Massless scalar fields and infrared divergences in the inflationary brane world

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Abstract. We study the quantum effects induced by bulk scalar fields in a model with a de Sitter (dS) brane in a flat bulk (the Vilenkin–Ipser–Sikivie model) in more than four dimensions. In ordinary dS space, it is well known that the stress tensor in the dS invariant vacuum for an effectively massless scalar ($m_{\text{eff}}^2 = m^2 + \xi \mathcal{R} = 0$ with $\mathcal{R}$ the Ricci scalar) is infrared divergent except for the minimally coupled case. The usual procedure for taming this divergence is replacing the dS invariant vacuum by the Allen–Follaci (AF) vacuum. The resulting stress tensor breaks dS symmetry but is regular. Similarly, in the brane world context, we find that the dS invariant vacuum generates $\langle T_{\mu\nu} \rangle$ divergent everywhere when the lowest lying mode becomes massless except for the massless minimal coupling case. A simple extension of the AF vacuum to the present case avoids this global divergence, but $\langle T_{\mu\nu} \rangle$ remains divergent along a timelike axis in the bulk. In this case, singularities also appear along the light cone emanating from the origin in the bulk, although they are so mild that $\langle T_{\mu\nu} \rangle$ stays finite except for non-minimal coupling cases in four or six dimensions. We discuss implications of these results for bulk inflaton models. We also study the evolution of the field perturbations in dS brane world. We find that perturbations grow linearly with time on the brane, as in the case of ordinary dS space. In the bulk, they are asymptotically bounded.

Keywords: extra dimensions, cosmology with extra dimensions, quantum field theory on curved space, physics of the early universe

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1. Introduction

The brane world (BW) scenario [1, 2] has been intensively studied in recent years. Little is known yet concerning the quantum effects from bulk fields in cosmological models [3]–[6]. Quite generically, one expects to find that local quantities such as $\langle T_{\mu\nu} \rangle$ and $\langle \phi^2 \rangle$ can be large close to the branes, due to the well known divergences appearing in Casimir energy density computations. This has been confirmed for example in [7, 8] for flat branes. These divergences are of ultraviolet (UV) nature and do not contribute to the force. Hence, they are ignored in Casimir force computations. However, they are relevant to the BW scenario since they may induce a large back-reaction, and are worthy of investigation.

In this paper, we shall shed light on another aspect of objects such as $\langle T_{\mu\nu} \rangle$ in the BW. We shall point out that they can suffer from infrared (IR) divergences as well. These divergences arise when there is a zero mode in the spectrum of bulk fields in brane models of RSII type with a dS brane [1, 9]. The situation is analogous to the case in dS space without a brane. It is well known that light scalars in dS space develop an IR divergence in the dS invariant vacuum. The main purpose of this article is to explore the effects of scalar fields with light modes in a BW cosmological set-up of the RSII type [1]. Considering
the massless limit of a scalar field in the inflating BW is especially well motivated in the context of ‘bulk inflaton’ models [5, 6], [10]–[12], in which the dynamics of a bulk scalar drives inflation on the brane. In the simplest realizations, the brane geometry is close to the dS one and the bulk scalar is nearly massless.

Let us recall what happens in the usual dS case [13]. For light scalars with \( m_{\text{eff}} \ll H \) (with \( H \) the Hubble constant) in dS space, \( \langle \phi^2 \rangle \) and \( \langle T_{\mu\nu} \rangle \) in the dS invariant vacuum develop a global IR divergence, \( \sim 1/m_{\text{eff}}^2 \). To be precise, this depends on whether the field is minimally coupled or not. What we have in mind is a generic situation in which the effective mass \( m_{\text{eff}}^2 = m^2 + \xi R \) is small, and \( \xi \neq 0 \). In these cases \( \langle T_{\mu\nu} \rangle \) diverges as mentioned. The point is that in the generic massless limit, another vacuum must be chosen to avoid the global IR divergence. This process breaks dS invariance [14], but this will not really bother us. The simplest choice is the Allen–Follaci (AF) vacuum, in which the stress tensor is globally finite and everywhere regular. The massless minimally coupled case is special [15], and it accepts a different treatment which gives finite \( \langle T_{\mu\nu} \rangle \) without violating dS invariance.

In the BW scenario [1], the bulk scalar is decomposed into a continuum of KK modes and bound states. Here we consider the case where there is a unique bound state with mass \( m_d \). If \( m_d \) is light, \( \langle \phi^2 \rangle \) and \( \langle T_{\mu\nu} \rangle \) for the dS invariant vacuum will also diverge like \( 1/m_d^2 \). In this case, again, one will be forced to take another vacuum state such as the AF vacuum. Then one naive question is what the behaviour of the stress tensor is in such a vacuum in the BW. Also, one might expect singularities on the light cone emanating from the centre (the fixed point under the action of the dS group) if we recall that the field perturbations for a massless scalar in dS space grow like \( \langle \phi^2 \rangle \sim \chi \), where \( \chi \) is the proper time in dS space [16]–[19] (see also [20]). The light cone in the RSII model corresponds to \( \chi \to \infty \).

Before we start our discussion, we should mention previous calculations given in [21]. In that paper the stress tensor for a massless minimally coupled scalar was obtained in four dimensions, in the context of open inflation. Montes showed that \( \langle T_{\mu\nu} \rangle \) can be regular everywhere except on the bubble. As we will see, these properties hold as well in other dimensions, but only for massless minimal coupling fields.

For simplicity, we consider one extremal case of the RSII model [1] in which the bulk curvature and hence the bulk cosmological constant is negligible. We take into account the gravitational field of the brane by imposing Israel’s matching conditions. The resulting spacetime can be constructed by the ‘cut-and-paste’ technique. Imposing mirror symmetry, one cuts the interior of a dS brane in Minkowski space and pastes it to a copy of itself (see figure 1). Such a model was introduced in the context of bubble nucleation by Vilenkin [22] and by Ipser and Sikivie [23], and we will refer to it as ‘the VIS model’.

This paper is organized as follows. In section 2, we describe the VIS model and introduce a bulk scalar field with generic bulk and brane couplings. The Green function is obtained first for the case where the bound state is massive, \( m_d > 0 \). The form of \( \langle T_{\mu\nu} \rangle \) in the limit \( m_d \to 0 \) is also obtained. In section 3, we consider an exactly massless bound state \( m_d = 0 \), and we present the divergences of the AF vacuum. The case where the bulk mass vanishes is technically simpler and explicit expressions for \( \langle T_{\mu\nu} \rangle \) can be obtained. This is done in section 4. Using the results, we describe the evolution of the field perturbations in section 5, and conclude in section 6.
2. Scalar fields in the VIS model

In this section we consider a generic scalar field propagating in the VIS model, describing a gravitating brane in an otherwise flat space \([22,23]\). Specifically, the spacetime consists of two copies of the interior of the brane glued at the brane location, as illustrated in figure 1.

In the usual Minkowski spherical coordinates the metric is \(ds^2 = -dT^2 + dR^2 + R^2 d\Omega^2_{(n)}\), where \(d\Omega^2_{(n)}\) stands for the line element on a unit \(n\) sphere. In view of the symmetry, it is convenient to introduce another set of coordinates. The Rindler coordinates, defined by

\[
R = r \cosh \chi, \\
T = r \sinh \chi,
\]

cover the exterior of the light cone emanating from \(R = T = 0\). In terms of them the brane location is simply \(r = r_0\), and the metric looks like

\[
\begin{align*}
\text{d}s^2 &= dr^2 + r^2 dS^2_{(n+1)},
\end{align*}
\]

(1)

where \(dS^2_{(n+1)}\) is the line element of de Sitter (dS) space of unit curvature radius. Thus, the Hubble constant on the brane is \(H = 1/r_0\). In order to cover the interior of the light cone, we introduce the ‘Milne’ coordinates according to

\[
R = \tau \sinh \psi, \\
T = \tau \cosh \psi.
\]

In these coordinates, the metric is \(ds^2 = -d\tau^2 + \tau^2 [d\psi^2 + \sinh^2 \psi \, d\Omega^2_{(n)}]\). Note that we can go from the Rindler to the expanding (contracting) Milne regions, making the continuation...
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\[ r = \pm i \tau \text{ and } \chi = \psi \mp (\pm)i\pi/2. \] Here upper and lower signatures correspond to \( +i\epsilon \) and \(-i\epsilon\) prescriptions, respectively.

We consider a scalar field even under \( Z_2 \) symmetry, with generic couplings described by the action

\[
S = -\frac{1}{2} \int_{\text{Bulk}} \left[ (\partial \phi)^2 + (M^2 + \xi R)\phi^2 \right] - \int_{\text{Brane}} [\mu + 2\xi \text{tr} K] \phi^2, \tag{2}
\]

where \( M \) and \( \mu \) are the bulk and brane masses, \( R \) is the Ricci scalar and \( \text{tr} K \) is the trace of the extrinsic curvature. The latter arises because the curvature scalar contains \( \delta \) function contributions on the brane, and the factor 2 in front of it is due to the \( Z_2 \) symmetry. The stress tensor for a classical field configuration can also be split into bulk and surface parts as \([13,24]\)

\[
T_{\mu\nu}^{\text{bulk}} = (1 - 2\xi)\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1 - 2\xi}{2}[(\partial\phi)^2 + (M^2 + \xi R)\phi^2]g_{\mu\nu} - 2\xi R_{\mu\nu}\phi^2 - 2\xi\phi \nabla_\mu \nabla_\nu \phi \tag{3}
\]

\[
T_{ij}^{\text{brane}} = [(4\xi - 1)\mu_{\text{eff}}h_{ij} - 2\xi K_{ij}]\delta^2(r - r_0),
\]

where \( h_{ij} \) is the induced metric on the brane, \( R_{\mu\nu} \) the Ricci tensor and the equation of motion has been used\(^3\). Here we have introduced an effective brane mass as

\[
\mu_{\text{eff}} \equiv \mu + 2(n + 1)\xi H, \tag{4}
\]

where \( H = 1/r_0 \) is the Hubble constant on the brane. Then, the v.e.v. \( \langle T_{\mu\nu} \rangle \) in point splitting regularization is computed as\(^4\)

\[
\langle T_{\mu\nu} \rangle^{\text{bulk}} = \frac{1}{2} T_{\mu\nu}[G^{(1)}(x, x')], \tag{5}
\]

with

\[
T_{\mu\nu} = \lim_{x' \to x} \left\{ (1 - 2\xi)\partial_\mu \partial'_\nu - \frac{1 - 2\xi}{2} g_{\mu\nu}(g^{\lambda\sigma} \partial_\lambda \partial'_\sigma + M^2) - 2\xi \nabla_\mu \nabla_\nu \right\}, \tag{6}
\]

and

\[
\langle T_{ij} \rangle^{\text{brane}} = \frac{1}{2}[(4\xi - 1)\mu_{\text{eff}}h_{ij} - 2\xi K_{ij}]\delta(r - r_0)G^{(1)}(x, x')|_{x' = x}, \tag{7}
\]

where \( \partial'_\mu = \partial/\partial x'^\mu \). This expression is extended to the case with a non-zero bulk cosmological constant by replacing \( M^2 \) with \( M^2 + \xi R \) and recovering the Ricci tensor term in equation (3).

\(^3\) Generically, surface terms are irrelevant for the Casimir force, but are essential for relating the vacuum energy density and the Casimir energy; see \([7], [24]-[27]\).

\(^4\) We omit the anomaly term since it vanishes in the bulk for odd dimension, and on the brane it can be absorbed in a renormalization of the brane tension.
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Figure 2. The shaded area corresponds to the values of \( \mu_{\text{eff}} \) and \( M^2 \) for which the bound state is normalizable (\( m_d < n/2 \)) and non-tachyonic, \( m_d \geq 0 \). The thick (red) line corresponds to the massless case. The plot is for \( n = 3 \).

2.1. The spectrum

The Klein–Gordon equation following from the action (2) is separable into radial and dS parts so we introduce the mode decomposition \( \phi = \sum \int U_p(r)Y_{p\ell m}(\chi, \Omega) \), where \( m \) is a multiple index. The radial equation is

\[
\left[ \partial_r^2 + \frac{n+1}{r} \partial_r + \frac{1}{r^2} \left( p^2 + \frac{n^2}{4} \right) - M^2 \right] U_p(r) = 0,
\]

while the brane terms can be encoded in the boundary condition

\[
(\partial_r + \mu_{\text{eff}})U_p|_{r=r_0} = 0,
\]

where \( Z_2 \) symmetry has been imposed and the effective brane mass \( \mu_{\text{eff}} \) is given in equation (4).

The de Sitter part satisfies

\[
[\Box_{n+1} - (n/2)^2 - p^2]Y_{p\ell m} = 0.
\]

Thus one obtains a tower of modes \( Y_{p\ell m} \) in dS space with masses \( m_{\text{KK}}^2 = (n/2)^2 + p^2 \) in units of \( H \). The mass spectrum determined by the Schrödinger problem defined by equations (8) and (9). It consists of a bound state plus a continuum of KK states with \( p \geq 0 \) (\( m_{\text{KK}} \geq n/2 \)). The radial part for the KK modes is of the form

\[
U_{p\ell m}^{\text{KK}}(r) = r^{-n/2}[A_p I_{ip}(Mr) + B_p I_{-ip}(Mr)],
\]

with \( A_p \) and \( B_p \) determined by the boundary condition (9) and continuum normalization,

\[
2 \int dr r^{n-1}U_{p\ell m}^{\text{KK}}(r)U_{p\ell m}^{\text{KK}}(r) = \delta(p-p').
\]

The mass of the discrete spectrum is

\[
m_d^2 = (n/2)^2 + p_d^2 < (n/2)^2,
\]

and hence \( p_d \) is pure imaginary. The normalizability implies that its wavefunction is

\[
U_{\text{bs}}^{\text{bs}}(r) = N_d r^{-n/2} I_{-ip_d}(Mr),
\]
with \(-ip_d > 0\). The boundary condition (9) determines \(p_d\) in terms of \(M\) and \(\mu_{\text{eff}}\) according to

\[
\nu I_{-ip_d}(Mr_0) - Mr_0 I'_{-ip_d}(Mr_0) = 0,
\]

where we have introduced the combination

\[
\nu \equiv \frac{n}{2} - \frac{\mu_{\text{eff}}}{H}.
\]

In the limit \(Mr_0 \ll 1\) and \(\mu_{\text{eff}}r_0 \ll 1\), equation (12) implies that the mass of the bound state is

\[
(Hm_d)^2 = nH\mu_{\text{eff}} + \frac{n}{n+2}M^2 + \mathcal{O}(\mu_{\text{eff}}^2, \mu_{\text{eff}}M^2, M^4),
\]

which agrees with the results of [10, 28]. Figure 2 shows the values of \(M^2\) and \(\mu_{\text{eff}}\) for which there exists a non-tachyonic \((-ip_d \leq n/2)\) bound state. In this paper, we are mostly interested in the situation where the bound state is massless. This happens whenever equation (12) with \(-ip_d\) replaced by \(n/2\) holds, that is when \(\nu\) reaches the ‘critical’ value

\[
\nu_c = \frac{Mr_0 I'_{n/2}(Mr_0)}{I_{n/2}(Mr_0)}.
\]

### 2.2. The Green function

The renormalized \(D\)-dimensional Green function can be split into the bound state and KK contributions,

\[
G_{(\text{ren})}^{(1)} = G_{\text{KK}} + G_{\text{bs}}^{(1)},
\]

with

\[
G_{\text{bs}}^{(1)} = \mathcal{U}_{\text{bs}}^{(r)}(r)\mathcal{U}_{\text{bs}}^{(r')}G_{pd(dS)}^{(1)},
\]

\[
G_{\text{KK}}^{(1)} = \int_0^\infty dp \left[ U_{p}\mathcal{U}_{p}\mathcal{U}_{p}\right]^{\text{ren}}G_{p(dS)}^{(1)},
\]

where \(G_{p(dS)}^{(1)}\) denotes the Green function of a field with mass \((n/2)^2 + p^2\) in \((n+1)\)-dimensional dS space with \(H = 1\). It depends on \(x\) and \(x'\) through the invariant distance in dS space, which we call \(\zeta\). Its precise form is given in appendix A. The ‘renormalized’ product of the KK mode functions in equation (18) is

\[
[U_{p}\mathcal{U}_{p}(r)\mathcal{U}_{p}(r')]^{\text{ren}} \equiv U_{p}\mathcal{U}_{p}(r)\mathcal{U}_{p}(r') - U_{p}\mathcal{U}_{p}(r)\mathcal{U}_{p}(r')
\]

\[
= \frac{ip\nu K_{-ip}(Mr_0) - Mr_0 K'_{-ip}(Mr_0)}{2\pi(rr')^{n/2}} - \frac{\nu I_{-ip}(Mr_0) - Mr_0 I'_{-ip}(Mr_0)}{2\pi} I_{-ip}(Mr)I_{-ip}(Mr') + (p \rightarrow -p).
\]

Here, \(U_{p}\mathcal{U}_{p}(r)\propto K_{ip}(Mr)\) is the Minkowski counterpart of (10). This effectively removes the UV divergent contribution to the Green function and guarantees that the renormalized Green function (16) is finite in the coincidence limit.
Since equation (19) is even in \( p \), equation (18) can be cast as \( G^{\text{KK}} = \int_{-\infty}^{\infty} dp \left[ U^{\text{KK}}_p(r)U^{\text{KK}}_p(r') \right]_1^{\text{ren}} G_{p(\text{ds})}^{(1)} \), where \( U^{\text{KK}}_p(r)U^{\text{KK}}_p(r') \) stands for the first term in equation (19) only. This can be evaluated by summing the residues. From equation (A.5), the poles in \( G_{p(\text{ds})}^{(1)} \) in the upper \( p \) plane are at \( p = i(q + n/2) \), with \( q = 0, 1, 2 \ldots \) (see figure 3). From equations (19) and (12), we see that the KK radial part has a pole at the value of \( p \) corresponding to the bound state, \( p = p_d \). We will now show that the residue is related to the bound state wavefunction as

\[
2\pi i \text{Res}[U^{\text{KK}}_p(r)U^{\text{KK}}_p(r')]_1^{\text{ren}}|_{p_d} = -\mathcal{U}^{\text{bs}}(r)\mathcal{U}^{\text{bs}}(r'). \tag{20}
\]

Using the Wronskian relation \( K'_{\lambda}(z)I'_{\lambda}(z) - K_{\lambda}(z)I_{\lambda}(z) = 1/z \) and equation (12), it is straightforward to show that

\[
2\pi i \text{Res}[U^{\text{KK}}_p(r)U^{\text{KK}}_p(r')]_1^{\text{ren}}|_{p_d} = - \frac{p_d/I_{-ip_d}(Mr_0)}{\partial_p (\nu I_{-ip}(Mr_0) - Mr_0 I'_{-ip}(Mr_0))}_{p=p_d}. \tag{21}
\]

The overall constant in the rhs is nothing but the normalization constant in the bound state wavefunction (19), up to the sign. Using the Schrödinger equation (8) and integrating by parts, we have

\[
(p^2 - p_d^2) \int_0^{r_0} \frac{dr}{r} I_{-ip_d}(Mr)I_{ip}(Mr) = I_{-ip}(Mr_0)Mr_0 I'_{-ip}(Mr_0) - I_{-ip_d}(Mr_0)Mr_0 I'_{-ip}(Mr_0). \]

Setting \( p = p_d \) after differentiation with respect to \( p \), we find

\[
\frac{1}{N^2_d} = 2 \int_0^{r_0} \frac{dr}{r} |I_{-ip_d}(Mr)|^2 = \frac{1}{p_d} I_{-ip_d}(Mr_0)\partial_p (\nu I_{-ip}(Mr_0) - Mr_0 I'_{-ip}(Mr_0))|_{p=p_d}, \tag{21}
\]

where we used equation (12). From this and the form of the bound state wavefunction (11), it is clear that equation (20) holds. Equation (20) implies that no term of the form \( \mathcal{U}^{\text{bs}}(r)\mathcal{U}^{\text{bs}}(r')G_{p_d}^{(1)} \) survives in the result. This is ‘fortunate’ because close to \( r = 0 \), \( \mathcal{U}^{\text{bs}} \sim r^{-n/2-ip_d} \) which is divergent. Thus, equation (20) guarantees that \( \langle T_{\mu\nu} \rangle \) is regular on the light cone.

Since only the poles from \( G_{p(\text{ds})}^{(1)} \) contribute to \( G^{\text{ren}} \), it can be written as the integral over a contour \( C \) that runs above \( p_d \) (see figure 3). Equation (A.5) leads to an expression appropriate for seeing the coincidence limit,

\[
G^{\text{ren}} = \int_C dp \left[ U^{\text{KK}}_p(r)U^{\text{KK}}_p(r') \right]_1^{\text{ren}} G_{p(\text{ds})}^{(1)} = - \frac{S^{(u)}_{(n)}}{2^{n-1}\Gamma((n+1)/2)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\nu K_{n/2+k+j}(Mr_0) - Mr_0 K'_{n/2+k+j}(Mr_0)}{\nu I_{n/2+k+j}(Mr_0) - Mr_0 I'_{n/2+k+j}(Mr_0)} \times \frac{(n/2 + k + j)}{(rr')^{n/2}} \times \frac{(-1)^k \Gamma(n + 2k + j)}{j!k!\Gamma((n+1)/2+k)} \left( 1 - \cos \zeta/2 \right)^k, \tag{22}
\]
and we recall that $\zeta$ is the invariant distance in dS space. Each term comes from the pole at $p = i((k+j)+n/2)$. Setting $k = 0$, we find that the term with a large $j$ is unsuppressed for $r = r' = r_0$ for $\zeta = 0$. Hence the Green function in the coincidence limit is divergent on the brane. This is the usual UV ‘Casimir’ divergence near the boundary. Since we are interested in the IR behaviour, we will not comment further on this UV divergence.

The term with $k = j = 0$ in equation (22) renders the Green function globally IR divergent in the limit where the bound state is massless, $\nu \to \nu_c$ (see equation (15)). One can show that this term comes from the homogeneous ($\ell = 0$) mode of the bound state. Using equations (C.2) and (C.3), the leading behaviour of equation (22) in the massless limit $m_d \to 0$ can be written as

$$G^{(\text{ren})(1)} = \frac{2}{S_{n+1}} \frac{H^2}{m_d^2} U_0^{\text{bs}}(r)U_0^{\text{bs}}(r') + O(m_d^0),$$

where $U_0^{\text{bs}}(r) = N_0 I_{n/2}(Mr)/r^{n/2}$ is the wavefunction of the bound state (11) for the exactly massless case.

The divergence (23) appears because in the massless limit, the wavefunction of the homogeneous mode of the bound state in the dS invariant vacuum broadens without bound [15]. It can be removed by considering another vacuum with finite width, which implies breaking dS symmetry [14]. Later, we will take the Allen–Follaci vacuum [14]. We will find in section 3 that in the brane world context, this process removes the global IR divergence, but a localized singularity within the bulk remains.

Let us now examine the behaviour of $\langle T_{\mu\nu} \rangle$ in the massless limit, $m_d \to 0$. In this limit, the stress tensor is given by

$$\langle T_{\mu\nu} \rangle^{\text{bulk}} \simeq \frac{H^2}{S_{n+1}m_d^2} T_{\mu\nu}[U_0^{\text{bs}}(r)U_0^{\text{bs}}(r')],$$

$$\langle T_{ij} \rangle^{\text{brane}} \simeq \frac{H^2}{S_{n+1}m_d^2} [(4\xi - 1)\mu_{\text{eff}} - 2H\xi|U_0^{\text{bs}}|_{r_0})^2\delta(r - r_0)h_{ij},$$

where $T_{\mu\nu}$ is the differential operator given in equation (6) and $h_{ij}$ is the induced metric on the brane. For $M \neq 0$, $U_0^{\text{bs}}$ is not constant. Therefore equation (24) explicitly shows the presence of a global IR divergence in $\langle T_{\mu\nu} \rangle$ for the dS invariant vacuum in the $m_d \to 0$ limit. For $M = 0$, $U_0^{\text{bs}}$ is constant. Hence, the bulk part is finite. However, if $\xi \neq 0$ then the surface term (25) diverges [15]. Thus, in the limit $m_d = 0$ we are forced to consider another quantum state. This is the subject of section 3. We will mention that there is a possibility for avoiding this IR divergence without modifying the choice of dS invariant vacuum state. In the present case the divergence is constant and is localized on the brane. Hence, it can be absorbed by changing the brane tension. We may therefore have a model in which this singular term is appropriately renormalized so as not to diverge in the $m_d \to 0$ limit. Of course, such a model is a completely different model from the original one without this IR renormalization.

The massless minimally coupled limit, $M = \mu_{\text{eff}} = \xi = 0$, is exceptional. Both bulk and brane parts of stress tensors (24) and (25) are finite in this limit, though here there

\[ \text{For } M = 0, \mu_{\text{eff}}(\phi^2) \text{ is finite because } \langle \phi^2 \rangle \sim 1/\mu_{\text{eff}}; \text{ see equation (14)}. \]
is a slight subtlety. The limiting values depend on how we fix the ratios among $M, \mu_{\text{eff}}$ and $\xi$. For example, using equation (14), the surface term is given by

$$\lim_{M, \mu_{\text{eff}}, \xi \to 0} \langle T_{ij} \rangle_{\text{brane}} \simeq -\frac{n + 2}{n} \frac{H^2 (U_0^m |r_0|^2)}{S_{(n+1)}} \frac{\mu_{\text{eff}} + 2\xi H}{(n + 2)H \mu_{\text{eff}} + M^2} \delta(r - r_0),$$  

(26)

where we used the approximate mass of the bound state,

$$m_d^2 \simeq n\mu H + 2n(n + 1)\xi H^2 + \frac{n}{n + 2} M^2,$$

which is valid in the massless minimal coupling limit (see equation (14)). Hence, in the absence of any fine tuning ($m_d^2 \approx \max(M^2, \xi H^2, H\mu)$), it is clear that the contribution (25) is not large, even though the Green function (23) is. Thus, only in the case where the parameters are ‘fine tuned’ according to equation (15) is the stress tensor (25) large. From equation (3), if the Green function is free from IR divergence, it is clear that the brane stress tensor must be zero in the massless minimally coupled case. The direction that reproduces this result is the one along which $\langle T_{ij} \rangle_{\text{brane}}$ already vanishes (in the massive case), that is $\mu_{\text{eff}} = -2\xi H$. Note that this feature is analogous to what happens in dS space [15].

The bulk stress tensor (24) also has a similar but slightly more complicated feature. The operator (6) has terms which do not manifestly involve a small quantity, such as $M^2, \xi$ or $\mu$. However, these terms are associated with derivative operators. In the limit $M^2 \to 0$, we can expand the dependence on $r$ of the term with $k = j = 0$ in equation (22) as

$$\frac{I_{n/2}(Mr)}{r^{n/2}} \approx \frac{1}{2^{n/2}\Gamma(n/2 + 1)} \left(1 + \frac{M^2 r^2}{2(n + 2)} + \cdots\right).$$  

(27)

Then, the leading term in the above expansion, which is not suppressed by a factor $M^2$, vanishes in (24). The remaining terms are finite unless the parameters are fine tuned, as in the case of the brane stress tensor.

Finally, equation (24) also shows that $\langle T_{\mu\nu} \rangle$ is perfectly regular on the light cone for $m_d \neq 0$. As mentioned before, this happens thanks to the KK modes. Note as well that for $M \gtrsim H$, equation (24) is exponentially localized on the brane, because the bound state is localized in this case.

3. The exactly massless bound state

In the preceding section, we have seen that de Sitter (dS) invariant vacuum causes divergence in the limit where the bound state is massless. The divergence is caused by the $\ell = 0$ homogeneous mode in the bound state. In this section we consider a different choice of vacuum state for this mode aiming at resolving the problem of divergence, following the standard methods used in dS space [14, 15]. For simplicity, we concentrate on the case of an exactly massless bound state $m_d = 0$, that is $p_d = ni/2$, although more general cases would be treated in a similar way. The case with $m_d = 0$ includes not only a massless minimally coupled scalar, $M = \mu = \xi = 0$, but also other fine tuned cases.
3.1. The Green function

Here, we should split the Green function into KK and bound state contributions, \( G^{\text{ren}(1)} = G^{\text{KK}(1)} + G^{\text{bs}(1)} \). We leave the quantum state for the KK contribution untouched, and change only the contribution from the bound state. In the integral representation for the Green function in (22), we have used the integration contour given in figure 3. This choice of contour automatically takes into account the bound state contribution simultaneously. In the present case, we consider the contour that runs below the pole at \( p = p_d \) to exclude the bound state contribution. For \( m_d = 0 \), the integrand has a double pole at \( p = ni/2 \) because the pole in the radial modes coincides with one of the poles in the dS Green function. Hence the integral with the contour given in figure 3, which runs through these merging poles, is not well defined. But the integral with the new contour, which picks up the contribution from the KK modes only, is well behaved, and it can be cast as

\[
G^{\text{KK}(1)} = 2\pi i \left\{ \sum_{\text{simple poles}} \text{Res}\left(\left[\mathcal{U}_p^{\text{KK}}(r)\mathcal{U}_p^{\text{KK}}(r')\right]^{\text{ren}}_1 G^{(\text{dS})}(1)\right) \right. \\
+ \text{Res}\left(\left[\mathcal{U}_p^{\text{KK}}(r)\mathcal{U}_p^{\text{KK}}(r')\right]^{\text{ren}}_1 \partial_r(p - p_d)\right)\left|_{p_d}\right. \\
+ \left. \text{Res}\left(\mathcal{G}_p^{(\text{dS})}(1)\partial_r(p - p_d)\right)\left|_{p_d}\right.\right\}. \tag{28}
\]

As before, \( \left[\mathcal{U}_p^{\text{KK}}(r)\mathcal{U}_p^{\text{KK}}(r')\right]^{\text{ren}}_1 \) denotes the first term in equation (19). In the first term, ‘simple poles’ means poles at \( p = i(q + n/2) \) with \( q = 1, 2, \ldots \) (see figure 3). That is, it is obtained by removing the term with \( k = j = 0 \) from equation (22). The last two terms are contributions from the double pole at \( p = ni/2 \).

Next, we consider the contribution from the bound state. A massless bound state behaves as a massless scalar from the viewpoint of \((n + 1)\)-dimensional dS space. In dS space it is well known that the dS invariant Green function diverges in the massless limit.
because of the \( \ell = 0 \) homogeneous mode [14,15]. The usual procedure is treating the \( \ell = 0 \) mode separately from the rest. It is easy to show that (see appendix B)

\[
G^{bs(1)} = U^{bs}(r)U^{bs}(r')\left\{ \sum_{\ell \geq 0} Y_{p\ell m}(\chi)Y^*_{p\ell m}(\chi) + \hat{\chi}_{AF}(\chi)\hat{\chi}^*_{AF}(\chi) \right\} + \text{c.c.}
\]

\[
= U^{bs}(r)U^{bs}(r')\left\{ \partial_p[(p - p_d)\gamma_{p00}(\chi)]|_{p_d} - \partial_p[(p - p_d)\gamma^*_{p00}(\chi)]|_{p_d} + \hat{\chi}_{AF}(\chi)\hat{\chi}^*_{AF}(\chi) \right\} + \text{c.c.},
\]

where \( Y_{p\ell m}(\chi) \) are the positive frequency dS invariant vacuum modes with mass \( p^2 + \frac{1}{2} \). The last term in equation (29) is the contribution from the homogeneous mode in the appropriate state, which we will take to be the Allen–Follaci (AF) vacuum [14] (see equation (36)).

At this stage, we note that the second term of equation (28) cancels the first term of equation (29), due to equation (20). This cancellation resembles the one that occurred in the previous case between the KK modes and the bound state. In that case, it guaranteed the absence of light cone divergences. In the present case, the terms that cancel in equations (28) and (29) are already regular. We show below (see the discussion after equation (32)) that instead the last term in equation (28) and the second term in equation (29) diverge on the light cone. However, when added up, they render \( G^{\text{ren}} \) finite on the light cone in odd dimensions. (In even dimensions, \( G^{\text{ren}} \) is finite but its derivatives are not.)

The fact that the dS invariant Green function diverges because of the homogeneous mode implies that \( \text{Res}(G^{(dS)})|_{p_d} = \lim_{p \rightarrow p_d} (p - p_d)\gamma_{p00}(\chi) + \text{c.c.} \). Using this and equation (20), we can rewrite the total Green function in the more convenient form

\[
G^{(\text{ren})}(1) = G^{(1)} + G_{\text{LC}} + \hat{G}_{AF},
\]

with

\[
G_{\text{LC}} = 2\pi i \partial_p((p - p_d)\gamma_{p00}(\chi) - (p - p_d)\gamma^*_{p00}(\chi))|_{p_d} + \text{c.c.},
\]

\[
\hat{G}_{AF} = U^{bs}(r)U^{bs}(r')\hat{\chi}_{AF}\hat{\chi}^*_{AF} + \text{c.c.},
\]

and \( G^{(1)}_{\text{simple}} \) is the contribution from the first term in equation (28). As is manifest from the expression (22) with the \( k = j = 0 \) term removed, \( G^{(1)}_{\text{simple}} \) is regular in the coincidence limit everywhere except for the ‘Casimir’ divergences on the brane, and depends on the points \( x \) and \( x' \) through \( r, r' \) and \( \zeta(x, x') \) (the invariant distance in dS space between the projections of points \( x \) and \( x' \)), and hence it is dS invariant. The second term, \( G_{\text{LC}} \), contains the two contributions in equations (28) and (29) that are separately divergent on the light cone mentioned in the previous paragraph. The last term, \( \hat{G}_{AF} \), encodes the choice of vacuum for the zero mode of the bound state.

Let us examine the contribution potentially divergent on the light cone \( G_{\text{LC}} \). The derivative of the \( \chi \) dependent part is obtained from equations (B.4) and (B.6). The derivative of the radial part follows from equation (19) and \( \partial_\lambda I_\lambda(z) = I_\lambda(z) \log z + f(z) \), where \( f(z) \) is a regular function. It is easy to see that \( G_{\text{LC}} \) takes the form

\[
G_{\text{LC}}(x, x') = \frac{2}{nS_{n+1}}U^{bs}(r)U^{bs}(r') \text{Re}[F(x) + F(x')] + \text{regular},
\]

where
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\[ F(x) = \log r + n \int_0^x \frac{d\chi_1}{\cosh^n \chi_1} \int_{-(i\pi/2)}^{\chi_1} d\chi_2 \cosh^n \chi_2, \]  

(33)

where the indicated regular term depends on \( r \) and \( r' \) only. The double integral in (33) grows linearly with \( \chi \) for large \( \chi \) and eventually blows up on the light cone. (The first integral grows asymptotically like \( e^{n\chi} / n \), and therefore the integrand in the second integral goes to a constant.) This is the expected behaviour from the massless bound state. On the other hand, the KK modes contribute the \( \log r \) term, which cancels the light cone divergence. To see this, we integrate by parts to obtain

\[ F(x) = \log r + \log \sinh \chi + \int_0^x \frac{d\chi_1}{\cosh^n \chi_1} \int_{-(i\pi/2)}^{\chi_1} d\chi_2 \cosh^n \chi_2 \sinh^2 \chi_2, \]  

(34)

and now the double integral is bounded. The first two terms are simply \( \log T \). Thus, the leading divergence in \( G_{LC} \) on the light cone cancels between contributions from the bound state and those from the KK modes, although it still diverges logarithmically at infinity.

This statement has to be qualified for even dimension. In this case, the derivatives of \( G_{LC} \) diverge on the light cone because of the last term in equation (34). To see this, note that the integrand in equation (34) can be expanded in exponentials. Then, the integral is a sum of exponential terms except for one, of the form \( \chi e^{-n\chi} \), if \( n \) is even. In terms of the null coordinates \( U \) and \( V \), this is \( \sim (V/U)^{n/2} \log(V/U) \) for \( \chi \to +\infty \) (for \( \chi \to -\infty \) make the replacement \( U \leftrightarrow V \)). Even though the Green function is regular at \( V = 0 \), the stress tensor develops a singularity which behaves like \( \sim 1/V \) or \( 1/U \) on the light cone in four dimensions \( (n = 2) \) and like \( \sim \log V \) or \( \log U \) in six dimensions \( (n = 4) \) if \( \xi \neq 0 \).

For reference, we show the explicit form of \( F(x) \) for dimensions 4 and 5,

\[ F(x) = \begin{cases} 
U \log U + V \log |V| & \text{for } n = 2, \\
\log(T + \tau) - \frac{(T - \tau)\tau}{R^2} & \text{for } n = 3,
\end{cases} \]  

(35)

where \( \tau^2 = T^2 - R^2 \). Note that despite appearances, equation (35) is regular at \( R = 0 \). This is guaranteed since the \( \chi \) dependent part of \( F(x) \) is related to the dS invariant vacuum modes (see equation (B.6)), which are regular at \( R = 0 \) \( (\xi = -\pi i/2) \) by construction. The expressions (35) are appropriate in the Milne region, and we have already taken the real part, which is the relevant part for \( G^{(1)} \).

The last term \( \tilde{G}_{AF} \) in equation (30) corresponds to the choice of vacuum for the \( \ell = 0 \) mode. This mode is peculiar because it behaves like a free particle rather than an oscillator. The eigenstates of the Hamiltonian, and in particular the ground state, are plane waves in field space. However, such states are not normalizable. One can construct well defined states as wavepackets. The simplest option is a Gaussian packet. This is the Allen–Follaci vacuum \([14]\), which in fact is a two-parameter family of vacua. Its mode function is given by \([14]\)

\[ \tilde{y}_{AF}(\chi) = \frac{1}{\sqrt{S_{(n)}}} \left[ \frac{1}{2\alpha} + i\beta - i\alpha \int_0^\chi \frac{d\chi'}{\cosh^n \chi'} \right], \]  

(36)
where $\alpha > 0$ and $\beta$ are the above-mentioned free (real) parameters. We will impose that the vacuum is time reversal symmetric. This translates into $\mathcal{V}_\alpha(-\chi) = \mathcal{V}_\alpha^*(\chi)$, and implies $\beta = 0$. Because of the time dependence, it breaks dS symmetry. For this vacuum,

$$\tilde{G}_\alpha(x, x') = \frac{U^{bs}(r)U^{bs}(r')}{S(n)} \left\{ \frac{1}{2\alpha^2} + 2\alpha^2 \text{Re}[F_\alpha(x)F_\alpha^*(x')] - \text{Im}[F_\alpha(x) + F_\alpha(x')] \right\}$$

(37)

with

$$F_\alpha = \int d\chi \cosh^{-n}\chi.$$

In order to obtain the analytic continuation of $F_\alpha$ to the Milne region, it is better to write it as an integral over a constant $T$. It is straightforward to see that

$$F_\alpha(x) = -T \int^R dR' \frac{R'^2 - T^2}{R'^{n-1}}.$$

It is transparent now that $F_\alpha$ behaves like $1/R^{n-1}$ near $R = 0$. It is also clear that it is regular on the light cone and is bounded at infinity. Moreover, it gets an imaginary part in the Milne region for odd dimensions (in the Rindler region it is always real). More specifically, we have

$$F_\alpha(x) = \begin{cases} \frac{T}{R}, & \text{for } n = 2, \\ i \frac{T^2}{2} - \ln \left( i \frac{T + \tau}{R} \right) & \text{for } n = 3. \end{cases}$$

(38)

3.2. The divergent stress tensor

Now we discuss the form of the expectation value of the stress tensor for the possible values of $M, \mu_{\text{eff}}$ and $\xi$ in which the bound state is massless. The bulk part of the stress tensor is most conveniently separated into

$$\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle_0 + \langle T_{\mu\nu} \rangle_{\text{simple}}$$

(39)

where $\langle T_{\mu\nu} \rangle_0$ contains the contributions from $G_{\text{LC}}$ and $\tilde{G}_\alpha$, whereas $\langle T_{\mu\nu} \rangle_{\text{simple}}$ is the contribution from $G^{(1)}_{\text{simple}}$. All the IR irregularities are contained in $\langle T_{\mu\nu} \rangle_0$. From equations (32) and (37), we obtain the bulk part of $\langle T_{\mu\nu} \rangle_0$ as

$$\langle T_{\mu\nu} \rangle_0^{\text{bulk}} = \frac{1}{2S(n)} T_{\mu\nu} \left[ U_\alpha(r)U_\alpha(r') \left\{ \frac{1}{2\alpha^2} + 2\alpha^2 \text{Re}[F_\alpha(x)F_\alpha^*(x')] \right\} \right]$$

$$- 2\xi \nabla_\mu \nabla_\nu U_\alpha(r)U_\alpha(r') \left[ \frac{\text{Re} F(x)}{nS(n+1)} - \frac{\text{Im} F_\alpha(x)}{2S(n)} \right].$$

(40)

The term proportional to $\alpha^2$ diverges at $R = 0$ like $1/R^{2n}$ for any value of $\xi$. $R = 0$ corresponds to a timelike axis in the bulk passing through the centre of symmetry (see figure 5). The last term also diverges like $1/R^{n+1}$ for odd dimensions, but vanishes for even dimensions. The piece involving $\text{Re} F$ diverges on the light cone for $n = 2$ or 4.
Let us begin with the most general case with $M \neq 0$. In this case $U_{bs}(r)$ is not constant, and the contribution from the term inversely proportional to $\alpha^2$ in equation (40) does not vanish. This term is analogous to equation (24), and it diverges globally in the $\alpha \to 0$ limit. Hence, this state cannot be taken, and one has to be content with the AF vacuum with some non-zero $\alpha$. Hence, the term proportional to $\alpha^2$ is unavoidable. But this is very noticeable since it contains a singularity at $R = 0$ of the form $\sim \alpha^2/R^{2n}$, present even for minimal coupling. The main point is that in the presence of a bulk mass, $\langle T_{\mu\nu} \rangle$ contains a quite severe bulk singularity even after we get rid of the global IR divergence. There is of course the possibility that a different choice of vacua for the KK modes could cancel it out. In this case, it seems that the vacuum choice should not be dS invariant; otherwise the singular zero-mode contribution that is not dS invariant could not be compensated. We leave this issue for future research.

Now we turn to the case with $M = 0$. Let us first consider the non-minimal case $\xi \neq 0$. Since $M = 0$, $U_{bs}(r)$ is constant and the bulk stress tensor does not diverge globally in the $\alpha \to 0$ limit. Divergences in the other two $\alpha$ independent terms in (40) also vanish in even dimensions with $n \geq 6$. However, we cannot avoid the divergence in the brane stress tensor in the $\alpha \to 0$ limit. From equation (7) and taking into account that $\mu_{\text{eff}} = 0$ (see equation (14)), we have

$$
\langle T_{ij} \rangle_{\text{brane}} = -2\xi \langle \phi^2 \rangle_{\text{AF}} K_{ij} \delta(r - r_0)
$$

$$
= -\frac{\xi n}{r_0^{n+1}} \left( \frac{2(\alpha)^2 + \alpha^2 F_{\text{AF}}^2(x)}{S_n} \right) h_{ij} \delta(r - r_0),
$$

where we used that $U_{bs}^2 = n/2r_0^n$. Thus, one is forced to take the AF vacuum again. As a result, the bulk stress tensor develops the same singularity at $R = 0$ as before in the $M \neq 0$ case. The globally divergent $(1/\alpha^2)$ term in equation (41) is proportional to the induced metric. One might wonder whether this effect is physical or not, since it could be simply absorbed in the brane tension as before. We think that such a procedure is not justified here because $\alpha$ is a state dependent parameter, and renormalization should be done independently of the choice of the quantum state.

Before we examine the stress tensor in the massless minimal coupling case, let us comment on the relation between the AF vacuum and the Garriga–Kirsten (GK) vacuum. The latter was introduced in [15] for a massless minimally coupled scalar in dS space. It corresponds to the plane wave state with zero momentum in field space. This is intrinsically ill defined, giving a ‘constant infinite’ contribution to the Green function. However, the point is that, since it is an eigenstate of the Hamiltonian, it does not depend on $\chi$ so it is dS invariant. Divergence in the Green function can be accepted in the massless minimally coupled case. Our reasoning is as follows. In this case, the action has the symmetry $\phi \to \phi + \text{constant}$. If we consider it as a ‘gauge’ symmetry, all the observables are to be constructed from derivatives (or differences) of the field. In fact the stress tensor operator (6) only contains derivatives of the type $\partial_\mu \partial'_\nu G^{(1)}(x, x')$ in this case. Then we will find that the constant contributions in the Green function are irrelevant\footnote{Taking this vacuum is analogous to performing a Gupta–Bleuler quantization [29].}. We need a practical way of computing quantities in this vacuum. If we follow the argument by Kirsten and Garriga, it will be given by the limit of the AF vacuum

$$
|0\rangle_{\text{GK}} \equiv \lim_{\alpha \to 0} |0\rangle_{\text{AF}}.
$$

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From equation (41), it follows that the brane term $\langle T_{ij} \rangle_{\text{brane}}$ vanishes in the GK vacuum. The first term in equation (40), which is responsible for the ‘infinite constant’, vanishes for $M = 0$, while the second term proportional to $\alpha^2$ also vanishes in the $\alpha \to 0$ limit. We note that the last two terms in equation (40) independent of $\alpha$ are also zero for $\xi = 0$. Hence, we are left with $\langle T_{\mu\nu} \rangle_{\text{simple}}$, which is manifestly dS invariant, and is finite (aside from the ‘Casimir’ divergences on the brane). Thus, the total stress tensor in the GK vacuum is given by a simple formula presented below in equation (50) with $\xi = 0$. In contrast to the massless minimal coupling limit discussed in the preceding section, here we do not have any ambiguity.

4. Zero bulk mass

In the absence of bulk mass $M$, the Green function and the stress tensor can be obtained explicitly. We will thus discuss this case now.

4.1. Generic mass of the bound state

First we discuss the case where the bound state is not massless. The other case is postponed until section 4.2.

For the bound state, the boundary condition (9) fixes $p_d = i\nu$ and $m_d$ is given by equation (14) with $M = 0$. Its wavefunction is proportional to $r^{-i\nu_d-n/2} = r^{-\mu_{\text{eff}}/H}$, which is constant if $m_d = 0$ (see equation (14)). From the discussion in section 2.1, the bound state is normalizable and non-tachyonic for $n/2 \geq \nu > 0$ (i.e. $0 \leq \mu_{\text{eff}} < (n/2)H$); see figure 2.

The normalized radial KK modes are

$$U_{p}^{KK}(r) = \frac{2}{\pi r_0^{n}(1+(\nu/p)^2)} \left( \cos(p \ln r) + \frac{\nu}{p} \sin(p \ln r) \right),$$

and the renormalized product of mode functions in equation (18) is

$$[U_{p}^{KK}(r)U_{p'}^{KK}(r')]_{\text{ren}} = \frac{1}{2\pi r^{n/2}} \left[ \frac{p + i\nu}{p - i\nu} (rr')^{-i\nu} + \frac{p - i\nu}{p + i\nu} (rr')^{i\nu} \right].$$

Proceeding as in equation (22), we can explicitly perform the integration over the KK modes and in this case, we obtain a simpler expression,

$$D_{(\text{ren})}^{(1)}(x, x') = \frac{1}{(2r_0)^n S(n)} \frac{n/2 + \nu + k + j}{n/2 - \nu + k + j} \frac{(-1)^k \Gamma(n+2k+j)}{\Gamma((n+1)/2 + k)}$$

$$\times \left( \frac{rr'}{r_0^2} \right)^{k+j} \frac{1 - \cos \zeta}{2}^k, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
$$
In order to compute the stress tensor, it is convenient to rewrite equation (44) in a more compact form. For a conformally coupled field, \( \nu \) vanishes, and the Green function actually takes the simple form:

\[
D_{\nu=0}^{(\text{ren})}(1) (x, x') = \frac{1}{n S_{(n+1)}} \left( \frac{1}{r_0^2 + (rr'/r_0)^2 - 2rr' \cos \zeta} \right)^{n/2}.
\] (45)

This expression can be obtained by the method of images. It corresponds to the potential induced by a source of unit charge at \( x' \) together with an image source located at \( r_0' = r_0^2 / r' \) with a charge \( q_i' = (r_0^2 / r_0)^n \). From the form of (44), the Green function for \( \nu \neq 0 \) can be obtained from equation (45) by applying the integral operator

\[
D_{\nu=0}^{(\text{ren})}(1) = \left[ 1 + 4\nu r_0^{-2\nu} \int_{r_0}^{\infty} \frac{d\tilde{r}_0^2}{\tilde{r}_0^2} \right] D_{\nu=0}^{(\text{ren})}(1).
\] (46)

Borrowing intuition from the method of images, equation (46) can be interpreted as the potential induced by the image charge mentioned in the previous paragraph together with a string stretching from \( r_0^2 / r' \) to infinity along the radial direction defined by \( x' \) with a charge line density given by \( \lambda(r) = 4\nu (r/r_0)^{2\nu-1} / r' \). To obtain the stress tensor, we can first compute for the case \( \nu = 0 \) and then apply the same operator as in equation (46). The general result with \( M = 0 \) and \( m_d \neq 0 \) is

\[
(m_d \neq 0) \langle T_{ij} \rangle = (\xi - \xi_c) \frac{(-1)^n (n+1)}{S_{(n+1)} r_0^{n+2}}
\]
\[
\times \left\{ \frac{r_0^{2n+2}}{(r_0^2 - r^2)^{n+1}} + \frac{4\nu}{n + 2 - 2\nu} \right. 2F_1 \left( n + 1, \frac{n}{2} + 1 - \nu; \frac{n}{2} + 2 - \nu; \frac{r^2}{r_0^2} \right) \left. \right\} ,
\]

\[
(m_d \neq 0) \langle T_{ij} \rangle = (\xi - \xi_c) \frac{(-1)^n (n+1)}{S_{(n+1)} r_0^{n+2}}
\]
\[
\times \left\{ \frac{r_0^{2n+2} (r_0^2 + r^2)}{(r_0^2 - r^2)^{n+2}} + \frac{4\nu}{n + 2 - 2\nu} \right. 2F_1 \left( n + 1, \frac{n}{2} + 1 - \nu; \frac{n}{2} + 2 - \nu; \frac{r^2}{r_0^2} \right) \left. \right\}
\]
\[
\times \frac{8\nu}{n + 4 - 2\nu} \frac{r^2}{r_0^2} 2F_1 \left( n + 2, \frac{n}{2} + 2 - \nu; \frac{n}{2} + 3 - \nu; \frac{r^2}{r_0^2} \right) \delta_{ij},
\] (47)

where \( \xi_c = n/4(n+1) \). As an aside, we will note also that for conformal coupling, \( \langle T_{\mu\nu} \rangle_{\text{bulk}} = 0 \) even with a non-zero boundary mass \( \mu \). This is a consequence of conservation and tracelessness of \( T_{\mu\nu} \).

4.2. Massless bound state

We consider now the case \( m_d = 0 \), that is \( \nu = n/2 \). We can proceed as in equation (30) and decompose \( D^{\text{ren}} = D_{\text{simple}} + G_{\text{LC}} + G_{\text{AF}} \), where \( D_{\text{simple}} \) is the contribution from the simple poles \( p = i(n/2 + k) \) with \( k = 1, 2, \ldots \) in equation (28). Thus, it is given by the terms in equation (44) with non-vanishing \( k \) and \( j \). The integral representation analogous to equation (46) is now

\[
D_{\text{simple}}^{(1)} = \left[ 1 + \frac{2n}{r_0^2} \int_{r_0}^{\infty} \frac{d\tilde{r}_0^2}{\tilde{r}_0^2} \right] \left( D_{\nu=0}^{(\text{ren})} - \frac{1}{n S_{(n+1)} r_0^{n+2}} \right).
\] (48)

\footnote{From equation (44), the expression for a Dirichlet scalar \( (\nu \to \infty) \) is the same with opposite sign.}
where we subtract the constant to remove the \( j = k = 0 \) term. The explicit expression in four dimensions was obtained in [21]\(^8\), and we will not reproduce it here. The case of main interest for us is \( n = 3 \) and we find, up to a finite constant, \( D^{(1)}_{\text{simple}}(x, x') = \text{Re} \left[ \frac{1}{8\pi^2} \frac{1}{r_0^3} \frac{1}{\Delta^{3/2}} + \frac{3}{8\pi^2} \frac{1}{r_0^3} \cos \frac{2\zeta}{\sin^2 \zeta} - \frac{\cos 2\zeta - (rr'/r_0^2) \cos \zeta}{\sin^2 \zeta \Delta^{1/2}} \right. \]

\[ - \log \left[ 1 - \frac{rr'}{r_0^2} \cos \zeta + \Delta^{1/2} \right] \right\}, \tag{49} \]

where \( \Delta = 1 + (rr'/r_0^2)^2 - 2(rr'/r_0^2) \cos \zeta \). As mentioned above, in the coincidence limit this contribution is regular except on the brane. We will note also that it grows logarithmically at infinity.

The contribution \( \langle T_{\mu\nu}\rangle_{\text{bulk}}^{\text{simple}} \) is easily found by exploiting the integral representation (48), as before. The only difference between (48) and (46) in the limit \( \nu = n/2 \) (i.e. \( m_d = 0 \)) is a constant, which does not affect \( \langle T_{\mu\nu}\rangle \) in this case. Thus, \( \langle T_{\mu\nu}\rangle_{\text{bulk}}^{\text{simple}} \) reduces to (47) with \( \nu = n/2 \). This gives a simple expression in terms of elementary functions. In four dimensions it is given in [21] (for \( \xi = 0 \)). In five dimensions, we obtain

\[
\begin{align*}
\langle T^\nu_{\ v}\rangle_{\text{simple}}^{\text{bulk}} &= -\frac{9}{32\pi^2} \frac{\xi - \xi_c}{r_0^3} \left\{ \frac{1}{(r_0 - r^2)^4} + \frac{r^4 - 3r_0^2r^2 + 3r_0^4}{(r_0^2 - r^2)^3} \right\}, \\
\langle T^i_{\ j}\rangle_{\text{simple}}^{\text{bulk}} &= -\frac{9}{32\pi^2} \frac{\xi - \xi_c}{r_0^3} \left\{ \frac{r^2 + r_0^2}{(r_0^2 - r^2)^5} + \frac{r^2 - 4r_0^2r^2 + 6r_0^4}{2(r_0^2 - r^2)^4} + \frac{r^4 - 3r_0^2r^2 + 3r_0^4}{(r_0^2 - r^2)^3} \right\} \delta^{ij} \tag{50} \end{align*}
\]

which is dS invariant and regular everywhere except on the brane.

5. Perturbations

We will now discuss the form of the field perturbations, focusing on the bulk massless case with \( n = 3 \), since we have obtained a closed form expression for the Green function and the stress tensor in the preceding section. We begin with the case with generic coupling, and we consider \( \langle \phi^2(x) \rangle \) in the Allen–Follaci vacuum. The massless minimally coupled case in the de Sitter (dS) invariant Garriga–Kirsten vacuum requires a different treatment, which will be discussed in section 5.2.

5.1. Generic coupling

The renormalized expectation value of \( \phi^2(x) \) is given by

\[ \langle \phi^2(x) \rangle_{\text{AF}}^{\text{ren}} = \frac{1}{2} G^{(1)}_{\text{AF}}(x, x), \]

with \( G^{(1)}_{\text{AF}} \) given by (30). From equations (37) and (38), it diverges at \( R = 0 \) because of \( G_{\text{AF}} \). It is also clear that this contribution is bounded at (null) infinity. From equation (32), the \( G_{\text{LC}} \) contribution is regular on the light cone. Equation (49) shows that the KK contribution \( D_{\text{simple}} \) diverges on the brane. In the bulk, both \( D_{\text{simple}} \) and \( G_{\text{LC}} \) grow logarithmically at infinity. However, the growing terms cancel out. Indeed, equation (49)

\(^8\) In the reference, \( Z_2 \) reflection symmetry is not assumed for the field.
the first two terms are UV divergent, we will rather consider

\[ \frac{1}{2} D^{(1)}_{\text{simple}}(x, x) \sim - \frac{3}{16 \pi^2 r_0^2} \log \left| 1 - \frac{r^2}{r_0^2} \right|, \]

in the limit \( x \to \infty \). As for \( G_{\text{LC}} \), since the wavefunction of the bound state in the \( M = 0 \) case is \( \mathcal{U}_{\text{bs}} = n/2r_0^n \), we find

\[ \frac{1}{2} G_{\text{LC}}(x, x) = \frac{3}{8 \pi^2 r_0^3} \text{Re} \left[ \log(T - i\tau) \right] + \mathcal{O}(1), \]

where we used equation (35). It is clear that the logarithmic terms cancel and as a result \( \langle \phi^2(x) \rangle_{\text{AF}}^{\text{ren}} \) is bounded at infinity. This is expected, because the bulk is flat. Intuitively, in four-dimensional dS space, the perturbations grow because when the modes are stretched to a super-horizon scale, they freeze out. Since modes of ever smaller scales are continuously being stretched, they pile up at a constant rate [19]. Since this effect is due to the local curvature of the spacetime, it should not happen in a flat bulk.

Accordingly, on the brane we recover the same behaviour as in de Sitter space. Indeed, restricting (52) on the brane, we obtain \( G^{(1)}_{\text{LC}} \sim \chi \). We have mentioned before that \( D^{(1)}_{\text{simple}}(x, x) \) is UV divergent on the brane. Since this happens because the point splitting regularization used here does not operate on the brane, this object needs UV regularization and renormalization. This can be done in a variety of ways, e.g. with dimensional regularization\(^9\), introducing a finite brane thickness, smearing the field etc. The point is that the renormalized value must be a constant simply because \( D^{(1)}_{\text{simple}}(x, x) \) is dS invariant and is a function of \( x \) only. Thus, equations (51) and (52) imply that on the brane, \( \langle \phi^2 \rangle \) grows in time, as in dS space.

### 5.2. Massless minimal coupling

Now, we will see that essentially the same features arise for a massless minimally coupled scalar in the GK vacuum, in which case everything can be obtained in a dS invariant way [15]. Because of the shift symmetry \( \phi \to \phi + \text{constant} \), \( \langle \phi^2 \rangle \) is not an observable in this case. Still, it is possible to define a ‘shift invariant’ notion for the field perturbations. Following [15], one introduces the correlator

\[ \mathcal{G}(x, y) \equiv \langle [\phi(x) - \phi(y)]^2 \rangle_{\text{GK}}, \]

which can be thought of as the combination \( [G^{(1)}(x, x) + G^{(1)}(y, y) - 2G^{(1)}(x, y)]/2 \). Since the first two terms are UV divergent, we will rather consider

\[ \mathcal{G}^{\text{ren}}(x, y) = \frac{1}{2} [G^{\text{ren}}(1)(x, x) + G^{\text{ren}}(1)(y, y) - 2G^{\text{ren}}(1)(x, y)]. \]

The main point is that all the terms of the form \( f(x) + f(y) \) in \( G^{\text{ren}}(1) \) cancel out in the combination \( \mathcal{G}^{\text{ren}}(x, y) \). All the terms in equations (32) and (37) that do not vanish in the

\(^9\) It is illustrative to consider the massless conformal case. The Green function (45) for \( x \) and \( x' \) on the brane is simply \( D^{(\text{ren})}_{\nu=0} (x, x') \sim 1/(x - x')^{n/2} \). If \( n \) is negative, this is clearly zero in the coincidence limit, and hence the continuation to \( n = 3 \) is zero as well. In the non-conformal case, we can use the integral form (46) to compute \( \langle \phi^2 \rangle \delta(r - r_0) \) in arbitrary dimension. Upon continuation to \( n = 3 \), one obtains a pole \( \sim 1/(n - 3) \) and a non-zero finite part. The pole can be absorbed in the brane tension and \( \langle \phi^2(r_0) \rangle^{\text{ren}} \) is a finite constant.
limit $\alpha \to 0$ are of this form because $U^{\text{bs}}$ is constant in the massless minimally coupled case. Thus, this correlator is well defined for the GK vacuum \cite{15} and we can readily write

$$G^\text{ren}(x, y) = \frac{1}{2} [D^{(1)}_{\text{simple}}(x, x) + D^{(1)}_{\text{simple}}(y, y) - 2D^{(1)}_{\text{simple}}(x, y)], \quad (55)$$

with $D_{\text{simple}}(x, y)$ given by (49). The behaviour of the perturbations in the GK vacuum for $x$ and $y$ distant is thus described by the asymptotic behaviour of $D_{\text{simple}}(x, x)$ and $D_{\text{simple}}(x, y)$. The former is summarized in equation (51). As mentioned before, for finite $x$, $D_{\text{simple}}(x, x)$ is regular everywhere except on the brane (in which case $y_I = y$).

Before describing the asymptotic form of $D_{\text{simple}}(x, y)$, we need to discuss the singularities that it contains. The combination that we called $\Delta$ in equation (49) is

$$\Delta(x, y) = |x - y_I|^2/|y_I|^2,$$

where $|x|^2 = \eta_{\mu\nu}x^\mu x^\nu$ and $y_I^\mu = (r_0^2/|y|^2)y^\mu$ is the image of $x$ (see the comments around equation (46) and figure 5). Note that when $y$ is in one of the Milne regions, $y_I$ is in the other one. Rather than an ‘image charge’, it represents the point where the light cone focuses (see figure 4(b)). For $x \not= y$ and both finite, $D^{(1)}_{\text{simple}}(x, y)$ has singularities in two types of situation. One is when $x$ is on the light cone of $y_I$ (then, $\Delta = 0$). The other case arises when the argument of the logarithmic term in equation (49) vanishes. It is convenient to rewrite this term as

$$-\frac{3}{8\pi^2 r_0^4} \log \left| \frac{(|x - y_I| + |y_I|)^2 - |x|^2}{2|y_I|^2} \right|. \quad (56)$$

The argument vanishes when they are aligned with respect to the origin and $|x|^2 > |y_I|^2$. This condition defines an ‘image string’ stretching from $y_I$ to infinity. Hence, this singularity occurs only when two points are in different Milne regions. This is consistent with the interpretation of equation (46), that the Green function can be constructed with a mirror image and a linear charge distribution over such a string; see figure 4. These two situations correspond to the coincidence of $x$ with the image of $y$ or its ‘light cone’. Therefore both are of UV nature. As an aside, let us comment on the singularities present in the bound state contribution—specifically, in the term $\partial_{p_i}[(p - p_d)G_p^{D-1}]$ present in equation (29). We recall that this term is cancelled by the KK contribution (28). Hence, it plays no role in the Green function. From \cite{14,15} we know that for $x$ and $y$ timelike related and far apart, $\zeta$ is large and pure imaginary, and it behaves like

$$\partial_{p_i}[(p - p_d)G_p^{D-1}] \sim \log |1 - \cos \zeta| = \log \left| \frac{(|x| - |x_I|)^2 - |x - x_I|^2}{2|x||x_I|} \right|. \quad (57)$$

This presents divergences on the light cone whenever $x$ and $y_I$ (or equivalently, $y$) are aligned. This condition reduces to the coincidence limit on the brane, but in the bulk one does not expect singularities to appear at all these points. The above-mentioned cancellation guarantees that these unphysical singularities are not present in $G^{\text{ren}}(x, y)$ once we include the KK contribution.

\footnote{Note that (55) is completely regular on the light cone.}

\footnote{In four dimensions \cite{21}, the logarithmic term is $\log ||x - y||/|y_I||^2$. Thus, this string singularity does not appear, despite the same interpretation of equation (46) holding.}
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Figure 4. The singularities in $D_{\text{simple}}(x, y)$ are located on the light cone emanating from $y_I$, the image point of $y$, and also when $x$ is on the light wedge emanating from the ‘image string’ stretching from $y_I$ to infinity. Here, we plot the image point (smaller filled circle) with the image string attached to it. In (a), $y$ is spacelike, in (b) null, in (c) timelike and in (d) at null infinity.

The asymptotic form of $D_{\text{simple}}(x, y)$ for distant points follows from equation (56) because then, the inverse powers of $\Delta$ in (49) are finite. Taking $y$ fixed and $x \to \infty$ (with $x$ not on the light cone from $y_I$ nor on the image string nor on its light wedge), one obtains

$$D_{\text{simple}}^{(1)}(x, y) \simeq -\frac{3}{8\pi^2 r_0^3} \log |x|. \quad (57)$$

However, since $D_{\text{simple}}(x, x)$ grows twice as fast (see equation (51)), the combination (54) is bounded in the bulk.

We can consider also both $x$ and $y$ approaching null infinity in the bulk. In this case, the image $y_I$ approaches the light cone, as illustrated in figure 4(d). Then the logarithm in equation (56) is $\sim \log(|x - y_I|/|y_I|^2) \simeq \log(|x|/|y|)$; hence the combination $G^{\text{ren}}$ is bounded again.

On the brane, the situation is very different. As mentioned before, $D_{\text{simple}}(x, x)$ is divergent but, when properly renormalized, it is simply a constant because of dS symmetry. Then, $G^{\text{ren}}(x, y)$ behaves like $D_{\text{simple}}(x, y)$ and from equation (56) for large separation and $|x| = |y| = r_0$, one obtains

$$G^{\text{ren}}(x, y) \simeq \frac{3}{4\pi^2 r_0^3} \log |x - y|. \quad (58)$$

Since $|x - y|^2 = 2r_0^2(1 - \cos \zeta)$, this corresponds to linear growth with the invariant dS time interval $\zeta(x, y)$ between $x$ and $y$ [15].

Finally, we will mention that the generalization of equation (53),

$$\left\langle \left[ \frac{\phi(x)}{U_{\text{bs}}(x)} - \frac{\phi(y)}{U_{\text{bs}}(y)} \right]^2 \right\rangle^{\text{ren}},$$

is completely regular in the GK vacuum when $M \neq 0$, and its asymptotic behaviour at infinity parallels that of (53).
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Figure 5. Divergences in $\langle \phi^2 \rangle_{\text{ren}} = G(x, x)_{\text{ren}}$ and $\langle T_{\mu\nu} \rangle_{\text{ren}}$ when the bound state is massless. For the AF vacuum, both the Green function and the stress tensor diverge at $R = 0$, as indicated with a thick dashed line. In addition, if $\xi \neq 0$ the stress tensor diverges on the light cone like $1/U$ or logarithmically in four and six dimensions respectively (thin dashed line). Besides this, we also have the ultraviolet ‘Casimir’ divergence on the brane, represented by the plain thick line. In the Garriga–Kirsten vacuum (for the massless minimally coupled scalar), $\langle T_{\mu\nu} \rangle_{\text{ren}}$ presents only this Casimir-type divergence on the brane.

6. Conclusions

Our main results can be summarized as follows. In analogy with what happens in de Sitter (dS) space [14, 15], scalar fields with a massless bound state in the spectrum do not have a well defined dS invariant vacuum, except for the massless minimally coupled case. (The case of vanishing bulk mass with non-vanishing curvature coupling has a little subtlety, though.) The Green function and the v.e.v. of the energy–momentum tensor diverge everywhere. The simplest alternative from the analogy to dS space is to take the Allen–Follaci vacuum. However, in this vacuum, divergences in the stress tensor are not removed completely within the bulk. Figure 5 illustrates the location of the IR singularities in the Green function and the stress tensor for the AF vacuum. It remains to clarify whether or not it is possible to avoid these singularities by choosing a vacuum for the KK modes other than the dS invariant vacuum.

When the bound state is very light (but not exactly massless) because $M$, $\mu$ and/or $\xi$ are fine tuned according to equation (15), the stress tensor in dS invariant vacuum takes the form of (24). The stress tensor in this case is smooth, but it becomes very large. Hence, even when the bound state mass is not exactly zero, the dS invariant vacuum looks problematic because of large back-reaction. Note that the situation here is different from the usual dS case in two respects. In dS space the large v.e.v. in the stress tensor for the dS invariant vacuum is a constant proportional to the metric. Hence, it might
be absorbed by IR renormalization of the cosmological constant. In our case, the stress tensor given by equation (24) is a non-local expression and cannot be ‘renormalized away’. On the other hand, if one does not want to make any IR renormalization, in the dS case one can take the AF vacuum, and \( \langle T_{\mu\nu} \rangle \) stays regular. In the brane world, we do not know the prescription for removing this large v.e.v. by changing the vacuum state. Choosing non-dS invariant vacuum will lead to not only the above-mentioned divergence in the bulk, but also a new singularity on the light cone, when the bound state mass \( m_d \) is not exactly zero. In this case, the radial function for the bound state behaves like \( r^{-m_d^2/\ell} \) near the light cone at \( r = 0 \). Hence if you single out the contribution to the Green function from the bound state, it is singular and its derivatives diverge at \( r = 0 \). Hence, as long as we restrict the change of quantum state to the bound state, we will not be able to remove the large v.e.v. of the stress tensor without spoiling its regularity.

A light bound state is compatible with a well behaved and not large stress tensor only in a situation ‘close’ to the massless minimally coupled case. More precisely, here we consider the cases where the bulk mass \( M \), the brane mass \( \mu \) and the non-minimal coupling \( \xi \) are all small (see equation (14)). This corresponds to having a light bound state without accidental cancellations. That is, the squared bound state mass \( m_d^2 \) is of the order of the largest among \( M^2 \), \( H\mu \) and \( H^2\xi \), where \( H \) is the Hubble constant on the brane. In this case, the dangerous terms proportional to \( m_d^{-2} \) are always associated with some small factor \( M^2 \), \( \mu \) or \( \xi \), and therefore none of them becomes large.

The case with \( M^2 \approx H\mu \ll H^2\xi \) has a little subtlety. In this case, the large v.e.v. in \( \langle T_{\mu\nu} \rangle \) appears only in the brane part, and it is a constant proportional to the induced metric. Hence, we might be able to consider a model with an appropriate IR renormalization. Only in such a modified model can the stress tensor in the dS invariant vacuum escape the appearance of a large v.e.v.

Application to the bulk inflaton-type models [5, 6] is a part of the motivation of the present study. In these models there must be a light bound state of a bulk scalar field. In order to explain the smallness of the bound state mass it will be natural to assume that it is due to smallness of all the bulk and brane parameters without fine tuning. Therefore we will not have to seriously worry about the back-reaction of the inflaton in the context of bulk inflaton-type models.

We have a few words to add on the massless minimally coupled case. If we consider this case as a limiting situation close to the massless minimally coupled case, the results depend on how we fix the ratio amongst \( M \), \( H\mu \) and \( H^2\xi \), and hence there remains ambiguity. However, this limiting case has the shift symmetry \( \phi \rightarrow \phi + \text{constant} \). If this shift symmetry is one of the symmetries that are to be gauged, there is no ambiguity because the problematic homogeneous mode does not exist in the theory from the beginning. In this set-up, undifferentiated \( \phi \) is not an observable. In fact, \( \langle T_{\mu\nu} \rangle \) automatically does not contain undifferentiated \( \phi \). Hence, \( \langle T_{\mu\nu} \rangle \) is unambiguously defined although the Green function is not well defined. We can compute \( \langle T_{\mu\nu} \rangle \) in this model by applying the idea of the Garriga–Kirsten vacuum (equivalent to a limiting case of AF vacuum), and we confirmed it to be dS invariant and regular as is expected.

We have also discussed the form of the field perturbations \( \langle \phi^2 \rangle \) when the bound state is massless. The main point is that on the brane \( \langle \phi^2 \rangle \) grows linearly with dS time \( \chi \) while in the bulk it is bounded, as expected since the bulk is flat. Aside from this, we derived in closed form the \( \langle T_{\mu\nu} \rangle \) for a generic field with zero bulk mass; see equation (47).
The same discussion applies in the RSII model with little modification, which mainly comes from the fact that the Ricci tensor term in the bulk stress tensor \( \sim \xi (\phi^2) R_{\mu\nu} \) is not zero in the RSII model. Thus, the bulk part of \( \langle T_{\mu\nu} \rangle \) is finite in the limit of the massless bound state only for massless minimal coupling.

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**Appendix A: The massive Green function in de Sitter space**

In this appendix we obtain the form of the Green function for a massive field propagating in \( D - 1 = (n + 1) \)-dimensional de Sitter (dS) space. The metrics for dS space and its Euclidean version are

\[
\text{d}S^2_{(n+1)} = -\text{d}t^2 + \cosh^2 \chi \text{d}x^2 + \sinh^2 \chi \text{d}x^2, \quad \text{d}\Omega^2_{(n)} = \text{d}\chi^2 + \sin^2 \chi \text{d}\Omega^2_{(n)},
\]

where \( \text{d}\Omega^2_{(n)} \) is the metric of a unit \( n \)-dimensional sphere. The Euclidean time is given by \( \chi_E = i \chi + \pi/2 \). The Euclidean version of the Hadamard function in dS space can be found from the equation

\[
\left[ \partial_{\chi_E}^2 + n \cot \chi_E \partial_{\chi_E} - \left( p^2 + \frac{n^2}{4} \right) \right] G_p^{(\text{dS})}(x, x') = -\delta^{(n+1)}(x - x'). \tag{A.2}
\]

From the symmetry, we can choose \( x' \) to be at the pole so that \( G_p^{(\text{dS})} \) depends on \( \chi_E \) only. In terms of \( F = (\sin(\chi_E - \pi))(n-1/2)G_p^{(\text{dS})}(x, x') \), this becomes

\[
\left[ (1 - w^2)\partial^2_w - 2w \partial_w - \left( p^2 + \frac{1}{4} \right) - \frac{(n - 1)^2}{4(1 - w^2)} \right] F = 0, \tag{A.3}
\]

where \( w = \cos(\chi_E - \pi) \), and this is solved to give

\[
F = N \exp\left( \frac{(n - 1)\pi i}{2} \right) \Gamma\left( \frac{n + 1}{2} \right) P_{ip-1/2}^{-(n-1)/2}(\cos(\chi_E - \pi)). \tag{A.4}
\]

Hence, the Green function will be given by the replacement of \( \chi_E \) by the proper distance between the points \( x \) and \( x' \) in dS space, which we call \( \zeta(x, x') \). \( G_p^{(\text{dS})} \) is guaranteed to be regular at \( \zeta \to \pi \). To see the behaviour in the \( \zeta \to 0 \) limit, the alternative expression

\[
G_p^{(\text{dS})}(1) = \frac{N}{(1 - \cos \zeta)^{(n-1)/2}} \left\{ F\left( -ip + \frac{1}{2}, ip + \frac{1}{2}, \frac{1}{2} ; \frac{n}{2} \right) \right. \\
+ \frac{\Gamma(n/2 - ip)\Gamma(n/2 + ip)\Gamma((n - 1)/2)}{\Gamma(1/2 - ip)\Gamma(1/2 + ip)\Gamma((n - 1)/2)} \\
\times \left( \frac{1 - \cos \zeta}{2} \right)^{(n-1)/2} F\left( -ip + \frac{n}{2}, ip + \frac{n}{2}, \frac{n + 1}{2} ; \frac{1 - \cos \zeta}{2} \right) \right\}. \tag{A.5}
\]
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is relevant, and here

\[ \tilde{N} = \frac{\Gamma((n + 1)/2) \Gamma((n - 1)/2)}{\Gamma(n/2 - ip) \Gamma(n/2 + ip)} N. \quad (A.6) \]

In the coincidence limit, the Green function must behave like

\[ G(1)(n + 1) \approx \frac{1}{n - 1} \zeta(n) \left( s - \frac{1}{2} \right), \]

where \( s \) is the area of a \( n \)-dimensional unit sphere. The limiting behaviour is controlled by the first term. Hence we have

\[ \tilde{N} = 2^{-(n-1)/2}/(n - 1) S(n). \]

Appendix B: The massless Green function in de Sitter space

Here, we compute the Green function for the massless scalar in dS space. In the massless limit \( (p \to in/2) \), the de Sitter (dS) invariant Green function diverges because of the contribution from the \( \ell = 0 \) mode, \( \mathcal{Y}_{p00} \). The idea is to construct a modified Green function by replacing this mode by another one that is finite in the limit \( p \to in/2 \). In other words,

\[ G^{(dS)(+)}_{(m=0)} = G^{(+)}_{(\ell>0)} + \tilde{\mathcal{Y}} \tilde{\mathcal{Y}}^*, \]

where \( \tilde{\mathcal{Y}} \) corresponds to the \( \ell = 0 \) mode and

\[ G^{(+)}_{(\ell>0)} = \sum_{\ell>0} \mathcal{Y}_{p\ell m}(\chi) \mathcal{Y}_{p\ell m}(\chi'), \]

where \( \mathcal{Y}_{p\ell m} \) are the positive frequency dS invariant vacuum modes. The latter can be obtained as follows. Since \( G^{(dS)}_{p} \) diverges because of the \( \ell = 0 \) mode, the divergent terms in the Laurent expansions for \( G^{(dS)}_{p} \) and \( \mathcal{Y}_{p00} \mathcal{Y}_{p00}^* \) coincide. We show below that this series contains a simple pole in \( p - in/2 \). Hence, we can write

\[ G^{(+)}_{(\ell>0)} = \lim_{p \to in/2} \left[ G^{(dS)(+)}_{p} - \mathcal{Y}_{p00} \mathcal{Y}_{p00}^* \right] = \partial_p \left[ \left( p - \frac{i n}{2} \right) G^{(dS)(+)}_{p} \right]_{p=in/2} - \partial_p \left[ \left( p - \frac{i n}{2} \right) \mathcal{Y}_{p00} \mathcal{Y}_{p00}^* \right]_{p=in/2}. \quad (B.1) \]

The explicit expression for the first term in the rhs is unnecessary because it is cancelled by the KK contribution (28). The dS invariant vacuum mode for \( \ell = 0 \) with \(-ip < n/2\) is

\[ \mathcal{Y}_{p00} = C_p^{2(n-1)/2} \frac{\Gamma((n + 1)/2)}{\cosh(\chi)^{(n+1)/2}} P_{-ip-1/2}^{-1/2}(i \sinh(\chi)), \quad (B.2) \]

and the normalization constant has been chosen so that \( \lim_{p \to in/2} \mathcal{Y}_{p00}/C_p = 1 \). The Klein–Gordon normalization requires

\[ |C_p|^2 = \frac{\Gamma(ip + n/2) \Gamma(-ip + n/2)}{2^n \Gamma((n + 1)/2)^2 S(n)}. \quad (B.3) \]

Expanding around \( p = in/2 \) we find

\[ |C_p|^2 = \frac{1}{inS(n+1)} \frac{1}{p - in/2} + O[(p - in/2)^0], \quad (B.4) \]
where $S_{n+1}$ is the area of an $(n + 1)$-dimensional sphere of unit radius. The behaviour of $\mathcal{V}_{p00}$ near $p = in/2$ is most easily found using the equation that it solves,

$$\left[\partial^2_\chi + n \tanh \chi \partial_\chi + \left(p^2 + \frac{n^2}{4}\right)\right] \mathcal{V}_{p00} = 0. \quad (B.5)$$

Bearing in mind that the positive frequency function for the dS invariant vacuum is determined by the regularity when the function is continued to the Euclidean region on the side that contains $\chi = -\pi i/2$, one finds

$$\mathcal{V}_{p00}(\chi) = C_p \left[1 - \left(p^2 + \frac{n^2}{4}\right) \int d\chi_1 \frac{1}{\cosh^n \chi_1} \int_{-\pi i/2}^{\pi i/2} d\chi_2 \cosh^n \chi_2 + \mathcal{O}((p - in/2)^2)\right]. \quad (B.6)$$

From this result, the last term in equation (B.1) can be readily evaluated, and equation (32) follows.

**Appendix C: Divergence in the Green function and the $\ell = 0$ mode**

Here, we show that the IR divergence in the Green function (22) or (44) is due to the homogeneous $\ell = 0$ mode of the bound state. Using equations (11), (21), (B.4) and (B.6), the contribution from the $\ell = 0$ mode of the bound state for $p_d$ close to $in/2$ is found to be

$$G_{\ell=0, p_d}^{(1)} = \mathcal{U}^{\text{in}}(r) \mathcal{U}^{\text{in}}(r') \mathcal{V}_{p_d00}(\chi) \mathcal{V}^*_{p_d00}(\chi') + \text{c.c.} \simeq 2 \mathcal{U}^{\text{in}}(r) \mathcal{U}^{\text{in}}(r') \left| C_{pd} \right|^2 \left[1 + \mathcal{O}(p_d - in/2)\right] \approx \frac{1}{S_{n+1}} \frac{1}{p_d - in/2} \frac{I_{n/2}(Mr)I_{n/2}(Mr')/(rr')^{n/2}}{I_{n/2}(Mr_0)\partial_p(\nu_\ell I_{-\nu_\ell}(Mr_0) - Mr_0 I'_{-\nu_\ell}(Mr_0))|_{p=in/2}} + \cdots \quad (C.1)$$

where the dots denote higher order in $p_d - in/2$. In the same limit, $\nu$ is close to $\nu_\ell$. The $j = k = 0$ term in (22) is

$$G_{j=0, k=0}^{(\text{ren}) (1)} = -\frac{n \Gamma(n)}{2^n \Gamma((n + 1)/2)^2 S_{n+1}} \frac{\nu K_{n/2}(Mr_0) - Mr_0 K'_{n/2}(Mr_0) I_{n/2}(Mr) I_{n/2}(Mr')/(rr')^{n/2}}{I_{n/2}(Mr_0) (\nu - \nu_\ell)} \times \frac{I_{n/2}(Mr)I_{n/2}(Mr')}{(rr')^{n/2}} \simeq -\frac{1}{S_{n+1}} \frac{I_{n/2}(Mr)I_{n/2}(Mr')/(rr')^{n/2}}{I_{n/2}^2(Mr_0) (\nu - \nu_\ell)} + \mathcal{O}((\nu - \nu_\ell)^0) \quad (C.2)$$

where in the second and third lines we used equation (15) and the Wronskian relation $K_{n/2}(z)I'_{n/2}(z) - K'_{n/2}(z)I_{n/2}(z) = 1/z$. Using equation (12), it is easy to see that

$$\nu - \nu_\ell = \partial_{pd} p_d \bigg|_{pd = in/2} \left(p_d - \frac{n}{2}\right) + \cdots \quad (C.3)$$
so (C.1) and (C.2) agree in the limit $p_d \to \infty /2$. Note that since in this limit this is the dominant contribution and $m_d \simeq \infty (p_d - \infty /2)$, the total Green function (22) can be rewritten in the simple form

$$G_{(\text{ren})}(x, x') \simeq \sum_{n=1} \frac{U_{0}^{\text{bs}}(r) U_{0}^{\text{bs}}(r')}{r^2 m_d^2} + O(m_d^2),$$

where $U_{0}^{\text{bs}}(r) = N_0 I_{\infty /2}(r)/r^{\infty /2}$ is the wavefunction of the bound state for the exactly massless case.

References

[1] Randall L and Sundrum R, 1999 Phys. Rev. Lett. 83 4690 [SPIRES] [hep-th/9906064]
[2] Randall L and Sundrum R, 1999 Phys. Rev. Lett. 83 3370 [SPIRES] [hep-ph/9905221]
[3] Naylor W and Sasaki M, 2002 Phys. Lett. B 542 289 [SPIRES] [hep-th/0205277]
Elizalde E, Nojiri S, Odintsov S D and Ogushi S, 2003 Phys. Rev. D 67 063515 [SPIRES] [hep-th/0209242]
Moss I G, Naylor W, Santiago-German W and Sasaki M, 2003 Phys. Rev. D 67 125010 [SPIRES] [hep-th/0302143]
Brevik I, Milton K A, Nojiri S and Odintsov S D, 2001 Nucl. Phys. B 599 305 [SPIRES] [hep-th/0010205]
[4] Nojiri S and Odintsov S D, 2003 J. Cosmol. Astropart. Phys. JCAP06(2003)004 [SPIRES] [hep-th/0205221]
[5] Naylor W and Sasaki M, 2002 Phys. Lett. B 542 289 [SPIRES] [hep-th/0205277]
Elizalde E, Nojiri S, Odintsov S D and Ogushi S, 2003 Phys. Rev. D 67 063515 [SPIRES] [hep-th/0209242]
Moss I G, Naylor W, Santiago-German W and Sasaki M, 2003 Phys. Rev. D 67 125010 [SPIRES] [hep-th/0302143]
Brevik I, Milton K A, Nojiri S and Odintsov S D, 2001 Nucl. Phys. B 599 305 [SPIRES] [hep-th/0010205]
[6] Nojiri S and Odintsov S D, 2003 J. Cosmol. Astropart. Phys. JCAP06(2003)004 [SPIRES] [hep-th/0205221]
[7] Naylor W and Sasaki M, 2002 Phys. Lett. B 542 289 [SPIRES] [hep-th/0205277]
Elizalde E, Nojiri S, Odintsov S D and Ogushi S, 2003 Phys. Rev. D 67 063515 [SPIRES] [hep-th/0209242]
Moss I G, Naylor W, Santiago-German W and Sasaki M, 2003 Phys. Rev. D 67 125010 [SPIRES] [hep-th/0302143]
