Black-box optimization on Hyper-rectangle using Recursive Modified Pattern Search and Application to Matrix Completion Problem with Non-convex Regularization

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Abstract In this paper, a pattern search based optimization technique is developed to optimize any black-box function on a hyper-rectangle. This algorithm consists of a series of ‘runs’ and inside each ‘run’ iterations are performed until a convergence criteria is satisfied following the principle which is similar to that of Generalized Pattern Search. During an iteration, jumps are made along the co-ordinates of the parameter one at a time with varying step-sizes within the restricted parameter space to search for the best direction to move. While solving a problem on $n$-dimensional hyper-rectangle, inside each iteration the objective function is evaluated at $2n$ independent directions. Hereby

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parallel computing can be easily incorporated using up to $2n$ (i.e., in the order of $n$) threads which is very convenient for GPU computing. Unlike other existing black-box optimization techniques (e.g., Genetic Algorithm (GA), Simulated Annealing (SA)), the prior knowledge of convexity about the objective function can be exploited and in that case it can be solved in lesser time. The comparative study of the performances of the proposed algorithm, GA and SA have been provided for several low-dimensional, a few moderate and high-dimensional benchmark functions with corresponding computation times. Simulation study has also been performed for moderate and high-dimensional cases where the solution is a boundary point in the domain. This black-box optimization technique has been used to solve matrix completion problem with non-convex regularization incorporating parallel computing.

**Keywords** Pattern search · Black-box optimization · Non-convex optimization · SCAD penalty · Matrix completion problem

1 Introduction

In the field of science and engineering, a black-box is considered as a device (Figure 1), system or object which can only be observed in terms of inputs and outputs. In other words, someone working with black-box is not aware of the internal process of the system; but the output result can only be observed for any given input. In the field of optimization, black-box optimization is one of the challenging problems. Suppose a black-box function needs to be minimized. Note that, a black-box function might have multiple minimums.
Due to unknown form of the explicit function, derivatives cannot be evaluated analytically at any point. Which makes this problem harder than the non-convex minimization problem where unlike black-box functions derivatives can be evaluated analytically.

In the field of convex optimization, ‘Gradient descent (GD)’ method ([1]), ‘Trust Region Reflective (TRF)’ algorithm ([2,3,4]), ‘Interior-point (IP)’ algorithm ([5,6,7,8]) and ‘Sequential Quadratic Programming (SQP)’ algorithm ([8,9,10]) are most common and widely used nowadays. Most of these algorithms (e.g., GD, TRF, SQP) use derivatives to find the direction of the movement while minimizing any objective function. Another problem with these convex algorithms is they look for local solution and hence they are more likely to get stuck at any local minimum in case the objective function has multiple minimums. Hereby due to requirement of analytical derivative (for most of them) and the tendency to stop iterations after reaching a local solution, convex optimization algorithms are not suitable for minimizing black-box functions. For low dimensional non-convex optimization problems,
the strategy of using convex optimization techniques with multiple starting points might be affordable, but with increasing dimension of the parameter space, this strategy proves to be computationally very expensive since with increasing dimension, the requirement of the number of starting points increases exponentially.

In many types of existing optimization algorithms, the main motivation is to minimize the number of function evaluations required to find a reasonable local minimum (e.g., [11]) which might not be desirable if finding the global minimum is our main objective. In the last century, many non-convex global optimization strategies were proposed among which ‘Genetic algorithm (GA)’ (see [12,13,14]) and ‘Simulated annealing (SA)’ (see [15,16]) remained quite popular and are being widely used. But as mentioned in [17], one of the problems of GA is it does not scale well with complexity because in higher dimensional optimization problems there is often an exponential increase in search space size. Besides, one major problem with these two afore-mentioned methods is they might be much expensive in case we use these methods for optimizing simple convex functions without the knowledge of its convexity (has been shown in the simulation study). Among other methods, one of the most commonly used black-box optimization technique ‘Particle swarm optimization (PSO)’ was first proposed in [18]. A few modification of this method can be found in [19][20][21][22].
To solve black-box functions, various derivative-free and coordinate search based techniques have evolved. In 1952, [23] proposed a very simple but effective coordinate search algorithm for minimizing unconstrained black-box functions. While minimizing a function, the main principle was to set a step-size and then move each coordinate by that step size in positive and negative direction one at a time, thus evaluating the objective function value at $2^n$ points in the neighborhood. Out of these $2^n + 1$ points (including the starting point of the iteration), the point with minimum function value is considered as the updated solution. If the solution stops improving, the same steps are repeated by decreasing the step-size to the half of its previous value. Thus it allows a finer search. Generalized pattern search (GPS) algorithm ([24]) can be noted as more general version of the afore-mentioned algorithm. Later, a few other coordinate search based and derivative free methods have been proposed in [25,26,27,28,29]. Some other derivative-free optimization methods have been proposed in [30,31,32,33,34,35,36].

In this paper a derivative-free pattern search based method is proposed for minimizing a black-box function on a hyper-rectangular domain. Similar to the Fermi’s principle ([23], Figure 2) and the principle of GPS ([24]), in the proposed algorithm in each iteration, the value of the objective function is evaluated at $2^n$ neighboring points which are obtained by making $2^n$ (where $n$ is the dimension of the parameter space) coordinate-wise movements with step-sizes. But unlike the Fermi’s principle and GPS, in the proposed method inside
an iteration the coordinate-wise movement step-sizes are modified for each co-
ordinates and directions and hereby their values might not be the same during
an iteration. In the proposed method, during an iteration, the coordinate-wise
step-size is changed only if the corresponding jump yields a point outside the
domain. Thus before evaluating the function value at some point, it is verified
that the point is within the domain of search. Again, unlike Fermi’s method
and GPS, another strategy of the proposed algorithm is to restart the search
procedure starting from the obtained local solution with large step sizes again.
The algorithm terminates when two consecutive restarts yield the same solu-
tion. This principle of restart helps to jump out of the local solution. One of
the biggest advantage of the proposed algorithm is that the objective func-
tions can be evaluated parallely in $2n$ directions since once the step-size for
the iteration is fixed, the jumps and the functional value evaluation steps in $2n$
possible directions are independent of each other. Another great benefit of this
technique is that in this way the requirement of parallel computing increases
in the order of the number of parameters which is very convenient for GPU
computing. The proposed algorithm is termed as ‘Recursive Modified Pattern
Search (RMPS)’.
2 Algorithm

Suppose we have a objective function $Y = f(x)$ where $x = (x_1, \cdots, x_n)$ is the parameter of dimension $n$. Our objective is to

$$\text{minimize} : f(x)$$

$$\text{subject to} : x \in S \subset \mathbb{R}^n$$

where $S = \prod_{j=1}^{n} I_j$, $I_j = [a_j, b_j]$ are closed and bounded intervals on $\mathbb{R}$ for $i = 1, \cdots, n$. Now consider the bijection

$$g : S \mapsto [0, 1]^n$$

where $g(z) = (g_1(z), \ldots, g_n(z)) \in [0, 1]^n$ is such that $g_i(z) = \frac{z_i - a_i}{b_i - a_i}$. So, without loss of generality, we can assume the domain of $x$ to be $S = [0, 1]^n$.

The proposed algorithm consists of several runs. Each run is an iterative procedure and a run stops based on some convergence criteria (see below). At the end of each run a solution is returned. After the first run, following runs will start from the solution obtained by the run just before it. For example, $4^{th}$ run will start from the solution obtained by $3^{rd}$ run. So the user should set the starting point for the first run (see Figure 3) only. Each run tries to improve the solution in a ‘greedy’ manner by making coordinate-wise jumps through a sequence of decreasing step sizes gradually decreasing to zero, around the solution obtained by the previous run (see below for details). Thus, with each run, the solution either gets improved or remains unchanged. If two consecutive runs give the same solution up to some fixed decimal place round, factor
(e.g., if accuracy up to 3 decimal place of the solution point is desired, user should set \textit{round factor} = 3) set by the user, the algorithm stops returning the current value of the solution (which is the final solution).

In the proposed algorithm, each run is similar except the values of the tuning parameters which can be reset after each run. Inside a run, we have three tuning parameters which are initial \textit{global step size} \( s_{\text{initial}} \), \textit{step decay rate} \( \rho \) (It is either equal to \( \rho_1 \) or \( \rho_2 \), see below for details), \textit{step size threshold} \( \phi \) respectively. For the first run, we set \( \rho = \rho_1 \) and for following runs, we set \( \rho = \rho_2 \). Other tuning parameters are kept same of all the runs. In every iteration we have a parameter called \textit{global step size} (denoted by \( s^{(j)} \) for \( j \)-th iteration) and 2n local parameters called \textit{local step sizes} (denoted by \( \{s^+_i\}_{i=1}^n \) and \( \{s^-_i\}_{i=1}^n \)). In a run, we start the first iteration setting the \textit{global step size} \( s^{(1)} = s_{\text{initial}} \). Within every iteration, the value of the \textit{global step size} is kept unchanged throughout all the operations. But at the end of the iteration, based on a convergence criteria (see step (6) of STAGE 1) either it is kept same or decreased by a factor of \( \rho \). The next iteration is started with that new value of \textit{global step size}. So the value of \( s^{(j+1)} \) (i.e., the \textit{global step size} at \( j + 1 \)-th iteration) can be either \( s^{(j)} \) or \( \frac{s^{(j)}}{\rho} \). At the beginning of any iteration, the \textit{local step size} \( \{s^+_i\}_{i=1}^n \) and \( \{s^-_i\}_{i=1}^n \) are set to be equal to the \textit{global step size} of the corresponding iteration. For example, at the beginning of \( j \)-th iteration, we set, \( s^+_i = s^-_i = s^{(j)} \) for \( i = 1, \cdots, n \). The \textit{local step sizes} which generate points outside the domain are updated so that all the new
2n points obtained by moving with the local step sizes \( \{s_i^+\}_{i=1}^n \) and \( \{s_i^-\}_{i=1}^n \) are within the domain. Assume the current value of \( x \) at the \( j \)-th iteration is \( x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)}) \). If moving \( x_i^{(j)} \) by \( s_i^+ \) (\( s_i^+ = s^{(j)} \) at the beginning) in the positive direction generates a point outside the domain (i.e., \( x_i^{(j)} + s_i^+ > 1 \)), then \( s_i^+ \) is updated to the value \( \frac{s^{(j)}}{\rho_f} \) where \( f \) is the smallest possible integer such that \( x_i^{(j)} + \frac{s^{(j)}}{\rho_f} < 1 \). Similarly if moving \( x_i^{(j)} \) by \( s_i^- \) (\( s_i^- = s^{(j)} \) at the beginning) in the negative direction generates a point outside the domain (i.e., \( x_i^{(j)} - s_i^- < 0 \)), then \( s_i^- \) is updated to the value \( \frac{s^{(j)}}{\rho_f} \) where \( f \) is the smallest possible integer such that \( x_i^{(j)} - \frac{s^{(j)}}{\rho_f} > 0 \). It should be noted that while choosing \( f \) for any given coordinate and given direction, it is made sure that the updated local step size is greater than step size threshold \( \phi \). In case such a \( f \) is not feasible (for example, if \( 1 - x_i^{(j)} < \phi \), then no such \( f \) exists such that \( x_i^{(j)} + \frac{s^{(j)}}{\rho_f} < 1 \) and \( \frac{s^{(j)}}{\rho_f} > \phi \)), that particular coordinate is not updated in the corresponding direction. So in short, in an iteration, there will be always only one global step size and \( 2n \) local step sizes which are initialized within the iteration being equal to global step size and at the end of the iteration, each of them end up being less than or equal to the global step size of that iteration. Again it should be noted that in a run, the global step size might decrease or remain same after each iteration. On the other hand, the local step sizes have memory-less properties since their values do not depend of their old values in the previous iteration. A run ends when global step size becomes smaller than \( \phi \).
2.1 Tuning parameters

- **step decay rate** \((\rho)\): \(\rho\) determines the rate of change of *global step size* at the end of each iteration. So it is understandable that the value of \(\rho\) must be greater than 1. Taking smaller values of \(\rho\) will make the decay of step sizes slower, which would allow finer search within the domain at the cost of more computation time. Once we get a solution from first run setting \(\rho = \rho_1\), to incorporate finer search, we again start following runs with smaller decay rate \(\rho_2\). It is noted that a reasonable range for \(\rho_1\) and \(\rho_2\) are \([1.05, 4]\) and \([1.01, \rho_1]\) respectively. But based on simulation experiments, it is noted that \(\rho_1 = 2, \rho_2 = 1.05\) yields satisfactory performance for a wide range of benchmark functions of lower, moderate and higher dimensions.

- **step size threshold** \((\phi)\): \(\phi\) controls the precision of the solution. This is the minimum possible value that the *global step size* and the *local step sizes* can take. Once the *global step size* goes below \(\phi\), the *run* stops. Setting the value of \(\phi\) to be smaller results in better precision in the cost of higher computation time. The default value of *step size threshold* \(\phi\) is taken to be \(10^{-6}\). In case more precision is required, or if there is knowledge of possibility of multiple local minimas within a very small neighborhood, \(\phi\) can be taken to be smaller.
• **max_iter**: `max_iter` denotes the maximum number of iterations allowed inside a `run`. Its default values is set to be `max_iter = 50000`.

• **max_runs**: `max_runs` denotes the maximum number of `runs` allowed in the algorithm. Its default values is set to be `max_runs = 1000`.

• **tol_fun**: `tol_fun` is another precision parameter which determines the minimum amount of movement after an iteration in the solution which is required to keep the value of `global step size` unchanged. In other words, if the sum of squares of differences of solutions obtained in two consecutive iterations is less than `tol_fun`, the improvement is not considered to be significant and the `global step size` is decreased for a finer search. Its default value has been taken to be `tol_fun = 10^{-15}`. It should be noted that taking smaller value of this parameter would yield finer local solution in the cost of more computation time.

• **round_factor**: The second `run` onwards, whenever a `run` ends, it is checked whether the solution returned by the current `run` is the same or different with the solution returned by the previous `run`. However, to check whether they are exactly equal, they need to be matched up to several decimal places depending on the type of storage variable and the type of software used. Thus it might result into a lot of extra `runs` just to improve the solution at distant decimal places which might be unnecessary. Hereby, the value
of \textit{round\_factor} should be fixed by the user and if the solution returned by two consecutive \textit{runs} match up to this many decimal places, the final solution is returned. Its default value is taken to be 6. Based on simulation studies, it is recommended to choose \textit{round\_factor} \geq -\log_{10} \phi. Choosing a larger value of \textit{round\_factor} results in finer solution up to a larger number of decimal places in the cost of higher number of function evaluations.

2.2 Algorithm steps

As mentioned in Section 2, the algorithm consists of a series of \textit{runs} and each \textit{run} is similar except the values of the tuning parameters. Below the algorithm has been described dividing into two stages. In STAGE 1, the internal mechanism of a single \textit{run} has been described. In STAGE 2, it is decided whether to exit the algorithm or to perform the next \textit{run} based on the value of current solution. Before going through STAGE 1 for the very first time, we set \( R = 1, \rho = \rho_1 \) and initial guess of the solution \( \mathbf{x}^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)}) \).

STAGE 1:

1. Set \( j = 1. \) Set \( s^{(j)} = s_{\text{initial}} (= 1) \) Go to step (2).

2. If \( j > \text{max\_iter}, \) set \( \tilde{x} = \mathbf{x}^{(j-1)} \), go to step (8). Else, set \( s_i^+ = s_i^- = s^{(j)} \) and \( f_i^+ = f_i^- = Y^{(j)} = f(\mathbf{x}^{(j)}) \) for all \( i = 1, \ldots, n. \) Set \( i = 1 \) and go to step (3).
3. If $i > n$, set $i = 1$ and go to step (4). Else evaluate vector $q_i^+ = (q_{i1}^+, \ldots, q_{in}^+)$ such that

$$q_{il}^+ = x_i^{(j)} + s_i^+ \quad \text{for } l = i$$
$$= x_i^{(j)} \quad \text{otherwise}$$

If $q_{ii}^+ < 1$, go to step (3d). If $q_{ii}^+ > 1$ and $(1 - x_i^{(j)}) > \phi$, go to step (3a). Else go to step (3b).

(a) Set $s_i^+ = \frac{s_i^+}{\rho_f}$ where $f = \left\lceil \log_{\rho_f} \frac{s_i^+}{(1 - x_i^{(j)})} \right\rceil + 1$ and go to step (3c) (Here $\lceil \cdot \rceil$ denotes the greatest smaller integer function, e.g., $\lceil 1.23 \rceil = 1$).

(b) Set $s_i^+ = 0, i = i + 1$ and go to step (3).

(c) Set $q_i^+ = x_i^{(j)} + s_i^+$ and go to step (3d).

(d) Evaluate $f_i^+ = f(q_i^+)$. Set $i = i + 1$ and go to step (3).

4. If $i > n$, set $i = 1$ and go to step (5). Else evaluate vector $q_i^- = (q_{i1}^-, \ldots, q_{in}^-)$ such that

$$q_{il}^- = x_i^{(j)} - s_i^- \quad \text{for } l = i$$
$$= x_i^{(j)} \quad \text{otherwise}$$

If $q_{ii}^- > 0$, go to step (4d). If $q_{ii}^- < 0$ and $x_i^{(j)} > \phi$, go to step (4a). Else go to step (4b).

(a) Set $s_i^- = \frac{s_i^-}{\rho_f}$ where $f = \left\lceil \log_{\rho_f} \frac{s_i^-}{x_i^{(j)}} \right\rceil + 1$ and go to step (4c).

(b) Set $s_i^- = 0, i = i + 1$ and go to step (4).

(c) Set $q_i^- = x_i^{(j)} - s_i^-$ and go to step (4d).

(d) Evaluate $f_i^- = f(q_i^-)$. Set $i = i + 1$ and go to step (4).
5. Set \( k_1 = \arg\min_{1 \leq l \leq m} f^+_l \) and \( k_2 = \arg\min_{1 \leq l \leq m} f^-_l \). If \( \min(f^+_{k_1}, f^-_{k_2}) < Y^{(j)} \), go to step (5a). Else, set \( x^{(j+1)} = x^{(j)} \) and \( Y^{(j+1)} = Y^{(j)} \), set \( j = j + 1 \). Go to step (6).

(a) If \( f^+_{k_1} < f^-_{k_2} \), set \( x^{(j+1)} = q^+_{k_1} \), else (if \( f^+_{k_1} \geq f^-_{k_2} \)), set \( x^{(j+1)} = q^-_{k_2} \). Set \( j = j + 1 \). Go to step (6).

6. If \( \sum_{i=1}^{n}(x^{(j)}(i) - x^{(j-1)}(i))^2 < \text{tol}_\text{fun} \), set \( s^{(j)} = s^{(j-1)}/\rho \). Go to step (7).

Else, set \( s^{(j)} = s^{(j-1)} \). Go to step (2).

7. If \( s^{(j)} \leq \phi \) (Convergence Criteria 1), set \( \hat{x} = x^{(j)} \). Go to step (8).

8. STOP execution. Set \( z^{(R)} = \hat{x} \). Set \( R = R + 1 \). Go to STAGE 2.

STAGE 2 :

1. If \( R > \text{max\_runs} \) or \( \text{round}(z^{(R)}, \text{round\_factor}) = \text{round}(z^{(R-1)}, \text{round\_factor}) \), STOP, RETURN \( \hat{x} \) as final solution and EXIT. Else go to step (2).

2. Set \( \rho = \rho_2 \) keeping other tuning parameters (\( \phi \) and \( s_{\text{initial}} \)) intact. Repeat algorithm described in STAGE 1 setting \( x^{(1)} = z^{(R-1)} \). If Else repeat step (1).

2.3 Restart strategy

Setting step size threshold to a sufficiently small value, it can be shown that if the function is differentiable and convex then the the value of the objective function at the solution is the global minimum (see Appendix : A). However for non-convex functions, there is no way to ensure whether that is a global minimum or not. To increase the likelihood of reaching the global minimum,
The STAGE 1 of the algorithm is repeated starting from the solution obtained from the last run until the solution returned by two consecutive runs are same. In the first run, we set \( \rho = \rho_1 \) and for the following runs, we set \( \rho = \rho_2 \). After first run, setting smaller value of \( \rho \) results in slower decay of step size in the iterations of the following runs. Thus, we look for better solution in the domain changing each co-ordinate one at a time for a sequence of finite step sizes slowly decaying to zero.

Note that the jump-start strategy at the end of each run is performed to ensure that the solution does not get stuck in some local solution. But in case it is known that the objective function is convex, clearly, there is no need to jump-start since each run is designed in such a way that it returns a local solution (see Appendix : A) which is the global solution in case of convex objective function. So for minimizing a convex function, only one run is sufficient. Thus using the prior knowledge of convexity, computation time can be
reduced while minimizing a convex function (see Section 4).

2.4 Comparison with GPS

It should be noted that although this idea of coordinate-wise movement with a given step size in possible $2^n$ directions is similar to that proposed in [23] and ‘Generalized Pattern Search (GPS)’ ([24]), there are several novel strategies and modifications in the proposed algorithm which makes it quite different. Firstly, the restart strategy with smaller step-size decay rate is something which is possibly proposed for the first time in the context of Pattern search to the best of our knowledge. Specially, using finer search induced by the smaller step-size decay rate from the second run and onwards has been noted to work very well for various ranges of benchmark functions in simulation study. This strategy makes the proposed algorithm quite different from all existing Direct Search (DS) [25,26,28,30,35,36] and Pattern Search (PS) ([29],[24]) algorithms. Secondly, unlike algorithm 1 of [24], instead of unconstrained minimization, the proposed algorithm minimizes the black-box function on a hyper-rectangle. In [24] and [24], the coordinate-wise jump sizes were kept equal inside an iteration while in the proposed algorithm, the domain of each coordinate being bounded, in every iteration, local-step sizes are modified separately for each coordinates in each direction as required. In GPS, each coordinate-wise jump step-sizes are evaluated using ‘exploratory moves algorithm’ (see [24]) while in the proposed algorithm it’s straightforward and
does not use ‘exploratory moves algorithm’. While optimizing a function on
a hyper-rectangle, since the domain is transformed into unit hyper-cube, the
global step-size is kept same for each coordinate. So while determining the
step-sizes of coordinate-wise movements, the proposed algorithm uses differ-
ent strategy than the ‘exploratory moves algorithm’. In GPS, the step size
(which plays the same role as the global step size in the proposed algorithm) is
decreased if there is no improvement in the objective function. While in the
proposed algorithm, the value of global step size is reduced only if there is no
‘significant’ change in the solution between two consecutive iterations which
is determined by the tuning parameter tol.fun. Introduction of the tuning
parameter tol.fun plays an important role. There might be a scenario when
moving with a given step size is improving very slowly. Since improvements
are there (even if relatively very small), the iterations will be still going on
with same step size in GPS. While in that scenario, the proposed algorithm
would reduce the global step size earlier instead of wasting function evalua-
tion costs for very small improvements. And also that amount of ‘significant’
improvement, which is controlled by tol.fun, should be defined by the user in
the proposed algorithm.

At the beginning of each run, the strategy of making jumps within the
unit cube domain with varying global step-sizes is the most unique feature of
the proposed algorithm and to the best of our knowledge, have never been
proposed before. Most of the global optimization methods (e.g., GA, PSO)
can be thought as a combination of movement around the sample space and local minimization around the potential solutions found. In the proposed algorithm, at the beginning of each run, in search of a better solution starting from a local minimum, the sample space is searched in such a way that at each iteration, the number of new points checked is $2n$ which is of order $n$. Also it can be easily verified from the algorithm described in Section 2 that the number of operations performed in each iteration is also of order $n$. Once the algorithm gets stuck at some local solution (as it happens at the end of each run), to avoid higher order or exponential rated search for better solution, the above-mentioned step-size decaying strategy has been considered since using this technique, the sample space can be traversed making operations of order $n$ only at each iteration. Thus it makes it very convenient for high-dimensional optimization. Starting from several initial points, using this strategy results in very high proportions of returned solutions near the true solution even in 1000-5000 dimensional challenging benchmark problems unlike existing and well recognized global optimization methods like GA, SA.

3 Comparative study on Benchmark functions

In this section, we compare the performance of the proposed algorithm with well known ‘Simulated annealing (SA)’ (15), and ‘genetic algorithm (GA)’ (13). Both of these algorithms are available in Matlab R2014a (The Mathworks) via the Optimization Toolbox functions simulannealbnd (for SA), and
| Function names          | RMPS (min. value) | GA (min. value) | SA (min. value) | RMPS (avg. time) | GA (avg. time) | SA (avg. time) |
|-------------------------|------------------|----------------|----------------|------------------|----------------|----------------|
| Ackley Function         | 0.00012          | 4.93E-06       | 2.46E-06       | 0.072            | 0.312          | 0.548          |
| Bukin Function N6       | 0.101226         | 0.021039       | 0.00896        | 0.053            | 0.255          | 0.505          |
| Cross-in-Tray Function  | -2.00261         | -2.06261       | -2.06257       | 0.063            | 0.156          | 0.404          |
| Drop-Wave Function      | -1               | -1             | -0.93625       | 0.053            | 0.153          | 0.571          |
| Eggholder Function      | -0.956.41        | -0.95691       | -0.88839       | 0.062            | 0.256          | 0.505          |
| Gramacy & Lee (2012) Function (d=1) | -0.86901       | -0.86901       | -0.86901       | 0.016            | 0.147          | 0.249          |
| Griewank Function       | 2.25E-07         | 0.007396       | 0.007525       | 0.057            | 0.146          | 0.501          |
| Holder Table Function   | -19.2855         | -19.2085       | -19.2085       | 0.074            | 0.163          | 0.51           |
| Langermann Function     | -4.15581         | -4.15581       | -4.15546       | 0.087            | 0.176          | 0.404          |
| Levy Function           | 9.9E-11          | 4.6E-12        | 1.46E-06       | 0.084            | 0.166          | 0.549          |
| Levy Function N 13      | 1.6E-10          | 4.5E-11        | 7.47E-07       | 0.075            | 0.192          | 0.401          |
| Rastrigin Function      | 8.63E-09         | 4.79E-10       | 1.42E-05       | 0.069            | 0.179          | 0.476          |
| Schaffer Function N2    | 1.66E-11         | 1.39E-13       | 0.005998       | 0.062            | 0.184          | 0.425          |
| Schaffer Function N4    | 0.299279         | 0.299289       | 0.298868       | 0.288            | 0.164          | 0.375          |
| Schewofl Function       | 2.55E-05         | 2.55E-05       | 1.18E-04       | 0.068            | 0.263          | 0.503          |
| Sinsbrett Function      | -186.731         | -186.731       | -186.731       | 0.052            | 0.173          | 0.439          |
| Bohachevskov Functions 1 | 3.98E-07         | 7.49E-12       | 8.74E-09       | 0.076            | 0.202          | 0.54           |
| Bohachevskov Functions 2 | 0.218313         | 1.63E-10       | 2.24E-07       | 0.07             | 0.201          | 0.401          |
| Bohachevskov Functions 3 | 9.01E-08         | 8.53E-11       | 1.65E-06       | 0.062            | 0.209          | 0.471          |
| Perm Function 0, d, β (= 10) | 1.42E-08         | 3.14E-09       | 8.68E-08       | 0.057            | 0.207          | 0.472          |
| Rotated Hyper-Ellipsoid Function | 1.07E-08         | 2.75E-12       | 3.62E-08       | 0.072            | 0.18           | 0.589          |
| Sphere Function         | 4.36E-11         | 1.99E-11       | 7.88E-08       | 0.054            | 0.162          | 0.429          |
| Sum of Different Powers Function | 8.32E-13       | 2.81E-15       | 1.05E-06       | 0.068            | 0.149          | 0.486          |
| Sum Squares Function    | 6.54E-11         | 5.16E-13       | 5.98E-08       | 0.072            | 0.145          | 0.466          |
| Traj Function           | -2               | -2             | -2             | 0.047            | 0.158          | 0.411          |
| Booth Function          | -3.3E+07         | -3.3E+07       | -3.3E+07       | 0.031            | 0.227          | 0.299          |
| Matyas Function         | 2.27E-11         | 2.18E-13       | 5.98E-08       | 0.062            | 0.183          | 0.457          |
| McCormack Function      | -1.91122         | -1.91122       | -1.91122       | 0.066            | 0.157          | 0.421          |
| Power Sum Function      | 2.06E+05         | 0.001365       | 0.000538       | 3.008            | 0.825          | 0.892          |
| Zakharov Function       | 1.13E-10         | 6.53E-12       | 4.53E-07       | 0.071            | 0.171          | 0.497          |
| Three-Hump Camel Function | 4.61E-11         | 2.93E-11       | 6.57E-07       | 0.067            | 0.167          | 0.499          |
| Six-Hump Camel Function | -1.01163         | -1.01163       | -1.01163       | 0.07             | 0.17           | 0.469          |
| Dixon-Price Function    | 1.6E+10          | 2.71E-10       | 6.29E-08       | 0.054            | 0.191          | 0.389          |
| Rosenbrock Function     | 6.57E-06         | 0.000567       | 0.000014       | 1.202            | 0.416          | 0.459          |
| De Jong Function N5     | 0.996004         | 0.996004       | 0.996004       | 0.076            | 0.145          | 0.483          |
| Easom Function          | -1               | -1             | -1.5E-09       | 0.053            | 0.157          | 0.262          |
| Michalewicz Function    | -1.8013          | -1.8013        | -1.8013        | 0.079            | 0.163          | 0.464          |
| Beale Function          | -0.15509         | -0.15509       | -0.15509       | 0.072            | 0.186          | 0.41           |
| Branin Function         | 0.397887         | 0.397887       | 0.397888       | 0.066            | 0.17           | 0.477          |
| Colville Function (d=4) | -469.313         | -469.313       | -469.313       | 0.204            | 0.251          | 1.177          |
| Forroster et al. (2006) Function | -6.02074         | -6.02074       | -6.02074       | 0.017            | 0.154          | 0.182          |
| Goldstein-Price Function | 4.04E+11         | 2.34E+10       | 9.22E-07       | 0.067            | 0.24           | 0.359          |
| Powell Function (d=4)   | 4.48E+07         | 2.36E+05       | 5.69E-05       | 0.069            | 0.561          | 0.991          |
| Styblinski-Tang Function | -7.83233         | -7.83233       | -7.83233       | 0.076            | 0.14           | 0.451          |

Table 1: Comparison of minimum values achieved and the average computation time (in seconds, computed in MATLAB R2014a) for minimizing benchmark functions with RMPS, GA and SA starting from 10 starting points in each cases. The dimension of all the problems are 2 unless the value of the dimension (i.e., $d$) is mentioned with the name of the function. Please refer to the appendix for the domain of search for each function.
ga (for GA) respectively. In our comparative study, we set the maximum number of allowed iterations and evaluations of objective function to be infinity for simulannealbnd function. In case of ga, we use the default values. Our proposed algorithm Recursive Modified Pattern Search (RMPS) is implemented in Matlab 2014a and the values of the tuning parameters have been taken to be default (as mentioned in Section 2). We consider 45 benchmark functions and each test function is minimized starting from 10 randomly generated points (under 10 consecutive random number generating seeds in MATLAB) within the considered domain of solution space using RMPS, GA and SA. The domains of the search regions can be found in Table 7 (in Appendix : B). All simulation studies in this paper have been performed in a machine with 64-Bit Windows 8.1, Intel i7 3.60GHz processors and 32GB RAM. In Table 1 the obtained minimum values in each occasions have been noted for all considered algorithms with average computation time. It is noted that RMPS and GA perform more or less better than SA. It should be also noted that RMPS yields reasonable solution in lesser time than GA and SA in most of the cases. Using RMPS up to 9 and 15 folds improvement in computation times are obtained over GA ans SA respectively. Although GA yielded better solutions in some of the cases at the cost of more computation time, it should be noted that in the case of RMPS, taking the value of step size threshold ($\phi$) smaller than its default value (i.e., $10^{-6}$), we can obtain finer solution with more computation time.
4 Exploiting convexity

As mentioned in Section 2.3 to minimize any convex function, in general, it is sufficient to perform only one run. Therefore the prior knowledge of convexity can be used to save computational time. Along with that, it is noted that using $\rho = 4$ makes it even faster (since the function is convex, in general, steep decrease in global step size should not affect the result). In Table 2 a comparison study of performances of RMPS, and changed RMPS with the prior knowledge of convexity (RMPS(c)), GA and SA has been provided for minimizing Sphere and Sum squares function for various dimensions starting from 10 randomly generated starting points (under 10 consecutive random number generating seeds in MATLAB) in each cases. Unlike RMPS, in RMPS(c) we perform only 1 run and take $\rho = 4$. Values of all other tuning parameters in RMPS(c) are kept same as that of RMPS. It is noted that in each cases, RMPS(c) performs faster than RMPS. Also in terms of computation times, using RMPS(c) we get up to 40 folds improvement with respect to GA and up to 92 folds improvement with respect to SA.

5 Comparative study for High-dimensional Benchmark problems

To compare the performance of RMPS with that of GA and SA, 100 and 1000 dimensional Ackleys function, Griewank function, Rastrigin function, Schwefel function, Sphere function and the Sum of square function have been considered. The domains on which these functions are minimized are $[-5,5]^d$,
Table 2 Comparative study of RMPS, RMPS(c), GA and SA for solving convex problems
(with average computation times in seconds).

| Functions          | RMPS(c) | RMPS | GA | SA | RMPS(c) | RMPS | GA | SA |
|--------------------|---------|------|----|----|---------|------|----|----|
|                    | (min value) | (max value) | (min value) | (min value) | (avg. time) | (avg. time) | (avg. time) | (avg. time) |
| Sphere (d = 4)     | 1.62E-11 |  6.55E-11 |  1.03E-10 |  1.41E-04 |  0.012 |  0.03 |  0.207 |  0.945 |
| Sphere (d = 20)    | 1.71E-10 |  2.22E-10 |  1.22E-06 |  4.12E-00 |  0.054 |  0.116 |  2.303 |  5.462 |
| Sphere (d = 40)    | 2.66E-10 |  3.67E-10 |  2.09E-05 |  2.00E+01 |  0.186 |  0.299 |  6.07  |  9.301 |
| Sphere (d = 100)   | 6.33E-10 |  8.94E-10 |  4.49E-04 |  8.02E+01 |  1.07  |  1.393 |  45.923 |  47.568 |
| Sum squares (d = 4) | 1.66E-11 |  1.91E-10 |  5.38E-10 |  1.21E-04 |  0.013 |  0.038 |  0.219 |  0.885 |
| Sum squares (d = 20) | 1.19E-09 |  2.20E-09 |  1.83E-06 |  5.79E+01 |  0.076 |  0.172 |  3.085 |  4.487 |
| Sum squares (d = 40) | 5.44E-09 |  7.45E-09 |  3.71E-05 |  7.45E-01 |  0.297 |  0.457 |  8.512 | 13.083 |
| Sum squares (d = 100) | 3.45E-08 |  4.62E-08 |  3.83E-04 |  1.63E-00 |  1.763 |  2.221 |  75.519 |  92.489 |

\([-10, 10]^d\), \([-5.12, 5.12]^d\), \([-500, 500]^d\), \([-5.12, 5.12]^d\) and \([-5.12, 5.12]^d\) respectively. Schwefel function attains the global minimum value 0 at \(x = (420.9687, \ldots, 420.9687)\) within the corresponding domain while all other functions attain the global minimum value 0 at the origin. We use \textit{ga} and \textit{simulannealbnd} of Matlab R2014a (The Mathworks) for GA and SA with tuning parameter values as described in Section 3. As mentioned in the earlier section, RMPS has been implemented in MATLAB and default parameter values (as mentioned in Section 2) has been used. While solving a particular function on a given domain, 10 distinct randomly generated (under 10 consecutive random number generating seeds in MATLAB) initial points has been considered and that same set of 10 starting points are used for GA, SA and RMPS for a fair
comparison. For GA and SA, the smallest of the 10 obtained objective values have been noted along with average computation times (Table 5). For RMPS, both the maximum and the minimum of the objective values and the average computation times have been noted down for each cases. In Table 5, it is noted that RMPS generally outperforms GA and SA. In all the cases, the maximum value of the obtained solutions by RMPS is better than the minimum values obtained by GA and SA. A significant improvement for using RMPS over GA and SA is visible especially in the case of solving the 1000 dimensional problems. Using RMPS we get up to 32 folds improvement in computation over GA (in case of 100-dimensional Sum squares function) and 368 folds improvement in computation over SA (in case of 1000-dimensional Sum squares function). In Figures 4, 5, 6, 7, 8, 9, a comparative study of RMPS, GA and SA has been made based on the improvement of the value of the 1000 dimensional objective functions in first 30 minutes. For all the functions except Sum squares function, the objective values have been plotted in absolute scale (Figures 4, 5, 6, 7, 8), while for Sum squares function (Figure 9), the objective values have been plotted in natural log scale.

To compare the performances of RMPS for the case where the solution is at a boundary point, we repeat the above-mentioned simulation study with changed domains. In this case, the domains of Ackleys function, Griewank function, Rastrigin function, Schwefel function, Sphere function and the Sum of square function are taken to be $[0, 5]^d$, $[0, 10]^d$, $[0, 5.12]^d$, $[0, 420.97]^d$, $[0, 5.12]^d$.
Table 3 Comparative study of RMPS, GA and SA in high-dimensional problems (with average computation times in seconds).

| Functions       | RMPS (min) | RMPS (max) | GA (min) | SA (min) | GA (avg. time) | SA (avg. time) |
|-----------------|------------|------------|----------|----------|----------------|----------------|
| Ackley’s (d = 100) | 1.03E-05   | 1.17E-05   | 3.75E-04 | 7.87E-00 | 3.5            | 68.1           |
| Griewank (d = 100) | 7.86E-11   | 1.17E-05   | 1.00E-03 | 1.45E-00 | 5.3            | 30.0           |
| Rastrigrin (d = 100) | 3.65E-07   | 4.14E-07   | 4.14E-07 | 5.01E+02 | 20.1           | 62.9           |
| Schwefel (d = 100) | 1.27E-03   | 1.27E-03   | 7.86E+03 | 1.93E+04 | 43.7           | 58.9           |
| Sphere (d = 100) | 6.33E-10   | 8.31E-10   | 4.49E-04 | 8.02E+01 | 1.9            | 47.3           |
| Ackley’s (d = 1000) | 3.45E-08   | 4.62E-08   | 3.83E-04 | 1.63E-00 | 2.2            | 72.2           |
| Griewank (d = 1000) | 9.98E-11   | 1.48E-10   | 2.49E-00 | 7.45E-00 | 655.7          | 2580.3         |
| Rastrigrin (d = 1000) | 3.86E-06   | 3.95E-06   | 2.79E+03 | 7.63E+03 | 3938.4         | 6391.2         |
| Schwefel (d = 1000) | 1.27E-02   | 1.27E-02   | 2.15E+05 | 1.95E+05 | 8664.6         | 9646.3         |
| Sphere (d = 1000) | 7.77E-09   | 8.36E-09   | 1.13E+03 | 1.70E+03 | 225.5          | 3971.4         |
| Sum squares (d = 1000) | 3.89E-06   | 4.21E-06   | 1.86E+05 | 1.34E-00 | 256.6          | 4805.3         |

In Table 3, we note down the maximum and minimum values of the obtained solutions after minimizing the 5000 dimensional objective functions with RMPS starting from 3 distinct randomly generated (under 3 consecutive random number generating seeds in MATLAB) initial points. The average computation time in each case has been also noted down in Table 3. Due to

and \([0, 5, 12]^d\) respectively. Note that, in each cases, the true solution is a boundary point. In Table 5, it is noted that in this case also RMPS generally outperforms GA ans SA. Using RMPS we get up to 40 folds improvement in computation over GA (in case of 100-dimensional Sum squares function) and 77 folds improvement in computation over SA (in case of 1000-dimensional Sum squares function).
requirement of excessive amount of time to solve these problems using GA and SA, we could not evaluate their performances in this case.

6 Application to Matrix Completion Problem

The problem of recovering an unknown matrix from only a given fraction of its entries is known as matrix completion problem. [37] first proposed a method
Table 4 Comparative study of RMPS, GA and SA in high-dimensional problems where the true solution is a boundary point. The average computation time in each case has been noted down in seconds.

| Functions    | RMPS (min) | RMPS (max) | GA (min)  | SA (min)  | RMPS (avg. time) | GA (avg. time) | SA (avg. time) |
|--------------|------------|------------|-----------|-----------|------------------|----------------|----------------|
| Ackley’s (d = 100) | 8.55E-06   | 1.16E-05   | 2.29E-00  | 8.75E-00  | 4.8              | 39.5           | 40.0           |
| Griewank (d = 100) | 5.24E-11   | 1.23E-02   | 3.28E-04  | 1.46E-00  | 5.6              | 50.2           | 45.0           |
| Rastrigin (d = 100) | 4.79E-08   | 9.29E-08   | 2.19E+01  | 6.81E+00  | 3.4              | 56.4           | 64.0           |
| Schwefel (d = 100) | 1.27E-03   | 1.27E-03   | 7.36E+03  | 1.91E+04  | 5.3              | 26.0           | 87.5           |
| Sphere (d = 100)   | 7.32E-10   | 8.76E-10   | 5.22E-04  | 1.72E+02  | 2.5              | 72.4           | 55.8           |
| Sum squares (d = 100) | 3.47E-08   | 4.58E-08   | 1.30E-03  | 2.29E+01  | 2.7              | 108.0          | 66.7           |
| Ackley’s (d = 1000) | 9.8E-06    | 1.12E-05   | 5.73E-00  | 9.98E-00  | 557.1            | 6079.5         | 4498.3         |
| Griewank (d = 1000) | 9.33E-11   | 1.52E-10   | 4.47E-00  | 7.25E-00  | 643.5            | 203.6          | 6644.6         |
| Rastrigin (d = 1000) | 6.77E-07   | 7.83E-07   | 4.14E+03  | 9.03E+03  | 528.5            | 6179.8         | 5673.5         |
| Schwefel (d = 1000) | 1.27E-02   | 1.27E-02   | 1.57E+05  | 1.99E+05  | 739.5            | 19898.0        | 10826.0        |
| Sphere (d = 1000)   | 7.7E-09    | 8.20E-09   | 3.48E+03  | 3.00E+03  | 238.3            | 321.0          | 6172.5         |
| Sum squares (d = 1000) | 3.8E-06    | 4.12E-06   | 1.20E+06  | 1.21E+02  | 280.7            | 534.1          | 21576.0        |

Table 5 Performance of RMPS in solving 5000 dimensional benchmark functions (with average computation times in minutes).
to recover a matrix from a few given entries solving a convex optimization problem. Later to solve this problem, [38] minimized nuclear norm of the matrix subject to the constraint that the given entries of the matrix should be the same. In other words, suppose we have a matrix $\mathbf{Y} = (y_{ij})_{n \times n}$ with some missing values. In that case, as mentioned in [38], the complete matrix $\mathbf{Y}$ can be obtained by solving the following problem,

$$
\text{minimize} : ||\mathbf{X}||_*, \\
\text{subject to} : x_{ij} = y_{ij} \text{ for all observed } (i, j),
$$

where $||\mathbf{M}||_* = \sum_i \sigma_i(\mathbf{M})$ denotes the nuclear norm, $\sigma_i(\mathbf{M})$ being the $i$-th singular value of matrix $\mathbf{M}$. This problem can be solved using convex optimization technique. On a closer look, it is noticeable that minimizing nuclear norm in this fashion is similar to the LASSO [39] penalty term. [40] proposed Smoothly Clipped Absolute Deviation (SCAD) penalty which was shown to have more desirable properties compared to LASSO for solving shrinkage based variable selection problems. But unlike LASSO, SCAD penalty is not a convex minimization (or concave maximization) problem. In this section, the matrix completion problem has been solved using the SCAD penalty with RMPS. The matrix completion problem using SCAD penalty can be re-formulated as

$$
\text{minimize} : \sum_i f(\sigma_i(\mathbf{X})) \\
\text{subject to} : x_{ij} = y_{ij} \text{ for all observed } (i, j),
$$

(3)
where $\sigma_i(X)$ are singular values and $f_i$ is the SCAD penalty function dependent of tuning parameters $\lambda$ and $a (= 3.7)$ (see [40]).

We consider a picture (Figure 10) with $61 \times 61$ pixels where approximately half (1877 to be precise) of its pixels are missing. The problem given by Equation (3) can been seen as a black-box function of dimension of 1877 (i.e., the number of missing pixels). It is also known that the numerical value of grey level of each pixel must be between 0 and 255. This problem is solved using RMPS method. We fit the model for 30 values of $\lambda$ which are $\{100, 200, \ldots, 3000\}$ and we obtain the best visual output for $\lambda = 900$ (Figure 11).

Unlike the functions considered in the previous simulations studies, it should be noted that the evaluation of SCAD penalty based on the singular values of the matrix is more computationally intensive. Therefore, the buffer time required of initialization and result collection part (see Section 7) while
Table 6 Computation times (in seconds) required for first 50, 100 and 200 iterations of RMPS using single thread and 4 parallel threads.

| Iterations | 1 thread (time) | 4 threads (time) | Improvement (folds) |
|------------|----------------|-----------------|---------------------|
| 50         | 1325.93        | 391.38          | 3.39                |
| 100        | 3189.97        | 881.75          | 3.62                |
| 200        | 7622.24        | 2162.56         | 3.52                |

using parallel threading is comparatively less in this case compared to the time required to perform the operations at each iteration. Thus, in this case using parallel computing is beneficial. It should be noted that this is a 1877 dimensional problem and therefore up to 3754 parallel threads can be used while solving it using RMPS algorithm. We use 4 parallel threads to derive the complete image given in Figure 11. For comparison of computation time required by single threading and parallel threading with 4 threads, the required computation times for first 50, 100 and 200 iterations have been provided for either cases in Table 6. We note more than 3 folds improvement in time for using parallel threading (with 4 threads) instead of single threading.

7 Discussion

As mentioned in Section 2, the proposed algorithm is parallelizable and up to $2d$ parallel threads can be used while solving a $d$-dimensional black-box problem. However in the simulation study part, the time required for optimizing each function has been noted down without using parallel computing. In
MATLAB, in case `parfor` loop is used instead of `for` loop to perform parallel computing, depending on the operations performed within the loops, a scenario might arise where `for` loop works faster than `parfor` loop. Because at the beginning of the `parfor` loop, an amount of time is spent in shipping the parallelizable works to different workers and at the end again, some time is spent in collecting the results. But this additional time is not spent in case of using `for` loop. So, in case the amount of tasks or operations performed in each loop is not much computationally intensive compared to the amount of buffer time required by `parfor` loop, it may actually spend more time than that required by `for` loop. In case of our test functions, it is observed that `parfor` loop takes more time than `for` loop which is why all the final computation times have been noted down using `for` loop only.

While using parallel computing, the time required for allocation and collection of works sent to different workers varies for different softwares. So, in case some software takes comparatively less time for these operations, parallel computing might work faster than single thread computation even for the considered benchmark functions in this paper. In case the black-box function is really complicated (e.g., it’s evaluation involves linear search, differentiation or integration, matrix inversion, products etc) parallelization should work much faster than single thread computing. Thus benefits of parallelizable algorithm would be more visible and much useful. In Section 6 it has been shown that the
parallel threading can improve the computation time by while solving complex black-box functions using RMPS.

8 Conclusion

This paper presents an efficient derivative-free algorithm for solving black-box function on a hyper-rectangle. Unlike GA, SA and most of the other meta-heuristic black-box optimization techniques, RMPS is a deterministic algorithm and therefore while minimizing any objective function, it returns the same solution in any software under any random number generating seed if the same starting point is considered. For several benchmark functions, it is noted that RMPS generally outperforms GA and SA in terms of accuracy of the solution or the computation time or simultaneously in both perspectives. In Section 4 it is shown that the prior knowledge of convexity of the objective function can be exploited and in that case solution can be found in less computation time. In Section 5 it is noted that using RMPS we get up to 40 folds improvement over GA and up to 368 fold improvement over SA in terms of computation time. Boundary solution cases are also considered for moderate and high-dimensional benchmark functions and in that case also RMPS generally outperforms GA and SA .

As seen in the simulation studies of Sections 4 and 5 the worst solution obtained using RMPS, after starting point 10 randomly generated starting points, is quite accurate in all the cases and better than the best solution
obtained by GA and SA for those cases. Therefore, it is noted that RMPS is less dependent on the starting point and it returns almost equally good solution starting from any initial guess. It is more prominent from the simulation results for 5000-dimensional cases given in Table 5.

Another important feature of RMPS is that the number of required function evaluations in each iteration for this proposed algorithm increases only in the order of the dimension of the black-box function. $2d$ parallel threads can be used while solving a $d$-dimensional problem. In Section 6, RMPS is used to solve the matrix completion problem using SCAD penalty over the sum of singular values. The benefits of parallel implementation of this algorithm is also noted.
Appendix A : Theoretical properties

It is well known that it is not possible for any algorithm to reach to a global minimum of any black-box function every time. But under restrictive conditions like convexity, it is desirable for any optimization algorithm to return the global minimum. In this section we show that under some regularity conditions, RMPS algorithm reaches a global minimum of the objective function. Consider the following theorem

**Theorem .1** Suppose $f : [0, 1]^n \to \mathbb{R}$ is a differentiable convex function. Consider a point $u = (u_1, \ldots, u_n) \in [0, 1]^n$. Consider a sequence $\delta_k = \frac{s}{\rho^k}$ for $k \in \mathbb{N}$ and $s > 0, \rho > 1$. Define $u_k^{(+)} = (u_1, \ldots, u_{i-1}, u_i + \delta_k, u_{i+1}, \ldots, u_n)$ and $u_k^{(-)} = (u_1, \ldots, u_{i-1}, u_i - \delta_k, u_{i+1}, \ldots, u_n)$ for $i = 1, \ldots, n$. If for all $k \in \mathbb{N}$, $f(u) \leq f(u_k^{(+)})$ and $f(u) \leq f(u_k^{(-)})$ for all $i = 1, \ldots, n$, the global minimum of $f$ occurs at $u$.

**Proof (Proof of Theorem 1)** Take an open neighborhood $U \subset [0, 1]^n$ w.r.t. $l_\infty$-norm containing $u$ at the center. So, there exists $r > 0$ such that $U = \prod_{i=1}^n U_i$ where $U_i = (u_i - r, u_i + r)$ for $i = 1, \ldots, n$. For some $i \in \{1, \ldots, n\}$, define $g_i : U_i \to \mathbb{R}$ such that $g_i(z) = f(u_1, \ldots, u_{i-1}, z, u_{i+1}, \ldots, u_n)$. Since $f$ is convex on $U$, it can be easily shown that $g_i$ is also convex on $U_i$. We claim that $g_i(u_i) \leq g_i(z)$ for all $z \in U_i$.

Suppose there exist a point $u_i^* \in U_i$ such that $g_i(u_i^*) < g_i(u_i)$. Take $d = |u_i^* - u_i|$. Clearly $0 < d < r$. Without loss of generality, assume $u_i^* > u_i$. Hence $u_i^* = u_i + d$. Since $\delta_k$ is a strictly decreasing sequence going to 0, there exists a $N$ such that for all $k \geq N$, $\delta_k < d$. Now we have $u_i < u_i + \delta_N < u_i + d$. Now there exists a $\lambda \in (0, 1)$ such that
\[ u_i + \delta_N = \lambda u_i + (1 - \lambda)(u_i + d). \]

From convexity of \( g_i \), we have
\[
\begin{align*}
  f(u_i + \delta_N) &= f(\lambda u_i + (1 - \lambda)(u_i + d)) \\
  &\leq \lambda f(u_i) + (1 - \lambda)f(u_i + d) \\
  &= \lambda f(u_i) + (1 - \lambda)f(u^*_i) \\
  &= \lambda f(u_i) + (1 - \lambda)f(u_i) + (1 - \lambda)(f(u^*_i) - f(u_i)) \\
  &= f(u_i) - (1 - \lambda)(f(u_i) - f(u^*_i)) \\
  &< f(u_i) \quad \text{(since } f(u^*_i) < f(u_i)).
\end{align*}
\]

But, we know \( f(u_i) \leq f(u_i + \delta_N) \). Hence it is a contradiction.

Since partial derivatives of \( f \) exist at \( x = u_i \), \( g_i \) is differentiable at \( z = u_i \). Since \( u_i \) is a local minima of \( g_i \) in \( U_i \), we have \( g'_i(u_i) = 0 \). So we have \( \frac{\partial}{\partial x_j} f(x)|_{x = u} = 0 \). By similar argument it can be shown that \( \frac{\partial}{\partial x_j} f(x)|_{x = u} = 0 \) for all \( j = 1, \ldots, n \). Since \( f \) is convex and \( \nabla f(u) = 0, u \) is a local minima in \( U \). Now, since \( U \subset [0,1]^n \), \( u \) is also a local minima of \( [0,1]^n \). But \( f \) is convex on \( [0,1]^n \). Since any local minimum of a convex function is necessarily global minimum, the global minimum of \( f \) occurs at \( u \).

Suppose the objective function is convex and all the partial derivatives exist at the obtained solution \( u \in [0,1]^n \) which is an interior point. The proposed algorithm terminates when two consecutive runs yield the same solution. It implies in the last run, the objective function values at all the sites obtained by making jumps of sizes \( \delta_k = \frac{s_{\text{max}}}{\rho^k} \) (until \( \delta_k \) gets smaller than step size threshold) around \( u \), i.e. \( f(u^{k+}_i) \) and \( f(u^{k-}_i) \), are greater than or equal
to $f(u)$ for $i = 1, \ldots, n$. So, taking step size threshold sufficiently small, our algorithm will reach the global minimum under the assumed regularity conditions of the objective function.

From Theorem 1 it is concluded that if the objective function is convex and differentiable then taking step size threshold sufficiently small yields the global minimum. It is to be noted that if the function is convex and differentiable and it takes minimum value at some interior point, evaluation of only the first run is sufficient to obtain the solution.
Appendix B: Domains of benchmark functions

| Function name | Domain | True minimum |
|---------------|--------|--------------|
| Ackley Function | $[-32768, 32768]^D$ | 0 |
| Bukin Function N1 | $[-15, -5] \times [-3, 3]$ | 0 |
| Crowe-Toy Function | $[-10, 10]^D$ | -2.0821 |
| Drop-Wave Function | $[-5, 12.5]^D$ | -1 |
| Eggholder Function | $[-512, 512]^D$ | -959.6407 |
| Gramacy & Lee (2012) Function (d=1) | $[0.5, 2.5]$ | unknown |
| Griewank Function | $[-600, 600]^D$ | 0 |
| Holder Table Function | $[-10, 10]^D$ | -19.2085 |
| Langermann Function | $[0, 10]^D$ | unknown |
| Levy Function | $[-10, 10]^D$ | 0 |
| Levy Function N 13 | $[-10, 10]^D$ | 0 |
| Rastrigin Function | $[-5, 12]^D$ | 0 |
| Schwefel Function | $[-500, 500]^D$ | 0 |
| Bohachevsky Functions 1 | $[-100, 100]^D$ | 0 |
| Bohachevsky Functions 2 | $[-100, 100]^D$ | 0 |
| Bohachevsky Functions 3 | $[-100, 100]^D$ | 0 |
| Perm Function $0, d (= 2), \beta (= 10)$ | $[-2, 2]^D$ | 0 |
| Rotated Hyper-Ellipsoid Function | $[-65.536, 65.536]^D$ | 0 |
| Sphere Function | $[-5, 12.5]^D$ | 0 |
| Sum of Different Powers Function | $[-1, 1]^D$ | 0 |
| Sum Squares Function | $[-5, 12.5]^D$ | 0 |
| Trid Function | $[-4, 4]^D$ | 0 |
| De Jong Function N5 | $[-65.536, 65.536]^D$ | unknown |
| Easom Function | $[-100, 100]^D$ | 0 |
| Michalewicz Function | $[0, \pi]^D$ | unknown |
| Beale Function | $[-4, 4]^D$ | 0 |
| Branin Function | $[-5, 10] \times [0, 15]$ | 0.397887 |
| Power Sum Function ($d = 4$) | $[0, 4]^D$ | unknown |
| Zabolov Function | $[-5, 10]^D$ | 0 |
| Three-Hump Camel Function | $[-2, 2]^D$ | 0 |
| Six-Hump Camel Function | $[-3, 3] \times [-2, 2]$ | -1.0316 |
| Drezic-Pinto Function | $[-10, 10]^D$ | 0 |
| Rosebrook Function | $[-5, 10]^D$ | 0 |
| De Jong Function N3 | $[-65.536, 65.536]^D$ | unknown |
| Eason Function | $[-100, 100]^D$ | 0 |
| Michalewicz Function | $[0, \pi]^D$ | 0.6513 |
| Dele Function | $[-4.5, 4.5]^D$ | 0 |
| Branin Function | $[-5, 10] \times [0, 15]$ | 0.397887 |
| Césile Function ($d = 4$) | $[-10, 10]^D$ | 0 |
| Forrester et al. (2009) Function | $[0, 1]$ | unknown |
| Goldberg-Prieto Function | $[-4.24, 3]$ | 0 |
| Penn Function ($d = 2), (\beta = 1.5)$ | $[-4.24, 3]$ | 0 |
| Powell Function ($d = 4$) | $[-4.24, 3]$ | 0 |
| Styblinski-Tang Function | $[-5, 5]^D$ | -37.1233 |

Table 7 Domains of search regions of the benchmark functions considered in Section 3.
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