ON A REMARKABLE EXAMPLE OF F. ALMGREN AND H. FEDERER IN THE GLOBAL THEORY OF MINIMIZING GEODESICS

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To Luis Caffarelli in his 70th birthday, with admiration.

Abstract. We present an exposition of a remarkable example attributed to Frederick Almgren Jr. in [30, Section 5.11] to illustrate the need of certain definitions in the calculus of variations.

The Almgren-Federer example, besides its intended goal of illustrating subtle aspects of geometric measure theory, is also a problem in the theory of geodesics. Hence, we wrote an exposition of the beautiful ideas of Almgren and Federer from the point of view of geodesics.

In the language of geodesics, the Almgren-Federer example constructs metrics in $S^1 \times S^2$, with the property that none of the Tonelli geodesics (geodesics which minimize the length in a homotopy class) are Class-A minimizers in the sense of Morse (any finite length segment in the universal cover minimizes the length between the end points; this is also sometimes given other names). In other words, even if a curve is a minimizer of length among all the curves homotopic to it, by repeating it enough times, we get a closed curve which does not minimize in its homotopy class.

In that respect, the example is more dramatic than a better known example due to Hedlund of a metric in $T^3$ for which only 3 Tonelli minimizers (and their multiples) are Class-A minimizers.

For dynamics, the example also illustrates different definitions of “integrable” and clarifies the relation between minimization and hyperbolicity and its interaction with topology.

1. Introduction. The paper [30], lays the foundation of the theory of flat chains. In Section 5.11 it presents a very remarkable example (attributed there to F. Almgren Jr. and which we will henceforth call Almgren-Federer example).

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The role of the Almgren-Federer example in [30] is to illustrate the need of considering rather general objects (flat chains) in the calculus of variations in geometric measure theory. The theory developed in [30] shows that certain functionals have minimizers in the set of flat chains. The example shows that, in general, the minimizers cannot be much simpler objects.

This Almgren-Federer example is a geometric problem and its properties are illuminating also for several other questions in the calculus of variations (e.g. for the theory of geodesics and for problems in Hamiltonian dynamics). It can also serve as motivation for some of the definitions in the theory of minimizing measures [43, 40]. Tentatively, we also believe it sheds some light in the problem of homogenization on quasi-periodic media or the problems of statistical mechanics in quasi-crystals.

The extremely precise and beautiful (but multi-layered) language of geometric measure theory may be an impediment for many readers to appreciate the beauty of the example and to appreciate its role on other contexts in which it is also relevant. The goal of this paper is to present the ideas behind the example in a way that it is accessible to practitioners in other fields (e.g. mechanics or geodesic flows) for which the example is highly relevant. In particular, we present applications to the theory of geodesics.

In the language of the theory of geodesics, the Almgren-Federer example constructs metrics in $S^1 \times S^2$, for which none of the Tonelli geodesics (periodic geodesics which minimize the length in a homotopy class, see Appendix C) is a Class-A minimizer in the sense of [47] (any finite length segment of the lift of the orbit to the universal cover minimizes the length between the two end points of the segment.) See Appendix B.

It is interesting to compare the Almgren-Federer example with the better known Hedlund example [31, 36]. In the Hedlund example, three Tonelli geodesics (and their multiples) are Class-A, whereas in the Almgren-Federer example, none of the Tonelli minimizers is Class-A. As a matter of fact, in the Almgren-Federer example, we give a characterization of the Class-A geodesics, none of which is periodic. Of course, the mechanism in Almgren-Federer example is very different from the mechanism in [31].

The paper [36] studies the global dynamical properties of Hedlund example. This study shows that there are geodesics with surprising properties (e.g. remaining close to each of the minimizers for very long segments). The paper [6] presents a reworking of the results of [47, 31] on 2-D manifolds from a new very powerful point of view that also allows to obtain many results on twist mappings and solid state models that were originally obtained by [41, 5]. We also call attention to [7] which develops relations between the theory of geodesics and the theory of 1-currents and to [8].

1.1. **Organization of the paper.** In Section 2 we present the Almgren-Federer example and call attention to some geometric properties. In Section 3 we state formally the main results of the paper: the fact that Tonelli orbits are not Class-A

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1 The Wikipedia entry for H. Federer says about [29] “the book’s unique style exhibits a rare and artistic economy that still inspires admiration, respect–and exasperation”.

2 In the literature, this property is referred by many other names such as minimizer, local minimizer, global minimizer, etc. Riemannian geometers call them “lines”. Unfortunately, these names are not used consistently for the same concepts by different authors, so we prefer to revert to the names used in [47].
and the characterization of Class-A geodesics for the example. We also provide some heuristic intuition on why the results should be true.

The proofs of the results are presented in Sections 4 and 5. Given that the aim of the paper is pedagogical, besides the precise proofs we have included many heuristic comments. We hope that they are not too distracting, but if they are, the reader is encouraged to skip them.

In the final Section 6 we indulge in some heuristic explanation of the possible relevance of the ideas in Almgren-Federer example in other areas such as quasiperiodic media.

In Appendix A, we collect some of the theory of geodesic flows. It turns out that the Almgren-Federer example also illustrates the relation between different notions of integrability in Hamiltonian systems.

In Appendices B and C, we present the classical definitions of Tonelli and Morse and some of the classical results on global calculus of variations of paths. This material is, of course classic, but we have included notes. In this paper, we will use mainly methods of calculus of variations, but in some points, we will make reference to the geodesic flow. Standard references are [51, 1, 4]. Some analysis at the Almgren-Federer example based on properties of the geodesic flow is in Appendix D.

It is a pleasure to dedicate this paper to Luis A. Caffarelli, master of geometric arguments in the calculus of variations. “Vamos a hacer cuatro rayas”. We thank V. Bangert and I. Belegradek for comments that improved the paper. The anonymous referee made several suggestions which improved the paper.

2. Geometric description of Almgren-Federer example. Following the presentation in [30] we consider the sphere \( S^2 \), written explicitly as:

\[
\{ (y, z) \mid y \in [-1, 1], z \in \mathbb{C}, y^2 + |z|^2 = 1 \}.
\]

(1)

We also consider the circle \( S^1 \) on which we define the coordinate \( x \).

We consider the manifold \( \mathcal{M} = \mathbb{R} \times S^2 \), as well as the quotient manifold

\[
\mathcal{N} = \mathbb{R} \times S^2 / \sim
\]

where

\[
(x + 1, y, z) \sim (x, y, z e^{2\pi i \omega})
\]

(2)

for some \( \omega \in \mathbb{R} - \mathbb{Q} \).

Clearly, the manifold \( \widetilde{\mathcal{M}} \) is the universal cover of \( \mathcal{M} \) and \( \mathcal{M} \) is diffeomorphic to \( S^1 \times S^2 \). The only homology (or homotopy) of \( \mathcal{M} \) is associated to the \( S^1 \) factor.

Remark 1. For dynamicists, the manifold \( \mathcal{M} \) is the suspension manifold \([33, 55]\) of the rotation in \( S^2 \) given in the coordinates in (1) by

\[
(y, z) \mapsto (y, z e^{2\pi i \omega}).
\]

(3)

Following this intuition, for people with background in dynamics, it may be useful to imagine \( \mathcal{M} \) as \( S^2 \) evolving in time – time is the variable \( x \) in the \( S^1 \) factor – but coming back to itself rotated by the irrational rotation (3). Of course, other physical interpretations may be useful.

The effects of the example, may be understood in this interpretation because we are trying to approximate a problem involving an irrational rotation by periodic orbits.
Remark 2. We can consider $x, y, z$ coordinates of the manifolds also as functions ([30] makes a distinction between the coordinates and the functions returning the value of the coordinates, but we will not make this distinction).

Remark 3. The paper [30] formulates the quotient (2) slightly differently. It fixes $\omega = 1/(2\pi)$.

To describe the geometry in pictorial terms it is convenient to think of $S^2$ embedded in $\mathbb{R}^3$ in such a way that $y$ is the vertical coordinate. We can use the geographical notation and we will call the set $\{y = 0\}$, the equator and the sets $\{y = \text{cte}\}$, the parallels. We will also find it convenient to introduce the longitude $\sigma$ by $z = (1 - y^2)^{1/2}e^{i\sigma}$. Note that in a parallel of constant $y$, the longitude ranges over the interval $[0, 2\pi)$, with both ends identified.

We will think of the coordinate $x$ as a time that advances.

The paper [30] endows $\mathbb{R} \times S^2$, the universal cover of $\mathcal{M}$, with a length element (the square root of a Riemannian metric).

\[(1 + y^2)(dx^2 + dy^2 + dz^2)^{1/2}. \tag{4}\]

The metric (4) is invariant under rotations around the $y$-axis. It is also symmetric under translations in the $x$ axis. Hence, it is invariant under the equivalence (2) and, hence, it defines a metric on $\mathcal{M}$.

Note that the coordinates $x, y, z$ are an orthogonal system of coordinates.

The important property for us is that for a fixed $x$, the resulting $S^2$ has an "hourglass" shape. The length of a parallel at height $y \in [-1, 1]$, is

\[2\pi(1 + y^2)\sqrt{1 - y^2} = 2\pi\sqrt{(1 + y^2)(1 + y^2)(1 - y^2)} = 2\pi\sqrt{1 + y^2 - y^4 - y^6}.\]

The length of the parallel starts growing with $y$ from the equator and then decreases. The same property will happen if we consider the standard metric in $S^2$ multiplied by a suitable conformal factor depending on $y$. For the purposes of the analysis here, the hourglass shape is the most important property of the example. The conclusions are very robust since they depend mostly on this property and the fact that the geodesics in the equator cannot be periodic.

3. Statement of results. The main result on the example in [30] are the following. We refer to Appendices B and C for the standard definitions of Tonelli orbits (minimizers of length in a homotopy class) and the Class-A minimizing orbits (orbits that, in the universal cover are minimizers of distance between any pair of points in the orbit). See Definitions 24 and 20.

**Theorem 4.** None of the Tonelli minimizing periodic orbits in Example (4) are Class-A minimizers.

**Remark 5.** [31] constructed an example for which there are only three Tonelli Class-A minimizers.

It is elementary that a periodic orbit which is Class-A is necessarily a Tonelli minimizer. Nevertheless, the main point of Almgren-Federer example is that Tonelli orbits may fail to be Class-A for some metrics.

The reason for Tonelli minimizers failing to be Class-A is that the definition of Tonelli minimizers, requires that the length is minimal along the orbits in the same homotopy class. Nevertheless, when we go to the universal cover, we could consider
multiples of the curve which wrap around many times. If some of these multiples is not a Tonelli minimizer (i.e. one can deform the multiple to get a lower length orbit), then the orbit is not a Class-A minimizer. Slightly more informally, a periodic orbit is Tonelli when it is stable (the length cannot be decreased) under perturbations of the same period. To be Class-A it has to be stable under perturbations of whatever period.

The idea of the proof of Theorem 4 is to show that if $\gamma$ is a Tonelli minimizer of period $T$, it is also a periodic curve of period $nT$ for every $n \in \mathbb{N}$. We will show that there is always an $n$ so that the $nT$ periodic orbit obtained by repeating the orbit $n$ times can be shortened by perturbations of period $nT$. In other words: the orbits whose length cannot be lowered by perturbations of a certain period are such that their length can be lowered by perturbations of longer period.

Actually, we will prove a more general result than Theorem 4.

**Theorem 6.** The only Class-A minimizing geodesics in Example (4) are the geodesics given in the universal cover by:

$$x(t) = t, \quad y(t) = 0, \quad z(t) = z_0.$$  \hspace{1cm} (5)

Note that each of the orbits in (5) is dense on the equator of $\mathcal{M}$. Note that, every time we increment $x$ by 1, the identification (2) makes the orbit to come back identified with an irrational rotation. Therefore, repeating the argument, we obtain that the orbits are dense in the equator. On the other hand, the speed is constant and concentrated in the direction of incrementing only $x$.

Note that the Class-A orbits in (5) form a one-dimensional family. In particular, they are not isolated and hence, cannot be hyperbolic.

This is in contrast with the Class-A orbits in Hedlund example which are hyperbolic and have very interesting shadowing properties. The Class-A geodesics in generalized Hedlund examples are characterized in [8].
Taking advantage of the hyperbolicity properties of the Hedlund orbits [36] constructed many periodic orbits using shadowing arguments. The periodic orbits thus produced are not guaranteed to be Class-A geodesics. Variational shadowing methods based on locally minimal orbits – which also do not guarantee Class-A orbits, appear in [10, 12, 24, 23].

The relation between hyperbolicity and minimization is a very interesting question since both properties lead to shadowing, see [50]. We also mention that relations between minimizing sets and hyperbolicity have been explored for generic systems specially in low dimensions [35, 20, 2].

**Remark 7.** It is worth remarking for experts that our considerations differ in some aspects from those of [30] and this affects some of the proofs and some of the considerations.

We consider geodesics, whereas [30] considers flat chains. The main result on existence of minimizers for geodesic is Tonelli’s Theorem which asserts the existence of minimizers in a homotopy class.

Note that in contrast, in other theories of minimization among more complicated objects (e.g. measures in J. Mather’s theory or flat chains in geometric measure theory), one cannot talk of homotopy, but only about homology. See Appendix D and [7].

For manifolds like the ones we consider in the Almgren-Federer example, homotopy and homology coincide, but in more general cases, the homotopy and homology can be very different.

Our proof of Theorem 23 is significantly simpler than the proof of the corresponding results in [30]. The reason why the flat chains that minimize a fixed period could, in principle be more complicated objects than closed geodesics is that the symmetry breaking requires more sophisticated constructions. See [30, Section 5.11]. Of course, the ideas of the argument presented in this exposition come from [30].

4. **Proof of Theorem 4.** We will assume that there indeed exists a Tonelli Class-A minimizer and then prove that it so has several contradictory properties. Hence, no such a thing exists.

If there were \( \gamma_m \), a curve of homotopy \( m \in \mathbb{Z} \) which is Tonelli and Class-A, by repeating it enough, we could get an orbit \( n\gamma_m \) that moves in the universal cover from \( x = 0, y = y_0, z = z_0 \) to \( x = nm, y = y_0, z = z_0 e^{2\pi i m \omega} \).

We can compare this curve with the curve going from \( (0, y_0, z_0) \) to \( (0, 0, z_0) \) along a meridian, then going from \( (0, 0, z_0) \) to \( (nm, 0, z_0) \) and then going to a \( (nm, y_0, z_0) \) along a meridian. We see that the length of this curve is less than \( nm + 2B \), where \( B \) is an upper bound of the length of the segments of meridian used above as the initial and final segments.

The length of the \( n \) multiple of \( \gamma_m \), the Tonelli orbit of homotopy \( m \) is \( n|\gamma_m| \).

If the Tonelli orbit was a Class-A we would have that the length of its \( n \) cover would be smaller than the length of the trial orbit described above. Hence, if the Tonelli orbit \( \gamma_m \) were Class-A, we would have

\[
n|\gamma_m| \leq n |m| + 2B
\]

where \( B \) is as above.

Therefore, taking limits as \( n \) grows to infinity, we obtain \( |\gamma_m| \leq |m| \).
On the other hand, we have that $\gamma_m$ increases the $x$ coordinate by $m$. Since the metric (4) is bigger than $(1 + g^2)(dx^2)^{1/2}$ we obtain that the $|\gamma_m| \geq |m|$ and that this bound can only be saturated when the segment is contained in the equator. See later Proposition 12 for a more detailed argument.

The lift of equator $\mathcal{E}$ in the universal cover of $\mathcal{M}$, is a cylinder $\tilde{\mathcal{E}} = \mathbb{R} \times S^1$. It is a flat cylinder (the universal cover of $\mathcal{E}$ is the flat plane). So that the orbits of minimal length are precisely the straight lines in the plane. In particular, the minimizing geodesics connecting two points are unique.

Now, we can give several arguments to show that such $\gamma_m$ does not exist.

We note that, because $\omega \notin \mathbb{Q}$, the straight line connecting $(0,0,z_0)$ to $(m,0,z_0e^{2\pi i m \omega})$ has length strictly bigger than $m$.

Alternatively, observe that because the rotation $\omega$ is irrational, we have that straight lines connecting $(0,0,z_0)$ and $(m,0,z_0e^{2\pi i m \omega})$ are different for different $m$, so that the multiples of the minimizing orbit do not agree with each other.

Remark 8. Note that the above arguments use essentially that the equator, in spite of being a cylinder, carries no topology in the angle, since the turns around the angle can be undone by moving over the pole so that all the straight lines in the equator are homotopic in $\mathcal{M}$.

If it was not because of the possibility of using homotopy in the two dimensional sphere we would not be able to verify the properties of the example.

It would be tempting to try to do a similar construction in $T^2$ by adding a twist in one of the coordinates. This does not work because the topology $T^2$ makes its lines with different slopes non-homotopic. Of course, such construction would contradict the results of [31].

Remark 9. Given the characterization of the Tonelli Class-A minimizers, we can take $n$ so that the $e^{2\pi i n m \omega}$ is arbitrarily close to 1. When this factor is very close to 1, the orbit connecting a point $(0,0,z_0)$ to its translate $(nm,0,z_0e^{2\pi i n m \omega})$, can be made shorter than the cover of the Tonelli orbit.

This makes concrete the fact that every Tonelli orbit has a cover which is not a minimizer.

Note that if indeed $e^{2\pi i n m \omega}$ is very close to 1, then, $e^{2\pi i (n+1) m \omega}$ will be close to $e^{2\pi i m \omega}$, so that we can arrange also that the minimizers in the $n+1$ cover approximate the original Tonelli orbit in segments of long length.

Remark 10. One can think heuristically that the mechanism at play is the conflict between the natural minimizers which are the straight lines in the equator and the fact that there is a twist given by (2) which prevents the straight lines from being periodic.

4.1. Relations between the topology and the existence of Class-A geodesics. The following result provides a further tie between Tonelli’s theory, Morse’s theory of Class-A geodesics and the topology of the manifold.

Theorem 11. Assume that $(\mathcal{M},g)$ is a Riemannian manifold with a non-trivial homotopy class of loops. Then, it admits a Class-A geodesic.

Proof. We denote by $\gamma_n : [0,T_n] \to \tilde{\mathcal{M}}, n \in \mathbb{N}$ the Tonelli minimizer corresponding to $n$ times the homotopy class. Note that $nT_1 \geq T_n$ and that $T_n \to \infty$ as $n \to \infty$.

\[3\]Note that $\tilde{\mathcal{E}}$ is topologically non-trivial and has homotopically non-trivial paths. Of course, these paths are homotopically trivial in $\tilde{\mathcal{M}}$. 
If we fix $M_0$ a fundamental domain in $\widetilde{M}$, we can find Deck transformations $D_n$ in $\pi_1(M)$ such that $D_n(\gamma_n(-\frac{T_n}{2})) \in M_0$. Denote

$$\eta_n(t) = D_n\left(\gamma_n(t - \frac{T_n}{2})\right)$$

(a transformation in space and time such that $\eta_n(0) \in M_0, \eta_n : [-\frac{T_n}{2}, \frac{T_n}{2}] \to \widetilde{M}$.)

Passing to a subsequence, we can obtain that, at the same time, we have:

$$\eta_{n_j}(0) \to \eta(0), \quad \eta'_{n_j}(0) \to \eta'(0).$$

The $\eta(0), \eta'(0)$ determine a unique geodesic $\eta$ by solving the Euler-Lagrange equations.

By the standard theorem of dependence on initial conditions for ordinary differential equations, we obtain that

$$\eta_{n_j} \to \eta, \quad \eta'_{n_j} \to \eta' \text{ uniformly on intervals } [-A, A].$$

Since $T_n \to \infty$, we have that for fixed $A$, large $j$, $\eta_{n_j}$ minimizes the length between $\eta_{n_j}(t_1), \eta_{n_j}(t_2)$ for any $t_1, t_2 \in [-A, A]$.

Now we can use a classical argument of [47] to show that $\eta$ is Class-A: Assume $\eta$ was not Class-A, we could find $\tilde{\eta}, B \in \mathbb{R}_+$ such that

$$|\tilde{\eta}|_{[-B, B]} \leq |\eta|_{[-B, B]} - \delta \quad \text{for some } \delta > 0.$$

By the uniform $C^1$ convergence of $\eta_{n_j}$, we have that for all $j > j_0$

$$|\eta_{n_j}|_{[-B, B]} \geq |\tilde{\eta}|_{[-B, B]} + \frac{\delta}{2}.$$

This contradicts the fact that, for sufficiently large $j$, $\eta_{n_j}$ is a minimizer of the length between $\eta_{n_j}(t_1), \eta_{n_j}(t_2)$ for all the points $t_1, t_2 \in [-B, B]$.

One could also get a proof of existence of Class-A geodesics using variational methods. The paper [43] shows that orbits in the support of a minimizing measure are Class A. See also [9, 7]. Note however, that the long term behavior of the orbits may be unrelated to the homology properties of the measure (e.g. in a Hedlund example, minimizing measures could be linear combination of periodic orbits. The orbits of a point in the support will be periodic).

5. Characterization of Class-A orbits in the Almgren-Federer example.

Proof of Theorem 6. Theorem 6 will be proved by a sequence of propositions that keep on restricting the possibilities of Class-A geodesics.

The argument is very typical of similar arguments in calculus of variations: We first show that there is a finite total budget for deviations. Secondly, we prove that the fluctuations have to be concentrated, each of them takes a substantial amount of the budget, hence there are only finitely many of them. Finally, we show that the fluctuations are none.

5.1. A finite total budget for deviations.

**Proposition 12.** Let $\eta_n$ be a segment of a Class-A geodesic joining a point of the form $(0, y_1, z_1)$ to another point $(n, y_2, z_2)$. Then, $|\eta_n| \leq n + 2B$ where $B = \text{diam}\{(0, y, z) : y \in [-1, 1], y^2 + |z|^2 = 1\}$. 
**Proof.** Just consider the curve formed by three segments as follows: The first segment joins \((0, y_1, z_1)\) to \((0, 0, z_1)\), the second one joins \((0, 0, z_1)\) to \((n, 0, z_1)\) and the third segment joins \((n, 0, z_1)\) to \((n, y_2, z_2)\).

The first and third segments have length less than \(B\) and the second one has length \(n\).

Since \(\eta_n\) is part of a Class-A geodesic, its length should be less than the trial segment. 

**Proposition 13.** Let \(\eta_n(t) = (x(t), y(t), z(t))\) be a segment of a Class-A geodesic as in Proposition 12. Let \(\delta > 0\). We denote \(O_{\delta,n} = \{x(t) : |y(t)| \geq \delta\} \subset \mathbb{R}\) and \(U_{\delta,n} = \{t : x(t) \in O_{\delta,n}\}\). Then, the set \(O_{\delta,n}\) has measure

\[
|O_{\delta,n}| \leq 2 \frac{B \delta^2}{\delta^2}, \tag{6}
\]

where \(|O_{\delta,n}|\) denotes the Lebesgue measure of the set \(O_{\delta,n}\).

**Proof.** It will be important that the bound (6) is uniform in \(n\). The proof is just to observe that

\[
|\dot{\eta}_n(t)|_y \geq (1 + y^2(t)) |\dot{x}(t)|.
\]

By Proposition 12, we have, denoting by \(T\) the time of the minimizing orbit

\[
n + 2B \geq \int_0^T |\dot{\eta}_n(t)|_y \ dt \geq \int_0^T (1 + y^2(t)) |\dot{x}(t)| \ dt = (1 + \delta^2) \int_{U_{\delta,n}} |\dot{x}(t)| \ dt + \int_{[0,T] \setminus U_{\delta,n}} |\dot{x}(t)| \ dt \geq n + \delta^2 |O_{\delta,n}|. \tag{7}
\]

**5.2. Possible fluctuations for a finite number of intervals.** The next step shows that if there is a deviation from the equator it has to be a substantial one. This is similar to density estimates in measure theory.

The key idea is that separating from the equator makes one to pay for deviating from the equator in the \(\dot{x}\) component. The only way that this can pay off is if the penalty by measuring the weight of \(\dot{x}\) is offset by derivatives in \(\dot{z}\) component. This only happens if we get close to the north/south pole.

The following elementary side calculation makes more precise the *hourglass shape* of the manifold \(\mathcal{M}\). We recall that the length of a parallel at height \(y\) is \(2\pi(1 + y^2)\sqrt{1 - y^2}\). Write \(u = y^2\) and let \(f(u) = (1 + u)\sqrt{1 - u}\) and then,

\[
f'(u) = \sqrt{1 - u} - \frac{1}{2} \frac{1 + u}{\sqrt{1 - u}} = \frac{1}{\sqrt{1 - u}} \left(1 - u - \frac{1}{2}u - \frac{1}{2}\right) = \frac{1}{\sqrt{1 - u}} \left(\frac{1}{2} - \frac{3}{2}u\right)
\]

So the function \(f\) is increasing when \(u \leq \frac{1}{3}\), that is, \(|y| \leq \sqrt{3}/3\).

We find it convenient to write \(z = \sqrt{1 - y^2}e^{i\sigma}\) where \(\sigma\) is the circle and it can be thought of as the longitude. Then,

\[
\dot{z}(t) = \sqrt{1 - y^2}e^{i\sigma(t)} \dot{\sigma}(t) - \frac{y \dot{y}(t)}{\sqrt{1 - y^2}}e^{i\sigma(t)}. \tag{7}
\]
the Class-A geodesics will have to include infinite intervals where 

$$|\dot{z}(t)| = \sqrt{\frac{y^2}{1-y^2} \dot{y}^2(t) + (1-y^2) \dot{\sigma}(t)^2}$$

and the length is

$$\sqrt{(1+y^2)^2|\dot{x}|^2 + \frac{(1+y^2)^2}{1-y^2} |\dot{y}|^2 + (1+y^2)^2(1-y^2) \dot{\sigma}^2}. \quad (8)$$

**Proposition 14.** Let $0 < \delta \ll 1$. Consider any segment $\eta(t) = (x(t), y(t), z(t))$ that joins $(x_1, y = \delta, z_1)$ to $(x_2, y = \delta, z_2)$. Assume $\delta \leq y(t) < \sqrt[3]{2}$ for all $t$ and that for some $t^*$, $y(t^*) > \delta$. Then, the segment $\eta(t)$ is not part of a Class-A geodesic.

The proof of Proposition 14 is very simple. We consider the rearranged path $\tilde{\eta}(t)$ defined by

$$\tilde{x}(t) = x(t); \quad \tilde{y}(t) = \delta; \quad \tilde{z}(t) = \sqrt{1-\delta^2} e^{i\sigma(t)}.$$  

It is clear that the rearranged path $\tilde{\eta}(t)$ and the original path $\eta(t)$ have the same original and final points and that the coefficients of the derivatives in the expression of the length have decreased. Note that this uses the hourglass shape and the assumption that the orbit remains in the region where the length element is monotone with respect to $y$. \hfill \Box

**Proposition 15.** Let $\eta(t) = (x(t), y(t), z(t))$ be a Class-A geodesic. If there is time $t^*$ for which $y(t^*) = 2\delta > 0$, then there are times $t$ for which $y(t) \geq \sqrt[3]{2}$.\hfill \Box

We argue by contradiction assuming that the conclusion is false. Note that $x(t)$ is unbounded. Due to Proposition 13, there have to be arbitrarily large $x(t)$’s and so arbitrarily large times for which $y(t)$ is smaller than $\delta$ and, by continuity since $y(t^*) > \delta$, one could find an interval $I = (t_1, t_2)$ with $t^* \in I$, $y(t_1) = y(t_2) = \delta$ and $|y| \geq \delta$.

If the conclusions of Proposition 15 were false, we could apply Proposition 14 and conclude that $\eta(t)$ cannot be a Class-A geodesic. \hfill \Box

In other words, if a Class-A geodesic leaves the region $|y| \leq \delta$, then it has to reach $|y| \geq \sqrt[3]{2}$. Since the Class-A geodesics satisfy the geodesic equation, these excursions have to take a time that is bounded from below. Note that this time is independent of $\delta$, this gives a lower bound on the length of the excursion.

Therefore, so far, we have showed that all the possible Class-A geodesics stay within $|y| \leq \delta$ except for a finite number of intervals, in which they reach $|y| \geq \sqrt[3]{2}$. This number is independent of the length considered. Therefore, for any $\delta > 0$, the Class-A geodesics will have to include infinite intervals where $|y| \leq \delta$ and the intervals will be of the form $[a_+, \infty)$ and $(-\infty, a_-]$.

### 5.3. Non-existence of fluctuations

We have shown that all orbits have to be spent a large time close to the equator and have ruled out fluctuations that stay in a region near the equator. The only possibility that needs to be excluded is the existence of some very big bumps that move substantially away from the equator.

In the following, we will rule out the bumps using the conserved quantities of the geodesic flow.
This part of the argument uses slightly the geodesic flow and the dynamics language. See Appendix A. Roughly, we note that if an orbit has to reach the large $y$ region it has to have some angular momentum, but since this is a conserved quantity, it forces that it keeps on circulating, which violates our previous a-priori bounds. Finally, we exclude the existence of minimizers of zero angular momentum. Note that, in contrast with the other parts of the argument, which use very general variational methods, in this part of the argument we use dynamical arguments and the symmetry of the metric. It would be interesting to substitute the present dynamical argument by more variational ones.

We use $L$ for the angular momentum (see Appendix A) and then we have:

**Proposition 16.** There are no Class-A orbits corresponding to $L \neq 0$.

**Proof.** We first note that $E \neq 0$ since all the points in the manifold are fixed points when the energy is zero.

We know that all the orbits in the future (or in the past) stay in the region $|y| < \delta$. Note that this implies that $|\dot{\sigma}|$ is bounded from below in that region, that is

$$|\dot{\sigma}| \geq \frac{|LE|}{(1 + \delta^2)^2} =: \sigma^* > 0. \quad (10)$$

For simplicity, we will only discuss the orbits in the future, the orbits in the past are obtained by changing $t$ into $-t$. The lower bound of angular velocity (10) implies that after time $T \geq \frac{N\pi}{\sigma^*}$, the orbit has gone around the equator $N$ times. If $N$ is large enough, we can compare the presumed Class-A orbit $\eta(t)$ with a segment obtained by joining the initial point to the equator (length $\leq C\delta$ for some $C > 0$) then joining $\dot{x} = 1, \dot{y} = 0, \dot{z} = 0$ and then joining again.

This orbit is shorter when $\frac{N}{\sigma^*} \geq 2C\delta$. Therefore, the orbit $\eta(t)$ is not Class-A. \qed

The final part of the argument is:

**Proposition 17.** The only Class-A geodesics with $L = 0$ are the straight orbits given in Theorem 6.

The reason why this is true is intuitively clear. The only reason why it pays off to depart from the equator is to gain by effecting the changes in $z$-direction around the poles.

If we have no angular momentum, there is no change in $z$-coordinate and therefore, leaving the equator is harmful. More formally, we note that if $\dot{\sigma} = 0$, we are faced with a 2-dimensional system and the motion happens in a plane. Any excursion from the equator will deviate from the straight line. \qed

6. Some possible extensions of the argument and some applications.

6.1. **Building more complicated examples.** Given a manifold $M$, with universal cover $\tilde{M}$ and fundamental group $\pi_1(M)$ (so that $M = \tilde{M}/\pi_1(M)$). Let $\ell : \pi_1(M) \to \mathbb{R}$ be a cocycle of the fundamental group (i.e., $\ell(\gamma_1 \circ \gamma_2) = \ell(\gamma_1) + \ell(\gamma_2)$). We can construct a manifold

$$S^2 \times \tilde{M}/\sim_\ell$$

where $S^2$ is parameterized as in [30]. We say that $(x, y, z) \sim_\ell (\tilde{x}, \tilde{y}, \tilde{z})$ if and only if we have that

$$\tilde{x} = \gamma x, \quad \tilde{y} = y$$
\[ \tilde{z} = e^{i\ell(\gamma)}z \quad (11) \]

for some Deck transformation \( \gamma \). Given a Riemannian metric \( g \) on \( M \), we can define a metric on \( \tilde{M} \) by \((1 + y^2)g\).

It is easy to check that if \( \ell(\gamma) \in \mathbb{R} \setminus \mathbb{Q} \) for some \( \gamma \in \pi_1 \), we have the same phenomenon as in the example of [30]: All the Tonelli geodesics in the classes \( \gamma^n \) are not Class-A.

In particular, if

\[ \ell(\gamma) \notin \mathbb{Q} \quad \forall \gamma \in \pi_1(M) \]

then there are no Tonelli geodesics which are Class-A. Hence there are no periodic Class-A orbits.

An amusing example of this construction is when we take \( M = \mathbb{T}^3 \) endowed with the metric of Hedlund’s example.

Note that there is a relation between cocycle of the fundamental group and the cohomology of the manifold. See [14]. Hence, the language in terms of cocycles is equivalent to the language in terms of cohomology.

It would be interesting to study the possibility of constructing more systematically examples in more general manifolds.

6.2. Tentative applications in statistical mechanics. There is an interesting physical interpretation of the Almgren-Federer example, which also makes it relevant for some problems in statistical mechanics.

We recall that the famous classical XYZ model of Heisenberg [46] consists of a system of particles in the line. Each of this particle occupies a state described by three coordinates \( u^1, u^2, u^3 \) constrained to \((u^1)^2 + (u^2)^2 + (u^3)^2 = 1\). That is \( u = (u^1, u^2, u^3) \in \mathbb{S}^2 \).

The state of the whole system (called configuration) is determined by describing the state \( u_i \) of the \( i \) particle for all particles. So, a configuration is just a mapping \( u : \mathbb{Z} \to \mathbb{S}^2 \).

These particles interact with their neighbors and the substratum so that the system is described by an energy given by the functional:

\[ E(u) := \sum_{j \in \mathbb{Z}} S(u_j, u_{j+1}) \quad (12) \]

where \( S \) is the interaction energy among next nearest neighbors. The sum in (12) is merely formal and is not meant to converge.

What physicists call “ground state” is identical to the Class-A minimizers of the calculus of variations. They are configurations whose energy cannot be lowered by changing a finite number of sites. Note that the definition of Class-A/ground state does not need that the sum in (12) converges.

The example in [30] can be recast in the language of the XYZ model. We have that \( \mathbb{S}^2 \times \mathbb{S}^1 \), the universal cover of \( \mathbb{S}^2 \times \mathbb{S}^1 \), is \( \mathbb{S}^2 \times \mathbb{R} \).

A configuration \( u_j \) can be thought of as sequence of points in \( \mathbb{S}^2 \times \mathbb{R} \), given by \((u_j, j)\). If we define

\[ S(u_j, u_{j+1}) := \text{dist}((u_j, j), (\mathcal{R}u_{j+1}, j+1)) \quad (13) \]

where \( \mathcal{R} \) is the rotation by \( \alpha \) in the identification. We see that the minimizers of the functional \( E \) will be geodesics.

The results of [30] can be expressed in statistical mechanics jargon as saying that imposing periodic boundary condition, we never obtain a ground state. Sliding
the boundary condition further, will always lower the energy and the system with periodic boundary condition never becomes a ground state. A configuration corresponding to a Tonelli orbit will become destabilized under fluctuations of longer and longer periods. It would be interesting to understand better the dynamics of these relaxations.

The form of the interaction (13) is natural in quasi-periodic media so that advancing one index $i$ by 1 is equivalent to rotating an internal phase.

The statistical mechanics interpretation, makes it natural to consider several extra features such as many-body interactions, long range interactions whose consequences are interesting to explore.

Appendix A. The dynamical point of view: Geodesic flows. The problem of existence of geodesics with certain properties can be recast in a more dynamical language leading to geodesic flows [18, 51, 1, 4]. In this paper, we rely much more on the methods of the calculus of variations, but we want to remark that the Almgren-Federer example also has interesting properties from the dynamical point of view.

A.1. Basic definitions. Given a function $L : T\mathcal{M} \to \mathbb{R}$ one can consider the functional

$$ S_{t_1}^{t_2}[\gamma] = \int_{t_1}^{t_2} L(\gamma(t), \dot{\gamma}(t)) \, dt. $$

Under appropriate regularity conditions – we refer to the references above – the functional $S$ is differentiable among paths that satisfy $\gamma(t_1) = a; \gamma(t_2) = b$. A path $\gamma$ is a critical point of the functional if and only if it satisfies the Euler-Lagrange equation.

$$ D_1 L - \frac{d}{dt} D_2 L = 0. $$

The first order differential equation (14) in $T\mathcal{M}$ can be interpreted as a second order equation in $\mathcal{M}$, which is considered as an evolution equation. Again, under appropriate conditions on the regularity, growth, of $L$ and on the manifold $\mathcal{M}$, the flow is complete (given any initial condition in $T\mathcal{M}$, there is a solution of (14) defined for all times.)

Much of the theory works also for time-dependent Lagrangians, but we will not consider this.

A.1.1. The geodesic flows. When we take $L_1(\gamma, \dot{\gamma}) = g_\gamma(\dot{\gamma}, \dot{\gamma})^{1/2}$ the functional $S$ is just the length. However, much of the theory of geodesic flows is obtained taking the Lagrangian $L_2(\gamma, \dot{\gamma}) = g_\gamma(\dot{\gamma}, \dot{\gamma})$. The Lagrangian $L_2$, quadratic in the velocity is more natural in Mechanics and, as we will see later, the superlinear growth of the Lagrangian in the velocity is important for the Mather theory of homology minimizing measures (see Appendix D and the references there).

As it turns out, the solutions of the Euler-Lagrange equations for both systems are the same.

The reason is that, using that $L_2$ is homogeneous in the velocity, one can show that $L_2$ is a conserved quantity for the Euler-Lagrange flow corresponding to $L_2$. For physicists, the $L_2$ is the kinetic energy. The fact that the energy is the same as the Lagrangian depends on the fact that the energy is homogeneous in the velocity.

---

4In Mechanics, it is customary to multiply it by a factor $\frac{1}{2}$. 
If we consider a Lagrangian \( \tilde{L} = F(L_2) \) with \( F \) a smooth function (we omit some details on regularity, etc.) we obtain that the Euler-Lagrange equations corresponding to it are

\[
F'(L_2)D_1L_2 - \frac{d}{dt}(F'(L_2)D_2L_2) = 0. \tag{15}
\]

We can see that if \( \gamma(t) \) is a solution of the Euler-Lagrange equation for \( L_2 \), then, because \( L_2 \) is conserved for these solutions, then \( \gamma(t) \) solves (15). Since these solutions can match all the initial conditions, they are all the solutions.

A.2. Conserved quantities of the geodesic flow of the metric (4). The metric (4) has 3 local symmetries (invariance with respect to time, horizontal translations and rotations along the longitude. According to Noether’s theorem, this means that the system has 3 conserved quantities.

It is interesting to note that, due to the gluing in (2) even if we have infinitesimal symmetries, the symmetry of translation along the \( x \) axis does not have compact leaves. This is closely related to the fact that there are no strict minimizers.

As remarked above, we could apply Noether’s theorem either to the Lagrangian \( L_1 \) or to the \( L_2 \), the conserved quantities obtained are different in both cases, but they are functions of each other.

**Proposition 18.** The geodesic flow corresponding to the Lagrangian \( L_1 \) corresponding to the length element (4) preserves the following conserved quantities:

- **energy:**
  \[
  E = (1 + y^2) \sqrt{x^2 + y^2 + |z|^2} \\
  = (1 + y^2) \sqrt{x^2 + \frac{y^2}{1 - y^2} + (1 - y^2) \dot{\sigma}^2}
  \]

- **momentum:**
  \[
  P = (1 + y^2) \frac{\dot{x}}{\sqrt{x^2 + \frac{y^2}{1 - y^2} + (1 - y^2) \dot{\sigma}^2}} \\
  = \frac{(1 + y^2) \dot{x}}{E}
  \]

- **angular momentum:**
  \[
  L = \frac{(1 + y^2)(\dot{z}_1 z_2 - \dot{z}_2 z_1)}{2 \sqrt{x^2 + y^2 + |z|^2}} = \frac{(1 + y^2)^2 (1 - y^2) \dot{\sigma}}{E}
  \]

**Proof.** (i). The energy is conserved because of the well-known fact that the length is preserved by the geodesic flow denoted by \( \phi^t \). In fact, note that applying Noether’s theorem (see, for example [3]) for the one-parameter group of diffeomorphisms \( \phi^t \) (the geodesic flow) produces a time conserved quantity:

\[
\frac{\partial E}{\partial \dot{q}} \dot{q} = E
\]

where \( q = (x, y, \sigma) \).

(ii). The momentum corresponds to the translation:

\[
x \rightarrow x + \epsilon \\
y \rightarrow y \\
z \rightarrow z.
\]
Applying the general rule of Noether’s theorem, we obtain the first integral \( \frac{\partial E}{\partial \dot{\sigma}} = P \).

(iii). The angular momentum corresponds to the transformation:
\[
\begin{align*}
  x & \rightarrow x \\
  y & \rightarrow y \\
  z & \rightarrow z e^{i\alpha} \leftrightarrow \sigma \rightarrow \sigma + \alpha.
\end{align*}
\] (17)

Applying Noether’s theorem, we obtain the first integral \( \frac{\partial E}{\partial \dot{\sigma}} = L \).

Note that these three first integrals will allow us to integrate the equations in the universal cover. For example, we can obtain \( \dot{y} \) as a function of \( y \) and the conserved quantities:
\[
E^2 = (1 + y^2)^2 \left[ \dot{x}^2 + \frac{\dot{y}^2}{1 - y^2} + (1 - y^2)\dot{\sigma}^2 \right]
\]
\[
= (1 + y^2)^2 \left[ \frac{P^2 E^2}{(1 + y^2)^4} + \frac{\dot{y}^2}{1 - y^2} + \frac{L^2 E^2}{(1 + y^2)^4(1 - y^2)} \right].
\]

Even if one can get the explicit solutions, it is not easy to understand the minimizing properties of the orbits.

Remark 19. It is interesting to remark that the symmetry under translation is a differentiable symmetry which leads to conserved quantity by Noether’s theorem, but it does not allow to make the quantities descend to the manifold. (The orbits of the symmetry are dense.) Hence, we cannot find quotient manifolds corresponding to the system.

This situation happens very often in quasi-periodic systems (see for example [25, 57]).

A.3. On the notion of integrability and the integrability of the geodesic flow for the Almgren-Federer example. The word integrability is used often very loosely in Hamiltonian mechanics and, a system can be integrable or not depending on the precise meaning given to integrability.

One of the loosest definition is that the system is integrable if it has as many conserved “functionally independent” conserved quantities as degrees of freedom. In this sense, the geodesic flow of the Almgren-Federer example is integrable.

On the other hand a more strict definition of integrability is that the conserved quantities commute with each other and be independent everywhere. It is known by the Liouville-Arnold theorem [32, p. 278] that this implies that the phase space should be of the form \( T^d \times \mathbb{R}^d \). Clearly, the phase space of the Almgren-Federer example (in the Hamiltonian formalism) is \( T^* (S^1 \times S^2) \), which is not of this form, so it cannot be integrable in this stronger sense.

It seems that the Hamiltonian flow of the Almgren-Federer example satisfies this definition of integrability, in some open sets but that there are singular leaves. See [13] for a discussion the topological issues involved in integrable systems with singular leaves.

The variables produced in the Liouville-Arnold theorem are called action-angle variables. They can be obtained by integrals. We note however that, even for polynomial systems, the action variables have complex singularities (even for an an-harmonic oscillator or a pendulum), so that in many computations it is advantageous to have methods that do not rely on the action angle variables.
In some notions of integrability, it is also required that the integrals or the action-angle are algebraic functions of the coordinates or obtained through some specific method, etc.

It is unfortunate that in many discussions of integrable systems, precise definitions are omitted.

Appendix B. H. M. Morse’s theory of globally minimizing geodesics. In this section, we present a summary of the main concepts and results of [47]. The paper [47] developed the global theory of minimizing geodesics in dimension 2 except for the torus and the sphere. The global theory on the torus was developed in [31] and in the sphere it is meaningless.

The effect of the example [30, Section 5.11] (and of Hedlund’s example [31]) is to show that this theory of [47] on geodesics does not generalize to higher dimensions.

Let \( \gamma : [a, b] \to \mathcal{M} \) be an absolutely continuous curve in a Riemannian manifold \((\mathcal{M}, g)\); then the length is defined as follows:

\[
|\gamma|_g := \int_a^b |\dot{\gamma}(t)|_g dt = \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.
\]

Definition 20 (Class-A geodesics). A geodesic \( \gamma \) parameterized by the arc-length \( s \in \mathbb{R} \) is said to be Class-A with respect to the Riemannian metric \( g \) if for any point \( s_1 < s_2 \) and for any absolutely continuous geodesic segment \( \tilde{\gamma} \) with end points \( \gamma(s_1) \) and \( \gamma(s_2) \) homotopic to \( \gamma([s_1, s_2]) \) on \( \mathcal{M} \), we have

\[
|\gamma|_{[s_1, s_2]} \leq |\tilde{\gamma}|_g.
\]

One of the main results of [47] is the following

Theorem 21. Let \( \mathcal{M} \) be a 2-dimensional Riemannian manifold of genus \( g \geq 2 \). Then, by the uniformization theorem, the universal cover \( \tilde{\mathcal{M}} \) is a disk and the metric in \( \tilde{\mathcal{M}} \) is conformal to the standard Poincaré metric.

There exists a constant \( C > 0 \) such that for every geodesic \( \alpha \) of the standard Poincaré metric in the Poincaré disk, there is Class-A geodesic \( \gamma_\alpha \) in \( \tilde{\mathcal{M}} \) such that \( \text{dist}(\alpha, \gamma_\alpha) \leq C \).

If \( \mathcal{M} = \mathbb{T}^2 \), for any straight line \( ℓ \) in the universal cover (which we identify with \( \mathbb{R}^2 \)) there exists a Class-A geodesic \( \gamma_\ell \) such that \( \text{dist}(\ell, \gamma_\ell) \leq C \).

The proof of the above results in [47] is very readable. It is based on the fundamental lemma that two minimizing geodesics can only cross once. (This lemma generalizes to many other contexts and is an important inspiration for the analysis in [5] of the ground states in Frenkel-Kontorova model. See [6] for a very general point of view that uncovers the deep relation among these seemingly disparate areas.

Remark 22. The examples of [31, 30] show that the results of the theory of geodesics do not generalize to geodesics in higher dimensions.

On the other hand, as pointed out in [48, 49] there are rather satisfactory generalizations to Objects of codimension 1. Of course, the analysis becomes much more complicated since one has to deal with objects of higher dimension which require tools more sophisticated than ODE’s. This has lead to many results by different schools.

Many results in elliptic PDE’s on tori, appear in [54]. The paper [15] considers minimal surfaces of codimension 1 (and even elliptic integrands) and allows rather general manifolds (the fundamental group has to be residually finite). The notion of
minimizers there is related to cocycles, which by [14] is also related to cohomology. The survey [7] considers also relations to geometric measure theory. The papers [11, 34, 17, 26, 16] consider different generalizations to statistical mechanics. Many other codimension one contexts have been obtained in the literature, including nonlocal operators.

Appendix C. Tonelli theory of minimizing periodic orbits. The work of Tonelli introduced many of the now standard semi-continuity arguments in the calculus of variations.

**Theorem 23** (Tonelli). Let $\mathcal{M}$ be a compact Riemannian manifold without boundary.

For every homotopy class $c$, there is a closed curve $\gamma_c$ of homotopy class $c$ such that

$$|\gamma_c| = \min_{\gamma \text{ has homotopy class } c} \{|\gamma|\}. \quad (18)$$

**Definition 24.** We will refer to the curves that satisfy (18) as Tonelli curves or Tonelli minimizers.

By today’s standard, the argument is rather elementary. There are many variants of the proof. But all have more or less the same ingredients:

1. The set of curves in a homotopy class with a finite total length less than $A$ is pre-compact in $C^0$ topology. (Note that bounding the total length gives a bound in the oscillation).

   Of course, the set is not empty, for $A$ sufficiently large.

   There are many precise formulations and can use some versions of Sobolev embedding theorem, Ascoli-Arzelà, broken geodesics, etc.

2. The length of curves is lower semi-continuous with respect to $C^0$ convergence.

   This is a very simple argument if one defines the length as the supremum of the sum of the lengths of all the piecewise geodesics joining points in the curve which are at sufficiently small distance (say, one half of the distance given by the Hopf-Rinow theorem).

3. The passing to the limit in a minimizing sequence does not change the homotopy class.

Many more details can be found in [40, 43, 45, 27].

Appendix D. J. Mather’s theory of homological minimizing measures. An important point of view in minimization problem for geodesics (and more generally Lagrangians) was introduced by J. Mather in the 80’s and 90’s [42, 43, 44]. Important generalizations were obtained by R. Mañé [40, 37, 38, 39, 21, 22]. See also [45, 56].

Of course we cannot even summarize the main concepts of this rich theory and we refer to the references above. (The theory applies also to periodic Lagrangians, we omit even mention many important concepts such as the critical value, dual formulation, ergodic characterizations, etc.)

In this section, we will only discuss the minimizing measures point of view and how it relates to the Almgren-Federer example. There are some resemblances between the motivation of [30] and [42]: the need to consider minimizers in dual spaces, the role of cohomology.
D.1. Some definitions in the theory of $c$ minimizing measures. In this section, we briefly go over the main definitions. We will assume that all functions are differentiable enough in this informal presentation. We refer to the references above for precise treatments. Our main goal in this paper is just to present the main concepts as applied to the Almgren-Federer example and we are omitting many important precisions (regularity assumptions, precise convexity assumptions, completeness of the flow on $TM$ generated by the Euler-Lagrange equations).

In this section it will be crucial that we use the Lagrangian $L_2$ from Appendix A since the superlinear growth will be important.

We recall that a one form $\eta$ is a function $\eta : TM \rightarrow \mathbb{R}$, which is linear on the tangent directions. Hence we can add a form to a Lagrangian.

$$L_\eta = L - \eta.$$ (Of course, the minus sign above is a convention, which simplifies some of the calculations).

It is well known that if $\eta$ is a closed form, the critical points of $L_\eta$ are the same as those of $L$. Indeed, if $\eta$ is a closed one form, the difference between the action of $L$ and $L_\eta$ are terms that depend only on the ends of $\eta$, which are fixed in the variational process.\footnote{In the Physics literature this is sometimes described as “adding a complete differential”}

Hence, the Euler-Lagrange equations for $L_\eta$ are the same as those for $L$.

In other words, the paths that are critical points for the action corresponding to $L_\eta$ are the same as the paths that are critical points for the action corresponding to $L$. On the other hand, the paths minimizing the action of $L_\eta$ can be different from those of $L$.

The main concept of $[42, 43, 40]$ is to consider variational principles not on orbits but on probability measures. The effective action in measure is defined as

$$A_\eta[\mu] = \int_{TM} L_\eta \, d\mu.$$ (19)

One conceptually very crucial observation of $[42, 43]$ is that $A_\eta$, depends only on $c = [\eta]$ the cohomology class of $\eta$.

Hence, we will henceforth use the notation $A_c$, where $c$ is a cohomology class.

Remark 25. The consideration in dynamics of variational principles on measures can be justified heuristically by arguing that if the orbits are very long, the integral along the orbit (divided by the time) should approximate the integral of a measure. This heuristic argument, is put forward in $[52, 53]$ for quasi-periodic orbits and justified in this context in $[41]$. See $[58]$ for an exposition of this circle of ideas.

From the point of view of developing a calculus of variations, we note that the Lagrangian $L_2$ (or $L_2 - \eta$) is coercive and convex in the fibers. In such a way, one obtains easily lower semi-continuous under weak-* convergence, so that it is easy to obtain existence of minimizers in several contexts.

The theory of $[43]$ was based on characterizing the measures achieving

$$\min_{\mu \in \mathcal{I}_L} A_c[\mu]$$ (20)

where $\mathcal{I}_L$ are probability measures that are invariant under the Euler Lagrange evolution.
The paper [37] studies the minimizers of
\[
\min_{\mu \in \mathcal{H}} A_c[\mu]
\]
where \(\mathcal{H}\) is the set of probability holonomic measures. These are measures \(\mu\) so that
\[
\int_{T^*M} \|v\| \, d\mu < +\infty
\]
and, more importantly, for us
\[
\int \eta \, d\mu = 0 \quad \forall \eta = d\phi
\]
with \(\phi\) a differentiable function.

The paper [37] shows that the minimizers in the class of holonomic measures are invariant and that the two minimizing problems (20) and (21) lead to the same minimizers. The advantage of the formulation in (21) over that in (20) is that the class of measures that enter in the minimization process is the same for all the Lagrangians, hence one can discuss rather comfortably the dependence on the minimizers on \(L\). This allows to discuss the dependence of the minimizers on the Lagrangian and, in particular study properties of minimizers for generic (in some appropriate sense) Lagrangians.

An important lemma in [37] is that holonomic measures can be characterized as weak-* limits of measures concentrated on loops.

**Remark 26.** To make more explicit the relation with geometric measure theory, we note that we are treating holonomic measures as dual of 1-forms, which are called 1-“currents” in geometric measure theory. In geometric measure theory one defines the differential by duality and hence holonomic measures are the same as closed 1-currents. The approximation of closed currents by simplices is treated in great generality in [30, 2.6] [29, 4.2.20]. The results in geometric measure theory mentioned above are much more refined than what we note, since they also include regularity considerations.

The paper [7] provides a comparison between the theories of minimizers based on measures and the theories based on closed 1-currents in measure theory.

In both of the problems (20), (21) it is straightforward to prove existence of minimizers. The growth at infinity makes the functional coercive. Hence suffices to consider a weak-* compact set of measures, the convexity makes it lower semi-continuous under weak-* convergence. Of course, the deep results are obtained by studying the geometric and dynamical properties of the minimizers and relating them to other mathematical questions such as solutions of Hamilton-Jacobi equations, transport properties, etc.

**Remark 27.** There is a dual formulation. One can prescribe a rotation number for a measure (which is an element in homology) and one can consider the minimization over measures with this homology. See [19, 56, 27]. From our point of view – computing the minimizers in a concrete example – it seemed easier to use the unconstrained minimizers, since minimizing with the constraint is harder to write up.

A very important function in Mather’s theory is
\[
-\alpha(c) = \inf_{\mu \in \mathcal{L}_c} A_c[\mu].
\]

Note that for a fixed \(\mu\), \(A_c[\mu]\) is linear in \(c\), so that \(-\alpha(c)\), being the infimum of linear functions is a concave function.
D.2. Characterization of Mather’s $c$ minimizers for the Almgren-Federer example. In the Almgren-Federer example, the space of cohomology is just the reals. A good representative of the cohomology class is $c dx$ and any element of the class can be written as $c dx + dS$ where $S$ is a function on the manifold.

In this section, we want to characterize the $c$ minimizing measures in the Almgren-Federer example. We will first characterize the minimizers when the infimum is taken over all the probability measures. Then, we will show that among all the minimizers found in this class, there is one which is a holonomic measure (and hence invariant). It follows immediately, that this is a minimizer in the more restricted class of holonomic measures. It also follows that there are no other minimizers in the class of holonomic measures since they would have to have an action not larger than that of our minimizer and, hence be minimizers in the class of all measures.

Our reason to follow this strategy is that the characterization of minimizers in the class of all measures can use standard tools such as rearrangements, which are well known in the class of all measures, but are not clear in the case of invariant or holonomic probability measures. When we find a holonomic or invariant measure among the minimizers among all probability measures, it is clearly a minimizer among the measures of this class.

**Lemma 28.** In the Lagrangian $L_2$ corresponding to the metric (4), we have

(i) when $c \neq 0$, the $c$ minimizing measures (among all probability measures) are the measures concentrated on the set $y = 0$, $L_2 = \frac{1}{2} c^2$ and the velocity is only along the $x$ direction, i.e. $v_x = \sqrt{L_2}$.

Moreover, among these minimizing measures (among all probability measures) there is a unique one which is holonomic—the Lebesgue measure supported on the equator.

(ii) when $c = 0$, the $c$ minimizing measures are those with velocity zero and any distribution in the position variables.

**Proof.** In this case, the $L_2$ is a conserved quantity of the geodesic flow. For a number $E$, the set of points in $T_M$ that satisfy $L_2 = E$ is a manifold. We disintegrate the measure along the level surfaces of $L_2$—which is natural since it is a conserved quantity—and furthermore disintegrate along $y$ levels. We write the problem as

$$
\int d\alpha(E) \int d\beta (y_0) \int_{L_2 = E; y = y_0} dv_{E, y_0} (L_2 - cv_x) \tag{22}
$$

and we choose the conditional measures $\alpha, \beta, \nu$ so as to minimize the integral. Note that there is some ambiguity in the integration. We can multiply by a factor $\nu$ and by the inverse factor $\beta$, etc. So, we can assume that $\nu_{E, y_0}$ is either zero or a probability measure. We can assume that it is always a probability measure and that the measures $\beta, \alpha$ give no weight to the $E, y_0$ for which $\nu$ is zero.

(i). For $c > 0$, since $dv_{E, y_0}$ is in the level set $L_2 = E$ the integrand reduces to $E - cv_x$. If we had any measure $\nu$ we could do better by another measure which is concentrated as much as possible in the value of $v_x$ that makes the integrand smaller. For $c > 0$—which is the case that we will consider for the moment—we want to take $v_x$ as large as possible consistent with being in the energy surface and a parallel. Since $E = (1 + y_0^2)(v_x^2 + v_y^2 + v_z^2)$, we see that the optimal value to concentrate $v_x$ is $\sqrt{E/(1 + y_0^2)}$.

6We abuse the letter $E$, since we had also used it in Appendix A. Also, we note for physicists that there is a factor 2 compared with customary definitions of the kinetic energy.
Hence, we can assume that for the minimizing measure \( \nu \), we have:
\[
\int_{L_2=E; y=y_0} (L_2 - cv_x) \, d\nu_{E,y_0} = E - c\sqrt{E}/(1 + y_0^2).
\]

Now we consider the measure \( d\beta_{E}(y_0) \). Since \( E, c \) are fixed positive numbers, we can again assume that \( d\beta \) is concentrated on the \( y_0 \) for which the integrand is the smallest. That is, \( y_0 = 0 \).

Hence, we are reduced to studying
\[
\int \left( E - c\sqrt{E} \right) \, d\alpha(E).
\]

Again, clearly, it is good to concentrate the measure in the values of \( E \) that make the integrand the smallest. An elementary calculation gives that this minimum is reached for \( E = \frac{1}{4}c^2 \).

The calculation for \( c < 0 \) is very similar and we omit the details.

Now that we have that the minimizers among all the probability measures are the measures concentrated on the equator and with a fixed speed, we note that the geodesic flow on the equator is an irrational rotation, which, according to Weyl’s theorem is uniquely ergodic, so there is only one measure invariant under the geodesic flow, which is the Lebesgue measure.

(ii). As for the case \( c = 0 \), we see that the minimization of the integrand happens precisely when all the velocities are zero. In this case the distribution in the space variables does not matter.

\[\square\]

D.3. Weak KAM theorem. There is a close connection between Mather’s theory and weak KAM solutions of Hamilton-Jacobi equation:
\[
H_2(x, du(x) + c) = \hat{H}(c), \quad x \in \mathcal{M}
\]
where \( H_2 \) is the Hamiltonian associated to \( L_2 \). This equation is a degenerate PDE equation of first order with two unknowns \((\hat{H}(c), u)\). The constant \( \hat{H}(c) \) is unique for any given \( c \) and is called the effective Hamiltonian (also called Mather’s \( \alpha \)-function). The function \( u \) defined on \( \mathcal{M} \) is \( C^0 \) but may not be unique. One can refer [28] for the weak KAM theory. Roughly speaking, the minimal geodesics can be embedded into the characteristic fields of Hamilton-Jacobi equation. See also [7] for the relations with geometric measure theory.

**Definition 29** (Calibrated curve). We say that a curve \( \gamma : \mathbb{R} \to \mathcal{M} \) is calibrated if for any \( t, t' \in I \) with \( t \leq t' \), we can find a real function \( u \) such that
\[
u(\gamma(t')) - \nu(\gamma(t)) = \int_t^{t'} L_\eta(\gamma(s), \dot{\gamma}(s)) \, ds + \hat{H}(c)(t' - t).
\]

**Proposition 30.** The Class-A geodesics in the Almgren-Federer example are calibrated.

We note that the closed form \( \eta \) is given by \( \eta = cdx + dS \) where \( S \) is a function on the manifold \( \mathcal{M} \).

Since the energy is conserved,
\[
\int_t^{t'} L_\eta(\gamma(s), \dot{\gamma}(s)) \, ds = E(t - t') - cx(t - t') - S(\gamma(t')) + S(\gamma(t)).
\]

One of the consequences of the general theory [28] is that the calibrated curves are Class-A.
REFERENCES

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978, Second edition, revised and enlarged, With the assistance of Tudor Ratiu and Richard Cushman.

[2] M.-C. Arnaud, Lyapunov exponents for conservative twisting dynamics: A survey, in *Ergodic theory*, De Gruyter, Berlin, 2016, 108–133.

[3] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, vol. 60 of Graduate Texts in Mathematics, Springer-Verlag, New York, [1989?], Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[4] V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics*, Translated from the French by A. Avez, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[5] S. Aubry and P. Y. Le Daeron, The discrete Frenkel-Kontorova model and its extensions. I. Exact results for the ground-states, *Phys. D*, 8 (1983), 381–422.

[6] V. Bangert, Mather sets for twist maps and geodesics on tori, *Dynam. Report. Ser. Dynam. Systems Appl.*, Wiley, Chichester, 1 (1988), 1–56.

[7] V. Bangert, Minimal measures and minimizing closed normal one-currents, *Geom. Funct. Anal.*, 9 (1999), 263–286.

[8] V. Bangert, Minimal geodesics, *Ergodic Theory Dynam. Systems*, 10 (1990), 263–286.

[9] P. Bernard, Connecting orbits of time dependent Lagrangian systems, *Ann. Inst. Fourier (Grenoble)*, 52 (2002), 1533–1568, http://aif.cedram.org/item?id=AIF_2002__52_5_1533_0.

[10] U. Bessi, L. Chierchia and E. Valdinoci, Upper bounds on Arnold diffusion times via Mather theory, *J. Math. Pures Appl.* (9), 80 (2001), 105–129.

[11] M. L. Blank, Metric properties of minimal solutions of discrete periodical variational problems, *Nonlinearity*, 2 (1989), 1–22. http://stacks.iop.org/0951-7715/2/1.

[12] S. Bolotin, Homoclinic trajectories of invariant sets of Hamiltonian systems, *NoDEA Nonlinear Differential Equations Appl.*, 4 (1997), 359–389.

[13] A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian Systems*, Chapman & Hall/CRC, Boca Raton, FL, 2004, Geometry, topology, classification, Translated from the 1999 Russian original.

[14] K. S. Brown, *Cohomology of Groups*, vol. 87 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.

[15] L. A. Caffarelli and R. de la Llave, Planelike minimizers in periodic media, *Comm. Pure Appl. Math.*, 54 (2001), 1403–1441.

[16] L. A. Caffarelli and R. de la Llave, Interfaces of ground states in Ising models with periodic coefficients, *J. Stat. Phys.*, 118 (2005), 687–719.

[17] A. Candel and R. de la Llave, On the Aubry-Mather theory in statistical mechanics, *Comm. Math. Phys.*, 192 (1998), 649–669.

[18] C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order*, San Francisco-London-Amsterdam, 1965.

[19] M. J. D. Carneiro, On minimizing measures of the action of autonomous Lagrangians, *Nonlinearity*, 8 (1995), 1077–1085.

[20] G. Contreras, A. Figalli and L. Rifford, Generic hyperbolicity of Aubry sets on surfaces, *Invent. Math.*, 200 (2015), 201–261.

[21] G. Contreras, J. Delgado and R. Iturriaga, Lagrangian flows: The dynamics of globally minimizing orbits. II, *Bol. Soc. Brasil. Mat. (N.S.)*, 28 (1997), 155–196.

[22] G. Contreras and R. Iturriaga, Global Minimizers of Autonomous Lagrangians, 22º Colóquio Brasileiro de Matemática. [22nd Brazilian Mathematics Colloquium], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1999.

[23] G. Contreras and G. P. Paternain, Connecting orbits between static classes for generic Lagrangian systems, *Topology*, 41 (2002), 645–666.

[24] X. Cui, C.-Q. Cheng and W. Cheng, Existence of infinitely many homoclinic orbits to Aubry sets for positive definite Lagrangian systems, *J. Differential Equations*, 214 (2005), 176–188.

[25] R. de la Llave and N. P. Petrov, Theory of circle maps and the problem of one-dimensional optical resonator with a periodically moving wall, *Phys. Rev. E* (3), 59 (1999), 6637–6651.

[26] R. de la Llave and E. Valdinoci, Ground states and critical points for Aubry-Mather theory in statistical mechanics, *J. Nonlinear Sci.*, 20 (2010), 153–218.
A. Fathi, Solutions KAM faibles conjuguées et barrières de Peierls, C. R. Acad. Sci. Paris Sér. I Math., 325 (1997), 649–652.

A. Fathi, Weak KAM Theorem in Lagrangian Dynamics, 2008.

H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

H. Federer, Real flat chains, cochains and variational problems, Indiana Univ. Math. J., 24 (1974/75), 351–407.

G. A. Hedlund, Geodesics on a two-dimensional Riemannian manifold with periodic coefficients, Ann. of Math. (2), 33 (1932), 719–739.

H. Hofer and E. Zehnder, Symplectic Invariants and Hamiltonian Dynamics, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2011, Reprint of the 1994 edition.

A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, vol. 54 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza.

H. Koch, R. de la Llave and C. Radin, Aubry-Mather theory for functions on lattices, Discrete Contin. Dynam. Systems, 3 (1997), 135–151.

P. Le Calvez, Les ensembles d’Aubry-Mather d’un difféomorphisme conservatif de l’anneau déviant la verticale sont en général hyperboliques, C. R. Acad. Sci. Paris Sér. I Math., 306 (1988), 51–54.

M. Levi, Shadowing property of geodesics in Hedlund’s metric, Ergodic Theory Dynam. Systems, 17 (1997), 187–203.

R. Mañé, On the minimizing measures of Lagrangian dynamical systems, Nonlinearity, 5 (1992), 623–638.

R. Mañé, Generic properties and problems of minimizing measures of Lagrangian systems, Nonlinearity, 9 (1996), 273–310.

R. Mañé, Lagrangian flows: The dynamics of globally minimizing orbits, Bol. Soc. Brasil. Mat. (N.S.), 28 (1997), 141–153.

R. Mañé, Global Variational Methods in Conservative Dynamics, Instituto de Matemática pura e aplicada, 1990.

J. N. Mather, Existence of quasiperiodic orbits for twist homeomorphisms of the annulus, Topology, 21 (1982), 457–467.

J. N. Mather, Minimal measures, Comment. Math. Helv., 64 (1989), 375–394.

J. N. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z., 207 (1991), 169–207.

J. N. Mather and G. Forni, Action minimizing orbits in Hamiltonian systems, in Transition to Chaos in Classical and Quantum Mechanics (Montecatini Terme, 1991), vol. 1589 of Lecture Notes in Math., Springer, Berlin, 1994, 92–186.

M. Mazzucchelli, Critical Point Theory for Lagrangian Systems, vol. 293 of Progress in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2012.

B. M. McCoy, Advanced Statistical Mechanics, vol. 146 of International Series of Monographs on Physics, Oxford University Press, Oxford, 2010.

H. M. Morse, A fundamental class of geodesics on any closed surface of genus greater than one, Trans. Amer. Math. Soc., 26 (1924), 25–60, http://dx.doi.org/10.2307/1989225.

J. Moser, Minimal solutions of variational problems on a torus, Ann. Inst. H. Poincaré Anal. Non Linéaire, 3 (1986), 229–272, http://www.numdam.org/item?id=AIHPC_1986__3_3_229_0.

J. Moser, Selected Chapters in the Calculus of Variations, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2003, Lecture notes by Oliver Knill.

D. Offin, Hyperbolic minimizing geodesics, Trans. Amer. Math. Soc., 352 (2000), 3323–3338.

G. P. Paternain, Geodesic Flows, vol. 180 of Progress in Mathematics, Birkhäuser/Springer, Boston, Inc., Boston, MA, 1999.

I. C. Percival, Variational principles for the invariant toroids of classical dynamics, J. Phys. A, 7 (1974), 794–802.

I. C. Percival, A variational principle for invariant tori of fixed frequency, J. Phys. A, 12 (1979), L57–L60.

P. H. Rabinowitz and E. W. Stredulinsky, Extensions of Moser-Bangert Theory, vol. 81 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser/Springer, New York, 2011, Locally minimal solutions.
[55] R. C. Robinson, *An Introduction to Dynamical Systems—Continuous and Discrete*, vol. 19 of Pure and Applied Undergraduate Texts, 2nd edition, American Mathematical Society, Providence, RI, 2012.

[56] A. Sorrentino, *Action-minimizing Methods in Hamiltonian Dynamics*, vol. 50 of Mathematical Notes, Princeton University Press, Princeton, NJ, 2015, An introduction to Aubry-Mather theory.

[57] X. Su and R. de la Llave, KAM theory for quasi-periodic equilibria in one-dimensional quasi-periodic media, *SIAM J. Math. Anal.*, 44 (2012), 3901–3927.

[58] X. Su and R. de la Llave, Percival Lagrangian approach to the Aubry-Mather theory, *Expo. Math.*, 30 (2012), 182–208.

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