Comparing Abs-Normal NLPs to MPECs

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We show that the class of unconstrained NLPs in abs-normal form is a subclass of the class of MPECs and that the class of NLPs with general constraints in abs-normal form is equivalent to the class of MPECs. Moreover, we compare constraint qualifications and stationarity concepts of these problem classes and observe close relations between them.

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1 Introduction

Nonsmoothness arises in many practical optimization problems, for example in engineering and economics. Typical problem classes are MPECs and abs-normal NLPs. In this paper we consider very briefly the relations between these two classes. An overview about MPECs can be found in [1], for prerequisites of the abs-normal form see [2, 3].

2 Unconstrained Abs-Normal NLP

We take interest in problems of the form

$$\min_{x \in \mathbb{R}^n} \varphi(x)$$

with $\varphi$ in abs-normal form [2, 3]. Then, these problems can be formulated as unconstrained abs-normal NLPs.

**Definition 2.1** (Unconstrained Abs-Normal NLP) Let $D^x$ be an open subset of $\mathbb{R}^n$. A nonsmooth unconstrained optimization problem is called an **unconstrained abs-normal NLP** if functions $f \in \mathcal{C}^2(D^x, |z|, \mathbb{R})$ and $c_Z \in \mathcal{C}^2(D^x, |z|, \mathbb{R}^s)$ for $D^x, |z| = D^x \times D^{1|z|}$ and $d \geq 1$ exist such that the NLP can equivalently be stated as

$$\min_{x,z} f(x, |z|) \quad \text{s.t.} \quad c_Z(x, |z|) - z = 0,$$

where $0 \in D^{1|z|}$ and $\partial_Z c_Z(x, |z|)$ is strictly lower triangular. The variables $z_i, i = 1, \ldots, s$ are called switching variables.

We can write $|z| = \Sigma z$ with $\Sigma = \text{diag}(\text{sign}(z))$. By the implicit function theorem the system $z = c_Z(x, \Sigma z)$ has a locally unique solution $z(x)$ for fixed $\Sigma$, with Jacobian $\partial_Z z(x) = [I - \partial c_Z(x, |z|)\Sigma^{-1}\partial_1 c_Z(x, |z|)]$. With this prerequisite we can define kink qualifications for (2).

**Definition 2.2** (LIKQ and MFKQ) We say that a point $x \in D^x$ satisfies LIKQ if the matrix $[\partial_1 z(x)]_{i=1}^s$ has full row rank. Herein, we use the active switching set $\alpha = \{i \in \{1, \ldots, s\} : z_i(x) = 0\}$.

Set $\tilde{\alpha} = \text{sign}(z_i(x))$ for $i \notin \alpha$ and choose $\tilde{\alpha} \in \{+1, -1\}$ for $i \in \alpha$. We say that a point $x \in D^x$ satisfies MFKQ if for all $\Sigma = \text{diag}(\tilde{\alpha})$ the system $[\Sigma \partial_1 z(x)]_{i=1}^s w > 0$ admits a solution $w \in \mathbb{R}^n$, unless $[\Sigma \partial_1 z(x)]_{i=1}^s w \geq 0$ admits only the solution $w = 0$.

Further, we can rewrite problem (2) as an MPEC. To this end we define variable vectors $u = [z]^+ = \max(0, z)$ and $v = [z]^- = \max(-z, 0)$ and replace $|z|$ by $u + v$ and $z$ by $u - v$. Moreover, we need to enforce complementarity of $u$ and $v$ so that the representations of $|z|$ and $z$ hold.

**Definition 2.3** (Counterpart MPEC) **The counterpart MPEC** of (2) reads

$$\min_{y, u, v} f(y, u + v) \quad \text{s.t.} \quad u - v - c_z(y, u + v) = 0, \quad 0 \leq u \perp v \geq 0.$$  

Thus, unconstrained abs-normal NLPs are a subclass of MPECs. In the following we compare regularity conditions and transfer stationarity concepts from MPECs. It turns out that LIKQ and MPEC-LICQ are equivalent.

**Proposition 2.4** (Equivalence of LIKQ and MPEC-LICQ) A feasible point $(\hat{x}, \hat{z})$ of (2) satisfies LIKQ if and only if the point $(\hat{y}, \hat{u}, \hat{v}) = (\hat{x}, [\hat{z}]^+, [\hat{z}]^-)$ of (3) satisfies MPEC-LICQ.

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Proposition 2.5 (Strongly Stationary Points and Minimizers of the Abs-Normal Form) Let (2) satisfy LIKQ. If \((\hat{x}, \hat{z})\) is a local minimizer of (2), then \((\hat{y}, \hat{u}, \hat{v}) = (\hat{x}, [\hat{z}]^+, [\hat{z}]^-)\) is a strongly stationary point of (3).

In contrast, MFKQ is weaker then MPEC-MFCQ because MPEC-MFCQ is equal to MPEC-LICQ in the specialized MPEC (3). It is part of ongoing research to find an equivalent concept to MPEC-MFCQ in a general setting.

Proposition 2.6 (MPEC-MFCQ implies MFKQ) A feasible point \((\hat{x}, \hat{z})\) of (2) satisfies MFKQ if the point \((\hat{y}, \hat{u}, \hat{v}) = (\hat{x}, [\hat{z}]^+, [\hat{z}]^-)\) of (3) satisfies MFCQ for all MPEC branch problems.

Note that Abadie’s constraint qualification (MPEC-ACQ) holds in this setting without any prerequisites. Key is the strictly lower triangular structure of \(\partial_{Z\subseteq R} (x, |z|)\) and the absence of additional constraints from (3).

Proposition 2.7 (MPEC-ACQ holds) Any feasible point \((\hat{y}, \hat{u}, \hat{v})\) of (3) satisfies MPEC-ACQ.

Proposition 2.8 (M-Stationary Points and Minimizers of the Abs-Normal Form) If a feasible point \((\hat{x}, \hat{z})\) of (2) is a local minimizer, then the point \((\hat{y}, \hat{u}, \hat{v}) = (\hat{y}, [\hat{z}]^+, [\hat{z}]^-)\) of (3) is an M-stationary point.

3 Abs-Normal NLP

Now, we consider generally constrained abs-normal NLPs and their counterpart MPECs.

Definition 3.1 (Abs-Normal NLP) Let \(D^Z\) be an open subset of \(\mathbb{R}^n\). We say that a non-smooth NLP is in abs-normal form if functions \(f \in C^1(D^x\times|z|, \mathbb{R}), g \in C^1(D^x\times|z|, \mathbb{R}^m), h \in C^1(D^x\times|z|, \mathbb{R}^n), \) and \(c_Z \in C^1(D^Z \times \mathbb{R}_{Z < 0}, \mathbb{R}^p)\) with \(\partial_Z c_Z (x, |z|)\) strictly lower triangular exist such that the problem reads

\[
\min_{x,z} f(x, |z|) \quad \text{s.t.} \quad g(x, |z|) = 0, \quad h(x, |z|) \geq 0, \quad c_Z(x, |z|) - z = 0. \tag{4}
\]

Definition 3.2 (Counterpart MPEC of Abs-Normal NLP) The counterpart MPEC of (4) reads

\[
\min_{y,u,v} f(y, u, v) \quad \text{s.t.} \quad g(y, u, v) = 0, \quad h(y, u + v) \geq 0, \quad u - v - c_Z(y, u, v) = 0, \quad 0 \leq u \perp v \geq 0.
\]

These problem classes are equivalent: with \(0 = \min(u, v) = \frac{1}{2} (u + v - (u - v))\) the complementarity condition is posed in abs-normal form.

4 Conclusion and Outlook

We have considered unconstrained abs-normal NLPs and we have studied their relations with MPECs; more details can be found in [5]. In [6], the LIKQ and optimality conditions for the general abs-normal NLP (4) are studied. The comparison of these concepts to the theory of MPECs is a subject of ongoing research.

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