

Classical conformal blocks and isomonodromic deformations

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1. Introduction

The classical limit of conformal field theories is of interest for various reasons. It gives the link between a Lagrangian description of a CFT and the abstract representation-theoretic definition of its correlation functions provided by the bootstrap approach. It is also crucial for understanding several aspects of the geometry encoded in the correlation function of conformal field theory. This is relevant in particular for various models of holographic correspondences between two-dimensional CFT and three-dimensional quantum gravity investigated in the context of the AdS$_3$–CFT$_2$-correspondence.

We are here going to demonstrate that conformal field theory is related to the isomonodromic deformation problem in the limit $c \to \infty$. The Schlesinger system describes monodromy preserving deformations of $2 \times 2$ first order matrix differential equations on $\mathbb{P}^1$ with $n$ regular singularities. It has an alternative description known as the Garnier system describing the isomonodromic deformations of the second order ODE naturally associated to the first order matrix differential equation. We refer to [IKSY] for a review and further references. It will be shown that one may describe the leading asymptotics of Virasoro conformal blocks with a suitable number of insertions of degenerate representations in terms of the generating function for a change of coordinates between two natural sets of Darboux coordinates for the Garnier system. One set of coordinates is natural for the Hamiltonian formulation of the Garnier system [IKSY], the other coordinates will be called complex Fenchel-Nielsen coordinates parameterising the space of monodromy data of the differential equation on $C_{0,n} = \mathbb{P}^1 \setminus \{z_1, \ldots, z_n\}$.

The results of this paper characterise the leading classical asymptotics of Virasoro conformal
blocks completely, and clarify in which sense conformal field theory represents a quantisation of the isomonodromic deformation problem.

2. The Garnier system

2.1 Basic definitions

The Garnier system describes monodromy preserving deformations of the differential equations
\[(\partial_y^2 + t(y))\psi(y) = 0\]
with \(t(y)\) of the form
\[t(y) := \sum_{r=1}^{n} \left( \frac{\delta_r}{(y-z_r)^2} - \frac{H_r}{y-z_r} \right) - \sum_{k=1}^{n-3} \left( \frac{3}{4(y-u_k)^2} - \frac{v_k}{y-u_k} \right).\]  \tag{2.1}

The differential equation \((\partial_y^2 + t(y))\psi(y) = 0\) has regular singular points at \(y = z_r\) and \(y = u_k\). The parameters \(\delta_r\) will be fixed once and for all. The singular points at \(y = u_k\) are special. They are called apparent singularities if the parameters \((u_r, v_r, H_r)\) are not independent but related through the constraints
\[v_k^2 + \dot{t}_k(u_k) = 0, \quad \dot{t}_k(u_k) := \lim_{y \to u_k} \left( t(y) + \frac{3}{4(y-u_k)^2} - \frac{v_k}{y-u_k} \right), \tag{2.2}\]

These constraints imply that the monodromy around \(y = y_k\) is \(-i\text{id}\). Indeed, having monodromy \(-i\text{id}\) is easily seen to be equivalent to the fact that there exists a solution \(\psi\) of \((\partial_y^2 + t(y))\psi(y) = 0\) which has the form \(\psi(y) = \exp \left( -\frac{3}{4} \int y \, dy^2 \eta'(y') \right)\), with \(\eta(y)\) of the local form \(\eta(y) = \sum_{l=0}^{\infty} (y-u_k)^{l-1} \eta_l\) satisfying the Ricatti equation \(t(y) = -\frac{3}{4} \eta^2 + \frac{1}{2} \eta'\). The Ricatti equation determines the coefficients \(\eta_l\) recursively in terms of the expansion coefficients of \(t(y) = \sum_{l=0}^{\infty} t_l(y-u_k)^{l-2}\). When \(t_0 = -\frac{3}{4}\) one finds the relation \(v^2 + \dot{t}_k(u_k) = 0\) as necessary and sufficient condition for the existence of a solution to the recursion relations following from the Ricatti equation.

In order to define the Garnier system we will choose the number of apparent singularities to be \(d = n-3\). More general values of \(d\) will be discussed later. We shall furthermore assume that the differential equation \((\partial_y^2 + t(y))\chi(y) = 0\) is regular at \(y = \infty\), which implies
\[\sum_{r=1}^{n} z_r^l (z_r H_r - (l + 1) \delta_r) - \sum_{k=1}^{n-3} u_k^l (u_k v_k - (l + 1) \frac{3}{4}) = 0. \tag{2.3}\]

where \(l = -1, 0, 1\). The constraints (2.3) determine three of the \(H_r\), in what follows usually chosen to be \(H_n, H_{n-1}\) and \(H_{n-2}\) in terms of \(H_r, r = 1, \ldots, n-3\). Equations (2.2) can then be solved allowing us to express \(H_r\) as a function \(H_r = H_r(u, v, z)\), of \(u = (u_1, \ldots, u_{n-3})\), \(v = (v_1, \ldots, v_{n-3})\) and \(z = (z_1, \ldots, z_n)\). The “potential” \(t(y)\) thereby gets determined as a function \(t(y) = t(y|u, v, z)\).
It can be shown [Ok, IKSY] that the Hamiltonian equations of motion
\[
\frac{\partial u_r}{\partial z_s} = \frac{\partial H}{\partial v_r}, \quad \frac{\partial v_r}{\partial z_s} = -\frac{\partial H}{\partial u_r}.
\] (2.4)
ensure that the monodromy of the differential equation \((\partial_y^2 + t(y))\chi(y) = 0\) stays constant under variations of the parameters \((u, v)\) satisfying (2.4). The coordinates \((u, v)\) are Darboux coordinates for the natural symplectic structure of the Garnier system.

### 2.2 Relation to the Schlesinger system

More widely known than the Garnier system may be the Schlesinger system describing isomonodromic deformations of holomorphic \(sl_N\)-connections of the form \(\partial_y - A(y)\) on \(C_{0,n} = \mathbb{P}^1 \setminus \{z_1, \ldots, z_n\}\), with matrix-valued functions \(A(y)\) of the form
\[
A(y) = \sum_{r=1}^{n} \frac{A_r}{y - z_r}.
\] (2.5)
We will assume that \(A_1, \ldots, A_n\) satisfy \(\sum_{k=1}^{n} A_k = 0\). Allowing that the residues \(A_r\) depend on the parameters \(Z = (z_1, \ldots, z_n)\) in a suitable way, one may ensure that the monodromy of \(\partial_y - A(y)\) does not depend on \(Z\). The Schlesinger system is the system of nonlinear partial differential equations describing how to cancel variations of \(z_r\) by corresponding variations of the residues \(A_s\), \(s = 1, \ldots, n\).

The Garnier system is nothing but the Schlesinger system for \(N = 2\) in disguise. The relation between these two dynamical systems is found by representing the holomorphic connection \(\partial_y - A(y)\) containing the dynamical variables of the Schlesinger system in the form \(g^{-1}(\partial_y - B(y))g\), with \(B(y) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}\). It is straightforward to find a matrix function \(g = g(y)\) relating \(B(y)\) to \(A(y) = \begin{pmatrix} A_0 & A_+ \\ A_- & A_0 \end{pmatrix}\) in this way, provided one allows \(g(y)\) to have square-root branch points at the zeros of \(A_-\). It is furthermore straightforward to show that the function \(t(y)\) representing the only non-constant matrix element of \(B(y)\) will have a singularity of the form
\[
t(y) = -\frac{3}{4(y-u)^2} + \frac{v}{y-u} + \hat{t}(u) + O((u-v)^1),
\] (2.6)
near each simple zero \(u\) of \(A_-(y)\). The coefficients \(v\) and \(\hat{t}(u)\) appearing in the Laurent expansion (2.6) must satisfy the relation \(v^2 + \hat{t}(u) = 0\), which is the necessary and sufficient condition for the solutions to \((\partial_y^2 + t(y))\psi = 0\) to have monodromy proportional to \(-1\) around \(y = u\). Denoting the \(n-3\) zeros \(A_-(y)\) generically has by \(u = (u_1, \ldots, u_{n-3})\) and the corresponding residues by \(v = (v_1, \ldots, v_{n-3})\) one recovers exactly the form of the function \(t(y)\) considered in the theory of the Garnier system.

The gauge transformation from \(A(y)\) to \(B(y)\) described above defines a map from the Schlesinger system to the Garnier system. It is known that this map relates the natural symplectic structures [DM].
2.3 Complex-Fenchel-Nielsen coordinates for moduli spaces of flat connections

The monodromy data represent the conserved quantities which remain constant in the Hamiltonian flows of the Garnier system, by definition. The goal of this subsection is to introduce useful coordinates for the space of monodromy data.

Holonomy map and Riemann-Hilbert correspondence between flat connections $\partial y - A(y)$ and representations $\rho : \pi_1(C_{0,n}) \to \text{SL}(2, \mathbb{C})$ relate the moduli space $\mathcal{M}_{\text{flat}}(C_{0,n})$ of flat sl$_2$-connections on $C_{0,n}$ to the so-called character variety $\mathcal{M}_{\text{char}}(C_{0,n}) = \text{Hom}(\pi_1(C_{0,n}), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$. One set of useful sets of coordinates for $\mathcal{M}_{\text{flat}}(C_{0,n})$ is given by the trace functions $L_\gamma := \text{tr} \rho(\gamma)$ associated to simple closed curves $\gamma$ on $C_{0,n}$.

Minimal sets of trace functions that can be used to parameterise $\mathcal{M}_{\text{flat}}(C_{0,n})$ can be identified using pants decompositions. A pants decomposition is defined by cutting $C_{0,n}$ along $n - 3$ simple non-intersecting closed curves $\gamma_k$. Each curve $\gamma_k$ separates two pairs of pants, the union of which will be a four-holed sphere $C_{0,4}^k$. As $\dim(\mathcal{M}_{\text{flat}}(C_{0,n})) = 2(n - 3)$ it suffices to introduce two coordinates for the flat connections on each $C_{0,4}^k$.

Let us therefore restrict attention to the case $n = 4$ in the following. Conjugacy classes of irreducible representations of $\pi_1(C_{0,4})$ are uniquely specified by seven invariants

$$
L_a = \text{Tr} M_a = 2 \cos 2\pi m_a, \quad a = 1, \ldots, 4, \quad (2.7a)
$$

$$
L_s = \text{Tr} M_1 M_2, \quad L_t = \text{Tr} M_1 M_3, \quad L_u = \text{Tr} M_2 M_3, \quad (2.7b)
$$

generating the algebra of invariant polynomial functions on $\mathcal{M}_{\text{char}}(C_{0,4})$. The monodromies $M_r$ are associated to the curves $\gamma_r$ depicted in Figure 1. These trace functions satisfy the quartic equation

$$
L_1 L_2 L_3 L_4 + L_s L_t L_u + L_1^2 + L_2^2 + L_3^2 + L_4^2 =
$$

$$
= (L_1 L_2 + L_3 L_4) L_s + (L_1 L_3 + L_2 L_4) L_t + (L_2 L_3 + L_1 L_4) L_u + 4. \quad (2.8)
$$

The affine algebraic variety defined by (2.8) is a concrete representation for the character variety of $C_{0,4}$. For fixed choices of $m_1, \ldots, m_4$ in (2.7a) equation (2.8) describes the character variety as a cubic surface in $\mathbb{C}^3$.

Figure 1: Basis of loops of $\pi_1(C_{0,4})$ and the decomposition $C_{0,4} = C_{0,3}^L \cup C_{0,3}^R$. 

This surface admits a parameterisation in terms of coordinates \((\lambda, \kappa)\) of the form

\[
\begin{align*}
L_s &= 2 \cos 2\pi \lambda, \\
L_t &= C^+_t(\lambda) e^{i\kappa} + C^0_t(\lambda) + C^-_t(\lambda) e^{-i\kappa}, \\
L_u &= C^+_u(\lambda) e^{i\kappa} + C^0_u(\lambda) + C^-_u(\lambda) e^{-i\kappa},
\end{align*}
\] (2.9)

where

\[
\begin{align*}
C^\pm_u(\lambda) &= -4 \prod_{s = \pm 1} \sin \pi(\lambda + s(m_1 \mp m_2)) \sin \pi(\lambda + s(m_3 \mp m_4)), \tag{2.10a} \\
C^0_u(\lambda) &= 2 \left[ \cos 2\pi m_2 \cos 2\pi m_3 + \cos 2\pi m_1 \cos 2\pi m_4 \right] \\
&\quad - 2 \cos 2\pi \lambda \left[ \cos 2\pi m_1 \cos 2\pi m_3 + \cos 2\pi m_2 \cos 2\pi m_4 \right]. \tag{2.10b}
\end{align*}
\]

together with similar formulae for \(C^k_t, k = \pm, 0\).

Using pants decompositions as described above one may define trace coordinates \(L_{k,s}, L_{k,t}\) and \(L_{k,u}\) for each four-holed sphere \(C_{0,4}\) defined above. In this way one may define a pair of coordinates \((\kappa_k, \lambda_k)\) associated to each cutting curve \(\gamma_k, k = 1, \ldots, n - 3\). Taken together, the tuples \(k = (\kappa_1, \ldots, \kappa_{n-3})\) and \(l = (\lambda_1, \ldots, \lambda_{n-3})\) form a system of coordinates for \(\mathcal{M}_{\text{flat}}(C)\). It is known that the coordinates \((k, l)\) are a set of Darboux coordinates for the moduli space \(\mathcal{M}_{\text{flat}}(C)\) of flat SL(2, \(\mathbb{C}\))-connections on \(C_{0,n}\) [NRS], bringing the natural symplectic structure on this space to the simple form

\[
\Omega = \frac{1}{4\pi} \sum_{r=1}^{n-3} d\kappa_r \wedge d\lambda_r. \tag{2.11}
\]

We note that the trace functions \(L_k = 2 \cos(2\pi \lambda_k)\) are globally well-defined, and that the Hamiltonian flows generated by the functions \(\lambda_k\) are linear in the variables \(\kappa_k\). One may therefore view the coordinates \((k, l)\) as action-angle variables making the integrable structure of the character variety manifest.

### 3. Classical limit of Virasoro conformal blocks

We are now going to explain how the Garnier system arises in the limit \(c \rightarrow \infty\) of certain Virasoro conformal blocks with degenerate field insertions.

#### 3.1 Conformal blocks with degenerate fields

Conformal blocks, the holomorphic building blocks of physical correlation functions in conformal field theories, can be defined as solutions to the conformal Ward identities [BPZ]. Our discussion will be brief, referring for the relevant background on conformal field theory to reviews such as [T17a]. Conformal blocks can be defined for all punctured Riemann surfaces,
with representations of the Virasoro algebra assigned to each puncture. We will consider highest weight representations $V_\alpha$ of the Virasoro algebra with central charge $c = 1 + 6Q^2$ with highest weight vector $e_\alpha$ satisfying $L_0 e_\alpha = \delta_{n,0} \Delta_\alpha e_\alpha$ for $n \geq 0$, where $\Delta_\alpha = \alpha(Q - \alpha)$. It will be convenient to represent the parameter $Q$ as $Q = b + b^{-1}$.

We will consider Virasoro conformal blocks on $C_{0,n-3+d+2}$ with generic representations $V_{\alpha+1/2b}$ assigned to the punctures at $z_r$, $r = 1, \ldots, d$. Representation $V_{\alpha_r}$ are assigned to the remaining punctures $z_r$, $r = d+1, \ldots, n$, and we will assume that $z_{n-2} = 1$, $z_{n-1} = 0$, $z_n = \infty$ to simplify some formulae. Degenerate representations $V_{-1/2b}$ are associated to $d$ punctures at $u_k$, $k = 1, \ldots, d$, and degenerate representations $V_{-b/2}$ are associated to the punctures at $y$ and $z_0$, respectively. The corresponding chiral partition functions satisfy the following differential equations

\[
\left( \frac{1}{b^2} \frac{\partial^2}{\partial y^2} + T(y) \right) Z(u, \hat{z}; y) = 0, \tag{3.12a}
\]

\[
T(y) := \sum_{r=0}^{n} \left( \frac{\Delta_r}{(y - z_r)^2} + \frac{1}{y - z_r} \frac{\partial}{\partial z_r} \right) - \sum_{k=1}^{d} \left( \frac{3b^{-2} + \frac{2}{y - u_k}}{4(y - u_k)^2} - \frac{1}{y - u_k} \frac{\partial}{\partial u_k} \right),
\]

\[
\left( b^2 \frac{\partial^2}{\partial u_k^2} + \tilde{T}_k(u_k) - \frac{3b^2 + 2}{4(u_k - y)^2} + \frac{1}{u_k - y} \frac{\partial}{\partial y} \right) Z(u, \hat{z}; y) = 0, \quad k = 1, \ldots, d, \tag{3.12b}
\]

\[
\tilde{T}_k(u_k) := \lim_{y \to u_k} \left( T(y) + \frac{3b^2 + 2}{4(y - u_k)^2} - \frac{1}{y - u_k} \frac{\partial}{\partial u_k} \right),
\]

\[
\sum_{r=0}^{n} y^l \left( z_r \frac{\partial}{\partial z_r} + (l + 1) \Delta_r \right) + \sum_{k=1}^{n-3} u_k^l \left( u_k \frac{\partial}{\partial u_k} + (l + 1) \Delta_{-b/2} \right) Z(u, \hat{z}; y) = 0, \quad l = -1, 0, 1, \tag{3.12c}
\]

using tuple notations $\hat{z} = (z_0, z_1, \ldots, z_{n-3})$, $u = (u_1, \ldots, u_d)$. Equation (3.12a) reflects the decoupling of the null-vector in the Verma-module associated to the representation $V_{-b/2}$ assigned to the point $y$, while (3.12b) is equivalent to the decoupling of the null-vectors in the representations $V_{-1/2b}$ associated to the punctures $u_k$, $k = 1, \ldots, d$. Equation (3.12c) simply reflects the global $\text{SL}(2)$-invariance on the sphere. It turns out that the insertion of the representations $V_{-b/2}$ modifies the conformal blocks only mildly in the limit $b \to 0$. We will use the representation at $y$ as a “probe”, exploiting the information provided by the associated differential equation, and the analytic continuation of its solutions. The representation at $z_0$ will only serve the task to define a convenient “base-point”.

### 3.2 Gluing construction of conformal blocks

Useful bases for the spaces of conformal blocks can be constructed by means of the gluing construction. This construction allows one to construct conformal blocks on arbitrary Riemann
surfaces \( C \) from the conformal blocks associated to the three-punctured spheres appearing in a pants decomposition of \( C \). For each of the simple closed curves used to define a given pants decomposition one has to specify a representation of the Virasoro algebra. The gluing of the conformal blocks on the pairs of pants to a conformal block on \( C \) is performed by summing over bases of the representations assigned to the cutting curves.

To be specific, let us start from \( C_{0,n} = \mathbb{P}^1 \setminus \{ z_1, \ldots, z_n \} \) with a fixed pants decomposition defined by cutting \( C_{0,n} \) along \( n - 3 \) non-intersecting simple closed curves \( \gamma_1, \ldots, \gamma_{n-3} \). Out of \( C_{0,n} \) with the given pants decomposition let us construct a \( n + d \)-punctured sphere by first cutting \( d \) sufficiently small non-intersecting discs \( D_k \) around \( z_k, k = 1, \ldots, d \), out of the pairs of pants appearing in the given pants decomposition for \( C_{0,n} \). Then glue twice-punctured discs \( D_k' \) back in such a way that the resulting surface has punctures at \( z_k \) and \( u_k, k = 1, \ldots, d \). The boundary of \( D_k' \) will be denoted by \( \mu_k \). In the pair of pants containing \( z_n \) let us finally replace a disc \( D_\infty \) around \( z_n \) by a three times punctured disc containing \( z_n, z_0 \) and \( y \), with \( z_0 \) and \( y \) contained in a smaller disc \( D_0 \) inside of \( D_\infty \). The boundaries of \( D_\infty \) and \( D_0 \) will be denoted \( \nu_\infty \) and \( \nu_0 \), respectively. The result of this construction is a sphere with \( n + d + 2 \) punctures \( z_1, \ldots, z_n; u_1, \ldots, u_d; z_0, y \) and a fixed pants decomposition.

The conformal blocks resulting from the gluing construction are determined uniquely up to normalisation by the assignment of representations to the curves \( \gamma_r \) and \( \mu_k \) with \( r = 1, \ldots, n-3 \) and \( k = 1, \ldots, d \), as well as \( \nu_0 \) and \( \nu_\infty \). We will assign representations \( V_{Q+2i\nu_k} \) to the curves \( \gamma_r, r = 1, \ldots, n-3 \), and representations \( V_{\alpha_k} \) to the curves \( \mu_k, k = 1, \ldots, d \). To the remaining curves \( \nu_0 \) and \( \nu_\infty \) we will assign representations \( V_0 \) and \( V_\infty \), respectively.

The chiral partition functions of the conformal blocks defined in this way will be denoted \( Z(p, u, z; y) \) with \( p = (p_1, \ldots, p_{n-3}) \). \( Z \) depends holomorphically on all of its variables.

### 3.3 Classical limit of null vector decoupling equations

We will consider the limit \( b \to 0 \) with \( \delta_r = b^2 \Delta_r \) and \( \lambda_r = bp_r \) kept fixed. One may notice that the differential equations (3.12) can be solved in the following form

\[
Z(b^{-1}, u, z; y) = e^{-\frac{2}{b}} W(1, u, z) \chi'(y|1, u, z) \chi''(z_0|1, u, z)(1 + O(b^2)),
\]

(3.13)

where \( z = (z_1, \ldots, z_{n-3}) \) and \( l = (\lambda_1, \ldots, \lambda_{n-3}) \). This ansatz will solve equation (3.12a) provided \( \chi' \) and \( \chi'' \) are two solutions of the differential equation \((\partial_y^2 + t(y))\chi(y) = 0\), with \( t(y) \) of the form (2.1), where the parameters \( v_r \) and \( H_r \) are obtained from \( W(1, u, z) \) as follows

\[
v_r = -\frac{\partial}{\partial u_r} W(1, u, z), \quad H_r = \frac{\partial}{\partial z_r} W(1, u, z).
\]

(3.14)

The equations (3.12b) furthermore imply that the parameters \( (u_r, v_r, H_r) \) are not independent but related through the constraints (2.2). Equations (3.12c) finally reproduce (2.3).
If \( d = n - 3 \), one has just as many equations as one needs to determine \( H_r, r = 1, \ldots, n \) as functions of \( u, v \), as was done in the definition of the Garnier system. This is how the kinematics of the Garnier system is recovered from the classical limit of Virasoro conformal blocks in this case, as first observed in [T10]. Similar observations have been exploited in [LLNZ]. The cases with \( d < n - 3 \) will be discussed later.

3.4 Verlinde loop operators

Useful additional information is provided by the action of the Verlinde loop operators studied in [AGGTV, DGOT] on spaces of conformal blocks, see [T17a, Section 2.7] for a brief review. The basic idea behind the definition of the Verlinde loop operators is as follows. Given a conformal block \( f \) on surface \( C \), there is a canonical way to define a conformal block \( f' \) on a surface \( C' \) having an extra puncture \( y_0 \) with vacuum representation assigned to \( y_0 \), and vice-versa. A conformal block \( f \) on \( C \) similarly defines a conformal block \( f'' \) on a surface \( C'' \) obtained by replacing a disc in \( C \) by a disc \( D_0 \) containing two punctures at \( y \) and \( z_0 \) with representations \( V_{-b/2} \) assigned to both punctures, and vacuum representation assigned to the boundary of \( D_0 \). If \( f \) is a conformal block defined by the gluing construction one may use the null vector decoupling equations (3.12a) to compute the analytic continuation \( \gamma.f'' \) of \( f'' \) along contours \( \gamma \) starting and ending at \( y \). The contribution to \( \gamma.f'' \) which has vacuum representation assigned to the boundary of \( D_0 \) may be canonically identified with a conformal block \( L_\gamma f \) on the original surface \( C \). This defines an operator \( L_\gamma \) on the space of conformal blocks associated to the surface \( C \). The algebra generated by the operators \( L_\gamma \) is a non-commutative deformation of the Poisson algebra of trace functions \( L_\gamma \) on \( M_{\text{flat}}(C) \) [DGOT, TV].

We had previously associated trace functions \( L_{k,i} \), \( i = s, t, u \) to each of the curves \( \gamma_k \) defining the pants decomposition of \( C_{0,n} \). The Verlinde loop operators associated to the contours defining \( L_{k,i} \), \( i = s, t, u \) will be denoted by \( L_{k,i} \), for \( i = s, t, u \) and \( k = 1, \ldots, n - 3 \), respectively.

We will need the results of [DGOT] for the Verlinde loop operators \( L_{k,i} \). In the case of the operators \( L_{k,s} \) a simple diagonal result was found\(^1\)

\[
L_{k,s} \mathcal{Z}(p, u, \hat{z}; y) = 2 \cosh(2\pi bp_k) \mathcal{Z}(p, u, \hat{z}; y). \tag{3.15a}
\]

The expressions for the operators \( L_{k,i} \) are more complicated for \( i = t, u \). They take the form

\[
L_{k,i} = \sum_{\nu = -1}^{1} C_{k,i}^\nu(p) e^{\nu ib \partial_p \theta}. \tag{3.15b}
\]

Explicit formulae for the coefficients \( C_{k,i}^\nu(p) \) can be found in [AGGTV, DGOT].

\(^1\)Comparing to [DGOT] we changed the definition of \( L_{k,i} \) slightly to absorb a factor of \( 2 \cos(\pi(1 + b^2)) \).
3.5 Classical limit of Verlinde loop operators

Using the results of the explicit calculations of these operators from [AGGTV, DGOT] we will now identify the variables $\lambda_k = b p_k$ and $\kappa_k = 4 \pi i \partial_{\lambda_k} W$ with complex Fenchel-Nielsen coordinates on the character variety $\mathcal{M}_{\text{char}}(C_{0,n})$ in the limit $b \to 0$ considered above.

In this limit one may compute the Verlinde loop operators in two different ways. One may, on the one hand, use the factorisation (3.13) in order to show that the classical limit of the Verlinde loop operators can be identified with traces of the monodromies of the differential operator $\partial^2_y + t(y)$. This can be compared to the classical limit of the explicit formulae (3.15) for the Verlinde loop operators which turn out to be identical to the expressions for the trace functions given in Section 2.3 when the conformal blocks are normalised appropriately. In this way it may be shown that $W(l, u, z)$ satisfies

$$\kappa_r = -4 \pi i \frac{\partial}{\partial \lambda_r} W(l, u, z), \quad r = 1, \ldots, n - 3,$$

(3.16)

with $\kappa_r = \kappa_r(l, u, z)$ being the value of the coordinate defined via (2.9).

3.6 Classical limit of conformal blocks as generating function

Recall that that the coordinates $(k, l)$ are a set of Darboux coordinates for the moduli space $\mathcal{M}_{\text{flat}}(C)$ of flat $\text{SL}(2, \mathbb{C})$-connections on $C_{0,n}$. Given the function $W(l, u, z)$ one may invert the relations (3.16) to define $u = u(k, l; z)$, and then define $v = v(k, l; z)$ using (3.14). This is just the standard procedure to define a canonical transformation in Hamiltonian mechanics in terms of generating functions. The coordinates $(u, v)$ defined in this way will therefore be another set of Darboux coordinates for the natural symplectic structure on $\mathcal{M}_{\text{flat}}(C)$ which is related to the natural symplectic structures of the Garnier system as

$$\Omega = \frac{1}{i} \sum_{r=1}^{n-3} dv_r \wedge du_r = \frac{1}{4 \pi} \sum_{r=1}^{n-3} d\kappa_r \wedge d\lambda_r.$$

(3.17)

The Riemann-Hilbert correspondence defines a $z$-dependent change of variables from $(l, k)$ to $(u, v)$. Fixing $(l, k)$ by imposing the condition of constant monodromy defines commuting flows of the variables $(u, v) \equiv (u(z), v(z))$. The Hamiltonian form (2.4) of the differential equations governing these flows can be found as follows. We are considering a canonical transformation in a non-autonomous Hamiltonian system generated by the function $W$. Having dynamics in the variables $(u, v)$ described in Hamiltonian form (2.4) is equivalent to having dynamics in the variables $(l, k)$ generated by the Hamiltonian $\tilde{H}_r = H_r - \partial_{z_r} W$. For describing isomonodromic deformations we choose $\tilde{H}_r \equiv 0$, implying that the functions $H_r$ related

---

2Changing the normalisation of the three-point conformal blocks will change the form of the coefficients $C^{(p)}_{k,i}(p)$ in (3.15b). There exists a choice of normalisation reproducing the corresponding coefficients in (2.9).
to \( \mathcal{W}(l, u, z) \) via (3.14) are the Hamiltonians to be used when representing the isomonodromic flows in the Hamiltonian form (2.4). The fact that the functions \( H_r \) defined from \( \mathcal{W} \) in (3.14) must coincide with the Hamiltonians of the Garnier system follows from the observation that both are uniquely determined by the system of linear equations (2.2). We recover, in an independent way, the fact that the coordinates \((u, v)\) are Darboux coordinates for the natural Poisson structure of the Garnier system.

In this way we have fully reproduced the Hamiltonian representation of the Garnier system describing the isomonodromic deformations of the differential equation \((\partial_y^2 + t(y))\chi(y) = 0\) with \(t(y)\) of the form (2.1) from conformal field theory.

### 4. Comparison with similar results

We would here like to compare our results to some known results of a similar nature.

#### 4.1 Genus zero analog of Kawai’s theorem

To begin with, let us discuss the cases where the number of degenerate fields \(d\) is less than \(n-3\). The classical limit can be analysed in the same way as before, the only change being a lower number of apparent singularities in the differential equation \((\partial_y^2 + t(y))\psi(y) = 0\). In this case we can only determine a subset of the parameters \(H_r\) using the constraints (2.2), determining, for example, \(H_1, \ldots, H_d\) as function of the parameters \((u, v)\). The total number of independent variables in the differential equation is \(n-3+d\).

In the extreme case \(d = 0\) we do not have any apparent singularities, the only parameters left are the \(n-3\) independent variables \(H = (H_1, \ldots, H_{n-3})\). These parameters can be identified as coordinates on the cotangent fibres of the moduli space \(\mathcal{M}_{0,n}\) of \(n\)-punctured spheres \(C^\delta_{0,n}\) having conical singularities at \(z_r, r = 1, \ldots, n\) with deficit angles determined by the parameters \(\delta_r\) in (2.1) [TZ]. Together with the positions \(z_1, \ldots, z_{n-3}\) one gets a system of Darboux coordinates for the total space of the cotangent bundle \(T^*\mathcal{M}_{0,n}\).

The holonomy of the corresponding connection defines a map from from \(T^*\mathcal{M}_{0,n}\) to the character variety \(\mathcal{M}_{\text{char}}(C_{0,n})\). Parameterising points on \(\mathcal{M}_{\text{char}}(C_{0,n})\) by the complex Fenchel-Nielsen coordinates introduced in Section 2.3 one obtains a change of coordinates from \((z, H)\) to \((k, l)\).

In the case \(d = 0\) presently under consideration we will still find a semiclassical asymptotics of the form (3.13). However, the function \(\mathcal{W}\) characterising the leading term will now be \(u\)-independent, \(\mathcal{W} = \mathcal{W}(l, z)\). This function will satisfy the relations (3.16) and the second relation in (3.14), as before. These relations identify \(\mathcal{W}(l, z)\) as generating function for the change of coordinates from \((z, H)\) to \((k, l)\) in the sense of symplectic geometry. The existence of such a generating function shows that the change of coordinates defined by the holonomy map preserves the natural symplectic structures. A similar result was obtained in [Ka] for the
The Liouville action is the functional of $\varphi$ functions of the residues $H$ function $S$. One may note, on the other hand, that the holonomy of $t$ function $t$ function $W$ argue that the function $W$ functional. The metric of constant negative curvature $u$ odomric deformation flows when $u_k$ approach $z_k$, $k = 1, \ldots, n - 3$.

The function $W(1, z)$ can furthermore be used to characterise the spectrum of the $SL(2, \mathbb{C})$-Gaudin model $[T10, T17b]$. In the context of the Nekrasov-Shatashvili program relating supersymmetric gauge theories to integrable models similar relations have been proposed in $[NRS]$.

### 4.2 Relation to Liouville theory

The observations made in Section 4.1 are related to the results obtained in $[TZ]$ describing the Weil-Petersson symplectic form on the Teichmüller spaces $T_{0,n}$ in terms of the Liouville action functional. The metric of constant negative curvature $d^2 s = e^{2\varphi(y, \bar{y})} dy d\bar{y}$ on $C^\delta_{0,n}$ defines a function $t_u(y)$ of the form (2.3) with $d = 0$ via $t_u(y) = -(\partial_y \varphi_u)^2 + \partial^2_y \varphi_u$. The particular values of the residues $H_r$ that are found when $t(y) = t_u(y)$ will be denoted as $E_r$.

The Liouville action is the functional of $\varphi(y, \bar{y})$ having an extremum when $d^2 s = e^{2\varphi(y, \bar{y})} dy d\bar{y}$ has constant negative curvature. Evaluating the Liouville action at this extremum defines a function $S_L = S_L(z, \bar{z})$. The residues $E_r$ of $t_u(y)$ are related to $S_L$ as $E_r = \partial_{z_r} S_L(z, \bar{z})$ $[TZ]$. One may note, on the other hand, that the holonomy of $t_u(y) = -(\partial_y \varphi_u)^2 + \partial^2_y \varphi_u$ defines functions $(k_u(z, \bar{z}), l_u(z, \bar{z}))$. Based on the semiclassical limit of conformal field theory we will argue that the function $W_u(z, \bar{z})$ defined by restriction of $W(1, z)$ to $l = l_u(z, \bar{z})$,

$$W_u(z, \bar{z}) = W(1_u(z, \bar{z}), z), \quad (4.18)$$

satisfies

$$\partial_{z_r} \text{Re}(W_u(z, \bar{z})) = H_r(z, \bar{z}), \quad r = 1, \ldots, n - 3. \quad (4.19)$$

Indeed, the correlation functions of Liouville theory, $\langle \prod_{r=1}^n e^{2\alpha_r \varphi(z_r, \bar{z}_r)} \rangle$, can be decomposed into conformal blocks as $[ZZ, T01]$

$$\langle \prod_{r=1}^n e^{2\alpha_r \varphi(z_r, \bar{z}_r)} \rangle = \int d\mu(p) \ |Z(p, z)|^2. \quad (4.20)$$

In the semiclassical limit one may use (3.13). The integral over $p = (p_1, \ldots, p_{n-3})$ in (4.20) will be dominated by a saddle point determined by the condition that

$$\partial_{\lambda_k} \text{Re}(W_u(1, z))|_{1=1_u(z, \bar{z})} = 0, \quad k = 1, \ldots, n - 3. \quad (4.21)$$

This condition ensures that $W_u(z, \bar{z})$ defined in (4.18) satisfies (4.19). It follows that the Liouville action $S_L$ coincides with $\text{Re}(W_u)$ up to a constant. This observation is related to the characterisation of the spectrum of the $SL(2, \mathbb{C})$-Gaudin model in terms of the function $W$ $[T10, T17b]$.
5. Conclusions: CFT as quantisation of the Garnier system

The observations above relate conformal field theory to the quantisation of the isomonodromic deformation problem. In order to quantise the Garnier system one may start by observing that both (u, v) and (1, k) represent Darboux coordinates for the moduli spaces of flat connections, \([3,17]\). One may therefore consider two quantisation schemes in which \(u_r\) and \(\lambda_r\) both get represented as multiplication operators, whereas the operators associated to \(v_r\) and \(\kappa_r\) are \(v_r = b^2 \frac{\partial}{\partial u_r}\) and \(l_r = 4\pi \frac{b^2}{r} \frac{\partial}{\partial \chi_r}\), acting on suitable spaces of function \(\Psi(u)\) and \(\Phi(1)\), respectively.

The next step will be to quantize the Hamiltonians \(H_r\). We will define the quantum counterparts \(H_r\) of \(H_r\) as solutions to the following set of \(n\) constraints,

\[
\begin{align*}
\frac{b^2 \partial^2}{\partial u_k^2} + \tilde{t}_k(u_k) &= 0, \quad \tilde{t}_k(u_k) := \lim_{y \to u_k} \left( \frac{3 + 2b^2}{4(y - u_k)^2} - \frac{b^2}{y - u_k} \frac{\partial}{\partial u_k} \right), \\
\tilde{t}(y) := & \sum_{r=1}^{n} \left( \frac{b^2 \Delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r} \right) - \sum_{k=1}^{d} \left( \frac{3 + 2b^2}{4(y - u_k)^2} - \frac{b^2}{y - u_k} \frac{\partial}{\partial u_k} \right), \quad k = 1, \ldots, d, \\
\sum_{r=1}^{n} \lambda_r \left( z_r H_r - (l + 1)b^2 \Delta_r \right) + \sum_{k=1}^{n-3} \left( b u_k \frac{\partial}{\partial u_k} - (l + 1)b^2 \Delta_r \frac{1}{\Psi} \right) &= 0, \quad l = -1, 0, 1.
\end{align*}
\]

As in the classical case one may solve these constraints to define second order differential operators \(H_r\) in the variables \(u\) if \(d = n - 3\). There are additional terms of order \(b^2\) which ensure that \([H_r, H_s] = 0\) for all \(r, s = 1, \ldots, n\). It is then natural to require that the quantum Hamiltonians generate the evolution with respect to the “time” variables \(z_r\),

\[
\frac{b^2 \partial}{\partial z_r} \Psi(u; z) = H_r \Psi(u; z), \quad r = 1, \ldots, n.
\]

These equations are easily seen to be equivalent to the null vector decoupling equations satisfied by the chiral partition functions \(Z(p, u, z)\) of the conformal blocks on \(C_{0,2n-3}\) obtained from the conformal blocks introduced in Section 3.1 by removing the punctures at \(z_0\) and \(y\), as was first pointed out in [10]. Generalising [GT Section 5.2] it is possible to show that these equation define the series expansion of \(Z(p, u, z)\) associated to suitable gluing patterns uniquely, with exponents of the leading terms in the expansions specified in terms of the variables \(p\).

It had previously been observed [Re] that the Knizhnik-Zamolodchikov equations appearing in CFTs with affine Lie algebra symmetry can be interpreted as the time-dependent Schrödinger equations that would appear in the quantisation of the isomonodromic deformation problem. Our observations in Section 3.3 are close analogs of the results in [Re] for the Virasoro case. Both are related to each other through a variant of Sklyanin’s Separation of Variables method [11]. The consideration of the classical limit of the Verlinde loop operators in Section 3.5 adds the crucial other side of the coin needed to get a precise characterisation of the classical conformal blocks as generating functions.
The quantisation of this classical integrable system yields equations characterising the conformal blocks of the Virasoro algebra completely. Describing the conformal blocks in this way suggests to reinterpret the chiral partition functions $Z(p, u, z)$ as the wave-functions intertwining the representations for the quantised Garnier system in terms of functions $\Psi(u)$ and $\Phi(l)$ introduced above.

We may furthermore note that conformal field theory is related to the isomonodromic deformation problem in two limits, the limit $c \to \infty$ discussed here and the limit $c = 1$ considered in [ILTe], see [T17a] for a review. In the case $c = 1$ one may identify the isomonodromic tau-function with a Fourier-transformation of Virasoro conformal blocks. This is remarkable, and deserves to be better understood.

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