THE BOSONIC FOCK REPRESENTATION AND A GENERALIZED SHALE THEOREM

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ABSTRACT. We detail a new approach to the bosonic Fock representation of a complex Hilbert space $V$: our account places the bosonic Fock space $S[V]$ between the symmetric algebra $SV$ and its full antidual $SV'$; in addition to providing a context in which arbitrary (not necessarily restricted) real symplectic automorphisms of $V$ are implemented, it offers simplified proofs of many standard results of the theory.

0. INTRODUCTION

Traditionally, the bosonic Fock representation of a complex Hilbert space $V$ is founded in symmetric Fock space $S[V]$: the Hilbert space completion of the symmetric algebra $SV$ relative to a canonical product. Again traditionally, the various operators of interest (such as the number operator, field operators, creators and annihilators) are initially defined on the symmetric algebra and then extended to their maximal domains in Fock space. An unfortunate aspect of this traditional approach is that these extended operators are defined implicitly rather than by explicit formulae, a circumstance that often entails the use of awkward and indirect arguments.

A celebrated theorem of Shale asserts that a symplectic automorphism $g$ of $V$ is unitarily implemented in the Fock representation on $S[V]$ if and only if the commutator $[g, i] = gi - ig$ is a Hilbert-Schmidt operator. A standard proof of this theorem involves first developing an essentially figurative expression for the corresponding displaced vacuum and then showing that the Hilbert-Schmidt condition is necessary and sufficient for this figurative expression to define an element of $S[V]$. It is reasonable to ask for a context in which such figurative expressions are strictly legitimate: a setting that accommodates displaced vacua for all symplectic automorphisms.

Our purpose in these notes is to present a new approach to the bosonic Fock representation that addresses each of the issues just mentioned. In spirit, ours is a variant of the rigged Hilbert space approach and places $S[V]$ between a suitable subspace and its antidual. In fact, we follow the simplest route: the canonical inner product embeds $SV$ in its full (purely algebraic) antidual $SV'$ comprising all antilinear functionals $SV 	o \mathbb{C}$; Fock space $S[V]$ is realized as the subspace of bounded antilinear functionals, whence the triple $SV \subset S[V] \subset SV'$. An important feature of this approach is that the antidual $SV'$ is itself a commutative associative algebra: indeed, the canonical product on $SV'$ arises from the canonical coproduct on $SV$ after the fashion familiar from Hopf algebra theory.

When $v \in V$ the Fock field operator $\pi(v)$ is defined in terms of the creator $c(v)$ and annihilator $a(v)$ according to the usual prescription $\sqrt{2} \pi(v) = c(v) + a(v)$. These operators are initially defined on $SV$ and then extend to $SV'$ by antiduality: thus, if $\Phi \in SV'$ and $\psi \in SV$ then $[c(v)\Phi](\psi) = \Phi(a(v)\psi)$ and $[a(v)\Phi](\psi) = \Phi(c(v)\psi)$ so that $[\pi(v)\Phi](\psi) = \Phi(\pi(v)\psi)$. The various operators restrict from $SV'$ to the usual domains in $S[V]$: for instance, $\pi(v)$ restricts from $SV'$ to define an operator that is selfadjoint on the natural domain $\{ \Phi \in S[V] : \pi(v)\Phi \in S[V] \}$; it is not necessary to establish that $\pi(v)$ is essentially selfadjoint on $SV$ and form the unique selfadjoint...
extension. We remark that in this context, the canonical commutation relations in Heisenberg form hold without qualification on $SV'$: thus, if $x, y \in V$ then $[\pi(x), \pi(y)] = i \mathrm{Im} < x|y > I$.

The universal implementability of symplectic automorphisms may be established rather directly within this formalism. By definition, a (generalized) Fock implementer for the symplectic automorphism $g$ of $V$ is a (nonzero) linear map $U : SV \rightarrow SV'$ that intertwines $\pi(v) \in \text{End}SV$ with $\pi(gv) \in \text{End}SV'$ in the sense $v \in V \Rightarrow U\pi(v) = \pi(gv)U$. It transpires that each symplectic automorphism $g$ of $V$ admits a (generalized) Fock implementer $U$ that is unique up to scalar multiples; moreover $U$ may be recovered from the corresponding displaced vacuum, which is a Gaussian (the exponential of a quadratic) in $SV'$. Of course, if the commutator $[g, i]$ is of Hilbert-Schmidt class then the Gaussian displaced vacuum lies in $S[V]$ and (when scaled appropriately) $U$ determines a unitary operator on $S[V]$ that implements $g$ in the usual sense.

Of course, the technique of placing a Hilbert space $\mathcal{E}$ between a suitable subspace $E$ and its antidual $E'$ so as to form a triple $E \subset \mathcal{E} \subset E'$ is well established, though the subspace $E$ is typically provided with extra structure (such as that of a nuclear space) and the antidual $E'$ respects this. The case in which $L^2(\mathbb{R}^n)$ is placed between the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is prototypical, of course. Of more direct relevance to the present paper is work of the Hida group and others on the White Noise Calculus: here, $\mathcal{E}$ is the $L^2$ space of a Gaussian measure on the dual of a nuclear space, $E$ the space of test white noise functionals and $E'$ the space of generalized white noise functionals; see [5] and [6] for detailed accounts.

Traditional approaches to the bosonic Fock representation may be found in [2] [3] [4]; traditional approaches to the classical Shale theorem may be found in [1] [2] [3] [4] [9] [10] [11] [12]. The approach taken in these notes, placing bosonic Fock space between the symmetric algebra and its full antidual, is both natural and elegant. The virtues of placing fermionic Fock space directly within this formalism. By definition, a (generalized) Fock implementer for the symplectic automorphism $g$ of $V$ is a (nonzero) linear map $U : SV \rightarrow SV'$ that intertwines $\pi(v) \in \text{End}SV$ with $\pi(gv) \in \text{End}SV'$ in the sense $v \in V \Rightarrow U\pi(v) = \pi(gv)U$. It transpires that each symplectic automorphism $g$ of $V$ admits a (generalized) Fock implementer $U$ that is unique up to scalar multiples; moreover $U$ may be recovered from the corresponding displaced vacuum, which is a Gaussian (the exponential of a quadratic) in $SV'$. Of course, if the commutator $[g, i]$ is of Hilbert-Schmidt class then the Gaussian displaced vacuum lies in $S[V]$ and (when scaled appropriately) $U$ determines a unitary operator on $S[V]$ that implements $g$ in the usual sense.

The task of presenting a similar treatment for Fock spaces over indefinite inner product spaces will be left to a subsequent paper.

1. Symmetric Fock spaces

Let $V$ be a complex Hilbert space with $< \cdot | \cdot >$ as its complex inner product and $J = i \cdot$ as its complex structure. Denote by

$$SV = \bigoplus_{d \in \mathbb{N}} S^d V$$

its graded symmetric algebra and by $P^d : SV \rightarrow S^d V$ projection on the summand of homogeneous degree $d \in \mathbb{N}$. Recall that $SV$ carries a standard complex inner product $< \cdot | \cdot >$ relative to which the homogeneous summands are mutually perpendicular: $1 \in \mathbb{C} = S^0 V$ is a unit vector and if $x_1, \ldots, x_d, y_1, \ldots, y_d \in V$ then

$$< x_1 \cdots x_d | y_1 \cdots y_d > = \text{Per}[< x_a | y_b >] = \sum_{\pi} \prod_{k=1}^{d} < x_k | y_{\pi(k)} >$$

where $\text{Per}$ denotes the permanent of a square matrix and $\pi$ runs over the group comprising all permutations of $1, \ldots, d$. In particular, if $x, y \in V$ then

$$< x^d | y^d > = d! < x | y >^d$$

and if $v, x_1, \ldots, x_a, y_1, \ldots, y_b \in V$ then

$$\left\langle \frac{v^{a+b}}{(a+b)!} | x_1 \cdots x_a y_1 \cdots y_b \right\rangle = \left\langle \frac{v^a}{a!} | x_1 \cdots x_a \right\rangle \left\langle \frac{v^b}{b!} | y_1 \cdots y_b \right\rangle$$
whence bilinearity implies that if \( v \in V \) and \( \phi, \psi \in SV \) then
\[
\left< \frac{v^a}{a!} \frac{v^b}{b!} \phi \psi \right> = \left< \frac{v^a}{a!} \phi \right> \left< \frac{v^b}{b!} \psi \right>.
\]

**Theorem 1.1.** If \( V \) contains \( \{v_1, \ldots, v_m\} \) as a unitary set then \( SV \) contains \( \{v^D : D \in \mathbb{N}^m\} \) as a unitary set, where if \( D = (d_1, \ldots, d_m) \in \mathbb{N}^m \) then
\[
v^D = \frac{v_1^{d_1} \cdots v_m^{d_m}}{\sqrt{d_1! \cdots d_m!}}.
\]

**Proof.** If \( A, B \in \mathbb{N}^m \) are distinct then \( <v^A|v^B> = 0 \); either \( v^A \) or \( v^B \) have distinct degrees or each term in the permanent expansion of \( <v^A|v^B> \) contains a vanishing inner product. If \( D = (d_1, \ldots, d_m) \) then the permanent expansion of \( <v^D|v^D> \) has exactly \( d_1! \cdots d_m! \) nonvanishing terms each of which equals \( <v_1|v_1>d_1 \cdots <v_m|v_m>d_m \). \( \square \)

For future reference, we remark that \( S^dV \) is spanned by the vectors \( \{v^d : u \in V\} \); indeed, \( S^dV \) is certainly spanned by \( \{u_1 \cdots u_d : u_1, \ldots, u_d \in V\} \) and polarization yields
\[
2^d d! u_1 \cdots u_d = \sum_{\pm \cdots \pm} (\pm u_1 \cdots \pm u_d)^d.
\]

It proves convenient to introduce the set \( F(V) \) comprising all finite-dimensional complex subspaces of \( V \) directed by inclusion. Note that \( SV \) is the union of its subalgebras \( SM \) as \( M \) runs over \( F(V) \):
\[
SV = \bigcup \{SM : M \in F(V)\}.
\]

When \( M \in F(V) \) we write \( P_M : V \to M \) for orthogonal projection and write
\[
F_M(V) = \{N \in F(V) : M \subset N\}.
\]

**Theorem 1.2.** If \( M \in F(V) \) then the functorial extension of \( P_M : V \to M \) is precisely the orthogonal projection \( P_M : SV \to SM \).

**Proof.** Formulae of the type (1.3) show that if \( v_1, \ldots, v_d \in V \) and \( z \in M \) then
\[
<z|v_1 \cdots v_d |v_1 \cdots v_d > = d! |z|P_Mv_1 > \cdots <z|P_Mv_d = <z|v_1 \cdots v_d |
\]
whence the remark following Theorem 1.1 shows that \( (P_Mv_1) \cdots (P_Mv_d) - v_1 \cdots v_d \) is perpendicular to \( S^dM \). \( \square \)

Denote by \( SV' \) the full antidual of the symmetric algebra, comprising all antilinear functionals \( SV \to \mathbb{C} \). Note that the standard complex inner product \( < \cdot | \cdot > \) linearly embeds \( SV \) in \( SV' \) via the canonical inclusion
\[
SV \to SV' : \phi \mapsto < \cdot | \phi >.
\]

When \( \Phi \in SV' \) and \( d \in \mathbb{N} \) we may consider \( \Phi^d := \Phi \circ P^d \) as an element of either \( SV' \) or \( (S^dV)' \) as convenient. Note that if \( \Phi \in SV' \) then
\[
\Phi = \sum_{d \in \mathbb{N}} \Phi^d
\]
for if also \( \psi \in SV' \) then each sum is actually finite in the following calculation:
\[
\Phi(\psi) = \Phi\left( \sum_{d \in \mathbb{N}} P^d \psi \right) = \sum_{d \in \mathbb{N}} \Phi(P^d \psi) = \sum_{d \in \mathbb{N}} \Phi^d(\psi).
\]

Note also that if to each \( d \in \mathbb{N} \) is associated an element \( \phi^d \in S^dV \) then the formal series \( \sum_{d \in \mathbb{N}} \phi^d \) determines an element of \( SV' \).
Now, let $\Phi \in SV'$. If $M \in F(V)$ and $d \in \mathbb{N}$ then the finite-dimensionality of $S^d M$ guarantees the existence of a unique $\Phi^d_M \in S^d M$ such that $\Phi|S^d M = \langle \cdot \Phi^d_M \rangle$. If also $N \in F_M(V)$ then $P_M \Phi^d_N = \Phi^d_M$ for if $\psi \in S^d M$ then $P_M \psi = \psi$ and therefore

$$\langle \psi|\Phi^d_M \rangle = \langle \psi|\Phi^d_N \rangle = \langle \psi|P_M \Phi^d_N \rangle.$$

In the opposite direction is the following description of the antidual.

**Theorem 1.3.** If to each $M \in F(V)$ and $d \in \mathbb{N}$ is associated an element $\Phi^d_M \in S^d M$ satisfying the consistency condition

$$N \in F_M(V) \Rightarrow P_M \Phi^d_N = \Phi^d_M$$

then there exists a unique $\Phi \in SV'$ such that if $M \in F(V)$ and $d \in \mathbb{N}$ then

$$\Phi|S^d M = \langle \cdot \Phi^d_M \rangle.$$

**Proof.** For $\psi \in SV$ we define

$$\Phi(\psi) = \sum_{d \in \mathbb{N}} \langle P^d \psi|\Phi^d_M \rangle$$

where $M \in F(V)$ is chosen so that $\psi \in SM$. The choice of $M \in F(V)$ is immaterial: if also $N \in F(V)$ and $\psi \in SN$ then each of $\langle P^d \psi|\Phi^d_M \rangle$ and $\langle P^d \psi|\Phi^d_N \rangle$ equals $\langle P^d \psi|\Phi^d_{M+N} \rangle$ by consistency. The rest of the proof is clear. \(\square\)

In fact, the antidual $SV'$ is naturally an algebra. The most elegant way to see this rests on the fact that $SV$ itself is naturally a coalgebra: the diagonal map $V \rightarrow V \oplus V$ induces an algebra homomorphism $SV \rightarrow S(V \oplus V)$ which when followed by the canonical isomorphism $S(V \oplus V) \rightarrow SV \otimes SV$ yields the (cocommutative) coproduct $\Delta : SV \rightarrow SV \otimes SV$. In these terms, the natural (commutative) product in $SV'$ is defined by the rule that if $\Phi, \Psi \in SV'$ and $\theta \in SV$ then

$$[\Phi \Psi](\theta) = [\Phi \otimes \Psi](\Delta \theta).$$

**Theorem 1.4.** The natural product in $SV'$ is weakly continuous.

**Proof.** Explicitly, if $(\Phi_\lambda : \lambda \in \Lambda)$ and $(\Psi_\lambda : \lambda \in \Lambda)$ are nets in $SV'$ converging weakly to $\Phi \in SV'$ and $\Psi \in SV'$ respectively then the net $(\Phi_\lambda \Psi_\lambda : \lambda \in \Lambda)$ converges to $\Phi \Psi$ in the same sense: if $\theta \in SV$ then

$$\lim_{\lambda \in \Lambda} [\Phi_\lambda \Psi_\lambda](\theta) = [\Phi \Psi](\theta)$$

for if $\Delta \theta = \sum_{k=1}^K \xi_k \otimes \eta_k$ then

$$[\Phi_\lambda \Psi_\lambda](\theta) = [\Phi_\lambda \otimes \Psi_\lambda](\Delta \theta) = \sum_{k=1}^K \Phi_\lambda(\xi_k)\Psi_\lambda(\eta_k)$$

which as $\lambda$ runs over $\Lambda$ converges to

$$\sum_{k=1}^K \Phi(\xi_k)\Psi(\eta_k) = [\Phi \otimes \Psi](\Delta \theta) = [\Phi \Psi](\theta).$$

\(\square\)

Note that the canonical inclusion $SV \rightarrow SV'$ is an algebra homomorphism: it is enough to see that if $\phi \in S^d V$ and $\psi \in S^b V$ then $\langle \cdot \phi \psi \rangle = \langle \cdot \phi \rangle \langle \cdot \psi \rangle$; this follows from (1.4) and the remark immediately after Theorem 1.3. When identified with its image, $SV$ is weakly dense in $SV'$. 
Theorem 1.5. \( \Phi \in SV' \) is the weak limit in \( SV' \) of the net \( (\Phi_M^0 + \cdots + \Phi_M^d : M \in \mathcal{F}(V), d \in \mathbb{N}) \) in \( SV \).

Proof. For \( \psi \in SV \) choose \( M_\psi \in \mathcal{F}(V) \) and \( d_\psi \in \mathbb{N} \) so that \( \psi \in SM_\psi \) and \( \psi^d = 0 \) when \( d > d_\psi \). If \( M \in \mathcal{F}(V) \) contains \( M_\psi \) and \( d \in \mathbb{N} \) exceeds \( d_\psi \) then plainly \( \Phi(\psi) = \langle \psi | \Phi^0_M + \cdots + \Phi^d_M \rangle \). \( \square \)

The following less elegant formulation of the product in \( SV' \) is occasionally useful.

Theorem 1.6. Let \( \Phi \) and \( \Psi \) lie in \( SV' \). If \( M \in \mathcal{F}(V) \) and \( d \in \mathbb{N} \) then

\[
[\Phi \Psi]_M^d = \sum_{a+b=d} \Phi^a_M \Psi^b_M.
\]

Proof. This follows from (1.4) and the remark after Theorem 1.1: if \( v \in M \) then as

\[
\Delta(v^d) = (v \otimes 1 + 1 \otimes v)^d = \sum_{a+b=d} \frac{d!}{a! b!} v^a \otimes v^b
\]

so

\[
\frac{d}{d!} [\Phi \Psi]_M^d = [\Phi \Psi] \left( \frac{d}{d!} \right) = (\Phi \otimes \Psi) \left( \sum_{a+b=d} \frac{v^a}{a!} \otimes \frac{v^b}{b!} \right)
\]

\[
= \sum_{a+b=d} \Phi \left( \frac{a}{a!} \right) \Psi \left( \frac{b}{b!} \right) = \sum_{a+b=d} \left| \frac{a}{a!} \Phi^a_M \right| \left| \frac{b}{b!} \Psi^b_M \right|
\]

\[
= \frac{d}{d!} \sum_{a+b=d} \Phi^a_M \Psi^b_M.
\]

\( \square \)

We remark that this actually provides an alternative construction of the product in \( SV' \): if \( \Phi, \Psi \in SV' \) then the assignment

\[
M \in \mathcal{F}(V), d \in \mathbb{N} \Rightarrow [\Phi \Psi]_M^d = \sum_{a+b=d} \Phi^a_M \Psi^b_M
\]

is readily confirmed to be consistent in the sense of Theorem 1.3.

Now, let \( \Phi \in SV' \). To each \( M \in \mathcal{F}(V) \) we associate the formal sum

\[
\Phi_M : = \sum_{d \in \mathbb{N}} \Phi_M^d \in SV'
\]

and to this formal sum we associate the number

\[
\|\Phi_M\| : = \sqrt{\left\{ \sum_{d \in \mathbb{N}} \|\Phi_M^d\|^2 \right\}} \in [0, \infty].
\]

Let also \( N \in \mathcal{F}(V) \): if \( d \in \mathbb{N} \) then the consistency condition \( \Phi_M^d = P_M \Phi_N^d \) implies that \( \|\Phi^d_M\| \leq \|\Phi^d_N\| \); consequently, summation yields \( \|\Phi_M\| \leq \|\Phi_N\| \). It follows that the net \( (\|\Phi_N\| : N \in \mathcal{F}(V)) \) in \([0, \infty]\) is increasing, with the same supremum as its subnet \( (\|\Phi_N\| : N \in \mathcal{F}(V)) \) for each \( M \in \mathcal{F}(V) \). Define

\[
\|\Phi\| : = \sup_N \|\Phi_N\| = \lim_N \|\Phi_N\|.
\]

Theorem 1.7. If \( \Phi \in SV' \) then \( \|\Phi\| \) is its operator norm as an antilinear functional on \( SV \) in the sense

\[
\|\Phi\| = \sup\{\|\Phi(\psi)\| : \psi \in SV, \|\psi\| \leq 1\}.
\]
Proof. Let $\psi \in SV$ be a unit vector: if $M \in F(V)$ and $d \in \mathbb{N}$ are chosen so that $\psi = \psi^0 + \cdots + \psi^d \in SM$ then
\[
|\Phi(\psi)| = \left| \langle \psi \rangle \sum_{a=0}^{d} \Phi_M^a \right| \leq \left| \sum_{a=0}^{d} \Phi_M^a \right| \leq \|\Phi\|
\]
so the operator norm of $\Phi$ is at most $\|\Phi\|$. Let $M \in F(V)$ and $d \in \mathbb{N}$: if $\Phi_M^0 + \cdots + \Phi_M^d$ is nonzero then the unit vector
\[
\psi : = \left( \sum_{a=0}^{d} \Phi_M^a \right) / \left( \sum_{a=0}^{d} \Phi_M^a \right)
\]
satisfies
\[
\Phi(\psi) = \left\langle \psi \right| \sum_{a=0}^{d} \Phi_M^a \rangle = \left| \sum_{a=0}^{d} \Phi_M^a \right|\]
whence the arbitrary nature of $M$ and $d$ implies that the operator norm of $\Phi$ is at least $\|\Phi\|$.

We are now in a position to introduce symmetric Fock space as
\[
S[V] = \{ \Phi \in SV' : \|\Phi\| < \infty \}.
\]
Plainly, $S[V]$ is a complex vector space upon which $\| \cdot \|$ defines a norm. In fact, this norm is induced by a complex inner product: indeed, if $\Phi, \Psi \in S[V]$ and $M \in F(V)$ then the parallelogram law in homogeneous summands of $SM$ yields
\[
\| (\Phi - \Psi)_M \|^2 + \| (\Phi + \Psi)_M \|^2 = 2 \{ \| \Phi_M \|^2 + \| \Psi_M \|^2 \}
\]
whence passage to the supremum as $M$ runs over $F(V)$ yields
\[
\| \Phi - \Psi \|^2 + \| \Phi + \Psi \|^2 = 2 \{ \| \Phi \|^2 + \| \Psi \|^2 \}
\]
so the parallelogram law holds in $S[V]$. Accordingly, $\| \cdot \|$ is induced by the inner product $\langle \cdot | \cdot \rangle$ defined by the rule that if $\Phi, \Psi \in S[V]$ then
\[
\langle \Phi | \Psi \rangle = \frac{1}{4} \sum_{p=0}^{3} i^{-p} \| \Phi + i^p \Psi \|^2.
\]

Theorem 1.8. If $\Phi \in S[V]$ and $M \in F(V)$ then
\[
\|\Phi\|^2 = \| \Phi - \Phi_M \|^2 + \| \Phi_M \|^2.
\]
Proof. Let $N \in F_M(V)$. If $d \in \mathbb{N}$ then consistency and the Pythagorean law in $S^d N$ yield
\[
\|\Phi_N^d\|^2 = \|\Phi_N^d - \Phi_M^d\|^2 + \|\Phi_M^d\|^2 = \| (\Phi - \Phi_M)_N^d \|^2 + \| (\Phi_M)_N^d \|^2
\]
whence summation yields
\[
\|\Phi_N\|^2 = \| (\Phi - \Phi_M)_N \|^2 + \| (\Phi_M)_N \|^2.
\]
Passage to the supremum as $N$ runs over $F_M(V)$ concludes the argument.

As is readily checked, it is also the case that if $\Phi \in S[V]$ then
\[
\|\Phi\|^2 = \sum_{d \in \mathbb{N}} \|\Phi^d\|^2. \tag{1.6}
\]
In fact, $SV$ is dense in the inner product space $S[V]$. 

Theorem 1.11. If $\Phi \in S[V]$ then the net $(\Phi^0_M + \cdots + \Phi^d_M : M \in F(V), d \in \mathbb{N})$ in $SV$ converges to $\Phi$ in $S[V]$.

Proof. Let $\varepsilon > 0$. Choose $M_\varepsilon \in F(V)$ so that $\|\Phi_{M_\varepsilon}\|^2 > \|\Phi\|^2 - \varepsilon^2$ and choose $d_\varepsilon \in \mathbb{N}$ so that $\|\Phi^0_{M_\varepsilon} + \cdots + \Phi^d_{M_\varepsilon}\|^2 > \|\Phi\|^2 - \varepsilon^2$. If $M \in F(M_\varepsilon)$ and $d \in \mathbb{N}$ is at least $d_\varepsilon \in \mathbb{N}$ then

$$\|\Phi - \sum_{a=0}^d \Phi^a_M\|^2 = \sum_{a > d} \|\Phi^a\|^2 + \sum_{a=0}^d \|\Phi^a - \Phi^a_M\|^2$$

$$= \sum_{a \in \mathbb{N}} \|\Phi^a\|^2 - \sum_{a=0}^d \|\Phi^a_M\|^2$$

$$\leq \|\Phi\|^2 - \sum_{a=0}^d \|\Phi^a_M\|^2 < \varepsilon^2.$$ 

Further, $S[V]$ is actually the Hilbert space completion of $SV$.

Theorem 1.10. The inner product space $S[V]$ is complete.

Proof. Let $(\Phi : j \in \mathbb{N})$ be a Cauchy sequence in $S[V]$. If $M \in F(V)$ and $d \in \mathbb{N}$ then $(\Phi^d_M : j \in \mathbb{N})$ is (by domination) a Cauchy sequence in the finite-dimensional (hence complete) space $S^dM$ so we may define $\Phi^d_M = \lim_j \Phi^d_M$. If also $N \in F(M)$ then $P^d_M \Phi^d_N = \Phi^d_M$ so that continuity of $P_M : S^dN \to S^dM$ implies $P_M \Phi_N = \Phi^d_M$. Now Theorem 1.3 furnishes a unique $\Phi \in SV$ such that if $M \in F(V)$ and $d \in \mathbb{N}$ then $\Phi|S^dM = \langle \cdot | \Phi^d_M \rangle$. Let $\varepsilon > 0$ and choose $j_\varepsilon \in \mathbb{N}$ so that if $p, q \geq j_\varepsilon$ then $\|\Phi^p - \Phi^q\| \leq \varepsilon$. If $M \in F(V)$ then $\|\Phi^p_M - \Phi^q_M\| \leq \varepsilon$ so that (upon inspection of homogeneous summands) letting $p = j \geq j_\varepsilon$ and $q \to \infty$ results in $\|\Phi^p_M\| \leq \varepsilon$; as $M$ is arbitrary, it follows that if $j \geq j_\varepsilon$ then $\|\Phi - \Phi^j\| \leq \varepsilon$. This places $\Phi$ in $S[V]$ as the limit of $(\Phi : j \in \mathbb{N})$. 

Of course, the canonical inclusion $SV \to S[V]$ is isometric.

We shall have occasion to use the following assertion of compatibility.

Theorem 1.11. If $\Phi \in S[V]$ and $\psi \in SV$ then $\Phi(\psi) = \langle \psi | \Phi \rangle$.

Proof. Select $M_\psi \in F(V)$ and $d_\psi \in \mathbb{N}$ so that $\psi = \psi^0 + \cdots + \psi^d \in SM_\psi$. If $M \in F(M_\psi)$ and $d \in \mathbb{N}$ exceeds $d_\psi$ then $\psi = \psi^0 + \cdots + \psi^d \in SM$ so

$$\Phi(\psi) = \langle \psi | \Phi^0_M + \cdots + \Phi^d_M \rangle.$$ 

An application of Theorem 1.9 ends the proof. 

We shall also have need for the subspace of $S[V]$ comprising all elements of homogeneous degree $d \in \mathbb{N}$:

$$S^d[V] = \{ \Phi \in S[V] : \Phi \circ P^d = \Phi \}.$$ 

Theorem 1.12. If $d \in \mathbb{N}$ then $S^d[V]$ is precisely the closure of $S^dV$ in $S[V]$.

Proof. Plainly, $S^dV \subset S^d[V]$ and (1.6) implies that the map $S[V] \to S[V] : \Phi \mapsto \Phi \circ P^d$ is continuous, so $S^dV \subset S^d[V]$. For the reverse inclusion, apply Theorem 1.9. 

Note that (1.6) shows that $S[V]$ is the Hilbert space direct sum of its homogeneous subspaces:

$$S[V] = \bigoplus_{d \in \mathbb{N}} S^d[V].$$
This notion of homogeneity may be conveniently reformulated and applies to the full antidual $SV'$. Explicitly, the group $\mathbb{R}^+$ of positive reals has a scaling action $\sigma$ on $V$ given by

$$t \in \mathbb{R}^+, v \in V \Rightarrow \sigma_tv = tv$$

which extends to $SV'$ by functoriality and then to $SV'$ by antiduality, so that

$$t \in \mathbb{R}^+, \Phi \in SV', \psi \in SV \Rightarrow [\sigma_t\Phi](\psi) = \Phi(\sigma_t\psi).$$

In these terms, the elements of homogeneous degree $d \in \mathbb{N}$ are those on which $\sigma_t$ acts as multiplication by $t^d$ whenever $t \in \mathbb{R}^+$.

In the sequel, our interest will centre largely on quadratics: elements of $S^2V'$. It is convenient to discuss these a little more fully here. Let us say that the (antilinear) map $Z : V \rightarrow V$ is symmetric precisely when

$$x, y \in V \Rightarrow <y|Zx> = <x|ZY>.$$

More generally, let us say that the (antilinear) map $Z : V \rightarrow V'$ is symmetric precisely when

$$x, y \in V \Rightarrow Z(x(y) = Z(y(x)).$$

Plainly, the space $S^2V'$ of all quadratics $\zeta$ is canonically isomorphic to the space of all symmetric antilinear maps $Z : V \rightarrow V'$ via the rule

$$(1.7) \quad x, y \in V \Rightarrow \zeta(xy) = Zx(y).$$

**Theorem 1.13.** $S^2[V]$ is canonically isomorphic to the space $\Sigma^2[V]$ comprising all Hilbert-Schmidt symmetric antilinear maps $V \rightarrow V$: explicitly, $\zeta \in S^2[V]$ and $Z \in \Sigma^2[V]$ correspond when

$$x, y \in V \Rightarrow \zeta(xy) = <y|Zx>.$$

**Proof.** Let $M \in \mathcal{F}(V)$ have $(v_1, \ldots, v_m)$ as unitary basis. Note that $\zeta_M \in S^2 M$ corresponds canonically to the symmetric antilinear map $Z_M : M \rightarrow M : v \mapsto (Zv)_M$. Accordingly, from Theorem 1.4 it follows that

$$\zeta_M = \frac{1}{2} \sum_{a,b} <v_a|v_b\zeta_M = v_a\zeta b = \frac{1}{2} \sum_{a,b} <v_a|Z_Mv_b > v_a\zeta b$$

whence

$$\|\zeta_M\|^2 = \frac{1}{2} \sum_{a,b} |<v_a|Z_Mv_b>|^2 = \frac{1}{2} \|Z_M\|_H^2.$$

Passage to the supremum as $M$ runs over $\mathcal{F}(V)$ now shows not only that $Z$ maps $V$ to itself but also that $Z : V \rightarrow V$ is Hilbert-Schmidt with $\|Z\|_H = \sqrt{2}\|\zeta\|. \square$

2. **Exponentials, creators and annihilators**

Let $v \in V$. We define the creator $c(v) : SV \rightarrow SV$ to be the operator of left (equivalently, right) multiplication by $v$:

$$\phi \in SV \Rightarrow c(v)\phi = v\phi.$$

We define the annihilator $a(v) : SV \rightarrow SV$ to be the unique linear derivation such that $a(v)1 = 0$ and such that if $w \in V$ then $a(v)w = <v|w>$. Recall that for $a(v)$ to be a derivation means that if $\phi, \psi \in SV$ then

$$a(v)[\phi\psi] = [a(v)\phi]\psi + \phi[a(v)\psi]$$

so that if $v_1, \ldots, v_m \in V$ then

$$a(v)[v_1 \cdots v_m] = \sum_{k=1}^m <v|v_k > v_1 \cdots \hat{v}_k \cdots v_m$$

where the circumflex $\hat{\cdot}$ signifies omission as usual.
Theorem 2.1. If \( v \in V \) then \( c(v) \) and \( a(v) \) are mutually adjoint on \( SV \) in the sense that if \( \phi, \psi \in SV \) then
\[
< a(v) \phi | \psi > = < \phi | c(v) \psi > .
\]

Proof. It is enough to verify the equality when \( \phi = x_0 x_1 \cdots x_m \) and \( \psi = y_1 \cdots y_m \) for vectors \( x_0, x_1, \ldots, x_m, y_1, \ldots, y_m \) in \( V \); in this case, verification amounts to an elementary permanent expansion.

We extend the definition of creators and annihilators to the antidual \( SV' \) by antiduality. Explicitly, let \( v \in V \): for \( \Phi \in SV' \) and \( \psi \in SV \) we define
\[
[c(v) \Phi](\psi) = \Phi(a(v) \psi) \\
[a(v) \Phi](\psi) = \Phi(c(v) \psi).
\]
In both the original and this extended context, creators and annihilators satisfy the canonical commutation relations in the following form.

Theorem 2.2. If \( x, y \in V \) then
\[
[a(x), a(y)] = 0 \\
[a(x), c(y)] = < x | y > I \\
c(x), c(y)] = 0.
\]

Proof. Validity on \( SV' \) follows at once by antiduality from validity on \( SV \). Here, the last identity is plain from commutativity of \( SV \) while the first then follows by Theorem 2.1; the central identity holds since if \( \phi \in SV \) then
\[
a(x) c(y) \phi = a(x) [y \phi] = [a(x) y] \phi + y [a(x) \phi] = < x | y > \phi + c(y) a(x) \phi.
\]

When \( SV' \) is given the topology of pointwise convergence, the extended creators and annihilators are continuous.

Theorem 2.3. If \( v \in V \) then \( c(v) \) and \( a(v) \) are weakly continuous on \( SV' \).

Proof. Let \( (\Phi_\lambda : \lambda \in \Lambda) \) be a net converging weakly to \( \Phi \) in \( SV' \): if \( \psi \in SV \) then as \( \lambda \) runs over \( \Lambda \)
\[
[c(v) \Phi_\lambda](\psi) = \Phi_\lambda(a(v) \psi) \to \Phi(a(v) \psi) = [c(v) \Phi](\psi)
\]
whence \( c(v) \Phi_\lambda \to c(v) \Phi \) weakly and \( a(v) \Phi_\lambda \to a(v) \Phi \) similarly.

The extended creators and annihilators are indeed extensions of the originals relative to the canonical inclusion \( SV \to SV' \): let \( v \in V \) and \( \phi \in SV \); if also \( \psi \in SV \) then by Theorem 2.1 it follows that
\[
[c(v) < | \phi | \psi >](\psi) = < | \phi | (a(v) \psi) = < |c(v) \phi | (\psi)
\]
whence \( c(v) < | \phi | \phi > = < |c(v) \phi | \phi > \) and \( a(v) < | \phi | \phi > = < |a(v) \phi | \phi > \) likewise. Further, the extended creators and annihilators inherit the following properties from the originals.

Theorem 2.4. If \( v \in V \) then \( c(v) : SV' \to SV' \) is multiplication by \( < | v > \) and \( a(v) : SV' \to SV' \) is a derivation.

Proof. For the annihilator, let \( \Phi \) and \( \Psi \) lie in \( SV' \): Theorem 1.5 furnishes nets \( (\phi_\lambda : \lambda \in \Lambda) \) and \( (\psi_\lambda : \lambda \in \Lambda) \) in \( SV \) converging weakly to \( \Phi \) and \( \Psi \) respectively, so letting \( \lambda \) run over \( \Lambda \) in
\[
a(v)[\phi_\lambda \psi_\lambda] = [a(v) \phi_\lambda] \psi_\lambda + \phi_\lambda[a(v) \psi_\lambda]
\]
yields the desired equality
\[
a(v)[\Phi \Psi] = [a(v) \Phi] \Psi + \Phi[a(v) \Psi]
\]
on account of Theorem 1.4 and Theorem 2.2. For the creator, argue by weak continuity or let \( \Phi \in SV' \); if \( u \in V \) then
\[
[c(v)\Phi]_M = <c(v)\Phi|d> = \sum_{a+b=d} \frac{d!}{a!b!} <v|a|b> (u^a \otimes u^b)
\]
whence the discussion after Theorem 1.1 implies that \( c(v)\Phi = <v|\Phi > \).

Another familiar inherited property concerns the Fock vacuum \( 1 \in C = S^0V \).

**Theorem 2.5.** The antifunctionals in \( SV' \) killed by each annihilator are exactly the scalar multiples of \( <1|> \).

**Proof.** By definition, each annihilator vanishes on \( 1 \in SV \) and hence on \( <1|> \in SV' \). Conversely, let \( \Phi \in SV' \) lie in the kernel of each annihilator. If \( v_0, v_1, \ldots, v_m \in V \) then
\[
0 = [a(v_0)\Phi](v_1 \cdots v_m) = \Phi(v_0v_1 \cdots v_m)
\]
so that \( \Phi \) vanishes on \( \oplus d>0 S^dV \) and is therefore proportional to \( <1|> \). \[\square\]

Similarly or otherwise, it is easily checked that each creator is actually injective.

In order to consider creators and annihilators as operators in symmetric Fock space \( SV[V] \) we investigate their relationship to \( \| \cdot \| \). As preparation, let \( v \in V \) and let
\[
\Phi = \sum_{d \in N} \Phi^d \in SV'.
\]
Plainly, if \( d \in N \) then \( (c(v)\Phi)^{d+1} = c(v)\Phi^d \) and \( (a(v)\Phi)^d = a(v)\Phi^{d+1} \). Let also \( M \in F(V) \) contain \( v \); if \( \psi \in S^{d+1}V \) then
\[
(c(v)\Phi)^{d+1}(\psi) = c(v)\Phi^d(\psi) = \Phi^d(a(v)\psi)
\]
whence
\[
(c(v)\Phi)^{d+1}_M = c(v)\Phi^{d+1}_M
\]
and similarly
\[
(a(v)\Phi)^d_M = a(v)\Phi^{d+1}_M.
\]

**Theorem 2.6.** If \( v \in V \) and \( \Phi \in SV' \) then
\[
\|c(v)\Phi\|^2 = \|a(v)\Phi\|^2 + \|v\|^2\|\Phi\|^2.
\]

**Proof.** Of course, both sides of the putative equality are numbers in \( [0, \infty) \). If \( M \in F(V) \) then Theorem 2.2 and the canonical commutation relations in Theorem 2.2 imply that
\[
\|c(v)\Phi^d_M\|^2 = \|c(v)\Phi^d_M|c(v)\Phi^d_M\| > = \|c(v)\Phi^d_M|a(v)c(v)\Phi^d_M\| > = \|a(v)\Phi^d_M\|^2 + \|v\|^2\|\Phi^d_M\|^2
\]
whence the formulae derived prior to the Theorem imply that if \( M \) contains \( v \) then
\[
\|(c(v)\Phi)^{d+1}_M\|^2 = \|(a(v)\Phi)^{d+1}_M\|^2 + \|v\|^2\|\Phi^d_M\|^2.
\]
Summation over $d > 0$ together with the evident equalities $\|(c(v)\Phi)_{\lambda_d}\| = \|c(v)\Phi_{\lambda_d}\| = \|v\|\|\Phi_{\lambda_d}\|$ and $(c(v)\Phi)_{\lambda_d} = 0$ yields
\[
\|c(v)\Phi\| = \|a(v)\Phi\| + \|v\|\|\Phi\|.
\]
Passage to the supremum as $M$ runs over $F(V)$ while containing $v$ concludes the proof. \hfill \Box

When $v \in V$ we may now consider $c(v)$ and $a(v)$ as operators in $S[V]$: thus, $c(v)$ has natural domain $\{\Phi \in S[V] : c(v)\Phi \in S[V]\}$ and $a(v)$ has natural domain $\{\Phi \in S[V] : a(v)\Phi \in S[V]\}$. Note that these domains coincide by Theorem 1.11 and plainly contain $S V$.

**Theorem 2.7.** When $v \in V$ the operators $c(v)$ and $a(v)$ in $S[V]$ are mutual adjoints: $c(v)^* = a(v)$ and $a(v)^* = c(v)$.

**Proof.** To see that $a(v) \subset c(v)^*$ let $\Phi$ and $\Psi$ lie in the domain of $a(v)$ and $c(v)$. If $d > 0$ and $M \in F(V)$ contains $v$ then
\[
\left(\sum_{a=0}^{d-1} \Psi_M^a \Phi \right) = \left(\sum_{a=0}^{d-1} \Psi_M^a \Phi \right) = \Phi \left(\left(\sum_{a=0}^{d-1} \Psi_M^a \Phi \right) \right) = \Phi \left(\left(\sum_{a=0}^{d-1} \Psi_M^a \Phi \right) \right) = \sum_{a=0}^{d-1} \left(\sum_{a=0}^{d-1} \Psi_M^a \Phi \right),
\]
whence it follows by Theorem 1.9 that
\[
\langle \Psi | a(v) \Phi \rangle = \langle c(v) \Psi | \Phi \rangle.
\]
To see that $c(v)^* \subset a(v)$ let $\Phi$ lie in the domain of $c(v)^*$. If $\psi \in SV$ then Theorem 1.11 implies that
\[
\langle c(v)^* \Phi | \psi \rangle = \langle \psi | c(v)^* \Phi \rangle = \langle c(v) \psi | \Phi \rangle = \psi | \Phi \rangle = a(v) \Phi.
\]
whence $a(v) \Phi = c(v)^* \Phi \in S[V]$. Thus $c(v)^* = a(v)$; likewise $a(v)^* = c(v)$. \hfill \Box

As a corollary, the operators $c(v)$ and $a(v)$ in $S[V]$ are closed: more directly, this may be seen as follows. Let $(\Phi_j : j \in \mathbb{N})$ be a sequence in the domain of $c(v)$ such that as $j \to \infty$ both $\Phi_j \to \Phi$ and $c(v)\Phi_j \to \Psi$ in $S[V]$. On the one hand, as $\Phi_j \to \Phi$ in $S[V]$ so $\Phi_j \to \Phi$ in $SV'$ by Theorem 1.11 and therefore $c(v)\Phi_j \to c(v)\Phi$ in $SV'$ by Theorem 1.3 on the other hand, $c(v)\Phi_j \to \Psi$ in $S[V]$ and hence in $SV'$. Thus $c(v)\Phi = \Psi \in S[V]$ and so $c(v)$ is closed.

We shall require certain precise estimates for the norms of a creator and its powers on homogeneous elements of $S[V]$. For these, let $v \in V$ be (without loss) a unit vector. Let $\phi \in S^d V$ and choose $M \in F(V)$ so that $v \in M$ and $\phi \in S^d M$. Extend $v = v_0$ to a unitary basis $(v_0, v_1, \ldots, v_m)$ for $M$. From Theorem 1.11 it follows that
\[
\phi = \sum_D \phi_D \frac{v_0^{d_0} v_1^{d_1} \cdots v_m^{d_m}}{\sqrt{d_0! d_1! \cdots d_m!}},
\]
where summation extends over all multiindices $D = (d_0, d_1, \ldots, d_m)$ with $d_0 + d_1 + \cdots + d_m = d$. Now
\[
v\phi = \sum_D \sqrt{(d_0 + 1)!} \phi_D \frac{v_0^{d_0+1} v_1^{d_1} \cdots v_m^{d_m}}{\sqrt{(d_0 + 1)! d_1! \cdots d_m!}}.
\]
so
\[ \|v\phi\|^2 = \sum_D (d_0 + 1)|\phi_D|^2 \leq (d + 1)\|\phi\|^2. \]

This elementary estimate is the basis for the following result.

**Theorem 2.8.** Let \( v \in V \) and let \( a \in \mathbb{N} \). If \( b \in \mathbb{N} \) and \( \Phi \in S^b[V] \) then
\[ \|v^a\Phi\|^2 \leq \frac{(a + b)!}{a! b!}\|v^a\|^2\|\Phi\|^2. \]

**Proof.** Allowing \( v \) to have arbitrary norm, the inequality immediately prior to the Theorem shows that if \( \phi \in S^bV \) then
\[ \|v\phi\|^2 \leq (b + 1)\|v\|^2\|\phi\|^2 \]
whence induction shows that
\[ \|v^a\phi\|^2 \leq (a + b)\cdots(1 + b)\|v\|^{2a}\|\phi\|^2 = \frac{(a + b)!}{a! b!}\|v^a\|^2\|\phi\|^2. \]

Thus, if \( \Phi \in S^b[V] \) and \( M \in \mathcal{F}(V) \) contains \( v \) then
\[ \|(v^a\Phi)_M\|^2 \leq \frac{(a + b)!}{a! b!}\|v^a\|^2\|\Phi_M\|^2 \]
and so passage to the supremum confirms the claimed equality. \( \square \)

In fact, if \( v \in V \) and \( a, b \in \mathbb{N} \) then the operator norm of \( c(v)^a : S^b[V] \to S^{a+b}[V] \) is exactly \( \sqrt{(a + b)!/a! b!}\|v^a\| \) as may be checked by computing \( \|c(v)^a v^b\| \).

Regarding exponentials let us begin simply, considering first the exponentials in \( SV' \) of vectors in \( V \). To be precise, when \( z \in V \) we define
\[ (2.1) \quad e^z := \sum_{n \in \mathbb{N}} \frac{z^n}{n!} \in SV'. \]

As usual, this formal power series is (in the first instance) weakly convergent, for individual elements of \( SV \) vanish in sufficiently high degrees.

These simple exponentials are called coherent vectors; they are common eigenvectors for the annihilators.

**Theorem 2.9.** If \( v \) and \( z \) lie in \( V \) then
\[ a(v)[e^z] = <v|z > e^z. \]

**Proof.** As \( a(v) \) is a derivation, if \( n \in \mathbb{N} \) then \( a(v)[z^n] = n <v|z > z^{n-1} \) so
\[ a(v)\left( \frac{z^n}{n!} \right) = <v|z > \frac{z^{n-1}}{(n-1)!} \]
from which the Theorem follows upon summation by virtue of the weak continuity expressed in Theorem 2.3. \( \square \)

In fact, these simple exponentials converge not only in \( SV' \) but also in \( S[V] \).

**Theorem 2.10.** If \( z \in V \) then the coherent vector \( e^z \) lies in \( S[V] \) and
\[ \|e^z\|^2 = e\|z\|^2. \]

**Proof.** If \( M \in \mathcal{F}(V) \) contains \( z \) then of course \( (e^z)_M = e^z \) and
\[ \|(e^z)_M\|^2 = \sum_{n \in \mathbb{N}} \frac{\|z^n\|^2}{(n!)^2} = \sum_{n \in \mathbb{N}} \frac{\|z\|^{2n}}{n!} = e\|z\|^2. \]
Now pass to the supremum as \( M \) runs over \( \mathcal{F}(V) \) while containing \( z \). \( \square \)
More generally, if \( x, y \in V \) then the coherent vectors \( e^x \) and \( e^y \) have inner product
\[
\langle e^x | e^y \rangle = e^{\langle x | y \rangle}.
\]

**Theorem 2.11.** The coherent vectors \( \{ e^z : z \in V \} \) constitute a linearly independent total set in \( S[V] \).

**Proof.** Let \( z_1, \ldots, z_m \in V \) be distinct and assume that \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \) are such that
\[
\lambda_1 e^{z_1} + \cdots + \lambda_m e^{z_m} = 0
\]
whence the taking of homogeneous components yields
\[
d \in \mathbb{N} \Rightarrow \lambda_1 z_1^d + \cdots + \lambda_m z_m^d = 0.
\]
Select \( v \in V \) outside the finite union
\[
\bigcup \{ \ker \langle \cdot | z_q - z_p \rangle : 1 \leq p < q \leq m \}
\]
of hyperplanes, so that the complex numbers \( \langle v | z_1 \rangle, \ldots, \langle v | z_m \rangle \) are distinct. Now
\[
d \in \mathbb{N} \Rightarrow \langle v^d | \lambda_1 z_1^d + \cdots + \lambda_m z_m^d \rangle = 0
\]
so
\[
d \in \mathbb{N} \Rightarrow \langle v | z_1 \rangle^d \lambda_1 + \cdots + \langle v | z_m \rangle^d \lambda_m = 0.
\]
This Vandermonde system forces the vanishing of \( \lambda_1, \ldots, \lambda_m \). This proves that the coherent vectors are linearly independent; we prove that their linear span is dense in \( S[V] \) as follows.

Let
\[
\Phi = \sum_{d \in \mathbb{N}} \Phi^d \in S[V]
\]
and suppose that \( \langle \Phi | e^z \rangle = 0 \) whenever \( z \in V \). If \( z \in V \) is fixed and \( \lambda \in \mathbb{C} \) varies then
\[
0 = \langle \Phi | e^{\lambda z} \rangle = \sum_{d \in \mathbb{N}} \langle \Phi^d | z^d \rangle \lambda^d d!
\]
whence equating coefficients shows that if \( d \in \mathbb{N} \) then \( \langle \Phi^d | z^d \rangle = 0 \). As \( z \in V \) is arbitrary, so \( \Phi \) vanishes in each degree, on account of Theorem 1.12 and the discussion following Theorem 1.1.

Of special importance are Gaussians: the exponentials of quadratics. Let the quadratic \( \zeta \in S^2 V' \) correspond to the symmetric antilinear map \( Z : V \to V' \) according to (1.7). We define the associated Gaussian by
\[
e^Z = \exp(\zeta) = \sum_{n \in \mathbb{N}} \frac{\zeta^n}{n!} \in SV'
\]
where the formal series converges weakly because individual elements of \( SV \) vanish in sufficiently high degree.

On Gaussians, annihilators act essentially as creators.

**Theorem 2.12.** If \( v \in V \) and if \( Z : V \to V' \) is symmetric antilinear then
\[
a(v) e^Z = (Zv) e^Z.
\]

**Proof.** Let \( Z \) correspond to the quadratic \( \zeta \in S^2 V' \) as usual: the rule (1.7) implies that \( a(v) \zeta = Zv \); hence Theorem 2.4 implies that if \( n \in \mathbb{N} \) then \( a(v) \zeta^n = n(Zv) \zeta^{n-1} \) so Theorem 1.4 and Theorem 2.3 imply that \( a(v) \exp \zeta = (Zv) \exp \zeta \). □

Contrary to the case for coherent vectors, Gaussians do not automatically lie in symmetric Fock space: in fact, we claim that \( e^Z \) lies in \( S[V] \) precisely when \( Z \) is of Hilbert-Schmidt class and has operator norm strictly less than unity.
In order to establish this claim, it is convenient to begin by supposing that $V$ is finite-dimensional. In this case, let $Z : V \to Z$ be a symmetric antilinear map and note that $Z^2$ is then a selfadjoint (indeed, positive) complex-linear map:

$$v \in V \Rightarrow < v | Z^2 v > = \| Z v \|.$$  

By diagonalization, $V$ has a unitary basis $(v_1, \ldots, v_m)$ such that if $1 \leq k \leq m$ then $Z v_k = \lambda_k v_k$ with $\lambda_k \geq 0$; in these terms,

$$\operatorname{Det}(I - Z^2) = (1 - \lambda_1^2) \cdots (1 - \lambda_m^2)$$

$$\| Z \| = \max(\lambda_1, \ldots, \lambda_m).$$

The quadratic $\zeta \in S^2 V$ to which $Z$ corresponds canonically is given by

$$\zeta = \frac{1}{2} \sum_{k=1}^{m} \lambda_k v_k^2$$

so that if $n \in \mathbb{N}$ then

$$\zeta^n = \sum_{N} \left( \begin{array}{c} n \\ (n_1 \cdots n_m) \end{array} \right) \left( \frac{\lambda_1}{2} \right)^{n_1} \cdots \left( \frac{\lambda_m}{2} \right)^{n_m} v_1^{2n_1} \cdots v_m^{2n_m}$$

and

$$\frac{\| \zeta^n \|^2}{(n!)^2} = \sum_{N} \left( \begin{array}{c} 2n_1 \\ n_1 \end{array} \right) \cdots \left( \begin{array}{c} 2n_m \\ n_m \end{array} \right) \left( \frac{\lambda_1}{2} \right)^{2n_1} \cdots \left( \frac{\lambda_m}{2} \right)^{2n_m}$$

where summation takes place over all multiindices $N = (n_1, \ldots, n_m) \in \mathbb{N}^m$ for which $n = n_1 + \cdots + n_m$. Consequently,

$$\| \exp \zeta \|^2 = \sum_{n \in \mathbb{N}} \frac{\| \zeta^n \|^2}{(n!)^2}$$

$$= \sum_{n_1 \in \mathbb{N}} \left( \begin{array}{c} 2n_1 \\ n_1 \end{array} \right) \left( \frac{\lambda_1}{2} \right)^{2n_1} \cdots \sum_{n_m \in \mathbb{N}} \left( \begin{array}{c} 2n_m \\ n_m \end{array} \right) \left( \frac{\lambda_m}{2} \right)^{2n_m}$$

$$= (1 - \lambda_1^2)^{-\frac{1}{2}} \cdots (1 - \lambda_m^2)^{-\frac{1}{2}}$$

$$= \operatorname{Det}^{\frac{1}{2}}(I - Z^2)^{-1}$$

provided that each of the nonnegative numbers $\lambda_1, \ldots, \lambda_m$ is strictly less than unity.

We may now establish the claim in full generality.

**Theorem 2.13.** If $Z \in \Sigma^2[V]$ and $\| Z \| < 1$ then $e^Z \in S[V]$ and

$$\| e^Z \|^2 = \operatorname{Det}^{\frac{1}{2}}(I - Z^2)^{-1}.$$  

**Proof.** Let $\zeta \in S^2[V]$ be the canonical correspondent to $Z \in \Sigma^2[V]$ as in (1.7). If $M \in \mathcal{F}(V)$ and if $\zeta_M \in S^2 M$ corresponds to $Z_M : M \to M$ then (by Theorem 1.6 say) $(\exp \zeta)_M = \exp(\zeta_M)$ so that $(e^Z)_M = e^{Z_M}$ while $\| Z_M \| \leq \| Z \| < 1$. The finite-dimensional calculation prior to the Theorem yields

$$\| (e^Z)_M \|^2 = \operatorname{Det}^{\frac{1}{2}}(I - Z_M^2)^{-1}.$$  

On the one hand, the net $(\| (e^Z)_M \| : M \in \mathcal{F}(V))$ is increasing by the discussion prior to Theorem 1.7 and indeed converges to $\| e^Z \|$ by definition; on the other hand, the limit of the net $(\operatorname{Det}(I - Z_M^2) : M \in \mathcal{F}(V))$ is $\operatorname{Det}(I - Z^2)$ by trace-norm continuity (or very definition) of the Fredholm determinant. \qed
Conversely, let $Z : V \to V'$ be symmetric antilinear with correspondent $\zeta \in S^2V'$ and suppose that $e^Z \in S^2V$. If $M \in \mathcal{F}(V)$ then the proof of Theorem \ref{Theorem1.13} yields

$$||Z_M||_{HS} = 2||\zeta_M||^2 < 2\exp \zeta_M||^2 = 2||e^{Z_M}||^2 \leq 2||e^Z||^2$$

whence $Z \in \Sigma^2[V]$. Further, $||Z|| < 1$ if $||Z|| \geq 1$ then let $u \in V$ be an eigenvector for $Z$ with eigenvalue $\lambda \geq 1$; setting $M = \mathbb{C}u \in \mathcal{F}(V)$ yields

$$||(e^Z)_M||^2 = \sum_{n \in \mathbb{N}} \left(\frac{2n}{n}\right) \left(\frac{\lambda}{2}\right)^{2n} = \infty$$

which places $e^Z$ outside $S[V]$.

More generally, we may explicitly compute the inner product between a pair of Gaussians in symmetric Fock space as follows.

**Theorem 2.14.** If $X$ and $Y$ in $\Sigma^2[V]$ have operator norms strictly less than unity then

$$<e^X|e^Y> = \text{Det}^\frac{1}{2}(I - XY)^{-1}.$$  

**Proof.** It follows from Theorem \ref{Theorem1.13} by the principle of analytic continuation that if $M \in \mathcal{F}(V)$ then

$$<e^{X_M}|e^{Y_M}> = \text{Det}^\frac{1}{2}(I - Y_MX_M)^{-1}$$

since both sides are respectively (antiholomorphic, holomorphic) in $(X_M, Y_M)$ and agree when $X_M = Y_M$. Now pass to the limit as $M$ runs over $\mathcal{F}(V)$ while taking into account Theorem \ref{Theorem1.8} and continuity of the determinant. \hfill \Box

Incidentally, it is perhaps worth recording a related formula. Let $z \in V$ and let $\zeta \in S^2[V]$ correspond to $Z \in \Sigma^2[V]$ with $||Z|| < 1$. By induction, if $n \in \mathbb{N}$ then

$$<z^{2n}|\zeta^n> = (2n)! \left(\frac{1}{2} < z|zz>\right)^n$$

so that by summation

$$<e^z|e^Z> = \exp \left(\frac{1}{2} < z|zZ>\right).$$

**Theorem 2.15.** Let $Z \in \Sigma^2[V]$ and let $||Z|| < 1$. If $\phi \in SV$ then $\phi e^Z \in S[V]$.

**Proof.** As usual, linearity and polarization grant us the right to suppose that $\phi = v^n$ for $v \in V$ a unit vector and $n \in \mathbb{N}$. Let $Z$ correspond to $\zeta \in S^2[V]$ and choose $s > 1$ so that $||sZ|| < 1$. From Theorem \ref{Theorem2.8} it follows at once that

$$||v^n e^Z||^2 \leq \sum_{k \in \mathbb{N}} (2k + n) \cdot (2k + 1) \frac{||\zeta^n||^2}{(k!)^2}.$$  

Now, the power series

$$\sum_{k=0}^{\infty} \frac{||\zeta^n||^2}{(k!)^2} t^{2k}$$

and

$$\sum_{k=0}^{\infty} (2k + n) \cdot (2k + 1) \frac{||\zeta^n||^2}{(k!)^2} t^{2k}$$

have the same radius of convergence; the former converges when $t = s > 1$ so the latter necessarily converges at $t = 1$. \hfill \Box

We can say a little more about $e^Z$ when $Z \in \Sigma^2[V]$ and $||Z|| < 1$: from Theorem \ref{Theorem2.15} it follows that $e^Z$ lies in the domain of each creator polynomial; in fact, by Theorem \ref{Theorem2.14} it follows further that $e^Z$ lies in the domain of each polynomial in creators and annihilators.
So far as symmetric Fock space itself is concerned, there is little point to considering the exponentials of homogeneous elements in SV' having degree greater than two: such exponentials do lie in SV' of course, but they only lie in S[V] when the homogeneous element is zero.

**Theorem 2.16.** Let \( \zeta \in S^d V' \) be homogeneous of degree \( d > 2 \). If \( \exp \zeta \) lies in \( S[V] \) then \( \zeta = 0 \).

**Proof.** If \( \exp \zeta \) lies in \( S[V] \) then of course its degree \( d \) component \( \zeta \) lies in \( S^d[V] \). Let \( v \in V \) be a unit vector and let \( M = \mathbb{C} v \in \mathcal{F}(V) \) so that \( \zeta_M = \lambda v^d \) for some \( \lambda \in \mathbb{C} \): from

\[
\| \exp(\lambda v^d) \|^2 = \sum_{n \in \mathbb{N}} \frac{\| (\lambda v^d)^n \|^2}{(n!)^2} = \sum_{n \in \mathbb{N}} \frac{(dn)!}{(n!)^2} |\lambda|^{2n}
\]

and

\[
\| \exp \zeta_M \| \leq \| \exp \zeta \| < \infty
\]

it follows that \( \lambda = 0 \) whence

\[
<v^d|\zeta> = <v^d|\zeta_M> = <v^d|\lambda v^d> = d! \lambda = 0.
\]

To complete the proof, invoke Theorem 1.12 in conjunction with the remark following Theorem 1.11.

\[ \square \]

### 3. Generalized Fock implementation

The imaginary part \( \Omega \) of the complex inner product \( < \cdot | \cdot > \) on \( V \) is a real symplectic form: an alternating real-bilinear form that is (strongly) nonsingular in the sense that the correspondence \( v \leftrightarrow \Omega(v, \cdot) \) is an isomorphism between \( V \) and its real dual. The corresponding symplectic group \( \text{Sp}(V) \) comprises all real-linear automorphisms \( g \) of \( V \) that are symplectic in the sense

\[
x, y \in V \Rightarrow \Omega(gx, gy) = \Omega(x, y).
\]

Note that each \( g \in \text{Sp}(V) \) is automatically bounded: as may be verified by direct calculation, its adjoint relative to the real inner product \( (\cdot | \cdot) = \text{Re} < \cdot | \cdot > \) on \( V \) is given by \( g^* = -J g^{-1} J \).

As is the case for any real-linear endomorphism of a complex vector space, each \( g \in \text{Sp}(V) \) decomposes uniquely as \( g = C_g + A_g \) where \( C_g = \frac{1}{2}(g - JgJ) \) is complex-linear and \( A_g = \frac{1}{2}(g + JgJ) \) is antilinear.

**Theorem 3.1.** If \( g \in \text{Sp}(V) \) then \( C_g^* = C_{g^{-1}} \) and \( A_g^* = -A_{g^{-1}} \) where adjunction is relative to the real inner product \( (\cdot | \cdot) \) on \( V \).

**Proof.** This follows at once from the formulae for \( C_g \) and \( A_g \) displayed prior to the Theorem, since \( J \) is skew-adjoint and \( g^* = -J g^{-1} J \).

In terms of the complex inner product \( < \cdot | \cdot > \) itself, if \( g \in \text{Sp}(V) \) and \( x, y \in V \) then

\[
< C_g x | y > = < x | C_g^{-1} y >
\]

\[
< x | A_g y > + < y | A_g^{-1} x > = 0.
\]

**Theorem 3.2.** If \( g \in \text{Sp}(V) \) then

\[
C_g^{-1} C_g + A_g^{-1} A_g = I
\]

\[
A_g^{-1} C_g + C_g^{-1} A_g = O.
\]

**Proof.** This is actually valid for any real-linear automorphism \( g \) of \( V \) and follows upon taking complex-linear and antilinear parts in

\[
(C_g^{-1} + A_g^{-1})(C_g + A_g) = g^{-1} g = I.
\]

\[ \square \]
Note that Theorem 3.1 and Theorem 3.2 together imply that if \( g \in \text{Sp}(V) \) then
\[
C_g^*C_g - A_g^*A_g = I
\]
\[
A_g^*C_g = C_g^*A_g.
\]
Thus, \( C_g^*A_g \) is real self-adjoint and if \( v \in V \) then
\[
\|C_g v\|^2 = \|A_g v\|^2 + \|v\|^2.
\]

**Theorem 3.3.** If \( g \in \text{Sp}(V) \) then its complex-linear part \( C_g \) is invertible.

**Proof.** The formula immediately prior to the Theorem shows that \( C_g \) is injective and indeed bounded below by unity. Similarly \( C_g^{-1} \) is injective, so Theorem 3.1 implies that \( C_g \) has dense range. Together, these facts force \( C_g \) to be invertible. \(\square\)

This justifies associating to each \( g \in \text{Sp}(V) \) the antilinear operator
\[
Z_g = -A_g C_g^{-1} = C_g^{-1} A_g^{-1}
\]
which is symmetric antilinear and has operator norm strictly less than unity by virtue of the formulae recorded after Theorem 3.2.

We shall find it convenient to introduce transformed creators and annihilators. Thus, let \( g \in \text{Sp}(V) \): for \( v \in V \) we define
\[
c_g(v) = c(C_g v) + a(A_g v)
\]
\[
a_g(v) = a(C_g v) + c(A_g v)
\]
as operators on \( SV \) and \( SV' \). These transformed creators and annihilators continue to satisfy the canonical commutation relations.

**Theorem 3.4.** If \( g \in \text{Sp}(V) \) and \( x, y \in V \) then
\[
[a_g(x), a_g(y)] = 0
\]
\[
[a_g(x), c_g(y)] = <x|y>I
\]
\[
[c_g(x), c_g(y)] = 0.
\]

**Proof.** Simple application of Theorem 3.1 and Theorem 3.2 to the canonical commutation relations of Theorem 2.2 taking the central identity for example,
\[
[a_g(x), c_g(y)] = [a(C_g x), c(C_g y)] + [c(A_g x), a(A_g y)]
\]
\[
= \{<C_g x|C_g y> - <a_g y|A_g x>\}I
\]
\[
= \{<x|C_g^{-1}C_g y> + <x|A_g^{-1}A_g y>\}I
\]
\[
= <x|y>I.
\]
\(\square\)

We remark further from Theorem 3.2 with \( g \in \text{Sp}(V) \) replaced by its inverse that if \( v \in V \) then
\[
c(v) = c_g(C_g^{-1} v) + a_g(A_g^{-1} v)
\]
\[
a(v) = a_g(C_g^{-1} v) + c_g(A_g^{-1} v).
\]
Now, the generalized Fock representation of $V$ is set up as follows. For $v \in V$ we define $\pi(v)$ as a complex-linear endomorphism of either the symmetric algebra $SV$ or its full antidual $SV'$ by the rule
\begin{equation}
\pi(v) = \frac{1}{\sqrt{2}} \{ c(v) + a(v) \}
\end{equation}
whence if $\Phi \in SV'$ and $\psi \in SV$ then
\[ [\pi(v)\Phi](\psi) = \Phi(\pi(v)\psi) \]
Note that if $v \in V$ then as $c(Jv) = ic(v)$ and $a(Jv) = -ia(v)$ so
\[ c(v) = \frac{1}{\sqrt{2}} \{ \pi(v) - i\pi(Jv) \} \]
\[ a(v) = \frac{1}{\sqrt{2}} \{ \pi(v) + i\pi(Jv) \} \]

The generalized Fock representation $\pi$ of $V$ is projective: it satisfies the Heisenberg form of the canonical commutation relations on $SV$ and $SV'$ (without qualification) as follows.

**Theorem 3.5.** If $x, y \in V$ then
\[ [\pi(x), \pi(y)] = i\Omega(x, y)I. \]

**Proof.** That the displayed equations hold on both $SV$ and $SV'$ follows at once from the canonical commutation relations in Theorem 2.2:
\[ [\pi(x), \pi(y)] = \frac{1}{2} [a(x), c(y)] + \frac{1}{2} [c(x), a(y)] \]
\[ = \frac{1}{2} \{ <x|y> - <y|x> \} I \]
\[ = i\Omega(x, y)I. \]

The generalized Fock representation is also weakly irreducible.

**Theorem 3.6.** If the linear map $T : V \to V'$ commutes with $\pi$ in the sense
\[ v \in V \Rightarrow T\pi(v) = \pi(v)T \]
then $T$ is a scalar (multiple of the canonical inclusion).

**Proof.** Here, $\pi(v) \in$ End $SV$ on the left and $\pi(v) \in$ End $SV'$ on the right. Taking complex-linear and antilinear parts in the hypothesized condition, if $v \in V$ then $Tc(v) = c(v)T$ and $Ta(v) = a(v)T$. Now
\[ v \in V \Rightarrow a(v)T1 = Ta(v)1 = 0 \]
whence Theorem 2.5 yields $\lambda \in \mathbb{C}$ such that $T1 = \lambda 1$. Finally, if $v_1, \ldots, v_n \in V$ then
\[ T(v_1 \cdots v_n) = Tc(v_1) \cdots c(v_n)1 = c(v_1) \cdots c(v_n)1 = \lambda v_1 \cdots v_n \]
and linearity concludes the argument.

Now, let $g \in \text{Sp}(V)$. The transformed representation $\pi \circ g$ of $V$ on $SV'$ given by
\begin{equation}
v \in V \Rightarrow \pi \circ g(v) = \pi(gv) = \frac{1}{\sqrt{2}} \{ c_g(v) + a_g(v) \}
\end{equation}
also satisfies the Heisenberg form of the canonical commutation relations: this may be seen by applying Theorem 3.4 (rather than Theorem 2.2) in the proof of Theorem 3.5. Accordingly, it is reasonable to ask whether the representations $\pi \circ g$ and $\pi$ are equivalent in any sense.
By a generalized Fock implem eter for $g \in \text{Sp}(V)$ we shall mean a (nonzero) linear map $U : SV \to SV'$ that intertwines $\pi$ and $\pi \circ g$ in the sense

$$v \in V \Rightarrow U\pi(v) = \pi(gv)U$$

where $\pi(v) \in \text{End} SV$ and $\pi(gv) \in \text{End} SV'$.

**Theorem 3.7.** The linear map $U : SV \to SV'$ is a generalized Fock implementer for $g \in \text{Sp}(V)$ precisely when

$$v \in V \Rightarrow \begin{cases} Uc(v) = c_g(v)U \\ Ua(v) = a_g(v)U. \end{cases}$$

**Proof.** In the one direction, taking complex-linear and antilinear parts in the equation (3.4) defining $U$ as a generalized Fock implementer yields the displayed equations; in the other direction, adding the displayed equations reveals $U$ as a generalized Fock implementer in view of (3.2) and (3.3). \hfill \Box

It follows easily by the observation after Theorem 3.4 that $U : SV \to SV'$ is a generalized Fock implementer for $g \in \text{Sp}(V)$ exactly when

$$v \in V \Rightarrow \begin{cases} Uc^{-1}(v) = c(v)U \\ Ua^{-1}(v) = a(v)U. \end{cases}$$

By a generalized Fock vacuum for $g \in \text{Sp}(V)$ we shall mean a (nonzero) vector $\Phi \in SV'$ such that

$$v \in V \Rightarrow \{\pi(gv) + i\pi(gJv)\} \Phi = 0$$

or equivalently

$$v \in V \Rightarrow a_g(v)\Phi = 0.$$

**Theorem 3.8.** If $g \in \text{Sp}(V)$ then the rule $\Phi = U\Phi_U$ sets up a bijective correspondence between the set of all generalized Fock vacua $\Phi \in SV'$ for $g$ and the set of all generalized Fock implementers $U : SV \to SV'$ for $g$.

**Proof.** On the one hand, if $U$ is a generalized Fock implementer and if $v \in V$ then Theorem 3.7 implies that

$$a_g(v)U1 = Ua(v)1 = 0$$

whence $U1$ is a generalized Fock vacuum. On the other hand, if $\Phi$ is a generalized Fock vacuum then the canonical commutation relations in Theorem 3.4 enable us to define a generalized Fock implementer $U$ by $U1 = \Phi$ and the rule that if $v_1, \ldots, v_n \in V$ then

$$U(v_1 \cdots v_n) = c_g(v_1) \cdots c_g(v_n)\Phi.$$

Finally, it is plain that $\Phi \leftrightarrow U$ is a bijective correspondence. \hfill \Box

Recall that in (3.1) we associated to each $g \in \text{Sp}(V)$ the symmetric antilinear operator $Z_g = -A_gC_g^{-1}$ with operator norm strictly less than unity; denote the corresponding quadratic by $\zeta_g \in S^2V'$ so that

$$v \in V \Rightarrow a(v)\zeta_g = Z_gv.$$

**Theorem 3.9.** The generalized Fock vacua for $g \in \text{Sp}(V)$ are precisely the scalar multiples of the Gaussian

$$e^{Z_g} = \exp(\zeta_g) \in SV'.$$
Proof. Let \( \Phi = \sum_{d \in \mathbb{N}} \Phi^d \) be a generalized Fock vacuum for \( g \). Upon taking homogeneous components, the generalized Fock vacuum condition following (3.5) on \( \Phi \) yields that if \( v \in V \) then \( a(C_g v)\Phi^1 = 0 \) (the \( d = 0 \) equation) while
\[
d > 0 \Rightarrow a(C_g v)\Phi^{d+1} + c(A_g v)\Phi^{d-1}.
\]
By Theorem 3.3 it follows that if \( v \in V \) then \( a(v)\Phi^1 = 0 \) (the \( d = 0 \) equation) while \( d > 0 \Rightarrow a(v)\Phi^{d+1} = c(Z_g v)\Phi^{d-1} \).

The \( d = 0 \) equation forces \( \Phi^1 \) to vanish and the even \( d > 0 \) equations then force all odd-degree components of \( \Phi \) to vanish by Theorem 2.5. The \( d = 1 \) equation forces \( \Phi^2 \) to equal \( \Phi^0 \zeta_g \) and the odd \( d > 0 \) equations then force \( \Phi^d \), \( \Phi = \Phi^0 \) \( \exp(\zeta_g) \) by induction. In the opposite direction, each scalar multiple of \( \exp(\zeta_g) \) is a generalized Fock vacuum for \( g \) either by essentially the same argument or by Theorem 2.12.

We are now able to establish the unconditional existence of generalized Fock implementers.

**Theorem 3.10.** The generalized Fock implementers for \( g \in \text{Sp}(V) \) are precisely the scalar multiples of \( U_g : SV \rightarrow SV' \) defined by \( U_g 1 = e^{Z_g} \) and the rule that if \( v_1, \ldots, v_n \in V \) then
\[
U_g (v_1 \cdots v_n) = c_g(v_1) \cdots c_g(v_n)e^{Z_g}.
\]

Proof. Of course, this is an immediate consequence of Theorem 3.8 and Theorem 3.9.

We remark that if \( g \in \text{Sp}(V) \) then the specific generalized Fock implementer \( U_g : SV \rightarrow SV' \) so defined is distinguished by having generalized vacuum expectation value unity in the sense that \( [U_g 1](1) = 1 \).

By extension of the usual notion, if \( T : SV \rightarrow SV' \) is a linear map then its adjoint is the linear map \( T^* : SV \rightarrow SV' \) defined by
\[
\phi, \psi \in SV \Rightarrow [T^* \phi](\psi) = [\overline{T\psi}](\overline{\phi}).
\]

**Theorem 3.11.** If \( g \in \text{Sp}(V) \) then \( U_g^* = U_{g^{-1}} \).

Proof. This proceeds with the aid of Theorem 3.7 and the remark thereafter: if \( v \in V \) and \( \phi, \psi \in SV \) then
\[
[U_g^* a(v) \phi](\psi) = [U_g \psi](a(v) \phi) = [c(v) U_g \psi](\phi) = [U_g c_{g^{-1}}(v) \psi](\phi) = [U_g^* \phi](c_{g^{-1}}(v) \psi) = [a_{g^{-1}}(v) U_g^* \phi](\psi)
\]
whence
\[
U_g^* a(v) = a_{g^{-1}} U_g^*
\]
while
\[
U_g^* c(v) = c_{g^{-1}} U_g^*
\]
similarly; finally,
\[
[U_g^* 1](1) = [U_g 1](1) = 1.
\]

Now traditionally, the Fock representation and Fock implementers act in symmetric Fock space \( S[V] \). The relationships between our generalized notions and the traditional ones are as follows.
First of all, let \( v \in V \). The generalized Fock operator \( \pi(v) : SV' \to SV' \) restricts to define in \( S[V] \) an operator also denoted by \( \pi(v) \) having natural domain
\[
\{ \Phi \in S[V] : \pi(v)\Phi \in S[V] \}.
\]
An argument along similar lines to that for Theorem 2.7 shows that this traditional Fock operator \( \pi(v) \) with the above domain is self-adjoint: \( \pi(v)^* = \pi(v) \). We point out that Theorem 3.5 is not true for these traditional Fock operators without qualification: domain technicalities enter into the (Heisenberg) canonical commutation relations, thus
\[
x, y \in V \Rightarrow [\pi(x), \pi(y)] \subset i\Omega(x, y)I.
\]
Again let \( g \in \text{Sp}(V) \). In the traditional context, it is natural to seek conditions necessary and sufficient for the existence of a unitary operator \( U : S[V] \to S[V] \) such that
\[
v \in V \Rightarrow U\pi(v) = \pi(gv)U.
\]
As Theorem 3.10 furnishes a linear map \( U_g : SV \to SV' \) such that
\[
v \in V \Rightarrow U_g\pi(v) = \pi(gv)U_g
\]
it is clear that the problem to solve now is essentially one of normalization.

**Theorem 3.12.** If \( g \in \text{Sp}(V) \) is such that \( A_g \) is of Hilbert-Schmidt class then the prescription
\[
U(g) : = \|e^{Z_g}\|^{-1}U_g
\]
determines a unitary operator on \( S[V] \).

**Proof.** Let \( A_g \) be Hilbert-Schmidt. The symmetric antilinear operator \( Z_g \) is now Hilbert-Schmidt also; as \( \|Z_g\| < 1 \) already, Theorem 2.13 places \( e^{Z_g} \) in \( S[V] \) with
\[
\|e^{Z_g}\|^4 = \text{Det}(I - Z_g^2)^{-1}.
\]
Normalizing, define \( U(g) = \|e^{Z_g}\|^{-1}U_g \) as announced. The corresponding generalized Fock vacuum \( \Phi(g) = U(g)1 = \|e^{Z_g}\|^{-1}e^{Z_g}1 \in S[V] \) is a unit vector in the domain of every creator-annihilator polynomial, on account of the remark after Theorem 2.13. From the definition of \( U_g \) in Theorem 3.10 it now follows that \( U(g) : SV \to S[V] \). To see that \( U(g) : SV \to S[V] \) is isometric, let \( x_1, \ldots, x_r, y_1, \ldots, y_s \in V \); the canonical commutation relations in Theorem 3.4 yield
\[
\langle U(g)(x_1 \cdots x_r) | U(g)(y_1 \cdots y_s) \rangle = \langle c_g(x_1) \cdots c_g(x_r) \Phi(g) | c_g(y_1) \cdots c_g(y_s) \Phi(g) \rangle
\]
\[
= \langle \Phi(g) | a_g(x_r) \cdots a_g(x_1) c_g(y_1) \cdots c_g(y_s) \Phi(g) \rangle
\]
\[
= \langle x_1 \cdots x_r | y_1 \cdots y_s \rangle
\]
by virtue of Theorem 2.7. Of course, parallel remarks apply to \( U(g^{-1}) \) because \( Z_{g^{-1}} = C_g^{-1}A_g \) is Hilbert-Schmidt. To see that the isometric extension \( \overline{U(g)} : S[V] \to S[V] \) is unitary, note that
\[
U(g^{-1}) = U(g)^* \text{ by Theorem 3.11 and the fact that } I - Z_{g^{-1}}^2 = C_g^{-1}(I - Z_g^2)C_g \text{ from Theorem 3.2.}
\]
Now, if \( \phi, \psi \in SV \) then Theorem 1.11 shows that
\[
\langle \phi | U(g)\psi \rangle = \|U(g)\| \langle \phi | \psi \rangle = \langle \overline{U(g)}^* \phi | \psi \rangle = \langle U(g^{-1})\phi | \psi \rangle
\]
whence if \( \Phi, \Psi \in S[V] \) then Theorem 1.9 shows that
\[
\langle \Phi | U(g)\Psi \rangle = \langle \overline{U(g)}^* \Phi | \Psi \rangle.
\]
Thus the Hilbert space adjoint of \( U(g) \) is the isometry \( \overline{U(g^{-1})} \).

Conversely, if \( U_g \) may be rescaled so as to produce a unitary operator on \( S[V] \) then in particular the Gaussian \( e^{Z_g} = U_g1 \) lies in \( S[V] \) and therefore \( A_g = -Z_gC_g \) is of Hilbert-Schmidt class.
Now by definition, the restricted symplectic group \( \text{Sp}_{\text{res}}(V) \) comprises precisely all those \( g \in \text{Sp}(V) \) for which \( A_g \) is of Hilbert-Schmidt class. When \( g \in \text{Sp}_{\text{res}}(V) \) we shall denote by

\[
(3.6) \quad \overline{U}(g) = \text{Det}^{\frac{1}{2}}(I - Z_g^2) U_g
\]

the extension of \( U(g) = \text{Det}^{\frac{1}{2}}(I - Z_g^2) U_g \) to a unitary operator on \( S[V] \). By definition, the resulting map

\[
(3.7) \quad \overline{U} : \text{Sp}_{\text{res}}(V) \rightarrow \text{Aut} S[V]
\]

is the metaplectic representation. This is indeed a projective representation, whose cocycle may be derived explicitly as follows.

**Theorem 3.13.** If \( g, h \in \text{Sp}_{\text{res}}(V) \) then

\[
\overline{U}_g \overline{U}_h = \delta(g, h) \overline{U}_{gh}
\]

where

\[
\delta(g, h) = \text{Det}^{\frac{1}{2}}(I - Z_h Z_g^{-1})^{-1}.
\]

**Proof.** Introduce a linear map \( \tilde{U}_{gh} : SV \rightarrow SV' \) by the rule

\[
\phi \in SV \Rightarrow \tilde{U}_{gh}(\phi) = \overline{U}_g \overline{U}_h(\phi) = \overline{U}_g(U_h \phi).
\]

If \( v \in V \) then it follows by Theorem 1.11 and Theorem 2.7 with the proof of Theorem 3.12 that

\[
[\tilde{U}_{gh} c(v) \phi](\psi) = <\psi|\overline{U}_g U_h c(v) \phi > = < U_{gh}^{-1} \psi | c_h(v) U_h \phi > = < a_h(v) U_{gh}^{-1} \psi | U_h \phi > = < a_{gh}(v) \psi | U_{gh} \phi > = [\tilde{U}_{gh} \phi](a_{gh}(v) \psi) = [c_{gh}(v) \tilde{U}_{gh} \phi](\psi).
\]

Accordingly, if \( v \in V \) then

\[
\tilde{U}_{gh} c(v) = c_{gh}(v) \tilde{U}_{gh}
\]

and similarly

\[
\tilde{U}_{gh} a(v) = a_{gh}(v) \tilde{U}_{gh}.
\]

Thus Theorem 3.7 and Theorem 3.10 imply that \( \tilde{U}_{gh} \) and \( U_{gh} \) are proportional, so \( \overline{U}_g \overline{U}_h \) and \( \tilde{U}_{gh} \) are proportional. All that remains is to compare normalizations: on the one hand, \( [\overline{U}_{gh} 1](1) = 1 \) by definition; on the other hand, Theorem 1.11 and Theorem 2.14 with the proof of Theorem 3.12 yield

\[
[\overline{U}_g \overline{U}_h 1](1) = < 1 | \overline{U}_g (U_h 1) > = < U_{gh}^{-1} 1 | U_h 1 > = < e^{Z_h^{-1}} | e^{Z_h} > = \text{Det}^{\frac{1}{2}}(I - Z_h Z_g^{-1})^{-1}.
\]

\[\Box\]

**4. Remarks**

In this final section, we make a number of remarks concerning the approach adopted in these notes.

Firstly, the approach via the antidual is decidedly elegant and offers a natural environment in which to develop the theory. It facilitates clean proofs: indeed, we have taken this opportunity to present simple proofs for several theorems difficult to locate in the literature. Thus, the handling of creators and annihilators is improved: for example, the proofs that if \( v \in V \) then \( c(v)^* = a(v) \) and \( a(v)^* = c(v) \) are particularly straightforward; field operators and the number operator are similarly transparent. Also, exponentials are manipulated with ease: among other things, we mention the effect of creators and annihilators on Gaussians and the fact that the exponentials of nonzero cubics do not lie in symmetric Fock space. Of course, the antidual is especially appropriate for the discussion of generalized Fock implementation.
As another example, let us outline a proof of the fact that if \( Z \in \Sigma^2[V] \) and \( \| Z \| < 1 \) then the Gaussian \( e^Z \in S[V] \) is cyclic for creators alone. Observe that \( I - Z^2 \) is an invertible positive operator, so we may define \( C := (I - Z^2)^{-1} \); the operator \( g := (I - Z)C \) then lies in \( \text{Sp}(V) \) and indeed in \( \text{Sp}_{\text{res}}(V) \) since \( Z_y = Z \) is Hilbert-Schmidt. Now, the unitary operator \( \overline{U(g)} \) on \( S[V] \) defined in Theorem 3.12 has the property that if \( v_1, \ldots, v_n \in V \) then
\[
\overline{U(g)}(v_1 \cdots v_n) = \| e^Z \|^{-1} c_g(v_1) \cdots c_g(v_n) e^Z
\]
whence Theorem 2.12 implies that
\[
\overline{U(g)}(v_1 \cdots v_n) \in \{ \phi e^Z : \phi \in SV \}.
\]
As the (possibly empty) products of vectors from \( V \) span \( SV \) and as \( \overline{U(g)} \) is unitary, so \( \{ \phi e^Z : \phi \in SV \} \) is dense in \( S[V] \). Otherwise said, \( e^Z \) is cyclic for creators alone.

Next, we ought at least to mention the direct construction of the bosonic Fock representation in Weyl form. Coherent states are especially well-suited for this purpose, so let us introduce a complex vector space \( EV \) with basis \( \{ \varepsilon^z : z \in V \} \) and inner product given by the rule that if \( x, y \in V \) then \( \langle \varepsilon^x | \varepsilon^y \rangle := \langle e^{x|y} \rangle \). Notice that Theorem 2.10 and Theorem 2.11 permit us to identify \( EV \) with the span of the coherent vectors \( \{ \varepsilon^z : z \in V \} \). Along with \( EV \) itself we naturally consider its full antidual \( EV' \) whose subspace \( E[V] \) of bounded antilinear functionals on \( EV \) is identified with \( S[V] \). Certain other subspaces of \( EV' \) are also important: for example, that comprising all \( \Phi \in EV' \) for which the function \( V \to C : z \mapsto \Phi(\varepsilon^z) \) is antiholomorphic in one of several senses, such as the usual sense on finite-dimensional subspaces.

To each \( v \in V \) we associate the linear automorphism \( W(v) \) of \( EV \) defined by the rule
\[
z \in V \Rightarrow W(v) \varepsilon^z = (\| e^v \| e^{<v|z>} )^{-1} e^{v+z}
\]
and extend it to \( EV' \) by antiduality according to the prescription
\[
\Phi \in EV', \psi \in EV \Rightarrow [W(v)\Phi](\psi) = \Phi(W(-v)\psi).
\]
Direct computation reveals that \( W(v) \) is unitary on \( EV \) and indeed on \( E[V] \). The resulting map \( W : V \to \text{Aut}E[V] \) is a regular projective representation: it is regular, for if \( x, y, v \in V \) then the inner product
\[
\langle \varepsilon^x | W(tv)\varepsilon^y \rangle = \exp \{ \langle x|y \rangle + \langle x|v \rangle - \langle v|y \rangle \} t - \frac{1}{2} \| v \| ^2 t^2
\]
depends continuously on \( t \in \mathbb{R} \); it is projective, its cocycle being readily verified to have the Weyl form
\[
x, y \in V \Rightarrow W(x)W(y) = \exp\{-i\Omega(x,y)\}W(x+y).
\]
In this formalism, a generalized Fock implementer for \( g \in \text{Sp}(V) \) is a (nonzero) linear map \( U : EV \to EV' \) that intertwines \( W \) on \( EV \) with \( W \circ g \) on \( EV' \) in the sense
\[
v \in V \Rightarrow UV(v) = W(gv)U.
\]
The intertwiner \( U \) may be required to satisfy further restrictions, such as that \( \langle \varepsilon^x | U\varepsilon^y \rangle \) be (antiholomorphic, holomorphic) in \( (x, y) \in V \times V \). With this definition, a specific generalized Fock implementer \( U_g : EV \to EV' \) is given explicitly by the rule that if \( x, y \in V \) then
\[
[U_g \varepsilon^y](\varepsilon^z) = \exp \left( \frac{1}{2} \langle x|C^{-1}_g(y - A_g^{-1}x) + \frac{1}{2} \langle x - A_g y | x \rangle \right)
\]
The proof of this fact is entirely routine: as the action of \( W \) passes from \( EV \) to \( EV' \) by antiduality, it is enough to argue algebraically that if \( x, y, v \in V \) then
\[
[U_g W(v)\varepsilon^y](\varepsilon^x) = [U_g \varepsilon^y](W(-gv)\varepsilon^x).
\]
Of course, if \( g \in \text{Sp}_{\text{res}}(V) \) then \( \text{Det}^{\frac{1}{2}}(I - Z_g^2)U_g \) determines a unitary intertwining operator on \( E[V] \). We remark that [7] presents a more detailed analysis, incorporating \((- , +)\) holomorphicity restrictions in terms of the complex-wave representation.
Lastly, the elegance of the approach adopted here suggests that it should be adopted elsewhere. As a matter of fact, in [8] we have already discussed an analogous treatment for the fermionic Fock representation of $V$: we placed fermionic Fock space $\mathcal{F}[V]$ between the exterior algebra $\wedge V$ and its full antidual $\wedge' V$ while simultaneously developing the Berezin calculus in arbitrary dimensions. In the fermionic context, it transpires that an orthogonal transformation $g \in O(V)$ admits a generalized Fock implementer precisely when the complex-linear part $C_g$ has finite-dimensional kernel; again, if the antilinear part $A_g$ is Hilbert-Schmidt then a suitably normalized implementer determines a unitary intertwining operator on $\mathcal{F}[V]$. Of course, it is natural to attempt a similar treatment for the Fock representation of an indefinite inner product space: when this is a Krein space the Hilbert space machinery may be employed, but even then it is not of primary importance; thus an approach by way of the antidual shows promise. Such matters will be addressed in a future publication.

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