A Variational Approach in the Dissipative Nonlinear Schrödinger Equation

Dagoberto S Freitas∗
Departamento de Física, Universidade Estadual de Feira de Santana, 44031-460, Feira de Santana, BA, Brazil.

Jairo R de Oliveira†
Departamento de Física e Matemática, Universidade Federal Rural de Pernambuco, 52171-030, Recife, PE, Brazil and Departamento de Física, Universidade Federal de Pernambuco, 50670-910, Recife, PE, Brazil.

(November 11, 2018)

*dfreitas@uefs.br
†jrocha@lftc.ufpe.br
Abstract

The properties of pulse propagation in a nonlinear fiber including linear damped term added in the usual nonlinear Schrödinger equation is analyzed analytically. We apply variational modified approach based on the lagrangian that describe the dynamic of system and with a trial function we obtain a solution which is more accuracy when compared with a pertubative solution. As a result, the problem of pulse propagation in a fiber with loss can be described in good agreement with exact results.

In recent years the propagation of optical pulses in fibers has obtained a great attention not only from the theoretical as well as from the experimental point of view. The nonlinear Schrödinger equation (NLSE) has been employed to explain a variety of effects in the propagation of optical pulses. As is well known the balance between the self-phase modulation (SPM) and group velocity dispersion (GVD) leads to the so called solitons solutions for the NLSE \[1,2\]. Solitary wave solutions have been known to exist in a variety of nonlinear and dispersive media for many years. In the context of optical communications, Hasegawa and Tappent \[3\] first made the important observation that a pulse propagating in an optical fiber with Kerr-law nonlinearity can form an envelope soliton. This offered the potential for undistorted pulse transmission over very long distances. Just as a balance between self-phase-modulation and group-velocity dispersion can lead to the formation of temporal solitons in single-mode fibers, it is also possible to have the analogous spatial soliton, where diffraction and self-focusing can compensate for each other \[4\]. The importance of studying optical solitons is from the fact that their have potential applications in optical transmission and all-optical processing. A soliton is a particular solution of the nonlinear-wave equation. Since analytical solution are known for only a few cases, investigations into the properties solutions are normally performed numerically using such approaches. However, it is often desirable to have an analytical model describing the dynamics of pulse propagation in a fiber.
In the theoretical treatment of these problems, considerable attention has been given to the variational approach \([4,6]\). A variational approach was employed in \([6]\) deriving information about the various parameters that characterize the beam, which are qualitatively as well as quantitatively, in good agreement with numerical results. This result invalidates the possibility of pulse compression without external gratings which is erroneous and is only an artifact of the paraxial approximation. In the same sense Anderson \([7]\) described the main characteristics of the temporal soliton as determined by NLSE. The discussion above does not consider the presence of the loss in the medium. It is well known that in real materials, the medium will not be purely transparent and the nonlinearity will not be of pure Kerr-law form, but will saturate. The problem of describing the physical properties of dissipative systems has been the subject of lengthily discussions \([14,15]\). These results were recently applied to the problem of propagation of cw (continuous wave) Gaussian beams in a saturable medium with loss \([13]\). In that work \([13]\) the diffraction is limited to one transverse solution. After that this problem was analyzed using a variational modified approach \([16]\).

In this paper, we will analyze the dynamics interplay between nonlinearity and dispersion through optical medium with loss using a variational modified approach \([16]\). Exact analytical expressions for the behavior of the pulse are determined.

The starting point of our analysis is the Nonlinear Schrödinger Equation that describes the propagation of a pulse envelope in a nonlinear loss medium,

\[
i \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + u |u|^2 = -i \Gamma u
\]  

(1)

where \(u(\zeta, \tau)\) is the normalized amplitude of the pulse, \(\xi\) is the normalized coordinate, \(\tau\) is the normalized time, and \(\Gamma\) is the normalized loss parameter of the medium.

Now we can handle Eq. (1) adequately in the form,

\[
\frac{\partial}{\partial \tau} \frac{\partial}{\partial u^*_\zeta} \left(e^{i \xi L}\right) + \frac{\partial}{\partial \xi} \frac{\partial}{\partial u^*_\zeta} \left(e^{i \xi L}\right) - \frac{\partial}{\partial u} \left(e^{i \xi L}\right) = 0
\]

(2)

where

\[
L = \frac{1}{2} \frac{\partial u}{\partial \tau}^2 + i(u \frac{\partial u^*}{\partial \xi} - u^* \frac{\partial u}{\partial \xi}) - \frac{1}{2} |u|^4
\]
and $u^*$ is complex conjugate of $u$ and subindexes are the differentiation with respect to $\tau$ and $\xi$. $L$ is the lagrangian of system without loss. The Eq. (2) is the Euler-Lagrange equation in the modified form that describe the propagation of the pulse in the medium with loss, and can be written in the form of the modified Hamilton’s principle \[15,16\],

$$
\delta \int_0^\infty \int_0^\infty e^{\Gamma \xi} L d\xi d\tau = 0 \tag{3}
$$

Assuming a trial functional of the form

$$
u (\xi, \tau) = A (\xi) \text{sech} \left( \frac{\tau}{w (\xi)} \right) \exp (i\phi (\xi)) , \tag{4}
$$

where $A$ is the amplitude of the pulse propagated, $w$ is the width and $\phi$ phase term. Using Eq. (4) into the variational formulation, Eq. (3), we can integrate the $\tau$ dependence explicit to obtain

$$
\delta \int_0^\infty e^{\Gamma \xi} \langle L \rangle d\xi = 0 , \tag{5}
$$

where

$$
\langle L \rangle = \frac{|A|^2}{3w} + 2iw \left( A \frac{dA^*}{d\xi} - A^* \frac{dA}{d\xi} \right) + 4w |A|^2 \frac{d\phi}{d\xi} - 2w |A|^4 \frac{3}{3} \tag{6}
$$
is the average of $L$ in the time. Then, from the standard calculus, deriving $e^{\Gamma \xi} \langle L \rangle$ with respect to $A$, $A^*$, $w$ and $\phi$ we obtain the following system of coupled ordinary differential equations

$$
\frac{d}{d\xi} \left( w |A|^2 \right) = -\Gamma w |A|^2 \tag{7}
$$

$$
w^2 |A|^2 = 1 \tag{8}
$$

$$
8w |A|^2 \frac{d\phi}{d\xi} = \frac{8w |A|^4}{3} - \frac{2 |A|^2}{3w} - 4iw \left( A \frac{dA^*}{d\xi} - A^* \frac{dA}{d\xi} \right) . \tag{9}
$$

The equations above describe the characteristics of the pulse and solving these equations we will obtain the full dynamics of the pulse through the medium. It is obvious that once
Eq. (7) and Eq. (8) are solved for $w$ and $|A|^2$, the phase $\phi$ is easily obtained from Eq. (9). In particular, if the longitudinal phase of the amplitude $A$ is introduced by writing $A = |A| e^{i\theta(\xi)}$ the Eq. (9) can be written as

$$\frac{d}{d\xi} (\phi + \theta) = \frac{1}{4w^2}.$$  

(10)

from it we obtain

$$\phi (\xi) + \theta (\xi) = \frac{1}{8\Gamma w (0)^2} \left( 1 - e^{-2\Gamma \xi} \right).$$

(11)

The equation above describe the regularized phase of the pulse. This system of equation has analytic solution. From Eq. (7) we obtain

$$w (\xi) |A (\xi)|^2 = w (0) |A (0)|^2 e^{-\Gamma \xi}.$$  

(12)

The compatibility of Eqs. (12) and (8) is possible when

$$|A (\xi)| = |A (0)| e^{-\Gamma \xi}$$

(13)

and

$$w (\xi) = w (0) e^{\Gamma \xi},$$

(14)

where was used the relation $|A (0)|^2 = 1/w^2 (0)$, with $A (0)$ and $\omega (0)$ is the initial amplitude and width of pulse, respectively. Now we can write the amplitude

$$A (\xi) = |A (0)| e^{-\Gamma \xi} e^{i\theta(\xi)}.$$  

(15)

Using the result above into the trial functional, Eq. (4), we can write the pulse in form

$$u (\xi, \tau) = |A (0)| e^{-\Gamma \xi} \sec \left( \frac{\tau}{w (\xi)} \right) \exp \left[ i (\phi (\xi) + \theta (\xi)) \right],$$

(16)

where the regularized phase $\phi (\xi) + \theta (\xi)$ is given by Eq. (11) and width $w (\xi)$ by Eq. (14).

As would expect, the fiber loss is detrimental simply because the peak power decreases exponentially with the fiber length. As a result, the pulse width of the fundamental soliton
also increase with propagation, as seen in the figure. However, these results are qualitatively better than the results obtained by using the inverse scattering method where $\Gamma$ is treated as a weak perturbation. The our results foresee that the amplitude as well as the width of the pulse suffer a smaller effect of the fiber loss that thought, and approximate more of exact numerical solution by a factor of 2 in the exponent of the exponentials $[3, 18, 19]$.

In conclusion, the propagation of a pulse in a nonlinear loss medium has been analysed using a variational modified approach. This modified approach describes in a more consistent way the behavior of pulse in a dissipative system. The our results are more accuracy when compared with a pertubative solution where $\Gamma$ is treated as a weak perturbation.

ACKNOWLEDGMENTS

One of us (J.R.O) thanks the financial support by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.
REFERENCES

[1] G. P. Agrawal, Nonlinear Fiber Optics (Academic, San Diego, 1989)

[2] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118(1971) [Sov. Phys. JETP 34, 62 (1972)].

[3] A. Hasegawa and F. Tappert, Appl. Phys. Lett. 23, 142 (1973).

[4] W. J. Firth, Opt. Commun. 22, 226 (1977).

[5] D. Anderson, M. Bonnedal and M. Lisak, Phys. Fluids. 22, 1838 (1979).

[6] M. Karlsson, D. Anderson, M. Desaix and M. Lisak, Opt. Lett. 16, 1973 (1991).

[7] D. Anderson, Phys. Rev. A27, 3135 (1983).

[8] R. Y. Chiao, E. Garmire, and C. H. Yownes, Phys. Rev. Lett. 13, 479 (1964).

[9] C. E. Max, Phys. Fluids. 19, 74 (1976).

[10] M. S. Sodha and V. K. Tripathi, Phys. Rev. A16, 201 (1977).

[11] J. T. Manassah, P. L. Baldeck and R. R. Alfano, Opt. Lett. 13, 1090 (1988).

[12] J. T. Manassah, P. L. Baldeck and R. R. Alfano, Opt. Lett. 13, 589 (1988).

[13] Z. Jovanoski and R. A. Sammut, Phys. Rev. E50, 4087 (1994).

[14] J. R. Ray, Am. J. Phys. 47, 626 (1979).

[15] L. Herrera, L. Núñez, A. Patiño and H. Rago, Am. J. Phys. 54, 273 (1986).

[16] D. S. Freitas, J. R. de Oliveira and M. A. de Moura, J. Phys. A: Math. Gen. 30 (1997).

[17] H. Kogelnik, T. Li, Appl. Opt.5, 1550 (1966).

[18] J. Satsuma and N. Yajima, Prog. Theor. Phys. Suppl. 55, 284 (1974).

[19] G. P. Agrawal, Nonlinear Fiber Optics (Academic, San Diego, 1989) [see ch. 5.4]
FIG. 1. Plot illustrate the pulse propagation for $\xi = 0$, 2 and 4, taking $\Gamma = 0.035$