Directed polymer in a random medium of dimension $1+1$ and $1+3$: weights statistics in the low temperature phase

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Abstract. We consider the low temperature $T < T_c$ disorder-dominated phase of the directed polymer in a random potential in dimension $1+1$ (where $T_c = \infty$) and $1+3$ (where $T_c < \infty$). To characterize the localization properties of the polymer of length $L$, we analyse the statistics of the weights $w_L(\vec{r})$ of the last monomer as follows. We numerically compute the probability distributions $P_1(w)$ of the maximal weight $w_L^{\text{max}} = \max_{\vec{r}} [w_L(\vec{r})]$, the probability distribution $\Pi(Y_2)$ of the parameter $Y_2(L) = \sum_{\vec{r}} w^2_{L}(\vec{r})$ as well as the average values of the higher-order moments $Y_k(L) = \sum_{\vec{r}} w^k_{L}(\vec{r})$. We find that there exists a temperature $T_{\text{gap}} < T_c$ such that (i) for $T < T_{\text{gap}}$, the distributions $P_1(w)$ and $\Pi(Y_2)$ present the characteristic Derrida–Flyvbjerg singularities at $w = 1/n$ and $Y_2 = 1/n$ for $n = 1, 2, \ldots$ In particular, there exists a temperature-dependent exponent $\mu(T)$ that governs the main singularities $P_1(w) \sim (1 - w)^{\mu(T)-1}$ and $\Pi(Y_2) \sim (1 - Y_2)^{\mu(T)-1}$ as well as the power-law decay of the moments $\bar{Y}_k(i) \sim 1/k^{\mu(T)}$. The exponent $\mu(T)$ grows from the value $\mu(T = 0) = 0$ up to $\mu(T_{\text{gap}}) \sim 2$. (ii) For $T_{\text{gap}} < T < T_c$, the distribution $P_1(w)$ vanishes at some value $w_0(T) < 1$, and accordingly the moments $\bar{Y}_k(i)$ decay exponentially as $(w_0(T))^k$ in $k$. The histograms of spatial correlations also display Derrida–Flyvbjerg singularities for $T < T_{\text{gap}}$. Both below and above $T_{\text{gap}}$, the study of typical and averaged correlations is in full agreement with the droplet scaling theory.

Keywords: disordered systems (theory)

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# Directed polymer in a random medium of dimension \(1+1\) and \(1+3\)

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## 1. Introduction

A convenient way to characterize disorder-dominated phases is through the statistics of some appropriate ‘weights’. In mean-field models, these weights represent either weights of pure states, as in the replica analysis of the Sherrington–Kirkpatrick model [1], or weights of microscopic configurations, as in the Random Energy Model [2] or in the directed polymer model on the Cayley tree [3]. It turns out that, in these three cases, the weights statistics is the same as in Lévy sums with some index \(0 < \mu < 1\) [4], where the index \(\mu\) depends on the temperature: for instance in the Random Energy Model [2] or in the directed polymer model on the Cayley tree [3], it is simply \(\mu(T) = T/T_c\). In [5], the corresponding probability distributions of the weights were found to exhibit characteristic singularities at some integer inverses. Similar Derrida–Flyvbjerg singularities also occur in many other contexts, such as randomly broken objects [5, 6], in population genetics [7]–[9] and in random walk excursions or loops [4, 10, 11].

For disordered systems in finite dimensions, it seems appropriate to consider the weights associated with a local degree of freedom, to characterize to what extent it is frozen. To the best of our knowledge, this idea has first been introduced to characterize the freezing of a folded polymer with random self-interactions [12]. It was then used in

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the context of secondary structures of random RNA to analyse the fraction of frozen pairs between degenerate ground states [13], and to characterize the freezing transition [14].

In this paper, we study the statistics of the weights $w_L(\mathbf{r})$ of the end-point of a directed polymer in a random potential [15]. We focus here on the low-temperature $T < T_c$ disorder-dominated phase both in dimension $1 + 1$ (where $T_c = \infty$) and $1 + 3$ (where $T_c < \infty$), since we have studied elsewhere [16] the weights statistics at criticality in $d = 3$, where multifractal behaviour occurs. We are not aware of previous studies on these weights in the physics literature. On the contrary, in the mathematical literature, the weight of the favourite site has been considered as a localization criterion [17,18], and a more detailed description of end-point weights was then given via the notion of $\epsilon$-atoms [19].

The paper is organized as follows. The model and observables are introduced in section 2. We then present a detailed study of the weights of the end-point of the directed polymer both in dimensions $1 + 1$ and $1 + 3$. For clarity, the statistical properties of the weights alone, independently of the distances involved, are described in section 3, whereas the study of spatial properties is postponed to section 4. We summarize our results in section 5.

2. Model and observables

2.1. Model definition

In this paper, we present numerical results for the random bond version of the model defined by the recursion relation on a cubic lattice in $d = 1$ and 3

$$Z_{t+1}(\mathbf{r}) = \sum_{j=1}^{2^d} e^{-\beta \epsilon_t(\mathbf{r} + \mathbf{e}_j, \mathbf{r})} Z_t(\mathbf{r} + \mathbf{e}_j).$$

(1)

The bond energies $\epsilon_t(\mathbf{r} + \mathbf{e}_j, \mathbf{r})$ are random independent variables drawn from the Gaussian distribution

$$\rho(\epsilon) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon^2/2}.$$  

(2)

In this paper, we consider the following boundary conditions. The first monomer is fixed at $\mathbf{r} = 0$, i.e. the initial condition of the recurrence of equation (1) is

$$Z_{t=0}(\mathbf{r}) = \delta_{\mathbf{r},0}.$$  

(3)

The last monomer is free, i.e. the full partition function of the polymer of length $L$ is then obtained by summing over all possible positions $\mathbf{r}$ at $t = L$

$$Z_L^{\text{tot}} = \sum_{\mathbf{r}} Z_L(\mathbf{r}).$$  

(4)

This model has attracted a lot of attention because it is directly related to non-equilibrium properties of growth models [15]. Within the field of disordered systems, it is also very interesting on its own because it represents a ‘baby-spin-glass’ model [3,4,15,20,21]. At low temperature, there exists a disorder-dominated phase, where the order parameter is an ‘overlap’. In finite dimensions, a scaling droplet theory

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was proposed [21, 22], in direct correspondence with the droplet theory of spin glasses [23], whereas in the mean-field version of the model on the Cayley tree, a freezing transition very similar to the one occurring in the Random Energy Model was found [3]. The phase diagram as a function of space dimension \(d\) is the following [15]. In dimension \(d \leq 2\), there is no free phase, i.e. any initial disorder drives the polymer into the strong disorder phase, whereas for \(d > 2\), there exists a phase transition between the low temperature disorder-dominated phase and a free phase at high temperature [24, 25].

In the following, we will focus on the statistical properties of the weights

\[
w_L(\vec{r}) = \frac{Z_L(\vec{r})}{Z_{tot}^L}\tag{5}
\]
normalized to (equation (4))

\[
\sum_{\vec{r}} w_L(\vec{r}) = 1.\tag{6}
\]

The numerical results given below have been obtained using polymers of various lengths \(L\), with corresponding numbers \(n_s(L)\) of disordered samples with the values

\[
L = 50, 100, 200, 400, 800 \tag{7}
\]
\[
n_s(L) = 13 \times 10^7, 35 \times 10^6, 9 \times 10^6, 225 \times 10^4, 57 \times 10^4 \tag{8}
\]
in \(d = 1\), and the values

\[
L = 6, 12, 18, 24, 36, 48, 60 \tag{9}
\]
\[
n_s(L) = 10^8, 10^7, 2 \times 10^6, 8 \times 10^5, 2 \times 10^5, 5 \times 10^4, 3 \times 10^4 \tag{10}
\]
in \(d = 3\). In the following, \(\overline{A}\) denotes the average of \(A\) over the disorder samples.

### 2.2. Characterization of the weights statistics

In analogy with the weight statistics in Lévy sums and in the Random Energy Model [4, 5], we have numerically computed the probability distribution \(P_1(w)\) of the maximal weight (equation (5))

\[
w_L^{\text{max}} = \max_{\vec{r}} \{w_L(\vec{r})\}\tag{11}
\]
as well as the probability distribution \(P_2(w)\) of the second maximal weight. Another useful way to characterize the statistical properties of the weights [4, 5] is to consider the moments of arbitrary order \(k\)

\[
Y_k(L) = \sum_{\vec{r}} w_L^k(\vec{r})\tag{12}
\]
which represents the probability that the last monomer of the polymer of length \(L\) is at the same point in \(k\) different thermal configurations of the same disordered sample. We have measured the probability \(\Pi(Y_2)\) of the parameter

\[
Y_2(L) = \sum_{\vec{r}} w_L^2(\vec{r})\tag{13}
\]
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Figure 1. $d=1$: probability distributions $P_1(w)$ and $P_2(w)$ of the largest and second-largest weight seen by the last monomer (see equation (11)) (a) at $T=0.1$ ($T<T_{\text{gap}}$) for $L=50,100,200$: the characteristic Derrida–Flyvbjerg singularities at $w=1$ and $1/2$ are clearly visible. (b) At $T=1$. ($T>T_{\text{gap}}$) for $L=50,100,200,400$: the distribution $P_1(w)$ does not reach $w=1$ anymore and the distribution $P_2(w)$ does not reach $w=1/2$ anymore.

as well as the moments $\overline{Y_k}(L)$ for $2 \leq k \leq 100$. Finally, we have also computed the weights density

$$f_L(w) = \sum_{\vec{r}} \delta(w - w_L(\vec{r}))$$

(14)

giving rise to the moments

$$\overline{Y_k}(L) = \int_0^1 dw \, w^k \, f_L(w).$$

(15)

The normalization condition for the density $f_L(w)$ is

$$\overline{Y_1}(L) = \int_0^1 dw \, w \, f_L(w) = 1.$$  

(16)

In the following, we will also present histograms of the associated entropy

$$s_L = -\sum_{\vec{r}} w_L(\vec{r}) \ln w_L(\vec{r}).$$

(17)

3. Study of the weights statistics

3.1. Probability distribution $P_1(w)$ of the largest weight

The probability distributions $P_1(w)$ and $P_2(w)$ of the largest (equation (11)) and second-largest weights of the last monomer in dimension $1+1$ are shown in figure 1 for two temperatures. These curves show that there exists a temperature $T_{\text{gap}}(d=1) \sim 0.7$ such that
(i) for $T < T_{\text{gap}}$ (see figure 1(a)) the distribution $P_1(w)$ reaches the point $w \rightarrow 1$ with a singularity parametrized by a temperature-dependent exponent

$$P_1(w) \propto (1 - w)^{\mu(T) - 1}.$$ \hfill (18)

Beyond this main singularity, $P_1(w)$ also present characteristic Derrida–Flyvbjerg singularities at $w = 1/2, 1/3, \ldots 1/n, \ldots$ [5]: in particular, the singularity of $P_1(w)$ at $w = 1/2$ is clearly visible on figure 1(a). Similarly, the distribution $P_2(w)$ reaches the point $w \rightarrow 1/2$ (see figure 1(a))

(ii) for $T > T_{\text{gap}}$ (see figure 1(b)) the distribution $P_1(w)$ does not reach $w = 1$ anymore, but vanishes at some maximal value $0 < w_0(T) < 1$

$$P_1(w) \propto (w_0(T) - w)^\sigma$$ \hfill (19)

with some exponent $\sigma$. Similarly, the distribution $P_2(w)$ does not reach the point $w = 1/2$ (see figure 1(b)).

This temperature $T_{\text{gap}}$ in $1 + 1$ where $T_c = \infty$ also exists in $1 + 3$ where $T_c$ is finite, as shown in figure 2. In figure 2(a), the distribution $P_1(w)$ is shown for three temperatures below $T_{\text{gap}}$ with exponents $\mu(T = 0.1) < 1$, $\mu(T = 0.3) \sim 1$ and $1 < \mu(T = 0.4) < 2$. In figure 2(b), the distribution $P_1(w)$ is shown for temperature $T \sim 0.6$ in the region $T_{\text{gap}}(d = 3) \sim 0.5 < T < T_c \sim 0.79$.

Note that here we only describe the low-temperature phase $T < T_c$ and we refer to [16] for a detailed study of the weights statistics at criticality $T_c = 0.79$ where multifractal behaviour occurs.

### 3.2. Probability distribution $G_L(s)$ of the entropy $s_L$ of the entropy $s_L = -\sum_{\vec{r}} w_L(\vec{r}) \ln w_L(\vec{r})$

The Derrida–Flyvbjerg singularities found for $T < T_{\text{gap}}$ at $w = 1/n$ for the weights translate into singularities in the histograms of the last-monomer entropy (equation (17))

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Figure 2. $d = 3$: probability distribution $P_1(w)$ of the largest weight seen by the last monomer (see equation (11)). (a) At $T = 0.1$ ($\mu < 1$), $T = 0.3$ ($\mu \sim 1$), $T = 0.4$ ($1 < \mu < 2$) for $L = 12, 24$. (b) At $T = 0.6 > T_{\text{gap}}$ for $L = 12, 24$. 

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Figure 3. $d = 1$: probability distribution $G_L(s)$ of the last monomer entropy $s_L = -\sum_{\vec{r}} w_L(\vec{r}) \ln w_L(\vec{r})$ (a) at three temperatures below $T_{\text{gap}}$, namely $T = 0.1$ ($\mu < 1$), $T = 0.4$ ($\mu \sim 1$) and $T = 0.5$ ($1 < \mu < 2$) for $L = 200$: the Derrida–Flyvbjerg singularities at $s = 0$, $s = \ln 2$ and $s = \ln 3$ are clearly visible. (b) At $T = 1. > T_{\text{gap}}$ for $L = 50, 100, 200, 400, 600, 800$: the histogram does not reach $s = 0$. In particular, the main singularity of $P_1(w)$ for $w \to 1$ (equation (18)) yields a corresponding singularity at $s \to 0$

$$G(s) \propto s^{\mu(T)-1}$$

for $0 < T < T_{\text{gap}}$, whereas a gap $s_{\text{min}}$ appears for $T > T_{\text{gap}}$. This is shown in figure 3 for $d = 1$ and in figure 4 for $d = 3$.

3.3. Probability distribution $\Pi(Y_2)$ of the parameter $Y_2$

The parameter $Y_2$ defined in equation (13) can reach the value $Y_2 \to 1$ only if the maximal weight $w_{\max}$ also reaches $w_{\max} \to 1$. As a consequence, the probability distribution $\Pi(Y_2)$ has the same singularity near $Y_2 \to 1$ as in equation (18)

$$\Pi(Y_2) \propto (1 - Y_2)^{\mu(T)-1}$$

for $0 < T < T_{\text{gap}}$. Beyond this main singularity, the distribution $\Pi(Y_2)$ presents the characteristic Derrida–Flyvbjerg singularities at $Y_2 = 1/n$ as shown in figure 5(a) for $d = 1$ and in figure 6(a) for $d = 3$. Again for $T > T_{\text{gap}}$, a gap appears as shown in figure 5(b) for $d = 1$ and in figure 6(b) for $d = 3$.

3.4. Density $f(w)$

The density $f(w)$ introduced in equation (14) is shown in figures 7 and 8 for $d = 1$ and $d = 3$, respectively.

By construction, this density coincides with the maximal weight distribution $P_1(w)$ for $w > 1/2$, with the sum $(P_1(w) + P_2(w))$ of the two largest weight distributions for $d = 1$ and $d = 3$, respectively.

\[ \text{Figure 3. } \]
Figure 4. $d = 3$: probability distribution $G_L(s)$ of the last monomer entropy $s_L = -\sum r \ln w_L(r)$ (a) at $T = 0.1 < T_{\text{gap}}$ for $L = 12, 24$: the Derrida–Flyvbjerg singularities at $s = 0$, $s = \ln 2$ and $s = \ln 3$ are clearly visible. (b) At $T = 0.6 > T_{\text{gap}}$ for $L = 6, 12, 24, 36, 48, 60$: the histogram does not reach $s = 0$.

Figure 5. $d = 1$: probability distribution $\Pi(Y_2)$ of the parameter $Y_2 = \sum r w^2_L(r)$ (a) at $T = 0.1 < T_{\text{gap}}$ for $L = 50, 100, 200$: the Derrida–Flyvbjerg singularities at $Y_2 = 1$, $Y_2 = 1/2$ and $Y_2 = 1/3$ are clearly visible. (b) At $T = 1. > T_{\text{gap}}$ for $L = 50, 100, 200, 400$: the histogram does not reach $Y_2 = 1$.

$1/3 < w < 1/2$, and so on [5]. As a consequence, $f(w)$ has the same singularity near $w \to 1$ as $P_t(w)$ (equation (18)) and the same gap (equation (19)) as long as $w_0(T) > 1/2$. The only other singularity is near $w \to 0$ where $f(w)$ diverges in a non-integrable manner, because in the $L \to \infty$, there is an infinite number of vanishing weights (only the product $(w f(w))$ has to be integrable at $w = 0$ as a consequence of the normalization condition of equation (16)).

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Figure 6. $d = 3$: probability distribution $\Pi(Y_2)$ of the parameter $Y_2 = \sum_r w_r^2(\vec{r})$ (a) for $T = 0.1$ ($L = 12, 18, 24$) and $T = 0.3$ ($L = 12, 24$): the Derrida–Flyvbjerg singularities at $Y_2 = 1$, $Y_2 = 1/2$ and $Y_2 = 1/3$ are clearly visible. (b) For $T = 0.5 \sim T_{\text{gap}}$ with $L = 12, 24, 48$.

Figure 7. $d = 1$: weight density $f(w)$ (a) at $T = 0.1, 0.2, 0.3, 0.4$ where $\mu(T) < 1$ for $L = 200$: the weight density $f(w)$ diverges at $w \to 0$ and $w \to 1$. (b) At $T = 0.5, 0.6, 0.7, 0.8, 0.9, 1, 5$. for $L = 200$: these curves show the temperature-dependent singularity at $w = 1$.

3.5. Moments $\overline{Y_k}$

For $0 < T \leq T_{\text{gap}}$, where $P_1(w)$ and $f(w)$ behaves near $w \to 1$ as in equation (18), the decay in $k$ of the averaged moments $\overline{Y_k(i)}$ (equation (12)) follow a power-law of exponent $\mu(T)$

$$\overline{Y_k(i)} \propto \frac{1}{k^{\mu(T)}} \quad \text{for } T \leq T_{\text{gap}}.$$  

(22)
Figure 8. $d = 3$: weight density $f(w)$ (a) for $T = 0.1 \ (L = 12, 18, 24)$ and $T = 0.2 \ (L = 12, 24)$ where $\mu(T) < 1$: the weight density $f(w)$ diverges at $w \to 0$ and $w \to 1$. (b) For $T = 0.3, 0.4, 0.5, 0.6, 0.7 \ (L = 24)$: these curves show that the gap appears around $T_{\text{gap}}(d = 3) \sim 0.5$.

Figure 9. $d = 1$: (a) decay of the moments $\overline{Y_k}$ of equation (22) as a function of $k \leq 100$ for $L = 800$ and $T = 0.1, 0.2, 0.3, 0.4, 0.5$; (b) exponent $\mu(T)$ as measured from the slope of the log-log decay in the asymptotic region.

The behaviour of $\overline{Y_k}$ for $k \leq 100$ is shown in figure 9(a) for $d = 1$ and in figure 10(a) for $d = 3$. The corresponding exponent $\mu(T)$ are shown in figure 9(b) for $d = 1$ and figure 10(b) for $d = 3$.

For $T > T_{\text{gap}}$ where there exists a gap $w_0(T)$ for $P_1(w)$, the behaviour of equation (19) also applies to $f(w)$ as long as $w_0(T) > 1/2$ (since $f(w) = P_1(w)$ for $w > 1/2$ as mentioned above) and thus the decay is then exponential

$$\overline{Y_k}(t) \propto \frac{(w_0(T))^k}{k^{1+\sigma}} \quad \text{for} \quad T > T_{\text{gap}}. \quad (23)$$

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Figure 10. $d = 3$: (a) decay of the moments $Y_k$ of equation (22) as a function of $k \leq 100$ for $L = 48$ and $T = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$; (b) exponent $\mu(T)$ as measured from the slope of the log–log decay in the asymptotic region.

4. Study of spatial properties

In the previous section, we have studied in detail the statistics of the weights independently of their spatial organization. In this section, we study the statistics of the transverse spatial correlation

$$C(r) = w(\vec{r}_{\text{pref}})w(\vec{r}_{\text{pref}} + \vec{r})$$

centred on the preferred position $\vec{r}_{\text{pref}}$ of maximal weight (equation (11)). We first recall the predictions of the droplet scaling analysis [21, 22] that will be useful to analyse our numerical results.

4.1. Reminder on the droplet scaling analysis

The droplet theory for directed polymers [21, 22] is very similar to the droplet theory of spin glasses [23]. It is a scaling theory that can be summarized as follows.

4.1.1. Statistics of low energy excitations above the ground state. At very low temperature $T \to 0$, all observables are governed by the statistics of low energy excitations above the ground state. An excitation of large length $l$ costs a random energy

$$\Delta E(l) \sim l^\theta u$$

where $u$ is a positive random variable distributed with some law $Q_0(u)$ having some finite density at the origin $Q_0(u = 0) > 0$. The exponent $\theta$ is the exponent governing the fluctuation of the energy of the ground state is exactly known in one dimension $\theta(d = 1) = 1/3$ [26]–[29] and for the mean-field version on the Cayley tree $\theta(d = \infty) = 0$ [3]. In finite dimensions $d = 2, 3, 4, 5, \ldots$, the exponent $\theta(d)$ has been numerically measured, and we only quote here the results of the most precise study we are aware of [30] for dimensions $d = 2, 3$: $\theta(d = 2) = 0.244$ and $\theta(d = 3) = 0.186$. 

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From (25), the probability distribution of large excitations \( l \gg 1 \) is within the droplet theory

\[
\mathrm{d}l \rho(E = 0, l) \sim \frac{\mathrm{d}l}{l} e^{-\beta \Delta E(l)} \sim \frac{\mathrm{d}l}{l} e^{-\beta u}
\]

(26)

where the factor \( \mathrm{d}l/l \) comes from the notion of independent excitations [23]. In particular, its average over the disorder follows the power law

\[
\overline{\mathrm{d}l \rho(E = 0, l)} \sim \int_0^{+\infty} \mathrm{d}u Q_0(u) \frac{\mathrm{d}l}{l} e^{-\beta u} = T Q(0) \frac{\mathrm{d}l}{l^{1+\theta}}.
\]

(27)

This prediction describes very well the numerical data in the regime \( 1 < l < L \) in dimensions \( d = 1, 2, 3 \) [31].

Since correlation functions at large distance are directly related to the probability of large excitations, we already see that the low temperature phase is very non-trivial from the point of view of correlation lengths: the typical exponential decay (26) indicates a finite typical correlation length \( \xi_{\text{typ}}(T) \), whereas the averaged power-law behaviour (27) means that the averaged correlation length \( \xi_{\text{av}}(T) \) is actually infinite in the whole low temperature phase

\[
\xi_{\text{av}}(0 < T \leq T_c) = \infty.
\]

(28)

Note that within the droplet theory of spin glasses [23], the correlation length \( \xi_{\text{av}}(T) \) is also infinite in the whole low temperature phase for the same reasons.

4.1.2. Low temperature phase governed by a zero-temperature fixed point. According to the droplet scaling theory [21, 22] the whole low temperature phase \( 0 < T < T_c \) is governed by a zero-temperature fixed point. However, many subtleties arise because the temperature is actually ‘dangerously irrelevant’. The main conclusions of the droplet analysis [21, 22] can be summarized as follows. The scaling (25) governs the free energy cost of an excitation of length \( l \), provided one introduces a longitudinal correlation length \( \xi_{\parallel}(T) \) to rescale the length \( l \)

\[
\Delta F(l) = \left( \frac{l}{\xi_{\parallel}(T)} \right)^\theta u.
\]

(29)

Here as before, \( u \) denotes a positive random variable distributed with some law \( Q(u) \) having some finite density at the origin \( Q(u = 0) > 0 \). As a consequence, the probability of a droplet of size \( l \gg 1 \) follows the scaling form

\[
\mathrm{d}l \rho(l) \sim \frac{\mathrm{d}l}{l} e^{-u l/\xi_{\parallel}(T)} \theta.
\]

(30)

In particular, the typical behaviour follows an exponential decay with exponent \( \theta \)

\[
\overline{\ln \rho(l)} \sim - \left( \frac{l}{\xi_{\parallel}(T)} \right)^\theta \int_0^{+\infty} \mathrm{d}u u Q(u)
\]

(31)

whereas the average over the disorder follows the power law

\[
\overline{\mathrm{d}l \rho(l)} \sim Q(0) \frac{\mathrm{d}l}{l} \left( \frac{\xi_{\parallel}(T)}{l} \right)^\theta.
\]

(32)

This average is governed by the rare events having \( u \sim 0 \).
A droplet of longitudinal size \( l \) corresponds to a transverse distance \( r \sim l^{\zeta} \), where

\[
\zeta = \frac{1 + \theta}{2}
\]  

is the roughness exponent of the low temperature phase [15]. Via the change of variable \( r \sim l^{\zeta} \), the droplet distribution of equation (30) translates into the following scaling form for the correlation at large distance \( r \) [21, 22]

\[
dr r^{d-1} C(r) = dl\rho(l) = \frac{dr}{r} e^{-u(r/\xi(\ell(T)))^{\theta/\zeta}}
\]  

where the transverse correlation length is \( \xi(\ell(T)) \sim [\xi(\ell(T))]^{\zeta} \), i.e. finally

\[
C(r) = \frac{1}{r^d} e^{-u(r/\xi(\ell(T)))^{\theta/\zeta}}.
\]  

As a consequence, the typical behaviour follows an exponential decay with exponent \( \theta/\zeta \)

\[
\ln C(r) \sim - \left( \frac{r}{\xi(\ell(T))} \right)^{\theta/\zeta} - d \ln r
\]  

whereas the average over the disorder is governed by the rare events and follows the power law

\[
\overline{C(r)} \sim \frac{1}{r^d} \left( \frac{\xi(\ell(T))}{r} \right)^{\theta/\zeta}.
\]  

We now describe our numerical data and compare with these predictions.

4.2. Disorder-averaged correlation \( C_L(r) \)

The previous predictions concern the limit \( L \to \infty \). To compare with our numerical data, we now recall the corresponding finite-size behaviour within the droplet theory [22]. The power-law behaviour (equation (37)) for the averaged correlation translates into the following scaling form for finite \( L \)

\[
\overline{C_L(r)} \sim \frac{1}{L^{d\zeta+\theta}} \Phi \left( x = \frac{r}{L^{\zeta}} \right)
\]  

where the scaling function \( \Phi(v) \) behaves as the following power law

\[
\Phi(v) \propto \frac{1}{v^{d+(\theta/\zeta)}}
\]  

to recover equation (37) as \( L \to \infty \).

We show in figure 11 the finite-size scaling analysis of equation (38) in \( d = 1 \) for two temperatures, one below and one above \( T_{\text{gap}} \). In both cases, the agreement with the droplet scaling ansatz is very good, confirming the zero-temperature fixed point picture. The corresponding finite-size scaling analysis for the disorder-averaged correlation in \( d = 3 \) is shown in figure 12.
Figure 11. Disorder-averaged correlation $\overline{C_L(r)}$ in $d = 1$: finite-size scaling of equation (38): $\ln \Phi = \ln(L\overline{C_L(r)})$ as a function of $\ln x = \ln(r/L^{2/3})$ for $L = 50 (\bigcirc), 100 (\bigodot), 200 (\bigtriangledown), 400 (\triangle), 800 (\blacktriangledown)$ (a) for $T = 0.2 < T_{\text{gap}}$; (b) for $T = 1. > T_{\text{gap}}$.

Figure 12. $d = 3$: disorder-averaged correlation $\overline{C_L(r)}$ in $d = 3$ finite-size scaling of equation (38): (equation (38)): $\ln \Phi = \ln(L^{d/2} \overline{C_L(r)})$ as a function of $\ln x = \ln(r/L^\zeta)$ for $T = 0.1 L = 24 (\bigodot), 36 (\bigtriangleup), 48 (\triangle), 60 (\blacktriangleup)$.

4.3. Typical correlation $\ln \overline{C_L(r)}$

The typical correlation of equation (36) is shown in figure 13(a) for $d = 1$ and in figure 14(a) for $d = 3$, respectively: the collapse for small $r$ is satisfactory. To take into account the $L$-dependent cut-off for large $r$, we have tried the following finite-size scaling form

$$\ln \overline{C_L(r)} + d \ln r \sim L^\theta \Psi \left( x = \frac{r}{L^\zeta} \right)$$

(40)
Directed polymer in a random medium of dimension $1+1$ and $1+3$.

**Figure 13.** Typical correlation $\ln C_L(r)$ in $d = 1$ for $T = 1$ and sizes $L = 50$ (○), 100 (□), 200 (○), 400 (△), 800 (▷): (a) $(\ln C_L(r) + \ln(r))$ as a function of $r^{1/2}$ (see equation (36)); (b) finite-size scaling of the same data: $\Psi = (\ln C_L(r) + \ln(r))/L^{1/3}$ as a function of $x^{1/2} = (r/L^{2/3})^{1/2}$ (see equation (40)).

**Figure 14.** Typical correlation $\ln C_L(r)$ in $d = 3$ for $T = 0.1$ and sizes $L = 24$ (○), 36 (○), 48 (△), 60 (▷): (a) $(\ln C_L(r) + d\ln(r))$ as a function of $r^{\theta/\zeta} = 0.314$ (see equation (36)); (b) finite-size scaling of the same data: $\Psi = (\ln C_L(r) + d\ln(r))/L^\theta$ as a function of $x^{\theta/\zeta} = (r/L^\zeta)^{\theta/\zeta}$ (see equation (40)).

where the scaling function $\Psi(v)$ behaves as the following power law

$$\Psi(v) \propto v^{\theta/\zeta}$$

(41)

to recover equation (36) as $L \to \infty$.

The results for $d = 1$ and 3 are shown in figures 13(b) and 14(b), respectively.
Directed polymer in a random medium of dimension $1 + 1$ and $1 + 3$

Figure 15. $d = 1$: probability distribution $D_r(C)$ of the correlation $C(r)$ (equation (24)) for $L = 200$: (a) for $T = 0.2 < T_{\text{gap}}$ the distribution $D_r(C)$ reaches the maximal value $C_{\text{max}} = 1/4$ for any distance $r$: here $r = 2, 4, 6, 10, 20, 30, 40$; (b) probability distribution $D_{r=2}(C)$ of the correlation of two neighbouring sites for $T = 0.2$ where $\mu(T) < 1$, $T = 0.5$ where $1 < \mu(T) < 2$ and $T = 1. > T_{\text{gap}}$ where the histogram does not reach the maximal possible value $C_{\text{max}} = 1/4$.

4.4. Histograms of correlation function

Since typical and disorder-averaged correlations are very different, we have also studied in $d = 1$ the probability distribution $D_r(C)$ of the correlation $C(r)$ (equation (24)) between the preferred position and a site at transverse distance $r$. For $T < T_{\text{gap}}$, the distribution $D_r(C)$ reaches the maximal possible value $C_{\text{max}} = 1/4$ for any distance $r$ as shown in figure 15(a). This maximal value $C_{\text{max}} \sim 1/4$ corresponds to the case where the maximal weight and second-maximal weight are both of order $w \sim 1/2$ and are at distance $r$. For large $r \sim L^{2/3}$, this corresponds to the droplet excitations.

For $T > T_{\text{gap}}$, the distribution $D_r(C)$ does not reach the maximal possible value $C_{\text{max}} = 1/4$ anymore, as shown in figure 15(b).

5. Conclusion

We have studied the weight statistics in the low temperature $T < T_c$ disorder-dominated phase of the directed polymer in a random potential in dimension $1 + 1$ (where $T_c = \infty$) and $1 + 3$ (where $T_c < \infty$). In particular, we have found a temperature $T_{\text{gap}} < T_c$ with the following properties. For $T < T_{\text{gap}}$, the histograms of weight observables and of spatial correlation display characteristic Derrida–Flyvbjerg singularities. In particular, there exists a temperature-dependent exponent $\mu(T)$ that governs the main singularities of $P_i(w) \sim (1 - w)^{\mu(T)-1}$, $\Pi(Y_2) \sim (1 - Y_2)^{\mu(T)-1}$ and $G(s) \sim s^{\mu(T)-1}$, as well as the power-law decay of the moments $Y_k(i) \sim 1/k^{\mu(T)}$. The exponent $\mu(T)$ grows from the value $\mu(T = 0) = 0$ up to $\mu(T_{\text{gap}}) \sim 2$. For $T_{\text{gap}} < T < T_c$, the distribution $P_i(w)$ vanishes at some value $w_0(T) < 1$, and accordingly the moments $Y_k(i)$ decay exponentially as

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Finally, our numerical results concerning typical and averaged correlations are in full agreement with the droplet scaling theory both below and above $T_{\text{gap}}$.

Together with our previous study on the freezing transition of random RNA secondary structures [14], this shows that the weight statistics is an efficient tool to characterize to what extent local degrees of freedom are frozen. Moreover, the position of $T_{\text{gap}}$ with respect to $T_c$ gives a better understanding of the transition. In the RNA case where $T_c < T_{\text{gap}}$, the interpretation is that there exist frozen pairs in the high temperature phase, but only of finite size [14]. Here, in the directed polymer case where $T_{\text{gap}} < T_c$, there exist frozen monomers only below $T_{\text{gap}}$, whereas for $T_{\text{gap}} < T < T_c$, the localization occurs on a tube of finite extent $\xi_\perp(T)$.

Note that the Derrida–Flyvbjerg singularities can only exist for discrete models, in which the weight $w_0 L^{m}$ of a single spatial position can bear almost the whole normalization. As a consequence, $T_{\text{gap}}$ does not exist in the continuous version of the directed polymer, where one has to consider a spatial probability density.

Finally, we point out that, in the mean-field version of the directed polymer model on the Cayley tree [3], the two temperatures coincide $T_{\text{gap}} = T_c$. This induces different properties for the weight statistics at criticality, as explained in detail in [16] and [32].

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