On vertex Ramsey graphs with forbidden subgraphs

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Abstract

A classical vertex Ramsey result due to Nešetřil and Rödl states that given a finite family of graphs $F$, a graph $A$ and a positive integer $r$, if every graph $B \in F$ has a 2-vertex-connected subgraph which is not a subgraph of $A$, then there exists an $F$-free graph which is vertex $r$-Ramsey with respect to $A$. We prove that this sufficient condition for the existence of an $F$-free graph which is vertex $r$-Ramsey with respect to $A$ is also necessary for large enough number of colours $r$.

We further show a generalisation of the result to a family of graphs and the typical existence of such a subgraph in a dense binomial random graph.

1 Introduction

Let $A$ be a graph and let $r$ be a positive integer. We say that a graph $G$ is (vertex) $r$-Ramsey with respect to $A$ if in every colouring of the vertices of $G$ in $r$ colours there exists a monochromatic copy of $A$. The existence of $r$-Ramsey graphs is straightforward: the complete graph $K_n$ is $r$-Ramsey with respect to $A$ for every $n \geq r(|V(A)| - 1) + 1$. It is thus natural to ask about the existence of sparse Ramsey graphs. One of the ways to define sparseness is to avoid copies of a given graph $B$ (or more generally of any graph from a given finite graph family $F$) in $G$. Let us call a graph $G$ $F$-free if it does not contain a subgraph isomorphic to $B$ for every $B \in F$.

Perhaps the most studied case is when both $A$ and $B$ are complete graphs on $s$ and $t$ vertices, respectively, where $t > s \geq 2$. Denote by $f_{s,t}(n)$ the minimum over all $K_t$-free graphs $G$ on $[n] := \{1, \ldots, n\}$ of the maximum number of vertices in an induced $K_s$-free subgraph of $G$. Erdős and Rogers [5] proved that, for a certain $\varepsilon = \varepsilon(s) > 0$, $f_{s,s+1}(n) \leq n^{1-\varepsilon}$ (note that this implies that for every $s \geq 2$ and $r \geq 2$, there exists a $K_{s+1}$-free graph $G$ which is $r$-Ramsey with respect to $K_s$). The result of Erdős and Rogers was subsequently refined by Bollobás and Hind [1] and Krivelevich [6]. Let us also mention that subsequent works by Dudek, Retter and Rödl [3] and by Dudek and Rödl [4] determined $f_{s,s+1}(n)$ up to a power of $\log n$ factor.

*Research supported in part by USA-Israel BSF grant 2018267.
†Research supported in part by Israel ISF grant 2110/22.
strengthened the known bounds for $f_{s,s+2}(n)$, and further improved the bounds for $f_{s,s+k}(n)$ when $s,k$ are large enough.

Considering general graphs $A$ and $B$ (and in fact, a family of graphs $B$), Nešetřil and Rödl [7] proved the following (see also [2]):

**Theorem 1.1** ([7]). Let $F$ be a finite family of graphs and let $A$ be a graph. Let $r \geq 2$ be an integer. If every graph from $F$ has a 2-vertex-connected subgraph which is not a subgraph of $A$, then there exists an $F$-free graph which is vertex $r$-Ramsey with respect to $A$.

See [9, 10, 11] for additional results on vertex-Ramsey graphs with forbidden subgraphs.

Our main result shows that the above sufficient condition is also necessary for large enough number of colours $r$. We say that $B$ is an $A$-forest of size $\ell$ if $B = \bigcup_{i=1}^{\ell} B_i$, where for every $1 \leq i \leq \ell$, $B_i$ is isomorphic to a subgraph of $A$, and for every $i \geq 2$, $|V(B_i) \cap V \left( \bigcup_{j=1}^{i-1} B_j \right) | \leq 1$.

**Theorem 1.** Let $\ell > 0$ be an integer. Let $B$ be an $A$-forest of size $\ell$. Let $r > 0$ be an integer such that $r \geq \ell \left( 2(|V(A)| - 1)(|V(B)| - 2) + 1 \right)$, and let $G$ be an $r$-Ramsey graph with respect to $A$. Then $G$ contains a copy of $B$.

Let us first note that since $\ell \leq |V(B)|$, it suffices to take $r = O(|V(A)||V(B)|^2)$. Furthermore, observe that the above implies the necessity of the condition in Theorem 1.1, for $r$ large enough. Indeed, let us say that a graph $B$ is $A$-degenerate, if every 2-vertex-connected subgraph of it is a subgraph of $A$. Note that any $A$-degenerate graph can be constructed recursively: (1) any subgraph of $A$ is $A$-degenerate; (2) if $B$ is an $A$-degenerate graph, then a union of $B$ with a subgraph of $A$ that shares with $B$ at most 1 vertex is $A$-degenerate as well. Theorems 1.1 and 1 can be formulated in terms of $A$-degenerate graphs: there exists an $F$-free graph which is $r$-Ramsey with respect to $A$ for all large enough $r$ if and only if every graph from $F$ is not $A$-degenerate.

Note that the case that $B$ consists of $\ell$ vertex-disjoint components, each isomorphic to a subgraph of $A$, is easy since if $G$ is $r$-Ramsey with respect to $A$ then it contains a large enough family of vertex-disjoint copies of $A$. On the other hand, if the components of $B$ are not disjoint, we can proceed by induction, deleting a component $B_i$ intersecting other components, finding a copy of $B - B_i$ using inductive hypothesis and then adjoining to it a correctly placed copy of $B_i$, see details in Section 2.

In the next section, we provide a short proof of Theorem 1.1 for the sake of completeness, followed by the proof of Theorem 1. In Section 3, we discuss generalisations of Theorem 1.1 to a family of graphs (instead of $A$), and the existence of an $F$-free graph which is $r$-Ramsey with respect to $A$ in a dense enough binomial random graph.

### 2. Proofs of Theorems 1.1 and 1

We say that a graph $G$ is $\varepsilon$-dense with respect to a graph $A$ if every induced subgraph of $G$ on $\lfloor \varepsilon |V(G)| \rfloor$ vertices contains a copy of $A$. Clearly, if $G$ is $1/r$-dense with respect to $A$, then it is also $r$-Ramsey with respect to $A$. Theorem 1.1 follows immediately from Theorem 2.1.
Theorem 2.1. Let $\mathcal{F}$ be a finite family of graphs. If there are no $A$-degenerate graphs in $\mathcal{F}$, then there exists a $\delta = \delta(A, \mathcal{F}) > 0$ such that for all large enough $n$, there exists an $\mathcal{F}$-free $n^{-\delta}$-dense graph on $[n]$ with respect to $A$.

Proof. Let $a := |V(A)|$. Let $\epsilon > 0$ be small enough and set $p = n^{1-a+\epsilon}$. Consider a hypergraph with vertex set $\binom{[n]}{2}$ whose edge set consists of all possible copies of $A$ on $[n]$. Let $H_A(n, p)$ be its binomial subhypergraph where each copy of $A$ is chosen independently and with probability $p$, and let $G_A(n, p)$ be the random graph constructed as follows: an edge belongs to $G_A(n, p)$ if and only if this edge belongs to a copy of $A$ in $H_A(n, p)$. We shall prove that it suffices to remove $O(\sqrt{n})$ vertices of $G_A(n, p)$ to get the desired graph whp.

Let $\delta_0 = \frac{\epsilon}{2(\alpha-1)}$. Let us show that whp $G_A(n, p)$ is $n^{-\delta_0}$-dense with respect to $A$. Set $N = \lceil n^{1-\delta_0} \rceil$. Then the expected number of $N$-sets containing no copy of $A$ in $G_A(n, p)$ is at most the expected number of $N$-subsets $U \subseteq [n]$ such that $\binom{U}{2}$ does not contain any copy of $A$ in $H_A(n, p)$ that equals to

$$\binom{n}{N}(1-p)^{\binom{N}{2} \frac{n}{\text{aut}(A)}} \leq \exp \left[ N \left( \delta_0 \ln n + 1 - p \frac{N^{a-1}}{\text{aut}(A)} \right) (1 + o(1)) \right]$$

$$\leq \exp \left[ N \left( \delta_0 \ln n - \frac{n^{1-a+\epsilon}(a-1)(1-\delta_0)}{\text{aut}(A)} \right) (1 + o(1)) \right]$$

$$\leq \exp \left[ -Nn^{\epsilon/2} \left( \frac{1}{\text{aut}(A)} - o(1) \right) \right] \to 0.$$

By the union bound, whp every $N$-set contains at least one copy of $A$ in $G_A(n, p)$, that is, whp $G_A(n, p)$ is $n^{-\delta_0}$-dense.

Let $\delta = \delta_0/2$ and let $C > 0$. Note that whp the deletion of any $C\sqrt{n}$ vertices from $G_A(n, p)$ leads to an $\tilde{n}^{-\delta}$-dense graph on $\tilde{n}$ vertices. Indeed, if $G_A(n, p)$ is $n^{-\delta_0}$-dense, then, since $\tilde{n}^{1-\delta} = (n - C\sqrt{n})^{1-0.5\delta_0} \geq n^{1-\delta_0}$, every set of $\tilde{n}^{1-\delta}$ vertices in the new graph has at least $n^{1-\delta}$ vertices and thus contains a copy of $A$. Therefore, it suffices to prove that whp we can remove $O(\sqrt{n})$ vertices from $G_A(n, p)$ and get an $\mathcal{F}$-free graph.

Given a graph $B$ and graphs $A_1, \ldots, A_m$ isomorphic to $A$, we say that $A_1 \cup \ldots \cup A_m$ is an inclusion-minimal cover of the edges of $B$ if $E(B) \subseteq E(A_1 \cup \ldots \cup A_m)$ but $E(B) \not\subseteq E(A_1 \cup \ldots \cup A_{i-1} \cup A_{i+1} \cup \ldots \cup A_m)$ for every $i \in [m]$. For every $B \in \mathcal{F}$, consider $B' \subset B$ such that every inclusion-minimal cover $A_1 \cup \ldots \cup A_m$ of the edges of $B'$ satisfies $|(A_i \cap \cup_{j \neq i} A_j) \cap B'| \geq 2$ for every $i \in [m]$. By Claim 2.2 (stated below), whp the number of copies of $B'$ in $G_A(n, p)$ is at most $\sqrt{n}$. We can now delete a single vertex from each such copy, and obtain a set of $\tilde{n} \geq n - |\mathcal{F}|\sqrt{n}$ vertices that induces an $\mathcal{F}$-free graph, as required. \qed

We note that a slight adjustment of the proof of Theorem 2.1 allows one to argue for the existence of $\mathcal{F}$-free $\epsilon$-dense graph for induced copies of $A$.

Claim 2.2. Whp the number of copies of $B'$ in $G_A(n, p)$ is at most $\sqrt{n}$.

Proof. Let $b := |V(B')|$ and let $k := |E(B')|$. Let $X_{B'}$ be the number of copies of $B'$ in $G_A(n, p)$. We shall bound $\mathbb{E}X_{B'}$ from above.

A copy of $B'$ may appear in $G_A(n, p)$ only through hyperedges $A_1, \ldots, A_{\ell} \in E(H_A(n, p))$ such that $B' \subset A_1 \cup \ldots \cup A_{\ell}$. For any possible inclusion-minimal cover of edges of $B'$ by
copies $A_1, \ldots, A_\ell$ of $A$, let us denote by $v_i$ the number of vertices in the intersection of $A_i$ and $B'$. Then, each $A_i$ in this cover contributes a factor of $O(n^{a-v_i} p)$ to $EX_{B'}$. More formally, if $B' = B_1 \cup \ldots \cup B_\ell$, where each $B_i$ is a subgraph of a copy of $A$, then, for every $i$,

$$\mathbb{P}(\exists A' \in H_A(n, p): A' \supset B_i) = O(n^{a-v_i} p).$$

Since there are $O(n^{b})$ choices of $B'$ in $K_n$, we get that

$$EX_{B'} = O \left( n^b \max_{A_1 \cup \ldots \cup A_\ell \supset B'} n^{a-b-\ldots-v_i} p^\ell \right) = O \left( n^{b+\max_{A_1 \cup \ldots \cup A_\ell} (\ell(1+\varepsilon)-v_1-\ldots-v_\ell)} \right),$$

where the maximum and minimum are taken over all inclusion-minimal covers $A_1 \cup \ldots \cup A_\ell$ of edges of $B'$ by copies of $A$.

Let $A_1 \cup \ldots \cup A_\ell$ be an inclusion-minimal cover of the edges of $B'$ by copies of $A$, and let $V_i$ be the set of vertices in the intersection of $A_i$ with $B'$ (as above, we let $v_i = |V_i|$). Since each $V_i$ has at least two common vertices with $\cup_{j \neq i} V_j$ and $\ell \geq 2$, we get

$$\sum_{i=1}^\ell v_i \geq |V_1 \cup \ldots \cup V_\ell| + \ell = b + \ell,$$

that is $\sum_{i=1}^\ell v_i \geq b + \ell$. Indeed, for every $i$, let $S_i = V_i \cap (\cup_{j \neq i} V_j)$, $s_i = |S_i| \geq 2$. Then $V_1 \cup \ldots \cup V_\ell = S_1 \cup \ldots \cup S_\ell \cup \Sigma$, where $\Sigma$ is the set of vertices that are covered once. Then $|\Sigma| = \sum_{i=1}^\ell (v_i - s_i)$, and $|S_1 \cup \ldots \cup S_\ell| \leq \frac{1}{2} \sum_{i=1}^\ell s_i$, since each vertex in this union is covered at least twice. We thus obtain,

$$|V_1 \cup \ldots \cup V_\ell| = |\Sigma| + |S_1 \cup \ldots \cup S_\ell| \leq \sum_{i=1}^\ell v_i - \frac{1}{2} \sum_{i=1}^\ell s_i \leq \sum_{i=1}^\ell v_i - \ell,$$

where the last inequality follows since each $s_i$ is at least 2.

We may assume that $\varepsilon < \frac{1}{2k}$. Due to (1) and (2), we get

$$EX_{B'} = O(n^{k\varepsilon}) = o(\sqrt{n}).$$

By Markov’s inequality, whp we have less than $\sqrt{n}$ copies of $B'$ in $G_A(n, p)$. □

We now turn to the proof of our main theorem.

**Proof of Theorem 1.** Let $a := |V(A)|$ and $b := |V(B)|$. If $a = 1$, that is, $A = K_1$, then note that $B$ is the empty graph on $b$ vertices and thus every graph on at least $b$ vertices contains a copy of $B$. If $a = b = 2$, then we have $B \subseteq A$ and thus every graph which is $r$-Ramsey with respect to $A$ contains a copy of $B$.

We assume that $a \geq 2, b \geq 3$. We enumerate the vertices of $A$: $V(A) = \{v_1, \ldots, v_a\}$. Given a graph $G$, and a copy $A'$ of $A$ in $G$, we define a mapping $\phi_{A'}: V(A) \rightarrow V(G)$ such that for every $v_i \in V(A)$, we set $\phi_{A'}(v_i)$ to be the vertex $v \in V(G)$ which is in the role of $v_i$ in the copy $A'$ of $A$. Given $v \in V(G)$, we denote by $A_i(v)$ the set of copies $A'$ of $A$ in $G$ for which
\( \phi_A(v_i) = v \). Furthermore, we denote by \( s_i(v) \) the maximal size of a subset of \( \mathcal{A}_i(v) \), in which every two copies of \( A \) in \( G \) intersect only at \( v \).

We prove by induction on \( \ell \), the minimum size of an \( A \)-forest of \( B \), where the base case \( \ell = 1 \) is trivial.

We now consider two cases separately. First, assume that in an \( A \)-forest of \( B \) of size \( \ell \) all components \( B_i \) are disjoint. Let \( M \) be the maximum size of a family of vertex disjoint copies of \( A \) in \( G \). Then, we can colour each copy in a maximum family of vertex disjoint copies of \( A \) in two separate colours, and colour all the other vertices in \((2M + 1)\)-th colour, without producing a monochromatic copy \( A \). As \( G \) is \( r \)-Ramsey with respect to \( A \), we conclude that \( r < 2M + 1 \). Since \( r \geq 2\ell \), we find \( \ell \) disjoint copies of \( A \) in \( G \), and therefore a copy of \( B \) in \( G \).

We now turn to the case where, without loss of generality, \( B_\ell \) intersects \( \cup_{i=1}^{\ell-1} B_i \) in an \( A \)-forest of \( B \). Let \( \tilde{B} := \cup_{i=1}^{\ell-1} B_i \), and let \( \{x\} := V(\tilde{B}) \cap V(B_\ell) \). We may further assume that for \( A \supseteq B_\ell \), \( x \) corresponds to \( v_k \) in \( A \), for some \( 1 \leq k \leq a \).

Let \( U = \{v \in V(G) : s_k(v) \leq b - 2\} \). We require the following claim.

**Claim 2.3.** \( G[U] \) can be coloured in \( 2(a - 1)(b - 2) + 1 \) colours, without a monochromatic copy of \( A \).

**Proof.** For every \( v \in U \), let \( S_k(v) \) be a maximal by inclusion subfamily of \( \mathcal{A}_k(v) \) composed of copies of \( A \) in \( G[U] \), where every two copies of \( A \) in the subfamily intersect only at \( v \), and let \( S_k(v) = \cup_{A' \in S_k(v)} V(A') \). By definition of \( U \), \( |S_k(v)| \leq b - 2 \) and \( |S_k(v)| \leq (a - 1)(b - 2) + 1 \).

Define an auxiliary directed graph \( \Gamma \) on the vertices of \( U \), where for every \( v \) and for every \( u \in S_k(v) \setminus \{v\} \), \( \Gamma \) contains a directed edge from \( v \) to \( u \). We thus have that \( \Delta^+(\Gamma) \leq (a - 1)(b - 2) \). Hence, the underlying undirected graph \( \Gamma \) is \( (2(a - 1)(b - 2)) \)-degenerate. Indeed, consider \( V' \subseteq V(\Gamma) \). We will show that in the induced subgraph \( \Gamma[V'] \) there exists a vertex of degree at most \( 2(a - 1)(b - 2) \). We have

\[
\sum_{v \in V'} d_\Gamma[V'](v) = 2|E(\Gamma[V'])| \leq 2 \sum_{v \in V'} d_\Gamma^+(v) \leq 2(a - 1)(b - 2)|V'|,
\]

and thus there must be at least one vertex \( v \in V' \) with \( d_\Gamma[V'](v) \leq 2(a - 1)(b - 2) \). Therefore, \( \Gamma \) is \( (2(a - 1)(b - 2) + 1) \)-colourable. We colour \( G[U] \) according to this colouring.

Suppose towards contradiction that there is a monochromatic copy \( A' \) of \( A \) in \( G[U] \), and let \( w = \phi_A(v_k) \). Since \( A' \) is monochromatic, it does not have common vertices with \( S_k(w) \) other than \( w \) — however this contradicts the maximality of \( S_k(w) \).

Recalling that \( G \) is \( r \)-Ramsey with respect to \( A \), and that \( G[U] \) can be coloured in \( 2(a - 1)(b - 2) + 1 \) colours without containing a monochromatic copy of \( A \), we have that \( G[V \setminus U] \) is \( (r - (2(a - 1)(b - 2) + 1)) \)-Ramsey with respect to \( A \). Observing that

\[
r - (2(a - 1)(b - 2) + 1) \geq (\ell - 1)(2(a - 1)(b - 2) + 1),
\]

we have by induction that \( G[V \setminus U] \) contains a copy of \( \tilde{B} \). Let \( v \) be the vertex in this copy of \( \tilde{B} \) that corresponds to \( x \). Since \( v \notin U \) we have that \( s_k(v) \geq b - 1 \), and hence there is a subset of size at least \( b - 1 \) in \( \mathcal{A}_k(v) \) such that every two copies \( A' \) of \( A \) in this subset intersect only at
v. Noting that \(|V(\tilde{B})| \leq b - 1\), we have that at least one copy \(A'\) of \(A\) in this subset completes \(\tilde{B}\) to \(B\), that is, \(\tilde{B} \cup A'\) contains a copy of \(B\) (see Figure 1 for an illustration).

Figure 1: The subgraph \(G[V \setminus U]\) and a copy of \(\tilde{B}\) in it. A copy \(A_i\) of \(A\) together with \(\tilde{B}\) contain a copy of \(B\). Note that some of the \(A_j\)'s may have vertices outside \(V \setminus U\).

### 3 Remarks and observations

Let us finish with two remarks.

**Remark 1.** Theorems 2.1 and 1 can be generalised to families of graphs instead of a single graph \(A\). Let \(A, F\) be two finite graph families, and \(\varepsilon > 0\). The proof of Theorem 1 is quite similar. For the proof of Theorem 2.1, let us say that a graph \(G\) is an \(F\)-free \(\varepsilon\)-dense with respect to \(A\) if it is \(B\)-free for every \(B \in F\), and every induced subgraph of \(G\) on exactly \(\lfloor \varepsilon |V(G)| \rfloor\) vertices contains a copy of every \(A \in A\). A graph \(B\) is \(A\)-degenerate, if every 2-vertex-connected subgraph of it is isomorphic to a subgraph of some \(A \in A\). If every \(B \in F\) is not \(A\)-degenerate, then there exists an \(F\)-free \(\varepsilon\)-dense graph with respect to \(A\) — indeed, let \(A\) be the disjoint union of the graphs from \(A\), and apply Theorem 2.1.

**Remark 2.** For a non-\(A\)-degenerate family \(F\) (consisting of graphs that are not \(A\)-degenerate) and sufficiently small \(\delta > 0\), we claim the likely existence of an \(F\)-free \(n^{-\delta}\)-dense subgraph in the binomial random graph \(G(n, n^{-2/a+\delta})\), where \(a\) is the number of vertices in \(A\). Indeed, consider the hypergraph with vertex set \(\binom{n}{a}\), and edge set being all the possible cliques of size \(a\), \(K_a\), on \([n]\). Let \(\mathcal{H}_a(n, p')\) be its binomial subgraph. Let us first show that there exists a coupling between \(\mathcal{H}_a(n, p')\), and the graph considered in the proof of sufficiency of Theorem 2.1, \(\mathcal{H}_A(n, p)\), such that \(p = \Theta(p')\) and \(\mathcal{H}_A(n, p) \subseteq \mathcal{H}_a(n, p')\). Indeed, let \(p = n^{1-a+\delta(\varepsilon)}\). Consider an \(a\)-set, and let \(p'\) be the probability that at least one copy of \(A\) appears on this \(a\)-set. Clearly, \(p = \Theta(p')\). Let \(Q\) be the conditional distribution of a binomial random hypergraph of copies of \(A\) on \([a]\), under the condition that at least one such copy exists. We can now draw \(\mathcal{H}_A(n, p)\) as
follows. We first choose every \( a \)-set with probability \( p' \), and then in every set that we chose we construct a random \( A \)-hypergraph with distribution \( Q \), independently for different \( a \)-sets. We thus have that \( \mathcal{H}_A(n, p) \subseteq \mathcal{H}_a(n, p') \), and we can continue the proof in the same manner as in Theorem 2.1. Now, we take \( q \) such that \( q^{(2)} = p' \). Therefore, by the above coupling between \( \mathcal{H}_a(n, p') \) and \( \mathcal{H}_A(n, p) \) and by Theorem 3.1 stated below, \textbf{whp} \( G(n, q) \supset G_{K_a}(n, p') \supset G_A(n, p) \).

**Theorem 3.1** (Riordan [8]). Let \( \varepsilon > 0 \) be small enough and \( q \leq n^{-\frac{\varepsilon}{2}} + \varepsilon \), \( p \sim q^{(2)} \). Then there exists a coupling between \( G(n, q) \) and \( \mathcal{H}_a(n, p) \) such that \textbf{whp} for every edge of \( \mathcal{H}_a(n, p) \) there exists a copy of \( K_a \) in \( G(n, q) \) with the same vertex set.

We note that Riordan in [8, Section 5] discusses a coupling between \( G(n, q) \) and \( \mathcal{H}_A(n, p) \), and provides sufficient conditions for its existence for some \( A \), however here we settle for higher values of \( q(n) \) with respect to \( p(n) \), thus making such coupling simpler.

**Acknowledgement.** The authors wish to thank Benny Sudakov for helpful remarks.

**References**

[1] B. Bollobás, H. R. Hind, *Graphs without large triangle-free subgraphs*, Discrete Mathematics, 87 (1991) 119–131.

[2] B. Bollobás, D. B. West, *A note on generalized chromatic number and generalized girth*, Discrete Mathematics, 213 (2000) 29–34.

[3] A. Dudek, T. Retter, V. Rödl, *On generalized Ramsey number of Erdős and Rogers*, Journal of Combinatorial Theory, Series B, 109 (2014) 213–227.

[4] A. Dudek, V. Rödl, *On \( K_s \)-free subgraphs in \( K_{s+k} \)-free graphs and vertex Folkman numbers*, Combinatorica, 31 (2011) 39–53.

[5] P. Erdős, C. A. Rogers, *The construction of certain graphs*, Canadian Journal of Mathematics, 14 (1962) 702–707.

[6] M. Krivelevich, *Bounding Ramsey numbers through large deviation inequalities*, Random Structures & Algorithms, 7 (1995) 145–155.

[7] J. Nešetřil, V. Rödl, *Partitions of vertices*, Commentationes Mathematicae Universitatis Carolinae, 17 (1976) 85–95.

[8] O. Riordan, *Random cliques in random graphs and sharp thresholds for \( F \)-factors*, Random Structures & Algorithms, 61 (2022) 619–637.

[9] V. Rödl, N. Sauer, *The Ramsey property for families of graphs which exclude a given graph*, Canadian Journal of Mathematics, 44 (1992) 1050–1060.

[10] V. Rödl, N. Sauer, X. Zhu, *Ramsey families which exclude a graph*, Combinatorica, 15 (1995) 589–596.

[11] N. Sauer, X. Zhu, *Ramsey families which exclude a graph*, Colloquia Mathematica Societatis János Bolyai 60, Sets, Graphs and Numbers, (1991) 631–636.