An elementary quantum mechanics calculation for the Casimir effect in one dimension

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**Abstract**

We obtain the Casimir effect for the massless scalar field in one dimension based on the analogy between the quantum field and the continuum limit of an infinite set of coupled harmonical oscillators.

## 1 Introduction

A well known fact in quantum mechanics is that, even though the classical system admits a zero minimal energy, this does not generally hold for its quantum counterpart. The typical example is the \( \frac{1}{2} \hbar \omega \) value for the non-relativistic harmonic linear oscillator, where \( \hbar \) is the Planck constant and \( \omega \) its proper frequency. More generally, if the system behaves as a collection of such oscillators, the minimal (or zero point) energy is

\[
E_0 = \frac{\hbar}{2} \sum_n \omega_n, \tag{1}
\]
where the sum extends over all proper frequencies \( \omega_n \). As often pointed out in quantum field theory textbooks\(^1\,^2\), non-interacting quantized fields can be pictured this way, in the limit of an infinite spatial density of oscillators. In particular, for the scalar field the analogy with a set of coupled oscillators can be constructed in a precise manner\(^1\), as we shall also sketch below. We shall use here the oscillator model to obtain the Casimir effect for the massless field, in the case of one spatial dimension. The calculation is a simple exercise in non-relativistic quantum mechanics.

What is usually referred to as the Casimir effect\(^3\) is the attraction force between two conducting parallel uncharged plates in vacuum. The phenomenon counts as a direct evidence for the zero point energy of the quantized electromagnetic field: assuming the plates are perfect conductors, the energy to area ratio reads\(^1\) (\(c\) is the speed of light and \(L\) is the plates separation)

\[
\frac{E_0}{A} = -\frac{\pi^2 hc}{720 L^3},
\]

from which the attraction force can be readily derived. Qualitatively, the \(L\) dependence in \(E_0\) is naturally understood as originating in that displayed by the proper frequencies of the field between the plates.

Actually, by summing over frequencies as in eq. (1) one obtains a divergent energy. This is a common situation in quantum field theory, being remedied by what is called renormalization: one basically subtracts a divergent quantity to render the result finite, with the justification that only energy differences are relevant\(^a\). Unfortunately, computational methods used to handle infinities to enforce this operation\(^b\) present themselves, rather generally, as a piece of technicality with no intuitive support; for the unaccustomed reader, they might very well leave the impression that the result is just a mathematical artifact. The oscillator analogy comes to provide a context to do the calculations within a physically transparent picture, with no extra mathematical input required.

\(^a\)In the assumption of neglecting gravitational phaenomena, see e.g. Ref 4.
\(^b\)i.e. regularization methods. An example follows next paragraph.
2 Quantum field theory calculation

We briefly review first the field theoretical approach. Consider the uncharged massless scalar field in one dimension \(-\infty < x < \infty\),

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \varphi(x, t) = 0,
\]

subjected to the conditions

\[
\varphi(0, t) = \varphi(L, t) = 0
\]

for some positive \(L\). We are interested in the zero point energy as a function of \(L\). We shall focus on the field in the “box” \(0 < x < L\). It is intuitively clear that the result for the exterior regions follows by making \(L \to \infty\). Note that by eqs. \(\textbf{H}\) the field in the box is causally disconnected from that in the exterior regions, paralleling thus the situation for the electromagnetic field in the previous chapter.

Eqs. (3) and (4) define the proper frequencies as

\[
\omega_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \infty,
\]

obviously making \(E_0\) a divergent quantity. A convenient way\(^5\) to deal with this is by introducing the damping factors

\[
\omega_n \to \omega_n \exp(-\lambda \omega_n/c), \quad \lambda > 0,
\]

and to consider \(E_0 = E_0(L, \lambda)\) in the limit \(\lambda \to 0\). Performing the sum one obtains

\[
E_0(L, \lambda) = \frac{\pi \hbar c}{8L} \left( \frac{\cosh^2 \frac{\pi \lambda}{2L} - 1}{\lambda^2} \right).
\]

Using the expansion

\[
\cosh z = \frac{1}{z} + \frac{z}{3} + O(z^3),
\]

one finds

\[
E_0(L, \lambda) = \frac{\hbar c}{2\pi \lambda^2} L - \frac{\pi \hbar c}{24L} + O \left( \frac{\lambda}{L} \right).
\]
Now, it is immediate to see that the $\lambda^{-2}$ term can be assigned to an infinite energy density corresponding to the case $L \to \infty$. The simple but essential observation is that, when considering also the energy of the exterior regions, the divergences add to an $L$-independent quantity, which makes them mechanically irrelevant. Renormalization amounts to ignore them. Thus one can set

$$E_0(L) = -\frac{\pi \hbar c}{24 L},$$

which stands as the analogous result of eq. (2).

3 Quantum mechanics calculation

Consider the one dimensional system of an infinite number of coupled oscillators described by the Hamiltonian (all notations are conventional)

$$H = \sum_k \frac{p_k^2}{2m} + \sum_k \frac{k}{2} (x_{k+1} - x_k)^2.$$  \hspace{1cm} (11)

$x_k$ measures the displacement of the $k$th oscillator from its equilibrium position, supposed equally spaced from the neighbored ones by distance $a$. Canonical commutations assure that the Heisenberg operators

$$x_k(t) = e^{+\frac{i}{\hbar}Ht} x_k e^{-\frac{i}{\hbar}Ht}$$

obey the classical equation

$$m \frac{d^2 x_k(t)}{dt^2} - k(x_{k+1}(t) + x_{k-1}(t) - 2x_k(t)) = 0.$$  \hspace{1cm} (13)

Let us consider the parameters $m$ and $k$ scaled such that

$$a^2 \frac{m}{k} = \frac{1}{c^2}.$$  \hspace{1cm} (14)

As familiar from wave propagation theory in elastic media, eq. (13) becomes the d’Alembert equation (3) with the correspondence

$$x_k(t) \to \varphi(ka, t),$$

$$\omega = \sqrt{\frac{k}{m}}$$

$\rho$
and letting \( a \to 0 \). \( x_k, p_m \) commutations can be also shown to translate into the equal-time field variables commutations required by canonical quantization\(^1\). One can thus identify the quantum field with the continuum limit of the quantum mechanical system.

Our interest lies in the oscillator analogy when taking into account conditions (I). It is transparent from eq. (13) that they formally amount to set in \( H \)

\[
x_0 = x_N = 0, \quad p_0 = p_N = 0,
\]
with \( N \) some natural number. In other words, the 0th and the \( N \)th oscillator are supposed fixed. As in the precedent paragraph, we shall calculate the zero point energy of the oscillators in the “box” \( 1 \leq k \leq N - 1 \).

The first step is to decouple the oscillators by diagonalizing the quadratic form in coordinates in eq. (11). Equivalently, one needs the eigenvalues \( \lambda_n \) of the \( N - 1 \) dimensional square matrix \( V_{km} \) with elements

\[
V_{k,k} = 2, \quad V_{k,k+1} = V_{k,k-1} = -1,
\]
and zero in rest. One easily checks they are

\[
\lambda_n = 4 \sin^2 \frac{n\pi}{N}, \quad n = 1, 2, \ldots N - 1,
\]
with \( \lambda_n \) corresponding to the (unnormalized) eigenvectors \( x_{n,k} = \sin \frac{nk}{N} \). It follows

\[
E_0(N, a) = \frac{\hbar c}{a} \sum_{n=1}^{N-1} \sin \frac{n\pi}{2N}.
\]

To make connection with the continuous picture, we assign to the system the length

\[
L = aN
\]
measuring the distance between the fixed oscillators, and eliminate \( N \) in favour of \( a \) and \( L \) in eq. (19). After summing the series one obtains

\[
E_0(L, a) = \frac{\hbar c}{2a} \left( \text{ctg} \frac{\pi a}{4L} - 1 \right).
\]
With an expansion similar to eq. (8)

\[ \text{ctg } z = \frac{1}{z} - \frac{z}{3} + \mathcal{O}(z^3), \quad (22) \]

it follows for \( a \ll L \)

\[ E_0(L, a) = \left( \frac{2hcL}{\pi a^2} - \frac{hc}{2a} \right) - \frac{\pi}{24} \frac{hc}{L} + \mathcal{O} \left( \frac{a}{L} \right). \quad (23) \]

The result is essentially the same with that in eq. (9). The \( a \) independent term reproduces the renormalized value (10). An identical comment applies to the \( a \to 0 \) diverging terms. Note that the \( L \to \infty \) energy density can be equally obtained by making \( N \to \infty \) in eq. (19) and evaluating the sum as an integral. Physically put, this corresponds to an infinite crystal with vibration modes characterized by a continuous quasimomentum in the Brillouin zone

\[ 0 \leq k < \frac{\pi}{a}, \quad (24) \]

and dispersion relation

\[ \omega(k) = \frac{2c}{a} \sin \frac{ka}{2}. \quad (25) \]

Note also that the second term, with no correspondent in eq. (9), can be absorbed into the first one with an irrelevant readjustment of the box length \( L \to L - \frac{\pi a}{4} \).

4 Quantum field vs oscillator model: quantitative comparison and a speculation

Let us define for \( a > 0 \) the subtracted energy \( E_0^S(L, a) \) as the difference between \( E_0(L, a) \) and the paranthesis in eq. (23), so that

\[ \lim_{a \to 0} E_0^S(L, a) = E_0(L). \quad (26) \]

One may ask when the oscillator model provides a good approximation for the quantum field, in the sense that

\[ \frac{E_0^S(L, a)}{E_0(L)} = -3 \left\{ \left( \frac{4L}{\pi a} \right) \text{ctg} \left( \frac{\pi a}{4L} \right) - \left( \frac{4L}{\pi a} \right)^2 \right\} \quad (27) \]
is close to unity. Note that by eq. (20) expression above is a function of $N$ only. The corresponding dependence is plotted in Fig.1. One sees, quite surprisingly, that already a number of around twenty oscillators suffices to assure a relative difference smaller than $10^{-4}$. More precisely, one has that the curve asymptotically approaches zero as

$$\frac{\pi^2}{240} \frac{1}{N^2}. \quad (28)$$

We end with a bit of speculation. Suppose there exists some privileged scale $l$ (say, the Plank scale) which imposes a universal bound for lengths measurements, and consider the oscillator system with the spacing given by $l$. The indeterminacy in $L$ will cause an indeterminacy in energy (we assume $L \gg l$)

$$\frac{\Delta E^S}{E^S_0} \sim \frac{\Delta E_0}{E_0} \sim \frac{l}{L}. \quad (29)$$

On the other hand, the asymptotic expression (28) implies

$$\frac{E^S_0 - E_0}{E_0} \sim \left(\frac{l}{L}\right)^2. \quad (30)$$

We are led thus to the conclusion that, as far as Casimir effect measurements are considered, one could not distinguish between the “real” quantum field and its oscillator model.

**References**

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Fig. 1. Relative difference between $E_0^S$ and $E_0$ as a function of the N-1 oscillators in the box.