A note on a problem in communication complexity

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Abstract

In this note, we prove a version of Tarui’s Theorem in communication complexity, namely $\mathsf{PH}^* \subseteq \mathsf{BP} \cdot \mathsf{PP}^*$. Consequently, every measure for $\mathsf{PP}^*$ leads to a measure for $\mathsf{PH}^*$, subsuming a result of Linial and Shraibman that problems with high mc-rigidity lie outside the polynomial hierarchy. By slightly changing the definition of mc-rigidity (arbitrary instead of uniform distribution), it is then evident that the class $\mathcal{M}^*$ of problems with low mc-rigidity equals $\mathsf{BP} \cdot \mathsf{PP}^*$. As $\mathsf{BP} \cdot \mathsf{PP}^* \subseteq \mathsf{PSPACE}^*$, this rules out the possibility, that had been left open, that even polynomial space is contained in $\mathcal{M}^*$.

1. Introduction

This note is a contribution to the field of communication complexity. We refer the reader to Kushilevitz & Nisan (1997) for an excellent introduction. We are concerned with ideas circling around the $\mathsf{PH}^*$-vs.-$\mathsf{PSPACE}^*$ problem, a long-standing open problem in structural communication complexity, first posed in Babai et al. (1986).

For each computation model there exists a corresponding structural complexity theory. The study of structural complexity theory began by considering circuit classes and the Turing machine model, see e.g., Balcázar et al. (1990, 1995); Du & Ko (2000); Hemaspaandra & Ogihara (2002) for good introductions. A prominent result in this area, influenced by Toda (1991), is Tarui’s Theorem, see Tarui (1991), relating the polynomial hierarchy to probabilistic computation modes.

Starting with Babai et al. (1986), communication complexity classes were defined and their relationships were studied. In contrast to the Turing-machine model, much is known about the relationships between the set of standard classes for the communication model (Yao’s model, Yao 1979). Unfortunately, the difficulties start with the second level of the polynomial hierarchy and, as said before, it is a long-standing open problem, whether or not the polynomial hierarchy, $\mathsf{PH}^*$, and polynomial space, $\mathsf{PSPACE}^*$, differ.

Several strategies have been proposed to tackle this problem. Razborov’s strategy is based on the rigidity of finite-field rank (see Razborov 1989, Wunderlich 2010). Lokam uses ideas of Tarui to reduce the problem to rigidity problems, where these rigidities are defined via rank over the field of real numbers (Lokam 2001). Linial & Shraibman 2009 establish a connection to learning theory. They define the notion of mc-rigidity and show that high mc-rigidity yields problems outside the polynomial hierarchy. Furthermore, they conjecture that families of Hadamard matrices have high mc-rigidity. If true, this would yield the desired separation.

In this note, we prove a version of Tarui’s Theorem in communication complexity, namely $\mathsf{PH}^* \subseteq \mathsf{BP} \cdot \mathsf{PP}^*$. Consequently, every measure for $\mathsf{PP}^*$ leads to a measure for $\mathsf{PH}^*$, subsuming one of the results of Linial and Shraibman mentioned above. We slightly change the definition of mc-rigidity. In our terminology, we apply the BP-operator on margin complexity. Now, an arbitrary probability distribution is allowed in the definition of margin rigidity. In contrast, in the original definition, the uniform distribution was used. (Hence, it is possible to consider unbalanced communication matrices as candidates for high mc-rigidity, too.) It is then evident that the class $\mathcal{M}^*$ of problems with low mc-rigidity equals $\mathsf{BP} \cdot \mathsf{PP}^*$. As $\mathsf{BP} \cdot \mathsf{PP}^* \subseteq \mathsf{PSPACE}^*$, this rules out the possibility, that had been left open by prior work, that even polynomial space is contained in $\mathcal{M}^*$.

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in $\mathcal{M}^{cc}$. In other words, the possibility that mc-rigidity defines a communication complexity class, which is too big to be useful for the desired separation result between $\mathcal{PH}^{cc}$ and $\mathcal{PSpace}^{cc}$, is ruled out.

2. Structural complexity theory

2.1. On $\mathcal{PP}^{cc}$. In the setting of communication complexity, formal languages are defined a bit differently than in the Turing-machine world. Let $\mathbb{B} := \{0,1\}$ denote the Boolean alphabet. The set of pairs of strings of equal length is denoted by $\mathbb{B}^* := \{(x,y) \mid x, y \in \mathbb{B}^*, |x| = |y|\}$. A (formal) language $L$ is a subset of $\mathbb{B}^*$, its $n$-bit section $L_n$ is the set of all pairs $(x,y) \in L$ of $n$-bit words $x,y$. A communication complexity class is a set of languages.

Communication complexity classes were first defined in Babai et al. (1986), in particular, the analog of probabilistic polynomial time, $\mathcal{PP}^{cc}$. In this subsection, we recall basic definitions and properties related to this class.

We define a guess protocol $\Pi$ (over domain $X \times Y$ with range $\mathbb{B}$) as a finite sequence $\Pi = (\Pi_1, \ldots, \Pi_l)$ of deterministic protocols $\Pi_i$ (over domain $X \times Y$ with range $\mathbb{B}$). We say that $\Pi$ uses $l$ guesses.

The number of accepting guesses of $\Pi$ on input $(x,y)$ is defined as

$$\text{acc}_\Pi(x,y) := |\{i \in [l] \mid f_{\Pi_i}(x,y) = 1\}| = \sum_{i \in [l]} f_{\Pi_i}(x,y)$$

where $f_{\Pi_i}$ denotes the function computed by the deterministic protocol $\Pi_i$.

The number of rejecting guesses, $\text{rej}_\Pi(x,y)$, is defined analogously. Clearly, we have $\text{acc}_\Pi(x,y) + \text{rej}_\Pi(x,y) = l$.

An acceptance mode is a two-ary predicate. The only acceptance mode of interest in this work is the $\text{PP acceptance mode}$, $\text{PP}(\text{acc}, \text{rej}) := (\text{acc} > \text{rej})$.

A guess protocol $\Pi$ computes a Boolean function $f$ in acceptance mode $\Xi$, if

$$f(x,y) = 1 \iff \Xi(\text{acc}_\Pi(x,y), \text{rej}_\Pi(x,y))$$

Given a guess protocol $\Pi$ and an acceptance mode $\Xi$, we denote by $f^\Xi_\Pi$ the Boolean function computed by $\Pi$ in acceptance mode $\Xi$.

The (worst-case) communication cost, $\text{PP}(\Pi)$, of a guess protocol $\Pi$ is defined as $\text{PP}(\Pi) := \lceil \log l \rceil + \max_{i \in [l]} D(\Pi_i)$, where $D(\Pi_i)$ denotes the worst-case communication cost of the deterministic protocol $\Pi_i$.

The (worst-case) PP communication complexity, $\text{PP}(f)$, of a Boolean function $f$ is defined as the minimum worst-case communication cost of a guess protocol computing $f$ in PP acceptance mode.

A family of guess protocols $\Pi = (\Pi_n)_{n \geq 1}$ is called efficient, if the communication cost of $\Pi_n$ is $\text{polylog}(n)$.

The set $\#\mathcal{PP}^{cc}$ is defined as the set of function families $f = (f_n)_{n \geq 1}$ such that there exists an efficient family of guess protocols $\Pi = (\Pi_n)_{n \geq 1}$ with $f_n = \text{acc}_{\Pi_n}$.

The communication complexity class $\mathcal{PP}^{cc}$ is then defined as the set of languages with efficient PP communication complexity, i.e.,

$$\mathcal{PP}^{cc} := \{L \mid \text{PP}(L_n) = \text{polylog}(n)\}.$$
\textbf{Lemma 2.2.} Let \( \Pi := (\Pi_n)_{n \geq 1} \) be an efficient family of guess protocols \( \Pi_n := (\Pi_1^{(n)}, \ldots, \Pi_l^{(n)}) \) computing \( L \) in PP acceptance mode. Efficiency implies \( l_n \leq 2^{polylog(n)} \). Define \( f_n := \text{acc}_{\Pi_n} \) and \( g_n := \lfloor l_n/2 \rfloor \). Then \( f := (f_n)_{n \geq 1} \notin \#P^c \text{ and} \)
\[
(x, y) \in L_n \iff \text{acc}_{\Pi_n}(x, y) > \text{rej}_{\Pi_n}(x, y) \iff \text{acc}_{\Pi_n}(x, y) > l_n - \text{acc}_{\Pi_n}(x, y) \iff f_n(x, y) > g_n.
\]

Let \( f = (f_n)_{n \geq 1} \notin \#P^c \) and \( g = (g_n)_{n \geq 1} \), \( g_n \leq 2^{polylog(n)} \) be given. Then there exists an efficient family \( \Pi := (\Pi_n)_{n \geq 1} \) of guess protocols \( \Pi_n := (\Pi_1^{(n)}, \ldots, \Pi_l^{(n)}) \) such that \( f_n = \text{acc}_{\Pi_n} \). W.l.o.g., we can assume that \( l_n \geq 2g_n \). Otherwise, we add \( (2g_n - l_n) \) many trivial, always-rejecting protocols to \( \Pi_n \). We define an efficient family \( \tilde{\Pi} := (\tilde{\Pi}_n)_{n \geq 1} \) of guess protocols as follows. The protocol \( \tilde{\Pi}_n \) consists of the sequence of deterministic protocols in \( \Pi_n \) plus \( (l_n - 2g_n) \) many trivial, always-accepting protocols. Clearly, we have
\[
\text{acc}_{\tilde{\Pi}_n}(x, y) = \text{acc}_{\Pi_n}(x, y) + l_n - 2g_n,
\]
\[
\text{rej}_{\tilde{\Pi}_n}(x, y) = \text{rej}_{\Pi_n}(x, y) = l_n - \text{acc}_{\Pi_n}(x, y).
\]
As a consequence, we obtain
\[
\text{acc}_{\tilde{\Pi}_n}(x, y) > \text{rej}_{\tilde{\Pi}_n}(x, y) \iff \text{acc}_{\Pi_n}(x, y) + l_n - 2g_n > l_n - \text{acc}_{\Pi_n}(x, y) \iff \text{acc}_{\Pi_n}(x, y) > g_n \iff (x, y) \in L_n,
\]
where the last equivalence is by the assumption. Hence, \( \tilde{\Pi} \) is an efficient family of guess protocols computing \( L \) in PP acceptance mode, i.e., \( L \in \#P^{cc} \).

In the remaining part of this subsection, we transfer results of [Beigel et al. 1991] to communication complexity.

Given a guess protocol \( \Pi := (\Pi_1, \ldots, \Pi_l) \), we consider the guess protocol \( \Pi' := (\Pi_1, \Pi_1, \ldots, \Pi_l, \Pi_l, 0) \), where 0 denotes the always-rejecting protocol. Then \( \Pi' \) has the property that \( \text{acc}_{\Pi'}(x, y) \neq \text{rej}_{\Pi'}(x, y) \) for all input pairs \((x, y)\), and \( \Pi' \) computes the same function as \( \Pi \) in PP acceptance mode. Hence, w.l.o.g. we can assume that every guess protocol has the above property.

We adapt the convenient notation of [Fenner et al. 1991] and define \( \text{gap}_{\Pi}(x, y) := \text{acc}_{\Pi}(x, y) - \text{rej}_{\Pi}(x, y) \). Then, we have
\[
f_{\Pi}^{PP}(x, y) = 1 \implies \text{gap}_{\Pi}(x, y) > 0,
\]
\[
f_{\Pi}^{PP}(x, y) = 0 \implies \text{gap}_{\Pi}(x, y) < 0.
\]

For a deterministic protocol \( \Pi \) with range \( B \), we define its \textit{complement} \( \overline{\Pi} \) as the protocol which accepts iff \( \Pi \) rejects. Given a guess protocol \( \Pi := (\Pi_1, \ldots, \Pi_l) \), we define its \textit{complement}, \( \overline{\Pi} \), as \( \overline{\Pi} := (\overline{\Pi_1}, \ldots, \overline{\Pi_l}) \). Clearly, we have \( \text{gap}_{\Pi} = -\text{gap}_{\overline{\Pi}} \).

Given two guess protocols \( \Pi := (\Pi_1, \ldots, \Pi_l) \) and \( \Pi' := (\Pi'_1, \ldots, \Pi'_l) \), respectively, we define their \textit{sum}, \( \Pi + \Pi' \), as \( \Pi + \Pi' := (\Pi_1 + \Pi'_1, \Pi_1 + \Pi'_1, \ldots, \Pi_l + \Pi'_l) \). Here, we have \( \text{gap}_{\Pi + \Pi'} = \text{gap}_{\Pi} + \text{gap}_{\Pi'} \).

Let \( \Pi \) and \( \Pi' \) be two deterministic protocols. We define their \textit{product}, \( \Pi \cdot \Pi' \), as the deterministic protocol, which runs as follows. First, \( \Pi \) is executed. If \( \Pi \) accepts, then \( \Pi' \) is executed, else \( \overline{\Pi'} \) is executed. Given two guess protocols \( \Pi := (\Pi_1, \ldots, \Pi_l) \) and \( \Pi' := (\Pi'_1, \ldots, \Pi'_l) \), respectively, we define their \textit{product}, \( \Pi \cdot \Pi' \), as \( \Pi \cdot \Pi' := (\Pi_1 \cdot \Pi'_1, \ldots, \Pi_1 \cdot \Pi'_1, \ldots, \Pi_l \cdot \Pi'_l) \). In this case, \( \text{gap}_{\Pi \cdot \Pi'} = \text{gap}_{\Pi} \cdot \text{gap}_{\Pi'} \).

The following lemma corresponds to [Beigel et al. 1991, Lemma 5].

\textbf{Lemma 2.2.} Let \( \Pi_1, \ldots, \Pi_l \) be guess protocols using at most \( l \) guesses and having communication cost at most \( c \). Let \( p(z_1, \ldots, z_k) \) be a polynomial of degree \( d \) with integer coefficients bounded above in absolute value by \( M \). Then there exists a guess protocol \( \Pi \) such that
\[
\text{gap}_{\Pi}(x, y) = p\left(\text{gap}_{\Pi_1}(x, y), \ldots, \text{gap}_{\Pi_k}(x, y)\right),
\]
\( M^f(d + k)^{k+1} \) many guesses, and has communication cost bounded above by
\[ [\log M + d \log l + (k + 1) \log(d + k)] + cd. \]

**Proof.** The guess protocol \( \Pi \) first guesses a monomial of \( p \), say \( \pm cz_1^{m_1} \cdots z_k^{m_k} \), with \( c > 0 \). As \( p \) has at most \( \sum_{i=0}^{d} \binom{d}{i} k \) monomials, this requires using \( \leq (d + k)^{k+1} \) guesses. Then \( \Pi \) guesses one of \( k \) branches, computes the product as described above, and complements if necessary. Here \( \Pi \) uses \( M \cdot l^d \) additional guesses.

The degree of a rational function is defined as the maximum of the degrees of its numerator and denominator. The following lemma corresponds to [Beigel et al. 1991, Lemma 6].

**Lemma 2.3.** Let \( \Pi_1, \ldots, \Pi_k \) be guess protocols using at most \( l \) guesses and having communication cost at most \( c \). Let \( r(z_1, \ldots, z_k) \) a rational function of degree \( d \) with integer coefficients bounded above in absolute value by \( M \). Then there exists a guess protocol \( \Pi \) such that \( \text{gap}_{\Pi}(x, y) \) and
\[ r (\text{gap}_{\Pi_1}(x, y), \ldots, \text{gap}_{\Pi_k}(x, y)) \]
have the same sign for all \( (x, y) \) where the latter is defined, \( \Pi \) uses at most
\[ (M^d(2d + k)^{k+1})^2 \]
many guesses, and has communication cost bounded above by
\[ 2 \left( [\log M + d \log l + (k + 1) \log(2d + k)] + cd \right). \]

**Proof.** Let \( r = p/q \). We just apply [Lemma 2.2] with polynomial \( p \cdot q \). The degree of this polynomial is at most \( 2d \) and the absolute values of its coefficients are bounded above by \( M^2 \cdot (2d + k) \).

The following functions are defined and studied in [Beigel et al. 1991].
\[
P_m(z) := (z - 1) \prod_{i=1}^{m} (z - 2^i)^2, \\
S_m^{(k)}(z) := \frac{(P_m(z))^b(k)}{(P_m(z))^h(k)} - \frac{(P_m(-z))^h(k)}{(P_m(-z))^b(k)}, \\
T_m^{(k)}(z_1, \ldots, z_k) := 2S_m^{(2k)}(z_1) + \cdots + 2S_m^{(2k)}(z_k) + 1.
\]

Here, \( h(k) \) denotes the least odd integer greater than or equal to \( \log(2k + 1) \).

The following proposition corresponds to [Beigel et al. 1991, Lemma 9 and 10].

**Proposition 2.4.**

(i) The degree of \( P_m^{(h(k))} \) is \( h(k)(2m + 1) \) and the absolute value of each of its coefficients is bounded by \( 2^{2h(k) \log h(k)) + 2h(k)m \log(2m+1)} \).

(ii) If \( 1 \leq z \leq 2^m \) then \( 1 \leq S_m^{(k)}(z) \leq 1 + 1/k \). If \( -2^m \leq z \leq -1 \) then \( -1 - 1/k < S_m^{(k)}(z) \leq -1 \). The rational function \( S_m^{(k)}(z) \) has degree \( h(k)(2m + 1) \) and the absolute value of each of its coefficients is bounded by \( 2^{1+2h(k) \log h(k)) + 3h(k)m \log(2m+1)} \).

(iii) Assume that \( 1 \leq |z_i| \leq 2^m \) for \( 1 \leq i \leq k \). Then \( T_m^{(k)} \) is a rational function that is positive if at least half of the \( z_i \)’s are positive, and negative otherwise. The degree of \( T_m^{(k)} \) is \( h(2k)(2m+1) \), and the absolute value of each of its coefficients is bounded by \( 2^{3h(2k) \log h(2k)) + m \log(2m+1 + 1)} \).

A proof is given in the appendix.
Proposition 2.5. Let $\Pi_1, \ldots, \Pi_k$ be PP protocols with communication cost at most $c$. Then there exists a PP protocol $\Pi$ with communication cost at most $O(k \log(k)(\log c) + c^2(\log k))$ such that $\text{gap}_{\Pi}(x, y)$ and $T^k_c(\text{gap}_{\Pi_1}(x, y), \ldots, \text{gap}_{\Pi_k}(x, y))$ have the same sign for all input pairs $(x, y)$.

Proof. We apply Lemma 2.3 on $\Pi_1, \ldots, \Pi_k$ and $r := T^k_c$ of degree $d := \deg r \leq h(2k)(2c + 1)$, where the absolute value of each of the coefficients in $r$ is bounded by

$$M \leq 2^{3h(2k)(\log h(2k) + c \log(2c + 1) + 1)}.$$ 

The protocols use at most $c$ guesses. We obtain the desired protocol $\Pi$ with communication cost bounded above by

$$2 \left(\log M + d \log l + (k + 1) \log(2d + k)\right) + cd \leq 2 \left(3h(2k)(\log h(2k) + c \log(2c + 1) + 1) + h(2k)(2c + 1)(\log c) + (k + 1) \log(2h(2k)(c + 1) + k) + h(2k)c(2c + 1) + 1\right)$$

$$= O(k \log(k)(\log c) + c^2(\log k)).$$

This lays the ground for the probability amplification result for randomized PP-protocols stated in the next section.

2.2. On $BP \cdot PP^{cc}$. Assume we are given a computation model $C$ together with a cost function measuring the resources consumed during a computation. First of all, this gives us a complexity measure $D$ by taking the infimum of the cost function over all “$C$-machines” in the computation model. In addition, if we are given a notion of efficiency, we can define a complexity class $C$ including all decision problems $L$ with a complexity $D(L)$ that is considered efficient. It is interesting to study the power of randomization by enriching the computation model $C$ with random bits. This can be done by defining a random $C$-machine as a probability distribution over $C$-machines, together with an acceptance mode, e.g., bounded error. Again, we have a cost measure, we can define a complexity measure $R$, and thus, we can also define a complexity class $\tilde{C}$ including all decision problems $L$ with a complexity $R(L)$ that is considered efficient. In computational complexity theory it has proven useful to define complexity class operators, e.g., the BP-operator. In our case, this operator describes the relationship between the complexity classes $C$ and $\tilde{C}$, namely $\tilde{C} = BP \cdot C$. In communication complexity, we can even go one step further. Here, it is possible to express the complexity measure $R$ as a perturbation of $D$, i.e., $R = BP \cdot D$. Hence, we arrive at an equation like $BP \cdot \tilde{C} = \{L \mid (BP \cdot D)(L) \text{ is efficient}\}$. In the following, we work out all the details to obtain a precise statement of this kind for the class $BP \cdot PP^{cc}$.

A randomized PP-protocol $\Pi$ (over domain $X \times Y$ with range $B$) is defined as a probability distribution $\alpha: A \rightarrow [0, 1]$ over a finite set $\{\Pi_a \mid a \in A\}$ of PP-protocols $\Pi_a$ (each over domain $X \times Y$ with range $B$).

We say that $\Pi$ computes a Boolean function $f$ with (two-sided) $\epsilon$-error, if for all $n$-bit input pairs $(x, y)$ we have

$$Pr^\alpha_{\Pi_a}[f_{\Pi_a}^PP(x, y) \neq f(x, y)] \leq \epsilon.$$ 

The (worst-case) communication cost of a randomized PP-protocol is defined as the maximum worst-case communication cost over all PP-protocols with non-zero probability, i.e., $\text{BP-PP}(\Pi) := \max\{\text{PP}(\Pi_a) \mid a \in A, \alpha(a) > 0\}$.

The (worst-case) $\epsilon$-error BP-PP communication complexity, $\text{BP-PP}_\epsilon(f)$, of a Boolean function $f$ is defined as the minimum worst-case communication cost of a randomized PP-protocol computing $f$ with two-sided $\epsilon$-error. If $\epsilon$ is not mentioned, we assume $\epsilon = 1/3$.

Probability amplification is possible for randomized PP-protocols. As a prerequisite for a proof of this, we need the following Chernoff-like result, which can be found in Köbler et al. 1993, p. 70, Lemma 2.14).
FACT 2.6. Let \( E \) be an event that occurs with probability \( \frac{1}{2} + \epsilon \), \( 0 < \epsilon \leq \frac{1}{2} \). Then \( E \) occurs within \( t \) independent trials (\( t \) odd) at least \( t/2 \) times with probability at least \( 1 - \frac{1}{2} \cdot (1 - 4 \cdot \epsilon^2)^{t/2} \).

THEOREM 2.7 (Probability amplification). For every Boolean function \( f \) and every \( \epsilon \in [0, 1/2] \) we have

\[
\text{BP-PP}_{\epsilon} \left( 1 - 4 \cdot \epsilon^2 \right)^{t/2}(f) \leq O \left( k(\log k) \text{BP-PP}_{\epsilon}^2(f) \right).
\]

PROOF. Apply Proposition 2.5 and Fact 2.6. □

We recall the definition of the BP-operator \( \mathcal{BP} \cdot \mathcal{C} \) for communication complexity classes \( \mathcal{C} \) given in Wunderlich (2010).

A language \( L \) is in \( \mathcal{BP} \cdot \mathcal{C} \) if there exist a language \( L' \in \mathcal{C} \) and a polynomially bounded function \( q \) such that for all \( n \)-bit input pairs \( (x, y) \) we have

\[
(x, y) \notin L \implies \left\{ r \in B^{[q(\log n)]} \mid \langle (x, r), (y, r) \rangle \in L' \right\} / 2^{[q(\log n)]} \leq 1/3,
\]

\[
(x, y) \in L \implies \left\{ r \in B^{[q(\log n)]} \mid \langle (x, r), (y, r) \rangle \in L' \right\} / 2^{[q(\log n)]} \geq 2/3.
\]

CLAIM 2.8.

\[
\mathcal{BP} \cdot \mathcal{PP}^{cc} = \{ L \mid \text{BP-PP}(L_n) = \text{polylog}(n) \}.
\]

PROOF. The proof of the \( \subseteq \)-inclusion is trivial. The other inclusion is an application of a result of Newman, see e.g., (Kushilevitz & Nisan 1997, Theorem 3.14), which allows us to replace the arbitrarily large set \( A \) and the distribution \( \alpha \) of a randomized PP-protocol by a uniform distribution on \( \text{polylog}(n) \) bits. We have to pay for this by increasing the error slightly, but this is not a real problem any longer, because by Theorem 2.7 probability amplification is possible to reduce the error to less than one-third again. □

Let \( \Lambda : \mathcal{M}_n(\{0, 1\}) \rightarrow \mathbb{R} \) be a mapping, assigning to each Boolean matrix \( f \) a real number \( \Lambda(f) \). In the sequel, it will be the communication-complexity measure PP. Let \( \epsilon \in [0, 1/2] \). The BP-operator applied on \( \Lambda \) is defined as

\[
(\text{BP}_\epsilon \cdot \Lambda)(f) := \max_\mu \min_{\tilde{f} : \mu(\tilde{f} \neq f) \leq \epsilon} \Lambda(\tilde{f}),
\]

where \( \mu \) denotes a probability distribution on the matrix entries of \( f \). Again, if \( \epsilon \) is not mentioned, we assume that \( \epsilon = 1/3 \).

We remark that the BP-operator may be considered as a perturbation operator that tests how much the measure \( \Lambda \) deviates from the value \( \Lambda(f) \) when \( f \) is altered by an \( \epsilon \)-fraction of its entries.

CLAIM 2.9. For every Boolean function \( f \) and every \( \epsilon \in [0, 1/2] \) we have

\[
\text{BP-PP}_\epsilon(f) = (\text{BP}_\epsilon \cdot \text{PP})(f).
\]

PROOF. This is just an application of Yao’s Minimax-principle, see e.g., (Kushilevitz & Nisan 1997, Theorem 3.20). Here, the PP-protocols take the role of the deterministic protocols in the original proof. □

Combining Claim 2.8 and Claim 2.9, we obtain

PROPOSITION 2.10. For every language \( L \) we have

\[
L \in \mathcal{BP} \cdot \mathcal{PP}^{cc} \iff (\text{BP}_\epsilon \cdot \text{PP})(L_n) = \text{polylog}(n),
\]

where \( \epsilon \in [0, 1/2] \) is an arbitrary but fixed constant.

A result of Klauck gives a characterization of PP-complexity via discrepancy, \( \text{disc'}(B) \), defined for Boolean matrices \( B \).
FACT 2.11 (Klauck 2001). Fact 6 in [Klauck 2003]. For every language L we have

\[ \log \frac{1}{\text{disc}(L_n)} \leq \text{PP}(L_n) \leq O \left( \log \frac{1}{\text{disc}(L_n)} + \log n \right) . \]

A main result of [Linial & Shraibman 2009] is a tight relationship between margin complexity, mc(A), and discrepancy, disc(A), defined for sign matrices A.

FACT 2.12. [Linial & Shraibman 2009, Theorem 3.4] For a sign matrix A the ratio between discrepancy \( \text{disc}(A)^{-1} \) and margin complexity \( \text{mc}(A) \) is a factor of at most eight.

The relationship between the Boolean and sign matrix version of discrepancy is given by \( \text{disc}(B) = \text{disc}(J - 2B) \), where J is the all-ones matrix.

Hence, combining Fact 2.11 and Theorem 2.14, we can rule out the possibility that polynomial space is strictly contained in the communication complexity class implicitly defined by low mc-rigidity, here, defined as \((\text{BP} \cdot \text{mc})'()\), where \(\text{mc}'(B) := \text{mc}(J - 2B)\) maps Boolean matrices to sign matrices.

COROLLARY 2.13. Let \( M^{cc} := \{L \mid (\text{BP} \cdot \text{mc})(L_n) = \text{polylog}(n)\} \). Then \( M^{cc} = \text{BP} \cdot \text{P}^{cc} \).

Hence, the statement \( \text{PSPACE} \subseteq \text{BPP} \) is not true.

2.3. On Toda’s Theorem. A remarkable result in structural complexity theory is Toda’s Theorem, which tells us that the polynomial hierarchy \( \text{PH} \) is contained in \( \text{BP} \cdot \text{⊕P} \), see [Toda 1991]. Using the concept of randomized polynomials, [Tarui 1991] extended this further by showing

FACT 2.14 (Tarui 1991). \( \text{PH} \subseteq \text{BP} \cdot \text{P} \).

In fact, he even showed a stronger statement.

Often there are several pitfalls when one tries to transfer a result from structural complexity theory to communication complexity. In case of Toda’s Theorems, see [Wunderlich 2010], the use of complexity class operators was essential to avoid problems with relativization. Establishing a communication complexity version of Toda’s result is a bit tricky, too. Indeed, it is not at all clear how to transfer the proof of Theorem 4.1 in [Tarui 1991] to Yao’s model. Instead, we express communication protocols as generalized \( \text{AC}^0 \) circuits and then apply Tarui’s randomized polynomial approximations for such circuits. As far as we know, the observation that languages in \( \text{PH}^{cc} \) can be expressed by \( \text{AC}^0 \) circuits is from [Razborov 1989]. This was used by [Lokam 1995] together with Tarui’s result to prove upper bounds for weak rigidities for languages in \( \text{PH}^{cc} \). (Hence, lower bounds for weak rigidities would give us languages outside \( \text{PH}^{cc} \).) In the same vein, [Linial & Shraibman 2009] utilized this insight in the proof of their result that languages with high mc-rigidity lie outside of the polynomial hierarchy. Coming back again to randomized polynomial approximations, these objects are not \( \# \text{P}^{cc} \) functions yet. Hence, we have to apply the same trick to handle the negative terms as in [Tarui 1991, Theorem 3.2].

THEOREM 2.15. \( \text{PH}^{cc} \subseteq \text{BP} \cdot \text{P}^{cc} \).

PROOF. Let L be a language in \( \text{PH}^{cc} \). It was observed in [Razborov 1989] and also [Lokam 1995, Proof of Theorem 4.1] that there exist an \( \text{AC}^0 \) circuit family \( \{C_{t_n}\}_{n \geq 1} \) of \( \lor, \land \) circuits \( C_{t_n} \) of size \( \leq 2^{\text{polylog}(n)} \) with \( t_n \leq 2^\text{polylog}(n) \) many variables, and families \( f_1^{(n)}, \ldots, f_{t_n}^{(n)}, g_1^{(n)}, \ldots, g_{t_n}^{(n)} \) of Boolean functions such that for all n-bit input pairs \((x, y)\) we have

\[ (x, y) \in L_n \iff C_{t_n} \left( f_1^{(n)}(x) \land g_1^{(n)}(y), \ldots, f_{t_n}^{(n)}(x) \land g_{t_n}^{(n)}(y) \right) = 1 . \]

In [Tarui 1991, Theorem 3.1] it was shown that such a circuit family can be approximated with bounded error (\( \epsilon = 1/3 \)) by a family of randomized polynomials \( \Phi_{t_n} \) over \( \mathbb{Z} \) such that the degree \( d_n \) of \( \Phi_{t_n} \) is \( d_n \leq \text{polylog}(t_n) = \text{polylog}(n) \), the absolute value of each coefficient is bounded by \( 2^{\text{polylog}(t_n)} = 2^{\text{polylog}(n)} \), \( \Phi_{t_n} \) uses \( \text{polylog}(t_n) = \text{polylog}(n) \) many random
bits, and that $\Phi_{t_n}$ computes $C_{t_n}$ with two-sided error and Boolean guarantee. We can write $\Phi_n(x,y) := \Phi_{t_n} \left( f_1^{(n)}(x) \land g_1^{(n)}(y), \ldots, f_m^{(n)}(x) \land g_m^{(n)}(y) \right)$ as

$$\Phi_n(x,y) = \sum_{S \subseteq \{1,\ldots,m\}, |S| \leq d_{t_n}} \alpha_S^{(n)} \cdot \prod_{s \in S} f_s^{(n)}(x) g_s^{(n)}(y),$$

where $f_s^{(n)}(x) := \prod_{t \in S} f_t^{(n)}(x)$, and similarly for $g_s^{(n)}(y)$.

Define $g_n$ as the sum of the absolute values of the negative coefficients in $\Phi_n(x,y)$, and define $\Psi_n(x,y) := \Phi_n(x,y) + g_n$. Then

$$\Psi_n(x,y) = \sum_{S \subseteq \{1,\ldots,m\}, |S| \leq d_{t_n}} \sum_{1\leq \alpha \leq \alpha_S^{(n)}} f_s^{(n)}(x) g_S^{(n)}(y)$$

$$+ \sum_{S \subseteq \{1,\ldots,m\}, |S| \leq d_{t_n}} \sum_{1\leq \alpha \leq \alpha_S^{(n)}} \left( 1 - f_s^{(n)}(x) g_S^{(n)}(y) \right)$$

and $(\Psi_n)_{n \geq 1}$ is clearly a family of randomized $\#\mathcal{P}_{cc}$ functions, because the number of terms is bounded by

$$\max_S |\alpha_S^{(n)}| \cdot \left( \frac{t_n}{d_{t_n}} \right) \leq 2^{\text{polylog}(n)} \cdot 2^{\text{polylog}(n) \cdot \text{polylog}(n)} = 2^{\text{polylog}(n)}.$$

In addition, for every $n$-bit input pair $(x,y)$ with high probability we have

$$(x,y) \in L_n \iff \Psi_n(x,y) > g_n.$$

Applying Proposition 2.1 on $(\Psi_n)_{n \geq 1}$ and $(g_n)_{n \geq 1}$ yields a randomized family of guess protocols computing $L$ in PP acceptance mode with bounded error. Hence, $L \in \mathcal{BP} \cdot \mathcal{PP}^{cc}$.  

**Corollary 2.16.** Any lower-bound method $\mu$ for $\mathcal{PP}^{cc}$ leads to a lower-bound method $\mathcal{BP} \cdot \mu$ for $\mathcal{PH}^{cc}$ via perturbation with the BP-operator.

In particular, this holds for instances such as the discrepancy method, the $\gamma^c_2$-norm and margin complexity, the latter subsuming a result of Linial and Shraibman.

Let $\overline{L} := \{ (x,y) \in \mathbb{B}^* \mid (x,y) \notin L \}$ denote the complement of a language $L$, and let $\text{co} \cdot C$ denote the class of all complements of languages from $C$.  

Define the $\mathcal{RP}$-operator, $\mathcal{RP} \cdot C$, analogously to the BP-operator but with one-sided error. Finally, define the Las-Vegas-operator, $\mathcal{ZP}$, as $\mathcal{ZP} \cdot C := \mathcal{RP} \cdot C \cap \text{co} \cdot \mathcal{RP} \cdot C$.

**Open Question 2.17.** Do we have $\mathcal{PH}^{cc} \subseteq \mathcal{ZP} \cdot \mathcal{PP}^{cc}$?

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**A. Proof of Proposition 2.4**

Denote by $\text{lc} P(z)$ the absolute value of the largest coefficient of a univariate polynomial $P(z)$. Clearly, we have

$$\text{lc} P(z) Q(z) \leq (\deg P(z) + \deg Q(z)) (\text{lc} P(z)) (\text{lc} Q(z)) .$$

By induction, we obtain

$$\text{lc} P^n(z) \leq n! (\deg P(z) \cdot \text{lc} P(z))^n.$$

1. We have $\deg P^h(z) = h(k) \deg P_m(z) = h(k)(1 + m \cdot 2)$. Furthermore, $\lc P_m(z) = 1 \cdot \lc \Pi_{i \in [m]}(z - 2^i) \leq \lc (z - 2^m)^{2m} \leq (2m)!/(2m)^{2m}$. Hence,

$$\lc P^h(z) \leq h(k)! (h(k)(2m + 1) \cdot (2m)!/(2m)^{2m})^{h(k)} \leq 2^{2h(k) \log h(k) + 3h(k)m \log(2m + 1)}.$$

2. We have $\deg S^k_m(z) = \deg P^h(z) = h(k)(2m + 1)$ and

$$\lc S^k_m(z) \leq 2 \lc P^h(z) \leq 2^{1 + 2h(k) \log h(k) + 3h(k)m \log(2m + 1)}.$$

3. Finally, we have $\deg T^k_m(z) = \deg S^{(2k)}_m(z) = h(2k)(2m + 1)$ and

$$\lc T^k_m(z) \leq 2 \lc S^{(2k)}_m(z) \leq 2^{3h(k)(\log h(k) + m \log(2m + 1) + 1)}.$$

$\square$