Decomposition of a second-order linear time-varying differential system as the series connection of two first order commutative pairs

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Abstract: Necessary and sufficiently conditions are derived for the decomposition of a second order linear time-varying system into two cascade connected commutative first order linear time-varying subsystems. The explicit formulas describing these subsystems are presented. It is shown that a very small class of systems satisfies the stated conditions. The results are well verified by simulations. It is also shown that its cascade synthesis is less sensitive to numerical errors than the direct simulation of the system itself.

Keywords: Differential equations, Analogue control, Equivalent circuits, Feedback circuits, Feedback control systems, Robust control

MSC: 34H99, 93A30, 93B35, 93C05, 93C15, 93C99

1 Introduction

The realization of many engineering systems consists of cascade connection of subsystems of simple orders, which is very important in design of electrical and electronic systems [1–5]. Although the order of connection of the subsystems mainly depends on the special design approach, engineering ingenuity, traditional synthesis methods, when sensitivity, stability, linearity, noise disturbance, robustness effects are considered the change of the order of connection without changing the main function of the total system (commutativity) may lead positive results. Therefore the commutativity is very important from the practical point of view.

Consider two linear time-varying differential systems

\[ A: \sum_{n=0}^{N} a_n(t) \frac{d^n}{dt^n} y_1(t) = x_1(t), \quad \text{with} \quad y_1^{(n)}(0), \quad n = 0, 1, \ldots, N - 1, \quad (1) \]

\[ B: \sum_{m=0}^{M} b_m(t) \frac{d^m}{dt^m} y_2(t) = x_2(t), \quad \text{with} \quad y_2^{(m)}(0), \quad m = 0, 1, \cdots, M - 1, \quad (2) \]

where \( a_n(t) \) and \( b_m(t) \) are continuous functions \( R \rightarrow R \); the superscripts in the parenthesis denote the derivatives with respect to time \( t \geq 0 \); \( a_N(0) \neq 0, b_M(0) \neq 0 \) where \( N, M \geq 1 \) indicate the order of the differential systems \( A \) and \( B \), respectively. These equations are assumed to describe the models of physical systems, also referred as \( A \) and \( B \), with inputs \( x_i(t) \) and outputs \( y_i(t), i = 1, 2. \)
When the output $y_1$ of $A$ is fed (equal to) the input $x_2$ of $B$, the combined system with input $x = x_1$ and output $y = y_2$ is called the cascade connection of $AB$; $BA$ is defined similarly. If the cascade connections $AB$ and $BA$ have identical input-output pairs for all time $t \geq 0$, then $A$ and $B$ are said to be commutative subsystems, that is $AB = BA$. It is very simple to show that $y(t) = x(t)/a_0(t)b_0(t)$ for the cascade connections of $AB$ and $BA$ when $A$ and $B$ are zero-order (scalar) subsystems; hence all zero-order linear time-varying systems are commutative; for this trivial result the case of $N = 0$ and $M = 0$ are excluded in Eqs. (1) and (2) and in the content of this paper.

The first record about the commutativity in the literature is known to be [6] where it is proved that a time-varying system can be commutative only with another time-varying system (excluding a scalar constant gain system), which is the first basic general conclusion about the commutativity of linear time-varying systems.

After several publications about the commutativity of low-order systems [7–10], an exhaustive study was introduced in 1988 [11]. This paper had been the main reference for many years since it covers the most general necessary and sufficient conditions for the commutativity of systems of any order but without initial conditions. The previous results for commutativity conditions of first-order [6], second-order [7], third-order [9], and fourth-order [10] systems were shown to be deduced from the main theorem of [11]. The paper also includes results concerning the commutativity properties of feed-back control systems and Euler differential systems.

Another tutorial work about the subject appeared in 2011 [12], where the explicit commutativity conditions for linear time-varying differential systems with non-zero initial conditions [13], the effects of commutativity on system sensitivity [14], and the explicit commutativity conditions for the fifth-order systems were derived for the first time.

Although the commutativity study of linear systems has recently extended to discrete time-varying systems [15], this case is out of scope of this paper which focuses on the decomposition of a second-order continuous time linear time-varying systems into two cascade (chain) connected first-order systems. After this introductory section, Section II presents the main theorem and its proof. Illustrative examples and simulation results are outlined in Section III. Finally, Section IV includes concluding remarks.

## 2 Main Theorem

Consider the second-order linear time-varying system $C$ described by the differential equation

$$
C: \quad c_2(t)\ddot{y}(t) + c_1(t)\dot{y}(t) + c_0(t)y(t) = x(t); \quad y(0), \dot{y}(0); \quad t \geq 0; \quad c_2(t) \neq 0,
$$

where $c_2(t)$ and its first ($\dot{c}_2$) and second ($\ddot{c}_2$) time derivatives, $c_1(t)$ and $c_0(t)$ are continuous functions $R^+ \rightarrow R$; and $y(0), \dot{y}(0) \in R$ are initial conditions; $x(t)$ and $y(t)$ being the input (independent exciting) and output (dependent response) variables. Let $A$ and $B$ be two first-order linear time-varying systems with the inputs $x_1(t)$ and $x_2(t)$, and outputs $y_1(t)$ and $y_2(t)$, respectively, and they are defined by

$$
A: \quad a_1(t)\dot{y}_1(t) + a_0(t)y_1(t) = x_1(t); \quad y_1(0), t \geq 0; \quad a_1(t) \neq 0,
$$

and

$$
B: \quad b_1(t)\dot{y}_2(t) + b_0(t)y_2(t) = x_2(t); \quad y_2(0), t \geq 0; \quad b_1(t) \neq 0.
$$

Then, the following theorem holds.

**Theorem 2.1.** For the decomposition of system (3) as the cascade connected first-order commutative pair of subsystems (4) and (5), that is $C = AB = BA$, it is necessary and sufficient that

$$
a_1(t) = \sqrt{\frac{c_2(t)}{k_1}},
$$

$$
a_0(t) = \frac{2c_1(t) - \dot{c}_2(t)}{4\sqrt{k_1c_2(t)}} - \frac{k_0}{2k_1},
$$

$$
b_1(t) = \sqrt{k_1c_2(t)}.
$$
where $k_0$ and $k_1$ are arbitrary constants in $\mathbb{R}$ satisfying

$$
\frac{3(\dot{c}_2(t))^2}{16c_2(t)} - \frac{\ddot{c}_2(t)}{4} + \frac{c_1(t) - 2\dot{c}_2(t)}{4c_2(t)} + \frac{\dot{c}_1(t)}{2} - \frac{k_1^2}{4k_1} = c_0(t). 
$$

Further if the equivalence is to be valid under non-zero initial conditions $y(0), \dot{y}(0)$, the constraint

$$
y_1(0) = y_2(0) = y(0)
$$

between the initial conditions, and the relation

$$
k_1 + k_0 = 1
$$

between the arbitrary constants should be included in the necessary and sufficient conditions in (6), (7), (8), (9), (10).

Proof of the Main Theorem. Consider the connection $AB$; that is the input $x$ of $AB$ is that of $A$, the output of $A$ is equal to the input of $B$, the output of $B$ is that of $AB$; which can be expressed as

$$
x = x_1, 
$$

$$
y_1 = x_2, 
$$

$$
y_2 = y. 
$$

Differentiating (5), we obtain

$$
\dot{b}_1 \dot{y}_2 + b_1 \ddot{y}_2 + \dot{b}_0 y_2 + b_0 \dot{y}_2 = \dot{x}_2. 
$$

From (14), and then from (4)

$$
\dot{x}_2 = \dot{y}_1 = \frac{x_1 - a_0 y_1}{a_1}. 
$$

Inserting the value $x_1 = x$ from (13), and the value

$$
y_1 = x_2 = b_1 \dot{y}_2 + b_0 y_2 
$$

from (14), and (5) all in to (17), we obtain

$$
\dot{x}_2 = \frac{x - a_0 (b_1 \dot{y}_2 + b_0 y_2)}{a_1}. 
$$

Finally, substituting this expression in (16) and letting $y_2 = y$ from (15), (16) is transformed to

$$
b_1 a_1 \ddot{y} + \left[ b_1 a_1 + b_1 a_0 + b_0 a_1 \right] \dot{y} + \left[ \dot{b}_0 a_1 + b_0 a_0 \right] y = x 
$$

after rearrangements. This equation describes the interconnection $AB$ with the initial conditions

$$
y(0) = y_2(0),
$$

$$
\dot{y}(0) = \dot{y}_2(0) = \frac{x_2(0) - b_0(0) y_2(0)}{b_1(0)} = \frac{y_1(0) - b_0(0) y_2(0)}{b_1(0)},
$$

where the first equation follows from (15); the subsequent equalities in the second equation follow from (15), (5), and (14).
When the interconnection $BA$ is considered, a similar procedure can be followed to derive
\[ a_1b_1\ddot{y} + [\hat{a}_1b_1 + a_1b_0 + a_0b_1]\dot{y} + [\hat{a}_0b_1 + b_0a_0]y = x, \]  \hfill (23)
\[ y(0) = y_1(0), \]  \hfill (24)
\[ \dot{y}(0) = \frac{y_2(0) - a_0(0)y_1(0)}{a_1(0)}. \]  \hfill (25)

These equations can also be obtained straightforwardly from (20), (21), and (22), respectively, by interchanging $a_i \leftrightarrow b_i$, $(i = 0, 1)$ and $y_1 \leftrightarrow y_2$.

The first commutativity condition for subsystems $A$ and $B$ is reduced from the general theorem (Koksal 1) of [12] as
\[ \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ a_0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_0 \end{bmatrix}, \]  \hfill (26)
where $k_1$ and $k_0$ are arbitrary constants. And the second commutativity condition is reduced from Eq. 3.2a of [12] as
\[ \begin{bmatrix} y_1(0) \\ y_2(0) - a_0(0)y_1(0) \end{bmatrix} = \begin{bmatrix} y_2(0) \\ y_1(0) - b_0(0)y_2(0) \end{bmatrix}. \]  \hfill (27)

For the case of first-order subsystems considered in the present proof, (27) results from the comparison of (21), (22), and (24), (25). Equation (26) is obvious by equating the coefficients of (20) and (23). In fact the equality of the coefficients of the first derivatives of $\dot{y}$ yields
\[ \hat{a}_1b_1 = b_1a_1 \rightarrow b_1 = k_1a_1, \]  \hfill (28)
which is the first line equation in (26). The second line equation can be obtained by equating the coefficients of $y$ in (20) and (23), which yields together with (28)
\[ \hat{a}_0b_1 = b_0a_1 \rightarrow b_0 = k_1a_0 + k_0. \]  \hfill (29)

Before proceeding further, we focus on the second line equation in (27). For nonzero initial conditions, this equation together with the first line equation and (26) yields
\[ \frac{1 - a_0(0)}{a_1(0)}y_1(0) = \frac{1 - b_0(0)}{b_1(0)}y_1(0) = \frac{1 - a_0(0)k_1 - k_0}{a_1(0)k_1}y_1(0), \]  \hfill (30)
which results in
\[ k_1 + k_0 = 1 \]  \hfill (31)
for the nonzero initial conditions $y_1(0) = y_2(0)$.

The intermediate step of the proof is to eliminate the coefficients $b_1(t)$, $b_0(t)$ and the initial condition $y_2(0)$ from the description (20), (21), (22) of the interconnection $AB$. In fact, using (26) and (27) in (20), (21), (22), we obtain
\[ a_1^2k_1\ddot{y} + [\hat{a}_1a_1k_1 + 2a_1a_0k_1 + a_1k_0]\dot{y} + \left[ a_1\hat{a}_0k_1 + a_0^2k_1 + a_0k_0 \right]y = x, \]  \hfill (32)
\[ y(0) = y_1(0), \]  \hfill (33)
\[ \dot{y}(0) = \frac{1 - k_0 - a_0(0)k_1}{a_1(0)k_1}y_1(0) = \frac{1 - a_0(0)}{a_1(0)}y_1(0) = \frac{1 - a_0(0)}{a_1(0)}y(0). \]  \hfill (34)

Since it is well known that (3) and (32) will have identical input-output pairs $(x(t), y(t))$ for all inputs $x(t)$ and the initial conditions $y(0)$ and $\dot{y}(0)$ if and only if they have the same continuous time-varying coefficients, for their equivalence it is required that
\[ a_1^2k_1 = c_2. \]  \hfill (35)
\[ \dot{a}_1 a_1 k_1 + 2 a_1 a_0 k_1 + a_1 k_0 = c_1, \quad (36) \]

\[ a_1 \dot{a}_0 k_1 + a_0^2 k_1 + a_0 k_0 = c_0. \quad (37) \]

In fact (35) yields (6). Moreover, (36) yields (7) since \( \dot{a}_1 = \frac{\dot{c}_2}{2 c_1 k_1} \) from (35). Inserting (6) and (7) in (26) yields (8) and (9) respectively. Finally, substituting (6) and (7) in (37) yields (10).

Note that (11) and (12) have already been proved as seen in (21), (24) and (31), respectively. Hence the proof of the theorem is completed.

Theorem 2.1 and its proof reveal some important results which are expressed by the following corollaries.

**Corollary 2.2.** For the synthesis of a second-order linear time-varying system in the form (3) with zero initial conditions as the cascade connection of two first-order linear time-varying subsystems in the forms (4) and (5), it is necessary and sufficient that the coefficient \( c_0(t) \) of the system is expressible in terms of the first two coefficients \( c_2(t) \) and \( c_1(t) \) as in (10) where \( k_1 \) and \( k_0 \) are arbitrary constants.

This corollary is a direct consequence or a simpler expression of Theorem 2.1.

The next corollary is for the case of nonzero initial conditions.

**Corollary 2.3.** For the synthesis of the second-order linear time-varying system considered in Corollary 2.2 but with non-zero initial conditions as the cascade connection of two first-order linear time-varying subsystems in the forms (4) and (5), it is necessary and sufficient that the condition of Corollary 2.2 is valid with the exposed condition (12) and further

\[ \dot{y}(0) = \left[ \frac{k_1 + 1}{2 \sqrt{k_1 c_2(0)}} + \frac{\dot{c}_2(0) - 2 c_1(0)}{4 c_2(0)} \right] y(0). \quad (38) \]

**Proof.** Substitutions \( y_2(0) = y(0) \) from (21) and \( y_1(0) = y(0) \) from (24) in (25), writing the expressions in (6), (7) for \( a_1(0) \) and \( a_0(0) \) in, and finally using (12) yield (38).

Although the necessity and sufficiency of the coefficient \( c_0(t) \) being expressible in terms of the coefficients \( c_2(t), c_1(t) \), and some of their derivatives as in (10) is enough under zero initial conditions, the second corollary states that the constants \( k_1 \) and \( k_0 \) should sum up to 1 as in (12) and \( k_1 \) should satisfy (38) with the arbitrarily given initial conditions \( \dot{y}(0) \) and \( y(0) \).

**3 Examples**

To illustrate the results of the paper, some examples are introduced in this section. The simulations conducted with MATLAB R2012a and obtained by a PC Intel® Core® i3 CPV, 2.13 GHz, 3.86 GB of RAM well verify the results.

**3.1 Example 1**

Consider the system defined by

\[ \ddot{y}(t) + \sin t \dot{y}(t) + \frac{1}{16} \left( 4 \sin^2 t + 8 \cos t - 9 \right) y(t) = x(t); \quad y(0) = 1, \quad \dot{y}(0) = 1.25, \quad t \geq 0. \quad (39) \]

For the synthesis of this second-order linear time-varying system in the form of two first-order cascade connected commutative subsystems, the constant \( k_1 \) is determined first by using (38). Since

\[ c_2(t) = 1; \quad c_1(t) = \sin t; \quad c_0(t) = \frac{1}{16} \left( 4 \sin^2 t + 8 \cos t - 9 \right) ; \quad y(0) = 1, \quad \dot{y}(0) = 1.25 \]

(38) yields

\[ \sqrt{k_1} = 2, \quad k_1 = 4. \quad (40) \]
\[
\sqrt{k_1} = 0.5, \ k_1 = 0.25
\]

with two possibilities for \( k_1 \), and (12) yields the following two values for \( k_0 \), respectively

\[
k_0 = -3,
\]

\[
k_0 = 0.75.
\]

Considering the first possibility, (10) is obviously satisfied with \( k_1 = 4, \ k_0 = -3 \). It is further seen that this equation is also satisfied for the second possibility \( k_1 = 0.25, \ k_0 = 0.75 \). Hence, the given second-order system seems to have two commutative decompositions of order one. For the first case with \( k_1 = 4, \ k_0 = -3 \), the commutative subsystems \( A \) and \( B \) of the given system \( C \) in (39) are found from (6), (7) and (8), (9) with the initial conditions in (11);

\[
a_1(t) = \frac{1}{2}, \ a_0(t) = \frac{1}{8} (8 + 2 \sin t), \ y_1(0) = 1; \text{ hence}
\]

\[
A: \quad \frac{1}{2} \dot{y}_1(t) + \frac{1}{8} (3 + 2 \sin t) y_1(t) = x_1(t), \ y_1(0) = 1.
\]

\[
b_1(t) = 2, \ b_0(t) = \frac{1}{2} (-3 + 2 \sin t), \ y_2(0) = 1; \text{ hence}
\]

\[
B: \quad 2 \dot{y}_2(t) + \frac{1}{2} (-3 + 2 \sin t) y_2(t) = x_2(t), \ y_2(0) = 1.
\]

The above computed synthesis of the given second-order system \( C \) in the form of cascade connections \( AB \) and \( BA \) of the subsystems \( A \) and \( B \) are tested by MATLAB simulations for the input

\[
x(t) = 10 \sin 10t.
\]

The outputs of the original second-order system \( C \), and its synthesis \( AB \) and \( BA \) by cascade connected commutative first-order systems (\( A \) and \( B \)) are all observed to be identical as shown in Fig. 1 (\( C = AB = BA \)).

**Fig. 1.** Output of \( C \) and its cascade synthesis \( AB \) and \( BA \) for Example 1.

It can easily be shown that \( A \) and \( B \) can be found in a similar manner as above for the second case of values \( k_1 = 0.25, \ k_0 = 0.75 \); the result is not different from that of the first case except the subsystems \( A \) and \( B \) found in (45), (46) and (47), (48) are interchanged \( A \leftrightarrow B \). When commutativity is of concern this does not necessitate any
change in the statements of the main theorem and its corollaries. But it deserves to note that although the quadratic equation (38) yields two solutions for $\sqrt{k_1}$ they both yield the same result effectively.

To show the validity of Corollary 2.3, the same example is simulated by changing $\dot{y}(0) = 0.25$ which is a case (38) is not satisfied. The decompositions $AB$ and $BA$ are not valid in this case since $C$ does not have the same output as shown in Fig. 1 (C1).

Simulation results reveal that $C$ and its subsystem $B$ are unstable systems. Although unlike linear time-invariant systems, stability of linear time-varying systems does not depend on only whether the characteristic value lies in the left half complex plane [16–18], the characteristic value of $B$ remains always on the right hand side, and those of $C$ enter in the right hand side some times; hence the instability of $B$ and $C$ is expected. The next example is chosen as a stable second-order system.

### 3.2 Example 2

In this example, the constants $k_1$ and $k_0$ which are arbitrarily selectable when initial conditions are zero are tried to be chosen so that both the second-order system $C$ and its commutative subsystems $A$ and $B$ are all stable. Let $C$ be in the form

$$C: \ddot{y}(t) + 2(2 - \cos t)\dot{y}(t) + c_0(t)y(t) = x(t).$$

(50)

Obviously, $c_2(t) = 1$ and $c_1(t) = 4 - 2\cos t$. Hence, (10) yields that $c_0(t)$ will be in the form

$$c_0(t) = (2 - \cos t)^2 + \sin t - \frac{k_0^2}{4k_1^1}.$$

(51)

Using (6), (7) and (8), (9), the subsystems $A$ and $B$ become

$$A: \frac{1}{\sqrt{k_1}}\dot{y}_1(t) + \left(\frac{2 - \cos t}{\sqrt{k_1}} - \frac{k_0}{2k_1}\right)y_1(t) = x_1(t).$$

(52)

$$B: \sqrt{k_1}\dot{y}_2(t) + \left(2 - \cos t\right)\frac{1}{\sqrt{k_1}} - \frac{k_0}{2}y_2(t) = x_2(t).$$

(53)

The characteristic values of $A$ and $B$ are obviously

$$D_A = \frac{k_0}{2\sqrt{k_1}} - 2 + \cos t,$$

(54)

$$D_B = \frac{k_0}{2\sqrt{k_1}} - 2 - \cos t.$$  

(55)

It can be shown very easily that both of these values remain negative for all $t \geq 0$ if

$$-2\sqrt{k_1} < k_0 < 2\sqrt{k_1}.$$  

(56)

Hence, when $(k_0, k_1)$ is chosen inside of the parabola $k_1 = \frac{k_0^2}{4} (k_0 = \mp 2\sqrt{k_1})$ as shown in Fig. 2, both $D_A$ and $D_B$ will always remain negative (this may not be sufficient for stability [16, 18]) and $A$ and $B$ most likely be stable subsystems, so is $C = AB = BA$. When the initial conditions are not zero, the additional equation (12) should also be satisfied for commutativity, the line $k_0 + k_1 = 1$ is also shown in Fig. 2. For two different choices of $k_0, k_1$, the outputs of the systems $C$, $AB$ and $BA$ are observed on the MATLAB Scope and plotted as shown in Fig. 3.

In Case I, $k_1 = 4$, $k_0 = -3$ (point $P$ in Fig. 2), the subsystems $A$, $B$ and their cascade connections $C = AB = BA$ as found from (52), (53) and (51) become

$$A: \frac{1}{2}\dot{y}_1(t) + (1.375 - 0.5\cos t) y_1(t) = x_1(t), \quad y_1(0) = 1.$$  

(57)

$$B: 2\dot{y}_2(t) + (2.5 - 2\cos t) y_2(t) = x_2(t), \quad y_2(0) = 1.$$  

(58)
C: \[ \ddot{y}(t) + (4 - 2 \cos t) \dot{y}(t) + (4 \cos t - 0.5 \cos 2t - \sin t - 3.9375) y(t) = x(t); \]

\[ y(0) = 1, \ \dot{y}(0) = 1. \] \hfill (59)

Fig. 2. Different regions of \( k_1 - k_0 \) plane for possibly stable choices of \( C \) and its stable subsystems \( A, B \).

Note that \( y_1(0) = y_2(0) = y(0) \) and \( \dot{y}(0) \) for \( C \) is computed from (38); hence, all the conditions of the main theorem and its corollaries are satisfied. All of the systems \( C, AB \) and \( BA \) are excited by \( x(t) = -5 + 10 \sin t \) superimposed by a square wave of amplitude 20, period 5 and pulse with \% 5. It is observed in Fig. 3 that all the outputs are the same (\( C1 = ABI = BAI \)) and verify the theory. It is pertinent to note that when the simulation time is extended far beyond \( t = 10 \), all the systems’ outputs remain bounded, that is they are all stable as expected.
In Case II, \( k_1 = 4, k_0 = 1 \) as shown in Fig. 2 by point \( Q \). For this case, the subsystems \( A, B \) and the second-order system \( C \) become

\[
A: \quad \frac{1}{2} \ddot{y}_1(t) + \frac{1 - \cos t}{2} y_1(t) = x_1(t), \quad y_1(0) = 1. \tag{60}
\]

\[
B: \quad 2 \ddot{y}_2(t) + (6 - 2 \cos t) y_2(t) = x_2(t), \quad y_2(0) = 0, \tag{61}
\]

\[
C: \quad \ddot{y}(t) + 2(2 - \cos t) \dot{y}(t) + \left( 3 + \cos^2 t - \sin t - 4 \cos t \right) y(t) = x(t);
\]

\[
y(0) = \dot{y}(0) = 0, \tag{62}
\]

which are found as mentioned in the previous case. For the same input of Case I, all the outputs are observed to be equal as in the first case and are shown in Fig. 3 (\( CII0 = ABI0 = BAI0 \)). Note that \( k_1 + k_0 \neq 1 \) for this case, which is not a necessary condition for commutativity in the case of zero initial conditions. We observe in this figure that due to the strong stability of the system, the initial condition response dies away quickly so that the total response of Case I approaches to that of the input response (response with zero initial conditions) of Case II.

Case III is exactly the same as Case II except nonzero initial conditions are assigned to \( A, B \) and \( C \). In fact, \( A, B \), and \( C \) are assigned to have the same initial conditions for their outputs, that is \( y_1(0) = y_2(0) = y(0) = 4 \) and hence, (11) of the main Theorem is satisfied; \( \dot{y}(0) = 2 \) as to satisfy the condition (38) of Corollary 2.3. Since the condition of (12) of this Corollary is not satisfied (\( k_1 + k_0 \neq 1 \)), we should not have equal outputs for \( C, AB \) and \( BA \); see Fig. 4 (\( CIII \neq ABI \neq BAI \)). We note again that due to the strong stability of the systems \( C, AB, BA \), the initial condition responses die away quickly and the outputs become equal as \( t \to \infty \). This is because of the decomposition of \( C \) into the commutative pair \( AB \) (or \( BA \)) is valid under zero initial conditions.

**Fig. 4. Outputs of Example 2 for Case III.**

3.3 Example 3

This example is presented to exhibit the numerical accuracies in simulations while validating the main theorem and its corollaries stated in the paper. The second-order linear time-varying system \( C \) to be synthesized as the commutative cascade connected pair of two first-order linear time-varying subsystems \( A \) and \( B \) are described by

\[
C: \quad \ddot{y}(t) + 2t^2 \dot{y}(t) + \left( t^4 + 2t - 0.5626 \right) y(t) = x(t), \quad y(0) = 1, \quad \dot{y}(0) = 1.25. \tag{63}
\]
respectively. Note that all the conditions of Theorem 2.1 are satisfied with $k_1 = 0.25$, $k_0 = 0.75$, where $k_1 + k_0 = 1$. The input is chosen as $x(t) = -5 + 10\sin 4t$. The simulation results for all outputs of $C$, $AB$ and $BA$ are equal as shown in Fig. 5 ($C = AB = BA$). The small discrepancies due to the numerical errors in the simulations are observed in the same figure as ($C$ and $AB$) and ($AB$ and $BA$). Note that the outputs of $C$, $AB$ and $BA$ can not be differed, though there is a difference between the results of $C$ and $AB$ in the order of smaller than $10^{-5}$ in magnitude. On the other hand, the difference between $AB$ and $BC$ are almost smaller than $10^{-15}$ in magnitude. This means that the cascade connections $AB$ and $BA$ are much more insensitive to the numerical errors than the whole system $C$. This shows the numerical stability of cascade synthesis procedure over the direct synthesis.

3.4 Example 4

This example is designed to show the importance of the orders $AB$ or $BA$ from the effect of the noise point of view interfered at the interconnection. The systems considered are

\begin{align}
A: \quad 2\dot{y}_1(t) + \left(2t^2 - 1.5\right)y_1(t) &= x_1(t), \quad y_1(0) = 1, \quad (64)
\end{align}

\begin{align}
B: \quad \frac{1}{2}\dot{y}_2(t) + \left(0.5t^2 + 0.375\right)y_2(t) &= x_2(t), \quad y_2(0) = 1, \quad (65)
\end{align}

The given decomposition of $C$ into the cascade connections $AB$ and $BA$ satisfies all the conditions of Theorem 2.1 where $k_1 = 0.64$, $k_0 = 0.36$. The input $x(t)$ is chosen as $10\sin 4t - 5$ plus a square wave of amplitude 20, period 1 and 20 pulse width. After validating that all outputs of $C$, $AB$ and $BA$ are the same (see Fig. 6, $C = AB = BA$), a noise signal of a square wave of period 1 ($5$ for $0 \leq t < 0.25$, $-5$ for $0.25 \leq t < 0.5$, $0$ for $0.5 \leq t \leq 1$) is added as disturbance at the interconnection of subsystems $A$ and $B$. The effect of this noise at the output of the cascade connection is also shown in Fig. 5 ($BA$ noise and $AB$ noise). It is apparently true that the cascade connection $BA$
is preferable since it causes almost half of the noise effect of produced in the connection $AB$. Hence among the alternative orders $AB$ and $BA$, $BA$ is preferred for the cascade synthesis of $C$ since it is less sensitive to distortive signals occurring at the interconnection point.

**Fig. 6.** The outputs of the system $C$ and its cascade synthesis $AB$ and $BA$ with and without noise.

![Graph showing system outputs with and without noise](image)

### 4 Conclusions

The necessary and sufficient conditions are derived for the decomposition of a second-order linear time-varying system into the cascade connection of two first-order commutative linear time-varying subsystems. The results of the main theorem and its corollaries are well verified by MATLAB simulations. It is shown that the cascade synthesis is more robust against the numerical errors that the direct simulation of the system. Although the order of the subsystems is not important from the view point of input-output relations, the paper shows that one of the sequences can be preferable when the effects of disturbance at the interconnection of two subsystems are considered.

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