An overdense plasma layer irradiated by an intense light can exhibit dramatic nonlinear-optical effects due to a relativistic mass-effect of free electrons: highly-multiple hysteresises of reflection and transition, and emergence of gigantic “rogue waves”. Those are trapped quasi-soliton field spikes inside the layer, sustained by an incident radiation with a tiny fraction of their peak intensity once they have been excited by orders of magnitude larger pumping. The phenomenon persists even in the layers with ”soft” boundaries, as well as in a semi-infinite plasma with low absorption.

In this Letter we discover, however, that even the most basic, 1D-case, reveals a large and apparently little known phenomenon of highly-multistable nonlinear EM-propagation and the emergence of trapped “rogue” (R) waves, with intensity exceeding the incident one by orders of magnitude. They may be released by plasma (R) waves, with intensity exceeding the incident one by orders of magnitude – varying by orders of magnitude – depending of the history of pumping. We assume a stationary, cw, or long pulse mode, and use only RL-nonlinearity in a cold plasma. While this model is greatly simplified vs various kinetic approaches, it allows us to keep the basic features necessary to elucidate new results, and have the theory applicable to other systems.

Even a few-λ-thick plasma layer can produce the effect, so that the absorption of light (including nonlinear self-focusing) would not affect the propagation significantly. Furthermore, while abrupt boundaries contribute to the effect, they are not essential: a layer with ”soft” boundaries still exhibits all the major features of the phenomenon. The phenomenon persists even for semi-infinite plasma with a small absorption, which also develops a strong retro-reflection. The ”Sommerfeld condition” (no wave comes from the ”infinity”) is to be revisited here: a wave is back-reflected deep inside the plasma and returns to the boundary. Imposed on a forward wave, it results in a semi-standing wave, trapped quasi-soliton R-waves, and multistability, same as in a finite layer. The energy accumulated in R-waves and excited free electrons, can then be released if plasma density reduces, as it may happen, e.g. in an astrophysical environment.

The wave propagation here is governed by the same so called nonlinear Klein-Gordon-Fock equation:

$$[\nabla^2 - \partial^2/(\nu \partial t)^2] \psi = k_0^2 f(|\psi|^2)\psi; \quad f(0) = 1. \quad (1)$$

where $f(|\psi|^2)$ is a function responsible for nonlinearity. Here, a generic variable, $\psi$, could be a scalar (e.g. a wave function in RQM), or a field vector in EM-wave propagation, $\mathbf{v}$ is a scale velocity, $(\nu = c$ for plasma and in RQM), and $k_0^{-1}$ is a spatial scale of the problem, (for RQM, $k_0 = m_0 c/\hbar = 2\pi/\lambda_C$, where $\lambda_C$ is the Compton wavelength, and $m_0$ is the rest-mass of an electron); for plasma $k_0 = \omega_{pl}/c$, where $\omega_{pl}$ is a plasma frequency due to free-charge density, $\rho_e$, and in the X-ray physics, $\omega_{ph} =$
$E_{\text{ph}}/\hbar$ is related to photo-ionization limit, $E_{\text{ph}}$, of atoms. For an $\omega$-monochromatic wave $\psi = \tilde{E}(\vec{r}) e^{-i\omega t}/2 + \text{c.c.}$, where $\tilde{E}(\vec{r})$ is a complex amplitude, and using $u^2 = |\xi|^2$, (1) is reduced to a nonlinear Helmholtz equation:

$$\nabla^2 \tilde{E}' + k^2 \epsilon(u^2)|\tilde{E}'| = 0; \quad \epsilon_{\text{pl}} = 1 - |\nu^2(\nu^2)|/\omega^2$$  \hspace{1cm} (2)

where $k = \omega/c$, and $\omega^2 = 4\pi\epsilon_0/e/m$, $m$ – electron mass, and $\mathcal{E} = E/E_{\text{nl}}$ with $E_{\text{nl}}$ being some characteristic nonlinear scale; for a RL-mass-effect it is $E_{\text{nl}} = \omega m q e/c$. In general, nonlinear $\epsilon$ may have various origins: a varying ionization rate, plasma waves, ponderomotive force, etc. Assuming fully ionized gas, $\rho_e = \text{const}$, and a circularly-polarized wave, $\mathcal{E}(\zeta) (\epsilon_x + i\epsilon_y) e^{-i\omega t}/2 + \text{c.c.}$, that has very negligible high-harmonics generation and minimal longitudinal plasma waves excitation, the most basic remaining source of nonlinearity is a field-induced RL mass-effect of electron: $m = m_0 + \gamma$, with a relativistic factor $\gamma = \sqrt{1 + (p/m_0c)^2} = \sqrt{1 + u^2}$ [see e. g. [6]] where $p$ is the momentum of electron, so that

$$\epsilon_{\text{el}} = 1 - |\nu^2\gamma(u^2)|^{-1} \quad \text{with} \quad \nu = \omega/(\omega_{\text{ps})0}$$  \hspace{1cm} (3)

where $(\omega_{\text{ps})0}$ is a linear plasma frequency with $m = m_0$. Since $m = m(u^2)$, a single electron exhibits large hysteretic cyclotron resonance predicted in [6a] and observed in [7]. The mass-effect has also become one of the major players of light-plasma interaction [3], e. g. in RL self-focusing, and in acceleration of electrons by the beat-wave and wake-field. The EM-propagation could also be accompanied by RL-intrinsic bistability [8].

In a 1D-case, letting a plane EM-wave propagate in the $z$-axis, we have $\nabla^2 = d^2/dz^2$. For a boundary between two dielectrics, a EM-wave incident from a dielectric with $\epsilon_{\text{in}} > 0$ under the angle $\theta$ onto a material of $\epsilon_{\text{NL}} > 0$, $\epsilon$ in (2) is replaced by $\epsilon_{\text{in}}(\epsilon_{\text{NL}})/\epsilon_{\text{in}}(\epsilon_{\text{NL}}) - \sin^2\theta$. For a mwa waveguide with a critical frequency $\omega_{\text{cr}}$, $\epsilon$ in (2) is replaced by $\epsilon_{\text{cr}}(1 - \nu^2\omega^2/\omega^2)$. The crossover point is attained at $\epsilon = 0$. In this approximation, (2) reduces to

$$\mathcal{E}'' + \epsilon(\zeta, u^2)\mathcal{E} = 0,$$  \hspace{1cm} (4)

where $\zeta = kz$, and "prime" denotes $d/d\zeta$; in general, we do not assume $\epsilon$ uniform in $\zeta$-axis. In a weakly-nonlinear media one can break the field into counter-propagating traveling waves and find their amplitudes via boundary conditions. However, near a crossover point one in general cannot distinguish between those waves. To make no assumptions about the wave composition, we represent the field using real variables $u$, and phase (eikonal), $\phi$, as $\mathcal{E} = u(\zeta) \text{exp}(i\phi(\zeta))$. Since $\mathcal{E}$ is in general complex, while $\epsilon = c(u^2)$, Eq. (4) is isomorphous to a 3-rd order eqn for $u$; yet, it is fully integrable in quadratures. Its first integral is a scaled momentum flux of EM-field, $P = u^2\phi' = \text{inv}$. In a lossless media $P$ is conserved over the entire space $\zeta < \infty$, even if the medium is non-uniform, multi-layered, linear and/or nonlinear, etc. If a layer borders a dielectric of $\epsilon = \epsilon_{\text{ex}}$ at the exit, we have $P = u_{\text{ex}}^2 \sqrt{c_{\text{ex}}}$, where $u_{\text{ex}}^2$ is the exit wave intensity. Eq. (2) is reduced then to a 2-nd order equation for $u$:

$$u'' + u[\epsilon(\zeta, u^2) - P^2/u^4] = 0,$$  \hspace{1cm} (5)

which makes an unusual yet greatly useful tool. By addressing only a real amplitude, while using flux $P$ as a parameter, (5) is nonlinear even for a linear propagation, yet is still analytically solvable if a density $\rho_e$ is uniform across the layer ($\partial\epsilon/\partial\zeta = 0$). A full-energy-like invariant of (5) is $u^2 + U(u^2) = W = \text{inv}$, with $U = \int_0^\zeta \epsilon(u^2) d(u^2) + P^2/u^4/2$ , where $u^2/2$ is "kinetic", and $U - \text{"potential" energies. For a RL-nonlinearity (3),}$

$$U(u^2) = (u^2 + P^2/u^4)/2 - [\gamma(u^2) - 1]/u^2.$$  \hspace{1cm} (6)

Here $W$ is a scaled free EM energy density of $\epsilon$-nonlinear medium $W = \int [H^2 + \int c d(E^2)]/(2E_{\text{nl}}^2)$, where $H$ is magnetic field. If a layer exit wall is a dielectric, one has $W = U(u_{\text{ex}}^2)$, since then $u_{\text{ex}} < 0$ (see below). For a metallic mirror, $W = u_{\text{ex}}^2/2$, since now $u_{\text{ex}} = 0$; and $W = 0$ for an evanescent wave in a semi-infinite medium. The implicit solution for spatial dynamics of $u$ in general case is found now as $\zeta = \int [2W - U(u^2)]^{1/2} du$.

Boundary conditions at the borders with linear dielectrics at the entrance, $\zeta = 0$, with $\epsilon_{\text{in}}$, and at the exit, $\zeta = d$, with $\epsilon_{\text{ex}}$, result in complex amplitudes of incident, $\epsilon_{\text{in}}$, and reflected, $\epsilon_{\text{refl}}$, waves at $\zeta = 0$:

$$\epsilon_{\text{in,refl}} = \left[ u \pm \frac{1}{\epsilon_{\text{ex}}}(P/u - iv')/2 \right],$$  \hspace{1cm} (6)

where "+" corresponds to $\epsilon_{\text{in}}$, and "-" to $\epsilon_{\text{refl}}$, and the exit point, $\zeta = d$: $u = \epsilon_{\text{ex}} \equiv u_{\text{ex}}$; $u' = 0$; and $\phi' = \sqrt{\epsilon_{\text{ex}}}$. A condition $P = 0$ corresponds to full reflection, resulting in either strictly standing wave, or nonlinear evanescent wave in a semi-infinite plasma, in particular in a "standing" soliton-like solution (see below). If $\epsilon(u^2) = 0$, there are no linear traveling waves; yet a purely traveling nonlinear wave may exist at sufficiently strong intensity $u^2 = u_{\text{trav}}^2 = \text{const}$, such that $u(\zeta) > 0$:

$$\epsilon_{\text{trav}} = u_{\text{trav}} \exp(i\phi'(\zeta)), \quad \phi' = \pm \sqrt{\epsilon(u_{\text{trav}}^2)},$$  \hspace{1cm} (7)

propagating either forward (+) or backward (−). However, if $\epsilon(u) = 0$, it is strongly unstable. A non-periodic solution of (5) with $P = 0$ is a nonlinear evanescent wave that forms a standing, trapped soliton. In low-RL case, one needs a small detuning from crossover point, $\delta \equiv \nu - 1 \ll 1$, to attain the effect at low laser intensity, $u^2 \ll 1$, so that the the dielectric constant (3) is Kerr-like and small: $\epsilon_{\text{trav}} \approx -2\delta + u^2/2$, $|\epsilon_{\text{trav}}| \ll 1$. A full solution of (5) with $P = 0$ and $u \rightarrow 0$ at $\zeta \rightarrow \infty$ yields then a standing soliton with a familiar intensity profile:

$$u^2 = 8\delta/\cosh^2[(\zeta - \zeta_{\text{pk}})/\sqrt{2}\delta]$$  \hspace{1cm} (8)

where the peak location $\zeta_{\text{pk}}$ is an integration constant. For an arbitrary frequency, $\nu < 1$, the soliton peak intensity is $u_{\text{trav}}^2 = 4(1 - \nu^2)/\nu^3$, instead of $8\delta$ as in (8);
dealing with apriori unknown number of multi-solutions. Besides, the latter one is very unreliable when
This provides a very fast numerical simulation of solution with conditions set at two boundaries.

vs \( u \)

\[ \epsilon (u_{pk}^2) = (1 - \nu^2)/(2 - \nu^2) \geq 0. \] When \( \nu^2 < 1/2 \), it is a strongly-RL soliton, \( u_{sol}^2 \gg 1 \), and its peak narrows down to a half-wave: \( u^2 \approx u_{sol}^2 \cos^2 (\zeta - \zeta_{pk}) \) at \( u^2 > 1 \).

In a finite layer one has a mix of standing/evanescent and traveling waves, with \( u_{min}^2 = u_{ex}^2 = P > 0 \). A full integration of (5) with nonlinearity \( \epsilon_{RL} \approx -2 \delta + u^2/2 \) yields then elliptic integrals of imaginary argument of the first kind; more importantly, eq. (5) and its invariant avails themselves to detailed analysis. The numerical simulations are needed, however, to find a solution for (a) strongly-RL field [using (5) and its integrals], or (b) non-uniform plasma density in (5), or (c) plasma with absorption [eq. (4)]. It then is found by an "inverse propagation" procedure, whereby we essentially back-track the propagation from a purely traveling exit wave back to the entrance. One sets first a certain magnitude of \( u_{ex}^2 = P \), \( u_{ex}^2 = 0 \) at the exit, numerically computes an amplitude profile \( u(\zeta) \) back to the entrance and incident and reflected intensities \( u_{ex}^2 \) and \( u_{ex}^2 \) using (6), and then maps \( u_{ex}^2 \) and \( u_{ex}^2 \) vs incident intensity, \( u_{in}^2 \). A data set \( u_{in}^2 (P) \) and \( u_{ex}^2 (P) \) for any given \( P \) is found then with a single run, vs a so called multi-shooting commonly used in search of solution with conditions set at two boundaries. This provides a very fast numerical simulation vs multi-shooting; besides, the latter one is very unreliable when dealing with apriori unknown number of multi-solutions.

For a fully-RL simulation with a \( L = 10 \lambda \), where \( L \) is the layer thickness, Fig. 1 show the emergence of large number, \( N_{hs} \), of huge hysteretic loops of the transmission (same as in reflection, not shown here), which bounces between full transparency (near the points touching an FT line) to nearly full reflection (near the points touching an envelope NFR). In general, \( N_{hs} = O(L/\lambda) \). In an unbound plasma, the solution of (5) with a traveling component, \( P > 0 \), is a spatially periodic and positively defined, with the intensity, \( u^2(\zeta) \), bouncing between two limits, \( u_{ex}^2 \), and \( u_{pk}^2 \). If \( P/16 \ll \delta^2 \ll 1 \), we have

\[ P = u_{ex}^2 \leq u^2 \leq u_{pk}^2 \approx 8 \delta + P/2 \delta \] (9)

i. e. the peaks are relatively large, \( u_{pk}^2 \gg u_{ex}^2 \) and form a train of well-separated quasi-solitons nearly coinciding with a standing soliton (8) of the peak intensity \( u_{pk}^2 \approx 8 \delta \). As \( P \) and \( u_{in}^2 \) increase, they grow larger and closer to each other. The spatial period, \( \Lambda \), of this structure is:

\[ \Lambda/\lambda \approx \ln (16 \delta/\sqrt{P})/(2 \pi \sqrt{2 \delta}) \] (10)

In a strongly RL case, \( P > 1 \), we have \( u_{pk}^2/P \approx 1 + (1 + P)^{-2} \), and \( \Lambda \approx \lambda/2 \), as for a standing, albeit inhibited wave in free space, while traveling wave emerges dominant, resulting in self-induced transparency.

Hysteric jumps occur when either valley or peak of the intensity profile coincide with an entrance, \( \zeta = 0 \). The valley, \( u_0^2 = u_{ex}^2 \), marks an off-jump in Fig. 1, and the peak, \( u_0^2 = u_{pk}^2 \), an on-jump. Suppose that in a layer of \( L > \lambda/(2 \pi \sqrt{2 \delta}) \), the incident intensity \( u_{in}^2 \) is ramped up from zero. When \( u_{in}^2 < 2 \delta \), Fig. 1, point 1, the amplitude is almost exponentially decaying, \( \zeta = 0 \), as \( u \approx 2 u_{in} \exp (-\sqrt{2 \delta} \zeta) \), i. e. is a nearly-linear evanescent wave, curve 1 in Fig. 1 inset; the layer is strongly refractive, and the transmission is low. As \( u_{in}^2 \) increases, the front end of that profile swells up, becoming a semi-bell-like curve, close to (8) with \( \zeta_{pk} \approx 0 \). With further slight increase of pumping, it gets unsustainable, and the field configuration has to jump up to the next stable branch of excitation, whereby it forms a steady R-wave at the back of the layer. If after that \( u_{in}^2 \) is pulled down adiabatically slow, the R-wave moves to the middle of the layer (Fig. 1, point 2, curve 2 in the inset). Finally, when it is exactly at the midlayer (Fig. 1, point 4, curve 4 in the inset), both valleys are at the borders of the layer, the pumping is nearly minimal to support an R-wave; below it the profile is unsustainable again, and the system jumps down to a regular nearly-evanescent wave and almost full reflection.

At this remarkable point, the layer is fully transparent, i. e. all the (very low) incident power is transmitted through, while a giant R-wave of peak intensity \( (u_{in})_{pk}^2 \approx 2 \delta \) inside the layer is sustained by a tiny incident power, \( (u_{in})_{min}^2 \). If \( L/2 \delta /N \lambda > 1 \), the contrast ratio – essentially a nonlinear resonator’s finesse, \( Q \), is

\[ Q = \frac{(u_{in})_{pk}^2}{(u_{in})_{min}^2} \approx \frac{\exp (2 \pi L \sqrt{2 \delta}/N \lambda)}{128 \delta} \gg 1 \] (11)

where \( N \) is the number of R-waves in a layer; the one with \( N = 1 \) occurs after the first jump-up. In the example for Fig. 1 (\( \delta = 0.02, L = 10 \lambda \)), \( Q \sim 10^7 \). In semi-infinite plasma, \( Q \) is limited by absorption, see below. It also

FIG. 1: Hysteretic transmission of light through a plasma layer of thickness \( L \). FT and NTR are full transparency and near total reflection limits. Points 1, 2, 3 mark a linear evanescent wave, 1-st, and 2-nd upper stable states respectively, and 4 – a R-wave sustained by a very low pumping. Arrows indicate direction of jumps within the lowest hysteretic loop. Inset: spatial amplitude profiles of waves corresponding to points 1-4 in the main plot.

develops oscillations due to rising standing wave, which eventually becomes a train of trapped R-waves, the last one being a quasi-soliton close to (8), and then vanishes exponentially. Reducing $\alpha$ pushes that last R-wave further back, but does not extinguish retro-reflection from the R-wave train at the crossover area deep inside plasma, keeping the condition $u \rightarrow 0$ at $\zeta \rightarrow \infty$.

Lab observation of the phenomena in plasma could be set up with e.g. jets of gas irradiated by a powerful CO$_2$ laser, with a gas density controlled to reach a crossover point at $\lambda_{CO_2} \approx 10\mu m$; the recent experiments [12] may actually indicate R-waves released by plasma expansion. This process may be naturally occurring in astrophysical environment in plasma sheets expelled from a star (e.g. the Sun); part of the star’s radiation spectrum below the initial plasma frequency is powerful enough to penetrate into the layer and be trapped as R-waves. When the layer expands, they can be released as a burst of radiation. It is also conceivable that the R-waves trapping and consequent release may be part of the physics of ball-lighting subjected to a powerful radiation emitted by the main lighting discharge in mw and far infrared domains. The R-waves might be used e.g. for laser fusion to deposit laser power much deeper into the fusion pallets; or for heating the ionosphere layers by a powerful rf radiation.

In conclusion, optical multi-hysteresises may emerge in a plasma near critical plasma frequency due to fundamental relativistic mass-effect of electrons. They may result in huge trapped, or standing rogue waves with the intensity greatly exceeding that of pumping radiation.

This work is supported by the US AFOSR.

[1] A. E. Kaplan, JETP Lett. 24, 114 (1976); also Sov. Physics JETP 45, 896 (1977).
[2] P. W. Smith, et al, Appl. Phys. Lett. 35 846 (1979); also IEEE JQE, 17 846 (1981); P. W. Smith and W. J. Tomlinson, IEEE JQE, 20 30 (1984); S. De Nicola, et al, Appl. Phys. B 49, 441 (1989).
[3] G. A. Mourou, T. Tajima, S. V. Bulanov, Rev. Mod. Phys., 78, 309 (2006)
[4] W. Chen and D. L. Mills, Phys. Rev. Lett. 58, 160 (1987), J. E. Sipe and H. G. Winful, Opt. Lett. 13, 132 (1988), D. N. Christodoulides and R. I. Joseph, Phys. Rev. Lett. 62, 1746 (1989); B. J. Eggleton, et al, ibid, 76, 1627 (1996); D. Mandelik, et al, ibid, 90, 053902 (2003).
[5] O. Zobay, S. Potting, P. Meystre, E. M. Wright, Phys. Rev. A 59, 643 (1999)
[6] A. E. Kaplan, Phys. Rev. Lett. 48, 138 (1982); ibid, 56, 456 (1986); A. E. Kaplan and P. L. Shkolnikov, Phys. Rev. Lett. 88, 074801, (2002).
[7] G. Gabrielse, H. Dehmelt, and W. Kells, Phys. Rev. Lett. 54, 537 (1985).
[8] R. J. Noble, Phys. Rev. A 32, 460 (1985); B. M. Ashkinadze and V. I. Yudson, Phys. Rev. Lett. 83, 812 (1999); G. Shvets, ibid, 93, 195004 (2004).
[9] B. Y. Zeldovich, Brief Comm, Lebedev Inst, 2, 20 (1970).
[10] Weakly nonlinear Fabri-Perot resonator has been used to first demonstrate optical bistability (first-order hysteresis), H. M. Gibbs, S. L. McCall, and T. N. C. Venkatesan, Phys. Rev. Lett. 36, 1135 (1976); H. M. Gibbs, Optical Bistability (Academic Press, New York, 1985).

[11] This suggests an explanation of the hysteresis lacking in

[2b]: an artificial nonlinearity used there was due to small dielectric particles suspended in a liquid and had large dissipation because of strong light scattering, vs the experiments [2a] and [2c] with nearly transparent fluids.

[12] P. L. Shkolnikov, private communication.