ON TYPES OF KKT POINTS IN POLYNOMIAL OPTIMIZATION

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Abstract. Let $f$ be a real polynomial function with $n$ variables and $S$ be a basic closed semialgebraic set in $\mathbb{R}^n$. In this paper, we are interested in the problem of identifying the type (local minimizer, maximizer or not extremum point) of a given isolated KKT point $x^*$ of $f$ over $S$. To this end, we investigate some properties of the tangency variety of $f$ on $S$ at $x^*$, by which we introduce the definition of faithful radius of $f$ over $S$ at $x^*$. Then, we show that the type of $x^*$ can be determined by the global extrema of $f$ over the intersection of $S$ and the Euclidean ball centered at $x^*$ with a faithful radius. Finally, we propose an algorithm involving algebraic computations to compute a faithful radius of $x^*$ and determine its type.

1. Introduction

Consider the following constrained polynomial optimization problem

$$\begin{cases}
\min_{x \in \mathbb{R}^n} f(x) \\
\text{s.t. } g_i(x) = 0, \ldots, g_l(x) = 0, \\
h_1(x) \geq 0, \ldots, h_m(x) \geq 0,
\end{cases}$$

where $f(x)$, $g_i(x)$'s, $h_j(x)$'s $\in \mathbb{R}[x]$ are polynomials in $x = (x_1, \ldots, x_n)$ with real coefficients. Denote by $S$ the feasible set of (1), which is a basic closed semialgebraic set in $\mathbb{R}^n$.

Let $x^* \in S$ be a Karush–Kuhn–Tucker (KKT for short) point of (1), i.e., the first order necessary optimality conditions

$$\nabla f(x^*) - \sum_{i=1}^l \lambda_i \nabla g_i(x^*) - \sum_{j \in J(x^*)} \nu_j \nabla h_j(x^*) = 0 \quad \text{and} \quad \nu_j \geq 0, \ j \in J(x^*),$$

hold at $x^*$ for some Lagrange multipliers $\lambda_i$'s, $\nu_j$'s $\in \mathbb{R}$, where $J(x^*)$ denotes the active set at $x^*$. Our goal in this paper is to determine the type of $x^*$. In other words, is $x^*$ a local minimizer, maximizer or not extremum point of (1)?

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The motivation of our research is as follows. Many nonlinear programming algorithms are designed to generate a sequence of points which, under certain conditions, converges to a KKT point. However, there is no theoretical guarantee that the obtained KKT point is local minimizer. Some algorithms are purely based on solving the system of first order optimality conditions of the optimization problem. Hence, the obtained KKT point may even be a maximizer. Unfortunately, we will see that testing the type of a KKT point may be a hard problem as shown by Murty and Kabadi in [22]. In fact, for (1), to decide the type of $x^*$, we may first consider the second-order necessary condition for $x^*$ to be a local minimizer (resp., maximizer). Let $H$ be the Hessian matrix of the Lagrange function of (1) with respect to $x$ at $x^*$. We need to check whether $y^T H y$ is nonnegative (resp., nonpositive) for all $y \in Y$, where

$$Y := \{ y \in \mathbb{R}^n : \begin{cases} \nabla g_i(x^*)^T y = 0, & i = 1, \ldots, l, \\ \nabla h_j(x^*)^T y = 0, & \text{for all } j \in J(x^*) \text{ with } \nu_j > 0 \\ \nabla h_j(x^*)^T y \geq 0, & \text{for all } j \in J(x^*) \text{ with } \nu_j = 0 \end{cases} \}.$$ 

If the second-order necessary condition holds, we may further consider the second-order sufficient condition. That is, to decide if $y^T H y$ is positive (resp., negative) for all $0 \neq y \in Y$. However, when $H$ is not positive semidefinite and the set $\{ j \mid j \in J(x^*), \nu_j = 0 \}$ is nonempty, Murty and Kabadi showed that if the entries in $H$, $\nabla g_i(x^*)$’s, $\nabla h_j(x^*)$’s are rational, then checking whether the second-order sufficient condition holds is co-NP-complete (c.f. [22, Theorem 4]). Even if we are able to check the second-order sufficient condition, if it is not satisfied, there is no straightforward and simple method to determine whether $x^*$ is a local minimizer (resp., maximizer) by present theory. In particular, Murty and Kabadi showed (c.f. [22, Theorem 2]) that checking if the KKT point 0 is not a local minimizer for the nonconvex quadratic problem $\min_{x \geq 0} x^T D x$ is NP-complete, where $D$ is a not positive matrix.

To the best of our knowledge, there is very little related work in the literature addressing this issue, even in the unconstrained case. If there is no constraint, the problem reduces to determining the type of a degenerate real critical point of $f$, i.e., a point at which the gradient $\nabla f$ vanishes and the Hessian matrix $\nabla^2 f$ is singular. To solve it, it is intuitive to consider the higher order partial derivatives of $f$. However, it is difficult to take into account only the higher order derivatives of $f$, to systematically solve this problem. When $f$ is a sufficiently smooth function (not necessarily a polynomial), some partial answers to this problem were given in [8, 12] under certain assumptions on its Taylor expansion at the point. When $f$ is a multivariate real polynomial, Qi investigated its critical points and extrema structures in [25] without giving a method to determine their types. Nie gave a numerical method in [24] to compute all $H$-minimizers (critical points at which the Hessian matrices are positive
semidefinite) of a polynomial by semidefinite relaxations. However, there is no completed procedure in [24] to verify that a $H$-minimizer is a saddle point.

Very recently, Guo and Pham [18] proposed a method to determine the type of an isolated degenerate real critical point of a multivariable real polynomial. They showed that the type of the critical point can be determined by the global extrema of the polynomial over the Euclidean ball centered at the critical point with the so-called faithful radius. An algorithm involving algebraic computations to compute a faithful radius of the critical point is given in [18]. To decide the type, instead of computing the extrema of the polynomial over the ball which itself is NP-hard, they presented an algorithm to identify the type by computing isolating intervals for each real root of a zero-dimensional polynomial system, which can be carried out efficiently (c.f. [1, 14–16, 26]). In this paper, we extend the method in [18] to constrained case (1). We generalize the definition of faithful radius to an isolated KKT point $x^*$ by means of the tangency variety of $f$ at $x^*$ over the constraint $S$ and derive analogue strategies as proposed in [18] to decide the type of $x^*$.

To end this section, we would like to point out that determining the type of the KKT point $x^*$ is a special case of the quantifier elimination problem. Precisely, determining the type of $x^*$ is equivalent to checking the truth of the following first-order sentences

$$\forall r \exists x (r = 0) \lor (\|x - x^*\|^2 \leq r^2) \land (g_1(x) = 0) \land \cdots \land (g_l(x) = 0) \land (h_1(x) \geq 0) \land \cdots \land (h_m(x) \geq 0) \land (f(x) > f(x^*)),$$

$$\forall r \exists x (r = 0) \lor (\|x - x^*\|^2 \leq r^2) \land (g_1(x) = 0) \land \cdots \land (g_l(x) = 0) \land (h_1(x) \geq 0) \land \cdots \land (h_m(x) \geq 0) \land (f(x) < f(x^*)),$$

where $\lor$ and $\land$ respectively denote the logical connectives “or” and “and”. These decision problems can be solved by algorithms based on the cylindrical algebraic decomposition (CAD) [4, 8]. However, the arithmetic complexity for solving them by the CAD is $((l + m + 3)D)^{O(n+1)}$ where $D \geq 2$ is a bound for the degrees of $f$, $g_i$’s and $h_j$’s [4, Exercise 11.7]. The complexity is doubly exponential in $n$ and limits its practical application to nontrivial problems. Indeed, a cylindrical decomposition of the whole space seems to be superfluous for determining the local extremality of $x^*$. Comparatively, by investigating the local values of $f$ on its tangency variety on $S$ at $x^*$ (Definition 3.4), the approach proposed in this paper enjoys a lower complexity at least in certain circumstances (see discussions in Section A.2), which can be observed from the numerical experiments in Section 5.

The paper is organized as follows. Some notation and preliminaries used in this paper are given in Section 2. We study some properties of the set of KKT points and tangency varieties in Section 3. In Section 4, we define the faithful radius of an isolated KKT point, by which we show how to decide the type of the KKT point. Some computational aspects are
investigated in Section 5, where the algorithm for determining its type of an isolated KKT point are presented. For a better readability, we put the correctness proof and complexity discussions of the algorithm in the Appendix A.

2. Preliminaries

We use the following notation and terminology. The symbol $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers. We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ denotes the ring of polynomials in variables $x = (x_1, \ldots, x_n)$ with real coefficients. The Euclidean space $\mathbb{R}^n$ is equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. For convenience, let $\|x\|^2 := x_1^2 + \cdots + x_n^2$ for any $x \in \mathbb{C}^n$. Denote $\mathbb{R}^{n \times n}$ (resp., $\mathbb{C}^{n \times n}$) as the set of $n \times n$ matrices with real (resp., complex) number entries. Denote by $\|A\|$ the 2-norm of a matrix $A \in \mathbb{R}^{n \times n}$. For $R > 0$, denote by $B_R(x)$ (resp., $B_R$) the closed ball with center $x$ (resp., 0) and radius $R$. For a subset $S \subset \mathbb{R}^n$, the interior and closure of $S$ in Euclidean topology is denoted by $\text{int}(S)$ and $\bar{S}$, respectively. The notation $C^p$ means $p$-times continuously differentiable; $C^\infty$ is infinitely continuously differentiable. If $f, g$ are two functions with suitably chosen domains and codomains, then $f \circ g$ denotes the composite function of $f$ and $g$.

2.1. Semialgebraic geometry. Let us recall some notion and results from semialgebraic geometry (see, for example, [5, 7, 28]) which we need.

Definition 2.1. (i) A subset of $\mathbb{R}^n$ is said semialgebraic if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) = 0, i = 1, \ldots, k; \ f_i(x) > 0, i = k + 1, \ldots, p\},$$

where all $f_i$'s are in $\mathbb{R}[x]$.

(ii) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semialgebraic sets. A map $F: A \to B$ is said to be semialgebraic if its graph

$$\{(x, y) \in A \times B \mid y = F(x)\}$$

is a semialgebraic subset in $\mathbb{R}^n \times \mathbb{R}^m$.

The class of semialgebraic sets is closed under taking finite intersections, finite unions, and complements; a Cartesian product of semialgebraic sets is a semialgebraic set. Moreover, a major fact concerning the class of semialgebraic sets is its stability under linear projections (see, for example, [3, 4, 28]).

Theorem 2.1 (Tarski–Seidenberg Theorem). The image of a semialgebraic set by a semialgebraic map is semialgebraic.
By the Tarski–Seidenberg Theorem, it is not hard to see that the closure and the interior of a semialgebraic set are semialgebraic sets.

Recall the Curve Selection Lemma which will be used in this paper (see, for example, [20, 21]).

**Lemma 2.1 (Curve Selection Lemma).** Let $A$ be a semialgebraic subset of $\mathbb{R}^n$, and $u^* \in \overline{A} \setminus A$. Then there exists a real analytic semialgebraic curve

$$\phi: (-\epsilon, \epsilon) \to \mathbb{R}^n$$

with $\phi(0) = u^*$ and with $\phi(t) \in A$ for $t \in (0, \epsilon)$.

In what follows, we will need the following useful results (see, for example, [28]).

**Lemma 2.2 (Monotonicity Lemma).** Let $a < b$ in $\mathbb{R}$. If $f: [a, b] \to \mathbb{R}$ is a semialgebraic function, then there is a partition $a =: t_1 < \cdots < t_N := b$ of $[a, b]$ such that $f|_{(t_l, t_{l+1})}$ is $C^1$, and either constant or strictly monotone, for $l \in \{1, \ldots, N-1\}$.

The next theorem (see [7, 28]) uses the concept of a cell whose definition we omit. We do not need the specific structure of cells described in the formal definition. For us, it will be sufficient to think of a $C^p$-cell of dimension $r$ as of an $r$-dimensional $C^p$-manifold, which is the image of the cube $(0,1)^r$ under a semialgebraic $C^p$-diffeomorphism. As follows from the definition, an $n$-dimensional cell in $\mathbb{R}^n$ is an open set.

**Theorem 2.2 (Cell Decomposition Theorem).** Let $A \subset \mathbb{R}^n$ be a semialgebraic set. Then, for any $p \in \mathbb{N}$, $A$ can be represented as a disjoint union of a finite number of cells of class $C^p$.

By Cell Decomposition Theorem, for any $p \in \mathbb{N}$ and any nonempty semialgebraic subset $A$ of $\mathbb{R}^n$, we can write $A$ as a disjoint union of finitely many semialgebraic $C^p$-manifolds of different dimensions. The dimension $\dim A$ of a nonempty semialgebraic set $A$ can thus be defined as the dimension of the manifold of highest dimension of its decomposition. This dimension is well defined and independent of the decomposition of $A$. By convention, the dimension of the empty set is taken to be negative infinity. We will need the following result (see [7, 28]).

**Proposition 2.1.** (i) Let $A \subset \mathbb{R}^n$ be a semialgebraic set and $f: A \to \mathbb{R}^m$ a semialgebraic map. Then, $\dim f(A) \leq \dim A$.

(ii) Let $A \subset \mathbb{R}^n$ be a nonempty semialgebraic set. Then, $\dim(\overline{A} \setminus A) < \dim A$. In particular, $\dim(\overline{A}) = \dim A$.

(iii) Let $A, B \subset \mathbb{R}^n$ be semialgebraic sets. Then,

$$\dim(A \cup B) = \max\{\dim A, \dim B\}.$$
Combining Theorems 2.4.4, 2.4.5 and Proposition 2.5.13 in [4], it follows that

**Proposition 2.2.** Let $A$ be a semialgebraic set of $\mathbb{R}^n$. The following statements hold.

(i) $A$ has a finite number of connected components which are closed in $A$.

(ii) $A$ is connected if and only if it is path connected.

Hence, in the rest of this paper, by saying that a semialgebraic subset of $\mathbb{R}^n$ is connected, we also mean that it is path connected.

Next we state a semialgebraic version of Sard’s theorem with the parameter in a simplified form which is sufficient for the applications studied here. Given a differentiable map between manifolds $f : X \to Y$, a point $y \in Y$ is called a regular value for $f$ if either $f^{-1}(y) = \emptyset$ or the derivative map $Df(x) : T_x X \to T_y Y$ is surjective at every point $x \in f^{-1}(y)$, where $T_x X$ and $T_y Y$ denote the tangent spaces of $X$ at $x$ and of $Y$ at $y$, respectively. A point $y \in Y$ that is not a regular value of $f$ is called a critical value. The following result is also called Thom’s weak transversality theorem.

**Theorem 2.3** (Sard’s theorem with parameter). Let $f : X \times \mathcal{P} \to Y$ be a differentiable semialgebraic map between semialgebraic submanifolds. If $y \in Y$ is a regular value of $f$, then there exists a semialgebraic set $\Sigma \subset \mathcal{P}$ of dimension smaller than the dimension of $\mathcal{P}$ such that, for every $p \in \mathcal{P} \setminus \Sigma$, $y$ is a regular value of the map $f_p : X \to Y, x \mapsto f(x, p)$.

**Proof.** For a proof, we refer the reader to [17] or [20, Theorem 1.10].

2.2. **Algebraic geometry.** A subset $I \subseteq \mathbb{R}[x]$ is said an ideal if $0 \in I$, $I + I \subseteq I$ and $p \cdot q \in I$ for all $p \in I$ and $q \in \mathbb{R}[x]$. For $g_1, \ldots, g_s \in \mathbb{R}[x]$, denote $\langle g_1, \ldots, g_s \rangle$ as the ideal in $\mathbb{R}[x]$ generated by $g_i$’s, i.e., the set $g_1 \mathbb{R}[x] + \cdots + g_s \mathbb{R}[x]$. An ideal is radical if $f^m \in I$ for some integer $m \geq 1$ implies that $f \in I$. The radical of an ideal $I \subseteq \mathbb{R}[x]$, denoted $\sqrt{I}$, is the set $\{f \in \mathbb{R}[x] \mid f^m \in I$ for some integer $m \geq 1\}$. An affine variety (resp., real affine variety) is a subset of $\mathbb{C}^n$ (resp., $\mathbb{R}^n$) that consists of common zeros of a set of polynomials. For an ideal $I \subseteq \mathbb{R}[x]$, denote $V_\mathbb{C}(I)$ and $V_\mathbb{R}(I)$ as the affine varieties defined by $I$ in $\mathbb{C}^n$ and $\mathbb{R}^n$, respectively. For a polynomial $g \in \mathbb{R}[x]$, respectively replace $V_\mathbb{C}(\langle g \rangle)$ and $V_\mathbb{R}(\langle g \rangle)$ by $V_\mathbb{C}(g)$ and $V_\mathbb{R}(g)$ for simplicity. Given a set $V \subseteq \mathbb{C}^n$, denote $I(V) \subseteq \mathbb{R}[x]$ as the vanishing ideal of $V$ in $\mathbb{R}[x]$, i.e., the set of all polynomials in $\mathbb{R}[x]$ which equal zero at every point in $V$. For an ideal $I \subseteq \mathbb{R}[x]$, denote dim$(I)$ as the Hilbert dimension of $I$, i.e., the degree of the affine Hilbert polynomial of $I$. For an ideal $I \subseteq \mathbb{R}[x]$, the decomposition $I = I_1 \cap \cdots \cap I_s$ is said the equidimensional decomposition of $I$ if each ideal $I_i$ is pure dimensional, i.e., all its associated primes have the same dimension. For an affine variety $V \subseteq \mathbb{C}^n$, denote dim$(V) = \text{dim}(I(V))$ as its dimension. When $V_\mathbb{C}(I)$ is finite, the ideal $I$ is said to be zero-dimensional. For any subset $S \subseteq \mathbb{C}^n$, denote $\overline{S}^\mathbb{Z}$ as the Zariski closure of $S$ in $\mathbb{C}^n$, i.e., $\overline{S}^\mathbb{Z} = V_\mathbb{C}(I(S))$. 


Recall the polynomials \( f, g_i, h_j \in \mathbb{R}[x] \) in (1) and the basic closed semialgebraic set
\[
S = \{ x \in \mathbb{R}^n \mid g_i(x) = 0, \ i = 1, \ldots, l, \ h_j(x) \geq 0, \ j = 1, \ldots, m \}.
\]

Let \( x^* \in S \) be a fixed KKT point in the rest of this paper. We also assume that \( x^* \) is not an isolated point of \( S \).

**Definition 3.1.**
(i) The point \( x^* \) is said to be a *local minimizer* of \( f \) on \( S \) if there is an open neighborhood \( U \) of \( x^* \) such that
\[
f(x^*) \leq f(x) \quad \text{for all } x \in S \cap U.
\]
(ii) The point \( x^* \) is said to be a *local maximizer* of \( f \) on \( S \) if there is an open neighborhood \( U \) of \( x^* \) such that
\[
f(x^*) \geq f(x) \quad \text{for all } x \in S \cap U.
\]
(iii) The point \( x^* \) is not an *extremum point* of \( f \) on \( S \) if for any open neighborhood \( U \) of \( x^* \), there exist \( u,v \in S \cap U \) such that
\[
f(u) < f(x^*) < f(v).
\]

### 3. KKT points and tangencies

3.1. **KKT points.** As is well known, most numerical optimization methods targeting local (including global) minimizers are often based on the following KKT optimality conditions:

\[
\nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0,
\]
\[
g_i(x) = 0, \ i = 1, \ldots, l, \ h_j(x) \geq 0, \ j = 1, \ldots, m,
\]
\[
\nu_j h_j(x) = 0, \ \nu_j \geq 0, \ j = 1, \ldots, m,
\]

where the variables \( \lambda_i, \nu_j \in \mathbb{R} \) are said to be Lagrange multipliers and \( \nabla f \) denotes the vector whose components are the partial derivatives of \( f \).

Sometimes the above KKT system fails to hold at some minimizers. Hence, we usually make an assumption said a *constraint qualification* to ensure that the KKT system holds. Such a constraint qualification—probably the one most often used in the design of algorithms—is defined as follows:

**Definition 3.2.** We say that the *linearly independent constraint qualification* (\((\text{LICQ})\) for short) holds at \( x \in S \) if the system of the vectors \( \nabla g_i(x), i = 1, \ldots, l, \nabla h_j(x), j \in J(x) \), is linearly independent, where \( J(x) \) is the set of indices \( j \) for which \( h_j \) vanishes at \( x \).
Remark 3.1. (i) Note that (LICQ) is generally satisfied, for a proof see [20, Theorem 6.1].

(ii) Since $x^*$ is not isolated in $S$, we can see that if (LICQ) holds at $x^*$, then $n - l - \#J(x^*) \geq 1$ where $\#J(x^*)$ denotes the number of elements in $J(x^*)$ and so $n - l \geq 1$.

Lemma 3.1. If (LICQ) holds at $x^* \in S$, then there exists a real number $R > 0$ such that (LICQ) holds at every $x \in S \cap B_R(x^*)$.

Proof. Since (LICQ) holds at $x^*$, then the system of the vectors $\nabla g_i(x^*), i = 1, \ldots, l, \nabla h_j(x^*), j \in J(x^*)$ is linearly independent. By continuity, for all $x$ near to $x^*$, $J(x) \subset J(x^*)$ and the system of the vectors $\nabla g_i(x), i = 1, \ldots, l, \nabla h_j(x), j \in J(x)$, is linearly independent. 

The following lemma says that if (LICQ) holds at $x^* \in S$, then the set $S$ intersects transversally the sphere $\{x \in \mathbb{R}^n \mid \|x - x^*\| = R\}$ for all $R > 0$ small enough.

Lemma 3.2. If (LICQ) holds at $x^* \in S$, then there exists a real number $R > 0$ such that the vectors $\nabla g_i(x), i = 1, \ldots, l, \nabla h_j(x), j \in J(x)$, and $x - x^*$ are linearly independent for all $x \in S \cap B_R(x^*) \setminus \{x^*\}$.

Proof. Without loss of generality, assume $x^* = 0$. Suppose that the lemma is not true, then there exists a sequence $\{x^k\} \subset S$ tending to 0 such that $x^k \neq 0$ and the system of the vectors $\nabla g_i(x^k), i = 1, \ldots, l, \nabla h_j(x^k), j \in J(x^k)$, and $x^k$ is linearly dependent for all $k$, i.e., there exist $\lambda^k_i, \nu^k_j$ and $\mu^k \in \mathbb{R}$ such that

$$\sum_{i=1}^{l} \lambda^k_i \nabla g_i(x^k) + \sum_{j \in J(x^k)} \nu^k_j \nabla h_j(x^k) + \mu^k x^k = 0, \text{ and}$$

$$\sum_{i=1}^{l} (\lambda^k_i)^2 + \sum_{j \in J(x^k)} (\nu^k_j)^2 + (\mu^k)^2 = 1.$$  

By passing to a subsequence, if necessary, we may assume that $J(x^k) = J \subset \{1, 2, \ldots, m\}$ for all $k$, and there exist the following limits

$$\lambda^*_i := \lim_{k \to \infty} \lambda^k_i, \quad \nu^*_j := \lim_{k \to \infty} \nu^k_j, \quad \text{and} \quad \mu^* := \lim_{k \to \infty} \mu^k.$$
Let

\[ A := \{(x, \lambda, \nu, \mu) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{#J} \times \mathbb{R} \mid g_i(x) = 0, \ i = 1, \ldots, l, \ h_j(x) = 0, \ j \in J, \]

\[ \sum_{i=1}^{l} \lambda_i \nabla g_i(x) + \sum_{j \in J} \nu_j \nabla h_j(x) + \mu x = 0, \]

\[ \sum_{i=1}^{l} \lambda_i^2 + \sum_{j \in J} \nu_j^2 + \mu^2 = 1 \}.\]

Then \( A \) is a semialgebraic set and \((0, \lambda^*, \nu^*, \mu^*)\) is a limit point of the set \( \{(x, \lambda, \nu, \mu) \in A \mid x \neq 0\} \) which is also semialgebraic. Using the Curve Selection Lemma 2.1, there exist a smooth semialgebraic curve \( \varphi(t) \) and semialgebraic functions \( \lambda_i(t), \nu_j(t), \mu(t), \ t \in (-\epsilon, \epsilon) \), such that

(a1) \( (\varphi(t), \lambda(t), \nu(t), \mu(t)) \in A \) and \( \varphi(t) \neq 0 \) for \( t \in (0, \epsilon) \);

(a2) \( \varphi(t) \to 0 \) as \( t \to 0^+ \).

It follows from (a1) that

\[ 0 = \sum_{i=1}^{l} \lambda_i(t) \langle \nabla g_i(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle + \sum_{j \in J} \nu_j(t) \langle \nabla h_j(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle + \mu(t) \langle \varphi(t), \frac{d\varphi(t)}{dt} \rangle \]

\[ = \sum_{i=1}^{l} \lambda_i(t) \frac{d}{dt} (g_i \circ \varphi)(t) + \sum_{j \in J} \nu_j(t) \frac{d}{dt} (h_j \circ \varphi)(t) + \frac{\mu(t) d\|\varphi(t)\|^2}{2} \frac{dt}{dt} \]

\[ = \frac{\mu(t) d\|\varphi(t)\|^2}{2} \frac{dt}{dt} \]

holds for each \( t \in (0, \epsilon) \). Applying the Monotonicity Lemma 2.2 and shrinking \( \epsilon \) (if necessary), we may assume that the functions \( \mu(t) \) and \( \|\varphi(t)\| \) are either constant or strictly monotone. Then, (a1) implies that for each \( t \in (0, \epsilon) \), \( \mu(t) = 0 \) and hence the vectors \( \nabla g_i(\varphi(t)), \ i = 1, \ldots, l, \nabla h_j(\varphi(t)), j \in J \subseteq J(\varphi(t)) \), are linearly dependent. By (a2), it contradicts Lemma 3.1.

**Definition 3.3.** The set of KKT points of \( f \) on \( S \) is defined as follows:

\[ \Sigma(f, S) := \{ x \in S \mid \text{there exist } \lambda_i, \nu_j \in \mathbb{R} \text{ such that} \]

\[ \nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \text{ and} \]

\[ \nu_j h_j(x) = 0, \ j = 1, \ldots, m \}. \]

**Remark 3.2.** By the Tarski–Seidenberg Theorem 2.1, \( \Sigma(f, S) \) is a semialgebraic (possibly empty) set and so it has a finite number of connected components. Moreover it is not hard to
see that if (LICQ) holds at every point in $S$ then $f(\Sigma(f, S))$ is a finite set (see, for example, [20, Theorem 2.3]).

The following statement is well known (see, for example, [6]).

**Theorem 3.1 (KKT necessary optimality conditions).** Assume that (LICQ) holds at $x^* \in S$. If $x^*$ is a local minimizer (or maximizer) of $f$ on $S$, then $x^* \in \Sigma(f, S)$.

**Corollary 3.1.** Assume that (LICQ) holds at $x^* \in S$ and $x^*$ is an isolated KKT point of $f$ on $S$. Then the restriction of $f$ on $S$ is nonconstant in some neighborhood of $x^*$.

**Proof.** This follows immediately from Lemma 3.1 and Theorem 3.1.

By the above corollary, we can see that if (LICQ) holds at $x^* \in S$ and $x^*$ is an isolated KKT point of $f$, then $x^*$ is a local minimizer (resp., maximizer) of $f$ if and only if it is an isolated local minimizer (resp., maximizer) of $f$.

### 3.2. Tangencies.

**Definition 3.4.** [19] The tangency variety of $f$ on $S$ at $x^*$ is defined as follows:

$$
\Gamma(f, S, x^*) := \{ x \in S \mid \text{there exist real numbers } \kappa, \lambda_i, \nu_j, \mu, \text{ not all zero, such that } \kappa \nabla f(x) - \sum_{i=1}^{t} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) - \mu (x - x^*) = 0, \text{ and } \nu_j h_j(x) = 0, \ j = 1, \ldots, m \}.
$$

**Lemma 3.3.** The following statements hold:

(i) $\Sigma(f, S) \subset \Gamma(f, S, x^*)$;
(ii) $\Gamma(f, S, x^*)$ is a nonempty, closed and semialgebraic set; in particular, it has a finite number of connected components;

If the restriction of $f$ on $S$ is nonconstant in some neighborhood of $x^*$, then

(iii) $x^* \in \Gamma(f, S, x^*)$ and it is a limit point of $\Gamma(f, S, x^*) \setminus \Sigma(f, S)$;
(iv) For any $R > 0$, $\dim (\Gamma(f, S, x^*) \setminus \Sigma(f, S)) \cap B_R(x^*) \geq 1$.

**Proof.** Without loss of generality, we assume $x^* = 0$.

(i) This is clear by definition.
(ii) For each $t \geq 0$, let

$$
S_t := S \cap \{ x \in \mathbb{R}^n \mid \| x \| = t \}.
$$

Since $x^*$ is not an isolated point of $S$, there exists $\epsilon > 0$ such that $S_t$ is a nonempty and compact set for all $t \in [0, \epsilon)$. The set $\Gamma(f, S, x^*)$ is nonempty because it contains all extremal points of $f$ on $S_t$ for all $t \in [0, \epsilon)$ by the Fritz-John necessary optimality conditions (see,
for example, \([\text{6}]\). The closedness of \(\Gamma(f, S, x^*)\) follows immediately from the definition. By the Tarski–Seidenberg Theorem\([\text{2}].\text{1}\), \(\Gamma(f, S, x^*)\) is a semialgebraic set, and so it has a finite number of connected components (due to Proposition\([\text{2}].\text{2}\)).

(iii) It is clear that \(x^* \in \Gamma(f, S, x^*)\) by definition. For the real number \(\epsilon > 0\) defined above, we can find two semialgebraic curves \(\varphi, \psi: [0, \epsilon) \to \mathbb{R}^n\) such that

(b1) \(\varphi(t)\) and \(\psi(t)\) are the minimizer and maximizer of \(f\) on \(S_t\) for \(t \in [0, \epsilon)\), respectively;

(b2) \(\|\varphi(t)\| = \|\psi(t)\| = t \to 0\) as \(t \to 0^+\).

By the Fritz-John necessary optimality conditions (see, for example, \([\text{6}]\)), \(\varphi(t), \psi(t) \in \Gamma(f, S, x^*)\). Hence, \(x^*\) is not isolated in \(\Gamma(f, S, x^*)\). By the Monotonicity Lemma \([\text{2}].\text{2}\) we may assume that \(\varphi\) and \(\psi\) are differentiable on \((0, \epsilon)\) (perhaps after reducing \(\epsilon\)).

Suppose that \(\varphi(t) \in \Sigma(f, S)\) for all \(t \in (0, \epsilon)\). Then there exist semialgebraic functions \(\lambda_i, \nu_j: (0, \epsilon) \to \mathbb{R}\) such that

(b3) \(\nabla f(\varphi(t)) - \sum_{i=1}^l \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^m \nu_j(t) \nabla h_j(\varphi(t)) \equiv 0\).

(b4) \(\nu_j(t) h_j(\varphi(t)) \equiv 0, \quad j = 1, \ldots, m\).

Since the functions \(\nu_j\) and \(h_j \circ \varphi\) are semialgebraic, for \(\epsilon > 0\) small enough, these functions are either constant or strictly monotone (thanks to the Monotonicity Lemma\([\text{2}].\text{2}\)). Then, by (b4), we can see that either \(\nu_j(t) \equiv 0\) or \((h_j \circ \varphi)(t) \equiv 0\) on \((0, \epsilon)\); in particular,

\[\nu_j(t) \frac{d}{dt}(h_j \circ \varphi)(t) \equiv 0, \quad j = 1, \ldots, m.\]

It follows from (b3)-(b4) that

\[0 = \langle \nabla f(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle - \sum_{i=1}^l \lambda_i(t) \langle \nabla g_i(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle - \sum_{j=1}^m \nu_j(t) \langle \nabla h_j(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle = \frac{d}{dt}(f \circ \varphi)(t) - \sum_{i=1}^l \lambda_i(t) \frac{d}{dt}(g_i \circ \varphi)(t) - \sum_{j=1}^m \nu_j(t) \frac{d}{dt}(h_j \circ \varphi)(t) = \frac{d}{dt}(f \circ \varphi)(t).\]

Consequently, \(f \circ \varphi\) is a constant function on \((0, \epsilon)\).

Similarly, suppose that the curve \(\psi(t)\) lies in \(\Sigma(f, S)\) for all \(t \in (0, \epsilon)\). Then \(f \circ \psi\) is a constant function on \((0, \epsilon)\). Since \(f\) is continuous, we have \(f \circ \varphi \equiv f \circ \psi \equiv f(0)\). It follows from (b1) that \(f\) is constant on \(\mathbb{R}_\epsilon\), a contradiction.

Therefore, for any \(0 < \epsilon' < \epsilon\), there exists \(t \in (0, \epsilon')\) such that either \(\varphi(t) \in \Gamma(f, S, x^*) \setminus \Sigma(f, S)\) or \(\psi(t) \in \Gamma(f, S, x^*) \setminus \Sigma(f, S)\). This, together with (b2), implies (iii).

(iv) This follows from (iii). \(\square\)
**Lemma 3.4.** Assume that (LICQ) holds at $x^*$. Then there exists $R > 0$ such that for all $x \in \Gamma(f, S, x^*) \cap B_R(x^*) \setminus \{x^*\}$, there exist real numbers $\lambda_i, \nu_j, \mu$ such that

$$
\nabla f(x) - \sum_{i=1}^l \lambda_i \nabla g_i(x) - \sum_{j=1}^m \nu_j \nabla h_j(x) - \mu (x - x^*) = 0,
$$

$$
\nu_j h_j(x) = 0, \quad j = 1, \ldots, m.
$$

**Proof.** This follows directly from Lemma 3.2. \qed

We will show that in general, in some neighbourhood of $x^*$, $\Gamma(f, S, x^*) \setminus \Sigma(f, S)$ is a curve. To see this, it suffices to change the Euclidean norm $\| \cdot \|$ by a “generic” one. More precisely, let $\mathcal{P}$ be the set of symmetric positive definite $n \times n$ matrices. Clearly, $\mathcal{P}$ is an open semialgebraic subset of $\mathbb{R}^{n(n+1)/2}$, where we identify $P = (p_{ij})_{n \times n} \in \mathcal{P}$ with $(p_{11}, \ldots, p_{1n}, p_{22}, \ldots, p_{2n}, \ldots, p_{nn}) \in \mathbb{R}^{n(n+1)/2}$. For each $P \in \mathcal{P}$, let

$$
\Gamma_P(f, S, x^*) := \{ x \in S \mid \text{there exist real numbers } \kappa, \lambda_i, \nu_j, \mu, \text{ not all zero, such that } \}
$$

$$
\kappa \nabla f(x) - \sum_{i=1}^l \lambda_i \nabla g_i(x) - \sum_{j=1}^m \nu_j \nabla h_j(x) - \mu P (x - x^*) = 0, \quad \text{and}
$$

$$
\nu_j h_j(x) = 0, \quad j = 1, \ldots, m\}.
$$

**Theorem 3.2.** Assume that (LICQ) holds at $x^*$ and the restriction of $f$ on $S$ is nonconstant in some neighbourhood of $x^*$. Then there exists an open and dense semialgebraic set $\mathcal{U}$ in $\mathcal{P}$ such that for each $P \in \mathcal{U}$, the set $(\Gamma_P(f, S, x^*) \setminus \Sigma(f, S)) \cap \text{int}(B_{R_P}(x^*))$ is a one-dimensional manifold for some $R_P > 0$ depending on $P$.

**Proof.** Without loss of generality, we assume $x^* = 0$. Choose a $R > 0$ satisfying the conditions in Lemmas 3.1 and 3.2. For each subset $J := \{ j_1, \ldots, j_k \}$ of $\{1, \ldots, m\}$, let $\nu := (\nu_j)_{j \in J} \in \mathbb{R}^{\# J}$, and

$$
X_J := \{ (x, \kappa, \lambda, \nu, \mu) \in \mathbb{R}^{n} \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{\# J} \times \mathbb{R} \mid \}
$$

$$
\kappa^2 + \sum_{i=1}^l \lambda_i^2 + \sum_{j \in J} \nu_j^2 + \mu^2 = 1, \mu \neq 0,
$$

$$
0 < \|x\| < R, \quad h_j(x) > 0 \text{ for } j \notin J\}.
$$

Clearly, $X_J$ is a semialgebraic manifold of dimension $n + l + \# J + 1$. Assume that $X_J \neq \emptyset$. We define the semialgebraic map $\Phi_J : X_J \times \mathcal{P} \to \mathbb{R}^{n} \times \mathbb{R}^l \times \mathbb{R}^{\# J}$ by

$$
\Phi_J(x, \kappa, \lambda, \nu, \mu, P) := \left( \kappa \nabla f(x) - \sum_{i=1}^l \lambda_i \nabla g_i(x) - \sum_{j \in J} \nu_j \nabla h_j(x) - \mu P x, \right.
$$

$$
g_1(x), \ldots, g_l(x), h_{j_1}(x), \ldots, h_{j_k}(x) \bigg).$$
Take any \((x, \kappa, \lambda, \nu, \mu, P) \in \Phi^{-1}_J(0)\). Then \(x \neq 0\). Without loss of generality, we assume that \(x_1 \neq 0\). Note that \(p_{ij} = p_{ji}\). Then, a direct computation shows that

\[
\left( D_x \Phi_J \mid D_{(p_{11}, \ldots, p_{1n})} \Phi_J \right) = \begin{pmatrix}
\begin{bmatrix}
x_1 & x_2 & \cdots & x_n \\
0 & x_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_1
\end{bmatrix} & -\mu \\
0 & \vdots & \ddots & \vdots \\
0 & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_1
\end{pmatrix}
\begin{pmatrix}
[\nabla g_1(x)]^T \\
\vdots \\
[\nabla g_l(x)]^T \\
[\nabla h_{j_1}(x)]^T \\
\vdots \\
[\nabla h_{j_k}(x)]^T
\end{pmatrix}
\end{pmatrix},
\]

where \(D_x \Phi_J\) and \(D_{(p_{11}, \ldots, p_{1n})} \Phi_J\) denote the derivative of \(\Phi_J\) with respect to \(x\) and \((p_{11}, \ldots, p_{1n})\), respectively. It follows from Lemma 3.1 that the rank of the Jacobian matrix of the map \(\Phi_J\) is \(n + l + \#J\) and hence 0 is a regular value of \(\Phi_J\). By the Sard theorem with parameter 2.3 there exists a semialgebraic subset \(\Sigma_J \subset \mathcal{P}\) of dimension \(< \dim \mathcal{P}\) such that for each \(P \in \mathcal{P} \setminus \Sigma_J\), 0 is a regular value of the map

\[\Phi_{J,P} : X_J \rightarrow \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}, \quad (x, \kappa, \lambda, \nu, \mu) \mapsto \Phi_J(x, \kappa, \lambda, \nu, \mu, P).\]

Thus, \(\Phi^{-1}_{J,P}(0)\) is either empty or an one-dimensional submanifold of \(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{\#J} \times \mathbb{R}\). By Proposition 2.1, \(\dim \pi_J(\Phi^{-1}_{J,P}(0)) \leq 1\), where \(\pi_J : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{\#J} \times \mathbb{R} \rightarrow \mathbb{R}^n\) is the projection on the first \(n\) coordinates. Let \(\mathcal{U} := \bigcap_J (\mathcal{P} \setminus \Sigma_J)\), where the intersection is taken over all subsets \(J\) of \(\{1, \ldots, m\}\) with \(X_J \neq \emptyset\). Taking any \(P \in \mathcal{U}\), by Lemma 3.2 it is easy to check that

\[\Delta_P := (\Gamma_P(f, S, x^*) \setminus \Sigma(f, S)) \cap \text{int}(\mathbb{B}_R) \subseteq \bigcup_J \pi_J(\Phi^{-1}_{J,P}(0)).\]

Hence \(\dim \Delta_P \leq 1\). On the other hand, by Lemma 3.3 (iv), we can see that \(\dim \Delta_P \geq 1\). Therefore, \(\dim \Delta_P = 1\). By Cell Decomposition Theorem 2.2 \(\Delta_P\) is a finite disjoint union of one-dimensional manifolds and points. Now we can choose a \(R_P \leq R\) to exclude the finitely many points in the union by \(\text{int}(\mathbb{B}_{R_P})\). As the remaining one-dimensional manifolds in \(\text{int}(\mathbb{B}_{R_P})\) are disjoint, the proof is complete.

We next show that after a generic linear change of coordinates, \(\Gamma(f, S, x^*) \setminus \Sigma(f, S)\) is indeed a curve in a neighborhood of \(x^*\). Let \(\mathcal{S}^{n \times n}\) be the set of all invertible \(n \times n\) matrices.
in \( \mathbb{R}^{n \times n} \). For \( p \in \mathbb{R}[x] \) and \( A \in \mathcal{S}^{n \times n} \), denote \( p^A := p(Ax) \) the polynomial obtained by applying the change of variables \( A \) to \( p \) and
\[
S^A := \{ x \in \mathbb{R}^n \mid g_i^A(x) = 0, \; i = 1, \ldots, l, \; h_j^A(x) \geq 0, \; j = 1, \ldots, m \}.
\]

**Remark 3.3.** Note that the extremality of \( x^* \) as a KKT point of \( f \) on \( S \) remains the same after an invertible linear change of coordinates, which means that we can equivalently consider the extremality of \( f^A \) at \( A^{-1}x^* \) over the set \( S^A \) for any invertible matrix \( A \in \mathbb{R}^{n \times n} \).

**Corollary 3.2.** Assume that \((\text{LICQ})\) holds at \( x^* \). Then there exists a non-empty Zariski open set \( \mathcal{E} \) in \( \mathbb{R}^{n \times n} \) such that for each \( A \in \mathcal{E} \), the set \( (\Gamma(f^A, S^A, A^{-1}x^*) \setminus \Sigma(f^A, S^A)) \cap \text{int}(\mathcal{B}_{R_A}(A^{-1}x^*)) \) is a manifold of dimension one for some \( R_A > 0 \) depending on \( A \).

**Proof.** Since \( \nabla f^A(x) = A^T \nabla f(Ax) \) for any \( A \in \mathcal{S}^{n \times n} \), it is easy to check that \( \Sigma(f^A, S^A) = A^{-1}(\Sigma(f, S)) \) and \( \Gamma(f^A, S^A, A^{-1}x^*) = A^{-1}(\Gamma_{A^{-T}A^{-1}}(f, S, x^*)) \). Let \( \mathcal{U} \) be the open and dense semialgebraic set in \( \mathcal{P} \) as described in Theorem 3.2. Let \( \mathcal{U}^{-1} := \{ P^{-1} \in \mathcal{P} \mid P \in \mathcal{U} \} \). As \( \mathcal{U}^{-1} \) is also an open and dense semialgebraic set in \( \mathcal{P} \), by [20, Lemma 1.4], there exists a non-constant polynomial \( \mathcal{F}: \mathbb{R}^{n(n+1)} \to \mathbb{R} \) such that \( \mathcal{U}^{-1} \supseteq \{ P \in \mathcal{P} \mid \mathcal{F}(P) \neq 0 \} \). Let \( \mathcal{E} := \{ A \in \mathcal{S}^{n \times n} \mid \mathcal{F}(AA^T) \neq 0 \} \), then \( \mathcal{E} \) is a non-empty Zariski open set in \( \mathbb{R}^{n \times n} \). For each \( A \in \mathcal{E} \), we have \( A^{-T}A^{-1} \in \mathcal{U} \). By Theorem 3.2 \( (\Gamma_{A^{-T}A^{-1}}(f, S, x^*) \setminus \Sigma(f, S)) \cap \text{int}(\mathcal{B}_{R'_{A}}(x^*)) \) is a manifold of dimension one for some \( R'_{A} > 0 \) depending on \( A \). Set \( R_A = R'_{A}/\|A\| \). Then we can verify that \( \mathcal{B}_{R_A}(A^{-1}x^*) \subseteq A^{-1}(\mathcal{B}_{R'_{A}}(x^*)) \). Consequently, \( (\Gamma(f^A, S^A, A^{-1}x^*) \setminus \Sigma(f^A, S^A)) \cap \text{int}(\mathcal{B}_{R_A}(A^{-1}x^*)) \subseteq A^{-1}((\Gamma_{A^{-T}A^{-1}}(f, S, x^*) \setminus \Sigma(f, S)) \cap \text{int}(\mathcal{B}_{R'_{A}}(x^*))) \) is a manifold of dimension one. \( \square \)

In Appendix A.1 we will prove that the complex version of Corollary 3.2 still holds, which is crucial in the design of algorithms for testing the extremality of \( x^* \).

### 4. Faithful radii and types of KKT points

In this section, we first define the so-called faithful radius of \( f \) on \( S \) at \( x^* \) by means of the tangency variety \( \Gamma(f, S, x^*) \). Then, we show that the type of \( x^* \) can be determined by the global extrema of \( f \) over the intersection of \( \Gamma(f, S, x^*) \) and the ball centered at \( x^* \) with a faithful radius.

#### 4.1. On faithful radii.

**Definition 4.1.** We say that a real number \( R > 0 \) is a faithful radius of \( f \) on \( S \) at \( x^* \) if the following conditions hold:

(i) \( \Sigma(f, S) \cap \mathcal{B}_R(x^*) = \{ x^* \} \);
(ii) \( \Gamma(f, S, x^*) \cap \mathcal{B}_R(x^*) \) is connected; and
(iii) \( \Gamma(f, S, x^*) \cap \{ x \in \mathbb{R}^n \mid f(x) = f(x^*) \} \cap B_R(x^*) = \{ x^* \} \).

**Theorem 4.1.** Assume that (LICQ) holds at \( x^* \). The point \( x^* \in S \) is an isolated KKT point of \( f \) on \( S \) if and only if there is a faithful radius \( R \) of \( f \) on \( S \) at \( x^* \).

**Proof.** Sufficiency. This is clear.

**Necessity.** Without loss of generality, we assume that \( x^* = 0 \) and \( f(x^*) = 0 \). As \( 0 \) is an isolated KKT point, there exists \( R_1 > 0 \) such that \( \Sigma(f, S) \cap B_{R_1} = \{ 0 \} \).

By Theorem 2.2, \( \Gamma(f, S, x^*) \) is a disjoint union of finitely many submanifolds \( \Gamma_1, \ldots, \Gamma_s \), each diffeomorphic to an open hypercube \((0,1)^{\dim(\Gamma_j)}\). Consider the map \( \Phi : x \mapsto \sum_{i=1}^n x_i^2 \) on these manifolds. By the semialgebraic version of Sard’s theorem [7, Theorem 9.6.2], there are finitely many critical values of the map \( \Phi \) on \( \Gamma_1, \ldots, \Gamma_s \). Fix a \( R_2 \in \mathbb{R}_+ \) to be the smallest one, then \( \Gamma(f, S, x^*) \cap B_{R} \) is connected for any \( 0 < R < \sqrt{R_2} \). To see this, note that by Proposition 2.2, \( \Gamma(f, S, x^*) \cap B_{R} \) has finitely many connected components \( C_1, \ldots, C_l \) which are closed in \( \mathbb{R}^n \). To the contrary, suppose that \( l \geq 2 \) and \( 0 \not\in C_2 \). As \( C_2 \) is closed and bounded, the function \( \sum_{i=1}^n x_i^2 \) reaches its minimum on \( C_2 \) at a minimizer \( u \). Since \( C_2 \subseteq \Gamma_i \) for some \( i, u \) is a critical point of \( \Phi \) on \( \Gamma_i \), a contradiction.

Finally, we show that \( \Gamma(f, S, x^*) \cap \{ x \in \mathbb{R}^n \mid f(x) = 0 \} \cap B_{R} = \{ 0 \} \) for some \( R > 0 \). If this is not the case, then by Lemma 3.4 and the Curve Selection Lemma 2.1, there exist smooth nonconstant semialgebraic curve \( \varphi(t) \) and semialgebraic functions \( \lambda_i(t), \nu_j(t), \mu(t), t \in (0, \epsilon) \), such that

\begin{enumerate}[(d1)]
    \item \( \varphi(t) \in S \) and \( f(\varphi(t)) = 0 \), for \( t \in (0, \epsilon) \);
    \item \( \| \varphi(t) \| \to 0 \) as \( t \to 0^+ \);
    \item \( \nabla f(\varphi(t)) - \sum_{i=1}^l \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^m \nu_j(t) \nabla h_j(\varphi(t)) - \mu(t) \varphi(t) \equiv 0 \); and
    \item \( \nu_j(t) h_j(\varphi(t)) \equiv 0, j = 1, \ldots, m \).
\end{enumerate}

As is shown in the proof of Lemma 3.3 (iii), we may assume that

\[
\nu_j(t) \frac{d}{dt} (h_j \circ \varphi)(t) \equiv 0, \quad j = 1, \ldots, m.
\]

It follows from (d3) that

\[
0 = \langle \nabla f(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle - \sum_{i=1}^l \lambda_i(t) \langle \nabla g_i(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle - \sum_{j=1}^m \nu_j(t) \langle \nabla h_j(\varphi(t)), \frac{d\varphi(t)}{dt} \rangle - \mu(t) \langle \varphi(t), \frac{d\varphi(t)}{dt} \rangle
\]

\[
= \frac{d}{dt} (f \circ \varphi)(t) - \sum_{i=1}^l \lambda_i(t) \frac{d}{dt} (g_i \circ \varphi)(t) - \sum_{j=1}^m \nu_j(t) \frac{d}{dt} (h_j \circ \varphi)(t) - \mu(t) \frac{d\|\varphi(t)\|^2}{dt}
\]

\[
= - \frac{\mu(t)}{2} \frac{d\|\varphi(t)\|^2}{dt}
\]
holds for each \( t \in (0, \epsilon) \). By the Monotonicity Lemma 2.2, there exists \( \epsilon' \in (0, \epsilon) \) such that for each \( t \in (0, \epsilon') \), it holds that \( \mu(t) = 0 \). Hence, (d3) implies that \( \varphi(t) \in \Sigma(f, S) \) for \( t \in (0, \epsilon') \), a contradiction. □

Now, we present some sufficient conditions to guarantee a \( R > 0 \) being a faithful radius, which will be used for computing a faithful radius of \( x^* \) in Section 5. For a given \( \mathcal{R} \in \mathbb{R}_+ \), consider the following condition

**Condition 4.2.** (i) \( \mathcal{R} \) is an isolation radius of \( x^* \) in the sense that \( \Sigma(f, S) \cap \mathbb{B}_\mathcal{R}(x^*) = \{x^*\} \).

(ii) The vectors \( \nabla g_i(x), i = 1, \ldots, l, \nabla h_j(x), j \in J(x) \), and \( x - x^* \) are linearly independent for all \( x \in S \cap \mathbb{B}_\mathcal{R}(x^*) \setminus \{x^*\} \);

(iii) For any \( u \in \Gamma(f, S, x^*) \cap \text{int}(\mathbb{B}_\mathcal{R}(x^*)) \) with \( u \neq x^* \), there exist a neighborhood \( \mathcal{O}_u \subset \mathbb{B}_\mathcal{R}(x^*) \) of \( u \), a differentiable map \( \phi: (-\epsilon, \epsilon) \to \mathbb{R}^n \) such that \( \phi((0, \epsilon)) = \Gamma(f, S, x^*) \cap \mathcal{O}_u \), \( \phi(0) = u \) and \( \frac{\|\phi\|}{dt}(0) \neq 0 \).

**Remark 4.1.** Assume that (LICQ) holds at \( x^* \) and \( x^* \) is an isolated KKT point, in particular, Condition 4.2 (i) holds for all \( \mathcal{R} > 0 \) sufficiently small. In light of Lemma 3.2 and the Cell Decomposition Theorem 2.2, up to a generic linear change of coordinates, \( \Gamma(f, S, x^*) \setminus \{x^*\} \cap \text{int}(\mathbb{B}_R(x^*)) \) is a one-dimensional smooth manifold for some \( R > 0 \). Then, due to Sard’s theorem (see, for example, [20, Corollary 1.1]), Condition 4.2 (iii) holds for all \( \mathcal{R} > 0 \) small enough. Furthermore, in Section 5, we shall see that a \( \mathcal{R} > 0 \) satisfying Condition 4.2 can be computed by some algebraic computations implemented in the current computer algebra systems, like MAPLE.

**Theorem 4.3.** Suppose that \( \mathcal{R} \in \mathbb{R}_+ \) satisfies Condition 4.2. Then, any \( R \in \mathbb{R}_+ \) with \( R < \mathcal{R} \) is a faithful radius of \( x^* \).

**Proof.** Without loss of generality, we assume \( x^* = 0 \). We first show that Condition 4.2 (iii) implies that \( \Gamma(f, S, x^*) \cap \mathbb{B}_R \) is connected. Otherwise, there is a connected component \( C \) such that \( 0 \notin C \). Since \( \Gamma(f, S, x^*) \cap \mathbb{B}_R \) is closed, \( C \) is closed by Proposition 2.2. Then, the function \( \|x\|^2 \) reaches its minimum on \( C \) at a minimizer \( u \in C \). By the assumption, there exist a neighborhood \( \mathcal{O}_u \) of \( u \) and a differentiable mapping \( \phi: (-\epsilon, \epsilon) \to \mathbb{R}^n \) such that \( \phi((-\epsilon, \epsilon)) = \Gamma(f, S, x^*) \cap \mathcal{O}_u \) and \( \phi(0) = u \). By choosing \( \epsilon \) small enough, we may assume that \( \phi((-\epsilon, \epsilon)) \subseteq C \cap \mathcal{O}_u \). Then, the function \( \|\phi\|^2 \) reaches its local minimum at 0, which contradicts Condition 4.2 (iii). Hence, \( \Gamma(f, S, x^*) \cap \mathbb{B}_R \) is connected.

Now assume to the contrary that there exists \( 0 \neq v \in \Gamma(f, S, x^*) \cap \{x \in \mathbb{R}^n \mid f(x) = 0\} \cap \mathbb{B}_R \). Since \( \Gamma(f, S, x^*) \cap \mathbb{B}_R \) is connected, there exists a path connecting 0 and \( v \). Then, \( f \) has a
local extremum on a relative interior of this path, say \( u \). By the assumption, there exists a differentiable and semialgebraic mapping \( \phi \) on \( (-\epsilon, \epsilon) \) as described in Condition 4.2 (iii). Then the differentiable function \( f \circ \phi \) reaches a local extremum at 0. By the mean value theorem,

\[
0 = \frac{d(f \circ \phi)}{dt}(0).
\]

On the other hand, by Condition 4.2 (ii) and (iii), there exist semialgebraic functions \( \lambda_i(t), \nu_j(t), \mu(t), t \in (-\epsilon, \epsilon) \), such that

1. \( \nabla f(\phi(t)) - \sum_{i=1}^{l} \lambda_i(t) \nabla g_i(\phi(t)) - \sum_{j=1}^{m} \nu_j(t) \nabla h_j(\phi(t)) - \mu(t) \phi(t) \equiv 0; \)
2. \( \nu_j(t) h_j(\phi(t)) \equiv 0, j = 1, \ldots, m. \)

As is shown in the proof of Lemma 3.3 (iii), we may assume that

\[
\nu_j(t) \frac{d}{dt}(h_j \circ \phi)(t) \equiv 0, j = 1, \ldots, m.
\]

It follows from (e1) that

\[
0 = \nabla f(\phi(t)) \cdot \frac{d\phi(t)}{dt} - \sum_{i=1}^{l} \lambda_i(t) \nabla g_i(\phi(t)) \cdot \frac{d\phi(t)}{dt} - \sum_{j=1}^{m} \nu_j(t) \nabla h_j(\phi(t)) \cdot \frac{d\phi(t)}{dt} - \mu(t) \frac{d\|\phi(t)\|^2}{dt}
\]

Let \( t \) tend to 0, it follows that

\[
0 = \frac{d}{dt}(f \circ \phi)(0) = \frac{\mu(0) d\|\phi\|^2}{dt}(0).
\]

Since \( \phi(0) \not\in \Sigma(f, S) \) by Condition 4.2 (i), we have \( \mu(0) \neq 0 \) and so

\[
\frac{d\|\phi\|^2}{dt}(0) = 0,
\]

which contradicts Condition 4.2 (iii). Therefore \( \Gamma(f, S, x^*) \cap \{x \in \mathbb{R}^n \mid f(x) = 0\} \cap B_R = \{0\} \), and so \( R \) is a faithful radius of 0.

4.2. On types of isolated KKT points. For each \( R > 0 \), let

\[
\begin{align*}
    f_R^{\min} &:= \min\{f(x) \mid x \in S \cap B_R(x^*)\}, \\
    f_R^{\max} &:= \max\{f(x) \mid x \in S \cap B_R(x^*)\}.
\end{align*}
\]
Proposition 4.1. For any $R \in \mathbb{R}_+$, we have

$$f^\text{min}_R = \min\{f(x) \mid x \in \Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*)\},$$

$$f^\text{max}_R = \max\{f(x) \mid x \in \Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*)\}.$$  

Proof. This follows immediately from the Fritz-John necessary optimality conditions (see, for example, [6]).

Remark 4.2. Assume that (LICQ) holds at $x^*$ and $x^*$ is an isolated KKT point. By Corollary 3.1, the following statements hold:

(i) If $x^*$ is a local minimizer of $f$ on $S$, then there is a $R > 0$:

$$f^\text{max}_R > f(x^*) = f^\text{min}_R.$$  

(ii) If $x^*$ is a local maximizer of $f$ on $S$, then there is a $R > 0$:

$$f^\text{max}_R = f(x^*) > f^\text{min}_R.$$  

(iii) If $x^*$ is not an extremum point of $f$ on $S$, then for any $R > 0$,

$$f^\text{max}_R > f(x^*) > f^\text{min}_R.$$  

Conversely, the next theorem shows that the type of $x^*$ can be determined by the global extrema of $f$ over the intersection of $\Gamma(f, S, x^*)$ and the ball centered at $x^*$ with a faithful radius.

Theorem 4.4. Assume that (LICQ) holds at $x^*$. Suppose that $R \in \mathbb{R}_+$ is a faithful radius of $f$ on $S$ at $x^*$, then the following statements hold:

(i) the point $x^*$ is a local minimizer of $f$ on $S$ if and only if $f^\text{max}_R > f(x^*) = f^\text{min}_R$;

(ii) the point $x^*$ is a local maximizer of $f$ on $S$ if and only if $f^\text{max}_R = f(x^*) > f^\text{min}_R$;

(iii) the point $x^*$ is not an extremum point of $f$ on $S$ if and only if $f^\text{max}_R > f(x^*) > f^\text{min}_R$.

Proof. By Remark 4.2, (i) and (ii) are clear if we can prove (iii).

(iii) Necessity. This is clear by Remark 4.2.

Sufficiency. By Proposition 4.1, there exists a point $u \in \Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*)$ such that $f(u) = f^\text{min}_R < f(x^*)$. Since $R$ is a faithful radius of $f$ on $S$, the semialgebraic set $\Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*)$ is connected, and so, is path connected by Proposition 2.2. Consequently, there exists a continuous and semialgebraic mapping $\phi: [0, 1] \to \Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*)$ such that $\phi(0) = x^*$ and $\phi(1) = u$. Thanks to the Monotonicity Lemma 2.2, we may assume that $\phi(t) \neq x^*$ for all $t \in (0, 1)$. We have $f(\phi(t)) < f(x^*)$ for all $t \in (0, 1]$. Otherwise, by the continuity, there exists $\bar{t} \in (0, 1)$ such that $f(\phi(\bar{t})) = f(x^*)$. Since the radius $R$ is faithful, we have $\phi(\bar{t}) = x^*$ by the definition, a contradiction.
Similarly, let $f_{\max}^R > f(x^*)$ be reached at $v \in \Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*)$. Then there exists a continuous and semialgebraic mapping $\psi: [0, 1] \to \Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*)$ such that $x^* \notin \psi((0, 1))$, $\psi(0) = x^*$, $\psi(1) = v$ and $f(\psi(t)) > f(x^*)$ for all $t \in (0, 1]$. Therefore, $x^*$ is not an extremum point of $f$.

Remark that computing the extrema $f_{\min}^R$ and $f_{\max}^R$ in (3) is NP-hard (c.f. [23]). Moreover, in practice, it is difficult to certify the equalities in Theorem 4.4 due to numerical errors.

For any $R \in \mathbb{R}_+$, comparing with Proposition 4.1, define

$$f_R^- := \min\{f(x) \mid x \in \Gamma(f, S, x^*) \cap S_R(x^*)\},$$

$$f_R^+ := \max\{f(x) \mid x \in \Gamma(f, S, x^*) \cap S_R(x^*)\},$$

where $S_R(x^*) = \{x \in \mathbb{R}^n \mid \|x - x^*\|^2 = R^2\}$. Then, we have the following criterion to determine the type of $x^*$.

**Theorem 4.5.** Suppose that $\mathcal{R} \in \mathbb{R}_+$ satisfies Condition 4.2. Then for any $0 < R < \mathcal{R}$, it holds that

(i) the point $x^*$ is a local minimizer if and only if $f_R^- > f(x^*)$;
(ii) the point $x^*$ is a local maximizer if and only if $f_R^+ < f(x^*)$;
(iii) the point $x^*$ is not an extremum point if and only if $f_R^+ > f(x^*) > f_R^-$. 

**Proof.** By Theorem 4.3 $R$ is a faithful radius of $x^*$. According to Theorem 4.3 and Definition 4.1 (iii), the “only if” parts in (i), (ii) and the “if” part in (iii) are clear.

(i) “if” part. For any $u \in \Gamma(f, S, x^*) \cap \mathbb{B}_R(x^*) \backslash \{x^*\}$, by Condition 4.2 (iii), it is easy to see that $u$ is path connected with $S_R(x^*)$. By continuity and the definition of faithful radius, $f(u) > f(x^*)$ which implies that $f_{\max}^R > f(x^*) = f_{\min}^R$. For details, see [18, Proposition 5.1 and Theorem 5.2].

Similarly, we can prove (ii) and then (iii) follows. □

Consequently, Theorem 4.5 shows that we need not check equalities to determine the type of $x^*$ as in Theorem 4.4. Moreover, computing $f_R^-$ and $f_R^+$ can be reduced to solving a zero-dimensional polynomial system and the inequalities in Theorem 4.5 can be certified by real root isolation of the polynomial system. See Section 5.2 for details.

5. Computational Aspects

In this section, according to the sufficient Condition 4.2 and Theorem 4.5, we give an algorithm to determine the type of an isolated KKT point $x^*$ of (1). By adding extra

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variables \(z_j, j = 1, \ldots, m\), consider the equality-constrained problem
\[
\min_{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m} f(x) \\
\text{s.t. } g_1(x) = 0, \ldots, g_l(x) = 0, \\
h_1(x) - z_1^2 = 0, \ldots, h_m(x) - z_m^2 = 0.
\] (4)

Then it is easy to see that (LICQ) holds at \(x^*\) and \(z^*\) is an isolated KKT point of \(f\) in (1) if and only if (LICQ) holds at \((x^*, z^*)\) and \((x^*, z^*)\) is an isolated KKT point of \(f\) in (1), where \(z_j^* = h_j(x^*), j = 1, \ldots, m\). Furthermore, \(x^*\) is a local minimizer (resp., maximizer or not extremum point) of (1) if and only if \((x^*, z^*)\) is a local minimizer (resp., maximizer or not extremum point) of (1). Hence, without loss of generality, we assume in the following that \(S\) is defined by equalities only, i.e.,
\[
S := \{x \in \mathbb{R}^n \mid g_1(x) = \cdots = g_l(x) = 0\}.
\]

5.1. **Algorithm.** Recall that \(l \leq n - 1\) by Remark 3.1 (ii). Let \(I_{\Sigma}\) be the ideal in \(\mathbb{R}[x]\) generated by the union of \(\{g_1, \ldots, g_l\}\) and the set of maximal minors of
\[
\begin{bmatrix}
\nabla f(x) & \nabla g_1(x) & \cdots & \nabla g_l(x)
\end{bmatrix}.
\]

Note that \(\Sigma(f, S) \subseteq V_{\mathbb{R}}(I_{\Sigma})\) and \(\dim(I_{\Sigma}) = 0\) is a sufficient condition for the isolatedness of the KKT point \(x^*\). Similarly, if \(l = n - 1\), let \(I_T := \langle g_1, \ldots, g_l \rangle\); otherwise, let \(I_T\) be the ideal in \(\mathbb{R}[x]\) generated by the union of \(\{g_1, \ldots, g_l\}\) and the set of maximal minors of
\[
\begin{bmatrix}
\nabla f(x) & \nabla g_1(x) & \cdots & \nabla g_l(x) & x - x^*
\end{bmatrix}.
\]

Clearly, it holds that \(\Gamma(f, S, x^*) = V_{\mathbb{R}}(I_T)\). Let \(I_L\) be the ideal in \(\mathbb{R}[x]\) generated by the union of \(\{g_1, \ldots, g_l\}\) and the set of maximal minors of
\[
\begin{bmatrix}
\nabla g_1(x) & \cdots & \nabla g_l(x) & x - x^*
\end{bmatrix}.
\]

By Lemma 3.2, \(V_{\mathbb{R}}(I_L) \cap B_R(x^*) = \{x^*\}\) for some \(R > 0\). Denote the vanishing ideal \(G := I \left( V_C(I_T) \setminus V_C(I_{\Sigma})^2 \right) \). Theorem A.1 shows that \(\dim(G) = 1\) up to a generic linear change of coordinates, which does not change of the type of the KKT point.

**Algorithm 5.1.** Type\((f, g_1, \ldots, g_l, x^*)\)

Input: \(f, g_1, \ldots, g_l \in \mathbb{R}[x]\) with \(x^*\) being an isolated KKT point of \(f\) on \(S\) defined by \(g_i\)'s

Output: The type of \(x^*\) as a KKT point of \(f\) over \(S\).

1. If \(\dim(I_T) = 1\), then let \(I = I_T\); else if \(\dim(G) = 1\), then let \(I = G\); otherwise, make a generic linear change of coordinates and proceed to step 1;
2. Compute a \(R_1 \in \mathbb{R}_+\) such that \(V_{\mathbb{R}}(I_{\Sigma}) \cap B_{R_1}(x^*) = \{x^*\}\) and \(V_{\mathbb{R}}(I_L) \cap B_{R_1}(x^*) = \{x^*\}\);
3. Compute a $R_2 \in \mathbb{R}_+$ such that $V_R(\mathcal{I}) \cap B_{R_2}(x^*) \setminus \{x^*\}$ is one-dimensional smooth manifold and there is no critical point of the map
\[ V_R(\mathcal{I}) \cap B_{R_2}(x^*) \setminus \{x^*\} \to \mathbb{R}, \quad x \mapsto \|x - x^*\|^2. \]
4. Fix a positive real number $r < \min\{R_1, R_2\}$. Compare $f^{-}_r$ and $f^{+}_r$ with $f(x^*)$, respectively.
5. If $f^{-}_r > f(x^*)$, return “local minimizer”; if $f^{+}_r < f(x^*)$, return “local maximizer”; if $f^{-}_r < f(x^*) < f^{+}_r$, return “not an extremum point”.

**Theorem 5.2.** Algorithm 5.1 runs successfully and is correct. In particular, any positive real number $r < \min\{R_1, R_2\}$ is a faithful radius of $x^*$ satisfying Condition 4.2.

*Proof.* See Appendix A.1. □

5.2. **Implementations.** Now we show some strategies to implement Step 2, 3 and 4 in Algorithm 5.1. We remark that the way to implement each step is not unique, while the ones are specified below in order to facilitate the complexity discussions in Section A.2. We use the following subroutines from the literature.

- **Num** [4, Algorithm 10.14 and 10.15]: For univariate polynomials $u, v \in \mathbb{R}[t]$, $\text{Num}(u, v)$ returns the number of elements in the set $\{u(t) > 0 \mid t \in \mathbb{R}, v(t) = 0\}$.

- **RUR** [26, Sec. 5.1]: For an ideal $I \subseteq \mathbb{R}[x]$ with $\dim(I) = 0$, $\text{RUR}(I)$ returns the rational univariate representation (RUR) of the points in $V_I$, i.e. univariate polynomials $v_0, v_1, u_i \in \mathbb{R}[t], i = 1, \ldots, n$, such that $x \in V_I$ if and only if $v_0(t) = 0, x_i = \frac{u_i(t)}{v_0(t)}$, $i = 1, \ldots, n$, for some $t \in \mathbb{R}$.

- **AlgSamp** [4, Algorithm 12.16]: For $p \in \mathbb{R}[x]$ with $p(x) \geq 0$ on $\mathbb{R}^n$ and $V_R(p)$ bounded, $\text{AlgSamp}(p)$ returns the rational univariate representation of a set of points which meets every connected component of $V_R(p)$.

**Step 2:** If $\dim(\mathcal{I}_\Sigma) = \dim(\mathcal{I}_L) = 0$, then consider the ideals $\mathcal{I}_\Sigma = \mathcal{I}_L + \langle \|x - x^*\|^2 - x_{n+1} \rangle$ and $\mathcal{I}_L = \mathcal{I}_L + \langle \|x - x^*\|^2 - x_{n+1} \rangle$. Apply the subroutine $\text{RUR}$ on $\mathcal{I}_\Sigma$ and obtain $v_0, v_i, u_i \in \mathbb{R}[t], i = 1, \ldots, n + 1$. Choose a number $R_\Sigma > 0$ such that $\text{Num}(u_{n+1}v_{n+1}, v_0) = \text{Num}(u_{n+1}v_{n+1} - R_\Sigma v_{n+1}^2, v_0)$ which holds for all $R_\Sigma > 0$ small enough. In the same way, choose a number $R_L > 0$ for $\mathcal{I}_L$. Then, we can set $R_1 = \min\{R_\Sigma, R_L\}$.

Otherwise, $R_1$ may be obtained by computing the distance of each connected component of $V_R(\mathcal{I}_\Sigma)$ and $V_R(\mathcal{I}_L)$ to $x^*$ using the critical point method proposed in [2]. We refer the readers to [18, Section 4.3] for the details.

**Step 3:** Compute the radical ideal $\sqrt{\mathcal{I}} = \langle \phi_1, \ldots, \phi_l \rangle$. Denote $\mathcal{D}$ as the set of the determinants of the Jacobian matrices $\text{Jac}(\phi_{i_1}, \ldots, \phi_{i_{n-1}}, \|x - x^*\|^2)$ for all $\{i_1, \ldots, i_{n-1}\} \subset$
{1, \ldots, t}. (Note that \( t \geq n - 1 \) because \( \dim \mathcal{I} = 1 \).) Define \( \Delta_\mathcal{I} := \{ \phi_1, \ldots, \phi_t \} \cup \Phi \) and
\[
\mathcal{R}_\mathcal{I} := \inf \{ r \in \mathbb{R}_+ \setminus \{0\} \mid \exists x \in V_\mathbb{R}(\Delta_\mathcal{I}), \text{s.t. } \|x - x^*\|^2 = r^2 \},
\]
Note that \( \mathcal{R}_\mathcal{I} \) are set to be \( \infty \) for convenience if no such \( x \) in the definition exists.

**Proposition 5.1.** \( \mathcal{R}_\mathcal{I} > 0 \) and any number \( R_2 \in (0, \mathcal{R}_\mathcal{I}) \) satisfies the condition in Step 3.

**Proof.** Let \( \mathcal{I} = \mathcal{I}^{(0)} \cap \mathcal{I}^{(1)} \) be the equidimensional decomposition of \( \mathcal{I} \), where \( \dim(\mathcal{I}^{(i)}) = i, i = 0, 1 \). Then, \( V_\mathbb{C}(\Delta_\mathcal{I}) \) contains three parts: \( V_\mathbb{C}(\mathcal{I}^{(0)}) \), the singular locus of \( V_\mathbb{C}(\mathcal{I}^{(1)}) \) and the set of critical points of the map
\[
V_\mathbb{C}(\mathcal{I}^{(1)}) \rightarrow \mathbb{C}, \quad x \mapsto \sum_{i=1}^{n} (x_i - x_i^*)^2.
\]
As the singular locus of \( \dim(\mathcal{I}^{(1)}) \) is zero-dimensional and there are finitely many critical values of the above map by Sard’s theorem, we obtain \( \mathcal{R}_\mathcal{I} > 0 \). For the second statement, see [18, Lemma 4.10] and [18, Theorem 4.11] for the details of the proof. \( \square \)

Let \( q \in \mathbb{R}[x] \) be the sums of squares of the elements in \( \Delta_\mathcal{I} \), then \( V_\mathbb{R}(\Delta_\mathcal{I}) = V_\mathbb{R}(q) \). Apply the subroutine \( \text{AlgSamp} \) on \( q + (\|x - x^*\|^2 - x_{n+1}^2) \) and obtain \( v_0, v_i, u_i \in \mathbb{R}[t], i = 1, \ldots, n+1 \). We can set \( R_2 > 0 \) to be a number satisfying \( \text{Num}(u_{n+1}v_{n+1}, v_0) = \text{Num}(u_{n+1}v_{n+1} - R_2^2 v_{n+1}^2, v_0) \) which holds for all \( R_2 > 0 \) small enough by the proof of Proposition 5.1.

**Step 4:** As \( \dim(\mathcal{I}) = 1 \), the ideal \( \tilde{\mathcal{I}} = \mathcal{I} + (\|x - x^*\|^2 - r^2) \) is zero-dimensional for any \( 0 < r < \mathcal{R}_\mathcal{I} \) (c.f. [18, Proposition 5.3]). Apply the subroutine \( \text{RURr} \) on \( \tilde{\mathcal{I}} + \langle f(x) - x_{n+1} \rangle \subset \mathbb{R}[x, x_{n+1}] \) and obtain \( v_0, v_i, u_i \in \mathbb{R}[t], i = 1, \ldots, n + 1 \). Compute the numbers \( n_1 = \text{Num}(u_{n+1}v_{n+1} - f(x^*)v_{n+1}^2, v_0) \) and \( n_2 = \text{Num}(f(x^*)v_{n+1}^2 - u_{n+1}v_{n+1}, v_0) \). If \( n_1 = 0 \), then \( f_r^- < f(x^*) \). If \( n_2 = 0 \), then \( f_r^+ > f(x^*) \). If \( n_1 > 0 \) and \( n_2 > 0 \), then \( f_r^- < f(x^*) < f_r^+ \).

**Remark 5.1.** (i) For a zero-dimensional ideal \( I \subset \mathbb{R}[x] \), the command \texttt{isolate} in MAPLE, which can employ the RUR algorithm [26] as a subroutine, is available to compute isolating intervals for each point in \( V_\mathbb{R}(I) \). Hence, in practice, we can use this command in Step 2 and Step 4, as well as in Step 3 if \( \dim(\langle \Delta_\mathcal{I} \rangle) = 0 \) (see Example 5.1). However, the whole arithmetic complexity of the command \texttt{isolate} is not known to us.

(ii) Although Algorithm 5.1 works in theory for any problem of the form (1) with an isolated KKT point \( x^* \in \mathbb{R}^n \) and \( f, g_i \)'s, \( h_j \)'s \( \in \mathbb{R}[X] \), we would like to remark that if either some coordinate of \( x^* \) is not rational or some of \( f, g_i \)'s, \( h_j \)'s are not in \( \mathbb{Q}[X] \), then the defining polynomials of \( \Gamma(f, S, x^*) \) may not be in \( \mathbb{Q}[X] \) and some difficulties may arise in implementing the algorithms proposed in this paper. That is because many algebraic computations which can be done in the current computer algebra systems, like MAPLE, is more efficient or only available in the rational number field.
Example 5.1. We consider three optimization problems in the following. For each one, it is easy to check that 0 is a KKT point with the second-order necessary optimality condition holds but the second-order sufficient condition does not. Hence, we can not decide the type of 0 by linear algebra.

For each problem, we can check in MAPLE that dim(IΣ) = dim(IL) = dim(⟨ΔΣ⟩) = 0, dim(IT) = 1 and IT itself is radical. Hence, I = IT without linear change of coordinates. The command isolate in MAPLE enables us to compute R1 in Step 2 and R2 in Step 3, as well as intervals [a1, b1] and [a2, b2] such that \( f^- \in [a_1, b_1], f^+ \in [a_2, b_2] \) and \( f(0) \not\in [a_1, b_1] \cup [a_2, b_2] \) in Step 4.

Then, by Theorem 4.5, it is easy to see that 0 is a local minimizer if \( a_1 > f(0), 0 \) is a local maximizer if \( b_2 < f(0) \) and 0 is not an extremum point if \( a_1 < f(0) < b_2 \). The whole process for each problem takes only a few seconds on a laptop with two 1.3 GHz cores and 8GB RAM.

(1) Consider the optimization problem

\[
\min_{x \in \mathbb{R}^3} 2x_2^4 + x_3^4 - 4x_1^2 \quad \text{s.t.} \quad x_2x_3 - x_3^2 + 2x_1 = 0.
\]

As \( \varepsilon \to 0 \), the two sequence of feasible points \((\varepsilon^2, \varepsilon, \varepsilon)\) and \((0, \varepsilon, 0)\) imply that 0 is not an extremum point of (5).

We get that \( R_1 \) can be chosen to be any positive number and \( R_2 = \frac{745607067994313975}{2305843009213693952} \approx 5.1511 \) in Step 2 and 3. Let \( r = 1 \) in Step 4, we obtain \( a_1 = -1 \) and \( b_2 = 2 \), which certify that 0 is not an extremum point of (5).

(2) Consider the optimization problem

\[
\min_{x \in \mathbb{R}^3} x_1^2 + x_2^2 + x_3^2 \quad \text{s.t.} \quad x_1^2 + x_2x_3 + x_2^2 + x_1 = 0
\]

As \( \varepsilon \to 0 \), the two sequence of feasible points \((0, 0, \varepsilon)\) and \((0, 0, -\varepsilon)\) imply that 0 is not an extremum point of (6).

We get \( R_1 = \sqrt{\frac{32794211686594305343}{7378697629483260464}} \approx 1.7982 \) and \( R_2 = \sqrt{\frac{923372036854373203}{36893488147419103232}} \approx 0.6667 \) in Step 2 and 3. Let \( r = \frac{1}{2} \) in Step 4, we obtain \( a_1 = -\frac{1}{8} \) and \( b_2 = \frac{923372036854373203}{36893488147419103232} \approx 0.2500 \), which certify that 0 is not an extremum point of (6).

(3) Consider the optimization problem

\[
\min_{x \in \mathbb{R}^3} \begin{cases} x_1x_2^4 - x_3x_2^4 + x_1^4 + 2x_1^2x_3^2 + x_3^4 - 4x_1^3x_3 - 4x_3^3 + x_1^2 + 4x_3^2 \\ \text{s.t.} \ x_1^2 + x_2^2 + x_3^2 - 2x_3 = 0. \end{cases}
\]

Note that the objective can be rewritten as \( x_1^2 + (-x_1^2 - x_3^2 + 2x_3)^2 - x_1x_2^4 - x_3x_2^4 \). Then, the value of the objective at any feasible point \( u \in \mathbb{R}^3 \) with \( \|u\|^2 < \frac{1}{4} \) is \( u_1^2 + (1 - u_1 - u_3)u_2^2 > 0 \), which implies that 0 is a strict local minimizer of (7).
We get \( R_1 = \sqrt{\frac{32743693424004235509}{36893488147419103232}} \approx 0.9420 \) and \( R_2 \) can be chosen to be any positive number less than 2 in Step 2 and 3. Let \( r = \frac{1}{2} \) in Step 4, we obtain \( a_1 = \frac{584702772733370915}{36893488147419103232} \approx 0.1585 \) and \( b_2 = \frac{1983399238052796449}{36893488147419103232} \approx 0.5376 \), which certify that 0 is a minimizer.

As mentioned in Section 1, the type of the KKT point \( x^* \) can be determined as a quantifier elimination problem by CAD based algorithms. We apply the \texttt{QuantifierElimination} command (c.f. [10, 11]) of the \texttt{RegularChains} library in MAPLE to the above problems by determining the truth of the sentences (2). For the problem (5), it took about 17 hours to determine the truth of (2). For the problems (6) and (7), the \texttt{QuantifierElimination} command kept running for days without any output, which shows the efficiency of our method.

6. Conclusions

By investigating some properties of the set of KKT points in (1) and the tangency variety of \( f \) at the isolated KKT point \( x^* \) over \( S \), we give the definition of faithful radius of \( x^* \) and show that the type of \( x^* \) can be determined by the global extrema of \( f \) over the intersection of \( S \) and the Euclidean ball centered at \( x^* \) with a faithful radius. An algorithm involving algebraic computations for determining the type of \( x^* \) is presented.

If \( x^* \) is a non-isolated KKT point, then the method proposed in this paper does not apply. In particular, since the condition (iii) in Definition 4.1 does not hold for any \( R \), we can not determine the local extremality of \( x^* \) by investigating the local values of \( f \) on its tangency variety on \( S \) at \( x^* \) as in Theorem 4.4 and 4.5. The extension of our method in the present paper to the non-isolated case will be studied in future work.

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**Appendix A. On Algorithm [5.1]**

### A.1. Correctness

For a given invertible matrix $A \in \mathcal{S}^{n \times n}$, replace $f$, $g_i$’s, $x^*$ by $f^A$, $g_i^A$’s, $A^{-1}x^*$ in the definition of $\mathcal{I}_\Gamma$, $\mathcal{I}_\Sigma$ and denote the resulting ideals by $\mathcal{I}_\Gamma^A$, $\mathcal{I}_\Sigma^A$, respectively.

**Theorem A.1.** There exists a non-empty Zariski open set $\mathcal{E} \subset \mathbb{C}^{n \times n}$ such that for all $A \in \mathcal{E} \cap \mathbb{R}^{n \times n}$, the Zariski closure $\overline{V_C(\mathcal{I}_\Gamma^A)} \setminus V_C(\mathcal{I}_\Sigma^A)$ is a one-dimensional algebraic variety in $\mathbb{C}^n$. 

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Proof. Since $\nabla f^A(x) = A^T \nabla f(Ax)$ for any $f(x) \in \mathbb{R}[x]$ and $A \in S_n^{i_{\infty n}}$, we have $V_C(I_A) = A^{-1}(V_C(I_A))$. Let $S_C = V_C((g_1, \ldots, g_l))$. Denote $S_{C_n}^{i_{\infty n}}$ as the set of symmetric matrices in $\mathbb{C}^{n \times n}$, which can be identified with the space $S_{\mathbb{C}^{n \times n}}^{i_{\infty n}}$. For any $P \in S_{C_n}^{i_{\infty n}}$, define

$$
\Gamma_{C,P}(f, S_C, x^*) := \{x \in S_C \mid \exists \kappa, \lambda, \mu \in \mathbb{C} \text{ not all zeros, s.t.} \}
$$

$$
\kappa \nabla f(x) - \sum_{i=1}^l \lambda_i \nabla g_i(x) - \mu P(x - x^*) = 0 \}.
$$

Then, it is easy to check that $V_C(I_A^1) = A^{-1}(\Gamma_{C,A^{-1}}(f, S_C, x^*))$ and hence $V_C(I_A^1) \setminus V_C(I_A^1) = A^{-1}(\Gamma_{C,A^{-1}}(f, S_C, x^*) \setminus V_C(I_A^1))$.

Define the map $\Phi: (\mathbb{C}^n \setminus V_C(I_A^1)) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}$ by

$$
\Phi(x, \kappa, \lambda, P) := (\kappa \nabla f(x) - \sum_{i=1}^l \lambda_i \nabla g_i(x) - P(x - x^*), g_1(x), \ldots, g_l(x)).
$$

Here, we rescale the coefficient $\mu$ to be 1 since $x$ in the domain of $\Phi$ is from $\mathbb{C}^n \setminus V_C(I_A^1)$. Similar to the proof of Theorem \[3.2\], it can be shown that 0 is a regular value of $\Phi$. Then according to the algebraic version of Thom’s weak transversality theorem (c.f. \[13, Ch. 3,\] Theorem 3.7.4, [3, \[27, Proposition B.3\]], there exists a Zariski closed subset $V_C \subset S_{C_n}^{i_{\infty n}}$ such that for all $P \in S_{C_n}^{i_{\infty n}} \setminus V_C$, 0 is a regular value of the map

$$
\Phi_P: (\mathbb{C}^n \setminus V_C(I_A^1)) \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}, \quad (x, \kappa, \lambda) \mapsto \Phi(x, \kappa, \lambda, P).
$$

It follows that $\Phi_P^{-1}(0)$ is either empty or a one-dimensional quasi-affine set of $\mathbb{C}^{n+1}$. Note that $\Gamma_{C,P}(f, S_C, x^*) \setminus V_C(I_A^1)$ is the projection of $\Phi_P^{-1}(0)$ on the first $n$ coordinates. Then we have $\dim(\Gamma_{C,P}(f, S_C, x^*) \setminus V_C(I_A^1)) \leq 1$ for all $P \in S_{C_n}^{i_{\infty n}} \setminus V_C$. As the Zariski closure

$$
V_{C}^{-1} := \{P \in S_{C_n}^{i_{\infty n}} \mid P^{-1} \in V_C \}^{\mathbb{C}}
$$

is an algebraic set in $\mathbb{C}^{n+1}$, the set

$$
\{A \in S_{C_n}^{i_{\infty n}} \mid AA^T \in V_{C}^{-1} \}
$$

is an algebraic set in $S_{C_n}^{i_{\infty n}}$. Let $S_{C_n}^{i_{\infty n}}$ be the set of invertible matrices in $\mathbb{C}^{n \times n}$. It follows that $E := \{A \in S_{C_n}^{i_{\infty n}} \mid AA^T \notin V_{C}^{-1} \} \cup S_{C_n}^{i_{\infty n}}$ is a non-empty Zariski open set of $\mathbb{C}^{n \times n}$. Then, for all $A \in E \cap R^{n \times n}$, $V_C(I_A) = A^{-1}(\Gamma_{C,A^{-1}}(f, S_C, x^*) \setminus V_C(I_A^1))$ is a one-dimensional algebraic variety in $\mathbb{C}^n$.

**Proposition A.1.** Suppose that (LICQ) holds at $x^*$ and $x^*$ is an isolated KKT point. Then, there exists a $R_1 \in \mathbb{R}_+$ such that $V_R(I_A) \cap B_{R_1}(x^*) = \{x^*\}$ and $V_R(I_L) \cap B_{R_1}(x^*) = \{x^*\}$. For the ideal $I$ in Algorithm \[5.1\] it holds that $\Gamma(f, S, x^*) \cap B_R(x^*) = V_R(I) \cap B_R(x^*)$ any $0 < R < R_1$.
Proof. By Lemma 3.2 there exists a \( \tilde{R} \in \mathbb{R}_+ \) such that \( V_R(I_L) \cap B_R(x^*) = \{x^*\} \). It is obvious that \( \Sigma(f,S) \subseteq V_R(I_S) \). Due to Lemma 3.1 there exists a \( \tilde{R} \in \mathbb{R}_+ \) such that \( \Sigma(f,S) \cap B_{\tilde{R}}(x^*) = V_R(I_S) \cap B_{\tilde{R}}(x^*) \). As \( x^* \) is an isolated KKT point, by shrinking \( \tilde{R} \) if necessary, we have \( V_R(I_S) \cap B_{\tilde{R}}(x^*) = \{x^*\} \). Consequently, we can let \( R_1 = \min\{\tilde{R}, \tilde{R}\} \).

As \( \Gamma(f,S,x^*) = V_R(I_T) \), it is clear that \( \Gamma(f,S,x^*) \cap B_R(x^*) = V_R(I) \cap B_R(x^*) \) for any \( R \in \mathbb{R}_+ \) if \( I = I_T \). Thus, we consider the case when \( I = \mathcal{G} \). As \( R < R_1 \), we have \( \Sigma(f,S) \cap B_R(x^*) = V_R(I_S) \cap B_R(x^*) = \{x^*\} \). Then, it is easy to see that

\[
\Gamma(f,S,x^*) \cap B_R(x^*) = \overline{\Gamma(f,S,x^*) \setminus \Sigma(f,S)} \cap B_R(x^*)
\]

\[
\subseteq \overline{V_R(I_T)} \setminus V_R(I_S) \cap B_R(x^*) \subseteq V_R(\mathcal{G}) \cap B_R(x^*).
\]

It is clear that \( \Gamma(f,S,x^*) \cap B_R(x^*) \supseteq V_R(\mathcal{G}) \cap B_R(x^*) \) and thus the conclusion follows. \( \square \)

Proof of Theorem 5.2 As \( \dim(I) = 1 \), by Sard’s theorem (see, for example, [20], Corollary 1.1]), a real number \( R_2 > 0 \) in Step 3 of Algorithm 5.1 always exists. Then, due to Theorem A.1 and Proposition A.1 Algorithm 5.1 can run successfully. It is clear that the number \( R_1 \) satisfies Condition 4.2 (i) and (ii). By the second statement of Proposition A.1 Condition 4.2 (iii) holds for any \( 0 < r < \min\{R_1, R_2\} \) which implies that \( r \) is a faithful radius. Then, the correctness of Algorithm 5.1 follows by Theorem 4.5. \( \square \)

A.2. Discussions on complexity. As the implementations involve algebraic computations of vanishing ideals, critical point method, the radical of an ideal and so on, the complexity in our algorithms depends heavily on these corresponding algorithms. Hence, we leave the complete complexity analysis of Algorithm 5.1 as our future work. Instead, we present a general discussion on the complexity under the assumption

Assumption A.1. \( \dim(I_S) = \dim(I_L) = 0 \) and the ideal \( I_T \) is radical.

We first recall the arithmetic complexity of the following subroutines from the literature.
- Num [4, Algorithm 10.14 and 10.15]: \( O(\deg(v)^2 \deg(u) + \deg(v)^4 \log_2(\deg(v))) \)
- RURr [26, Sec. 5.1]: \( d^{O(n)} \) where \( d \) is the maximal degree of the generators of \( I \)
- AlgSamp [4, Algorithm 12.16]: \( \deg(p)^{O(n)} \)

Denote by \( D \) the maximal degree of \( f, g_1, \ldots, g_l \). For simplicity, we use \( Dl + D \) to bound the degrees of the generators in \( I_T, I_S \) and \( I_L \). Note that \( n \geq 2 \), otherwise the problem is trivial.

Step 1: As \( \dim(I_S) = 0 \), we have \( I = I_T \) without computing the vanishing ideal \( \mathcal{G} \).

Step 2: As \( \dim(I_S) = \dim(I_L) = 0 \), the arithmetic complexity of applying RURr on \( I_S \) and \( I_L \) is \( (Dl + D)^{O(n+1)} \). As the numbers of the points in \( V_R(I_S) \) and \( V_R(I_L) \) are both bounded
by the Bézout number \((Dl + D)^n\), the degrees of \(v_0, u_i, v_i\) returned by \textsc{RURr}\ are bounded by \((Dl + D)^n\) \[26\]. Therefore, the arithmetic complexity of applying the subroutine \textsc{Num} is \(O(n(Dl + D)^{4n}\log_2(Dl + D))\).

**Step 3:** Recall that \(I = \mathcal{I}_F\) which is assumed to be radical. Clearly, the degrees of polynomials in \(\Delta I\) is bounded by \(n(Dl + D)\). As \(2n(Dl + D) \geq 4\), the arithmetic complexity of applying the subroutine \textsc{AlgSamp} is \((2n(Dl + D))^{O(n+1)}\) and the degrees of \(v_0, u_i, v_i\) returned by \textsc{AlgSamp} are bounded by \(O(2n(Dl + D))^{n+1}\) \[4\ pp. 493\]. Therefore, the arithmetic complexity of applying the subroutine \textsc{Num} is \(O((2n(Dl + D))^{4(n+1)}(n + 1)\log_2(2n(Dl + D)))\).

**Step 4:** As the maximal degree of the generators of \(\tilde{I} + \langle f(x) - x_{n+1}\rangle\) is bounded by \(Dl + D\), the arithmetic complexity of applying \textsc{RURr} on it is \((Dl + D)^{O(n+1)}\). As the number of the points in \(V_\mathbb{K}(\tilde{I})\) is both bounded by the Bézout number \((Dl + D)^n\), the degrees of \(v_0, u_i, v_i\) returned by \textsc{RURr} are bounded by \((Dl + D)^n\) \[26\]. Therefore, the arithmetic complexity of applying the subroutine \textsc{Num} is \(O(n(Dl + D)^{4n}\log_2(Dl + D))\).

**Remark A.1.** Recall that the arithmetic complexity for solving the quantifier elimination problems \([2]\) by the CAD is \(((l + 3)D)^{O(1)^n+1}\) if \(m = 0\) \[4\ Excercise 11.7\], which is doubly exponential in \(n\). Comparitively, under Assumption \[A.1\] the method proposed in this paper has a lower complexity, which can be observed from the numerical experiments in Section \[3\].

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