JENSEN’S FUNCTIONAL EQUATION ON THE SYMMETRIC GROUP $S_n$

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Abstract. Two natural extensions of Jensen’s functional equation on the real line are the equations $f(xy) + f(xy^{-1}) = 2f(x)$ and $f(xy) + f(y^{-1}x) = 2f(x)$, where $f$ is a map from a multiplicative group $G$ into an abelian additive group $H$. In a series of papers [1], [2], [3], C. T. Ng has solved these functional equations for the case where $G$ is a free group and the linear group $GL_n(R)$, $R = \mathbb{Z}, \mathbb{R}$, a quadratically closed field or a finite field. He has also mentioned, without detailed proof, in the above papers and in [4] that when $G$ is the symmetric group $S_n$ the group of all solutions of these functional equations coincides with the group of all homomorphisms from $(S_n, \cdot)$ to $(H, +)$. The aim of this paper is to give an elementary and direct proof of this fact.

1. Introduction

On the real line Jensen’s functional equation can be written in the following form

$$f(x + y) + f(x - y) = 2f(x), \forall x, y \in \mathbb{R}.$$  

Let $(G, \cdot)$ be a group with the neutral element $e$, $(H, +)$ an abelian group with the zero element 0, $f : G \to H$ any map. The following natural extensions of Jensen’s functional equation were considered by C.T. Ng ([1], [2], [3]):

$$f(xy) + f(xy^{-1}) = 2f(x), \text{ for all } x, y \in G; \quad (1.1)$$

$$f(xy) + f(y^{-1}x) = 2f(x), \text{ for all } x, y \in G. \quad (1.2)$$

By considering $g(x) := f(x) - f(e)$ for all $x \in G$ we may assume that

$$f(e) = 0. \quad (1.3)$$

Denote by $S_1(G, H)$ resp. $S_2(G, H)$ the set of all solutions of the functional equation (1.1) resp. (1.2) with the normalized condition (1.3).

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Denote by $\text{Hom}(G, H)$ the set of all homomorphisms from $G$ to $H$. These sets are abelian additive groups and it is clear that

$$\text{Hom}(G, H) \leq S_1(G, H) \quad (1.4)$$

and

$$\text{Hom}(G, H) \leq S_2(G, H). \quad (1.5)$$

The equalities in (1.4) and (1.5) occur just in some special cases, for example, when $G$ is abelian and $H$ has no element of order 2; $G$ a free group; $G$ the linear group $GL_n(R)$, $n \geq 2$, $R = \mathbb{Z}, \mathbb{R}$, a quadratically closed field or a finite field. For the case where $G$ is the symmetric group $S_n$, $n \geq 1$, C.T. Ng mentioned in [3] and [4] that the above equalities also hold, however the author has not given a direct proof for this case. In this paper we give an elementary and direct proof for the equalities in (1.4) and (1.5) when $G = S_n, n \geq 1$.

2. The functional equation $f(xy) + f(xy^{-1}) = 2f(x)$

In this section we consider the functional equation (1.1) with the normalized condition (1.3). Denote by $S_n, n \geq 1$, the symmetric group on $n$ elements. The following theorem is the main result in this section.

**Theorem 2.1.** $S_1(S_n, H) = \text{Hom}(S_n, H)$.

To prove Theorem 2.1 we need the following formulae.

**Proposition 2.1** (11, Theorem 2). Let $(G, \cdot)$ and $(H, +)$ be groups. Then for each $f \in S_1(G, H)$ and for all $n \in \mathbb{Z}, x, y, z \in G$ we have

$$f(xyz) + f(zyx) = 2f(xy) + 2f(xz) - 2f(x); \quad (2.1)$$

$$f(xy) + f(yx) = 2f(xz) + 2f(yz) - 2f(z); \quad (2.2)$$

$$2f(xyz) = 2f(xy) + 2f(xz) + 2f(yz) - 2f(x) - 2f(y) - 2f(z); \quad (2.3)$$

$$f(xyz) - f(xyz) = 2f(yz) - 2f(y) - 2f(z); \quad (2.4)$$

$$f(xy^n z) = nf(xyz) - (n - 1)f(xz). \quad (2.5)$$

In particular, we have

$$f(x^n) = nf(x), \text{ for all } x \in G. \quad (2.6)$$

**Lemma 2.1.** Let $\sigma = (a b)$ and $\tau = (b c)$ be transpositions in $S_n$ ($a \neq c$). Then for each $f \in S_1(S_n, H)$ we have

$$f(\sigma \tau) = f(\sigma) + f(\tau) = 0.$$
Proof. Denote by $\delta$ the transposition $(a\ c)$. It is easy to verify that
\[
(\sigma \tau)^4 = \sigma \tau; \quad (2.7)
\]
\[
\sigma \delta \tau = \tau \delta \sigma = \delta; \quad (2.8)
\]
\[
\tau \delta \sigma = \delta \sigma \tau = \sigma; \quad (2.8)
\]
\[
\delta \tau \sigma = \sigma \tau \delta = \tau.
\]
Substituting $x$ by $\delta$, $y$ by $\sigma \tau$ in (1.1), noting that $(\sigma \tau)^{-1} = \tau \sigma$, we have
\[
f(\delta \sigma \tau) + f(\delta \tau \sigma) = 2f(\delta).
\]
It follows from (2.8) and (2.6) that
\[
f(\sigma) + f(\tau) = f(\delta^2) = f(e) = 0.
\]
On the other hand, substituting $x$ by $\sigma \tau$, $y$ by $\delta$ in (1.1), noting that $\delta^{-1} = \delta$, we have
\[
f(\sigma \tau \delta) + f(\sigma \tau \delta) = 2f(\sigma \tau).
\]
It follows from (2.7), (2.6) and (2.8) that
\[
f(\sigma \tau) = f((\sigma \tau)^4) = 4f(\sigma \tau) = 2[2f(\sigma \tau)] = 2[2f(\tau)] = 2f(\tau^2) = 0.
\]
Therefore, $f(\sigma \tau) = 0 = f(\sigma) + f(\tau)$.  \qed

**Lemma 2.2.** Let $\sigma = (a\ b)$ and $\tau = (c\ d)$ be transpositions in $S_n$ ($a, b, c$ and $d$ distinguished). Then for each $f \in S_1(S_n, H)$ we have
\[
f(\sigma \tau) = f(\sigma) + f(\tau) = 0.
\]

Proof. It follows from (2.8) that $\sigma \tau = (a\ b)[(c\ d)(a\ d)]$. Substituting $x$ by $(a\ b)(a\ c)$, $y$ by $(c\ d)(a\ d)$ in (1.1), noting that $((c\ d)(a\ d))^{-1} = (a\ d)(c\ d)$ and $(a\ b)(a\ c)(a\ d)(c\ d) = (a\ d)(d\ b)$, we have
\[
f(\sigma \tau) + f((a\ d)(d\ b)) = 2f((a\ b)(a\ c)).
\]
It follows from Lemma 2.1 that
\[
f((a\ d)(d\ b)) = f((a\ b)(a\ c)) = 0.
\]
Therefore $f(\sigma \tau) = 0$.

On the other hand, it follows also from Lemma 2.1 that
\[
f((a\ b)) + f((b\ c)) = 0;
\]
\[
f((b\ c)) + f((c\ d)) = 0.
\]
Taking the summation of these two equations, noting that $2f((b\ c)) = f((b\ c)^2) = f(e) = 0$, we obtain
\[
f(\sigma) + f(\tau) = 0.
\]
Therefore, $f(\sigma \tau) = 0 = f(\sigma) + f(\tau)$.  \qed
Lemma 2.3. The product of two arbitrary transpositions in $S_n$ has always a square root.

Proof. Let $\sigma$ and $\tau$ be transpositions in $S_n$. If $\sigma = (a \ b)$ and $\tau = (b \ c)$, it follows from (2.7) that

$$\sigma \tau = (\sigma \tau)^4 = [(\sigma \tau)^2]^2.$$ 

On the other hand, if $\sigma = (a \ b)$ and $\tau = (c \ d)$ then it is easy to verify that

$$\sigma \tau = (a \ b)(c \ d) = [(a \ c)(c \ b)(b \ d)]^2.$$ 

\[\Box\]

Lemma 2.4. For each $f \in S_1(S_n, H)$, $2f(x) = 0$ for all $x \in S_n$.

Proof. The statement is trivial for $S_1 = \{e\}$. For $S_2$, $x^2 = e$ for all $x \in S_2$. Therefore it follows from (2.6) that

$$2f(x) = f(x^2) = f(e) = 0, \forall x \in S_2.$$ 

Now we consider the case where $n \geq 3$. Since each permutation in $S_n$ can be written as a product of transpositions (see, for example, [3, Corollary 1, p. 293]), for each $x \in S_n$,

$$x = \sigma_1 \sigma_2 \cdots \sigma_r, \text{ each } \sigma_i \text{ is a transposition in } S_n.$$ 

We prove the lemma by induction on $r$.

For $r = 1$, it follows from (2.6) that $2f(x) = f(x^2) = f(e) = 0$. Assume that the lemma holds for $r \geq 1$. We show that the lemma also holds for $r + 1$. In fact, it follows from (2.3) that

$$2f(x) = 2f((\sigma_1 \cdots \sigma_{r-1}) \cdot \sigma_r \cdot \sigma_{r+1})$$

$$= 2f((\sigma_1 \cdots \sigma_{r-1}) \cdot \sigma_r) + 2f((\sigma_1 \cdots \sigma_{r-1}) \cdot \sigma_{r+1}) +$$

$$+ 2f(\sigma_r \sigma_{r+1}) - 2f(\sigma_1 \cdots \sigma_{r-1}) - 2f(\sigma_r) - 2f(\sigma_{r+1}).$$

The conclusion for $r + 1$ follows from Lemma 2.1, Lemma 2.2 and the induction hypothesis. \[\Box\]

Corollary 2.1. For each $f \in S_1(S_n, H)$ and for every $x, y, z \in S_n$, we have

$$f(xyz) = f(xzy) = f(yxz).$$

Therefore we may re-arrange the order of the transpositions in each permutation $x \in S_n$ in such a way that the value $f(x)$ does not change.

Proof. It follows from (2.1), (2.2), (2.4) and Lemma 2.4 that

$$f(xyz) + f(yxz) = f(xyz) + f(xzy) = f(xyz) - f(xzy) = 0.$$ 

This proves the corollary. \[\Box\]
Lemma 2.5. For each $f \in S_1(S_n, H)$ and for each $x \in S_n$, the following holds:

(i) If the permutation $x$ is even then $f(x) = 0$.
(ii) If $x$ is odd then $f(x) = -f(\sigma_r)$, where $\sigma_r$ is the last transposition in a decomposition of $x$.

Proof. Let $x \in S_n$. Then $x$ can be written as $x = \sigma_1 \cdots \sigma_r$, where each $\sigma_i$ is a transposition. If the number of transpositions $r$ is even, namely $r = 2k$, it follows from Lemma 2.3 that there exist permutations $\tau_1, \cdots, \tau_k$ such that

$$x = \tau_1^2 \cdots \tau_k^2.$$ 

Then it follows from Lemma 2.4 that

$$f(x) = f((\tau_1 \cdots \tau_k)^2) = 2f(\tau_1 \cdots \tau_k) = 0.$$ 

If $r$ is odd, namely $r = 2k + 1$, we can write $x = \tau_1^2 \cdots \tau_k^2 \sigma_r$. Then it follows from (2.3) and Lemma 2.4 that

$$f(x) = f((\tau_1 \cdots \tau_k)^2 \sigma_r) = 2f((\tau_1 \cdots \tau_k)\sigma_r) - f(\sigma_r) = -f(\sigma_r).$$ 



Proof of Theorem 2.1 Let $f$ be an arbitrary element of $S_1(S_n, H)$, $x$ and $y$ any permutations in $S_n$. Then these permutations can be written as $x = \sigma_1 \cdots \sigma_r$ and $y = \tau_1 \cdots \tau_s$, where each $\sigma_i$ and $\tau_j$ are transpositions in $S_n$. We have the following cases:

The first case: both $r$ and $s$ are even. Then the product $xy$ is an even permutation, hence it follows from Lemma 2.5 that $f(x) = f(y) = f(xy) = 0$. Thus $f(xy) = f(x) + f(y)$.

The second case: $r$ is even, $s$ is odd. Then the product $xy$ is odd, hence it follows from Lemma 2.5 that $f(x) = 0, f(y) = -f(\tau_s)$ and

$$f(xy) = f((\sigma_1 \cdots \sigma_r)(\tau_1 \cdots \tau_{s-1})\tau_s) = -f(\tau_s) = f(x) + f(y).$$ 

The third case: $r$ is odd, $s$ is even. Similarly to the second case, we have

$$f(xy) = -f(\sigma_r) = f(x) + f(y).$$ 

The last case: both $r$ and $s$ are odd. Then $f(x) = -f(\sigma_r), f(y) = -f(\tau_s)$, while $f(xy) = 0$ since the product $xy$ is even. It follows from the proof of Lemma 2.4 and Lemma 2.2 that $f(\sigma_r) + f(\tau_s) = 0$. Therefore in this case we have also $f(xy) = f(x) + f(y)$.

Thus in any cases we always have $f(xy) = f(x) + f(y)$, i.e., $f \in \text{Hom}(S_n, H)$. This proves the theorem.
3. The functional equation $f(xy) + f(y^{-1}x) = 2f(x)$

In this section we consider the functional equation (1.2) with the normalized condition (1.3) and show that the equality in (1.5) occurs for the group $G = S_n$. The proof for this equality follows step-by-step the one given for the equation (1.1) in Section 2.

**Proposition 3.1** ([3, Theorem 2.1]). Let $(G, \cdot)$ and $(H, +)$ be groups. Then for each $f \in S_2(G, H)$ and for all $n \in \mathbb{Z}, x, y, z \in G$ we have

\[
f(x^n) = nf(x);
\]

\[
f(xyz) + f(xzy) = 2f(xy) + 2f(xz) - 2f(x);
\]

\[
f(xyz) + f(yxz) = 2f(xz) + 2f(yz) - 2f(z);
\]

\[
f(xyz) - f(xzy) = 2f(yz) - 2f(y) - 2f(z);
\]

\[
2f(xyz) = 2f(xy) + 2f(xz) + 2f(yz) - 2f(x) - 2f(y) - 2f(z);
\]

\[
f(xy^2z) = f(xz) + 2f(y).
\]

**Lemma 3.1.** Let $\sigma = (a \ b)$ and $\tau = (b \ c)$ be transpositions in $S_n$ ($a \neq c$). Then for each $f \in S_2(S_n, H)$ we have

\[
f(\sigma \tau) = f(\sigma) + f(\tau) = 0.
\]

**Proof.** Denote $\delta = (a \ c)$. Substituting $x$ by $\sigma \tau$ and $y$ by $\delta$ in (1.2), noting that $\delta^{-1} = \delta$ and using (2.8) we have

\[
f(\tau) + f(\sigma) = 2f(\sigma \tau).
\]

It follows from (3.7), (2.7) and (3.1) that

\[
f(\sigma \tau) = f((\sigma \tau)^4) = 4f(\sigma \tau) = 2f(\sigma) + 2f(\tau) = f(\sigma^2) + f(\tau^2) = f(e) + f(e) = 0.
\]

Substituting $f(\sigma \tau) = 0$ into the equation (3.7) we have

\[
f(\sigma) + f(\tau) = 2f(\sigma \tau) = 0 = f(\sigma \tau).
\]

\[\square\]

**Lemma 3.2.** Let $\sigma = (a \ b)$ and $\tau = (c \ d)$ be transpositions in $S_n$ ($a, b, c$ and $d$ distinguished). Then for each $f \in S_2(S_n, H)$ we have

\[
f(\sigma \tau) = f(\sigma) + f(\tau) = 0.
\]

**Proof.** Substituting $x = (a \ b)(a \ c)$ and $y = (c \ d)(a \ d)$ in (1.2), noting that

\[
xy = (a \ b)(a \ c)(c \ d)(a \ d) = (a \ b)[(a \ c)(c \ d)(a \ d)] = (a \ b)(c \ d) = \sigma \tau,
\]

\[\Rightarrow ((c \ d)(a \ d))^{-1} = (a \ d)(c \ d),\]
and \( y^{-1}x = (a \ d)(c \ d)(a \ b)(a \ c) = (b \ d)(d \ c) \), we have
\[
f(\sigma \tau) + f((b \ d)(d \ c)) = 2f((a \ b)(a \ c)).
\]
Hence it follows from Lemma 3.1 that \( f(\sigma \tau) = 0 \).
By a similar argument to the proof of the second part of Lemma 2.2 we have also \( f(\sigma) + f(\tau) = 0 = f(\sigma \tau) \).

**Lemma 3.3.** For each \( f \in S_2(S_n, H) \), \( 2f(x) = 0 \) for all \( x \in S_n \).

**Proof.** Since we have also the equation (3.5) which is the same as the equation (2.3), the proof for this lemma is the same as the one given in Lemma 2.4.

**Corollary 3.1.** For each \( f \in S_2(S_n, H) \) and for every \( x, y, z \in S_n \), we have
\[
f(xyz) = f(xzy) = f(yxz).
\]
Therefore we may re-arrange the order of the transpositions in each permutation \( x \in S_n \) in such a way that the value \( f(x) \) does not change.

**Proof.** The corollary follows from (3.2), (3.3), (3.4) and Lemma 3.3.

**Lemma 3.4.** For each \( f \in S_2(S_n, H) \) and for each \( x \in S_n \), the following holds:

(i) If the permutation \( x \) is even then \( f(x) = 0 \).

(ii) If \( x \) is odd then \( f(x) = f(\sigma_r) \), where \( \sigma_r \) is the last transposition in a decomposition of \( x \).

**Proof.** The proof for (i) is the same as the one given in Lemma 2.5 (i), using Lemma 3.3. The proof for (ii) is also similar to the one given in Lemma 2.5 (ii), using the equation (3.6) instead of the equation (2.5).

Therefore, by a similar argument to the proof of Theorem 2.1, using Lemma 3.4 instead of Lemma 2.5 we obtain the main theorem of this section.

**Theorem 3.1.** \( S_2(S_n, H) = \text{Hom}(S_n, H) \).

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