Static Spherically Symmetric Solution of the Einstein-aether Theory

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We present the static spherically symmetric solution in the Einstein-aether theory by using of the Euler-Lagrange equations. The solution is similar to the Reissner-Nordstrom-de Sitter solution in that it has an inner Cauchy horizon, an outer black hole event horizon and a "cosmic horizon". But different from the Reissner-Nordstrom-de Sitter solution, its "cosmic horizon" locates not at some finite physical radius but at the spatial infinity. The resulting electric potential is regular in the whole spacetime except for the curvature singularity. On the other hand, the magnetic potential is divergent on both Cauchy horizon and the outer event horizon.

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I. INTRODUCTION

The Einstein-aether theory [1, 2] belongs to the vector-tensor theories in nature. Besides the ordinary matters and the metric tensor $g_{\mu\nu}$, the fundamental field in the theory is a timelike vector field $A_{\mu}$. Different from the usual vector-tensor theories, $A_{\mu}$ is constrained to have a constant norm. So the vector field $A_{\mu}$ cannot vanish anywhere. Therefore, a preferred frame is defined and the Lorentz symmetry is violated. The vector field is referred to as the "aether". The Einstein-aether theory has become an interesting theoretical laboratory to explore both the Lorentz violation effects and the preferred frame effects. Up to now, the Einstein-aether theory has been widely studied in literature in various ways: the analysis of classical and quantum perturbations [3–8], the cosmologies [9, 10], the gravitational collapse [12], the Einstein-aether waves [13], the radiation damping [14] and so on.

The purpose of the present paper is to seek for the static spherically symmetric solution of the Einstein-aether theory. The black hole solutions in the Einstein-aether theory have been investigated in Refs. [15–18]. These investigations mainly focus on the numerical analysis of the solutions due to the complication of the Einstein equations. To our knowledge, one have not yet find the exact, static and spherically symmetric solution in the Einstein-aether theory. In this paper, instead of solving the Einstein equations, we are going to solve the Euler-Lagrange equations in order to derive the static spherically symmetric solution. We find it is relatively simple in the calculations. We shall use the system of units in which $16\pi G = c = \hbar = 4\pi\epsilon_0 = 1$ and the metric signature (−, +, +, +) throughout the paper.

II. STATIC SPHERICALLY SYMMETRIC SOLUTION

In the context of spherical symmetry and after the redefinitions of metric $g_{\mu\nu}$ and aether field $A_{\mu}$, the Lagrangian density of the Einstein-aether theory can be written as

$$\mathcal{L} = -R - \frac{c_1}{2} F_{\mu\nu} F^{\mu\nu} - c_2 (\nabla_{\mu} A^{\mu})^2 + \lambda (A_{\mu} A^{\mu} + m^2) ,$$

(1)

with the field strength tensor

$$F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} .$$

(2)

Here $R$ is the Ricci scalar and the $c_i$ are dimensionless constants. $\lambda$ is the Lagrange multiplier field which has the dimension of the square of inverse length, $l^{-2}$. $m$ is a positive dimensionless constant which has the physical meaning of the squared norm for the aether field. The requirement of $m^2 > 0$ ensures the aether to be timelike.
The static and spherically symmetric metric can always be written as

\[ ds^2 = -U(r) \, dt^2 + \frac{1}{U(r)} \, dr^2 + f(r)^2 \, d\Omega^2. \] (3)

Instead of solving the Einstein equations, we prefer to deal with the Euler-Lagrange equations from the Lagrangian Eq. (1) for simplicity in calculations. Because of the static and spherically symmetric property of the spacetime, the vector field \( A_\mu \) takes the form

\[ A_\mu = \left[ \phi(r), \frac{1}{\psi(r)}, 0, 0 \right], \] (4)

where \( \phi \) and \( \psi^{-1} \) correspond to the electric and magnetic part of the electromagnetic potential. Then we have

\[ F_{\mu\nu}F^{\mu\nu} = -2\phi'^2, \quad \nabla_\mu A^\mu = \left( \frac{U'}{\psi} \right)' + 2\frac{f'}{f\psi} \frac{f'U}{f\psi}. \] (5)

The prime here and in what follows denotes the derivative with respect to \( r \). Taking into account the Ricci scalar, \( R \), we have the total Lagrangian as follows

\[ \mathcal{L} = -U'' - 4U' \frac{f'}{f} - 4U \frac{f''}{f} + \frac{2}{f^2} - 2U \frac{f'^2}{f^2} + c_1\phi'^2 - c_2 \left[ \left( \frac{U}{\psi} \right)' + 2\frac{f'U}{f\psi} \right]^2 + \lambda \left( -\frac{1}{U} \phi^2 + \frac{U}{\psi^2} + m^2 \right). \] (6)

Let

\[ \psi = \frac{UF^2}{K}, \] (7)

we can rewrite the Lagrangian, Eq. (6), as follows

\[ \mathcal{L} = -U'' - 4U' \frac{f'}{f} - 4U \frac{f''}{f} + \frac{2}{f^2} - 2U \frac{f'^2}{f^2} + c_1\phi'^2 - c_2 \frac{K'^2}{f^4} + \lambda \left( -\frac{1}{U} \phi^2 + \frac{K}{UF^4} + m^2 \right). \] (8)

Now there are \( U, f, \phi, K, \lambda \) five variables in the Lagrangian which correspond to five equations of motion. Then using the Euler-Lagrange equation, we obtain the equation of motion for \( \lambda \),

\[ -\frac{1}{U} \phi^2 + \frac{K^2}{UF^4} + m^2 = 0, \] (9)

for \( \phi \),

\[ c_1UF \phi'' + 2c_1U \phi' f' + \lambda f = 0, \] (10)

for \( K \),

\[ c_2UFK'' - 2c_2UK f' + \lambda K f = 0, \] (11)
for $U$,  

$$-2U^2f^3f'' - \lambda K^2 + \lambda \phi^2 f^4 = 0,$$  

(12)

and for $f$,  

$$-c_1 \phi'^2 U f^4 - c_2 U K'^2 + \lambda \phi^2 f^4 + U f^4 U'' - \lambda m^2 U f^4$$

$$+ 2U f^3 U' f' + 2U^2 f^3 f'' + \lambda K^2 = 0,$$  

(13)

respectively. We have five independent differential equations and five unknown variables, $U$, $f$, $\phi$, $K$, $\lambda$. So the system of equations is closed.

From Eq. (9) and Eq. (10), we obtain  

$$U = \frac{\phi^2 f^4 - K^2}{m^2 f^4},$$  

(14)

and  

$$\lambda = -\frac{c_1 U \left( f \phi'' + 2f' f' \right)}{f \phi},$$  

(15)

respectively. Substituted Eq. (14) and Eq. (15) into Eq. (12), then Eq. (12) becomes  

$$2\phi f'' + c_1 m^2 f \phi'' + 2c_1 m^2 \phi' f' = 0.$$  

(16)

The norm of the aether field is usually constrained by the Lagrange multiplier to be unity, $m = 1$. But in this paper, we would constrain the norm to meet  

$$c_1 m^2 = 2,$$  

(17)

for convenience. We note that this choice is consistent with the perturbation analysis of Lim [3]. He showed that in order to have a positive definite Hamiltonian, $c_1$ should satisfy  

$$c_1 > 0.$$  

(18)

Then Eq. (16) gives the solution as follows  

$$\phi = \frac{\phi_0 + \phi_1 f}{f},$$  

(19)

where $\phi_0$, $\phi_1$ are two integration constants. $\phi_1$ is dimensionless while $\phi_0$ has the dimension of length.

Keeping Eqs. (14), (15) and (19) in mind, we find Eq. (11) and Eq. (13) are reduced to the following form  

$$2c_2 m^2 K' f' - c_2 m^2 f K'' - 2K f'' = 0,$$  

(20)

and  

$$2f^2 K'^2 + 12K^2 f'^2 - 12f K f' K' + c_2 m^2 f^2 K'^2$$

$$-6f K^2 f'' + 2K f^2 K'' = 0,$$  

(21)

respectively. Putting  

$$c_2 = \frac{1}{\alpha m^2},$$  

(22)

with $\alpha$ a new dimensionless parameter, we obtain from Eq. (20) and Eq. (21).
$$6\alpha f K^2 f'' - 12\alpha K^2 f' + 8\alpha K f f' K' + 4\alpha^2 f K^2 f' - 2\alpha f K f' + f^2 K' = 0. \quad (23)$$

In order that the spin-0 field does not propagate superluminally, Lim constrained $c_2$ to meet

$$c_2 > 0, \quad \text{and} \quad \frac{c_2}{c_1} \leq 1. \quad (24)$$

Taking account of Eq. (18), we conclude that $\alpha$ should satisfy

$$\alpha \geq \frac{1}{2}. \quad (25)$$

Solving the differential equation, we obtain

$$K = K_0 \exp \left\{ \int \frac{1}{(1+2\alpha)} \left[ 4\alpha f' + \sqrt{2(2\alpha + 3)(2\alpha f f'' + f f' - 2f' - 2)} \right] \, dr \right\}, \quad (26)$$

where $K_0$ is an integration constant which has the dimension of the square of the length, $l^2$. We may assume $K_0 > 0$. Substituting Eq. (26) into Eq. (21), we obtain

$$\left( 16\alpha f f'' + 8ff'' - 16f' \right) \sqrt{2(2\alpha + 3)(2\alpha f f'' + f f' - 2f' - 2)}$$

$$+ \sqrt{2} \left( 4\alpha^2 f^2 f'' + 12\alpha^2 f f' f'' - 8\alpha f^3 + 4\alpha^2 f f' - 9f f f' f'' + f^2 f' f'' + 12f^3 \right) = 0. \quad (27)$$

Then $f$ is found to be

$$f = f_0 \left( 1 - k^2 r^2 \right)^{2\alpha + 1} \frac{2\alpha + 3}{2\alpha - 1} \frac{1}{e^{\frac{\sqrt{6\alpha + 4\alpha^2} k}{2\alpha - 1}}} \tanh^{-1} kr, \quad (28)$$

where $f_0$ and $k$ are integration constants. Both $f_0$ and $k^{-1}$ have the dimension of length. Without the loss of generality, the third integration constant with the dimension of length has been absorbed by $r$ and

$$\arg \tanh kr \equiv \tanh^{-1} kr. \quad (29)$$

So $kr$ is forced to satisfy

$$-1 \leq kr \leq 1. \quad (30)$$

Up to this point, we could present all the unknown variables:
\[ f = f_0 \left(1 - k^2 r^2\right) e^{\frac{2\alpha + 4\alpha^2}{2m^2 f_0^2} \tanh^{-1} kr}, \]

\[ K = K_0 \left(1 - kr\right)^{\frac{4\alpha + \sqrt{4\alpha^2 + 4\alpha}}{2m^2 f_0^2}} \left(1 + kr\right)^{\frac{4\alpha + \sqrt{4\alpha^2 + 4\alpha}}{2m^2 f_0^2}} e^{\frac{2\alpha + 4\alpha^2}{2m^2 f_0^2} \tanh^{-1} kr}, \]

\[ \phi = \frac{1}{f_0} (\phi_0 + \phi_1 r) \left(1 - k^2 r^2\right) - \frac{4\alpha + \sqrt{4\alpha^2 + 4\alpha}}{2m^2 f_0^2} \tanh^{-1} kr, \]

\[ U = \frac{1}{m^2 f_0^2} (\phi_0 + \phi_1 r)^2 \left(1 - k^2 r^2\right)^2 - \frac{2\alpha + 1}{2m^2 f_0^2} e^{-\frac{2\alpha + 4\alpha^2}{2m^2 f_0^2} \tanh^{-1} kr}, \]

\[ \psi = \frac{(\phi_0 + \phi_1 r)^2 \left(1 - k^2 r^2\right)^2 - \frac{4\alpha + \sqrt{4\alpha^2 + 4\alpha}}{2m^2 f_0^2} \tanh^{-1} kr}{K_0 m^2} \]

\[ \lambda = -\frac{2k^2 (\phi_0 + \phi_1 r)^2}{m^4 f_0^2} \left(2\alpha - 1\right)^2 \left(1 - k^2 r^2\right)^2 \left[4kr \sqrt{4\alpha^2 + 4\alpha} - 6\alpha - 2k^2 r^2 - 1 - 4\alpha k^2 r^2\right] e^{-\frac{2\alpha + 4\alpha^2}{2m^2 f_0^2} \tanh^{-1} kr}, \]

\[ \cdot \left(1 - kr\right)^{\frac{4\alpha + \sqrt{4\alpha^2 + 4\alpha}}{2m^2 f_0^2} \tanh^{-1} kr} \cdot \left(1 + kr\right)^{\frac{4\alpha + \sqrt{4\alpha^2 + 4\alpha}}{2m^2 f_0^2} \tanh^{-1} kr} e^{-\frac{2\alpha + 4\alpha^2}{2m^2 f_0^2} \tanh^{-1} kr}. \]

If we define

\[ \phi_0 = \alpha_0 f_0, \quad \phi_1 = \alpha_1, \quad k = \alpha_2 \frac{1}{f_0}, \quad K_0 = \alpha_3 f_0^2, \]

then \( \alpha_i \) are dimensionless constants. Together with \( m \) and \( \alpha \), we have totally six dimensionless constants and one dimensional parameter, \( f_0 \).

Since \( f \) is the physical length, we should rewrite the metric as follows

\[ ds^2 = -U(r) \, dt^2 + \frac{1}{V(r)} \, d\Omega^2 + f(r)^2 \, d\Omega^2, \]

with

\[ V(r) = U(r) f^2. \]

Now \( f \) plays the role of physical radius (proper length) of the static spherically symmetric space.

As an example, we put the dimensional constant \( f_0 = 1 \) (for example, \( f_0 \) equals to one Schwarzschild radius). Five dimensionless constants are put \( m = 1, \quad \alpha_0 = \alpha_2 = \alpha_3 = 1, \quad \alpha = 3/2 \). As for \( \alpha_1 \), we let \( \alpha_1 = 0.15, \ 0.1, \ 0, \ -0.2, \ -0.4 \), respectively.

In Fig. 1 and Fig. 2 we plot the evolution of \( U \) and \( V \) with respect to the physical radius \( \ln f \) for different \( \alpha_1 \). The figures show that there are commonly three horizons in the spacetime. One of them is the inner Cauchy horizon and the other is the black hole event horizon. The third horizon is the “cosmic horizon”. This is very similar to the spacetime of Reissner-Nordstrom-de Sitter solution. On the other hand, the cosmic horizon of the Reissner-Nordstrom-de Sitter spacetime locate at finite radius. But in this solution, the “cosmic horizon” locates at the spatial infinity. With the increasing of \( \alpha_1 \), the event horizon is shrinking. When \( \alpha_1 = 0 \), the inner Cauchy horizon and the black hole event horizon coincide and the solution corresponds to the extreme solution.

In Fig. 3 we plot the evolution of the electric potential \( \phi \) with respect to the physical radius \( f \) for different \( \alpha_1 \). It shows that \( \phi \) is regular in the spacetime except for \( f = 0 \) (curvature singularity). The potential \( \phi \) is divergent at \( f = 0 \) and asymptotically approaches zero in the infinity of space. This behavior is the same as the electric potential in Reissner-Nordstrom-de Sitter solution. In order to show \( \phi = 0 \) is the curvature singularity, as an example, we plot
the evolution of the Ricci scalar $R$ with respect to the physical radius $f$ in Fig. 4 with $m = 1$, $\alpha_0 = \alpha_2 = \alpha_3 = 1$, $\alpha = 3/2$, $\alpha_1 = -0.4$. It is apparent $R$ is divergent at $f = 0$. This reveals $f = 0$ is indeed the curvature singularity.

In Fig. 5 we plot the evolution of the inverse of magnetic potential $\psi$ with respect to the physical radius $\ln f$ for different $\alpha_1$. It shows that the magnetic potential $\psi^{-1}$ is divergent on both horizons while asymptotically approaches zero in both the infinity of space and the curvature singularity.

In Fig. 6, we plot the evolution of $U$ with respect to the physical radius $\ln f$ with values $m = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\alpha = 3/2$. As for $\alpha_0$, we let $\alpha_0 = 0.15$, $0.1$, $0$, $-0.2$, $-0.4$, respectively. Comparing with Fig. 1, we find the black hole event horizon is pushed to infinity in this case. We are left with only the inner Cauchy horizon. Keep the constants ($\alpha_0$, $\alpha_1$, $\alpha_2$, $m$, $f_0$) to be fixed and verify $\alpha_3$ or $\alpha$, we find the figures are similar to Fig. 1 or Fig. 6.

III. CONCLUSION AND DISCUSSION

In conclusion, the static spherically symmetric solution in the Einstein-aether is obtained. Due to the complication of the Einstein equations, we prefer to deal with the Euler-Lagrange equations. This method is relatively simple and the same as the Einstein equations in nature. By this way, the exact solution is constructed.

The solution is similar to the Reissner-Nordstrom-de Sitter solution in that it has an inner Cauchy horizon, an outer black hole event horizon and a “cosmic horizon”. But different from the Reissner-Nordstrom-de Sitter solution, its “cosmic horizon” locates not at some finite physical radius but at the spatial infinity. The resulting electric potential is regular in the whole spacetime except for the curvature singularity. On the other hand, the magnetic potential is
FIG. 3: The evolution of the electric potential $\phi$ with respect to the physical radius $f$.

FIG. 4: The evolution of the Ricci scalar $R$ with respect to the physical radius $f$.

FIG. 5: The evolution of the inverse of magnetic potential $\psi$ with respect to the physical radius $\ln f$. 
FIG. 6: The evolution of $U$ with respect to the physical radius $\ln f$.

divergent on both Cauchy horizon and the outer event horizon.

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