INVERTIBLE FUNCTIONS ON NON-ARCHIMEDEAN
SYMMETRIC SPACES

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Abstract. Let \( u \) be a nowhere vanishing holomorphic function on the Drinfeld space \( \Omega^r \) of dimension \( r - 1 \), where \( r \geq 2 \). The logarithm \( \log |u| \) of its absolute value may be regarded as an affine function on the attached Bruhat-Tits building \( BT^r \). Generalizing a construction of van der Put in case \( r = 2 \), we relate the group \( \mathcal{O}(\Omega^r)^* \) of such \( u \) with the group \( H(BT^r, \mathbb{Z}) \) of integer-valued harmonic 1-cochains on \( BT^r \). This also gives rise to a natural \( \mathbb{Z} \)-structure on the first (\( \ell \)-adic or de Rham) cohomology of \( \Omega^r \).

0. Introduction

The non-archimedean symmetric spaces \( \Omega = \Omega^r \) introduced by Drinfeld [4] have shown great importance in the theories of modular and automorphic forms and of Shimura varieties, in the analytic uniformization of algebraic varieties, in the representation theory of \( \text{GL}(r, K) \), in the local Langlands correspondence, and in several other topics of the arithmetic of non-archimedean local fields \( K \). An incomplete list of a few references is [14], [15], [9], [17], [13], [2].

For a complete non-archimedean local field \( K \) with finite residue class field \( F \) and completed algebraic closure \( C \), the space \( \Omega \) is defined as the complement of the \( K \)-rational hyperplanes in \( P^{r-1}(C) \). It carries a natural structure as a rigid-analytic space defined over \( K \), and is supplied with an action of the group \( \text{PGL}(r, K) \). In contrast with the case of real symmetric spaces, it fails to be simply connected (in the tame topology), but has a rich cohomological structure. Its cohomology (for cohomology theories satisfying the usual axioms) has been calculated by Schneider and Stuhler [17], see also [2] and [11].

Suppose for the moment that \( r = 2 \). In this case, \( \Omega = \Omega^2 \) has dimension 1, and a coarse combinatorial picture is provided by the Bruhat-Tits tree \( T \) of \( \text{PGL}(2, K) \), a \((q+1)\)-regular tree, where \( q = \#(F) \) is the residue class cardinality of \( K \). A map \( \varphi \) from the set \( A(T) \) of oriented 1-simplices (“arrows”) of \( T \) to \( \mathbb{Z} \) that satisfies

\[
(A) \quad \varphi(e) + \varphi(\overline{e}) = 0 \quad \text{for each } e \in A(T) \text{ with inverse } \overline{e}, \text{ and}

(B) \quad \sum \varphi(e) = 0 \quad \text{for each vertex } v \text{ of } T, \text{ where } e \text{ runs through the arrows emanating from } v,
\]

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is called a (\(\mathbb{Z}\)-valued) harmonic cochain on \(\mathcal{T}\). The group \(\mathcal{H}(\mathcal{T}, \mathbb{Z})\) of all such yields upon tensoring with \(\mathbb{Z}_\ell\) (\(\ell\) a prime coprime with \(q\)) the first tale cohomology group \(H^{1}_\ell(\Omega^2, \mathbb{Z}_\ell)\) of \(\Omega^2\) ([1] Proposition 10.2). In 1981 Marius van der Put ([18], see also [5] I.8.9) established a short exact sequence

\[
1 \longrightarrow C^* \longrightarrow O(\Omega^2)^* \overset{P}{\longrightarrow} \mathcal{H}(\mathcal{T}, \mathbb{Z}) \longrightarrow 0
\]

of \(\text{PGL}(2, K)\)-modules, where \(O(\Omega^2)\) is the \(C\)-algebra of holomorphic functions on \(\Omega^2\) with multiplicative group \(O(\Omega^2)^*\). The van der Put transform \(P(u)\) of an invertible function \(u\) is a substitute for the logarithmic derivative \(u'/u\), and (0.1) provides the starting point for a study of the “Riemann surface” \(\Gamma \setminus \Omega^2\), where \(\Gamma \subset \text{PGL}(2, K)\) is a discrete subgroup ([9], [10]).

The proof requires the construction of certain functions \(u\) of \(\Omega^2\) tensoring with \(\mathbb{Z}_\ell\). It is the aim of the present paper to develop a higher-rank (i.e., \(r > 2\)) analogue of (0.1). In [8] it was shown that the absolute value \(|u|\) of \(u \in O(\Omega^2)^*\) factors over the building map

\[
\lambda: \Omega^r \longrightarrow BT^r
\]

and that its logarithm \(\log_q |u|\) defines an affine map on \(BT^r(\mathbb{Q})\). Here \(BT^r\) is the Bruhat-Tits building of \(\text{PGL}(r, K)\) (the higher-dimensional analogue of \(BT^2 = \mathcal{T}\)) and \(BT^r(\mathbb{Q})\) is the set of \(\mathbb{Q}\)-points of its realization \(BT^r(\mathbb{R})\). This makes it feasible that \(u \mapsto \log_q |u|\) gives rise to a construction of \(P\) generalizing van der Put’s in the case \(r = 2\). The transform \(P(u)\) of \(u\) will be a \(\mathbb{Z}\)-valued function on the set of arrows \(A(BT^r)\) of \(BT^r\) subject to (obvious generalizations of) the conditions (A) and (B) above.

Our first result, Proposition 3.4, is that \(P(u)\) satisfies one more relation (condition (C) in Corollary 2.9) not visible if \(r = 2\). We then define \(\mathcal{H}(BT^r, \mathbb{Z})\) as the group of those \(\varphi: A(BT^r) \to \mathbb{Z}\) which satisfy (A), (B) and (C).

The principal result of the present paper is the fact that the set of these relations is complete:

**Theorem 3.10:** The map \(P: O(\Omega^r)^* \to \mathcal{H}(BT^r, \mathbb{Z})\) is surjective, and the van der Put sequence

\[
1 \longrightarrow C^* \longrightarrow O(\Omega^r)^* \longrightarrow \mathcal{H}(BT^r, \mathbb{Z}) \longrightarrow 0
\]

is an exact sequence of \(\text{PGL}(r, K)\)-modules.

The proof requires the construction of certain functions \(u = f_{H, H', n}\) whose transforms \(P(u)\) have a prescribed behavior on the finite subcomplex \(BT^r(n)\) of \(BT^r\), and a crucial technical result (Proposition 3.9) which solely refers to the geometry of \(BT^r\).

Still, \(\mathcal{H}(BT^r, \mathbb{Z})\) is a torsion-free abelian group of complicated appearance. However, as a further consequence of Proposition 3.9 we are able to describe it in Theorem 1.16:

- either as \(\mathcal{H}(\mathcal{T}_{\text{co}}, \mathbb{Z})\), where \(\mathcal{T}_{\text{co}}\) is a subcomplex of dimension 1 of \(BT^r\) (in fact, a tree, which for \(r = 2\) agrees with the Bruhat-Tits tree \(\mathcal{T} = BT^2\)), and where only conditions (A) and (B) are involved,
- or as the group \(D(\mathbb{P}(\mathbb{V}^r), \mathbb{Z})\) of \(\mathbb{Z}\)-valued distributions of total mass 0 on the compact space \(\mathbb{P}(\mathbb{V}^r)\) of hyperplanes of the \(K\)-vector space \(\mathbb{V} = K^r\).

As the corresponding group \(D(\mathbb{P}(\mathbb{V}^r), A)\) with coefficients in some ring \(A\) depending on the cohomology theory used (e.g., \(A = \mathbb{Z}_\ell\) for tale cohomology, or \(A = K\) for de Rham cohomology) has been shown to agree with the first cohomology \(H^1(\mathbb{Q}^r, A)\) ([17], Section 3, Theorem 1), we get in particular a natural integral structure on \(H^1(\mathbb{Q}^r, A)\) along with a concrete arithmetic interpretation.
1. Background

1.1. Throughout, \( K \) denotes a non-archimedean local field with ring \( O \) of integers, a fixed uniformizer \( \pi \), and finite residue class field \( O/(\pi) = F = F_q \) of cardinality \( q \). Hence \( K \) is a finite extension of either a \( p \)-adic field \( Q_p \) or of a Laurent series field \( F_p((X)) \). We normalize its absolute value \(||\) by \(|\pi| = q^{-1}\), and let \( C = \overline{K} \) be its completed algebraic closure with respect to the unique extension of \(||\) to \( \overline{K} \). Further, \( \log: C^* \to \mathbb{Q} \) is the map \( z \mapsto \log_q |z| \).

1.2. Given a natural number \( r \geq 2 \), the Drinfeld symmetric space \( \Omega = \Omega^r \) of dimension \( r - 1 \) is the complement \( \Omega = \mathbb{P}^{r-1} \setminus \bigcup H \) of the \( K \)-rational hyperplanes \( H \) in projective space \( \mathbb{P}^{r-1} \). Hence the set of \( C \)-valued points of \( \Omega \) (for which we briefly write \( \Omega \)) is

\[ \Omega = \{ (\omega_1, \ldots, \omega_r) \in \mathbb{P}^{r-1}(C) \mid \text{the } \omega_i \text{ are } K \text{-linearly independent} \} . \]

If not indicated otherwise, we always suppose that projective coordinates \((\omega_1, \ldots, \omega_r)\) are unimodular, that is \( \max_i |\omega_i| = 1 \). The set \( \Omega \) carries a natural structure as a rigid-analytic space defined over \( K \) (see [4], [3], [17]); in fact, it is an admissible open subspace of \( \mathbb{P}^{r-1} \), and even a Stein domain ([17], Section 1, Proposition 14; see [12] for the notion of non-archimedean Stein domain).

1.3. Let \( G \) be the group scheme \( \text{GL}(r) \) with center \( Z \); hence \( G(K) = \text{GL}(r,K) \), \( Z(K) \cong K^* \), etc. The Bruhat-Tits building \( \mathbb{B}T = BT^r \) of \( G(K)/Z(K) = \text{PGL}(r,K) \) is a contractible simplicial complex with set of vertices

\[ V(BT) = \{ [L] \mid L \text{ an } O\text{-lattice in } V \} , \]

where \( L \) runs through the set of \( O \)-lattices in the \( K \)-vector space \( V = K^r \) and \( [L] \) is the similarity class of \( L \). (An \( O\text{-lattice} \) is a free \( O \)-submodule of rank \( r \) of \( V \), two such, \( L \) and \( L' \), are similar if there exists \( 0 \neq c \in K \) such that \( L' = cL \).) The classes \([L_0], \ldots, [L_s]\) form an \( s \)-simplex if and only if they are represented by lattices \( L_0 \) such that

\[ L_0 \supseteq L_1 \supseteq \cdots \supseteq L_s \supseteq \pi L_0 . \]

The combinatorial distance \( d(v, v') \) of two vertices \( v, v' \in V(BT) \) is the length of a shortest path connecting them in the 1-skeleton of \( BT \). It is easily verified that

\[ d(v, v') = \min \left\{ n \mid \exists \text{ representatives } L, L' \text{ for } v, v' \text{ such that } L \supset L' \supset \pi^n L \right\} . \]

The star \( \text{st}(v) \) of \( v \in V(BT) \) will always denote the full subcomplex of \( BT \) with set of vertices

\[ V(\text{st}(v)) = \{ w \in V(BT) \mid d(v, w) \leq 1 \} . \]

We regard \( V \) as a space of row vectors, on which \( G(K) \) acts as a matrix group from the right. Hence \( G(K) \) acts also from the right on \( BT \). If the syntax requires a left action, we shift this action to the left by the usual formula \( \gamma x := x \gamma^{-1} \).

1.4. The relationship between \( \Omega \) and \( BT \) is as follows: By the Goldman-Iwahori theorem [10], the realization \( BT(R) \) of \( BT \) is in a natural one-to-one correspondence with the set of similarity classes of real-valued non-archimedean norms on \( V \), where a vertex \( v = [L] \in V(BT) = BT(Z) \) corresponds to the class of a norm with unit ball \( L \subset V \). Now the building map

\[ \lambda: \Omega \to BT(R) \]

\[ \omega = (\omega_1, \ldots, \omega_r) \mapsto [\nu \omega] \]
is well-defined, where the norm $\nu_\omega$ maps $x = (x_1, \ldots, x_r) \in V$ to
\[
\nu_\omega(x) = \left| \sum_{1 \leq i \leq r} x_i \omega_i \right|.
\]
and $[\nu_\omega]$ is its similarity class. According to the value group $|C^*| = q^Q$, $\lambda$ maps to $BT(Q)$, and is in fact onto $BT(Q)$, the set of points of $BT(R)$ with rational barycentric coordinates. $G(K)$ acts from the left on the set of norms via
\[
(1.4.2) \quad \gamma \nu(x) := \nu(x \gamma)
\]
for $x \in V$, a norm $\nu$, and $\gamma \in G(K)$; the reader may verify that $\lambda$ is $G(K)$-equivariant, where the action on $\Omega$ is the standard one through left matrix multiplication. The pre-images under $\lambda$ of simplices of $BT$ yield an admissible covering of $\Omega$, see e.g. [2] (6.2) and (6.3). We therefore consider $BT$ as a combinatorial picture of $\Omega$.

We cite the following results from [7] and [8].

**Theorem 1.5** ([8], Theorem 2.4): Let $u$ be an invertible holomorphic function on $\Omega$. Then $|u(\omega)|$ depends only on the image $\lambda(\omega)$ of $\omega \in \Omega$ in $BT(Q)$.

**Theorem 1.6** ([8], Theorem 2.6): Let $u$ be an invertible holomorphic function on $\Omega$. Then $\log q = \log |u|$ regarded as a function on $BT(Q)$ is affine, that is, interpolates linearly in simplices.

**1.5.1.** We thus define the spectral norm $\|u\|_x$ as the common absolute value $|u(\omega)|$ for all $\omega \in \lambda^{-1}(x)$, where $x \in BT(Q)$.

**Theorem 1.6** ([8], Theorem 2.6): Let $u$ be an invertible holomorphic function on $\Omega$. Then $\log u = \log q |u|$ regarded as a function on $BT(Q)$ is affine, that is, interpolates linearly in simplices.

**1.7.** Let $A(BT)$ be the set of arrows, i.e., of oriented 1-simplices of $BT$. For each arrow $e = (v, v') = ([L], [L'])$ we write
\[
(1.7.1) \quad o(e) = \text{origin of } e := v, \quad t(e) = \text{terminus of } e := v',
\]
and $\text{type}(e) := \dim F(L/L')$.

where $L, L'$ are representatives with $L \supset L' \supset \pi L$. Then $1 \leq \text{type}(e) \leq r - 1$ and $\text{type}(e) + \text{type}(e') = r$, where $e' = (v', v)$ is $e$ with reverse orientation. We let
\[
(1.7.2) \quad A_v = \bigcup_{1 \leq t \leq r - 1} A_{v, t}
\]
be the arrows $e$ with $o(e) = v$, grouped according to their types $t$. For an invertible function $u$ on $\Omega$ and an arrow $e = (v, w)$, define the van der Put value $P(u)(e)$ of $u$ on $e$ as
\[
(1.7.3) \quad P(u)(e) = \log q \|u\|_w - \log q \|u\|_v
\]
with the spectral norm of $1.5.1$.

**Proposition 1.8** ([7], Proposition 2.9): The van der Put transform
\[
P(u) : A(BT) \rightarrow Q
\]
\[
e \mapsto P(u)(e)
\]
of $u$ has in fact values in $\mathbb{Z}$ and satisfies
\[
(1.8.1) \quad \sum_{e \in A_v} P(u)(e) = 0
\]
for all $v \in V(BT)$. Here the sum is over the arrows $e$ with $o(e) = v$ and $\text{type}(e) = 1$. 
1.8.2. For later use, we describe how (1.8.1) comes out. The canonical reduction $\lambda^{-1}(v)$ of the affinoid $\lambda^{-1}(v)$ is a variety over the residue class field $F = \mathbb{O}/(\pi)$ isomorphic with

$$
\Omega_F := \mathbb{P}^{r-1}/F \setminus \bigcup \Pi_e,
$$

where $\Pi$ runs through the hyperplanes defined over $F$. Assume $u$ is scaled such that $\|u\|_v = 1$. Its reduction $\overline{u}$ is a rational function on $\lambda^{-1}(v)$ without zeroes or poles. The boundary hyperplanes $\Pi$ of $\lambda^{-1}(v)$ correspond canonically to the elements of $A_{v,1}$ by $e \mapsto \Pi_e$, say. Let $m_e$ be the vanishing order of $\overline{u}$ along $\Pi_e$ (negative, if $\overline{u}$ has a pole along $\Pi_e$) and let $\ell_e$ be a linear form on $\mathbb{P}^{r-1}/F$ with vanishing set $\Pi_e$. Then $P(u)(e) = -m_e$ and, since $\overline{u}$ up to a multiplicative constant equals $\prod_{e \in A_{v,1}} \ell_e^{m_e}$, we find

$$
- \sum_{e \in A_{v,1}} m_e = \text{weight of the form } \overline{u} = 0.
$$

Remarks 1.9:
(i) In the case $r = 2$, the results [13, 17, 18] have been known for quite some time: see [18] and e.g. [13] 1.8.9. For general $r$, they are shown in [17] and [18] in the framework of these papers, where $\text{char}(K) = \text{char}(\mathbb{F}) = p$. However, the proofs make no use of this assumption, and are therefore valid for $\text{char}(K) = 0$, too.

(ii) The three cited results are local in the sense that they do not require $u$ to be a global unit. If, e.g., $u$ is a holomorphic function without zeroes on the affinoid $\lambda^{-1}(x)$ with $x \in B\mathcal{T}(\mathbb{Q})$, then $|u(x)|$ is constant on $\lambda^{-1}(x)$; if $u$ is invertible on $\lambda^{-1}(\sigma)$ with a closed simplex $\sigma$ of $B\mathcal{T}$, then $\log u$ is affine there, and if $u$ is invertible on $\lambda^{-1}(\text{st}(v))$, where $\text{st}(v)$ is the star of $v \in V(B\mathcal{T})$ (see [13,3]), then $P(u)(e)$ is defined for all $e \in A_v$ and satisfies (1.8.1).

(iii) It is immediate from definitions that for invertible functions $u, u'$ and arrows $e$,

$$
P(u)(e) + P(u)(\overline{e}) = 0,
$$

and more generally

$$
\sum_{e \text{ runs through the arrows of a closed path in } B\mathcal{T}} P(u)(e) = 0,
$$

as well as

$$
P(uu') = P(u) + P(u').
$$

Hence the van der Put transform $P : u \mapsto P(u)$ is a homomorphism from the multiplicative group $O(\Omega)^*$ of invertible holomorphic functions on $\Omega$ to the additive group of maps $\varphi : A(B\mathcal{T}) \to \mathbb{Z}$ that satisfy (1.9.1), (1.9.2) and (1.8.1). Moreover, for $\gamma \in G(K)$,

$$
P(u)(e\gamma) = P(u \circ \gamma^{-1})(e),
$$

i.e., $\gamma(P(u)) = P(\gamma u) := P(u \circ \gamma^{-1})$ holds, whence $P$ is $G(K)$-equivariant.

In Theorem 3.10 we will find exact conditions that characterize the image of $P$. This will yield the exact sequence (0.2) of $G(K)$-modules that generalizes (0.1).

2. Evaluation of $P$ on elementary rational functions

2.1. Let $U$ be a subspace of $V = K^r$ of codimension $t$, where $1 \leq t \leq r - 1$. We define the shift toward $U$ on $V(B\mathcal{T})$ by

$$
\tau_U : V(B\mathcal{T}) \to V(B\mathcal{T}),
$$

$$
v = [L] \mapsto [L']
$$

where $L' = (L \cap U) + \pi L$. Obviously, $e = (v, \tau_U(v))$ is a well-defined arrow of type $\text{type}(e) = \text{codim}_V(U) = t$. We say that $e$ points to $U$. 

2.1.2. For a local ring $R$ (in practice: $R = K$, or $O$, or a finite quotient $O_n := O/(\pi^n)$) and a free $R$-module $F$ of finite rank, let $Gr_{K,t}(F)$ be the Grassmannian of direct summands $F'$ such that $\text{rank}_K(F/F') = t$. Fixing $v = [L] \in V(\mathcal{B}T)$, there is a natural surjective map
\[
Gr_{K,t}(V) \to A_{v,t}
\]
\[
U \mapsto (v, \tau_U(v))
\]
and a canonical bijection
\[
A_{v,t} \xrightarrow{\cong} Gr_{F,t}(L/\pi L)
\]
given by $e = (v, w) = ([L], [M]) \mapsto \overline{M} := M/\pi L$, where $L \supset M \supset \pi L$. We denote the image of $e$ by $\overline{e}$ and the pre-image of $\overline{e}$ in $A_{v,t}$ by $e_{\overline{e}}$.

2.1.5. For two arrows $e = e_{\overline{e}}$ and $e' = e_{\overline{e'}}$ with the same origin, we write $e \prec e'$ ($e'$ dominates $e$) if and only if $\overline{M} \subset \overline{M'}$.

2.1.6. Fix $n \in \mathbb{N}$ and let $O_n$ be the ring $O/(\pi^n)$. Then, as a generalization of the above, $U \mapsto (v, \tau_U(v), \ldots, \tau_v^n(v))$ is surjective from $Gr_{K,t}(V)$ onto the set $A_{v,t,n}$ of paths of length $n$ in $\mathcal{B}T$ which emanate from $v$, are composed of arrows of type $t$, and whose endpoints $w$ have distance $d(v, w) = n$ (e.g., $A_{v,t,1} = A_{v,t}$). The set $A_{v,t,n}$ corresponds one-to-one to $Gr_{O_n,t}(L/\pi^n L)$, where the composite map from $Gr_{K,t}(V)$ to $Gr_{O_n,t}(L/\pi^n L)$ is given by $U \mapsto ((L \cap U) + \pi^n L)/\pi^n L$. This yields in the limit the canonical bijections
\[
Gr_{K,t}(V) \xrightarrow{\cong} \lim_{n} A_{v,t,n} = \lim_{n} Gr_{O_n,t}(L/\pi^n L) = Gr_{O,t}(L),
\]
whose composition is simply $U \mapsto U \cap L$. Let $e$ be an arrow of type $t$. Then
\[
Gr_{K,t}(e) := \{ U \in Gr_{K,t}(V) \mid e \text{ points to } U \}
\]
is compact and open in the compact space $Gr_{K,t}(V)$, and it follows from the considerations above that the set of all $Gr_{K,t}(e)$, where $v, t$ are fixed and $e$ belongs to $A_{v,t,n}$ for some $n \in \mathbb{N}$, form a basis for the topology on $Gr_{K,t}(V)$.

2.2. Given a hyperplane $H$ in $V$, we let $\ell_H : V \to K$ be a linear form with kernel $H$. We denote by the same symbol its extension $\ell_H : V \otimes_K C = C' \to C$. The quotients
\[
\ell_{H,H'} := \ell_H/\ell_{H'}
\]
of two such are rational functions on $\mathbb{P}^{r-1}(C)$ without zeroes or poles on $\Omega \to \mathbb{P}^{r-1}(C)$. Note that $\ell_H$ is determined up to multiplication by a non-zero scalar in $K$; hence $P(\ell_{H,H'})$ depends only on $H$ and $H'$, but not on the scaling of $\ell_H$ and $\ell_{H'}$. Our first task will be to describe $P(\ell_{H,H'})$.

2.3. We start with some local considerations around the vertex $v_0 = [L_0]$, where $L_0$ is the standard lattice $O'$ in $V$. Let us first recall the easily verified fact (where the unimodularity normalization of $\omega \in \Omega$ is used):

\[
\lambda^{-1}(v_0) = \{ \omega \in \Omega \mid \nu_0 \text{ has unit ball } L_0 \} = \{ \omega \in \Omega \mid \text{ the } \omega_i \text{ are orthogonal and } |\omega_i| \text{ for } 1 \leq i \leq r \}.
\]

$(z_1, \ldots, z_n \in C$ are orthogonal if and only if $\sum_{1 \leq i \leq n} a_i z_i = \max_{|a_i|} |a_i z_i|$ for arbitrary coefficients $a_i \in K$.) Hence the canonical reduction of $\lambda^{-1}(v_0)$ equals
\[
\overline{\lambda^{-1}(v_0)} = \mathbb{P}^{r-1}/F \setminus \bigcup \overline{H},
\]
where $\overline{H}$ runs through the hyperplanes defined over $O/(\pi) = F$. 
2.4. Write $\langle \cdot , \cdot \rangle$ for the standard bilinear form on $V$ given by
\[
\langle x', x \rangle = \sum_{1 \leq i \leq r} x'_i x_i,
\]
which we extend to a form $\langle \cdot , \cdot \rangle$ on $C^r$. Each hyperplane $H$ of $V$ is given as the kernel of a linear form
\[
\ell_H = \ell_y : x \mapsto \langle y, x \rangle
\]
with some $y \in L_0 - \pi L_0$. The arrow $(v_0, \tau_H(v_0)) \in A_{v_0, 1}$ equals $e \cdot c$ with
\[
\overline{H} = ((L_0 \cap H) + \pi L_0)/\pi L_0 = ((L_0 \cap \ker(\ell_y)) + \pi L_0)/\pi L_0.
\]
Two such vectors $y, y'$ give rise to the same $e \cdot c$ if and only if $y' \equiv c \cdot y \pmod{\pi}$ with some unit $c \in O^*$. More generally, $y$ and $y'$ give rise to the same path $(v_0, \tau_H(v_0), \ldots, \tau_H^n(v_0)) \in A_{v_0, 1, n}$ if and only if
\[
y' \equiv c \cdot y \pmod{\pi^n}
\]
with $c \in O^*$. In this case we call $y$ and $y'$ $n$-equivalent; the respective equivalence classes are briefly the $n$-classes of $y, y'$.

2.5. Let now hyperplanes $H, H'$ of $V$ be given by $y, y'$ as above. The function $\ell_{H, H'} = \ell_y/\ell_{y'}$ has constant absolute value 1 on $\lambda^{-1}(v_0)$ and therefore, by reduction, gives a rational function $t_{H, H'}$ on $\lambda^{-1}(v_0) \hookrightarrow \mathbb{P}^{r-1}/\mathbb{F}$. Put
\[
\overline{H} = ((L_0 \cap H) + \pi L_0)/\pi L_0,
\]
and ditto $\overline{H}'$. By definition, it is an $\mathbb{F}$-subvector space of $L_0/\pi L_0 \cong \mathbb{F}^r$. Abusing language, we denote by the same symbol the corresponding $\mathbb{F}$-rational linear subvariety of $\mathbb{P}^{r-1}/\mathbb{F}$ that appears e.g. in (2.3.2). Suppose that $\overline{H}$ differs from $\overline{H}'$. Then $t_{H, H'}$ has vanishing order 1 along $\overline{H}$, vanishing order $-1$ along $\overline{H}'$, and vanishing order 0 along the other hyperplanes in the boundary of $\lambda^{-1}(v_0)$ (see (2.3.2)). If however $\overline{H} = \overline{H}'$, then $t_{H, H'}$ has neither zeroes nor poles along the boundary (and is therefore constant). According to the recipe discussed in [1,8.2] we find the following description.

**Proposition 2.6:** Let $e$ be an arrow in $A_{v_0, 1}$. Then
\[
P(\ell_{H, H'})(e) = \begin{cases} -1, & e = (v_0, \tau_H(v_0)) \neq (v_0, \tau_{H'}(v_0)), \\ 1, & e = (v_0, \tau_H(v_0)) \neq (v_0, \tau_{H'}(v_0)), \\ 0, & \text{otherwise}. \end{cases}
\]

Formula (1.9.4) implies
\[
P(\ell_{H, H'})(\gamma e) = P(\ell_{H^{-1,1}, H'^{-1,1}})(e)
\]
for arrows $e$ and $\gamma \in G(K)$. As $G(K)$ acts transitively on $V(BT)$, we may transfer \[2.60\] to arbitrary arrows of type 1, and thus get:

**Corollary 2.7:** Let $e \in A_{v, 1}$ be an arrow of type 1 with arbitrary origin $v \in V(BT)$. Write $e_H$ (resp. $e_{H'}$) for the arrow $(v, \tau_H(v))$ (resp. $(v, \tau_{H'}(v))$). Then
\[
P(\ell_{H, H'})(e) = \begin{cases} -1, & e = e_H \neq e_{H'}, \\ 1, & e = e_{H'} \neq e_H, \\ 0, & \text{otherwise}. \end{cases}
\]

Next, we deal with arrows of arbitrary type.
Proposition 2.8: Given hyperplanes $H, H'$ of $V$ and an arrow $e$ of $\mathcal{BT}$ with origin $v \in V(\mathcal{BT})$, let $e_H$ (resp. $e_{H'}$) be the arrow with origin $v$ pointing to $H$ (resp. to $H'$). The transform $P(\ell_{H,H'})$ evaluates on $e$ as follows:

$$P(\ell_{H,H'})(e) = \begin{cases} -1, & e \prec e_H, e \neq e_{H'}, \\ +1, & e \prec e_{H'}, e \neq e_H, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $L$ be a lattice with $[L] = v$ and $e = c_M$, where $M$ is a subspace of $L/\pi L$ of codimension $t = \text{type}(e)$. Without restriction, $t \geq 2$. Suppose that $e \prec e_H$, i.e.,

$$M \subset \mathcal{M} = ([L \cap H] + \pi L)/\pi L \subset L/\pi L.$$

Let $M_0 = L/\pi L \supseteq M_1 = \mathcal{M} \supseteq \cdots \supseteq M_t = \mathcal{M}$ be a complete flag connecting $L/\pi L$ to $M$, where $\text{codim}_{L/\pi L}(M_i) = i$ for $0 \leq i \leq t$. It corresponds to a path $(v_0, v_1, \ldots, v_t)$ in $\mathcal{BT}$, where $v_0 = v = [L]$, $v_i = t(c_M)$, and all the arrows $e_1 = (v_0, v_1), \ldots, e_t = (v_{t-1}, v_t)$ of type 1. As $\{v_0, \ldots, v_t\}$ is a $t$-simplex, $d(v_0, v_i) = 1$ for $1 \leq i \leq t$, and therefore no $e_i$ different from $e_1 = e_H$ points to $H$. Suppose that moreover $e \neq e_{H'}$, that is,

$$\mathcal{M} \not\subset ([L \cap H'] + \pi L)/\pi L.$$

Then none of the $e_i$ $(1 \leq i \leq t)$ points to $H'$, so

$$P(\ell_{H,H'})(e) = \sum_{1 \leq i \leq t} P(\ell_{H,H'})(e_i) = P(\ell_{H,H'})(e_1) = -1$$

by (1.6.2) and Corollary 2.7. If $e \prec e_{H'} \neq e_H$, then we can arrange the flag $M_0 \supseteq \cdots \supseteq M_t$ such that as before $e_1$ points to $H$, $e_2$ points to $H'$, and no $e_i$ $(3 \leq i \leq t)$ points to $H$ or $H'$. In this case

$$P(\ell_{H,H'})(e) = P(\ell_{H,H'})(e_1) + P(\ell_{H,H'})(e_2) = -1 + 1 = 0.$$ 

If $e \prec e_H = e_{H'}$, then

$$P(\ell_{H,H'})(e) = P(\ell_{H,H'})(e_1) = 0$$

by 2.7.

If neither $e \prec e_H$ nor $e \prec e_{H'}$, neither of the arrows $e_i$ $(1 \leq i \leq t)$ corresponding to a flag $M_0 = L/\pi L \supseteq \cdots \supseteq M_t = \mathcal{M}$ points to $H$ or to $H'$, and so $P(\ell_{H,H'})(e) = 0$ results. The case $e \prec e_{H'}$, $e \neq e_H$ comes out by symmetry. \[\square\]

Corollary 2.9: Let $H_1, \ldots, H_n$ be finitely many hyperplanes of $V$ with corresponding linear forms $\ell_i = \ell_{H_i}$, $\ker(\ell_i) = H_i$, and multiplicities $m_i \in \mathbb{Z}$ such that $\sum_{1 \leq i \leq n} m_i = 0$. The function

$$u := \prod_{1 \leq i \leq n} \ell_i^{m_i}$$

is a unit on $\Omega$, whose van der Put transform $P(u)$ satisfies the condition:

(C) For each arrow $e \in \mathcal{A}(\mathcal{BT})$ with $o(e) = v \in V(\mathcal{BT})$,

$$P(u)(e) = \sum_{e' \in \mathcal{A}_{e,1} \atop e \prec e'} P(u)(e').$$

Proof. (C) is satisfied for $u = \ell_{H,H'} = \ell_H/\ell_{H'}$ by 2.7 and 2.8. The general case follows as condition (C) is linear (it holds for $u \cdot u'$ if it holds for $u$ and $u'$) and $\prod \ell_i^{m_i}$ is a product of functions of type $\ell_{H,H'}$. \[\square\]
3. The van der Put sequence

**Proposition 3.1:** Let \( u \) be an invertible holomorphic function on \( \Omega \). Then its van der Put transform \( P(u) \) satisfies condition (C) from Corollary 2.7.

**Proof.** Again by (1.9.4) we may suppose that the origin \( o(e) \) of the arrow in question equals to \( v_0 = [L_0] \). So \( e = c_{\mathcal{M}} \) with some non-trivial \( \mathbb{F} \)-subspace \( \mathcal{M} \) of \( L_0/\pi L_0 \). As in 2.5 we use the same letter \( \mathcal{M} \) for the corresponding linear subvariety of \( \mathbb{P}^{r-1}/\mathbb{F} \) of codimension \( t = \text{type}(e) = \text{codim}_{L_0/\pi L_0}(\mathcal{M}) \).

Multiplying \( u \) by suitable functions of type \( \ell_{H',H} \) (which doesn’t alter the (non)-validity of (C) for \( u \)), we may assume that \( P(u)(e') = 0 \) for all \( e' \in A_{v_0,t} \) dominating \( e \). Then we must show, that \( P(u)(e) = 0 \), too. Let \( u \) be normalized such that \( \|u\|_{v_0} = 1 \), and let \( \mathcal{M} \) be its reduction as a rational function on \( \mathbb{P}^{r-1}/\mathbb{F} \), see (2.3.2).

If \( P(u)(e) < 0 \) then \( |u| \) decays along \( e = c_{\mathcal{M}} \) and \( \mathcal{M} \) vanishes along \( \mathcal{M} \). Correspondingly, if \( P(u)(e) > 0 \) then \( (\mathcal{M})^{-1} = (\mathcal{M}^{-1}) \) vanishes along \( \mathcal{M} \). Hence it suffices to show that, under our assumptions, \( \mathcal{M} \) restricts to a well-defined rational function on \( \mathcal{M} \), i.e., \( \mathcal{M} \) is neither contained in the vanishing locus \( V(\mathcal{M}) \) nor in \( V(\mathcal{M}^{-1}) \). But the latter is obvious: With a suitable constant \( c \neq 0 \), we have

\[
\mathcal{M} = c \cdot \prod_{e} (\ell_{\mathcal{M}}^{m(\mathcal{M})}),
\]

where \( \mathcal{H} \) runs through the boundary components of \( \lambda^{-1}(v_0) \) as in (2.3.2), \( \ell_{\mathcal{M}} \) is a linear form vanishing on \( \mathcal{H} \), \( \sum m(\mathcal{H}) = 0 \), and \( m(\mathcal{H}) = \overline{P(u)}(e) = 0 \) if \( \mathcal{M} \subset \mathcal{H} \). Hence neither the rational function \( \mathcal{M} \) nor its reciprocal vanishes identically on \( \mathcal{M} \). \( \square \)

3.2. The proposition motivates the following definition. Let \( A \) be any additively written abelian group. The group of **\( A \)-valued harmonic 1-cochains** \( \mathbf{H}(B^T, A) \) is the group of maps \( \varphi : A(B^T) \to A \) that satisfy

(A) \( \sum \varphi(e) = 0 \), whenever \( e \) ranges through the arrows of a closed path in \( B^T \);
(B) for each type \( t, 1 \leq t \leq r-1 \), and each \( v \in V(B^T) \), the condition

(B_1) \( \sum_{e \in A_{v,t}} \varphi(e) = 0 \)

(C) for each \( v \in V(B^T) \) and each \( e \in A_{v,t} \),

\( \sum_{e' \in A_{v,t} : e \prec e'} \varphi(e') = \varphi(e) \).

**Remarks 3.3:**

(i) In the case where the coefficient group \( A \) equals \( \mathbb{Z} \), condition (A) is (1.9.2), (B_1) is (1.8.1), and (C) is the condition dealt with in 2.9 and 3.1. (A) in particular implies that \( \varphi \) is alternating, i.e., \( \varphi(\mathcal{M}) = -\varphi(e) \). Further, (B_1) together with (C) implies (B_1) for all types \( t \), as

\[
\sum_{e \in A_{v,t}} \varphi(e) = \sum_{e' \in A_{v,t}} \varphi(e') \# \{ e \in A_{v,t} \mid e \prec e' \},
\]

where \( \# \{ \cdots \} \), the cardinality of some finite Grassmannian, is independent of \( e' \).

(ii) Note that the current \( \mathbf{H}(B^T, \mathbb{Z}) \) differs from the group defined in [7], as condition (C) is absent there.

(iii) Proposition 3.1 together with the preceding considerations shows that

\[
P : \mathbf{O}(\Omega)^* \to \mathbf{H}(B^T, \mathbb{Z})
\]

\( u \mapsto P(u) \).
is well-defined. Its kernel consists of the invertible holomorphic functions on \( \Omega \) with constant absolute value, which equals the constants \( C^* \), as \( \Omega \) is a Stein domain. Hence, by \( \text{Proposition } 4 \), we have the exact sequence of \( G(K) \)-modules

\[ 1 \longrightarrow C^* \longrightarrow \mathcal{O}(\Omega)^* \longrightarrow P \longrightarrow H(\mathcal{B}T, \mathbb{Z}). \]

In fact, we will show that \( P \) is also surjective.

(iv) Beyond the natural coefficient domains \( A = \mathbb{Z} \) or \( \mathbb{Q} \) for \( H(\mathcal{B}T, A) \), at least the torsion groups \( A = \mathbb{Z}/(N) \) deserve interest. For example, in the case \( r = 2 \) and \( \text{char}(C) = p \), the invariants \( H(\mathcal{B}T, \mathbb{F}_p)^k \) under an arithmetic subgroup \( \Gamma \subset G(K) \) differ in general from \( H(\mathcal{B}T, \mathbb{Z})^k \otimes \mathbb{F}_p \), see Section 6. The coefficient rings \( A = \mathbb{Z}_\ell (\ell \text{ prime number}) \) and \( A = K \) come into the game by relating \( H(\mathcal{B}T, \mathbb{Z}) \) with the first cohomology of \( \Omega \), see Remark 5.5.

3.4. The strategy of proof of the surjectivity of \( P \) will be to approximate a given \( \varphi \in H(\mathcal{B}T, \mathbb{Z}) \) by elements \( P(u) \), where \( u \) is a function \( \ell_{H,H'} \), or a relative of it.

Given two hyperplanes \( H \neq H' \) of \( V \) and \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \), define

\[ (3.4.1) \quad f_{H,H',n} := 1 + \pi^n \ell_{H,H'}. \]

Here \( \ell_{H,H'} = \ell_H/\ell_{H'} = \ell_y/\ell_{y'} \), where \( y,y' \in L_0 - \pi L_0, H = \ker(\ell_y), H' = \ker(\ell_{y'}). \) Like \( \ell_{H,H'}, f_{H,H',n} \) is a unit on \( \Omega \). We denote by

\[ (3.4.2) \quad \mathcal{B}T(n) \subset \mathcal{B}T \]

the full subcomplex with vertices \( V(\mathcal{B}T(n)) = \{ v \in V(\mathcal{B}T) \mid d(v_0, v) \leq n \} \). Hence \( \mathcal{B}T(0) = \{ v_0 \}, \mathcal{B}T(1) = st(v_0), \) etc. Further,

\[ (3.4.3) \quad \Omega(n) := \lambda^{-1}(\mathcal{B}T(n)). \]

Then \( \Omega(n) \) is an admissible affinoid subspace of \( \Omega \) and \( \Omega = \bigcup_{n \geq 0} \Omega(n) \). (In [17] Section 1, Proposition 4, \( \Omega(n) \) is called \( \mathfrak{T}_n \), and a system of affinoid generators is constructed.)

**Lemma 3.5:** For \( n \in \mathbb{N}_0 \), the following hold on \( \Omega(n) \):

(i) \( \log \ell_{H,H'} \leq n \);  
(ii) \( |f_{H,H',n}| = 1 \) if \( n > 0 \).

**Proof.** (i) By our normalization, \( |\ell_{H,H'}(\omega)| = 1 \) for \( \omega \in \lambda^{-1}(v_0) \). Then by \( \| \ell_{H,H'}(\omega) \|_v \leq q^n \) for \( v \in V(\mathcal{B}T) \) whenever \( d(v_0, v) \leq n \), which gives the assertion.

(ii) \( |f_{H,H',n}(\omega)| = |1 + \pi^n \ell_{H,H'}(\omega)| \leq 1 \) on \( \Omega(n) \) by (i), with equality at least if \( \omega \) doesn’t belong to \( \lambda^{-1}(v) \), where \( v \) is a vertex with \( d(v_0, v) = n \), since in this case \( \log \ell_{H,H'}(\omega) < n \). But the equality must also hold for \( \omega \) with \( \lambda(\omega) = \text{such a } v \), due to the linear interpolation property \( \| f_{H,H',n} \|_x \) for \( x \) belonging to an arrow \( e = (v', v) \) with \( d(v_0, v') = n - 1 \). \( \square \)

**Definition 3.6:** A vertex \( v \in V(\mathcal{B}T) \) is called \( \text{n-special} \) \( (n \in \mathbb{N}_0) \) if there exists a (necessarily uniquely determined) path \( (v_0, v_1, \ldots, v_n = v) = \mathbf{A}_{v_0,1,n} \), i.e., the arrows \( e_i = (v_{i-1}, v_i), i = 1, 2, \ldots, n \) all have type 1, and \( d(v_0, v) = n \). (By definition, \( v_0 \) is 0-special.) An arrow \( e \in \mathbf{A}(\mathcal{B}T) \) is \( \text{n-special} \) \( (n \in \mathbb{N}) \) if \( o(e) \) is \( (n-1) \)-special and \( t(e) \) is \( n \)-special, that is, if it appears as some \( e_n \) as above. Also, the path \( (v_0, \ldots, v_n) = (e_1, \ldots, e_n) \) is called \( \text{n-special} \). An arrow \( e \) with \( d(v_0, o(e)) = n \) is \( \text{inbound} \) \( (\text{of level } n) \) if it belongs to \( \mathcal{B}T(n) \), and \( \text{outbound} \) otherwise. That is, \( e \) is inbound \( \iff d(v_0, t(e)) \leq n \).
3.7. Next, we describe the restriction of $P(f_{H,H,n})$ to $(n+1)$-special arrows $e$. Let $n \in \mathbb{N}$, and choose hyperplanes $H, H'$ of $V$, given as $H = \ker(\ell_y), H' = \ker(\ell_{y'})$ as in (3.4.1). Assume that $y$ and $y'$ are not 1-equivalent \(\text{(2.4.2)},\) that is, $\tau_H(v_0) \neq \tau_{H'}(v_0)$.

(i) According to Corollary 2.7 \(\ell_{H,H'} = \ell_y/\ell_{y'}\) has the property that $\log \ell_{H,H'}$ grows by 1 in each step of the $(n+1)$-special path

\[ (v_0, v_1, \ldots, v_n, v_{n+1}) = (e_1, e_2, \ldots, e_{n+1}) \]

from $v_0$ toward $H'$. Together with \(3.5\)(ii), this implies that $P(f_{H,H',n})(e_{n+1}) = 1$.

(ii) On the other hand, again by Corollary 2.7 \(\log \ell_{H,H'} < n\) on $\lambda^{-1}(v)$ for each $n$-special $v$ different from $v_n$. By a variation of the linear interpolation argument in the proof of \(3.5\)(ii), $P(f_{H,H',n})(e) = 0$ for each $(n+1)$-special arrow $e$ with $o(e) \neq v_n$.

(iii) The function $u := f_{H,H',n} = (\ell_{y'} + \pi^n \ell_y)/\ell_{y'}$ satisfies $\|u\|_{v_n} = 1$. Its reduction $\overline{\Pi}$ as a rational function on the reduction

\[ \lambda^{-1}(v_n) \cong \mathbb{P}^{r-1}/\mathbb{F} \setminus \bigcup \mathcal{H} \] (see 2.3.2)

of $\lambda^{-1}(v_n)$ has a simple pole along the hyperplane $\overline{H}e_{n+1}$ of $\mathbb{P}^{r-1}/\mathbb{F}$ corresponding to the arrow $e_{n+1}$, a simple zero along a unique $\overline{H}e_1$, where $e = (v_n, w)$, and neither zeroes nor poles along other hyperplanes that appear in (3.7.1). The hyperplane $\overline{H}_e$ is the vanishing locus in $\mathbb{P}^{r-1}/\mathbb{F}$ of the reduction of the form $\ell_{y'} + \pi^n \ell_y = \ell_{y'}$; accordingly, $w = \tau_{H''}(v_n)$, where $H'' = \ker(\ell_{y''})$ and

\[ y'' = y' + \pi^n y. \]

(iv) If $y'$ is fixed and $y$ runs through the elements of $L_0 \setminus \pi L_0$ not 1-equivalent with $y'$, then the corresponding $y''$ are $n$-equivalent but not $(n+1)$-equivalent with $y'$ (cf. (2.4.2))

In this way we get all the $(n+1)$-classes with this property, that is, all the $(n+1)$-special paths $(e_1, e_2, \ldots, e_n, e)$ which agree with the path $(e_1, \ldots, e_n, e_{n+1})$ of (3.7.1) except for the last arrow. We collect what has been shown.

Proposition 3.8:

(i) Let $H, H'$ be two hyperplanes in $V$ with $\tau_H(v_0) \neq \tau_{H'}(v_0)$ and $n \in \mathbb{N}$. Put $v_i := (\tau_{H'}^i)(v_0)$. If $e$ is an $(n+1)$-special arrow then

\[ P(f_{H,H',n})(e) = \begin{cases} +1, & \text{if } e = (v_n, v_{n+1}), \\ -1, & \text{if } e = (v_n, w), \\ 0, & \text{otherwise.} \end{cases} \]

Here $w = \tau_{H''}(v_n) \neq v_{n+1}$, where $H''$ is the hyperplane $\ker(\ell_{y''})$ with

\[ y'' = y' + \pi^n y. \]

as described in \(3.7\), notably in \(3.9\).

(ii) If $H'$ is fixed, each $(n+1)$-special arrow $e \neq (v_n, v_{n+1})$ with $o(e) = v_n$ occurs through a suitable choice of $H$ as the arrow $e = (v, w)$ where $P(f_{H,H',n})$ evaluates to $-1$. \(\square\)

The next result, technical in nature, is crucial for the proof of Theorem 3.10. Its proof is postponed to the next section.

Proposition 3.9: Let $n \in \mathbb{N}_{0}$ and $\varphi \in H(BT, \mathbb{Z})$ be such that $\varphi(e) = 0$ for arrows $e$ that either belong to $BT(n)$ or are $(n+1)$-special. Then $\varphi(e) = 0$ for all arrows $e$ of $BT(n+1)$.

Now we are able to show (modulo 3.9) the principal result.

Theorem 3.10: The van der Put map $P : \theta(\Omega)^* \to H(BT, \mathbb{Z})$ is surjective, and so the sequence

\[ 0 \longrightarrow 1 \longrightarrow C^* \longrightarrow \theta(\Omega)^* \longrightarrow H(BT, \mathbb{Z}) \longrightarrow 0 \]

is a short exact sequence of $G(K)$-modules.
Proof. (i) Let $\varphi \in \mathbf{H}(\mathbf{BT}, \mathbf{Z})$ be given. By successively subtracting $P(u_n)$ from $\varphi$, where $(u_n)_{n \in \mathbb{N}}$ is a suitable series of functions in $\mathcal{O}(\Omega)^*$ with $u_n \to 1$ locally uniformly, we will achieve that

$$\varphi - P\left( \prod_{1 \leq i \leq n} u_i \right) \equiv 0 \quad \text{on } \mathbf{BT}(n).$$

Then $\varphi = P(u)$, where $u = \lim_{n \to \infty} \prod_{1 \leq i \leq n} u_i$ is the limit function.

(ii) From condition (B1) for $\varphi$ and Proposition 2.8 we find a function $u_1$, namely a suitable finite product of functions of type $\ell_{H,H'}$, such that $\varphi - P(u_1) \equiv 0$ for each $e \in \mathbf{A}_{v_0,1}$. By condition (C), $\varphi - P(u_1)$ vanishes on all $e \in \mathbf{A}_{v_0,1}$, and thus by (A) on all $e$ that belong to $\mathbf{BT}(1) = \text{st}(v_0)$.

(iii) Suppose that $u_1, \ldots, u_n \in \mathcal{O}(\Omega)^*$ are constructed $(n \in \mathbb{N})$ such that for $1 \leq i \leq n$

(a) $P(u_i) \equiv 0$ on $\mathbf{BT}(i-1)$,

(b) $u_i \equiv 1 \pmod{\pi^{i(i-1)/2}}$ on $\mathbf{BT}(\lceil (i - 1)/2 \rceil)$; here $\lceil \cdot \rceil$ is the Gau bracket;

(c) $\varphi - P(\prod_{1 \leq i \leq n} u_i) \equiv 0$ on $\mathbf{BT}(n)$

hold. (Condition (a) is empty for $i = 1$ and therefore trivially fulfilled.) We are going to construct $u_{n+1}$ such that $u_1, \ldots, u_{n+1}$ fulfill the conditions on level $n+1$.

(iv) From (c) and (B1) we have for $n$-special vertices $v$ and $\psi := \varphi - P(\prod_{1 \leq i \leq n} u_i) \in \mathbf{H}(\mathbf{BT}, \mathbf{Z})$:

$$\sum_{e \in \mathbf{A}_{v,e_1}} \psi(e) = \sum_{e \in \mathbf{A}_{v,e_1}} \psi(e) = 0.$$ 

(v) According to Proposition 3.9 we find $u_{n+1}$, viz, a suitable product of functions $f_{H,H',n}$, such that

$$(\varphi - P(u_{n+1}))(e) = \left( \varphi - P\left( \prod_{1 \leq i \leq n+1} u_i \right) \right)(e) = 0$$

on all $(n+1)$-special arrows $e$. Furthermore, that $u_{n+1}$ (like the functions $f_{H,H',n}$, see Lemma 5.5 (ii)) satisfies $P(u_{n+1}) \equiv 0$ on $\mathbf{BT}(n)$, i.e., condition (a), and condition (b): $u_{n+1} \equiv 1 \pmod{\pi^{n/2}}$ on $\mathbf{BT}(\lfloor n/2 \rfloor)$. Hence $\varphi - P(\prod_{1 \leq i \leq n+1} u_i)$ vanishes on arrows which belong to $\mathbf{BT}(n)$ or are $(n+1)$-special. Using Proposition 3.9 $\varphi - P(\prod_{1 \leq i \leq n+1} u_i)$ vanishes on $\mathbf{BT}(n+1)$. That is, conditions (a), (b), (c) hold for $u_1, \ldots, u_{n+1}$, and we have inductively constructed an infinite series $u_1, u_2, \ldots$ with (a), (b) and (c) for all $n$.

(vi) It follows from (b) that the infinite product

$$u = \prod_{i \in \mathbb{N}} u_i$$

is normally convergent on each $\Omega(n)$ and thus defines a holomorphic invertible function $u$ on $\Omega$. Its van der Put transform $P(u)$ restricted to $\mathbf{BT}(n)$ depends only on $u_1, \ldots, u_n$, due to (c), and thus agrees with $\varphi$ reduced to $\mathbf{BT}(n)$. Therefore, $\varphi = P(u)$, and the result is shown. \qed

4. The group $\mathbf{H}(\mathbf{BT}, \mathbf{Z})$

4.1. We start with the

Proof of Proposition 3.9

(i) The requirements of Proposition 3.9 for $\varphi \in \mathbf{H}(\mathbf{BT}, \mathbf{Z})$ on level $n \in \mathbb{N}_0$ will be labelled by $R(n)$.

(ii) Suppose that $R(n)$ holds for $\varphi$. Then $\varphi$ vanishes on all arrows $\mathbf{A}_{v,e_1}$ whenever $v$ is $n$-special, since such an $e$ is either $(n+1)$-special or belongs to $\mathbf{BT}(n)$. Hence by conditions (C) and (A) of 3.2, $\varphi(e) = 0$ whenever $e$ is contiguous with $v$, i.e., if $e$ belongs to $\text{st}(v)$. This shows, in particular, that Proposition 3.9 holds for $n = 0$. 

(iii) Let \( v \in \mathbf{V}(\mathcal{B}T) \) have distance \( d(v_0, v) = n \), but it is not necessarily \( n \)-special. For the same reason as in (ii), \( \varphi \) vanishes identically on \( st(v) \) if it vanishes on all outbound arrows \( e \in A_{v,1} \). Hence if suffices to show

\[ \varphi(e) = 0 \]

for outbound arrows \( e \) of type 1 and level \( n \).

(iv) For a vertex \( v \) with \( d(v_0, v) = n \), we let \( s(v) \) be the distance to the next \( w \in \mathbf{V}(\mathcal{B}T) \) which is \( n \)-special. We are going to show assertion (O) by induction on \( s(o(e)) \).

(v) By \( R(n) \), (O) holds if \( s = s(o(e)) = 0 \), i.e., if \( o(e) \) is \( n \)-special. Therefore, suppose that \( s > 0 \). By the preceding we are reduced to showing

\[ \text{(P)} \]

Let \( e \) be an outbound arrow of type 1, level \( n \), and with \( s = s(o(e)) > 0 \).

Then \( e \) belongs to \( st(\tilde{v}) \), where \( d(v_0, \tilde{v}) = n \) and \( s(\tilde{v}) < s \).

(vi) We reformulate (P) in lattice terms. Representing \( v \in \mathbf{V}(\mathcal{B}T) \) correspond one-to-one to sublattices \( L \) of full rank \( r \) which satisfy

\[ \text{dim}_F(L/L') = 1 \]

and \( L \subset L_0 \), with \( L \not\subset L_0 \). Then, as \( \text{dim}_F(L/L') = 1 \) and \( d(v_0, v) = n + 1 \), \( (n' = n + 1, n_2, \ldots, n_r) \) is the sed of \( L_0/L' \). This means that \( L_0 \) has an ordered \( O \)-basis \( \{x_1, \ldots, x_r\} \) such that \( \{\pi^{n_1}x_1, \pi^{n_2}x_2, \ldots, \pi^{n_r}x_r\} \) is a basis of \( L' \) and \( \{\pi^n x_1, \pi^{n_2}x_2, \ldots, \pi^{n_r}x_r\} \) is a basis of \( L \). Assume that \( k \) with \( 2 \leq k \leq r \) is maximal with \( n_k = n_k \). Let \( M \) be the sublattice of \( L \) with basis \( \{\pi^n x_1, \pi^{n_2}x_2, \ldots, \pi^{n_r}x_r\} \). Then \( w = [M] \) is \( n \)-special and \( s(v) = d(v, w) = n_2 \), which by assumption is positive. Put \( \tilde{L} \) for the lattice with basis \( \{\pi^n x_1, \pi^{n_2}x_2, \ldots, \pi^{n_k}x_k, \pi^{n_{k+1}}x_{k+1}, \ldots, \pi^{n_r}x_r\} \). The vertex \( \tilde{v} := [\tilde{L}] \) satisfies

\[ d(v_0, \tilde{v}) = n, \quad d(v, \tilde{v}) = 1 = d(v', \tilde{v}) \quad \text{and} \quad s(\tilde{v}) = d(w, \tilde{v}) = n_2 - 1 = s(v) - 1. \]

Hence \( e = (v, v') \) belongs to \( st(\tilde{v}) \), where \( \tilde{v} \) is as wanted for assertion (P).

This finishes the proof of Proposition 3.9.

Corollary 4.2: Let \( \varphi \in \mathbf{H}(\mathcal{B}T, \mathbb{Z}) \) be such that \( \varphi(e) = 0 \) for all \( i \)-special arrows \( e \), where \( 1 \leq i \leq n \). Then \( \varphi \equiv 0 \) on \( \mathcal{B}T(n) \).

Proof. This follows by induction from 3.9.

4.3. Let \( v \) be an \( n \)-special vertex \( (n \geq 1) \), \( v^* \) its predecessor on the uniquely determined \( n \)-special path \( (v_0, v_1, \ldots, v_{n-1} = v^*, v) \) from \( v_0 \) to \( v \), and \( e' \) the \( n \)-special arrow \( (v', v) \). Its inverse \( e' = (v, v') \) belongs to \( A_{v,1} \).

Lemma 4.4: In the given situation, \( e \in A_{v,1} \) is inbound if and only if \( e' \prec e \).

Proof. As the stabilizer \( GL(r, O) \) of \( L_0 = O' \) acts transitively on \( n \)-special vertices or arrows, we may suppose that \( v = [L_n] \), where \( L_n \) is the \( O \)-lattice with basis \( \{\pi^n x_1, \ldots, x_r\} \), and thus \( v^* = [L_{n-1}] \). (Here \( \{x_1, \ldots, x_r\} \) is the standard basis of \( L_0 \).) Under (2.1.4), \( e' \) corresponds to the one-dimensional subspace \( \pi L_{n-1}/\pi L_n \) of the \( r \)-dimensional \( \mathbb{F} \)-space \( L_n/\pi L_n \), which has \( \pi x_1 = \pi^n x_1 \) (mod \( \pi L_n \)) as a basis vector. Let \( H \) be a hyperplane
in \( L_n/\pi L_n \) with pre-image \( H \) in \( L_n \), and let \( e_{\overline{\varphi}} = (v, v_{\overline{\varphi}}) \) be the arrow of type 1 determined by \( \overline{\varphi} \). Then \( v_{\overline{\varphi}} = [H] \) and

\[
\overline{v} < e_H \iff (\pi^n x_1) \in \overline{\Pi} \iff \pi^n x_1 \in H
\]

\[
\iff \pi^n L_0 \subset H \iff d(v_0, v_{\overline{\varphi}}) \leq n \iff e_{\overline{\varphi}} \text{ is inbound.}
\]

\[\square\]

4.5. We may now reformulate condition (B_1) for \( \varphi \in H(BT, \mathbb{Z}) \) at the \( n \)-special vertex \( v \) of level \( n \geq 1 \) as follows: Splitting 4.5.1

\[
\mathbf{A}_{v,1} = \mathbf{A}_{v,1,\text{in}} \cup \mathbf{A}_{v,1,\text{out}}
\]

into the subsets of inbound / outbound arrows (note that \( e \in \mathbf{A}_{v,1} \) is outbound if and only if it is \((n+1)\)-special), (B_1) reads

\[
0 = \sum_{e \in \mathbf{A}_{v,1,\text{in}}} \varphi(e) = \sum_{e \in \mathbf{A}_{v,1,\text{out}}} \varphi(e) + \sum_{e \in \mathbf{A}_{v,1,\text{out}}} \varphi(e) = \varphi(\overline{e}^*) + \sum_{e \in \mathbf{A}_{v,1,\text{out}}} \varphi(e)
\]

(where we used [14] and condition (C) for \( \varphi(\overline{e}^*) \)), i.e., as the flow condition

\[
(4.5.2) \quad \varphi(e^*) = \sum_{e \in \mathbf{A}_{v,1,\text{out}}} \varphi(e).
\]

The number of terms in the sum is

\[
(4.5.3) \quad \# \mathbf{A}_{v,1,\text{out}} = \# \mathbf{A}_{v,1} - \# \mathbf{A}_{v,1,\text{in}} = \# \mathbf{F}^{r-1}(F) - \# \mathbf{F}^{r-2}(F) = q^{r-1}
\]

4.6. Let \( T_{v_0} \) be the full subcomplex of \( BT \) composed of the \( n \)-special vertices \((n \in \mathbb{N}_0)\) along with the 1-simplices connecting them. In other words, \( T_{v_0} \) is the union of the paths \( \mathbf{A}_{v_0,1,n} \), where \( n \in \mathbb{N}_0 \), see [2.1.6]. It is connected, one-dimensional and cycle-free, hence a tree. The valence (= number of neighbors) of \( v_0 \) is \# \( \mathbf{F}^{r-1}(F) = (q^r - 1)/(q - 1) \), the valence of each other vertex \( v \neq v_0 \) is \( q^{r-1} + 1 \), as we read off from 4.5.3. Let further \( T_{v_0}(n) := T_{v_0} \cap BT(n) \).

4.6.1. We define \( H(n) \) as the image of \( H(BT, \mathbb{Z}) \) in \( \{ \varphi : \mathbf{A}(BT(n)) \to \mathbb{Z} \} \) obtained by restriction. Hence

\[
(4.6.2) \quad H(BT, \mathbb{Z}) = \lim_{\substack{n \to \infty \in \mathbb{N}}} H(n).
\]

Put further

\[
(4.6.3) \quad H'(n) := \left\{ \varphi : \mathbf{A}(T_{v_0}(n)) \to \mathbb{Z} \mid \begin{array}{l}
\varphi \text{ is subject to (4.6.4) and (4.6.5)}(\forall) \\
\text{for each } i\text{-special } v, 0 \leq i < n
\end{array} \right\}.
\]

Here \( \mathbf{A}(S) \) is the set of arrows (oriented 1-simplices) of the simplicial complex \( S \), and the conditions are

\[
(4.6.4) \quad \varphi(e) + \varphi(\overline{e}) = 0 \quad \text{for each arrow } e \text{ with inverse } \overline{e};
\]

\[
(4.6.5)(v) \quad \sum_{e \in \mathbf{A}(T_{v_0})} \varphi(e) = 0.
\]
4.7. Equality \((4.5.2)\) together with the condition \((B_1)\) at \(v_0\) states that the restriction of \(\varphi \in H(BT, Z)\) to \(T_0(n)\) is an element of \(H'(n)\). Therefore, restriction defines homomorphisms \(r_n: H(n) \to H'(n)\), which make the diagram (with natural maps \(q_n, q'_n\))

\[
\begin{array}{cccc}
H(n + 1) & \xrightarrow{r_{n+1}} & H'(n + 1) \\
\downarrow q_n & & \downarrow q'_n \\
H(n) & \xrightarrow{r_n} & H'(n)
\end{array}
\]

(4.7.1)

commutative. Note that both \(q_n\) and \(q'_n\) are surjective. Corollary \(4.2\) may be rephrased as

**Proposition 4.8:** \(r_n\) is surjective.

**Lemma 4.9:** \(r_n\) is also surjective.

\[\tag*{\text{□}}\]

**Proof.** For \(n = 1\), this is implicit in the proof of Theorem \(3.10\) (i.e., one may arbitrarily prescribe the value of \(\varphi \in H(BT, Z)\) on \(e \in A_{w_0,1}\), subject only to \((B_1)\) at \(v_0\)).

For \(n \geq 1\), let \(Q_{n+1}\) (respectively \(Q'_{n+1}\)) be the kernel of \(q_n\) (resp. \(q'_n\)). Then \(r_{n+1}(Q_{n+1}) \subset Q'_{n+1}\), and we have the commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \longrightarrow & Q_{n+1} & \longrightarrow & H(n + 1) & \longrightarrow & H(n) & \longrightarrow & 0 \\
\downarrow & & \downarrow r_{n+1} & & \downarrow r_n & & \downarrow & & \downarrow & & \downarrow \phi \\
0 & \longrightarrow & Q'_{n+1} & \longrightarrow & H'(n + 1) & \longrightarrow & H'(n) & \longrightarrow & 0.
\end{array}
\]

By induction hypothesis, \(r_n\) is surjective, so the surjectivity of \(r_{n+1}\) is implied by

\[(*)\]

\(r_{n+1}(Q_{n+1}) = Q'_{n+1}\)

But

\[Q_{n+1} = \{ \varphi \in H(n + 1) \mid \varphi \equiv 0 \text{ on } BT(n) \}\]

\[Q'_{n+1} = \{ \varphi \in H'(n + 1) \mid \varphi \equiv 0 \text{ on } T_0(n) \},\]

so \((*)\) follows from the existence of sufficiently many elements of \(Q_{n+1}\) (e.g., the classes in \(H(n + 1)\) of the \(P(f_{H', n})\)) which have sufficiently independent values on the arrows in \(T_0(n + 1)\) not in \(T_0(n)\). See also the proof of Theorem \(3.10\) steps (iv) and (v). \[\tag*{\text{□}}\]

4.10. Let \(H(T_0, Z) = \lim_{\leftarrow \leftarrow} H'(n)\) be the group of functions \(\varphi: A(T_0) \to Z\) which satisfy \((4.6.3)\) and \((4.6.5)(v)\) for all vertices \(v\) of \(T_0\). Similarly, we define \(H(T_0, A)\) for an arbitrary abelian group \(A\) instead of \(Z\). That is, elements of \(H(T_0, A)\) are characterized by conditions analogous with \((A)\) and \((B)\) of \(5.2\) while \((C)\) is not applicable. Putting together the considerations of \(4.5\) with \(4.8\) and \(4.9\) we find

\[
H(BT, Z) \xrightarrow{\phi} H(T_0, Z),
\]

where the canonical isomorphism is given by restricting \(\varphi \in H(BT, Z), \varphi: A(BT) \to Z\) to the subset \(A(T_0)\) of \(A(BT)\).

In what follows, \(A\) is an arbitrary abelian group. The next result is a consequence of the above.

**Proposition 4.12:** The canonical maps

\[
H(BT, A) \otimes A \longrightarrow H(BT, A)
\]

and

\[
H(T_0, Z) \otimes A \longrightarrow H(T_0, A)
\]

are bijective, and \((4.11)\) yields

\[
H(BT, A) \xrightarrow{\phi} H(T_0, A).
\]

\[(4.12.1)\]
Proof. As \( H(n) \) and \( H'(n) \) are finitely generated free \( \mathbb{Z} \)-modules, their tensor products with \( A \) are isomorphic with the similarly defined groups of \( A \)-valued maps. Then \( 4.12.1 \) follows from \( 1.6.2 \) and \( 1.11 \). \( \square \)

4.13. Recall that an \( A \)-valued distribution on a compact totally disconnected topological space \( X \) is a map \( \delta: U \to \delta(U) \in A \) from the set of compact-open subspaces \( U \) of \( X \) to \( A \) which is additive in finite disjoint unions. We call \( \delta(U) \) the volume of \( U \) with respect to \( \delta \). The total mass (or volume) of \( \delta \) is \( \delta(X) \).

We apply this to the situation (see 2.1.6 - 2.1.8) where \( \mathbb{A} \) with \( \mathbb{G} \)-equivariant \( \mathbb{A} \) are isomorphic with the similarly defined groups of \( \mathbb{G} \)-valued maps. Then (4.6.2) and (4.11) follows from \( 4.13.3 \).

4.13.2. Let \( \mathbb{A} \) be the group \( \mathbb{A} = \mathbb{A}(\mathbb{T}_{\mathbb{N}}) \) of \( \mathbb{A} \)-valued distributions on \( \mathbb{P}(\mathbb{V}) \) with sub-group \( \mathbb{D}(\mathbb{P}(\mathbb{V}), \mathbb{A}) \) of distributions with total mass 0. By \( 2.1.8 \), the sets \( \mathbb{P}(\mathbb{V})(e) = \mathbb{G}_{\mathbb{K},1}(e) \), where \( e \) runs through the outbound arrows of \( \mathbb{A}_{0,1,\alpha} \) \((\alpha \in \mathbb{N})\), i.e., through the set

\[
\mathbb{A}^+(\mathbb{T}_{\mathbb{N}}) = \{ e \in \mathbb{A}(\mathbb{T}_{\mathbb{N}}) \mid e \text{ oriented away from } v_0 \},
\]

form a basis for the topology on \( \mathbb{P}(\mathbb{V}) \). Therefore, an element \( \delta \) of \( \mathbb{D}(\mathbb{P}(\mathbb{V}), \mathbb{A}) \) is an assignment

\[
\delta: \mathbb{A}^+(\mathbb{T}_{\mathbb{N}}) \to \mathbb{A}
\]

(we interpret \( \delta(e) \) as the volume of \( \mathbb{P}(\mathbb{V})(e) \) with respect to \( \delta \)) subject to the requirement

\[
\delta(e^*) = \sum_{e \in \mathbb{A}^+(\mathbb{T}_{\mathbb{N}}), o(e) = t(e^*)} \delta(e)
\]

for each \( e^* \in \mathbb{A}^+(\mathbb{T}_{\mathbb{N}}) \). The total mass of \( \delta \) is

\[
\delta(\mathbb{P}(\mathbb{V})) = \sum_{e \in \mathbb{A}^+(\mathbb{T}_{\mathbb{N}}), o(e) = v_0} \delta(e) = \sum_{e \in \mathbb{A}_{0,1}} \delta(e).
\]

In view of \( 1.6.2 \) and (4.6.5)(v0), we find that

\[
\mathbb{D}(\mathbb{P}(\mathbb{V}), \mathbb{A}) \overset{\approx}{\to} \mathbb{H}(\mathbb{T}_{\mathbb{N}}, \mathbb{A}),
\]

where some \( \delta: \mathbb{A}^+(\mathbb{T}_{\mathbb{N}}) \to \mathbb{A} \) in the left hand side is completed to a map on \( \mathbb{A}(\mathbb{T}_{\mathbb{N}}) \) by \( 1.6.1 \), i.e., by \( \varphi(\mathbb{T}) = -\varphi(e) \).

While both isomorphisms in \( 1.11 \) (or \( 1.12.1 \) and \( 1.13 \)) fail to be \( \mathbb{G}(\mathbb{K}) \)-equivariant (as \( \mathbb{G}(\mathbb{K}) \) fixes neither \( v_0 \) nor \( \mathbb{T}_{\mathbb{N}} \)), the resulting isomorphism

\[
\mathbb{H}(\mathbb{B} \mathbb{T}, \mathbb{A}) \overset{\approx}{\to} \mathbb{D}(\mathbb{P}(\mathbb{V}), \mathbb{A})
\]

\[
\varphi \mapsto \tilde{\varphi}
\]

is. Here the distribution \( \tilde{\varphi} \) evaluates on \( \mathbb{P}(\mathbb{V})(e) \) as \( \varphi(e) \) whenever \( e \) is an arrow of \( \mathbb{B} \mathbb{T} \) of type 1 and \( \mathbb{P}(\mathbb{V})(e) \) is the compact-open subset of hyperplanes \( H \) of \( V \) such that \( e \) points to \( H \).

We summarize what has been shown.
Theorem 4.16: Let $A$ be an arbitrary abelian group. Restricting the evaluation of $\varphi \in \mathcal{H}(BT, A)$ to arrows of $T_{v_0}$ (resp. arrows of type 1 of $BT$) yields canonical isomorphisms

$$\mathcal{H}(BT, A) \xrightarrow{\cong} \mathcal{H}(T_{v_0}, A)$$

resp.

$$\mathcal{H}(BT, A) \xrightarrow{\cong} \mathcal{D}^0(\mathcal{P}(V^\wedge), A).$$

The second of these is equivariant for the natural actions of $G(K) = GL(r, K)$ on both sides, while the first isomorphism is equivariant for the actions of the stabilizer $G(O)Z(K)$ of $v_0 \in G(K)$. Each of the three modules $\mathcal{H}(BT, A), \mathcal{H}(T_{v_0}, A), \mathcal{D}^0(\mathcal{P}(V^\wedge), A)$ is the tensor product with $A$ of the same module with coefficients in $\mathbb{Z}$.

As a direct consequence of the first isomorphism, i.e., of (4.12.1) we find the following Corollary, which is in keeping with the fact that bounded holomorphic functions on $\Omega$ are constant.

Corollary 4.17: If $\varphi \in \mathcal{H}(BT, A)$ has finite support, it vanishes identically.

Proof. Suppose that $\varphi$ has support in $BT(n)$ with $n \in \mathbb{N}$. Then its restriction to $T_{v_0}(n+1)$ satisfies (4.6.4) and (4.6.5) at all vertices $v$ of $T_{v_0}(n + 1)$. As $T_{v_0}(n+1)$ is a finite tree, this forces $\varphi$ to vanish identically on $T_{v_0}(n+1)$, thus on $BT$. □

5. Concluding remarks

5.1. Ehud de Shalit in [2] Section 3.1 postulated four conditions $A, B, C, D$ for what he calls harmonic $k$-cochains on $BT$. These conditions specialized to $k = 1$ are essentially our conditions (A), (B), (C) from 3.2. Grosso modo, de Shalit’s $B$ corresponds to (B), $C$ to (C) and $D$ to (A), while $A$ is a special case of (A).

5.2. In fact, the relationship with de Shalit’s work is as follows. Suppose that $\text{char}(K) = 0$, and consider the diagram

$$\begin{array}{ccc}
\mathcal{O}(\Omega)^* & \xrightarrow{P} & \mathcal{H}(BT, \mathbb{Z}) \\
\downarrow \text{closed 1-forms} & \text{res} & \downarrow \text{res} \\
d\log u = u^{-1}du & \mathcal{H}(BT, K) (= \mathcal{C}^1_{\text{har}} \text{ of [2]}) & \mathcal{H}(T, \mathbb{C})
\end{array}$$

(5.2.1)

where “res” is de Shalit’s residue mapping. Its commutativity follows for $u = \ell_{u, w}$ from Corollary 7.6 and Theorem 8.2 of [2] (along with the explanations given there, and our description of $P(u)$), and may be verified for general $u$ by approximating. Hence the van der Put transform $P$ yields a concrete description of the residue mapping on logarithmic 1-forms.

5.3. Now suppose that $\text{char}(K) = p > 0$, and that moreover $r = 2$. Then $BT$ is the Bruhat-Tits tree $T$, and the residue mapping

$$\text{res}: \{1\text{-forms on } \Omega = \Omega^2 \} \longrightarrow \mathcal{H}(T, \mathbb{C})$$

(see [6] 1.8) is such that the diagram analogous with (5.2.1)

$$\begin{array}{ccc}
\mathcal{O}(\Omega)^* & \xrightarrow{P} & \mathcal{H}(T, \mathbb{Z}) \\
\downarrow \{1\text{-forms on } \Omega \} & \text{res} & \downarrow \text{res} \\
d\log u & \mathcal{H}(T, \mathbb{C}) & \mathcal{H}(T, \mathbb{C})
\end{array}$$

(5.3.1)
commutes, with remarkable arithmetic consequences (loc. cit., Sections 6 and 7). A similar residue map for \( r > 2 \) unfortunately lacks so far. In any case, we should regard \( P \) as a substitute for the logarithmic derivation operator

\[
u \mapsto d \log u = u^{-1}du
\]

in characteristic 0.

5.4. In [17], Peter Schneider and Ulrich Stuhler described the cohomology \( H^*(\Omega, A) \) of \( \Omega = \Omega^r \) with respect to an abstract cohomology theory, where \( A = H^0(\text{Sp}(K)) \). That theory is required to satisfy four natural axioms, loc. cit., Section 2. As they explain, these axioms are fulfilled at least

- for the \( \ell \)-adic cohomology of rigid-analytic spaces over \( K \), where \( \ell \) is a prime different from \( p = \text{char}(F) \), and \( A = \mathbb{Z}_\ell \), and
- for the de Rham cohomology (where one must moreover assume that \( \text{char}(K) = 0 \)); here \( A = K \).

Their result is stated loc. cit. Section 3, Theorem 1, which in dimension 1 is (in our notation)

\[
H^1(\Omega^r, A) \xrightarrow{\sim} D^0(\mathbb{P}(V^\wedge), A).
\]

Theorem 8.2 in [2] gives that (in the case where \( \text{char}(K) = 0 \) and \( H^* = H^*_{\text{dR}} \) is the de Rham cohomology)

\[
H^k_{\text{dR}}(\Omega^r) \xrightarrow{\sim} C^k_{\text{bar}},
\]

where \( C^k_{\text{bar}} \) is our \( \mathbf{H}(BT, K) \). Hence our Theorems 3.10 and 4.16 refine the above in the case \( k = 1 \) and moreover provide natural \( \mathbb{Z} \)-structures on the \( H^1 \)-groups.

5.5. Let now \( \Gamma \) be a discrete subgroup of \( G(K) \). The most interesting cases are those where the image of \( \Gamma \) in \( G(K)/Z(K) = \text{PGL}(r, K) \) has finite covolume with respect to Haar measure, or is even cocompact. Examples are given as Schottky groups in \( \text{PGL}(2, K) \) [9] or as arithmetic subgroups of \( G(K) \) of different types, when \( K \) is the completion \( k_{\infty} \) of a global field \( k \) at a non-archimedean place \( \infty \) [4], [16]. Then often the quotient analytic space \( \Gamma \setminus \Omega \) is the set of \( C \)-points of an algebraic variety [10], [4], [15], which may be studied via a spectral sequence relating the cohomologies of \( \Omega \) and \( \Gamma \) with that of \( \Gamma \setminus \Omega \) (17 Section 5). For \( r = 2 \), this essentially boils down to a study of the \( \Gamma \)-cohomology sequence of \( \Omega \) (17 Section 5). But also for \( r > 2 \), (12) with its \( \Gamma \)-action will be useful, which is the topic of ongoing work.
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