1 Introduction

According to a famous conjecture of S. Lang’s, if $K$ is a number field, then the set of $K$-rational points of any variety of general type defined over $K$ is not dense in the Zariski topology. In a recent paper, entitled Uniformity of rational points ([CHM]), L. Caporaso, J. Harris, and B. Mazur show that this conjecture implies the existence of a uniform bound on the number of $K$-rational points over all smooth curves of genus $g$ defined over $K$, for some fixed $g \geq 2$. Their bound depends on the genus $g$ and on the number field $K$.

D. Abramovich has proved an extension of this result. In his paper, [A], he proves that, assuming Lang’s conjecture, the bound $B(K,g)$ of [CHM] remains bounded as $K$ varies over all quadratic number fields, or as $K$ varies over all quadratic extensions of a fixed number field.

It is this result of Abramovich’s that we shall generalize in this paper. We will prove that given a number field $K$, Lang’s conjecture implies the existence of a uniform bound on the number of $L$-rational points over all smooth curves of a fixed genus $g > 1$ defined over $L$, as $L$ varies over all extensions of $K$ of degree $d$ for any positive integer $d$. This bound will depend on $K$, $d$, and $g$, but is independent of the actual number field $L$.

**Theorem 1.1** Assume that Lang’s conjecture regarding varieties of general type is true. Let $g \geq 2$ and $d \geq 1$ be integers, and let $K$ be a number field. Then there exists an integer $B_K(d,g)$, which, for a given $K$ depends only $d$ and $g$, such that for any extension $L$ of $K$ of degree $d$, and any curve $C$ of genus $g$ defined over $L$, it follows that

$$\#C(L) \leq B_K(d,g).$$

By letting $K = \mathbb{Q}$ we have the following:

**Corollary 1.2** Assume Lang’s conjecture is true. Let $g \geq 2$ and $d \geq 1$ be integers. Then there exists a bound $B(d,g)$, depending only on $d$ and $g$, such that for any number field $L$ of degree $d$, and for any curve $C$ of genus $g$ defined over $L$, it follows that

$$\#C(L) \leq B(d,g).$$
Since any extension $L$ of $K$ of degree $d$ is a number field of some fixed degree $d'$, Theorem 1.1 and Corollary 1.2 are equivalent. We state Theorem 1.1 separately, however, as it might be interesting to study the dependence of the bound on $K$ in later work, perhaps to see what happens assuming Lang's so-called strong conjecture.

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2 Definitions, Notation, and Ideas

Lang's conjecture concerns varieties of general type. Recall the definitions:

Definition 2.1 A line bundle $L$ on a variety $Y$ is said to be big if for high values of $k$, $H^0(Y, L)$ has enough sections to induce a birational map to projective space.

Definition 2.2 A smooth projective variety $Y$ is of general type if its dualizing sheaf $\omega_Y$ is big. An arbitrary projective variety is of general type if a desingularization of it is.

The main idea we will be working with is that given a family of curves $X \to B$, we can study the symmetric $d$-th power of the $n$-th fibered product (for some sufficiently large $n$),

$$Sym^d(X^n_B) = (X^n_B)^d/S_d.$$ 

For our purposes we will take the family $X \to B$ that we work with to be a tautological family. This is a family $X \to B$ of stable curves along with a finite surjective map $\phi: B \to \overline{M}_g$, where $\overline{M}_g$ is the moduli space of smooth stable curves of genus $g$. The map $\phi$ is assumed to have the property that $\phi(b) = [X_b]$ for all $b \in B$. See [CHM], §5.1 for a proof that such a family exists. The reason we work with a tautological family is quite simple; our theorem asserts that there is a bound on the number of rational points on any smooth curve of genus $g$ - hence we want a family in which every such curve appears as a fiber, possibly after a field extension of bounded degree.

Assume $d \geq 1$ and $g \geq 2$ to be fixed integers throughout the remainder of the paper, where $d$ represents the degree of an extension of number fields $K \subseteq L$, and $g$ represents the genus of the curves we will be looking at. Following the notation in [A], write

$$Y_n = Sym^d(X^n_B).$$
Let \( L \) be an extension of \( K \) of degree \( d \), and let \( \sigma_1, \ldots, \sigma_n \) be the \( d \) embeddings of \( L \) in \( \overline{K} \) fixing \( K \). Then if \( b \in B(L) \), and \( (P_1, \ldots, P_n) \in X_b(L) \), we obtain a \( K \)-rational point \( y_{(P_1, \ldots, P_n)} \) of \( Y_n \):
\[
y_{(P_1, \ldots, P_n)} = \{(P_1, \ldots, P_n)^{\sigma_1}, \ldots, (P_1, \ldots, P_n)^{\sigma_d}\}
\]

**Definition 2.3** For a variety \( V \) defined over \( K \) and a field extension \( K \subset L \), let \( V(L, K) \) be the set of those points lying on the variety \( V \) which are defined over \( L \) but are not defined over \( K \) nor any other intermediate field between \( K \) and \( L \).

**Definition 2.4** Let \( m \geq n \). Again following [A], we call a point \( y_{(P_1, \ldots, P_n)} \in Y_n(K) \) \( m \)-prolongable if there exists a number field \( L \), \( [L:K] = d \), and a prolongation \( y_{(P_1, \ldots, P_n)} \in Y_m(K) \) such that \( P_i \in X(L, K) \) for all \( 1 \leq i \leq m \), and if \( 1 \leq i \neq j \leq m \) then \( P_i \neq P_j \). In other words, \( y_{(P_1, \ldots, P_n)} \in Y_n \) is \( m \)-prolongable if the \( n \) points \( P_1, \ldots, P_n \) can be extended to \( m \) distinct points \( P_1, \ldots, P_m \in X(L, K) \), producing a point \( y_{(P_1, \ldots, P_m)} \in Y_m \).

We will call \( E_n^{(m)} \) the set of \( m \)-prolongable points in \( Y_n \), and will denote by \( F_{n}^{(m)} \) the Zariski closure \( \overline{E_n^{(m)}} \).

**Lemma 2.5** For all \( n \geq 1 \), there exists an integer \( m(n) \) such that for all positive integers \( k \),
\[
F_{n}^{(m(n)+k)} = F_{n}^{(m(n))}.
\]

**Proof:** Since \( F_{n}^{(m+1)} \subseteq F_{n}^{(m)} \), the \( F_{n}^{(m)} \)'s form a decreasing sequence of closed sets in the noetherian space \( Y_n \), which must eventually stop. So for all \( n \) there is an integer \( m(n) \) such that \( F_{n}^{(m(n)+k)} = F_{n}^{(m(n))} \) for all positive integers \( k \). Q.E.D.

To ease notation we will write \( F_n \) for \( F_n^{(m(n))} \).

Our goal is to show that each \( F_n \) is empty. For each \( n \), \( F_n \) is the closure of those \( n \)-tuples of distinct points defined over \( L \) which can be extended to arbitrarily long \( m \)-tuples of distinct points over the same field. If we show that each \( F_n \) is empty, this shows that there must be a bound of the desired type.

We want to look at some of the properties of the \( F_n \)'s, and study the maps between them. The ideas in the remainder of this section are generalizations of those found in [A].

**Lemma 2.6** Suppose \( n' > n \) and \( I \subseteq \{1, 2, \ldots, n'\} \) is an \( n \)-tuple. Then the projection \( \pi_I : F_{n'} \to F_n \) is surjective.

**Proof:** This is clearly true for the \( E_n^{(m)} \)'s by definition. Thus the same holds for the \( F_n \)'s. Q.E.D.

We have a natural finite map
\[
\pi_{n,k} : F_{n+k} \to (F_{n+1})_{F_n}^k.
\]
To see this, look at the case where \( k = 2 \). If we consider an element \( y \in F_{n+2} \), then \( y \) can be written as:

\[
y = \{(P_{1,1}, \ldots, P_{n,1}, P_{n+1,1}, P_{n+2,1}), \ldots, (P_{1,d}, \ldots, P_{n,d}, P_{n+1,d}, P_{n+2,d})\}
\]

The map \( \pi_{n,2} \) is defined by sending \( y \) to the following two elements of \( F_{n+1} \):

\[
\{(P_{1,1}, \ldots, P_{n,1}, P_{n+1,1}), \ldots, (P_{1,d}, \ldots, P_{n,d}, P_{n+1,d})\}
\]

and

\[
\{(P_{1,1}, \ldots, P_{n,1}, P_{n+2,1}), \ldots, (P_{1,d}, \ldots, P_{n,d}, P_{n+2,d})\}
\]

These two elements together form an element of \( (F_{n+1})^2_{F_n} \), thus defining \( \pi_{n,2}(y) \).

Similarly, an induction argument shows that \( F_{n+k} \) maps finitely to \( (F_{n+1})^k_{F_n} \).

Notice that by definition the \( E_n \)'s and the \( F_n \)'s are not contained in the \textit{big diagonal} in \( Y_n \); by the big diagonal, we mean the set of \( n \)-tuples for which at least 2 entries agree. Consider the following lemma from [A]:

\textbf{Lemma 2.7 (see [A], Lemma 1)} Let \( D \to Z \) be a generically finite morphism, and let \( \Delta_n \) be the big diagonal in \( D^n_Z \). Then there exists an integer \( n \) for which the \( n \)-th fiber product of \( D \) over \( Z \), \( D^n_Z \backslash \Delta_n \to Z \), is not dominant.

If we let \( D = F_{n+1}, Z = F_n \), then this lemma shows that if \( y \in F_n \) then the dimension of the fiber above \( y \) in \( F_{n+1} \) is at least 1. This means that any component of \( F_n \) has a component of \( F_{n+1} \) above it of positive dimension. Moreover, the lemma shows that if \( y \) is in \( F_n \) and if \( k \) is a positive integer, then the dimension of the fiber above \( y \) in \( F_{n+k} \) is at least \( k \).

We want to perform an induction argument on the relative dimension of \( F_{n+1} \) over \( F_n \) to show that \( F_n \) is empty for all \( n \). We know that this dimension can’t be greater than \( d \), as this is the relative dimension of \( Y_{n+1} \) over \( Y_n \), and we also know that it is at least 1, by the lemma above. What we will do is show that, for any \( l \), if the relative dimension of \( F_{n+1} \) over \( F_n \) is at least \( l \) then it must be at least \( l + 1 \). Eventually we will get to \( l = d \) where the argument must stop, and we will have to conclude that each \( F_n \) is empty.

Assume that for all \( n \) and for all \( y \in F_n \), the dimension of the fiber above \( y \) in \( F_{n+1} \) is greater than or equal to \( l \). Suppose there exists an element \( y \in F_n \) with the dimension of the fiber above \( y \) in \( F_{n+1} \) exactly \( l \). By the semicontinuity of the fiber dimension of projective maps, there is an irreducible component of \( F_n \), call it \( M_n \), with the properties that \( M_n \) contains \( y \), and the general fibers in \( F_{n+1} \) over \( M_n \) have dimension equal to \( l \). It also follows by induction that for any integer \( k \) the dimensions of the fibers in \( F_{n+k} \) above \( M_n \) have dimension equal to \( kl \).

Consider the following diagram where, as you may recall, the top map is finite and birational.

\[
\begin{array}{ccc}
F_{n+k} & \rightarrow & (F_{n+1})^k_{F_n} \\
\downarrow & & \downarrow \\
F_n & & 
\end{array}
\]
Because the map \( F_{n+k} \to (F_{n+1})^k_{F_n} \) is finite, there exists at least one irreducible component \( H_k \) of \( F_{n+k} \) which dominates a component of \((F_{n+1})^k_{F_n}\); this component of \((F_{n+1})^k_{F_n}\) dominates \( M_n \) and has maximal relative dimension \( kl \) over \( F_n \).

Later in this paper we will prove the following important proposition:

**Proposition 2.8** For large values of \( k \) and \( n \) every component of the fiber product \((F_{n+1})^k_{F_n}\) of maximal relative dimension is a variety of general type.

Therefore, for large \( k \), \( H_k \) dominates a variety of general type, and because the map from \( F_{n+k} \) to \((F_{n+1})^k_{F_n}\) is birational, we can conclude that for large enough \( k \) and \( n \), \( F_{n+k} \) is also a variety of general type. Remember, however, that by definition the set of rational points in \( F_{n+k} \) is dense. Thus Lang’s conjecture combined with the work above implies a contradiction. Thus we have discovered that for all \( n \) and for all \( y \in F_n \), the dimension of the fiber above \( y \) in \( F_{n+1} \) must be at least \( l + 1 \).

Continuing by induction on this relative dimension, we will be forced to conclude that if Lang’s conjecture is true, we have a contradiction unless \( F_{n+k} \) is empty for large values of \( k \) and \( n \). Since we have surjective projections from \( F_{n+1} \to F_n \) for all \( n \), however, this implies that all of the \( F_n \)’s must be empty, hence proving Theorem 1.1.

The remainder of the paper will be devoted to proving Proposition 2.8.

### 3 Background Lemmas and Definitions

In definition 2.1, we defined a big line bundle; now we provide an extension of that definition which shall be quite useful.

**Definition 3.1** If \( L \) is a line bundle on a variety \( Z \) and \( \mathcal{J} \) is an ideal sheaf on \( Z \), then we define \( L \otimes \mathcal{J} \) to be big if for high values of \( k \), \( H^0(L^\otimes k \otimes \mathcal{J}^k) \) induces a birational map to projective space \( P^N \) for some \( N \).

**Lemma 3.2** Assume \( Z \) is a projective irreducible variety of dim \( l > 0 \), and that

\[
Z \subseteq C_1 \times \cdots \times C_d
\]

where the \( C_i \)'s are irreducible projective curves. Suppose further that for some \( i \in \{1, \ldots, d\} \) the projection map \( \pi_i : Z \to C_i \) is surjective. Then there exists a set \( J \subseteq \{1, \ldots, d\} \) such that \( i \in J \), \( \# J = l \), and the projection map

\[
\pi_J : Z \to \prod_{j \in J} C_j
\]

is surjective.
**Proof:** We proceed with induction on \( d \). If \( d = 1 \), we simply have \( Z \subseteq C_1 \); hence it must be that both \( i = 1 \) and \( l = 1 \). We use the fact that the inclusion map from \( Z \) to \( C_1 \) must be either constant or surjective, but as \( Z \) has positive dimension, it can’t be constant. Therefore the lemma holds, with \( J = \{1\} \).

Assume the result true for \( d - 1 \) curves, and suppose that we have \( Z \subseteq C_1 \times \cdots \times C_d \), with \( Z \) surjecting onto \( C_i \) for some \( 1 \leq i \leq d \). Choose \( I \subseteq \{1, \ldots, d\} \) to be a subset of cardinality \( d - 1 \) with the property that \( i \in I \), and let \( \pi_I \) be the projection map

\[
\pi_I : Z \to \prod_{i \in I} C_i.
\]

Define \( Z_I = \pi_I(Z) \). We know that \( Z_I \) is an irreducible projective variety, with dimension either \( l \) or \( l - 1 \).

If \( Z_I \) has dimension \( l \), the induction hypothesis produces a set \( J \subseteq I \) such that \( i \in J \), \#\( J = l \), and the projection map

\[
\pi_J : Z_I \to \prod_{j \in J} C_j
\]

is surjective. Clearly \( Z \) surjects to \( Z_I \), so by composition we obtain a surjection

\[
Z \to \prod_{j \in J} C_j.
\]

Now suppose that \( Z_I \) has dimension \( l - 1 \). The induction hypothesis gives a set \( J \subseteq I \) such that \( i \in J \), \#\( J = l - 1 \), and the projection map

\[
\pi_J : Z_I \to \prod_{j \in J} C_j
\]

is surjective. Let \( k \in \{1, \ldots, d\} \) be the one element which is not in the set \( I \). Consider \( Z_I \times C_k \). A simple dimension argument shows that this is equal to \( Z \). Therefore we have a surjection

\[
Z = Z_I \times C_k \to \prod_{j \in J} C_j \times C_k.
\]

Therefore \( Z \) surjects onto a product of \( l \) curves, as desired. Q.E.D.

**Lemma 3.3** Lemma \[22\] above still holds if we replace \( Z \subseteq C_1 \times \cdots \times C_d \) with a fiber product of families of curves \( Z \subseteq C_1 \times_B \cdots \times_B C_d \) where \( B \) is a projective variety, the \( C_i \) form a family of curves over \( B \), and that for some \( i \) the projection map from \( Z \) the family \( C_i \) is surjective.

**Proof:** Let \( \eta \in B \) be a generic point. Because \( Z \) surjects to each family \( C_i \), it follows that each generic fiber of \( Z \) surjects to generic fibers of curves. In other words, we may apply lemma \[32\] to the fiber \( Z_\eta \). We obtain a set \( J \) of size \( l \) with \( i \in J \) such that we have a surjective and generically finite projection map

\[
\pi_J : Z_\eta \to \prod_{j \in J} C_i,\eta
\]
where the product above is a fiber product over \( B \). Because generic points and fibers are dense, we know that the projection map

\[
\pi_J : Z \to \prod_{j \in J} C_i
\]

is generically finite (where the above product is once again a fiber product over \( B \)). The properness of the projection map implies the surjectivity of \( \pi_J \). Q.E.D.

**Lemma 3.4** Suppose we have two surjective maps \( g_i : Z_i \to B \) for \( i = 1, 2 \). Then the two natural maps \( f_i : Z_1 \times_B Z_2 \to Z_i, \ i = 1, 2, \) are also surjective.

**Proof:** To show that \( f_1 \) is surjective, choose an element \( z_1 \in Z_1 \). Let \( b = g_1(z_1) \). Then there exists a \( z_2 \in Z_2 \) such that \( g_2(z_2) = b \), as \( g_2 \) is surjective. It follows that \( (z_1, z_2) \in Z_1 \times_B Z_2 \) and \( f_1((z_1, z_2)) = z_1 \). A similar argument shows that the map \( f_2 \) is also surjective. Q.E.D.

**Lemma 3.5** Let \( Y \subseteq Z_1 \times Z_2 \) be a variety, and let \( f_i : Y \to Z_i \) be surjective projection maps for \( i = 1, 2 \). Suppose \( L_1 \) and \( L_2 \) are big line bundles on \( Z_1 \) and \( Z_2 \) respectively. Then the line bundle

\[
f_1^*L_1 \otimes f_2^*L_2
\]

is big. Further, if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are ideal sheaves on \( Z_1 \) and \( Z_2 \), such that each \( L_i \otimes \mathcal{J}_i \) is big on \( Z_i \), then it follows that

\[
f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(\mathcal{J}_1) \cdot f_2^{-1}(\mathcal{J}_2))
\]

is big on \( Y \). Finally, the above still holds if we replace \( Y \subseteq Z_1 \times Z_2 \) by a variety \( Y \) which maps generically finitely to \( Z_1 \times Z_2 \),

\[
h : Y \to h(Y) \subseteq Z_1 \times Z_2
\]

provided that the projections from \( h(Y) \) to \( Z_i \) are still surjective.

**Proof:** The first statement is a consequence of the second; hence we just prove the second.

By assumption, since each \( L_i \otimes \mathcal{J}_i \) is big on \( Z_i \), there exists open sets \( U_i \subseteq Z_i \) such that for large \( k \), global sections of \( H^0(Z_i, L_i^k \otimes \mathcal{J}_i^k) \) separate points in \( U_i \). Let \( U = f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \), and let \( P = (P_1, P_2), Q = (Q_1, Q_2) \in U \). We claim that we can produce sections of

\[
H^0(Z_1 \times Z_2, f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(\mathcal{J}_1) \cdot f_2^{-1}(\mathcal{J}_2)))
\]

for large \( k \) which separate \( P \) and \( Q \). In other words, there is a section vanishing at \( P \) and not at \( Q \), and vice versa. This will show that there are enough sections to induce a birational map to projective space, hence this will suffice to show that

\[
f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(\mathcal{J}_1) \cdot f_2^{-1}(\mathcal{J}_2))
\]

is big.

We have the following map:

\[
f_1^* \otimes f_2^* : H^0(L_1^\otimes k \otimes \mathcal{J}_1^k) \otimes H^0(L_2^\otimes k \otimes \mathcal{J}_2^k) \to H^0(f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(\mathcal{J}_1) \cdot f_2^{-1}(\mathcal{J}_2)))
\]
Therefore, sections of $H^0(Z_1 \times Z_2, f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(J_1) \cdot f_2^{-1}(J_2)))$ have the form

$$f_1^*(s_i) \otimes f_2^*(r_j)$$

where $\{s_i\}$ and $\{r_j\}$ form bases for $H^0(Z_1, L_1^ {\otimes k} \otimes J_1^ k)$ and $H^0(Z_2, L_2^ {\otimes k} \otimes J_2^ k)$, respectively.

Because $P_1$ and $Q_1$ are elements of $U_1$, it follows that there exists a section $s \in H^0(Z_1, L_1^ {\otimes k} \otimes J_1^ k)$ such that $s(P_1) = 0$ but $s(Q_1) \neq 0$. Similarly, there exists a section $r \in H^0(Z_2, L_2^ {\otimes k} \otimes J_2^ k)$ such that $r(P_2) = 0$, but $r(Q_2) \neq 0$. Consider the section $f_1^*s \times f_2^*r$. It follows that

$$f_1^*s \times f_2^*r(P) = (s(P_1), r(P_2)) = 0$$

and

$$f_1^*s \times f_2^*r(Q) = (s(Q_1), r(Q_2)) \neq 0.$$ 

Similarly, we can find a section of $H^0(Z_1 \times Z_2, f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(J_1) \cdot f_2^{-1}(J_2)))$, which vanishes on $Q$ but not on $P$. Therefore, for large $k$, sections of $H^0(Z_1 \times Z_2, f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(J_1) \cdot f_2^{-1}(J_2)))$ generically separate points. Hence, we have that

$$f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(J_1) \cdot f_2^{-1}(J_2))$$

is big.

We now prove the last statement of the lemma. Since $h(Y)$ surjects both $Z_1$ and $Z_2$, the work above shows that $f_1^*(L_1) \otimes f_2^*(L_2) \otimes (f_1^{-1}(J_1) \cdot f_2^{-1}(J_2))$ is big on $h(Y)$. Now $h : Y \to h(Y)$ is a generically finite surjective morphism; hence the pullback of a big sheaf on $h(Y)$ will be big on $Y$. Q.E.D.

4 The Proof

Recall that for a given $n$, $F_n$ is contained in $Y_n = Sym^d(X_B^n)$. Let $X_n = (X_B^n)^d$. Then we have a map $\sigma : X_n \to Y_n$, namely the quotient map given by action of $S_d$. Now we can look at the inverse image of $F_n$ under $\sigma$, which is contained in $X_n$. Call it $G_n$.

These varieties $G_n$ will be central to our proof. We can think of the $G_n$’s as forming a tower: we have projection maps $\pi : G_{n+1} \to G_n$ such that

$$\pi(G_{n+1}) = G_n.$$ 

Fix $n$. Let $l$ be the relative dimension of $G_{n+1}$ over $G_n$. Let $\phi$ be the projection of $G_n$ down to $B^d$, and for $i = 1, \ldots, d$, denote by $\pi_i$ the $d$ projections from $B^d$ to $B$. Notice that because the image of $G_n$ in $(X_B^{n-1})^d$ is $G_{n-1}$, it follows that no matter which $n$ we start with the image $\phi(G_n)$ in $B^d$ is the same. Define $B_i \subseteq B$ to be the image of an irreducible component of $G_n$ in $B$ under the map $\pi_i \circ \phi$. We shall see later that it is enough to restrict our attention to one component of $G_n$, hence, we do so now.

Choose

$$\mathcal{B} \subseteq B_1 \times B_2 \times \cdots \times B_d$$
to be a component of $\phi(G_n)$ such that $\mathcal{B}$ surjects to each component $B_i$. Let $\tilde{\mathcal{B}}$ be a desingularization of $\mathcal{B}$. By Hironaka’s Theorem we know we can choose this desingularization such that the discriminant locus is a divisor with normal crossings.

We have the following diagram:

\[ \tilde{\mathcal{B}} \to \mathcal{B} \subseteq B_1 \times \ldots \times B_d \subseteq B^d \to B \]

Notice that we have $d$ choices for the last map in the above diagram, namely the $d$ projections $\pi_i$. Let $g_i : \tilde{\mathcal{B}} \to B$ be the composition of the above where $\pi_i$ is chosen as the last mapping.

Define families of curves

\[ C_i = g_i^*(X \to B) = X \times_B \tilde{\mathcal{B}}. \]

\[ C_i \longrightarrow X \]

\[ \downarrow \quad \downarrow \]

\[ \tilde{\mathcal{B}} \overset{g_i}{\longrightarrow} B \]

We now pull back the varieties $G_n$ to $\tilde{\mathcal{B}}$; to simplify matters, we shall keep the same notation $G_n$. We then also need to pull back the families $C_i$ to $G_n$. Again, it is easiest to abuse notation, and still refer to them as $C_i$. In our new situation we have the following containment:

\[ G_{n+1} \subseteq C_1 \times_{G_n} \cdots \times_{G_n} C_d. \]

Our goal is to work with these varieties $G_n$ to show that for large $n$, $F_n$ is of general type, which will lead to our contradiction. Our main difficulty will be in avoiding the singularities of $G_n$. To do this, we will eventually be working with products of stable curves over the smooth base $\tilde{\mathcal{B}}$. These curves will have canonical singularities, which are easier to control. We would like to have $G_{n+1}$ surject onto each of these families of curves $C_i$. While this is not always so, we will show that we may reduce to this case.

For $i = 1, \ldots, d$, define projection maps $p_i : G_{n+1} \to C_i$. To illustrate the situation, assume for a moment that $G_{n+1}$ is irreducible. Then for each $i$, $p_i(G_{n+1})$ is an irreducible subvariety of $C_i$. Suppose that for one index $i$, the map is not surjective; without loss of generality assume that $p_1(G_{n+1}) = P$ where $P \subseteq C_i$ and has degree $k$ over $G_n$. Now we go up $k$ steps and use the fact that

\[ G_{n+k+1} \subseteq (G_{n+1})^{k+1}_{G_n}. \]

In other words,

\[ G_{n+k+1} \subseteq P^{k+1}_{G_n} \times_{G_n} C_2^{k+1} \times_{G_n} \cdots \times_{G_n} C_d^{k+1}. \]

Since $P$ consists of $k$ points over $G_n$, the pigeon-hole principle says that two of the coordinates of $P^{k+1}_{G_n}$ must agree, contradicting the fact that $G_{n+k}$ is not contained in the big diagonal.

Thus if $G_{n+1}$ is irreducible, we see that it is contained inside a product of $d$ families of curves fibered over $G_n$ and that it surjects onto each family.
Now suppose that $G_{n+1}$ consists of $m$ irreducible components:

$$G_{n+1} = V_1 \cup V_2 \cup \cdots \cup V_m \subseteq C_1^m \times_G \cdots \times_G C_d^m.$$  

Define projection maps $p_{i,j} : V_i \to C_j$ for $1 \leq i \leq m$ and $1 \leq j \leq d$. We want to show that at least one $V_i$ surjects onto all $d$ curves. So suppose for contradiction that

$$p_{1,j(1)}(V_1) = S_1, \ldots, p_{m,j(m)}(V_m) = S_m$$

where each $S_i \subseteq C_{j(i)}$ maps generically finitely to $G_n$ with degree $r_i$.

Therefore, we know

$$G_{n+1} \subseteq (p_{1,1}(V_1) \times_{G_n} \cdots \times_{G_n} p_{1,d}(V_1)) \cup \cdots \cup (p_{m,1}(V_m) \times_{G_n} \cdots \times_{G_n} p_{m,d}(V_m))$$

and

$$G_{n+k} \subseteq (G_{n+1})_{G_n}^k$$

So $G_{n+k}$ is contained in the above union fibered $k$ times over $G_n$.

In taking elements of

$$(p_{1,1}(V_1) \times_{G_n} \cdots \times_{G_n} p_{1,d}(V_1))$$

we have only $r_1$ distinct choices for the $j(1)$-st component. Continuing in this spirit, in taking an element of

$$(p_{m,1}(V_m) \times_{G_n} \cdots \times_{G_n} p_{m,d}(V_m))$$

we have only $r_m$ distinct choices for the $j(m)$-th entry.

Thus, if we take $k \geq r_1 + \cdots + r_m$, an element of $G_{n+k}$ must have 2 coordinates equal, contradicting the fact that $G_{n+k}$ is not contained in the big diagonal.

Therefore we know that if $G_{n+1}$ at least one component of $G_{n+1}$ surjects onto all $d$ families of curves $C_i$.

Suppose once again that $G_{n+1} = V_1 \cup \cdots \cup V_m$. Define $W_1$ to be the union of those $V_i$ which do not surject to each family $C_i$, or are not dominant over $G_n$. Define $W_2$ to be the union of those components which do surject onto each $C_i$; by our work above, $W_2$ is not empty. Hence

$$G_{n+1} \subseteq W_1 \cup W_2$$

Choose a component $G$ of $G_{n+k}$ which is irreducible and dominant over $G_n$ with relative dimension $kl$. We then have

$$G \subseteq G_{n+k} \subseteq (G_{n+1})_{G_n}^k \subseteq (W_1 \cup W_2)_{G_n}^k.$$  

If we let $J_1$ and $J_2$ run over all subsets of $\{1, \ldots, k\}$ such that $J_1 \cup J_2 = \{1, \ldots, k\}$, then it follows that

$$G \subseteq \cup_{J_1, J_2} (W_1)^{J_1}_{G_n} \times_{G_n} (W_2)^{J_2}_{G_n}$$

Because $W_1$ and $W_2$ consist of unions of distinct components, and $G$ is a component, it follows that for one choice of $J_1$ and $J_2$, say $J_{1,k}$ and $J_{2,k}$ that

$$G \subseteq (W_1)^{J_{1,k}}_{G_n} \times_{G_n} (W_2)^{J_{2,k}}_{G_n}.$$
Recall that $G_{n+k}$ is not contained in the big diagonal. Therefore, applying the same reasoning as earlier, we conclude that there exists an integer $r$ such that $\#J_{1,k} \leq r$ for all $k$. This is because, by definition, the set $W_1$ consists of elements of $G_{n+1}$ which do not surject to all of the families of curves $C_i$. Since we know that $\#J_{1,k} + \#J_{2,k} = k$, it follows that $\#J_{2,k} \geq k - r$, and so as $k$ grows, the size of the sets $J_{2,k}$ also grows. Since, by definition, components of $W_2$ surject to each of the families $C_i$, we can obtain a surjection

$$(W_2)^{J_{2,k}}_{G_n} \to C_1^{k_1} \times G_n \cdots \times G_n C_d^{k_d}$$

where we can make the exponents $k_i$ as large as we wish, by taking larger values of $k$, since the size of $J_{2,k}$ grows with $k$.

Because $G$ is a component of $G_{n+1}$ of maximal dimension which is dominant over $G_n$, it follows that $G$ dominates a component $W$ of $(W_2)^{J_{2,k}}_{G_n}$. This component $W$ is contained inside a product of, say, $c$ of the components of $W_2$. As $k$ grows, so does $k - r$; for large values of $k - r$ at least one of those $c$ components, call it $W'$, will appear at least $\kappa = \frac{k - r}{c}$ times, where $\kappa$ also grows with $k$. Therefore $G$ dominates $(W')^\kappa_{G_n}$.

As $\kappa$ grows, $(W')^\kappa_{G_n}$ will be mapping surjectively to higher and higher powers of the families of curves $C_i$. In the rest of this paper we shall prove that this is exactly what is needed to show that $(W')^\kappa_{G_n}$ is a variety of general type. Moreover, we shall show that for $\kappa$ large enough (i.e., for $k$ large enough) $(W')^\kappa_{G_n}$ modulo the action of the symmetric group is a variety of general type.

Because $G$ dominates $(W')^\kappa_{G_n}$ we obtain a family of varieties

$$G \to (W')^\kappa_{G_n}.$$ 

Because $G_n$ is not contained in the fixed point locus, over each general point in $(W')^\kappa_{G_n}$, the fiber in $G$ consists of a product of curves of genus $g$ (as opposed to a quotient of products of curves). In other words, each fiber is a variety of general type, as is the base. We then utilize a theorem of Viehweg ([V], Satz III) to conclude that $G$ itself is a variety of general type. Viehweg’s theorem states that if $Z \to B$ is a family of varieties of general type where the base $B$ is also of general type, then it follows that $Z$ is of general type.

Let $F$ be the image of $G$ in $F_{n+k}$. Since $G$ is of maximal dimension in $G_{n+k}$ it follows that $F$ is of maximal dimension in $F_{n+k}$ Since $(W')^\kappa_{G_n}$ modulo the group action is of general type, we may again apply Viehweg’s theorem as above to see that $F$ also is of general type, using that $F$ is a family of varieties of general type over the image of $(W')^\kappa_{G_n}$ modulo the group action. Therefore, we’ve shown that for large $n+k$, a component of $F_{n+k}$ of maximal dimension is a variety of general type, proving proposition 2.8.

Our work above proves the following proposition.

**Proposition 4.1** In order to prove Proposition 2.8, it suffices to prove it for the case where each $G_n$ is irreducible.

For the remainder of the paper, we shall assume that for all large $n$ $G_{n+1}$ consists of one irreducible component, which projects surjectively onto each family of curves.
Because we are assuming that $G_{n+1}$ projects surjectively onto each $C_i$, we may apply lemma 3.3. We thus obtain $d$ generically finite maps $\pi_i$ from $G_{n+1}$ to a product of the families of curves $C_i$ fibered over $G_n$. In other words, for each $i = 1, \ldots, d$ there exists a subset $J_i \subseteq \{1, \ldots, d\}$ such that $\#J_i = l$, $i \in J_i$, and

\[ \pi_{J_i} : G_n \to \prod_{j \in J_i} C_j. \]

Recall, however, that we can project $G_n$ down to $\tilde{B}$; this allows us to obtain $d$ generically finite surjective maps from $G_{n+1}$ to a product of families of curves fibered over $\tilde{B}$.

Since each $G_{n+k}$ is contained in a fiber product of $G_{n+1}$ over $G_n$, we can apply the maps $\sigma_i$ to each component of $G_{n+1}$ in this product, and we can produce a generically finite map from $G_{n+k}$ to a product of powers of the $C_i$:

\[ G_{n+k} \to C_1^{k_1} \times \tilde{B} \cdots \times \tilde{B} C_d^{k_d}. \]

By taking $k$ large as necessary, we can increase the exponents $k_i$, making them as large as we wish.

To ease notation, for large $n$ let

\[ V_n = C_1^{k_1} \times \tilde{B} \cdots \times \tilde{B} C_d^{k_d}. \]

So we have a generically finite map $G_n \to V_n$ where the exponents appearing in $V_n$ can be made larger by increasing $n$.

Recall the statement of proposition 2.8, which says that for large values of $n$ and $k$, every component of the fiber product $(F_{n+1})^k_{F_n}$ of maximal dimension is a variety of general type. We are ready to begin in earnest the proof of this proposition.

First, we prove two more lemmas which will be of assistance to us. The first statement in the following lemma is a well known fact, but we include it here, as we will utilize it later.

**Lemma 4.2** Let $f : V \to B$ be a flat morphism of irreducible projective varieties with irreducible general fiber. Let $L$ be a big line bundle on $V$, and $\mathcal{J} \neq 0$ an ideal sheaf on $V$. Let

\[ \pi_i : V_B^k \to V \]

be projections from the fiber product to the $i$-th factor. Define $L_k$ and $\mathcal{J}_k$ by

\[ L_k = \otimes_i \pi_i^* L \]

and

\[ \mathcal{J}_k = \sum_i \pi_i^{-1} \mathcal{J}. \]

Then:

1. There exists an integer $m$ such that $L^\otimes m \otimes \mathcal{J}$ is big on $V$;
2. For high enough values of $k$, it follows that

\[ L_k \otimes \mathcal{J}_k \]

is big on $V_B^k$. 

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**Proof:** Consider the following exact sequence of cohomology groups:

$$0 \to H^0(V, \mathcal{L}^m \otimes \mathcal{J}) \to H^0(V, \mathcal{L}^m) \to H^0(V, \mathcal{L}^m / \mathcal{L}^m \otimes \mathcal{J}) \to 0$$

Because $L$ is big, we know that the dimension of $H^0(V, \mathcal{L}^m)$ is $O(m^{\dim V})$. Since the zero set of a variety has a lower dimension, we have that the dimension of $H^0(V, \mathcal{L}^m / \mathcal{L}^m \otimes \mathcal{J})$ is less than the dimension of $H^0(V, \mathcal{L}^m)$. Therefore, the dimension of $H^0(V, \mathcal{L}^m / \mathcal{L}^m \otimes \mathcal{J})$ is less than or equal to $O(m^{\dim V - 1})$. By counting the dimensions in the exact sequence, we can conclude that the dimension of $H^0(V, \mathcal{L}^m \otimes \mathcal{J})$ is greater than or equal to $O(m^{\dim V} - m^{\dim V - 1})$. This growth in the dimension of $H^0(V, \mathcal{L}^m \otimes \mathcal{J})$ shows that there exists an $m$ such that $\mathcal{L}^m \otimes \mathcal{J}$ is big on $V$.

Because $\otimes \pi_i^* \mathcal{L}^m = \mathcal{L}_k^m$, we obtain a map

$$\otimes \pi_i^* H^0(V, \mathcal{L}^m) \to H^0(V, \otimes \pi_i^* \mathcal{L}^m).$$

The indices $i$ range from 1 to $k$. The inclusions

$$\otimes \pi_i^* H^0(V, \mathcal{L}^m \otimes \mathcal{J}) \hookrightarrow \otimes \pi_i^* H^0(V, \mathcal{L}^m)$$

and

$$H^0(V_B^k, \mathcal{L}_k^m \otimes \mathcal{J}_k^k) \hookrightarrow H^0(V_B^k, \mathcal{L}_k^m)$$

produce a commutative diagram inducing a map

$$\otimes \pi_i^* H^0(V, \mathcal{L}^m \otimes \mathcal{J}) \hookrightarrow H^0(V_B^k, \mathcal{L}_k^m \otimes \mathcal{J}_k^k).$$

Notice also that if we choose $k > m$, we also have the inclusion

$$H^0(V_B^k, \mathcal{L}_k^m \otimes \mathcal{J}_k^k) \hookrightarrow H^0(V_B^k, \mathcal{L}_k^m \otimes \mathcal{J}_k^m)$$

since $\mathcal{J}_k^k \subseteq \mathcal{J}_k^m$.

$$\otimes \pi_i^* H^0(\mathcal{L}^m) \quad \longrightarrow \quad H^0(\mathcal{L}_k^m)$$

$$\uparrow \quad \quad \quad \quad \quad \quad \uparrow$$

$$\otimes \pi_i^* H^0(\mathcal{L}^m \otimes \mathcal{J}) \quad \longrightarrow \quad H^0(\mathcal{L}_k^m \otimes \mathcal{J}_k^k) \quad \longrightarrow \quad H^0(\mathcal{L}_k^m \otimes \mathcal{J}_k^m)$$

As $\mathcal{L}^m \otimes \mathcal{J}$ is big, we have a birational map from $V$ to $P^n$, induced by $H^0(V, \mathcal{L}^m \otimes \mathcal{J})$. The same reasoning as in Lemma 3.3 lets us conclude that we can use the sections of $H^0(V, \mathcal{L}^m \otimes \mathcal{J})$ to generically separate points of $V_B^k$. Thus for large values of $k$ it follows that $\mathcal{L}_k \otimes \mathcal{J}_k$ is big on $V_B^k$.

Q.E.D.

**Lemma 4.3** For large enough $n$, the relative dualizing sheaf

$$\omega_{V_n/\overline{B}}$$

is big, and furthermore, the dualizing sheaf

$$\omega_{V_n}$$

is also big.
Proof: Recall the following diagram:

\[ \bar{B} \rightarrow B \hookrightarrow B_1 \times \cdots \times B_d \subseteq B^d \rightarrow B \]

Define \( C_{i,0} \) to be the pullback of the family \( X \rightarrow B \) to \( B_i \). Lemma 3.4 in [CHM] states that the relative dualizing sheaf \( \omega_{C_{i,0}/B_i} \) is big.

Choose \( n \) to be large, so that we have a generically finite map

\[ G_n \rightarrow C_1^{k_1} \times \cdots \times B^d \subseteq B^d = V_n. \]

Now for \( i = 1, \ldots, d \) consider the family

\[ (C_{i,0})_{B_i}^{k_i} \rightarrow B_i. \]

Call this map \( f_i \). Notice that

\[ (C_{i,0})_{B_i}^{k_i} \subseteq (C_{i,0})_{B}^{k_i}. \]

Now, \( C_{i,0} \) maps surjectively to \( B_i \) by definition. Hence lemma 3.4 implies that each projection

\[ (C_{i,0})_{B_i}^{k_i} \rightarrow C_{i,0} \]

is surjective. Thus lemma 3.5 tells us that since \( \omega_{C_{i,0}/B_i} \) is big on \( C_{i,0} \), it follows that \( \omega_{f_i} \) is big on \( (C_{i,0})_{B_i}^{k_i} \).

Let \( \phi \) be the map

\[ \phi : B \hookrightarrow B_1 \times \cdots \times B_d. \]

Let \( V_{n,0} \) be the pullback under \( \phi \) of the family

\[ (C_{1,0})_{B_1}^{k_1} \times \cdots \times (C_{d,0})_{B_d}^{k_d} \rightarrow B_1 \times \cdots \times B_d \]

Thus

\[ V_{n,0} \hookrightarrow (C_{1,0})_{B_1}^{k_1} \times \cdots \times (C_{d,0})_{B_d}^{k_d} \]

We also have surjective projections

\[ V_{n,0} \rightarrow (C_{i,0})_{B_i}^{k_i} \]

They are surjective because \( B \) was chosen so as to project onto each component of \( B_1 \times \cdots \times B_d \). Therefore, lemma 3.3 tells us that \( \omega_{V_{n,0}/B} \) is big, since each \( \omega_{f_i} \) is big on \( (C_{i,0})_{B_i}^{k_i} \).

Let \( \psi \) be the map

\[ \bar{B} \rightarrow B. \]

We have that \( V_n \) is the pullback of \( V_{n,0} \) under \( \psi \). Thus we know that

\[ \omega_{V_n/B} = \psi^* \omega_{V_{n,0}/B} \]

and since \( \psi \) is a generically finite map, it follows that \( \omega_{V_n/B} \) is big.

It remains to show that the dualizing sheaf \( \omega_{V_n} \) is big. Recall that

\[ \omega_{V_n} = \omega_{V_n/B} \otimes \omega_{\bar{B}} \]
Choose $\mathcal{I}$ to be an ideal on the base $\mathcal{B}$ such that there is an injection

$$\mathcal{I} \hookrightarrow \omega_{\mathcal{B}}.$$  

It will be sufficient to show that

$$\omega_{V_n/\mathcal{B}} \otimes \mathcal{I}$$  

is big.

To do this, define $Z$ to be the restriction of $X^d$ to $\mathcal{B}$. In other words,

$$Z = C_1 \times_\mathcal{B} \cdots \times_\mathcal{B} C_d.$$  

By Lemma 3.4 in [CHM] we know that $\omega_{Z/\mathcal{B}}$ is big. Recall that by choosing $n$ large enough we can control the size of the exponents in $V_n$, making them as large as we wish. Therefore, for any integer $m$ there exists an integer $n_m$ such that if $n > n_m$, all the $k_i$ are greater than $m$. Write

$$k_i - m = r_i > 0.$$  

Then we can rewrite

$$V_n = C_1^{r_1} \times_\mathcal{B} \cdots \times_\mathcal{B} C_d^{r_d} \times_\mathcal{B} C_1^m \times_\mathcal{B} \cdots \times_\mathcal{B} C_d^m$$  

$$= C_1^{r_1} \times_\mathcal{B} \cdots \times_\mathcal{B} C_d^{r_d} \times_\mathcal{B} Z_m^m$$  

$$= V' \times_\mathcal{B} Z_m^m$$  

where we define $V'$ to equal $C_1^{r_1} \times_\mathcal{B} \cdots \times_\mathcal{B} C_d^{r_d}$.

The first part of this lemma shows that for large enough $n$, $\omega_{V'/\mathcal{B}}$ is big on $V'$. Lemma 4.2 states that, since $\omega_{Z/\mathcal{B}}$ is big on $Z$, for large enough $m$,

$$\omega_{Z_m^m/\mathcal{B}} \otimes \mathcal{I}$$  

is big on $Z_m^m$.

We now apply Lemma 3.3 to see that

$$\omega_{V'/\mathcal{B}} \otimes (\omega_{Z_m^m/\mathcal{B}} \otimes \mathcal{I})$$  

is big on $V' \times_\mathcal{B} Z_m^m$. In other words,

$$\omega_{V_n/\mathcal{B}} \otimes \mathcal{I}$$  

is big, as desired; hence $\omega_{V_n}$ is big on $V_n$. Q.E.D.

We’re almost ready to prove Proposition 2.8. Let us set up some notation first.

Let $G_{n+k}$ be an equivariant desingularization of $G_{n+k}$. In other words, the action of a subgroup of the symmetric group $S_d$ on $\mathcal{B}$ and $G_{n+k}$ lifts to $G_{n+k}$. Let $r$ be the map from $G_{n+k}$ to $G_{n+k}$ which gives the resolution of singularities. Let $\Phi_{n+1}$ and $\Phi_{n+k}$ denote the set of points of $G_{n+1}$ and $G_{n+k}$, respectively, which are fixed by the group action.
Recall that we have a generically finite map, call it $\sigma_0$, from $G_n$ to $V_n$:

$$\sigma_0 : G_n \to V_n = C_1^{k_1} \times \bar{B} \cdots \times \bar{B} C_d^{k_d}.$$ 

For $i = 1, \ldots, d$, denote by $\sigma_i$ the $d$ projections from $G_{n+1}$ to a product of $l$ families of curves, where $l$ is the relative dimension of $G_{n+1}$ over $G_n$. Let $Z_i$ be the image of $G_{n+1}$ under $\sigma_i$. We can write

$$Z_i = C_j^{1(i)} \times \bar{B} \cdots \times \bar{B} C_{j_l}^{l(i)}$$

where one of $j_1(i), \ldots, j_l(i)$ is equal to $i$.

Define varieties $V_{n+1,i}$ by

$$V_{n+1,i} = V_n \times \bar{B} Z_i.$$ 

Then for any positive integer $k$, we have

$$V_{n+k,i} = V_n \times \bar{B} (Z_i)^k.$$ 

Let $\pi_0$ be the map from $G_{n+k}$ to $G_n$ given by projection to the first $n$ coordinates, and for $j = 1, \ldots, k$, define projection maps

$$\pi_j : G_{n+k} \to G_{n+1}$$

as follows: If $(P_1, \ldots, P_n, \ldots, P_{n+k}) \in G_{n+k}$ then

$$\pi_j((P_1, \ldots, P_n, \ldots, P_{n+k}) = (P_1, \ldots, P_n, P_{n+j}).$$

Recall that $G_{n+k} \subseteq (G_{n+1})^k_G$. We use this to define maps

$$(\sigma_0, \sigma_i, \ldots, \sigma_i) : G_{n+k} \to V_{n+k,i}.$$ 

They act in the following manner: $\sigma_0$ is applied to $G_n$, while the $\sigma_i$ are applied to the copies of $G_{n+1}$. In other words, if $P = (P_1, \ldots, P_n, P_{n+1}, \ldots, P_{n+k}) \in G_{n+k}$, then

$$(\sigma_0, \sigma_i, \ldots, \sigma_i)(P) = (\sigma_0(P_1, \ldots, P_n), \sigma_i(P_{n+1}), \ldots, \sigma_i(P_{n+k})).$$

We have the following diagram:

\[
\begin{array}{cccc}
G_{n+k} & \xrightarrow{(\sigma_0, \sigma_i, \ldots, \sigma_i)} & V_{n+k,i} \\
\downarrow \pi_j & & \downarrow \\
G_{n+1} & \rightarrow & V_{n+1,i} \\
\downarrow & & \downarrow \\
G_n & \xrightarrow{\sigma_0} & V_n
\end{array}
\]

Recall from the proof of Lemma 4.3 that we defined $Z$ to be the restriction of $X^d$ to $\tilde{B}$.

$$Z = C_1 \times \bar{B} \cdots \times \bar{B} C_d.$$
We know that $G_{n+1}$ maps to $Z$ and in fact, the $\sigma_i$’s factor through this map. So let 
\[ \tau : G_{n+1} \rightarrow W \]
be the map from $G_{n+1}$ to $Z$, where $W$ is the image of $G_{n+1}$ in $Z$. Let 
\[ e : W \rightarrow Z \]
be the inclusion map from $W$ to $Z$. Finally, define the projection maps 
\[ \rho_i : Z \rightarrow Z_i \]
where 
\[ \rho_i(C_1 \times \tilde{B} \ldots \times \tilde{B} C_d) = C_{j_1(i)} \times \tilde{B} \ldots \times \tilde{B} C_{j_d(i)}. \]

We have the diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{e} & Z \\
\uparrow & & \downarrow \rho_i \\
G_{n+1} & \rightarrow & Z_i
\end{array}
\]

An important fact to notice is that the image in $W$ of the fixed points $\Phi_{n+1}$ is not all of $W$. Why is this? Well, let’s look at what the map $\tau : G_{n+1} \rightarrow W \subseteq Z$ does. Remember that $G_{n+1} \subseteq (\overline{X^{n+1}})^d$ and that $Z = X^d|_{\tilde{B}}$. We can write a $K$-rational element, $g$, of $G_{n+1}$ as 
\[ g = ((P_1, \ldots, P_{n+1})^{\alpha_1}, \ldots, (P_1, \ldots, P_{n+1})^{\alpha_d}) \]
where $\alpha_1, \ldots, \alpha_d$ are the embeddings of $L$ into $\overline{K}$ fixing $K$. Then 
\[ \tau(g) = (P_{n+1}^{\alpha_1}, \ldots, P_{n+1}^{\alpha_d}). \]
Therefore $\tau(g)$ is actually a Galois orbit. Recall that the $P_{n+1}$’s are defined over $L$ but not any smaller field between $L$ and $K$. Hence all of $P_{n+1}^{\alpha_1}, \ldots, P_{n+1}^{\alpha_d}$ are distinct. Therefore, $\tau(G_{n+1}) = W$ is the closure of a set of points which are not fixed by the group action.

This fact that the fixed points do not surject to $W$ will soon be very important.

We want to prove that every component of $F_{n+k}$ of maximal dimension is a variety of general type. This means that if $\overline{F_{n+k}}$ is a resolution of singularities of $F_{n+k}$, we need to show that the dualizing sheaf $\omega_{\overline{F_{n+k}}}$ is big.

The main idea is to pull back sections of dualizing sheaves along our various projections. We know that both $\omega_{V_n}$ and $\omega_{V_{n+k,i}}$ are big for large $n$. ($\omega_{V_{n+k,i}}$ is big since the families of the curves $C_i$ appear in $V_{n+k,i}$ with even larger exponents.) We would like to utilize the fact that $\omega_{V_{n+k,i}}$ is big to pull back a lot of sections to $\overline{F_{n+k}}$ which vanish along the fixed points of the group action (for
we will see that this is what suffices for $F_{n+k}$ to be of general type). Unfortunately we can’t pull back sections along any one projection of $G_{n+k}$ to $V_{n+k,i}$; the sections pulled back in this way will not vanish along the fixed points. To overcome this difficulty we will pull back sections via all the various projections, tensor them together, and use our earlier lemmas to obtain that $\omega_{\widetilde{G}_{n+k}}$ is big, with lots of sections vanishing on the fixed points.

Recall that $G_{n+k}$ is contained in a fiber power of families of stable curves over $\overline{B}$, and, moreover, $\overline{B}$ was chosen to be smooth and irreducible, with the property that its discriminant locus is a divisor of normal crossings. We appeal to Lemma 3.3 of [CHM], which states that this implies that any singularities of $V_{n+k,i}$ are canonical. This means that the singularities do not prohibit the extension of pluricanonical sections from the smooth locus to any desingularization. Therefore we may pull back sections of $V_{n+k,i}$ to $\overline{G}_{n+k}$. We have the following injection:

$$r^*(\sigma_0, \sigma_i, \ldots, \sigma_i)^*\omega_{V_{n+k,i}} \hookrightarrow \omega_{\overline{G}_{n+k}}$$

and so

$$r^*(\otimes_i ((\sigma_0, \sigma_i, \ldots, \sigma_i))^*\omega_{V_{n+k,i}} \hookrightarrow (\omega_{\overline{G}_{n+k}})^{\otimes d}$$

Now, the left-hand side above can be rewritten as:

$$r^*(\pi_0^*\sigma_0^*\omega_V) \otimes r^*(\pi_1^* \otimes_i \sigma_i^*\omega_{Z_i/B}) \otimes \cdots \otimes r^*(\pi_k^* \otimes_i \sigma_i^*\omega_{Z_i/B})$$

Notice in the second diagram above that

$$\sigma_i = \rho_i \circ e \circ \tau$$

Hence,

$$\sigma_i^* = \tau^* e^* \rho_i^*.$$  

Let’s look at one of the last $k$ terms above (take some $j$ such that $1 \leq j \leq k$):

$$r^*(\pi_j^* \otimes_i \sigma_i^*\omega_{Z_i/B}) = r^*(\pi_j^* (\otimes_i \tau^* e^* \rho_i^*\omega_{Z_i/B}))$$

$$= r^*(\pi_j^* (\otimes_i \tau^* e^* (\omega_{C_{j1(i)}/\overline{B}} \otimes \cdots \otimes \omega_{C_{j1(i)}/\overline{B}})))$$

$$= r^*(\pi_j^* \tau^* e^* (\otimes_i (\omega_{C_{j1(i)}/\overline{B}} \otimes \cdots \otimes \omega_{C_{j1(i)}/\overline{B}})))$$

$$= r^*(\pi_j^* \tau^* e^* (\omega_{C_{j1}/B} \otimes \cdots \otimes \omega_{C_{j1}/B}))$$

The exponents $l_i$ are defined by the last line above, and we are guaranteed that each one is positive. We know that for each $i$, one of the $j_k(i)$’s is equal to $i$; hence we when we tensor over all $i$ we know get each family $C_i$ appearing some positive number $l_i$ times.

Let $M$ denote the line bundle

$$M = \omega_{C_{j1}/B}^{l_1} \otimes \cdots \otimes \omega_{C_{jd}/B}^{l_d}.$$  

We claim that $M$ is a big line bundle on $Z$, and if we let $M_W$ denote $e^*M$, then, in fact, $M_W$ is big on $W$.

Recall that

$$M = \otimes_i \rho_i^*\omega_{Z_i/B}.$$  

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In other words, 

\[ M = \omega_{C_1/B}^{\otimes l_1} \otimes \cdots \otimes \omega_{C_d/B}^{\otimes l_d} \]

where each \( l_i \) is positive.

As in the proof of Lemma 4.3, let \( C_i,0 \) be the pullback of \( X \to B \) to \( B_i \subseteq B \); recall that \( \omega_{C_i,0/B_i} \) is big. Moreover, using techniques from the proof of Lemma 4.3, we know that \( \omega_{C_i,0/B_i}^{\otimes l_i} \) is big.

We have

\[ Z \subseteq C_{1,0} \times \cdots \times C_{d,0} \to B_1 \times \cdots \times B_d \]

and each \( Z \) surjects to each \( B_i \). Since each \( \omega_{C_i,0/B_i} \) is big, it follows that each \( \omega_{C_i,0/B_i}^{\otimes l_i} \) is also big.

Therefore applying Lemma 3.5 we see that the pullback of \( \otimes_i \omega_{C_i,0/B_i}^{\otimes l_i} \) to \( Z \) is big. This pullback is equal to

\[ \omega_{C_1/B}^{\otimes l_1} \otimes \cdots \otimes \omega_{C_d/B}^{\otimes l_d} = M \]

so \( M \) is big on \( Z \). Now \( W \subseteq Z \) and \( W \) surjects to each \( C_i \) since \( W \) is the image of \( G_{n+1} \). Hence applying lemma 3.5 once again, we conclude that \( M_W \) is big on \( W \).

Putting all of the above together, we see that

\[ r^*((\pi^n_0)^*\sigma^n_0^*\omega_{V_n}) \otimes \cdots \otimes r^*((\pi^n_k)^*\tau^*M_W) \to \omega_{\widetilde{G}_{n+k}}^{\otimes d}. \]

Recall that the image of the fixed points \( \Phi_{n+1} \) in \( W \) is not all of \( W \). Therefore this image forms a proper subvariety of \( W \), and we obtain a non-zero ideal \( \mathcal{I} \) of sections of \( M_W \) which vanish on this subvariety.

We now apply the following lemma, borrowed from [CHM].

**Lemma 4.4** [see [CHM], Lemma 4.1] Suppose that \( X \) is a projective variety with \( G \) a finite group of order \( l \) acting on \( X \). Let \( \theta \) be an \( m \) canonical form on \( X \) which is invariant under the group action of \( G \). Then if \( \theta \) vanishes to order \( m(l-1) \) on the locus of all points of \( X \) fixed by the group action, then \( \theta \) descends to smooth form on \( X/G \).

We use Lemma 4.4 to show that smooth pluricanonical forms on \( \widetilde{G}_{n+k} \) descend to smooth forms on \( \widetilde{F}_{n+k} \) modulo the group action, if the forms vanish to a prescribed order on the fixed point locus. Bear in mind that \( G_{n+k} \) modulo the group action is nothing more than \( F_{n+k} \), a desingularization of \( F_{n+k} \). So to show that \( \omega_{\widetilde{F}_{n+k}} \) is big for large \( k \), we will show that \( \omega_{\widetilde{G}_{n+k}} \) not only is big, but also has lots of sections vanishing to arbitrarily high order on the fixed point locus.

By the first statement of Lemma 4.2 we know that for some \( s \) the line bundle \( M_W^{\otimes s} \otimes \mathcal{I} \) is big. Look at the following diagram:

\[
\begin{array}{ccc}
G_{n+k} & \xrightarrow{r} & G_{n+k} \\
\downarrow & & \downarrow \\
V_n \times W \times \cdots \times W & & W
\end{array}
\]
Let $\mathcal{J}_\Phi$ be the locus of fixed points in $G_{n+k}$. Then

$$\sum_j \pi_j^{-1} \tau_j^{-1} \mathcal{J} \subset \mathcal{J}_\Phi$$

and so

$$\prod_j \pi_j^{-1} \tau_j^{-1} \mathcal{J} \subset \mathcal{J}_\Phi^k$$

Therefore,

$$r^*(\pi_0^* \sigma_0^* \omega_{\mathcal{V}_n}) \otimes ds \otimes r^*(\bigotimes_j \pi_j^* \tau_j^* M_W^s \otimes (\prod_j \pi_j^{-1} \tau_j^{-1} \mathcal{J})) \hookrightarrow \omega_{G_{n+k}}^s \otimes \mathcal{J}_\Phi^k$$

We know that $\omega_{\mathcal{V}_n}$ is big, and as in the proof of lemma 4.2, as long as $k > s$, $M_W^s \otimes (\prod_j \pi_j^{-1} \tau_j^{-1} \mathcal{J})$ is big. Therefore, for $k > s$ the entire left hand side above is big, so we have lots of sections of $\omega_{G_{n+k}}^s$ vanishing to high order, and the proof is complete.

References

[A] D. Abramovich, *Uniformité des points rationnels des courbes algébriques sur les extensions quadratiques et cubiques*, C.R. Acad. Sci. Paris, t. 321, Série I, p. 755-758, 1995.

[CHM] L. Caporaso, J. Harris, B. Mazur, *Uniformity of rational points*, J. Amer. Math. Soc., to appear
ftp://ftp.math.harvard.edu/pub/uniformityofrationalpoints.tex

[V] E. Viehweg, *Die additivität der Kodaira dimension für projektive Faserräume über varietäten des allgemeinen typos*, Jour. reine und angew. Math. 330, p. 132-142, 1982.