Observation of the resonance frequencies of a stable torus of fluid

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We report the first quantitative measurements of the resonance frequencies of a torus of fluid confined in a horizontal Hele-Shaw cell. By using the unwetting property of a metal liquid, we are able to generate a stable torus of fluid with an arbitrary aspect ratio. When subjected to vibrations, the torus displays azimuthal patterns at its outer periphery. These lobes oscillate radially, and their number depends on the forcing frequency. We report the instability “tongues” of the patterns up to n = 25. These resonance frequencies are well explained by adapting to a fluid torus the usual drop model of Lord Rayleigh. This approach could be applied to the modeling of large-scale structures arisen transiently in vortex rings in various domains.

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Vortex rings are ubiquitous in Nature. They occur at different scales in various domains: hydrodynamics [1], plasma physics [2, 3], geophysics during volcanic eruptions [4], quantum gravity [5], and biophysics such as bioconvection [6], or underwater bubble rings produced by dolphins [7]. Since their first mathematical analysis in 1858 [8], their formation, dynamics, or collision have been extensively studied [9–11]. Indeed, it is common to generate transient vortex rings (such as smoke rings) in a laboratory by pushing a fluid out of a tube or of a hole between two transparent plates, at the periphery of a cylinder, and has an elliptical section of semi-major and semi-minor axes a = (R − Ri)/2 = 3 mm and b = h/2 (see Fig. 1). The aspect ratio of the torus χ ≡ (R + Ri)/(2a) = 6, roughly twice the one of typical doughnut confectionery.

FIG. 1: Top: Experimental setup. Plate forcing: A sin(2πft). Bottom: top view of fluid torus deformations (mode 7).

A schematic view of the experimental setup is shown in Fig. 1. A volume of mercury (V = 1.5 mL) is injected, between two transparent plates, at the periphery of a cylinder to form a stable torus of fluid of outer (resp. inner) radius R = 21 mm (resp. Ri = 15 mm). The distance between the top and bottom plates is h = 1.5 mm (except otherwise stated). Due to gravity, the torus is flattened and has an elliptical section of semi-major and semi-minor axes a = (R − Ri)/2 = 3 mm and b = h/2 (see Fig. 1). The aspect ratio of the torus χ ≡ (R + Ri)/(2a) = 6, roughly twice the one of typical doughnut confectionery.
The photodiode measures the amplitude of the horizontal oscillations of the torus outer periphery over time. By comparing both amplitudes oscillating initially in phase at \( f \), one can define accurately the onset of the instability when the fluid oscillations become slightly modulated in amplitude as a precursor of the subharmonic behavior at \( f/2 \) of the fluid in response to the forcing at \( f \) (see Fig. 1 in Supp. Material [31]). Then, we note the corresponding amplitude of the shaker, \( A_1 \), and the mode number, \( n \), of the pattern observed from the camera. Indeed, above this critical amplitude, \( A_1 \), an azimuthal pattern is observed in the horizontal plane at the torus outer periphery (see Fig. 2): lobes oscillate radially at \( f/2 \), i.e., half the forcing frequency. When \( f \) is increased, the number \( n \) of oscillating lobes increases as shown in Figs. 2a-b. Movies and more pictures are shown in Supp. Material [31].

Similar experiments are also carried out without the solid cylinder located in the center of the cell, to form a flattened puddle of fluid (see Figs. 2c-d) with an outer diameter adjusted to the previous value for the torus. The pattern observed for a torus is found to have the same properties as the one for a puddle, for the same \( f \) (see Figs. 2a and c or Figs. 2b and d). Moreover the evolution of \( n \) with \( f \) is similar. No azimuthal oscillation in the inner diameter of the torus occurs. This means that the instability occurs only near the ring outer periphery and the bulk plays no major role.

The derivation of the eigenfrequency \( f_n \) of an inviscid fluid torus has to be independent of the nature of the forcing and arises from the interplay between inertia and surface tension effects. We consider small radial deformations of the outer peripheral surface of amplitude \( \eta_n(t) \approx R \). In polar coordinates, this reads \( r(\theta, t) = R + \eta_n(t) \cos(n\theta) \) (see Fig. 1). In the limit \( 2R \gg h \), the torus shape is approximated by a thin hollow cylinder. The radial amplitude of the lobes \( \eta_n(t) \) is then governed by a harmonic oscillator equation, \( d^2\eta_n(t)/dt^2 + \omega_n^2\eta_n(t) = 0 \), of eigenfrequency \( f_n \) (see Supp. Material [31]).

\[
\omega_n^2 = \frac{\gamma}{\rho R^2} n(n^2 - 1) \frac{1 - (R_i/R)^{2n}}{1 + (R_i/R)^{2n}} \text{ for } n > 1
\]

with \( \omega_n = 2\pi f_n \). To wit, we have adapted the Rayleigh derivation valid for a puddle \( (\omega_n^2 = \frac{\gamma}{\rho h R^2} n(n^2 - 1)) \) [28, 29] to a fluid torus. A correction factor \( (\leq 1) \) thus occurs that tends to 1 for large \( n \). For our torus aspect ratio, both models are almost similar (1.3% difference for \( n = 2 \) and 0.2% for \( n = 3 \)). In addition to these azimuthal modes, an axisymmetric mode due to the flattening of the confined torus on the top plate can be also derived (see Supp. Material [31]).

\[
\omega_1^2 = \frac{\gamma}{\rho h R^2} \left( \frac{2(R^2 - R_i^2)}{R^2 - R_i^2} \ln \frac{R}{R_i} \right)\]

The vertical vibrations of the substrate force the fluid parametrically, leading thus to a Mathieu equation for
We now define \( f_{\text{min}} \) as the forcing frequency for which the \( n \)th tongue is minimum in Fig. 3. The resonance frequency \( f_n \) of the azimuthal mode \( n \) is thus inferred from Fig. 3 as \( f_n = f_{\text{min},n} / 2 \). We plot in Fig. 4 these experimental resonance frequencies for a torus of fluid, and for a puddle, as a function of \( n(n^2 - 1)/R^3 \) in order to compare with the prediction of Eq. (1) (see solid line). If one takes into account the flattening axisymmetric mode of Eq. (2) (also called breathing mode for a puddle), the agreement with \( f_n^2 + f_m^2 \) is excellent on 2 decades with no fitting parameter and regardless of the fluid geometry. Such a nontrivial coupling would deserve further studies. As shown in the Supp. Material, we reiterate numerous experiments to measure \( f_n \) for different experimental parameters (notably \( R, h, \) plates fixed together or not), and the law in \( n(n^2 - 1)/R^3 \) is still found to be valid.

As quantified above, the resonance frequencies of a torus are expected to be the same as the ones of a puddle for large enough \( n \). Physically, it occurs when the azimuthal oscillations occurring at the outer boundary of the torus do not feel the presence of the solid central cylinder. This is the case when their typical wavelength is much smaller than the torus horizontal thickness, i.e. \( \lambda \ll 4\pi a \). From the torus outer perimeter, one has \( 2\pi R \simeq n\lambda \), and thus the condition reads \( n \gg R/(2a) = (1 + \chi)/2 = 3.5 \). The mode number \( n \) should thus be much larger than the torus aspect ratio. Here, this condition is almost validated since \( n \in [5, 25] \). We cannot reach smaller values of \( n \) on Fig. 3 since the shaker amplitude is limited. Moreover, no break up of the torus by Rayleigh-Plateau (RP) instability is observed here due to the presence of the cylinder. Indeed, RP should occur for \( \lambda \geq 2\pi a \), that is for \( n \leq R/a = 7 \).

We now define \( A_{\text{min}} \) as the minimum amplitude of the \( n \)th tongue in Fig. 3. These minima also display a minimum with \( f \) (see Fig. 3). Figure 5 then shows \( 1/A_{\text{min}} \) as a function of \( f \) in the case of a fluid torus or of a puddle. Similar behavior is observed regardless of the geometry: Much less energy is needed to reach the instability threshold near \( f = f_0 \). This comes from the occurrence of a confined poloidal mode (oscillating at \( f \)) observed before each azimuthal mode and resonating at \( f_0 \) [see Supp. Material (Movie a13.mp4) for a puddle]. For different confinement depths \( h \), one finds \( f_0^2 \sim h^{-3} \), as displayed in...
Fig. 5, the error bar being due to the depth measurement uncertainty (0.1 mm). This poloidal mode is thus clearly related to the modulation of the confinement and not of the gravity one since \( h \lesssim 2 l_c \), the acceleration is weak (< 0.5g) and the fluid never loses contact with the plates. This poloidal mode can be explained by the oscillations (in phase opposition) of the outer free surface of the fluid outer boundary to be half a typical wavelength in the vertical plane, \( l \sim \alpha h \) (see inset of Fig. 5b). One has from geometry \( l = \alpha h \), with \( \alpha = \pi / 2 \approx 1.57 \) for a half-circle, or an adjustable parameter experimentally quantifying the more realistic ellipsoidal section of the ring. One has thus \( \lambda_2 = 2 \alpha h \). Using the usual dispersion relation of capillary waves \( \omega^2_2 = (\gamma / \rho) k^2_2 \) with \( k_2 = 2 \pi / \lambda_2 \), one has thus \( f_2^2 = 2 \pi (\gamma / \rho) / (2 \alpha h)^2 \). Figure 5 shows that this law is in better agreement with experiments for \( \alpha = 1.9 \) (dashed line) than for \( \pi / 2 \) (dash-dotted line). Indeed, the torus is not of circular section due to the fluid-plate interactions.

The theoretical frequency response of a simple harmonic oscillator \( \sim 1 / \sqrt{f_0^2 - f^2} \), with \( f_0 \) given above, is then in good agreement with the data, as shown in Fig. 5a.

Finally, an indirect measurement of the geometric properties of the section of the elliptic torus is inferred. Balancing the half-ellipse perimeter \( \approx \pi / 2 \sqrt{2(a^2 + b^2)} \) with \( l \), and using its eccentricity \( e \equiv \sqrt{1 - (b/a)^2} \), with \( a \) its semi-major and \( b = h / 2 \) its semi-minor axes, one finds \( a = \frac{h}{\sqrt{8} \alpha^2 / \pi^2 - 1} \) and \( e = \sqrt{1 - 8 \alpha^2 / \pi^2 - 1} \). For a circle (\( \alpha = \pi / 2 \)), one has \( a / b = 1 \) and \( e = 0 \), as expected. For \( \alpha = 1.9 \), as inferred experimentally from Fig. 5b, it corresponds thus to an ellipsoidal section of the ring with \( a / b \approx 1.4 \) and \( e \approx 0.7 \). These experiments were made with top and bottom plates fixed together, distant of \( h \), and vibrating thus in phase (see empty symbols in Fig. 5b), in order to avoid the fluid flattening with a fixed top plate and a vibrating bottom one (\( \alpha = 1.71 \)).

In conclusion, we reported the first quantitative measurements of the azimuthal patterns on a stable torus of fluid. Using analytical calculations, we showed that they correspond to the eigenmodes of a thin hollow cylinder provided \( n \) is much larger than the torus aspect ratio. This approach could be applied to the modeling of the transient large-scale structures in vortex rings in various domains. Our experimental configuration can be easily used to study a stable vortex ring including (poloidal) vorticity by just applying a Lorentz force to the liquid metal. The solid internal confinement could be also replaced by a toroidal potential. This should reveal more precisely the origin of these transient large-scale structures in vortex rings.

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[1] T. J. Maxworthy, J. Fluid Mech. 51, 15 (1972).
[2] M. Gharib, S. Mendozaa, M. Rosenfeldb, M. Beizaia, and F. J. Alves Pereira, Proc. Nat. Acad. Sci. 114, 12657 (2017).
[3] C. D. Cothran, M. R. Brown, T. Gray, M. J. Schaffer, and G. Marklin, Phys. Rev. Lett. 103, 215002 (2009).
[4] J. Taddeucci, M. A. Alatorre-Ibaraguengoitia, D. M. Paladino, P. Scarlato, and C. Camaldo, Geophys. Res. Lett. 42, 6253 (2015).
[5] M. A. Abramowicz, O. M. Blaes, J. Horák, W. Kluźniak,
and P. Rebusco, Classical and Quantum Gravity 23, 1689 (2006).

[6] R. Simkus, V. Kirejev, R. Meskiene, and R. Meskys, Exp. Fluids 46, 365 (2009).

[7] K. Marten, K. Shariﬁ, S. Psarakos, and D. White, Scientiﬁc American 275, 22 (1996).

[8] H. von Helmholtz, J. für die reine und angewandte Mathematik 55, 22 (1958).

[9] G. K. Batchelor, An Introduction to Fluid Dynamics (Cambridge University Press, London, 1967).

[10] P. G. Saffman, Vortex Dynamics (Cambridge University Press, Cambridge, 1992).

[11] P. G. Saffman, J. Fluid Mech. 84, 625 (1978).

[12] P. G. Saffman, J. Fluid Mech. 84, 625 (1978).

[13] P. G. Saffman, J. Fluid Mech. 84, 625 (1978).

[14] J. Deng, J. Xue, X. Mao, and C. P. Caulﬁeld, Phys. Rev. F 2, 022701(R) (2017).

[15] Y. Renardy, S. Popinet, L. Duchemin, M. Renardy, S. Zaleski, C. Josserand, M. A. Drumright-Clarke, D. Richard, C. Clanet, and D. Quéré, J. Fluid Mech. 484, 69 (2003).

[16] N. Baumann, D. D. Joseph, P. Mohr, and Y. Renardy, Phys. Fluids 4, 567 (1992).

[17] M. C. Sostarecz and A. Belmonte, J. Fluid Mech. 497, 03 (2003).

[18] E. Zoueshtiagh, H. Caps, F. Legendre, N. Vandewalle, P. Petitjeans, and P. Kurowski, Eur. Phys. J. E 20, 317 (2006).

[19] M. A. Worthington, Proc. R. Soc. Lond. 30, 49 (1880).

[20] E. Pairam and A. Fernandez-Nieves, Phys. Rev. Lett. 102, 234501 (2009).

[21] J. C. Bird, R. de Ruiter, L. Courbin, and H. A. Stone, Nature 465, 759 (2010).

[22] J. D. McGraw, J. Li, D. L. Tran, A.-C. Shia, and K. Dahoki-Veress, Soft Matter 6, 1258 (2010).

[23] A. A. Fragiopoulos and A. Fernandez-Nieves, Phys. Rev. E 95, 033122 (2017).

[24] P. Orlandi and R. Verzicco, J. Fluid Mech. 256, 615 (1993).

[25] M. Cheng, J. Lou, and L.-S. Luo, J. Fluid Mech. 660, 430 (210).

[26] S. Perrard, Y. Couder, E. Fort, and L. Limat, EPL (Europhysics Letters) 100, 54006 (2012).

[27] E. Pairam, J. Vallamkondu, V. Koning, B. C. van Zuijlen, P. W. Ellis, M. A. Bates, V. Vitelli, and A. Fernandez-Nieves, Proc. Nat. Acad. Sci. 110, 9295 (2013).

[28] H. Lamb, Hydrodynamics (Dover, New York, 1932), 6th ed.

[29] L. Rayleigh, Proc. R. Soc. London 29, 71 (1879).

[30] Ultra Ever Dry® spray — See http://ultraeverdrytap.com/index.html.

[31] See Supplemental Material [url] for experimental details, movies, pictures, further data analyses and derivation details, which includes Refs. 32–39.

[32] N. Yoshiyasu, K. Matsuda, and R. Takaki, J. Phys. Soc. Jpn. 65, 2068 (1996).

[33] T. Jamin, Y. Djama, J.-C. Bacri, and E. Falcon, Phys. Rev. Fluids 1, 021901(R) (2016).

[34] J. Mathews and R. L. Walker, Mathematical Methods of Physics (Addison-Wesley Pub., New York, 1969), 2nd ed.

[35] X. Ma and J. C. Burton, J. Fluid Mech. 846, 263 (2018).

[36] For the bottom conical plate, the depth between the two plates is 2.4 mm at the center, and 1.5 mm at the tore/puddle periphery ($R = 21$ mm).

[37] F. Celestini and R. Kofman, Phys. Rev. E 73, 041602 (2006).

[38] C.-T. Chang, J. B. Bostwick, P. H. Steen, and S. Daniel, Phys. Rev. E 88, 023015 (2013).

[39] X. Noblin, A. Buguin, and F. Brochard-Wyart, Phys. Rev. Lett. 94, 166102 (2005).