TOWARD ABHYANKAR’S INERTIA CONJECTURE FOR
\(\text{PSL}_2(\ell)\)

by

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Abstract. — For \(\ell \neq p\) odd primes, we examine \(\text{PSL}_2(\ell)\)-covers of the projective line branched at one point over an algebraically closed field \(k\) of characteristic \(p\), where \(\text{PSL}_2(\ell)\) has order divisible by \(p\). We show that such covers can be realized with a large variety of inertia groups. Furthermore, for each inertia group realized, we can realize all “sufficiently large” higher ramification filtrations.

1. Introduction

Over an algebraically closed field \(k\) of characteristic 0, finite algebraic branched covers \(Y \to \mathbb{P}^1\) with \(n\) fixed branch points are in one-to-one correspondence with finite topological branched covers of the Riemann sphere with \(n\) fixed branch points. Both correspond to finite index subgroups of \(\pi_1(\mathbb{P}^1_k \setminus \{x_1, \ldots, x_n\})\), the free (profinite) group on \(n-1\) generators. In particular, there exist no nontrivial covers of \(\mathbb{P}^1_k\) branched at one point (hereafter called “one-point covers”).

If \(k\) is algebraically closed of characteristic \(p\), the situation differs in two important ways. First, there exist many covers which do not have topological analogs. In particular, as a consequence of Abhyankar’s conjecture (proven by Raynaud (Ray94) in the case of the affine line and Harbater (Har94) in general), it follows that a finite group \(G\) can be realized as the Galois group of an \(n\)-point cover of \(\mathbb{P}^1_k\) exactly when \(G/p(G)\) can be generated by \(n-1\) elements (here, \(p(G)\) is the subgroup of \(G\) generated by all of the \(p\)-Sylow subgroups). Thus, \(G\) occurs as the Galois group of a one-point cover iff \(G = p(G)\). Such a group is called quasi-\(p\). The second major difference is that, unlike in characteristic zero, the genus of \(Y\) is not determined by the degree,

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ramification points, and ramification indices of \( f : Y \to \mathbb{P}^1 \), but also depends on wild ramification behavior at the ramification points. This behavior is encoded in the higher ramification filtration \((\mathfrak{mol})\).

While Abhyankar's conjecture guarantees existence of certain one-point \( G \)-covers, it does not provide examples. In particular, the question of what subgroups \( I \subseteq G \) can occur as the inertia group of a \( G \)-Galois one-point cover has been answered in only a few cases. By basic ramification theory, such a group \( I \) must be of the form
\[
P \rtimes \mathbb{Z}/m
\]
where \( P \) is a \( p \)-group and \( p \nmid m \). Furthermore, \( I \) must also generate \( G \) as a normal subgroup (this is automatic, for instance, when \( G \) is simple). Abhyankar's inertia conjecture \([\text{Abh01}]\) states that any \( I \) satisfying these properties occurs as the inertia group of a one-point \( G \)-cover. This is known to be true for \( G = \text{PSL}_2(p) \) and \( G = A_p \), for \( p \geq 5 \) \([\text{BP03}]\), and for \( A_{p+2} \) when \( p \equiv 2 \pmod{3} \) is an odd prime \([\text{MPT0}]\). It is clearly true in some trivial cases, for instance when \( G \) is abelian. However, outside of these cases, our knowledge is limited. Abhyankar has constructed many examples of one-point \( G \)-covers with various inertia groups where \( G \) is simple, but all of these examples are in cases where \( G \) is alternating, sporadic, or of Lie type over a field of characteristic \( p \). In particular, if \( G \) is a simple group with a cyclic \( p \)-Sylow subgroup of order greater than \( p \) (such a group cannot be alternating, sporadic, or of Lie type over a field of characteristic \( p \)), then as far as I know, no examples of one-point \( G \)-covers have been constructed, let alone with any particular inertia group.

In this paper, we investigate Abhyankar’s inertia conjecture for certain simple groups \( G = \text{PSL}_2(\ell) \), where \( p \mid |G| \) and \( \ell \neq p \) are odd primes. In this case, the group \( \text{PSL}_2(\ell) \) has a cyclic \( p \)-Sylow subgroup of order \( p^a \), where \( a = v_p(\ell^2 - 1) \). While we are not able to prove the conjecture in full (see Remark 3.8), we exhibit many examples with a wide variety of inertia groups:

**Corollary 3.6** — Let \( k \) be an algebraically closed field of characteristic \( p \geq 7 \) and \( G \cong \text{PSL}_2(\ell) \), where \( p \mid |G| \). Suppose \( I \) is either a cyclic group of order \( p^r \) or a dihedral group \( D_{2p^r} \) of order \( 2p^r \), with \( 1 \leq r \leq v_p(|G|) \). Then there exists a \( G \)-cover \( f : Y \to \mathbb{P}^1_k \) branched at one point with inertia groups isomorphic to \( I \).

This result generalizes \([\text{BP03}]\) Theorem 3.6], which deals with the case \( a = 1 \). We also investigate the higher ramification behavior of one-point \( \text{PSL}_2(\ell) \)-covers with inertia group \( I \). We show that, in the situation of Corollary 3.6, any sufficiently "large" higher ramification filtration must occur. See Corollary 3.5 for a more specific result.

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2. Preliminaries

Notation: If $G$ is a group with cyclic $p$-Sylow subgroup $P$, we write $m_G$ for $|N_G(P)/Z_G(P)|$, the normalizer modulo the centralizer. If $R$ is a local ring or a discretely valued field, then $\hat{R}$ is the usual completion. If $x$ is a point of a scheme $X$, then $\mathcal{O}_{X,x}$ is the local ring of $X$ at $x$.

2.1. Higher ramification filtrations. — We recall some facts from [Ser79, IV].

Let $K$ be a complete discrete valuation field with algebraically closed residue field $k$ of characteristic $p > 0$. If $L/K$ is a finite Galois extension of fields with Galois group $G$, then $L$ is also a complete discrete valuation field with residue field $k$. Here $G$ is of the form $P \rtimes \mathbb{Z}/m$, where $P$ is a $p$-group and $m$ is prime to $p$. The group $G$ has a filtration $G \supseteq G^i$ ($i \in \mathbb{R}_{\geq 0}$) called the higher ramification filtration for the upper numbering. If $i < j$, then $G^j \supseteq G^i$ (see [Ser79, IV, §1, §3]). The subgroup $G^i$ is known as the $i$th higher ramification group for the upper numbering. One knows that $G^0 = G$, and that for sufficiently small $\epsilon > 0$, $G^\epsilon = P$. For sufficiently large $i$, $G^i = \{id\}$. Any $i$ such that $G^i \supseteq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an upper jump of the extension $L/K$. If $i$ is an upper jump and $i > 0$, then $G^i/G^{i+\epsilon}$ is an elementary abelian $p$-group for sufficiently small $\epsilon$. The greatest upper jump (i.e., the greatest $i$ such that $G^i \neq \{id\}$) is called the conductor of higher ramification of $L/K$.

If $P \cong \mathbb{Z}/p^r$ is cyclic, then $G$ must have $r$ different positive upper jumps $u_1 < \cdots < u_r$. Since the sequence $(u_1, \ldots, u_r)$ encodes the entire higher ramification filtration, we will simply say that such an extension (or the inertia group of such an extension) has upper higher ramification filtration $(u_1, \ldots, u_r)$ in this case.

The higher ramification filtration is important because if $f : Y \to X$ is a branched cover of curves in characteristic $p$, and $y \in Y$ is a ramification point and $f(y) = x$, then the higher ramification filtration for $\mathcal{O}_{Y,y}/\mathcal{O}_{X,x}$ figures into the ramification divisor term in the Hurwitz formula. Specifically, let $f : Y \to \mathbb{P}^1$ be a one-point $G$-cover with inertia groups $I \cong \mathbb{Z}/p^r \rtimes \mathbb{Z}/m$ with $p \nmid m$ and higher ramification filtration $(u_1, \ldots, u_r)$. Then, by [Pr06, Lemma 1], the genus of $Y$ is $1 - |G| + \frac{|G| \deg(R)}{2mp^r}$, where $R$ is the ramification divisor, which has degree

$$mp^r - 1 + (p - 1)m(u_1 + pu_2 + \cdots + p^{r-1}u_r).$$

2.2. Stable reduction. — Let $X \cong \mathbb{P}^1_K$, where $K$ is a characteristic zero complete discretely valued field with algebraically closed residue field $k$ of characteristic $p > 0$ (e.g., $K$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$). Let $R$ be the valuation ring of $K$.

Let $f : Y \to X$ be a $G$-Galois cover defined over $K$, with $G$ any finite group, such that the branch points of $f$ are defined over $K$ and their specializations do not collide on the special fiber of $X_R$. Assume that $2g_X - 2 + r \geq 1$, where $g_X$ is the genus of $X$ and $r$ is the number of branch points of $f$. By a theorem of Deligne and
Mumford ([DM69, Corollary 2.7]), combined with work of Raynaud ([Ray99] and Liu ([Liu06]), there is a minimal finite extension $K^{st}/K$ with ring of integers $R^{st}$, and a unique model $Y^{st}$ of $Y_{K^{st}}$ (called the stable model) such that

- The special fiber $\overline{Y}$ of $Y^{st}$ is semistable (i.e., it is reduced, and has only nodes for singularities).
- The ramification points of $f_{K^{st}} = f \times_K K^{st}$ specialize to distinct smooth points of $\overline{Y}$.
- Any genus zero irreducible component of $\overline{Y}$ contains at least three marked points (i.e., ramification points or points of intersection with the rest of $\overline{Y}$).

Since the stable model is unique, it is acted upon by $G$, and we set $X^{st} = Y^{st}/G$. Then $X^{st}$ is semistable ([Ray90]) and can be naturally identified with a blowup of $X \times_R R^{st}$ centered at closed points. The map $f^{st} : Y^{st} \to X^{st}$ is called the stable model of $f$. The special fiber $\overline{f} : \overline{Y} \to \overline{X}$ of $f^{st}$ is called the stable reduction of $f$.

From now on, assume that $f$ has bad reduction (i.e., $\overline{X}$ is reducible). Any irreducible component $\overline{X}_b$ of $\overline{X}$ that intersects the rest of $\overline{X}$ in only one point is called a tail. If $G$ acts without inertia above the generic point of $\overline{X}_b$, we call $\overline{X}_b$ an étale tail. If, furthermore, $\overline{X}_b$ contains the specialization of a branch point of $f$, then $\overline{X}_b$ is called primitive. If not, $\overline{X}_b$ is called new. On an étale tail $\overline{X}_b$, branching of $\overline{f}$ can only take place at the point where $\overline{X}_b$ intersects the rest of $\overline{X}$ and, if $\overline{X}_b$ is primitive, where the branch point of $f$ specializes.

Suppose $G$ has a cyclic $p$-Sylow subgroup $P$. Consider an étale tail $\overline{X}_b$ of $\overline{X}$. Let $\overline{x}_b$ be the unique point at which $\overline{X}_b$ intersects the rest of $\overline{X}$. Let $\overline{y}_b$ be a component of $\overline{Y}$ lying above $\overline{X}_b$, and let $y_b$ be a point lying above $\overline{x}_b$. Then the effective ramification invariant $\sigma_b$ is the conductor of higher ramification of the extension $\text{Frac}(\hat{O}_{\overline{X}_b, \overline{x}_b})/\text{Frac}(\hat{O}_{\overline{X}, \overline{x}_b})$.

The most important result that we will use is a special case of the vanishing cycles formula. This formula relates the stable reduction of $f$ to the genus of $X$ and the branching behavior of $f$. In particular, it places a limit on how many tails $\overline{X}$ can have and how large their effective ramification invariants can be. If $|G|$ is prime to $p$, then $f$ has good reduction, and there is no need for such a formula. The original version of the vanishing cycles formula, proved by Raynaud ([Ray99, §3.4.2 (5)]), requires that $G$ has a $p$-Sylow subgroup of order $p$. A generalized version was proven by the author in [Obu10a, Theorem 3.14], which applies when $G$ has a cyclic $p$-Sylow subgroup of arbitrary order (Theorem 2.1 below is a special case, corresponding to [Obu10a, Corollary 3.15]). The formula will be essential for us to exhibit a one-point cover in characteristic $p$ whose Galois group has large cyclic $p$-Sylow group but whose inertia groups are small.

**Theorem 2.1 (Vanishing cycles formula).** — Let $f : Y \to X \cong \mathbb{P}^1_K$ be a three-point $G$-Galois cover with bad reduction, where $G$ has a cyclic $p$-Sylow subgroup. Let
Let $B_{new}$ be an indexing set for the new étale tails and let $B_{prim}$ be an indexing set for the primitive étale tails. Then

\begin{equation}
1 = \sum_{b \in B_{aux}} (\sigma_b - 1) + \sum_{b \in B_{prim}} \sigma_b.
\end{equation}

**Remark 2.2.** — Note that, by [Obu10a, Lemma 4.2 (i)], each term on the right hand side of (2.1) is at least $\frac{1}{m_G}$. In particular, each term is positive.

### 2.3. The auxiliary cover

— Retain the assumptions from the beginning of §2.2 (in particular, $G$ need not have a cyclic $p$-Sylow subgroup). Assume that $f : \tilde{Y} \to X$ is a $G$-cover defined over $\mathbb{K}$ as in §2.2 with bad reduction, so that $\mathbb{T}$ is not just the original component. By [Obu10b, §2.6] (see also [Ray99, §3.2]), we can construct an auxiliary cover $f_{aux} : Y_{aux} \to X$ over some finite extension $K'$ of $K$. The cover $f_{aux}$ is a (connected) $G_{aux}$-cover, for some subgroup $G_{aux} \subseteq G$.

The only properties we will need of $f_{aux}$ are the following:

**Proposition 2.3.** — (i) For each branch point of $f$ of (branching) index $e$ with $v_p(e) \geq 1$, the cover $f_{aux}$ has a branch point of index $e'$ with $v_p(e') = v_p(e)$. For each étale tail of $\mathbb{T}$, the cover $f_{aux}$ may have a branch point of prime-to-$p$ index. There are no other branch points of $f_{aux}$.

(ii) If a $p$-Sylow subgroup of $G$ is cyclic, then the group $G_{aux}$ has a normal subgroup $N$ of prime-to-$p$ order such that $G/N \cong \mathbb{Z}/p' \times \mathbb{Z}/m$, with $p \nmid m$.

**Proof.** — Part (i) follows from the construction in [Obu10b, §2.6]. Part (ii) follows from [Obu10b, Proposition 2.12] and [Obu10a, Proposition 2.4(i)].\[\square\]

### 3. A one-point cover

We maintain the notations $K$, $R$, and $k$ of §2. We write $\overline{K}$ for the algebraic closure of $K$.

**Lemma 3.1.** — Let $\ell \neq p$ be odd primes such that $v_p(\ell^2 - 1) =: a \geq 2$. If $G \cong PSL_2(\ell)$, then there exists a three-point $G$-cover $f : \tilde{Y} \to X = \mathbb{P}^1_{\overline{K}}$ whose branch points have branching indices $e_1$, $e_2$, and $e_3$, with $0 = v_p(e_1) < v_p(e_2) < v_p(e_3)$.

**Proof.** — If $H \cong SL_2(\ell)$, it suffices to exhibit a three-point $H$-cover with branching indices $e_1'$, $e_2'$, and $e_3'$ satisfying $0 = v_p(e_1') < v_p(e_2') < v_p(e_3')$. We obtain the desired $G$-cover by quotienting out by $\{\pm 1\}$ (as for $i \in \{1, 2, 3\}$, we will have $e_i = e_i'$ or $e_i = e_i'/2$, depending on the action of $\{\pm 1\}$ above the relevant branch point). As in [BW04, §3], pick an $(\ell - 1)$st root of unity $\zeta \in \mathbb{F}_\ell$ and an $(\ell + 1)$st root of unity $\zeta \in \mathbb{F}_{\ell^2}$. Then for $0 < i < \frac{\ell - 1}{2}$ (resp. $0 < i < \frac{\ell + 1}{2}$), we write $C(i)$ (resp. $\tilde{C}(i)$) for the unique conjugacy class of elements of $H$ with eigenvalues $\zeta^{\pm i}$ (resp. $\tilde{\zeta}^{\pm i}$). In
particular, if \( x \in C(i) \) (resp. \( \bar{C}(i) \)), then \( v_p(\text{ord}(x)) = \max(0, v_p((\ell - 1)/i)) \) (resp. \( \max(0, v_p((\ell + 1)/i)) \)).

We construct a triple of conjugacy classes \( C = (C_1, C_2, C_3) \) giving rise to the desired cover. If \( v_p(\ell - 1) = a \), then pick \( C = (\bar{C}(\frac{\ell + 1}{2} - 1), C(\frac{\ell - 1}{2} - p), C(\frac{\ell - 1}{2} - 1)) \). If \( v_p(\ell + 1) = a \), then pick \( C = (C(\frac{\ell - 1}{2} - 1), \bar{C}(\frac{\ell + 1}{2} - p), \bar{C}(\frac{\ell + 1}{2} - 1)) \). Note that since \( p^2 \) divides either \( \ell - 1 \) or \( \ell + 1 \), and \( p \) and \( l \) are odd, then \( 2p^2 \leq \ell + 1 \). In particular, one easily checks \( 2p + 4 < \ell - 1 \). Now, [BW04 Proposition 5.6] shows that there exist two isomorphism classes of three-point \( H \)-covers with inertia groups over the three branch points lying in \( C_1, C_2, \) and \( C_3 \) (our \( \ell \) is called \( p \) in [BW04]). Checking the ramification indices yields \( v_p(e'_1) = 0, v_p(e'_2) = a - 1, \) and \( v_p(e'_3) = a \). \( \square \)

**Lemma 3.2.** The stable reduction of the \( G \)-cover \( f \) in Lemma 3.1 has one new étale tail and one primitive étale tail. The new tail \( X_{\text{aux}} \) has effective ramification invariant \( \sigma_{\text{aux}} = \frac{3}{2} \).

**Proof.** By [Obu10a Proposition 2.15] and [Ray99 Proposition 2.4.8], there exists exactly one primitive étale tail of \( X \). We note that \( m_G = 2 \), so Remark 2.2 shows that the effective ramification invariant of a new étale tail must be at least \( \frac{3}{2} \), whereas the invariant of a primitive étale tail must be at least \( \frac{1}{2} \). If we can show that there exists a new étale tail, then the vanishing cycles formula (2.1) shows that there is only one, and it has invariant \( \frac{1}{2} \). So we need only show that there exists a new étale tail.

Consider the auxiliary cover \( f_{\text{aux}} : Y_{\text{aux}} \to X \) of \( f \). Assume, for a contradiction, that the primitive tail is the only étale tail. Then, by Proposition 2.3(i), the cover \( f_{\text{aux}} \) is branched at two points with indices \( e_2 \) and \( e_3 \) such that \( 0 < v_p(e_2) < v_p(e_3) \), and possibly at a third point with index \( e_1 \) such that \( p \nmid e_1 \). In fact, \( f_{\text{aux}} \) must be branched at this third point, because if it were branched at only two points, the Hurwitz formula would imply \( e_2 \neq e_3 \). By the basic theory of the fundamental group in characteristic zero, the Galois group \( G_{\text{aux}} \) of \( f_{\text{aux}} \) can be generated by an element of order \( e_1 \) and an element of order \( e_2 \). By Proposition 2.3(ii), \( G_{\text{aux}} \) has a quotient of the form \( \mathbb{Z}/p^r \times \mathbb{Z}/m \), for some \( r \) and \( m \), where \( r = v_p((G_{\text{aux}})) \geq v_p(e_3) > v_p(e_2) \). But such a group cannot be generated by an element of order dividing \( e_2 \) and an element of order dividing \( e_1 \). This is a contradiction. \( \square \)

**Remark 3.3.** One can also prove Lemma 3.2 using a rigidity argument, along the lines of [BP03 Proposition 2.8].

**Theorem 3.4.** Let \( p \geq 7 \) be an odd prime and \( \ell \) an odd prime such that \( v_p(\ell^2 - 1) = a \geq 1 \). Take \( G = \text{PSL}_2(\ell) \). Then for any algebraically closed field \( k \) of characteristic \( p \), there exists a \( G \)-cover of \( \mathbb{P}_k^1 \) branched at one point whose inertia groups are dihedral of order \( 2p \) with higher ramification filtration (\( \frac{3}{2} \)).

**Proof.** If \( a = 1 \) this is [BP03 Proposition 2.8]. So assume \( a \geq 2 \). By Lemmas 3.1 and 3.2 there exists a three-point \( G \)-cover \( f : Y \to X = \mathbb{P}_k^1 \) whose stable reduction
has a new tail $\overline{X}_b$ with effective ramification invariant $\sigma_b = \frac{3}{2}$. Let $\overline{Y}_b$ be a component above $\overline{X}_b$, and let $I$ be an inertia group of $\overline{Y}_b \to \overline{X}_b$ above the unique branch point. We know that $I$ is of the form $\mathbb{Z}/p^r \times \mathbb{Z}/m$ for some $m$ with $p \nmid m$. As $m\mathbb{Z} = 2$, Lemma 4.2(iii) shows that $\sigma_b \geq \frac{2}{m-1}$. Since $p \geq 7$, we conclude that $r = 1$. Furthermore, since $\sigma_b \notin \mathbb{Z}$, the Hasse-Arf theorem shows that $I$ is not abelian. By Huppert II, Hauptsatz 8.27, the only nonabelian subgroup of $G$ of the form $\mathbb{Z}/p \times \mathbb{Z}/m$ is the dihedral group $D_p$ of order $2p$. So $I \cong D_p$.

It remains to show that the Galois group of $\overline{Y}_b \to \overline{X}_b$ is $G$, as then our cover will be given by $\overline{Y}_b \to \overline{X}_b$. But, since $p \geq 7$, then Huppert II, Hauptsatz 8.27 shows that the only quasi-$p$ subgroup of $G$ containing a dihedral group of order $2p$ is $G$ itself. So we are done.

We know that $b \tau_b \rightarrow b$ has a new tail with effective ramification invariant $\sigma_b = \frac{3}{2}$. Let $\overline{Y}_b$ be a component above $\overline{X}_b$, and let $I$ be an inertia group of $\overline{Y}_b \to \overline{X}_b$ above the unique branch point. We know that $I$ is of the form $\mathbb{Z}/p^r \times \mathbb{Z}/m$ for some $m$ with $p \nmid m$. As $m\mathbb{Z} = 2$, Lemma 4.2(iii) shows that $\sigma_b \geq \frac{2}{m-1}$. Since $p \geq 7$, we conclude that $r = 1$. Furthermore, since $\sigma_b \notin \mathbb{Z}$, the Hasse-Arf theorem shows that $I$ is not abelian. By Huppert II, Hauptsatz 8.27, the only nonabelian subgroup of $G$ of the form $\mathbb{Z}/p \times \mathbb{Z}/m$ is the dihedral group $D_p$ of order $2p$. So $I \cong D_p$.

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**Remark 3.5.** — By Dirichlet’s theorem on arithmetic progressions, there are infinitely many $\ell$ satisfying the condition of Theorem 3.3 for any given $p$.

**Corollary 3.6.** — Let $k$ be an algebraically closed field of characteristic $p \geq 7$ and $G \cong PSL_2(\ell)$, where $p \mid |G|$. Suppose $I$ is either a cyclic group of order $\ell^r$ or a dihedral group $D_{\ell^r}$ of order $(\ell^r)2$, with $1 \leq r \leq \ell_p(|G|)$. Then there exists a $G$-cover $f : Y \to \mathbb{P}^1_k$ branched at one point with inertia groups isomorphic to $I$.

**Proof.** — If $\ell_p(|G|) = 1$, then this is [BP03] Theorem 3.6. Suppose $\ell_p(|G|) > 1$. In the case $I \cong D_{\ell^r}$, such a cover is given by Theorem 3.3. If $I \cong D_{\ell^r}$ with $r \geq 1$, the existence of such a cover follows from [Har03] Theorem 3.6 (taking $G = H$, $H = D_{\ell^r}$, and $H' = D_{\ell'^r}$ in that theorem). Consider a one-point $G$-cover $f : Y \to \mathbb{P}^1_k$ with inertia groups isomorphic to $D_{\ell^r}$. Assume, without loss of generality, that the branch point is $\infty$. If we base change this cover by the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $y = x^2$, then Abyanak’s Lemma ([SGA1] Lemma X.3.6]) shows that the inertia groups of the new cover are cyclic of order $\ell^{r'}$.

**Remark 3.7.** — Applying Abyanak’s lemma does not change the higher ramification filtration of the $\mathbb{Z}/\ell^r$-subextension. In particular, if $I \cong \mathbb{Z}/\ell$, we can take the higher ramification filtration to be (3). Alternatively, one can use [BP03] Theorem 3.1 to exhibit a cover with inertia groups $I \cong \mathbb{Z}/\ell$ and higher ramification filtration (3). Furthermore, if $\ell \equiv \pm 1 \pmod{8}$ and $I \cong \mathbb{Z}/\ell$, then the exact same proof as [BP03] Proposition 2.9 (with our vanishing cycles formula replacing the vanishing cycles formula used there) shows that there is a cover with higher ramification filtration (2).

**Remark 3.8.** — This does not fully prove Abyanak’s inertia conjecture for groups $PSL_2(\ell)$ as in Theorem 3.3 because we do not realize cyclic groups of non-$p$-power order as inertia groups (cf. [BP03] end of §3.1). But note that if $I \cong \mathbb{Z}/\ell^r$ or $D_{\ell^r}$, then all subgroups of $PSL_2(\ell)$ isomorphic to $I$ are conjugate. So the conjecture will
be proven in this case if we can find covers with all possible isomorphism classes of cyclic inertia groups.

4. Higher ramification filtrations

In addition to the inertia groups, one wants to understand the higher ramification filtrations of one-point covers of \( \mathbb{P}^1 \), for the reasons mentioned in (§2.1). More generally, let \( f : Y \to X \) be a one-point \( G \)-cover of curves over an algebraically closed field \( k \) of characteristic \( p \). Assume that a \( p \)-Sylow subgroup of \( G \) is cyclic and the inertia groups of \( f \) are isomorphic to 
\[
I \cong \mathbb{Z}/p^r \rtimes \mathbb{Z}/m,
\]
where \( p \nmid m \). Recall that 
\[
m_I := |N_I(\mathbb{Z}/p^r)/\mathbb{Z}/p^r|.
\]
Recall also from (§2.1) that the higher ramification filtration for the upper numbering at any ramification point is determined by its sequence of positive upper jumps \( (u_1, \ldots, u_r) \). The following theorem places restrictions on this sequence:

**Theorem 4.1** ([OP10], Theorem 1.1). — With notations as above, the sequence \( (u_1, \ldots, u_r) \) occurs as the upper higher ramification filtration of \( I \) if and only if:

1. \( u_i \in \frac{1}{m} \mathbb{N} \) for \( 1 \leq i \leq r \);
2. \( \gcd(m, mu_1) = m/m_1 \);
3. \( p \nmid mu_1 \) and, for \( 1 < i \leq r \), either \( u_i = pu_{i-1} \) or both \( u_i > pu_{i-1} \) and \( p \nmid mu_i \);
4. \( mu_i \equiv mu_1 \pmod{m} \) for \( 1 \leq i \leq r \).

Call a sequence \( (u_1, \ldots, u_r) \) \( I \)-admissible if it can occur as a sequence of positive upper jumps for \( I \), based on Theorem 4.1. There is a partial order on \( I \)-admissible sequences given by \( (u_1, \ldots, u_r) \leq (u'_1, \ldots, u'_r) \) iff \( u_i \leq u'_i \) for all \( 1 \leq i \leq r \). The main result of this section is the following:

**Proposition 4.2.** — Let \( f : Y \to X \) be a one-point \( G \)-cover as above with inertia groups isomorphic to \( I \) having upper higher ramification filtration \( \Sigma = (u_1, \ldots, u_r) \). Let \( \Sigma' = (u'_1, \ldots, u'_r) \geq \Sigma \) be an \( I \)-admissible sequence such that \( mu_1 \equiv mu'_1 \pmod{m} \). Then there exists a one-point \( G \)-cover \( f' : Y' \to X \) with inertia groups isomorphic to \( I \) having the sequence \( \Sigma' \) of positive upper jumps in the higher ramification filtration.

The key step is to show that there is a “singular deformation” (defined below, or see [Pr06, §3]) of our original cover in a formal neighborhood of the branch point to a germ of a cover with the new ramification filtration \( \Sigma' \). In the case \( r = 1 \), the deformation result we need is given by [Pr04, Proposition 2.2.1], whereas in the case \( m = 1 \), we can use [Pr06, Proposition 22]. One can adapt the proof of [Pr06] Proposition 22 for general \( m \), but we will present a different proof based on the explicit equations of [OP10].
4.1. Explicit equations. — We first write down the explicit form of any \( I \)-extension of the complete local ring of a point on \( X \).

**Theorem 4.3.** — Let \( k \) be an algebraically closed field of characteristic \( p \) and let \( I \cong \mathbb{Z}/p^r \rtimes \mathbb{Z}/m \), with \( p \nmid m \). Then any \( I \)-extension of \( k((u)) \) can be given by the following equations:

\[
\begin{align*}
(x_m &= \frac{1}{u} , \\
y_i^p - y_i &= f_i(y_1, \ldots, y_{i-1}, x_1, \ldots, x_i), 1 \leq i \leq r,
\end{align*}
\]

where the \( x_i \) are polynomials in \( k[x] \) with the degrees of all terms lying in a common residue class modulo \( m \) and prime to \( p \). The \( f_i \) are the specific polynomials in \( \mathbb{F}_p[y_1, \ldots, y_{i-1}, x_1, \ldots, x_i] \) from [OP10] top of p. 568. The lone term of \( f_i \) involving \( x_i \) is \( x_i \) itself. Any element \( c \) of order \( m \) in \( I \) satisfies \( c(x)/x = \zeta \), where \( \zeta \) is an \( m \)th root of unity. Furthermore, for all \( i \), \( c(x_i)/x_i = c(y_i)/y_i = \zeta^i \), where \( \zeta^i \) is an \( m \)th root of unity. There is an element \( \sigma \) of order \( p^m \) in \( I \) such that \( \sigma(y_i) = y_i + f_i(y_1, \ldots, y_{i-1}, 1, 0, \ldots, 0) \).

Conversely, if \( L/k((u)) \) is given by the equations (4.1) and (4.2), then it can be made an \( I \)-Galois extension under the action of \( c \) and \( \sigma \) described above. Its upper higher ramification filtration is \( (u_1, \ldots, u_r) \), where the \( u_i \) are defined inductively by \( u_1 = \text{deg}_{x_i}/m \) and \( u_i = \max(\text{deg}_{x_i}, pu_{i-1}) \).

**Proof.** — This summarizes the content of [OP10] §3, 4, 5. \( \Box \)

4.2. Deformation. — We follow the notation and definitions of [Pr06] §3. Recall that \( f : Y \to X \) is a \( G \)-cover branched at one point with inertia groups isomorphic to \( I \cong \mathbb{Z}/p^r \rtimes \mathbb{Z}/m \). Let \( R = k[[t]] \), and \( U_k = \text{Spec } k[[u]] \), and let \( U_R = U_k \times_k R = \text{Spec } k[[t, u]] \). Let \( \hat{\phi} : \hat{Z} \to U_k \) be the germ of \( f \) at a ramification point \( y \) of \( f \). A singular deformation of \( \hat{\phi} \) is an \( I \)-Galois cover \( \hat{\phi}_R : \hat{Y} \to U_R \) of normal irreducible germs of \( R \)-curves, with branch locus \( u = 0 \), such that the normalization of the subscheme \( t = 0 \) of \( \hat{\phi}_R \) is isomorphic to \( \hat{\phi} \) away from \( t = u = 0 \).

**Proposition 4.4.** — In the above situation, suppose \( \hat{\phi} \) has upper higher ramification filtration \( \Sigma = (u_1, \ldots, u_r) \). If \( \Sigma' = (u'_1, \ldots, u'_s) \geq \Sigma \) is an \( I \)-admissible sequence with \( mu_1 \equiv mu'_1 \) (\( \text{mod } m \)), then there is a singular deformation \( \hat{\phi}_R \) of \( \hat{\phi} \) such that \( \hat{\phi}_R \times_R k((t)) \) has upper higher ramification filtration \( \Sigma' \).

**Proof.** — By Theorem 4.3, the morphism \( \hat{\phi} \) is given by the normalization of \( U_k \) in a field generated by the equations (4.1) and (4.2). Now, consider the \( I \)-Galois extension \( \hat{\phi}_R : \hat{Y} \to U_R \) given by normalizing \( U_R \) in the function field generated by the equations (4.1) and (4.2), but with \( x_i \) replaced by \( x'_i := x_i + tx\mu_i' \) if \( u'_i > pu'_{i-1} \) and \( u'_i > u_i \). The only singular point of \( \hat{Y} \) lies above \( u = t = 0 \). It is unramified away from \( u = 0 \), because the right-hand sides of (4.1) and (4.2) have poles only at \( u = 0 \). Away from \( t = 0 \), our equations for \( \hat{\phi} \) give an extension in the form of Theorem 4.3 with the \( x_i \).
replaced by $x'_i$. It is easy to see that the degrees of the monomials in $x'_i$ (as functions of $x$) satisfy the conditions of Theorem 4.3 when $t \neq 0$.

We prove that, when $t \neq 0$, the $i$th upper jump $u''_i$ is equal to $u'_i$ by induction. For $i = 1$, it is clear, as both are equal to $\deg_x(x'_i)/m$. For general $i$, assume first that $u'_i > pu'_{i-1}$. Then $\deg_x(x'_i)/m = u'_i > pu'_{i-1}$, and $u'_{i-1} = u''_{i-1}$ by the inductive hypothesis. So by Theorem 4.3 we have $u''_i = u'_i$. Now assume $u'_i = pu'_{i-1}$. Then $\deg_x(x'_i)/m = \deg_x(x_i)/m \leq u_i \leq u'_i = pu'_{i-1} = pu''_{i-1}$. So by Theorem 4.3 we have $u''_i = pu''_{i-1} = pu''_{i-1} = u'_i$. 

**Proof of Proposition 4.2** — The proof is identical to that of [Pr04 Proposition 2.2.2], with our Proposition 4.1 substituting for [Pr04 Proposition 2.2.1], so we only give a sketch. Let $\xi$ be the branch point of $f : Y \to X$. Construct the singular deformation at the formal neighborhood $U_k$ of $\xi$ as in Proposition 4.1. Then induce this $I$-Galois cover up to $G$, forming a disconnected cover. This gives the data for a relative $G$-Galois thickening problem, which has a solution by [HS99 Theorem 4]. Namely, there is a $G$-cover $f_R : Y_R \to X_R$, where $R = k[[t]]$, where $X_R = X \times_k R$, where $f_R$ is isomorphic to the trivial deformation of $f$ away from $\xi \times_k R$, and where $f_R$ is our induced singular deformation above a formal neighborhood of $\xi \times_k R$. Since the construction is finite in nature, we can construct a cover $f_{R'} : Y_{R'} \to X_{R'}$ over a subring $R' \subset R$ that is of finite type over $k$ such that $f_{R'} \times_R R \cong f_R$. Then Spec $R'$ has infinitely many $k$-points, and for a generic $k$-point $u$, we have that $f_{R'} \times_R \{u\}$ satisfies the requirements of the proposition.

Combining Proposition 4.2 with the results of [3] we have the following:

**Corollary 4.5.** — Let $k$ be an algebraically closed field of characteristic $p \geq 7$, and let $G \cong \text{PSL}_2(\ell)$, such that $\ell$ is odd and $v_p(|G|) = a > 1$. Let $I \cong \mathbb{Z}/p^r$ or $D_{2r}$. Then there exists an $I$-admissible sequence $\Sigma$ for $I$ such that, for any $I$-admissible sequence $\Sigma' \geq \Sigma$, there is a one-point cover $f : Y' \to X = \mathbb{P}^1$ over $k$ with inertia groups $I$ and upper higher ramification filtration $\Sigma'$. If $I \cong D_{2r}$, we can take $\Sigma = (\frac{1}{2})$. If $I = \mathbb{Z}/p$, we can take $\Sigma = (3)$ (or $\Sigma = (2)$ if $\ell \equiv 1 \pmod{8}$).

**Proof.** — By Corollary 3.6 there is a one-point $G$-cover $f : Y \to X$ with inertia groups $I$. Let $\Sigma = (u_1, \ldots, u_r)$ be its upper higher ramification filtration. Suppose $\Sigma' = (u'_1, \ldots, u'_r) \geq \Sigma$ is any $I$-admissible sequence. By Proposition 4.2 there exists a one-point $G$-cover $f' : Y' \to X$ with inertia groups $I$ and upper higher ramification filtration $\Sigma'$, so long as, if $I \cong D_{2r}$, we have $2u_1 \equiv 2u'_1 \pmod{2}$. This is true because, according to Theorem 4.1 (b), $2u_1$ and $2u'_1$ must both be odd.

Theorem 3.4 shows that we can take $\Sigma = (\frac{1}{2})$ if $I \cong D_{2r}$. Remark 3.7 shows that we can take $\Sigma = (3)$ if $I \cong \mathbb{Z}/p$, and $\Sigma = (2)$ if, in addition, $\ell \equiv 1 \pmod{8}$. 

**Remark 4.6.** — One would like to find an explicit minimal choice of $\Sigma$ for general $I$. However, the result [Har03 Theorem 3.6] used in the proof of Corollary 3.6 is not
constructive as stated. It would be interesting to look carefully through the proof of [Har03, Theorem 3.6] to see if it can be made constructive.

References

[Abh01] S. Abhyankar, Resolution of singularities and modular Galois theory, *Bull. Amer. Math. Soc. (N. S.)* 38 (2001), no. 2, 131-169.

[BP03] I. Bouw and R. Pries, Rigidity, ramification, and reduction, *Math. Ann.* 326 (2003), no. 4, 803–824.

[BW04] I. Bouw and S. Wewers, Stable reduction of modular curves, *Modular curves and abelian varieties*, Progr. Math., 24, Birkhäuser, Basel, 2004, 1–22.

[DM69] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), 75–109.

[SGA1] A. Grothendieck, Revêtement étale et groupe fondamental (SGA 1), *Lecture Notes in Math.* 224, Springer-Verlag, Berlin-New York, 1971.

[Har93] D. Harbater, Formal patching and adding branch points, *Amer. J. Math.* 115 (1993), no. 3, 487–508.

[Har94] D. Harbater, Abhyankar’s conjecture on Galois groups over curves, *Invent. Math.* 117 (1994), no. 1, 1–25.

[Har03] D. Harbater, Abhyankar’s conjecture and embedding problems, *J. Reine Angew. Math.* 559 (2003), 1–24.

[HS99] D. Harbater and K. Stevenson, Patching and thickening problems, *J. Algebra* 212 (1999) no. 1, 272–304.

[Hup67] B. Huppert, *Endliche Gruppen*, Springer-Verlag, Berlin-New York, 1967.

[Liu06] Q. Liu, Stable reduction of finite covers of curves, *Compos. Math.* 142 (2006), 101–118.

[MP10] J. Muskat and R. Pries, Alternating group covers of the affine line, to appear in *Israel J. Math.*

[Obu10a] A. Obus, Vanishing cycles and wild monodromy, preprint. Available at http://arxiv.org/abs/0910.0676v2

[Obu10b] A. Obus, Fields of moduli of three-point G-covers with cyclic p-Sylow, II, preprint. Available at http://arxiv.org/abs/1001.3723v3

[OP10] A. Obus and R. Pries, Wild tame-by-cyclic extensions, *J. Pure Appl. Algebra* 214 (2010), no. 5, 565–573.

[Pr04] R. Pries, Conductors of wildly ramified covers. III, *Pacific J. Math.* 211 (2003) no. 1, 163–182.

[Pr06] R. Pries, Wildly ramified covers with large genus, *J. Number Theory* 119 (2006), no. 2, 194–209.

[Ray90] M. Raynaud, p-Groupes et réduction semi-stable des courbes, *the Grothendieck Festschrift, Vol. III*, Progr. Math., 88, Birkhäuser Boston, Boston, MA, 1990, 179–197.

[Ray94] M. Raynaud, Revêtements de la droite affine en caractéristique p > 0 et conjecture d’Abhyankar, *Invent. Math.* 116 (1994), 425–462.

[Ray99] M. Raynaud, Specialization des revêtements en caractéristique p > 0, *Ann. Sci. École Norm. Sup.*, 32 (1999), 87–126.

[Ser79] J.-P. Serre, *Local fields*, Springer-Verlag, New York, 1979.
