Counterposed phase velocity and energy–transport velocity vectors in a dielectric–magnetic uniaxial medium

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Abstract: When a plane wave is launched from a plane surface in a linear, homogenous, dielectric–magnetic, uniaxial medium, we show that its phase velocity and the energy–transport velocity vectors can be counterposed (i.e., lie on different sides of the surface normal) under certain circumstances.

Keywords: Anisotropy; Energy–transport velocity; Phase velocity;

1 Introduction

In any linear homogeneous medium, two distinct plane waves can propagate in any direction (except in very rare circumstances [1, 2], which are ignored here). With each plane wave are associated a phase velocity vector and an energy–transport velocity vector [3]. These two vectors are parallel to each other in isotropic mediums, but not in anisotropic mediums.

While examining a recently reported experimental result [4], we came across the following question: If a plane wave is launched from an infinite plane — possibly, either by reflection or refraction — into a linear, homogeneous, anisotropic medium, can the phase velocity and

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the energy–transport velocity vectors be counterposed (i.e., lie on different sides of the surface normal), as shown in Figure 1? Although we suspected an affirmative answer to the question, we were unable to find any treatment of the question in standard textbooks. Therefore, we undertook an investigation, the results of which are reported here.

2 Analysis

Let us consider a dielectric–magnetic uniaxial medium whose relative permittivity and relative permeability dyadics are denoted by

\[ \varepsilon_r = \varepsilon_a \mathbb{I} + (\varepsilon_b - \varepsilon_a) \mu_c \mu_c, \]  
\[ \mu_r = \mu_a \mathbb{I} + (\mu_b - \mu_a) \epsilon_c \epsilon_c, \]  

respectively, where \( \mathbb{I} \) is the identity dyadic and \( \epsilon_c \) is a unit vector parallel to the distinguished axis of the medium.

In this medium, two distinct plane waves can propagate in any given direction, as detailed elsewhere [5]. The wavenumbers of the two plane waves are obtained as

\[ k_1 = k_0 \left( \frac{\mu_a \varepsilon_a \varepsilon_b}{\epsilon_k \cdot \varepsilon_k \cdot \epsilon_k} \right)^{1/2}, \]  
\[ k_2 = k_0 \left( \frac{\varepsilon_a \mu_a \mu_b}{\mu_k \cdot \mu_k \cdot \mu_k} \right)^{1/2}, \]  

where \( \epsilon_k \) is a unit vector denoting the direction of propagation while \( k_0 \) is the free–space wavenumber. The electric and magnetic field phasors associated with the two plane waves are known, their expressions not being needed for the present purposes.

But we do need expressions for the phase velocity and the energy–transport velocity vectors. With the assumption that the imaginary parts of \( \varepsilon_{a,b} \) and \( \mu_{a,b} \) are negligibly small, we obtain [5]

\[ \varepsilon_{\ell} = \frac{\omega}{k_\ell} \epsilon_k, \quad (\ell = 1, 2), \]  

2
for the phase velocity vectors, and

\[ v_{e1} = c_0 \frac{k_1}{k_0} \frac{\mu}{\varepsilon} \cdot \frac{\mu}{\varepsilon} \]
\[ v_{e2} = c_0 \frac{k_2}{k_0} \frac{\varepsilon}{\mu} \cdot \frac{\varepsilon}{\mu} \]

for the energy–transport velocity vectors of the two plane waves, with \( c_0 \) denoting the speed of light in free space. Note that \( v_{e1,2} \) are co–parallel with the respective time–averaged Poynting vectors; and they are also co–parallel with the respective group velocity vectors in the absence of dispersion [3, Sec. 3.6].

Let us now suppose that a plane wave is launched into the half–space \( z > 0 \) from the plane \( z = 0 \). We say that the phase velocity vector \( v_p \) and the energy–transport velocity vector \( v_{e} \) of the \( \ell \)-th plane wave, \((\ell = 1, 2)\), are counterposed if the two vectors are pointed on the opposite sides of the +z axis.

Without loss of generality, we set

\[ \mathbf{u}_x = \sin \xi \mathbf{u}_x + \cos \xi \mathbf{u}_z \]
\[ \mathbf{u}_k = \sin \theta \mathbf{u}_x + \cos \theta \mathbf{u}_z \]

where \( \mathbf{u}_x \) and \( \mathbf{u}_z \) are unit cartesian vectors, while the angles \( \theta \in [-90^\circ, 90^\circ] \) and \( \xi \in [-90^\circ, 90^\circ] \). Then, the expressions

\[ v_{e1} = \frac{k_1}{k_0} \frac{c_0}{2\varepsilon_a \varepsilon_b} \left\{ (\varepsilon_a + \varepsilon_b) (\sin \theta \mathbf{u}_x + \cos \theta \mathbf{u}_z) \right. \\
+ (\varepsilon_a - \varepsilon_b) [\sin(\theta - 2\xi) \mathbf{u}_x - \cos(\theta - 2\xi) \mathbf{u}_z] \} \]

and

\[ v_{e2} = \frac{k_1}{k_0} \frac{c_0}{2\mu_a \mu_b} \left\{ (\mu_a + \mu_b) (\sin \theta \mathbf{u}_x + \cos \theta \mathbf{u}_z) \right. \\
+ (\mu_a - \mu_b) [\sin(\theta - 2\xi) \mathbf{u}_x - \cos(\theta - 2\xi) \mathbf{u}_z] \} \]

emerge from (6).

Let us define angles \( \psi_\ell \), \((\ell = 1, 2)\), through the relation \( v_{e\ell} = v_{e1,2} (\sin \psi_\ell \mathbf{u}_x + \cos \psi_\ell \mathbf{u}_z) \); hence,

\[ \tan \psi_\ell = \frac{\sin \theta - \delta_\ell \sin(\theta - 2\xi)}{\cos \theta + \delta_\ell \cos(\theta - 2\xi)} \], \((\ell = 1, 2)\), (10)
where the degree of uniaxiality

\[
\delta_\ell = \begin{cases} 
\frac{\epsilon_b - \epsilon_a}{\epsilon_b + \epsilon_a}, & \text{if } \ell = 1 \\
\frac{\mu_b - \mu_a}{\mu_b + \mu_a}, & \text{if } \ell = 2
\end{cases}
\]  \hspace{1cm} (11)

The counterposition condition then amounts to

\[
(sin \theta) (tan \psi_\ell) < 0 . \hspace{1cm} (12)
\]

Alternatively, the two velocity vectors of the \(\ell\)-th plane wave are counterposed if \(\psi_\ell \leq 0^\circ\) when \(\theta \geq 0^\circ\).

Figure 2 shows computed values of \(\psi_\ell\) for \(\theta > 0^\circ\), when the degree of uniaxiality is positive (i.e., \(\epsilon_b > \epsilon_a\) for \(\ell = 1\), and \(\mu_b > \mu_a\) for \(\ell = 2\)). Figure 3 shows the computed values for negative uniaxiality (i.e., \(\epsilon_b < \epsilon_a\) for \(\ell = 1\), and \(\mu_b < \mu_a\) for \(\ell = 2\)). The latter figure can, in fact, be deduced from Figure 2 via the substitution \(\{\delta_\ell \rightarrow -\delta_\ell, \xi \rightarrow \xi \pm \pi/2\}\), but has been included for completeness. Quite clearly, a wide \(\xi\)–range exists for very small angles \(\theta\) for which the counterposition condition is satisfied. As \(\theta\) increases, the \(\xi\)–range for counterposition diminishes and eventually vanishes. The higher the degree of uniaxiality in magnitude, the larger is the portion of the \(\theta\xi\) space wherein the counterposition condition is satisfied.

### 3 Conclusion

When a plane wave is launched from a plane surface — possibly by refraction or reflection — in a linear, homogenous, dielectric–magnetic, uniaxial medium, we have shown here that its phase velocity and the energy–transport velocity vectors may be counterposed. An excellent experimental example has been furnished by Zhang et al. [4].

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Figure 1: Schematic to explain counterposition of the phase velocity and the energy transport velocity vectors.
Figure 2: Top: Computed values of $\psi_\ell$ for $\theta \in [0^\circ, 90^\circ]$, when $\delta_\ell = 1/9$. Bottom: Negative values of $\psi_\ell$ are isolated in this graph.
Figure 3: Same as Figure 2, but for $\delta_{\ell} = -1/9$. 