A primal-dual algorithm with optimal stepsizes and its application in decentralized consensus optimization

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Abstract We consider a primal-dual algorithm for minimizing $f(x) + h(Ax)$ with differentiable $f$. The primal-dual algorithm has two names in literature: Primal-Dual Fixed-Point algorithm based on the Proximity Operator (PDFP$^2$O) and Proximal Alternating Predictor-Corrector (PAPC). In this paper, we extend it to solve $f(x) + h(l(Ax))$ with differentiable $l^*$ and prove its convergence under a weak condition (i.e., under a large dual step size). With additional assumptions, we show its linear convergence. In addition, we show that this condition is optimal and can not be weaken. This result recovers the recent proposed positive-indefinite linearized augmented Lagrangian method.

Then we consider the application of this primal-dual algorithm in decentralized consensus optimization. We show that EXact first-ordeR Algorithm (EXTRA) and Proximal Gradient-EXTRA (PG-EXTRA) can be consider as the primal-dual algorithm applied on a problem in the form of $h(l(Ax))$. Then, the optimal upper bound of the stepsize for EXTRA/PG-EXTRA is derived. It is larger than the existing work on EXTRA/PG-EXTRA. Furthermore, for the case with strongly convex functions, we proved linear convergence under the same condition for the stepsize.

Keywords linearized augmented Lagrangian · primal-dual · decentralized consensus optimization

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1 Introduction

Minimizing the sum of two functions has applications in a variety of areas including image processing, machine learning, and decentralized consensus optimization \[2,3,12,19\]. In this paper, we aim to minimize the sum of two functions in the following form:

\[
x^* = \arg \min_{x \in X} f(x) + h\Box l(Ax),
\]

where \(X\) and \(S\) are two Hilbert spaces; \(f: X \rightarrow (-\infty, +\infty], h: S \rightarrow (-\infty, +\infty]\) are proper lower semi-continuous (lsc) convex functions; \(h\Box l\) is the infimal convolution of \(h\) and \(l\) that is defined as

\[
\inf_{t \in S} h(t) + l(s - t);
\]

the linear operator \(A: X \rightarrow S\) is bounded. In addition, we assume that \(f\) and \(l^*\) (the conjugate function of \(l\)) have Lipschitz continuous gradients and the proximal operator of \(h\), which is defined as

\[
\text{prox}_{\lambda h}(t) = (I + \lambda \partial h)^{-1}(t) := \arg \min_{s \in S} h(s) + \frac{1}{2\lambda} ||s - t||^2,
\]

has a closed-form solution or can be easily computed.

Let \(l\) be the indicator function \(\iota_{\{0\}}\) that returns 0 if \(s = 0\) and +\(\infty\) otherwise. Its conjugate function is \(l^*(s) = 0\). Then the infimal convolution \(h\Box l\) degenerates to \(h\), and the problem (1) becomes

\[
x^* = \arg \min_{x \in X} f(x) + h(Ax).
\]

It is equivalent to the following saddle-point problem

\[
\min_{x \in X} \max_{s \in S} f(x) + \langle Ax, s \rangle - h^*(s).
\]

In order to solve (2) (or (3)), a primal-dual algorithm was proposed in different fields under different names \[5,9,16\]. Loris and Verhoeven \[16\] focused on a particular smooth function \(f(x) = \frac{1}{2}||Kx - y||^2\), where \(K\) is a linear operator. Chen, Huang, and Zhang \[5\] considered the general problem (2) and proposed a Primal-Dual Fixed-Point algorithm based on the Proximity Operator (PDFP\(_2\)O). Then the same algorithm was rediscovered under the name Primal Alternating Predictor-Corrector (PAPC) in \[9\] to solve (2) and its extension to a finite sum of composite functions when \(h\) is separable.

One iteration of the algorithm is

\[
\begin{align*}
\lambda s^{k+1} & \in \left(\frac{1}{\lambda}I + \partial h^*(t)\right)^{-1} \left(\frac{\gamma}{\lambda} (I - \lambda AA^T)s^k + A (x^k - \gamma \nabla f(x^k))\right), \\
x^{k+1} & = x^k - \gamma \nabla f(x^k) - \gamma A^T s^{k+1}.
\end{align*}
\]

Here \(\lambda\) and \(\gamma\) are two positive parameters, and the convergence of this algorithm is shown when \(\lambda \leq 1/\lambda_{\text{max}}(AA^T)\) and \(\gamma < 2/L\), where \(L\) is the Lipschitz constant of \(\nabla f\).
There are many other algorithms for solving (2) and its extensions. For example, Condat-Vu [4, 7, 21] solves a more general problem than (2) with an additional non-differential function. However, the corresponding parameters $\lambda$ and $\gamma$ have to satisfy $\lambda \cdot \lambda_{\text{max}}(AA^\top) + 2\gamma/L \leq 1$, and Condat-Vu converges slower than PAPC in solving (2). When $f = 0$, Condat-Vu reduces to Chambolle-Pock [2]. There are several other primal-dual algorithms for minimizing the sum of three functions with one differentiable function [6, 22]. Interested readers are referred to [13, 22] for the comparison of different primal-dual algorithms for minimizing the sum of three functions.

When there is only one function $f(x)$, i.e., $h = 0$, we let $A = 0$, and the primal-dual algorithm reduces to the gradient descent with stepsize $\gamma$. Therefore, the condition $\gamma < 2/L$ can not be relaxed. Then whether the condition $\lambda \leq 1/\lambda_{\text{max}}(AA^\top)$ can be relaxed?

In [5, Fig. 1], the authors numerically showed that a larger stepsize (e.g., $\lambda = 4/(3\lambda_{\text{max}}(AA^\top))$) gives a better performance than stepsizes satisfying the condition $\lambda \leq 1/\lambda_{\text{max}}(AA^\top)$. However, there is no theoretical result for the convergence under this large stepsize.

For linearized Augmented Lagrangian Method (ALM) [23]–a special case of the primal-dual algorithm (4)–the positive definiteness is relaxed in [10]. Consider the constrained optimization problem

$$\begin{align*}
\text{minimize} & \quad h^*(s), \\
\text{subject to} & \quad - A^\top s = b.
\end{align*}$$

Its dual problem is

$$\begin{align*}
\text{minimize} & \quad b^\top x + h(Ax),
\end{align*}$$

which is the problem in (2) with $f(x) = b^\top x$. The linearized ALM is

$$\begin{align*}
\begin{bmatrix}
    s^{k+1} = \arg \min\limits_s & h^*(s) + \frac{\beta}{2} \| s - s^k - \frac{1}{\beta} A(x^k - \gamma(A^\top s^k + b)) \|^2, \\
    x^{k+1} = x^k - \gamma(A^\top s^{k+1} + b).
\end{bmatrix}
\end{align*}$$

(6a) (6b)

It is exactly the primal-dual algorithm (4) with $\beta = \gamma/\lambda$. Note that the step in (6a) can be written as

$$\begin{align*}
\arg \min\limits_s & \quad h^*(s) - \langle x^k, A^\top s + b \rangle + \frac{\gamma}{2} \| A^\top s + b \|^2 + \frac{\beta}{2} \| s - s^k \|^2 - (\langle x^k, A^\top s + b \rangle - \gamma/\beta)AA^\top.
\end{align*}$$

In [23], positive-definiteness of $I - (\gamma/\beta)AA^\top$ is required for proving the convergence. Then the authors in [10] showed that the matrix $I - (\gamma/\beta)AA^\top$ can be positive-indefinite. More specifically, $(\gamma/\beta) \leq 4/(3\lambda_{\text{max}}(AA^\top))$ is the necessary and sufficient condition for the convergence of linearized ALM. This result motivates us showing the convergence of (4) under a weak condition. In this paper, we extend the result to the primal-dual algorithm (4) by providing a necessary and sufficient condition on $\lambda$ for its convergence. The extension is
nontrivial because the differentiable function \( f(x) = b^T x \) in linearized ALM and the Lipschitz constant of \( \nabla f \) is 0.

Furthermore, we consider the more general problem (1) with infimal convolution instead of (2). There are few applications using infimal convolution [8] in image processing [11] and motion planning for robotics [14]. In this paper, we present another application in decentralized consensus optimization with the sum of smooth and nonsmooth functions. For details about decentralized consensus optimization, please see [17, 18, 15] and references therein.

In decentralized consensus on an undirected network, node \( i \) has one part of \( s \) (for simplicity, we assume that \( s_i \) is stored and computed on node \( i \)). The final purpose is to make sure that the values on all nodes are consensual, i.e., \( s_1 = \cdots = s_n \), and to obtain the minimizer of the objective function

\[
\sum_{i=1}^n (h_i^*(x_i) + l_i^*(x_i)).
\]

The consensus condition can be enforced by \( A^T \bar{s} = 0 \) with \( \ker(A^T) \) spanned by \( \{1\} \), and the problem can be rewritten in the following form:

\[
\min_{\bar{s}} h^*(\bar{s}) + l^*(\bar{s}) + \iota_{\{0\}}(-A^T \bar{s}).
\]

Its corresponding dual problem is

\[
\min_{x} h(\nabla l(Ax)) \quad \text{subject to } A^T x = 0.
\]

Therefore, the decentralized consensus problem is also a special case of (1) with \( f(x) = 0 \). We will show in Section 3 that the famous decentralized algorithm—Proximal Gradient EXact first-order Algorithm (PG-EXTRA) [19] is exactly the primal-dual algorithm in (7), which is a generalization of (4) for solving (1).

In addition, we relax the parameter \( \lambda \) in the primal-dual algorithm and provide a optimal upper bound for \( \lambda \) that is verified by an example from decentralized optimization. Note the convergence of EXTRA under a large stepsize is demonstrated numerically in [20] without theoretical analysis.

The contributions of this paper can be summarized as follows:

– We extend PDFP²O/PAPC to solve the problem (1) with an infimal convolution term.
– We relax the parameter for the primal-dual algorithm and provide an optimal bound for the parameters. This results recovers the positive-indefinite ALM in [10].
– For decentralized consensus, we show that PG-EXTRA is equivalent to the primal-dual algorithm applied to the dual problem and provide optimal bounds for its parameters that are larger than those gave in [19]. Then we prove the linear convergence of EXTRA under the same weak condition for the stepsize and an additional assumption for the smooth functions.

The rest of this paper is organized as follows. In Section 2, we present the algorithm to solve (1). We show its convergence for the general case in Section 2.2 and linear convergence rate under additional assumptions in Section 2.3. In Section 3, we build the connection between the proposed algorithm with PG-EXTRA in decentralized consensus optimization and provide an optimal bound for its parameters. Then we end this paper with a short conclusion.
2 New convergence results

2.1 A primal-dual algorithm

In this paper, we extend the existing primal-dual algorithm \( (4) \) for solving \( (1) \) with an infimal convolution term and show its new convergence results. Given \( x^k \) and \( s^k \), one iteration of the primal-dual algorithm is

\[
\begin{align*}
    s^{k+1} &\in \left( \frac{\gamma}{\lambda} D + \partial h^* \right)^{-1} \left( \frac{1}{\gamma} M s^k + A (x^k - \gamma P^{-1} \nabla f(x^k)) - \nabla l^*(s^k) \right), \quad (7a) \\
    x^{k+1} &= x^k - \gamma P^{-1} \nabla f(x^k) - \gamma P^{-1} A^T s^{k+1}, \quad (7b)
\end{align*}
\]

where \( M = \frac{\gamma^2}{\lambda} (D - \lambda A P^{-1} A^T) \). Here \( \lambda \) and \( \gamma \) are two positive parameters, and \( P \) and \( D \) are two positive definite operators defined on \( \mathcal{X} \) and \( \mathcal{S} \), respectively.

Let \( I \) be the identity operator defined on a Hilbert space. For simplicity, we do not specify the space on which it is defined when it is clear from the context. When \( \nabla l^* \equiv 0 \) and \( P = D = I \), the iteration reduces to \( (4) \), which is an existing primal-dual algorithm proposed in [5, 9, 16]. Its convergence is shown if \( I - \lambda AA^T \) is positive semidefinite and \( \gamma < \frac{2}{L} \) with \( L \) being the Lipschitz constant of \( \nabla f \).

Except extending this existing primal-dual algorithm to \( (7) \) for solving the problem \( (1) \) with an infimal convolution, we also show its convergence with a larger \( \lambda \). Specifically, we show that we can choose \( \lambda \) such that \( D - \frac{4}{3} \lambda A P^{-1} A^T \) is positive definite, i.e., the upper bound for \( \lambda \) is increased by \( 1/3 \).

For convenience, we introduce two operators as

\[
M_1 := \frac{\gamma^2}{\lambda} (D - \theta \lambda A P^{-1} A^T), \quad M_2 := \gamma^2 (1 - \theta) A P^{-1} A^T.
\]

Here \( \theta \in (3/4, 1] \) is chosen such that \( M_1 \) is positive definite and \( M_2 \) is positive semidefinite. We can find such \( \theta \) when \( \lambda \leq \frac{4}{(3\lambda_{\text{max}}(D^{-1/2} A P^{-1} A^T D^{-1/2}))} \).

Then we define

\[
C_1 = \lambda_{\text{max}}(M_1^{-1/2} M_2 M_1^{-1/2}) \geq 0.
\]

With these two operators, we have \( M = M_1 - M_2 \). In addition, we define a positive definite operator as follows

\[
\tilde{M} := M_1 + M_2.
\]

We let \( (s, t)_{\tilde{M}} := (s, \tilde{M} t) \) and \( \|s\|_{\tilde{M}}^2 = \langle s, \tilde{M} s \rangle \) for any self-adjoint operator \( \tilde{M} \). Note that \( \|s\|_{\tilde{M}}^2 \) can be negative if \( \tilde{M} \) is not positive semidefinite. When \( \tilde{M} \) is positive definite, we further define the induced norm as \( \|s\|_{\tilde{M}} = \sqrt{(s, s)_{\tilde{M}}} \).

For \( (x, s) \in \mathcal{X} \times \mathcal{S} \), we define \( \|(x, s)\|_{\tilde{M}}^2 = \|x\|_{\tilde{P}}^2 + \|s\|_{\tilde{M}}^2 \).
Assumption 1 Let \((x^*, s^*)\) be any fixed point of (7). For any \(x \in X\) and \(s \in S\), we have
\[
\langle x - x^*, \nabla f(x) - \nabla f(x^*) \rangle \geq \beta \|\nabla f(x) - \nabla f(x^*)\|_\mathcal{P}^2,
\]
(8)
\[
\langle s - s^*, q_h(s) - q_h(s^*) \rangle \geq 0,
\]
(9)
\[
\langle s - s^*, \nabla l^*(s) - \nabla l^*(s^*) \rangle \geq \beta \|\nabla l^*(s) - \nabla l^*(s^*)\|_{\mathcal{M}^{-1}}^2,
\]
(10)
for some \(\beta > 0\). Here \(q_h(s) \in \partial h^*(s)\) and \(q_h(s^*) \in \partial h^*(s^*)\).

This assumption is satisfied if both \(f\) and \(l^*\) have Lipschitz continuous gradients. For example, (8) is satisfied with \(\mathcal{P} = I\) if \(f\) has a \(1/\beta\) Lipschitz continuous gradient [1, Theorem 18.15]. Also, if \(\nabla f\) (or \(\nabla l^*\)) is fixed for all \(x\) (or \(s\)), e.g., the linear \(f\) in linearized ALM, then (8) (or (10)) is satisfied with any \(\beta > 0\).

Assumption 2 Let \((x^*, s^*)\) be any fixed point of (7). There exist \(\tau_f \geq 0\), \(\tau_h \geq 0\), and \(\tau_l \geq 0\), such that, for any \(x \in X\) and \(s \in S\),
\[
\langle x - x^*, \nabla f(x) - \nabla f(x^*) \rangle \geq \tau_f \|x - x^*\|_\mathcal{P}^2,
\]
(11)
\[
\langle s - s^*, q_h(s) - q_h(s^*) \rangle \geq \tau_h \|s - s^*\|^2_{\mathcal{M}^1},
\]
(12)
\[
\langle s - s^*, \nabla l^*(s) - \nabla l^*(s^*) \rangle \geq \tau_l \|s - s^*\|^2_{\mathcal{M}^1},
\]
(13)
where \(q_h(s) \in \partial h^*(s)\) and \(q_h(s^*) \in \partial h^*(s^*)\).

The assumption is satisfied if functions \(f\), \(h^*\), and \(l^*\) are convex, and in this case, \(\tau_f = \tau_h = \tau_l = 0\).

2.2 Convergence for general convex functions

First of all, we find a subgradient of \(h^*\) at \(s^{k+1}\):
\[
q_h(s^{k+1}) := \frac{1}{2} \mathcal{M}s^k - \frac{1}{2} \mathcal{M}s^{k+1} + A x^{k+1} - \nabla l^*(s^k) \in \partial h^*(s^{k+1}).
\]
(14)
It can be easily obtained from (7), and its proof is omitted here. Let \((x^*, s^*)\) be any fixed point of (7), and we have a subgradient of \(h^*\) at \(s^*\):
\[
q_h(s^*) := A x^* - \nabla l^*(s^*) \in \partial h^*(s^*).
\]
(15)

Lemma 1 (fundamental inequality) Let \((x^*, s^*)\) be any fixed point of (7). Then we have
\[
\|(x^{k+1}, s^{k+1}) - (x^*, s^*)\|_{\mathcal{P}, \mathcal{M}}^2
\leq \|(x^k, s^k) - (x^*, s^*)\|_{\mathcal{P}, \mathcal{M}}^2 - \|s^k - s^{k+1}\|_{\mathcal{M}}^2
- 2\gamma (s^{k+1} - s^*, q_h(s^{k+1}) - q_h(s^*) + \nabla l^*(s^k) - \nabla l^*(s^*))
+ 2\gamma (\nabla f(x^k) - \nabla f(x^*), x^k - x^* + (4\theta - 3)(x^k - x^{k+1}))
- (4\theta - 3)\|x^k - x^{k+1}\|_{\mathcal{P}} + 4(1 - \theta) \gamma^2 \|\nabla f(x^k) - \nabla f(x^*)\|_{\mathcal{P}^{-1}}^2.
\]
(16)
Proof The definitions of \( q_0(s^{k+1}) \) and \( q_0(s^*) \) in (14) and (15), respectively, and the update of \( x^{k+1} \) in (7b) show

\[
2\gamma(s^{k+1} - s^*, q_0(s^{k+1}) - q_0(s^*) + \nabla l^*(s^k) - \nabla l^*(s^*)) = 2\gamma(s^{k+1} - s^*, \frac{1}{\gamma} M s^k - \frac{1}{\gamma} M s^{k+1} + A x^{k+1} - A x^*)
\]

\[
= 2(s^{k+1} - s^*, s^k - s^{k+1})_M + 2\gamma(s^{k+1} - s^*, A x^{k+1} - A x^*)
\]

\[
= 2(s^{k+1} - s^*, s^k - s^{k+1})_M + 2\gamma(A^T s^{k+1} - A^T s^*, x^{k+1} - x^*)
\]

\[
= 2(s^{k+1} - s^*, s^k - s^{k+1})_M + 2(x^k - x^{k+1}, x^{k+1} - x^*)_P - 2\gamma(\nabla f(x^k) - \nabla f(x^*), x^{k+1} - x^*)
\]

\[
= ||s^k - s^*||^2_M - ||s^{k+1} - s^*||^2_M - ||s^k - s^{k+1}||^2_M + ||x^k - x^*||^2_P - ||x^k - x^{k+1}||^2_P
\]

\[
+ 2\gamma(\nabla f(x^k) - \nabla f(x^*), x^* - x^{k+1})
\]

where we expand the first two terms in (17) with \( 2\langle a, b \rangle = ||a + b||^2 - ||a||^2 - ||b||^2 \) to obtain the last equality. Therefore, we have

\[
\| (x^{k+1}, s^{k+1}) - (x^*, s^*) \|^2_P = 2\gamma(\nabla f(x^k) - \nabla f(x^*), x^* - x^{k+1})
\]

\[
- \|s^k - s^{k+1}\|^2_M - \|s^k - s^{k+1}\|^2_M - \|s^k - s^{k+1}\|^2_M + \|x^k - x^*\|^2_P - \|x^k - x^{k+1}\|^2_P
\]

\[
+ 2\gamma(\nabla f(x^k) - \nabla f(x^*), x^* - x^{k+1})
\]

The fact that \( M = M_1 - M_2 \) gives us an upper bound for the last term of (18).

\[
- \|s^k - s^{k+1}\|^2_M = - \|s^k - s^{k+1}\|^2_M + \|s^k - s^{k+1}\|^2_M
\]

\[
= - \|s^k - s^{k+1}\|^2_M + \|s^k - s^* + s^* - s^{k+1}\|^2_M
\]

\[
\leq - \|s^k - s^{k+1}\|^2_M + 2\|s^k - s^*\|^2_M + 2\|s^{k+1} - s^*\|^2_M.
\]  

Adding \( 2\|s^{k+1} - s^*\|^2_M \) onto both sides of (18), recalling that \( \tilde{M} = M_1 + M_2 = M + 2M_2 \), and combining (19) and (18), we have

\[
\| (x^{k+1}, s^{k+1}) - (x^*, s^*) \|^2_P \leq 2\gamma(\nabla f(x^k) - \nabla f(x^*), x^* - x^{k+1})
\]

\[
- 2\gamma(s^{k+1} - s^*, q_0(s^{k+1}) - q_0(s^*) + \nabla l^*(s^k) - \nabla l^*(s^*))
\]

\[
+ \| (x^k, s^k) - (x^*, s^*) \|^2_P\tilde{M} - \|x^k - x^{k+1}\|^2_P - \|s^k - s^{k+1}\|^2_M,
\]

\[
+ 4\|s^{k+1} - s^*\|^2_M.
\]  

(20)
With the definition of $M_2$, the last term in (20) can be written as

\[
4\|s^{k+1} - s^*\|_{M_2}^2 = 4(1 - \theta)\|\gamma P^{-1}A^T s^{k+1} - \gamma P^{-1}A^T s^*\|_P^2
\]

\[
= 4(1 - \theta)\|x^k - \gamma P^{-1}\nabla f(x^k) - x^{k+1} + \gamma P^{-1}\nabla f(x^*)\|_P^2
\]

\[
= 4(1 - \theta)\|x^k - x^{k+1}\|_P^2 + 4(1 - \theta)\gamma^2\|\nabla f(x^k) - \nabla f(x^*)\|_{P^{-1}}^2
\]

\[
- 8(1 - \theta)\gamma(x^k - x^{k+1}, \nabla f(x^k) - \nabla f(x^*)) ,
\]

(21)

where the second equality comes from (7b). Then, we plug (21) into (20) and obtain

\[
\| (x^{k+1}, s^{k+1}) - (x^*, s^*) \|_{\tilde{P}, \tilde{M}}^2
\]

\[
\leq 2\gamma(\nabla f(x^k) - \nabla f(x^*), x^k - x^* + (4\theta - 3)(x^k - x^{k+1}))
\]

\[
- 2\gamma(x^k - s^*, q_h(s^{k+1}) - q_h(s^*), + \nabla l^*(s^k) - \nabla l^*(s^*) \|_{\tilde{P}, \tilde{M}}^2
\]

\[
+ \| (x^k, s^k) - (x^*, s^*) \|_{\tilde{P}, \tilde{M}}^2
\]

\[
- (4\theta - 3)\|x^k - x^{k+1}\|_P^2 + 4(1 - \theta)\gamma^2\|\nabla f(x^k) - \nabla f(x^*)\|_{P^{-1}}^2.
\]

The result is proved. \qed

**Lemma 2** Let (10) be satisfied, then

\[
- \|s^k - s^{k+1}\|_{M_1}^2 - 2\gamma(x^{k+1} - s^*, \nabla l^*(s^k) - \nabla l^*(s^*))
\]

\[
\leq - (1 - \gamma/(2\beta))\|s^k - s^{k+1}\|_{M_1}^2,
\]

**Proof** Because $M_1$ is positive definite, we have

\[
- \|s^k - s^{k+1}\|_{M_1}^2 - 2\gamma(x^{k+1} - s^*, \nabla l^*(s^k) - \nabla l^*(s^*))
\]

\[
= - \|s^k - s^{k+1}\|_{M_1}^2 - 2\gamma(s^{k+1} - s^*, \nabla l^*(s^k) - \nabla l^*(s^*))
\]

\[
- 2\gamma(s^k - s^*, \nabla l^*(s^k) - \nabla l^*(s^*))
\]

\[
\leq - \|s^k - s^{k+1}\|_{M_1}^2 + \frac{\gamma}{2\beta}\|s^k - s^{k+1}\|_{M_1}^2
\]

\[
+ 2\gamma\|\nabla l^*(s^k) - \nabla l^*(s^*)\|_{M_1}^2
\]

\[
= - \|s^k - s^{k+1}\|_{M_1}^2 + \frac{\gamma}{2\beta}\|s^k - s^{k+1}\|_{M_1}^2,
\]

where the inequality comes from the Cauchy-Schwarz inequality and (10). \qed

**Theorem 1** Let Assumption 1 hold, $\theta \in (3/4, 1)$, and $\gamma \in (0, 2\beta)$. For any fixed point $(x^*, s^*)$ of (7), we have

\[
\| (x^{k+1}, s^{k+1}) - (x^*, s^*) \|_{\tilde{P}, \tilde{M}}^2
\]

\[
\leq - \frac{1}{2\beta} \|s^k - s^{k+1}\|_{M_1}^2 - \frac{(4\theta - 3)(2\beta - \gamma)}{2\beta^2 - 4(1 - \theta)\gamma} \|x^k - x^{k+1}\|_P^2.
\]

(22)
Proof Applying Lemma 2 and (9) to the inequality (16) in Lemma 1 gives
\[
\|(x^{k+1}, s^{k+1}) - (x^*, s^*)\|^2_{P,M} \\
\leq \|(x^k, s^k) - (x^*, s^*)\|^2_{P,M} - (1 - \gamma/(2\beta)) \|s^k - s^{k+1}\|^2_M \\
+ 2\gamma(\nabla f(x^k) - \nabla f(x^*), x^* - x^k) + 4(1 - \theta)\gamma^2 \|\nabla f(x^k) - \nabla f(x^*)\|^2_{P^{-1}} \\
- (4\theta - 3)\|x^k - x^{k+1}\|^2_P + 2\gamma(4\theta - 3)(\nabla f(x^k) - \nabla f(x^*), x^k - x^{k+1}).
\]
(23)

Next we bound terms A and B. For term A, the assumption (8) implies
\[2\gamma(\nabla f(x^k) - \nabla f(x^*), x^* - x^k) \leq -2\gamma\beta\|\nabla f(x^k) - \nabla f(x^*)\|^2_{P^{-1}},\] (24)
and the Cauchy-Schwarz inequality applied to term B implies
\[2\gamma(4\theta - 3)(\nabla f(x^k) - \nabla f(x^*), x^k - x^{k+1}) \leq (2\gamma\beta - 4(1 - \theta)\gamma^2)\|\nabla f(x^k) - \nabla f(x^*)\|^2_{P^{-1}} \\
+ \frac{\gamma(4\theta - 3)^2}{2\beta - 4(1 - \theta)\gamma} \|x^k - x^{k+1}\|^2_P,\] (25)
where \(\theta \in (3/4, 1]\) and \(\gamma \in (0, 2\beta).\) Plugging (24) and (25) into (23), we have
\[
\|(x^{k+1}, s^{k+1}) - (x^*, s^*)\|^2_{P,M} \\
\leq \|(x^k, s^k) - (x^*, s^*)\|^2_{P,M} - (1 - \gamma/(2\beta)) \|s^k - s^{k+1}\|^2_M \\
- (4\theta - 3)\|x^k - x^{k+1}\|^2_P + \frac{\gamma(4\theta - 3)^2}{2\beta - 4(1 - \theta)\gamma} \|x^k - x^{k+1}\|^2_P \\
= \|(x^k, s^k) - (x^*, s^*)\|^2_{P,M} - (1 - \gamma/(2\beta)) \|s^k - s^{k+1}\|^2_M \\
- \frac{\gamma(4\theta - 3)(2\beta - \gamma)}{2\beta - 4(1 - \theta)\gamma} \|x^k - x^{k+1}\|^2_P.
\]
The inequality (22) is proved. \(\square\)

Remark 1 When \(\beta = +\infty,\) i.e., the Lipschitz constant of \(\nabla f\) and \(\nabla l^*\) is 0, then (22) becomes
\[
\|(x^{k+1}, s^{k+1}) - (x^*, s^*)\|^2_{P,M} - \|(x^k, s^k) - (x^*, s^*)\|^2_{P,M} \\
\leq - \|s^k - s^{k+1}\|^2_M - (4\theta - 3)\|x^k - x^{k+1}\|^2_P.
\]
This is the key result in [10, Theorem 3.1] for linearized ALM.

Note that \(\lambda\) is the product of the primal stepsize \(\gamma\) and the dual stepsize. Larger \(\lambda\) means larger dual stepsize. So this result shows that we can choose a larger dual stepsize.

Theorem 2 Assume that \(X\) and \(S\) are finite dimensional. Under the assumptions in Theorem 1, the sequence \(\{(x^k, s^k)\}\) converges to a fixed point of (7).
Proof The inequality (22) shows that \( \{(x^k, s^k)\} \) is a bounded sequence. The finite dimensionality of \( \mathcal{X} \) and \( \mathcal{S} \) yields the compactness of \( \mathcal{X} \) and \( \mathcal{S} \). Then there exists a subsequence \( \{(x^{k_n}, s^{k_n})\} \) that converges to \((\bar{x}^*, \bar{s}^*)\). In addition, we have \( \lim_{k \to \infty} \| (x^k, s^k) - (x^{k+1}, s^{k+1}) \|_{P,M_1} = 0 \) from the inequality (22), and the subsequence \( \{(x^{k_n+1}, s^{k_n+1})\} \) converges to the same point \((\bar{x}^*, \bar{s}^*)\). Therefore, \((\bar{x}^*, \bar{s}^*)\) is a fixed point of (7).

Letting \((\bar{x}^*, \bar{s}^*)\) in (22) be \((\bar{x}^*, \bar{s}^*)\), we have

\[
((x^{k+1}, s^{k+1}) - (\bar{x}^*, \bar{s}^*))^2 \leq ((x^k, s^k) - (\bar{x}^*, \bar{s}^*))^2_{P,M_1}
\]

Therefore, the sequence \( \{(x^k, s^k)\} \) converges to \((\bar{x}^*, \bar{s}^*)\), which is a fixed point of (7). \(\square\)

2.3 Linear convergence

In this subsection, we prove the linear convergence of the sequence \( \{(x^k, s^k)\} \) in Theorem 3 under the additional Assumption 2. This general linear convergence result requires \( \tau_f > 0 \). Then, for the special case when \( f = 0 \) and \( h = 0 \), we show the linear convergence of the sequence in Theorem 4, and this result will be applied to obtain a stronger result for EXTRA than previous work.

Before showing the linear convergence, we prove the following lemma, which is different from Lemma 2.

Lemma 3 Let (10) and (13) be satisfied, then

\[
-\|s^k - s^{k+1}\|^2_{M_1} - 2\gamma(\|s^{k+1} - s^*\|_1, \nabla l^*(s^k) - \nabla l^*(s^*)) \\
\leq -\|M_2(s^{k+1} - s^k) + \gamma A x^{k+1} - \gamma A x^* - \gamma q_h(s^{k+1}) + \gamma q_h(s^*)\|^2_{M_1} \\
- (2\gamma - \gamma^2/\beta) \eta\|s^k - s^*\|^2_{M_1}.
\] (26)

Proof Because \( M_1 \) is positive definite, we have

\[
-\|s^k - s^{k+1}\|^2_{M_1} - 2\gamma(\|s^{k+1} - s^*\|_1, \nabla l^*(s^k) - \nabla l^*(s^*)) \\
= -\|s^k - s^{k+1}\|^2_{M_1} - 2\gamma(\|M_1^{1/2}(s^{k+1} - s^k), M_1^{-1/2}(\nabla l^*(s^k) - \nabla l^*(s^*))\) \\
- 2\gamma(\|s^k - s^*\|_1, \nabla l^*(s^k) - \nabla l^*(s^*)) \\
= -\|M_1^{1/2}(s^{k+1} - s^k) + M_1^{-1/2}\gamma(\nabla l^*(s^k) - \nabla l^*(s^*))\|^2_{M_1^{-1}} \\
+ \gamma^2\|\nabla l^*(s^k) - \nabla l^*(s^*)\|^2_{M_1^{-1}} - 2\gamma(\|s^k - s^*\|_1, \nabla l^*(s^k) - \nabla l^*(s^*)).
\] (27)

The first term on the right hand side of (27) becomes

\[
-\|M_1^{1/2}(s^{k+1} - s^k) + M_1^{-1/2}\gamma(\nabla l^*(s^k) - \nabla l^*(s^*))\|^2_{M_1^{-1}} \\
= -\|M_1(s^{k+1} - s^k) + \gamma(\nabla l^*(s^k) - \nabla l^*(s^*))\|^2_{M_1} \\
= -\|M_2(s^{k+1} - s^k) + M(s^{k+1} - s^k) + \gamma(\nabla l^*(s^k) - \nabla l^*(s^*))\|^2_{M_1^{-1}} \\
\tag{14} \tag{15} \leq -\|M_2(s^{k+1} - s^k) + \gamma A x^{k+1} - \gamma A x^* - \gamma q_h(s^{k+1}) + \gamma q_h(s^*)\|^2_{M_1^{-1}}.
\]
where the second equality comes from $M = M_1 - M_2$.

For the other two terms on the right hand side of (27), we have

$$
\begin{align*}
\gamma^2 \|\nabla l^*(s^k) - \nabla l^*(s^*)\|_{M_1^{-1}}^2 & - 2\gamma (s^k - s^*, \nabla l^*(s^k) - \nabla l^*(s^*)) \\
= \gamma^2 \|\nabla l^*(s^k) - \nabla l^*(s^*)\|_{M_1^{-1}}^2 - (\gamma^2 / \beta) (s^k - s^*, \nabla l^*(s^k) - \nabla l^*(s^*)) \\
& - (2\gamma - \gamma^2 / \beta) (s^k - s^*, \nabla l^*(s^k) - \nabla l^*(s^*)) \\
& \leq (10,13) - (2\gamma - \gamma^2 / \beta) \|s^k - s^*\|_{M_1}^2.
\end{align*}
$$

Combining both together with (27) gives (26).

\begin{proof}
Applying Lemma 3 to (16) in Lemma 1 gives

$$
\begin{align*}
\|(x^{k+1}, s^{k+1}) - (x^*, s^*)\|_{P, M}^2 & \leq \|(x^k, s^k) - (x^*, s^*)\|_{P, M}^2 - (2\gamma - \gamma^2 / \beta) \tau_l \|s^k - s^*\|_{M_1}^2 \\
& - 2\gamma (s^{k+1} - s^*, q_k(s^{k+1}) - q_k(s^*)) \\
& + 2\gamma \langle \nabla f(x^k) - \nabla f(x^*), x^* - x^k + (4\theta - 3)(x^k - x^{k+1}) \rangle \\
& - (4\theta - 3) \|x^k - x^{k+1}\|_{P}^2 + 4(1 - \theta)\gamma^2 \|\nabla f(x^k) - \nabla f(x^*)\|_{P-1}^2 \\
& \leq \|(x^k, s^k) - (x^*, s^*)\|_{P, M}^2 - (2\gamma - \gamma^2 / \beta) \tau_l \|s^k - s^*\|_{M_1}^2 \\
& - 2\gamma (s^{k+1} - s^*, q_k(s^{k+1}) - q_k(s^*)) \\
& - 2\gamma \langle \nabla f(x^k) - \nabla f(x^*), x^k - x^* \rangle + \gamma^2 \|\nabla f(x^k) - \nabla f(x^*)\|_{P-1}^2 \\
& - (4\theta - 3) \|x^k - x^{k+1}\|_{P}^2 - \gamma \|\nabla f(x^k) - \nabla f(x^*)\|_{P}^2.
\end{align*}
$$

Note that

$$
\begin{align*}
- 2\gamma \langle \nabla f(x^k) - \nabla f(x^*), x^k - x^* \rangle + \gamma^2 \|\nabla f(x^k) - \nabla f(x^*)\|_{P-1}^2 \\
& \leq (8) - (2\gamma - \gamma^2 / \beta) \|\nabla f(x^k) - \nabla f(x^*), x^k - x^*\|_{P}^2 \\
& \leq (11) - (2\gamma - \gamma^2 / \beta) \tau_f \|x^k - x^*\|_{P}^2.
\end{align*}
$$

The sequence $\{(x^k, s^k)\}$ converges linearly to the fixed point $(x^*, s^*)$ with rate $\rho_1 < 1$ if $\gamma \in (0, 2\beta)$, $\tau_l + \tau_l > 0$, and $\tau_f > 0$.

\end{proof}
Then we have, together with (12),
\[
\|(x^{k+1}, s^{k+1}) - (x^*, s^*)\|^2_{\mathbf{P}, \mathcal{M}} \\
\leq \|(x^k, s^k) - (x^*, s^*)\|^2_{\mathbf{P}, \mathcal{M}} - (2\gamma - \gamma^2/\beta) \tau_j \|s^k - s^*\|^2_{\mathcal{M}} \\
- 2\gamma \tau_h \|s^{k+1} - s^*\|^2_{\mathcal{M}} - (2\gamma - \gamma^2/\beta) \tau_f \|x^k - x^*\|^2_{\mathbf{P}}.
\]

That is
\[
\|x^{k+1} - x^*\|^2_{\mathbf{P}} + \|s^{k+1} - s^*\|^2_{(1+2\gamma \tau_h)\mathcal{M}_1 + \mathcal{M}_2} \\
\leq (1 - (2\gamma - \gamma^2/\beta) \tau_f) \|x^k - x^*\|^2_{\mathbf{P}} + \|s^k - s^*\|^2_{(1-(2\gamma - \gamma^2/\beta) \tau_j)\mathcal{M}_1 + \mathcal{M}_2}.
\]

(29)

For the last term on the right hand of (29), we have
\[
\|s^k - s^*\|^2_{(1-(2\gamma - \gamma^2/\beta) \tau_j)\mathcal{M}_1 + \mathcal{M}_2} \\
\leq \|M_1^{1/2}(s^k - s^*)\|^2_{(1-(2\gamma - \gamma^2/\beta) \tau_j)\mathcal{I} + \mathcal{M}_1^{-1/2} \mathcal{M}_2 \mathcal{M}_1^{-1/2}} \\
\leq \frac{1-(2\gamma - \gamma^2/\beta) \tau_j + C_1}{1+2\gamma \tau_h + C_1} \|s^k - s^*\|^2_{(1-(2\gamma - \gamma^2/\beta) \tau_j)\mathcal{I} + \mathcal{M}_1^{-1/2} \mathcal{M}_2 \mathcal{M}_1^{-1/2}} \\
\leq \frac{1-(2\gamma - \gamma^2/\beta) \tau_j + C_1}{1+2\gamma \tau_h + C_1} \|s^k - s^*\|^2_{(1+(2\gamma \tau_h)\mathcal{M}_1 + \mathcal{M}_2)}.
\]

Therefore, the inequality (28) is proved.

This theorem requires both \(\tau_f > 0\) and \(\tau_h + \gamma > 0\) for the linear convergence. Thus it does not cover EXTRA, which is the case when \(f = 0\) and \(h = 0\). For the case when \(f = 0\) and \(h = 0\), we have linear convergence when \(\gamma > 0\). The result is shown in Theorem 4, while the connection to EXTRA will be explained in details in the next section. For simplicity, we let \(\mathbf{D} = \mathbf{I}\) and \(\mathbf{P} = \mathbf{I}\).

**Theorem 4** Let \(f = 0\), \(h = 0\), \(\mathbf{D} = \mathbf{I}\), and \(\mathbf{P} = \mathbf{I}\). Let \((x^*, s^*)\) be a fixed point of (7) and assumptions (10) and (13) hold. Define
\[
\mathcal{M} := \left\{ \begin{array}{ll}
\mathcal{M}_1 + \frac{\gamma^2 - 1}{2(1-\theta)} \mathcal{M}_2, & \text{when } \theta < 1, \\
\mathcal{M}_1, & \text{when } \theta = 1,
\end{array} \right.
\]
and we have
\[
(1 + \frac{(4\theta - 3)C_2}{4\theta - 3 + 4(1-\theta)C_1}) \|x^{k+1} - x^*\|^2 + \|s^{k+1} - s^*\|^2_{\mathcal{M}} \\
\leq \rho_2 \left( 1 + \frac{(4\theta - 3)C_2}{4\theta - 3 + 4(1-\theta)C_1} \|x^k - x^*\|^2 + \|s^k - s^*\|^2_{\mathcal{M}} \right),
\]

(30)

where
\[
C_2 = \frac{\lambda_{\min}(\mathbf{A}\mathbf{A}^\top)}{1 - \theta \lambda_{\min}(\mathbf{A}\mathbf{A}^\top)}
\]
and
\[
\rho_2 = \max \left\{ \frac{4\theta - 3 + 4(1-\theta)C_1}{4\theta - 3 + 4(1-\theta)C_1}, \frac{1 - (2\gamma - \gamma^2/\beta) \tau_j + \frac{2\gamma}{2\gamma - \gamma^2/\beta} \mathcal{C}_1}{1 + \frac{2\gamma}{2\gamma - \gamma^2/\beta} \mathcal{C}_1} \right\}.
\]

Here \(\lambda_{\min}(\mathbf{A}\mathbf{A}^\top)\) is the smallest nonzero eigenvalue of \(\mathbf{A}\mathbf{A}^\top\). The sequence \(\{(x^k, s^k)\}\) converges linearly to the fixed point \((x^*, s^*)\) with rate \(\rho_2 < 1\) if \(\gamma \in (0, 2\beta)\) and \(\tau_j > 0\).
An optimal primal-dual algorithm

Proof Because $x^{k+1} - x^*$ is in the range of $A^T$, let $x^{k+1} - x^* = A^T u$ for some $u$. In addition, let $AA^T = V\Sigma V^T$ be its eigendecomposition with orthonormal $V$ and diagonal $\Sigma$. Then we have

$$
\| \gamma A(x^{k+1} - x^*)\|_{M_1^{-1}}^2 \leq \| A^T u \|_{\Sigma(I - \theta \lambda A^T)^{-1} A A^T}^2 = \lambda \| V^T u \|_{\Sigma(I - \theta \lambda A^T)^{-1} \Sigma}^2 \geq \frac{\lambda \lambda_{\text{min}}(AA^T)}{1 - \theta \lambda_{\text{min}}(AA^T)} \| V^T u \|_{\Sigma}^2 = C_2 \| u \|_{AA^T}^2 = C_2 \| x^{k+1} - x^* \|^2. \tag{31}
$$

(1) We consider the case with $\theta = 1$ first. The equation (18) becomes

$$
\| x^{k+1} - x^* \|^2 + \| s^{k+1} - s^* \|^2_{M_1} = \| x^k - x^* \|^2 + \| s^k - s^* \|^2_{M_1} - 2\gamma (s^{k+1} - s^*, \nabla I^*(s^k) - \nabla I^*(s^*)) - \| x^k - x^{k+1} \|^2 - \| s^k - s^{k+1} \|^2_{M_1} \leq (1 + C_2) \| x^k - x^* \|^2 + \| s^k - s^* \|^2_{M_1}
$$

(26)

$$
\leq (1 - \gamma^2 / \beta) \gamma \| s^k - s^* \|^2_{M_1} - \| x^k - x^{k+1} \|^2 - 2\gamma (s^{k+1} - s^*, \nabla I^*(s^k) - \nabla I^*(s^*)) - \| x^k - x^* \|^2 + (1 - \gamma^2 / \beta) \gamma \| s^k - s^* \|^2_{M_1} \leq (1 + C_2) \| x^k - x^* \|^2 + \| s^k - s^* \|^2_{M_1}
$$

(31)

Therefore, we have

$$
(1 + C_2) \| x^k - x^* \|^2 + \| s^k - s^* \|^2_{M_1} \leq \rho_2 \left( (1 + C_2) \| x^k - x^* \|^2 + \| s^k - s^* \|^2_{M_1} \right)
$$

with

$$
\rho_2 = \max \left\{ \frac{1}{1 + C_2}, 1 - \left( \frac{\gamma^2}{\beta} \right) \gamma \right\}.
$$

(2) Then we consider the case with $\theta < 1$. The definition of $M_2$ and (7b) give

$$
\| x^k - x^{k+1} \|^2 = \| A^T s^{k+1} - \gamma A^T s^* \|^2 = \frac{1}{\theta^2} \| s^{k+1} - s^* \|^2_{M_2}. \tag{32}
$$

From (18) and (32), we have

$$
\| x^{k+1} - x^* \|^2 + \| s^{k+1} - s^* \|^2_{M_1} = \| x^k - x^* \|^2 + \| s^k - s^* \|^2_{M_1} - 2\gamma (s^{k+1} - s^*, \nabla I^*(s^k) - \nabla I^*(s^*)) - \frac{1}{\theta^2} \| s^{k+1} - s^* \|^2_{M_2} - \| s^k - s^{k+1} \|^2_{M_1} \leq (1 - \gamma^2 / \beta) \gamma \| s^k - s^* \|^2_{M_1} - \frac{1}{\theta^2} \| s^{k+1} - s^* \|^2_{M_2} + \| s^k - s^{k+1} \|^2_{M_2}
$$

(26)

$$
\leq (1 - \gamma^2 / \beta) \gamma \| s^k - s^* \|^2_{M_1} - \frac{1}{\theta^2} \| s^{k+1} - s^* \|^2_{M_2} + \| s^k - s^{k+1} \|^2_{M_2} - (2\gamma^2 / \beta) \gamma \| s^k - s^* \|^2_{M_1} - \| M_2 (s^{k+1} - s^k) + \gamma A(x^{k+1} - x^*) \|^2_{M_1^{-1}}. \tag{33}
$$
In addition, we have
\[ \|s^{k+1} - s^*\|_{M_2}^2 \leq 2\|s^{k+1} - s^*\|_{M_2}^2 + 2\|s^k - s^*\|_{M_2}^2 - \frac{1}{2}\|s^{k+1} - s^k\|_{M_2}^2. \] (34)

Adding (34) multiplied by \( \frac{1}{2(1-\beta)} \) onto (33) gives
\[
\begin{align*}
\|x^{k+1} - x^*\|^2 + \|s^{k+1} - s^*\|_{M_2}^2 &\leq \|x^k - x^*\|^2 + \|s^k - s^*\|_{M_2}^2 - \frac{4\beta}{2(1-\beta)}\|s^{k+1} - s^k\|_{M_2}^2 \\
&\quad - \frac{4\beta}{4(1-\beta)}\|s^{k+1} - s^k\|_{M_2}^2 - (2\gamma - \gamma^2/\beta)\|s^k - s^*\|_{M_2}^2 \\
&\quad - \|M_2(s^{k+1} - s^k) + \gamma A(x^{k+1} - x^*)\|_{M_2^{-1}}^2.
\end{align*}
\] (35)

We have the following inequality
\[
\begin{align*}
\|x^{k+1} - x^*\|^2 + \|s^{k+1} - s^k\|_{M_2}^2 &\leq \|x^k - x^*\|^2 + \|s^k - s^*\|_{M_2}^2 - \gamma A(x^{k+1} - x^*)\|_{M_2^{-1}}^2 \\
&\quad - \|s^{k+1} - s^k\|_{M_2}^2 - 2\gamma A(x^{k+1} - x^*)\|_{M_2}^2 \\
&\quad - \|s^k - s^*\|_{M_2}^2 - (2\gamma - \gamma^2/\beta)\|s^k - s^*\|_{M_2}^2 \\
&\quad - \|M_2(s^{k+1} - s^k) + \gamma A(x^{k+1} - x^*)\|_{M_2^{-1}}^2 \\
&\quad - \|s^{k+1} - s^k\|_{M_2}^2 + \|s^k - s^*\|_{M_2}^2 + \|s^k - s^*\|_{M_2}^2 \\
&\quad - (2\gamma - \gamma^2/\beta)\|s^k - s^*\|_{M_2}^2.
\end{align*}
\]

Then, (35) becomes
\[
\begin{align*}
(1 + \frac{(4\beta-3)(2)}{4(1-\beta)(C_1)}\|x^{k+1} - x^*\|^2 + \|s^{k+1} - s^k\|_{M_2}^2 + \frac{2\beta-1}{2(1-\beta)}\|s^{k+1} - s^k\|_{M_2}^2 \\
\leq \|x^k - x^*\|^2 + \|s^k - s^*\|_{M_2}^2 + \frac{2\beta-1}{2(1-\beta)}\|s^{k+1} - s^k\|_{M_2}^2 \\
&\quad - (2\gamma - \gamma^2/\beta)\|s^k - s^*\|_{M_2}^2.
\end{align*}
\]

Therefore, (30) is proved.

3 Application in decentralized consensus optimization

In this section, we show that the algorithm (7) recovers PG-EXTRA [19] for decentralized consensus optimization. Then we prove its convergence under a weak condition that is more general than that in [19] and provide an optimal bound for the stepsize.

We use the same notation as [19]. The decentralized consensus problem is
\[
\min_{x \in \mathbb{R}^p} \sum_{i=1}^n s_i(x) + r_i(x),
\]
where \( s_i : \mathbb{R}^p \rightarrow \mathbb{R} \) and \( r_i : \mathbb{R} \rightarrow (-\infty, +\infty) \) are proper lsc convex functions held privately by node \( i \) to encode the node’s objective function. The objective of decentralized consensus is minimizing the sum of all private objective
functions while using information exchange between neighboring nodes in a
network. Here $s_i$ has a Lipschitz continuous gradient with parameter $L > 0$
and the proximal mapping of $r_i$ is simple. We let $x_i$ be one copy of $x$ kept
at node $i$. These $\{x_i\}_{i=1}^n$ are not the same in general, and we say that it is
consensual if they are the same. Stacking all the copies together, we define

$$
x := \begin{pmatrix}
-x_1^T \\
-x_2^T \\
\vdots \\
-x_n^T
\end{pmatrix} \in \mathbb{R}^{n \times p},
$$

and

$$
s(x) = \sum_{i=1}^n s_i(x_i), \quad r(x) = \sum_{i=1}^n r_i(x_i).
$$

Then the decentralized consensus problem becomes

$$
\min_x s(x) + r(x), \text{ subject to } x_1 = x_2 = \cdots = x_n.
$$

The gradient of $s$ at $x$ is written in the following matrix form:

$$
\nabla s(x) := \begin{pmatrix}
-(\nabla s_1(x_1))^T \\
-(\nabla s_2(x_2))^T \\
\vdots \\
-(\nabla s_n(x_n))^T
\end{pmatrix} \in \mathbb{R}^{n \times p},
$$

and $\| \cdot \|_F$ is the Frobenius norm for a matrix in $\mathbb{R}^{n \times p}$. One iteration of PG-
EXTRA reads as

$$
\begin{align}
\mathbf{z}^{k+1} &= \mathbf{z}^k - \mathbf{x}^k + \frac{I + \mathbf{W}}{2}(2\mathbf{z}^k - \mathbf{x}^{k-1}) - \alpha \nabla s(\mathbf{x}^k) + \alpha \nabla s(\mathbf{x}^{k-1}), \\
\mathbf{x}^{k+1} &= \arg \min_{\mathbf{x}} r(\mathbf{x}) + \frac{1}{2\alpha} \| \mathbf{x} - \mathbf{z}^{k+1} \|^2_F,
\end{align}
$$

(36a)

(36b)

where $\alpha$ is the stepsize and $\mathbf{W}$ is a symmetric doubly stochastic matrix that
represents information exchange between neighboring nodes. Thus $I - \mathbf{W}$ is
positive semidefinite, and we can find $\mathbf{A}$ such that $I - \mathbf{W} = \mathbf{A}^\top \mathbf{A}$. In addition,
we assume that $\text{Null}(\mathbf{A}^\top) = \text{Null}(I - \mathbf{W}) = \text{span}(1_{n \times 1})$, which means that
$\mathbf{A}^\top \mathbf{x} = \mathbf{0}$ is equivalent to $x_1 = x_2 = \cdots = x_n$. Therefore, the decentralized
consensus problem becomes

$$
\min_x s(x) + r(x) \text{ subject to } \mathbf{A}^\top \mathbf{x} = \mathbf{0}.
$$

Its dual problem, in the form of (1), is

$$
\min_y r^\star \mathbf{a}^\star(\mathbf{A} y),
$$

(37)
where \( r^* \) and \( s^* \) are convex conjugate functions of \( r \) and \( s \), respectively. We apply (7) to (37) \( (h \Rightarrow r^*, l \Rightarrow s^*, x \Rightarrow y, s \Rightarrow x) \) and arrive at

\[
\begin{align*}
  z^{k+1} &= (I - \lambda AA^\top)x^k + \frac{\gamma}{\lambda}Ay^k - \frac{\gamma}{\lambda} \nabla s(x^k), \\
  x^{k+1} &= \arg \min_x \{ r(x) + \frac{\gamma}{2\lambda} \| x - z^{k+1} \|_F^2 \}, \\
  y^{k+1} &= y^k - \gamma A^\top x^{k+1}.
\end{align*}
\]

(38a)

(38b)

(38c)

Combining (38a) and (38c), we get

\[
\begin{align*}
  z^{k+1} &= z^k - x^k - (I - \lambda AA^\top)(2x^k - x^{k-1}) - \frac{\lambda}{\gamma} \nabla s(x^k) + \frac{\lambda}{\gamma} \nabla s(x^{k-1}).
\end{align*}
\]

(39)

We let \( \lambda = \frac{1}{2} \) and \( \gamma = \frac{1}{2\beta} \), then (39) is exactly (36a). Because \( M = 2\gamma^2(I - (1/2)AA^\top) = \gamma^2(I + \mathbf{W}) \) is positive definite, we can let \( M_1 = M \). If \( \{ \nabla s_i(x) \}_{i=1}^n \) are Lipschitz continuous with constant \( L > 0 \), the other condition for convergence is

\[
\gamma < 2\beta \leq \frac{2\gamma^2}{L} \lambda_{\min}(M_1) = \frac{2\gamma^2}{L} \lambda_{\min}(I + \mathbf{W}),
\]

where the second inequality comes from

\[
\langle \nabla s(\tilde{x}) - \nabla s(x), \tilde{x} - x \rangle \geq \frac{1}{L} \| \tilde{x} - x \|_2^2 \geq \frac{1}{L} \lambda_{\min}(M_1) \| \tilde{x} - x \|^2_{M_1^{-1}}.
\]

Therefore, we obtain the condition on the stepsize

\[
\alpha = \frac{1}{2\gamma} < \lambda_{\min}(I + \mathbf{W}) / L.
\]

This is exactly the upper bound in [19].

The previous upper bound is obtained with \( \theta = 1 \). By letting \( \theta = 3/4 + \epsilon \) with an arbitrary small \( \epsilon > 0 \), we have \( M_1 = \gamma^2(2I - (3/4 + \epsilon)AA^\top) \) and \( M_2 = (1/4 + \epsilon)\gamma^2AA^\top \). Then a larger upper bound for the stepsize

\[
\alpha = \frac{1}{2\gamma} \leq \lambda_{\min}(2I - (3/4 + \epsilon)AA^\top) / L
\]

\[
< \lambda_{\min}(2I - (3/4)AA^\top) / L = ((3/4)\lambda_{\min}(I + \mathbf{W}) + 1/2) / L,
\]

is derived.

In addition, the condition that \( \mathbf{W} = I - AA^\top \) being doubly stochastic can be relaxed. The new condition is that \( M_1 = \gamma^2(2I - (3/4 + \epsilon)AA^\top) = \gamma^2((5/4 - \epsilon)I + (3/4 + \epsilon)\mathbf{W}) \) is positive definite. That is \( 5I + 3\mathbf{W} \) is positive definite. The comparison for both convergence conditions is in Table 1. Since EXTRA [18] is a special case of PG-EXTRA when \( r(x) = 0 \), then the results in Table 1 also apply to EXTRA. Note that this is also the stepsize for linear convergence of EXTRA when the functions \( s(x) \) satisfy (13) with \( l^* \) being \( s \).
An optimal primal-dual algorithm

| λ(W) | stepsize | linear convergence |
|------|----------|--------------------|
| [18, 19] | α < λ_{\min}(I + W)/L | α < μ_\gamma λ_{\min}(I + W)/L* |
| our result | α < (\frac{1}{2} λ_{\min}(I + W) + \frac{1}{L}) | α < (\frac{1}{2} λ_{\min}(I + W) + \frac{1}{L})/L |

Table 1 The comparison of convergence conditions for EXTRA/PG-EXT RA with respect to the eigenvalues of W and the upper bound of stepsize α. μ_\gamma is the restricted strongly convex constant of s(x) + \frac{1}{4\gamma}(x^2 - x_W/2) with respect to x^*, which is difficult to find and depends on α. Our result is better than that in [18, 19], and it is optimal.

3.1 Optimal stepsize

In this subsection, we show that the upper bound of the stepsize α in Table 1 is optimal. We consider a special problem with r(x) = 0 and s(x) = ∑_{i=1}^{n} \frac{1}{2}\|x_i - y_i\|^2. Note that the Lipschitz constant L = 1 in this case. Then the iteration reads as

\[ x^{k+1} = (I + W)x^k - \frac{I + W}{2}x^{k-1} - αx^k + αx^{k-1}. \]

It can be formulated as the following fixed point problem

\[
\begin{bmatrix}
  x^k \\
  x^{k+1}
\end{bmatrix} = M_F \begin{bmatrix}
  x^{k-1} \\
  x^k
\end{bmatrix},
\]

where

\[
M_F = \begin{bmatrix}
  -\frac{I + W}{2} + αI & I \\
  -\frac{I + W}{2} + αI & I + W - αI
\end{bmatrix}.
\]

A necessary condition for the convergence of the iteration is λ(M_F) ∈ (-1, 1], which implies the determinant det(M_F + dI) > 0 for all d ≥ 1 because the dimension of M_F is 2n × 2n. Since

\[
\det(M_F + dI) = \left| \begin{array}{cc}
  \frac{I + W}{2} + αI & dI + (I + W) - αI \\
  dI + (I + W) - αI & \frac{I + W}{2} + αI
\end{array} \right| = d^2 \left| \begin{array}{cc}
  dI + (1 + \frac{1}{2d})(I + W) - α(1 + \frac{1}{d})I \\
  dI + (1 + \frac{1}{2d})(I + W) - α(1 + \frac{1}{d})I
\end{array} \right|.
\]

The eigenvalues of dI + (1 + \frac{1}{2d})(I + W) increase continuously to infinity as d increases, while α(1 + \frac{1}{d}) decreases. Therefore, det(M_F + dI) > 0 for all d ≥ 1 implies λ_{\min}(I + (1 + \frac{1}{d})(I + W)) > 2α. Thus the step-size α < (\frac{1}{2} λ_{\min}(I + W) + \frac{1}{L}) is also a necessary condition, which shows that the upper bound given in Table 1 is optimal.

4 Conclusion

In this paper, we extend the primal-dual algorithm in [5,9,16] to solve the problem f(x) + h(\triangledown l(x)) and show its convergence under an optimal condition.
The condition for the primal stepsize is the same, and the dual stepsize can be increased by $1/3$. We provide an example to show that this condition cannot be weakened. This result recovers and is more general than the positive-indefinite linear ALM proposed in [10]. Then we apply this result to decentralized consensus optimization and extend the stepsize in PG-EXTRA/EXTRA for both convergence and linear convergence. For EXTRA, the stepsize for the linear convergence is the same as the stepsize for convergence and larger than the previous result.

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