The Parallel-Sequential Duality : Matrices and Graphs.

Abstract. Usually, mathematical objects have highly parallel interpretations. In this paper, we consider them as sequential constructors of other objects. In particular, we prove that every reflexive directed graph can be interpreted as a program that builds another and is itself built by another. That leads to some optimal memory computations, codings similar to modular decompositions and other strange dynamical phenomenons.

1. Introduction.

In this paper, we will deal with matrices and finite directed graphs $G = (V, A)$ where $V$ is a finite list $(x_1, x_2, \ldots, x_n)$ of vertices and $A$ is a subset of $V \times V$ of arcs.

Usually, in classical mathematics, matrices are highly parallel objects. For computing the transformation of a vector $X$ by a linear mapping $X := E(X)$, a mathematician first keeps the initial vector $X$ in mind or in a safe place, and then computes the successive images of each component of $X$ by the projections of $E$. Equivalently, he can also compute $Y := E(X)$ and then $X := Y$. In both cases, one seems to have to build a copy of the initial vector $X$ in order to preserve its initial values for the whole computation that will modify these values. Hence, the data size for this kind of computation processes is twice the size of the input data $X$. A motivation of this paper is to show that one can compute the transformation $X := E(X)$ without any copy. In [1], we proved a similar result for the computation of boolean mappings. Here, we will interpret matrices as sequential programs that perform sequences of assignments on the components of the initial vector. In general, the mapping computed by this way is not the usual linear mapping represented. Now, a matrix admits a usual parallel interpretation and a sequential interpretation. We are going to investigate this duality.

Definition. (matrices). Let $K$ be a field. Given a square matrix $M$ of $M_{n,n}(K)$, $M_i$ is the $i$-th row vector of $M$ and $M_{i,j}$ is the $j$-th component of the row $M_i$. The matrix $M$ is said regular if it only has 1s on its diagonal, i.e., $M_{i,i} = 1$ for every $i$. A matrix $M'$ of $M_{n,n}(K)$ is similar to $M$ if it has the same values than $M$ excepted possibly on the diagonal, i.e., $M'_{i,j} \neq M_{i,j} \implies i = j$.

First, we recall the classical interpretation of a matrix in linear algebra.

Definition. (parallel mapping). Let $K$ be a field. Every square matrix $M$ of $M_{n,n}(K)$ represents a linear mapping $M^\#$ from $K^n$ to $K^n$ that we will call the parallel mapping of $M$ and defined by the following. For every $X = (x_1, \ldots, x_n) \in K^n$, the parallel image of $X$ by $M$ is the vector

$M^\#(X) = (M_1.X, M_2.X, \ldots, M_n.X)$
This mapping \( M^\# \) can be computed by the following straight-line program:

**Input**: \( X = (x_1, \ldots, x_n) \)

For \( i \) from 1 to \( n \) do
\[
y_i := \sum_{j=1}^{n} M_{i,j} x_j
\]

**Output**: \( Y = (y_1, \ldots, y_n) \)

2. **Sequentializing.**

Now, we define a dual interpretation of a matrix in linear algebra such that the image of a vector \( X \) can be computed via a sequence of linear transformations of this vector \( X \) and only using this vector for the whole computation. We obtain an "in situ" straight-line program.

**Definition. (sequential mapping).** Let \( K \) be a field. Every square matrix \( M \) of \( M_{n,n}(K) \) represents a linear mapping \( M^\downarrow \) from \( K^n \) to \( K^n \) called the sequential mapping of \( M \) defined by the following. For every \( X = (x_1, \ldots, x_n) \in K^n \), the sequential image of \( X \) by \( M \) is the resulting vector computed by the following straight-line program denoted \( M^\pi \):

**Input**: \( X = (x_1, \ldots, x_n) \)

For \( i \) from 1 to \( n \) do
\[
x_i := \sum_{j=1}^{n} M_{i,j} x_j
\]

**Output**: \( X = (x_1, \ldots, x_n) \)

Denote \( M^s \) the matrix such that \((M^s)^\# = M^\downarrow\).

As we said above, one only uses the input vector \( X \) for the computation of the sequential image, whereas in the standard parallel interpretation, one uses a second vector \( Y \) (in order to preserve the values of the input vector \( X \)).

For example, for \( K = \mathbb{R} \), the sequential mapping \( M^\downarrow \) of the matrix

\[
M = \begin{bmatrix}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8
\end{bmatrix}
\]

maps every vector \((a, b, c)\) to \((b + 2c, 7b + 11c, 55b + 97c)\) since the corresponding sequential program \( M^\pi \) is:

\[
\begin{align*}
x_1 &= M_{11} x_1 + M_{12} x_2 + M_{13} x_3 \\
x_2 &= M_{21} x_1 + M_{22} x_2 + M_{23} x_3 \\
x_3 &= M_{31} x_1 + M_{32} x_2 + M_{33} x_3
\end{align*}
\]
a := 0.a + 1.b + 2.c (= b + 2c) 

b := 3.a + 4.b + 5.c (= 3(b + 2c) + (4b) + (5c) = 7b + 11c) 

c := 6.a + 7.b + 8.c (= 6(b + 2c) + 7(7b + 11c) + 8c = 55b + 97c) 

and one has:

\[
M^s = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 7 & 11 \\ 0 & 55 & 97 \end{bmatrix}
\]

The sequential interpretation of a matrix leads to a natural notion in the context of directed graphs.

**Definition. (graph construction).** Let \( G = (V, A) \) be a finite directed graph where \( V \) is a finite list \( (x_1, x_2, \ldots, x_n) \) of vertices and \( A \) is a subset of \( V \times V \) of arcs. We say that \( G \) **sequentially constructs** a directed graph \( G' \) when their respective adjacency matrices \( M, M' \) in \( M_{n,n}(F_2) \) satisfy:

\[
M^s = M'
\]

That is a kind of decomposition of a graph. Instead of working with a graph \( G \), one can consider a sequential constructor of \( G \) or the graph that \( G \) constructs. We will see in the sequel some advantages. We are going to study the relations between these graphs and first of all, their existence. Given a graph \( G \), is there another graph \( G' \) which is a sequential constructor of \( G \)? We will see that the answer is NO. However, if one assumes the graph \( G \) to be reflexive, the answer is YES.

In the formalism of matrices, observe that by definition, the parallel interpretation of \( M^s \) is equal to the sequential interpretation of \( M \). For the other direction, a natural question is:

*given a matrix \( M \), is there a matrix \( P \) with a sequential interpretation equal to the parallel interpretation of \( M \)?*

Unfortunately, the answer is NO in general. A minimal example for \( K = F_2 \) is:

\[
M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

If there were a matrix \( P \) such that \( P^\dagger = M^\#$, since \( P \) and \( M \) have necessarily the same first rows (i.e., \( P_1 = M_1 \)), the matrix \( P \) must be some

\[
\begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}
\]
and the sequential program \( P^\pi \):
\[
a := 0 \\
b := k_1.a + k_2.b
\]
should transform every vector \((a, b)\) in \((0, a)\) : that is not possible.

However, we are not very far from a positive answer with the following preliminary result.

**Theorem 1.** Let \( K \) be a field. For every linear mapping \( E \) from \( K^n \) to \( K^n \) with \( n > 0 \), the assignment \( X := E(X) \) is computed by a straight-line program \( P \) made of at most \( 2n - 1 \) linear assignments of the components of \( X \). Moreover, the \( n \) first steps of \( P \) form a program \( M^\pi \) for a matrix \( M \) in \( M_{n,n}(K) \).

**Proof.** We construct the code of the program \( P \) in two parts.

Part 1.
Let \( M \) be the matrix such that \( M^# = E \) and let \( P \) be the empty program.
For \( i \) from 1 to \( n \) do

**case 1.** If \( M_{ii} \neq 0 \)
Let \( r_i := 0 \)
Add to \( P \) the instruction \([x_i := M_i.X]\)
For every \( k > i \) with \( M_{ki} \neq 0 \) do
\[
M_k := M_k + M_{ki}.M_{ii}^{-1}.(x_i - M_i)
\]

**case 2.** If \( M_{ii} = 0 \) and for every \( j > i \), \( M_{ji} = 0 \)
Let \( r_i := 0 \)
Add to \( P \) the instruction \([x_i := M_i.X]\)

**case 3.** If \( M_{ii} = 0 \) and there exists some \( j > i \) with \( M_{ji} \neq 0 \)
Let \( r_i := j \) for such a chosen \( j \).
Add to \( P \) the instruction \([x_i := (M_i - M_j).X]\)
For every \( k > i \) with \( M_{ki} \neq 0 \) do
\[
M_k := M_k + M_{ki}.(-M_{ji})^{-1}.(x_i - M_i + M_j)
\]

Part 2.
for \( i \) from \( n - 1 \) downto 1 do
If \( r_i = j > i \) then add to \( P \) the instruction \([x_i := x_i + x_j]\).
We prove the correction of this method, that is to say, that the program $P$ produced computes the transformation $X := E(X)$ for every $X \in K^N$.

In Part 1, assume that at each step $i = 1, \ldots, n$ the rows $M_i, \ldots, M_n$ represent the respective images by $E$ of the components $x_1, \ldots, x_N$ according to their current values at step $i$. That is true for $i = 1$ because $M^0 = E$. At step $i$, the program makes an assignment of $x_i$.

In case 1, $M_{ii} \neq 0$ and $x_i$ receives the new value $a.x_i + b$ where $a = M_{ii} \neq 0$. Hence, the initial value of $x_i$ is not lost and was $a^{-1} \cdot (x_i - b)$ where $x_i$ is its new value. In order to preserve the hypothesis on $M$, one has to replace the references to the initial $x_i$ by this new linear function $a^{-1} \cdot (x_i - b)$ in every $M_k$ for $k > i$. Hence every $M_k = M_{ki}x_i + F$ with $k > i$ and $M_{ki} \neq 0$ should become

$$M_{ki} \cdot a^{-1} \cdot (x_i - b) + F = M_{ki} \cdot a^{-1} \cdot (x_i - M_i + a.x_i) + F$$

$$= M_{ki} \cdot a^{-1} \cdot (x_i - M_i) + M_{ki} \cdot x_i + F$$

$$= M_{ki} \cdot M_{ii}^{-1} \cdot (x_i - M_i) + M_k$$

In case 2, $M_{ii} = 0$ and $M_{ii}^{-1}$ does not exist. Hence, the previous adjustment is not possible. However, this operation is not necessary since the old value of $x_i$ is used in no $M_j$ for $j > i$.

Case 3 is more critical. The old value of $x_i$ will be used again in some $M_j$ with $j > i$ but a direct assignment of $x_i$ would make impossible to recover it because $M_{ii} = 0$. The adopted solution here is to first transform $M_i$ in $M_i' = M_i - M_j$ in order to fail in case 1 since $M_i'_{ii} = -M_{ji} \neq 0$. The correction will be made in Part 2.

In Part 2, assume that for each $i = n - 1, \ldots, 1$, the components $x_{i+1}, \ldots, x_n$ have received their correct values. That is true for $i = n - 1$, since the assignment of $x_n$ in Part 1 cannot fail in case 3 because $i = n$ is maximal (there is no $j > i$). Hence $x_n$ has received its correct value by hypothesis in part 1. If for $i$ such that $r_i = j > i$, by hypothesis, $x_j$ has received its correct value $E_j$. Hence, the assignment $x_i := x_i + x_j$ will perform $x_i := x_i + E_j = E_i - E_j + E_j = E_i$ and $x_i$ also receives its correct value.

Observe that the Part 1 of the produced program $P$ is exactly the sequential program $M^+$ and in Part 2 all the assignments are quite simple and have the form

$$x_i := x_i + x_j$$

Hence, one can code Part 1 with the matrix $M$ and Part 2 with the list of numbers $[r_1, r_2, \ldots, r_n]$.

For example, let us compute the program for the linear mapping $E : \mathbb{R}^3 \mapsto \mathbb{R}^3$ with $E(a, b, c) = (c, a + b + c, 3a + 3b + 2c)$

For $i = 1$, one has $M_{11} = 0$ and we fall in case 3 since $M_{21} \neq 0$ (and also $M_{31} \neq 0$). One chooses for instance $j = 2$ and remember that for Part 2 with
\( r_1 := 2 \). Then performs \( M_1 = M_1 - M_2 = [-1, -1, 0] \). The first assignment of the program is \( a := -a - b \). Then \( M_2 \) becomes \([-1, 0, +1]\) and \( M_3 \) becomes \([-3, 0, 2]\).

For \( i = 2 \), \( M_{22} = 0 \) and we fall in case 2 since \( M_{32} = 0 \) too. The second assignment of the program is \( b := -a + c \). And \( M_3 \) remains equal to \([-3, 0, 2]\).

For \( i = 3 \), we are in case 1 and the third assignment of the program is \( c := -3a + 2c \).

In part 2, the only assignment added is \( a := a + b \). Finally, the computation program \( P \) is

\[
\begin{align*}
a &:= -a - b \\
b &:= -a + c \\
c &:= -3a + 2c \\
a &:= a + b
\end{align*}
\]

One can verify that \( P \) computes \( X := E(X) \).

Begin with \( X = (a, b, c) \).

\[
\begin{array}{ccc}
P & | & X \\
a := -a - b & | & a \\
b := -a + c & | & b \\
c := -3a + 2c & | & c \\
a := a + b & | & c
\end{array}
\]

From every \( X = (a, b, c) \in \mathbb{R}^3 \), the program ends with \( X = (c, a + b + c, 3a + 3b + 2c) \) as expected.

The only problem for sequentializing is Case 3 that forces an adaptation in Part 2. Hence, one needs a larger structure than matrices themselves in order to code this sequentializing: for instance, a matrix AND the list of numbers \( r_i \). In the previous example, a possible coding of the sequentializing of

\[
M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}
\]

is:

\[
\begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ -3 & 0 & 2 \end{bmatrix} [2, 0, 0]
\]

For another example, the sequentializing of

\[
M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]
leads to the following straight-line program:

\[
\begin{align*}
a &:= -a \\
b &:= -a \\
a &:= a + b
\end{align*}
\]

and a possible coding is:

\[
\begin{bmatrix}
-1 & 0 \\
-1 & 0
\end{bmatrix} [2,0]
\]

Now we consider matrices up to reflexivity (i.e., up to their diagonals). With this restriction, the previous sequentializing method can be simplified such that only Cases 1 occur. Hence, the sequentializing program is only made of \( n \) steps where \( n \) is the dimension of the given matrix (or the graph).

**Theorem 2.** Let \( K \) be a field. Let \( M \) be a matrix in \( M_{n,n}(K) \). There exists a regular matrix \( dM \) in \( M_{n,n}(K) \) such that \( dM^t \) is similar to \( M \).

**Proof.** The construction is similar than in the previous proof. However, the adjustments consist in the replacement at step \( i \) of a critical case \( M_{i,i} = 0 \) by forcing \( M_{i,i} := 1 \). We explicit now this procedure in the particular case \( K = F_2 \).

Begin with \( dM = M \) and \( P \) is the empty program.

For \( i \) from 1 to \( N \) do

Set \( dM_{i,i} := 0 \).

Add to \( P \) the instruction \( [x_i := x_i + dM_{i,X}] \)

For every \( k > i \) with \( dM_{ki} = 1 \) do

\[dM_k := dM_k + dM_i\]

Set \( dM_{i,i} := 1 \).

As previously, the construction implies that at each step \( i \), the rows \( dM_i, \ldots, dM_n \) represent the images (up to reflexivity) of the initial vector according to its current values.

For example, for

\[
M = \begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

one obtains:
for $i = 1$, one adds to $P$ the instruction $[a := a + b + c]$ and $^dM$ becomes

$$^dM = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

for $i = 2$, one adds to $P$ the instruction $[b := a + b + c]$ and $^dM$ becomes

$$^dM = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

for $i = 3$, one adds to $P$ the instruction $[c := b + c]$

We have obtained the regular matrix

$$^dM = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

where

$$^dM^s = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

is similar to $M$.

3. Dynamics.

Some strange dynamical systems can be deduced from the above. Let $G$ be a reflexive directed graph. Hence its adjacency matrix $M$ is boolean and regular. Let us construct the sequence of matrices $M = M_0, M_1, M_2, \ldots$ where $M_{i+1} = ^dM_i$. By construction, these matrices are all boolean and regular. Since the set containing these matrices is finite, there must exists some cycle, i.e. some pair $0 \leq p < q$ with $M_p = M_q$ and where $p$ is taken minimal.

We claim that $p = 0$.

Observe that this sequence of matrices can be constructed in the other direction. From $M_{i+1}$, build the matrix $M_{i+1}^s$. With the previous result, $M_{i+1}^s = ^dM_i^s$ is similar to $M_i$. But $M_i$ is regular. Hence, by putting only 1s on the diagonal of $M_{i+1}^s$, one obtains a deterministic way to transform $M_{i+1}$ to $M_i$. Denote $\phi$ this mapping on regular matrices that takes a matrix $M_i$, computes $M_i^s$ and puts 1s on its diagonal.

Now assume that $p > 0$. The matrix $M_p$ would satisfy $\phi(M_p) = M_{p-1}$ and $\phi(M_p) = M_q$. Since $\phi$ is a mapping, one obtains $M_{p-1} = M_q$ : a contradiction with the minimality of $p$.

For example, the maximal cycle on matrices of $M_{4,4}(F_2)$ has length 18 and from the matrix:
one obtains a cycle of length 13122.

4. Relations.

We have seen that there exist matrices $M$ that do not admit some matrix $P$ so that $P^e = M$. However, every matrix $M$ has a unique sequential mapping $M^\downarrow$. Hence there exist pairs of matrices $(P, P')$ that construct the same matrix. Let us investigate in this paragraph these relations.

**Definition. (equivalence).** Two matrices $M, W$ are sequentially equivalent if their sequential mappings $M^\downarrow$ and $W^\downarrow$ are equal.

For example, the two matrices of $M_{4,4}(\mathbb{R})$:

$$
M = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\quad W = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
$$

are sequentially equivalent since the two mappings $M^\downarrow$ and $W^\downarrow$ are equal to

$$(a, b, c, d) \mapsto (a + 2b + 3c + 4d, a + 2b + 3c + 4d, a + 2b + 3c + 4d, a + 2b + 3c + 4d)$$

However, their sequential programs are different:

$$
\begin{align*}
M^\pi & a := a + 2b + 3c + 4d & a & := a + 2b + 3c + 4d \\
b & := a & b & := a \\
c & := b & c & := a \\
d & := c & d & := a
\end{align*}
$$

Of course, this notion can be exported in the context of directed graphs.

**Definition. (equivalence).** Two directed graph $G, G'$ are sequentially equivalent if they sequentially construct the same graph. In that case, we write $G \equiv G'$.
**Proposition (chains).** Every directed graph $G = (V = (x_1, \ldots, x_n), A)$ that admits an induced subgraph $H$ on vertices $(x_p, x_{p+1}, \ldots, x_q)$ which is a chain (i.e., $(x_i, x_{i+1}) \in A$ for $p \leq i < q$), is sequentially equivalent to the graph $G'$ where an arc $(x_i, x_{i+1})$ of this chain is replaced by an arc $(x_i, x_j)$ with $p \leq j < i + 1$.

**Proof.** The corresponding sequential program performs:

$x_{p+1} := x_p$
$x_{p+2} := x_{p+1}$
$x_{p+3} := x_{p+2}$
\[ \ldots \]
$x_q := x_{q-1}$

and computes the same mapping than the program:

$x_{p+1} := x_p$
$x_{p+2} := x_p$
$x_{p+3} := x_p$
\[ \ldots \]
$x_q := x_p$  ■

For example:

![Diagram](image)

The complexity of graphs can be considerably reduced up to sequential equivalence. For instance, when a graph contains a maximal directed acyclic subgraph.

**Proposition (linear orders).** Every directed graph $G = (V = (x_1, \ldots, x_n), A)$ that admits an induced subgraph $H$ on vertices $(x_p, x_{p+1}, \ldots, x_q)$ which is a linear order (i.e., $(x_i, x_j) \in A$ for $p \leq i < j \leq q$), is sequentially equivalent to the graph $G'$ where one removes all the arcs of $H$ excepted the arc $(x_{p+1}, x_p)$.

**Proof.** The corresponding sequential program performs:

$x_{p+1} := x_p$
$x_{p+2} := x_{p+1} + x_p$
$x_{p+3} := x_{p+2} + x_{p+1} + x_p$
\[ \ldots \]
and computes the same mapping than the program:

\[
\begin{align*}
x_{p+1} &:= x_p \\
x_{p+2} &:= 0 \\
x_{p+3} &:= 0 \\
&\ldots \\
x_q &:= 0\quad \blacksquare
\end{align*}
\]

For example:

5. Conclusion.

The sequential interpretation of matrices we presented here is a powerful tool for computations of linear transformations \( X := E(X) \) using minimal memory: one only uses the input data memory \( X \) for that computation. In the case of graphs, and more generally directed graphs, the sequential interpretation defines a graph decomposition: roughly speaking, every vertex of rank \( i \) can "use" the connecting work of vertices of ranks \( j < i \). Moreover, sequential equivalences in graphs enable some simplifications in the codings. A complete axiomatization of these relations could be done.

Under a dynamical system point of view, iterating sequential interpretations leads to strange phenomena which should also merit particular investigations.

Let us mention that other choices of construction are possible. Here, the sequential constructor \( dM \) of a matrix \( M \) is chosen to have only 1s on its diagonal. However, every sequence of invertible elements in the field \( K \) is possible. Moreover, similar constructions can be done on finite sets in general. We have seen here two sequential decompositions of a linear mapping on \( K^n \). The first one consists in \( 2n-1 \) steps where the last \( n-1 \) steps are quite minimal and coded by assignments \( x_i := x_i + x_j \) with \( j > i \) or by a sequence of integers \( [r_1, r_2, \ldots, r_{n-1}] \). The second one just consists in \( n \) steps but the diagonal of the matrix can be modified. Let us propose a third possibility, still in \( n \) steps, where each row of the matrix is completely preserved. However, the order of the rows may change: during the process, when \( M_{i,i} = 0 \) and there exists \( j > i \) with \( M_{j,i} \neq 0 \), then exchange the two rows \( M_i \) and \( M_j \) and proceed like in the two other methods by substitution. Now, the coding of the result is a
new matrix and a permutation $\sigma$ of $S_n$. This method preserves the informative structure of the mapping. The similar operation on directed graphs, up to a permutation of vertices, preserves the numbers $d^-(x)$ of incoming arcs to each vertex $x$. More precisely, there exists a permutation of vertices $\sigma$ such that every arc $(x, y)$ is transformed in an arc $(\sigma(x), y)$. The semantic of this operation also merits other investigations.

References.

[1] S. Burckel, *Closed Iterative Calculus*, Theoretical Computer Science **158** (1996) 371–378.

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