An investigation into the source of stability of the electron spin projections

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Abstract
We propose that the stability of the two projections of the electron spin along the direction of an applied magnetic field, is an effect of high-frequency vibrations acting on the spin magnetic moment. The source of the high-frequency vibrations is to be found in the zero-point radiation field.

1 Introduction
In contrast with a classical compass needle, which points invariably towards the North pole, in the presence of an external magnetic field the spin of an electron orients itself either parallel or antiparallel to the direction of the field. This is such a well-established fact in quantum theory that nobody ever seems to wonder about it, let alone try to find an explanation for it. The double sign of the spin projection is taken for granted, even though the parallel orientation appears as classically counterintuitive, as it is energetically less favourable than any other orientation of the spin with respect to the field: it corresponds in the classical case to a position of unstable equilibrium.

Yet also in classical mechanics there exist systems that can have two positions of stable equilibrium in opposite directions. The best known instance of such a system is the inverted pendulum, which is used as a benchmark in control theory and finds numerous technical applications, some as popular as the self-balancing scooter. Without going so far, we refer to the experience of balancing a broom that stands upside down on the palm of our hand, by making small, rapid movements with the hand to keep it upright. Such remarkable state of motion is the result of adding to the pendulum a vibrating motion of high frequency applied to the supporting point [1]. This would suggest the idea that something similar can be occurring in the case of the spin magnetic moment. But such an idea would seem not to work at all, if no high-frequency source is available.
Here we address this enigma by appealing to the existence of the vacuum or zero-point radiation field, as suggested by stochastic electrodynamics (SED). In this theory, proposed as a foundation for quantum mechanics, a central role is played by the zero-point radiation field (ZPF), taken as a Maxwellian field that fills the whole space and covers the entire frequency spectrum (see e.g. [2] and references therein). In particular, the high-frequency modes (of Compton’s frequency) acting on the electron have been shown elsewhere to be related to the origin of de Broglie’s wave [3] and to (a nonrelativistic version of) the zitterbewegung [4].

It is therefore natural within SED to consider that also the magnetic moment of the electron is subject to the action of these ZPF modes. This is what we do in the present paper, with the purpose of finding an explanation for the two positions of stable equilibrium of the (quantum) electron spin. In section 2 we present the model of the electron spin as it emerges from SED, to be used for this purpose. In section 3 we establish the equations of motion for the electron subject to an applied magnetic field in addition to the ZPF, and solve them for the one-dimensional case by separating the fast variables from the slow ones, which leads to a direct demonstration of the stability of the two spin states, as seen in section 4. The extra vibrations of the magnetic moment of the electron due to its interaction with the high-frequency magnetic component of the ZPF result in an effective potential with two deep minima. As shown in section 5, under conditions that hold in real experiments these minima correspond to the two positions of stable equilibrium of the spin magnetic moment: spin ‘up’ and spin ‘down’. In section 6 the same conclusion is seen to apply in the two-dimensional case, that takes into account the Larmor precession. The paper concludes with a brief recapitulation.

2 The spin and the zero-point field

Before proceeding with the calculations let us briefly present the model of the electron spin to be used in this paper. In quantum theory the electron spin is represented with the aid of the Pauli matrices, which automatically introduce all the required quantum properties. The situation is quite different in SED, a theory based on the hypothesis that quantum mechanics is the result of the action of the radiation ZPF on an otherwise classical system. This theory has evolved in the course of time to reach a well-developed status (see e.g. ref. [2]) that includes, in particular, a theory of the electron spin, just the one we take here as the basis for our proposal.

The most immediate effect of the presence of the ZPF is that an electron, which normally is part of an atomic system, acquires a stochastic motion. The ZPF covers the entire spectrum; but not all frequencies have the same importance for the atomic system. As is well known, Dirac’s equation for the free electron reveals the existence of the zitterbewegung, a helicoidal motion of frequency of the order of $\omega_C$ and amplitude of the order of $\lambda_C$, where $\omega_C = mc^2/\hbar$ represents the Compton frequency of the electron of mass $m$, and $\lambda_C$ the corresponding
wavelength $\lambda_C = h/mc$. In the textbooks on quantum mechanics, $\lambda_C$ appears normally only in association with the Compton effect and related considerations. When looking beyond the quantum formalism, $\lambda_C$ reappears, with an important though indirect role as the source of de Broglie’s wavelength ($[2, 3]$). In SED—as in nonrelativistic QED—$\omega_C$ is used frequently as a convenient cutoff to regularize integrals that determine e.g. the average properties of dynamical variables. This leads to the emergence of particle oscillations of frequencies of order $\omega_C$. The introduction of the cutoff is thus of physical significance rather than a simple mathematical device, meaning that the electron (as all matter) becomes transparent to the radiation field of very high frequencies.

The physical meaning assigned to the cutoff leads to an also physically meaningful finite effective size for the electron, of the order of $\lambda_C$. Such effective size has only a statistical (or coarse-grained) sense, since it is the result of the fluctuations or rapid oscillations impressed upon the particle by the field, the original electron being still a point particle; it is the interaction with the field that dresses it. This view coincides with the QED picture, where the electron acquires through its electromagnetic interactions an effective size of the order of $\lambda_C$ (see e.g. [4]).

We thus arrive at a consistent picture of the electron as a small sphere of effective radius of order $\lambda_C$ realizing a kind of nonrelativistic *zitterbewegung* of frequency of order $\omega_C$. As a consequence of the torque exerted by the electric field modes of a given circular polarization, the electron describes a helicoidal motion, which results in a mean intrinsic angular momentum of value $\hbar/2$, mean square angular momentum $3\hbar^2/4$, and an associated magnetic moment with $g$-factor of 2. The spin is thus identified as an emergent property generated by the action of the *zpf*, as shown in detail in refs. [5]-[7].

The approach used in SED may seem (to some) too classical to reproduce the quantum properties of matter. However, as shown e.g. in chapter 5 of ref. [2], when the SED system transits to a state in which it has acquired ergodic properties (and also energy balance holds), the appropriate description of the dynamics becomes one in which the variables are represented by matrices and the whole Hilbert-space formalism of quantum mechanics applies—including the electron spin with its two projections. In brief, a *qualitative* change of state occurs in the behaviour of the system (as a sort of phase transition), that demands a leap in its description from classical to quantum. The present paper is intended to shed some light on the physical mechanism leading precisely to the two spin projections.

Such abrupt changes in the description are not unknown to physics. A traditional example is the transition (in both, classical and quantum physics) in the description of a system of a few particles to a system with a huge number of them, which requires a statistical treatment. A more recent example is that of nonlinear dissipative systems, in which the phenomenon of deterministic chaos characteristically takes place, contrary in principle to the traditional classical behaviour. In both cases new notions become indispensable in replacement of the older ones.
3 The effective potential

Let us consider an electron with its spin and magnetic moment $\mathbf{\mu}$ forming an angle $\theta$ with the $z$-axis; the applied magnetic field is $\mathbf{B} = \hat{z}B$, with $B$ constant for simplicity. The corresponding interaction energy is $V = -\mathbf{\mu} \cdot \mathbf{B}$. Since $\mu = e\hbar/2mc = -\mu_0 < 0$, the energetically stable orientation of the spin is opposite to the direction of the magnetic field. However, quantum theory considers also the direction along the field to correspond to a stable solution, as has been experimentally proven again and again.

With the purpose of finding an explanation for the stability of the spin orientation in both directions, we take into account that the above system is subject to the action of the ZPF of high frequency. For our present purposes it is enough to consider a mode of this field of a sufficiently high frequency $\gamma$, uniformly distributed on the $xy$-plane and directed along the $z$-axis; we therefore write the magnetic component of this mode as $\mathbf{B}_0(t) = \hat{e}_z B_0 \cos \gamma t$.

In spherical coordinates, the Lagrangian for the problem is

$$L = \frac{1}{2} I \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) - \mu_0 (B \cos \theta + B_0 \cos \theta \cos \gamma t), \quad (1)$$

with $I$ an effective moment of inertia to be determined below. The corresponding equations of motion are

$$\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta = \frac{\mu_0 B}{I} \sin \theta + \frac{\mu_0 B_0}{I} \sin \theta \cos \gamma t, \quad (2)$$

$$\dot{\phi} \sin^2 \theta = \text{constant}. \quad (3)$$

Note that the second term on the l.h.s. is a kinematic term due to the use of spherical variables; it does not disappear in the absence of the magnetic field. Further, the variable $\phi$ is ignorable, and the angular velocity about the $z$-axis, given by $\dot{\phi}$, is arbitrary. Let us, for clarity, assume in a first instance that initially $\dot{\phi} = 0$ and $\phi = 0$; then according to eq. (3) the magnetic moment vector remains on the $xz$-plane, and (2) reduces to

$$\ddot{\theta} = \frac{\mu_0 B}{I} \sin \theta + \frac{\mu_0 B_0}{I} \sin \theta \cos \gamma t. \quad (4)$$

Now we follow the usual procedure [1] to determine the effect of the high-frequency term on the orientation of $\mathbf{\mu}$, by separating the fast terms from the slowly varying component of the motion. This procedure applies when $\gamma \gg \omega_0$, where $\omega_0$ is a frequency parameter associated with the external field,

$$\omega_0 \equiv \sqrt{\frac{\mu_0 B}{I}}. \quad (5)$$

Under this condition, a first-order determination of the effects of the high-frequency vibration is sufficient, as will become clear below (see also [1], [8]). Thus we write $\theta = \Theta + \xi$, with $\Theta$ the slow (dominant) motion component and
the fast (small, high-frequency) correction. A Taylor series expansion of eq. (4) gives to first order in $\xi$

$$\ddot{\Theta} + \ddot{\xi} = \omega_0^2 (\sin \Theta + \xi \cos \Theta) + \frac{\omega_0^2 B_0}{B} (\sin \Theta \cos \gamma t + \xi \cos \Theta \cos \gamma t).$$  \hfill (6)

This equation contains both the smooth and the rapidly vibrating terms, which must be separately equal. For the latter we get

$$\ddot{\xi} = \frac{\omega_0^2 B_0}{B} \sin \Theta \cos \gamma t + O(\xi),$$  \hfill (7)

which gives to first order

$$\xi = -\frac{\omega_0^2 B_0}{\gamma^2 B} \sin \Theta \cos \gamma t.$$  \hfill (8)

Substituting this in (6) and averaging over the short period $2\pi/\gamma$, an interval during which $\Theta$ remains virtually the same, we obtain an equation for the slow motion,

$$\ddot{\Theta} = \omega_0^2 \sin \Theta - 2\Omega^2 \frac{B}{B_0} \sin \Theta \cos \Theta \cos \gamma t + \frac{\omega_0^2 B_0}{B} \sin \Theta \cos \gamma t$$

$$-2\Omega^2 \sin \Theta \cos \Theta \cos^2 \gamma t,$$  \hfill (9)

which simplifies into

$$\ddot{\Theta} = \omega_0^2 \sin \Theta - \Omega^2 \sin \Theta \cos \Theta,$$  \hfill (10)

with the frequency parameter $\Omega$ defined as

$$\Omega \equiv \frac{\mu_0 B_0}{\sqrt{2I\gamma}}.$$  \hfill (11)

Since in what follows we are interested in the slow motion only, described by eq. (10), we shall go back to the original notation for the angle, i.e., $\Theta \rightarrow \theta$. Equating now the angular acceleration $I\ddot{\theta}$ to (minus) the derivative of an effective potential energy $V_{\text{eff}}$ associated with the magnetic moment, we obtain

$$V_{\text{eff}} = I\omega_0^2 \cos \theta + \frac{1}{2} I\Omega^2 \sin^2 \theta.$$  \hfill (12)

This potential is represented in fig. 1, for two values of $a/2 = \omega_0^2/\Omega^2 < 1$ (see below).

4 Stable equilibrium positions

As is well known from the theory of high-frequency excitations (1), a first-order consequence of these excitations is an additional (mean) force applied on the
Figure 1: Effective potential \( U_{\text{eff}} = (2/I\Omega^2)V_{\text{eff}} = a \cos \theta + \sin^2 \theta \), for \( a = 10^{-2} \) (solid line) and \( a = 10^{-1} \) (dashed line). Current experimental values are \( a \ll 10^{-6} \). The energy gap is \( \Delta E = 2a \).

system, able to produce peculiar effects. Here the additional force derives from the second term in \( V_{\text{eff}} \) (see eq. (10)), and the effect of it is the emergence, under appropriate conditions, of two positions of stable equilibrium for \( \mu \) at the minima of the potential curve represented in fig. 1. This situation is closely analogous to the behaviour of the inverted pendulum (see e.g. [1], [8], [9]). Indeed, a comparison of these two theories shows that eq. (10) is common to both, with the appropriate parameters and dynamical variables in each case.

To determine the stable equilibrium positions we look for the solutions of the equation

\[
\frac{dV_{\text{eff}}}{d\theta} = 0 \quad \text{or} \quad (-\omega_0^2 + \Omega^2 \cos \theta) \sin \theta = 0,
\]

whence both \( \theta = 0, \pi \) are equilibrium solutions. Since

\[
\frac{d^2V_{\text{eff}}}{d\theta^2} = I \left( -\omega_0^2 \cos \theta + \Omega^2 (\cos^2 \theta - \sin^2 \theta) \right),
\]

it is clear from the outset that \( \theta = \pi \) is a stable solution for any value of the parameters. This represents the more energetically favourable equilibrium position (the only stable one in the classical case). For \( \theta = 0 \) the second derivative \( d^2V_{\text{eff}}/d\theta^2 \) has positive values if and only if

\[
\omega_0^2 < \Omega^2.
\]

When this condition is satisfied, also the solution \( \theta = \pi \) is stable. In terms of the magnetic fields, eq. (15) rewrites as
which means that for a very strong applied magnetic field \( B \) the stability of this solution may be lost.

To estimate the frequencies \( \omega_\pm \) of oscillation around the equilibrium points we make the usual small-amplitude approximation

\[
\frac{d^2V_{\text{eff}}}{d\theta^2}\bigg|_{\theta=0,\pi} = I\omega^2_\pm,
\]

with the plus sign corresponding to \( \theta = \pi \). Combining with eq. (14) and using (5) and (11), this gives

\[
\omega^2_\pm = \Omega^2 \pm \omega^2_0.
\]

(18)

It is clear from fig. 1 that for \( \omega^2_0 \ll \Omega^2 \), the two equilibrium positions are virtually equally stable. The difference between the two frequencies is given in this case, according to eq. (18), by

\[
\Delta \omega = \omega_+ - \omega_- = \Omega \left[ \sqrt{1 + \left( \frac{\omega_0}{\Omega} \right)^2} - \sqrt{1 - \left( \frac{\omega_0}{\Omega} \right)^2} \right] \approx \frac{\omega^2_0}{\Omega},
\]

(19)

the more energetically favourable one being slightly higher.

Further, notice that the energy difference between the two stable solutions is determined solely by the value of the applied field,

\[
\Delta E = 2\mu_0 B,
\]

(20)

with independence of the values of the parameters associated with the ZPF.

5 Stability of the electron spin projections

In order to apply the above results to the specific case of the electron spin we must assign values to the various physical parameters involved. According to the discussion in section 2, we take \( \omega_C \) as the frequency of the vibrating mode of the ZPF acting on the spin magnetic moment. This gives for the moment of inertia, defined as \( I \simeq \hbar/\gamma \) for an angular momentum of the order of \( \hbar \), the (approximate) value

\[
I \simeq \frac{\hbar}{\omega_C} = \frac{\hbar^2}{mc^2}.
\]

(21)

By writing the moment of inertia in terms of the mass and the effective radius of the electron acquired as a result of the zitterbewegung, \( I \simeq mr_{\text{eff}}^2 \), we obtain for the latter the following value

\[
r_{\text{eff}} \simeq \sqrt{\frac{I}{m}} \simeq \frac{\hbar}{mc} = \frac{\lambda_C}{2\pi},
\]

(22)
i.e., the effective size of the electron turns out to be of the order of Compton’s wavelength, in agreement with the standard literature (see e.g. ref. [4]) and our previous discussion.

We need also an estimate for the magnetic field amplitude $B_0$, which is related to the energy density of the zpf mode of Compton’s frequency. This energy density is given by the total energy of this mode, $(1/2)\hbar\omega_C$, divided by the effective volume occupied by the electron, $V_{eff} = (4/3)\pi r_{eff}^3$, which gives

$$B_0^2 = \frac{3}{2} \left(\frac{2\pi}{\lambda_C}\right)^3 \hbar\omega_C; \quad (23)$$

whence we obtain from eq. (11), with $\mu_0 = |e| \hbar/2mc$,

$$\Omega^2 = \frac{3}{8} \alpha \omega_C^2, \quad (24)$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant. With these order-of-magnitude estimates the condition (15) for the existence of two stable equilibrium positions reads

$$\omega_0 < \frac{3}{8} \alpha \omega_C^2, \quad (25)$$

or in terms of the magnitude of the applied field $B$, according to eq. (5),

$$\mu_0 B < \frac{3}{8} \alpha \hbar \omega_C = \frac{3}{8} \alpha mc^2. \quad (26)$$

For a comparison with experiment it is useful to rewrite (26) as a condition on the Larmor frequency,

$$\omega_L = \frac{\mu_0 B}{\hbar} < \frac{3}{8} \alpha \omega_C. \quad (27)$$

Electron paramagnetic resonance (EPR) measurements are usually carried out with microwaves of frequencies of the order of $\omega_L \approx 10^{10} \text{ s}^{-1}$, whilst the Compton frequency of the electron is $\omega_C \approx 10^{21} \text{ s}^{-1}$. Such Larmor frequencies correspond to magnetic field strengths of the order of 0.35 T. Magnetic fields of extremely high intensity, of the order of 10 T, correspond to Larmor frequencies close to $\omega_L \approx 10^{13} \text{ s}^{-1}$, still eight orders of magnitude smaller than $\omega_C$. Therefore, under present experimental situations the stability condition (15) is amply satisfied. This means that the stability of the two equilibrium positions $\theta = 0, \pi$ is well guaranteed for a wide range of values of the applied magnetic field.

Further, note that according to eq. (19), the relative difference between the two frequencies of oscillation around the points of equilibrium is of the order of

$$\frac{\Delta \omega}{\Omega} \approx \left(\frac{\omega_0}{\Omega}\right)^2 = \frac{8}{3\alpha} \frac{\omega_L}{\omega_C}, \quad (28)$$

which represents an insignificant deviation for usual magnetic field strengths. fig. 1 shows that the (near harmonic) potential wells are indeed very similar in width, and deep enough to sustain the stability of both solutions.
6 Inclusion of the Larmor precession

It is possible to extend the analysis carried out above to the more general case in which the magnetic moment rotates in two dimensions. This allows us to take into account the effect of the torque exerted on the magnetic moment by the magnetic force,

\[ \tau = \mu \times B = \frac{d\mu}{dt}. \] (29)

As is well known from classical electrodynamics, this torque gives rise to the Larmor precession, which is a rotation of \( \mu \) about the \( z \)-axis, with angular velocity given by \( \dot{\phi} = \omega_L = \mu_0 B/\hbar \) (see eq. (27)), independent of the zenithal angle \( \theta \). Since this movement of precession is orthogonal to the motion along the \( xz \)-plane described above (with \( \theta \) as the only variable), and \( \phi \) is not a rapidly oscillating variable, the stability behaviour should be essentially the same as above. To show that this is the case, we rewrite the complete equation (2) in terms of the Larmor frequency for \( \dot{\phi} \),

\[ \ddot{\theta} = \omega_0^2 \sin\theta + \omega_L^2 \sin\theta \cos\theta + \frac{\omega_0^2 B_0}{B} \sin\theta \cos\gamma. \] (30)

By separating the slow motion from the fast terms in this equation, in analogy with the 1-D case, one obtains for the effective potential, using once more (11),

\[ V_{\text{eff}} = I \omega_0^2 \cos\theta + \frac{I}{2} \left( \Omega^2 - \omega_L^2 \right) \sin^2\theta. \] (31)

As before, both values \( \theta = 0, \pi \) correspond to points of stability. From

\[ \frac{d^2V_{\text{eff}}}{d\theta^2} = -I \omega_0^2 \cos\theta + I \left( \Omega^2 - \omega_L^2 \right) \left( \cos^2\theta - \sin^2\theta \right) \] (32)

we see that for

\[ \omega_0^2 \left( 1 + \frac{\omega_L^2}{\omega_0^2} \right) < \Omega^2, \] (33)

again both equilibrium positions are stable. Since, for the values of the parameters given in section 5 (see the discussion following eq. (27)),

\[ \frac{\omega_L^2}{\omega_0^2} = \frac{\mu_0 B}{\hbar \omega_C} \ll 1, \] (34)

essentially the same stability condition (15) holds in the more general case that includes the Larmor precession.

7 Conclusions

Stochastic electrodynamics shows us that an electron spin free to rotate under the combined action of an applied field \( B \) and the high-frequency oscillatory mode \( B_0 \cos \gamma t \) of the ZPF parallel to \( B \), has two stable equilibrium positions, parallel and antiparallel to \( B \), just as described in quantum mechanics. The condition for these two spin projections, expressed in (15), is largely satisfied for present experimental values of the magnetic field intensity.
References

[1] J. J. Thomsen, Vibrations and Stability. Advanced theory, Analysis, and Tools, Second ed. (Springer, 2003).

[2] L. de la Peñ a, A. M. Cetto and A. Valdés-Hernández, The Emerging Quantum (Springer, 2015).

[3] L. de la Peñ a and A. M. Cetto, The Quantum Dice (Kluwer, 1996)

[4] P. W. Milonni, The Quantum Vacuum. An Introduction to Quantum Electrodynamics (Academic Press, 1994).

[5] A. M. Cetto, L. de la Peñ a and A. Valdés-Hernández, Emergence of quantization: the spin of the electron, JPCS 504 (2014) 012007

[6] A. M. Cetto and L. de la Peñ a, Electron system correlated by the zero-point field: physical explanation for the spin-statistics connection, JPCS 701 (2015) 012008

[7] A. M. Cetto, L. de la Peñ a and A. Valdés-Hernández, Proposed physical explanation for the electron spin and related antisymmetry, Quantum Stud: Math. Found. (2017), DOI:101007/s40509-017-0152-8

[8] L. D. Landau and E. M. Lifschitz, Mechanics, Addison-Wesley § 5.

[9] E. Butikov, On the dynamic stabilization of an inverted pendulum. Am. J. Phys. 69 (2001) 1.