Persistence in a Simple Model for the Earth’s Atmosphere Temperature Fluctuations

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Abstract

The effect caused by the presence of a number of distinct time scales in a simple stochastic model for the Earth’s atmosphere temperature fluctuations is studied. The model is described by a dissipative dynamics consisting of a set of coupled stochastic evolution equations. The system shows features that resemble recent observations. In contrast to other approaches, like autoregressive models, the fluctuations of the atmosphere’s temperature depend on parameters with clear physical meaning. A reduced version of the model is constructed and its temporal autocovariance function is explicitly written.

1 Introduction

Interaction among processes with several length and time scales is common to a variety of complex systems. For instance, the long-range temporal correlations found in signals from a variety of fields can be associated with an
interplay of a number of time scales \[1, 2, 3, 4, 5\]. In particular, it is an extended belief that the persistence observed in the temperature fluctuations of the Earth’s atmosphere is a consequence of its feedback with slower dynamical components in the climate system, like the oceans and Earth’s surface \[3\]. Persistence at short time scales is related to the everyday life observation where similar weather conditions are likely to be experienced over a given region in a time scale of a few days. The existence of these short-term correlations make weather forecasting possible. The climate’s persistence is also found for larger time scales, however its characterization is a more difficult task \[6\]. In some recent experiments, the temperature records from different places around the globe have been analyzed. The observations indicate the existence of universal power-laws \(C(t) \sim t^{-r}\) describing the correlations of the temperature fluctuations around its mean seasonal value. Although there is some disagreement about the value of the exponent \(r\), it has been firmly established that the persistence in temperature fluctuations can indeed be characterized by power-law autocorrelation functions \[6\]. It has been reported by some authors that for time scales that range from \(\sim 1\) to \(\sim 25\) years, correlations measured on data from meteorologic stations placed on small islands decay with an exponent \(r \sim 0.4\), while for continental stations data are closer to \(r \sim 0.7\) \[7\]. According to other authors \[8\], the persistence is even more pronounced for the oceanic regions (being roughly characterized by a \(1/f\) noise), while in the inner continents \(r \sim 1\) (in terms of the power spectrum, a white noise at low frequencies), with a transition region in the coastal zones in which \(r \sim 0.7\).

The emergence of long-range temporal correlations is a non-trivial feature that can be used to test models of the Earth’s climate \[7\]. The understanding of the long-range temporal correlations is fundamental because they characterize the interaction among the different climate components \[8\]. There is some controversy with respect to the description made by large scale models of the atmospheric temperature variability \[8, 9, 10, 11, 6\]. The purpose of the present Letter is to introduce a conceptual stochastic model for the fluctuations of the Earth’s atmospheric temperature, or more precisely, its radiated energy (for a survey on conceptual climate models see, for instance, Imkeller and Monahan \[12\]). It will be shown that the model displays qualitative features that closely resemble observations. In contrast with other simple stochastic models of atmospheric temperature fluctuations, like autoregressive models \[13, 14\], all the parameters of the presented model have direct physical interpretation. Additionally, as it will be discussed below, the
model introduced here also reveals statistical features that are reminiscent of recent observations on the spatial structure of the climate network. Therefore, this work is intended to be a contribution towards the construction of realistic and unexpensive algorithms for Earth’s climate simulation.

The model is based on energy balance [15]. On Earth, as in other planets with a solid crust, the influx of solar radiation is balanced by the outflow from the surface and the atmosphere. In the simplest description of this process, the Earth is treated as a single point. Let us denote by \( y \) and \( x \) the global averages of the radiation emitted by the atmosphere and by the surface (oceans and land), respectively. A fraction of the Sun’s total radiation is immediately reflected back into space and another is absorbed by the atmosphere. The remainder of the flux is transmitted through the atmosphere and reaches the surface, which in turn absorbs some of the radiation and reflects the rest. The radiation absorbed by the surface is then radiated back as heat. The surface radiates the absorbed energy in the infrared (IR) region of the spectra. It turns out that the atmosphere is not transparent to IR radiation, essentially due to the presence of the so-called greenhouse gases. Let us denote by \( a \) the fraction of IR radiation absorbed by the atmosphere. A fraction \( b \) of the total radiation absorbed by the atmosphere is directed towards the surface and the rest is finally lost into space. All these considerations are put together in the well known zero dimensional energy balance model:

\[
\begin{align*}
y &= A + ax, \\
x &= B + by,
\end{align*}
\]  

where constants \( A \) and \( B \) are the net contribution made to \( y \) and \( x \) by the solar radiation flux, taking into account that some heat is removed by water evaporation from the surface. The constant \( a \) is called the infrared (IR) absorption coefficient. All the constants in Eqs. (1) are calculated by averaging over a year and over the entire Earth’s surface. Assuming a blackbody process, the average atmosphere’s temperature is given by \( y = \nu T^4 \), where \( \nu \) is the Stefan–Boltzmann constant. In spite of its simplicity, the zero dimensional energy balance model is capable of predicting with very good accuracy the mean Earth’s surface temperature. Another interesting prediction of the zero dimensional energy balance model is the increment of the mean temperature as the coupling parameters \( a \) and \( b \) grow.

In this Letter a spatially extended and time-dependent generalization of model (1) is introduced. The Letter is organized as follows: in Sec. 2 the
model is introduced and the temporal and spatial correlations are discussed numerically. A reduced version of the model is constructed and formally solved in the framework of the Langevin approach. A discussion of the statistical properties of the solution is given. Some conclusions and future directions are discussed in Sec. 3.

2 The Model

Model (1) is generalized by the assumption that energy balance is satisfied locally and a transient time is necessary in order to achieve a stationary state. A set of $N$ cells is considered. In each cell, atmosphere interacts with the surface through the local atmosphere’s IR absorption coefficient $a_n$ and the local fraction of heat that the atmosphere returns towards the surface, $b_n$. Each component, atmosphere and surface, has its own intrinsic local response time. The cells of each component interchange radiation via a diffusive process. The model is written as

$$
\dot{y}_n = d_1 \Delta y_n - \lambda_n [y_n - (A_n + a_n x_n)] + \varepsilon_n(t),
$$

$$
\dot{x}_n = d_2 \Delta x_n - \gamma_n [x_n - (B_n + b_n y_n)].
$$

In this equation $(\lambda_n)^{-1}$ and $(\gamma_n)^{-1}$ are the local response times of the atmosphere and the surface, respectively. The symbol $\Delta$ is the discrete Laplacian and $d_1$, $d_2$ represent the diffusion coefficients of each component. The term $\varepsilon_n(t)$ is a Gaussian white noise, without correlations between different cells. The meaning of the rest of the terms follow from the zero dimensional energy balance model (1). In particular, $y_n$ gives the radiation emitted by the atmosphere in the site $n$ at a given time. The atmosphere is expected to have shorter intrinsic response times than that of the surface. The radiation emitted by the surface at time $t$ in cell $n$ is represented by $x_n$. The noise reflects the more rapid variations, or weather. Periodic boundary conditions are taken. The constants $\lambda_n$, $\gamma_n$, $A_n$, $B_n$, $a_n$ and $b_n$ are assumed to be independent variables, such that averages over index (in the limit $N \to \infty$) give the corresponding values of the parameters of the zero dimensional energy balance model, $A$, $B$, $a$, $b$; and the effective inverse response times for the atmosphere and the surface. The first important thing to notice with model (2) is that it recovers the zero dimensional energy balance model. This can be seen by averaging Eq. (2) over the cells in the limit $N \to \infty$. The following
reduced version of Eq. (2) is obtained:

\[
\dot{y} = -\lambda[y - (A + ax)] + \varepsilon(t), \\
\dot{x} = -\gamma[x - (B + by)].
\]

(3)

From the fact that Eq. (3) represents an overdamped dynamics in a parabolic potential, the system converges to a stationary state with mean value given by the solution of the zero dimensional energy balance model. This result shows that model (2) is capable of representing with good approximation the mean behavior of energy in the coupled atmosphere – surface system. Now it will be argued that Eq. (2) can give realistic descriptions of temperature fluctuations as well. A large variability over the intrinsic characteristic times of the surface around the globe is expected to exist, as a consequence of the different response times present in the geosphere, hydrosphere, cryosphere and biosphere. At first instance, this situation is modeled by treating the \(\gamma_n\)'s like independent random variables taken from a uniform probability distribution. On the other hand, under the basis of the relative homogeneity of the atmosphere composition, it will be assumed by now that the response time to perturbations of the atmosphere is the same in all cells. In what follows, the value \(\lambda_n = \lambda = 1\) is used. Under these assumptions, the time unit is defined as the mean atmospheric response time. The model represents the interaction between a hierarchy of time scales, ranging from minutes to days to geological times. From this point of view, it would be therefore reasonable to assume that \(1/\lambda\) lies in the intermediate scales, roughly in the range from weeks to months. This and other important aspects about the definition of the parameters in the model are intended to be refined by the author in the near future by close cooperation with climate experts. At this point the main goal is limited to explore the capabilities of the model (2), in order to give qualitatively realistic statistical descriptions of the temperature fluctuations present in the Earth’s atmosphere.

Figure 1(a) is a log–log plot of the power spectrum of the time series of the atmosphere’s temperature in a particular cell. This time series is obtained from the corresponding time series for \(y_n(t)\) under the assumption of a blackbody process. The parameters \(a_n, \ b_n, \ A_n\) and \(B_n\) are left constant and set to their experimental global averages: \(a_n = a = 0.96, \ b_n = b = 0.61, \ A_n = A = 179.36\ \text{W/m}^2\) and \(B_n = B = 47.82\ \text{W/m}^2\). The values \(\gamma_n\) are drawn from a uniform distribution on the range \((0, 0.2)\). The noise values \(\varepsilon_n(t)\) are uniformly distributed over the interval \((-1, 1)\)\(W/m^2\). The diffusion
coefficients are taken as $d_1 = d_2 = 1$. System size is $N = 50$. The power spectrum is consistent with a power law at low frequencies, with exponent $\sim -0.25$. This exponent of the power spectrum implies a power-law decay of correlations at large times, $C(t) \propto t^{-r}$, with $r \sim 0.75$. An alternative approach is given in Fig. 1(b). In order to characterize the correlations, the sum of the temperature values on time at a given cell is studied, resulting in:

$$Y_n(t) = \sum_{\tau=1}^{t} T_n(\tau),$$

where $T_n(\tau)$ stands for the atmosphere’s temperature at cell $n$ and time $\tau$. The signal $Y_n(t)$ is then compared to a random walk. In Fig. 1(b) the standard deviation of $Y_n(t)$ as a function of time is plotted. For large times (roughly greater than 50 time units), $\sigma \sim t^{0.648}$, which implies a power-law decay of the autocorrelation function at large times, with exponent $\sim 0.7$. This result is consistent with Fig. 1(a).

An analysis of the radiation spectrum has been carried out for the same experimental setup as above. Results indicate that radiation and temperature spectra are basically equivalent, differing at most by a normalization factor. This numerical finding is consistent with recent studies that suggest that a strongly correlated signal preserves its correlation properties after an even polynomial transformation [17].

The model is also capable of showing scaling behavior for surface temperature. As an example, the power spectrum of surface temperature fluctuations of an individual cell is shown in Fig. 2. The system has the same parameter values as before, but $d_2 = 0$. Notice that with this choice of parameters, the coupling with the atmosphere is essential for the emergence of scaling in the surface temperature.

The model displays an interesting spatial structure. In Fig. 3 a case is considered in which the system size is $N = 200$ and the other parameters are the same as in the case presented in Fig. 1. The spatial autocorrelation function $C(n)$ is inferred from the power spectrum of the temperature values vector at a fixed time. A least-square fit of the power spectrum to a power-law function indicates that $S(k) \sim k^{-0.22}$, which implies that the spatial autocorrelation function can be roughly characterized by $C(n) \sim n^{-0.78}$ for spatial scales in the range from one to 200 cells. Further numerical analysis of the same model setup indicates that if larger spatial scales are considered, the power spectrum displays a crossover to white noise at low frequencies.
Figure 1: (a): Log–log plot of the power spectrum of a temperature signal generated by the term $y_3(t)$ of model (2). The parameters are as discussed in the text. The power spectrum is consistent with the power-law $S(f) \sim f^{-0.25}$. This indicates a power-law decay of the autocorrelation function given by $C(t) \sim t^{-0.75}$ for time scales between five and 1000 time units. (b): An alternative way to estimate the autocorrelation function for the same situation as in (a). The standard deviation as a function of time of the sum of the temperature signal is plotted in a log–log graph. The graph shows a clear difference with respect to the behavior expected from a random walk. The standard deviation is consistent with $\sigma(t) \sim t^{-0.65}$, which indicates $C(t) \sim t^{-0.7}$ for time scales up to 1000 time units.

Figure 2: Log–log plot of the power spectrum of a temperature signal generated by the term $x_3(t)$ of model (2). The parameters are the same as in Fig. 1, except for $d_2 = 0$. A power-law $S(f) \propto f^{-0.6}$ is plotted for comparison.
Figure 3: Log–log plot of the power spectrum of the spatial vector of temperature values vector at a fixed time. The parameters are as discussed in the text. For this situation, in which there is no local variability in the parameters (besides \( \gamma_n \)), the power spectrum indicates a power-law decay of the spatial autocorrelation function for scales in the range from one to 200 cells, characterized by \( C(n) \sim n^{-0.78} \).

In a more realistic description, the parameters (besides \( \gamma_n \)) must have some local variability due, for instance, to differences in the Earth’s albedo and in the solar radiation flux over different regions. As an example, a case in which \( A_n = A + \epsilon_n \) and \( B_n = B + \epsilon'_n \) is discussed in Fig. 4(a). The \( \epsilon \)'s are independent random variables uniformly distributed in the range \((-1, 1)\) \( W/m^2 \). The other parameters are chosen as before. The power spectrum presented in Fig. 4(a) indicates a crossover between two different scaling regimes. For scales from 50 to 1000 cells the correlation function is consistent with \( C(n) \sim n^{-0.75} \), while for shorter scales \( C(n) \sim n^{-0.2} \). A situation in which the coupling parameters have also local variability is presented in Fig. 4(b). The values of \( b_n \) are set as \( b_n = b + \rho_n \), where \( \rho_n \) is uniformly distributed over the interval \((-0.15, 0.15)\). The parameters \( a_n \) are taken as \( a_n = 0.98 \) for all \( n \), which implies a stronger mean coupling. The other parameters are the same as in Fig. 4(a). In Fig. 4(b) the correlation decay is faster for the short scales and slower for the large scales in comparison to Fig. 4(a). This effect can be interpreted as an increment of large scale coherence as the mean coupling grows, while the spatial coherence at short scales decreases due to the increment in the local variability. The scaling of the spatial autocorrelation function displayed by the model is reminiscent of recent observations on the spatial structure of the climate network, which indicate that nodes in
the climate system conform to a network with the small-world property \[16\]. This property is related to the presence of significant correlations between distant nodes.

![Graph](image)

Figure 4: This situation is similar to Fig. 3 except from the fact that some parameters besides the $\gamma_n$’s have local variability. In (b) the local variability and the mean coupling parameter $a$ have larger values than in (a).

The numerical findings strongly suggest that model (2) shows qualitative features that are close to the observations. However, a more precise definition of the parameters is needed. For instance, a different choice of the scale separation between the $\gamma$’s and $\lambda$ in general lead to different properties of the autocorrelations. Another aspect to be refined is concerned with the already mentioned spatial variability of the parameters. This question is closely related to the definition of the size associated to cells. In a realistic model setup, the parameter values come from spatial averages over the region $n$. In the simple $1d+1$ situation discussed here, those would be global averages over a given latitudinal interval. As already mentioned, these and other relevant questions are intended to be investigated by the author in the near future, working in close contact with climate experts.

In order to gain insight into model (2) it is discussed analytically its reduced version given by Eq. \[2\]. Without loss of generality, the constants $A$ and $B$ are chosen equal to zero. The term $\varepsilon(t)$ is a Gaussian white noise, defined through the moments $\langle \varepsilon(t) \rangle = 0$, $\langle \varepsilon(t)\varepsilon(t') \rangle = D\delta(t - t')$ and with all higher moments equal to zero. The diffusion constant is a parameter that measures the strength of the noise. The function $\delta(t - t')$ is a Dirac’s delta. In the absence of coupling $x$ and $y$ simply converge exponentially to
the stationary state $\langle y \rangle = x = 0$, $\langle y^2 \rangle = \frac{D}{2K}$ with characteristic times $\frac{1}{\lambda}$ and $\frac{1}{\gamma}$. It is assumed that $\lambda > \gamma$ so one of the dynamics is fast and the other is slow. Strictly speaking, the system has three time scales, the third one being associated with the noise. However this time scale has an infinite separation with respect to the other two. In the language of control theory, we can view $y$ like an output system with uncertainties (noise) that has feedback with an input whose response time is different from the time scale of $y$. Applying a Laplace transform over time to Eq. (3) and assuming for simplicity the initial conditions $x(0) = y(0) = 0$, one gets

\begin{align*}
sy(s) &= -\lambda y(s) + \lambda ax(s) + \varepsilon(s), \\
sx(s) &= -\gamma x(s) + \gamma by(s).
\end{align*}

Solving Eq. (5) for $y(s)$ and $x(s)$, and by the use of the Faltung theorem, a formal solution for $y(t)$ in terms of the noise is found. This solution can be used to write explicitly the covariance function, that describes the fluctuations around the mean value.\footnote{This covariance function can also be derived from a general initial value problem for a linear system of stochastic differential equations of arbitrary dimension.} For large times, the covariance function is given by the following expression:

\begin{equation}
\langle y(t)y(t+T) \rangle = q_1 e^{\mu_1 T} + q_2 e^{\mu_2 T},
\end{equation}

where

\begin{align*}
q_1 &= \frac{D}{16K^2} (\lambda - \gamma + 2K) \left( \frac{\lambda - \gamma + 2K}{\lambda + \gamma + 2K} + \frac{\gamma - \lambda + 2K}{\gamma + \lambda} \right), \\
q_2 &= \frac{D}{16K^2} (\gamma - \lambda + 2K) \left( \frac{\gamma - \lambda + 2K}{\gamma + \lambda - 2K} + \frac{\lambda - \gamma + 2K}{\gamma + \lambda} \right),
\end{align*}

\begin{align*}
\mu_1 &= -\gamma + \frac{\lambda}{2} - K, \\
\mu_2 &= -\gamma + \frac{\lambda}{2} + K,
\end{align*}
\[ K = \sqrt{\left(\frac{\gamma + \lambda}{2}\right)^2 - \gamma \lambda (1 - ab)}. \quad (11) \]

In the region of interest of the parameter space, correlations decay monotonically with a characteristic time

\[ \tau = \frac{1}{\left(\frac{\gamma + \lambda}{2}\right) - \sqrt{\left(\frac{\gamma + \lambda}{2}\right)^2 - \gamma \lambda (1 - ab)}}. \quad (12) \]

It must be realized that \( \tau \) is greater than any of the two intrinsic times. Therefore, because of the feedback there is an emergence of large memory. The characteristic time becomes infinitely large when \( \gamma \rightarrow 0 \) with \( \lambda \) finite. This result is consistent with previous works on reduced models of temperature fluctuations [18].

Figure 5: Log–log plot of the covariance function Eq. (6) with parameters as discussed in the text. The covariance cannot be fitted to a single exponential for a time interval greater than any of the two intrinsic times.

The reduced model typically shows a region in which none of the two intrinsic time scales is dominant and correlations cannot be adequately fitted by a single exponential. For instance, with \( \gamma = 0.2, \lambda = 1, a = 0.96, b = 0.64 \) and \( D = 1(\frac{W}{\alpha})^2 \), the covariance function is not exponential for time scales approximately an order of magnitude greater than the intrinsic time of the fast variable, as Fig. 5 shows. Approximation to a power-law or other types of slow decay by a sum of exponentials with different characteristic times
has been discussed in several fields [19, 20]. In particular, this mechanism has been already proposed in [14], in order to explain the persistence found in the atmosphere’s temperature record, by fitting the coefficients of a 3d AR(1) type process to data.

3 Conclusions

The features shown by the spatially extended stochastic model presented here motivates the construction of realistic and simple algorithms for the prediction of the Earth’s temperature distribution and fluctuations. In this spatially extended situation the parameters vary locally, so there is a number of characteristic times. In order to gain insight on the extended model, a reduced version of it has been constructed and the covariance function explicitly written.

One of the future directions of the work is to conduct a more general study of the spatially extended model, in close connection to climate research to have plausible parameter values. A study of the presented model in the context of general systems with several time scales is also intended. The study of such systems is important in fields like control theory, inhomogeneous media and predator–prey systems among others [1, 2, 3, 4, 5].

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