Pinning of stripes in cuprate superconductors

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We examine the effects of disorder on striped phases in high-temperature superconductors and related materials. In the presence of quenched disorder, pinning by the atomic lattice – which might give rise to commensuration effects – is irrelevant for the stripe array on large length scales. As a consequence, the stripes have divergent displacement fluctuations and topological defects are present at all temperatures. Therefore the positional order of the stripe array is short ranged, with a finite correlation length even at zero temperature. Thus lock-in phenomena can exist only as crossovers but not as transitions. In addition, this implies the glassy nature of stripes observed in recent experiments.

PACS numbers: 63.70.+h, 71.45.Lr, 74.72.-h

1. INTRODUCTION

During the last decade evidence emerged from theoretical work and from experiments on cuprates and closely related nickelates for the existence of striped structures within the MO$_2$ ($M$ = Cu, Ni) planes. These stripes are highly correlated states of holes which are introduced into the planes by doping and which order into a unidirectional charge-density wave (CDW, wave length $a$) that may be accompanied by a simultaneous spin-density wave of period $2a$ in the sublattice magnetization of the antiferromagnetic metallic spins. Qualitatively, one may think of stripes as parallel strings of holes that constitute an antiphase boundary for spin order.

Particular interest in these stripes arises from the possible interplay between these stripes and superconductivity. Thereby it is important to distinguish between “dynamic” and “static” stripes. While there is evidence that superconductivity can coexist with both dynamic and static stripes, static stripes tend to suppress superconductivity in contrast to dynamic stripes. Therefore the study of the structure and dynamics of stripes is of principal importance.

Various phenomenological pictures have been developed for the theoretical description of stripes. While charge and spin order are naturally described within a Landau theory, the aspect that stripes act as magnetic domain walls suggests to describe them as string-like objects. In the ideally ordered case these strings form a periodic array. Dynamic fluctuations are generated by thermal and quantum effects, whereas potentials tend to suppress dynamic fluctuations while they may reduce or increase static conformatons of the stripes.

In such phenomenological models, the crystalline structure of the underlying atomic lattice (period $b$) has to be taken into account by a periodic potential, which tends to increase the positional order of the stripe array since is can be the source of lock-in effects. On the other hand, the spatially inhomogeneous distribution of dopants provides a disorder potential for the stripes because of the (screened) Coulomb interaction between dopants and holes. For low enough temperatures, these dopants can be considered as quenched.

Pinning by the periodic potential is of particular interest since lock-in effects might explain the special role of certain values for the stripe spacing. In the cuprate system La$_{2-x}$Sr$_x$CuO$_4$, the mismatch $\delta$ of magnetic Bragg satellite peaks has a nonzero value only beyond a threshold doping $x_c \approx 0.04$, where $\delta \approx x$ holds. Eventually a saturation value $\delta = \frac{1}{8}$ is observed for $x \gtrsim 0.12$. Since the mismatch $\delta$ is related to the lattice and CDW periods via $\delta = b/2a$, it allows for a natural explanation of the “$x = \frac{1}{8}$ problem”. Since at $\delta = \frac{1}{8}$ the periods of the CDW and the Cu spacings have an integer ratio $p := a/b = 4$ this saturation could be a commensuration effect. Similarly, the nickelate system La$_{2-x}$Sr$_x$NiO$_{4+y}$ shows anomalous thermodynamic behavior at the values $p = 2,3,4,5$ which seem to be stable over certain respective ranges of the hole concentration $x + 2y$.

Even more, evidence was reported for plateaus of the mismatch as a function temperature at rational values $\delta = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ in La$_{2}$NiO$_{4.125}$.

On the other hand, the stripe array can also be pinned by disorder. Therefore it is important to take a closer look at the competitive pinning by the periodic atomic lattice and by disorder in order to understand to what extent lock-in effects can persist. A first step in this direction was made by Hasselmann et al. who focus on a single stripe. However, since a single stripe and a stripe array differ in dimensionality one expects qualitatively distinct behavior of the response of the system to disorder.

The purpose of this paper is to determine the effects of periodic and disorder potentials on the structural order of the two-dimensional stripe array on large scales. Quantum fluctuations turn out to be irrelevant in the presence of disorder, i.e. the system behaves essentially as a classical stripe array. We find that a locked state with long-ranged positional order can exist only in the absence of disorder. Disorder always leads to unlocking
and a state with short-ranged positional order where dislocations proliferate.

The outline of this article is as follows. In Sec. II we establish an elastic model for the quantum stripe array with periodic and pinning potentials. The effects of these potentials are discussed in Sec. III in the classical limit. In Sec. IV we demonstrate the irrelevance of quantum fluctuations. Our conclusions are drawn in Sec. V where we discuss the relation of our work to previous work, the role of topological defects and implications for experiments.

II. THE MODEL

We describe the stripe system as an array of interacting quantum strings. The strings are assumed to be aligned in $y$ direction and to have an average spacing $a$ in $x$ direction. In the following we ignore topological defects in the array, the role of which will be discussed in Sec. V. Then the stripe array can be considered as an elastic system. The displacement field $u$ represents a bosonic collective mode of the electron system. Its fluctuations are governed by a (“reduced”) dynamic action

$$S = \frac{1}{\hbar} \int_0^{\hbar/T} d\tau \left\{ \int d^2r \left( \frac{\mu}{2} (\partial_x u)^2 + H \right) \right\}. \quad (1)$$

The action is formulated using the imaginary time (we set $k_B = 1$). We identify $r = (x, y)$, $\mu$ is a mass density and the Hamiltonian $H$ has a contribution from the elastic energy

$$H_{el} = \int d^2r \frac{\gamma}{2} (\nabla u)^2 \quad (2)$$

with a stiffness constant $\gamma$ which includes the line tension of the strings as well as their interaction. Besides a contributions of entropic nature, the main contribution to this interaction will stem from the Coulomb interaction between the stripes. A further contribution arises from a crystal field that aligns the stripes in $y$ direction. A priori, the stiffness can be anisotropic with an elastic energy density $\propto \gamma_x (\partial_x u)^2 + \gamma_y (\partial_y u)^2$. To simplify the analysis, such an anisotropy can be removed by rescaling the $y$ coordinate. Then the effective isotropic stiffness constant is related to the original anisotropic constants through $\gamma = \gamma_x/\gamma_y$. Note that $\gamma_y$ is dependent on the stripe spacing $a$, i.e., on doping. With increasing distance $a$ between the stripes $\gamma$ will shrink.

We will examine the coupling of the stripe array to the periodic potential $U(x)$ generated by the atomic structure as well as to a random potential $V(x, y)$ due to the interaction between the holes and the dopants which may be considered quenched at low temperatures. The corresponding energy contributions read

$$H_U = \int d^2r \rho(r) U(x), \quad H_V = \int d^2r \rho(r) V(x, y) \quad (3)$$

in terms of the stripe density

$$\rho(r, \tau) \simeq \frac{1}{a} \left\{ \sum_m e^{iQ_m |x-u(r, \tau)|} - \partial_x u(r, \tau) \right\} \quad (4)$$

where $Q_m = 2\pi m/a$ are reciprocal lattice vectors of the stripe array. The elastic and disorder pinning energies of the stripe array are similar to those of vortex lines in planar type-II superconductors. A recent review of the latter system can be found in Ref. 34.

$U$ is assumed to be periodic, $U(x) = U(x + b)$, with a period $b < a$ (the modulation along the stripes is negligible for our purposes). For simplicity we take $U(x)$ as an even function (this restriction is for the simplicity of our analysis but not essential for the results)

$$U(x) = -\sum_{n \geq 1} U_n \cos(pQ_n x) \quad (5)$$

with $p > 1$. We assume that the random potential $V(x, y)$ is Gaussian distributed with zero average and a variance

$$\overline{V(r)V(0)} = \frac{\Delta}{\sqrt{2\pi} \xi} e^{-\xi^2/2\xi^2} \delta(y) \quad (6)$$

with a correlation length $\xi$ and a weight $\Delta$.

Subsequently we will establish the global phase diagram for the total system with a partition sum

$$Z = \int \mathcal{D}[u] \ e^{-S}. \quad (7)$$

Since the system without pinning provides an important reference point we start with a brief discussion of thermal and quantum fluctuations of the displacement. There is a characteristic length scale $\ell_T := \sqrt{\hbar^2/\gamma/2\mu}$ beyond which thermal fluctuations dominate over quantum fluctuations. A related temperature scale $T_a := \sqrt{\hbar^2/\gamma/2a^2\mu}$ is defined by the coincidence $\ell_T = a$. In terms of these scales, the displacement fluctuations in systems with a large size $L \gg \ell_T$ are obtained as

$$\langle u^2 \rangle \simeq \begin{cases} \frac{T}{2\pi\gamma} \ln \frac{L}{a} & \text{for } T \gg T_a, \\ \frac{T}{2\pi\gamma} \ln \frac{L}{\ell_T} + \frac{\hbar}{2a\sqrt{\pi\gamma}\mu} & \text{for } T \ll T_a. \end{cases} \quad (8)$$

Thus, while the unpinned stripe array is flat (i.e., $\langle u^2 \rangle$ is finite for $L \to \infty$) at $T = 0$, it is logarithmically rough (i.e., $\langle u^2 \rangle \propto \ln L$ for $L \to \infty$) at any finite temperature.
III. CLASSICAL LIMIT \((h = 0)\)

For the analysis of the effects of the potentials it is convenient to examine the various limiting cases defined by the relative strength of thermal fluctuations, quantum fluctuations, periodic pinning and disorder pinning. We start from the consideration of the classical limit \(h \to 0\) acting on the \(h\) appearing explicitly in Eq. (3) but not on possible implicit dependences of other model parameters. In this limit temporal fluctuations become negligible and one has to examine the system governed by the Hamiltonian

\[
\mathcal{H} = \mathcal{H}_{cl} + \mathcal{H}_{U} + \mathcal{H}_{V}.
\]

In the absence of the potentials \(U\) and \(V\) thermal fluctuations lead to an average displacement that diverges logarithmically with the system size \(L\) [cf. Eq. (8) for \(T_a = 0\)] which means that the stripe structure has only quasi-long-range order in the position of the stripes.

**A. Periodic potential only \((V = 0)\)**

To analyze the relevance of a periodic pinning potential we focus on commensurabilities of low order with integer \(p\). In this case the stripe structure can lock into the periodic potential at low temperatures while it unlocks at large temperatures. The transition between these two states is analogous to the roughening transition of crystal surfaces. We follow the standard analysis of the roughening transition (see Ref. 35 and references therein) in order to obtain the transition temperature \(T_R\). Combining Eqs. (3), (4) and (8) we find an average potential energy

\[
\langle \mathcal{H}_U \rangle \simeq -L^2 a^2 \sum_{n \geq 1} U_n \exp \left(-p^2 Q_n^2 (w^2)/2 \right)
\]

\[
\simeq -L^2 a^2 \sum_{n \geq 1} U_n \left(\frac{L}{a}\right)^{-p^2 Q_n^2 T/4\pi \gamma}
\]

for an infinitesimally weak periodic potential. The lowest harmonic \(n = 1\) gives the most relevant contribution to this energy. The stripes are locked when the average potential energy does not vanish in the limit \(L \to \infty\), i.e., for temperatures below

\[
T_R = \frac{2\gamma a^2}{\pi p^2}.
\]

The transition temperature increases with increasing strength of the potential since \(\gamma\) is renormalized to larger values. The effective parameters on large length scales \(L = a e^x\) are described by renormalization group (RG) flow equations.

**B. Pinning potential only \((U = 0)\)**

In order to discuss the relevance of disorder pinning, we start from the Hamiltonian \(\mathcal{H}_V\) as given in Eq. (3), discard rapidly oscillating terms that are irrelevant on scales much larger than \(a\) and keep only the most relevant term \(m = 1\) in the sum over harmonics for the density (see Ref. 34 for intermediate steps). After averaging over disorder we find the effective replica pinning Hamiltonian

\[
\mathcal{H}_{V, \text{eff}} \simeq \sum_{\alpha, \beta} \int d^2 r \left\{ -\frac{\gamma^2 \sigma}{2T} \nabla u^\alpha \nabla u^\beta - \frac{\Delta}{a^2 T} \cos \frac{2\pi}{a} [u^\alpha(r) - u^\beta(r)] \right\}
\]

Disorder couples to the \(\partial_{x_x} u\) term in the density Eq. (4) as a random field and gives rise to the first term in Eq. (13).
Below the transition temperature \( T < T_{SR} \), \( \Delta \) flows to a finite fixed-point value and \( d\sigma/dl \) becomes constant. \( \sigma \) thus asymptotically has a logarithmic dependence on the scale \( L \) and gives the dominant contribution to the fluctuations
\[
\langle |u(r) - u(0)|^2 \rangle \sim a^2 \ln^2 \frac{r}{a},
\]
(18)

Slightly below \( T_{SR} \), \( \chi \propto (1 - T/T_{SR})^2 \). This squared-logarithmic roughness \([18]\) defines the superrough (SR) phase. In this phase thermal fluctuations can still give a logarithmic contribution to the correlator \([18]\) which – however – is sub-dominant.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Schematic representation of the renormalization group flow describing the superrouhing transition according to Eqs. \([14]\). Arrows indicate the RG flow. The dashed line is the phase boundary between the superrouhing and rough phase. Crosses represent fixed points (note, however, that \( \sigma \) flows to infinity for \( T < T_{SR} = \gamma a^2/\pi \)).}
\end{figure}

\section*{C. With complete potential}

In the presence of both a disorder potential and a periodic pinning potential it is not immediately clear which one will prevail in what part of the phase diagram. At first sight it seems to be possible to have a flat phase, a ln-rough phase, or a ln²-rough phase. We assume that \( U \) and \( V \) are weak such that a renormalization of the stiffness \( \gamma \) as well as of \( T_R \) and \( T_{SR} \) is negligible. Then one always has \( T_R < T_{SR} \) for \( p > \sqrt{2} \). Thus, at high temperatures \( T > T_{SR} \) the system will be logarithmically rough since both \( U \) and \( V \) are irrelevant. At intermediate temperatures in the interval \( T_R < T < T_{SR} \), \( V \) is relevant while \( U \) is irrelevant with respect to thermal fluctuations. This suggests that the system is superrouhing. Although we ultimately find this to be true (see below), the argument needs to be refined since it is no longer sufficient that \( U \) is irrelevant with respect to thermal fluctuations. Instead, we one has to argue that \( U \) is irrelevant at the disorder-dominated, superrouhing fixed point. Eventually, for \( T < T_R \) one might expect to find a superrough phase.

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\[^{[13]}\] with a value \( \sigma = \frac{\Delta}{\gamma} \). (Note that strictly speaking this term should contain only \( \partial_x u^\alpha \partial_x u^\beta \) in the unrenormalized Hamiltonian; the form written in Eq. \([13]\) anticipates that renormalization generates a random field coupling to both components of the gradient.) Similar to the estimate for the roughening temperature above, one can estimate the relevance of \( H_V^{rep} \) from its average with respect to \( H_{el} \),
\[
\langle H_V^{rep} \rangle \simeq - \sum_{\alpha, \beta = 1}^N L^2 \frac{\Delta}{a^2 T} \exp(-2\pi^2 \langle [u^\alpha - u^\beta]^2 \rangle/a^2)
\]
\[
= - L^2 N(N-1) \frac{\Delta}{a^2 T} \left( \frac{L}{a} \right)^{-2\pi^2} T, \tag{14}
\]
where we used \( \langle u^\alpha(r)u^\beta(\mathbf{r}) \rangle = \delta^{\alpha\beta} \frac{L}{a} \ln \frac{L}{a} \). For temperatures above
\[
T_{SR} = \frac{\gamma a^2}{\pi} \tag{14}
\]
the average disorder energy vanishes on large scales. Below, disorder shows to be relevant and its effects have to be calculated by renormalization group techniques. Note that
\[
T_{SR} \simeq \frac{\gamma a^2}{2} T_R. \tag{15}
\]
This relation becomes an identity if the renormalization of \( \gamma \) due to the presence of the potentials can be neglected. Then \( T_{SR} > T_R \) for \( p > \sqrt{2} \).

Cardy and Ostlund \([34]\) were the first to derive RG equations near the transition and Villain and Fernandez \([34]\) studied the flow of parameters to their large-scale values at zero temperature. A concise summary of these two approaches is given in Ref. \([34]\). We combine the flow equations for these two temperature ranges by the interpolation
\[
\frac{d\sigma}{dl} = c_1 T^2 \gamma^2 a^4 + \Delta (\gamma^2 a^4 + c_2 \Delta), \tag{16a}
\]
\[
\frac{d\Delta}{dl} = \left( 2 - \frac{2\pi T}{\gamma a^2} \right) \Delta - 2 \frac{c_2 \Delta^2}{\gamma^2 a^4 + c_2 \Delta}. \tag{16b}
\]
The numbers \( c_1 \) and \( c_2 \) are of order unity and depend only weakly on temperature. \( \gamma \) is not renormalized due to a statistical symmetry \([34]\) just like \( \sigma \) does not feed back to \( \Delta \). This holds for the replica Hamiltonian \([13]\) which is a good approximation on large length scales. Smaller scales will weakly renormalize the stiffness \( \gamma \) to larger values and generate additional irrelevant terms. From the flow equations, \( T_{SR} = \gamma a^2/\pi \) is identified as the temperature above which \( \Delta \) is renormalized to zero. Nevertheless, disorder is marginal for \( T > T_{SR} \) since \( \sigma \) takes a finite fixed-point value. Thus, here one has displacement fluctuations
\[
\langle u^2 \rangle = \frac{1}{2\pi} \left( \frac{T}{\gamma} + \sigma \right) \ln \frac{L}{a}. \tag{17}
\]
if $U$ is weak compared to $V$ and a flat phase if $V$ is weak compared to $U$. However, the following arguments show that a flat phase is not stable in the presence of disorder and that superroughness should persist for all $T < T_{SR}$.

In order to argue that a flat phase cannot exist for arbitrarily weak $V$ and that the stripe array is superrough for all $T < T_{SR}$, we show (i) that weak periodic potentials ($U \ll V$) are irrelevant at the superrough fixed point and (ii) that arbitrarily weak disorder ($V \ll U$) will roughen the stripe array even for $T < T_{R}$, i.e., that disorder is relevant at the flat-phase fixed point, which implies that $U$ is renormalized to zero and the system is superrough on large scales.

Consideration (i). For $U \ll V$ the irrelevance of a periodic potential follows from an analysis in analogy to the one in Sec. III A. We assume that the disorder-induced fluctuations of the displacement field are Gaussian with a correlator $u^2 \sim \frac{1}{a^2} \chi(L/a)$ [which is implied by Eq. (18)]. This replaces the thermal correlator in Eq. (10a). This now leads to

$$\langle H_U \rangle \sim -U_1 \frac{L^2}{a} \sum_{n \geq 1} U_n \left( \frac{L}{a} \right)^{-\pi^2 n^2 p^2 \chi \ln(L/a)}$$

and the average periodic pinning energy vanishes for large system sizes.

Consideration (ii). The situation $V \ll U$ is more subtle. We assume that the stripes are locked into a flat state. We neglect thermal fluctuations which would renormalize $U$ to smaller values. For simplicity of our argument we assume that $p = 2$ although the argument can be generalized to any $p > 1$. In addition, we retain only the most relevant, lowest harmonic of the periodic pinning energy,

$$H_U \simeq -\frac{U_1}{a} \int \cos \frac{2\pi p}{a} u(r).$$

In the absence of $V$ all flat states $u(r) = nb$ with some integer $n$ would be equivalent. Disorder certainly breaks this degeneracy and one might expect the stripe array to find a ground state where $u(r)$ fluctuates only weakly around $na$ with some particular $n$, say $n = 0$. However, one can show that the ground state is not given by small fluctuations within this particular “valley” $n = 0$ but that solitons (i.e., local areas where $u(r) \approx \delta nb$ with a shift $\delta n \neq 0$, cf. Fig. 3) are preferred energetically. The proliferation of a large number of such solitons implies the irrelevance of $U$ and hence the superroughness of the stripe array on large scales.

We now examine a disk-like soliton of Radius $R$ and estimate its elastic energy cost and typical gain of pinning energy in order to decide whether such solitons are favorable. From an energetical point of view the creation of $\delta n = 1$ solitons is equivalent to the creation of a magnetic domain in a random-field Ising model. For a strong periodic pinning potential, $U_1 \gg \gamma a$, the soliton has a narrow border of width $\ell_U \approx \sqrt{a\gamma/U_1 b}$ and of an energy per unit length $\epsilon \approx \sqrt{U_1 a\gamma/p}$. Thus the elastic energy cost is proportional to the border length, $E_{el} \propto \epsilon R$. In the area of the soliton the strings are exposed to a different disorder potential $V$ (we assume the disorder correlation length to be small, $\xi \lesssim a$). Then the typical energy gain is proportional to the square root of the area $E_{g} \propto -\sqrt{\Delta R^2/a\gamma}$. This gain is larger than the elastic energy cost for solitons of all sizes for a disorder strength beyond a threshold value

$$\Delta_{c} \propto a \xi \epsilon^2.$$  

(21)

However, for large solitons one has to take into account that the soliton border will be roughened by disorder. The equivalent roughening of domain walls in the random field Ising model was studied in Refs. [14,15]. As a consequence, the border line tension $\epsilon$ is renormalized to zero on a finite length scale

$$\ell_{\epsilon} \sim a \exp \left( \frac{a \xi \epsilon^2}{\Delta} \right),$$

(22)

where $c$ is a constant of order unity. The creation of solitons of a size $R$ larger than $\ell_{\epsilon}$ is thus energetically favorable. Overlapping solitons of unbounded size imply a roughness of the stripe array, provided the sum of the shifts $\sum_{i} n_{i}$ at a given position increases with the number of solitons (enumerated by the index $i$) that include this site. In principle, the interaction between solitons could lead to a compensation of the shifts between pairs of solitons. However, the interaction between the solitons is short-ranged on the scale $\ell_{U}$ and it cannot compete with a disorder energy that discriminates between a shift $\delta n_{i} = 1$ and a $\delta n_{i} = -1$ for each soliton. Although the pinning energy of these two states is identical in the bulk area, the disorder energy of these two solitons is different in the border region and leads to an energy contribution proportional to $\sqrt{R}$. Therefore the multiple creation of large and overlapping solitons leads to uncorrelated contributions to shifts $\delta n$ and therefore implies the roughness of the stripe array on scales beyond $\ell_{\epsilon}$. In this sense $U$ is irrelevant on large scales in the presence of arbitrarily weak $V$, although on small scales the stripes will be confined to valleys of $U$. Since $U$ is irrelevant, the stripe array will be superrough as in the absence of $U$.

Although the case $p = 1$ appears not to be of physical relevance for striped systems, we add as a side remark that for $p = 1$ a flat phase can exist. In this case the elastic energy cost $\propto R$ of a soliton cannot be compensated by a disorder energy which no longer has a bulk contribution $\propto R$ but only a border contribution $\propto \sqrt{R}$.

Strictly speaking, we have shown the irrelevance of $U$ only for integer values of $p$. Since noninteger rational values of $p$ would correspond to commensurabilities of higher order, they are even more susceptible to the destruction of long-range order by $U$.

It is interesting to note that the absence of a flat phase for $p > 1$ is peculiar to two dimensions. In dimensions
2 < d < 4 a flat phase is stable for disorder weaker than a threshold value.\[4\] Therefore, a stack of planar stripe arrays with a finite coupling between the planes will exhibit also a flat phase for a disorder strength below a certain threshold value.

![Diagram](image)

**FIG. 3.** Illustration of a soliton of radius $R$ for $p = 2$. Full lines represent the strings of holes of average spacing $a$, dashed lines are minima of $U$ with spacing $b = a/p$. The soliton width is $\ell_u$.

**IV. ADDING QUANTUM FLUCTUATIONS**

Now we finally study how the potentials $U$ and $V$ affect the elastic string model taking into account its quantum mechanical nature. In order to analyze whether these potentials are relevant at all, the scaling arguments used in section III can be applied analogously to the dynamic action. Since the displacement correlations of the unpinned array are qualitatively different for $T = 0$ (where the array is flat) and $T > 0$ (where the array is rough) these cases will be discussed separately.

**A. At zero temperature**

In this section we focus on the case with disorder but without a periodic potential. Before we turn to the analysis of this case, it is worthwhile to point out that this system is essentially a two-dimensional “Bose glass.” Because of an analogy between a $d$-dimensional bosonic problem at zero temperature and a classical $(d+1)$-dimensional problem a finite temperature superconducting vortices in the presence of columnar pinning centers provide another equivalent system that has received a lot of recent attention. The problem at hand, a bosonic one-component displacement field at $T = 0$, is equivalent to a stack of classical elastic layers which was studied by Balents.\[4\] According to his analysis – which is restricted to dimensions close to $d = 4$ – the displacement field is logarithmically rough. We now focus on two dimensions because characteristic modifications leading to superroughness are to be expected.

The analysis of the disorder pinning is most convenient using the replicated action

\[
S_{\text{rep}}^U \simeq \sum_{\alpha, \beta} \int d\tau d\tau' d^2 r \left\{ -\frac{\gamma^2}{2} \nabla u^\alpha(r, \tau) \nabla u^\beta(r, \tau') - \frac{\Delta}{a^2 \hbar^2} \Delta [u^\alpha(r, \tau) - u^\beta(r, \tau')] \right\}
\]

(23)

where

\[
\Delta(u) = \cos \frac{2\pi}{a} u
\]

(24)

if we retain only the lowest harmonic of the stripe density as most relevant term as in Eq. \[13\].

For a scaling analysis of this action contribution, we consider a rescaling of space, time, displacement and action quantum according to

\[
r = e^\ell r, \quad \tau = e^\ell \tau, \quad u = e^{\ell t} u, \quad \hbar = e^{\ell} \hbar
\]

(25)

with a dynamical exponent $z$, roughness exponent $\eta$ and action scaling exponent $\eta$. In order to analyze the relevance of quantum fluctuation later on we allow for a rescaling of \( \hbar \). We note that due to a statistical tilt symmetry the flow equation of $\gamma$ consists only of the scaling part

\[
\frac{d}{dt} \gamma = (z - \eta + 2\zeta) \gamma
\]

(26)

which implies

\[
\eta = z + 2\zeta
\]

(27)

at any possible fixed point. If $\eta > 0$ quantum fluctuations are irrelevant on large scales and the fixed point can be called a “classical” ($\hbar_\infty = 0$) fixed point in analogy to the irrelevance of thermal fluctuations for classical systems.\[4\]

To establish the relevance of pinning we note that the action of the unpinned system $S_0$ is invariant under a rescaling with $z = 1$, $\eta = 0$ and $\zeta = -\frac{\eta}{2}$. According to Eq. \[8\] the displacement fluctuations are finite for $T = 0$ (this is reflected by the negative value of $\zeta$) which implies that $\langle \cos \frac{2\pi}{a} [u^\alpha(r, \tau) - u^\beta(r, \tau')] \rangle$ is finite since in the unpinned system $\langle [u^\alpha(r, \tau) - u^\beta(r, \tau')]^2 \rangle \simeq \hbar/(a\sqrt{\pi\gamma})$ for $l \to \infty$ and thus disorder is strongly relevant, $S_{\text{rep}}^U \propto e^{4\ell}$.

Disorder can be taken into account on a crude level (the “random field” approximation) by retaining in action \[23\] only the harmonic parts bilinear in $u$. Then this action part is found to be scale invariant together with $S_0$ for $z = 1$, $\zeta = 1$, and $\eta = 3$. To gain qualitative insight into the spatio-temporal correlations we calculate the displacement fluctuations in this random-field approximation. We find

\[
\langle [u^\alpha(r, \tau) - u^\beta(0, 0)]^2 \rangle = C_1^{\alpha\beta}(r, \tau) + C_2(r)
\]

(28a)
with a contribution $C_1$ from quantum fluctuations and a disorder contribution $C_2$. $C^{\alpha \beta}_1(r, \tau)$ vanishes for $\alpha = \beta$ and $r = 0$ and $\tau = 0$ and takes a finite value

$$C^{\alpha \beta}_1(r, \tau) \simeq \frac{h}{a \sqrt{\pi} \gamma \nu}$$

(28b)

for $\alpha \neq \beta$ or $r \gg a$ or $\tau \gg \sqrt{\mu/\gamma a}$. $C_2(r)$ is rough; its roughness

$$C_2(r) \simeq \frac{\sigma}{\pi} \ln \frac{r}{a} + \frac{4\pi \Delta}{\gamma^2 a^4} r^2$$

(28c)

is dramatically overestimated in this approximation. From this correlation function we recognize that while disorder roughens the displacement in spatial directions (consider large $r$ for $\tau = 0$), it preserves the flatness in temporal directions (consider large $\tau$ for $r = 0$). These qualitative properties should hold even after renormalization effects due to the anharmonic terms in action (23) are taken into account.

Although a systematic renormalization group analysis is very intricate and beyond the scope of this article, we present arguments in favor of $\eta > 0$ at the true fixed point. First, we use the fact that a fixed point with a periodic correlator $\Delta(u)$ can exist only for $\zeta = 0$. Thus

$$\eta = z$$

(29)

from Eq. (27) and it is sufficient to show $z > 0$ for the irrelevance of quantum fluctuations. It is natural to assume that the dynamic exponent is positive, if not diverging on large scales as is typical of glassy systems where the dynamics is governed by tunneling through divergent barriers.

The difficulty to calculate $z$ is related to the fact that $\Delta(u)$ may become nonanalytic near $u = 0$. While such a nonanalyticity follows from a functional renormalization group analysis near $d = 4$ [9] it is not clear whether such a nonanalyticity is present in $d = 2$. According to the analysis summarized in Sec. III B, such a nonanalyticity is absent at finite temperature near $T_{SR}$. At zero temperature, the strong coupling analysis of Villain and Fernandez misses such a possible nonanalyticity. Therefore we consider both possibilities.

We first assume analyticity of the function $\Delta(u)$. In this case the “dynamic stiffness” will be renormalized according to

$$\frac{d}{dt} \mu = (2 - z + 2\zeta - \eta) \mu + \beta_{\mu}[\Delta].$$

(30)

The functional $\beta_{\mu}[\Delta]$ represents the vertex corrections arising from the $S^V_{\text{rep}}$. These corrections are expected to be positive since disorder pins the stripes at minima of $V$, thereby confining the temporal fluctuations which amounts to an increase of the renormalized $\mu$. If $\beta_{\mu}[\Delta]$ is finite at the fixed point, then $\eta = z = 1 + \beta_{\mu}[\Delta]/(2\mu^*)$ ($\mu^*$ denotes the fixed-point value of $\mu$). Then $\eta > 0$ and the fixed point will be classical.

In case the fixed-point correlator is nonanalytic the functional $\beta_{\mu}[\Delta]$ diverges. This signals a qualitative increase of the dynamic stiffness which should be described by a kinetic term of the form

$$S_{\text{kin}} = \frac{1}{\hbar} \int_0^{k/T} d\tau \int d^2 r \nu |\partial_\tau u|$$

(31)

which is more relevant than the original kinetic term in Eq. (10). The coefficient $\nu$ would flow according to

$$\frac{d}{dt} \nu = (2 + \zeta - \eta) \nu + \beta_{\nu}[\Delta]$$

(32)

with another functional $\beta_{\nu}[\Delta] > 0$. This would imply an even larger dynamical exponent $z = \eta = 2 + \beta_{\nu}[\Delta]/\nu^*$ and even stronger irrelevancy of quantum fluctuations.

Thus, in any case $\eta > 0$ and the system flows to the classical fixed point value for $V \neq 0 = U$. Quantum effects will result only in a finite renormalization of the parameters in the classical system. The most important renormalization effect concerns an increase of the dynamic stiffness with a possible generation of $\nu$. Although there is no way of handling a kinetic action of the form (11) we expect quantum fluctuations on small scales to induce a flat but finite quantum contribution $C_1$ to the displacement correlation. The classical contribution $C_2$ will be renormalized as in the absence of quantum fluctuations (the proper correlation can be obtained from equation (28c) by inserting the scale-dependent values of $\Delta$ and $\sigma$ as obtained from the flow equations (16) without rescaling; in the superrough phase $\sigma \propto \ln r$ and $\Delta \propto (\ln r)/r^2$).

Since the presence of the disorder potential implies the irrelevance of quantum fluctuations, they cannot be expected to modify the competition between $U$ and $V$ as was analyzed in Sec. III C. Thus, the quantum array has superrough spatial correlations at $T = 0$.

B. At finite temperature

An inspection of the correlator (3) of the unpinned system suggests that thermal fluctuations dominate over quantum fluctuations on large scales. In fact, quantum fluctuations are irrelevant at the classical fixed point also for $T > 0$.

This can be seen from the action as follows. The classical fixed points (with both thermal roughness or superroughness) are described by $\zeta = 0$. The finiteness of the time integral implies $z = 0$ and according to Eq. (27) also $\eta = 0$. Then the effective “dynamical stiffness” flows to infinity according to Eq. (30) or Eq. (32). Thereby temporal fluctuations of the displacement are suppressed on large scales, on which the system is described by the static limit.
Thus quantum fluctuations will lead only to a renormalization of the parameters in the classical description. Therefore the scaling arguments that were applied in section II to the Hamiltonian hold also for the dynamic action. Thus the system in disorder will be superrough at low temperatures – without or with an additional periodic potential – while it will be thermally rough at high temperatures.

V. SUMMARY AND DISCUSSION

So far, we have analyzed the stripe array in the elastic approximation, i.e., neglecting dislocations. Before we discuss the relevance of dislocations, we summarize our results. In general, we found quantum fluctuations to be irrelevant in the presence of thermal and/or disorder-induced fluctuations, i.e., to renormalize the classical elastic model only weakly.

In the absence of disorder (\(\Delta = 0\)), the stripe array locks into a commensurate periodic potential below a roughening temperature \(T_R = 2\gamma b^2/\pi = 2\gamma a^2/\pi p^2\). This transition temperature is proportional to the stiffness \(\gamma\) which is implicitly temperature dependent due to an entropic contribution \(\gamma_{\text{entr}} = 2\gamma a^2/\pi p^2\). If there were only this entropic contribution, the system would always be unlocked (\(T > T_R\)) at finite temperatures for \(p > \sqrt{2}\). However, a temperature independent contribution to \(\gamma\), which arises from the Coulomb interaction, leads to a lock-in transition at a finite temperature for \(p > \sqrt{2}\), where the translational order changes from long ranged to quasi-long ranged (with logarithmic roughness).

The large-scale structure of the stripe array is dramatically influenced by the presence of disorder (\(\Delta > 0\)). If there were no periodic potential, the stripe array would undergo a superroughening transition at \(T_{\text{SR}} = \gamma a^2/\pi\). For \(T > T_{\text{SR}}\) the array would be unpinned with inroughness, while it would be pinned and superrough for \(T < T_{\text{SR}}\). The same scenario holds also in the presence of the periodic potential, from which the stripe array always unlocks (on sufficiently large scales even for arbitrarily weak disorder). Therefore, the array is rough at all temperatures for \(\Delta > 0\). However, for weak disorder and strong periodic potential the crossover length scale from flat to superrough correlations will be exponentially large, cf. Eq. (23).

Hasselmann et al. previously proposed a phase diagram for a single stripe with flat and disordered phases. To our understanding, the disorder was effectively assumed to have long-ranged correlations, which allows for the existence of a flat phase even for a single stripe. In contrast to this we consider disorder with short-ranged correlations and find that it always roughens the stripe array. Because a single stripe represents an elastic system of lower dimensionality than the stripe array, our finding implies also the roughness of a single stripe in disorder with short-ranged correlations.

Now we come back to discuss the relevance of topological defects in the stripe array, starting with the simplest situation for \(U = V = 0\). At low temperatures the stripe array can be considered as a “smectic” with quasi-long-range translational order and long-range orientational order. At a temperature

\[
T_m = \frac{\gamma a^2}{8\pi},
\]

it would melt due to a proliferation of dislocations into a “nematic” liquid with short-range translational order and quasi-long-range orientational order, before the proliferation of disclinations drives a second transition into an isotropic liquid with short-ranged orientational order. While this scenario is well known for classical systems (for a review see e.g. Ref. 32), its relevance to doped Mott insulators was pointed out by Kivelson, Fradkin, and Emery.

It is instructive to compare the melting temperature to the other characteristic temperatures related to the potentials. As pointed out by José et al. distinct melting and lock-in transitions can exist only for \(p \geq 4\) since \(T_R = (4/p)^2 T_m\). For \(p < 4\) they will merge to a single phase transition. The effect of disorder is quite virulent: In the elastic approximation, it makes the system superrough at temperatures below \(T_{\text{SR}} = 8T_m\). However, superroughness implies that dislocations become energetically favorable [this follows from the flow equations (10), see e.g. Ref. 2]. Therefore, their density will be finite even at zero temperature. Nevertheless, for weak disorder and at low temperatures the length scale where free dislocation appear can be extremely large because of a \(U\) that tends to lock the stripes up to exponentially large scales. Since melting in the absence of disorder occurs at a temperature, \(T_m < T_{\text{SR}}\), free dislocations will be present at all temperatures.

The presence of dislocations reduces the quasi-long-range order (where the satellite peaks have an algebraic singularity due to the ln-roughness; ln\(^2\)-roughness corresponds to an algebraic singularity with an exponent that depends weakly on the wave vector) to short-range order with a correlation length of the order of the distance between (free) dislocations. Thus, the saturation of the dislocation density at low temperatures implies also the saturation of the correlation length at a maximum value. This conclusion is consistent with experimental observations in the cuprates (here it was observed in magnetic ordering, which can be destroyed due to a coupling to disorder even without a distortion of the stripe array) as well as in the nickelates (here the correlation length is explicitly that of charge order).

Note that the true correlation length of charge order \(\xi^C\) (which we identify with the distance between free dislocations) can be related to the measured width of peaks in the structure function only if the wave-vector resolution
is much smaller than $1/\xi_C$. Thus, a system without dislocations (with ln- or ln$^2$-roughness) has an infinite correlation length and the apparent correlation length deduced from experiments would be resolution limited. Zachar recently proposed an explanation of the observed apparent correlation lengths in terms of chaotic fluctuations of the distance between neighboring stripes (see also Ref. [2]), excluding explicitly a key role of dislocations. However, this argument is based on the assumption of non-integer $p$ and cannot account for the finite correlation length observed for integer $p$ in the nickelates and the cuprates. It also cannot explain why the transverse correlation length (along the stripes) -- which apparently is not resolution limited -- can be finite. Even more, the fact that longitudinal and transverse correlation lengths are roughly of the same size is consistent with their relation to the dislocation density.

As both the roughening transition is washed out by disorder and the superroughening transition is washed out by the presence of dislocations, no sharp transition will exist in the thermodynamic limit. Nevertheless, crossover phenomena may be observable. For weak disorder and/or strong lattice potential the stripe array appears to be locked to the lattice potential up to very large length scales. It "unlocks" also on finite scales with increasing temperature or decreasing lattice potential. An apparent lock-in transition was observed in La$_2$-$x$Nd$_x$Sr$_x$CuO$_4$ where the strength of the lattice potential corresponds to the Nd concentration. Since $T_{SR} \gg T_m$, the density of free dislocations will be high near $T_{SR}$ so that reminiscences on finite scales of the superroughening transition are unlikely to survive.

Since the array has short-range order at all temperatures for $\Delta > 0$ we expect the absence of commensurate/incommensurate transitions (identified from fluctuations on asymptotically large length scales). Nevertheless, the experimental observations of anomalies at particular values of $p$ can be related to the effects of $U$ on finite length scales.

In view of our conclusion about the absence of commensuration effects we comment on the $\delta = \frac{1}{8}$ problem, i.e., the observation that $\delta = \frac{1}{2}p$ apparently saturates at $\delta = \frac{1}{4}$ near a doping $x = 0.125$ of the cuprates. If there were a commensuration effect as suggested previously, one would expect to observe the value $\delta = \frac{1}{8}$ around $x = \frac{1}{8}$, i.e., above ($x > \delta$) and below ($x < \delta$) the matching $x = \delta$. To the best of our knowledge (cf. the data collected in Fig. 7 of Ref. [3]), there is no evidence for data with $x < \delta$. As pointed out by Yamada et al. [3], the saturation of $\delta$ coincides with a saturation of the effective hole concentration in the CuO$_2$ planes beyond a certain doping level $x \approx 0.12$. Thus it is conceivable that other mechanisms limit the effective hole concentration and lead to a plateau in $\delta$.

In conclusion, we have pointed out the relevance of disorder for a stripe array even in states where its period is commensurate with the atomic structure. We found that on large scales pinning by disorder dominates over pinning by the atomic structure. This induces the super-roughness of the array and, on sufficiently large scales, the presence of free dislocations even at low temperatures, which explains the saturation of the correlation length observed in experiments.

ACKNOWLEDGMENTS

The authors are grateful to T. Nattermann for stimulating discussions and valuable hints. They acknowledge financial support by Deutsche Forschungsgemeinschaft through SFB341.

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Note that our usage of the notions commensurate and incommensurate follows the traditional usage in the context of commensurate-incommensurate transitions. We distinguish incommensurability from a mismatch $\delta$ of Bragg peaks in the structure factor, whereas in the stripe literature incommensurability is used synonymously to mismatch. Our notions allow to distinguish incommensurate and commensurate states with finite mismatch.

Note also that in the nickelates the charge correlation length is larger than the spin correlation length by roughly a factor 4, while it should be smaller in the absence of topological defects according to Ref. 56.