Abstract

We analyze in the four-generation model the first measurement of the branching ratio of rare kaon decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, using the constraints from $\Delta m_K, \varepsilon_K, B_d - \bar{B}_d$ mixing, $\Gamma(b \rightarrow s\gamma), B_s - \bar{B}_s$ mixing, $D^0 - \bar{D}^0$ mixing, $B(K_L \rightarrow \pi^0 \nu \bar{\nu})$ and $B(K_L \rightarrow \mu \bar{\mu})$, and study its effects on the unitarity triangle. With the results of searching for the maximum mixing of the fourth generation, we predict that $D^0 - \bar{D}^0$ mixing $\Delta m_D$ and the branching ratio of "direct" CP-violating decay process $K_L \rightarrow \pi^0 \nu \bar{\nu}$ could attain the values 1-2 orders of magnitude larger than the predictions of the Standard Model.

1e-mail: hattori@ias.tokushima-u.ac.jp
2e-mail: hasuike@anan-nct.ac.jp
3e-mail: wakaizum@medsci.tokushima-u.ac.jp
Recently, the branching ratio of the flavor-changing neutral current (FCNC) process, $K^+ \rightarrow \pi^+\nu\bar{\nu}$, has been measured for the first time by the USA-Japan-Canada Collaboration at the Brookhaven National Laboratory, and it has turned out to be $B = (4.2^{+0.7}_{-3.5}) \times 10^{-10}$ [1]. The central value seems to be 4-6 times larger than the predictions of the Standard Model $B = (0.6 - 1.5) \times 10^{-10}$ [2].

This process had already been studied by Gaillard and Lee in 1974 and they obtained a branching ratio of $\sim 10^{-10}$ by using the "short-distance" $W - W$ box and $Z^0$-penguin diagrams in the "4-quark" model [3]. After that in 1981, Inami and Lim obtained the rigorous expressions for these and other related diagrams relevant to the FCNC processes and studied the effects of superheavy quarks and leptons in $K_L \rightarrow \mu\bar{\mu}$, $K^+ \rightarrow \pi^+\nu\bar{\nu}$ and $K^0 - \bar{K}^0$ mixing [4], before the top-quark is discovered.

In this work, we analyze the new branching ratio of $K^+ \rightarrow \pi^+\nu\bar{\nu}$ in the four-generation model [5], since the above-mentioned factor 4-6 of the new measurement seems to imply the existence of a fourth generation with roughly the same mixing as for the third generation. We will search for the maximum mixing for the "hypothetical" fourth generation by imposing the constraints from $\Delta m_K$, $\varepsilon_K$, $B_d - \bar{B}_d$ mixing, $\Gamma(b \rightarrow s\gamma)$, $B_s - \bar{B}_s$ mixing, $D^0 - \bar{D}^0$ mixing, $B(K_L \rightarrow \pi^0\nu\bar{\nu})$ and $B(K_L \rightarrow \mu\bar{\mu})$, and study its effects on CP violation in neutral $B$ meson decays and the unitarity triangle.

For the unitary $4 \times 4$ quark mixing matrix, we will use the Hou-Soni-Steger parametrization [6], which has a simple form in the third column; $(V_{ub}, V_{cb}, V_{tb}) = (s_x c_u e^{-i\phi_1}, s_y c_z c_u, c_y c_z c_u)$, in the fourth row; $(V_{t'd}, V_{t's}, V_{t'\beta}, V_{t'\gamma}) = (-c_u c_v s_w e^{i\phi_3}, -c_u s_v e^{i\phi_2}, -s_u, c_u c_v c_w)$ and $V_{us} = s_x c_u c_v - s_z s_u s_v e^{i(\phi_2 - \phi_1)}$, where the three mixing angles $s_x (\equiv \sin \theta_x)$, $s_y$ and $s_z$ give the elements $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$, respectively as in the Standard Model, the phase $\phi_1$ corresponds to the Kobayashi-Maskawa(KM) CP-violating phase $\delta_{KM}$ [7], and $s_u, s_v$ and $s_w$ are the new mixing angles and $\phi_2$.
and $\phi_3$ are the new phases, $t'$ and $b'$ being the fourth generation up- and down-quark, respectively.

As an input, we use the following values\[^{[2]}\]

\begin{align}
    s_x &= 0.22, \\
    s_y &= 0.040 \pm 0.003, \\
    s_z/s_y &= 0.08 \pm 0.02, \quad (1)
\end{align}

in the same way as in the Standard Model, since the magnitude of the three elements $V_{us}, V_{cb}$ and $V_{ub}$ are experimentally determined from the semileptonic decays of hyperons and $B$ mesons, and the existence of a fourth generation would not affect the determination. We search for the maximum mixing of the fourth generation by testing the three cases of $(s_w, s_v, s_u) = (\lambda^4, \lambda^3, \lambda^2), (\lambda^3, \lambda^2, \lambda)$ and $(\lambda^2, \lambda^2, \lambda)\[^{[8]}\]$, where $\lambda \equiv 0.22 \simeq \sin \theta_C$ is the expansion parameter used in the Wolfenstein parametrization of the $3 \times 3$ KM matrix. The constraints we impose on the model are the following; $K_L - K_S$ mass difference $\Delta m_K = (3.522 \pm 0.016) \times 10^{-12}$ MeV\[^{[3]}\], CP-violating parameter in the neutral kaon system $\varepsilon_K = (2.28 \pm 0.02) \times 10^{-3} \[^{[4]}\]$, $\Delta m_{B_d} = (3.12 \pm 0.20) \times 10^{-10}$ MeV\[^{[5]}\]$ for $B_d - \bar{B}_d$ mixing, $B(b \to s\gamma) = (2.32 \pm 0.67) \times 10^{-4} \[^{[6]}\]$ for the inclusive radiative $b$ decay, $B(K^+ \to \pi^+\nu\bar{\nu}) = (4.2^{+9.7}_{-3.5}) \times 10^{-10} \[^{[7]}\]$, $x_s > 10.5 \[^{[1]}\]$ for $B_s - \bar{B}_s$ mixing strength, $\Delta m_D < 1.4 \times 10^{-10}$ MeV\[^{[12]}\]$ for $D^0 - \bar{D}^0$ mixing, $B(K_L \to \pi^0\nu\bar{\nu}) < 5.8 \times 10^{-5} \[^{[13]}\]$ and $B(K_L \to \mu^+\mu^-)_{\text{SD}} < 4.4 \times 10^{-9}$, where the short-distance(SD) contribution to $B(K_L \to \mu^+\mu^-)$ is taken to be the value two times larger than the one by Bélanger and Geng\[^{[14]}\] as a loose constraint.

Each of the above-mentioned nine constraints is studied in the following.

(i) $K_L - K_S$ mass difference, $\Delta m_K$

The short-distance part of $\Delta m_K$ comes from the well-known $W - W$ box diagram with $c, t$ and $t'$ as internal quarks and the contribution is expressed, for example, for the box with two $c$-quarks as follows,

\begin{align}
    \Delta m_K(c, c) &= \frac{G_F^2 M_W^2}{12\pi^2} f_K^2 B_K m_K \text{Re}[V_{us} V_{cd}^*] \eta_{cc} S(x_c), \quad (2)
\end{align}
where $S(x)$ is the Inami-Lim box function, $x_c \equiv m_c^2/M_W^2$, $m_c$ being the charm-quark mass, $\eta^K_{cc}$ is the QCD correction factor including the next-to-leading order effects, and $f_K$ and $B_K$ are the decay constant and bag parameter of the kaon, respectively. By taking for these parameters the values of $m_c = 1.3$ GeV, $\eta^K_{cc} = 1.38$, $f_K = 0.16$ GeV and $B_K = 0.75 \pm 0.15$, we obtain from the inputs of eq.(1) the $(c,c)$ contribution $\Delta m_K(c,c) = (2.6 - 3.9) \times 10^{-12}$ MeV, which is already consistent by itself with the measured value. Numerically, the $(c,t)$ and $(t,t)$ contributions are very small as compared with the $(c,c)$ contribution, so we take a constraint for the fourth-generation contributions to be

$$\frac{\Delta m_K(c,t) + \Delta m_K(t,t') + \Delta m_K(t',t')}{\Delta m_K(c,c)} < 1$$

as a loose constraint, since there are a large amount of long-distance contributions.

(ii) CP-violating parameter in neutral kaon system, $\varepsilon_K$

The quantity $\varepsilon_K$ is expressed by the imaginary part of hadronic matrix element of the effective Hamiltonian with $\Delta S = 2$ between $K^0$ and $\bar{K}^0$, to which the short-distance contribution comes from the $W - W$ box diagram as in $\Delta m_K$. The box contribution with $c$ and $t$ quarks gives an expression of

$$\varepsilon_K(c,t) = \frac{1}{\sqrt{2} \Delta m_K} \frac{G_F M_W^2}{6 \pi^2} f_K^2 B_K m_K \text{Im}[V_{cs} V_{ct}^* V_{ts} V_{tt}^*] \eta^K_{ct} S(x_c, x_t).$$

If we take the QCD correction factor including the next-to-leading order as $\eta^K_{ct} = 0.47$, the dominant term in the $(c,t)$-box contribution leads to $\varepsilon_K(c,t) \simeq 2.83 \times 10^{-3} B_K \sin \phi_1$ for $m_t = 180$ GeV, where $\phi_1$ is the CP-violating phase. Since this magnitude of $\varepsilon_K(c,t)$ is close to the measured value, we take the constraint from $\varepsilon_K$ that the sum of the contributions from $c,t$ and $t'$ quarks should be within the $1\sigma$ error of the measured value,

$$\sum_{i,j=c,t,t', i \leq j} \varepsilon_K(i,j) = (2.28 \pm 0.02) \times 10^{-3}.$$
The theoretical uncertainty in the bag parameter $B_K = 0.75 \pm 0.15$ is taken into consideration.

(iii) $B_d - \bar{B}_d$ mixing, $\Delta m_{B_d}$

The mass difference between the two mass-eigenstates of $B_d - \bar{B}_d$ system is given by the $W - W$ box diagram, and the $(t,t)$-box contribution is expressed by

$$\Delta m_{B_d}(t,t) = \frac{G_F^2 M_W^2}{12\pi^2} f_B B_B m_{B_d} |V_{tb} V_{td}^*|^2 \eta_{tt}^B S(x_t),$$

where $f_B$ and $B_B$ are the decay constant and the bag parameter for $B_d$ meson, respectively, and $\eta_{tt}^B$ is the QCD correction factor including the next-to-leading order effects. By taking for these parameters the values of $\sqrt{B_B} f_B = (0.20 \pm 0.04)$ GeV[2] and $\eta_{tt}^B = 0.55[2]$ and by using the inputs of eq.(1), we obtain the $(t,t)$ contribution; $\Delta m_{B_d}(t,t) = (1.75 - 3.95) \times 10^{-10}$ MeV, of which range includes the measured value. Since $(c,c)$ and $(c,t)$ contributions are numerically very small as compared with the $(t,t)$ contribution, we take the constraint from $\Delta m_{B_d}$ that the sum of the contributions from $t$ and $t'$ should be within the 1$\sigma$ error of the measured value, $\Delta m_{B_d} = (3.12 \pm 0.20) \times 10^{-10}$ MeV[9].

(iv) $B(b \to s\gamma)$

The dominant contribution to the inclusive radiative $b$ decay, $b \to s\gamma$, comes from the electromagnetic penguin diagram with $t$- and $t'$-quark exchange in the four-generation model. The partial decay width is given by[13]

$$\Gamma(b \to s\gamma) = \frac{\alpha G_F^2 m_b^5}{128\pi^3} |V_{tb} V_{ts}^* c_7(m_b) + V_{t'b} V_{t's}^* c'_7(m_b)|^2,$$

where $\alpha$ is the fine-structure constant and $c_7(m_b)$ and $c'_7(m_b)$ are the Wilson coefficients for the electromagnetic dipole operator, calculated via leading-logarithmic evolution equation with the electromagnetic penguin functions at the electroweak scale[3] down to the renormalization scale $\mu = m_b(= 4.5$ GeV)[16] for the $t$- and $t'$-exchange diagrams, respectively. We take the constraint from $B(b \to s\gamma)$ that
the sum of $t$ and $t'$ contributions to the decay width of eq.(7) should be within the 1σ error of $\Gamma(b \rightarrow s\gamma) = (9.54 \pm 2.76) \times 10^{-17}$ GeV, calculated from the branching ratio and the lifetime of $B_d$ meson, $\tau_{B_d} = 1.60$ ps.[9].

(v) $B(K^+ \rightarrow \pi^+\nu\bar{\nu})$

The short-distance contributions to the rare decay $K^+ \rightarrow \pi^+\nu\bar{\nu}$ are from the $W-W$ box diagram and $Z^0$-penguin diagram. The branching ratio is given by

$$B(K^+ \rightarrow \pi^+\nu\bar{\nu}) = \kappa_+ \left| \frac{V_{cd}^* V_{cs}}{\lambda} P_0 + \frac{V_{td}^* V_{ts}}{\lambda^5} \eta_t X_0(x_t) + \frac{V_{t'd}^* V_{t's}}{\lambda^5} \eta_{t'} X_0(x_{t'}) \right|^2,$$

where $\kappa_+ = 4.57 \times 10^{-11}$, $P_0$ is the sum of charm contributions to the two diagrams including the next-to-leading order QCD corrections[17], $X_0(x_t)$ and $X_0(x_{t'})$ the sum of the $W-W$ box and $Z^0$-penguin functions for $t$- and $t'$-quark exchange[4], respectively, $\eta_t (= 0.985)$ is the next-to-leading order QCD correction to the $t$-exchange calculated by Buchalla and Buras[18] and we will take $\eta_{t'} = 1.0$ for $t'$-exchange. The constraint is that the branching ratio of eq.(8) should be consistent with the measured value of branching ratio $B = (4.2^{+9.7}_{-3.3}) \times 10^{-10}$[1], since the long-distance contribution is estimated to be very small ($B \sim 10^{-13}$)[19]. We do not assume the mixing in the leptonic sector.

(vi) $B_s - \bar{B}_s$ mixing, $x_s$

The dominant contribution to $B_s - \bar{B}_s$ mixing is the $W-W$ box diagram with $t$- and $t'$-exchange as in $B_d - \bar{B}_d$ mixing. We take the constraint that the sum of $(t,t), (t,t')$ and $(t',t')$ contributions to the mixing strength should be larger than the present experimental lower bound $x_s > 10.5$[11], where $x_s \equiv \Delta m_{B_s}/\Gamma_{B_s}$, $\Delta m_{B_s}$ being the mass difference of the two mass eigenstates of $B_s - \bar{B}_s$ system.

(vii) $D^0 - \bar{D}^0$ mixing, $\Delta m_D$

The dominant contribution to $D^0 - \bar{D}^0$ mixing in the four-generation model is the $W-W$ box diagram with fourth-generation down-quark $b'$ exchange[20]. We take the constraint that this contribution to the mass difference between the two mass-
eigenstates of $D^0 - \bar{D}^0$ system should be smaller than the present experimental upper bound\cite{12}, $\Delta m_D(b', b') < 1.4 \times 10^{-10}$MeV, since the Standard Model box contribution of two $s$-quarks exchange \cite{21} and the long-distance contributions\cite{22} are estimated to be three to four orders of magnitude smaller than the upper bound.

(viii) $B(K_L \to \pi^0 \nu\bar{\nu})$

The process $K_L \to \pi^0 \nu\bar{\nu}$ is a "direct" CP-violating decay \cite{23} and the rate is expressed by the imaginary part of sum of the same $W - W$ box and $Z^0$-penguin diagram amplitudes as in $K^+ \to \pi^+ \nu\bar{\nu}$\cite{3}. We take the constraint that the sum of $t$ and $t'$ contributions to the branching ratio should be smaller than the experimental upper bound\cite{13} $B(K_L \to \pi^0 \nu\bar{\nu}) < 5.8 \times 10^{-5}$.

(ix) $B(K_L \to \mu^0 \bar{\mu})_{SD}$

The process $K_L \to \mu^0 \bar{\mu}$ is a CP-conserving decay. The short-distance(SD) contribution is given by the $W - W$ box and $Z^0$-penguin diagrams and the branching ratio for this part is expressed as \cite{2}

$$B(K_L \to \mu^0 \bar{\mu})_{SD} = \kappa_{\mu} \left[ \frac{\text{Re} \left( V_{cd} V_{cs}^* \right)}{\lambda} P'_0 + \frac{\text{Re} \left( V_{cd} V_{ts}^* \right)}{\lambda^5} Y_0(x_t) + \frac{\text{Re} \left( V_{t'd} V_{t's}^* \right)}{\lambda^5} Y_0(x_{t'}) \right]^2,$$

where $\kappa_{\mu} = 1.68 \times 10^{-9}$, $P'_0$ the sum of charm contributions to the two diagrams including the next-to-leading order QCD corrections\cite{17} and $Y_0(x_t)$ and $Y_0(x_{t'})$ are the sum of the $W - W$ box and $Z^0$-penguin functions for $t$- and $t'$-exchange, respectively\cite{4}. We take the constraint that the branching ratio of eq.(9) should be smaller than the upper bound of the short-distance contribution as stated before, $B(K_L \to \mu^0 \bar{\mu})_{SD} < 4.4 \times 10^{-9}$.

In order to find the maximum mixing for the fourth generation consistent with the above nine constraints, we study the following three cases of $(|V_{t'd}|, |V_{t's}|, |V_{t'b}|) \simeq (s_w, s_v, s_u) = (\lambda^4, \lambda^3, \lambda^2), (\lambda^3, \lambda^2, \lambda)$ and $(\lambda^2, \lambda^2, \lambda)$, where $\lambda = 0.22$ is the Cabibbo
Table 1: Combinations of relevant quark mixing matrix elements for $\Delta m_B, b \to s \gamma, K^+ \to \pi^+ \nu\bar{\nu}$ and $(K_L \to \mu\bar{\mu})_{SD}$ for the third generation and the three cases of fourth generation mixing.

| Mixing          | $\Delta m_B$ | $b \to s \gamma$ | $K^+ \to \pi^+ \nu\bar{\nu}$ | $(K_L \to \mu\bar{\mu})_{SD}$ |
|-----------------|--------------|-------------------|-------------------------------|-----------------------------|
| $(V_{td}, V_{ts}, V_{tb})$ | $(\lambda^3, \lambda^2, 1)$ | $V_{td}V_{tb}$ | $V_{ts}V_{tb}$ | $V_{td}V_{ts}$ | $V_{td}V_{ts}$ |
| $(V'_{td}, V'_{ts}, V'_{tb})$ | $(\lambda^4, \lambda^3, \lambda^2)$ | $V'_{td}V'_{tb}$ | $V'_{ts}V'_{tb}$ | $V'_{td}V'_{ts}$ | $V'_{td}V'_{ts}$ |
| $(\lambda^3, \lambda^2, \lambda)$ | $\lambda^5$ | $\lambda^5$ | $\lambda^5$ | $\lambda^5$ |
| $(\lambda^2, \lambda^2, \lambda)$ | $\lambda^4$ | $\lambda^4$ | $\lambda^4$ | $\lambda^4$ |

We tentatively take the mass of the fourth generation quarks $(t', b')$ as $m_{t'} = 400$ GeV and $m_{b'} = 350$ GeV so as to satisfy the constraints obtained from the analyses with the oblique parameters $S, T$ and $U$. Strong constraints come from $\Delta m_K, \varepsilon_K, B_d - \bar{B}_d$ mixing, $b \to s \gamma, K^+ \to \pi^+ \nu\bar{\nu}$ and $(K_L \to \mu\bar{\mu})_{SD}$. In the Standard Model, the largest contribution comes from the top-quarks for $B_d - \bar{B}_d$ mixing, $b \to s \gamma, K^+ \to \pi^+ \nu\bar{\nu}$ and $(K_L \to \mu\bar{\mu})_{SD}$, and the combination of the relevant quark mixing matrix elements is $V_{td}V_{tb} \sim \lambda^3$ for $B_d - \bar{B}_d$ mixing, $V_{ts}V_{tb} \sim \lambda^2$ for $b \to s \gamma$, and $V_{td}V_{ts} \sim \lambda^5$ for $K^+ \to \pi^+ \nu\bar{\nu}$ and $(K_L \to \mu\bar{\mu})_{SD}$. The combinations of the corresponding matrix elements for $t'$-quark are shown in Table 1 for each of the above three cases. By comparing these combinations between the Standard Model and the four-generation model, the numerical analyses give the following results: the case of $(\lambda^4, \lambda^3, \lambda^2)$ gives almost the same predictions to the above-mentioned nine processes as in the Standard Model and the contributions of the fourth generation are very small. So, this case is not interesting. For the case of $(\lambda^3, \lambda^2, \lambda)$, almost all the processes satisfy the constraints with only one exception of $B(K_L \to \mu\bar{\mu})_{SD}$, for which this mixing gives a value almost seven times larger than the upper bound. The last case of $(\lambda^2, \lambda^2, \lambda)$ predicts too large values for $B(K^+ \to \pi^+ \nu\bar{\nu})$ and $B(K_L \to \mu\bar{\mu})_{SD}$. 

8
Table 2: Comparison of $B(K^+ \to \pi^+ \nu \bar{\nu})$, $x_s(B_s - \bar{B}_s$ mixing), $\Delta m_D$ and $B(K_L \to \pi^0 \nu \bar{\nu})$ among the experimental values, Standard Model(SM) predictions and four-generation model predictions with maximum mixing.

|                  | $B(K^+ \to \pi^+ \nu \bar{\nu})$ | $x_s$ | $\Delta m_D$(MeV) | $B(K_L \to \pi^0 \nu \bar{\nu})$ |
|------------------|-----------------------------------|-------|-------------------|-----------------------------------|
| Experiment       | $(4.2^{+9.7}_{-3.5}) \times 10^{-10}$ | $>10.5$ | $<1.4 \times 10^{-10}$ | $<5.8 \times 10^{-5}$ |
| SM               | $(0.6 - 1.5) \times 10^{-10}$ | $19 - 27$ | $\sim 10^{-14}$ | $(1.1 - 5.0) \times 10^{-11}$ |
| 4-generation     | $(0.7 - 4.4) \times 10^{-10}$ | $19 - 29$ | $(0.7 - 2.1) \times 10^{-12}$ | $(0.05 - 10) \times 10^{-10}$ |

These results imply that the mixing $(\lambda^3, \lambda^2, \lambda)$ is a little large for the fourth generation and it turns out that a mixing with $s_w$ and $s_v$ reduced by 20%, that is, $(s_w, s_v, s_u) = (0.8\lambda^3, 0.8\lambda^2, \lambda)$ satisfies all of the nine constraints as a maximum mixing.

We can obtain the following predictions from this maximum mixing; the branching ratio of $K^+ \to \pi^+ \nu \bar{\nu}$ takes a range from the Standard Model(SM) values to the central value of the new measurement as $B = (0.7 - 4.4) \times 10^{-10}$, the strength for $B_s - \bar{B}_s$ mixing is $19 \leq x_s \leq 29$, $\Delta m_D$ of $D^0 - \bar{D}^0$ mixing could have a value $(0.7 - 2.1) \times 10^{-12}$ MeV, about two orders of magnitude larger than the SM prediction ($\sim 10^{-14}$ MeV[21]), and the branching ratio of $K_L \to \pi^0 \nu \bar{\nu}$ takes a range of $(0.05 - 10) \times 10^{-10}$, from the SM values to the ones two orders of magnitude larger than the SM prediction ($(1.1 - 5.0) \times 10^{-11}$[2]). These results are summarized in Table 2. The branching ratios of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ are correlated with each other as shown in Fig.1 for the maximum mixing, the area of the correlation resulting from the freedom of the three phases $\phi_1, \phi_2$ and $\phi_3$. In the region of $1.2 \times 10^{-10} \leq B(K^+ \to \pi^+ \nu \bar{\nu}) \leq 4.5 \times 10^{-10}$, the correlation is around the line $B(K_L \to \pi^0 \nu \bar{\nu}) = 4.5B(K^+ \to \pi^+ \nu \bar{\nu}) - 1.3 \times 10^{-10}$, which is caused by the positive collaboration of third and fourth generations. On the other hand, in the region of $0.7 \times 10^{-10} \leq B(K^+ \to \pi^+ \nu \bar{\nu}) \leq 1.2 \times 10^{-10}$, an interference of second- and third-generation contributions with fourth-generation
ones brings this range of SM values of \( B(K^+ \to \pi^+ \nu \bar{\nu}) \) and a broad range of \( B(K_L \to \pi^0 \nu \bar{\nu}) = (0.05 - 4) \times 10^{-10} \).

The maximum mixing gives an interesting effect on CP-asymmetry of the decay rates of the "gold-plate" mode of \( B_d \) meson, \( B_d \to J/\psi K_S \). The asymmetry is given by

\[
C_f = \frac{\Gamma(B_d \to J/\psi K_S) - \Gamma(\bar{B}_d \to J/\psi K_S)}{\Gamma(B_d \to J/\psi K_S) + \Gamma(\bar{B}_d \to J/\psi K_S)},
\]

and it is expressed as\[^{25}\]

\[
C_f = -\frac{x_d}{1 + x_d^2} \text{Im} \Lambda, \quad \Lambda \equiv \sqrt{\frac{M_{12}^* A(\bar{B}_d \to J/\psi K_S)}{M_{12} A(B_d \to J/\psi K_S)}},
\]

where \( x_d \) is the mixing strength for \( B_d - \bar{B}_d \) mixing, \( M_{12} \) the off-diagonal element of the mass matrix in \( B_d - \bar{B}_d \) system and \( A \) is the decay amplitude. In the Standard Model\[^{26}\], the quantity \( C_f \) takes a positive sign as \( 0.18 \leq C_f \leq 0.37 \), which results from the phase range \( 0 < \phi_1 < \pi \), constrained from the positive sign of \( \varepsilon_K \). However, in the four-generation model\[^{27}\], \( C_f \) can take also a negative sign as \( -0.38 \leq C_f \leq 0.40 \), since the phase \( \phi_1 \) can take the whole range of \( 0 < \phi_1 < 2\pi \) due to the two more new phases \( \phi_2 \) and \( \phi_3 \) and the maximum mixing of the fourth generation. Although in the four-generation model the penguin diagrams could affect the decay amplitude, they would cause at most several percent change of the value of \( C_f \), even if they happen to have a magnitude as large as 50\% of the tree amplitude.

Second, the unitarity triangle in the Standard Model transforms into unitarity quadrangle in the four-generation model\[^{28}\]. For the maximum mixing obtained here, some of the typical quadrangles are shown in Fig.2. The fourth side of the quadrangle, \( V_{td}V_{tb}^* \), is of order \( \lambda^4 \), while the other three sides are of order \( \lambda^3 \). The first example of Fig.2(a) is for positive value of \( C_f \). The second one of Fig.2(b) is for negative value of \( C_f \) and the quadrangle is reversed with respect to the base line of \( V_{cd}V_{cb}^* \), since \( \phi_1 > \pi \).
Summarizing, we find a maximum mixing of the fourth generation \((V'_{td}, V'_{ts}, V'_{tb}) \simeq (0.8\lambda^3, 0.8\lambda^2, \lambda)\), which is consistent with the nine constraints from \(\Delta m_K, \varepsilon_K, B_d - \bar{B}_d\) mixing, \(b \to s\gamma, K^+ \to \pi^+\nu\bar{\nu}, B_s - \bar{B}_s\) mixing, \(D^0 - \bar{D}^0\) mixing, \(K_L \to \pi^0\nu\bar{\nu}\) and \(K_L \to \mu\bar{\mu}\). The mass difference \(\Delta m_D\) from \(D^0 - \bar{D}^0\) mixing and the branching ratio of \(K_L \to \pi^0\nu\bar{\nu}\) could reach the values two orders of magnitude larger than the Standard Model predictions, and CP asymmetry of the decay rates of \(B_d \to J/\psi K_S\) could take a value of opposite sign to the SM one.

We are grateful to Takeshi Komatsubara, Minoru Tanaka, Takeshi Kurimoto, Xing Zhi-Zhong, Masako Bando, C.S. Lim, and Morimitsu Tanimoto for helpful discussions.

References

[1] S. Adler et al., Phys. Rev. Lett. 79, 2204 (1997).

[2] G. Buchalla, A.J. Buras and M.E. Lautenbacher, Rev. Mod. Phys. 68, 1125 (1996), and references therein.

[3] M.K. Gaillard and B.W. Lee, Phys. Rev. D 10, 897 (1974).

[4] T. Inami and C.S. Lim, Prog. Theor. Phys. 65, 297 (1981).

[5] U. Türke, Phys. Lett. B 168, 296 (1986); I.I. Bigi and S. Wakaizumi, Phys. Lett. B 188, 501 (1987); G. Eilam, J.L. Hewett and T.G. Rizzo, Phys. Lett. B 193, 533 (1987).

[6] W.-S. Hou, A. Soni and H. Steger, Phys. Lett. B 192, 441 (1987).

[7] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).

[8] T. Hayashi, M. Tanimoto and S. Wakaizumi, Prog. Theor. Phys. 75, 353 (1986).
[9] Particle Data Group, R.M. Barnett et al., Phys. Rev. D 54, 1 (1996).

[10] M.S. Alam et al., Phys. Rev. Lett. 74, 2885 (1995).

[11] W. Adam et al., Phys. Lett. B 414, 382 (1997).

[12] E.M. Aitala et al., Phys. Rev. Lett. 77, 2384 (1996).

[13] M. Weaver et al., Phys. Rev. Lett. 72, 3758 (1994).

[14] G. Bélanger and C.G. Geng, Phys. Rev. D 43, 140 (1991).

[15] J.L. Hewett, Phys. Lett. B 193, 307 (1987); Report "Top ten models constrained by $b \rightarrow s\gamma$, SLAC-PUB-6521 (May 1994).

[16] B. Grinstein, R. Springer and M.B. Wise, Phys. Lett. B 202, 138 (1988).

[17] G. Buchalla and A.J. Buras, Nucl. Phys. B 412, 106 (1994).

[18] G. Buchalla and A.J. Buras, Nucl. Phys. B 398, 285 (1993); Nucl Phys. B 400, 225 (1993).

[19] D. Rein and L.M. Sehgal, Phys. Rev. D 39, 335 (1989).

[20] K.S. Babu, X.-G. He, X.-Q. Li and S. Pakvasa, Phys. Lett. B 205, 540 (1988).

[21] A. Datta and D. Kumbhakar, Z. Phys. C 27, 515 (1985).

[22] L. Wolfenstein, Phys. Lett. B 164, 170 (1985); J. Donoghue, E. Golowich, B.R. Holstein and J. Trampetić, Phys. Rev. D 33, 179 (1986).

[23] L.S. Littenberg, Phys. Rev. D 39, 3322 (1989).

[24] T. Inami, T. Kawakami and C.S. Lim, Mod. Phys. Lett. A 10, 1471 (1995).

[25] A.B. Carter and A.I. Sanda, Phys. Rev. D 23, 1567 (1981); I.I. Bigi and A.I. Sanda, Nucl. Phys. B 193, 85 (1981).
[26] I. Dunietz and J.L. Rosner, Phys. Rev. D 34, 1404 (1986).

[27] T. Hasuike, T. Hattori, T. Hayashi and S. Wakaizumi, Mod. Phys. Lett. A 4, 2465 (1989); Phys. Rev. D 44, 3582 (1991).

[28] Y. Nir and H.R. Quinn, in B Decays, edited by S. Stone (world Scientific, Singapore, 1992).
Figure captions

Fig.1. The correlation of $B(K^+ \to \pi^+ \nu \bar{\nu})$ and $B(K_L \to \pi^0 \nu \bar{\nu})$ for the maximum mixing $(s_w, s_v, s_u) = (0.8\lambda^2, 0.8\lambda^2, \lambda)$ in the four-generation model. The hatched area is the allowed region for the branching ratios. The rectangle surrounded by the dashed lines is the prediction of the Standard Model.

Fig.2. Typical examples of the unitarity quadrangle. (a) $\phi_1 = \frac{1}{3}\pi, \phi_2 = \frac{11}{6}\pi, \phi_3 = \frac{19}{12}\pi; C_f(B_d \to J/\psi K_S) = 0.39, B(K^+ \to \pi^+ \nu \bar{\nu}) = 2.1 \times 10^{-10}$, (b) $\phi_1 = \frac{10}{12}\pi, \phi_2 = \pi, \phi_3 = \frac{3}{4}\pi; C_f(B_d \to J/\psi K_S) = -0.35, B(K^+ \to \pi^+ \nu \bar{\nu}) = 2.7 \times 10^{-10}$. 
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9804412v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9804412v1