CONSTANT ANGLE SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. In last years, the study of the geometry of surfaces in product spaces $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ is developed by a large number of mathematicians. In [1] the authors study constant angle surfaces in $S^2 \times \mathbb{R}$, namely those surfaces for which the unit normal makes a constant angle with the tangent direction to $\mathbb{R}$. In this paper we classify constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane.

1. Preliminaries

Let $\tilde{M} = \mathbb{H}^2 \times \mathbb{R}$ be the Riemannian product of $(\mathbb{H}^2(-1), g_H)$ and $\mathbb{R}$ with the standard Euclidean metric. Denote by $\tilde{g}$ the product metric and by $\tilde{\nabla}$ the Levi Civita connection of $\tilde{g}$. Denote by $t$ the (global) coordinate on $\mathbb{R}$ and hence $\partial_t = \frac{\partial}{\partial t}$ is the unit vector field in the tangent bundle $T(\mathbb{H}^2 \times \mathbb{R})$ that is tangent to the $\mathbb{R}$-direction. Hence, the product metric can be written as

$$\tilde{g} = g_H + dt^2.$$ 

The Riemann-Christoffel curvature tensor $\tilde{R}$ of $\mathbb{H}^2 \times \mathbb{R}$ is given by

$$\tilde{R}(X, Y, Z, W) = -g_H(X_H, W_H)g_H(Y_H, Z_H) + g_H(X_H, Z_H)g_H(Y_H, W_H)$$

for any $X, Y, Z, W$ tangent to $\mathbb{H}^2 \times \mathbb{R}$. If $X$ is a tangent vector to $\mathbb{H}^2 \times \mathbb{R}$ we put $X_H$ its projection to the tangent space of $\mathbb{H}^2$.

Let $M$ be a surface in $\tilde{M} = \mathbb{H}^2 \times \mathbb{R}$. If $\xi$ is a unit normal to $M$, then the shape operator is denoted by $A$. We have the formulas of Gauss and Weingarten

\begin{align*}
(\text{G}) & \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \\
(\text{W}) & \quad \tilde{\nabla}_X \xi = -AX,
\end{align*}

for every $X$ and $Y$ tangent to $M$. Here $\nabla$ is the Levi Civita connection on $M$ and $h$ is a symmetric $(1, 2)$-tensor field taking values in the normal bundle and called the second fundamental form of $M$. We have $\tilde{g}(h(X, Y), \xi) = g(X, AY)$ for all $X, Y$ tangent to $M$, where $g$ is the restriction of $\tilde{g}$ to $M$.

Since $\partial_t$ is of unit length, we decompose $\partial_t$ as

$$\partial_t = T + \cos \theta \ \xi$$

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where $T$ is the projection of $\partial_t$ on the tangent space of $M$ and $\theta$ is the angle function, defined by

$$\cos \theta = \tilde{g}(\partial_t, \xi).$$

If $X, Y$ are tangent to $M$, then we have the following relation

$$g_H(X_H, Y_H) = g(X, Y) - g(X, T)g(Y, T).$$

Thus, if $R$ is the Riemannian curvature on $M$ the equation of Gauss can be written as

$$R(X, Y, Z, W) = g(AX, W)g(AY, Z) - g(AX, Z)g(AY, W) - g(X, W)g(Y, Z) + g(Y, Z)g(X, W) - g(X, Z)g(Y, T)g(Z, T) - g(Y, W)g(X, T)g(Z, T)$$

for every $X, Y, Z, W \in T(M)$.

Using the expression of the curvature $\tilde{R}$ of $\mathbb{H}^2 \times \mathbb{R}$, after a straightforward computation we write the equation of Codazzi

$$\nabla_X AY - \nabla_Y AX - A[X, Y] = \cos \theta (g(X, T)Y - g(Y, T)X)$$

for all $X, Y \in T(M)$.

Now we give the following

**Proposition 1.1.** Let $X$ be a tangent vector to $M$. We have

$$\begin{cases}
\nabla_X T = \cos \theta AX \\
X(\cos \theta) = -g(AX, T).
\end{cases}$$

**Proof.** For any $X$ tangent to $M$ we can write

$$X = X_H + g(X, T) \partial_t.$$

We have

$$\tilde{\nabla}_X \partial_t = \tilde{\nabla}_{X_H} \partial_t + g(X, T) \tilde{\nabla}_{\partial_t} \partial_t = 0.$$

On the other hand,

$$\tilde{\nabla}_X \partial_t = \tilde{\nabla}_X T + \tilde{\nabla}_X (\cos \theta) \xi = \nabla_X T + h(X, T) + X(\cos \theta) \xi - (\cos \theta) AX.$$

Identifying the tangent and the normal part respectively, one gets

$$\nabla_X T = \cos \theta AX \quad \text{and} \quad X(\cos \theta) \xi = -h(X, T).$$

Hence the conclusion.

If $\theta \in [0, \pi)$ is a constant angle, from the previous proposition the following relation holds: $g(AX, T) = 0$ for every $X$ tangent to $M$ (at $p$), which is equivalent to

$$g(AX, T) = 0, \quad \forall X \in T_p(M).$$

This means that, if $T \neq 0$, $T$ is a principal direction with principal curvature 0.

**Remark 1.2.** If $T = 0$ on $M$, then $\partial_t$ is always normal so, $M \subseteq \mathbb{H}^2 \times \{t_0\}$, for $t_0 \in \mathbb{R}$. 
If $T \neq 0$, we consider

\begin{equation}
    e_1 = \frac{1}{||T||} T,
\end{equation}

where $||T|| = \sin \theta$.

Let $e_2$ be a unit vector tangent to $M$ and perpendicular to $e_1$. Then the shape operator $A$ takes the following form

\[ S = \begin{pmatrix}
    0 & 0 \\
    0 & \lambda
\end{pmatrix} \]

for a certain function $\lambda$ on $M$. Hence we have

\begin{equation}
    h(e_1, e_1) = 0, \; h(e_1, e_2) = 0, \; h(e_2, e_2) = \lambda \xi.
\end{equation}

**Proposition 1.3.** If $M$ is a constant angle surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant angle $\theta \neq 0$, then $M$ has constant Gaussian curvature $K = -\cos^2 \theta$ and the projection $T$ of $\frac{\partial}{\partial t}$ is a principal direction with principal curvature 0.

**Proof.** We have to prove only the first part of this statement. To do this, we decompose $e_1, e_2 \in T(M)$ as

\begin{equation}
    e_1 = E_1 + \sin \theta \partial_t, \quad e_2 = E_2
\end{equation}

with $E_1, E_2 \in \chi(\mathbb{H}^2)$. We immediately have

\[ g_H(E_1, E_1) = \cos^2 \theta, \; g_H(E_1, E_2) = 0, \; g_H(E_2, E_2) = 1. \]

Combining (6) with Gauss’ equation we find for Gaussian curvature of $M$

\begin{equation}
    K = -\cos^2 \theta.
\end{equation}

We conclude this section with the following

**Proposition 1.4.** The Levi Civita connection of $g$ on $M$ is given by

\begin{equation}
    \nabla_{e_1} e_1 = 0, \; \nabla_{e_2} e_1 = \lambda \cot \theta \; e_2, \\
    \nabla_{e_1} e_2 = 0, \; \nabla_{e_2} e_2 = -\lambda \cot \theta \; e_1.
\end{equation}

**Proof.** Direct computation from (3). \hfill \blacksquare

2. Characterization of constant angle surfaces

In this section we want to classify the constant angle surfaces $M$ in $\mathbb{H}^2 \times \mathbb{R}$. There exist two trivial cases, namely $\theta = 0$ and $\theta = \frac{\pi}{2}$. As we have already seen, in the first case one has that $\frac{\partial}{\partial t}$ is always normal and hence $M$ is an open part of $\mathbb{H}^2 \times \{t_0\}$, while in the second case $\frac{\partial}{\partial t}$ is always tangent. This corresponds to the Riemannian product of a curve in $\mathbb{H}^2$ and $\mathbb{R}$.

We can take coordinates $(u, v)$ on $M$ such that the metric $g$ on $M$ has the form

\begin{equation}
    g = du^2 + \beta^2(u, v) \; dv^2
\end{equation}

with \( \partial_u := \frac{\partial}{\partial u} = e_1 \) and \( \partial_v := \frac{\partial}{\partial v} = \beta e_2 \), where \( \beta \) is a smooth function on \( M \). This can be done since \([e_1, e_2]\) is collinear with \( e_2 \). We have
\[
0 = [\partial_u, \partial_v] = [\partial_u, \beta e_2] = \beta_u e_2 + \beta [e_1, e_2] = (\beta_u - \beta \lambda \cot \theta) e_2
\]
and hence \( \beta \) satisfies the following PDE
\[
(11) \quad \beta_u = \beta \lambda \cot \theta.
\]
We can now write the Levi Civita connection of \( g \) on \( M \) in terms of coordinates \( u \) and \( v \), namely
\[
(12) \quad \nabla_{\partial_u} \partial_v = 0, \quad \nabla_{\partial_v} \partial_u = \nabla_{\partial_u} \partial_v = \lambda \cot \theta \partial_v,
\]
\[
\nabla_{\partial_v} \partial_v = -\beta \beta_u \partial_u + \beta \partial_v.
\]

**Proposition 2.1.** The two functions \( \lambda \) and \( \beta \) are given by
\[
(13) \quad \lambda(u, v) = \sin \theta \tanh (u \cos \theta + C(v))
\]
\[
(14) \quad \beta(u, v) = D(v) \cosh (u \cos \theta + C(v)),
\]
or
\[
(15) \quad \lambda(u, v) = \pm \sin \theta
\]
\[
(16) \quad \beta(u, v) = D(v) e^{\pm u \cos \theta}
\]
where \( C \) and \( D \) are smooth functions depending on \( v \), \( D(v) \neq 0 \) for any \( v \).

**Proof.** From the equation of Codazzi (EC), if we put \( X = e_1 \) and \( Y = e_2 \) one obtains that \( \lambda \) must satisfy the following PDE
\[
(17) \quad \lambda_u = \sin \theta \cos \theta - \lambda^2 \cot \theta.
\]
By integration, one gets (13) or (15). Now, solving (11) we obtain \( \beta \).

There are many models for the hyperbolic plane. In the following we will deal with the Minkowski model or the hyperboloid model \( \mathcal{H} \) of \( \mathbb{H}^2 \), namely the upper sheet \( \{z > 0\} \) of the hyperboloid
\[
\{(x, y, z) \in \mathbb{R}^3_1 : x^2 + y^2 - z^2 = -1\}.
\]
We denoted by \( \mathbb{R}^3_1 \) the Minkowski 3-space with Lorentzian metric tensor
\[
< \cdot, \cdot > = dx^2 + dy^2 - dz^2.
\]
The unit normal to \( \mathcal{H} \) in a point \( p \in \mathbb{R}^3_1 \) is \( N = \pm p \). We will take the external normal \( N = p \) and we have \( < N, N > = -1 \).

We recall the notion of the Lorentzian cross-product (see e.g. [3]):
\[
\boxtimes : \mathbb{R}^3_1 \times \mathbb{R}^3_1 \longrightarrow \mathbb{R}^3_1, ([a_1, a_2, a_3], [b_1, b_2, b_3]) \mapsto [a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_2 b_1 - a_1 b_2].
\]
Let \( M \) be a 2-dimensional surface in \( \mathcal{H} \times \mathbb{R} \subset \mathbb{R}^3_1 \times \mathbb{R} \). On the ambient space we consider the product metric: \( g_o = dx^2 + dy^2 - dz^2 + dt^2 \). Denote by \( \tilde{\nabla} \) the Levi Civita connection on \( \mathbb{R}^3_1 \times \mathbb{R} \) and let \( D^\perp \) be the normal connection of \( M \) in \( \mathbb{R}^3_1 \times \mathbb{R} \). If \( \tilde{\xi} \)
is the unit normal to $\tilde{M}$, then $\tilde{\xi}(p_1, p_2, p_3, p_4) = (p_1, p_2, p_3, 0)$. The shape operator w.r.t. $\tilde{\xi}$ is denoted by $\tilde{A}$.

**Theorem 2.2.** A surface $M$ in $\mathcal{H} \times \mathbb{R}$ is a constant angle surface if and only if the position vector $F$ is, up to isometries of $\mathcal{H} \times \mathbb{R}$, locally given by

\[(18) \quad F(u, v) = \left( \cosh(u \cos \theta) f(v) + \sinh(u \cos \theta) f(v) \mathbb{B} f'(v), u \sin \theta \right),\]

where $f$ is a unit speed curve on $\mathcal{H}$.

**Remark 2.3.** This result is similar to that given in Theorem 2 in [1].

**Proof of the Theorem.**

First we have to prove that the given immersion [18] is a constant angle surface in $\mathcal{H} \times \mathbb{R}$. To do this we compute the tangent vectors (in an arbitrary point on $M$)

\[
F_u(u, v) = \left( \cos \theta [\sinh(u \cos \theta)f(v) + \cosh(u \cos \theta)f(v) \mathbb{B} f'(v)], \sin \theta \right)
\]

\[
F_v(u, v) = \left( \cosh(u \cos \theta)f'(v) + \sinh(u \cos \theta)f(v) \mathbb{B} f''(v), 0 \right) = \left( \cosh(u \cos \theta) - \kappa(v) \sinh(u \cos \theta) \right) f'(v), 0,\]

where $\kappa$ is the geodesic curvature of the curve $f$. This follows from the identity $f''(v) = f(v) + \kappa(v)f(v) \mathbb{B} f'(v)$ which implies $f(v) \mathbb{B} f''(v) = -\kappa(v)f'(v)$, since $f$ is a unit speed curve on $\mathcal{H}$.

We will calculate now both $\xi$ and $\tilde{\xi}$. The second normal vector is nothing but the position vector where we take the last component to be 0, namely we have

\[
\tilde{\xi}(u, v) = \left( \cosh(u \cos \theta)f(v) + \sinh(u \cos \theta)f(v) \mathbb{B} f'(v), 0 \right).
\]

Looking for the expression of the unitary normal $\xi$ as linear combination of $f$, $f'$ and $f \mathbb{B} f'$ we find after some easy computations that

\[
\xi(u, v) = \left( -\sin \theta [\sinh(u \cos \theta)f(v) + \cosh(u \cos \theta)f(v) \mathbb{B} f'(v)], \cos \theta \right).
\]

It follows $\langle \xi, \partial_t \rangle = \cos \theta$ (which is a constant).

Conversely, consider a surface $M$ in $\mathcal{H} \times \mathbb{R}$ with constant angle function $\theta$. If $M$ is one of the trivial cases (see page 3), then it can be parameterized by [18]. Suppose from now on that $\theta \notin \{0, \frac{\pi}{2}\}$.

We have

\[
(F_4)_u = \tilde{g}(F_u, \partial_t) = \tilde{g}(F_u, T + \cos \theta \xi) = g(\partial_u, T) = \sin \theta
\]

and

\[
(F_4)_v = \tilde{g}(F_v, \partial_t) = g(\partial_v, T) = 0.
\]

These relations and the initial condition $F_4(0, 0) = 0$ yield

\[(19) \quad F_4 = u \sin \theta.\]

If $X = (X_1, X_2, X_3, X_4)$ is tangent to $M$, then $\nabla_X \tilde{\xi} = (X_1, X_2, X_3, 0)$. It follows

- $D_X \tilde{\xi} < (X_1, X_2, X_3, 0), \xi > \xi = -\cos \theta < X, T > \xi$
- $D_X \tilde{\xi} = \cos \theta < X, T > \xi$. 

Then \( \tilde{A} \) can be expressed in the basis \( \{ \partial_u, \partial_v \} \) by

\[
\tilde{A} = \begin{pmatrix}
-\cos^2 \theta & 0 \\
0 & -1
\end{pmatrix}.
\]

From (11) one has \( \xi_j = -\tan \theta (F_j)_u \) for all \( j = 1, 2, 3 \).

From the previous relations, the formula of Gauss can be written as:

1. \( (F_j)_{uu} = \cos^2 \theta F_j \)
2. \( (F_j)_{uv} = \lambda \cot \theta (F_j)_v \)
3. \( (F_j)_{vv} = -\beta \beta_u (F_j)_u + \frac{\beta_v}{\beta} (F_j)_v - \lambda \beta^2 \tan \theta (F_j)_u + \beta^2 F_j \).

Case 1: \( \lambda \) satisfies (13). Integrating (21) one gets

\[ (F_j)_v = H_j(v) \cosh(u \cos \theta + C(v)) \]

Hence

\[
F_j = \int_0^v \cosh(u \cos \theta + C(\tau)) H_j(\tau) d\tau + I_j(u).
\]

Substituting in (20) we obtain

\[
(*) \quad I_j = K_j \cosh(u \cos \theta) + L_j \sinh(u \cos \theta)
\]

with \( K_j \) and \( L_j \) real constants.

We define the following functions

\[
f_j = K_j + \int_0^v \cosh C(\tau) H_j(\tau) d\tau, \quad (j = 1, 2, 3)
\]

\[
g_j = L_j + \int_0^v \sinh C(\tau) H_j(\tau) d\tau, \quad (j = 1, 2, 3).
\]

Case 2: \( \lambda \) satisfies (15). One gets

\[
F_j = e^{\pm u \cos \theta} \int_0^v H_j(\tau) d\tau + I_j(u)
\]

with \( I_j \) having the same form as in (*).

In this case we put

\[
f_j = K_j + \int_0^v H_j(\tau) d\tau, \quad (j = 1, 2, 3)
\]

\[
g_j = L_j \pm \int_0^v H_j(\tau) d\tau, \quad (j = 1, 2, 3).
\]

Let \( f = (f_1, f_2, f_3) \) and \( g = (g_1, g_2, g_3) \).
To summarize, in both cases $F$ is of the following form:

\[(23) \quad F = (\cosh(u \cos \theta)f + \sinh(u \cos \theta)g, u \sin \theta).\]

Let $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$. Then we have

\[1. \sum \epsilon_j F_j^2 = -1, \quad 2. \sum \epsilon_j F_{j,u}^2 = \cos^2 \theta, \quad 3. \sum \epsilon_j F_{j,u} F_{j,v} = 0, \quad 4. \sum \epsilon_j F_{j,v}^2 = \beta^2.\]

From 1. and 2. one obtains

\[\sum \epsilon_j f_j^2 - \sum \epsilon_j g_j^2 = -2. \quad (i)\]

Now, relations 1. and 3. can be written as

\[\sum \epsilon_j f_j^2 \cosh^2(u \cos \theta) + \sum \epsilon_j g_j^2 \sinh^2(u \cos \theta) + 2 \sum \epsilon_j f_j g_j \sinh(u \cos \theta) \cosh(u \cos \theta) = -1. \quad (ii)\]

and

\[\left(\sum \epsilon_j f_j f_j' + \sum \epsilon_j g_j g_j'\right) \sinh(u \cos \theta) \cosh(u \cos \theta) + \sum \epsilon_j f_j' g_j \cosh^2(u \cos \theta) + \sum \epsilon_j f_j g_j' \sinh^2(u \cos \theta) = 0. \quad (iii)\]

By derivation in 1. one has

\[\sum \epsilon_j f_j f_j' \cosh^2(u \cos \theta) + \sum \epsilon_j g_j g_j' \sinh^2(u \cos \theta) + \left(\sum \epsilon_j f_j g_j' + \sum \epsilon_j f_j' g_j\right) \sinh(u \cos \theta) \cosh(u \cos \theta) = 0 \quad (iv)\]

\[\left(\sum \epsilon_j f_j^2 + \sum \epsilon_j g_j^2\right) \sinh(u \cos \theta) \cosh(u \cos \theta) + \sum \epsilon_j f_j g_j (\cosh^2(u \cos \theta) + \sinh^2(u \cos \theta)) = 0. \quad (v)\]

Finally (i), (ii) and (v) yield

\[\sum \epsilon_j f_j^2 = -1, \quad \sum \epsilon_j g_j^2 = 1, \quad \sum \epsilon_j f_j g_j = 0. \]

Moreover, it follows

\[\sum \epsilon_j f_j f_j' = 0, \quad \sum \epsilon_j g_j g_j' = 0, \quad \sum \epsilon_j f_j g_j' + \sum \epsilon_j f_j' g_j = 0.\]

From (iii) we get

\[(24) \quad \sum \epsilon_j f_j g_j' = 0 \quad \text{and} \quad \sum \epsilon_j f_j' g_j = 0.\]

Hence, the relation (iv) is identically satisfied.

Let’s write these last equations in another way:

\[<f,f> = -1 \quad <g,g> = 1 \quad <f,g> = 0 \quad \text{and} \quad <f,f'> = 0 \quad <g,g'> = 0 \quad <f,g'> = 0\]

\[(25)\]

We still have to develop the relation 4. This yields

\[<H(v),H(v)> = <f',f'> - <g',g'> = D^2(v), \quad H = (H_1, H_2, H_3).\]

Remark that $f$ can be thought as a curve on $\mathcal{H}$ (while $g$ not).
Since \( <f', f'> \geq 0 \) (it can be easily proved), one can change the \( v \)-coordinate such that \( f \) becomes a unit speed curve in \( \mathcal{H} \); this corresponds to \( D(v)^2 \cosh^2 C(v) = 1 \) or \( D(v)^2 = 1 \) (this depends of the value of \( \lambda \)).

We have \( g \perp f \) and \( g \perp f' \). This means that \( g \) is collinear to \( f \times f' \). We have \( <g, g> = 1 \) and \( <f \times f', f \times f'> = 1 \) and hence \( g = \pm f \times f' \). We can assume that \( g = f \times f' \). Then \( F \) is given by \([18]\) as we wanted to prove.

\[ \text{Remark 2.4.} \] Looking for all minimal constant angle surfaces in \( \mathcal{H} \times \mathbb{R} \), these must be totally geodesic in \( \mathcal{H} \times \mathbb{R} \). Hence we obtain the following surfaces:

1. \( \mathcal{H} \times \{t_0\}, \ t_0 \in \mathbb{R} \)
2. \( f \times \mathbb{R} \) with \( f \) a geodesic line in \( \mathcal{H} \).

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