On the Malgrange isomonodromic deformations of non-resonant meromorphic \((2 \times 2)\)-connections

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Abstract

We study the tau-function and theta-divisor of an isomonodromic family of linear differential \((2 \times 2)\)-systems with non-resonant irregular singularities. In some particular case the estimates for pole orders of the coefficient matrices of the family are applied.

1 Introduction

Consider a meromorphic linear \((2 \times 2)\)-system on the Riemann sphere \(\mathbb{C}\), i.e., a system of two linear ordinary differential equations with singularities \(a_1^0, \ldots, a_n^0 \in \mathbb{C}\) and possibly \(\infty\). By a conformal mapping one can always arrange that all the singularities are in the complex plane only. This means that one can reduce the system to the form

\[
\frac{dy}{dz} = B(z) y, \quad B(z) = \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{B^{0}_{ij}}{(z - a_i^0)^j},
\]

where \(y(z) \in \mathbb{C}^2, B^{0}_{ij}\) are \((2 \times 2)\)-matrices and \(\sum_{i=1}^{n} B^{0}_{i1} = 0\), to ensure that \(\infty\) is not a singular point.

The non-negative integers \(r_1, \ldots, r_n\) are called the Poincaré ranks of the singularities \(a_1^0, \ldots, a_n^0\) respectively. One can assume that the Poincaré ranks \(r_1, \ldots, r_m\) are positive and \(r_{m+1} = \ldots = r_n = 0\) (that is, the singular points \(a_{m+1}^0, \ldots, a_n^0\) are Fuchsian) for some \(0 \leq m \leq n\).

We consider the non-resonant case. This means that the leading term \(B^{0}_{i,r_i+1}\) of each non-Fuchsian singularity \(a_i^0, i = 1, \ldots, m\), has two distinct eigenvalues. In that case the singular points \(a_1^0, \ldots, a_m^0\) are irregular.

The system (1) can be thought of as a meromorphic connection \(\nabla^0\) (more precisely, as an equation for horizontal sections with respect to this connection) on a holomorphically trivial vector bundle \(E^0\) of rank 2 over \(\mathbb{C}\). As known (see [9, §21]), in a neighbourhood of each (non-resonant) irregular singularity \(a_i^0\), the local connection form \(\omega^0 = B(z) dz\) of \(\nabla^0\) is formally equivalent to the 1-form

\[
\omega_{\Lambda^0_i} = \sum_{j=1}^{r_i+1} \frac{\Lambda^0_{ij}}{(z - a_i^0)^j} dz,
\]

where \(\Lambda^0_{11}, \ldots, \Lambda^0_{i,r_i+1}\) are diagonal matrices and the leading term \(\Lambda^0_{i,r_i+1}\) is conjugated to \(B^{0}_{i,r_i+1}\). This means that there is an invertible matrix formal Taylor series \(\tilde{F}(z)\) in \((z - a_i^0)\) such that the transformation \(\tilde{y} = \tilde{F}^{-1}(z)y\) transforms the 1-form \(\omega^0\) into \(\omega_{\Lambda^0_i}\):

\[
\omega_{\Lambda^0_i} = \tilde{F}^{-1} \omega^0 \tilde{F} - \tilde{F}^{-1}(d\tilde{F}).
\]
One should note that formally equivalent systems in a neighbourhood $O_{a_i^0}$ of an irregular singularity $a_i^0$ are not necessary holomorphically or meromorphically equivalent. The system (1) has in $O_{a_i^0}$ a formal fundamental matrix of the form

$$\hat{Y}(z) = \hat{F}(z)(z - a_i^0)^{\Lambda_i}e^{Q(z)}, \quad Q(z) = \sum_{j=1}^{r_i} \frac{\Lambda_{ij}^0}{j} (z - a_i^0)^{-j}. \quad (2)$$

One can cover $O_{a_i^0}$ by a set of sufficiently small sectors $S_1, \ldots, S_N$ with vertices at $a_i^0$ such that in each $S_k$ there exists a unique fundamental matrix $Y_k(z) = F_k(z)(z - a^0_i)^{\Lambda_{i}^0}e^{Q(z)}$ of the system with $F_k(z)$ having $\hat{F}(z)$ as an asymptotic series in $S_k$ (see [9, §21]). In every intersection $S_k \cap S_{k+1}$ the fundamental matrices $Y_k(z)$, $Y_{k+1}(z)$ are connected by a constant matrix $C_k$: $Y_{k+1}(z) = Y_k(z)C_k$, which is called a Stokes’ matrix. If $a_i^0$ is a non-resonant singularity, then two formally equivalent systems are holomorphically equivalent in $O_{a_i^0}$ if and only if they have the same sets of Stokes’ matrices (see [9, §21] again).

Further we will focus on deformations of the system (1) (of the pair $(E^0, \nabla^0)$) that allow the local formal equivalence class

$$\omega_{\Lambda_i} = \sum_{j=2}^{r_i-1} \frac{\Lambda_{ij}}{(z - a_i)^j} dz + \frac{\Lambda_{i1}^0}{z - a_i} dz, \quad i = 1, \ldots, m,$$

to vary in the sense that the diagonal matrices $\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}$ vary in a neighbourhood of $\Lambda_{i2}^0, \ldots, \Lambda_{i,r_i+1}^0$ with $\Lambda_{i1}^0$ held fixed. Thus for the set $\Lambda_i = \{\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}\}$ of $r_i$ diagonal matrices we denote by $\nabla_{\Lambda_i}$ the meromorphic connection on the holomorphically trivial vector bundle of rank 2 over $O_{a_i}$ whose 1-form is $\omega_{\Lambda_i}$. To describe the required deformations in more details let us begin with a deformation space.

For $k \in \mathbb{N}$ we denote by $Z^k$ the subset of the space $\mathbb{C}^k$ whose points have pairwise distinct coordinates. Then $Z^n$ will be the space of pole locations and

$$C_i = \overline{\mathbb{C}^2 \times \cdots \times \mathbb{C}^2 \times \mathbb{Z}^2}, \quad i = 1, \ldots, m,$$

will be the space of local formal equivalence classes at the pole $a_i$ (any class is determined by $r_i - 1$ diagonal matrices $\Lambda_{i2}, \ldots, \Lambda_{i,r_i}$ and a diagonal matrix $\Lambda_{i,r_i+1}$ whose eigenvalues are pairwise distinct). Define the deformation space $\mathcal{D}$ as the universal cover

$$\mathcal{D} = \tilde{Z}^n \times \tilde{C}_1 \times \cdots \times \tilde{C}_m$$

of the Cartesian product $Z^n \times C_1 \times \cdots \times C_m$.

One has the standard projections

$$a = (a_1, \ldots, a_n) : \mathcal{D} \to Z^n,$$

$$\Lambda_i = (\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}) : \mathcal{D} \to C_i, \quad i = 1, \ldots, m.$$ 

For $t \in \mathcal{D}$ we denote by $a_i(t)$ the $i$-th coordinate of the image of $t$ under the first projection and by $\Lambda_i(t)$ the image of $t$ under the second one. Denote then by $t^0$ the base point of the deformation space $\mathcal{D}$ corresponding to the system (1) (to the initial connection $\nabla^0$), i.e., $a(t^0) = (a_1^0, \ldots, a_n^0)$, $\Lambda_i(t^0) = (\Lambda_{i2}^0, \ldots, \Lambda_{i,r_i+1}^0)$. Consider also the singular hypersurfaces

$$X_i = \{(z, t) \in \overline{\mathbb{C} \times \mathcal{D}} \mid z = a_i(t)\} \subset \overline{\mathbb{C} \times \mathcal{D}}, \quad i = 1, \ldots, n.$$
Now for \( i = 1, \ldots, m \), consider the fibre bundle \( \mathcal{M}_i \to \mathcal{C}_i \), whose fiber over each point \( \Lambda_i \in \mathcal{C}_i \) is the moduli space of local holomorphic equivalence classes of connections that are all formally equivalent to the connection \( \nabla_{\Lambda_i} \). A point of this fiber (a holomorphic equivalence class of connections) is determined by a corresponding set of Stokes’ matrices. Let \( \sigma_0^i \in \mathcal{M}_i \) denote the holomorphic equivalence class of the connection \( \nabla^{0}_{O_0^i} \sim \nabla^{0}_{\Lambda_i^0} \) and let \( \sigma_i \) denote the unique horizontal section of the fibre bundle \( \mathcal{M}_i \to \mathcal{C}_i \) such that \( \sigma_1(\Lambda_i^0) = \sigma_0^i \).

Due to B. Malgrange [11, Th. 3.1] (see also [13, Th. 2.9]) the following statement holds.

**Theorem 1.** There exists a unique\(^1\) isomonodromic deformation \( (E, \nabla) \) of the pair \((E^0, \nabla^0)\), that is, the rank 2 holomorphic vector bundle \( E \) over \( \mathbb{C} \times \mathcal{D} \) and integrable meromorphic connection \( \nabla \) on \( E \) with a simple type \( r \) singularity\(^2\) along \( X_i \), \( i = 1, \ldots, n \), satisfying the following properties:

- the restriction of \( (E, \nabla) \) to \( \mathbb{C} \times \{t^0\} \) is isomorphic to \( (E^0, \nabla^0) \);

- for any \( t \in \mathcal{D} \) the restriction of \( \nabla \) to \( \mathbb{C} \times \{t\} \) is formally equivalent to the local connection \( \nabla_{\Lambda_i(t)} \) near \( z = a_i(t), i = 1, \ldots, m \), and belongs to the local holomorphic equivalence class \( \sigma_i(\Lambda_i(t)) \in \mathcal{M}_i \).

The deformation described above will be referred to as the Malgrange isomonodromic deformation of the pair \((E^0, \nabla^0)\).

According to the Malgrange–Helmink–Palmer theorem (see [13, §3] or [11, §3]) the set

\[
\Theta = \{t \in \mathcal{D} \mid E|_{\mathbb{C} \times \{t\}} \text{ is non-trivial}\}
\]

is either empty or \( \Theta \subset \mathcal{D} \) is an analytic subset of codimension one (which is usually called the Malgrange \( \Theta \)-divisor). If the latter holds, there exists a function \( \tau \) (called the \( \tau \)-function of the isomonodromic deformation) holomorphic on the whole space \( \mathcal{D} \) whose zero set coincides with \( \Theta \).

Thus the Malgrange isomonodromic deformation of the pair \((E^0, \nabla^0)\) determines an isomonodromic deformation

\[
\frac{dy}{dz} = \left( \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{B_{ij}(t)}{(z-a_i(t))^j} \right) y, \quad B_{ij}(t^0) = B_{ij}^0,
\]

of the system \( (1) \) for \( t \in D(t^0) \), where \( D(t^0) \) is a neighbourhood of the point \( t^0 \) in the space \( \mathcal{D} \). The matrix functions \( B_{ij}(t) \), holomorphic in \( D(t^0) \), can be extended meromorphically to the whole space \( \mathcal{D} \) and have \( \Theta \) as a polar locus.

Recall that for a Fuchsian system (the case of \( m = 0 \))

\[
\frac{dy}{dz} = \left( \sum_{i=1}^{n} \frac{B_{i0}^0}{z-a_i^0} \right) y
\]

\(^1\)Under some additional assumption we discuss later on.

\(^2\)That is, near \( X_i \) the local connection 1-form \( \Omega \) of \( \nabla \) looks like

\[
\Omega = \frac{B_i(z,t)}{(z-a_i(t))^r_i+1} \, dz - a_i(t) + \sum_{k} \frac{C_{ik}(z,t)}{(z-a_i(t))^r_i} \, dt_k,
\]

where the matrices \( B_i, C_{ik} \) are holomorphic and (for \( i = 1, \ldots, m \)) the eigenvalues of \( B_i(a_i(t),t) \) are pairwise distinct.
the best known isomonodromic deformation has been described by L. Schlesinger [14], [15]. Starting from the initial conditions 
\[ B_i(a^0) = B_0^i, \quad a^0 = (a_0^0, \ldots, a_0^n), \]
the residue matrices \( B_i(a) \) vary satisfying the Schlesinger equation
\[
dB_i(a) = - \sum_{j=1 \atop j \neq i}^n \left[ B_i(a), B_j(a) \right] \frac{d(a_i - a_j)}{a_i - a_j}, \quad i = 1, \ldots, n,
\]
and they are extended as meromorphic matrix functions to the deformation space \( \tilde{\mathbb{Z}}^n \) from a neighbourhood \( D(a^0) \) of the initial point \( a^0 \).

A. A. Bolibruch [2] has obtained the following result concerning pole orders of the matrices \( B_i(a) \).

**Theorem 2.** Let the monodromy of the \((2 \times 2)\)-system (4) be irreducible and let \( a^* \in \Theta \) be a point of the \( \Theta \)-divisor such that the restriction \( E|_{\mathbb{C} \times \{a^*\}} \) is of the form
\[
E|_{\mathbb{C} \times \{a^*\}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1).
\]
Then in a neighbourhood \( D(a^*) \) of \( a^* \) the \( \Theta \)-divisor is an analytic submanifold and the matrix functions \( B_i(a) \) have poles of at most second order along \( D(a^*) \cap \Theta \).

The latter means that \( \tau^2(a)B_i(a) \) are holomorphic matrix functions in \( D(a^*) \). The proof of Theorem 2 (formulated in a more general setting) also contains in [5].

Adapting Bolibruch’s ideas to the case of linear systems with irregular singularities we propose a local description of the \( \Theta \)-divisor of the Malgrange isomonodromic deformation and generalization of Theorem 2 when the initial system has at most two irregular singularities and their Poincaré ranks are equal to 1 (Theorem 3).

### 2 Holomorphic vector bundles and the Riemann–Hilbert problem for irregular systems

The fact \( t^* \in \Theta \) means that the restriction \( E|_{\mathbb{C} \times \{t^*\}} \) of the holomorphic vector bundle \( E \) described in Theorem 1 is not holomorphically trivial. This restriction belongs to the family \( \mathcal{F} \) of holomorphic vector bundles over the Riemann sphere endowed with meromorphic connections which occurs in the investigation of the corresponding Riemann–Hilbert problem. The latter is the question on existence of a global meromorphic linear system with the singular points \( a^*_1 = a_1(t^*), \ldots, a^*_n = a_n(t^*) \) of Poincaré ranks \( r_1, \ldots, r_n \) respectively that

1) has the same monodromy as the initial one and
2) is meromorphically equivalent to the local system
\[
dy = \omega_i^* y
\]
determined by the local holomorphic equivalence class \( \sigma_i(\Lambda_i(t^*)) \) near each irregular singular point \( a_i^* \).

The Riemann–Hilbert problem under consideration has a positive answer (it is sufficient one of the irregular singularities to be non-resonant for positive solution in the two-dimensional case, see [4]). This means there is a holomorphic vector bundle (not \( E|_{\mathbb{C} \times \{t^*\}} \)) in the family \( \mathcal{F} \) that is holomorphically trivial. Thus we are coming to the point where it is naturally to recall briefly the construction of the family \( \mathcal{F} \) (see details in [4]).
By the monodromy representation (generated by the monodromy matrices $G_1, \ldots, G_n$) of the initial system (11) one constructs over the punctured Riemann sphere $\mathbb{C} \setminus \{a_1^*, \ldots, a_m^*\}$ a holomorphic vector bundle $\tilde{F}$ of rank 2 with a holomorphic connection $\tilde{\nabla}$ having the prescribed monodromy. This bundle is defined by a set $\{U_\alpha\}$ of sufficiently small discs covering $\mathbb{C} \setminus \{a_1^*, \ldots, a_m^*\}$ and set $\{g_{\alpha\beta}\}$ of constant matrices defining a gluing cocycle. A connection $\tilde{\nabla}$ is defined by a set $\{\omega_\alpha\}$ of matrix differential 1-forms $\omega_\alpha \equiv 0$. So in the intersections $U_\alpha \cap U_\beta \neq \emptyset$ the gluing conditions

$$\omega_\alpha = (dg_{\alpha\beta})g^{-1}_{\alpha\beta} + g_{\alpha\beta}\omega_\beta g^{-1}_{\alpha\beta}$$

hold.

Further one extends the pair $(\tilde{F}, \tilde{\nabla})$ to the whole Riemann sphere. In neighbourhoods $O_{a_i^*}$ of the irregular singular points $a_i^*$, $i = 1, \ldots, m$, the extension of $\tilde{\nabla}$ is determined by the corresponding local matrix differential 1-forms $\omega_i^*$ of the coefficients of the systems (5), while in neighbourhoods $O_{a_i^*}$ of the Fuchsian singular points $a_i^*$, $i = m + 1, \ldots, n$, the extension of $\tilde{\nabla}$ is determined by the matrix differential 1-forms $E_i dz/(z - a_i^*)$. Here $E_i = 1/(2\pi\sqrt{-1}) \ln G_i$ is a normalized logarithm of the monodromy matrix $G_i$ and its branch is chosen so that the eigenvalues $\rho_i^k$ of $E_i$ satisfy the condition

$$0 \leq \text{Re} \rho_i^k < 1.$$  

This is the so-called canonical extension $(\tilde{F}, \tilde{\nabla})$ of the pair $(\tilde{F}, \tilde{\nabla})$ in the sense of Malgrange [12] (and Deligne [6], for the Fuchsian case).

Finally, consider a formal fundamental matrix (see [2])

$$\tilde{Y}_i(z) = \tilde{F}_i(z)(z - a_i^*)^{A_i^0}e^{Q_i(z)},$$

$$Q_i(z) = \sum_{j=1}^{\nu_i} \frac{A_{i,j+1}}{j}(z - a_i^*)^{-j}, \quad A_{i,j+1} = \Lambda_{i,j+1}(t^*),$$

of each local irregular system (5), $i = 1, \ldots, m$, and write it in the form

$$\tilde{Y}_i(z) = \tilde{F}_i(z)(z - a_i^*)^{A_i^0}(z - a_i^*)E_i e^{Q_i(z)}, \quad A_i^0 = [\text{Re} \Lambda_i^0].$$

The diagonal elements of the integer-valued matrix $A_i^0$ are referred to as the formal valuations of the system. As follows, the diagonal elements $\rho_i^k$ of the matrix $\tilde{F}_i$ satisfy the condition (6). By an analogue of Sauvage’s lemma (see [3, L. 11.2]) for formal matrix series, for any diagonal integer-valued matrix $A_i$ there exists a matrix $\Gamma_i'(z)$ meromorphically invertible in $O_{a_i^*}$ such that

$$\Gamma_i'(z)\tilde{F}_i(z)(z - a_i^*)^{A_i^0-A_i} = (z - a_i^*)\tilde{A}_i\tilde{H}_i(z),$$

where $\tilde{A}_i$ is a diagonal integer-valued matrix and $\tilde{H}_i(z)$ is an invertible formal (matrix) Taylor series in $z - a_i^*$.

Now one constructs the family $\mathcal{F}$ of extensions of the pair $(\tilde{F}, \tilde{\nabla})$ replacing the form $\omega_i^*$ in the construction of $(\tilde{F}, \tilde{\nabla})$ by the form

$$\omega^i = (d\Gamma_i)\Gamma_i^{-1} + \Gamma_i \omega_i^* \Gamma_i^{-1}, \quad \Gamma_i(z) = (z - a_i^*)^{-\tilde{A}_i}\Gamma_i'(z), \quad i = 1, \ldots, m,$$

and the form $E_i dz/(z - a_i^*)$ by the form

$$\omega^i = (d\Gamma_i)\Gamma_i^{-1} + \Gamma_i \frac{E_i dz}{z - a_i^*} \Gamma_i^{-1}, \quad \Gamma_i(z) = (z - a_i^*)^A_i S_i, \quad i = m + 1, \ldots, n,$$
where $A_i = \text{diag}(d_i^1, d_i^2)$ is a diagonal integer-valued matrix whose diagonal elements satisfy the condition $d_i^1 \geq d_i^2$, and $S_i$ is a non-singular matrix reducing the matrix $E_i$ to an upper-triangular form $E_i' = S_i E_i S_i^{-1}$. As follows from \cite{17}, \cite{3}, a formal fundamental matrix of the local irregular system $dy = \omega^{A_i}y$, $i = 1, \ldots, m$, is of the form

$$
\hat{Y}_i'(z) = \Gamma_i(z)\hat{Y}_i(z) = \hat{H}_i(z)(z - a_i^*)^{A_i}A_i(z - a_i^*)^{-A_i}E_i e^{Q_i(z)}. \quad (9)
$$

Its singular point $z = a_i^*$ is of Poincaré rank $r_i$ again. At the same time, the local system $dy = \omega^{A_i}y$, $i = m + 1, \ldots, n$, is Fuchsian:

$$
\omega^{A_i} = \left(\frac{A_i}{z - a_i^*} + (z - a_i^*)^{A_i} \frac{E_i'}{z - a_i^*} (z - a_i^*)^{-A_i}\right) dz.
$$

Let us call the matrices $A_1, \ldots, A_n$, $S_{m+1}, \ldots, S_n$ involved in the construction above, the \textit{admissible} matrices. Thus the family $\mathcal{F}$ consists of the pairs $(F^{A,S}, \nabla^{A,S})$ obtained by all sets $(A,S) = \{A_1, \ldots, A_n, S_{m+1}, \ldots, S_n\}$ of admissible matrices. Though the matrices $\Gamma_i'(z), \ldots, \Gamma_m'(z)$ (see \cite{13}) are also involved in the construction of the pair $(F^{A,S}, \nabla^{A,S})$, one should note that in our (non-resonant) case the bundle $F^{A,S}$ does not depend on them (for a fixed $(A,S)$).

Now the restriction $(E, \nabla)|_{\Sigma \times \{t^*\}}$ can be thought of as an element of the family $\mathcal{F}$:

$$
(E, \nabla)|_{\Sigma \times \{t^*\}} \cong (F^{A_0, S_0}, \nabla^{A_0, S_0}),
$$

where the matrices $A_0, \ldots, A_n$ are defined in \cite{17}, and the sets of the (admissible) matrices $A_{m+1}, \ldots, A_n$ and $S_{m+1}, \ldots, S_n$ come from the \textit{Levett} decompositions \cite{10} of a fundamental matrix $Y(z)$ of the initial system (9) at the corresponding Fuchsian singularities $a_{m+1}, \ldots, a_n$:

$$
Y(z) = U_i(z)(z - a_i^0)^{A_0}S_i^0(z - a_i^0)^{-A_i}, \quad i = m + 1, \ldots, n,
$$

where the matrix $U_i(z)$ is holomorphically invertible at the point $a_i^0$. The matrices $A_{m+1}, \ldots, A_n$ are preserved along the deformation (see \cite{3}). And one requires the matrices $S_{m+1}, \ldots, S_n$ to be also preserved, to ensure that the Malgrange deformation is a unique isomonodromic deformation of the pair $(E^0, \nabla^0)$ (see Theorem 1).

### 3 Theorem on $\Theta$-divisor

Now let us consider a linear meromorphic $(2 \times 2)$-system with $n$ singular points such that $m \leq 2$ of them are irregular and their Poincaré ranks are equal to 1, i. e., the system of the form (11), where $r_1, r_2 \leq 1$, $r_3 = \ldots = r_n = 0$:

$$
\frac{dy}{dz} = \left(\frac{B_{12}^0}{(z - a_1^0)^2} + \frac{B_{22}^0}{(z - a_2^0)^2} + \sum_{i=1}^{n} \frac{B_{1i}^0}{z - a_i^0}\right) y. \quad (10)
$$

The $\Theta$-divisor and the coefficient matrices $B_{ij}(t)$ of the Malgrange isomonodromic deformation \cite{14} of such system possess the following properties.

**Theorem 3.** Let the monodromy representation of the $(2 \times 2)$-system (10) be irreducible and let $t^* \in \Theta$ be a point of the $\Theta$-divisor such that

$$
E|_{\Sigma \times \{t^*\}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1).
$$
Then in a neighbourhood $D(t^*)$ of $t^*$ the $\Theta$-divisor is an analytic submanifold and the matrix functions $B_{ij}(t)$ have poles of at most second order along $D(t^*) \cap \Theta$.

Before proving this theorem let us recall a calculation algorithm for the local $\tau$-function of the Malgrange isomonodromic deformation $(E, \nabla)$ of the system (10).

Consider a point $t^* \in \Theta$. Though the corresponding pair $(E, \nabla)|_{t^*} \cong (F^{A^0}, S^0, \nabla A^0, S^0)$ is such that the bundle

$$F^{A^0, S^0} \cong O(-1) \oplus O(1)$$

is not holomorphically trivial, one can construct an auxiliary linear meromorphic system

$$\frac{dy}{dz} = \left( \frac{B_{12}^{*0}(t)}{(z - a_1^*)^2} + \frac{B_{22}^{*0}(t)}{(z - a_2^*)^2} + \sum_{i=1}^{n} \frac{B_{i1}^{*0}(t)}{z - a_i^*} \right) y, \quad (11)$$

with irregular non-resonant singular points $a_1^* = a_1(t^*), a_2^* = a_2(t^*)$ of Poincaré rank 1 and Fuchsian singular points $a_3^* = a_3(t^*), \ldots, a_n^* = a_n(t^*)$. This system is holomorphically equivalent to the local systems determined by the connection $\nabla A^0, S^0$ in neighbourhoods of the corresponding singular points, but it has an apparent Fuchsian singularity at the infinity (i.e., the monodromy at this point is trivial). Its fundamental matrix is of the form $Y^+(z) = U(z) z^K$ near the infinity, where

$$U(z) = I + U_1 \frac{1}{z} + U_2 \frac{1}{z^2} + \ldots, \quad K = \text{diag}(-1, 1). \quad (12)$$

Therefore, the residue matrix at the infinity is equal to $-K$, and $\sum_{i=1}^{n} B_{i1}^{*0} = K$ (existence of such a system in the Fuchsian case is explained, for example, in the proof of Theorem 2 from [16]; an explanation here is the same).

The columns of the fundamental matrix $Y^+(z)$ of the system (11) under $\mathbb{C}$ determine a basis of sections of the bundle $F^{A^0, S^0}$ horizontal with respect to $\nabla A^0, S^0$. Consider a matrix $V(z)$ holomorphically invertible in a neighbourhood $O_\infty$ of the infinity whose columns determine this basis under $O_\infty$. Then the quotient $Y^+(z) V^{-1}(z) = g_{0\infty}$ is a cocycle of the bundle $F^{A^0, S^0}$, which is $z^K$, i.e.

$$U(z) z^K = z^K V(z). \quad (13)$$

Let us include the auxiliary system (11) into the Malgrange isomonodromic family

$$\frac{dy}{dz} = \left( \frac{B_{12}^*(t)}{(z - a_1(t))^2} + \frac{B_{22}^*(t)}{(z - a_2(t))^2} + \sum_{i=1}^{n} \frac{B_{i1}^*(t)}{z - a_i(t)} \right) y, \quad B_{ij}^*(t^*) = B_{ij}^{*0}. \quad (14)$$

An appropriate matrix meromorphic differential 1-form determining this family (see [7], Ch.4, §§1) has the form

$$\omega = \sum_{i=1}^{2} \frac{B_{i2}^*(t)}{(z - a_i(t))^2} \, d(z - a_i(t)) + \sum_{i=1}^{n} \frac{B_{i1}^*(t)}{z - a_i(t)} \, d(z - a_i(t)) + (d\Lambda) \text{-part}. \quad (15)$$

Observe that the equality $\sum_{i=1}^{n} B_{i1}^*(t) = K$ holds. Indeed, the differential 1-form $\omega$ satisfies the Frobenius integrability condition, i.e., $d\omega = \omega \wedge \omega$. One can directly check that the residue (in the sense of Leray) of $\omega \wedge \omega$ along $\{z = \infty\}$ is equal to zero and the residue of $d\omega$ along $\{z = \infty\}$ is equal to $d\sum_{i=1}^{n} B_{i1}^*(t)$. 

7
Let \( Y(z, t) \) be the fundamental matrix of the Pfaffian system \( dy = \omega y \) of the form

\[
Y(z, t) = U(z, t)z^K, \quad U(z, t) = I + U_1(t)\frac{1}{z} + U_2(t)\frac{1}{z^2} + \ldots,
\]

at the infinity, and \( Y(z, t^*) = Y^*(z) \) (by analogy with the Fuchsian case \([1]\)).

As follows from \([15]\),

\[
\frac{\partial Y}{\partial a_i}Y^{-1} = -\sum_{j=1}^{r_i+1} \frac{B_{ij}^*(t)}{(z - a_i)^j} = -\sum_{j=1}^{r_i+1} \frac{B_{ij}^*(t)}{z^j(1 - \frac{a_i}{z})^j}.
\]

Expanding into series the left and the right sides of \((17)\) near the infinity, one gets

\[
\frac{\partial U_1(t)}{\partial a_i} \frac{1}{z} + o(z^{-1}) = \left(-B_{i1}^*(t)\frac{1}{z} + o(z^{-1})\right)\left(I + U_1(t)\frac{1}{z} + o(z^{-1})\right),
\]

dependence

\[
\frac{\partial U_1(t)}{\partial a_i} = -B_{i1}^*(t), \quad i = 1, \ldots, n.
\]

From the relation

\[
\frac{\partial Y}{\partial z}Y^{-1} = \sum_{i=1}^{r_i+1} \frac{B_{ij}^*(t)}{z^j(1 - \frac{a_i}{z})^j}
\]

one gets

\[
-U_1(t)\frac{1}{z^2} + o(z^{-2}) + \left(I + U_1(t)\frac{1}{z} + o(z^{-1})\right) \frac{K}{z} =
\]

\[
= \left(\frac{K}{z} + \left(\sum_{i=1}^{n} B_{i1}^*(t)a_i + B_{12}^*(t) + B_{22}^*(t)\right)\frac{1}{z^2} + o(z^{-2})\right)\left(I + U_1(t)\frac{1}{z} + o(z^{-1})\right).
\]

Hence

\[-U_1 + [U_1, K] = \sum_{i=1}^{n} B_{i1}^*(t)a_i + B_{12}^*(t) + B_{22}^*(t).
\]

Thus the upper-right element \( u_1(t) \) of the matrix \( U_1(t) \) coincides with the same element of the matrix \( \sum_{i=1}^{n} B_{i1}^*(t)a_i + B_{12}^*(t) + B_{22}^*(t) \).

**Lemma 1.** The function \( u_1(t) \) is not equal to zero identically and vanishes at the point \( t = t^* \).

**Proof.** Since the matrix \( U_1(t^*) \) is that from the decomposition \([12]\), the vanishing of \( u_1(t) \) at the point \( t^* \) follows from the relation \([13]\).

Now let us explain that the function \( u_1(t) \) is not equal to zero identically. We denote by \( b_{ij}(t) \) the upper-right elements of the matrices \( B_{ij}^*(t) \). Then

\[
u_1(t) = b_{12}(t) + b_{22}(t) + \sum_{i=1}^{n} b_{i1}(t)a_i
\]

and as follows from \([18]\),

\[
\frac{\partial u_1(t)}{\partial a_i} = -b_{i1}(t), \quad i = 1, \ldots, n.
\]
Arguing by contradiction, suppose that \( u_1(t) \equiv 0 \). Then the following equalities should be true:

\[
\begin{align*}
b_{i1}(t) &= 0, \quad i = 1, \ldots, n, \\
b_{12}(t) + b_{22}(t) &= 0.
\end{align*}
\]

We will show that \( b_{12}(t) = b_{22}(t) \equiv 0 \) as well, which contradicts irreducibility of the monodromy of the family [14].

To use the fact that \( z = \infty \) is an apparent singularity of the family [14], let us turn to a new independent variable \( \xi = z^{-1} \) and examine the matrix differential 1-form \( B^*(z,t)dz \) of the coefficients of this family near the point \( \xi = 0 \):

\[
B^*(z,t)dz = -\frac{B^*(\xi^{-1},t)}{\xi^2} d\xi, \quad -\frac{B^*(\xi^{-1},t)}{\xi^2} = -\sum_{i=1}^{2} \frac{B^*_{i2}(t)}{(1-a_i\xi)^2} - \sum_{i=1}^{n} \frac{B^*_i(t)}{\xi(1-a_i\xi)} =
\]

\[
\begin{align*}
&= -\frac{1}{\xi} \left( K + \sum_{i=1}^{n} B_{i1}(t)a_i\xi + \sum_{i=1}^{n} B_{i1}(t)a_i^2\xi^2 + o(\xi^2) \right) - \left( \sum_{i=1}^{2} B_{i2}(t) + 2 \sum_{i=1}^{n} B_{i2}(t)a_i\xi + o(\xi) \right) = \\
&= -\frac{1}{\xi} - K - \left( \sum_{i=1}^{n} B_{i1}(t)a_i + 2 \sum_{i=1}^{n} B_{i2}(t) \right) - \left( \sum_{i=1}^{2} B_{i1}(t)a_i^2 + 2 \sum_{i=1}^{n} B_{i2}(t)a_i \right) \xi + o(\xi) = \\
&= \left( \begin{array}{cc} 1 & 0 \\
0 & -1 \end{array} \right) \frac{1}{\xi} + \left( \begin{array}{cc} * & 0 \\
* & * \end{array} \right) + \left( \begin{array}{cc} * & -2 \sum_{i=1}^{2} b_{i2}(t)a_i \\
* & * \end{array} \right) \xi + o(\xi).
\end{align*}
\]

The gauge transformation \( \tilde{y} = \xi^K y \) changes the latter matrix into a new one having the form

\[
\frac{1}{\xi} \left( \begin{array}{cc} 0 & -2 \sum_{i=1}^{2} b_{i2}(t)a_i \\
0 & 0 \end{array} \right) + O(1).
\]

The monodromy matrix of the Fuchsian singular point \( \xi = 0 \) of the transformed system is identity. On the other hand, both eigenvalues of its residue matrix are zeros. Thus the monodromy matrix is equal to the exponent of the residue matrix, i.e.,

\[
\exp 2\pi \sqrt{-1} \left( \begin{array}{cc} 0 & -2 \sum_{i=1}^{2} b_{i2}(t)a_i \\
0 & 0 \end{array} \right) = I.
\]

Then the equality \( b_{12}(t)a_1 + b_{22}(t)a_2 \equiv 0 \) holds, which (together with the equality \( b_{12}(t) + b_{22}(t) \equiv 0 \)) implies \( b_{12}(t) = b_{22}(t) \equiv 0 \). \( \square \)

**Lemma 2.** The function \( u_1(t) \) is a local \( \tau \)-function of the Malgrange isomonodromic deformation of the system [10], i.e., it locally determines the \( \Theta \)-divisor near the point \( t^* \in \Theta \).

**Proof.** If \( u_1(t) \neq 0 \), then we can consider a holomorphically invertible (with respect to \( z \)) in \( \mathbb{C} \) matrix

\[
\Gamma_1^*(z,t) = \left( \begin{array}{cc} \frac{1}{-u_1(t)} & 0 \\
\frac{z}{-u_1(t)} & 1 \end{array} \right).
\]

By the construction the matrix \( U'(z,t) = \Gamma_1^*(z,t)U(z,t) \) is of the form

\[
U'(z,t) = \left( U'_0(t) + U'_1(t) \frac{1}{z} + \ldots \right) z^{-K}, \quad U'_0(t) = \left( \begin{array}{c} 0 \\
-\frac{1}{u_1(t)} \frac{u_1(t)}{f(t)} \frac{1}{u_1(t)} \end{array} \right),
\]

where \( f(t) \) is a holomorphic function at the point \( t = t^* \).
The gauge transformation

$$y_1 = \Gamma_1(z,t)y, \quad \Gamma_1(z,t) = U_0'(t)^{-1}\Gamma_1'(z,t),$$

(19)

transforms the system (14) into a new one, with a fundamental matrix

$$Y^1(z,t) = \Gamma_1(z,t)Y(z,t)$$

(20)

that is holomorphically invertible at the infinity. As the columns of the matrix $Y(z,t)$ form a basis of horizontal (with respect to the restriction of the connection $\nabla$ on $\mathbb{C} \times \{t\}$) sections of the bundle $E_{\mathbb{C} \times \{t\}}$ over $\mathbb{C}$, the relation (20) implies a holomorphic triviality of this bundle.

If $u_1(t) = 0$, then the matrix

$$V_\infty(z) = z^{-K}U(z,t)z^K = z^{-K}\left(I + \left( \begin{array}{cc} * & 0 \\ * & * \end{array} \right) \frac{1}{z} + \ldots \right)z^K$$

is holomorphically invertible at the infinity, hence $Y(z,t) = z^KV_\infty(z)$ and $E|_{\mathbb{C} \times \{t\}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$.

**Proof of Theorem 3.** First we explain that $du_1(t^*) \neq 0$. Indeed, in the opposite case the following equalities should be true:

$$b_{i1}(t^*) = 0, \quad i = 1, \ldots, n,$$

$$b_{i2}(t^*) + b_{22}(t^*) = 0.$$

Then similarly to the proof of Lemma 1 one gets the relations $b_{12}(t^*) = b_{22}(t^*) = 0$, which contradict the monodromy irreducibility. Thus the $\Theta$-divisor of the Malgrange isomonodromic deformation of the system (10) is an analytic submanifold in a neighbourhood $D(t^*)$ of the point $t^*$.

Now let us estimate the pole orders of the matrices $B_{i1}(t), B_{i2}(t), B_{22}(t)$ along $\Theta \cap D(t^*)$. Return to the proof of Lemma 2. The family obtained from (14) via the gauge transformation (19), coincides with the Malgrange isomonodromic deformation (for $t \in D(t^*) \setminus \Theta$) of the initial system (10). (Indeed, this transformation does not change connection matrices at the Fuchsian singular points and it also does not change holomorphic equivalence classes of the family at the irregular singularities.) Therefore the coefficient matrix of the Malgrange isomonodromic deformation of the initial system (10) has the form

$$\frac{\partial \Gamma_1}{\partial z}\Gamma_1^{-1} + \Gamma_1\left(\frac{B_{i2}(t)}{(z - a_1(t))^2} + \frac{B_{22}(t)}{(z - a_2(t))^2} + \sum_{i=1}^n \frac{B_{ii}(t)}{z - a_i(t)}\right)\Gamma_1^{-1}.$$

As the matrix $\Gamma_1(z,t)$ is holomorphically invertible (with respect to $z$) in $\mathbb{C}$, one has

$$B_{i1}(t) = \Gamma_1(a_i(t),t) B_{i1}^*(t)\Gamma_1^{-1}(a_i(t),t), \quad i = 3, \ldots, n,$$

and for $i = 1, 2$ one has

$$B_{i2}(t) = \Gamma_1(a_i(t),t) B_{i2}^*(t)\Gamma_1^{-1}(a_i(t),t),$$

$$B_{i1}(t) = \frac{\partial \Gamma_1}{\partial z}(a_i(t),t) B_{i2}^*(t)\Gamma_1^{-1}(a_i(t),t) + \Gamma_1(a_i(t),t) B_{i2}^*(t)\Gamma_1^{-1}(a_i(t),t) +$$

$$+ \Gamma_1(a_i(t),t) B_{i2}^*(t)\frac{\partial \Gamma_1^{-1}}{\partial z}(a_i(t),t).$$
Since
\[
\Gamma_1(z,t) = U_0'(t)^{-1} \Gamma_1(z,t) = \begin{pmatrix}
\frac{f(t)}{u_1(t)} & -u_1(t) \\
\frac{-z}{u_1(t)} & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{-z}{u_1(t)} & 1
\end{pmatrix} = \begin{pmatrix}
z + \frac{f(t)}{u_1(t)} & -u_1(t) \\
\frac{1}{u_1(t)} & 0
\end{pmatrix}
\]
and the matrices \(B_{ij}^*(t)\) are holomorphic near the point \(t = t^*\), one sees that the same holds for all the matrices \((u_1(t))^2 B_{ij}(t)\).

\[\square\]

**Remark.** Recall that the Painlevé III and V equations can be described in terms of isomonodromic deformations satisfying Theorem 3 (see details in [7, Ch. 5, §§4,5]): for P_{III} one has \(m = n = 2\) and for P_{V} one has \(m = 1, n = 3\). If \(t^* \in \Theta\) and \(E_{\mathbb{C} \setminus \{t^*\}} \cong O(-k) \oplus O(k)\), then the estimate \(2k \leq m + n - 2\) holds [4] when the monodromy of a connection is irreducible. Thus \(2k \leq 2\) and hence \(k = 1\) in the both cases.

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