A CHARACTERIZATION OF THE ELECTROMAGNETIC STRESS-ENERGY TENSOR

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Abstract

In [1], we pointed out how a dimensional analysis of the stress-energy tensor of the gravitational field allows to derive the field equation of General Relativity.

In this note, we comment an analogous reasoning in presence of a 2-form, that allows to characterize the so called electromagnetic stress-energy tensor.

Natural tensors associated to a metric and a 2-form

Let $X$ be an $n$ dimensional smooth manifold. Let us recall the rigorous definition of an “intrinsic and local construction from a metric and a 2-form” (for a more detailed account in the case of solely a metric, see [1] and references therein).

Let $S^2_+ T^* X \to X$ be the fibre bundle of semi-Riemannian metrics of a given signature on $X$, $\Lambda^2 T^* X \to X$ be the vector bundle of 2-forms on $X$ and $\otimes^p T^* X \otimes \otimes^q T X \to X$ be the vector bundle of $(p, q)$-tensors on $X$. Let us also denote their sheaves of smooth sections by Metrics, Forms and Tensors, respectively.

A morphism of sheaves $T: \text{Metrics} \times \text{Forms} \to \text{Tensors}$ is said to be natural if it is equivariant with respect to the action of local diffeomorphisms of $X$; that is, if for each diffeomorphism $\tau: U \to V$ between open sets of $X$ and for each metric $g$ and each 2-form $F$ on $V$, the following condition is satisfied:

$$T(\tau^* g, \tau^* F) = \tau^* (T(g, F)). \quad (0.0.1)$$

Definition 0.1. A tensor naturally constructed from a metric $g$ and a 2-form $F$ (or a natural tensor associated to $(g, F)$) is a tensor of the type $T(g, F)$, where $T: \text{Metrics} \times \text{Forms} \to \text{Tensors}$ is a natural morphism of sheaves. [1]

The main examples of tensors naturally constructed from a metric are the curvature tensor of the metric, its covariant derivatives and tensor products and contractions of these.

The main example of a 2-covariant tensor $E_2(g, F)$ naturally constructed from a metric $g$ and a 2-form $F$ is the so called electromagnetic stress-energy tensor, that in local coordinates is written as:

$$E_{ij} = F_i^k F_{kj} - \frac{1}{4} F_{kl} F^{kl} g_{ij}$$

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1There is also a regularity hypothesis for $T$, that amounts to saying that the construction is “differentiable”, see [1] or [3].
Nevertheless notice that, by definition, the coefficients of a natural tensor $T(g, F)$ are only assumed to be locally constructed from $(g, F)$ and therefore need not necessarily be polynomial functions on the coefficients of $g$, $g^{-1}$, $F$ and its derivatives.

**Dimensional analysis** In the theory of General Relativity, space-time is a smooth manifold $X$ of dimension 4 endowed with a Lorentz metric $g$, called the time metric. The proper time of a particle following a trajectory in $X$ is defined to be the length of that curve using the metric $g$. So that if the metric $g$ is changed by a proportional one $\lambda^2 g$, with $\lambda \in \mathbb{R}^+$, then the proper time of particles is multiplied by the factor $\lambda$. Therefore, replacing the metric $g$ by $\lambda^2 g$ amounts to a change in the time unit.

On the other hand, the presence of an electromagnetic field on the space-time $X$ is represented with a 2-form $F$. The endomorphism $\bar{F}$ metrically equivalent to $F$ assigns to each observer $u$ carrying the unit of charge, the acceleration that he measures due to the electromagnetic field, $\nabla u u$ (recall the Lorentz equation). So, if we change the time unit and therefore replace the time metric $g$ by $\lambda^2 g$, then the observer $u$ becomes $\lambda^{-1} u$; the endomorphism $\bar{F}$ becomes $\lambda^{-1} \bar{F}$ and the 2-form $F$ changes to $\lambda F$.

These reasonings lead to consider a special kind of homogeneity for the natural constructions:

**Definition 0.2.** A morphism of sheaves $T$: Metrics × Forms → Tensors is said to be homogenous of weight $w \in \mathbb{R}$ if it satisfies:

$$T(\lambda^2 g, \lambda F) = \lambda^w T(g, F) \quad \forall g, F \quad \forall \lambda > 0 .$$

If the morphism $T$ has weight 0, it is said to be independent of the unit of time.

A tensor $T(g, F)$ naturally constructed from a metric and a 2-form is homogenous of weight $w$ if the corresponding morphism of sheaves $T$ is homogenous of weight $w$.

**Description of the vector space of natural tensors**

To describe the space of homogenous natural tensors $T(g, F)$ of weight $w$ (that results to be a finite dimensional $\mathbb{R}$-vector space), it is useful to introduce normal tensors at a point.

Fix any point $x \in X$. In the case of a metric, these vector spaces of normal tensors $N_r$ are described in [1]. In order to deal with a 2-form, we also introduce:

$$\Lambda_l := \Lambda^2 T^*_x X \otimes S^l T^*_x X$$

If $l = 0$, $\Lambda_0$ is defined to be the space of 2-forms at $x \in X$.

On a semi-riemannian manifold, the germ on $x$ of a 2-form $F$ produces a sequence of tensors in $\Lambda_l$, $l \geq 0$. To see this, fix normal coordinates in a neighbourhood of $x$ and write:

$$F^l_x := \sum_{i j k_1 \ldots k_l} F_{i j, k_1 \ldots k_l} dx_i \wedge dx_k \otimes dx_{k_1} \otimes \ldots \otimes dx_{k_l} \in \Lambda_l$$

where:

$$F_{i j, k_1 \ldots k_l} := \frac{\partial^l F_{i j}}{\partial x_{k_1} \ldots \partial x_{k_l}}(x)$$

Moreover, let $O := O(n^+, n^-)$ be the orthogonal group of $(T_x X, g_x)$ and let $T^q_{p,x}$ be the space of $p$-covariant and $q$-contravariant tensors at $x$.

The vector space of homogenous natural tensors $T(g, F)$ is described in the following theorem, whose proof is completely analogous to the one referred in [1]:
Theorem 0.3. There exists an $\mathbb{R}$-linear isomorphism:

$$\{ (p,q)\text{-Tensors of weight } w \text{ naturally constructed from } (g,F) \}$$

$$\bigoplus_{\{d_i,c_j\}} \text{Hom}_O(S^{d_2}N_2 \otimes \cdots \otimes S^{d_r}N_r \otimes S^{c_0}A_0 \otimes \cdots \otimes S^{c_l}A_l, T^q_{p,x})$$

where the summation is over all sequences of non-negative integers $\{d_2, \ldots, d_r, c_0, \ldots, c_l\}$, $r \geq 2$, satisfying the equation:

$$2d_2 + \ldots + r d_r + c_0 + 2c_1 + \ldots + (s + 1)c_s = p - q - w.$$ \hfill (0.0.2)

If this equation has no solutions, the above vector space reduces to zero.

The isomorphism of the above theorem is explicit (see Remarks in [1]) and can be used, in simple cases, to make exhaustive computations. As an application, it follows the announced characterization.

**A characterization of the electromagnetic stress-energy tensor**

Let us consider the natural tensor $E_2(g, F)$ defined in local coordinates as:

$$E_{ij} = F_{ik}F_{kj} - \frac{1}{4} F_{kl}F^{kl}g_{ij}$$

There already exists a geometric characterization of this tensor (see [2]). The following one uses homogeneity (condition (a) below) to get rid of the assumptions imposed in [2] on the derivatives of the metric and the 2-form that appear on the tensor.

Theorem 0.4. The tensor $E_2$ is the only (up to constant multiples) 2-covariant tensor naturally constructed from a metric and a 2-form satisfying:

(a) It is independent of the unit of time (i.e., $E_2(\lambda^2g, \lambda F) = E_2(g, F)$, $\forall \lambda > 0$).

(b) If $dF = \delta F = 0$, then $\text{div} E_2 = 0$.

(c) If $F = 0$, then $E_2 = 0$ (i.e., there are no addends independent of $F$).

Proof. Using Theorem 0.3, we only have to analyze the solutions to:

$$2d_2 + \ldots + r d_r + c_0 + 2c_1 + \ldots + (s + 1)c_s = 2$$

Condition (c) says that we cannot consider solutions where $c_0 = c_1 = \ldots c_s = 0$. So there are only two possibilities:

- $c_1 = 1, c_i = d_j = 0$. In this case we have to compute the $O$-invariant linear maps (see Remarks in [1]):

$$\Lambda^2 T^*_x X \otimes T^*_x X \otimes T^*_x X \otimes T^*_x X \rightarrow \mathbb{R}$$

This space of maps is generated, as an $\mathbb{R}$-vector space, by successive contractions. As the tensor space has in this case an odd number of indices, there only exists the null map.
\[ c_0 = 2, c_i = d_j = 0. \] For this case we have to compute the \( O \)-invariant linear maps:

\[ S^2(\Lambda^2 T^*_x X) \otimes T^*_x X \otimes T^*_x X \to \mathbb{R} \]

Due to the symmetries, there are only two independent generators:

\[ (13)(24)(56) \quad \text{and} \quad (13)(25)(46) \]

where \((ij)\) denotes contraction of that pair of indices. These maps correspond, respectively, to the tensors:

\[ F_{kl} F^{kl} g_{ij} \quad \text{and} \quad F^k_i F_{kj} \]

Now, the divergence condition \((b)\) implies, by a standard computation, that the only possible natural tensors are constant multiples of \( E_2 \).

\[ \square \]

**Super-energy tensor of a \( p \)-form:** The above characterization can be readily generalized to the super-energy tensor of a \( p \)-form (see [4]), although the dimensional analysis is no longer made a priori, as it has been done before in the case of a 2-form.

**Theorem 0.5.** Let \( \omega_p \) be a \( p \)-form on a semi-riemannian manifold \((X, g)\). The super-energy tensor of the \( p \)-form \( T_2(g, \omega_p) \) is the only (up to constant multiples) 2-covariant tensor naturally constructed from \( g \) and \( \omega_p \) satisfying:

- \( T_2(\lambda^2 g, \lambda^{p-1} \omega_p) = T_2(g, \omega_p), \forall \lambda \in \mathbb{R}. \)
- If \( dF = \delta F = 0 \), then \( \text{div} T_2 = 0. \)
- \( F = 0 \) if and only if \( T_2 = 0. \)

**References**

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