Consistent estimation of the critical current density of a superconductor from hysteresis loops

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Abstract

It has been noticed that the critical current density $J_c$ of some of the superconducting samples, calculated on the basis of Bean model, increases with increasing magnetic field $H$ up to a significant range above $H=0$. This is an inconsistent behavior of $J_c$ since the theory of Kim and the theories based on vortex dynamics, all, lead to decreasing $J_c$ with increasing $H$ for $H > 0$. It has been argued that a realistic variation of $J_c$ for low $H$ may be obtained within Bean framework by redefining the width of the hysteresis loop. The new definition of the loop width is guided by the requirement that $J_c$ stays as close to the $J_c$ of the theory of Kim as possible. Illustrative calculations of $J_c$ show its considerable enhancement over the Bean values.

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1. Introduction

In a recent publication Cheng et al have presented values of the critical current density \( J_c \) of various samples of the MgB\(_2\) superconductor [1]. The \( J_c \) of one of these samples, namely that with 8\% nano diamond (ND), decreases at 10 K with decreasing magnetic field \( H \) below about 0.5 T. This is an unexpected behavior of the \( H \)-dependence of \( J_c \). In order to see why it is so, we first consider the lower critical field \( (H_{c1}) \) of the MgB\(_2\) superconductor. By using the values of the penetration depth, \( \lambda = 140 \) nm [2], and coherence length, \( \xi = 5.2 \) nm [2] in Eq. 3.56 of de Gennes [3] we find that \( H_{c1} \) turns out to be about 0.03 T, which is practically zero on the scale of the irreversibility field \( (H_{\text{irr}} \sim 6 - 10 \) T [1]). Thus, as soon as the magnetic field is applied to the superconductor from \( H = 0 \), vortices will start to enter in the system. The movement of these vortices will increase with increasing magnetic field [4] so that \( J_c \) will decrease with increasing \( H \). This will happen at least for low \( H \) irrespective of the model used for describing the vortex dynamics (cf. Eqs. 6, 7, 13, 16, 25, 31 and 40 of Wordenweber [4]). In fact, in general, this will continue up to higher \( H \) also, but for the case like that of collective pinning [5] \( J_c \) will increase for a portion of \( H \) near \( H_{\text{irr}} \). From the work of Chen and Goldfarb [6] it becomes clear that in the Kim’s theory also \( J_c \) decreases with increasing \( H \).

Vajpayee et al [7] have also made a study of \( J_c \) of ND-doped MgB\(_2\) samples. The 5\% ND and 7\% ND samples of these authors show increasing \( J_c \) with \( H \) at 10 K below about 1.5 T. In fact, this behavior of \( J_c \) has been encountered earlier also, for example, by Niu and Hampshire [8] in the sample numbers 6 and 7 of PbMo\(_6\)S\(_8\) below about 0.5 T. (On the basis of Ref. [8] it may be shown that \( H_{c1} \) of PbMo\(_6\)S\(_8\) will be of order of 0.0002 T, while \( H_{\text{irr}} \) is of order of 25 T.) Many
more references may be found out in literature where $J_c$ increases with $H$ in the low-$H$ regime. However, for specificity, we shall limit to the references [1], [7] and [8] only.

The above-mentioned increasing $J_c$ with increasing $H$ in the low magnetic field regime arises due to the Bean’s formulation [9] since all of the above authors have used this method for estimating $J_c$ from the $M$-$H$ curves. From Fig. 8 of Chen and Goldfarb it becomes clear that the Bean’s $J_c$ shows increasingly larger deviation from the Kim’s $J_c$ near $H = 0$ when the magnetic moment corresponds to increasingly stronger dependence on $H$. Since the Bean’s theory is a special case of the Kim’s theory [6], the latter is more realistic than the Bean’s theory. In this sense a deviation of the Bean’s and Kim’s $J_c$ values for low $H$ implies inadequacy of the Bean’s theory for this region of low $H$. Thus there is a need for a realistic method for estimating $J_c$ from hysteresis loops. It may be noted that knowledge of the realistic $J_c$ is important not only from the view-point of its magnitude, but also from the view-point of pinning mechanism. This is because the pinning mechanism is understood by looking at the variation of the pinning force density

$$F_p = \mu_0 H J_c(H) \propto \left(\frac{H}{H_{c1}}\right)^p \left(1 - \frac{H}{H_{c1}}\right)^q$$

with $H$ [1]. (Here $\mu_0$ is free space permeability.) If $J_c(H)$ is not realistic, the values of $p$ and $q$ may be unrealistic too, leading to a misleading interpretation of the vortex dynamics. Thus a realistic way for estimating $J_c$ from hysteresis loops is highly desired.

This task has been performed in the present article. We suggest a method for extracting values of the critical current density by redefining the width of the hysteresis loop at a particular $H$ such that the resulting $J_c$ stays as close to the Kim’s $J_c$ as possible.
2. Formalism

Let $M^+(H)$ and $M^-(H)$ denote respectively the positive and negative parts of the magnetic moment of a hysteresis loop. Then, according to Bean’s formulation, $J_c$ is given by [6]

$$J_c(H) = G[M^+(H) - M^-(H)].$$  

(1)

Here $G$ is a geometric factor.

In order to see why Bean’s formula results in an inconsistent behavior of $J_c$ for low $H$, we proceed as follows. In the Bean’s theory the critical current density is considered to be independent of the magnetic field [6]. In the sense of Eq. (1) this means

$$\frac{dM^+(H)}{dH} \approx 0; \quad \frac{dM^-(H)}{dH} \approx 0$$  

(2)

On the other hand, the magnetic moment of MgB$_2$ and PbMo$_6$S$_8$ superconductors changes rapidly with $H$. That is to say,

$$\frac{dM^+(H)}{dH} \gg 0; \quad \frac{dM^-(H)}{dH} \gg 0$$  

(3)

This can be seen, for example, from Fig. 2 of Vajpayee et al [7] and from the inset of Fig. 9 of Niu and Hampshire [8].

The situation of Eq. (3) enhances the possibility of the variation of the right-hand side of Eq. (1) with $H$ to be positive. That is to say, for

$$\frac{dM^+(H)}{dH} - \frac{dM^-(H)}{dH} \gg 0$$  

(4)
to be satisfied for low $H$. When this occurs, Eq. (1) will lead to a $J_c$ which increases with $H$ for low $H$. Thus, the inadequacy of the Bean’s formalism at low $H$ arises due to fast variation of the moments $M^+(H)$ and $M^-(H)$ with $H$.

In order to clarify Eq. (4) we consider the positions of the maximum (minimum) of the moment $M^+$ ($M^-$). The maximum of $M^+(H)$ lies in the negative-$H$ side, while that of $M^-(H)$ lies on the positive-$H$ side such that both are equidistance from $H=0$. (cf. Fig. 6e of Chen and Goldfarb.) In this sense let $-H_{\text{max}} (H_{\text{max}} \geq 0)$ be the position of the maximum of $M^+(H)$, then $H_{\text{max}}$ will be the position of the minimum of $M^-(H)$. If $H_{\text{max}} = 0$, condition of Eq. (4) will never be satisfied because the left-hand side of this equation will be essentially negative. In fact, for $H_{\text{max}} = 0$, $\frac{dM^+(H)}{dH}$ will be negative, while $\frac{dM^-(H)}{dH}$ will be positive. When $H_{\text{max}}$ increases beyond 0, $\frac{dM^+(H)}{dH}$ will remain negative, but $\frac{dM^-(H)}{dH}$ will change sign from positive to negative between $H=0$ and $H= H_{\text{max}}$. Thus if $H_{\text{max}}$ is sufficiently away from $H=0$, a situation will arise when $|\frac{dM^-(H)}{dH}|$ will become larger than $|\frac{dM^+(H)}{dH}|$. When it happens so, Eq. (4) will be satisfied and, according to Eq. (1), $J_c$ will increase with $H$ up to $H = H_{\text{max}}$.

Thus it is the value of $H_{\text{max}}$, which leads to values of $J_c$ as found by Cheng et al [1], Vajpayee et al [7], and Niu and Hampshire [8] for low $H$ for some of the superconducting samples. The value of $H_{\text{max}}$ is $0.083 H_{\text{MPP}}$ for the Fig. 6e of Chen and Goldfarb [6]. Cheng et al [1], Vajpayee et al [7], and Niu and Hampshire [8] have not given hysteresis loops for the above-mentioned samples. So, it is difficult to estimate accurate values of $H_{\text{max}}$ for these samples. However, on the basis of the variation of $J_c$ of these samples a rough estimate can be made. We find $H_{\text{max}} = 0.5$ T,
1.5 T, 1.5 T, 0.5 T and 0.5 T respectively for the 8%ND sample [1], 5% ND sample [7], 7%ND sample [7], sample number 6 [8] and sample number 7 [8].

Let us see what the Kim’s theory, of which Bean’s theory is a special case [6], say about the behavior of $J_c$ for the situation of Eq. (3). Looking at the various parts of Fig. 6 of Chen and Goldfarb [6] we find that the condition of Eq. (3) is most satisfied for Fig. 6e. Fig. 8e shows the values of $J_c$ corresponding to this figure. We see that there is perfect agreement between the Bean’s $J_c$ and Kim’s $J_c$ for $H > 0$, where $H_p$ is full penetration field. Below $H_p$ the difference between these two theories become increasingly larger with decreasing $H$. While the Bean’s $J_c$ changes curvature below $H_p$ so that it bends downward near $H=0$, the Kim’s $J_c$ continues the same curvature down to $H=0$ tending to infinity for $H=0$. In fact, for the situation of Fig. 8e Kim’s $J_c$ behaves like $1/H$.

The above comparison of the Bean’s $J_c$ and Kim’s $J_c$ makes it clear that for the situation of Eq. (3) these two critical currents move in opposite directions. Because of the change in curvature the Bean’s $J_c$ becomes lower than the realistic $J_c$ near $H=0$. On the other hand, because of the fact that $J_c \to \infty$ for $H \to 0$, the Kim’s $J_c$ will be larger than the realistic $J_c$ near $H=0$. So the realistic $J_c$ will lie in between the Bean’s $J_c$ and Kim’s $J_c$. Below we describe a method to estimate this realistic $J_c$.

We take the magnetic moments $M^+(H)$ and $M^-(H)$ as input, but redefine the width of the hysteresis loop, $\Delta M(H)$, at the magnetic field $H$. For this purpose we, first of all, note that for $H > H_p$ Bean’s $J_c$ and Kim’s $J_c$ lead to the same set of values [6]. So, we take

$$\Delta M(H > H_p) = M^+(H) - M^-(H).$$

(5)
We now consider the $H=0$ point. Since the sought-for $J_c$ is required to have positive curvature for all $H$, it will be maximum at $H=0$. Moreover, we require that the sought-for $J_c$ remains as close to the Kim’s $J_c$ as possible. The maximum possible value of $\Delta M(H=0)$ from the moments $M^+(H)$ and $M^-(H)$ is given by

$$\Delta M(H=0) = M^+(-H_{\text{max}}) - M^-(+H_{\text{max}}). \quad (6)$$

We are now left with the values of $\Delta M(H)$ for $0 < H < H_p$. For this purpose we compress the values of $M^+(H)$ from a range of width $H_{\text{max}} + H_p$ ($-H_{\text{max}} \leq H \leq H_p$) to a shorter range of width $H_p$ ($0 \leq H \leq H_p$). Such a task is performed in a practically convenient way by the function

$$s(H) = \exp \left( -\frac{H}{H_p} \right) \quad (7)$$

such that

$$e^{-\mu} \approx 0. \quad (8)$$

From Eq. (8) we can see that $s(0)=1$ and $s(H > H_p) \approx 0$. Using the function $s(H)$ we can stretch the moment values $M^-$ from the shorter range of width $H_p$ ($H_{\text{max}} \leq H \leq H_p$) to a range of width $H_p$ ($0 \leq H \leq H_p$). The loop width $\Delta M(H)$ for $0 < H < H_p$ is now expressed in terms of the compressed moment $M^+$ and stretched moment $M^-$ as given by

$$\Delta M(H) = M^+(H - H_{\text{max}}) - M^-(H + H_{\text{max}}). \quad (9)$$
This equation tends to Eq. (6) for $H=0$, and to Eq. (5) for $H \gg H_p$. We replace the Bean’s loop width by this new width so that Eq. (1) for the critical current density is modified to

$$I_c(H) = G\Delta M(H).$$ \hspace{1cm} (10)

For $H_{\text{mean}} \approx 0$ this equation tends to the Bean’s formula (Eq. 1).

3. Results and discussion

In order to clarify the importance of Eqs. (9) and (10) we have calculated $I_c(H)$ using the hysteresis loop of Fig. 6e of Chen and Goldfarb [6]. The results are shown in the third column of Table 1 for various values of $H/H_p$. In the calculations we have taken $u=7$, which guarantees that Eq. (5) gets satisfied to within an error of 0.001. The second (fourth) column of this table corresponds to the Bean’s (Kim’s) $I_c$ read from Fig. 8e of Chen and Goldfarb. From table 1 we see that the present values of $I_c$ matches with that of Kim’s $I_c$ down to $H = 0.6H_p$, while that of the Bean’s $I_c$ matches with the Kim’s $I_c$ down to $H = H_p$ only. This shows that the present method leads to indeed more realistic $I_c$ than the Bean’s $I_c$. The deviation of the present $I_c$ from Kim’s $I_c$ below $H = 0.6H_p$ occurs because the latter diverges at $H=0$, while the present $I_c$ is limited to a finite value by the width of the hysteresis loop.

The value of $I_c$ obtained by using Eqs. (9) and (10) for $H=0$ is $4.40I_c(H_p)$ (cf. Table 1). This is significantly larger than the corresponding Bean’s value, $3.21I_c(H_p)$.

Apart from this quantitative difference, the main difference lies in the qualitative sense. While the present $I_c$
continues increasing for decreasing $H$ down to $H=0$, the Bean’s $I_c$ changes curvature at $H = 0.8 H_p$ (cf. Fig. 6e of Ref. [6]). We emphasize that this change of curvature in the Bean’s method is responsible for lower $I_c$ values near $H=0$. The value of $H_{\text{max}}$ for Fig. 6e of Chen and Goldfarb, as mentioned above, is $0.083 H_{\text{prv}}$. If the value of $H_{\text{max}}$ increases further then after some stage we expect that Eq. (4) gets satisfied. When it happens so, $I_c$ will decrease for decreasing $H$ near $H=0$ in the Beans model, but according to the present estimation (Eqs. 9 and 10) $I_c$ will continue increasing for decreasing $H$. An important result of this illustration is that Bean’s $I_c$ will need modification if $H_{\text{max}} > 0$, irrespective of how $I_c$ varies near $H=0$. Moreover, near $H=0$ the present method will be more realistic than the Kim’s method also because the latter gives diverging $I_c$ at $H=0$. In fact, a superconductor can never support an infinite $I_c$. The upper limit of $I_c$ will be $I_{c,\text{max}} = n^* e^2 h / m^*$ where $n^*$ is superfluid density, $e$ is magnitude of electron’s charge, $h$ is reduced Planck’s constant, and $m$ is electron’s mass. For the MgB$_2$ superconductor $I_{c,\text{max}}$ will be of the order of $10^8$ A/cm$^2$.

Hysteresis loops are not available in the articles of Cheng et al [1], Vajpayee et al [7] and Niu and Hampshire [8]. So, we are not in a position to present values like those in table 1 for the samples considered in these references. However, from the variation of $I_c$ we can get rough idea about $H_{\text{max}}$ and variations of the magnetic moments with $H$ near $H=0$. Such values of $H_{\text{max}}$ and $I_c(H)$ are given in table 2. It is clear from this table that the present method modifies $I_c(H)$ considerably.
It may be noted that the enhancement of $J_c(H)$ in the present method will shift the critical force density $F_p$ towards $H = 0$. This will lower the peak position, $p/(p+q)$, of $F_p$ versus $H/H_{c2}$ curve, thereby affecting the nature of pinning mechanism.

4. Conclusions

In the present paper we have pointed out cases where the critical current density of some samples of MgB$_2$ and PbMo$_6$S$_8$ increases with magnetic field for low $H$. Since in these systems the values of the lower critical field are practically zero on the scale of the irreversible field, this trend of $J_c(H)$ is unrealistic. We have identified the origin of this behavior of $J_c(H)$ for low $H$ in the sharp variation of the magnetic moment with $H$ (Eq. 3) combined with significantly larger values of $H_{c2}$. For a consistent extraction of the critical current density from the hysteresis loops we have suggested a method, Eq. (9) and (10), which is governed by the condition that the new values of $J_c(H)$ stay as close to the Kim’s values as possible. Although the present method is motivated by increasing $J_c(H)$ with $H$, it has a larger range of applicability in that it is suitable for any superconductor satisfying Eq. (3) and having sufficient value of $H_{c2}$ irrespective of whether $J_c(H)$ increases or decreases with $H$ for low $H$. 
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Table 1: Values of the relative critical current density $J_c(H)/J_c(H_p)$ for various values of $H/H_p$ in the present case, Bean’s formalism and Kim’s theory. The latter two sets of values are read from the Fig. 8e of Chen and Goldfarb [6], while the present values are obtained on the basis of the hysteresis loop of Fig. 6e of Ref. [6].

| $H/H_p$ | $J_c(H)/J_c(H_p)$  |
|---------|-------------------|
|         | Bean  | Present | Kim  |
| 0.0     | 3.21  | 4.40    | $\infty$ |
| 0.2     | 3.13  | 3.13    | 5.18  |
| 0.4     | 2.90  | 2.23    | 2.45  |
| 0.6     | 2.18  | 1.67    | 1.67  |
| 0.8     | 1.45  | 1.18    | 1.18  |
| 1.0     | 1.00  | 1.00    | 1.00  |
Table 2: Values of $H_{\text{max}}$ and $I_c(H)$ for different superconducting samples. The Bean’s values are read from the respective references.

| Sample            | Ref. | $\mu_0 H_{\text{max}}$ (T) | $\mu_0 H$ (T) | $(10^5 \text{ A/cm}^2)$ |
|-------------------|------|-----------------------------|---------------|-------------------------|
| 8%ND MgB$_2$      | 1    | 0.5                         | 0.0           | 3.98                    | 5.75 |
| 5%ND MgB$_2$      | 7    | 1.5                         | 1.0           | 2.01                    | 3.19 |
| 7%ND MgB$_2$      | 7    | 1.5                         | 1.0           | 0.60                    | 1.59 |
| No. 6 PbMo$_6$S$_8$ | 8    | 0.5                         | 0.0           | 2.03                    | 2.54 |
| No. 7 PbMo$_6$S$_8$ | 8    | 0.5                         | 0.0           | 1.04                    | 1.30 |