Fusion Algebras Induced by Representations of the Modular Group

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Abstract

Using the representation theory of the subgroups $SL_2(\mathbb{Z}_p)$ of the modular group we investigate the induced fusion algebras in some simple examples. Only some of these representations lead to 'good' fusion algebras. Furthermore, the conformal dimensions and the central charge of the corresponding rational conformal field theories are calculated. Two series of representations which can be realized by unitary theories are presented. We show that most of the fusion algebras induced by admissible representations are realized in well known rational models.
1. Introduction

Starting with the seminal work of A.A. Belavin, A.M. Polyakov and A.M. Zamolodchikov in 1984 [1] there have been several Ansätze to classify rational conformal field theories (RCFT). In one approach extensions of the Virasoro algebra are obtained either by the free field construction starting from Kac-Moody algebras [2–4] or by other direct explicit methods (see [5, 6] and references therein). For extensions of the Virasoro algebra the modular properties of the highest weight representations and their fusion algebras have been investigated [7, 8]. In another approach one deals with abstract fusion algebras and tries to classify them [9].

In this paper we exclusively deal with fusion algebras induced by representations of the modular group where we call a fusion algebra induced if it can be calculated from a representation of the modular group using the Verlinde formula having chosen a vacuum state in the representation space. The aim of this approach is the classification of all physically relevant representations of the modular group and thus a classification of RCFTs. So far it seems that all representations which satisfy some natural criteria for ”good” fusion rules are realized in RCFT.

This paper is organized as follows. In chapter two we will discuss fundamental properties of RCFT which are important for later discussions. In chapter three we give a short review of the representations of special subgroups of the modular group. Then we investigate these representations in detail and show that most of the physically relevant representations are realized by some RCFT. Finally we summarize our results and point out some open questions.

2. Fundamental properties of RCFT

Let $\mathcal{R}$ be a rational conformal field theory with (extended) symmetry algebra $\mathcal{W}$ such that $\mathcal{W}$ contains the Virasoro algebra as a subalgebra. The finite set of $\mathcal{W}$-primary fields $\{\phi_i\}_{i=1}^n$ of $\mathcal{R}$ correspond to the highest weight representations $\mathcal{H}_i$ of $\mathcal{W}$. Let $h_i$ be the conformal dimensions of the primary fields $\phi_i$ and denote the vacuum representation by $\mathcal{H}_\nu$ with $h_\nu = 0$. The conformal characters of the representations $\mathcal{H}_i$ read

$$\chi_i(\tau) := tr_{\mathcal{H}_i}(q^{L_0 - c/24}) \quad \text{with} \quad q := e^{2\pi i \tau}$$  \hspace{0.8cm} (2.1)

where $c$ is the central charge of the theory and $\tau$ is the modular parameter. These conformal characters form a finite dimensional (right) representation $r$ of the modular group $SL_2(\mathbb{Z})$. This group is generated by the two transformations

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$  \hspace{0.8cm} (2.2)

and acts on the characters as

$$(r(A)\chi_i)(\tau) := \chi_i\left(\frac{a\tau + b}{c\tau + d}\right) \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$  \hspace{0.8cm} (2.3)
The modular transformation $T$ is represented by a diagonal matrix with nonvanishing elements and $S^2$ by the identity matrix, if one chooses the conformal characters (2.1) as a basis of the representation space. Note that $r(S)^2 = \mathbb{I}$ is only true if one considers the characters as $q$-dimensions of the representations $\mathcal{H}_i$ (if the characters additionally take account of charge quantum numbers, $r(S)^2$ equals the 'conjugation' in the fusion algebra which is a matrix of order 2 (see e.g. [9]).

Since we are considering a rational theory the dimensions $h_i$ of the primary fields and the central charge $c$ are rational [10]. This implies the existence of a positive integer $m$ such that

$$r(T^m) = \mathbb{I}. \quad (2.4)$$

If the kernel of the representation $r$ of the modular group contains a principal congruence subgroup $\Gamma_m$ it factorizes to a representation of $SL_2(\mathbb{Z}/m)$ and (2.4) is evident. This is true for all explicitly known examples so that we will restrict ourselves to representations of $SL_2(\mathbb{Z}/m)$.

If the representation $r$ is known explicitly one can calculate the fusion algebra

$$\phi_i \times \phi_j = \sum_{k=1}^n N_{ij}^k \phi_k. \quad (2.5)$$

using the famous Verlinde formula [8]

$$N^{(\nu)}_{ij} = \sum_{m=1}^n \frac{r(S)_{i,m} r(S)_{j,m} r(S^{-1})_{m,k}}{r(S)_{\nu,m}} \quad (2.6)$$

where we have written the dependence of the fusion coefficients on the vacuum state explicitly. In order to be able to interpret the fusion coefficients $N^{(\nu)}_{ij}$ as dimensions of the corresponding intertwiner spaces these coefficients should be nonnegative integers. This is what we call a 'good' fusion algebra.

Assume that the conformal dimensions $h_i$ of the primary fields are nondegenerate ($i \neq j \Rightarrow h_i - h_j \notin \mathbb{Z}$). Denote the lowest energy representation by $\mathcal{H}_\lambda$ ($i \neq \lambda \Rightarrow h_\lambda < h_i$), which in the unitary case is the vacuum representation. Then the quantum dimensions of the representations $\mathcal{H}_i$ with respect to $\mathcal{H}_\lambda$ are given by

$$\delta_k = \lim_{q \to 1} \frac{\chi_k(q)}{\chi_\lambda(q)} = \lim_{\tau \to 0} \frac{\chi_k(\tau)}{\chi_\lambda(\tau)} = \lim_{\tau \to i \infty} \frac{\chi_k(-\frac{1}{2})}{\chi_\lambda(-\frac{1}{2})} = \lim_{\tau \to i \infty} \frac{(r(S) \chi_k)(\tau)}{(r(S) \chi_\lambda)(\tau)} = \frac{r(S)_{\lambda,k}}{r(S)_{\lambda,\lambda}}. \quad (2.7)$$

We have used that the conformal characters are dominated in the large $Im(\tau)$ limit by the leading term $q^{h_i - \frac{c}{2 h_i}}$ and that the main contribution is given by the lowest energy state. Since the quantum dimensions describe the relative dimensions of the representations $\mathcal{H}_i$ they are positive reals. For unitary theories one knows (see e.g. [11]) that for

$$\delta_k < 2 \Rightarrow \delta_k \in \{2 \cos\left(\frac{\pi}{j+2}\right)\}_{j \in \mathbb{N}}. \quad (2.8)$$
Furthermore, the quantum dimensions $\delta_k$ satisfy the fusion algebra

$$\delta_i \delta_j = \sum_{k=1}^{n} N_{ij}^k \delta_k. \quad (2.9)$$

Starting from this setup one possible way to classify RCFT arises from the representation theory of $SL_2(\mathbb{Z}_m)$: given a representation of $SL_2(\mathbb{Z}_m)$, what RCFT corresponds to it, if any? In particular, the following questions occur:

- Is it possible to identify the lowest energy representation such that the resulting quantum dimensions are positive reals and lead for the case of unitary theories to the ones given by (2.8)?
- Is there an element of the representation space corresponding to the vacuum, such that the fusion coefficients defined by (2.6) are positive integers?

The irreducible representations of $SL_2(\mathbb{Z}_m)$ can be obtained using Weil representations (c.f. [12]). A Weil representation is a representation of $SL_2(\mathbb{Z})$ in the space of complex-valued functions over a finite $\mathbb{Z}_m$ module. In these representations $T$ is represented by a pointwise multiplication and $S$ by a discrete Fourier transformation. For more details as well as for the mathematical definition of Weil representations we refer to [12]. Because it is always possible to represent $r(T)$ unitarily one can choose a basis such that $r(T)$ becomes diagonal.

If for a special representation the answers to these two questions are positive we will call the representation "conformally admissible". From now on this kind of representations will just be called admissible.

In the following we will concentrate to the case of irreducible representations of $SL_2(\mathbb{Z}_m)$. Though there are well known RCFT which lead to reducible representations of the modular group a reduction to the irreducible case might be possible. For example, RCFT with nondegenerate conformal dimensions $h$ lead to irreducible representations of the modular group. Even some representations with degenerate conformal dimensions like e.g. the Virasoro minimal models corresponding to nondiagonal partition functions contain irreducible admissible subrepresentations. The restriction to irreducible representations is in this case justified because the subrepresentations are realized on $W$-algebra characters [6, 17]. Similarly, for particular fractional level Kac-Moody algebras with reducible representations (cf. the example in [13]) one irreducible part is admissible and realizes a Virasoro minimal model.

The irreducible representations of $SL_2(\mathbb{Z}_m)$ are known. To construct them it is sufficient to know the irreducible representations of $SL_2(\mathbb{Z}_p)$ where $p$ is a prime power. Since the investigation of fusion algebras induced by representations of $SL_2(\mathbb{Z}_p)$ ($p$ an odd prime) is very simple for representations with nondegenerate conformal dimensions we will further restrict on this case in the next chapters.
3. Irreducible representations of $SL_2(\mathbb{Z}_p)$ with $p$ an odd prime

The irreducible representations of $SL_2(\mathbb{Z}_p)$ can be classified by their characters (characters are understood in the group theoretical sense, and have to be distinguished from the conformal characters (2.1)). Using the notation of L. Dornhoff [15] there are the following inequivalent representations:

| representation | $\mathbb{I}$ | $\psi$ | $\chi_i$ | $\theta_j$ | $\xi_1, \xi_2$ | $\eta_1, \eta_2$ |
|----------------|------------|--------|-----------|-----------|-------------|-------------|
| dimension      | 1          | $p$    | $p + 1$   | $p - 1$   | $\frac{1}{2}(p + 1)$ | $\frac{1}{2}(p - 1)$ |
| $r(S^2)$       | $\mathbb{I}$ | $\mathbb{I}$ | $(-1)^i \mathbb{I}$ | $(-1)^j \mathbb{I}$ | $(-1)^{\frac{p-1}{2}} \mathbb{I}$ | $(-1)^{\frac{p+1}{2}} \mathbb{I}$ |

with $1 \leq i \leq \frac{1}{2}(p - 3)$, $1 \leq j \leq \frac{1}{2}(p - 1)$. (3.1)

To avoid confusion we tolerate by convention that both the conformal characters and certain irreducible representations of $SL_2(\mathbb{Z}_p)$ have been denoted by $\chi_i$. The table shows that there are altogether $p + 4$ inequivalent irreducible representations of $SL_2(\mathbb{Z}_p)$. We have also listed representations with $r(S^2) \neq \mathbb{I}$ since we will consider in the sequel equivalence classes of representations which differ only by a one dimensional representation of the modular group. These one dimensional representations can be realized by powers of the Dedekind eta function (for a detailed discussion see chapter 4).

The trivial representation $\mathbb{I}$ is of no real interest since the corresponding quantum dimension and fusion algebra is trivial. We will now consider some of the nontrivial representations in general and verify that most of the admissible representations are realized in some RCFT. To this end we have worked out the explicit form of $r(T)$ and $r(S)$ where $r$ is one of the representations in the table. This is done by using quadratic modules and the Weil representations (for the general outline see [12]). The explicit form of $r(T)$ shows for all representations except $\chi_i$ that in the corresponding RCFT the conformal dimensions $h_j$ cannot be degenerate, because the diagonal elements of $r(T)$ are pairwise different. The particular form of the matrices $r(T)$ and $r(S)$ for the representations is given below.

Because $T$ should be represented by a diagonal matrix and $S$ by a unitary one the only remaining freedom in the choice of a basis in the nondegenerate case is a conjugation with a diagonal matrix whose entries are roots of unity. If one fixes a possible lowest energy state ($\lambda$) of the representation space corresponding to $H_\lambda$ with

$$r(S)_{k,\lambda} \neq 0 \quad 1 \leq k \leq n,$$

(3.2)

the basis is completely determined by the requirement of positive real quantum dimensions (condition (3.2) is necessary in order to give well defined quantum dimension (c.f. (2.5))).

The set of all ($\lambda$) which obey (3.2) will be denoted by $F_r$. In the next step one can fix another element ($\nu$) of the representation space corresponding to the vacuum representation $H_\nu$ and ask whether this choice leads to ”good” fusion rules defined by the Verlinde formula. This is the program we will follow for some simple examples in the next chapter.
4. Explicit results

We now investigate which irreducible representations of $SL_2(\mathbb{Z}_p)$, where $p$ is an odd prime, are admissible.

The representations $\eta_{1,2}$

Consider the representations $\eta_{1,2}$. The dimensions of the representation spaces are $n = \frac{1}{2}(p - 1)$ and the representations are given by the matrices

$$r(T)_{k,k} = e^{\frac{2\pi i}{p}ak^2}$$
$$r(S)_{k,j} = \frac{2i}{\sqrt{p}}\left(\frac{a}{p}\right)\epsilon(p)\sin\left(\frac{2\pi}{p}2akj\right)$$

$$1 \leq k, j \leq n$$

where $\epsilon(p) = \begin{cases} 1, & p \equiv 1 \ (mod \ 4) \\ i, & p \equiv 3 \ (mod \ 4) \end{cases}$

and $(\frac{a}{p})$ is the Legendre symbol ($(\frac{a}{p})$ is 1 if $a$ is a square in $\mathbb{Z}_p$ and -1 otherwise). The two different representations are obtained for two values of $a$ namely $\eta_1$ if $a$ is a square and $\eta_2$ if $a$ is no square in $\mathbb{Z}_p$. Note that for $p$ odd but not prime and $a \in \mathbb{Z}_p$ invertible, eq. (4.1) also defines a representation of $SL_2(\mathbb{Z}_p)$.

The choice of an element $(\lambda)$ which obeys (3.2) is free, but fixes already the element $(\nu)$, if one requires the fusion coefficients to be positive integers. If the pair $(\lambda,\nu)$ leads to non-negative fusion coefficients the exchanged pair $(\nu,\lambda)$ also gives positive fusion coefficients. The corresponding conformal dimensions $h_i$ for a given vacuum state $(\nu)$ are obtained from (4.1)

$$h_k \equiv \frac{a}{p}(k^2 - \nu^2) \ (mod \ \mathbb{Z}).$$

Using the well known relation between the central charge $c$ and the conformal dimensions $h_i$ the central charge is given by [16]

$$c \in \frac{24}{n} \sum_{i=1}^{n} h_i - 2(n - 1) + \frac{4}{n}(\mathbb{N}\setminus\{1\}) = 3 - p - \frac{24a}{p}\nu^2 + \frac{48}{p - 1}\mathbb{Z} + \frac{4}{n}(\mathbb{N}\setminus\{1\})$$

In the following table we list the possible pairs $(\lambda,\nu)$ for $p = 3, 5$ and 7:

| $p$ | $\eta_1$       | $\eta_2$       |
|-----|----------------|----------------|
| 3   | (1,1)          | (1,1)          |
| 5   | (1,1), (2,2)   | (1,2), (2,1)   |
| 7   | (1,2), (2,1), (3,3) | (1,3), (2,2), (3,1) |
where we choose $a = 1$ to define $\eta_1$ and $a = \min\{m \in \mathbb{N}| \left(\frac{m}{p}\right) = -1\}$ to define $\eta_2$.

A representation can only be realized by a unitary RCFT if the states $(\lambda)$ and $(\nu)$ are equal, so that for example for $p = 5$ the representation $\eta_2$ cannot be realized in any unitary RCFT. For the above admissible representations which could be realized by unitary RCFT the quantum dimensions are

\[
\begin{align*}
p = 3: & \quad \delta_{1,2}^{\eta_{1,2}} = 1 = 2\cos\left(\frac{\pi}{3}\right) \\
p = 5: & \quad \delta_{1,2}^{\eta_{1,2}} \in \{1, 2\cos\left(\frac{\pi}{5}\right)\} \\
p = 7: & \quad \delta_{1,2}^{\eta_{1,2}} \in \{1, 2\cos\left(\frac{\pi}{7}\right), \beta\}
\end{align*}
\]

and $2 < \beta$ is the largest positive root of $x^3 - 2x^2 + 1$.

This implies that all of the above representation with $(\lambda) = (\nu)$ can be realized by a unitary theory.

Looking for candidates of unitary theories one finds for every $p$ exactly two possible ones. These series are given by (4.1) with $a = 1$ and $a = -1$ for $\lambda = \nu = \frac{1}{2}(p - 1)$. It is easy to calculate the corresponding quantum dimensions

\[
\begin{align*}
\delta_{1/2(p-1)} = 1 \quad , \quad \delta_1 = 2\cos\left(\frac{\pi}{p}\right) \quad , \quad \delta_k > 2 \quad \text{for} \quad 1 < k < \frac{1}{2}(p - 1). \quad (4.5)
\end{align*}
\]

The cases $p = 3, 5, 7$ are in complete agreement with the results of [9] and the possible fusion algebras in the above table lead exactly to the $h$- and $c$-values given in the classification of fusion algebras corresponding to RCFT with one, two and three conformal characters in [9].

There are two remarkable facts concerning these fusion algebras. First, the fusion algebras induced by $\eta_1$ and $\eta_2$ are equivalent. This is seen using the simple recurrence relation

\[
\mathcal{N}(\nu)_{i+n,j+v}^{\hat{k}} = \mathcal{N}(\nu)_{i,j}^{\hat{k}} + \delta_{0,i+j+v-k} - \delta_{0,i+j+v+k}
\]

with

\[
\hat{k} = \begin{cases} k, & k \leq n \\ p-k, & \text{else} \end{cases} \quad \delta_{a,b} = \begin{cases} 1, & a \equiv b \ (mod \ p) \\ 0, & \text{else} \end{cases}.
\]

which follows from standard trigonometric identities. This relation is independent of $a$ so that the representations $\eta_1$ and $\eta_2$ induce the same fusion algebra.

Secondly, one finds that all fusion algebras for different elements $(\nu)$ of the representations $\eta_{1,2}$ are equivalent

\[
\mathcal{N}(1)_{i,j}^{\hat{k}} = \pm \mathcal{N}(\nu)_{\nu_i, \nu_j}^{\hat{k}} \quad \forall \nu \in \mathcal{F}_{\eta_{1,2}} \quad (4.7)
\]
where $F_{\eta_1,2}$ is given by

$$\{a \mid 1 \leq a \leq n \ , \ (a, p) = 1 \} \tag{4.8}$$

which clearly equals $M_n = \{1, \ldots, n\}$ if $p$ is a prime.

Similar symmetry properties will occur for most of the representations we consider.

We have seen that the representations $\eta_1, 2$ are indeed admissible, but: are they realized by a RCFT? Looking closer at $r(S)$ and $r(T)$ in (4.1) one observes that for $2a \equiv 1 \pmod{p}$ these matrices are up to a phase equal to the ones for the Virasoro minimal models with central charge $c = c(2, p)$. The phase can be understood in the following way. It is always possible to write the conformal characters as

$$\chi_j = \eta^{-\alpha} \Lambda_j \tag{4.9}$$

where $\eta$ is the Dedekind eta function and the $\Lambda_j$ are cusp forms of weight $\frac{\alpha}{2} \in \mathbb{N}$ if $\alpha$ is large enough. There are two possible natural choices for $\alpha$. Either $\alpha$ is chosen minimal or such that the representation of the modular group on the $\Lambda_j$ factorizes to a representation of $SL_2(\mathbb{Z}_m)$ with minimal $m$. In both cases the representation $r$ in the space of conformal characters factorizes into a one-dimensional one acting on $\eta^{-\alpha}$ and an $n$-dimensional one acting on the modular forms $\Lambda_j$. Writing the conformal characters of the Virasoro minimal models $c(2, p)$ as in (4.9) with

$$\alpha = \alpha(p) = \begin{cases} 4, & p \equiv 7 \pmod{8} \\ 10, & p \equiv 5 \pmod{8} \\ 16, & p \equiv 3 \pmod{8} \\ 22, & p \equiv 1 \pmod{8} \end{cases} \tag{4.10}$$

leads to a representation on the $\Lambda_j$ given by (4.1) with $2a \equiv 1 \pmod{p}$. This generalizes: even for $p$ odd but not prime and $2a \equiv 1 \pmod{p}$ (4.1) leads to the correct $r(S)$ and $r(T)$ matrices for the Virasoro minimal models with central charge $c = c(2, p)$. Using the notation of [9] this implies that the representations $\eta_1, 2$ of $SL_2(\mathbb{Z}_p)$ induce fusion algebras of type $B_{\frac{p-3}{2}}$.

For $p = 5$ and $a = 1, -1$ there are two possible candidates for unitary RCFT as can be seen from table 2. These two are realized by the level 1 WZW models corresponding to $G_2$ and $F_4$ with $\alpha = 22$ and 10, respectively. Both theories lead to the same representations of the modular group but their $h$-values differ (see eq. (4.2)). We have not yet found a realization for the other members of the two unitary series.

The representations $\xi_{1,2}$

The dimension of the representation spaces is $n = \frac{1}{2}(p + 1)$ and $r(T)$ and $r(S)$ are given
by 
\[ r(T)_{k,k} = e^{2\pi i a k^2/p} \]
\[ r(S)_{k,l} = \frac{2}{\sqrt{p}} \epsilon(p) \left( \frac{a}{p} \right) \cos \left( \frac{2\pi}{p} 2akl \right) \quad 1 \leq k, l \leq n - 1 \]
\[ r(S)_{0,l} = r(S)_{l,0} = \frac{2}{\sqrt{p}} \epsilon(p) \left( \frac{a}{p} \right) \quad l \neq 0 \]
\[ r(S)_{0,0} = \frac{1}{\sqrt{p}} \epsilon(p). \]

In complete analogy to \( \eta_{1,2} \) one obtains the two different representations \( \xi_{1,2} \) with two values of \( a \) namely \( \xi_{1} \) with \( \left( \frac{a}{p} \right) = 1 \) and \( \xi_{2} \) with \( \left( \frac{a}{p} \right) = -1 \). Furthermore (4.11) also gives representations for nonprime odd \( p \).

Using standard trigonometric identities one obtains for the fusion coefficient with respect to \( \nu = 0 \)
\[ \mathcal{N}(0)_{0,j}^k = \delta_{jk} \]
\[ \mathcal{N}(0)_{i,j}^k = \begin{cases} \frac{1}{\sqrt{2}}, & i \pm j \pm k \equiv 0 \ (mod \ p) \\ 0, & \text{otherwise} \end{cases} \text{ if } i, j, k \neq 0. \] (4.12)

Note that these equations are independent of \( a \). For \( \nu \neq 0 \) one obtains
\[ \mathcal{N}(\nu)_{ij}^k \in \{0, \pm 1, \pm \frac{1}{\sqrt{2}}, \pm \sqrt{2}\} \] (4.13)

and
\[ \mathcal{N}(1)_{ij}^k = \pm \mathcal{N}(\nu)_{i\nu, j\nu}^{\nu k} \quad \forall \nu \neq 0. \] (4.14)

The calculation of the fusion algebras shows that this series of representations cannot be realized by RCFT.

One can show, however, that direct sums of two representations \( \xi \) exist such that the resulting fusion algebra is integer valued. In order to ensure the fusion coefficients to be integers one has to use two representations for odd numbers \( p \) and \( \tilde{p} \) with \( p - \tilde{p} = 4 \) and perform a unitary change of basis mixing the two states labelled by 1. These fusion algebras are realized by the parabolic \( \mathcal{W}(2, 8k) \) algebras with \( 4k \) odd [14].

The representations \( \theta_j \)

To give the explicit form of the matrices \( r(T) \) and \( r(S) \) we have to fix some notation. Let \( \mathbb{F}_p \) the field obtained from \( \mathbb{Z}_p \) by quadratic extension, \( \epsilon \) be a generator of the cyclic group \( \mathbb{F}_p^* \) and \( \rho_j = e^{2\pi i j/ p+1} \). Using this notation the matrices \( r(T) \) and \( r(S) \) are given by
\[ r(T)_{k,k} = e^{2\pi i a k^2/p} \]
\[ r(S)_{k,l} = \frac{1}{p} \sum_{m=0}^{p} \rho_j^m e^{2\pi i l/ p} e^{2\pi i k/ p} Re(e^{i(p-1)m} e^{i k/ p}) \] (4.15)
\[ 0 \leq k, l \leq p - 2. \]
The inequivalent representations $\theta_j$ are obtained for $1 \leq j \leq \frac{1}{2}(p - 1)$. Calculating the fusion coefficients for these representations for $p = 3, 5, 7, 11$ leads to admissible representations only if $p \equiv 5 \pmod{6}$ and $j = \frac{1}{3}(p + 1)$. Continuing this series one finds admissible representations for $p = 17$ and 23. As in the case of the representations $\eta_{1,2}$ the choice of an element $(\lambda)$ already fixes the state $(\nu)$. Since these states are always different, these representations cannot be realized by a unitary RCFT. The corresponding $h$- and $c$-values can be read off from (4.15) using (4.3). Furthermore, all fusion algebras obtained for different elements $(\nu)$ from these representations are equivalent, since

$$\mathcal{N}(1)_{i,j}^k = \pm \mathcal{N}(\nu)_{\sigma_{\nu}(i)\sigma_{\nu}(j)}^{\sigma_{\nu}(k)}$$

$$\sigma_{\nu}(k) = k + \nu \pmod{p - 1}. \quad (4.16)$$

In contrast to the representations $\eta_{1,2}$ there seems to be no canonical extension of these four examples since calculations show that for $23 < p \leq 167$ the resulting fusion algebras are not integer valued.

The case $p = 3$ corresponds to a theory with a single character. These theories have been classified completely in ref. [21]. For $p = 5$ one obtains that $\theta_2 = \eta_1 \otimes \eta_2$ such that this representation is realized in at least two RCFT. It is known that the representations with $p = 11, 17, 23$ are realized by the exceptional $\mathcal{W}(2, \delta)$ algebras with $\delta = 4, 6, 8$ and $c_{\text{eff}} = \frac{4(\delta - 1)}{3\delta - 1}$ [17, 18, 19]. For the realization of the conformal characters of the corresponding $\mathcal{W}(2, \delta)$ algebras as well as for more details we refer to [19]. The explicit calculations of the fusion algebras for $p = 5, 11, 17, 23$ show that the maximal fusion coefficients for the corresponding representations are $1, 3, 6, 12$.

Since the other representations $\theta_j$ for $p = 3, 5, 7, 11$ and all $\theta_j$ for $p = 29$ are not admissible, we expect that the ones mentioned above are the only admissible representations.

**The representation $\psi$**

The dimension of the representation space is $n = p$ and the representation is given by

$$r(T)_{k,k} = e^{2\pi i k}$$

$$r(S)_{k,l} = \frac{1}{p} \sum_{a=1}^{p-1} e^{2\pi i \frac{ak(a^{-1} + 1)}{p}} 1 \leq k, l \leq p - 1$$

$$r(S)_{k,0} = r(S)_{0,k} = \sqrt{p + 1} \quad k \neq 0$$

$$r(S)_{0,0} = -\frac{1}{p}. \quad (4.17)$$
A simple calculation leads to the fusion coefficients with respect to \( \nu = 0 \)

\[
\mathcal{N}(0)_{0,j}^k = \delta_{jk}
\]

\[
\mathcal{N}(0)_{i,j}^k = \frac{1}{\sqrt{p+1}}(f_{i,j}^k - 1) \quad \text{if } i, j, k \neq 0
\]

with

\[
f_{k,l}^m = \frac{1}{p} \left( \sum_{a, b, c \in \mathbb{Z}_p^* \atop a + b + c \equiv 0 \pmod{p}} e^{2\pi i (a^{-1}k + b^{-1}l + c^{-1}m)} - 2 \right) \in \{0, \pm 1\}.
\]

\[\text{(4.18)}\]

For \( \nu = 1 \) one obtains

\[
\mathcal{N}(1)_{i,j}^k \in \mathbb{Q}\left( \frac{1}{\sqrt{p+1}} \right)
\]

so that the smallest field containing the fusion coefficients is \( \mathbb{Q}\left( \frac{1}{\sqrt{p+1}} \right) \). Furthermore we have

\[
\mathcal{N}(1)_{i,j}^k = \pm \mathcal{N}(\nu)_{i,j}^k \quad \forall \nu \neq 0.
\]

\[\text{(4.19)}\]

This shows that there are no RCFT corresponding to the representation \( \psi \).

The representations \( \chi_l \)

Let \( b \) be a generator of the cyclic group \( \mathbb{Z}_p^* \) and \( \rho_l = e^{2\pi i \frac{l}{p-1}} \). With this notation the matrices \( r(T) \) and \( r(S) \) for the representations \( \chi_l \) read

\[
r(T)_{j,j} = e^{2\pi i \frac{j}{p}}
\]

\[
r(S)_{j,k} = \frac{1}{p} \sum_{m=1}^{p-1} \rho_l^m e^{2\pi i \frac{b^m j + b^{-m} k}{p}} \quad \text{for } 0 \leq j, k \leq p - 1
\]

\[\text{(4.21)}\]

\[
r(S)_{p,k} = (-1)^k r(S)^*_{k,p} = \frac{1}{p} \sum_{m=1}^{p-1} \rho_l^m e^{2\pi i \frac{b^{-m}}{p}} \quad \text{for } 0 \leq k \leq p - 1
\]

\[\text{and } r(S)_{p,p} = 0.
\]

The diagonal elements of \( r(T) \) are not pairwise different, so that corresponding RCFT must have degenerate \( h \)-values. We have not investigated this case in detail, but we found no admissible representations for \( p = 5, 7, 11 \).

We should remark that the new symmetry property (4.7), (4.14), (4.16), (4.20) arises not only in these four examples but also for fractional level Kac-Moody algebras (cf. the example presented in [13]) and for parabolic \( W \)-algebras [20].
5. Summary and open questions

We have presented a new approach to the classification of RCFT starting from irreducible representations of the modular group. For the case of $SL_2(\mathbb{Z}_p)$ with $p$ an odd prime we determined the possible physically relevant representations and found that most of these cases are realized by well-known RCFT. Furthermore, we observed a new symmetry property of fusion algebras induced by representations of the modular group.

In the future more general investigations of physically relevant representations might lead to a better understanding of the representations of the modular group and RCFT. In particular, it should be possible to derive a classification of all representations corresponding to RCFT with nondegenerate conformal dimensions. In this context it would be interesting to understand the connection between known RCFT and reducible representations of the modular group.

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