{\epsilon} q(f_n, x)\right) - \left(n - \frac{4n}{\epsilon} q(f_n, y)\right)\right]$$

$$= \frac{4n}{\epsilon} \left[q(f_n, y) - q(f_n, x)\right]$$

$$\leq \frac{4n}{\epsilon} q(x, y).$$

Therefore, we have that the function $g$ is semi-Lipschitz with $k = \frac{4n}{\epsilon}$. \qed

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