We solve self-consistently the coupled equations of motion for trapped particles and the field of a one-dimensional optical lattice. Optomechanical coupling creates long-range interaction between the particles, whose nature depends crucially on the relative power of the pump beams. For asymmetric pumping, traveling density wave-like collective oscillations arise in the lattice, even in the overdamped limit. Increasing the lattice size or pump asymmetry these waves can destabilize the lattice.

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Optical lattices (OL) are perfectly periodic arrays of particles trapped by the standing wave interference pattern of several laser beams. They have important applications as model systems for solid state physics as well as for quantum information science. The back-action of the trapped particles on the light trap is carefully avoided in most OL experiments. However, it is known to give rise to intriguing phenomena in related systems, e.g., cavity cooling [1], mirror cooling [2], and optical binding [3]. For OL’s this back-action has been predicted [4] and observed [5, 6] to reduce the lattice constant compared to the naive expectation.

In this Letter we consider the dynamical effects of optical back-action in a one-dimensional OL, brought about by tuning a hitherto neglected parameter, the asymmetry in the intensities of the lattice beams. Due to the back-action the trap light mediates an interaction between the particles, which is substantially altered by this asymmetry. Net energy and momentum flow is induced through the OL, relating it to crystals driven far from equilibrium, e.g., arrays of vortices in a type-II superconductor [7], and trains of water drops dragged by oil [8]. The phonon-like traveling density waves characteristic of these systems become the elementary excitations of the OL as well, and can destabilize it, even in the presence of arbitrarily strong viscous damping. The excitations arise resonantly at specific values of the asymmetry, which allows for tuning the dispersion relation of the lattice. Moreover, the light-mediated interaction in the OL is of infinite range, and thus all these effects depend heavily on the size of the lattice. As absorption inevitably leads to pumping asymmetry, this dynamic instability limits the size of any near-resonant OL.

We consider a dipole trap formed by two counterpropagating phase locked laser beams with frequency $\omega = \omega_0$. The waist of the trap is much larger than the wavelength $\lambda = 2\pi/k$, so the light field is essentially 1 dimensional along $x$. The two beams have unequal intensities: the electric field incident from the left is $E(x) = E_0 e^{i k x - i \omega t}$, from the right, $E(x) = E_1 e^{-i k x - i \omega t}$, with $E_1 = \epsilon_0\sqrt{\delta}E_0$, and $\phi$ the relative phase. Besides the pump power $P > 1$, we introduce another measure of the asymmetry: $A = |E_1/E_0| - |E_0/E_1|$. We consider particles of linear polarizability $\alpha$ and mass $m_A$ pre-cooled down to very low temperatures (possibly pre-trapped) and trapped by the dipole force in the light field. These can be atoms, the lasers being detuned to the red of a specific transition so far that it is not saturated and spontaneous emission can be ignored. Additionally, they can be plastic beads trapped in water, as in, e.g., [9, 10], albeit of size well below $\lambda$ so that complications of Mie scattering are avoided. If the particles are cold enough, they gather at the antinodes, forming $N$ disk-shaped clouds of axial size much smaller than $\lambda$. For simplicity we assume that each cloud has the same number of particles, and thus identical surface density $\eta$, surface mass density $m = \eta m_A$, and dimensionless polarizability $\zeta = k\eta\alpha/(2\epsilon_0)$. The setup is sketched in Fig. 1.

We now take the back-action of the particles on the light field into account. Following [4], this is achieved by solving the scalar Helmholtz equation, with the $N$ clouds represented by Dirac-$\delta$ distributions of linearly polarizable material,

$$ (\partial_x^2 + k^2)E(x) = -2kE(x) \sum_{j=1}^{N} \zeta \delta(x - x_j). $$

(1)

Throughout this Letter we assume $\zeta \in \mathbb{R}$, neglecting spontaneous emission and scattering into other transverse modes, justified as long as the laser beams are far detuned from any resonance of the trapped particles. Note that although these approximations can be relaxed by setting $\zeta \in \mathbb{C}$, very close to resonance the reabsorption of spontaneously emitted photons plays an important role in the dynamics [11], and this is not
The solution of Eq. (1) between two clouds is a superposition of plane waves, 
\[ E(x_{j-1} < x < x_j) = A_j e^{i k (x-x_j)} + B_j e^{-i k (x-x_j)} = C_{j-1} e^{i k (x-x_{j-1})} + D_{j-1} e^{-i k (x-x_{j-1})}. \]

The clouds constitute boundary conditions for the field:
\[ E(x = x_j - 0) = E(x = x_j + 0); \]
\[ \partial_x E(x = x_j - 0) = \partial_x E(x = x_j + 0) + 2 k \zeta E(x_j). \]

This amounts to representing each cloud as a beam splitter (BS) at \( x = x_j \) with reflection and transmission coefficients \( r = i \zeta/(1 - i \zeta) \) and \( t = 1/(1 - i \zeta) \), so that \( \zeta = -ir/t \).

Since \( E(x) \) is not differentiable at the cloud position, as given by Eq. (2b) and shown in Fig. 1 we need to calculate the dipole force on the cloud carefully. Integrating the force over a finite cloud and then taking the Dirac-\( \delta \) limit, we obtain
\[ F_j = \frac{\eta \alpha}{8} (\partial_x E^2(x_j - 0) + \partial_x E^2(x_j + 0)) \]

for the force on a unit surface of the cloud, averaged over an optical period. This formula can also be derived based on the amount of momentum transferred to the cloud by the field, via the Maxwell stress tensor, as in [1,2].

For a single cloud at steady state, both \( F_j = 0 \) and Eqs. (3) must hold, which is only possible if
\[ \zeta A < 2. \]

This simple equilibrium criterion can be intuitively understood in the following way. If \( |E_0|^2 < |E_1|^2 \), more photons are incident on the right of the BS than the left, giving a force on it. If enough light is transmitted (\( |t| > \frac{1}{2} |r| A \)), and the interference is favourable (depending on the position of the BS), the imbalance in the outgoing number of photons is enough to counteract this force, leading to a steady state.

For several clouds trapped by the same light, at steady state \( F_j \) has to vanish on all components of the system, which with Eqs. (2) and (3) implies that \( E(x) \) and \( \partial_x E(x) \) are the same to the left and right of any component. As a result, \( |E^2(x)| = |E_0|^2 + |E_1|^2 + 2 |E_0 E_1| \cos(2 k x - \Phi(x)) \) everywhere in the sample, the clouds only contribute to the phase: \( \Phi(x_j < x < x_{j+1}) = \sum_{j=1}^N \chi_l \), the phase slip at the \( l \)th cloud depending on the polarizability \( \chi_l \) of the cloud as
\[ \cos \chi_l(\zeta, A) = \frac{\sqrt{4 - \zeta_l^2 A^2 - \zeta_l^2 \sqrt{A^2 + 4}}}{2(1 + \zeta_l^2)}. \]

Thus, at steady state, \( |A_1| = \ldots = |A_N| = |C_1| = \ldots = |C_N| \), and \( |B_1| = \ldots = |B_N| = |D_1| = \ldots = |D_N| \), i.e., the pump lasers fill the structure unattenuated.

Now consider the steady state of \( N > 1 \) identical, purely dispersive trapped clouds, with \( \zeta_1 = \ldots = \zeta_N = \zeta < 2/A \).

Since at every cloud \( |C_j|/|B_j| = |B_j/C_j| = A \), the phase slips are all equal: \( \chi_1 = \ldots = \chi_N = \chi \). Thus the equilibrium configuration is an equidistant lattice, \( x_j = x_j^{(0)} + (j-1) d \). The lattice constant \( d \) is clearly independent of \( N \), and a decreasing function of the phase shift \( \chi \) – see Fig. 1 and the introduction of [4] –, explicitly
\[ d = \frac{\lambda}{2 \pi} (\pi - \chi(\zeta, A)). \]

FIG. 2: (color online) The lattice constant as a function of the asymmetry is shown in thick (red) curves for \( \zeta = 0.01, \zeta = 0.1, \zeta = 0.5, \zeta = 1, \zeta = 2 \). Shaded (green) areas indicate regions of stability (see page 4), for \( N \leq 800 \) (darkest shade), \( N \leq 100, N \leq 10 \) and \( N \leq 2 \) (lightest shade). The white area is unstable, see Eq. (4). The orange circle marks the parameter regime of Fig. 4.

For \( A = 0 \) this gives \( d_{\text{symm}} = \frac{\lambda}{4 \pi} (1 - 2 \tan(\zeta)/\pi) \) as in [4]. For a given \( \zeta \), increasing \( A \) causes the phase shift \( \chi \) to increase, and \( d \) to be reduced, as illustrated in Fig. 2 (thick red lines). For \( A > 2/\zeta \), the inequality (4) is violated, the stronger beam pushes all the particles away. At \( A = 2/\zeta \) the lattice constant \( d \) is, remarkably, exactly half of \( d_{\text{symm}} \):
\[ d_{\min}(\zeta) = \frac{\lambda}{4 \pi}(\pi - 2 \tan \zeta). \]

The fact that an equilibrium lattice configuration exists is only physically relevant if this equilibrium is dynamically stable. The dynamics of an OL is given by
\[ m \ddot{x}_j = -\mu \dot{x}_j + F_j(x_1 \ldots x_N), \]

where in addition to the light-induced dipole force \( F_j \) from Eq. (5), we include viscous friction with coefficient \( \mu \) (related to the single-particle friction coefficient \( \mu_A \) by \( \mu = N \mu_A \)). For plastic beads immersed in water, \( \mu \) follows from the Stokes law; for atoms in vacuum, it can represent some laser cooling mechanism. This equation is nonlinear, as its solution involves integrating (1) to obtain the electric field for the force. We proceed by linearizing Eq. (8) around an equilibrium configuration. For \( \xi_j = x_j^{(0)} - x_j \ll \lambda \) we have
\[ m \ddot{x}_j = -\mu \dot{x}_j + \sum_{l=1}^N D_{jl} \xi_l, \]

where the matrix \( \mathbf{D} \) is defined by
\[ D_{jl} = \frac{\partial}{\partial x_l} F_j(x_n = 0, n = 1 \ldots N). \]

Stability analysis requires finding the eigenvectors of \( \mathbf{D} \) and determining their dynamics. Details of this calculation are involved and will be published elsewhere, we outline the procedure below. The key tool is the transfer matrix (TM) method, as used in [4]. The TM of the whole optical lattice is a product of the TM’s of a single block of the lattice, which consist
of the BS transformation followed by free propagation over length \( d \). Since losses are neglected, the two eigenvalues of the TM of a single block are \( e^\pm \imath \Theta \) with \( \Theta \in \mathbb{C} \). The parameter \( \Theta \), related to the quasimomentum, is given by the solution of \( \cos \Theta = \cos \kappa d - \zeta \sin \kappa d \). We solve this equation explicitly and obtain the surprisingly simple result

\[
\sin \Theta = \zeta A/2; \quad \pi/2 < \Theta < \pi.
\]

We next apply the TM method to a perturbed OL where the \( l \)th cloud is displaced by an infinitesimal amount. The calculations lead to explicit formulas for the matrix \( \mathbf{D} \) which we omit here for the sake of brevity. Two important properties of \( \mathbf{D} \) must be mentioned. First, \( D_{ij} \) depends only on \( l - j \); \( \mathbf{D} \) is Toeplitz matrix. In particular, for \( D_{ij} < 0 \) all clouds are trapped in identical wells. Second, \( \mathbf{D} \) is not symmetric. This shows that \( F_j \) is not a conservative force: if it were, \( F_j = -\partial/\partial x_j V(x_1, \ldots, x_N) \) would imply that \( \mathbf{D} \) is a Hessian matrix, symmetric by Young’s theorem. Note that reflection symmetry of the system is broken by the pump asymmetry.

The eigenvalue problem of a nonsymmetric real matrix is in general not trivial. We have found, however, that a generalized Fourier transformation with complex wavenumbers diagonalizes \( \mathbf{D} \) exactly. The analytical formulas for the eigenvectors \( \mathbf{v}_b \) and eigenvalues \( z_b \) of \( \mathbf{D} \), with \( b = 0, \ldots, N - 1 \), read

\[
\begin{align*}
[\mathbf{v}_b] &= (\mathbf{P} e^{\pm \imath \beta b})_{j/N}, \\
\frac{z_b}{\beta} &= \sqrt{\mathbf{P}} \cos \Theta \left[ 1 + \frac{4 \sqrt{\mathbf{P}} \sin^2 \Theta}{\left( \sqrt{\mathbf{P}} e^{\pm \imath \beta b/N} - e^{-\imath \beta b/N} \right)^2} \right]^{-1},
\end{align*}
\]

where \( \beta = 8k\zeta I_0/e \) is related to the oscillation frequency \( \omega_0 \) of a single cloud in a symmetric (incident laser intensities \( I_0 = I_1 = e_0 |E_0|^2 c/2 \)) trapped by \( m\omega_0^2 = \beta \). Due to the pump asymmetry the eigenmodes \( \{12\} \) of the lattice are complex, except for \( b = 0 \), which is a distorted center-of-mass mode and, if \( N \) is even, \( b = N/2 \), the density wave of highest wavenumber possible (\( \pi/d \)). These two modes are always stable, as \( z_{N/2} = z_0 < 0 \). Since \( \mathbf{D} \) is real, all other modes form conjugate pairs: \( z_b = z_{N-b}^* \) and \( \mathbf{v}_b = \mathbf{v}_{N-b}^* \). We briefly discuss the meaning of these eigenmodes below.

Consider a pair of complex eigenvalues \( z_b = z_{N-b}^* \) with \( 0 < b < N/2 \), and the corresponding eigenvectors \( \mathbf{v}_b = \mathbf{v}_{N-b}^* \). Both \( \text{Re}(\mathbf{v}_b) \) and \( \text{Im}(\mathbf{v}_b) \) describe density waves of wavelength \( N d/b \), modulated so that their amplitude increases towards the stronger beam. Now time evolution by \( \{9\} \) does not lead out of the subspace of \( \mathbb{R}^N \) spanned by these modes: for any superposition \( \xi = p \text{Re}(\mathbf{v}_b) + q \text{Im}(\mathbf{v}_b) \) with \( p, q \in \mathbb{R} \), Eq. \( \{9\} \) is equivalent to a single complex homogeneous second-order linear differential equation, whose general solution is

\[
p + \imath q = c_+ e^{(\kappa_+ + \imath \omega_+ t)} + c_- e^{(\kappa_+ + \imath \omega_- t)}.
\]

Here \( c_{\pm} = p_{\pm} + \imath q_{\pm} \) are arbitrary constants, and

\[
(\kappa_{\pm} + \imath \omega_{\pm}) = -\frac{\mu \pm \sqrt{\mu^2 + 4mz_0^2}}{2m}.
\]

with \( \kappa_- < \kappa_+ \) to fix notation. This corresponds to two superimposed density waves of wavelength \( N d/b \), one copropagating with the stronger beam (\( \omega_- < 0 \)), and one counterpropagating (\( \omega_+ > 0 \)). Their phase velocities are given by \( Nd/|\omega_{\pm}| \). The copropagating wave is exponentially damped with constant \( \kappa_- < 0 \), but the counterpropagating wave can be either damped or amplified. Thus, this pair of modes is stable if \( \kappa_+ < 0 \), which corresponds to

\[
m (\text{Im} z_b)^2 < -\mu^2 \text{Re} z_b.
\]

For symmetric pumping \( A = 0 \), the matrix \( \mathbf{D} \) is symmetric, its eigenmodes \( \{12\} \) are the Fourier components, and the eigenvalues \( \{13\} \) are all real and negative, thus the lattice is stable. Almost all modes have the same frequency as a single trapped cloud, \( z_1 = z_2 = \ldots = z_N = -\beta \), except the center-of-mass mode, with \( z_0 = -\beta/(1 + N^2\zeta^2) \), which becomes soft if \( N \to \infty \).

With the introduction of a pump asymmetry \( A > 0 \), the eigenmodes and the eigenvalues acquire imaginary parts, and as \( A \) is increased, the real parts of the eigenvalues turn positive one by one. The first two eigenvalues are shown as functions of \( A \) for two examples in Fig. 3. In the “strong collective coupling”, \( N\zeta \gg 1 \) limit (Fig. 3a), we observe clearly separated resonances. In this limit, whenever \( \pi - \Theta \lesssim \pi/N \), we have \( \sqrt{\mathbf{P}} \approx 1 \), and the denominator of \( \{13\} \) is approximately \( 1 - \sin^2 \Theta/\sin^2(\pi b/N) \), which, with Eq. \( \{11\} \), places the resonance for mode \( b \) at \( A \approx A_b = 2b\pi/(N\zeta) \). We remark that \( A = 2\pi/(N\zeta) \) fits the boundaries between the shaded green areas of Fig. 2 almost perfectly for \( A < 1 \). Outside of the strong collective coupling regime (Fig. 3b), the resonances are not well resolved. It may even happen (as in the plotted example) that mode \( b = 2 \) becomes absolutely unstable (Re \( z_2 > 0 \)) at lower \( A \) than mode \( b = 1 \). This causes the “shoulder” in the \( N = 10 \) instability limit on Fig. 2. At the
critical asymmetry \( A = 2/\zeta = 20 \), we have \( \Theta = \pi/2 \) and all eigenvalues are 0; for \( A > 20 \) all modes are unstable.

A few remarks about the nature of these eigenmodes and the instability are in order. Two timescales govern the dynamics of the OL: \( \tau_o = \sqrt{\mu/|\text{Re}(z_b)|} \) of the oscillations and \( \tau_d = \mu/\mu \) of damping. For weak damping \( \tau_0 \ll \tau_d \), modes with nonzero \( \text{Im}(z_b) \) are potentially unstable, but damping can restore their stability, cf. Eq.\((16)\). At the other extreme, in the overdamped limit \( \tau_d \ll \tau_o \), the dynamics is effectively first-order, and the copropagating mode disappears (is “damped out”), for the counterpropagating mode we have \( \omega_+ = -\text{Im}(z_b)/\mu \) and \( \kappa_+ = \text{Re}(z_b)/\mu \). Even with arbitrarily strong damping, the OL becomes unstable if \( \text{Re}(z_b) > 0 \), as the rhs of \((16)\) is negative. This “absolute instability” is used to define the shaded areas of Fig.\(2\). We illustrate the dynamics close to the absolute instability limit in Fig.\(4\) showing the results of numerical integration of Eq.\((8)\) in the overdamped regime near this limit.

Dynamical instabilities resulting from asymmetric pumping have been observed in a far-detuned OL where atom-light interaction was amplified by a ring cavity \([13]\). In free space near-resonant light has to be used (detunings of a few tens of atomic linewidths seem realistic), and thus the influence of spontaneous resonant photons poses serious experimental limitations. We checked via simulation that the dissipative scattering force induces quantitative, but no qualitative changes as long as \( |\text{Im}(\zeta)| < |\text{Re}(\zeta)|/100 \). However, spontaneous emission also heats the clouds, putting an upper limit on the timescale accessible by an experiment, and complicating the very creation of the OL. One possible way to circumvent the latter problem could be creating the OL at larger detuning, where spontaneous heating is negligible, and then continuously decreasing the detuning of the trap beams down to the desired value. As for the timescale of an experiment, we estimate that, e.g., for a cold gas of Rb atoms in a dipole trap detuned by \( \Delta = -10\gamma \ldots -20\gamma \), forming \( N = 100 \) disk-shaped clouds, at pump power ratio \( \rho = 10 \), the destabilization rate \( \kappa_+ \) can exceed the heating rate by orders of magnitude if the surface density of the clouds is \( \eta > 1/(2\lambda^2) \).

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{(color online) Time dependence of position distortions \( \xi \) (in color coding, in units of \( 10^{-3}\lambda \)) in an asymmetrically pumped overdamped optical lattice of \( N = 100 \) clouds with polarizability \( \zeta = 0.1 \), after excitation of mode \( \text{Re}(\nu_1) \) at \( t = 0 \) with amplitude \( 10^{-3}\lambda \). The continuous grey contour line is \( \xi = 0 \). In a), the system is subcritical: \( A = 0.632 \), and \( z_1/\beta = -0.55 - 6.88i \). The excitation results in a density wave propagating towards the stronger beam, and dying out. In b), at supercritical asymmetry \( A = 0.655 \) the eigenvalue is \( z_1/\beta = 1.48 - 5.94i \). The density wave is now amplified, and at \( t \approx 0.5\mu\lambda c/I_0 \approx 2.5\mu/\beta \) we leave the linear regime. Then a local drop in the lattice constant develops at \( x \approx 30\lambda \), which will result in two clouds coalescing, and eventually all particles will be pushed away by the stronger beam (not shown in figure).}
\end{figure}

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