Large-$N$ supersymmetric $\beta$-functions

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We present calculations of the leading and $O(1/N)$ terms in a large-$N$ expansion of the $\beta$-functions for various supersymmetric theories: a Wess-Zumino model, supersymmetric QED and a non-abelian supersymmetric gauge theory. In all cases $N$ is the number of a class of the chiral superfields in the theory.
Coupling constant perturbation theory has constituted the main approach to quantum field theories since their introduction, but it has its limitations. Therefore any approach which reaches beyond it is worthy of attention: one such is large-$N$ expansions, where $N$ denotes the number of fields (or some subset thereof). Simple theories such as $O(N)$-symmetric $\phi^4$ have been much studied [1] as has QCD at both large $N_c$ [2] and $N_f$ [3] [4]. In this paper we calculate the $\beta$-functions for a Wess-Zumino model, and also for various supersymmetric gauge theories—supersymmetric QED in the limit of large $N_f$, i.e. for a large number of chiral superfields, together with an abelian gauged Wess-Zumino model and a more general non-abelian theory. In all cases the leading contribution is a simple one-loop calculation, and things become interesting at $O(1/N)$, when bubble sums are involved. It turns out that the supersymmetric D-algebra part of the calculation is quite straightforward; the resulting Feynman integral calculation appears formidable, but simplifies in miraculous fashion, as observed for similar (non-supersymmetric) calculations in Ref. [5].

1. The Bubble Sums

In this section we describe the Feynman integral calculations. We do all calculations with zero external momentum, using supersymmetric dimensional regularisation (with $d = 4 - 2\epsilon$) and minimal subtraction (DRED). By performing subtractions at the level of the Feynman integrals we completely separate the calculation of the (subtracted) Feynman integrals from the details of the theory under consideration. It is convenient to redefine the $d$-dimensional integration measure so that

$$\int \frac{d^d k}{k^2 (k - p)^2} = \pi^2 \frac{1}{\epsilon} (p^2)^{-\epsilon}. \tag{1.1}$$

Three diagrams of the kind we will require are shown in Figure 1; these will in fact suffice to derive all the results we present, except for those in section 5. Let us consider Fig. 1(B). After subtracting all sub-divergences, the $n$-bubble (i.e. $(n+1)$-loop) contribution to this diagram is given by the expression:

$$B_n = \frac{\kappa^n}{\epsilon^{n+1}} G(\epsilon) \sum_{r=1}^{n+1} r^{-1} (1 - r\epsilon) \Gamma(1 + r\epsilon) \Gamma(1 - r\epsilon) \left( \frac{n}{r - 1} \right) (-1)^{r+1} x^{r\epsilon} \tag{1.2}$$

where

$$G(\epsilon) = \frac{\Gamma(2 - 2\epsilon)}{\Gamma(2 - \epsilon) \Gamma(1 - \epsilon)^2 \Gamma(1 + \epsilon)} \tag{1.3}$$
and } x = \mu^2, \mu \text{ being the regulator mass. The parameter } \kappa \text{ subsumes any constant factors which will recur on a bubble-by-bubble basis.}

![Feynman diagrams representing the bubble sums A, B and C. The black dots denote squared propagators](image)

**Fig.1:** Feynman diagrams representing the bubble sums A, B and C. The black dots denote squared propagators

We now write

\[(1 - r\epsilon)\Gamma(1 + r\epsilon)\Gamma(1 - r\epsilon)x^{r\epsilon} = \sum_{j=0}^{\infty} L_j(r\epsilon)^j. \quad (1.4)\]

Substituting in Eq. (1.2), and using the identity

\[\Delta_j = \sum_{r=1}^{n+1} r^{j-1} \left( \binom{n}{r-1} (-1)^r = 0 \quad \text{when } j = 1, 2, \ldots n\right. \]

\[\Delta_j = -(n+1)^{-1} \quad \text{when } j = 0 \quad (1.5)\]

we find that the pole terms in } B \text{ are given by the expression

\[B_{n}^{\text{pole}} = \frac{\kappa^n}{(n+1)\epsilon^{n+1}} \sum_{i=0}^{n} G_i \epsilon^i \quad (1.6)\]

where we have written } G(\epsilon) = \sum G_n \epsilon^n. \text{ The identity Eq. (1.5) removes all the non-local (i.e. ln } x \text{-dependent) counter-terms. Now we want to sum over } n. \text{ In a } \beta \text{-function or anomalous dimension calculation, the result will be given by the coefficient of the simple pole in } \epsilon \text{ in the quantity } \sum (n+1)B_{n}^{\text{pole}}, \text{ which is easily seen to give}

\[B = \sum_{n=0}^{\infty} G_n \kappa^n = G(\kappa). \quad (1.7)\]

Similar calculations give:

\[A = -\kappa^{-1} \left[ G(\kappa) - 1 + 2 \int_{0}^{\kappa} G(x) \, dx \right] \quad (1.8)\]

and

\[C = -2\kappa^{-1} \left[ G(\kappa) - 1 + \int_{0}^{\kappa} (1 + 2x)G(x) \, dx \right]. \quad (1.9)\]

Thus all the bubble sums relevant to our calculations depend on the function } G(x), \text{ which has a zero at } x = 1 \text{ and a simple pole at } x = \frac{3}{2}. \text{ We may therefore anticipate that our results in subsequent sections will have a finite radius of convergence in the appropriate coupling constant, because of this pole. We turn now to explicit models.}
2. The large-N Wess-Zumino model

The superpotential of the model is

$$W = \frac{\lambda}{\sqrt{N}} \sum_{i=1}^{N} \phi_\xi \chi_i. \quad (2.1)$$

At leading order, it is trivial to see that $\beta_\lambda$ is determined by the one loop contribution to $\gamma_\phi$. The Feynman diagrams contributing at $O(1/N)$ are shown in Fig. 2.

![Feynman diagrams](a) ![Feynman diagrams](b)

*Fig.2: The Feynman diagrams for section 2. Dashed lines are $\phi$-propagators, and solid lines are $\xi$ or $\chi$ propagators.*

We find

$$\gamma_\xi = \gamma_\chi = \frac{y}{N} B(y) = \frac{1}{N} y G(y),$$

$$\gamma_\phi = y \left[ 1 + 2 \frac{y}{N} A(y) \right] = y + \frac{2y}{N} \left[ 1 - G(y) - 2 \int_{0}^{y} G(x) \, dx \right]. \quad (2.2)$$

where $y = \lambda^2/16\pi^2$. These results (and all our subsequent results for $\beta$-functions and $\gamma$-functions) are correct to $O(1/N)$. Our result for $\beta_\lambda = \lambda [2\gamma_\xi + \gamma_\phi]$ is thus

$$\beta_\lambda = \lambda y \left[ 1 + \frac{2}{N} H(y) \right] \quad (2.3)$$

where

$$H(y) = 1 - 2 \int_{0}^{y} G(x) \, dx. \quad (2.4)$$

It is quite straightforward to verify that Eq. (2.3) reproduces the relevant terms in the existing four-loop calculation [6] for a generalised Wess-Zumino model.

3. Supersymmetric QED

In this section we consider supersymmetric QED with $M = 2N$ charged chiral superfields $\xi, \chi$, with pairs of charges $\pm g/\sqrt{N}$, for large $N$. As for the WZ model, the dominant contribution to $\beta_g$ is one-loop. The graphs for the $O(1/N)$ calculation are shown in Fig. 3.
Fig. 3: Feynman diagrams for section 3. Wavy lines are vector propagators, and solid lines are $\xi$ or $\chi$ propagators. Blobs denote a chain of $\xi$ or $\chi$ bubbles.

In Refs. [7], [8] we found $\beta_g$ for an abelian theory to four loops, by calculating the vector superfield self-energy in the Feynman gauge (note that in this gauge this suffices in the abelian case). For details of our technique for dealing with the D-algebra part of the calculation we refer the reader to Ref. [8]; the upshot is that we have simply to replace the Feynman integrals $A, B, C$ of that reference with the corresponding bubble summed quantities $A, B, C$ from this one. Our result for $\beta_g$ is

$$\beta_g = gK \left[ 1 + \frac{2}{N} \int_0^K (1 - 2x)G(x) \, dx \right]$$  \hspace{1cm} (3.1)$$

where $K = g^2/8\pi^2$, while for the anomalous dimension of each chiral superfield, $\gamma(g)$, we obtain

$$\gamma(g) = -\frac{K}{N}G(K).$$  \hspace{1cm} (3.2)$$

It is interesting at this point to compare these results with the NSVZ all orders formula [9] for $\beta_g$, which for our theory reads:

$$\beta_g^{NSVZ} = gK \left[ 1 - 2\gamma^{NSVZ} \right].$$  \hspace{1cm} (3.3)$$

We see that our results for $\beta_g$ and $\gamma(g)$ do not satisfy this relation. This is not surprising, because it was shown explicitly in Ref. [7] that the DRED and NSVZ $\beta$-functions part company at three loops. It is straightforward to construct order by order in $g$ the coupling constant redefinition that connects the two schemes. Because in this explicit example we have no non-trivial tensor structure, interesting constraints on the nature of the redefinition of the kind exploited in Refs. [7], [8] do not occur.
4. General Abelian Theory

Here we present results for a general theory produced by a $U_1$ gauging of the model defined by Eq. (2.1). It is easy to see that for $N > 1$ the constraints of gauge invariance of $W$ and anomaly cancellation mean that the most general gauging such that $q_{\chi_i}$ and $q_{\xi_i}$ are independent of $i$ is given by $q_{\phi} = 0$ and $q_{\xi} = -q_{\chi} = q$; we will set $q = 1$. For the special case $\lambda = \sqrt{2g}$ we have $\mathcal{N} = 2$ supersymmetry. In Fig. 4 we show the new Feynman diagrams we require for $\beta_g$, beyond those calculated in the previous section.

![Feynman diagrams](image)

Fig.4: Additional Feynman diagrams for section 4. Blobs denote a chain of $\xi, \chi$ bubbles.

The result is

$$\beta_g = gK \left[ 1 + \frac{2}{N} \int_y^K (1 - 2x)G(x) \, dx \right]. \quad (4.1)$$

We also find

$$\gamma_{\xi} = \gamma_{\chi} = \frac{1}{N} \left[ yG(y) - KG(K) \right],$$

$$\gamma_{\phi} = y + \frac{2y}{N} \left[ G(K) - G(y) + 2 \int_y^K G(x) \, dx \right]. \quad (4.2)$$

It is again easy to verify that our result agrees with the three and four loop calculations presented in Ref. [8]. Moreover, for $\mathcal{N} = 2$ (which corresponds to $y = K$) we have $\beta_g = \gamma_{\phi} = 0$ beyond one loop, and $\gamma_{\xi} = \gamma_{\chi} = 0$ to all orders, in accordance with Ref. [10].

5. General Non-Abelian Theory

We now consider a non-abelian theory with gauge group $\mathcal{G}$ and superpotential

$$W = \frac{\lambda}{\sqrt{N}} \phi^a \sum \xi_i^T S_a \chi_i, \quad (5.1)$$

where $\xi_i, \chi_i, \phi$ are multiplets transforming under the $S, S^*$ and adjoint representations of $\mathcal{G}$ respectively. For notational simplicity we take the representation $S$ to be irreducible.
In addition to diagrams similar in form to those computed earlier in the abelian case, the two-point function for the vector superfield includes the additional diagrams depicted in Fig. 5, because the \( \phi \) field now has gauge interactions, as well as further diagrams involving the gauge coupling \( g \) only.

\[
\begin{align*}
\text{Fig.5: Additional Feynman diagrams for section 5.}
\end{align*}
\]

The diagrams in Fig. 5 give rise to bubble sums similar to \( A, B \) and \( C \) calculated earlier. The fact that these diagrams do not contain vector superfield propagators suggests that they correctly determine the corresponding contributions in the non-abelian case. (Note that the graph similar to Fig. 5(b) but with only one \( \chi \) or \( \xi \) loop gives no simple pole.) We can then infer the non-abelian result by using the afore-mentioned fact that there are no divergences beyond one loop for \( N = 2 \). The result is

\[
\beta_g = gK \left[ T(S) + \frac{2\text{tr}[C(S)^2]}{rNT(S)} \int \hat{K} (1 - 2x)G(x) \, dx \right] + \frac{gK}{N} \left[ \int r\hat{K} G(x) \, dx - 1 \right] C(G).
\]

For the chiral superfield anomalous dimensions we find:

\[
\begin{align*}
\gamma_\xi = \gamma_\chi &= \frac{1}{N} \left[ yG(\hat{y}) - KG(\hat{K}) \right] C(S), \\
\gamma_\phi &= \hat{y} + \frac{2y\text{tr}[C(S)^2]}{rNT(S)} \left[ G(\hat{K}) - G(\hat{y}) + 2 \int \hat{K} G(x) \, dx \right] + \frac{1}{N} \left( y - K \right) G(\hat{K}) C(G) - \frac{y}{N} C(G)
\end{align*}
\]

where \( \hat{y} = yT(S) \) and \( \hat{K} = KT(S) \). For definitions of the (fairly standard) group theory factors \( C(S), T(S) \) and \( C(G) \) see for instance Ref. [7]; \( r \) is the number of generators of the group. The above results contain as special cases all those presented in previous sections. Once again one can check compatibility with the three and four-loop calculations from Ref. [8] and Ref. [11].
6. Discussion

In a recent paper\cite{6}, we argued that the $\beta$-functions in simple Wess-Zumino models (such as the single field case) suggest that at $L$ loops one has $\beta^L \sim (-1)^{L+1} L!$ behaviour. Hence we suggested that they are susceptible to Padé-Borel summation, and we argued that this was favourable for the quasi-infra-red fixed point scenario. The large $N$ regime dealt with here is clearly quite different in that we find that through $O(1/N)$ we have a finite radius of convergence in the coupling constant(s), caused by the pole at $x = \frac{3}{2}$ in $G(x)$.

It would clearly be interesting to see whether the finite radius of convergence mentioned above persists at higher orders in $1/N$. One might well expect, in fact, the $O(1/N^2)$ term to depend on $G^2$, or some convolution thereof. Perhaps the critical methods of Ref. \cite{12} could facilitate such calculations, as they have in the non-supersymmetric case.

Since the above calculations include contributions to all orders in perturbation theory, it behoves us, more than usual, to consider the issue of the potential ambiguities in DRED raised in Refs.\cite{13}, \cite{14}. The central tenets of these papers have not, to our knowledge, been challenged; and yet DRED has remained, by and large, the regularisation of choice for higher order supersymmetric calculations.

Although we do here include all orders of perturbation theory, the DRED ambiguities of \cite{13}, \cite{14} do not, in fact, arise because of the “bubble chain” structure of the graphs. This would suggest that our calculation of the $O(1/N)$ contribution is well defined, but may not seem entirely satisfactory, since if there are ambiguous contributions at any order of $1/N$ one may question the consistency of the regulator and the significance of the results. (This objection could, of course, also be made to any of the many conventional perturbative DRED computations.) We believe, however, that it should be possible to formulate the DRED ambiguities in a way which demonstrates them to be equivalent to scheme dependence ambiguities. In support of this conjecture, consider \cite{15}, \cite{16}, which dealt with the metric and torsion $\beta$-functions for two-dimensional supersymmetric $\sigma$-models. In \cite{16} it was explicitly verified that if one requires the two-dimensional alternating tensor $\epsilon^{\mu\nu}$ to satisfy the equation:

$$\epsilon^{\mu\nu}\epsilon_{\nu\rho} = (1 + c\epsilon)g^{\mu\rho}$$

(6.1)

(where here $\epsilon = 2 - d$) then although the $\beta$-functions do depend on $c$ at two loops, this dependence can be removed by field redefinitions. This $c$-dependence is associated with a

\footnote{In fact relations of this type were first explored in the non-supersymmetric context: see \cite{17}.}
two-dimensional version of the ambiguity noted (in the four-dimensional case) by Siegel [13];
we conjecture, therefore, that the four dimensional case may be dealt with in a similar way,
with coupling constant redefinitions instead of field redefinitions.

We hope to flesh out this idea, and also consider applications of our results, in future
publications.

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References

[1] M. Campostrini and P. Rossi, Int. J. Mod. Phys. A7 (1992) 3265;  
   G. Eyal et al, Nucl. Phys. B470 (1996) 369;  
   D.J. Broadhurst, J.A. Gracey and D. Kreimer, hep-th/9607174, Z. Phys. C in press  
[2] E. Witten, Ann. Phys. 128 (1980) 363  
[3] J.A. Gracey, Int. J. Mod. Phys. A8 (1993) 2465; Phys. Lett. B373 (1996) 173  
[4] J.A. Gracey, Nucl. Phys. B414 (1994) 614  
[5] A. Palanques-Mestre and P. Pascual, Comm. Math. Phys. 95 (1984) 277  
[6] P.M. Ferreira, I. Jack, and D.R.T. Jones, Phys. Lett. B392 (1997) 376  
[7] I. Jack, D.R.T. Jones and C.G. North, Phys. Lett. B386 (1996) 138  
[8] I. Jack, D.R.T. Jones and C.G. North, hep-ph/9609325, Nucl. Phys. B in press  
[9] V. Novikov et al, Nucl. Phys. B229 (1983) 381;  
   V. Novikov et al, Phys. Lett. B166 (1986) 329;  
   M. Shifman and A. Vainstein, Nucl. Phys. B277 (1986) 456;  
   A. Vainstein, V. Zakharov and M. Shifman, Sov. J. Nucl. Phys. 43 (1986) 1028;  
   M. Shifman, A. Vainstein and V. Zakharov Phys. Lett. B166 (1986) 334  
[10] P.S. Howe, K.S. Stelle and P. West, Phys. Lett. B124 (1983) 55;  
    P.S. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. B236 (1984) 125  
[11] I. Jack, D.R.T Jones and C.G. North, Nucl. Phys. B473 (1996) 308  
[12] A.N Vasil’ev, Yu.M. Pis’mak and J.R. Honkonen, Theor. Math. Phys. 46 (1981) 157;  
    ibid 47 (1981) 291  
[13] W. Siegel, Phys. Lett. B94 (1980) 37  
[14] L.V. Avdeev, G.A. Chochia and A.A. Vladimirov, Phys. Lett. B105 (1981) 272  
[15] D.R.T. Jones, Phys. Lett. B192 (1987) 391  
[16] R.W. Allen and D.R.T. Jones, Nucl. Phys. B303 (1988) 271  
[17] C.M. Hull and P.K. Townsend, Phys. Lett. B191 (1987) 115;  
    R.R. Metsaev and A.A. Tseytlin, Phys. Lett. B191 (1987) 354;  
    I. Jack and D.R.T. Jones, Phys. Lett. B200 (1988) 453  
