A vanishing theorem for quadratic intersection multiplicities

Niels Feld

Abstract

We study intersection theoretic problems in the setting of Chow-Witt groups with coefficients in a fixed Milnor-Witt cycle algebra over a perfect field. We prove that the product maps on such groups satisfy the following property: given two points in a regular local scheme with supports which do not intersect properly, their product vanishes. This gives an analogue of Serre’s vanishing result for intersection multiplicities.

MSC—14C17, 14C35, 11E81

1 Introduction

1.1 Current work

Classical intersection theory [Ful98] stands on the study of cycles in Chow groups. Thus, many classical results could be reinterpreted in the new context given by Chow-Witt groups. For example, if $R$ is a regular local ring of dimension $d$, and $M$ and $N$ are two $R$-modules of finite type such that the product $M \otimes N$ has a finite length, then Serre defines an intersection multiplicity

$$\chi_R(M, N) = \sum_{i=0}^{d} (-1)^i \lg_R(Tor_i^R(M, N)),$$

where $\lg_R$ is the length of an $R$-module of finite type. Serre vanishing conjecture states that if $\dim M + \dim N < d$, then $\chi_R(M, N) = 0$ (see the work of Roberts [Rob85], and Gillet-Soulé [GS85, GS87] for a proof in the general case). The appeal of Serre’s multiplicities comes from the fact that they can be used to compute the product of cycles in the Chow ring of a variety.

Following ideas of [FS08], we would like to have a similar description of the intersection product defined for Milnor-Witt cycle modules in [Fel20, Section 11]. In particular: does the intersection multiplicities of Serre have a quadratic interpretation? The question is difficult. Nevertheless, keeping in mind Serre vanishing conjecture and [FS08] Conjecture 1, the following result seems plausible:

Conjecture 1. Let $(R, \mathfrak{m})$ a regular local ring of dimension $n$. Let $Z$ and $T$ be two closed subsets of Spec $R$ such that $\dim Z + \dim T < n$ and $Z \cap T = \mathfrak{m}$. Then the multiplication [Fel20, Section 11]

$$\overline{CH}_Z^i(R) \times \overline{CH}_T^j(R) \to \overline{CH}_{i+j}^\mathfrak{m}(R)$$
is zero for any natural numbers $i, j \in \mathbb{N}$.

As evidence, we have the following theorem (see Subsection 1.2 and Appendix A for the notations and more details about the definitions).

**Theorem 2** (see Theorem 2.2.1). Let $M$ be a Milnor-Witt cycle algebra over a fixed perfect field $k$. Let $X$ be a regular local scheme of dimension $n$ over $k$ and denote by $x_0$ its closed point. Let $V_X$ be a virtual vector bundle over $X$. Let $Z$ and $T$ be closed subsets of $X$ such that $Z \cap T = \{x_0\}$. Then the intersection product

$$A^i_Z(X, M, V_Z) \times A^j_T(X, M, V_T) \to A^{i+j}_{x_0}(X, M, V_{x_0})$$

is zero for any $i, j \in \mathbb{Z}$.

In particular for $M = K^{MW}$, this is true for the intersection product defined on the Chow-Witt ring of $X$. More generally, the theorem apply to any ring spectrum $M$ according to Theorem A.2.2.

In the future, we hope to extend Theorem 2 to more general schemes $X/k$ (the theory of Milnor-Witt cycle modules can be defined over a large class of base scheme $S$, see [DFJ22]). Another direction would be to consider effective MW-cycle modules in the sense of [Fel21b].

Finally, we add that Conjecture 1 remains unclear if we do not assume the existence of a base field. Nevertheless, following the ideas of Gillet-Soulé [GS85], a proof may still be obtained by studying analogues of the Adams operations (see [FH20]).

### 1.2 Notation

Throughout the paper, we fix a perfect field $k$.

We denote by $\text{Grp}$ and $\text{Ab}$ the categories of (abelian) groups.

We consider only schemes that are essentially of finite type over $k$. All schemes and morphisms of schemes are defined over $k$. The category of smooth $k$-schemes of finite type is denoted by $\text{Sm}_k$ and is endowed with the Nisnevich topology (thus, sheaf always means sheaf for the Nisnevich topology).

Let $X$ be a scheme and $x$ a point of $X$. We define the codimension of $x$ in $X$ to be $\dim(O_{X, x})$, the dimension of the localisation ring of $x$ in $X$ (see also [Sta18, TAG 02l2]). If $n$ a natural number, we denote by $X(n)$ (resp. $X^{(n)}$) the set of point of dimension $n$ (resp. codimension $n$) of $X$ (this makes sense even if $X$ is not smooth).

By a field $E$ over $k$, we mean a $k$-finitely generated field $E$. Since $k$ is perfect, notice that $\text{Spec } E$ is essentially smooth over $S$. We denote by $\mathcal{F}_k$ the category of such fields.

Let $f : X \to Y$ be a (quasi)projective lci morphism of schemes (e.g. a morphism between smooth schemes). Denote by $\mathcal{L}_f$ (or $\mathcal{L}_{X/Y}$) the virtual vector bundle over $Y$ associated with the cotangent complex of $f$ defined as follows: if $p : X \to Y$ is a smooth morphism, then $\mathcal{L}_p$ is (isomorphic to) $\Omega_{X/Y}$ the space of (Kähler) differentials. If $i : Z \to X$ is a regular closed immersion, then $\mathcal{L}_i$ is the normal cone $\mathcal{N}_Z X$. If $f$ is the composite $Y \longrightarrow \mathbb{P}^n_X \overset{p} \longrightarrow X$ with $p$ and $i$ as previously (in other words, if $f$ is lci projective), then $\mathcal{L}_f$ is isomorphic to the virtual tangent bundle $i^*\Omega_{\mathbb{P}^n_X/X} \simeq \mathcal{N}_Y (\mathbb{P}^n_X)$ (see also [Fel20, Section 9]). Denote by $\omega_f$ (or $\omega_{X/Y}$) the determinant of $\mathcal{L}_f$. 

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Let $X$ be a scheme and $x \in X$ a point, we denote by $L_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^{\vee}$ and $\omega_x/X = \omega_x$ its determinant. Similarly, let $v$ a discrete valuation on a field, we denote by $\omega_v$ the line bundle $(\mathfrak{m}_v/\mathfrak{m}_v^2)^{\vee}$.

Let $E$ be a field (over $k$) and $v$ a valuation on $E$. We will always assume that $v$ is discrete. We denote by $\omega$ its valuation ring, by $\mathfrak{m}_v$ its maximal ideal and by $\kappa(v)$ its residue class field. We consider only valuations of geometric type, that is we assume: $k \subset \mathfrak{O}_v$, the residue field $\kappa(v)$ is finitely generated over $k$ and satisfies $\text{tr.deg}_k(\kappa(v)) + 1 = \text{tr.deg}_k(E)$.

Let $E$ be a field. We denote by $\text{GW}(E)$ the Grothendieck-Witt ring of symmetric bilinear forms on $E$. For any $a \in E^*$, we denote by $\langle a \rangle$ the class of the symmetric bilinear form on $E$ defined by $(X, Y) \mapsto aXY$ and, for any natural number $n$, we put $n\varepsilon = \sum_{i=1}^{n} (-1)^{i-1}$. Recall that, if $n$ and $m$ are two natural numbers, then $(nm)\varepsilon = n\varepsilon m\varepsilon$.

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2 Main theorem

We fix $M$ a Milnor-Witt cycle algebra over the perfect base field $k$. We start with proving some geometric lemmas about the Chow-Witt groups with coefficients in $M$.

2.1 Geometric lemmas

Lemma 2.1.1. Let $g : Y \to X$ be a smooth morphism of finite type schemes of constant fiber dimension 1, $\sigma : X \to Y$ a section of $g$ and $\mathcal{V}_X$ a virtual vector bundle over $X$. Let $i : Z \to X$ be a closed immersion and consider $\bar{Z} = g^{-1}(Z)$ the pullback along $g$. The induced map $\bar{\sigma} : Z \to \bar{Z}$ is such that the pushforward $\bar{\sigma}^* : C_\bullet(Z, M, \mathcal{V}_Z) \to C_\bullet(\bar{Z}, M, \mathcal{V}_{\bar{Z}})$ is zero on homology.

Proof. See [Fel21a, Lemma 4.1.5].

Definition 2.1.2. If $k[x_1', \ldots, x_m'] \to k[[x_1, \ldots, x_n]]$ is a morphism between power series algebras over $k$ induced by $x'_i \to \sum a_{ij}x_j$ for some $a_{ij} \in k$, we call the induced map $\text{Spec}(k[x_1', \ldots, x_m']) \to \text{Spec}(k[[x_1, \ldots, x_n]])$ a linear projection. Such linear projection are determined by points in $\mathbb{A}_k^{mn}(k)$.

Lemma 2.1.3. Let $X = \text{Spec}(k[x_1, \ldots, x_n])$ and $Y = \text{Spec}(k[z_1, \ldots, z_{n-1}])$. Let $Z, T \subset X$ be closed subsets such that $\dim Z + \dim T < \dim X$ and $Z \cap T$ is supported on the closed point. Then for any sufficiently general\footnote{See [Mat86] Theorem 14.14} linear projection $p : X \to Y$, we have:

- $Z \neq p^{-1}(p(Z))$
• $p^{-1}(p(Z)) \cap T$ is also supported on the closed point.

Proof. See [FS08, Corollary 2.4].

**Lemma 2.1.4.** Let $X = \text{Spec}(k[x_1, \ldots, x_n])$ and $Y = \text{Spec}(k[z_1, \ldots, z_{n-1}])$. Let $Z \subset X$ be a proper closed subset. Then for any integer $i$ and any sufficiently general linear projection $p : X \to Y$, then extension of support

$$A^i_Z(X, M, *) \to A^i_{p^{-1}(p(Z))}(X, M, *)$$

is zero.

Proof. (see also [FS08]) As $Z$ is a proper closed subset of $X$, there exists a nonzero non-unit $t \in k[x_1, \ldots, x_n]$ such that $Z \subset V(t)$. Let $j : V(t) \to X$ be the inclusion. Any sufficiently general linear projection $p : X \to Y$ is flat and has the property that $p|_{V(t)} : V(t) \to Y$ is finite. Consider the following fibre product:

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
p' \downarrow & & \downarrow p \\
V(t) & \xrightarrow{j} & Y.
\end{array}
\]

The inclusion $j : V(t) \to X$ induces a closed immersion $i' : V(t) \to X'$ such that $f i' = j$. Observe that $V(t)$ is also a principal divisor in $X'$ (see [Sri96, Theorem 5.23]). As closed subsets, we have $p^{-1}(p(Z)) = f p'^{-1}(Z)$ and then it is enough to show that

$$A^i_Z(X, M, *) \to A^i_{p'^{-1}(p(Z))}(X, M, *)$$

is zero to get the result. The following diagram

\[
\begin{array}{ccc}
A^i_Z(X, M, *) & \xrightarrow{\epsilon} & A^i_{f(p')^{-1}(Z)}(X, M, *) \\
\downarrow j_* & & \downarrow f_* \\
A^i_{p^{-1}}(V(t), M, *) & \xrightarrow{(i')_*} & A^i_{p'^{-1}(Z)}(X', M, *).
\end{array}
\]

is commutative, where $\epsilon$ is the extension of support. We can see that $j_*$ is an isomorphism. Therefore, Lemma 2.1.4 shows that

$$A^i_{p'(Z)}(X', M, *) \to A^i_{p'^{-1}(Z)}(X', M, *)$$

is zero.

□

**Lemma 2.1.5.** Let $X$ be a scheme over $k$ and $g : X_{k(t)} \to X$ be the (smooth) base change. Then

$$g^* : A^*(X, M, *) \to A^*(X_{k(t)}, M, - \mathcal{L}_{X_{k(t)}/X} + *)$$

is injective.

Proof. See [Fel20, Theorem 8.3].
2.2 Proof of the main theorem

**Theorem 2.2.1.** Let $M$ be a Milnor-Witt cycle algebra. Let $X$ be a regular local scheme of dimension $n$ over $k$ and denote by $x_0$ its closed point. Let $V_X$ be a virtual vector bundle over $X$. Let $Z$ and $T$ be closed subsets of $X$ such that $Z \cap T = \{x_0\}$. Then the intersection product

$$A^j_Z(X, M, V_Z) \times A^j_T(X, M, V_T) \to A^{i+j}_{x_0}(X, M, V_{x_0})$$

is zero for any $i, j \in \mathbb{Z}$.

In particular for $M = k^{MW}$, this is true for the intersection product defined on the Chow-Witt ring of $X$.

**Proof.** Let $\hat{X}$ be the completion of the local ring $X$ (for the $x_0$-adic valuation). By definition, we have $A^n_{x_0}(X, M, V_{x_0}) \simeq A^n_{\hat{x}_0}(\hat{X}, M, V_{x_0})$ for any integer $n$, and the following diagram

$$\begin{array}{ccc}
A^j_Z(X, M, V_Z) \times A^j_T(X, M, V_T) & \to & A^{i+j}_{x_0}(X, M, V_{x_0}) \\
\downarrow & & \downarrow \\
A^j_Z(\hat{X}, M, V_Z) \times A^j_T(\hat{X}, M, V_T) & \to & A^{i+j}_{\hat{x}_0}(\hat{X}, M, V_{x_0})
\end{array}$$

is commutative, where the vertical arrows are induced by the completion. Hence, it is enough to prove the result for a complete regular local scheme and we may assume that $X$ is the spectrum of the ring $A = k[x_1, \ldots, x_n]$.

By Lemma 2.1.5, we may also assume that $k$ is infinite. Now, put $B = k[z_1, \ldots, z_{n-1}]$, and apply Lemma 2.1.3 and 2.1.4 there exists a linear projection $p : X \to \text{Spec}(B)$ such that:

1. The extension of support $\epsilon : A^j_Z(X) \to A^j_{p^{-1}(\rho(Z))}(X)$ is zero.
2. $p^{-1}(\rho(Z)) \cap T = x_0$.

The conclusion follows from the following commutative diagram:

$$\begin{array}{ccc}
A^j_Z(X, M, V_Z) \times A^j_T(X, M, V_T) & \to & A^{i+j}_{x_0}(X, M, V_{x_0}) \\
\downarrow_{\epsilon \times \text{id}} & & \downarrow \\
A^i_{p^{-1}(\rho(Z))}(X, M, V_{p^{-1}(\rho(Z))}) \times A^j_T(X, M, V_T) & \to & A^{i+j}_{x_0}(X, M, V_{x_0}).
\end{array}$$

\[\square\]

A Recollection in motivic homotopy theory

A.1 Milnor-Witt cycle modules

We denote by $\mathcal{S}_k$ the category whose objects are the couple $(E, V_E)$ where $E$ is a field over $k$ and $V_E \in \mathfrak{V}(E)$ is a virtual vector space (of finite dimension over $F$). A morphism $(E, V_E) \to (F, V_F)$ is the data of a morphism $E \to F$ of fields over $k$ and an isomorphism $V_E \otimes E F \simeq V_F$ of virtual $F$-vector spaces.

A morphism $(E, V_E) \to (F, V_F)$ in $\mathcal{S}_k$ is said to be finite (resp. separable) if the field extension $F/E$ is finite (resp. separable).
We recall that a Milnor-Witt cycle modules $M$ over $k$ is a functor from $\mathfrak{G}_k$ to the category $\mathcal{A}b$ of abelian groups equipped with data.

**D1** (restriction maps) Let $\varphi : (E, V_E) \to (F, V_F)$ be a morphism in $\mathfrak{G}_k$. The functor $M$ gives a morphism $\varphi_* : M(E, V_E) \to M(F, V_F)$.

**D2** (corestriction maps) Let $\varphi : (E, V_E) \to (F, V_F)$ be a morphism in $\mathfrak{G}_k$ where the morphism $E \to F$ is finite. There is a morphism $\varphi^* : M(F, \Omega_{F/k} + V_F) \to M(E, \Omega_{E/k} + V_E)$.

**D3** (Milnor-Witt K-theory action) Let $(E, V_E)$ and $(E, W_E)$ be two objects of $\mathfrak{G}_k$. For any element $x$ of $\mathbb{K}^{MW}(E, W_E)$, there is a morphism

$$\gamma_x : M(E, V_E) \to M(E, W_E + V_E)$$

so that the functor $M(E, -) : \mathfrak{G}(E) \to \mathcal{A}b$ is a left module over the lax monoidal functor $\mathbb{K}^{MW}(E, -) : \mathfrak{G}(E) \to \mathcal{A}b$ (see [Yet03, Definition 39]).

**D4** (residue maps) Let $E$ be a field over $k$, let $v$ be a valuation on $E$ and let $V$ be a virtual projective $\mathcal{O}_v$-module of finite type. Denote by $V_E = V \otimes_{\mathcal{O}_v} E$ and $V_{\kappa(v)} = V \otimes_{\mathcal{O}_v} \kappa(v)$. There is a morphism

$$\partial_v : M(E, V_E) \to M(\kappa(v), -N_v + V_{\kappa(v)}),$$

and satisfying compatibility rules (R1a), . . . , (R4a). Moreover, a Milnor-Witt cycle module $M$ satisfies axioms FD (finite support of divisors) and C (closedness) that enable us to define a complex $(C_p(X, M, V_X), d_p)_{p \in \mathbb{Z}}$ for any scheme $X$ and virtual bundle $V_X$ over $X$ where

$$C_p(X, M, V_X) = \bigoplus_{x \in X(p)} M(\kappa(x), \Omega_{\kappa(x)/k} + V_x).$$

**Example A.1.1.** The main example of MW-cycle module is given by Milnor-Witt K-theory $\mathbb{K}^{MW}$ (see [Fe20, Fe21c] for more details).

A.1.2. The complex $(C_p(X, M, V_X), d_p)_{p \in \mathbb{Z}}$ is called the Milnor-Witt complex of cycles on $X$ with coefficients in $M$ and we denote by $A_p(X, M, V_X)$ the associated homology groups (called Chow-Witt groups with coefficients in $M$). We can define five basic maps on the complex level (see [Fe20, Section 4]):

**Pushforward** Let $f : X \to Y$ be a $k$-morphism of schemes, let $V_Y$ be a virtual bundle over the scheme $Y$. The data $[D2]$ induces a map

$$f_* : C_p(X, M, V_X) \to C_p(Y, M, V_Y).$$

**Pullback** Let $g : X \to Y$ be an essentially smooth morphism of schemes. Let $V_Y$ a virtual bundle over $Y$. Suppose $X$ connected (if $X$ is not connected, take the sum over each connected component) and denote by $s$ the relative dimension of $g$. The data $[D1]$ induces a map

$$g^* : C_p(Y, M, V_Y) \to C_{p+s}(X, M, -L_{X/Y} + V_X).$$

**Multiplication with units** Let $X$ be a scheme of finite type over $k$ with a virtual bundle $V_X$. Let $a_1, \ldots, a_n$ be global units in $\mathcal{O}_X$. The data $[D3]$ induces a map

$$[a_1, \ldots, a_n] : C_p(X, M, V_X) \to C_p(X, M, (n) + V_X).$$
**Multiplication with \( \eta \)** Let \( X \) be a scheme of finite type over \( k \) with a virtual bundle \( \mathcal{V}_X \). The Hopf map \( \eta \) and the data \( [D3] \) induces a map

\[
\eta : C_p(X, M, \mathcal{V}_X) \to C_p(X, M, -\mathbb{A}^1_X + \mathcal{V}_X).
\]

**Boundary map** Let \( X \) be a scheme of finite type over \( k \) with a virtual bundle \( \mathcal{V}_X \), let \( i : Z \to X \) be a closed immersion and let \( j : U = X \setminus Z \to X \) be the inclusion of the open complement. The data \( [D4] \) induces a map

\[
\partial = \partial^j_i : C_p(U, M, \mathcal{V}_U) \to C_{p-1}(Z, M, \mathcal{V}_Z).
\]

These maps satisfy the usual compatibility properties (see \cite{Fel20} Section 5). In particular, they induce maps \( f_*, g^*, [u], \eta, \partial^j_i \) on the homology groups \( A_*(X, M, \cdot) \).

**Definition A.1.3.** A pairing \( M \times M' \to M'' \) of MW-cycle modules over \( k \) is given by bilinear maps for each \( (E, \mathcal{V}_E), (E, \mathcal{W}_E) \) in \( \mathfrak{A}_k \)

\[
M(E, \mathcal{V}_E) \times M'(E, \mathcal{W}_E) \to M''(E, \mathcal{V}_E + \mathcal{W}_E)
\]

\[
(\rho, \mu) \mapsto \rho \cdot \mu
\]

which respect the \( K^{MW} \)-module structure and which are compatible with the data \( D1, D2, D3 \) and \( D4 \) in the sense of \cite{Fel20} Definition 3.21].

A ring structure on a MW-cycle module \( M \) is a pairing

\[
M \times M \to M
\]

(in the sense of \cite{Fel20} Definition 3.21]) which induces on

\[
\bigoplus_{\mathcal{V}_E \in \mathfrak{V}(E)} M(E, \mathcal{V}_E)
\]

an associative and \( \varepsilon \)-commutative ring structure. In that case, we may say that \( M \) is an algebra.

**Example A.1.4.** By definition, a Milnor-Witt cycle module \( M \) comes equipped with a pairing \( K^{MW} \times M \to M \). When \( M = K^{MW} \), this defines a ring structure on \( M \).

**A.1.5. Cross Products.** Let \( M \) be a Milnor-Witt cycle module with a ring structure \( M \times M \to M \) (see \cite{Fel20} Definition 3.21]). Let \( Y \) and \( Z \) be two essentially smooth schemes over \( k \) equipped with virtual vector bundles \( \mathcal{V}_Y \) and \( \mathcal{W}_Z \). We define the cross product

\[
\times : C_p(Y, M, \mathcal{V}_Y) \times C_q(Z, M', \mathcal{W}_Z) \to C_{p+q}(Y \times Z, M'' \mathcal{V}_Y \times \mathcal{W}_Z + \mathcal{W}_Y \times Z)
\]

as follows. For \( y \in Y \), let \( Z_y = \text{Spec} \kappa(y) \times Z \), let \( \pi_y : Z_y \to Z \) be the projection and let \( i_y : Z_y \to Y \times Z \) be the inclusion. For \( z \in Z \) we understand similarly \( Y_z, \pi_z : Y_z \to Y \) and \( i_z : Y_z \to Y \times Z \). We give the following two equivalent definitions:

\[
\rho \times \mu = \sum_{y \in Y(q)} (i_y)_*(\rho_y \cdot \pi_y^*(\mu)),
\]

\[
\rho \times \mu = \sum_{z \in Z(q)} (i_z)_*(\pi_z^*(\rho) \cdot \mu).
\]

The cross product satisfies the expected properties (associativity, graded-commutativity, chain rule; see \cite{Fel20} Section 11]).
A.1.6. Intersection. For \( X \) smooth, the product induces a map
\[
A^p(X, M, V_X) \times A^q(X, M, W_X) \to A^{p+q}(X \times X, M, V_{X \times X} + W_{X \times X}).
\]

By composing with the Gysin morphism
\[
\Delta^* : A^{p+q}(X \times X, M, V_{X \times X} + W_{X \times X}) \to A^{p+q}(X, M, -\mathcal{T}_\Delta + V_X + W_X)
\]
induced by the diagonal \( \Delta : X \to X \times X \), we obtain the map
\[
A^p(X, M, V_X) \times A^q(X, M, W_X) \to A^{p+q}(X, M, -\mathcal{T}_\Delta + V_X + W_X).
\]

The preceding considerations and the functoriality of the Gysin maps prove the following theorem.

**Theorem A.1.7.** If \( M \) is a MW-cycle module with a ring structure and \( X \) a smooth scheme over \( k \), the intersection product turns
\[
\bigoplus_{V_X \in \mathcal{V}(X)} A^*(X, M, V_X)
\]
into a graded commutative associative algebra over
\[
\bigoplus_{V_X \in \mathcal{V}(X)} A^*(X, K^{MW}, V_X).
\]

In particular, we obtain a product on the Chow-Witt ring \( \widetilde{\text{CH}}(X) \) which coincides with the intersection product (defined in [Fas20, §3.4], see also [Fas08]). Indeed, our construction of Gysin morphisms follows the classical one (using deformation to the normal cone) and our cross products correspond to the one already defined for the Milnor-Witt K-theory (see [Fas20, §3]).

### A.2 Homotopy modules

A.2.1. We denote by \( \text{SH}(k) \) the stable homotopy category over \( k \). It is equipped with the *homotopy* t-structure given by the full subcategory \( \text{SH}_{\geq 0}(k) \) (resp. \( \text{SH}_{\leq 0}(k) \)) consisting of \( \mathbb{P}^1 \)-spectra \( M \) with
\[
\pi_n(M)_m = 0
\]
for each \( m \in \mathbb{Z} \) and \( n < 0 \) (resp. \( n > 0 \)) (see [Mor03, §5.2]).

The heart \( \text{SH}(k)^{\heartsuit} \) of this t-structure is equivalent to the category of homotopy modules which, by definition, are Nisnevich sheaves from the category of smooth schemes over \( k \) to the category of \( \mathbb{Z} \)-graded abelian groups satisfying the \( \mathbb{A}^1 \)-homotopy invariance property. The main theorem of [Fel21a] is the following:

**Theorem A.2.2.** The category of Milnor-Witt cycle modules over \( k \) (denoted by \( \mathcal{M}^{MW}(k) \)) is equivalent to the category of homotopy modules (or, equivalently, the heart of the stable homotopy category equipped with its homotopy t-structure):
\[
\mathcal{M}^{MW}(k) \simeq \text{SH}(k)^{\heartsuit}.
\]
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