Dynamics with unitary phase operator: implications for Wigner’s problem

Ramandeep S. Johal

Institut für Theoretische Physik,
Technische Universität Dresden,
01062 Dresden, Germany.

Ph.: + 49 (0351) 463 35582
Fax: + 49 (0351) 463 37299

Abstract

We show that for general deformations of $SU(2)$ algebra, the dynamics in terms of ladder operators is preserved. This is done for a system of precessing magnetic dipole in magnetic field, using the unitary phase operator which arises in the polar decomposition of $SU(2)$ operators. It is pointed out that there is a single phase operator dynamics underlying the dynamics of usual and deformed ladder operators.

PACS code(s): 03.65.Fd, 02.20.Uw, 42.50.D.

Keywords: Wigner’s Problem, deformed algebras, quantum dynamics, unitary phase operator.
1 Introduction

Wigner’s problem [1] usually formulated for the case of quantum harmonic oscillator states that the equations of motion do not determine a unique set of commutation relations for the observables. In classical mechanics also, it is known [2]-[4] that same dynamical equations may be obtained using alternative hamiltonians and definitions of Poisson brackets. Parastatistics is another example of such a nonuniqueness [5]. Recently, Wigner’s problem for a precessing magnetic dipole (with dynamical algebra $SU(2)$) was discussed [6] and a class of modified commutation relations were shown to be compatible with the same dynamical equations. In this paper, we point out that invariance of dynamics under general deformations of the $SU(2)$ algebra, can be understood in a unified manner as an underlying dynamics in terms of unitary phase operator, that arises in the polar decomposition of the ladder operators.

The polar decomposition procedure referred to above is the operator analogue of factorising a complex number into a real argument and an exponential phase. For an operator the factors should be a hermitian part and a unitary phase operator. The unitary phase operator ($e^{i\phi}$) in turn defines a hermitian phase operator $\phi$. For the purpose of $SU(2)$ algebra, the phase or angle operator is conjugate to angular momentum component, though the canonical conjugacy is modified when, as in this case, the operators are bounded [7, 8]. Moreover, dynamics in terms of unitary phase operator also helps to understand the concept of angular velocity in finite space quantum mechanics [9].

In section 2, we first review the dynamics and algebraic structure of the precessing magnetic dipole in the presence of magnetic field, in terms of generators of $SU(2)$ algebra. Then we describe the polar decomposition procedure for the ladder operators of this algebra and the dynamics is cast in terms of the unitary
phase operator. It is shown that dynamics for standard ladder operators can be derived from the dynamical equation for phase operator. In section 3, we consider general deformations of the $SU(2)$ algebra and show that dynamics for deformed ladder operators also follows from the same dynamical equation for phase operator. Other classes of deformations are discussed in Section 4. Section 5 presents some concluding remarks.

2 Precessing magnetic dipole

The hamiltonian for a magnetic dipole precessing in a magnetic field is given by $H = -\mu(\vec{J}.\vec{B})$. For simplicity, let us choose the magnetic field to be along the $z$-axis, so that

$$H = -\mu BJ_z,$$

where $J_z$ is the $z$ component of angular momentum operator $\vec{J}$. In terms of the ladder operators defined by $J_\pm = (J_x \pm iJ_y)/\sqrt{2}$, and $J_0 = J_z$, which are generators of $SU(2)$ algebra

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0,$$

the equations of motion are given as

$$\frac{dJ_\pm}{dt} = \mp i\mu BJ_\pm,$$

$$\frac{dJ_0}{dt} = 0.$$

We choose the set of basis states to be the standard angular momentum states $\{|j,-j\rangle, |j,-j+1\rangle, \ldots, |j,j-1\rangle, |j,j\rangle\}$, which define a $(2j+1)$-dimensional irreducible representation for $J_\pm, J_0$:

$$J_\pm = \sum_{m=-j}^{+j} \sqrt{(j+m)(j+1-m)}|j, m\pm 1\rangle\langle j, m|,$$

$$J_0 = \sum_{m=-j}^{+j} m|j m\rangle\langle j m|.$$
The casimir for this algebra is given by

\[ C \equiv \vec{J}^2 = J_+ J_- + J_0 (J_0 + 1) = J_+ J_- + J_0 (J_0 - 1). \] (7)

In the following we write the equations of motion using the unitary phase operator, which arises in the polar decomposition procedure \[10\] of ladder operators

\[ J_+ = \sqrt{J_+ J_-} e^{i\phi} \sqrt{J_- J_+}, \] (8)

\[ J_- = \sqrt{J_- J_+} e^{-i\phi} \sqrt{J_+ J_-}. \] (9)

The exponential phase operator \( e^{i\phi} \) is unitary \((e^{i\phi} e^{-i\phi} = e^{-i\phi} e^{i\phi} = 1)\) and is given as

\[ e^{i\phi} = \sum_{m=-j}^{j-1} |j, m+1\rangle \langle j, m| + e^{i(2j+1)\theta_0} |j, -j\rangle \langle j, j|. \] (10)

In other words, operator \( \phi \) is hermitian. Here \( \theta_0 \) is an arbitrary phase angle, which defines the domain of phase operator \( \phi \) to be \([\theta_0, \theta_0 + 2\pi)\).

We can write the equation of motion for \( e^{i\phi} \)

\[ \frac{d}{dt} e^{i\phi} = \frac{1}{i\hbar} [e^{i\phi}, H], \] (11)

using the following commutator \[11\]

\[ [e^{\pm i\phi}, J_0] = \pm \hbar \{ -e^{\pm i\phi} + (2j + 1)e^{\pm i(2j+1)\theta_0} | \pm (j)\langle \pm j| \}. \] (12)

Now to get equation of motion for \( J_+ \), we just multiply Eq. \[12\] with \( \sqrt{J_- J_+} \) on the left (or with \( \sqrt{J_- J_+} \) on the right) and use the fact that \( J_+ J_- (J_- J_+) \) commutes with \( J_0 \) and hence with the hamiltonian. Also we use \( J_+ |j, j\rangle = \langle j, j| J_- = 0 \).

Similarly, we can obtain Eq. \[13\] corresponding to \( J_- \) starting with equation of motion for \( e^{-i\phi} \) and using \( J_- |j, -j\rangle = \langle j, -j| J_+ = 0 \).
3 Dynamics with deformed ladder operators

In this section, we show how starting from the equation of motion for $e^{\pm i\phi}$, we can preserve the linear dynamics in terms of deformed ladder operators. Specifically, we consider general deformations of SU(2) algebra \[1\]

\[
[\tilde{J}_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm, \quad [\tilde{J}_+, \tilde{J}_-] = f(\tilde{J}_0). \tag{13}
\]

$f(z)$ is a real parameter-dependent analytic function of its argument, holomorphic in the neighbourhood of zero and goes to $2z$ for certain limiting value of the parameter. Also define a function $g$ through

\[
f(\tilde{J}_0) = g(\tilde{J}_0) - g(\tilde{J}_0 - 1). \tag{14}
\]

The function $g(\tilde{J}_0)$ is not unique and is determined up to any periodic function of unit period. The casimir for this algebra is $\tilde{C} = \tilde{J}_- \tilde{J}_+ + g(\tilde{J}_0) = \tilde{J}_+ \tilde{J}_- + g(\tilde{J}_0 - 1)$.

A generalized map which does not preserve hermitian conjugation between $\tilde{J}_+$ and $\tilde{J}_-$ may be given as

\[
\tilde{J}_+ = J_+ A(C, J_0), \quad \tilde{J}_- = B(C, J_0) J_-, \quad \tilde{J}_0 = J_0. \tag{15}
\]

It is easy to verify the first relation in Eq. (13). To satisfy the second relation, the following condition must hold

\[
A(J_0 - 1)B(J_0 - 1)(C - J_0(J_0 - 1)) - B(J_0)A(J_0)(C - J_0(J_0 + 1)) = f(J_0). \tag{16}
\]

Assuming that $A$ and $B$ commute, the above condition implies

\[
A(J_0 - 1)B(J_0 - 1)(C - J_0(J_0 - 1)) = -g(J_0 - 1) + p(J_0), \tag{17}
\]

where $p(J_0)$ is some periodic function of period unity. Note that this only fixes the product $A(J_0)B(J_0)$. Different choices of these functions as well as the function $p$, produce a variety of realizations for the deformed algebra.
Now we observe that

\[ \tilde{J}_+ = J_+ A(C, J_0) = e^{i\phi} \sqrt{J_- J_+} A(C, J_0) \]
\[ \equiv e^{i\phi} G(C, J_0), \]

(18)

(19)

where Eqs. (8) and (7) have been used. Then, multiplying Eq. (11) on the right by \( G(C, J_0) \) and using the fact that \( G \) commutes with the Hamiltonian, we obtain the dynamical equation for \( \tilde{J}_+ \)

\[ \frac{d\tilde{J}_+}{dt} = -i\mu B \tilde{J}_+. \]

(20)

Similarly we can write

\[ \tilde{J}_- = B(C, J_0) J_- = B(C, J_0) \sqrt{J_- J_+} e^{-i\phi} \]
\[ \equiv K(C, J_0) e^{-i\phi}. \]

(21)

(22)

Again multiplying the equation of motion for \( e^{-i\phi} \) on the left by \( K(C, J_0) \) and using the fact that \( K \) commutes with the Hamiltonian, we obtain the dynamical equation for \( \tilde{J}_- \)

\[ \frac{d\tilde{J}_-}{dt} = i\mu B \tilde{J}_-. \]

(23)

Thus we see that equations of motion for \( \tilde{J}_\pm \) are identical in form to those for \( J_\pm \), Eqs. (3). This can also be proved without using the unitary phase operator, as was done in [6]. But the idea here is to point out that underlying the identical dynamics of the usual and deformed ladder operators, there is a single unitary phase operator dynamics. Note that it is not possible to obtain the dynamics of unitary phase operator by going in the opposite fashion, i.e. starting with the equation of motion for ladder operators and using the polar decomposition. This way the second term on the right hand side of Eq. (10) cannot be reproduced.
4 Alternate deformations

It is clear from above that the existence of a mapping function which commutes with the hamiltonian (for the present purpose the mapping function is a function of operators $C$ and $J_0$) ensures to preserve the dynamics in terms of operators $J_{\pm,0}$ and $\tilde{J}_{\pm,0}$. In this section, we study other $q$-deformations of the $SU(2)$ algebra which can be treated with the above scheme.

When $\tilde{J}_- = \tilde{J}_+^\dagger$ is imposed, we can express the deformed generators in terms of those of $SU(2)$ algebra as the following maps

$$\tilde{J}_+ = \sqrt{\frac{f(J_0 + j)f(J_0 - 1 - j)}{(J_0 + j)(J_0 - 1 - j)}} J_+, \quad \tilde{J}_- = \tilde{J}_+^\dagger, \quad \tilde{J}_0 = J_0. \quad (24)$$

The Drinfeld-Jimbo deformation of $SU(2)$ algebra, denoted by $SU_q(2)$ [12, 13], is a special case of the above deformed algebras, where $f(x) = (q^x - q^{-x})/(q - q^{-1}) \equiv [x]_q$ with $q$ as real. Its $(2j + 1)$-dimensional representation is given by

$$\tilde{J}_\pm = \sum_{m=-j}^{+j} \sqrt{[j + m]_q[j \pm m + 1]_q} j, m \pm 1\langle jm|, \quad (25)$$

$$\tilde{J}_0 = J_0 = \sum_{m=-j}^{+j} m|jm\rangle\langle jm|. \quad (26)$$

Note that action of $\tilde{J}_0$ is not deformed. The casimir of this algebra is $\tilde{C} = \tilde{J}_-\tilde{J}_+ + [\tilde{J}_0]_q[\tilde{J}_0 + 1]_q = \tilde{J}_+\tilde{J}_- + [\tilde{J}_0]_q[\tilde{J}_0 - 1]_q$. For $q$ as phase factor, the above representation suffers from problem of negative norm. To ensure positive norm, a modified representation may be taken, as considered in [14].

Similarly, deforming maps for well known quantum algebras corresponding to $SU(2)$ can be given [15], and they can be discussed within this framework. As another example, consider Witten’s second deformation generated by $\{W_\pm, W_0\}$ and given as follows [16]

$$[W_0, W_+]_r \equiv rW_0W_+ - \frac{1}{r}W_+W_0 = W_+, \quad (27)$$

$$[W_0, W_-]_r \equiv rW_0W_- - \frac{1}{r}W_-W_0 = W_-,$$
\[ [W_+, W_-]_r^{1/r^2} = W_0, \]  
(28)  
\[ [W_0, W_-]_r = W_- . \]  
(29)  

Here \( W_{\pm,0} \) is equivalent to \( \tilde{J}_{\pm,0} \). The deforming map in terms of generators of \( SU(2) \) is given by

\[
W_0 = \frac{1}{r-1/r} \left( 1 - \frac{J_0^2 + 1 + r^{-2}}{r + 1/r} r^{-2} \right),
\]  
(30)
\[
W_+ = r^{-J_0} \sqrt{\frac{2r}{r + 1/r}} \left[ J_0 + j \right]_{r} \left[ J_0 - 1 - j \right]_{r} J_+, \]  
(31)
\[
W_- = W_+^\dagger. \]  
(32)

Now we find that for \( H = -\mu BJ_0 \),

\[
\frac{dW_\pm}{dt} = -i\mu B [W_\pm, J_0] = \mp i\mu BW_\pm, \]  
(33)
and

\[
\frac{dW_0}{dt} = 0, \]  
(34)

which is identical in form with Eqs. (3) and (4).

Clearly, adopting the polar decomposition procedure for the ladder operators \( J_{\pm} \) of \( SU(2) \) algebra, the dynamical equations for deformed operators, identical in form to Eqs. (20) and (23), can be recovered from dynamics of the unitary phase operator.

Finally, we argue that realization of deformed ladder operators as proposed in [6] is a special case of the mapping in Eq. (15). Following [6], a non-linear deformation of ladder operators may be defined through an arbitrary function \( F(C, J_0) \) as

\[
\tilde{J}_+ = J_+ F, \tilde{J}_- = J_- F, \tilde{J}_0 = J_0 F. \]  
(35)

The transformation leads to the following deformed algebra

\[
[\tilde{J}_0, \tilde{J}_\pm] = \left\{ 1 - \frac{F(C, J_0 \mp 1)}{F(C, J_0)} \right\} \tilde{J}_0 \tilde{J}_\pm \pm F(C, J_0 \mp 1) \tilde{J}_\pm, \]  
(36)
\[ [\tilde{J}_+, \tilde{J}_-] = \left\{ 1 - \frac{F(C, J_0 + 1)}{F(C, J_0 - 1)} \right\} \tilde{J}_+ \tilde{J}_- + 2F(C, J_0 + 1)\tilde{J}_0. \]  

(37)

Now although the operator \( \tilde{J}_0 \) is modified as compared to \( J_0 \) in the above algebra, the hamiltonian is expressed in terms of the usual operator \( J_0 \). Thus the commutator \([\tilde{J}_0, \tilde{J}_\pm] \) does not play any role in the dynamics considered in [6]. In fact, if we choose not to deform \( J_0 \) as considered above in our approach, Eq. (36) just reduces to \([J_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm \). Secondly, the commutator in Eq. (37) follows if we choose \( A(C, J_0) = F(C, J_0) \) and \( B(C, J_0) = F(C, J_0 + 1) \), in Eqs. (15) and (16).

5 Concluding Remarks

We close with a few remarks on the analogous problem for the harmonic oscillator. It is known that a unitary phase operator does not exist for quantum harmonic oscillator, the system for which Wigner originally formulated his problem. A unitary phase operator was defined for the finite \((s+1)\)-dimensional harmonic oscillator as [17]

\[ e^{i\Phi} = \sum_{n=0}^{s} |n-1\rangle\langle n| + e^{i(s+1)\phi_0} |s\rangle\langle 0|, \]  

(38)

where \( \{|n\rangle \}_{0, 1, \ldots, s} \) are the eigenstates of the number operator \( N = \sum_{n=0}^{s} n|n\rangle\langle n| \).

Then with the oscillator Hamiltonian \( H = Nh\omega \), we can write the following equation of motion

\[ \frac{d}{dt} e^{i\Phi} = -i\omega \{ e^{i\Phi} - e^{i(s+1)\phi_0} (s+1)|s\rangle\langle 0| \}. \]  

(39)

Multiplying the above equation from right with \( \sqrt{N} \) and noting that annihilation operator \( a = e^{i\Phi}\sqrt{N} \), we get

\[ \frac{da}{dt} = -i\omega a. \]  

(40)
Similarly the equation of motion for creation operator $a^\dagger$ can be derived
\[
\frac{da^\dagger}{dt} = i\omega a^\dagger.
\] (41)

These results match with the corresponding equations valid for usual infinite dimensional harmonic oscillator. Now $q$-deformed oscillator with $q = e^{i2\pi/(s+1)}$ also admits finite dimensional realizations and the above hermitian or unitary phase operator can be used \[18\]. Particularly, from the point of view of $q$-deformed $SU(2)$ algebra, we can adopt the representation which is manifestly free of negative norm
\[
a_q = \sqrt{[1-n_0]_q + [n_0]_q} |0\rangle\langle 1| + \cdots + \sqrt{[1-n_0]_q + [n_0]_q} |s-1\rangle\langle s|, \quad (42)
\]
\[
a_q^\dagger = (a_q)^\dagger, \quad (43)
\]
\[
N' \equiv N - n_0 = -n_0|0\rangle\langle 0| + (1 - n_0)|1\rangle\langle 1| + \cdots + (s - n_0)|s\rangle\langle s|, \quad (44)
\]
and $n_0 = \frac{(s+1)}{4}, \quad [14]$. It is easy to see that deformed annihilation and creation operators $a_q$ and $a_q^\dagger$ also satisfy the identical linear relations, Eqs. (40) and (41) respectively, and they can be derived from the dynamics of unitary phase operator Eq. (39) and polar decomposition, $a_q = e^{i\Phi} \sqrt{[N - n_0]_q + [n_0]_q}$.

Now consider two such type of commuting $q$-deformed oscillators, \{\[^a_q, \, a_q^\dagger, \, N_1\}\} and \{\[^b_q, \, b_q^\dagger, \, N_2\}\}. They can be used to realize generators of $q$-deformed $SU(2)$ algebra \[14\], by the generalized Jordan-Schwinger mapping \[13\]
\[
\tilde{J}_+ = a_q^\dagger b_q, \quad \tilde{J}_- = b_q^\dagger a_q, \quad J_0 = (N_1 - N_2)/2. \quad (45)
\]

Then taking the hamiltonian of a two-dimensional harmonic oscillator $H = \hbar(N_1\omega_1 + N_2\omega_2)$, we again see that the dynamics for $\tilde{J}_{\pm,0}$ is identical to that of $J_{\pm,0}$ operators i.e. Eqs. (3) and (4), if we impose the condition $\omega_2 - \omega_1 = \mu B$. In this sense, under the mapping in Eq. (45), The dynamics of $\tilde{J}_{\pm,0}$ follows
from the dynamics of unitary phase operator for a finite dimensional $q$-deformed oscillator.

Concluding, the Wigner problem for a precessing magnetic dipole has been analyzed through the polar decomposition of ladder operators. We have argued that a dynamics in terms of unitary phase operator can be assigned to various types of deformations of ladder operators.

**Acknowledgements**

The author would like to acknowledge Alexander von Humboldt Foundation, Germany for grant of financial support.

**References**

[1] E. P. Wigner, Phys. Rev. 77 (1950) 711.

[2] G. Marmo, E. J. Saletan, A. Simoni and B. Vitale, *Dynamical Systems: A Differential Geometric Approach to Symmetry and Reduction* (Wiley, New York, 1985).

[3] S. Okubo, Phys. Rev. A 23 (1981) 2776.

[4] V. V. Dodonov, V. I. Man'ko and V. D. Skarzhinsky, Hadronic J. 4 (1981) 1734.

[5] H. S. Green, Phys. Rev. 90 (1953) 270.

[6] R. Lopez-Peña, V. I. Man'ko and G. Marmo, Phys. Rev. A 56 (1997) 1126.

[7] S. M. Barnett and D. T. Pegg, Phys. Rev. A 41 (1990) 3427.

[8] T. S. Santhanam, Phys. Lett. A 56 (1976) 345.
[9] R. S. Johal, Phys. Lett. A 263 (1999) 62.

[10] D. Ellinas, J. Math. Phys. 32 (1991) 135.

[11] A. P. Polychronakos, Mod. Phys. Lett. A 5 (1990) 2325; M. Rocek, Phys. Lett. B 255 (1991) 554; D. Bonatsos, C. Daskaloyannis and P. Kolokotronis, J. Phys. A: Math. Gen., 26 (1993) L871.

[12] M. Jimbo, Lett. Math. Phys. 11 (1986) 247

[13] L. C. Biedenharn, J. Phys. A 22 (1989) L873; A. Macfarlane, *ibid.* 4581.

[14] K. Fujikawa, H. Kubo and C.H. Oh, Mod. Phys. Lett. A 12 (1997) 403.

[15] T. L. Curtright and C. K. Zachos, Phys. Lett. B 243 (1990) 237.

[16] E. Witten, Nucl. Phys. B 330 (1990) 285.

[17] D. T. Pegg and S. M. Barnett, Phys. Rev. A 39 (1989) 1665.

[18] D. Ellinas, Phys. Rev. A 45 (1992) 3358.