Existence of bound states of $N$-body problem in an optical lattice

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Abstract

We provide sufficient conditions to have at least one $N$-particle bound state below the essential spectrum of a large class of $N$-particle discrete Schrödinger operators $H(K)$, $K \in \mathbb{T}^d$, $d \geqslant 1$, associated with the Hamiltonian of (not necessarily identical) $N$ particles, moving on a lattice $\mathbb{Z}^d$ and interacting via short-range pair potentials. We also describe the essential spectrum of $H(K)$.

Keywords: dispersion functions, short-range pair potentials, HVZ theorem, essential spectrum, cluster operators, bound states, Schrödinger operator

Introduction

One of the remarkable results in the spectral theory of multi-particle continuous Schrödinger operators is the description of essential spectrum (the HVZ theorem in honour of Hunziker [14], van Winter [40], and Zhislin [47]): the essential spectrum of an $N$-particle Hamiltonian (in the center-of-mass frame) is a half-line whose lowest bound is the lowest possible energy which two independent subsystems can have. Since then, the result has been substantially improved and extended to various classes of operators (see the survey [15] and references therein).

Few results are available in the literature on essential spectra of discrete Schrödinger operators associated with many-body systems in an optical lattice $\mathbb{Z}^d$, $d \geqslant 1$. The essential spectrum of a three-body problem on $\mathbb{Z}^3$ with analytic dispersion functions was described in [3]; a four-body HVZ theorem with discrete Laplacian and zero-range potentials in $\mathbb{Z}^3$ was proved in
[1, 28]; see also [27] and references therein for other results related to spectral properties of multi-particle lattice Schrödinger operators.

One of the fundamental differences between the multi-particle continuous Hamiltonian in $\mathbb{R}^d$, $d \geq 1$, and the discrete Hamiltonian in $\mathbb{Z}^d$ is that the latter is not rotationally invariant. However, using a technique of separation of variables—a lattice analogue of the center-of-mass frame [10, 20, 44] (see also section I of the present paper), the discrete Hamiltonian can be decomposed into fibers, i.e. it can be represented as a direct integral of a family of discrete Schrödinger operators $H(K)$, parameterized by the so-called $N$-particle quasi-momentum $K \in \mathbb{T}^d$, where $\mathbb{T}^d$ is the $d$-dimensional torus [26, 27]. In contrast to the continuous case, fibers non-trivially depend on the quasi-momentum $K$, and therefore their spectra are quite sensitive to the change of $K$: even in the two-particle case, the essential spectrum may collapse to a point, and hence it is not absolutely continuous [23, remark 2.1]. Moreover, by virtue of the boundedness of $H(K)$, its essential spectrum is no longer a half-line on the real axis but is rather, at most, a countable union of closed segments (see theorem 3.1), and in turn this may allow the Efimov effect to appear not only at the lower edge of the essential spectrum, but also at the edges of gaps between those segments [29].

Discrete Schrödinger operators in lattice and their applications in solid-state physics were duly stressed, for instance, in [10, 26, 27, 34]; we also refer to [16, 25, 41, 42] and references therein for experimental and theoretical results in the theory of ultracold atoms on optical lattices.

In the first part of this paper, we consider the discrete Schrödinger operator $H(K)$, $K \in \mathbb{T}^d$, associated with the Hamiltonian of a system of $N \geq 2$ particles moving on a $d$-dimensional lattice $\mathbb{Z}^d$ and interacting via short-range pair potentials. We prove an analogue of the HVZ theorem using the diagrammatic method for a large class of potentials and dispersion relations that are not necessarily of compact support. More precisely, we show that under Hypotheses (1.2)–(1.3), the essential spectrum of $H(K)$ is a union of spectra of all two-cluster operators and consists of an at most countable union of disjoint closed segments; the only accumulation points (if any) of eigenvalues of $H(K)$ outside the essential spectrum are the edges of those segments (theorem 3.1).

One of the practical applications of the HVZ theorem is that it allows us to use variational techniques more efficiently to study eigenvalues and eigenvectors of $H(K)$. Further eigenvectors of $H(K)$ will also be called $N$-particle bound states. There is a considerable literature devoted to the finiteness of bound states of continuous Schrödinger operators with short-range pair potentials (see e.g. [6, 19, 36, 37, 45] and references therein). Nevertheless, apart from the Efimov effect (see e.g. [5, 12, 38, 39, 43] and references therein), not much seems to be known of the existence of bound states; some sufficient conditions to have an $N$-particle bound state can be found, for example, in [8, 9, 32], and spectral properties (including existence and non-existence of discrete spectrum) of $2 + 1$ fermionic trimers with contact interactions have been studied in [6]; see also [48].

In the discrete case, the Efimov effect was studied, for instance, in [2, 20]. The existence of three-particle bound states of purely attractive and repulsive systems of three identical bosons, interacting via zero-range pair potentials in $\mathbb{Z}^d$, $d = 1, 2$, has been recently established in [22]; the same results still hold for the purely attractive or purely repulsive system of $2 + 1$ fermionic trimers in $\mathbb{Z}^d$, $d = 1, 2$, interacting via zero-range pair potentials [24].

In the second part of the present paper, we provide sufficient conditions to have at least one $N$-particle bound state of a purely attractive (resp. purely repulsive) system of particles. The main result here is that if any two-particle subsystem has a bound state below (resp. above) its essential spectrum, then the discrete spectrum below (resp. above) the essential spectrum
of the $N$-particle Schrödinger operator is nonempty (theorems 4.2 and 4.3). To the best of our knowledge, such a result has not been published yet in the continuous case.

An advantage of this result in applications is that spectral properties of two-particle discrete Schrödinger operators in lattice have been sufficiently well studied (see e.g. [4, 23] and references therein), in particular, in $\mathbb{Z}^d$, $d = 1, 2$, a two-particle bound state always exists, and hence our result generalizes also the results of [22] (see section 5). However, recall that we provide only sufficient conditions: one counterexample would be the Efimov effect.

The universality of our existence results is that we claim the existence of bound states in every purely attractive or repulsive system of arbitrary (finite) numbers of particles as soon as pairwise interactions between particles are strong enough. Such universality is less obvious, for example, in the Efimov physics (see [30, section 1.2]), since currently the Efimov effect is known only for three-particle systems [11] and not every system can allow it. Moreover, our conditions to have a bound state can be derived using only the controllable parameters of the system, e.g. the lattice geometry, masses, and dispersion relations of particles and two-body potentials (see examples in section 5) and we do not need to know the exact or numerical solutions of two-particle problems. Recall that because of the controllability of collision properties of ultracold atoms, a stable repulsive bound pair of $^{87}$Rb atoms has been experimentally observed in the optical lattice $\mathbb{Z}^3$ [42]; ultracold heteronuclear molecules assembled from fermionic $^{40}$K and bosonic $^{87}$Rb atoms in $\mathbb{Z}^3$ have been produced at a heteronuclear Feshbach resonance on both the attractive and repulsive sides of the resonance [31]; see also [13, 46].

The present work is organized as follows. In section 1, we introduce the notation, the $N$-particle Hamiltonian $\mathbf{H}$, and decompose it into the direct integral of Schrödinger operators $\mathbf{H}(K)$. Cluster operators and their spectra are studied in section 2. In section 3, we prove the HVZ theorem. Section 4 is devoted to results related to the existence of bound states and we provide some examples in section 5. Finally, section 6 contains some discussions and comments on the main results.

1. $N$-particle discrete Schrödinger operator in lattice $\mathbb{Z}^d$

Notation. Let $N \geq 2$ denote the number of particles in the system, $\mathbb{Z}^d$ be a $d \geq 1$-dimensional lattice and $\mathbb{T}^d = \mathbb{R}^d/(2\pi \mathbb{Z})^d = (-\pi, \pi]^d$ be the $d$-dimensional torus (the first Brillouin zone, the dual group of $\mathbb{Z}^d$) equipped with a Haar measure. We use the symbol $dq$ to mean that the integration is upon the variable $q$ over $(\mathbb{T}^d)^m$ (or sometimes over its submanifold) for some $m \in \mathbb{N}$. Elements of $\mathbb{Z}^d$ and $\mathbb{R}^d$ will be denoted by $x, y, s, \ldots$ and

$$|x| := |x^{(1)}| + \cdots + |x^{(d)}|, \quad x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{Z}^d \text{ (or } \mathbb{R}^d)$$

denotes the norm of $x$. The elements of $\mathbb{T}^d$ are usually denoted by $p, q, t, \ldots$ and the norm of $p \in \mathbb{T}^d$ is

$$|p| := \sqrt{|p^{(1)}|^2 + \cdots + |p^{(d)}|^2}, \quad p := (p^{(1)}, \ldots, p^{(d)}).$$

For $p \in \mathbb{T}^d$ and $x \in \mathbb{Z}^d$, we define the duality between $p$ and $x$ as

$$p \cdot x := \sum_{i=1}^d p^{(i)} x^{(i)}.$$ 

Given $m \geq 1$, elements of $(\mathbb{Z}^d)^m$ will be denoted by bold letters $\mathbf{x} := (x_1, \ldots, x_m)$, $x_j \in \mathbb{Z}^d$, $j = 1, \ldots, m$. The notation is analogous for $(\mathbb{T}^d)^m$. Moreover,
Given $k \in \mathbb{T}^d$ and $m \in \mathbb{N}$, we also set

$$F^m_k := \{ \mathbf{q} = (q_1, \ldots, q_m) \in (\mathbb{T}^d)^m : q_1 + \cdots + q_m = k \} ;$$

when $m = N$, we write shortly $\mathbb{G}^N_k := F_k$. By $\ell^2(\mathbb{Z}^d)^m$ (resp. $L^2(\mathbb{T}^d)^m$) we denote the Hilbert space of square-summable (resp. dk-square-integrable) functions defined on $(\mathbb{Z}^d)^m$ (resp. $(\mathbb{T}^d)^m$), $m \geq 1$. For simplicity, set $\ell^2(\mathbb{Z}^d)^0 = L^2(\mathbb{T}^d)^0 = \mathbb{C}$. Elements of $L^2$ is written as $f, g, \ldots$ whereas the notation $\hat{f}, \hat{g}, \ldots$ is used for functions on $\ell^2$. The symbol $\sigma(A)$ stands for the spectrum of a linear operator $A$, and its essential and discrete spectra are indicated by $\sigma_{\text{ess}}(A)$ and $\sigma_{\text{disc}}(A)$, respectively. By the standard Fourier transform, we mean the operator $\mathcal{F}_m : \ell^2(\mathbb{Z}^d)^m \to L^2(\mathbb{T}^d)^m$,

$$(\mathcal{F}_m \hat{f})(\mathbf{p}) := (2\pi)^{-dm/2} \sum_{\mathbf{x} \in (\mathbb{Z}^d)^m} \hat{f}(\mathbf{x}) e^{i \mathbf{p} \cdot \mathbf{x}}.$$  

Similarly, we introduce the Hilbert space of $\ell^2$ (resp. $L^2$)—functions defined on a sublattice (resp. submanifold) of $(\mathbb{Z}^d)^m$ (resp. $(\mathbb{T}^d)^m$). By $\delta$, we denote the Dirac-delta function in $\mathbb{T}^d$ concentrated at 0, defined formally as

$$\delta(q) := (2\pi)^{-d/2} \sum_{\mathbf{x} \in \mathbb{T}^d} e^{i q \cdot \mathbf{x}}, \quad q \in \mathbb{T}^d.$$  

11. $N$-particle Hamiltonian

In coordinate representation, the total Hamiltonian $\hat{H}$, associated with a system of $N \geq 2$ particles moving in the $d$-dimensional lattice $\mathbb{Z}^d$ and interacting via short-range pair potentials $\hat{V}_{ij}$, is defined in the Hilbert space $\ell^2((\mathbb{Z}^d)^N)$ as

$$\hat{H} = \hat{H}_0 - \hat{V},$$

$$\hat{H}_0 = \frac{1}{m_1} \hat{\Delta}_1 \otimes \hat{I}_d \otimes \cdots \otimes \hat{I}_d + \cdots + \frac{1}{m_N} \hat{I}_d \otimes \cdots \otimes \hat{I}_d \otimes \hat{\Delta}_N,$$

and

$$\hat{V} := \sum_{1 \leq i < j \leq N} \hat{V}_{ij}.$$  

Here, $m_i \in (0, +\infty]$ is the mass of particle $i$, $\hat{I}_d$ is the identity map in $\ell^2(\mathbb{Z}^d)$, and $\hat{\Delta}_i$ is a generalized Laplacian multidimensional Laurent–Toeplitz-type operator in $\ell^2(\mathbb{Z}^d)$:

$$\hat{\Delta}_i f(x) := \sum_{s \in \mathbb{Z}^d} \hat{\varepsilon}_i(s) f(x + s), \quad f \in \ell^2(\mathbb{Z}^d), \quad i = 1, \ldots, N.$$  

We assume that

$$\left\{ \begin{array}{l}
\varepsilon_i(y) = \varepsilon_i(-y), \quad y \in \mathbb{Z}^d, \\
\exists \gamma > 0 : \sum_{y \in \mathbb{Z}^d} |y|^\gamma |\varepsilon_i(y)| < +\infty,
\end{array} \right. \quad i = 1, \ldots, N. \quad (1.2)$$
The real-valued continuous function $\varepsilon_i := \mathcal{F}_N \tilde{\varepsilon}_i$, $i = 1, \ldots, N$, is called the dispersion relation of the $i$th normal mode, associated with the free particle $i$.

The pair potential $\mathbf{V}_y$ is the multiplication operator by a function $\tilde{v}_y(x_i - x_j)$ in $\ell^2((\mathbb{Z}^d)^N)$:

$$(\mathbf{V}_y f)(x_1, \ldots, x_N) = \tilde{v}_y(x_i - x_j) \tilde{f}(x_1, \ldots, x_N).$$

We suppose that

$$\tilde{v}_y \in \ell^1(\mathbb{Z}^d)$$

and is an even function, $1 \leq i < j \leq N$. (1.3)

Under assumptions (1.2)–(1.3), the total Hamiltonian $\hat{H}$ is a bounded self-adjoint operator in $\ell^2((\mathbb{Z}^d)^N)$ (see e.g. [17] for $N = 2$).

In the momentum space $L^2((\mathbb{T}^d)^N)$, $\hat{H}$ is represented as

$$\mathbf{H} = \mathbf{H}_0 - \mathbf{V},$$

where $\mathbf{H}_0 = \mathcal{F}_N \hat{\mathbf{H}}_0 \mathcal{F}_N^{-1}$, $\mathbf{V} = \mathcal{F}_N \hat{\mathbf{V}} \mathcal{F}_N^{-1}$, and $\mathcal{F}_N$ is the inverse Fourier transform.

The free Hamiltonian $\mathbf{H}_0$ is the multiplication operator

$$(\mathbf{H}_0 f)(\mathbf{p}) = \mathcal{E}(\mathbf{p}) f(\mathbf{p}),$$

by the function

$$\mathcal{E}(\mathbf{p}) = \sum_{i=1}^N \frac{1}{m_i} \varepsilon_i(p_i), \quad \mathbf{p} := (p_1, \ldots, p_N) \in (\mathbb{T}^d)^N.$$ (1.4)

The perturbation $\mathbf{V}$ is the sum of partial integral operators $\mathbf{V}_{ij}$, $i, j = 1, \ldots, N$, $i < j$:

$$(\mathbf{V}_{ij} f)(\mathbf{p}) = (2\pi)^{-d/2} \int_{(\mathbb{T}^d)^2} \mathcal{V}_{ij}(p_i - q_j) \delta(p_i + q_j - q_i) \times f(p_1, \ldots, q_i, \ldots, q_j, \ldots, p_N) d\mathbf{q},$$

where the kernels $\mathcal{V}_{ij} = \mathcal{F}_N \tilde{v}_{ij}$,

$$\mathcal{V}_{ij}(\mathbf{p}) = (2\pi)^{-d/2} \sum_{x \in \mathbb{Z}^d} \tilde{v}_{ij}(x) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad \mathbf{p} \in \mathbb{T}^d.$$ (1.5)

are real-valued continuous functions on $\mathbb{T}^d$.

### 1.2. Decomposition of $\mathbf{H}$ and representations of fiber operators

Let $\{\mathbf{U}_s\}_{s \in \mathbb{Z}^d}$ be the Abelian group of discrete translations in $\ell^2((\mathbb{Z}^d)^N)$:

$$(\mathbf{U}_s f)(x_1, \ldots, x_N) = f(x_1 + s, \ldots, x_N + s), \quad x_1, \ldots, x_N, s \in \mathbb{Z}^d.$$

Via the Fourier transform $\mathcal{F}_N$, the family $\{\mathbf{U}_s\}_{s \in \mathbb{Z}^d}$ is unitary-equivalent to the family of unitary multiplication operators $\{\mathbf{U}_s\}_{s \in \mathbb{Z}^d}$ acting in $L^2((\mathbb{T}^d)^N)$ as

$$(\mathbf{U}_s f)(\mathbf{p}) = \exp(-i(p_1 + \cdots + p_N) \cdot s) f(\mathbf{p}), \quad f \in L^2((\mathbb{T}^d)^N).$$

Let $\pi_j : (\mathbb{T}^d)^N \rightarrow (\mathbb{T}^d)^{N-1}$, $j = 1, \ldots, N$, be the projection map defined as

$$\pi_j(p_1, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_N) = (p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N),$$

and let $\pi_{jk}$ be the restriction of $\pi_j$ to $\mathbb{F}_k$, where $\mathbb{F}_k$ is defined in (1.1) with $N$ in place of $m$. As $\pi_{jk} : \mathbb{F}_k \rightarrow (\mathbb{T}^d)^{N-1}$, $K \in \mathbb{T}^d$, is bijective with the inverse mapping given by
\[(\pi_K)^{-1}(p_1, \ldots, p_{j-1}, p_j, \ldots, p_N) = (p_1, \ldots, p_{j-1}, K - \sum_{i=1, i \neq j}^{N} p_i, p_{j+1}, \ldots, p_N),\]

where \(\mathbb{F}_K\) is homeomorphic to \((\mathbb{T}^d)^{N-1}\).

The decomposition of the space \(L^2((\mathbb{T}^d)^N)\) into the direct integral
\[L^2((\mathbb{T}^d)^N) = \int_{K \in \mathbb{T}^d} L^2(\mathbb{F}_K) dK\] yields the corresponding decomposition of the unitary representation \(U_s, s \in \mathbb{Z}^d\), into the direct integral
\[U_s = \int_{K \in \mathbb{T}^d} U_s(K) dK,\]
where
\[U_s(K) = e^{-iK \cdot s I_{L^2(\mathbb{F}_K)}}\]
and \(I_{L^2(\mathbb{F}_K)}\) is the identity operator on the Hilbert space \(L^2(\mathbb{F}_K)\). The Hamiltonian \(H\) obviously commutes with \(U_s, s \in \mathbb{Z}^d\), and hence by [34, theorem XIII.84], the operator \(H\) is also decomposed into the von Neumann integral
\[H = \int_{K \in \mathbb{T}^d} \tilde{H}(K) dK,\]
associated with the decomposition (1.6).

In the physical literature, the parameter \(K \in \mathbb{T}^d\) is called the \(N\)-particle quasi-momentum and corresponding operators \(\tilde{H}(K), K \in \mathbb{T}^d\), are called the fiber operators. Observe that given \(K \in \mathbb{T}^d\), the fiber operator \(\tilde{H}(K)\) acts in \(L^2(\mathbb{F}_K)\) as
\[\tilde{H}(K) = \tilde{H}_0(K) - \tilde{V},\]
where
\[\tilde{H}_0(K) f(p) = \left( \frac{1}{m_1} \varepsilon_1(p_1) + \cdots + \frac{1}{m_N} \varepsilon_N(p_N) \right) f(p), \quad p \in \mathbb{F}_K,\]
and
\[\tilde{V} = \sum_{1 \leq i < j \leq N} \tilde{V}_{ij}\]
with
\[(\tilde{V}_{ij})(p) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v_{ij}(t) f(p_1, \ldots, p_{i-1}, t, p_i + t, p_{i+1}, \ldots, p_{N}) dt.\] (1.8)

Using the unitary operator \(U_{jK} : L^2(\mathbb{F}_K) \to L^2((\mathbb{T}^d)^{N-1})\), \(U_{jK} f = f \circ (\pi_{jK})^{-1}\), \(j = 1, \ldots, N\), we define the momentum representation of \(\tilde{H}(K)\) as
\[H_j(K) = U_{jK} \tilde{H}(K) U_{jK}^{-1}.\]
For simplicity, we write \( H(K) := H_N(K) \),
\[
H(K) = H_0(K) - V,
\]
where \( H_0(K) \) is the multiplication operator in \( L^2((\mathbb{T}^d)^{N-1}) \) by the continuous function \( \xi_K : (\mathbb{T}^d)^{N-1} \to \mathbb{R} : \frac{1}{m_1} \xi_1(p_1) + \cdots + \frac{1}{m_{N-1}} \xi_{N-1}(p_{N-1}) + \frac{1}{m_N} \xi_N(K - p_1 - \ldots - p_{N-1}) \).

and the perturbation \( V = \sum_{i,j} V_{ij} \) acts in \( L^2((\mathbb{T}^d)^{N-1}) \) with
\[
(V_{ij}(f)(p_1, \ldots, p_{N-1}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v_{ij}(t)f(p_1, \ldots, p_i - t, \ldots, p_j + t, \ldots, p_{N-1}) \, dt
\]
for \( 1 \leq i < j < N \), and
\[
(V_{jj}(f)(p_1, \ldots, p_{N-1}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v_{jj}(t)f(p_1, \ldots, p_i - t, \ldots, p_j + t, \ldots, p_{N-1}) \, dt
\]
for \( 1 \leq i < N \).

The operator
\[
\tilde{H}(K) := \mathcal{F}_{N-1}^{-1} H(K) \mathcal{F}_{N-1}
\]
is called the coordinate representation of \( \tilde{H}(K) \) in \( \ell^2((\mathbb{Z}^d)^{N-1}) \).

In what follows, any operator unitarily equivalent to \( \tilde{H}(K) \), will be called the \( N \)-particle discrete Schrödinger operator. We use its various representations according to their convenience in applications.

2. Cluster operators

**Definition 2.1.** A partition \( \mathcal{C} \) of the set \( \{1, \ldots, N\} \) into nonintersecting subsets \( C_1, C_2, \ldots, C_\ell \) is called a cluster decomposition. Each \( C_\nu \) is called a cluster.

Given a cluster decomposition \( \mathcal{C} = \{C_1, C_2, \ldots, C_\ell\} \), we write \( |C_\nu| \) to denote the number of particles in \( C_\nu \); the symbol \( ij \in \mathcal{C} \) means \( i, j \in C_\nu \) for some \( 1 \leq \nu \leq \ell \); analogously, the symbol \( ij \notin \mathcal{C} \) denotes the situation in which particles \( i \) and \( j \) are in different clusters (i.e. \( i \in C_\alpha \) and \( j \in C_\beta \) with \( \alpha \neq \beta \)) and \( \#\mathcal{C} \) denotes the number of elements in \( \mathcal{C} \), i.e. \( \#\mathcal{C} = \ell \); besides, set \( V^\mathcal{C} := \sum_{ij \in \mathcal{C}} V_{ij}, \ I^\mathcal{C} := \sum_{ij \notin \mathcal{C}} V_{ij} = V - V^\mathcal{C} \).

**Definition 2.2.** The operator
\[
H^\mathcal{C}(K) = H(K) + I^\mathcal{C}, \quad K \in \mathbb{T}^d,
\]
is called the cluster operator corresponding to a cluster decomposition \( \mathcal{C} \). We write \( \tilde{H}^\mathcal{C}(K) \) (resp. \( \tilde{H}^\mathcal{C}(K) \)) for the fiber (resp. coordinate) representation of \( H^\mathcal{C}(K) \).

2.1. The discrete Schrödinger operator corresponding to a cluster

Let \( C_\nu, 1 \leq \nu \leq l \), be a cluster in a decomposition \( \mathcal{C} = \{C_1, \ldots, C_l\} \) and \( n_\nu := |C_\nu| \). Suppose \( C_\nu = \{\alpha_1, \ldots, \alpha_{n_\nu}\} \subseteq \{1, \ldots, N\} \). For \( k \in \mathbb{T}^d \) set
Recall that \( \mathbb{P}^n_k \) is homeomorphic to \((T^d)^{n−1}\). In view of (1.7), the operator \( \widehat{H}^C(k) : L^2(\mathbb{P}^n_k) \to L^2(\mathbb{P}^n_k) \), defined as
\[
\widehat{H}^C(k) = \widetilde{h}_0^C(k) - \tilde{v}^C_n,
\]
where
\[
(\tilde{h}_0^n(k)f)(p) = \sum_{\alpha_i \in C} \frac{1}{m_{\alpha_i}} \varepsilon_{\alpha_i}(p_i)f(p), \quad p \in \mathbb{P}^n_k,
\]
and
\[
\tilde{v}^C_n = \sum_{\alpha_i \in C, \alpha_{i<j}} \tilde{v}_{\alpha_i\alpha_j}
\]
with
\[
(\tilde{v}_{\alpha_i\alpha_j}f)(p) = (2\pi)^{-d} \int_{\mathbb{T}^d} \nu_{\alpha_i\alpha_j}(t)f(p_{\alpha_i} - t, \ldots, p_{\alpha_j} + t, \ldots, p_{\alpha_n}) dt,
\]
is the \( n_v \)-particle discrete Schrödinger operator associated with the Hamiltonian of the particle system \( C_v \).

### 2.2. Spectrum of cluster operators

Let \( K \in T^d \) and \( C = \{C_1, \ldots, C_l\}, l \geq 2 \). Set \( n_v := |C_v|, \nu = 1, \ldots, l \). It is easy to see that
\[
\mathbb{P}^n_K = \bigcup_{\kappa_1, \ldots, \kappa_l \in T^d, \sum \kappa_i = K} \mathbb{P}^n_{k_1} \times \cdots \times \mathbb{P}^n_{k_l} := \bigcup_{\kappa_1, \ldots, \kappa_l \in T^d, \sum \kappa_i = K} \mathbb{F}(k_1, \ldots, k_l).
\]
Hence the Hilbert space \( L^2(\mathbb{P}^n_K) \) is decomposed into the von Neumann direct integral
\[
L^2(\mathbb{P}^n_K) = \int_{\kappa_1, \ldots, \kappa_l = K} \mathbb{P}^n_{k_1} \times \cdots \times \mathbb{P}^n_{k_l})dk,
\]
where, as stated in the Notation subsection, \( dk \) denotes the restriction of the Haar measure in \((T^d)^l\) to the manifold \( \{k = (k_1, \ldots, k_l) \in (T^d)^l : k_1 + \cdots + k_l = K\} \), along which the integration is taken.

Since the fiber cluster operator \( \widehat{H}^C(K) \) commutes with the decomposable Abelian group of multiplication operators by functions \( \phi_k : \mathbb{F}_K \to \mathbb{C}, s = (s_1, \ldots, s_l) \in (\mathbb{Z}^d)^l \),
\[
\phi_k(q) = \exp(i \sum_{\alpha \in C_1} q_{\alpha} \cdot s_1) \times \cdots \times \exp(i \sum_{\alpha \in C_l} q_{\alpha} \cdot s_l),
\]
the decomposition (2.2) yields the decomposition of \( \widehat{H}^C(K) \) into the direct integral
\[
\widehat{H}^C(K) = \int_{\kappa_1, \ldots, \kappa_l = K} \widehat{h}^C(k_1, \ldots, k_l)dk.
\]
The fiber operator \( \widehat{h}^C(k_1, \ldots, k_l), (k_1, \ldots, k_l) \in (T^d)^l \) acts in the Hilbert space
\[
L^2(\mathbb{P}^n_{k_1}) \otimes \cdots \otimes L^2(\mathbb{P}^n_{k_l})
\]
as follows:
\[
\tilde{h}^C(k_1, \ldots, k_l) = \tilde{h}^{C_1}(k_1) \otimes \tilde{f}^{C_2}(k_2) \otimes \cdots \otimes \tilde{f}^{C_l}(k_l) \\
+ \cdots + \tilde{f}^{C_1}(k_1) \otimes \tilde{f}^{C_2}(k_2) \otimes \cdots \otimes \tilde{h}^C(k_l),
\]
(2.5)
where \(\tilde{f}^{C_i}(k_i)\) is the identity operator in \(L^2(T^d)\) and \(\tilde{h}^C(k_i)\) is the \(n_i\)-particle Schrödinger operator given by (2.1). We denote by \(h^{C_i}(k_i)\) its momentum representation acting in \(L^2(T^d)^{n_i-1}\) so that \(\tilde{h}^C(k_1, \ldots, k_l)\) is unitarily equivalent to
\[
h^C(k_1, \ldots, k_l) = h^{C_1}(k_1) \otimes f^{C_2}(k_2) \otimes \cdots \otimes f^{C_l}(k_l) \\
+ \cdots + f^{C_1}(k_1) \otimes f^{C_2}(k_2) \otimes \cdots \otimes h^C(k_l),
\]
where \(k_l := K - k_1 - \ldots - k_{l-1}\) and \(f^{C_i}(k_i)\) is the identity operator in \(L^2(T^d)^{n_i-1}\).

**Theorem 2.1.** Assume (1.2) and (1.3) and let \(C\) be a cluster decomposition with \(\#C \geq 2\). Then the operator \(H^C(K)\) has only essential spectrum and
\[
\sigma(H^C(K)) = \sigma_{\text{ess}}(H^C(K)) = \bigcup_{k_1, \ldots, k_l \in T^d, k_1 + \ldots + k_l = K} \sigma(\tilde{h}^C(k_1, \ldots, k_l)).
\]

**Proof.** We use the fiber representation and show \(\sigma_{\text{disc}}(H^C(K)) = \emptyset\). Indeed, if \(\lambda \in \sigma_{\text{disc}}(\tilde{H}^C(K))\), by [34, theorem XIII.85(e)], there would exist a set \(A \subset \mathbb{R}(k_1, \ldots, k_l)\) of positive measure such that \(\lambda \in \sigma_{\text{disc}}(\tilde{h}^C(k_1, \ldots, k_l))\) for any \((k_1, \ldots, k_l) \in A\). Let \(A = \bigcup_m A_m\) where \(\{A_m\}\) is a pairwise disjoint partition of \(A\) into sets of positive measure. For each \((k_1, \ldots, k_l) \in A\), let us choose an associated normalized eigenvector \(\psi(k_1, \ldots, k_l)(\cdot) \in L^2(\mathbb{R}(k_1, \ldots, k_l))\) and define
\[
\Psi_m := \int_{A_m} \psi(k_1, \ldots, k_l)(\cdot) \, dk.
\]
By construction, \(\{\Psi_m\}\) is an orthonormal system. Moreover, by the definition of the decomposition,
\[
\tilde{H}^C(K)\Psi_m = \int_{A_m} \tilde{h}^C(k_1, \ldots, k_l)\psi(k_1, \ldots, k_l)(\cdot) \, dk \\
= \lambda \int_{A_m} \psi(k_1, \ldots, k_l)(\cdot) \, dk = \lambda \Psi_m.
\]
Hence, \(\lambda\) is an eigenvalue of infinite multiplicity, i.e. \(\lambda \notin \sigma_{\text{disc}}(\tilde{H}^C(K))\), so that \(\sigma(\tilde{H}^C(K)) = \sigma_{\text{ess}}(\tilde{H}^C(K))\).

Let us show now
\[
\sigma(\tilde{H}^C(K)) = \bigcup_{k_1, \ldots, k_l \in T^d, k_1 + \ldots + k_l = K} \sigma(\tilde{h}^C(k_1, \ldots, k_l)).
\]
For shortness, given \(k := (k_1, \ldots, k_{l-1}) \in (T^d)^{l-1}\) let
\[
h^C(k) := h^C(k_1, \ldots, k_{l-1}, K - k_1 - \ldots - k_{l-1})
\]
and \( U := \bigcup_{\sigma} \sigma(h^c(\mathbf{k})) \). By [34, theorem XIII.85 (d)], \( \sigma(H^c(K)) \subseteq U \). On the other hand, if \( \lambda \in U \), there exists \( \mathbf{k}_0 \in (\mathbb{T}^d)^{\lambda-1} \) such that \( \lambda \in \sigma(h^c(\mathbf{k}_0)) \) and by the norm-continuity of \( h^c(\mathbf{k}) \), there exists a neighborhood \( O \subset (\mathbb{T}^d)^{\lambda-1} \) of \( \mathbf{k}_0 \) such that
\[
(\lambda - \eta, \lambda + \eta) \cap \sigma(h^c(\mathbf{k})) \neq \emptyset \quad \forall \mathbf{k} \in O, \quad \forall \eta > 0.
\]
Hence, by virtue of [34, theorem XIII.85 (d)], \( \lambda \in \sigma(H^c(K)) \). Theorem is proved.

Given cluster decompositions \( C = \{C_1, \ldots, C_m\} \) and \( D = \{D_1, D_2, \ldots, D_n\} \), we say that \( C \) is a refinement of \( D \) if each \( D_i \) is a union of some \( C_j \).

**Theorem 2.2.** Assume (1.2)–(1.3) and let \( C \) be a refinement of \( D \). Then, \( \sigma(H^c(K)) \subseteq \sigma_{\text{ess}}(H^D(K)) \). In particular, \( \sigma(H^c(K)) \subseteq \sigma_{\text{ess}}(H(K)) \) for any cluster decomposition \( C \) with \( \#C \geq 2 \).

**Proof.** We use the coordinate representation. For simplicity, set \( \tilde{H}^c := \tilde{H}^c(K) \) and \( \tilde{H}_0 := \tilde{H}_0(K) \). Note that \( \#C \geq 2 \) since \( C \neq D \). Let \( \lambda \in \sigma(\tilde{H}^c) = \sigma_{\text{ess}}(\tilde{H}^c) \). By the Weyl criterion there exists a sequence \( \{\tilde{f}_n\} \subset \ell^2((\mathbb{Z}^d)^{N-1}) \) weakly converging to 0 such that \( \|\tilde{f}_n\| = 1 \) and \( \|\tilde{H}^c - \lambda\tilde{f}_n\| \to 0 \) as \( n \to \infty \). Now by means of \( \tilde{f}_n \), we build the sequence \( \{\tilde{g}_r\} \subset \ell^2((\mathbb{Z}^d)^{N-1}) \) weakly converging to 0 such that \( \|\tilde{g}_r\| = 1 \) and \( \|\tilde{H}^D - \lambda\tilde{g}_r\| \to 0 \) as \( r \to \infty \).

Let \( \psi \in C^{0, \gamma}([0, \infty)) \) (recall that \( \gamma \) is given in (1.2)) be such that \( |\psi(t)| \leq 1 \) and
\[
\psi(t) = \begin{cases} 
1, & t \geq 2, \\
0, & 0 \leq t < 1.
\end{cases}
\]
Define
\[
\rho : (\mathbb{R}^d)^{N-1} \to \mathbb{R}, \quad \rho(y_1, \ldots, y_{N-1}) = \prod_{i \notin C} \psi(|y_i - y_j|)
\]
with \( y_N = 0 \in \mathbb{R}^d \). Since \( \psi \) is bounded and H"older continuous of order \( \gamma \), so is \( \rho \). Let \( c_\rho \) be a H"older constant of \( \rho \).

For \( r \in \mathbb{N} \) we define the function
\[
\rho_r : (\mathbb{Z}^d)^{N-1} \to \mathbb{R}, \quad \rho_r(y) = \rho(y/r).
\]
Let \( R_r \) denote the multiplication operator by \( \rho_r \) in \( \ell^2((\mathbb{Z}^d)^{N-1}) \). Observe that \( \text{supp}(1 - \rho_r) \) is finite, and therefore the operator \( I - R_r \) is compact for any \( r \in \mathbb{N} \).

Since \( \tilde{f}_n \) weakly converges to 0, there exists \( N(r) \) such that \( \|(I - R_r)\tilde{f}_n\| < 1/2 \) for all \( n > N(r) \). This and the relation
\[
1 = \|\tilde{f}_n\| \leq \|(I - R_r)\tilde{f}_n\| + \|R_r\tilde{f}_n\|
\]
implies that \( \|R_r\tilde{f}_n\| \geq 1/2 \) for all \( n > N(r) \). We can assume that \( r \to N(r) \) is strictly increasing.

Now choose a sequence of natural numbers \( n_1 < n_2 < \ldots \) such that \( n_r > N(r) \) and consider the sequence \( \tilde{g}_r = R_r\tilde{f}_n/\|R_r\tilde{f}_n\| \) in \( \ell^2((\mathbb{Z}^d)^{N-1}) \). Note that for any \( f \in \ell^2((\mathbb{Z}^d)^{N-1}) \), we have...
\[(g_r, f) \leq 2|(\mathcal{R}_n f_n, \hat{f})| \leq 2|\hat{f}_n| |\mathcal{R}_n f| = 2|\mathcal{R}_n f| \to 0\]
as \(r \to \infty\), and hence \(g_r\) weakly converges to 0.

By the definition,
\[
\hat{H}^D = \hat{H}^C - \sum_{ij \in D, ij \not\in C} \hat{V}_{ij},
\]

Note that \(\hat{V}_D = \hat{R}_D \hat{V}^D\) as \(\hat{V}_r \hat{R}_r = \hat{R}_r \hat{V}_r\). Therefore
\[
\|\mathcal{R}_n f_n\| (\hat{H}^D - \lambda) \hat{g}_r = \mathcal{R}_r (\hat{H}^C - \lambda) \hat{f}_n + \hat{[H_0, \mathcal{R}_r] f_n} - \sum_{ij \in D, ij \not\in C} \hat{V}_{ij} \mathcal{R}_n f_n,
\]

where \([A, B] = AB - BA\). Since \(\|\mathcal{R}_n f_n\| \geq 1/2\), we have
\[
\|\mathcal{R}_n f_n\| (\hat{H}^D - \lambda) \hat{g}_r \leq 2\|\mathcal{R}_r (\hat{H}^C - \lambda) \hat{f}_n\| + 2\|\hat{[H_0, \mathcal{R}_r] f_n}\| + 2 \sum_{ij \in D, ij \not\in C} \|\hat{V}_{ij} \mathcal{R}_n f_n\|. \tag{2.6}
\]

Observe that
\[
\|\hat{H}_0, \mathcal{R}_r]\| \leq \frac{c_N}{r} \sum_{y \in Z^d} \sum_{y \not\in Z^r} |y| \gamma |\hat{v}_i(y)|,
\]

\[
\|\hat{V}_r \mathcal{R}_r\| \leq \|\rho\|_\infty \sup_{|y| \geq r} |\hat{v}_i(y)|, \quad \forall ij \in D, \quad ij \not\in C,
\]

and
\[
\|\mathcal{R}_n (\hat{H}^C - \lambda) f_n\| \leq \|\rho\|_\infty \|\hat{H}^D - \lambda\| \hat{f}_n\|. \tag{2.7}
\]

Now the choice \(\hat{f}_n\), assumptions (1.2), and (1.3) and inequalities (2.6) and (2.7) imply
\[
\lim_{r \to \infty} \|\mathcal{R}_n (\hat{H}^D - \lambda) \hat{g}_r\| = 0.
\]

Since \(\hat{g}_r\) weakly converges to 0, by the Weyl criterion, \(\lambda \in \sigma_{\text{ess}}(\hat{H}^D)\).

\[\Box\]

3. HVZ theorem for \(H(K)\)

The main result of this section is the following analogue of the HVZ theorem.

**Theorem 3.1.** Assume (1.2)–(1.3). For any \(K \in \mathbb{T}^d\) the essential spectrum of \(H(K)\) is an at most countable union of disjoint closed segments; more precisely, it consists of the union of the spectra of all two-cluster operators:
\[
\sigma_{\text{ess}}(H(K)) = \bigcup_{\mathcal{D} \in \Xi, \# \mathcal{D} = 2} \sigma(H^D(K)),
\]

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where $\Xi$ is the set of all cluster decompositions. Moreover, $\sigma_{\text{disc}}(H(K))$ can accumulate only at the edges of the constituent segments.

Recall that analogous statements for essential spectra of discrete Schrödinger operators have been shown, for instance, in [3] ($N = 3$, proven studying the two-particle channels) and in [1, 28] ($N = 4$, proven using Faddeev–Yakubovskiy equations). We prove theorem 3.1 using diagrammatic techniques of Hunziker [14, 34].

**Proof of theorem 3.1.** The first assertion follows directly from the spectral theory of self-adjoint operators, as the family of spectral projections associated with $H(K)$ is increasing and $\sigma_{\text{ess}}(H(K))$ is its continuity points [33].

We may suppose that at least one $V_0$ is non-zero, otherwise the result is trivial. By virtue of theorem 2.2,

$$\bigcup_{D \in \Xi, \#D \geq 2} \sigma(H^D(K)) = \bigcup_{D \in \Xi, \#D \geq 2} \sigma(H^D(K)) \subseteq \sigma_{\text{ess}}(H(K)).$$

Under the terminology and notation of [14], the ‘difficult’ part,

$$\sigma_{\text{ess}}(H(K)) \subseteq \bigcup_{\#D \geq 2} \sigma(H^D),$$

and the final assertion of the theorem are proved essentially the same as in [14] provided that the terms of the theorem expansion

$$G(z) = \sum_{n=0}^{\infty} \sum_{(j_1), \ldots, (j_N)} G_0(z) V_{n,j_1} G_0(z) V_{n,j_2} \ldots G_0(z) V_{n,j_N} G_0(z). \quad (3.1)$$

corresponding to connected graphs, are compact. Here $G_0(z) = (H_0(K) - z)^{-1}$, $G(z) = (H(K) - z)^{-1}$, and $|z|$ is large enough that the series (3.1) converges absolutely. For the convenience of the reader we prove this fact in two steps.

**Step 1.** Let $u \in C^0((\mathbb{T}^d)^{N-1})$ be a non-identically zero function with Fourier coefficients $\hat{u} \in \ell^1((\mathbb{Z}^d)^{N-1})$. Assume that $E_0$ is the multiplication operator by the function $u(\cdot)$ in $L^2((\mathbb{Z}^d)^{N-1})$ and let $T = V_{\alpha_1, \beta_1} E_0 \ldots E_0 V_{\alpha_N, \beta_N}$. $\alpha_j, \beta_j \in \{1, \ldots, N\}$, $\alpha_j < \beta_j$. If the graph corresponding to this operator is connected, then $T$ is compact.

It suffices to show that $\hat{T}(B)$ is relatively compact in $\ell^2((\mathbb{Z}^d)^{N-1})$, where $B$ is the unit ball in $\ell^2((\mathbb{Z}^d)^{N-1})$, and

$$\hat{T} = \mathcal{F}_{N-1}^{-1} T \mathcal{F}_{N-1} := \hat{V}_{\alpha_1, \beta_1} \hat{E}_0 \ldots \hat{E}_0 \hat{V}_{\alpha_N, \beta_N}$$

with $\hat{E}_0 = \mathcal{F}_{N-1}^{-1} E_0 \mathcal{F}_{N-1}$. By the Kolmogorov criterion, we need to prove that for every $\eta > 0$ there exists $R_\eta > 0$ such that

$$Q_R(\hat{f}) := \sum_{|x| > R} |(\mathcal{F}(\hat{f}))(x)|^2 \leq \eta, \quad R \geq R_\eta, \quad \hat{f} \in B. \quad (3.2)$$

We recall that $\hat{V}_{ji}, 1 \leq i < j \leq N$, is the multiplication operator by $\hat{v}_{ji}(x_i - x_j)$ in $\ell^2((\mathbb{Z}^d)^{N-1})$ with the convention $x_N = 0$. Since $\|\hat{f}\| \leq 1$, from definitions of $\hat{E}_0$ and $\hat{V}_{ji}$ one can easily verify that

$$\sum_{|x| > R} |(\mathcal{F}(\hat{f}))(x)|^2 \leq \sum_{|x| > R} \left| \sum_{j=1}^{N} \hat{S}_{ji}(\hat{f}) \right|^2 \leq \sum_{|x| > R} \sum_{j=1}^{N} \left| \hat{S}_{ji}(\hat{f}) \right|^2 \leq \sum_{|x| > R} \sum_{j=1}^{N} \left| \hat{V}_{ji}(x_j - x_i) \right|^2 \leq \sum_{|x| > R} \sum_{j=1}^{N} \left| \hat{v}_{ji}(x_j - x_i) \right|^2.$$
\[ Q_R(\hat{f}) \leq \sup_{|x| > R} \left| \sum_{y_1, \ldots, y_{n-1} \in (\mathbb{Z}^d)^{n-1}} \prod_{j=1}^{n-1} |\hat{u}(y_j)| \times \prod_{j=1}^{n} |\hat{v}_{\alpha_j \beta_j} (x_{\alpha_j} - x_{\beta_j} + \sum_{i=1}^{j-1} (y_i)_{\alpha_i} - (y_i)_{\beta_i})|^2 \right. \]

Let \( M = \max_{\hat{g}} \sup_x |\hat{v}_{\hat{g}}(x)| \) (clearly, \( M \in (0, +\infty) \)), and given \( m \in \mathbb{N} \),
\[ \mathbb{E}^m_L := \{ y = (y_1, \ldots, y_m) \in ((\mathbb{Z}^d)^{N-1})^m : |y_i| < L, i = 1, \ldots, m, \} \quad L > 0. \]

By assumption on \( \hat{u} \), \( \sum_{y_1, \ldots, y_{n-1}} \prod_{j=1}^{n-1} |\hat{u}(y_j)| \leq \| \hat{u} \|_{p_1}^n < +\infty \), and thus, given \( \eta > 0 \), there exists \( L_\eta > 0 \) such that
\[ \sum_{y \in ((\mathbb{Z}^d)^{N-1})^{n-1} \setminus \mathbb{E}^m_L} \prod_{j=1}^{n-1} |\hat{u}(y_j)| \leq \frac{\eta}{2M^{2n}}. \]

In particular, for such \( L \),
\[ \sum_{y \in ((\mathbb{Z}^d)^{N-1})^{n-1} \setminus \mathbb{E}^m_L} \prod_{j=1}^{n-1} |\hat{u}(y_j)| \prod_{j=1}^{n} |\hat{v}_{\alpha_j \beta_j} (x_{\alpha_j} - x_{\beta_j} + \sum_{i=1}^{j-1} (y_i)_{\alpha_i} - (y_i)_{\beta_i})|^2 \leq \frac{\eta}{2}. \] (3.3)

Now let us estimate the finite sum
\[ A := \sum_{y \in \mathbb{E}_L^{n-1}} \prod_{j=1}^{n-1} |\hat{u}(y_j)| \prod_{j=1}^{n} |\hat{v}_{\alpha_j \beta_j} (x_{\alpha_j} - x_{\beta_j} + \sum_{i=1}^{j-1} (y_i)_{\alpha_i} - (y_i)_{\beta_i})|^2. \]

By (1.3) there exists \( r_\eta > 0 \) such that for any \( r > r_\eta \),
\[ \sup_{|y| > r} |\hat{v}_{\alpha_j \beta_i}(y)|^2 < \frac{\eta}{2\|\hat{u}\|_{p_1}^2 M^{2n-2}}, \quad j = 1, \ldots, n. \]

Since the graph corresponding to \( T \) is connected, if \( |x| \to \infty \), then at least one of \( |x_{\alpha_j} - x_{\beta_j}| \) tends to \( \infty \). Moreover, if \( y \in \mathbb{E}_L^{n-1} \), for each \( x \in (\mathbb{Z}^d)^{N-1} \) with \( |x| > Nr + nL \), there is \( j_0 = j_0(x) \) such that \( |x_{\alpha_{j_0}} - x_{\beta_{j_0}} + \sum_{i=j_0}^{j-1} ((y_i)_{\alpha_i} - (y_i)_{\beta_i})| \geq r \), and hence for such \( x \),
\[ \prod_{j=1}^{n} |\hat{v}_{\alpha_j \beta_j} (x_{\alpha_j} - x_{\beta_j} + \sum_{i=1}^{j-1} (y_i)_{\alpha_i} - (y_i)_{\beta_i})|^2 \leq \sup_{|y| > r} \left| \hat{v}_{\alpha_{j_0} \beta_{j_0}} (y_{\alpha_{j_0}} - y_{\beta_{j_0}} + \sum_{i=1}^{j_0-1} ((y_i)_{\alpha_i} - (y_i)_{\beta_i})) \right|^2 \times \prod_{j=1}^{n} \left| \hat{v}_{\alpha_j \beta_j} (x_{\alpha_j} - x_{\beta_j} + \sum_{i=1}^{j-1} (y_i)_{\alpha_i} - (y_i)_{\beta_i})) \right|^2 \leq \frac{\eta}{2\|\hat{u}\|_{p_1}^2}, \]
whence $A \leq \eta/2$. Now (3.2) follows from this and (3.3) with the choice $R_\eta = Nr_\eta + nL_\eta$.

**Step 2.** The operator $T = G_0(z) V_{\alpha_1 \beta_1} G_0(z) \ldots G_0(z) V_{\alpha_n \beta_n}, 1 \leq \alpha_j < \beta_j \leq N, j = 1, \ldots, n$, is compact if and only if the corresponding graph is connected.

Assume that the corresponding graph is not connected and the associated cluster decomposition $D = \{D_1, \ldots, D_l\}$ with $l \geq 2$. Without loss of generality, we may assume that $N \in D_i$. Define the Abelian group of unitary operators $U_j, s \in \mathbb{Z}^d$ in $L^2((\mathbb{T}^d)^{N-1})$ as follows:

$$(U_j f)(p) = \exp(i \sum_{a \in D_i} p_a \cdot s) f(p).$$

Clearly, $T$ commutes with $U_j$ as $V_{\alpha_1 \beta_1}, \alpha_j \beta_j \in D, j = 1, \ldots, n,$ and $G_0(z)$ commutes with $U_j$. Choose $f_0 \in L^2((\mathbb{T}^d)^{N-1})$ such that $Tf_0 \neq 0$. Since $U_i \to 0$ weakly as $s \to \infty, U_i f_0 \to 0$ in $L^2((\mathbb{T}^d)^{N-1})$ as $s \to \infty$, but

$$\|T(U_j f_0)\| = \|U_j(T f_0)\| = \|T f_0\| \neq 0.$$ 

Hence, $T$ is not compact.

Now assume that the graph is connected. Recall that $G_0(z)$ is the multiplication operator by (the non-zero function) $u = \{E_k - z\}^{-1}, E_k(p_1, \ldots, p_{N-1}) = E(p_1, \ldots, p_{N-1}, K - p_1, \ldots, p_{N-1})$ and $E$ is defined in (1.4). Since $u \in C^0(\mathbb{T}^{N-1})$, there exists a sequence of non-zero trigonometric polynomials $u_j : (\mathbb{T}^d)^{N-1} \to \mathbb{C}, j \in \mathbb{N},$ such that $u_j$ converges uniformly to $u$ as $j \to \infty$. Let $E_j$ be the multiplication operator by the function $u_j$ in $L^2((\mathbb{T}^d)^{N-1})$ note that $E_j \to G_0(z)$ in the operator norm as $j \to +\infty$. Since $\hat{u}_j = F_{N-1}^{-1} u_j \in L^1((\mathbb{Z}^d)^{N-1})$, by step 1 the operators

$$T_j = E_j V_{\alpha_1 \beta_1} E_j \ldots V_{\alpha_n \beta_n}, j = 1, 2, \ldots,$$

are compact. Now, as $G_0(z)$ and $V_{\beta_j}$ are bounded operators and $n$ is finite, $T_j$ converges to $T$ in the operator norm, implying that $T$ is also compact. \square

4. Existence of $N$-particle bound states

Unless otherwise stated, in this section we always suppose

$$\begin{cases} 
(a) \quad \hat{\epsilon}_i, i = 1, \ldots, N, \text{ satisfies (1.2)}, \\
(b) \quad \hat{\nu}_j \in L^1(\mathbb{Z}^d) \text{ is a nonnegative even function}, \\
(c) \quad \sum_{\gamma \in \mathbb{Z}^d} \nu_j(\gamma) > 0, 1 \leq i < j \leq N, 
\end{cases} \quad (4.1)$$

so that the system of particles is purely attractive. Our aim is to provide some sufficient conditions to have at least one $N$-particle bound state below the lowest edge of the essential spectrum $\Sigma := \Sigma(K) = \inf \sigma_{es}(H(K)), K \in \mathbb{T}^d$. Recall that by theorems 2.1 and 3.1 there exists a cluster decomposition $D = \{D_1, D_2\}$ such that $\Sigma = \min_{k \in \mathbb{T}^d} \sigma(h^D(k))$, where

$$h^D(k) = h^{D_1}(k) \otimes I^{D_2}(K - k) + I^{D_1}(K - k) \otimes h^{D_2}(K - k), \quad k \in \mathbb{T}^d. \quad (4.2)$$

Define

$$\Sigma(k) := \min \sigma(h^D(k)).$$
By the norm-continuity of \( k \mapsto h_D(k) \), the map \( k \mapsto \Sigma(k) \) is uniformly continuous. Hence, there exists \( k_0 \in \mathbb{T}^d \) such that \( \Sigma(k_0) = \Sigma \).

**Lemma 4.1.** Let \( z_i(k) := \inf \sigma(h^D_i(k)), i = 1, 2. \) Then \( \Sigma = z_1(k_0) + z_2(K - k_0) \).

**Proof.** Let \( n_\nu := |D_\nu|, \) \( \nu = 1, 2. \) For \( \Psi(p, q) = \phi(p) \psi(q), \) \( \phi \in L^2(\mathbb{T}^d)^{n_1-1}, \psi \in L^2(\mathbb{T}^d)^{n_2-1} \), we have

\[
(h^D_i(k_0) \Psi, \Psi) = (h^D_i(k_0) \phi, \phi) \| \psi \|_{L^2}^2 + (h^D_i(K - k_0) \psi, \psi) \| \phi \|_{L^2}^2.
\]

Now, choosing \( \| \phi \|_{L^2} = \| \psi \|_{L^2} = 1 \) and taking infimum over \( \phi \) and \( \psi \), we deduce

\[
\Sigma \leq \inf_{\Psi} (h^D_i(k_0) \Psi, \Psi) \leq \inf_{\phi} (h^D_i(k_0) \phi, \phi) + \inf_{\psi} (h^D_i(K - k_0) \psi, \psi)
= z_1(k_0) + z_2(K - k_0).
\]

On the other hand, for any \( \eta > 0 \), let us choose \( \Phi(p, q) = \sum_{i=1}^{M} \phi_i(p) \psi_i(q) \in L^2(\mathbb{T}^d)^{n_1-1} \otimes L^2(\mathbb{T}^d)^{n_2-1} \) such that both \( \{ \phi_i \} \) and \( \{ \psi_i \} \) are orthonormal systems, and

\[
\Sigma + \eta > (h^D_i(k_0) \Phi, \Phi) \| \phi_i \|_{L^2}^2 + (h^D_i(K - k_0) \psi_i, \psi_i) \| \phi_i \|_{L^2}^2.
\]

But as \( \| \phi_i \|_{L^2}^2 = M, \) and \( (h^D_i(k_0) \phi_i, \phi_i) \geq z_1(k_0), \)

\[
(h^D_i(K - k_0) \psi_i, \psi_i) \geq z_2(K - k_0),
\]

we have

\[
\Sigma + \eta > z_1(k_0) + z_2(K - k_0) \sum_{i=1}^{M} \| \phi_i \|_{L^2}^2 \| \psi_i \|_{L^2}^2 = z_1(k_0) + z_2(K - k_0).
\]

Since \( \eta > 0 \) is arbitrary, the assertion of the lemma follows. \( \square \)

An essential tool in the proof of the existence of bound states is the following.

**Theorem 4.1.** Suppose \( (4.1), K \in \mathbb{T}^d, \) and \( \Sigma = \Sigma(K) = \inf \sigma_{\text{res}}(H(K)). \) Let \( D = (D_1, D_2) \) be a cluster decomposition such that the cluster operator \( h_D^2(k) \), defined as in \( (4.2) \), satisfies

\[
\Sigma = \min_{k \in \mathbb{T}^d} \min \sigma(h_D^2(k))
\]

and let \( z_1(k_0) \) and \( z_2(K - k_0) \), given in \( \text{Lemma 4.1} \), be isolated eigenvalues of \( h^D_i(k_0) \) and \( h^D_i(K - k_0) \), respectively. Then \( \sigma_{\text{res}}(H(K)) \cap (-\infty, \Sigma) \) is nonempty.

**Proof.** Without loss of generality assume that \( D_1 = \{1, \ldots, n\} \) and \( D_2 = \{n + 1, \ldots, N\}. \) By the norm-continuity of \( k \mapsto h^D_i(k) \) and \( k \mapsto h^D_i(K - k) \), and by the assumption of the theorem, \( z_1(k) \) and \( z_2(K - k) \), \( k \in U_{\rho}(k_0) \subset \mathbb{T}^d \) are still eigenvalues of \( h^D_i(k) \) and \( h^D_i(K - k) \) respectively, where \( U_{\rho}(k_0) \) is a sufficiently small \( \rho \)-neighborhood of \( k_0. \) Now we work in spaces \( L^2(\mathbb{T}_K^d), L^2(\mathbb{T}_K^d) \) and \( L^2(\mathbb{T}_K^{d-n}) \). Let \( \tilde{\phi}(k; \cdot) \in L^2(\mathbb{T}_K^d) \) (resp. \( \tilde{\psi}(K - k; \cdot) \in L^2(\mathbb{T}_K^{d-n}) \)) be a normalized eigenfunction of \( h^D_i(k) \) (resp. \( h^D_i(K - k) \)) associated with \( z_1(k) \) (resp. \( z_2(K - k) \)), \( k \in U_{\rho}(k_0). \) Notice that we have also

\[
\tilde{h}^D_i(k)\tilde{\phi}(k; \cdot) = z_1(k) \tilde{\phi}(k; \cdot) \quad \text{and} \quad \tilde{h}^D_i(K - k)\tilde{\psi}(K - k; \cdot) = z_2(K - k) \tilde{\psi}(K - k; \cdot)
\]
for any \( k \in U_\rho(k_0) \). Hence, we can suppose that both \( \phi(k, \cdot) \) and \( \psi(k, \cdot) \) are real-valued. We extend them to 0 for \( k \in \mathbb{R}^d \setminus U(k_0) \),

For a sequence \( \rho_l \in (0, \rho) \) converging to 0, define \( \Psi_l \in L^2(\mathbb{R}_K^d) \), \( l \in \mathbb{N} \), as

\[
\Psi_l(p_1, \ldots, p_N) := |U_{\rho_l}(k_0)|^{-1/2} \chi_{U_{\rho_l}(k_0)} \left( \sum_{i=1}^n p_i \right) \phi \left( \sum_{i=1}^n p_i p_1, \ldots, p_n \right) \psi \left( \sum_{i=n+1}^N p_i p_{n+1}, \ldots, p_N \right),
\]

where \( \chi_A \) is the characteristic function of a set \( A \). Then, by (2.2) and (2.4),

\[
(\Psi_l, \Psi_l) = |U_{\rho_l}(k_0)|^{-1} \int_{U_{\rho_l}(k_0)} dk \int_{\mathbb{R}^d} \left| \phi \left( k; p_1, \ldots, p_n \right) \right|^2 dp \\
\times \int_{\mathbb{R}^d} \left| \psi \left( K - k; p_{n+1}, \ldots, p_N \right) \right|^2 dp = 1.
\]

Moreover, by virtue of (2.3), (2.5), and (1.8), as well as the definitions of \( \phi \) and \( \psi \), we have

\[
(\tilde{H}^{\nu_l}(K) \Psi_l, \Psi_l) = |U_{\rho_l}(k_0)|^{-1} \int_{U_{\rho_l}(k_0)} dk \left( z_1(k) + z_2(K - k) \right)
\]

and

\[
(\tilde{V}_{\nu_l} \Psi_l, \Psi_l) = |U_{\rho_l}(k_0)|^{-1} \int_{\mathbb{R}^d} v_{\nu_l}(t) dt \int_{U_{\rho_l}(k_0)} dk \int_{\mathbb{R}^d} \phi \left( k - t; p_1, \ldots, p_n - t \right) \phi \left( k; p_1, \ldots, p_n \right) dp \\
\times \int_{\mathbb{R}^d} \psi \left( K - k + t; p_{n+1}, \ldots, p_N + t \right) \psi \left( K - k; p_{n+1}, \ldots, p_N \right) dp.
\]

By assumption (4.1), \( \tilde{V}_l \geq 0 \), and thus

\[
(\tilde{H}(K) \Psi_l, \Psi_l) \leq (\tilde{H}^{\nu_l}(K) \Psi_l, \Psi_l) - (\tilde{V}_{\nu_l} \Psi_l, \Psi_l).
\]

Since the maps \( k \mapsto z_1(k) + z_2(K - k) \), \( k \mapsto \phi(k, \cdot) \), and \( k \mapsto \psi(K - k, \cdot) \) are continuous in \( U_{\rho_l}(k_0) \), taking \( \liminf \) as \( l \to +\infty \) in (4.4), and using the relations \( \Sigma = z_1(k_0) + z_2(K - k_0) \) and \( \phi(k, \cdot) = 0 \), \( k \in \mathbb{R}^d \setminus U_{\rho_l}(k_0) \), we obtain

\[
\inf \sigma(H(K)) \leq \Sigma - \int_{U_{\rho_l}(k_0)} v_{\nu_l}(t) dt \int_{\mathbb{R}^d} \phi \left( k_0 - r; p_1, \ldots, p_n - t \right) \phi \left( k_0; p_1, \ldots, p_n \right) dp \\
\times \int_{\mathbb{R}^d} \psi \left( K - k_0 + r; p_{n+1}, \ldots, p_N + t \right) \psi \left( K - k_0; p_{n+1}, \ldots, p_N \right) dp.
\]

Since

\[
v_{\nu_l}(0) = (2\pi)^{-d/2} \sum_{x \in \mathbb{Z}^d} \tilde{v}_{\nu_l}(x) > 0 \quad \text{(recall (1.5) and (4.1))},
\]

\[
\int_{\mathbb{R}^d} \phi \left( k_0; p_1, \ldots, p_n \right) \phi \left( k_0; p_1, \ldots, p_n \right) dp = 1,
\]

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and
\[ \int_{\mathbb{R}^{n-1} \times k_0} \psi(K - k_0; p_{n+1}, \ldots, p_N) \psi(K - k_0; p_{n+1}, \ldots, p_N) \, dp = 1, \]
by the continuity, possibly taking smaller \( \rho \), we may suppose that \( v_n(t) > 0 \),
\[ \int_{\mathbb{R}^{n-1} \times k_0} \phi(k_0 - t; p_1, \ldots, p_n - t) \phi(k_0; p_1, \ldots, p_n) \, dp > 0, \]
and
\[ \int_{\mathbb{R}^{n-1} \times k_0} \psi(K - k_0 + t; p_{n+1}, \ldots, p_N + t) \psi(K - k_0; p_{n+1}, \ldots, p_N) \, dp > 0, \]
for all \( t \in U_\rho(0) \). Then \( \inf \sigma(\tilde{H}(K)) < \Sigma \), whence \( \inf \sigma(\tilde{H}(K)) \in \sigma_{\text{disc}}(\tilde{H}(K)) \).

**Remark 4.1.** According to the proof of theorem 4.1, its assertion is still valid if we replace the hypothesis (4.1(c)) with a weaker assumption
\[ \sum_{x \in \mathbb{Z}^2} \sum_{i < j, i \in D_1, j \in D_2} \tilde{v}_{ij}(x) > 0. \]

**Remark 4.2.** The proof of theorem 4.1 may fail if there are no intercluster interactions, i.e. the hypothesis (4.1(c)) is replaced by
\[ \sum_{x \in \mathbb{Z}^2} \sum_{i < j, i \in D_1, j \in D_2} \tilde{v}_{ij}(x) = 0. \]

For example, consider an attractive three-particle system in \( \mathbb{Z}^2 \) in which particles 1 and 2 interact via a zero-range pair potential and particle 3 does not interact with 1 and 2, i.e. \( \tilde{v}_{12} = \mu \delta_{x_0}, \) \( \mu > 0, \) and \( \tilde{v}_{13} = \tilde{v}_{23} = 0. \) It is known that the ground state energy \( z(k), k \in \mathbb{T}^2, \) of the two-particle operator \( \mathbf{h}_{12}^0(k), \) acting in \( L^2(\mathbb{T}^2) \) as in (4.7), is always an eigenvalue. Therefore, by lemma 4.1 the lowest edge \( \Sigma(K), K \in \mathbb{T}^2, \) of the essential spectrum of the three-particle discrete Schrödinger operator \( H(K) \) is given by
\[ \Sigma(K) = \min_{k \in \mathbb{T}^2} \left( z(k) + \varepsilon_3(K - k) \right), \]
where \( \varepsilon_3 \) is the dispersion relation of the third particle. Furthermore, there exists \( k_0 \in \mathbb{T}^2 \) such that \( \Sigma(K) = z(k_0) + \varepsilon_3(K - k_0), \) so that for the cluster decomposition \( C = \{ \{1, 2\}, \{3\} \} \) all assumptions (but (4.1(c)) of theorem 4.1 hold. However, since \( V_{13} = V_{23} = 0, \) for the function \( \Psi_I, \) defined in (4.3), one has
\[ (\tilde{H}(K)\Psi_I, \Psi_I) = (\tilde{H}^0(K)\Psi_I, \Psi_I), \]
and therefore if we pass to the limit in (4.5) as \( l \to +\infty \) we get the trivial inequality
\[ \inf \sigma(\tilde{H}(K)) \leq \Sigma(K), \]
which does not give any useful information about \( \inf \sigma(\tilde{H}(K)) \).
More practical sufficient condition to have at least one \( N \)-particle bound state is given in the following corollary of theorem 4.1.

**Theorem 4.2.** Suppose (4.1) and

\[
\sigma_{\text{disc}}(h^l(k)) \cap (-\infty, \min \sigma_{\text{ess}}(h^l(k))) \neq \emptyset \quad \text{for all } k \in \mathbb{T}^d,
\]

where

\[
h^l(k) = h^l_0(k) - v_{ij}.
\]

\( h^l(k) \) is the multiplication operator by \( E^l(k, q) := \frac{1}{m_j} \epsilon_j(q) + \frac{1}{m_i} \epsilon_i(k - q) \) in \( L^2(\mathbb{T}^d) \) and \( F_1^{-1} v_{ij} F_1 \) is the multiplication operator by \( \hat{v}_{ij} \) in \( L^2(\mathbb{Z}^d) \). Then \( \sigma_{\text{disc}}(H(K)) \cap (-\infty, \Sigma(K)) \neq \emptyset \) for any \( K \in \mathbb{T}^d \).

**Proof.** The assertion of the theorem follows from the following

**Claim.** Let \( \mathcal{M} \) be an attractive system of particles moving in \( \mathbb{Z}^d \) for which (4.1) holds and given a subsystem \( \mathcal{S} \subseteq \mathcal{M} \), let \( H_\mathcal{S}(K) \), \( K \in \mathbb{T}^d \), denote the Schrödinger operator, associated to the Hamiltonian of \( \mathcal{S} \). Suppose that

\[
\inf \sigma(H_\mathcal{S}(K)) \subseteq \sigma_{\text{disc}}(H_\mathcal{S}(K))
\]

for every \( K \in \mathbb{T}^d \) and two-particle subsystem \( \mathcal{S} \subseteq \mathcal{M} \). Then

\[
\inf \sigma(H_\mathcal{U}(K)) \subseteq \sigma_{\text{disc}}(H_\mathcal{U}(K))
\]

for any \( \mathcal{U} \subseteq \mathcal{M} \).

We prove (4.8) by an induction argument on the number of particles \( \# \mathcal{U} \) in \( \mathcal{U} \).

If \( \# \mathcal{U} = 1 \), i.e. \( \mathcal{U} \) consists of a single particle \( \alpha \), by the definition of the direct integral (1.6) and the fiber operator (1.7), \( H_\mathcal{U}(K) \) is a multiplication operator by \( \epsilon_\alpha(k) \) in \( \mathbb{C} \), where \( \epsilon_\alpha \) is the dispersion relation of the particle \( \alpha \). Therefore,

\[
\sigma(H_\mathcal{U}(K)) = \sigma_{\text{disc}}(H_\mathcal{U}(K)) = \{ \epsilon_\alpha(K) \}
\]

and (4.8) holds. Recall that, by assumption, (4.8) is true if \( \# \mathcal{U} = 2 \).

Suppose that \( \# \mathcal{U} \geq 3 \), and

\[
\inf \sigma(H_\mathcal{S}(k)) \subseteq \sigma_{\text{disc}}(H_\mathcal{S}(k)), \quad k \in \mathbb{T}^d,
\]

for any \( \mathcal{S} \subseteq \mathcal{U} \). We prove that (4.9) holds also for \( \mathcal{S} = \mathcal{U} \).

Indeed, fix \( K \in \mathbb{T}^d \). In view of section 2.1 and by theorem 3.1 and lemma 4.1, there exist a cluster decomposition \( \{ \mathcal{D}_1, \mathcal{D}_2 \} \) of \( \mathcal{U} \) and \( k_0 \in \mathbb{T}^d \) such that

\[
\inf \sigma_{\text{ess}}(H_\mathcal{U}(K)) = \inf \sigma(H_{\mathcal{D}_1}(k_0)) + \inf \sigma(H_{\mathcal{D}_2}(K - k_0)).
\]

By (4.9), applied with \( \mathcal{U} = \mathcal{D}_i, i = 1, 2, k = k_0 \) and \( k = K - k_0 \), we have

\[
\inf \sigma(H_{\mathcal{D}_1}(k_0)) \subseteq \sigma_{\text{disc}}(H_{\mathcal{D}_1}(k_0)) \quad \text{and} \quad \inf \sigma(H_{\mathcal{D}_2}(K - k_0)) \subseteq \sigma_{\text{disc}}(H_{\mathcal{D}_2}(K - k_0)),
\]
therefore, by theorem 4.1, \(\inf \sigma(H_{\Omega}(K)) \in \sigma_{\text{disc}}(H_{\Omega}(K))\). The claim is proved.

Under the notation of theorem 4.2, define the Birman–Schwinger operator \(B_{ij}(k, z)\), associated to \(h_{ij}^{(z)}(k)\), the nonnegative compact and self-adjoint operator in \(L^2(\mathbb{T}^d)\) as

\[
B_{ij}(k, z) := v_{ij}^{-1/2}(h_{ij}^{(z)}(k) - z)^{-1}v_{ij}^{-1/2}, \quad z < \min_q \varepsilon_q(k, q), \quad 1 \leq i < j \leq N. 
\]

(4.10)

The following corollary of theorem 4.2 provides a condition, which implies (4.6), and will be used in the following section.

**Corollary 4.1.** Let

\[
\sup_{k \in \mathbb{T}^d} \lim_{z \to \min_q \varepsilon_q(k, q) - 0} \frac{1}{\|B_{ij}(k, z)\|} < 1, \quad 1 \leq i < j \leq N. 
\]

(4.11)

Then \(\sigma_{\text{disc}}(H(K)) \cap (-\infty, \Sigma(K)) \neq \emptyset\) for any \(K \in \mathbb{T}^d\).

**Proof.** By the celebrated Birman–Schwinger principle [7, 35], \(z < \min_q \varepsilon_q(k, q)\) is an eigenvalue of \(h_{ij}^{(z)}(k)\) if and only if 1 is an eigenvalue of \(B_{ij}(k, z)\). Since \(B_{ij}(k, \cdot)\) is increasing and continuous on \((-\infty, \min_q \varepsilon_q(k, q))\), and

\[
\begin{cases} 
\lim_{z \to -\infty} \|B_{ij}(k, z)\| = 0, \\
\lim_{z \to \min_q \varepsilon_q(k, q) - 0} \|B_{ij}(k, z)\| > 1, \quad k \in \mathbb{T}^d, \quad \text{(by assumption (4.11)),}
\end{cases}
\]

there exists \(z(k) < \min_q \varepsilon_q(k, q)\) such that \(\|B_{ij}(k, z(k))\| = 1\). By compactness of \(B_{ij}(k, z(k))\), 1 is its eigenvalue; hence, by the Birman–Schwinger principle, \(z(k)\) is an eigenvalue of \(h_{ij}^{(z)}(k)\) below the essential spectrum. Now an application of theorem 4.2 implies the non-emptiness of \(\sigma_{\text{disc}}(H(K)) \cap (-\infty, \Sigma(K))\) for any \(K \in \mathbb{T}^d\).

In the case of a repulsive system of particles (i.e. \(\hat{v}_{ij} \leq 0\), applying theorem 4.2 to \(\hat{H}(K)\), we get the following analogue of theorem 4.2.

**Theorem 4.3.** Suppose that

\[
\begin{cases} 
\varepsilon_i, i = 1, \ldots, N, \text{satisfies (1.2)}, \\
\hat{v}_{ij} \in L^1(\mathbb{T}^d) \text{ is a non-positive even function and } \sum_{\varepsilon_i} \hat{v}_{ij}(x) \leq 0, 1 \leq i < j \leq N,
\end{cases}
\]

and \(\sigma_{\text{disc}}(h_{ij}^{(z)}) \cap (\max \sigma_{\text{ess}}(h_{ij}^{(z)}), +\infty) \neq \emptyset\) for all \(K \in \mathbb{T}^d\), where

\[
h_{ij}^{(z)}(k) = h_{ij}^{(z)}(k) - \hat{v}_{ij}.
\]

\(h_{ij}^{(z)}(k)\) is the multiplication operator by \(\varepsilon_{ij}(k, q) := \frac{1}{m} \varepsilon_i(q) + \frac{1}{m} \varepsilon_j(k - q)\) in \(L^2(\mathbb{T}^d)\) and \(\mathcal{F}_1^{-1}v_{ij}\mathcal{F}_1\) is the multiplication operator by \(\hat{v}_{ij}\) in \(\ell^2(\mathbb{Z}^d)\), \(1 \leq i < j \leq N\). Then \(\sigma_{\text{disc}}(H(K)) \cap (\Theta(K), +\infty) \neq \emptyset\) for any \(K \in \mathbb{T}^d\), where

\[
\Theta(K) := \sup \sigma_{\text{ess}}(H(K)), \quad K \in \mathbb{T}^d.
\]

**Remark 4.3.** Theorem 4.3 is not observed in the continuous case. As in the latter, the essential spectrum is a half line.
Remark 4.4.

(a) The sign definiteness of $\hat{v}_{ij}$ is essential in our proofs since it provides that all perturbations $V_{ij}$ have the same sign so that inequality (4.4) is valid.

(b) The results hold true also in the presence of several particles of infinite mass.

5. Some examples

In this section we provide some sufficient conditions to have at least one $N$-particle bound state in purely attractive systems. By virtue of corollary 4.1, we need just to provide some conditions for dispersion relations $\hat{\varepsilon}_i$ and pair potentials $\hat{v}_{ij}$ which ensure the validity of (4.11).

5.1. $N$-particle system with zero-range potentials

Let $\mathbf{h}^{\hat{v}}(k), k \in T^d$ be as in theorem 4.2 with $\hat{v}_{ij} = \mu_{ij} \hat{\delta}_x$, $\mu_{ij} > 0, 1 \leq i < j \leq N$, and let $\mathcal{B}_ij^0(k,z)$ be the Birman–Schwinger operator, associated with $\mathbf{h}^{\hat{v}}(k)$ defined as in (4.10). Then, for any $k \in T^d$ and $z < \min_p \hat{\varepsilon}_i(k,p)$,

$$\|\mathcal{B}_ij^0(k,z)\|^2 = \frac{\mu_{ij}}{(2\pi)^{d/2}} \int_{T^d} \frac{dq}{\hat{\varepsilon}_ij(k,q) - z}.$$ 

Set

$$\hat{\varepsilon}^{\text{min}}_ij(k) := \min_q \hat{\varepsilon}_ij(k,q), \quad \hat{\varepsilon}^{\text{max}}_ij(k) := \max_q \hat{\varepsilon}_ij(k,q), \quad k \in T^d.$$ 

Since

$$\begin{cases} \hat{\varepsilon}^{\text{min}}_ij(k) \geq m_o := \frac{1}{m_i} \min_q \varepsilon_i(q) + \frac{1}{m_j} \min_q \varepsilon_j(q) \\ \hat{\varepsilon}^{\text{max}}_ij(k) \leq m^o := \frac{1}{m_i} \max_q \varepsilon_i(q) + \frac{1}{m_j} \max_q \varepsilon_j(q) \end{cases} \quad \text{for all } k \in T^d,$$

we have\(^5\)

$$A_{ij}(k) := \frac{1}{(2\pi)^{d/2}} \int_{T^d} \frac{dq}{\hat{\varepsilon}_ij(k,q) - \hat{\varepsilon}^{\text{min}}_ij(k)} \in \left[ \frac{(2\pi)^{d/2}}{m^o - m_o}, +\infty \right], \quad k \in T^d,$$

so that $\sup_{k \in T^d} A_{ij}(k)^{-1} \leq (2\pi)^{-d/2}(m^o - m_o) < +\infty$. Therefore, supposing

$$\mu_{ij} > \sup_{k \in T^d} A_{ij}(k)^{-1}, \quad (5.1)$$

we get (4.11). Now corollary 4.1 implies that the corresponding $N$-particle system has a non-empty discrete spectrum below the lower edge of its essential spectrum.

5.2. $N$-particle system with arbitrary potentials

Let $\mathbf{h}^{\hat{v}}(k), k \in T^d$, be as in theorem 4.2 with $\hat{v}_{ij} \geq \mu_{ij} \hat{\delta}_x$, and $\mu_{ij}$ satisfies (5.1), $1 \leq i < j \leq N$,

\(^5\) We suppose $(2\pi)^{d/2} = +\infty$ if $m_o = m^o$ (i.e. when $\hat{\varepsilon}_i$ is identically constant).
and let $h^{ij}(k)$ be as in section 5.1. Since $h^{ij}(k) \leq h^{ij}(k)$, by the classical Weyl theorem on the essential spectrum [34, section XIII.4],

$$\sigma_{\text{ess}}(h^{ij}(k)) = \sigma_{\text{ess}}(h^{ij}(k)) = [E^{\text{min}}_{ij}(k), E^{\text{max}}_{ij}(k)].$$

Then applying the Birman–Schwinger principle to $h^{ij}(k)$ and using (5.1) we get

$$\inf \sigma(h^{ij}(k)) \leq \inf \sigma(h^{ij}(k)) < \inf \sigma_{\text{ess}}(h^{ij}(k)) = E^{\text{min}}_{ij}(k) = \inf \sigma_{\text{ess}}(h^{ij}(k)), \quad k \in \mathbb{T}^d.$$  

Now, by virtue of theorem 4.2, $\sigma_{\text{disc}}(H(K)) \cap (-\infty, \Sigma(K)) \neq \emptyset$ for any $K \in \mathbb{T}^d$.

When the dispersion functions are of the form $\varepsilon_i = a_i \varepsilon$, $a_i \in \mathbb{R}$, $i = 1, \ldots, N$, where $\varepsilon : \mathbb{T}^d \to \mathbb{R}$ is a conditionally negative definite function (see e.g. [4, 34]) satisfying (4.1), by [21, proofs of theorems 1 and 2], one has $A_{ij}(k) \geq A_{ij}(0) > 0$ for all $k \in \mathbb{T}^d$, and thus, for such systems, the sufficiency assumption (5.1) reads as

$$\mu_{ij} > A_{ij}(0)^{-1}. \quad (5.2)$$

5.3. The standard discrete Laplacian case

Suppose that $m_i = 1$, $\hat{\varepsilon}_i = \hat{\varepsilon}$, $i = 1, \ldots, N$, where

$$\hat{\varepsilon}(x) := \begin{cases} d & x = 0, \\ \frac{-1}{2} |x| = 1, \\ 0 & |x| > 1, \end{cases}$$

and $\hat{\delta}_{ij} = \mu \delta_{ij}$, $\mu > 0$, $1 \leq i < j \leq N$. Then

$$\varepsilon(p) = \sum_{l=1}^{d} (1 - \cos p^{(l)})$$

is conditionally negative definite and $A_{ij}(k) = A(k)$ for all $ij$, where

$$A(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} \frac{dq}{2 \sum_{l=1}^{d} \cos \frac{\varepsilon^{(l)}}{2} (1 - \cos (\frac{\varepsilon^{(l)}}{2} - q))} \in (0, +\infty].$$

By [23] we have

$$A(k) = +\infty$$

if $d = 1, 2$, or if $d \geq 3$ and at most two coordinates of $k$ are not equal to $\pi$, and

$$A(k) \in (0, +\infty)$$

if $d \geq 3$ and at least three coordinates of $k$ are not equal to $\pi$. Obviously, $A(k) \geq A(0)$ for any $k \in \mathbb{T}^d$, and if $d \geq 3$, condition (5.2) becomes

$$\mu > A(0)^{-1} > 0.$$  

When $d = 1, 2$, (5.2) always holds since $A(0) = +\infty$. This shows that a bound state of an attractive system of $N$-particles moving in $\mathbb{Z}^d$, $d = 1, 2$, always exists. In particular, from here we recover the existence results of [22].
6. Discussion and conclusion

One of the interesting problems in the theory of Schrödinger operators is an appearance of bound states. In the discrete case, this phenomenon is sufficiently well understood in (purely attractive or purely repulsive) two-particle systems: if $d = 1, 2$, a bound state always exists as soon as particles interact, and if $d \geq 3$, bound states come out either from the threshold resonances or from the threshold eigenvalues [4]. Moreover, in this case the compactness of the pair potential allows us to establish necessary and sufficient conditions for the existence or nonexistence of bound states by means of Birman–Schwinger operators defined in (4.10) (see e.g. [4, 23] and references therein).

In multi-particle systems, an appearance of bound states and a necessary condition for their existence are not yet well understood. As recalled in the introduction, in three-particle systems, the Efimov effect provides one sufficient condition: if none of three two-particle subsystems has a bound state below the threshold of the essential spectrum, and at least two of them have threshold resonances, then the three-particle system has infinitely many bound states below the threshold [2, 5, 12, 38, 39, 43], see also review [30] and references therein for the Efimov physics. Besides the Efimov effect, in the continuous case some sufficient conditions to pair potentials ensuring the existence of at least one $N$-particle bound state can be found, for example, in [32, 48]. Moreover, in [6], the authors provide a complete description of spectra of a family of Schrödinger operators, associated with a system of two fermions and one different particle, moving on $\mathbb{R}^3$ and interacting via zero range pair potentials. In particular, using variational techniques efficiently in certain ranges of parameters (e.g. the mass of the different particle and the coupling constant of fermions), they have proved that the three-particle discrete spectrum is non-empty.

The existence of a bound state of a system of three identical bosons and of a system of two identical fermions and one different particle, moving in $\mathbb{Z}^d$, $d = 1, 2$, and interacting via zero-range pair potentials (of the same sign), has been recently established [22, 24]. In both works the main method is the use of Fredholm determinants associated with certain Birman–Schwinger-type operators.

The main contribution of the present work is to show that in a purely attractive (resp. purely repulsive) system of particles, a multi-particle bound state always exists below (resp. above) the essential spectrum provided that any two-particle subsystem has a bound state below (resp. above) its essential spectrum (theorems 4.2 and 4.3). Recall that our method heavily relies on the sign definiteness and non-vanishing of pair potentials (remark 4.2). We expect analogous results to be valid for an attractive (resp. repulsive) system of identical bosonic systems. This and some other extensions of results of the current paper for systems with some particle symmetries will be presented in an upcoming paper [18]. An analysis of the appearance of bound states of systems with arbitrary potentials will be addressed in these future investigations.

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