Variational speed selection for the interface propagation in superconductors

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We study the interface propagation in superconductors by means of a variational method. We compute the lower and upper bounds for which the planar front speed propagation is valid. To take into account delay or memory effects in the front propagation, an hyperbolic differential equation is introduced as an extension of the model.

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In recently years the magnetic field penetration or its expulsion from Superconducting samples have attracted the attention of different research groups. The understanding of the different particularities of this phenomenon has been a major challenge.

In Ref[1] the authors have obtained the interface speed by using heuristic methods such as Marginal stability hypothesis(MSH) and Reduction order.

In this paper, we board the interface propagation speed from the variational point of view. The variational speed selection or BD method was proposed by Benguria and Depassier\textsuperscript{2,4} in order to study the reaction-diffusion equations. Using a trial function $g(x)$ in the procedure, one may find accurate lower and upper bounds for the speed $c$. The function $g(x)$ must satisfy that $g(x) > 0$ and $g'(x) < 0$ in $(0,1)$. Only if the lower and the upper bounds coincide can the value of $c$ be determined without any uncertainty.

Our start point are the Ginzburg-Landau equations\textsuperscript{5}, which comprise a coupled equations for the density of superconducting electrons and the local magnetic field. In order to describe the evolution of the system between two homogeneous steady states, we assume a SC sample embedded in a stationary applied magnetic field equal to the critical $H = H_c$. The magnetic field is rapidly removed, so the unstable normal-superconducting planar interface propagates toward the normal phase so as to expel any trapped magnetic flux, leaving the sample in Meissner state. Also, we have considered that the interface remains planar during all the process.

To take into account the delay effect in the interface propagation, due to, for example, imperfections and non-homogeneous superconducting properties in the material, we have included the delay time $\tau$ and indeed introduce the hyperbolic differential(HD) equation. This type of equation has been recently applied in biophysics to model the spread of humans\textsuperscript{6}, bistable systems\textsuperscript{7}, forest fires\textsuperscript{8}, and in population dynamics\textsuperscript{8}.

Traveling wave solutions. We are interested in finding traveling wave solutions for our model. To start we use the one-dimensional time-dependent Ginzburg-Landau equations(TDGL), which in dimensionless units\textsuperscript{2} are

$$\partial_t f = \frac{1}{\kappa^2} \partial_x^2 f - q^2 f + f - f^3,$$

$$\bar{\sigma} \partial_t q = \partial_x^2 q - f^2 q,$$

where $f$ is the magnitude of the superconducting order parameter, $q$ is the gauge-invariant vector potential (such that $h = \partial_t q$ is the magnetic field), $\bar{\sigma}$ is the dimensionless normal state conductivity (the ratio of the order parameter diffusion constant to the magnetic field diffusion constant) and $\kappa$ is the Ginzburg-Landau parameter which determines the type of superconducting material; $\kappa < 1/\sqrt{2}$ describes what are known as type-I superconductors, while $\kappa > 1/\sqrt{2}$ describes what are known as type-II superconductors.

In our analysis we will search for steady traveling waves solutions of the TDGL equations of the form $f(x,t) = s(x - ct)$ and $q(x,t) = n(x - ct)$, where $z = x - ct$ with $c > 0$. Then the equations become

$$\frac{1}{\kappa^2} s_{zz} + c s_z - n^2 s + s - s^3 = 0,$$

$$n_{zz} + \bar{\sigma} c n_z - s^2 n = 0,$$

I. VARIATIONAL ANALYSIS

Vector potential $q = 0$. In this section, we assume $q = 0$ for the TDGL equations,

$$\partial_t f = \frac{1}{\kappa^2} \partial_x^2 f + f - f^3.$$

Then, there exists a front $f = s(x - ct)$ joining $f = 1$, the state corresponding to the whole superconducting phase to $f = 0$ the state corresponding to the normal phase. Both states may be connected by a traveling front with speed $c$. The front satisfies the boundary conditions $\lim_{s \to -\infty} f = 1$, $\lim_{s \to -\infty} f = 0$. Then Eq.(3) can be written as,

$$s_{zz} + c \kappa^2 s_z + \mathfrak{F}_k(s) = 0,$$

where $\mathfrak{F}_k$ is given by $\mathfrak{F}_k = \kappa^2 s(1 - s^2)$.
We define \( p(s) = -ds/dz \), where the minus sign is included so that \( p \) is positive. One finds that the front is solution of

\[
P(s) \frac{dp(s)}{ds} - c \kappa^2 p(s) + \tilde{g}_k(s) = 0, \tag{5}
\]

with \( p(0) = 0, p(1) = 0, p > 0 \) in \((0,1)\).

Let \( g \) be any positive function in \((0,1)\) such that \( h = -dg/ds > 0 \). Multiplying Eq.\((5)\) by \( g(s) \) and integrating by parts between \( s = 0 \) and \( s = 1 \) and taking into account \( h p + (g \tilde{g}_k/p) \geq 2 \sqrt{g h \tilde{g}_k} \), we obtain that,

\[
c \geq \frac{2}{\kappa} \int_0^1 (g h \tilde{g})^{1/2} ds/\int_0^1 g ds, \tag{6}
\]

As a trial function we have chosen \( g(s) = (1-s)^2 \). Then one finds that,

\[
c \geq \frac{2}{\kappa} \left[ \int_0^1 s(1-s)^2 ds \right]^{1/2} \int_0^1 (1-s)^2 ds \tag{7}
\]

after integration the speed is given by,

\[
c \geq \frac{3}{64 \kappa} \left[ 124 + 37 \sqrt{2} \log(3 - 2 \sqrt{2}) \right]. \tag{8}
\]

Notice that \( c \leq 2/\kappa \), where \( 2/\kappa \) is the result obtained by using the MSH method. In Fig.1, the graphic shows that for values \( \kappa > 1.4 \) the MSH speed tends to the BD value, but for \( \kappa < 1.4 \) the variational speed selection provides a better lower bound.

**Vector potential** \( q = 1 - f \). For a set of parameters \( \kappa = 1/\sqrt{2} \) and \( \sigma = 1/2 \), we have that \( s(z) = n(z) = 1 \), then Eq.\((6)\) takes the form,

\[
s_{zz} + \frac{c}{2} s_z + \bar{\mathcal{F}}(s) = 0, \tag{9}
\]

With this in mind, we look for solutions of the form \( s(z) = 1 - n(z) \). Proceeding as in Eq.\((6)\) we have that,

\[
c \geq 2 \sqrt{2} \int_0^1 (g h \bar{\mathcal{F}})^{1/2} ds/\int_0^1 g ds, \tag{10}
\]

then,

\[
c \geq 2 \sqrt{2} \int_0^1 \left[ n^2(1-n)^2(1-n)(2-2n) \right]^{1/2} ds \int_0^1 (1-n)^2 ds. \tag{11}
\]

Finally, for the Eq.\((11)\) we arrive to \( c \geq 1 \), which is a better lower bound than the \( \sqrt{2} \) predicted by the MSH method.

**II. FRONT FLUX EXPULSION WITH DELAY**

An import feature phenomena is the existence of a delay time. In systems with interface propagation, this can be taken into account by resorting to the hyperbolic differential equation seen in Section I, which generalizes the parabolic equation. The aim of this section is to study the interface speed problem in superconducting samples by means of the HD equations.

Our starting point is the HD equation,

\[
\tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) + \tau \frac{\partial f(u)}{\partial t}. \tag{12}
\]

In the absence of a delay time \( (\tau = 0) \), this reduces to the classical equation \( u_t = u_{xx} + f(u) \).

**Vector potential** \( q = 0 \). Taking into account the Eqs.\((1)\) and \((12)\) we can write the following expression,

\[
\kappa^2 \tau \frac{\partial^2 f}{\partial t^2} + \kappa^2 \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \kappa^2 \bar{\mathcal{F}} + \kappa^2 \tau \frac{\partial \bar{\mathcal{F}}}{\partial t}, \tag{13}
\]

where \( \bar{\mathcal{F}} = s(1-s^2) \).

It has been proved that Eq.\((13)\) has traveling wave fronts with profile \( s(x-ct) \) and moving with speed \( c > 0 \). Then we can write Eq.\((13)\) as follows,

\[
(1 - a c^2) s_{zz} + c [\kappa^2 - a \bar{\mathcal{F}}(s)] s_z + \bar{\mathcal{F}}_k(s) = 0, \tag{14}
\]

where \( z = x-ct \), \( a = \kappa^2 \tau \), \( \bar{\mathcal{F}}_k = \kappa^2 \bar{\mathcal{F}} \), and with boundary conditions \( \lim_{z \to -\infty} s = 0 \), \( \lim_{z \to +\infty} s = 1 \), and \( s_z < 0 \) in \((0,1)\); \( s_z \) vanishes for \( z \to \pm \infty \).

For the variational analysis we define \( p(s) = -s_z \) with \( p(0) = p(1) = 0 \) and \( p > 0 \) in \((0,1)\). Then the Eq.\((14)\) may be written as

\[
(1 - a c^2) p \frac{dp}{ds} - c [\kappa^2 - a \bar{\mathcal{F}}(s)] p + \bar{\mathcal{F}}_k(s) = 0. \tag{15}
\]

Multiplying Eq.\((15)\) by \( g/p \) where \( g \) is an arbitrary positive function and integrating by parts, we have that

\[
c \kappa^2 \int_0^1 g[1 - \frac{a}{\kappa^2} \bar{\mathcal{F}}] ds = \int_0^1 [(1 - a c^2) h p + \frac{g \bar{F}_k}{p}] ds. \tag{16}
\]

where we have used the relation

\[
(1 - a c^2) h p + \frac{g \bar{F}_k}{p} \geq 2 \sqrt{1 - ac^2} \sqrt{g h \bar{F}_k}, \tag{17}
\]
and \( h = -g' > 0 \).

\[
\frac{c}{\sqrt{1 - ac^2}} \geq 2\kappa \frac{\int_0^1 (g h \tilde{\delta})^{1/2} ds}{\int_0^1 g(\kappa^2 - a \tilde{\delta}') ds}. \tag{18}
\]

The maximum is attained for a \( g \). Thus, the expression for the velocity is given by

\[
c \geq 2\kappa \frac{I_1}{[I_2^2 + 4\kappa^2 a I_2^2]^{1/2}}. \tag{19}
\]

\[
I_1 = \int_0^1 \sqrt{gh\tilde{\delta}} ds, \quad I_2 = \int_0^1 g(\kappa^2 - a \tilde{\delta}') ds, \tag{20}
\]

Notice that if the delay time is neglected \( a = 0 \), this reduces to Eq. \((19)\).

The lower bound. To compute the lower bound we start with the trial function given by \( g(s) = (1 - s)^2 \) and the expression for \( \tilde{\delta}_k \), which both are substituted in Eq. \((19)\). Then,

\[
I_1 = \int_0^1 \left[ 2n(n^2 - 1)(n - 1)^3 \right]^{1/2} dn,
\]

\[
I_2 = \int_0^1 (1 - n)^2 \left[ \kappa^2 - a(1 - 3n^2) \right] dn. \tag{21}
\]

from Eq. \((19)\) we have that,

\[
c \geq 2\kappa \frac{\mathcal{J}}{[1 + 4\kappa^2 a \mathcal{J}^2]^{1/2}}, \tag{22}
\]

where

\[
\mathcal{J} = \frac{15 \left[ 124 + 37\sqrt{2} \log (3 - 2\sqrt{2}) \right]}{64(10k^2 - 2a)}. \tag{23}
\]

The upper bound. The upper bound can be computed by using the Jensen’s inequality,

\[
\int_0^1 \mu(s) \sqrt{\alpha(s)} ds \leq \sqrt{\int_0^1 \mu(s) \alpha(s) ds / \int_0^1 \mu(s) ds}, \tag{24}
\]

where \( \mu(s) > 0 \) and \( \alpha(s) \geq 0 \). We define \( \mu(s) = g(\kappa^2 - a \tilde{\delta}') \) and \( \alpha(s) = \tilde{\delta} h/g(\kappa^2 - a \tilde{\delta}')^2 \). Then we can write

\[
\int_0^1 (g h \tilde{\delta})^{1/2} ds \leq \left[ \int_0^1 \left[ h \tilde{\delta}/(\kappa^2 - a \tilde{\delta}') \right] ds \right]^{1/2}, \tag{25}
\]

where

\[
\int_0^1 h \tilde{\delta} ds = \int_0^1 n [1 - n^2 (2 - 2n)] d\mathfrak{g},
\]

\[
\int_0^1 g(\kappa^2 - a \tilde{\delta}') ds = \int_0^1 (1 - n)^2 [\kappa^2 - a(1 - 3n^2)] d\mathfrak{g}. \tag{26}
\]

then we have that,

\[
\frac{c}{\sqrt{1 - ac^2}} \geq \frac{2 \kappa \int_0^1 (g h \tilde{\delta})^{1/2} ds}{\int_0^1 g(\kappa^2 - a \tilde{\delta}') ds}. \tag{27}
\]

The Eq. \((27)\) gives a better upper bound than the one predicted by linear stability, i.e. \( c = c_{max} = 1/\sqrt{a} \).

In Fig. 2 we have plotted the results of the BD method given by Eqs. \((22)\) and \((27)\) as well as the bound proposed by linear stability (LS) methodology. The interface speed propagation can be predicted in a precisely way by using our trial function. On the other hand the difference with linear stability result is notable.

Vector potential \( q = 1 - f \). Taking into account the Eqs. \((11)\) and \((12)\) we can write the following expression,

\[
\frac{\tau}{2} \frac{\partial^2 f}{\partial t^2} + \frac{1}{2} \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \tilde{\delta} + \frac{\tau}{2} \frac{\partial \tilde{\delta}}{\partial t}, \tag{30}
\]

where \( \tilde{\delta} = s^2(1 - s) \).

Then we can write Eq. \((30)\) as follows,

\[
(1 - ac^2) s_x x + c [\kappa^2 - a \tilde{\delta}^2(s)] s_z + \tilde{\delta}_k(s) = 0, \tag{31}
\]

where we have assumed \( \tilde{\delta}_k = (1/2) \tilde{\delta} \) and \( a = \tau/2 \).

The expression for the velocity is given by

\[
\frac{c}{\sqrt{1 - ac^2}} \geq \frac{2 \kappa \int_0^1 (g h \tilde{\delta})^{1/2} ds}{\int_0^1 g(\kappa^2 - a \tilde{\delta}') ds}. \tag{32}
\]
Proceeding as in Eq. (18), we get the following expression,

\[ c \geq 2 \sqrt{2} \frac{J_1}{(J_2^2 + 8aJ_1^3)^{1/2}} \]  

(33)

where

\[ J_1 = \int_0^1 \sqrt{gb\bar{\xi}}ds, \quad J_2 = \int_0^1 g(1 - 2a\bar{\zeta})ds, \]  

(34)

The lower bound. As mentioned before, one may obtain lower bound for the interface speed by means of our trial function \( g(n) \). Taking into account Eqs. (34), the integral functions can be written as,

\[ J_1 = \int_0^1 [n^2(2 - 2n)(1 - n)^3]^{1/2} dn, \]

\[ J_2 = \int_0^1 (1 - n)^2 [1 - 2a(2n - 3n^2)] dn. \]  

(35)

Then the velocity takes the form,

\[ c \geq \sqrt{5}(15 - a)^{-1/2}, \]  

(36)

The upper bound. To compute the upper bound we have used the expression Eq. (25) but with \( \bar{\zeta} = s^2(1 - s) \) and \( \kappa^2 = 1/2 \), then

\[ \frac{c}{\sqrt{1 - ac^2}} \leq 2 \sqrt{2} \frac{\int_0^1 \frac{\bar{\zeta}}{[1 - 2a\bar{\zeta}]} ds}{\int_0^1 \frac{1}{[1 - 2a\bar{\zeta}^2]} ds}. \]  

(37)

\[ \int_0^1 \frac{\bar{\zeta}}{[1 - 2a\bar{\zeta}]} ds = \int_0^1 \frac{1}{[1 - 2a\bar{\zeta}^2]} ds, \]

\[ \int_0^1 g(1 - 2a\bar{\zeta})ds = \int_0^1 (1 - n)^2 [1 - 2a(2n - 3n^2)] dn. \]  

(38)

After integrating and do some algebra, the expression for the velocity is given by

\[ c \geq 2 \sqrt{2} \frac{\mathcal{B}}{[\frac{4}{3} - \frac{4\alpha}{\mathcal{B}}] + 8a\mathcal{B}^2}^{1/2}. \]  

(39)

where,

\[ \mathcal{B} = \frac{\beta_1}{18\sqrt[4]{a^{3/2}}} \left( \frac{2\arctan \alpha + \arctan 4\alpha}{\sqrt{6 - 4a}} + \frac{\beta_2}{\sqrt{2 - a}} \right)^{1/2}, \]  

(40)

\[ \alpha \equiv \sqrt{a\left(\frac{3}{2} - a\right)/(3 - a)}, \]  

(41)

\[ \beta_1 \equiv 8a^2 + 6a + 9, \]

\[ \beta_2 \equiv 2\sqrt{a}[3(3 + 4a) + (3 + 2a)\log(1 + 2a)]. \]

In Fig. 3 we have plotted the results of the BD method given by Eqs. (36) and (39) as well as the bound proposed by LS method. The interface speed propagation can be predicted in a precisely way by using this trial function. On the other hand the difference with LS result is notable.

Conclusion. Throughout this work, we have performed analytical analyses on the superconducting-normal interface propagation speed problem in parabolic and hyperbolic equations. We have made use of the variational analysis to obtain the lower and upper bounds for the speed in each case.