HODGE NUMBERS OF HYPERSURFACES IN $\mathbb{P}^4$ WITH ORDINARY TRIPLE POINTS

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Abstract. We give a formula for the Hodge numbers of hypersurfaces in $\mathbb{P}^4$ with ordinary triple points.

Introduction

Let $X$ be a degree $d$ hypersurface in the projective four-space $\mathbb{P}^4$ with ordinary triple points as the only singularities. Denote by $\Sigma = \text{Sing}(X)$ the singular locus of $X$ and by $\tilde{X}$ the natural crepant resolution of singularities of $X$ by the blow–up of the singular locus $\Sigma \subset \mathbb{P}^4$.

The main goal of the present paper it to proof the following formula for the Hodge numbers of $\tilde{X}$ (Cor. 4)

$$h^{1,1}(\tilde{X}) = 1 + \mu + \delta, \quad h^{1,2}(\tilde{X}) = h^{1,2}((X_{\text{smooth}}) - 11\mu + \delta,$$

where $X_{\text{smooth}}$ is a smooth hypersurface in $\mathbb{P}^4$ of the same degree, $\mu = \#\Sigma$ is the number of singularities of $X$ and $\delta$ is non–negative integer called the defect. Our main result is an extension of Werner’s defect formula for nodal hypersurfaces ([12]) as explained in Remark 1.

Three dimensional varieties with ordinary triple points admit crepant divisorial resolution of singularities, they are very suitable for explicit constructions. Probably the main obstacle in applications was the lack of formulas for the invariants (Betti numbers or Hodge numbers). Our formula can be also used (via a triple cyclic covering, cf. Rem. 2) to study surface of degree divisible by three with ordinary triple points as the only singularities, surfaces of a small degree with triple points have been studied in [11, 5].

The proof of the main theorem is based on a study of exact sequences of cohomology groups of differential forms (with logarithmic poles), cf. [2]. The main difference is that in the present paper it is not sufficient to determine the dimensions of various cohomology groups but we need to study the rank of the natural map $H^1(\Omega^3_{\mathbb{P}^4}(\tilde{X}) \rightarrow H^1(\Omega^3_X(\tilde{X})))$ which was possible because of an explicit representation of basis of various cohomology group (based on f.i. [3]).

1. Logarithmic differential forms

Throughout the paper $X$ is a degree $d$ hypersurface in the projective space $\mathbb{P}^4$ with ordinary triple points as the only singularities. Let $\Sigma := \{P_1, \ldots, P_\mu\}$ be the singular locus of $X$ and $F \in \mathbb{k}[X_0, \ldots, X_4]$ the homogeneous equation of $X$. The
strict transform \( \tilde{X} \) of \( X \) under the blow-up \( \sigma : \mathbb{P}^4 \to \tilde{\mathbb{P}}^4 \) is a crepant resolution of singularities. Let \( E_i := \sigma^{-1}(P_i) \) be the exceptional divisor of \( \sigma \) over a singular point \( P_i \) and denote \( E := \sigma^{-1}(\Sigma) = E_1 + \cdots + E_\mu \). In this situation the following formulas hold (cf. [2]).

**Proposition 1.**

1. \( \sigma_* \mathcal{O}_{\mathbb{P}^4}(-mE) \cong \mathcal{J}_m, \quad \text{for } m \geq 0, \)
2. \( R^i \sigma_* \mathcal{O}_{\mathbb{P}^4}(-mE) = 0, \quad \text{for } i \neq 0, m \geq 0, \)
3. \( H^i(\mathcal{O}_{\tilde{X}}) = 0, \quad \text{for } i = 1, 2, \)
4. \( H^1(\Omega^3_{\mathbb{P}^4}) = 0 \) for \( i \leq 2, \)
5. \( H^1(\Omega^3_{\mathbb{P}^4}(\tilde{X})) \cong H^1(\Omega^3_{\mathbb{P}^4}(X)), \)
6. \( H^1(\Omega^3_{\mathbb{P}^4}(2\tilde{X})) \cong H^1(\Omega^3_{\mathbb{P}^4}(2X) \otimes \mathcal{J}_\Sigma). \)

**Lemma 2.** \( h^{1,2}(\tilde{X}) = h^0(\Omega^3_{\mathbb{P}^4}(\tilde{X}))-h^0(\Omega^3_{\mathbb{P}^4}(\tilde{X}))+\dim \ker(\Omega^3_{\mathbb{P}^4}(\tilde{X}) \to \Omega^3_{\mathbb{P}^4}(\tilde{X})) \)

**Proof.** By Serre duality, \( H^0\Omega^3_{\mathbb{P}^4} = H^2(\mathcal{O}_{\tilde{X}}) = 0. \) Since \( \tilde{X} \) is smooth the following sequence is exact ([6, 2.3(b)])

\[
0 \to \Omega^3_{\mathbb{P}^4} \to \Omega^3_{\mathbb{P}^4}(\log \tilde{X}) \to \Omega^2_{\tilde{X}} \to 0
\]

which implies

\[
H^0\Omega^3_{\tilde{X}} \cong H^0\Omega^3_{\mathbb{P}^4}(\log \tilde{X}) = 0
\]

\[
H^1\Omega^2_{\tilde{X}} \cong H^1\Omega^3_{\mathbb{P}^4}(\log \tilde{X}).
\]

Similarly, the following sequence is exact ([6, 2.3(c)])

\[
0 \to \Omega^3_{\mathbb{P}^4}(\log \tilde{X}) \to \Omega^3_{\mathbb{P}^4}(\tilde{X}) \to \Omega^1_{\tilde{X}}(\tilde{X}) \to 0.
\]

and the derived long exact sequence

\[
0 \to H^0\Omega^3_{\mathbb{P}^4}(\tilde{X}) \to H^0\Omega^3_{\mathbb{P}^4}(\tilde{X}) \to H^1\Omega^3_{\mathbb{P}^4}(\tilde{X}) \to H^1\Omega^3_{\mathbb{P}^4}(\tilde{X}) \to H^1\Omega^3_{\tilde{X}}(\tilde{X})
\]

implies the assertion of the lemma. \( \square \)

**Corollary 3.** The following sequence is exact

\[
H^0\Omega^3_{\mathbb{P}^4}(X) \to H^0(\Omega^3_{\mathbb{P}^4}(X) \otimes \mathcal{E}) \to H^1\left(\Omega^3_{\mathbb{P}^4}(\tilde{X})\right) \to 0.
\]

**Proof.** By direct computations \( \sigma^*\Omega^3_{\mathbb{P}^4} = \Omega^3_{\mathbb{P}^4}(\log E)(-3E), \) so we have an exact sequence ([6, 2.3(c)])

\[
0 \to \sigma^*\Omega^3_{\mathbb{P}^4}(X) \otimes \mathcal{E}(\tilde{X}) \to \Omega^3_{\mathbb{P}^4}(\tilde{X}) \to \Omega^3_{\mathbb{P}^4}(\tilde{X}) \to 0,
\]

applying the direct image yields

\[
\sigma_*\Omega^3_{\mathbb{P}^4}(\tilde{X}) = \Omega^3_{\mathbb{P}^4}(\tilde{X}) \otimes \mathcal{J}_\Sigma, \quad R^i \sigma_*\Omega^3_{\mathbb{P}^4}(\tilde{X}) = 0 \quad \text{for } i > 0.
\]

From the Leray spectral sequence we get

\[
H^1\Omega^3_{\mathbb{P}^4}(\tilde{X}) = H^1(\Omega^3_{\mathbb{P}^4}(X) \otimes \mathcal{J}_\Sigma)
\]

and the assertion follows from the cohomology derived sequence associated to

\[
0 \to \Omega^3_{\mathbb{P}^4}(X) \otimes \mathcal{J}_\Sigma \to \Omega^3_{\mathbb{P}^4}(X) \to \Omega^3_{\mathbb{P}^4}(X) \otimes \mathcal{O}_\Sigma \to 0.
\]

\( \square \)
Corollary 4. The following sequence is exact

\( H^0(\Omega^4_{\tilde{P}^4}(2X)) \rightarrow H^0(\Omega^4_{\tilde{P}^4}(2X) \otimes \mathcal{O}_{3\Sigma}) \rightarrow H^1(\Omega^3_{\tilde{X}}(\tilde{X})) \rightarrow 0 \)

Proof. By the adjunction formula \( \Omega^3_{\tilde{X}}(\tilde{X}) \cong \Omega^4_{\tilde{P}^4}(2\tilde{X})|_{\tilde{X}} \), so we have an exact sequence

\[
0 \rightarrow \Omega^4_{\tilde{P}^4}(\tilde{X}) \rightarrow \Omega^4_{\tilde{P}^4}(2\tilde{X}) \rightarrow \Omega^3_{\tilde{X}}(\tilde{X}) \rightarrow 0,
\]

the derived long exact sequence and the Proposition [1] yield

\[
H^1(\Omega^3_{\tilde{X}}(\tilde{X})) \cong H^1(\Omega^4_{\tilde{P}^4}(2X) \otimes \mathcal{J}_{3\Sigma}).
\]

The assertion follows now from the exact sequence

\[
0 \rightarrow \Omega^4_{\tilde{P}^4}(2X) \otimes \mathcal{J}_{3\Sigma} \rightarrow \Omega^4_{\tilde{P}^4}(2X) \rightarrow \Omega^4_{\tilde{P}^4}(2X) \otimes \mathcal{O}_{3\Sigma} \rightarrow 0.
\]

2. Main result

We keep the notation introduced in the previous section.

Definition 1. Define the equisingular ideal of \( X \) as

\[
I_{\text{eq}} := \bigcap_{i=1}^{\mu} (m_i^3 + \text{Jac } F),
\]

where \( m_i \) is the (maximal) ideal of \( P_i \) and \( \text{Jac } F \) is the jacobian ideal of \( F \).

Let \( S = \bigoplus_{d=0}^{\infty} S_d = k[X_0, \ldots, X_4] \) be the graded ring of polynomials in five variables, for a homogeneous ideal \( I \subset S \) we denote by \( I^{(d)} := I \cap S^d \) the degree \( d \) graded summand of \( I \).

Theorem 5.

\[
\begin{align*}
& h^{1,1}(\tilde{X}) = \dim(I^{(2d-5)}_{\text{eq}}) - \binom{2d-1}{4} + 12\mu + 1 \\
& h^{1,2}(\tilde{X}) = \dim(I^{(2d-5)}_{\text{eq}}) - 5\binom{d}{4}
\end{align*}
\]

Proof. Consider the following commutative diagram with exact rows [1] and [2]

\[
\begin{array}{ccc}
(S^{d-4})^{\oplus 5} / S^{d-5} & \xrightarrow{k^{4\mu}} & S^{d-5} \\
\downarrow \cong & & \downarrow \cong \\
H^0(\Omega^3_{\tilde{P}^4}(X)) & \xrightarrow{\theta} & H^0(\Omega^4_{\tilde{P}^4}(X) \otimes \mathcal{O}_{3\Sigma}) \\
\downarrow \cong & & \downarrow \cong \\
H^0(\Omega^3_{\tilde{P}^4}(2X)) & \xrightarrow{\gamma} & H^1(\Omega^3_{\tilde{X}}(\tilde{X})) \\
\downarrow \cong & & \downarrow \phi \\
S^{2d-5} & \xrightarrow{\alpha} & k^{15\mu}
\end{array}
\]
We shall describe explicitly all maps in the above diagram. Denote by $K_j$ the contraction with the vector field $\frac{\partial}{\partial X_j}$ and by $\Omega$ the 4-form $\sum_{i=0}^4 (-1)^i dX_0 \wedge \cdots \wedge dX_i$. The two vertical isomorphisms in the first column are given by

$$(S^{d-4})^\oplus \ni (A_0, \ldots, A_4) \mapsto \sum_{i=0}^k \frac{A_i}{F^5} \kappa_i \Omega \in H^0(\Omega^3_4(X))$$

(with the inclusion $S^{d-5} \ni A \mapsto (AX_0, \ldots, AX_4) \in (S^{d-4})^\oplus$). and

$$S^{(2d-5)} \ni A \mapsto \frac{A}{F^5} \Omega \in H^0(\Omega^1_4(2X))$$

In terms of these isomorphisms the homomorphism $\eta$ associates to a quintuple $A_0, \ldots, A_4$ the values at singular points $P_j$ in the vector space $(k^4)^n$ identified with $\sum_{j=1}^n (k^5/P_j k)$. Finally $\xi$ is the exterior derivative $\omega \mapsto d\omega$ so

$$d \left( \sum_{i=0}^4 \frac{A_i}{F^5} K_i \Omega \right) = \frac{1}{F^2} \sum_{i=0}^4 (F \frac{\partial A_i}{\partial X_i} - A_i \frac{\partial F}{\partial X_i}) \Omega$$

and consequently $\beta(A_0, \ldots, A_4, \mu)$ is given by $3$–jets of $\sum_{i=0}^4 A_i(P_i) \frac{\partial F}{\partial X_i}$ at singular points $P_1, \ldots, P_\mu$. As the polynomial $F$ has an ordinary triple point at $P_j$ partial derivatives of $F$ at $P_j$ are linearly independent modulo the third power $m_j^3$ of the maximal ideal $m_j$, and consequently the map $\beta$ is injective. In particular $\dim \text{Im} (\beta) = 4\mu$. Diagram chasing with simple linear algebra yields

$$\dim \text{Ker } \Phi = h^1(\Omega^3_4(X)) - \dim(\text{Im } \beta) + \dim(\text{Im } \eta \cap \text{Im } \beta).$$

The local description shows that $\text{Im}(\eta) \cap \text{Im}(\beta) \cong (I_{eq}/(\bigcap_{j=1}^\mu m_j^3))^{(2d-5)}$. Using the Lemma 2 we get formula for $h^{1,2}$.

Observe that the Milnor number at an ordinary triple point is 16, resolution replaces this point with a smooth cubic surface with the Euler number 9, finally $e(\tilde{X}) = -d^4 + 5d^3 - 10d^2 + 10d + 24\mu$. As the resolution of $X$ is crepant we have $h^{0,1}(\tilde{X}) = (d-1)$ and the formula for $h^{1,1}$ follows. \hfill $\square$

Since the point $P_i$ ($i = 1, \ldots, \mu$) is an ordinary triple point, the codimension of the ideal $(m_i^3 + \text{Jac } F)$ equals 11, consequently the expected dimension of $I_{eq}^{(2d-5)}$ is $(2d-1) - 11\mu$. We shall call the difference between “the actual dimension” and “the expected dimension” the defect.

**Definition 2.** Define the defect of the hypersurface $X$ as the integer

$$\delta := \dim(I_{eq}^{(2d-5)}) - \left( \binom{2d-1}{4} - 11\mu \right).$$

**Corollary 6.**

$$h^{1,1}(\tilde{X}) = 1 + \mu + \delta, \quad h^{1,2}(\tilde{X}) = h^{1,2}(X_{\text{smooth}}) - 11\mu + \delta$$

A hypersurface $X$ is $\mathbb{Q}$–factorial iff it has no defect.

**Remark 1.** The above definition of defect is a direct generalization the definition of the defect of a hypersurface with A–D–E singularities in [2] Def. 2.1.
Remark 2. Our main theorem generalizes (with the same proof) to the case of a degree \(d\) hypersurface \(X\) in a weighted projective space \(\mathbb{P}(w_0, \ldots, w_d)\) with ordinary triple points provided \(X \cap \text{Sing}(\mathbb{P}(w_0, \ldots, w_d)) = \emptyset\). The last condition implies in particular that the weights \(w_i\) are pairwise co-prime and divide the degree \(d\).

In this situation we define the **defect** as

\[ \delta := \dim(I_{\text{eq}}^{(2d-\lfloor w \rfloor)}) - (\dim S^{2d-\lfloor w \rfloor} - 11\mu) \]

and the same arguments (using [4]) yield the following weighted version of the main theorem

\[ h^{1,1}(\tilde{X}) = \dim(I_{\text{eq}}^{(2d-\lfloor w \rfloor)}) - \dim S^{2d-\lfloor w \rfloor} + 12\mu + 1 = 1 + \mu + \delta \]

\[ h^{1,2}(\tilde{X}) = \dim(I_{\text{eq}}^{(2d-\lfloor w \rfloor)}) - \sum_{i=0}^{k} \dim S^{d+w_i-\lfloor w \rfloor} = h^{1,2}(X_{\text{smooth}}) - 11\mu + \delta \]

The most important example is a triple solid, if \(D \subset \mathbb{P}^3\) is a surface in projective three space of degree divisible by three than there exists a triple cyclic cover \(\pi : X \to \mathbb{P}^3\) branched along \(D\). The singularities of \(X\) corresponds one-to-one to the singularities of \(D\), in particular an ordinary triple point on \(D\) gives an ordinary triple point on \(X\). The threefold \(X\) is given in \(\mathbb{P}(1,1,1,1,d/3)\) by an equation \(x_4^3 = g(x_0, \ldots, x_3)\), where \(g\) is an equation of \(D\). We get a formula analogous to the Clemens defect formula for double solid (cf. [1]) with defect defined by the degree \(d/3 - 4\) component of the equisingular ideal.

**Example 2.1.** We shall study a degree six surface in \(\mathbb{P}^3\) with ten ordinary triple points constructed in [11] as an element of a three dimensional family. Let

\[
\begin{align*}
K_1 &= 2x_1^2 - (\varepsilon + 2)x_3 + \varepsilon^2 x_1 x_3, \\
K_2 &= -x_2^2 + 2\varepsilon x_1 + x_2 + \varepsilon^2 x_1 x_2, \\
K_3 &= 2x_3^2 - 2\varepsilon^2 x_2 + (6\varepsilon + 2)x_3 + 4\varepsilon^2 x_2 x_3, \\
Q &= - (\varepsilon + 2 - x_1)(\varepsilon - x_2)(\varepsilon^2 + x_3) + x_1(x_2 - 1)(x_3 + 3\varepsilon + 1).
\end{align*}
\]

where \(\varepsilon\) is a third root of unity. Then the degree six polynomial

\[ 27K_1K_2K_3 + 2Q^3 = 0 \]

defines an element of three dimensional family of sextic surfaces with ten (maximal possible) number of ordinary triple points. Moreover for \(p = 67\) with \(\varepsilon = -30\) all singular points are defined over the base field (11). Computations conducted with **Singular** yield \(\dim(I_{\text{eq}}^{(6)}) = 30, \mu = 10, \delta = 10, h^{1,1} = 21, h^{1,2} = 3\).

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