On matrix realizations of the Lie superalgebra $D(2, 1; \alpha)$

Elena Poletaeva

Department of Mathematics,  
University of Texas-Pan American,  
Edinburg, TX 78539  
Electronic mail: elenap@utpa.edu

Abstract. We obtain a realization of the Lie superalgebra $D(2, 1; \alpha)$ in differential operators on the supercircle $S^{1|2}$ and in $4 \times 4$ matrices over a Weyl algebra. A contraction of $D(2, 1; \alpha)$ is isomorphic to the universal central extension $\hat{\mathfrak{psl}}(2|2)$ of $\mathfrak{psl}(2|2)$. We realize it in $4 \times 4$ matrices over the associative algebra of pseudodifferential operators on $S^1$. Correspondingly, there exists a three–parameter family of irreducible representations of $\hat{\mathfrak{psl}}(2|2)$ in a $(2|2)$–dimensional complex superspace.

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1. Introduction

Recall that $D(2, 1; \alpha)$, where $\alpha \in \mathbb{C}\setminus\{0, -1\}$, is a one-parameter family of classical simple exceptional Lie superalgebras of dimension 17 [1]. The bosonic part of $D(2, 1; \alpha)$ is $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, and the action of $D(2, 1; \alpha)_0$ on $D(2, 1; \alpha)_1$ is the product of two-dimensional representations. This family has a very interesting relation to the simple classical Lie superalgebra $\mathfrak{psl}(2|2)$, which is the only basic classical Lie superalgebra having a nontrivial universal central extension with three central elements [2]. Note that this central extension can be obtained as a contraction of $D(2, 1; \alpha)$ when $\alpha \to -1$. The Lie superalgebra $\mathfrak{psl}(2|2)$ and its central extensions play an important rôle in the AdS/CFT correspondence [3–6].

In [7] M. Scheunert gave an equivalent description of the superalgebra $D(2, 1; \alpha)$ as the family of superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_i$ are nonzero complex numbers.
such that $\sigma_1 + \sigma_2 + \sigma_3 = 0$. Note that $\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong D(2, 1; \alpha)$, where $\alpha = \sigma_1/\sigma_2$. We use the notation of [7].

In this work we consider the standard embedding of $\Gamma(2, -1 - \alpha, \alpha - 1)$, where $\alpha \in \mathbb{C}$, into the Poisson superalgebra of differential operators on the supercircle $S^{1|2}$ with two odd variables and into its deformation. This allows us to realize $\Gamma(2, -1 - \alpha, \alpha - 1)$ in $4 \times 4$ matrices over the Weyl algebra $\mathcal{W} = \sum_{i \geq 0} \mathbb{C}[t, t^{-1}]d^i$, where $d = \frac{\partial}{\partial t}$. This realization differs from the realization that we obtained in [8, 9], where we essentially used pseudodifferential operators on $S^{1|2}$.

Note that in [10, 11] we realized the superconformal algebras $\hat{K}'(4)$ and $\text{CK}_6$ in matrices over $\mathcal{W}$ of size 4 and 8, respectively. A superconformal algebra naturally has a $\mathbb{Z}$-grading (see [12]). The zeroth part of the $\mathbb{Z}$-grading of $\hat{K}'(4)$ is isomorphic to a central extension of $\mathfrak{psl}(2|2)$ with two central elements. The zeroth part of the $\mathbb{Z}$-grading of $\text{CK}_6$ is isomorphic to the universal central extension $\hat{P}(3)$ of the classical simple Lie superalgebra $P(3)$ (see [1] for notation). C. Martinez and E. I. Zelmanov obtained the analogous realization of $\hat{P}(3)$ in matrices of size 8 over $\mathcal{W}$ by a different approach using Jordan superalgebras; see [13].

In [14], C. Martinez, I. Shestakov and E. Zelmanov considered Cheng-Kac Jordan superalgebras $\text{CK}(Z, D)$, where $Z$ is an associative commutative algebra with a derivation $D : Z \rightarrow Z$. They obtained an embedding of $\text{CK}(Z, D)$ into the $4 \times 4$ matrices over the algebra $\mathcal{W}$ of differential operators on $Z$ with the $\mathbb{Z}/2\mathbb{Z}$-gradation. This embedding extends the King-McCrimmon embedding of the Kantor double of the vector type bracket $\{a, b\} = D(a)b - aD(b)$ into $2 \times 2$ matrices over $\mathcal{W}$; see [15, 16]. The Cheng-Kac Lie superalgebra introduced in [17] and denoted by $\text{CK}_6$ is isomorphic to the universal central extension $\hat{P}(3)$ of the Jordan superalgebra $\text{CK}(\mathbb{C}[t, t^{-1}], d/dt)$. In [14], the authors obtained, in particular, an embedding of the Cheng-Kac Lie superalgebra into a superalgebra of $8 \times 8$ matrices over the Weyl algebra $\mathcal{W}$.

The methods of [14] can be used to obtain an embedding of $\Gamma(\sigma_1, \sigma_2, \sigma_3)$. On the Jordan level, the problem is equivalent to an embedding of four-dimensional Jordan superalgebras $D(\alpha)$ into $2 \times 2$-matrices over $\mathcal{W}$. Let $R = \mathbb{C}[t]$ and $d = d/dt$. Consider the algebra $\hat{R} = R + Rw$, $w^2 = 1$ with the derivation $d$, and consider the vector type bracket $\{a, b\} = d(a)b - ad(b)$. Then the Kantor double $K(\hat{R}, d) = \hat{R} + \hat{R}w$ contains $\mathbb{C} + \mathbb{C}w + \mathbb{C}v + \mathbb{C}[(1 - \alpha)tw + (1 + \alpha)t]v \cong D(\alpha)$.

K. McCrimmon proved in [18] that a Kantor double of a vector type bracket can be embedded into $2 \times 2$ matrices over the corresponding Weyl algebra. Then there exists an embedding of $D(\alpha)$ into $2 \times 2$-matrices over $\mathcal{W} \oplus \mathcal{W}$, and also an embedding of $D(\alpha)$ into $2 \times 2$ matrices over $\mathcal{W}$, since $D(\alpha)$ is simple.

A contraction of $\Gamma(2, -1 - \alpha, \alpha - 1)$ when $\alpha \rightarrow \pm 1$ is isomorphic to the universal central extension $\mathfrak{psl}(2|2)$ of $\mathfrak{psl}(2|2)$. We realize it in $4 \times 4$ matrices over the associate algebra of pseudodifferential operators on $S^1$, which contains $\mathcal{W}$ as a subalgebra.
Correspondingly, we obtain a three-parameter family of irreducible representations of \( \hat{\mathfrak{psl}}(2|2) \) in \((2|2)\)-dimensional complex superspace.

2. Embedding of \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \) into the Poisson superalgebra on \( S^{1|2} \)

Recall the definition of \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \) [7]. Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra, where \( \mathfrak{g}_0 = sp(\psi_1) \oplus sp(\psi_2) \oplus sp(\psi_3) \) and \( \mathfrak{g}_1 = V_1 \otimes V_2 \otimes V_3 \), where \( V_i \) are two-dimensional vector spaces, and \( \psi_i \) is a non-degenerate skew-symmetric form on \( V_i, \ i = 1, 2, 3 \). A representation of \( \mathfrak{g}_0 \) on \( \mathfrak{g}_1 \) is the tensor product of the standard representations of \( sp(\psi_i) \) in \( V_i \). Consider the \( sp(\psi_i) \)-invariant bilinear mapping

\[
P_i : V_i \times V_i \rightarrow sp(\psi_i), \quad i = 1, 2, 3, \tag{1}
\]

given by

\[
P_i(x_i, y_i)z_i = \psi_i(y_i, z_i)x_i - \psi_i(z_i, x_i)y_i \tag{2}
\]

for all \( x_i, y_i, z_i \in V_i \). Let \( \mathcal{P} \) be a mapping

\[
\mathcal{P} : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_0 \tag{3}
\]

given by

\[
\mathcal{P}(x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3) = \\
\sigma_1 \psi_1(x_2, y_1)\psi_3(x_3, y_3)\mathcal{P}_1(x_1, y_1) + \\
\sigma_2 \psi_1(x_1, y_1)\psi_3(x_3, y_3)\mathcal{P}_2(x_2, y_2) + \\
\sigma_3 \psi_1(x_1, y_1)\psi_2(x_2, y_2)\mathcal{P}_3(x_3, y_3) \tag{4}
\]

for all \( x_i, y_i \in V_i, i = 1, 2, 3 \), where \( \sigma_1, \sigma_2, \sigma_3 \) are some complex numbers. The super-Jacobi identity is satisfied if and only if \( \sigma_1 + \sigma_2 + \sigma_3 = 0 \). In this case \( \mathfrak{g} \) is denoted by \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \). Superalgebras \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \) and \( \Gamma(\sigma'_1, \sigma'_2, \sigma'_3) \) are isomorphic if and only if there exists a nonzero element \( k \in \mathbb{C} \) and a permutation \( \pi \) of the set \( \{1, 2, 3\} \) such that

\[
\sigma'_i = k \cdot \sigma_{\pi i}, \text{ for } i = 1, 2, 3. \tag{5}
\]

Superalgebras \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \) are simple if and only if \( \sigma_1, \sigma_2, \sigma_3 \) are all different from zero. Note that \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \cong D(2, 1; \alpha) \) (see [1]) where \( \alpha = \sigma_1/\sigma_2 \).

The Poisson algebra \( P \) of pseudodifferential operators on the circle is formed by the formal series

\[
A(t, \tau) = \sum_{-\infty}^{n} a_i(t)\tau^i, \tag{6}
\]
where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the even variable $\tau$ corresponds to $\partial_t$, see [19]. The Poisson bracket is defined as follows:

$$\{A(t, \tau), B(t, \tau)\} = \partial_\tau A(t, \tau) \partial_t B(t, \tau) - \partial_t A(t, \tau) \partial_\tau B(t, \tau).$$

(7)

An associative algebra $P_h$, where $h \in (0, 1]$, is a deformation of $P$, see [20]. The multiplication in $P_h$ is given as follows:

$$A(t, \tau) \circ_h B(t, \tau) = \sum_{n \geq 0} \frac{h^n}{n!} \partial^n_\tau A(t, \tau) \partial^n_t B(t, \tau).$$

(8)

The Lie algebra structure on the vector space $P_h$ is given by

$$[A, B]_h = \frac{1}{h} (A \circ_h B - B \circ_h A),$$

(9)

and so

$$\lim_{h \to 0} [A, B]_h = \{A, B\}. \quad (10)$$

Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$ with the parity $p(\xi_i) = p(\eta_i) = \bar{1}$. The Poisson superalgebra of pseudodifferential operators on $S^{1|N}$ is $P(2N) = P \otimes \Lambda(2N)$. The Poisson bracket is defined as follows:

$$\{A, B\} = \partial_\tau A \partial_t B - \partial_t A \partial_\tau B + (-1)^{p(A)+1} \sum_{i=1}^N (\partial_{\xi_i} A \partial_{\eta_i} B + \partial_{\eta_i} A \partial_{\xi_i} B).$$

(11)

Let $\Lambda_h(2N)$ be an associative superalgebra with generators $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$ and relations

$$\xi_i \xi_j = -\xi_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \xi_j = h \delta_{i,j} - \xi_j \eta_i.$$

(12)

Let $P_h(2N) = P_h \otimes \Lambda_h(2N)$ be a superalgebra with the product given by

$$(A_1 \otimes X)(B_1 \otimes Y) = (A_1 \circ_h B_1) \otimes (XY),$$

(13)

where $A_1, B_1 \in P_h$ and $X, Y \in \Lambda_h(2N)$. The Lie bracket of $A = A_1 \otimes X$ and $B = B_1 \otimes Y$ is

$$[A, B]_h = \frac{1}{h} (AB - (-1)^{p(X)p(Y)} BA),$$

(14)

and (10) holds. $P_h(2N)$ is called the Lie superalgebra of pseudodifferential operators on $S^{1|N}$; see [10, 11]. We also consider the subalgebras of differential operators $P^+ \subset P$ and $P^+_h \subset P_h$ formed by $\sum_{i \geq 0} a_i(t) \tau^i$, and their superanalogues $P^+(2N)$ and $P^+_h(2N)$.
Proposition 2.1. For each $\alpha \in \mathbb{C}$ there exists an embedding

$$\rho_\alpha : \Gamma(2, -1 - \alpha, \alpha - 1) \subset P^+(4).$$

(15)

$\Gamma_\alpha = \rho_\alpha(\Gamma(2, -1 - \alpha, \alpha - 1))$ is spanned by the following elements:

$$E^1_\alpha = t^2, \quad F^1_\alpha = \tau^2 - 2\alpha t^{-2}\xi_1\xi_2\eta_1\eta_2, \quad H^1_\alpha = t\tau,$$
$$E^2_\alpha = \xi_1\xi_2, \quad F^2_\alpha = \eta_1\eta_2, \quad H^2_\alpha = \xi_1\eta_1 + \xi_2\eta_2,$$
$$E^3_\alpha = \xi_1\eta_2, \quad F^3_\alpha = \xi_2\eta_1, \quad H^3_\alpha = \xi_1\eta_1 - \xi_2\eta_2,$$
$$T^1_\alpha = t\eta_1, \quad T^2_\alpha = t\eta_2, \quad T^3_\alpha = t\xi_1, \quad T^4_\alpha = t\xi_2,$$
$$D^1_\alpha = \tau\xi_1 + \alpha t^{-1}\xi_1\xi_2\eta_1\eta_2, \quad D^2_\alpha = \tau\xi_2 - \alpha t^{-1}\xi_1\xi_2\eta_1, \quad D^3_\alpha = \tau\eta_1 + \alpha t^{-1}\xi_2\eta_1\eta_2, \quad D^4_\alpha = \tau\eta_2 - \alpha t^{-1}\xi_1\eta_1\eta_2.$$

(16)

Proof. Note that if $\alpha = 0$, then $\Gamma(2, -1, -1) \cong \mathfrak{sp}(2|4)$, and $\rho_\alpha$ is the standard embedding of $\mathfrak{sp}(2|4)$ into $P^+(4)$. Let

$$V_1 = \text{Span}(e_1, e_2), \quad V_2 = \text{Span}(f_1, f_2), \quad V_3 = \text{Span}(h_1, h_2),$$

(17)

and

$$\psi_1(e_1, e_2) = -\psi_1(e_2, e_1) = 1,$$
$$\psi_2(f_1, f_2) = -\psi_2(f_2, f_1) = 1,$$
$$\psi_3(h_1, h_2) = -\psi_3(h_2, h_1) = 1.$$

(18)

Explicitly an embedding $\rho_\alpha$ is given as follows:

$$\rho_\alpha(\mathfrak{sp}_1(e_1, e_2)) = -E^1_\alpha, \quad \rho_\alpha(\mathfrak{sp}_1(e_2, e_2)) = -F^1_\alpha, \quad \rho_\alpha(\mathfrak{sp}_1(e_1, e_2)) = -H^1_\alpha,$$
$$\rho_\alpha(\mathfrak{sp}_2(f_1, f_1)) = -2F^2_\alpha, \quad \rho_\alpha(\mathfrak{sp}_2(f_2, f_2)) = -2E^2_\alpha, \quad \rho_\alpha(\mathfrak{sp}_2(f_1, f_2)) = H^2_\alpha,$$
$$\rho_\alpha(\mathfrak{sp}_3(h_1, h_1)) = -2F^3_\alpha, \quad \rho_\alpha(\mathfrak{sp}_3(h_2, h_2)) = 2E^3_\alpha, \quad \rho_\alpha(\mathfrak{sp}_3(h_1, h_2)) = H^3_\alpha,$$
$$\rho_\alpha(e_1 \otimes f_1 \otimes h_1) = \sqrt{2}iT^1_\alpha, \quad \rho_\alpha(e_1 \otimes f_1 \otimes h_2) = \sqrt{2}iT^2_\alpha,$$
$$\rho_\alpha(e_1 \otimes f_2 \otimes h_1) = -\sqrt{2}iT^4_\alpha, \quad \rho_\alpha(e_1 \otimes f_2 \otimes h_2) = \sqrt{2}iT^3_\alpha,$$
$$\rho_\alpha(e_2 \otimes f_1 \otimes h_1) = \sqrt{2}iD^3_\alpha, \quad \rho_\alpha(e_2 \otimes f_1 \otimes h_2) = \sqrt{2}iD^4_\alpha,$$
$$\rho_\alpha(e_2 \otimes f_2 \otimes h_1) = -\sqrt{2}iD^2_\alpha, \quad \rho_\alpha(e_2 \otimes f_2 \otimes h_2) = \sqrt{2}iD^1_\alpha.$$

(19)

Thus $\mathfrak{sp}(\psi_i) \cong \text{Span}(E^i_\alpha, H^i_\alpha, F^i_\alpha)$ for $i = 1, 2, 3$. □
3. Matrices over a Weyl algebra

By definition, a Weyl algebra is

$$\mathcal{W} = \sum_{i \geq 0} A d^i,$$

(20)

where $\mathcal{A}$ is an associative commutative algebra and $d : \mathcal{A} \to \mathcal{A}$ is a derivation of $\mathcal{A}$, with the relations

$$da = d(a) + ad, \quad a \in \mathcal{A},$$

(21)

see [13]. Set

$$\mathcal{A} = \mathbb{C}[t, t^{-1}], \quad d = \frac{\partial}{\partial t}.$$  

(22)

Note that $\mathcal{W}$ is isomorphic to the associative algebra $P^{+}_{h=1}$. Let $\text{End}(\mathcal{W}^{2|2})$ be the complex Lie superalgebra of $4 \times 4$ matrices over $\mathcal{W}$.

**Theorem 3.1.** For each $\alpha \in \mathbb{C}$ there exists an embedding

$$\bar{\rho}_\alpha : \Gamma(2, -1 - \alpha, \alpha - 1) \to \text{End}(\mathcal{W}^{2|2})$$

(23)

given as follows:

$$\bar{\rho}_\alpha(T^1_\alpha) = \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix}, \quad \bar{\rho}_\alpha(T^2_\alpha) = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\rho}_\alpha(T^3_\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\rho}_\alpha(T^4_\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\rho}_\alpha(D^1_\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d + \alpha t^{-1} & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\rho}_\alpha(D^2_\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d - \alpha t^{-1} \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\rho}_\alpha(D^3_\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\rho}_\alpha(D^4_\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d + \alpha t^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{pmatrix},$$

(24)
\[ \rho_{\alpha}(E^1_{\alpha}) = t^2 1_{|4|4} \]
\[ \rho_{\alpha}(F^1_{\alpha}) = \begin{pmatrix} (d^2 + \alpha t^{-1}d) 1_{2|2} \\ 0 \end{pmatrix} \]
\[ \rho_{\alpha}(H^1_{\alpha}) = (td + \frac{1 + \alpha}{2}) 1_{|4|4} \]
\[ \rho_{\alpha}(E^2_{\alpha}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ \rho_{\alpha}(F^2_{\alpha}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ \rho_{\alpha}(H^2_{\alpha}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ \rho_{\alpha}(E^3_{\alpha}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ \rho_{\alpha}(F^3_{\alpha}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
\[ \rho_{\alpha}(H^3_{\alpha}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

**Proof.** For each \( h \in (0, 1] \) and each \( \alpha \in \mathbb{C} \) there exists an embedding

\[ \rho_{\alpha, h} : \Gamma(2, -1 - \alpha, \alpha - 1) \rightarrow P_h^+(4). \]  

(25)

\[ \Gamma_{\alpha, h} = \rho_{\alpha, h}(\Gamma(2, -1 - \alpha, \alpha - 1)) \] is spanned by the following elements:

\[ E^1_{\alpha, h} = t^2, \quad H^1_{\alpha, h} = t\tau + \frac{\alpha + 1}{2} h, \]
\[ F^1_{\alpha, h} = \tau^2 - \alpha(2t^{-2}\xi_1\xi_2\eta_1\eta_2 + t^{-2}(\xi_1\eta_1 + \xi_2\eta_2)h - t^{-1}\tau h), \]
\[ E^2_{\alpha, h} = \xi_1\xi_2, \quad H^2_{\alpha, h} = \xi_1\eta_1 + \xi_2\eta_2 - h, \quad F^2_{\alpha, h} = \eta_1\eta_2, \]
\[ E^3_{\alpha, h} = \xi_1\eta_2, \quad H^3_{\alpha, h} = \xi_1\eta_1 - \xi_2\eta_2, \quad F^3_{\alpha, h} = \xi_2\eta_1, \]
\[ T^1_{\alpha, h} = t\eta_1, \quad T^2_{\alpha, h} = t\eta_2, \quad T^3_{\alpha, h} = t\xi_1, \quad T^4_{\alpha, h} = t\xi_2, \]
\[ D^1_{\alpha, h} = \tau\xi_1 + \alpha^{-1}\xi_1\xi_2\eta_2, \quad D^2_{\alpha, h} = \tau\xi_2 - \alpha^{-1}\xi_1\xi_2\eta_1, \]
\[ D^3_{\alpha, h} = \tau\eta_1 + \alpha^{-1}\eta_1\eta_2\xi_2, \quad D^4_{\alpha, h} = \tau\eta_2 - \alpha^{-1}\eta_1\eta_2\xi_1, \]

and so

\[ \lim_{h \to 0} \Gamma_{\alpha, h} = \Gamma_{\alpha} \subset P^+(4). \]  

(27)

We fix \( h = 1 \). Let \( V = \mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2) \). Define a representation of \( \Gamma(2, -1 - \alpha, \alpha - 1) \) in \( V \) according to the embedding \( \rho_{\alpha, h=1} \). Namely, \( \xi_i \) is the operator of multiplication in \( \Lambda(\xi_1, \xi_2) \), \( \eta_i \) is identified with \( \partial_{\xi_i} \), and \( 1 \in P^+_{h=1}(4) \) acts by the identity operator.

Consider the following basis in \( V \):

\[ v^0_m = t^m, \quad v^1_m = t^m \xi_1, \]
\[ v^2_m = t^m \xi_2, \quad v^3_m = t^m \xi_1 \xi_2 \] for all \( m \in \mathbb{Z} \).  

(28)
Explicitly, the action of $\Gamma(2, -1 - \alpha, \alpha - 1)$ on $V$ is given as follows

\[
T_\alpha^1(v_m^3) = v_{m+1}^2, \quad T_\alpha^1(v_m^1) = v_{m+1}^0, \quad T_\alpha^2(v_m^3) = -v_{m+1}^1, \quad T_\alpha^2(v_m^2) = v_{m+1}^0, \\
T_\alpha^3(v_m^3) = v_{m+1}^1, \quad T_\alpha^3(v_m^2) = v_{m+1}^3, \quad T_\alpha^4(v_m^3) = v_{m+1}^2, \quad T_\alpha^4(v_m^1) = -v_{m+1}^3, \\
D_\alpha^1(v_m^0) = mv_{m-1}^1, \quad D_\alpha^1(v_m^2) = (m + \alpha)v_{m-1}^3, \\
D_\alpha^2(v_m^0) = v_{m-1}^2, \quad D_\alpha^2(v_m^1) = -(m + \alpha)v_{m-1}^0, \\
D_\alpha^3(v_m^3) = mv_{m-1}^2, \quad D_\alpha^3(v_m^1) = (m + \alpha)v_{m-1}^0, \\
D_\alpha^4(v_m^3) = -mv_{m-1}^1, \quad D_\alpha^4(v_m^2) = (m + \alpha)v_{m-1}^0, \\
E_\alpha^1(v_m^0) = v_{m+2}^0, \quad E_\alpha^1(v_m^3) = v_{m+2}^3, \quad E_\alpha^1(v_m^1) = v_{m+2}^1, \quad E_\alpha^1(v_m^2) = v_{m+2}^1, \\
F_\alpha^1(v_m^0) = m(m - 1 + \alpha)v_{m-2}^0, \quad F_\alpha^1(v_m^3) = m(m - 1 + \alpha)v_{m-2}^3, \\
F_\alpha^1(v_m^1) = (m + \alpha)(m - 1)v_{m-2}^1, \quad F_\alpha^1(v_m^2) = (m + \alpha)(m - 1)v_{m-2}^2, \\
H_\alpha^1(v_m^i) = (m + \alpha + 1)v_m^i, \quad i = 0, 1, 2, 3, \\
E_\alpha^2(v_m^0) = v_m^3, \quad F_\alpha^2(v_m^3) = -v_m^0, \quad H_\alpha^2(v_m^0) = -v_m^0, \quad H_\alpha^2(v_m^3) = v_m^3, \\
E_\alpha^3(v_m^3) = v_m^2, \quad F_\alpha^3(v_m^1) = v_m^2, \quad H_\alpha^3(v_m^3) = v_m^1, \quad H_\alpha^3(v_m^2) = -v_m^2.
\]

Thus we obtain the above-mentioned embedding $\tilde{\rho}_\alpha$ of $\Gamma(2, -1 - \alpha, \alpha - 1)$ into $\text{End}(\mathfrak{w}^{2|2})$.

\[\square\]

**Remark 3.2.** Note that the superalgebras $\Gamma(2, -1 - \alpha, \alpha - 1)$ are not simple, when $\alpha = 1$ or $-1$. Correspondingly, we have the following realizations of $\mathfrak{psl}(2|2)$ as a subsuperalgebra of $\text{End}(\mathfrak{w}^{2|2})$.

If $\alpha = 1$, then

\[
\text{Span}(E_i^\alpha, H_i^\alpha, F_i^\alpha, T_j^\alpha, D_j^\alpha) \mid i = 1, 2 \text{ and } j = 1, \ldots, 4 \cong \mathfrak{psl}(2|2), \\
\Gamma(2, -2, 0)/\mathfrak{psl}(2|2) \cong \mathfrak{sl}(2).
\]

If $\alpha = -1$, then

\[
\text{Span}(E_i^\alpha, H_i^\alpha, F_i^\alpha, T_j^\alpha, D_j^\alpha) \mid i = 1, 3 \text{ and } j = 1, \ldots, 4 \cong \mathfrak{psl}(2|2), \\
\Gamma(2, 0, -2)/\mathfrak{psl}(2|2) \cong \mathfrak{sl}(2).
\]

4. **Contractions of $\Gamma(2, -1 - \alpha, \alpha - 1)$**

We denote the associative algebra $R_{h=1}$ by $\tilde{\mathfrak{w}}$. The product $A(t, \tau) \circ B(t, \tau)$ in $\tilde{\mathfrak{w}}$ is defined as in (8) where $h = 1$. Clearly, $\mathfrak{w} \subset \tilde{\mathfrak{w}}$. In this section we consider
a contraction $\Gamma$ of $\Gamma(2, -1 - \alpha, \alpha - 1)$ when $\alpha \to 1$ (or $\alpha \to -1$), which is a Lie superalgebra isomorphic to the universal central extension of $\mathfrak{psl}(2|2)$ (cf. [6]). We obtain an embedding of $\Gamma$ into $4 \times 4$ matrices over $\mathbb{W}$. In this realization we essentially use pseudodifferential operators. This allows us to construct a three-parameter family of irreducible representations of the universal central extension of $\mathfrak{psl}(2|2)$ in $(2|2)$-dimensional complex superspace.

The nonzero commutation relations in $\Gamma(2, -1 - \alpha, \alpha - 1)$ are as follows:

\begin{align}
[T^i_\alpha, T^j_\alpha] &= E^1_\alpha, \quad [D^i_\alpha, D^j_\alpha] = F^1_\alpha, \quad [E^1_\alpha, D^j_\alpha] = -2T^j_\alpha, \\
[E^1_\alpha, D^i_\alpha] &= -2T^i_\alpha, \quad [F^1_\alpha, T^i_\alpha] = 2D^i_\alpha, \quad [F^1_\alpha, T^j_\alpha] = 2D^j_\alpha,
\end{align}

where $i = 1, j = 3$ or $i = 2, j = 4$;

\begin{align}
[E^2_\alpha, T^i_\alpha] &= \mp T^i_\alpha, \quad [E^2_\alpha, D^i_\alpha] = \pm D^i_\alpha, \\
[F^2_\alpha, T^i_\alpha] &= \pm T^i_\alpha, \quad [F^2_\alpha, D^i_\alpha] = \mp D^i_\alpha,
\end{align}

where $i = 1, j = 4$ or $i = 2, j = 3$;

\begin{align}
[E^1_\alpha, F^i_\alpha] &= -4H^1_\alpha, \quad [E^2_\alpha, F^i_\alpha] = -H^2_\alpha, \\
[H^i_\alpha, E^1_\alpha] &= 2E^i_\alpha, \quad [H^i_\alpha, F^j_\alpha] = -2F^i_\alpha, \quad i = 1, 2, \\
[H^1_\alpha, T^i_\alpha] &= T^i_\alpha, \quad [H^1_\alpha, D^i_\alpha] = -D^i_\alpha, \quad i = 1, \ldots, 4 \\
[H^2_\alpha, T^i_\alpha] &= \mp T^i_\alpha, \quad [H^2_\alpha, D^i_\alpha] = \pm D^i_\alpha, \quad i = 1, 2 \text{ or } i = 3, 4.
\end{align}

\begin{align}
[E^3_\alpha, T^i_\alpha] &= \mp T^i_\alpha, \quad [E^3_\alpha, D^i_\alpha] = \pm D^i_\alpha, \\
[F^3_\alpha, T^i_\alpha] &= \mp T^i_\alpha, \quad [F^3_\alpha, D^i_\alpha] = \pm D^i_\alpha,
\end{align}

where $i = 1, j = 2$, or $i = 4, j = 3$;

\begin{align}
[H^3_\alpha, T^i_\alpha] &= \mp T^i_\alpha, \quad [H^3_\alpha, D^i_\alpha] = \pm D^i_\alpha, \quad i = 1, 4 \text{ or } i = 2, 3 \\
[E^3_\alpha, F^3_\alpha] &= -H^3_\alpha, \quad [H^3_\alpha, E^3_\alpha] = 2F^3_\alpha, \quad [H^3_\alpha, F^3_\alpha] = -2F^3_\alpha.
\end{align}

\begin{align}
[D^i_\alpha, T^j_\alpha] &= \pm (1 + \alpha)E^2_\alpha, \quad [T^i_\alpha, D^j_\alpha] = \mp (1 + \alpha)F^2_\alpha, \text{ where } i = 1, j = 4 \text{ or } i = 2, j = 3, \\
[T^i_\alpha, D^j_\alpha] &= \pm (\alpha - 1)E^3_\alpha, \quad [T^i_\alpha, D^j_\alpha] = \pm (\alpha - 1)F^3_\alpha, \text{ where } i = 3, j = 4 \text{ or } i = 2, j = 1, \\
[T^1_\alpha, D^1_\alpha] &= H^1_\alpha + \left(\frac{1 + \alpha}{2}\right)H^2_\alpha + \left(\frac{1 - \alpha}{2}\right)H^3_\alpha, \quad [T^2_\alpha, D^2_\alpha] = H^1_\alpha + \left(\frac{1 + \alpha}{2}\right)H^2_\alpha + \left(\frac{\alpha - 1}{2}\right)H^3_\alpha \\
[T^3_\alpha, D^3_\alpha] &= H^1_\alpha - \left(\frac{1 + \alpha}{2}\right)H^2_\alpha + \left(\frac{\alpha - 1}{2}\right)H^3_\alpha, \quad [T^4_\alpha, D^4_\alpha] = H^1_\alpha - \left(\frac{1 + \alpha}{2}\right)H^2_\alpha + \left(\frac{1 - \alpha}{2}\right)H^3_\alpha.
\end{align}

Let

\begin{align}
E^3_\alpha = (\alpha - 1)^{-1}C_+, \quad H^3_\alpha = (\alpha - 1)^{-1}C, \quad F^3_\alpha = (\alpha - 1)^{-1}C_-.
\end{align}
If $\alpha \to 1$, then $\Gamma(2,-1-\alpha,\alpha-1)$ contracts to the Lie superalgebra $\Gamma$, so

$$\Gamma = \text{Span}(T_i, D_i, E_j, H_j, F_j, C, C_+, C_-) \quad \text{where } i = 1, \ldots, 4, j = 1, 2.$$ \hspace{1cm} (36)

Note that the nonzero commutation relations in $\Gamma$ are as in (32) and as follows:

$$[T^1_a, D^1_a] = -2F_a^2, \quad [T^2_a, D^3_a] = 2F_a^2, \quad [D^1_a, T^1_a] = 2E_a^2, \quad [D^2_a, T^3_a] = -2E_a^2,$$

where $\alpha = C_+ - C_-$. If $\alpha$ is such that

$$[T^1_a, D^1_a] = H^1_a + H^2_a - \frac{C}{2}, \quad [T^2_a, D^2_a] = H^1_a + H^2_a + \frac{C}{2},$$

then $\Gamma(2,1,\alpha) \to \Gamma(2,-1-\alpha,\alpha-1)$ contracts to the Lie superalgebra

$$\text{Span}(T_i, D_i, E_j, H_j, F_j, C, C_+, C_-) \quad \text{where } i = 1, \ldots, 4, j = 1, 3,$$ \hspace{1cm} (39)

which is isomorphic to $\Gamma$.

Let $a, b \in \mathbb{C}$ be such that $ab \neq 0, \pm 1$.

**Remark 4.1.** Let

$$E_a^2 = (\alpha + 1)^{-1}C_+, \quad H_a^2 = (\alpha + 1)^{-1}C, \quad F_a^2 = (\alpha + 1)^{-1}C_-. \hspace{1cm} (38)$$

If $\alpha \to -1$, then $\Gamma(2,-1-\alpha,\alpha-1)$ contracts to the Lie superalgebra

$$\text{Span}(T_i, D_i, E_j, H_j, F_j, C, C_+, C_-) \quad \text{where } i = 1, \ldots, 4, j = 1, 3,$$ \hspace{1cm} (39)

which is isomorphic to $\Gamma$.

**Theorem 4.2.** The Lie superalgebra $\Gamma$ is isomorphic to a subsuperalgebra of $\text{End}(\tilde{W}_{2\mid 2})$ spanned by the following matrices:

$$\tilde{C}_+ = (t\tau)1_{2\mid 2}, \quad \tilde{C} = 1_{2\mid 2}, \quad \tilde{C}_- = (\tau^{-1} \circ t^{-1})1_{2\mid 2},$$

$$\tilde{E}_1 = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \tilde{F}_1 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \tilde{H}_1 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

$$\tilde{E}_2 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \tilde{F}_2 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \quad \tilde{H}_2 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right),$$

$$\tilde{T}_3 = \left( \begin{array}{ccc} 0 & 0 & t\tau \\ 0 & 0 & 0 \\ 0 & a & 0 \end{array} \right), \quad \tilde{T}_2 = \left( \begin{array}{ccc} 0 & 0 & -t\tau \\ 0 & 0 & 0 \\ a & 0 & 0 \end{array} \right),$$

$$\tilde{T}_3 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{array} \right), \quad \tilde{T}_2 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{array} \right).$$

\hspace{1cm} (39)
\[ \tilde{D}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t\tau \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -t\tau & 0 \\ a & 0 & 0 & 0 \end{pmatrix}. \]

\[ \tilde{T}_1 = \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & \tau^{-1} \circ t^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{T}_4 = \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\tau^{-1} \circ t^{-1} & 0 & 0 \end{pmatrix}. \] (40)

**Proof.** Let \( q = \sqrt{\frac{2}{1+ab}} \)

An embedding \( \varphi : \Gamma \rightarrow \text{End}(\tilde{W}^{2|2}) \) is given as follows:

\[ \varphi(T_1) = -qi\tilde{T}_1, \quad \varphi(T_2) = q\tilde{T}_2, \quad \varphi(T_3) = qi\tilde{T}_3, \quad \varphi(T_4) = q\tilde{T}_4, \]

\[ \varphi(D_1) = qi\tilde{D}_1, \quad \varphi(D_2) = -q\tilde{D}_2, \quad \varphi(D_3) = qi\tilde{D}_3, \quad \varphi(D_4) = q\tilde{D}_4, \]

\[ \varphi(E_1) = 2\tilde{E}_1, \quad \varphi(H_1) = \tilde{H}_1, \quad \varphi(F_1) = -2\tilde{F}_1, \]

\[ \varphi(E_2) = i\tilde{E}_2, \quad \varphi(H_2) = -\tilde{H}_2, \quad \varphi(F_2) = i\tilde{F}_2, \]

\[ \varphi(C) = q^2(1-ab)\tilde{C}, \quad \varphi(C_+) = q^2ai\tilde{C}_+, \quad \varphi(C_-) = -q^2bi\tilde{C}_- \] (42)

Recall that \( \text{sl}(2|2) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A, B, C, D \text{ are } 2 \times 2 \text{ matrices over } \mathbb{C} \text{ such that } trA = trD \} \),

\[ \text{psl}(2|2) = \text{sl}(2|2)/ < 1_{2|2} >, \] (43) (44)

see [1]. Let \( B_{ij}, C_{ij} \) be elementary \( 2 \times 2 \) matrices. Let \( \tilde{B}_{ij}, \tilde{C}_{ij} \) and \( \tilde{E}_i, \tilde{F}_i, \tilde{H}_i \) (see (40)), where \( i = 1, 2 \), be the corresponding elements of \( \text{psl}(2|2) \).

Recall that a central extension of a Lie superalgebra \( \mathfrak{g} \) is a pair \( (U, \psi) \) consisting of a Lie superalgebra \( U \) and a surjective homomorphism \( \psi : U \rightarrow \mathfrak{g} \) such that \( [\text{Ker}\psi, U] = 0 \).
A central extension \((U, \psi)\) of a Lie superalgebra \(g\) is universal if the following conditions hold (see [2]):

1. \([g, g] = g\).
2. for any central extension \((W, \phi)\) of \(g\) there exists a homomorphism \(\nu : U \rightarrow W\) such that \(\phi \circ \nu = \psi\).

**Remark 4.3.** Let \(g = \mathfrak{psl}(2|2)\). Note that \(\dim H^2(g, \mathbb{C}) = 3\) [2].

Let \(\epsilon = \text{Span}(c_+, c, c_-) \cong \mathbb{C}^3\). The universal central extension of \(g\) is given by an exact sequence of Lie superalgebras

\[
0 \rightarrow \epsilon \rightarrow \hat{g} \rightarrow g \rightarrow 0,
\]

so that

\[
[(g_1, x_1), (g_2, x_2)] = ([g_1, g_2], f(g_1, g_2))
\]

where \(g_1, g_2 \in g\) and \(x_1, x_2 \in \text{Span}(c_+, c, c_-)\). The corresponding 2–cocycle \(f\) is given as follows:

\[
f(\tilde{B}_{12}, \tilde{C}_{21}) = f(\tilde{C}_{12}, \tilde{B}_{21}) = f(\tilde{C}_{22}, \tilde{B}_{22}) = f(\tilde{B}_{11}, \tilde{C}_{11}) = c,
\]

\[
f(\tilde{C}_{22}, \tilde{C}_{11}) = -f(\tilde{C}_{12}, \tilde{C}_{21}) = c_+,
\]

\[
f(\tilde{B}_{12}, \tilde{B}_{21}) = -f(\tilde{B}_{11}, \tilde{B}_{22}) = c_-.
\]

Let \(\omega \in C^2(g, \mathbb{C})\), and let \(X_1, X_2, X_3 \in g\) be such that \(p(X_1) = 0\), \(p(X_2) = p(X_3) = 1\). Then

\[
d\omega(X_1, X_2, X_3) = -\omega([X_1, X_2], X_3) - \omega([X_1, X_3], X_2) - \omega([X_2, X_3], X_1).
\]

One can check that \(df = 0\); hence \(f \in Z^2(g, \mathbb{C})\). On the other hand, if \(\omega \in C^1(g, \mathbb{C})\) and \(p(X_1) = p(X_2) = 1\), then \(d\omega(X_1, X_2) = -\omega([x_1, x_2])\). Since \([\tilde{C}_{22}, \tilde{C}_{11}] = 0\), then \(f \notin B^2(g, \mathbb{C})\). Hence \(f \in H^2(g, \mathbb{C})\), and it is easy to see that the conditions (1) and (2) hold.

**Theorem 4.4.** There exists a three-parameter family of irreducible representations of the universal central extension \(\hat{\mathfrak{psl}}(2|2)\) in \((2|2)\)–dimensional superspace depending on \(\lambda, a, b \in \mathbb{C}\) such that \(\lambda \neq 0\) and \(ab \neq -1\).

**Proof.** Let \(\lambda \neq 0\). Let \(V^\lambda\) be a complex \((2|2)\)–dimensional superspace spanned by even vectors \((t^\lambda, 0, 0, 0), (0, t^\lambda, 0, 0)\) and odd vectors \((0, 0, t^\lambda, 0), (0, 0, 0, t^\lambda)\). The natural action of \(\varphi(\Gamma)\) on \(V^\lambda\) is given by the matrices \(\tilde{E}_i, \tilde{F}_i, \tilde{H}_i\), where \(i = 1, 2\), the matrices \(\tilde{C}_+^\lambda = (\lambda)1_{2|2}, \tilde{C}_-^\lambda = (\lambda^{-1})1_{2|2}\) and \(\bar{C} = 1_{2|2}\), and the matrices \(\tilde{T}_i^\lambda, \tilde{D}_i^\lambda\), where \(i = 1, \ldots, 4\):
\[ \tilde{T}_3^\lambda = \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad \tilde{T}_2^\lambda = \begin{pmatrix} 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \tilde{D}_4^\lambda = \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{D}_1^\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}, \]

\[ \tilde{T}_1^\lambda = \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{T}_4^\lambda = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\lambda^{-1} & 0 & 0 \end{pmatrix}, \]

\[ \tilde{D}_2^\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ \lambda^{-1} & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{D}_3^\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

These matrices span a superalgebra isomorphic to a central extension of \( \mathfrak{psl}(2|2) \). Assume that \( ab \neq -1 \). Let \( s = \frac{1}{\sqrt{ab + 1}} \). Explicitly a family of irreducible representations \( \theta \) of the universal central extension \( \hat{\mathfrak{psl}}(2|2) \) in \( (2|2) \)–dimensional complex superspace is given as follows:

\[ \theta(\tilde{C}_{11}) = s\tilde{D}_4^\lambda, \quad \theta(\tilde{C}_{12}) = s\tilde{T}_2^\lambda, \quad \theta(\tilde{C}_{22}) = s\tilde{T}_3^\lambda, \quad \theta(\tilde{C}_{21}) = s\tilde{D}_1^\lambda, \]

\[ \theta(\tilde{B}_{11}) = s\tilde{T}_4^\lambda, \quad \theta(\tilde{B}_{12}) = s\tilde{T}_1^\lambda, \quad \theta(\tilde{B}_{22}) = s\tilde{D}_3^\lambda, \quad \theta(\tilde{B}_{21}) = s\tilde{D}_2^\lambda, \]

\[ \theta(\tilde{E}_i) = \tilde{E}_i, \quad \theta(\tilde{F}_i) = \tilde{F}_i, \quad \theta(\tilde{H}_i) = \tilde{H}_i, \quad \theta(c) = \frac{s^2(ab - 1)}{2}1_{2|2}, \quad \theta(c_+) = (s^2\lambda a)1_{2|2}, \quad \theta(c_-) = (s^2\lambda^{-1}b)1_{2|2}. \]  

\[ \square \]

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