Numerical studies of the Bethe–Salpeter equation for a two-fermion bound state

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Abstract. Some recent advances on the solution of the Bethe-Salpeter equation (BSE) for a two-fermion bound system directly in Minkowski space are presented. The calculations are based on the expression of the Bethe-Salpeter amplitude in terms of the so-called Nakanishi integral representation and on the light-front projection (i.e. the integration of the light-front variable $k^- = k^0 - k^3$). The latter technique allows for the analytically exact treatment of the singularities plaguing the two-fermion BSE in Minkowski space. The good agreement observed between our results and those obtained using other existing numerical methods, based on both Minkowski and Euclidean space techniques, fully corroborate our analytical treatment.

1. Introduction

More than half a century ago, in a seminal work [1] Salpeter and Bethe presented a dynamical equation for describing bound systems within the relativistic field theory. In the subsequent years, there has been a large number of applications of their integral equation, but mainly adopting Euclidean variables or effective reduction to a 3D space. More recently, a method based on the so-called Nakanishi integral representation (NIR) of the Bethe-Salpeter (BS) amplitude (see, e.g., Ref. [2] and references therein), has allowed to make substantial steps forward in obtaining accurate numerical solutions of the actual Bethe-Salpeter equation (BSE). With massive-boson exchanges, it has been investigated: (i) two-scalar bound and zero-energy states [3, 4, 5, 6, 7] as well as two-fermion ground states [8, 9], with a ladder kernel, governing, as well-known, the tail of the momentum distributions; (ii) a two-scalar system, with a cross-ladder kernel [10].

In this contribution, we present the formally exact integration of the singularities that prevent a straightforward application of the NIR for solving the two-fermion ladder BSE in Minkowski space, as it was accomplished in the case of two-scalar systems [4, 5, 6, 7]. Then, after exactly transforming BSE in a coupled eigen-equation system, we compare our eigenvalues with both (i) the ones still obtained in Minkowski space [8], but introducing an auxiliary smoothing function, and (ii) outcomes in Euclidean space [11].

This contribution is so organized. In Sect. 2, we recall the general properties of the BS amplitudes as derived from its transformation properties under various symmetries. In Sect. 3, we present the method of the NIR, the light-front projection, and some of the obtained results.
Finally, in Sect 4, we report the conclusions and also discuss the perspectives of the present approach.

2. Properties of the Bethe-Salpeter amplitude

In this Section, we discuss about some properties of the BS amplitude for spin 1/2 fermions. To be definite, let us consider these spin 1/2 particles to be quarks. Then, the BS amplitude $\Phi_{\alpha,\beta}(k, p)$ of a bound quark-antiquark system (a meson) is defined to be

$$\Phi_{\alpha,\beta}(k, p) = \int d^4x_1 \ d^4x_2 e^{i(k+\bar{k})x_1+i(k-\bar{k})x_2}(0|T\{\psi^H_\alpha(x_1)\bar{\psi}^H_\beta(x_2)|M\}$$

where $|M\rangle$ is the state describing the bound system (the various quantum numbers defining it, as spin, parity, etc., are understood) and $\psi^H_\alpha(x)$ the quark fields in Heisenberg picture. The index $\alpha$ specifies one of the 4 Dirac component of the field, since it describes a spin 1/2 particle. The amplitude $\Phi_{\alpha,\beta}(k, p)$ is then a $4 \times 4$ matrix in the space of the Dirac components, and in general it can be decomposed in terms of the elements of the Clifford algebra. However, we have to take into account also the transformation properties of the Dirac fields under Lorentz, parity, and charge conjugation, as well as the quantum numbers of the meson bound state.

Let us consider first parity. Denoting with $U_P$ the unitary operator inducing the parity transformation in the Hilbert space, it is well known that [12]

$$U_P \psi^H_\alpha(x) U_P^\dagger = \eta_P (\gamma^0)_{\alpha,\alpha'} \psi^H_{\alpha'}(\bar{x}) ,$$

where $\eta_P$ is the quark intrinsic parity and $\bar{x} \equiv (t, -\vec{x})$. Using this property, we find

$$\Phi_{\alpha,\beta}(k, p) = \eta_P (\gamma^0)_{\alpha,\alpha'} \Phi_{\alpha',\beta}(\bar{k}, \bar{p})(\gamma^0)_{\beta',\beta} = \eta_P \gamma^0 \Phi(\bar{k}, \bar{p}) \gamma^0 ,$$

where $\eta_P$ is the parity of state $|M\rangle$, namely $U_P |M\rangle = \eta_P |M\rangle$.

Let us consider now the charge conjugation operator $U_C$. We know that [12]

$$U_C \psi^H_\alpha(x) U_C^\dagger = -i \left( \bar{\psi}^H(x) C \right)_\alpha^T ,$$

where with $X^T$ we denote the transpose of the quantity $X$, and $C = \gamma^0\gamma^2$. Assuming $U_C |M\rangle = \eta_C |M\rangle$, the BS amplitude has to satisfy

$$\Phi_{\alpha,\beta}(k, p) = \eta_C (C)_{\alpha,\alpha'} \Phi_{\alpha',\beta'}(-k, p)(C)_{\beta',\beta} .$$

For a $q\bar{q}$ system, charge conjugation is equivalent to the exchange of the two particles. Assuming that the $q\bar{q}$ system is in a state of relative orbital quantum number $L$ and total spin $S$ ($= 0, 1$), then the exchange produces a factor $(-)^{L+S}$, therefore $\eta_C = (-)^{L+S}$. For example, for a pion we know that $L = S = 0$, therefore $\eta_C = +1$, while for a $\rho$-meson $L = 0$, $S = 1$, and consequently $\eta_C = -1$, etc.

Finally, let us discuss the properties of the BS amplitude under an orthochronous Lorentz transformation $\Lambda$. Let us denote the operator inducing the Lorentz transformation as $U_\Lambda$. First of all,

$$U_\Lambda \psi^H_\alpha(x) U_\Lambda^{-1} = S(\Lambda)_{\alpha,\alpha'} \psi^H_{\alpha'}(\Lambda x) ,$$

where $S(\Lambda)$ are well known $4 \times 4$ matrices which can be written in terms of the $\gamma$ matrices [12]. The only property we need to know is the following

$$S(\Lambda) \gamma^\mu S(\Lambda)^{-1} = \Lambda^\mu_\nu \gamma^\nu .$$
Moreover, \( U_\Lambda|M,J,J_z\rangle = D_{J_z,J}^f(W)|M,J,J_z\rangle \), where \( W \) is the Wigner rotation [13] related to the Lorentz transformation \( \Lambda \) and now we have explicitly specified the total angular momentum quantum number \( J \) and its z-component of the meson state. The BS amplitude is found to verify

\[
\Phi_{\alpha,\beta}(k,p) = D_{J_z,J}^f(W)S(\Lambda)_{\alpha,\alpha'}\Phi_{\alpha',\beta'}(\Lambda k, \Lambda p)S(\Lambda)^{-1}_{\beta',\beta} ,
\]

(8)

For example, for a scalar particle \( D_{J_z,J}^f(W) = 1 \), and the above expression reads

\[
\Phi_{\alpha,\beta}(k,p) = S(\Lambda)_{\alpha,\alpha'}\Phi_{\alpha',\beta'}(\Lambda k, \Lambda p)S(\Lambda)^{-1}_{\beta',\beta} ,
\]

(9)

namely \( \Phi_{\alpha,\beta}(k,p) \) must be a scalar quantity. The most general expression verifying this condition is [14]

\[
\Phi(k,p) = A + B\gamma^5 + C\not{p} + D\not{k} + E\not{p}\gamma^5 + F\not{k}\gamma^5 + G\sigma^{\mu\nu}p_{\mu}k_{\nu} + H\sigma^{\mu\nu}p_{\mu}k_{\nu}\gamma^5 ,
\]

(10)

where \( A, B, \ldots \) are functions of the scalar quantities \( p^2, p \cdot k, \) and \( k^2 \), namely \( A \equiv A(k,p) \equiv A(p^2, p \cdot k, k^2) \), etc. Moreover, if we specialize to the BS amplitude of a pion, for which \( \eta_p = -1 \) and \( \eta_C = +1 \), we find that the parity condition (3) is verified only if \( A = C = D = G = 0 \). The charge conjugation condition (5) imposes

\[
B(k,p) = +B(-k,p) , \quad E(k,p) = +E(-k,p) ,
\]

\[
F(k,p) = -F(-k,p) , \quad H(k,p) = +H(-k,p) .
\]

(11)

Therefore, the BS amplitude for a scalar particle of negative parity and positive charge conjugation can be written as a sum of four terms,

\[
\Phi(k,p) = B\gamma^5 + E\not{p}\gamma^5 + F\not{k}\gamma^5 + H\sigma^{\mu\nu}p_{\mu}k_{\nu}\gamma^5 ,
\]

(12)

with \( B, E, \) and \( H \) (\( F \)) even (odd) under the exchange \( k \rightarrow -k \).

3. Solution and results

We discuss here the calculation of the BS amplitude for a \( q\bar{q} \) system in a \( J^z = 0^- \), positive charge-conjugation state. As in [8], the BS amplitude is written as

\[
\Phi(k,p) = \phi_1(k,p)S_1 + \phi_2(k,p)S_2 + \phi_3(k,p)S_3 + \phi_4(k,p)S_4 ,
\]

(13)

where

\[
S_1 = \gamma_5 , \quad S_2 = \frac{\not{p}}{M} \gamma_5 , \quad S_3 = \frac{k \cdot p}{M^2} \not{p} \gamma_5 - \frac{1}{M} \not{k} \gamma_5 , \quad S_4 = \frac{i}{M^2} \sigma^{\mu\nu}p_{\mu}k_{\nu} \gamma_5 ,
\]

(14)

and \( M^2 = p^2 \). The amplitudes \( \phi_i \) are again function of \( p^2, k \cdot p, \) and \( k^2 \), and are combinations of the amplitudes \( B, E, F, \) and \( H \) introduced earlier, see Eq. (12). In particular

\[
\phi_1 = B , \quad \phi_2 = ME + \frac{k \cdot p}{M^2} MF , \quad \phi_3 = -MF , \quad \phi_4 = M^2 H .
\]

(15)

The choice to express the BS amplitude as in Eq. (13) is related to the fact that the quantities \( S_i \) are “orthogonal” to each other, i.e. \( T_{\gamma}(S_i S_j) = N_i(k,p) \delta_{ij} \).

In the following, we consider the fermions interacting via the exchange of either a scalar or a pseudoscalar boson of mass \( \mu \). The corresponding interaction Lagrangians is \( \mathcal{L} = g\bar{\psi}(x)\phi(x)\psi(x) \)
in case of a scalar boson and $\mathcal{L} = ig\bar{\psi}(x)\gamma^5\phi(x)\psi(x)$ for a pseudoscalar boson. The BS amplitude in both cases satisfies the following homogeneous BSE \[8\]

$$
\Phi(k, p) = S(k + p/2) \left[ \int d^4k' F^2(k - k') i\mathcal{K}(k, k') \Gamma_1 \Phi(k', p) \Gamma_2 \right] S(k - p/2), \tag{16}
$$

where $S(q) = i\frac{q + m}{q^2 - m^2 + i\epsilon}$ is the propagator of a particle of spin 1/2 (and mass $m$) and $\Gamma_1 = \Gamma_2 = 1/(\gamma_3)$ in case of the scalar (pseudoscalar) vertex. Finally, $F(k - k') = (\mu^2 - \Lambda^2)/[(k - k')^2 - \Lambda^2 + i\epsilon]$ is a vertex form factor. The dimensionless coupling constant, $g$, and the momentum-dependent part of the exchanged-boson propagator are contained in $i\mathcal{K}$. In ladder approximation, in both scalar and pseudoscalar case, the kernel is $\mathcal{K} = \pm g^2/[(k - k')^2 - \mu^2 + i\epsilon]$ (plus for the scalar case and minus for the pseudoscalar case). In Eq. (16), $\Gamma_2 = -C \Gamma_1 C^{-1}$, where $C = \gamma^0\gamma^2$ is the charge conjugation matrix defined above.

Inserting Eq.(13) in (16) and taking the trace, a system of four coupled integral equations are obtained:

$$
\phi_i(k, p) = ig^2 \sum_j \int \frac{d^4k''}{(2\pi)^4} \frac{\phi_j(k'', p)}{(k - k'')^2 - \mu^2 + i\epsilon} \left[ \frac{c_{ij}(k, k'', p) F^2(k - k'')}{(\frac{p}{2} + k + m)^2 - m^2 + i\epsilon} \right] \tag{17}
$$

with $i, j = 1, 2, 3, 4$. The coefficients

$$
c_{ij}(k, k', p) = \frac{1}{N(k, p)} \text{Tr} \left\{ S_i(k) \left( \frac{\not{p} + \not{k} + m}{2} \right) \Gamma_1 S_j(k'') \Gamma_2 \left( \frac{\not{p} - \not{k} - m}{2} \right) \right\}, \tag{18}
$$

are explicitly given in Ref. [8], for both couplings. Notably, the numerator of each $c_{ij}(k, k'', p)$ can contain the third power of the four-momentum $k$, at the most.

In complete analogy with the two-scalar interacting system, where only one amplitude is present $[3, 4, 5, 6, 7, 8, 10]$, one can introduce the NIR for each amplitudes $\phi_i$, viz

$$
\phi_i(k, p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g_i(\gamma', z')}{[k^2 + z'q \cdot k - \gamma' - \kappa^2 + i\epsilon]^3}, \tag{19}
$$

where $\kappa^2 = m^2 - M^2/4$ and $g_i(\gamma', z')$ are unknown real functions, called the Nakanishi weight functions, to be numerically determined through the solutions of the eigen-problem, formally generated after inserting the above NIR in the BSE. Another ingredient that greatly facilitates the numerical solution of the BSE is obtained by introducing the light-front (LF) components for the involved momenta, i.e. $k^\pm = k^0 \pm k^3$ and $k_\perp [2, 5, 6, 7, 9]$, and integrating both sides of Eq. (17) over $k^-$.

First of all, let us consider the integration of the LHS. One gets, see Ref.[9] (in preparation) for details:

$$
LHS = \int \frac{dk^-}{2\pi} \phi_i(k, p) = \frac{i}{M} \int_{0}^{\infty} d\gamma' \frac{g_i(\gamma', z)}{\left( \gamma + \gamma' + m^2 z^2 + \kappa^2 (1 - z^2) - i\epsilon \right)^2}, \tag{20}
$$

where $p^\mu = (M, 0, 0, 0)$ (we put ourselves in the CM), and

$$
\gamma = (k_\perp)^2, \quad z = -\frac{2}{M} k^+. \tag{21}
$$

Regarding the RHS, first we use the generalized Feynmann trick

$$
\frac{1}{A^m B^n C^\ell} = \frac{\Gamma(m+n+\ell)}{\Gamma(m)\Gamma(n)\Gamma(\ell)} \int_{0}^{1} dv \int_{0}^{1} dw \frac{w^{m-1}w^{n-1}(1-w-v)^{\ell-1} \theta(1-v-w)}{[Av + Bw + C(1-w-v)]^{m+n+\ell}} \tag{22}
$$

\[\int_{0}^{1} dv \int_{0}^{1} dw \frac{w^{m-1}w^{n-1}(1-w-v)^{\ell-1} \theta(1-v-w)}{[Av + Bw + C(1-w-v)]^{m+n+\ell}} \tag{22}\]
then we shift the integration over \( k'' \) to \( k'' + k(1 - v) + vp'z'/2 \). Now it turns out that all \( c_{ij}(k, k'', p) \) are either independent on \( k'' \) or linearly depending on it. Therefore,

\[
c_{ij}(k, k'', p) = \tilde{c}_{ij}(k, p) + \tilde{c}_{ij}(k, p)\mu(k'')\mu \rightarrow \tilde{c}_{ij}(k, p) + \tilde{c}_{ij}(k, p)\mu \left( k'' + k(1 - v) + \frac{vp'z'}{2} \right)^{\mu} \tag{23}
\]

Integrating over \( k'' \), the term linear in \( k'' \) gives a vanishing contribution. Moreover it can be proved that \( \tilde{c}_{ij}(k, p)\mu p^{\mu} = 0 \). Therefore, in practice we can assume that \( c_{ij}(k, k'', p) \equiv c_{ij}(k(1 - v), p) \). After the integration over \( k'' \), the RHS reads

\[
RHS = \sum_j \int_{-1}^{+1} dz' \int_0^\infty dv' g_j(\gamma', z') \frac{-3g^2(\Lambda^2 - \mu^2)^2}{(4\pi)^2} \frac{1}{\left[ (p/2 - k)^2 - m^2 + i\epsilon \right]} \int \frac{dk}{2\pi} \int_0^1 dv \int_0^{1-v} dw \frac{\tilde{c}_{ij}(k, k(1 - v), p)}{\left( \beta k^* + A + i\epsilon \right)^4}, \tag{24}
\]

where \( \beta = v(1 - v) \frac{M}{2}(z' - z) \) and

\[
A = -v(1 - v)\gamma - v^2 \frac{M^2}{4}(z')^2 - v(1 - v) \frac{M^2}{4} z^2' - \mu^2(1 - v) - v(\kappa^2 + \gamma') - w(\Lambda^2 - \mu^2). \tag{25}
\]

Moreover,

\[
\left( \frac{p}{2} + k \right)^2 - m^2 + i\epsilon = \frac{M}{2} (1 - z) (k^* - k_L^*), \quad k^*_L = \frac{\gamma + m^2 - \frac{M^2}{2}(1 - z) - i\epsilon}{\frac{M}{2} (1 - z)}, \tag{26}
\]

\[
\left( \frac{p}{2} - k \right)^2 - m^2 + i\epsilon = -\frac{M}{2} (1 + z) (k^* - k_U^*), \quad k_U^* = -\frac{\gamma - m^2 + \frac{M^2}{2}(1 + z) + i\epsilon}{\frac{M}{2} (1 + z)}, \tag{27}
\]

with \( \Im(k_U^*) > 0 \) and \( \Im(k_L^*) < 0 \), and we can write in general

\[
c_{ij}(k, k(1 - v), p) = \sum_{n=0}^{3} c^{(n)}_{ij}(v, \gamma, z)(k^*)^n. \tag{28}
\]

At this point, we perform the integration over \( k^* \), which has to be carefully done due to the extra powers of \( k^* \) in the numerator. In Ref. [8], a numerical treatment has been employed, i.e. by multiplying both sides of the BSE equation by a non-negative smoothing function of \( k^* \) before performing the integration. Here, following Ref. [9], we present an alternative method, where the singular terms arising from the direct integration over \( k^* \) are correctly taken into account. This can be achieved by using the following results (here \( n = 0, 1, 2, 3 \),

\[
\int \frac{dk}{2\pi} \frac{1}{k^* - k_{U}^*} \frac{1}{k^* - k_{L}^*} \frac{(k^*)^n}{(\beta k^* + A + i\epsilon)^4} = \theta(\beta) \frac{(k_{U}^*)^n}{k_{U}^* - k_{L}^*} \frac{(k_{L}^*)^n}{(\beta k_{L}^* + A + i\epsilon)^4} + \theta(-\beta) \frac{(k_{L}^*)^n}{k_{U}^* - k_{L}^*} \frac{(k_{U}^*)^n}{(\beta k_{U}^* + A + i\epsilon)^4}
\]

\[
- \frac{i}{3} \delta(\beta) \frac{(k_{U}^*)^{n-1} - (k_{L}^*)^{n-1}}{k_{U}^* - k_{L}^*} \frac{1}{(A + i\epsilon)^3} \delta_{n \geq 2}
\]

\[
- \frac{i}{6} \delta(\beta) \frac{(k_{L}^*)^{n-2} - (k_{U}^*)^{n-2}}{k_{U}^* - k_{L}^*} \frac{1}{(A + i\epsilon)^2} \delta_{n \geq 3}, \tag{29}
\]
where $\delta_{n=m} = 1$ for $n \geq m$ and 0 otherwise, obtaining

$$
RHS = \sum_{j} \int_{-1}^{1} dz' \int_{0}^{\infty} dv' g_{j}(\gamma', z') \frac{-3ig^{2}(\Lambda^{2} - \mu^{2})^{2}}{(4\pi)^{2}} \int_{0}^{1} dv v^{2}(1-v)^{2} \left\{ \theta(z' - z)(1 + z)^{4} \frac{\sum_{n=0}^{3} c_{ij}^{(n)}(v, \gamma, z)(k_{U})^{n}}{MD_{0}} \frac{3D_{-} + (1-v)(1+z)(\Lambda^{2} - \mu^{2})}{6D_{-}^{2} (D_{-} + (1-v)(1+z)(\Lambda^{2} - \mu^{2}))^{3}} \right.
$$

$$
+ \theta(z - z')(1 - z)^{4} \frac{\sum_{n=0}^{3} c_{ij}^{(n)}(v, \gamma, z)(k_{L})^{n}}{MD_{0}} \frac{3D_{+} + (1-v)(1-z)(\Lambda^{2} - \mu^{2})}{6D_{+}^{2} (D_{+} + (1-v)(1-z)(\Lambda^{2} - \mu^{2}))^{3}}
$$

$$
+ \frac{\delta(z' - z)}{v(1-v)} c_{ij}^{(2)}(v, \gamma, z) + c_{ij}^{(3)}(v, \gamma, z)(k_{U}^{-1} + k_{L}^{3}) \frac{1}{6D^{2} (D_{-} + (1-z)(\Lambda^{2} - \mu^{2}))^{2}} - \frac{M^{2}}{4} (1-z) v
$$

$$
+ \frac{d\delta(z'-z)}{dz'} c_{ij}^{(3)}(v, \gamma, z) \frac{1}{6(\Lambda^{2} - \mu^{2})^{2}} - \frac{M^{2}}{4} (1-z^{2}) \left[ \ln \left( 1 + \frac{\tilde{B}}{A} \right) - \frac{\tilde{B}}{A + \tilde{B}} \right] \right\}.
$$

where we have used the relation $\delta(\beta)/\beta = -d\delta(\beta)/d\beta$. Moreover, $\tilde{B} = (1-v)(\Lambda^{2} - \mu^{2})$,

$$
D_{0} = \gamma + m^{2}z^{2} + \kappa^{2}(1-z^{2}),
$$

$$
D_{+} = v(1-v)(1-z')\gamma + vM^{2} \left[ (1-v)(1-z')z^{2} + vz'(1-z) \right] + \frac{v\kappa^{2}}{2} \left[ (1-z)(1-z') + v(z-z') + (1-z)[(1-v)\mu^{2} + vz'] \right],
$$

$$
\tilde{A} = v(1-v)\gamma + v^{2}M^{2} \left[ z^{2} + v(1-v) \frac{M^{2}}{4} z' + m^{2}z^{2} + \kappa^{2}z^{2} + \mu^{2}(1-v) + \kappa^{2} + vz' \right],
$$

and $D_{-} = D_{+}(\gamma, -z, \gamma', -z', v)$. The final expression for the coupled equations is:

$$
D_{0} \int_{0}^{1} dv' g_{j}(\gamma', z) \frac{d\gamma'}{(\gamma' + m^{2}z^{2} + \kappa^{2}(1-z^{2}) - i\epsilon)^{2}}
$$

$$
= -\frac{g^{2}}{4\pi} \frac{m^{2}(\Lambda^{2} - \mu^{2})^{2}}{2\pi} \sum_{j} \int_{-1}^{1} dz' \int_{0}^{\infty} dv' g_{j}(\gamma', z') \int_{0}^{1} dv v^{2}(1-v)^{2}
$$

$$
\left\{ \theta(\beta)(1 + z)^{4} \frac{\sum_{n=0}^{3} c_{ij}^{(n)}(v, \gamma, z)(k_{U})^{n}}{4m^{2}} \frac{3D_{-} + (1-v)(1+z)(\Lambda^{2} - \mu^{2})}{D_{-}^{2} (D_{-} + (1-v)(1+z)(\Lambda^{2} - \mu^{2}))^{3}} \right. 
$$

$$
+ \theta(-\beta)(1 - z)^{4} \frac{\sum_{n=0}^{3} c_{ij}^{(n)}(v, \gamma, z)(k_{L})^{n}}{4m^{2}} \frac{3D_{+} + (1-v)(1-z)(\Lambda^{2} - \mu^{2})}{D_{+}^{2} (D_{+} + (1-v)(1-z)(\Lambda^{2} - \mu^{2}))^{3}}
$$

$$
- \frac{\delta(z' - z)}{v(1-v)} \frac{D_{0}}{m^{2}M^{2}(1-z^{2})} \left[ D_{-}(\tilde{D} + (1-v)(\Lambda^{2} - \mu^{2}))^{2} \right] 
$$

$$
- \frac{d\delta(z'-z)}{dz'} \frac{D_{0}}{m^{2}M^{3}(1-z^{2})} \left[ \ln \left( 1 + \frac{\tilde{B}}{A} \right) - \frac{\tilde{B}}{A + \tilde{B}} \right] \right\}.
$$

The last term can be integrated by parts taking into account that $g_{j}(\gamma', z' = \pm 1) = 0$, obtaining a term depending on $dg_{j}/dz'$. It is easy to prove that $v \rightarrow 1$ the integrands are not singular. The
Table 1. (After Ref. [9]) The squared scalar coupling constant vs the binding energy for two masses of the exchanged particle \( \mu/m = 0.15 \) and \( \mu/m = 0.50 \). First column: binding energy. Second column: coupling constant \( g^2 \) for \( \mu/m = 0.15 \), obtained in Ref. [9] using the method discussed in this contribution. Third column: results for \( \mu/m = 0.15 \), from Ref. [8] with a numerical treatment of the singularities. Fourth column: the same as the second one, but for \( \mu/m = 0.50 \). Fifth column: the same as the third one, but for \( \mu/m = 0.50 \). Sixth column: results in Euclidean space from Ref. [11]. In the vertex form factor it is taken \( \Lambda = 2 \), as in [8] and [11].

| \( B_e/m \) | \( g_{\text{PFSV}}^2 \) | \( g_{\text{CK}}^2 \) | \( g_{\text{PFSV}}^2 \) | \( g_{\text{CK}}^2 \) | \( g_{\text{E}}^2 \) |
|---|---|---|---|---|---|
| 0.01 | 7.844 | 7.813 | 25.327 | 25.23 | - |
| 0.02 | 10.040 | 10.05 | 29.487 | 29.49 | - |
| 0.04 | 13.675 | 13.69 | 36.183 | 36.19 | 36.19 |
| 0.05 | 15.336 | 15.35 | 39.178 | 39.19 | 39.18 |
| 0.10 | 23.122 | 23.12 | 52.817 | 52.82 | - |
| 0.20 | 38.324 | 38.32 | 78.259 | 78.25 | - |
| 0.40 | 71.060 | 71.07 | 130.177 | 130.7 | 130.3 |
| 0.50 | 88.964 | 86.95 | 157.419 | 157.4 | 157.5 |
| 1.00 | 187.855 | - | 295.61 | - | - |
| 1.40 | 254.483 | - | 379.48 | - | - |
| 1.80 | 288.31 | - | 421.05 | - | - |

coupled equations reported above correctly take into account of the singular behavior related to the \( k^- \) integration.

The solution of the coupled integral equations (35) is obtained in our case by expanding the Nakanishi weight functions \( g_i(\gamma', z') \) on a suitable basis. As in Ref. [5] for two-scalar bound states, we have chosen a basis composed by Laguerre and Gegenbauer polynomials (with the needed weights). The derivative \( dg_j/dz' \) is easily obtained in terms of derivatives of the Gegenbauer polynomials. In our basis, we have up to 44 Laguerre polynomials (with the same parameters as in Ref. [5]) and 44 Gegenbauer ones. In the studies of BSE, it is customary to assign a value to the binding energy \( B_e = 2m - M \), and, in correspondence, look for an eigenvalue \( g^2 \). Then, an eigen-problem of the type \( B G = g^2 A G \), (with \( B \) and \( A \) suitable matrices and \( G \), a vector of coefficients to be determined) is obtained.

If the eigenvalue exists then the whole procedure is validated. Tables 1 (scalar coupling) and 2 (pseudoscalar coupling) show the comparison between the values of \( g^2 \) obtained within our approach, where the singularities have been singled out and analytically evaluated, and both (i) the calculations by Ref. [8], where a non trivial numerical treatment of the singular behaviors was introduced (without recognizing the possibility of a systematic analysis of the singularities as in [15]) and (ii) the available numerical results in Euclidean space [11], with a suitable number of digits. Notably, we were also able to extend our calculation up to \( B_e/m \sim 2 \), namely when the expected critical behavior of a \( \phi^3 \) theory manifests itself [16].

4. Conclusion and perspectives

The achieved final agreement, within the adopted numerical accuracy, strongly supports the validity of our analytical method for treating the singularities that plague the ladder BSE, when an interacting two-fermion system is considered. The most severe singularity is met when the third power of \( k^- \) appears in the numerator of the kernel in Eq. (24). The powers of \( k^- \) are generated only by the external propagators and the structure of the BS amplitude, present in
Table 2. (After Ref. [9]) The same as in Table 1, but for a pseudoscalar coupling.

| $\mu/m = 0.15$ | $\mu/m = 0.50$ |
|----------------|----------------|
| $B_e/m$       | $\gamma^2_{PFSV}$ | $\gamma^2_{CK}$ |
| 0.01          | 225.7           | 224.8           |
| 0.02          | 233.2           | 232.9           |
| 0.04          | 243.1           | 243.1           |
| 0.05          | 247.1           | 247.0           |
| 0.10          | 262.1           | 262.1           |
| 0.20          | 282.9           | 282.9           |
| 0.40          | 311.7           | 311.8           |
| 0.50          | 322.9           | 323.1           |
| 1.00          | 362.3           | -               |
| 1.40          | 380.1           | -               |
| 1.80          | 388.7           | -               |

the RHS of (16). For instance, in the case of a two-vector system, this simple counting rule leads to expect derivatives of the Dirac delta-function not too high ($\geq 2$, depending only upon the complexity of the BS amplitude, like in Eq. (13)), and therefore still manageable within our approach.

The robustness of the technique based on NIR for solving the BSE with spin degrees of freedom encourages to extend this novel tool to many areas, since old limitations constraining the calculations to an unphysical space can be removed. The approach can deal with further dynamical effects, since the analytical structure of BS kernels, truncated at any power of the coupling constant, is made explicit as in the ladder case (see, e.g., [2] for the half-off-shell T-matrix), allowing the LF projection.

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