Ferrofluid drop rolling on the surface of a liquid

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Abstract. We report on the theoretical analysis of an experiment where a ferrofluid drop swimming on top of a non-magnetic fluid layer is propelled forward by means of a rotating magnetic field. The drop is modelled first as a solid sphere with a Navier slip boundary condition, then as a liquid (half-)sphere with its own inner flow field. In both cases an analytical expression for the drop speed in terms of the experimentally accessible parameters is obtained. While the solution of the Navier slip model contains an unknown parameter, the slip length, the result of the liquid half-sphere model is completely free of fitting parameters and is shown to represent the experimentally measured dependencies very well.

1. Introduction

When a drop of ferrofluid is brought into a rotating magnetic field, the suspended magnetic particles start to move together with this field. Due to the viscous coupling of the particles to their surrounding carrier liquid the angular momentum is transferred to the whole drop and an abundance of phenomena is observed. Drops of ferrofluid which are rotated in the horizontal plane while being completely immersed in another (nonmagnetic) liquid of comparable density have been examined both experimentally and theoretically \cite{1, 2}. By contrast, we have investigated a drop of ferrofluid which is swimming on top of a layer of nonmagnetic liquid and is rotated in a vertically oriented plane. This setup makes it possible to manoeuvre the drop over the fluid surface by tuning of the magnetic field, thereby constituting a pumping device without mechanical parts.

In the experiment, which was carried out at the University of Bayreuth, the constant translation velocity of a mm-sized drop of kerosene-based ferrofluid has been measured in dependence of the drop volume, the magnetic field amplitude and its frequency \cite{3}. Motivated by this experiment, we have developed a simplifying theoretical model which gives an analytical description of a spherical magnetized object that is half-way immersed in a liquid, rotates with constant angular velocity $\Omega$ and moves with a constant speed $v_{\text{drop}}$. We have treated two approaches, one in which the drop is approximated by a solid magnetic sphere with a Navier slip boundary condition, and another which describes the full magnetohydrodynamics of a liquid (half-)sphere.

The theoretical description of the 'real' setup poses a very complicated boundary value problem - mainly due to the moving contact line - which would have to be solved by numerical methods. In order to extract the essence of the effect we make some simplifying assumptions.
Figure 1. The droplet is considered to be a spherical object half-way immersed into a liquid with an otherwise perfectly flat surface. The liquid below is supposed to cover the infinite half-space. The problem is treated from the reference frame where the sphere is spinning on the spot. In order to ensure stationarity in this frame, the overall forces and torques acting on the sphere must cancel out. After the velocity field of the surrounding liquid has been determined, its asymptotic value at infinity will give the negative translation velocity of the sphere in the laboratory frame.

which lead to an analytical solution. The basic simplifications and the geometry of the setup are explained in figure 1.

The velocity field of the non-magnetic liquid bearing the sphere (and later also the flow field within the ferrofluid drop) is expanded in vector spherical harmonics according to [4]. Thus the basic differential equations, i.e., the continuity and Stokes equations, $\nabla \cdot \mathbf{v} = 0$ and $\nabla^2 (\nabla \times \mathbf{v}) = 0$, respectively, are transformed into ordinary differential equations which can be solved by a potential ansatz and the appropriate boundary conditions [3].

2. Solid sphere and Navier slip condition

We first treat the case of a solid sphere where the main problem lies in determining the viscous torque acting on it, since it has to compensate for the magnetic torque. In order to avoid a logarithmic divergence due to the large stresses at the moving contact line, a Navier slip condition is employed at the sphere surface [5, 6]. This is a linear relation between the flow field $\mathbf{v}$ and its derivative at the sphere surface $r = R$ and the velocity $U$ of the sphere surface:

$$\left[ \frac{\partial}{\partial r} - \frac{1}{R} \right] \mathbf{v}(r = R) = \frac{\mathbf{v}(r = R) - U}{L_s} \tag{1}$$

The parameter $L_s$ is called the slip length and denotes the amount of slippage between fluid and solid. This leads to a viscous torque $T_{\text{vis}} = -\pi \eta \Omega R^3 (L_s) \mathbf{e}_y$ on the sphere [7], where $\eta$ and $R$ denote fluid viscosity and sphere radius, respectively, while $\Omega$ is the yet unknown angular velocity of the drop. The abbreviation $\Sigma(L_s)$ includes the infinite series from the expansion in vector spherical harmonics, depends on the slip length and cannot be computed analytically in closed form.

Setting the viscous torque equal to the magnetic torque of the homogeneously magnetized sphere [8, 9]

$$T_{\text{mag}} = \frac{4\pi}{3} \mu_0 H^2 \chi'' R^3 (1 + \chi'/3)^2 + (\chi''/3)^2 \mathbf{e}_y, \tag{2}$$

the resulting flow field (which is quite a lengthy expression, cf. [3]) at infinity gives the speed

$$\mathbf{v}_{\text{drop}}^{\text{sol}} = \frac{2}{3} \left[ (1 + \chi'/3)^2 + (\chi''/3)^2 \right] \frac{\mu_0 H^2 \chi'' R}{\eta \Sigma(L_s)} \mathbf{e}_x \tag{3}$$

with which the drop ‘rolls’ over the surface. Here, $H$ denotes the amplitude of the external magnetic field, and $\chi'$ and $\chi''$ are the frequency-dependent real and imaginary parts, respectively, of the ferrofluid’s magnetic susceptibility.

Unfortunately, the series in $\Sigma(L_s)$ converges very slowly, making it difficult to determine $L_s$ from experimental data. Therefore, it would be of advantage to have an expression for the drop speed that does not depend on such a parameter.
3. Liquid half-sphere

We now treat the case of a magnetic liquid sphere whose inner flow field is computed along with the flow field of the liquid below. The Navier slip condition and the equality of viscous and magnetic torque are replaced by the continuity of tangential velocities and stresses at the interface of the two fluids.

As a consequence of the requirement of a flat surface it is not possible to obtain a spherical inner velocity field. More precisely, when the vertical velocity component is demanded to vanish at the whole contact line circle, this results in it being zero throughout the whole drop section \( z = 0 \).

However, when the boundary conditions are applied after the mirror image of the setup with respect to the fluid surface has been added (which is in order since the equations are linear), a flow field is obtained which proves to be very useful. As the field becomes completely horizontal within the plane of symmetry, it is suggested that only the lower half-sphere is identified with the ferrofluid drop, i.e., after solving the mirror image setup, the whole upper half-space is neglected, resulting in the flow field displayed in figure 2.

The translation velocity of this liquid half-drop now reads [3]

\[
\mathbf{v}_{\text{drop}} = -\frac{1}{4} \frac{\mu_0 H^2 \chi'' R}{(1 + \chi'/3)^2 + (\chi''/3)^2}[2\eta^{(o)} + 3\eta^{(i)}] \mathbf{e}_x \quad (4)
\]

with \( \eta^{(i)} \) and \( \eta^{(o)} \) denoting the viscosities of inner (magnetic) and outer (nonmagnetic) liquid. This expression looks very similar to the previous one (3), but has the desired advantages, i.e., it purely consists of parameters that are experimentally measurable or tunable, and there is no need to calculate numerically an infinite series, so that this result can easily be compared to experimental data.

4. Comparison with the experiment

Figure 3 shows the comparison of the measured values (blue full circles) with the theoretical result (4). The required magnitudes of real part (\( \chi' \)) and imaginary part (\( \chi'' \)) of the ferrofluid’s magnetic susceptibility were measured in an independent experiment as functions of the rotation frequency \( f \) of the external field \( H \). The data points for the radius dependence were obtained...
by determining the volume of the drop and assuming a spherical shape. This assumption is well justified for drops up to volumes of about 12 µl. Larger drops get considerably deformed by the magnetic field, as can be seen in a movie attached to [3].

Despite this discrepancy and the simplifications of our half-sphere model, the magnetic field dependence is in excellent agreement and that of the sphere radius is rather good. For a discussion of the frequency dependence the reader is referred to [3].

5. Conclusions
The experiment has shown that rotating fields can transport ferrofluidic drops. With our analytical modelling we have succeeded in explaining quantitatively the experimental results without any free fitting parameters. Moreover, the theory provides an explicit solution of the flow fields both for a rotating solid magnetic sphere and a spherical ferrofluid drop half-way immersed in a nonmagnetic liquid. The similarity of the final results of both approaches demonstrates the equivalence of Navier slip at a solid surface on the one hand and the continuity of tangential stresses at a fluid-fluid boundary on the other hand.

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