Unique prime factorization for infinite tensor product factors

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Abstract

In this article, we investigate a unique prime factorization property for infinite tensor product factors. We provide several examples of type II and III factors which satisfy this property, including all free product factors with diffuse free product components. In the type III setting, this is the first classification result for infinite tensor product non-amenable factors. Our proof is based on Popa’s intertwining techniques and a characterization of relative amenability on the continuous cores.

1 Introduction

The tensor product construction is a fundamental tool in the von Neumann algebra theory. It has been used to construct interesting examples of von Neumann algebras. In particular, infinite tensor product factors $\otimes_{n \in \mathbb{N}}(M_n, \varphi_n)$, where $M_n$ are type I factors equipped with faithful normal states $\varphi_n$, attracted strong attention since it appears in the quantum field theory. We call such factors Araki–Woods factors. All Araki–Woods factors are classified in terms of $(M_n, \varphi_n)$ [Po67, AW68] and this led to the celebrated classification of amenable factors due to Connes [Co75] (see also [Kr75, Ha85]).

It is then natural to consider a classification problem of infinite tensor products constructed from non-amenable factors. More precisely, we are interested in thinking about a classification of factors $\otimes_{n \in \mathbb{N}}(M_n, \varphi_n)$ in terms of non-amenable factors $(M_n, \varphi_n)$.

To investigate this problem, we should require some rigidity of $M_n$. Indeed any (non-type I) infinite tensor product factor $\otimes_{n \in \mathbb{N}}(M_n, \varphi_n)$ is known to be McDuff, meaning that it is stable under taking tensor products with the hyperfinite II$_1$ factor $R$. Then using the decomposition $R = \otimes_{n \in \mathbb{N}}(R, \tau)$ where $\tau$ is the trace on $R$, one has

$$\otimes_{n \in \mathbb{N}}(M_n, \varphi_n) \cong \otimes_{n}(M_n, \varphi_n) \otimes R \cong \otimes_{n \in \mathbb{N}}(M_n \otimes R, \varphi_n \otimes \tau).$$

Observe that tensor components determine up to tensor products with $R$. Thus it is not easy to pick up information of $M_n$ directly. To avoid this situation, in this article, we will assume that each $M_n$ is a prime factor; meaning that for any tensor decomposition $M_n = P \otimes Q$, we have either $P$ or $Q$ is of type I. In this case, $M_n$ is not isomorphic to $M_n \otimes R$ and we may treat $M_n$ as tensor components. We mention that in the Araki–Woods factor case, all type I factors $M_n$ are prime by definition.

Here we briefly review the study of prime factors and related results. Examples of prime factors were first discovered by Popa [Po83] and then by Ge [Ge96], in which they proved that any free group factor $L\mathbb{F}_n$ ($n \geq 2$, possibly infinite) is prime. Ozawa established a completely new and much simpler proof, using $C^*$-algebraic techniques [Oz03]. Based on this Ozawa’s new proof and combined with Popa’s intertwining techniques (see Section

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Ozawa and Popa obtained a remarkable structural theorem for tensor product factors [OP03]. They proved that, whenever we consider a tensor product of finitely many free group factors, then the resulting tensor product factor remembers its tensor components in the following precise sense. Let $M_i$ and $N_j$ be some free group factors and assume that $\otimes_{i=1}^n M_i$ and $\otimes_{j=1}^m N_j$ are stably isomorphic for some $n, m \in \mathbb{N}$, then $n = m$ and, after permutation of indices, $M_i$ and $N_j$ are stably isomorphic for all $i$. Here $P$ and $Q$ are stably isomorphic if $P \otimes \mathbb{B}(\ell^2)$ and $Q \otimes \mathbb{B}(\ell^2)$ are isomorphic.

This should be called the unique prime factorization for free group factors. Thus the classification of such tensor product factors is completely reduced to the one of each tensor component. This is a complete answer for the aforementioned classification problem for tensor products of finitely many free group factors.

Many new examples of type II and III factors satisfying the unique prime factorization have been discovered then. However, all of such results treat only tensor products of finitely many tensor components. The first example of unique prime factorization for infinite tensor product factors are given by ourselves [Is16b], but they are all type II$_1$ factors. The aim of this article is to investigate the unique prime factorization for infinite tensor product factors that include type III factors. It is a more challenging problem since infinite tensor product factors $\otimes_{n \in \mathbb{N}} (M_n, \varphi_n)$ depend on the choice $\varphi_n$, and this dependence does not appear in both the finite tensor product case and the infinite tensor product II$_1$ factor case.

To introduce our main theorem, we need to prepare some notation and terminology. We say that an inclusion $B \subset M$ of von Neumann algebras is with expectation if there is a faithful normal conditional expectation from $M$ onto $B$. We say that a von Neumann algebra $M$ admits a large centralizer if there is a finite von Neumann algebra $N \subset M$ with expectation such that $N' \cap M \subset N$. We define the following class of factors which satisfy a practical condition. See Section 8 and 9 for symbols $\preceq$ and $\preceq$ respectively.

**Definition 1.1.** We say that a factor $M$ is in the class $\mathcal{P}$ if it is a separable factor with a large centralizer and it satisfies the following condition:

- for any separable factors $B$, $P$, $Q$ such that $B \bar{\otimes} M = P \bar{\otimes} Q$ and that $P, Q$ have large centralizers, we have either $P \preceq_{\mathbb{M}} B$ or $Q \preceq_{\mathbb{M}} B$.

We say that a factor $M$ is semiprime if for any tensor decomposition $M = P \bar{\otimes} Q$, we have either $P$ or $Q$ is amenable. In the main theorem, we will use semiprimeness instead of primeness, thus we will consider semiprime (and non-amenable) factors which are in the class $\mathcal{P}$. As concrete examples, we will show that the following factors are semiprime, non-amenable, and contained in the class $\mathcal{P}$. See Section 4 for details.

- Any non-amenable factor that satisfies condition $(AO)^+$ and has the W*CBAP.
- Any free product factor $(M_1, \varphi_1) * (M_2, \varphi_2)$, where each $M_i$ is a diffuse von Neumann algebra with a faithful normal state $\varphi_i$.

We mention that in the finitely many tensor components case, the unique prime factorization of condition (AO) factors are proved in [OP03, Is14, HI15] and the one of free product II$_1$ factors are proved in [Pe06].

Now we introduce the main theorem of this article.

**Theorem A.** Let $X, Y \subset \mathbb{N}$ be subsets and let $M_m$ and $N_n$ be non-amenable separable factors for $m \in X$ and $n \in Y$. Assume that each $M_m$ is in the class $\mathcal{P}$ and that each $N_n$ admits a large centralizer. If there are faithful normal states $\varphi_m$ on $M_m$, $\psi_n$ on $N_n$, amenable separable factors $M_0$ and $N_0$ (which are possibly trivial) such that

$$M := \otimes_{m \in X} (M_m, \varphi_m) \bar{\otimes} M_0 \simeq \otimes_{n \in Y} (N_n, \psi_n) \bar{\otimes} N_0,$$

then
then there is an injective map \( \sigma : Y \to X \) such that \( M_{\sigma(n)} \preceq_M N_n \) for all \( n \in Y \).

If we further assume that all \( N_n \) are semiprime, then \( \sigma \) is bijective and there are projections \( p_n \in M_{\sigma(n)} \), \( q_n \in N_n \) and amenable factors \( R_n \) such that

\[
p_n M_{\sigma(n)} p_n \otimes R_n \simeq q_n N_n q_n \quad \text{for all } n \in Y.
\]

By assuming that all factors are prime and belong to the class \( \mathcal{P} \), we obtain the following unique prime factorization result. This is the first classification result for infinite tensor product type III factors in the non-amenable setting. We mention that, regarding free product type III factors, it is new even for finite sets \( X, Y \).

**Corollary B.** Let \( X, Y \subset \mathbb{N} \) and let \( M_m \) and \( N_n \) be prime factors in the class \( \mathcal{P} \) for all \( m \in X \) and \( n \in Y \). The following statements are equivalent.

- There are faithful normal states \( \varphi_m \) on \( M_m \) and \( \psi_n \) on \( N_n \) and amenable factors \( M_0 \) and \( N_0 \) with separable preduals such that \( M_{\sigma(n)} \otimes M_0 \) and \( N_0 \otimes N_0 \) are stably isomorphic for all \( n \in Y \).

- There is a bijection \( \sigma : Y \to X \) such that \( M_{\sigma(n)} \) and \( N_n \) are stably isomorphic for all \( n \in Y \).

The organization of this paper is as follows. In Section 2 we recall some known facts on infinite tensor product factors and large centralizer conditions.

In Section 3 and Section A, we define and study relative amenability for general von Neumann algebras as a generalization of [AD93]. The main observation here is a characterization of relative amenability in terms of continuous cores (Theorem 3.2). Using this, we prove two lemmas (Lemma 3.7 and 3.8) for tensor product factors which are key ingredients of the proof of our main theorem.

In Section 4 we provide several examples of type II and III factors which are in the class \( \mathcal{P} \). They are proved by combinations of known techniques which are established in [Oz03, IPP05]. We will use variants of them introduced in [Is12a, Is16b, Io12, HU15b].

In Section 5 we prove the main theorem. Using the condition of the class \( \mathcal{P} \) and lemmas in Section 3 we essentially reduce our problem to tensor product factors with finitely many tensor components. Then using techniques developed in [Is13a, HI15] for type III factors, we will finish the proof.

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A Relative amenability for bimodules
Notation

Throughout the paper, we will use the following notation. Let $M$ be a von Neumann algebra and $\varphi$ a faithful normal semifinite weight on $M$. The modular operator, conjugation, and action are denoted by $\Delta_\varphi$, $J_\varphi$, and $\sigma_\varphi$ respectively. The continuous core is the crossed product von Neumann algebra $M \rtimes \sigma_\varphi \mathbb{R}$ and is denoted by $C_\varphi(M)$. The centralizer algebra $M_\varphi$ is a fixed point algebra of the modular action. The norm $\| \cdot \|_\infty$ is the operator norm of $M$, while $\| \cdot \|_{2,\varphi}$ is the $L^2$-norm by $\varphi$. The GNS representation of $\varphi$ is denoted by $L^2(M,\varphi)$ and sometimes we omit $\varphi$ regarding as a standard representation. See [Ta01] for definitions of these objects.

For a tensor product von Neumann algebra $M \otimes N$, we always regard $M$ and $N$ as subalgebras in $M \otimes N$ via identifications $M = M \otimes C \subset M \otimes N$, $N = C \otimes N \subset M \otimes N$. For a von Neumann subalgebra $A \subset M$ with unit $1_A$, we will write as $A \otimes N$ the von Neumann subalgebra (with unit $1_A \otimes 1_N$) generated by $a \otimes x$ for $a \in A$ and $x \in N$.

2 Preliminaries

In this section, we recall basic properties of infinite tensor product factors. We particularly focus on the large centralizer condition of them. All results in this section should be known to experts but we could not find them in the literature. So we include all proofs for reader’s convenience.

We refer the reader to [Co72] and [Ta01, Chapter XII] for definitions and basic facts of type III$_\lambda$ factors for $0 \leq \lambda < 1$.

Lemma 2.1. The following statements hold true.

1. Any semifinite factor and any type III$_\lambda$ factor for $0 \leq \lambda < 1$ admit large centralizers.

2. For any type III$_1$ factor with separable predual, $M$ admits a large centralizer if and only if there is a faithful normal state $\varphi$ on $M$ such that $M_\varphi' \cap M = \mathbb{C}$.

Proof. For the first statement, the finite factor case is trivial. For the semifinite and infinite case, we have only to observe that $\mathbb{B}(\ell^2(I))$ for any set $I$ admits an atomic masa $\ell^\infty(I)$ that is with expectation. For the type III case, Connes proved that any type III$_\lambda$ factor for some $0 \leq \lambda < 1$ has a maximal abelian subalgebra with expectation [Co72, THÉORÈM 4.2.1(a) and 5.2.1(a)].

For the second statement, if $M$ admits a large centralizer, then it has a maximal abelian subalgebra $A \subset M$ with expectation [Po81, Theorem 3.3]. Then the conclusion holds by [HI15, Corollary 3.6].

Lemma 2.2. Let $M$ be a $\sigma$-finite factor of not type III$_1$ and $\varphi$ a faithful normal state on $M$. Then for any $\varepsilon > 0$, there exist a matrix unit $\{e_{i,j}\}_{i,j=1}^n$ in $M$ (possibly $n = \infty$) with the decomposition $M = eMe \otimes \mathbb{B}(\ell^2_n)$, where $e := e_{1,1}$, and faithful normal states $\psi$ on $eMe$ and $\omega$ on $\mathbb{B}(\ell^2_n)$ such that $\| \varphi - \psi \otimes \omega \| < \varepsilon$, where $n$ and $\psi$ are taken as:

- if $M$ is a type II$_1$ factor, then $n < \infty$ and $\psi$ is the trace on $eMe$;

- if $M$ is a type II$_\infty$ factor, then $n = \infty$, $e$ is a finite projection, and $\psi$ is the trace on $eMe$;
• if $M$ is a type $\text{III}_\lambda$ factor for some $0 < \lambda < 1$, then $n < \infty$ and $\psi$ is a generalized trace in the sense that $(eMe)^\prime_\psi \cap eMe = \mathbb{C};$

• if $M$ is a type $\text{III}_0$ factor, then $n = 1$, $(eMe)_\psi$ is of type $\Pi_1$ and $(eMe)^\prime_\psi \cap eMe \subset (eMe)_\psi$.

**Proof.** We first study the type $\text{III}_\lambda$ case. Since the $\text{T}$-set of $M$ is $2\pi \mathbb{Z}/\log(\lambda)$, by [Co72] THÉORÈM 1.3.2 there is a faithful normal state $\psi$ and a positive invertible operator $h \in \mathcal{Z}(M_\psi)$ such that $\psi = \varphi(h \cdot)$ and $\sigma_\lambda^\psi = \text{id}$, where $T := 2\pi/\log(\lambda)$. By [Co72] THÉORÈM 4.2.6, it holds that $M_\psi' \cap M = \mathbb{C}$ and so $M_\psi$ is a type $\Pi_1$ factor. Observe $h^{-1} \in L^1(M_\psi, \psi)$ since $\psi(h^{-1}) = \varphi(1) = 1 < \infty$. We can find a family of mutually equivalent and orthogonal projections $(\epsilon_i)_{i=1}^\infty$ from spectral projections of $h^{-1}$ such that $\|h^{-1} - \sum_{i=1}^n \epsilon_i\|_{1,\psi} < \varepsilon$ for some $\mu_i > 0$ (possibly $\mu_i = \mu_j$). Observe that

$$\left\| \sum_{i=1}^n \psi(\epsilon_i \cdot) - \varphi \right\| = \left\| \psi \left( \sum_{i=1}^n \epsilon_i \cdot \right) - \psi(h^{-1} \cdot) \right\| < \varepsilon.$$

Let $\{\epsilon_{i,j}\}_{i,j=1}^n$ be a matrix unit in $M_\psi$ such that $\epsilon_{i,i} = e_i$ for all $i$. Then putting $\omega$ as the vector functional by $\sum_{i=1}^n \mu_i^{1/2} e_i$ (which is well defined by $\| \sum_i \mu_i^{1/2} e_i \|_{2,\psi} = \psi(\sum_i \mu_i e_i) \sim \varphi(1) = 1$), one has

$$(\psi|_{eMe} \otimes \omega)(x) = \sum_{i=1}^n \mu_i \psi(\epsilon_{i,1} x \epsilon_{i,1}) = \sum_{i=1}^n \mu_i \psi(\epsilon_{i,1} \epsilon_{i,1} x) = \sum_{i=1}^n \mu_i \psi(\epsilon_i x)$$

for all $x \in M$ and therefore

$$\|\psi|_{eMe} \otimes \omega - \varphi\| = \left\| \sum_{i=1}^n \psi(\mu_i \epsilon_i \cdot) - \varphi \right\| < \varepsilon.$$

Finally since $|\psi|_{eMe} \otimes \omega(1) - 1| < \varepsilon$, up to normalizing $\psi|_{eMe} \otimes \omega$ and up to replacing $\varepsilon$ small, we obtain the desired matrix unit and states.

Type $\Pi_2$ factor cases follow from the same argument as in the type $\text{III}_\lambda$ case, since any $\varphi$ is a perturbation of a trace.

Finally we study the type $\text{III}_0$ factor case. By [Co72] THÉORÈM 5.2.1(a), any faithful normal state $\psi$ on $M$ satisfies $M_\psi' \cap M \subset M_\psi$. So we only study the property that $M_\psi$ is of type $\Pi_1$.

By [Co72], LEMME 5.2.4, there is a projection $e \in M_\varphi$ and an invertible positive element $h \in eM_\varphi e$ such that $\psi_e := \varphi(h \cdot)$ is lacunary on $eMe$, that is, 1 is isolated in the spectrum of $\Delta_\psi$. In this case $(eMe)_\psi$ is of type $\Pi_1$ (indeed, Connes proved that $(eMe_\psi \times \mathbb{B}(l^2))_\psi \otimes \text{Tr}$ is of type $\Pi_\infty$, see the last part of the proof of [Co72] THÉORÈM 5.3.1)). By replacing $h$ if necessary, we may assume that $\psi_e(e) = \varphi(e)$. Using Zorn’s lemma, take mutually orthogonal projections $(\epsilon_i)_i$ and self-adjoint elements $h_i \in e_i Me_i$ for each $i$ such that $\sum_i e_i = 1$ and each $e_i$ and $h_i$ are as above. Define $k := \sum_i h_i$ as an unbounded operator affiliated with $M_\varphi$ and a faithful normal state $\psi$ on $M$ given by

$$\psi(x) := \varphi(kx) = \sum_i \varphi(h_i x) = \sum_i \psi_{e_i}(x), \quad x \in M^+.$$

Observe that $M_\psi$ is of type $\Pi_1$ since it contains $\sum_i (e_i Me_i)_{\psi_{e_i}}$ as a unital subalgebra. Since $\psi(k^{-1}) = \varphi(1) = 1 < \infty$, $k^{-1}$ is contained in $L^1(M_\psi, \psi)$ and therefore there is
Lemma 2.3. always choose canonical states as \((\varphi^\infty)_n\) with given states \((\varphi,\psi_n)_n\). Let \(M_n\) be the algebraic tensor product of \(\{M_n\}_n\) which is a dense *-subalgebra in both of \(\overline{\varrho}\equiv\{M_n,\varphi_n\}\) and \(\varrho\equiv\{M_n,\psi_n\}\). Tensor product states \(\varphi:=\bigotimes_n\varphi_n\) and \(\psi:=\bigotimes_n\psi_n\) are well defined on \(M_0\). We show that \(\psi\) is well defined on \(\overline{\omega}\equiv\overline{\omega_n(M_n,\varphi_n)}\).

To see this, consider faithful normal states \(\omega_n\) on \(\overline{\omega_n(M_n,\varphi_n)}\) for \(n\in\mathbb{N}\) given by

\[
\omega_n := \psi_1\otimes\cdots\otimes\psi_n\otimes\varphi_{n+1}\otimes\varphi_{n+2}\otimes\cdots.
\]

Observe that for \(n < m\),

\[
\|\omega_n - \omega_m\| \leq \|\varphi_{n+1}\otimes\cdots\otimes\varphi_m - (\psi_{n+1}\otimes\cdots\otimes\psi_m)\| \\
\leq \sum_{k=n+1}^{m} \|\varphi_k - \psi_k\|.
\]

So \((\omega_n)_n\) is a Cauchy sequence in the predual of \(\overline{\omega_n(M_n,\varphi_n)}\) and converges to a normal state \(\omega\). By construction, this coincides with \(\psi\) on \(M_0\). This means \(\psi\) is well defined on \(\overline{\omega_n(M_n,\varphi_n)}\).

Finally applying the GNS construction for \(\psi\), we have a *-homomorphism

\[
\pi: \overline{\omega_n(M_n,\varphi_n)} \rightarrow \overline{\omega_n(M_n,\psi_n)}
\]

which is the identity on \(M_0\). By exchanging the roles of \(\varphi\) and \(\psi\), we get an inverse map of \(\pi\) and therefore \(\pi\) is a desired *-isomorphism.

The following proposition clarifies relations between infinite tensor product factors with given states \((\varphi_n)_n\) and the one with canonical states. As the proposition says, we can always choose canonical states as \((\varphi_n)_n\), up to tensor products with Araki–Woods factors.

**Proposition 2.4.** The following statements hold true.

1. If all \(M_n\) are type III\(_1\) factors, then the infinite tensor product \(\overline{\omega_n(M_n,\varphi_n)}\) does not depend on the choice of \((\varphi_n)_n\).

2. Let \((\lambda_n)_n \in (0,1]^{\mathbb{N}}\). If each \(M_n\) is a type III\(_{\lambda_n}\) factor for \(0 < \lambda_n < 1\), then there are faithful normal states \(\psi_n\) on \(M_n\) for all \(n\in\mathbb{N}\) such that \((M_n)^{\psi_n} \cap M_n = C\) and

\[
\overline{\omega_n(M_n,\varphi_n)} \simeq \overline{\omega_n(M_n,\psi_n)} \overline{\omega} R,
\]

where \(R\) is an Araki–Woods factor (possibly trivial).
(3) If all $M_n$ are type III$_0$ factors, then there are faithful normal states $\psi_n$ on $M_n$ for all $n \in \mathbb{N}$ such that $(M_n)_{\psi_n}$ is of type I$_1$, $(M_n)_{\psi_n}' \cap M_n \subset (M_n)_{\psi_n}'$ and

\[ \overline{\otimes}_{n \in \mathbb{N}} (M_n, \varphi_n) \cong \overline{\otimes}_{n \in \mathbb{N}} (M_n, \psi_n). \]

(4) If all $M_n$ are type II$_\infty$ factors, then there are finite projections $p_n \in M_n$ for all $n \in \mathbb{N}$ such that

\[ \overline{\otimes}_{n \in \mathbb{N}} (M_n, \varphi_n) \cong \overline{\otimes}_{n \in \mathbb{N}} (p_n M_n p_n, \tau_n) \otimes R, \]

where $\tau_n$ are traces and $R$ is an Araki–Woods factor (which must be properly infinite).

(5) If all $M_n$ are type II$_1$ factors, then there are projections $p_n \in M_n$ for all $n \in \mathbb{N}$ such that

\[ \overline{\otimes}_{n \in \mathbb{N}} (M_n, \varphi_n) \cong \overline{\otimes}_{n \in \mathbb{N}} (p_n M_n p_n, \tau_n) \otimes R, \]

where $\tau_n$ are traces and $R$ is an Araki–Woods factor (possibly trivial).

Proof. Statement (1) is a straightforward consequence of Lemma 2.3 and Connes–Størmer’s transitivity [CS76, THEOREM 4], while all others are of Lemma 2.3 and 2.2. Note that for (2), we need the fact that every projections in a $\sigma$-finite type III factor are equivalent.

We have two corollaries. We will use the first one in the proof of the main theorem.

Corollary 2.5. If $M_n$ has a large centralizer for all $n \in \mathbb{N}$, then $\overline{\otimes}_{n \in \mathbb{N}} (M_n, \varphi_n)$ has a large centralizer.

Proof. Put $(M, \varphi) := \overline{\otimes}_{n \in \mathbb{N}} (M_n, \varphi_n)$. Since any Araki–Woods factor has a large centralizer, by Proposition 2.4 we may assume that $(M_n)_{\varphi_n}' \cap M_n \subset (M_n)_{\varphi_n}'$ for all $n \in \mathbb{N}$. Then it is easy to see that $M_{\varphi}' \cap M \subset M_{\varphi}$.

Corollary 2.6. If $M_n$ has separable predual for all $n \in \mathbb{N}$, then $\overline{\otimes}_{n \in \mathbb{N}} (M_n, \varphi_n)$ is McDuff (unless it is of type I).

Proof. Since any (non-type I) Araki–Woods factor is McDuff, by Proposition 2.4 we may assume that $(M_n)_{\varphi_n}$ is of type II$_1$ for all $n \in \mathbb{N}$. Then the conclusion follows easily.

3 Relative amenability for subalgebras

In this section, we define and study relative amenability for general inclusions of von Neumann algebras. The goal of this section is to prove two lemmas, which are necessary for our main theorem. For this, we prove a characterization of relative amenability in terms of continuous cores. Since results in Appendix will be used, we refer the reader to the appendix section before starting this section.

The following definition is a generalization of [OP07] in which they treat only finite von Neumann algebras.

Definition 3.1. Let $B \subset M$ be von Neumann algebras, $p \in M$ a projection and $A \subset pMp$ a von Neumann subalgebra.

(1) Let $z$ be the central support projection of $p$ in $M$. We say that $A$ is semidiscrete relative to $B$ in $M$ if we have

\[ M_z L^2(zMp)_A \prec M_z L^2(zM) \otimes_{B_z} L^2(zMp)_A. \]
Let \( E_A : M \to A \) be a faithful normal conditional expectation. We say that the pair \((A, E_A)\) is injective relative to \( B \) in \( M \) if there exists a conditional expectation from \( p(M, B)p \) onto \( A \) which restricts to \( E_A \) on \( pMp \). In this case we write as \((A, E_A) \preceq_M B\).

Observe that \((A, E_A)\) is injective relative to \( M \) in \( B \) if and only if the \( pMp-B\)-bimodule \( pL^2(M) \) is left \((A, E_A)\)-injective (see Appendix). Also \( A \) is semidiscrete relative to \( M \) in \( B \) if and only if \( pMp-B\)-bimodule \( L^2(pM) \) is left \( A\)-semidiscrete (to see the if part, use Lemma [A.2]). So Definition 3.1 is a special case of Definition [A.1].

In item (1) above, the projection \( z \) is necessary to get injectivity of the left \( M \)-action of the bimodule \( _ML^2(Mp)_A \). We sometimes write this condition as, by omitting \( z \),

\[
ML^2(Mp)_A \preceq_M L^2(M) \otimes_B L^2(Mp)_A.
\]

In item (2) above, as will be explained in Remark [3.3] the relative injectivity does not depend on the choice of \( E_A \) if \( B \subset M \) is with operator valued weight, that is, there is a faithful normal operator valued weight from \( M \) onto \( B \). We refer the reader to [Ha77a, Ha77b] for the theory of operator valued weights. In this case, we will simply write as \( A \preceq_M B \).

For \( M, B, p, A, E_A \) as in Definition [3.1](2) and assuming \( B \subset M \) is with operator valued weight \( E_B \), we will use the following notation. Let \( \varphi_B \) and \( \psi_A \) be faithful normal semifinite weights on \( B \) and \( A \) respectively, and put \( \varphi := \varphi_B \circ E_B \) and \( \psi := \psi_A \circ E_A \). We further extend \( \psi \) on \( M \) by adding a faithful normal semifinite weight on \((1-p)M(1-p)\), so that \( \sigma_t^\psi(p) = p \) for all \( t \in \mathbb{R} \). We have \( \sigma_t^\varphi|_B = \sigma_t^{\varphi_B} \) and \( \sigma_t^\psi|_A = \sigma_t^{\psi_A} \) for all \( t \in \mathbb{R} \), and therefore there are inclusions

\[
C_\varphi(B) \subset C_\varphi(M), \quad C_\psi(A) \subset C_\psi(pMp).
\]

Note that the second inclusion depends only on \( \psi|_{pMp} \). Let \( \Pi_{\varphi,\psi} : C_\psi(M) \to C_\varphi(M) \) be the canonical \(*\)-isomorphism, which is the identity on \( M \).

The following theorem establishes the equivalence of the relative injectivity of the inclusion and the one in the continuous core. Condition (2) below is particularly important to us and will be used later in this section. We note that condition (4) below is new, since it does not appear when \( A = M \).

**Theorem 3.2.** Let \( M, B, p, A, E_A \) be as in Definition [3.1](2) and assume that \( B \subset M \) is with operator valued weight \( E_B \). Then using the notation introduced above, the following conditions are equivalent.

1. We have \((A, E_A) \preceq_M B\).
2. We have \( C_\psi(A) \preceq_{C_\varphi(M)} B \).
3. We have \( \Pi_{\varphi,\psi}(C_\varphi(A)) \preceq_{C_\varphi(M)} C_\varphi(B) \).
4. There is a ucp map \( \Psi : p(M, B)p \to \langle pMp, A \rangle \) such that \( \Psi(x) = x \) for all \( x \in pMp \).

**Proof.** Observe first that if the central support projection \( z \) of \( p \) in \( M \) is not 1, then all statements in this theorem is equivalent to the same statements but for the inclusions \( A \subset pMzp \) and \( Bz \subset Mz \). Hence up to replacing \( z \) with \( 1_M \), without loss of generality, we may assume \( z = 1 \).

Before starting the proof, we mention that, since there is an operator valued weight \( E_B \), there is also an operator valued weight from \( \langle M, B \rangle \) into \( M \). This follows from [Ha77b Theorem 5.9],

(1)\(\iff\)(2) This is exactly the equivalence of (1) and (2) in Theorem [A.7] by using Lemma [A.2].
Proof. Let $\Psi: \mathcal{L}_{C_\varphi}(B)(\rho L^2(C_\varphi(M))) \rightarrow \langle pC_\varphi(M)p, \Pi_{\varphi,\psi}(C_\varphi(A)) \rangle$ such that $\Psi(x) = x$ for all $x \in pC_\varphi(M)p$. Observe that there are identifications

$$\langle pC_\varphi(M)p, \Pi_{\varphi,\psi}(C_\varphi(A)) \rangle \simeq \langle C_\psi(pMp), C_\psi(A) \rangle = \langle pMp, A \rangle \times_\beta \mathbb{R},$$

where $\alpha_t = \text{Ad} \Delta^A_\psi$ and $\beta_t = \text{Ad} \Delta^B_\psi$ for $t \in \mathbb{R}$, and they canonical contain $p\langle M, B \rangle p$ and $\langle pMp, A \rangle$ respectively. By restriction, we have a map $\Psi: \langle p(M, B)p \rangle \rightarrow \langle pMp, A \rangle \times_\beta \mathbb{R}$ such that $\Psi(x) = x$ for all $x \in pMp$. Finally composing this $\Psi$ with a conditional expectation from $\langle pMp, A \rangle \times_\beta \mathbb{R}$ onto $\langle pMp, A \rangle$ and we get a desired ucp map.

(4)$\Rightarrow$(1) This is trivial by composing the compression map by the Jones projection of $E_A$.

Remark 3.3. In this theorem, condition (4) does not depend on the choice of $E_A$. Hence under the assumption that $B \subset M$ is with operator valued weight, the relative injectivity does not depend the choice of $E_A$. More precisely, if $(A, E_A) \lessdot_M B$ for some $E_A$, then we have $(A, E'_A) \lessdot_M B$ for any other faithful normal conditional expectation $E'_A$.

The following corollary is an immediate consequence of condition (4) above. It is a generalization of [OP07, Proposition 2.4(3)]. Our proof here is much simpler and can be applied to non tracial von Neumann algebras.

Corollary 3.4. Let $B \subset M$ and $A \subset pMp$ be von Neumann algebras with expectations $E_A, E_B$ and let $N \subset M$ be a von Neumann subalgebra with an operator valued weight. If $(A, E_A) \lessdot_M B$ and $(B, E_B) \lessdot_N N$, then $(A, E'_A) \lessdot_M N$.

Proof. Let $\Psi: \langle N, M \rangle \rightarrow \langle M, B \rangle$ and $\Phi: \langle p(M, B)p \rangle \rightarrow \langle pMp, A \rangle$ be ucp maps as in Theorem 3.2(4). Then the composition $\Phi \circ \Psi$ works.

We also prove the following useful properties.

Proposition 3.5. Let $M, B, p, A, E_A$ be as in Definition 3.1(2) and assume that $B \subset M$ is with operator valued weight. If $(A, E_A) \lessdot_M B$, then there is a conditional expectation $E: \langle p(M, B)p \rangle \rightarrow A$ which restricts to $E_A$ on $M$ and which is approximated by normal ccp maps from $\langle p(M, B)p \rangle$ to $A$ in the point $\sigma$-weak topology.

Proof. Since the inclusion $pMp \subset \langle p(M, B)p \rangle$ is with operator valued weight, we can apply Corollary 3.1(2) and get the conclusion.

Corollary 3.6. For $i = 1, 2$, let $M_i, B_i, p_i, A_i, E_{A_i}$ be as in Definition 3.1(2) and assume that $B_i \subset M_i$ is with operator valued weight. If $(A_i, E_{A_i}) \lessdot_{M_i} B_i$ for $i = 1, 2$, then we have $(A_1 \otimes A_2, E_{A_1} \otimes E_{A_2}) \lessdot_{M_1 \otimes M_2} B_1 \otimes B_2$.

Proof. By the previous proposition, for each $i$, take a net of normal ccp maps $\varphi_{\lambda_i}$ from $p_i(M_i, B_i)p_i$ to $A_i$ which converges to a conditional expectation whose restriction is $E_{A_i}$ on $M_i$. As normal ccp maps on $p_i(M_i)p_i$, consider duals $\varphi^*_{\lambda_i}: (p_i(M_i)p_i)_s \rightarrow (p_i(M_i)p_i)_s$ and then, up to convex combinations, we may assume that $\|\varphi^*_{\lambda_i}(\omega) - (E_{A_i})^*(\omega)\| \rightarrow 0$ for all $\omega \in (p_i(M_i)p_i)_s$. Since each $\varphi_{\lambda_i}$ is normal, we can define a net of normal ccp maps $\varphi_{\lambda_1} \otimes \varphi_{\lambda_2}$ from $p_1(M_1, B_1)p_1 \otimes p_2(M_2, B_2)p_2$ to $A_1 \otimes A_2$. Let $\Phi$ be a cluster point of
\(\varphi_\lambda \otimes \varphi_\mu\) in the point \(\sigma\)-weak topology. Then an easy computation, together with the above convergence condition on \((p_i M_i p_i)_*,\) implies that \(\Phi|_{p_1 M_1 p_1 \otimes p_2 M_2} = E_{A_1} \otimes E_{A_2}\). Hence \(\Phi\) is a conditional expectation onto \(A_1 \otimes A_2\) which restricts to \(E_{A_1} \otimes E_{A_2}\). Finally using the identification

\[
p_1(M_1, B_1)p_1 \otimes p_2(M_2, B_2)p_2 = (p_1 \otimes p_2)(M_1 \otimes M_2, B_1 \otimes B_2)(p_1 \otimes p_2),
\]

we get the conclusion. \(\square\)

**Some lemmas for tensor product factors**

We next prove two lemmas for tensor product factors. They are indeed key lemmas for the proof of the main theorem. We will use condition (2) of Theorem 3.2.

Let \(X \subset \mathbb{N}\) and let \(M_n\) be von Neumann algebras with faithful normal states \(\varphi_n\) for \(n \in X\). Put \((M, \varphi) := \bigotimes_{n \in X}(M_n, \varphi_n)\). For any subset \(F \subset X\), we write

\[M_F := \bigotimes_{n \in F}M_n \subset M, \quad M_F^\perp := \bigotimes_{n \in X \setminus F}M_n \subset M.\]

Observe that \(M = M_F \overline{\otimes} M_F^\perp\) for any \(F \subset X\). Let \(p \in M\) be a projection and \(P \subset pMp\) a von Neumann subalgebra with expectation \(E_p\). Let \(\psi\) be a faithful normal state on \(M\) such that \(\psi \circ E_p = \psi\) on \(pMp\) and \(p \in M_\psi\). Put \(\tilde{F} := P \subset pMp\) and \(\tilde{M} := C_\varphi(M)\) and \(\tilde{M}_F := C_\varphi(M_F)\). We write as \(\text{Tr}\) the canonical semifinite trace on \(\tilde{M}\).

The first lemma is a variant of [PV11, Proposition 2.7]. Since their proof does not work for non-finite von Neumann algebras, we prove it with a different way under a much stronger assumption.

**Lemma 3.7.** Keep the notation and assume \(X = \{1, 2, 3\}\). If \((P, E_P)\) is injective relative to both \(Q_1 := M_1 \otimes \mathbb{C} \otimes M_3\) and \(Q_2 := \mathbb{C} \otimes M_2 \otimes M_3\) in \(M\), then \((P, E_P)\) is injective relative to \(Q_1 \cap Q_2 = \mathbb{C} \otimes \mathbb{C} \otimes M_3\) in \(M\).

**Proof.** As in the proof of Theorem 3.2, we may assume the central support of \(p\) in \(M\) is 1. Then by Theorem 3.2(2), AD93 and Lemma AD2, our assumption is equivalent to

\[\begin{align*}
\overline{\tilde{M}}L^2((\tilde{M}, Q_1)) & \prec \overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \otimes_{\overline{\tilde{M}}} L^2((\tilde{M}, Q_1)) = \overline{\tilde{M}}L^2((\tilde{M}, Q_1)p), \\
\overline{\tilde{M}}L^2((\tilde{M}, Q_2)) & \prec \overline{\tilde{M}}L^2((\tilde{M}, Q_2)) \otimes_{\overline{\tilde{M}}} L^2((\tilde{M}, Q_2)p).
\end{align*}\]

for \(i = 1, 2\). Using AD93 Lemma 1.7, we apply \(\overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \otimes_{\overline{\tilde{M}}} \overline{\tilde{M}}L^2((\tilde{M}, Q_2))\) from the left side and get that

\[\overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \otimes_{\overline{\tilde{M}}} \overline{\tilde{M}}L^2((\tilde{M}, Q_2)) \prec \overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \otimes_{\overline{\tilde{M}}} L^2((\tilde{M}, Q_2)p).\]

Observe that, as \(\tilde{M} - \tilde{P}\)-bimodules, the left hand side satisfies

\[
\begin{align*}
\overline{\tilde{M}}L^2((\tilde{M}, Q_1)) & \otimes_{\overline{\tilde{M}}} \overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \\
& \simeq \overline{\tilde{M}}L^2((\tilde{M}) \otimes Q_1) \otimes_{\overline{\tilde{M}}} \overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \\
& \simeq \overline{\tilde{M}}L^2((\tilde{M}) \otimes Q_1) \otimes_{\overline{\tilde{M}}} \overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \\
& \simeq \overline{\tilde{M}}L^2((\tilde{M}) \otimes Q_1) \otimes_{\overline{\tilde{M}}} \overline{\tilde{M}}L^2((\tilde{M}, Q_2)p).
\end{align*}
\]

Hence we obtain

\[
\overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \prec \overline{\tilde{M}}L^2((\tilde{M}, Q_1)) \otimes_{\overline{\tilde{M}}} L^2((\tilde{M}, Q_2)p).
\]
Next we claim that the right hand side is actually a multiple of \( L^2(\widetilde{M}) \otimes_{M_3} L^2(\widetilde{M}) \). Indeed, by [Is16a, Proposition 2.3], as \( \widetilde{M} \)-bimodules, we have

\[
\begin{align*}
\mathbb{M} L^2(\langle \widetilde{M}, Q_1 \rangle) & \otimes_{\mathbb{M}} L^2(\langle \widetilde{M}, Q_2 \rangle) \\
& \simeq \mathbb{M} \left( L^2(\widetilde{M}) \otimes_{Q_1} L^2(\widetilde{M}) \right) \otimes_{\mathbb{M}} \left( L^2(\widetilde{M}) \otimes_{Q_2} L^2(\widetilde{M}) \right) \\
& \simeq \mathbb{M} L^2(\widetilde{M}) \otimes_{Q_1} \left[ L^2(\widetilde{M}) \otimes_{\mathbb{M}} \left( L^2(\widetilde{M}) \otimes_{Q_2} L^2(\widetilde{M}) \right) \right] \\
& \simeq \mathbb{M} L^2(\widetilde{M}) \otimes_{Q_1} \left[ L^2(\widetilde{M}) \otimes_{Q_2} L^2(\widetilde{M}) \right] \\
& \simeq \mathbb{M} L^2(\widetilde{M}) \otimes_{Q_1} \left[ L^2(\mathbb{R}) \otimes L^2(M_1) \otimes L^2(Q_2) \otimes L^2(M_1) \otimes L^2(\mathbb{R}) \right].
\end{align*}
\]

In the final line, we have a copy of

\[
L^2(\langle \widetilde{M}, Q_1 \rangle) = L^2(\widetilde{M}) \otimes_{Q_1} L^2(\mathbb{R}) \otimes L^2(M_1) \otimes L^2(Q_2).
\]

We again apply [Is16a, Proposition 2.3] to this part and then the above bimodule is isomorphic to

\[
\begin{align*}
& \simeq \mathbb{M} L^2(\mathbb{R}) \otimes L^2(M_2) \otimes L^2(Q_1) \otimes L^2(M_2) \otimes L^2(\mathbb{R}) \otimes L^2(M_1) \otimes L^2(\mathbb{R}) \\
& = \mathbb{M} L^2(\mathbb{R}) \otimes L^2(M_1) \otimes L^2(M_2) \otimes L^2(M_3) \otimes L^2(\mathbb{R}) \otimes L^2(M_2) \otimes L^2(M_1) \otimes L^2(\mathbb{R}).
\end{align*}
\]

Here we are using symbols \( \ell \) and \( r \) at the bottom of Hilbert spaces, which means the given left (resp. right) action acts on Hilbert spaces with the symbol \( \ell \) (resp. \( r \)). Note that there is no actions on \( L^2(\mathbb{R}) \), so we can ignore this part. We finally apply again [Is16a, Proposition 2.3] to this bimodule and then it is isomorphic to

\[
\simeq \bigoplus \mathbb{M} L^2(\widetilde{M}) \otimes_{M_3} L^2(\widetilde{M}_{\mathbb{M}}),
\]

where \( \bigoplus \) comes from the above \( L^2(\mathbb{R}) \) on which there is no actions. Thus the claim is proven and we obtain

\[
\mathbb{M} L^2(\widetilde{M}) \otimes_{M_3} L^2(\widetilde{M}_{\mathbb{M}}) \preceq \mathbb{M} L^2(\widetilde{M}) \otimes_{M_3} L^2(\widetilde{M}_{\mathbb{M}}).
\]

This exactly means \( \widetilde{P} \) is semidiscrete relative to \( M_3 \). By Theorem 3.2, this is equivalent to the conclusion.

The next lemma will be used to solve a problem that arises from infiniteness of tensor product components.

**Lemma 3.8.** Assume that \( X = \mathbb{N} \). If \((P,E_P)\) is injective relative to \( M^*_F \) for all finite subsets \( F \subset \mathbb{N} \setminus \{1\} \), then \((P,E_P)\) is injective relative to \( M_1 \).

**Proof.** As before, we may assume the central support of \( p \) in \( M \) is 1. Then by Theorem 3.2, A.5 and Lemma A.2, our assumption is equivalent to that \( \widetilde{P} \) is semidiscrete relative to \( M^*_F \) in \( \widetilde{M} \) for all finite subsets \( F \subset \mathbb{N} \setminus \{1\} \). We will show that \( \widetilde{P} \) is semidiscrete relative to \( M_1 \) in \( \widetilde{M} \), that is equivalent to the conclusion by Theorem 3.2.

To see this, we have only to show that \( \tau \widetilde{P} r \) is semidiscrete relative to \( M_1 \) for all Tr-finite projections \( r \in P \). So we will indeed prove the following more general statement: let \( p \in \widetilde{M} \) be a projection with \( \text{Tr}(p) < \infty \) and \( P \subset p \widetilde{M} p \) be a von Neumann subalgebra. If \( P \) is semidiscrete relative to \( M^*_F \) for all finite subsets \( F \subset \mathbb{N} \setminus \{1\} \), then \( P \) is semidiscrete relative to \( M_1 \).
Fix a finite subset \( \mathcal{F} \subset \mathbb{N} \setminus \{1\} \). By assumption we have a weak containment

\[
\tilde{M}L^2(\tilde{M}p)_P \prec \tilde{M}L^2(\tilde{M}) \otimes_{M_F} L^2(\tilde{M}p)_P,
\]

where we omit the support projection of \( p \), as explained before. Here we claim that \( L^2(\tilde{M}) \otimes_{M_F} L^2(\tilde{M}) \) is, as a \( M_{\mathcal{F} \cup \{1\}} \)-module, a multiple of \( L^2(\tilde{M}) \otimes_{M_1} L^2(\tilde{M}) \), so that we indeed obtain

\[
\tilde{M}_{\mathcal{F} \cup \{1\}}L^2(\tilde{M}p)_P \prec \tilde{M}_{\mathcal{F} \cup \{1\}}L^2(\tilde{M}) \otimes_{M_1} L^2(\tilde{M}p)_P.
\]

We prove the claim. Since \( M = M_{\mathcal{F}} \otimes M_F^c \), by \cite[Proposition 2.3]{ls16}, we have a canonical \( \tilde{M} \)-bimodule isomorphism

\[
L^2(\tilde{M}) \otimes_{M_F} L^2(\tilde{M}) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(M_{\mathcal{F}}) \otimes L^2(M_{\mathcal{F}}) \otimes L^2(\mathbb{R})
\]

\[
= L^2(\mathbb{R}) \otimes L^2(M_{\mathcal{F}}) \otimes L^2(M_{\mathcal{F}}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R})
\]

If we write as \( L^2(\mathbb{R}) \otimes H_1 \otimes H_2 \otimes H_3 \otimes H_4 \otimes L^2(\mathbb{R}) \) the Hilbert space in the final line, then the left \( \tilde{M}_{\mathcal{F} \cup \{1\}} \)-action is the one on \( L^2(\mathbb{R}) \otimes H_1 \otimes H_2 \) and the right-one is on \( H_2 \otimes H_3 \otimes H_4 \otimes L^2(\mathbb{R}) \). We also consider by \cite[Proposition 2.3]{ls16}

\[
L^2(\tilde{M}) \otimes_{M_1} L^2(\tilde{M}) = L^2(\mathbb{R}) \otimes L^2(M_{\mathcal{F} \cup \{1\}}) \otimes L^2(\mathbb{R}) \otimes L^2(M_{\mathcal{F} \cup \{1\}}) \otimes L^2(\mathbb{R})
\]

and observe that the difference of \( L^2(M) \otimes_{M_F} L^2(M) \) and \( L^2(M) \otimes_{M_{\mathcal{F}}} L^2(M) \) is only the component \( L^2(M_{\mathcal{F} \cup \{1\}}) \), on which there is no left-right actions as \( \tilde{M}_{\mathcal{F} \cup \{1\}} \)-module. Thus we obtain the desired result, and the claim is proven.

Since the resulting weak containment holds for all finite subsets \( \mathcal{F} \subset \mathbb{N} \setminus \{1\} \), if we put

\[
\tilde{M}_{\text{fin}} := \text{the norm closure of } \bigcup_{\mathcal{F} \subset \mathbb{N}, \text{ finite}} \tilde{M}_{\mathcal{F}} \subset \tilde{M},
\]

which is a \( C^* \)-algebra, then we have

\[
\tilde{M}_{\text{fin}}L^2(\tilde{M}p)_P \prec \tilde{M}_{\text{fin}}L^2(\tilde{M}) \otimes_{M_1} L^2(\tilde{M}p)_P.
\]

Let \( \pi \) denote the left \( \tilde{M} \)-action and \( \theta \) the right \( \tilde{P} \)-action on \( \tilde{M}_P \otimes_{M_1} \tilde{M}p_P \). Let \( \nu \) be the algebraic \( * \)-homomorphism corresponding to the above weak containment. We define an algebraic positive linear functional

\[
\Omega: *_{\text{alg}}\{\pi(\tilde{M}), \theta(\tilde{P}^{op})\} \to \mathbb{C}; \quad \Omega(a \otimes_{M_1} b^{op}) := \text{Tr}(ab), \quad a \in \tilde{M}, \quad b \in P.
\]

This is indeed a positive linear functional, since it is a composition of \( \nu \) and the vector state by \( p \in L^2(\tilde{M}) \). We know that \( \Omega \) is bounded on \( *_{\text{alg}}\{\pi(\tilde{M}_{\text{fin}}), \theta(\tilde{P}^{op})\} \). We claim that \( \Omega \) is bounded on the whole domain. By Lemma \cite[3]{A} this is equivalent to that \( P \) is semidiscrete relative to \( M_1 \) in \( \tilde{M} \), which is our conclusion.

We prove the claim. For any subset \( \mathcal{F} \subset \mathbb{N} \), let \( E_{\mathcal{F}} \) denotes the canonical conditional expectation from \( \tilde{M} \) onto \( \tilde{M}_{\mathcal{F}} \). Observe that \( \text{id}_{\tilde{M}} = \lim_{\mathcal{F}} E_{\mathcal{F}} \) in the point strong topology, where the limit is taken over all \( \text{finite} \) subsets of \( \mathbb{N} \). Using the fact \( E_{\mathcal{F}}(\tilde{M}) \subset \tilde{M}_{\text{fin}} \) and writing as \( C > 0 \) the bound of \( \Omega \) on the dense domain, we compute that for any \( a_i \in \tilde{M}, \)
Thus we obtain the boundedness of the desired map. \(\Box\)

4 Factors in the class \(\mathcal{P}\)

Popa’s intertwining techniques

We recall Popa’s intertwining techniques \cite{Po01, Po03}. We introduce the one formulated in \cite{HI15} for general \(\sigma\)-finite von Neumann algebras.

Definition 4.1. Let \(M\) be any \(\sigma\)-finite von Neumann algebra, \(1_A\) and \(1_B\) any nonzero projections in \(M\), \(A \subset 1_A M 1_A\) and \(B \subset 1_B M 1_B\) any von Neumann subalgebras with faithful normal conditional expectations \(E_A : 1_A M 1_A \to A\) and \(E_B : 1_B M 1_B \to B\) respectively.

We will say that \(A\) embeds with expectation into \(B\) inside \(M\) and write \(A \preceq_M B\) if there exist projections \(e \in A\) and \(f \in B\), a nonzero partial isometry \(v \in f M e\) and a unital normal *-homomorphism \(\theta : e A e \to f B f\) such that the inclusion \(\theta(e A e) \subset f B f\) is with expectation and \(v a = \theta(a)v\) for all \(a \in e A e\).

We prove some lemmas.

Lemma 4.2 (\cite{HI15} Lemma 4.9). Keep the notation as in the previous definition. If \(A \preceq_M B\), then \(B' \cap 1_B M 1_B \preceq_M A' \cap 1_A M 1_A\).

Lemma 4.3. Keep the notation as in the previous definition and let \(N_0 \subset N\) be any inclusion of \(\sigma\)-finite von Neumann algebras with expectation \(E_{N_0}\). Then \(A \preceq_M B\) if and only if \(A \otimes N_0 \preceq_{M \otimes N} B \otimes N\).

Proof. The case \(A\) finite is proved in \cite{HI15} Lemma 4.6]. Assume that \(A \preceq_M B\) and take \(p, q, \theta, v\) as in the definition. Then \(p \otimes 1, q \otimes 1, \theta \otimes \text{id}, v \otimes 1\) work for \(A \otimes N_0 \preceq_{M \otimes N} B \otimes N\).

Assume next that \(A \otimes N_0 \preceq_{M \otimes N} B \otimes N\). By \cite{BH16} Theorem 2(ii)], take a nonzero positive element \(d \in (A \otimes N_0)' \cap 1_A (M \otimes N, B \otimes N) 1_A\) such that \(d 1_A J 1_B J = d\) and \(T(d) \in M \otimes N\), where \(J\) is the modular conjugation for \(L^2(M \otimes N)\), \(\tilde{B}\) is the unitization of \(B\) in \(M\), and \(T\) is the operator valued weight from \((M \otimes N, B \otimes N)\) to \(M \otimes N\) corresponding to \(E_{\tilde{B}} \otimes \text{id}_N\). Let \(\psi\) be a faithful normal state on \(N\) such that \(\psi \circ E_{N_0} = \psi\). Observe that

\[
(A \otimes N_0)' \cap 1_A (M \otimes N, B \otimes N) 1_A = (A' \cap 1_A (M, B) 1_A) \otimes (N_0' \cap N)
\]
and hence \( d_0 := (\text{id} \otimes \psi)(d) \) is a nonzero positive element in \( A' \cap 1_A(M, B) 1_A \) satisfying \( d_0 1_A J_B J = d_0 \). Observe that \( \langle M \overline{\otimes} N, B \overline{\otimes} N \rangle = \langle M, B \rangle \overline{\otimes} N \) and \( T \) is of the form \( T_0 \otimes \text{id}_N \), where \( T_0 \) is the operator valued weight corresponding to \( E_B \). Hence we have

\[
T_0(d_0) = (T \otimes \psi)(d) = (\text{id} \otimes \psi)(T(d)) \in M.
\]

By [BH16, Theorem 2(ii)], we obtain \( A \precsim M B \). \( \square \)

**Lemma 4.4.** Let \( M, N \) be \( \sigma \)-finite von Neumann algebras, \( p \in M \) a projection and \( A \subset pMp \) a diffuse von Neumann subalgebra with expectation. Then we have \( A \not\precsim M \overline{\otimes} N \).

**Proof.** This is actually proved in the last part of the proof of [HI15, Theorem 5.6]. Since \( A \) is diffuse, there is a diffuse abelian von Neumann subalgebra \( A_0 \subset A \) with expectation. Using [HI15, Lemma 4.8], up to replacing \( A \) with \( A_0 \), we may assume \( A \) is abelian. Let \( (u_n)_n \) be a sequence of unitaries in \( A \) such that \( u_n \to 0 \) weakly. Then a simple computation yields that \( E_N((a \otimes b) u_n(c \otimes d)) \to 0 \) strongly for all \( a, c \in M \) and \( b, d \in N \), where \( E_N \) is a faithful normal conditional expectation given by \( E_N = \varphi \otimes \text{id}_N \) for a faithful normal state \( \varphi \) on \( M \). This implies the conclusion by [HI15, Theorem 4.3(5)].

**Factors in the class \( \mathcal{P} \)**

We show examples mentioned in Introduction are indeed contained in the class \( \mathcal{P} \). For this we prepare a few lemmas.

**Lemma 4.5.** Let \( M, N \) be separable factors, \( p \in M, q \in N \) projections, and let \( R_1, R_2 \) be amenable separable factors. If \( pMp \overline{\otimes} R_1 \precsim qNq \overline{\otimes} R_2 \) and if \( M \) is in the class \( \mathcal{P} \), then \( N \) is in the class \( \mathcal{P} \).

**Proof.** Let \( R_\infty \) be the Araki–Woods factor of type \( \text{III}_1 \). Put \( R_i := R_i \overline{\otimes} R_\infty \) for \( i = 1, 2 \) and observe that they are of type \( \text{III}_1 \) and therefore properly infinite. We get \( M \overline{\otimes} R_i \precsim N \overline{\otimes} R_i \). Let \( N \overline{\otimes} B = P \overline{\otimes} Q \) be as in Definition 1.4 and we will show \( P \precsim N \overline{\otimes} B \) or \( Q \precsim N \overline{\otimes} B \). By tensoring \( R_2 \), we have

\[
R_2 \overline{\otimes} P \overline{\otimes} Q = R_2 \overline{\otimes} N \overline{\otimes} B = M \overline{\otimes} R_2 \overline{\otimes} B.
\]

Since \( M \) is in the class \( \mathcal{P} \), we have either

\[
\text{(i) } (R_2 \overline{\otimes} P) \precsim_M R_2 \overline{\otimes} B R_\infty; \text{ or (ii) } Q \precsim_M R_2 \overline{\otimes} B R_\infty \overline{\otimes} B.
\]

Assume (i). Since \( C \overline{\otimes} P \subset R_2 \overline{\otimes} P \) with expectation, we have \( C \overline{\otimes} \overline{\otimes} P \precsim_M R_2 \overline{\otimes} B \overline{\otimes} R_\infty \overline{\otimes} B \) by [HI15, Lemma 4.8]. Lemma 4.3 implies \( C \overline{\otimes} P \precsim_M R_2 \overline{\otimes} B \overline{\otimes} C \). Using the isomorphism \( M \overline{\otimes} R_2 \overline{\otimes} B = N \overline{\otimes} R_2 \overline{\otimes} B \) and again applying Lemma 4.3 we can remove \( R_2 \) and obtain \( P \precsim_N B \).

Assume (ii). Since \( R_2 \) is amenable, it holds that \( R_2 \precsim B \). Combined with a trivial condition \( B \precsim_M B \) and using Corollary 3.3, we get \( R_2 \overline{\otimes} B \precsim_M R_2 \overline{\otimes} B \). The assumption (ii) and Corollary 3.3 then implies \( Q \precsim_M R_2 \overline{\otimes} B \). Since \( M \overline{\otimes} R_2 \overline{\otimes} B = N \overline{\otimes} R_2 \overline{\otimes} B \), there is a conditional expectation \( \Phi \) from \( N \overline{\otimes} R_2 \overline{\otimes} B, B \) onto \( Q \) which is faithful and normal on \( N \overline{\otimes} R_2 \overline{\otimes} B, B \). Using the identification

\[
\langle N \overline{\otimes} R_2 \overline{\otimes} B, B \rangle = B(L^2(N)) \overline{\otimes} B(L^2(\mathbb{R})) \overline{\otimes} B \supset B(L^2(N)) \overline{\otimes} C \overline{\otimes} B = \langle N \overline{\otimes} B, B \rangle,
\]

the restriction of \( \Phi \) on \( \langle N \overline{\otimes} B, B \rangle \) verifies that \( Q \) is amenable relative to \( B \) inside \( N \overline{\otimes} B \). \( \square \)

**Lemma 4.6.** A separable factor \( M \) is in the class \( \mathcal{P} \) if and only if it satisfies the condition in Definition 1.4 by assuming that \( B, P, Q \) are type \( \text{III}_1 \) factors.
Proof. We show the ‘if’ direction. Let \( B, P, Q \) be as in Definition 1.1 and assume that \( P \not\preceq_{B \otimes M} B \). We will show that \( Q \preceq_{B \otimes M} B \). Let \( R_\infty \) be the Araki–Woods factor of type \( \text{III}_1 \) and decompose it as \( R_\infty = R_1 \otimes R_2 \), where \( R_1 \simeq R_2 \simeq R_\infty \). Consider

\[
\tilde{B} \otimes M = \tilde{P} \otimes Q, \quad \text{where} \quad \tilde{B} := R_\infty \otimes B, \quad \tilde{P} := R_1 \otimes P, \quad \tilde{Q} := R_2 \otimes Q.
\]

The assumption \( P \not\preceq_{B \otimes M} B \) is equivalent to \( \tilde{P} \not\preceq_{\tilde{B} \otimes M} \tilde{B} \) by Lemma 4.3. Hence if \( M \) satisfies the ‘if’ condition of the statement, since \( B, P, Q \) are type \( \text{III}_1 \) factors, we get that \( \tilde{Q} \preceq_{\tilde{B} \otimes M} \tilde{B} \). This means there is a conditional expectation \( E : \langle \tilde{B} \otimes M, \tilde{B} \rangle \to \tilde{Q} \) which is faithful and normal on \( \tilde{B} \otimes M \). Finally using the isomorphism \( \langle \tilde{B} \otimes M, \tilde{B} \rangle = R_\infty \otimes \langle B \otimes M, B \rangle \), we obtain \( Q \preceq_{B \otimes M} B \).

\[
\square
\]

Lemma 4.7. Assume that a separable factor \( M \) satisfies the following condition:

- for any separable type \( \text{III}_1 \) factor \( B \) and an abelian von Neumann subalgebra \( A \subset B \otimes M \) with expectation, we have either \( A \preceq_{B \otimes M} B \) or \( A' \cap (B \otimes M) \preceq_{B \otimes M} B \).

Then \( M \) is in the class \( \mathcal{P} \).

Proof. Let \( B, P, Q \) be as in Definition 1.1 and assume that \( P \not\preceq_{B \otimes M} B \). We will show that \( Q \preceq_{B \otimes M} B \). Thanks to Lemma 4.6, we may assume that \( B, P, Q \) are type \( \text{III}_1 \) factors.

Since \( P \) has a large centralizer and is of type \( \text{III}_1 \), by Lemma 2.1, it has a type \( \text{II}_1 \) subfactor \( N \subset P \) with expectation such that \( N' \cap P = \mathbb{C} \). Observe that we have \( N \not\preceq_{B \otimes M} B \) by Lemma 4.3 (indeed \( N \preceq_{B \otimes M} B \) implies \( P \preceq_{B \otimes M} B \) by taking relative commutant two times). Using [HI15, Corollary 4.7], there is an abelian von Neumann subalgebra \( A \subset N \) with expectation such that \( A \not\preceq_{B \otimes M} B \). Now we apply the assumption of \( M \) in the statement and get that \( A' \cap M \preceq_{B \otimes M} B \). Since \( Q \subset A' \cap M \) with expectation, we conclude that \( Q \preceq_{B \otimes M} B \).

\[
\square
\]

Theorem 4.8. The following factor \( M \) is prime and belongs to the class \( \mathcal{P} \):

(i) A free product von Neumann algebra \( M := (M_1, \varphi_1) * (M_2, \varphi_2) \), where \( (M_i, \varphi_i) \) are diffuse von Neumann algebras with separable predual equipped with faithful normal states.

(ii) A non-amenable separable factor \( M \) that satisfies condition \((\text{AO})^+\) in the sense of [Is12a, Definition 3.1.1] and has the \( W^* \) CBAP (e.g. [BO08, §12.3]). This includes the following examples (see also [HI15, Remarks 2.7(3)]):

- any group von Neumann algebra \( L\Gamma \), where \( \Gamma \) is an ICC, non-amenable and weakly amenable discrete group which is bi-exact in the sense of [BO08, §15.1];

- any compact quantum group von Neumann algebra \( L^\infty (\hat{G}) \) that is a non-amenable factor, where \( \hat{G} \) is weakly amenable and bi-exact (see [Is15, Theorem C]);

- any free Araki–Woods factor [HR10, HI15, Appendix C].

Proof. The second statement follows from Ozawa’s celebrated solidity theorem [Oz03]. The primeness is a direct consequence of the solidity result, while the large centralizer condition is verified in [HI15, Theorem 3.7] also by the solidity. Proceeding as in the proof of [Is16b], we can prove the condition in Lemma 4.7 and therefore \( M \) is in the class \( \mathcal{P} \). See also the proof of [Is12b, Theorem 5.3.3] which treats Ozawa’s proof for type \( \text{III} \) factors.

We see the first statement. The factoriality, primeness, and the large centralizer condition are proved in [Ce10, Theorem 3.4], [CH08, Theorem 1.2], and [HU15a, Theorem A.1] respectively. So we will check only the condition in Definition 1.1.
Let $M$ be the free product as in the statement and let $B, P, Q$ be as in Definition 1.1. By Lemma 4.6 we assume that $B, P, Q$ are type III factors. As in the proof of Lemma 4.7 we can find type II$_1$ subfactors $P_0 \subset P$ and $Q_0 \subset Q$ with expectations and with trivial relative commutants, and an abelian subalgebra $A \subset P_0$ such that $A \nsubseteq B \varprojlim M$. We will show that $Q \nsubseteq B \varprojlim M$.

Let $\varphi_M$ be the free product state on $M$ and $\varphi_B, \varphi_P, \varphi_Q$ faithful normal states on $B, P, Q$ respectively. We may assume $P_0 = P_{\varphi_P}$ and $Q_0 = Q_{\varphi_Q}$. We put $N := B \varprojlim M$, $\varphi := \varphi_B \otimes \varphi_M, \psi := \varphi_P \otimes \varphi_Q$, and consider continuous cores $\tilde{N} := C^\varphi(N), \tilde{B} := C^\varphi(B)$, $\tilde{Q} := \Pi_{\varphi,\psi}(C^\varphi(Q)), \tilde{P} := \Pi_{\varphi,\psi}(C^\varphi(P))$ and $\tilde{A} := \Pi_{\varphi,\psi}(C^\varphi(A))$. We write as $\text{Tr}$ the canonical trace on $\tilde{N}$. Observe that $\tilde{A}$ is abelian containing $A$ and the inclusion $A \subset \tilde{N}$ is with expectation. For any Tr-finite projection $e \in \tilde{A}$, we have $e \tilde{A} \nsubseteq \tilde{N}$ by [BHR12 Proposition 2.10]. Observe that there is the amalgamated free product structure

$\tilde{N} = C^\varphi(B \varprojlim M_1) \ast_{\tilde{B}} C^\varphi(B \varprojlim M_2)$.

We apply [HUI15, Theorem A.4] and get the following result: for any Tr-finite projection $e \in \tilde{A}$, we have either one of the following conditions:

(i) $N_{e\tilde{N}}(Ae)'' \nsubseteq \tilde{B}$; or (ii) $N_{e\tilde{N}}(Ae)'' \leq \tilde{N} C^\varphi(B \varprojlim M_1)$ for some $i \in \{1, 2\}$.

**Suppose first that** (i) $N_{e\tilde{N}}(Ae)'' \nsubseteq \tilde{B}$ for all such $e$. Observe that $e\tilde{A}e \subset N_{e\tilde{N}}(Ae)''$, where $\tilde{A} := \Pi_{\varphi,\psi}(C^\varphi(A' \cap N))$. We have $\tilde{A} \nsubseteq \tilde{N} \tilde{B}$ which turns out that $A \nsubseteq \tilde{N} B$ by Theorem 4.2. Hence we obtain $Q \nsubseteq B$ and get the conclusion.

**Suppose next that** (ii) $N_{e\tilde{N}}(Ae)'' \leq \tilde{N} C^\varphi(B \varprojlim M_1)$ for some $i \in \{1, 2\}$ and for a projection $e$. We have $\tilde{A}e \leq \tilde{N} C^\varphi(B \varprojlim M_i)$ and hence $\tilde{A}e \leq \tilde{N} C^\varphi(B \varprojlim M_i)$ for some $i$. For simplicity we assume $i = 1$. Using [HIL15, Lemma 4.8] and since the inclusion $Q_0 \subset \tilde{A}e$ is with expectation, it holds that $Q_0 \leq \tilde{N} C^\varphi(B \varprojlim M_1)$. In this setting, we consider the following two cases:

(ii-a) $Q_0 \leq \tilde{N} \tilde{B}$; or (ii-b) $Q_0 \nsubseteq \tilde{N} \tilde{B}$.

**Assume that** (ii-a) $Q_0 \leq \tilde{N} \tilde{B}$. Then [BHR12 Proposition 2.10] implies $Q_0 \leq \tilde{N} B$. Using $Q_0' \cap B = \mathbb{C}$ and applying Lemma 4.2 two times, we indeed get $Q \leq \tilde{N} B$. Applying [HIL15, Lemma 4.13] and since $B, Q$ are type III factors, we can take a partial isometry $v$ such that $qBq = \upsilon Q_0 \upsilon^* \varprojlim L$, where $\upsilon = \upsilon^* \in B' \cap N = M$, $p = \upsilon^* \upsilon \in Q' \cap N = P$, and $L$ is a factor. Since $P$ is a type III factor, $p$ is equivalent to 1, so we may assume $\upsilon^* \upsilon = 1$. Since $\upsilon Q_0 \upsilon^* \subset Bq$ and $q \in B' \cap N$, we get that $\upsilon Q_0 \upsilon^*$ is injective relative to $B$ inside $N$. Applying the conjugacy map $Ad \upsilon^*$, we obtain $Q \leq\tilde{N} B$ that is the desired condition.

**Assume that** (ii-b) $Q_0 \nsubseteq \tilde{N} \tilde{B}$ and we will deduce a contradiction. Combined with the assumption $Q_0 \leq \tilde{N} C^\varphi(B \varprojlim M_1)$ and using (the proof of) [HUI15, Lemma 2.6], there are $p \in Q_0, q \in C^\varphi(B \varprojlim M_1), \theta : pQ_0 p \to qC^\varphi(B \varprojlim M_1)q$, $v \in \tilde{N}$ such that they witness $Q_0 \leq \tilde{N} C^\varphi(B \varprojlim M_1)$ and that $\theta(pQ_0 p) \nsubseteq \tilde{N} \tilde{B}$. Using the proof of [HIL15, Theorem 4.3(1)\Rightarrow(2-a)], up to replacing $q$ with a slightly smaller projection, we may assume $\text{Tr}(q) < \infty$. Observe that the condition $\theta(pQ_0 p) \nsubseteq \tilde{N} \tilde{B}$ implies $\theta(pQ_0 p) \nsubseteq C^\varphi(B \varprojlim M_1) \tilde{B}$ and hence by [CH08, Theorem 2.4], it holds that $\theta(pQ_0 p)' \cap q\tilde{N}q \subset qC^\varphi(B \varprojlim M_1)q$. Since $\upsilon \upsilon^* \in \theta(pQ_0 p)' \cap q\tilde{N}q \subset qC^\varphi(B \varprojlim M_1)q$, up to replacing $q$ with $\upsilon \upsilon^*$ (and $q$ with $\theta(q)\upsilon \upsilon^*$), we may assume $q = \upsilon \upsilon^*$. Observer that $p$ is a $\text{I}_{\infty}$ factor with the trace $\text{Tr}$, and $\text{Tr}$ is semifinite on the diffuse subalgebra $P_0 := \Pi_{\varphi,\psi}(C^\varphi(P_0))$. So any projection in $P$ is equivalent to a projection in $P_0$. Since $\upsilon \upsilon^* v \in (pQ_0 p)' \cap p\tilde{N}p = \tilde{P}p$, it is equivalent to a projection in $\tilde{P}_0 p$. 

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Up to replacing, we may assume that \( vv^* = q \in C_\varphi(B \otimes M_1) \), \( v^*v \in \tilde{P}_0p \), together with the inclusion
\[
vQ_0v^* = \theta(pQ_0p) \subset qC_\varphi(B \otimes M_1)q.
\]
The assumption \( Q_0 \not\preceq \tilde{N} \) and \( B \) and \( \chi \) for all \( i \) implies \( v^*vQ_0v^*v \not\preceq \tilde{N} \) \( B \). This means \( vQ_0v^* \not\preceq \tilde{N} \) \( B \) and so \( vQ_0v^* \not\preceq C_\varphi(B \otimes M_1) \), \( \tilde{B} \), and therefore \( \chi \) for all \( i \) implies
\[
v\tilde{P}v^* = (vQ_0 \cap \tilde{N})v^* = (vQ_0)^* \cap q\tilde{N}q \subset qC_\varphi(B \otimes M_1)q.
\]
Recall that we first assumed \( A \not\preceq B \). By \( \chi \) for all \( i \), this implies \( A \not\preceq \tilde{B} \). Since \( A \subset \tilde{P}_0 \), we have \( \tilde{P}_0 \not\preceq \tilde{B} \) by \( \chi \) for all \( i \). Since \( v^*v \in \tilde{P}_0p \), by \( \chi \) for all \( i \), we have \( v^*v\tilde{P}_0v^*v \not\preceq \tilde{N} \) \( B \), which is equivalent to \( v\tilde{P}_0v^* \not\preceq \tilde{N} \) \( B \). This implies \( v\tilde{P}_0v^* \not\preceq C_\varphi(B \otimes M_1) \), \( \tilde{B} \) and therefore \( \chi \) for all \( i \) again can be applied, so that
\[
\tilde{Q}v^* = (P_0 \cap \tilde{N})v^* = (v\tilde{P}_0v^*)^* \cap q\tilde{N}q \subset qC_\varphi(B \otimes M_1)q.
\]
Summary we obtain
\[
q\tilde{N}q = v\tilde{N}v^* = (\tilde{P} \vee \tilde{Q})v^* \subset qC_\varphi(B \otimes M_1)q.
\]
This implies \( \tilde{N} \preceq \tilde{N} C_\varphi(B \otimes M_1) \) and hence \( C_\varphi(M_2) \preceq \tilde{N} C_\varphi(B \otimes M_1) \) by \( \chi \) for all \( i \). Let \( C \subset M_2 \) be any diffuse abelian von Neumann algebra with expectation and let \( \omega \) be a faithful normal state on \( N \) such that \( \omega \circ E_{M_2} = \omega \) and \( C \subset (M_2)_\omega \). We have that \( \Pi_\varphi_\omega(C_\varphi(M_2)) \preceq \tilde{N} C_\varphi(B \otimes M_1) \) and hence \( \Pi_\varphi_\omega(C_\varphi(C)) \preceq \tilde{N} C_\varphi(B \otimes M_1) \) by \( \chi \) for all \( i \). We apply \( \chi \) for all \( i \), then implies \( C \preceq M_1 \). Since \( C \subset M_2 \) is diffuse, which is equivalent to \( C \preceq M_2 \) \( \mathbb{C} \), we obtain \( C \preceq M_1 \) by \( \chi \) for all \( i \), that is a contradiction. \( \square \)

5 Proof of Theorem \[A\]

Proof of Theorem \[A\]. Fix faithful normal states \( \varphi_0 \) and \( \psi_0 \) on \( M_0 \) and \( N_0 \) respectively. As in previous sections, we use the following notation:

\[
(M, \varphi) := \bigotimes_{m \in \{0\} \cup X} (M_m, \varphi_m), \quad (N, \psi) := \bigotimes_{n \in \{0\} \cup Y} (N_n, \psi_n);
\]
\[
M_F := \bigotimes_{n \in F} M_n \subset M, \quad M'_F := \bigotimes_{n \in F} M_n \subset M, \quad \text{for all } F \subset \{0\} \cup X.
\]

We use similar notations for \( N_0 \), such as \( N_F \) for \( F \subset \{0\} \cup Y \). We identify \( M = N \) for simplicity.

Fix \( j \in Y \) and we first find \( i \in X \) such that \( M_i \preceq_M N_j \). For this, suppose by contradiction that \( M_i \not\preceq_M N_j \) for all \( i \in X \). By Lemma \[B\], this is equivalent to \( N^*_j \not\preceq_M M_i^c \) for all \( i \), and by Corollary \[C\] factors \( N_j \) and \( N^*_j \) have large centralizers. Since \( M_i \) is in the class \( P \), we have \( N_j \leq_M M^c \) for all \( i \in X \). Applying Lemma \[D\] we can take intersections of \( M^c \) for finitely many \( i \in X \), that is, we have \( N_j \leq_M M^c \) for all finite subsets \( F \subset X \). We then apply Lemma \[E\] and get \( N_j \leq_M M_0 \). Since \( M_0 \) is amenable, we conclude that \( N_j \) is amenable which contradicts our assumption. Thus we have proved that for any \( j \in Y \), there is \( i \in X \) such that \( M_i \preceq_M N_j \). We can then define a map \( \sigma : Y \to X \) such that \( M_{\sigma(j)} \preceq_M N_j \) for all \( j \in Y \).

We show that \( \sigma \) is injective. Assume that \( \sigma(j) = \sigma(j') \). By \( \chi \) for all \( i \), take a partial isometry \( v \in M \) such that \( vM_{\sigma(j)}v^* \subset vv^*N_jv^* \) with expectation and that \( vv^* = qq' \) for projections \( q \in N_j \) and \( q' \in N^*_j \). Since \( vv^*N_jv^* \simeq qN_jq \), we can find a diffuse abelian subalgebra \( A \subset qN_jq \) with expectation such that \( Aq' \subset vM_{\sigma(j)}v^* \). Since

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For each $j$ isometry, then each tensor component is stably isomorphic. $q$ a nonzero projection where $M_v$ and 4.5. Then consider the inclusion $v^*Aq'v \subset v^*vM_{\sigma(j)}v^*v$ and apply [HI15 Lemma 4.8], so that $v^*Aq'v \preceq M_{M_j}$. We get $Aq' \preceq M_{M_j}$ and hence $A \preceq M_{M_j}$ by [HI15 Remark 4.2(2)]. This implies $j = j'$ by Lemma 4.4 (if $j \neq j'$, we have $A \preceq M_{M_j'}$, a contradiction), and we obtain injectivity of $\sigma$.

Next we assume that $N_j$ is semiprime for all $j \in Y$ and prove the surjectivity of $\sigma$. For each $j \in Y$, using [HI15 Lemma 4.13] and semiprimeness of $N_j$, there is a partial isometry $v \in M$ such that $vM_{\sigma(j)}v^* \otimes R_j = vv^*N_jv^*$ where $R_j$ is an amenable factor. By Lemma 4.5, $N_j$ is also in the class $\mathcal{P}$ for all $j \in Y$. So by exchanging the roles of $M_i$ and $N_j$ and using the result we already proved, there is an injective map $\tau : X \to Y$ such that $N_{\tau(i)} \preceq M_i$ for all $i \in X$. We fix $i \in X$ and observe that we have $M_{\sigma(\tau(i))} \preceq M_{N_{\tau(i)}}$ and $N_{\tau(i)} \preceq M_i$. Since $N_{\tau(i)} \cap N$ is a factor, [HI15 Lemma 4.12(2) and 4.12(3)] imply $M_{\sigma(\tau(i))} \preceq M_i$. By Lemma 4.4, we get $\sigma(\tau(i)) = i$. This implies $\sigma$ is surjective and $\tau$ is the inverse of $\sigma$. Since we already constructed amenable factors $R_j$ for all $j \in Y$, we finish the proof.

Proof of Corollary A Since all $N_j$ are prime, amenable factors $R_j$ in the last statement of Theorem A become finite dimensional. Hence if tensor product factors are isomorphic, then each tensor component is stably isomorphic.

Conversely assume that each tensor component is stably isomorphic. For simplicity we assume that $M_n \otimes \mathbb{B}(\ell^2) = N_n \otimes \mathbb{B}(\ell^2)$ for all $n \in X = Y$. If $M_n$ and $N_n$ are properly infinite, then we have $M_n = N_n$, so we take any faithful normal state $\varphi_n$ on $M_n$ and $\psi_n$ on $N_n$ such that $\varphi_n$ and $\psi_n$ coincide via $M_n = N_n$. If $M_n$ is finite and $N_n$ is properly infinite, then we have $M_n \otimes \mathbb{B}(\ell^2) = N_n$. Take any product state $\varphi_n \otimes \omega$ on $M_n \otimes \mathbb{B}(\ell^2)$ and define $\psi_n$ on $N_n$ using $M_n \otimes \mathbb{B}(\ell^2) = N_n$. Define similarly if $M_n$ is properly infinite and $N_n$ is finite. Finally if both $M_n$ and $N_n$ are finite, we have $M_n \otimes M_k(\mathbb{C}) = q_nN_nq_n \otimes M_l(\mathbb{C})$ for a nonzero projection $q_n \in N_n$ and $k, l \in \mathbb{N}$. By choosing appropriate $k, l \in \mathbb{N}$, we may assume that the trace value of $q_n$ is sufficiently close to 1. Define $\varphi_n$ and $\psi_n$ as traces on $M_n$ and $q_nN_nq_n$ respectively. Summary we have the following isomorphism

$$\varinjlim_{n \in X}(M_n, \varphi_n) \otimes M_0 = \varinjlim_{n \in Y}(q_nN_nq_n, \psi_n) \otimes N_0,$$

where $M_0$ and $N_0$ are some Araki–Woods factors and $q_n \in N_n$ are projections (which is 1$_{N_n}$ unless both $M_n$ and $N_n$ are finite). To consider the effect of $q_n$, for simplicity we assume that all $M_n$ and $N_n$ are II$_1$ factors. Let $\tau_n$ be the trace for $N_n$. Observe that since we can control the value $\tau_n(q_n)$ for all $n$, we may assume that the element $q := q_1 \otimes q_2 \otimes q_3 \otimes \cdots$ defines a nonzero projection in $\varinjlim_{n \in Y}(N_n, \tau_n)$. Hence with a suitable choice of $(q_n)_n$, it is not hard to see that

$$q(\varinjlim_{n \in Y}(N_n, \tau_n)) q \simeq \varinjlim_{n \in Y}(q_nN_nq_n, \psi_n).$$

Thus we obtain the desired stable isomorphism.

A Relative amenability for bimodules

In this Appendix, we define and investigate relative amenability for bimodules. All of our studies are based on the work of Connes [Co75] on amenability and the one of Anantharaman-Delaroche [AD93] on co-amenability. Although most of our results here are straightforward generalizations, we give detailed proofs for reader’s convenience.

Throughout the appendix, we use the following notation. For any von Neumann algebras $M$ and $B$, an $M$-$B$-module $H = _MM_B$ is a Hilbert space equipped with faithful
normal unital *-homomorphisms $\pi_H: M \to \mathbb{B}(H)$ and $\theta_H: B^{\text{op}} \to \mathbb{B}(H)$ such that $\pi_H(M)$ and $\theta_H(B^{\text{op}})$ commute. All opposite items are denoted with circles, such as $B^\circ = B^{\text{op}}$. The conjugate module $\overline{H}$ is the conjugate Hilbert space of $H$ equipped with the $B$-$M$-module structure given by
\[
\pi_{\overline{H}}(b)\theta_{\overline{H}}(x^\circ)\xi := \overline{\pi(x)\theta((b^\circ)^*)}\xi, \quad x \in M, \, b \in B, \, \xi \in H.
\]
The set of all $B^\circ$-module maps on $H$ will be denoted by
\[
\mathcal{L}_B(H_B) := \theta_H(B^\circ)' \cap \mathbb{B}(H).
\]
We always have $\pi_H(M) \subset \mathcal{L}_B(H_B)$. We denote by $\nu_H$ the *-homomorphism from the algebraic *-algebra generated by $M$ and $B^\circ$ into the $C^*$-algebra generated by $\pi_H(M)$ and $\theta_H(B^\circ)$. For $M$-$B$-modules $H$ and $K$, we will write $H \prec K$ if we have a weak containment, that is, representations $\nu_H$ and $\nu_K$ satisfy $\|\nu_H(x)\|_\infty \leq \|\nu_K(x)\|_\infty$ for all $x$ in the *-algebra of $M$ and $B^\circ$.

Let $B \subset M$ be von Neumann algebras with operator valued weight, $p \in M$ a projection, and $A \subset pMp$ a von Neumann subalgebra with expectation. Consider $H = L^2(pM)$ as a $pMp$-$B$-module. Then we have
\[
\mathcal{L}_B(H_B) = \theta_H(B^\circ)' \cap \mathbb{B}(H) = p(M,B)p
\]
and the $B$-$pMp$-module $L^2(pM)$ is canonically identified with the standard $B$-$pMp$-module $L^2(Mp)$, via the modular conjugation $J$ of $L^2(M)$: $\overline{L^2(M)} \ni \xi \mapsto J\xi \in L^2(M)$. From these points of view, the study on bimodules in this appendix will be used in the study of relative amenability in Section 3.

The following definition is a generalization of [PV11], in which they treat only finite von Neumann algebras. We introduce two notions of relative amenability which are equivalent for finite von Neumann algebras.

**Definition A.1.** Let $B$ and $M$ be von Neumann algebras, $A \subset M$ a von Neumann subalgebra, and $H = MHB$ an $M$-$B$-module.

1. We say that $H = MHB$ is left $A$-semidiscrete if we have a weak containment
\[
_M L^2(M)_A \prec_M H \otimes_B \overline{H}_A,
\]
where $\otimes_B$ is the Connes’ relative tensor product (e.g. [Ta01], Chapter IX. §3).

2. Assume that $A \subset M$ is with expectation $E_A$. We say that $MHB$ is left $(A, E_A)$-injective if there exists a conditional expectation
\[
E: \mathcal{L}_B(H_B) \to \pi_H(A) \simeq A \quad \text{such that} \quad E(\pi_H(x)) = E_A(x) \quad \text{for all} \ x \in M.
\]

Before starting our work on the relative amenability, we prepare several lemmas.

**Lemma A.2.** Let $M, N$ be von Neumann algebras and $MHN$, $MK_N$ be $M$-$N$-bimodules. Let $p \in M$, $q \in N$ be any projections such that central supports of $p$ in $M$ and $q$ in $N$ are $1_M$ and $1_N$ respectively.

1. There are canonical identifications
\[
_M L^2(Mp) \otimes_{pMp}(pH)_N \simeq_M MHN, \quad (M(qH) \otimes_{qNq} L^2(qN)_N) \simeq_M MHN.
\]

2. We have that $MHN \prec_M MK_N$ if and only if $pMp(qH)qNq \prec_{pMp(pKq)qNq}$.
Proof. We note that the left $M$-action on $L^2(Mp)$ is faithful if and only if the central support projection of $p$ in $M$ is $1_M$.

(1) It is easy to see that $ML^2(Mp) \otimes_{pMp} L^2(pM)_M \simeq ML^2(M)_M$. We have

\[ ML^2(Mp) \otimes_{pMp} (pH)_N \simeq ML^2(Mp) \otimes_{pMp} L^2(pM) \otimes_M H_N \simeq ML^2(M) \otimes_M H_N \simeq MH_N. \]

The same argument works for $M(Hq) \otimes_{qNq} L^2(qN)_N$.

(2) The only if direction is trivial. To see the if part, using [AD93, Lemma 1.7], apply $ML^2(Mp) \otimes_{pMp}$ from the left and $\otimes_{qNq}L^2(qN)$ from the right side. \hfill \Box

Lemma A.3. Let $M, N$ be von Neumann algebras and $M, K_N$ be $M$-$N$-bimodules. If $M, K_N \simeq M, K_N$, then there is a ucp map $\Psi: L_N(K_N) \to L_N(H_N)$ such that $\Psi(x) = \pi_H(x)$ for all $x \in M$.

Proof. Let $\nu$ be the bounded $*$-homomorphism for $M, K_N \simeq M, K_N$, namely, it sends $\nu_K(\epsilon)$ into $\nu_H(\epsilon)$. We extend $\nu$ on $C^* \{ L_N(K_N), \theta_K(N^o) \} \subset B(K)$ as a u.c.p. map to $B(H)$ by Arveson’s extension theorem. Define $\Psi: L_N(K_N) \to B(H)$ by $\Psi(T) := \nu(T)$. Obviously $\Psi(\pi_K(\epsilon)) = \pi_H(\epsilon)$ for $\epsilon \in M$. We have to show that $\text{Im} \Psi \subset L_N(H_N)$, which means $\text{Im} \Psi$ commutes with $\theta_H(N^o)$. For any $\nu \in U(N)$ and $T \in L_N(K_N)$, since $\theta_K(\nu)$ is contained in the multiplicative domain of $\nu$ (e.g. [BO05 Proposition 1.5.7]), we have

\[ \Psi(T)\theta_H(\nu) = \nu(T)\theta_K(\nu) = \nu(T\theta_K(\nu)) = \nu(\theta_K(\nu)T) = \theta_H(\nu)\Psi(T). \]

Hence $\Psi(T)$ commutes with $\theta_H(\nu)$ for all $\nu \in U(N)$, and $\Psi$ is a desired ucp map. \hfill \Box

Lemma A.4. Let $M, N$ be von Neumann algebras and $M, K_N$ be $M$-$N$-bimodules. Assume that there is a cyclic vector $\xi \in M, H_N$, that is, $\pi_H(M)\theta_H(N^o)\xi \subset H$ is dense. Then $M, H_N \simeq M, K_N$ if and only if the linear functional

\[ B(K) \ni *\text{-alg}\{ \pi_K(M), \theta_K(N^o) \} \ni \pi_K(\epsilon)\theta_K(\eta) \mapsto \langle \pi_H(\epsilon)\theta_H(\eta), \xi \rangle_H \in \mathbb{C} \]

is bounded (with respect to the norm in $B(K)$).

Proof. The only if’ part is trivial. For the converse, observe that the given linear functional is positive on the $*$-algebra, so it is extended on $C^* \{ \pi_K(M), \theta_K(N^o) \}$ as a positive linear functional. Then since $\xi$ is cyclic, the Hilbert space of the GNS representation of this functional is identified as $H$. In particular the GNS representation is identified with the $*$-homomorphism $\pi_K(\epsilon)\theta_K(\eta) \mapsto \pi_H(\epsilon)\theta_H(\eta)$, and hence it is bounded. \hfill \Box

Characterizations of left injectivity/semidiscreteness

We start our work with proving well-known characterizations of relative amenability, which are generalizations of a part of [Co75 Theorem 5.1] and [AD93, Section 3]. For finite von Neumann algebras, they are proved in [OP07 Theorem 2.1] and [PV11 Proposition 2.4].

Theorem A.5. Let $B, M, A, E_A$ and $M, H_B$ as in Definition A.1(2) and consider the following conditions.

(1) The bimodule $M, H_B$ is left $(A, E_A)$-semidiscrete.

(2) There is a $B$-$A$-bimodule $K$ such that $ML^2(M)_A \simeq M \otimes_B K_A$.

(3) There is a ucp map $\Psi: L_B(H_B) \to (M, A)$ such that $\Psi(\pi_H(x)) = x$ for all $x \in M$.

(4) The bimodule $M, H_B$ is left $(A, E_A)$-injective.
Then we have (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (4). If we further assume A is semifinite, then (4) \(\Rightarrow\) (1) holds, so all conditions are equivalent.

**Proof.** The implication (1) \(\Rightarrow\) (2) is trivial and (2) \(\Rightarrow\) (3) follows from Lemma A.6. To see (3) \(\Rightarrow\) (4), apply the compression map by the Jones projection \(e_A\) of \(E_A\).

(4) \(\Rightarrow\) (1) Assume first that A is finite with a faithful normal trace \(\tau_A\). Put \(\psi := \tau_A \circ E_A\). We apply Lemma A.6 below for \(N := L_B(H_B)\), and get a net of unit vectors \((\xi_i)_i \in L^2(N)\) such that \((\pi_H(x)\xi_i, \xi_i) \rightarrow \psi(x)\) for all \(x \in M\) and \(\|\pi_H(u)J\pi_H(u)J\xi_i - \xi_i\|_2 \rightarrow 0\) for all \(u \in \mathcal{U}(A)\), where J is the modular conjugation for \(L^2(N)\). Observe by [Sa81, PROPOSITION 3.1] that

\[
L^2(N) = L^2(\theta_H(B^\circ)) \simeq H \otimes_B \overline{\Pi}
\]
as \(N\)-bimodules, and hence as \(M\)-bimodules. Note that the \(M\)-bimodule structure of \(H \otimes_B \overline{\Pi}\) here is given by the left action \(\pi_H(x) \otimes_B 1\) and the right one \(1 \otimes_B \theta_{\overline{\Pi}}(y^\circ)\) for all \(x, y \in M\). We regard \((\xi_i)_i\) as vectors in \(H \otimes_B \overline{\Pi}\). Then the second condition on \((\xi_i)_i\) is translated as follows: for any \(a \in \mathcal{U}(A)\)

\[
\|(\pi_H(a) \otimes_B 1)\xi_i - (1 \otimes_B \theta_{\overline{\Pi}}(a^\circ))\xi_i\|_2 = \|(\pi_H(a) \otimes_B \theta_{\overline{\Pi}}(a^\circ))\xi_i - \xi_i\|_2 \rightarrow 0.
\]

Using the first condition on \((\xi_i)_i\), together, we obtain

\[
\langle (\pi_H(x) \otimes_B \theta_{\overline{\Pi}}(a^\circ))\xi_i, \xi_i \rangle \rightarrow \psi(xa) = \langle xa^\circ \xi_\psi, \xi_\psi \rangle_{\psi}\quad (x \in M, a \in A),
\]

where \(\xi_\psi \in L^2(M, \psi)\) is the canonical cyclic vector. In particular the linear functional \(\pi_H(x) \otimes_B \theta_{\overline{\Pi}}(a^\circ) \mapsto (xa^\circ \xi_\psi, \xi_\psi)_{\psi}\) is bounded. So by Lemma A.4 we get \(M L^2(M, \psi)_A \lesssim M H \otimes_B \overline{\Pi}_A\), which is our conclusion.

We next show the general case. So assume that A is semifinite. Let \(\nu\) be the algebraic \(*\)-homomorphism for the weak containment \(M L^2(M)_A \lesssim M H \otimes_B \overline{\Pi}_A\). We will show that \(\nu\) is bounded. For this we fix \(x \in *-\text{alg}\{\pi_H(M) \otimes_B 1, 1 \otimes_B \theta_{\overline{\Pi}}(A^\circ)\}\) and we will show \(\|x\|_\infty \leq \|\nu(x)\|_\infty\).

Let \(p \in A\) be a finite and \(\sigma\)-finite projection. Then the assumption (4) implies that \(p M p \theta_{\overline{\Pi}}(p)\) is left \((p A p, E_{p A p})\)-injective. Since \(p A p\) has a faithful normal trace, by the result we already proved, \(p M p \theta_{\overline{\Pi}}(p)\) is left semidiscrete. Put \(p_H := \pi_H(p) \theta_{\overline{\Pi}}(p^\circ)\) and \(\tilde{p} := pp^\circ\), and observe that left semidiscreteness of \(p M p \theta_{\overline{\Pi}}(p)\) implies

\[
\|\tilde{p} \nu(x) \tilde{p}\|_\infty = \|\nu(p_H x p_H)\|_\infty \leq \|p_H x p_H\|_\infty.
\]

Next take a net \((p_i)_i\) of finite and \(\sigma\)-finite projections in \(A\) which converges to \(1_A\) strongly. Taking the supremum of such \(p_i\), we obtain

\[
\|\nu(x)\|_\infty = \sup_i \|\tilde{p}_i \nu(x) \tilde{p}_i\|_\infty \leq \sup_i \|p_i x p_i H\|_\infty = \|x\|_\infty.
\]

Here we used the following fact: for any projections \(p_i \in \mathbb{B}(K)\) converging to 1 strongly on any Hilbert space \(K\), we have \(\|X\|_\infty = \sup_i \|p_i X p_i\|_\infty\) for any \(X \in \mathbb{B}(K)\). Thus we get the boundedness of \(\nu\) and this is the desired condition.

The lemma below was used in the proof of the previous theorem. We note that the condition (2) below is more general than the relative injectivity, which corresponds to the case \(N = L_B(H_B)\).

**Lemma A.6.** Let \(A \subseteq M\) be \(\sigma\)-finite von Neumann algebras with expectation \(E_A\), and let \(N\) be a von Neumann algebra containing \(M\). Assume that \(A\) is finite. We fix a trace \(\tau_A\) on \(A\) and put \(\psi := \tau_A \circ E_A\). Then the following conditions are equivalent.

\[\Box\]
Theorem IX.1.2(iv) shows that \( L \) topology of replacing with convex combinations, the net \((u_\psi)_\psi\) is crucial to obtain the equivalence of semidiscreteness and injectivity.

Proof. The proof is almost identical to the one of [OP07, Theorem 2.1]. Hence we give a sketch of the proof.

(1) \( \Rightarrow \) (2) Using the A-centrality, we have

\[
|\tilde{\psi}(ax)| \leq \|a\|_{1,\tau_A}\|x\|_\infty \quad \text{for all } a \in A, \ x \in N.
\]

For any \( x \in N \), define a functional \( T_x: A \to \mathbb{C} \) by \( T_x(a) := \tilde{\psi}(ax) \). This is bounded on \( L^1(A, \tau_A) \) and so there is a unique element \( \Phi(x) \in A \) such that \( \tau_A(a\Phi(x)) = \tilde{\psi}(ax) \) for all \( a \in A \). This \( \Phi \) is a desired conditional expectation.

(2) \( \Rightarrow \) (1) Compose \( \psi \) with the given conditional expectation.

(1) \( \Rightarrow \) (3) Let \((\psi_i)_i\) be a net of normal states on \( N \) converging to \( \tilde{\psi} \) weakly. This satisfies that for any \( u \in \mathcal{U}(A) \), the net \((w_\psi u^* - \psi_i)_i\) converges to zero weakly, where \( w_\psi u^* := x u \psi_i \). So by the Hahn–Banach separation theorem, one has that, up to replacing with convex combinations, the net \((w_\psi u^* - \psi_i)_i\) converges to zero in the norm topology of \( N \), for all \( u \in \mathcal{U}(A) \). For each \( i \), let \( \xi_i \) be the unit vector in the positive cone of \( L^2(N) \) such that the vector state of \( \xi_i \) is \( \psi_i \). Then the Powers–Størmer inequality [Ta01, Theorem IX.1.2(iv)] shows that

\[
\|uJ_N u J_N \xi_i - \xi_i\|_2^2 \leq \|w_\psi u^* - \psi_i\|_{N_*} \to 0 \quad \text{for all } u \in \mathcal{U}(A).
\]

(3) \( \Rightarrow \) (1) Define a state on \( N \) by \( \tilde{\psi}(x) := \lim_i \langle x \xi_i, \xi_i \rangle \). \( \square \)

Continuous core approach

We next study relative amenability using continuous cores. The use of the continuous core is natural in our context because, as observed in Theorem A.3, the tracial condition is crucial to obtain the equivalence of semidiscreteness and injectivity.

We fix the following setting. Let \( B,M \) be von Neumann algebras and \( A \subset M \) a von Neumann subalgebra with expectation \( E_A \). Let \( \psi_A \) be a faithful normal semifinite weight on \( A \). Put \( \psi := \psi_A \circ E_A \) and recall that continuous cores have an embedding \( C_\psi(A) \subset C_\psi(M) \). Let \( H = M H_B \) be an \( M \)-\( B \)-bimodule and define a \( C_\psi(M) \)-\( B \)-bimodule \( MB \) by \( K := C_{\psi(M)} L^2(C_\psi(M)) \otimes_M H_B \). Note that, under the isomorphism

\[
C_{\psi(M)} L^2(C_\psi(M)) \otimes_M H_B \simeq C_{\psi(M)} (L^2(\mathbb{R}) \otimes L^2(M)) \otimes_M (L^2(M) \otimes_M H_B)
\]

\[
\simeq C_{\psi(M)} L^2(\mathbb{R}) \otimes L^2(M) \otimes_M H_B
\]

\[
\simeq C_{\psi(M)} L^2(\mathbb{R}) \otimes H_B,
\]

our actions are of the forms: for \( x \in M \), \( t \in \mathbb{R} \), \( y \in B \),

\[
\pi_K(x) := \pi_{\sigma_\psi}(x), \quad \pi_K(\lambda t) := \lambda t \otimes 1, \quad \rho_K(y^\text{op}) := 1 \otimes \theta_H(y^\text{op}),
\]

where \( \pi_{\sigma_\psi}(x) \) is the usual representation in crossed products given by \( (\pi_{\sigma_\psi}(x) \xi)(s) = \sigma_\psi(x) \xi(s) \) for all \( x \in M \) and \( \xi \in L^2(\mathbb{R}) \otimes H = L^2(\mathbb{R}, H) \).

The following theorem establishes an equivalence of semidiscreteness and injectivity, using the bimodule \( MB \). This equivalence will be used in the proof of the main theorem.
Theorem A.7. Keep the setting as above and consider the following conditions.

1. The bimodule $M H_B$ is left $(A, E_A)$-injective.

2. The bimodule $C_{\psi(M)} K_B$ is left $C_{\psi(A)}$-semidiscrete, that is,
   \[ C_{\psi(M)} L^2(C_{\psi(M)}) C_{\psi(A)} \lesssim C_{\psi(M)} K \otimes_B K_{C_{\psi(A)}}. \]

3. The bimodule $M K_B$ is left $A$-semidiscrete, that is,
   \[ M L^2(M)_A \lesssim M K \otimes_B K_A. \]

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If we further assume that there is an operator valued weight from $\mathcal{L}_B(H_B)$ to $\pi_H(M)$, then we have (3) $\Rightarrow$ (1), so all conditions are equivalent.

Proof. (1) $\Rightarrow$ (2) Let $e_A$ be the Jones projection for $E_A$. Observe that the composition map by $A$ and $C$ and the inclusion $C \hookrightarrow A$.

We denote this expectation again by $E_A$. We will show that $C_{\psi(M)} K_B$ is left $(C_{\psi(A)}, E_A)$-injective which is equivalent to (2) by Theorem A.5. By assumption there is a conditional expectation $E: \mathcal{L}_B(H_B) \to A$ which restricts to $E_A$, where we omit $\pi_H$.

We can then construct a conditional expectation
\[
\tilde{E}: \mathcal{L}_B(H_B) \otimes \mathbb{B}(L^2(\mathbb{R})) \to A \otimes \mathbb{B}(L^2(\mathbb{R}))
\]
which restricts to $E_A \otimes id$ on $M \otimes \mathbb{B}(L^2(\mathbb{R}))$. To see this, take finite rank projections $p_n \in \mathbb{B}(L^2(\mathbb{R}))$ which converges to 1 strongly. Then since $E \otimes id$ is defined on $\mathcal{L}_B(H_B) \otimes p_n \mathbb{B}(L^2(\mathbb{R})) p_n$, we can define $\tilde{E}$ as a cluster point of maps $x \mapsto (E \otimes id)((1 \otimes p_n)x(1 \otimes p_n))$.

One can directly check that $\tilde{E}|_{M \otimes \mathbb{B}(L^2(\mathbb{R}))} = E_A \otimes id$.

Observe that, by omitting $\pi_H$, $\mathcal{L}_B(H_B) \otimes \mathbb{B}(L^2(\mathbb{R}))$ and $A \otimes \mathbb{B}(L^2(\mathbb{R}))$ contain $C_{\psi(M)}$ and $C_{\psi(A)}$ respectively, and the restriction of $\tilde{E}$ on $C_{\psi(M)}$ is $E_A$. Observe next that $A \otimes \mathbb{B}(L^2(\mathbb{R})) \simeq C_{\psi(A)} \rtimes \mathbb{R}$ by the Takesaki duality [Ta01, Theorem X.2.3], so there is a conditional expectation from $A \otimes \mathbb{B}(L^2(\mathbb{R}))$ onto $C_{\psi(A)}$.

By composing this expectation with $\tilde{E}$, we obtain a conditional expectation from $\mathcal{L}_B(H_B) \otimes \mathbb{B}(L^2(\mathbb{R}))$ onto $C_{\psi(A)}$ which restricts to $E_A$ on $C_{\psi(M)}$.

Finally, observe that $\theta_K(B^\circ'y) = \mathcal{L}_B(H_B) \otimes \mathbb{B}(L^2(\mathbb{R}))$ and the inclusion $C_{\psi(M)} \subset \mathcal{L}_B(H_B) \otimes \mathbb{B}(L^2(\mathbb{R}))$ mentioned above coincides with the one of $\pi_K(C_{\psi(M)}) \subset B^\circ'y$. Hence we have constructed a conditional expectation from $\theta_K(B^\circ'y)$ onto $C_{\psi(A)}$ which restricts to $E_A$ on $C_{\psi(M)}$. This is the desired condition.

(2) $\Rightarrow$ (3) By definition, we have $M L^2(C_{\psi(M)})_A \lesssim M K \otimes_B K_A$. We claim $M L^2(M)_M \lesssim M L^2(C_{\psi(M)})_M$, which obviously implies (3).

To see the claim, we have to show for any $x_i, y_i \in M$, $i = 1, \ldots, n$,
\[
\left\| \sum_i x_i y_i^\circ \right\|_\infty \leq \left\| \sum_i \pi_{\sigma^\circ}(x_i)(y_i^\circ \otimes 1) \right\|_\infty.
\]

Since $\pi_{\sigma^\circ}(M)$ and $M^\circ \otimes 1$ are contained in $\mathbb{B}(L^2(M)) \otimes L^\infty(\mathbb{R}) \simeq L^\infty(\mathbb{R}, \mathbb{B}(L^2(M)))$, the right hand side in this inequality coincides with
\[
\text{ess- sup} \left\| \sum_i \sigma^\circ_t(x_i)y_i^\circ \right\|_\infty.
\]
Since the map $\mathbb{R} \ni t \mapsto \sum_i \sigma_i^\psi(x_i)y_i^\psi$ is strongly continuous, the map $\mathbb{R} \ni t \mapsto \left\| \sum_i \sigma_i^\psi(x_i)y_i^\psi \right\|_\infty$ is lower semi-continuous. Hence for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left\| \sum_i x_iy_i^\psi \right\|_\infty - \varepsilon \leq \left\| \sum_i \sigma_i^\psi(x_i)y_i^\psi \right\|_\infty$$

for all $|t| < \delta$, and therefore

$$\left\| \sum_i x_iy_i^\psi \right\|_\infty - \varepsilon \leq \left\| \sum_i \pi_{\sigma^\psi}(x_i)(y_i^\psi \otimes 1) \right\|_\infty.$$

Letting $\varepsilon \to 0$, the claim is proven.

(3) $\Rightarrow$ (1) Assume that there is an operator valued weight $E_M : \mathcal{L}_B(H_B) \to M$. Define a faithful normal semifinite weight on $\mathcal{L}_B(H_B)$ by $\hat{\psi} := \psi \circ E_M$. It then holds that $\sigma_i^\hat{\psi} |_{M} = \sigma_i^\psi$ and hence there is an inclusion $C_\psi(M) \subset C_\hat{\psi}(\mathcal{L}_B(H_B))$. By assumption and Theorem [A.5] $M K_B$ is left $A$-injective, so there is a conditional expectation

$$E : \mathcal{L}_B(K_B) = \mathcal{L}_B(H_B) \otimes \mathbb{B}(L^2(\mathbb{R})) \to \pi_K(A) = \pi_{\sigma^\psi}(A)$$

which restricts to $E_A$ on $\pi_{\sigma^\psi}(M)$. Observe that $\mathcal{L}_B(H_B) \otimes \mathbb{B}(L^2(\mathbb{R}))$ contains $C_\hat{\psi}(\mathcal{L}_B(H_B))$. By restriction, we have a conditional expectation from $\pi_{\sigma^\psi}(\mathcal{L}_B(H_B))$ onto $\pi_{\sigma^\psi}(A)$ which restricts to $E_A$. This means (1). \hfill \Box

In the case $A = M$, the following corollary is well known to experts but it is not explicitly written in [AD93]. The corollary states that a conditional expectation can be approximated by normal ccp maps up to Morita equivalence.

**Corollary A.8.** Let $A \subset M \subset N$ be von Neumann algebras. Assume that there is a conditional expectation $E : N \to A$ which restricts to a faithful normal conditional expectation $E_A$ on $M$. Let $\psi_A$ be a faithful normal semifinite weight and put $\psi := \psi_A \circ E_A$. Let $\pi_{\sigma^\psi} : M \to M \rtimes \psi \subset N \otimes \mathbb{B}(L^2(\mathbb{R}))$ be the canonical embedding.

1. There is a conditional expectation from $N \otimes \mathbb{B}(L^2(\mathbb{R}))$ onto $\pi_{\sigma^\psi}(A)$ which restricts to $E_A$ on $\pi_{\sigma^\psi}(M)$ and which is approximated by normal ccp maps from $N \otimes \mathbb{B}(L^2(\mathbb{R}))$ to $\pi_{\sigma^\psi}(A)$ in the point $\sigma$-weak topology.

2. Assume further that there is an operator valued weight from $N$ to $M$. Then there is a conditional expectation from $N$ onto $A$ which restricts to $E_A$ on $M$ and which is approximated by normal ccp maps from $N$ to $A$ in the point $\sigma$-weak topology.

**Proof.** (1) Fix $N \subset \mathbb{B}(H)$ and put $B := (N')^\circ$ and $H = M H_B$. By assumption, there is a conditional expectation $\mathcal{L}_B(H_B) \to A$ which restricts to $E_A$. So $M H_B$ is left $(A, E_A)$-injective. By Theorem [A.7] (1)$\Rightarrow$(3), we have $M L^2(M)_A \prec_M K \otimes_B \overline{K}_A$. Since $K \otimes_B \overline{K}$ is the standard representation of $\theta_K(B^2)' = N \otimes \mathbb{B}(L^2(\mathbb{R})) =: N$ [Sa81] **PROPOSITION 3.1], we have

$$M L^2(M)_A \prec_M K \otimes_B \overline{K}_A = M L^2(\tilde{N})_A = M L^2(\tilde{N}) \otimes_{\tilde{K}} L^2(\tilde{N})_A.$$

This means $M L^2(\tilde{N})_A$ is left $A$-semidiscrete. For any right $A$-module $L = L_A$, we denote by $X(L) := \text{Hom}_A(L^2(A), L)$ the set of all bounded linear maps from $L^2(A)$ to $L$ which commute with right $A$-actions and define an $A$-valued inner product by $(T, S)_{X(L)} := T^* S \in (A')^\circ = A$. See [AD93] **Preliminaries** for the relation of bimodules and $W^*$-Hilbert
modules as well as for general theories of them. Then, the above weak containment is equivalent to
\[ M X(L^2(M)) \prec M X(L^2(\tilde{N})), \]
where we are thinking them as Hilbert $A$-modules with left $M$-actions. Here $X(L^2(M))$ is identified as $M$ with the inner product $(x, y)_{X(L^2(M))} = E_A(x^* y)$ for $x, y \in M$. By the weak containment, for the vector $1_M \in X(L^2(M))$, any $\sigma$-weak neighborhood $\mathcal{V}$ of 0, and any finite subset $\mathcal{E} \subset M$, there are vectors $\eta_i \in X(L^2(\tilde{N})), i = 1, \ldots, n$, such that
\[
\langle x 1_M, 1_M \rangle_{X(L^2(M))} - \sum_{i=1}^{n} \langle x \eta_i, \eta_i \rangle_{X(L^2(\tilde{N}))} \in \mathcal{V}
\]
for all $x \in \mathcal{E}$. We define a normal completely positive map from $\tilde{N}$ into $A$ by
\[
\varphi_{(\mathcal{V}, \mathcal{E})}(T) := \sum_{i=1}^{n} \langle T \eta_i, \eta_i \rangle_{X(L^2(\tilde{N}))}.
\]
Observe that for any $x \in \mathcal{E}$,
\[
E_A(x) - \varphi_{(\mathcal{V}, \mathcal{E})}(x) \in \mathcal{V}.
\]
Hence letting $\mathcal{E}$ larger and $\mathcal{V}$ smaller, we have that $\varphi_{(\mathcal{V}, \mathcal{E})}(x) \to E_A(x)$ $\sigma$-weakly for any fixed $x \in M$. By [AD93, Lemma 1.6], regarding $\varphi_{(\mathcal{V}, \mathcal{E})}$ as cp maps from $M$ into $A$, up to convex combinations and up to transforms $\varphi_{(\mathcal{V}, \mathcal{E})} \mapsto b \varphi_{(\mathcal{V}, \mathcal{E})} b^*$ for $b \in A$, we may assume that $\varphi_{(\mathcal{V}, \mathcal{E})}(1) \leq E_A(1) = 1$, hence ccp maps. Since the resulting ccp maps are still finite sums of $M \ni x \mapsto (x \eta, \eta)_{X(L^2(\tilde{N}))}$ for $\eta \in X(L^2(\tilde{N}))$, we can again regard $\varphi_{(\mathcal{V}, \mathcal{E})}$ as normal ccp maps from $\tilde{N}$ to $A$, which are ccp by conditions $\varphi_{(\mathcal{V}, \mathcal{E})}(1) \leq E_A(1) = 1$. Finally any cluster point of $\varphi_{(\mathcal{V}, \mathcal{E})}$ is a conditional expectation from $\tilde{N}$ onto $A$ which restricts to $E_A$. Hence we can find a desired net of ccp maps as a subnet of $(\varphi_{(\mathcal{V}, \mathcal{E})})_{(\mathcal{V}, \mathcal{E})}$. (2) Take a faithful normal semifinite weight $\hat{\psi}$ on $N$ such that $\sigma_i^\hat{\psi}|_M = \sigma_i^\psi$, so that we have inclusions $C_\psi(M) \subset C_\hat{\psi}(N) \subset N \boxtimes B(L^2(\mathbb{R}))$. Then take a conditional expectation constructed in (1) and restrict it on $\pi_\psi(N)$. We get a conditional expectation from $\pi_\psi(N)$ onto $\pi_\psi(A)$ which restricts to $E_A$ on $\pi_\psi(M)$ and which is approximated by normal ccp maps. This is the conclusion. \hfill \Box

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