Toponium as Nonabelian Positronium?

W. Kummer and W. Mödritsch

Institut für Theoretische Physik, TU-Wien
1040 Wien, Wiedner Hauptstraße 8-10/136
FAX: (01143) 222-567760

Abstract
The expected large mass $m_t$ of the top quark provides for the first time a chance to discuss the bound-states of the corresponding quantum system as the nonabelian generalization of positronium, using the full power of relativistic quantum field theoretic methods which are available for weakly bound systems. Thus our approach differs in principle from the one used in the vast phenomenological literature on quarkonium potentials. We emphasize especially the corrections of energy levels which are of order $\alpha^4 m_t$ or numerically comparable to that order, and which have no counterpart in the 'relativistic' corrections of QED. In contrast to previous computations we give analytic expressions for all contributions considered in our present work, hopefully preparing the ground for further similar calculations.

Vienna, June 1993
PACS: 11.10S, 11.15, 12.10D, 12.38, 14.40G
1 Introduction

Perturbative expansions in the coupling constant in quantum field theory possess two types of applications, the calculation of scattering processes and the computation of processes involving weakly bound systems. Many of the successes of quantum electrodynamics (QED) are, in fact, related to positronium, i.e. to the second one of the aforementioned applications. The proper starting point for any bound-state calculation in quantum field theory is an integral equation, comprising an infinite sum of Feynman graphs. The Bethe-Salpeter (BS) equation [1] fulfills this task and it is well known that in the limit of binding energies of $O(\alpha^2 m)$ the Schrödinger equation with static Coulomb attraction is obtained. The computation of higher order corrections to the Bohr-levels, however, turned out to be far from trivial. It was recognized, though, relatively late that, at least conceptually, substantial progress with respect to a systematic treatment results from a consistent use of a perturbation theory geared to the original BS equation [2]. In that manner, at the same time, nonrelativistic expansions as implied by Hamiltonian approaches with successive Fouldy-Wouthuysen transformations [3] are avoided. Within the BS-technique, however, it is desirable to have an exactly solvable zero order equation different from the Schrödinger equation, otherwise the approximation procedure for the wave function lacks sufficient transparency, especially in higher orders. One of the advantages of the BS approach to perturbation theory is the freedom to select a different zero order equation. Of course, in that case, certain corrections included already at the zero level are to be properly subtracted out in higher orders. An especially useful zero order equation has been proposed some time ago by Barbieri and Remiddi (BR equation [4]). Still, one of the most annoying features of all bound-state calculations remains the pivotal rule played by the Coulomb gauge. In other gauges, e.g. already the (in QED vanishing) corrections $O(\alpha^3 m)$ of the Bohr levels imply to take into account an infinite set of Feynman graphs [5]. Only in very special cases, when certain subsets of graphs can be shown to represent together a gauge-independent correction, another more suitable gauge may be chosen. By contrast to QED the vast literature on bound state problems in quantum chromodynamics (QCD) adheres to a description of the quark-antiquark system by the Schrödinger equation with corrections 'motivated' by QCD [6]. As long as a relatively small number of parameters suffices for an adequate phenomenological description of observed quantum levels, this approach undoubtedly has an ample practical justification. However, again and again certain deviations from such phenomenological description are reported [7]. Thus
also for this reason a return to more rigorous QCD arguments remains as desirable as ever. The standard literature on this subject almost exclusively is based on nonrelativistic expansions [3] or on the calculation of purely static Coulomb forces [8]. Also very often potentials with higher order corrections as determined from on-shell quarkonia scattering are used [9]. In that cases relevant off-shell effects may be even lost which are typical for higher order corrections. On the other hand, from the point of view of relativistic quantum field theory as elaborated in the abelian case of positronium, a similar, more systematic approach seems desirable, the more so because the basic techniques are well developed. In addition, at least in one case, namely the decay of S-wave quarkonium, the result of a full BS-perturbation calculation [10], including the QCD corrections to the bound state wave function, yields a result very different from the one which considered only the corrections to the quark antiquark annihilation alone [11].

In this connection the relatively large size of the running coupling constant even at high energies represents a well known problem, together with large coefficients from a perturbative expansion. For this reason e.g. problems arise in the comparison of the coupling constant as determined from scattering experiments within the minimal subtraction scheme (\(\overline{\text{MS}}\)), with the coupling constant to be used in a consistent weak bound-state approach. The philosophy within our present work will be that the orders of magnitude, as determined from \(\alpha_{\overline{\text{MS}}}\) will be used for estimates, but that we shall imply a determination of \(\alpha_s\) by some physical observable (e.g. energy levels, cf. the remarks after eq. (4.31) ) of the quarkonium system itself. In that way delicate correlations of 'genuine' orders of \(\alpha_s\) from \textit{basically} different types of experiments are avoided.

Of course, the quarkonium system also differs profoundly from positronium because of the confinement of quarks and gluons. However, the phenomenological success of the nonrelativistic quarkonium model can be explained by the fact that the bound states of heavy quarkonia are deep in the Coulomb funnel and thus sufficiently far away from the confinement part of the potential. Estimates in the early 80-s [12] of nonperturbative effects, describing the 'tail' of confinement by a gluon condensate [13] suggested that only with quark masses well above about 50\(\text{GeV}\) the importance of such nonperturbative effects for low Bohr quantum numbers may decrease sufficiently to make perturbative 'field theoretical' level corrections competitive and observable. From high precision electro-weak experiments of the LEP collaborations, the mass range of the top quark now seems to be established to lie in the range 100-180\(\text{GeV}\) [14]. Thus for the first time a nonabelian bound state quarkonium system seems to fullfil the high mass requirement for a genuine
field theoretic approach. Unfortunately the drawback of this situation is that the weak decay $t \to b + W$ broadens the energy levels for increasing mass $m_t$ so that above $m_t \approx 140 GeV$ individual levels effectively disappear. Even for $m_t \approx 100 GeV$ this broadening of energy levels certainly already makes the resolution of different Bohr levels of $O(ma^2/4n^2) \approx 1 GeV$ (for $\alpha \approx 0.2$) difficult to observe. The first field theoretic correction for nonabelian QCD already starts at $O(ma^3)$ [8, 16], and the corresponding shift depends on the principal and the angular momentum quantum number. Although one thus has to fully acknowledge the problems related to experimental observations of corrections to $O(\alpha^4 m)$ for the energy levels of the toponium system, we have been tempted to trust the ingenuity of experimentalists to eventually arrive at sufficiently precise data [7], together with the hope that $m_t$ does not lie too far above $100 GeV$. In that case, for the first time, genuine field-theoretic consequences of QCD could be tested at something like a nonabelian positronium system. Of course, the well known ‘relativistic’ corrections, corresponding to the same type of graphs as in the abelian case are present here as well. However, already considering only vertex corrections, a difference to the abelian case at $O(\alpha^4 m)$ was discovered a long time ago [10]. In addition typical other nonabelian contributions may appear at this level as well. In contrast to scattering processes, the determination of the order to which a certain graph contributes in relativistic bound state perturbation theory in each case requires a special analysis. Other terms (from QED and weak interaction), incidentally, may be of the same numerical order $O(\alpha^4 m)$. The Coulomb gauge also entails peculiar additional nonlocal interaction terms in the effective action, appearing e.g. in the path integral formulation [18]. We also investigate the effect of those terms here.

In section 2 we recall some basic facts about BS-perturbation theory and about the BR equation for nonabelian weakly bound onium-systems.

Tree graphs leading among others to the well known $O(\alpha^4 m)$ corrections are discussed in sect. 3. As indicated already above, the one loop vacuum polarization (sect. 4.1) provides a correction term $O(\alpha^3 m)$ in the nonabelian case. Here we take the opportunity to point out that effects from some of the lighter quarks (charm, bottom) in the toponium system must be treated more carefully than it is usually done by including the quarks only in the number of flavours of a running coupling constant.

In sect. 4.2 we revisit the old calculation of Duncan [16] within our present formalism, avoiding some approximations made in this early computation which allows us to make even some statements on the term $O(\alpha^5 m)$. Finally sect. 5 is devoted to an exploration of
possible further corrections of the same numerical order of magnitude as $O(\alpha^4 m)$. We list several candidates of relevant QCD graphs and we show in cases of most simple two loop graphs that such corrections may well be relevant. Beside these graphs from QCD, also corrections from the weak interaction and QED may turn out to contribute to this order, but the nonlocal Schwinger-Christ-Lee type graphs, peculiar to the Coulomb gauge, are irrelevant to $O(\alpha^4 m)$ as shown by an explicit calculation.

## 2 BS-Perturbation of the BR Equation

A correct formulation of QCD in Coulomb gauge entails not only Faddeev-Popov-ghost terms but also the inclusion of nonlocal interaction terms [18]. Therefore, the full Lagrangian reads ($a=1,...,8$ for SU(3)):

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_{j=1}^f \bar{\Psi}_j (i\gamma^\mu D - m_j) \Psi_j + B^a (\partial_j A_j^a) - \eta^a \partial^i (\delta_{ab} \partial_i + gf_{abc} A_i^c) \eta^b + v_1 + v_2 \quad (2.1)
$$

where the Lagrange multiplier $B^a$ guarantees the Coulomb gauge, and where

$$
D_\mu = \partial_\mu - ig T^a A_\mu^a, \quad (2.2)
$$

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc} A_\mu^b A_\nu^c. \quad (2.3)
$$

$v_1$ and $v_2$ are given in [18] and are discussed more explicitely below. The above Lagrangian will include all effects of the strong interaction, but, as we will show, QED and weak corrections may also give a contribution within the numerical order of our main interest ($O(\alpha^4)$).

In our notation the BS equation in terms of Feynman amplitudes for $K$ and $S$ reads as

$$
\chi_{ij}^{BS}(p; P) = -iS_{ii'}(\frac{P}{2} + p)S_{jj'}(-\frac{P}{2} + p) \int \frac{d^4 p'}{(2\pi)^4} K_{i'j',i'j'}(P, p, p') \chi_{i'j'}^{BS}(p'; P). \quad (2.4)
$$

$\chi$ denotes the BS wave function, $S$ the exact fermion propagators, and $K$ is the sum of all two fermion irreducible graphs. Furthermore, we have introduced relative momenta $p$ and $p'$, a total momentum $P = p_1 - p_2$, and we choose a frame where $P = (P_0, \vec{0})$.

The notation can be read off the pictorial representation in fig.1. $i, j$ are collective indices for spin (noted $\sigma, \rho$) and colour (noted $\alpha, \beta$). It is well known that the dominant
part in $K$ for weak binding ($\alpha \to 0$) is the one-Coulomb gluon exchange which results in an ordinary Schrödinger equation with static Coulomb potential. This result is even independent of the chosen gauge in the ladder approximation by a simple scaling argument $p_0 \approx O(m\alpha^2), |\vec{p}| \approx O(m\alpha), P_0 \approx 2m - O(m\alpha^2)$ \cite{19}:

$$K \to K_c := -\frac{4\pi\alpha}{(|\vec{p} - \vec{p}'|)^2} \gamma^0_{\sigma\sigma'} \gamma^0_{\rho'\rho}$$

(2.5)

In equation (2.5) we have already used the fact that only colour singlet states can form bound states because the Coulomb potential is repulsive for colour octets. The colour trace will always be understood to be already done, leading to the definition

$$\alpha \equiv \frac{4g^2}{34\pi} = \frac{4}{3} \alpha_s.$$  

(2.6)

in terms of the usual strong coupling constant $\alpha_s$. Because the above mentioned nonrelativistic limit of the BR equation contains the projection operators $\lambda^\pm$, defined below, it is awkward to calculate the so-called relativistic corrections in a straightforward way within the framework of BS perturbation theory, starting from (2.4) with (2.3). Therefore, we use the BR equation \cite{1} instead of the Schrödinger equation. It is obtained from (2.4) by the substitutions

$$S \to \frac{1}{\gamma p - m},$$

$$K \to K_{BR} \equiv [\gamma_0 \Lambda^+ \lambda^+ \Lambda^+ \gamma_{\sigma',\sigma'} \gamma_{\rho',\rho'} \tilde{K},$$

(2.7)

with the projection operators

$$\Lambda^\pm(\vec{p}) \equiv \frac{E_p + \alpha\vec{p} \pm \beta m}{2E_p}, \quad E_p \equiv \sqrt{\vec{p}^2 + m^2},$$

(2.8)

$$\lambda^\pm \equiv \frac{1}{2}(1 \pm \gamma_0)$$

and

$$\tilde{K} \equiv -\frac{4\pi\alpha}{(|\vec{p} - \vec{p}'|)^2} m \frac{2E_p}{E_p + m} \frac{2E_{p'}}{E_{p'} + m} \frac{2}{\sqrt{P_0 + 2E_p}} \frac{2}{\sqrt{P_0 + 2E_{p'}},}$$

(2.9)

$m$ denotes the mass of the (heavy) quark. The solutions of the equations obtained in this manner are exact. The (colour singlet, normalized \cite{20}) eigenfunctions read ($\omega := E_p - \frac{P_0}{2}$)
\[
\chi(p) = -i \frac{\Lambda^+ \lambda^+ \Gamma \lambda^- \Lambda^-}{(p_0 - \omega + i\epsilon)(p_0 + \omega - i\epsilon)} \frac{E_p \sqrt{2E_p + P_0(P_0 - 2E_p)}}{\sqrt{P_0(E_p + m)}} \phi(\tilde{p})
\]

\[
\bar{\chi}(p) = i \frac{\gamma_0(\Lambda^+ \lambda^+ \Gamma \lambda^- \Lambda^-)^* \gamma_0}{(p_0 - \omega + i\epsilon)(p_0 + \omega - i\epsilon)} \frac{E_p \sqrt{2E_p + P_0(P_0 - 2E_p)}}{\sqrt{P_0(E_p + m)}} \phi^*(\tilde{p})
\]

and belong to the spectrum of bound states

\[
P_0 = M^0_n = 2m \sqrt{1 - \frac{\alpha^2}{4n^2}} \approx 2m - m \frac{\alpha^2}{4n^2} - m \frac{\alpha^4}{64n^4} + O(\alpha^6).
\]

In eqs. (2.10) \( \Gamma \) is a constant \( 4 \times 4 \) matrix which represents the spin state of the particle-antiparticle system:

\[
\Gamma = \begin{cases} 
\gamma_5 \lambda^- : S=0 \\
\bar{a}_m \gamma \lambda^- : S=1.
\end{cases}
\]

\( \phi \) is simply the normalized solution of the Schrödinger equation in momentum space, depending on the usual quantum numbers \( (n, l, m) \) \[21\], \( a_\pm, a_0 \) describe the triplet states.

In the following it will often be sufficient to use the nonrelativistic approximations of eqs. (2.10)

\[
\chi(p)^{nr} = \frac{\sqrt{2i\omega}}{p_0^2 - \omega^2 + i\epsilon} \phi(\tilde{p}) \Gamma,
\]

\[
\bar{\chi}(p)^{nr} = \frac{\sqrt{2i\omega}}{p_0^2 - \omega^2 + i\epsilon} \phi^*(\tilde{p})(-\gamma_0 \Gamma^* \gamma_0).
\]

Perturbation theory for the BS equation starts from an exactly solvable equation, in our case the BR equation for the Green function \( G_0 \) of the scattering of two fermions \[22\]

\[
iG_0 = -D_0 + D_0 K_0 G_0.
\]

\( D_0 \) is the product of two zero order propagators, \( K_0 \) the corresponding kernel. The exact Green function may be represented as

\[
G = \sum_l \frac{\chi^{BS}_{nl}}{P_0 - M_n} \tilde{\chi}^{BS}_{nl} + G_{reg} = G_0 \sum_{\nu=0}^{\infty} \nu (HG_0)^\nu,
\]
where the corrections are contained in the insertions $H$. Bound state poles $M_n$ contribute, of course, only for $P_0 < 2m$. It is easy to show how $H$ can be expressed by the full kernel $K$ and the full propagators $D$:

\[ H = -K + K_0 + iD^{-1} - iD_0^{-1}. \]  

(2.17)

Since the corrections to the external propagators contribute only to $O(\alpha^5)$ [23], the perturbation kernel is essentially the negative difference of the exact BS-kernel and of the zero order approximation.

Expanding both sides of equation (2.16) in powers of $P_0 - M_n^0$, with $M_n^0$ from (2.11) the mass shift is obtained [2]:

\[ \Delta M = \langle h_0 \rangle (1 + \langle h_1 \rangle) + \langle h_0 g_1 h_0 \rangle + O(h^3). \]  

(2.18)

Here the BS-expectation values are defined as e.g.

\[ \langle h \rangle \equiv \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4p'}{(2\pi)^4} \bar{\chi}_{ij}(p)h_{ii'}(p, p')\chi_{jj'}(p'), \]  

(2.19)

We emphasize the four-dimensional $p$-integrations which correspond to the generic case, rather than the usual three dimensional ones in a completely nonrelativistic expansion.

In (2.18) $h_i$ and $g_i$ represent the expansion coefficients of $H$ and $G_0$, respectively, i.e.

\[ H = \sum_{n=0}^{\infty} h_n (P_0 - M_n^0)^n \]  

(2.20)

\[ G_0 = \sum_{n=0}^{\infty} g_n (P_0 - M_n^0)^{n-1} \]  

(2.21)

3 QCD Tree Diagrams

The contributions stemming from the tree diagrams 2.a to 2.c are well-known from the abelian case. Fig. 2.a is peculiar for the use of a different zero order equation than the Schrödinger equation. It contains the difference between the exact one Coulomb-gluon exchange and the BR-Kernel (2.7). The exchange of one transverse gluon is represented by graph 2.b, and fig. 2.c shows the annihilation graph. The latter does not contribute in our nonabelian case.

For the sake of completeness and in order to illustrate the present formalism, we exhibit first the results for the tree graphs as well. The perturbation kernel for the Coulomb-gluon
exchange
$$-iH_c := -ig^2 T^\alpha_{\alpha'} T^\beta_{\beta'} \gamma^0_{\sigma\sigma'} \gamma^0_{\rho\rho'} \frac{1}{(\vec{p} - \vec{p}')^2}, \quad (3.22)$$
is needed for the calculation of the energy shift induced by fig. 2.a using Eqs. (2.18), (2.17), (2.19) and (2.7) to (2.9). For the spin-singlet we have:

$$\Delta M_{1,a} = \langle H_c + K_{BR} \rangle = \frac{\alpha^2}{16n^2} \langle K_\zeta \rangle + \frac{\pi \alpha}{P_0 m} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} [\phi^* \phi - 2\phi^* \frac{\vec{p}\vec{p}'}{(\vec{p} - \vec{p}')^2} \phi] + O(\alpha^6) = m\alpha^4 (\delta_{00} + \frac{q}{16n^4} - \frac{1}{16n^3(l + 1/2)}) + O(\alpha^6). \quad (3.23)$$

The contribution from the transverse gluon (fig. 1.b)

$$-iH_{1,b} = i4\pi \alpha \frac{\delta_{j\sigma}}{q^2} \frac{1}{q^2} (g_{ik} + q_j q_k), \quad (3.24)$$

with

$$q \equiv \rho' - p, \quad (3.25)$$
gives rise to a spin singlet-triplet (magnetic hyperfine) splitting. Because of the $\gamma^j$ matrices, the $\lambda^\pm$ projectors from both wave functions annihilate (3.24). This means that two factors $\vec{p} \vec{q}$, contained in $\Lambda^\pm$, are needed for a nonzero result which in turn gives rise to two extra orders of $\alpha$. By this mechanism we arrive at the well known contribution $O(\alpha^4)$ from this graph. For the spin-singlet the mass shift reads

$$\Delta M_{1,b,s=0} = \frac{2\pi \alpha}{m^2} \left[-|\Psi(0)|^2 + 2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \phi^* (\vec{p}) (\vec{p}') (\frac{\vec{p}\vec{q}}{q^2} - \frac{\vec{p}'\vec{q}'}{q'^2}) \phi (\vec{p}) + O(\alpha^6) \right] = m\alpha^4 (\frac{1}{8n^4} - \frac{3}{16n^3(l + 1/2)}) + O(\alpha^6). \quad (3.26)$$

The evaluation of the singlet-triplet splitting requires some awkward Dirac-algebra, but the final result may be brought in a quite transparent form (where one recognizes this expression as the well known spin-spin and spin-orbit interaction, adapted to the present problem, cf. e.g. [25])

$$\Delta M_{ortho,m} = \Delta M_{para} = \frac{2\pi \alpha}{m^2} \int d^3p \int d^3p' \phi^* (\vec{p}) (1 + \frac{|q\vec{a}_m|^2}{q^2}) - 3 (\vec{p}' \times \vec{p}) (\vec{a}_m \times \vec{a}_m) \phi (\vec{p}) = \frac{2\pi \alpha}{m^2} \left( -\frac{\delta(\vec{r})}{3\delta(\vec{r})} + \frac{1}{4\pi} \frac{\vec{r}^2 - 3|\vec{r}\vec{a}_m|^2}{|\vec{r}|^5} - \frac{3i(\vec{r} \times \vec{p}) (\vec{a}_m \times \vec{a}_m)}{4\pi |\vec{r}|^3} \right). \quad (3.27)$$
For our purpose it is sufficient to calculate the singlet-triplet splitting for the ground state given by the term proportional to $\delta(\vec{r})$:
\[
\Delta M_{ortho} - \Delta M_{para} = \frac{8\pi\alpha}{3m^2} |\Psi(0)|^2 = m\alpha^4 \frac{\delta l_0}{3n^3}.
\] (3.28)

It should be mentioned that this result differs from the positronium case, because the graph 2.c gives no contribution here. Moreover, we want to emphasize again that according to our 'purist' field theoretic point of view, integrals like (3.27) and (3.28), of course, are not to be evaluated between phenomenological wave-functions.

4 One Loop Corrections

4.1 One Loop Vacuum Polarization

In the case of positronium no massless particles can contribute to vacuum polarization and so this effect is only of order of magnitude $\alpha^5$. In contrast to this, QCD contains massless gluons and light quark flavours which may contribute significantly to the spectrum.

Using standard techniques for evaluating the corresponding integrals one obtains for the loop in fig. 3.a ($N=3$ for QCD):
\[
\begin{align*}
\pi_{3,a}^{ab} &\equiv 4g^2 \delta^{ab} \int \frac{d^{D} r \ q^2 - \frac{(\vec{q}\vec{r})^2}{r^2}}{(2\pi)^D r^2(\vec{q} - \vec{r})^2} \\
&= -\frac{ig^2 N\delta^{ab}}{3\pi^2} q^2 \left[ \mathcal{D} - \ln q^2 + \frac{7}{3} - 2 \ln 2 - \epsilon \left( \frac{7}{3} - 2 \ln 2 + \ln 4\pi \right) \ln q^2 + \frac{\epsilon}{2} \ln^2 q^2 + \epsilon \cdot const + O(\epsilon^2) \right]
\end{align*}
\] (4.29)

with
\[
\mathcal{D} = \frac{1}{\epsilon} - \gamma + \ln 4\pi, \quad \epsilon = \frac{4 - D}{2}
\] (4.30)

We have written down this result correct to orders $\epsilon \cdot f(q)$ in order to make it applicable in the two loop calculation. The graph 3.b is more difficult to calculate, because it contains $q_0$ terms. These terms can be completely avoided if we carry out the $p_0$ integrations first (cf. (2.19) and (2.18)). The result can be expanded in powers of $\vec{p}$ and $\alpha$ in order to show that the effect of $q_0$ is of $O(\alpha^5)$. With this simplification graph 3.b is exactly calculable:
\[
\pi_{3,b}^{ab} = \frac{ig^2 N\delta^{ab}}{96\pi^2} \left[ 10q^2 (\mathcal{D} - \ln q^2) + 16(7 - 8 \ln 2)q^2 \right] + O(\epsilon)
\] (4.31)
Our renormalization prescription consists of a subtraction at the point \( q = (0, \mu) \), where \( \mu \) has to be of the order \( \alpha m \) to avoid large logarithmic contributions from higher orders. This seems to be the natural renormalization prescription for bound state problems, because also in the BS expectation values \([2,19]\), the Bohr momentum \( \alpha m \) together with \( p_0 \approx O(\alpha^2 m) \) provides the dominant parts of the integrals.

After renormalization, the contribution from the gluonic vacuum polarization (with the colour trace already done) reads \([16]\)

\[
-iH_g = -i\gamma_0 \otimes \gamma_0 \tilde{H}_g \quad \text{(4.32)}
\]

\[
\tilde{H}_g = -\frac{11\alpha^2 N}{4\tilde{q}^2} \ln \frac{\tilde{q}^2}{\mu^2}.
\]

The expectation value of this expression can be obtained by performing the Fourier transformation into coordinate space, where the integrations can be done analytically (see Appendix A). Our surprisingly simple result is

\[
\Delta M_g = \langle H_g \rangle = -m\alpha^3 \frac{11N}{16\pi^2} \Psi_1(n + l + 1) + \gamma + \ln \frac{\mu n}{\alpha m} + O(\alpha^5) \quad \text{(4.33)}
\]

where \( \Psi_n \) is the \( n \)-th logarithmic derivation of the gamma function and \( \gamma \) denotes Euler’s constant. The closed form of Eq. (4.33) was not obtained in previous calculations.

Now we turn to the contribution from the fermion loops fig. 3.c. In the literature the lighter quarks are usually taken as massless (and ‘absorbed’ in the number of flavours appearing in the running coupling constant) or even ignored \([16, 3]\), but we will show that they do contribute within the order of interest and, furthermore, the explicit dependence on the masses of the lighter quarks is important. As pointed out already in the introduction, this is due to the fact that the top quark is expected to lie above 100GeV \([24]\) and therefore the bottom and charm quark can neither be taken as relatively massless nor as relatively super-heavy as compared to the natural mass scale \( \alpha m \).

The finite part of the self energy in fig. 3.c is a well known quantity \([25]\) for arbitrary masses \( m_j \) of the quark:

\[
\Pi_F = -i g^2 \delta^{ab} \frac{q^2}{4\pi^2} I(q^2, m_j^2). \quad \text{(4.34)}
\]

We approximate in the exact solution

\[
I(q^2, m_j^2) = \int_0^1 dx x(1 - x) \ln [x(1 - x)q^2 + m_j^2] = \frac{1}{6} \ln m_j^2 - \frac{5}{18} + f(\rho), \quad \text{(4.35)}
\]
\[ f(\rho) = \frac{2\rho}{3} + \frac{1}{6}(1 - 2\rho)\sqrt{1 + 4\rho \ln \frac{\sqrt{4\rho + 1} + 1}{\sqrt{4\rho + 1} - 1}}, \quad (4.36) \]

\[ \rho := \frac{m_j^2}{q^2}, \quad (4.37) \]

for later convenience \( f(\rho) \) by

\[ f(\rho) \approx \frac{1}{6} \ln \left( \frac{1}{\rho} + e^{\frac{5}{6}} \right). \quad (4.38) \]

This agrees with the original \( f(\rho) \) better than 1% within the whole integration region.

It seems instructive to transform into coordinate space in order to obtain the potential, effectively produced by this fermionic vacuum polarization:

\[ \Delta M_F = \langle H_F \rangle, \]

\[ H_F(r) = -\frac{\alpha^2}{4\pi r} \left[ \text{Ei}(-r m_j e^{\frac{5}{6}}) - \frac{5}{6} + \frac{1}{2} \ln \left( \frac{\mu^2}{m_j^2} + e^{\frac{5}{6}} \right) \right]. \quad (4.39) \]

The mass shift can be obtained from (4.39) in closed form using the integral formula \[26\]

\[ \int_0^x e^{-\beta x} \text{Ei}(-\alpha x) dx = -\frac{1}{\beta} \left[ e^{-\beta x} \text{Ei}(-\alpha x) + \ln(1 + \frac{\beta}{\alpha}) - \text{Ei}(-(\alpha + \beta)x) \right]. \quad (4.40) \]

Thus a useful expression for the energy shift induced by fermionic vacuum polarization with arbitrary masses \( m_j \) reads

\[ \Delta M_{F}^{l} = -\frac{m \alpha^3}{8\pi n^2} \left\{ \sum_{k=0}^{2n-2-l-2} b_{nl}^k \left[ \left( -\frac{d}{d\beta} \right)^{2l+1+k} \left[ -\frac{1}{\beta} \ln \left( 1 + \beta \frac{\alpha m}{nm_j e^{\frac{5}{6}}} \right) \right] \right] - \right. \]

\[ \left. -\frac{5}{6} + \frac{1}{2} \ln \left( \frac{\mu^2}{m_j^2} + e^{\frac{5}{6}} \right) \right\}, \quad (4.41) \]

with

\[ b_{nl}^k := \frac{(n - l - 1)!}{k!(n + l)!} \left( \frac{d}{d\rho} \right)^k \left[ I_{n-l-1}^{2l+1}(\rho) \right]_{\rho=0}. \quad (4.42) \]

For states up to \( n = 3 \) we write this result more explicitly as

\[ \Delta M_{F,nl} = \frac{m \alpha^3}{8\pi n^2} \left\{ A_{nl} \left( \frac{nm_j}{\alpha m} \right) + \ln \left( \frac{\mu^2}{m_j^2} + e^{\frac{5}{6}} \right)^{\frac{1}{2}} \right\}, \quad (4.43) \]

with \( A_{nl} \) from Tab.2,
\begin{align*}
\begin{array}{|c|c|}
\hline
n & l & A_{nl}(\frac{nm_j}{\alpha m}) \\
\hline
1 & 0 & a \\
2 & 0 & a^3 - \frac{1}{2}a^2 + a \\
2 & 1 & \frac{1}{2}a^3 + \frac{1}{2}a^2 + a \\
3 & 0 & 2a^5 - \frac{7}{2}a^4 + \frac{10}{3}a^3 - a^2 + a \\
3 & 1 & a^5 - \frac{3}{4}a^4 + \frac{1}{3}a^3 + \frac{1}{2}a^2 + a \\
3 & 2 & \frac{1}{5}a^5 + \frac{1}{4}a^4 + \frac{1}{3}a^3 + \frac{1}{2}a^2 + a \\
\hline
\end{array}
\end{align*}

Tab. 2

using the shorthand

\[ a^{-1} := 1 + \frac{nm_je^{\frac{\alpha}{6}}}{\alpha m} \]  \hspace{1cm} (4.44)

Only for \( m_j >> \alpha m \) this gives an \( O(\alpha^5) \) Uehling term, modified by off-shell subtraction, but for \( m_j \rightarrow 0 \) it becomes an \( O(\alpha^3) \) contribution, which means that eq. (4.43) interpolates numerically in a range of two orders in \( \alpha \). Therefore (4.43) must be definitely taken into account for quarks with \( m_j \approx \alpha m \) at \( O(\alpha^4) \).

### 5 Vertex Corrections

The one loop corrections to the vertex together with self-energy insertions into the fermion lines (fig. 3.d) in the abelian case (positronium), are known to provide corrections only of \( O(\alpha^5) \). The reason for this is the "classical" Ward identity which continues to relate those contributions in such a way that the sum of these terms vanishes at \( |\vec{q}| \rightarrow 0 \). This Ward identity happens to continue to hold even in the Coulomb gauge and even in the nonabelian case [3], but only for the vertex corrections referring to the Coulomb component of the gauge field. However, the presence of the gluon splitting graphs 3.e and 3.f produces a contribution already to \( O(\alpha^4) \) [16]. A simple dispersion theoretic argument allows to understand this difference: In the abelian case the first graph 3.d in the variable \( |\vec{q}| \) for \( q_0 = 0 \) has a cut for \( \text{Re}|\vec{q}| < 2m \). Thus corrections in \( \vec{q} \), e.g. in the electron form factor \( F_1 \), for symmetry reasons must be of order \( \vec{q}^2 \), because \( F_1(\vec{q}^2) \) is regular at \( \vec{q} \rightarrow 0 \). This is no longer the case with the mass zero intermediate state allowed in 3.e and 3.f. The first - and to our knowledge only- computation of the nonabelian vertex corrections in the sense of our present approach was performed in ref. [16]. This work contains certain approximations which we wanted to avoid in order to prepare the ground
for future calculations even at the level $O(\alpha^5)$. We thus make a systematic expansion and solve the remaining integrals analytically which contain contributions of the order of interest. The vertex correction of fig. 3.e after performing the colour trace becomes

$$-iH_{3,e} = \frac{36\pi^2 \alpha^2 \vec{q}_i}{\vec{q}^2} \int \frac{d^4r}{(2\pi)^4} \frac{1}{r^2 (\vec{r} - \vec{q})^2} \left( -\delta_{ki} + \frac{r_k r_i}{\vec{r}^2} \right) \times \left[ \gamma^k \gamma^0 - \gamma^0 \gamma^0 - \frac{1}{\gamma^0} \gamma^k \right]$$

$$= \frac{36\pi^2 \alpha^2 \vec{q}_i}{\vec{q}^2} (\gamma^k v_{ki}(1,p) \gamma^0 - \gamma^0 v_{ki}(-1,p') \gamma^k)$$  \hspace{1cm} (5.1)

where ($\epsilon = \pm 1$)

$$v_{ki}(\epsilon, p) := \int \frac{d^3r}{(2\pi)^3} \frac{1}{r^2 (\vec{r} - \vec{q})^2} (-\delta_{ki} + \frac{r_k r_i}{\vec{r}^2}) \frac{\gamma^0 p + \epsilon \gamma^0 m}{(p + \epsilon r)^2 - m^2}$$  \hspace{1cm} (5.2)

After the $r_0$ integration it proves useful to proceed with the $p_0$ integrations (cf. Eq. (2.19)), where in contrast to ref. [16], who approximates already at this point, we took into account also the pole arising from the denominator of Eq. (5.2). This results in

$$v_{ki}(\epsilon, p) = -i \int \frac{d^3r}{(2\pi)^3} \frac{1}{r^2 (\vec{r} - \vec{q})^2} (-\delta_{ki} + \frac{r_k r_i}{\vec{r}^2}) \frac{\gamma^0 p + \epsilon \gamma^0 m}{(p + \epsilon r)^2 - m^2} F(\vec{p}, \vec{r})$$  \hspace{1cm} (5.3)

$$F(\vec{p}, \vec{r}) = \frac{[(r + E_{p+\epsilon r}) (m - \vec{p} - \epsilon \vec{r}) + \gamma_0 E_{p+\epsilon r} \frac{p_\parallel}{r_\parallel} (r + E_{p+\epsilon r} + \omega) - \gamma_0 E_{p+\epsilon r} \frac{p_\parallel}{r_\parallel} \omega]}{2r E_{p+\epsilon r} (r + E_{p+\epsilon r}) [\frac{p_\parallel}{r_\parallel} (r + E_{p+\epsilon r} + \omega)]}$$  \hspace{1cm} (5.4)

The calculation of the $\gamma$ trace to lowest order requires the inclusion of the $\gamma^\parallel \vec{p}$-terms in the wave functions (2.11) stemming from the projection operators (2.8). After performing the $\gamma$ trace we expand in terms of $\vec{p}$ and $m\alpha^2$ which enables us to combine both terms in eq. (5.1):

$$H_{3,e} = \frac{36\pi^2 \alpha^2 \vec{q}_i}{\vec{q}^2} \int \frac{d^3r}{(2\pi)^3} \frac{\vec{q}^2 - (\vec{p}^\parallel)^2}{(\vec{r} - \vec{q})^2} \frac{1}{r^2 E_r} \text{ + higher orders}$$  \hspace{1cm} (5.5)

Now Eq. (5.3) may be evaluated exactly in terms of dilogarithms and the result has a cut for $|\vec{q}| < 0$, but no pole at $|\vec{q}| = 0$. It can be formally expanded for $|\vec{q}| > 0$ to $O(|\vec{q}|)$:

$$H_{3,e} = \frac{9\alpha^2}{|\vec{q}| m} (\frac{\pi^2}{8} - \frac{2|\vec{q}|}{3} + O(\vec{q}^2))$$  \hspace{1cm} (5.6)

The first term in this expansion has been obtained in [16], the second one is the expected contribution to $O(\alpha^5)$. The BS - expectation value of (5.6) becomes to $O(\alpha^4)$:

$$\Delta M_{3,e} = \frac{9m\alpha^4}{36n^2} \frac{1}{n(l + \frac{1}{2})}$$  \hspace{1cm} (5.7)
It remains to calculate the vertex corrections with two transverse gluons, graph 3.f. At first sight it seems that this graph would give a contribution to order $\alpha^3$ because no $\vec{p}\vec{\gamma}$ terms are needed from the wave functions. This, as we will see, is not true because the leading (constant) term vanishes as a consequence of renormalization and accordingly graph 3.f has been estimated to be of order $O(\alpha^5)$ in ref. [16]. Here we use an approach which explicitly provides at least part of the exact result of this contribution. The vertex part of the graph 3.f reads:

$$V_{3,f} = \frac{-Ng^3T^b}{4} \int \frac{d^D r}{(2\pi)^D} \gamma^i \gamma^k \left[ (p - r)^2 - m^2 \right] \delta^{ik} - \left( q_i - r_i \right) \left( q_k - r_k \right) \gamma^i \gamma^k \left( (p - r) - 2r^0 \right) \frac{(q_k - r_k)}{(q - r)^2} \left( \delta^{ik} - \frac{r_i r_k}{r^2} \right) \left( \delta^{ik} - \frac{r_i}{r^2} \right) \left( \delta^{ik} - \frac{r_i (q - r)}{r^2} \right) \left( \delta^{ik} - \frac{r_i}{r^2} \right)$$

(5.8)

With the gamma-trace to relative $O(\alpha^2)$ and using Feynman parametrization, the effective vertex part becomes

$$V_{3,f}^{\text{eff}} = \frac{-ig^3NT^c}{16} \int \frac{d^{D-1} k}{(2\pi)^{D-1}} \int_0^1 dx \int_0^{1-x} dy \left[ L - \frac{3x^2 m^2}{L} \right] \left( 1 + \frac{[k(q - r)]^2}{k^2(q - r)^2} \right)$$

(5.9)

where

$$L = (yq_0 - xp_0)^2 + k^2 + 2xp\vec{k} - 2yq\vec{k} - x(p^2 - m^2) - yq^2.$$

(5.10)

For our estimate it is sufficient to consider the part stemming from the constant term in the second bracket in Eq. (5.9). The $\vec{k}$ integration can be done easily leaving us with a finite part

$$V_{3,f,1}^{\text{eff}} = \frac{-ig^3NT^c}{32\pi^2} \int_0^1 dx \int_0^{1-x} dy \left[ \ln M + \frac{2x^2 m^2}{M} \right]$$

(5.11)

where

$$M = x^2 m^2 - w,$$

(5.12)

$$w = y(1 - y)q^2 + x(1 - x)(p^2 - m^2) + 2xypq.$$

(5.13)

This expression cannot be expanded in terms of $w$ because this would yield a spurious linear divergence in the next order from the $q^2$ term which would indicate an equally spurious $O(\alpha^4)$ contribution. Therefore, we expand in terms of $(w + y(1 - y)q^2)$ and solve
the leading part analytically in terms of dilogarithms. Expanding the result in terms of $\vec{q}^2$, one has

$$V_{\text{eff}}^{3,e} = \frac{ig^3 NT^c}{32\pi^2} \{2 - \frac{1}{4} \left[1 + \ln\left(\frac{\vec{q}^2}{m^2}\right)\right] \frac{\vec{q}^2}{m^2}\} + \text{other } O(\alpha^2) \quad (5.14)$$

Since (5.14) does not contain a term $\propto |\vec{q}|$, the vertex correction due to two transverse gluons does not contribute to $O(\alpha^4)$.

### 5.1 Other QCD graphs

To one loop order also the box graph 3.g occurs in the correction to the BR kernel. It possesses an exact counterpart in QED and is known to contribute only to $O(\alpha^5)$ \[27\]. Furthermore we have also investigated the two-loop vertex-correction depicted in fig. 3.h. Of course this correction is but one of several two loop vertex corrections. The renormalization must take into account the whole set of this graphs. Nevertheless, it seems that this graph, after proper renormalization, yields an $O(\alpha^5)$ contribution.

### 6 Other Corrections

#### 6.1 Two Loop Vacuum Polarization

As pointed out already in subsection 4.1, the usual renormalization group arguments relying on massless quarks in the running coupling constant do not consistently include the effect of 'realistic' quark masses in the toponium system, when a systematic BS perturbation is attempted. However, in the one loop case finite quark masses gave terms of numerical order $O(\alpha^4)$, therefore the same may be expected here, leading to corrections of $O(\alpha^5)$. On the other hand, in a full calculation of effects of $O(\alpha^4)$ two loops with gluons cannot be neglected. Although it is enough to consider the two loop vacuum polarization for Coulomb gluons only, the computation of all those graphs is beyond the scope of our present paper. We just want to indicate how already the graphs 4.a-4.c yield contributions of this order. The result of the self energy part from 4.a to 4.c are given in tab.3. Let us start with graph 4.a. Performing the zero component integrations of momenta results in (including a symmetry factor $1/2$)

$$-iH_{4,a} = -i\gamma_0 \otimes \gamma_0 g^6 N^2 2\vec{q}^4 \left(\Pi^{(1)} + \Pi^{(2)}\right) \quad (6.1)$$
with
\[
\Pi^{(1)} = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{\vec{p}^2|k||\vec{q} - \vec{k} - \vec{p}|}, 
\]
\[
\Pi^{(2)} = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{[(\vec{q} - \vec{k} - \vec{p})]^2}{\vec{p}^2|k|^3|q - k - p|^3}. 
\]

By using dimensional regularization, Feynman-parametrization and usual integration formulas \[28\] we arrive at
\[
\Pi^{(1)} = \frac{\Gamma(-\frac{1}{2} + \epsilon)\Gamma(1 - \epsilon)\Gamma(2 - 2\epsilon)}{\Gamma^{2}(\frac{1}{2})(4\pi)^{2\epsilon}\Gamma(2 - 2\epsilon)} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{\vec{p}^2[(\vec{q} - \vec{p})^2]^{\frac{1}{2} + \epsilon} = (6.4) 
\]
\[
= q^2 \frac{(4\pi)^{2\epsilon}(q^2)^{-2\epsilon}\Gamma(1 - \epsilon)\Gamma(-1 + 2\epsilon)\Gamma(\frac{1}{2} - \epsilon)}{\Gamma(\frac{5}{2} - 3\epsilon)} 
\]
and
\[
\Pi^{(2)} = \frac{3\Gamma(\frac{3}{2} - \epsilon)\Gamma(1 - \epsilon)\Gamma(2\epsilon)}{4(4\pi)^{3-2\epsilon}(\frac{3}{2} - 2\epsilon)(-1 + 2\epsilon)\Gamma(\frac{5}{2})\Gamma(\frac{1}{2} - 3\epsilon)(q^2)^{1-2\epsilon} + 
\]
\[
+ \frac{3\Gamma(\frac{1}{2} - \epsilon)\Gamma(2\epsilon)\Gamma(2 - 2\epsilon)(q^2)^{1-2\epsilon}}{2(4\pi)^{3-2\epsilon}\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} \times (6.5) 
\]
\[
\left\{ \frac{\Gamma(2\epsilon)\Gamma(\frac{7}{2} - \epsilon)\Gamma(2 - \epsilon)}{(-1 + 2\epsilon)\Gamma(\frac{3}{2} - \epsilon)\Gamma(\frac{9}{2} - 3\epsilon)} + \frac{\Gamma(2\epsilon)}{3 - 4\epsilon} \left[ \frac{\Gamma(1 - \epsilon) - 4\Gamma(2 - \epsilon)}{\Gamma(\frac{5}{2} - 3\epsilon)} - \frac{4\Gamma(3 - \epsilon)}{\Gamma(\frac{5}{2} - 3\epsilon)} \right] 
\right. 
\]
\[
\left. - 2\frac{\Gamma(2\epsilon)\Gamma(\frac{5}{2} - \epsilon)\Gamma(2 - \epsilon)}{\Gamma(\frac{9}{2} - 3\epsilon)} + \frac{\Gamma(2\epsilon)\Gamma(1 + 2\epsilon)}{\Gamma(\frac{9}{2} - 3\epsilon)} \right\}. 
\]

From eqs. (6.4) and (6.5) the finite, renormalized contribution to the perturbation kernel may be extracted as
\[
H_{4,a} = \gamma_0 \otimes \gamma_0 \frac{81\alpha^3}{8\pi q^2} \ln \frac{q^2}{\mu^2}. 
\]

Eq. (6.6) differs from eq. (1.32) by a simple factor proportional to \(\alpha\) and therefore results in the mass shift
\[
\Delta M_4 = \frac{81m\alpha^4}{64\pi^2n^2}(\Psi_1(n + l + 1) + \gamma + \ln \frac{\mu n}{\alpha m}). 
\]

The contribution from fig. 4.b is similar. After performing the integrations over the zero components we have
\[
\Pi_{4,b} = -\frac{i g^4 N^2}{2} \delta_{ab} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{(\vec{q} - \vec{k})^2|\vec{p}||k|((\vec{q} - \vec{p})^2 \times (6.8) 
\]
\[
\times \left[ \frac{\vec{q}^2}{\vec{k}^2} \frac{\vec{q}^2}{\vec{p}^2} \frac{\vec{q}^2}{\vec{p}^2} + \frac{\vec{q}^2}{\vec{p}^2} \frac{\vec{q}^2}{\vec{p}^2} \right]. 
\]
This expression can be written in terms of integrals already solved in the course of the one loop calculation, and the outcome is

\[
\Pi_{4.b} = -\frac{ig^4\vec{q}^2N^2\delta_{ab}}{24\pi^4}[D^2 + 2D(\frac{7}{3} - 2\ln 2) - 2D\ln \vec{q}^2 + 2(\gamma - \ln 4\pi - \frac{14}{3} + 4\ln 2) \ln \vec{q}^2 + 2\ln 2\vec{q}^2 + \text{const} + O(\epsilon)].
\]

Eq. (6.9) contains overlapping divergences and two graphs like 3.a with one or the other vertex replaced by a counterterm have to be added. After that, only an additive infinity survives which is subtracted by our usual renormalization prescription. Graph 4.b has a net contribution which is proportional to \(\ln^2\):

\[
\Pi^{\text{ren}}_{4.b} = -\frac{iN^2g^4\vec{q}^2}{24\pi^4} \ln \frac{\vec{q}^2}{\mu^2}.
\]

The last two loop correction we are considering is the graph in fig. 4.c, whose contribution to the Coulomb gluon propagator can be written as
\[ \Pi_{4,c} = -6ig^4N^2 \int \frac{d^D r}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} (q - k) \cdot q^l \frac{1}{(q - \vec{k})^2(q - \vec{k} - \vec{r})^2r^2k^2} \times \]
\[ \times \left( \delta^r l - \frac{k^r k^l}{k^2} - \frac{r^r r^l}{r^2} + \frac{r^r (\vec{r}\vec{k}) k^l}{r^2 k^2} \right). \] (6.11)

Eq. (6.11) can be evaluated entirely in terms of gamma functions by a somewhat lengthy calculation, but following the same steps as above. The analytic result is
\[ \Pi_{4,c} = -ig^4 \frac{3N^2}{2(8\pi^2)^2} (\vec{q}^2)^{1-2\epsilon} (4\pi)^2 \frac{\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(\frac{5}{2}-2\epsilon)\Gamma(\frac{5}{2}-3\epsilon)\Gamma(1+\epsilon)(1-2\epsilon)(1-4\epsilon)}. \] (6.12)

Expanding in terms of \( \epsilon \) and properly renormalizing the result we finally obtain
\[ \Pi_{4,c} = \frac{i g^4 N^2}{48\pi^4} \left( -\frac{4}{3} \ln \frac{\vec{q}^2}{\mu^2} + \ln^2 \frac{\vec{q}^2}{\mu^2} \right) \] (6.13)

In table 3 we collect the results for the self energy parts of fig. 4.a to 4.c, apart from a factor \(-i\frac{N^2 g^4}{24 \pi^4} \vec{q}^2\)

| graph | \( \Pi \) |
|-------|-------|
| 4.a   | \( \frac{3}{4} \ln \frac{\vec{q}^2}{\mu^2} \) |
| 4.b   | \( \ln^2 \frac{\vec{q}^2}{\mu^2} \) |
| 4.c   | \( \frac{2}{3} \ln \frac{\vec{q}^2}{\mu^2} - \frac{1}{2} \ln^2 \frac{\vec{q}^2}{\mu^2} \) |

Tab. 3

The full calculation of the gluon self energy to two loops seems to be very involved in the Coulomb gauge. This is the more so, if massive fermions are included. However, in view of the results from the one loop calculation with massive fermions we may expect that nonvanishing masses tend to decrease the importance of such terms in practice to something that would be de facto numerically equivalent to \( O(\alpha^5) \).

We try to circumvent these problems, for the time being, by the following argument, which also includes the three 'massless' quarks u,d,s. Because of the Ward identity for the Coulomb-vertex, it is clear from the theory of renormalization group that the same corrections can be obtained by expanding the running coupling constant with a two loop (gluons+u,d,s) input for the latter which provides also 'nonleading' logarithmic contributions. Our present calculation in any case illustrates on the one hand the procedure to
be followed in a systematic BS perturbation theory. On the other hand, we believe that especially the computation methods for the notoriously difficult Coulomb gauge may be useful elsewhere.

The beta function to two loops is renormalization scheme independent for massless quarks \[28\] and its two loop part has been calculated some time ago \[29\]:

\[
\beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \ldots \tag{6.14}
\]

\[
\beta_0 = \frac{1}{(4\pi)^2} (11 - \frac{2}{3} n_f) \tag{6.15}
\]

\[
\beta_1 = \frac{1}{(4\pi)^4} (102 - \frac{38}{3} n_f). \tag{6.16}
\]

Here \(n_f\) is the number of effective (massless) flavours and \(\beta(g)\) is the solution of

\[
\ln \sqrt{-q^2/\mu} = \int_g^{\bar{g}} \, \frac{d\bar{g}^4}{\beta(\bar{g})}, \tag{6.17}
\]

which reads up to two loops

\[
\ln \sqrt{-q^2/\mu} = \frac{1}{2\beta_0} \left[ \frac{1}{\bar{g}^2} - \frac{1}{g^2} + \frac{\beta_1}{\beta_0} \ln \frac{\bar{g}^2 (\beta_0 + \beta_1 g^2)}{g^2 (\beta_0 + \beta_1 g^2)} \right]. \tag{6.18}
\]

Considering this as an equation for \(\bar{g} = g(q^2)\) we 'undo' the renormalization group improvement by expanding with 'small' \(g^2 \propto \alpha\) (cf. eq. (2.6)):

\[
\alpha(q^2) = \alpha \left\{ 1 - \alpha \frac{33 - 2n_f}{16\pi} \ln \frac{\bar{q}^2}{\mu^2} + \right. \tag{6.19}
\]

\[
\left. + \frac{\alpha^2}{(16\pi)^2} \left[ (33 - 2n_f)^2 \ln^2 \frac{\bar{q}^2}{\mu^2} - 9(102 - \frac{38}{3} n_f) \ln \frac{\bar{q}^2}{\mu^2} \right] \right\}
\]

Clearly the one loop term agrees with the calculation in subsect. 4.1 in the limit \(m_j \to 0\). Nevertheless, this term should not be considered within the present argument because its error has the same order of magnitude as contributions of \(O(\alpha^4)\). For the computation of the rest we need the expectation value of \(\ln \frac{\bar{q}^2}{q^2}\). This integral can be done analytically (Appendix A) and the result is:

\[
\langle \ln \frac{\bar{q}^2}{q^2} \rangle = \frac{m \alpha}{2\pi n^2} \left\{ \frac{\pi^2}{12} + \Psi_2(n + l + 1) + s_{nl} + \gamma + \ln \frac{\mu n}{m} \right\} \tag{6.20}
\]

with

\[
s_{nl} = \frac{2(n - l - 1)!}{(n + l)!} \sum_{k=0}^{n-l-2} \frac{(2l + 1 + k)!}{k!(n - l - 1 - k)^2}.
\]
With eq. (6.20) we obtain for the mass shift, induced by the leading logs of the two loop vacuum polarization of the Coulomb gluon a contribution:

\[
\Delta M^{\text{2 loop}} = -\frac{m\alpha^4}{128\pi^2 n^2} \left\{ \frac{27}{12} + \Psi_2(n + l + 1) + s_{nl} + (\Psi_1(n + l + 1) + \gamma + \ln \frac{\mu n}{\alpha m})^2 \right\} + 288(\Psi_1(n + l + 1) + \gamma + \ln \frac{\mu n}{\alpha m}) \right\}.
\]

(6.21)

In this expression we have set the number of effective flavours equal to 3 as dictated by the number of sufficiently light quarks. Whether eq. (6.21) really represents the full two loop quark- gluon vacuum polarization, numerically consistent with other terms \(O(\alpha^4)\), must still be checked in a calculation of the Coulomb gluon’s self-energy to two loop order in the Coulomb gauge, i.e. going beyond the sample calculation here.

### 6.2 QCD 2-loop Box Graphs

It would be incorrect to extrapolate from the QED case the absence of corrections of \(O(\alpha^4)\), because gluon splitting allows new types of graphs. Our first example of a QCD box graph is fig. 5.a. Between the nonrelativistic projectors \(\lambda^\pm\) of the wave functions the perturbation kernel from this graph can be written effectively as

\[
-i H_{5.a} = -12 i g^6 \int \frac{d^4k}{(2\pi)^4} \frac{(k_0 + p_1^0 + m)(q^0 - t^0 + p_2^0 - m)}{[(p_1 - k)^2 - m^2][(p_2 - t)^2 - m^2][k^2(\vec{q} - \vec{k})^2(t - \vec{k})^2]} \times \frac{1}{(t - k)^2} \left( -(\vec{q} - \vec{k})\vec{k} + \frac{[\vec{t} - \vec{k})(\vec{q} - \vec{k})][\vec{k}(\vec{t} - \vec{k})]}{(t - k)^2} \right).
\]

(6.22)

After performing the integrations over the zero components \(t^0\) and \(k^0\) the external momenta can be set to \((m, \vec{0})\). This is justified a posteriori by the finiteness of the remaining terms. The resulting expression will thus only depend on \(\vec{q}\) and \(m\). The leading contribution seems to be

\[
H_{5.a} \propto \frac{\alpha^3}{|\vec{q}|^2},
\]

(6.23)

but a really reliable estimate or even a calculation of the coefficient of the leading term is not available yet. Supplementing the usual three powers of \(\alpha\) from the wave functions for the computation of energy levels, we see that the graph 5.a indeed gives a contribution \(O(\alpha^4)\). The qualitative result (6.23) had been noted already in [16], [8] and [3]. It should be noted, however, that also e.g. the graph 5.b may yield a contribution of the same structure.
A similar graph with crossed Coulomb lines (fig. 5.c) does not contribute because of the fact that the group theoretic factor vanishes.

Now we investigate the 'X' graph in fig. 5.d for possible new contributions. As in the calculation of fig. 5.a it can be simplified to give

\[ -iH_{5,d} = 3i g^6 \int \frac{d^4t}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{(k_0 + p_1^0 + m)(t_0 + p_2^0 - m)}{[(k + p_1)^2 - m^2][(t + p_2)^2 - m^2]k^2(q - \vec{k})^2} \times \]

\[ \times \frac{1}{t^2(q - t)^2} \left( 1 + \frac{[\vec{t}(\vec{q} - \vec{t})]^2}{\vec{t}^2(\vec{q} - \vec{t})^2} \right). \tag{6.24} \]

The integration over \( k \) yields a \( 1/|q| \) divergence if \( \vec{q} \) and \( \vec{p} \) tend to zero. Contributions within the order of interest can therefore only come from possible poles after the \( t \)-integration. For simplicity we consider the part of eq. (6.24) from the factor one in the second line:

\[ I_t = \int \frac{d^4t}{(2\pi)^4} \frac{t_0 + p_2^0 + m}{[(t + p_2)^2 - m^2]t^2(q - t)^2} = \]

\[ = \frac{-i}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{yq_0 - x p_2^0 + p_0^0 + m}{(yq - xp)^2 - yq^2 - x(p^2 - m^2)} \approx \]

\[ \approx \frac{-i}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{x}{x^2m^2 + y(1 - x - y)q^2} + O(\alpha) = \]

\[ = \frac{-i}{(4\pi)^2m} \ln \frac{|q|}{m} + O(\alpha). \tag{6.25} \]

Thus the part of graph 5.d, specified above, has a leading contribution of

\[ H_{5,d,1} = \frac{g^6N}{32(4\pi)^2|q|m} \ln \frac{|q|}{m} \]

as \( \vec{q} \to 0 \) and therefore contributes to \( O(\alpha^5 \ln \alpha) \). The second part of graph 5.c gives a similar contribution, only with a different numerical factor. We conclude that - in contrast to the QED case [27] - box graphs may contribute to \( O(\alpha^4) \). As illustrated by the explicit calculations above, to \( O(\alpha^5) \) beside abelian QED type corrections [23, 27], a host of further non-abelian contributions can be foreseen.

### 6.3 QED Correction

As a rule, one is not forced to consider also electromagnetic effects in QCD calculations, but at high energies the strong coupling decreases, and in the case of toponium we expect
\( \alpha_s^2 \) to be of the same order as \( \alpha_{QED} \). We can obtain this contribution by solving the BR equation for the sum of an QED and an QCD Coulomb gluon which results in the energy levels

\[
P_0 = M_n^0 = 2m \sqrt{1 - \frac{(\alpha + \alpha_{QED}Q)^2}{4n^2}} \approx 2m - \frac{\alpha^2}{4n^2} - \frac{m\alpha\alpha_{QED}Q^2}{2n^2} - m\frac{\alpha^4}{64n^4} - \frac{m^2\alpha_{QED}Q^4}{4n^2} + O(\alpha^6),
\]

where \( Q \) is the electric charge of the heavy quark, that means 2/3 for toponium. Clearly even the 'leading' third term can only be separated from the effect of the second one to the extent that \( \alpha(\mu) \) and \( \alpha_{QED}(\mu) \) can be studied separately with sufficient precision.

### 6.4 Weak Corrections

While also weak interactions can usually be neglected in QCD calculations, this is not true in the high energy region. This is because the weak coupling scales like \( \sqrt{G_F m^2} \) and this becomes comparable to the strong coupling if the fermion mass \( m \) is large. Even bound states through Higgs exchange may be conceivable [30]. Therefore we have to consider weak corrections and especially the exchange of a single Higgs or Z particle, assuming for simplicity the standard model with minimal Higgs sector.

The Higgs boson gives rise to the kernel

\[
-i H_{Higgs} = -i \sqrt{2G_F m^2} \frac{1}{q^2 - m_H^2} \approx i \sqrt{2G_F m^2} \frac{1}{q^2 + m_H^2},
\]

where the notation should be obvious.

Since we do not know the ratio \( \alpha m / m_H \) which would allow some approximations if that ratio is small, we calculate explicitly the level shifts by transforming into coordinate space. As in Appendix A, we express the Laguerre polynomials in terms of differentiations of the generating function, do the integration and perform the differentiation afterwards to obtain an expression valid for arbitrary levels and Higgs boson masses:

\[
\Delta M_{Higgs} = -m \frac{G_F m^2 \alpha}{4\sqrt{2\pi}} I_{nl}(\frac{\alpha m}{nm_H})
\]

with

\[
I_{nl}(a_n) \equiv \frac{a_n^{2l+2}}{n^2(1 + a_n)^{2n}} \sum_{k=0}^{n-l-1} \binom{n + l + k}{k} \binom{n - l - 1}{k} (a_n^2 - 1)^{n-l-1-k}
\]
As an illustration some explicit results for the lowest levels are given in tab.4.

| n | 1 | 2 | 2 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|
| l | 0 | 0 | 1 | 0 | 1 | 2 |
| $I_{nl}(a_n)$ | $\frac{a^2_1}{(1+a_1)^4}$ | $\frac{a^2_2(2+a_2^2)}{4(1+a_2)^4}$ | $\frac{a^4_2}{4(1+a_2)^4}$ | $\frac{a^2_2(3+6a_2^2+a_4^2)}{9(1+a_1)^4}$ | $\frac{a^4_2(4+a_2^2)}{9(1+a_3)^4}$ | $\frac{a^6_3}{9(1+a_3)^4}$ |

Tab. 4

It is evident that Eq. (6.28) will give a contribution of order $G_F m^2 \alpha^3$ if the Higgs mass is comparable to the mass of the heavy quark and should therefore be taken into account in a consistent treatment of heavy quarkonia spectra to numerical order $O(\alpha^4)$.

Next we consider the contributions of the neutral current, the single Z-exchange and Z-annihilation.

For the Z-exchange we obtain

$$H_Z = -4\sqrt{2} G_F m_Z^2 [\gamma^\mu (g_1 + g_2 \gamma_5)]_{\sigma\rho} \frac{g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{m_Z^2}}{q^2 - m_Z^2} [\gamma^\nu (g_1 + g_2 \gamma_5)]_{\rho\sigma}$$

(6.30)

with

$$g_1 = \frac{1}{2} T^f_3 - Q_f \sin^2 \Theta_w,$$

(6.31)

$$g_2 = -\frac{1}{2} T^f_3,$$

(6.32)

where $T^f_3$ is the eigenvalue of the diagonal SU(2) generator for the fermion $f$. If $f$ is the top quark then $T^f_3 = 1$.

Because $m_Z$ is expected to be larger than $\alpha m$, it is possible to write the leading term of the mass shift resulting from (6.30) in the form

$$\Delta M_Z = -4\sqrt{2} G_F m_Z^2 \left( \frac{g_2^2(3 - 2S^2)}{q^2 - m_Z^2} - g_1^2 \right) =$$

(6.33)

$$= m G_F m^2 \alpha^3 \frac{\sqrt{2\pi n^3}}{2} (g_2^2(3 - 2S^2) - g_1^2) \delta_{10}$$

where $S$ is total spin of the quark-antiquark system. Therefore this expression gives rise to a singlet triplet splitting within the order of interest.

In contrast to the gluon annihilation there is no colour structure involved in the Z-annihilation graph. Therfore this process does contribute. The corresponding energy shift

23
is easily evaluated:

\[
\Delta M_{S=0} = \frac{6G_F m^2 g_2^2 m \alpha^3}{n^3} \delta_{l0} \\
\Delta M_{S=1} = -g_1^2 \frac{m_Z^2}{g_2^2} \frac{m^2}{4m^2 - m_Z^2} \Delta M_{S=0}
\]

Therefore this contribution also yields a singlet triplet splitting.

### 6.5 Schwinger-Christ-Lee Terms

As mentioned in section (2), nonlocal interactions have to be added to the Lagrangian of QCD in Coulomb gauge. We are not aware of any previous attempt to look whether these terms give contributions to bound state problems. In fact, by explicit calculation we found though that the Schwinger-Christ-Lee terms \(v_1\) and \(v_2\) in (2.1) only give rise to corrections to the propagator of the transverse gluon, and therefore can actually be neglected to \(O(\alpha^4)\).

By analogy to the second ref. [18] we calculate the \(v_1\) term for SU(N) to \(O(g^4)\)

\[
v_1 = -g_1^4 \frac{N^2}{16} \int d^3r d^3r' d^3r'' A_i^c(\vec{r}') K_{ij}(\vec{r} - \vec{r}') K_{jk}(\vec{r} - \vec{r}'') A_k^c(\vec{r}'')
\]

\[
K_{ij}(\vec{r}) = \frac{1}{4\pi|\vec{r}|} \left[ \delta_{ij} - \frac{1}{4\pi|\vec{r}|^3} (3\rho_i \rho_j - \vec{r}^2 \delta_{ij}) \right].
\]

This corrects the gluon propagator by

\[
\delta G_{\mu\nu}^{ab}(x_1, x_2) = -\frac{1}{Z[0]} \frac{\delta^2}{\delta J_{\mu}(x_1) \delta J_{\nu}(x_2)} \frac{ig^4 N^2}{16} \int d^4x \int d^3r d^3r' d^3r'' \frac{\delta}{\delta J_i^c} K_{ij} K_{jk} \frac{\delta}{\delta J_k^c} Z_0[J].
\]

In momentum space \(\delta G\) can be calculated by using dimensional regularization to give

\[
\delta G_{mn}^{ab}(q, q') = (2\pi)^4 \delta^{ab} \delta(q - q') \frac{g^4 N^2 q^2}{8^9} \frac{1}{q^2 q'^2} \left( -\delta_{mn} + \frac{q_m q_n}{q^2} \right),
\]

which means that we have a mass shift with the same structure as the one transverse gluon exchange (cf. sect. (3)), only suppressed by two more orders in \(\alpha\).

Since the second term \(v_2\) also represents a correction to the propagator of the transverse gluon, it can be estimated by the same method to contribute only in higher orders of \(\alpha\) as well.
A Calculation of expectation values

In sect. 4 and 6 we needed the expectation values of logarithmic potentials between Schrödinger wave functions. They can be obtained by

\[
\langle \ln \frac{n^r}{r} \rangle = \left. \frac{d^n}{d\lambda^n} \langle r^{\lambda-1} \rangle \right|_{\lambda=0}, \quad (A.1)
\]

if the expectation value \( \langle r^{\lambda-1} \rangle \) is known analytically. This expression can be calculated by using the representation

\[
L_{n-l-1}^{2l+1}(\rho) = \lim_{z \to 0} \frac{1}{(n-l-1)!} \frac{d^{n-l-1}}{dz^{n-l-1}} (1-z)^{-2l-2} e^{\rho z^2} \quad (A.2)
\]

of the Laguerre polynomials. This allows an easy evaluation of the integrations and the remaining differentiations can be done with some care:

\[
\langle r^{\lambda-1} \rangle = \frac{(\alpha m)^{1-\lambda}}{2n^{2-\lambda}} \frac{(n-l-1)!}{(n+l)!} \Gamma(2l+2+\lambda) \sum_{k=0}^{n-l-1} \binom{n-l-1}{k} \frac{\lambda}{2} \frac{(-2l-2-\lambda)}{k} (-1)^k \quad (A.3)
\]

Using now the Fourier transformations \[31\]

\[
F[\ln \frac{\vec{q}^2}{\mu^2}] = -\frac{\gamma + \ln \mu r}{2\pi r} \quad (A.4)
\]

\[
F[\ln^2 \frac{\vec{q}^2}{\mu^2}] = \frac{1}{2\pi r} \left[ \frac{\pi^2}{6} + 2(\gamma + \ln \mu r)^2 \right] \quad (A.5)
\]

one arrives immediately at eq.(4.33) and (6.20), respectively.
References

[1] E.E. Salpeter, H.A. Bethe, Phys.Rev. 84 (1951) 1232.
    M. Gell-Mann, F.E. Low, Phys.Rev. 84 (1951) 350.

[2] G. Lepage, Phys. Rev. A 16 (1977) 863,
    W. Kummer, Nucl. Phys. B179 (1981) 365.

[3] F.L. Feinberg, Phys.Rev. D 17 (1978) 2659.

[4] R. Barbieri, E. Remiddi, Nucl.Phys. B141 (1978) 413
    W.E Caswell, G.P. Lepage, Phys.Rev.A 18 (1978).

[5] S. Love, Ann.Phys. (N.Y), 113 (1978) 153.

[6] For recent reviews see:
    J.H. Kühn, P.W. Zerwas, Phys.Rep. 167 (1988) 321,
    W. Lucha, F.F. Schöberl, D. Gromes, Phys.Rep. 209 (1991) 127

[7] P.J. Franzini, Phys.Lett B296 (1992) 199,195

[8] T. Appelquist, M. Dine,I.J. Muzinich, Phys.Lett. 69B (1977) 231;
    W. Fischler, Nucl.Phys. B129 (1977) 157;
    I.J. Muzinich and F.E. Paige, Phys.Rev. D21 (1980) 1151

[9] J. Pantaleone, S.-H. H. Tye, Y.J. Ng, Phys.Rev. D 33 (1986) 777 ;
    F. Halzen et. al. Phys.Rev. D47 (1993) 3013.

[10] W. Kummer, G. Wirthumer, Nucl. Phys. B185 (1981) 41

[11] R. Barbieri, G. Curci, E. d'Emilio, E. Remiddi, Nucl.Phys. B154 (1979) 535;
    R. Barbieri, M. Caffo, R.Gatto, E. Remiddi, Phys.Lett. B95 (1980) 93;
    K. Hagiwara, C.B. Kim, T. Yoshino, Nucl.Phys. B177 (1981) 461

[12] H. Leutwyler, Phys.Lett.B 98 (1981) 447

[13] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl.Phys. B147 (1978) 385, 448.

[14] ALEPH,DELPHI,L3, OPAL, Phys.Lett. B276 (1992) 247
[15] J.H. Kühn, 'The Toponium Scenario' Karlsruhe preprint TTP92-40; M. Jażabek, J.H. Kühn, 'The Top Width', Karlsruhe preprint TTP 93-4

[16] A. Duncan, Phys.Rev.D 13 (1976) 2866.

[17] G. Pancheri, J.-P. Revol, C. Rubbia, Phys.Lett. B277 (1992) 518

[18] J. Schwinger, Phys. Rev. 127, (1962) 324; N.H. Christ, T.D. Lee, Phys.Rev. 22 (1980) 939.

[19] G. Tiktopoulos, Journ. Math.Phys. 6 (1965) 573; W. Kummer, Nucl. Phys. B179 (1981) 365.

[20] S. Mandelstam, Proc.Roy.Soc A233 (1955) 248; D. Lurié, A.J. Macfarlane, Y. Takahashi, Phys.Rev. 140 (1965) B 1091.

[21] G. Wirthumer, PhD thesis, TU-Wien 1984

[22] W. Buchmüller, E. Remiddi, Nucl.Phys. B162 (1980) 250.

[23] J. Malenfant, Phys.Rev.D 35 (1987) 1525.

[24] A. Barbaro-Galteni, Nucl. Phys. B (Proc.Suppl.) 27 (1992) 187

[25] L.D. Landau, E.M. Lifschitz; 'Lehrbuch der Theoretischen Physik Bd.IV, Quantenelektrodynamik', Akademie Verlag Berlin 1986.

[26] I.S. Gradstein und I.M. Rhyshik; 'Summen, Produkt und Integraltafeln', Harri Deutsch, Thun 1981.

[27] T. Fulton, P. Martin, Phys.Rev. 95 (1954) 811.

[28] T. Muta, 'Foundations of Quantum Chromodynamics', World Scientific Lecture Notes in Physics, Vol.5 (1986).

[29] D.R.T. Jones, Nucl.Phys. B75 (1974) 531, W.E. Caswell, Phys.Rev.Lett. 33 (1974) 244.

[30] P. Jain et.al., Phys.Rev.D 46 (1992) 4029 and refereces therein.

[31] I.M. Gel’fand, G.E. Shilov; 'Generalized Functions' Vol.1, Academic Press Inc., New York 1964
Figure Captions

Fig.1: BS-equation for bound states

Fig.2: Tree graphs (broken lines represent Coulomb gluons, curly lines depict transverse gluons, wavy lines represent a general gluon and solid lines stand for fermions).

Fig.3: One loop graphs and vertex corrections

Fig.4: Two loop vacuum polarization

Fig.5: Two loop box graphs
\[ \chi(i, p_1, j, p_2) = \chi(i, p_1, j, p_2) \]

Fig. 1

\[ \sigma \quad \sigma' \quad \sigma \quad \sigma' \]
\[ \rho \quad \rho' \quad \rho \quad \rho' \]

2.a

Fig. 2

2.b

2.c
Fig. 3
Fig. 4
