Particle-Field Theory and Its Relativistic Generalization II (Relativistic Generalization of Micro Harmonic Oscillator and Hydrogen Atom)

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Abstract

As a serious attempt for constructing a new foundation for describing micro-entities from a causal standpoint, it was explained before in \cite{1,2,3} that by unifying the concepts of information, matter and energy, each micro-entity is assumed to be composed of a probability field joined to a particle called a particle-field or PF system. The relativistic generalization of this theory and its invariance under Lorentz transformation has been proved.

In this essay, based on the relativistic generalization of Schrödinger equation derived in \cite{4}, we solve the relativistic Schrödinger equation for relativistic micro-harmonic oscillator to find its energy. Also we obtain the energy spectrum of Hydrogen atom that is the main purpose of this paper. We see that the result is completely consistent with the relativistic correction to the Hydrogen’s energy in first-order perturbation theory.

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1 Introduction

Quantum theory is the theoretical basis of modern physics that explains the nature and behavior of matter and energy on the atomic and subatomic level. It is undoubtedly one of the most important and experimentally accurate scientific theories in the history of science. It continues to yield novel and unexpected results, in technology as well as in all scientific fields, including physics,

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biology, chemistry, and so on. However, in spite of all empirical and mathematical success of quantum mechanics, due to its broad and ambiguous conceptual framework, until now, all attempts to reach a satisfactory understanding of its meaning have been remained unpalatable. One can never expect an understanding of quantum mechanics that is similar in clarity and intelligibility to the one provided by other domains of physics. Its problems are of a fundamental nature, or they are more fundamental than other cases. Although refined theoretical arguments and new experimental techniques have produced a substantial advance in the last recent decades, the famous sentence of Feynman, who said “nobody really understand quantum mechanics” [5] is still marked.

We have attempted to construct a new foundation for describing micro-events from a deterministic causal standpoint, in which a micro-entity is supposed to be an allied particle-field system, instead of composing of a particle and (or) a field (wave) [1, 2, 3]. It has been explained in the first essay of this series that in the microworld, one encounters an unified concept of information, matter and energy [1]. In this new approach, the principles of realism and causality based on the classic-like equations of motion are presumed. A particle-field system is not composed of a particle and a wave. Instead, it is a unified system for which the particle and the wave notions are only abstract constructions without real manifestation.

Here, one may pose the question that what the differences are between this approach and Bohmian [6, 7] account for a micro-system. The point is that, a system is neither a particle nor a wave, not also a combination of these two entities. It is a totality of both wave and particle notions, so that one can imagine it as a field that enfolds a particle. We abstract the notions particle and field, the PF system to describe more elaborately. So, it seems that particle and field construct the PF system and when the energy of the field approaches zero, the classical particle appears. Yet, in reality, there are no distinct entities such as the particle and the field. Only when the PF system loses its all holistic nature, it reduces to a known classical particle. Thus, it looks like we have two different energies, one for the particle and the other for the field and the latter causes the quantum behavior of the system.

This important feature of a PF system enables one, e.g., to show why the squared modulus of the wave function behaves like a probability density in spatial coordinates. While, in Bohmian theory Born postulate is accepted a priori. Considering a PF system allows one to explain the origin of the Schrödinger equation, since here we assume that the underlying dynamics of a supposed field is influenced by an oscillatory force which could be approximated to a harmonic one in the first order [1]. Taking into account anharmonic effects, one can obtain non-linear forms of the Schrödinger equation. In addition, from a fundamental point of view, these two theories explain bizarre quantum phenomena like the measurement problem, tunneling effect and double-slit experiment in completely different directions. The interested reader can follow the corresponding fashions of explanation in each model in [1, 2, 3] and [6, 7].

Moreover, one of the most important matters is that the PF theory is not in contradiction with Special Relativity in its origin. The Lorentz-invariant forms of equations in PF theory to obtain relativistic Schrödinger equation has been presented [4].

Here, we are going to solve the relativistic Schrödinger equation found in [4] for relativistic micro-harmonic oscillator and relativistic Hydrogen atom to find
their energy spectrum.

The paper is organized as follows: In section 2, we review the basic elements of the PF theory for a one-particle one-dimensional microsystem and its relativistic generalization. In section 3, we solve the relativistic Schrödinger equation for micro-harmonic oscillator and in section 4, we find the energy of Hydrogen atom and show that it is consistent with the result of relativistic correction discussed in text book [8]. In section 5, the whole content of our paper is discussed and concluded.

2 Review of Basic Elements

In this section, we give a brief review of basic elements of the PF theory and its relativistic generalization. More details are available in [1, 2, 3, 4].

For a one-dimensional, one-particle microsystem, three physical entities are introduced:

1. A particle with mass \( m \) and position \( x(t) \) whose dynamics is given by the Newton’s second law:
\[
m \frac{d^2 x(t)}{dt^2} = f_P,
\]
where \( f_P \) is the force defined for the particle. For the conservative forces, the particle possesses a conserved energy \( E_P = V_P + K_P \), where \( K_P = \frac{p_P^2}{2m} \) is the kinetic energy and \( p_P \) is the linear momentum of the particle.

2. Like the particle aspect of the PF system, there is a field denoted by \( X(x(t), t) \) with velocity \( v_F = \frac{dX}{dt} = \dot{X} \) along the positive direction of \( x \), where
\[
\dot{X} = \left( \frac{\partial X}{\partial x} \right) v_P + \left( \frac{\partial X}{\partial t} \right),
\]
and \( v_P \) is the velocity of the particle along the same direction. The amplitude of the field has a dimension of length. Similar to the particle, we assume that the field obeys a Newton-like dynamics too in the same direction,
\[
m \frac{d \dot{X}}{dt} = f_F,
\]
where \( f_F \) is the force the field is subjected to. If the particle is subjected to a conservative force \( f_P \), we shall consider \( X = \chi(x(t)). \) Then, one can show that
\[
f_F = m v_F^2 \frac{d|\chi|^2}{dx} + |\chi|^2 f_P.
\]
From a physical point of view, the field \( X \) merely enfolds the particle. It experiences its own mechanical-like force introduced as \( f_F \) in [3]. Although the presence of the particle is essential for defining the force of the field. If there is no particle, there will not be any associated field too. The existence of the field depends on the existence of the particle, but the opposite is not true, because \( X \) is a function of particle’s position, not vice versa.

For a conservative field subjected to the force \( f_F \) in [4], one can define the energy \( E_F = V_F + K_F \) where \( K_F = \frac{1}{2} m v_F^2 = K_P |\chi'|^2 \). The kinetic energy of the field includes the kinetic energy of the particle. Here, one can’t separate the meaning of \( K_F \) from \( K_P \).
In the quantum domain, the quantities $E_P$ and $E_F$ are not practically discernible, but the total energy $E = E_P + E_F$ is an observable property. One can write the total energy as:

$$E = V_P + (E_F + \frac{p^2}{2m}),$$

$$= V_P + \frac{p^2}{2m} \quad (5)$$

where $\frac{p^2}{2m} = (E_F + \frac{p^2}{2m})$, and $V_P$ is the particle’s potential.

Unfortunately, the form of the force $f_F$ in (4) is complicated and unknown a priori, and so it is not possible to obtain it from (3) or (4). Accordingly, we postulate that for stationary states for which the energy is conserved, the form of $X = \chi(x(t))$ could be obtained from the time-independent Schrödinger equation:

$$\chi'' = -k^2 \chi \quad (6)$$

where

$$k^2 = \frac{p^2}{k^2} = \frac{2m}{\hbar}(E - V_P)$$

and

$$\chi'' = \frac{d^2\chi(x)}{dx^2}$$

For stationary states in which $\chi$ is a real function, one can rewrite relation (4) as

$$f_F = -m\overline{m^2} \chi + f_P \chi' \quad (7)$$

in which we have used (6) and $\overline{m^2} = v_P^2 k^2$. This shows that when $\chi$ is real and relying on $x(t)$ only, the field-at least partly-experiences an oscillatory force. Here, $\overline{m}$ depends on $x(t)$ and the first term in (7) does not actually describe an oscillating force, but has the same form.

3. Neither the particle, nor the field representation alone is adequate for explaining the physical behavior of a microsystem, comprehensively. What really gives us a thorough understanding of the nature of a quantum system is a holistic depiction of both particle and its associated field which we call here a PF system. The kinetic energy of a PF system is proportional to $K_P + K_F$, but its total energy is the same as $E$ in (5). Let us define the kinetic energy of a PF system as $K_{PF} \propto K_P + K_F$, or

$$q^2 = g_{PF}^2 (\dot{x}^2 + |\dot{X}|^2) \quad (8)$$

where $g_{PF}$ is a proportionality factor and $\dot{x} = v_P$. For many problems, this factor is equal to one, but the non-oneness of its value in general is crucial in some other problems [1, 2].

From The above relation, one can obtain the trajectories of a PF system:

$$q(x, t) = g_{PF} \int dx \sqrt{\left(1 + \left|\frac{dX(x, t)}{dx}\right|^2\right)} \quad (9)$$

The relation (9) shows that while we expect the particle to move along the infinitesimal displacement $dx$ in the $x$ direction, the displacement of the whole
system is equal to $dq$, not $dx$. The difference here is due to the existence of the associated field which adds a new term, in addition to the direction the particle moves along. Hence, the PF system indeed keeps going through an integrated path determined by the whole action of the particle and its associated field.

Using the relation (9), one can obtain the finite displacement $q$ of a PF system in terms of the particle’s location $x(t)$ and time, when the field $X(x,t)$ is known. Then, if the form of dependence of $x$ to $t$ is also known for a given physical problem, it is possible to write $q$ totally in terms of $t$. For stationary states, however, $q = q(x(t))$ and there is no explicit time-dependency. Therefore, one can see that the time variable could be kept concealed in equations of motions, so that the spatial direction $x$ would be sufficient for illustrating the behavior of $q$.

The dynamics of the PF system can also be described according to a Newtonian equation. So, we have

$$m \frac{d^2q}{dt^2} = f_{PF}, \quad (10)$$

where $f_{PF}$ is the force the PF system is subjected to.

To show that this theory is consistent with the theory of special relativity, we need to find a unified concept of spacetime that is invariant under Lorentz transformation. Using the definition of relativistic kinetic energy of the stationary field and the PF system, we have found the relativistic generalization of (9) as [4],

$$\dot{q} = c \left(1 - \frac{1}{[(\gamma_p - 1)(1 + \chi^2) + 1]^2}\right)^{-\frac{1}{2}}, \quad (11)$$

where $c$ is speed of light, $\gamma_p = \frac{1}{\sqrt{1 - \frac{v_p^2}{c^2}}}$ and $\chi' = \frac{d\chi}{dx}$. Using (11), then, we have shown that

$$ds^2 = c^2 dt^2 - dq^2 \quad (12)$$

is invariant under Lorentz transformation [4].

Now, one can derive new form of the relativistic Schrödinger equation. The physical structure of the PF formalism which has constitutional similarities to classical equations of motion permits us to derive a well-defined relativistic Schrödinger equation for stationary fields, regardless of the spin variable.

The dynamics of a stationary real field in one dimension (denoted by $\chi = \chi(x(t))$) in the relativistic regime can be represented as

$$\frac{d(m_p\dot{\chi})}{dt} = f_{rF}, \quad (13)$$

where $f_{rF}$ is the force defined for the field under the relativistic conditions and $m_p$ is the relativistic mass of the particle:

$$m_p = \gamma_p m_0; \quad \gamma_p = \left(1 - \frac{v_p^2}{c^2}\right)^{-\frac{1}{2}}, \quad (14)$$

where $m_0$ is the rest mass, as before and $v_P$ is the velocity defined for the particle. The stationary field $\chi(x(t))$ does not explicitly depend on time. So,
one can find out that
\[ f_{RF} = f_{RP} \chi' + \gamma_P m_0 v_P^2 \chi'' \] (15)
where \( \chi' = \frac{dx}{dt} \) and \( f_{RP} = m_0 \dot{v}_P \gamma_P^3 \) is the force exerted on the particle. It has been argued that for stationary real fields, there exists an oscillating-like term in the force expression (denoted by the second term in (15), when \( \gamma_P \to 1 \)) from which the non relativistic time-independent Schrödinger equation can be resulted [1]. Here, we suppose that the same situation holds true under the relativistic conditions. That is, for stationary real fields, we postulate the following equality as a general rule:
\[ -m_P \ddot{w}^2 \chi = \gamma_P m_0 v^2_P \chi'' \] (16)
where \( m_P \) was defined in relation (14) and \( \ddot{w}^2 = k^2 v^2_P \). Here again, we define \( k = \frac{p}{\hbar} \), where \( p \) is the relativistic de Broglie momentum. From the relation (16), it is immediately concluded that
\[ -\hbar^2 \chi'' = p^2 \chi, \] (17)
which has the same form as the non-relativistic Schrödinger equation. After some calculations to find an appropriate relation for \( p^2 \) in relation (17), one can derive relativistic Schrödinger equations for the cases that the potential energy of the particle includes the relativistic mass as [4]:
\[ -\frac{\hbar^2}{2m_0} \chi'' + \frac{1}{2} m_0 c^2 \chi = \frac{E^2}{2m_0 c^2} \left( 1 + \frac{V_{nrP}}{m_0 c^2} \right)^{-2} \chi, \] (18)
and for potential energy that is independent of mass as [4]:
\[ -\frac{\hbar^2}{2m_0} \chi'' + \frac{1}{2} m_0 c^2 \chi = \frac{1}{2m_0 c^2} (E - V_{nrP})^2 \chi. \] (19)
In the following, we consider the problem of one-dimensional harmonic oscillator and the relativistic Hydrogen and we solve equations (18) or (19) to find their energy spectrum.

3 Relativistic Micro-Harmonic Oscillator

The relativistic Generalization of the classical or quantum harmonic oscillator has been debated in literature (see, e. g., [9,10]). Yet, there has been provided no unique definition of the relativistic harmonic oscillator, at least in micro-domain [11]. The main difficulty is that using a Lorentz-invariant variational principle in classical domain, one can show that the mass is potential-dependent [10]. So, the total energy of the system should be expressed as (18). Strictly speaking, the relativistic energy of the particle can be defined as:
\[ E_{rP} = \gamma_P \left( \frac{1}{2} m_0 \omega^2 x^2 + m_0 c^2 \right) \] (20)
where \( \gamma_P \) is defined in [14] and \( \omega_0 \) is the spring-frequency of the oscillator. Since the potential energy of the particle includes the relativistic mass (and so is a function of the velocity of particle), its definition as a Hermitian operator faces a
difficulty. Thus, in perturbation methods as well as the Klein-Gordon solutions of the relativistic quantum harmonic oscillator, the potential-dependency of mass is usually ignored and similar to the relation (19), the relativistic energy of the particle is considered as

$$E_{rP} = \frac{1}{2} m_0 w_0^2 x^2 + \gamma_P m_0 c^2. \quad (21)$$

Using Lagrangian approach leading to the Klein-Gordon equation of a harmonic oscillator in which the potential of the oscillator is assumed to be nonrelativistic, the relativistic energy of the system can be obtained as [10]:

$$E_n = \pm m_0 c^2 (1 + \frac{2E_0}{m_0 c^2})^{\frac{1}{2}}$$

$$= \pm (m_0 c^2 + E_n^0 - \frac{E_n^0}{2m_0 c^2} + ...) \quad (23)$$

where $E_n^0 = \hbar w_0 (n + \frac{1}{2})$. Here, we see that $E_0$ appears as the first order correction of $E_n$, not as a correction-free term.

Now, we examine the equation (18) and (19) for solving the same problem, using the relations (20) and (21), respectively. To solve the equation (18), we first use the following approximation, having attention that for a simple harmonic oscillator we can ignore higher powers of $x$ (power greater than 2) in the potential expression. Hence,

$$(1 + \frac{V_{nrP}}{m_0 c^2})^{-2} \simeq (1 - \frac{2V_{nrP}}{m_0 c^2}) \quad (24)$$

where $V_{nrP} = \frac{1}{2} m_0 w_0^2 x^2$. Then, the equation (18) can be rearranged as:

$$\chi''(x) + (\beta_1 - \alpha_1^2 x^2) \chi(x) = 0 \quad (25)$$

where

$$\beta_1 = \frac{E^2 - m_0^2 c^4}{\hbar^2 c^4}; \quad \alpha_1^2 = \frac{E^2 w_0^2}{\hbar^2 c^4} \quad (26)$$

The equation (25) has precisely the form of the time-independent Schrödinger equation for nonrelativistic harmonic oscillator. So, one at once can obtain the energy of the relativistic harmonic PF system. Here, we have:

$$2\alpha_1 (n + \frac{1}{2}) = \beta_1 \quad (27)$$

leading to:

$$E^2 - 2EE_n^0 - m_0^2 c^4 = 0, \quad (28)$$

thus, we obtain:

$$E_n = E_n^0 \pm m_0 c^2 (1 + \frac{E_n^0}{m_0^0 c^2})^{\frac{1}{2}}. \quad (29)$$

Choosing the plus sign, the energy $E_n$ can be expanded as

$$E_n = E_n^0 + m_0 c^2 + \frac{E_n^0}{2m_0 c^2} + ... \quad (30)$$
This result has not been recorded in literature so far. The normalized eigenfunctions (corresponding to the stationary fields $\chi_n(x)$ in (80)) can be written as:

$$\psi_n(x) = (d^n n!)^{-\frac{1}{2}} \left( \frac{\alpha_1 n}{\pi} \right) \exp\left(\frac{\alpha_1 x^2}{2}\right) H_n\left(\sqrt{\alpha_1} x\right)$$

(31)

where $H_n\left(\sqrt{\alpha_1} x\right)$ are the Hermite polynomials. Assuming that $E_0^0 \ll m_0 c^2$ and considering $E_n \approx m_0 c^2$ in $\alpha_1 n$, we get $\alpha_1 \approx \frac{m_0}{\hbar}$ like its nonrelativistic definition.

On the other hand, taking into account the relation (21), we can write the equation (19) as:

$$\chi''(x) + (\beta_2 - \alpha_2^2 x^2) \chi(x) = 0$$

(32)

where

$$\beta_2 = \beta_1 = \frac{E^2 - m_0^2 c^4}{\hbar^2 c^2}; \quad \alpha_2^2 = \frac{E m_0 u_0^2}{\hbar^2 c^2}$$

(33)

In reaching the equation (32), we have assumed that $(E - V_{np})^2 \approx (E^2 - 2E V_{np})$. The coefficients $\alpha_2$ and $\beta_2$ satisfy the similar relation as (27). Accordingly, we get:

$$E^2 - 2 \frac{c}{u_0} \left( \sqrt{m_0 u_0^2 E} E_0^0 - m_0^2 c^4 \right) = 0$$

(34)

To solve the above equation, we assume that $E_0^0 \ll m_0^2 c^2$, so that one can put inside the radical $E \approx m_0^2 c^2$. Then, we obtain:

$$E^2 = m_0^2 c^4 \left( 1 + \frac{2E_0^0}{m_0 c^2} \right)$$

(35)

from which the same solution as (22) is resulted. In effect, the solution of our relativistic Schrödinger equation (19) for the case of nonrelativistic harmonic potential coincides with the answer obtained by Lagrangian approach leading to the Klein-Gordon equation.

4 Relativistic Schrödinger Equation For Hydrogen Atom

The Hydrogen atom consists of a heavy, essential motionless proton (we may as well put it at the origin), of charge $e$, together with a much lighter electron (charge $-e$) that orbits around it, bound by the mutual attraction of opposite charges. From Coulomb’s law, the potential energy (in SI units) is

$$V(r) = -\frac{1}{4\pi \varepsilon_0} \frac{e^2}{r}.$$

(36)

Since the potential of Hydrogen atom is dependent of mass, the field, $\chi$, and the energy of the PF system, $E_{PF}$, satisfies relation (19). Our problem is to solve this equation for $\chi$, and determine the allowed energies, $E$.

The generalization to three dimension of relation (19) is straightforward. Typically, the potential is a function only of the distance from the origin. In that
case it is natural to adopt spherical coordinates \((r, \theta, \phi)\). In spherical coordinates the relation \((19)\) takes the form
\[
- \frac{\hbar^2}{2m_0} \frac{1}{r^2} \left[ \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{\sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right) \right] \chi(r) + \frac{1}{2} m \omega^2 \chi(r) = \frac{1}{2m_0c^2} (E - V_{nrp}(r))^2 \chi(r).
\]

In relation \((37)\), the potential \(V_{nrp}(r)\) is the classical potential defined by \((56)\). We begin by looking for solutions that are factorable in \(r\) and angular variables \(\theta\) and \(\phi\):
\[
\chi(r) = R(r)Y(\theta, \phi).
\]
Putting this into relation \((37)\), we have
\[
- \frac{\hbar^2}{2m_0} Y(\theta, \phi) \frac{d}{dr} \left( r^2 \frac{d}{dr} R(r) \right) - \frac{\hbar^2}{2m_0} \frac{R(r)}{r^2 \sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right) Y(\theta, \phi) + \frac{1}{2} m \omega^2 R(r)Y(\theta, \phi) = \frac{1}{2m_0c^2} (E - V_{nrp}(r))^2 R(r)Y(\theta, \phi).
\]

Dividing by \(R(r)Y(\theta, \phi)\) and multiplying by \(-\frac{2m_0}{\hbar^2}\), one gets:
\[
\left[ \frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{r^2}{\hbar^2 c^2} (E - V_{nrp}(r))^2 - \frac{r^2}{\hbar^2} m \omega^2 c^2 \right] \frac{Y(\theta, \phi)}{r^2 \sin \theta} + \left[ \frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) - \frac{r^2}{\hbar^2 c^2} (E - V_{nrp}(r))^2 + \frac{r^2}{\hbar^2} m \omega^2 c^2 \right] Y(\theta, \phi) = 0.
\]

The term in the first curly bracket depends only on \(r\), where as the reminder depends only on \(\theta\) and \(\phi\). Accordingly, each must be a constant. We will write separation constant in the form \(l(l+1)\) \([8]\),
\[
\frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{r^2}{\hbar^2 c^2} (E - V_{nrp}(r))^2 - \frac{r^2}{\hbar^2} m \omega^2 c^2 = l(l+1) \quad (40)
\]
\[
\frac{1}{Y(\theta, \phi)} \frac{1}{\sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right) Y(\theta, \phi) = -l(l+1) \quad (41)
\]
The angular part of the wave function, \(Y_l^m(\theta, \phi)\), is the same for all spherically symmetric potentials and it has been solved in most of text books of quantum mechanics. So, the solution of the angular part of Hydrogen atom given by \((41)\) is \(Y_l^m(\theta, \phi)\), that are called spherical harmonic \([8]\). On the other hand the actual shape of the potential, \(V(r)\) affects only the radial part of the wave function \(R(r)\) which is determined by relation \((40)\). We rewrite it as
\[
\frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{r^2}{\hbar^2 c^2} (E^2 - 2EV + V^2)R(r) - \frac{r^2}{\hbar^2} m \omega^2 c^2 R(r) = l(l+1)R(r) \quad (42)
\]
or
\[
\frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{r^2}{\hbar^2 c^2} (E^2 - m_0^2 c^4 - 2EV + V^2)R(r) = l(l+1)R(r) \quad (43)
\]
Let
\[ u(r) = rR(r), \] (44)
so one can rewrite the relation (43) as
\[ \frac{d^2 u(r)}{dr^2} + \frac{1}{\hbar^2 c^2} (E^2 - m_0^2 c^4 - 2EV + V^2)u(r) = \frac{l(l+1)}{r^2} u(r). \] (45)

Putting the relation (36) in (45), we find
\[ \frac{d^2 u(r)}{dr^2} + \frac{1}{\hbar^2 c^2} (E^2 - m_0^2 c^4) + \frac{2E}{\hbar^2 c^2} \frac{e^2}{4\pi \varepsilon_0} \frac{1}{r} \]
\[ + \left( \frac{1}{\hbar^2 c^2} \frac{e^2}{4\pi \varepsilon_0} \right)^2 - \frac{l(l+1)}{r^2} u(r) = 0 \] (46)

For simplifying the form of the relations, we write relation (46) as
\[ \frac{d^2 u}{dr^2} = \left[ -\frac{A}{r} + \frac{B(B+1)}{r^2} + D^2 \right] u, \] (47)
which \( A, G, B \) and \( D \) are defined as:
\[ A = \frac{2E}{\hbar^2 c^2} \frac{e^2}{4\pi \varepsilon_0}, \] (48a)
\[ G^2 = \left( \frac{e^2}{4\pi \varepsilon_0} \right)^2 \frac{1}{\hbar^2 c^2}, \] (48b)
\[ B(B+1) = l(l+1) - G^2, \] (48c)
\[ D^2 = \frac{1}{\hbar^2 c^2} (m_0^2 c^4 - E^2). \] (48d)

Since \( m_0^2 c^4 \gg E^2 \), we always have \( D^2 > 0 \). We introduce \( \rho = Dr \) and \( \frac{A}{D} = \rho_0 \), so the equation (47) is derived as
\[ \frac{d^2 u}{d\rho^2} + \left[ 1 - \frac{\rho_0}{\rho} \right] \frac{B(B+1)}{\rho^2} u = 0. \] (49)

This equation can be solved like the way the radial part of the Schrödinger equation of Hydrogen atom is solved. We introduce the new function \( v(\rho) \) as:
\[ u(\rho) = \rho^{B+1} e^{-\rho} v(\rho). \] (50)

In terms of \( v(\rho) \), the relation (47) reads as
\[ \rho \frac{d^2 v}{d \rho^2} + 2(B+1 - \rho) \frac{dv}{d\rho} + \left[ \rho_0 - 2(B+1) \rho \right] v = 0. \] (51)

Finally, we assume the solution \( v(\rho) \) can be expressed as a power series in \( \rho \):
\[ v(\rho) = \sum c_j \rho^j \] (52)
Now, our problem is to determine the coefficient \( c_j \). Inserting this into the equation (51), we have
\[ c_{j+1} = \frac{2(j+B+1) - \rho_0}{(j+1)(j+2B+2)} c_j. \] (53)
This recursion formula determines the coefficients, and hence the function \( v(\rho) \). For large values of \( j \), the solutions aren’t normalizable. To get rid of this dilemma, the series must terminate. There must occur some maximal integer, \( j_{\text{max}} \), such that

\[ c_{j_{\text{max}}+1} = 0. \]

Evidently

\[ 2(j_{\text{max}} + B + 1) - \rho_0 = 0 \]

and

\[ \rho_0 = 2(j_{\text{max}} + B + 1). \]

Using the relation (48c) we find

\[ B^2 + B - l(l+1) + G^2 = 0, \]

So,

\[
B = \frac{1}{2} \left( -1 \pm \sqrt{1 - 4[-l(l+1) + G^2]} \right) = \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - G^2}, \tag{54}
\]

and \( \rho_0 \) is obtained as

\[
\rho_0 = 2 \left( j_{\text{max}} + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - G^2} \right). \tag{55}
\]

Then from the relation (53) we have

\[
c_{j+1} = \frac{2(j + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - G^2}) - \rho_0}{(j+1)(j+2\sqrt{(l + \frac{1}{2})^2 - G^2) + 1}} c_j. \tag{56}
\]

Now, with substituting \( A \) and \( D \), from the relations (48a) and (48b) and having \( \rho_0 = \frac{A}{D} \), one obtains:

\[
\rho_0 = \frac{A}{D} = 2 \left( j_{\text{max}} + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - G^2} \right) = \frac{2E - c^2}{\sqrt{\frac{1}{\hbar^2 c^2} (m_0^2 c^4 - E^2)}}
\]

So, with a little calculation, we can obtain the value of \( E \) as,

\[
E = \pm \frac{m_0 c^2 \left( j_{\text{max}} + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - G^2} \right)}{\sqrt{G^2 + \left( j_{\text{max}} + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - G^2} \right)^2}} \tag{57}
\]

The relation (57) shows the energy of the PF system related to Hydrogen atom. Using the relation (48d), the value of \( G \) is obtained from:

\[ G^2 = \left( \frac{e^2}{4\pi \varepsilon_0} \right)^2 \frac{1}{\hbar^2 c^2} \approx 5.4 \times 10^{-5}. \]
Since \( \frac{G^2}{(l + \frac{1}{2})^2} \ll 1 \), we can simplify the relation (57) as follows. Considering

\[
\left[ (l + \frac{1}{2})^2 - G^2 \right]^{\frac{1}{2}} = \left[ (l + \frac{1}{2})^2 \left( 1 - \frac{G^2}{(l + \frac{1}{2})^2} \right) \right]^{\frac{1}{2}} \\
\simeq (l + \frac{1}{2}) - \frac{G^2}{2(l + \frac{1}{2})} - \frac{1}{8} \frac{G^4}{(l + \frac{1}{2})^3},
\]

and

\[
\left[ G^2 + (j' + l' \sqrt{1 - \frac{G^2}{l'^2}}) \right]^{\frac{1}{2}} = \left[ j'^2 + l'^2 + 2j'l' \sqrt{1 - \frac{G^2}{l'^2}} \right]^{-\frac{1}{2}} \\
\simeq (j' + l')^{-1} \left[ 1 - \frac{j''(j' + l')^2}{l'^2} \left( \frac{G^2}{l'^2} \right) + \frac{1}{4} \left( \frac{G^2}{l'^2} \right)^2 + \ldots \right]^{-\frac{1}{2}} \\
\simeq \frac{1}{n} \left[ 1 + \frac{j''}{n^2} \frac{G^2}{2l'^2} + \frac{1}{8} \frac{j''}{n^2} (1 + 3 \frac{j''}{n^2}) \left( \frac{G^2}{l'^2} \right)^2 + \ldots \right],
\]

where we have ignored the higher power of \( G^4 \). \( n, j' \) and \( l' \) are defined as:

\[
n = j_{\text{max}} + l + 1, \\
j' = j_{\text{max}} + \frac{1}{2}, \\
l' = l + \frac{1}{2}
\]

So, putting relations (60), (59) and (58) in (57), we have:

\[
E = \frac{m_0 c^2}{2n^2} \left[ 1 - (\frac{l + \frac{1}{2}}{2})^2 \frac{G^2}{(l + \frac{1}{2})^2} + \frac{48(l + \frac{1}{2})^4}{128n^4} \left( \frac{G^2}{(l + \frac{1}{2})^2} \right)^2 \right] \\
= \frac{m_0 c^2}{2n^2} \left[ 1 - \frac{G^2}{2n^2} + \frac{48G^4}{128n^4} - \frac{64G^4}{128n^4 (l + \frac{1}{2})} \right].
\]

Eliminating \( G \) by using the relation (48b), and having into account that \( E_n = -\frac{m_0 c^2}{2n^2} \left( \frac{G^2}{4\pi n^2} \right)^2 \frac{4n}{n} \), it follows:

\[
E = m_0 c^2 + E_n - \frac{E_n}{2m_0 c^2} \left[ \frac{4n}{n} \sqrt{(l + \frac{1}{2})} - 3 \right]
\]

As we see, the relation (62) is exactly the same as relativistic correction to the energy levels of Hydrogen atom obtained by the first-order time independent perturbation theory.

Finally, the spatial wave function labeled by three quantum numbers, \( n, l \) and \( m \) is

\[
\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi),
\]

where, referring back to (44) and (50) for \( R(r) \),

\[
R_{nl}(r) = \frac{1}{r} r^{B+1} e^{-\rho} v(\rho)
\]

in which \( v(\rho) \) is a polynomial of degree \( j_{\text{max}} \) whose coefficients are determined by the recursion formula, (56).
5 Conclusion

It is indeed one of the main successes of our theory to provide us with a coherent way for describing conservative systems. In [4] we derived the relativistic Schrödinger equation for the case that the potential energy of the particle includes the relativistic mass and the case that it is independent of it. In this essay, we solved the relativistic Schrödinger equation to find the energy spectrum of a relativistic micro-harmonic oscillator and relativistic Hydrogen atom.

Considering the relativistic micro-harmonic oscillator, we examined relations (18) and (19) for solving the same problem, using the relations (20) and (21), respectively. The solution of the relativistic Schrödinger equation (19) for the case of nonrelativistic harmonic potential coincides with the answer obtained by Lagrangian approach leading to the Klein-Gordon equation.

Moreover, we found the energy spectrum of relativistic Hydrogen atom using the relativistic Schrödinger equation for potential energy that is independent of mass. It is one of our achievements that the result of energy spectrum in (62) for relativistic Hydrogen atom is completely consistent with the relativistic correction to the energy levels of Hydrogen atom obtained by the first-order time independent perturbation theory.

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