Radiation hydrodynamics in Kerr space–time: equations without coordinate singularity at the event horizon

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ABSTRACT
Equations of fully general relativistic radiation hydrodynamics around a rotating black hole are derived by using the Kerr–Schild coordinate where there is no coordinate singularity at the event horizon. Since the radiation interacts with matter moving with relativistic velocities near the event horizon, the interplay between the radiation and the matter should be described fully relativistically. In the formalism used in this study, while the interactions between matter and radiation are introduced in the comoving frame, the equations and the derivatives for the description of the global evolution of both the matter and the radiation are given in the Kerr–Schild frame which is a frame fixed to the coordinate describing the central black hole. As a frame fixed to the coordinate, we use the locally non-rotating reference frame representing a radially falling frame when the Kerr–Schild coordinate is used. Around the rotating black hole, both the matter and the radiation are affected by the frame-dragging effects.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – radiative transfer – relativity.

1 INTRODUCTION
It is widely believed that the accretion flow on to black holes plays one of the essential roles in the active phenomena in the Universe. Relativistic effects in accretion discs around black holes drive some major activities of astrophysical black holes, such as active galactic nuclei, Galactic black hole candidates and possibly gamma-ray bursts. For mass accretion rates near or over the Eddington mass accretion rate, the interactions between radiation and matter in the accretion disc are important. While for the supercritical accretion flows considered in e.g. Seyfert galaxies or black hole binaries the photons interact with matter, for the hypercritical accretion flows considered in the central engine of gamma-ray bursts the neutrinos interact with matter around the central black hole. In such situations, the dynamics and the energy balance in matter and radiation are affected by each other.

So far, general relativistic radiative transfer has been investigated by many authors (e.g. Lindquist 1966; Anderson & Spiegel 1972; Schmid-Burgk 1978; Thorne 1981; Schindler 1988; Turolla & Nobili 1988; Anile & Romano 1992; Cardall & Mezzacappa 2003; Park 2006). Lindquist (1966) gives a general treatment of the radiation transfer equation and the radiation hydrodynamic equations derived by using a comoving Lagrangian frame of reference. Thorne (1981) derives general relativistic moment equations up to an arbitrary order by introducing projected symmetric trace-free tensors. Since these formalisms are based on the comoving frames, the physical quantities in terms of matter and radiation and the directional derivatives are represented in the comoving frame. In some astrophysical objects, the radiation interacts with matter moving at relativistic velocities. In such cases, the interaction between the radiation and the matter should be described fully relativistically. This is easily done if both radiation field and matter are evaluated in the comoving frame which is a frame where the element of matter is at rest. Based on this idea, Mihalas (1980) introduces the radiation hydrodynamic equations in the Eulerian framework. In this approach, the physical quantities in terms of matter and radiation and the derivatives are introduced in the frame fixed to the coordinate of the central object, e.g. a black hole, while the interactions between matter and radiation are calculated in the comoving frame. That is, the local processes like the interaction between the matter and the radiation are evaluated in the comoving frame, while the derivatives which are used for the calculations of the global dynamics of both the matter and the radiation are derived in the frame fixed to the coordinate describing the central object.

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The dynamical equations for the matter and the radiation field are described as

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \]

\[ \frac{\partial}{\partial t} (\rho u^\alpha) + \nabla \cdot (\rho u^\alpha \mathbf{u}) + \nabla P = 0, \]

\[ \frac{\partial}{\partial t} \rho_h + \nabla \cdot (\rho_h \mathbf{u}) = 0, \]

\[ \frac{\partial}{\partial t} (\rho_h \mathbf{u}) + \nabla \cdot (\rho_h \mathbf{u} \mathbf{u}) + \nabla P_h = 0, \]

\[ \frac{\partial}{\partial t} \rho_\gamma + \nabla \cdot (\rho_\gamma \mathbf{u}) = 0, \]

\[ \frac{\partial}{\partial t} (\rho_\gamma \mathbf{u}) + \nabla \cdot (\rho_\gamma \mathbf{u} \mathbf{u}) + \nabla P_\gamma = 0, \]

where \( \mathbf{u} \) is the four-velocity, \( \rho \) is the rest-mass density, \( h \) is the relativistic specific enthalpy, \( P \) is the pressure, \( \rho_h \) is the specific enthalpy of the gas, \( P_h \) is the pressure of the gas, \( \rho_\gamma \) is the specific energy density of the gas, \( \mathbf{h} \) is the four-velocity, \( \epsilon \) is the energy density of the gas, \( \mathbf{v} \) is the comoving flow velocity, \( I(\mathbf{x}; \mathbf{n}, \nu) \) is the specific intensity, \( T^{ab} \) is the energy–momentum tensor, \( P \) is the pressure of the gas, \( \rho \) is the particle number density measured in the comoving frame, \( n \) is the mass of the gas particle, \( \chi \) is the opacity, and \( \eta \) is the emissivity.

2 COVARIANT EQUATIONS FOR GENERAL RELATIVISTIC RADIATION HYDRODYNAMICS

Before the explicit expressions for the equations of the radiation hydrodynamics in Kerr space–time are given, here we briefly summarize the basic equations of the general relativistic radiation hydrodynamics in covariant form (e.g. Mihalas & Mihalas 1984). We assume the orthonormal tetrads fixed to the coordinate (Section 3.1), the radiation moments (Section 3.2) and the radiation four-force (Section 3.3) are derived. In Section 4, we also give the basic equations for the radiation hydrodynamics including the continuity equation (Section 4.1), the hydrodynamic equations (Section 4.2) and the radiation moment equations (Section 4.3). Concluding remarks are given in the last section. In this paper, we assume \( c = 1 \) in most equations except in a few cases where \( c \) is explicitly used for clarity. Latin and Greek indices denote spatial components and space–time components, respectively. \( \nabla_a \) denotes the covariant derivative with respect to \( g_{\mu\nu} \).

The energy–momentum tensor for matter of an ideal gas described as

\[ T^{ab} = \rho u^a u^b + P g^{ab}, \]

\( P \) is the pressure of the gas, \( u^a \) is the four-velocity, \( \rho \) is the rest-mass density, \( \epsilon \) is the energy density of the gas. Here, the fluid quantities \( h, \epsilon, P \) and \( \rho \) are all being measured in the comoving frame of the fluid. The radiation stress–energy tensor is calculated from the specific intensity \( I(\mathbf{x}; \mathbf{n}, \nu) \) as

\[ R^{ab} = \int \int I(\mathbf{n}, \nu)n^a n^b d\nu d\Omega, \]

where \( n^a = (hv)/(1, \mathbf{n}) \) is the photon (or neutrino) four-momentum and \( n^\nu = P^\nu/\epsilon \). Here, the specific intensity \( I(\mathbf{x}; \mathbf{n}, \nu) \) is defined for photons (or neutrinos) moving in direction \( \mathbf{n} \) with the frequency \( \nu \). The particle number conservation equation in the absence of particle creation and annihilation and the conservation equation for the total energy–momentum of gas plus radiation are given as

\[ \nabla_a (\rho u^a) = 0, \]

\[ \nabla_a (T^{ab} + R^{ab}) = 0, \]

respectively. Here \( n \) is the particle number density measured in the comoving frame, \( n \) is related to \( \rho \) as \( n = \rho m_\gamma \) where \( m_\gamma \) is the mass of the gas particle. On the other hand, the radiation four-force density acting on the matter is given as

\[ G^a = \frac{1}{c} \int \int \chi \nu I(\mathbf{n}, \nu) - \eta \nu n^a d\nu d\Omega, \]

where \( \chi \) and \( \eta \) are the opacity and the emissivity, respectively. The invariant emissivity and invariant opacity are \( \eta \nu/\nu^2 \) and \( \nu \chi \), respectively. The dynamical equations for the matter and the radiation field are described as

\[ \nabla_a T^{ab} = G^a, \]

\[ \nabla_a R^{ab} = -G^a. \]
3 RADIATION HYDRODYNAMICS DESCRIBED BY THE KERR–SCHILD COORDINATE

In this study, we assume the background geometry around the rotating black hole written by the Kerr–Schild coordinate where there is no coordinate singularity at the event horizon. By using this coordinate, the metric around a rotating black hole is described as
\[
dx^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),
\]
where \(i, j = r, \theta, \phi\) and the non-zero components of the lapse function \(\alpha\), the shift vector \(\beta^i\) and the metric in three-dimensional spatial hypersurface \(\gamma_{ij}\) are given in geometric units as
\[
\alpha = \left(1 + \frac{2mr}{\Sigma}\right)^{-1/2}, \quad \beta^\mu = \frac{2mr/\Sigma}{1 + 2mr/\Sigma},
\]
\[
\gamma_{rr} = 1 + \frac{2mr}{\Sigma}, \quad \gamma_{\theta\theta} = \Sigma, \quad \gamma_{\phi\phi} = \frac{A \sin^2 \theta}{\Sigma}, \quad \gamma_{\theta\phi} = \gamma_{\phi\theta} = -a \sin^2 \theta \left(1 + \frac{2mr}{\Sigma}\right).
\]
Here, we use the geometric mass \(m = GM/c^2\), \(\Sigma = r^2 + a^2 \cos^2 \theta\), \(A = (r^2 + a^2) - a^2 \Delta \sin^2 \theta = \Sigma \Delta + 2mr(r^2 + a^2) = \Sigma^2 + a^2 \sin^2 \theta (\Sigma + 2mr)\) and \(\Delta = r^2 - 2Mr + a^2\), where \(M\) is the black hole mass, \(G\) is the gravitational constant and \(c\) is the speed of light. The position of the outer and inner horizon, \(r_\pm\), is calculated from \(\Delta = 0\) as \(r_\pm = m \pm (m^2 - a^2)^{1/2}\). The angular velocity of the frame dragging due to the black hole’s rotation is calculated as \(\omega = -g_{\phi\theta}/g_{\theta\theta} = 2mar/A\).

3.1 Reference frames and orthonormal tetrads

Three reference frames are used in this study: (1) the Kerr–Schild frame (KSF) which is the frame based on the Kerr–Schild coordinate describing the metric, (2) the locally non-rotating reference frame (LNRF) which is the orthonormal frame fixed to the coordinate and (3) the comoving frame where the element of the fluid is at rest. The interactions between the matter and the radiation are introduced in the comoving frame. The LNRF is calculated from a stationary congruence formed by observers with a future-directed unit vector orthogonal to \(t = \text{constant}\). The components of the four-velocity for such observers are given as (Frolov & Novikov 1998)
\[
u_\mu = -\alpha \delta_\mu^r, \quad \text{and} \quad u^\mu = a^{-1}, \quad u^i = -a^{-1} \beta^i (i = r, \theta, \phi),
\]
respectively. Since the vorticity tensor vanishes for this congruence, the reference frame formed by this congruence is locally non-rotating. Then, the tetrad vectors, \(e^\mu_0\) and \(e^\mu_r\), for the LNRF are given as
\[
e^\mu_0 = [\alpha, \ 0, \ 0, \ 0],
\]
\[
e^\mu_r = \left[\beta^r (\gamma^{rr})^{-1/2}, \ (\gamma^{rr})^{-1/2}, \ 0, \ 0\right],
\]
\[
e^\mu_\theta = [0, \ 0, \ (\gamma_{\theta\theta})^{1/2}, \ 0],
\]
\[
e^\mu_\phi = \left[\beta^\phi \gamma_{\phi\phi} (\gamma_{\phi\phi})^{-1/2}, \ \gamma_{\phi\phi} (\gamma_{\phi\phi})^{-1/2}, \ 0, \ (\gamma_{\phi\phi})^{1/2}\right],
\]
and
\[
e^\mu_\theta = \left[\alpha^{-1}, \ -\beta^r \alpha^{-1}, \ 0, \ 0\right],
\]
\[
e^\mu_\phi = \left[0, \ (\gamma^{\phi\phi})^{1/2}, \ 0, \ (\gamma^{\phi\phi})^{-1/2}\right],
\]
\[
e^\mu_\phi = \left[0, \ 0, \ (\gamma_{\phi\phi})^{1/2}, \ 0\right],
\]
\[
e^\mu_\phi = \left[0, \ 0, \ (\gamma_{\phi\phi})^{-1/2}\right].
\]
Here, the hat denotes the physical quantities measured in the LNRF. The base \(e_\mu = \partial/\partial x^\mu\) of the LNRF can be expressed by the coordinate base \(\partial/\partial x^\mu\) as
\[
\frac{\partial}{\partial \tau} = \frac{1}{\alpha} \left(\frac{\partial}{\partial t} - \beta^r \frac{\partial}{\partial r}\right), \quad \frac{\partial}{\partial \phi} = \sqrt{\gamma^{rr}} \left(\frac{\partial}{\partial \phi} - \frac{\gamma_{\phi\phi}}{\gamma_{\theta\theta}} \frac{\partial}{\partial \theta}\right), \quad \frac{\partial}{\partial \theta} = \frac{1}{\sqrt{\gamma^{\phi\phi}}} \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial \phi} = \frac{1}{\sqrt{\gamma^{\phi\phi}}} \frac{\partial}{\partial \phi},
\]
where \(\mu = t, r, \theta, \phi\). Here, \((x^0, x^1, x^2, x^3) = (t, \ r, \ \theta, \ \phi)\). The components of the fluid’s three-velocity measured by a fiducial observer who is fixed with respect to the coordinates in the LNRF are calculated as
\[
\vec{v}^\mu = \frac{u^\mu}{u^0}, \quad (i = r, \ \theta, \ \phi),
\]
where \(u^0\) is the four-velocity in the LNRF. The components of the three-velocity are explicitly calculated as
\[
\vec{v}^r = \frac{1}{\alpha \sqrt{\gamma^{rr}}} \left(\frac{u^r}{u^0} + \beta^r\right), \quad \vec{v}^\theta = \frac{1}{\alpha \sqrt{\gamma^{\theta\theta}}} \left(\frac{u^\theta}{u^0}\right), \quad \vec{v}^\phi = \frac{\sqrt{\gamma_{\phi\phi}}}{\alpha} \left[\Omega + \frac{\gamma_{\phi\phi}}{\gamma_{\theta\theta}} \left(\frac{u^\theta}{u^0} + \beta^\phi\right)\right],
\]
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where $\Omega \equiv \Omega^0 / u^0$ is the angular velocity and we have used $u^i = -u_i = \alpha u'^i$. The Lorentz factor $\tilde{\gamma}$ for this three-velocity is calculated as
\begin{equation}
\tilde{\gamma} = (1 + \tilde{v}^2)^{-1/2} = \alpha u',
\end{equation}
where $\tilde{v}^2 = v \cdot \tilde{v} = \tilde{v}_i \tilde{v}_i + \tilde{v}_\phi^2$.

A tetrad base for the comoving frame $\partial / \partial x^a$ is calculated by the Lorentz transformation as
\begin{equation}
\frac{\partial}{\partial x^a} = \Lambda_{\tilde{p}}^a(v) \frac{\partial}{\partial x^{\tilde{p}}}.
\end{equation}
where the bar denotes the physical quantities measured in the comoving frame. The components of the Lorentz transformation $\Lambda_{\tilde{p}}^a(v)$ are given as $\Lambda_{\tilde{p}}^i = \tilde{\gamma} \tilde{v}_i$, $\Lambda_{\tilde{p}}^j = \tilde{v}_j$, and $\Lambda_{\tilde{p}}^j = \delta^j + \tilde{v}_i \tilde{v}_j (1 + \gamma) (i, j = r, \phi, \theta)$. Here, $(x^0, x^1, x^2, x^3) = (\tilde{t}, \tilde{r}, \tilde{\phi}, \tilde{\theta})$. The components of the base of the comoving tetrad $\partial / \partial x^a$ can be expressed by the coordinate base $\partial / \partial x^{\tilde{p}}$ as
\begin{equation}
\frac{\partial}{\partial x^a} = \frac{\partial}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{r}} + \frac{\partial}{\partial \tilde{\phi}} + \frac{\partial}{\partial \tilde{\theta}}.
\end{equation}
\begin{equation}
\frac{1}{\gamma} \frac{\partial}{\partial \tilde{t}} = \gamma \left[ 1 - \frac{\beta'}{\alpha} \left( \frac{\tilde{v}_r \gamma_{\tilde{p}} - \tilde{v}_r \gamma_{\phi}}{\gamma_{\tilde{p}}} \right) \right] \frac{\partial}{\partial \tilde{t}} + \left[ -\tilde{v}_r + \frac{\beta'}{\alpha} \frac{\gamma_{\tilde{p}}}{\gamma_{\phi}} \left( 1 + \frac{\tilde{v}_r^2}{\gamma_{\phi}} \right) \right] \frac{\partial}{\partial \tilde{r}}
\end{equation}
\begin{equation}
= -\gamma \left[ \frac{\tilde{v}_r + \tilde{v}_\phi \gamma_{\tilde{p}}}{\gamma_{\tilde{p}}} \right] \frac{\partial}{\partial \tilde{t}} + \left[ 1 + \frac{\tilde{v}_r^2}{\gamma_{\phi}} \right] \gamma_{\tilde{p}} \frac{\partial}{\partial \tilde{r}}
\end{equation}
\begin{equation}
= -\gamma \left[ \tilde{v}_r + \tilde{v}_\phi \gamma_{\tilde{p}} \frac{\partial}{\partial \tilde{t}} + \left( 1 + \frac{\tilde{v}_r^2}{\gamma_{\phi}} \right) \frac{\partial}{\partial \tilde{r}} + \frac{\tilde{v}_r^2 \tilde{v}_\phi}{\gamma_{\phi}} \frac{\partial}{\partial \tilde{\phi}} \right].
\end{equation}

3.2 Radiation moments

The radiation energy density $E$, the radiation flux $F^i$ and the radiation pressure tensor $P^{ij}$ are defined as the zeroth, the first and the second moments of the specific intensity $I_i(x^a, \mathbf{n})$, respectively. We denote the radiation moments as
\begin{equation}
\bar{E} = \int \int \bar{I}_{\mathbf{n}} d\Omega, \quad \bar{F}^i = \int \int \bar{I}_{\mathbf{n}} \bar{v}^i d\Omega, \quad \bar{P}^{ij} = \int \int \bar{I}_{\mathbf{n}} \bar{v}^i \bar{v}^j d\Omega,
\end{equation}
when measured in the LNRF, and
\begin{equation}
\hat{E} = \int \int \hat{I}_{\mathbf{n}} d\Omega, \quad \hat{F}^i = \int \int \hat{I}_{\mathbf{n}} \hat{v}^i d\Omega, \quad \hat{P}^{ij} = \int \int \hat{I}_{\mathbf{n}} \hat{v}^i \hat{v}^j d\Omega.
\end{equation}
when measured in the comoving frame. Correspondingly, the radiation stress tensors for the LNRF and the comoving frame are given as

$$R^{\alpha \beta} = \begin{pmatrix} E & F^r & F^\theta & F^\phi \\ F^r & \check{P}^r & \check{P}^\theta & \check{P}^\phi \\ F^\theta & \check{P}^\theta & \check{P}^r & \check{P}^\phi \\ F^\phi & \check{P}^\phi & \check{P}^\phi & \check{P}^r \end{pmatrix}$$

and

$$\check{R}^{\alpha \beta} = \begin{pmatrix} \check{E} & \check{F}^r & \check{F}^\theta & \check{F}^\phi \\ \check{F}^r & \check{\check{P}}^r & \check{\check{P}}^\theta & \check{\check{P}}^\phi \\ \check{F}^\theta & \check{\check{P}}^\theta & \check{\check{P}}^r & \check{\check{P}}^\phi \\ \check{F}^\phi & \check{\check{P}}^\phi & \check{\check{P}}^\phi & \check{\check{P}}^r \end{pmatrix},$$

respectively. The contravariant components of the radiation stress tensor $R^{\alpha \beta}$ are calculated from $\check{R}^{\alpha \beta}$ by the transformation as

$$R^{\alpha \beta} = \frac{\partial x^\alpha}{\partial \check{x}^\alpha} \frac{\partial x^\beta}{\partial \check{x}^\beta} \check{R}^{\alpha \beta},$$

and explicitly given as

$$R^{\alpha \beta} = \begin{pmatrix} \frac{\check{E}}{\alpha^2} & \frac{1}{\alpha} \left( \sqrt{\gamma^r r} \check{F}^r - \frac{\beta^r}{\alpha} \check{E} \right) & \frac{1}{\alpha} \left( \gamma^r r \check{P}^r - \frac{\beta^r}{\alpha} \check{P}^r \right) & 0 \\ \frac{1}{\alpha} \left( \sqrt{\gamma^r r} \check{F}^r + \frac{\beta^r}{\alpha} \check{E} \right) & \frac{1}{\alpha} \left( \sqrt{\gamma^r r} \check{P}^r + \frac{\beta^r}{\alpha} \check{P}^r \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^r - \beta^r \check{P}^r \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^r + \beta^r \check{P}^r \right) \\ \frac{1}{\alpha} \left( \sqrt{\gamma^r r} \check{P}^\theta - \frac{\beta^\theta}{\alpha} \check{P}^\theta \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^\theta - \beta^\theta \check{P}^\theta \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^\theta + \beta^\theta \check{P}^\theta \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^\theta - \beta^\theta \check{P}^\theta \right) \\ \frac{1}{\alpha} \left( \sqrt{\gamma^r r} \check{P}^\phi + \frac{\beta^\phi}{\alpha} \check{P}^\phi \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^\phi + \beta^\phi \check{P}^\phi \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^\phi - \beta^\phi \check{P}^\phi \right) & \frac{1}{\alpha} \left( \gamma^r r \check{\check{P}}^\phi + \beta^\phi \check{P}^\phi \right) \end{pmatrix}.$$
where $\tilde{\chi}$ and $\tilde{\eta}$ are the mean opacity and the emissivity measured in the comoving frame, respectively. The time component $G^t$ has the dimension $c^{-1}$ times the net rate of the radiation energy per unit volume, and the spatial component $G^i$ has the dimension of the net rate of the momentum exchange between the matter and the radiation. The time component of the radiation four-force density measured in the comoving frame can be calculated as (Mihalas & Mihalas 1984; Park 2006)

$$G^t = \Gamma - \Lambda,$$

where the heating function $\Gamma$ and the cooling function $\Lambda$ are defined as

$$\Gamma \equiv \frac{1}{c} \int \tilde{\chi} \bar{I}_d d\Omega, \quad \Lambda \equiv \frac{1}{c} \int \tilde{\eta} d\Omega.$$  

(41)

The components of the radiation force $G^a$ are calculated from those in the comoving frame $G^b$ by the transformation

$$G^a = \frac{\partial x^a}{\partial x^b} G^b,$$

and explicitly given as

$$G^t = \frac{\dot{\gamma}}{\alpha} \left( G^t + \hat{v}_t G^i \right),$$

$$G^i = \sqrt{\gamma''} G^i + \dot{\gamma} \left( \sqrt{\gamma''} \hat{v}_i - \frac{\beta''}{\alpha} \right) G^t + \frac{\dot{\gamma} \hat{v}_t}{\gamma + 1} \hat{v}_i G^t,$$

$$G^\phi = \frac{\dot{\gamma}}{\sqrt{\gamma\phi}} G^\phi + \frac{\gamma''\phi}{\sqrt{\gamma\phi}} G^\phi + \frac{\dot{\gamma} \hat{v}_\phi}{\gamma + 1} \left( \hat{v}_\phi + \frac{\gamma''\phi}{\sqrt{\gamma\phi}} \hat{v}_t \right) \left( G^t + \frac{\dot{\gamma}}{\gamma + 1} \hat{v}_t G^i \right).$$

(43)

4 RADIATION HYDRODYNAMIC EQUATIONS

4.1 Continuity equation

As a continuity equation, now we consider the particle number conservation, $(\nu a^\nu)_{,\nu} = 0$. This equation can be calculated as

$$\frac{\partial}{\partial t} \left( \alpha \sqrt{\gamma} n u^t \right) + \frac{\partial}{\partial x^i} \left( \alpha \sqrt{\gamma} n u^i \right) = 0,$$

(44)

where $\gamma \equiv \det g_{ij}$. In the case of the Kerr metric written by the Boyer–Lindquist coordinate $\gamma = \Sigma (\Sigma + 2mr) \sin^2 \theta$.

4.2 Hydrodynamic equations

The relativistic Euler equations are obtained by the transformation of the energy momentum conservation $\nabla_\beta T^{\alpha\beta} = G^\alpha$ on the specific directions by using the projection tensor $P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ as $P^{\alpha}_{\beta} \nabla_\beta T^{\alpha\beta} = P^\alpha_{\beta} G^\alpha$. From this, we can obtain $\rho_0 h_0 u^\beta \nabla_\beta u^\alpha + (g^{\alpha\beta} + u^\alpha u^\beta) \partial_\beta P^\alpha_{\phi} = G^\alpha + u^\alpha u^\beta G^\beta$.

The Euler equations in $r, \theta$ and $\phi$ directions are given as

$$\rho_0 h_0 \frac{du^r}{dt} + \rho_0 h_0 \frac{Du^r}{Dt} + \rho_0 h_0 \Gamma^{r}_{\rho\gamma} u^\rho u^\gamma + (g^{r\sigma} + u^r u^\sigma) \frac{dP^r_{\phi}}{dt} + (g^{r\sigma} + u^r u^\sigma) \frac{dP^r_{\phi}}{dx^i} = G^r + u^r u^\beta G^\beta,$$

(45)

$$\rho_0 h_0 \frac{du^\theta}{dt} + \rho_0 h_0 \frac{Du^\theta}{Dt} + \rho_0 h_0 \Gamma^{\theta}_{\rho\gamma} u^\rho u^\gamma + (g^{\theta\sigma} + u^\theta u^\sigma) \frac{dP^\theta_{\phi}}{dt} + (g^{\theta\sigma} + u^\theta u^\sigma) \frac{dP^\theta_{\phi}}{dx^i} = G^\theta + u^\theta u^\beta G^\beta,$$

(46)

$$\rho_0 h_0 \frac{du^\phi}{dt} + \rho_0 h_0 \frac{Du^\phi}{Dt} + \rho_0 h_0 \Gamma^{\phi}_{\rho\gamma} u^\rho u^\gamma + (g^{\phi\sigma} + u^\phi u^\sigma) \frac{dP^\phi_{\phi}}{dt} + (g^{\phi\sigma} + u^\phi u^\sigma) \frac{dP^\phi_{\phi}}{dx^i} = G^\phi + u^\phi u^\beta G^\beta,$$

(47)

where

$$\Gamma^{r}_{\rho\gamma} u^\rho u^\gamma = \Gamma^{r}_{\rho\gamma} (u^\rho)^2 + \Gamma^{r}_{\rho\gamma} u^\rho u^\gamma + \Gamma^{r}_{\rho\gamma} u^\gamma + \Gamma^{r}_{\rho\gamma}(u^\rho)^2 + \Gamma^{r}_{\rho\gamma}(u^\gamma)^2$$

$$+ 2 \Gamma^{r}_{\rho\gamma} u^\rho u^\gamma + 2 \Gamma^{r}_{\rho\gamma} u^\sigma u^\gamma + 2 \Gamma^{r}_{\rho\gamma} u^\sigma u^\gamma + 2 \Gamma^{r}_{\rho\gamma} u^\rho u^\gamma + 2 \Gamma^{r}_{\rho\gamma} u^\sigma u^\gamma + 2 \Gamma^{r}_{\rho\gamma} u^\sigma u^\gamma,$$

(48)

and now $\alpha = r, \theta, \phi$. The Christoffel symbols are given in Appendix A. $u_\alpha G^\alpha$ is calculated by the heating and cooling function defined in the comoving frame as

$$u_\alpha G^\alpha = -G^\alpha = \tilde{\Lambda} - \tilde{\Gamma}.$$  

(49)

This is also calculated as

$$u_\alpha G^\alpha = \left[ -\alpha + \beta^r \beta^r G^r + \gamma_\phi G^\phi \right] u^r + \left[ \gamma_r (G^r + \beta^r G^r) + \gamma_\phi G^\phi \right] u^\phi + \gamma_\theta (G^\theta + \beta^\theta G^\theta),$n

(50)
The radiation energy equation is obtained from the local energy conservation as

\[-n \frac{\partial}{\partial t} \left( \frac{\rho_{b} h_{b}}{n} \right) - n u^{\gamma} \frac{\partial}{\partial x^{\gamma}} \left( \frac{\rho_{b} h_{b}}{n} \right) = \frac{\partial P_{x}}{\partial t} + \frac{\partial P_{\phi}}{\partial x^{\phi}} + u^{\gamma} P_{\gamma} = \mu_{b} G^{\beta}. \]  

(51)

### 4.3 Radiation moment equations

The radiation moment equation \( V_{\beta} R^{\beta \phi} = - G^{\phi} \) gives the equation for the energy density, the radiation flux and the radiation pressure tensor. The radiation energy equation is obtained from the \( \alpha \)-component of this equation \( V_{\alpha} R^{\alpha} = - G \) calculated as

\[
\frac{\partial R^{\alpha}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial r} \left( \sqrt{r} \left( \sqrt{r} R^{\alpha} - \frac{\beta^{\alpha}}{\alpha} \hat{E} \right) \right) + \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \theta} \left( \sqrt{\theta} \sqrt{\phi} R^{\alpha} \right) + \frac{1}{\alpha} \frac{\partial}{\partial \phi} \left( \sqrt{\gamma} \sqrt{\phi} R^{\alpha} \right) + \frac{1}{\alpha} \frac{\partial}{\partial \hat{h}} \left( \sqrt{r} \sqrt{\phi} R^{\alpha} \right) + \frac{1}{\alpha} \frac{\partial}{\partial \hat{r}} \left( \sqrt{r} \sqrt{\phi} R^{\alpha} \right) = - G^{\alpha}. \]  

(52)

The radiation momentum equation in the \( r \) direction \( V_{\alpha} R^{\alpha} \) is calculated as

\[
\frac{\partial R^{\alpha}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial r} \left( \sqrt{r} \left( \sqrt{r} R^{\alpha} - \frac{\beta^{\alpha}}{\alpha} \hat{E} \right) \right) + \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \theta} \left( \sqrt{\theta} \sqrt{\phi} R^{\alpha} \right) + \frac{1}{\alpha} \frac{\partial}{\partial \phi} \left( \sqrt{\gamma} \sqrt{\phi} R^{\alpha} \right) = - G^{\alpha}. \]  

(53)

In the similar manner, the radiation momentum equation in the \( \theta \) direction \( V_{\alpha} R^{\alpha} \) is calculated as

\[
\frac{\partial R^{\alpha}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial r} \left( \sqrt{r} \left( \sqrt{r} R^{\alpha} - \frac{\beta^{\alpha}}{\alpha} \hat{E} \right) \right) + \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \theta} \left( \sqrt{\theta} \sqrt{\phi} R^{\alpha} \right) + \frac{1}{\alpha} \frac{\partial}{\partial \phi} \left( \sqrt{\gamma} \sqrt{\phi} R^{\alpha} \right) = - G^{\alpha}. \]  

(54)

The radiation momentum equation in the \( \phi \) direction \( V_{\alpha} R^{\alpha} \) is calculated as

\[
\frac{\partial R^{\alpha}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial r} \left( \sqrt{r} \left( \sqrt{r} R^{\alpha} - \frac{\beta^{\alpha}}{\alpha} \hat{E} \right) \right) + \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \theta} \left( \sqrt{\theta} \sqrt{\phi} R^{\alpha} \right) + \frac{1}{\alpha} \frac{\partial}{\partial \phi} \left( \sqrt{\gamma} \sqrt{\phi} R^{\alpha} \right) = - G^{\alpha}. \]  

(55)

Finally, by inserting the components of the radiation stress tensor given by equation (30) into equations (52)–(55), we obtain the radiation moment equations as

\[
\frac{\partial}{\partial t} \left[ \frac{1}{\sqrt{r}} \sqrt{\gamma} \left( \sqrt{r} \hat{E}^{\alpha} - \frac{\beta^{\alpha}}{\alpha} \hat{E} \right) \right] + \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \theta} \left[ \sqrt{\theta} \sqrt{\phi} \hat{E}^{\alpha} \right] + \frac{1}{\alpha} \frac{\partial}{\partial \phi} \left[ \sqrt{\gamma} \sqrt{\phi} \hat{E}^{\alpha} \right] + \frac{1}{\alpha} \frac{\partial}{\partial \hat{h}} \left[ \sqrt{r} \sqrt{\phi} \hat{E}^{\alpha} \right] + \frac{1}{\alpha} \frac{\partial}{\partial \hat{r}} \left[ \sqrt{r} \sqrt{\phi} \hat{E}^{\alpha} \right] = - G^{\alpha}. \]  

(56)

\[
\frac{\partial}{\partial t} \left[ \frac{1}{\sqrt{r}} \sqrt{\gamma} \left( \sqrt{r} \hat{E}^{\alpha} - \frac{\beta^{\alpha}}{\alpha} \hat{E} \right) \right] + \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \theta} \left[ \sqrt{\theta} \sqrt{\phi} \hat{E}^{\alpha} \right] + \frac{1}{\alpha} \frac{\partial}{\partial \phi} \left[ \sqrt{\gamma} \sqrt{\phi} \hat{E}^{\alpha} \right] + \frac{1}{\alpha} \frac{\partial}{\partial \hat{h}} \left[ \sqrt{r} \sqrt{\phi} \hat{E}^{\alpha} \right] + \frac{1}{\alpha} \frac{\partial}{\partial \hat{r}} \left[ \sqrt{r} \sqrt{\phi} \hat{E}^{\alpha} \right] = - G^{\alpha}. \]  

(57)
\[
\frac{\partial}{\partial t} \left( \frac{\hat{F}^\phi}{\alpha \sqrt{g_{\theta\theta}}} \right) + \frac{\partial}{\partial \phi} \left[ \frac{\alpha \sqrt{\gamma}}{\sqrt{g_{\theta\theta}}} \left( \sqrt{\gamma^r} \hat{p}^\phi - \frac{\beta_r}{\alpha} \hat{F}^\phi \right) \right] + \frac{\partial}{\partial t} \left[ \frac{\alpha \sqrt{\gamma}}{\sqrt{g_{\theta\theta}}} \left( \sqrt{\gamma^r} \hat{p}^\phi - \frac{\beta_r}{\alpha} \hat{F}^\phi \right) \right] + \frac{\partial}{\partial \phi} \left[ \frac{1}{\alpha \sqrt{g_{\phi\phi}}} \left( \sqrt{\gamma^r} \hat{p}^\phi - \frac{\beta_r}{\alpha} \hat{F}^\phi \right) \right]
\]

\[
+ \Gamma_{\phi r}^\phi \left[ \frac{\hat{F}^\phi}{\alpha \sqrt{g_{\phi\phi}}} \right] + \frac{\partial}{\partial r} \left[ \frac{\alpha \sqrt{\gamma}}{\sqrt{g_{\theta\theta}}} \left( \sqrt{\gamma^r} \hat{F}^\phi - \frac{\beta_r}{\alpha} \hat{E} \right) + \frac{\partial}{\partial \phi} \left[ \frac{\alpha \sqrt{\gamma}}{\sqrt{g_{\phi\phi}}} \right] \right]
\]

\[
+ \frac{\partial}{\partial t} \left[ \frac{1}{\alpha \sqrt{g_{\phi\phi}}} \left( \sqrt{\gamma^r} \hat{F}^\phi - \frac{\beta_r}{\alpha} \hat{E} \right) + \frac{\partial}{\partial \phi} \left[ \frac{\alpha \sqrt{\gamma}}{\sqrt{g_{\theta\theta}}} \right] \right]
\]

\[
+ \Gamma_{\phi r}^\phi \left[ \sqrt{\gamma^r} \hat{p}^\phi - \frac{2\beta_r}{\alpha} \sqrt{\gamma^r} \hat{F}^\phi - \left( \frac{\beta_r}{\alpha} \hat{E} \right)^2 \right] + \frac{\partial}{\partial \phi} \left[ \frac{\alpha \sqrt{\gamma}}{\sqrt{g_{\phi\phi}}} \right]
\]

\[
+ 2\Gamma_{\phi r}^\phi \left[ \frac{1}{\alpha \sqrt{g_{\phi\phi}}} \left( \sqrt{\gamma^r} \hat{p}^\phi - \frac{\beta_r}{\alpha} \hat{F}^\phi \right) \right] + 2\Gamma_{\phi r}^\phi \left[ \sqrt{\gamma^r} \hat{p}^\phi + \left( \frac{\beta_r}{\alpha} \hat{F}^\phi \right)^2 \right] = -G^\phi.
\]

where the Christoffel symbols are given in Appendix A.

### 5 CONCLUDING REMARKS

In this study, we have derived equations of fully general relativistic radiation hydrodynamics around a rotating black hole by using the Kerr–Schild coordinate where there is no coordinate singularity at the event horizon. Both the matter and the radiation are affected by the frame-dragging effects due to the black hole’s rotation. Since the radiation usually interacts with matter moving at relativistic velocities near the event horizon, the interaction between the radiation and the matter should be treated fully relativistically. This can be done if the interplay between the radiation and the matter is evaluated in the comoving frame where the matter is at rest. In this approach while the interactions between matter and radiation are introduced in the comoving frame, the equations and the derivatives for the description of the global evolution of both matter and the radiation are given in the KSF which is a frame fixed to the coordinate describing the central black hole. As a frame fixed to the coordinate, we use the LNRF which is a radially falling frame when the Kerr–Schild coordinate is used. We can see in the derived equations that both the matter and the radiation are affected by the frame-dragging effects due to the black hole’s rotation. It is widely known that the moment equations truncated at the finite order do not produce the complete system of equations, i.e. the number of variables is larger than the number of equations. So, additional equations are required to close the system of equations (Chandrasekhar 1960; Mihalas 1970; Pomraning 1973; Rybicki & Lightman 1979; Mihalas & Mihalas 1984; Shu 1991). These additional equations should be also described fully relativistically when the radiation interacts with the matter moving at relativistic velocities. As one possible way with respect to such additional equations, the flux-limited diffusion approximation proposed by Levermore & Pomraning (1981) is sometimes used. The covariant theories for the flux-limited diffusion approximation are presented for radiation propagating through inhomogeneous and non-stationary media (Anile & Romano 1992). It is noted that constructing the closure relations which are essentially required for the radiation hydrodynamical calculations is non-trivial when the photons and the matter are relatively relativistically moving. This is mainly because the truncation of the moment equations at the finite order cannot be performed frame-independently. So, in such cases, the equations given in this article are of no use if the sufficient closure relations cannot be constructed. In the case of the calculations based on the Kerr–Schild coordinate, the fluid velocity is usually smaller than the speed of light even at the event horizon (Takahashi 2007a), for example, the gamma factor of the fluid at the horizon \( \gamma < 1 \text{--} 100 \). So, if the photon motion has a speed similar to the fluid’s speed and the closure relations which are correct for such motions cannot be constructed, the equations given in this study can be used for the description of such radiation hydrodynamic flows. It is also noted that when effects of the radiation stress cannot be neglected, the dissipation effects cannot be correctly treated so far. This is mainly because we do not know the covariant theory describing the local dissipation effects such as viscosity. One promising approach is given by a method based on the
extended causal thermodynamics such as the Israel–Stewart theory (Israel & Stewart 1979) where the causality violating infinite signal speeds are eliminated (see also Anile, Pavón & Romano 1998). However, while the hydrodynamical equations based on such a theory are formulated by Peitz & Appl (1998), there are no formulations based on such a theory for the radiation hydrodynamic or magnetohydrodynamic equations. This is one of the important studies in terms of relativistic radiation hydrodynamics in the future.

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REFERENCES

Anderson J. L., Spiegel E. A., 1972, ApJ, 171, 127
Anile A. M., Romano V., 1992, ApJ, 386, 325
Anile A. M., Pavón D., Romano V., 1998, preprint (gr-qc/9810014)
Cardall C. Y., Mezzacappa A., 2003, Phys. Rev. D, 68, 023006
Chandrasekhar S., 1960, Radiative Transfer. Dover Press, New York
Cook G. B., 2000, Max-Planck-Gesellschaft Living Reviews Series, No. 2000-5
Flammang R. A., 1982, MNRAS, 199, 833
Flammang R. A., 1984, MNRAS, 206, 589
Font J. A., 2000, Max-Planck-Gesellschaft Living Reviews Series, No. 2000-2
Font J. A., Ibáñez J. M., Papadopoulos P., 1998, ApJ, 507, 67
Frolow V. P., Novikov I. D., 1998, Black Hole Physics: Basic Concepts and New Developments. Kluwer, Dordrecht
Gammie C., Popham R., 1998, ApJ, 498, 313
Gammie C. F., McKinney J. C., Tóth G., 2003, ApJ, 598, 444
Gammie C. F., Shapiro S. L., McKinney J. C., 2004, ApJ, 602, 312
Israel W., Stewart J. M., 1979, Ann. Phys., 118, 341
Komissarov S. S., 2001, MNRAS, 326, L41
Komissarov S. S., 2004, MNRAS, 350, 1431
Lindquist R. W., 1966, Ann. Phys., 37, 487
Mihalas D., 1970. Stellar Atmosphere. Freeman & Co., San Francisco
Mihalas D., 1980, ApJ, 237, 574
Miharas D., Mihalas B. W., 1984, Foundations of Radiation Hydrodynamics. Oxford Univ. Press, Oxford
Papadopoulos P., Font J. A., 1998, Phys. Rev. D, 58, 024005
Park M.-G., 1993, A&A, 274, 642
Park M.-G., 2006, MNRAS, 367, 1739
Park M.-G., Miller G. S., 1991, ApJ, 371, 708
Peitz J., Appl S., 1997, MNRAS, 296, 231
Pomraning G. C., 1973, The Equations of Radiation Hydrodynamics. Pergamon Press, Oxford
Rybicki G. B., Lightman A. P., 1979, Radiative Processes in Astrophysics. Wiley, New York
Schindler P. J., 1988, Phys. Rev. D, 38, 1673
Schindler P. J., Bludman S. A., 1989, ApJ, 346, 350
Schmid-Burgk J., 1978, Ap&SS, 56, 191
Shu F. H., 1991, The Physics of Astrophysics Vol. 1, Radiation. University Science Books, Mill Valley, CA
Takahashi R., 2000, MNRAS, 382, 567
Takahashi R., 2000b, MNRAS, 382, 1041
Thorne K. S., 1981, MNRAS, 194, 439
Turolla R., Nobili L., 1988, MNRAS, 235, 1273

APPENDIX A: DIFFERENTIAL VALUES OF METRIC COMPONENTS AND CHRISTOFFEL SYMBOLS

Here, we present the explicit expressions for the metric components and their differential values with respect to $r$ and $\theta$ used in this paper.

Non-zero components of the metric are given as

$$g_{rr} = \left(1 - \frac{2mr}{\Sigma}\right), \quad g_{\theta\theta} = \frac{2mr}{\Sigma}, \quad g_{\phi\phi} = -\frac{2mar sin^2 \theta}{\Sigma}, \quad g_{\phi\phi} = 1 + \frac{2mr}{\Sigma},$$

$$g_{r\phi} = g_{\phi r} = -a sin^2 \theta \left(1 + \frac{2mr}{\Sigma}\right), \quad g_{\theta\phi} = \frac{A sin^2 \theta}{\Sigma}, \quad g_{\phi\theta} = \frac{A sin^2 \theta}{\Sigma}.$$
The differential values of the metric components are given as

\[ g'^{\theta} = g^{r} = 0, \quad g'^{\phi} = g^{\theta} = \frac{2m r}{r^2 - \Sigma^2}, \quad g'^{tt} = g^{rr} = \frac{\Delta}{\Sigma}, \quad g'^{t\phi} = g^{\theta\phi} = \frac{a}{\Sigma}, \quad g'^{\phi\phi} = \frac{1}{\Sigma \sin^2 \theta}. \]  

(A2)

The Christoffel symbols \( \Gamma^\alpha_{\beta\gamma} \) are given as

\[ \Gamma^r_{\theta r} = \frac{2mr}{r^2 - \Sigma^2} \left( 1 - \frac{r^2}{\Sigma^2} \right), \quad \Gamma^r_{\phi r} = \frac{2mr}{r^2 - \Sigma^2} \left( 1 + \frac{mr}{\Sigma} \right) \left( 1 - \frac{2r^2}{\Sigma^2} \right), \]

\[ \Gamma^r_{\theta\phi} = \frac{\Sigma^2}{m} \left( \frac{1}{\Sigma} - \frac{2r^2}{\Sigma^2} \right), \quad \Gamma^r_{\phi\theta} = \frac{\Sigma^2}{m} \left( \frac{1}{\Sigma} - \frac{2r^2}{\Sigma^2} \right), \quad \Gamma^r_{\phi\phi} = \frac{\Sigma^2}{m} \sin^2 \theta \left( 1 - \frac{2r^2}{\Sigma^2} \right), \]

\[ \Gamma^r_{t\theta} = \frac{\Delta}{\Sigma} \left( 1 - \frac{2r^2}{\Sigma^2} \right), \quad \Gamma^r_{t\phi} = \frac{\Delta}{\Sigma} \left( 1 - \frac{2r^2}{\Sigma^2} \right), \quad \Gamma^r_{\theta\theta} = \frac{\Delta}{\Sigma} \sin^2 \theta \left( 1 - \frac{2r^2}{\Sigma^2} \right), \]

\[ \Gamma^r_{\theta\phi} = \frac{\Delta}{\Sigma} \sin^2 \theta \left( 1 - \frac{2r^2}{\Sigma^2} \right), \quad \Gamma^r_{\phi\theta} = \frac{\Delta}{\Sigma} \sin^2 \theta \left( 1 - \frac{2r^2}{\Sigma^2} \right), \quad \Gamma^r_{\phi\phi} = \frac{\Delta}{\Sigma} \sin^2 \theta \left( 1 - \frac{2r^2}{\Sigma^2} \right), \]

(A3)

where we have used

\[ \Delta = 2r, \quad \Delta = 2 \left[ 2r \Sigma + (r + m) a^2 \sin^2 \theta \right], \quad \frac{\partial r}{\partial \phi} = -a \sin 2\theta \left[ 1 + 2(mr^2 + a^2) \right], \quad \frac{\partial \phi}{\partial \theta} = \frac{\Delta + 2mr \left( r^2 + a^2 \right)^2}{\Sigma^2} \sin 2\theta, \]

\[ \frac{\partial \phi}{\partial \phi} = \frac{\Delta + 2mr \left( r^2 + a^2 \right)^2}{\Sigma^2} \sin 2\theta, \]

\[ \frac{\partial \phi}{\partial r} = \frac{\Delta + 2mr \left( r^2 + a^2 \right)^2}{\Sigma^2} \sin 2\theta, \]

\[ \frac{\partial \phi}{\partial \theta} = \frac{\Delta + 2mr \left( r^2 + a^2 \right)^2}{\Sigma^2} \sin 2\theta, \]

\[ \frac{\partial \phi}{\partial \phi} = \frac{\Delta + 2mr \left( r^2 + a^2 \right)^2}{\Sigma^2} \sin 2\theta. \]

(A5)
\[ \Gamma_{\theta\theta}^\phi = - \frac{1}{2} g^{\theta\phi} \partial_r g_{\theta\theta} = - \frac{a r}{\Sigma}, \quad \Gamma_{\phi\phi}^\phi = - \frac{1}{2} g^{\phi\phi} \partial_r g_{\phi\phi} = - \frac{a \Sigma}{\Sigma} \sin^2 \theta \left[ r + \frac{ma^2 \sin^2 \theta}{\Sigma} \left( 1 - \frac{2r^2}{\Sigma} \right) \right]. \]

\[ \Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = \frac{1}{2} g^{\phi\phi} \partial_r g_{r\phi} = - \frac{ma}{\Sigma^2} \left( 1 - \frac{2r^2}{\Sigma} \right), \quad \Gamma_{\phi\theta}^\phi = \Gamma_{\theta\phi}^\phi = \frac{1}{2} \left( g^{\theta\theta} \partial_r g_{\phi\theta} + g^{\phi\phi} \partial_r g_{\theta\phi} \right) = - \frac{2ma}{\Sigma^2} \cot \theta, \]

\[ \Gamma_{r\theta}^\phi = \Gamma_{\phi\theta}^\phi = - \frac{1}{2} g^{\phi\phi} \partial_r g_{r\theta} = \frac{ma^2 \sin^2 \theta}{\Sigma^2} \left( 1 - \frac{2r^2}{\Sigma} \right), \quad \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \frac{1}{2} \left( g^{\theta\theta} \partial_r g_{\phi\theta} + g^{\phi\phi} \partial_r g_{\theta\phi} \right) = - \frac{a}{\Sigma} \left( 1 + \frac{2mr}{\Sigma} \right) \cot \theta. \]

\[ \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \frac{1}{2} \left( g^{\theta\theta} \partial_r g_{\phi\theta} + g^{\phi\phi} \partial_r g_{\theta\phi} \right) = \left[ 1 + \frac{2mr}{\Sigma} \left( \frac{r^2 + a^2}{\Sigma} - 1 \right) \right] \cot \theta. \] (A6)

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