Functionally graded structural member with three longitudinal circular cylindrical cracks loaded in tension: a fracture analysis

V Rizov
Department of Technical Mechanics, University of Architecture, Civil Engineering and Geodesy, 1046 Sofia, Bulgaria
E-mail: v_rizov_fhe@uacg.bg

Abstract. The present paper is concerned with fracture analysis of a structural member of circular cross-section with three longitudinal circular cylindrical cracks. The cracks are located arbitrary in radial direction of the structural member. Beside, the three cracks have different lengths. A circular notch is cut-out in the mid-span. The structural member is loaded in tension by two axial forces applied at the ends. The material is functionally graded in radial direction. The mechanical behaviour of the material is non-linear elastic. The fracture is studied in terms of the strain energy release rate. The solutions to the strain energy release rate derived for three cracks are verified by considering the balance of the energy. A parametric investigation of the fracture is carried-out. The effects of locations of cracks and material gradient on the fracture are evaluated.

1. Introduction
The properties of functionally graded structural materials change continuously along one or more directions in the volume of structural member [1-3]. The functionally graded materials are widely used in aeronautics and automotive engineering since they combine the best qualities of their constituent materials [4]. Fracture strongly influences the functioning of structural members and components made of functionally graded materials [5]. Thus, analysis of fracture is an important part of the design process of functionally graded components.

The present paper is devoted to the problem of longitudinal fracture of a functionally graded non-linear elastic structural member of circular cross-section with three circular cylindrical longitudinal cracks loaded in tension. It should be noted that the previous papers deal with fracture analyses of functionally graded members with one longitudinal crack [6, 7]. However, the change of material properties in radial direction is a premise for appearance of more longitudinal cracks. Hence, the present paper contributes towards this topic by developing a fracture analysis of a structural member with three concentric longitudinal cracks.

2. Deriving of the strain energy release rate for the longitudinal cracks
A functionally graded structural member with three longitudinal cracks is shown in figure 1. The cross-section of the structural member is a circle of radius, \( R \). The structural member is loaded in centric tension by axial forces, \( F \), applied at the ends. The length of the structural member is \( 2l \). The member is functionally graded in radial direction. The appearance of more longitudinal cracks is
related to the fact that the material properties change in radial direction. The material has non-linear elastic behaviour. The cracks are concentric circular cylindrical surfaces. The lengths of cracks 1, 2 and 3 are $2a_1$, $2a_2$ and $2a_3$, respectively. The radiuses of the fronts of cracks 1, 2 and 3 are $R_1$, $R_2$ and $R_3$, respectively. The three cracks are located symmetrically with respect to the middle of the beam (figure 1). The cracked portion of the structural member is divided by the three cracks into four concentric crack arms (a crack arm is part of the beam located between two neighboring concentric circular cracks). The cross-section of crack arm 1 is a circle of radius, $R_1$. The length of crack arm 1 is $2a_1$. The crack arm 2 has a ring-shaped cross-section of internal and external radiuses, $R_1$ and $R_2$, respectively (this crack arm is located between cracks 1 and 2). The length of crack arm 2 is $2a_1$. The crack arm 3 has a ring-shaped cross-section of internal and external radiuses, $R_2$ and $R_3$, respectively (this crack arm is between cracks 2 and 3). The length of crack arm 3 is $2a_2$.

![Figure 1. Geometry and loading of a functionally graded structural member of circular cross-section with three longitudinal cracks.](image)

The crack arm 4 has a ring-shaped cross-section of internal and external radiuses, $R_3$ and $R_4$, respectively (this crack arm is between crack 3 and the surface of the structural member). The length of crack arm 4 is $2a_3$. A circular notch of depth, $R_4 - R_3$, is cut-out in the crack arm 4 in the middle of the structural member. The notch divides the crack arm 4 into two symmetric parts each of length, $a_3$. It is obvious that the two parts of crack arm 4 are free of stresses.

The longitudinal fracture behaviour is studied in terms of the strain energy release rate, $G$. Only half of the structural member, $l \leq x \leq 2l$, is analyzed due to the symmetry (figure 1). First, the strain energy release rate is obtained assuming an elementary increase, $da_1$, of the length of crack 1. For this purpose, the strain energy release rate is written as [8]:

$$G = 2 \frac{dU^*}{l_{cf1}da_1},$$

where $U^*$ is the complementary strain energy, $l_{cf1}$ is the length of the front of crack 1. It should be noted that the right-hand side of (1) is doubled in view of the symmetry (figure 1). The length of the front of crack 1 is expressed as
By combining (1) and (2), one obtains

\[ l_{c1} = 2\pi R_1. \]  

(2)

The complementary strain energy is written as

\[ U^* = U_1^* + U_2^* + U_3^* + U_{P_1P_1}^* + U_{P_2P_2}^* + U_{P_3P_3}^* , \]  

(4)

where \( U_1^* \), \( U_2^* \), \( U_3^* \), \( U_{P_1P_1}^* \), \( U_{P_2P_2}^* \) and \( U_{P_3P_3}^* \) are the complementary strain energies cumulated in crack arms 1, 2, 3, in the internal parts of structural portions, \( P_2P_3 \) and \( P_3P_4 \), and in the un-cracked structural portion, \( P_4P_5 \), respectively (figure 1). It should be specified that the internal part of structural portion, \( P_2P_3 \), has a circular cross-section of radius, \( R_2 \). The length of structural portion, \( P_2P_3 \), is \( a_2 - a_1 \). The cross-section of the internal part of structural portion, \( P_3P_4 \), is a circle of radius, \( R_3 \). The length of structural portion, \( P_3P_4 \), is \( a_3 - a_2 \). The length of the un-cracked structural portion is \( l - a_3 \). The complementary strain energy in crack arm 1 is expressed as

\[ U_1^* = a_1 \int_0^R u_{01}^* 2\pi R dR, \]  

(5)

where \( u_{01}^* \) is the complementary strain energy density in this crack arm. The following formula is used to calculate \( u_{01}^* \) [6]:

\[ u_{01}^* = \sigma \varepsilon - u_{01}, \]  

(6)

where \( \sigma \) is the stress, \( \varepsilon \) is the strain, \( u_{01} \) is the strain energy density.

In the present paper, the mechanical behaviour of the material is treated by using the following non-linear stress-strain relation [9]:

\[ \sigma = \frac{\varepsilon}{b + f\varepsilon}, \]  

(7)

where \( b \) and \( f \) are material properties. The distribution of \( b \) in radial direction is described as

\[ b = b_0 + \frac{b_1 - b_0}{R_4^g} R^g, \]  

(8)

where \( b_0 \) and \( b_1 \) are the values of \( b \) in the centre of the member cross-section and the surface of the member, \( g \) controls the material inhomogeneity in radial direction.

In principle, the strain energy density is equal to the area enclosed by the stress-strain curve. Thus, the strain energy density in the crack arm 1 is obtained by integrating of (7)

\[ u_{01} = \frac{1}{f} \left[ \varepsilon \frac{b}{f} - \ln \left( \frac{\varepsilon + b}{f} \right) + \frac{b}{f} \ln \frac{b}{f} \right]. \]  

(9)

By substituting of (7) and (9) in (6), one derives

\[ u_{01}^* = \frac{\varepsilon^2}{b + f\varepsilon} - \frac{1}{f} \left[ \varepsilon \frac{b}{f} \ln \left( \frac{\varepsilon + b}{f} \right) + \frac{b}{f} \ln \frac{b}{f} \right]. \]  

(10)
The magnitude of \( \varepsilon \) is determined by using the following equation for equilibrium of the axial forces in the crack arms:

\[
N_1 + N_2 + N_3 = F,
\]

where \( N_1, N_2 \) and \( N_3 \) are axial forces in crack arms 1, 2 and 3, respectively. The axial forces are expressed as

\[
N_i = \int_0^{R_i} \sigma_{cai} 2\pi RdR, \quad N_2 = \int_{R_1}^{R_2} \sigma_{ca2} 2\pi RdR, \quad N_3 = \int_{R_2}^{R_3} \sigma_{ca3} 2\pi RdR,
\]

where \( \sigma_{cai}, \sigma_{ca2} \) and \( \sigma_{ca3} \) are the stresses in crack arms 1, 2 and 3, respectively. After substituting of (7), (8) and (12) in (11), the equation for equilibrium is solved with respect to \( \varepsilon \) by using the MatLab computer program.

The complementary strain energies in crack arms 2 and 3, in the internal parts of member portions, \( P_2P_3 \) and \( P_3P_4 \), and in the un-cracked member portion, \( P_4P_2 \), are obtained as

\[
U_{2*} = a_1 \int_0^{R_2} u_{02}^* 2\pi RdR,
\]

\[
U_{3*} = a_2 \int_0^{R_3} u_{03}^* 2\pi RdR,
\]

\[
U_{P_2P_3}^* = (a_2 - a_1) \int_0^{R_2} u_{0P_2P_3}^* 2\pi RdR,
\]

\[
U_{P_3P_4}^* = (a_3 - a_2) \int_0^{R_3} u_{0P_3P_4}^* 2\pi RdR,
\]

\[
U_{P_4P_2}^* = (l - a_3) \int_0^{R_4} u_{0P_4P_2}^* 2\pi RdR,
\]

where the complementary strain energy densities are found by (10). For this purpose, \( \varepsilon \) is replaced with the strain in the corresponding part of the member.

By substituting of (4), (5), (13), (14), (15), (16) and (17) in (3), one derives the following solution to the strain energy release rate:

\[
G = \frac{2}{R_1} \left( \int_0^{R_1} u_{01}^* RdR + \int_{R_1}^{R_2} u_{02}^* RdR - \int_{R_2}^{R_3} u_{0P_2P_3}^* RdR \right).
\]

The integration in (18) is carried-out by the MatLab computer program.

Formula (3) is applied also to drive the strain energy release rate for crack 2. For this purpose, \( da_1 \) and \( R_1 \) are replaced with \( da_2 \) and \( R_2 \), respectively. Analogically, the strain energy release rate for crack 3 is found by replacing of \( da_1 \) and \( R_1 \) with \( da_3 \) and \( R_3 \).

The strain energy release rate is derived also by considering the balance of the energy for verification. The longitudinal displacement of the application point of force, \( F \), that is needed when analyzing the balance of the energy is determined by the integrals of Maxwell-Mohr. It should be noted that the strain energy release rates obtained by considering the balance of the energy are exact.
matches of these found by using (1). This fact is a verification of the fracture analysis developed in the present paper.

3. Numerical results

The solutions to the strain energy release rate obtained in previous section of the paper are applied here in order to evaluate the effect of material gradient on the longitudinal fracture of the structural member. The strain energy release rate is presented in normalized form by using the formula \( G_N = G b_0 / R_a \). The material gradient is characterized by \( b_1 / b_0 \) ratio (this is the ratio of the values of \( b \) at the surface and the centre of the member cross-section). The locations of cracks 1, 2 and 3 in radial direction are characterized by \( R_1 / R_a \), \( R_2 / R_a \) and \( R_3 / R_a \) ratios, respectively. It is assumed that \( R_a = 0.010 \text{ m}, \ f = 0.3 b_0 \) and \( g = 0.6 \).

![Figure 2](image.png)

**Figure 2.** The strain energy release rate in normalized form plotted against \( b_1 / b_0 \) ratio (curve 1 – at \( R_1 / R_a = 0.1 \), curve 2 – at \( R_1 / R_a = 0.2 \) and curve 3 – at \( R_1 / R_a = 0.3 \)).

The strain energy release rate in normalized form is plotted against \( b_1 / b_0 \) ratio in figure 2 at three \( R_1 / R_a \) ratios by using the solution to the strain energy release rate for crack 1.

One can observe in figure 2 that the strain energy release rate increases with increasing of \( b_1 / b_0 \) ratio. It can also be observed in figure 2 that the strain energy release rate decreases with increasing of \( R_1 / R_a \) ratio.

The effect of location of crack 2 on the fracture behaviour is evaluated too. For this purpose, the strain energy release rate in normalized form is plotted against \( R_2 / R_a \) ratio in figure 3 at three values of \( F \) by applying the solution to the strain energy release rate for crack 2. The curves in figure 3 indicate that the strain energy release rate decreases with increasing \( R_2 / R_a \) ratio.

The effect of the location of crack 3 on the fracture behaviour of the beam is illustrated in figure 4 where the strain energy release rate in normalized form is plotted against \( R_3 / R_a \) ratio by using the solution to the strain energy release rate for crack 3. The curves in figure 4 show that the strain energy release rate decreases with increasing \( R_3 / R_a \) ratio.
Figure 3. The strain energy release rate in normalized form plotted against $R_3/R_4$ ratio (curve 1 – at $F = 10$ N, curve 2 – at $F = 12$ N and curve 3 – at $F = 14$ N).

Figure 4. The strain energy release rate in normalized form plotted against $R_3/R_4$ ratio (curve 1 – at non-linear behaviour of the material and curve 2 – at linear-elastic behaviour).

The strain energy release rate obtained assuming linear-elastic behaviour of the functionally graded material is also plotted in figure 4. The linear-elastic solution is derived by substituting $f = 0$ in the non-linear solution since at $f = 0$ the non-linear stress-strain relation (7) transforms in the Hook’s law assuming that $1/b$ is the modulus of elasticity (therefore, the non-linear solution can be applied
also for linear-elastic behaviour as a particular case). It can be observed in figure 4 that the material non-linearity leads to increase of the strain energy release rate.

4. Conclusions
Fracture behaviour of a structural member of circular cross-section with tree longitudinal circular cylindrical cracks loaded in centric tension is analyzed in terms of the strain energy release rate (the appearance of more longitudinal cracks is related to the fact that the material properties change in radial direction). The structural member is functionally graded in radial direction. The material has non-linear elastic behaviour. The solutions to the strain energy release rate derived in the present paper are used to evaluate the effect of the locations of the cracks on the fracture behaviour. It is found that the strain energy release rate decreases with increasing of \( R_1/R_4 \), \( R_2/R_4 \) and \( R_3/R_4 \) ratios. The effect of material gradient is evaluated too. The analysis reveals that the strain energy release rate decreases with increasing of \( b_1/b_0 \) ratio. These findings can be applied in preliminary design of functionally graded non-linear elastic structural members with considering of the longitudinal fracture. The analysis of the strain energy release rate developed in the present paper can be used also to check for growth of existing longitudinal cracks in structural members. For this purpose, the calculated strain energy release rate has to be compared with the fracture toughness. A crack growth will occur if the calculated strain energy release rate exceeds the fracture toughness.

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