Rotating regular black holes and rotating stars with a Tolman type potential as a regular interior for the Kerr metric

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We obtain a new class of singularity-free axisymmetric spacetimes by applying a slightly modified version of the Newman-Janis algorithm to a spherically symmetric seed in order to generate a rotating core that may be smoothly matched to the Kerr metric. The starting spherically symmetric configuration represents a nonisotropic de-Sitter type fluid whose radial pressure $p_r$ satisfies an state equation of the form $p_r = -\rho$, where the energy density $\rho$ is chosen to be the Tolman type-VII energy density [R. C. Tolman, Phys. Rev. 55, 364 (1939)]. The resulting rotating metric is then smoothly matched to the exterior Kerr metric, and the main properties of the obtained geometries are investigated. Depending on the relative values of the total mass $m$ and rotation parameter $a$, the resulting rotating spacetimes represent different kinds of rotating compact objects such as regular stars, regular black holes, and extreme regular black holes.

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I. INTRODUCTION

General Relativity theory developed into a powerful tool for studying the structure of massive and compact bodies and the gravitational effects generated by such objects. However, the same tool results not sufficient when trying to explain a particular property of one of the most beautiful predictions of the theory, the black holes, more specifically the singularities associated to black holes. This pathology is predicted by the singularity theorems [1], and is present in the final state of collapsing massive matter, and is inevitable if the matter satisfies some reasonable energy conditions. A simple way of characterizing a singularity is by means of the curvature scalars, whenever some of the curvature scalars diverge a gravitational singularity occurs. Curvature singularities are intrinsic features of the manifold in the sense that it is impossible to eliminate them by any change of coordinates. In order to get rid of this kind of divergences in blackhole solutions, several alternative approaches have been proposed, the result being the regular black holes. These are compact objects with horizons but free of singularities. We may say that the search for regular black holes started with the work by Bardeen [2], and with the passing of time many other models were built, see Refs. [3–14] for a small sample of regular blackhole models in general relativity, see also [15] for an interesting review and [16] for more references on the subject. In these models, the usual way to avoid singularities is by assuming a kind of matter content that violates the energy conditions, such as perfect fluids with negative pressures, so that the singularity theorems do not apply.

Most of the regular blackhole models found in the literature, including the ones mentioned in the last paragraph, are restricted to nonrotating configurations. Even though they represent idealized configurations, many interesting properties of very compact objects in general relativity have been exploited in such studies. On the other hand, observational data show that astrophysical objects are spinning. In this aspect, even though an analogous of the Birkhoff theorem for rotating spacetimes is missing, the Kerr geometry [17] is accepted as an appropriate or the most realistic among the several blackhole solutions that can be used to describe collapsed matter with angular momentum. However, the Kerr metric brings with it certain complications such as a singularity in form of a ring, causality violation and closed timelike curves (see, e.g., [18]). A simple solution to these issues is to replace the problematic region with a regular matter source in the same way as done for regular nonrotating blackhole configurations. This approach leads us to one of the most interesting problems in relativistic astrophysics, that is finding a metric which solves the Einstein equations, that can be used as interior geometry for a rotating compact object, and that can be smoothly matched to the exterior Kerr solution. In fact, many works tried to obtain an interior solution for the Kerr geometry (see e.g. Refs. [19, 20] for reviews), but several solutions reported are valid only in the slow rotation limit (see for instance [21–27]). Other proposed models for rotating compact objects in which the interior solution is smoothly matched to the Kerr metric, i.e., without any further approximation and avoiding the presence of surface layers bearing any kind of matter, are characterized by configurations presenting some nonphysical or unexpected properties for a rotating star such as negative mass [28], a singular behavior [29], or require some exotic matter such as nonisotropic fluids with negative pressure profiles [30]. More recently, some promising models have been investigated, see e.g. [31, 32] and references therein.

In general, finding an exact solution to the Einstein
equations that describes a rotating spacetime is an arduous and complicate work, that is why many people opt for the numerical approach. Nevertheless, apparently there is an alternative way by means of an appropriate complex coordinate transformation, an interesting trick revealed by Newman and Janis [33] when obtaining the Kerr metric in Boyer-Lindquist coordinates from the Schwarzschild metric. This is the Newman-Janis algorithm (NJA), an approach that has been also used to derive the Kerr-Newman geometry from the static Reissner-Nordström metric (see, e.g., [34] for more details). In this way, starting from a known regular static and spherically symmetric spacetime, the NJA generates a rotating geometry and this procedure has been used also to obtain exact solutions for regular rotating black holes, see e.g. [35–42]. Similarly to the case static regular blackhole models, these rotating regular black holes violate some of the energy conditions.

Initially the NJA was show to work well for the Kerr and Kerr-Newman vacuum solutions, but it was later adapted to generate solutions with rotating matter distributions. For instance, in [43] the Newman-Janis algorithm was applied to a generalized version of the interior Schwarzschild solution as an attempt to generate a regular source for the Kerr metric. In principle, the interior rotating solution could be matched smoothly to the Kerr metric as long as the arbitrary functions introduced satisfy suitable constraints. The resulting source is a nonisotropic fluid, but the authors conclusion is that the model investigated there is “quite imperfect to be considered as the interior Kerr metric”. In Ref. [44] the NJ algorithm was generalized to be applied to any static spherically symmetric spacetime, the smooth boundary conditions to join the interior rotation metric to the exterior Kerr metric through an oblate spheroidal surface were explicitly formulated. In [45] the NJA was applied to the interior Schwarzschild solution as in Ref. [43] and the possible matter source for the geometry in slowly rotating limit was considered. In the same work [45] a second example of rotating interior solution for Kerr metric was investigated by applying the NJA to the static spherically symmetric anisotropic fluid sphere found in Ref [46].

Following the approaches of the works mentioned in the last paragraph, the chief goal of the present work is constructing a rotating interior solution that can be smoothly matched to the exterior Kerr metric in such a way that the resulting axisymmetric spacetime is free of singularities. Our strategy is to apply the Newman-Janis algorithm to the Tolman type-VII potential [47], which is a well behaved static spherically symmetric matter distribution. The motivation for using the Tolman VII ansatz as the seed solution in the NJ procedure is that the metric is singularity-free, the potential is a continuous and well-behaved function inside of the compact object. The corresponding energy density presents a quadratic falloff profile, decreasing monotonically from the center toward the edge of the spherical distribution of matter, what represents a realistic physical model [48].

The present work is organized as follows. In Sec. II the Newman-Janis algorithm is described brief. In Sec. III we present a new rotating metric that represents rotating spheroidal compact objects, and the smoothly matching condition to the exterior Kerr metric is performed. Section IV is dedicated to the study of the main physical properties of new rotating anisotropic spheroids. In this section we investigate the matter sources of the geometries and this procedure has been used also to obtain surfaces. In Sec. V we conclude.

II. THE NEWMAN-JANIS ALGORITHM

The strategy here is to employ the Newman-Janis algorithm (NJA) [33] to generate new rotating solutions of the Einstein field equations that can be used as interior metric and matched to the exterior Kerr metric. The algorithm starts with a static, spherically symmetric spacetime ansatz,

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2, \]

where \( d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2) \) and \( f(r) \) is an arbitrary function of the radial coordinate alone.

When the starting point is a static spherically symmetric metric in Schwarzschild coordinates as (1), the NJA consists on four steps. The first step is to change the timelike coordinate \( t \) to the advanced Eddington-Finkelstein coordinate \( u \) by means of the transformation

\[ du = dt - f^{-1}(r)dr. \]

In the new \( \{u, r, \theta, \phi\} \) coordinate system, the metric (1) takes the form

\[ ds^2 = -f(r)du^2 - 2du dr + r^2d\Omega^2. \]

The second step is to write the metric (in the advanced Eddington-Finkelstein coordinate system) in terms of the lightlike (null) tetrad \( Z_{\mu}^{\nu} = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu) \), according to the Newman-Penrose formalism [49],

\[ g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu, \]

where \( l^\mu \) and \( n^\mu \) are two real lightlike vectors, \( m^\mu \) and \( \bar{m}^\mu \) are two complex lightlike vectors, with \( \bar{m}^\mu \) being the complex conjugate of \( m^\mu \). The basis vectors are normalized and satisfy the conditions \( l^\mu \ell^\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0 \), \( l^\mu n_\mu = -m^\mu \bar{m}_\mu = -1 \). For the metric (3), one finds

\[ l^\mu = \delta^\mu_t, \]

\[ n^\mu = \delta^\mu_\nu - \frac{f(r)}{2} \delta^\mu_t, \]

\[ m^\mu = \frac{1}{\sqrt{2r^2}} \left( \delta^\mu_\theta + \frac{i}{\sin\theta} \delta^\mu_\phi \right), \]

where \( \delta^\mu_\nu \) stands for the Kronecker delta tensor.
The third step is a complex transformation in the $r-u$ plane as follows

$$
egin{align*}
    r &\rightarrow r' = r + ia \cos \theta, \\
    u &\rightarrow u' = u - ia \cos \theta.
\end{align*}
$$

(6)

As it is widely known, this step introduces ambiguities in the Newman-Janis procedure. In fact, the direct substitution $r \rightarrow r'$ brings the metric potential $f(r)$ into a complex function of the coordinates $r$ and $\theta$. Since the metric potentials are required to be real functions of the coordinates, a further instruction must be furnished in order to result with a real function $F(r, \theta)$. Here we follow the type-I complexification of Ref. [37], write the function $f(r)$ as $f(r) = 1 - 2M(r)/r$, complexify the $1/r$ term without altering the mass term $M(r)$, and then take the real part of $1/r'$, as done in the original works on the Schwarzschild metric [33, 50]. With such a procedure, it follows

$$
\frac{M(r)}{r} \rightarrow \frac{M(r)}{2} \left( \frac{1}{r'} + \frac{1}{r} \right) = \frac{r M(r)}{\Sigma},
$$

(7)

where

$$
\Sigma = \Sigma(r, \theta) = r^2 + a^2 \cos^2 \theta.
$$

(8)

Therefore, the transformed $f(r)$ function gives

$$
f(r) \rightarrow F(r, \theta) = 1 - \frac{2r M(r)}{\Sigma}.
$$

(9)

After applying the complexification (6), followed by the substitution (7), the tetrad (5) takes the form

$$
\begin{align*}
    t^u &= \delta^u, \\
    n^u &= \delta_u^u - \frac{1}{2} \left( \frac{1}{r'} + \frac{1}{r} \right) \delta^u_r, \\
    m^u &= \frac{1}{\sqrt{2\Sigma}} \left( ia \sin \theta (\delta_u^\theta - \delta^\theta_u) + \delta_u^\phi + \frac{i}{\sin \theta} \delta^\phi_u \right).
\end{align*}
$$

(10)

Substituting Eq. (10) into Eq. (4), the rotating version of metric components $g^{\mu\nu}$ in the coordinates $\{u, r, \theta, \phi\}$ is found.

The fourth and final step is to change from the Eddington-Finkelstein coordinate $\{u, r, \theta, \phi\}$ back to the Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$, what is accomplished by the following transformations,

$$
du = dt - \frac{r^2 + a^2}{\Delta(r)} dr, \\
d\phi = d\phi - \frac{a}{\Delta(r)} dr,
$$

(11)

where

$$
\Delta(r) = r^2 + a^2 - 2r M(r).
$$

(12)

It is found that the rotating version of the metric (1) in the Boyer-Lindquist coordinate is

$$
\begin{align*}
    ds^2 &= -\left( 1 - \frac{2r M(r)}{\Sigma} \right) dt^2 + \Sigma \left[ dr^2 + \sin^2 \theta \left( r^2 + a^2 + \frac{2r M(r) a^2 \sin^2 \theta}{\Sigma} \right) d\phi^2 \\
    &+ \frac{4r M(r) a \sin^2 \theta}{\Sigma} dt d\phi + \Sigma d\theta^2 \right] \\
    &= -\left( 1 - \frac{2r M(r)}{\Sigma} \right) dt^2 + \Sigma \left[ dr^2 + \sin^2 \theta \left( r^2 + a^2 + \frac{2r M(r) a^2 \sin^2 \theta}{\Sigma} \right) d\phi^2 \\
    &+ \frac{4r M(r) a \sin^2 \theta}{\Sigma} dt d\phi + \Sigma d\theta^2 \right].
\end{align*}
$$

(13)

Let us mention that the a line element in this form with an arbitrary mass function $M(r)$ was found in Ref. [51].

III. GENERATING A NEW ROTATING SOLUTION

A. The interior solution

Inspired in the work by Tolman [47], we consider the metric function $f(r)$ of the static spherically symmetric spacetime (1) in the form

$$
f(r) = 1 - \frac{r^2}{R} + \frac{r^4}{A^4},
$$

(14)

where $R$ and $A$ are two arbitrary constant parameters. This is the Tolman metric potential associated to his type-VII solution. With this choice, the mass function $M(r)$ is given by

$$
M(r) \equiv M_{\text{tr}}(r) = \frac{1}{2} \left( \frac{r^3}{R^2} - \frac{r^5}{A^4} \right).
$$

(15)

The mass function (15) and the static metric (1) correspond to an anisotropic fluid whose energy density $\rho$, radial pressure $p_r$, and tangential pressure $p_t$ take respectively the forms

$$
\begin{align*}
    8\pi \rho(r) &= \frac{3}{R^2} - \frac{5r^2}{A^4}, \\
    8\pi p_r(r) &= -8\pi \rho(r) = -\left( \frac{3}{R^2} - \frac{5r^2}{A^4} \right), \\
    8\pi p_t(r) &= -\left( \frac{3}{R^2} - \frac{10r^2}{A^4} \right).
\end{align*}
$$

(16)

The main difference here is that we assume a nonisotropic fluid, while Tolman dealt just with perfect fluid models. In fact, the above solution can be viewed also as a particular case of the anisotropic spheres studied in Ref. [52].

As commented above, in the present analysis, when applying the NJA to obtain a rotating spacetime, the complexification (7) does not modify the original form of mass function $M(r)$. This strategy works perfectly well in obtaining the Kerr and Kerr-Newman rotating geometries respectively from the Schwarzschild and Reissner-Nordstöm static metrics. After applying the NJA, performing the four steps, the result is a new metric representing a rotating spacetime solution. In the Boyer-Lindquist coordinates, the resulting metric is given by the general form (13) with the mass function $M(r)$ given in Eq. (15). The main properties of this rotating solution are investigated in the next section.

B. The exterior solution

The exterior solution is the Kerr metric which, in the Boyer-Lindquist coordinates, is given by (13) with a con-
The two relations in (18) imply the constraints related to Eq. (17). See Appendix A for more details.

Eq. (15), while

\[ M(r) = M_{Kerr} = m = \text{constant}. \]

This exterior metric is to be smoothly matched to the interior metric presented in Sec. III A.

C. The matching conditions

The goal of this section is finding the smooth junction conditions between the exterior Kerr metric and the interior axisymmetric metric. For this end we employ the matching conditions of Darmois-Israel [53], see also Drake and Turolla [44] for an interesting example of application of these junction conditions to rotating spacetimes.

As a first step, we need to identify the boundary surface \( S \) that separates the interior from the exterior spacetime regions. Such a surface is chosen to be \( S : r = r_0 = \text{constant} \), which is the simplest choice also suggested in Refs. [43, 44, 51]. Hence, the interior region is defined by values of the radial Boyer-Lindquist coordinate \( r \) such that \( r < r_0 \), while the exterior region, where the Kerr metric applies, is for \( r \) in the interval \( r_0 < r < \infty \).

The next step is to assume that the transition between the two solutions is smooth, what requires that the first fundamental form on \( S \) (the projection of the metric tensor at \( r = r_0 \)) and that the second fundamental form (the extrinsic curvature tensor of \( S \)) be continuous functions across the boundary. Such conditions avoid singular energy-momentum tensor at the matching surface (for more details see, e.g., [44]). Thus, with the interior and exterior metrics given by the line element (13), the smooth matching conditions imply the mass function \( M(r) \) must be \( C^1 \). As a result of these assumptions, the boundary conditions become

\[ M_{in}(r_0) = M_{Kerr}(r_0) = m, \]

\[ \frac{dM_{in}(r)}{dr} \bigg|_{r=r_0} = \frac{dM_{Kerr}(r)}{dr} \bigg|_{r=r_0} = 0, \quad (18) \]

where \( M_{in} \) stands for the inner mass function, given by Eq. (15), while \( M_{Kerr} \) denotes the exterior mass parameter, related to Eq. (17). See Appendix A for more details. The two relations in (18) imply the constraints

\[ r_0^3 = 5m R^2, \quad A^4 = \frac{5r_0^2 R^2}{3}, \quad (19) \]

so that the complete solution is given by the line element (13) with the mass function in the form

\[ M(r) = \frac{5m r^3}{2r_0^3} \left( 1 - \frac{3r^2}{5r_0^2} \right) \Theta(r_0 - r) + m \Theta(r - r_0), \quad (20) \]

where \( \Theta(x) \) is the step (Heaviside) function.

IV. ROTATING ANISOTROPIC SPHEROIDS AND ROTATING REGULAR BLACK HOLES

A. General properties

The present model has five constant parameters, namely, \( R, A, r_0, m, \) and \( a \), and two constraints between them given by relations (19). Hence, only three of them are free parameters. Here we choose to work with \( m, a \), and \( r_0 \) as the relevant free parameters. Moreover, without loss of generality, all parameters carrying dimensions of length may be normalized by \( r_0 \). Therefore, in the numerical analysis, we take \( m/r_0, a/r_0 \) as the true free parameters of the model.

In order to investigate the properties of the solution, we study the behavior of the some curvature scalars of the spacetime metric (13), with the mass \( M(r) \) defined in Eq. (20). Looking for the existence of regular rotating black holes, we investigate the presence of horizons and ergospheres. The energy conditions are also analyzed next.

B. Energy-momentum tensor

By using the Einstein field equations we obtain the energy-momentum tensor compatible with the axisymmetric spacetime metric under study. Here, we refer Ref. [51] to verify that the matter source for the spacetime geometry (13) is an anisotropic fluid that can be cast into the form

\[ T^{\mu\nu} = \epsilon u^\mu u^\nu + p_r e_r^\mu e_r^\nu + p_\theta e_\theta^\mu e_\theta^\nu + p_\phi e_\phi^\mu e_\phi^\nu, \quad (21) \]

where \( \{ u^\mu, e_r^\mu, e_\theta^\mu, e_\phi^\mu \} \) is an orthonormal tetrad basis given by

\[ u^\mu = \frac{1}{\sqrt{\pm \Delta \Sigma}} \left( r^2 + a^2 \delta_\Sigma^\mu + a \delta_\phi^\mu, \right), \]

\[ e_r^\mu = \frac{1}{\sqrt{\pm \Delta \Sigma}} \delta_r^\mu, \quad e_\theta^\mu = \frac{1}{\sqrt{\Sigma}} \delta_\theta^\mu, \quad (22) \]

\[ e_\phi^\mu = \frac{1}{\sqrt{\Sigma \sin \theta}} \left[ a \sin^2 \theta \delta_\phi^\mu + \delta_\phi^\mu \right], \]

in which the plus (+) sign goes for positive \( \Delta(r) \), while the minus (−) sign goes for negative \( \Delta(r) \), respectively. In spacetime regions where \( \Delta(r) \) is positive, the plus sign is chosen and it results that \( u^\mu \) is a timelike vector while the remaining are spacelike vectors. In spacetime regions where \( \Delta(r) \) is negative, the minus sign has to be chosen and the vectors \( u^\mu \) and \( e_r^\mu \) change places, with \( e_\phi^\mu \) being the timelike vector of the tetrad.

The quantities \( \epsilon, p_r, p_\theta, \) and \( p_\phi \) are the energy density, the radial, and the tangential pressures of the fluid, respectively, given by the relations

\[ 8\pi \epsilon = -8\pi p_r = \frac{15m}{r_0^2} \left( 1 - \frac{r^2}{r_0^2} \right) \Theta(r_0 - r), \quad (23) \]
\[
8\pi p_0 = 8\pi p_\phi = \frac{15m}{r_0} \left[ \frac{r^2}{\Sigma} \left( 1 - \frac{r^2}{r_0^2} \right) - \left( 2 - 3\frac{\Sigma}{r_0^2} \right) \right] \Theta (r_0 - r).
\] (24)

It is straightforward to verify that the energy density \(\epsilon\) assumes only nonnegative values. For instance, in the equatorial plane (\(\theta = \pi/2\)) it assumes the maximum value \(8\pi \epsilon = 15m/r_0^3 = 3/R^2\) at \(r = 0\), and decreases monotonically as \(r\) grows, finally vanishing at the surface \(r = r_0\). For \(\theta \neq \pi/2\), \(\epsilon\) vanishes at \(r = 0\) and at the surface \(r = r_0\), being finite and positive everywhere else in the interval \(0 < r < r_0\).

The radial pressure \(p_r\) presents a similar behavior as energy density \(\epsilon\) but with opposite sign, assuming non-positive values alone.

The tangential pressures \(p_\theta\) and \(p_\phi\) are identical. In the equatorial plane, they are growing functions of \(r\), taking the negative value \(8\pi p_\theta = 8\pi p_\phi = -15m/r_0^3 = -3/R^2\) at \(r = 0\), increasing as \(r\) grows, passing through zero at \(r = r_0\sqrt{2}/2\), and then taking positive values till reaching the maximum value \(8\pi p_\theta = 8\pi p_\phi = 15m/r_0^3 = 3/R^2\) at the surface \(r = r_0\). For \(\theta \neq \pi/2\), the tangential pressures start with zero value at \(r = 0\), decrease to a minimum (negative) value, then grow back and reach zero value at an intermediate value of \(r\), attaining the maximum positive value at the boundary surface.

Other interesting conditions that a fluid is usually tested for are the standard energy conditions. Here we need to check just the weak energy condition (WEC). The WEC applied to a nonisotropic fluid implies that the fluid quantities must be restricted by \(\epsilon + p_t \geq 0\) and \(\epsilon + p_r \geq 0\) \((i = r, \theta, \phi)\) [1]. According to Eq. (23), the energy density satisfies the first constraint in the whole interval \(0 \leq r \leq r_0\). The second constraint also holds for the radial \(p_r\), i.e., one has \(\epsilon + p_r \geq 0\) for all \(r \in [0, r_0]\). On the other hand, as we see from Eq. (24), such an inequality does not hold for the tangential pressures \(p_\theta\) and \(p_\phi\). In fact, the second constraint holds in the equatorial plane but this is not the case in other planes. For \(\theta \neq \pi/2\), the constraint \(\epsilon + p_\theta \geq 0\) is satisfied in the interval \(r_b \leq r \leq r_0\), but it is violated close to \(r = 0\), in the interval \(0 < r < r_b\), where \(r_b\) is such that \(r_b^2 = a^2 \cos^2 \theta \left( \sqrt{1 + 8\pi/r_0^2/a^2 \cos^2 \theta} - 1 \right)/2\).

As it is seen, the present model of rotating compact objects violates the WEC for all \(a \neq 0\). This result agrees with Refs. [54, 55] where the authors claim that the addition of rotation into regular blackhole solutions inevitably leads to violation of the WEC, as it happens for several rotating regular black holes reported in the literature, see e.g. [37, 39–42].

C. Curvature scalars

It is well known that a spacetime presents a curvature or a physical singularity when the metric coefficients tend to infinity at a specific point, and which cannot be removed by introducing an appropriate coordinate system. This type of divergence can be located by analyzing the behavior of the relevant curvature invariants (scalars), such as the Ricci scalar \(R = g^\mu\nu R_{\mu\nu}\), the Ricci squared \(R_2 = R_{\mu\nu}R^{\mu\nu}\), and the Kretschmann scalar \(K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\). For the metric (13), these curvature scalars take, respectively, the forms

\[
\mathcal{R} = 2\left(2M' + M''\right),
\]

\[
\mathcal{R}_2 = \frac{1}{\Sigma^2} \left[ \left(8r^4 + 3a^4 \cos^4 \theta \right)M^2 + r^3 \Sigma^2 M'^2 + 4r \Sigma M' M'' a^2 \cos^2 \theta \right],
\]

\[
\mathcal{K} = \frac{4}{\Sigma^6} \left[ 384 r^6 M^2 - 192 r^4 M \Sigma (3M + r M') \left(2M' + M''\right) + \Sigma^4 (2M' + r M'')^2 + 8r^2 \Sigma^2 (27M^2 + 4r^2 M'^2 + 28r M M' + 2r^2 M M'') - 4 \Sigma^6 (12r M' + 3M^2 + 3r^2 M M'') + 7r^2 M'^2 + 2r^3 M M'' \right].
\]

In the case of the Kerr solution, \(M'(r) = 0\) implying that \(\mathcal{R}\) and \(\mathcal{R}_2\) are identically zero, but the Kretschmann scalar is not. This scalar diverges at the location where the function \(\Sigma = r^2 + a^2 \cos^2 \theta\) vanishes, i.e., at \((r, \theta) = (0, \pi/2),\) what represents a ring-shaped singularity. In other situations, since \(M'(r)\) and its derivatives are regular functions, the curvature singularities could also occur at the spacetime region where \(\Sigma(r, \theta) = 0\) as in the Kerr metric. However, a careful analysis shows that this is not the case for the metric (13) with the mass function given by Eq. (20). In fact, by approaching the region \(r = 0\) along the equatorial plane, i.e., by putting \(\theta = \pi/2\) and then taking the limit \(r \rightarrow 0\), the scalar invariants (25) reduce respectively to

\[
\lim_{r \rightarrow 0} \left( \lim_{\theta \rightarrow \pi/2} \mathcal{R} \right) = \frac{12}{R^2} = \frac{60m}{r_0^3},
\]

\[
\lim_{r \rightarrow 0} \left( \lim_{\theta \rightarrow \pi/2} \mathcal{R}_2 \right) = \frac{36}{R^4} = \frac{900m^2}{r_0^6},
\]

\[
\lim_{r \rightarrow 0} \left( \lim_{\theta \rightarrow \pi/2} \mathcal{K} \right) = \frac{24}{R^6} = \frac{600m^2}{r_0^9}.
\]

Additionally, by approaching the region \(r = 0\) from outside the equatorial plane, that is, \(r \rightarrow 0\) with \(\theta \neq \pi/2\), we have that all the curvature invariants vanish. This means that the curvature invariants have finite values at the origin of the Boyer-Lindquist coordinates, and then they are regular everywhere in the spacetime. Therefore, we conclude that, differently from the Kerr metric, the geometry generated by the Newman-Janis algorithm with the mass function (20) is free of curvature singularities.

D. Horizons

In Boyer-Lindquist coordinates, metric (13) presents horizons at the spacetime regions where the function
\[ \Delta(r) \text{ vanishes. Thus, from Eq. (12) it follows that the} \]
\[ \text{horizons radii } r_i \text{ are the real and positive roots of the} \]
\[ \text{polynomial equations} \]
\[ a^2 + r^2 - \frac{5m}{r_0} \left(1 - \frac{3r^2}{5r_0} \right) \frac{r^4}{r_0^2} = 0, \quad r < r_0, \quad (27) \]
\[ a^2 + r^2 - 2mr = 0, \quad r \geq r_0, \]
\[ \text{with the supplemental conditions that the horizons are} \]
\[ \text{determined by the solutions of the first equation when} \]
\[ r_i \leq r_0, \text{ but are given by the solutions of the second} \]
\[ \text{equation when } r_i \geq r_0. \]
\[ \text{A simple analysis shows that each one of the polynomial} \]
\[ \text{equations (27) may have two, one, or none real} \]
\[ \text{positive roots, depending on the relative values of the} \]
\[ \text{parameters } a, m, \text{ and } r_0, \text{ and recalling from Eq. (19) that} \]
\[ r_0 \text{ and } m \text{ are related by } r_0^2 = 5m R^2. \]
\[ \text{In the cases with two horizons, one of them is the event (outer) horizon, at} \]
\[ r = r_+, \text{ and the other one is a Cauchy (inner) horizon, at} \]
\[ r = r_-, \text{ the solutions are regular black holes. In the} \]
\[ \text{cases with one horizon, the horizon is extremal, i.e., it is} \]
\[ \text{a double horizon for which } r_+ = r_- \text{ and the solutions} \]
\[ \text{are extreme regular black holes. The solutions with no} \]
\[ \text{horizons are regular stars. The critical case that sep}-\]
\[ \text{arates situations with none horizon from situations with} \]
\[ \text{two horizons is the extreme case, in which } r_+ = r_- \text{ and} \]
\[ \text{Eq. (27) yields the extremal mass } m_c. \]
\[ \frac{m_c}{r_0} = \frac{1}{1000 a^2} \sqrt{5 + \frac{3a^2}{r_0^2}} \left[ 5 + \frac{27a^2}{r_0^2} \right] \]
\[ + \frac{243}{1000} \frac{a^2}{r_0^4} - \frac{25}{1000} \frac{a^2}{r_0^2} + \frac{27}{100} \quad r_c < r_0, \quad (28) \]
\[ \frac{m_c^2}{r_0^2} = \frac{a^2}{r_0^2} \quad r_c > r_0. \quad (29) \]
\[ \text{Notice that } m_c/r_0 \text{ grows with the rotation parameter } a. \]
\[ \text{Configurations with masses } m \text{ such that } 0 < m/r_0 < m_c/r_0 \text{ have no horizons being identified as regular stars.} \]
\[ \text{Configurations with masses } m \text{ such that } m/r_0 = m_c/r_0 \text{ present one extreme horizon being identified as regular} \]
\[ \text{extreme black holes. Configurations with masses } m \text{ such that} \]
\[ m/r_0 > m_c/r_0 \text{ present two horizons being identified} \]
\[ \text{as regular black holes. The extremal mass given by the} \]
\[ \text{first solution, i.e., given by (28), applies to the cases in} \]
\[ \text{which the double horizon is located in the inner region of} \]
\[ \text{the spacetime. This first solution presents a minimum} \]
\[ \text{extremal mass given by } m_c/r_0 = 12/25 = 0.48 \text{ at } a = 0. \]
\[ \text{This means that, differently from the Kerr metric,} \]
\[ \text{extremal (double horizon) black holes may occur even} \]
\[ \text{for rotation parameter smaller that unity. The second} \]
\[ \text{solution, given in (29), applies to the cases in which} \]
\[ \text{the double horizon is located in the exterior region. More} \]
\[ \text{details on this point are reported in Sec. IVF.} \]
\[ \text{Figure 1 shows the behavior of } \Delta(r) \text{ as a function of} \]
\[ r/r_0 \text{ for different values of the ratio } a/m \text{ and of the total} \]
\[ \text{mass } m/r_0 \text{ (see also Figs. 4, 3, and 5). The dotted} \]
\[ \text{lines are associated to the exterior Kerr metric, and are} \]
\[ \text{drawn for constant } M(r), \text{i.e., for } M(r) = m, \text{ while} \]
\[ \text{the solid lines correspond to the complete solution with } M(r) \text{ given by (20). As seen from the figures, the solid} \]
\[ \text{and dotted lines are the same in the vacuum region, i.e., for} \]
\[ r/r_0 \geq 1. \text{ The intersections of each solid curve with the} \]
\[ \text{horizontal axis determine the existence and the position} \]
\[ \text{of the horizons } r_- \text{ (the smallest root, that corresponds} \]
\[ \text{to the inner (Cauchy) horizon) and } r_+ \text{ (the largest root,} \]
\[ \text{that corresponds to the event horizon).} \]
\[ \text{Panel (a) of Fig. 1 displays the behavior of the horizon} \]
\[ \text{function } \Delta(r) \text{ for the case without rotation, } a/m = 0. \text{ In this} \]
\[ \text{case, the exterior metric (for } r \geq r_0) \text{ is the exterior} \]
\[ \text{Schwarzschild solution. Depending on the relative values of} \]
\[ \text{the free parameters, the complete solution (indicated by the} \]
\[ \text{solid lines) may correspond to a regular star, a regular} \]
\[ \text{extreme black hole, or a regular non-extreme black hole.} \]
\[ \text{In fact, the spacetime does not present horizons for} \]
\[ \text{masses in the interval } 0 < m/r_0 < m_c/r_0, \text{ and the} \]
\[ \text{corresponding configurations represent regular stars.} \]
\[ \text{As mentioned above, the minimum value of the extremal} \]
\[ \text{mass in order to get an extreme black hole solution is} \]
\[ m_c/r_0 = 12/25 = 0.48, \text{ and this minimum value holds} \]
\[ \text{true for the non-rotating solutions, i.e., the extremal} \]
\[ \text{mass is bounded from below by } m_c/r_0 \geq 0.48 \text{ and the} \]
\[ \text{lower bound occurs exactly in the nonrotating case. The} \]
\[ \text{solution for the extremal mass } m/r_0 = m_c/r_0 \text{ presents a} \]
\[ \text{double horizon and corresponds to a regular nonrotating} \]
\[ \text{extreme black hole. On the other hand, for larger masses,} \]
the extremal mass \( m/r_0 > m_c/r_0 \), each corresponding spacetime presents two horizons characterizing a regular nonrotating black hole.

Panel (b) of Fig. 1 displays the behavior of the horizon function \( \Delta(r) \) for the case with rotation parameter such that \( a/m = 0.6 \), and for a few representative values of the mass parameter \( m/r_0 \). The effects of rotation on the horizon radii may be seen in that panel by comparison to panel (a) of the same figure. An interesting effect is the increase in the maximum masses of solutions representing regular stars, as it is seen by comparing the three panels of Fig. 1. The extremal mass \( m_c \) grows from \( m_c/r_0 = 12/25 \) for \( a/m = 0 \) to \( m_c/r_0 \approx 0.5386 \) for \( a/m = 0.6 \), and to \( m_c/r_0 = 1.0 \) in the extremely rotating case, for which \( a/m = 1 \). Another observed effect is a decreasing of the distance between the horizons radii values with the rotation parameter while the mass parameter is kept fixed. For a given mass \( m \), the two horizons radii coalesce at some value of the rotation parameter that depends on the matching surface radius \( r_0 \) (besides on the mass parameter).

Panel (c) of Fig. 1 shows the behavior of \( \Delta(r) \) in the extremely rotating case, for which \( a/m = 1 \). In this case the extremal mass \( m_c/r_0 \) is equal to unity, \( m_c/r_0 = 1.0 \). Configurations with small masses, such that \( m/r_0 < m_c/r_0 \), have no horizons being identified as extremely spinning stars. Configurations whose masses assume the extremal value, \( m/r_0 = m_c/r_0 \), have parameters that satisfy the relationships \( a/r_0 = r_-/r_0 = r_+/r_0 = m/r_0 = 1 \). This situation represents an extremely rotating black hole with a lightlike matching, meaning that the matching surface coincides with the double horizon. Configurations with large masses, such that \( m/r_0 > m_c/r_0 \), are all regular extremely rotating black holes with timelike matching, in the sense that they present a double horizon \( r_-/r_0 = r_+/r_0 \) and where the central regular core is matched to the exterior Kerr metric at a surface located inside the Cauchy horizon, \( r_0 < r_- \). The horizons of these configurations are determined by the exterior Kerr metric, and so they have parameters that satisfy the relationships \( a/r_0 = r_-/r_0 = r_+/r_0 = m/r_0 > 1 \).

Overspinning solutions \( (a/m > 1) \) present no horizons and, of course, no black hole solutions are possible. All the configurations are regular rotating stars whose masses are unbounded.

The horizons of the present rotating spacetime solutions are also displayed in Figs. 4, 3, and 5 by dotted red lines for some values of the parameter \( a/m \). The corresponding ergosurfaces and matching surfaces are also shown in that figures by solid blue and dashed black lines, respectively.

E. Ergosurfaces and ergoregions

Other important surfaces when considering rotating solutions are the surfaces of stationary limit, the so-called ergosurfaces. In the present case, such limiting surfaces correspond to spacetime regions where the Killing vector \( \chi^\mu = (1, 0, 0, 0) \) becomes lightlike, i.e., it is such that \( g_{\mu\nu}\chi^\mu\chi^\nu = 0 \). This condition implies in \( g_{tt} = 0 \), leading to the following equations,

\[
a^2 \cos^2 \theta + r^2 - \frac{m}{r_0} \left(1 - \frac{3r^2}{5r_0^2}\right) = 0, \quad r < r_0,
\]

\[
a^2 \cos^2 \theta + r^2 - 2mr = 0, \quad r \geq r_0.
\]

As in the case of the Kerr geometry, these equations may have two real positive roots that define two ergosurfaces delimiting the ergoregions. We denote the two roots by \( r_e^+ \).

As it is seen from the two relations in (30), for \( \theta = 0 \) and \( \theta = \pi \), the ergosurfaces coincide with the horizons of the geometry. Then, it is convenient to investigate the roots of that equations for a different polar angle, here we choose \( \theta = \pi/4 \). With that choice, the dependency of the metric coefficient \( g_{tt} \) defined in Sec. III as a function of the normalized radial coordinate \( r/r_0 \) is shown in Fig. 2. Each panel is drawn for a different value of the relative rotation parameter \( a/m \) and for a few values of the total mass \( m/r_0 \), as indicated by the line labels. The conventions are the same as in Fig. 1. The intersections of the solid curves with the horizontal axis determine the positions of the ergosurfaces. Depending on the relative value of the rotation parameter \( a/m \), for sufficiently large masses \( m/r_0 \), Eq. (30) shows two real positive solutions, \( r_e^- \) (the smallest root) and \( r_e^+ \) (the largest root), corresponding respectively to the inner and outer surfaces of stationary limit.

Panel (a) of Fig. 2 confirms that the ergosurfaces for the nonrotating geometries, with \( a/m = 0 \), coincide with the horizons shown in panel (a) Fig. 1. This is an expected result since Eq. (30) reduces to Eq. (27) in the case \( a = 0 \).

The inclusion of rotation makes the surfaces of stationary limit to differ from the horizons, as verified in panels (b) and (c) of Fig. 2. In the present model, such surfaces appear for sufficiently large masses and, for a fixed rotation parameter \( a/m \) and for \( \theta \neq 0, \pi \), the separation between the radii of the inner and outer surfaces increases with the mass parameter \( m/r_0 \). For a given pair of values of the parameters \( a/m \) and \( m/r_0 \), in all situations where ergosurfaces and horizons are present, the constraints \( r_e^- \leq r_- < r_+ \leq r_e^+ \) are satisfied, as expected. For instance, in the case with \( a/m = 0.6 \) and \( m/r_0 = 0.700 \), which is plotted in panel (b) of Fig. 2, the ergosurfaces are located at \( r_e^-/r_0 \approx 0.6881 \) and \( r_e^+/r_0 \approx 1.334 \), while the corresponding horizons are located at \( r_-/r_0 \approx 0.7537 \) and at \( r_+/r_0 \approx 1.260 \), respectively.

It is easy to verify also that two ergosurfaces may be present in cases where horizons are absent. This happens for rotating stars with sufficiently high rotation parameter \( a/m \) and for masses close to the extremal mass \( m_c \), as it is clearly seen by comparing the curve for \( m/r_0 = 0.800 \) in panel (c) of Fig. 2 with the corresponding curve for the horizons, in panel (c) of Fig. 1.
The ergosurfaces of the present rotating spacetime solutions are also displayed in Figs. 4, 3, and 5 by solid blue lines for some values of the parameter $a/m$. The matching surfaces and the horizons (when present) are also shown in these figures respectively by dashed black lines and dotted red lines. A more detailed description of these figures is presented next.

F. On the causal character of the matching surface and classification of the solutions

1. On the causal character of the matching surface

After having determined the horizons and ergoregions, the study of some properties of the matching surface gets simplified, and we turn attention to this subject here. The junction between the two spacetime regions may be of three different kinds according to the causal character of the matching surface being timelike, lightlike or spacelike. Such a character may be determined by calculating the normal vector to the boundary surface $n^\nu$, whose norm is $N = \Sigma \Delta / T$ (see Appendix A). Hence, in order to determine the kind of matching, one needs to find the particular region in the parameter space where the norm $N$ changes of sign. Since the functions $\Sigma$ and $T$ are non-negative in the region of interest, the causal character of the matching surface is fully determined by the function $\Delta(r)$. It alone, and it may be positive, zero, or negative. The critical values of the parameters are calculated by the condition $\Delta(r = r_0) = 0$, what furnishes a critical mass given by

$$\frac{m_h}{r_0} = \frac{1}{2} \left( 1 + \frac{a^2}{r_0^2} \right),$$  \hspace{1cm} (31)

which represents a line in the parameter space spanned by $m/r_0$ and $a/r_0$.

Since the equation $\Delta(r) = 0$ also determines the existence (or not) of horizons, the extreme mass $m_c$ given by relations (28) and (29) is also an important quantity for the present analysis. Moreover, considering a fixed pair of parameters $r_0$ and $a$ while varying the mass parameter $m$, when the mass reaches the critical value $m_h$, the transition from the inner to the exterior region of the spacetime is made at one of the horizons, i.e., $r_0$ coincides with $r_+$, with $r_-$, or with both in the extremal case.

Note that extremal masses depend just on the ratio $a^2/r_0^2$, and that the inequality $m_c/m_h \leq 1$ holds for all values of $a/r_0$, with the equality being valid at $a^2/r_0^2 = 1$. In fact, the first solution for $m_c$ given by Eq. (28), that holds for $r_0 < r_-$, is such that the ratio $m_c/m_h$ is restricted to the interval $0.96 \leq m_c/m_h \leq 1$ for $a^2/r_0^2 \in [0, 1]$, and is restricted to $1 \geq m_c/m_h \geq 0.972$ for $a^2/r_0^2 \in [1, \infty]$. In turn, the second solution for $m_c$ given by (29), that holds for $r_0 > r_+$, is such that the ratio $m_c/m_h$ is restricted to the interval $0 \leq m_c/m_h \leq 1$ for $a^2/r_0^2 \in [0, 1]$, and is restricted to $1 \geq m_c/m_h \geq 0$ for $a^2/r_0^2 \in [1, \infty]$.

The properties of the matching surface in terms of the free parameters of the model $a/r_0$ and $m/r_0$ are considered next. The analysis is split into three distinct cases depending on the values of the normalized rotation parameter $a/m$, namely, $0 \leq a/m < 1$, $a/m = 1$, and $a/m > 1$.

2. Underspanning objects: For $0 \leq a/m < 1$

This is the richest region of the space of parameters because it presents several classes of objects that depend on the relative values of the masses in comparison to the extremal mass $m_c$ defined in Eqs. (28) and (29), and the critical mass $m_h$ defined in Eq. (31). In fact, twelve different configurations are seen in Figs. 3 and 4. For the sake of convenience we split the description of these objects into nine cases, as follows.

(i) For small masses, $0 < m/r_0 \leq 12/25 = 0.48$:

Independently of the values of the rotation parameter $a/r_0$, for masses such that $0 < m/r_0 \leq 12/25 = 0.48$ the solutions present no horizons nor ergospheres, and the corresponding configurations are regular rotating (static in the case $a = 0$) stars where the matching is done on a timelike surface. Typical examples of these
configurations are shown by the upper lines in Figs. 1 and 2, and in panel (a) of Fig. 3.

(ii) For masses in the interval 12/25 < m/r_0 < m_c/r_0:

Independently of the values of the rotation parameter a/r_0, the solutions present no horizons but ergospheres are formed. The configurations represent regular stars with timelike matching. Within this interval of masses, depending on the value of m_c/r_0 being smaller or larger than 1/2, two distinct configurations of ergospheres are found, as follows.

(ii.a) For m_c/r_0 ≤ 1/2:

This condition also means 12/25 < m/r_0 < m_c/r_0 ≤ 1/2 and implies 0 < a/r_0 < \frac{1}{9} \sqrt{(7\sqrt{7} - 10)} /3 \simeq 0.1873.

In such a case there are two disjoint ergoregions located completely inside the matter distribution, r_{c\pm} < r_0. A typical example of these configurations is shown in panel (b) of Fig. 3. In the limiting case, m/r_0 = 1/2, the external ergosurface (r_{e+}) reaches the matching surface r_0 at the equatorial plane. A typical example of these configurations is drawn in panel (a) of Fig. 4.

(ii.b) For m_c/r_0 > 1/2:

This condition also means 1/2 < m/r_0 < m_c/r_0 and \frac{1}{9} \sqrt{(7\sqrt{7} - 10)} /3 < a/r_0. In such a case the external ergosurface r_{e+} reaches the region outside matter close to the equatorial plane. A typical example of these configurations is drawn in panel (b) of Figs. 4.

Figure 3: Ergospheres, horizons, and the matching surfaces in the x–z plane, where x = (r/r_0) \sin \theta and z = (r/r_0) \cos \theta, for a/m = 0.3 and three values of m/r_0, as indicated by the labels in each plot. The solid blue lines represent the ergosurfaces, the dotted red lines are the horizons, and the dashed lines indicate the junction surfaces at r/r_0 = 1.

Figure 4: Ergospheres, horizons, and the matching surface in the x–z plane, where x = (r/r_0) \sin \theta and z = (r/r_0) \cos \theta. The solid blue lines represent the ergosurfaces, the dotted red lines are the horizons, and the dashed lines indicate the junction surfaces r/r_0 = 1. The first six plots are drawn with a/m = 0.6. The last two plots are drawn for a/m = 0.96, i.e., with a larger rotation parameter to better visualize the cases where the matching is at and inside the Cauchy horizon.
(iii) The extremal case, for \( m/r_0 = m_c/r_0 \):

In this case the two horizons coincide and the matching surface can be timelike or lightlike depending on the value of \( m_c/r_0 \) being smaller, equal or larger than 1. Three distinct configurations are found, as follows.

(iii.a) For \( m/r_0 = m_c/r_0 \) with \( m_c/r_0 < 1 \):

In the extremal case and for configurations with masses such that \( m/r_0 < 1 \), which also obeys \( a/r_0 < 1 \) and \( 0 < a/m < 1 \), the two horizons coincide and the matching surface is timelike and it is located outside the horizon, \( r_0 > r_+ = r_− \). The double horizon and the ergosurfaces coincide at the poles. The two ergosurfaces join each other at the poles forming a unique ergoregion. The exterior ergosurface is inside the matter distribution (\( r_+ < r_0 \)) close to the poles, but may it enter the region outside the matter distribution (\( r_0 > r_+ \)) close to the equator. The corresponding spacetimes are extreme regular rotating black holes. A typical case where the double horizon and the ergoregions are formed inside the matter region is shown in panel (c) of Fig. 3. A typical case of the situation where the double horizon is located inside the matter region, while the outer ergosurface enters the exterior region is shown by the curve labeled by \( m/r_0 = 0.5386 \) in panels (b) of Figs. 1 and 2, whose ergosurfaces and horizons are shown in panels (c) of Fig. (4).

(iii.b) For \( m/r_0 = m_c/r_0 \) with \( m_c/r_0 \geq 1 \):

The extremal cases with \( m/r_0 = m_c/r_0 \geq 1 \) also obey \( m/r_0 = a/r_0 \geq 1 \), the matching surface is timelike and it is located inside the double horizon, \( r_0 < r_− = r_+ \), so that Eq. (29) provides the relation between the extremal mass \( m_c \) and the rotation parameter \( a \). I.e., this kind of configurations occur for \( a/m = 1 \) and then a more detailed description is presented in Sec. IV F 3.

(iv) For intermediate masses, \( m_c/r_0 \leq m/r_0 < m_h/r_0 \):

In the case of intermediate masses, when the total mass takes values in the range \( m_c/r_0 \leq m/r_0 < m_h/r_0 \), two horizons appear and the matching is done on a timelike surface located at a radius \( r_0 \) larger than the event horizon radius, \( r_0 > r_+ \). The two horizons are placed inside the inner region and the geometries correspond to regular rotating black holes. Typical examples of this situation are shown by panels (d) of Figs. 3 and 4. For \( m_c/r_0 < m/r_0 \leq 0.5 \), the ergosurfaces are completely inside the matching surface, cf. panel (d) of Fig. 3. For \( m_c/r_0 < m/r_0 > 0.5 \), the outer ergosurfaces reaches the region outside the matching surface, cf. panel (d) of Fig. 4.

(v) For the critical mass, \( m/r_0 = m_h/r_0 \):

In this critical case the matching surface is lightlike and it is located at one of the horizons, i.e., at the event horizon \( r_0 = r_+ \), or at the Cauchy horizon \( r_0 = r_− \). The first case occurs for \( m_h/r_0 < 1 \), in which the Cauchy horizon \( r_− \) is located in the inner region of the spacetime. The second case occurs for \( m_h/r_0 > 1 \) so the matching surface coincides with the Cauchy horizon. In both cases the solutions are regular rotating black holes.

A typical example of the situation where \( r_0 = r_+ \) is shown in panel (e) of Fig. 4, while a typical case of the situation where \( r_0 = r_− \) is shown in panel (g) of Fig. 4.

(vi) For \( m/r_0 > m_h/r_0 \) with \( m_h/r_0 < 1 \):

In the case of larger masses, when \( m/r_0 > m_h/r_0 \) but with \( m_h/r_0 < 1 \), which also obeys \( 0 < a/r_0 < 1 \), the matching is done on a spacelike surface located between the two horizons, \( r_− < r_0 < r_+ \). The Cauchy horizon is located in the inner region and the solutions correspond to regular rotating black holes. A typical case of this situation is shown by the curve labeled by \( m/r_0 = 0.700 \) in panel (b) of Fig. 1 and in panel (f) of Fig. 4.

(vii) For \( m/r_0 > m_h/r_0 \) and \( m_h/r_0 > 1 \):

For the case of very large masses, when \( m/r_0 > m_h/r_0 \) and with \( m_h/r_0 > 1 \), which also means \( a/r_0 > 1 \), the matching is done on a timelike surface located inside the Cauchy horizon, \( r_0 < r_− < r_+ \). The two horizons are placed outside the matter region and the solutions correspond to regular rotating black holes. A typical case of this kind of configurations is shown in panel (h) of Fig. 4.

(viii) Further comments on some special cases:

Finally we notice that there are a few particular configurations with special characteristics that occur for small values of the rotation parameters, more specifically, for \( a/m \) in the interval \( 0 < a/m \leq 2/\sqrt{(7\sqrt{7} - 10)}/3 \approx 0.3745 \). The reason for the appearance of special cases in this region of the parameter space is related to the smallness of the extremal mass \( m_c/r_0 \), cf. Eq. (28). In fact, for configurations with rotation parameters in the interval \( 0 < a/m \leq 0.3745 \), the minimal extremal masses are small and bounded by \( 12/25 < m_c/r_0 \leq 1/2 \) so that the horizons and the ergosurfaces may be formed entirely within the matter region. A few representative configurations of this kind are shown in Fig. 3. For masses in the interval \( 0 < m/r_0 \leq 12/25 \), the solutions are similar to the cases with larger \( a/m \) commented in item (i) above, in this section, and correspond to rotating star without ergoregions. For \( 12/25 < m/r_0 < m_c/r_0 < 1/2 \), the ergosurfaces are formed entirely inside the matter region, representing regular star models similar to the cases studied in item (ii) of the last subsection. In cases with masses constrained by \( m_c/r_0 \leq m/r_0 \leq 1/2 \), the horizons and the ergoregions are formed entirely inside the matter region, this is seen in panels (c) and (d) of Fig. 3. The cases with larger masses, i.e., for \( m/r_0 > 1/2 \), are similar to the cases shown in panels (d)-(g) of Fig. 3 and we omit their descriptions here.

3. Extremely spinning objects: For \( a/m = 1 \)

Configurations satisfying the condition \( a/m = 1 \) are extremely rotating objects. Moreover, such a condition implies that the extremal and the critical
masses $m_{c}/r_{0}$ and $m_{h}/r_{0}$ given by Eqs. (28), (29) and (31), respectively, take the same value which is equal to unity i.e., $m_{c}/r_{0} = m_{h}/r_{0} = 1$. This case is conveniently separated in three disjoint classes of objects.

(i) For small masses, $0 < m/r_{0} < m_{c}/r_{0}$: The configurations with masses in the interval $0 < m/r_{0} < m_{c}/r_{0}$ present no horizons, corresponding to regular rotating stars where the matching is done on a timelike surface. The properties of the solutions in respect to the ergospheres are basically the same as in the nonextremely rotating cases with $0 < a/m < 1$, see panels (a) and (b) in Fig. 5.

(ii) For the extremal mass, $m_{0}/r_{0} = m_{c}/r_{0}$: In case that the total mass equals the extremal mass, which also means $m/r_{0} = m_{h}/r_{0}$ and $m/r_{0} = 1$, the matching is done on a lightlike surface located at the extreme (double) horizon, $r_{0} = r_{-} = r_{+}$, and the solution corresponds to a regular extreme black hole. This situation is shown by the curves with label $m/r_{0} = 1.000$ in panels (c) of Figs. 1 and 2, and the corresponding ergospheres and horizons are shown in panel (d) of Fig. 5.

(iii) For large masses, $m/r_{0} > m_{c}/r_{0}$: In cases with large masses, i.e., for masses larger than the extremal mass, $m/r_{0} > m_{c}/r_{0}$, which also means $m/r_{0} > m_{h}/r_{0}$ and $m/r_{0} > 1$, the matching is done on a timelike surface located at a radius smaller than the extreme horizon, $r_{0} < r_{-} = r_{+}$, and the solutions are regular black holes. A typical example of this situation is shown by the curves with label $m/r_{0} = 1.200$ in panels (c) of Fig. 1 and 2, and the corresponding ergospheres and horizons are shown in panel (d) of Fig. 5.

4. For $a/m > 1$

In this regime of overspinning solutions no horizons are formed, the matching is done on a timelike surface and all configurations correspond to regular rotating stars, see the bottom panels of Fig. 5. The properties of the solutions in respect to the ergospheres are basically the same as for smaller rotation parameters in the cases with no horizons. The main difference is the presence of two disjoint ergoregions. As seen from the figure, once the ergospheres are formed with the growing mass, two disjoint ergoregions appear and hold for all larger values of the mass and rotation parameters. As the ratio $a/m$ grows the ergoregions become more similar to the case of the Kerr geometry.

V. CONCLUSION

Models for rotating compact objects that may be smoothly matched to the exterior Kerr solution are obtained by applying the Newman-Janis algorithm to a static spherically symmetric metric. The static metric is obtained by using the same horizon function $q_{rr} = 1/(1 - r^{2}/R^{2} + r^{4}/A^{4})$ as the type VII exact solution by Tolman [47], representing a deformed de-Sitter type
metric potential with the energy density satisfying a quadratic profile. In the present work the deformation on the de Sitter metric is also through the inclusion of a non-isotropic fluid, with the tangential pressures being different from the radial pressure. The resulting models bear three free parameters, the mass \( m \), the rotation \( a \), and a third parameter \( r_0 \) that carries dimension of length and corresponds to the radius of the boundary surface of the matter distribution in Boyer-Lindquist coordinates. Among the possible solutions of the model we find rotating regular black holes and other configurations representing rotating regular stars. The energy-momentum tensor associated with these solutions is represented by an anisotropic fluid that violates the weak energy condition.

The analysis of the exact solution shows that there is a lower bound on the value of the normalized mass parameter \( m/cr_0 \) for spacetimes to present horizons, i.e., spacetimes with mass parameter \( m/r_0 \) such that \( m/r_0 < m/cr_0 \) present no horizons for any value of the rotation parameter \( a \), and these solutions are interpreted as rotating regular stars. For values of mass above the extremum mass \( m/r_0 > m/cr_0 \) and with \( a/m < 1 \), the corresponding geometries exhibit two horizons, representing rotating regular black holes. Since the radii of the horizons increase with the mass parameter faster than the matching surface radius, one has that the boundary surface may be timelike, lightlike, or spacelike. It is timelike in three situations. The first one is for relatively low masses when no horizons are present and the solutions are regular stars. The second is for intermediate masses when two horizons are present, the solutions are regular black holes and the matching occurs outside the event horizon. The third situations is for very large masses when there are two horizons, the solutions are also regular black holes and the matching occurs inside the Cauchy horizon. It is lightlike when the solutions are regular black holes and matching surface coincides with one of the horizons. It is spacelike when the solutions are regular black holes and the matching occurs between the two horizons.

In the present work we focused on finding the exact solutions for rotating objects and studying the main properties of such solutions, with chief interest on regular rotating black holes. The next step is to study the maximal analytic extension of the regular black hole configurations (work in progress). A further important study related to the present models is to investigate the stability of the relevant objects.

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Appendix A: The matching conditions on rotating a lightlike surface

To deal with the junction conditions in the particular case where the matching surface coincides with a horizon, we follow the strategy of Ref. [57] in which the authors adapted the Darmois-Israel formalism [53] so that it can be applied to a rotating lightlike surface. In this way, the matching condition on the horizon can be analyzed without following the strategy defined in Ref. [58], that makes use of the transverse extrinsic curvature constructed from a transverse lightlike normal vector.

In the present case, the matching parameter is the radial Boyer-Lindquist coordinate \( r \). More specifically, the matching surface is a hypersurface defined by \( S : r = r_0 = \text{constant} \). Thus, the normal vector to \( S \) is given by \( n^\mu = \delta^\mu r \sqrt{N\Delta/\Sigma} \), with \( N \) being the norm of \( n^\mu \) which is chosen to be the lapse function,

\[
N \equiv n_\mu n^\mu = \frac{\Sigma \Delta}{\Upsilon},
\]

where \( \Upsilon = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \).

Notice that, differently from the Darmois-Israel formalism, here the norm \( N \) is not unity, \( N \neq \pm 1 \). Since the functions \( T \) and \( \Sigma \) are both positive in the region of interest (\( r > 0 \)), the norm \( N \) is positive (negative) according to \( \Delta(r) \) being positive (negative), and it vanishes as a horizon (\( \Delta \to 0 \)) is approached. As we shall show explicitly below, the use of this non-unit normal vector allows to apply the matching conditions even on a lightlike surface.

The induced metric on the surface \( S \) is

\[
ds_2^2 = -N dt^2 + g_{\phi \phi} \left( d\phi + \frac{g_{\phi t}}{g_{\phi \phi}} dt \right)^2 + g_{\theta \theta} d\theta^2
\]

with the intrinsic coordinates being \( \xi^a = (t, \theta, \phi) \). The interior and exterior metrics are given by line element (13) with the mass function \( M(r) \) defined respectively by (15) for the for interior spacetime region, and by (17) for the exterior region.

Let us assume initially that the matching surface \( S \) is a timelike (or a spacelike) boundary surface, which means \( N^{(\pm)}(r_0) > 0 \) (or \( N^{(\mp)}(r_0) < 0 \)) and, consequently, \( \Delta^{(\pm)}(r_0) > 0 \) (or \( \Delta^{(\pm)}(r_0) < 0 \)). The continuity of the first fundamental form at the boundary implies in \( N^+ = N^- \), \( g^{\phi t}_{+} = g^{\phi t}_{-} \), \( g^{\phi \phi}_{+} = g^{\phi \phi}_{-} \), and \( g^{\theta \phi}_{+} = g^{\theta \phi}_{-} \). Those conditions are satisfied by the alone relation \( M_{in}(r_0) = M_{Kerr}(r_0) \), which, after using Eqs. (15) and (17), results in

\[
\frac{r_0^3}{F^2} - \frac{r_0^3}{A^2} = 2m.
\]

The extrinsic curvature components take the form

\[
K_b^{(\pm)} = -g^{ad}_{(\pm)} g^{(\pm)}_{r b d}.
\]
with the Latin indexes spanning the intrinsic coordinates on \( S \): \( a, b = t, \theta, \phi \). The non identically zero components of \( K_b^{c^{(\pm)}} \) are

\[
K_t^{(\pm)} = -\frac{\sqrt{T}}{2\Sigma} \left( \frac{\partial g_{tt}}{\partial r} - \frac{g_{\phi\phi}}{g_{\phi\phi}} \frac{\partial g_{t\phi}}{\partial r} \right) \biggr|^{(\pm)} , \tag{A6}
\]

\[
K_\phi^{(\pm)} = \frac{\sqrt{T}}{2\Sigma} \left( \frac{g_{\phi\phi}}{g_{\phi\phi}} \frac{\partial g_{t\phi}}{\partial r} - \frac{g_{\phi\phi}}{g_{\phi\phi}} \frac{\partial g_{t\phi}}{\partial r} \right) \biggr|^{(\pm)} , \tag{A7}
\]

\[
K_\theta^{(\pm)} = -\frac{|\Delta|}{2\Sigma \sqrt{T}} \frac{\partial g_{\theta\theta}}{\partial r} \biggr|^{(\pm)} , \tag{A8}
\]

\[
K_\phi^{(\pm)} = \frac{g_{\phi\phi} \sqrt{T}}{2g_{\phi\phi}} \left( \frac{\partial g_{t\phi}}{\partial r} - \frac{g_{\phi\phi}}{g_{\phi\phi}} \frac{\partial g_{t\phi}}{\partial r} \right) \biggr|^{(\pm)}, \quad + \frac{|\Delta|}{2g_{\phi\phi} \sqrt{T}} \frac{\partial g_{\phi\phi}}{\partial r} \biggr|^{(\pm)}, \tag{A9}
\]

and where the component \( K_t^{(\pm)} = g_{tt} g^{\phi\phi} K_\phi^{(\pm)} \) was not written.

The above expressions could be further simplified by noticing that the partial derivatives of \( g_{t\phi} \) and \( g_{\phi\phi} \) with respect to \( r \) may be written in terms of \( \partial g_{tt}/\partial r \) as follows,

\[
\frac{\partial g_{t\phi}}{\partial r} = -2a \sin^2 \theta \frac{\partial g_{tt}}{\partial r} , \tag{A10}
\]

\[
\frac{\partial g_{\phi\phi}}{\partial r} = 2r \sin^2 \theta + 2a^2 \sin^2 \theta \frac{\partial g_{tt}}{\partial r} , \tag{A11}
\]

with \( \partial g_{tt}/\partial r \) given by

\[
\frac{\partial g_{tt}}{\partial r} = 2r \frac{dM(r)}{dr} \frac{\Sigma}{\Sigma} + \frac{2M(r)}{\Sigma} - \frac{4r^2 M(r)}{\Sigma^2} . \tag{A12}
\]

Therefore, the smooth junction conditions of the extrinsic curvature on the surface \( S \), \( K_b^{c^{(\pm)}} = K_b^{c^{(\pm)}} \), reduce to only one condition, namely, \( \partial g_{tt}/\partial r \biggr|^{(\pm)} = \partial g_{tt}/\partial r \biggr|^{(\pm)} \), what depends on the mass function \( M(r) \) given in Eq. (20) alone. This condition implies in

\[
M_{in}(r_0) = M_{Kerr} = m , \tag{A13}
\]

\[
dM_{in} \biggr|_{r_0} = \frac{dM_{Kerr}}{dr} \biggr|_{r_0} = 0. \tag{A14}
\]

The first constraint (A13) implies in relation (A4), while the constrain (A14) leads to

\[
\frac{5r_0^2}{R^2} - \frac{3}{A^2} = 0. \tag{A15}
\]

By combining Eqs. (A4) and (A15) it follows the two relations shown in Eq. (19).

Now it is a straightforward task to verify that all the above functions are well defined in the limit \( \Delta \to 0 \). In fact, in such a limit the induced metric on \( S \) in the case \( r_0 \) coincides with an horizon is given by Eq. (A3) with \( N = 0 \), and is well defined everywhere on such a surface. Then, the continuity of the first fundamental form at the horizon \( r = r_0 = r_- , r_+ \) is fully satisfied by the constraint (A4). Additionally, the components of the extrinsic curvature given by Eqs. (A6)–(A9) are also well defined with only the component \( K_\phi^{(\pm)} \) vanishing in the limit \( \Delta \to 0 \). Then, the above analysis holds also in such a limit, and the continuity of the second fundamental form is fully satisfied by the conditions (A4) and (A15), meaning that the smooth match is well behaved even when the matching surface coincides with one of the horizons.

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space and Time* (Cambridge University Press, Cambridge, England, 1973).
[2] J. M. Bardeen, Non-singular general-relativistic gravitational collapse, in *Proceedings of GR5* (Tbilisi, URSS, 1968).
[3] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Through a black hole into a new universe!?, Phys. Lett. B 216, 272 (1989).
[4] I. G. Dymnikova, Vacuum nonsingular black hole, Gen. Relativ. Gravit. 24, 235 (1992).
[5] E. Ayón-Beato and A. García, Regular black hole in general relativity coupled to nonlinear electrodynamics, Phys. Rev. Lett. 80, 5056 (1998), arXiv:gr-qc/9911046.
[6] K. A. Bronnikov, Regular magnetic black holes and monopoles from nonlinear electrodynamics, Phys. Rev. D 63, 044005 (2001), arXiv:0006014 [gr-qc].
[7] J. P. S. Lemos, V. T. Zanchin, Regular black holes: Electrically charged solutions, Reissner-Nordström outside a de Sitter core, Phys. Rev. D 83, 124005 (2011), arXiv:1104.4790 [gr-qc].
[8] S. A. Hayward, Formation and evaporation of nonsingular black holes, Phys. Rev. Lett. 96, 031103 (2006), arXiv:gr-qc/0506126.
[9] K. A. Bronnikov and J. C. Fabris, Regular phantom black holes, Phys. Rev. Lett. 96, 251101 (2006), arXiv:gr-qc/0511109.
[10] K. A. Bronnikov, R. A. Konoplya, and A. Zhidenko, Instabilities of wormholes and regular black holes supported by a phantom scalar field, Phys. Rev. D 86,
024028 (2012), arXiv:1205.2224v3 [gr-qc].

[11] J. P. S. Lemos, V. T. Zanchin, Regular black holes: Guilloyle electrically charged solutions with a perfect fluid phantom core, Phys. Rev. D 93, 124012 (2016), arXiv:1603.07359v2 [gr-qc].

[12] A. Simpson and M. Visser, Regular black holes with asymptotically Minkowski cores, Universe 6, 8 (2019), arXiv:1911.01020 [gr-qc].

[13] T. Berry, A. Simpson, and M. Visser, Photon spheres, ISCOs, and OSCOs: Astrophysical observables for regular black holes with asymptotically Minkowski cores, Universe 7, 2 (2020), arXiv:2008.13308 [gr-qc].

[14] K. A. Bronnikov and J. C. Fabris, Regular phantom black holes Phys. Rev. Lett. 96, 251101 (2006), arXiv:gr-qc/0511109 [gr-qc].

[15] S. Ansoldi, Spherical black holes with regular center: A Review of existing models including a recent realization with Gaussian sources, arXiv:0802.0330 [gr-qc].

[16] Angel D. Masa, Enessson S. de Oliveira, and Vilson T. Zanchin, Stability of regular black holes and other compact objects with a charged de Sitter core and a surface matter layer, Phys. Rev. D 103, 104051 (2021), arXiv:2009.10948v3 [gr-qc].

[17] R. P. Kerr, Gravitational field of a spinning mass as an example of algebraically special metrics, Phys. Rev. Lett. 11, 237 (1963).

[18] S. M. Carroll, Spacetime and Geometry: An Introduction to General Relativity (Addison Wesley, London, England, 2004).

[19] A. Krasiński, Ellipsoidalspace-times, Annals Phys. 112, 22 (1978).

[20] F. Z. Majidi, Another Kerr interior solution, arXiv:1705.00584 [gr-qc].

[21] J. M. Cohen, Note on the Kerr metric and rotating masses, J. Math. Phys. 8, 1477 (1967).

[22] W. C. Hernandez, Material sources for the Kerr metric, Phys. Rev. 159, 1070 (1967).

[23] V. De La Cruz and W. Israel, Spinning shell as a source of the Kerr metric, Proc. Roy. Soc. London A 333, 1215 (1975).

[24] J. B. Hartle and K. S. Thorne, Slowly rotating relativistic stars: II. Models for neutron stars and supermassive stars, Astrophys. J. 153, 807 (1968).

[25] P. Florides, A rotating sphere as a possible source of the Kerr metric, Nuovo Cimento 13B, 1 (1973).

[26] P. Florides, A rotating spheroid as a possible source of the Kerr metric, Nuovo Cimento 25B, 251 (1975).

[27] N. Uchikata and S. Yoshida, Slowly rotating regular black holes with a charged thin shell, Phys. Rev. D 90, 064042 (2014), arXiv:1506.06478 [gr-qc].

[28] W. Israel, Source of the Kerr metric, Phys. Rev. D 2, 641 (1970).

[29] V. H. Hamuty, An “interior” of the Kerr metric, Phys. Lett. A 56, 78 (1976).

[30] P. Collas and J. K. Lawrence, Trapped null geodesics in a rotating interior metric, Gen. Relativ. Gravit. 7, 715 (1976).

[31] A. Simpson and M. Visser, The eye of the storm: A regular Kerr black hole, arXiv:2111.12329 [gr-qc].

[32] A. Simpson and M. Visser, Astrophysically viable Kerr-like spacetime – into the eye of the storm, arXiv:2112.04647 [gr-qc].

[33] E. T. Newman and A. I. Janis, Note on the Kerr spinning particle metric, J. Math. Phys. 6, 915 (1965).

[34] S. P. Drake and P. Szekeres, Uniqueness of the Newman-Janis algorithm in generating the Kerr-Newman metric, Gen. Relativ. Gravit. 32, 445 (2000), arXiv:gr-qc/0007001.

[35] A. Smajlić and E. Spallucci, “Kerr” black hole: The lord of the string, Phys. Lett. B 688, 82 (2010), arXiv:1003.3918 [hep-th].

[36] L. Modesto and P. Nicolini, Charged rotating noncommutative black holes, Phys. Rev. D 82, 104035 (2010), arXiv:1005.5605v3 [gr-qc].

[37] C. Bambi and L. Modesto, Rotating regular black holes, Phys. Lett. B 721, 329 (2013).

[38] M. Azreg-Ainou, Generating rotating regu lar black hole solutions without complexification, Phys. Rev. D 90, 064041 (2014), arXiv:1405.2569 [gr-qc].

[39] B. Toshmatov, B. Ahmedov, A. Abdujab-barov, and Z. Stuchlík, Rotating regular black hole solution, Phys. Rev. D 89, 104017 (2014), arXiv:1404.6443 [gr-qc].

[40] J. C. S. Neves and A. Saa, Regular rotating black holes and the weak energy condition, Phys. Lett. B 734, 44 (2014), arXiv:1402.2694 [gr-qc].

[41] I. Dymnikova and E. Galaktionov, Regular rotating electrically charged black holes and solitons in non-linear electrodynamics minimally coupled to gravity, Classical Quant. Grav. 32, 165015 (2015), arXiv:1510.01353 [gr-qc].

[42] S. G. Ghosh, M. Amir, and S. D. Maharaj, Ergosphere and shadow of a rotating regular black hole, Nucl. Phys. B 957, 115088 (2020), arXiv:2006.07570 [gr-qc].

[43] L. Herrera and J. Jimenez, The complexification of a nonrotating sphere: an extension of the Newman-Janis algorithm, J. Math. Phys. 23, 2339 (1982).

[44] S. P. Drake and R. Turolla, The application of the Newman-Janis algorithm in obtaining interior solutions of the Kerr metric, Classical Quant. Grav. 14, 1883 (1997), arXiv:gr-qc/9703084.

[45] S. Viaggiu, Interior Kerr solutions with the Newman-Janis algorithm starting with static physically reasonable spacetimes, Int. J. Mod. Phys. D 15, 1441 (2006), arXiv:gr-qc/0603036.

[46] B. W. Stewart, Conformally flat, anisotropic spheres in general relativity, J. Phys. A: Math. Gen. 15 2419 (1982).

[47] L. C. Tolman, Static solutions of Einstein’s field equations for spheres of fluid, Phys. Rev. 55, 364 (1939).

[48] A. M. Raghoonundun and D. W. Hobill, Possible physical realizations of the Tolman VII solution, Phys. Rev. D 92, 124005 (2015), arXiv:1506.05813v2 [gr-qc].

[49] E. Newman and R. Penrose, An approach to gravitational radiation by a method of spin coefficients, J. Math. Phys. 3, 566 (1962).

[50] E. T. Newman, R. Couch, K. Chinnapparad, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. 6, 918 (1965).

[51] M. Gurses and F. Gursey, Lorentz covariant treatment of the Kerr-Schild metric, J. Math. Phys. 16, 2985 (1975).

[52] S. D. Maharaj and M. Chasi, Equation of state for anisotropic spheres, Phys. Lett. B 330, 1723 (2006).

[53] W. Israel, Singular hypersurfaces and thin shells in general relativity, Nuovo Cimento 44B, 1 (1966); and corrections in 48B, 463 (1967).

[54] A. Burinskii, E. Elizalde, S. R. Hildebrandt, and G. Magli, Regular sources of the Kerr-Schild class for rotating and nonrotating black hole solutions, Phys. Rev. D 65, 064039 (2002), arXiv:gr-qc/0109085.
[55] R. Torres and F. Fayos, On regular rotating black holes, Gen. Relativ. Gravit. 49, 2 (2017), arXiv:1611.03654 [gr-qc].

[56] R. Penrose and R. Floyd, Extraction of rotational energy from a black hole, Nature 229, 177 (1971).

[57] P. Beltracchi, P. Gondolo and E. Mottola, Surface stress tensor and junction conditions on a rotating null horizon, Phys. Rev. D 105, 024001 (2022), arXiv:2103.05074v1 [gr-qc].

[58] C. Barrabès and W. Israel, Thin shells in general relativity and cosmology: The lightlike limit, Phys. Rev. D 43, 1129 (1991).