UNIFORM CONVERGENCE OF FOURIER-BESSEL SERIES ON A Q-LINEAR GRID

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Abstract. We study Fourier-Bessel series on a \(q\)-linear grid, defined as expansions in complete \(q\)-orthogonal systems constructed with the third Jackson \(q\)-Bessel function, and obtain sufficient conditions for uniform convergence. The convergence results are illustrated with specific examples of expansions in \(q\)-Fourier-Bessel series.

1. Introduction

Based on the orthogonality relation

\[
\int_0^1 J_\nu(j_\nu m t)J_\nu(j_\nu n t)dt = 0,
\]

if \(m \neq n\), where \(J_\nu\) stands for the Bessel functions of order \(\nu\) and \(j_\nu n\) is their \(n\)th positive zero, a theory of Fourier-Bessel series was developed in a close parallelism to the classical theory of Fourier series. G. H. Hardy proved that, within some boundaries, the Bessel functions are the most general functions satisfying such an orthogonality “with respect to their own zeros”, giving no space for generalizations of the theory of Fourier-Bessel series, in the scope of Lebesgue measure.

However, for a certain \(q\)-analogue of the Bessel function, such an extension is possible, when considering the proper measure. The third Jackson \(q\)-Bessel function \(J^{(3)}_\nu(z; q) \equiv J_\nu(z; q)\) (we drop the superscript for notational convenience) or, for some authors, the Hahn-Exton \(q\)-Bessel function, is defined as

\[
J^{(3)}_\nu(z; q) \equiv J_\nu(z; q) := z^\nu(q^{\nu+1}; q)_\infty \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q^{\nu+1}; q)_k(q; q)_k} z^{2k},
\]

where \(\nu > -1\) is a real parameter. When \(q \to 1^-\) we recover the Bessel function from \(J_\nu(z; q)\), after a normalization. It is a well known fact that this function satisfies the orthogonality relation

\[
\int_0^1 x J_\nu(j_\nu \nu qx; q^2) J_\nu(j_\nu \nu qx; q^2) d_q x = \eta_{\nu, \nu} \delta_{\nu, \nu},
\]

\[
\eta_{\nu, \nu} = \frac{q - 1}{2} q^{\nu - 1} J^{(4)}_{\nu+1}(q j_\nu \nu; q^2) J'_\nu(j_\nu \nu; q^2),
\]

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where \( j_{\nu}(q^2) \equiv j_{\nu} \) are the positive zeros of \( J_{\nu}(z; q^2) \) arranged in ascending order of magnitude, \( j_{\nu_1} < j_{\nu_2} < j_{\nu_3} < \cdots \), and \( d_q \xi \) stands for the measure of the Jackson \( q \)-integral.

It is our purpose in this paper to develop a theory of \( q \)-Fourier-Bessel series, based on the above orthogonality relation (1.1), on results about the completeness of these system [5], and on the localization of the zeros \( j_{\nu} \) [4]. Since it was proved [2], under the same general conditions imposed by Hardy, that the above orthogonality relation characterizes the functions \( J_{\nu}(z; q^2) \), this is the most general Fourier theory based on functions \( q \)-orthogonal with respect to their own zeros. Another interest in such a \( q \)-Fourier-Bessel theory is the existence of a \( q \)-analogue of the Hankel transform with an inversion formula, introduced in [36], whose kernel is the third Jackson \( q \)-Bessel function. Such a transform was instrumental in the sampling and Paley Wiener type theory associated with the third Jackson \( q \)-Bessel function [11, 3].

In the papers [19], [20] and [21], a theory of Fourier series on a \( q \)-linear grid was developed, using a \( q \)-analogue of the exponential function and the corresponding \( q \)-trigonometric functions introduced by Exton [26]. This was motivated by Bustoz-Suslov orthogonality and completeness results of \( q \)-quadratic Fourier series [18]. Later a simple argument has been found to prove such completeness and results [18], which is likely to also adapt to prove the completeness and orthogonality of \( q \)-linear systems, using the expansions from [7].

In this work we will first prove that pointwise convergence associated with orthogonal discrete systems always holds and we obtain sufficient conditions on the function for uniform convergence of the corresponding \( q \)-Fourier-Bessel series. It should be emphasized that Ismail stimulated a considerable research activity by conjecturing properties of the zeros of \( q \)-Bessel functions, confirmed in [4] and [28]. First, as documented in [17], the asymptotic expansion for the zeros of \( q \)-difference equations has been conjectured in a letter from Ismail to Hayman. Then, in a preprint that circulated in the early 2000’s [31], Ismail conjectured properties of the positive zeros of \( q \)-Bessel functions. Several results followed, among which we can single out [17] and [28], the bounds for the zeros of the third Jackson \( q \)-Bessel function [4], the asymptotic results of [11] and the recent improvement in the corresponding accuracy [38, Prop. A.3]. All these results are contributions to the intriguing topic of functions of order zero, whose investigation started in Littlewood’s PhD thesis, published in [37]. A new interesting direction is the study of radii of starlikeness of functions of order zero [15, 8]. The notation \( J_{\nu}^{(k)}, k = 1, 2, 3 \), from [30] and is used to distinguished the three \( q \)-analogues of the Bessel function defined by Jackson. As it was suggested in the very beginning of the introduction, since the only \( q \)-Bessel function to appear on the text is \( J_{\nu}^{(3)} \), we will drop the superscript for shortness of the notation and simply write \( J_{\nu}(z; q^2) = J_{\nu}^{(3)}(z; q^2) \). This function shows up naturally in the study of the quantum group of plane motions [34]. We will adhere to the notations of [33] and [12]. Our
$q$-Fourier series will be defined in terms of orthogonal sets of the form

$$u_n^{(\nu)}(x) = x^{\frac{1}{2}} J_{\nu}(j_{\nu q} qx; q^2).$$

Systems of the above form were used to obtain sampling theorems in [1]. In this paper we will look in more detail to the $q$-Fourier series expansions in $L^2_q[0, 1]$ associated with a function $f$

$$S_q^{(\nu)}[f](x) = \sum_{k=1}^{\infty} a_k^{(\nu)}(f) x^{\frac{1}{2}} J_{\nu}(j_{\nu q} x; q^2).$$

Since the measure of $L^2_q[0, 1]$ is discrete, pointwise convergence is an easy consequence of the completeness results of [5, 6]. We will make some comments about this in section 5 of the paper. Our main result is the following sufficient conditions for uniform convergence of (1.2). We will use the notation $V_q^+ = \{q^n : n = 0, 1, 2, \ldots\}$.

**Theorem 1.1.** If the function $f$ is $q$-linear Hölder of order $\alpha > 1$ in $V_q^+ \cup \{q^{-1}\}$ and such that $t^{1-\frac{\alpha}{2}} f(t) \in L^2_q[0, 1]$ and the limit $\lim_{x \rightarrow 0^+} f(x) = f(0^+)$ is finite then, the correspondent basic Fourier-Bessel series $S_q^{(\nu)}[f](x)$ converges uniformly to $f$ on $V_q^+$ whenever $\nu > 0$.

The paper is organized as follows. In the next section, we collect the main definitions and preliminary results. The third section is devoted to the evaluation of a few finite sum. The fourth section contains a brief introduction to the $q$-Fourier-Bessel series and the fifth section discusses pointwise convergence for systems associated with discrete orthogonality relations. We prove our main result in section 6, starting with some auxiliary Lemmas, including estimates for the coefficients of basic Fourier-Bessel series. In the last section of the paper, two examples of basic Fourier-Bessel expansions are provided.

### 2. Definitions and Preliminary Results

Following the standard notations in [27], consider $0 < q < 1$, the $q$-shifted factorial for a finite positive integer $n$ is defined as

$$(a; q)_n = (1 - q)(1 - aq) \cdots (1 - aq^{n-1}), \quad (a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n}$$

and the zero and infinite cases as

$$(a; q)_0 = 1, \quad (a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n.$$

The symmetric $q$-difference operator acting on a suitable function $f$ is defined by

$$\delta_q f(x) = f(q^{1/2}x) - f(q^{-1/2}x),$$

with $q > 1$. For $q < 1$, $\delta_q f(x)$ is defined as

$$\delta_q f(x) = f(q^{1/2}x) - f(q^{-1/2}x),$$

and

$$\Rightarrow \delta_q f(x) = f(q^{1/2}x) - f(q^{-1/2}x).$$
hence, the symmetric $q$-derivative becomes

$$\delta_q f(x) = \begin{cases} \frac{f(q^{\frac{1}{2}}x) - f(q^{-\frac{1}{2}}x)}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})x} & \text{if } x \neq 0, \\ f'(0) & \text{if } x = 0 \text{ and } f'(0) \text{ exists}. \end{cases}$$

On the opposite direction, the $q$-integral in the interval $(0, a)$ is defined by

$$\int_a^0 f(t) \delta_q t = (1 - q) \sum_{k=0}^{\infty} f(aq^k)aq^k.$$  

Using this definition we may consider an inner product by setting

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\delta_q t,$$

The resulting Hilbert space is commonly denoted by $L^2_q(0, 1)$. The space $L^2_q(0, 1)$ is a separable Hilbert space [9]. For the properties of the more general spaces $L^p_q(a, b)$ and $L^p_q(a, b)$, with $p \geq 1$, see [24].

We will also need the following straightforward formula of $q$-integration by parts [22, Lemma 2, p. 5], valid for $a, b \in \mathbb{R}$ assuming the involved limits exist:

$$\int_a^b g(q^{\frac{1}{2}}x)\delta_q f(x)\delta_q x = \int_a^b f(q^{\frac{1}{2}}x)\delta_q g(x)\delta_q x + q^{\frac{1}{2}} \left\{ \left[ (fg)\left((aq^{-\frac{1}{2}}) - (fg)(aq^{\frac{1}{2}}) \right) - \lim_{n \to +\infty} (fg)(bq^{\frac{k}{2}} + n) - \lim_{n \to -\infty} (fg)(aq^{\frac{k}{2}} + n) \right] \right\}. $$

The third Jackson $q$-Bessel function has a countable infinite number of real and simple zeros, as it was shown in [35]. In [4, Theorem 2.3] it was proved that, when $q^{2\nu+2} < (1 - q^2)^2$, the positive zeros $j_{k\nu} = \omega_k^{(\nu)}(q^2)$ of the function $J_{\nu}(z; q^2)$ satisfy

$$j_{k\nu} = q^{-k + \epsilon_{k}^{(\nu)}(q^2)},$$

with

$$0 < \epsilon_{k}^{(\nu)}(q^2) < \alpha_{k}^{(\nu)}(q^2),$$

where

$$\alpha_{k}^{(\nu)}(q^2) = \frac{\log \left( 1 - q^{2(k+\nu)}/(1 - q^{2k}) \right)}{2\log q}.$$  

Using Taylor expansion one proves that, as $k \to \infty$,

$$\alpha_{k}^{(\nu)}(q^2) = \mathcal{O}\left(q^{2k}\right).$$

Moreover [4, Remark 2.5, page 4247], the above restriction on $q$ can be dropped if $k$ is chosen large enough. This is a consequence of the fact that (2.5)-(2.7) remain valid for every $k \geq k_0$ if $q^{2(k_0+\nu)} \leq (1 - q^2)(1 - q^{2k_0})$. Hence, the following theorem holds.
**Theorem A** For every \( q \in ]0,1[ \), \( k_0 \in N \) exists such that, if \( k \geq k_0 \) then

\[
j_{kn} = q^{-k+\epsilon_k(q^2)},
\]

where \( 0 < \epsilon_k(q^2) < \alpha_k(q^2) \) and \( \alpha_k(q^2) \) is given by (2.7).

From now on, in order to simplify the notation, we will consider \( \epsilon_k(q^2) = \epsilon_k(q^2) \).

We will refer often to the following theorem from [5]:

**Theorem B** The orthonormal sequence \( \{u_k\}_{k \geq 1} \) defined by

\[
u_k(x) = \frac{x^{\frac{k}{2}} J_{\nu}(j_{kn} qx; q^2)}{\|x^{\frac{k}{2}} J_{\nu}(j_{kn} qx; q^2)\|}
\]

is complete in \( L_2^2(0,1) \).

This means that, whenever a function \( f \) is in \( L_2^2(0,1) \), if \( \int_0^1 f(x)u_k(x)d_qx = 0, \ k = 1, 2, 3, \ldots \), then \( f(q^k) = 0, \ k = 0, 1, 2, \ldots \).

The following theorem was recently proved [23] and will be of fundamental importance to obtain sufficient conditions for the uniform convergence of the basic Fourier-Bessel series.

**Theorem C** For large values of \( k \),

\[
|J_{\nu}(q_{kn}; q^2)| \leq \frac{(-q^2, -q^{2(\nu+1)}; q^2)_\infty q^{(k+\nu)(k-1)}}{(q^2; q^2)_\infty}.
\]

### 3. Identities for finite sums in q-calculus

In this section we intent to present some identities that will be needed in other sections, but first, we refer the following identity

\[
(aq^m; q)_k = (aq^k; q)_m
\]

valid for any \( a \neq q^{-j}, \ j = 0, 1, 2, \ldots \), and \( m \) and \( k \) nonnegative integers, which is a consequence of the following trivial identity

\[
(a; q)_m (aq^m; q)_k = (a; q)_k (aq^k; q)_m
\]

which holds for every \( a \) and for every integers \( m \) and \( k \).

The following two propositions and of the first lemma can be proved by induction.

**Proposition 3.1.** For \( i = 0, 1, 2, \ldots \),

\[
\sum_{k=0}^{i} q^k \left( \frac{q^j; q}{q; q} \right)_k = \left( \frac{q^{i+j}; q}{q; q} \right)_i.
\]
Proposition 3.2. For \( \lambda = 0, 1, 2, \cdots, \)
\[
\sum_{k=0}^{i} q^{2k} \frac{(q^{j-1}; q)_{k}}{(q; q)_{k}} \left( q^{1+i+i+k}; q \right)_{1} = (q; q)_{1}^{1} \left( q^{i+1}; q \right)_{i} + (q^{\lambda}; q)_{i} q^{1+i} \left( q^{j}; q \right)_{i}.
\]

In the proof of the next lemma, it is used Proposition 3.2 with \( \lambda = 0 \).

Lemma 3.3. For each fixed non-negative integer \( i \), the identity
\[
\sum_{k=0}^{i} q^{2k} \frac{(q^{j-1}; q)_{k}}{(q; q)_{k}} \left( q^{1+i+i+k}; q \right)_{1} = (q^{1+i+j}; q)_{n+i} \left( q^{1+n+i}; q \right)_{1}
\]
holds for \( n = 0, 1, 2, \cdots \).

Remark 3.4. We point out that, identity \((3.1)\) enables one to rewrite the previous results in an apparently different way: for instance, Proposition 3.1 can look like
\[
\frac{(q^{1+i+j}; q)_{n+i} \left( q^{1+n+i}; q \right)_{1}}{(q; q)_{n+i}}
\]
using relations \((3.2)\), \((3.3)\) and properties of the sums, one proves the following lemma, where, for \( i \), \([x]\) denotes the greatest integer which does not exceed \( x \).

Lemma 3.5. Given a sequence of numbers \( \{\gamma_{\lambda}\} \) then, for \( m = 0, 1, 2, \cdots, \)
\[
\sum_{\lambda=0}^{m} a_{0}^{(\lambda, \nu)} a_{0}^{(m-\lambda, \nu)} \gamma_{\lambda} = \sum_{\theta=0}^{[m/2]} a_{0}^{(m-2\theta, \nu)} \left( \sum_{\lambda=\theta}^{m-\theta} \gamma_{\lambda} \right).
\]

Proof. Using induction, it is easy to prove that, for \( m = 0, 1, 2, \cdots, \)
\[(3.2)\]
a_{0}^{(\lambda, \nu)} a_{0}^{(m-\lambda, \nu)} = \sum_{\theta=0}^{\lambda} a_{0}^{(m-2\theta, \nu)} \text{ if } 0 \leq \lambda \leq \left[ \frac{m}{2} \right]
\]
and, as a consequence,
\[(3.3)\]
a_{0}^{(\lambda, \nu)} a_{0}^{(m-\lambda, \nu)} = \sum_{\theta=0}^{m-\lambda} a_{0}^{(m-2\theta, \nu)} \text{ if } \left[ \frac{m}{2} \right] + 1 \leq \lambda \leq m
\]
Then, writing
\[
\sum_{\lambda=0}^{m} a_{0}^{(\lambda, \nu)} a_{0}^{(m-\lambda, \nu)} \gamma_{\lambda} = \sum_{\lambda=0}^{\left[ m/2 \right]} a_{0}^{(\lambda, \nu)} a_{0}^{(m-\lambda, \nu)} \gamma_{\lambda} + \sum_{\lambda=\left[ m/2 \right]+1}^{m} a_{0}^{(\lambda, \nu)} a_{0}^{(m-\lambda, \nu)} \gamma_{\lambda},
\]
using relations \((3.2)\), \((3.3)\) and properties of the sums, one proves the following lemma, where, for \( x \in \mathbb{R} \), \([x]\) denotes the greatest integer which does not exceed \( x \). \( \square \)
4. Fourier-Bessel Series on a $q$-Linear Grid

Using the orthogonal relation (1.1), we may consider the Fourier Bessel series on a $q$-linear grid associated with $f$ as the sum

$$S_q^{(v)}[f](x) = \sum_{k=1}^{\infty} a_k^{(v)}(f) x^{\frac{1}{q^2}} J_{\nu}(q j_{k\nu}; q^2),$$

with the coefficients $a_k^{(v)}$ given by

$$a_k^{(v)}(f) = \frac{1}{\eta_{k,\nu}} \int_0^1 t^{\frac{1}{q^2}} f(t) J_{\nu}(q j_{k\nu}; q^2) d_q t,$$

or, more conveniently,

$$S_q^{(v)}[f](x) = \sum_{k=1}^{\infty} a_k^{(v)}(f) J_{\nu}(q j_{k\nu}; q^2),$$

with the coefficients $a_k^{(v)}$ given by

$$a_k^{(v)}(f) = \frac{1}{\eta_{k,\nu}} \int_0^1 t f(t) J_{\nu}(q j_{k\nu}; q^2) d_q t,$$

where

$$\eta_{k,\nu} = \int_0^1 \left[t^{\frac{1}{q^2}} J_{\nu}(q j_{k\nu}; q^2)\right]^2 d_q t = -\frac{q^{\nu-1}}{2} J_{\nu+1}(q j_{k\nu}; q^2) J_{\nu}'(q j_{k\nu}; q^2)$$

$$= -\frac{(1 - q) q^{\nu-2}}{2 j_{k\nu}} J_{\nu}(q j_{k\nu}; q^2) J_{\nu}'(q j_{k\nu}; q^2),$$

being the last equality valid by identity (vii)

$$J_{\nu}(q j_{k\nu}; q^2) = q j_{k\nu} J_{\nu+1}(q j_{k\nu}; q^2)$$

of [22, Prop. 5, p. 8].

With respect to the series (4.1), our goal is to establish pointwise convergence at each of the points $x \in V_q^+ = \{q^n : n = 0, 1, 2, \cdots \}$ and to obtain sufficient conditions for uniform convergence on $V_q^+$.

5. Pointwise Convergence

5.1. A general set-up. With a view to study pointwise convergence of the series (4.1) when $x \in V_q^+ = \{q^n : n = 0, 1, 2, \cdots \}$, we first establish a general result concerning the pointwise convergence of these series. The setting to be used in this section is a very general one, designed to cover not only the convergence of $q$-Fourier-Bessel series but also other Fourier systems based on discrete orthogonality relations, as in [19], [20], [21] and [10]. There is no real novelty in this section and we are aware that the pointwise convergence can be extracted from the mean convergence by using known results from linear analysis. However, we believe that the reader may benefit from the following
Let \( N = \{ a_n \mid n \in \mathbb{N} \} \) be a numerable space and let \( \mu \) be a positive measure on \( N \) such that \( \mu_n = \mu(\{a_n\}) > 0 \). We will denote by \( L^2_\mu \), the space of all functions \( f : N \rightarrow \mathbb{C} \), such that
\[
\|f\|_{L^2_\mu}^2 = \sum_{n=1}^{\infty} |f(a_n)|^2 \mu_n < +\infty.
\]
In such a space, the scalar product \( \langle f, g \rangle \) of two functions is defined by
\[
\langle f, g \rangle = \sum_{n=1}^{\infty} f(a_n)\overline{g(a_n)} \mu_n.
\]
The sequence of functions \((e_n)_{n \geq 1}\) defined on \( N \) by
\[
e_n(a_k) = \begin{cases} 
\mu_n^{-1/2}, & k = n, \\
0, & k \neq n.
\end{cases}
\]
is a complete orthonormal system in \( L^2_\mu \). To check this fact, notice that the function \( g_N, N \in \mathbb{N} \), defined by
\[
g_N = f - \sum_{n=1}^{N} \langle f, e_n \rangle \mu e_n, \quad f \in L^2_\mu,
\]
is such that \( g_N(a_k) = 0 \) for all \( k \leq N \) and \( g_N(a_k) = f(a_k) \) for all \( k > N \). Therefore,
\[
\|g_N\|_{L^2_\mu}^2 = \sum_{n=N+1}^{\infty} |f(a_n)|^2 \mu_n \rightarrow 0, \quad \text{as} \quad N \rightarrow \infty.
\]
Thus, for an arbitrary \( f \in L^2_\mu \), we have
\[
f = \sum_{n=1}^{\infty} \langle f, e_n \rangle \mu e_n,
\]
with convergence in norm \( \| \cdot \|_{L^2_\mu}^2 \). This is also true for any other complete orthonormal system \((u_n)_{n \geq 1}\), i.e., for an arbitrary \( f \in L^2_\mu \) one has
\[
f = \sum_{n=1}^{\infty} \langle f, u_n \rangle \mu u_n,
\]
with convergence in norm \( \| \cdot \|_{L^2_\mu}^2 \). It remains only to check when the convergence of the above series is pointwise. The answer to this question is in the following lemma.

**Lemma 5.1.** Let \((u_n)_{n \geq 1}\) be a complete orthonormal system in \( L^2_\mu \). Then for any arbitrary \( f \in L^2_\mu \)
\[
f(a_k) = \lim_{N \rightarrow \infty} \sum_{n=1}^{N} \langle f, u_n(a_k) \rangle, \quad \forall a_k \in N.
\]
Proof. Let $a_k$ be an arbitrary element of $\mathcal{N}$. Then, the function $d_k := \mu^{-1/2}_k e_k$, where $e_k$ is the function given in (5.1), satisfies the property

$$\langle f, d_k \rangle = \langle f, \mu^{-1/2}_k e_k \rangle = f(a_k) \mu^{-1/2}_k e_k(a_k) \mu_k = f(a_k).$$

In particular, $\langle u_n, d_k \rangle = u_n(a_k)$. Then,

$$f(a_k) = \langle f, d_k \rangle = \left( \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, u_n \rangle u_n, d_k \right) = \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, u_n \rangle \langle u_n, d_k \rangle,$$

and, therefore,

$$f(a_k) = \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, u_n \rangle u_n(a_k).$$

$\blacksquare$

5.2. Application to $q$-Fourier-Bessel series. Let be $\mathcal{N} = V_q^+$ and in $V_q^+$ define the measure $\mu$ associated to the Jackson $q$-integral (2.3). Let $L^2_q[0,1]$ be the corresponding $L^2_\mu$ space. Since the set of functions

$$u_n^{(\nu)}(x) = \frac{x^{\frac{\nu}{2}} J_{\nu}(j_n \nu qx; q^2)}{\| x^{\frac{\nu}{2}} J_{\nu}(j_n \nu qx; q^2) \|_{L^2_q[0,1]}},$$

is a complete orthonormal system in $L^2_q[0,1]$, then, for an arbitrary $f \in L^2_q[0,1]$, i.e., $f$ such that

$$\int_0^1 |f(x)|^2 d_q x < +\infty,$$

we have the equality

$$f(q^k) = \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, u_n^{(\nu)} \rangle u_n^{(\nu)}(q^k), \quad k = 0, 1, 2, \ldots,$$

where

$$\langle f, u_n^{(\nu)} \rangle = \int_0^1 f(t) u_n^{(\nu)}(t) d_q t.$$

This summarizes in the following theorem.

**Theorem 5.2.** If $f \in L^2_q[0,1]$, then the $q$-Fourier-Bessel series (4.1) converges to the function $f$ at every point $x \in V_q^+$.

**Remark 5.3.** Let us mention that in the case of the standard trigonometric series the equivalent result of Lemma 5.1 ($L^2_\mu$ convergence implies pointwise convergence) is not true. In fact this problem leads to the celebrated Carleson Theorem (see e.g. [14]). The main difference between these two cases is that, contrary to the case of the discrete space $L^2_\mu$ (see the function $d_k$ used in the proof of Lemma 5.1), for functions $f \in L^2([0,2\pi])$ and for every $a \in [0,2\pi]$, there not exists functions $f_a$ such that $\langle f, f_a \rangle = f(a)$. 
Remark 5.4. In [20], some convergence theorems of $q$-Fourier series associated with the $q$-trigonometric orthogonal system \( \{1, C_q(q^2 \omega_0 x), S_q(q \omega_0 x)\} \) were established, where the $q$-cosines $C_q$ and $q$-sines $S_q$ can be defined in terms of the third $q$-Bessel functions by the identities
\[
C_q(z) = q^{-3/8} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} z^{1/2} J_{-1/2}(q^{-3/4} z; q^2), \quad S_q(z) = q^{1/8} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} z^{1/2} J_{1/2}(q^{-1/4} z; q^2),
\]
being \( \{\omega_k\} \) the sequence of positive zeros of the function $S_q$, arranged in ascending order of magnitude. Notice that, since this orthogonal system is a complete system \([19]\) in $L^2_q[-1, 1]$, then one can derive in a similar way that the $q$-trigonometric Fourier series converge to $f \in L^2_q[-1, 1]$ at every point of $V_q = \{\pm q^n : n = 0, 1, 2, \ldots\}$, i.e., for every fixed $x \in V_q$,
\[
f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k C_q(q^2 \omega_k x) + b_k S_q(q \omega_k x) \right\},
\]
with $a_0 = \int_{-1}^{1} f(t) d_q t$ and, for $k = 1, 2, 3, \ldots$,
\[
a_k = \frac{1}{\tau_k} \int_{-1}^{1} f(t) C_q(q^2 \omega_k t) d_q t, \quad b_k = \frac{q^2}{\tau_k} \int_{-1}^{1} f(t) S_q(q \omega_k t) d_q t,
\]
where,
\[
\tau_k = (1 - q) C_q(q^{1/2} \omega_k) S_q'(\omega_k).
\]
Thus, the corresponding open problem posed in the concluding remarks section of [20] is completely solved.

Remark 5.5. In [10] a rigorous theory of $q$-Sturm-Liouville systems was developed. In particular it was shown that the set of all normalized eigenfunctions forms an orthonormal basis for $L^2_q[0, a]$. Therefore Lemma 5.1 can be used to show that the Fourier expansions in terms of the eigenfunctions of $q$-Sturm-Liouville systems are pointwise convergent.

6. Uniform Convergence

By (4.1) and (4.2) one may write, with $\eta_{k, \nu}$ given by (4.3),
\[
S_q^{(\nu)}[f](q^n) = \sum_{k=1}^{\infty} a_k^{(\nu)}(f) J_\nu(q^{n+1} j_{k\nu}; q^2)
\]
(6.1)
\[
= \sum_{k=1}^{\infty} \left( \frac{1}{\eta_{k, \nu}} \int_{0}^{1} t f(t) J_\nu(q j_{k\nu} t; q^2) d_q t \right) J_\nu(q^{n+1} j_{k\nu}; q^2).
\]

6.1. Behavior of $J_\nu(q^{n+1} j_{k\nu}; q^2)$. The study of the factor $J_\nu(q^{n+1} j_{k\nu}; q^2)$ will be crucial. We begin with the basic difference relation (2.12) of [35] p. 693, make the shift $q \to q^2$,
\[
J_\nu(q^2 x; q^2) + q^{-\nu}(q^2 x^2 - 1 - q^{2\nu}) J_\nu(q x; q^2) + J_\nu(x; q^2) = 0,
\]
and then use induction on $n$ to prove the following proposition.
Proposition 6.1. For \( n = 0, 1, 2, \ldots \),
\[
J_\nu(q^{n+1} j_{kv}; q^2) = J_\nu(q j_{kv}; q^2) P_n(j_{kv}^2; q) \quad , \quad n = 0, 1, 2, \ldots
\]
where \( \{P_n(x; q)\}_n \) is a sequence of polynomials such that, for each \( n = 0, 1, 2, \ldots \), \( P_n(x; q) \) has degree \( n \) in the variable \( x \) and
\[
\begin{align*}
P_{n+1}(j_{kv}^2; q) &= \left\{ (q^{\nu} + q^{-\nu}) - q^{-\nu+2(n+1)} j_{kv}^2 \right\} P_n(j_{kv}^2; q) - P_{n-1}(j_{kv}^2; q), \\
P_0(j_{kv}^2; q) &= 1, \quad P_{-1}(j_{kv}^2; q) = 0.
\end{align*}
\]

Let
\[
P_n(j_{kv}^2; q) := \sum_{j=0}^{n} a_j^{(n, \nu)}(q^2) j_{kv}^j.
\]

We have the following recurrence relation for the polynomial coefficients \( a_j^{(n, \nu)} \equiv a_j^{(n, \nu)}(q) \):
\[
\begin{align*}
a_j^{(n+1, \nu)} &= (q^{\nu} + q^{-\nu}) a_j^{(n, \nu)} - q^{-\nu+2(n+1)} a_{j-1}^{(n, \nu)} - a_j^{(n-1, \nu)}, \quad j \leq n, \\
a_{-1}^{(0, \nu)} &= 0, \quad a_0^{(0, \nu)} = 1, \\
a_j^{(n, \nu)} &= 0 \quad \text{whenever} \quad j > n,
\end{align*}
\]

Moreover, it follows from (6.3) that, for every integer \( n \),
\[
a_0^{(n, \nu)} = q^{-n\nu} \sum_{i=0}^{n} (q^{2\nu})^i \quad \text{and} \quad a_n^{(n, \nu)} = (-1)^n q^{n(n+1-\nu)}.
\]

From (6.3) one also deduces that
\[
\begin{align*}
a_j^{(n, \nu)} &= -q^{2\nu} \sum_{\lambda=0}^{n-j} q^{2(n-1-\lambda)} a_0^{(\lambda, \nu)} (a_j^{(n-1-\lambda, \nu)}), \quad j \leq n \\
a_{-1}^{(0, \nu)} &= 1 \\
a_j^{(n, \nu)} &= 0 \quad \text{whenever} \quad j > n,
\end{align*}
\]

Now we are able to prove the following result for the \( a_j^{(n, \nu)} \).

Proposition 6.2. An explicit expression for the polynomial coefficients \( a_j^{(n, \nu)} \) is given by
\[
a_j^{(n, \nu)} = (-1)^j q^{j(j+1-\nu)} \sum_{i=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} a_0^{(n-j-2i, \nu)} q^{2i} \frac{(q^2)^i; q^2}{(q^2; q^2)_i} \frac{(q^{2\nu})^{1+i}; q^{2}}{(q^2; q^2)_{n-j-2i}} \frac{(q^{2n+2i}; q^2)^2}{(q^2; q^2)_{n-j-i}} \\
= (-1)^j q^{j(j+1-\nu)} \sum_{i=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} a_0^{(n-j-2i, \nu)} q^{2i} \frac{(q^2)^i; q^2}{(q^2; q^2)_i} \frac{(q^{2\nu})^{1+i}; q^{2}}{(q^2; q^2)_{n-j-2i}} \frac{(q^{2n+2i}; q^2)^2}{(q^2; q^2)_{n-j-i}}.
\]
with \(0 \leq j \leq n\), \(n = 0, 1, 2, \cdots\).

**Proof.** The proof is carried out by induction and is rather long and technical, thus we will simply present a sketch.

When \(n = 0\) the corresponding proposition is true. We point out that Proposition \([6.2]\) is clearly true for every \(n\) when \(j = 0\). Let us, now, admit that

\[
a^{(k,\nu)}_i = (-1)^{k} q^{i(1+\nu)} \sum_{l=0}^{\lfloor k/l \rfloor} a_0^{(k-l-2i,\nu)} q^{2i} \left( \frac{(q^2)^j; q^2}{(q^2; q^2)^i} \right) \]

holds true for \(k = 0, 1, 2, \cdots, n-1\): and : \(0 < l \leq k\). Then, using \([6.5]\) and the hypothesis, one gets, for \(0 < j \leq n\),

\[
a^{(n,\nu)}_j = -q^{2-\nu} \sum_{\lambda=0}^{n-j} q^{2(n-1-\lambda)} a_0^{(\lambda,\nu)} (-1)^{j-1} q^{(j-1)(j-\nu)} \sum_{i=0}^{\lfloor n-j-\lambda \rfloor} a_0^{(n-j-\lambda-2i,\nu)} c_{\lambda,i}
\]

with : \(c_{\lambda,i} = q^{2i} \left( \frac{(q^2)^j; q^2}{(q^2; q^2)^i} \right) \frac{(q^2)^{n-\lambda-2i}}{(q^2)^{n-j-\lambda-2i}} \frac{(q^2)^2}{(q^2)^2} \), hence

\[
a^{(n,\nu)}_j = (-1)^{j} q^{i(1-j+\nu)} \sum_{\lambda=0}^{n-j} \left( \sum_{i=0}^{\lfloor n-j-\lambda \rfloor} a_0^{(\lambda,\nu)} a_0^{(n-j-\lambda-2i,\nu)} c_{\lambda,i} \right).
\]

Considering \(\gamma_{\lambda,i} = (q^2)^{n-j-\lambda-2i} c_{\lambda,i}\) in the last identity and, then, using Lemma \([3.5]\) it results

\[
ad^{(n,\nu)}_j = (-1)^{j} q^{i(1-j+\nu)} \sum_{\lambda=0}^{n-j} \left( \sum_{\theta=0}^{n-j-2i} \frac{(q^2)^{n-j-\lambda-2\theta,\nu}}{(q^2)^2} \right) \left( \sum_{\lambda=0}^{n-j-2i} \gamma_{\lambda,i} \right),
\]

which can be transformed into

\[
ad^{(n,\nu)}_j = (-1)^{j} q^{i(1-j+\nu)} \sum_{\theta=0}^{n-j-2i} \frac{(q^2)^{n-j-2\theta,\nu}}{(q^2)^2 \theta} \left( \sum_{\lambda=0}^{n-j-2i} \frac{(q^2)^{\lambda}}{(q^2)^2} \frac{(q^2)^{1+i+\lambda-\theta}}{(q^2)^2} \right),
\]

and the proposition follows by Lemma \([3.3]\) \(\square\)

**Remark 6.3.** Notice that Proposition \([6.2]\) holds true for every nonnegative integers \(n\) and \(j\) since, when \(j > n\), then, by \([6.3]\), both members become identically zero.

We also need the following result.
Lemma 6.4. For $n = 0, 1, 2, \ldots$, the sequence $\{J_\nu(q^{1+n}j_{kv};q^2)\}_{k \in \mathbb{N}}$ is uniformly bounded (with respect to $n$) whenever we fix $\nu > 0$.

Proof. One must show that there exists $C$, independent of $k$ and $n$, such that

$$|J_\nu(q^{1+n}j_{kv};q^2)| \leq C$$

for every $k$ and $n$. Using Theorem A we may write, for $k$ large enough,

$$J_\nu(q^{1+n}j_{kv};q^2) = J_\nu(q^{1+n-k+\epsilon_k};q^2).$$

Thus, by (2.6) and (2.8), the last equality puts in evidence that $n - k$ will play a crucial role on the behavior of $J_\nu(q^{1+n}j_{kv};q^2)$.

We will separate the proof in two parts.

(i) When $k - n > 0$ is sufficiently large then, by Corollary 3 of [23],

$$j_{k-n-1,\nu} = q^{1+n-k+\epsilon_k^{(\nu)} - 1} < q^{1+n-k+\epsilon_k} = q^{1+n-j_{kv}} < q^{1+n-k}.$$ 

Therefore, being $n \in \mathbb{N}$ then $q^{1+n-j_{kv}} \in ]j_{k-n-1,\nu}, q^{1+n-k}[$, whenever $k - n > 0$ is sufficiently large. On one hand, by the definition of $j_{k-n-1,\nu}$,

$$(6.6) \quad J_\nu(j_{k-n-1,\nu};q^2) = 0,$$

and, on the other hand, by (12) of [16, p. 1205], for large (positive) values of $k - n$,

$$J_\nu(q^{1+n-k};q^2) \leq \frac{(-q^2, -q^{2(\nu+1)};q^2)_\infty}{(q^2, q^2)_\infty} q^{(k-n+\nu)(k-n-1)}.$$ 

However, again by Theorem A,

$$]j_{k-n-1,\nu}, q^{1+n-k}[ \subset ]q^{1+n-k+\epsilon_k^{(\nu)} - 1}, q^{1+n-k}[$$ 

Thus, by Corollary 2 of [23], the function $J_\nu(x; q^2)$ is monotone in the interval $]j_{k-n-1,\nu}, q^{1+n-k}[$, hence, by (6.6) and (6.7), $|J_\nu(q^{1+n}j_{kv};q^2)|$ is bounded whenever $k - n$ is sufficiently large (positive).

(ii) Now, let us consider all the other possible cases for $k - n$, i.e., the cases where $k - n$ is bounded above. Then $n - k$ is bounded below, thus, if $\nu > 0$, $|J_\nu(q^{1+n}j_{kv};q^2)| = |J_\nu(q^{1+n-k+\epsilon_k};q^2)|$ is trivially bounded for all such cases.

Joining both cases (i) and (ii) and since all the possible cases for $k$ and $n$ were considered, the lemma follows.  \[\square\]
Lemma 6.5. For large values of $k$,

$$J'_\nu(j_{kv}; q^2) = A_\nu(q) q^{-(k+\frac{\nu}{2} - 1 - \epsilon_k)} S_k,$$

where $A_\nu(q) = \frac{2(q^{2(\nu+1)}; q^2)_\infty}{(q^2; q^2)_\infty} q^{\frac{(\nu-1)(\nu-3)}{4}}$ and $\lim_{k \to \infty} |S_k| > 0$.

Proof. We will present only the main steps of the proof. Computing the derivative of the function $J_\nu(z; q^2)$ and considering $z = j_{kv}$,

$$J'_\nu(j_{kv}; q^2) = \frac{2(q^{2(\nu+1)}; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n n q^{n(n+1)} (j_{kv})^{2n+\nu-1}.$$

By Theorem A we may write $j_{kv} = q^{-k+\epsilon_k}$ so, the above identity becomes

$$J'_\nu(j_{kv}; q^2) = A_\nu(q) q^{-(k+\frac{\nu}{2} - 1 - \epsilon_k)} S_k,$$

where $A_\nu(q) = \frac{2(q^{2(\nu+1)}; q^2)_\infty}{(q^2; q^2)_\infty} q^{\frac{(\nu-1)(\nu-3)}{4}}$ and $S_k = \sum_{n=0}^{\infty} (-1)^n n q^{n(n+1/2+\epsilon_k)} S(n-k+1/2+\epsilon_k)$. Considering $m = n-k$, straightforward manipulations give

$$(-1)^k S_k = \sum_{m=-k}^{\infty} (-1)^m m q^{m+1/2+\epsilon_k} S(m+1/2+\epsilon_k).$$

Thus

$$(-1)^k S_k = \sum_{m=-k}^{-(2p+2)} F_{m,k}(q) - \sum_{m=-(2p+1)}^{2p} F_{m,k}(q) + \sum_{m=2p+1}^{\infty} F_{m,k}(q)$$

with

$$F_{m,k}(q) = (-1)^m m q^{m+1/2+\epsilon_k} S(m+1/2+\epsilon_k).$$

Hence,

$$|S_k| \geq \left| \sum_{m=-(2p+1)}^{2p} F_{m,k}(q) \right| - \left| \sum_{m=-k}^{-(2p+2)} F_{m,k}(q) \right| + \left| \sum_{m=2p+1}^{\infty} F_{m,k}(q) \right|.$$

Since $p$ is an arbitrary positive integer and since by (2.6)-(2.8), $\epsilon_k(q) = q^{j(i+1)} \to 0$ when $k \to \infty$, we have

$$|S_k| \geq \frac{q^{1/4}}{(q^{2(\nu+1)}; q^2)_\infty} \left| \sum_{i=0}^{\infty} (-1)^i (2i+1) q^{j(i+1)} \right|.$$

6.2. Behavior of $J'_\nu(j_{kv}; q^2)$. The asymptotic behavior of $J'_\nu(j_{kv}; q^2)$ when $k \to \infty$ was recently obtained [23, Lemma 1]. The proof combines the asymptotic properties of the infinite $q$-shifted factorial ($z; q$) (or the $q$-Pochhammer symbol) from [25] with the ideas developed in [38]. Nevertheless, we present the next lemma which is equivalent to the aforementioned result and provides a different direct proof based only on the definition of the Hahn-Exton $q$-Bessel function and its derivative.
Identity (10.4.9) of Corollary 10.4.2 due to Jacobi [13, page 500] guarantees that

\[(6.9) \quad \sum_{i=0}^{\infty} (-1)^i (2i + 1) q^{i(i+1)} = \prod_{i=1}^{\infty} (1 - q^{2i})^3 > 0 \]

thus, by (6.8)-(6.9), the Lemma follows. \( \square \)

Notice that from the above Lemma it follows that \( J_{\nu}'(j_{k\nu}; q^2) = \mathcal{O}\left(q^{-k(k+\nu-2)}\right) \) as \( k \to \infty \).

6.3. **Sufficient conditions.** With the previous notation \( V_q^+ = \{q^n : n = 0, 1, 2, \ldots\} \), which coincides with the support points of the \( q \)-integral (2.3) in \([0, 1]\), we will consider the following \( q \)-linear H"older concept [21, p. 103] adapted to the set of points \( V_q^+ \).

**Definition.** If two constants \( M \) and \( \lambda \) exist such that

\[ |f(q^{n-1}) - f(q^n)| \leq M q^{\lambda n}, \quad n = 0, 1, 2, \ldots, \]

then the function \( f \) is said to be \( q \)-linear H"older of order \( \lambda \) (in \( V_q^+ \cup \{q^{-1}\} \)).

In [22] the following upper bound for basic Fourier-Bessel coefficient (4.2) has been obtained.

**Theorem.** If the function \( f \) is \( q \)-linear H"older of order \( \alpha > 0 \) in \( V_q^+ \) and such that \( t^{-\frac{1}{2}} f(t) \in L_q^2(0, 1) \) and the limit \( \lim_{x \to 0^+} f(x) = f(0^+) \) is finite then

\[ \left| \int_0^1 t f(t) J_\nu(q_{k\nu} t; q^2) d_q t \right| \leq \frac{(1 - q)q^{\nu-1}}{j_{k\nu}} \left| f(q^{-1}) J_{\nu+1}(q_{k\nu}; q^2) \right| + \frac{(1 - q)q^{\nu-3}}{j_{k\nu}^2} \eta_{k\nu} \left( \frac{q^{\nu+1} M_1}{(1 - q)q^2(1 - q^{2\alpha})^{\frac{1}{2}}} + \frac{q^{\nu} - q^{-\nu}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}} \sqrt{M_2}} \right), \]

where \( M_1 \) and \( M_2 \) are independent of \( k \) and \( \eta_{k\nu} \) is given by (1.3).

However, the conditions on the function \( f \) stated in this theorem seem to be not sufficient to obtain the uniform convergence of its basic Fourier-Bessel expansion and we will need to impose the slightly more restrictive conditions of the Theorem [14] First we need the following lemma.

**Lemma 6.6.** If \( f \) is a function such that \( \lim_{x \to 0^+} f(x) < +\infty \) and \( \nu > 0 \) then

\[ \int_0^1 t f(t) J_\nu(q_{k\nu} t; q^2) d_q t = \frac{(1 - q)q^{\nu-2} f(q^{-1}) J_{\nu}(q_{k\nu}; q^2)}{j_{k\nu}^2} - \frac{(1 - q^2)q^{\nu-3}}{j_{k\nu}^2} \times \left[ (q^\frac{\nu}{2} - q^{-\frac{\nu}{2}}) \left( q^\frac{\nu}{2} \int_0^1 J_\nu(q_{k\nu} t; q^2) \frac{f(q) t}{t} d_q t - q^{-\frac{\nu}{2}} \int_0^1 J_\nu(q_{k\nu} t; q^2) \frac{f(t) t}{t} d_q t \right) - q^\frac{\nu}{2} \left( q^\frac{\nu}{2} \int_0^1 J_\nu(q_{k\nu} t; q^2) \frac{f(q t) - f(t) t}{t} d_q t - q^{-\frac{\nu}{2}} \int_0^1 J_\nu(q_{k\nu} t; q^2) \frac{f(t) - f(t/q) t}{t} d_q t \right) \right], \]

whenever the involved \( q \)-integrals exist.
Proof. Using first (3.7) of [22, Proposition 4, p. 7]
\[
\frac{\delta_q}{\delta_q x} \left[ x^\nu J_\nu(x; q^2) \right] = \frac{q^{-\frac{x}{2}}}{1-q} x^\nu J_{\nu-1}(q^{-\frac{x}{2}} x; q^2),
\]
then, the \( q \)-integration by parts formula (2.4) and the hypothesis \( \lim_{x \to 0^+} f(x) = f(0^+) \) one gets,
\[
\begin{align*}
\int_0^1 t f(t) J_\nu(q j_{kv} t; q^2) d_q t &= \frac{(1 - q) q^{\nu - 1} f(q^{-1}) J_{\nu+1}(q j_{kv}; q^2)}{j_{kv}} - \frac{(1 - q) q^{\nu + \frac{x}{2}}}{j_{kv}} \\
&\quad \times \left[ q^{\frac{x}{2}} - q^{-\frac{x}{2}} \int_0^1 J_{\nu+1}(q j_{kv} t; q^2) f(t) d_q t + q^{\frac{x}{2}} \int_0^1 J_{\nu+1}(q j_{kv} t; q^2) f_2(t) d_q t \right].
\end{align*}
\]
(6.10)
where
\[
f_2(t) := \frac{t \delta_q f(q^{-\frac{x}{2}} t)}{\delta_q t},
\]
and the operator \( \delta_q \) is given in (2.1). For more details in the previous calculations see the proof of Theorem 1 [22, p. 10].

(i) Step 1: for the first \( q \)-integral that figures in the right side of (6.10): using (3.8) of [22, Proposition 4, p. 7] and again [22, Lemma 2, p. 5] together with both hypothesis \( \lim_{x \to 0^+} f(x) = f(0^+) \) and \( \nu > 0 \), it results
\[
\int_0^1 J_{\nu+1}(q j_{kv} t; q^2) f(t) d_q t = \frac{(1 - q) q^{\nu + \frac{x}{2}}}{j_{kv}} \int_0^1 J_\nu(q j_{kv} t; q^2) \frac{t^{\nu - \delta_q} f(q^{\frac{x}{2}} t)}{\delta_q t} d_q t,
\]
which, by (2.2), becomes
\[
\begin{align*}
\int_0^1 J_{\nu+1}(q j_{kv} t; q^2) f(t) d_q t &= \frac{(1 - q) q^{\nu + \frac{x}{2}}}{q^\frac{x}{2} - q^{-\frac{x}{2}} j_{kv}} \\
&\quad \times \left[ q^\frac{x}{2} \int_0^1 J_\nu(q j_{kv} t; q^2) \frac{f(q t)}{t} d_q t - q^{-\frac{x}{2}} \int_0^1 J_\nu(q j_{kv} t; q^2) \frac{f(t)}{t} d_q t \right].
\end{align*}
\]
(6.12)

(i) Step 2: in a similar way, for the second \( q \)-integral that figures in the right side of (6.10): using the same hypothesis and a similar procedure used in Step 1, together with (6.11), we obtain
\[
\begin{align*}
\int_0^1 J_{\nu+1}(q j_{kv} t; q^2) f_2(t) d_q t &= \frac{(1 - q) q^{\nu + \frac{x}{2}}}{q^\frac{x}{2} - q^{-\frac{x}{2}} j_{kv}} \\
&\quad \times \left[ q^\frac{x}{2} \int_0^1 J_\nu(q j_{kv} t; q^2) \frac{f(q t) - f(t)}{t} d_q t - q^{-\frac{x}{2}} \int_0^1 J_\nu(q j_{kv} t; q^2) \frac{f(t) - f(q t)}{t} d_q t \right].
\end{align*}
\]
(6.13)
The lemma follows by introducing (6.12) and (6.13) into (6.10) and using identity (4.4). \( \square \)

Remark 6.7. Sufficient conditions to guarantee the existence of all the \( q \)-integrals involved in the previous lemma coincide with the sufficient conditions of Theorem 1.7 for the uniform convergence of the \( q \)-Fourier-Bessel series.
We are now in conditions to prove our main result, where sufficient conditions on the function \( f \) are given in order that its correspondent \( q \)-Fourier-Bessel series converges uniformly to the function itself in the set \( V_q^+ = \{ q^n : n = 0, 1, 2, \ldots \} \).

**Proof of Theorem 1.1** Under the assumptions on \( f \) we can use Lemma 6.6 to obtain

\[
\left| \int_0^1 t f(t) J_\nu(q_{jk}t; q^2) dq_t \right| \leq \frac{(1 - q) q^{\nu - 2} J_\nu(q_{jk}q^2)}{j_{k\nu}^2} \left[ (1 - q)^2 q^{\nu - 3} \right]^{1/2} \times \frac{(q^{\frac{1}{\nu}} - q^{-\frac{1}{\nu}})^2}{j_{k\nu}^2} \left( \int_0^1 f(t) \frac{f(t)}{t} dq_t \right) \left( \int_0^1 \frac{f^2(t)}{t^3} dq_t \right)^{1/2} \leq \eta_{k\nu} \left( \int_0^1 \frac{f^2(t)}{t^3} dq_t \right)^{1/2},
\]

(6.14)

\[
\left| \int_0^1 J_\nu(q_{jk}t; q^2) f(qt) \frac{f(t)}{t} dq_t \right| \leq \left( \int_0^1 t J_\nu^2(q_{jk}t; q^2) dq_t \right)^{1/2} \left( \int_0^1 \frac{f^2(t)}{t^3} dq_t \right)^{1/2} \leq \eta_{k\nu} \left( \int_0^1 \frac{f^2(t)}{t^3} dq_t \right)^{1/2},
\]

(6.15)

\[
\left| \int_0^1 J_\nu(q_{jk}t; q^2) f(qt) - f(t) \frac{f(t)}{t} dq_t \right| \leq \eta_{k\nu} \left( \int_0^1 \frac{(f(qt) - f(t))^2}{t^3} dq_t \right)^{1/2},
\]

(6.16)

\[
\left| \int_0^1 J_\nu(q_{jk}t; q^2) f(t) - f(t) \frac{f(t)}{t} dq_t \right| \leq \eta_{k\nu} \left( \int_0^1 \frac{(f(t) - f(t))^2}{t^3} dq_t \right)^{1/2}.
\]

(6.17)

Now, using the definition (2.3), since \( t^{-\frac{3}{2}} f(t) \in L^2_q[0, 1] \),

\[
\int_0^1 \frac{f^2(qt)}{t^3} dq_t = (1 - q) \sum_{n=0}^{\infty} \frac{q^n f^2(q^{n+1})}{q^{3n}} = (1 - q) q^2 \sum_{n=0}^{\infty} \left( \frac{f(q^{n+1})}{q^{n+1}} \right)^2 = S < +\infty,
\]

(6.19)

and

\[
\int_0^1 \frac{f^2(t)}{t^3} dq_t = (1 - q) \sum_{n=0}^{\infty} \left( \frac{f(q^n)}{q^n} \right)^2 = T < +\infty.
\]

(6.20)
Moreover, since \( f \) is \( q \)-linear Hölder of order \( \alpha > 1 \) in \( V_q^* \cup \{q^{-1}\} \), there exist \( M \) and \( N \) such that

\[
\int_0^1 \frac{(f(qt) - f(t))^2}{q^3} dt = (1 - q) \sum_{n=0}^{\infty} \frac{(f(q^{n+1}) - f(q^n))^2}{q^{2n}} \leq (1 - q) M^2 \sum_{n=0}^{\infty} \frac{q^{2n}}{q^{2n}} = \frac{(1 - q) M^2}{1 - q^{2(\alpha - 1)}},
\]

and

\[
\int_0^1 \frac{(f(t) - f(t/2))^2}{q^3} dt = (1 - q) \sum_{n=0}^{\infty} \frac{(f(q^n) - f(q^{n-1}))^2}{q^{2n}} \leq (1 - q) N^2 \sum_{n=0}^{\infty} \frac{q^{2n}}{q^{2n}} = \frac{(1 - q) N^2}{1 - q^{2(\alpha - 1)}}.
\]

The constants \( S \equiv S_q(f), T \equiv T_q(f), M \equiv M_q(f) \) and \( N \equiv N_q(f) \) are independent of \( k \).

Introducing inequalities (6.19), (6.20), (6.21) and (6.22) into inequalities (6.15), (6.16), (6.17) and (6.18), respectively, and the resulting ones into (6.14), gives:

\[
\left| \int_0^1 tf(t) J_\nu(q_j k; t, q_t^2) dt \right| \leq \frac{(1 - q) q^{\nu - 2} \left| f \left( \frac{1}{q} \right) \right| J_\nu(q_j k; q_t^2)}{J_{\nu, q}^2} + \frac{(1 - q)^2 q^{\nu - 3} \eta_k}{\left( q^2 - q^{-1} \right)^2 J_{\nu, q}^2} \times \left[ q^\nu q^{-\nu} \left( q^\nu \sqrt{S} + q^{-\nu} \sqrt{T} \right) + q^{\nu} \left( q^\nu \sqrt{1 - q M^2} + q^{-\nu} \sqrt{1 - q N^2} \right) \right],
\]

or, equivalently,

\[
\left| \int_0^1 tf(t) J_\nu(q_j k; t, q_t^2) dt \right| \leq \frac{1}{J_{\nu, q}^2} \left\{ C_1 \left| J_\nu(q_j k; q_t^2) \right| + C_2 \left| \eta_k \right| \right\},
\]

where \( C_1 \) and \( C_2 \) depend on \( f, \nu \) and \( q \) but are independent of \( k \). Thus, the absolute value of the \( k \)th term of the infinite sum (6.1) verifies the following inequality

\[
|a_k(f) J_\nu(q^{n+1} j_k; q_t^2)| = \left| \left( \frac{1}{\eta_k} \int_0^1 tf(t) J_\nu(q_j k; t, q_t^2) dt \right) J_\nu(q^{n+1} j_k; q_t^2) \right| \leq \left\{ \frac{C_1}{J_{\nu, q}^2 |\eta_k|} + \frac{C_2}{J_{\nu, q}^2 |\eta_k|} \right\} |J_\nu(q^{1+n} j_k; q_t^2)|.
\]

Therefore, by (6.3), \( |a_k(f) J_\nu(q^{n+1} j_k; q_t^2)| \) is bounded by

\[
\left\{ \frac{q C_1}{J_{\nu, q}^2 |J_\nu(q_j k; q_t^2)|} + \frac{q^2 C_2}{J_{\nu, q}^2 |J_\nu(q_j k; q_t^2)|} \right\} |J_\nu(q^{1+n} j_k; q_t^2)|.
\]
Remark 6.8. We should point out that the accomplish for all $n \epsilon \mathbb{N}$ where we have used Theorem 6.9. If 

\[
\frac{|J_\nu(q^{1+n} j_{k\nu}; q^2)|}{J_{k\nu}|J'_\nu(j_{k\nu}; q^2)|} \leq Mq^k.
\]

Now the second term. By Proposition 6.1,

\[
\frac{|J_\nu(q^{1+n} j_{k\nu}; q^2)|}{J_{k\nu}|J'_\nu(j_{k\nu}; q^2)|} \leq \frac{1}{2} \frac{|J_\nu(q^2 j_{k\nu}; q^2)|}{J_{k\nu}|J'_\nu(j_{k\nu}; q^2)|} \frac{1}{2} |P_n(j_{k\nu}^2; q)|.
\]

Using the expression (6.2) as well as Proposition 6.2 and Eq. (6.4) it follows that, for all given $M > 0$, independent of $k$, such that

\[
\lim_{k \to \infty} \frac{P_n(j_{k\nu}^2; q)}{a_n(2)(j_{k\nu}^2)^2} = 1,
\]

and so, $P_n(j_{k\nu}^2; q) = O(q^{n(n+1-\nu)/2})$ as $k \to \infty$.

Substituting the above expression and combining Theorems A and C with Lemmas 6.4 and 6.5 leads to the following bound for the second term in (6.23):

\[
Aq^{k^2+\nu k+n(n+1-\nu)-2kn} = Aq^{(n-k-\nu+1^2)/2}q^k \leq Bq^k,
\]

where we have used $\epsilon_k^{(\nu)}(q) > 0$ and $\epsilon_k^{(\nu)}(q) = O(q^k)$ when $k \to \infty$ (see (2.8) and (2.6)). The constants $A$ and $B$ are positive and independent of $k$. Therefore,

\[
\left|a_k(f) J_\nu(q^{n+1} j_{k\nu}; q^2)\right| = O(q^k) \quad \text{as} \quad k \to \infty,
\]

which proves the uniform convergence of the basic Fourier-Bessel series (1.1) on the set $V_q^+$.

Remark 6.8. We should point out that the $q$-linear Hölder condition does not need to be strictly accomplish for all $n = 0, 1, 2, \cdots$. It suffices that $f$ satisfies the almost $q$-linear Hölder condition [21] p. 105, which means that the condition only needs to be satisfied for all integers $n$ such that $n \geq n_0$, where $n_0$ is a positive integer.

The next result is a corollary of Theorem B and Theorem 1.1.

Theorem 6.9. If $f \in L_q^2[0, 1]$ and the $q$-Fourier-Bessel series $S_q^{(\nu)}[f](x)$ converges uniformly on $V_q^+ = \{q^n : n = 0, 1, 2, \ldots\}$, then its sum is $f(x)$ whenever $x \in V_q^+$.

Proof. Let $g(x)$ denote the sum of $S_q^{(\nu)}[f](x)$, the $q$-Fourier series of the function $f$ given by (1.1)-(1.3):

\[
\sum_{k=1}^{\infty} a_k^{(\nu)}(f) J_\nu(q j_{k\nu}; q^2) = g(x).
\]
Multiplying both sides of (6.24) by \( x J_\nu(q_j \nu t; q^2) \), \( k \geq 1 \), using the uniform convergence and integrating termwise, it results, by the orthogonality relations (1.1),

\[
a_k^{(\nu)}(f) = \frac{1}{\eta_{k,\nu}} \int_0^1 t g(t) J_\nu(q_j k \nu t; q^2) d_q t.
\]

Hence,

\[
a_k^{(\nu)}(f) = a_k^{(\nu)}(g),
\]

so, applying in \( L^2_q[0,1] \) the completeness Theorem \( B \) to the function \( h(x) = f(x) - g(x) \), we may conclude that \( h(q^k) = 0, k = 0, 1, 2, \ldots \) which means that

\[
f(q^k) = g(q^k), k = 0, 1, 2, \ldots.
\]

\( \square \)

7. Examples

We conclude with some explicit examples of uniformly convergent Fourier-Bessel series on a \( q \)-linear grid.

Example 7.1. Consider \( f(x) := x^\nu \). Using the power series expansion of \( J_\nu(x; q^2) \) and the definition of the \( q \)-integral, a calculation shows that

\[
\int_0^1 t^{\nu+1} J_\nu(q_j k \nu t; q^2) d_q t = \frac{1 - q^{-j} q_j k \nu}{q_j k \nu J_\nu(q_j k \nu; q^2)}
\]

and \( \text{(4.2)}-(\text{4.3}) \) gives

\[
a_n(x^\nu) = -\frac{2}{q^{\nu} j \nu} \frac{1}{J_\nu'(j \nu; q^2)}.
\]

It is trivial to check that the function \( f(x) = x^\nu \) is \( q \)-linear Hölder of order \( \nu \) and, if \( \nu > 1 \),

\[
x^{-\frac{3}{2}} f(x) = x^{\nu - \frac{3}{2}} \in L^2_q[0,1]
\]

and

\[
\lim_{x \to 0^+} x^\nu = 0,
\]

thus, by Theorem \( \text{1.1} \), we may conclude that the \( q \)-Fourier-Bessel series \( S_q^{(\nu)} \int x^\nu \) converges uniformly on \( V_q^+ = \{ q^n : n = 0, 1, 2, \ldots \} \) whenever \( \nu > 1 \), hence, under this restriction, by Theorem \( \text{6.9} \)

\[
x^\nu = -2 q^{-\nu} \sum_{k=1}^{\infty} \frac{J_\nu(q_j k \nu x; q^2)}{J_\nu'(j \nu; q^2)}
\]

for all \( x = q^n, n = 0, 1, 2, \ldots \).

The convergence of the expansion of \( x^\nu \) in the classical Fourier-Bessel series is studied in \( \text{39, 18.22} \) using contour integral methods.
Example 7.2. Consider \( g_{\nu,\mu}(x; q) \equiv g(x; q) := x^\nu \frac{(x^2 q^\mu; q^\nu)_\infty}{(x^2 q^{\mu-2\nu}; q^\nu)_\infty} \), with \(|x| < 1\) and \( \mu > \nu > -\frac{1}{2} \).

Using the q-binomial theorem \cite[(1.3.2)]{27} we have

\[
g(x; q) = \frac{1}{1-x^2} \left[ \sum_{n=0}^{\infty} \frac{(q^{2\mu-2\nu}; q^2)_n}{(q^2; q^2)_n} x^{2n} \right]^{-1}.
\]

Since

\[
\lim_{q \to 1^-} \sum_{n=0}^{\infty} \frac{(q^{2\mu-2\nu}; q^2)_n}{(q^2; q^2)_n} x^{2n} = \sum_{n=0}^{\infty} \frac{(2\mu - 2\nu)_n}{n!} x^{2n} = (1 - x^2)^{-2\mu + 2\nu},
\]

it becomes clear that \( g(x; q) \) is a q-analogue of \( g(x) = x^\nu (1 - x^2)^{2\mu - 2\nu - 1} \).

We can expand \( g(x; q) \) in uniform convergent q-Fourier-Bessel series. Setting \( x = q j_{kv} \) in formula \((4.11)\) from \cite{1}, it is easy to see, following \cite{4.2}, that

\[
\int_{0}^{1} t g(t; q) J_{\nu}(q j_{kv} t; q^2) dt = (1 - q)(q j_{kv})^{\nu - \mu} \frac{(q^2; q^2)_\infty}{(q^{2\mu-2\nu}; q^2)_\infty} J_{\mu}(q j_{kv}; q^2),
\]

therefore, \((4.2)-(4.3)\) enables one to write

\[
a_k^{(\nu)}(g(x; q)) = -2q^{1-\mu}(j_{kv})^{\nu - \mu} \frac{(q^2; q^2)_\infty}{(q^{2\mu-2\nu}; q^2)_\infty} J_{\nu+1}(q j_{kv}; q^2) J_{\nu}(j_{kv}; q^2).
\]

It is easy to check that \( g(x; q) \) is q-linear Hölder of order \( \nu + 2 \) and, if \( \nu > 1 \), \( x^{-\frac{3}{2}} g(x; q) \in L^2_q[0, 1] \) and \( \lim_{x \to 0^+} g(x; q) = 0 \), thus, by Theorem \cite[5.1]{1} we may conclude that the q-Fourier series \( S^{(\nu)}(g(x; q)) \) converges uniformly on \( V^+_q = \{ q^n : n = 0, 1, 2, \ldots \} \) whenever \( \nu > 1 \), hence, by Theorem \cite[6.9]{5.1}

\[
x^\nu \frac{(x^2 q^2; q^2)_\infty}{(x^2 q^{2\mu-2\nu}; q^2)_\infty} = -2q^{1-\mu} \frac{(q^2; q^2)_\infty}{(q^{2\mu-2\nu}; q^2)_\infty} \sum_{k=1}^{\infty} \left( \frac{(j_{kv})^{\nu - \mu} J_{\nu}(q j_{kv}; q^2)}{J_{\nu+1}(q j_{kv}; q^2)} \right) J_{\nu}(q j_{kv} x; q^2).
\]

We note that choosing \( \mu = \nu + 1 \) in the latter example one obtains the first one.

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