New proof of a Theorem on k-hypertournament losing scores

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AMS Subject Classification: 05C

Abstract. In this paper, we give a new and short proof of a Theorem on k-hypertournament losing scores due to Zhou et al. [7].

1. Introduction

An edge of a graph is a pair of vertices and an edge of a hypergraph is a subset of the vertex set, consisting of at least two vertices. An edge in a hypergraph consisting of k vertices is called a k-edge, and a hypergraph all of whose edges are k-edges is called a k-hypergraph.

A k-hypertournament is a complete k-hypergraph with each k-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. In other words, given two non-negative integers n and k with \( n \geq k > 1 \), a k-hypertournament on n vertices is a pair \((V, A)\), where V is a set of vertices with \(|V| = n\) and A is a set of k-tuples of vertices, called arcs, such that any k-subset S of V, A contains exactly one of the \( k! \) k-tuples whose entries belong to S. If \( n < k \), \( A = \emptyset \) and this type of hypertournament is called a null-hypertournament. Clearly, a 2-hypertournament is simply a tournament.

Let \( e = (v_1, v_2, ..., v_k) \) be an arc in a k-hypertournament H. Then \( e(v_i, v_j) \) represents the arc obtained from \( e \) by interchanging \( v_i \) and \( v_j \).

The following result due to Landau [4] characterizes the score sequences in tournaments.

**Theorem 1.** A sequence of non-negative integers \([ s_1, s_2, ..., s_n ]\) in non-decreasing order is a score sequence of some tournament if and only if

\[
\sum_{i=1}^{j} s_i \geq \binom{j}{2}, \quad 1 \leq j \leq n,
\]
Now, there exist several proofs of Landau's theorem and a survey of these can be found in Reid [5]. There are stronger inequalities on the scores in tournaments which are due to Brualdi and Shen [1].

Instead of scores of vertices in a tournament, Zhou et al. [7] considered scores and losing scores of vertices in a k-hypertournament, and derived a result analogous to Landau’s theorem [4]. The score \( s(v_i) \) or \( s_i \) of a vertex \( v_i \) is the number of arcs containing \( v_i \) and in which \( v_i \) is not the last element, and the losing score \( r(v_i) \) or \( r_i \) of a vertex \( v_i \) is the number of arcs containing \( v_i \) and in which \( v_i \) is the last element. The score sequence (losing score sequence) is formed by listing the scores (losing scores) in non-decreasing order.

For two integers \( p \) and \( q \), \( \binom{p}{q} = \frac{p!}{q!(p-q)!} \) and \( \binom{p}{q} = 0 \) if \( p < q \).

The following characterization of losing score sequence in k-hypertournaments are due to Zhou et al. [7].

**Theorem 2.** Given two non-negative integers \( n \) and \( k \) with \( n \geq k > 1 \), a non-decreasing sequence \( R = [r_1, r_2, ..., r_n] \) of non-negative integers is a losing score sequence of some k-hypertournament if and only if for each \( j \),

\[
\sum_{i=1}^{j} r_i \geq \binom{j}{k},
\]

with equality when \( j = n \).

Koh and Ree [3] have given a different proof of Theorem 2. Some more results on scores of k-hypertournaments can be found in [2, 6]. The following is the new and short proof of Theorem 2.

**Proof.** The necessity part is obvious.

We prove sufficiency by contradiction. Assume all sequences of non-negative integers in non-decreasing order of length fewer than \( n \), satisfying conditions (1) be the losing score sequences. Let \( n \) be the smallest length and \( r_1 \) be the smallest possible with that choice of \( n \) such that \( R = [r_1, r_2, ..., r_n] \) is not a losing score sequence.

Consider two cases, (a) equality in (1) holds for some \( j < n \), and (b) each inequality in (1) is strict for all \( j < n \).

**Case (a).** Assume \( j \ (j < n) \) is the smallest such that

\[
\sum_{i=1}^{j} r_i \geq \binom{j}{k}.
\]

By the minimality of \( n \), the sequence \( [r_1, r_2, ..., r_j] \) is the losing score sequence of some k-hypertournament \( H_1 \). Also,
\[
\sum_{i=1}^{m} \left[ r_{j+i} - \left( \frac{1}{m} \right) \sum_{i=1}^{k-1} \binom{j}{i} \left( \binom{n-j}{k-i} \right) \right]
= \sum_{i=1}^{m+j} r_i - \left( \frac{j}{k} \right) - \sum_{i=1}^{k-1} \binom{j}{i} \left( \binom{n-j}{k-i} \right)
\geq \left( \frac{m+j}{k} \right) - \left( \frac{j}{k} \right) - \sum_{i=1}^{k-1} \binom{j}{i} \left( \binom{n-j}{k-i} \right)
= \binom{m}{k}.
\]

for each \( m, 1 \leq m \leq n-j \), with equality when \( m = n-j \).

Let \( \frac{1}{m} \sum_{i=1}^{k-1} \binom{j}{i} \left( \binom{n-j}{k-i} \right) = \alpha \). Therefore, by the minimality of \( n \), the sequence \([r_{k+1} - \alpha, r_{k+2} - \alpha, ..., r_n - \alpha]\) is the losing score sequence of some \( k \)-hypertournament \( H_2 \). Taking disjoint union of \( H_1 \) and \( H_2 \), and adding all \( m \alpha \) arcs between \( H_1 \) and \( H_2 \) such that each arc among \( m \alpha \) has the last entry in \( H_2 \) and each vertex of \( H_2 \) gets equal shares from these \( m \alpha \) last entries, we obtain a \( k \)-hypertournament with losing score sequence \( R \), which is a contradiction.

**Case (b).** Let each inequality in (1) is strict when \( j < n \), and in particular \( r_1 > 0 \). Then the sequence \([r_1 - 1, r_2, ..., r_n + 1]\) satisfies (1), and therefore by minimality of \( r_1 \), is the losing score sequence of some \( k \)-hypertournament \( H \), a contradiction. Let \( x \) and \( y \) be the vertices respectively with losing scores \( r_{n+1} \) and \( r_{1-1} \). If there is an arc \( e \) containing both \( x \) and \( y \) with \( y \) as the last element in \( e \), let \( e' = (x, y) \). Clearly, \( (H-e) \cup e' \) is the \( k \)-hypertournament with losing score sequence \( R \), again a contradiction. If not, since \( r(x) > r(y) \) there exist two arcs of the form \( e_1 = (w_1, w_2, ..., w_{l-1}, u, w_l, ..., w_{k-1}) \) and \( e_2 = (w'_1, w'_2, ..., w'_{l-1}, v) \), where \( (w'_1, w'_2, ..., w'_{k-1}) \) is a permutation of \( (w_1, w_2, ..., w_{k-1}) \), \( x \notin \{w_1, w_2, ..., w_{k-1}\} \) and \( y \notin \{w_1, w_2, ..., w_{k-1}\} \). Then, clearly \( R \) is the losing score sequence of the \( k \)-hypertournament \( (H-\langle e_1 \cup e_2 \rangle) \cup \langle e'_1 \cup e'_2 \rangle \) where \( e'_1 = (u, w_{k-1}) \), \( e'_2 = (w'_1, v) \) and \( t \) is the integer with \( w'_t = w_{k-1} \). This again contradicts the hypothesis. Hence, the result follows.

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