Deformations of Coxeter Hyperplane Arrangements

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Abstract

We investigate several hyperplane arrangements that can be viewed as deformations of Coxeter arrangements. In particular, we prove a conjecture of Linial and Stanley that the number of regions of the arrangement

\[ x_i - x_j = 1, \quad 1 \leq i < j \leq n, \]

is equal to the number of alternating trees on \( n + 1 \) vertices. Remarkably, these numbers have several additional combinatorial interpretations in terms of binary trees, partially ordered sets, and tournaments. More generally, we give formulae for the number of regions and the Poincaré polynomial of certain finite subarrangements of the affine Coxeter arrangement of type \( A_{n-1} \). These formulae enable us to prove a “Riemann hypothesis” on the location of zeros of the Poincaré polynomial. We also consider some generic deformations of Coxeter arrangements of type \( A_{n-1} \).

1 Introduction

The Coxeter arrangement of type \( A_{n-1} \) is the arrangement of hyperplanes given by

\[ x_i - x_j = 0, \quad 1 \leq i < j \leq n. \] (1.1)

This arrangement has \( n! \) regions. They correspond to \( n! \) different ways of ordering the sequence \( x_1, \ldots, x_n \).

In the paper we extend this simple, nevertheless important, result to the case of a general class of arrangements which can be viewed as deformations of the arrangement (1.1).
One special case of such deformations is the arrangement given by
\[ x_i - x_j = 1, \quad 1 \leq i < j \leq n. \] (1.2)

We will call it the Linial arrangement. This arrangement was first considered by N. Linial and S. Ravid. They calculated its number of regions and the Poincaré polynomial for \( n \leq 9 \). On the basis of this numerical data the second author of the present paper made a conjecture that the number of regions of (1.2) is equal to the number of alternating trees on \( n + 1 \) vertices (see [25]). A tree \( T \) on the vertices \( 1, 2, \ldots, n + 1 \) is alternating if the vertices in any path in \( T \) alternate, i.e., form an up-down or down-up sequence. Equivalently, every vertex is either less than all its neighbors or greater than all its neighbors. These trees first appeared in [10], and in [23] a formula for the number of such trees on \( n + 1 \) vertices was proved. In this paper we provide a proof of the conjecture on the number of regions of the Linial arrangement.

In fact, we prove a more general result for truncated affine arrangements, which are certain finite subarrangements of the affine hyperplane arrangement of type \( \tilde{A}_{n-1} \) (see Section 9). As a byproduct we get an amazing theorem on the location of zeros of Poincaré polynomials of these arrangements. This theorem says that in one case all zeros are real, whereas in the other case all zeros have the same real part.

The paper is organized as follows. In Section 2 we give the basic notions of hyperplane arrangement, number of regions, Poincaré polynomial, and intersection poset. In Section 3 we describe the arrangements we will be concerned with in this paper—deformations of the arrangement (1.1). In Section 4 we review several general theorems on hyperplane arrangements. Then in Section 5 we apply these theorems to deformed Coxeter arrangements. In Section 6 we consider a “semi-generic” deformation of the braid arrangement (the Coxeter arrangement of type \( A_{n-1} \)) related to the theory of interval orders. In Section 7 we study the hyperplane arrangements which are related, in a special case, to interval orders (cf. [25]) and the Catalan numbers. We prove a theorem that establishes a relation between the numbers of regions of such arrangements. In Section 8 we formulate the main result on the Linial arrangement. We introduce several combinatorial objects whose numbers are equal to the number of regions of the Linial arrangement: alternating trees, local binary search trees, sleek posets, semiacyclic tournaments. We also prove a theorem on characterization of sleek posets in terms of forbidden subposets. At last, in Section 9 we study truncated affine arrangements. We prove a functional equation for the generating function for the numbers of regions of such arrangements, deduce a formula for these numbers, and the theorem on the location of zeros of the characteristic polynomial.

2 Arrangements of Hyperplanes

First, we give several basic notions related to arrangements of hyperplanes. For more details, see [31, 16, 17].

A hyperplane arrangement is a discrete collection of affine hyperplanes in a vector space. We will be concerned here only with finite arrangements. Let \( \mathcal{A} \) be a finite
hyperplane arrangement in a real finite-dimensional vector space $V$. It will be convenient to assume that the vectors dual to hyperplanes in $A$ span the vector space $V^*$. Denote by $r(A)$ the number of regions of $A$, which are the connected components of the space $V - \bigcup_{H \in A} H$. We will also consider the number $b(A)$ of (relatively) bounded regions of $A$.

These numbers have a natural $q$-analogue. Let $A_C$ denote the complexified arrangement $A$. In other words, $A_C$ is the collection of the hyperplanes $H \otimes \mathbb{C}$, $H \in A$, in the complex vector space $V \otimes \mathbb{C}$. Let $C_A$ be the complement to hyperplanes of $A_C$ in $V \otimes \mathbb{C}$. Then one can define the Poincaré polynomial $\text{Poin}_A(q)$ of $A$ as

$$\text{Poin}_A(q) = \sum_{k \geq 0} \dim H^k(C_A, \mathbb{C}) q^k,$$

the generating function for the Betti numbers of $C_A$.

The following theorem, proved in the paper of Orlik and Solomon [16], shows that the Poincaré polynomial generalizes the number of regions $r(A)$ and the number of bounded regions $b(A)$.

**Theorem 2.1** We have $r(A) = \text{Poin}_A(1)$ and $b(A) = \text{Poin}_A(-1)$.

Orlik and Solomon gave a combinatorial description of the cohomology ring $H^*(C_A, \mathbb{C})$ (cf. Section 8.3) in terms of the intersection poset $L_A$ of the arrangement $A$.

The intersection poset is defined as follows: The elements of $L_A$ are nonempty intersections of hyperplanes in $A$ ordered by reverse inclusion. The poset $L_A$ has a unique minimal element $\hat{0} = V$. This poset is always a meet-semilattice for which every interval is a geometric lattice. It will be a (geometric) lattice if and only if $L_A$ contains a unique maximal element, i.e., the intersection of all hyperplanes in $A$ is nonempty. In fact, $L_A$ is a geometric semilattice in the sense of Wachs and Walker [28], and thus for instance is a shellable and hence Cohen-Macaulay poset.

The characteristic polynomial of $A$ is defined by

$$\chi_A(q) = \sum_{z \in L_A} \mu(\hat{0}, z) q^{\dim z},$$

(2.1)

where $\mu$ denotes the Möbius function of $L_A$ (see [24, Section 3.7]).

Let $d$ be the dimension of the vector space $V$. Note that it follows from the properties of geometric lattices [24, Proposition 3.10.1] that the sign of $\mu(\hat{0}, z)$ is equal to $(-1)^{d - \dim z}$.

The following simple relation between the (topologically defined) Poincaré polynomial and the (combinatorially defined) characteristic polynomial was found in [16]:

$$\chi_A(q) = q^d \text{Poin}_A(-q^{-1}).$$

(2.2)

Sometimes it will be more convenient for us to work with the characteristic polynomial $\chi_A(q)$ rather than the Poincaré polynomial.
A combinatorial proof of Theorem 2.1 in terms of the characteristic polynomial was earlier given by T. Zaslavsky in [31].

The number of regions, the number of (relatively) bounded regions, and, more generally, the Poincaré (or characteristic) polynomial are the most simple numerical invariants of a hyperplane arrangement. In this paper we will calculate these invariants for several hyperplane arrangements related to Coxeter arrangements.

3 Coxeter Arrangements and their Deformations

Let $V_{n-1}$ denote the subspace (hyperplane) in $\mathbb{R}^n$ of all vectors $(x_1, \ldots, x_n)$ such that $x_1 + \cdots + x_n = 0$. All hyperplane arrangements that we consider below lie in $V_{n-1}$. The lower index $n - 1$ will always denote dimension of an arrangement.

The braid arrangement or Coxeter arrangement (of type $A_{n-1}$) is the arrangement $A_{n-1}$ of hyperplanes in $V_{n-1} \subset \mathbb{R}^n$ given by

$$x_i - x_j = 0, \quad 1 \leq i < j \leq n. \quad (3.1)$$

It is clear that $\mathcal{A}$ has $r(A_{n-1}) = n!$ regions (called Weyl chambers) and $b(A_{n-1}) = 0$ bounded regions. Arnold [1] calculated the cohomology ring $H^*(C_{A_n}, \mathbb{C})$. In particular, he proved that

$$\text{Poin}_{A_{n-1}}(q) = (1 + q)(1 + 2q) \cdots (1 + (n-1)q). \quad (3.2)$$

In this paper we will study deformations of the arrangement (3.1), which are hyperplane arrangements in $V_{n-1} \subset \mathbb{R}^n$ of the following type:

$$x_i - x_j = a_{ij}^{(1)}, \ldots, a_{ij}^{(m_{ij})}, \quad 1 \leq i < j \leq n. \quad (3.3)$$

where $m_{ij}$ are nonnegative integers and $a_{ij}^{(k)} \in \mathbb{R}$.

One special case is the arrangement given by

$$x_i - x_j = a_{ij}, \quad 1 \leq i < j \leq n. \quad (3.4)$$

The following hyperplane arrangements of type (3.3) worth mentioning:
• The **generic arrangement** (see the end of Section 5) given by
  \[ x_i - x_j = a_{ij}, \quad 1 \leq i < j \leq n, \]
  where the \(a_{ij}\)'s are generic real numbers.

• The **semigeneric arrangement** \(G_n\) (see Section 6) given by
  \[ x_i - x_j = a_i, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n, \quad i \neq j, \]
  where the \(a_i\)'s are generic real numbers.

• The **Linial arrangement** \(L_{n-1}\) (see [25] and Section 8) given by
  \[ x_i - x_j = 1, \quad 1 \leq i < j \leq n. \]
  (3.5)

• The **Shi arrangement** \(S_{n-1}\) (see [22, 23, 25] and Section 9.2) given by
  \[ x_i - x_j = 0, \quad 1 \leq i < j \leq n. \]
  (3.6)

• The **extended Shi arrangement** \(S_{n-1,k}\) (see Section 9.2) given by
  \[ x_i - x_j = -k, -k + 1, \ldots, k + 1, \quad 1 \leq i < j \leq n. \]
  (3.7)

  where \(k \geq 0\) is fixed.

• The **Catalan arrangements** (see Section 7)
  \(C_{n-1}(1)\) given by
  \[ x_i - x_j = -1, 1, \quad 1 \leq i < j \leq n, \]
  (3.8)

  and \(C_{n-1}^0(1)\) given by
  \[ x_i - x_j = -1, 0, 1, \quad 1 \leq i < j \leq n. \]
  (3.9)

• The **truncated affine arrangement** \(A_{n-1}^{ab}\) (see Section 9) given by
  \[ x_i - x_j = -a + 1, -a + 2, \ldots, b - 1, \quad 1 \leq i < j \leq n, \]
  (3.10)

  where \(a\) and \(b\) are fixed integers such that \(a + b \geq 2\).

One can define analogous arrangements for any root system. Let \(V\) be a real \(d\)-dimensional vector space, and let \(R\) be a **root system** in \(V^*\) with a chosen set of **positive** roots \(R_+ = \{\beta_1, \beta_2, \ldots, \beta_N\}\) (see, e.g., [7, Ch. VI]). The **Coxeter arrangement** \(R\) of type \(R\) is the arrangement of hyperplanes in \(V\) given by

\[ \beta_i(x) = 0, \quad 1 \leq i \leq N. \]

(3.11)

Brieskorn [1] generalized Arnold’s formula (3.2). His formula for the Poincaré polynomial of (3.11) involves the exponents \(e_1, \ldots, e_d\) of the corresponding Weyl group \(W\):

\[ \text{Poin}_R(q) = (1 + e_1q)(1 + e_2q) \cdots (1 + e_dq). \]

Consider the hyperplane arrangement given by

\[ \beta_i(x) = a_{i}^{(1)}, \ldots, a_{i}^{(m_i)}, \quad 1 \leq i \leq N, \]

(3.12)

where \(x \in V\), \(m_i\) are some nonnegative integers, and \(a_{i}^{(k)} \in \mathbb{R}\). Many of the results of this paper have a natural counterpart in the case of an arbitrary root system. We will briefly outline several related results and conjectures. In more detail they will appear elsewhere.
4 Whitney’s formula and the NBC theorem

In this section we review several essentially well-known results on hyperplane arrangements that will be useful in the what follows.

Consider the arrangement $\mathcal{A}$ of hyperplanes in $V \cong \mathbb{R}^d$ given by equations

$$h_i(x) = a_i, \quad 1 \leq i \leq N,$$

(4.1)

where $x \in V$, the $h_i \in V^*$ are linear functionals on $V$, and the $a_i$ are real numbers.

We call a subset $I$ in $\{1, 2, \ldots, N\}$ central if the intersection of the hyperplanes $h_i(x) = a_i$, $i \in I$, is nonempty. For a subset $I = \{i_1, i_2, \ldots, i_l\}$, denote by $\text{rk}(I)$ the dimension (rank) of the linear span of the vectors $h_{i_1}, \ldots, h_{i_l}$.

The following statement is a generalization of a classical formula of Whitney \[29\].

**Theorem 4.1** The Poincaré and characteristic polynomials of the arrangement $\mathcal{A}$ are equal to

$$\text{Poin}_\mathcal{A}(q) = \sum_I (-1)^{|I| - \text{rk}(I)} q^{\text{rk}(I)},$$

(4.2)

$$\chi_\mathcal{A}(q) = \sum_I (-1)^{|I|} q^{d - \text{rk}(I)},$$

(4.3)

where $I$ ranges over all central subsets in $\{1, 2, \ldots, N\}$. In particular,

$$r(\mathcal{A}) = \sum_I (-1)^{|I| - \text{rk}(I)}$$

(4.4)

$$b(\mathcal{A}) = \sum_I (-1)^{|I|}.$$

We also need the well-known cross-cut theorem (see, \[24\, Corollary 3.9.4\]).

**Theorem 4.2** Let $L$ be a finite lattice with minimal element $\hat{0}$ and maximal element $\hat{1}$, and let $X$ be a subset of vertices in $L$ such that (a) $\hat{0} \not\in X$, and (b) if $y \in L$ and $y \neq \hat{0}$, then $x \leq y$ for some $x \in X$ (such elements are called atoms). Then

$$\mu_L(\hat{0}, \hat{1}) = \sum_k (-1)^k n_k,$$

(4.5)
where \( n_k \) is the number of \( k \)-element subsets in \( X \) with join equal to \( \hat{1} \).

Now we can easily deduce Theorem 4.1.

**Proof.** Let \( z \) be any element in the intersection poset \( L_A \), and let \( L(z) \) be the subposet of all elements \( x \in L_A \) such that \( x \leq z \), i.e., the subspace \( x \) contains \( z \). In fact, \( L(z) \) is a geometric lattice. Let \( X \) be the set of all hyperplanes from \( A \) which contain \( z \). If we apply Theorem 4.2 to \( L = L(z) \) and sum (4.5) over all \( z \in L_A \), we get the formula (4.3). Then, by (2.2), we get (4.2). \( \square \)

A cycle is a minimal subset \( I \) such that \( \text{rk}(I) = |I| - 1 \). In other words, a subset \( I = \{i_1, i_2, \ldots, i_l\} \) is a cycle if there exists a nonzero vector \( (\lambda_1, \lambda_2, \ldots, \lambda_l) \), unique up to a nonzero factor, such that \( \lambda_1 h_{i_1} + \lambda_2 h_{i_2} + \cdots + \lambda_l h_{i_l} = 0 \). It is not difficult to see that a cycle \( I \) is central if, in addition, we have \( \lambda_1 a_{i_1} + \lambda_2 a_{i_2} + \cdots + \lambda_l a_{i_l} = 0 \). Thus, if \( a_1 = \cdots = a_N = 0 \) then all cycles are central, and if the \( a_i \) are generic then there are no central cycles.

A subset \( I \) is called acyclic if \( |I| = \text{rk}(I) \), i.e., \( I \) contains no cycles. It is clear that any acyclic subset is central.

**Corollary 4.3** In the case when the \( a_i \) are generic, the Poincaré polynomial is given by

\[
Poin_A(q) = \sum_I q^{\text{rk}(I)},
\]

where the sum is over all acyclic subsets \( I \) of \( \{1, 2, \ldots, N\} \). In particular, the number of regions \( r(A) \) is equal to the number of acyclic subsets.

Indeed, in this case a subset \( I \) is acyclic if and only if it is central.

**Remark 4.4** The word “generic” in the corollary means that no \( k \) distinct hyperplanes in (4.1) intersect in an affine subspace of codimension less than \( k \). For example, if \( A \) is defined over \( \mathbb{Q} \) then it is sufficient to require that the \( a_i \) be linearly independent over \( \mathbb{Q} \).

Let us fix a linear order \( \rho \) on the set \( \{1, 2, \ldots, N\} \). We say that a subset \( I \) in \( \{1, 2, \ldots, N\} \) is a broken central circuit if there exists \( i \not\in I \) such that \( I \cup \{i\} \) is a central cycle and \( i \) is the minimal element of \( I \cup \{i\} \) with respect to the order \( \rho \).

The following, essentially well-known, theorem gives us the main tool for the calculation of Poincaré (or characteristic) polynomials. We will refer to it as the No Broken Circuit (NBC) Theorem.

**Theorem 4.5** We have

\[
Poin_A(q) = \sum_I q^{|I|},
\]

where the sum is over all acyclic subsets \( I \) of \( \{1, 2, \ldots, N\} \) without broken central circuits.
Proof. We will deduce this theorem from Theorem 4.1 using the involution principle. In order to do this we construct an involution $\iota : I \rightarrow \iota(I)$ on the set of all central subsets $I$ with a broken central circuit such that for any $I$ we have $\text{rk}(\iota(I)) = \text{rk}(I)$ and $|\iota \cdot I| = |I| \pm 1$.

This involution is defined as follows: Let $I$ be a central subset with a broken central circuit, and let $s(I)$ be the set of all $i \in 1, \ldots, N$ such that $i$ is the minimal element of a broken central circuit $J \subset I$. Note that $s(I)$ is nonempty. If the minimal element $s_*^*$ of $s(I)$ lies in $I$, then we define $\iota(I) = I \setminus \{s_*^*\}$. Otherwise, we define $\iota(I) = I \cup \{s_*^*\}$.

Note that $s(I) = s(\iota(I))$, thus $\iota$ is indeed an involution. It is clear now that all terms in (4.2) for $I$ with a broken central circuit cancel each other and the remaining terms yield the formula in Theorem 4.5. □

Remark 4.6 Note that by Theorem 4.5 the number of subsets $I$ without broken central circuits does not depend on the choice of the linear order $\rho$.

5 Deformations of Graphic Arrangements

In this section we show how to apply the results of the previous section to arrangements of type (3.3) and to give an interpretation of these results in terms of (colored) graphs.

With the hyperplane $x_i - x_j = a_{ij}^{(k)}$ in (3.3) one can associate the edge $(i, j)$ that has the color $k$. We will denote this edge by $(i, j)^{(k)}$. Then a subset $I$ of hyperplanes corresponds to a colored graph $G$ on the set of vertices $\{1, 2, \ldots, n\}$. According to the definitions in Section 4, a circuit $(i_1, i_2)^{(k_1)}, (i_2, i_3)^{(k_2)}, \ldots, (i_l, i_1)^{(k_l)}$ in $G$ is central if $a_{i_1,i_2}^{(k_1)} + a_{i_2,i_3}^{(k_2)} + \cdots + a_{i_l,i_1}^{(k_l)} = 0$. Clearly, a graph $G$ is acyclic if and only if $G$ is a forest.

Fix a linear order on the edges $(i, j)^{(k)}$, $1 \leq i < j \leq n$, $1 \leq k \leq m_{ij}$. We will call a subset of edges $C$ a broken A-circuit if $C$ is obtained from a central circuit by deleting the minimal element (here $A$ stands for the collection $\{a_{ij}^{(k)}\}$). Note that it should not be confused with the classical notion of a broken circuit of a graph, which corresponds to the case when all $a_{ij}^{(k)}$ are zero.

We summarize below several special cases of the NBC Theorem (Theorem 4.3). Here $|F|$ denotes the number of edges in a forest $F$.

Corollary 5.1 The Poincaré polynomial of the arrangement (3.3) is equal to

$$\text{Poin}_A(q) = \sum_{F} q^{|F|},$$

where the sum is over all colored forests $F$ on the vertices $1, 2, \ldots, n$ (an edge $(i, j)$ can have a color $k$, where $1 \leq k \leq m_{ij}$) without broken A-circuits. The number of regions of arrangement (3.3) is equal to the number of such forests.

In the case of the arrangement (3.4) we have:
Corollary 5.2 The Poincaré polynomial of the arrangement (3.4) is equal to

\[ \text{Poin}_A(q) = \sum_F q^{|F|}, \]

where the sum is over all forests on the set of vertices \( \{1, 2, \ldots, n\} \) without broken A-circuits. The number of regions of the arrangement (3.4) is equal to the number of such forests.

In the case when the \( a_{ij}^{(k)} \) are generic these results become especially simple.

For a forest \( F \) on vertices 1, 2, \ldots, \( n \) we will write

\[ m_F := \prod_{(i,j) \in F} m_{ij}, \]

where the product is over all edges \( (i, j) \), \( i < j \), in \( F \). Let \( c(F) \) denote the number of connected components in \( F \).

Corollary 5.3 Fix nonnegative integers \( m_{ij}, 1 \leq i < j \leq n \). Let \( A \) be an arrangement of type (3.3) where the \( a_{ij}^{(k)} \) are generic. Then

1. \( \text{Poin}_A(q) = \sum_F m^F q^{|F|} \),
2. \( r(A) = \sum_F m^F \),

where the sums are over all forests \( F \) on the vertices 1, 2, \ldots, \( n \).

Corollary 5.4 The number of regions of the arrangement (3.4) with generic \( a_{ij} \) is equal to the number of forests on \( n \) labelled vertices.

This corollary is “dual” to the following known result (see, e.g., [24, Exercise 4.32(a)]).

Proposition 5.5 Let \( P_n \) be the permutohedron, i.e., the polyhedron with vertices \( (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n \), where \( \sigma_1, \ldots, \sigma_n \) ranges over all permutations of 1, \ldots, \( n \). Then the number of integer points in \( P_n \) is equal to the number of forests on \( n \) vertices.

The connected components of the \( \left( \begin{array}{c} n \\ 2 \end{array} \right) \)-dimensional space of all arrangements (3.3) correspond to (coherent) zonotopal tilings of the permutohedron \( P_n \), i.e., certain subdivisions of \( P_n \) into parallelopipeds. The regions of a generic arrangement (3.4) correspond to the vertices of the corresponding tiling, which are all integer points in \( P_n \).

6 A semigeneric deformation of the braid arrangement.

Define the “semigeneric” deformation \( \mathcal{G}_n \) of the braid arrangement (3.1) to be the arrangement

\[ x_i - x_j = a_i, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n, \quad i \neq j, \]
where the $a_i$’s are generic real numbers (e.g., linearly independent over $\mathbb{Q}$). The significance of this arrangement to the theory of interval orders is discussed in [25, §3]. In [25, Thm. 3.1 and Cor. 3.3] a generating function for the number $r(G_n)$ of regions and for the characteristic polynomial $\chi_{G_n}(q)$ of $G_n$ is stated without proof. In this section we provide the proofs.

**Theorem 6.1** Let

$$z = \sum_{n \geq 0} \frac{r(G_n)}{n!} x^n = 1 + x + \frac{3}{2!} x^2 + \frac{19}{3!} x^3 + \frac{195}{4!} x^4 + \frac{2831}{5!} x^5 + \frac{53703}{6!} x^6 + \cdots.$$

Define a power series

$$y = 1 + x + \frac{5}{2!} x^2 + \frac{46}{3!} x^3 + \frac{631}{4!} x^4 + \frac{11586}{5!} x^5 + \cdots$$

by the equation

$$1 = y(2 - e^{xy}).$$

Then $z$ is the unique power series satisfying

$$\frac{z'}{z} = y^2, \quad z(0) = 1.$$

**Proof.** We use the formula (4.4) to compute $R(G_n)$. Given a central set $I$ of hyperplanes $x_i - x_j = a_i$ in $G_n$, define a directed graph $G_I$ on the vertex set $1, 2, \ldots, n$ as follows: let $i \to j$ be a directed edge of $G_I$ if and only if the hyperplane $x_i - x_j = a_i$ belongs to $I$. (By slight abuse of notation, we are using $I$ to denote a set of hyperplanes, rather than the set of their indices.) Note that $G_I$ cannot contain both the edges $i \to j$ and $j \to i$, since the intersection of the corresponding hyperplanes is empty. If $k_1, k_2, \ldots, k_r$ are distinct elements of $\{1, 2, \ldots, n\}$, then it is easy to see that if $r$ is even then there are exactly two ways to direct the edges $k_1k_2, k_2k_3, \ldots, k_{r-1}k_r, k_rk_1$ so that the hyperplanes corresponding to these edges have nonempty intersection, while if $r$ is odd then there are no ways. It follows that $G_I$, ignoring the direction of edges, is bipartite (i.e., all circuits have even length). Moreover, given an undirected bipartite graph on the vertices $1, 2, \ldots, n$ with blocks (maximal connected subgraphs that remain connected when any vertex is removed) $B_1, \ldots, B_s$, there are exactly two ways to direct the edges of each block so that the resulting directed graph $G$ is the graph $G_I$ of a central set $I$ of hyperplanes. In addition, $\text{rk}(I) = n - c(G)$, where $c(G)$ is the number of connected components of $G$. Letting $e(G)$ be the number of edges and $b(G)$ the number of blocks of $G$, it follows from equation (1.3) that

$$\chi_{G_n}(q) = \sum_{G} (-1)^{e(G)} 2^{b(G)} q^{c(G)},$$

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where $G$ ranges over all bipartite graphs on the vertex set $1, 2, \ldots, n$. This formula appears without proof in [23, Thm. 3.2]. In particular, putting $q = -1$ gives

$$r(G_n) = (-1)^n \sum_G (-1)^{e(G) + c(G)} 2^{b(G)}. \tag{6.1}$$

To evaluate the generating function $z = \sum r(G_n) x^n / n!$, we use the following strategy.

(a) Compute $A_n := \sum_G (-1)^{e(G)}$, where $G$ ranges over all (undirected) bipartite graphs on $1, 2, \ldots, n$.

(b) Use (a) and the exponential formula to compute $B_n := \sum_G (-1)^{e(G)}$, where now $G$ ranges over all connected bipartite graphs on $1, 2, \ldots, n$.

(c) Use (b) and the block-tree theorem to compute the sum $C_n := \sum_G (-1)^{e(G)} 2^{b(G)}$, where $G$ ranges over all bipartite blocks on $1, 2, \ldots, n$.

(d) Use (c) and the block-tree theorem to compute the sum $D_n := \sum_G (-1)^{e(G)} 2^{b(G)}$, where $G$ ranges over all connected bipartite graphs on $1, 2, \ldots, n$.

(e) Use (d) and the exponential formula to compute the desired sum (6.1).

We now proceed to steps (a)–(e).

(a) Let $b_k(n)$ be the number of $k$-edge bipartite graphs on the vertex set $1, 2, \ldots, n$. It is known (e.g., [26, Exercise 5.5]) that

$$\sum_{n \geq 0} \sum_{k \geq 0} b_k(n) q^k \frac{x^n}{n!} = \left[ \sum_{n \geq 0} \frac{n}{n!} (1 + q)^{(n-1)} \binom{n}{i} \right] \frac{x^n}{n!} \right]^{1/2}.$$

Put $q = -1$ to get

$$\sum_{n \geq 0} A_n \frac{x^n}{n!} = \left( 1 + \sum_{n \geq 1} 2 \frac{x^n}{n!} \right)^{1/2} = (2e^x - 1)^{1/2}.$$

(b) According to the exponential formula [12, p. 166], we have

$$\sum_{n \geq 1} B_n \frac{x^n}{n!} = \log \sum_{n \geq 0} A_n \frac{x^n}{n!} = \frac{1}{2} \log(2e^x - 1).$$

(c) Let $B'_n$ denote the number of rooted connected bipartite graphs on $1, 2, \ldots, n$. Since $B'_n = nB_n$, we get

$$\sum_{n \geq 1} B'_n \frac{x^n}{n!} = x \frac{d}{dx} \sum_{n \geq 1} B_n \frac{x^n}{n!} = \frac{x}{2 - e^{-x}}. \tag{6.2}$$
Suppose now that $B$ is a set of nonisomorphic blocks $B$ and $w$ is a weight function on $B$, so $w(B)$ denotes the weight of the block $B$. Let

$$T(x) = \sum_{B \in B} w(B) \frac{x^{p(B)}}{p(B)!},$$

where $p(B)$ denotes the number of vertices of $B$. Let

$$u(x) = \sum_G \left( \prod_B w(B) \right) \frac{x^{p(G)}}{p(G)!},$$

where $G$ ranges over all connected graphs whose blocks are rooted and are isomorphic (as unrooted graphs) to elements of $B$, and where $B$ ranges over all blocks of $G$. The block-tree theorem \cite[(1.3.3)]{13}, Ch. 5 Exercises\] asserts that

$$u = xe^{T(u)}.$$ \hfill (6.3)

If we take $B$ to be the set of all nonisomorphic bipartite blocks, $w(B) = (-1)^{e(B)}$, and $u = x/(2 - e^{-x})$, then it follows from (6.2) that

$$T(x) = \sum_{n \geq 1} C_n \frac{x^n}{n!}.$$ \hfill (6.4)

(d) Let $D'_n$ be defined like $D_n$, except that $G$ ranges over all rooted connected bipartite graphs on $1, 2, \ldots, n$, so $D'_n = nD_n$. Let $v(x) = \sum_{n \geq 1} D'_n \frac{x^n}{n!}$. By the block-tree theorem we have

$$v = xe^{2T(v)},$$

where $T(x)$ is given by (6.4). Substitute $v^{-1}$ for $x$ and use (6.3) to get

$$x = v^{-1}(x)e^{2T(x)}$$

$$= v^{-1}(x) \left( \frac{x}{u^{-1}(x)} \right)^2.$$

Substitute $v(x)$ for $x$ to obtain

$$x v(x) = u^{-1}(v(x))^2.$$

Take the square root of both sides and compose with $u(x) = x/(2 - e^{-x})$ on the left to get

$$\frac{\sqrt{xv}}{2 - e^{-\sqrt{xv}}} = v.$$ \hfill (6.5)

(e) Equation (6.4) and the exponential formula show that

$$z = \exp \left( - \sum_{n \geq 1} (-1)^n D_n \frac{x^n}{n!} \right)$$

$$= \exp \left( - \int \frac{v(-x)}{x} \right),$$ \hfill (6.6)
where \( \int \) denotes the formal integral, i.e., \( \int \sum a_n \frac{x^n}{n!} = \sum a_n \frac{x^{n+1}}{(n+1)!} \). (The first minus sign in \( \text{(6.6)} \) corresponds to the factor \((-1)^c(G)\) in \( \text{(6.1)} \).)

Let \( v(-x) = -xy^2 \). Equation \( \text{(6.5)} \) becomes (taking care to choose the right sign of the square root)

\[
1 = y(2 - e^{xy}),
\]

while \( \text{(6.6)} \) shows that \( z'/z = -v(-x)/x = y^2 \). This completes the proof. \( \square \)

**Note.** The semigeneric arrangement \( G_n \) satisfies the hypotheses of [25, Thm. 1.2]. It follows that

\[
\sum_{n\geq 0} \chi g_n(q) \frac{x^n}{n!} = z(-x)^{-q},
\]

as stated in [25, Cor. 3.3]. Here \( z \) is as defined in Theorem 6.1.

An arrangement closely related to \( G_n \) is given by

\[
G'_n : \quad x_i - x_j = a_i, \quad 1 \leq i < j \leq n,
\]

where the \( a_i \)’s are generic. The analogue of equation \( \text{(6.1)} \) is

\[
r(G'_n) = (-1)^n \sum_G (-1)^{e(G) + c(G) + 2b(G)},
\]

where now \( G \) ranges over all bipartite graphs on the vertex set \( 1, 2, \ldots, n \) for which every block is alternating, i.e., every vertex is either less than all its neighbors or greater than all its neighbors. We don’t see, however, how to use this formula to obtain a generating function for \( r(G'_n) \) analogous to Theorem 6.1.

### 7 Catalan Arrangements and Semiorders

Let us fix distinct real numbers \( a_1, a_2, \ldots, a_m > 0 \), and let \( A = (a_1, \ldots, a_m) \). In this section we consider the arrangement \( C_{n-1} = C_{n-1}(A) \) of hyperplanes in the space \( V_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\} \) given by

\[
x_i - x_j = a_i, a_2, \ldots, a_m, \quad i \neq j.
\]

(7.1)

We consider also the arrangement \( C^0_{n-1} = C^0_{n-1}(A) \) obtained from \( C_{n-1} \) by adjoining the hyperplanes \( x_i = x_j, \) i.e., \( C^0_n \) is given by

\[
x_i - x_j = 0, a_1, a_2, \ldots, a_m, \quad i \neq j.
\]

(7.2)

Let

\[
f_A(t) = \sum_{n\geq 0} r(C_{n-1}) \frac{t^n}{n!},
\]

\[
g_A(t) = \sum_{n\geq 0} r(C^0_{n-1}) \frac{t^n}{n!}
\]

be the exponential generating functions for the numbers of regions of the arrangements \( C_{n-1} \) and \( C^0_{n-1} \).

The main result of this section is the following:
Theorem 7.1 We have \( f_A(t) = g_A(1 - e^{-t}) \) or, equivalently,
\[
r(C_{n-1}^0) = \sum_{k \geq 0} c(n, k) r(C_{k-1}),
\]
where \( c(n, k) \) is the signless Stirling number of the first kind, i.e., the number of permutations of \( 1, 2, \ldots, n \) with \( k \) cycles.

Let us have a closer look at two special cases of arrangements (7.1) and (7.2). Consider the arrangement of hyperplanes in \( V_{n-1} \subset \mathbb{R}^n \) given by the equations
\[
x_i - x_j = \pm 1, \quad 1 \leq i < j \leq n.
\]
(7.3)
Consider also the arrangement given by
\[
x_i - x_j = 0, \quad \pm 1, \quad 1 \leq i < j \leq n.
\]
(7.4)
It is not difficult to check the following result directly from the definition.

Proposition 7.2 The number of regions of the arrangement (7.4) is equal to \( n! C_n \), where \( C_n \) is the Catalan number \( C_n = \frac{1}{n+1} \binom{2n}{n} \).

Theorem 7.1 then gives a formula for the number of regions of the arrangement (7.3).

Let \( R \) be a region of the arrangement (7.3), and let \( (x_1, \ldots, x_n) \in R \) be any point in the region \( R \). Consider the poset \( P \) on the vertices \( 1, \ldots, n \) such that \( i >_P j \) if and only if \( x_i - x_j > 1 \). Clearly, distinct regions correspond to distinct posets. The posets that can be obtained in such a way are called semiorders. See [25] for more results on the relation between hyperplane arrangements and interval orders (which are a generalization of semiorders).

The symmetric group \( S_n \) naturally acts on the space \( V_{n-1} \) by permuting the coordinates \( x_i \). Thus it also permutes the regions of the arrangement (7.4). The region \( x_1 < x_2 < \cdots < x_n \) is called the dominant chamber. Every \( S_n \)-orbit consists of \( n! \) regions and has a unique representative in the dominant chamber. It is also clear that the regions of (7.4) in the dominant chamber correspond to unlabelled (i.e., nonisomorphic) semiorders on \( n \) vertices. Hence, Proposition 7.2 is equivalent to a well-known result of Wine and Freund [30] that the number of nonisomorphic semiorders on \( n \) vertices is equal to the Catalan number. In the special case of the arrangements (7.3) and (7.4), i.e., \( A = (1) \), Theorem 7.1 gives a formula for the number of labelled semiorders on \( n \) vertices which was first proved by Chandon, Lemaire, and Pouget [8].

The following theorem, due to Scott and Suppes [21], presents a simple characterization of semiorders (cf. Theorem 8.4).

Theorem 7.3 A poset \( P \) is a semiorder if and only if it contains no induced subposet of either of the two types shown on Figure 3.
Lemma 7.4 The number of regions of $C^0_{n-1}$ is equal to $n!$ times the number of $\mathfrak{S}_n$-orbits in $R_{n-1}$.

Indeed, the number of regions of $C^0_{n-1}$ is $n!$ times the number of those in the dominant chamber. They, in turn, correspond to $\mathfrak{S}_n$-orbits in $R_{n-1}$. As was shown in [25], the regions of $C_{n-1}$ can be viewed as (labelled) generalized interval orders. On the other hand, the regions of $C^0_{n-1}$ that lie in the dominant chamber correspond to unlabelled generalized interval orders. The statement now is tautological, that the number of unlabelled objects is the number of $\mathfrak{S}_n$-orbits.

Now we can apply the following well-known lemma of Burnside (actually first proved by Cauchy and Frobenius).

Lemma 7.5 Let $G$ be a finite group which acts on a finite set $M$. Then the number of $G$-orbits in $M$ is equal to
\[ \frac{1}{|G|} \sum_{g \in G} \text{Fix}(g, M), \]
where $\text{Fix}(g, M)$ is the number of elements in $M$ fixed by $g \in G$.

By Lemmas 7.4 and 7.5, we have
\[ r(C^0_{n-1}) = \sum_{\sigma \in \mathfrak{S}_n} \text{Fix}(\sigma, C_{n-1}), \]
where $\text{Fix}(\sigma, C_{n-1})$ is the number of regions of $C_{n-1}$ fixed by the permutation $\sigma$.

Theorem 7.1 now follows easily from the following lemma.

Lemma 7.6 Let $\sigma \in \mathfrak{S}_n$ be a permutation with $k$ cycles. Then the number of regions of $C_{n-1}$ fixed by $\sigma$ is equal to the total number of regions of $C_{k-1}$.

Indeed, by Lemma 7.6, we have
\[ r(C^0_{n-1}) = \sum_{\sigma \in \mathfrak{S}_n} \text{Fix}(\sigma, C_{n-1}) = \sum_{k \geq 0} c(n, k) r(C_{k-1}), \]
which is precisely the claim of Theorem 7.1.

Proof of Lemma 7.6 We will construct a bijection between the regions of $C_{n-1}$ fixed by $\sigma$ and the regions of $C_{k-1}$. 

Figure 3: Forbidden subposets for semiorders.
Let $R$ be any region of $\mathcal{C}_{n-1}$ fixed by a permutation $\sigma \in \mathfrak{S}_n$, and let $(x_1, \ldots, x_n)$ be any point in $R$. Then for any $i, j \in \{1, \ldots, n\}$ and any $s = 1, \ldots, m$ we have $x_i - x_j > a_s$ if and only if $x_{\sigma(i)} - x_{\sigma(j)} > a_s$.

Let $\sigma = (c_{i1} c_{i2} \cdots c_{i_1}) (c_{i2} c_{i2} \cdots c_{i_{2l}}) \cdots (c_{k_1} c_{k_2} \cdots c_{k_l})$ be the cycle decomposition of the permutation $\sigma$, where $X_i = (x_{c_{i1}}, x_{c_{i2}}, \ldots)$ for $i = 1, \ldots, k$. We will write $X_i - X_j > a$ if $x_i - x_j > a$ for any $x_i \in X_i$ and $x_j \in X_j$. The notation $X_i - X_j < a$ has an analogous meaning. We will show that for any two classes $X_i$ and $X_j$ and for any $s = 1, \ldots, m$ we have either $X_i - X_j > a_s$ or $X_i - X_j < a_s$.

Let $x_{i^*}$ be the maximal element in $X_i$ and let $x_{j^*}$ be the maximal element in $X_j$. Suppose that $x_{i^*} - x_{j^*} > a_s$. Since $R$ is $\sigma$-invariant, for any integer $p$ we have the inequality $x_{\sigma^p(i^*)} - x_{\sigma^p(j^*)} > a_s$. Then, since $x_{i^*}$ is the maximal element of $X_i$, we have $x_{i^*} - x_{\sigma^p(j^*)} > a_s$. Again, for any integer $q$, we have $x_{\sigma^q(i^*)} - x_{\sigma^q(j^*)} > a_s$, which implies that $X_i - X_j > a_s$.

Analogously, suppose that $x_{i^*} - x_{j^*} < a_s$. Then for any integer $p$ we have $x_{\sigma^p(i^*)} - x_{\sigma^p(j^*)} < a_s$. Since $x_{j^*} \geq x_{\sigma^p(i^*)}$, we have $x_{\sigma^p(j^*)} - x_{j^*} < a_s$. Finally, for any integer $q$ we obtain $x_{\sigma^q(i^*)} - x_{\sigma^q(j^*)} < a_s$, which implies that $X_i - X_j < a_s$.

If we pick an element $x_{i^*}$ in each class $X_i$ we get a point $(x_{i'}, x_{i''}, \ldots, x_{k'})$ in $\mathbb{R}^k$. This point lies in some region $R'$ of $\mathcal{C}_{k-1}$. The construction above shows that the region $R'$ does not depend on the choice of $x_{i^*}$ in $X_i$.

Thus we get a map $\phi : R \to R'$ from the regions of $\mathcal{C}_{n-1}$ invariant under $\sigma$ to the regions of $\mathcal{C}_{k-1}$. It is clear that $\phi$ is injective. To show that $\phi$ is surjective, let $(x_{1'}, \ldots, x_{k'})$ be any point in a region $R'$ of $\mathcal{C}_{k}$. Pick the point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $x_{c_{i1}} = x_{c_{i2}} = \cdots = x_{i'}, x_{c_{i2}} = x_{c_{i2}} = \cdots = x_{i''}, \ldots, x_{c_{k1}} = x_{c_{k2}} = \cdots = x_{k'}$. Then $(x_1, \ldots, x_n)$ is in some region $R$ of $\mathcal{C}_{n-1}$ (here we use the condition $a_1, \ldots, a_m \neq 0$). According to our construction, we have $\phi(R) = R'$. Thus $\phi$ is a bijection.

This completes the proof of Lemma 7.6 and therefore also of Theorem 7.1. \hfill \Box

8 The Linial Arrangement.

As before, $V_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$. Consider the arrangement $\mathcal{L}_{n-1}$ of hyperplanes in $V_{n-1}$ given by the equations

$$x_i - x_j = 1, \quad 1 \leq i < j \leq n. \quad (8.1)$$

Recall that $r(\mathcal{L}_{n-1})$ denotes the number of regions of the arrangement $\mathcal{L}_{n-1}$. This arrangement was first considered by Nati Linial and Shmuel Ravid. They calculated the numbers $r(\mathcal{L}_{n-1})$ and the Poincaré polynomials $\text{Poin}_{\mathcal{L}_{n-1}}(q)$ for $n \leq 9$.

In this section we give an explicit formula and several different combinatorial interpretations for the numbers $r(\mathcal{L}_{n-1})$.

8.1 Alternating trees and local binary search trees

We call a tree $T$ on the vertices $0, 1, 2, \ldots, n$ alternating if the vertices in any path $i_1, \ldots, i_k$ in $T$ alternate, i.e., we have $i_1 < i_2 > i_3 < \cdots i_k$ or $i_1 > i_2 < i_3 > \cdots i_k$. In other words, there are no $i < j < k$ such that both $(i, j)$ and $(j, k)$ are edges in $T$.\hfill 16
Equivalently, every vertex is either greater than all its neighbors or less than all its
neighbors. Alternating trees first appear in [10] and were studied in [20], where they
were called intransitive trees (see also [25]).

![Figure 4: An alternating tree.](image)

Let $f_n$ be the number of alternating trees on the vertices $0, 1, 2, \ldots, n$, and let

$$f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

be the exponential generating function for the sequence $f_n$.

A plane binary tree $B$ on the vertices $1, 2, \ldots, n$ is called a local binary search
tree if for any vertex $i$ in $T$ the left child of $i$ is less than $i$ and the right child of $i$
is greater than $i$. These trees were first considered by Ira Gessel [11]. Let $g_n$ denote
the number of local binary search trees on the vertices $1, 2, \ldots, n$. By convention,
$g_0 = 1$.

![Figure 5: A local binary search tree.](image)

The following result was proved in [20] (see also [10, 25]).

**Theorem 8.1** For $n \geq 1$ we have

$$f_n = g_n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (k + 1)^{n-1}$$

and $f = f(x)$ satisfies the functional equation

$$f = e^{x(1+f)/2}.$$
The first few numbers $f_n$ are given in the table below.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| $f_n$ | 1 | 1 | 2 | 7 | 36 | 246 | 2104 | 21652 | 260720 | 3598120 | 56010096 |

The main result on the Linial arrangement is the following:

**Theorem 8.2** The number $r(L_{n-1})$ of regions of $L_{n-1}$ is equal to the number $f_n$ of alternating trees on the vertices $0, 1, 2, \ldots, n$, and thus to the number $g_n$ of local binary search trees on $1, 2, \ldots, n$.

This theorem was conjectured by the second author (thanks to the numerical data provided by Linial and Ravid) and was proved by the first author. A different proof was later given by C. Athanasiadis [3].

In Section 9 we will prove a more general result (see Theorems 9.1 and Corollary 9.9).

### 8.2 Sleek posets and semiacyclic tournaments

Let $R$ be a region of the arrangement $L_{n-1}$, and let $(x_1, \ldots, x_n)$ be any point in $R$. Define $P = P(R)$ to be the poset on the vertices $1, 2, \ldots, n$ such that $i <_P j$ if and only if $x_i - x_j > 1$ and $i < j$ in the usual order on $\mathbb{Z}$.

We will call a poset $P$ on the vertices $1, 2, \ldots, n$ sleek if $P$ is the intersection of a semiorder (see Section 7) with the chain $1 < 2 < \cdots < n$.

The following proposition immediately follows from the definitions.

**Proposition 8.3** The map $R \mapsto P(R)$ is a bijection between regions of $L_{n-1}$ and sleek posets on $1, 2, \ldots, n$. Hence the number $r(L_{n-1})$ is equal to the number of sleek posets on $1, 2, \ldots, n$.

There is a simple characterization of sleek posets in terms of forbidden induced subposets (compare Theorem 7.3).

**Theorem 8.4** A poset $P$ on the vertices $1, 2, \ldots, n$ is sleek if and only if it contains no induced subposet of the four types shown on Figure 4, where $a < b < c < d$.

In the remaining part of this section we prove Theorem 8.4.

First, we give another description of regions in $L_{n-1}$ (or, equivalently, sleek posets). A tournament on the vertices $1, 2, \ldots, n$ is a directed graph $T$ without loops such that for every $i \neq j$ either $(i, j) \in T$ or $(j, i) \in T$. For a region $R$ of $L_{n-1}$ construct a tournament $T = T(R)$ on the vertices $1, 2, \ldots, n$ as follows: let $(x_1, \ldots, x_n) \in R$. If $x_i - x_j > 1$ and $i < j$, then $(i, j) \in T$; while if $x_i - x_j < 1$ and $i < j$, then $(j, i) \in T$.

Let $C$ be a directed cycle in the complete graph $K_n$ on the vertices $1, 2, \ldots, n$. We will write $C = (c_1, c_2, \ldots, c_m)$ if $C$ has the edges $(c_1, c_2), (c_2, c_3), \ldots, (c_m, c_1)$. By
convention, \( c_0 = c_m \). An ascent in \( C \) is a number \( 1 \leq i \leq m \) such that \( c_{i-1} < c_i \). Analogously, a descent in \( C \) is a number \( 1 \leq i \leq m \) such that \( c_{i-1} > c_i \). Let \( \text{asc}(C) \) denote the number of ascents and \( \text{des}(C) \) denote the number of descents in \( C \). We say that a cycle \( C \) is ascending if \( \text{asc}(C) \geq \text{des}(C) \). For example, the following cycles are ascending: \( C_0 = (a, b, c) \), \( C_1 = (a, c, b, d) \), \( C_2 = (a, d, b, c) \), \( C_3 = (a, b, d, c) \), \( C_4 = (a, c, d, b) \), where \( a < b < c < d \). These cycles are shown on Figure 7.

We call a tournament \( T \) on \( 1, 2, \ldots, n \) semiacyclic if it contains no ascending cycles. In other words, \( T \) is semiacyclic if for any directed cycle \( C \) in \( T \) we have \( \text{asc}(C) < \text{des}(C) \).

**Proposition 8.5** A tournament \( T \) on \( 1, 2, \ldots, n \) corresponds to a region \( R \) in \( L_{n-1} \), i.e., \( T = T(R) \), if and only if \( T \) is semiacyclic. Hence \( r(L_{n-1}) \) is the number of semiacyclic tournaments on \( 1, 2, \ldots, n \).

This fact was independently found by Shmulik Ravid.

For any tournament \( T \) on \( 1, 2, \ldots, n \) without cycles of type \( C_0 \) we can construct a poset \( P = P(T) \) such that \( i <_P j \) if and only if \( i < j \) and \( (i, j) \in T \). Now the four ascending cycles \( C_1, C_2, C_3, C_4 \) in Figure 7 correspond to the four posets on Figure 6. Therefore, Theorem 8.4 is equivalent to the following result.

**Theorem 8.6** A tournament \( T \) on the vertices \( 1, 2, \ldots, n \) is semiacyclic if and only if it contains no ascending cycles of the types \( C_0, C_1, C_2, C_3, \) and \( C_4 \) shown in Figure 7, where \( a < b < c < d \).

**Remark 8.7** This theorem is an analogue of a well-known fact that a tournament \( T \) is acyclic if and only if it contains no cycles of length 3. For semiacyclicity we have obstructions of lengths 3 and 4.
Proof. Let $T$ be a tournament on $1, 2, \ldots, n$. Suppose that $T$ is not semiacyclic. We will show that $T$ contains a cycle of type $C_0$, $C_1$, $C_2$, $C_3$, or $C_4$. Let $C = (c_1, c_2, \ldots, c_m)$ be an ascending cycle in $T$ of minimal length. If $m = 3$, or 4 then $C$ is of type $C_0$, $C_1$, $C_2$, $C_3$, or $C_4$. Suppose that $m > 4$.

Lemma 8.8 We have $\text{asc}(C) = \text{des}(C)$.

Proof. Since $C$ is ascending, we have $\text{asc}(C) \geq \text{des}(C)$. Suppose $\text{asc}(C) > \text{des}(C)$. If $C$ has two adjacent ascents $i$ and $i+1$ then $(c_{i-1}, c_{i+1}) \in T$ (otherwise we have an ascending cycle $(c_{i-1}, c_i, c_{i+1})$ of type $C_0$ in $T$). Then $C' = (c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m)$ is an ascending cycle in $T$ of length $m - 1$, which contradicts the fact that we chose $C$ to be minimal. So for every ascent $i$ in $C$ the index $i + 1$ is a descent. Hence $\text{asc}(C) \leq \text{des}(C)$, and we get a contradiction. □

We say that $c_i$ and $c_j$ are on the same level in $C$ if the number of ascents between $c_i$ and $c_j$ is equal to the number of descents between $c_i$ and $c_j$.

Lemma 8.9 We can find $i, j \in \{1, 2, \ldots, m\}$ such that (a) $i$ is an ascent and $j$ is a descent in $C$, (b) $i \not\equiv j \pm 1 \pmod{m}$, and (c) $c_i$ and $c_{j-1}$ are on the same level (see Figure 8).

Proof. We may assume that for any $1 \leq s \leq m$ the number of ascents in $\{1, 2, \ldots, s\}$ is greater than or equal to the number of descents in $\{1, 2, \ldots, s\}$ (otherwise take some cyclic permutation of $(c_1, c_2, \ldots, c_m)$). Consider two cases.

1. There exists $1 \leq t \leq m - 1$ such that $c_i$ and $c_m$ are on the same level. In this case, if the pair $(i, j) = (1, t)$ does not satisfy conditions (a)–(c) then $t = 2$. On the other hand, if the pair $(i, j) = (t + 1, m)$ does not satisfy (a)–(c) then $t = m - 2$. Hence, $m = 4$ and $C$ is of type $C_1$ or $C_2$ shown in Figure 7.

2. There is no $1 \leq t \leq m - 1$ such that $c_i$ and $c_m$ are on the same level. Then $2$ is an ascent and $m - 1$ is a descent. If the pair $(i, j) = (2, m - 2)$ does not satisfy (a)–(c) then $m = 4$ and $C$ is of type $C_3$ or $C_4$ shown on Figure 4. □

Now we can complete the proof of Theorem 8.6. Let $i, j$ be two numbers satisfying the conditions of Lemma 8.9. Then $c_{i-1}$, $c_i$, $c_{j-1}$, $c_j$ are four distinct vertices such that (a) $c_{i-1} < c_i$, (b) $c_{j-1} > c_j$, (c) $c_i$ and $c_{j-1}$ are on the same level, and (d) $c_{i-1}$ and $c_j$ are on the same level (see Figure 8). We may assume that $i < j$.

If $(c_{j-1}, c_{i-1}) \in T$ then $(c_{i-1}, c_i, \ldots, c_{j-1})$ is an ascending cycle in $T$ of length less than $m$, which contradicts the requirement that $C$ is an ascending cycle on $T$ of minimal length. So $(c_{i-1}, c_{j-1}) \in T$. If $c_{i-1} < c_{j-1}$ then $(c_{j-1}, c_{j-2}, \ldots, c_m, c_1, \ldots, c_{i-1})$ is an ascending cycle in $T$ of length less than $m$. Hence, $c_{i-1} > c_{j-1}$.

Analogously, if $(c_i, c_j) \in T$ then $(c_j, c_{j+1}, \ldots, c_p, c_1, \ldots, c_i)$ is an ascending cycle in $T$ of length less than $m$. So $(c_j, c_i) \in T$. If $c_i > c_j$ then $(c_i, c_{i+1}, \ldots, c_j)$ is an ascending cycle in $T$ of length less than $m$. So $c_i < c_j$.

Now we have $c_{i-1} > c_{j-1} > c_j > c_i > c_{i-1}$, and we get an obvious contradiction.

We have shown that every minimal ascending cycle in $T$ is of length 3 or 4 and thus have proved Theorem 8.6. □
8.3 The Orlik-Solomon algebra

In [16] Orlik and Solomon gave the following combinatorial description of the cohomology ring of an arbitrary hyperplane arrangement. Consider a complex arrangement $A$ of affine hyperplanes $H_1, H_2, \ldots, H_N$ in the complex space $V \cong \mathbb{C}^n$ given by

$$H_i : f_i(x) = 0, \quad i = 1, \ldots, N,$$

where $f_i(x)$ are linear forms on $V$ (with a constant term).

We say that hyperplanes $H_{i_1}, \ldots, H_{i_p}$ are independent if the codimension of the intersection $H_{i_1} \cap \cdots \cap H_{i_p}$ is equal to $p$. Otherwise, the hyperplanes are dependent.

Let $e_1, \ldots, e_N$ be formal variables associated with the hyperplanes $H_1, \ldots, H_N$. The Orlik-Solomon algebra $\text{OS}(A)$ of the arrangement $A$ is generated over the complex numbers by $e_1, \ldots, e_N$ subject to the relations:

$$e_i e_j = -e_j e_i, \quad 1 \leq i < j \leq N, \quad (8.2)$$

$$e_{i_1} \cdots e_{i_p} = 0, \quad \text{if } H_{i_1} \cap \cdots \cap H_{i_p} = \emptyset, \quad (8.3)$$

$$\sum_{j=1}^{p+1} (-1)^j e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_{p+1}} = 0, \quad (8.4)$$

whenever $H_{i_1}, \ldots, H_{i_{p+1}}$ are dependent. (Here $\widehat{e_{i_j}}$ denotes that $e_{i_j}$ is missing.)

Let $C_A = V - \bigcup_i H_i$ be the complement to the hyperplanes $H_i$ of $A$, and let $H^*_{\text{DR}}(C_A; \mathbb{C})$ denote de Rham cohomology of $C_A$.

**Theorem 8.10** (Orlik, Solomon [16]) The map $\phi : \text{OS}(A) \to H^*_{\text{DR}}(C_A; \mathbb{C})$ defined by

$$\phi : e_i \mapsto \left[\frac{df_i}{f_i}\right]$$

is an isomorphism.

Here $[df_i/f_i]$ is the cohomology class in $H^*_{\text{DR}}(C_A; \mathbb{C})$ of the differential form $df_i/f_i$.

We will apply Theorem 8.10 to the Linial arrangement. In this case hyperplanes $x_i - x_j = 1, i < j$, correspond to edges $(i, j)$ of the complete graph $K_n$. 
Proposition 8.11 The Orlik-Solomon algebra $\text{OS}(\mathcal{L}_{n-1})$ of the Linial arrangement is generated by $e_{vw} = e_{(v,w)}$, $1 \leq v < w \leq n$ subject to relations (8.2), (8.3), and also to the following relations:

\begin{align*}
  e_{ab}e_{bc}e_{ac} - e_{ab}e_{bc}e_{cd} + e_{ac}e_{bc}e_{ad} - e_{bc}e_{bd}e_{ad} &= 0, \\
  e_{ac}e_{bc}e_{bd} - e_{ac}e_{bc}e_{ad} + e_{ac}e_{bd}e_{ad} - e_{bc}e_{bd}e_{ad} &= 0.
\end{align*}

where $1 \leq a < b < c < d \leq n$. (cf. Figure 7).

Proof. Let $C = (c_1, c_2, \ldots, c_p)$ be a cycle in $K_n$. We say that $C$ is balanced if $\text{asc}(C) = \text{des}(C)$. We may assume that in equation (8.4) $i_1, i_2, \ldots, i_p$ are edges of a balanced cycle $C$. We will prove (8.4) by induction on $p$. If $p = 4$ then $C$ is of type $C_1, C_2, C_3$, or $C_4$ (see Figure 7). Thus $C$ produces one of the relations (8.5). If $p > 4$, then we can find $r \neq s$ such that both $C' = (c_r, c_{r+1}, \ldots, c_s)$ and $C'' = (c_s, c_{s+1}, \ldots, c_r)$ are balanced. Equation (8.4) for $C$ is the sum of the equations for $C'$ and $C''$. Thus the statement follows by induction. $\Box$

Remark 8.12 This proposition is an analogue to the well-known description of the cohomology ring of the Coxeter arrangement (3.1), due to Arnold [1]. This cohomology ring is generated by $e_{vw} = e_{(v,w)}$, $1 \leq v < w \leq n$, subject to relations (8.2), (8.3) and also the following “triangle” equation:

\[ e_{ab}e_{bc} - e_{ab}e_{ac} + e_{bc}e_{ac} = 0, \]

where $1 \leq a < b < c \leq n$.

9 Truncated affine arrangements

In this section we study a general class of hyperplane arrangements which contains, in particular, the Linial and Shi arrangements.

Let $a$ and $b$ be two integers such that $a + b \geq 2$. Consider the hyperplane arrangement $\mathcal{A}_{n-1}^{ab}$ in $V_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$ given by

\[ x_i - x_j = -a + 1, -a + 2, \ldots, b - 1, \quad 1 \leq i < j \leq n. \]  

We call $\mathcal{A}_{n-1}^{ab}$ truncated affine arrangement because it is a finite subarrangement of the affine arrangement of type $\tilde{A}_{n-1}$ given by $x_i - x_j = k$, $k \in \mathbb{Z}$.

As we will see the arrangement $\mathcal{A}_{n-1}^{ab}$ has different behavior in the balanced case ($a = b$) and the unbalanced case ($a \neq b$).

9.1 Functional equations

Let $f_n = f_n^{ab}$ be the number of regions of the arrangement $\mathcal{A}_{n-1}^{ab}$, and let

\[ f(x) = \sum_{n \geq 0} f_n x^n/n! \]  

be the exponential generating function for $f_n$. 

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Theorem 9.1 Suppose \( a, b \geq 0 \).

1. The generating function \( f = f(x) \) satisfies the following functional equation:

\[
f^{b-a} = e^x \cdot \frac{f^a - f^b}{1 - f}.
\]  

(9.3)

2. If \( a = b \geq 1 \), then \( f = f(x) \) satisfies the equation:

\[
f = 1 + x f^a,
\]  

(9.4)

Note that the equation (9.4) can be formally obtained from (9.3) by l'Hôpital's rule in the limit \( a \to b \).

In the case \( a = b \) the functional equation (9.4) allows us to calculate the numbers \( f_{nn}^a \) explicitly. The following statement was proved by P. Headley [14].

Corollary 9.2 The number \( f_{nn}^a \) is equal to

\[
an(n-1) \cdots (an-n+2).
\]

The functional equation (9.3) is especially simple in the case \( a = b-1 \). We call the arrangement \( A_{n-1}^{a,a+1} \) the extended Shi arrangement. In this case we get:

Corollary 9.3 Let \( a \geq 1 \). The number \( f_n \) of regions of the hyperplane arrangement in \( \mathbb{R}^n \) given by

\[
x_i - x_j = -a+1, -a+2, \ldots, a, \quad i < j,
\]

is equal to \( f_n = (a n+1)^{n-1} \), and the exponential generating function \( f = \sum_{n \geq 0} f_n \frac{x^n}{n!} \) satisfies the functional equation \( f = e^{x} f^a \).

In order to prove Theorem 9.1 we need several new definitions. A graded graph is a graph \( G \) on a set \( V \) of vertices labelled by natural numbers together with a function \( h : V \to \{0, 1, 2, \ldots\} \), which is called a grading. For \( r \geq 0 \) the vertices \( v \) in \( G \) such that \( h(v) = r \) form the \( r \)th level of \( G \). Let \( e = (u, v) \) be an edge in \( G \), \( u < v \). We say that the type of the edge \( e \) is the integer \( t = h(v) - h(u) \) and that a graded graph \( G \) is of type \((a, b)\) if the types of all edges in \( G \) are in the interval \([-a+1, b-1] = \{-a+1, -a+2, \ldots, b-1\}\).

Choose a linear order on the set of all triples \((u, t, v)\), \( u, v \in V \), \( t \in [-a+1, b-1] \). Let \( C \) be a graded cycle of type \((a, b)\). Every edge \((u, v)\) in \( C \) corresponds to a triple \((u, t, v)\), where \( t \) is the type of the edge \((u, v)\). Choose the edge \( e \) in \( C \) with the minimal triple \((u, t, v)\). We say that \( C \setminus \{e\} \) is a broken circuit of type \((a, b)\).

Let \((F, h)\) be a graded forest. We say that \((F, h)\) is grounded or that \( h \) is a grounded grading on the forest \( F \) if each connected component in \( F \) contains a vertex on the 0th level.

Proposition 9.4 The number \( f_n \) of regions of the arrangement (9.1) is equal to the number of grounded graded forests of type \((a, b)\) on the vertices 1, 2, \ldots, \( n \) without broken circuits of type \((a, b)\).
Proof. By Corollary 5.3, the number $f_n$ is equal to the number of colored forests $F$ on the vertices $1, 2, \ldots, n$ without broken $A$-circuits. Every edge $(u,v)$, $u < v$, in $F$ has a color which is an integer from the interval $[-a + 1, b - 1]$. Consider the grounded grading $h$ on $F$ such that for every edge $(u,v)$, $u < v$, in $F$ of color $t$ we have that $t = h(v) - h(u)$ is the type of $(u,v)$. It is clear that such a grading is uniquely defined. Then $(F,h)$ is a grounded graded forest of type $(a,b)$. Clearly, this gives a correspondence between colored and graded forests. Then broken $A$-circuits correspond to broken graded circuits. The proposition easily follows. \hfill \Box

From now on we fix the lexicographic order on triples $(u,t,v)$, i.e., $(u,t,v) < (u',t',v')$ if and only if $u < u'$, or $(u = u'$ and $t < t'$), or $(u = u'$ and $t = t'$ and $v < v')$. Note the order of $u$, $t$, and $v$. We will call a graded tree $T$ solid if $T$ is of type $(a,b)$ and $T$ contains no broken circuits of type $(a,b)$.

Let $T$ be a solid tree on $1, 2, \ldots, n$ such that vertex 1 is on the $r$-th level. If we delete the minimal vertex 1, then the tree $T$ decomposes into connected components $T_1, T_2, \ldots, T_m$. Suppose that each component $T_i$ is connected with 1 by an edge $(1,v_i)$ where $v_i$ is on the $r_i$-th level.

Lemma 9.5 Let $T, T_1, \ldots, T_m, v_1, \ldots, v_m$, and $r_1, \ldots, r_m$ be as above. The tree $T$ is solid if and only if (a) all $T_1, T_2, \ldots, T_m$ are solid, (b) for all $i$ the $r_i$-th level is the minimal nonempty level in $T_i$ such that $-a + 1 \leq r_i - r \geq b - 1$, and (c) the vertex $v_i$ is the minimal vertex on its level in $T_i$.

Proof. First, we prove that if $T$ is solid then the conditions (a)–(c) hold. Condition (a) is trivial, because if some $T_i$ contains a broken circuit of type $(a,b)$ then $T$ also contains this broken circuit. Assume that for some $i$ there is a vertex $v_i'$ on the $r_i'$-th level in $T_i$ such that $r_i' < r_i$ and $r_i' - r \geq -a + 1$. Then the minimal chain in $T$ that connects vertex 1 with vertex $v_i'$ is a broken circuit of type $(a,b)$. Thus condition (b) holds. Now suppose that for some $i$ vertex $v_i$ is not the minimal vertex $v_i''$ on its level. Then the minimal chain in $T$ that connects vertex 1 with $v_i''$ is a broken circuit of type $(a,b)$. Therefore, condition (c) holds too.

Now assume that conditions (a)–(c) are true. We prove that $T$ is solid. Suppose not. Then $T$ contains a broken circuit $B = C \setminus \{e\}$ of type $(a,b)$, where $C$ is a graded circuit and $e$ is its minimal edge. If $B$ does not pass through vertex 1 then $B$ lies in $T_i$ for some $i$, which contradicts condition (a). We can assume that $B$ passes through vertex 1. Since $e$ is the minimal edge in $C$, $e = (1,v)$ for some vertex $v'$ on level $r'$ in $T$. Suppose $v \in T_i$. If $v'$ and $v_i$ are on different levels in $T_i$ then, by (b), $r_i < r$. Thus the minimal edge in $C$ is $(1,v_i)$ and not $(1,v')$. If $v'$ and $v_i$ are on the same level in $T_i$, then by (c) we have $v_i < v'$. Again, the minimal edge in $C$ is $(1,v_i)$ and not $(1,v')$. Therefore, the tree $T$ contains no broken circuit of type $(a,b)$, i.e., $T$ is solid. \hfill \Box

Let $s_i$ be the minimal nonempty level in $T_i$, and let $l_i$ be the maximal nonempty level in $T_i$. By Lemma 9.5, the vertex 1 can be on the $r$-th level, $r \in \{s_i - b + 1, s_i - b + 1, \ldots, l_i + a - 1\}$, and for each such $r$ there is exactly one way to connect 1 with $T_i$. \vspace{1em}
Let \( p_{nkr} \) denote the number of solid trees (not necessarily grounded) on the vertices \( 1, 2, \ldots, n \) which are located on levels \( 0, 1, \ldots, k \) such that vertex 1 is on the \( r \)th level, \( 0 \leq r \leq k \).

Let
\[
p_{kr}(x) = \sum_{n \geq 1} p_{nkr} \frac{x^n}{n!}, \quad p_k(x) = \sum_{r=0}^{k} p_{kr}(x).
\]

By the exponential formula (see [12, p. 166]) and Lemma 9.5, we have
\[
p'_{kr}(x) = \exp(b_{kr}(x)), \tag{9.5}
\]
where
\[
b_{kr}(x) = \sum_{n \geq 1} b_{nkr} x^n n!
\]
and \( b_{nkr} \) is the number of solid trees \( T \) on \( n \) vertices located on the levels \( 0, 1, \ldots, k \) such that at least one of the levels \( r-a+1, r-a+2, \ldots, r+b-1 \) is nonempty, \( 0 \leq r \leq k \). The polynomial \( b_{kr}(x) \) enumerates the solid trees on levels \( 1, 2, \ldots, k \) minus trees on levels \( 1, \ldots, r-a \) and trees on levels \( r+b, \ldots, k \). Thus we obtain
\[
b_{kr}(x) = p_k(x) - p_{r-a}(x) - p_{k-r-b}(x).
\]

By (9.5), we get
\[
p'_{kr}(x) = \exp(p_k(x) - p_{r-a}(x) - p_{k-r-b}(x)),
\]
where \( p_{-1}(x) = p_{-2}(x) = \cdots = 0, p_0(x) = x, p_k(0) = 0 \) for \( k \in \mathbb{Z} \). Hence
\[
p'_{k}(x) = \sum_{r=0}^{k} \exp(p_k(x) - p_{r-a}(x) - p_{k-r-b}(x)).
\]

Equivalently,
\[
p'_{k}(x) \exp(-p_k(x)) = \sum_{r=0}^{k} \exp(-p_{r-a}(x)) \exp(-p_{k-r-b}(x)).
\]

Let \( q_k(x) = \exp(-p_k(x)) \). We have
\[
q'_k(x) = - \sum_{r=0}^{k} q_{r-a}(x) q_{k-r-b}(x), \tag{9.6}
\]
\( q_{-1} = q_{-2} = \cdots = 1, q_0 = e^{-x}, q_k(0) = 1 \) for \( k \in \mathbb{Z} \).

The following lemma describes the relation between the polynomials \( q_k(x) \) and the number of regions of the arrangement \( A_{n-1}^{ab} \).

**Lemma 9.6** The quotient \( q_{k-1}(x)/q_k(x) \) tends to \( \sum_{n \geq 0} f_n \frac{x^n}{n!} \) as \( k \to \infty \).

**Proof.** Clearly, \( p_k(x) - p_{k-1}(x) \) is the exponential generating function for the numbers of grounded solid trees of height less than or equal to \( k \). By the exponential formula (see [12, p. 166]) \( q_{k-1}(x)/q_k(x) = \exp(p_k(x) - p_{k-1}(x)) \) is the exponential
generating function for the numbers of grounded solid forests of height less than or equal to \( k \). The lemma obviously follows from Proposition 9.4. \( \square \)

All previous formulae and constructions are valid for arbitrary \( a \) and \( b \). Now we will take advantage of the condition \( a, b \geq 0 \). Let

\[
q(x, y) = \sum_{k \geq 0} q_k(x) y^k.
\]

By (9.6), we obtain the following differential equation for \( q(x, y) \):

\[
\frac{\partial}{\partial x} q(x, y) = - (a_y + y^a q(x, y)) \cdot (b_y + y^b q(x, y)),
\]

\[
q(0, y) = (1 - y)^{-1},
\]

where \( a_y := (1 - y^a)/(1 - y) \).

This differential equation has the following solution:

\[
q(x, y) = \frac{b_y \exp(-x \cdot b_y) - a_y \exp(-x \cdot a_y)}{y^a \exp(-x \cdot a_y) - y^b \exp(-x \cdot b_y)}.
\] (9.7)

Let us fix some small \( x \). Since \( Q(y) := q(x, y) \) is an analytic function of \( y \), then \( \gamma = \gamma(x) = \lim_{k \to \infty} q_{k-1}/q_k \) is the pole of \( Q(y) \) closest to 0 (\( \gamma \) is the radius of convergence of \( Q(y) \) if \( x \) is a small positive number). By (9.7), \( \gamma^a \exp(-x \cdot a_y) - \gamma^b \exp(-x \cdot b_y) = 0 \). Thus, by Lemma 9.6, \( f(x) = \sum_{n \geq 0} f_n x^n/n! = \gamma(x) \) is the solution of the functional equation

\[
f^a e^{-x \cdot \frac{1-f^a}{1-f}} = f^b e^{-x \cdot \frac{1-f^b}{1-f}},
\]

which is equivalent to (9.3).

This completes the proof of Theorem 9.1. \( \square \)

### 9.2 Formulae for the characteristic polynomial

Let \( \mathcal{A} = \mathcal{A}_{n-1}^{ab} \) be the truncated affine arrangement given by (9.1). Consider the characteristic polynomial \( \chi^{ab}(q) \) of the arrangement \( \mathcal{A}_n^{ab} \). Recall that \( \chi_n^{ab}(q) = q^{n-1} \text{Poin}_{\mathcal{A}_{n-1}^{ab}}(-q^{-1}) \).

Let \( \chi^{ab}(x, q) \) be the exponential generating function

\[
\chi^{ab}(x, q) = 1 + \sum_{n>0} \chi_n^{ab}(q) \frac{x^n}{n!}.
\]

According to [23, Theorem 1.2], we have

\[
\chi^{ab}(x, q) = f(-x)^{-q},
\] (9.8)

where \( f(x) = \chi^{ab}(-x, -1) \) is the exponential generating function (9.2) for numbers of regions of \( \mathcal{A}_{n-1}^{ab} \).

Let \( S \) be the shift operator \( S : f(q) \mapsto f(q-1) \).
Theorem 9.7 Assume that $0 \leq a < b$. Then
\[ \chi_n^{ab}(q) = (b-a)^{-n}(S^a + S^{a+1} + \cdots + S^{b-1})^n \cdot q^{n-1}. \]

**Proof.** The theorem can be deduced from Theorem 9.1 and (9.8) (using, e.g., the Lagrange inversion formula). □

In the limit $b \to a$, using l'Hospital's rule, we obtain
\[ \chi_n^{aa}(q) = \left( \frac{S^a \log S}{1-S} \right)^n \cdot q^{n-1}. \]

In fact, there is an explicit formula for $\chi_n^{aa}(q)$. The following statement easily follows from Corollary 9.2 and appears in [14, ??][9, proof of Prop. 3.1].

**Theorem 9.8** We have
\[ \chi_n^{aa}(q) = (q + 1 - an)(q + 2 - an) \cdots (q + n - 1 - an). \]

There are several equivalent ways to reformulate Theorem 9.7, as follows:

**Corollary 9.9** Let $r = b - a$.

1. We have
\[ \chi_n^{ab}(q) = r^{-n} \sum (q - \phi(1) - \cdots - \phi(n))^{n-1}, \]
where the sum is over all functions $\phi : \{1, \ldots, n\} \to \{a, \ldots, b - 1\}$.

2. We have
\[ \chi_n^{ab}(q) = r^{-n} \sum_{s, l \geq 0} (-1)^l(q - s - an)^{n-1} \binom{n}{l} \binom{s + n - rl - 1}{n - 1}. \]

3. We have
\[ \chi_n^{ab}(q) = r^{-n} \sum \binom{n}{n_1, \ldots, n_r} (q - an_1 - \cdots - (b - 1)n_r)^{n-1}, \]
where the sum is over all nonnegative integers $n_1, n_2, \ldots, n_r$ such that $n_1 + n_2 + \cdots + n_r = n$.

**Examples:**

1. ($a = 1$ and $b = 2$) The Shi arrangement $S_{n-1}$ given by (3.6) is the arrangement $A_{n-1}^{12}$. By Corollary 9.3.1, we get the following formula of Headley [14, ??] (generalizing the formula $r(S_{n-1}) = (n + 1)^{n-1}$ due to Shi [22, ??][23]):
\[ \chi_n^{12}(q) = (q - n)^{n-1}. \]
2. \((a \geq 1 \text{ and } b = a + 1)\) More generally, for the extended Shi arrangement \(\mathcal{S}_{n-1, k}\) given by (1.7), we have (cf. Corollary 3.3)
\[
\chi_{n}^{a, a+1}(q) = (q - an)^{n-1}.
\]

3. \((a = 0 \text{ and } b = 2)\) In this case we get the Linial arrangement \(\mathcal{L}_{n-1} = A_{n-1}^{02}\) (see Section 8). By Corollary 9.3, we have (cf. Theorem 8.2)
\[
\chi_{n}^{0, 2}(q) = 2^{n} \sum_{k=0}^{n} \binom{n}{k} (q - k)^{n-1},
\]
(9.9)

4. \((a \geq 0 \text{ and } b = a + 2)\) More generally, for the arrangement \(A_{n-1}^{a, a+2}\), we have
\[
\chi_{n}^{a, a+2}(q) = 2^{n} \sum_{k=0}^{n} \binom{n}{k} (q - an - k)^{n-1}.
\]

Formula (9.9) for the characteristic polynomial \(\chi_{n}^{0, 2}(q)\) was earlier obtained by C. Athanasiadis [3, Theorem 5.2]. He used a different approach based on a combinatorial interpretation of the value of \(\chi_{n}(q)\) for sufficiently large primes \(q\).

[asymptotic behavior of \(\chi_{n}^{ab}(q)\) — to be inserted]

9.3 Roots of the characteristic polynomial

Theorem 9.11 has one surprising application concerning the location of roots of the characteristic polynomial \(\chi_{n}^{ab}(q)\).

We start with the case \(a = b\). One can reformulate Theorem 9.8 in the following way:

**Corollary 9.10** Let \(a \geq 1\). The roots of the polynomial \(\chi_{n}^{aa}(q)\) are the numbers \(an - 1, an - 2, \ldots, an - n + 1\) (each with multiplicity 1). In particular, the roots are symmetric to each other with respect to the point \((2a - 1)n/2\). Now assume that \(a \neq b\).

**Theorem 9.11** Let \(a + b \geq 2\). All the roots of the characteristic polynomial \(\chi_{n}^{ab}(q)\) of the truncated affine arrangement \(A_{n-1}^{ab}\), \(a \neq b\), have real part equal to \((a + b - 1) n/2\). They are symmetric to each other with respect to the point \((a + b - 1)n/2\).

Thus in both cases the roots of the polynomial \(\chi_{n}^{ab}(n)\) are symmetric to each other with respect to the point \((a + b - 1)n/2\), but in the case \(a = b\) all roots are real, whereas in the case \(a \neq b\) the roots are on the same vertical line in the complex plane \(\mathbb{C}\). Note that in the case \(a = b - 1\) the polynomial \(\chi_{n}^{ab}(q)\) has only one root \(an = (a + b - 1)n/2\) with multiplicity \(n - 1\).

The following lemma is implicit in a paper of Auric [5] and also follows from a problem posed by Pólya [18] and solved by Obreschkoff [15] (repeated in [19, Problem V.196.1, pp. 70 and 251]). For the sake of completeness we give a simple proof.
Lemma 9.12 Let $P(q) \in \mathbb{C}[q]$ have the property that every root has real part $a$. Let $z$ be a complex number satisfying $|z| = 1$. Then every root of the polynomial $R(q) = (S + z)P(q) = P(q - 1) + zP(q)$ has real part $a + \frac{1}{2}$.

Proof. We may assume that $P(q)$ is monic. Let 

$$P(q) = \prod_j (q - a_j), \quad b_j \in \mathbb{R}.$$ 

If $R(w) = 0$, then $|P(w)| = |P(w - 1)|$. Suppose that $w = a + \frac{1}{2} + c + di$, where $c, d \in \mathbb{R}$ and $i = \sqrt{-1}$. Thus

$$\left| \prod_j \left( \frac{1}{2} + c + (d - b_j)i \right) \right| = \left| \prod_j \left( -\frac{1}{2} + c + (d - b_j)i \right) \right|.$$ 

If $c > 0$ then $\left| \frac{1}{2} + c + (d - b_j)i \right| > \left| -\frac{1}{2} + c + (d - b_j)i \right|$. If $c < 0$ then we have strict inequality in the opposite direction. Hence $c = 0$, so $w$ has real part $a + \frac{1}{2}$. □

Proof of Theorem 9.11. All the roots of the polynomial $q^{n-1}$ have real part 0. The operator $T = (S^a + S^{a+1} + \cdots + S^{b-1})^n$ can be written as

$$T = S^{an} \prod_{j=1}^{b-1-a} (S - z_j)^n,$$

where each $z_j$ is a complex number of absolute value one (in fact, a root of unity). The proof now follows from Theorem 9.7 and Lemma 9.12. □

9.4 Other root systems.

The results of Subsections 9.1-9.3 extend, partly conjecturally, to all the other root systems, as well as to the nonreduced root system $BC_n$ (the union of $B_n$ and $C_n$, which satisfies all the root system axioms except the axiom stating that if $\alpha$ and $\beta$ are roots satisfying $\alpha = c\beta$, then $c = \pm 1$). Henceforth in this section when we use the term “root system,” we also include the case $BC_n$.

Given a root system $R$ in $\mathbb{R}^n$ and integers $a$ and $b$ satisfying $a + b \geq 2$, we define the truncated $R$-affine arrangement $A^{ab}(R)$ to be the collection of hyperplanes

$$\langle \alpha, x \rangle = -a + 1, -a + 2, \ldots, b - 1,$$

where $\alpha$ ranges over all positive roots of $R$ (with respect to some fixed choice of simple roots). Here $\langle , \rangle$ denotes the usual scalar product on $\mathbb{R}^n$, and $x = (x_1, \ldots, x_n)$. As in the case $R = A_{n-1}$ we refer to the balanced case ($a = b$) and unbalanced case ($a \neq b$).

The characteristic polynomial for the balanced case was found by Edelman and Reiner [3, proof of Prop. 3.1] for the root system $A_{n-1}$ (see Theorem 9.8), and conjectured (Conjecture 3.3) by them for other root systems. This conjecture was
proved by Athanasiadis [3, Cor. 7.2.3 and Thm. 7.7.6] for types $A$, $B$, $C$, $BC$, and $D$. For types $A$, $B$, $C$ and $D$ the result is also stated in [3, Thm. 5.5]. We will not say anything more about the balanced case here.

For the unbalanced case, we have considerable evidence (discussed below) to support the following conjecture.

**Conjecture 9.13** Let $R$ be an irreducible root system in $\mathbb{R}^n$. Suppose that the unbalanced truncated affine arrangement $\mathcal{A} = \mathcal{A}^{ab}(R)$ has $h(\mathcal{A})$ hyperplanes. Then all the roots of the characteristic polynomial $\chi_{\mathcal{A}}(q)$ have real part equal to $h(\mathcal{A})/n$.

Note. (a) If all the roots of $\chi_{\mathcal{A}}(q)$ have the same real part, then this real part must equal $h(\mathcal{A})/n$, since for any arrangement $\mathcal{A}$ in $\mathbb{R}^n$ the sum of the roots of $\chi_{\mathcal{A}}(q)$ is equal to $h(\mathcal{A})$.

(b) Conjecture 9.13 implies the “functional equation”

$$
\chi_{\mathcal{A}}(q) = (-1)^n \chi_{\mathcal{A}}(-q + 2h(\mathcal{A})/n).
$$

(9.10)

Thus $\chi_{\mathcal{A}}(q)$ is determined by around half of its coefficients (or values).

(c) Let $a+b \geq 2$ and $R = A_n$, $B_n$, $C_n$, or $D_n$. Athanasiadis [2, Thms. 7.2.1 and 7.2.4] has shown that except possibly when both $a = 1$ and $R = C_n$ we have

$$
\chi_{R}^{ab}(q) = \chi_{0,b-a}^{0}(q - ak),
$$

(9.11)

where $k$ denotes the Coxeter number of $R$. Presumably this equation also holds for the missing case $a = 1$ and $R = C_n$. For $BC_n$ there is a similar result of Athanasiadis [2, Thm. 7.2.4]. These results and conjectures reduce Conjecture 9.13 to the case $a = 0$ when $R$ is a classical root system. A similar reduction is likely to hold for the exceptional root systems.

(d) Conjecture 9.13 is true for all the classical root systems ($A_n$, $B_n$, $C_n$, $BC_n$, $D_n$). This follows from explicit formulas found for $\chi_{R}^{ab}(q)$ by Athanasiadis [3] together with Lemma 9.12. The result of Athanasiadis is the following.

**Theorem 9.14** Up to a constant factor, we have the following characteristic polynomials of the indicated arrangements. (If the formula has the form $F(S)q^n$ or $F(S)(q-1)^n$, then the factor is $1/F(1)$.)

\[
\begin{align*}
\mathcal{A}^{0,2k+2}(B_n) & : (1 + S^2 + \cdots + S^{2k})^2(1 + S^2 + \cdots + S^{4k+2})^{n-1}(q - 1)^n \\
\mathcal{A}^{0,2k+2}(C_n) & : \text{ same as for } \mathcal{A}^{0,2k+2}(B_n) \\
\mathcal{A}^{0,2k+1}(B_n) & : (1 + S + \cdots + S^{2k})^2(1 + S^2 + \cdots + S^{4k})^{n-1}q^n \\
\mathcal{A}^{0,2k+1}(C_n) & : \text{ same as for } \mathcal{A}^{0,2k+1}(B_n) \\
\mathcal{A}^{0,2k+2}(D_n) & : (1 + S^2)(1 + S^2 + \cdots + S^{2k})^4(1 + S^2 + \cdots + S^{4k+2})^{n-3}(q - 1)^n \\
\mathcal{A}^{0,2k+1}(D_n) & : (1 + S + \cdots + S^{2k})^4(1 + S^2 + \cdots + S^{4k})^{n-3}q^n \\
\mathcal{A}^{0,2k+2}(BC_n) & : (1 + S^2 + \cdots + S^{2k})(1 + S^2 + \cdots + S^{4k+2})^{n}(q - 1)^n \\
\mathcal{A}^{0,2k+1}(BC_n) & : (1 + S + \cdots + S^{2k})(1 + S^2 + \cdots + S^{4k})^{n}q^n.
\end{align*}
\]
We also checked Conjecture 9.13 for the arrangements $\mathcal{A}^{02}(F_4)$ and $\mathcal{A}^{02}(E_6)$ (as well as the almost trivial case $\mathcal{A}^{03}(G_2)$, $a \neq b$). The characteristic polynomials are

$$\mathcal{A}^{02}(F_4) : q^4 - 24q^3 + 258q^2 - 1368q + 2917$$

$$\mathcal{A}^{02}(E_6) : q^6 - 36q^5 + 630q^4 - 6480q^3 + 40185q^2 - 140076q + 212002.$$

The formula for $\chi^{02}_{F_4}(q)$ has the remarkable alternative form:

$$\mathcal{A}^{02}(F_4) : \frac{1}{8}((-q - 1)^4 + 3(q - 5)^4 + 3(q - 7)^4 + (q - 11)^4) - 48.$$

Note that the numbers 1, 5, 7, 11 are the exponents of the root system $F_4$. For $E_6$ the analogous formula is given by

$$\mathcal{A}^{02}(E_6) : \frac{1}{1008}P(q) - 210,$$

where

$$P(q) = 61(q - 1)^6 + 352(q - 4)^6 + 91(q - 5)^6 + 91(q - 7)^6 + 352(q - 8)^6 + 61(q - 11)^6,$$

which is not as intriguing as the $F_4$ case. It is not hard to see that the symmetry of the coefficient sequences (1, 3, 3, 1) and (61, 352, 91, 91, 352, 61) is a consequence of equation (9.10) and the fact that if $e_1 < e_2 < \cdots < e_n$ are the exponents of an irreducible root system $R$, then $e_i + e_{n+1-i}$ is independent of $i$.

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