LONG-TIME BEHAVIOR OF A NONLOCAL CAHN-HILLIARD EQUATION WITH REACTION

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Abstract. In this paper we study the long-time behavior of a nonlocal Cahn-Hilliard system with singular potential, degenerate mobility, and a reaction term. In particular, we prove the existence of a global attractor with finite fractal dimension, the existence of an exponential attractor, and convergence to equilibria for two physically relevant classes of reaction terms.

1. Introduction. In this paper we aim to study the long-time behavior of a nonlocal Cahn-Hilliard system with singular potential, degenerate mobility, and a reaction term given by

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot (\mu(u)\nabla v) &= g(u) \text{ in } Q, & (1a) \\
v &= f'(u) + w \text{ in } Q, & (1b) \\
w(x,t) &= \int_{\Omega} K(|x-y|)(1-2u(y,t))dy \text{ for } (x,t) \in Q, & (1c) \\
n \cdot \mu \nabla v &= 0 \text{ on } \Gamma, & (1d) \\
u(x,0) &= u_0(x), \quad x \in \Omega, & (1e) \\
f(u) &= u \log u + (1-u) \log(1-u), & (1f) \\
\mu(u) &= \frac{1}{f''(u)} = u(1-u). & (1g)
\end{align*}
\]

where the spatial domain \( \Omega \subset \mathbb{R}^d, \quad d \leq 3, \) is assumed to be bounded and with Lipschitz boundary \( \partial \Omega, \quad Q = \Omega \times (0, +\infty), \quad \Gamma = \partial \Omega \times (0, +\infty) \) and \( n \) denotes the outward normal unit vector on \( \partial \Omega. \)

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The Cahn-Hilliard equation [6, 7] arises in the context of phase transitions. These are defined as the changes of a system from one regime or state to another exhibiting very different properties. Two types of models have been traditionally adopted in the literature to describe their occurrence: sharp-interface and phase-field models. The first ones describe the interface between two components in a mixture (e.g., liquid/solid or two chemical species) as a \((d - 1)\)-dimensional hypersurface.

In phase-field models the sharp interface is replaced by a thin transition region in which a mixture of the two components is present. The main unknown is here represented by a real valued function \(u\) which describes the local concentration of one of the two components. In contrast with sharp-interface models, \(u\) is allowed to vary continuously in the interval between the pure concentration values (say 0 and 1). This approach allows us to avoid enforcing complicated boundary conditions across the interface as well as being concerned with regularity issues.

The Cahn-Hilliard equation has been originally introduced as a phase-field model to study the phenomena of spinodal decomposition (loss of mixture homogeneity and formation at a fine scale of pure phase regions) and coarsening (aggregation of pure phase regions into larger domains) in a binary alloy. The original model is a gradient-flow (in the \(H^{-1}\) metric) of the free energy functional given by [7]

\[
E_{CH}(u) = \int_{\Omega} \frac{\tau^2}{2} |\nabla u|^2 + F(u),
\]

where \(\tau\) is a small positive parameter related to the transition region thickness and \(F\) is a double well potential attaining its two global minima in correspondence of the pure phases (represented here by the values 0 and 1). The related evolution problem is then given by the \(H^{-1}\)-gradient flow associated with the energy functional (2)

\[
\frac{\partial u}{\partial t} + \nabla \cdot J_{CH} = 0,
\]

\[
J_{CH} = -\mu(u) \nabla v_{CH},
\]

\[
v_{CH} = \frac{\delta E_{CH}(u)}{\delta u} = F'(u) - \tau^2 \Delta u,
\]

where the function \(\mu(u)\) is known as mobility.

Though quite successful and largely studied, the Cahn-Hilliard model cannot be rigorously derived as a macroscopic limit of microscopic models of interacting particles. Considering the hydrodynamic limit of such a microscopic model, Giacomin and Lebowitz [25] derived a nonlocal energy functional of the form

\[
E(u) = \int_{\Omega} \int_{\Omega} \tilde{K}(x - y)(u(x) - u(y))^2 dx dy + \eta \int_{\Omega} F(u(x)) dx,
\]

where \(\tilde{K}\) is a positive and symmetric convolution kernel and \(\eta\) is a positive parameter. The associated evolution problem is a nonlocal variant of the Cahn-Hilliard system

\[
\frac{\partial u}{\partial t} + \nabla \cdot J = 0,
\]

\[
J = -\mu(u) \nabla v,
\]

\[
v = \frac{\delta E(u)}{\delta u} = \eta F'(u) + \tilde{k}(x)u - \tilde{K} * u,
\]

where \(\tilde{k}(x) := \int_{\Omega} \tilde{K}(x - y) dy\). We note that (under suitable choices of \(F, \eta, \) and \(\tilde{K}\)) the system can be rewritten in the form
\[ \frac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla v) = 0, \quad (4a) \]
\[ v = f'(u) + w, \quad (4b) \]
\[ w(x,t) = \int_{\Omega} K(x-y)(1-2u(y,t))dy. \quad (4c) \]

Here $K$ is again a symmetric positive convolution kernel and the potential $f$ is convex.

Local and nonlocal systems (along with several variants) have been widely studied in the last years from different perspectives, e.g., well posedness [4, 14, 20, 12, 32], qualitative properties [31, 10], numerical aspects [24, 26], applications [5, 3, 27], long-time behavior [30, 1, 21, 33, 11], and asymptotics [2, 34, 23, 29, 28], just to mention a few. At first glance, the local and nonlocal equations seem to differ considerably: the first one is a fourth-order PDE, while the nonlocal one is an integro-differential parabolic equation. Nevertheless, many fundamental features are shared by the two systems, such as the gradient flow structure, the lack of comparison principles, and the separation of the solution from the pure phases [31, 10]. Moreover, formal computations show that, with suitably choices of convolution kernels, the nonlocal energy functional $E$ converges to the local one $E_{CH}$ [1]. In addition, the two energy functionals admit the same $\Gamma$-limit for vanishing interface thickness [34, 2] (see also [28, 23] for the sharp interface limit of the local Cahn-Hilliard equation).

Including a reaction term can be crucially significant in applications; some examples are image processing [5, 8], chemistry [3, 33], and biological models [27, 11, 16]. We note that its introduction in the equations sensibly influences their properties. It is well known that solutions of the Cahn-Hilliard system without reaction (i.e. $g = 0$) entail conservation of the total mass ($\frac{d}{dt} \int_{\Omega} u = 0$) and separate uniformly from the pure phases, i.e., $k \leq u(t) \leq 1-k$ for all $t \geq t_0$ for some positive $k = k(t_0)$ independent of $t$. These properties are no longer satisfied in the case $g \neq 0$, as a trivial consequence of the relation $\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} g(u)$. Moreover, the system with reaction loses its gradient-flow structure, or, in other words, does not admit (to the best of the authors’ knowledge) a Lyapunov functional. Finally, uniqueness of stationary solutions is lost. Consequently, the study of the asymptotic behavior of solutions to (1) seems to be of particular interest.

Although the non-local Cahn-Hilliard equation is often simplified in the literature by considering a regular (polynomial) potential and a nondegenerate (i.e. bounded away from zero) mobility, here, as in [32, 22], we consider a singular (logarithmic) potential and a degenerate mobility. This setting is indeed more physically feasible relevant for it allows us to prove that the solution $u$ remains bounded between the pure phases 0 and 1, i.e. $0 \leq u(t) \leq 1$ for all $t \geq 0$. Due to the lack of a comparison principle for both local and nonlocal Cahn-Hilliard equations, this property cannot be proved (and in general does not hold true) in the case of regular potential and nondegenerate mobility.

As a consequence of (1g), system (1a)-(1b) can be rewritten as a nonlocal perturbation of a parabolic equation

\[ \begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot (\mu \nabla v) &= 0, \\
v &= f'(u) + w, \\
w(x,t) &= \int_{\Omega} K(x-y)(1-2u(y,t))dy.
\end{align*} \]

The local term $\tau |\nabla u|^2$ can be obtained as the formal limit of the corresponding nonlocal term with kernel $K(x,y) = m^{d+2} J(|m(x-y)|^2)$ as $m \to \infty$, where $J$ is a nonnegative function with compact support.
\[
\frac{\partial u}{\partial t} - \Delta u - \nabla \cdot (\mu \nabla w) = g(u). \tag{1a*}
\]
 Existence, uniqueness, regularity, and separation from pure phases have been already proved in [32]. Here, we address questions concerning the asymptotic behavior of solutions. In particular, we first prove the existence of the global attractor with finite fractal dimension. Let us recall that the global attractor \( A \) describes the asymptotic dynamics of the system. In simple words, every trajectory will be eventually close to the set \( A \). Knowing that \( A \) has finite dimension is thus relevant as it implies that a finite number of variables can describe the long-time dynamics of the system with good approximation. This is fundamental, e.g., in numerical analysis.

Secondly, we show the existence of an exponential attractor \( A' \). Even though uniqueness and invariance properties cannot be proved in this case, we gain an important information about convergence, which was lacking for the global attractor. Finally, we prove existence of equilibria, i.e., weak solutions to the equation

\[-\nabla \cdot (\mu \nabla v) = g(u), \quad 0 \leq u \leq 1.\]

Most importantly, we get convergence results to steady states for specific classes of reaction terms \( g \). In the literature there exist several results about convergence to equilibria for the reaction-free Cahn-Hilliard equation. Most of these results (cf. [1] for the local case and [30, 31] for the nonlocal case) make use of the gradient-flow structure of the equation. More precisely, the existence of a Lyapunov functional is crucial to prove convergence to equilibria, for instance by using a technique based on the Lojasiewicz-Simon theorem [17]. In our case, however, the existence of a Lyapunov-type functional is unknown. Therefore, traditionally used techniques cannot be applied in our setting. We obtain convergence results for reaction terms \( g \) with a definite sign (namely, \( g \geq 0 \) or \( g \leq 0 \)) or strictly monotone decreasing. In the first case, we observe that the total mass is monotone as a function of time. Combined with the boundedness of \( u \), this proves the convergence of the solution to one of the pure phases. In the second case we exploit a linearization technique around the equilibrium point. We emphasize that these choices include a wide spectrum of reaction terms particularly relevant in applications.

**Plan of the paper.** The paper is structured as follows. In Section 2 we present the assumptions, revise some preliminary results, and state the main theorems. Section 3 is devoted to a detailed proof of the existence and finite fractal dimension of the global attractor. Existence of an exponential attractor is illustrated in Section 4. Finally, existence and convergence to steady states are explored in Section 5, where the two different reaction-term scenarios are analyzed separately.

2. Assumptions and main results.

2.1. **Assumptions.** We assume that the given functions \( K \), \( u_0 \) and \( g \) fulfill the following conditions:

\( \text{(K):} \) The convolution kernel \( K : \mathbb{R}^d \to \mathbb{R} \) satisfies

\[ K(x) = K(-x) \text{ for a.a. } x \in \mathbb{R}^d, \]

\[ \sup_{x \in \Omega} \int_{\Omega} |K(x - y)| \, dy < +\infty, \tag{K1} \]

\[ \forall p \in [1, +\infty] \ \exists r_p > 0 \text{ such that } \|K * \rho\|_{W^{1,p}(\Omega)} \leq r_p \|\rho\|_{L^p(\Omega)} \ \forall \rho \in L^p(\Omega), \tag{K2} \]

\[ \exists C > 0 \text{ such that } \|K * \rho\|_{H^1(\Omega)} \leq C \|\rho\|_{H^1(\Omega)} \ \forall \rho \in H^1(\Omega); \tag{K3} \]

\[ \exists C > 0 \text{ such that } \|K * \rho\|_{H^2(\Omega)} \leq C \|\rho\|_{H^1(\Omega)} \ \forall \rho \in H^2(\Omega). \tag{K4} \]
(U0): The initial datum $u_0 \in L^\infty(\Omega)$ is such that

$$0 \leq u_0(x) \leq 1 \quad \text{for a.a. } x \in \Omega,$$

(U01)

$$0 < \bar{u}_0 < 1,$$

(U02)

where we use the notation $\bar{z} = \frac{1}{|\Omega|} \int_\Omega z$.

(G): The reaction term $g : (x, s) \in \Omega \times [0, 1] \mapsto g(x, s) \in \mathbb{R}$ is such that

$$g$$

is measurable in $x$ and uniformly $C^{1,1}$ in $s$,

$$g(x, 0) \geq 0 \geq g(x, 1) \quad \text{for a.a. } x \in \Omega.$$  

(G1)

(G2)

Finally, we recall that $f(u) = u \log u + (1 - u) \log(1 - u)$ and $\mu(u) = u(1 - u)$. In particular, $\mu(u) = 1/f''(u)$.

Examples of convolution kernels $K$ satisfying conditions (K) are given by

- Gaussian kernels: $K(|x|) = C \exp(- |x|^2 / \lambda)$,

- Mollifiers: $K(x) = \begin{cases} C \exp(-h^2 |x|^2) / \lambda & \text{if } |x| < h, \\ 0 & \text{if } |x| \geq h, \end{cases}$

- Newton potentials: $K(|x|) = k_d |x|^{2-d}$ for $d > 2$, $K(|x|) = -k_2 \ln |x|$ for $d = 2$,

where $h, \lambda, k_d > 0$.

Hypotheses (G) cover a wide range of reaction terms occurring in applications; some relevant examples are given by

$$g(x, u) = \alpha(x)u(1 - u),$$

(5)

$$g(x, u) = -\sigma(x)u,$$

(6)

$$g(x, u) = \beta(x)(h(x) - u),$$

(7)

where $\alpha, \beta, h, \sigma \in L^\infty(\Omega)$ are positive functions and $h \leq 1$ a.e. in $\Omega$. The function defined by (5) is a space-dependent logistic map largely employed in population dynamics (cf. [27]). Equation (1) with (6), i.e. the so-called Cahn-Hilliard-Oono equation, was introduced in the context of diblock copolymers [3, 33]. Finally, if $\beta$ is the nonnegative function $\beta(x) = \lambda_0\chi_{D}(x)$, equation (1) together with (7) is known as the Cahn-Hilliard-Bertozzi equation and is used in image inpainting, where $D \subset \Omega$ is the inpainting region [5, 9, 8].

2.2. Preliminaries. In our work we deal with weak solutions to equation (1). This notion is made precise by the following definition:

**Definition 2.1.** Let assumptions (K), (U0), (G) be satisfied. $u$ is a weak solution to (1a)-(1e) on $[0, +\infty)$ if

$$u \in L^2_{\text{loc}}(0, +\infty; H^1(\Omega)) \cap H^1_{\text{loc}}(0, +\infty; (H^1(\Omega))^*)$$

(8)

$$0 \leq u \leq 1 \text{ a.e. in } Q,$$

(9)

$$w = K * (1 - 2u) \text{ a.e in } Q,$$

(10)

$$w \in C \left([0, +\infty); W^{1,\infty}(\Omega)\right)$$

(11)

$$u(0) = u_0 \text{ a.e. in } \Omega,$$

(12)

and the following equality is satisfied a.e. in $(0, +\infty)$ and for every $\psi \in H^1(\Omega)$

$$\langle \dot{u}, \psi \rangle_{H^1(\Omega)} + (\mu(u)\nabla w, \nabla \psi)_{L^2(\Omega)} + (\nabla u, \nabla \psi)_{L^2(\Omega)} = (g(u), \psi)_{L^2(\Omega)}.$$
Several important properties of the solutions such as existence, uniqueness, continuous dependence on the initial datum, and separation from pure phases have been proved in [32]:

**Theorem 2.2** (Existence, uniqueness and separation properties, [32]). Let assumptions (K), (U0), (G) be satisfied. Then:

(i) There exists a unique weak solution \( u \) to (1);

(ii) If \( u_1, u_2 \) are two solutions of (1) corresponding to initial data \( u_{01}, u_{02} \) respectively, the continuous dependence estimate

\[
\|u_1 - u_2\|_{L^\infty(0,T;L^2(\Omega))} \leq e^{Ct}\|u_{01} - u_{02}\|_{L^2(\Omega)}
\]

holds true for some constant \( C = C(K,\Omega) > 0 \);

(iii) For every \( 0 < t_0 < T, \) \( u \in L^\infty(t_0,T;H^2(\Omega)) \);

(iv) For every \( 0 < t_0 < T, \) there exist \( k_1 = k_1(\bar{u}_0, t_0, T) > 0 \) and \( k_2 = k_2(\bar{u}_0, t_0, T) > 0 \) such that

\[
k_1 \leq u(x,t) \leq 1 - k_2 \quad \text{for a.a. } x \in \Omega \text{ and } t_0 \leq t \leq T;
\]

(v) If \( g \geq 0 \) a.e. in \( \Omega \times [0,1] \), \( k_1 \) is independent of \( T \). Similarly, if \( g \leq 0 \) a.e. in \( \Omega \times [0,1] \), \( k_2 \) is independent of \( T \).

Theorem 2.2 allows us to define a dynamical system \((X,S(t))\), where the state space

\[
X = \{ u \in L^2(\Omega) : 0 \leq u \leq 1 \text{ a.e. in } \Omega \}
\]

is equipped with the \( L^2(\Omega) \)-topology and \( S(t) \) is the (strongly continuous) solution operator associated with (1) at time \( t \), i.e., \( S(t)u_0 = u(t) \), where \( u \) is the solution to (1).

2.3. **Main results.** Our first result shows the existence of the global attractor for problem (1). The notion of global attractor (see Def. 10.4, [36]) is one of the most discussed and widely used tools to investigate the long-time behavior of an evolution equation. The global attractor \( A \) is the maximal compact totally-invariant subset of \( X \), i.e., such that

\[
S(t)A = A \quad \text{for all } t \geq 0,
\]

or equivalently the minimal set attracting all bounded sets \( B \), i.e. such that

\[
\text{dist}_H(S(t)B, A) \to 0 \quad \text{as } t \to \infty. \tag{13}
\]

Here, \( \text{dist}_H \) is the so-called Hausdorff semidistance defined by

\[
\text{dist}_H(B, C) = \sup_{b \in B} \inf_{c \in C} \|b - c\|_{L^2(\Omega)},
\]

for all \( B \) and \( C \) nonempty subsets of \( X \). As a consequence of (13), one can say that the long-time dynamics of the system is well described by the dynamics close to the attractor. Although the existence of the global attractor is already a valuable information, often the set \( A \) is not explicitly described and (though compact) might be large and difficult to characterize. This motivates us in proving that the global attractor has finite fractal dimension.

The notion of fractal dimension is a generalization of the standard notion of Hausdorff dimension. Following (Def. 13.1, [36]) we define
Definition 2.3 (Fractal dimension). If \( \bar{Y} \) is compact in a Hilbert space \( B \), the *fractal dimension* of \( Y \) with respect to the topology of \( B \) is defined as

\[
d_{\text{frac}}^B(Y) = \limsup_{\varepsilon \to 0} \frac{\log N(Y, \varepsilon)}{\log(1/\varepsilon)},
\]

(14)

where \( N(Y, \varepsilon) \) denotes the minimum number of \( B \)-balls of radius \( \varepsilon \) needed to cover \( Y \).

The theorem hence reads as follows:

**Theorem 2.4 (Global Attractor).** Let assumptions (K), (U0), (G) be satisfied. Then, the global attractor for the dynamical system \((X, S(t))\)

(i) exists and is connected;

(ii) has finite fractal dimension.

We prove Theorem 2.4 in Section 3.

The results concerning the global attractor, however, do not give us any indication about the rate of convergence of the solutions. Hence we investigate the existence of a finite-dimensional exponential attractor \( A' \) for system (1) with respect to the \( L^2(\Omega) \) metric.

**Definition 2.5.** A compact set \( A' \) is called *exponential attractor* with respect to the \( L^2(\Omega) \) metric if there exist some positive constants \( c, C > 0 \) independent of \( t \) and \( u_0 \) such that

\[
d_{L^2(\Omega)}(S(t)u_0, A) = \sup_{a \in A} \|S(t)u_0 - a\|_{L^2(\Omega)} \leq Ce^{-tc} \quad \text{for all } u_0 \in X \text{ and } t \geq 0.
\]

Our main result concerning exponential convergence is then summarized in the following theorem:

**Theorem 2.6 (Exponential attractor).** Let assumptions (K), (U0), (G) be satisfied. Then, there exists an exponential attractor with respect to the \( L^2(\Omega) \)-metric for the dynamical system \((X, S(t))\).

We remark that analogous results have been already proved in the reaction-free case in [22] (see also [18, 19] for a system coupled with Navier-Stokes equations). Theorems 2.4 and 2.6 can be interpreted as an extension of these results in several directions. First, we allow for a nontrivial reaction term. Secondly, the dynamical system considered in [22] has as a state space the set of functions \( 0 \leq u \leq 1 \) whose spatial average \( \bar{u} \) is bounded away from the pure phases 0 and 1 by an arbitrarily small but fixed constant. The reason for this restriction is that uniform (in time) separation from pure phases ensures better estimates, in particular \( L^\infty \) estimates on the chemical potential, that are used to prove the exponential convergence to the attractor. Moreover, the finite dimension of the attractor is proved in [22] with respect to a metric weaker than the \( L^2 \) metric. The restriction on \( \bar{u} \) is of course unnatural in the case \( g \neq 0 \) for the quantity \( \bar{u} \) is no longer preserved in time. This forced us to find a different way to derive estimates. By applying a generalization of the Gronwall Lemma (see Lemma 5.2 in the Appendix) we can bound \( \nabla u \) in \( L^\infty(t_0, \infty, L^2(\Omega)) \) (cf. estimate (22)). This strong estimate provides sufficient regularity to obtain the stronger statement of Theorem 2.4.

We now focus our attention on the steady states of system (1). These correspond to solutions of the equation

\[
- \nabla \cdot (\mu \nabla v) = g(u).
\]

(15)
In the reaction-free case (i.e., \( g = 0 \)), the uniform-in-time separation properties proved in [31] allow us to explicitly obtain equilibrium triples \((u^*, v^*, w^*)\), where
\[
\begin{align*}
    u^* &= \frac{1}{1 + \exp(w^* - v^*)}, \\
    v^* &= \text{const}, \\
    w^* &= K^* (1 - 2u^*).
\end{align*}
\]

On the other hand, in the case \( g \neq 0 \) the separation properties are not uniform in time as already observed in [32]. In particular, the chemical potential \( V \) is not well defined. This suggests to rewrite equation (15) as
\[
\begin{align*}
    -\Delta u - \nabla \cdot (\mu \nabla w) &= g(u) \quad \text{in } \Omega, \quad (16a) \\
    n \cdot (\mu \nabla w + \nabla u) &= 0 \quad \text{on } \partial\Omega, \quad (16b) \\
    w &= K^* (1 - 2u) \quad \text{in } \Omega. \quad (16c)
\end{align*}
\]

More precisely, we give the following definition.

**Definition 2.7.** Let \((K), (U0), (G)\) be satisfied. Then, \(u\) is called equilibrium point for (1) or weak solution to (16) if
\[
\begin{align*}
    u &\in H^1(\Omega), \\
    0 &\leq u \leq 1 \text{ a.e. in } \Omega, \text{ and it satisfies}
\end{align*}
\]
\[
(\nabla u + \mu \nabla w, \nabla \psi)_{L^2(\Omega)} = (g(u), \psi)_{L^2(\Omega)} \quad \forall \psi \in H^1(\Omega), \quad (17)
\]
where \(w\) satisfies (16c) pointwise a.e. in \(\Omega\).

Differently from the reaction-free case (i.e., \( g = 0 \)), existence of stationary solutions for (1) is not trivial and has to be proved directly. Moreover, uniqueness is false in the general case (see Remark 3). We recall that, as we do not know any Lyapunov functional for system (1), convergence to equilibria is hard to obtain. However, restricting the class of reaction terms \(g\) we can prove convergence to equilibria. More precisely, we consider functions \(g\) which either have a sign (see Case 1 in Theorem 2.8) or are monotone decreasing (see Case 2).

Our result reads as follows.

**Theorem 2.8 (Equilibria).** Let assumptions \((K), (U0), (G)\) hold. Then, we have the following:

**Existence:** There exists at least one equilibrium point \(u \in H^1(\Omega)\) such that \(0 \leq u \leq 1\) a.e. in \(\Omega\), i.e., a solution of (17).

**Convergence - Case 1:** If in addition \(g\) is such that
\[
g(x,s) \geq 0 \quad \text{for a.a. } x \in \Omega \text{ and } \forall s \in [0,1], \quad (A1)
\]
\[
\forall \kappa > 0 \exists m_\kappa > 0 \text{ s.t. } g(x,s) \geq -m_\kappa (s - 1) \quad \forall s \in [\kappa,1], \quad \text{for a.a. } x \in \Omega, \quad (A2)
\]
then, \(u(t) \to 1\) exponentially fast in \(L^2(\Omega)\) for every \(u_0 \in X\) if the measure of the set \(\{ x \in X \text{ s.t. } g(x,0) > 0 \}\) is positive and for every \(u_0 \neq 0\) if \(g(\cdot,0) \equiv 0\).

Similarly, if \(g\) is such that
\[
g(x,s) \leq 0 \quad \text{for a.a. } x \in \Omega \text{ and } \forall s \in [0,1], \quad (B1)
\]
\[
\forall \kappa > 0 \exists m_\kappa > 0 \text{ s.t. } g(x,s) \leq -m_\kappa s \quad \forall s \in [0,1 - \kappa], \quad \text{for a.a. } x \in \Omega, \quad (B2)
\]
then \(u(t) \to 0\) exponentially fast in \(L^2(\Omega)\) for every \(u_0 \in X\) if the measure of the set \(\{ x \in X \text{ s.t. } g(x,1) < 0 \}\) is positive and for every \(u_0 \neq 1\) if \(g(\cdot,1) \equiv 0\).
**Convergence - Case 2:** If in addition $g$ is uniformly differentiable in the second variable and
\[
\partial_s g(x,s) \leq -\lambda < 0 \text{ for a sufficiently large } \lambda = \lambda(\Omega, K) \text{ for a.a. } (x,s) \in \Omega \times [0,1],
\]
then there exists a unique equilibrium $u^*$ and $u(t) \to u^*$ exponentially fast in $L^2(\Omega)$ for $t \to +\infty$.

Many physically relevant reaction terms [27, 3] are included in these scenarios, such as the ones illustrated in (5)-(6).

**Corollary 1** (Cahn-Hilliard with logistic reaction term). Let $g$ be as in (5) with $\alpha(x) \geq \alpha_0 > 0$ a.e. in $\Omega$. Then, the solution $u$ to (1) converges to 1 exponentially fast in $L^2(\Omega)$ for every initial datum $u_0$ not identically 0. Moreover, $u = 0$ is an equilibrium point.

**Corollary 2** (Cahn-Hilliard-Oono). Let $g$ be as in (6) with $\sigma(x) \geq \sigma_0 > 0$ a.e. in $\Omega$. Then, the solution $u$ converges to 0 exponentially fast in $L^2(\Omega)$.

**Corollary 3.** Let $g$ be as in (7) with $\beta(x) \geq \beta_0 > 0$ a.e. in $\Omega$ for $\beta_0 = \beta_0(\Omega, K)$ sufficiently large. Then, the solution $u$ converges to the unique stationary point $u^*$ exponentially fast in $L^2(\Omega)$.

**Remark 1.** We note that the case covered by Corollary 3 does not correspond to the Cahn-Hilliard-Bertozzi equation [5, 9, 8], since the fidelity term does not satisfy the assumption $\beta(x) \geq \beta_0 > 0$ a.e. in $\Omega$.

3. **The global attractor: Proof of Theorem 2.4.**

3.1. **Part (i): Existence.** The existence of the global attractor is based on proving that $S(t)$ is dissipative and possesses a compact absorbing set (cf. Theorem 10.5 in [36]). The first property holds since $X$ is $\|\cdot\|_{L^2(\Omega)}$-bounded. In order to show the existence of a compact absorbing set, we simply derive uniform $L^2$-estimates on the gradient of $\nabla u$. To this aim, we start by testing equation (1a) with $u$ and estimate, by using boundedness of $u$ and $g$ and assumption (1g) on $\mu$,
\[
\frac{d}{dt} \frac{1}{2} \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + (\mu \nabla w, \nabla u)_{L^2(\Omega)} \leq \int_{\Omega} g(u)u \leq C.
\]
Thus, integrating over $[t, t+1]$ and using $0 \leq u \leq 1$, we get
\[
\int_t^{t+1} \left( \|\nabla u\|^2_{L^2(\Omega)} + (\mu \nabla w, \nabla u)_{L^2(\Omega)} \right) \leq C + \frac{1}{2} \|u(t)\|^2_{L^2(\Omega)} \leq C + \frac{1}{2} |\Omega|.
\]
Hence
\[
\sup_{t \in [0, +\infty)} \int_t^{t+1} \left( \|\nabla u\|^2_{L^2(\Omega)} + (\mu \nabla w, \nabla u)_{L^2(\Omega)} \right) < C_1
\]
where $C$ does not depend on $u_0$.

Testing now equation (1a) with $\dot{u}$ we get
\[
\|\dot{u}\|^2_{L^2(\Omega)} + \frac{d}{dt} \frac{1}{2} \|\nabla u\|^2_{L^2(\Omega)} + (\mu \nabla w, \nabla \dot{u})_{L^2(\Omega)} = \int_{\Omega} g(u) \dot{u}.
\]
Note that $(\mu \nabla w, \nabla \dot{u})_{L^2(\Omega)} = -\int_{\Omega} \frac{d}{dt} (\mu \nabla w) \nabla u + \frac{d}{dt} \int_{\Omega} \mu \nabla w \nabla u$. Using properties of $u$, $\mu$ and $K$, we estimate
\[
\left| \int_{\Omega} \frac{d}{dt} (\mu \nabla w) \nabla u \right| \leq C \|\dot{u}\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq C \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2} \|\dot{u}\|^2_{L^2(\Omega)}.
\]
where we used the fact that
\[
\left\| \frac{d}{dt} (\mu \nabla w) \right\|_{L^2(\Omega)} \leq \left\| \mu \nabla w \right\|_{L^2(\Omega)} + \left\| \frac{d}{dt} (\nabla w) \right\|_{L^2(\Omega)} \\
\leq \left\| \nabla w \right\|_{L^\infty(\Omega)} \left\| \hat{u} \right\|_{L^2(\Omega)} + \frac{1}{4} \left\| \frac{d}{dt} (\nabla w) \right\|_{L^2(\Omega)} \\
\leq \left\| \nabla (K \ast (1 - 2u)) \right\|_{L^\infty(\Omega)} \left\| \hat{u} \right\|_{L^2(\Omega)} + \frac{1}{4} \left\| \nabla (K \ast \frac{d}{dt} (1 - 2u)) \right\|_{L^2(\Omega)} \\
\leq C \left\| \hat{u} \right\|_{L^2(\Omega)}.
\]
By substituting into (19) and using Young’s inequality, we get
\[
\frac{1}{2} \| \hat{u} \|^2_{L^2(\Omega)} + \frac{d}{dt} \left( \frac{1}{2} \| \nabla u \|^2_{L^2(\Omega)} + \int_\Omega \mu \nabla w \cdot \nabla u \right) \leq \int_\Omega g(u) \hat{u} + C \| \nabla u \|^2_{L^2(\Omega)} \\
\leq C + C \| \nabla u \|^2_{L^2(\Omega)} + \frac{1}{4} \| \hat{u} \|^2_{L^2(\Omega)}.
\]
Note that, thanks to (1g), (K3), using Hörder and Young’s inequalities,
\[
\| \nabla u \|^2_{L^2(\Omega)} = \| \nabla u \|^2_{L^2(\Omega)} + 2 \int_\Omega \mu \nabla w \cdot \nabla u - 2 \int_\Omega \mu \nabla w \cdot \nabla u \\
\leq \| \nabla u \|^2_{L^2(\Omega)} + 2 \int_\Omega \mu \nabla w \cdot \nabla u + \frac{1}{2} \| \nabla u \|^2_{L^2(\Omega)} + C.
\]
Thus,
\[
\frac{1}{2} \| \nabla u \|^2_{L^2(\Omega)} \leq \| \nabla u \|^2_{L^2(\Omega)} + 2 \int_\Omega \mu \nabla w \cdot \nabla u + C.
\]
By substituting into (20), we finally have
\[
\frac{d}{dt} \left( \frac{1}{2} \| \nabla u \|^2_{L^2(\Omega)} + \int_\Omega \mu \nabla w \cdot \nabla u \right) \leq C_2 \left( \frac{1}{2} \| \nabla u \|^2_{L^2(\Omega)} + \int_\Omega \mu \nabla w \cdot \nabla u \right) + C_3.
\]
Applying the Uniform Gronwall Lemma (Lemma 5.2, see Appendix) with
\[
\eta(t) = \frac{1}{2} \| \nabla u(t) \|^2_{L^2(\Omega)} + \int_\Omega \mu \nabla w \cdot \nabla u,
\]
or = 1, t_0 = 0, φ(t) = a_1 = C_2, ψ(t) = a_2 = C_3 and a_3 = C_1 we get
\[
\frac{1}{2} \| \nabla u(t) \|^2_{L^2(\Omega)} + \int_\Omega \mu(t) \nabla w(t) \cdot \nabla u(t) \leq C \quad \forall t \geq 1.
\]
This yields, as a consequence of (21), the following relevant estimate
\[
\| \nabla u(t) \|^2_{L^2(\Omega)} \leq C_4 \quad \forall t \geq 1,
\]
where C_3 does not depend on u_0. This proves that B_1 = \{ v \in H^1(\Omega) \subset X : \| \nabla v \|^2_{L^2(\Omega)} \leq C_3 \} is an absorbing set for (X, S(t)) which is also compact in L^2(Ω). By virtue of Theorem 10.5 of [36], there exists the global attractor A for (X, S(t)).

3.2. Part (ii): Finite fractal dimension. In this section, we show that d_{frac}(A) is finite. To this aim we adopt a standard strategy presented e.g. in [36]. The underlying heuristic idea is the following. Consider an arbitrary infinitesimal n-dimensional set V, (e.g. a cube) and follow its evolution under the action of the semigroup S(t). Suppose that for all V the n-dimensional volume vanishes for large times, i.e.,
\[
\text{Vol}_n(S(t)V) \to 0 \quad \text{as } t \to \infty.
\]
Since the global attractor \( \mathcal{A} \) is the union of all the \( \omega \)-limit sets, this means that \( \mathcal{A} \) contains no \( n \)-dimensional subsets. Thus \( d_{\text{Haus}}(\mathcal{A}) \leq n \).

In order to do this in a rigorous way we first need a technical condition ensuring us that the flow is smooth enough to linearize the equation around a trajectory:

**Lemma 3.1** (Uniform differentiability). \( S(t) \) is uniformly differentiable, i.e., for all \( u_0 \in \mathcal{A} \) there exists a linear operator \( \Lambda(t, u_0) : L^2(\Omega) \rightarrow L^2(\Omega) \) such that for all \( t \geq 0 \),

\[
\sup_{v_0 : 0 < \|v_0 - v_0\|_{L^2(\Omega)} \leq \varepsilon} \frac{\|S(t)v_0 - S(t)u_0 - \Lambda(t, u_0)(v_0 - u_0)\|_{L^2(\Omega)}}{\|u_0 - v_0\|_{L^2(\Omega)}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,
\]

and

\[
\sup_{u_0 \in \mathcal{A}} \|\Lambda(t, u_0)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} < \infty \quad \text{for all } t \geq 0.
\]

Moreover, \( \Lambda(t, u_0) \) is compact.

**Proof.** Let \( v(t) = S(t)v_0, u(t) = S(t)u_0 \) and \( U(t) = \Lambda(t, u_0)(v_0 - u_0) \) where \( \Lambda(t, u_0) \) is the solution operator associated to the linearized equation

\[
\dot{U} - \Delta U - \nabla \cdot (\mu(u)\nabla w(U) + \mu(u)\nabla \hat{w}(U)) - g'(u)U = 0 \quad \text{in } \Omega \times (0, \infty), \quad (25)
\]

\[
(\nabla U + \mu'(u)\nabla w(u)U + \mu(u)\nabla \hat{w}(U)) \cdot n = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad (26)
\]

where \( \hat{w}(U) = K * (-2U) \). We can prove that for all \( u_0, v_0 \in \mathcal{A} \), there exists a unique solution \( U \) to the linearized equation (25)-(26) such that \( U(0) = v_0 - u_0 \) with the following regularity

\[
U \in H^1(0, T; (H^1(\Omega))^*) \cap L^2(0, T; H^1(\Omega)) \quad \text{for all } T > 0.
\]

Moreover, for all \( t > 0 \) there exists a constant \( c := c(t) \) independent on \( u_0 \) and \( v_0 \) such that

\[
\|\nabla U\|_{L^2(\Omega)} \leq c. \quad (27)
\]

Aiming at clarity, we postpone the proof of these results to the Appendix (Lemma 5.4).

Taking the difference between equations (1a) for \( v \), (1a) for \( u \) and (25) and testing with \( v - u - U \), we get

\[
0 = \frac{1}{2} \frac{d}{dt} \|v - u - U\|_{L^2(\Omega)}^2 + \|\nabla (v - u - U)\|_{L^2(\Omega)}^2
\]

\[
+ \int_{\Omega} (\mu(v)\nabla w(v) - \mu(u)\nabla w(u) - \mu'(u)\nabla w(u)U - \mu(u)\nabla \hat{w}(U)) \nabla (v - u - U)
\]

\[
+ \int_{\Omega} (g(v) - g(u) - g'(u)U)(v - u - U).
\]

Note that

\[
\int_{\Omega} (g(v) - g(u) - g'(u)U)(v - u - U)
\]

\[
\leq \int_{\Omega} g'(u)(v - u - U)^2 + \int_{\Omega} \frac{1}{2} \sup_{s \in [0, t]} |g''(s)||v - u|^2 |v - u - U|
\]

\[
\leq C \|v - u - U\|_{L^2(\Omega)}^2 + C \left( \int_{\Omega} |v - u|^4 + \int_{\Omega} |v - u - U|^2 \right)
\]

\[
\leq C \|v - u - U\|_{L^2(\Omega)}^2 + C \|v - u\|_{L^4(\Omega)}^4.
\]
By adding and subtracting first $\mu(u)\nabla w(v)$ and then $\mu'(u)(v-u)\nabla w(u)$ we also have
\[
\int_{\Omega} (\mu(v)\nabla w(v) - \mu(u)\nabla w(u) - \mu'(u)\nabla w(u)U - \mu(u)\nabla \bar{w}(U)) \nabla (v-u-U) \\
= \int_{\Omega} ((\mu(v) - \mu(u)) \nabla w(v) - \mu(u)(\nabla w(u) - \nabla w(v)) - \mu'(u)\nabla w(u)U \\
- \mu(u)\nabla \bar{w}(U)) \nabla (v-u-U) \\
= \int_{\Omega} \mu'(u)(v-u-U)\nabla w(u)\nabla (v-u-U) + \int_{\Omega} \mu'(u)(v-u)\nabla \bar{w}(v-u)\nabla (v-u-U) \\
+ \int_{\Omega} \mu(u)\nabla \bar{w}(v-u-U)\nabla (v-u-U) \\
+ \int_{\Omega} \frac{1}{2} \sup_{s \in [0,1]} |\mu'(s)|(u-v)^2\nabla w(v)\nabla (v-u-U).
\]

We estimate
\[
\int_{\Omega} (\mu'(u)(v-u-U)\nabla w(u)\nabla (v-u-U) \\
\leq \sup_{s \in [0,1]} |\mu'(s)| r_{\infty} \|u\|_{L^\infty(\Omega)} \|v-u-U\|_{L^2(\Omega)} \|\nabla(v-u-U)\|_{L^2(\Omega)} \\
\leq \frac{1}{2} \|\nabla(v-u-U)\|^2_{L^2(\Omega)} + C \|v-u-U\|^2_{L^2(\Omega)},
\]

Using assumption (K3)
\[
\int_{\Omega} (\mu'(u)(v-u)\nabla \bar{w}(v-u)\nabla (v-u-U) \\
\leq C \sup_{s \in [0,1]} |\mu'(s)|^2 \int_{\Omega} |v-u|^2 |\nabla \bar{w}(v-u)|^2 + \frac{1}{4} \|\nabla(v-u-U)\|^2_{L^2(\Omega)} \\
\leq C \left( \int_{\Omega} |v-u|^4 \right)^{1/2} \left( \int_{\Omega} |\nabla \bar{w}(v-u)|^4 \right)^{1/2} + \frac{1}{4} \|\nabla(v-u-U)\|^2_{L^2(\Omega)} \\
\leq Cr_4^2 \int_{\Omega} |v-u|^4 + \frac{1}{4} \|\nabla(v-u-U)\|^2_{L^2(\Omega)},
\]

\[
\int_{\Omega} \mu(u)\nabla \bar{w}(v-u-U)\nabla (v-u-U) \\
\leq \sup_{s \in [0,1]} \mu(s)r_2 \|v-u-U\|_{L^2(\Omega)} \|\nabla(v-u-U)\|_{L^2(\Omega)} \\
\leq C \|v-u-U\|^2_{L^2(\Omega)} + \frac{1}{8} \|\nabla(v-u-U)\|^2_{L^2(\Omega)},
\]

\[
\int_{\Omega} \frac{\mu''(c)}{2} (u-v)^2 \nabla w(v)\nabla (v-u-U) \\
\leq \|\nabla w(v)\|_{L^\infty(\Omega)} \int_{\Omega} |u-v|^2 |\nabla (v-u-U)| \\
\leq C \int_{\Omega} |u-v|^4 + \frac{1}{16} \|\nabla(v-u-U)\|^2_{L^2(\Omega)}.
\]

Combining the above estimates we get
\[
\frac{d}{dt} \|v-u-U\|^2_{L^2(\Omega)} + \frac{1}{2} \|\nabla(v-u-U)\|^2_{L^2(\Omega)} \\
\leq C \|v-u\|^4_{L^2(\Omega)} + C \|v-u-U\|^2_{L^2(\Omega)},
\]

(28)
We now recall that \( u_0 \) and \( v_0 \) belong to the global attractor \( \mathcal{A} \). Fix \( t > t_0 > 0 \). Since \( \mathcal{A} \) is invariant, \( u, v \in \mathcal{A} \) and for every \( t_0 \in (0, t+1) \), \( u \) is bounded in \( L^\infty((t_0, t+1), H^2(\Omega)) \) (cf. (iii), Thm. 2). Using the Gagliardo-Nirenberg interpolation inequality \([35]\), we have that
\[
\|u - v\|_{L^4(\Omega)} \leq C \|u - v\|_{H^2(\Omega)}^{\alpha} \|u - v\|_{L^2(\Omega)}^{1 - \alpha},
\]
where \( \alpha = d/8 \). Therefore, as a consequence of the continuous dependence from initial data, we have
\[
\|u(t) - v(t)\|_{L^4(\Omega)} \leq C(t_0, t+1) \|u(t) - v(t)\|_{L^2(\Omega)}^\alpha \leq C(t_0, t+1)C^t \|u_0 - v_0\|_{L^2(\Omega)}^{\frac{8-d}{2}}.
\]
Substituting into inequality \((28)\) we get
\[
\frac{d}{dt}\|v - u - U\|_{L^2(\Omega)}^2 \leq C(t_0, t+1) \|u_0 - v_0\|_{L^2(\Omega)}^2 + C \|v - u - U\|_{L^2(\Omega)}^2
\]
and hence, applying the Gronwall lemma and recalling that \( v_0 - u_0 - U(0) = 0 \)
\[
\|v - u - U\|_{L^2(\Omega)} \leq C(t_0, t+1) \|u_0 - v_0\|_{L^2(\Omega)}^{\frac{8-d}{2}}.
\]
As a consequence, for every \( t > 0 \) fixed we have
\[
\sup_{v_0 : 0 < \|u_0 - v_0\|_{L^2(\Omega)} \leq \varepsilon} \frac{\|S(t)v_0 - S(t)u_0 - \Lambda(t, u_0)(v_0 - u_0)\|_{L^2(\Omega)}}{\|u_0 - v_0\|_{L^2(\Omega)}} = \sup_{v_0 : 0 < \|u_0 - v_0\|_{L^2(\Omega)} \leq \varepsilon} \frac{\|v(t) - u(t) - U(t)\|_{L^2(\Omega)}}{\|u_0 - v_0\|_{L^2(\Omega)}} \leq C(t) \|u_0 - v_0\|_{L^2(\Omega)}^{\frac{8-d}{4}} \leq C(t) \varepsilon^{\frac{4-d}{4}}.
\]
As long as \( d < 4 \), the right hand side of \((29)\) goes to zero as \( \varepsilon \to 0 \). This proves \((23)\).
The last step of our proof, i.e., the boundedness and compactness of \( \Lambda(u_0, t) \) for any fixed \( t \), follows from estimate \((27)\).
\[
\square
\]
As already mentioned, in order to prove that \( d_{\text{frac}}(\mathcal{A}) \) is finite we are interested in keeping track of the evolution of the volume of an infinitesimal cube. More precisely, we focus our attention on the following quantity:
\[
\mathcal{T}R_{\lambda}(A) = \sup_{u_0 \in A} \sup_{P^{(n)}(0)} \left\langle \text{Tr}(L(t; u_0)P^{(n)}(t)) \right\rangle,
\]
where \( \left\langle \cdot \right\rangle \) denotes \( \limsup_{t \to \infty} \frac{1}{t} \int_0^t \langle h(s) \rangle ds \). Here \( L(t, u_0) : L^2(\Omega) \to (L^2(\Omega))^* = L^2(\Omega) \) (whose domain is \( H^1(\Omega) \)) is the linearized evolution operator at time \( t \) associated with the initial condition \( u_0 \in A \) given by
\[
(L(t, u_0)\phi, \psi)_{L^2(\Omega)} = -\int_{\Omega} \nabla \phi \nabla \psi + (\mu'(u) \nabla u(u) \phi + \mu(u) \nabla \tilde{w}(\phi)) \nabla \psi - g'(u) \phi \psi,
\]
where \( u(t) = S(t)u_0 \) and \( P^{(n)} : L^2(\Omega) \to L^2(\Omega) \) is a rank \( n \) projection operator onto the subspace of \( L^2(\Omega) \).
We now take advantage of the following result proved in \([36]\):

**Theorem 3.2** ([36, Thm.13.16]). Suppose that \( S(t) \) is uniformly differentiable on \( \mathcal{A} \) and that there exists a \( t_0 \) such that \( \Lambda(t, u_0) \) is compact for all \( t \geq t_0 \). If \( \mathcal{T}R_{\lambda}(A) < 0 \) then \( d_{\text{frac}}(\mathcal{A}) \leq n \).
We prove that in our case $\mathcal{T}R_n(A) < 0$ for $n$ big enough. To this aim, we start by fixing orthonormal vectors $\phi_j$, $j = 1, \ldots, n$, and defining $P^{(n)} : L^2(\Omega) \to \text{span} \{\{\phi_j\}\}$ as the standard orthonormal projection. We then compute for every $\delta > 0$ and some $C_\delta > 0$

$$\langle \text{Tr}(L(t, u_0)P^{(n)}) \rangle = -\langle \text{Tr}(-\Delta P^{(n)}) \rangle + \sum_{j=1}^{n} \int_{\Omega} \phi_j^2 g'(u)$$

$$- \sum_{j=1}^{n} \int_{\Omega} (\mu'(u)\nabla w(u)\phi_j + \mu(u)\nabla \tilde{w}(\phi_j)) \nabla \phi_j$$

$$\leq - \langle \int_{\Omega} |\nabla \phi_j|^2 \rangle + \|g'(u)\|_{L^\infty(0, \infty; L^\infty(\Omega))} \left( \sum_{j=1}^{n} \left( C\delta \int_{\Omega} \phi_j^2 + \delta \int_{\Omega} |\nabla \phi_j|^2 \right) \right)$$

$$+ \|\mu'(u)\nabla w(u)\|_{L^\infty(0, \infty; L^\infty(\Omega))} \left( \sum_{j=1}^{n} \left( C\delta \int_{\Omega} \|\nabla \tilde{w}(\phi_j)\|^2 + \delta \int_{\Omega} |\nabla \phi_j|^2 \right) \right)$$

$$\leq - \langle \int_{\Omega} |\nabla \phi_j|^2 \rangle + C\delta \left( \sum_{j=1}^{n} \int_{\Omega} \phi_j^2 \right) + C\delta \left( \sum_{j=1}^{n} \int_{\Omega} |\nabla \phi_j|^2 \right).$$

Note that the constant $C_\delta$ depends on $u$ and $u_0$ only through their $L^\infty$-norm. Recalling that for all $u_0 \in A$, $0 \leq u, u_0 \leq 1$ one can assume without loss of generality (by possibly choosing a larger constant $C$) that $C_\delta$ is independent on $u$ and $u_0$. Choosing $\delta$ sufficiently small, and recalling that $\int_{\Omega} \phi_j^2 = 1$ we have

$$\langle \text{Tr}(L(t, u_0)P^{(n)}) \rangle \leq -\frac{1}{2} \left( \int_{\Omega} |\nabla \phi_j|^2 \right) + Cn$$

$$= -\frac{1}{2} \langle \text{Tr}(-\Delta P^{(n)}) \rangle + Cn.$$

Applying [36, Lemma 13.17], we get

$$\langle \text{Tr}(L(t, u_0)P^{(n)}) \rangle \leq -\frac{1}{2} \langle \text{Tr}(-\Delta P^{(n)}) \rangle + Cn \leq -\frac{1}{2} Cn^{\frac{d+2}{d}} + Cn.$$

Thus, there exists $N \in \mathbb{N}$ such that $\langle \text{Tr}(L(t, u_0)P^{(n)}) \rangle$ is negative for all $n \geq N$. Moreover $N$ is independent of the choice of the vectors $\phi_j$ and of the initial condition $u_0 \in A$. By virtue of Thm. 3.2 and Lemma 3.1 (see Appendix), we conclude that the fractal dimension of $A$ is smaller than $N$, hence finite.

4. Exponential attractor: Proof of Theorem 2.6. We now prove the existence of a finite-dimensional exponential attractor $A'$. The idea we follow is presented in [13] to show an analogous result in the case $g = 0$. To this aim, we first need to prove the following:

**Lemma 4.1.** Let $u_{01}, u_{02} \in X$ and let $u_1, u_2$ be the solutions of (1) corresponding to the initial conditions $u_1(0) = u_{01}, u_2(0) = u_{02}$. Then, there exists $C$ independent
of \( t \) such that
\[
\|u_1(t) - u_2(t)\|_{L^2(\Omega)} \leq C e^{-t} \|u_{01} - u_{02}\|_{L^2(\Omega)} + C \|u_1 - u_2\|_{L^2(0,t;L^2(\Omega))},
\]
\[
\|u_1 - u_2\|_{H^1(0,t;H^1_0(\Omega)) \cap L^2(0,t;H^1(\Omega))} \leq C \|u_{01} - u_{02}\|_{L^2(\Omega)}.
\]

**Proof.** Equation (31) comes from the following. Taking the difference of the equation (1a) for \( u_1 \) and \( u_2 \) and testing it with \( u_1 - u_2 \) one can easily obtain (see [32, Sec. 3.1] for details)
\[
\frac{d}{dt} \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega)}^2 \leq C \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2.
\]
In particular,
\[
\frac{d}{dt} \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 + \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 \leq (C + 1) \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2
\]
which yields, thanks to the Gronwall Lemma,
\[
\|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 \leq e^{-t} \|u_{01,0} - u_{02,0}\|_{L^2(\Omega)}^2 + (C + 1) \int_0^t \|u_1 - u_2\|_{L^2(\Omega)}^2,
\]
that is equivalent to (31). In order to derive (32), we first recall that [32, Sec. 3.1]
\[
\|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)}^2 \leq C e^{Ct} \|u_{01,0} - u_{02,0}\|_{L^2(\Omega)}^2.
\]
Moreover, by testing the difference of equation (1a) for \( u_1 \) and \( u_2 \) with a test function \( \psi \in H^1(0,T;H^1(\Omega)) \), and using the Lipschitz property of \( g \) and \( \mu \) and the linearity of the convolution, we get:
\[
(\partial_t u_1 - \partial_t u_2, \psi)_{H^1(\Omega)}
\]
\[
= (\nabla (u_1 - u_2) + \mu(u_1) \nabla w(u_1) - \mu(u_2) \nabla w(u_2), \nabla \psi)_{L^2(\Omega)} + (g(u_1) - g(u_2), \psi)_{L^2(\Omega)}
\]
\[
\leq C \|u_1 - u_2\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}.
\]
Then, using this result together with equation (34) we finally obtain
\[
\|\partial_t u_1 - \partial_t u_2\|_{L^2(0,t;H^1(\Omega))^*} + \int_0^t \|u_1 - u_2\|_{H^1(\Omega)}^2 \leq C e^{Ct} \|u_{01,0} - u_{02,0}\|_{L^2(\Omega)}^2.
\]
\[\square\]

Now we can proceed with proving Theorem 2.6. We choose \( T > 0 \) such that \( \gamma := C e^{-T} < \frac{1}{2} \) and define
\[
\mathcal{H}_1 := H^1(0,t;H^1_0(\Omega)) \cap L^2(0,t;H^1(\Omega)),
\]
\[
\mathcal{H} := L^2(0,t;L^2(\Omega)),
\]
\[
\mathcal{T} : u_0 \in \mathcal{X} \mapsto u \in \mathcal{H},
\]
where \( u \) solves (1) with \( u(0) = u_0 \). Note that equations (31) and (32) can be rewritten as
\[
\|S(T)u_{01} - S(T)u_{02}\|_{L^2(\Omega)} \leq \gamma \|u_{01} - u_{02}\|_{L^2(\Omega)} + \tilde{C} \|\mathcal{T}u_{01} - \mathcal{T}u_{02}\|_{\mathcal{H}},
\]
\[
\|\mathcal{T}u_{01} - \mathcal{T}u_{02}\|_{\mathcal{H}_1} \leq C e^{CT} \|u_{01} - u_{02}\|_{L^2(\Omega)},
\]
for some \( \tilde{C} \) independent of \( T \).

We now build a discrete exponential attractor for the (discrete) system \((X,S(T))\).
Define \( X_0 = \{0\} \). By definition of \( X \) we have that \( X \subseteq \mathcal{B}_R^{L^2(\Omega)}(0) \), where \( R = |\Omega| \)
and $B_R^{L^2(\Omega)}(0)$ denotes the $L^2(\Omega)$-ball of radius $R$ centered in 0. As a consequence of (38) we have

$$T(X) \subseteq B_H^{\mathcal{C}}(T(0)).$$

(39)

Fix now $\theta > 0$ such that $2(\gamma + \theta) < 1$. Since $\mathcal{H}_1$ is compactly embedded in $\mathcal{H}$, there exists a finite number $n_1$ of $\mathcal{H}$-balls of radius $R_\mathcal{H}^{\frac{\theta}{C}}$ that cover $T(X)$, i.e.,

$\exists X_1 = \{x_1, \ldots, x_{n_1}\} \subseteq X^{n_1}$ such that

$$T(X) \subseteq \bigcup_{x \in X_1} B_{\mathcal{H}}(T(x)).$$

(40)

Without loss of generality we assume that $n_1$ is the minimum number for which (40) holds. Note that, as a consequence of (38), $n_1$ can be estimated by the minimum number of $\mathcal{H}$-balls of radius $R_\mathcal{H}^{\frac{\theta}{C}}$ necessary to cover $B_{\mathcal{H}}^{\mathcal{H}_1}(R_{\mathcal{C}}e^{\mathcal{C}T})(0)$. Thus, $n_1 \leq N(\theta, T)$, where $N(\theta, T)$ depends on $\theta, T, \mathcal{C}$ but not on $R$.

We now observe that the family of $L^2(\Omega)$-balls with radius $2(\gamma + \theta)R$ and centers $S(T)x_i \in X_1$ covers $S(T)X$. Indeed, let $x \in S(T)X$. Then, there exists $u_0 \in X$ such that $T(u_0) = x$. Thanks to (37), we estimate

$$\|S(T)u_0 - S(T)x_i\|_{L^2(\Omega)} \leq \gamma \|u_0 - x_i\|_{L^2(\Omega)} + \tilde{C}\|u_0 - T x_i\|_{\mathcal{H}} \leq 2(\gamma + \theta)R.$$

Here we used that (recalling (40))

$$\tilde{C}T(u_0) \in \bigcup_{x \in X_1} B_{\mathcal{H}}^{\mathcal{H}_1}(T(x)) \subseteq \bigcup_{x \in X_1} B_{\mathcal{H}}^{\mathcal{H}_1}(x).$$

Hence,

$$S(T)X \subseteq \bigcup_{x \in X_1} B_2^{2(\gamma + \theta)R}(x).$$

Applying the same procedure to each ball in the covering $\{B_2^{2(\gamma + \theta)R}(x) : x \in X_1\}$ we can find a new covering of $S(T)X$ with at most $N^2(\theta, T) L^2(\Omega)$-balls of radius $(2(\gamma + \theta))^2R$, i.e., there exists $X_2$, $\#X_2 \leq N^2(\theta, T)$, such that

$$S(2T)X = S(T)^2X \subseteq \bigcup_{x \in X_2} B_2^{2(\gamma + \theta)R}(x).$$

Iterating this argument for all $K \in \mathbb{N}$ we can show that there exists a set $X_K \in X$ such that $\#X_K \leq N^K(\theta, T)$ and

$$S(KT)X = S(T)^KX \subseteq \bigcup_{x \in X_K} B_2^{2(\gamma + \theta)K R}(x).$$

In particular,

$$d_{L^2(\Omega)}(S(T)^KX, X_K) \leq R(2(\gamma + \theta))^K.$$

The set $A_d = \bigcup_{K = 1}^{+\infty} X_K$ is then an exponential attractor for the discrete system $(S(T), X)$.

Moreover, $A_d$ has finite fractal dimension. Indeed, fix $\varepsilon > 0$ and let $N_\varepsilon$ be the minimum number of $L^2(\Omega)$-balls of radius $\varepsilon$ needed to cover $A_d$. Let $K \in \mathbb{N}$ be the integer part of $\log_2(2(\gamma + \theta)) = \frac{\log \varepsilon}{\log(2(\gamma + \theta))}$, i.e., $(2(\gamma + \theta))^{K+1} < \varepsilon \leq (2(\gamma + \theta))^K$. Thus, $N_\varepsilon \leq N_\varepsilon(2(\gamma + \theta)) \leq (N(\theta, T))^{K+1}$. Consequently

$$\limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}} \leq (K + 1) \frac{\log N(\theta, T)}{\log 2(\gamma + \theta)} = -\frac{\log N(\theta, T)}{\log 2(\gamma + \theta)} < +\infty. \quad (41)$$

As a consequence of Definition 2.3, $d_{\text{frac}} A_d$ is finite.
Thus, there exists a discrete exponential attractor \( \mathcal{A}_d \), i.e.,
\[
\|d_{L^2(\Omega)}((S(T))^n u_0, \mathcal{A}_d) = d_{L^2(\Omega)}(S(nT)u_0, \mathcal{A}_d) \leq C e^{-nc}
\]
for some positive constant \( C \). We now construct the exponential attractor for the semigroup \( S(t) \). To this aim, we define
\[
\mathcal{A}' = U_{t \in [0,T]} S(t) \mathcal{A}_d.
\]
For all \( t > 0 \) we can find \( n \) such that \( t = nT + \hat{t} \) where \( 0 \leq \hat{t} < T \). For all \( \varepsilon > 0 \) we can find \( a_{n,u_0,\varepsilon} \in \mathcal{A}_d \) such that
\[
\|S(nT)u_0 - a_{n,u_0,\varepsilon}\|_{L^2(\Omega)} \leq (C + \varepsilon) e^{-nc}.
\]
By virtue of estimate (34), we have that
\[
\|S(t)u_0 - S(\hat{t})a_{n,u_0,\varepsilon}\|_{L^2(\Omega)} \leq C e^{C \hat{t}} \|S(nT)u_0 - a_{n,u_0,\varepsilon}\|_{L^2(\Omega)} \leq C e^{CT} (C + \varepsilon) e^{-nc}.
\]
Since \( T \) is fixed, by suitably renaming the constants we have
\[
\|S(t)u_0 - S(\hat{t})a_{n,u_0,\varepsilon}\|_{L^2(\Omega)} \leq (C + \varepsilon) e^{-ct}.
\]
Since, by construction \( S(\hat{t})a_{n,u_0,\varepsilon} \in \mathcal{A}' \), by the arbitrariness of \( \varepsilon \) we conclude that \( \mathcal{A}' \) is an exponential attractor for \( S(t) \).

**Remark 2.** The finite fractal dimension of the global attractor proved in Section 3.2 can also be derived as a direct consequence of the existence of an exponential attractor, which has finite fractal dimension and contains the global attractor by definition. The argument used in Section 3.2, however, allowed us to obtain a different upper bound on the fractal dimension of the global attractor (cf. Thm. 3.2) with respect to the upper bound found for the exponential attractor in (41).

5. Equilibria: Proof of Theorem 2.8.

5.1. Existence. Since we are looking for solutions \( 0 \leq u \leq 1 \), we can assume without loss of generality (and using hypothesis (G))
\[
\mu(s) = 0 \quad \forall s \notin [0,1] \quad (42)
\]
\[
g(s) = g(0) \geq 0 \quad \text{for} \ s < 0 \quad \text{and} \quad g(s) = g(1) \leq 0 \quad \text{for} \ s > 1 \quad \text{a.e. in} \ \Omega. \quad (43)
\]
The proof is based on a regularization procedure and a fixed point argument. We first consider the regularization problem parametrized by small \( \varepsilon > 0 \):
\[
-\Delta u - \nabla \cdot (\mu \nabla w) + \varepsilon u = g(u), \quad (44a)
\]
\[
n \cdot (\mu \nabla w + \nabla u) = 0, \quad (44b)
\]
\[
w = K * (1 - 2u). \quad (44c)
\]
We define then the mapping \( \Gamma : z \in L^2(\Omega) \rightarrow u \in L^2(\Omega) \) where \( u \) is weak solution to
\[
-\Delta u + \varepsilon u = \nabla \cdot (\mu(z) \nabla (K * (1 - 2z))) + g(z). \quad (45)
\]
The hypotheses on \( \mu, g \) and \( K \) ensure that the right hand side of (45) is in \( (H^1(\Omega))^\star \) and \( \Gamma \) is well defined thanks to the Lax-Milgram theorem. In order to prove the existence of a fixed point, we apply Schaefer’s fixed point Theorem (Thm. 5.1, see Appendix). The continuity of \( \Gamma \) is again guaranteed by the assumptions on \( K \) and the fact that \( \mu \) and \( g \) are Lipschitz and bounded. Compactness of \( \Gamma \) can be proved by
testing (45) with \( u \) and using \( \|(\mu(z)\nabla(K * (1 - 2z))) + g(z)\|_{L^2(\Omega)} \leq C\|z\|_{L^2(\Omega)} + C \) to obtain the estimate
\[
\|u\|_{H^1(\Omega)} \leq C\|z\|_{L^2(\Omega)} + C\varepsilon.
\]

In order to apply Schaefer’s fixed point theorem, we are now only left with proving that the set \( \{ u \in L^2(\Omega) : u = \alpha \Gamma(u) \text{ for some } \alpha \in [0, 1]\} \) is bounded. This is equivalent to showing that the set
\[
A := \{ u \in L^2(\Omega) : u/\alpha = \Gamma(u) \text{ for some } \alpha \in (0, 1]\}
\]
is bounded. To this end, let \( u \in A \). Then, there exists \( \alpha \in (0, 1] \) such that
\[
-\Delta u/\alpha + \varepsilon u/\alpha = \nabla \cdot (\mu(u) \nabla (K * (1 - 2u))) + g(u). \tag{46}
\]
By testing the equation with \( \alpha(u - 1)^+ \) and using assumptions (42)-(43), we get
\[
\int_{u \geq 1} |\nabla u|^2 + \varepsilon \int_{u \geq 1} (u^2 - u) = -\alpha \int_{u \geq 1} \mu \nabla w \nabla u + \alpha \int_{u \geq 1} g(u)(u - 1) \leq 0.
\]
This yields \( u \leq 1 \) a.e. in \( \Omega \) for every \( u \in A \). Similarly, it can be shown that \( u \geq 0 \) a.e. in \( \Omega \) for every \( u \in A \). This implies that \( A \) is bounded in \( L^2(\Omega) \) for every \( \varepsilon \) fixed. Thus, the mapping \( \Gamma \) has a fixed point \( u \) which solves (44) and such that \( 0 \leq u \leq 1 \) a.e. in \( \Omega \).

Our final step is the passage to the limit \( \varepsilon \to 0 \). Let \( u_{\varepsilon} \) be a solution of the regularized problem (44); hence, \( 0 \leq u_{\varepsilon} \leq 1 \). Test (44a) with \( u_{\varepsilon} \), and estimate
\[
\int_{\Omega} |\nabla u_{\varepsilon}|^2 + \varepsilon |u_{\varepsilon}|^2 = \int_{\Omega} \mu(u_{\varepsilon}) \nabla w_{\varepsilon} \nabla u_{\varepsilon} + \int_{\Omega} g(u_{\varepsilon}) u_{\varepsilon} \\
\leq C + C\|\nabla u_{\varepsilon}\|_{L^2(\Omega)}.
\]
This implies \( \|\nabla u_{\varepsilon}\|_{L^2(\Omega)} \leq C \). Thanks to \( 0 \leq u_{\varepsilon} \leq 1 \), we get \( \|u_{\varepsilon}\|_{H^1(\Omega)} \leq C \), where \( C \) is independent of \( \varepsilon \). Consequently, for a not relabeled subsequence we get
\[
u_{\varepsilon} \to u \text{ weakly in } H^1(\Omega),
\]
\[
u_{\varepsilon} \to u \text{ strongly in } L^2(\Omega) \text{ and pointwise a.e. in } \Omega.
\]
Moreover, using the continuity and boundedness of \( \mu \) and \( g \) we have
\[
\mu(u_{\varepsilon}) \to \mu(u) \text{ strongly in } L^2(\Omega),
\]
\[
g(u_{\varepsilon}) \to g(u) \text{ strongly in } L^2(\Omega).
\]
Finally, thanks to the continuity of the convolution and assumption (K3),
\[
w_{\varepsilon} = K * (1 - 2u_{\varepsilon}) \to w = K * (1 - 2u) \text{ strongly in } H^1(\Omega),
\]
\[
\mu(u_{\varepsilon}) \nabla w_{\varepsilon} \to \mu(u) \nabla w \text{ weakly in } H^1(\Omega).
\]
This allows us to pass to the limit in the weak formulation of (44) and prove that \( u \) solves (16) as well as \( 0 \leq u \leq 1 \).

**Remark 3.** Uniqueness of equilibrium points is not guaranteed. As an example, consider a function \( g \) such that \( g(0) = g\left(\frac{1}{2}\right) = g(1) = 0 \). The constant functions \( u^*(x) = 0, u^*(x) = \frac{1}{2} \) and \( u^*(x) = 1 \) are then all possible solutions of (16).
5.2. **Convergence to equilibria.** As stated in the introduction, many known results about convergence to equilibria for the reaction-free Cahn-Hilliard equation rely on the existence of a Lyapunov functional [1, 30, 31]. The presence of a reaction term, however, does not allow us to apply the same techniques. Therefore, we adopt two different strategies to obtain the convergence to equilibria in the following cases:

1. \( g \) has sign, i.e. \( g \geq 0 \), or \( g \leq 0 \) (more precisely, case 1 of Theorem 2.8)
2. \( g \) is monotone decreasing and \( g' \leq -\lambda \) for a sufficiently large positive constant \( \lambda \) (case 2 of Theorem 2.8).

In the first case, we observe that the total mass is monotone as a function of time. Combined with the boundedness of \( u \), this proves the convergence of the solution to one of the pure phases. In the second case we exploit a linearization technique around the equilibrium point.

We emphasize that these conditions include a wide spectrum of reaction terms notably relevant in applications, such as (5)-(6). In particular we prove convergence to equilibria for the Cahn-Hilliard-Oono [3] equation and for the Cahn-Hilliard equation with the reaction term (7), as well as for a Cahn-Hilliard equation coupled with a logistic-type reaction term [27].

### 5.2.1. Case 1.

We outline the details for the convergence to \( u = 1 \). A similar approach can be applied to show the second part of the theorem (i.e., the convergence to \( u = 0 \)). Hypothesis (A1) and Theorem 2.2 guarantee that \( u \) separates from 0 uniformly in time, namely \( \forall T_0 \exists k_1(\bar{u}_0, T_0) > 0 \) such that \( u(t) \geq k_1 \) for all \( t \geq T_0 \) a.e. in \( \Omega \). \(^2\) Using hypothesis (A2) with \( k_1 = \kappa \) and defining \( r_\kappa(s) := -m_\kappa(s - 1) \), we have that \( g(x, s) \geq r_\kappa(s) \forall s \in [\kappa, 1] \).

Testing equation (1a) with \( \frac{1}{|\Omega|} \int_\Omega g(x, u) \geq \frac{1}{|\Omega|} \int_\Omega r_\kappa(u) = r_\kappa(\bar{u}) \).

Recalling \( r_\kappa(s) = -m_\kappa(s - 1) \), we can rewrite the last inequality as \( \frac{d}{dt} (\bar{u} - 1) \geq -m_\kappa(\bar{u} - 1) \). Since \( 0 \leq u \leq 1 \) (cf. Theorem 2.2), we have then \( 0 \geq \bar{u}(t) - 1 \geq (\bar{u}_0 - 1) \exp(-m_\kappa t) \). Hence, \( \bar{u}(t) \to 1 \). Finally, applying Lemma 5.3 to \( U = 1 - u \) we get \( u(t) \to 1 \) in \( L^2(\Omega) \).

This proves \( u(t) \to 1 \) exponentially fast in \( L^2(\Omega) \) for every \( u_0 \in \Omega \) if the measure of the set \( \{ x \in \Omega \text{ s.t. } g(x, 0) > 0 \} \) is positive and for every \( u_0 \neq 0 \) if \( g(\cdot, 0) = 0 \). We recall that, if \( g(0) = 0, u = 0 \) is an equilibrium point.

The previous result can be applied to show the convergence to equilibria for the Cahn-Hilliard equation with logistic reaction term (5) (cf. Corollary (1)).

### 5.2.2. Case 2.

Let \( u \) be a solution to (1a) and \( u^* \) be an equilibrium point. Taking the difference between equations (1a) and (16a) and testing it with \( U = u - u^* \) gives

\[
\frac{d}{dt} \frac{1}{2} \|U\|_{L^2(\Omega)}^2 + \|\nabla U\|_{L^2(\Omega)}^2 + \int_{\Omega} (\mu \nabla w - \mu^* \nabla w^*) \nabla U = \int_{\Omega} (g(u) - g(u^*)) U, \tag{47}
\]

\(^2\) Note that \( k_1 \) is strictly positive for every \( u_0 \) if \( g(x, 0) > 0 \) on a set of positive measure, and for every \( u_0 \neq 0 \) (a.e. in \( \Omega \)) if \( g(x, 0) = 0 \) a.e. in \( \Omega \).
where \( \mu^* = \mu(u^*) \) and \( w^* = K \ast (1 - 2u^*) \). Thanks to (K3), we estimate
\[
\int \Omega (\mu \nabla w - \mu^* \nabla w^*) \nabla U = \int \Omega (\mu \nabla (w - w^*)) \nabla U + \int \Omega (\mu - \mu^*) \nabla w^* \nabla U \\
\leq \frac{1}{4} r_2 \| U \|_{L^2(\Omega)} \| \nabla U \|_{L^2(\Omega)} + \int \Omega (1 - u - u^*) \nabla w^* \nabla U \\
\leq \left( \frac{r_2}{4} + r_\infty \right) \| U \|_{L^2(\Omega)} \| \nabla U \|_{L^2(\Omega)} \\
\leq \frac{1}{2} \| \nabla U \|_{L^2(\Omega)}^2 + \frac{1}{2} \left( \frac{r_2}{4} + r_\infty \right)^2 \| U \|_{L^2(\Omega)}^2.
\]
Here we used the fact that \( \sup_{s \in [0, 1]} \mu(s) \leq \frac{1}{4} \). By assumption (18) we have
\[
\int \Omega (g(u) - g(u^*)) U \leq -\lambda \| U \|_{L^2(\Omega)}^2.
\]
Thus, by substituting into (47), we get
\[
\frac{d}{dt} \frac{1}{2} \| U \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla U \|_{L^2(\Omega)}^2 \leq (C_1 - \lambda) \| U \|_{L^2(\Omega)}^2,
\]
where \( c_1 = \frac{1}{2} \left( \frac{r_2}{4} + r_\infty \right)^2 \). By applying the Gronwall Lemma one gets
\[
\| U(t) \|_{L^2(\Omega)}^2 \leq e^{2(C_1 - \lambda)t} \| u_0 - u^* \|_{L^2(\Omega)}^2.
\]
Thus, for \( \lambda > C_1 \) we have \( U(t) \to 0 \) in \( L^2(\Omega) \). This implies \( u(t) \to u^* \) in \( L^2(\Omega) \).

This result can be applied to show the convergence to equilibria for the Cahn-Hilliard-Oono [3] equation and for the Cahn-Hilliard equation with reaction term \( g \) as in (7).

Appendix. In the following we include some auxiliary results we use throughout our proofs.

Theorem 5.1 (Schaefer [15, Thm.4, Sec.9.2]). Let \( X \) be Banach and let \( \Gamma : X \to X \) be continuous, compact, and such that the set \( \{ x \in X : x = \alpha \Gamma(x) \text{ for some } \alpha \in [0, 1] \} \) is bounded. Then, \( \Gamma \) has a fixed point.
The existence of solutions can be obtained by using a Galerkin approximation. Proof. Moreover, for all \( t > t_0 \) where \( \tilde{U} \) (Uniform Gronwall Lemma, [36, Ex.11.2])

Lemma 5.2 \([\bar{\Omega}, \Omega]\) we have that

\[
\|u(t)\|_{L^2(\Omega)} \leq 2|\Omega|\bar{u}(t) \to 0.
\]

This concludes the proof of the lemma.

Proof. Let \( u, v \in \mathcal{A} \). There exists a unique solution \( U \) to the equation

\[
\dot{U} - \Delta U - \nabla \cdot (\mu'(u)\nabla w(u)U + \mu(u)\nabla \bar{w}(U)) - g'(u)U = 0 \text{ in } \Omega \times (0, \infty),
\]

\[
(\nabla U + \mu'(u)\nabla w(u)U + \mu(u)\nabla \bar{w}(U)) \cdot n = 0 \text{ on } \partial\Omega \times (0, \infty),
\]

\[
U(0) = v_0 - u_0,
\]

where \( \bar{w}(U) = K \ast (-2U) \) with the following regularity:

\[
U \in H^1(0, T; (H^1(\Omega))^*) \cap L^2(0, T; H^1(\Omega)) \text{ for all } T > 0.
\]

Moreover, for all \( t > 0 \) there exists a constant \( c := c(t) \) independent of \( u_0 \) and \( v_0 \) such that

\[
\|\nabla U\|_{L^2(\Omega)} \leq c.
\]

Proof. The existence of solutions can be obtained by using a Galerkin approximation scheme and by virtue of the following estimate obtained by testing the equation with \( U \):

\[
\frac{1}{2} \frac{d}{dt} \|U\|_{L^2(\Omega)}^2 + \|\nabla U\|_{L^2(\Omega)}^2 \leq \int_\Omega (\mu'(u)\nabla w(u)U + \mu(u)\nabla \bar{w}(U)) \nabla U + g'(u)U^2 \leq C \int_\Omega |U| |\nabla U| + |\nabla \bar{w}(U)||\nabla U| + |U|^2 \leq C \|U\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla U\|_{L^2(\Omega)}^2.
\]
Here we used boundedness of $\mu$, $\mu'$, $g'$, and assumption (K3). From estimate (53) it follows, thanks to the Gronwall lemma,
\[ \|U\|^2_{L^2(\Omega)} \leq \|U(0)\|^2_{L^2(\Omega)} + Ce^{Ct}, \]
\[ \int_0^t \|\nabla U\|^2_{L^2(\Omega)} \leq Cte^{Ct}, \]
and, by comparison in the equation (48), $\dot{U} \in L^2(0, T; (H^1(\Omega))^*)$.

Uniqueness is a consequence of estimate (54) and of the linearity of the equation. We now prove (52). Fix $t > 0$ and integrate (53) between 0 and $t$, getting
\[ \frac{1}{2} \int_0^t \|\nabla U\|^2_{L^2(\Omega)} \leq C \int_0^t \|U\|^2_{L^2(\Omega)} + \|U(0)\|^2_{L^2(\Omega)} . \]

We now test equation (48) with $(-\Delta U)$. Using boundedness of $u$, $\mu$, $\mu'$, $\mu''$, assumption (K3) and (K4), the continuous embedding of $H^1(\Omega)$ into $L^4(\Omega)$ (that holds true for $d \leq 3$), and the fact that $u_0 \in A$ implies $u \in L^\infty(0, t; H^2(\Omega))$ for all $t > 0$ (cf. [32]), we obtain
\[ \frac{1}{2} \frac{d}{dt} \|\nabla U\|_{L^2(\Omega)}^2 + \|\Delta U\|_{L^2(\Omega)}^2 \leq \int_\Omega \nabla (\mu'(u)\nabla w(u)U + \mu(u)\nabla \tilde{w}(U)) (-\Delta U) + g'(u)U(-\Delta U) \]
\[ \leq \frac{1}{2} \|\Delta U\|_{L^2(\Omega)}^2 + C \|U\|^2_{L^2(\Omega)} + C \|\nabla (\mu'(u)\nabla w(u)U + \mu(u)\nabla \tilde{w}(U))\|_{L^2(\Omega)}^2 \]
\[ \leq \frac{1}{2} \|\Delta U\|_{L^2(\Omega)}^2 + C \|U\|^2_{L^2(\Omega)} + C \|\nabla U\|_{L^2(\Omega)}^2 . \]

In particular, we have estimated
\[ \|\nabla (\mu'(u)\nabla w(u)U + \mu(u)\nabla \tilde{w}(U))\|_{L^2(\Omega)} \]
\[ \leq \|\mu''(u)\nabla w(u)U\nabla u\|_{L^2(\Omega)} + \|\mu'(u)\Delta w(u)U\|_{L^2(\Omega)} \]
\[ + \|\mu'(u)\nabla w(u)\nabla U\|_{L^2(\Omega)} + \|\mu'(u)\nabla u\nabla \tilde{w}(U)\|_{L^2(\Omega)} \]
\[ + \|\mu(u)\Delta \tilde{w}(U)\|_{L^2(\Omega)} , \]
\[ \|\mu'(u)\nabla w(u)U\nabla u\|_{L^2(\Omega)} \leq \|\mu''(u)\nabla w(u)\|_{L^\infty(\Omega)} \|U\|_{L^1(\Omega)} \|\nabla u\|_{L^4(\Omega)} \]
\[ \leq C \|U\|_{H^1(\Omega)} \|\nabla u\|_{H^1(\Omega)} \leq C \|U\|_{H^1(\Omega)} , \]
\[ \|\mu'(u)\Delta w(u)U\|_{L^2(\Omega)} \leq \|\mu'(u)\|_{L^\infty(\Omega)} \|\Delta w(u)\|_{L^2(\Omega)} \|U\|_{L^2(\Omega)} \leq C \|U\|_{H^1(\Omega)} , \]
\[ \|\mu'(u)\nabla w(u)\nabla U\|_{L^2(\Omega)} \leq \|\mu'(u)\|_{L^\infty(\Omega)} \|\nabla w(u)\|_{L^2(\Omega)} \|\nabla U\|_{L^2(\Omega)} \leq C \|U\|_{H^1(\Omega)} , \]
\[ \|\mu'(u)\nabla u\nabla \tilde{w}(U)\|_{L^2(\Omega)} \leq \|\mu'(u)\|_{L^\infty(\Omega)} \|\nabla u\|_{L^4(\Omega)} \|\nabla \tilde{w}(U)\|_{L^4(\Omega)} \]
\[ \leq C \|U\|_{H^1(\Omega)} \leq C \|U\|_{H^1(\Omega)} , \]
\[ \|\mu(u)\Delta \tilde{w}(U)\|_{L^2(\Omega)} \leq \|\mu(u)\|_{L^\infty(\Omega)} \|\Delta \tilde{w}(U)\|_{L^2(\Omega)} \leq C \|U\|_{H^1(\Omega)} . \]

Note that the above estimates are just formal, however they can be justified rigorously by mean of an approximation procedure.
Integrating (57) between $s$ and $t$ for some $0 < s < t$, we get
\[
\|\nabla U(t)\|_{L^2(\Omega)}^2 \leq \|\nabla U(s)\|_{L^2(\Omega)}^2 + C \int_s^t \|U\|_{L^2(\Omega)}^2 + C \int_s^t \|\nabla U\|_{L^2(\Omega)}^2
\]
\[
\leq \|\nabla U(s)\|_{L^2(\Omega)}^2 + C \int_0^t \|U\|_{L^2(\Omega)}^2 + C \int_0^t \|\nabla U\|_{L^2(\Omega)}^2
\]
Integrating now between 0 and $t$ with respect to $s$, and using (56), we obtain
\[
t \|\nabla U(t)\|_{L^2(\Omega)}^2 \leq (1 + Ct) \int_0^t \|\nabla U\|_{L^2(\Omega)}^2 + Ct \int_0^t \|U\|_{L^2(\Omega)}^2
\]
\[
\leq (1 + Ct)C \int_0^t \|U\|_{L^2(\Omega)}^2 + \|U(0)\|_{L^2(\Omega)}^2.
\]
The right hand side is bounded for every $t > 0$ fixed. This concludes the proof of the lemma.

REFERENCES

[1] H. Abels and M. Wilke, Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy, *Nonlinear Anal.*, **67** (2007), 3176–3193.
[2] G. Alberti and G. Bellettini, A non-local anisotropic model for phase transitions: Asymptotic behaviour of rescaled energies, *European J. Appl. Math.*, **9** (1998), 261–284.
[3] M. Bahiana and Y. Oono, Cell dynamical system approach to block copolymers, *Phys. Rev.* **9** (1998), 27–41.
[4] P. W. Bates and J. Han, The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation, *J. Math. Anal. Appl.*, **311** (2005), 289–312.
[5] A. L. Bertozzi, S. Esedoḡlu and A. Gillette, Inpainting of binary images using the Cahn-Hilliard equation, *IEEE Trans. Image Process.*, **16** (2007), 285–291.
[6] J. W. Cahn, On spinodal decomposition, *Acta Metall.*, **9** (1961), 795–801.
[7] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.*, **28** (1958), 258–267.
[8] L. Cherfils, H. Fakih and A. Miranville, Finite-dimensional attractors for the Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation in image inpainting, *Inverse Probl. Imaging*, **9** (2015), 105–125.
[9] L. Cherfils, H. Fakih and A. Miranville, On the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation with logarithmic nonlinear terms, *SIAM J. Imaging Sci.*, **8** (2015), 1123–1140.
[10] L. Cherfils, A. Miranville and S. Zelik, The Cahn-Hilliard equation with logarithmic potentials, *Milan J. Math.*, **79** (2011), 561–596.
[11] L. Cherfils, A. Miranville and S. Zelik, On a generalized Cahn-Hilliard equation with biological applications, *Discrete Contin. Dyn. Syst. Ser. B.*, **19** (2014), 2013–2026.
[12] F. Della Porta and M. Grasselli, Convective nonlinear Cahn-Hilliard equations with reaction terms, *Discrete Contin. Dyn. Syst. Ser. B*, **20** (2015), 1529–1553, *arXiv:math/0605406*.
[13] M. Efendiev, A. Miranville and S. Zelik, Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$, *C. R. Math. Acad. Sci.*, **330** (2000), 713–718.
[14] C. M. Elliott and H. Garcke, On the Cahn-Hilliard equation with degenerate mobility, *SIAM J. Math. Anal.*, **27** (1996), 404–423.
[15] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, New York, 1998.
[16] H. Fakih, A Cahn-Hilliard equation with a proliferation term for biological and chemical applications, *Asymptot. Anal.*, **94** (2015), 71–104.
[17] E. Feireisl, F. Issard-Roch and H. Petzeltová, A non-smooth version of the Lojasiewicz–Simon theorem with applications to non-local phase-field systems, *J. Differential Equations*, **199** (2004), 1–21.
[18] S. Frigeri and M. Grasselli, Global and trajectory attractors for a nonlocal Cahn-Hilliard-Navier-Stokes system, *J. Dynam. Differential Equations*, **24** (2012), 827–856.
[19] S. Frigeri, M. Grasselli and E. Rocca, A diffuse interface model for two-phase incompressible flows with non-local interactions and non-constant mobility, *Nonlinearity*, **28** (2015), 1257–1293.
[20] H. Gajewski and K. Zacharias, On a nonlocal phase separation model, *J. Math. Anal. Appl.*, 286 (2003), 11–31.
[21] C. G. Gal and M. Grasselli, Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27 (2010), 401–436.
[22] C. G. Gal and M. Grasselli, Longtime behavior of nonlocal Cahn-Hilliard equations, *Discrete Contin. Dyn. Syst.*, 34 (2014), 145–179.
[23] H. Garcke, B. Nestler and B. Stoth, On anisotropic order parameter models for multi-phase systems and their sharp interface limits, *Phys. D*, 115 (1998), 87–108.
[24] H. Garcke, M. Rumpf and U. Weikard, The Cahn-Hilliard equation with elasticity-finite element approximation and qualitative studies, *Interfaces Free Bound.*, 3 (2001), 101–118.
[25] G. Giacomin and J. L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits, *J. Stat. Phys.*, 87 (1997), 37–61.
[26] Z. Guan, J. S. Lowengrub, C. Wang and S. M. Wise, Second order convex splitting schemes for periodic nonlocal Cahn-Hilliard and Allen-Cahn equations, *J. Comput. Phys.*, 277 (2014), 48–71.
[27] E. Khain and L. M. Sander, Generalized Cahn-Hilliard equation for biological applications, *Phys. Rev. E*, 77 (2008), 051129.
[28] N. Q. Le, A Gamma-convergence approach to the Cahn-Hilliard equation, *Calc. Var. Partial Differential Equations*, 32 (2008), 499–522.
[29] M. Liero and S. Reichelt, Homogenization of Cahn-Hilliard-type equations via evolutionary Γ-convergence, *NoDEA Nonlinear Differential Equations Appl.*, 25 (2018), Art. 6, 31 pp.
[30] S. Londen and H. Petzeltová, Convergence of solutions of a non-local phase-field system, *Discrete Contin. Dyn. Syst. Ser. S*, 4 (2011), 653–670.
[31] S. Londen and H. Petzeltová, Regularity and separation from potential barriers for a non-local phase-field system, *J. Math. Anal. Appl.*, 379 (2011), 724–735.
[32] S. Melchionna and E. Rocca, On a nonlocal Cahn-Hilliard equation with a reaction term, *Adv. Math. Sci. Appl.*, 24 (2014), 461–497.
[33] A. Miranville, Asymptotic behavior of the Cahn-Hilliard-Oono equation, *J. Appl. Anal. Comput.*, 1 (2011), 523–536.
[34] L. Modica and S. Mortola, Un esempio di Γ-convergenza, *Boll. Unione Mat. Ital. B (5)*, 14 (1977), 285–299.
[35] L. Nirenberg, On elliptic partial differential equations, in *Il Principio Di Minimo E Sue Applicazioni Alle Equazioni Funzionali*, Springer, (2011), 1–48.
[36] J. C. Robinson, *Infinite-dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Vol. 28, Cambridge University Press, 2001.

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