Two-Armed Bandit Problem, Data Processing, and Parallel Version of the Mirror Descent Algorithm

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Abstract—We consider the minimax setup for the two-armed bandit problem as applied to data processing if there are two alternative processing methods available with different a priori unknown efficiencies. One should determine the most effective method and provide its predominant application. To this end we use the mirror descent algorithm (MDA). It is an effective method and provides its predominant application. To improve significantly the theoretical estimate of the factor using Monte-Carlo simulations. Then we propose a parallel version of the MDA which allows processing of data by packets in a number of stages. The usage of parallel version of the MDA ensures that total time of data processing depends mostly on the number of packets but not on the total number of data. It is unexpectedly that the parallel version behaves unlike the ordinary one even if the number of packets is large. Moreover, the parallel version considerably improves control performance because it provides significantly smaller value of the minimax risk. We explain this result by considering another parallel modification of the MDA which behavior is close to behavior of the ordinary version. Our estimates are based on invariant descriptions of the algorithms. All estimates are obtained by Monte-Carlo simulations.

It’s worth noting that parallel version performs well only for methods with close efficiencies. If efficiencies differ significantly then one should use the combined algorithm which at initial sufficiently short control horizon uses ordinary version and then switches to the parallel version of the MDA.

Keywords—two-armed bandit problem, control in a random environment, minimax approach, robust control, mirror descent algorithm, parallel processing.

1. Introduction

We consider the two-armed bandit problem (see, e.g. [1], [2]) which is also well-known as the problem of expedient behavior in a random environment (see, e.g. [3], [4]) and the problem of adaptive control in a random environment (see, e.g. [5], [6]) in the following setting. Let \( \xi_n, n = 1, \ldots, N \) be a controlled random process which values are interpreted as incomes, depend only on currently chosen actions \( y_n \) (\( y_n \in \{1, 2\} \)) and have probability distributions

\[
Pr(\xi_n = 1|y_n = \ell) = p_\ell, \quad Pr(\xi_n = 0|y_n = \ell) = q_\ell
\]

where \( p_\ell + q_\ell = 1, \ell = 1, 2. \) So, this is the so-called Bernoulli two-armed bandit. Such bandit is described by a parameter \( \theta = (p_1, p_2) \) with the set of values \( \Theta = \{ \theta : 0 \leq p_\ell \leq 1; \ell = 1, 2 \} \). It is well-known that mathematical expectation and variance of one-step income are equal to

\[
m_\ell = E(\xi_n|y_n = \ell) = p_\ell, \quad D_\ell = Var(\xi_n|y_n = \ell) = p_\ell q_\ell
\]

The goal is to maximize (in some sense) the total expected income. Control strategy \( \sigma \) at the point of time \( n \) assigns a random choice of the action \( y_n \) depending on the current history of the process, i.e. responses \( x^{n-1} = x_1, \ldots, x_{n-1} \) to applied actions \( y^{n-1} = y_1, \ldots, y_{n-1} \):

\[
\sigma_\ell(y^{n-1}, x^{n-1}) = Pr(y_n = \ell|y^{n-1}, x^{n-1}), \quad \ell = 1, 2
\]

The most general set of strategies is denoted here by \( \Sigma_\ell \). However, there may be some additional restrictions on the set of strategies. For example, one can consider the set of strategies \( \Sigma_2 \) described by the mirror descent algorithm presented below. More restrictive is additional requirement to strategy to allow parallel processing, this is the set \( \Sigma_\parallel \) below. In the sequel we define some additional sets of strategies.

If parameter \( \theta \) is known then the optimal strategy should always apply the action corresponding to the largest value of \( m_1, m_2 \). The total expected income would thus be equal to \( N(m_1 \lor m_2) \). If parameter is unknown then the regret

\[
L_N(\sigma, \theta) = N(m_1 \lor m_2) - E_{\sigma, \theta} \left( \sum_{n=1}^{N} \xi_n \right)
\]

describes expected losses of total income with respect to its maximal possible value due to incomplete information. Here \( E_{\sigma, \theta} \) denotes the mathematical expectation calculated with respect to measure generated by strategy \( \sigma \) and parameter \( \theta \).
According to the minimax approach the maximal value of the regret on the set of parameters $\Theta$ should be minimized on the set of strategies. The value
\[
R_N^{(0)}(\Theta) = \inf_{\Sigma_0} \sup_{\theta} L_N(\sigma, \theta)
\]
is called the minimax risk corresponding to the most general set of strategies $\Sigma_0$ and the optimal strategy $\sigma^M_0$ is called the minimax strategy. Application of the minimax strategy ensures that the following inequality holds
\[
L_N(\sigma^M_0, \theta) \leq R_N^{(0)}(\Theta)
\]
for all $\theta \in \Theta$ and this implies robustness of the control.

The minimax approach to the problem was proposed by H. Robbins in [7]. This article caused a significant application of the main theorem of the theory of games for all $\theta \in \Theta$.

**Theorem 1.** The following estimates of the minimax risk (1) hold as $N \to \infty$ for Bernoulli two-armed bandit:
\[
0.612 \leq \left( \frac{D}{N} \right)^{-1/2} R_N^{(0)}(\Theta) \leq 0.752
\]
with $D = 0.25$ being the maximal possible variance of one-step income.

Presented here the lower estimate was obtained in [9]. The upper estimate was obtained in [8] for the following strategy.

**Thresholding strategy.** Use actions turn-by-turn until the absolute difference between total incomes for their applications exceeds the value of the threshold $\alpha(DN)^{1/2}$ or the control horizon expires. If the threshold has been achieved and the control horizon has not expired then at the residual control horizon use only the action corresponding to the larger value of total initial income.

The optimal value of $\alpha$ is $\alpha \approx 0.584$ and the maximal value of the regret corresponds to $|p_1 - p_2| \approx 3.78(D/N)^{1/2}$ with additional requirement that $p_1$, $p_2$ are close to 0.5.

This approach is generalized in [11, 12] for Gaussian (or Normal) two-armed bandit, i.e. described by the probability distribution density of incomes
\[
f_D(x|m) = (2\pi D)^{-1/2} \exp \left\{ -\frac{(x - m)^2}{2D} \right\}
\]
if $y_\ell = \ell (\ell = 1, 2)$. Gaussian two-armed bandit can be described by a vector parameter $\theta = (m_1, m_2)$. The set of parameters is assumed to be the following
\[
\Theta = \{ \theta : |m_1 - m_2| \leq 2C \},
\]
where $0 < C < \infty$. Restriction $C < \infty$ ensures the boundedness of the regret on $\Theta$.

In [11, 12], according to the main theorem of the theory of games the minimax risk for Gaussian two-armed bandit is sought for as Bayesian one corresponding to the worst-case prior distribution for which Bayesian risk attains its maximal value. The Bayesian approach allows to write recursive Bellman-type equation for numerical determination of both Bayesian strategy and Bayesian risk. However, a direct application of the main theorem of the theory of games is virtually impossible because of its high computational complexity. Therefore, at first a description of the worst-case prior distribution is done. It is shown that the worst-case prior is symmetric and asymptotically uniform and this allows significantly to simplify the Bellman-type equation. In [12] the estimates (2) are improved as follows.

**Theorem 2.** The following estimate of the minimax risk (1) holds for Gaussian two-armed bandit
\[
\lim_{N \to \infty} \left( \frac{DN}{N} \right)^{-1/2} R_N^{(0)}(\Theta) = r_0
\]
with $r_0 \approx 0.637$.

**Remark 1.** In [11, 12] the case $D = 1$ is considered. However, all reasoning can be easily extended to distributions with arbitrary $D$.

Let’s explain the choice of Gaussian distribution of incomes. We consider the problem as applied to group control of processing a large amount of data. Let $N = TM$ data items be given that can be processed using either of the two alternative methods. The result of processing of the $n$-th item of data is $\xi_n = 1$ if processing is successful and $\xi_n = 0$ if it is unsuccessful. Probabilities $Pr(\xi_n = 1|y_t = \ell) = p_\ell$, $\ell = 1, 2$ depend only on selected methods (actions). Let’s assume that $p_1$, $p_2$ are close to $p$ ($0 < p < 1$). We partition the data into $T$ packets of $M$ data in each packet and use the same method for data processing in the same packet. For the control, we use the values of the process $\eta_\ell = M^{-1/2} \sum_{n=(t-1)M+1}^{tM} \xi_n$, $t = 1, \ldots, T$. According to the central limit theorem probability distributions of $\eta_\ell$, $\ell = 1, \ldots, T$ are close to Gaussian and their variances are close to $D = p(1-p)$ as in considered setting.

Note that data in the same packet may be processed in parallel. In this case, the total time of data processing depends on the number of packets rather than on the total number of data. However, there is a question of losses in the control performance as the result of such aggregation. It was shown in [11, 12] that if $T$ is large enough (e.g. $T \geq 30$) then parallel control is close to optimal. Therefore, say 30000 items of data can be processed in 30 stages by packets of 1000 data with almost the same maximal losses as if the data were processed optimally one-by-one. However, one should ensure the closeness of $p_1$, $p_2$ in this case. Otherwise parallel processing causes large losses at the initial stage of the control when both actions are applied turn-by-turn. In [11, 12] this requirement is discussed in more details and an adaptive algorithm which is optimal for both close and distant $p_1$, $p_2$ is proposed.

**Remark 2.** The estimates (2) can be easily extended to Gaussian two-armed bandit with a glance that the maximal value of the regret corresponds to $|m_1 - m_2| \approx 3.78(D/N)^{1/2}$ in this case. In particular, this implies that thresholding strategy allows parallel processing. The estimate (3) can be in turn extended to Bernoulli two-armed bandit by usage of parallel processing of data.
Remark 3. Parallel control in the two-armed bandit problem was first proposed for treating a large group of patients by either of the two alternative drugs with different unknown efficiencies. Clearly, the doctor cannot treat the patients sequentially one-by-one. Say, if the result of the treatment will be manifest in a week and there is a thousand of patients, then one-by-one treatment would take almost twenty years. Therefore, it was proposed to give both drugs to sufficiently large test groups of patients and then the more effective one to give to the rest of them. As the result, the entire treatment takes two weeks. Note that the two-armed bandit problem, as applied to medical trials, was usually considered in Bayesian setting and for sufficiently small number of stages (two, and sometimes three or four treatment stages). So, the results depend on the prior which is often specifically chosen and the control quality is less than for sufficiently large number of stages. The discussion and bibliography of the problem can be found, for example, in [13].

There are some different approaches to robust control in the two-armed bandit problem, see, e.g. [6], [15], [16], [17]. In these articles, stochastic approximation method and mirror descent algorithm are used for the control. Instead of minimax risk, the authors often consider the equivalent attitude called the guaranteed rate of convergency. The order of the minimax risk for these algorithms is $N^{1/2}$ or close to $N^{1/2}$. However, more precise estimates were not presented. The versions for parallel processing were not proposed as well.

The goal of this paper is to investigate the mirror descent algorithm (MDA) for the two-armed bandit problem proposed in [16]. For this algorithm the minimax risk has the order $N^{1/2}$ and theoretical estimate of the factor (or normalized minimax risk) is $r_1 \leq 4.710$. We improve this estimate by Monte-Carlo simulations as $r_1 \leq 2.0$. Then we propose a parallel version of the algorithm which partitions application of actions in the packet in proportion to corresponding probabilities. For this parallel version of the MDA, we obtain invariant description which does not depend on the size of the packet. We show that corresponding minimax risk has the order $N^{1/2}$ and estimate the value of the factor as $r_2 \approx 1.1$ using Monte-Carlo simulations. It is quite unexpectedly that parallel versions behave unlike the ordinary one even if the number of packets is sufficiently large. Moreover, it provides significantly smaller value of the minimax risk. We explain this result by considering another parallel version of the MDA which partitions actions in the packet sequentially with probabilities determined at the beginning of packet processing. This version of the MDA behaves like the ordinary one if the number of packets is large enough. For this version of MDA, we obtain invariant description as well.

It is important that parallel versions of the MDA perform well only for close values of probabilities $p_1, p_2$. For distant probabilities there may be significant expected losses caused by processing of the first packet. To avoid this, combined versions of the MDA are proposed. These algorithms at initial sufficiently short stage apply the ordinary MDA and then switch to the parallel version. These algorithms perform well for all probabilities $p_1, p_2$.

The structure of the paper is the following. In Section 2 we present the description of the algorithm from [16] and improve the estimate of the minimax risk by Monte-Carlo simulations. In Section 3 we propose the version of this algorithm which allows parallel processing and propose the invariant description of the algorithm. In Section 4 we propose another parallel version of the MDA which behaves like the ordinary algorithm. Combined algorithms are presented in Section 5. Section 6 contains a conclusion. Note that some results were presented in [13].

2. Description and Properties of the MDA for Bernoulli Two-Armed Bandit

![Figure 1. Normalized regret for Algorithm 1. $\beta = 2.2; \rho = 0.5; N = 500, 2000, 8000, 16000, 32000$.](image)

In this section, we provide a description of the MDA proposed in [16] for Bernoulli two-armed bandit problem. Note that the idea of the mirror descent originates from [18] for multy-armed bandit.

Let’s introduce probability vectors $\pi_n = (p_n^{(1)}, p_n^{(2)})$ s.t. $p_n^{(1)} \geq 0, p_n^{(2)} \geq 0, p_n^{(1)} + p_n^{(2)} = 1$, dual vectors $\zeta_n = (\zeta_n^{(1)}, \zeta_n^{(2)})$ and stochastic gradient vectors $\pi_n = (\pi_n^{(1)}, \pi_n^{(2)})$. Gibbs distribution is defined as follows

$$
\overline{G}_\beta(\zeta) = \{S_\beta(\zeta)\}^{-1} \left( e^{-\zeta^{(1)}/\beta}, e^{-\zeta^{(2)}/\beta} \right)
$$

where

$$
S_\beta(\zeta) = e^{-\zeta^{(1)}/\beta} + e^{-\zeta^{(2)}/\beta}.
$$

MDA for the two-armed bandit is defined recursively. Algorithm 1.
1) Fix some \( \bar{\gamma}_0 \) and \( \zeta_0 \).

2) For \( n = 1, 2, \ldots, N \):

   a) Draw an action \( y_n \) distributed as follows:
      \[
      \Pr(y_n = \ell) = \frac{\ell}{N}, \quad \ell = 1, 2;
      \]
    b) Apply the action \( y_n \) and get random income \( \xi_n \) distributed as follows:
      \[
      \Pr(\xi_n = 1|y_n = \ell) = p_\ell, \quad \Pr(\xi_n = 0|y_n = \ell) = q_\ell,
      \]
      \( \ell = 1, 2 \);
   c) Compute the thresholded stochastic gradient \( \hat{\pi}_n(\bar{\pi}_{n-1}) \):
      \[
      \hat{\pi}_n(\bar{\pi}_{n-1}) = \begin{cases} 
        \left( \frac{1 - \xi_n}{p_{n-1}}, 0 \right), & \text{if } y_n = 1, \\
        \left( 0, \frac{1 - \xi_n}{p_{n-1}} \right), & \text{if } y_n = 2;
      \end{cases}
      \]
   d) Update the dual and probability vectors
      \[
      \bar{\zeta}_n = \bar{\zeta}_{n-1} + \hat{\pi}_n(\bar{\pi}_{n-1}),
      \hat{\pi}_n = \mathcal{O}_{\sigma_n}(\bar{\zeta}_n).
      \]

Let’s denote by \( \Sigma_1 \) the set of strategies described by the MDA and by

\[
R_N^{(1)}(\Theta) = \inf_{\Sigma_1} \sup_{\Theta} L_N(\sigma, \theta)
\]

(4)
corresponding minimax risk. The following theorem holds \([16]\).

**Theorem 3.** Consider Algorithm 1. Let \( \beta_n = \beta^* \cdot \{D(n + 1)\}^{1/2} \) with \( \beta^* = (8/\log 2)^{1/2} \approx 3.397, D = 0.25 \). Then for any horizon \( N \geq 1 \) for the minimax risk \([4]\) the estimate holds

\[
R_N^{(1)}(\Theta) \leq r_1^* \{D(N + 1)\}^{1/2},
\]

(5)

with \( r_1^* = 4(2 \log 2)^{1/2} \approx 4.710 \).

**Remark 4.** Our description of the algorithm differs from the original in some details. The algorithm in \([16]\) is proposed for the problem of minimization of the total expected income; it is done for multi-armed bandit with arbitrary finite number of actions and for 2nd moment rather than variance \( D \) of incomes.

The estimate \([5]\) was obtained theoretically. It is approximately 7.39 times worse than the estimate \([5]\). However, it may be improved by Monte-Carlo simulations. To this end, the following normalized regret was calculated:

\[
\hat{L}_N^{(1)}(\beta, p, d) = (D N)^{-1/2} L_N(\sigma_N, \theta_N),
\]

where \( \theta_N \) and \( d \) are related as \( \theta = (p + d(D/N)^{-1/2}, p - d(D/N)^{1/2}) \), where \( 0 < p < 1, D = p q, q = 1 - p \) and \( \sigma_N \) stands for Algorithm 1 with \( \beta_n = \beta D(n + 2) \). Here and below we put \( p_0^{(1)} = p_0^{(2)} = 0.5, s_0^{(1)} = s_0^{(2)} = 0 \). The number of Monte-Carlo simulations is always 10000.

On figure 2 we present \( \hat{L}_N^{(1)}(\beta, p, d) \) calculated for different horizons \( N \) by Monte-Carlo simulations if \( \beta = 2.2, p = 0.5 \) and \( 1 \leq d \leq 10 \). Results are presented for \( N = 500, 2000, 5000, 16000, 32000 \). One can see that \( \hat{L}_N^{(1)}(\beta, p, d) \) converges to some limiting function as \( N \to \infty \).

One can guess that the limiting function \( \hat{L}_N^{(1)}(\beta, p, d) \) does not depend on \( p \) if \( 0 < p < 1 \) just like the results of \([11, 12]\). However, this is not the case for MDA. On figure 2 we present \( \hat{L}_N^{(1)}(\beta, p, d) \) calculated by Monte-Carlo simulations if \( \beta = 2.2, N = 2000 \) and \( 0 \leq d \leq 10 \). Results are presented for \( p = 0.1, 0.3, 0.5, 0.7, 0.9 \). One can see that the the lines are not similar and maximal expected losses are attained for the smallest \( p \).

Therefore, we calculate the following normalized regret

\[
L_N^{(1)}(\beta, p, d) = (D N)^{-1/2} L_N(\sigma_N, \theta_N),
\]

where \( \theta_N \) and \( d \) are related as \( \theta = (p + d(D/N)^{-1/2}, p - d(D/N)^{1/2}) \) where \( 0 < p < 1, D = 0.25 \) and \( \sigma_N \) stands for Algorithm 1 with \( \beta_n = \beta D(n + 1) \). First, we fix \( p = 0.1 \) and calculate \( L_N^{(1)}(\beta, p, d) \) by Monte-Carlo simulations if \( N = 2000 \) and \( 0 \leq d \leq 10 \). Results are presented on figure 3 for \( \beta = 1, 1.5, 2, 2.0, 2.5, 2.8 \). One can see that \( \beta = 2.0 \) is approximately optimal because it provides the least maximal normalized regret in this case. More careful calculations give that \( \beta \approx 2.2 \) is approximately optimal.

Finally we calculate \( L_N^{(1)}(\beta, p, d) \) if \( \beta = 2.2, N = 2000 \) for different \( p \). Results are presented on figure 4 for \( p = 0.1, 0.3, 0.5, 0.7, 0.9 \). One can see that maximal values of \( L_N^{(1)}(\beta, p, d) \) are attained if \( p = 0.1 \). Hence, the value \( \beta \approx 2.2 \) is approximately optimal and

\[
r_1 = \inf_{\beta > 0} \max_{1 \leq d \leq 10, 0.1 < p < 0.9} L_N^{(1)}(\beta, p, d) \approx 2.0.
\]

This estimate is approximately 2.37 times better than theoretical estimate \([5]\).
The following parallel version of the MDA assigns \([M_t^{(1)}], [M_t^{(2)}]\) in \(t\)-th packet in proportion to \(p_{t-1}^{(1)}, p_{t-1}^{(2)}\) and then applies the first and the second actions \([M_t^{(1)}]\) and \([M_t^{(2)}]\) times respectively.

**Algorithm 2.**

1) Fix some \(\overline{p}_0\) and \(\zeta_0\).

2) For \(t = 1, 2, \ldots, T;\)
   a) Let \(M_t^{(2)} = p_{t-1}^{(2)} \times M, \ell = 1, 2;\)
   b) For \(n \in [(t-1)M + 1, tM]\) apply the \(\ell\)-th action \([M_t^{(\ell)}]\) times getting random income \(\eta_t^{(\ell)} = \sum_{n} (1 - \xi_n) y_n = \ell)\),
      where
      \[
      \Pr (\xi_n = 1 | y_n = \ell) = p_{r},
      \Pr (\xi_n = 0 | y_n = \ell) = q_{r},
      \]
      \(\ell = 1, 2;\)
   c) Compute the thresholded stochastic gradient \(\pi_t(\overline{p}_{t-1})\):
      \[
      \pi_t(\overline{p}_{t-1}) = \left( \frac{\eta_t^{(1)}}{p_{t-1}^{(1)}}, \frac{\eta_t^{(2)}}{p_{t-1}^{(2)}} \right),
      \]
   d) Update the dual and probability vectors 
      \[
      \begin{align*}
      \zeta_t &= \zeta_{t-1} + \pi_t(\overline{p}_{t-1}), \\
      \overline{p}_t &= G_{\beta_t}(\zeta_t), \\
      \overline{p}_t &= \mathcal{P}_\rho(\overline{p}_t);
      \end{align*}
      \]

The projection operator is used because \(M_t^{(\ell)} < 1\) if corresponding \(p_t^{(\ell)}\) is small enough. We define the normalized regret

\[
\hat{L}_N^{(2)}(\beta, p, d) = (\hat{D} N)^{-1/2} L_N(\sigma_N, \theta_N)
\]

where \(\theta_N = (p_d + d(\hat{D} N)^{1/2}, p_d - d(\hat{D} N)^{1/2})\), and \(\sigma_N\) stands for **Algorithm 2** with \(\beta_t = \beta(\hat{D} M(t + 0.5))^{1/2}, 0 < p < 1, q = 1 - p, \hat{D} = pq\).

Note that if \(M\) is large enough then according to the central limit theorem all \(\{\eta_t^{(\ell)}\}\) are approximately normally distributed with parameters

\[
\begin{align*}
E(\eta_t^{(\ell)}) &= q_t[M_t^{(\ell)}], \\
\text{Var}(\eta_t^{(\ell)}) &= \hat{D}[M_t^{(\ell)}],
\end{align*}
\]

\(\ell = 1, 2.\) For normally distributed incomes, it is convenient to set a control problem with a continuous time. Namely,
if the $\ell$-th action is applied for a duration $\Delta M$, which is not obligatory integer, then it generates normally distributed income with mathematical expectation $q_\ell \times \Delta M$ and variance $D \times \Delta M$ and independent from all previously obtained incomes. Let's present corresponding modified algorithm.

**Algorithm 3.**

1) Fix some $\overline{p}_0$ and $\overline{\tau}_0$.
2) For $t = 1, 2, \ldots, T$:
   a) Let $M_t^{(\ell)} = p_{t-1}^{(\ell)} \times M$, $\ell = 1, 2$;
   b) Apply the $\ell$-th action for a duration $M_t^{(\ell)}$ getting random income $\eta_t^{(\ell)}$ which is normally distributed with $E(\eta_t^{(\ell)}) = q_t \times M_t^{(\ell)}$, $\text{Var}(\eta_t^{(\ell)}) = D \times M_t^{(\ell)}$, $\ell = 1, 2$ and independent from all previous incomes;
   c) Compute the thresholded stochastic gradient $\overline{\tau}_t(p_{t-1})$:
      $$\overline{\tau}_t(p_{t-1}) = \left( \frac{\eta_t^{(1)}}{p_t^{-1}} , \frac{\eta_t^{(2)}}{p_t^{-1}} \right),$$
   d) Update the dual and probability vectors $\overline{\tau}_t = \overline{\tau}_{t-1} + \overline{\tau}_t(p_{t-1})$, $\overline{p}_t = G_{\beta}(\overline{\tau}_t)$, $p_t = p_0(p_{t-1})$;
   e) Update the dual and probability vectors $\overline{\tau}_t = \overline{\tau}_{t-1} + \overline{\tau}_t(p_{t-1})$, $\overline{p}_t = G_{\beta}(\overline{\tau}_t)$, $p_t = p_0(p_{t-1})$;

We define the normalized regret

$$I_N^{(3)}(\beta, p, d) = (\hat{D}N)^{-1/2}L_N(\sigma_N, \theta_N)$$

where $\theta_N = (p + d(\hat{D}N)^{1/2}, p - d(\hat{D}N)^{1/2})$, and $\sigma_N$ stands for Algorithm 3 with $\beta_\ell = \beta(\hat{D}M(t + 0.5))^{1/2}$, $0 < p < 1$, $q = 1 - p$, $\hat{D} = pq$.

**Theorem 4.** Consider Algorithm 3 with a fixed number $T$ of packet processing stages and arbitrary $\hat{D} > 0$. Then normalized loss function $I_N^{(3)}(\beta, p, d)$ does not depend on $N$, $M$, $p$, $\hat{D}$.

**Proof.** Consider $q_t = \omega t(\hat{D}/N)^{1/2}$, $\ell = 1, 2$, so as $q_1 - q_2 = -2d(\hat{D}/N)^{1/2}$, $d > 0$. Let $X_t^{(\ell)}(D_t^{(\ell)})$ denote independent normally distributed random variables s.t.

$$E(X_t^{(\ell)}(D_t^{(\ell)})) = 0, \text{Var}(X_t^{(\ell)}(D_t^{(\ell)})) = D_t^{(\ell)},$$

$\ell = 1, 2$; $t = 1, 2, \ldots, T$. Then

$$\eta_t^{(\ell)} = \omega t(\hat{D}/N)^{1/2}M_t^{(\ell)} + X_t^{(\ell)}(\hat{D}M_t^{(\ell)})$$

$$= \varepsilon \omega t(\hat{D}/N)^{1/2}p_{t-1}^{(\ell)} + X_t^{(\ell)}(\hat{D}N\varepsilon p_{t-1}^{(\ell)}).$$

Here $\varepsilon = M/N$. Next, we obtain

$$\eta_t^{(\ell)} = \varepsilon \omega t(\hat{D}/N)^{1/2} + X_t^{(\ell)}\left(\frac{\hat{D}N\varepsilon}{p_{t-1}^{(\ell)}}\right).$$

So,

$$\zeta_t^{(\ell)} = \tau\omega t(\hat{D}N)^{1/2} + \sum_{i=1}^t X_i^{(\ell)}\left(\frac{\hat{D}N\varepsilon}{p_{i-1}^{(\ell)}}\right).$$

where $\tau = t/T$. Recall that $\beta_t = \beta(\hat{D}M(t + 0.5))^{1/2}$. Hence

$$\frac{\zeta_t^{(\ell)}}{\beta_t} = \frac{1}{\beta} \left(\frac{\tau\omega t}{(\tau + 0.5)^{1/2}} + X^{(\ell)}(t)\right)$$

with

$$Y^{(\ell)}(t) = \sum_{i=1}^t X_i^{(\ell)}\left(\frac{\varepsilon}{(\tau + 0.5)p_{i-1}^{(\ell)}}\right).$$

So,

$$\frac{\zeta_t^{(1)}}{\beta_t} - \frac{\zeta_t^{(2)}}{\beta_t} = -\frac{2d\tau}{\beta(\tau + 0.5)^{1/2}} + \frac{1}{\beta} \left(Y^{(1)}(t) - Y^{(2)}(t)\right).$$

Note that (7) depends only on parameters of packet processing. Since

$$p_{t-1}^{(1)} = \frac{\exp\left(-\frac{\zeta_t^{(1)} - \zeta_t^{(2)}}{\beta_t}\right)}{\exp\left(-\frac{\zeta_t^{(1)} - \zeta_t^{(2)}}{\beta_t}\right) + 1}$$

and $p_{t-1}^{(2)} = 1 - p_{t-1}^{(1)}$ then all $\{p_{t-1}\}$ depend only on the parameters of packet processing. The regret $L_N^{(3)}(\beta, p, d)$ can be expressed as follows

$$L_N^{(3)}(\beta, p, d) = (\hat{D}N)^{-1/2}(p_1 - p_2)\sum_{t=1}^T M E\left(p_{t-1}^{(2)}\right)$$

$$= 2d\sum_{t=1}^T \varepsilon E\left(p_{t-1}^{(2)}\right).$$

This expression does not depend on $N$, $M$, $p$, $\hat{D}$ but only on the parameters of packet processing. This proves theorem 4.

**Remark 5.** Denote by $\{\eta_t^{(\ell)}\}$ incomes corresponding to Algorithm 2. Let $\delta_t^{(\ell)}$ be independent random variables s.t.

$$E(\delta_t^{(\ell)}) = O(M^{-1}), \text{Var}(\delta_t^{(\ell)}) = O(M^{-1}).$$

Then $\eta_t^{(\ell)} = \eta_t^{(\ell)}(1 + \delta_t^{(\ell)}), \ell = 1, 2$; $t = 1, 2, \ldots, T$, because deviations $\{\delta_t^{(\ell)}\}$ are caused by nonintegral values $M_t^{(\ell)}$ in Algorithm 2. Since the number of stages $T$ is finite and fixed it means that normalized loss function $L_N^{(2)}(\beta, p, d)$ is close to $L_N^{(3)}(\beta, p, d)$ in (9) if $M$ is large enough.

**Remark 6.** It seems very likely that Algorithm 3 converges as $N \to \infty$ and $\varepsilon = M/N \to 0$. Let’s put $s_N^{(\ell)}(\tau) = (\hat{D}N)^{-1/2} \zeta_t^{(\ell)}$, $F_N^{(\ell)}(\tau) = p_{t-1}^{(\ell)}$, $\ell = 1, 2$
Then limiting normalized regret is equal to

\[ L^{(3)}(\beta, p, d) = 2d \int_0^1 \mathbb{E}\left(P^{(2)}(\tau)\right) d\tau. \]

However, we do not have a rigorous proof of this result.

In view of remark 5 we present simulations of Algorithm 2 but expect to observe the properties of Algorithm 3. On figure 6 we present \( \hat{L}^{(2)}_N(\beta, p, d) \) calculated for different sizes of packet \( M \) by Monte-Carlo simulations if \( \beta = 1.0, p = 0.5, \varrho = 0.02, T = 100 \) and \( 1 \leq d \leq 25 \). Results are presented for \( M = 50, 200, 500, 1000 \). One can see that \( \hat{L}^{(2)}_N(\beta, p, d) \) is almost independent from the size of packet \( M \).

On figure 7 we present \( \hat{L}^{(2)}_N(\beta, p, d) \) calculated for different \( T \) by Monte-Carlo simulations if \( \beta = 1.0, p = 0.5, \varrho = 0.02, M = 100 \) and \( 1 \leq d \leq 25 \). Results are presented for \( T = 50, 100, 200, 500, 1000 \). One can see that \( \hat{L}^{(2)}_N(\beta, p, d) \) converges as \( T \to \infty \).

According to theorem 4 the limiting function \( \hat{L}^{(2)}_N(\beta, p, d) \) as \( N \to \infty \) does not depend on \( p \) if \( 0 < p < 1 \). On figure 8 we present \( \hat{L}^{(2)}_N(\beta, p, d) \) calculated by Monte-Carlo simulations if \( \beta = 1.0, M = 100, T = 500, \varrho = 0.02 \) and \( 1 \leq d \leq 25 \). Results are presented for \( p = 0.1, 0.3, 0.5, 0.7, 0.9 \). One can see that these lines are close to each other.

Then we determine the optimal \( \beta \). We define the normalized regret

\[ \hat{L}^{(2)}_N(\beta, p, d) = (DN)^{-1/2} L_N(\sigma_N, \theta_N) \]
where $\theta_N = (p + d(D/N)^{1/2}, p - d(D/N)^{1/2})$ and $\sigma_N$ stands for Algorithm 2 with $\beta_i = (\beta(DM(t + 0.5)))^{1/2}$, $0 < p < 1$, $D = 0.25$. First, we fix $p = 0.5$ and calculate $L^{(2)}_N(\beta, p, d)$ by Monte-Carlo simulations if $M = 100$, $T = 300$, $\varrho = 0.02$ and $0 \leq d \leq 25$. Results are presented on figure 8 for $\beta = 0.5, 0.75, 1.25, 1.5, 2.0$. One can see that $\beta = 1.0$ is approximately optimal because it provides the least maximal normalized regret $L^{(2)}_N(\beta, p, d) < 1.05$ if $d < 20$.

Finally we calculate $L^{(2)}_N(\beta, p, d)$ if $\beta = 1.0$, $M = 100$, $T = 300$, $\varrho = 0.02$ and $0 \leq d \leq 25$. Results are presented on figure 9 for $p = 0.1, 0.3, 0.5, 0.7, 0.9$. One can see that maximal values of $L^{(2)}_N(\beta, p, d)$ are attained if $p = 0.5$. Hence, the value $\beta = 1.0$ is approximately optimal and

$$r_2 = \inf_{\beta > 0} \max_{1 \leq d \leq 20, \; 0.1 < p < 0.9} L^{(2)}_N(\beta, p, d) \approx 1.1.$$  

This estimate is even better than $r_1$. However, it is attained for close $p_1, p_2$ because $\varrho > 0$.

4. Another Parallel Version of the MDA

Let’s consider now another parallel version of the MDA which behaves closely to the ordinary version.

Algorithm 4.

1) Fix some $\overline{p}_0$ and $\overline{y}_0$.
2) For $t = 1, 2, \ldots, T$:
   a) i) Put $\chi^{(1)}_t = \chi^{(2)}_t = 0$.
      ii) For $n = (t - 1) \times M + 1, \ldots, t \times M$:
         A) Draw an action $y_{nt}$ distributed as follows:
            $\Pr(y_{nt} = \ell) = p_{nt}^{(\ell)}$, $\ell = 1, 2$;
         B) Apply the action $y_{nt}$, get random income $\xi_n$ distributed as follows:
            $\Pr(\xi_n = 1 | y_{nt} = \ell) = p_\ell$, 
            $\Pr(\xi_n = 0 | y_{nt} = \ell) = q_\ell$,
            and update:
            $$\chi^{(\ell)}_t \leftarrow \chi^{(\ell)}_t + (1 - \xi_n)$$
            if $y_{nt} = \ell$, $\ell = 1, 2$;
   b) Compute the thresholded stochastic gradient $\overline{\pi}_t(\overline{p}_{t-1})$:
      $$\overline{\pi}_t(\overline{p}_{t-1}) = \left( \frac{\chi^{(1)}_t}{p^{(1)}_{t-1}}, \frac{\chi^{(2)}_t}{p^{(2)}_{t-1}} \right),$$
   c) Update the dual and probability vectors
      $$\overline{\alpha}_t = \overline{\alpha}_{t-1} + \overline{\pi}_t(\overline{p}_{t-1}),$$
      $$\overline{p}_t = \overline{\alpha}_t(\overline{\alpha}_t);$$

It is straightforward to check that

$$\chi^{(\ell)}_t = \frac{\sum_{n=(t-1)M+1}^{tM} (1 - \xi^{(\ell)}_n)}{t},$$

where $\{\xi^{(\ell)}_n\}$ are i.i.d. variables distributed as

$$\Pr(1 - \xi^{(\ell)}_n) = 1 = p^{(\ell)}_{t-1}q_\ell,$$
$$\Pr(1 - \xi^{(\ell)}_n) = 0 = 1 - p^{(\ell)}_{t-1}q_\ell$$
and $\{\xi^{(1)}_n\}, \{\xi^{(2)}_n\}$ are independent from each other. Hence

$$E(\chi^{(\ell)}_t) = MP^{(\ell)}_{t-1}q_\ell,$$
$$\text{Var}(\chi^{(\ell)}_t) = MP^{(\ell)}_{t-1}q_\ell(1 - p^{(\ell)}_{t-1}q_\ell), \ell = 1, 2.$$  \hfill (10)

Let’s put $p_1 = p + d(D/N)^{1/2}$, $p_2 = p - d(D/N)^{1/2}$, $D = 0.25$ and $q_\ell = 1 - p_\ell$, $\ell = 1, 2$. Note that if $N$ is large enough then distributions of $\chi^{(\ell)}_t$, $\ell = 1, 2$ are close to gaussian ones. We define the normalized regret

$$L^{(4)}_{N}(\beta, p, d) = (D/N)^{-1/2}L_N(\sigma_N, \theta_N),$$
where $\theta_N = (p + d(D/N)^{1/2}, p - d(D/N)^{1/2})$, and $\sigma_N$ stands for Algorithm 4 with $\beta_i = (\beta(DM(t + 0.5)))^{1/2}$.

**Theorem 5.** Consider Algorithm 4 with a fixed number $T$ of packet processing stages. Assume that $\chi^{(\ell)}_t, \ell = 1, 2$ are normally distributed with mathematical expectations and variances given by (10). Then asymptotically (as $N \to \infty$) normalized loss function $L^{(4)}_{N}(\beta, p, d)$ does not depend on $N$, $M$, but does depend on on $q$, $D$ and parameters of packet processing.

**Proof.** Let’s put $q_\ell = w_{\ell}(D/N)^{1/2}$, $\ell = 1, 2$, so that $q_1 - q_2 = -2d(D/N)^{1/2}, d > 0$. Let $X^{(\ell)}_t(D^{(\ell)})$ denote independent normally distributed random variables s.t.

$$E(X^{(\ell)}_t(D^{(\ell)})) = 0, \quad \text{Var}(X^{(\ell)}_t(D^{(\ell)})) = D^{(\ell)},$$

$\ell = 1, 2; t = 1, 2, \ldots, T$. Then

$$X^{(\ell)}_t = w_{\ell}(D/N)^{1/2}M^{(\ell-1)} + X^{(\ell)}_t(M^{(\ell-1)}q_\ell(1 - p^{(\ell)}_{t-1}q_\ell))$$
$$= w_{\ell}(D/N)^{1/2}M^{(\ell-1)} + X^{(\ell)}_t(\varepsilon N^{(\ell)}_{t-1}q_\ell(1 - p^{(\ell)}_{t-1}q_\ell)).$$
Here $\varepsilon = M/N$. Next, we obtain

$$
\frac{X_{t}^{(2)}(t)}{p_{l-1}} = \varepsilon w_{t}(DN)^{1/2} + X_{t}^{(2)} \left( \frac{\varepsilon N q_{t}(1-p_{l-1}q_{t})}{p_{l-1}} \right).
$$

So,

$$
\zeta_{t}^{(l)} = \tau w_{t}(DN)^{1/2} + \sum_{i=1}^{l} X_{t}^{(l)} \left( \frac{\varepsilon N q_{t}(1-p_{l-1}q_{t})}{D(\tau + 0.5\varepsilon)p_{l-1}} \right),
$$

where $\tau = t/T$. Recall that $\beta_{l} = \beta(DM(t + 0.5))^{1/2} = \beta(DN(\tau + 0.5\varepsilon))^{1/2}$. Hence

$$
\frac{\zeta_{t}^{(l)}}{\beta_{l}} = 1 \beta \left( \frac{\tau w_{t}}{(\tau + 0.5\varepsilon)^{1/2}} + Y_{l}^{(l)}(t) \right)
$$

with

$$
Y_{l}^{(l)}(t) = \sum_{i=1}^{l} X_{t}^{(l)} \left( \frac{\varepsilon q_{t}(1-p_{l-1}q_{t})}{D(\tau + 0.5\varepsilon)p_{l-1}} \right).
$$

Note that $q_{t} \to q$ as $M \to \infty$, $l = 1, 2$. Hence, $Y_{l}^{(l)}(t) \to Y_{l}^{(l)}(t)$ as $M \to \infty$ where

$$
Y_{l}^{(l)}(t) = \sum_{i=1}^{l} X_{t}^{(l)} \left( \frac{\varepsilon q_{t}(1-p_{l-1}q_{t})}{D(\tau + 0.5\varepsilon)p_{l-1}} \right).
$$

As the number of stages $T$ is fixed, then asymptotically (as $M \to \infty$)

$$
\frac{\zeta_{l}^{(1)} - \zeta_{l}^{(2)}}{\beta_{l}} = \frac{-2d\tau}{\beta(\tau + 0.5\varepsilon)^{1/2}} + \frac{1}{\beta} \left( Y^{(1)}(t) - Y^{(2)}(t) \right).
$$

Note that (12) depends only on $q$, $D$ and $T$. Since

$$
p_{l-1}^{(2)} = \frac{\exp \left( -\frac{\zeta_{l}^{(1)} - \zeta_{l}^{(2)}}{\beta_{l}} \right)}{\exp \left( -\frac{\zeta_{l}^{(1)} - \zeta_{l}^{(2)}}{\beta_{l}} \right) + 1}
$$

and $p_{l-1}^{(2)} = 1 - p_{l-1}^{(1)}$ then all $\{p_{l-1}^{(l)}\}$ depend only on $q$, $D$ and $T$. The regret $L_{N}^{(4)}(\beta, p, d)$ can be expressed as follows

$$
L_{N}^{(4)}(\beta, p, d) = (DN)^{-1/2}(p_{1} - p_{2})\sum_{i=1}^{T} M E \left( p_{i-1}^{(2)} \right)
$$

$$
= 2d\sum_{i=1}^{T} \varepsilon E \left( p_{i-1}^{(2)} \right).
$$

This expression does not depend on $N, M$, but does depend on $q, D$ and $T$. This proves theorem 4.

**Remark 7.** Like in section 3 we present a limiting description of Algorithm 4 as $M \to \infty$ and $\varepsilon = M/N \to 0$. Let’s put $\zeta_{N}^{(l)}(\tau) = (DN)^{-1/2} \zeta_{l}^{(l)}$, $P_{N}^{(l)}(\tau) = p_{l-1}^{(l)}$, $\ell = 1, 2$ where $\tau = t/T$. Let $W_{l}(\tau)$, $\ell = 1, 2$ be independent Wiener processes. Denote by $\zeta_{N}^{(l)}(\tau)$, $P_{N}^{(l)}(\tau)$ corresponding limiting random processes as $N \to \infty$ and $\varepsilon = M/N \to 0$ and $L_{N}^{(4)}(\beta, p, d)$ the limit of $L_{N}^{(4)}(\beta, p, d)$. Using (11), (13) and (14) a limiting description may be presented as

$$
d\zeta_{N}^{(l)}(\tau) = w_{l}d\tau + \frac{\left( q_{l} - q_{l}P_{N}^{(l)}(\tau) \right)}{D(\tau + 0.5\varepsilon)p_{l-1}^{(l)}} dW_{l}(\tau),
$$

$$
P_{N}^{(l)}(\tau) = \frac{\exp \left( -\frac{\zeta_{l}^{(1)}(\tau) - \zeta_{l}^{(2)}(\tau)}{\beta_{l}} \right)}{\exp \left( -\frac{\zeta_{l}^{(1)}(\tau) - \zeta_{l}^{(2)}(\tau)}{\beta_{l}} \right) + 1},
$$

where $w_{l} - w_{l} = 2d, P_{N}^{(2)}(\tau) = 1 - P_{N}^{(1)}(\tau), \tau \in [0, 1]$. Initial conditions are given by

$$
\zeta_{l}^{(1)}(0) = \zeta_{l}^{(2)}(0) = 0.
$$

Then limiting normalized regret is equal to

$$
L_{N}^{(4)}(\beta, p, d) = 2d \int_{0}^{1} E \left( P_{N}^{(2)}(\tau) \right) d\tau.
$$

We do not have a rigorous proof of this result as well.

On figure 10 we present $L_{N}^{(4)}(\beta, p, d)$ calculated for different $M$ by Monte-Carlo simulations if $\beta = 2.2$, $p = 0.5$, $M = 10000$ and $1 \leq d \leq 25$. Results are presented for $M = 20, 50, 100, 200$ (accordingly $T = 500, 200, 100, 50$). The case $M = 1$ corresponds to ordinary MDA and $L_{N}^{(1)}(\beta, p, d)$. One can see that $L_{N}^{(4)}(\beta, p, d)$ is close to $L_{N}^{(1)}(\beta, p, d)$ if $T$ is large enough.

5. Combined Algorithms

One can see on figure 6 and figure 10 that larger sizes of packets correspond to larger sizes of normalized regret if $d$ is large enough. It is caused by equal applications of both actions to initial packet. To avoid this effect of initial packet processing one can take initial packets of smaller sizes. The simplest decision is to use the ordinary algorithm at initial short stage and then to switch to parallel algorithm.

First, we combine Algorithm 1 and Algorithm 4 as follows.

**Algorithm 5.**

1. Apply Algorithm 1 at the initial horizon $n = 1, \ldots, M_{0}$. Get $p_{M_{0}}$ and $\zeta_{M_{0}}$.
2. Apply Algorithm 4 at the residual horizon $n = M_{0} + 1, \ldots, N$.

On figure 11 we present $L_{N}^{(5)}(\beta, p, d)$ calculated by Monte-Carlo simulations for Algorithm 5 if $\beta = 2.2$; $N = 20000; M_{0} = 600; M = 200; p = 0.1, 0.3, 0.5, 0.7, 0.9$. One
can see that results are close to those presented on figure 4 for ordinary MDA.

To take advantage of Algorithm 2, we combine Algorithm 1 and Algorithm 2 as follows.

**Algorithm 6.**

1) Apply Algorithm 1 at the initial horizon $n = 1, \ldots, M_0$ with $\beta = \beta_1$. Get $p_{M_0}$ and $\zeta_{M_0}$.

2) Apply Algorithm 2 at the residual horizon $n = M_0 + 1, \ldots, N$ with $\beta = \beta_2$.

However, Algorithm 2 provides large normalized regret if $d$ is large enough (see figure 5, 6) because it applies both actions with probabilities no less than $\varrho$. Therefore we consider the following combined algorithm.

**Algorithm 7.**

If $\kappa$ is appropriately chosen this algorithm for small $d$ switches mostly to Algorithm 2. For large $d$ it switches mostly to Algorithm 4. On figure 12 we present comparative results for $L_{N}^{(6)}(\beta, p, d)$ and $L_{N}^{(7)}(\beta, p, d)$ if $\beta_1 = 2.2; \beta_2 = 1.0; N = 30000; p = 0.5; M_0 = 30000; \varrho = 0.02; \kappa = 0.2$.

Maximal values of $L_{N}^{(7)}(\beta, p, d)$ are larger than those for $L_{N}^{(5)}(\beta, p, d)$ but smaller than those for $L_{N}^{(5)}(\beta, p, d)$.

6. Conclusion

Two parallel versions of the mirror descent algorithm (MDA) for the two-armed bandit problem are proposed. The usage of parallel versions of the MDA ensures that total time of data processing depends mostly on the number of packets but not on the total number of data. Monte-Carlo simulations show that maximal expected losses for parallel versions are not more than for ordinary version which processes data one-by-one. However, it is true only for close mathematical expectations. For distant mathematical expectations the effect of the first packet processing, when both actions are equally applied, causes significant losses if the size of the packet is large enough. This effect may be avoided if at the sufficiently short initial stage the ordinary mirror descent algorithm is used and then switched to the parallel version.
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