Invariant Means, Complementary Averages of Means, and a Characterization of the Beta-Type Means

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Abstract: We prove that whenever the selfmapping \((M_1, \ldots, M_p) : I^p \to I^p, (p \in \mathbb{N} \text{ and } M_i \text{-s are } p\text{-variable means on the interval } I)\) is invariant with respect to some continuous and strictly monotone mean \(K : I^p \to I\) then for every nonempty subset \(S \subseteq \{1, \ldots, p\}\) there exists a uniquely determined mean \(K_S : I^p \to I\) such that the mean-type mapping \((N_1, \ldots, N_p) : I^p \to I^p\) is \(K\)-invariant, where \(N_i := K_S\) for \(i \in S\) and \(N_i := M_i\) otherwise. Moreover \(\min(M_i ; i \in S) \leq K_S \leq \max(M_i ; i \in S)\). Later we use this result to: (1) construct a broad family of \(K\)-invariant mean-type mappings, (2) solve functional equations of invariant-type, and (3) characterize Beta-type means.

Keywords: invariant means; complementary averages of means; characterizations; beta-type means

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1. Preliminaries

In the whole paper \(I \subset \mathbb{R}\) stands for an interval, \(p \in \mathbb{N}, p > 1\) is fixed, and \(\mathbb{N}_p := \{1, \ldots, p\}\). A function \(M : I^p \to I\) is called a mean on \(I\) if

\[
\min \left( x_1, \ldots, x_p \right) \leq M \left( x_1, \ldots, x_p \right) \leq \max \left( x_1, \ldots, x_p \right), \quad x_1, \ldots, x_p \in I,
\]

or, briefly, if

\[
\min x \leq M (x) \leq \max x, \quad x = (x_1, \ldots, x_p) \in I^p.
\]

The mean \(M\) is called strict, if for all nonconstant vectors \(x\) these inequalities are sharp; and symmetric, if \(M \left( x^{(1)}, \ldots, x^{(p)} \right) = M \left( x_1, \ldots, x_p \right)\) for all \(x_1, \ldots, x_p \in I\) and all permutations \(\sigma\) of the set \(\mathbb{N}_p\). Mean \(M\) is monotone if it is increasing in each of its variables. It is important to emphasize that every strictly monotone mean is strict.

A mapping \(M : I^p \to I^p\) is referred to as mean-type if there exist some means \(M_i : I^p \to I, i \in \mathbb{N}_p\), such that \(M = (M_1, \ldots, M_p)\). For a mean-type mapping \(M : I^p \to I^p\) we denote a projection onto the \(i\)-th coordinate by \([M]_i : I^p \to I\). In this case we obviously have \([M]_i = M_i\) for all \(i \in \mathbb{N}_p\).

We say that a function \(K : I^p \to \mathbb{R}\) is invariant with respect to \(M\) (briefly \(M\)-invariant), if \(K \circ M = K\).

**Theorem 1** (Invariance Principle, [1]). If \(M : I^p \to I^p, M = (M_1, \ldots, M_p)\) is a continuous mean-type mapping such that

\[
\max M (x) - \min M (x) < \max (x) - \min (x), \quad x \in I^p \setminus \Delta (I^p),
\]
where $\Delta(I^p) := \{ x = (x_1, \ldots, x_p) \in I^p : x_1 = \ldots = x_p \}$ then there is a unique $M$-invariant mean $K : I^p \to I$ and the sequence of iterates $(M^n)_{n \in \mathbb{N}}$ of the mean-type mapping $M$ converges to $K := (K, \ldots, K)$ pointwise on $I^p$.

2. A Family of Complementary Means

Let us recall the following result.

Remark 1 (Matkowski [2]). Assume that $K : I^2 \to I$ is a symmetric mean which is continuous and monotone. Then

(i) for an arbitrary mean $M_1 : I^2 \to I$ there is a unique mean $M_2 : I^2 \to I$ such that $K$ is $(M_1, M_2)$-invariant,

\[ K \circ (M_1, M_2) = K; \]

$K_{M_1} := M_2$, is referred to as a $K$-complementary mean for $M_1$; and we have

\[ K_{M_1} = M_2 \iff K_{M_2} = M_1; \]

(ii) if $M_1, M_2 : I^2 \to I$ are means such that $K$ is $(M_1, M_2)$-invariant, then there exists a unique mean $M : I^2 \to I$ such that

\[ \min (M_1, M_2) \leq M \leq \max (M_1, M_2) \]

and

\[ K \circ (M, M) = K; \]

moreover

\[ M = K. \]

In this case, $p \geq 3$ the counterpart of part (i) of Remark 1 is false which shows the following

Example 1. Let $I = \mathbb{R}$, and $K = A$ where $A(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3}$. The functions $M_1(x_1, x_2, x_3) = \frac{x_1 + x_2}{2}$, $M_2(x_1, x_2, x_3) = x_2$ are means in $\mathbb{R}$. Then it is easy to see that there is no mean $M_3$ such that

\[ A \circ (M_1, M_2, M_3) = A, \]

but a partial counterpart of part (ii) holds true.

The main result of this section reads as follows.

Theorem 2. Let $M_1, \ldots, M_p : I^p \to I$ be means. Assume that $K : I^p \to I$ is a continuous and monotone mean which is invariant with respect to the mean type mapping $M := (M_1, \ldots, M_p)$.

Then for every nonempty subset $S \subseteq \mathbb{N}_p$ there exists a unique mean $K_S(M) : I^p \to I$ such that $K$ in $K_S(M)$-invariant, where $K_S(M) : I^p \to I^p$ is given by

\[ [K_S(M)]_i := \begin{cases} 
K_S(M) & \text{for } i \in S, \\
M_i & \text{for } i \in \mathbb{N}_p \setminus S.
\end{cases} \quad (1) \]

Moreover, $\min(M_i : i \in S) \leq K_S(M) \leq \max(M_i : i \in S)$.

Proof. In the case $S = \mathbb{N}_p$ the $K_S(M)$-invariance of $K$ implies $K_S(M) = K$ and the statement is obvious. From now on we assume that $S \neq \mathbb{N}_p$. 

}\]
Denote briefly $M_\vee := \max\{M_i : i \in S\}$ and $M_\wedge := \min\{M_i : i \in S\}$. Fix $x \in \mathbb{R}^p$ arbitrarily. Define a function $T: I \to I^p$ by

$$[T(a)]_i := \begin{cases} M_i(x) & i \in \mathbb{N}_p \setminus S, \\ \alpha & i \in S. \end{cases}$$

and $f: I \to I$ by $f(a) := K \circ T(a)$. Then, as $K$ is continuous and strictly increasing, so is $f$. Therefore in view of the equality $K \circ M(x) = K(x)$ we obtain $f(M_\vee(x)) \geq K(x)$ and $f(M_\wedge(x)) \leq K(x)$. Thus, there exists unique number $a_\alpha \in [M_\wedge(x), M_\vee(x)]$ such that $f(a_\alpha) = K(x)$. Now, as $x \in \mathbb{R}^p$ was arbitrary we define $K_S(M)(x) := a_\alpha$.

Then we have

$$K(x) = f(a_\alpha) = f(K_S(M)(x)) = (K \circ T \circ (K_S(M)))(x) = (K \circ (K_S(M)))(x),$$

which shows that $K$ is $K_S(M)$-invariant.

Now we need to show that $K_S(M)$ is uniquely determined. Assume that $K$ is $K_S(M)$-invariant and $K_S(M)(x) \neq a_\alpha$ for some $x \in \mathbb{R}^p$. Then, as $f$ is a monomorphism we obtain

$$K \circ (K_S(M))(x) = (K \circ T \circ (K_S(M)))(x) \neq f(a_\alpha) = K(x)$$

contradicting the $K_S(M)$-invariance. □

Let us underline that we do not assume that means $M_1, \ldots, M_p$ are continuous. This is relatively recent approach to invariant property which was studied by the authors in [3].

The intuition beyond this theorem is the following. Once we have a continuous and monotone mean $K$ such that $M$ is $K$-invariant mean we can unite a subfamily $(M_i)_{i \in S}$ into a single mean (denoted by $K_S(M)$) to preserve the $K$-invariance. In view of Theorem 1, such a mean is unique. In this connection we propose the following

**Definition 1.** Let $K: \mathbb{R}^p \to I$ be a continuous and monotone mean which is invariant with respect to the mean type mapping $M := (M_1, \ldots, M_p)$.

For each set $S \subseteq \mathbb{N}_p$

(i) the mean $K_S(M)$ is called a $K$-complementary averaging of the means $(M_i : i \in S)$ with respect to the invariant mean-type mapping $M = (M_1, \ldots, M_p)$;

(ii) the mean-type mapping $K_S(M)$ given by (1) is called a $K$-complementary averaging of the mean-type mapping $M = (M_1, \ldots, M_p)$ with respect to the means $(M_i : i \in S)$.

Moreover, the set

$$\mathcal{R}(K, M) := \{K_S(M) : S \subseteq \mathbb{N}_p, S \neq \emptyset\}$$

is called the family of all $K$-complementary averaging of the mean-type mapping $M = (M_1, \ldots, M_p)$.

We can now reapply this result to the complementary of the establish a $K$-complementary of $K_S(M)$ for the set $\mathbb{N}_p \setminus S$. More precisely we obtain

**Corollary 1.** Under the assumptions of Theorem 2 there exists unique mean $K_S^*(M): \mathbb{R}^p \to I$ such that $K$ is $K_S^*(M)$-invariant, where $K_S^*(M): \mathbb{R}^p \to \mathbb{R}^p$ is given by

$$[K_S^*(M)]_i := \begin{cases} K_S(M) & \text{for } i \in S, \\ K_S^*(M) & \text{for } i \in \mathbb{N}_p \setminus S. \end{cases}$$

Moreover, $\min(M_i : i \in \mathbb{N}_p \setminus S) \leq K_S^*(M) \leq \max(M_i : i \in \mathbb{N}_p \setminus S)$. 

Let us underline that the value $K^*_S(M)$ does not depend on $M$ explicitly. The whole system of dependences is illustrated in Figure 1.

Observe that, as the mean $K^*_S(M)$ is uniquely determined, we obtain

$$K^*_S(M) \in \mathcal{R}(K, M) \iff \text{all means } (M_i : i \in \mathbb{N}_p \setminus S) \text{ are equal to each other.}$$

**Figure 1.** Map of dependencies between means. Rectangle vertexes are dependent on $S$. Dotted line means that there could exists more $M$-invariant mean satisfying the conditions of Theorem 2.

## 3. Application in Solving Functional Equations

**Theorem 3.** Let $M = (M_1, \ldots, M_p)$ be a mean-type mapping such that $M_1, \ldots, M_p : (0, \infty)^p \to (0, \infty)$ are strictly monotonic and homogeneous. Then

(i) the sequence $(M^n : n \in \mathbb{N})$ of iterates of $M$ converge uniformly on compact subsets to a mean-type mapping $K = (K, \ldots, K)$, where $K$ is a unique $M$-invariant mean.

(ii) $K$ is monotone, homogeneous and for every $S \subset \mathbb{N}_p$ the iterates of $K^*_S(M)$ converge uniformly on compact subsets to a mean-type mapping $K = (K, \ldots, K)$;

(iii) a function $F : (0, \infty)^p \to \mathbb{R}$ is continuous on the diagonal $\Delta((0, \infty)^p) := \{(x_1, \ldots, x_p) \in (0, \infty)^p : x_1 = \cdots = x_p\}$ and satisfies the functional equation

$$F \circ M = F$$

if and only if $F = \varphi \circ K$, where $\varphi : (0, \infty) \to \mathbb{R}$ is an arbitrary continuous function of a single variable;

(iv) a function $F : (0, \infty)^p \to \mathbb{R}$ is continuous on the diagonal $\Delta((0, \infty)^p)$ and satisfies the simultaneous system of functional equations

$$F \circ K^*_S(M) = F_S, \quad S \subset \mathbb{N}_p;$$

if and only if $F = \varphi \circ K$, where $\varphi : (0, \infty) \to \mathbb{R}$ is an arbitrary continuous function of a single variable (so (2) and (3) are equivalent).

**Proof.** The homogeneity and monotonicity of $M_1, \ldots, M_p$ imply their continuity (cf. ([4], Theorem 2)), so the invariance principle implies (i).

Now we prove that $K$ is monotone. Indeed, take two vectors $v, w \in (0, \infty)^p$ such that $v_i \leq w_i$ for all $i \in \mathbb{N}_p$ and $v_{i_0} < w_{i_0}$ for certain $i_0 \in \mathbb{N}_p$. Then, as each $M_i$ is monotone, there exists a constant $\theta \in (0, 1)$ such that $M_i(v) \leq \theta M_i(w)$ for all $i \in \mathbb{N}_p$. 


Then for all \( n \in \mathbb{N} \) and \( i \in \mathbb{N}_p \) we have

\[
[M^n(v)]_i = [M^{n-1}(M_1(v), \ldots, M_p(v))]_i \leq [M^{n-1}(\theta M_1(w), \ldots, \theta M_p(w))]_i = \theta [M^n(w)]_i.
\]

In a limit case as \( n \to \infty \) in view of the first part of this statement we obtain \( K(v) \leq \theta K(w) < K(w) \).

Thus, \( K \) is monotone, which is (ii).

(iii) Assume first that \( F : (0, \infty)^p \to \mathbb{R} \) that is continuous on the diagonal \( \Delta \left( (0, \infty)^p \right) \) and satisfies Equation (2). Hence, by induction,

\[
F = F \circ M^n, \quad n \in \mathbb{N},
\]

By (ii) the sequence \( (M^n : n \in \mathbb{N}) \) converges to \( K = (K, \ldots, K) \). Since \( F \) is continuous on \( \Delta \left( (0, \infty)^p \right) \), we hence get for all \( x \in (0, \infty)^p \),

\[
F(x) = \lim_{n \to \infty} F(M^n(x)) = F \left( \lim_{n \to \infty} M^n(x) \right) = F(K(x)) = F(K(x), \ldots, K(x)).
\]

Setting

\[
\varphi(t) := F(t, \ldots, t), \quad t \in (0, \infty),
\]

we hence get \( F(x) = \varphi(K(x)) \) for all \( x \in (0, \infty)^p \).

To prove the converse implication, take an arbitrary function \( \varphi : I \to \mathbb{R} \) and put \( F := \varphi \circ K \). Then, for all \( x \in (0, \infty)^p \), making use of the \( K \)-invariance with respect to \( M \), we have

\[
F(M(x)) = (\varphi \circ K)(M(x)) = \varphi(K(M(x))) = \varphi(K(x)) = F(x),
\]

which completes the proof of (iii).

(iv) we omit similar argument. \( \square \)

Part (ii) of this result gives rise to the following extension.

### 3.1. General Complementary Process

Once we have a mean-type \( M : I^p \to I^p \) and a continuous and monotone mean \( K : I^p \to I \) which is \( M \)-invariant let \( \mathfrak{R}^+(M,K) \) be the smallest family of mean-type mappings containing \( M \) which is closed under \( K \)-complementary averaging.

More precisely, for every \( X \in \mathfrak{R}^+(M,K) \) and nonempty subset \( S \subseteq \mathbb{N}_p \) we have \( K_S(X) \in \mathfrak{R}^+(M,K) \), too. We also define a family of means

\[
\mathfrak{R}_0(M,K) := \{ [X]_i : X \in \mathfrak{R}^+(M,K) \text{ and } i \in \mathbb{N}_p \}
\]

Obviously using notions from Theorem 2 and Corollary 1 we have

\[
\mathfrak{R}^+(M,K) \supseteq \mathfrak{R}^+(K_S(M),K) \supseteq \mathfrak{R}^+(K^*_S(M),K)
\]

Furthermore, we have the following.

**Proposition 1.** Given an interval \( I \subseteq \mathbb{R} \), \( p \in \mathbb{N} \), and a mean-type mapping \( M := (M_1, \ldots, M_p) : I^p \to I^p \) which is invariant with respect to some continuous and monotone mean \( K : I^p \to I \). Then

\[
\min(M_1, \ldots, M_p) \leq X \leq \max(M_1, \ldots, M_p) \quad \text{for all } X \in \mathfrak{R}_0(M,K).
\]

Its inductive proof is obvious in view of Theorem 2 (moreover part). Now we prove that complementary means preserve symmetry.
Proposition 2. If a continuous and monotone mean $K: I^p \rightarrow I$ is invariant with respect to a mean-type mapping $M := (M_1, \ldots, M_p): I^p \rightarrow I^p$ such that all $M_i$-s are symmetric, then $K$ and all means in $\mathcal{F}_0(M, K)$ are symmetric.

Proof. Fix a nonconstant vector $x \in I^p$ and a permutation $\sigma$ of $\mathbb{N}_p$. First observe that $K(x) = K \circ M(x) = K \circ M(x \circ \sigma) = K(x \circ \sigma)$, which implies that $K$ is symmetric.

As the family $\mathcal{F}_0(M, K)$ is generating by complementing, we need to show that symmetry is preserved by a single complement. Therefore, it is sufficient to show that the mean $K_S(M)$ defined in Theorem 2 is symmetric. However, using the notions therein, we have

$$K \circ (K_S(M))(x) = K(x) = K(x \circ \sigma) = K \circ (K_S(M))(x \circ \sigma).$$

By monotonicity of $K$, if $K_S(M)(x) < K_S(M)(x \circ \sigma)$ we would have

$$K \circ (K_S(M))(x) = K(x) < K \circ (K_S(M))(x \circ \sigma)$$

contradicting the above equality. Similarly we exclude the case $K_S(M)(x) > K_S(M)(x \circ \sigma)$. Therefore $K_S(M)(x) = K_S(M)(x \circ \sigma)$ which, as $x$ and $\sigma$ were taken arbitrarily, yields the symmetry of $K_S(M)$. □

4. An Applications to Beta-Type Means

Following [5], for a given $k \in \mathbb{N}$ we define a $p$-variable Beta-type mean $B_p: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ by

$$B_p(x_1, \ldots, x_p) := \left( \frac{px_1 \cdots x_p}{x_1 + \cdots + x_p} \right)^{\frac{1}{p-1}}.$$

This is a particular case of so-called bila planar-combinatoric means (Media biplana combinatoria) defined in Gini [6] and Gini–Zappa [7].

In order to formulate the next results, we adapt the notation that $A, G$ and $H$ are arithmetic, geometric and harmonic means of suitable dimension, respectively.

In [8], the invariance $G \circ (A, H) = G$, equivalent to the Pythagorean proportion, has been extended for arbitrary $p \geq 3$. In case $p = 3$ it takes the form $G \circ (A, F, H) = G$, where

$$F(x_1, x_2, x_3) := \frac{x_2x_3 + x_3x_1 + x_1x_2}{x_1 + x_2 + x_3}, \quad x_1, x_2, x_3 > 0,$$

and $H \leq F \leq A$. Hence, making use of Corollary 1 with $p = 3$, $K = G$, $S = \{1, 2\}$ we obtain the following.

Remark 2. For all $x_1, x_2, x_3$, the following inequality holds

$$H(x_1, x_2, x_3) \leq B_3(x_1, x_2, x_3) \leq A(x_1, x_2, x_3),$$

and the inequalities are sharp for nonconstant vectors $x = (x_1, x_2, x_3) \in (0, \infty)^3$.

Passing to the main part of this section, first observe the following easy-to-see lemma.

Lemma 1. Let $p \in \mathbb{N}$, $p \geq 2$. Then there exists exactly one mean $M: I^p \rightarrow I$ such that $G \circ (A, M_1, \ldots, M_p) = G$. Furthermore $M = B_p$.

Its simple proof which is a straightforward implication of Theorem 2 is omitted. Based on this lemma it is natural to define a mean-type mapping $B: I^p \rightarrow I^p$ by $[B]_1 := A$ and $[B]_i := B_p$ for all

$$[\sigma]\cdot [B]^{(p-1)}$$

where $\sigma$ is a permutation of $1, 2, \ldots, p$.
\( i \in \{2, \ldots, p\} \). Then we have \( G \circ B = G \), which implies that the geometric mean is the unique \( B \)-invariant mean.

We are now going to establish the set \( \mathbb{R}^+ (B, G) \). It is quite easy to observe that all means in \( \mathcal{R}_0 (B, G) \) are of the form \( \mathcal{H}_{p,a} : I^p \to I \)\((a \in \mathbb{R})\) given by

\[
\mathcal{H}_{p,a}(x_1, \ldots, x_p) := (x_1 \cdots x_p)^{\frac{1-a}{p}} \left( \frac{x_1 + \cdots + x_p}{p} \right)^a
\]

including \( B_p = \mathcal{H}_{p, \frac{1}{p-1}} \). In the next lemma we show some elementary properties of the family \( \{\mathcal{H}_{p,a}\} \).

**Lemma 2.** Let \( p \in \mathbb{N} \). Then

1. \( \mathcal{H}_{p,a} \) is reflexive for all \( a \in \mathbb{R} \), that is \( \mathcal{H}_{p,a}(x, \ldots, x) = x \) for all \( x \in \mathbb{R}_+ \),
2. \( \mathcal{H}_{p,a} \) is continuous for all \( a \in \mathbb{R} \) (as a \( p \)-variable function),
3. \( \mathcal{H}_{p,a} \) is a strict mean for all \( a \in [-\frac{1}{p-1}, 1] \),
4. \( \mathcal{H}_{p,a} \) is a symmetric function for all \( a \in \mathbb{R} \), that is \( \mathcal{H}_{p,a}(x \circ \sigma) = \mathcal{H}_{p,a}(x) \) for all \( x \in \mathbb{R}_+^p \) and a permutation \( \sigma \) of \( \mathbb{N}_p \),
5. \( \mathcal{H}_{p,1} \) and \( \mathcal{H}_{p,0} \) are \( p \)-variable arithmetic and geometric means, respectively,
6. \( \mathcal{H}_{p,a} \) is increasing with respect to \( a \), that is \( \mathcal{H}_{p,a}(x) < \mathcal{H}_{p,\beta}(x) \) for every nonconstant vector \( x \in \mathbb{R}_+^p \) and \( a, \beta \in \mathbb{R} \) with \( a < \beta \).

**Proof.** By the definition of \( \mathcal{H}_{p,a} \) we can easily verify that (1), (2), (4) and (5) holds.

From now on fix a nonconstant vector \( x = (x_1, \ldots, x_p) \in \mathbb{R}_+^p \). By (4) we may assume without loss of generality that \( x_1 \leq \cdots \leq x_p \). Denote briefly

\[
\mathcal{g} := \sqrt[p]{x_1 \cdots x_p} \quad \text{and} \quad a := \frac{x_1 + \cdots + x_p}{p}.
\]

By Cauchy inequality we have \( x_1 < \mathcal{g} < a < x_p \). Moreover, by the definition \( \mathcal{H}_{p,a}(x) = a^a \mathcal{g}^{1-a} \).

Thus, for all \( a < \beta \) we have

\[
\mathcal{H}_{p,\beta}(x) = a^\beta \mathcal{g}^{1-\beta} = a^a \mathcal{g}^{1-a} (\mathcal{g}^{\frac{\beta}{a}})^{\beta-a} > a^a \mathcal{g}^{1-a} = \mathcal{H}_{p,a}(x),
\]

which completes the proof of (6). The only remaining part to be proved is (3). However, applying (6), it is sufficient to show that

\[
\mathcal{H}_{p,1}(x) < \max(x) = x_p \quad \text{and} \quad \mathcal{H}_{p,\frac{1}{p-1}}(x) > \min(x) = x_1.
\]

By (5) we immediately obtain \( \mathcal{H}_{p,1}(x) = a < x_p \). For the second part observe that

\[
\mathcal{H}_{p,\frac{1}{p-1}}(x) = p^{-1} \sqrt[p]{\frac{\mathcal{g}^p}{a}} = p^{-1} \sqrt[p]{\frac{x_1 \cdots x_p}{a}} > p^{-1} \sqrt[p]{\frac{x_1 \cdots x_p}{x_p}} = p^{-1} \sqrt[x_1 \cdots x_{p-1}]{x_1} \geq x_1,
\]

which completes the proof. \( \square \)

Now we generalize Lemma 1 to the following form

**Lemma 3.** Let \( p \in \mathbb{N}, p \geq 2 \) and \( a \in \mathbb{R}^p \). Then \( G \circ (\mathcal{H}_{p,a_1}, \ldots, \mathcal{H}_{p,a_p}) = G \), if and only if \( a_1 + \cdots + a_p = 0 \).

Its proof is obvious in view of the identity \( G \circ (\mathcal{H}_{p,a_1}, \ldots, \mathcal{H}_{p,a_p}) = \mathcal{H}_{p,\frac{1}{p-1}(a_1 + \cdots + a_p)} \). Having this proved, let us show the next important result.
Theorem 4. Let $p \geq 2$. Then
\[
\mathcal{R}^+ \left( (A, B_p, \ldots, B_p), G \right) \subseteq \left\{ (\mathcal{H}_{p,a_1}, \ldots, \mathcal{H}_{p,a_p}) \mid a_1, \ldots, a_p \in \mathbb{Q} \cap \left[-\frac{1}{p-1}, 1\right] \text{ and } a_1 + \cdots + a_p = 0 \right\}.
\]
In particular
\[
\mathcal{R}_0 = \left( (A, B_p, \ldots, B_p), G \right) \subseteq \left\{ \mathcal{H}_{p,a} \mid a \in \mathbb{Q} \cap \left[-\frac{1}{p-1}, 1\right] \right\}.
\]

Proof. First observe that in view of Lemma 1 $G$ is $\mathcal{B}$-invariant, and thus the set $\mathcal{R}^+ (\mathcal{B}, G)$ is well-defined. Now denote briefly
\[
\Lambda := \{ (\mathcal{H}_{p,a_1}, \ldots, \mathcal{H}_{p,a_p}) \mid a_1, \ldots, a_p \in \mathbb{Q} \cap \left[-\frac{1}{p-1}, 1\right] \text{ and } a_1 + \cdots + a_p = 0 \}.
\]

Obviously $\mathcal{B} \in \Lambda$, so it is sufficient to prove that $\Lambda$ is closed with respect to $G$-complementary averaging. To this end take an arbitrary vector $(a_1, \ldots, a_p)$ real numbers such that $H := (\mathcal{H}_{p,a_1}, \ldots, \mathcal{H}_{p,a_p}) \in \Lambda$ and a nonempty subset $S \subseteq \mathbb{N}_p$.

By Theorem 2 there exists exactly one mean $G_S (H) : I^p \to I$ such that $G$ is $G_S (H)$-invariant, where $G_S (H) : I^p \to I^p$ is given by
\[
\left[ G_S (H) \right]_i := \begin{cases} 
\mathcal{H}_{p,a_i} & \text{for } i \in \mathbb{N}_p \setminus S, \\
G_S (H) & \text{for } i \in S.
\end{cases}
\]

On the other hand, in view of Lemma 3 we obtain that $G$ is invariant with respect to the mean-type mapping $H_0 : I^p \to I^p$ given by
\[
\left[ H_0 \right]_i := \begin{cases} 
\mathcal{H}_{p,a_i} & \text{for } i \in \mathbb{N}_p \setminus S, \\
H_{p,\beta} & \text{for } i \in S,
\end{cases}
\]
where $\beta = \frac{1}{|S|} \sum_{i \in S} a_i$.

As $G$ is both $G_S (H)$-invariant and $H_0$-invariant we obtain $G_S (H) = H_{p,\beta}$, and consequently $G_S (H) = H_0$. Observe that $H_0 \in \Lambda$ is a straightforward implication of the equality
\[
\sum_{i \in \mathbb{N}_p \setminus S} a_i + |S| \beta = \sum_{i \in \mathbb{N}_p} a_i = 0.
\]

Now we show that $\beta \in \mathbb{Q} \cap \left[-\frac{1}{p-1}, 1\right]$, which would complete the proof. However, this is easy in view of the definition of $\beta$ and the analogous property $a_i \in \mathbb{Q} \cap \left[-\frac{1}{p-1}, 1\right]$, which is valid for all $i \in S$. \qed

Remark 3. Let us emphasize that inclusions in the above theorem are strict. More precisely, we can prove by simple induction that the denominator of $a$ in irreducible form has no prime divisors greater than $p$.

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