Computing the Ehrhart Quasi-Polynomial of a Rational Simplex

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Abstract. We present a polynomial time algorithm to compute any fixed number of the highest coefficients of the Ehrhart quasi-polynomial of a rational simplex. Previously such algorithms were known for integer simplices and for rational polytopes of a fixed dimension. The algorithm is based on the formula relating the $k$th coefficient of the Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by affine lattice subspaces parallel to $k$-dimensional faces of the polytope. We discuss possible extensions and open questions.

1. Introduction and Main Results

Let $P \subset \mathbb{R}^d$ be a rational polytope, that is, the convex hull of a finite set of points with rational coordinates. Let $t \in \mathbb{N}$ be a positive integer such that the vertices of the dilated polytope

$$tP = \{tx : x \in P\}$$

are integer vectors. As is known (see, for example, Section 4.6 of [27]), there exist functions $e_i(P; \cdot) : \mathbb{N} \rightarrow \mathbb{Q}$, $i = 0, \ldots, d$, such that

$$e_i(P; n + t) = e_i(P; n) \quad \text{for all } n \in \mathbb{N}$$

and

$$|nP \cap \mathbb{Z}^d| = \sum_{i=0}^{d} e_i(P; n)n^i \quad \text{for all } n \in \mathbb{N}.$$ 

The function on the right-hand side is called the Ehrhart quasi-polynomial of $P$. It is clear that if $\dim P = d$, then $e_d(P; n) = \text{vol } P$. In this paper, we are interested in the computational complexity of the coefficients $e_i(P; n)$.

If the dimension $d$ is fixed in advance, the values of $e_i(P; n)$ for any given $P$, $n$, and $i$ can be computed in polynomial time by interpolation, as implied by a polynomial time algorithm to count integer points in a polyhedron of a fixed dimension [4], [6].

If the dimension $d$ is allowed to vary, it is an NP-hard problem to check whether $P \cap \mathbb{Z}^d \neq \emptyset$, let alone to count integer points in $P$. This is true even when $P$ is a rational simplex, as exemplified by the knapsack problem; see, for example, Section 16.6 of [25]. If the polytope $P$ is integral, then the coefficients $e_i(P; n) = e_i(P)$
do not depend on $n$. In that case, for any $k$ fixed in advance, computation of the Ehrhart coefficient $e_{d-k}(P)$ reduces in polynomial time to computation of the volumes of the $(d-k)$-dimensional faces of $P$. The algorithm is based on efficient formulas relating $e_{d-k}(P)$, volumes of the $(d-k)$-dimensional faces, and cones of feasible directions at those faces; see [22], [6], and [23]. In particular, if $P = \Delta$ is an integer simplex, there is a polynomial time algorithm for computing $e_{d-k}(\Delta)$ as long as $k$ is fixed in advance.

In this paper, we extend the last result to rational simplices (a $d$-dimensional rational simplex is the convex hull in $\mathbb{R}^d$ of $(d+1)$ affinely independent points with rational coordinates).

Let us fix an integer $k \geq 0$. The paper presents a polynomial time algorithm, which, given an integer $d \geq k$, a rational simplex $\Delta \subset \mathbb{R}^d$, and a positive integer $n$, computes the value of $e_{d-k}(\Delta; n)$.

We present the algorithm in Section 7 and discuss its possible extensions in Section 8.

This is in contrast to the case of an integral polytope, for a general rational polytope $P$ computation of $e_i(P; n)$ cannot be reduced to computation of the volumes of faces and some functionals of the “angles” (cones of feasible direction) at the faces. A general result of McMullen [19] (see also [21] and [20]) asserts that the contribution of the $i$-dimensional face $F$ of a rational polytope $P$ to the coefficient $e_i(P; n)$ is a function of the volume of $F$, the cone of feasible directions of $P$ at $F$, and the translation class of the affine hull $\text{aff}(F)$ of $F$ modulo $\mathbb{Z}^d$.

Our algorithm is based on a new structural result, Theorem 1.1 below, relating the coefficient $e_{d-k}(P; n)$ to volumes of sections of $P$ by affine lattice subspaces parallel to faces $F$ of $P$ with $\dim F \geq d-k$. Theorem 1.1 may be of interest in its own right.

1.1. Valuations and polytopes. Let $V$ be a $d$-dimensional real vector space and let $\Lambda \subset V$ be a lattice, that is, a discrete additive subgroup which spans $V$. A polytope $P \subset V$ is called a $\Lambda$-polytope or a lattice polytope if the vertices of $P$ belong to $\Lambda$. A polytope $P \subset V$ is called $\Lambda$-rational or just rational if $tP$ is a lattice polytope for some positive integer $t$.

For a set $A \subset V$, let $[A] : V \rightarrow \mathbb{R}$ be the indicator of $A$:

$$[A](x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

A complex-valued function $\nu$ on rational polytopes $P \subset V$ is called a valuation if it preserves linear relations among indicators of rational polytopes:

$$\sum_{i \in I} \alpha_i [P_i] = 0 \implies \sum_{i \in I} \alpha_i \nu(P_i) = 0,$$

where $P_i \subset V$ is a finite family of rational polytopes and $\alpha_i$ are rational numbers. We consider only $\Lambda$-valuations or lattice valuations $\nu$ that satisfy

$$\nu(P + u) = \nu(P) \quad \text{for all } u \in \Lambda;$$

see [21] and [20].

A general result of McMullen [19] states that if $\nu$ is a lattice valuation, $P \subset V$ is a rational polytope, and $t \in \mathbb{N}$ is a number such that $tP$ is a lattice polytope, then
there exist functions $\nu_i(P; \cdot) : \mathbb{N} \rightarrow \mathbb{C}$, $i = 0, \ldots, d$, such that

$$\nu(nP) = \sum_{i=0}^{d} \nu_i(P; n)n^i \quad \text{for all } n \in \mathbb{N}$$

and

$$\nu_i(P; n + t) = \nu_i(P; n) \quad \text{for all } n \in \mathbb{N}.$$  

Clearly, if we compute $\nu(mP)$ for $m = n, n + t, \ldots, n + td$, we can obtain $\nu_i(P; n)$ by interpolation.

We are interested in the counting valuation $E$, where $V = \mathbb{R}^d$, $\Lambda = \mathbb{Z}^d$, and

$$E(P) = |P \cap \mathbb{Z}^d|$$

is the number of lattice points in $P$.

The idea of the algorithm is to replace valuation $E$ by some other valuation, so that the coefficients $e_d(P; n), \ldots, e_{d-k}(P; n)$ remain intact, but the new valuation can be computed in polynomial time on any given rational simplex $\Delta$, so that the desired coefficient $e_{d-k}(\Delta; n)$ can be obtained by interpolation.

### 1.2. Valuations $E_L$.

Let $L \subset \mathbb{R}^d$ be a lattice subspace, that is, a subspace spanned by the points $L \cap \mathbb{Z}^d$. Suppose that $\dim L = k$ and let $pr : \mathbb{R}^d \rightarrow L$ be the orthogonal projection onto $L$. Let $P \subset \mathbb{R}^d$ be a rational polytope, let $Q = pr(P)$, $Q \subset L$, be its projection, and let $\Lambda = pr(\mathbb{Z}^d)$. Since $L$ is a lattice subspace, $\Lambda \subset L$ is a lattice.

Let $L^\perp$ be the orthogonal complement of $L$. Then $L^\perp \subset \mathbb{R}^d$ is a lattice subspace. We introduce the volume form $\text{vol}_{d-k}$ on $L^\perp$ which differs from the volume form inherited from $\mathbb{R}^d$ by a scaling factor chosen so that the determinant of the lattice $\mathbb{Z}^d \cap L^\perp$ is 1. Consequently, the same volume form $\text{vol}_{d-k}$ is carried by all translations $x + L^\perp$, $x \in \mathbb{R}^d$.

We consider the following quantity

$$E_L(P) = \sum_{m \in \Lambda} \text{vol}_{d-k}(P \cap (m + L^\perp)) = \sum_{m \in Q \cap \Lambda} \text{vol}_{d-k}(P \cap (m + L^\perp))$$

(clearly, for $m \notin Q$ the corresponding terms are 0).

In words, we take all lattice translates of $L^\perp$, select those that intersect $P$, and add the volumes of the intersections.

Clearly, $E_L$ is a lattice valuation, so

$$E_L(nP) = \sum_{i=0}^{d} e_i(P, L; n)n^i$$

for some periodic functions $e_i(P, L; \cdot)$. If $tP$ is an integer polytope for some $t \in \mathbb{N}$, then

$$e_i(P, L; n + t) = e_i(P, L; n) \quad \text{for all } n \in \mathbb{N}$$

and $i = 0, \ldots, d$.

Note that if $L = \{0\}$, then $E_L(P) = \text{vol} P$ and if $L = \mathbb{R}^d$, then $E_L(P) = |P \cap \mathbb{Z}^d|$, so the valuations $E_L$ interpolate between the volume and the number of lattice points as $\dim L$ grows.

We prove that $e_{d-k}(P; n)$ can be represented as a linear combination of $e_{d-k}(P, L; n)$ for some lattice subspaces $L$ with $\dim L \leq k$. 
Theorem 1.1. Let us fix an integer $k \geq 0$. Let $P \subset \mathbb{R}^d$ be a full-dimensional rational polytope and let $t$ be a positive integer such that $tP$ is an integer polytope. For a $(d-k)$-dimensional face $F$ of $P$ let $\text{lin}(F) \subset \mathbb{R}^d$ be the $(d-k)$-dimensional subspace parallel to the affine hull $\text{aff}(F)$ of $F$ and let $L^F = (\text{lin} F)^{\perp}$ be its orthogonal complement, so $L^F \subset \mathbb{R}^d$ is a $k$-dimensional lattice subspace.

Let $\mathcal{L}$ be a finite collection of lattice subspaces which contains the subspaces $L^F$ for all $(d-k)$-dimensional faces $F$ of $P$ and is closed under intersections. For $L \in \mathcal{L}$ let $\mu(L)$ be integer numbers such that the identity
\[
\left[ \bigcup_{L \in \mathcal{L}} L \right] = \sum_{L \in \mathcal{L}} \mu(L)[L]
\]
holds for the indicator functions of the subspaces from $\mathcal{L}$.

Let us define
\[
\nu(nP) = \sum_{L \in \mathcal{L}} \mu(L)E_L(nP) \quad \text{for } n \in \mathbb{N}.
\]
Then there exist functions $\nu_i(P; \cdot) : \mathbb{N} \rightarrow \mathbb{Q}$, $i = 0, \ldots, d$, such that

\begin{align*}
(1) & \quad \nu(nP) = \sum_{i=0}^{d} \nu_i(P; n)n^i \quad \text{for all } n \in \mathbb{N}, \\
(2) & \quad \nu_i(P; n + t) = \nu_i(P; n) \quad \text{for all } n \in \mathbb{N}, \\
(3) & \quad e_{d-i}(P; n) = \nu_{d-i}(P; n) \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad i = 0, \ldots, k.
\end{align*}

We prove Theorem 1.1 in Section 4 after some preparations in Sections 2 and 3.

Remark 1.2. Valuation $E$ clearly does not depend on the choice of the scalar product in $\mathbb{R}^d$. One can observe that valuation $\nu$ of Theorem 1.1 admits a dual description which does not depend on the scalar product. Instead of $\mathcal{L}$, we consider the set $\mathcal{L}'$ of subspaces containing the subspaces $\text{lin}(F)$ and closed under taking sums of subspaces, and for $L \in \mathcal{L}'$ we define $E_L^\vee(\cdot)$ as the sum of the volumes of sections of the polytope by the lattice affine subspaces parallel to $L$. Then
\[
\nu = \sum_{L \in \mathcal{L}'} \mu^\vee(L)E_L^\vee,
\]
where $\mu^\vee$ are some integers computed from the set $\mathcal{L}'$, partially ordered by inclusion.

However, using the explicit scalar product turns out to be more convenient.

The advantage of working with valuations $E_L$ is that they are more amenable to computations.

- Let us fix an integer $k \geq 0$. We present a polynomial time algorithm, which, given an integer $d \geq k$, a $d$-dimensional rational simplex $\Delta \subset \mathbb{R}^d$, and a lattice subspace $L \subset \mathbb{R}^d$ such that $\dim L \leq k$, computes $E_L(\Delta)$.

We present the algorithm in Section 6 after some preparations in Section 5.
1.3. **The main ingredient of the algorithm to compute** \(e_{d-k}(\Delta; n)\). Theorem 1.1 allows us to reduce the computation of \(e_{d-k}(\Delta; n)\) to that of \(E_L(\Delta)\), where \(L \subset \mathbb{R}^d\) is a lattice subspace and \(\dim L \leq k\). Let us choose a particular lattice subspace \(L \subset R^d\) with \(\dim L = j \leq k\).

If \(P = \Delta\) is a simplex, then the description of the orthogonal projection \(Q = pr(\Delta)\) of \(\Delta\) onto \(L\) can be computed in polynomial time. Moreover, one can compute in polynomial time a decomposition of \(Q\) into a union of non-intersecting polyhedral pieces \(Q_i\), such that \(\text{vol}_{d-j}(pr^{-1}(x))\) is a polynomial on each piece \(Q_i\). Thus computing of \(E_L(\Delta)\) reduces to computing of the sum

\[
\sum_{m \in Q \cap \Lambda} \phi(m),
\]

where \(\phi\) is a polynomial with \(\deg \phi = d - j\), \(Q_i \subset L\) is a polytope with \(\dim Q_i = j \leq k\), and \(\Lambda \subset L\) is a lattice. The sum is computed by applying the technique of “short rational functions” for lattice points in polytopes of a fixed dimension; cf. [7], [6], and [12].

The algorithm for computing the sum of a polynomial over integer points in a polytope is discussed in Section 5.

2. **The Fourier expansions of** \(E\) **and** \(E_L\)

Let \(V\) be a \(d\)-dimensional real vector space with the scalar product \(\langle \cdot, \cdot \rangle\) and the corresponding Euclidean norm \(\| \cdot \|\). Let \(\Lambda \subset V\) be a lattice and let \(\Lambda^* \subset V\) be the dual or the reciprocal lattice

\[\Lambda^* = \{ x \in V : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda \}.\]

For \(\tau > 0\), we introduce the theta function

\[
\theta_{\Lambda}(x, \tau) = \tau^{d/2} \sum_{m \in \Lambda} \exp \{ -\pi \tau \|x - m\|^2 \} = (\det \Lambda)^{-1} \sum_{l \in \Lambda^*} \exp \left\{ -\pi \|l\|^2/\tau + 2\pi i \langle l, x \rangle \right\}, \text{ where } x \in V.
\]

The last inequality is the reciprocity relation for theta series (essentially, the Poisson summation formula); see, for example, Section 69 of [9].

For a polytope \(P\), let \(\text{int} P\) denote the relative interior of \(P\) and let \(\partial P = P \setminus \text{int} P\) be the boundary of \(P\).

**Lemma 2.1.** Let \(P \subset V\) be a full-dimensional polytope such that \(\partial P \cap \Lambda = \emptyset\). Then

\[
|P \cap \Lambda| = \lim_{\tau \to +\infty} \int_P \theta_{\Lambda}(x, \tau) \, dx = (\det \Lambda)^{-1} \lim_{\tau \to +\infty} \sum_{l \in \Lambda^*} \exp \left\{ -\pi \|l\|^2/\tau \right\} \int_P \exp\{2\pi i \langle l, x \rangle\} \, dx.
\]

**Proof.** As is known (cf., for example, Section B.5 of [17]), as \(\tau \to +\infty\), the function \(\theta_{\Lambda}(x, \tau)\) converges in the sense of distributions to the sum of the delta-functions concentrated at the points \(m \in \Lambda\). Therefore, for every smooth function \(\phi : \mathbb{R}^d \to \mathbb{R}\) with a compact support, we have

\[
\lim_{\tau \to +\infty} \int_{\mathbb{R}^d} \phi(x) \theta_{\Lambda}(x, \tau) \, dx = \sum_{m \in \Lambda} \phi(m).
\]
Since $\partial P \cap \Lambda = \emptyset$, we can replace $\phi$ by the indicator function $[P]$ in (2.1).

Remark 2.2. If $\partial P \cap \Lambda \neq \emptyset$, the limit still exists but then it counts every lattice point $m \in \partial P$ with the weight equal to the “solid angle” of $m$ at $P$, since every term $\exp \{-\pi \tau \|x - m\|^2\}$ is spherically symmetric about $m$. This connection between the solid angle valuation and the theta function was described by the author in the unpublished paper [2] (the paper is very different from paper [5] which has the same title) and independently discovered by Diaz and Robins [13]. Diaz and Robins used a similar approach based on Fourier analysis to express coefficients of the Ehrhart polynomial of an integer polytope in terms of cotangent sums [14]. Banaszczyk [1] obtained asymptotically optimal bounds in transference theorems for lattices by using a similar approach with theta functions, with the polytope $P$ replaced by a Euclidean ball.

The formula of Lemma 2.1 can be considered as the Fourier expansion of the counting valuation.

We need a similar result for valuation $E_L$ defined in Section 1.2.

Lemma 2.3. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope and let $L \subset \mathbb{R}^d$ be a lattice subspace with $\dim L = k$. Let $\text{pr} : \mathbb{R}^d \rightarrow L$ be the orthogonal projection onto $L$, let $Q = \text{pr}(P)$, and let $\Lambda = \text{pr}(\mathbb{Z}^d)$, so $\Lambda \subset L$ is a lattice in $L$. Suppose that $\partial Q \cap \Lambda = \emptyset$.

Then

$$E_L(P) = \lim_{\tau \rightarrow +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \{-\pi \|l\|^2/\tau\} \int_{P} \exp \{2\pi i \langle l, x \rangle\} \, dx.$$ 

Proof. We observe that $L \cap \mathbb{Z}^d = \Lambda^*$. For a vector $x \in \mathbb{R}^d$, let $x_L$ be the orthogonal projection of $x$ onto $L$. Applying the reciprocity relation for theta functions in $L$, we write

$$\sum_{l \in L \cap \mathbb{Z}^d} \exp \{-\pi \|l\|^2/\tau + 2\pi i \langle l, x \rangle\} = \sum_{l \in L \cap \mathbb{Z}^d} \exp \{-\pi \|l\|^2/\tau + 2\pi i \langle l, x_L \rangle\} = (\det \Lambda)^{k/2} \sum_{m \in \Lambda} \exp \{-\pi \tau \|x_L - m\|^2\}.$$ 

As is known (cf., for example, Section B.5 of [17]), as $\tau \rightarrow +\infty$, the function

$$g_\tau(x) = \tau^{k/2} \sum_{m \in \Lambda} \exp \{-\pi \tau \|x_L - m\|^2\}$$

converges in the sense of distributions to the sum of the delta-functions concentrated on the subspaces $m + L^\perp$ (this is the set of points where $x_L = m$) for $m \in \Lambda$.

Therefore, for every smooth function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ with a compact support, we have

$$(2.2) \lim_{\tau \rightarrow +\infty} \int_{\mathbb{R}^d} \phi(x) g_\tau(x) \, dx = \sum_{m \in \Lambda} \int_{m + L^\perp} \phi(x) \, d_{L^\perp} x,$$

where $d_{L^\perp} x$ is the Lebesgue measure on $m + L^\perp$ induced from $\mathbb{R}^d$. 

Since \( \partial Q \cap \Lambda = \emptyset \), each subspace \( m + L^\perp \) for \( m \in \Lambda \) either intersects the interior of \( P \) or is at least some distance \( \epsilon = \epsilon(P, L) > 0 \) away from \( P \). Hence we may replace \( \phi \) by the indicator \([P]\) in (2.2).

Recall from Section 1.2 that measuring volumes in \( m + L^\perp \), we scale the volume form in \( L^\perp \) induced from \( \mathbb{R}^d \) so that the determinant of the lattice \( L^\perp \cap \mathbb{Z}^d \) is 1. One can observe that \( \det \Lambda \) provides the required normalization factor, so

\[
(\det \Lambda) \int_{m+L^\perp} [P](x) \ dx = \text{vol}_{d-k} (P \cap (m + L^\perp)).
\]

The proof now follows. \( \square \)

**Remark 2.4.** If \( \partial Q \cap \Lambda \neq \emptyset \), the limit still exists, but then for \( m \in \partial Q \cap \Lambda \) the volume \( \text{vol}_{d-k} (P \cap (m + L^\perp)) \) is counted with the weight defined as follows: we find the minimal (under inclusion) face \( F \) of \( P \) such that \( m + L^\perp \) is contained in \( \text{aff}(F) \) and the weight is equal to the solid angle of \( P \) at \( F \).

3. **Exponential valuations**

Let \( V \) be a \( d \)-dimensional Euclidean space, let \( \Lambda \subset V \) be a lattice, and let \( \Lambda^* \) be the reciprocal lattice. Let us choose a vector \( l \in \Lambda^* \) and let us consider the integral

\[
\Phi_l(P) = \int_P \exp\{2\pi i(l, x)\} \ dx,
\]

where \( dx \) is the Lebesgue measure in \( V \). Note that for \( l = 0 \) we have \( \Phi_l(P) = \Phi_0(P) = \text{vol} \ P \). We have

\[
\Phi_l(P + a) = \exp\{2\pi i(l, a)\} \Phi_l(P) \quad \text{for all } a \in V.
\]

It follows that \( \Phi_l \) is a \( \Lambda \)-valuation on rational polytopes \( P \subset V \).

If \( l \neq 0 \), then the following lemma (essentially, Stokes’ formula) shows that \( \Phi_l \) can be expressed as a linear combination of exponential valuations on the facets of \( P \). The proof can be found, for example, in [3].

**Lemma 3.1.** Let \( P \subset V \) be a full-dimensional polytope. For a facet \( \Gamma \) of \( P \), let \( d_{\Gamma^*} \) be the Lebesgue measure on \( \text{aff}(\Gamma) \), and let \( p_\Gamma \) be the unit outer normal to \( \Gamma \). Then, for every \( l \in V \setminus 0 \), we have

\[
\int_P \exp\{2\pi i(l, x)\} \ dx = \sum_{\Gamma} \frac{(l, p_\Gamma)}{2\pi i ||l||^2} \int_\Gamma \exp\{2\pi i(l, x)\} \ d_{\Gamma^*}x,
\]

where the sum is taken over all facets \( \Gamma \) of \( P \).

Let \( F \subset P \) be an \( i \)-dimensional face of \( P \). Recall that by \( \text{lin}(F) \) we denote the \( i \)-dimensional subspace of \( \mathbb{R}^d \) that is parallel to the affine hull \( \text{aff}(F) \) of \( F \). We need the following result.

**Theorem 3.2.** Let \( P \subset V \) be a rational full-dimensional polytope and let \( t \) be a positive integer such that \( tP \) is a lattice polytope. Let \( \epsilon \geq 0 \) be a rational number and let \( a \in V \) be a vector. Let us choose \( l \in \Lambda^* \). Then there exist functions \( f_i(P, \epsilon, a, l; \cdot) : \mathbb{N} \rightarrow \mathbb{C}, i = 0, \ldots, d, \) such that

\[
\Phi_l((n + \epsilon)P + a) = \sum_{i=0}^d f_i(P, \epsilon, a, l; n) n^i \quad \text{for all } n \in \mathbb{N}
\]

and
that \( \Gamma \) is a lattice polytope, for every facet \( \Gamma \) there is a vector \( t \in \mathbb{R}^d \).

Applying Lemma 3.1 to \( (\Gamma, \epsilon, l; n) \), we get

\[
\Phi_t(P) = \exp \{ 2\pi i l \cdot a \} \Phi_t(P),
\]

without loss of generality we assume that \( a = 0 \). We will denote \( f_i(P, \epsilon, 0, l; n) \) just by \( f_i(P, \epsilon, l; n) \).

We proceed by induction on \( d \). For \( d = 0 \) the statement of the theorem obviously holds. Suppose that \( d \geq 1 \). If \( l = 0 \), then \( \Phi_t((n + \epsilon)P) = (n + \epsilon)^d \det P \) and the statement holds as well.

Suppose that \( l \neq 0 \). For a facet \( \Gamma \) of \( P \), let \( \Lambda_\Gamma = \Lambda \cap \text{lin}(\Gamma) \) and let \( l_\Gamma \) be the orthogonal projection of \( l \) onto \( \text{lin}(\Gamma) \). Thus \( \Lambda_\Gamma \) is a lattice in the \((d-1)\)-dimensional Euclidean space \( \text{lin}(\Gamma) \) and \( l_\Gamma \in \Lambda_\Gamma \), so we can define valuations \( \Phi_{t_\Gamma} \) on \( \text{lin}(\Gamma) \). Since \( tP \) is a lattice polytope, for every facet \( \Gamma \) there is a vector \( u_\Gamma \in V \) such that

\[
\text{lin}(\Gamma) = \text{aff}(t\Gamma) - tu_\Gamma \quad \text{and} \quad tu_\Gamma \in \Lambda.
\]

Let \( \Gamma' = \Gamma - u_\Gamma \), so \( \Gamma' \subset \text{lin}(\Gamma) \) is a \( \Lambda_\Gamma \)-rational \((d-1)\)-dimensional polytope such that \( t\Gamma' \) is a \( \Lambda_\Gamma \)-polytope. We have

\[
(n + \epsilon)\Gamma = (n + \epsilon)\Gamma' + (n + \epsilon)u_\Gamma.
\]

Applying Lemma 3.1 to \((n + \epsilon)P\), we get

\[
\Phi_t((n + \epsilon)P) = \sum_{\Gamma} \psi(\Gamma, l; n) \Phi_{t_\Gamma}((n + \epsilon)\Gamma'),
\]

where

\[
\psi(\Gamma, l; n) = \frac{\langle l, p_\Gamma \rangle}{2\pi i \|l\|^2} \exp \{ 2\pi i (n + \epsilon)\langle l, u_\Gamma \rangle \}
\]

and the sum is taken over all facets \( \Gamma \) of \( P \).

Since \( tu_\Gamma \in \Lambda \) and \( l \in \Lambda^* \), we have

\[
\psi(\Gamma, l; n + t) = \psi(\Gamma, l; n) \quad \text{for all} \quad n \in \mathbb{N}.
\]

Hence, applying the induction hypothesis, we may write

\[
f_i(P, \epsilon, l; n) = \sum_{\Gamma} \psi(\Gamma, l; n) f_i(\Gamma', \epsilon, l_\Gamma; n) \quad \text{for all} \quad n \in \mathbb{N}
\]

and \( i = 0, \ldots, d - 1 \) and \( f_d(P, \epsilon, l; n) \equiv 0 \). Hence (1)–(2) follows by the induction hypothesis.

If \( f_{d-k}(P, \epsilon, l; n) \neq 0 \), then there is a facet \( \Gamma \) of \( P \) such that \( f_{d-k}(\Gamma', \epsilon, l_\Gamma; n) \neq 0 \).

By the induction hypothesis, there is a face \( F' \) of \( \Gamma' \) such that \( \dim F' = d - k \), and \( l_\Gamma \) is orthogonal to \( \text{lin}(F') \). Then \( F = F' + u_\Gamma \) is a \((d-k)\)-dimensional face of \( P \), \( \text{lin}(F') = \text{lin}(F) \), and \( l \) is orthogonal to \( \text{lin}(F) \), which completes the proof.
4. Proof of Theorem 1.1

First, we discuss some ideas relevant to the proof.

4.1. Shifting a valuation by a polytope. Let $V$ be a $d$-dimensional real vector space, let $\Lambda \subset V$ be a lattice, and let $\nu$ be a $\Lambda$-valuation on rational polytopes. Let us fix a rational polytope $R \subset V$. McMullen [19] observed that the function $\mu$ defined by

$$\mu(P) = \nu(P + R)$$

is a $\Lambda$-valuation on rational polytopes $P$. Here “+” stands for the Minkowski sum:

$$P + R = \{ x + y : x \in P, y \in R \}.$$

This result follows since the transformation $P \mapsto -P + R$ preserves linear dependencies among indicators of polyhedra; cf. [21].

Let $t$ be a positive integer such that $tP$ is a lattice polytope. McMullen [19] deduced that there exist functions $\nu_i(P, R; \cdot) : \mathbb{N} \to \mathbb{C}$, $i = 0, \ldots, d$, such that

$$\nu(nP + R) = \sum_{i=0}^{d} \nu_i(P, R; n)n^i$$

for all $n \in \mathbb{N}$ and

$$\nu_i(P, R; n + t) = \nu_i(P, R; n)$$

for all $n \in \mathbb{N}$.

4.2. Continuity properties of valuations $E$ and $E_L$. Let $R \subset \mathbb{R}^d$ be a full-dimensional rational polytope containing the origin in its interior. Then for every polytope $P \subset \mathbb{R}^d$ and every $\epsilon > 0$ we have $P \subset (P + \epsilon R)$. We observe that

$$|(P + \epsilon R) \cap \mathbb{Z}^d| = |P \cap \mathbb{Z}^d|,$$

for all sufficiently small $\epsilon > 0$. If $P$ is a rational polytope, the supporting affine hyperplanes of the facets of $nP$ for $n \in \mathbb{N}$ are split among finitely many translation classes modulo $\mathbb{Z}^d$. Therefore, there exists $\delta = \delta(P, R) > 0$ such that

$$|(nP + \epsilon R) \cap \mathbb{Z}^d| = |nP \cap \mathbb{Z}^d|$$

for all $0 < \epsilon < \delta$ and all $n \in \mathbb{N}$.

We also note that for every rational subspace $L \subset \mathbb{R}^d$, we have

$$\lim_{\epsilon \to 0^+} E_L(P + \epsilon R) = E_L(P).$$

We will use the perturbation $P \mapsto P + \epsilon R$ to push valuations $E$ and $E_L$ into a sufficiently generic position, so that we can apply Lemmas 2.1 and 2.3 without having to deal with various boundary effects. This is somewhat similar in spirit to the idea of [8].

4.3. Linear identities for quasi-polynomials. Let us fix positive integers $t$ and $d$. Suppose that we have a possibly infinite family of quasi-polynomials $p_l : \mathbb{N} \to \mathbb{C}$ of the type

$$p_l(n) = \sum_{i=0}^{d} p_i(l; n)n^i$$

for all $n \in \mathbb{N}$, where functions $p_i(l; \cdot) : \mathbb{N} \to \mathbb{C}$, $i = 0, \ldots, d$, satisfy

$$p_i(l; n) = p_i(l; n + t)$$

for all $n \in \mathbb{N}$. 
Suppose further that \( p : \mathbb{N} \rightarrow \mathbb{C} \) is yet another quasi-polynomial
\[
p(n) = \sum_{i=0}^{d} p_i(n)n^i \quad \text{where} \quad p_i(n + t) = p_i(n) \quad \text{for all} \quad n \in \mathbb{N}.
\]
Finally, suppose that \( c_l(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{C} \) is a family of functions and that
\[
p(n) = \lim_{\tau \rightarrow +\infty} \sum_l c_l(\tau)p_l(n) \quad \text{for all} \quad n \in \mathbb{N}
\]
and that the series converges absolutely for every \( n \in \mathbb{N} \) and every \( \tau > 0 \).

Then we claim that for \( i = 0, \ldots, d \) we have
\[
p_i(n) = \lim_{\tau \rightarrow +\infty} \sum_l c_l(\tau)p_i(l; n) \quad \text{for all} \quad n \in \mathbb{N}
\]
and that the series converges absolutely for every \( n \in \mathbb{N} \) and every \( \tau > 0 \).

This follows since \( p_i(n) \), respectively \( p_i(l; n) \), can be expressed as linear combinations of \( p(m) \), respectively \( p_l(m) \), for \( m = n, n + t, \ldots, n + td \) with the coefficients depending on \( m, n, t, \) and \( d \) only.

Now we are ready to prove Theorem 1.1.

4.4. **Proof of Theorem 1.1.** Let us fix a rational polytope \( P \subset \mathbb{R}^d \) as defined in the statement of the theorem. For \( L \in \mathcal{L} \) let \( P_L \subset L \) be the orthogonal projection of \( P \) onto \( L \) and let \( \Lambda_L \subset L \) be the orthogonal projection of \( \mathbb{Z}^d \) onto \( L \).

Let \( a \in \text{int} P \) be a rational vector and let
\[
R = P - a.
\]
Hence \( R \) is a rational polytope containing the origin in its interior. Let \( R_L \) denote the orthogonal projection of \( R \) onto \( L \).

Since \( P \) is a rational polytope and \( \mathcal{L} \) is a finite set of rational subspaces, there exists \( \delta = \delta(P, R) > 0 \) such that for all \( 0 < \epsilon < \delta \) and all \( n \in \mathbb{N} \), we have
\[
(nP + \epsilon R) \cap \mathbb{Z}^d = nP \cap \mathbb{Z}^d \quad \text{and} \quad \partial(nP + \epsilon R) \cap \mathbb{Z}^d = \emptyset \quad \text{for all} \quad n \in \mathbb{N}
\]
and for all \( L \in \mathcal{L} \), we have
\[
(nP_L + \epsilon R_L) \cap \Lambda_L = nP_L \cap \Lambda_L \quad \text{and} \quad \partial(nP_L + \epsilon R_L) \cap \Lambda_L = \emptyset \quad \text{for all} \quad n \in \mathbb{N};
\]
cf. Section 4.2. Let us choose any rational \( 0 < \epsilon < \delta \).

Because of (4.1), we can write
\[
\|nP + \epsilon R\|_{\mathbb{Z}^d} = \sum_{i=0}^{d} e_i(P; \epsilon)n^i \quad \text{for all} \quad n \in \mathbb{N}
\]
and by Lemma 2.1 we get
\[
\|nP + \epsilon R\|_{\mathbb{Z}^d} = \lim_{\tau \rightarrow +\infty} \sum_{l \in \mathbb{Z}^d} \exp\{-\pi\|l\|^2/\tau\} \Phi_l(nP + \epsilon R),
\]
where \( \Phi_l \) are the exponential valuations of Section 3.

Since \( \Phi_l \) is a \( \mathbb{Z}^d \)-valuation, by Section 4.1 there exist functions \( f_i(P, \epsilon, l; \cdot) : \mathbb{N} \rightarrow \mathbb{C}, i = 0, \ldots, d, \) such that
\[
\Phi_l(nP + \epsilon R) = \sum_{i=0}^{d} f_i(P, \epsilon, l; n)n^i \quad \text{for} \quad n \in \mathbb{N}
\]
and

\begin{equation}
(4.6) \quad f_i(P, \varepsilon, l; n + t) = f_i(P, \varepsilon, l; n) \quad \text{for all } n \in \mathbb{N}.
\end{equation}

Moreover, we can write

\[ nP + \varepsilon R = nP + \varepsilon (P - a) = (n + \varepsilon)P - \varepsilon a. \]

Therefore, by Theorem 3.2, for \( i \leq k \) we have \( f_{d-i}(P, \varepsilon, l; n) = 0 \) unless \( l \in L^F \) for some face \( F \) of \( P \) with \( \dim F = d - k \). Therefore, combining (4.3)–(4.6) and Section 4.3, we obtain for all \( 0 \leq i \leq k \) and all \( n \in \mathbb{N} \)

\[ e_{d-i}(P; n) = \lim_{\tau \to +\infty} \sum_{l \in \mathbb{Z}^d} \exp \left\{ -\pi \|l\|^2 / \tau \right\} f_{d-i}(P, \varepsilon, l; n) \]

\[ = \lim_{\tau \to +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{ -\pi \|l\|^2 / \tau \right\} f_{d-i}(P, \varepsilon, l; n), \]

since vectors \( l \in \mathbb{Z}^d \) outside of subspaces \( L \in \mathcal{L} \) contribute 0 to the sum. Therefore, for \( 0 \leq i \leq k \) and all \( n \in \mathbb{N} \)

\begin{equation}
(4.7) \quad e_{d-i}(P; n) = \lim_{\tau \to +\infty} \sum_{L \in \mathcal{L}} \mu(L) \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{ -\pi \|l\|^2 / \tau \right\} f_{d-i}(P, \varepsilon, l; n) \]
\end{equation}

On the other hand, because of (4.2), by Lemma 2.3 we get for all \( L \in \mathcal{L} \) and all \( n \in \mathbb{N} \)

\begin{equation}
(4.8) \quad E_L(nP + \varepsilon R) = \lim_{\tau \to +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{ -\pi \|l\|^2 / \tau \right\} \Phi_l(nP + \varepsilon R).
\end{equation}

Since \( E_L \) are \( \mathbb{Z}^d \)-valuations, by Section 4.1 there exist functions \( e_i(P, \varepsilon, L; \cdot) : \mathbb{N} \to \mathbb{Q}, \) \( i = 0, \ldots, d, \) such that

\begin{equation}
(4.9) \quad E_L(nP + \varepsilon R) = \sum_{i=0}^{d} e_i(P, \varepsilon, L; n)n^i \quad \text{for all } n \in \mathbb{N}
\end{equation}

and

\begin{equation}
(4.10) \quad e_i(P, \varepsilon, L; n + t) = e_i(P, \varepsilon, L; n) \quad \text{for all } n \in \mathbb{N}.
\end{equation}

Combining (4.5)–(4.6) and (4.8)–(4.10), by Section 4.3 we conclude

\[ e_{d-i}(P, \varepsilon, L; n) = \lim_{\tau \to +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{ -\pi \|l\|^2 / \tau \right\} f_{d-i}(P, \varepsilon, l; n) \quad \text{for all } n \in \mathbb{N}. \]

Therefore, by (4.7), for \( 0 \leq i \leq k \) we have

\begin{equation}
(4.11) \quad e_{d-i}(P; n) = \sum_{L \in \mathcal{L}} \mu(L)e_{d-i}(P, \varepsilon, L; n) \quad \text{for all } n \in \mathbb{N}.
\end{equation}

Since \( E_L \) is a \( \mathbb{Z}^d \)-valuation, there exist functions \( e_i(P, L; \cdot) : \mathbb{N} \to \mathbb{Q}, \) \( i = 0, \ldots, d, \) such that

\begin{equation}
(4.12) \quad E_L(nP) = \sum_{i=0}^{d} e_i(P, L; n)n^i \quad \text{for all } n \in \mathbb{N}
\end{equation}

and

\[ e_i(P, L; n + t) = e_i(P, L; n) \quad \text{for all } n \in \mathbb{N}. \]

Let us choose an \( m \in \mathbb{N} \). Substituting \( n = m, m+1, \ldots, m+td \) in (4.12), we obtain \( e_i(P, L; m) \) as a linear combination of \( E_L(nP) \) with coefficients depending on \( n, m, \)
t, and \(d\) only. Similarly, substituting \(n = m, m + t, \ldots, m + td\) in (4.9), we obtain \(e_i(P, \epsilon, L; m)\) as the same linear combination of \(E_L(nP + \epsilon R)\). Since volumes are continuous functions, in view of (4.2) (see also Section 4.2), we get

\[
\lim_{\epsilon \to 0^+} E_L(nP + \epsilon R) = E_L(nP) \quad \text{for } n = m, m + t, \ldots, m + td.
\]

Therefore,

\[
\lim_{\epsilon \to 0^+} e_i(P, \epsilon, L; m) = e_i(P, L; m) \quad \text{for all } m \in \mathbb{N}.
\]

Taking the limit as \(\epsilon \to 0^+\) in (4.11), we obtain for \(0 \leq i \leq k\)

\[
e_d-i(P; n) = \sum_{L \in L} \mu(L) e_d-i(P, L; n) \quad \text{for all } n \in \mathbb{N}.
\]

To complete the proof, we note that

\[
\nu_d-i(P, L; n) = \sum_{L \in L} \mu(L) e_d-i(P, L; n).
\]

5. Summing up a polynomial over integer points in a rational polytope

Let us fix a positive integer \(k\) and let us consider the following situation. Let \(Q \subset \mathbb{R}^k\) be a rational polytope, let \(\text{int } Q\) be the relative interior of \(Q\), and let \(f : \mathbb{R}^k \to \mathbb{R}\) be a polynomial with rational coefficients. We want to compute the value

\[
(5.1) \quad \sum_{m \in \text{int } Q \cap \mathbb{Z}^k} f(m).
\]

We claim that as soon as the dimension \(k\) of the polytope \(Q\) is fixed, there is a polynomial time algorithm to do that. We assume that the polytope \(Q\) is given by the list of its vertices and the polynomial \(f\) is given by the list of its coefficients.

For an integer point \(m = (\mu_1, \ldots, \mu_k)\), let

\[
x^m = x_1^{\mu_1} \cdots x_k^{\mu_k} \quad \text{for } x = (x_1, \ldots, x_k)
\]

be the Laurent monomial in \(k\) variables \(x = (x_1, \ldots, x_k)\). We use the following result [3].

5.1. The short rational function algorithm. Let us fix \(k\). There is a polynomial time algorithm, which, given a rational polytope \(Q \subset \mathbb{R}^k\), computes the generating function (Laurent polynomial)

\[
S(Q; x) = \sum_{m \in \text{int } Q \cap \mathbb{Z}^k} x^m
\]

in the form

\[
S(Q; x) = \sum_{i \in I} \epsilon_i \frac{x^{a_i}}{(1 - x^{b_{i1}}) \cdots (1 - x^{b_{ik}})},
\]

where \(a_i \in \mathbb{Z}^k, b_{ij} \in \mathbb{Z}^k \setminus \{0\}\), and \(\epsilon_i \in \mathbb{Q}\). In particular, the number \(|I|\) of fractions is bounded by a polynomial in the input size of \(Q\).

Our first step is computing the generating function

\[
S(Q, f; x) = \sum_{m \in \text{int } Q \cap \mathbb{Z}^k} f(m)x^m.
\]
Our approach is similar to that of [12], although we obtain better complexity bounds (our algorithm is polynomial in \( \deg f \) whereas the algorithm of [12] is exponential in \( \deg f \)).

5.2. The algorithm for computing \( S(Q, f; x) \). We observe that

\[
S(Q, f; x) = f \left( x_1 \frac{\partial}{\partial x_1}, \ldots, x_k \frac{\partial}{\partial x_k} \right) S(Q; x).
\]

We compute \( S(Q; x) \) as in Section 5.1.

Let \( a = (\alpha_1, \ldots, \alpha_k) \) be an integer vector, let \( b_j = (\beta_{j1}, \ldots, \beta_{jk}) \) be non-zero integer vectors for \( j = 1, \ldots, k \), and let \( \gamma_1, \ldots, \gamma_k \) be positive integers. Then

\[
\left( x_i \frac{\partial}{\partial x_i} \right) \frac{x^a}{(1 - x^{b_1})^{\gamma_1} \cdots (1 - x^{b_k})^{\gamma_k}}
= \alpha_i \frac{x^a}{(1 - x^{b_1})^{\gamma_1} \cdots (1 - x^{b_k})^{\gamma_k}} + \sum_{j=1}^k \gamma_j \beta_{j1} x^{a + b_1} \cdots \frac{1}{\prod_{s \neq j} (1 - x^{b_s})^{\gamma_s}}.
\]

Consecutively applying the above formula and collecting similar fractions, we compute

\[
\prod_{j} \frac{x^{a_j}}{(1 - x^{b_1})^{\gamma_{1j}} \cdots (1 - x^{b_k})^{\gamma_{kj}}},
\]

where \( \rho_j \in \mathbb{Q}, \gamma_{1j}, \ldots, \gamma_{kj} \) are non-negative integers satisfying \( \gamma_{1j} + \cdots + \gamma_{kj} \leq k + \deg f \) and \( a_j \) are vectors of the type

\[
a_j = a + \mu_1 b_1 + \cdots + \mu_k b_k,
\]

where \( \mu_i \) are non-negative integers and \( \mu_1 + \cdots + \mu_k \leq \deg f \). The number of terms in (5.2) is bounded by \( \deg f O(k) \), which shows that for a \( k \) fixed in advance, the algorithm runs in polynomial time.

Consequently, \( S(Q, f; x) \) is computed in polynomial time.

Formally speaking, to compute the sum (5.1), we have to substitute \( x_i = 1 \) into the formula for \( S(Q, f; x) \). This, however, cannot be done in a straightforward way since \( x = (1, \ldots, 1) \) is a pole of every fraction in the expression for \( S(Q, f; x) \). Nevertheless, the substitution can be done via efficient computation of the relevant residue of \( S(Q, f; x) \) as described in [4] and [7].

5.3. The algorithm for computing the sum. The output of Algorithm 5.2 represents \( S(Q, f; x) \) in the general form

\[
S(Q, f; x) = \sum_{i \in I} \epsilon_i \frac{x^{a_i}}{(1 - x^{b_{i1}})^{\gamma_{i1}} \cdots (1 - x^{b_{ik}})^{\gamma_{ik}}},
\]

where \( \epsilon_i \in \mathbb{Q} \), \( a_i \in \mathbb{Z}^k \), \( b_{ij} \in \mathbb{Z}^k \setminus \{0\} \), and \( \gamma_{ij} \in \mathbb{N} \) are such that \( \gamma_{i1} + \cdots + \gamma_{ik} \leq k + \deg f \) for all \( i \in I \).

Let us choose a vector \( l \in \mathbb{Q}^k \), \( l = (\lambda_1, \ldots, \lambda_k) \), such that \( (l, b_{ij}) \neq 0 \) for all \( i, j \) (such a vector can be computed in polynomial time; cf. [4]). For a complex \( \tau \), let

\[
x(\tau) = (e^{\tau \lambda_1}, \ldots, e^{\tau \lambda_k}).
\]
We want to compute the limit

$$\lim_{\tau \to 0} G(\tau) \quad \text{for} \quad G(\tau) = S(Q, f; x(\tau)).$$

In other words, we want to compute the constant term of the Laurent expansion of $G(\tau)$ around $\tau = 0$.

Let us consider a typical fraction

$$\frac{x^\alpha}{(1 - x^{b_1})^{\gamma_1} \cdots (1 - x^{b_k})^{\gamma_k}}.$$

Substituting $x(\tau)$, we get the expression

$$(5.3) \quad \frac{e^{\alpha \tau}}{(1 - e^{\tau \beta_1})^{\gamma_1} \cdots (1 - e^{\tau \beta_k})^{\gamma_k}},$$

where $\alpha = \langle a, l \rangle$ and $\beta_i = \langle b_i, l \rangle$ for $i = 1, \ldots, k$. The order of the pole at $\tau = 0$ is $D = \gamma_1 + \cdots + \gamma_k \leq k + \deg f$. To compute the constant term of the Laurent expansion of (5.3) at $\tau = 0$, we do the following.

We compute the polynomial $q(\tau) = \sum_{i=0}^{D} \frac{\alpha^i}{i!} \tau^i$ that is the truncation at $\tau^D$ of the Taylor series expansion of $e^{\alpha \tau}$. For $i = 1, \ldots, k$ we compute the polynomial $p_i(\tau)$ with $\deg p_i = D$ such that

$$\frac{\tau}{1 - e^{\tau \beta_i}} = p_i(\tau) + \text{terms of higher order in } \tau$$

at $\tau = 0$. Consecutively multiplying polynomials mod $\tau^{D+1}$, we compute a polynomial $u(\tau)$ such that

$$q(\tau) p_1^\gamma(\tau) \cdots p_k^{\gamma_k}(\tau) \equiv u(\tau) \mod \tau^{D+1}.$$

The coefficient of $\tau^D$ in $u(\tau)$ is the desired constant term of the Laurent expansion.

6. Computing $E_L(\Delta)$

Let us fix a positive integer $k$. Let $\Delta \subset \mathbb{R}^d$ be a rational simplex given by the list of its vertices and let $L \subset \mathbb{R}^d$ be a rational subspace given by its basis and such that $\dim L = k$. In this section, we describe a polynomial time algorithm for computing the value of $E_L(\Delta)$ as defined in Section 1.2.

Let $pr : \mathbb{R}^d \rightarrow L$ be the orthogonal projection. We compute the vertices of the polytope $Q = pr(\Delta)$ and a basis of the lattice $\Lambda = pr(\mathbb{Z}^d)$. For basic lattice algorithms see [25] and [16].

As is known, as $x \in \Delta$ varies, the function

$$\phi(x) = \vol_{d-k}(P_x) \quad \text{where} \quad P_x = (\Delta \cap (x + L^\perp))$$

is a piecewise polynomial on $Q$. Our first step consists of computing a decomposition

$$Q = \bigcup_i C_i$$

such that $C_i \subset Q$ are rational polytopes (chambers) with pairwise disjoint interiors and polynomials $\phi_i : L \rightarrow \mathbb{R}$ such that $\phi_i(x) = \phi(x)$ for $x \in C_i$.

We observe that every vertex of $P_x$ is the intersection of $x + L^\perp$ and some $k$-dimensional face $F$ of $\Delta$. 

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For every face $G$ of $\Delta$ with $\dim G = k - 1$ and such that $\text{aff}(G)$ is not parallel to $L^\perp$, let us compute
$$A_G = \{ x \in L : x + L^\perp \cap \text{aff}(G) \neq \emptyset \}.$$Then $A_G$ is an affine hyperplane in $L$. The number of different hyperplanes $A_G$ is $dO(k)$ and hence they cut $Q$ into at most $dO(k^2)$ polyhedral chambers $C_i$; cf. Section 6.1 of [18]. As long as $x$ stays within the relative interior of a chamber $C_i$, the strong combinatorial type of $P_x$ does not change (the facets of $P_x$ move parallel to themselves) and hence the restriction $\phi_i$ of $\phi$ onto $C_i$ is a polynomial; cf. Section 5.1 of [24]. Since in the $(d-k)$-dimensional space $x + L^\perp$ the polytope $P_x$ is defined by $d$ linear inequalities, $\phi_i$ can be computed in polynomial time; see [15] and [3].

The decomposition (6.1) gives rise to the formula
$$|Q| = \sum_j |Q_j|,$$where $Q_j$ are open faces of the chambers $C_i$ (the number of such faces is bounded by a polynomial in $d$); cf. Section 6.1 of [18]. Hence we have
$$E_L(\Delta) = \sum_j \sum_{m \in Q_j \cap \Lambda} \phi(m).$$Each inner sum is the sum of a polynomial over lattice points in a polytope of dimension at most $k$. By a change of the coordinates, it becomes the sum over integer points in a rational polytope and we compute it as described in Section 5.

7. Computing $e_{d-k}(\Delta; n)$

Let us fix an integer $k \geq 0$. We describe our algorithm, which, given a positive integer $d \geq k$, a rational simplex $\Delta \subset \mathbb{R}^d$ (defined, for example, by the list of its vertices), and a positive integer $n$, computes the number $e_{d-k}(\Delta; n)$.

We use Theorem 1.1.

7.1. Computing the set $\mathcal{L}$ of subspaces. We compute subspaces $L$ and numbers $\mu(L)$ described in Theorem 1.1. Namely, for each $(d-k)$-dimensional face $F$ of $\Delta$, we compute a basis of the subspace $L^F = (\text{lin} F)^\perp$. Hence $\dim L^F \leq k$. Clearly, the number of distinct subspaces $L^F$ is $dO(k)$. We let $\mathcal{L}$ be the set consisting of the subspaces $L^F$ and all other subspaces obtained as intersections of $L^F$. We compute $\mathcal{L}$ in $k$ (or fewer) steps. Initially, we let
$$\mathcal{L} := \{ L^F : F \text{ is a } (d-k)\text{-dimensional face of } \Delta \}.$$Then, at every step, we consider the previously constructed subspaces $L \in \mathcal{L}$, consider the pairwise intersections $L \cap L^F$ as $F$ ranges over the $(d-k)$-dimensional faces of $\Delta$, and add the obtained subspace $L \cap L^F$ to the set $\mathcal{L}$ if it is not already there. If no new subspaces are obtained, we stop. Clearly, in the end of this process, we will obtain all subspaces $L$ that are intersections of different $L^F_i$. Since $\dim L^F_i = k$, each subspace $L \in \mathcal{L}$ is an intersection of some $k$ subspaces $L^F_i$. Hence the process stops after $k$ steps and the total number $|\mathcal{L}|$ of subspaces is $dO(k^2)$.

Having computed the subspaces $L \in \mathcal{L}$, we compute the numbers $\mu(L)$ as follows.
For each pair of subspaces \( L_1, L_2 \in \mathcal{L} \) such that \( L_1 \subset L_2 \), we compute the number \( \mu(L_1, L_2) \) recursively: if \( L_1 = L_2 \), we let \( \mu(L_1, L_2) = 1 \). Otherwise, we let
\[
\mu(L_1, L_2) = - \sum_{L \in \mathcal{L} \setminus L \subset L_1 \subset L_2 \setminus L} \mu(L, L_1).
\]
In the end, for each \( L \in \mathcal{L} \), we let
\[
\mu(L) = \sum_{L \subset L_1 \subset L} \mu(L, L_1).
\]
Hence \( \mu(L_i, L_j) \) are the values of the Möbius function on the set \( \mathcal{L} \) partially ordered by inclusion, so
\[
\left[ \bigcup_{L \in \mathcal{L}} L \right] = \sum_{L \in \mathcal{L}} \mu(L)[L]
\]
follows from the Möbius inversion formula; cf. Section 3.7 of [27].

Now, for each \( L \in \mathcal{L} \) and \( m = n, n + t, \ldots, n + td \) we compute the values of \( E_L(m\Delta) \) as in Section 6, compute
\[
\nu(m\Delta) = \sum_{L \in \mathcal{L}} \mu(L)E_L(m\Delta),
\]
and find \( \nu_{d-k}(\Delta; n) = e_{d-k}(\Delta, n) \) by interpolation.

8. Possible extensions and further questions

8.1. Computing more general expressions. Let \( P \subset \mathbb{R}^d \) be a rational polytope, let \( \alpha \geq 0 \) be a rational number, and let \( u \in \mathbb{R}^d \) be a rational vector. One can show (cf. Section 4.1) that
\[
\left| (n + \alpha)P + u \right| \cap \mathbb{Z}^d = \sum_{i=0}^{d} e_i(P, \alpha, u; n)n^i \quad \text{for all } n \in \mathbb{N},
\]
where \( e_i(P, \alpha, u; \cdot) : \mathbb{N} \to \mathbb{Q}, i = 0, \ldots, d \), satisfy
\[
e_i(P, \alpha, u; n + t) = e_i(P, \alpha, u; n) \quad \text{for all } n \in \mathbb{N},
\]
provided \( t \in \mathbb{N} \) is a number such that \( tP \) is an integer polytope. As long as \( k \) is fixed in advance, for given \( \alpha, u, n, \) and a rational simplex \( \Delta \subset \mathbb{R}^d \), one can compute \( e_{d-k}(\Delta, \alpha, u; n) \) in polynomial time. Similarly, Theorem 1.1 and its proof extend to this more general situation in a straightforward way.

8.2. Computing the generating function. Let \( P \subset \mathbb{R}^d \) be a rational polytope. Then, for every \( 0 \leq i \leq d \), the series
\[
\sum_{n=1}^{+\infty} e_i(P; n)t^n
\]
converges to a rational function \( f_i(P; t) \) for \(|t| < 1\).

It is not clear whether \( f_{d-k}(\Delta; t) \) can be efficiently computed as a “closed form expression” in any meaningful sense, although it seems that by adjusting the methods of Sections 5–7, for any given \( t \) such that \(|t| < 1\) one can compute the value of \( f_{d-k}(\Delta; t) \) in polynomial time (again, \( k \) is assumed to be fixed in advance).
8.3. Extensions to other classes of polytopes. If \( k \) is fixed in advance, the coefficient \( e_{d-k}(P; n) \) can be computed in polynomial time, if the rational polytope \( P \subset \mathbb{R}^d \) is given by the list of its \( d+c \) vertices or the list of its \( d+c \) inequalities, where \( c \) is a constant fixed in advance. A similar result holds for rational parallelepipeds \( P \), that is, for Minkowski sums of \( d \) rational intervals that do not lie in the same affine hyperplane in \( \mathbb{R}^d \).

8.4. Possible applications to integer programming and integer point counting. If \( P \subset \mathbb{R}^m \) is a rational polytope given by the list of its defining linear inequalities, the problem of testing whether \( P \cap \mathbb{Z}^m = \emptyset \) is a typical problem of integer programming; see [16] and [25]. Moreover, a general construction of “aggregation” (see Section 16.6 of [25] and Section 2.2 of [29]) establishes a bijection between the sets \( P \cap \mathbb{Z}^m \) and \( \Delta \cap \mathbb{Z}^d \) provided \( P \) is defined by \( d+1 \) linear inequalities. Here \( \Delta \subset \mathbb{R}^d \) is a rational simplex whose definition is computable in polynomial time from that of \( P \). It would be interesting to find out whether approximating valuation \( E \) by valuation \( \nu \) of Theorem 1.1 for some \( k \ll d \) and applying the algorithm of this paper to compute \( \nu(\Delta) \) can be of any practical use to solve higher-dimensional integer programs and integer point counting problems. It could complement existing software packages [11] and [10] based on the “short rational functions” calculus.

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