Nonsubsampled Graph Filter Banks and Distributed Implementation

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Abstract

In this paper, we consider nonsubsampled graph filter banks (NSGFBs) to process data on a graph in a distributed manner. Given an analysis filter bank with small bandwidth, we propose algebraic and optimization methods of constructing synthesis filter banks such that the corresponding NSGFBs provide a perfect signal reconstruction in the noiseless setting. Moreover, we prove that the proposed NSGFBs can control the resonance effect in the presence of bounded noise and they can limit the influence of shot noise primarily to a small neighborhood of its location on the graph. For an NSGFB on a graph of large size, a distributed implementation has a significant advantage, since data processing and communication demands for the agent at each vertex depend mainly on its neighboring topology. In this paper, we propose an iterative distributed algorithm to implement the proposed NSGFBs. Based on NSGFBs, we also develop a distributed denoising technique which is demonstrated to have satisfactory performance on noise suppression.

Keywords: Graph signal processing, Graph filter bank, Distributed algorithm, Noise suppression, Random geometric graph, Laplacian matrix.

I. Introduction

Spatially distributed networks (SDNs) have an agent at each location equipped with some data processing and communication abilities, and they have been widely used in wireless sensor networks, power grids and many real world applications ([1]–[5]). Data collected by an SDN resides naturally on vertices of a graph. Graph signal processing provides an innovative framework to process data on graphs. Many concepts, such as the Fourier transform, wavelet transform and filter banks, in classical signal processing, have been extended to graph settings in recent years. However there are still lots of fundamental problems unexplored or not completely answered ([6]–[11]).

The wavelet transform is one of the most prominent techniques to process signals in regular domains ([12]–[14]). During the past decades, graph wavelet transforms have been introduced and some of them are designed using the eigenvalue and eigenspace information of the graph Laplacian matrix ([15]–[19]). Graph wavelet transform is under the same theoretical structure with graph filter banks and the corresponding wavelet filter bank carries a down and up-sampling procedure ([7], [8], [20]–[27]). Several forms of the down and up-sampling procedure have been defined by the partitioned graph coloring in [20], the maximum spanning tree structure of the graph in [25], and the SVD decomposition of the graph Laplacian matrix in [26]. A proper definition of the down and up-sampling procedure is not obvious especially when the residing graph is of large size and complicated topological structure. This motivates us to consider a nonsubsampled graph filter bank (NSGFB) that contains an analysis filter bank \((H_0, H_1)\) and a synthesis filter bank \((G_0, G_1)\), see Figure 1 for its block diagram. The analysis filter bank decomposes a graph signal into two components carrying different frequency information. The nonsubsampled structure in an NSGFB greatly simplifies the design of analysis filter banks for spectral decomposition and synthesis filter banks for signal reconstruction.

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Filter banks can be implemented either in a centralized system or a cooperative decentralized (distributed) system. In a centralized system, a central facility receives data from agents at vertices, performs designed data processing and sends the processed data back to agents at vertices. In a decentralized system, the agent at each vertex has certain data processing ability to perform designed data processing, and data collected from an agent at each vertex is shared only with neighboring vertices. Most filter banks on graphs are designed for centralized processing, however for the implementation of filter banks on a graph of large size, a centralized system may suffer from high computational burden and call for significant efforts to create a data exchange network. For signal processing on an SDN or a graph of large size, a distributed implementation provides an indispensable tool. It has been used for signal sampling and reconstruction on an SDN in [5], graph signal inpainting in [28] and economic dispatch in power networks [29]. The reader may refer to [5], [28]–[34] and references therein on distributed implementation of signal processing on graphs. In a distributed implementation of the analysis and synthesis procedures of an NSGFB, signal information on each vertex is transmitted only to neighboring vertices, which dramatically reduces the computational cost and calls for low energy consumption. In this paper, we study NSGFBs on a cooperative decentralized system from design to distributed implementation, and then to distributed signal denoising.

A. Main contributions

An important concept for an NSGFB is the perfect reconstruction condition, i.e., the output \( \tilde{x} \) in Figure 1 is always the same as the input \( x \), which can be characterized by the following matrix equation,

\[
G_0 H_0 + G_1 H_1 = I,
\]

(1.1)

where \((H_0, H_1)\) and \((G_0, G_1)\) are its analysis and synthesis filter bank respectively. Given an analysis filter bank, the existence of synthesis filter banks is theoretically guaranteed so that the corresponding NSGFB satisfies the perfect reconstruction condition (1.1) ([13], [14]). The first contribution of this paper is that we introduce two methods to construct localized synthesis filter banks. In the first approach, the synthesis filter bank is obtained by solving a Bezout identity for polynomials. Its bandwidth could be no larger than the bandwidth of the analysis filter bank. In the second approach, the synthesis filter bank is the solution of a constrained optimization problem. It does not necessarily have small bandwidth, however it has an exponential off-diagonal decay. Consequently, the output of the corresponding localized NSGFB suffers primarily in a small neighborhood of vertices where agents lose data processing ability and/or communication capability.

In some real world applications of an NSGFB, the input \( x \) is the original signal \( x_o \) corrupted by an additive noise \( \epsilon \). In addition, the subband signals \( z_0 = H_0 x \) and \( z_1 = H_1 x \) are usually processed via some (non)linear procedure, such as hard/soft thresholding and quantization. Then the output

\[
\tilde{x} = G_0 \Psi_0(H_0(x_o + \epsilon)) + G_1 \Psi_1(H_1(x_o + \epsilon))
\]

(1.2)

of the NSGFB is no longer the original signal \( x_o \), where \( \epsilon \) is the input noise, and \( \Psi_0, \Psi_1 \) are subband processing operators. The robustness of an NSGFB is of paramount importance. For an SDN, an agent at each vertex operates almost independently and the noise that arises at each vertex of the graph is usually contained in some range [5]. So we may use a bounded deterministic/random noise model for NSGFBs on a distributed system. A reasonable fidelity measure to assess the robustness of an NSGFB is the bounded
In Section IV, we discuss the analysis filter bank \( (\ell, H) \) that bounded filters with finite bandwidth are graph filters on graph \( G \). In Section III, we introduce the concept of graph filters on \( \ell_B \). Organization denoising technique that has satisfactory performance on noise suppression. As an application of NSGFBs, we develop a distributed problem can be locally approximated by solutions of local optimization problems, when the objective function and constraints are well localized [5]. For the distributed implementation of an NSGFB, we require that the residing graph \( G \) with all entries taking value \( t \) and \( t^+ \) be its sign, integral part and positive part respectively, and \( \text{sgn}(t) \) and \( \lfloor t \rfloor \) and \( t_+ \) be its sign, integral part and positive part respectively, and \( t \) be the vector of appropriate size with all entries taking value \( t \). For a set \( F \), denote its cardinality and indicator function by \( \# F \) and \( \chi_F \) respectively.

B. Organization

In Section II, we briefly review some fundamental concepts of graphs and introduce an overlapping graph decomposition (II.1). In Section III, we introduce the concept of graph filters on \( \ell^p \), see Definition III.1 and Proposition III.3. In Section IV, we discuss the analysis filter bank \((H_0, H_1)\) of an NSGFB which are required to have small bandwidth, to pass/block the normalized constant signal, and to have stability on \( \ell^2 \), see Assumptions IV.1, IV.2 and IV.4. We show that analysis filter banks have stability on \( \ell^p \) for all \( 1 \leq p \leq \infty \), with an estimate on their lower and upper \( \ell^p \)-stability bounds independent on the size of the graph, see Theorem IV.6. In Section V, we propose an algebraic design of synthesis filter banks \((G_0, G_1)\) when analysis filters \( H_0 \) and \( H_1 \) are polynomials of the symmetric normalized Laplacian on the graph, see Theorem V.1. In Section VI, we consider the construction of synthesis filter banks \((G_0, G_1)\) by solving the constrained optimization problem (VI.1) and (VI.2) with the objective function consisting of Frobenius norms of \( G_0 \) and \( G_1 \), see (VI.4) and Theorem VI.1. In Section VII, we propose an exponentially convergent iterative algorithm (VII.9) and (VII.10) to implement the synthesis procedure, where each iteration can be implemented in a distributed manner, see Theorem VII.2 and Algorithm VII.1. In Section VIII, we create a distributed denoising technique associated with spline NSGFBs, see Figure 5 and demonstrate its performance for signal denoising on graphs of large size. All proofs are collected in the appendices.

C. Notation

We use the common convention of representing matrices and vectors with bold letters and scalars with normal letters. For a matrix \( A \), denote its transpose, trace, Frobenius norm and operator norm on \( \ell^p \), \( 1 \leq p \leq \infty \), by \( A^T \), \( \text{tr}(A) \), \( \| A \|_F \) and \( \| A \|_{\ell^p} \) respectively. For a graph \( G \), denote its adjacency matrix and degree matrix by \( A_G \) and \( D_G \) respectively, and define its Laplacian matrix by \( L_G := D_G - A_G \) and its symmetric normalized Laplacian matrix by \( L_G^{\text{sym}} := D_G^{-1/2}L_G D_G^{-1/2} \). For a scalar \( t \), let \( \text{sgn}(t) \), \( [t] \) and \( t_+ \) be its sign, integral part and positive part respectively, and \( t \) be the vector of appropriate size with all entries taking value \( t \). For a set \( F \), denote its cardinality and indicator function by \( \# F \) and \( \chi_F \) respectively.

II. Preliminaries on graphs

Let \( G := (V, E) \) be a graph, where \( V = \{1, 2, \cdots, N\} \) is the set of vertices and \( E \) is the set of edges ([5], [6]). For the distributed implementation of an NSGFB, we require that the residing graph \( G \) has certain global features.
Assumption II.1. Throughout the paper, we consider simple graphs $G$, i.e., they are undirected and unweighted, and they do not contain self-loops and multiple edges.

Take $r \geq 0$. For a graph $G = (V, E)$ satisfying Assumption II.1, we define the $r$-neighborhood and the $r$-neighboring subgraph of $i \in V$ by $B(i, r) := \{ j \in V : \rho(i, j) \leq r \}$ and $G_{i,r} := (B(i, r), E(i, r))$ respectively, where $E(i, r)$ contains all edges of the graph $G$ with endpoints in $B(i, r)$, and $\rho(i, j)$ is the geodesic distance between vertices $i$ and $j$ in $V$. Then for $r \geq 1$, we can decompose the graph $G$ into a family of overlapping subgraphs $G_{i,r}, i \in V$, of diameters at most $2r$,

$$G = \bigcup_{i \in V} G_{i,r}. \tag{II.1}$$

For distributed implementation for an NSGFB, we presume that numbers of vertices in the $r$-neighborhood of any vertex are dominated by a polynomial about $r$.

Assumption II.2. Throughout the paper, we consider graphs $G$ with the counting measure $\mu$ having polynomial growth, i.e., there exist positive constants $D_1(G)$ and $d$ such that

$$\mu(B(i, r)) \leq D_1(G)(r + 1)^d \tag{II.2}$$

for all $i \in V$ and $r \geq 0$, where $\mu(F) := \#F$ for all $F \subset V$.

The minimal constants $d$ and $D_1(G)$ in (II.2) are called as Beurling dimension and density of the graph $G$ respectively [5].

The decomposition (II.1) plays a crucial role in the proposed distributed implementation for an NSGFB, and the selection of the radius parameter $r$ in (II.1) depends on Beurling dimension $d$ and density $D_1(G)$ of the graph $G$, see Theorem VII.2. Accordingly, we expect that the Beurling dimension $d$ and density $D_1(G)$ of the graph $G$ are much smaller than (or even independent on) the size of the graph, which implies that the graph $G$ should be sparse. Shown in Figure 2 are two representative graphs that satisfy Assumptions II.1 and II.2:

- The Minnesota traffic graph with 2642 vertices, where each vertex represents a spatial location in the state of Minnesota equipped with a traffic monitoring sensor and each edge denotes a direct communication link between monitoring sensors ([20], [21]).
- The random geometric graph $RGG_N$ with $N$ vertices randomly deployed in the region $[0,1]^2$ and an edge existing between two vertices if their physical distance is not larger than $\sqrt{2N^{-1/2}}$ ([26], [37], [38]).

Fig. 2. Plotted on the left is the Minnesota traffic graph that has Beurling dimension 2 and density 2.1378. On the right is a random geometric graph with $N = 4096$, which has Beurling dimension 2 and density 3.0775.
III. GRAPH SIGNAL AND FILTERING

Let \( \mathcal{G} = (V, E) \) satisfy Assumptions \[ \text{II.1} \] and \[ \text{II.2} \]. A signal \( x \) residing on the graph \( \mathcal{G} \) is a vector \( (x_i)_{i \in V} \), where \( x_i \) refers to the signal value at vertex \( i \in V \). In SDNs and many real world applications, data collected belongs to some sequence space \( \ell^p, 1 \leq p \leq \infty \) ([5], [35], [36]).

A filter \( A \) on the graph \( \mathcal{G} \) is a linear transformation from one signal \( x \) on \( \mathcal{G} \) to another signal \( y = Ax \) on \( \mathcal{G} \), which is usually represented by a matrix \( A = (a(i, j))_{i,j \in V} \). A filter \( A \) is expected to map a signal with finite energy to another signal with finite energy and a bounded signal to another bounded signal. A quantitative description of the above filtering procedure is

\[
\|Ax\|_p \leq C\|x\|_p \quad \text{for all} \quad x \in \ell^p,
\]

where \( 1 \leq p \leq \infty \) and \( C \) is a positive constant.

**Definition III.1.** Let \( 1 \leq p \leq \infty \). We say that \( A \) is a graph filter on \( \ell^p \) if (III.1) is satisfied, and we call the minimal constant \( C \) for (III.1) to hold, denoted by \( \|A\|_{\ell^p} \), the filter bound on \( \ell^p \).

In some practical applications ([8], [20], [21], [22], [26], [39]), a graph filter \( A \) is a polynomial \( P(t) := \sum_{l=0}^{L} p_l t^l \) of the symmetric normalized Laplacian \( \mathbf{L}^\text{sym}_G \) on \( \mathcal{G} \), i.e.,

\[
A = P(\mathbf{L}^\text{sym}_G) = p_0 \mathbf{I} + \sum_{l=1}^{L} p_l (\mathbf{L}^\text{sym}_G)^l.
\]

Let \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq 2 \) be eigenvalues of the symmetric normalized Laplacian \( \mathbf{L}^\text{sym}_G \) and write

\[
\mathbf{L}^\text{sym}_G = \mathbf{U}^T \mathbf{A} \mathbf{U},
\]

where \( \mathbf{U}^T = [u_1, \ldots, u_N] \) is an orthogonal matrix and \( \mathbf{A} = \text{diag}(\lambda_1, \ldots, \lambda_N) \) is a diagonal matrix. Then

\[
A = \mathbf{U}^T P(\mathbf{A}) \mathbf{U},
\]

and the filter bound \( \|A\|_{\ell^2} \) can be evaluated explicitly,

\[
\|A\|_{\ell^2} = \sup_{1 \leq n \leq N} |P(\lambda_n)| \leq \sup_{0 \leq t \leq 2} |P(t)|.
\]

To estimate \( \|A\|_{\ell^p}, p \neq 2 \), of a graph filter \( A = (a(i, j))_{i,j \in V} \), we define the bound of \( A \) by

\[
\|A\|_{\infty} = \sup_{i,j \in V} |a(i, j)|.
\]

A graph filter \( A \) on \( \ell^p, 1 \leq p \leq \infty \), has bounded entries and

\[
\|A\|_{\infty} \leq \sup_{j \in V} \|A e_j\|_p \leq \|A\|_{\ell^p}, 1 \leq p \leq \infty,
\]

where the last inequality is obtained from (III.1) by replacing \( x \) by the standard unit vector \( e_j \) with \( j \)-th component taking value one while all others are zero.

For the distributed implementation for an NSGFB, bounded filters with finite bandwidth will be used as analysis filters, see Assumption [IV.1].

**Definition III.2.** The bandwidth \( \sigma := \sigma(A) \) of a graph filter \( A = (a(i,j))_{i,j \in V} \) is the minimal nonnegative integer such that \( a(i,j) = 0 \) for all \( i,j \in V \) with \( \rho(i,j) > \sigma \). For a filter pair \( (A, B) \), we define its bandwidth \( \sigma := \sigma(A, B) \) by \( \max(\sigma(A), \sigma(B)) \).

In the following proposition, we show that a bounded filter with finite bandwidth is a graph filter on \( \ell^p, 1 \leq p \leq \infty \), with filter bound dominated by some constant, independent of the size of the graph \( \mathcal{G} \).
Proposition III.3. Let $1 \leq p \leq \infty$, and $G$ be a graph satisfying Assumptions [II.1] and [II.2]. Then for any bounded graph filter $A$ with bandwidth $\sigma$, we have
\[
\|A\|_{\infty} \leq \|A\|_{B_p} \leq D_1(G)(\sigma + 1)^d\|A\|_{\infty},
\] (III.8)
where $d$ and $D_1(G)$ are the Beurling dimension and density of the graph $G$ respectively.

For $n \geq 1$, we define spline filters $H_{0,n}^{spln}$ and $H_{1,n}^{spln}$ of order $n$ by
\[
H_{0,n}^{spln} = \left( I - \frac{1}{2} L_{G}^{\text{sym}} \right)^n \quad \text{and} \quad H_{1,n}^{spln} = \left( \frac{1}{2} L_{G}^{\text{sym}} \right)^n,
\] (III.9)
see [39] in circulant graph setting. Spline filters $H_{0,n}^{spln}$ and $H_{1,n}^{spln}$, $n \geq 1$, have bandwidth $n$ and their filter bounds on $\ell^2$ dominated by one, i.e.,
\[
\|H_{l,n}^{spln}\|_{B_2} \leq 1, \ l = 0, 1.
\] (III.10)
For $p \neq 2$, we obtain from Proposition [III.3] that
\[
\|H_{l,n}^{spln}\|_{B_p} \leq \|H_{l,1}^{spln}\|_{B_p} \leq \left( 2^d D_1(G) \|H_{1,1}^{spln}\|_{\infty} \right)^n = (2^{d-1} D_1(G))^n, \ l = 0, 1,
\] (III.11)
where $d$ and $D_1(G)$ are the Beurling dimension and density of the graph $G$ respectively. Therefore our representative spline filters $H_{0,n}^{spln}$ and $H_{1,n}^{spln}$, $n \geq 1$, are graph filters on $\ell^p$, $1 \leq p \leq \infty$, with filter bounds dominated by some constants independent of the size of the graph $G$.

IV. Analysis Filter Banks

The analysis filter bank decomposes the input signal on a graph into two components carrying frequency information. In this section, we design the analysis filter bank $(H_0, H_1)$ of an NSGFB to have small bandwidth, to pass/block the normalized constant signal, and to have stability on $\ell^2$, see Assumptions [IV.1] [IV.2] and [IV.4]. In this section, we also show that analysis filter banks have stability on $\ell^p$ for all $1 \leq p \leq \infty$, with an estimate on their lower and upper $\ell^p$-stability bounds independent of the size of the graph, see Theorem [IV.6].

Let $G = (V, E)$ be a graph satisfying Assumptions [II.1] and [II.2] and $(H_0, H_1)$ be the analysis filter bank of an NSGFB. For the distributed implementation of an NSGFB, we make the following assumption for its analysis filter bank $(H_0, H_1)$.

Assumption IV.1. The analysis filter bank $(H_0, H_1)$ has bandwidth $\sigma \geq 1$.

Given an input graph signal $x = (x_i)_{i \in V}$, outputs of analysis procedure are
\[
z_0 = H_0 x \quad \text{and} \quad z_1 = H_1 x.
\] (IV.1)
Write $z_l = (z_l(i))_{i \in V}$ and $H_l = (h_l(i, j))_{i, j \in V}$, $l = 0, 1$. Then it follows from (IV.1) and Assumption [IV.1] that component values of the outputs $z_0$ and $z_1$ at each vertex $k \in V$ are weighted sums of values of the input $x$ in a $\sigma$-neighborhood of $k$,
\[
z_l(k) = \sum_{\rho(i, k) \leq \sigma} h_l(k, i) x(i), \ i \in V.
\] (IV.2)
Thus the analysis procedure of an NSGFB can be implemented in a distributed manner.

To apply an NSGFB to some real world applications, such as noise suppression and abnormal phenomenon detection, its analysis filter bank should constitute certain spectral decomposition ([20], [21], [23], [45]). Throughout the paper, we also make the following assumption for the analysis filter bank $(H_0, H_1)$. 
Assumption IV.2. The filter $H_0$ passes the normalized constant signal $D_{G}^{1/2}1$, and the filter $H_1$ blocks the normalized constant signal $D_{G}^{1/2}1$, i.e.,

$$H_0D_{G}^{1/2}1 = D_{G}^{1/2}1 \quad \text{and} \quad H_1D_{G}^{1/2}1 = 0. \quad \text{(IV.3)}$$

The frequency partition of an analysis filter bank on an arbitrary graph $G$ is not as obvious as that in classical setting. For the case that $H_0 = P_0(L_{sym}^G)$ and $H_1 = P_1(L_{sym}^G)$ (IV.4), one may verify that Assumption IV.2 is satisfied if and only if

$$P_0(0) = 1 \quad \text{and} \quad P_1(0) = 0. \quad \text{(IV.5)}$$

The above equivalence follows from the fact that $D_{G}^{1/2}1$ is an eigenvector of the symmetric normalized Laplacian $L_{sym}^G$ associated with eigenvalue zero.

The spline filter banks $(H_0^{spln}, H_1^{spln})$, $n \geq 1$, are of the form (IV.4) with $P_0(t) = (1 - t/2)^n$ and $P_1(t) = (t/2)^n$, and they satisfy Assumption IV.2 by (IV.5), i.e.,

$$H_0^{spln}D_{G}^{1/2}1 = D_{G}^{1/2}1 \quad \text{and} \quad H_1^{spln}D_{G}^{1/2}1 = 0. \quad \text{(IV.6)}$$

Spline filter banks in the circulant graph setting are known in [39] as graph-spline wavelet transform. Shown in Figure 3 is local smoothing/blocking phenomenon of the spline filter bank $(H_0^{spln}, H_1^{spln})$ to a blockwise constant signal on the Minnesota traffic graph and a blockwise smooth signal on the random geometric graph $RGG_{4096}$ in Figure 2. It is observed that the lowpass filtered signal is very close to the original signal except near the boundary between different blocks, and that the highpass filtered signal essentially vanishes except around the region where the original signal exhibits sharp local variation.

Fig. 3. Plotted on the top (resp. at the bottom), from left to right, are the original signal $x$ on the Minnesota traffic graph (resp. on the random geometric graph $RGG_{4096}$ in Figure 2), the lowpass filtered signal $H_0^{spln}x$ and the highpass filtered signal $H_1^{spln}x$. The signal $x$ on the top is a blockwise constant function that has only two values $\pm 1$ on three blocks with one block only containing a vertex ([20], [21]), and the signal $x$ at the bottom is a blockwise polynomial consisting of four strips and imposing the polynomial $0.5 - 2c_x$ on the first and third diagonal strips and $0.5 + c_y$ on the second and fourth strips respectively, where $(c_x, c_y)$ are the coordinates of vertices ([26]).

Robustness is a fundamental requirement in the context of filter bank to control the signal dynamic range and to regulate the input noise. For the robustness of an NSGFB on $\ell^p$, $1 \leq p \leq \infty$, we introduce stability of a graph filter pair on $\ell^p$.

**Definition IV.3.** Let $1 \leq p \leq \infty$. We say that $(H_0, H_1)$ is stable on $\ell^p$ if there are two positive constants $C_p$ and $D_p$ such that

$$C_p\|x\|_p \leq (\|H_0x\|_p^p + \|H_1x\|_p^p)^{1/p} \leq D_p\|x\|_p \quad \text{for all} \quad x \in \ell^p. \quad \text{(IV.7)}$$
hold for all $x \in \ell^p$ if $1 \leq p < \infty$, and
\[ C_\infty \|x\|_\infty \leq \max(\|H_0 x\|_\infty, \|H_1 x\|_\infty) \leq D_\infty \|x\|_\infty \]  \tag{IV.8}
hold for all $x \in \ell^\infty$ if $p = \infty$. The optimal constants $C_p$ and $D_p$ for the inequalities in (IV.7) and (IV.8) to hold are called as lower and upper stability bounds of the graph filter bank $(H_0, H_1)$ on $\ell^p$ respectively.

Given an NSGFB with the analysis filter bank $(H_0, H_1)$ and synthesis filter bank $(G_0, G_1)$ such that the perfect reconstruction condition (I.1) holds, we have that
\[ \text{for all } x \in \ell^2 \text{, So throughout the paper, we assume that the analysis procedure is stable on } \ell^2. \]

**Assumption IV.4.** The analysis filter bank $(H_0, H_1)$ is stable on $\ell^2$.

For any $x \in \ell^2$, direct calculation leads to
\[ \|H_0 x\|^2_2 + \|H_1 x\|^2_2 = x^T (H_0^T H_0 + H_1^T H_1) x. \]  \tag{IV.9}
Thus we have the following characterization to Assumption [IV.4]

**Proposition IV.5.** Let $G$ satisfy Assumptions [I.1] and [I.2]. Then $(H_0, H_1)$ satisfies Assumption [IV.4] if and only if $H := H_0^T H_0 + H_1^T H_1$ is positive definite. Moreover, the optimal constants $C_2$ and $D_2$ for (IV.7) to hold can be evaluated by
\[ C_2^2 = (\|H^{-1}\|_{B_2})^{-1} \text{ and } D_2^2 = \|H\|_{B_2}. \]  \tag{IV.10}

For graph filters $H_0$ and $H_1$ of the form (IV.4), we obtain from (III.3) that
\[ H_0^T H_0 + H_1^T H_1 = U^T ((P_0(\Lambda))^2 + (P_1(\Lambda))^2) U. \]  \tag{IV.11}
Hence we can evaluate the optimal constants $C_2$ and $D_2$ for (IV.7) to hold explicitly:
\[ \inf_{1 \leq m \leq N} (P_0(\lambda_m))^2 + (P_1(\lambda_m))^2 \]
\[ = \inf_{\|x\|_2=1} \|H_0 x\|^2_2 + \|H_1 x\|^2_2 \leq \sup_{\|x\|_2=1} \|H_0 x\|^2_2 + \|H_1 x\|^2_2 \]
\[ = \sup_{1 \leq m \leq N} (P_0(\lambda_m))^2 + (Q_0(\lambda_m))^2. \]  \tag{IV.12}
Set $R_n(t) = (1-t/2)^{2n} + (t/2)^{2n}$, $n \geq 1$. Then
\[ \inf_{0 \leq t \leq 2} R_n(t) = 2^{-2n+1} \text{ and } \sup_{0 \leq t \leq 2} R_n(t) = 1. \]  \tag{IV.13}
Taking $P_0(t) = (1-t/2)^n$ and $P_1(t) = (t/2)^n$ in (IV.12) and applying (IV.13), we get
\[ 2^{-2n+1} \|x\|^2_2 \leq x^T ((H_{\text{spln}}^0, n)^T H_{\text{spln}}^0 + (H_{\text{spln}}^1, n)^T H_{\text{spln}}^1) x \]
\[ = \|H_{\text{spln}}^0 x\|^2_2 + \|H_{\text{spln}}^1 x\|^2_2 \leq \|x\|^2_2 \]  \tag{IV.14}
for all $x \in \ell^2$. Therefore spline filter banks $(H_{\text{spln}}^0, n)$ of order $n \geq 1$ satisfy Assumption [IV.4] with lower bound $2^{-n+1/2}$ and upper bound 1 by (IV.12) and (IV.14).

Filters in a stable filter bank on $\ell^p$ are graph filters on $\ell^p$, $1 \leq p \leq \infty$. In the following theorem, we show that analysis filter banks are stable on $\ell^p$, $1 \leq p \leq \infty$, with quantitative estimates on their lower and upper stability bounds by some constants independent of the size of the graph.

\[ \text{for all } x \in \ell^2. \]
**Theorem IV.6.** Let $G$ be a graph satisfying Assumptions [II.1] and [II.2]. $H_0$ and $H_1$ have bandwidth $\sigma \geq 1$, and set $H := H_0^T H_0 + H_1^T H_1$. If $(H_0, H_1)$ is stable on $\ell^2$, then it is stable on $\ell^p$ for all $1 \leq p \leq \infty$. Moreover, we have the following estimates for its lower and upper stability bounds $C_p$ and $D_p$:

$$C_p \gtrless \frac{\|H\|_{B_2}^{1/2}}{d!2^{d+1}(D_1(G))^2(\sigma + 1)^{2d}\kappa^{d+2}}$$  \hspace{1cm} (IV.15)$$

and

$$D_p \leq 2D_1(G)(\sigma + 1)^d\|H\|_{B_2}^{1/2},$$  \hspace{1cm} (IV.16)

where $d$ and $D_1(G)$ are the Beurling dimension and density of the graph $G$ respectively, and

$$\kappa = \|H^{-1}\|_{B_2}\|H\|_{B_2}$$  \hspace{1cm} (IV.17)

is the condition number of the matrix $H$.

Combining (IV.14) and Theorem IV.6, the spline filter banks $(H_{0,n}^{sph}, H_{1,n}^{sph}), n \geq 1$, are stable on $\ell^p, 1 \leq p \leq \infty$, and their lower and upper stability bounds $C_p$ and $D_p$ satisfy

$$\frac{1}{2^{(d+2)n-1}d!(n+1)^{2d}(D_1(G))^2} \leq C_p \leq D_p \leq 2D_1(G)(n+1)^{d+1}.$$  \hspace{1cm} (V.1)

We finish this section with a remark on stability bounds of a graph filter bank on the space $\ell^2$ and on the spaces $\ell^p, p \neq 2$.

**Remark IV.7.** For a finite graph $G = (V, E)$, a stable filter bank $(H_0, H_1)$ on $\ell^2$ is also stable on $\ell^p, 1 \leq p \leq \infty$, and the lower stability bounds $C_2$ and $C_p$ satisfy

$$N^{-|1/p-1/2|} \leq \frac{C_2}{C_p} \leq N^{1/p-1/2},$$  \hspace{1cm} (IV.18)

where $N = \#V$ is the size of the graph $G$. The above estimation is unfavorable when the graph $G$ has large size, however it cannot be improved if there is no restriction on the filter bank $(H_0, H_1)$. As our analysis filter bank $(H_0, H_1)$ has small bandwidth $\sigma$, we obtain the following estimate independent of the size $N$ of the graph $G$ from Proposition IV.5 and Theorem IV.6

$$\frac{1}{2D_1(G)(\sigma + 1)^d\kappa^{1/2}} \leq \frac{C_2}{C_p} \leq d!2^{d+1}(D_1(G))^2(\sigma + 1)^{2d}\kappa^{d+3/2},$$  \hspace{1cm} (IV.19)

where $\kappa$ is given in (IV.17). The reader may refer to [5] and [40]-[44] for historical remarks and various estimates on the ratio between stability bounds on $\ell^p$ and $\ell^q, 1 \leq p, q \leq \infty$, for matrices with certain off-diagonal decay.

**V. SYNTHESIS FILTER BANKS AND BEZOUT IDENTITY**

Let $G = (V, E)$ be a graph satisfying Assumptions [II.1] and [II.2], and $(H_0, H_1)$ be a graph filter bank satisfying Assumptions [IV.1] [IV.2] and [IV.4]. In this section, we propose an algebraic method to construct graph filters $G_0$ and $G_1$ so that the NSGFB with the analysis filter bank $(H_0, H_1)$ and synthesis filter bank $(G_0, G_1)$ satisfies the perfect reconstruction condition (I.1) and the bandwidth of synthesis filter bank $(G_0, G_1)$ is no larger than the bandwidth of the analysis filter bank $(H_0, H_1)$. The proposed approach applies for filter banks $(H_0, H_1)$ being polynomials of the symmetric normalized Laplacian on the graph $G$, i.e.,

$$H_0 = P_0(L^\text{sym}_G) \text{ and } H_1 = P_1(L^\text{sym}_G)$$  \hspace{1cm} (V.1)

for some polynomials $P_0$ and $P_1$. 
Theorem V.1. Let $\mathcal{G}$ be a graph satisfying Assumptions IV.1 and IV.2, $(H_0, H_1)$ be a graph filter bank satisfying Assumptions IV.1 IV.2 and IV.4 and being of the form (IV.1), and let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq 2$ be eigenvalues of the symmetric normalized Laplacian $L^\text{sym}_\mathcal{G}$. If polynomials $Q_0$ and $Q_1$ satisfy

$$P_0(\lambda_m)Q_0(\lambda_m) + P_1(\lambda_m)Q_1(\lambda_m) = 1, \quad 1 \leq m \leq N, \quad (V.2)$$

then the NSGFB with the analysis filter bank $(H_0, H_1)$ and synthesis filter bank $(G_0, G_1)$ satisfies the perfect reconstruction condition (I.1), where

$$G_0 = Q_0(L^\text{sym}_\mathcal{G}) \quad \text{and} \quad G_1 = Q_1(L^\text{sym}_\mathcal{G}). \quad (V.3)$$

The filter $G_0$ in (V.3) passes the normalized constant signal $D^1/2_\mathcal{G}1$, since $G_0D^1/2_\mathcal{G}1 = Q_0(0)D^1/2_\mathcal{G}1 = D^1/2_\mathcal{G}1$, where the last equation follows from (IV.5) and (V.2). However, the filter $G_1$ in (V.3) may not block the normalized constant signal, as $G_1D^1/2_\mathcal{G}1 = Q_1(0)D^1/2_\mathcal{G}1$ is not necessarily a zero signal. In this case, we can construct a new synthesis filter bank by lifting,

$$\tilde{G}_0 = G_0 + Q_0(0)H_1 \quad \text{and} \quad \tilde{G}_1 = G_1 - Q_1(0)H_0, \quad (V.4)$$

which satisfies $\tilde{G}_0D^1/2_\mathcal{G}1 = D^1/2_\mathcal{G}1$ and $\tilde{G}_1D^1/2_\mathcal{G}1 = 0$.

A strong version of (V.2) is the Bezout identity

$$P_0(z)Q_0(z) + P_1(z)Q_1(z) = 1, \quad z \in \mathbb{C} \quad (V.5)$$

for polynomials $P_0, P_1, Q_0$ and $Q_1$. In the circulant graph setting, the above approach of constructing synthesis filter banks via solving Bezout identity (V.5) was discussed in [39]. Comparing with the Bezout identity (V.2) on the eigenvalue set of the symmetric normalized Laplacian $L^\text{sym}_\mathcal{G}$, the advantage of the approach (V.5) provides a tool to design synthesis filter banks without a priori knowledge of global topology of the residing graph and then it simplifies the design of synthesis filter banks for signal reconstruction. It is well known that the Bezout identity (V.5) is solvable if and only if polynomials $P_0$ and $P_1$ have no common root. Moreover, there is a unique solution pair $(Q_{0,B}, Q_{1,B})$ to the Bezout identity (V.5) such that $Q_{0,B}(0) = 1$, $Q_{1,B}(0) = 0$ and the degree of $Q_{0,B}$ (resp. $Q_{1,B}$) is no larger than the degree of $P_1$ (resp. $P_0$). Define

$$G_{0,B} = Q_{0,B}(L^\text{sym}_\mathcal{G}) \quad \text{and} \quad G_{1,B} = Q_{1,B}(L^\text{sym}_\mathcal{G}). \quad (V.6)$$

Then the bandwidth of the synthesis filter bank $(G_{0,B}, G_{1,B})$ is no larger than bandwidth of the analysis filter bank $(H_0, H_1)$. Moreover, for any synthesis filter bank $(G_0, G_1)$ there exists a polynomial $R$ such that

$$G_0 = G_{0,B} + R(L^\text{sym}_\mathcal{G})H_1 \quad \text{and} \quad G_1 = G_{1,B} - R(L^\text{sym}_\mathcal{G})H_0 \quad (V.7)$$

satisfies the perfect reconstruction condition (I.1). We remark that the above polynomials $R$ could be appropriately chosen for real world applications of an NSFGB.

Following (V.6), we define synthesis spline filters $G_{0,n}^{B,\text{spln}}$ and $G_{1,n}^{B,\text{spln}}$ of order $n \geq 1$ by

$$G_{0,n}^{B,\text{spln}} = Q_{0,n}^{B,\text{spln}}(L^\text{sym}_\mathcal{G}) \quad \text{and} \quad G_{1,n}^{B,\text{spln}} = Q_{1,n}^{B,\text{spln}}(L^\text{sym}_\mathcal{G}), \quad (V.8)$$

where

$$Q_{0,n}^{B,\text{spln}}(t) = \sum_{l=0}^{n-1} \binom{2n-1}{l} \left(1 - \frac{t}{2}\right)^{n-1-l} \left(\frac{t}{2}\right)^l$$

$$+ \binom{2n-1}{n-1} \left(\frac{t}{2}\right)^n$$
and

$$Q_{1,n}^{B,\text{spln}}(t) = \sum_{l=0}^{n-1} \binom{2n-1}{l} \left( \frac{t}{2} \right)^{n-1-l} \left( 1 - \frac{t}{2} \right)^l - \binom{2n-1}{n-1} \left( 1 - \frac{t}{2} \right)^n.$$ 

For $n \geq 1$, the filter $G_{0,n}^{B,\text{spln}}$ passes the normalized constant signal $D_g^{1/2}1$, the filter $G_{1,n}^{B,\text{spln}}$ blocks the normalized constant signal $D_g^{1/2}1$, and the NSGFB with the analysis spline bank $(H_{0,n}^{\text{spln}}, H_{1,n}^{\text{spln}})$ and synthesis filter bank $(G_{0,n}^{B,\text{spln}}, G_{1,n}^{B,\text{spln}})$ satisfies the perfect reconstruction condition (I.11). The first two results follow from $Q_{0,n}^{B,\text{spln}}(0) = 1$ and $Q_{1,n}^{B,\text{spln}}(0) = 0$, while the perfect reconstruction conclusion holds since

$$Q_{0,n}^{B,\text{spln}}(t) + Q_{1,n}^{B,\text{spln}}(t) = \sum_{l=0}^{n-1} \binom{2n-1}{l} (1-u)^{2n-1-l} u^l + \sum_{l=0}^{n-1} \binom{2n-1}{l} (1-u)^l u^{2n-1-l} = ((1-u) + u)^{2n-1} = 1,$$

where $u = t/2$.

In real world applications of an NSGFB such as the proposed distributed denoising in Section VIII, the subband signals $z_0$ and $z_1$ in (IV.1) are processed via some (non)linear procedure, such as hard/soft thresholding and quantization. In this case, the reconstructed signal $\tilde{x}$ is not necessarily the same as the original signal $x$. In the following theorem, we show that the difference is mainly dominated by the error caused by the subband processing.

**Proposition V.2.** Let the graph $G$, the analysis filter bank $(H_0, H_1)$ and the synthesis filter bank $(G_0, G_1)$ be as in Theorem [V.1]. Assume that the error caused by the subband processing $\Psi_l$ on subband signals $x_l = H_l x$, $l = 0, 1$, is dominated by $\epsilon$ for any input signal $x \in \ell^p$, i.e.,

$$\|z_l - \Psi_l(z_l)\|_p \leq \epsilon, \quad l = 0, 1,$$  \hspace{1cm} (V.9)

where $\epsilon \geq 0$ and $1 \leq p \leq \infty$. For the input signal $x \in \ell^p$, the reconstructed signal $\tilde{x} = G_0 \Psi_0(z_0) + G_1 \Psi_1(z_1)$ via the corresponding NSGFB belongs to $\ell^p$ as well. Moreover

$$\|\tilde{x} - x\|_p \leq D_1(G)(\bar{\sigma} + 1)^d (\|G_0\|_\infty + \|G_1\|_\infty) \epsilon,$$  \hspace{1cm} (V.10)

where $d$ and $D_1(G)$ are the Beurling dimension and density of the graph $G$ respectively, and $\bar{\sigma}$ is the bandwidth of the synthesis filter bank $(G_0, G_1)$.

We finish this section with a distributed implementation of the NSGFB with analysis/synthesis filter banks selected in Theorem [V.1]. Write $G_l = (g_l(i,j))_{i,j \in V}, l = 0, 1$. As the synthesis filters $G_0$ and $G_1$ have finite bandwidth $\bar{\sigma}$, the synthesis procedure can be implemented in a distributed manner,

$$\tilde{x}_k = \sum_{\rho(j,k) \leq \bar{\sigma}} (g_0(k,j) \tilde{z}_0(j) + g_1(k,j) \tilde{z}_1(j)), \quad k \in V,$$  \hspace{1cm} (V.11)

where $\tilde{x} = (\tilde{x}_i)_{i \in V}$ is the reconstructed signal and $\Psi_l(z_l) = (\tilde{z}_l(i))_{i \in V}, l = 0, 1$, are outputs of subband processing. Hence values of the reconstructed signals $\tilde{x}$ at each vertex $k \in V$ are weighted sums of values of the subband processed outputs $\Psi_0(z_0)$ and $\Psi_1(z_1)$ in a $\bar{\sigma}$-neighborhood of $k \in V$, cf. (IV.2) for distributed implementation of the analysis procedure.
Our representative subband processing procedures $\Psi$ are hard(soft) thresholding and uniform quantization. For those cases, the subband processing $\Psi$ is of the form $\Psi(z) = (\psi(z_i))_{i \in V}$ for $z = (z_i)_{i \in V}$, where $\psi$ is the hard(soft) thresholding and uniform quantization function. Thus the subband processing can be implemented in a distributed manner and the error resulted are bounded (i.e., (VI.9) holds for $p = \infty$) by the hard(soft) thresholding and quantization level. This together with (IV.2) and (V.11) implies that the NSGFB with analysis/synthesis filter banks in Theorem V.1 can be implemented in a distributed manner too, provided that the subband processing can be.

VI. SYNTHESIS FILTER BANK AND OPTIMIZATION

Let $G = (V, E)$ be a graph satisfying Assumptions II.1 and II.2 and $(H_0, H_1)$ be a graph filter bank satisfying Assumptions [IV.1] IV.2 and IV.4. In this section, we consider the construction of synthesis filter banks $(G_0, G_1)$ of an NSGFB by solving the minimization problem:

$$\text{minimize}_{G_0, G_1} \|G_0\|_F^2 + \|G_1\|_F^2$$

subject to the perfect reconstruction condition

$$G_0H_0 + G_1H_1 = I.$$  

(VI.2)

Define the Lagrange function $L$ of the constrained optimization problem (VI.1) and (VI.2) by

$$L(G_0, G_1, \Theta) = \|G_0\|_F^2 + \|G_1\|_F^2 - \text{tr}((G_0H_0 + G_1H_1 - I)\Theta^T).$$

By direct calculation, we have

$$\begin{align*}
\frac{\partial L}{\partial G_0} &= 2G_0 - \Theta H_0^T \\
\frac{\partial L}{\partial G_1} &= 2G_1 - \Theta H_1^T \\
\frac{\partial L}{\partial \Theta} &= G_0H_0 + G_1H_1 - I.
\end{align*}$$

(VI.3)

Set $H = H_0^T + H_1^T$. Solving

$$\frac{\partial L}{\partial G_0} = \frac{\partial L}{\partial G_1} = \frac{\partial L}{\partial \Theta} = 0$$

leads to the unique solution of the constrained optimization problem (VI.1) and (VI.2),

$$G_{0,L} = H^{-1}H_0^T$$

and

$$G_{1,L} = H^{-1}H_1^T.$$  

(VI.4)

The synthesis filter bank $(G_{0,L}, G_{1,L})$ in (VI.4) satisfies

$$G_{0,L}H_0 + G_{1,L}H_1 = I,$$

and the filter $G_{0,L}$ passes the normalized constant signal $D_g^{1/2}1$, since

$$G_{0,L}D_g^{1/2}1 = H^{-1}(H_0^TH_0 + H_1^TH_1)D_g^{1/2}1 = D_g^{1/2}1.$$  

We remark that $G_{1,L}$ may not block the normalized constant signal $D_g^{1/2}1$.

For the case that $H$ is a diagonal matrix, the synthesis filter bank $(G_{0,L}, G_{1,L})$ in (VI.4) has the same bandwidth as the analysis filter bank $(H_0, H_1)$, and

$$|g_{l,L}(i,j)| \leq \left\{ \begin{array}{ll} \|H^{-1}\|\mathcal{B}_z\|H_i\|_\infty & \text{if } \rho(i,j) \leq \sigma \\ 0 & \text{otherwise}, \end{array} \right.$$

(VI.5)

where $G_{l,L} := (g_{l,L}(i,j))_{l \in V, l = 0, 1}$.

Let $\kappa$ be the condition number of the matrix $H$ in (IV.17). It is well known that $\kappa > 1$ when $H$ is not a diagonal matrix. For $\kappa > 1$, the synthesis filter bank $(G_{0,L}, G_{1,L})$ in (VI.4) does not necessarily have a small bandwidth, however it always has an exponential off-diagonal decay.
Theorem VI.1. Let $G = (V, E)$ be a graph satisfying Assumptions [II.1] and [II.2], $(H_0, H_1)$ be a graph filter bank satisfying Assumptions [IV.1] [IV.2] and [IV.4], $\kappa$ be the condition number of the matrix $H := H_0^T H_0 + H_1^T H_1$, and let $G_{i,j} := (g_{i,j}(i, j))_{i,j \in V, l = 0, 1, 0, 1}$. Assume that $\kappa > 1$, then

$$|g_{i,j}(i, j)| \leq D_1(\mathcal{G})(\sigma + 1)^d(1 - 1/\kappa)^{-1/2} \times \|H^{-1}\|_{B_2}\|H_i\|_{\infty} \exp\left(-\frac{\theta}{2\sigma}\rho(i, j)\right)$$  \hspace{1cm} (VI.6)

hold for all $i, j \in V$ and $l = 0, 1$, where $\theta = \ln(\kappa/(\kappa - 1)), \sigma \geq 1$ is the bandwidth of the analysis filter bank $(H_0, H_1)$, and $d$ and $D_1(\mathcal{G})$ are the Beurling dimension and density of the graph $\mathcal{G}$ respectively.

Remark VI.2. Agents located at some vertices may lose data processing ability and/or communication capability. In that case, outputs of the analysis procedure of an NSGFB can be considered as being corrupted by shot noise. The exponential off-diagonal decay property in Theorem [VI.1] implies that the reconstructed signal suffers mainly in their neighborhood of limited size. This means that the proposed NSGFB can limit the influence of shot noise essentially to their small neighborhoods on the graph.

Remark VI.3. By the exponential off-diagonal decay property in Theorem [VI.1], the synthesis filters $(G_{0,L}, G_{1,L})$ are filters on $\ell^p, 1 \leq p \leq \infty$,

$$\|G_{i,j}\|_{B_p} \leq d!2^d(D_1(\mathcal{G}))^2(\sigma + 1)^{2d} \kappa^{d+1}(1 - 1/\kappa)^{-1/2} \times \|H^{-1}\|_{B_2}\|H_i\|_{\infty}, \ l = 0, 1.$$  \hspace{1cm} (VI.7)

The above conclusion with $p = \infty$ indicates that the NSGFB does not have a resonance effect.

Applying similar argument used in the proof of Proposition [V.2], we have

Corollary VI.4. Let $G, (H_0, H_1), (G_{0,L}, G_{1,L})$ be as in Theorem [VI.1], and let $p, \Psi_0, \Psi_1, \epsilon$ be as in Proposition [V.2]. Assume that the input signal $x$ of the corresponding NSGFB belongs to $\ell^p$, then the reconstructed signal $\hat{x} = G_{0,L}\Psi_0(H_0x) + G_{1,L}\Psi_1(H_1x)$ via the NSGFB belongs to $\ell^p$ and

$$\|\hat{x} - x\|_p \leq d!2^d(D_1(\mathcal{G}))^2(\sigma + 1)^{2d} \kappa^{d+1}\|H^{-1}\|_{B_2}\times(1 - 1/\kappa)^{-1/2}(\|H_0\|_{\infty} + \|H_1\|_{\infty})\epsilon.$$  \hspace{1cm} (VI.8)

Solving the constrained optimization program (VI.1) and (VI.2) associated with the analysis spline filter banks $(H_{0,L}^{\text{spln}}, H_{1,L}^{\text{spln}})$, we obtain the synthesis spline filter bank $(G_{0,L}^{\text{spln}}, G_{1,L}^{\text{spln}}), n \geq 1$, where

$$G_{i,j}^{\text{spln}} = \left((H_{0,L}^{\text{spln}})^2 + (H_{1,L}^{\text{spln}})^2\right)^{-1} H_{i,j}^{\text{spln}}, \ l = 0, 1.$$  \hspace{1cm} (VI.9)

The synthesis spline filters $G_{0,L}^{\text{spln}}$ and $G_{1,L}^{\text{spln}}, n \geq 1$, have full bandwidth, however they have exponential off-diagonal decay. Write $G_{i,j}^{\text{spln}} = (g_{i,j}^{\text{spln}}(i, j))_{i,j \in V, l = 0, 1}$. By (VI.6) and Theorem VI.1 we obtain that

$$|g_{i,j}^{\text{spln}}(i, j)| \leq 2^{3n-3/2}(2^{2n-1} - 1)^{-1/2}(n + 1)^d D_1(\mathcal{G}) \times \exp\left(-\frac{\ln(2^{2n-1}/(2^{2n-1} - 1))}{2n}\rho(i, j)\right)$$

hold for all $i, j \in V$ and $l = 0, 1$.

By (III.4), we may use $P(\lambda)$ to describe frequency response of a filter $A = P(L_{\text{sym}}^g)$ of the form (III.2), where the vector $\lambda = (\lambda_1, \ldots, \lambda_N)$ is composed of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq 2$ of the symmetric normalized Laplacian $L_{\text{sym}}^g$. Shown in Figure 4 are frequency responses of the analysis spline filter banks $(H_{0,L}^{\text{spln}}, H_{1,L}^{\text{spln}})$ of order $n$, the synthesis spline filter banks $(G_{0,L}^{\text{spln}}, G_{1,L}^{\text{spln}})$ in (V.8), and the synthesis spline filter banks $(G_{0,L}^{\text{spln}}, G_{1,L}^{\text{spln}})$ just constructed, where $n = 1, 2$. It is observed that the frequency responses of analysis spline filter banks $(H_{0,L}^{\text{spln}}, H_{1,L}^{\text{spln}})$ and synthesis spline filter banks $(G_{0,L}^{\text{spln}}, G_{1,L}^{\text{spln}})$ have certain complementary property, while the synthesis spline filter banks $(G_{B,L}^{\text{spln}}, G_{B,L}^{\text{spln}})$ constructed via solving a Bezout identity do not.
VII. ITERATIVE DISTRIBUTED ALGORITHM FOR SYNTHESIS PROCEDURE

For the NSGFB with synthesis filter banks in Theorem V.1, the distributed implementation of the corresponding synthesis procedure has been discussed in (V.11).

For the NSGFB with the analysis filter bank \((H_0, H_1)\) and synthesis filter bank \((G_0, L, G_1, L)\) obtained from solving the constrained optimization problem (VI.1) and (VI.2), output \(\tilde{x}\) of the synthesis procedure is

$$\tilde{x} = G_0, L \tilde{z}_0 + G_1, L \tilde{z}_1, \quad (VII.1)$$

where \(\tilde{z}_0\) and \(\tilde{z}_1\) be outputs of subband processing. As filters \(G_0, L\) and \(G_1, L\) may have full bandwidth, it is infeasible to evaluate \(G_0, L \tilde{z}_0\) and \(G_1, L \tilde{z}_1\) directly in a distributed manner. In this paper, we do not intend to find synthesis filters \(G_0, L\) and \(G_1, L\) explicitly, instead we propose an iterative distributed algorithm to implement the synthesis procedure (VII.1).

The proposed iterative distributed algorithm is based on two pivoting observations. The first observation is that the output signal \(\tilde{x}\) in (VII.1) is the unique solution of the following global least squares problem:

$$\min_x \|H_0 x - \tilde{z}_0\|^2_2 + \|H_1 x - \tilde{z}_1\|^2_2, \quad (VII.2)$$

which follows from (VI.4). To solve the global optimization problem (VII.2) in a distributed manner, we introduce a family of local least squares problems,

$$\min_x \|H_0 \chi_r^r x - \tilde{z}_0\|^2_2 + \|H_1 \chi_r^r x - \tilde{z}_1\|^2_2, \quad k \in V, \quad (VII.3)$$

where \(\chi_r^r, k \in V\), are truncation operators defined by

$$\chi_r^r : (x(i))_{i \in V} \mapsto (x(i)\chi_{B(k,r)}(i))_{i \in V}, \quad (VII.4)$$

and \(r \geq 1\) is a radius parameter to be determined later [5]. One may verify that given any \(k \in V\), the unique solution of the local optimization problem (VII.3) is given by

$$v_{k,r} = \chi_k^{2r} (\chi_k^{2r} H \chi_k^{2r})^{-1} \chi_k^{2r} (H_0^T \tilde{z}_0 + H_1^T \tilde{z}_1), \quad (VII.5)$$

where \(H = H_0^T H_0 + H_1^T H_1\). The second crucial observation is that the unique solution \(v_{k,r}\) of the local least squares problem (VII.3) in the \((2r)\)-neighborhood of the vertex \(k\) provides a local approximation

Fig. 4. Plotted on the top (resp. at the bottom) are the frequency responses of analysis/synthesis spline filters of order \(n\) on the Minnesota traffic graph (resp. on the random geometric graph \(RGG_{4096}\) in Figure 2), where \(n = 1\) for the left figure and \(n = 2\) for the right figure.
Based on (VII.5), (VII.6) and (VII.7), we propose the following iterative distributed algorithm with initials when the radius parameter $r$ to generate an approximation to the solution $\tilde{x}$ of the global least squares problem (VII.2) in $\ell^p$ norm, i.e., there exists $\delta_{r,\sigma} \in (0, 1)$ such that

$$\|v_r - \tilde{x}\|_p \leq \delta_{r,\sigma}\|\tilde{x}\|_p$$

when the radius parameter $r \geq 1$ is chosen appropriately. Set

$$J = \left( \sum_{k' \in V} \chi_{k'}^r \right)^{-1} \sum_{k \in V} \chi_k^r H \chi_k^{2r} - \chi_k^{2r}.$$  

(VII.8)

Based on (VII.5), (VII.6) and (VII.7), we propose the following iterative distributed algorithm with initials $\tilde{z}_0, \tilde{z}_1 \in \ell^p$:

$$\begin{cases}
    v^{(m)} = J(\tilde{z}_0^{(m-1)} - \tilde{z}_1^{(m-1)} - H_0 v^{(m)}) \\
    \tilde{z}_0^{(m)} = \tilde{z}_0^{(m-1)} - H_0 v^{(m)} \\
    \tilde{z}_1^{(m)} = \tilde{z}_1^{(m-1)} - H_1 v^{(m)} \\
    x^{(m)} = x^{(m-1)} + v^{(m)}
\end{cases}$$

(VII.9)

for $m \geq 1$, where

$$x^{(0)} = 0, \quad \tilde{z}_0^{(0)} = \tilde{z}_0, \quad \tilde{z}_1^{(0)} = \tilde{z}_1.$$  

(VII.10)

Remark VII.1. Decompose $H = D + R$ into a diagonal component $D$ and the remainder $R$. Then the classical Jacobi method to solve the linear system $Hx = H_0 \tilde{z}_0 + H_1 \tilde{z}_1$ is

$$x^{(m)} = D^{-1}(H_0^T \tilde{z}_0 + H_1^T \tilde{z}_1 - Rx^{(m-1)}), \quad m \geq 1.$$  

(VII.11)

The above iterative method converges when $H$ is diagonally dominated, which is not necessarily true for the case in our setting. We observe that for $r = 0$, the matrix $J$ in (VII.8) is equal to $D^{-1}$. Hence the sequence $x^{(m)}, m \geq 0$, in the proposed algorithm (VII.9) and (VII.10) with $r = 0$ is the same as the one in the Jacobi method (VII.11) with initial $x^{(0)} = 0$.

Write $H_l = (h_l(i, j))_{i,j \in V}$ and $\tilde{z}_l = (\tilde{z}_l(i))_{i \in V}$, $l = 0, 1$. For the distributed implementation of the iterative algorithm (VII.9) and (VII.10), each agent $k \in V$ is required to transmit information to its neighboring vertices in $B(k, 2r + 2\sigma)$, and to store the number $m_k = \mu(B(k, r))$ of its neighboring vertices in $B(k, r)$ and four matrices $H_{l,k} = (h_l(i, j))_{i \in B(k, 2r+\sigma), j \in B(k, 2r)}$ and $\tilde{H}_{l,k} = (h_l(i, j))_{i \in B(k, 2r+\sigma), j \in B(k, 2r+2\sigma), l = 0, 1}$. Shown in Algorithm (VII.11) is a distributed implementation of the iterative algorithm (VII.9) and (VII.10), where every vertex $k \in V$ is required to store data of size $O((r + \sigma)^{2d})$, to perform $O((r + \sigma)^{2d})$ algebraic manipulations in each iteration, and to transmit data to its $(2r + 2\sigma)$-neighborhood twice in each iteration.

In the next theorem, we further show that the iterative algorithm (VII.9) and (VII.10) converges exponentially when $r$ is appropriately selected.

Theorem VII.2. Let $1 \leq p \leq \infty$, $\mathcal{G}$ be a graph satisfying Assumptions II.1 and II.2, $(H_0, H_1)$ be a graph filter bank satisfying Assumptions IV.1, IV.2 and IV.4, $\kappa > 1$ be as (IV.17), the condition number of the matrix $H := H_0^T H_0 + H_1^T H_1$, and let $(G_{0,L}, G_{1,L})$ be as in (VI.4). Set

$$\delta_{r,\sigma} := \frac{(D_1(\mathcal{G}))^2(2\sigma + 1)^d\kappa^2}{\kappa - 1} \exp\left(-\frac{\theta}{2\sigma r}\right)(3r + 2\sigma + 1)^d,$$

(VII.12)

where $\theta := \frac{\theta}{2\sigma r}$.
Algorithm VII.1 Iterative Distributed Reconstruction Algorithm

Inputs: stop criterion $\varepsilon$ and observations $\tilde{z}_{l,k} = (\tilde{z}(i))_{i \in B(k, 2r+\sigma)}$ for $l = 0, 1$.

Operation: Compute $F_k = H_{0,k}^T H_{0,k} + H_{1,k}^T H_{1,k}$, find its inverse $(F_k)^{-1}$, and then compute $G_{l,L,k} := (F_k)^{-1} H_{l,k}^T$, $l = 0, 1$.

Initialization: $x_k^{(0)} = 0$, $\tilde{z}_0^{(0)} = \tilde{z}_{0,k}$ and $\tilde{z}_{1,k}^{(0)} = \tilde{z}_{1,k}$.

Iteration:
1) $u_k = G_{0,L,k} \tilde{z}_0^{(m)} + G_{1,L,k} \tilde{z}_{1,k}^{(m)}$ and write $u_k = (u_k(i))_{i \in B(k, 2r)}$.
2) Communicate to all vertices $i \in B(k, r) \setminus \{k\}$ to send data $u_k(i)$ and receive data $u_i(k)$.
3) Produce $v(k) = \frac{1}{m} \sum_{i \in B(k, r)} u_i(k)$.
4) Communicate to all vertices $i \in B(k, r+2\sigma) \setminus \{k\}$ to send data $v(k)$ and receive data $v(i)$, and then generate a vector $v_k = (v(i))_{i \in B(k, r+2\sigma)}$.
5) Update $x_k^{(m+1)} = x_k^{(m)} + v_k$ and $\tilde{z}_{l,k}^{(m+1)} = \tilde{z}_{l,k}^{(m)} - H_{l,k} v_k$, $l = 0, 1$.
6) Evaluate $\|v_k\|_\infty \leq \varepsilon$. If yes, terminate the iteration and output $x_k^{(m+1)}$. Otherwise, set $m = m + 1$.

Outputs: $x_k^{(m+1)}$.

where $\theta = \ln(\kappa/(\kappa - 1))$, $\sigma \geq 1$ is the bandwidth of the analysis filter bank $(H_0, H_1)$, and $d$ and $D_1(G)$ are the Beurling dimension and density of the graph $G$ respectively. Take $z_0, z_1 \in \ell^p$, and let $x^{(m)}$, $m \geq 0$, be as in (VII.9) and (VII.10). If the radius parameter $r$ is so chosen that

$$\delta_{r, \sigma} \in (0, 1),$$

then $x^{(m)}$, $m \geq 0$, converges to the least squares solution $\bar{x}$ in (VII.1) exponentially,

$$\|x^{(m)} - \bar{x}\|_p \leq (\delta_{r, \sigma})^m \|\bar{x}\|_p, \ m \geq 0.$$  \hfill (VII.14)

Remark VII.3. For $l = 0, 1$, we can apply (VII.9) to prove by induction on $m$ that

$$\tilde{z}_l^{(m)} - (z_l - H_l \bar{x}) = -H_l (x^{(m)} - \bar{x}), \ m \geq 0.$$  \hfill (VII.14)

This together with Theorem VII.2 implies that $\tilde{z}_l^{(m)}$, $m \geq 1$, in the iterative algorithm (VII.9) and (VII.10) converges to $z_l - H_l \bar{x}$ exponentially, where $l = 0, 1$.

By (VII.12) and (VII.14) in Theorem VII.2, the iterative distributed algorithm (VII.9) and (VII.10) has fast convergence rate when a large radius parameter $r$ is chosen. In that case, heavier burden arises at each iteration, which implies that each vertex in the graph $G$ should have more data storages, better computing abilities and stronger communication capacities in real world applications. Shown in Tables I and II are the average $E_{m, r}$ of $\|x^{(m)} - x\|_{\infty}/\|x\|_{\infty}$ over 50 trials versus the number $m \geq 1$ of iterations and the radius parameter $r \geq 0$, where $H_{0,n,1}^{\text{spin}}, H_{1,n}^{\text{spin}}$, $n = 1, 2$ are used as analysis filter banks, the signal $x$ in Tables I and II is randomly selected on the Minnesota traffic graph and on the random geometric graph RGG$_{4096}$ in Figure 2 respectively. This demonstrates that the iterative distributed algorithm (VII.9) and (VII.10) converges faster for larger radius $r$, and the original signal can be well approximated in one step when a large radius $r$ is chosen, see Tables I and II.

By (VII.12) and Theorem VII.2, there is a radius parameter $r_0$ such that the iterative distributed algorithm (VII.9) and (VII.10) converges exponentially whenever $r \geq r_0$. We can select the above radius parameter $r_0$ to be independent of the size of the graph $G$. Our simulation indicates that the iterative distributed algorithm (VII.9) and (VII.10) with $r = 0$, i.e. the Jacobi iterative method by Remark VII.1, diverges for some bounded inputs on the Minnesota traffic graph and on some random geometric graphs, see the first column of Tables I and II.
TABLE I

Performance of the iterative distributed reconstruction algorithm to recover signals on the Minnesota traffic graph

| m  | E_{m,r} | r | 0 | 1  | 2  | 3  | 4  | 5  |
|----|----------|---|----|----|----|----|----|----|
| 1  | .4155    | .2220|    | .0375| .0160| .0033| .0003|     |
| 2  | .1355    | .0238|    | .0007| .0001| .0000| .0000|     |
| 3  | .0547    | .0039|    | .0000| .0000| .0000| .0000|     |
| 4  | .0226    | .0006|    | .0000| .0000| .0000| .0000|     |
| 5  | .0098    | .0000|    | .0000| .0000| .0000| .0000|     |
| 10 | .0002    | .0000|    | .0000| .0000| .0000| .0000|     |

n = 2

| m  | E_{m,r} | r | 0 | 1  | 2  | 3  | 4  | 5  |
|----|----------|---|----|----|----|----|----|----|
| 1  | 1.1988   | .6563| .3187| .1523| .0725| .0178|     |
| 2  | 1.1662   | .2315| .0518| .0136| .0029| .0002|     |
| 3  | 1.4550   | .1162| .0125| .0017| .0002| .0000|     |
| 4  | 1.4697   | .0567| .0026| .0002| .0000| .0000|     |
| 5  | 2.5386   | .0296| .0006| .0000| .0000| .0000|     |
| 10 | 12.7921  | .0013| .0000| .0000| .0000| .0000|     |
| 14 | 52.4168  | .0001| .0000| .0000| .0000| .0000|     |

TABLE II

Performance of the iterative distributed reconstruction algorithm to recover signals on the random geometric graph RGG_{4096} in Figure 2

| m  | E_{m,r} | r | 0 |    | 1  | 2  | 3  | 4  | 5  |
|----|----------|---|----|----|----|----|----|----|----|
| 1  | .4182    | .1301| .0156| .0035| .0006| .0001|     |
| 2  | .1963    | .0083| .0002| .0000| .0000| .0000|     |
| 3  | .1143    | .0008| .0000| .0000| .0000| .0000|     |
| 4  | .0699    | .0000| .0000| .0000| .0000| .0000|     |
| 10 | .0050    | .0000| .0000| .0000| .0000| .0000|     |
| 19 | .0001    | .0000| .0000| .0000| .0000| .0000|     |

n = 1

| m  | E_{m,r} | r | 0 | 1  | 2  | 3  | 4  | 5  |
|----|----------|---|----|----|----|----|----|----|
| 1  | 1.5267   | .4674| .1487| .0437| .0159| .0049|     |
| 2  | 2.8586   | .1098| .0120| .0011| .0001| .0000|     |
| 3  | 6.6794   | .0374| .0014| .0000| .0000| .0000|     |
| 4  | 16.4089  | .0121| .0002| .0000| .0000| .0000|     |
| 5  | 40.9430  | .0041| .0000| .0000| .0000| .0000|     |
| 8  | 672.8632 | .0002| .0000| .0000| .0000| .0000|     |

n = 2

VIII. DISTRIBUTED DENOISING

Given an NSGFB with analysis filter bank \((H_0, H_1)\) and synthesis filter bank \((G_0, G_1)\), we propose a denoising technique with hard thresholding operator \(T_\tau\) applied to the high-pass subband signal, where \(T_\tau(t) = \text{sgn}(t)(|t| - \tau)_+\) is the hard thresholding function with threshold value \(\tau \geq 0\), cf. [26], [27], [45], [46]. Presented in Figure 5 is the block diagram of the proposed denoising procedure. In this section, we demonstrate the performance of the proposal denoising procedure associated with spline NSGFBs, which can be implemented in a distributed manner.
The Minnesota traffic graph is a test bed for various techniques in signal processing on graphs of medium size ([8], [20], [22], [26]). The denoising performance of the proposed spline NSGFBs on the Minnesota graph is presented in Table III, where the original signal \( x \) is the noisy input and \( \hat{x} \) is the denoised output.

![Fig. 5. Block diagram of the proposed denoising procedure, where \( x \) is the noisy input and \( \hat{x} \) is the denoised output.](image)

**TABLE III**

**DENOISING PERFORMANCE ON THE MINNESOTA TRAFFIC GRAPH MEASURED WITH THE STANDARD \( \ell^2 \)-SNR**

| \( \eta \)  | 1/32 | 1/16 | 1/8 | 1/4 | 1/2 | 1  |
|--------------|------|------|-----|-----|-----|----|
| Input \( \ell^2 \)-SNR | 34.89 | 28.85 | 22.83 | 18.82 | 10.81 | 4.75 |
| graphBior    | 34.43 | 28.91 | 24.06 | 18.21 | 12.79 | 7.39 |
| OSGFB        | 38.25 | 32.59 | 24.44 | 16.70 | 12.54 | 4.69 |
| PRT          | 35.31 | 29.41 | 23.74 | 18.46 | 15.45 | 12.77 |
| NSGFB-B1     | 37.50 | 31.45 | 25.43 | 18.91 | 13.18 | 7.39 |
| NSGFB-B2     | 37.25 | 30.74 | 24.95 | 18.53 | 13.32 | 7.68 |
| NSGFB-L1     | 38.49 | 32.44 | 26.42 | 19.25 | 13.82 | 8.34 |
| NSGFB-L2     | 37.25 | 30.67 | 24.91 | 18.16 | 13.33 | 7.88 |

In the simulations, the noisy input is

\[
x = x_o + \epsilon, \quad (VIII.1)
\]

where \( x_o = (x_{o,i})_{i \in V} \) is the original graph signal and the noise \( \epsilon = (\epsilon_i)_{i \in V} \) has value \( \epsilon_i \) at vertex \( i \in V \) randomly selected in the range \([-\eta, \eta]\). The spline NSGFBs have analysis spline filter banks \((H_{0,n}^{s,\text{spln}}, H_{1,n}^{s,\text{spln}})\) in (III.9) and synthesis spline filter banks being either \((G_{0,n}^{B,\text{spln}}, G_{1,n}^{B,\text{spln}})\) in (V.8) or \((G_{0,n}^{L,\text{spln}}, G_{1,n}^{L,\text{spln}})\) in (VI.9), where \( n \geq 1 \). They are abbreviated by NSGFB-B\(n\) and NSGFB-L\(n\) respectively. The denoising procedure is performed by retaining the low-pass subband signal \( z_0 = H_{0,n}^{s,\text{spln}}x \) and applying the hard thresholding operation \( T_{\tau} \) to the high-pass subband signal \( z_1 = H_{1,n}^{s,\text{spln}}x \), where \( \tau > 0 \) is chosen appropriately. Thus the denoised output is

\[
\hat{x} = G_{0,n}^{B,\text{spln}}z_0 + G_{1,n}^{B,\text{spln}}T_{\tau}(z_1)
\]

for NSGFB-B\(n\), and

\[
\hat{x} = G_{0,n}^{L,\text{spln}}z_0 + G_{1,n}^{L,\text{spln}}T_{\tau}(z_1)
\]

for NSGFB-L\(n\) respectively, where \( n \geq 1 \). For the above denoising procedure, we use \( 20 \log_{10} \|x_o\|_p/\|x-x_o\|_p \) to measure the input \( \ell^p \)-signal-to-noise ratio (\( \ell^p \)-SNR) in dB, and \( 20 \log_{10} \|x_o\|_p/\|\hat{x}-x_o\|_p \) to measure the output \( \ell^p \)-SNR in dB, where \( 1 \leq p \leq \infty \).

The Minnesota traffic graph is a test bed for various techniques in signal processing on graphs of medium size ([8], [20], [22], [26]). The denoising performance of the proposed spline NSGFBs on the Minnesota graph is presented in Table III, where the original signal \( x_o \) is the blockwise constant function in Figure 3 the threshold value \( \tau \) is selected to be \( 3\eta \), and the input and output \( \ell^2 \)-SNRs are the average values over 50 trials. Shown also in Table III are the performance comparison with the biorthogonal graph filter bank (graphBior) in [21], the \( M \)-channel oversampled graph filter bank (OSGFB) in [22], and the pyramid transform (PRT) in [26], where the corresponding output \( \ell^2 \)-SNRs are calculated from the accompanying codes in these references. It indicates that the spline NSGFBs and the OSGFB outperform other two methods in the small noise scenario, the spline NSGFBs have the best performance in the moderate noise environment, and the PRT stands out from the rest in the strong noisy case.

Presented in Tables IV and V are the denoising performance of spline NSGFBs and the performance comparison with the graphBior in [21], the OSGFB in [22], and the PRT in [26] on the random geometric graph RGG\(_N\), where \( N = 4096 \), the original signal \( x_o \) is the blockwise polynomial in Figure 3 the threshold value \( \tau \) is selected to be \( 3\eta \), and the input and output \( \ell^2 \)-SNRs in Table IV and the input and
TABLE IV
Denoising performance on the random geometric graph $RGG_{4096}$ measured with the standard $\ell^2$-SNR

| $\eta$ | 1/32 | 1/16 | 1/8 | 1/4 | 1/2 | 1 |
|-------|------|------|-----|-----|-----|---|
| Input $\ell^2$-SNR | 35.06 | 29.04 | 23.02 | 17.01 | 10.97 | 4.95 |
| graphBior | 33.82 | 28.61 | 23.27 | 18.20 | 13.21 | 8.34 |
| OSGFB | 31.69 | 26.37 | 20.79 | 16.40 | 13.40 | 11.13 |
| PRT | 32.89 | 27.51 | 22.44 | 17.70 | 14.21 | 11.81 |
| NSGFB-B1 | 37.43 | 31.40 | 25.34 | 19.31 | 13.47 | 7.62 |
| NSGFB-B2 | 36.65 | 30.63 | 24.89 | 19.37 | 13.80 | 8.25 |
| NSGFB-L1 | 38.86 | 32.87 | 26.61 | 20.45 | 14.91 | 9.40 |
| NSGFB-L2 | 36.08 | 29.97 | 24.27 | 19.16 | 13.92 | 9.05 |

TABLE V
Denoising performance on the random geometric graph $RGG_{4096}$ measured with the $\ell^\infty$-SNR

| $\eta$ | 1/32 | 1/16 | 1/8 | 1/4 | 1/2 | 1 |
|-------|------|------|-----|-----|-----|---|
| Input $\ell^\infty$-SNR | 34.90 | 28.88 | 22.85 | 16.83 | 10.81 | 4.79 |
| graphBior | 23.45 | 17.28 | 11.12 | 5.34 | 0.32 | -4.00 |
| OSGFB | 20.59 | 14.32 | 6.83 | 0.99 | -1.99 | -2.71 |
| PRT | 23.24 | 17.25 | 11.27 | 5.43 | 0.39 | -2.15 |
| NSGFB-B1 | 31.84 | 25.16 | 18.71 | 11.05 | 6.55 | 2.60 |
| NSGFB-B2 | 26.86 | 20.34 | 14.67 | 8.32 | 3.55 | 1.67 |
| NSGFB-L1 | 29.28 | 22.70 | 16.19 | 9.28 | 4.08 | 0.35 |
| NSGFB-L2 | 24.66 | 17.91 | 12.27 | 6.72 | 0.48 | -0.52 |

output $\ell^\infty$-SNRs in Table [V] are the average values over 50 trials. It is observed that the spline NSGFBs proposed in this paper outperform the graphBior, OSGFB and PRT in small and moderate noise scenario, and that the spline NSGFBs have comparable performance with the rest in the strong noisy case. Also from Tables [IV] and [V] we see that the differences between the input and output $\ell^p$-SNRs for $p = 2, \infty$ are in some range. This confirms the conclusions in Proposition [V.2] and Corollary [VI.4] that the output noise is dominated by a multiple of the input noise.

Shown in Figure [6] is the input noise $\epsilon$ with $\eta = 1/16$ and differences between the original signal $x_o$ and the denoised signal $x$ via the graphBior, OSGFB, PRT and spline NSGFBs, where a random geometric graph $RGG_{4096}$, original signal $x_o$ and noise $\epsilon$ are the same as in Tables [IV] and [V]. It indicates that all denoising techniques have satisfactory performance inside the same strip where the signal has small variation, and that the spline NSGFBs proposed in this paper achieve better performance visually on noise suppression than the other three methods do near the boundary of two adjacency strips where the signal has large variation.

The proposed NSGFBs can be implemented in a distributed manner and they are beneficial to (local) noise suppression on graphs of very large scale. Our simulations indicate that for random geometric graphs $RGG_N$ with large size $N$ and $1 \leq p \leq \infty$, the output $\ell^p$-SNRs of spline NSGFBs have invisible change for the same input noise level when the graph size $N$ increases.

APPENDIX

A. Proof of Proposition [III.3]

The first inequality follows from (III.7). Now we prove the second inequality. Write $A = (a(i,j))_{i,j \in V}$, and define its Schur norm by

$$\|A\|_S = \max \left( \sup_{i \in V} \sum_{j \in V} |a(i,j)|, \sup_{j \in V} \sum_{i \in V} |a(i,j)| \right).$$
It is well known that the filter bound \( \|A\|_{\mathcal{B}_p}, 1 \leq p \leq \infty \), of a graph filter \( A \) is dominated by its Schur norm,
\[
\|A\|_{\mathcal{B}_p} \leq \|A\|_S \text{ for all } 1 \leq p \leq \infty.
\] (A.1)

Then it suffices to prove
\[
\|A\|_S \leq D_1(\mathcal{G})(\sigma + 1)^d \|A\|_{\infty}.
\] (A.2)

For any \( i \in V \), we obtain
\[
\sum_{j \in V} |a(i, j)| = \sum_{\rho(i, j) \leq \sigma} |a(i, j)| \leq \|A\|_{\infty} \sum_{\rho(i, j) \leq \sigma} 1 \leq D_1(\mathcal{G})(\sigma + 1)^d \|A\|_{\infty},
\] (A.3)

where the second inequality follows from (II.2). Similarly for any \( j \in V \), we have
\[
\sum_{i \in V} |a(i, j)| \leq D_1(\mathcal{G})(\sigma + 1)^d \|A\|_{\infty}.
\] (A.4)

Combining (A.3) and (A.4) completes the proof.

**B. Proof of Theorem IV.6**

The upper bound estimate (IV.16) follows directly from Proposition [III.3](#) and the observation that
\[
\|H_0\|_{\mathcal{B}_2} + \|H_1\|_{\mathcal{B}_2} \leq 2\|H\|^{1/2}_{\mathcal{B}_2}.
\] (A.5)

Now we prove the lower bound estimate (IV.15). Set
\[
B = I - \frac{H}{\|H\|_{\mathcal{B}_2}}.
\] (A.6)

Then \( B \) has bandwidth \( 2\sigma \),
\[
\|B\|_{\mathcal{B}_2} \leq (\kappa - 1)/\kappa,
\] (A.7)

and
\[
H^{-1} = (\|H\|_{\mathcal{B}_2})^{-1} \sum_{n=0}^{\infty} B^n.
\] (A.8)
Write $H^{-1} = (g(i, j))_{i, j \in V}$. For $\kappa = 1$, we have
\[ H^{-1} = (\|H\|_{B_2})^{-1}I. \] (A.9)

Now we consider the case that $\kappa > 1$. Set $\theta = \ln(\kappa/(\kappa - 1))$, and for $i, j \in V$ let $n_0(i, j)$ be the minimal integer such that $2n_0(i, j) \geq \rho(i, j)/\sigma$. Then
\[
|g(i, j)| \leq (\|H\|_{B_2})^{-1} \sum_{n=n_0(i, j)}^{\infty} \|B^n\|_{\infty}
\leq (\|H\|_{B_2})^{-1} \sum_{n=n_0(i, j)}^{\infty} \|B\|^n_{B_2}
\leq (\|H\|_{B_2})^{-1} \kappa(1 - \kappa^{-1})n_0(i, j)
\leq \|H^{-1}\|_{B_2} \exp \left( -\frac{\theta}{2\sigma} \rho(i, j) \right),
\] (A.10)
where the first inequality follows from (A.8) and the observation that $B^n$ have bandwidth $2n\sigma$, the second one is true by (III.7), and the third one holds by (III.8) and (A.7).

From (A.9) we immediately get
\[
\|H^{-1}\|_{B_2} = \|H^{-1}\|_{B_2}
\] (A.11)
if $\kappa = 1$, and by (A.1) and (A.10), we have
\[
\|H^{-1}\|_{B_p} \leq \|H^{-1}\|_{B_2} \times
\sup_{i \in V} \sum_{n=0}^{\infty} \sum_{2n\sigma \leq \rho(i, j) < 2(n+1)\sigma} \exp \left( -\frac{\theta}{2\sigma} \rho(i, j) \right)
\leq \|H^{-1}\|_{B_2} \sup_{i \in V} \sum_{n=0}^{\infty} e^{-n\theta} \mu \left( B(i, 2(n+1)\sigma - 1) \right)
\leq (2\sigma)^d D_1(G) \|H^{-1}\|_{B_2} \sum_{n=0}^{\infty} (n+1)^d (1 - \kappa^{-1})^n
\leq (2\sigma)^d D_1(G) \|H^{-1}\|_{B_2} \left( \left( \frac{1}{1 - t} \right)^{(d)} \right)_{t=1-\kappa^{-1}}
\leq d!(2\sigma)^d D_1(G) \kappa^{d+1} \|H^{-1}\|_{B_2}
\] (A.12)
if $\kappa > 1$. Then
\[
\|x\|_p \leq \|H^{-1}\|_{B_p} \left( \|H_0^T\|_{B_p} \|H_0 x\|_p + \|H_0^T\|_{B_p} \|H_1 x\|_p \right)
\leq d!(2\sigma)^d D_1(G) \kappa^{d+1} \|H^{-1}\|_{B_2}
\times \left( \|H_0^T\|_{B_p} \|H_0 x\|_p + \|H_0^T\|_{B_p} \|H_1 x\|_p \right)
\leq d!2^d(\sigma + 1)^{2d} (D_1(G))^2 \kappa^{d+1} \|H^{-1}\|_{B_2}
\times \left( \|H_0\|_{B_2} \|H_0 x\|_p + \|H_1\|_{B_2} \|H_1 x\|_p \right)
\leq d!2^{d+1}(\sigma + 1)^{2d} (D_1(G))^2 \kappa^{d+2} \|H\|_{B_2}^{1/2}
\times \max \left( \|H_0 x\|_p, \|H_1 x\|_p \right),
\leq d!2^{d+1}(\sigma + 1)^{2d} (D_1(G))^2 \kappa^{d+2} \|H\|_{B_2}^{-1/2}
\times \left( \|H_0 x\|_p + \|H_1 x\|_p \right)^{1/2},
\] (A.13)
where the second inequality follows from (A.11) and (A.12), the third holds by Proposition (III.3) and the fourth one is true by (A.S) and (IV.17). This proves (IV.15) and completes the proof.
C. Proof of Theorem VII.2

By (III.3), (V.1), (V.2) and (V.3), we obtain
\[ G_0 H_0 + G_1 H_1 = Q_0 (L_0^{\text{sym}})P_0 (L_0^{\text{sym}}) + Q_1 (L_1^{\text{sym}})Q_1 (L_1^{\text{sym}}) = U^T (Q_0 (A)P_0 (A) + Q_1 (A)P_1 (A)) U = U^T U = I. \]

This completes the proof.

D. Proof of Proposition VII.2

Set \( z_0 = H_0 x \) and \( z_1 = H_1 x \). Then
\[
\| \hat{x} - x \|_p \leq \| G_0 (z_0 - \Psi_0 (z_0)) \|_p + \| G_1 (z_1 - \Psi_1 (z_1)) \|_p \\
\leq (\| G_0 \|_{B_p} + \| G_1 \|_{B_p}) \epsilon \\
\leq D_1 (G)(\sigma + 1)^d (\| G_0 \|_{\infty} + \| G_1 \|_{\infty}) \epsilon,
\]
where the first inequality follows from the perfect reconstruction condition (I.1) for the NSGFB constructed in Theorem VII.1, the second one holds by (V.I), and the last estimate is true by Proposition III.3.

E. Proof of Theorem VI.1

By (VI.4) and (A.10), we have
\[
|g_{l,l}(i,j)| \leq \| H^{-1} \|_{B_p} \| H_l \|_{\infty} \sum_{\rho(k,j) \leq \sigma} \exp \left( -\frac{\theta}{2\sigma} \rho(i,k) \right) \\
\leq D_1 (G) \| H^{-1} \|_{B_p} \| H_l \|_{\infty} (\sigma + 1)^d \\
\times \exp \left( -\frac{\theta}{2\sigma} \rho(i,j) + \frac{\theta}{2} \right), \ i,j \in V;
\]
where \( l = 0, 1 \). This proves (VI.6).

F. Proof of Theorem VII.2

Set \( y^{(m)} = \hat{x} - x^{(m)} \) and write \( y^{(m)} = (y^{(m)}(i))_{i \in V}, m \geq 0 \). We claim that
\[
y^{(m)} = H^{-1} (H_0^T \hat{z}_0^{(m)} + H_1^T \hat{z}_1^{(m)}), \ m \geq 0.
\]

The above claim holds for \( m = 0 \), since
\[
y^{(0)} = \hat{x} = H^{-1} (H_0^T \hat{z}_0^0 + H_1^T \hat{z}_1^0) = H^{-1} (H_0^T \hat{z}_0^0 + H_1^T \hat{z}_1^0)
\]
by (VII.1) and (VII.10). Inductively for \( m \geq 1 \), we have
\[
y^{(m)} = y^{(m-1)} - v^{(m)} \\
= H^{-1} (H_0^T \hat{z}_0^{(m-1)} + H_1^T \hat{z}_1^{(m-1)}) - v^{(m)} \\
= H^{-1} (H_0^T \hat{z}_0^{(m)} + H_1^T \hat{z}_1^{(m)}),
\]
where the first and third equalities follow from (VII.9) and the second equality holds by the inductive hypothesis. This completes the proof of Claim A.15.

Write \( (\chi_k^{2r} H \chi_k^{2r} )^{-1} = (g_k(i,j))_{i,j \in B(k,2r)} \) and
\[
\chi_k^{r} (\chi_k^{2r} H \chi_k^{2r} )^{-1} \chi_k^{2r} H (\chi_k^{2r+2r} - \chi_k^{2r}) = (\tilde{g}_k(i,j))_{i,j \in V, k \in V}.
\]
Following the argument used to prove (A.10), we have

\[ |g_k(i, j)| \leq \|H^{-1}\|_{B_2} \exp \left( -\frac{\theta}{2\sigma} \rho(i, j) \right) \]  

(A.17)

for all \( i, j \in B(k, 2r) \). By (III.7), (A.16) and (A.17), we obtain

\[ \tilde{g}_k(i, j) = 0 \]  

(A.18)

where either \( i \notin B(k, r) \) or \( j \notin B(k, 2r + 2\sigma) \backslash B(k, 2r) \), and

\[ |\tilde{g}_k(i, j)| \leq \|H^{-1}\|_{B_2} \|H\|_\infty \sum_{l \in B(j, 2\sigma)} \exp \left( -\frac{\theta}{2\sigma} \rho(i, l) \right) \]

\[ \leq D_1(G)(2\sigma + 1)^d \kappa \exp \left( -\frac{\theta}{2\sigma} r + \theta \right) \]  

(A.19)

where \( i \in B(k, r) \) and \( j \in B(k, 2r + 2\sigma) \backslash B(k, 2r) \).

Write \( v^{(m)}_k = (v^{(m)}_k(i))_{i \in V}, m \geq 1, k \in V \). By (VII.9), (A.15), we have

\[ \chi_k^r(v^{(m)}_k - y^{(m-1)}) = \chi_k^r(\lambda_{k}^{2\sigma} H \lambda_{k}^{2\sigma} - \lambda_{k}^{2\sigma}) \times H(\lambda_{k}^{2\sigma + 2\sigma} - \lambda_{k}^{2\sigma}) y^{(m-1)}. \]

Combining the above equation with (A.18) and (A.19), we get

\[ |v^{(m)}_k(i) - y^{(m-1)}(i)| = \left| \sum_{j \in B(k, 2r + 2\sigma)} \tilde{g}_k(i, j) y^{(m-1)}(j) \right| \]

\[ \leq D_1(G)(2\sigma + 1)^d \kappa \exp \left( -\frac{\theta}{2\sigma} r + \theta \right) \]

\[ \times \left( \sum_{j \in B(i, 3r + 2\sigma)} |y^{(m-1)}(j)| \right), \quad i \in B(k, r). \]  

(A.20)

This together with (VII.9) implies that

\[ |y^{(m)}(i)| = \left| v^{(m)}(i) - y^{(m-1)}(i) \right| \]

\[ \leq \frac{1}{\mu(B(i, r))} \sum_{k \in B(i, r)} |v^{(m)}_k(i) - y^{(m-1)}(i)| \]

\[ \leq D_1(G)(2\sigma + 1)^d \kappa \exp \left( -\frac{\theta}{2\sigma} r + \theta \right) \]

\[ \times \left( \sum_{j \in B(i, 3r + 2\sigma)} |y^{(m-1)}(j)| \right) \]  

(A.21)

for all \( i \in V \) and \( m \geq 1 \). Using the above componentwise estimate, we obtain

\[ \|y^{(m+1)}\|_p \leq \delta_{r, \sigma} \|y^{(m)}\|_p, \quad m \geq 0. \]  

(A.22)

Iteratively applying the above estimate proves (VII.14).
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