Minimum Cost Adaptive Submodular Cover

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Abstract

Adaptive submodularity is a fundamental concept in stochastic optimization, with numerous applications such as sensor placement, hypothesis identification and viral marketing. We consider the problem of minimum cost cover of adaptive-submodular functions, and provide a $4(1 + \ln Q)$-approximation algorithm, where $Q$ is the goal value. In fact, we consider a significantly more general objective of minimizing the $p^{th}$ moment of the coverage cost, and show that our algorithm simultaneously achieves a $(p + 1)^{p+1}(\ln Q + 1)^p$ approximation guarantee for all $p \geq 1$. All our approximation ratios are best possible up to constant factors (assuming $P \neq NP$). Moreover, our results also extend to the setting where one wants to cover multiple adaptive-submodular functions. Finally, we evaluate the empirical performance of our algorithm on instances of hypothesis identification.

1 Introduction

Adaptive stochastic optimization, where an algorithm makes sequential decisions while (partially) observing uncertainty, arises in numerous applications such as active learning [Das04], sensor placement [GKS05] and viral marketing [TWTD17]. Often, these applications involve an underlying submodular function, and the framework of adaptive-submodularity (introduced by [GK11]) has been widely used to solve these problems. In this paper, we study a basic problem in this context: covering an adaptive-submodular function at the minimum expected cost.

In some applications, such as sensor placement (or stochastic set cover [GV06]), the uncertainty can be captured by an independent random variable associated with each decision. However, there are also a number of applications where the random variables associated with different decisions are correlated. The adaptive-submodularity framework that we consider is also applicable in certain applications involving correlations.

One such application is the viral marketing problem, where we are given a social network and target $Q$, and the goal is to influence at least $Q$ users to adopt a new product. A user can be influenced in two ways (i) directly because the user is offered a promotion, or (ii) indirectly because some friend of the user was influenced and the friend influenced this user. We incur a cost only in case (i), which accounts for the promotional offer. A widely-used model for influence behavior is the independent cascade model [KKT15]. Here, each arc $(u, v)$ has a value $p_{uv} \in [0, 1]$ that represents the probability that user $u$ will influence user $v$ (if $u$ is already influenced). A solution is a sequential process that in each step, selects one user $w$ to influence directly, after which we get to observe which of $w$’s friends were influenced (indirectly), and which of their friends were influenced, and so on. So, the solution can utilize these partial observations to make decisions adaptively. Such an adaptive solution can be represented by a decision tree; however, it may require exponential space to store explicitly. We will analyze simple solutions that can be implemented in polynomial time.

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(and space), but our performance guarantees are relative to an optimal solution that can be very complex. Also, note that the random observations associated with different decisions (in the viral marketing problem) are highly correlated: the set of nodes that get (indirectly) influenced by any node \( w \) depends on the entire network (not just \( w \)).

While there has been extensive prior work on minimum cost cover of adaptive submodular functions \([\text{QV06}][\text{GKT1}]][\text{INvdZ16}][\text{HKP21}][\text{EKM21}]\), all these results focus on minimizing the expected cost, which is a risk-neutral objective. However, one may also be interested in minimizing a higher moment of the random cost, which corresponds to a risk-averse objective. We note that the quality of a solution may vary greatly depending on the chosen objective. For example, consider two solutions \( A \) and \( B \). Solution \( A \) has cost 1 with probability (w.p.) \( 1 - \frac{1}{M} \) and cost \( M \) w.p. \( \frac{1}{M} \). Solution \( B \) has cost \( M^{1/3} \) w.p. 1. The expected cost (i.e., first moment) of \( A \) is \( 2 - \frac{1}{M} \), whereas that of \( B \) is \( M^{1/3} \). On the other hand, the second moment of \( A \) is \( \approx M \), whereas that of \( B \) is \( M^{2/3} \).

Clearly, solution \( A \) is much better in terms of the expected cost, whereas solution \( B \) is much better in terms of the second moment.

Motivated by this, we consider the adaptive submodular cover problem under the more general objective of minimizing the \( p \)th moment cost, for any \( p \geq 1 \). Somewhat surprisingly, we show that there is a universal algorithm for adaptive-submodular cover that approximately minimizes all moments simultaneously. We note that our result is the first approximation algorithm for higher moments (\( p > 1 \)), even for widely studied special cases such as (independent) stochastic submodular cover \([\text{INvdZ16}][\text{HKP21}]\) and optimal decision tree \([\text{AH12}][\text{GB09}][\text{GNR17}]\).

1.1 Problem Definition

Random items. Let \( E \) be a finite set of \( n \) items. Each item \( e \in E \) corresponds to a random variable \( \Phi_e \in \Omega \), where \( \Omega \) is the outcome space (for a single item). We use \( \Phi = \langle \Phi_e : e \in E \rangle \) to denote the vector of all random variables (r.v.s). The r.v.s may be arbitrarily correlated across items. We use upper-case letters to represent r.v.s and the corresponding lower-case letters to represent realizations of the r.v.s. Thus, for any item \( e \), \( \phi_e \in \Omega \) is the realization of \( \Phi_e \); and \( \phi = \langle \phi_e : e \in E \rangle \) denotes the realization of \( \Phi \). Equivalently, we can represent the realization \( \phi \) as a subset \( \{ (e, \phi_e) : e \in E \} \subseteq E \times \Omega \) of item-outcome pairs.

A partial realization \( \psi \subseteq E \times \Omega \) refers to the realizations of any subset of items; \( \text{dom}(\psi) \subseteq E \) denotes the items whose realizations are represented in \( \psi \), and \( \psi_e \) denotes the realization of any item \( e \in \text{dom}(\psi) \). Note that a partial realization contains at most one pair of the form \( (e, *) \) for any item \( e \in E \). The (full) realization \( \phi \) corresponds to a partial realization with \( \text{dom}(\phi) = E \). For two partial realizations \( \psi, \psi' \subseteq E \times \Omega \), we say that \( \psi \) is a subrealization of \( \psi' \) (denoted \( \psi \preceq \psi' \)) if \( \psi \subseteq \psi' \); in other words, \( \text{dom}(\psi) \subseteq \text{dom}(\psi') \) and \( \psi_e = \psi'_e \) for all \( e \in \text{dom}(\psi) \). Two partial realizations \( \psi, \psi' \subseteq E \times \Omega \) are said to be disjoint if there is no full realization \( \phi \) with \( \psi \preceq \phi \) and \( \psi' \preceq \phi \); in other words, there is some item \( e \in \text{dom}(\psi) \cap \text{dom}(\psi') \) such that the realization of \( \Phi_e \) is different under \( \psi \) and \( \psi' \).

We assume that there is a prior probability distribution \( p(\phi) = \Pr[\Phi = \phi] \) over realizations \( \phi \). Moreover, for any partial realization \( \psi \), we assume that we can compute the posterior distribution \( p(\phi | \psi) = \Pr(\Phi = \phi | \psi \preceq \Phi) \).

Utility function. In addition to the random items (described above), there is a utility function \( f : 2^{E \times \Omega} \to \mathbb{R}_{\geq 0} \) that assigns a value to any partial realization. We will assume that this function is monotone, i.e., having more realizations can not reduce the value. Formally,
**Definition 1.1** (Monotonicity). A function \( f : 2^{E \times \Omega} \to \mathbb{R}_{\geq 0} \) is **monotone** if
\[
 f(\psi) \leq f(\psi') \quad \text{for all partial realizations } \psi \preceq \psi'.
\]

We also assume that the function \( f \) can always achieve its maximal value, i.e.,

**Definition 1.2** (Coverable). Let \( Q \) be the maximal value of function \( f \). Then, function \( f \) is said to be **coverable** if this value \( Q \) can be achieved under every (full) realization, i.e.,
\[
 f(\phi) = Q \quad \text{for all realizations } \phi \text{ of } \Phi.
\]

Furthermore, we will assume that the function \( f \) along with the probability distribution \( p(\cdot) \) satisfies a submodularity-like property. Before formalizing this, we need the following definition.

**Definition 1.3** (Marginal benefit). The **conditional expected marginal benefit** of an item \( e \in E \) conditioned on observing the partial realization \( \psi \) is:
\[
 \Delta(e|\psi) := \mathbb{E} \left[ f(\psi \cup (e, \Phi_e)) - f(\psi) \mid \psi \preceq \Phi \right] = \sum_{\omega \in \Omega} \Pr[\Phi_e = \omega \mid \psi \preceq \Phi] \cdot (f(\psi \cup (e, \omega)) - f(\psi)).
\]

We will assume that function \( f \) and distribution \( p(\cdot) \) jointly satisfy the adaptive-submodularity property, defined as follows.

**Definition 1.4** (Adaptive submodularity). A function \( f : 2^{E \times \Omega} \to \mathbb{R}_{\geq 0} \) is adaptive submodular w.r.t. distribution \( p(\cdot) \) if for all partial realizations \( \psi \preceq \psi' \), and all items \( e \in E \setminus \text{dom}(\psi') \), we have
\[
 \Delta(e|\psi) \geq \Delta(e|\psi').
\]

In other words, this property ensures that the marginal benefit of an item never increases as we condition on more realizations. Given any function \( f \) satisfying Definitions 1.1, 1.2 and 1.4, we can pre-process \( f \) by subtracting \( f(\emptyset) \), to get an equivalent function (that maintains these properties), and has a smaller \( Q \) value. So, we may assume that \( f(\emptyset) = 0 \).

**Min-cost adaptive-submodular cover (ASC).** In this problem, each item \( e \in E \) has a positive cost \( c_e \). The goal is to select items (and observe their realizations) sequentially until the observed realizations have function value \( Q \). The objective is to minimize the expected cost of selected items.

Due to the stochastic nature of the problem, the solution concept here is much more complex than in the deterministic setting (where we just select a static subset). In particular, a solution corresponds to a “policy” that maps observed realizations to the next selection decision. The observed realization at any point corresponds to a partial realization (namely, the realizations of the items selected so far). Formally, a **policy** is a mapping \( \pi : 2^{E \times \Omega} \to E \), which specifies the next item \( \pi(\psi) \) to select when the observed realizations are \( \psi \). The policy \( \pi \) terminates at the first point when \( f(\psi) = Q \), where \( \psi \subseteq E \times \Omega \) denotes the observed realizations so far. For any policy \( \pi \) and full realization \( \phi \), let \( C(\pi, \phi) \) denote the total cost of items selected by policy \( \pi \) under realization \( \phi \). Then, the expected cost of policy \( \pi \) is:
\[
 c_{\text{exp}}(\pi) = \mathbb{E}_\Phi \left[ C(\pi, \Phi) \right] = \sum_{\phi} p(\phi) \cdot C(\pi, \phi).
\]

\(^2\)Policies and utility functions are not necessarily defined over all subsets \( 2^{E \times \Omega} \), but only over partial realizations; recall that a partial realization is of the form \( \{(e, \phi_e) : e \in S\} \) where \( \phi \) is some full-realization and \( S \subseteq E \).
While minimizing the expected cost is the primary objective, we are also interested in minimizing higher moments of the cost. For any $p \geq 1$ and policy $\pi$, the $p^{th}$ moment of the policy’s cost is:

$$c_p(\pi) = \mathbb{E}_\Phi [C(\pi, \Phi)^p] = \sum_\phi p(\phi) \cdot C(\pi, \phi)^p.$$  

At any point in policy $\pi$, we refer to the cumulative cost incurred so far as the time. If $J_1, J_2, \cdots J_k$ denotes the (random) sequence of items selected by $\pi$ then for each $i \in \{1, 2, \cdots k\}$, we view item $J_i$ as being selected during the time interval $[\sum_{h=1}^{i-1} c(J_h), \sum_{h=1}^{i} c(J_h))$ and the realization of $J_i$ is only observed at time $\sum_{h=1}^{i} c(J_h)$. For any time $t \geq 0$, we use $\Psi(\pi, t) \subseteq E \times \Omega$ to denote the (random) realizations that have been observed by time $t$ in policy $\pi$. We note that $\Psi(\pi, t)$ only contains the realizations of items that have been completely selected by time $t$. Note that the policy terminates at the earliest time $t$ where $f(\Psi(\pi, t)) = Q$.

Given any policy $\pi$, we define its cost $k$ truncation by running $\pi$ and stopping it just before the cost of selected items exceeds $k$. That is, we stop the policy as late as possible while ensuring that the cost of selected items never exceeds $k$ (for any realization).

**Remark:** Our definition of the utility function $f$ is slightly more restrictive than the original definition [GK11]. In particular, the utility function in [GK11] is of the form $g : 2^E \times \Omega \rightarrow \mathbb{R}_{\geq 0}$, where the function value $g(\text{dom}(\psi), \Phi)$ for any partial realization $\psi$ is still random and can depend on the outcomes of unobserved items, i.e., those in $E \setminus \text{dom}(\psi)$. Nevertheless, our formulation (ASC) still captures most applications of the formulation studied in [GK11]. See Section 3 for details.

### 1.2 Adaptive Greedy Policy

Algorithm 1 describes a natural greedy policy for min-cost adaptive-submodular cover, which has also been studied in prior works [GK17, EKM21, HKP21].

**Algorithm 1** Adaptive Greedy Policy $\pi$.

1. selected items $A \leftarrow \emptyset$, observed realizations $\psi \leftarrow \emptyset$
2. while $f(\psi) < Q$ do
3. $e^* = \arg \max_{e \in E \setminus A} \frac{\Delta(e|\psi)}{c_e}$
4. add $e^*$ to the selected items, i.e., $A \leftarrow A \cup \{e^*\}$
5. select $e^*$ and observe $\Phi_{e^*}$
6. update $\psi \leftarrow \psi \cup \{(e^*, \Phi_{e^*})\}$

**Remark:** Note that the policy $\pi$ remains the same if we replace the greedy choice by

$$e^* = \arg \max_{e \in E \setminus A} \frac{\Delta(e|\psi)}{c_e \cdot (Q - f(\psi))}. \tag{1.1}$$

This is because the additional term $Q - f(\psi)$ is the same for each item $e \in E \setminus A$ (note that at any particular step, $\psi$ is a fixed partial realization). We will make use of this alternative greedy criterion in our analysis.
1.3 Results and Techniques

Our first main result is on the expected cost of the greedy policy.

**Theorem 1.5.** Consider any instance of minimum cost adaptive-submodular cover, where the utility function \( f : 2^{E \times \Omega} \to \mathbb{R}_{\geq 0} \) is monotone, coverable and adaptive-submodular w.r.t. the probability distribution \( p(\cdot) \). Suppose that there is some value \( \eta > 0 \) such that \( f(\psi) > Q - \eta \) implies \( f(\psi) = Q \) for all partial realizations \( \psi \subseteq E \times \Omega \). Then, the expected cost of the greedy policy is

\[
c_{\text{exp}}(\pi) \leq 4 \cdot (1 + \ln(Q/\eta)) \cdot c_{\text{exp}}(\sigma),
\]

where \( \sigma \) denotes the optimal policy.

This is an asymptotic improvement over the \( (1 + \ln(Q/\eta))^2 \)-approximation bound from [GK17] and the \( (1 + \ln(\frac{nQc_{\text{max}}}{\eta})) \)-approximation bound from [EKM21]; the maximum item cost \( c_{\text{max}} \) can even be exponentially larger than \( Q \). Our bound is the best possible, up to the constant factor of 4, because the set cover problem is a special case of ASC (where \( Q \) is the number of elements to cover and \( \eta = 1 \)). [DS14] showed that, assuming \( P \neq NP \), one cannot obtain a better than \( \ln(Q) \) approximation ratio for set cover.

As a consequence, we obtain the first \( O(\ln Q) \)-approximation algorithm for the viral marketing application mentioned earlier. We also obtain an improved bound for the optimal decision tree problem with uniform priors. See Section 3 for details.

In fact, we obtain Theorem 1.5 as a special case of the following general result on minimizing the \( p \)-th moment of the coverage cost.

**Theorem 1.6.** Consider any instance of minimum cost adaptive-submodular cover, where the utility function \( f \) is monotone, coverable and adaptive-submodular w.r.t. the probability distribution \( q(\cdot) \). Suppose that there is some value \( \eta > 0 \) such that \( f(\psi) > Q - \eta \) implies \( f(\psi) = Q \) for all partial realizations \( \psi \subseteq E \times \Omega \). Then, for every \( p \geq 1 \), the \( p \)-th moment cost of the greedy policy is

\[
c_p(\pi) \leq (p + 1)^{p+1} \cdot (1 + \ln(Q/\eta))^p \cdot c_p(\sigma_p),
\]

where \( \sigma_p \) denotes the optimal policy for the \( p \)-th moment objective.

This result is also best possible for any \( p \geq 1 \), up to the leading constant of \( (p + 1)^{p+1} \). We emphasize that the greedy policy \( \pi \) is oblivious to the choice of \( p \): so Theorem 1.6 provides a universal algorithm that achieves the stated approximation ratio for every value of \( p \). We are not aware of any previous result for minimizing higher moments, even for special cases such as (independent) stochastic submodular cover or optimal decision tree.

Our proof technique is very different from prior works on adaptive submodular cover [GK17, EKM21, HKP21]. The approaches in these papers were tailored to bound the expected cost, and seem difficult to extend to higher moments \( p > 1 \). We start by expressing the \( p \)-th moment objective of any policy as an appropriate integral of “non-completion probabilities” over all times. Then, we relate the non-completion probabilities in the greedy policy (at any time \( t \)) to that in the optimal policy (at a scaled time \( \frac{t}{\alpha} \)). In order to establish this relation, we consider the integral of the greedy criterion value over each suffix \((t, \infty)\) of time, and prove lower and upper bounds on this quantity. Finally, we relate the \( p \)-th moment objectives of greedy and the optimal policy by analyzing a double integral over all suffixes. The high-level approach of using non-completion probabilities has been used earlier for minimizing the expected cost in a number of stochastic covering problems, including the independent special case of ASC [NvdZ16]. We refine and improve this approach by (i) utilizing non-completion probabilities at all times (not just power-of-two times) and (ii) using...
a stronger upper bound on the greedy criterion value. Moreover, we extend this approach to $p^{th}$ moment objectives (prior work only looked at expected cost).

Additionally, our algorithm and analysis extend in a straightforward manner, to the setting with multiple adaptive-submodular functions, where the objective is the sum of $p^{th}$ moments of the “cover times” of all the functions. We obtain the same approximation ratio even for this more general problem. In fact, the multiple ASC problem with $Q = \eta = 1$ and an expected cost objective (i.e., $p = 1$) generalizes the min-sum set cover problem [FLT04], which is NP-hard to approximate better than factor 4. The constant factor $(p + 1)^{p+1}$ in our approximation ratio for this problem is also 4, which implies that our bound is tight in this case.

Finally, we provide computational results of our algorithm on real-world instances of optimal decision tree. We compare the performance of our algorithm on $p^{th}$ moment objectives for $p = 1, 2, 3$ to lower-bounds that we obtain (via Huffman coding). Our algorithm performs very well on the instances tested.

1.4 Related Work

Adaptive submodularity was introduced by [GK11], where they considered both the maximum-coverage and the minimum-cost-cover problems. They showed that the greedy policy is a $(1 - \frac{1}{e})$ approximation for maximum coverage, where the goal is to maximize the expected value of an adaptive-submodular function subject to a cardinality constraint. They also claimed that the greedy policy is a $(1 + \ln(\frac{Q}{\eta}))$ approximation for min-cost cover of an adaptive-submodular function. However, this result had an error [NS17], and a corrected proof [GK17] only provides a double-logarithmic $(1 + \ln(\frac{Q}{\eta}))^2$ approximation. Recently, [EKM21] obtained a single-logarithmic approximation bound of $(1 + \ln(\frac{n Q c_{\max}}{\eta}))$. However, this bound depends additionally on the number of items $n$ and their maximum cost $c_{\max}$. Our result shows that the greedy policy is indeed an $O(\ln(\frac{Q}{\eta}))$ approximation. As noted earlier, our definition of ASC is simpler and slightly more restrictive than the original one in [GK11], although most applications of adaptive-submodularity do satisfy our definition.

The special case of adaptive-submodularity where the random variables are independent across items, has also been studied extensively. For the maximum-coverage version, [AN16] obtained a $(1 - \frac{1}{e})$-approximation algorithm via a “non adaptive” policy (that fixes a subset of items to select upfront). Subsequent work [GNS17,BSZ19,ASW16] obtained constant factor approximation algorithms for a variety of constraints (beyond just cardinality). The minimum-cost cover problem (called stochastic submodular cover) was studied in [AAK19,HPK21,HH18,GGN22,NvdZ16]. In particular, an $O(\ln(\frac{Q}{\eta}))$ approximation algorithm follows from [NvdZ16], and recently [HKP21] proved that the greedy policy has a $(1 + \ln(\frac{Q}{\eta}))$ approximation guarantee. The latter guarantee is the best possible, even up to the constant factor: this also matches the best approximation ratio for the deterministic submodular cover problem [Wol82] and its special case of set cover [DS14]. The papers [AAK19,GGN22] studied stochastic submodular cover under limited “rounds of adaptivity”, and obtained smooth taredoffs between the approximation ratio and the number of rounds.

The (deterministic) submodular cover problem with multiple functions was introduced in [AG11], where they obtained an $O(\ln(\frac{Q}{\eta}))$ approximation algorithm for minimizing the sum of cover times. Subsequently, [NvdZ16] studied the stochastic submodular cover problem with multiple functions (which involved independent items), and obtained an $O(\ln(\frac{Q}{\eta}))$ approximation algorithm. The analysis in our paper is similar, at a high level, to the analysis in [NvdZ16], which also relied on the non-completion probabilities. However, we handle the more general adaptive-submodular setting (where items may be correlated), and we obtain a much better constant factor and extend the techniques to minimizing higher moments of the cost.
Minimizing higher moments of the covering cost has been studied previously in the deterministic setting by [GGKT08]. Specifically, they considered the \(L_p\) set cover problem, where given a collection of sets (items in our setting) and elements, the goal is to find a sequence of sets so as to minimize the total \(p^{th}\) moment of cover times over all elements. [GGKT08] showed that the greedy algorithm for \(L_p\) set cover achieves an approximation ratio of \((p + 1)^{p+1}\) for each \(p \geq 1\) simultaneously. We note that \(L_p\) set cover is a special case of the multiple ASC problem studied in our paper, where \(Q = \eta = 1\) and each function corresponds to an element in set cover. So, our approximation ratio for multiple ASC in this special case matches the best known bound for \(L_p\) set cover. Moreover, [GGKT08] showed that for any fixed value of \(p\), there is no approximation ratio better than \(\Omega(p^p)\) for \(L_p\) set cover, unless \(NP \subseteq DTIME(n^{\log \log n})\). So, the leading factor \((p + 1)^{p+1}\) in our bound for multiple ASC is nearly the best possible, for each \(p\). As noted before, ours is the first paper to consider higher moment objectives in the stochastic setting, even for special cases of ASC such as stochastic submodular cover and optimal decision tree. Our proof technique is also very different from that in [GGKT08].

A different (scenario based) model for correlations in adaptive submodular cover was studied in [NKN20, GHKL16]. Here, the utility function \(f\) is just required to be submodular (not adaptive-submodular), but the algorithm requires an explicit description of the probability distribution \(p(\cdot)\). In particular, [NKN20] obtained a greedy-style policy with approximation ratio \(O(\ln(mQ/\eta))\) where \(m\) is the support-size of distribution \(p(\cdot)\), and \(Q\) and \(\eta\) are as before. We believe that our proof technique may be useful in extending the expected-cost results in [NKN20] to higher moments and in improving the constant factor in their approximation ratio.

2 Analyzing the Greedy Policy

In this section, we prove our main result (Theorem 1.6), which bounds the \(p^{th}\) moment of the greedy policy cost. Note that setting \(p = 1\) in Theorem 1.6 implies the bound on expected cost (Theorem 1.5). We first show how to re-write the \(p^{th}\) moment objective in terms of “non-completion” probabilities. Then, we relate the non-completion probabilities in the greedy and optimal policies by analyzing the integral of the “greedy criterion value” (1.1) over time.

2.1 Non-completion Probabilities and the Moment Objective

Our analysis is based on relating the “non-completion” probabilities at different times in the greedy policy \(\pi\) and the optimal policy \(\sigma\). We first define these quantities formally.

**Definition 2.1** (Non-completion probabilities). For any time \(t \geq 0\), let

\[
o(t) := \Pr[\sigma \text{ does not terminate by time } t] = \Pr[C(\sigma, \Phi) > t] = \Pr[f(\Psi(\sigma, t)) < Q].\]

Similarly, for any \(t \geq 0\), let

\[
a(t) := \Pr[\pi \text{ does not terminate by time } t] = \Pr[C(\pi, \Phi) > t] = \Pr[f(\Psi(\pi, t)) < Q].\]

See Figure 1 for an example of the non-completion probabilities \(o(t)\). Clearly, \(o(t)\) and \(a(t)\) are non-increasing functions of \(t\). Moreover, \(o(0) = a(0) = 1\) and we have \(o(t) = a(t) = 0\) for all \(t \geq \sum_{e \in E} c_e\) (this is because any policy would have selected all items by this time).
It is easy to see that the expected cost of any policy is exactly the integral of the non-completion probabilities over time. That is,

\[
c_{\text{exp}}(\sigma) = \int_0^\infty \Pr[\sigma \text{ does not terminate by time } t] dt = \int_0^\infty o(t) dt.
\]

\[
c_{\text{exp}}(\pi) = \int_0^\infty \Pr[\pi \text{ does not terminate by time } t] dt = \int_0^\infty a(t) dt.
\]

Figure 1: Graph of a simple \(o(\cdot)\) function.

It turns out that we can also express the \(p^{th}\) moment objective of any policy as a suitable integral of the non-completion probabilities. This relies on the following result.

**Lemma 2.2.** Suppose that \(X\) is a non-negative, discrete and bounded random variable. For any \(p \geq 1\), the \(p^{th}\) moment of \(X\) is

\[
\mathbb{E}[X^p] = p \int_{t=0}^\infty t^{p-1} \cdot \Pr[X > t] dt.
\]

**Proof.** Let \(V\) denote the (finite) support of r.v. \(X\). Note that \(x^p = p \cdot \int_0^x t^{p-1} dt\) for all \(x \geq 0\). So, we obtain

\[
\mathbb{E}[X^p] = \sum_{x \in V} \Pr[X = x] \cdot x^p = p \sum_{x \in V} \Pr[X = x] \int_{t=0}^x t^{p-1} dt
\]

\[
= p \int_0^\infty t^{p-1} \sum_{x \in V: x > t} \Pr[X = x] dt = p \int_0^\infty t^{p-1} \cdot \Pr[X > t] dt.
\]

The third equality above switches the order of the integral and summation using Fubini’s theorem.

Now, we apply Lemma 2.2 with random variable \(X = C(\sigma, \Phi)\) where \(\sigma\) is the optimal policy and \(\Phi\) denotes the r.v.s in the ASC instance. Note that the cost \(C(\sigma, \Phi)\) is non-negative, discrete and bounded. Using the fact that \(o(t) = \Pr[C(\sigma, \Phi) > t]\), we obtain:

\[
c_p(\sigma) = \mathbb{E}_\Phi [C(\sigma, \Phi)^p] = p \int_0^\infty t^{p-1} \cdot o(t) dt. \tag{2.1}
\]

Similarly, applying Lemma 2.2 to the r.v. \(C(\pi, \Phi)\) for the greedy policy \(\pi\),

\[
c_p(\pi) = \mathbb{E}_\Phi [C(\pi, \Phi)^p] = p \int_0^\infty t^{p-1} \cdot a(t) dt. \tag{2.2}
\]
2.2 Using the Greedy Criterion

Our analysis of \( \pi \) relies on tracking the “greedy criterion value” defined in (1.1). The following definition formalizes this.

**Definition 2.3** (Greedy score). For \( t \geq 0 \) and any partial realization \( \psi \) observed at time \( t \), define

\[
\text{score}(t, \psi) := \begin{cases} \\
\frac{\Delta(e|\psi)}{c_e(Q-f(\psi))}, & \text{where } e \text{ is the item being selected in } \pi \text{ at time } t \\
0, & \text{if no item is being selected in } \pi \text{ at time } t \text{ when } \psi \text{ was observed.}
\end{cases}
\]

Note that conditioned on \( \psi \), the item \( e \) being selected in \( \pi \) at time \( t \) is deterministic.

The expression for \( \text{score} \) above is exactly the greedy criterion in (1.1). Moreover, the score may increase and decrease over time: see Figure 2 for an example.

In order to reduce notation, for any time \( t \), we use \( \Psi_t := \Psi(\pi, t) \) to denote the (random) partial realization observed by the greedy policy \( \pi \) at time \( t \); recall that this only includes items that have been completely selected by time \( t \).

**Definition 2.4.** (Potential gain) For any time \( i \geq 0 \), its potential gain is the expected total score accumulated after time \( i \),

\[
G_i := \int_{t=i}^{\infty} \mathbb{E}[\text{score}(t, \Psi_t)] \, dt.
\]

![Figure 2: Graph of a simple score(\( t, \psi \)) for illustration. \( e_1, e_2, ... \) are greedy selections and \( \psi_i \) is the partial realization just before selecting \( e_i \).](image)

The key part of the analysis lies in upper and lower bounding the potential gain starting from all time points. The upper-bound relates to the non-completion probabilities \( a(\cdot) \) in \( \pi \) and the lower-bound relates to the non-completion probabilities \( o(\cdot) \) in \( \sigma \). Putting the upper and lower bound together allows us to relate \( a(\cdot) \) and \( o(\cdot) \), which in turn will be used to upper-bound \( c_p(\pi) \) in terms of \( c_p(\sigma) \). For the upper-bound (Lemma 2.5), we view gains as the expected values over full-realizations, which allows us to write the total (conditional) gain as a harmonic series. For the
lower-bound (Lemma 2.6), we view the gain as an integral of expected contributions over time and prove a lower bound for each time step using the optimal policy $\sigma$ and the adaptive-submodularity property.

Below, let $L := 1 + \ln(Q/\eta)$ and $\beta > 1$ be some constant value (that will be fixed later).

**Lemma 2.5.** For any $i \geq 0$, the potential gain at time $i$ is

$$G_i = \int_i^\infty E[score(t, \Psi_t)] dt \leq L \cdot a(i).$$

**Proof.** We start by re-expressing the score and gain in terms of the full realization. For any time $t \geq 0$ and full realization $\phi$, let

$$S(t, \phi) := \left\{ \begin{array}{ll}
\frac{f(\psi, (e, \phi)) - f(\psi)}{\mathbb{E}[\sigma - f(\psi)]}, & \text{where } e \text{ is the item being selected in } \pi \text{ at time } t \text{ under } \phi, \\
0, & \text{if no item is being selected in } \pi \text{ at time } t \text{ under } \phi.
\end{array} \right.$$ 

Then, for any $t \geq 0$ and any partial realization $\psi$ observed at time $t$,

$$score(t, \psi) = E_{\Phi}[S(t, \Phi)|\psi \prec \Phi].$$

This uses the definition of $\Delta(e|\psi)$ and the fact that conditioned on $\psi$, the item $e$ (being selected at time $t$) is fixed. Hence, for any time-point $k \geq 0$, its potential gain

$$G_k = \int_k^\infty E[score(t, \Psi_t)] dt = \int_k^\infty E_{\Psi_t}[E_{\Phi}[S(t, \Phi)|\Psi_t \prec \Phi]] dt = \int_k^\infty E[S(t, \Phi)] dt.$$

Now, fix time $i \geq 0$ and condition on any (full) realization $\phi$.

Case 1: suppose that $\pi$ under $\phi$ terminates before ($<$) time $i$. Then, $S(t, \phi) = 0$ for all $t \geq i$, and so:

$$G_i(\phi) = \int_i^\infty S(t, \phi) dt = 0$$

Case 2: suppose that $\pi$ under $\phi$ terminates after ($\geq$) time $i$.

$$G_i(\phi) = \int_i^\infty S(t, \phi) dt = \int_i^\infty S(t, \phi) dt \leq \int_0^\infty S(t, \phi) dt \leq L,$$

where the last inequality is by Lemma A.1 (proved in Appendix A).

Note that case 2 above happens exactly with probability $a(i)$. So,

$$G_i = E_{\Phi}[G_i(\Phi) \mid \text{case 2 occurs under } \Phi] \cdot Pr[\text{case 2 occurs}] \leq L \cdot a(i),$$

which completes the proof.

**Lemma 2.6.** For any time $t \geq 0$,

$$E[\text{score}(t, \Psi_t)] \geq \frac{a(t) - o(t/(L\beta))}{t/(L\beta)}.$$

Hence, $G_i \geq \int_i^\infty \frac{a(t) - o(t/(L\beta))}{t/(L\beta)} dt$ for each time $i \geq 0$.

**Proof.** Note that the first statement in the lemma immediately implies the second statement. Indeed,

$$G_i = \int_i^\infty E[\text{score}(t, \Psi_t)] dt \geq \int_i^\infty \frac{a(t) - o(t/(L\beta))}{t/(L\beta)} dt.$$

We now prove the first statement. Henceforth, fix time $t \geq 0$. 

10
Truncated optimal policy  Let $\sigma$ denote the cost of policy $\sigma$ truncated at time $t/(L\beta)$. Note that the total cost of selected items in $\sigma$ is always at most $t/(L\beta)$. However, $\sigma$ may not fully cover the utility function $f$ (so it is not a feasible policy for min-cost adaptive submodular cover). We define the following random quantities associated with policy $\sigma$:

$I_k := \text{set of first } k \text{ items selected by } \sigma$, for $k = 0, 1, \cdots$.

$I_\infty := \text{set of all items selected by the end of } \sigma$.

$P_k := \{(e, \Phi_e) : e \in I_k\}$, i.e. partial realization of the first $k$ items selected by $\sigma$, for $k = 0, 1, \cdots$.

$P_\infty := \{(e, \Phi_e) : e \in I_\infty\}$, i.e. partial realization observed by the end of $\sigma$.

Note that $\sigma$ covers $f$ exactly when $f(P_\infty) = Q$. Moreover, by definition of the function $o(\cdot)$, we have $\Pr[\sigma \text{ covers } f] = 1 - o(t/(L\beta))$.

Conditioning on partial realizations in greedy  Let $\psi$ be any partial realization corresponding to $\Psi_t$ with $f(\psi) < Q$. In other words, (i) $\psi$ is the partial realization observed at time $t$ in some execution of policy $\pi$, and (ii) the policy has not terminated (under realization $\psi$) by time $t$. Let $R(\pi, t)$ denote the collection of such partial realizations. Note that the partial realizations in $R(\pi, t)$ are mutually disjoint, and the total probability of these partial realizations equals the probability that $\pi$ does not terminate by time $t$. We will show that:

$$\Pr[\psi \leq \Phi] \cdot \text{score}(t, \psi) \geq \frac{L\beta}{t} \cdot \Pr[\psi \leq \Phi \land (\sigma \text{ covers } f)], \quad \forall \psi \in R(\pi, t). \quad (2.3)$$

We first complete the proof of the lemma assuming (2.3).

$$\mathbb{E}[\text{score}(t, \Psi_t)] \geq \sum_{\psi \in R(\pi, t)} p(\psi) \cdot \text{score}(t, \psi) = \sum_{\psi \in R(\pi, t)} \Pr[\psi \leq \Phi] \cdot \text{score}(t, \psi) \geq \frac{L\beta}{t} \cdot \sum_{\psi \in R(\pi, t)} \Pr[\psi \leq \Phi \land (\sigma \text{ covers } f)] \quad (2.4)$$

$$= \frac{L\beta}{t} \cdot \Pr[(\pi \text{ doesn’t terminate by time } t) \land (\sigma \text{ covers } f)] \quad (2.5)$$

$$\geq \frac{L\beta}{t} \cdot (\Pr[\pi \text{ doesn’t terminate by time } t] - \Pr[\sigma \text{ does not cover } f]) \quad (2.6)$$

$$= \frac{L\beta}{t} \cdot (a(t) - o(t/(\beta L))) \quad (2.7)$$

Inequality (2.4) is by (2.3). The equality in (2.5) uses the definition of $R(\pi, t)$. Inequality (2.6) is by a union bound. Equation (2.7) is by definition of the functions $a(\cdot)$ and $o(\cdot)$.

Proof of (2.3)  Henceforth, fix any partial realization $\psi \in R(\pi, t)$. Our proof relies on the following quantity:

$$Z := \mathbb{E}_\Phi \left[ 1(\psi \leq \Phi) \cdot \frac{f(\psi \cup P_\infty) - f(\psi)}{Q - f(\psi)} \right] \quad (2.8)$$

In other words, this is the expected increase in policy $\sigma$’s function value (relative to the “remaining” target $Q - f(\psi)$) when restricted to (full) realizations $\Phi$ that agree with partial realization $\psi$. 11
For any partial realization $\psi'$ such that $\psi \lesssim \psi'$ and item $e \notin \text{dom}(\psi')$, let $X_{e,\psi,\psi'}$ denote the indicator r.v. that policy $\pi$ selects item $e$ at some point when its observed realizations are precisely $\psi' \cup \chi$ where $\chi \subseteq \psi$. That is, $X_{e,\psi,\psi'} = 1$ if policy $\pi$ selects $e$ at a point where (i) all items in $\text{dom}(\psi' \setminus \psi)$ have been selected and their realization is $\psi' \setminus \psi$, (ii) no item in $E \setminus \text{dom}(\psi')$ has been selected, and (iii) if any item in $\text{dom}(\psi)$ has been selected then its realization agrees with $\psi$. Note that conditioned on $\psi' \lesssim \Phi$, $X_{e,\psi,\psi'}$ is a deterministic value: the realizations of items $\text{dom}(\psi')$ are fixed by $\psi'$ and if any item in $E \setminus \text{dom}(\psi')$ is selected (before $e$) then $X_{e,\psi,\psi'} = 0$ irrespective of its realization.

We can write $Z$ as a sum of increments as follows:

$$Z = \frac{1}{Q - f(\psi)} \mathbb{E}_{\Phi} \left[ \mathbb{I}(\psi \lesssim \Phi) \cdot \sum_{k \geq 1} [f(\psi \cup P_k) - f(\psi \cup P_{k-1})] \right]$$

$$= \frac{1}{Q - f(\psi)} \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} \mathbb{E}_{\Phi} \left[ \mathbb{I}(\psi' \lesssim \Phi) \cdot X_{e,\psi,\psi'} \cdot [f(\psi' \cup (e, \Phi_e)) - f(\psi')] \right]$$

$$= \frac{1}{Q - f(\psi)} \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} \mathbb{E}_{\Phi} \left[ \mathbb{I}(\psi' \lesssim \Phi) \cdot X_{e,\psi,\psi'} \cdot [f(\psi' \cup (e, \Phi_e)) - f(\psi')] \mathbb{I}(\psi' \lesssim \Phi) \right]$$

$$= \frac{1}{Q - f(\psi)} \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} \mathbb{E}_{\Phi} \left[ \mathbb{I}(\psi' \lesssim \Phi) \cdot X_{e,\psi,\psi'} \cdot [f(\psi' \cup (e, \Phi_e)) - f(\psi')] \right]$$

$$= \frac{1}{Q - f(\psi)} \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} \mathbb{E}_{\Phi} \left[ \mathbb{I}(\psi' \lesssim \Phi) \cdot X_{e,\psi,\psi'} \cdot [f(\psi' \cup (e, \Phi_e)) - f(\psi')] \right]$$

$$\leq \frac{1}{Q - f(\psi)} \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} \mathbb{Pr}[\psi' \lesssim \Phi \land X_{e,\psi,\psi'} = 1] \cdot \Delta(e|\psi')$$

$$= \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} \mathbb{Pr}[\psi' \lesssim \Phi \land X_{e,\psi,\psi'} = 1] \cdot c_e \cdot \frac{\Delta(e|\psi')}{c_e(Q - f(\psi'))}$$

$$\leq \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} \mathbb{Pr}[\psi' \lesssim \Phi \land X_{e,\psi,\psi'} = 1] \cdot c_e \cdot \text{score}(t, \psi)$$

$$= \text{score}(t, \psi) \sum_{\psi' : \psi \lesssim \psi'} \sum_{e \notin \text{dom}(\psi')} c_e \cdot \mathbb{Pr}[\psi' \lesssim \Phi \land X_{e,\psi,\psi'} = 1]$$

$$= \text{score}(t, \psi) \sum_{e \in E \setminus \text{dom}(\psi)} c_e \cdot \sum_{\psi' : \psi \lesssim \psi'} \mathbb{Pr}[\psi' \lesssim \Phi \land X_{e,\psi,\psi'} = 1]$$

$$= \text{score}(t, \psi) \sum_{e \in E \setminus \text{dom}(\psi)} c_e \cdot \mathbb{Pr}[\psi \lesssim \Phi \land e \in I_\infty]$$

$$\leq \text{score}(t, \psi) \cdot \mathbb{E}_{\Phi} \left[ \mathbb{I}(\psi \lesssim \Phi) \cdot \sum_{e \in I_\infty} c_e \right]$$

$$\leq \text{score}(t, \psi) \cdot \text{score}(t, \psi) \cdot (t/(L\beta)) \cdot \mathbb{E}_{\Phi}[\mathbb{I}(\psi \lesssim \Phi)] = \text{score}(t, \psi) \cdot (t/(L\beta)) \cdot \mathbb{Pr}[\psi \lesssim \Phi].$$

The equality (2.9) uses the fact that $X_{e,\psi,\psi'}$ is deterministic when conditioned on $\psi' \lesssim \Phi$. Inequality (2.10) is by adaptive submodularity. (2.11) is by the greedy selection criterion. The
inequality in (2.13) uses the fact that the total cost of \( \tilde{\sigma} \)'s selections is always bounded above by \( t/(L\beta) \). Equation (2.12) uses the definition of \( I_\infty \) (all selected items in \( \tilde{\sigma} \)) and the following identity:

\[
\sum_{\psi' : \psi \not\in \psi'} 1(\psi' \not\in \Phi) \cdot X_{e,\psi',\psi'} = 1(\psi \not\in \Phi \land e \in I_\infty), \quad \forall e \in E \setminus \text{dom}(\psi).
\]

To see this, condition on any full realization \( \phi \). If \( \psi \not\in \phi \) then both the left-hand-side (LHS) and right-hand-side (RHS) are 0. If \( \psi \not\in \phi \) and \( e \) is not selected by \( \tilde{\sigma} \) under \( \phi \), then again LHS = RHS = 0. If \( \psi \not\in \phi \) and \( e \) is selected by \( \tilde{\sigma} \) under \( \phi \), then RHS = 1 and LHS is the sum of \( X_{e,\psi',\psi'} \) over \( \psi' \) such that \( \psi \not\in \psi' \not\in \phi \) and \( e \notin \text{dom}(\psi') \). In this case, \( X_{e,\psi',\psi'} = 1 \) for exactly one such partial realization \( \psi' \), namely \( \psi' = \psi \cup \kappa \) where \( \kappa \not\in \phi \) is the partial realization immediately before \( e \) is selected. So, LHS = RHS in all cases.

Note that whenever \( \tilde{\sigma} \) covers \( f \), we have \( f(P_\infty) = Q \). Combined with the monotone property of \( f \), we have \( f(\psi \cup P_\infty) = Q \) whenever \( \tilde{\sigma} \) covers \( f \). So, we have:

\[
Z \geq \Pr[(\psi \not\in \Phi) \land (\tilde{\sigma} \text{ covers } f)].
\]

Combining the above inequality with (2.13) finishes the proof of (2.3). \( \square \)

### 2.3 Wrapping Up

We are now ready to complete the proof of Theorem 1.6. Using Lemmas 2.5 and 2.6, we get:

\[
a(i)L \geq G_i \geq L\beta \int_{i}^{\infty} \frac{a(t) - o(t/(L\beta))}{t} dt, \quad \forall i \geq 0.
\]

Multiplying this inequality by \( t^{p-1} \) and integrating over all \( i \geq 0 \), we obtain:

\[
\int_{0}^{\infty} t^{p-1} a(i) di \geq \beta \int_{0}^{\infty} t^{p-1} \int_{t=i}^{\infty} \frac{a(t) - o(t/(L\beta))}{t} dt \cdot dt \cdot di. \tag{2.14}
\]

Now, using (2.2), the \( p \)th moment of the greedy cost is

\[
c_p(\pi) = p \cdot \int_{0}^{\infty} t^{p-1} a(i) di \geq \beta p \cdot \int_{0}^{\infty} t^{p-1} \int_{t=i}^{\infty} \frac{a(t) - o(t/(L\beta))}{t} dt \cdot dt \tag{2.15}
\]

\[
= \beta p \cdot \int_{0}^{\infty} t^{p-1} \cdot a(t)/t dt - \beta \cdot \int_{0}^{\infty} t^{p-1} \cdot o(t/L\beta) dt \tag{2.16}
\]

\[
= \frac{\beta}{p} \cdot c_p(\pi) - \beta \cdot \int_{0}^{\infty} t^{p-1} \cdot o(t/L\beta) dt \tag{2.17}
\]

\[
= \frac{\beta}{p} \cdot c_p(\pi) - \beta (L\beta)^p \cdot \int_{y=0}^{\infty} y^{p-1} o(y) dy = \frac{\beta}{p} \cdot c_p(\pi) - \frac{\beta}{p} \cdot (L\beta)^p \cdot c_p(\sigma). \tag{2.18}
\]

The inequality in (2.15) is by (2.14). The first equality in (2.16) is by interchanging the order of the integrals (by Fubini’s theorem). Equality (2.17) uses (2.2) for \( c_p(\pi) \). The first equality in (2.18) is by a change of variable \( y = \frac{t}{L\beta} \) in the integral, and the last equality uses (2.1) for \( c_p(\sigma) \).

It now follows that

\[
\left( \frac{\beta}{p} - 1 \right) \cdot c_p(\pi) \leq \frac{\beta}{p} \cdot (L\beta)^p \cdot c_p(\sigma).
\]
In order to minimize the approximation ratio, we choose $\beta = p + 1$ (note that this is only used in the analysis), which implies:

$$c_p(\pi) \leq (p + 1)^{p+1} \cdot L^p \cdot c_p(\sigma).$$

This completes the proof of Theorem 1.6. Setting $p = 1$, we also obtain Theorem 1.5

### 3 Applications

Here, we provide some concrete applications of our framework. These applications were already discussed in [GK17], but as noted in Section 1.1, the function definition in ASC is slightly more restrictive than the framework in [GK17].

**Stochastic Submodular Cover.** In this problem, there are $n$ stochastic items (for example, corresponding to sensors). Each item $e$ can be in one of many “states”, and this state is observed only after selecting item $e$. E.g., the state of a sensor indicates the extent to which it is working. The states of different items are independent. There is a utility function $\hat{f} : 2^{E \times \Omega} \rightarrow \mathbb{R}_{\geq 0}$, where $E$ is the set of items and $\Omega$ the set of states. It is assumed that $\hat{f}$ is monotone and submodular. For e.g., $\hat{f}$ quantifies the information gained from a set of sensors having arbitrary states. Each item $e$ is also associated with a cost $c_e$. Given a quota $Q$, the goal is to select items sequentially to achieve utility at least $Q$, at the minimum expected cost. We assume that the quota $Q$ can always be achieved by selecting adequately many items, i.e., $\hat{f}(\{(e, \phi_e) : e \in E\}) \geq Q$ for all possible states $\{\phi_e \in \Omega\}_{e \in E}$ for the items. This is a special case of ASC, where the items $E$ and states (outcomes) $\Omega$ remain the same. We define a new utility function $f(\psi) = \min \{\hat{f}(\psi), Q\}$ for all $\psi \subseteq E \times \Omega$. Note that $Q$ is the maximal value of function $f$ and this value is achieved under every possible (full) realization. Moreover, $f$ is also monotone and submodular. Clearly, the monotonicity property (Definition 1.1) holds. The adaptive-submodularity property also holds because the items are independent. Indeed, for any partial realizations $\psi \prec \psi'$ and $e \in E \setminus \text{dom}(\psi')$,

$$\Delta(e|\psi) = \sum_{\omega \in \Omega} \Pr[\Phi_e = \omega|\psi \preceq \Phi] \cdot (f(\psi \cup (e, \omega)) - f(\psi))$$

$$= \sum_{\omega \in \Omega} \Pr[\Phi_e = \omega] \cdot (f(\psi \cup (e, \omega)) - f(\psi))$$

$$\geq \sum_{\omega \in \Omega} \Pr[\Phi_e = \omega] \cdot (f(\psi' \cup (e, \omega)) - f(\psi'))$$

$$= \sum_{\omega \in \Omega} \Pr[\Phi_e = \omega|\psi' \preceq \Phi] \cdot (f(\psi' \cup (e, \omega)) - f(\psi')) = \Delta(e|\psi').$$

In particular, when $\hat{f}$ is integer-valued, Theorem 1.5 implies a $4(1 + \ln Q)$-approximation algorithm. We note that [HKP21] obtained a $(1 + \ln Q)$-approximation ratio, using a different analysis. The latter bound is the best possible, including the constant factor, as the problem generalizes set cover. However, our approach is more versatile and also provides a $(p + 1)^{p+1} \cdot (1 + \ln Q)^p$ approximation bound for the $p^{th}$ moment objective. Our result is the first approximation algorithm for higher moments ($p > 1$).

**Adaptive Viral Marketing.** This problem is defined on a directed graph $G = (V, A)$ representing a social network [KKT15]. Each node $v \in V$ represents a user. Each arc $(u, v) \in A$ is associated
with a random variable $X_{uv} \in \{0, 1\}$. The r.v. $X_{uv} = 1$ if $u$ will influence $v$ (assuming $u$ itself is influenced); we also say that arc $(u, v)$ is active in this case. The r.v.s $X_{uv}$ are independent, and we are given the means $\mathbb{E}[X_{uv}] = p_{uv}$ for all $(u, v) \in A$. When a node $u$ is activated/influenced, all arcs $(u, v)$ out of $u$ are observed and if $X_{uv} = 1$ then $v$ is also activated. This process then continues on $u$’s neighbors to their neighbors and so on, until no new node is activated. We consider the “full feedback” model, where after activating a node $w$, we observe the $X_{uw}$ r.v.s on all arcs $(u, v)$ such that $u$ is reachable from $w$ via a path of active arcs. Further, each node $v$ has a cost $c_v$ corresponding to activating node $v$ directly, e.g. by providing some promotional offer. Note that there is no cost incurred on $v$ if it is activated (indirectly) due to a neighbor $u$ with $X_{uv} = 1$. Given a quota $Q$, the goal is to activate at least $Q$ nodes at the minimum expected cost.

To model this as ASC, the items $E = V$ are all nodes in $G$. We add self-loops $A_o = \{(v, v) : v \in V\}$ that represent whether a node is activated directly. So, the new set of arcs is $A' = A \cup A_o$. The outcome $\Phi_w$ of any node $w \in V$ is represented by a function $\phi_w : A' \rightarrow \{0, 1, ?\}$ where $\phi_w((w, w)) = 1$, $\phi_w((v, v)) = 0$ for all $v \in V \setminus w$, and for any $(u, v) \in A$:

- $\phi_w(u, v) = 1$ if there is a $w - u$ path of active arcs and $X_{uw} = 1$ (i.e., $(u, v)$ is active).
- $\phi_w(u, v) = 0$ if there is a $w - u$ path of active arcs and $X_{uw} = 0$ (i.e., $(u, v)$ is not active).
- $\phi_w(u, v) = ?$ if there is no $w - u$ path of active arcs (so, the status of $(u, v)$ is unknown).

Let $\Omega$ denote the collection of all such functions: this represents the outcome space. Note that $\Phi_w$ depends on the entire network (and not just node $w$). So, the r.v.s $\{\Phi_w\}_{w \in V}$ may be highly correlated. Observe that $\Phi_w$ is exactly the feedback obtained when node $w$ is activated directly (by incurring cost $c_w$) at any point in a policy. Define function $\bar{f} : 2^{E \times \Omega} \rightarrow \mathbb{R}_{\geq 0}$ as:

$$\bar{f}(\psi) = \sum_{v \in V} \min \left\{ \sum_{u : \phi_w(u, v) \in A'} |\{w \in \text{dom}(\psi) : \psi_w(u, v) = 1\}|, 1 \right\}. \tag{3.1}$$

$\bar{f}$ is a sum of set-coverage functions, which is monotone and submodular. Then, utility function $f : 2^{E \times \Omega} \rightarrow \mathbb{R}_{\geq 0}$ is $f(\psi) = \min\{\bar{f}(\psi), Q\}$. Function $f$ is clearly monotone (Definition 1.1). The adaptive-submodularity property also holds: see Theorem 19 in [GK17].

Hence, Theorem 1.13 implies a $4(1 + \ln Q)$-approximation algorithm for adaptive viral marketing. This is an improvement over previous approximation ratios of $(1 + \ln Q)^2$ [GK17] and $(1 + \ln(mQc_{\text{max}}))$ [EKM21], where $n = |V|$ and $c_{\text{max}}$ is the maximum cost. We also obtain the first approximation algorithm for the $p^{\text{th}}$ moment objective.

**Optimal Decision Tree (uniform prior).** In this problem, there are $m$ hypotheses $H$ and $n$ binary tests $E$. Each test $e \in E$ costs $c_e$, and has a positive outcome on some subset $T_e \subseteq H$ of hypotheses (its outcome is negative on the other hypotheses). An unknown hypothesis $h^*$ is drawn from $H$ uniformly at random. The goal is to identify $h^*$ by sequentially performing tests, at minimum expected cost. This is a special case of ASC, where the items correspond to tests $E$ and the outcome space $\Omega = \{+,-\}$. The outcome $\Phi_e$ for any item $e$ is the test outcome under the (unknown) hypothesis $h^*$. For any test $e \in E$, define subsets $S_{e,+} = H \setminus T_e$ and $S_{e,-} = T_e$, corresponding to the hypotheses that can be eliminated when we observe a positive or negative outcome on $e$. The utility function is

$$f(\psi) = \frac{1}{|H|} \cdot \left| \bigcup_{e \in \text{dom}(\psi)} S_{e,\psi_e} \right|.$$

---

4Our results also extend to the case of multiway tests with non-binary outcomes.
The quota $Q = 1 - \frac{1}{|H|}$. Achieving value $Q$ means that $|H| - 1$ hypotheses have been eliminated, which implies that $h^*$ is identified. The function $f$ is again monotone and submodular. The monotonicity property (Definition 1.1) clearly holds. Moreover, using the fact that $h^*$ has a uniform distribution, it is known that $f$ is adaptive-submodular: see Lemma 23 in [GK17].

So, Theorem 1.3 implies a $4(1 + \ln(|H| - 1))$-approximation algorithm for this problem; we use $Q$ as above and $\eta = \frac{1}{|H|}$. The previous-best bounds for this problem were $(1 + \ln(|H| - 1))^2$ [GK17], $12 \cdot \ln |H|$ [GB09] and $(1 + \ln(n|H|_{\text{max}}))$ [BKM21]. Again, our result is the first approximation algorithm for this problem under $p^{th}$ moment objectives.

We note that [GK17] also obtained a $\left(\ln \frac{1}{p_{\text{min}}}\right)^2$-approximation for the optimal decision tree problem with arbitrary priors (where the distribution of $h^*$ is not uniform); here $p_{\text{min}} \leq \frac{1}{|H|}$ is the minimum probability of any hypothesis. This uses a different utility function that falls outside our ASC framework (as our definition of function $f$ is more restrictive). Moreover, there are other approaches [GNR17, NKN20] that provide a better $O(\ln |H|)$-approximation bound even for the problem with arbitrary priors.

4 Adaptive Submodular Cover with Multiple Functions

Here, we extend ASC to the setting of covering multiple adaptive-submodular functions. In the multiple adaptive-submodular cover (MASC) problem, there is a set $E$ of items and outcome space $\Omega$ as before. Each item $e \in E$ has a cost $c_e$: we will view this cost as the item’s processing time. Now, there are $k$ different utility functions $f_r : 2^E \times \Omega \rightarrow \mathbb{R}_{\geq 0}$ for $r \in [k]$. We assume that each of these functions satisfies the monotonicity, coverability and adaptive-submodularity properties. We also assume, without loss of generality (by scaling), that the maximal value of each function $\{f_r\}_{r=1}^k$ is $Q$. As for the basic ASC problem, a solution to MASC corresponds to a policy $\pi : 2^E \times \Omega \rightarrow E$, that maps partial realizations to the next item to select. Given any policy $\pi$, the cover time of function $f_r$ is defined as:

$$C_r(\pi) := \text{the earliest time } t \text{ such that } f_r(\Psi(\pi, t)) = Q.$$ 

Recall that $\Psi(\pi, t) \subseteq E \times \Omega$ is the partial realization that has been observed by time $t$ in policy $\pi$. Note that the cover time is a random quantity. The expected cost objective in MASC is the expected total cover time of all functions, i.e., $\sum_{r=1}^k E[C_r(\pi)]$. More generally, for any $p \geq 1$, the $p^{th}$ moment objective of policy $\pi$ is $\bar{c}_p(\pi) := \sum_{r=1}^k E[C_r(\pi)^p]$. When we have just $k = 1$ function, the MASC problem reduces to ASC.

Remark: One might also consider an alternative multiple-function formulation where we are interested in the expected maximum cover time of the functions. This formulation can be directly solved as an instance of ASC where we use the single adaptive-submodular function $g = \sum_{r=1}^k f_r$ with maximal value $Q' = kQ$.

We extend the greedy policy for ASC to MASC, as described in Algorithm 2. For each $r \in [k]$ and item $e \in E$, we use $\Delta_r(e|\psi)$ to denote the marginal benefit of $e$ under function $f_r$. Notice that the greedy selection criterion here involves a sum of terms corresponding to each un-covered function. A similar greedy rule was used earlier in the (deterministic) submodular function ranking problem [AG11] and in the independent special case of MASC in [INvdZ16].

**Theorem 4.1.** Consider any instance of adaptive-submodular cover with $k$ utility functions, where each function $f_r : 2^E \times \Omega \rightarrow \mathbb{R}_{\geq 0}$ is monotone, coverable and adaptive-submodular w.r.t. the same
Algorithm 2 Adaptive Greedy Policy $\pi$ for MASC.

1: selected items $A \leftarrow \emptyset$, observed realizations $\psi \leftarrow \emptyset$
2: while there exists function $f_r$ with $f_r(\psi) < Q$ do
3:    select item $e^* = \arg\max_{e \in E \setminus A} \frac{1}{c_e} \cdot \sum_{r \in [k]: f_r(\psi) < Q} \frac{\Delta_r(e|\psi)}{Q - f_r(\psi)}$, and observe $\Phi_{e^*}$.
4:    add $e^*$ to the selected items, i.e., $A \leftarrow A \cup \{e^*\}$
5: update $\psi \leftarrow \psi \cup \{(e^*, \Phi_{e^*})\}$

probability distribution $q(\cdot)$. Suppose that there is some value $\eta > 0$ such that $f_r(\psi) > Q - \eta$ implies $f_r(\psi) = Q$ for all partial realizations $\psi \subseteq E \times \Omega$ and $r \in [k]$. Then, for every $p \geq 1$, the $p^{th}$ moment of the cost of the greedy policy is

$$\bar{c}_p(\pi) \leq (p + 1)^{p+1} \cdot (1 + \ln(Q/\eta)) \cdot \bar{c}_p(\sigma_p),$$

where $\sigma_p$ is the optimal MASC policy for the $p^{th}$ moment objective.

In particular, setting $p = 1$, we obtain a $4(1 + \ln(Q/\eta))$ approximation algorithm for MASC with the expected cost objective. The proof of Theorem 4.1 is a natural extension of Theorem 1.6 for ASC. We use the same notations and definitions if not mentioned explicitly.

Definition 4.2 (MASC non-completion probabilities). For any time $t \geq 0$ and $r \in [k]$, let

$$\bar{o}_r(t) := \Pr[\sigma \text{ does not cover } f_r \text{ by time } t] = \Pr[C_r(\sigma) > t] = \Pr[f_r(\Psi(\sigma, t)) < Q].$$

$$\bar{a}_r(t) := \Pr[\pi \text{ does not cover } f_r \text{ by time } t] = \Pr[C_r(\pi) > t] = \Pr[f_r(\Psi(\pi, t)) < Q].$$

Also, define for any $t \geq 0$, $\bar{o}(t) := \sum_{r=1}^{k} \bar{o}_r(t)$ and $\bar{a}(t) := \sum_{r=1}^{k} \bar{a}_r(t)$.

For each $r \in [k]$, the functions $\bar{o}_r(t)$ and $\bar{a}_r(t)$ are non-increasing functions of $t$; moreover, $\bar{o}_r(0) = \bar{a}_r(0) = 1$ and $\bar{o}_r(t) = \bar{a}_r(t) = 0$ for all $t \geq \sum_{e \in E} c_e$ (this is because any policy would have selected all items by this time). So, the functions $\bar{o}_r(t)$ and $\bar{a}_r(t)$ share the same properties as $o(t)$ and $a(t)$ for ASC. As in (2.22) and (2.21), we can express the $p^{th}$ moment objective in MASC as:

$$\bar{c}_p(\sigma) = p \int_0^\infty t^{p-1} \cdot \bar{o}(t) dt, \quad \text{and} \quad \bar{c}_p(\pi) = p \int_0^\infty t^{p-1} \cdot \bar{a}(t) dt.$$

Definition 4.3 (MASC greedy score). For any $t \geq 0$ and partial realization $\psi$ observed at time $t$, we define

$$\bar{score}(t, \psi) := \frac{1}{\sum_{r \in [k]: f_r(\psi) < Q} \frac{\Delta_r(e|\psi)}{Q - f_r(\psi)}},$$

where $e$ is the item being selected in $\pi$ at time $t$ and $\psi$ was observed just before selecting $e$.

and $\bar{score}(t, \psi) := 0$ if no item is being selected in $\pi$ at time $t$ when $\psi$ was observed.

Definition 4.4 (MASC potential gain). For any time $i \geq 0$, its potential gain is the expected total score accumulated after time $i$,

$$\tilde{G}_i := \int_i^\infty E[\bar{score}(t, \Psi_t)] dt.$$
The next two lemmas lower and upper bound the potential gain.

**Lemma 4.5.** For any \( i \geq 0 \)
\[
\hat{G}_i \leq \sum_{r \in [k]} L \cdot \tilde{a}_r(t) = L \cdot \tilde{a}(t).
\]

Lemma 4.5 follows by applying Lemma 2.5 to each \( f_r \) and adding over \( r \in [k] \).

**Lemma 4.6.** For any \( t \geq 0 \),
\[
\mathbb{E}[\tilde{\text{score}}(t, \Psi_t)] \geq \sum_{r \in [k]} \tilde{a}_r(t) - \tilde{o}_r(t/(L\beta)) = \frac{\tilde{a}(t) - \tilde{o}(t/(L\beta))}{t/(L\beta)}.
\]

Hence, \( \hat{G}_i \geq L\beta \int_{i}^{\infty} \left( \frac{\tilde{a}(t)(t/(L\beta))}{t} \right) \)dt for each time \( i \geq 0 \).

**Proof Outline.** We replicate all the steps in Lemma 2.6 except that we redefine \( Z \) to be
\[
\hat{Z} := \mathbb{E}_{\Phi} \left[ 1(\psi \leq \Phi) \cdot \sum_{r \in [k] : f_r(\psi) < Q} \frac{f_r(\psi \cup P_\infty) - f_r(\psi)}{Q - f_r(\psi)} \right],
\]
which takes all \( k \) functions into account. Here, \( \psi \) is any partial realization observed at time \( t \). We then obtain:
\[
\tilde{\text{score}}(t, \psi) \cdot \frac{t}{L\beta} \cdot \mathbb{P}[\psi \leq \Phi] \geq \hat{Z} \geq \sum_{r \in [k] : f_r(\psi) < Q} \mathbb{P}[(\psi \leq \Phi) \land (\sigma \text{ covers } f_r)].
\]

Let \( R(\pi, t) \) denote all the possible partial realizations observed at time \( t \) in policy \( \pi \). Then,
\[
\mathbb{E}[\tilde{\text{score}}(t, \Psi_t)] = \sum_{\psi \in R(\pi, t)} \mathbb{P}[\psi \leq \Phi] \cdot \tilde{\text{score}}(t, \psi)
\geq \frac{L\beta}{t} \sum_{\psi \in R(\pi, t)} \sum_{r \in [k] : f_r(\psi) < Q} \mathbb{P}[(\psi \leq \Phi) \land (\sigma \text{ covers } f_r)]
= \frac{L\beta}{t} \sum_{r \in [k]} \mathbb{P}[(\pi \text{ doesn’t cover } f_r \text{ by time } t) \land (\sigma \text{ covers } f_r)]
\geq \frac{L\beta}{t} \sum_{r \in [k]} (\mathbb{P}[\pi \text{ doesn’t cover } f_r \text{ by time } t] - \mathbb{P}[\sigma \text{ doesn’t cover } f_r])
= \frac{L\beta}{t} \sum_{r \in [k]} (\tilde{a}_r(t) - \tilde{o}_r(t/(L\beta))) \geq L\beta \frac{\tilde{a}(t) - \tilde{o}(t/(L\beta))}{t}.
\]

Finally, we combine Lemma 4.5 and Lemma 4.6 as in Theorem 1.6 to obtain:
\[
\tilde{c}_p(\pi) \leq \frac{L^p \beta^{p+1}}{\beta - p} \cdot \tilde{c}_p(\sigma).
\]

Setting \( \beta = p + 1 \) to minimize the approximation ratio, we obtain Theorem 4.1.

We now list some applications of the MASC result.
• When items are deterministic, MASC reduces to the deterministic submodular ranking problem, for which an $O(\ln(Q/\eta))$ was obtained in [AG11]. We note that the result in [AG11] was only for unit costs, whereas our result holds for arbitrary costs. This problem generalizes the min-sum set cover problem, which is NP-hard to approximate better than factor 4 [FLT04]. For min-sum set cover, the parameters $Q = \eta = 1$: so Theorem 4.1 implies a tight 4-approximation algorithm for it.

• When the outcomes are independent across items, MASC reduces to stochastic submodular cover with multiple functions, which was studied in [INvdZ16]. We obtain an $4 \cdot (1 + \ln(Q/\eta))$ approximation ratio that improves the bound of $56 \cdot (1 + \ln(Q/\eta))$ in [INvdZ16] by a constant factor. Although [INvdZ16] did not try to optimize the constant factor, their approach seems unlikely to provide such a small constant factor.

• Consider the following generalization of adaptive viral marketing. Instead of a single quota on the number of influenced nodes, there are $k$ different quotas $Q_1 \leq Q_2 \leq \cdots \leq Q_k$. Now, we want a policy such that the average expected cost for achieving these quotas is minimized. Recall the function $\bar{f}$ defined in (3.1) for the single-quota problem. Then, corresponding to the different quotas, define functions $f_r(\psi) = \frac{1}{Q_r} \cdot \min\{\bar{f}(\psi), Q_r\}$ for each $r \in [k]$. Each of these functions is monotone, adaptive-submodular and has maximal value $Q_k = 1$. The parameter $\eta = 1/Q_k$, so we obtain a $4(1 + \ln Q_k)$-approximation algorithm.

5 Computational Results

In this section, we empirically evaluate the performance of the greedy policy on a real-world dataset of Optimal Decision Tree (ODT), which is one of the applications of ASC (see Section 3). This dataset has been used in a number of previous papers on ODT [BBS12, BAD+07, NKN20]. We also provide a lower-bound for the $p^{th}$ moment objective in ODT, and compare the greedy policy’s performance to this bound.

Recall that in ODT, there are $m$ hypotheses (with uniform probabilities) and $n$ binary tests. We assume that all tests have unit costs. The goal is to identify the realized hypothesis by performing sequential tests. The objective is to minimize the $p^{th}$ moment of the testing cost. The case $p = 1$ is the usual expected cost objective, which has been analyzed before. We also consider higher moments with $p = 2, 3$, for which our result provides the first approximation ratio.

Information theoretic lower bound for ODT. It is well-known that the expected testing cost for any instance of ODT (with uniform probability and cost) is at least $\log_2(m)$. This follows from the entropy lower bound for binary coding. In fact, we can get a stronger lower bound using Huffman coding [Huf52], which corresponds to the ODT instance with $m$ hypotheses and all possible binary tests. Viewed this way, any policy for this ODT instance is a binary tree $T$ with $m$ leaves $L(T)$: the $p^{th}$ moment objective is then $\frac{1}{m} \sum_{i \in L(T)} d(i)^p$ where $d(i)$ is the depth of leaf $i$. Below, we show that the Huffman coding construction also minimizes the $p^{th}$ moment: so it provides a lower bound for ODT with $p^{th}$ moment objectives.

We define the Huffman tree with $m$ leaves to be a binary tree with all non-leaf nodes having two children, and exactly $2^{[\log_2(m)]} - m$ leaves at depth $[\log_2(m)]$ and $2m - 2^{[\log_2(m)]}$ leaves at depth $[\log_2(m)]$. When $m$ is a power of two, the Huffman tree corresponds to the “complete balanced binary tree” with all $m$ leaves at depth $\log_2(m)$. The following lemma (proved in Appendix A) summarizes this lower bound.
Lemma 5.1. Among all binary trees with $m$ leaves, the Huffman tree minimizes the sum of $p^{th}$ powers of the leaf-depths, for all $p \geq 1$. Hence, the optimal $p^{th}$ moment for any ODT instance with $m$ hypotheses and unit-cost tests is at least:

$$\frac{1}{m} \cdot \left( 2^{\lceil \log_2(m) \rceil} - m \right) \cdot [\log(m)]^p + \frac{2}{m} \cdot \left( 2m - 2^{\lceil \log_2(m) \rceil} \right) \cdot [\log(m)]^p.$$ 

Instances. We use the WISER data (http://wiserc.nlm.nih.gov/), which lists 415 toxins and 79 symptoms. Each toxin is associated with a set of observed symptoms that are reported in the data set. This corresponds to an ODT instance where the goal is to identify the toxin by testing for symptoms. For some toxin/symptom combinations it is unknown whether exposure to a toxin definitely exhibits a certain symptom. For such cases, we fill in randomly either a positive or negative outcome for the symptom, and generate 5 such variations, labelled as datasets $A_1$ to $A_5$. Given that we fill in the unknown symptoms randomly, some hypotheses (i.e., toxins) are equivalent (i.e., have identical set of symptoms); so we remove all such duplicates. This is why the number of hypotheses $m$ varies in the instances.

Our results show that the greedy policy is optimal (or extremely close to the optimum) on all the datasets $A_1$ to $A_5$. This indicates that the WISER data set is highly structured with many balanced tests, and that the greedy policy is able to exploit this structure. In order to test the performance on less-structured instances, we also created 5 more instances (labeled $B_1$ to $B_5$) by restricting to a random subset of 15 (out of 79) tests. Again, we remove all duplicate hypotheses.

Results. In Table 1, we report the performance of the greedy policy on each of the 10 instances described above. For each instance, we report the number $m$ of hypotheses and the empirical approximation ratio (greedy objective divided by the lower-bound from Lemma 5.1) for $p = 1, 2, 3$. For the expectation objective ($p = 1$), we also report the objective value of the greedy policy and the two lower bounds: entropy-based (which was also used in prior work) and the Huffman bound (Lemma 5.1). We note that the Huffman bound is indeed better than the entropy-based lower-bound. For instances $A_1$ to $A_5$, the greedy policy’s cost matches the lower bound, resulting in an empirical approximation ratio of 1. For the modified instances $B_1$ to $B_5$, our algorithm’s performance is still very good: the empirical approximation for $p = 1, 2, 3$ is at most 1.04, 1.09 and 1.18 (respectively).

6 Conclusions

We studied the adaptive-submodular cover problem with $p^{th}$ moment objectives, and showed that the natural greedy policy simultaneously achieves a $(p+1)^{p+1} \cdot (\ln Q + 1)^p$ approximation guarantee for all $p \geq 1$. Even for the well-studied case of minimizing expected cost ($p = 1$), our result provides the first $O(\ln Q)$ approximation bound. Our results also extend to the setting with multiple adaptive-submodular functions. While our approximation ratios are the best possible up to constant factors, it would still be interesting to pin down the exact constant. For example, can we get a $\ln(Q)$ approximation ratio for minimizing the expected cost?
Table 1: Computational results on WISER dataset.

| Instance | $m$   | Entropy bound | Revised bound | Sum of costs | Approx. factor $p=1$ | Approx. factor $p=2$ | Approx. factor $p=3$ |
|----------|-------|---------------|---------------|--------------|----------------------|----------------------|----------------------|
| $A_1$    | 405   | 3508.02       | 3538          | 3538         | 1.0000               | 1.0000               | 1.0000               |
| $A_2$    | 395   | 3407.16       | 3438          | 3438         | 1.0000               | 1.0000               | 1.0000               |
| $A_3$    | 399   | 3447.46       | 3478          | 3479         | 1.0000               | 1.0001               | 1.0000               |
| $A_4$    | 399   | 3447.46       | 3478          | 3478         | 1.0000               | 1.0000               | 1.0000               |
| $A_5$    | 395   | 3407.16       | 3438          | 3439         | 1.0003               | 1.0007               | 1.0012               |
| $B_1$    | 207   | 1592.55       | 1607          | 1666         | 1.0367               | 1.0941               | 1.1784               |
| $B_2$    | 248   | 1972.64       | 1976          | 2019         | 1.0218               | 1.0525               | 1.0931               |
| $B_3$    | 249   | 1982.04       | 1985          | 2025         | 1.0202               | 1.0488               | 1.0868               |
| $B_4$    | 266   | 2142.71       | 2148          | 2211         | 1.0293               | 1.0715               | 1.1284               |
| $B_5$    | 274   | 2218.86       | 2228          | 2269         | 1.0184               | 1.0440               | 1.0775               |

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A Missing Proofs

A.1 Upper bounding the score

Below, we upper bound the total score for a single realization $\phi$. A similar fact and proof was used previously in [HKP21, InvdZ16, AG11].

Lemma A.1. For any (full) realization $\phi$,

$$\int_0^\infty S(t, \phi)dt \leq L = \ln(Q/\eta) + 1$$

Proof. Under $\phi$, let $e_1, e_2, ..., e_k$ be the sequence of items selected by $\pi$, let $\psi_i$ be the partial realization just before selecting $e_i$, and define $f_i := f(\psi_i)$. Note that $0 \leq f_1 \leq f_2 \leq \ldots \leq f_{k+1} = Q$ by monotonicity and the assumption that $f$ is always covered by $\pi$. Moreover, we have $f_k \leq Q - \eta$ by the definition of $\eta$: otherwise we would have $f_k = Q$ and $\pi$ would terminate before selecting $e_k$.

Define a function $g : [0, \infty) \to \mathbb{R}$ by

$$g(x) := \begin{cases} \frac{1}{Q-x}, & \text{for } x \in [f_i, f_{i+1}) \text{ and any } i = 1, 2, ..., k-1, \\ 0, & \text{otherwise.} \end{cases} \quad (A.1)$$

Note that $g(x) \leq \frac{1}{Q-x}$ for all $0 \leq x \leq Q - \eta$ (see Figure 3 for an example). So,

$$\int_0^\infty S(t, \phi)dt = \sum_{i=1}^{k} \frac{f_{i+1} - f_i}{Q - f_i} = \sum_{i=1}^{k-1} \frac{f_{i+1} - f_i}{Q - f_i} + 1 = \int_0^\infty g(x)dx + 1 \leq \int_0^{Q-\eta} \frac{1}{Q-x}dx + 1 = L$$

The first equality uses the definition of $S(t, \phi)$ and the second equality uses $f_{k+1} = Q$. \qed
A.2 Proof of Lemma 5.1

Let $T$ denote the binary tree with $m$ leaves that minimizes the sum of $p^{th}$ powers of the leaf-depths, for any $p \geq 1$. We will show that $T$ must be the Huffman tree. Let $D_p(T) := \sum_{i \in L(T)} d(i)^p$ be the objective of tree $T$.

Note that every non-leaf node in $T$ must have 2 children. Indeed, if node $v \in T$ has exactly one child $u$ then we can remove $v$ and hang the subtree rooted at $u$ below $v$’s parent: this reduces the objective $D_p(T)$, contrary to the optimality of $T$.

We now claim that the difference between the maximum/minimum depth of leaves in $T$ is at most one. Let $b$ denote a deepest leaf in $T$, and $b'$ its sibling. Note that $b'$ is also a leaf in $T$ because $b$’s parent (which is non-leaf) must have 2 children. Let $a \in T$ be a leaf of minimum depth.

Suppose, for a contradiction, that $d(a) \leq d(b) - 2$. Let tree $T'$ be obtained from $T$ by adding two children of $a$ as leaves and removing the leaves $\{b, b'\}$ (which makes their parent a leaf in $T'$). Now,

$$D_p(T') - D_p(T) = 2 \cdot (d(a) + 1)^p + (d(b) - 1)^p - (d(a)^p + 2 \cdot d(b)^p)$$
$$= 2 \cdot ((d(a) + 1)^p - d(b)^p) + (d(b) - 1)^p - d(a)^p$$
$$< (d(a) + 1)^p - d(b)^p + (d(b) - 1)^p - d(a)^p$$
$$= (d(a) + 1)^p - d(a)^p + (d(b) - 1)^p - d(b)^p \leq 0.$$  

The (strict) inequality uses $d(a) + 1 < d(b)$, and the last inequality is by convexity of $x^p$ and $d(a) + 1 \leq b(b) - 1$. This is a contradiction to the optimality of tree $T$.

Let $\ell$ be the maximum leaf-depth in $T$: so every leaf has depth $\ell - 1$ or $\ell$. Let $r$ be the number of leaves at depth $\ell - 1$. As there is no leaf at depth less than $\ell - 1$ and each non-leaf node has 2 children, tree $T$ has $2^{\ell-1} - r$ non-leaf nodes at depth $\ell - 1$. This also implies that there are exactly $2 \cdot (2^{\ell-1} - r)$ leaf nodes at depth $\ell$. As $T$ has $m$ leaves in total, we must have $r + 2 \cdot (2^{\ell-1} - r) = m$, which gives $r = 2^{\ell} - m$. Finally, using the fact that $0 \leq r \leq 2^{\ell-1}$, we get $2^{\ell-1} \leq m \leq 2^\ell$. So, $\ell = \lceil \log_2 m \rceil$, which means $T$ is the Huffman tree.