Quadratic Hedging for Sequential Claims
with Random Weights in Discrete Time

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Abstract

We study a quadratic hedging problem for a sequence of contingent claims with random weights in
discrete time. We obtain the explicit optimal hedging strategy in a recursive representation, without
imposing the nondegeneracy condition on the model and square integrability on hedging strategies. We
relate the results to hedging under random horizon and fair pricing in the quadratic sense.

Keywords: binomial model; quadratic hedging; pricing; sequential claims
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1 Introduction

Rooted from the classical mean-variance criterion in portfolio selection, quadratic hedging (also called
variance-optimal hedging) is an essential topic and a popular criterion in the literature. A pioneering work
in this area is Schweizer (1995), in which the author seeks an optimal strategy to minimize the expected
quadratic hedging error of a contingent claim in a discrete time model. In this paper, motivated by practical
problems in finance and insurance, we extend the work of Schweizer (1995) by considering the quadratic
hedging problem for a sequence of contingent claims with random weights.

Let us describe a standard quadratic hedging problem briefly. An investor faces the risk exposure of a
contingent claim $H_N$, with maturity at time $N$, and wants to construct a portfolio strategy $\xi$ from tradable
assets to hedge the claim $H_N$. The objective is to find an optimal strategy $\xi^*$ to the following problem:

$$V(c) := \min_{\xi \in \mathcal{A}} \mathbb{E}[(H_N - c - G_N(\xi))^2],$$  \hspace{1cm} (1)

where constant $c$ is the initial capital (or interpreted as the hedging cost), $G(\xi)$ is the cumulative gain
process under strategy $\xi$, and $\mathcal{A}$ is the set of admissible strategies. Note that $\xi^* = \xi^*(c)$ depends on the

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initial capital $c$. If the minimum hedging error $V(c) = 0$ for some $c$ in (1), then the portfolio $(c, \xi^* (c))$ replicates the claim $H_N$, and hence $c$ is the fair price of $H_N$ at time 0. However, in a general incomplete market, it is most likely $V(c) > 0$ for all $c$ and one may further consider $\min_{c \in \mathbb{R}} V(c)$ to find the “optimal” hedging cost $c^*$, along with the corresponding optimal strategy $\xi^*(c^*)$. In other words, an investor, with objective given by (1), always prefers the portfolio $(c^*, \xi^*(c^*))$ over any admissible portfolio $(c, \xi)$.

Problem (1) is first studied in continuous time by Duffie and Richardson (1991), in which $H_N$ is treated as a non-tradable asset and a hedger dynamically trades another correlated asset in a standard geometric Brownian model. Schweizer (1992) generalizes the work of Duffie and Richardson (1991) by considering a general claim that may depend on both assets. Further extension is done in Schweizer (1996) and Pham et al. (1998) under a general semimartingale framework. There is an extensive body of literature by now on this topic, and, to save space, we refer readers to Schweizer (2001, 2010) and the references therein for a detailed overview of pricing and hedging under a quadratic criterion in continuous time. We mention that static hedging under the quadratic criterion is also a popular topic, see, e.g., Carr and Madan (1998) and Leung and Lorig (2016).

In this article, we extend the classic quadratic hedging Problem (1) to a sequence of contingent claims $H = (H_n)_{n=0,1,\ldots,N}$ with random weights $\omega = (\omega_n)_{n=0,1,\ldots,N}$, where both $H$ and $\omega$ are adapted to a given filtration and $\omega_n \in [0, 1]$ for all $n$. Precisely, we solve the following problem:

$$V(c) := \min_{\xi \in \mathcal{A}} J(\xi; c) := \min_{\xi \in \mathcal{A}} \sum_{n=0}^{N} \mathbb{E} \left[ \omega_n (H_n - c - G_n(\xi))^2 \right],$$

(2)

where $c \in \mathbb{R}$ is the initial capital (hedging cost) and $\mathcal{A}$ is the set of admissible strategies. It is clear that the standard Problem (1) is a special case of our Problem (2), by simply taking $\omega_0 = \cdots = \omega_{N-1} = 0$ and $\omega_N = 1$. The extended problem arises naturally from practical finance and insurance concerns, such as hedging under random horizon and pricing path-dependent contingent claims. More economic interpretations are discussed in Subsection 2.1.

Solving Problem (1) in a discrete time model dates back to Schäl (1994) and Schweizer (1995). In Schäl (1994), the optimal hedging strategy is obtained under the assumption that $\mathbb{E}_t[\Delta S]/\sqrt{\mathbb{V}_t[\Delta S]}$ is bounded, where $\mathbb{E}_t[\Delta S]$ (resp. $\mathbb{V}_t[\Delta S]$) denotes the conditional mean (resp. variance) of the price changes of the risky asset $S$, which is equivalent to the so-called nondegeneracy (ND) condition in Schweizer (1995). Note that the trading strategies in Schäl (1994) are not necessarily self-financing, but only mean-self-financing (see Eq.(3.6) there). Schweizer (1995) presents a more complete and general analysis of Problem (1) under the ND condition, and obtains the optimal strategy in a recursive form. In both Schäl (1994) and Schweizer (1995) (and many works in continuous time), the existence of a solution to Problem (1) is obtained using the Hilbert projection theorem, which requires the subspace of $\{G_N(\xi) : \xi \in \mathcal{A}\}$ to be closed and in turn needs the ND condition; while on the other hand, finding the optimal strategy is based on the KunitaWatanabe decomposition. To overcome the restriction of the ND condition, Melnikov and Nechaev (1999) study the conditional version of Problem (1), replacing $\mathbb{E} [\cdot]$ by $\mathbb{E} [\cdot | \mathcal{F}_0]$, where $\mathcal{F}_0$ is the sigma field at time 0 and
may be non-trivial. Černý (2004) applies dynamic programming to study Problem (1). Černý and Kallsen (2009) utilizes the sequential regression method to derive the optimal strategy.

Our paper contributes to the literature in three ways. First, Problem (2) is general enough to including the standard Problem (1), and, to the best of our knowledge, has not been studied before. Second, we obtain the optimal hedging strategy \( \xi^* \), the value function \( V(c) \), and the minimum hedging cost \( c^* = \arg \min V(c) \) in closed forms. We do not impose the ND condition on the price process \( S \) or the square integrable condition on the hedging strategies \( \xi \). Third, we also consider a special quadratic hedging problem under random horizon \( \tau \), even the stopped market \( S^\tau \) may admit arbitrage opportunities.

In the remaining of the paper, we formulate the problem in Section 2, and present the main results in Section 3. Two examples are given in Section 5. Technical proofs are placed in Section 6 and Appendix.

## 2 The Problem

We consider a discrete time market model with \( N \) periods. To simplify notations, we introduce two index sets of time by \( \mathcal{T} = \{0, 1, 2, \ldots, N\} \) and \( \mathcal{T}^+ = \{1, 2, \ldots, N\} \). Let us fix a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_n)_{n \in \mathcal{T}}, \mathbb{P}) \), which supports all the random objects considered in the paper. We consider a representative investor who can trade a risk-free asset (bond) and a risky asset (stock) in this market. For convenience, we normalize the risk-free asset and set the interest rate to be zero. The price process of the risky asset is given by an \( \mathcal{F} \)-adapted and square-integrable process \( S := (S_n)_{n \in \mathcal{T}} \). Introduce \( L^2(\mathbb{P}) \) as the set of all square-integrable random variables under \( \mathbb{P} \). We denote the price increment \( \Delta S \) by \( \Delta S_n = S_n - S_{n-1} \), for all \( n \in \mathcal{T}^+ \). Denote \( \mathcal{P}(\mathcal{F}) \) the set of all \( \mathcal{F} \)-predictable processes, i.e., if \( \psi = (\psi_n)_{n \in \mathcal{T}^+} \in \mathcal{P}(\mathcal{F}) \), we have \( \psi_n \in \mathcal{F}_{n-1} \). We set \( \psi_0 = 0 \) for any predictable process \( \psi \) unless stated otherwise.

A hedging strategy is a predictable process \( \xi = (\xi_n)_{n \in \mathcal{T}^+} \in \mathcal{P}(\mathcal{F}) \), where \( \xi_n \) is the number of shares in the risky asset held by the investor from time \( (n-1) \) to time \( n \). The cumulative gain process of strategy \( \xi \) is denoted by \( G(\xi) = (G_n(\xi))_{n \in \mathcal{T}} \). A strategy \( \xi \) is called self-financing if

\[
G_n(\xi) = \sum_{i=1}^{n} \xi_i \Delta S_i, \quad n \in \mathcal{T}^+, \quad \text{and} \quad G_0(\xi) = 0. \tag{3}
\]

A strategy \( \xi \) is called admissible if it is predictable and self-financing. Denote the admissible set by \( \mathcal{A} \).

In our setup, the risk exposure the investor faces is modeled by a sequence of contingent claims \( H \) with random weights \( \omega \). Denote such a sequential risk by \( \mathcal{F} \)-adapted processes \( H = (H_n)_{n \in \mathcal{T}} \) and \( \omega = (\omega_n)_{n \in \mathcal{T}} \), where claim \( H_n \in L^2(\mathbb{P}) \) and weight \( \omega_n \in [0, 1] \) for all \( n \). We also call \( (H, \omega) \) a (contingent) claim.

**Problem 2.1.** The investor, with initial capital \( c \), aims to solve the quadratic hedging problem for a sequence of randomly weighted claim \( (H, \omega) \) formulated in Problem (2), i.e.,

\[
V(c) := \min_{\xi \in \mathcal{A}} J(\xi; c) := \min_{\xi \in \mathcal{A}} \sum_{n=0}^{N} \mathbb{E} \left[ \omega_n (H_n - c - G_n(\xi))^2 \right],
\]
We call a solution $\xi^* = \xi^*(c)$ to Problem (2) an optimal hedging strategy, and $V(c)$ the value function or the minimum hedging error.

Remark 2.2. We do not impose the ND condition on $S$ or square-integrability on $\xi$, which are required in almost all existing works (see, e.g., Schäl (1994); Schweizer (1995); Černý and Kallsen (2009)). One exception is Melnikov and Nechaev (1999), where the problem is instead assumed to be well posed. We show, without these conditions, that $V(c) < \infty$ for all $c \in \mathbb{R}$, and hence Problem (2) is well posed.

2.1 Interpretation of Problem 2.1

First, Problem 2.1 can be linked to quadratic hedging problems under random horizon. To wit, let $\tau$ denote a random time, a positive $\mathcal{F}$-measurable random variable taking values in $\mathcal{T}$. Consider a quadratic hedging problem with random horizon $\tau$ as follows:

$$\min_{\xi \in A} \mathbb{E}(H_\tau - c - G_\tau(\xi))^2, \quad H_n \in \mathcal{L}^2(\mathbb{P}) \text{ for all } n \in \mathcal{T}. \quad (4)$$

It is easy to see that Problem (4) is equivalent to Problem (2) given $\omega_n = \mathbb{P}(\tau = n|\mathcal{F}_n)$ for all $n \in \mathcal{T}$. But such an equivalence fails in general. Especially, if we interpret $\tau$ as an $(\mathcal{F}_n)$-stopping time, it is related to American option pricing. At first glance, Problem (4) may not have a solution since the stopped market $S_\tau$ may admit arbitrage opportunities (see, e.g., Aksamit et al. (2017, 2018); Choulli and Deng (2017)). In Example 5.2, we solve Problem 4 when the stopped market $S^\tau$ does have arbitrage.

Second, many practical problems in insurance can be formulated in the form of Problem (4). For example, we may interpret $H_\tau$ as the payment of a life insurance contract, liquidated at the random death time $\tau$, and consider an insurer who trades longevity bond to hedge such a risk in discrete time.

Third, the problem also arises from tracking a benchmark index ($H_n$) by trading available assets and evaluate the tracking performance using the quadratic criterion on a regular basis (say weekly) over a fixed period (say one year). It is also related to optimal execution under the market-on-close benchmark (see, e.g., Frei and Westray (2018)). Finally, Problem 2.1 can serve as an upper bound or estimate of the pricing of many exotic options, such as Bermuda and Asian options. For instance, if we treat $(H_n)$ as the underling asset ($S_n$) and consider an average Asian option, we have

$$\mathbb{E}\left(\frac{S_1 + S_2 + \cdots + S_N}{N} - c - G_N(\xi)\right)^2 \leq \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left[(S_n - c - \hat{G}_n(\xi))^2\right], \text{ with } \hat{G}_n(\xi) := N \Delta G_n(\xi).$$

3 Main Result

In this section, we present the main results of this paper, a closed-form solution to Problem (2), in Theorem 3.4. We first derive a sufficient optimality condition of Problem (2) in the following.
Proposition 3.1. An admissible hedging strategy $\xi^*$ is optimal to Problem (2) if it satisfies
\[ \mathbb{E} \left[ \left( \sum_{n=1}^{N} \omega_n (H_n - c - G_n(\xi^*)) \right) \Delta S_i \bigg| \mathcal{F}_{i-1} \right] = 0, \quad \text{for all } i \in T^+. \tag{5} \]

Proof. Let $\xi \in \mathcal{A}$ be an arbitrary admissible strategy, and $\xi^* \in \mathcal{A}$ satisfying condition (5). Noting $\xi + \xi^* \in \mathcal{A}$, and the linearity of $G$ by (3), we obtain
\[ J(\xi^* + \xi; c) = J(\xi^*; c) + \sum_{n=0}^{N} \mathbb{E} \left[ \omega_n G_n(\xi)^2 \right] - 2 \sum_{i=1}^{N} \mathbb{E} \left[ \xi_i \mathbb{E} \left[ \left( \sum_{n=1}^{N} \omega_n (H_n - c - G_n(\xi^*)) \right) \Delta S_i \bigg| \mathcal{F}_{i-1} \right] \right], \]
which leads to the desired result. \qed

Remark 3.2. The optimality condition (5) is equivalent to the one obtained under the Hilbert projection theorem, see Eq.(2.15) in Schweizer (1995). But here we do not need the closedness condition of the subspace $\{G_N(\xi)\}$.

To facilitate the presentation of the main results, we define the following predictable processes by
\[ \beta_n := \frac{\alpha_n}{\delta_n} \quad \text{and} \quad \rho_n := \frac{\eta_n}{\delta_n}, \quad \text{for all } n \in T^+, \tag{6} \]
where $\alpha = (\alpha_n)_{n \in T^+}$, $\eta = (\eta_n)_{n \in T^+}$, and $\delta = (\delta_n)_{n \in T^+}$ are given by
\[ \alpha_n := \mathbb{E} \left[ \Delta S_n \left( \sum_{i=n}^{N} \omega_i \left( \prod_{j=n+1}^{i} (1 - \beta_j \Delta S_j) \right) \right) \bigg| \mathcal{F}_{n-1} \right], \tag{7} \]
\[ \eta_n := \mathbb{E} \left[ \Delta S_n \left( \sum_{i=n}^{N} H_i \omega_i \left( \prod_{j=n+1}^{i} (1 - \beta_j \Delta S_j) \right) \right) \bigg| \mathcal{F}_{n-1} \right], \tag{8} \]
\[ \delta_n := \mathbb{E} \left[ \Delta S_n^2 \left( \sum_{i=n}^{N} \omega_i \left( \prod_{j=n+1}^{i} (1 - \beta_j \Delta S_j)^2 \right) \right) \bigg| \mathcal{F}_{n-1} \right]. \tag{9} \]
As convention, we set a sum over an empty set to zero, a product over an empty set to one, and 0/0 to zero. The above definitions are made via backward induction. Namely, we first define, at time $n = N$, that
\[ \alpha_N = \mathbb{E} [\omega_N \Delta S_N | \mathcal{F}_{N-1}], \quad \eta_N = \mathbb{E} [\omega_N H_N \Delta S_N | \mathcal{F}_{N-1}], \quad \delta_N = \mathbb{E} [\omega_N \Delta S_N^2 | \mathcal{F}_{N-1}], \quad \beta_N = \frac{\alpha_N}{\delta_N}, \quad \rho_N = \frac{\eta_N}{\delta_N}, \]
and use induction to complete the definitions for all $n = N - 1, N - 2, \ldots, 1$.

Remark 3.3. We show these processes are well defined by later using Proposition 6.1 and Corollary 6.2.

Theorem 3.4. For any fixed initial capital $c$, we define $\xi^*(c) := (\xi^*_n(c))_{n \in T^+}$ by
\[ \xi_n^*(c) := \rho_n - \beta_n (c + G_{n-1}(\xi^*(c))), \tag{10} \]
where $\rho = (\rho_n)_{n \in T^+}$ and $\beta = (\beta_n)_{n \in T^+}$ are given by (6). The following results hold true:
(a) \( V(c) = J(\xi^*; c) < \infty \) for all \( c \in \mathbb{R} \), and the strategy \( \xi^*(c) \) defined in (10) solves Problem (2).

(b) The value function of Problem (2) is given by

\[
V(c) = c^2 \sum_{n=0}^{N} \mathbb{E}(Z_n) - 2c \sum_{n=0}^{N} \mathbb{E}(Z_n H_n) + \sum_{n=0}^{N} \mathbb{E} \left( \omega_n \left( H_n - \sum_{i=1}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) \right)^2 \right),
\]

where \( Z = (Z_n)_{n \in T} \) is defined by

\[
Z_n := \omega_n \prod_{i=1}^{n} (1 - \beta_i \Delta S_i).
\]

(c) Fix an arbitrary but fixed non-negative integer \( n \) in \( T \), i.e., \( n = 0, 1, \cdots, N \). We have for all \( k = 0, 1, \cdots, n + 1 \) that (setting \( G_{-1} = 0 \))

\[
H_n - c - G_n(\xi^*(c)) = H_n - \sum_{i=k}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) - (c + G_{k-1}(\xi^*(c))) \prod_{i=k}^{n} (1 - \beta_i \Delta S_i).
\]

Corollary 3.5. If the random weights process \( \omega = (\omega_n)_{n \in T} \) degenerates into a sequence of constants and \( S \) is an \( \mathbb{F} \)-martingale, then we have

\[
\xi_n^*(c) = \rho_n = \frac{\sum_{i=n}^{N} \omega_i \mathbb{E} \left[ H_i \Delta S_n \left| \mathcal{F}_{n-1} \right. \right]}{\sum_{i=n}^{N} \omega_i \mathbb{E} \left[ \Delta S_n^2 \left| \mathcal{F}_{n-1} \right. \right]}, \quad n \in T^+,
\]

which is independent of the initial capital \( c \).

4 Connection with Pricing

In this section, we explore the connection between hedging and pricing of a sequence of contingent claims in the quadratic sense. Key results are presented in Theorem 4.1. Following the setup in Problem (2), we formulate the quadratic pricing of the claim \((H, \omega)\) as follows:

\[
V^* = \min_{c \in \mathbb{R}} V(c) = \min_{c \in \mathbb{R}^+, \xi \in \mathcal{A}} \sum_{n=0}^{N} \mathbb{E} \left[ \omega_n (H_n - c - G_n(\xi))^2 \right].
\]

The financial interpretation of Problem (14) is that one chooses an initial capital \( c \), along with a self-financing strategy \( \xi \), to minimize the quadratic hedging error of the contingent claim \((H, \omega)\).

Theorem 4.1. Problem (14) has an optimal solution \((c^*, \xi^*(c^*))\) given by

\[
c^* = \frac{\sum_{n=0}^{N} \mathbb{E}(Z_n H_n)}{\sum_{n=0}^{N} \mathbb{E}(Z_n)} \quad \text{and} \quad \xi^*(c^*) \text{ by (10)},
\]

where \( Z = (Z_n)_{n \in T} \) is defined in (12).
Proof. We have \( \sum_{n=0}^{N} \mathbb{E}(Z_n) \geq 0 \) from (29) in the proof of Theorem 3.4. The rest is obvious from Assertion (b) in Theorem 3.4. In fact, we have \( V^* = \mathcal{J}(\xi^*(c^*); c^*) \leq \mathcal{J}(\xi^*(c); c) \leq \mathcal{J}(\xi; c) \) for all \( c \in \mathbb{R} \) and \( \xi \in \mathcal{A} \). This ends the proof.

Similar to Problem (4), we can reformulate the quadratic pricing problem in (14) under a random horizon \( \tau \), and apply Theorem 4.1 to obtain the solution to such a problem.

**Corollary 4.2.** Let \( \tau \) be a random time and \((H, \omega)\) be a contingent claim, with \( \omega_n = \mathbb{P}(\tau = n|\mathcal{F}_n) \). The minimum capital \( c^* \), given in (15), solves the following pricing problem under random horizon:

\[
\min_{c \in \mathbb{R}} \min_{\xi \in \mathcal{A}} \mathbb{E} \left[ (H_{\tau} - c - G_\tau(\xi))^2 \right].
\]

Furthermore, if \( \tau \) is independent of \( S \) and \( S \) is an \( \mathbb{F} \)-martingale, the minimum capital \( c^* \) is equal to \( c^* = \sum_{n=1}^{N} \mathbb{E}(H_n) \cdot \mathbb{P}(\tau = n) \).

**Remark 4.3.** Denote \( \tilde{Z} = (\tilde{Z}_n)_{n \in \mathbb{N}} \), where \( \tilde{Z}_n = Z_n / \sum_{i=0}^{N} \mathbb{E}[Z_i] \). Then \( c^* \) given in (15) can be rewritten as \( c^* = \sum_{n=0}^{N} \mathbb{E}[\tilde{Z}_n H_n] \). Therefore, \( \tilde{Z} \) can be seen as a “fair” pricing measure for the contingent claim \((H, \omega)\). Furthermore, if \( \tau \) degenerates to a constant \( N \), Corollary 4.2 is reduced to Corollary 3.2 in Schweizer (1995) and \( \tilde{Z} \) is reduced to a signed probability measure absolutely continuous with respect to \( \mathbb{P} \).

5 Examples

Throughout this section, we consider a two-period binomial model \((\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,1,2}, \mathbb{P})\) specified as follows:

- \( \Omega = \{x_1, x_2, x_3, x_4\} \), \( \mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_1 = \sigma(\{x_1, x_2\}, \{x_3, x_4\}) \), and \( \mathcal{F}_2 = \mathcal{F} = 2^\Omega \).
- \( \mathbb{P}(x_1) = p^2, \mathbb{P}(x_2) = \mathbb{P}(x_3) = pq, \) and \( \mathbb{P}(x_4) = q^2 \), where \( 0 < p < 1 \) and \( q = 1 - p \).
- The stock price \( S \) evolves by: at time 0, \( S_0 = 1 \); at time 1, \( S_1(\{x_1, x_2\}) = u \) and \( S_1(\{x_3, x_4\}) = d \); at time 2, \( S_2(\{x_1\}) = u^2, S_2(\{x_2\}) = S_2(\{x_3\}) = ud, \) and \( S_2(\{x_4\}) = d^2 \), where \( 0 < d < 1 < u \).

5.1 Example 1

In this example, we consider a contingent claim \((H, \omega)\) given by

\[
H_0 = 0, \quad H_1 = a_1 \mathbb{1}_{\{x_3, x_4\}}, \quad H_2 = b_1 \mathbb{1}_{\{x_1\}} + b_2 \mathbb{1}_{\{x_2\}}; \quad \omega_0 = 0, \quad \omega_1 = 1, \quad \omega_2 = \mathbb{1}_{\{x_2\}}, \quad (16)
\]

where \( a_1, b_1 \) and \( b_2 \) are constants and \( \mathbb{1} \) denotes an indicator function.

Using (6)-(9), we first compute their values at time 2: \( \alpha_2 = \mathbb{E}[\omega_2 \Delta S_2 | \mathcal{F}_1] = u(d - 1)q \mathbb{1}_{\{x_1, x_2\}}, \eta_2 = \mathbb{E}[\omega_2 H_2 \Delta S_2 | \mathcal{F}_1] = b_2 u(d - 1)q \mathbb{1}_{\{x_1, x_2\}}, \) \( \delta_2 = \mathbb{E}[\omega_2 \Delta S^2_2 | \mathcal{F}_1] = u^2(d - 1)^2 q \mathbb{1}_{\{x_1, x_2\}}, \) and \( \eta_2 = \mathbb{E}[\omega_2 H_2 \Delta S_2 | \mathcal{F}_1] = b_2 u(d - 1)q \mathbb{1}_{\{x_1, x_2\}}, \) which imply

\[
\beta_2 = \frac{\alpha_2}{\delta_2} = \frac{\mathbb{1}_{\{x_1, x_2\}}}{u(d - 1)} \quad \text{and} \quad \rho_2 = \frac{\eta_2}{\delta_2} = \frac{b_2 \mathbb{1}_{\{x_1, x_2\}}}{u(d - 1)}. \quad (17)
\]
At time 1, notice that \( \omega_1(1 - \beta_2 \Delta S_2) \equiv 0 \). We then obtain
\[
\begin{align*}
\alpha_1 &= \mathbb{E}[\omega_1 \Delta S_1] = (u - 1)p + (d - 1)q, \\
\eta_1 &= \mathbb{E}[\omega_1 H_1 \Delta S_1] = (d - 1) a_1 q, \text{ and } \\
\delta_1 &= (u - 1)^2 p + (d - 1)^2 q, \text{ leading to }
\end{align*}
\]
and
\[
\begin{align*}
\beta_1 &= \frac{\alpha_1}{\delta_1} = \frac{(u - 1)p + (d - 1)q}{(u - 1)^2 p + (d - 1)^2 q} \quad \text{and} \\
\rho_1 &= \frac{\eta_1}{\delta_1} = \frac{a_1 (d - 1) q}{(u - 1)^2 p + (d - 1)^2 q}.
\end{align*}
\]
(18)

By (12), we get \( Z_0 = Z_2 = 0 \) and \( Z_1 = 1 - \beta_1 \Delta S_1 \). We then have:

**Corollary 5.1.** Let \((H, \omega)\) be given by (16). For any initial capital \( c \), the optimal quadratic hedging strategy \( \xi^* = \xi^*(c) \) to Problem (2) is given by
\[
\xi_1^* = \rho_1 - c \beta_1 \quad \text{and} \quad \xi_2^* = \rho_2 - \beta_2 (c + \xi_1 (u - 1)),
\]
where \( \beta_i \) and \( \rho_i, i = 1,2 \), are defined in (17) and (18). The minimum capital \( c^* \) to Problem (14) is given by
\[
c^* = \frac{a_1 (u - 1)(d - 1)}{(u + d - 2)^2}.
\]

**5.2 Example 2**

In the second example, we study an quadratic hedging problem under random horizon as formulated in (4). We set up the contingent claim \( H \) by
\[
H_0 = a_0, \quad H_1 = a_1 1_{\{x_1, x_2\}} + a_2 1_{\{x_3, x_4\}}, \quad H_2 = b_1 1_{\{x_1\}} + b_2 1_{\{x_2\}} + b_3 1_{\{x_3\}} + b_4 1_{\{x_4\}}, \quad (19)
\]
where all the \( a_i \) and \( b_i \)'s are constants. The random time \( \tau \) is defined by
\[
\tau = 0 \cdot 1_{\{x_1, x_2\}} + 1 \cdot 1_{\{x_3\}} + 2 \cdot 1_{\{x_4\}}.
\]
(20)

To use the results from Theorem 3.4, we require \( \omega_n = \mathbb{P}(\tau = n|\mathcal{F}_n) \) for all \( n = 0,1,2 \), which yields \( \omega_0 = p, \omega_1 = p \cdot 1_{\{x_3, x_4\}} \), and \( \omega_2 = 1_{\{x_4\}} \).

Similar to the previous example, we carry out calculations by (6)-(9) and obtain
\[
\begin{align*}
\alpha_2 &= d(d - 1) q 1_{\{x_3, x_4\}}; \\
\eta_2 &= d(d - 1) b_1 q 1_{\{x_3, x_4\}}; \quad \text{and} \\
\delta_2 &= d^2 (d - 1)^2 q 1_{\{x_3, x_4\}}, \quad \text{implying} \\
\beta_2 &= \frac{\eta_2}{\delta_2} = \frac{1_{\{x_3, x_4\}}}{d(d - 1)} \quad \text{and} \quad \\
\rho_2 &= \frac{\alpha_2}{\delta_2} = \frac{b_1 1_{\{x_3, x_4\}}}{d(d - 1)}. \quad \text{At time} \\
1, \quad \text{we compute} \\
\alpha_1 &= (d - 1)p q; \\
\eta_1 &= (d - 1) a_2 p q; \quad \beta_1 = (d - 1)^2 p q, \quad \text{leading to} \\
\beta_1 &= \frac{1}{d - 1} \quad \text{and} \quad \rho_1 = \frac{a_2}{d - 1}.
\end{align*}
\]

The sequence \( (Z_n)_{n=0,1,2} \), defined in (12), reads in this example as \( Z_0 = p \) and \( Z_1 = Z_2 = 0 \). In turn, we get \( \sum_{n=0}^{2} \mathbb{E}[Z_n] = p \) and \( \sum_{n=0}^{N} \mathbb{E}[Z_n H_n] = a_0 p \). An application of Theorems 3.4 and 4.1 yields:

**Corollary 5.2.** Let \( H \) and \( \tau \) be defined by (19) and (20), and \( \omega_n = \mathbb{P}(\tau = n|\mathcal{F}_n) \) for \( n = 0,1,2 \). For any initial capital \( c \), the optimal hedging strategy \( \xi^* = \xi^*(c) \) to Problem (2) is given by
\[
\xi_1^*(c) = \frac{a_2 - c}{d - 1} \quad \text{and} \quad \xi_2^*(c) = \frac{b_1 - a_2}{d(d - 1)} 1_{\{x_3, x_4\}}.
\]
(21)

The minimum capital \( c^* \) to Problem (14) is \( c^* = a_0 \).

**Proposition 5.3.** Let the assumptions in Corollary 5.2 hold. We have:
(a) The optimal strategy \( \xi^*(c^*) \) with initial capital \( c^* = a_0 \), where \( \xi^* \) is given by (21), replicates the contingent claim \( H = (H_0, H_1, H_2) \) on \( \Omega, \{x_3, x_4\}, \) and \( \{x_4\} \), respectively.

(b) The stopped market \( S^\tau \) admits arbitrage.

Proof. Assertion (a) can be verified by using (21) from Corollary 5.2. To show Assertion (b), take an admissible strategy \( \phi = (\phi_1, \phi_2) \) with \( \phi_1 = \phi_2 = -1 \). Then, by (3), we obtain

\[
G_1 = \phi_1 \Delta S_1^\tau = (1 - d)1_{\{x_3, x_4\}} \geq 0,
\]

\[
G_2 = \phi_1 \Delta S_1^\tau + \phi_2 \Delta S_2^\tau = (1 - d)1_{\{x_3, x_4\}} + d(1 - d)1_{\{x_4\}} \geq 0,
\]

and \( \mathbb{P}(G_2 > 0) = q > 0 \). Hence, \( \phi \) is an arbitrage strategy, which proves Assertion (b). \( \square \)

6 Proof to Theorem 3.4

In this section, we provide the proof to Theorem 3.4. To that end, we first present several preliminary results. We define processes \( A = (A_n)_{n \in T} \), \( B = (B_n)_{n \in T} \), \( C = (C_n)_{n \in T} \), and \( D = (D_n)_{n \in T} \) by

\[
A_n := \sum_{i=n}^{N} \omega_i \prod_{j=n+1}^{i} (1 - \beta_j \Delta S_j), \quad B_n := A_n \Delta S_n, \quad C_n := \beta_n B_n, \quad D_n := \sum_{i=n}^{N} \omega_i \prod_{j=n+1}^{i} (1 - \beta_j \Delta S_j)^2, \tag{22}
\]

where \( \beta \) is defined in (6). By (22) and the definition of \( \alpha = (\alpha_n)_{n \in T^+} \) in (7), we easily deduce that

\[
A_N = \omega_N \quad \text{and} \quad \mathbb{E}[A_n|\mathcal{F}_n] = \mathbb{E}[A_{n+1}|\mathcal{F}_n] + \omega_n - \alpha_n \beta_{n+1}, \quad n = 0, 1, \ldots, N - 1. \tag{23}
\]

Lemma 6.1. Let processes \( A, B, C, \) and \( D \) be defined by (22). We have: (1) \( A, B, \) and \( C \) are square integrable; and (2)

\[
\mathbb{E}[A_n|\mathcal{F}_n] = \mathbb{E}[D_n|\mathcal{F}_n] \leq N - n + 1, \quad \forall n \in T. \tag{24}
\]

Assertion (2) in Lemma 6.1 can be shown by backward induction, while Assertion (1) is proved by using the Cauchy-Schwarz inequality, \( \omega_n \in [0, 1] \) and (24). Please see Appendix A for the complete proof. The following is an immediate application of Lemma 6.1.

Corollary 6.2. The processes \( \beta, \rho, \alpha, \eta, \) and \( \delta \) given in (6)-(9) are well defined.

Lemma 6.3. We have, for all \( n \in T^+ \), that:

\[
\mathbb{E} \left[ \Delta S_n \left( \sum_{i=n}^{N} \omega_i \left( \sum_{j=n+1}^{i} \rho_j \Delta S_j \prod_{k=j+1}^{i} (1 - \beta_k \Delta S_k) \right) \right) \bigg| \mathcal{F}_{n-1} \right] = \mathbb{E} \left[ \Delta S_n \left( \sum_{i=n}^{N} \omega_i H_i \left( \sum_{j=n+1}^{i} \beta_j \Delta S_j \prod_{k=j+1}^{i} (1 - \beta_k \Delta S_k) \right) \right) \bigg| \mathcal{F}_{n-1} \right].
\]
Lemma 6.3 is proved by noticing $\beta_n \eta_n = \alpha_n \rho_n$ from (6), see Appendix B for details. We are now ready to show the three assertions in Theorem 3.4 and complete this task in four parts.

**Proof to Theorem 3.4. Part 1:** We first show (13) in Assertion (c) holds by backward induction. We fix an integer $n \in \mathcal{T}$. When $k = n + 1$, (13) is trivial. Next suppose (13) holds for all $k = n + 1, n, n - 1, \ldots, l + 1$, our goal is to verify that (13) also holds for $k = l$. To that end, we obtain (denoting $\xi^* = \xi^*(c)$)

$$H_n - c - G_n(\xi^*) = H_n - \sum_{i=l+1}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) - (c + G_l(\xi^*)) \prod_{i=l+1}^{n} (1 - \beta_i \Delta S_i)$$

$$= H_n - \sum_{i=l+1}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) - (c + G_{l-1}(\xi^*)) \prod_{i=l+1}^{n} (1 - \beta_i \Delta S_i)$$

$$- (\beta_l - \beta_l(c + G_{l-1}(\xi^*))) \Delta S_l \prod_{i=l+1}^{n} (1 - \beta_i \Delta S_i)$$

$$= H_n - \sum_{i=l}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) - (c + G_{l-1}(\xi^*)) \prod_{i=l}^{n} (1 - \beta_i \Delta S_i)$$

which arrives at the wanted result for $k = l$. To derive (25), we use the assumption that (13) holds for $k = l + 1$. To derive (26), we use $G_l(\xi^*) = G_{l-1}(\xi^*) + \xi^* \Delta S_l$ from (3) and the expression of $\xi^*$ from (10).

The last two equalities are due to straightforward calculations (e.g., distribute the last term in (26) and collect like terms).

Assertion (c) is now proved.

**Part 2:** We show Problem (2) is well posed. That is, we prove $V(c) = J(\xi^*; c) < \infty$ for any $c \in \mathbb{R}$, which is done by checking $\mathbb{E}[\omega_n(H_n - c - G_n(\xi^*))^2] < \infty$ for all $n \in \mathcal{T}$. To achieve this purpose, we obtain

$$\mathbb{E}\left(\omega_n (H_n - c - G_n(\xi^*))^2\right)$$

$$= \mathbb{E}\left(\omega_n \left(H_n - \sum_{i=0}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) - c \prod_{i=0}^{n} (1 - \beta_i \Delta S_i)\right)^2\right)$$

$$(\text{take } k = 0 \text{ in (13)})$$

$$\leq 3 \mathbb{E}\left(\omega_n H_n^2 + \omega_n \left(\sum_{i=0}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j)\right)^2 + \omega_n c^2 \prod_{i=0}^{n} (1 - \beta_i \Delta S_i)^2\right)$$

$$(\text{Cauchy-Schwartz})$$

$$\leq 3 \mathbb{E}(H_n^2) + 3 \mathbb{E}\left(\omega_n \left(\sum_{i=0}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j)\right)^2\right) + 3c^2 \mathbb{E}(D_0)$$

$$(\omega_n \in [0, 1] \text{ and (22)})$$
\[ \leq 3 \mathbb{E} (H_n^2) + 3(n + 1) \sum_{i=0}^{n} \mathbb{E} \left( \rho_i^2 \Delta S_i^2 \sum_{j=i}^{N} \omega_j \prod_{l=i+1}^{j} (1 - \beta_l \Delta S_l)^2 \right) + 3\epsilon^2(N + 1) \]  
(take \( \mathbb{E} \) for \( D_0 \) in (24))

\[ = 3 \mathbb{E} (H_n^2) + 3\epsilon^2(N + 1) + 3(n + 1) \sum_{i=0}^{n} \mathbb{E} \left( \frac{\eta_i^2}{\delta_i} \right) \]  
(use (6) and (9))

\[ \leq 3 \mathbb{E} (H_n^2) + 3\epsilon^2(N + 1) \]

\[ + 3(n + 1) \sum_{i=0}^{n} \mathbb{E} \left[ \frac{1}{\delta_i} \mathbb{E} \left[ \sum_{j=i}^{N} H_j^2 | \mathcal{F}_{i-1} \right] \cdot \mathbb{E} \left[ \Delta S_i^2 \sum_{j=i}^{N} \omega_j \prod_{k=i+1}^{j} (1 - \beta_k \Delta S_k)^2 | \mathcal{F}_{i-1} \right] \right] \]  
(8 and Hölder)

\[ = 3 \mathbb{E} (H_n^2) + 3\epsilon^2(N + 1) + 3(n + 1) \sum_{i=0}^{n} \sum_{j=i}^{N} \mathbb{E}[H_j^2] < +\infty. \]

In particular, we obtain \( V(c) = \mathcal{J}(\xi^*; c) < \infty \) without imposing the ND condition and \( \xi^* \in \mathcal{L}^2(\mathbb{P}). \)

**Part 3:** We show \( \xi^* = \xi^*(c) \) given by (10) satisfies the sufficient condition (5) in Proposition 3.1, and hence is optimal to Problem (2). The proof below is based on backward induction.

When \( n = N \), by using (3), we have

\[ \mathbb{E} [\omega_N (H_N - c - G_N(\xi^*)) \cdot \Delta S_N | \mathcal{F}_{N-1}] = \mathbb{E} [\omega_N H_N \Delta S_N | \mathcal{F}_{N-1}] - \xi_N^* \mathbb{E} [\omega_N \Delta S_N^2 | \mathcal{F}_{N-1}] \]

\[ - (c + G_{N-1}(\xi^*)) \mathbb{E} [\omega_N \Delta S_N | \mathcal{F}_{N-1}] = \eta_N - \delta_N \xi_N^* - \alpha_N (c + G_{N-1}(\xi^*)), \]

which vanishes with \( \xi_N^* = \rho_N - \beta_N (c + G_{N-1}(\xi^*)) \), where \( \rho_N = \eta_N/\delta_N \) and \( \beta_N = \alpha_N/\delta_N \) by (6).

Next suppose the desired statement is true for \( n = N, N-1, \ldots, k+1 \). We aim to prove the same statement holds for \( n = k \) as well. We first recall a useful identity (which can be proven by induction)

\[ \prod_{i=k+1}^{l} (1 - a_i) = 1 - \sum_{i=k+1}^{l} a_i \prod_{j=i+1}^{l} (1 - a_j), \quad (27) \]

where \( k \) and \( l \) are fixed integers, and \( a = (a_n) \) is any sequence. We then obtain

\[ \mathbb{E} \left[ \left( \sum_{n=k}^{N} \omega_n (H_n - c - G_n(\xi^*)) \right) \cdot \Delta S_k | \mathcal{F}_{k-1} \right] \]

\[ = \mathbb{E} \left[ \left( \sum_{n=k}^{N} \omega_n \left( H_n - \sum_{i=k+1}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) \right) \right) \Delta S_k | \mathcal{F}_{k-1} \right] \]

\[ - \mathbb{E} \left[ \left( \sum_{n=k}^{N} \omega_n (c + G_k(\xi^*)) \prod_{j=k+1}^{n} (1 - \beta_j \Delta S_j) \right) \Delta S_k | \mathcal{F}_{k-1} \right] \]  
(by (13))

\[ = \mathbb{E} \left[ \Delta S_k \left( \sum_{n=k}^{N} \omega_n H_n \right) | \mathcal{F}_{k-1} \right] - \mathbb{E} \left[ \Delta S_k \sum_{n=k}^{N} \omega_n \left( \sum_{i=k+1}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) \right) | \mathcal{F}_{k-1} \right] \]
\[-(c + G_{k-1}(\xi^*)) \mathbb{E} \left( \sum_{n=0}^{N} \omega_n \prod_{j=k+1}^{n} (1 - \beta_j \Delta S_j) \left[ \Delta S_k \mid F_{k-1} \right] \right) \]

\[-\xi_k \mathbb{E} \left( \sum_{n=0}^{N} \omega_n \prod_{j=k+1}^{n} (1 - \beta_j \Delta S_j) \Delta S_k^2 \left[ \Delta S_k \mid F_{k-1} \right] \right) \]

\[-\mathbb{E} \left( \Delta S_k \left( \sum_{n=0}^{N} \omega_n H_n \prod_{i=k+1}^{n} (1 - \beta_i \Delta S_i) \right) \left[ \Delta S_k \mid F_{k-1} \right] \right) \]

\[
\begin{align*}
\eta_k & - \alpha_k (c + G_{k-1}(\xi^*)) - \delta_k \xi_k^* = 0, \\
\text{which confirms the induction indeed holds for } n = k.
\end{align*}
\]

By definition (10), $\xi^*$ is predictable and self-financing, and hence solves Problem (2).

**Part 4:** We show that the value function $V(c)$ is given by (11). Taking $k = 1$ in (13) for all $n \in T$, we get

\[
\sum_{n=0}^{N} \mathbb{E}(\omega_n (H_n - c - G_{n}(\xi^*))^2) = \sum_{n=0}^{N} \mathbb{E} \left( \omega_n \left( H_n - \sum_{i=1}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) - c \prod_{i=1}^{n} (1 - \beta_i \Delta S_i) \right)^2 \right) \]

\[
= c^2 \mathbb{E} \left( \sum_{n=0}^{N} \omega_n \prod_{i=1}^{n} (1 - \beta_i \Delta S_i)^2 \right) - 2c \sum_{n=0}^{N} \mathbb{E} \left( \omega_n H_n \prod_{i=1}^{n} (1 - \beta_i \Delta S_i) \right) + 2c \cdot \text{Cross-Term}, \quad (28)
\]

where the Cross-Term $\text{CT} := \sum_{n=0}^{N} \mathbb{E} \left( \omega_n \left( \sum_{i=1}^{n} \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) \right) \prod_{k=1}^{n} (1 - \beta_k \Delta S_k) \right)$.

Using (23), we obtain that

\[
\mathbb{E} \left( \sum_{n=0}^{N} \omega_n \prod_{i=1}^{n} (1 - \beta_i \Delta S_i)^2 \right) = \mathbb{E}[D_0] = \mathbb{E}[A_0] = \sum_{n=0}^{N} \mathbb{E}[Z_n] \geq 0, \quad (29)
\]

where $Z_n$ is defined in (12). Also by (12), the second term in (28) becomes $2c \sum_{n=0}^{N} \mathbb{E}(H_n Z_n)$. By comparing with (11), we see that Assertion (b) is proved if $\text{CT} = 0$, which is done in the sequel:

\[
\text{CT} = \mathbb{E} \left[ \sum_{i=1}^{N} \prod_{k=1}^{i} (1 - \beta_k \Delta S_k) \cdot \left( \sum_{n=i}^{N} \omega_n \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j) \right) \prod_{k=i+1}^{n} (1 - \beta_k \Delta S_k) \mid F_{i-1} \right] \]

\[
= \mathbb{E} \left[ \sum_{i=1}^{N} \prod_{k=1}^{i} (1 - \beta_k \Delta S_k) \cdot \left( \sum_{n=i}^{N} \omega_n \rho_i \Delta S_i \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j)^2 \mid F_{i-1} \right) \right].
\]
\[
\begin{align*}
&= \mathbb{E}\left[ \sum_{i=1}^{N} \prod_{k=1}^{i} (1 - \beta_k \Delta S_k) \rho_i \mathbb{E}\left[ \Delta S_i \sum_{n=i}^{N} \omega_n \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j)^2 \bigg| \mathcal{F}_{i-1} \right] \right] \\
&= \mathbb{E}\left[ \sum_{i=1}^{N} \prod_{k=1}^{i} (1 - \beta_k \Delta S_k) \rho_i \beta_i \mathbb{E}\left[ \Delta S_i^2 \sum_{n=i}^{N} \omega_n \prod_{j=i+1}^{n} (1 - \beta_j \Delta S_j)^2 \bigg| \mathcal{F}_{i-1} \right] \right] \\
&= \mathbb{E}\left[ \sum_{i=1}^{N} \prod_{k=1}^{i} (1 - \beta_k \Delta S_k) \cdot \rho_i (\alpha_i - \beta_i \delta_i) \right] = 0,
\end{align*}
\]

where in the first line we have used \( \prod_{k=1}^{n} = \prod_{k=1}^{i} \prod_{k=i}^{n} \prod_{k=j+1}^{n} \) to simplify the computations on conditional expectation.

The proof to the main theorem, Theorem 3.4, is now complete. \( \Box \)

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A Proof to Lemma 6.1

Proof. We first prove Assertion (2) by induction. When \( n = N \), we get \( A_N = D_N = \omega_N \in [0, 1] \), so (24) holds trivially.

Next, suppose (24) is true for all \( N, N - 1, \ldots, n + 1 \), where \( n < N \). We need to show that (24) is also true for \( n \). To this purpose, we compute

\[
\mathbb{E}[D_n|\mathcal{F}_n] = \mathbb{E}[\omega_n + \sum_{i=n+1}^{N} \omega_i \sum_{j=n+1}^{i} (1 - \beta_j \Delta S_j)^2|\mathcal{F}_n] \quad \text{(by (22))}
\]

\[
= \mathbb{E}[\omega_n + (1 - \beta_{n+1} \Delta S_{n+1})^2 \cdot \mathbb{E}[D_{n+1}|\mathcal{F}_{n+1}]|\mathcal{F}_n] \quad \text{(by tower rule)}
\]

\[
= \omega_n + \mathbb{E}\left[(1 - 2\beta_{n+1} \Delta S_{n+1}) \mathbb{E}[D_{n+1}|\mathcal{F}_{n+1}]|\mathcal{F}_n\right] \quad \text{(by assumption)}
\]

\[
+ \beta_{n+1}^2 \mathbb{E}[\Delta S_{n+1}^2 D_{n+1}|\mathcal{F}_{n+1}] = \delta_{n+1} \quad \text{(by (9))}
\]

\[
= \omega_n + \mathbb{E}[A_{n+1}|\mathcal{F}_n] - 2\beta_{n+1} \alpha_{n+1} + \beta_{n+1}^2 \delta_{n+1} \quad \text{(by (7)-(8))}
\]

\[
= \omega_n + \mathbb{E}[A_{n+1}|\mathcal{F}_n] - \beta_{n+1} \alpha_{n+1} \quad \text{(by (6))}
\]

\[
= \mathbb{E}[A_n|\mathcal{F}_n]. \quad \text{(by (23))}
\]

Recall \( \alpha_{n+1} \beta_{n+1} = \beta_{n+1}^2 \delta_{n+1} \) and \( \delta_{n+1} \geq 0 \), and \( \mathbb{E}[A_{n+1}|\mathcal{F}_{n+1}] \leq N - n \) by assumption, we then have

\[
\mathbb{E}[A_n|\mathcal{F}_n] \leq \omega_n + \mathbb{E}[\mathbb{E}[A_{n+1}|\mathcal{F}_{n+1}]|\mathcal{F}_n] \leq N - n + 1,
\]

which, together with the above results, confirms (24) holds for all \( n \in \mathcal{T} \).

Our next objective is to show Assertion (1). To that end, we deduce

\[
\mathbb{E}\left[A_n^2|\mathcal{F}_n\right] = \mathbb{E}\left(\sum_{i=n}^{N} \omega_i \prod_{j=n+1}^{i} (1 - \beta_j \Delta S_j)^2 \right)^2 |\mathcal{F}_n) \quad \text{(by (22))}
\]

\[
\leq (N - n + 1) \mathbb{E}\left(\sum_{i=n}^{N} \omega_i \prod_{j=n+1}^{i} (1 - \beta_j \Delta S_j)^2 |\mathcal{F}_n) \quad \text{(By Cauchy-Schwarz and } \omega_i \in [0, 1])
\]

\[
= (N - n + 1) \mathbb{E}[D_n|\mathcal{F}_n] \quad \text{(by (22))}
\]

\[
\leq (N - n + 1)^2, \quad \text{(by (24))}
\]

which readily shows \( A_n \in \mathcal{L}^2(\mathbb{P}) \) for all \( n \in \mathcal{T} \). Using this result, we immediately obtain the square integrability of \( B \) by

\[
\mathbb{E}[B_n^2] = \mathbb{E}\left[A_n^2 \Delta S_n^2\right] = \mathbb{E}[\Delta S_n^2 \mathbb{E}[A_n^2|\mathcal{F}_n]] \leq (N - n + 1)^2 \mathbb{E}\left[\Delta S_n^2\right] < \infty, \quad \forall n \in \mathcal{T},
\]

where we have used the fact that \( S \in \mathcal{L}^2(\mathbb{P}) \). Lastly, to see \( C \) is also square integrable, we obtain

\[
\mathbb{E}\left[C_n^2\right] = \mathbb{E}\left[\beta_n^2 \Delta S_n^2 A_n^2\right] = \mathbb{E}\left[\beta_n^2 \Delta S_n^2 \mathbb{E}[A_n|\mathcal{F}_n] \mathbb{E}[D_n|\mathcal{F}_n]\right] \quad \text{(By equality in (24))}
\]

15
\[
\leq (N - n + 1) \mathbb{E} \left[ \beta_n^2 \delta_n \right] = (N - n + 1) \mathbb{E} \left[ \frac{\alpha_n^2}{\delta_n} \right] \quad \text{(By (24) and (9))}
\]
\[
\leq (N - n + 1)^2. \quad \text{(By Cauchy-Schwarz and } \omega_i \in [0, 1])
\]

The proof is now complete. \(\square\)

**B Proof to Lemma 6.3**

*Proof.* By definition (6), we readily see \(\beta_n \eta_n = \alpha_n \rho_n\), which reads as

\[
\mathbb{E} \left[ \beta_n \Delta S_n \sum_{i=n}^N \omega_i H_i \prod_{j=n+1}^i (1 - \beta_j \Delta S_j) \mid \mathcal{F}_{n-1} \right] = \mathbb{E} \left[ \rho_n \Delta S_n \sum_{i=n}^N \omega_i \prod_{j=n+1}^i (1 - \beta_j \Delta S_j) \mid \mathcal{F}_{n-1} \right],
\]

Using the above result, we derive

\[
\text{l.h.s.} = \mathbb{E} \left[ \Delta S_n \left( \sum_{j=n+1}^N \rho_j \Delta S_j \left( \sum_{i=j}^N \omega_i \prod_{k=j+1}^i (1 - \beta_k \Delta S_k) \right) \right) \mid \mathcal{F}_{n-1} \right] = \mathbb{E} \left[ \Delta S_n \left( \sum_{j=n+1}^N \beta_j \Delta S_j \left( \sum_{i=j}^N \omega_i H_i \prod_{k=j+1}^i (1 - \beta_k \Delta S_k) \right) \right) \mid \mathcal{F}_{n-1} \right] = \text{r.h.s.}
\]

The proof is now complete. \(\square\)