Classical description of bosonic many-body systems in terms of the reduced-state-of-the-field framework

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(Dated: February 10, 2022)

We discuss compatibility between various quantum aspects of bosonic many-body systems, relevant for quantum optics and quantum thermodynamics, and the mesoscopic formalism of reduced state of the field (RSF). In particular, we derive exact conditions under which Gaussian evolution can be cast into the RSF framework. In that regard, special emphasis is put on Gaussian thermal operations. To strengthen the link between the RSF formalism and the notion of classicality, we prove that RSF contains no information about entanglement in two-mode Gaussian states. For the same purpose, we show that the entropic characterisation of RSF by means of the von Neumann entropy is qualitatively the same as its description based on the Wehrl entropy. Based on our findings, we endorse RSF as a tool for semi-classical treatment of bosonic many-body phenomena.

I. INTRODUCTION

Between the physics of a single particle, given by the wave function, and the high particle number regime, successfully described by (often classical) statistical methods, is the intermediate world of few-to-many-body systems. Today, the current technological advancements allow for experiments with trapped ions \cite{1,2}, cold atoms \cite{3-5} and photons \cite{6,7} that lie precisely in this intersection between micro- and macroscopic ensembles (or in its vicinity).

In such situations, standard description in terms of the wave function or its second-quantized counterpart is often too complex, leading to the necessity of resorting to numerical simulations relying on the Hartree-Fock method or the density functional theory \cite{8}. On the other hand, statistical description breaks down for such, relatively small, particle numbers. A proper treatment requires a mesoscopic theory \cite{9} that takes into account only the relevant features of the system, simplifying the theory without damaging its accuracy.

Recently, one such theory in the form of the reduced state of the field (RSF) has been proposed \cite{10}. Relying solely on the first two moments of the creation and annihilation operators of the $N$-mode continuous variable system at hand, the description reduces the infinite-dimensional density operator to an $N$-dimensional matrix defining the aforementioned RSF. Importantly, RSF comes equipped with a definition of entropy and an evolution equation derived from the von Neumann entropy and the GKLS (Lindblad) equation, respectively.

Originally, RSF was designed as a description of macroscopic fields of a single particle type, e.g. photons \cite{10}. The main aim was to capture the potential quantum features of fields that are most often treated classically, e.g. gravitational or hydrodynamic waves. In particular, the formalism was successfully applied to thermal sources and polarization optics.

Here, we adopt the opposite perspective. The main purpose of this article is to analyze the applicability of the formalism with respect to quantum many-body phenomena, with the aim of capturing their classical features. To this end, we derive the exact conditions under which the reduced kinetic equations driving the evolution of RSF coincide with Gaussian evolution with (non-Gaussian) scattering \cite{11-13} which is of special importance in quantum optics, among others. We find that the RSF formalism describes such Gaussian evolution successfully provided it corresponds to passive transformations such as beam splitters and phase shifters.

To further strengthen the interpretation of RSF as a classical description of quantum fields, we investigate the issues of entropy and entanglement. In the case of entropy, we derive a competing measure of entropy of RSF, based on the Wehrl entropy \cite{14}, which we compare with the original entropy of RSF, based on the von Neumann entropy. We find both RSF entropies to be qualitatively similar to the Wehrl entropy, which is typically considered to be a semi-classical entropy of quantum states. In the case of entanglement, we prove that, despite being built upon non-local correlation functions, RSF contains no information about entanglement in two-mode Gaussian states.

These results motivate us to propose RSF as a semi-classical framework for bosonic many-body phenomena. A few examples of such phenomena, such as Gaussian thermal operations and scattering with quadratic generators, are investigated by us in the article.

This work is organized as follows. In Section II, we define and briefly summarize the relevant characteristics of RSF, including its entropy and time evolution. In Section III, we compare the reduced kinetic equations for RSF with Gaussian evolution. In Sections IV and V, we investigate the entropy and entanglement properties of RSF. We conclude in Section VI.
II. REDUCED STATE OF THE FIELD

In this section, we summarize the relevant information about the reduced state of the field (RSF), including its evolution and entropy. To this end, it is beneficial to briefly recall the notions of first and second quantizations.

In the first quantization, the $N$-dimensional single particle Hilbert space is spanned by a set of complex modes $f_k, k \in \{1, 2, \ldots, N\}$. Here, it is assumed that $N$ is finite, mainly for notational convenience. Each mode corresponds to an eigenstate of the system Hamiltonian and is, in turn, associated with a possible energy level of the particle.

An extension of this scenario to an arbitrary number of identical particles is given by the second quantization. The single-particle $N$-level Hilbert space is promoted to a tensor product of $N$ multi-particle Hilbert spaces, each associated with a different energy level (mode) of the original single-particle space. In the case of bosons, which are in the focus of this article, the new Hilbert space is conveniently described using a set of $N$ annihilation and creation operators $\hat{a}_k, \hat{a}_k^\dagger$, respectively, fulfilling the canonical commutation relations

$$[\hat{a}_k, \hat{a}_k^\dagger] = \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0.$$  \hspace{1cm} (1)

An arbitrary $n$-particle state in the many-body Hilbert space can be constructed by acting on the vacuum state with $n$ appropriate creation operators.

While elegant, second quantization can prove to be a significant challenge in calculations. One of the problems has its origin in the fact that each bosonic energy level can be in principle occupied by an infinite number of particles. Even a trivial case of a single mode, one-dimensional first-quantized Hilbert space gives rise to infinite-dimensional Hilbert space in the second quantization, resulting in a relatively complex description.

In the formalism of RSF, the infinite-dimensional many-particle Hilbert space is reduced back to $N$ dimensions, like in the first quantization, but is concerned with multi-particle systems, like in the second quantization. RSF of any density operator of the second-particle Hilbert space is reduced back to a single-particle Hilbert space, resulting in a relatively intuitive construction. Both RSF components are defined in such a way that a single creation / annihilation operator in the full Hilbert space corresponds to a single bra / ket in the reduced space. In particular, observables of the form

$$\hat{O} = \sum_{k,k'=1}^N o_{kk'} \hat{a}_k^\dagger \hat{a}_{k'}$$  \hspace{1cm} (4)

are reduced in a natural way to

$$o = \sum_{k,k'=1}^N o_{kk'} |k\rangle \langle k'|.$$  \hspace{1cm} (5)

Furthermore, the expectation values of such observables are preserved by the RSF formalism:

$$\text{Tr} \hat{O} = \text{tr} \hat{O}.$$  \hspace{1cm} (6)

A. Time evolution

The time evolution of quantum open systems is typically modeled by the GKLS (Lindblad) equation [15, 16], describing the most general Markovian evolution [17]:

$$\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_k \left( \hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}\} \right),$$  \hspace{1cm} (7)

where $\hat{H}$ denotes the system Hamiltonian and $\hat{L}_k$ are the Lindblad operators.

In the formalism of RSF, the field is treated as a set of individual particles subject to spontaneous decay and production, as well as interaction with coherent classical sources and random scattering by the environment. The GKLS evolution driven by such phenomena is given by [10]

$$\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_{k=1}^N \left[ \hat{\zeta}_k \hat{\rho} \hat{a}_k^\dagger - \frac{1}{2} \{\hat{a}_k^\dagger \hat{a}_k, \hat{\rho}\} \right]$$

$$+ \sum_{k,k'=1}^N \Gamma_k^{kk'} \left( \hat{a}_k \hat{\rho} \hat{a}_k^\dagger - \frac{1}{2} \{\hat{a}_k \hat{a}_k^\dagger, \hat{\rho}\} \right)$$

$$+ \int \mu(du) \left( \hat{\rho} \hat{U}^\dagger - \hat{\rho} \right).$$  \hspace{1cm} (8)
Here, the complex vector $\vec{\zeta}$ describes the coherent source of the field and the positive matrices $\Gamma_\downarrow$, $\Gamma_\uparrow$ contain particle annihilation and creation rates for random sources. The Hamiltonian has the form $\hat{\mathcal{H}} := \hbar \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k$, where $\omega_k > 0$ determine the energy levels. The last term describes random scattering parametrized by the positive measure $\mu(du)$ on the unitary group acting on the Hilbert space.

Tracing both sides of eq. (8) with $\hat{a}_\uparrow^\dagger \hat{a}_\downarrow$ and $\hat{a}_\downarrow$ yields the reduced kinetic equations for RSF. As the resulting equations slightly differ from the ones derived originally in [10], where minor errors appear [18] we provide them in full in the following proposition.

**Proposition 1.** The time evolution of RSF is given by the reduced kinetic equations:

$$
\frac{d}{dt} r = - \frac{i}{\hbar} [\mathcal{H}, r] + |\zeta\rangle \langle \alpha| + |\alpha\rangle \langle \zeta|
+ \frac{1}{2} \{\gamma_\uparrow - \gamma_\downarrow, r\} + \gamma_\uparrow
+ \int \mu(du) (uru^\dagger - r),
$$

$$
\frac{d}{dt} |\alpha\rangle = - \frac{i}{\hbar} [\mathcal{H}, |\alpha\rangle] + \frac{1}{2} (\gamma_\uparrow - \gamma_\downarrow) |\alpha\rangle + |\zeta\rangle
+ \int \mu(du) (u - 1) |\alpha\rangle.
$$

Here, $\mathcal{H} := \hbar \sum_{k=1}^N \omega_k |k\rangle \langle k|$, $|\zeta\rangle := \sum_{k=1}^N \zeta_k |k\rangle$, $\gamma_\uparrow := \sum_{k,k'=1}^N \Gamma_{k,k'}^\uparrow |k\rangle \langle k'|$, $\Gamma_{k,k'}^\downarrow$ are the single-particle counterparts to $\mathcal{H}$, $\zeta$ and $\Gamma_{k,k'}$, respectively. The unitary matrices $u$ are fixed by demanding the RSF of $\hat{U} \hat{\rho} \hat{U}^\dagger$ to be equal to $uru^\dagger$.

It is clear from construction that every evolution equation of the form (8) has an equivalent set of reduced kinetic equations. However, in principle, other quantum evolution equations can also be represented by simpler RSF alternatives. In Section III, we derive the reduced kinetic equations for Gaussian evolution.

**B. Entropy**

The standard choice for quantum (information) entropy is given by the von Neumann entropy

$$
S_N (\hat{\rho}) := -k_B \text{Tr} \hat{\rho} \ln \hat{\rho},
$$

where $k_B$ is the Boltzmann’s constant. Among all the quantum states with the same RSF, the von Neumann entropy is maximal for the thermal-like state [10], for which it equals

$$
k_B \text{Tr} [(r_\alpha + 1_N) \ln (r_\alpha + 1_N) - r_\alpha \ln r_\alpha].
$$

Here and throughout the rest of this work, $1_N$ denotes the identity matrix of size $N \times N$, while

$$
r_\alpha := r - |\alpha\rangle \langle \alpha| \geq 0,
$$

defines the correlation matrix associated with the RSF. In other words, the von Neumann entropy of an arbitrary state $\hat{\rho}$ with the reduced state of the field $(r, |\alpha\rangle)$ is upper bounded by eq. (11). According to the maximum entropy principle [19, 20], in the absence of any knowledge about the field other than its RSF, this upper bound is a meaningful measure of the state’s entropy. Thus, the entropy of RSF is defined by the reduced (von Neumann) entropy

$$
s_N (\hat{\rho}) := k_B \text{Tr} [(r_\alpha + 1_N) \ln (r_\alpha + 1_N) - r_\alpha \ln r_\alpha].
$$

The reduced entropy satisfies the natural condition $s_N (\hat{\rho}) \geq 0$, with equality if and only if the correlation matrix is equal to zero, which happens only when the density operator of the field is given by a coherent state.

In contrast, the von Neumann entropy vanishes for any pure state. In the original work [10], this departure is interpreted as a dependence of the reduced entropy on the complexity of description, which is determined by the assumed level of control over the system. For example, from the point of view of joint measurement of mean photon number and phase, the $n$-photon state, which has no definite phase, contains less information (more entropy) with respect to the selected measurement scheme than a coherent state with the same mean photon number. We investigate the reduced entropy in Section IV.

**III. GAUSSIAN EVOLUTION AND RSF**

One of the main goals of our work is to investigate under what circumstances second-quantized evolution can be replaced by the reduced kinetic equations for RSF. In this section, we are concerned with Gaussian evolution, defined here as the most general Markovian evolution that preserves the Gaussian property of states. Since it plays a significant role in the reduced kinetic equations, we extend this evolution to include an additional scattering term beyond Gaussian dynamics.

The section is organized as follows: firstly, we summarize the relevant information about Gaussian states and Gaussian evolution with (non-Gaussian) scattering. Then, we derive the conditions under which such evolution is equivalent to reduced kinetic equations and illustrate our findings on concrete examples. Finally, we discuss the classicality of the derived reduced kinetic equations.

**A. Preliminaries: Gaussian states and evolution**

Given the set of $N$ creation and annihilation operators, we construct a vector of $N$ pairs of mode quadratures

$$
\vec{\zeta} := (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_N, \hat{p}_N)^T,
$$
where $\hat{x}_k := \frac{1}{\sqrt{2}}(\hat{a}_k + \hat{a}_k^\dagger)$ and $\hat{p}_k := -\frac{i}{\sqrt{2}}(\hat{a}_k - \hat{a}_k^\dagger)$. Since the mode quadratures form a basis of operators acting in the $N$-mode Hilbert space, the state of the system is fully described by the complete collection of the values of correlation functions of the form

$$\langle \hat{\xi}_1, \ldots, \hat{\xi}_n \rangle := \text{Tr}[\hat{\rho} \hat{\xi}_1 \cdots \hat{\xi}_n].$$

(15)

In the case of Gaussian states, defined as states with normal (Gaussian) characteristic functions and quasiprobability distributions [11–13], the state is fully described by one- and two-point correlation functions, i.e. with $n = 1, 2$ in the equation above. The information about the former is contained in the vector

$$|\xi \rangle := \sum_{k=1}^{2N} \langle \hat{\xi}_k \rangle |k\rangle,$$

(16)

while the latter is encoded in the matrix of second moments

$$V := \frac{1}{2} \sum_{k,k'=1}^{2N} \langle \{\hat{\xi}_k, \hat{\xi}_{k'}\} \rangle |k\rangle \langle k'|\rangle.$$

(17)

The covariance matrix of a quantum state is just $V_{\text{cov}} = V - |\xi \rangle \langle \xi|$. Any valid covariance matrix has to fulfill the Heisenberg uncertainty principle:

$$\sqrt{\langle \hat{x}_k^2 \rangle - \langle \hat{x}_k \rangle^2} \sqrt{\langle \hat{p}_k^2 \rangle - \langle \hat{p}_k \rangle^2} \geq \frac{\hbar}{2},$$

(18)

where $k \in \{1, \ldots, N\}$, equivalent to [11]

$$V_{\text{cov}} + \frac{i}{2} J \succeq 0.$$

(19)

Here, $J$ is the symplectic form, defined as

$$J := -\sum_{k,k'=1}^{2N} \frac{i}{\hbar} \langle \hat{\xi}_k, \hat{\xi}_{k'} \rangle |k\rangle \langle k'|,$$

(20)

and explicitly equal to

$$J = \bigoplus_{k=1}^N J_2, \quad J_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(21)

The symplectic form defines the symplectic group $Sp(2N,\mathbb{R})$ consisting of matrices $S$ of size $2N \times 2N$, such that $SJS^T = J$.

As a matter of fact, the pair $(V, |\xi \rangle)$ contains the same information as $(V_{\text{cov}}, |\xi \rangle)$, and both in the same way define the symplectic picture of quantum states (sometimes referred to as covariance matrix picture). In this work, we choose to use the pair $(V, |\xi \rangle)$, since by construction it is closer to $(r, |\alpha \rangle)$ than the pair $(V_{\text{cov}}, |\xi \rangle)$.

We stress that the symplectic picture is complete (i.e. equivalent to the standard, density operator description) only in the case of Gaussian states. However, it is a valid partial description of all quantum states.

One of the main sources of motivation for studying the symplectic picture is that, due to technical limitations, in practice we are often restricted to Hamiltonians that are at most second-order in the mode quadratures:

$$\dot{H} = \frac{1}{2} \xi^T G \xi,$$

(22)

where $G$ is a $2N \times 2N$, real, symmetric matrix. In particular, the structure-preserving evolution of Gaussian states is driven by such quadratic Hamiltonians.

Similarly, to preserve Gaussianity of an initial state along the course of time evolution, the Lindblad operators need to be linear in mode quadratures [21]:

$$\hat{L}_k = \bar{c}_k \cdot \hat{\xi}, \quad \bar{c}_k \in \mathbb{C}^{2N},$$

(23)

so that the resulting dissipator is quadratic, like the Hamiltonian. However, this requirement immediately removes random scattering processes present in eq. (8), as unitary Lindblad operators are necessarily of infinite order [22].

Therefore, for the sake of comprehensiveness of the comparison we consider an extension beyond Gaussian evolution that includes such scattering [23, 24]. While this extension does not preserve Gaussian states, it results in a self-contained equation in the symplectic picture. It can be thus used to describe the evolution of the first two moments of any quantum state. Indeed, it is known to model various stochastic phenomena, such as e.g. Poisson processes [24]. From the perspective of standard Gaussian evolution, it may also be used to introduce non-Gaussian noise in evolution of (approximately) Gaussian states.

Below we make a clear distinction between evolution terms that preserve Gaussianity and terms that do not. Representing the standard quadratic component relevant for evolution of Gaussian states by functions $F_2$ and $f_2$, and the unitary Lindblad operators responsible for Gaussianity-breaking scattering by functions $F_\infty$ and $f_\infty$, we obtain

$$\frac{d}{dt} |\xi \rangle = F_2(V) + F_\infty(V),$$

$$\frac{d}{dt} |\xi \rangle = f_2(|\xi \rangle) + f_\infty(|\xi \rangle).$$

(24)

The functions $F_2, f_2$ are responsible for the most general quadratic evolution of Gaussian states [25, 26]:

$$F_2(V) := AV + VA^T + JRC^T, \quad f_2(|\xi \rangle) := A|\xi \rangle.$$

(25)

Here

$$A := J [G + I_C],$$

(26)

where $R_C \equiv \text{re} C^T C, I_C \equiv \text{im} C^T C$ and $C_{kl} := (\bar{c}_k)_l$. 

The functions \( F_{\infty}, f_{\infty} \) describe the Gaussianity-breaking evolution of infinite order and read [24]

\[
F_{\infty}(V) := \int \mu(dk) \left( KV K^T - V \right),
\]

\[
f_{\infty}(|\xi\rangle) := \int \mu(dk) \left( K|\xi\rangle - |\xi\rangle \right),
\]

where the measure \( \mu(dk) \) is restricted to the space of symplectic matrices (so that \( K \) are symplectic).

**B. Reduced kinetic equations for Gaussian evolution with scattering**

Our results regarding the connection between the evolution in eq. (24) and reduced kinetic equations are summarized in the following proposition, which is proved in Appendix A.

**Proposition 2.** Let \( (V,|\xi\rangle) \) denote the symplectic description of a system undergoing evolution given by eq. (24). The corresponding RSF evolves according to the reduced kinetic equations (9) if and only if

\[
0 = [J, G] = [J, IC]
\]

and for all \( K \) entering the integral (27)

\[
0 = TKT^T \quad \text{and} \quad TKT^\dagger \quad \text{is unitary.}
\]

Here,

\[
T := \frac{1}{\sqrt{2}} \sum_{k=1}^{N} |k\rangle \left([2k-1] + i(2k)\right)
\]

denotes the transfer matrix. The reduced kinetic equations are characterized by:

\[
h = i T G J T^\dagger, \quad |\xi\rangle = 0,
\]

\[
\gamma_{\tau} = \pm T (I_C J + JR_C J) T^\dagger,
\]

\[
u = TKT^\dagger, \quad \mu(du) = \mu(dk).
\]

We are in position to make a couple of observations. Firstly, our results let us assign a clear physical interpretation to the matrices \( R_C, I_C \), which play a rather ambiguous role from the point of view of Gaussian part of the evolution in (24). We find that the two matrices are responsible for the particle creation and annihilation rates as quantitatively described by eq. (31).

Secondly, while the coherent source vanishes in eq. (31), it is only a consequence of Gaussian Hamiltonian (22) being defined without terms linear in the mode quadratures. In general, such terms can be present in Gaussian evolution and would contribute to a non-zero coherent source in the reduced kinetic equations.

From eq. (29) we can see that also the scattering terms, which violate Gaussianity are restricted by the RSF formalism. Physical meaning of these constraints is explained in passing in Section III C, where major focus is put on quantum Gaussian evolution. While illustrating our results with two examples below we also restrict our attention to Gaussian dissipators.

**Example 1** (Gaussian thermal operations). Among the key ingredients in the resource-based approach to quantum thermodynamics are thermal operations, defined as energy-preserving operations on continuous variable systems coupled to a thermal environment. Due to the prevalence of quadratic Hamiltonians in experimental setups, special emphasis is put on Gaussian thermal operations (GTOs), which are thermal operations that preserve the set of Gaussian states.

Recently, a complete characterization of GTOs has been provided in [27]. Here, we focus on a natural subclass of GTOs generated by time-independent, non-degenerate Hamiltonians. Such GTOs are effectively reduced to single-mode transformations [27] of the form

\[
V \to S \left[ Q S^{-1} V (S^{-1})^T Q^T + P \right] S^T,
\]

where \( S \) is a constant \( 2 \times 2 \) symplectic matrix, \( P := (1-p)\nu I_2 \) and

\[
Q := \sqrt{p} \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}.
\]

Here, \( \nu := \coth \beta \omega/2, \omega = \text{const} \) is the Hamiltonian eigenvalue associated with the considered mode, \( \beta \) is the inverse temperature, while \( p \in [0,1] \) and \( \phi \in [0,2\pi) \) are in principle time-dependent.

In accordance with the adopted notation we get

\[
A = \frac{1}{2p} \frac{dp}{dt} + \frac{d\phi}{dt} SS^T J, \quad JR_C J = -\frac{1}{p} \frac{dp}{dt} \nu SS^T.
\]

According to Proposition 2 [using eq. (A6) from the Appendix], such operators can be reconciled with reduced kinetic equations if and only if

\[
0 = \frac{d\phi}{dt} \left[ J, SS^T \right].
\]

Discussion of this result is postponed to the next subsection.

**Example 2** (Stabilizability in two-mode entangled Gaussian systems). In quantum open systems, it is sometimes desirable to counteract the effects of dissipation by using an appropriate Hamiltonian. In the framework of stabilizability, once can check whether this is possible by solving a finite set of conditions rather than checking every Hamiltonian separately [28, 29].

Recently, stabilizability was used to investigate the robustness of two-mode Gaussian states against three classes of dissipation [30]. The maximum amount of entanglement was stabilized in the system when subject to evolution described by:
• in the case of local damping: $\hat{L}_k := \hat{a}_k$ and

$$\hat{H} = \hat{H}\text{sq} := -i\hbar\omega(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger),$$

(36)

where $\omega$ defines the energy levels of the system;

• in the case of dissipative squeezed-state preparation: $\hat{H} = \hat{H}\text{sq}$ and

$$\dot{\hat{L}}_1 := \cosh \alpha \hat{a}_1 - \sinh \alpha \hat{a}_2^\dagger,$$

$$\dot{\hat{L}}_2 := \cosh \alpha \hat{a}_2 - \sinh \alpha \hat{a}_1^\dagger,$$

(37)

where $\alpha > 0$ denotes the amount of squeezing;

• in the case of cascaded oscillators coupled to the vacuum: $\hat{L} := (\hat{a}_1 + \hat{a}_2)$ and

$$\dot{\hat{H}} = \dot{\hat{H}}\text{cas} := -i\hbar\frac{\omega}{2} \left[ (\hat{a}_1 + \hat{a}_2)^2 - (\hat{a}_1^\dagger + \hat{a}_2^\dagger)^2 \right].$$

(38)

It is straightforward to check that while all the dissipators fulfill eq. (28), neither Hamiltonian does, making neither evolution compatible with the reduced kinetic equations.

C. Classicality of Gaussian reduced kinetic equations

The conditions (28-29) under which the reduced kinetic equations and Gaussian evolution with scattering coincide, put restrictions on the Hamiltonian, the scattering/integral term and the remaining quadratic dissipator. Here, we view each condition from the perspective of classicality of RSF.

In the case of the Hamiltonian, using the explicit representation of the symplectic form (21) in eq. (28), we compute that the allowed Hamiltonians consist of $2 \times 2$ block matrices of the form

$$G_{kk'} = G_{kk'}^T = a_{kk'} \mathbb{1}_2 + (1 - \delta_{kk'}) b_{kk'} J_2,$$

(39)

where $k, k'$ enumerate the blocks and $a_{kk'}, b_{kk'} \in \mathbb{R}$. Making use of eq. (22) we check that eq. (39) allows only for particle number-preserving, or passive interactions.

In standard optical implementations, passive transformations correspond to experimental operations with classical analogues, such as beam splitters and phase shiftsers. According to standard notions of non-classicality, such as non-positivity of the Glauber P representation or the presence of entanglement, the output of passive transformations can be non-classical only if given non-classical input [31, 32].

The remaining active transformations, such as squeezing, have no classical analogues. Moreover, they can be a source of quantum advantage, e.g. in metrology [33, 34]. Such transformations are forbidden by eq. (39). In Example 2, this restriction disallowed entanglement-maximizing Hamiltonians.

Similar results concern the integral term, where the condition (29) is fulfilled only when the integration is over operations $K$, which are orthogonal in addition to being symplectic. From the physical point of view, they also correspond to passive transformations only [27].

Finally, in the case quadratic dissipation, we make use of Example 1. There, we found that condition (28) takes the form of eq. (35), which can be fulfilled in two cases. The first possibility is that, like above, the symplectic transformation $S$ is passive. The second possibility is that

$$\phi(t) = \phi(0) = 0,$$

(40)

where the rightmost equality is required by the initial condition on $V$. To understand this case, we come back to eq. (32), which now reads

$$V(t) = pV(0) + (1 - p) \nu SS^T.$$

(41)

For one mode, $SS^T \propto G$ [27], so the above equation simply tells us that the quantum state at hand evolves into a classical mixture of the initial state and the thermal state of the Hamiltonian, which, as shown before, has to be passive.

In summary, we find that Gaussian evolution of RSF can be written as reduced kinetic equations provided that active, non-classical transformations, such as squeezing, are forbidden. These findings suggest that the reduced kinetic equations for Gaussian evolution with scattering have a classical character.

IV. PROPERTIES OF REDUCED ENTROPY

To further reinforce the interpretation of RSF as a classical description of quantum fields, we investigate the formalism itself. In this section, we analyze the properties of the reduced entropy, especially in relation to the quantum von Neumann and Wehrl entropies. After their brief summary, we derive an analogue of the reduced entropy based on the Wehrl entropy, which we link to the original reduced (von Neumann) entropy. We conclude with short discussion.

A. Preliminaries: von Neumann and Wehrl entropies.

Contrary to the thermodynamic entropy, which is uniquely defined up to an additive constant, there exist many competing measures of quantum (information) entropy. In this work, we are concerned with two of the most prominently used measures: the von Neumann and the Wehrl entropies.

The von Neumann entropy (10) is a generalization of the Shannon entropy and is considered the standard quantum entropy [35]. Because of its information-theoretic origin, it is most easily interpreted as a measure of uncertainty about the state of the system. The von Neumann entropy is invariant under all unitary
transitions and it attains its minimum value – zero – for all pure states.

The Wehrl entropy [13] is defined as the continuous Shannon entropy of the Husimi Q representation of the quantum state:

\[ S_W(\hat{\rho}) := -k_B \int \frac{d^{2N} \vec{\beta}}{\pi^N} Q(\vec{\beta}) \ln Q(\vec{\beta}). \] (42)

Here, \( Q(\vec{\beta}) = \langle \vec{\beta} | \hat{\rho} | \vec{\beta} \rangle \) is the Husimi Q representation [36] of the state \( \hat{\rho} \), \( | \vec{\beta} \rangle \) is an \( N \)-mode coherent state and the integration is over the real and imaginary parts of every component of the complex vector \( \vec{\beta} \). The Wehrl entropy is often considered to be a semi-classical alternative to the von Neumann entropy, the reduced entropy (11) under the same unitary transformations. This is a direct consequence of the definition of RSF and the fact that the two entropies share more qualities, e.g. both are invariant under the same unitary transformations. This is a direct consequence of the definition of RSF and the fact that the two entropies depend solely on traces of analytic functions of the correlation matrix.

In addition to qualitative similarities, the two entropies can be linked quantitatively.

**Proposition 3.** The following relation between the reduced von Neumann and Wehrl entropies holds:

\[ 0 < s_w - s_v \leq k_B N. \] (45)

**Proof.** We begin with the l.h.s. inequality. Rearranging eq. (11),

\[ s_v = k_B \tr \{ r_\alpha \ln(r_\alpha + 1_N) - \ln r_\alpha \} + k_B \tr \ln(r_\alpha + 1_N). \] (46)

By definition of the reduced Wehrl entropy, the second term is equal to \(-k_B N + s_w\). In the first term, we apply the eigendecomposition \( r_\alpha = \sum_{k=1}^N \lambda_k |k\rangle \langle k|, \) where \( \lambda_k \geq 0 \). Using basic properties of the logarithm, we arrive at

\[ s_v = k_B \sum_{k=1}^N \ln(1 + 1/\lambda_k)^{\lambda_k} N - k_B N + s_w. \] (47)

Clearly, the first term is maximized in the limit \( \lambda_k \to \infty \), in which, by definition of the Euler’s number, it approaches \( k_B N \). Then, the first and second terms cancel, leaving \( s_v < s_w \) as in the l.h.s. inequality.

To prove the r.h.s. inequality we observe that, since \( r_\alpha \geq 0 \):

\[ s_v \geq k_B \tr \left[ (r_\alpha + 1_N) \ln(r_\alpha + 1_N) - r_\alpha \ln(r_\alpha + 1_N) \right] = s_w - k_B N, \] (48)

which is equivalent to the r.h.s. inequality. \( \square \)

We remark that for states with mean particle number much bigger than the effective number of modes \( \tr r = \langle \hat{n} \rangle \gg N \), the term \( k_B N \) is vanishing in comparison to \( s_w, s_v \). Therefore, it follows from eq. (45) that for most many-particle states, the two reduced entropies are effectively equal. Note that whether a similar result holds for the original two entropies is unknown. While the difference \( S_W - S_V \) is always positive, there is no known upper bound for it.

**C. Classicality of reduced entropy**

Both reduced entropies are minimized solely by coherent states and are invariant only under some unitary transformations. This makes them akin to the Wehrl entropy, which is considered to be a semi-classical entropy of quantum states. Especially the fact that the reduced entropy originating from the von Neumann entropy is qualitatively more similar to the Wehrl entropy is a strong argument for classicality of the RSF formalism.
V. ENTANGLEMENT AND RSF

The final issue that we want to investigate concerns entanglement in the formalism of RSF. To this end, we go back to two-mode Gaussian states (see Preliminaries in Section III for a brief summary) and show that the RSF of such states contains no information about entanglement, further reinforcing our argument about the classical character of RSF.

Proposition 4. RSF contains no information about entanglement in two-mode Gaussian states.

Proof. To prove our statement, we show that for the RSF of any two-mode Gaussian state corresponds to a separable symplectic description \((V, |\xi\rangle)\).

We start by observing that the vector of the first moments \(|\xi\rangle\) can be arbitrarily adjusted by means of local operations and as such contains no information about the entanglement in the state. In turn, the averaged field \(\langle \alpha \rangle\), which depends solely on \(|\xi\rangle\), must also contain no information about entanglement. Therefore, if such information is contained in RSF at all, it must be present in the correlation matrix \(r_\alpha\), which, by virtue of eqs (12, A2) does not depend on the first moments \(|\xi\rangle\).

Starting from eq. (12) and using the easy-to-check relation \(TJT^† = -iI\), we find that the positivity of the correlation matrix is equivalent to the Heisenberg uncertainty principle (19) [this is most easy to see by employing eq. (A2) from Appendix A]. Therefore, any valid correlation matrix corresponds to a valid covariance matrix \(V\) and vice versa.

In the particular case of two modes, any covariance matrix possesses a simple, unique form, called the standard form [11]:

\[
V_{sf} = \begin{bmatrix}
  a & 0 & c_+ & 0 \\
  0 & a & 0 & c_- \\
  c_+ & 0 & b & 0 \\
  0 & c_- & 0 & b
\end{bmatrix},
\]

(49)

where the parameters \(a, b \geq 1/2\) are related to the average number of particles / excitations in the modes and the coefficients \(c_{\pm} \in \mathbb{R}\) contain the information about the correlations between the modes. In particular, a necessary condition for the presence of entanglement in the state is given by [30]

\[
c_+ c_- < 0.
\]

(50)

Crucially, any two-mode covariance matrix can be brought into its standard form by means of local symplectic operations, which, similarly to local unitary operations on density matrices, do not change global properties of the state, such as entanglement.

For this reason, for the purposes of this proof we can, with no loss of generality, consider only covariance matrices in the standard form. Then

\[
r_\alpha = \frac{1}{2} \begin{bmatrix}
  2a - 1 & c_+ + c_- \\
  c_+ + c_- & 2b - 1
\end{bmatrix}.
\]

Clearly, for a fixed matrix \(r_\alpha\) the values of \(a, b\) are also fixed. However, the other two parameters only need to be related by \(c_+ + c_- = 2(r_\alpha)_{12}\). In particular, the relation is fulfilled by the choice \(c_+ = c_- = (r_\alpha)_{12}\). By virtue of eq. (50), such covariance matrix corresponds to a separable state. This completes the proof.

VI. CONCLUDING REMARKS

We studied the applicability of the reduced state of the field (RSF) to a variety of quantum phenomena. We derived exact conditions under which the evolution of RSF coincides with Gaussian evolution with non-Gaussian scattering. Furthermore, we derived a competing measure of entropy of RSF based on the Wehrl entropy, which we linked qualitatively and quantitatively to the original entropy of RSF, as well as the Wehrl entropy itself. Finally, we showed that RSF contains no information about entanglement in two-mode Gaussian states. Based on our findings, we suggest RSF as a tool for a semi-classical treatment of bosonic many-body systems.

Besides potential application of RSF to description of such systems, our work suggests the following directions for future research. To start with, our results can be generalized. It would be interesting to see if our result regarding entanglement and RSF extends to arbitrary quantum states and under what conditions the reduced kinetic equations can replace evolution families beyond Gaussian. Furthermore, the RSF formalism is based on one- and two-point correlation functions. Can a self-consistent mesoscopic framework based on higher-order correlations be designed? If so, what new insights does it offer?

ACKNOWLEDGMENTS

We would like to thank Paweł Mazurek, Marcin Karczewski, Gerd Leuchs and Stefano Cusumano for discussion. We acknowledge support by the Foundation for Polish Science (IRAP project, ICTQT, contract no. 2018/MAB/5, co-financed by EU within Smart Growth Operational Programme).

[1] R. Blatt, C. F. Roos, Quantum simulations with trapped ions, Nat. Phys. 8, 277 (2012).

[2] S. Ding, G. Maslennikov, R. Hablützel, H. Loh,
In this Appendix we prove Proposition 2, i.e. we derive the conditions under which Gaussian evolution is equivalent to the reduced kinetic equations.

To this end, it is useful to define an auxiliary field, \( \hat{X} \) is hermitian. Thus, excluding the case where \( \hat{X} \) is of order zero (for which \( \hat{U} \) reduces to a number), \( \hat{U} \) is of infinite order in mode quadratures.

Appendix A: Proof of Proposition 2

In this Appendix we prove Proposition 2, i.e. we derive the conditions under which Gaussian evolution is equivalent to the reduced kinetic equations.

To this end, it is useful to define an auxiliary field,
which we call \textit{conjugate RSF}:

\begin{equation}
    c := \sum_{k,k'=1}^{N} \text{Tr} \left[ \hat{\rho} \hat{a}_k \hat{a}_{k'} \right] |k\rangle \langle k'|,
\end{equation}

\begin{equation}
    |\alpha^*\rangle := \sum_{k=1}^{N} \text{Tr} \left[ \hat{\rho} \hat{a}_k^\dagger \right] |k\rangle.
\end{equation}

The key observation is that RSF is related to the first two moments of the mode quadratures by

\begin{equation}
    r = TVT^\dagger - \frac{1}{2}, \quad |\alpha\rangle = T|\xi\rangle,
\end{equation}

\begin{equation}
    c = TVT^T, \quad |\alpha^*\rangle = T^*|\xi\rangle.
\end{equation}

Furthermore, the transfer matrix fulfills

\begin{equation}
    T^\dagger T = \frac{1}{2} \left( I + iJ \right).
\end{equation}

Notably, \( T^\dagger T + T^T T^* = I \). Making extensive use of this identity in eq. (24), along with relations (A2), we obtain the corresponding evolution equations for RSF:

\begin{equation}
    \frac{d}{dt} r = yr + ry^\dagger + yc^\dagger + cz^\dagger + \frac{1}{2} (y + y^\dagger) + w
    + \int \mu(dk) \left[ qrg^\dagger + sr^\dagger s^\dagger \right] - r + qcs^\dagger + sc^\dagger q^\dagger
    + \frac{1}{2} \left( qq^\dagger + ss^\dagger - I_N \right),
\end{equation}

\begin{equation}
    \frac{d}{dt} |\alpha\rangle = y|\alpha\rangle + z|\alpha^*\rangle + \int \mu(dk) \left[ (q - 1)|\alpha\rangle + s|\alpha^*\rangle \right].
\end{equation}

(A4)

Here,

\begin{equation}
    y := TAT^\dagger, \quad z := TAT^T, \quad w := TJR_C J^T T^\dagger
    \quad q := TKT^\dagger, \quad s := TKT^T.
\end{equation}

(A5)

Unlike the reduced kinetic equations, Gaussian evolution equation for RSF couples it to the conjugate field. Therefore, if the two equations are to coincide for arbitrary input states, \( c \) cannot enter eq. (A4). This has two implications. Firstly, \( z = 0 \) and in turn \( 0 = T^\dagger z T^* \), which is equivalent to

\begin{equation}
    0 = \left[ J, A \right].
\end{equation}

(A6)

Using transposition we immediately find this condition to be identical to eq. (28). Secondly, \( s = 0 \), which is equivalent to the l.h.s. condition in eq. (29). The r.h.s. condition of eq. (29) follows immediately after upon comparison with the reduced kinetic equations.

It remains to show that the reduced kinetic equations are given by eq. (31). Provided eqs (28-29) are fulfilled, eq. (A4) becomes

\begin{equation}
    \frac{d}{dt} r = -\frac{i}{\hbar} \left[ -hy_i, r \right] + \frac{1}{2} \left( (w + y_r) - (w - y_r), r \right)
    + (w + y_r) + \int \mu(dk) \left( qqr^\dagger - r \right),
\end{equation}

\begin{equation}
    \frac{d}{dt} |\alpha\rangle = -\frac{i}{\hbar} \left[ -hy_i, |\alpha\rangle \right] + \frac{1}{2} \left( (w + y_r) - (w - y_r), |\alpha\rangle \right)
    + \int \mu(dk) \left( q|\alpha\rangle - |\alpha\rangle \right),
\end{equation}

(A7)

where we split \( y = y_r + iy_i \) with \( y_r, y_i \) hermitian (this can be done for any complex matrix). Equations (A7) have the same form as the reduced kinetic equations with

\begin{equation}
    h = -hy_i, \quad \gamma = \pm y_r, \quad |\xi\rangle = 0,
    \quad u = q, \quad \mu(du) = \mu(dk),
\end{equation}

(A8)

Using eqs (A5, 28) we arrive at the desired result (31).

\section*{Appendix B: Derivation of reduced \textit{Wehrl} entropy}

In this Appendix, we derive the reduced \textit{Wehrl} entropy (44), defined as the maximum \textit{Wehrl} entropy among all the states with a fixed RSF. The solution to the problem is equivalent to finding the extremum of the following functional with respect to \( Q \):

\begin{equation}
    S_W[Q] - \lambda f[Q] - \sum_{k,k'=1}^{N} \mu_{k,k'} g_{kk'}[Q]
    + \sum_{k=1}^{N} t^*_k h_k[Q] + \sum_{k=1}^{N} s_k h^*_k[Q],
\end{equation}

(B1)

where \( S_W \) is the \textit{Wehrl} entropy (42) and the three constraints

\begin{equation}
    f[Q] := \int \frac{d^N \beta}{\pi^N} \beta^\dagger \beta - 1 = 0,
\end{equation}

\begin{equation}
    g_{kk'}[Q] := \int \frac{d^N \beta}{\pi^N} \left( \beta_k \beta_{k'}^\dagger - \delta_{kk'} \right) Q(\beta) - r_{kk'} = 0,
\end{equation}

\begin{equation}
    h_k[Q] := \int \frac{d^N \beta}{\pi^N} \beta_k Q(\beta) - \alpha_k = 0,
\end{equation}

(B2)

fix the normalization and the RSF of the state to \( (r, |\alpha\rangle) \) [cf. eq. (43)]. Finally, \( \lambda, \mu_{k,k'}, t_k \) and \( s_k \) are the Lagrange multipliers. Note that the signs, as well as the notation (e.g. \( t^*_k \) instead of \( t_k \)) in eq. (B1) are completely arbitrary. Therefore, we made a choice that anticipates the final results best.

The solution to the variational problem is given by

\begin{equation}
    \tilde{Q}(\bar{\beta}) := Ae^{-\beta^\dagger \mu \beta + \bar{\beta}^\dagger \beta + \bar{\beta} \bar{\beta}} z,
\end{equation}

(B3)
where $A$ is a normalization constant. Substituting the solution into the three constraints (B2) and making use of the integration formula [40]

$$\int \frac{d^{2N} \vec{\beta}}{\pi^N} e^{-\vec{\beta}^\dagger \mu \vec{\beta} + \vec{\gamma} \vec{\beta} + \vec{\beta}^\dagger \vec{s}} = \frac{1}{\det \mu} e^{\vec{\gamma} \mu^{-1} \vec{s}}, \quad (B4)$$

eq. (B4) yields

$$A = \det \mu e^{-\vec{\gamma} \mu^{-1} \vec{s}}, \quad \mu^{-1} = r_{\alpha} + I_N, \quad \vec{r} = \vec{s} = \mu \vec{\alpha} \quad (B5)$$

and in turn

$$\hat{Q}(\vec{\beta}) = \frac{1}{\det (r_{\alpha} + I_N)} e^{-(\vec{\beta} - \vec{\alpha})^\dagger (r_{\alpha} + I_N)^{-1} (\vec{\beta} - \vec{\alpha})}. \quad (B6)$$

Plugging this into the definition of Wehrl entropy (42) leads to eq. (44).