Integral $F(R)$ Gravity and Saddle Point Condition as a Remedy for the $H_0$-tension

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In this work, we shall provide an $F(R)$ gravity theoretical framework for solving the $H_0$-tension. Specifically, by exploiting the $F(R)$ gravity correspondence with a scalar-tensor theory, we shall provide a condition in which when it is satisfied, the $H_0$-tension is alleviated. The condition that remedies the $H_0$-tension restricts the corresponding $F(R)$ gravity, and we present in brief the theoretical features of the constrained $F(R)$ gravity theory in both the Jordan and Einstein frames. The condition that may remedy the $H_0$-tension is based on the existence of a metastable de Sitter point that occurs for redshifts near the recombination. This metastable de Sitter vacuum restricts the functional form of the $F(R)$ gravity in the Jordan frame. We also show that by appropriately choosing the $F(R)$ gravity, along with the theoretical solution offered for the $H_0$-tension problem, one may also provide a unified description of the inflationary era with the late-time accelerating era, in terms of two extra de Sitter vacua. We propose a new approach to $F(R)$ gravity by introducing a new class of integral $F(R)$ gravity functions, which may be wider than the usual class expressed in terms of elementary $F(R)$ gravity functions. Finally, the Einstein frame inflationary dynamics formalism is briefly discussed.

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I. INTRODUCTION

The old story of the $H_0$-tension between measurements of the Hubble rate at high and low redshift seems to persist even to date. Indeed, as it is pointed out in the literature, there is an observational tension between the value of the Hubble constant inferred from small redshifts, as in the case of observational data coming from Type Ia supernova (SNIa) calibrated by Cepheid observations [1] and from large redshifts, as the cosmic microwave background (CMB) observational data [2]. There exist several proposals for eliminating or alleviating the $H_0$-tension, such as the early dark energy proposal [3, 4], see also [6, 7], but also radical physics changes before $70-150$ Myrs may also eliminate the $H_0$-tension, see for example [8, 11]. Also, the tension might be an artifact of the Cepheid calibration [see 8, 12] for example. For a mainstream of recent articles on the $H_0$-tension, see for example [13–18, 20–30, 30–32] and references therein.

In this paper, we shall provide a theoretical solution to the $H_0$-tension, exploiting the correspondence of the Jordan frame $F(R)$ gravity theory with a scalar-tensor theory in the Einstein frame. Already in the context of $F(R)$ gravity, the $H_0$-tension problem has been addressed [28, 29], but the result was not deemed too successful in terms of ordinary $F(R)$ gravity models. Hence, in this work, we shall propose a condition that may eliminate the $H_0$-tension, considered both in the Jordan and the Einstein frame. Specifically, $F(R)$ gravity in the Jordan frame can be rewritten in a scalar-tensor form in the Einstein frame. When we consider the inflationary era, the scalar field is the inflaton and when we consider the present-day accelerating expansion, the scalar field can be regarded as quintessence. Both the inflationary era and the present day accelerating expansion of the Universe, correspond to the almost flat potential of the scalar field. However, in the case of the inflationary potential, the potential is slightly unstable and in the case of the accelerating expansion of the present Universe, the potential could be stable. Although one needs to be careful about the frame because the expansion of the Universe in the Jordan frame, which may correspond to the physical frame observed by the observers, is different from that in the Einstein frame, the scalar-tensor description provides insights and an intuitive understanding of the properties of the cosmological expansion. In the scalar-tensor description of $F(R)$ gravity, the $H_0$-tension might be solved by the saddle point of the Einstein frame potential of the scalar field. We may assume a stationary point of the potential and if the stationary point is a saddle point,
the potential is unstable for one direction but stable for another direction. If the gradient of the potential at the vicinity of the saddle point is positive with respect to the variation of the scalar curvature, the curvature with a smaller value than the value of the saddle point becomes larger. After that, it reaches the saddle point and stays there for a limited amount of time, and after that, the scalar curvature becomes larger again. On the other hand, if the gradient is negative at the vicinity of the saddle point, the curvature with a value larger than that at the saddle point becomes smaller and reaches the saddle point. At that point, it stays there for a while and after that, the curvature becomes smaller again. Then near the saddle point, the potential plays the role of the cosmological constant. Then there is a saddle point after clear up of the Universe with a negative gradient at the vicinity of the saddle point, a short-lasting de Sitter era occurs, for which the Hubble rate may correspond to the Hubble constant obtained by the CMB observation. It is worth noting that a similar principles work has been developed in [30].

Going to more technical details, we shall consider the \( F(R) \) gravity model where the function \( F(R) \) is given by an integral of some function \( Q(R) \) of the scalar curvature. The function \( Q(R) \) reflects the properties of the potential in the scalar-tensor description and therefore the function \( Q(R) \) directly connects the scalar-tensor description with \( F(R) \) description. Therefore it becomes easier to construct a model for which the inflationary era and the late-time acceleration era can be described in a unified way [40] and simultaneously, the \( H_0 \)-tension problem is eliminated. With this article, we propose a newly introduced approach to standard \( F(R) \) gravity, by introducing a new class of integral \( F(R) \) gravity functions, which belong to a wider class of models, compared with conventional \( F(R) \) gravity models.

This article is organized as follows: In section II we present the theoretical formalism of \( F(R) \) gravity in both Jordan and Einstein frames and we introduce the saddle point condition which may explain the \( H_0 \)-tension. We analyze in depth the forms of \( F(R) \) gravity which may resolve and \( H_0 \)-tension and we also discuss the correspondence with the Einstein frame theory. In section III, we present in brief the inflationary features of the integral \( F(R) \) gravity in the Einstein which may explain the \( H_0 \)-tension. Finally, the conclusions follow at the end of the paper.

II. \( F(R) \) GRAVITY SADDLE POINT PROPOSAL AND THE EINSTEIN FRAME PICTURE

The action of \( F(R) \) gravity is given by replacing the scalar curvature \( R \) in the Einstein-Hilbert action which is,

\[
S_{\text{EH}} = \int d^4 x \sqrt{-g} \left( \frac{R}{2\kappa^2} + L_{\text{matter}} \right),
\]

by using some appropriate function of the scalar curvature, as follows,

\[
S_{F(R)} = \int d^4 x \sqrt{-g} \left( \frac{F(R)}{2\kappa^2} + L_{\text{matter}} \right).
\]

In Eqs. 1 and 2, \( L_{\text{matter}} \) is the Lagrangian density of the perfect matter fluids. In this section, we find the saddle point condition by using the Einstein frame picture of \( F(R) \) gravity and we propose theoretical solutions which may solve the problem of the \( H_0 \)-tension. We also present in brief the slow-roll inflation formalism in the Einstein frame for completeness.

A. General Properties of \( F(R) \) gravity

In this subsection, we review the general properties of \( F(R) \) gravity, especially the relations between the Jordan frame and the Einstein frame and the viability conditions that any \( F(R) \) gravity must satisfy.

By varying the action 2 with respect to the metric, we obtain the equation of motion for \( F(R) \) gravity theory as follows,

\[
G^F_{\mu\nu} \equiv \frac{1}{2} g_{\mu\nu} F - R_{\mu\nu} F_R - g_{\mu\nu} \Box F_R + \nabla_\mu \nabla_\nu F_R = -\kappa^2 T_{\mu\nu}.
\]

Here \( F_R \equiv \frac{dF(R)}{dR} \) and \( T_{\mu\nu} \) is the energy momentum tensor of the perfect matter fluids.

We can find several (in many cases exact) solutions of Eq. 3. Without the presence of matter, a simple solution is given by assuming that the Ricci tensor is covariantly constant, that is, \( R_{\mu\nu} \propto g_{\mu\nu} \). Then Eq. 3 is simplified to the following algebraic equation:

\[
0 = 2F - RF_R.
\]
If Eq. (11) has a solution, then the (anti-)de Sitter and/or Schwarzschild-(anti-)de Sitter space or the Kerr-(anti-)de Sitter space is an exact solution in a vacuum.

We should note that we can also rewrite $F(R)$ gravity in a scalar-tensor form. We introduce an auxiliary field $A$ and rewrite the action (2) of the $F(R)$ gravity in the following form,

$$ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F_R(A) \left( R - A \right) + F(A) \right\}. \tag{5} $$

We obtain $A = R$ by the variation of the action with respect to $A$ and by substituting the obtained equation $A = R$ into the action (5), we find that the action in (2) is reproduced. If we rescale the metric by a kind of a scale transformation,

$$ g_{\mu \nu} \to e^\sigma g_{\mu \nu}, \quad \sigma = - \ln F_R(A), \tag{6} $$

we obtain the action in the Einstein frame,

$$ S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{3}{2} g^\rho_\sigma \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right), $$

$$ V(\sigma) = e^\sigma g(e^{-\sigma}) - e^{2\sigma} f(g(e^{-\sigma})) = \frac{A}{F_R(A)} - \frac{F(A)}{F_R(A)^2}. \tag{7} $$

Here $g(e^{-\sigma})$ is given by solving the equation $\sigma = - \ln F_R(A)$ as $A = g(e^{-\sigma})$. The effective gravitational coupling is given by

$$ \frac{d^2V(\sigma)}{d\sigma^2} = \frac{3}{2} \left\{ \frac{A}{F_R(A)} - \frac{4F(A)}{F_R(A)^2} + \frac{1}{F_{RR}(A)} \right\}, \tag{8} $$

and if the mass $m_\sigma$ is not very large, there appears a large correction to the Newton law. Here $F_{RR}(A) \equiv \frac{d^2F(R)}{dR^2} \big|_{R=A}$.

We now also need to mention the problem of antigravity. Eq. (5) indicates that the effective gravitational coupling is given by $\kappa_{\text{eff}}^2 = \frac{\kappa^2}{F_R}$. Therefore when $F_R$ is negative, it is possible to have antigravity regions. Then, we need to require

$$ F_R > 0. \tag{9} $$

We should note that from the viewpoint of the field theory, the graviton becomes a ghost in the antigravity region.

It should be noted that the de Sitter or anti-de Sitter space solution in Eq. (4) corresponds to the extremum of the potential $V(\sigma)$. In fact, we find,

$$ \frac{dV(\sigma)}{dA} = \frac{F_{RR}(A)}{F_R(A)^3} \left(-AF_R(A) + 2F(A)\right). \tag{10} $$

Therefore, if Eq. (4) is satisfied, the scalar field $\sigma$ should be on the local maximum or local minimum of the potential and $\sigma$ can be a constant. When the condition (4) is satisfied, the mass given by (5) has the following form,

$$ m_\sigma^2 = \frac{3}{2F_R(A)} \left(-A + \frac{F_R(A)}{F_{RR}(A)}\right). \tag{11} $$

Therefore, in the case that the condition (4) for avoiding the antigravity holds true, the mass squared $m_\sigma^2$ is positive, showing that the scalar field is on the local minimum if

$$ -A + \frac{F_R(A)}{F_{RR}(A)} > 0. \tag{12} $$

On the other hand, the scalar field is on the local maximum of the potential if

$$ -A + \frac{F_R(A)}{F_{RR}(A)} < 0. \tag{13} $$

In this case, the mass squared $m_\sigma^2$ is negative. The condition (12) is nothing but the stability condition of the de Sitter space. Then the inflation may correspond to the unstable de Sitter space but the late-time accelerating expansion may correspond to the stable de Sitter space.
The Hubble tension might be solved by the de Sitter condition for the saddle point. Let the solution of (14) to be $R = R_0$. If $R = R_0$ is a saddle point, we find $-R_0 + \frac{F_R(R)}{F_R(R_0)} = 0$. If $\frac{dV(\sigma)}{dA} > 0$ at the vicinity of $A = R_0$, the curvature $R$ smaller than $R_0$ becomes larger and reaches $R = R_0$ and has this value for some limited time and after that $R$ becomes larger again. On the other hand, if $\frac{dV(\sigma)}{dA} < 0$ at the vicinity of $A = R_0$, the curvature $R$ larger than $R_0$ becomes smaller and reaches $R = R_0$ and has this value for some limited and after that $R$ becomes smaller again. Then if there is a saddle point $R = R_0$ after clear-up of the Universe with $\frac{dV(\sigma)}{dA} < 0$ at the vicinity of $A = R_0$, a short-lasting de Sitter era is realized, in which case, the Hubble rate corresponds to the Hubble constant obtained by the CMB observation. In this way, the Universe may have different Hubble rates for the CMB redshifts, while for small redshifts the Universe has a different Hubble rate. In Fig. 1 the typical behavior of $V(A = R)$ in the case $m_x^2 > 0$ corresponding to (12) is given in (a) and that in the case $m_x^2 < 0$ in (13) is given in (b). With $R_0$ we denote the de Sitter saddle point curvature. The qualitative behavior of $V(R)$ is given in detail in Fig. 1 where also the de Sitter saddle point solution appears in all plots. Also in Fig. 1 the typical behavior of $V(A = R)$ in the case that $R$ becomes larger is given in the subplot (c) and the case when $R$ becomes smaller is given in the subplot (d).

B. Integral $F(R)$ gravity models

The arguments of the previous subsection in the last subsection indicate that the model which solves the $H_0$-tension problem, and simultaneously may successfully describe inflation and the dark energy era, is given by the model which satisfies the equation,

$$\left. \frac{dV(\sigma)}{dA} \right|_{A=R} = \frac{d}{dR} \left( \frac{R}{F_R(R)} - \frac{F(R)}{F_R(R)^2} \right) = \frac{F_{RR}(R)}{F_R(R)^3} \left( -AF_R(R) + 2F(R) \right) = C(R) \left( R_{\text{inf}} - R \right) \left( R - R_{\text{CMB}} \right)^2 \left( R - R_{\text{late}} \right), \tag{14}$$

where $C(R)$ is a positive function of $R$ or positive constant, $R_{\text{inf}}$ is the curvature during the inflationary epoch, $R_{\text{CMB}}$ is the curvature corresponding to the Hubble rate after clear up of the Universe, and $R_{\text{late}}$ is the curvature in the epoch of the late-time accelerating expansion. Therefore $R_{\text{inf}} > R_{\text{CMB}} > R_{\text{late}}$. Then for the inflationary era $R = R_{\text{inf}}$ becomes an unstable de Sitter era, the late-time accelerating expansion corresponding to $R = R_{\text{late}}$ becomes an stable de Sitter era, and the epoch corresponding to $R = R_{\text{CMB}}$ becomes a saddle point with $\frac{dV(\sigma)}{dA} < 0$ at the vicinity of $A = R_0$. In Fig. 2 we present the qualitative behavior of $V(A = R)$ corresponding to (14) as a function of $R$ is given. In the plot, several important values of the curvature are also included, such as the values of the curvature during inflation, dark energy era and at recombination.

We now rewrite (14) as,

$$RF_R(R) - F(R) = Q(R)F_R(R)^2, \quad Q(R) \equiv \int dRC(R) \left( R_{\text{inf}} - R \right) \left( R - R_{\text{CMB}} \right)^2 \left( R - R_{\text{late}} \right), \tag{15}$$

which can be regarded as a differential equation for $F(R)$ with respect to $R$. Since Eq. (15) can be rewritten as,

$$\frac{d}{dR} \left( \frac{F(R)}{R} \right) = Q(R) \left( \frac{F_R(R)}{R} \right)^2, \tag{16}$$
we can solve Eq. (15) as follows,

$$F(R) = - \frac{R}{\int dRQ(R)}.$$  \hspace{2cm} (17)

As the simplest example, we consider the case that $C(R)$ is a positive constant $C(R) = C_0 > 0$. Then we find,

$$Q(R) = C_0 \left\{ - \frac{R^5}{5} + \frac{(R_{\text{inf}} + 2R_{\text{CMB}} + R_{\text{late}}) R^4}{4} - \frac{(R_{\text{inf}} R_{\text{late}} + 2R_{\text{CMB}} R_{\text{inf}} + 2R_{\text{CMB}} R_{\text{late}} + R_{\text{CMB}}^2) R^3}{3} \\
+ \frac{(2R_{\text{inf}} R_{\text{late}} R_{\text{CMB}} + R_{\text{CMB}}^2 R_{\text{inf}} + R_{\text{CMB}}^2 R_{\text{late}}) R^2}{2} - R_{\text{inf}} R_{\text{late}} R_{\text{CMB}}^2 R + C_1 \right\},$$  \hspace{2cm} (18)

where $C_1$ is a constant of the integration. Therefore we find,

$$F(R) = - \frac{R}{C_0} \left\{ - \frac{R^6}{30} + \frac{(R_{\text{inf}} + 2R_{\text{CMB}} + R_{\text{late}}) R^5}{20} - \frac{(R_{\text{inf}} R_{\text{late}} + 2R_{\text{CMB}} R_{\text{inf}} + 2R_{\text{CMB}} R_{\text{late}} + R_{\text{CMB}}^2) R^4}{12} \\
+ \frac{(2R_{\text{inf}} R_{\text{late}} R_{\text{CMB}} + R_{\text{CMB}}^2 R_{\text{inf}} + R_{\text{CMB}}^2 R_{\text{late}}) R^3}{6} - R_{\text{inf}} R_{\text{late}} R_{\text{CMB}}^2 R^2 + C_1 R + C_2 \right\}^{-1},$$  \hspace{2cm} (19)

were $C_2$ is a constant of the integration. For the theory to be reduced to the Einstein-Hilbert gravity $F(R) \to R$ in the weak curvature limit, we require,

$$-C_0 C_2 = 1.$$  \hspace{2cm} (20)

In Eq. (18), we considered the simplest case, where $C(R)$ is a constant. We can, however, consider the more complicated case where $C(R)$ is a non-trivial function of $R$ although it might be difficult to execute the integrations of $Q(R)$ in (15) and (18). If we choose $C(R)$ to be small at a stationary point corresponding to the inflationary era, the late-time accelerating expansion, or a short inflationary era after the clear up of the Universe, as it is clear from Eq. (14), the period that the Universe remains in the vicinity the stationary point becomes larger, and on the other hand, if we choose $C(R)$ to be large at a stationary point, the Universe remains at the stationary point for a short time.

C. Slow-roll parameters in integral $F(R)$ gravity

For completeness of our study, in this subsection we shall present the formalism of inflation in the Einstein frame for integral $F(R)$ gravity models. We shall focus on the final expressions of the first and second slow-roll parameters in the Einstein frame. These are defined as:

$$\epsilon_E = \frac{1}{6} \left( \frac{V''(\sigma)}{V(\sigma)} \right)^2 \sim \frac{\dot{H}_E}{H_E^2}, \quad \eta_E = \frac{V''(\sigma)}{3V(\sigma)} \sim \frac{\dot{H}_E}{H_E^2} - \frac{1}{2} \frac{\ddot{H}_E}{H_E^2}.$$  \hspace{2cm} (21)
Here $H_\text{E}$ is the Hubble rate in the Einstein frame. Since $\sigma = -\ln F_R(A)$, we find $\partial_\sigma = -\frac{F_R(A)}{F_{RR}(A)} \partial_A$. Then by using Eqs. (7), (8), and (14), we find,

$$V(\sigma) = \frac{R}{F_R(R)} - \frac{F(R)}{F_R(R)^2}, \quad V'(\sigma) = \frac{R}{F_R(R)} - \frac{2F(R)}{F_R(R)^2},$$
$$V''(\sigma) = \frac{1}{F_{RR}(R)} + \frac{R}{F_R(R)} - \frac{4F(R)}{F_R(R)^2},$$

and therefore,

$$\epsilon_\text{E} = \frac{1}{6} \left( 1 - \frac{F(R)}{RF_R(R) - F(R)} \right)^2, \quad \eta_\text{E} = \frac{1}{3} \left( 1 + \frac{-3F(R) + \frac{F_R(R)^2}{F_{RR}(R)}}{RF_R(R) - F(R)} \right).$$

By using Eq. (13), we find,

$$F(R) = -\frac{R}{\int \! dR Q(R)} , \quad F_R(R) = -\frac{1}{\int \! dR Q(R)} + \frac{RQ(R)}{(\int \! dR Q(R))^2},$$
$$F_{RR}(R) = \frac{2Q(R)}{(\int \! dR Q(R))^2} + \frac{RQ'(R)}{(\int \! dR Q(R))^2} - \frac{RQ(R)^2}{(\int \! dR Q(R))^3},$$

and also,

$$V(\sigma) = \frac{R (\int \! dR Q(R))^2}{\int \! dR Q(R) + RQ(R)} + \frac{R (\int \! dR Q(R))^3}{(-\int \! dR Q(R) + RQ(R))^2},$$
$$V'(\sigma) = \frac{R (\int \! dR Q(R))^2}{\int \! dR Q(R) + RQ(R)} + \frac{2R (\int \! dR Q(R))^3}{(-\int \! dR Q(R) + RQ(R))^2},$$
$$V''(\sigma) = \frac{(\int \! dR Q(R))^3}{(2Q(R) + RQ'(R)) \int \! dR Q(R) - RQ(R)^2} + \frac{R (\int \! dR Q(R))^2}{(-\int \! dR Q(R) + RQ(R))^2} - \frac{4R (\int \! dR Q(R))^3}{(-\int \! dR Q(R) + RQ(R))^2},$$

and in effect we have,

$$\epsilon_\text{E} = \frac{1}{6} \left( 1 + \frac{\int \! dR Q(R)}{RQ(R)} \right)^2,$$
$$\eta_\text{E} = \frac{1}{3} \left( 1 + \frac{\int \! dR Q(R) \left( (\int \! dR Q(R))^2 + R(4Q(R) + 3RQ'(R)) \int \! dR Q(R) - 5R^2Q(R)^2 \right)}{R^2Q(R) ((2Q(R) + RQ'(R)) \int \! dR Q(R) - 2RQ(R)^2)} \right).$$

If we require $\epsilon_\text{E} \sim \eta_\text{E} \sim 0$, we find,

$$-1 = \frac{\int \! dR Q(R)}{RQ(R)},$$
$$-1 = \frac{\int \! dR Q(R) \left( (\int \! dR Q(R))^2 + R(4Q(R) + 3RQ'(R)) \int \! dR Q(R) - 5R^2Q(R)^2 \right)}{R^2Q(R) ((2Q(R) + RQ'(R)) \int \! dR Q(R) - 2RQ(R)^2)}.$$

The first equation (27) indicates,

$$\int \! dR Q(R) \sim -RQ(R) \sim \frac{R_0}{R},$$

1 We should note that the kinetic term of $\sigma$ is given by $-\frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma$ as in [1].
and therefore,
\[
\int dRQ(R) \left( (\int dRQ(R))^2 + R(4Q(R) + 3RQ'(R)) \int dRQ(R) - 5R^2Q(R)^2 \right) / R^2Q(R) (2Q(R) + RQ'(R)) \int dRQ(R) - 2RQ(R)^2) ∼ -1.
\]  
Therefore the second equation in [27] is automatically satisfied if the first equation is satisfied.

### D. Solving H0-tension by saddle point condition

In this section, by using the outcomes of the previous subsections, we shall try to qualitatively describe the basic features of integral \( F(R) \) gravity models which may solve the \( H_0 \)-tension. Comparing Eq. [29] with [13], we get,

\[
(R - R_{\text{late}}) \sim \frac{2R_0}{R^3},
\]

which is difficult to be realized when \( R \sim R_{\text{inf}} \) or \( R \sim R_{\text{late}} \). Then instead of assuming that the inflationary and the late-time accelerating expansions correspond to stationary points, they correspond to almost flat potential satisfying the slow-roll condition [29] although there is a saddle point corresponding to the curvature after clear up of the Universe. As an example, instead of [13], we consider,

\[
Q(R) = \int dRQ_0(R - R_{\text{CMB}})^2 / R^5 + Q_1^2 R^3,
\]

where \( Q_0 \) and \( Q_1 \) are positive constants. Then when \( R \) is large, we find \( Q'(R) \sim Q_0 R_{\text{CMB}} \) and therefore \( Q(R) \sim -\frac{Q_0 R_{\text{CMB}}}{2R^2} \).

On the other hand, when \( R \) is small, we find \( Q'(R) \sim Q_0 R_{\text{CMB}}^2 Q_1^2 R^3 \) and therefore \( Q(R) \sim -\frac{Q_0 R_{\text{CMB}}^2 Q_1^2 R^2}{2Q_1^2 R^5} \). Therefore, both of the region where \( R \) is large and the region \( R \) is small, the slow-roll conditions are satisfied. Now let us try to realize specific cosmologies in \( F(R) \) gravity, satisfying the saddle point condition that remedies the \( H_0 \)-tension.

The field equation corresponding to the first FRW equation in \( F(R) \) gravity is given by,

\[
0 = -\frac{F(R)}{2} + 3 \left( H^2 + \dot{H} \right) F'(R) - 18 \left( 4H^2\dot{H} + H\ddot{H} \right) F''(R) + \kappa^2 \rho .
\]  
with \( R = 6\dot{H} + 12H^2 \). We now rewrite Eq. [32] by using a new variable (which is often called e-folding) instead of the cosmological time \( t \), \( N = \ln \frac{\tau_0}{a} \). The variable \( N \) is related with the redshift \( z \) by \( e^{-N} = \frac{a_0}{a} = 1 + z \). Since \( \frac{dt}{dN} = H \frac{dR}{dN} \) and therefore \( \frac{d^2}{dN^2} = H^2 \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN} \), one can rewrite [32] by

\[
0 = -\frac{F(R)}{2} + 3 \left( H^2 + H'^2 \right) F'(R) - 18 \left( 4H^3H' + H^2 (H')^2 + H^3 H'' \right) F''(R) + \kappa^2 \rho .
\]

Here \( H' \equiv dH/dN \) and \( H'' \equiv d^2H/dN^2 \). If the matter energy density \( \rho \) is given by a sum of the fluid densities with constant EoS parameter \( w_i \), we find

\[
\rho = \sum_i \rho_0 a^{-3(1+w_i)} = \sum_i \rho_0 a_0^{-3(1+w_i)} e^{-3(1+w_i)N} .
\]

Let assume that the Hubble rate is given in terms of \( N \), \( H = H(N) \). By defining \( G(N) \equiv H(N)^2 \), Eq. [33] can be rewritten as

\[
0 = -9G(N(R)) (4G' (N(R)) + G'' (N(R))) \frac{d^2F(R)}{dR^2} + \left( 3G(N(R)) + \frac{3}{2} G'(N(R)) \right) \frac{dF(R)}{dR} - \frac{F(R)}{2} + \sum_i \rho_0 a_0^{-3(1+w_i)} e^{-3(1+w_i)N(R)} .
\]

Because the scalar curvature is given by \( R = 3G'(N) + 12G(N) \), we have assumed that \( N \) can be solved with respect to \( R \), \( N = N(R) \). Hence, when we find \( F(R) \) satisfying the differential equation [35], such \( F(R) \) theory admits the solution \( H = H(N) \). Hence, such \( F(R) \) gravity realizes the cosmological solution. As an example, we reconstruct
the $F(R)$ gravity which reproduces the ΛCDM-era but without the presence of any perfect matter fluids. In the Einstein-Hilbert gravity case, the FRW equation for the ΛCDM cosmology is given by

$$\frac{3}{\kappa^2}H^2 = \frac{3}{\kappa^2}H_0^2 + \rho_0a^{-3} = \frac{3}{\kappa^2}H_0^2 + \rho_0a_0^{-3}e^{-3N}.$$  \hfill (36)

Here $H_0$ and $\rho_0$ are constants. The first term in the r.h.s. corresponds to the cosmological constant and the second term to the cold dark matter (CDM). The (effective) cosmological constant $\Lambda$ in the present Universe is given by $\Lambda = 12H_0^2$. Then one gets,

$$G(N) = H_0^2 + \frac{\kappa^2}{3}\rho_0a_0^{-3}e^{-3N},$$  \hfill (37)

and $R = 3G'(N) + 12G(N) = 12H_0^2 + \kappa^2\rho_0a_0^{-3}e^{-3N}$, which can be solved with respect to $N$ as follows,

$$N = -\frac{1}{3}\ln\left(\frac{R - 12H_0^2}{\kappa^2\rho_0a_0^{-3}}\right).$$  \hfill (38)

Eq. (35) takes the following form:

$$0 = 3\left(R - 9H_0^2\right)\left(R - 12H_0^2\right)\frac{d^2F(R)}{dR^2} - \left(\frac{1}{2}R - 9H_0^2\right)\frac{dF(R)}{dR} - \frac{1}{2}F(R).$$  \hfill (39)

By changing the variable from $R$ to $x$ by $x = \frac{R}{3H_0^2} - 3$, Eq. (39) reduces to the hypergeometric differential equation:

$$0 = x(1-x)\frac{d^2F}{dx^2} + (\gamma - (\alpha + \beta + 1)x)\frac{dF}{dx} - \alpha\beta F.$$  \hfill (40)

Here

$$\gamma = -\frac{1}{2}, \quad \alpha, \beta = -\frac{1}{2}, \frac{1}{3}.$$  \hfill (41)

Solution of (40) is given by Gauss’ hypergeometric function $2F_1(\alpha, \beta, \gamma; x)$:

$$F(x) = A_2F_1\left(-\frac{1}{2}, \frac{1}{3}, \frac{1}{2}; x\right) + Bx^2_2F_1\left(1, \frac{11}{6}, \frac{5}{2}; x\right).$$  \hfill (42)

Here $A$ and $B$ are constants. Thus, we demonstrated that modified $F(R)$ gravity may describe the ΛCDM epoch without the need to introduce an effective cosmological constant. Coming back to the saddle condition that resolves the $H_0$-tension problem, since,

$$F(R) = -\frac{R}{\int dRQ(R)},$$  \hfill (43)

and

$$2F_1'(\alpha, \beta, \gamma; x) = \frac{\alpha\beta}{\gamma}2F_1(\alpha + 1, \beta + 1, 1; x),$$  \hfill (44)

we find,

$$Q(R) = \left[A_2F_1\left(-\frac{1}{2}, \frac{1}{3}, -\frac{1}{2}; x\right) + Bx^2_2F_1\left(1, \frac{11}{6}, \frac{5}{2}; x\right) - \frac{AR}{9H_0^2}2F_1\left(\frac{1}{2}, \frac{4}{3}, \frac{1}{2}; x\right) + \frac{BR}{2H_0^2}x^2_2F_1\left(\frac{11}{6}, \frac{5}{2}; x\right) + \frac{11BR}{45H_0^2}x^7_2F_1\left(\frac{17}{6}, \frac{7}{2}; x\right)\right]$$

$$\times \left[A_2F_1\left(-\frac{1}{2}, \frac{1}{3}, -\frac{1}{2}; x\right) + Bx^2_2F_1\left(1, \frac{11}{6}, \frac{5}{2}; x\right)\right]^{-2}.$$  \hfill (45)

In the limit, $N \to \infty$, we find $H \to H_0$ and therefore $R \to 12H_0^2$ and $x \to 1$. Since,

$$2F_1(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}2F_1(\alpha, \beta, \alpha + \beta + 1 - \gamma; 1 - x)$$
Here we denote the quantities in the Einstein frame with the index "E".

In the Einstein frame (7), the FRW equations have the following form,

\[ Q(R) \rightarrow \left( A + \frac{B \Gamma \left( \frac{2}{9} \right) \Gamma (\frac{5}{9})}{\Gamma (\frac{8}{9})} \right)^{-1} \left( 4 - \frac{R}{3H_0^2} \right)^{-\frac{2}{9}} \frac{R}{9H_0^2}. \]  

Eq. (47) indicates that when \( N \) is very large, the contribution of the second term becomes very small and spacetime is almost described by a de Sitter evolution. Such a situation for which the difference from the de Sitter Universe is very small could occur even in the inflationary era during the early Universe.

### III. SLOW-ROLL PARAMETERS IN INTEGRAL FORM OF \( F(R) \) GRAVITY AND INFLATIONARY DYNAMICS FORMALISM

In this section, we try to find the expressions of the slow-roll parameters in the integral form of \( F(R) \) gravity. Since Eq. (6) indicates the relation between the metric \( g_{\mu \nu} \) in the Jordan frame and the metric \( g_{\mu \nu} \) in the Einstein metric, \( g_{\mu \nu} = e^x g_{\mu \nu} \), if we denote the cosmological time and the scale factor in the Einstein frame by \( t_E \) and \( a_E \), respectively, we find,

\[ ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2 = e^x ds^2 = e^\sigma \left( -dt_E^2 + a_E(t)^2 \sum_{i=1,2,3} (dx_i^E)^2 \right). \]  

Here we denote the quantities in the Einstein frame with the index "E" but we assume \( x_i^E = x_i \). Therefore, since \( dt = e^\sigma dt_E \) and \( a(t) = e^\sigma a_E(t) \), we obtain,

\[ H_E = \frac{1}{a_E} \frac{da_E}{dt_E} = e^{-\frac{\sigma}{2}} \frac{d}{dt} \left( e^{-\frac{\sigma}{2}} a \right) = e^{-\frac{\sigma}{2}} \left( -\frac{1}{2} \frac{d\sigma}{dt} + H \right), \]

and also,

\[ \frac{dH_E}{dt_E} = e^{\frac{\sigma}{2}} \left( \frac{1}{4} \frac{d\sigma}{dt} \right)^2 - \frac{1}{2} H \frac{d\sigma}{dt} - \frac{1}{2} \frac{d^2\sigma}{dt^2} + \frac{dH}{dt}. \]

\[ \frac{d^2H_E}{dt^2} = e^{\frac{3\sigma}{2}} \left( \frac{1}{4} \frac{d\sigma}{dt} \right)^3 - \frac{1}{2} H \left( \frac{d\sigma}{dt} \right)^2 - \frac{1}{2} \frac{dH}{dt} \frac{d^2\sigma}{dt^2} + \frac{d\sigma}{dt} \left( \frac{1}{2} \frac{dH}{dt} + \frac{d\sigma}{dt} \right) - \frac{1}{2} \frac{d^2H}{dt^2} \frac{d^2\sigma}{dt^2} - \frac{1}{2} \frac{d^3\sigma}{dt^3} + \frac{d^2H}{dt^2} \right). \]  

In the Einstein frame (7), the FRW equations have the following forms,

\[ 3H_E^2 = \frac{3}{2} \left( \frac{d\sigma}{dt_E} \right)^2 + V(\sigma), \quad -2 \frac{dH_E}{dt_E} - 3H_E^2 = \frac{3}{2} \left( \frac{d\sigma}{dt_E} \right)^2 - V(\phi). \]

which gives,

\[ -2 \frac{dH_E}{dt_E} = 3 \left( \frac{d\sigma}{dt_E} \right)^2, \quad \frac{dH_E}{dt_E} + 3H_E^2 = V(\sigma) \]

The equation for \( \sigma \) is given by,

\[ 0 = 3 \left( \frac{d^2\sigma}{dt_E^2} + 3H_E \frac{d\sigma}{dt_E} \right) + V'(\sigma). \]
By combining (52) and (53), we find,
\[ \frac{d^2 H_E}{dt_E^2} = \frac{d\sigma}{dt_E} \left( 9 H_E \frac{d\sigma}{dt_E} + V'(\sigma) \right). \]  
(54)

By using (53), we also obtain,
\[ 0 = 3 \left( \frac{d^3 \sigma}{dt_E^3} + 3 \frac{d H_E}{dt_E} \frac{d\sigma}{dt_E} - 9 H_E^2 \frac{d\sigma}{dt_E} - H_E V'(\sigma) \right) + V''(\sigma) \frac{d\sigma}{dt_E}. \]  
(55)

On the other hand, we find,
\[ H_E = \frac{1}{a_E} \frac{d\sigma}{dt_E} = \frac{e^{-\frac{2}{3} a} d (e^{-\frac{2}{3} a})}{d t} = e^{-\frac{2}{3}} \left( -\frac{1}{2} \frac{d\sigma}{dt} + H \right) = -\frac{1}{2} \frac{d\sigma}{dt_E} + e^{-\frac{2}{3} H}. \]  
(56)

and also,
\[ \frac{d H_E}{dt_E} = \frac{d\sigma}{dt_E} + \frac{1}{4} \left( \frac{d\sigma}{dt_E} \right)^2 + \frac{1}{6} V'(\sigma) + e^{-\sigma} \frac{d H}{dt}, \]
\[ \frac{d^2 H_E}{dt_E^2} = -\frac{1}{2} \frac{d^3 \sigma}{dt_E^3} - \frac{1}{2} \frac{d^2 \sigma}{dt_E^2} e^{-\frac{2}{3} H} + \frac{1}{4} \left( \frac{d\sigma}{dt_E} \right)^2 e^{-\frac{2}{3} H} - \frac{3}{2} \frac{d\sigma}{dt_E} e^{-\sigma} \frac{d H}{dt} + e^{-\frac{2}{3} a} \frac{d^2 H}{dt^2}. \]  
(57)

In the slow-roll limit,
\[ \left( \frac{d\sigma}{dt_E} \right)^2 \ll V(\phi), \quad \frac{d^2 \sigma}{dt_E^2} \ll \left| \frac{H_E}{dt_E} \right|, \]  
(58)

by using (51) and (52), we find,
\[ 3 H_E^2 \sim V(\sigma), \quad -9 H_E \frac{d\sigma}{dt_E} \sim V'(\sigma), \]
\[ \frac{d\sigma}{dt_E} \sim -\frac{V'(\sigma)}{9 \sqrt{V'(\sigma)}}, \quad \frac{d H_E}{dt_E} = \frac{V''(\sigma)}{18 V'(\sigma)}. \]  
(59)

In the slow-roll limit (58) and (59), Eqs. (56) and (57) give,
\[ H \sim e^{\frac{2}{3} \sqrt{\frac{V(\sigma)}{3}}}, \quad \frac{dH}{dt} \sim -\frac{1}{18} e^{\sigma} V'(\sigma), \quad \frac{d^2 H}{dt^2} \sim e^{\frac{2}{3} \sigma} \frac{V'(\sigma)V''(\sigma)}{18 \sqrt{3 V(\sigma)}}, \]  
(60)

where we have used Eq. (59). Then the slow-roll parameters are given by,
\[ \epsilon \equiv \frac{\dot{H}}{H^2} \sim -\frac{V'(\sigma)}{6 V'(\sigma)}, \quad \eta \equiv -\frac{1}{2} \frac{\dot{H}}{H H} \sim \frac{V''(\sigma)}{2 V'(\sigma)}. \]  
(61)

The expression of $V(\sigma)$, $V'(\sigma)$, and $V''(\sigma)$ in the integral form are given in [53]. The above relations can be used to explicitly check the inflationary viability of an appropriately chosen integral $F(R)$ gravity.

**IV. CONCLUSIONS**

The problem of the $H_0$-tension has not been fully addressed in the framework of vacuum $F(R)$ gravity. In this work, we considered a theoretical solution for the $H_0$-tension, using an $F(R)$ gravity framework. Specifically, using a saddle point condition for the $F(R)$ gravity, and its correspondence to the Einstein frame, we provided a theoretical framework for $F(R)$ gravity which may resolve the $H_0$-tension. This newly introduced framework is an entirely new approach, absent in other $F(R)$ gravity frameworks and models, like for example the Starobinsky $R^2$ model, which alone does not solve the $H_0$-tension problem. Most of the models that can describe inflation solely, or even inflation and dark energy in a unified way, cannot by themselves solve the $H_0$-tension problem. One must use the integral $F(R)$ gravity formalism we introduced in this paper to produce a saddle de Sitter point that may eventually
solve the $H_0$-tension problem. The drawback is the complicated functional form obtained, which is not so useful for analytic calculations of the inflationary dynamics for the model considered. The de Sitter saddle point condition has implications in the Einstein frame, and in the Jordan frame, the $F(R)$ gravity that achieves such a solution has an integral form. By suitably choosing the integral $F(R)$ gravity, apart from the $H_0$-tension, one may also unify the inflationary era and the late-time acceleration of the Universe. Finally, we provided the inflationary theoretical framework which governs the integral $F(R)$ gravity. Our theoretical solution to the $H_0$-tension problem is based on having a saddle de Sitter point for curvature being of the order of the CMB curvature. The formalism we introduced can be used for producing viable inflationary $F(R)$ gravity theories, and can also solve simultaneously in a theoretical way the $H_0$-tension problem. Also if the formalism is viewed as a reconstruction technique, one may manipulate appropriate functional forms of integral $F(R)$ gravity and eventually provide a unified model for dark energy and inflation, which simultaneously solves the $H_0$-tension. Work is in progress toward this research line.

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