WORST UNSTABLE POINTS OF A HILBERT SCHEME

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Abstract. In this paper, we describe the worst unstable points of a Hilbert scheme for some special Hilbert polynomials and ambient spaces using Murai’s work on Gotzmann monomial sets. We investigate the geometry of the projective schemes represented by worst unstable Hilbert points and see that in certain cases that they fail to be K-stable or attain maximal regularity.

1. Introduction

Let \( k \) be an algebraically closed field and let \( S = k[x_0, \ldots, x_r] \) be a polynomial ring over \( k \) where \( r \geq 1 \). Let \( P \) be the Hilbert polynomial of \( S/I \) for some homogeneous ideal \( I \) of \( S \). In this paper, \( g_P \) is the Gotzmann number associated to \( P \) defined in [7]. For \( d \geq g_P \) there are closed immersions

\[
\text{Hilb}^P \left( \mathbb{P}^r_k \right) \hookrightarrow \text{Gr}(S_d, Q_P(d)) \hookrightarrow \mathbb{P} \left( \bigwedge Q_P(d) S_d \right)
\]

which are compatible with the canonical linear action of the general linear group \( \text{GL}_{r+1}(k) \) where

\[
Q_P(d) = \begin{pmatrix} r + d \\ r \end{pmatrix} - P(d).
\]

Consider the GIT quotient \( \text{Hilb}^P \left( \mathbb{P}^r_k \right) \sslash_{d} \text{SL}_{r+1}(k) \) with respect to the above Plücker embedding corresponding to \( d \) and another GIT-quotient \( \mathbb{P} \left( \bigwedge Q_P(d) S_d \right) \sslash_{d} \text{SL}_{r+1}(k) \). We have the Hesselink stratification of \( \mathbb{P} \left( \bigwedge Q_P(d) S_d \right) \) described in [10, p. 9]. That is, there is a stratification of the unstable locus

\[
\mathbb{P} \left( \bigwedge^b S_d \right)_{\text{us}} = \coprod_{[\lambda], d'} E^{d,b}_{[\lambda], d'}
\]

for all \( d, b \in \mathbb{N} \). An unstable point \( x \) belongs to a stratum \( E^{d,b}_{[\lambda], d'} \) if the conjugacy class \( [\lambda] \) contains a 1-parameter subgroup that is adapted to \( x \) and the Kempf index \([11]\) of \( x \) is \( d' \). Setting \( b = Q_P(d) \) in \([1]\), we obtain the Hesselink stratification

\[
\text{Hilb}^P \left( \mathbb{P}^r_k \right)_{d} = \coprod_{[\lambda], d'} E^{d,Q_P(d)}_{[\lambda], d'} \cap \text{Hilb}^P \left( \mathbb{P}^r_k \right)
\]

of the Hilbert scheme \( \text{Hilb}^P \left( \mathbb{P}^r_k \right) \) with respect to the Plücker embedding into \( \mathbb{P} \left( \bigwedge Q_P(d) S_d \right) \). Now we are ready to define worst unstable points of a Hilbert scheme \( \text{Hilb}^P \left( \mathbb{P}^r_k \right) \) for an arbitrary choice of \( r \) and \( P \).

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Definition: For \( r, d \in \mathbb{N} \) and a Hilbert polynomial \( P \), let \( \Gamma(\text{SL}_{r+1}(k)) \) be the group of all 1-parameter subgroups of \( \text{SL}_{r+1}(k) \) and

\[
\sigma_{\text{max}}(r, d, P) := \max \{ \sigma \in \mathbb{R}_{>0} \mid \exists \lambda \in \Gamma(\text{SL}_{r+1}(k)) \text{ s.t. } E_{\lambda, \sigma}^{d, P(d)} \cap \text{Hilb}^P(\mathbb{P}^r_k) \neq \emptyset \}.
\]

A point \( x \in \text{Hilb}^P(\mathbb{P}^r_k) \) is a worst unstable point of \( \text{Hilb}^P(\mathbb{P}^r_k) \) with respect to \( d \) if \( x \in E_{\lambda, \sigma_{\text{max}}(r, d, P)}^{d, P(d)} \) for some 1-parameter subgroup \( \lambda \) of \( \text{SL}_{r+1}(k) \). If \( x \in \text{Hilb}^P(\mathbb{P}^r_k) \) is a worst unstable point of \( \text{Hilb}^P(\mathbb{P}^r_k) \) with respect to all but finitely many nonnegative integers, then let us call \( x \) as a worst unstable point of \( \text{Hilb}^P(\mathbb{P}^r_k) \) or a worst unstable Hilbert point, shortly. Also, let

\[
\sigma'_{\text{max}}(r, d, b) := \max \{ \sigma \in \mathbb{R}_{>0} \mid \exists \lambda \in \Gamma(\text{SL}_{r+1}(k)) \text{ s.t. } E_{\lambda, \sigma_{\text{max}}(r, d, b)}^{d, b, \lambda} \cap \text{Gr}(S_{d, b}, \lambda) \neq \emptyset \}
\]

where the Grassmannian \( \text{Gr}(S_{d, b}, \lambda) \) is considered as the closed subscheme of \( \mathbb{P}(\lambda^b S_d) \) via the Plücker embedding. A point \( y \in \text{Gr}(S_{d, b}, \lambda) \) is a worst unstable point of \( \text{Gr}(S_{d, b}, \lambda) \) if \( y \in E_{\lambda, \sigma'_{\text{max}}(r, d, b)}^{d, b, \lambda} \) for some 1-parameter subgroup \( \lambda \) of \( \text{SL}_{r+1}(k) \).

Describing worst unstable Hilbert points is a first step to understand the geometric meaning of the Hesselink stratification above. When \( r = 1 \) and \( P \) is a constant polynomial, a hypersurface defined by a homogeneous polynomial \( f \) of degree \( d \) is unstable if and only if there is a root of multiplicity \( m \geq d/2 \), which is explained in [15, p. 80]. It is natural to ask if a projective scheme represented by a worst unstable point has a unique closed point if \( P \) is a constant polynomial. We will show that this guess is true for arbitrary \( r \geq 1 \) (Theorem 4.3). On the other hand, there is a theorem on a semi-stable bi-canonical curve.

**Theorem 1.1** ([8] Corollary 4.5. and 2.5. on page 924)). Suppose that \( \text{char } k = 0 \). If \( C \subset \mathbb{P}^{3g-4}_k \) is a bi-canonical curve of genus \( g \geq 3 \) and is semi-stable for all but finitely many choices of Plücker embeddings, then \( O_C \) is 2-regular.

The preceding theorem means that there is an upper bound for Castelnuovo-Mumford regularity \( \text{reg } \mathcal{O}_C \leq 2 \) for asymptotically semi-stable bi-canonical curves \( C \subset \mathbb{P}^{3g-4}_{k} \) of genus \( g \geq 3 \). Since \( \text{reg } \mathcal{O}_C \leq \text{reg } \mathcal{I}_C \), it is reasonable to guess that every asymptotically worst unstable point attains maximal Castelnuovo-Mumford regularity, as lex-segment ideals do [2] (2.9). Actually, we will show that this guess is true in the case of plane curves, which does not hold for arbitrary \( r \) and \( P \) (Theorem 5.2).

We first try to describe the worst unstable points of \( \text{Gr}(S_{d, Q_P(d)}) \) for \( d \gg 0 \) using the asymptotic behavior (Lemma 3.3) of some functions associated to the Hilbert polynomial \( P \) (Lemma 3.3). Also, we prove that these points have constant Hilbert polynomial. That is, a worst unstable point of \( \text{Gr}(S_{d, Q_P(d)}) \) belongs to \( \text{Hilb}^P(\mathbb{P}^r_k) \) if and only if \( P \) is constant (Theorem 4.2). After that, we use [16, Proposition 8] to describe worst unstable Hilbert points when

\[
P(d) = \binom{r + d}{r} - \binom{r + d - \gamma}{r} + p
\]

for some \( p, \gamma \in \mathbb{N} \) (Theorem 4.3, Theorem 5.4). Note that (2) is always true when \( r = 2 \). As we described above, we need to choose a Plücker embedding to define the Hesselink stratification of a Hilbert scheme \( \text{Hilb}^P(\mathbb{P}^r_k) \). However, we will show that the set of all worst unstable Hilbert points with respect to \( d \) remains unchanged for all but finitely many choices of \( d \) when (2) is true (Theorem 5.5). Furthermore, these worst unstable Hilbert points are Borel-fixed, so that it is possible to compute the
Castelnuovo-Mumford regularity of an arbitrary worst unstable point of $\text{Hilb}^P(\mathbb{P}^r_k)$, which only depends on $r$ and $P$ (Theorem 6.2). Under the condition (2), we will see that the Hilbert polynomial $P$ satisfies one of the following properties

- $r = 2$ and $\gamma \neq 0$
- $p \leq 2$

if and only if a worst unstable Hilbert point of $\text{Hilb}^P(\mathbb{P}^r_k)$ attains the maximal regularity $g_P$ in the last paragraph of this paper.

By [3, Section 2.3], asymptotic GIT stability and $K$-stability are closely related. The worst unstable Hilbert points described in this paper are worst points in the sense of asymptotic Hilbert-Mumford stability. The value of the equation on [17, line 27, p. 11] for a fixed adapted 1-parameter subgroup $\lambda$ of a worst unstable Hilbert point $x$ of $\text{Hilb}^P(\mathbb{P}^r_k)$ has to be maximal at $x$ for all $n \gg 0$. The dimension of the fiber of a maximal value under the function which maps each Hilbert point to corresponding $F_0(\lambda)$ [17, p. 11] can be large enough so that we can expect that the projective scheme represented by an arbitrary worst unstable Hilbert point may have the largest $F_0(\lambda)$ and $F_1(\lambda)$ (which is a Donaldson-Futaki invariant defined in [17, p. 12]) so that such a scheme may not be $K$-stable. We will compute some Donaldson-Futaki invariants of the projective schemes represented by the worst unstable points described in this paper and their associated 1-parameter subgroup using the asymptotic behavior of the numerical functions we have found. In this way, we will see that the projective scheme represented by a worst unstable Hilbert point is not $K$-stable if the Hilbert polynomial is in the form (2) with the assumption $\gamma \neq 1$ in Theorem 6.1.

2. Preliminaries and Details in Computation

2.1. Castelnuovo-Mumford regularity and Gotzmann theorems. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_k$ is $m$-regular if

$$H^i(\mathbb{P}^r_k, \mathcal{F}(m-i)) = 0$$

for all $i > 0$. Let $\text{reg}(\mathcal{F}) := \min\{m \in \mathbb{Z}|\mathcal{F} \text{ is } m \text{ regular}\}$. For an arbitrary graded $S$-module $M$, the regularity of $M$, denoted $\text{reg } M$ is defined to be the least integer $m$ satisfying

$$\text{Ext}^i(M, S)_j = 0 \quad \text{for all } i + j < -m.$$ 

Equivalently, $\text{reg } M = \max_{j \geq 0}(b_j - j)$ where $b_j$ is the maximal degree of a minimal generator of $F_i$ for a minimal free resolution $\{F_i\}_{i=0}^\infty$ of $M$ [4, 20.5]. There is an ideal sheaf $\mathcal{I}$ on $\text{Proj } S \cong \mathbb{P}^r_k$ associated to $I$ when $I$ is a graded ideal of $S$. There is an equality

$$\text{reg } I = \text{reg } \mathcal{I}$$

for every saturated graded ideal $I$ of $S$, as stated in [5, Chapter 4]. The Gotzmann number $g_P$ (which is equal to $m(Q_P)$ under the notation of [2, p. 62]) is defined by the equation

$$g_P := \max \{\text{reg}(I)|I \text{ is a saturated graded ideal of } S, \text{ the Hilbert polynomial of } S/I \text{ is } P\}.$$ 

Gotzmann’s regularity theorem [2, (2.9)] implies that lex-segment ideals whose Hilbert polynomial is $Q_P$ attain maximal regularity $g_P$. Actually, every Hilbert
polynomial \( Q_P \) of a graded ideal has a Macaulay representation [7 (2.5)] of the form
\[
Q_P(d) = \sum_{i=0}^{n} \binom{r-i+d-a_i}{r-i}
\]
for an integer \( 0 \leq n < r \) and \( n+2 \) integers \( \{a_i\}_{i=1}^{n} \) satisfying
\[
1 = a_1 \leq a_0 \leq \ldots \leq a_n.
\]
Let \( M_d \) be the set of all monomials in \( S_d \) and fix the lexicographic ordering \( \leq_{\text{lex}} \) on \( M_d \) with respect to the term order \( x_0 > x_1 > \ldots > x_r \). Let \( I \) be the ideal generated by monomials greater than or equal to
\[
\mu = x_0^a_0 x_1^a_1 \ldots x_n^a_n - 1
\]
with respect to \( \leq_{\text{lex}} \). Then, \( I_d \) is spanned by monomials greater than or equal to \( \mu x_r^{a_r} \) with respect to \( \leq_{\text{lex}} \) for all \( d \geq a_n \). That is,
\[
I_d = \bigoplus_{i=0}^{n} \text{span}\{m \in M_d | \deg x_i m \geq a_i - a_i - 1 + 1 \text{ and } \deg x_j m = a_j - a_j - 1 \text{ for all } j < i\}
\]
for all \( d \geq g_P \). Thus we can directly compute
\[
Q_P(d) = \text{dim}_k I_d = \sum_{i=0}^{n} \binom{r-i+d-a_i}{r-i}
\]
for all \( d \geq g_P \). Equation (3) implies that \( I \) has a minimal generator containing \( \mu \) of degree \( a_n \). Also, \( I \) is saturated. Therefore, \( g_P \geq \text{reg} I \geq a_n \). Furthermore, \( g_P \leq a_n \) by [7 (2.9)] so that \( a_n = g_P \).

**Definition** Let \( b \in \mathbb{N} \). For \( W \in \text{Gr}(S_d, b) \), let \( SW \) denote the ideal of \( S \) generated by \( W \). If \( C \subset S_d \), we let \( \text{span} C \) denote the \( k \)-subspace of \( S_d \) spanned by \( C \). Define \( U_i(n) = \{v \in M_d | \deg x_j v \leq \deg x_j n \text{ for all } j \in \{0, 1, \ldots, r\} \setminus \{i\} \} \) for a monomial \( n \in S_d \) and \( i \in \{0, 1, \ldots, r\} \). A monomial ideal \( J \) is said to be Borel-fixed if \( \frac{\partial}{\partial x_i} m \) is not a monomial or is in \( J \), for all monomials \( m \in J \) and \( 0 \leq i < r \).

### 2.2. State polytopes and geometric invariant theory

Let \( T_r \) denote the maximal torus of \( \text{GL}_r(k) \), which consists of all diagonal matrices in \( \text{GL}_r(k) \). For any affine algebraic group \( G \), let \( X(G) \) (resp. \( \Gamma(G) \)) be the group of characters (resp. 1-parameter subgroups) of \( G \). Consider the canonical \( \text{GL}_{r+1}(k) \)-actions on a Plücker coordinate \( \mathbb{P} \left( \wedge^b S_d \right) \) and its affine cone \( \wedge^b S_d \), which are induced by the canonical \( \text{GL}_{r+1}(k) \)-action on \( S_1 \). These actions induce a \( T_{r+1} \)-action on \( \wedge^b S_d \), which has the character decomposition [13 Proposition 4.14]
\[
\wedge^b S_d = \bigoplus_{\chi \in X(T_{r+1})} \left( \wedge^b S_d \right)_{\chi}
\]
where
\[
\left( \wedge^b S_d \right)_{\chi} = \{v \in \wedge^b S_d | t.v = \chi(t)v \text{ for all } t \in T_{r+1}\}.
\]
Let us also fix a basis \( \{ \lambda_i \}_{i=0}^r \) of \( X(T_{r+1}) \) where \( \lambda_i(t) = t_{ii} \) for all \( 0 \leq i \leq r \) and \( t \in T_{r+1} \). Then, we can easily see that \( (\wedge^b S_d)_\chi \) is generated by

\[
\left\{ \wedge^b \{ m_i \mid m_i \in M_d \text{ for all } i \text{ and } \prod_{j=0}^r m_i = \prod_{j=0}^r x_j^d \} \right\}
\]

if \( \chi = \prod_{i=0}^r \chi_j^d_i \). It follows that \( \sum_{j=0}^r d_i = db \) if \( (\wedge^b S_d)_\chi \neq 0 \). There is a basis \( \{ \lambda_i \}_{i=0}^r \) of \( \Gamma(T_{r+1}) \) which is the dual basis of \( \{ \chi_i \}_{i=0}^r \) with respect to the pairing \( \langle , \rangle : X(T_{r+1}) \times \Gamma(T_{r+1}) \to \mathbb{Z} \) which satisfies

\[
\chi(\lambda(t)) = \langle \chi, \lambda \rangle
\]

for all \( t \in k \setminus \{ 0 \} \). We can identify \( X(T_{r+1}) \cong \mathbb{Z}^{r+1} \cong \Gamma(T_{r+1}) \) under this choice of basis. Let \( ||\cdot|| \) denote the Euclidean norm on \( X(T_{r+1}) \cong \mathbb{Z}^{r+1} \cong \Gamma(T_{r+1}) \) with respect to the basis \( \{ \lambda_i \}_{i=0}^r \) of \( X(T_{r+1}) \). There is a norm \( ||\cdot||_R \) on \( X(T_{r+1})_R = X(T_{r+1}) \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^{r+1} \cong \Gamma(T_{r+1}) \otimes_\mathbb{Z} \mathbb{R} = \Gamma(T_{r+1})_R \) induced by \( ||\cdot|| \). There is also a pairing \( \langle , \rangle : X(T_{r+1})_R \times \Gamma(T_{r+1})_R \to \mathbb{R} \) obtained from \( \langle , \rangle \) by the base change to \( \mathbb{R} \).

**Definition** An arbitrary \( v \in \wedge^b S_d \) has a decomposition

\[
v = \sum_{\chi \in X(T_{r+1})} v_\chi
\]

with \( v_\chi \in (\wedge^b S_d)_\chi \). The state polytope \( \Delta_{[v]} \) of the line \( [v] \in \mathbb{P}(\wedge^b S_d) \) through the origin and \( v \) is the set

\[
\Xi_{[v]} = \{ \chi \in X(T_{r+1}) | v_\chi \neq 0 \}.
\]

The state polytope \( \Delta_{[v]} \) of \( [v] \) is the convex hull of \( \Xi_{[v]} \otimes \mathbb{Z} 1 \) in \( X(T_{r+1})_R \).

For example, \( \Delta_{[v]} \) is a point if \( v \) is a wedge of monomials. Let \( \xi_{d,b} = \frac{db}{r+1} \mathbb{1} \) where \( \mathbb{1} \) is the all-1 vector of \( X(T_{r+1})_R \) with respect to the basis \( \{ \lambda_i \}_{i=0}^r \). If \( L \) is the line bundle on \( \text{Hilb}^P(\mathbb{P}^r_k) \) defined by the Plücker embedding corresponding to \( d \), \( L \) admits a canonical linearization by the canonical \( \text{SL}_{r+1}(k) \)-action and \( L^{\otimes (r+1)} \) also can be linearized by a similar way [15, p. 33]. We can see that the induced \( \text{GL}_{r+1}(k) \)-linearization on \( L^{\otimes (r+1)} \) twisted by the \( dQ_P(d) \)'th power of the determinant function on \( \text{GL}_{r+1}(k) \) (whose restriction on \( T_{r+1} \) is the character corresponding to \( (r+1)\xi_{d,Q_P(d)} \)) and \( \text{SL}_{r+1}(k) \)-linearization on \( L \) are equivalent in the sense of GIT; they defines the same stable locus, semi-stable locus, GIT-quotients and same numerical weight functions on \( \Gamma(\text{SL}_{r+1}(k)) \) up to constant [15, Definition 2.2, p. 49]. Note that \( \xi_{d,b} \) is the arithmetic mean of the set \( \{ \chi \in X(T_{r+1}) | (\wedge^b S_d)_\chi \neq 0 \} \otimes \mathbb{Z} 1 \).

For each \( v \in \wedge^b S_d \) satisfying \( \xi_{d,b} \notin \Delta_{[v]} \), there is a unique \( \lambda_{[v]} \in \Gamma(T_{r+1}) \) which satisfies the following properties.

- There is \( u \in \mathbb{R}^+ \) such that \( \lambda_{[v]} \otimes \mathbb{Z} u + \xi_{d,b} \in \Delta_{[v]} \) via the isomorphism \( X(T_{r+1}) \cong \Gamma(T_{r+1}) \) we defined above and \( ||\lambda_{[v]} \otimes \mathbb{Z} u ||_R \) is equal to the distance from \( \xi_{d,b} \) to \( \Delta_{[v]} \).
- There is no \( m \in \mathbb{N} \setminus \{ 0, 1 \} \) and \( \lambda \in \Gamma(T_{r+1}) \) such that \( m\lambda = \lambda_{[v]} \).

The image of such a 1-parameter subgroup \( \lambda_{[v]} \) is contained in \( \text{SL}_{r+1}(k) \) because the sum of all coefficients of \( \lambda_{[v]} \) is 0.

**Definition** Let \( \Delta_{[v]}[0] \) be the distance from \( \xi_{d,b} \) to \( \Delta_{[v]} \). The set of state polytopes \( \{ \Delta_{g,[v]} \mid g \in \text{GL}_{r+1}(k) \} \) determines the state of \([v]\) from the viewpoint of geometric invariant theory.
Theorem 2.1. For an arbitrary \( v \in \wedge^b S_d \), there is \( g \in \text{GL}_{r+1} \) such that

\[
|\Delta_{g, v}| = \max_{h \in \text{GL}_{r+1}(k)} |\Delta_{h, v}|. 
\]

For such \( g \), \([ v ] \in E^{d,b}_{|\Delta_{g, v}|, |\Delta_{g, v}|} \) if \( |\Delta_{g, v}| > 0 \). Otherwise, \([ v ] \) is semi-stable.

Proof. A generalized version of this theorem can be found in [2] 1.1.4. and 1.1.5]. See also [11], [9] and [13].

2.3. Computation of worst unstable points. For an arbitrary \( v \in \wedge^b S_d \), let \( |\Delta_{[v]}| \) denote the distance from the origin of \( X(T_{r+1})_k \) to \( \Delta_{[v]} \). Since \( \Delta_{[v]} \subseteq \Delta_{d,b} := \{ w \in X(T_{r+1})_k | (w, 1 \otimes 2) = dB \} \) and \( \xi_{d,b} \) is the point on \( H_{d,b} \) closest to the origin, \( |\Delta_{[v]}|^2 = |\Delta_{[v]}|^2 + \|\xi_{d,b}\|^2 = |\Delta_{[v]}|^2 + \frac{D}{2} \). Therefore, it is enough to consider the optimization problem on \( |\Delta_{[v]}| \) to describe worst unstable points.

Definition Define

\[
R(d, b, S) = \{ W \in \text{Gr}(S_d, b) | W \text{ is generated by monomials}, Z^b_d(S) = \{ W \in R(d, b, S) | |\Delta_W| \geq |\Delta_{W'}| \text{ for all } W' \in R(d, b, S) \} 
\]

and

\[
X^b_d(S) = \left\{ W \in R(d, Q_p(d), S) \cap \text{Hilb}_p^{P_k} \mid |\Delta_W| \geq |\Delta_{W'}| \text{ for all } W' \in R(d, Q_p(d), S) \cap \text{Hilb}_p^{P_k} \right\}. 
\]

We can see that both \( Z^b_d(S) \) (resp. \( X^b_d(S) \)) and \( \text{GL}_{r+1}(k).Z^b_d(S) \) (resp. \( \text{GL}_{r+1}(k).X^b_d(S) \)) are closed subschemes of \( \text{Gr}(S_d, b) \) (resp. \( \text{Hilb}_p^{P_k} \)) under some scheme structure using the argument explained in [9] (6.1(c), (6.2)(b)). Let \( \log W \in X(T_{r+1})_k \) denote the lattice points satisfying \( \Delta_W = \{ \log W \} \) for an arbitrary \( W \in R(d, b, S) \).

For all \( d, b \in \mathbb{N} \) and \( W \in R(d, b, S) \), let \( N(W) \) denote the monomial basis of \( W \). Let \( W^* \in \text{Gr}(S_d, b') \) be the \( k \)-subspace of \( S_d \) generated by \( M_d \setminus N(W) \) where \( b' = (r+1) - b \). If \( W \in R(d, b, S) \) then \( \log W \) records the exponent of a monomial \( \prod_{n \in N(W)} n \).

Now we are ready to state a theorem on a construction of the set of worst unstable points from from our \( Z^b_d(S) \) and \( X^b_d(S) \).

Theorem 2.2. Fix a Hilbert scheme \( \text{Hilb}_p^{P_k} \) and the Plücker embedding corresponding to an integer \( d \geq g_p \). Every worst unstable point of \( \text{Gr}(S_d, b) \) (resp. \( \text{Hilb}_p^{P_k} \)) is in the orbit of some element in \( Z^b_d(S) \) (resp. \( X^b_d(S) \)). In particular, the set of all worst unstable points of \( \text{Gr}(S_d, b) \) (resp. \( \text{Hilb}_p^{P_k} \)) with respect to \( d \) is a closed subscheme of \( \text{Gr}(S_d, b) \) (resp. \( \text{Hilb}_p^{P_k} \)) under some scheme structure.

Proof. It is clear by the construction of \( Z^b_d(S) \) and Theorem 2.1 that an arbitrary point in \( \text{GL}_{r+1}(k).Z^b_d(S) \) is a worst unstable point of \( \text{Gr}(S_d, b) \). Conversely, any worst unstable point of \( \text{Gr}(S_d, b) \) is in the orbit of \( W \in \text{Gr}(S_d, b) \) such that the cardinality of \( \Delta_W \) is 1. Otherwise, we can find \( v(g) \in R(d, b, S) \) satisfying \( |\Delta_{v(g)}| > |\Delta_{W}| \) for an arbitrary choice of \( g \in \text{GL}_{r+1}(k) \) and such a conclusion contradicts the maximality of \( \max_{g \in \text{GL}_{r+1}(k)} |\Delta_{g,W}| \). Fixing any total order of the monomial
basis of $S_d$, $W$ is the wedge of all row vectors in a matrix in echelon form. It directly follows that $W \in R(d, b, S)$ if the cardinality of $\Delta_W$ is 1. Therefore, $GL_{r+1}(k)Z_d^b(S)$ is equal to the set of worst unstable points of $Gr(S_d, b)$ as sets. Let $I$ be a worst unstable Hilbert point of $\text{Hilb}^P(\mathbb{P}^r_d)$ with respect to $d$. We may consider $I$ as a saturated graded ideal of $S$, whose Hilbert polynomial is $Q_P$ as an $S$-module. Without loss of generality, assume that $|\Delta_I| = \max_{g \in GL_{r+1}(k)}|\Delta_{g, I}|$. If the cardinality of $\Delta_I$ is not equal to 1, then we can choose a vertex $x$ of $\Delta_I$, satisfying $\|x\|_\mathbb{R} > |\Delta_I|$. Using the proof of \[1\] Theorem 3.1., we can show that there is a monomial order $\prec$ on $S_d$, such that $\Delta_{\text{in}_\prec I} = \{x\}$. The Hilbert polynomial of $\text{in}_\prec I$ is equal to $Q_P$ and $|\Delta_{\text{in}_\prec I}| = \|x\|_\mathbb{R} > |\Delta_I|$ so that $I$ is not a worst unstable point of $\text{Hilb}^P(\mathbb{P}^r_d)$ with respect to $d$ by Theorem \[2.1\], which contradicts the assumption on $I$, so that the cardinality of $\Delta_I$ is equal to 1. The remaining claims easily follow from the construction of $X^P_d(S)$ and Theorem \[2.1\].

From now on, we will concentrate on the computation of $Z_d^b(S)$ and $X^P_d(S)$. Suppose $W \in R(d, Q_P(d), S)$ and let $W = (c_0, c_1, \ldots, c_r)$. Then $\sum_{i=0}^r c_i = dQ_P(d)$ from the definition. Also, we can derive $|\Delta_W|^2 = \sum_{i=0}^r c_i^2$ and

$$\prod_{n \in N_W} n = \prod_{i=0}^r c_i^e_i.$$ 

The function

$$f(c_0, c_1, \ldots, c_r) = \sum_{i=0}^r c_i^2$$

defined on the set $\{c_i | i = 0 \in (\mathbb{R}^+)^{r+1}\}$ has a unique minimum at $dQ_P(d+1)$ by the convexity of $f$ and has a maximum at $(dQ_P(d), 0, \ldots, 0)$. Therefore, it is natural to guess that $W$ maximizing $\max_{0 \leq i \leq r} c_i$ also maximizes $|\Delta_W|$. It is straightforward to check that all lex-segment ideals maximize $\max_{0 \leq i \leq r} c_i$. However, it is not true that the orbit of a lex-segment ideal is the set of all worst unstable Hilbert points.

**Example** Let $S = k[x, y, z]$, $P(d) = 3$ and $d = 3$ so that $r = 2$ and $Q_P(d) = 7$. It is true that the orbit of $I = \langle x^3, x^2y, x^2z \rangle$ is the set of all worst unstable points of the Grassmannian $Gr(S_3, 3) \subset \mathbb{P}(S_3 \wedge S_3 \wedge S_3)$ because such a choice of $I$ maximizes the first coordinate of log $I$. Indeed,

$$(a + 1)^2 + (b + 1)^2 + (c + 1)^2 \leq (a + b + c + 1)^2 + 1 + 1 = 51$$

for all $a, b, c \geq 0$ satisfying $a + b + c = 6$ and equality holds if and only if two among $a, b$ and $c$ are zero. Therefore, $I$ is an element of the set

$$R(3, 3, S) \setminus (R(3, 3, k[x, y]) \cup R(3, 3, k[y, z]) \cup R(3, 3, k[z, x])),$$

which maximizes the function $|\Delta|$. Also, we can check that

$$(a + 1)^2 + (b + 3)^2 \leq (a + b + 1)^2 + 9 < 51$$

for all $a, b \geq 0$ satisfying $a + b = 5$; this means that the orbit of $I$ is the unique worst unstable orbit of $Gr(S_3, 3)$ by Theorem \[2.2\]. By the symmetry stated in Lemma \[3.1\] if

$$J = \langle xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3 \rangle$$
then the orbit of $J_3$ is the set of all worst unstable points of $\text{Gr}(S_3, 7)$. Moreover, $J$ has the minimal growth at degree 3. Thus, $J_3 \in \text{Hilb}^3(P^2_k)$ and it is a worst unstable Hilbert point.

In fact, we will generalize this observation; our computations so far can be generalized if we choose the Plücker embedding corresponding to an integer $d \gg 0$.

3. Worst unstable points of a Grassmannian containing a Hilbert scheme

As we discussed in the previous section, there is a relation between worst unstable points of $\text{Gr}(S_d, P(d))$ and worst unstable points of $\text{Gr}(S_d, Q_P(d))$.

**Lemma 3.1.** For $W \in R(d, P(d), S)$ we have that $W \in Z_{d}^{P(d)}(S)$ if and only if $W^* \in Z_{d}^{Q_P(d)}(S)$.

**Proof.** For any $V \in R(d, P(d), S)$, we have

$$\log V + \log V^* = \frac{d}{r + 1} \binom{r + d}{r} (1 \otimes Z_1).$$

Also, we have the equalities

$$\max_{V \in R(d, P(d), S)} \left\| \log V - \frac{dP(d)}{r + 1} (1 \otimes Z_1) \right\|_R^2 = \max_{V \in R(d, P(d), S)} \left\| \log V^* - \frac{dQ_P(d)}{r + 1} (1 \otimes Z_1) \right\|_R^2$$

and

$$\left\| \log W^* - \frac{dQ_P(d)}{r + 1} (1 \otimes Z_1) \right\|_R = \left\| \log W - \frac{dP(d)}{r + 1} (1 \otimes Z_1) \right\|_R$$

using (4). Then $W \in Z_d^{P(d)}(S)$ if and only if

$$\log W - \frac{dP(d)}{r + 1} (1 \otimes Z_1) \quad \text{if and only if} \quad \log W^* - \frac{dQ_P(d)}{r + 1} (1 \otimes Z_1)$$

because $\Delta_V \subseteq \{ v \in X(T)_R | (v, 1 \otimes Z_1)_R = dP(d) \}$ for all $V \in \text{Gr}(S_d, P(d))$. Similarly, $W^* \in Z_{d}^{Q_P(d)}(S)$ if and only if

$$\log W^* - \frac{dQ_P(d)}{r + 1} (1 \otimes Z_1) \quad \text{if and only if} \quad \log V - \frac{dP(d)}{r + 1} (1 \otimes Z_1).$$

This completes the proof. \qed

We will examine the asymptotic behavior of some functions associated to a Hilbert polynomial $P$ in Lemma 3.3. This observation (i.e., Lemma 3.3) makes it possible to describe worst unstable points of $\text{Gr}(S_d, Q_P(d))$ for $d \gg 0$. If $P = 0$ or $Q_P = 0$ then our problem becomes a trivial one. Let’s assume that $P \neq 0 \neq Q_P$.

**Definition** Let $\mu(t, d)$ be the $t$th greatest monomial of $M_d$ with respect to $\leq_{\text{lex}}$. Let $L(t, d, S)$ be the subspace of $S_d$ generated by generated by $\{ \mu(i, d) | 1 \leq i \leq t \}$ and $A(t, d, S)$ be the subspace of $S_d$ generated by $\{ \mu(i, d) | t + 1 \leq i \leq \binom{r + d}{d} \}$ We may
also consider $L(t, d, S)$ and $A(t, d, S)$ as points in the Grassmannian using Plücker embedding:

$$L(t, d, S) = \left[ \bigwedge_{i=1}^{t} \mu(i, d) \right] \in \text{Gr}(S_d, t)$$

and

$$A(t, d, S) = \left[ \bigwedge_{i=t+1}^{(r+d)} \mu(i, d) \right] \in \text{Gr} \left( S_d, \binom{r+d}{r} - t \right).$$

Consider $\text{Hilb}^P(\mathbb{P}^r)$ for a Hilbert polynomial $P \in \mathbb{Q}[t]$ where $t$ is a variable. We can define some numerical functions and constants corresponding to $P$.

**Definition** There is a function $\delta : \mathbb{N} \to \mathbb{N}$ such that

$$\binom{r + \delta(d) - 1}{r} < Q_P(d) \leq \binom{r + \delta(d)}{r}$$

for all $d \geq g_P$ because $\mathbb{Z}$ is well-ordered. The defining inequality of $\delta$ is equivalent to

$$x_0^{d-\delta(d)} x_r^{\delta(d)} \leq \text{lex} \mu(Q_P(d), d) < \text{lex} x_0^{d-\delta(d)+1} x_r^{\delta(d)-1}.$$

That is, $d - \delta(d) = \deg_{x_0} \mu(Q_P(d), d) \geq 0$ for all $d \geq g_P$. Also, $x_r \mu(Q_P(d), d) = \mu(Q_P(d+1), d+1)$ since the Hilbert polynomial of the ideal generated by $L(Q_P(d), d, S)$ is $Q_P$ for all $d \geq g_P$, as we can see in [7, (2.1), (2.5), (2.9)]. Therefore,

$$d + 1 - \delta(d + 1) = \deg_{x_0} \mu(Q_P(d + 1), d + 1)$$

$$= \deg_{x_0} x_r \mu(Q_P(d), d)$$

$$= \deg_{x_0} \mu(Q_P(d), d)$$

$$= d - \delta(d)$$

for all $d \geq g_P$ so that there’s an integer $\gamma \geq 0$ such that $\delta(d) = d - \gamma$ for all $d \geq g_P$.

There is a function $l : \mathbb{N} \to \mathbb{Z}$ such that

$$\dim_k L \left( \binom{r + l(d) - 1}{r}, d, S \right) = \sum_{0 \leq i \leq l(d) - 1} \binom{r + i - 1}{r - 1}$$

$$\leq P(d)$$

$$\leq \sum_{0 \leq i \leq l(d)} \binom{r + i - 1}{r - 1}$$

$$= \dim_k L \left( \binom{r + l(d)}{r}, d, S \right)$$

for $d \geq g_P$ because $\mathbb{Z}$ is well-ordered. Define

$$e(d) = \left\lfloor \log \left( \log \left( P(d), d, S \right), \sum_{i=1}^{r} \lambda_i \otimes \mathbb{Z} 1 \right) \right\rfloor_{\mathbb{R}}$$

$$= \frac{\left| \{ n \in N(L(P(d), d, S)) | \deg_{x_0 n} = d - l(d) \} \right|}{l(d)}$$

$$+ \sum_{0 \leq i \leq l(d) - 1} \frac{\left| \{ n \in N(L(P(d), d, S)) | \deg_{x_0 n} = d - i \} \right|}{i}$$

$$= \left[ P(d) - \sum_{0 \leq i \leq l(d) - 1} \binom{r + i - 1}{r - 1} \right] l(d) + \sum_{0 \leq i \leq l(d) - 1} \binom{r + i - 1}{r - 1} i.$$
That is,
\[(6) \quad \left\langle \log L(P(d), d, S), \lambda_0 \otimes \mathbb{Z} 1 \right\rangle_R = dP(d) - e(d).\]
Indeed, \(L(P(d), d, S)\) maximizes the function
\[
\max_{0 \leq i \leq r} (\log \lambda_i \otimes \mathbb{Z} 1)_R : R(d, P(d), S) \to \mathbb{N}.
\]

**Lemma 3.2.** \(dP(d) - e(d) \geq (\log W, \lambda_0 \otimes \mathbb{Z} 1)_R\) for all \(W \in R(d, P(d), S)\). In the case of equality, we have \(L((r+1)(d-1), d, S) \subset W\) and \(W \subset L((r+1)(d), d, S)\).

**Proof.** Fix \(W \in R(d, P(d), S)\). If \(L((r+1)(d-1), d, S) \not\subset W\) or \(W \not\subset L((r+1)(d), d, S)\), then we can find monomials \(n \in M_d \setminus N(W)\) and \(m \in N(W)\) such that \(\langle \log W', \lambda_0 \otimes \mathbb{Z} 1 \rangle_R > \langle \log W, \lambda_0 \otimes \mathbb{Z} 1 \rangle_R\) by the construction of \(l\), where \(W'\) is generated by \(N(W) \cup \{n\} \setminus \{m\}\).

If \(L((r+1)(d-1), d, S) \subset W\) and \(W \subset L((r+1)(d), d, S)\), then we have \(dP(d) - e(d) = \langle \log W, \lambda_0 \otimes \mathbb{Z} 1 \rangle_R\) by the construction of \(e\). \(\square\)

Before we state another lemma about asymptotic behavior of numerical functions defined above in this section, let’s define the first discriminant function \(\Phi\).

**Definition** Let \(\Phi : \mathbb{N} \to \mathbb{N}\) be the function satisfying
\[
\Phi(d) = d^2[P(d)]^2 - 4dP(d)e(d) + \frac{2(r + 1)}{r}[e(d)]^2
\]
for all \(d \in \mathbb{N}\).

Actually, \(\Phi\) is the discriminant of a quadratic inequality, which will be mentioned in Lemma 3.4.

**Lemma 3.3.** There is an integer \(D_P \geq g_P\) corresponding to the Hilbert polynomial \(P\) such that every integer \(d \geq D_P\) satisfies following properties.
\[
(7) \quad \Phi(d) > 0
\]
\[
(8) \quad \frac{dP(d) - \sqrt{\Phi(d)}}{2dP(d)} < \frac{1}{r + 1}
\]
\[
(9) \quad \left| dP(d) - 2e(d) - \sqrt{\Phi(d)} \right| < 2e(d)
\]
\[
(10) \quad \frac{e(d)}{dP(d)} \leq \frac{l(d)}{d} < \frac{1}{8}
\]
Furthermore, if \(P\) is a constant polynomial, then there’s \(D_P \in \mathbb{N}\) such that \(d \geq D_P\) implies (7), (8), (10) and
\[
(11) \quad \left| dP(d) - 2e(d) - \sqrt{\Phi(d)} \right| < 2.
\]

**Proof.** We claim that
\[
\lim_{d \to \infty} \frac{l(d)}{d} = 0
\]
and this implies
\[
\lim_{d \to \infty} \frac{e(d)}{dP(d)} = 0.
\]
These properties imply what we want to prove. For example, the left-hand sides of (8) and (11) tend to zero as $d \to \infty$ and the left-hand side of (7) tends to $\infty$ as $d \to \infty$ if these assumptions are true. By the definition of $l$, the value $l(d)$ is the smallest integer satisfying

$$P(d) \leq \sum_{i=0}^{l(d)} \binom{r+i-1}{r-1} = \binom{r + l(d)}{r}.$$ 

Let $\frac{\eta - 1}{r} < \eta < 1$. Note that

$$\lim_{d \to \infty} \sum_{i=\delta(d)}^{d} \frac{(r+i-1)}{(r+i+1)} = \lim_{d \to \infty} \sum_{i=0}^{\gamma} \frac{(r+d+\gamma+i-1)}{(r+i+1)} = 0.$$

The preceding formula means that

$$P(d) < \binom{r + d}{r} - \binom{r + \delta(d) - 1}{r}$$

$$= \sum_{i=\delta(d)}^{d} \binom{r+i-1}{r-1}$$

$$< \binom{r + [d^\eta]}{r}$$

for sufficiently large $d$. Therefore, $l(d) \leq [d^\eta]$ for $d \gg 0$. From the definition of $e$ we see that

$$e(d) = \sum_{0 \leq i \leq l(d)-1} \binom{r+i-1}{r-1} i + l(d) \left[ P(d) - \sum_{0 \leq i \leq l(d)-1} \binom{r+i-1}{r-1} \right]$$

$$\leq \sum_{0 \leq i \leq l(d)-1} \binom{r+i-1}{r-1} l(d) + l(d) \left[ P(d) - \sum_{0 \leq i \leq l(d)-1} \binom{r+i-1}{r-1} \right]$$

$$= l(d) P(d)$$

so that

$$0 \leq \lim_{d \to \infty} \frac{e(d)}{d P(d)} \leq \lim_{d \to \infty} \frac{l(d)}{d} \leq \lim_{d \to \infty} \frac{[d^\eta]}{d} = 0,$$

as desired. \hfill \Box

We can define the number $D_P$ corresponding to the Hilbert polynomial $P$ using Lemma 3.3.

**Definition** For each Hilbert polynomial $P$, let $D_P$ be the minimal integer satisfying the conditions in Lemma 3.3.

Now we are ready to state some properties of $Z_{d}^{P(d)}(S)$ for an arbitrary constant Hilbert polynomial $P$.

**Lemma 3.4.** Let $P$ be a constant Hilbert polynomial and suppose that $d \geq D_P$. Let $W \in Z_{d}^{P(d)}(S)$. If $\log W = (c_0, \ldots, c_r)$, then $\max_{1 \leq i \leq r} c_i = d P(d) - e(d)$. There is a permutation matrix $q \in GL_{r+1}(k)$ such that $S(q, W^*)$ is a Borel-fixed monomial ideal.
Proof. We can prove that $c_i \leq dP(d) - e(d)$ for all $0 \leq i \leq r$ using Lemma 3.2.

Let us apply an action on $W$ by a permutation matrix to assume that $c_i \geq c_{i+1}$ for every $0 \leq i \leq r - 1$, if necessary. We claim that if $c_0 \leq dP(d) - e(d) - 1$ then $W \notin Z_d^P(d)(S)$. If $c_0 = dP(d) - e(d)$ then $|\Delta_W|^2$ is at least $(dP(d) - e(d))^2 + \frac{1}{r}[e(d)]^2$ by the convexity of the square sum function defined on a simplex defined by the equation $\sum_{0 \leq i \leq r} c_i = e(d)$. Therefore it suffices to show that

$$\sum_{i=0}^{r} c_i^2 \leq c_0^2 + (dP(d) - c_0)^2 = 2c_0^2 - 2dP(d)c_0 + d^2[P(d)]^2$$

is obtained by the convexity of the square sum function defined on a simplex defined by the equation $\sum_{0 \leq i \leq r} c_i = e(d)$. Thus, the first statement is true. There is a permutation matrix $P$ such that $\sum_{0 \leq i \leq r} c_i = dP(d)$ because of the pigeonhole principle. The first inequality is equivalent to

$$dP(d) - \sqrt{\Phi(d)} < c_0 < \frac{dP(d) + \sqrt{\Phi(d)}}{2}$$

by Lemma 3.3. The equation 5 means that

$$\frac{dP(d) - \sqrt{\Phi(d)}}{2} < c_0 < \frac{dP(d) + \sqrt{\Phi(d)}}{2}.$$

Since $r \geq 1$, we get

$$\frac{dP(d) + \sqrt{\Phi(d)}}{2} \leq \frac{dP(d) + \sqrt{d^2[P(d)]^2 - 4dP(e(d)) + 4[e(d)]^2}}{2} = \frac{dP(d) - e(d)}{r + 1}.$$

Using (7), we see that the difference between the both sides of the preceding inequality is sufficiently small; that is,

$$\left|dP(d) - e(d) - \frac{dP(d) + \sqrt{\Phi(d)}}{2}\right| < 1$$

so that

$$dP(d) - e(d) - 1 < \frac{dP(d) + \sqrt{\Phi(d)}}{2} \leq dP(d) - e(d).$$

Thus, the first statement is true. There is a permutation matrix $q \in \text{GL}_{r+1}(k)$ such that $\log qW^* = (c'_0, \ldots, c'_r)$ satisfies $c'_i \geq c'_{i+1}$ for all $0 \leq i \leq r - 1$. Thus, $S(qW^*)$ is not Borel-fixed, then there are $n_1 \in N(qW^*)$ and $n_2 \in S_d \setminus N(qW^*)$ such that $x_i n_1 = x_j n_2$ for some $0 \leq i < j \leq r$. If $W' \in \text{Gr}(S_d, Q_P(d))$ is generated by $N(W) \cup \{n_2\} \setminus \{n_1\}$, then

$$|\Delta_{W'}|^2 - |\Delta_W|^2 = (c'_i + 1)^2 + (c'_j - 1)^2 - (c'_i)^2 - (c'_j)^2 > 0,$$

which contradicts the maximality of $|\Delta_W|$ by Lemma 3.1. 

There are some properties satisfied by an arbitrary worst unstable point of $\text{Gr}(S_d, P(d))$ for an arbitrary choice of $P \in \mathbb{Q}[t]$, which are weaker than the properties stated in Lemma 3.3.

Lemma 3.5. If $d \geq D_P$ then every $W \in Z_d^P(d)(S)$ satisfies the following properties.

- $\log W = (c_0, \ldots, c_r)$ satisfies $dP(d) - 2e(d) < \max_{0 \leq i \leq r} c_i \leq dP(d) - e(d)$. 

• $W$ is a $P(d)$-dimensional subspace of $S_d$ generated by monomials and $x^{\frac{d}{2}}_\beta$ divides any monomial in $W$ where $c_\beta = \max_{0 \leq i \leq r} c_i$.

• For any monomial $n \in W$, $U_\beta(n) \subset W$.

• There is a permutation matrix $q \in \mathrm{GL}_{r+1}(k)$ such that $S(q.W^*)$ is a Borel-fixed ideal.

**Proof.** It is possible to modify the proof of Lemma 3.4 in order to prove that the first bullet is true for any $d \geq D_P$; indeed, $\frac{dP(d)}{r+1} \leq c_0$ if we assume that $c_0 = \max_{0 \leq i \leq r} c_i$ after using an action by a permutation matrix. $c_0 \leq dP(d) - 2e(d)$ implies that

$$\frac{dP(d) - \sqrt{\Phi(d)}}{2} < \frac{dP(d)}{r+1} \leq c_0 \quad (\because (8), (9))$$

$$\leq dP(d) - 2e(d) < \frac{dP(d) + \sqrt{\Phi(d)}}{2} \quad (\because (10), (11))$$

so that

$$\sum_{i=0}^{r} c_i^2 \leq c_0^2 + (dP(d) - c_0)^2$$

$$= 2c_0^2 - 2dP(d)c_0 + d^2[P(d)]^2$$

$$< (dP(d) - e(d))^2 + \frac{1}{r}[e(d)]^2$$

$$\leq \|\log L(P(d), d, S)\|_2^2,$$

which contradicts the maximality of $\|\log W\|_R$. If there is a $W \in Z^{P(d)}_d(S)$ which does not satisfy the third bullet, then there’s a monomial $n \in W$ satisfying $U_\beta(n) \not\subset W$. Choose a monomial $m \in U_\beta(n) \setminus W$. For the $P(d)$-dimensional subspace $W'$ of $S_d$ spanned by $N(W) \cup \{m\} \setminus \{n\}$, there is a sequence $\{a_i\}_{i=0}^{r}$ of integers satisfying

$$\left\langle \log W', \lambda_i \otimes \mathbb{Z} 1 \right\rangle_{R} = c_i + a_i \quad \text{for all } 0 \leq i \leq r$$

so that $a_\beta > 0$, $a_i \leq 0$ for all $i \neq \beta$ and $\sum_{i=0}^{r} a_i = 0$ by the definition of $U_\beta(n)$. From these hypotheses, we can derive an inequality

$$|\Delta_{W'}|^2 - |\Delta_{W}|^2 = \sum_{i=0}^{r} 2c_i a_i + a_i^2 \geq \sum_{i=0}^{r} 2c_\beta a_i + a_i^2 \geq a_\beta^2 > 0$$

by the maximality of $c_\beta$, which leads us to a contradiction. Therefore, the third bullet holds for $d \geq D_P$. If there is $W \in Z^{P(d)}_d(S)$ which does not satisfy the second bullet, there are monomials $m' \in M_d \setminus W$ and $n' \in W$ such that $x^{d-\ell(d)}_\beta \mid m'$ and $x^{d-j}_\beta \mid n'$. Consider $W''$ spanned by $N(W) \cup \{m'\} \setminus \{n'\}$. There is a sequence $\{b_i\}_{i=0}^{r}$ of integers satisfying

$$\left\langle \log W'', \lambda_i \otimes \mathbb{Z} 1 \right\rangle_{R} = c_i + b_i \quad \text{for all } 0 \leq i \leq r.$$
By the defining property of $m'$ and (10), $b_β > \left\lfloor \frac{d}{3} \right\rfloor - l(d) > \frac{d}{3}$ and $b_i \geq -d$ for all $i \neq β$. Also, $\sum_{0 \leq i \leq r, i \neq β} c_i < 2e(d)$ by the first bullet we have proved. We see that

$$|\Delta_{W''}|^2 - |\Delta_W|^2 = \sum_{i=0}^{r} 2c_i b_i + b_i^2$$

$$\geq 2c_β b_β + \sum_{0 \leq i \leq r, i \neq β} 2c_i b_i$$

$$> 2(dP(d) - 2e(d)) \cdot \frac{d}{3} - 4de(d)$$

$$= 2(dP(d) - 8e(d)) \cdot \frac{d}{3} > 0$$

by (10); this leads us to another contradiction so that the second bullet holds for all $d \geq D_P$. We can prove the fourth bullet as we did in the proof of Lemma 3.3. □

4. Worst Unstable Hilbert Points for a Constant Hilbert Polynomial

We would like to investigate whether Lemma 3.4 and Lemma 3.5 are useful for finding worst unstable points (that is, whether, we have $Z_{g,P}(d)(S) = X_d^P(S)$ for all $d \geq g_P$). In this section, we compute the Hilbert polynomial of an arbitrary $W \in Z_d^P(S)$ to answer this question.

**Lemma 4.1.** Suppose $W \in R(d, Q_P(d), S)$. Assume that $W^* \in R(d, P(d), S)$ satisfies all bullets in Lemma 3.4 and $d \geq D_P$. Let $J$ be the graded ideal of $S$ generated by $W$. Then $\text{dim}_k(S/\text{SW})_t = P(d)$ for all $t \geq d$. In particular, If $W^* \in Z_d^P(S)$ then the Hilbert polynomial of $S/\text{SW}$ is constant, whose value at an arbitrary integer is equal to $P(d)$.

**Proof.** The set $L = N(W^*)$ satisfies $1 \leq \min_{m \in L} \text{deg}_x m$ by the second bullet of Lemma 3.4 and

$$L = \bigcup_{m \in L} U_0(m) = \bigcup_{m \in L} \{n | n \in k[x_1, \ldots, x_r], n \text{ divides } m \} \cap M_d$$

by the third bullet of Lemma 3.5. By definition, $J = S(M_d \setminus N(W^*)) = S(M_d \setminus L)$. We claim that the Hilbert polynomial of $S/J$ is constant. We have the inclusion

$$M_{d+1} \setminus J_{d+1} = M_{d+1} \setminus (S_1(M_d \setminus L)) = M_{d+1} \setminus M_1 \left( \bigcap_{m \in L} M_{d} \setminus U_0(m) \right) \supset M_{d+1} \setminus \left( \bigcap_{m \in L} M_1(M_{d} \setminus U_0(m)) \right).$$

It’s easy to check that for every $m \in M_d$, $M_1(M_{d} \setminus U_0(m)) = M_{d+1} \setminus U_0(x_0 m)$ if $x_0 | m$. Note that $x_0 | m$ for every $m \in L$; therefore,

$$M_{d+1} \setminus J_{d+1} \supset \bigcup_{m \in L} U_0(x_0 m).$$

If there is a monomial $n \in M_{d+1}$ such that $n \notin \bigcup_{m \in L} U_0(x_0 m)$ and $n \notin J_{d+1}$, then $\text{deg}_{x_0} n = 0$; otherwise, $\frac{n}{x_0} \notin \bigcup_{m \in L} U_0(m)$ so that $n/x_0 \notin J_d$, which implies that $n \in J_{d+1}$. Such a conclusion contradicts our assumption $n \notin J_{d+1}$. Note that $\text{deg}_{x_0=1} m \leq d - 1$ for every $m \in L$ by the second bullet of Lemma 3.5 so that
Lemma 4.1 and Lemma 3.5 imply that $d$ corresponding to $k$ and thus $\dim_k S/J_i = d$ corresponding to $W \in \text{Theorem } 4.2$. Using the same argument, we can show that $m$ divides $n/x_0$ such that $P$ is equal to $n/x$ for some $m \in L \cap M_{d+1}$. Note that this set and $n/x$ leads us to another contradiction against the assumption $n \notin J_{d+1}$. Therefore, 

$$M_{d+1} \setminus J_{d+1} = \bigcup_{m \in L} U_0(x_m).$$

Note that this set and $L$ have the same cardinality. We can dehomogenize both sides of the preceding equation by the first variable $x_0$ to compare their cardinality; that is,

$$\left| \bigcup_{m \in L} U_1(x_1m) \right|$$

$$= \left| \bigcup_{m \in L} \{nx_0^{d+1-\deg n} | n \in k[x_1, \ldots, x_r], n \text{ divides } m \, \mid x_0=1 \} \cap M_{d+1} \right|$$

$$= \left| \{nx_0^{d+1-\deg n} | n \in k[x_1, \ldots, x_r], n \text{ divides } m \, \mid x_0=1 \, \text{for some } m \in L \} \cap M_{d+1} \right|$$

$$= \left| \{nx_0^{d-\deg n} | n \in k[x_1, \ldots, x_r], n \text{ divides } m \, \mid x_0=1 \, \text{for some } m \in L \} \cap M_d \right|$$

$$= |L|.$$ 

Thus 

$$\dim_k (S/J)_{d+1} = \dim_k (S/J)_d = P(d).$$

Using the same argument, we can show that 

$$M_{d+i+1} \setminus J_{d+i+1} = \bigcup_{m \in L} U_0(x_0^{i+1}m)$$

if 

$$M_{d+i} \setminus J_{d+i} = \bigcup_{m \in L} U_0(x_0^i m)$$

and thus $\dim_k (S/J)_{d+i+1} = \dim_k (S/J)_{d+i}$ for all $i \in \mathbb{N}$. By induction, $\dim_k (S/J)_t = P(d)$ for all $t \geq d$. \hfill \Box

Now we explain when Lemma 3.5 is useful to find worst unstable Hilbert points.

**Theorem 4.2.** Suppose that $d \geq D_P$. Let us fix a Plücker embedding of $\text{Hilb}^P(\mathbb{P}_k^r)$ corresponding to $d$. The following statements are equivalent.

i) A worst unstable point of $\text{Gr}(Q_P(d), S_d)$ is in $\text{Hilb}^P(\mathbb{P}_k^r)$.

ii) Every worst unstable point of $\text{Gr}(Q_P(d), S_d)$ is in $\text{Hilb}^P(\mathbb{P}_k^r)$.

iii) $P$ is a constant polynomial.

**Proof.** i) $\implies$ ii) is trivial. Suppose $P$ is a constant polynomial and $W \in Z_{d}^{Q_P(d)}(S)$. Lemma 4.1 and Lemma 3.5 imply that $W \in \text{Hilb}^P(\mathbb{P}_k^r)$ under the Plücker embedding corresponding to $d \geq D_P \geq g_P$. Theorem 2.2 implies iii) $\implies$ ii). If i) is true, there is $W \in Z_{d}^{Q_P(d)}(S)$ such that $W \in \text{Hilb}^P(\mathbb{P}_k^r)$. Then the Hilbert polynomial of $S/\text{SW}$ is equal to $P$ by the definition. By Lemma 3.4, $W^* \in Z_{d}^{Q_P(d)}(S)$ so that the Hilbert polynomial of $S/\text{SW}$ is constant so that $P$ is a constant polynomial by Lemma 4.1. As a result, i) $\implies$ iii) is true. \hfill \Box

Consequently, we cannot use both Lemma 3.5 and Lemma 3.4 to describe worst unstable points of $\text{Hilb}^P(\mathbb{P}_k^r)$ when $P$ is non-constant while Lemma 3.4 is still useful when $P$ is constant.
Definition Let $S' = k[x_1, \ldots, x_r]$. Define $\rho : \mathbb{N} \to \mathbb{N}$

$$\rho(d) = \binom{r + l(d)}{r} - P(d) = Q_P(d) + \binom{r + l(d)}{r} - \binom{r + d}{r}$$

for all $d \geq g_P$. Note that $\rho$ is constant for $d \geq g_P$ if $P$ is constant.

Theorem 4.3. Suppose that $P$ is a constant polynomial and $d \geq D_P$. For an arbitrary worst unstable point $W$ of $\text{Hilb}^P(P^r_k)$ with respect to $d$, there are $g \in \text{GL}_{r+1}(k)$ and $W' \in Z^{p(d)}_{l(d)}(S')$ such that

$$g.W = x_0^{d-l(d)}W' + A\left(\binom{r + l(d)}{r}, d, S\right)$$

and $S(g.W)$ is a Borel-fixed ideal. In particular, the algebraic equation defined by the graded ideal $SW$ has a unique solution in $P^r_k$.

Proof. By Theorem 2.2, there is $g_0 \in \text{GL}_{r+1}(k)$ such that $g_0.W \in Z^{Q_P(d)}_{d}(S)$. By Lemma 3.4 and Theorem 4.2, there is a permutation matrix $q \in \text{GL}_{r+1}(k)$ such that $qg_{0}.W$ is Borel-fixed and $\langle \log(qg_{0}.W), \lambda_{0} \otimes \mathbb{Z}1 \rangle_{R} = dP(d) - e(d)$. Let $g = qg_{0}$.

By Lemma 3.2

$$L\left(\binom{r + l(d) - 1}{r}, d, S\right) \subset (g.W)^* \subset L\left(\binom{r + l(d)}{r}, d, S\right).$$

Applying $\ast$ to each subspace of $S_d$ in the preceding equation, we have

$$A\left(\binom{r + l(d) - 1}{r}, d, S\right) \supset g.W \supset A\left(\binom{r + l(d)}{r}, d, S\right).$$

By the construction of $A(t, d, S)$, $\deg_{x_0} m = d - l(d)$ for all $m \in N(g.W) \setminus N\left(A\left(\binom{r + l(d)}{r}, d, S\right)\right)$. Therefore, there is $W' \in R(l(d), \rho(d), S')$ such that

$$g.W = x_0^{d-l(d)}W' + A\left(\binom{r + l(d)}{r}, d, S\right).$$

We can prove that $W' \in Z^{p(d)}_{l(d)}(S)$ using the maximality of $\| \log g.W \|_{\mathbb{R}}$: indeed, we can express each component $\langle \log g.W, \chi_{i} \otimes \mathbb{Z}1 \rangle_{\mathbb{R}}$ of $\log g.W$ in terms of $\langle \log W', \chi_{i} \otimes \mathbb{Z} \rangle_{\mathbb{R}}$. 
We see that \( x \) has a unique closed point. Let’s prove the last sentence of the theorem. Lemma 3.5 implies that \( r \) sentence is true when \( A \)(minimal growth condition polynomials, however, we have to make use of the Theorem 5.1.

Hilbert points described in [16, (1) on page 844].

...satisfying \( U \) and \( c \) point of Hilb

every monomial in the Hilbert polynomial \( P \) such that \( h \cdot x \)

As mentioned in the introduction, Theorem 4.3 means that every zero dimensional \( 0 \).

\( □ \)

As mentioned in the introduction, Theorem 4.3 means that every zero dimensional projective scheme which is represented by a worst unstable Hilbert point has a unique closed point. If we set \( r = 2 \), we get the following consequence.

**Corollary 4.4.** Let \( c \) be a positive integer, which we view as a constant Hilbert polynomial. For every worst unstable point \( x \) of \( \text{Hilb}^r(P_2^k) \), there is \( h \in \text{GL}_3(k) \) such that \( h \cdot x \) represents a projective scheme corresponding to the ideal generated by \( A(c,g,c,S) \). In particular, the solution of the algebraic equation defined by the ideal of \( S \) generated by \( A(c,g,c,S) \) is unique so that the projective scheme represented by \( x \) has a unique closed point.

**Proof.** We see that \( h \cdot x = A(c,d,S) \) for some \( h \in \text{GL}_3(k) \) if \( x \) is a worst unstable point of \( \text{Hilb}^r(P_2^k) \) with respect to \( d \geq D_P \) by Theorem 1.3. We also see that \( A(c,g,c,S) \) and \( A(c,d,S) \) define the same projective scheme by [7] (1.2)(iii). \( □ \)

5. AN APPLICATION OF MURAI’S RESULT

We described the form of almost all monomials in a point \( W \in Z_d^{Q_P(d)}(S) \) when the Hilbert polynomial \( P \) is constant in Theorem 1.3. For non-constant Hilbert polynomials, however, we have to make use of the minimal growth condition of Hilbert points described in [16] (1) on page 844.

**Theorem 5.1.** Suppose \( M \subset M_d \) generates an ideal \( I \). Let \( P \) and \( Q_P \) be the Hilbert polynomials of \( S/I \) and \( I \), respectively. Assume \( d \geq g_P \). If \( [\wedge Q_P(d)I_d] \in \text{Hilb}^P(P_2^k) \) and \((r + \delta(d) - 1) < Q_P(d) \leq (r + \delta(d)) \) for some \( 0 \leq \delta(d) \leq d \) then there is a monomial \( m \) of degree \( d - \delta(d) \) such that \( m \) divides any monomial in \( M \) and there is \( 0 \leq i \leq r \) satisfying \( U_i(n) \subset M_d \setminus M \) for all \( n \in M_d \setminus M \). The converse of the preceding sentence is true when \( r = 2 \).
The preceding theorem is \cite{16} Proposition 8. Although Lemma \ref{lemma33} cannot be used to describe worst unstable points for general \( P \) by Theorem \ref{thm42}, such a fact means that describing the shape of a general Gotzmann set of monomials \cite{16} helps us to find general worst unstable Hilbert points. A property of an arbitrary Hilbert point which is a wedge of monomials has been described in Theorem \ref{thm51}. In this section, we modify Theorem \ref{thm43} to describe general worst unstable Hilbert points for some non-constant \( P \).

**Definition** Let us introduce some notation. Put \( p(d) = \left( r + \delta(d) \right) - Q_P(d) \) and \( \alpha(d) = P(d) - p(d) = \left( r + \delta(d) \right) - \left( r + \delta(d) \right) \). The polynomial \( Q_P \) is the Hilbert polynomial of the ideal \( S(L(P_P(d), d, S)) \) for all \( d \geq g_P \), as explained in \cite{7}. By the construction of \( \delta \), there is a relation \( x_0^{d - \delta(d)} x_r^{\delta(d)} \leq_{\text{lex}} \mu(P_P(d), d) <_{\text{lex}} x_0^{d - \delta(d) + 1} x_r^{\delta(d) - 1} \). The monomial \( \nu = Q_P(d, d) |_{x_0 = 1} \) is the \( Q_P(d) - \left( r + \delta(d) - 1 \right) \)th largest monomial in \( S_r(d) \), with respect to the lexicographic order of monomials in \( S' \) induced by the inclusion \( S' \subset S \). Thus, for any \( t \geq 0 \),

\[
\dim_k \left( \left( Q_P(d) - \left( r + \delta(d) - 1 \right) \right) \delta(d), S' \right)_{\delta(d + t)} = \left| \{ m \in S'_{\delta(d + t)} \mid m \text{ is a monomial and } m \geq_{\text{lex}} x_r^{\nu} \} \right|
\]

\[
= \left| \{ m \in M_{d+1} x_r^{\nu} \mu(P_P(d), d) \leq_{\text{lex}} m <_{\text{lex}} x_0^{d - \delta(d) + 1} x_r^{\delta(d + t) - 1} \} \right|
\]

\[
= Q_P(d + t) - \left( r + \delta(d + t) - 1 \right)
\]

\[
= \left( r + \delta(d + t) - 1 \right) - p(d + t).
\]

Therefore, \( \left( r + \delta(d + t) - 1 \right) - p(d + t) \) is the Hilbert polynomial of the ideal \( J \) of \( S' \) generated by \( L \left( Q_P(d) - \left( r + \delta(d) - 1 \right) \right) S' \). Since \( p(t + \gamma) \in Q[t] \) is a Hilbert polynomial, we can define \( l \) and \( e \) for \( p(t + \gamma) \) in \( S' \) as we did for \( P \) in \( S \) before. Let us denote these functions by \( l' \) and \( e' \) respectively. That is, \( l'(d) \) is the integer satisfying

\[
\sum_{0 \leq i \leq l'(d) - 1} \left( r + i - 2 \right) < p(d + \gamma) \leq \sum_{0 \leq i \leq l'(d)} \left( r + i - 2 \right)
\]

and

\[
e'(d) = \left[ p(d + \gamma) - \sum_{0 \leq i \leq l'(d) - 1} \left( r + i - 2 \right) \right] l'(d) + \sum_{0 \leq i \leq l'(d) - 1} \left( r + i - 2 \right) i.
\]

On the other hand, define

\[
\epsilon(d) = \sum_{i=\delta(d)+1}^{d} \left( r + i - 1 \right) i.
\]

We can easily see that

\[
\log A \left( \left( r + \delta(d) \right)_r, d, S' \right) = \left( d \alpha(d) - \epsilon(d), \frac{\epsilon(d)}{r}, \ldots, \frac{\epsilon(d)}{r} \right).
\]

If \( \gamma = 0 \) then \( P = p \) and \( \alpha = 0 = \epsilon \). Also, worst unstable points of \( \text{Gr}(Q_P(d), S_d) \) satisfy the conclusion of Theorem \ref{thm51} as we have seen in Lemma \ref{lemma33}. Let’s assume
that $\gamma \geq 1$. Define the second discriminant $\Omega : \mathbb{N} \to \mathbb{Z}$ and the sum $C : \mathbb{N} \to \mathbb{Z}$ as follows:

$\Omega(d) = \left( p(d)\gamma + d\alpha(d) - \epsilon(d) - \frac{\epsilon(d)}{r} - p(d)\delta(d) \right)^2 - 8p(d)\delta(d)e'(d) + 8[e'(d)]^2$

and

$C(d) = -p(d)\gamma - d\alpha(d) + \epsilon(d) + \frac{\epsilon(d)}{r} + p(d)\delta(d)$.

In Theorem 5.4, we will see that $\Omega$ and $C$ are actually a discriminant and the sum of the roots of some quadratic equation, respectively. These functions have the following asymptotic behavior.

**Lemma 5.2.** If $\gamma \neq 0$ then

(12) $\lim_{d \to \infty} \frac{p(d)\delta(d)}{C(d) + \sqrt{\Omega(d)}} = 0,$

(13) $\lim_{d \to \infty} \frac{p(d)}{P(d)} = 0$

and

(14) $\lim_{d \to \infty} \frac{e'(d)}{dp(d + \gamma)} = 0.$

Furthermore, if $p$ is a constant,

(15) $\lim_{d \to \infty} \frac{C(d) - \sqrt{\Omega(d)}}{2} = 0^+.$

**Proof.** Note that (14) follows from the proof of lemma 3.3. It suffices to show that

$$\lim_{d \to \infty} \frac{\alpha(d)}{P(d)} = 1$$

and

$$\lim_{d \to \infty} \frac{\epsilon(d)}{dP(d)} = 1.$$  

Recall that there is a lex-segment ideal $J$ generated by $J_{gP}$, whose Hilbert polynomial is $Q_P$. Suppose that $\mu$ is the last monomial of $J_{gP}$ with respect to the lexicographic order we fixed. We can write $\mu$ as follows:

$$\mu = x_0^{a_0-1}x_1^{a_1-a_0} \ldots x_n^{a_n-a_{n-1}+1}$$

where

$$1 = a_0 \leq a_0 \leq a_1 \leq \ldots \leq a_n = gP$$

and $n < r$. From the definition of $\gamma$, we can prove that $x_0^\gamma x_r^{d-\gamma} \leq_{\text{lex}} x_0^{d-gP} \mu <_{\text{lex}} x_0^{\gamma+1}x_r^{d-\gamma-1}$. Therefore, $a_0 = \gamma + 1$. Recall that $Q_P$ has a Macaulay representation

$$Q_P(d) = \sum_{i=0}^{n} \binom{r - i + d - a_i}{r - i}.$$  

As a result, the leading term of $P(d) = \binom{r+d}{r} - Q_P(d)$ is

(16) $$\frac{\gamma}{(r-1)!}d^{r-1}.$$
Note that for all $d \geq g_P$,
\[
\alpha(d) = \binom{r + d'}{r} - \binom{r + \delta(d)}{r} = \sum_{i=1}^{\gamma} \binom{r + d - \gamma + i - 1}{r - 1}
\]
so that $\alpha$ and $P$ have the same leading term. This proves that $\lim_{d \to \infty} \alpha(d)/P(d) = 1$. By the definition, $\epsilon(d)$ is a sum of $\gamma$ monic polynomials in $d$ of non-homogeneous degree $r$ so that $\lim_{d \to \infty} \epsilon(d)/(dP(d)) = 1$. \hfill \Box

**Definition** Suppose that $p$ is a constant polynomial and $\gamma \neq 0$. Let $D^p \in \mathbb{N}$ be the minimal integer satisfying $\Omega(d) > 0$, $D^p \geq g_P$, $D^p \geq D_P + \gamma$ and
\[
0 < \frac{C(d) - \sqrt{\Omega(d)}}{2} < 1 < p(d)\delta(d) < \frac{C(d) + \sqrt{\Omega(d)}}{2}
\]
for all $d \geq D^p$. Such $D^p$ exists by Lemma 5.2. Let $b'(d) = \binom{r+\delta(d)-1}{r-1} - p$ and let $I(W')$ denote the graded ideal of $S$ generated by
\[
x_0^\gamma \left( L \left( \binom{r + \delta(d) - 1}{r}, \delta(d), S \right) + W' \right)
\]
for an arbitrary $W' \in Z_{b'(d)} S$.

**Lemma 5.3.** Suppose that $d \geq D^p$ and (2) holds. For all $W' \in Z_{b'(d)} S$, the Hilbert polynomial of $I(W')$ is $Q_P$.

**Proof.** It is clear that
\[
L \left( \binom{r + t - 1}{r}, t, S \right) = \{ f \in S \mid x_0 \text{ divides } f \}
\]
for all $t \in \mathbb{N}$ by the construction of $L(t, d, S)$. Also,
\[
I(W')_{d+1} \supset x_0^\gamma \left( L \left( \binom{r + \delta(d + 1) - 1}{r}, \delta(d + 1), S \right) + (S'W')_{d+1} \right)
\]
by the construction of $I(W')$.

If $f \in x_0^\gamma L \left( \binom{r+\delta(d)}{r}, \delta(d), S \right)$, then $x_j f \in x_0^\gamma L \left( \binom{r+\delta(d+1)}{r}, \delta(d + 1), S \right)$ for all $0 \leq j \leq r$ by (18). If $f \in x_0^\gamma W'$, then $x_0 f \in x_0^\gamma L \left( \binom{r+\delta(d+1)}{r}, \delta(d + 1), S \right)$ by (18) and $x_j f \in (S'W')_{d+1}$ for all $1 \leq j \leq r$ by the definition. Therefore,
\[
I(W')_{d+1} = x_0^\gamma \left( L \left( \binom{r + \delta(d + 1) - 1}{r}, \delta(d + 1), S \right) + (S'W')_{d+1} \right).
\]

By (18), we see that
\[
L \left( \binom{r + \delta(d + 1) - 1}{r}, \delta(d + 1), S \right) \cap (S'W')_{d+1} = \{0\}
\]
and
\[
\dim_k L \left( \binom{r + \delta(d + 1) - 1}{r}, \delta(d + 1), S \right) = \binom{r + \delta(d + 1) - 1}{r}.
\]
By Lemma 151, \( \dim_k(S(W'))_{d+1} = b'(d + 1) \) since \( \delta(d) \geq D_p \) by the defining property of \( D^p \). Therefore,

\[
\dim_k I(W')_{d+1} = \dim_k(S(W'))_{d+1} + \dim_k L \left( \frac{r + \delta(d) - 1}{r}, \delta(d) + 1, S \right) = b'(d + 1) + \frac{r + \delta(d) - 1}{r}
\]

by the construction of \( p \). Similarly, we can show that \( \dim_k I(W')_d = Q^p(d) \) by the definition of \( I(W') \) and [153]. Now we can use the Gotzmann persistence theorem [17 Satz on p.61] to show that the Hilbert polynomial of \( I(W') \) is equal to \( Q^p \). □

**Theorem 5.4.** Suppose that the Hilbert polynomial \( P \) satisfies \( 2 \) for some \( \gamma \neq 0 \) and a natural number \( p \). If \( W \in Gr(S_d, Q^p(d)) \) is a worst unstable point of \( \text{Hilb}^P(\mathbb{P}^r_k) \) with respect to \( d \geq D^p \), then there are \( g \in GL_{r+1}(k) \) and \( W' \in Z^{b_p(d)}(S' \rangle \) such that

\[ g.W = I(W') \]

and the ideal \( S(g.W) \) of \( S \) is Borel-fixed.

**Proof.** There is \( g \in GL_{r+1}(k) \) such that \( g.W \in X^P_d(S) \) by Theorem 2.2. Let \( \log g.W = (c'_0, c'_1, \ldots, c'_{r}) \in X(T)_{\mathbb{R}} \). After \( g.W \) is acted by a suitable permutation matrix, we can assume that \( c'_i \geq c'_{i+1} \) for all \( 0 \leq i \leq r-1 \). By Theorem 5.1 there is \( m \in M_d \) such that \( m \) divides every monomial in \( g.W \) because \( g.W \in \text{Hilb}^P(\mathbb{P}^r_k) \).

There is \( V \in R_{\delta(d), Q^p(d)}(S) \) such that \( g.W = mV \). If \( m \neq x_0^\gamma \) then \( |\Delta_{g.W}| < |\Delta_{x_0^\gamma V}| \) by the maximality of \( c'_0 \). Therefore \( m = x_0^\gamma \). Let \( \log V^\ast = (c_0, c_1, \ldots, c_r) \). We can derive a relation between \( \log g.W^\ast \) and \( \log V^\ast \) as follows:

\[
\log g.W^\ast = \log [g.W^\ast \cap L \left( \frac{r + \delta(d)}{r}, d, S \right)] + \log [g.W^\ast \cap A \left( \frac{r + \delta(d)}{r}, d, S \right)] = \log [g.W^\ast \cap x_0^\gamma S_{\delta(d)}] + \log A \left( \frac{r + \delta(d)}{r}, d, S \right) = \log x_0^\gamma V^\ast + \log A \left( \frac{r + \delta(d)}{r}, d, S \right) = \log V^\ast + \left[ p\gamma + d\alpha(d) - \epsilon(d), \frac{\epsilon(d)}{r}, \ldots, \frac{\epsilon(d)}{r} \right].
\]

Thus, we can derive a relation between \( \|\log g.W^\ast\|_{\mathbb{R}}^2 \) and \( \|\log V^\ast\|_{\mathbb{R}}^2 \):

\[
\|\log g.W^\ast\|_{\mathbb{R}}^2 = \|\log V^\ast\|_{\mathbb{R}}^2 + 2 \left[ p\gamma + d\alpha(d) - \epsilon(d) - \frac{\epsilon(d)}{r} \right] c_0 + 2 \frac{\epsilon(d)}{r} \rho \delta(d) + (p\gamma + d\alpha(d) - \epsilon(d))^2 + \frac{[\epsilon(d)]^2}{r}.
\]

\[
(19)
\]
If $c_0 = 0$, then $V^* \in Z^{p(d)}_{\delta(d)}(S')$ by the maximality of $\|\log g.W^*\|$ (which is given by \textcolor{red}{[0]} and Lemma \textcolor{red}{5.3}) so that $W'$ mentioned in the statement exists by the construction of $I(W')$ and Lemma \textcolor{red}{5.1}. To prove that $c_0 = 0$, we can show that

$$2 \left( \gamma p(d) + d \alpha(d) - \epsilon(d) - \frac{\epsilon(d)}{r} \right) c_0 + \sum_{i=0}^{r} c_i^2 - (p(d) \delta(d) - 2 \epsilon'(d))^2 > 0$$

first. We can prove that

$$0 \leq \|\log g.W^*\|^2 - \left\| \log \left[ x_0^r(W')^* + A \left( \left( r + \delta(d) \right), d, S \right) \right] \right\|^2$$

$$= \sum_{i=0}^{r} c_i^2 + 2 \left( p(d) \gamma + d \alpha(d) - \epsilon(d) - \frac{\epsilon(d)}{r} \right) c_0 - \|\log (W')^*\|^2$$

for all $W' \in Z^{p(d)}_{\delta(d)}(S')$ from the maximality of $\|g.W^*\|$ using Lemma \textcolor{red}{5.3} and that

$$\|\log (W')^*\|^2 > (p(d) \delta(d) - 2 \epsilon'(d))^2$$

for all $W' \in Z^{p(d)}_{\delta(d)}(S')$ by Lemma \textcolor{red}{5.3}. We have the inequality

$$0 < 2 \left( p(d) \gamma + d \alpha(d) - \epsilon(d) - \frac{\epsilon(d)}{r} \right) c_0 + \sum_{i=0}^{r} c_i^2 - (p(d) \delta(d) - 2 \epsilon'(d))^2$$

$$\leq 2c_0^2 - 2C(d)c_0 + 4p(d) \delta(d)e'(d) - 4[e'(d)]^2.$$  

Being a quadratic inequality in $c_0$, the preceding inequality is equivalent to

$$\frac{C(d) - \sqrt{\Omega(d)}}{2} > c_0 \quad \text{or} \quad c_0 > \frac{C(d) + \sqrt{\Omega(d)}}{2}.$$  

By \textcolor{red}{17}, the second condition is redundant because $c_0 \leq p(d) \delta(d)$. That is, $c_0 = 0$ by \textcolor{red}{17} again.

If $S(g.W)$ is not Borel-fixed, then there are $n_1 \in N(g.W)$ and $n_2 \in S_d \setminus N(g.W)$ such that $x_i n_1 = x_j n_2$ for some $1 \leq i < j \leq r$. The existence of $W'$, Lemma \textcolor{red}{5.4} and Lemma \textcolor{red}{5.1} means that $i \neq 1$ and $j \neq r$. Therefore, consider the subspace $W'' \in \Gr(S_d, Q_p(d))$ generated by $N(g.W) \cup \{n_2\} \setminus \{n_1\}$ satisfies $(SW'')_t = (S(g.W))_t$ for all $t \geq d$ by Theorem \textcolor{red}{4.1}. That is, $W'' \in \Hilb^P_\kappa$. Furthermore, $\|\log W''\|^2 > \|\log g.W\|^2$ because $c_i \geq c_j$, which cannot happen by the maximality of $\|\log g.W\|_\kappa$.

\hfill $\Box$

Theorem \textcolor{red}{4.3} and Theorem \textcolor{red}{5.4} describes worst unstable Hilbert points when $d \gg 0$. Observe that these worst unstable Hilbert points remain unchanged for all but finitely many choices of $d$. We state this as a theorem.

**Theorem 5.5.** If $P$ satisfies $\textcolor{red}{2}$, then $X_P^d = X_{d'}^P$ for all $d, d' \gg 0$.

Compared with $\textcolor{red}{2}$, 0.2.3 and \textcolor{red}{12} Theorem 3.1., we can expect that this is true for an arbitrary Hesselink strata.

When $r = 2$, it is easy to check that any worst unstable point of $\Gr(S_{d'}^d, b'(d))$ is in the $\GL_{r+1}(k)$-orbit of a lex-segment ideal. We state this as a corollary of Theorem \textcolor{red}{5.4}.

**Corollary 5.6.** For an arbitrary Hilbert scheme $\Hilb^P_\kappa$ of 1-dimensional closed subschemes of $\mathbb{P}_k^2$, every worst unstable point of $\Hilb^P_\kappa$ is in $\GL_{r+1}(k)$-orbit of a lex-segment ideal.
Proof. When \( r = 2 \), \( S' = k[x_1, x_2] \) so that
\[
Z^{b'(d)}_{\delta(d)}(S') = \left\{ \left[ \bigwedge_{i=0}^{\gamma_1} x_1^i x_2^{\delta(i)-i} \right], \left[ \bigwedge_{i=0}^{\gamma_2} x_1^i x_2^{\delta(i)-i} \right] \right\}
\]
by the definition of \( Z^{b(d)}_{\delta(d)}(S') \). Whichever element of \( Z^{b(d)}_{\delta(d)}(S') \) has been chosen as \( W' \), \( I(W') \) is equal to a lex-segment ideal after an action by a suitable permutation matrix is applied to \( I(W') \). Now the claimed statement is clear by Theorem 5.4. \( \square \)

For all \( d \geq g \),
\[
P(d) = \sum_{i=1}^{\gamma} \left( \frac{r + d - i}{d - i + 1} \right) + \sum_{i=1}^{p} \left( \frac{d - \gamma - i + 1}{d - \gamma - i + 1} \right)
\]
if (2) holds. This is the Macaulay representation of \( P \) at \( d \). Let us define
\[
P_0(d) = \left( \frac{r + d}{r} \right) - \left( \frac{r + d - \gamma}{r} \right) = \sum_{i=1}^{\gamma} \left( \frac{r + d - i}{d - i + 1} \right) \quad \text{and} \quad P_1 = p = P - P_0.
\]

As we have seen, \( \gamma \) and \( p \) can be computed from the Macaulay representation of \( P \). Furthermore, we can decompose worst unstable point into other worst unstable points of minor Hilbert polynomials.

Corollary 5.7. Suppose that \( P \) is in the form (2) and \( \gamma \neq 0 \). For every worst unstable point \( z \) of \( \text{Hilb}^P_k(\mathbb{P}_k^r) \) and each \( 0 \leq i \leq 1 \) there is a worst unstable point \( z_i \) of \( \text{Hilb}^{P_i}_k(\mathbb{P}_k^{r-i}) \) satisfying the following conditions.

- The projective scheme represented by \( z \) is a union of two closed subschemes \( H \) and \( e \) of \( \mathbb{P}_k^r \), where \( H \) is represented by \( z_0 \) and \( e \) is a zero dimensional scheme.
- The reduced scheme \( H_{\text{red}} \) associated to the scheme \( H \) is isomorphic to \( \mathbb{P}_k^{r-1} \). Furthermore, the corresponding closed immersion \( H_{\text{red}} \hookrightarrow H \hookrightarrow \mathbb{P}_k^r \) is a hyperplane embedding.
- With above closed immersions, \( c \cap H_{\text{red}} \) is a closed subscheme of \( H_{\text{red}} \cong \mathbb{P}_k^{r-1} \), whose Hilbert polynomial is \( P_1 \). Also, \( z_1 \) represents \( c \cap H_{\text{red}} \).

Proof. The projective scheme represented by \( x \) is defined by the ideal generated by \( g.W \) in Theorem 5.4 after applying a change of coordinates if necessary. Let
\[
z_0 = x_0^\gamma \left( L \left( \left[ \frac{r + \delta(d)}{r} \right], \delta(d), S \right) \right) \in R \left( d, \left[ \frac{r + \delta(d)}{r} \right], S \right)
\]
and \( z_1 = W' \in Z_0^{b'(d)}(S') \). Then \( H \) is defined by the graded ideal generated by \( x_0^\gamma \). Let \( c \) be the closed subscheme of \( \mathbb{P}_k^r \) which is defined by the homogeneous ideal generated by \( L \left( \left[ \frac{r+\delta(d)-1}{r} \right], \delta(d), S \right) + W' \). Theorem 5.4 means that \( z_i \)'s are worst unstable points. Then the remaining claims easily follow. \( \square \)

6. K-stability and Castelnuovo-Mumford regularity of worst unstable points

Throughout this section, let’s assume (2). For these Hilbert polynomials, we described worst unstable Hilbert points. In Theorem 6.1, we will prove that a worst unstable projective scheme (that is, the projective scheme represented by a
worst unstable Hilbert point) whose Hilbert polynomial is \(2\) fails to be \(K\)-stable if \(\gamma \neq 1\). Suppose that \(I\) is a saturated monomial ideal of \(S\) and \(\lambda \in \Gamma(\text{SL}_{r+1})\). Assume that \(S/I\) has Hilbert polynomial \(P\). Define
\[
F_{I,\lambda}(d) = \frac{(\log I^*_\lambda, -\lambda \otimes Z_2)_R}{dP(d)}.
\]
Actually, \(F_{I,\lambda}\) admits the expansion [3, p.293]
\[
F_{I,\lambda}(d) = \sum_{i=0}^{\infty} B^{I,\lambda}_i \frac{1}{d^i}
\]
for sufficiently large \(d\). Let \(X_I\) denote the projective scheme defined by the ideal sheaf \(I\) of \(\mathbb{P}^r\). The Donaldson-Futaki invariant of the polarized scheme \((X_I, \mathcal{O}_X(1))\) and 1-parameter subgroup \(\lambda\) defined in [3, p.294] and [17, Definition 6.] is \(B^{I,\lambda}_1\).

**Theorem 6.1.** Suppose that a Hilbert polynomial \(P\) satisfies (2) and \(\gamma \neq 1\). If \(x\) is a worst unstable point of \(\text{Hilb}^P(\mathbb{P}^r_\lambda)\), then the projective scheme represented by \(x\) is not \(K\)-stable.

**Proof.** Fix \(\lambda = (r, -1, \ldots, -1) \in \Gamma(\text{SL}_{r+1}) \cong \mathbb{Z}^{r+1}\). If \(I\) is the saturation of the ideal generated by \(g.W\) in Theorem 4.3, we can compute that
\[
F_{I,-\lambda_1}(d) = \frac{(\log I^*_\lambda, -\lambda \otimes Z_2)_R}{dP(d)} = \frac{rdP(d) + (r + 1)e(d)}{dP(d)} = -r + \frac{(r + 1)e(d)}{P(d)} \frac{1}{d}
\]
and functions \(e\) and \(P\) are constant for \(d \geq g_P\) so that \(B^{I,-\lambda_1}_1 > 0\). This means that \(X_I\) is not \(K\)-stable in this case [3, p. 294]. If \(I\) is the saturation of the ideal generated by \(g.W\) in Theorem 5.4 then
\[
F_{I,\lambda_1}(d) = \frac{(\log I^*_\lambda, \lambda \otimes \mathbb{Z})_R}{dP(d)} = \frac{rp(d)\gamma + rd\alpha(d) - (r + 1)e(d) - p(d)\delta(d)}{dP(d)}.
\]
In the proof and the statement of Lemma 5.2 we have shown that
\[
\lim_{d \to \infty} \frac{\alpha(d)}{P(d)} = 1 = \lim_{d \to \infty} \frac{e(d)}{dP(d)}
\]
so that
\[
B^{I,\lambda_1}_0 = \lim_{d \to \infty} F_{I,-\lambda_1}(d) = -1
\]
and
\[
B^{I,\lambda_1}_1 = \lim_{d \to \infty} d \left( F_{I,\lambda_1}(d) - B^{I,\lambda_1}_0 \right)
\]
\[
= \lim_{d \to \infty} \frac{rp(d)\gamma + rd\alpha(d) - (r + 1)e(d) - p(d)\delta(d)}{P(d)}
\]
\[
= \lim_{d \to \infty} \frac{(r + 1)p(d)\gamma - (r + 1)e(d) - d\alpha(d))}{P(d)}
\]
\[
= \lim_{d \to \infty} \frac{(r + 1)p(d)\gamma}{P(d)} + \frac{r + 1}{P(d)} \sum_{i=1}^{\gamma} \left( \begin{array}{c} r + i + d - \gamma - 1 \\ r - 1 \end{array} \right) (\gamma - i).
\]
Recall that the leading term of $P$ is $(16)$ so that

$$B_1^{L, \lambda_r} = \frac{(r + 1)(\gamma - 1)}{2} > 0$$

if $\gamma \neq 1$. Such a conclusion means that the projective scheme represented by $x$ is not $K$-stable. □

As pointed out in the introduction, it is natural to ask if there is a positive relation between Castelnuovo-Mumford regularity and GIT stability. We will simply compute the regularities of each worst unstable Hilbert points we have described in this paper and compare it with $g_P$, the sharp upper bound for the regularity of an arbitrary Hilbert point in Hilb$_P^r$.\[7\]

If $p \neq 0$ and $\gamma = 0$, let $l_P$ be an integer satisfying

$$\sum_{0 \leq i \leq l_P - 1} \left( \frac{r + i - 1}{r - 1} \right) < p \leq \sum_{0 \leq i \leq l_P} \left( \frac{r + i - 1}{r - 1} \right).$$

If $p \neq 0$ and $\gamma \neq 0$, let $l_P$ be an integer satisfying

$$\sum_{0 \leq i \leq l_P - 1} \left( \frac{r + i - 2}{r - 2} \right) < p \leq \sum_{0 \leq i \leq l_P} \left( \frac{r + i - 2}{r - 2} \right).$$

If $p = 0$, let $l_P = -1$. Now we are ready to compute regularity of worst unstable points in Hilb$_P^r$.

**Theorem 6.2.** Given a Hilbert polynomial $P$ satisfying (2), the Castelnuovo-Mumford regularity of the ideal sheaf on $\mathbb{P}^r_k$ corresponding to $x$ is $l_P + \gamma + 1$ for every worst unstable point $x$ of Hilb$_P^r$.

**Proof.** Note that there is $g \in \text{GL}_{r+1}(k)$ such that the saturated ideal represented by $g.x$ has a minimal generator of maximal degree $l_P + \gamma + 1$ and is Borel-fixed by Theorem 5.4, Lemma 3.4 and Lemma 3.1. By [6, Proposition 2.11 in the part of Mark L. Green], the regularity of $x$ is $l_P + \gamma + 1$. □

Consider the lex-segment ideal $I$ generated by

$$\{ m \in M_{\gamma+p}|x_0^\gamma x_i^p |_{\leq \text{lex} m} \}.$$

For all $d \geq \gamma + p$, we can derive

$$I_d = \bigoplus_{i=0}^{r-2} x_0^\gamma x_i^p k[x_0, x_{i+1}, \ldots, x_r]_{d-\gamma-1} \oplus x_0^\gamma x_{r-1}^p k[x_{r-1}, x_r]_{d-p-\gamma}$$

so that

$$Q_P(d) = \dim_k I_d = \sum_{i=0}^{r-2} \binom{r - i + d - \gamma - 1}{r - i} + \binom{r - (r - 1) + d - p - \gamma}{1}.$$
This implies that $g_P = p + \gamma$ by \([7, (2.9)]\). On the other hand,

$$P(d) = \dim_k(S/I)_d$$

$$= \left| \{ m \in M_d | m <_{\text{lex}} x_0^\gamma x_{r-1}^p x_r^{d-\gamma-p} \} \right|$$

$$= \left| \{ m \in M_d | x_0^\gamma x_r^{d-\gamma} \leq_{\text{lex}} m <_{\text{lex}} x_0^\gamma x_{r-1}^p x_r^{d-\gamma-p} \} \right|$$

$$+ \left| \{ m \in M_d | m <_{\text{lex}} x_0^\gamma x_r^{d-\gamma} \} \right|$$

$$= p + \left( \frac{r + d}{r} \right) - \left( \frac{r + d - \gamma}{r} \right),$$

for all $d \geq g_P$. Therefore, $g_P = p + \gamma$ if \([2]\) holds. If $g_P = l_P + \gamma + 1$ and $r \geq 2$, then one of the following statements is true.

- $r = 2$ and $\gamma \neq 0$.
- $p \leq 2$.

In other words, there is no general positive relationship between the Castelnuovo-Mumford regularity and the Kempf index of a Hilbert point. However, we can still expect that there is a positive relationship between the Kempf index and the Castelnuovo-Mumford regularity of 1-dimensional projective scheme by the first bullet above.

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