INVERTIBILITY AND STABILITY FOR A GENERIC CLASS OF RADON TRANSFORMS WITH APPLICATION TO DYNAMIC INVERSE OPERATORS

SIAMAK RABIENIAHARATBAR

Abstract. Let \( X \) be an open subset of \( \mathbb{R}^2 \). We study the dynamic inverse operator, \( \mathcal{A} \), integrating over a family of level curves in \( X \) when the object changes between the measurement. We use analytic microlocal analysis to determine which singularities can be recovered by the data-set. Our results show that not all singularities can be recovered, as the object moves with speed lower than the X-ray source. We establish stability estimates and prove that the injectivity and stability are of a generic set if the dynamic inverse operator satisfies the visibility, no conjugate points, and local Bolker conditions. We also show this results can be implemented to Fan beam geometry.

1. Introduction

Tomography of moving objects has been attracting a growing interest recently, due to its wide range of applications in medical imaging, for example, X-ray of the heart or the lungs. Data acquisition and reconstruction of the object which changes its shape during the measurement is one of the challenges in computed tomography and dynamic inverse problems. The major difficulty in the reconstruction of images from the measurement sets is the fact that object changes between measurements but does not move fast enough compared to the speed of X-rays. This means that some singularities of the object might not be detectable even if the source fully rotates around the object. The application of known reconstruction methods (based on the inversion of the Radon transform) usually results in many motion artifacts within the reconstructed images if the motion is not taken into account. One extreme example will be the case when the object (or some small part of it) rotates with the same rate as the scanner. This leads to integration over the same family of rays, and therefore, one may not locally recover all the singularities.

Analytic techniques for reconstruction of dynamic objects, known as motion compensation, have been used widely for different types of motion, like affine deformation, see e.g. \([2, 3, 9, 18]\). In the case of non-affine deformations, there is no inversion formula. Iterative reconstructions, however, do exist in order to detect singularities by approximation of inversion formulas for the parallel and fan beam geometries \([14]\), as well as cone beam geometry \([15]\). In a recent work, Hahn and Quinto \([8]\) studied the dynamic inverse operator

\[
\mathcal{A}f(s, t) = \int_{\psi_t^{-1}(x) \cdot \omega(t) = s} \mu(t, z) f(\psi_t(z)) dS_z, \quad \omega(t) = (\cos t, \sin t),
\]

with a smooth motion where the limited data case has been analyzed, and characterization of visible and added singularities have been investigated.

Our work in this paper is motivated by these dynamic measurements. We first show this dynamic problem can be reduced to an integral geometry problem integrating over level curves. By an appropriate change of variable (see section 2), \( \mathcal{A} \) can be written as

\[
\mathcal{A}f(s, t) = \int_{\phi(t, x) = s} \hat{\mu}(t, x) f(x) dS.
\]

Therefore, we study the following general operator:

\[
\mathcal{A}f(s, t) = \int_{\phi(t, x) = s} \mu(t, x) f(x) dS_{s, t},
\]

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which allows us to study the original dynamic problem with a more general set of curves (see also \[13\],) and then transfer the result to a dynamic inverse operator \( \mathcal{A} \) given by \([7, 8]\).

The dynamic inverse operator \( \mathcal{A} \) formulated as above falls into the general microlocal framework studied by Beylkin \([3]\) (see also \([6, 7, 13]\)) which goes back to Guillemin and Sternberg \([6, 7]\) who studied the integral geometry problems with a more general platform from the microlocal point of view. See also \([1]\), where a weighted integral transform has been studied on a compact manifold with a boundary over a general set of curves (a smooth family of curves passing through every point in every direction).

The main novelty of our work, compared to previous works which are concentrated on the microlocal invertibility, is that for the dynamic problem, under some natural microlocal conditions, the actual uniqueness and stability results have been established. In fact, our imposed natural microlocal conditions guarantee that one can recover each singularity, and a functional analysis argument leads to stability results. We show that under these conditions, the dynamic inverse operator is stably invertible in a neighborhood of pairs \((\phi, \mu)\) in a generic set, and in particular, it is injective and stable for slow enough motion (which is not required to be a periodic motion model). This is the similar kind of stability result which has been studied in \([13]\) for the generalized Radon transform and in \([1]\) which coincide when the dimension is two. The data is cut (restricted) in a way to have the normal operator related to the localized dynamic inverse operator \( \mathcal{A} \) as a pseudodifferential operator (ΨDO) near each singularity. We do not analyze the case where these conditions are not satisfied globally, but our analysis (see also \([8]\)) shows that one may still recover the visible singularities in a stable way, and periodicity or non-periodicity plays no role in the reconstruction process. We also show that, due to the generality of our approach, our results can be implemented to other geometries, for instance, fan beam geometry.

This paper is organized as follow: Section one is an introduction. In section two, we state the definitions of Visibility, Local Bolker Condition, Semi-Global Bolker Condition, and our main result. Some preliminary results have been stated in section three. Section four is devoted to analytic microlocal analysis approach which is used to show that the operator \( \mathcal{A} \) is a Fourier Integral Operator (FIO). Then the canonical relation \( \mathcal{C} \) is computed and it is shown that it is a four-dimensional non-degenerated conic submanifold of the conormal bundle. In section five, it is shown that a certain localized version of the normal operator \( \mathcal{N} = \mathcal{A}^* \mathcal{A} \) is an elliptic pseudodifferential operator (ΨDO) under the visibility, and the local and semi-global Bolker conditions. In section six, we study the operators \( \mathcal{A} \) and \( \mathcal{N} \) globally, and show that uniqueness and stability (injectivity) are of a generic set with the corresponding topology. In the last section, we implement our results for the initial dynamic problem of scanning a moving object while changing its shape. We also show that our results can be applied to fan beam geometry by an appropriate choice of phase function \( \phi \).

### 2. Main Results

In this section, we first introduce the dynamic operator and then reduce it to an integral geometry problem integrating over level curves. After some necessary propositions, we state our main results.

**Definition 2.1.** Let \( X \) be a fixed open set in \( \mathbb{R}^2 \) and \( Y \) be the open sets of lines determined by \((s, t)\) in \( \mathbb{R}^2 \). For \( \mathcal{A} : C^\infty_0(X) \to C^\infty(Y) \), the operator of the dynamic inverse problem is defined by

\[
\mathcal{A} f(s, t) = \int_{x : \omega(t) = s} \mu(t) f(\psi_t(x)) dS_x,
\]

where \( \omega(t) = (\cos t, \sin t) \) and the function \( \mu \) is a non-vanishing smooth weight changing with respect to the variable \( t \) and the position \( x \).

Here \( \psi_t \) is a diffeomorphism in \( \mathbb{R}^2 \), which is identity outside \( X \), smoothly depending on the variable \( t \), and \( dS_x \) is the euclidean measure restricted to the lines parametrized by \( \{s = x \cdot \omega(t)\} \). Notice that each point (position) \( x \in X \), lies on the lines in \( Y \) parametrized by \((s, t)\).

The operator \( \mathcal{A} \) can be written in the following format:

\[
\mathcal{A} f(s, t) = \iint_{\mathbb{R}^2} \mu(t, x) f(\psi_t(x)) \delta(s - x \cdot \omega(t)) dx.
\]
Since $\psi_t$ is a diffeomorphism, by performing a change of variable $z = \psi_t(x)$, we get $x = \psi_t^{-1}(z)$ and therefore, we have

$$A f(s,t) = \int_{\mathbb{R}^2} J(t,z) \mu(t,\psi_t^{-1}(z)) f(z) \delta(s - \psi_t^{-1}(z) \cdot \omega(t)) \, dz.$$  

From now on, we do most of our analysis on the following general operator:

$$A f(s,t) = \int_{\phi(t,x) = s} \mu(t,x) f(x) dS_{s,t}, \tag{2.1}$$

where $\mu$ is a new positive weight and the map

$$x = (x^1, x^2) \rightarrow \phi(t,x),$$

is real-valued. Here $dS_{s,t}$ is the Euclidean measure of the level curves of function $\phi$, defined as

$$H(s,t) = \{ x \in \mathbb{R}^2 : s = \phi(t,x), s \in \mathbb{R}, t \in \mathbb{R} \}.$$  

We, first, need to show for any time $t$ and point $x$, there exists a curve passing through the point $x$ with direction $\omega(t)$.

**Proposition 2.1.** Let $H(s,t)$ be the level curves of $\phi$. Then, locally near $(s_0, t_0)$ and near a fixed $x_0 \in H(s,t)$ the followings are equivalent.

i) The map from the variable $t$ to the unit normal vector $\nu$ of the level curves $H(s,t)$:

$$t \rightarrow \nu(t,x) = \frac{\partial_x \phi(t,x)}{|\partial_x \phi(t,x)|}, \quad \partial_x \phi(t,x) \neq 0, \tag{2.2}$$

is a local diffeomorphism, where $\partial_x = (\partial_{x^1}, \partial_{x^2})$.

ii) The Local Bolker Condition:

$$h(t,x) = \det \left( \frac{\partial \phi}{\partial x^1}, \frac{\partial^2 \phi}{\partial x^1 \partial x^1} \right) \bigg|_{(t,x) = (t_0, x_0)} \neq 0, \tag{2.3}$$

holds locally near $(s_0, t_0)$ and near $x_0$.

**Remark 2.1.** i) The proof of Proposition 2.1 is postponed to the next section. In our setting, the equation \[2.3\] is the generalization of what it is known as a Bolker condition in [Theorem 3.4 (2), 8].

ii) One may always rotate the unit normal vector $\nu$ by $\frac{\pi}{2}$ (at a fixed point $x$ on the curve) to get the tangent vector at that fixed point. Now the first part in Proposition 2.1 implies that the map from the variable $t$ to the tangent vector at point $x$ on the level curve $H(s,t)$, is also a local diffeomorphism.

iii) We work locally near $(s_0, t_0)$ and a fixed $x_0$ on the level curve. Let $l_0$ denote the unit tangent (normal) vector at $x_0$. By the first part, for any unit tangent vector $l$ in some small neighborhood of $l_0$, the map from the variable $t$ to the unit tangent at a fixed point $x$ is a local diffeomorphism. Now the Implicit Function Theorem implies that for any given $t$, there exists a curve passing through the fixed point $x$ with a tangent vector $l$. This indeed is what to expect if we want the level curves to behave like the geodesic curves.

iv) The local Bolker condition requires that when the object moves by time, the curve changes its direction. The counterexample when the local Bolker condition does not hold is the case where an object and the scanner move with the same rate. In this situation, the object can be considered stationary where it is being scanned with stationary parallel rays. The above proposition guarantees that locally and microlocally this situation will not happen and the parameter $t$ changes the angle if we keep the object stationary. (i.e the movement is not going to be synchronized with the scanner)

v) Proposition 3.1 in the next section, shows that one may connect the local Bolker condition to Fourier Integral Operator (FIO) theory by extending the function $\phi$ to a homogeneous function of order one (see \[1\]), and therefore one may use the condition \[2.3\] for the analysis.

For main results, we first state the following definitions.

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**Remark 2.1.**
Definition 2.2. The function \( \phi \) satisfies the **Visibility condition** at \((x, \xi) \in T^*X \setminus 0\) if there exists a pair \((s, t)\) with property \( \phi(t, x) = s \), such that \( \partial_x \phi(t, x) \parallel \xi \). Here \( T^*X \) is the cotangent bundle of \( X \).

The visibility condition requires that at a point \( x \) and co-direction \( \xi \), locally, there exists a unique curve passing through \( x \) which is conormal to \( \xi \). As we pointed out in Remark 2.1, this property is a natural property of level curves as are expected to behave like geodesic curves. It also means that each singularity can be probed locally.

Definition 2.3. The function \( \phi \) satisfies the **Semi-Global Bolker Condition** in a neighborhood of a fixed \((x, (s, t)) \in X \times Y\) if

\[
\begin{align*}
\phi(t, x) &= \phi(t, y) = s \\
\partial_t \phi(t, x) &= \partial_t \phi(t, y)
\end{align*}
\]

\( \implies x = y \) for \( x, y \in X \).

The first equation in (2.4) implies that at instance \( t \), the measurement will not distinguish between points \( x \) and \( y \), when they are both on the same level curve. The second equation implies that the new measurement, due to perturbation in the variable \( t \), still cannot distinguish between these two points as they both belong to the same perturbed level curve. Note that, we only need our fixed point \( x \) to have no conjugate points along the curve passing through it with conormal \( \xi \). Also, note that we do not require any restriction on \( x \) and it can vary freely in that neighborhood.

We now are ready to state our main result for the operator \( A \) given by (2.1).

Theorem 2.1. Consider the operator \( A \) with a nowhere vanishing smooth weight \( \mu \). Let \( \Sigma \) be a set of all possible pairs \((\phi, \mu)\) which are smooth in some \( C^K \)-topology with \( K \) an arbitrary large natural number. Assume that for any \((x, \xi) \in T^*X \setminus 0\), (i) the visibility condition holds and (ii) the local and semi-global Bolker conditions are satisfied for some \((s, t)\) given by the visibility condition.

Then within \( \Sigma \), there exists a dense and open (generic) set \( \Lambda \) of pairs of \((\phi, \mu)\) such that locally near any pair in \( \Lambda \), the uniqueness results and therefore stability (injectivity) estimates given by Proposition 6.2 hold.

To formulate above result for the dynamic inverse operator \( A \) given by (2.1), we first state the visibility, and the local and semi-global Bolker conditions for \( A \).

Visibility. This condition implies that for \((x, \xi) \in T^*X \setminus 0\), the map

\[
t \to \frac{\xi}{|\xi|} \in S^1
\]

is locally surjective. Here the point \((s, t)\) lies on the level curve \( s = \psi^{-1}_t(x) \cdot \omega(t) \).

Local Bolker Condition. This condition (see Proposition 4.1) implies that

\[
h(t, x) = \det \left( \frac{\partial \psi^{-1}_t(x) \cdot \omega(t)}{\partial x^j}, \frac{\partial^2 \psi^{-1}_t(x) \cdot \omega(t)}{\partial x^j \partial x^j} \right) \neq 0.
\]

Semi-global Bolker condition (No conjugate points condition). By condition (2.4), semi-global Bolker condition holds if the map

\[
x \to \left( \psi^{-1}_t(x) \cdot \omega(t), \partial_t(\psi^{-1}_t(x) \cdot \omega(t)) \right)
\]

is one-to-one.

Now for the dynamic inverse operator \( A \) given by (2.1), we have the following result:

Theorem 2.2. Consider the dynamic inverse operator \( A \) with a nowhere vanishing smooth weight \( \mu \). Let \( \Sigma \) be a set of all possible pairs \((\psi, \mu)\) which are smooth in some \( C^K \)-topology with \( K \) an arbitrary large natural number. Assume that for any \((x, \xi) \in T^*X \setminus 0\), (i) the visibility condition \((2.7)\) holds and (ii) the local and semi-global Bolker conditions given by \((2.6)\) and \((2.7)\) are satisfied for some \((s, t)\) given by the visibility condition.

Then within \( \Sigma \), there exists a dense and open (generic) set \( \Lambda \) of pairs of \((\psi, \mu)\) such that locally near any pair in \( \Lambda \), the uniqueness results and therefore stability (injectivity) estimates hold.
Corollary 2.1. In particular, for a small perturbation of φ(t, x) = x · ω(t) where there is no motion or the motion is small enough (μ ≈ 1), we have the actual injectivity and invertibility as the set of pairs of (φ, μ) is included in Λ.

Remark 2.2. The Corollary 2.1 follows from the fact that the stationary Radon transform is analytic and for a small perturbation of phase function, the invertibility and injectivity still hold.

3. Preliminary Results

In this section, we first prove Proposition 2.1 and then connect the local Bolker condition \( \text{Proposition 3.1} \) to Fourier Integral Operator theory. At the end, we state some definitions which will be used in the following sections.

Definition 3.1. A set Σ is conic, if ξ ∈ Σ then rξ ∈ Σ for all r > 0.

Proof of Proposition 2.1. i) → ii) Fix \((t_0, x_0)\) and let \(φ(t_0, x_0) = s_0\). We work on some neighborhood of \((s_0, t_0)\) and \(x_0\). Since \(∂_φ φ(t, x) \neq 0\), the map \(\varphi\) is well-defined and there exists a tangent at a fixed time \(t\) when \(x\) varies. The map \(\psi\) is a local diffeomorphism, therefore \(∂_ν t, x \neq 0\) and its inverse exists with non-zero derivative in a conic neighborhood.

Assume now that \(h(t, x) = 0\). Then there exists a non-zero constant \(c\) such that

\[
(3.1) \quad ∂_t ∂_x φ(t, x) = c ∂_x φ(t, x).
\]

Plugging (3.1) into \(∂_t ν(t, x)\):

\[
∂_t ν(t, x) = \frac{∂_t ∂_x φ(t, x)}{|∂_x φ(t, x)|} - ∂_x φ(t, x) \frac{∂_x φ(t, x) · ∂_t ∂_x φ(t, x)}{|∂_x φ(t, x)|^3}
\]

we get \(∂_t ν(t, x) = 0\), which is a contradiction. Therefore \(h(t, x) \neq 0\).

ii) → i) Assume that \(\text{Proposition 3.2}\) is true. This in particular implies that \(∂_x φ(t, x)\) and \(∂_t ∂_x φ(t, x)\) are non-zero and linearly independent. For any \(t\), let \(ν(t, x) = \frac{∂_x φ(t, x)}{|∂_x φ(t, x)|}\) denotes the unit normal at a fixed point \(x\) on the curve. To show the map in \(\text{Proposition 3.2}\) is a local diffeomorphism, we need to show \(∂_ν t, x \neq 0\) in a conic neighborhood. Note that this map is well-defined as \(∂_x φ(t, x) \neq 0\). Assume that \(∂_ν t, x = 0\). Then

\[
\frac{∂_t ∂_x φ(t, x)}{|∂_x φ(t, x)|} = ∂_x φ(t, x) \frac{∂_x φ(t, x) · ∂_t ∂_x φ(t, x)}{|∂_x φ(t, x)|^3}
\]

which implies that

\[
∂_t ∂_x φ(t, x) = c ∂_x φ(t, x), \quad c = \frac{∂_x φ(t, x) · ∂_t ∂_x φ(t, x)}{|∂_x φ(t, x)|^2}.
\]

This contradicts with the fact that \(∂_x φ(t, x)\) and \(∂_t ∂_x φ(t, x)\) are linearly independent. Now by Inverse Function Theorem, the map \(\varphi\) is a local diffeomorphism as it is smooth and its Jacobian is nowhere vanishing. □

One may extend the function \(φ\) to a homogeneous function of order one as follow:

(3.2) \(φ(x, θ) = ψ_t^{-1}(x) · θ = |θ| ∂x (arg θ, x), \quad \text{where} \ θ = (θ^1, θ^2) = |θ|(cos t, sin t) \in \mathbb{R}^2 \setminus 0\).

As we pointed out above, we work locally in a conic neighborhood of \(t_0\) and \(s_0\). This guarantees that function \(arg θ\) is single-valued. To connect the local Bolker condition to Fourier Integral Operator theory, we have the following proposition.

Proposition 3.1. For the function \(φ\) defined by \(φ\) in \(\varphi\), the local Bolker condition \(\varphi\) holds if and only if

\[
\det \left( \frac{∂^2 φ}{∂θ^i ∂x^j} \right) \neq 0.
\]
Proof. Since $\partial_x \varphi = |\theta| \partial_x \phi \neq 0$, we have
\[
\frac{\partial^2 \varphi}{\partial \theta^1 \partial x^3} = \frac{\partial}{\partial \theta^1} \left( \frac{\partial \phi}{\partial x^3} \right) = \frac{\theta^1}{|\theta|} \frac{\partial \phi}{\partial x^3} - \frac{\theta^2}{|\theta|} \frac{\partial \phi}{\partial t \partial x^3},
\]
and
\[
\frac{\partial^2 \varphi}{\partial \theta^2 \partial x^3} = \frac{\partial}{\partial \theta^2} \left( \frac{\partial \phi}{\partial x^3} \right) = \frac{\theta^2}{|\theta|} \frac{\partial \phi}{\partial x^3} + \frac{\theta^1}{|\theta|} \frac{\partial \phi}{\partial t \partial x^3},
\]
where $t = \arg \theta$. Assume first that $\partial_x \varphi(t, x)$ and $\partial_x \partial_x \varphi(t, x)$ are linearly independent. We show that columns in the matrix $\frac{\partial^2 \varphi}{\partial \theta^i \partial x^j}$ are linearly independent for $i = 1, 2$. So let
\[
c_1 \frac{\partial^2 \varphi}{\partial \theta^1 \partial x^3} + c_2 \frac{\partial^2 \varphi}{\partial \theta^2 \partial x^3} = 0.
\]
Then we have
\[
(c_1 \frac{\theta^1}{|\theta|} + c_2 \frac{\theta^2}{|\theta|}) \frac{\partial \phi}{\partial x^3} + (-c_1 \frac{\theta^2}{|\theta|} + c_2 \frac{\theta^1}{|\theta|}) \frac{\partial^2 \varphi}{\partial t \partial x^3} = 0.
\]
Since $\partial_x \varphi(t, x)$ and $\partial_x \partial_x \varphi(t, x)$ are linearly independent, we have
\[
c_1 \theta^1 + c_2 \theta^2 = 0, \quad -c_1 \theta^2 + c_2 \theta^1 = 0,
\]
which simply implies that $c_1 = c_2 = 0$, and therefore $\frac{\partial^2 \varphi}{\partial \theta^i \partial x^j}$ are linearly independent for $i = 1, 2$.

Assume now that $\frac{\partial^2 \varphi}{\partial \theta^i \partial x^j}$ are linearly independent for $i = 1, 2$. We show that $\partial_x \varphi(t, x)$ and $\partial_x \partial_x \varphi(t, x)$ are linearly independent. We first rewrite $\partial_x \varphi(t, x)$ and $\partial_x \partial_x \varphi(t, x)$ as follow:
\[
\frac{\theta^1}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^1 \partial x^3} = \frac{(\theta^1)^2}{|\theta|} \frac{\partial \phi}{\partial x^3} - \frac{\theta^1 \theta^2}{|\theta|} \frac{\partial \phi}{\partial t \partial x^3},
\]
and
\[
\frac{\theta^2}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^2 \partial x^3} = \frac{(\theta^2)^2}{|\theta|} \frac{\partial \phi}{\partial x^3} + \frac{\theta^1 \theta^2}{|\theta|} \frac{\partial \phi}{\partial t \partial x^3}.
\]
Adding the last two equations we get
\[
\frac{\theta^1}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^1 \partial x^3} + \frac{\theta^2}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^2 \partial x^3} = \frac{\partial \phi}{\partial x^3}.
\]
Consider
\[
-\frac{\theta^2}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^1 \partial x^3} = -\frac{\theta^1 \theta^2}{|\theta|} \frac{\partial \phi}{\partial x^3} + \frac{(\theta^2)^2}{|\theta|} \frac{\partial \phi}{\partial t \partial x^3},
\]
and
\[
\frac{\theta^1}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^2 \partial x^3} = \frac{\theta^1 \theta^2}{|\theta|} \frac{\partial \phi}{\partial x^3} + \frac{(\theta^1)^2}{|\theta|} \frac{\partial \phi}{\partial t \partial x^3}.
\]
Adding the last two equations, we have
\[
-\frac{\theta^2}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^1 \partial x^3} + \frac{\theta^1}{|\theta|} \frac{\partial^2 \varphi}{\partial \theta^2 \partial x^3} = \frac{\partial \phi}{\partial x^3}.
\]
Now assume that
\[
\tilde{c}_1 \frac{\partial \phi}{\partial x^3} + \tilde{c}_2 \frac{\partial \phi}{\partial t \partial x^3} = 0.
\]
In a similar way as we showed above and using the fact that $\frac{\partial^2 \varphi}{\partial \theta^i \partial x^j}$ are linearly independent for $i = 1, 2$, we conclude that $\tilde{c}_1 = \tilde{c}_2 = 0$. This proves the proposition. □

In principle, Proposition 2.2 implies that we may use our analysis with $\tilde{c}_2 = 0$, see [3].

**Definition 3.2.** We say that $(x_0, \xi^0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is not in the Wave Front Set of $f \in D'(\mathbb{R}^n)$, WF $f$, if there exists $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$ so that for any $N$, there exists $C_N$ such that
\[
|\partial \phi f(\xi)| \leq C_N|1 + \xi|^N
\]
for $\xi$ in some conic neighborhood of $\xi^0$. This definition is independent of the choice of $\phi$. 

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Definition 3.3. For the case of a scalar-valued distribution, one may define the Analytic Wave Front Set, WF, as the complement of all \((x, \xi) \in T^* (\mathbb{R}^n \setminus 0)\) such that

\[
\int e^{i\lambda(x-y) \cdot \xi - \frac{1}{2}(x-y)^2} \chi(y) f(y) \, dy = O(e^{-\frac{\lambda}{C}}), \quad \lambda > 0
\]

with some \(C > 0\) and \(\chi \in C_0^\infty\) equal to 1 near \(x\). We recall that, there are three equivalent definitions of Analytic Wave Front Set in the literature.

Definition 3.4. We call \((x, \xi) \in T^* X \setminus 0\) a position singularity if \((x, \xi) \in WF f\). We call \((x, \xi) \in T^* X \setminus 0\) a visible singularity if \((x, \xi) \in WF f\) satisfies the visibility condition by Definition 2.2. Similarly, we call \((s, t, \sigma, \tau)\) a measurement singularity if \((s, t, \sigma, \tau) \in WF A\).

4. Microlocal Analyticity

In this section, we study the microlocal analyticity of operator \(A\) for a given \(f\). We first compute the adjoint operator.

Adjoint Operator \(A^*\). Let \(\phi \in C^\infty (\mathbb{R} \times \tilde{X})\) be given, where \(X\) is embedded in an open set \(\tilde{X}\). We extend our function \(f\) to be zero on \(\tilde{X} \setminus X\). Consider now the one-dimensional level curves

\[
H(s, t) = \{x \in \tilde{X} : s = \phi(t, x), \quad s \in \mathbb{R}, \quad t \in \mathbb{R}\},
\]

with Euclidean measure \(dS_x\) induced by the volume form \(dx\) in the domain \(X\). There exists a non-vanishing and smooth function \(J(t, x)\) such that

\[
dS_{s,t}(x) \wedge ds = J(t, x) dx.
\]

Therefore,

\[
\int_{T_1}^{T_2} \int_{\mathbb{R}} (Af) \tilde{g} ds dt = \int_{T_1}^{T_2} \int_{\mathbb{R}} \int_{H(s, t)} \mu(t, x) f(x) \tilde{g}(s, t) dS_{s,t} ds dt
\]

\[
= \int_{T_1}^{T_2} \int_{X} \mu(t, x) f(x) \phi(t, x) J(t, x) dx dt,
\]

where \(T_1 < t < T_2\) and \(0 < T_2 - T_1 \ll 2\pi\). In the second equality above, we used the fact that the double integral \(\int_{\mathbb{R}} \int_{H(s, t)}\) equals to an integral over \(\tilde{X}\), by Fubini’s Theorem. Thus, the adjoint of \(A\) in \(L^2(\mathbb{X}, dx)\) is

\[
A^* g(x) = \int_{\mathbb{R}} \tilde{\mu}(t, x) \tilde{J}(t, x) \phi(t, x) J(t, x) dt,
\]

where \(\mu\) is supported in \(\{t \in \mathbb{R} : T_1 < t < T_2\}\). In fact, the adjoint \(A^* g(x)\) is localized in \(t\) and is equal to the average of \(g(s, t)\) over all lines determined by \((s, t)\) passing through the point \(x\).

Schwartz Kernel. Now we compute the Schwartz kernel of the operator \(A\).

Lemma 4.1. The Schwartz kernel \(K_A\) of \(A\) is

\[
K_A(s, t, y) = \delta(s - \phi(t, y)) \mu(t, y) J(t, y),
\]

where \(J(t, y) = |d_y \phi| = (\sum |\partial_{y^i} \phi|^2)^{1/2}\).

Proof. Let \(\Phi(s, t, y) = s - \phi(t, y)\). By (6.1), we have

\[
Af(s, t) = \int_{\phi(y) = s} \mu(t, y) f(y) dS_{s,t} = \int_{\phi(y) = s} \mu(t, y) f(y) |d_y \Phi| |d_y \Phi|^{-1} dS_{s,t}.
\]

Since \(\partial_{y^i} \Phi = -\partial_{y^i} \phi\) and \(\partial_{y^i} \phi \neq 0\) when \(\Phi = 0\), by Theorem (6.1.5) Hörmander, we have

\[
|d_y \Phi|^{-1} dS_{s,t} = \Phi^* \delta_0.
\]

Here \(\Phi^*\) is pullback with \(\Phi^* \delta_0 = \delta_0 \circ \Phi\). The second integral above can be written as

\[
\int \Phi^* \delta_0 \mu(t, y) f(y) |d_y \Phi| dy = \langle \Phi^* \delta_0 \mu | d_y \Phi, f \rangle.
\]
Therefore, the Schwartz kernel of $\mathcal{A}$ is

$$K_{\mathcal{A}}(s,t,y) = \delta(s - \phi(t,y))\mu(t,y)|d_d\Phi|.$$

\[\square\]

**Remark 4.1.** One may compute the Schwartz kernel of $\mathcal{A}^*$ and $\mathcal{N} = \mathcal{A}^*\mathcal{A}$:

$$K_{\mathcal{A}^*}(s,t,x) = \delta(\phi(t,x) - s)\bar{\mu}(t,x)J(t,x),$$

$$K_{\mathcal{N}}(s,t,x,y) = \int_{\mathbb{R}} \delta(\phi(t,x) - \phi(t,y))\bar{\mu}(t,x)J(t,x)\mu(t,y)J(t,y)dt.$$

The following lemma shows that the operator $\mathcal{A}$ is an elliptic Fourier Integral Operator (FIO).

**Lemma 4.2.** Let $M = \{(s,t,x) : \Phi(s,t,x) = s - \phi(t,x) = 0\} \subset Y \times X$. Then the operator $\mathcal{A}$ is an elliptic FIO of order $-\frac{1}{2}$ associated with the conormal bundle of $M$:

$$N^*M = \{(s,t,x,\sigma,\tau,\xi) \in T^*(Y \times X) | (\sigma,\tau,\xi) = 0 \text{ on } T_{(s,t,x)}M\},$$

where $(s,t,\sigma,\tau)$ and $(x,\xi)$ are the coordinates on $T^Y$ and $T^X$, respectively.

**Proof.** By Lemma 3.1 the Schwartz kernel $K_{\mathcal{A}}$ has singularities conormal to the manifold $M$. Since $\dim X = \dim Y = 2$, the Schwartz kernel $K_{\mathcal{A}}$ is conormal type in the class $I^{-\frac{1}{2}}(Y \times X; M)$, see (Section 18.2, [11]). This shows that the operator $\mathcal{A}$ is an elliptic FIO of order $-\frac{1}{2}$ associated with the conormal bundle $N^*M$.

Note that $\sigma$ is a one-dimensional non-zero variable.

We now compute the canonical relation $\mathcal{C}$ and show it is a four-dimensional non-degenerated conic submanifold of $N^*M$ parametrized by $(t,x,\sigma)$. Note that $N^*M$ is a Lagrangian submanifold of $T^*(Y \times X)$.

**Proposition 4.1.** Let $\mathcal{C}$ be the canonical relation associated with $M$. Then

$$\mathcal{C} = \{(\phi(t,x),t,\sigma,-\sigma\partial_t\phi(t,x);x,\sigma\partial_x\phi(t,x)|(\phi(t,x),t,x) \in M, \quad 0 \neq \sigma \in \mathbb{R}\}.$$

Furthermore, the canonical relation $\mathcal{C}$ is a local canonical graph if and only if for any $t$, the map

$$x \to (\phi(t,x),\partial_t\phi(t,x)) \tag{4.1}$$

is locally injective and local Bolker condition \([2.3]\) holds.

**Proof.** The twisted conormal bundle of $M$:

$$\mathcal{C} = (N^*M \setminus 0)' = \{(s,t,\sigma,\tau;x,\xi) | (s,t,\sigma,\tau;x,\tau,\xi) \in N^*M\},$$

gives the canonical relation associated with $M$. We first calculate the differential of the function $\Phi(s,t,x) = s - \phi(t,x)$. We have

$$d\Phi(s,t,x) = ds - \partial_t\phi(t,x)dt - \partial_x\phi(t,x)dx.$$

Therefore, the canonical relation is given by

$$\mathcal{C} = \{(\phi(t,x),t,\sigma,-\sigma\partial_t\phi(t,x);x,\sigma\partial_x\phi(t,x)|(\phi(t,x),t,x) \in M, \quad 0 \neq \sigma \in \mathbb{R}\}.$$

Now consider the microlocal version of double fibration:

$$\begin{array}{ccc}
\pi_Y & \mathcal{C} & \pi_X \\
T^*(Y) & & T^*(X)
\end{array}$$
exists a tangent vector to each level curve $H$.  

**Proof.** If $d_{t,x,\sigma} \Pi_Y$ has rank equal to four, then the Bolker condition is locally satisfied. Indeed, this is true, as $d_{t,x,\sigma} \Pi_Y$ has rank equal to four if and only if the condition (2.3) holds. This implies that $\dim C = 4$. Since the map in (2.3) is one-to-one, the projection $\Pi_Y : C \to T^*(Y)$ is an injective immersion. Hence, $\Pi_Y$ is a local diffeomorphism. 

The following lemma states whether position singularities and measurement singularities can affect each other. We refer the reader to Definitions 3.3 and 3.4, for position and measurement singularities.

**Lemma 4.3.** Let $X$ be a fixed open set in $\mathbb{R}^2$ and $Y$ be the open sets of lines determined by $(s,t)$ in $\mathbb{R}^2$. Then, the map

$$
\Pi_X \circ \Pi_Y^{-1} : T^*(Y) \to T^*(X)
$$

is a local diffeomorphism. 

**Proof.** Consider the map $\Pi_Y : C \to T^*(Y)$. We show that for a given $(s, t, \sigma, -\sigma \partial_t \phi) \in T^*(Y)$, one can determine $(x, \xi) \in T^*(X)$. Since $\partial_t \phi$ is non-zero (and both $\sigma \partial_t \phi$ are non-zero), for a given $(s, t)$ there exists a tangent vector to each level curve $H(s, t)$. By Remark 2.1, one can find a non-zero normal vector $\partial_x \phi$ on each level curve, and therefore $\xi = \sigma \partial_x \phi$. On each level curve $H(s, t)$, we have $s = \phi(t, x)$. Since $\partial_x \phi \neq 0$, the Implicit Function Theorem implies that the variable $t$ determines $x$. Hence, the map $\Pi_Y$ is a local diffeomorphism.

Now consider the map $\Pi_X : C \to T^*(X)$. Our goal is to determine $(s, t, \sigma, -\sigma \partial_t \phi) \in T^*(Y)$, for a given $(x, \xi) = (x, \sigma \partial_x \phi) \in T^*(X)$. By Proposition 2.1, the map

$$
\frac{\xi}{|\xi|} = \frac{\partial_x \phi}{|\partial_x \phi|} \to t,
$$

is a local diffeomorphism for a fixed point $x$ provided that the condition (2.3) holds. Thus, $(x, \frac{\xi}{|\xi|})$ determines the variable $t$. In particular, for a given $(x, \xi)$ this implies that one can identify the level curve $H(s, t)$, as $(t, x)$ determines $\phi$, and therefore $s$ (on each level curve we have $s = \phi(t, x)$). Since $\xi = \sigma \partial_x \phi$ with $\xi \neq 0$, one can determine $\sigma = \frac{|\xi|}{|\partial_x \phi|}$. To determine the last variable $\sigma \partial_t \phi$, it is enough to take the partial derivative of $\phi$ with respect to the variable $t$. Thus, the map $\Pi_X$ is a local diffeomorphism. We remind that the above argument is valid when the condition (2.3) is satisfied.

Now since $\dim(Y) = \dim(X)$ and $\Pi_X : C \to T^*(X)$ and $\Pi_Y : C \to T^*(Y)$ are local diffeomorphisms, the map

$$
\Pi_X \circ \Pi_Y^{-1} : T^*(Y) \to T^*(X)
$$

$$(s, t, \sigma, \tau) \mapsto (x, \xi)$$

will be a local diffeomorphism. 

**Remark 4.2.** 1) Note that, by Proposition 4.1.4 (Hörmander [12]), if we show one of the maps $\Pi_Y$ or $\Pi_X$ is a local diffeomorphism, then the other map is also a local diffeomorphism as $\dim(Y) = \dim(X)$. We, however, in above lemma have shown that both maps are local diffeomorphisms, as the proof reveals whether each map will be a global diffeomorphism or not. In fact, for a fixed $(x, \xi) \in T^*(X)$ there might be more than one curve which resolves the same singularity.
ii) The map $\Pi_Y : \mathcal{C} \to T^*(Y)$ being a local diffeomorphism implies that one can always track the position singularities $(x, \sigma \partial_x \phi) \in \text{WF}(f)$ by having the measurement singularities $(\phi(t,x), t, \sigma, -\sigma \partial_t \phi(x, \xi)) \in \text{WF}(A)$.

iii) From the geometrical point of view, the map $\Pi_N : \mathcal{C} \to T^*(X)$ being a local diffeomorphism means that for any fixed position $x$ and covector $\xi$, there exists a curve (NOT necessarily unique) passing through $x$ perpendicular to $\xi$. This means singularities in data, i.e. $(x, \sigma \partial_x \phi) \in \text{WF}_A(f)$, can affect the measurement singularities, i.e. $(\phi(t,x), t, \sigma, -\sigma \partial_t \phi(x, \xi)) \in \text{WF}_A(A)$.

iv) Proposition 4.1 and Lemma 4.3 show the local surjectivity of the map $[T_1, T_2] \ni t \to \frac{\partial_x \phi(t,x)}{|\partial_x \phi(t,x)|} \in S^1$, for a fixed $x$.

Note that if the visibility condition holds, then we have the global surjectivity on $S^1$.

5. Global Bolker Condition

In this section, we study the normal operator $N = A^* A$ to prove a stability estimate. It is known that the normal operator is a $\Psi$DO if the projection $\Pi_Y : \mathcal{C} \to T^*(Y)$ is an injective immersion (see Proposition 8.2, [13]). For our analysis, we assume that the Semi Global Bolker Condition holds which is similar to the No Conjugate Points assumption for the geodesics ray transform studied in ([13], [17]).

Remark 5.1. There might be some points $(s, t)$ which do not satisfy the visibility, semi-global and local Bolker conditions. A point for which all above three conditions are satisfied is called a Regular point.

Our goal is to show that normal operator $N$ is a $\Psi$DO of order -1. From now on, we fix a covector $(x_0, \xi^0) \in T^*X \setminus 0$ and work in a small conic neighborhood of this covector.

Theorem 5.1. The normal operator $N$ is a classical $\Psi$DO of order $-1$ with principal symbol

$$p(x, \xi) = (2\pi)^{-1} W(x, x, \xi) + W(x, x, -\xi)$$

near $(x_0, \xi^0)$. Here the functions $W$ and $\tilde{h}$ are defined as

$$W(x, x, \xi) = |\mu(x, \xi)|^2 J^2(x, \xi), \quad \text{and} \quad \tilde{h}(x, \xi) = \frac{|\xi|}{|\partial_x \phi(t,x)|} h(x, \xi),$$

where $t = t(x, \xi)$ is well-defined locally by Lemma 4.3.

Proof. For the proof we mainly follow (Lemma 2, [13]) Considering the Schwartz kernel of $N$, we split the integration over $R$ into $\{\sigma > 0\}$ and $\{\sigma < 0\}$. So we get

$$K_N = \int_R \int_0^{+\infty} e^{i(\phi(t,x) - \phi(t,y))\sigma} W(t, x, y) \mathrm{d}\sigma \mathrm{d}t$$

$$+ \int_R \int_0^{+\infty} e^{-i(\phi(t,x) - \phi(t,y))\sigma} W(t, x, y) \mathrm{d}\sigma \mathrm{d}t = K_{N^+} + K_{N^-},$$

where $K_{N^+}$ and $K_{N^-}$ are the Schwartz kernels of the operators $N^+$ and $N^-$ with $N = N^+ + N^-$. We first consider $K_{N^+}$. Note that $K_{N^+}$ is localized as the function $\phi$ priori satisfies the local Bolker condition. By semi-global Bolker condition ([13]), we have

$$\left\{ \begin{array}{l} \phi(t, x) = \phi(t, y) = s \\
\partial_t \phi(t, x) = \partial_t \phi(t, y) \end{array} \right. \implies x = y.$$ 

Now a stationary phase method implies that $K_{N^+}$ is smooth away from the diagonal $\{x = y\}$. Since $\partial_x \phi(t, x) \neq 0$, for a fixed $x$ there exists a neighborhood $U$ on which we have normal vectors. We work on normal coordinates $(x', y')$ as coordinates on $U \times U$, with $x' = y'$. In these local coordinates, we may expand the phase function near the diagonal $\{x = y\}$. Let

$$\phi(t, x) - \phi(t, y) = (x - y) \cdot \xi(t, \sigma, x, y),$$

(5.1)
where $\xi(t, \sigma, x, y)$ is defined by the map

$$(t, \sigma) \mapsto \xi(t, \sigma, x, y) = \int_0^1 \sigma \partial_x \phi(t, x + \tau(y - x))d\tau.$$  

On the diagonal, we have $\xi(t, x, x) = \sigma \partial_x \phi(t, x) = \xi$ and the map is a smooth diffeomorphism as

$$\det\left( \frac{\partial \xi}{\partial(t, \sigma)} \right) \big|_{x=y} = \det \left( \frac{\partial \phi}{\partial x}, \sigma \frac{\partial^2 \phi}{\partial \tau \partial x} \right) = \sigma h(t, x) \neq 0.$$  

Notice that $\sigma = \frac{\mid l \mid}{\mid \partial_x \phi(t, x) \mid}$ and $t = t(x, \xi)$ is locally well-defined by Lemma 4.3. Therefore,

$$\tilde{h}(x, \xi) = \frac{\mid l \mid}{\mid \partial_x \phi \mid} h(x, \xi) \neq 0.$$  

Using the above change of variable $\xi$ on the diagonal yields

$$K_{\mathcal{N}^+}(s, t, x, y) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} W(x, y, \xi) |\tilde{h}(x, \xi)|^{-1}d\xi,$$

where the function $W$ is defined above. By restricting the amplitude to diagonal $\{x = y\}$, one can find the principal symbol of $K_{\mathcal{N}^+}$. Now the principal symbol of $K_{\mathcal{N}}$ is given by the sum of those for $K_{\mathcal{N}^+}$ and $K_{\mathcal{N}^-}$. Since the weight $\mu$ is nowhere vanishing, the normal operator $\mathcal{N}$ is a classical PDO with principal symbol $P(x, \xi)$ provided the function $\phi$ satisfies the local and semi-global Bolker condition. Now the operator $\mathcal{N}$ is an elliptic PDO if the visibility condition is satisfied. \hfill $\square$

## 6. Analysis of Global Problem and Stability

In previous sections, we studied the operators $A$ and $\mathcal{N}$. We showed that under the visibility, local and semi-global Bolker conditions, the normal operator $\mathcal{N}$ is a PDO of order $-1$ in a small conic neighborhood of a fixed covector $(x, \xi) \in T^* X \setminus 0$.

To reconstruct $f \in L^2(X)$ from its measurements $Af$ using the normal operator $\mathcal{N}$, we need to expand our results globally. For the analysis, we require the visibility, local and semi-global Bolker conditions as well as using cut-off functions. By visibility, for each covector $(x_0, \xi^0)$, there exists some $(s_0, t_0)$ such that $\phi(t_0, x_0) = s_0$ and $\partial_x \phi(t_0, x_0) \parallel \xi^0$; which means for each point $x_0$ and co-direction $\xi^0$, there exists a curve passing through $x_0$ where $\xi^0$ is normal to it. Note that the local and semi-global Bolker conditions should be satisfied for this $(s_0, t_0)$. We also remind that $(s_0, t_0)$ determines the line $l_0$ passing through the point $x_0$ with conormal $\xi^0$.

For a fixed $(x_0, \xi^0) \in T^* X \setminus 0$, there exists a pair of conic neighborhood $(V, \tilde{V})$ with property $(x_0, \xi^0) \in V$ and $\tilde{V} \subset V$ such that the visibility, local and semi-global Bolker conditions are satisfied for $\tilde{V}$. Let $\{V_\alpha\}$ be an open covering for $T^* Y \setminus 0$. By Proposition 4.1, since the map $\Pi_X \circ \Pi_Y^{-1} : T^* Y \longrightarrow T^* X$ is a local diffeomorph, $\{V_\alpha\}$ corresponds to an open covering $\{U_\alpha\} \ni (s_0, t_0, \sigma_0, \tau_0)$ for $T^* Y \setminus 0$, where $\{V_\alpha\}$ corresponds to $\{U_\alpha\}$ and $\{U_\alpha\} \subset \{\tilde{U}_\alpha\}$. Note that $\{U_\alpha\}$ is a conic neighborhood with respect to $(\sigma_0, \tau_0)$. Now a compactness argument implies that one may find a finite covering for $\{U_\alpha\}$ and therefore, a finite covering for $\{V_\alpha\}$. Consider the projection

$$(s, t, \sigma, \tau) \longmapsto (s, t).$$

We choose a family of smooth cut-off functions $\{\chi_i\}$ which are positive on $U_i$ and with property $\text{supp} \chi_i \subset \tilde{U}_i$. Set $\mathcal{N}$ to be a finite sum of operators of form $N_i = A^* \chi_i \mathcal{A}$, that is,

$$\mathcal{N} = \sum_i N_i.$$  

We, in fact, restrict the operator $A$ to small perturbations of $(x_0, \xi^0)$ where the visibility, local and semi-global Bolker conditions are satisfied. The restricted operator $A$ allows us to resolve the singularities. Note that, here the operator $\mathcal{N}$ is not the normal operator $A^* \mathcal{A}$ which we studied in the previous section. In fact,
it is a refined version of that as we excluded all non-regular points (see Remark 5.1). By Theorem 5.1 each term \( N_i \) is a \( \Psi DO \) of order \(-1\) supported in a conic neighborhood of \((x, \left[ \frac{\xi}{\xi_i} \right])\) with the principal symbol

\[ p_i(x, \xi) = \tilde{\chi}_i(x)p(x, \xi). \]

Here the cut-off function \( \tilde{\chi}_i(x) \) (correspondence to \( \chi_i(s, t) \)) via the local diffeomorphism \( \Pi_X \circ \Pi_X^{-1} \) is the projection of a smooth function defined on \( T^*X \setminus 0 \), where it is non-zero on \( \tilde{V} \) but vanishes on \( \check{V} \). Note also that the weight function \( \mu \) is nowhere vanishing and local Bolker condition \( \frac{\xi}{\xi_i} \) implies that \( h \neq 0 \).

Now for any \((x, \xi)\), there exists \( i \) such that \( \chi_i \neq 0 \) and all other terms are non-negative. Hence \( \sum N_i \) is elliptic, and therefore the operator \( N \) is a classical \( \Psi DO \) of order \(-1\) with principal symbol \( P(x, \xi) = \sum p_i(x, \xi) \).

**Remark 6.1.** It should be noted that in our analysis, the cut-off functions \( \chi_i \) are used for the \( C^\infty \) results. For the case of analytic arguments, one cannot use cut-off functions.

In the following proposition, we show that for any neighborhood of a fixed covector \((x_0, \xi^0) \in T^*X \setminus 0\), ellipticity holds along normals in a conic neighborhood of this covector.

**Proposition 6.1.** Assume that the dynamic operator \( A \) satisfies the visibility, the local and semi-global Bolker conditions. Let \( \mu \) be a non-vanishing smooth weight and \( f \in L^2(X) \) with supp \( f \subset X \). If \( Af = 0 \) in a neighborhood of some line, \( l_0 \), determined by \((s_0, t_0)\), then

\[ \text{WF}_A(f) \cap N^*(l_0) = \emptyset. \]

**Proof.** Let \((x_0, \xi^0) \in T^*X \setminus 0\) be fixed. By visibility condition, there exists \((s_0, t_0)\) such that \( \phi(t_0, x_0) = s_0 \) and \( \partial_x\phi(t_0, x_0) \parallel \xi^0 \). Now the proof follows directly from [Proposition 1, [13]] and applying it to all normals of the fixed curve \( l_0 \), determined by \((s_0, t_0)\). \( \square \)

**Corollary 6.1.** Let \( A \) satisfies the visibility, the local and semi-global Bolker conditions. If \( Af = 0 \) in an open set of lines determined by \((s, t)\) such that one line is in the exterior of supp \( f \), then \( A \) is injective.

**Proof.** Let \( \tilde{X} \) be an open set where the function \( f \) is extended to be zero on \( \tilde{X} \setminus X \) (\( X \) is embedded in the set \( \tilde{X} \)). By Proposition 6.1, \( f \) analytic in the interior of \( \tilde{X} \). Since \( f \) is identically zero on \( \tilde{X} \setminus X \), \( f \) must be identically zero on all of \( X \). Hence \( A \) is injective. \( \square \)

The following proposition is a standard stability estimate which follows from elliptic regularity see (Theorem 2, [13]) and (Proposition V.3.1, [20]).

**Proposition 6.2.** Let \( \mu \in C^\infty(\mathbb{R} \times X) \) be a non-vanishing smooth weight and let \( \phi \in C^\infty(\mathbb{R} \times X) \) be a function satisfying the visibility, the local and semi-global Bolker conditions. Then for all \( f \in L^2(X) \) and \( s > 0 \) there exists \( C > 0 \) and \( C_s > 0 \) depending on \( s \) such that

\[ \|f\|_{L^2(X)} \leq C \|Nf\|_{H^1(\tilde{X})} + C_s \|f\|_{H^{-s}}, \quad \forall s. \]

Moreover, if \( N : L^2(X) \rightarrow H^1(\tilde{X}) \) is injective, then there exists a stability estimate,

\[ \|f\|_{L^2(X)} \leq C' \|Nf\|_{H^1(\tilde{X})} \]

where \( C' > 0 \) is a constant.

**Proof.** The proof directly follows from Theorem 5.1 and above arguments. \( \square \)

**Remark 6.2.** Note that the way the parametrix is constructed in above proposition, one has control on how the constant \( C \) to be chosen. This, however, is not the case for \( C' \) in the second inequality.

In what follows, we perturb \( \phi \) and \( \mu \), and prove that the perturbation yields a small constant times an \( L^2 \)-norm of the function \( f \) which can be absorbed by the left-hand side of above estimate. The following lemma is in the spirit of [Lemma 4, [13]].
Lemma 6.1. Let \( \phi, \tilde{\phi} \) be two smooth real-valued functions and \( \mu, \tilde{\mu} \) be two non-vanishing smooth weights. Let \( N = \sum_i N_i = \sum_i A^* \chi_i A \) and \( \tilde{N} = \sum_i \tilde{N}_i = \sum_i \tilde{A}^* \tilde{\chi}_i \tilde{A} \) be two normal operators corresponding \( \mu \) and \( \tilde{\mu} \), respectively. There exist a \( K \gg 2 \) such that if

\[
\| \phi - \tilde{\phi} \|_{C^K(R \times \tilde{X})}, \quad \| \mu - \tilde{\mu} \|_{C^K(R \times \tilde{X})} < \delta \ll 1,
\]

then there exists \( C \geq 0 \) depending on the \( C^K(R \times \tilde{X}) \) norm of \( \phi \) and \( \mu \) such that

\[
\| (N - \tilde{N}) f \|_{H^1(\tilde{X})} \leq C \delta \| f \|_{L^2(\tilde{X})},
\]

Proof. By Theorem 5.1, we know that for each \( i \), the operators \( N_i = A^* \chi_i A \) and \( \tilde{N}_i = \tilde{A}^* \tilde{\chi}_i \tilde{A} \) are both elliptic \( \Psi \)DOs with symbols depending on \( \phi, \mu \) and \( \tilde{\phi}, \tilde{\mu} \), respectively.

Now let \( (x_0, \xi^0) \) be a fixed covector. By Lemma 4.3 and Remark 4.2, for any line \( l \) close to \( l_0 \) determined by \( (s_0, t_0) \), a perturbation of \( \phi \) results in the perturbation of the whole family of level curves near \( \phi(t_0, x) = s_0 \). Therefore, the cut-off functions can be chosen independently of perturbation of \( \phi \), since in prior, we assumed that \( \phi \) and \( \tilde{\phi} \) are \( \delta \)-close with \( C^K \)-topology. Hence, one may pick \( \{ \chi_i \} \) and \( \{ \tilde{\chi}_i \} \) in a way that they are the same in each neighborhood where the visibility, the local and semi-global Bolker conditions are satisfied. We now directly apply the argument on \( \| N_i - \tilde{N}_i \|_{L^2(\tilde{X})} - \| \tilde{N}_i \|_{L^2(\tilde{X})} \), to conclude that for each \( i \)

\[
\| N_i - \tilde{N}_i \|_{L^2(\tilde{X})} \to H^1(\tilde{X}) = O(\delta),
\]

and therefore,

\[
\| (N_i - \tilde{N}_i) f \|_{H^1(\tilde{X})} \leq C \delta \| f \|_{L^2(\tilde{X})}.
\]

Now the fact that the operator \( N \) is a finite sum of operators of the form \( N_i \), as well as using the triangle inequality

\[
\| (N - \tilde{N}) f \|_{H^1(\tilde{X})} \leq \sum_i \| (N_i - \tilde{N}_i) f \|_{H^1(\tilde{X})},
\]

concludes the results. □

Next result is a stability estimate for a generic class of dynamic inverse operators satisfying the visibility, the local and semi-global Bolker conditions.

Theorem 6.1. Let \( X \) be an open set of points (positions) \( x \) lying on lines in \( Y \), where \( Y \) is the open sets of lines determined by \( (s, t) \) in \( \mathbb{R}^2 \). Let \( \mathcal{A} : L^2(X) \to H^1(\tilde{X}) \), satisfying the visibility, the local and semi-global Bolker conditions, be an injective dynamic inverse operator defined by \( \phi \) with a non-vanishing smooth weight \( \mu \). Then

i) For any \( \tilde{\phi} \in \text{neigh}(\phi) \) and \( \tilde{\mu} \in \text{neigh}(\mu) \) with \( C^K \)-topology ( \( K \) an arbitrary large natural number) and for all \( f \in L^2(X) \), there exists \( C \geq 0 \) such that

\[
\| f \|_{L^2(X)} \leq C \| \tilde{N} f \|_{H^1(\tilde{X})}.
\]

In particular, the operator \( \tilde{A} \) is injective.

ii) The following stability estimate remains true for any perturbation of \( \phi \) and \( \mu \):

\[
\| f \|_{L^2(X)} / C \leq \| \tilde{N} f \|_{H^1(\tilde{X})} \leq C \| f \|_{L^2(X)}.
\]

Proof. i) \( \mathcal{A} \) is injective, thus by Proposition 6.2, we have the following stability estimate:

\[
\| f \|_{L^2(\tilde{X})} \leq C_1 \| N f \|_{H^1(\tilde{X})} = C_1 \| \tilde{N} f + (N - \tilde{N}) f \|_{H^1(\tilde{X})} \leq C_1 \| \tilde{N} f \|_{H^1(\tilde{X})} + C_1 \| (N - \tilde{N}) f \|_{H^1(\tilde{X})}.
\]

By Lemma 6.1, there exists a constant \( C_2 \geq 2 \) such that

\[
\| (N - \tilde{N}) f \|_{H^1(\tilde{X})} \leq C_2 \delta \| f \|_{L^2(\tilde{X})},
\]

and therefore,

\[
\| f \|_{L^2(\tilde{X})} \leq C_1 \| \tilde{N} f \|_{H^1(\tilde{X})} + C_1 C_2 \delta \| f \|_{L^2(\tilde{X})}.
\]
Letting \( \delta < \min\{ (2C_1 C_2)^{-1}, 1/2 \} \) yields
\[
\| f \|_{L^2(X)} \leq C \| \tilde{N} f \|_{H^1(X)} .
\]
Assume now that \( \tilde{A} f = 0 \). Then
\[
\tilde{N} f = \sum_i \tilde{A}^* X_i \tilde{A} f = 0, \quad \text{as } \tilde{A} f = 0.
\]
Above inequality in part (i) implies that \( f = 0 \). Hence, the operator \( \tilde{A} \) is injective.

ii) This part follows directly from the first part and the continuity of pseudodifferential operator \( \tilde{N} \).

\( \square \)

**Proof of Theorem 2.1.** The proof directly follows from Theorem 6.1.

\( \square \)

7. Analysis of the initial dynamic problem

In this section, we state the implications of our analysis for the partial case, where the dynamic inverse operator is given by \( \tilde{A} f(x,t) = \int \int_{\mathbb{R}^2} J(t,x) \mu(t, \psi^{-1}_t(x)) f(x) \delta(s-\psi^{-1}_t(x) \cdot \omega(t)) \, dx, \)

where \( \psi^{-1}_t(x) \cdot \omega(t) \) is the level curve corresponding to \( A \).

**Canonical relation.** Setting \( \Phi(s,t,x) = s-\psi^{-1}_t(x) \cdot \omega(t) \) in Proposition 4.1, the canonical relation \( C \) associated with \( A \) will be
\[
C = \{ (\psi^{-1}_t(x) \cdot \omega(t), t, \sigma, -\sigma(\partial_t \psi^{-1}_t(x) \cdot \omega(t) + \psi^{-1}_t(x) \cdot \omega(t)) ; x, \sigma \partial_x \psi^{-1}_t(x) \cdot \omega(t)) \} |(s,t,x) \in M \}.
\]
The microlocal version of double fibration is given by:
\[
\begin{array}{c}
\Pi_\gamma & \longrightarrow & C \\
& \leftarrow & \Pi_\gamma \\
T^*(Y) & \longrightarrow & T^*(X)
\end{array}
\]

where
\[
\Pi_X (\psi^{-1}_t(x) \cdot \omega(t), t, \sigma, -\sigma \partial_t (\psi^{-1}_t(x) \cdot \omega(t))) ; x, \sigma \partial_x \psi^{-1}_t(x) \cdot \omega(t)) = (x, \sigma \partial_x \psi^{-1}_t(x) \cdot \omega(t)),
\]
\[
\Pi_Y (\psi^{-1}_t(x) \cdot \omega(t), t, \sigma, -\sigma \partial_t (\psi^{-1}_t(x) \cdot \omega(t))) ; x, \sigma \partial_x \psi^{-1}_t(x) \cdot \omega(t)) = (\psi^{-1}_t(x) \cdot \omega(t), t, \sigma, -\sigma \partial_t (\psi^{-1}_t(x) \cdot \omega(t))).
\]

**Visibility.** The operator \( A \) satisfies in the visibility condition if for any \((x, \xi) \in T^* X \setminus 0 \), the map given by \( \xi \) is locally surjective.

**Local Bolker Condition.** As it is shown in Proposition 4.1, the projection \( \Pi_Y \) is an immersion if the matrix \( d_{t,x,\sigma} \Pi_Y \) has rank equal to four or equivalently \( \det(d_{t,x,\sigma} \Pi_Y) \neq 0 \). Since
\[
\det(d_{t,x,\sigma} \Pi_Y) = \det \left( \frac{\partial \psi^{-1}_t(x) \cdot \omega(t)}{\partial x^j}, \frac{\partial^2 \psi^{-1}_t(x) \cdot \omega(t)}{\partial x^j \partial x^k} \right) = h(t,x),
\]
the projection \( \Pi_Y \) being an immersion is equivalent to the condition \( h(t,x) \) being non-zero, i.e. \( h(t,x) \neq 0 \).

**Semi-global Bolker condition (No conjugate points condition).** By condition \( (2.3) \), \( \Pi_Y \) is injective if the map
\[
x \rightarrow \left( \psi^{-1}_t(x) \cdot \omega(t), \partial_t (\psi^{-1}_t(x) \cdot \omega(t)) \right)
\]
is one-to-one.

The normal operator \( N \) is a \( \Psi DO \) of order \(-1\). Under the local and the semi-global Bolker conditions,
Theorem 5.1 implies that the normal operator $N$ associated with the dynamic operator $A$ is a $ΨDO$ of order $-1$ with principal symbol $p(x, ξ)$ computed microlocally near each $(x_0, ξ^0)$. The principal symbol is given by

$$p(x, ξ) = (2π)^{-1} |∂_x ψ^{-1}_T(x)·ω(t)| \frac{|μ(x, ξ)|^2 J^2(x, ξ) + |μ(x, -ξ)|^2 J^2(x, -ξ)}{|ξ| h(x, ξ)},$$

where $t = t(x, ξ)$ is locally well-defined by Lemma 4.3.

**Remark 7.1.** Note that we do not require the function $φ(t, x)$ to be smoothly periodic.

**Fan Beam Geometry.** In previous sections, we have stated our results for the dynamic inverse operator $A$ with parallel beam geometry which also can be written in the following representation:

$$A_P f(s, β) = \int_R f(sω(β) + pω^⊥(β)) \, dp,$$

where $ω(β) = (cos β, sin β)$. Another common geometry which is much more applicable in numerical simulations is Fan beam geometry. Let lines along which the dynamic inverse operator of $f$ is known, are specified by $γ$ (the angle between the incident ray direction and the line from the source to the rotation center) and $t$ (the angular position of the source). Then the fan beam data at time $t$ is given by

$$A_F f(t, γ) = \int_0^∞ f(S(t) + pθ(γ)) \, dp, \quad θ(γ) ∈ S^1,$$

where $S(t)$ is the source at time $t$ which moves along the trajectory with radius $R$. Here $t$ is both a parameter along the source trajectory and the time variable. Note also that using the Parallel-Fan beam geometry relation (see figure 1)

**Figure 1.** Parallel-Fan beam geometry relation.

$$s = R \sin γ \quad \beta = t + γ - \frac{π}{2},$$
one may derive the fan beam dynamic inverse operator $A_F$, given by

$$A_F f(t, \gamma) = A_P f(R \sin \gamma, t + \gamma - \frac{\pi}{2}).$$

Since the Jacobian

$$\left| \frac{\partial(s, \beta)}{\partial(t, \gamma)} \right| = R \cos \gamma$$

is non-zero, the transformation between these two geometries is smooth.

To implement our results in fan beam geometry, we need to find appropriate level curves $\phi$. Let $S(t)$ be the source and $x$ be the point on the incident ray, see figure 1. We first set

$$\vec{\alpha} = x - S(t) = (x^1 - R \cos t, x^2 - R \sin t),$$

and then compute the perpendicular vector $\vec{\alpha}^\perp$ as follow:

$$\vec{\alpha}^\perp = \frac{\text{sgn}(x^1 - R \cos t)}{|\vec{\alpha}|}(R \sin t - x^2, x^1 - R \cos t) = (\cos \alpha^\perp, \sin \alpha^\perp).$$

We only work with one direction from two possible orientations for $\vec{\alpha}^\perp$; say the one with $\text{sgn}(x^1 - R \cos t) > 0$. For a fixed point $x$ on the incident ray and a specific time $t$, the polar angle $\alpha^\perp$ is determined by

$$\alpha^\perp = \text{arg}(\vec{\alpha}^\perp) = \tan^{-1}\left(\frac{x^1 - R \cos t}{R \sin t - x^2}\right).$$

We set

$$\phi(t, x) = \text{arg}(\vec{\alpha}^\perp).$$

Now our results are valid if the visibility, the local and semi-global Bolker conditions are satisfied for this choice of function $\phi$. Note that, here the function $\text{arg}$ is not globally defined but this does not affect the analysis, as our results are local and we have chosen the branch where $\text{sgn}(x^1 - R \cos t) > 0$. One may choose another branch of $\tan^{-1}$, however, this plays no role in differentiation which is involved in all above main three conditions.

**Remark 7.2.** What we discussed above is the Fan beam geometry for the case where there is no motion or the motion is small enough, i.e. $\psi_t = \text{Id}$. For the general case, one may replace $x = (x^1, x^2)$ by $\psi_t^{-1}(x) = (\psi_t^{-1}(x^1), \psi_t^{-1}(x^2))$ to get the main result.

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