The phase diagram of dense QCD

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Abstract
The current status of theoretical studies on the quantum chromodynamics (QCD) phase diagram at finite temperature and baryon chemical potential is reviewed with special emphasis on the origin of various phases and their symmetry breaking patterns. Topics include quark deconfinement, chiral symmetry restoration, order of the phase transitions, QCD critical point(s), colour superconductivity, various inhomogeneous states and implications from QCD-like theories.

(Some figures in this article are in colour only in the electronic version)

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1. Introduction

One of the most crucial properties in the non-Abelian gauge theory of quarks and gluons, quantum chromodynamics (QCD) [1], is the asymptotic freedom [2, 3]; the coupling constant runs towards a smaller value with increasing energy scale. It is hence a natural anticipation that QCD matter at high energy densities undergoes a phase transition from a state with confined hadrons to a new state of matter with on-shell (real) quarks and gluons.

There are two important external parameters for QCD in equilibrium, the temperature \( T \) and the baryon number density \( n_B \). (In the grand canonical ensemble, the quark chemical potential \( \mu_q = \mu_B/3 \) may be introduced as a conjugate variable to the quark number density \( n_q = 3n_B \).) Since the intrinsic scale of QCD is \( \Lambda_{\text{QCD}} \sim 200 \text{ MeV} \), it would...
be conceivable that the QCD phase transition should take place around \( T \sim \Lambda_{\text{QCD}} \sim O(10^{12}) \text{ K} \) or \( n_B \sim \Lambda_{\text{QCD}}^3 \sim 1 \text{ fm}^{-3} \). Experimentally, the heavy-ion collisions (HICs) in laboratories provide us with a chance to create hot and/or denser QCD matter and elucidate its properties. In particular, the Relativistic Heavy-Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) has conducted experiments to create hot QCD matter (a quark–gluon plasma or QGP in short) by Au–Au collisions with the highest collision energy \( \sqrt{s_{\text{NN}}} = 200 \text{ GeV} \). The Large Hadron Collider (LHC) at CERN will continue experiments along the same line with higher energies \([4, 5]\). Exploration of a wider range of the QCD phase diagrams with \( n_B \) up to several times of the normal nuclear matter density \( n_0 \sim 0.17 \text{ fm}^{-3} \) may be carried out by low-energy scans in HICs at RHIC as well as at the future facilities such as the Facility for Antiproton and Ion Research (FAIR) at GSI, the Nuclotron-based Ion Collider Facility (NICA) at JINR and the Japan Proton Accelerator Research Complex (J-PARC) at JAEA and KEK.

In nature, the deep interior of compact stellar objects such as neutron stars would be the relevant place where dense QCD matter at low temperature is realized (see [6] for a review). In fact, continuous efforts have been and are being made in the observations of neutron stars to extract information of the equation of state of dense QCD. If the baryon density is asymptotically high, weak-coupling QCD analyses indicate that the QCD ground state forms a condensation of quark Cooper pairs, namely the colour superconductivity (CSC). Since quarks have not only spin but also colour and flavour quantum numbers, the quark pairing pattern is much more intricate than the electron pairing in metallic superconductors.

In this review we will discuss selected topical developments in the QCD phase diagram with an emphasis on the phases at a finite \( \mu_B \). For the topics not covered in this paper, readers may consult other reviews \([7–19]\).

We organize this paper as follows. In section 2, after a brief introduction of two key features of hot/dense QCD, i.e. the deconfinement and chiral restoration, we show a conjectured QCD phase diagram in the \( \mu_B–T \) plane. In section 3 we introduce three major order parameters to characterize QCD matter at finite \( T \) and \( \mu_B \), i.e. the Polyakov loop, the chiral condensate and the diquark condensate. In section 4 we summarize the current status on the chiral phase transition at a finite \( T \) with \( \mu_B = 0 \). In section 5 we go into more details about the phase transitions in the density region up to a moderate \( \mu_B \) comparable to \( T \), which are classified according to different spontaneous chiral symmetry breaking (S\( \chi \)B) patterns. We then address the situation with \( \mu_B \) much greater than \( T \) in section 6. Even though the theoretical understanding has not been fully settled down yet, section 7 is devoted to a review over conjectured inhomogeneous states of QCD matter as well as some alternative scenarios. In section 8 we look over suggestive results and implications from QCD-like theories. Section 9 is devoted to the summary and concluding remarks.

2. QCD phase structure

We will run through the QCD phase transitions and associated phase structure here before detailed discussions in subsequent sections.

2.1. Deconfinement and chiral restoration

A first prototype of the QCD phase diagram in the \( T–n_B \) plane was conjectured in \([20]\). It was elucidated that one could give an interpretation of the Hagedorn limiting temperature in the Statistical Bootstrap Model (SBM) \([21]\) as a critical temperature associated with a second-order phase transition into a new state of matter. The weakly interacting quark matter at a large \( n_B \) due to the asymptotic freedom has also been recognized \([22]\). Historical summary of QCD phase diagram and its exploration in HIC experiments are given in \([23, 24]\).

- Deconfinement. In an early picture of hadron resonance gas at a finite temperature \([25]\), the density of (mostly mesonic) states \( \rho(m) \) as a function of the resonance mass \( m \) is proportional to \( \exp(m/T_H) \) where \( T_H \sim 0.19 \text{ GeV} \) is known from the Regge slope parameter. Such an exponentially growing behaviour of the density of states should be balanced by the Boltzmann factor \( \exp(–m/T) \) in the partition function. When \( T > T_H \), the integration over \( m \) becomes singular, so that \( T_H \) plays a role of the limiting temperature (Hagedorn temperature) above which the hadronic description breaks down. This argument is applied to estimate the critical value of \( \mu_B \) as well. The density of baryonic states, \( \rho_B(m_B) \propto \exp(m_B/T_H^B) \), is balanced by the Boltzmann factor \( \exp(–(m_B – \mu_B)/T) \), leading to the limiting temperature \( T = (1 – \mu_B/m_B)T_H^B \). We see equivalently that the critical \( \mu_B \) at \( T = 0 \) is given by \( m_B (\gtrsim 1 \text{ GeV}) \).

If one bears in mind a simple bag-model picture that hadrons are finite-size objects in which valence quarks are confined, it is conceivable to imagine that hadrons overlap with each other and start to percolate at the Hagedorn temperature \([26, 27]\), which is an intuitive portrayal of quark deconfinement. Although the Hagedorn/percolation picture is useful for practical objectivization, it is necessary to develop a field-theoretical definition of the quark-deconfinement in QCD. Global centre symmetry of pure gluonic sector of QCD gives such a definition as elucidated in section 3.1.

- Chiral restoration. The QCD vacuum should be regarded as a medium with full of quantum fluctuations that are responsible for the generation of non-perturbative quark mass. In hot and dense energetic matter, quarks turn bare due to asymptotic freedom. Therefore, one may expect a phase transition from a state with heavy constituent quarks to another state with light current quarks. Such a transition is called the chiral phase transition named after the underlying chiral symmetry of QCD. The QCD phase diagram at finite \( T \) and \( \mu_B \) was also conjectured from the point of view of chiral symmetry \([28]\). In this case, the order parameter is the chiral condensate \( \langle \bar{\psi}\psi \rangle \) which takes a value about \( –(0.24 \text{ GeV})^3 \) in the vacuum
and sets a natural scale for the critical temperature of chiral restoration. In the chiral perturbation theory ($\chi$PT) the chiral condensate for two massless quark flavours at low temperature is known to behave as $\langle \bar{\psi} \psi \rangle_T / \langle \bar{\psi} \psi \rangle = 1 - T^2/(8 f_\pi^2) - T^4/(384 f_\pi^4) - \cdots$ with the pion decay constant $f_\pi \approx 93$ MeV [29]. Although the validity of $\chi$PT is limited to low temperature, this is clear evidence of the melting of chiral condensate at a finite temperature.

At low baryon density, likewise, the chiral condensate decreases as $\langle \bar{\psi} \psi \rangle_m / \langle \bar{\psi} \psi \rangle = 1 - \sigma_{\pi N} n_B / (f_\pi^2 m_\pi^2) - \cdots$ [30–32] where $\sigma_{\pi N} \approx 40$ MeV is the $\pi - N$ sigma term. (For higher order corrections, see [33, 34].)

The chiral transition is a notion independent of the deconfinement transition. In section 3.2 we classify the chiral transition according to the $S\chi B$ pattern.

### 2.2. Conjectured QCD phase diagram

Figure 1 summarizes our state-of-the-art understanding on the phase structure of QCD matter including conjectures which are not fully established. At present, relatively firm statements can be made only in limited cases—phase structure at a finite $T$ with a small baryon density ($\mu_B \ll T$) and that at an asymptotically high density ($\mu_B \gg \Lambda_{\text{QCD}}$). Below we will take a closer look at figure 1 from a smaller to larger value of $\mu_B$ in order.

**Hadron-quark phase transition at $\mu_B = 0$.** The QCD phase transition at finite temperature with zero chemical potential has been studied extensively in the numerical simulation on the lattice. Results depend on the number of colours and flavours as expected from the analysis of effective theories on the basis of the renormalization group together with the universality [35, 36]. A first-order deconfinement transition for $N_c = 3$ and $N_f = 0$ has been established from the finite-size scaling analysis on the lattice [37], and the critical temperature is found to be $T_c \approx 270$ MeV. For $N_f > 0$ light flavours it is appropriate to address more on the chiral phase transition. Recent analyses on the basis of the staggered fermion and Wilson fermion indicate a crossover from the hadronic phase to the quark–gluon plasma for realistic $u, d$ and $s$ quark masses [38, 39].

The pseudo-critical temperature $T_{\text{pc}}$, which characterizes the crossover location, is likely to be within the range 150–200 MeV as summarized in section 4.2.

Even for the temperature above $T_{\text{pc}}$ the system may be strongly correlated and show non-perturbative phenomena such as the existence of hadronic modes or pre-formed hadrons in the quark–gluon plasma at $\mu_B = 0$ [28, 40] as well as at $\mu_B \neq 0$ [41–43]. Similar phenomena can be seen in other strong-coupling systems such as the high-temperature superconductivity and in the BEC regime of ultracold fermionic atoms [44].

**QCD critical points.** In the density region beyond $\mu_B \sim T$ there is no reliable information from the first-principles lattice QCD calculation. Investigation using effective models is a pragmatic alternative then. Most of the chiral models suggest that there is a QCD critical point located at $(\mu_B = \mu_E, T = T_E)$ and the chiral transition becomes first order (crossover) for $\mu_B > \mu_E$ ($\mu_B < \mu_E$) for realistic $u, d$ and $s$ quark masses [45–48] (see point E in figure 2). The criticality implies enhanced fluctuations, so that the search for the QCD critical point is of great experimental interest [49, 50].

There is also a possibility that the first-order phase boundary ends at another critical point in the lower-$T$ and higher-$\mu_B$ region whose location we shall denote by $(\mu_F, T_F)$ as shown by point F in figure 2. As discussed in section 6, the cold dense QCD matter with three degenerate flavours may have no clear border between superfluid nuclear matter and superconducting quark matter, which is called the quark–hadron continuity.

In reality, the fate of the above critical points ($E$ and $F$) depends strongly on the relative magnitude of the strange quark mass $m_s$ and the typical values of $T$ and $\mu_B$ at the phase boundary.
Figure 2. Characteristic points on the QCD phase diagram. E represents the so-called QCD critical point. F is another critical point induced by the quark–hadron continuity. G is the critical point associated with the liquid-gas transition of nuclear matter. H refers to a region which looks like an approximate triple point. See the text for details.

**Liquid–gas phase transition of nuclear matter.** Since the nucleon mass is \( m_N \approx 939 \text{ MeV} \) and the binding energy in isospin-symmetric nuclear matter is around 16 MeV, a non-vanishing baryon density of nuclear matter starts arising at \( \mu_B = \mu_{\text{NM}} \approx 924 \text{ MeV} \) at \( T = 0 \). At the threshold \( \mu_B = \mu_{\text{NM}} \), the density \( n_B \) varies from zero to the normal nuclear density \( n_0 = 0.17 \text{ fm}^{-3} \). For \( 0 < n_B < n_0 \) the nuclear matter is fragmented into droplets with \( n_B = n_0 \), so that \( n_B < n_0 \) is achieved on spatial average. This is a typical first-order phase transition of the liquid–gas type. The first-order transition weakens as \( T \) grows and eventually ends up with a second-order critical point at \((\mu_B, T_G)\) as indicated by point G in figure 2. Low-energy HIC experiments indicate that \( \mu_G \sim \mu_{\text{NM}} \) and \( T_G \sim 15–20 \text{ MeV} \) [51].

**Quarkyonic matter.** The Statistical Model is successful in reproducing the experimentally observable particle abundances at various \( 1/(N_{\text{c}}N_{\text{f}}) \) and thus various \( \mu_B \). This model assumes a thermally equilibrated gas of non-interacting mesons, baryons and resonances for a given \( T \) and \( \mu_B \) [52–54]. Within the model description one can extract \( T \) and \( \mu_B \) from the HIC data by fitting the particle ratios. The accumulation of extracted points makes a curve on the \( \mu_B-T \) plane, which is called the chemical freeze-out line.

The freeze-out line is not necessarily associated with any of the QCD phase boundaries. Nevertheless, there is an argument to claim that the sudden freeze-out of chemical compositions should take place close to the phase transition [55]. Along the freeze-out line the thermal degrees of freedom are dominated by mesons for \( \mu_B \ll m_N \). If more baryons are excited, the \( \mu_B \) becomes higher. This indicates that there must be a transitional change at \((T_H, \mu_H)\), where the importance of baryons in thermodynamics surpasses that of mesons. This happens around \( \mu_H \approx 350–400 \text{ MeV} \) and \( T_H \approx 150–160 \text{ MeV} \) according to the Statistical Model analysis.

It is interesting that such a phase structure is suggested from the large \( N_{\text{c}} \) limit of QCD. When \( N_{\text{c}} \) is large, quark loops are suppressed by \( 1/N_{\text{c}} \) as compared with gluon contributions [56, 57]. A finite baryon number density arises and the pressure grows of order \( N_{\text{c}} \) once \( \mu_B \) becomes higher than the lowest baryon mass \( M_B \). Such cold dense matter in the \( N_{\text{c}} = \infty \) world is named *quarkyonic matter* [58]. Then, the phase diagram of large-\( N_{\text{c}} \), QCD consists of three regions separated by first-order phase transitions, i.e. the confined, deconfined and quarkyonic phases. The meeting point of the three first-order phase boundaries is the *triple point* whose remnant for finite \( N_{\text{c}} \) is indicated by point H in figure 2, as suggested in [59]. We will revisit the idea of quarkyonic matter in section 8.1.

**Colour superconductivity.** If \( \mu_B \) is asymptotically large, i.e. \( \mu_B \gg \Lambda_{\text{QCD}} \), the ground state of QCD matter can be analysed in terms of the weak-coupling methods in QCD. Also we can count on the knowledge from condensed matter physics with quarks substituting for electrons. In this analogue between electrons in metal and quarks in quark matter, one may well anticipate that the ground state of QCD matter at low \( T \) should form Cooper pairs leading to colour superconductivity (CSC) [13, 15, 18, 19, 60, 61]. Theoretical characterization will be further elucidated in sections 3.3 and 6.

There are many patterns of Cooper pairing and thus many different CSC states. The search for the most stable CSC state remains unsettled except for \( \mu_B \gg \Lambda_{\text{QCD}} \) or \( m_s \to 0 \). It is in fact the strange quark mass \( m_s \) that makes the problem cumbersome. In the intermediate density region particularly, the Fermi surface mismatch \( \delta \mu_q \) of different quark flavours is given by \( \delta \mu_q \sim m_s^2/\mu_q \). When the gap energy \( \Delta \) is comparable to \( \delta \mu_q \), an inhomogeneous diquark condensation may have a chance to develop energetically. Such an inhomogeneous CSC state gives rise to a crystal structure with respect to \( \Delta(x) \), that is the crystalline CSC phase [62]. There are, at the same time, various candidates over the crystalline CSC phase in this intermediate density region and the true ground state has not been fully revealed there (see section 7).

So far we have quickly looked over the key phases labelled in figure 1 and important points specifically picked up in figure 2. In section 3 we will proceed to the theoretical framework to deal with the phase transitions of quark deconfinement and chiral restoration, respectively.

### 3. Order parameters for the QCD phase transition

QCD has (at least) three order parameters for quark deconfinement, chiral symmetry restoration and colour superconductivity as discussed below.

#### 3.1. Polyakov loop and quark deconfinement

The Polyakov loop which characterizes the deconfinement transition in Euclidean space–time is defined as [63, 64]

$$ L(x) = \mathcal{P} \exp \left[ -ig \int_0^\beta d\tau_4 A_4(x, \tau_4) \right], \quad (1) $$

which is an \( N_{\text{c}} \times N_{\text{c}} \) matrix in colour space. Here \( \beta \) is the inverse temperature \( \beta = 1/T \) and \( \mathcal{P} \) represents the path ordering. We will use \( \ell \) to represent the traced Polyakov loop:

$$ \ell = \frac{1}{N_{\text{c}}} \text{tr} L. \quad (2) $$

Let us consider the centre \( Z(N_{\text{c}}) \) of the colour gauge group \( \text{SU}(N_{\text{c}}) \): elements of the centre commute with all \( \text{SU}(N_{\text{c}}) \)
elements and can be written as $z_k I$ with $z_k = e^{2\pi ik/N_c}$, $(k = 0, 1, \ldots, N_c - 1)$ and $I$ being an $N_c \times N_c$ unit matrix. Under a non-periodic gauge transformation of the following form, $V_k(x) = [z_k I]^{x/k}$, the gauge fields receive a constant shift,$$
abla_A \rightarrow \nabla_A^k = V_k[\nabla_A - (ig)^{-1}A_k]V_k^\dagger = \nabla_A - \frac{2\pi k}{g N_c B}.
$$

so that the traced Polyakov loop transforms as $\ell \rightarrow z_k \ell$. Because $A_0^k$ still maintains the periodicity in $x_4$, such a non-periodic gauge transformation still forms the symmetry of the gauge action. This is called the centre symmetry [8, 35]. The quark action (with the quark field denoted by $\psi$) explicitly breaks the centre symmetry because the transformed field $V_k(x)\psi(x)$ does not respect the anti-periodic boundary condition any longer. Thus, the centre symmetry is an exact symmetry only in the pure gluonic theory where dynamical quarks are absent or quark masses are infinitely heavy ($m_q \rightarrow \infty$).

The expectation value of the Polyakov loop and its correlation in the pure gluonic theory can be written as [65–67]

$$
\Phi = \langle \ell(x) \rangle = e^{-\beta f_q},
\Phi = \langle \ell^\dagger(x) \rangle = e^{-\beta f_\bar{q}},
$$

$$
\langle \ell(x) \ell(y) \rangle = e^{-\beta f_\bar{q}(x-y)}.
$$

Here, the constant $f_q$ ($f_\bar{q}$) independent of $x$ is the excess free energy for a static quark (anti-quark) in a hot gluon medium. Also, $f_\bar{q}(x-y)$ is the excess free energy for an anti-quark at $x$ and a quark at $y$.

In the confining phase of the pure gluonic theory, the free energy of a single quark diverges ($f_q \rightarrow \infty$) and the potential between a quark and an anti-quark increases linearly at a long distance ($f_{qq}(r \rightarrow \infty) \rightarrow \sigma r$ with $r = |x - y|$), which leads to $\Phi \rightarrow 0$ and $\langle \ell^\dagger(\tau \rightarrow \infty) \ell(0) \rangle \rightarrow 0$. On the other hand, in the deconfined phase, the free energy of a single quark is finite ($f_q < \infty$). Also the potential between a quark and an anti-quark is of the Yukawa type at a long distance with a magnetic screening mass $m_M$ [68–70],

$$
f_{qq}(r \rightarrow \infty) \rightarrow f_q + f_{\bar{q}} + \alpha \frac{e^{-m_{sr}r}}{r},
$$

where $\alpha$ is a dimensionless constant. Note that the glueball exchange with mass of $O(g^2 T)$ which is smaller than the electric screening scale $O(g T)$ dominates over the long-range correlation at weak coupling. Therefore, $\Phi \neq 0$ and $\langle \ell^\dagger(\tau \rightarrow \infty) \ell(0) \rangle \neq 0$.

The qualitative behaviour of $\Phi = \langle \ell \rangle$ and $\langle \ell^\dagger \ell \rangle$ is summarized in table 1. We see that $\Phi$ can nicely characterize the state of matter in such a way that $\ell$ behaves like a magnetization in 3D classical spin systems [8, 35, 71]. The pure gluonic theory for $N_c = 2$ and 3 have been studied in lattice gauge simulations with the finite-size scaling analysis [37, 72]. It was shown that there is a second-order phase transition for $N_c = 2$ and a first-order phase transition for $N_c = 3$; for $N_c = 2$ the critical exponent is found to agree with the Z(2) Ising model in accordance with the universality. For $N_c = 3$ the Ginzburg–Landau free energy for the Polyakov loop has a cubic invariant, so that the first-order transition can naturally be induced [8, 37, 73]. Recent studies of the pure gluonic theory with $N_c = 4, 6, 8, 10$ indicate that the transition is of first order for $N_c \geq 3$ and becomes stronger as $N_c$ increases [74, 75]. This behaviour is similar to that of the 3D $N_c$-state Potts model [76].

The idea of constructing the deconfinement order parameter can be extended to a more general setup. Suppose that there is an arbitrary operator $O$ written in terms of gauge fields that is not invariant under centre transformation. (If we take $O$ as the quark propagator straight up along the imaginary-time direction, the following argument shall result in the standard definition of the Polyakov loop.) The expectation value of $O$ in the pure gluonic theory can be decomposed by the triality projection [77]; $\langle O[A]\rangle = \sum_{n=0}^{N_c-1} \langle O_n[A]\rangle$ with

$$
\langle O_n[A]\rangle = \frac{1}{N_c} \sum_{k=0}^{N_c-1} \langle O[A^k]\rangle e^{-2\pi i nk/N_c},
$$

where $A^k$ is defined in (3). Then, $\langle O_n[A]\rangle$ with $n \neq 0$ can be an order parameter sensitive to the spontaneous breaking of the centre symmetry. A particular choice $O = \psi \bar{\psi}$ [78, 79] (the so-called dual condensate) has some practical advantages [80].

Centre symmetry discussed so far is rigorously defined only in the pure glueion system as already mentioned: in the presence of dynamical quarks, the centre symmetry is broken explicitly, so that $\Phi$ and $\Phi$ always take finite values. This is analogous to a spin system under an external magnetic field, in which the magnetization is always non-vanishing [81]. Nevertheless, $\Phi$ may be estimated as $\Phi = e^{-\beta M}$ with a hadronic mass scale ($M \sim 0.8$ GeV) at low $T$ and thus the low-$T$ phase approximately preserves the centre symmetry.

3.2. Chiral condensate and dynamical breaking of chiral symmetry

In the QCD vacuum at $T = \mu_B = 0$, the chiral symmetry is spontaneously broken, which is the source of hadron masses. It is common wisdom that the chiral symmetry breaking is driven by the expectation value of an operator which transforms as $(N_1, N_2^*) + (N_1^*, N_2)$ under chiral symmetry. The simplest

| Table 1. Behaviour of the expectation value and the correlation of the Polyakov loop in the confined and deconfined phases in the pure gluonic theory. |
|----------------|-----------------|-----------------|-----------------|
|                | Confined (disordered) phase | Deconfined (ordered) phase |
| Free energy    | $f_q = \infty$   | $f_q < \infty$   | $\epsilon^{m_{sr}r}$ |
| Polyakov loop  | $\langle \ell \rangle = 0$ | $\langle \ell \rangle \neq 0$ | $\langle \ell^\dagger(\tau(0)) \rangle \rightarrow |\langle \ell^\dagger \ell \rangle|^2 \neq 0$ |

[3] Strictly speaking, the expectation value in the pure gluonic theory $\langle \ell(x) \rangle$ should be defined by taking the limit $m_q \rightarrow +\infty$ after taking the thermodynamic limit $V \rightarrow \infty$, so that it takes a real value. Alternatively, one may use $\langle \overline{(\langle \ell(x) \rangle)} \rangle$ to define the free energy of a single quark.

[4] Even when there are dynamical quarks, one may use the Polyakov loops, (4) and (5), to define the heavy-quark free energies.
choice of the order parameter for the chiral symmetry breaking is a bilinear form called the chiral condensate,

\[ \langle \bar{\psi} \psi \rangle = \langle \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \rangle, \]

where colour and flavour indices of the quark fields are to be summed. If the above chiral condensate is non-vanishing even after taking the limit of zero quark masses, the chiral symmetry is spontaneously broken according to the pattern \( G \rightarrow \mathcal{H} \) with

\[ G = \text{SU}(N_f) \times \text{SU}(N_f) \times U(1)_B \times \mathbb{Z}(2N_f)_A \]

\[ \mathcal{H} = \text{SU}(N_f)_V \times U(1)_B, \]

which leads to \( N_f^2 - 1 \) massless Nambu–Goldstone bosons for \( N_f > 1 \). The \( U(1)_A \) symmetry in the classical level of the QCD Lagrangian is broken down explicitly to \( \mathbb{Z}(2N_f)_A \) in the quantum level. Then, the \( U(1)_A \) current is no longer conserved if \( U(1)_A \) anomaly

\[ \bar{J}_\mu = \frac{g^2 N_f}{32\pi^2} \epsilon^{abc} F^a_{\mu\nu} F^b_{\mu\nu}. \]

The right-hand side of the above relation is nothing but the topological charge density. Thus, gauge configurations with non-trivial topology are microscopically responsible for the \( U(1)_A \) anomaly. In other words, the \( U(1)_A \) current could be approximately conserved if the gauge configurations are dominated by topologically trivial sectors. We will come back to this point later to address effective restoration of \( U(1)_A \) symmetry in the medium [83].

Note that (8) is not a unique choice for the order parameter [84–87]. For example, one may consider the following four-quark condensate:

\[ \langle \bar{\psi} \lambda^a \psi \rangle = \langle \bar{\psi}_R \lambda^a \psi_L + \bar{\psi}_L \lambda^a \psi_R \rangle. \]

If this is non-vanishing, the ground state breaks the chiral symmetry \( G \rightarrow \mathcal{H} \times \mathbb{Z}(N_f)_A \) where \( \mathbb{Z}(N_f)_A \) corresponds to a discrete axial rotation. If the bilinear condensate (8) is non-zero, the four-quark condensate (12) takes a finite value in general. However, a non-zero value of (12) does not necessarily enforce a finite value of (8). In fact, if \( \mathbb{Z}(N_f)_A \) symmetry is left unbroken, (8) must vanish. As long as the Dirac determinant in QCD is positive definite, possibility of unbroken \( \mathbb{Z}(N_f)_A \) symmetry has been ruled out by the exact QCD inequality [85]. However, it is not necessarily the case for finite \( \mu_B \) with which the positivity of the Dirac determinant does not hold.

3.3. Diquark condensate and colour superconductivity

QCD at high baryon density shows a novel mechanism of spontaneous chiral symmetry breaking [88]. The fundamental degrees of freedom in the CSC phase with three colours and three flavours are the diquarks defined as

\[ \langle \bar{\psi}_R \psi_L \rangle \sim \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} (\bar{\psi}_L^i \psi_R^j) (\psi_R^k), \]

\[ \langle \bar{\psi}_L \psi_R \rangle \sim \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} (\bar{\psi}_R^i \psi_L^j) (\psi_L^k), \]

where \( (i, j, k) \) are flavour indices and \( (\alpha, \beta, \gamma) \) are colour indices. Note that the charge conjugation matrix \( C = i\gamma^5 \gamma^0 \) is necessary to make \( \psi_L/R \) a Lorentz scalar. Then, \( \psi_L/R \) is a triplet both in colour and flavour, so that it transforms in the same way as the quark field \( \psi_L/R \).

Under certain gauge fixing, one may consider the expectation values of these operators. They are called the diquark condensates as will be discussed further in section 6. Instead, we can also construct an analogue of (8) in terms of the diquarks,

\[ \langle \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \rangle. \]

where colour and flavour indices are summed. This is a gauge-invariant four-quark condensate which characterizes the spontaneous chiral symmetry breaking in CSC [13]. Unlike the bilinear operator in (8), the four-quark operator here keeps the diquark condensates as will be discussed further in section 6. The six-quark operator here breaks \( U(1)_B \) with its \( \mathbb{Z}(6)_B \) subgroup maintained. If this condensate is non-zero, there appears an exactly massless Nambu–Goldstone boson because the baryon number conservation may be spontaneously broken. A colour singlet order parameter to detect such symmetry breaking can be

\[ \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} (\bar{\psi}_L \psi_R) \sim \langle \bar{\psi}_L \psi_R \rangle \sim \langle \bar{\psi}_R \psi_L \rangle. \]

4. Chiral phase transition at finite temperature

The chiral phase transition at a finite \( T \) with \( \mu_B = 0 \) has been and is being extensively studied by the renormalization group method near the critical point à la Ginzburg–Landau–Wilson and by the lattice-QCD simulations. In this section we will briefly summarize the current status of these studies. (See [39] for further details.)

4.1. Ginzburg–Landau–Wilson analysis

If the phase transition is of second order or of weak first order, one may write down the free-energy functional in terms of the order parameter field \( \Phi \) as a power series of \( \Phi/T_c \). The large fluctuation of \( \Phi \) near the critical point is then taken into account by the renormalization group method. This is called the Ginzburg–Landau–Wilson approach. For chiral phase transition in QCD, the relevant order parameter field is a
\( N_f \times N_f \) matrix in flavour space, \( \Phi_{ij} \sim (\bar{\psi}_j (1 - \gamma_5) \psi_i) \). Under the flavour chiral rotation \( U(N_f)_L \times U(N_f)_R \), \( \Phi \) transforms as \( \Phi \rightarrow V^a \Phi V^a_R \). Then the Ginzburg–Landau free energy in three spatial dimensions (\( D = 3 \)) with full \( U(N_f)_L \times U(N_f)_R \) symmetry up to the quartic order in \( \Phi_{ij} \) becomes \cite{36, 47}

\[
\Omega_{\text{sym}} = \frac{1}{2} \text{tr} \; \nabla \Phi^+ \nabla \Phi + \frac{a_0}{2} \text{tr} \; \Phi^+ \Phi + \frac{b_1}{4!} (\text{tr} \; \Phi^+ \Phi)^2 + \frac{b_2}{4!} \text{tr} (\Phi^+ \Phi)^2.
\]

The effects of temperature \( T \) enter through the parameters \( a_0, b_1 \) and \( b_2 \). Note that \( \Omega_{\text{sym}} \) is bounded from below as long as \( b_1 + b_2/N_f \geq 0 \) and \( b_2 > 0 \) are satisfied. The renormalization group analysis of \( \Omega_{\text{sym}} \) on the basis of the leading-order \( \epsilon = (4-D) \) expansion leads to a conclusion that there is no stable IR fixed point for \( N_f > \sqrt{3} \) \cite{36}. This implies that the thermal phase transition described by \( \Omega_{\text{sym}} \) is of the fluctuation-induced first order for two or more flavours.

In QCD, however, there is \( U(1)_A \) anomaly and the correct chiral symmetry is \( SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times Z(2N_f)_A \) for \( N_f \) massless quarks. The lowest dimensional operator which breaks the \( U(1)_A \) symmetry explicitly while keeping the rest of the chiral symmetry is the Kobayashi–Maskawa–’t Hooft (KMT) term \cite{89–92}.

\[
\Omega_{\text{anomaly}} = -\frac{c_0}{2} (\text{det} \; \Phi + \text{det} \; \Phi^+) \tag{17}
\]

The coefficient \( c_0 \), which is \( T \)-dependent in general, dictates the strength of the \( U(1)_A \) anomaly. In the instanton picture \cite{91, 92}, \( c_0(T = 0) \) is proportional to the instanton density \( n_{\text{inst}} \), which is perturbatively evaluated as \cite{7}

\[
c_0(T = 0) \propto n_{\text{inst}}(\rho, T = 0) = \left( \frac{8\pi^2}{g^2} \right)^{2N_f} e^{-8\pi^2/g^2} \rho^{-5}, \tag{18}\]

with \( \rho \) being a typical instanton size.

For \( N_f = 3 \), the KMT term becomes a cubic invariant in the order parameter. Hence, \( \Omega(\Phi) = \Omega_{\text{sym}} + \Omega_{\text{anomaly}} \) leads to the chiral phase transition of first order. For \( N_f = 2 \), on the other hand, the KMT term becomes a quadratic invariant. Also the chiral symmetry in this case is \( SU(2)_L \times SU(2)_R \cong SO(4) \). Such an effective theory with \( O(4) \) symmetry has a Wilson–Fisher type IR fixed point as long as the coefficient of the quartic term of \( \Phi \) is positive. Therefore, if the chiral phase transition of massless \( N_f = 2 \) QCD is of second order, its critical exponents would be the same as those in the 3D \( O(4) \) effective theory according to the notion of universality. In table 2 we summarize the Ginzburg–Landau–Wilson analysis from the chiral effective theory \cite{36}.

In the real world, none of the quarks are exactly massless: for example, \( m_u = 1.5 - 3.3 \text{ MeV}, m_d = 3.5 - 6.0 \text{ MeV} \) and \( m_s = 105^{+25}_{-35} \text{ MeV} \) at the renormalization scale of 2 GeV \cite{93}. Therefore, it is useful to draw a phase diagram by treating quark masses as external parameters. This is called the Columbia plot \cite{94} as shown in figure 3 where the isospin degeneracy is assumed \( m_d = m_s = m_u \). The first-order chiral transition and the first-order deconfinement transition at a finite \( T \) are indicated by the left-bottom region and the right-top region, respectively. The chiral and deconfinement critical lines, which separate the first-order and crossover regions, belong to a universality class of the 3D \( Z(2) \) Ising model except for special points at \( m_u = 0 \) or \( m_s = 0 \) \cite{95}.

If the chiral transition is of second order for the massless \( N_f = 2 \) case, the \( Z(2) \) chiral critical line meets the \( m_u = 0 \) axis at \( m_u = m_u^{\text{tricritical point}} \) and changes its universality to \( O(4) \) for \( m_s > m_s^{\text{tricritical point}} \) \cite{96}. The tricritical point at \( m_u = m_u^{\text{tricritical point}} \) is a Gaussian fixed point of the 3D \( \phi^6 \) model (that is, the critical dimension is not 4 but 3 at the tricritical point), so that the critical exponents take the classical (mean-field) values \cite{97}, which is confirmed in numerical studies of the chiral model \cite{98}.

### 4.2. Lattice QCD simulations

Although the critical properties expected from the Ginzburg–Landau–Wilson analysis discussed above are expected to be universal, the quantities such as the critical temperature and the equation of state depend on the details of microscopic dynamics. In QCD, only a reliable method known for microscopic calculation is the lattice-QCD simulation in which the functional integration is carried out on the space–time lattice with a lattice spacing \( a \) and lattice volume \( V \) by the method of importance sampling. In lattice-QCD simulations

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**Table 2. Order of the chiral phase transition conjectured from the chiral effective theory with massless \( N_f \) flavours with and without the \( U(1)_A \) anomaly.**

| \( N_f \) | \( U(1)_A \) symmetric \( (c_0 = 0) \) | Fluctuation-induced | First order | 1st order |
|---|---|---|---|---|
| 2 | | | 2nd order | [O(4) universality] |
| \( \geq 3 \) | Broken \( (c_0 \neq 0) \) | 1st order | | |

**Figure 3. Schematic figure of the Columbia phase diagram in 3-flavour QCD at \( \mu_3 = 0 \) on the plane with the light and heavy quark masses. The \( U(1)_A \) symmetry restoration is not taken into account. Near the left-bottom corner the chiral phase transition is of first order and turns to smooth crossover as \( m_u \) and/or \( m_s \) increase. The right-top corner indicates the deconfinement phase transition in the pure gluonic dynamics.**
there are at least two extrapolations required to obtain physical results; the extrapolation to the continuum limit \( (a \to 0) \) and the extrapolation to the thermodynamic limit \( (V \to \infty) \). Therefore, lattice results receive not only statistical errors due to the importance sampling but systematic errors due to the extrapolations also.

For nearly massless fermions in QCD, there is an extra complication to reconcile chiral symmetry and lattice discretization; the Wilson fermion and the staggered fermion have been the standard ways to define light quarks on the lattice, while the domain-wall fermion and the overlap fermion recently proposed have more solid theoretical ground although the simulation costs are higher. For various applications of lattice-QCD simulations to the system at finite \( T \) and \( \mu_B \), see a recent review [39].

Here we mention only two points relevant to the discussions below. (i) The thermal transition for physical quark masses is likely to be crossover as indicated by a star symbol in figure 3. This is based on the finite-size scaling analysis using a staggered fermion [38]. Confirmation of this result by other fermion formalisms is necessary, however. (ii) The (pseudo)-critical temperature \( T_{pc} \) with different types of fermions and with different lattice spacings are summarized in figure 4. In view of these data with error bars, we adopt a conservative estimate at present; \( T_{pc} = 150–200 \) MeV.\(^7\) It has been clarified recently that improvement of the staggered action with less taste-symmetry breaking favours a smaller value of \( T_{pc} \lesssim 170 \) MeV [107, 108].

### 5. Chiral phase transition at finite baryon density

Let us now introduce the baryon chemical potential \( \mu_B \) as an extra axis to the Columbia plot. In the so-called ’standard scenario’ the first-order region in the lower left corner of the Columbia plot is elongated with increasing \( \mu_B \) as written in the left panel of figure 5. If the physical point at \( \mu_B = 0 \) is in the crossover region, there arises a critical chemical potential \( \mu_E \) so that the system shows a first-order transition for \( \mu_B > \mu_E \). That is, we find a QCD critical point at \( (\mu_B, T_{pc}) \) on the QCD phase diagram in the \( \mu_B-T \) plane. In the right of figure 5, the so-called ’exotic scenario’ is sketched, in which the size of the first-order region shrinks as \( \mu_B \) increases. In this case, if the physical point at \( \mu_B = 0 \) is in the crossover region, it stays crossover for finite \( \mu_B \), so that there arises no critical point (at least for small \( \mu_B \)) in the QCD phase diagram in the \( \mu_B-T \) plane. In general, the critical surface in figure 5 can have a more complicated structure, which may allow for several QCD critical points in the \( \mu_B-T \) plane.

\(^7\) Possible uncertainties in \( T_{pc} \) stem from discretization errors, conversion from the lattice unit to the physical unit and the prescription of defining \( T_{pc} \). For crossover transition \( T_{pc} \) may depend on which susceptibility (either chiral susceptibility \( \chi_{su}(T) \) or Polyakov loop susceptibility \( \chi_{PL}(T) \)) is used, and may also depend on \( T \)-dependent normalization of the susceptibilities.

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**Figure 4.** Determination of the pseudo-critical temperature \( T_{pc} \) for thermal QCD transition(s) from recent lattice QCD simulations. (1) 169(12)(4) MeV for 2 + 1 flavours in the asqtad action with \( N_t \) up to 8 determined by \( \chi_{su}/T^2 \) (where \( \chi_{su} \) is the chiral susceptibility) [99]. (2) 192(7)(4) MeV for 2 + 1 flavours in the p4flav3 staggered action with \( N_t \) up to 6 determined by \( \chi_{su} \) and \( \chi_L \) (where \( \chi_L \) is the Polyakov loop susceptibility) [100]. (3) 151(3)(3) MeV and 176(3)(4) MeV for 2 + 1 flavours in the stout-link improved staggered action with \( N_t \) up to 10 determined by \( \chi_{su}/T^4 \) and \( \chi_L \), respectively [101]. (4) 172(7) MeV for two flavours in clover improved Wilson action with \( N_t \) up to 6 determined by \( \chi_L \) [102]. (5) 152(3)(3) MeV and 170(4)(3) MeV for 2 + 1 flavours in the asqtad and p4 actions with \( N_t \) up to 8 determined by \( \chi_{su} \) and \( \chi_L \) [104]. (7) 174(3)(6) MeV for two flavours in the improved Wilson action with \( N_t \) up to 12 determined by \( \chi_{su}/T^2 \) and \( \chi_L \), respectively [103]. (6) 185–195 MeV for 2 + 1 flavours in the stout-link improved staggered action with \( N_t \) up to 12 determined by \( \chi_{su}/T^4 \) and \( \chi_L/T^2 \) [106], (9) 147(2)(3) MeV and 165(5)(3) MeV for 2 + 1 flavours in the stout-link improved staggered action with \( N_t \) up to 16 determined by \( \chi_{su}/T^4 \) and \( \chi_L/T^2 \) (where \( \chi_L \) is the strange-quark susceptibility) [107].

**Figure 5.** Schematic evolution of the Columbia plot with increasing \( \mu_B \) in the standard scenario (left) and the exotic scenario (right).

#### 5.1. Lattice QCD at low baryon density

In the presence of \( \mu_q \neq 0 \), the QCD partition function on the lattice is written as

\[
Z(T, \mu) = \int [dU] \det(F(\mu_q)) \exp^{-\beta S_{YM}(U)},
\]

where \( U \) is the matrix-valued gauge field defined on the links. The Yang–Mills action is denoted by \( \beta S_{YM}(U) \) with \( \beta = 2N_c/g^2 \), while the quark contribution is denoted by the
the determinant of \( F(\mu) = D(\mu) + m_q \) with \( D(\mu) \) being the Euclidean Dirac operator. The quark mass \( m_q \) is real and positive. Let \( \psi \) be an eigenfunction of \( D(\mu) \) as \( D(\mu)\psi = \lambda_i\psi \), with an eigenvalue \( \lambda_i \). If \( \mu_q = 0 \) then \( D(0) \) is an anti-Hermitian operator and thus \( \lambda_i \) should be purely imaginary. In addition, \( \gamma^5\psi \) is an independent eigenfunction with an eigenvalue \( -\lambda_i^* \); because of \( \gamma^5\)-Hermiticity \( \gamma^5 D(0)\gamma^5 = -D(0) = D^*(0) \). Thus, we always have complex-conjugate pairs \( (\lambda_i, \lambda_i^*) \) and the determinant becomes real and positive.

\[
\det[D(0) + m_q] = \prod_i (\lambda_i + m_q)(\lambda_i^* + m_q) > 0. \tag{20}
\]

In this way, at \( \mu_q = 0 \), the integrand in \( \langle \gamma^5 \rangle \) is positive definite and the importance sampling method works fine to evaluate the functional integral.

For \( \mu_q \neq 0 \) the \( \gamma^5\)-Hermiticity is replaced by

\[
\gamma^5 D(\mu)\gamma^5 = D^*(-\mu_q^*). \tag{21}
\]

Therefore \( \det[D(\mu) + m_q] \) is no longer real unless \( \mu_q \) is purely imaginary. This requires us to deal with a severe cancellation of positive and negative numbers to evaluate the functional integral. This is the notorious sign problem whose difficulty grows exponentially as the lattice volume \( V \) increases.

In the following we shall briefly see some of the approaches in the lattice-QCD simulations at a finite \( \mu_q \). For more details, see the reviews [109–111] and references therein.

- **Multi-parameter reweighting method.** An expectation value of an operator \( \mathcal{O} \) at a finite \( \mu_q \) can be formally rewritten in terms of an ensemble average at \( \mu_q = 0 \):

\[
\langle \mathcal{O} \rangle_{\mu_q} = \langle \mathcal{O} \cdot R(\mu_q) \rangle_{\mu_q=0} \langle R(\mu_q) \rangle_{\mu_q=0}^{-1}, \tag{22}
\]

where \( R(\mu_q) \equiv \det F(\mu_q)/\det F(0) \) is called the reweighting factor [112]. The gauge configurations generated at \( \mu_q = 0 \) only occasionally sample the region where \( \mathcal{O} \cdot R(\mu_q) \) and \( R(\mu_q) \) are large (overlap problem). Also, they take complex values (sign problem). Therefore, such a simulation works only when \( \mu_q \) and \( V \) are small.

To have a better overlap, the reweighting may be generalized towards not only the \( \mu_q \) direction but also the \( \beta \) direction [113]:

\[
\langle \mathcal{O} \rangle_{\tilde{\mu}_q} = \langle \mathcal{O} \cdot R(\mu_q, \delta \beta) \rangle_{\tilde{\mu}_q=0} \langle R(\mu_q, \delta \beta) \rangle_{\tilde{\mu}_q=0}^{-1}, \tag{23}
\]

where the multi-parameter reweighting factor is defined as \( R(\mu_q, \delta \beta) \equiv R(\mu_q) \exp(-\delta \beta S) \) with \( \delta \beta = \beta - \beta_0 \). Here \( \beta_0 \) should be chosen to maximize the overlap. The physical temperature \( T \) and \( \beta = 2N_c/g^2 \) are implicitly related through the lattice spacing \( a \). Then, the reweighting along the phase boundary in the \( \mu_q-T(\beta) \) plane (as sketched in the right panel of figure 6) would have a better chance to probe into larger \( \mu_q \).

- **Taylor expansion method.** The full evaluation of the reweighting factor \( R(\mu_q) \) is not an easy task even on a computer. If \( \mu_q \) is small enough, the right-hand side of (22) can be expanded in terms of \( \mu_q/T \) [114–116]:

\[
\langle \mathcal{O} \rangle_{\mu_q} = \sum_{n=0}^{\infty} c_n \left( \frac{\mu_q}{T} \right)^n, \tag{24}
\]

The coefficients \( c_n \) are written in terms of the quark propagator and can be simulated at \( \mu_q = 0 \). Because the expansion is based on (22), the original overlap problem translates into the convergence problem in higher order terms; the Taylor expansion makes sense for \( \mu_q/T \) within a radius of convergence dictated by the singularity closest to the origin in the complex \( \mu_q/T \)-plane.

- **Imaginary chemical potential method.** If \( \mu_q \) is purely imaginary (which is denoted here by \( \tilde{\mu}_q \)), (21) reduces to the \( \gamma^5\)-Hermiticity, so that there is no sign problem [117,118]. Then, it is possible to perform lattice simulations to find physical observables by the analytic continuation back to the real chemical potential:

\[
\langle \mathcal{O} \rangle_{\tilde{\mu}_q} = \sum_{n=0}^{\infty} c_n \left( \frac{\tilde{\mu}_q}{\mu_0} \right)^n \rightarrow \langle \mathcal{O} \rangle_{\mu_q} = \sum_{n=0}^{\infty} c_n \left( \frac{-i\mu_q}{\mu_0} \right)^n. \tag{25}
\]

The applicability of this method is bounded by singularities or periodicity. For imaginary chemical potential, \( \tilde{\mu}_q/T \) plays a role of an angle variable which has naive periodicity by 2\( \pi \). However, the partition function is a function of \( \tilde{\mu}_q + gA_4 \) and a change in \( \tilde{\mu}_q \) by \( 2\pi/N_c \) can be absorbed by the centre transformation (3). Therefore, the actual period of \( \tilde{\mu}_q/T \) is \( 2\pi/N_c \) (Roberge–Weiss (RW) periodicity) [119]. In the deconfinement phase at high \( T \), in particular, there is a first-order phase transition at half of the RW point; \( \tilde{\mu}_q/T = \pi/N_c \sim 1 \) for \( N_c = 3 \). Thus the method of imaginary chemical potential works up to \( \tilde{\mu}_q/T \lesssim 1 \) at best, which is also confirmed in model studies [120].

- **Canonical ensemble method.** In the thermodynamic limit the canonical ensemble with a fixed particle number \( N_q \) is equivalent to the grand canonical ensemble with fixed chemical potential \( \mu_q \) (and the mean value of \( N_q \) is specified). To convert the grand canonical to the canonical description, the Fourier transform in terms of the
imaginary chemical potential is necessary [117, 121–125]:

\[ \langle O \rangle_{N_i} = \frac{1}{Z} \int d\phi e^{iN\phi} \langle O \rangle_{\mu_\infty=\phi T}. \tag{26} \]

In this case, the difficulty of the sign problem is transferred to the integration with respect to \( \phi = \mu_\infty/T \). This canonical approach works fine as long as the volume is not large, for which centre symmetry is forced to be restored for \( N_q \) that is a multiple of \( N_c \) [77]. It is a highly delicate procedure to take the correct thermodynamic limit \( V \to \infty \) in which not \( N_q \) but \( n_q = N_q/V \) should be kept fixed. In particular, the order of the phase transition is sensitive to how the thermodynamic limit is approached [126].

- **Density of states method.** There are several variants of the density of states method depending on the choice of a variable to rewrite the partition function. Here we take an example of a phase \( \theta \) of the Dirac determinant [127, 128]. (The plaquette \( P \) or energy \( E \) is useful as well [129–131].)

The density of states in this case reads as

\[ \rho(\theta) = \langle \delta(\theta - \theta(U)) \rangle_{\mu_\infty}. \tag{27} \]

where the expectation value is taken by the phase quenched simulation in which \( \det F(\mu_\infty) \) is replaced by \( \det F(\mu_\infty) \) to avoid the sign problem. Using \( \rho(\theta) \) one can express the expectation value of an operator \( O \) as

\[ \langle O \rangle_{\mu_\infty} = \frac{1}{Z} \int d\theta \rho(\theta) e^{i\theta} \langle O \rangle_{\theta}. \tag{28} \]

with \( Z = \int d\theta \rho(\theta) e^{i\theta} \) and

\[ \langle O \rangle_{\theta} = \frac{1}{\rho(\theta)} \langle O \cdot \delta(\theta - \theta(U)) \rangle_{\mu_\infty}. \tag{29} \]

Both \( \rho(\theta) \) and \( \langle O \rangle_{\theta} \) are quantities calculable in the lattice simulation. The difficulty of the sign problem is now translated into the precise determination of the density of states; (21) implies that the phase quenched simulation at \( \mu_\infty \neq 0 \) with two degenerate flavours is identical to the simulation at a finite isospin chemical potential \( \mu_1(=\mu_\infty) \):

\[ |\det F(\mu_\infty)|^2 = |\det F(\mu_\infty)\det F(-\mu_\infty)|. \tag{30} \]

Then, the phase quenched expectation value goes through a qualitative change as \( \mu_\infty \) increases; an exotic phase with pion condensation appears for \( \mu_\infty > m_\pi/2 \) [132]. For small \( \mu_\infty \), both lattice-QCD simulation [128] and an analysis in the \( \chi PT \) [133–135] show that \( \rho(\theta) \) behaves as Gaussian, while \( \rho(\theta) \) becomes Lorenzian for \( \mu_\infty > m_\pi/2 \) in the \( \chi PT \). In the latter case, extremely precise determination of \( \rho(\theta) \) and \( \langle O \rangle_{\theta} \) is crucially important. So far, the density of states method is applied for relatively small values of \( \mu_\infty/T \) combined with the Taylor expansion of the fermion determinant [128, 136].

- **Complex Langevin method.** The sign problem arises from the importance sampling to deal with the multi-dimensional functional integral. Therefore, it may not appear in different quantization schemes other than the functional integral. The stochastic quantization [137–141] formulated in terms of the Langevin dynamics with a fictitious time is a promising candidate for such an alternative. If we consider a scalar field theory defined by an action \( S[\phi] \), the Langevin equation reads as

\[ \frac{\partial \langle \phi(x, s) \rangle}{\partial s} = -\frac{\delta S[\phi]}{\delta \langle \phi(x, s) \rangle} + \eta(x, s), \tag{31} \]

where \( s \) is a fictitious time and \( \eta \) a Gaussian noise; \( \langle \eta(x, s) \rangle = 0 \) and \( \langle \eta(x, s)\eta(x', s') \rangle = 2\delta(x-x')\delta(s-s') \). The expectation value is taken over the \( \eta(x, s) \) distribution and the equilibrated value is obtained in the \( s \to \infty \) limit. At a finite \( \mu_\infty \) the action and noise are complex. The complex Langevin dynamics can evade the sign problem and correctly describe the system at a finite density for some simple models, while there are also cases where it may fall into a wrong answer [142, 143]. Whether this method is applicable to dense QCD or not is still an open question.

It would be an important milestone if the lattice-QCD simulation can show the existence/non-existence of the QCD critical point in the \( \mu_B-T \) plane. In [113, 144] the location of the Lee–Yang zero in the complex lattice-coupling (\( \beta = 2N_c/g^2 \)) plane has been investigated using the multi-parameter reweighting for 2 + 1-flavour staggered fermions. If the thermal transition is of first order, the Lee–Yang zero nearest to the real axis in a finite lattice box is expected to approach the real axis as the lattice volume increases, while there is no such tendency for the crossover transition. It was then concluded that the QCD critical point is located at \( (\mu_B, T) = (\mu_E, T_E) = (360 \pm 40 \text{ MeV}, 162 \pm 2 \text{ MeV}) \) for physical quark masses [144]. It has been argued, however, in [145] that the sign problem at a finite baryon density can fool the Lee–Yang zero near the real axis. It is also suggested to study the behaviour of a set of Lee–Yang zeros in the complex \( \beta \) plane to make a firm conclusion.

If we assume that the convergence of the Taylor expansion in terms of \( \mu_\infty/T \) is dictated by the singularity at the critical point, the radius of convergence is an indicator of the location of the critical point. In [116] it was reported that coefficients up to sixth order yield \( T_B/T_{pc} = 0.94 \pm 0.01 \) and \( \mu_E/T_{pc} = 1.8 \pm 0.1 \) for 2-flavour staggered fermions with \( m_\pi/m_p = 0.3 \). In the same way, with 2 + 1-flavour staggered fermions, the radius of convergence is discussed in [146], though the results are not conclusive. The idea of the radius of convergence has also been tested in a chiral effective model coupled with the Polyakov loop, in which the exact location of the critical point and the higher order coefficients are calculable in the mean-field approximation [147]. The result from the model test suggests that the relation between the convergence radius of the expansion and the location of the critical point is not clear.

In [148] the canonical partition function is extracted using the density of states method with an assumption of the Gaussian distribution of the phase \( \theta \) of the Dirac determinant. In this case the standard S-shaped curve in the \( \mu_B-\eta_B \) plane is a signature of the first-order transition. The location of the critical point...
is estimated to be $T_\phi/T_{pc} \approx 0.76$ and $\mu_B/T_{pc} \approx 7.5$ for 2-flavour staggered fermions with $m_\pi \simeq 770$ MeV. In [125] the canonical ensemble method is used to study the $S$ shape and it was found that $T_\phi/T_{pc} = 0.94 \pm 0.03$ and $\mu_B/T_{pc} = 3.01 \pm 0.12$ for 3-flavour clover fermions with $m_\pi \simeq 700$ MeV.

As shown in figure 5 one may consider the critical surface $\mu_B = \mu_B(m_{ad}, m_f)$ and its behaviour near $\mu_B = 0$ to check whether the critical point is located in the small $\mu_B$ region. In particular, on the SU(3)$_V$ symmetric line ($m_{ad} = m_f$), one can define the critical mass as a function of $\mu_B$ and make a Taylor expansion,

$$\frac{m_c(\mu_B)}{m_c(0)} = 1 + c_2 \left( \frac{\mu_B}{T_{pc}} \right)^2 + c_4 \left( \frac{\mu_B}{T_{pc}} \right)^4 + \cdots. $$ (32)

The sign of $c_2$ and $c_4$ determines whether the critical surface behaves like the standard scenario or the exotic scenario in figure 5. Simulations with 3-flavours of staggered fermions on a coarse lattice $(N_t = 4)$ with the imaginary chemical potential method showed $c_2 = -3.3(3)$ and $c_4 = -47(20)$ [149], which is consistent with the exotic scenario in figure 5. This does not, however, exclude a possibility that the critical point exists for a large value of $\mu_B/T$. Confirmation of this result with different fermion formulations with a smaller lattice spacing is necessary to draw a firm conclusion [150].

In summary, in any method, the validity of QCD simulations at a finite baryon density is still limited in the region $\mu_B/T < 1$ at present because of the sign problem. Although some of the lattice-QCD simulations suggest the existence of the QCD critical point in the $\mu_B$–$T$ plane, the results are to be taken with care if it is predicted at a large $\mu_B/T$.

5.2. Effective U(1)$_{\chi}$ symmetry restoration

For $N_t = 3$ the first-order transition at a finite $T$ is driven by the trilinear chiral condensates from the U(1)$_{\chi}$-breaking KMT-type interaction with the strength $c_0$ as discussed in section 4.1. In the instanton calculation $c_0$ is proportional to the instanton density (18) which is screened by the medium at high $T$ and $\mu_B$ as

$$n_{ad}(T, \mu_B) = \left( \frac{8\pi^2}{g^2} \right)^{2N_c} e^{-8\pi^2/g^2\rho^2} \rho^{-5} \times \exp \left[ -\pi^2 \rho^2 T^2 \left( \frac{2N_c}{3} + \frac{N_f}{3} \right) - N_f \rho^2 m_\pi^2 \right].$$ (33)

In the one-loop calculation with the instanton size whose empirical value is $\rho \sim 0.3$ fm [151]. It is thus expected that the U(1)$_{\chi}$-breaking interaction is exponentially suppressed as $T$ and/or $\mu_B$ grow larger. If one performs the $\rho$-integration (which is not IR divergent thanks to the exponential suppression), the suppression factor would be $T^{-14}$ or $\mu_B^{-14}$ for $N_c = N_t = 3$ by the dimensional reason. It is a non-trivial question whether the instanton-induced interaction drops by either exponential or power function [152]. The Columbia plot with an in-medium reduction of U(1)$_{\chi}$ breaking should have a smaller region for the first-order phase transition. This has been confirmed at $\mu_B = 0$ [153] in the Polyakov-loop extended chiral models such as the PNJL model [154, 155] and the PQM model [156]. Consequently the location of the QCD critical point at a finite $\mu_B$ would be very sensitive to the strength $c_0$ [153, 157, 158]. The negative coefficients $c_2$ and $c_4$ in (32) might be attributed to the effective U(1)$_{\chi}$ restoration at a larger $\mu_B$ [159]; it has been found that the chiral model analysis with an exponential ansatz, $c_0 \propto e^{-N_f^2 m_\pi^2}$, leads to a scenario similar to the right panel of figure 5.

5.3. QCD critical point search

In this subsection we summarize physical consequences of the critical point under the assumption that it exists in the QCD phase diagram.

Figure 7 is a schematic illustration of the shape of the effective potentials in the crossover and first-order transitions. If the critical point is approached along the first-order phase boundary, the associated potential shape appears like that of the second-order phase transition around a non-vanishing order parameter. The critical properties at the critical point can be mapped to the 3D Ising model with Z(2) symmetry with the (reduced) temperature $t$ and the external field $h$. On the $\mu_B$–$T$ plane in the QCD phase diagram the mapped $t$-direction is tangential to the first-order phase boundary because Z(2) symmetry at the critical point is locally preserved along this direction. The determination of the $h$-direction, on the other hand, requires microscopic calculations.

In the vicinity of the critical point along the $t$-direction the correlation length $\xi$, the chiral condensate $\sigma$ and the chiral susceptibility $\chi_\sigma$ have the following scaling:

$$\xi \sim t^{-\gamma}, \quad \sigma = \langle \bar{\psi} \psi \rangle \sim t^\beta, \quad \chi_\sigma \sim t^{-\gamma},$$ (34)

where $\nu \simeq 0.63$, $\beta \simeq 0.33$ and $\gamma \simeq 1.24$ are known critical exponents in the 3D Ising model. Because chiral symmetry is explicitly broken by $m_q \neq 0$, the chiral condensate at the critical point is non-vanishing and takes a finite $\langle \bar{\psi} \psi \rangle_0$. In the same way, along the $h$-direction, they scale as

$$\xi \sim h^{-\nu/\beta}, \quad \sigma \sim h^{1/\beta}, \quad \chi_\sigma \sim h^{-1+4/\beta}.$$ (35)
where $\delta \approx 4.8$. It is also possible to parametrize the singular part of the equation of state in a similar way as to the 3D Ising model [160].

The divergent correlation length $\xi$ can be interpreted physically as the vanishing screening mass in the $\sigma$-meson channel, which may have significant consequences in the HIC experiments [49, 50]. However, one should keep in mind that the pole mass in the $\sigma$ channel can never be massless since it is not directly related to $\xi$ [161, 162].

In a finite-density medium there is a mixing between the chiral condensate and the baryon density. Therefore the divergence in the chiral susceptibility can also be seen in the baryon number (36).

\[
\langle (\delta N)^2 \rangle_c \sim \xi^{(5-\eta)/2-3},
\]

where $\langle \cdots \rangle_c$ represents the connected piece of the correlation function. Naturally the higher moments have stronger divergence in the vicinity of the critical point where $\xi \to \infty$. On the other hand, they require a large subtraction like $\langle \cdots \rangle_c$.

5.4. First-order phase transition at high baryon density

We now turn to the chiral phase transition at low temperature and high baryon density where lattice simulations are not reachable yet. In this region, only qualitative analyses based on various chiral effective theories are available at present. Many of these models predict a first-order transition in cold and dense matter; we are going to extract some common features in these models based on a quasi-particle picture at $\mu_q \neq 0$.

Let us introduce a vacuum pressure $P_{\text{vac}}[M_q]$, which embodies the dynamical chiral symmetry breaking, as a function of $M_q$ (constituent quark mass) as

\[
P_{\text{vac}}[M_q] = -a(M_0^2 - M_q^2)^2,
\]

with a positive curvature parameter $a$. The maximum of $P_{\text{vac}}[M_q]$ is realized when $M_q = \pm M_0 \neq 0$. (If the current quark mass $m_q$ is non-zero, a linear term in $M_q$, which favours $M_q > 0$ at the maximum of $P_{\text{vac}}[M_q]$, should be present.) At zero temperature, as long as $\mu_q$ is smaller than $M_q$, nothing happens and the vacuum remains empty, while $n_B$ starts arising for $\mu_q > M_q$. The pressure from finite $\mu_q$ is expressed as

\[
P_{\mu}[M_q] = \int_0^{\mu_q^*} d\mu_q \langle n_q(\mu_q)\rangle
\]

\[
= -\frac{v}{6\pi^2} \int_0^{\mu_q^*} d\mu_q \langle (\mu_q^2 - M_0^2)^{3/2} \theta(\mu_q^2 - M_0^2)\rangle
\]

where $n_q(\mu_q)$ represents the fermion density in the quasi-particle approximation and $v = (\text{spin}) \times (\text{colour}) \times \widehat{\text{flavour}}$ is the fermion degeneracy. Because more particles can reside in the Fermi sphere for a smaller mass, $P_{\mu_q}[M_q]$ naturally has a maximum at $M_q = 0$ and goes to zero at $M_q = \mu_q$.

Here $P_{\text{vac}}[M_q]$ and $P_{\mu_q}[M_q]$ have a peak at $M_q = M_0$ and $M_q = 0$, respectively. Assuming that the total pressure is simply $P[M_q] = P_{\text{vac}}[M_q] + P_{\mu_q}[M_q]$, we can see that the existence of two separate peaks in $P[M_q]$ requires [166]

\[
a < \frac{v}{16\pi^2} \frac{\mu_q^2}{M_0^2} \lesssim \frac{v}{16\pi^2} \simeq 0.076,
\]

for 2-flavour case with $v = 12$. At the first-order critical point the peak at $M_q = 0$ is just as high as the second peak at $M_q \simeq M_0$, which determines the critical chemical potential,

\[
a \simeq \frac{v}{24\pi^2} \frac{\mu_c^4}{M_0^4}.
\]

Once $a$ satisfies (42), the chiral phase transition with increasing $\mu_q$ at $T = 0$ should be of first order at $\mu_q \simeq \mu_c$.

The actual value of $a$ is model-dependent: in the NJL model with two flavours and in the linear-$\sigma$ model, $a$ is estimated, respectively, as

\[
a = \begin{cases} \frac{1}{2M_0^2} \left( \frac{\nu A^2}{8\pi^2} - \frac{1}{4G_S} \right) = 0.067 & \text{(NJL model)}, \\ \frac{M_0^2 f_0^2}{8M_0^4} = 0.02 \sim 0.05 & \text{(linear-$\sigma$ model)}, \end{cases}
\]
where we used the standard NJL parameters; \( \Lambda = 631 \text{ MeV} \), \( G_S \Lambda^2 = 2.19 \) and the resultant \( M_0 = 336.2 \text{ MeV} \) [10]. The uncertainty in the linear-\( \sigma \) model comes from the choice of the \( \sigma \) meson mass. Then, in both cases, the estimated uncertainty in the linear-density-density type can totally wash out the first-order phase transition. The critical chemical potential deduced from (43) is, in the NJL model case, given by \( \mu_\text{c} = 1.07 M_0 \approx 360 \text{ MeV} \) which is consistent with that obtained numerically in the NJL model.

Let us now argue that the first-order transition obtained as above is rather sensitive to the choice of the model Lagrangian. Indeed, it has been known that the contribution to the pressure of the form \( +G_V n_\pi^2 \) induced by a quark interaction of density-density type can totally wash out the first-order transition [166–171]. For example, in the NJL model with the density-density interaction, condition (42) is changed to

\[
\alpha < \frac{\nu}{16\pi^2} \left( 1 - \frac{2\nu G_V M_0^2}{3\pi^2} \right),
\]

which implies that the first-order phase transition does not arise for \( G_V > 0.25 G_S \) [166, 169, 170].

### 6. Formation of the diquark condensate

Finding a ground state of quark matter at \( T \approx 0 \) with extremely large value of \( \mu_\text{q} \) is an interesting theoretical challenge. (We use \( \mu_\text{q} \) instead of \( \mu_\text{B} \) throughout this section, for our central interest is the quark degrees of freedom). Let us consider the Cooper’s stability test in quark matter [60, 61]. In the perturbative regime of QCD the one-gluon exchange potential is proportional to a product of the quark SU(3) charges:

\[
(t^a)_{\alpha \beta} (t^b)_{\alpha' \beta'} = - \frac{N_c + 1}{4N_c} (\delta_{ab} \delta_{\alpha \beta'} - \delta_{ab'} \delta_{\alpha \beta}) + \frac{N_c - 1}{4N_c} (\delta_{ab} \delta_{\alpha' \beta'} + \delta_{ab'} \delta_{\alpha' \beta}).
\]

The first term on the right-hand side of (46) with a negative sign is anti-symmetric under the exchange of colour indices; \( \alpha \leftrightarrow \alpha' \) or \( \beta \leftrightarrow \beta' \), so that a quark pair in the colour anti-triplet channel has attraction. The second term on the right-hand side of (46) with a positive sign is symmetric under the same exchange of colour indices, so that a quark pair in the colour sextet channel has repulsion. A Fermi system with two particles attracting each other on a sharp Fermi sphere has an instability towards the formation of Cooper pairs. Therefore, the normal quark matter inevitably becomes the colour-superconducting (CSC) phase with the diquark condensate at an asymptotically high density and at a sufficiently low temperature [60, 61, 172–174].

Since quarks carry not only spin but also colour and flavour, various pairing patterns are possible. Hereafter we use the following notation: the colour indices \( \alpha, \beta, \gamma \) run from 1 to 3 meaning \( r \) (red), \( g \) (green) and \( b \) (blue) in order, and in the same way the flavour indices \( i, j, k \) run from 1 to 3 meaning \( u \) (up), \( d \) (down) and \( s \) (strange) in order. We will focus our attention to quark matter below the charm threshold, i.e. \( \mu_\text{q} < m_\text{charm} \), so that we do not take into account heavy flavours (\( c, b \) and \( t \)).

#### Spin-zero condensate

The quark pairing with zero total spin is characterized by the following order parameter with \( 3 \times 3 \) matrix structure [88]:

\[
(d')_{\alpha i} \sim \epsilon_{\alpha \beta \gamma} \epsilon_{ijk} \langle \psi_5^\dagger \gamma_5 \psi_{5j} \rangle.
\]

The quark pair in this case is in the colour anti-symmetric (anti-triplet), flavour anti-symmetric (anti-triplet) and spin anti-symmetric channel to satisfy the Fermi statistics. The charge conjugation matrix \( C = i\gamma^5 \gamma^0 \) together with \( \gamma^5 \) makes the above condensate a Lorentz scalar. (The effect of instantons and also quark masses favours the scalar condensate instead of the pseudo-scalar condensate.) If the masses of \( u, d \) and \( s \) quarks are all degenerate, there is an exact flavour SU(3) symmetry. Then, one can always diagonalize the order parameter by the bi-unitary rotation in colour and flavour space to obtain \( (d')_{ia} = \delta_{ia} \Delta_i \). We will adopt this as an ansatz even when \( u, d \) and \( s \) quarks are not degenerate, which is in practice a good approximation [175, 176].

Let us introduce notation \( \Delta_{dS} = \Delta_1, \Delta_{SU} = \Delta_2, \Delta_{ud} = \Delta_3 \) to indicate which quarks are involved in the Cooper pairing. If all the gaps are non-vanishing \( (\Delta_{ud} \neq 0, \Delta_{dS} \neq 0 \text{ and } \Delta_{SU} \neq 0) \), such a state is called the colour-flavour locked (CFL) phase [88] because the colour and flavour degrees of freedom are entangled with each other. For \( \Delta_{dS} = 0 \) with other components non-vanishing, it is called the uSC phase since both \( \Delta_{ud} \) and \( \Delta_{SU} \) contain the \( u \) quark. The dSC and sSC phases are defined in the same way [176–178]. If only \( \Delta_{ud} \) is non-vanishing, it is called the 2SC (two-flavour superconducting) phase. In the case where we refer to similar phases with other flavour combinations, we write the 2SCds, 2SCsu or 2SCud(\( =2SC \)) phase [175, 179]. When all the pairing gaps are absent, the system is in the normal state of quark matter (NQM). We summarize these abbreviations in Table 3.

### Table 3. Classification of the colour-superconducting phases.

| Pairing | CFL | uSC | dSC | sSC | 2SC | 2SCds | 2SCsu | NQM |
|---------|-----|-----|-----|-----|-----|-------|-------|-----|
| \( \Delta_{ud} \) | \( \bigcirc \) | \( \bigcirc \) | \( \bigcirc \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \Delta_{dS} \) | \( \bigcirc \) | \( \times \) | \( \bigcirc \) | \( \bigcirc \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \Delta_{SU} \) | \( \bigcirc \) | \( \times \) | \( \bigcirc \) | \( \times \) | \( \times \) | \( \times \) | \( \bigcirc \) | \( \times \) |

8 A similar comment on the quotient groups \( (G') \) and \( (\mathcal{H}') \) to that given to (9) and (10) is applied here.
Here the baryon number symmetry is broken to leave its discrete subgroup $Z(2)_V$ which corresponds to a reflection $\psi \mapsto -\psi$. Also, (global) colour symmetry and chiral symmetry are broken simultaneously to leave their diagonal subgroup $\text{SU}(3)_{C\leftrightarrow R}$ intact. Note that the electromagnetic symmetry $U(1)_{\text{em}}$ associated with the electric charge $Q = \text{diag}(2/3, -1/3, -1/3)$ happens to be a subgroup of the vector symmetry $\text{SU}(3)_{L\leftrightarrow R}$. In the CFL phase $U(1)_{\text{em}}$ survives as a modified symmetry, $U(1)_{\text{em}+C}$, which contains simultaneous electromagnetic and colour rotation. As a result, seven gluons and one gluon–photon mixture acquire finite Meissner mass by the Anderson–Higgs mechanism, while the other photon–gluon mixture remains massless.

The 2SC phase is quite different from the CFL phase in the sense that only $u$ and $d$ quarks participate in the pairing with a chiral flavour-singlet combination. The symmetry breaking pattern in this case for massless 2-flavours with infinitely heavy strange quarks reads as [15]

$$\mathcal{G} = \text{SU}(3)_C \times \text{SU}(2)_L \times \text{SU}(2)_R \times U(1)_L \times Z(4)_A,$$

(50)

$$\mathcal{H} = \text{SU}(2)_C \times \text{SU}(2)_L \times \text{SU}(2)_R \times U(1)_{C\leftrightarrow B}. \quad (51)$$

Note that the chiral symmetry remains intact while the colour symmetry is broken, so that five gluons (4th to 8th) receive finite Meissner mass. In the 2SC phase, the baryon number symmetry $U(1)_B$ survives as a modified symmetry, $U(1)_{C\leftrightarrow B}$, which contains simultaneous electromagnetic and 8th colour rotation. The global electromagnetism, which is originally a combination of the baryon number symmetry and the isospin symmetry, also survives as a modified symmetry, $U(1)_{\text{em}+C}$. Therefore, the 2SC phase is neither a superfluid nor an electromagnetic superconductor. The modified symmetries discussed above remain unbroken in the uSC and dSC phases as well.

The CFL phase is reminiscent of what is called the B phase of superfluid $^3\text{He}$ in which fermionic $^3\text{He}$ atoms have a spin-triplet ($S = 1$) and $p$-wave ($L = 1$) pairing. The B phase is a state in which the orbital angular momentum $L$ and the spin $S$ are locked so that the symmetry breaking pattern is $\text{SO}(3)_S \times \text{SO}(3)_L \times U(1)_B \rightarrow \text{SO}(3)_{S+1}$, where $U(1)_B$ is a symmetry corresponding to $U(1)_{C/\text{em}}$ in quark matter.

**Spin-one condensate.** There is also a possibility of flavour symmetric Cooper pairs such as the pairing within the same flavour [172]. In this case $s$-wave Cooper pair must be spin-triplet ($J = L + S = 0 + 1 = 1$) to satisfy the Fermi statistics.

Various forms of the spin-one CSC states have been studied so far [179–182]. Since the Cooper pair is tripllet in both colour and spin, the order parameter becomes a complex $3 \times 3$ matrix, which is similar to the situation in the CFL phase where the Cooper pair is triplet in both colour and flavour. The explicit form would be

$$(\tilde{d})_{\alpha i} \sim \epsilon_{\alpha \beta \gamma} (\psi_{\beta}^\dagger C \gamma^\dagger \psi_{\gamma}),$$

(52)

where $i$ refers to not the flavour but the Lorentz index. The colour-spin locked (CSL) ansatz, $$(\tilde{d})_{\alpha i} \propto \delta_{i\alpha},$$

leads to a symmetry breaking pattern which is similar to the CFL phase in CSC and to the B phase in superfluid $^3\text{He}$:

$$\text{SU}(3)_C \times \text{SU}(2)_J \times U(1)_{\text{em}} \rightarrow \text{SU}(2)_C,$$  

(53)

On the other hand, if we take an ansatz, $$(\tilde{d})_{\alpha i} \propto \delta_{i\alpha}(\delta_{1\alpha} + i\delta_{2\alpha}),$$

we have a symmetry breaking pattern,

$$\text{SU}(3)_C \times \text{SU}(2)_J \times U(1)_{\text{em}} \rightarrow \text{SU}(2)_C \times U(1)_{\text{em}+C} \times \text{SU}(2)_C,$$  

(54)

which is analogous to the A phase of superfluid $^3\text{He}$ in which the symmetry breaking pattern is $\text{SO}(3)_S \times \text{SO}(3)_L \times U(1)_\phi \rightarrow U(1)_S \times U(1)_{C\leftrightarrow B}$. Other pairing patterns such as the polar phase, the planar phase and so on have also been investigated [182].

**Weak-coupling results.** In the asymptotically high-density region, the running coupling constant of the strong interaction becomes small enough to justify the weak-coupling calculations, so that the diquark condensate as well as the gap energy of quarks can be estimated from the first-principles QCD calculation. The driving force of the diquark condensate is the interaction between quarks due to gluon exchange. The chromoelectric part of the interaction in high-density quark matter is screened by the Debye mass $m_D = \sqrt{N_f/(2\pi^2)} g \mu_q$, while the chromomagnetic part of the interaction is screened only dynamically by Landau damping [183]. Therefore, instead of having the standard BCS form, $\Delta \sim \mu_q \exp(-\text{const.}/g^2)$, the gap energy at the Fermi surface is parametrically enhanced due to large forward scattering between quarks [180, 183–186]:

$$\Delta_F = 512(\pi^2/3) \left(\frac{2}{N_f}\right)^{5/2} e^{-(\pi^2 g^2)/(\lambda_1 \lambda_2^{5/3})} \lambda_{a1}^{-1/2} g \mu_q$$

$$\times \exp \left(-\frac{3\pi^2}{\sqrt{2g}}\right),$$

(55)

where $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_2 = 0$ for the 2SC phase, while $\lambda_1 = 4$ and $\lambda_2 = 1$ with $\lambda_1 = 1/3$ and $\lambda_2 = 2/3$ in the CFL phase. In the exponent of (55), an unusual dependence on the coupling $g^2$ instead of the standard dependence $g^{-2}$ enhances the gap substantially in the weak-coupling regime both for 2SC and CFL phases. We see that $\Delta_{2SC}$ in the 2SC phase is greater than that in the CFL phase as $\Delta_{2SC} = 2^{1/3} \Delta_{CFL}$, which holds not only in the weak-coupling QCD calculation but also in effective model calculations. Similarly to the BCS theory, the gap energy at zero temperature and the critical temperature are related to each other in the mean-field approximation [180]:

$$T_c = (\lambda_1^{a1} \lambda_2^{a2})^{1/2} \Delta_F \frac{\lambda_{a1}}{\pi},$$

(56)

Note that the factor $(\lambda_1^{a1} \lambda_2^{a2})$ on the right-hand side is cancelled by the same factor in (55), so that $\Delta_{2SC} > \Delta_{CFL}$ does not necessarily imply $T_c^{2SC} > T_c^{CFL}$.

### 6.1 Neutrality in electric and colour charges

For the bulk system to be stable, the total electric and colour charges must be zero due to the Gauss law [187–189]. In
some effective models, which do not have gauge fields as explicit degrees of freedom, the charge neutrality must be imposed by electric and colour chemical potentials, \( \mu_{ai} = \mu_q + \mu_Q (Q)_{ai} + \mu_3 (T)_{ai} + \mu_8 (T)_{ai} \). Their explicit forms read as

\[
\begin{align*}
\mu_{ru} &= \mu_q - \frac{1}{2} \mu_e + \frac{1}{2} \mu_3 + \frac{1}{2} \mu_8, \\
\mu_{gd} &= \mu_q + \frac{1}{2} \mu_e - \frac{1}{2} \mu_3 + \frac{1}{2} \mu_8, \\
\mu_{bs} &= \mu_q + \frac{1}{2} \mu_e - \frac{1}{2} \mu_3 - \frac{1}{2} \mu_8, \\
\mu_{rd} &= \mu_q - \frac{1}{2} \mu_e + \frac{1}{2} \mu_3 - \frac{1}{2} \mu_8, \\
\mu_{gu} &= \mu_q + \frac{1}{2} \mu_e - \frac{1}{2} \mu_3 + \frac{1}{2} \mu_8, \\
\mu_{rs} &= \mu_q + \frac{1}{2} \mu_e + \frac{1}{2} \mu_3 - \frac{1}{2} \mu_8, \\
\mu_{bu} &= \mu_q - \frac{1}{2} \mu_e + \frac{1}{2} \mu_3 + \frac{1}{2} \mu_8, \\
\mu_{gs} &= \mu_q + \frac{1}{2} \mu_e - \frac{1}{2} \mu_3 + \frac{1}{2} \mu_8, \\
\mu_{bd} &= \mu_q + \frac{1}{2} \mu_e - \frac{1}{2} \mu_3 - \frac{1}{2} \mu_8,
\end{align*}
\]

where the electron chemical potential \( \mu_e \) satisfies \( \mu_e = -\mu_Q \) due to \( \beta \)-equilibrium. These constraints add some variations in the physics of CSC because one needs to consider the Cooper pairing between quarks with mismatched Fermi surfaces.

For unpaired quark matter with \( m_{ud} = 0 \) and \( m_s \neq 0 \), the number of strange quarks is less than that of other quarks. There arise electrons in the system to maintain charge neutrality accordingly. The neutrality conditions obtained by the variation of the free energy, \( \partial \Omega_{unpaired} / \partial \mu_e = 0 \) and \( \partial \Omega_{unpaired} / \partial \mu_3 = \partial \Omega_{unpaired} / \partial \mu_8 = 0 \), lead to \( \mu_3 = \mu_8 = 0 \) and \( \mu_e = m_e^2 / (4 \mu_q) \). With the free dispersion relation at large chemical potential, \( \epsilon = \pm |p| \) and \( \epsilon = \sqrt{|p|^2 + m_e^2 + m_\eta^2 / (2 \mu_q)} \), the Fermi momenta have mismatches as

\[
\begin{align*}
p_{u}^d &= \mu_q = \frac{m_e^2}{6 \mu_q}, \\
p_{d}^d &= \mu_q = \frac{m_e^2}{12 \mu_q}, \\
p_{u}^s &= \mu_q = \frac{5 m_e^2}{12 \mu_q}.
\end{align*}
\]

This is illustrated in figure 8.

At a high density where the expansions in terms of \( m_s / \mu_q \) and \( \Delta / \mu_q \) are valid, model-independent conclusions can be drawn for the CSC quark matter at \( T = 0 \) [187, 188]. In the CFL phase the modified electric charge \( \tilde{Q} \) associated with the symmetry \( U(1)_{emc} \) is conserved. Also, the Cooper pairs have \( \tilde{Q} = 0 \), so that the CFL quark matter at \( T = 0 \) is a \( \tilde{Q} \)-insulator [190]. Thus, the CFL free energy \( \Omega_{\text{CFL}} \) is independent of the corresponding chemical potential except for a small contribution proportional to \( \mu_q^2 \) from \( \tilde{Q} \)-violating electrons. Under the neutrality conditions, \( \partial \Omega_{\text{CFL}} / \partial \mu_e = \partial \Omega_{\text{CFL}} / \partial \mu_3 = \partial \Omega_{\text{CFL}} / \partial \mu_8 = 0 \), we obtain

\[
\mu_3 = \mu_8 = 0, \quad \mu_e = \frac{m_e^2}{2 \mu_q}.
\]

It is an important property that the CFL phase (\( \Delta_{ud} \simeq \Delta_{ds} \simeq \Delta_{su} \) with \( \mu_e = 0 \)) is rigid against the Fermi surface mismatch [190] as long as the CFL phase is in the stable region; \( \Delta > m_\eta^2 / (2 \mu_q) \). We will discuss later what happens once this stability condition is violated in section 7.2.

In the 2SC phase the colour and charge neutralities lead to

\[
\mu_3 = \mu_8 = 0, \quad \mu_e = \frac{m_e^2}{2 \mu_q},
\]

so that electrons are required to be present with even larger chemical potentials than the normal quark matter. Also, the 2SC phase contains ungapped \( \tilde{Q} \)-carrying modes, and is therefore a \( \tilde{Q} \)-conductor.

6.2. Ginzburg–Landau approach

To identify the phase structure of CSC near second-order or weak first-order transitions at a finite \( T \), the Ginzburg–Landau approach is quite useful [61, 191]. If we assume that the gap energy is small compared with the critical temperature, the Ginzburg–Landau free energy with a power series expansion in terms of \( (d)_{\alpha} \) up to the quartic order reads as [177]

\[
\Omega = \alpha \text{tr} d d^\dagger + \beta_1 (|d d^\dagger|^2) + \beta_2 \text{tr}(d d^\dagger)^2 + \varepsilon e (|d d^\dagger|^2 + |d d^\dagger| |d d^\dagger|^2).
\]

where \( \text{tr} \) is taken in flavour space. For \( m_s = \mu_e = 0 \) the transition from the CFL phase to the normal quark matter is driven by the parameter \( \alpha \) changing the sign from negative to positive. The fourth term with \( \varepsilon \) takes care of the asymmetry between \( \{u, d\} \) and \( \{s, \bar{s}\} \) introduced by \( m_s \neq 0 \), while the fifth term with \( \eta \) represents the asymmetry between other quarks. From the charge neutrality effect by \( \mu_e \neq 0 \). The effect of colour neutrality is negligible near \( T_c \). Under the diagonal ansatz on the gap matrix, the free energy simplifies to

\[
\Omega = \alpha' \left( \Delta_{ud}^2 + \Delta_{ds}^2 + \Delta_{su}^2 \right) - \varepsilon \left( \Delta_{ud} - \Delta_{ds} - \Delta_{su} \right)^2 + \beta_2 \left( \Delta_{ud}^4 + \Delta_{ds}^4 + \Delta_{su}^4 \right)
\]

with \( \alpha' = \alpha + \eta \Delta_{ud} / 3 \). In the weak-coupling analysis one can show that \( \beta_1 = \beta_2 = 0 \) and \( \alpha = \alpha_0 (T - T_c) / T_c \), where \( \alpha_0 = 2 \mu_s^2 / \pi^2 \) and \( T_c \) is the critical temperature defined for \( \varepsilon = \eta = 0 \). In the weak-coupling limit of QCD with \( m_s \ll \mu_q \), it is shown that \( \varepsilon \approx 2 \mu_s \approx 2 \alpha_0 \delta \) with a parameter \( \delta \approx (m_s^2 / 2 \mu_q^2) \ln(\mu_q / T_c) \). Then the gap energies near \( T_c \) behave as

\[
\Delta_{ud}^2 (T) = \frac{\alpha_0}{8 \beta} \left( \frac{T_c - T}{T_c} + \frac{8}{3} \delta \right),
\]

\[
\Delta_{ds}^2 (T) = \frac{\alpha_0 \delta}{2 \beta}, \quad \Delta_{su}^2 (T) = \Delta_{ud}^2 - \frac{\alpha_0 \delta}{\beta}.
\]
The inequalities, $\Delta_{ud}(T) > \Delta_{ds}(T) > \Delta_{su}(T)$, imply sequential melting of CSC as $T$ increases at a high baryon density, i.e. CFL $\rightarrow$ dSC $\rightarrow$ 2SC $\rightarrow$ NQM. This scenario of melting pattern has been confirmed by explicit calculations in the NJL model [178].

The ordering $\Delta_{ud} > \Delta_{ds} > \Delta_{su}$ in (63) can be understood in an intuitive manner. Because $T$ is close to $T_c$, the Fermi surface is blurred and the mismatch of the Fermi momentum no longer imposes a pressure to take the pairing apart. The magnitude of the gap energy is then determined by the density of states. For example, the magnitude of the density of states for the pairing $\Delta_{ud}$ is related to the size of the ‘averaged’ Fermi surface over $u$ and $d$ quarks. The Fermi surface ordering as shown in figure 8 gives rise to the ordering of the averaged Fermi surfaces as shown in figure 9, which explains the ordering $\Delta_{ud} > \Delta_{ds} > \Delta_{su}$ near $T_c$.

In some model calculations in the literature, uSC instead of dSC appears even for extremely large $\mu_\Upsilon$. This can be caused by an artefact of the UV cutoff $\Lambda$ in the model: if $\Lambda$ is not chosen to be sufficiently large, $\epsilon$ and $\eta$ suffer from significant cutoff artefact, so that $\eta$ may even have an opposite sign from the QCD prediction [178]. On the other hand, as $\mu_\Upsilon$ becomes small and approaches $m_q$, the expansion in terms of $m_q/\mu_\Upsilon$ adopted for the computation of $\epsilon$ and $\eta$ in (61) is no longer valid, and the other melting pattern, CFL $\rightarrow$ uSC $\rightarrow$ 2SC $\rightarrow$ NQM, indeed becomes possible [178].

6.3. Quark–hadron continuity and $U(1)_A$ anomaly

The Ginzburg–Landau approach can be extended to incorporate the competition between the diquark condensate and the chiral condensate [192]. In this case it is necessary to distinguish the right-handed and left-handed diquark condensates $d_L$ and $d_R$ as given in (13) to construct properly the Ginzburg–Landau free energy with the symmetry $G$ in (48). The basic transformation properties of the fields $\Phi$, $d_L$, and $d_R$ are

$$
\Phi \rightarrow V_L \Phi V_R^\dagger, \quad d_L \rightarrow V_L d_L V_C, \quad d_R \rightarrow V_R d_R V_C',
$$

where $V_L$, $V_R$ and $V_C$ correspond to $U(3)_L$, $U(3)_R$ and $SU(3)_C$ rotations, respectively.

In a simple case where all $u$, $d$ and $s$ quarks are massless, the chiral part has a standard expansion in the same form as (16):

$$
\Omega_\Phi = \frac{\alpha}{2} \text{tr} \Phi^4 + \frac{\beta_1}{4!} (\text{tr} \Phi^1 \Phi^2)^2 + \frac{\beta_2}{4!} \text{tr}(\Phi^1 \Phi^2)^2 + \frac{\gamma}{2} \text{det} \Phi \text{det} \Phi^1.
$$

The term with the coefficient $c_0$ originates from the axial anomaly.

The diquark free energy up to the quartic order reads as

$$
\begin{align*}
\Omega_d &= \alpha \text{tr}(d_L d_L^\dagger + d_R d_R^\dagger) + \beta_1 \{[\text{tr} \ d_L d_L^\dagger]_2^2 + (\text{tr} \ d_R d_R^\dagger)\} \\
+ \beta_2 \{[\text{tr} \ d_L d_L^\dagger]_2^2 + (\text{tr} \ d_R d_R^\dagger)\} \\
+ \beta_3 \{[\text{tr} \ d_L d_L^\dagger d_L^\dagger]_1 + \beta_4 \text{tr} d_L d_L^\dagger d_R d_R^\dagger\}. \\
\end{align*}
$$

The transition from the CFL phase to normal quark matter is driven by the parameter $\alpha$ changing the sign from negative to positive. Unlike $\text{det} \Phi$ in (65), terms such as $d_L$ and $d_R$ are not allowed in (66) since they break $U(1)_B$.

The coupling between the diquark and the chiral condensates has the following general form up to the quartic order:

$$
\Omega_{\Phi d} = \gamma_1 \text{tr}(d_L d_L^\dagger \Phi + d_R d_R^\dagger \Phi^\dagger) \\
+ \lambda_1 \text{tr}(d_L d_L^\dagger \Phi^\dagger + d_R d_R^\dagger \Phi) + \lambda_R \text{tr} d_L d_L^\dagger \Phi + \lambda_3 (\text{det} \Phi \text{tr} d_L d_L^\dagger + \text{det} \Phi^\dagger \text{tr} d_R d_R^\dagger). \\
$$

The term with the coefficient $\gamma_1$ originates from the axial anomaly.

In the massless 3-flavour limit, one may assume $\Phi = \text{diag}(\sigma, \sigma, \sigma)$ and $d_L = -d_R = \text{diag}(\Delta, \Delta, \Delta)$ and then the sum of the above three pieces amounts to

$$
\Omega_{3f} = \left( a \sigma^2 - \frac{c}{3} \sigma^3 + \frac{b}{4} \sigma^4 \right) + \left( \frac{a}{2} \Delta^2 + \frac{\beta}{4} \Delta^4 \right)
$$

Here the $\sigma^3$ and the $\Delta^2\sigma$ terms originate from the axial anomaly. From the Fierz transform of the anomaly-induced KMT interaction in the quark level, the coefficients $c$ and $\gamma$ turn out to have the same sign [192]. Furthermore, the sign of $c$ and $\gamma$ can be taken to be positive without loss of generality because of the relation, $\Omega_{3f}(\sigma, \Delta; c, \gamma) = \Omega_{3f}(\sigma, \Delta; -c, -\gamma)$, which is a situation analogous to the quark mass term.

It should be noted that the $\Delta^2\sigma$ term with positive coefficient $\gamma$ favours the coexistence of the chiral and diquark condensates. Also, this term is linear in $\sigma$ and behaves as if it is an explicit chiral symmetry breaking term. In contrast, $\Delta^2\sigma^2$ term with positive coefficient $\lambda$ (the positive sign is supported by the weak-coupling calculation and the NJL model) disfavours the coexistence. Therefore, if the effect of $\gamma$ is sufficiently strong and $\Delta$ is sufficiently large, the first-order phase boundary, which normally separates the chiral symmetry breaking phase ($\sigma \neq 0$) and the CFL phase ($\sigma = 0$ and $\Delta \neq 0$), can be smeared out. In such a case the two phases are connected smoothly to each other and a critical point associated with this crossover appears as shown in figure 10.

The smooth crossover of the CFL phase and the Nambu–Goldstone (hadronic) phase may have a close connection to the idea of quark–hadron continuity [175, 193]. Since the axial anomaly tends to enhance (reduce) the first-order transition through the term proportional to $c$ ($\gamma$) in (68), it is a dynamical issue whether the phase diagram as sketched in figure 10 is realized or not in the real world [194–196].
### 6.4. Collective excitations

An interesting and related question associated with the U(1)$_\Lambda$ anomaly is the fate of collective excitations. Using the chiral effective Lagrangian approach [197], one finds the dispersion relations for the pions and kaons in the CFL phase with $m_\pi \neq 0$ and $\mu_\pi \neq 0$ [198, 199]:

$$\epsilon_{\pi(+)}(p) = \pm \mu_\pi + \sqrt{v^2 p^2 + M_\pi^2},$$

$$\epsilon_{K^0}(p) = \pm \mu_\pi + \frac{m^2}{2\mu_\pi} + \sqrt{v^2 p^2 + M_K^2},$$

$$\epsilon_{K^\pm}(p) = - \frac{m^2}{2\mu_\pi} + \sqrt{v^2 p^2 + M_K^2},$$

where $v^2 = 1/3$ at a high density. The CFL-meson masses are given by

$$M_{\pi^+}^2 = a(m_u + m_d)m_u + \chi (m_u + m_d),$$

$$M_{K^0}^2 = a(m_u + m_d)m_u + \chi (m_u + m_d),$$

$$M_{K^\pm}^2 = a(m_u + m_d)m_u + \chi (m_u + m_d).$$

Here $a = 3\Delta f_\pi^2/(\sigma^2 f_\pi^2)$ with $f_\pi^2 = (21 - 8\ln 2)\mu_\pi^2/(36\pi^2)$ at a high density and $\chi$ parametrizes the contribution of the U(1)$_\Lambda$ anomaly which generates $\langle \bar{\psi}\psi \rangle$ and therefore contributes to the CFL-meson masses.

In the absence of the U(1)$_\Lambda$-breaking term ($\chi = 0$), the energies for $K^+$ and $K^0$ become negative and kaon condensation occurs for $m_\pi \gtrsim m^{1/3}\Delta^{2/3}$ with $m$ being either $m_u$ or $m_d$. In particular, the electron contribution to the thermodynamic potential in the CFL phase favours the $K^0$ condensation (the CFL-$K^0$ phase) [200–203]. The phase structure with the CFL-$K^0$ state and its variants have been also investigated in the NJL-type model [204, 205]. The onset of the $K^0$ condensation depends on the strength of the U(1)$_\Lambda$ anomaly $\chi$ as evident from (70).

In view of (70) the meson masses have the ordering $M_{\pi^+} > M_{K^0} \approx M_{K^\pm}$ for $m_u \gg m_d \approx m_\pi$ and $\chi \approx 0$, which is inverse of the ordinary ordering in vacuum [198].

This is, however, natural from the diquark picture as already implied by the order parameter (14) in which CFL-$\sigma$ meson consists of two diquarks, $\bar{q}q$. The Nambu–Goldstone bosons are accordingly composed of $\bar{q}q$: CFL-$\pi^+$ contains a $d\bar{s}$ diquark that transforms like $u$ quark and an $su$ diquark like $\bar{d}q$, while CFL-$K^+$ a $d\bar{s}$ diquark and a $ud$ diquark like $\bar{s}q$. Therefore CFL-$K^+$ has a $d$ quark instead of an $s$ quark as compared with CFL-$\pi^+$ and thus it becomes lighter than CFL-$\pi^+$. The effect of $\chi \neq 0$ in (70), on the other hand, favours the standard ordering. This is because $\chi$ arises from the instanton interaction and induces a mixture of $\bar{q}q\bar{q}q$ and $\bar{q}q$, which is embodied in the $\Delta^2\sigma$ term in (68).

If the quark–hadron continuity is realized, not only the pseudo-scalar mesons but also the vector mesons and fermions would obey the spectral continuity. For example, the continuity of flavour-octet vector mesons at a low density and that of the colour-octet gluons at a high density together with the fate of the flavour-singlet vector meson have been investigated in the in-medium QCD sum rules [206].

In order to clarify the existence of another critical point associated with the quark–hadron continuity and the formation of $K^0$ condensation in the CFL phase, it is demanded as a theory task to quantify how much the U(1)$_\Lambda$ symmetry is effectively restored at finite $T$ and/or $\mu_q$.

### 7. Inhomogeneous states

In the intermediate density regions of the QCD phase diagram, the ground state may have inhomogeneity with respect to the condensates. The conventional $\pi^-$ and $\pi^0$ condensations induced by the $p$-wave pion–nucleon interaction in nuclear matter and in neutron matter are well-known examples [207]. In this section, we will address some of the proposed inhomogeneous phases associated with chiral transition and with colour superconductivity.

#### 7.1. Chiral-density waves

In various materials typically at low dimensions, the charge-density wave (CDW) [208] and the spin-density wave (SDW) [209] are realized as ground states. Their counterpart in quark matter is the chiral-density waves: such a possibility has been discussed in the large $N_c$ limit of QCD [210, 211] with a spatially modulated chiral condensate,

$$\langle \bar{\psi}(x)\psi(x) \rangle = \sigma \cos(2q \cdot x),$$

with $|q| = \mu_q$. Figure 11 is a schematic picture of the pairing of a quark and a quark-hole which makes an inhomogeneous condensate. The magnitude of the net momentum is given by $2\mu_q$. This pairing is favoured by the forward scattering with singular interaction $\sim 1/q^2$ at a high density. The conclusion of [211] is that this chiral-density wave state can be realized only for $N_c \gtrsim 1000N_f$. The reason why large $N_c$ is required is that the gluon interaction $\sim 1/q^2$ is IR screened (Thomas–Fermi screening and Landau damping) by the quark loops of $O(1/N_c)$.

In [212] a different ansatz in the 2-flavour case has been investigated:

$$\langle \bar{\psi}(x)\psi(x) \rangle = i\langle \bar{\psi}(x)i\gamma^5\tau_j\psi(x) \rangle = \sigma e^{2iq \cdot x}. \quad (72)$$

Here $\tau_j$ is a third component of the flavour Pauli matrices. This type of condensate is known to be realized as the chiral spiral...
in (1 + 1)-dimensional chiral models at a finite density [213]. Recently, the chiral spiral in QCD has been analysed with confinement effects taken into account [214]. The confining interaction $\sim 1/(q^2)^2$ is more IR singular than the perturbative interaction, so that the pairing, as the one drawn in figure 11, may be realized for a smaller $N_c$ interaction, where the magnitude of $\delta \mu_q$ exceeds this bound, a new form of colour superconductivity appears, which is a counterpart of what is known as the Sarma state in condensed matter physics. The Sarma state is, however, not stable as it is, and it has been argued that a number constraint such as the charge neutrality condition may help the stabilization [216]. In the QCD context [217] such superconducting states are called the gapless 2SC (g2SC) phase [218, 219] and gapless CFL (gCFL) phase [220, 221] in the 2-flavour and 3-flavour cases, respectively.

It has been found, however, that the gapless superconducting states suffer from another instability problem. In [222–226] the Meissner (magnetic screening) masses in the g2SC and gCFL phases have been calculated and turned out to be imaginary. That is, there appear negative eigenvalues from the mass-squared matrix in colour space, which is known as the chromomagnetic instability. The physics implication of the chromomagnetic instability can be nicely articulated using the Ginzburg–Landau description. A gauged kinetic term can be added to the free energy in the form of

$$\Omega[A, A] \sim \Omega_0[A] \sim -\kappa^{ab}[(\partial_i - i g A_i) A_j]^{gb}[(\partial^j - i g A^j) A^b],$$

where one can derive the Meissner mass-squared as $(m^2_M)^{ab} = 2\kappa^{ab}(T^a)_{cd}(T^b)_{bd}/\delta \mu_q < 0$ in the g2SC phase and also $\kappa^{11} = \kappa^{22} < 0$ leading to $(m^2_M)^{44} - (m^2_M)^{77} < 0$ whose onset is slightly delayed after the gapless onset. The remaining three gluons are unscreened. On the other hand, in the gCFL phase, all eight eigenvalues of the mass-squared matrix can potentially become negative [224, 226, 227]. The unstable regions are mapped out in the NJL model [227] onto the phase diagram, which is presented in figure 12. The shaded regions are unstable with respect to gluons with their colour labelled aside.

Now let us consider a modulation of the diquark condensate in the simplest form of the plane-wave type. This is written as

$$\Delta(x) = |\Delta| e^{i T^a q^a x},$$

which is known as the (coloured) Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) state [228, 229]. Then, the stability with respect to growth of $q^a \neq 0$ is deduced from the potential curvature:

$$\frac{1}{3} \sum_{i=3}^8 \frac{\partial \Omega}{\partial q_i^a} \frac{\partial \Omega}{\partial q_i^b} \sim (m^2_M)^{ab}.$$  

Therefore the chromomagnetic instability leads to the FFLO state. In the g2SC case with $\kappa^{33} < 0$ a non-coloured ($a = 0$) FFLO state may be enough to cure the chromomagnetic
Figure 12. An example of the mean-field model phase diagram with regions suffering from chromomagnetic instability [227]. At a high density the ordering is the CFL, dSC and 2SC phases from the bottom to the top as explained in figure 9. At a lower density the uSC phase is favoured because the Fermi surface mismatch between d and s quarks is the largest and \( \Delta_{ds} \) tends to melt first when the critical \( T \) is not large. The diagram is drawn with sufficiently large UV cutoff \( \Lambda = 1 \text{ GeV} \) and \( \Delta \lesssim 40 \text{ MeV} \) is chosen at \( \mu_q = 500 \text{ MeV} \), so that the results are robust against cutoff artefacts; otherwise unphysical structures would easily enter.

Figure 13. Pairing with a Fermi surface mismatch by \( \delta \mu_q \). (Left: BCS pairing with vanishing net momentum. One particle is lifted up from the inner Fermi surface by \( \delta \mu_q \), which costs an energy. (Right: FFLO pairing between particles having momenta \( p + q \) and \( -p + q \) with a net momentum \( 2q \). The energy costs are reduced around the dotted rings depicted in the figure.

Crystalline colour superconductivity. It is intuitively understandable that the plane-wave FFLO state is a likely alternative of the ground state if the Fermi surface mismatch grows large. Naturally one can anticipate that the plane-wave FFLO state is a likely alternative of the ground state if the Fermi surface mismatch grows large. Generally the instability analysis tells us a tendency towards the destination at best, but cannot go into the ground state. There are several candidates proposed so far, among which the free energy should be compared to sort out the most stable ground state, which is still an open question. Below some of the proposals are briefly reviewed.

### Gluonic phase

The most straightforward interpretation of the chromomagnetic instability would be the gluonic phase [236] in which gauge fields have a finite expectation value. If only one component of gluons condenses, it is equivalent with the single plane-wave FFLO state. In fact, an ansatz,

\[
\mu_8 = \frac{\sqrt{3}}{2} g(A_8^0), \quad \mu_3 = g(A_3^0), \quad g(A_3^0) \neq 0, \quad (81)
\]

is considered and named the gluonic cylindrical phase in [237]. Because only one spatial gluon condensate is involved, this state can be mapped to a FFLO-type pairing with a coloured phase factor. On the other hand, multi-gluon condensation cannot be transformed into a single plane-wave FFLO state. In the 2-flavour model the free energy of the preferred gluonic phase has turned out to be smaller than that of the FFLO state in a wide parameter region [238]. Such an example is the gluonic colour-spin locked (GCSL) phase that is defined by the following ansatz:

\[
\mu_8 = \frac{\sqrt{3}}{2} g(A_8^0), \quad g(A_6^0) = g(A_6^0) \neq 0, \quad (82)
\]

which is free from the chromomagnetic instability [239]. It is then found that these gluonic phases have a smaller free energy than the single plane-wave FFLO state and the unstable g2SC phase. In most regions the GCSL is the most stable apart from the vicinity of the first-order phase transition to normal quark matter where the gluonic cylindrical phase is energetically favoured. It has not been understood, however,
how to generalize the description of the gluonic phase to the 3-flavour case. Moreover, an extension of the gluonic condensation with spatial inhomogeneity may further decrease the free energy [240].

**Meson supercurrent state.** As we have already emphasized, the CFL phase spontaneously breaks the chiral symmetry, where low-energy excitations are described by a chiral effective Lagrangian expressed in terms of colour-singlet modes. Then it is possible to formulate the instability by using the chiral effective Lagrangian [241]. The origin of the instability should be common, but it is no longer the ‘chromomagnetic’ instability since gluons do not appear explicitly but all the physical degrees of freedom are Nambu–Goldstone bosons. Then, instead of gluons, one may expect the condensation of vector fields given by mesons, that is, the meson currents. Such a destination of the ground state is called the *meson supercurrent state*. The description looks different at a glance from the chromomagnetic instability and the single plane-wave FFLO state, but the underlying physics must be closely related. An advantage of using the chiral effective Lagrangian is that the inclusion of the $K^0$ condensate is straightforward, which is complicated in microscopic models. Such an interpretation as the supercurrent generation is also proposed in [242] not relying on the chiral effective Lagrangian but in terms of phase fluctuations around the diquark condensate.

**Mixed phase and phase separation.** All the states as we have seen so far should compete with a rather conventional possibility, i.e. the mixed phase. Neutrality conditions with respect to gauge charge enforce the Fermi surface mismatch, but these conditions may be relaxed by the formation of a mixture of CSC and NQM regions as discussed in the 2-flavour case in [243].

To clarify the mixed phase structure, however, the precise determination of the surface tension is indispensable. The balance between the surface tension and the Coulomb energy fixes the typical domain size of the mixed phase structure. Naturally, the larger the surface tension is, the larger the favoured domain size grows, and eventually the mixed phase should be rather regarded as the phase separation [244]. It is not easy to extract the information on reliable value of the surface tension in the intermediate density region, however. So far, the possibility of the mixed phase in 3-flavour CSC phase has not been studied seriously.

**8. Suggestions from QCD-like theories**

In the research towards the phase diagram of dense QCD, some knowledge from QCD-like theories would provide us with a useful hint to attack the QCD problem. There have been many such attempts and it is impossible to cover all of them in this paper. Here, we discuss some selected topics which are highly relevant to the understanding of the dense QCD phase diagram.

**8.1. Quarkyonic matter at large $N_c$**

The novel QCD phase structure in the large $N_c$ limit has been recently proposed [58] as schematically shown in figure 14.

![Figure 14. Schematic phase diagram of large-$N_c$ QCD. The pressure is $O(N_c^0)$ in the hadronic phase, $O(N_c)$ in the quarkyonic matter and $O(N_c^2)$ in the deconfined phase.](image)

![Figure 15. Intuitive picture of the Fermi sphere which accommodates quarkyonic matter—quark Fermi sea and baryonic Fermi surface.](image)

When $\mu_q$ is smaller than the threshold of the constituent quark mass $M_q \sim M_B/N_c \sim O(N_c^0)$, we have the hadronic phase with zero baryon density at a low temperature. As the temperature is increased, there appears the first-order deconfinement transition at $T_d \sim \Lambda_{QCD}$ at which the number of degrees of freedom and the pressure jump discontinuously from $O(N_c^0)$ to $O(N_c^2)$. Since quark loops are suppressed by $1/N_c$ as compared with gluon contributions, $T_d$ is independent of $\mu_q$ in this region as shown in figure 14.

When $\mu_q$ becomes greater than $M_q$, a non-zero baryon density is turned on. The pressure associated with this threshold at $\mu_q \approx M_q$ changes discontinuously from $O(N_c^0)$ to $O(N_c)$. Also $T_d$ does not change with increasing $\mu_q$ as long as $\mu_q \ll O(N_c^0)$, so that quarks are still confined in the right-bottom region ($\mu_q > M_q$ and $T < T_d$) in figure 14. The confining phase with the pressure of $O(N_c)$ is called the *quarkyonic matter* [58].

If we describe the quarkyonic matter as a weakly interacting quark system, the pressure of $O(N_c)$ is a natural consequence, but it is difficult to reconcile it with the confining feature. If we describe the quarkyonic matter as a baryonic system, it must be a strongly interacting matter where the pressure is dominated by baryon interactions of $O(N_c)$ rather than the kinetic pressure of $O(1/N_c)$. A possible idea to unify these two descriptions proposed in [58] is illustrated in figure 15; the quarks deep inside the Fermi sphere are weakly interacting, because it is hard to excite these quarks above...
the Fermi sea due to Pauli blocking. On the other hand, the quarks near the Fermi surface with a shell width $\sim \Lambda_{\text{QCD}}$ are not affected so much by the Pauli blocking and can interact strongly through IR singular gluons at a large $N_c$. Thus, the bulk thermodynamics such as the pressure, entropy and so on are dominated by the quarks inside the Fermi sphere, while the physical excitations on top of the Fermi surface are dominated by colour-singlet mesons and baryons.

Whether the remnant of quarkyonic matter at a large $N_c$ remains in the QCD phase diagram at $N_c = 3$ is an open question. Also, how the chiral transition takes place inside quarkyonic matter is an important problem to be studied. An interesting and plausible possibility comes from the fact that the gluon propagator may be non-perturbative and IR singular because of the confining nature. If this is the case, such IR singular interaction would induce an inhomogeneous chiral condensate as elucidated in section 7.1 or the quarkyonic chiral spiral [214].

### 8.2. QCD at $N_c = 2$

A QCD-like theory with $N_c = 2$ [245, 246] is also an interesting limit opposite to the case of quarkyonic matter; small $N_c$ instead of large $N_c$. Two-colour QCD is free from the sign problem if the number of degenerate flavours is even, so that Monte Carlo simulation at a finite baryon density is possible [247]. Two-colour QCD shares many non-perturbative features with $N_c = 3$ QCD such as confinement, chiral symmetry breaking and superfluidity.

In two-colour QCD, baryons consist of quark pairs $qq$ which are bosons. Therefore, nuclear matter composed of fermionic baryons in $N_c = 3$ is replaced by a superfluid state of bosonic baryons in $N_c = 2$. The order parameter of superfluidity is the colour-singlet baryon condensate $\langle q\bar{q} q\bar{q}\rangle$. To investigate the phase structure of two-colour QCD, such a baryon condensate has been computed both in numerical simulations and in analytical strong-coupling expansion [248, 249, 250]. There is also a suggestive result that supports the idea of quarkyonic matter in $N_c = 2$ from lattice simulations [251], which is consistent with the model analysis [252]. The correlation functions and excitation spectra can be studied both analytically and numerically for $N_c = 2$. In particular, the low-energy chiral Lagrangian at a finite baryon density as well as Leutwyler-Smilga type spectral sum rules have been derived [253]. In-medium hadron spectra as a function of the baryon chemical potential have been also investigated [254-256]. As seen in these examples, two-colour QCD continues to be a valuable testing ground for studying hot/dense QCD at $N_c = 3$.

### 8.3. Ultracold atoms

High density QCD matter and ultracold atomic systems, although differing by some twenty orders of magnitude in energy scales, share analogous physical aspects [257]. Phenomenological studies of QCD indicate a strong spin-singlet diquark correlation inside the nucleon [258]. Therefore, it may be a good starting point to model the transition from 2-flavour quark matter at a high density to nuclear matter at a low density in terms of a boson–fermion mixture, in which small-sized diquarks are the bosons, unpaired quarks the fermions and the extended nucleons are regarded as composite Bose–Fermi particles [259].

Recent advances in atomic physics have made it possible indeed to realize a boson–fermion mixture in the laboratory. In particular, tuning the atomic interaction via a Feshbach resonance allows the formation of heteronuclear molecules, as has recently been observed in a mixture of $^{87}\text{Rb}$ and $^{40}\text{K}$ atomic vapours in an optical dipole trap [260]. An analysis of such a non-relativistic mixture of cold atoms indicate that the BCS-like superfluidity of composite fermion $(N = b + f)$ with a small gap is a natural consequence of the strong boson–fermion attraction [259], which may explain the reason why the fermion gap in nucleon superfluidity can be an order of magnitude smaller than the gap in colour superconductivity. A possible correspondence between cold atoms and QCD is summarized in table 4. Fuller understanding, both theoretical and experimental, of the boson–fermion mixture as well as a mixture of three species of atomic fermions [261, 262] may further reveal properties of high-density QCD.

### 9. Summary and concluding remarks

In this paper we reviewed the current status of theoretical investigations to explore the QCD phase structure at a finite temperature ($T$) and a finite baryon chemical potential ($\mu_B$). There are (at least) three fundamental states of matter in QCD: the hadronic matter with broken chiral symmetry and quark confinement in the low-$T$ and low-$\mu_B$ region, the quark–gluon plasma at a high $T$ and the colour superconductivity at a low $T$ and a high $\mu_B$. On top of these states, some exotic phenomena have been conjectured, e.g. the chiral-density waves, the crystalline colour superconductivity, the gluonic phase, the quakyonic matter and the quark–hadron continuity, as covered in this paper, and even more possibilities are still developing. Lattice QCD simulations, the Ginzburg–Landau–Wilson approach and effective theories of QCD are useful theoretical tools to study these phenomena.

From the experimental point of view, the QCD phase transitions at a high $T$ with $\mu_B/T < 1$ can be studied using high-energy heavy-ion collisions at RHIC and LHC. The QCD phase diagram at a relatively low $T$ with $\mu_B/T \gtrsim 1$ may also be probed in the future facilities with lower-energy heavy-ion collisions.

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**Table 4.** Correspondence between the boson–fermion mixture in ultracold atoms (such as $^{87}\text{Rb}$ and $^{40}\text{K}$ mixture) and the diquark-quark mixture in 2-flavour QCD.

| Cold atoms | Dense QCD |
|------------|-----------|
| $b$ (bosonic atom) | $D$ (spin-0 diquark) |
| $N_{\text{q}}$ (fermionic atom) | $q_{\text{q}}$ (unpaired quark) |
| $N_{\text{q}}$ (b–f molecule) | $N_{\text{q}}$ (D–q bound state = nucleon) |
| $b$–f attraction | gluonic $D$–q attraction |
| b–BEC | 2SC |
| N–BCS | nucleon superfluidity |

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beams. In addition, recent attempts to determine the mass and the radius of neutron stars from x-ray bursts would be an alternative way to access the equation of state of dense QCD [263, 264]. Furthermore, gravitational waves [265] and neutrinos from supernova explosions [266] or cooling processes in the neutron star [267] carry valuable information of dense QCD matter. Dense QCD shall continue to be one of the most fascinating theoretical and experimental topics in particle, nuclear and astrophysics.

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