Conjugate Pairs of Subfactors and Entropy for Automorphisms

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Abstract

Based on the fact that, for a subfactor $N$ of a II$_1$ factor $M$, the first non-trivial Jones index is 2 and then $M$ is decomposed as the crossed product of $N$ by an outer action of $Z_2$, we study pairs $\{N, uN^*\}$ from a view point of entropy for two subalgebras of $M$ with a connection to the entropy for automorphisms, where the inclusion of II$_1$ factors $N \subset M$ is given as $M$ is the crossed product of $N$ by a finite group of outer automorphisms and $u$ is a unitary in $M$.

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1 Introduction

For two von Neumann subalgebras $A$ and $B$ of a finite von Neumann algebra $M$, in the previous paper [4] we gave a modified constant $h(A|B)$ of the Connes-Størmer relative entropy $H(A|B)$ in [7] (cf. [18]). The aim was to see the entropy for unistochastic matrices from the viewpoint of the operator algebras and we showed among others that $h(D|uDu^*) = H(b(u))$, where $D$ is the algebra of the diagonal matrices in the $n \times n$ complex matrices $M_n(\mathbb{C})$, and where $H(b(u))$ is the entropy in [21] for the unistochastic matrix $b(u)$ induced by a unitary $u$ in $M_n(\mathbb{C})$, and in general, it does not holds that $H(D|uDu^*) = H(b(u))$ (see, for example [17]).
In this paper, we replace the type I$_n$ factor $M_n(\mathbb{C})$ to a II$_1$ factor $M$. The above relation in [4] (cf. [15]) suggests us that if $A$ and $B$ are maximal abelian subalgebras of a II$_1$ factor $M$, then $h(A|B)$ is not necessarily finite. In order to discuss on a subalgebra $A$ of $M$ with $h(A|uAu^*) < \infty$ for all unitaries $u$ in $M$, we pick up here a subfactor $N \subset M$ with the Jones index $[M : N] < \infty$ ([11]) (cf. [8]).

Let $M$ be a type II$_1$ factor, and let $N$ be a subfactor of $M$ such that $[M : N] = 2$ which is the simplest, nontrivial unique subfactor of $M$ up to conjugacy. Then $M$ is decomposed into the crossed product of $N$ by an outer automorphism with the period 2. Based on this fact, we study the set of the values $h(N|uNu^*)$ for the inclusion of factor-subfactor $N \subset M$, with a connection to the inner automorphisms $Adu$, where $M$ is given as the crossed product $N \rtimes G$ of a type II$_1$ factor $N$ by a finite group $G$ with respect to an outer action $\alpha$ and $u$ is a unitary in $M$.

First, for two von Neumann subalgebras $A$ and $B$ of a finite von Neumann subalgebra $M$, we show, in Corollary 2.2.3 below, that if $E_AE_B = E_BE_A$ (which is called the commuting square condition in the sense of [9]) then $H(A|B) = h(A|B)$, where $E_A$ is the conditional expectation of $M$ onto $A$.

We give an extended notion $H_N(Adu)$ of $H(b(u))$ to the inner automorphisms $Adu$ in Definition 3.1.1, and we show that $h(N|uNu^*) \leq H_N(Adu)$ in Theorem 3.1.4.

The inner conjugacy class of $N$ is rich from the view point of the values of $h(N|uNu^*)$, that is, in the special case of $G = \mathbb{Z}_2$, keeping the fact that $h(N|uNu^*) \leq H(M|uNu^*) = \log 2$ for all unitary $u \in M$ in mind, we have that $\{h(N|uNu^*) : u \in M \text{ a unitary}\} = [0, \log 2]$ in Theorem 3.2.3.

2 Preliminaries

In this section, we summarize, for future reference, notations, terminologies and basic facts.

Let $M$ be a finite von Neumann algebra, and let $\tau$ be a faithful normal tracial state. For each von Neumann subalgebra $A$, there is a unique $\tau$ preserving conditional expectation $E_A : M \to A$. 

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2.1 Connes-Størmer relative entropy

Let $S$ be the set of all finite families $(x_i)$ of positive elements in $M$ with $1 = \sum x_i$. Let $A$ and $B$ be two von Neumann subalgebras of $M$. The relative entropy $H(A|B)$ is defined by Connes and Størmer ([7]) as

$$H(A \mid B) = \sup_{(x_i) \in S} \sum_i (\tau \eta E_B(x_i) - \tau \eta E_A(x_i)).$$

Here, $\eta$ is the function defined by

$$\eta(t) = -t \log t, \quad (0 < t \leq 1) \quad \text{and} \quad \eta(0) = 0.$$

Let $\phi$ be a normal state on $M$. Let $\Phi$ be the set of all finite families $(\phi_i)$ of positive linear functionals on $M$ with $\phi = \sum \phi_i$. The relative entropy $H_\phi(A|B)$ of $A$ and $B$ with respect to $\phi$ is given by Connes ([6]) as

$$H_\phi(A|B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i \mid_A, \phi \mid_A) - S(\phi_i \mid_B, \phi \mid_B))$$

and if $\phi = \tau$ then $H_\tau(A|B) = H(A \mid B)$. Here $S(\phi, \psi)$ is the relative entropy for positive linear functionals $\phi$ and $\psi$ on $M$ (cf. [14, 16]).

2.2 Modified relative entropy for two subalgebras.

We modified in [4] the Connes-Størmer relative entropy for a pair of subalgebras as follows:

Let $A$ and $B$ be two von Neumann subalgebras of $M$. Let $S$ be the set of all finite families $(x_i)$ of positive elements in $M$ with $1 = \sum x_i$. The conditional relative entropy $h(A \mid B)$ of $A$ and $B$ corresponding $H(A|B)$ is given as

$$h(A \mid B) = \sup_{(x_i) \in S} \sum_i (\tau \eta E_B(E_A(x_i)) - \tau \eta E_A(x_i)).$$

Let $S(A) \subset S$ be the set of all finite families $(x_i)$ of positive elements in $A$ with $1 = \sum x_i$. Then it is clear that

$$h(A \mid B) = \sup_{(x_i) \in S(A)} \sum_i (\tau \eta E_B(x_i) - \tau \eta (x_i)).$$
Let $S'(A) \subset S(A)$ be the set of all finite families $(x'_i)$ with each $x'_i$ a scalar multiple of a projection in $A$. Then

$$h(A \mid B) = \sup_{(x'_i) \in S'(A)} \sum_i (\tau \eta E_B(x'_i) - \tau \eta(x'_i)).$$

The conditional relative entropy of $A$ and $B$ with respect to $\phi$ corresponding $H_\phi(A|B)$ is given as

$$h_\phi(A|B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i \mid_A, \phi \mid_A) - S((\phi_i \circ E_A) \mid_B, (\phi \circ E_A) \mid_B)).$$

If we let $\Phi(A) \subset \Phi$ be the set of all finite families $(\phi'_i)$ in $\Phi$ with each $\phi'_i = \phi_i \circ E_A$, then

$$h_{\phi'}(A|B) = \sup_{(\phi'_i) \in \Phi(A)} \sum_i (S(\phi'_i \mid_A, \phi \mid_A) - S(\phi'_i \mid_B, (\phi \circ E_A) \mid_B)).$$

We give conditions for that $h_{\phi'}(A|B) = H_{\phi'}(A|B)$ in Corollary 2.2.4 below, and show relations for $h_{\phi'}(A|B), H_{\phi'}(A|B), H_{\phi'}(A)$ and $h_{\phi'}(A)$ in Theorem 2.2.2, where $h_{\phi'}(A)$ is given by modifing $H_{\phi'}(A)$ in [6] for a von Neumann subalgebra $A$ of $M$ (cf. [14]), that is

$$h_{\phi'}(A) = \sup_{(\phi_i) \in \Phi(A)} \sum_i (\eta(\phi_i(1)) + S(\phi_i \mid_A, \phi \mid_A))$$

and

$$H_{\phi'}(A) = \sup_{(\phi_i) \in \Phi} \sum_i (\eta(\phi_i(1)) + S(\phi_i \mid_A, \phi \mid_A)).$$

Cleary, we have that $0 \leq h_{\phi'}(A) \leq H_{\phi'}(A)$. In the case of $\phi$ is the trace $\tau$, $h_{\phi'}(A) = h(A)$, which is given as

$$h(A) = \sup_{(x_i) \in S(A)} \sum_i (\eta(\tau(x_i)) - \tau \eta(x_i)).$$

We need the following lemma in order to prove Theorem 2.2.2, in which we show relations among $H_{\phi'}(A), h_{\phi'}(A), H_{\phi'}(A|B)$ and $H_{\phi'}(A|B)$:

**Lemma 2.2.1.** Let $A$ and $B$ be von Neumann subalgebras of a finite von Neumann algebra $M$, and let $\psi, \phi$ be positive linear functionals on $M$. If $E_A E_B = E_B E_A$, then

$$S((\psi \circ E_A) \mid_B, (\phi \circ E_A) \mid_B) = S((\psi \circ E_A) \mid_{A \cap B}, (\phi \circ E_A) \mid_{A \cap B}).$$
Proof. Relative entropy for positive linear functionals $\psi, \phi$ on a unital $C^*$-algebra $C$ is expressed in [13] by

$$S(\psi, \phi) = \sup_{n \in \mathbb{N}} \sup_n \{\psi(1) \log n - \int_{1/n}^\infty \left( \psi(y(t)^*y(t)) + \frac{1}{t} \phi(x(t)x(t)^*) \right) \frac{dt}{t} \}$$

where $x(t) : (\frac{1}{n}, \infty) \to C$ is a step function with finite range, and $y(t) = 1 - x(t)$.

Let $x(t) : (\frac{1}{n}, \infty) \to B$ be a step function with finite range, then $E_A(x(t)) : (\frac{1}{n}, \infty) \to A \cap B$ is a step function with finite range because of that $E_A E_B = E_B E_A$. Since $E_A(x)^* E_A(x) \leq E_A(x^*x)$ for all $x \in M$, we have that

$$\psi(1) \log n - \int_{1/n}^\infty \left( \psi \circ E_A(y(t)^*y(t)) + \frac{1}{t} \phi \circ E_A(x(t)x(t)^*) \right) \frac{dt}{t} \leq \psi(1) \log n - \int_{1/n}^\infty \left( \psi(E_A(y(t))^*E_A(y(t))) + \frac{1}{t} \phi(E_A(x(t))E_A(x(t)^*)) \right) \frac{dt}{t} \leq S((\psi \circ E_A) |_{A \cap B}, (\phi \circ E_A) |_{A \cap B}).$$

This implies that

$$S((\psi \circ E_A) |_B, (\phi \circ E_A) |_B) \leq S((\psi \circ E_A) |_{A \cap B}, (\phi \circ E_A) |_{A \cap B}).$$

Since the opposite inequality is clear, we have the equality. \qed

Theorem 2.2.2. Let $M$ be a finite von Neumann algebra with a normal faithful tracial state $\tau$. Let $\phi$ be a normal state of $M$, and let $A, B$ be von Neumann subalgebras of $M$. Then

1. $h_\phi(A|B) \leq h_\phi(A|1) = h_\phi(A)$.

2. Assume that $E_A E_B = E_B E_A$. Then $h_\phi(A|B) = h_\phi(A|A \cap B)$.

Hence, if $A \cap B = \mathbb{C}$, then $h_\phi(A|B) = h_\phi(A)$.

3. If $E_A E_B = E_B E_A$ and if $\phi = \phi \circ E_A$, then $H_\phi(A|B) = H_\phi(A|A \cap B)$.

Especially, if $A \cap B = \mathbb{C}$, then $H_\phi(A|B) = H_\phi(A)$.

4. If $B \subset A$, then

$$H_\phi(A|B) = h_\phi(A|B).$$

Especially, $H_\phi(A) = h_\phi(A)$. 

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\textbf{Proof.} (1) Let $\phi, \psi$ be normal states of $M$, and let $(\phi_i)_i \in \Phi$. Then $(\phi \circ E_A) |_B$ and $\frac{1}{\phi_i(1)}(\phi_i \circ E_A) |_B$ are states of $B$ so that
\begin{align*}
S\left(\frac{1}{\phi_i(1)}(\phi_i \circ E_A) |_B, (\phi \circ E_A) |_B\right) &\geq 0 \\
S\left(\frac{1}{\phi_i(1)}(\phi_i \circ E_A) |_B, (\phi \circ E_A) |_B\right) &\geq \eta(\phi_i(1))\end{align*}
and
\begin{align*}
S\left(\frac{1}{\phi_i(1)}(\phi_i \circ E_A) |_B, (\phi \circ E_A) |_B\right) &= \frac{1}{\phi_i(1)}S(\phi_i \circ E_A |_B, (\phi \circ E_A) |_B) - \phi_i(1)\eta(\frac{1}{\phi_i(1)}) \\
S\left(\frac{1}{\phi_i(1)}(\phi_i \circ E_A) |_B, (\phi \circ E_A) |_B\right) &= \frac{1}{\phi_i(1)}S(\phi_i \circ E_A |_B, (\phi \circ E_A) |_B) - \log(\phi_i(1)).
\end{align*}
Hence
\begin{align*}
-S\left((\phi_i \circ E_A) |_B, (\phi \circ E_A) |_B\right) &\leq \eta(\phi_i(1)), \\
\text{and if } B = C_1 &\text{ then the equality holds because } \frac{1}{\phi_i(1)}\phi_i C_1 = \phi C_1. \end{align*}
These imply that
\begin{align*}
h_\phi(A|B) &= \sup_{(\phi_i) \in \Phi} \sum_i \left(S(\phi_i |_A, \phi |_A) - S((\phi_i \circ E_A) |_B, (\phi \circ E_A) |_B)\right) \\
&\leq \sup_{(\phi_i) \in \Phi} \sum_i \left(S(\phi_i |_A, \phi |_A) + \eta(\phi_i(1))\right) \\
&= h_\phi(A|A \cap B),
\end{align*}
and the equality holds if $B = C_1$.

(2) Assume that $E_A E_B = E_B E_A$, then by lemma 2.2.1, we have that
\begin{align*}
h_\phi(A|B) &= \sup_{(\phi_i) \in \Phi} \sum_i \left(S(\phi_i |_A, \phi |_A) - S((\phi_i \circ E_A) |_B, (\phi \circ E_A) |_B)\right) \\
&= \sup_{(\phi_i) \in \Phi} \sum_i \left(S(\phi_i |_A, \phi |_A) - S((\phi_i \circ E_A) |_{A \cap B}, (\phi \circ E_A) |_{A \cap B})\right) \\
&= h_\phi(A|A \cap B).
\end{align*}
Since $h_\phi(A|B)$ is decreasing in $B$, it implies that
\begin{align*}
h_\phi(A|B) &= h_\phi(A|A \cap B).
\end{align*}
Hence, if $A \cap B = \mathbb{C} 1$, then $h_\phi(A|B) = h_\phi(A|\mathbb{C}1) = h_\phi(A)$.

(3) Assume that $E_AE_B = E_BE_A$ and that $\phi \circ E_A = \phi$. Let $(\phi_i)_i \in \Phi$, then $(\phi_i \circ E_A)_i \in \Phi$ and we have that

$$H_\phi(A|B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|A, \phi|A) - S(\phi_i|B, \phi|B))$$

$$\geq \sup_{(\phi_i) \in \Phi} \sum_i (S((\phi_i \circ E_A)|A, \phi|A) - S((\phi_i \circ E_A)|B, \phi|B))$$

$$= \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|A, \phi|A) - S((\phi_i \circ E_A)|B, (\phi \circ E_A)|B))$$

$$= \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|A, \phi|A) - S(\phi_i|A \cap B, (\phi \circ E_A)|A \cap B))$$

$$= \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|A, \phi|A) - S(\phi_i|A \cap B, \phi|A \cap B))$$

$$= H_\phi(A|A \cap B).$$

In general, $H_\phi(A|A \cap B) \geq H_\phi(A|B)$ so that

$$H_\phi(A|B) = H_\phi(A|A \cap B).$$

Especially, $H(A|B) = H(A|A \cap B)$ (cf. [22]), and if $A \cap B = \mathbb{C}$, then $H_\phi(A|A \cap B) = H_\phi(A|\mathbb{C}1) = H_\phi(A)$ so that

$$H_\phi(A|B) = H_\phi(A).$$

(4) If $B \subset A$, then

$$H_\phi(A|B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|A, \phi|A) - S(\phi_i|B, \phi|B))$$

$$= \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|A, \phi|A) - S((\phi_i \circ E_A)|B, (\phi \circ E_A)|B))$$

$$= h_\phi(A|B).$$

By combining with (1), we have $H_\phi(A) = h_\phi(A)$. \qed
Corollary 2.2.3. Assume that $E_A E_B = E_B E_A$. Then $H(A|B) = h(A|B)$. Moreover, if $\phi = \phi \circ E_A$, then
\[
H_\phi(A|B) = h_\phi(A|B).
\]

Proof. Assume that $E_A E_B = E_B E_A$ and that $\phi = \phi \circ E_A$. Then by using Theorem 2.2.2 (3), (4) and (2), we have that
\[
H_\phi(A|B) = H_\phi(A|A \cap B) = h_\phi(A|A \cap B) = h_\phi(A|B).
\]
Since $E_A$ is the $\tau$-conditional expectation, $\tau = \tau \circ E_A$, hence
\[
H(A|B) = h(A|B).
\]

\[
\square
\]

3 Inner conjugate subfactors

Connes-Størmer defined the entropy $H(\alpha)$ for a trace preserving automorphism $\alpha$ of a finite von Neumann algebra in [7]. The definition is arivable for a trace preserving *-endomorphism too.

For a trace preserving *-endomorphism $\sigma$ of a finite von Neumann algebra $N$, it was shown a relation between the entropy $H(\sigma)$ for $\sigma$ and the relative entropy $H(N | \sigma(N))$ in the papers [11, 2, 3, 10, 20] (cf. [14]). The relation is, roughly speaking, that
\[
H(\sigma) = \frac{1}{2} H(N | \sigma(N))
\]
under a certain condition. Such a *-endomorphism $\sigma$ can be extended often to an automorphism $\alpha$ of a finite von Neumann algebra $M$ which contains $N$ as a von Neumann subalgebra. Some examples of such endomorphisms appeared in a relation to Jones index theory of subfactors. In [2], we studied a nice class of such a *-endomorphism $\sigma$ of a type II$_1$factor $N$ which is extendable to an automorphism $\alpha$ of the big type II$_1$factor $M$ obtained by the basic construction from $N \supset \sigma(N)$. We called such a $\sigma$ basic *-endomorphism and
showed that $H(\alpha) = \frac{1}{2}H(N|\sigma(N))$. Since $\sigma(N) \subset N$, we have by Theorem 2.2.2 (4) that $H(N|\sigma(N)) = h(N|\sigma(N))$ so that

$$H(\alpha) = \frac{1}{2}H(N|\sigma(N)) = \frac{1}{2}h(N|\sigma(N)) = \frac{1}{2}h(N|\alpha(N)).$$

This means that for an automorphism $\alpha$ of a $\text{II}_1$ factor $M$ we may be able to choose a subfactor $N \subset M$ such that the entropy for $\alpha$ is given from $h(N|\alpha(N))$.

Our study in this section is motivated by these results. The above automorphism $\alpha$ arising from a *-endomorphism as is outer. Here, we discuss by replacing the $\alpha$ to inner automorphisms $Adu$ and the entropy $H(\alpha)$ to the entropy $H_N(Adu)$ defined below.

### 3.1 Entropy for Inner Automorphisms with respect to Subfactors

Let $N$ be a type $\text{II}_1$ factor with the canonical trace $\tau$ and let $G$ be a finite group. Let $\alpha$ be an outer action of $G$ on $N$, so that for all $g \in G, g \neq 1$ if $\alpha_g(x)a = ax$ for all $x \in N$, then $a = 0$. Hereafter, we let $M$ be the crossed product of $N$ by $G$ with respect to $\alpha$:

$$M = N \rtimes_\alpha G.$$ 

We identify $N$ with the von Neumann subalgebra embedded in $M$, and denote by $v$ the unitary representation of $G$ in $M$ such that every $v_g$ is a unitary in $M$ with

$$\alpha_g(x) = v_g xv_g^*, \quad (x \in N, g \in G).$$

Then every $x \in M$ is written by the Fourier expansion

$$x = \sum_{g \in G} x_g v_g, \quad (x_g \in N)$$

and $x_g = E_N(xv_g^*)$. A $u \in M$ is a unitary if and only if

$$\sum_{g \in G} u_{hg}\alpha_h(u_g^*) = \delta_{h,1} \quad \text{and} \quad \sum_{g \in G} \alpha_g^{-1}(u_g^*u_{gh}) = \delta_{h,1},$$

where we denote the identity of $G$ by 1. This imply that $\sum_{g \in G} \tau(u_gu_g^*) = 1$, and we can put as the followings:
Definition 3.1.1. The entropy of the inner automorphism $Adu$ of $M$ with respect to $N$ is given by

$$H_N(Adu) = \sum_{g \in G} \eta \tau(u_g u_g^*) .$$

Comment 3.1.2. Each $x \in M$ is represented as the matrix $x = (x(g, h))_{gh}$ indexed by the elements of $G$. Here $x(g, h) \in N$ for all $g, h$ in $G$, and $x(g, h) = \alpha_g^{-1}(E_N(x_h^*)) = \alpha_g^{-1}(x_h)$. The entropy $H(b(u))$ defined in [21] is written as

$$H(b(u)) = \frac{1}{n} \sum_{i,j} \eta(|u(i, j)|^2),$$

when $b(u)$ is the unistochastic matrix induced by a unitary $u = (b(i, j))_{ij}$ in $M_n(\mathbb{C})$. A matrix representation for an $x$ in $M_n(\mathbb{C})$ is depend on the diagonal matrix algebra. In that sense, we consider the notion of $H_N(Adu)$ corresponds to the notion of the entropy for a unistochastic matrix.

Lemma 3.1.3. (1) If $Adu$ and $Adw$ are conjugate, then $H_N(Adu) = H_N(Adw)$.

(2) If $\theta = Adu$ for some unitary $u \in M$, then $H_N(\theta^{-1}) = H_N(\theta)$.

Proof. (1) Assume $Adu = \theta Adw\theta^{-1}$ for some automorphism $\theta$ of $M$. Then $\theta(w) = \lambda w$ for some complex number $\lambda$ with $|\lambda| = 1$ and so $\eta \tau(w_g w_g^*) = \eta \tau(u_g u_g^*)$ which implies that $H_N(Adu) = H_N(Adw)$.

(2) Let $w \in M$ be a unitary with $Adw = \theta^{-1}$, then $w = \gamma u^*$ for some $\gamma \in \mathbb{T}$. For the expression that $w = \sum_g w_g v_g$, we have that $w_g = \gamma \alpha_g(u_g^{* -1})$ for all $g \in G$ so that

$$H_N(\theta^{-1}) = \sum_g \eta \tau(w_g w_g^*) = \sum_g \eta \tau(u_g u_g^*) = H_N(\theta).$$

The $h(N|uNu^*)$ is bounded by $H_N(Adu)$ as follows:

Theorem 3.1.4. Assume that $N$ is a type II$_1$ factor, $G$ is a finite group and $M = N \rtimes_\alpha G$ with respect to the outer action $\alpha$. Then for each unitary $u \in M$, we have that

$$h(N|uNu^*) \leq H_N(Adu) = \sum_{g \in G} \eta \tau(u_g u_g^*) \leq \log |G| ,$$

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where $|G|$ is the cardinality of $G$.

**Proof.** Let $(\lambda_ip_i)_{i \in I} \in S'(N)$ be a finite partition of the unity, that is,

$$\sum_{i \in I} \lambda_ip_i = 1$$

where $(\lambda_i)_{i \in I}$ are positive numbers and $(p_i)_{i \in I}$ are projections in $N$. For a given $\varepsilon$, choose an $\epsilon > 0$ with $2|G|\eta(\epsilon) < \min\{\varepsilon, 1/e\}$. There exist mutually orthogonal projections $(q_{i,k})_{k \in N}$ and nonnegative numbers $(\alpha_{i,k})_{k}$ which satisfy that

$$p_i = \sum_k q_{i,k} \quad \text{and} \quad 0 \leq q_{i,k}u_g^*u_g q_{i,k} - \alpha_{i,k}q_{i,k} \leq \epsilon q_{i,k}.$$  

This is possible by the induction method of the spectral decompositions for $(p_iu_g^*u_g p_i)_{i \in I, g \in G}$ (see for example, [18, Proof of 4.3 Lemma]). In fact, letting

$$G = \{g_1, \ldots, g_m\}$$

and by the spectral decomposition for $p_iu_g^*u_g p_i \in p_iNp_i$, we have mutually orthogonal projections $(q_{i,k})_{k \in p_i Np_i}$ and nonnegative numbers $(\alpha_{i,k})_{k}$

$$p_i = \sum_k q_{i,k} \quad \text{and} \quad 0 \leq q_{i,k}u_g^*u_g q_{i,k} - \alpha_{i,k}q_{i,k} \leq \epsilon q_{i,k}.$$  

Next by the consideration for $q_{i,k}^1 u_g^* u_g q_{i,k}^1$, we have a partition $(q_{i,k}^2)_{k}$ of $q_{i,k}^1$ and $(\alpha_{i,k}^2)_{k}$. Put $\alpha_{i,k}^j = \alpha_{i,k,\ldots,k_j}$ and $q_{i,k} = q_{i,k,\ldots,k_m}$. Then these satisfy the desired conditions.

Since $\eta(x + y) \leq \eta(x) + \eta(y)$, $(x, y \in N)$, $\eta$ is increasing on $[0, 1/e]$, and the family $(q_{i,k})_{k}$ is mutually orthogonal, we have for $\epsilon$ with $\epsilon \leq 1/e$

$$\tau\eta(\sum_k q_{i,k}u_g^*u_g q_{i,k}) \leq \tau\eta(\sum_k (q_{i,k}u_g^*u_g q_{i,k} - \alpha_{i,k}^q q_{i,k})) + \tau\eta(\sum_k \alpha_{i,k}^q q_{i,k}) \leq \eta(\epsilon)\tau(p_i) + \sum_k \eta(\alpha_{i,k}^q)\tau(q_{i,k}).$$

Hence, by the condition that $\sum_{i,k} \lambda_i \tau(q_{i,k}) = 1$, the operator concavity of $\eta$
implies that
\[
\sum_{i,g} \lambda_i \tau \eta \left( \sum_k q_{i,k} u_g u_g^* q_{i,k} \right) \\
\leq \sum_{i,k,g} \lambda_i \eta(\alpha_{i,k}^g) \tau(q_{i,k}) + |G| \eta(\epsilon) \\
= \sum_{i,k} \lambda_i \tau(q_{i,k}) \sum_g \eta(\alpha_{i,k}^g) + |G| \eta(\epsilon) \\
\leq \sum_g \eta \left( \sum_{i,k} \lambda_i \tau(q_{i,k}) \alpha_{i,k}^g \right) + |G| \eta(\epsilon).
\]

Remark that for all \( g \in G \)
\[
\tau(u_g u_g^*) - \sum_{i,k} \lambda_i \alpha_{i,k}^g \tau(q_{i,k}) = \sum_{i,k} \lambda_i \tau(q_{i,k} u_g u_g^* q_{i,k} - \alpha_{i,k}^g q_{i,k})
\]
and that
\[
0 \leq \sum_{i,k} \lambda_i \tau(q_{i,k} u_g u_g^* q_{i,k} - \alpha_{i,k}^g q_{i,k}) \leq \epsilon.
\]

Then we have that
\[
0 \leq \tau(u_g u_g^*) - \sum_{i,k} \lambda_i \tau(q_{i,k}) \alpha_{i,k}^g \leq \epsilon, \quad (g \in G)
\]
so that
\[
\sum_g \eta \left( \sum_{i,k} \lambda_i \tau(q_{i,k}) \alpha_{i,k}^g \right) \leq \sum_g \eta \tau(u_g u_g^*) + |G| \eta(\epsilon).
\]

Here we used the following inequality in \cite[(2.8)]{14}
\[
|\eta(s) - \eta(t)| \leq \eta(s - t) \quad \text{for } 0 \leq s - t \leq \frac{1}{2}.
\]

Remark that \( \sum_g u_g u_g^* = 1 \) and that for all \( i \) the projections \( (q_{i,k})_k \) is mutually orthogonal. Hence by using the following fact that
\[
\tau \eta(u_g^* q_{i,k} u_g) = \tau \eta(q_{i,k} u_g u_g^* q_{i,k}),
\]
we have that
\[
\sum_{i,k}(\tau \eta E_N(u^* \lambda_i q_{i,k} u) - \tau \eta(\lambda_i q_{i,k}))
\]
\[
= \sum_{i,k}(\tau \eta(\sum_{g \in G} \alpha_g(u^*_g \lambda_i q_{i,k} u_g) - \eta(\lambda_i)\tau(q_{i,k})))
\]
\[
\leq \sum_{i,k} \sum_{g}(\tau \eta(u^*_g \lambda_i q_{i,k} u_g) - \eta(\lambda_i)\tau(q_{i,k}))
\]
\[
= \sum_{i,k,g}(\eta(\lambda_i)\tau(u^*_g q_{i,k} u_g) + \lambda_i \tau \eta(u^*_g q_{i,k} u_g) - \sum_{i,k}(\eta(\lambda_i)\tau(q_{i,k})))
\]
\[
= \sum_{i}(\eta(\lambda_i)\tau(p_i \sum_{g} u_g u^*_g) + \sum_{i,k}(\lambda_i \tau \eta(u^*_g q_{i,k} u_g) - \sum_{i}(\eta(\lambda_i)\tau(p_i))
\]
\[
= \sum_{i,k,g}(\lambda_i \tau \eta(q_{i,k} u_g u^*_g q_{i,k})
\]
\[
= \sum_{i,k,g}(\lambda_i \tau \eta(\sum_{k} q_{i,k} u_g u^*_g q_{i,k})
\]
\[
\leq \sum_{g}(\eta \tau(u_g u^*_g) + 2|G|\eta(\epsilon).
\]

Thus
\[
h(N|u Nu^*) = \sup_{(\lambda_i p_i) \in S'(N)} \sum_{i}(\tau \eta E_N(u^* \lambda_i p_i u) - \tau \eta(\lambda_i p_i))
\]
\[
= \sup_{(\lambda_i q_{i,k}, k)} \sum_{i,k}(\tau \eta E_N(u^* \lambda_i q_{i,k} u) - \tau \eta(\lambda_i q_{i,k}))
\]
\[
\leq \sum_{g}(\eta \tau(u_g u^*_g).
\]

Since \( \eta \) is a concave function, Theorem 3.1.4 implies the following:

**Corollary 3.1.5.** Assume that \( N, G, u \) be as in Theorem 3.1.4 and that \( h(N|u Nu^*) = \log |G| \). Then \( \tau(u_g u^*_g) = \frac{1}{|G|} \) for all \( g \in G \).

**Remark and Example 3.1.6.** Let \( A \) and \( B \) be subalgebras of a type II\(_1\) factor \( M \). Then \( h(A|B) \leq H(A|B) \leq H(M|B) \), and if \( B \) is a subfactor with
$B' \cap M = \mathbb{C}$ then $H(M | B) = \log[M : B]$ by [18] so that $h(A | B) \leq \log[M : B]$.

Størmer says that relative entropy can be viewed as a measure of distance between two subalgebras, which in the noncommutative case also measures their sizes and relative position.

Here, we give an example, which shows that $h(A | B)$ measures relative position and that some small size subalgebra $A_G$ can take the maximal value of $h(A | B)$, although the entropy $h(A | B)$ is increasing in $A$.

Assume that the finite group $G$ in Theorem 3.1.4 is abelian. Let $B = uNu^*$. By taking the inner automorphism $Adu^*$, we may consider $M$ as the crossed product of $B$ by $G$ so that $x \in M$ has a unique expansion $x = \sum_{g \in G} x_g v_g$, $(x_g \in B)$. Let $A_G$ be the von Neumann algebra generated by the unitary group $v_G$, (that is, $|G|$ dimensional abelian algebra). Then

$$h(A_G | B) = \log |G| = H(M | B) = \log[M : B].$$

In fact, it is clear that $h(A | B) \leq H(M | B) = \log[M : B] = \log |G|$. To show the opposite inequality, let $\hat{G}$ be the character group of $G$. Given $\chi \in \hat{G}$, let

$$p_\chi = \frac{1}{|G|} \sum_{g \in G} \chi_g v_g.$$ 

Then $\{p_\chi; \chi \in \hat{G}\}$ is a family of mutually orthogonal projections in $A_G$ with $\sum_{\chi \in \hat{G}} p_\chi = 1$. Hence

$$h(A_G | B) \geq \sum_{\chi \in \hat{G}} (\tau \eta E_B(p_\chi)) = \sum_{\chi \in \hat{G}} \eta \left( \frac{1}{|G|} \right) = |\hat{G}| \frac{1}{|G|} \log(|G|) = \log |G|,$$

and we have that

$$h(A_G | B) = \log |G|.$$

In the next section, we show that inner conjugacy classes of subfactors $N$ of type $\Pi_1$ factor can take the maximum value of $h(N | uNu^*)$. 

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3.2 Case of $G = \mathbb{Z}_n$

Here, we assume that the group $G$ in 3.1 is a finite cyclic group $\mathbb{Z}_n$, that is, $M$ is the crossed product $N \rtimes_{\alpha} \mathbb{Z}_n$ of a $\Pi_1$-factor $N$ by the group generated by an automorphism $\alpha$ on $N$ such that $\alpha^n$ is the identity and $\alpha^i$ is outer for $i = 1, \ldots, n - 1$. Such an automorphism $\alpha$ is called a minimal periodic automorphism (cf. [5]).

3.2.1 Matrix units for minimal periodic automorphisms

Let $\gamma$ be a primitive $n$-th root. Connes showed in the proof for the characterization of minimal periodic automorphisms ([5, Cor. 2.7]) that if $\alpha$ is minimal periodic, then there exists a set of matrix units $\{e_{ij}\}_{i,j=1}^n$ in $N$ such that

$$\alpha(e_{ij}) = \gamma^{i-j}e_{ij}, \quad (i, j = 1, \ldots, n).$$

Let $w = \sum_i e_{i+1,i}$. Then $w$ is a unitary in $N$ which satisfies that

$$w^j = \sum_i e_{i+j,i}, \quad w^{i*}e_{jj}w^i = e_{j-i,j-i} \quad \text{and} \quad \alpha(w) = \gamma w.$$

The following indicates that the inner conjugacy class of $N$ can take the max$L h(N|L)$, where $L$ is a subfactor of $M$ with $[M : L] = n$.

**Theorem 3.2.2.** Let $N \subset M$ be the above. Then there exists a unitary operator $u$ in $M$ which satisfies the following properties:

1. $h(N|uNu^*) = H_N(Ad u) = \log n$.

2. The conditional expectations $E_N$ and $E_{uNu^*}$ commute.

**Proof.** Let $w \in N$ be the unitary operator in 3.2.1. Let $v \in M$ be a unitary in $M$ implementing $\alpha$, that is, $\alpha(x) = vxv^*$ for all $x \in N$. We put

$$u = \frac{1}{\sqrt{n}} \sum_i w^i v^{i-1}.$$

Then $u$ is a unitary and $\sqrt{n}E_N(u) = w$. Since $H_N(u) = \log n$, we have by Theorem 3.1.4 that

$$h(N|uNu^*) \leq \log n.$$
As a finite partition of the unity, we choose \( \{p_j : p_j = e_{jj}, j = 1, \cdots, n\} \).

Then

\[
\begin{align*}
\frac{1}{n^2} \sum_j \sum_{i,l} \gamma^{2ki} w^k \alpha^{-l-i}(a) w^{-i} = E_N \left( \frac{1}{n} \sum_i \alpha^i(a) w^* \right) = E_N E_{uN}^* (a) = E_{uN}^* E_N (a).
\end{align*}
\]

Hence \( h(N|uNu^*) = \log n \).

(2) To show that \( E_{uN}^* E_N = E_N E_{uN}^* \), remark that for all \( a \in N \),

\[
E_{uN}^* (a v^k) = \frac{1}{n^2} \sum_j \sum_{i,l} \gamma^{2ki} w^k \alpha^{-l-i}(a) w^{-i} \]

Assume that \( k \neq 0 \). Then \( E_N (a v^k) = 0 \). On the other hand,

\[
E_N E_{uN}^* (a v^k) = \frac{1}{n^2} \sum_j \sum_{i,l} \gamma^{2ki} w^k \alpha^{-l-i}(a) w^{-i} \]

and

\[
\sum_i \sum_l \gamma^{2ki} w^{l-i} \alpha^{l-i}(a) w^{-i} = \sum_j \sum_{i,l} \gamma^{i} w^{i} \alpha^{j-i} (a) w^{j} = 0.
\]

Therefore,

\[
E_N E_{uN}^* (a v^k) = 0 = E_{uN}^* E_N (a v^k).
\]

for all \( a \in N \) and \( k = 1, \cdots, n - 1 \). Also for each \( a \in N \), we have that

\[
E_{uN}^* E_N (a) = E_N \left( \frac{1}{n} \sum_i \alpha^i(a) w^* \right) = E_N E_{uN}^* (a).
\]
These show that
\[ E_{uNu^*}E_N(x) = E_NE_{uNu^*}(x) \quad \text{for all} \quad x \in M. \]

\[ \square \]

3.2.3 A continuous family of subfactors with index 2

At the last, in the case of \( G = \mathbb{Z}_2 \), we show a result corresponding one in [4] for maximal abelian subalgebras of the type \( \text{I}_n \) factors \( M_n(\mathbb{C}) \).

**Theorem 3.2.3.** Let \( N \) be a type \( \text{II}_1 \) factor and let \( M \) be the crossed product \( N \rtimes_\alpha \mathbb{Z}_2 \) by an outer automorphism \( \alpha \) with the period 2. For the unitary \( w \in N \) in 3.2.1, let
\[
u(\lambda) = \sqrt{\lambda} w + \sqrt{1 - \lambda} v, \quad (0 \leq \lambda \leq 1).
\]
Then \( \nu(\lambda) \) is a unitary in \( M \) which satisfies the followings:

1. \( h(N \mid \nu(\lambda) N \nu(\lambda)^*) = H_N(\nu) = \eta(\lambda) + \eta(1 - \lambda). \)

and
\[
\{ h(N \mid \nu(\lambda) N \nu(\lambda)^*) : \lambda \in [0, 1] \} = [0, \log 2].
\]

2. \( N \subset M \cup \cup N \cap \nu(\lambda)Nu(\lambda)^* \subset \nu(\lambda)Nu(\lambda)^* \)

is a commuting square in the sense of [9] if and only if \( \lambda = \frac{1}{2} \).

3. \( h(N \mid \nu(\frac{1}{2}) N \nu(\frac{1}{2})^*) = H_N(Ad\nu(\frac{1}{2})) = \max_u h(N \mid uNu^*) = \log 2 \)

where \( u \) is a unitary in \( M \).
Proof. (1) It is clear that $H_N(Adu(\lambda)) = \eta(\lambda) + \eta(1-\lambda)$. Hence by Theorem 3.1.4, we have

$$h(N|u(\lambda)Nu(\lambda)^*) \leq \eta(\lambda) + \eta(1-\lambda).$$

We remark that for each $x \in N$,

$$E_{u(\lambda)Nu(\lambda)^*}(x) = u(\lambda)E_N(u(\lambda)^*xu(\lambda))u(\lambda)^* = \lambda w^*xw + (1-\lambda)\alpha(x).$$

Let $\{e_{ij}\}_{i,j=1,2} \subset N$ be the set of matrix units for $\alpha$ in 3.2.1. Then

$$E_{u(\lambda)Nu(\lambda)^*}(e_{ii}) = \lambda w^*e_{ii}w + (1-\lambda)\alpha(e_{ii}) = \lambda e_{i+1,i+1} + (1-\lambda)e_{ii}, \pmod{2}.$$ 

Hence, we have that for each $i = 1, 2$,

$$\tau\eta(E_{u(\lambda)Nu(\lambda)^*}(e_{ii})) = \frac{1}{2}(\eta(\lambda) + \eta(1-\lambda)), $$

so that

$$h(N|u(\lambda)Nu(\lambda)^*) \geq \eta(\lambda) + \eta(1-\lambda).$$

This implies that

$$h(N|u(\lambda)Nu(\lambda)^*) = \eta(\lambda) + \eta(1-\lambda).$$

(2) First we remember the following ; the diagram is a commuting square in the sense of [9] means that $E_N E_{uNu^*} = E_{uNu^*} E_N$.

Let $x \in N$. Since $\alpha(w) = -w^*$, we have that

$$E_N E_{u(\lambda)Nu(\lambda)^*}(x) = \lambda^2 x + 2\lambda(1-\lambda)w\alpha(x)w^* + (1-\lambda)^2 x$$

and

$$E_{uNu^*} E_N(x) = \lambda^2 x + 2\lambda(1-\lambda)w\alpha(x)w^* + (1-\lambda)^2 x + \sqrt{\lambda(1-\lambda)(2\lambda-1)}xwv$$

Hence $E_N E_{u(\lambda)Nu(\lambda)^*}(x) = E_{uNu^*} E_N(x)$ for all $x \in N$ if and only if $\lambda = 1/2$. Similarly, for all $x \in N$,

$$E_N E_{u(1/2)Nu(1/2)^*}(xv) = w\alpha(x_1) + x_1 w^* + x_1 \alpha(w) + \alpha(w^*x_1) = 0$$

and

$$E_{u(1/2)Nu(1/2)^*} E_N(xv) = 0.$$
These imply the conclusion.

(3) Since $N$ is a subfactor of $M$ with $[M : N] = 2$, we have that

$$h(N | uNu^*) \leq H(N | uNu^*) \leq H(M | N) = \log 2$$

so that

$$h(N | u(\frac{1}{2}) N u(\frac{1}{2})^*) = \log 2 = \max_u h(N | uNu^*).$$

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