RADIATIVE SEMILEPTONIC KAON DECAYS

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Abstract

We evaluate the matrix elements for the radiative kaon decays $K^+ \rightarrow l^+ \nu l \gamma$, $l^+ \nu l^+ l^−$ and $K \rightarrow \pi l \nu l \gamma$ $(l, l' = e, \mu)$ to next-to-leading order in chiral perturbation theory. We calculate total rates and rates with several kinematical cuts and confront the results with existing data. Measurements at future kaon facilities will allow for a more detailed comparison between theory and experiment.

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1 Introduction

There are two main reasons for undertaking a systematic study of radiative semileptonic kaon decays at this time:

- Very intense kaon beams will soon become available, which will increase the existing, rather meager, data sample dramatically. In fact, the present investigation was triggered by the experimental program foreseen at the Φ factory DAFNE at present under construction in Frascati [1].

- The standard model can be tested in those processes to next-to-leading order in the chiral expansion without any further assumptions. All low-energy constants involved have already been determined phenomenologically, allowing for unambiguous predictions for a large number of available channels. As is the case in general, there is of course no guarantee that effects of $O(p^6)$ or higher in chiral perturbation theory (CHPT) [2, 3, 4] are small. However, the full amplitudes of $O(p^4)$ for radiative semileptonic kaon decays can be directly confronted with experiment.

The amplitudes at leading order in CHPT coincide with the current algebra amplitudes of pre-QCD times. At next-to-leading order, the amplitudes of $O(p^4)$ carry additional dynamical information. The one-loop amplitudes entering at $O(p^4)$ are in general divergent and must be supplemented by local counterterm amplitudes depending on a number of (renormalized) low-energy coupling constants [4]. The last part is due to the chiral anomaly [5] which contributes to all the $K$-decay channels considered here. At the mesonic level, all anomalous amplitudes can be derived from the Wess–Zumino–Witten functional [6] without any free parameters. Radiative semileptonic decays thus offer a number of possibilities to experimentally investigate one of the fundamental aspects of the standard model.

The present study completes earlier work by Donoghue and Holstein [7, 8] who did not calculate the loop amplitudes. Although non-leading in $1/N_C$, where $N_C$ is the number of colours, the importance of loop contributions depends very much on the specific process under consideration. In fact, radiative semileptonic transitions offer a wide range of possibilities in this respect:

- Except for the renormalization of the kaon-decay constant, there is no loop amplitude for $K_{l2\gamma}$.

- For $K_{l2l^+l^−}$, the one-loop contribution is contained in the electromagnetic form factor of the charged kaon. This form factor is known to be dominated by the coupling constant $L_9$ at low energies [3].
• The loop amplitude is sizeable for $K_{l3\gamma}$ and interferes destructively with the local counterterm amplitude. It is essential for a meaningful comparison with experiment.

The outline of the paper is as follows. In Sect. 2 we recapitulate CHPT to next-to-leading order to the extent needed in subsequent sections. The decay amplitudes for $K_{l2\gamma}$ are calculated in Sect. 3 to $O(p^4)$ in CHPT. The decays $K_{l2e^+e^-}$, $\pi_{l2e^+e^-}$ and $K_{l3\gamma}$ are treated in Sects. 4 and 5, respectively. Conclusions are drawn in Sect. 6. Appendix A contains a summary of our notation and of integrals appearing in the various one-loop amplitudes. The decomposition of the tensor amplitude for $K_{l2\gamma}$ into invariant amplitudes is discussed in Appendix B. Finally, Appendix C contains a derivation of Ward identities without using formal manipulations of $T$-products and current commutators.

2 CHPT to next-to-leading order

Chiral perturbation theory is a systematic approach to formulate the standard model as a quantum field theory at the hadronic level. It is characterized by an effective chiral Lagrangian in terms of pseudoscalar meson fields (and possibly other low-lying hadronic states) giving rise to a systematic low-energy expansion of amplitudes [2, 3, 4].

In the formulation of Ref. [4], one considers the generating functional $Z[v, a, s, p]$ of connected Green functions of quark currents associated with the fundamental Lagrangian

$$L = L^0_{\text{QCD}} + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q.$$ (2.1)

Here, $L^0_{\text{QCD}}$ is the QCD Lagrangian with the masses of the three light quarks set to zero. The external fields $v_\mu, a_\mu, s$ and $p$ are Hermitian $3 \times 3$ matrices in flavour space. To describe electromagnetic and semileptonic interactions, the relevant external gauge fields of the standard model are $r_\mu$ and $l_\mu$:

$$r_\mu = v_\mu + a_\mu = -eQA_\mu$$

$$l_\mu = v_\mu - a_\mu = -eQA_\mu - \frac{e}{\sqrt{2}\sin\theta_W}(W_\mu^+T_+ + \text{h.c.})$$ (2.2)

$$Q = \frac{1}{3}\text{diag}(2, -1, -1), \quad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

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1We adopt the present conventions of the Particle Data Group [10].
where the $V_{ij}$ are Kobayashi–Maskawa matrix elements. The quark-mass matrix

$$
\mathcal{M} = \text{diag}(m_u, m_d, m_s)
$$

is contained in the scalar field $s(x)$.

The generating functional $Z$ admits an expansion in powers of external momenta and quark masses (CHPT). In the meson sector at leading order in CHPT, it is given by the classical action

$$
Z_2 = \int d^4x \mathcal{L}_2(U, v, a, s, p),
$$

with $\mathcal{L}_2$ the non-linear $\sigma$ model Lagrangian coupled to the external fields $v, a, s, p$:

$$
\mathcal{L}_2 = \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \rangle,
$$

where

$$
D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu, \quad \chi = 2B_0(s + ip),
$$

and $\langle A \rangle$ stands for the trace of the matrix $A$; $U$ is a unitary $3 \times 3$ matrix

$$
U^\dagger U = 1, \quad \text{det} \ U = 1,
$$

which incorporates the fields of the eight pseudoscalar Goldstone bosons. A convenient parametrization of it is:

$$
U = \exp (i\sqrt{2}\Phi/F), \quad \Phi = \begin{pmatrix}
\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & -\pi^+ & -K^+ \\
\pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & -K^0 \\
K^- & -\bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}}
\end{pmatrix}.
$$

The functional $Z_2$ is invariant under local $SU(3)_L \times SU(3)_R$ transformations:

$$
Z_2(v', a', s', p') = Z_2(v, a, s, p),
$$

with

$$
v'_\mu + a'_\mu = g_R(v_\mu + a_\mu)g_R^+ + ig_R\partial_\mu g_R^+, \\
v'_\mu - a'_\mu = g_L(v_\mu - a_\mu)g_L^+ + ig_L\partial_\mu g_L^+, \\
s' + ip' = g_R(s + ip)g_L^+.
$$

The matrix $U$ transforms as $U \rightarrow g_R U g_L^+$ under (2.9).

---

We follow the Condon–Shortley–de Swart phase conventions.
The parameters $F$ and $B_0$ are the only free constants at $O(p^2)$: $F$ is the pion decay constant in the chiral limit [cf. Eq. (2.17)]

$$F_\pi = F[1 + O(m_{quark})] = 93.2 \text{ MeV}$$

(2.10)

whereas $B_0$ is related to the quark condensate:

$$\langle 0 | \bar{u} u | 0 \rangle = -F^2 B_0 [1 + O(m_{quark})] ;$$

(2.11)

$B_0$ always appears multiplied by quark masses. At $O(p^2)$, the product $B_0 m_q$ can be expressed in terms of meson masses because

$$M_{\pi^+}^2 = B_0 (m_u + m_d) + O(M^2) .$$

(2.12)

The Lagrangian (2.5) is referred to as the effective chiral Lagrangian of $O(p^2)$. The chiral counting rules are the following: the field $U$ is of $O(p^0)$, the derivative $\partial_\mu$ and the external gauge fields $v_\mu, a_\mu$ are terms of $O(p)$, and the fields $s, p$ count as $O(p^2)$.

At order $p^4$ the generating functional consists of three different classes of contributions [4]:

i) The one-loop diagrams generated by the lowest-order Lagrangian (2.5).

ii) An explicit local action of order $p^4$.

iii) A contribution to account for the chiral anomaly.

The contributions from categories (i) and (ii) are invariant under (2.9), whereas those from (iii) are not. Below we will use the Bardeen form [5] of this non-invariant part. In this convention, the generating functional is still invariant under local gauge transformations generated by the vector currents,

$$Z(v', a', s', p') = Z(v, a, s, p) ; g_R = g_L .$$

(2.13)

The local action of $O(p^4)$ [class (ii)] is generated by the Lagrangian $L_4$ [4]:

$$\mathcal{L}_4 = \ldots + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle + L_5 \langle D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi) \rangle + \ldots - iL_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle ,$$

(2.14)

where

$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i [r^\mu, r^\nu]$$

$$F_L^{\mu\nu} = \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i [\ell^\mu, \ell^\nu]$$

(2.15)
Table 1: Phenomenological values \[^4\] for the coupling constants $L_i^r(M_\rho)$ ($i = 4, 5, 9, 10$). The quantities $\Gamma_i$ determine the scale dependence of the $L_i^r(\mu)$ according to Eq. (2.16).

| $i$ | $L_i^r(M_\rho) \times 10^3$ | $\Gamma_i$ |
|-----|----------------------------|------------|
| 4   | $-0.3 \pm 0.5$             | 1/8        |
| 5   | $1.4 \pm 0.5$              | 3/8        |
| 9   | $6.9 \pm 0.7$              | 1/4        |
| 10  | $-5.5 \pm 0.7$             | $-1/4$     |

and where we have only written down the terms necessary for our purposes. The low-energy couplings $L_4, L_5, L_9, L_{10}$ are divergent. They absorb the divergences of the one-loop graphs leading to amplitudes depending on renormalized, finite couplings $L_i^r(\mu)$ with the following scale dependence

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln \frac{\mu_1}{\mu_2}. \quad (2.16)$$

Observable quantities are independent of the scale $\mu$, once the loop contributions are included. The coefficients $\Gamma_i$ are displayed in Table 1 together with the phenomenological values of the renormalized coupling constants $L_i^r(M_\rho)$. In fact, the constants $L_4$ and $L_5$ will never appear explicitly in our amplitudes because we will use the relations \[^4\]

$$\frac{F_\pi}{F} = 1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{F^2} L_5^r(\mu) + \frac{4(M_\pi^2 + 2M_K^2)}{F^2} L_4^r(\mu)$$

$$\frac{F_K}{F} = 1 - \frac{3}{4} \mu_\pi - \frac{3}{2} \mu_K - \frac{3}{4} \mu_\eta + \frac{4M_K^2}{F^2} L_5^r(\mu) + \frac{4(M_\pi^2 + 2M_K^2)}{F^2} L_4^r(\mu)$$

$$\mu_P = \frac{M_P^2}{16\pi^2 F^2} \log \frac{M_P}{\mu}, \quad \frac{F_K}{F_\pi} = 1.22 \quad (2.17)$$

which are valid at order $p^4$ in the isospin symmetry limit $m_u = m_d$.

We now turn to point (iii) above. The contributions of the chiral anomaly to CHPT amplitudes of $O(p^4)$ can be derived from the Wess–Zumino–Witten functional \[^6\]. Here, we will only write down the pieces relevant for the radiative decays under consideration. These terms can be expressed as two anomalous Lagrangians of $O(p^4)$

$$L_{\text{anom}}(W\gamma) = -\frac{i\alpha}{4\sqrt{2}\pi \sin \theta_W} \varepsilon^{\mu\nu\alpha\beta} W_\mu^+ F_{\nu\alpha} \left\langle T_+ \left\{ U^\dagger D_\beta U, Q + \frac{1}{2} U^\dagger QU \right\} \right\rangle + \text{h.c.} \quad (2.18)$$
\[
\mathcal{L}_{\text{anom}}(\Phi^3) = -\frac{e}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}A_\sigma\langle Q \left[ \partial_\mu U \partial_\nu U^\dagger \partial_\rho U \partial_\sigma U^\dagger U \right] \rangle
\]
\[
= -\frac{ie\sqrt{2}}{4\pi^2 F^3}\varepsilon^{\mu\nu\rho\sigma}A_\sigma\langle Q \partial_\mu \Phi \partial_\nu \Phi \partial_\rho \Phi \rangle + \ldots ,
\] (2.19)

where \( D_\mu U = \partial_\mu U + ieA_\mu [Q, U] \) is the covariant derivative with respect to electromagnetism only and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field strength tensor.

We are now in a position to calculate the amplitudes for radiative semileptonic \( K \) and \( \pi \) decays to \( O(p^4) \) in CHPT. We will actually calculate the various amplitudes for an external \( W \), which then turns into a lepton and its associated neutrino. For the low energies that are relevant here, we can neglect the momentum dependence of the \( W \) propagator.

### 3 Radiative \( K_{l\gamma} \) decays

We consider the \( K_{l\gamma} \) decay

\[
K^+(p) \rightarrow l^+(p_l)\nu_l(p_\nu)\gamma(q) \quad [K_{l\gamma}] ,
\] (3.1)

where \( l \) stands for \( e \) or \( \mu \), and \( \gamma \) is a real photon with \( q^2 = 0 \). Processes where the (virtual) photon converts into a \( e^+e^- \) or \( \mu^+\mu^- \) pair are considered in the next section. The \( K^- \) mode is obtained from (3.1) by charge conjugation.

#### 3.1 Matrix elements and kinematics

The matrix element for \( K^+ \rightarrow l^+\nu\gamma \) has the structure

\[
T = -iG_F e V_{us}^* \epsilon^*_\mu \{ F_K L^\mu - H^{\mu\nu} l_\nu \} \]
(3.2)

with

\[
L^\mu = m_l \bar{u}(p_\nu) (1 + \gamma_5) \left( \frac{2p^\mu}{2pq} - \frac{2p_l^\mu + q_\gamma^\mu}{2pq} \right) v(p_l)
\]

\[
l^\mu = \bar{u}(p_\nu) \gamma^\mu (1 - \gamma_5) v(p_l)
\]

\[
H^{\mu\nu} = iV(W^2) \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta - A(W^2)(qW g^{\mu\nu} - W^\mu q^\nu)
\]

\[
W^\mu = (p - q)^\mu = (p_l + p_\nu)^\mu.
\] (3.3)

Here, \( \epsilon_\mu \) denotes the polarization vector of the photon with \( q^\mu \epsilon_\mu = 0 \), whereas \( A, V \) stand for two Lorentz-invariant amplitudes which occur in the general decomposition of the tensors.
\[ I^{\mu\nu} = \int dx e^{i q x + i W y} \langle 0 \mid TV_{em}(x) I_{4-\gamma 5}^{\mu}(y) \mid K^+(p) \rangle , \quad I = V, A. \quad (3.4) \]

The form factor \( A (V) \) is related to the matrix element of the axial (vector) current in (3.4). In Appendix B we display the general decomposition of \( A^{\mu\nu}, V^{\mu\nu} \) for \( q^2 \neq 0 \) and provide also the link with the notation used by the PDG [10] and in [11, 12].

The term proportional to \( L^\mu \) in (3.2) does not contain unknown quantities – it is determined by the amplitude of the non-radiative decay \( K^+ \to l^+ \nu_l \). This part of the amplitude is usually referred to as ”inner Bremsstrahlung (IB) contribution”, whereas the term proportional to \( H^{\mu\nu} \) is called ”structure-dependent (SD) part”.

The form factors are analytic functions in the complex \( W^2 \) plane cut along the positive real axis. The cut starts at \( W^2 = (M_K + 2M_\pi)^2 \) for \( A \) [at \( W^2 = (M_K + M_\pi)^2 \) for \( V \)]. In our phase convention, \( A \) and \( V \) are real in the physical region of \( K_{l2\gamma} \) decays,

\[ m_l^2 \leq W^2 \leq M_K^2. \quad (3.5) \]

The kinematics of (spin-averaged) \( K_{l2\gamma} \) decays needs two variables, for which we choose the conventional quantities

\[ x = 2pq/M_K^2, \quad y = 2pp_l/M_K^2. \quad (3.6) \]

In the \( K \) rest frame, the variable \( x \) (\( y \)) is proportional to the photon (charged lepton) energy,

\[ x = 2E_\gamma/M_K, \quad y = 2E_l/M_K, \quad (3.7) \]

and the angle \( \theta_{l\gamma} \) between the photon and the charged lepton is related to \( x \) and \( y \) by

\[ x = \frac{(1 - y/2 + A/2)(1 - y/2 - A/2)}{1 - y/2 + A/2 \cos \theta_{l\gamma}}; \quad A = \sqrt{y^2 - 4r_l}. \quad (3.8) \]

In terms of these quantities, one has

\[ W^2 = M_K^2(1 - x); \quad (q^2 = 0). \quad (3.9) \]

We write the physical region for \( x \) and \( y \) as

\[ 2\sqrt{r_l} \leq y \leq 1 + r_l \]

\[ 1 - \frac{1}{2}(y + A) \leq x \leq 1 - \frac{1}{2}(y - A) \quad (3.10) \]

or, equivalently, as

\[ 0 \leq x \leq 1 - r_l \]

\[ 1 - x + \frac{r_l}{(1 - x)} \leq y \leq 1 + r_l \quad (3.11) \]
where

\[ r_l = \frac{m_l^2}{M_K^2} = \begin{cases} 1.1 \cdot 10^{-6} & (l = e) \\ 4.6 \cdot 10^{-2} & (l = \mu) \end{cases} \]  

(3.12)

### 3.2 Decay rates

The partial decay rate is

\[ d\Gamma = \frac{1}{2M_K (2\pi)^5} \sum_{\text{spins}} |T|^2 dLIPS(p; p_l, p_\nu, q) \]  

(3.13)

The Dalitz plot density

\[ \rho(x, y) = \frac{d^2\Gamma}{dx dy} = \frac{M_K}{256\pi^3} \sum_{\text{spins}} |T|^2 \]  

(3.14)

is a Lorentz-invariant function which contains \( V \) and \( A \) in the following form [13],

\[
\rho(x, y) = \rho_{IB}(x, y) + \rho_{SD}(x, y) + \rho_{INT}(x, y)
\]

\[
\rho_{IB}(x, y) = A_{IB} f_{IB}(x, y)
\]

\[
\rho_{SD}(x, y) = A_{SD} M_K^2 [ (V + A)^2 f_{SD^+}(x, y) + (V - A)^2 f_{SD^-}(x, y) ]
\]

\[
\rho_{INT}(x, y) = A_{INT} M_K [ (V + A) f_{INT^+}(x, y) + (V - A) f_{INT^-}(x, y) ]
\]

(3.15)

where

\[
f_{IB}(x, y) = \left[ \frac{1 - y + r_l}{2x(1 - x)(1 - r_l)} \right] \left[ x^2 + 2(1 - x)(1 - r_l) - 2x r_l (1 - r_l) \right]
\]

\[
f_{SD^+}(x, y) = \left[ \frac{x + y - 1 - r_l}{(x + y - 1)(1 - x) - r_l} \right]
\]

\[
f_{SD^-}(x, y) = \left[ \frac{1 - y + r_l}{(1 - x)(1 - y) + r_l} \right]
\]

\[
f_{INT^+}(x, y) = \left[ \frac{1 - y + r_l}{x(x + y - 1 - r_l)} \right] \left[ (1 - x)(1 - y) + r_l \right]
\]

\[
f_{INT^-}(x, y) = \left[ \frac{1 - y + r_l}{x(x + y - 1 - r_l)} \right] \left[ x^2 - (1 - x)(1 - y) \right] - r_l
\]

(3.16)

and

\[
A_{IB} = 4r_l \left( \frac{F_K}{M_K} \right)^2 A_{SD}
\]

\[
A_{SD} = \frac{G^2 |V_{us}|^2 \alpha}{32\pi^2} M_K^5
\]

\[
A_{INT} = 4r_l \left( \frac{F_K}{M_K} \right) A_{SD}
\]

(3.17)
Table 2: The quantities $X_I, N_I$. SD$^\pm$ and INT$^\pm$ are evaluated with full phase space, IB with restricted kinematics (3.20).

|     | SD$^+$ | SD$^-$ | INT$^+$ | INT$^-$ | IB       | $K_{e2\gamma}$ |
|-----|--------|--------|--------|--------|----------|---------------|
| $X_I$ | $1.67 \cdot 10^{-2}$ | $1.67 \cdot 10^{-2}$ | $-8.22 \cdot 10^{-8}$ | $3.67 \cdot 10^{-6}$ | $3.58 \cdot 10^{-6}$ | $K_{e2\gamma}$ |
| $X_I$ | $1.18 \cdot 10^{-2}$ | $1.18 \cdot 10^{-2}$ | $-1.78 \cdot 10^{-3}$ | $1.23 \cdot 10^{-2}$ | $3.68 \cdot 10^{-2}$ | $K_{\mu2\gamma}$ |
| $N_I$ | 2      | 2      | 1      | 1      | 0        |               |

For later convenience, we note that

$$A_{SD} = \frac{\alpha}{8\pi r_l(1-r_l)^2} \left( \frac{M_K}{F_K} \right)^2 \Gamma(K \to l\nu_l).$$  

(3.18)

The subscripts IB, SD and INT stand respectively for the contribution from inner Bremsstrahlung, from the structure-dependent part, and from the interference term between the IB and the SD parts in the amplitude.

To get a feeling for the magnitude of the various contributions IB, SD$^\pm$ and INT$^\pm$ to the decay rate, we consider the integrated rates

$$\Gamma_I = \int_{R_I} dxdy\rho_I(x,y); \ I = \text{SD}^\pm, \text{INT}^\pm, \text{IB},$$  

(3.19)

where $\rho_{SD} = \rho_{SD^+} + \rho_{SD^-}$ etc. For the region $R_I$ we take the full phase space for $I \neq \text{IB}$, and

$$R_{IB} = 214.5 \text{ MeV}/c \leq p_l \leq 231.5 \text{ MeV}/c$$  

(3.20)

for the Bremsstrahlung contribution. Here $p_l$ stands for the modulus of the lepton three-momentum in the kaon rest system. We consider constant form factors $V, A$ and write for the rates and for the corresponding branching ratios

$$\Gamma_I = A_{SD} \{M_K(V \pm A)\}^{N_I} X_I$$

$$\text{BR}_I = \frac{\Gamma_I/\Gamma_{tot}}{N \{M_K(V \pm A)\}^{N_I} X_I}$$

(3.21)

with

$$N = A_{SD}/\Gamma_{tot} = 8.348 \cdot 10^{-2}.$$  

(3.22)

The values for $N_I$ and $X_I$ are listed in Table 2.

To estimate $\Gamma_I$ and $\text{BR}_I$, we note that the form factors $V, A$ are of order

$$M_K(V + A) \simeq -10^{-1}, \ M_K(V - A) \simeq -4 \cdot 10^{-2}.$$  

(3.23)

---

3 This cut has been used in [12] for $K_{\mu2\gamma}$, because this kinematical region is free from $K_{\mu3}$ background. We apply it here for illustration also to the electron mode $K_{e2\gamma}$. 
From this and from the entries in the table one concludes that for the above regions $R_I$, the interference terms $\text{INT}^\pm$ are negligible in $K_{e2\gamma}$, whereas they are important in $K_{\mu2\gamma}$. Furthermore, $\text{IB}$ is negligible for $K_{e2\gamma}$, because it is helicity-suppressed, as can be seen from the factor $m_l^2$ in $A_{\text{IB}}$. This term dominates however in $K_{\mu2\gamma}$.

### 3.3 Determination of $A(W^2)$ and $V(W^2)$

The decay rate contains two real functions

$$F^\pm(W^2) = V(W^2) \pm A(W^2)$$

as the only unknowns. The density distributions $f_{\text{IB}}, \ldots, f_{\text{INT}^\pm}$ have very different Dalitz plots [13, 14]. Therefore, in principle, one can determine the strength of each term by choosing a suitable kinematical region of observation. To pin down $F^\pm$, it would be sufficient to measure at each photon energy the interference term $\text{INT}^\pm$. This has not yet been achieved, either because the contribution of $\text{INT}^\pm$ is too small (in $K_{e2\gamma}$), or because too few events have been collected (in $K_{\mu2\gamma}$). On the other hand, from a measurement of $\text{SD}^\pm$ alone one can determine $A, V$ only up to a fourfold ambiguity:

$$\text{SD}^\pm \to \{(V, A); -(V, A); (A, V); -(A, V)\}.$$  \hspace{1cm} (3.25)

In terms of the ratio

$$\gamma_K = A/V$$

this ambiguity amounts to

$$\text{SD}^\pm \to \{\gamma_K; 1/\gamma_K\}.$$  \hspace{1cm} (3.27)

Therefore, in order to pin down the amplitudes $A$ and $V$ uniquely, one must measure the interference terms $\text{INT}^\pm$ as well.

### 3.4 Previous experiments

$K^+ \to e^+\nu_e\gamma$

The PDG uses data from two experiments [11, 13], both of which have been sensitive mainly to the $\text{SD}^+$ term in (3.16). In [13], 56 events with $E_\gamma > 100$ MeV, $E_{e^+} > 236$ MeV and $\theta_{e^+\gamma} > 120^\circ$ have been identified, whereas the later experiment [11] has collected 51 events with $E_\gamma > 48$ MeV, $E_{e^+} > 235$ MeV, and $\theta_{e^+\gamma} > 140^\circ$. In these kinematical regions, background from $K^+ \to e^+\nu_e\pi^0$ is
absent because \( E_{e}^{\max}(K_{e3}) = 228 \text{ MeV} \). The combined result of both experiments is

\[
\Gamma(\text{SD}+)/\Gamma(K_{\mu2}) = (2.4 \pm 0.36) \cdot 10^{-5}. \tag{3.28}
\]

For \( \text{SD}^- \), the bound

\[
\Gamma(\text{SD}^-)/\Gamma(\text{total}) < 1.6 \cdot 10^{-4} \tag{3.29}
\]

has been obtained from a sample of electrons with energies \( 220 \text{ MeV} \leq E_e \leq 230 \text{ MeV} \) \cite{11}. Using (3.21, 3.22), the result (3.28) leads to

\[
M_K|V + A| = 0.105 \pm 0.008. \tag{3.30}
\]

The bound (3.29) on the other hand implies \cite{11}

\[
|V - A| / |V + A| < \sqrt{11}, \tag{3.31}
\]

from where it is concluded \cite{11} that \( \gamma_K \) is outside the range \(-1.86 \) to \(-0.54 \),

\[
\gamma_K \notin [-1.86, -0.54]. \tag{3.32}
\]

As we already mentioned, the interference terms \( \text{INT}^\pm \) in \( K \rightarrow e\nu_e\gamma \) are small and can hardly ever be measured. As a result of this, the amplitudes \( A \), \( V \) and the ratio \( \gamma_K \) determined from \( K_{e2\gamma} \) are subject to the ambiguities (3.25), (3.27).

\[ K^+ \rightarrow \mu^+\nu_\mu\gamma \]

Here, the interference terms \( \text{INT}^\pm \) are non-negligible in appropriate regions of phase space, see \cite{13, 14}. Therefore, this decay allows one in principle to pin down \( V \) and \( A \). The PDG uses data from two experiments \cite{12, 16}. In \cite{12}, the momentum spectrum of the muon was measured in the region (3.20). In total \( 2 \pm 3.44 \text{ SD}^+ \) events have been found with \( 216 \text{ MeV/c} < p_\mu < 230 \text{ MeV/c} \) and \( E_\gamma > 100 \text{ MeV} \), which leads to

\[
M_K|V + A| < 0.16. \tag{3.33}
\]

In order to identify the effect of the \( \text{SD}^- \) terms, the region \( 120 \text{ MeV/c} < p_\mu < 150 \text{ MeV/c} \) was searched. Here, the background from \( K_{\mu3} \) decays was very serious. The authors found 142 \( K_{\mu\nu\gamma} \) candidates and conclude that

\[
-1.77 < M_K(V - A) < 0.21. \tag{3.34}
\]

The result (3.33) is consistent with (3.30), and the bound (3.34) is worse than the result (3.31) obtained from \( K_{e2\gamma} \). The branching ratios which follow \cite{12} from (3.33, 3.34) are displayed in Table 3, where we also show the \( K_{e2\gamma} \) results \cite{15, 11}. The entry \( SD^- + \text{INT}^- \) for \( K_{\mu2\gamma} \) is based on additional constraints from \( K_{e2\gamma} \) \cite{12}.

\footnote{In all four experiments \cite{15, 11, 12, 16} discussed here and below, the form factors \( A \) and \( V \) have been treated as constants.}
Table 3: Measured branching ratios $\Gamma(K \rightarrow l\nu l\gamma)/\Gamma_{\text{total}}$. The $K_{e2\gamma}$ data are from [13, 11], the $K_{\mu2\gamma}$ data from [12, 14]. The last column corresponds [12] to the cut (3.20).

|        | SD$^+  $ | SD$^-  $ | INT$^+  $ | SD$^-  $ + INT$^-  $ | total               |
|--------|----------|----------|----------|----------------------|-------------------|
| $K_{e2\gamma}$ | (1.52 ± 0.23) · 10$^{-5}$ | < 1.6 · 10$^{-4}$ | | |                     |
| $K_{\mu2\gamma}$ | < 3 · 10$^{-5}$ | < 2.7 · 10$^{-3}$ (modulus) | < 2.6 · 10$^{-4}$ (modulus) | | (3.02 ± 0.10) · 10$^{-3}$ |

3.5 Theory

The amplitudes $A(W^2)$ and $V(W^2)$ have been worked out in the framework of various approaches such as current algebra, PCAC, resonance exchange, dispersion relations, . . . For a rather detailed review together with an extensive list of references up to 1976, see [17]. Here, we concentrate on the predictions of $V, A$ in the framework of CHPT.

A) Chiral expansion to one loop

At leading order in the low-energy expansion, one has

$$ A = V = 0 \quad (3.35) $$

and $F_K$ replaced by $F$ in Eq. (3.2). The rate is therefore entirely given by the IB contribution at leading order. The loop effects manifest themselves only in the replacement $F \rightarrow F_K$ in Eq. (3.2). The local terms of order $p^4$ give

$$
A = - \frac{4}{F} (L_0^r + L_{10}^r), \\
V = - \frac{1}{8\pi^2} \frac{1}{F}, \\
\gamma_K = 32\pi^2 (L_0^r + L_{10}^r),
$$

(3.36)

where $L_0^r$ and $L_{10}^r$ are the renormalized low-energy couplings evaluated at the scale $\mu$ (the combination $L_0^r + L_{10}^r$ is scale-independent). The vector form factor stems from the Wess–Zumino term [3].

Remarks:

5 The relevant Feynman diagrams are displayed and discussed in the next section in connection with the decays $K^\pm \rightarrow l^\pm \nu l'^+l'^-$ where the photon is virtual.
Table 4: Chiral prediction at order $p^4$ for the branching ratios $\Gamma(K \to l\nu\gamma)/\Gamma_{\text{total}}$. The cut used in the last column is given in Eq. (3.20).

|           | SD$^+$  | SD$^-$  | INT$^+$ | INT$^-$ | total   |
|-----------|---------|---------|---------|---------|---------|
| $K_{e2}\gamma$ | $1.30 \cdot 10^{-6}$ | $1.95 \cdot 10^{-6}$ | $6.64 \cdot 10^{-10}$ | $-1.15 \cdot 10^{-8}$ | $2.34 \cdot 10^{-6}$ |
| $K_{\mu2}\gamma$ | $9.24 \cdot 10^{-6}$ | $1.38 \cdot 10^{-6}$ | $1.44 \cdot 10^{-9}$ | $-3.83 \cdot 10^{-9}$ | $3.08 \cdot 10^{-9}$ |

1. At this order in the low-energy expansion, the form factors $A, V$ do not exhibit any $W^2$ dependence. A non-trivial $W^2$ dependence only occurs at the next order in the energy expansion (two-loop effect, see the discussion below). Note that the available analyses of experimental data of $K \to l\nu\gamma$ decays [13, 11, 12, 16] use constant form factors throughout.

2. Once the combination $L_9 + L_{10}$ has been pinned down from other processes, Eq. (3.36) allows one to evaluate $A, V$ unambiguously at this order in the low-energy expansion. Using $L_9 + L_{10} = 1.4 \cdot 10^{-3}$ and $F = F_\pi$, one has

$$
M_K(A + V) = -0.097 \\
M_K(V - A) = -0.037 \\
\gamma_K = 0.45 .
$$

The result for the combination $(A + V)$ agrees with (3.30) within the errors, while $\gamma_K$ is consistent with (3.32).

3. At this order in the low-energy expansion, the form factors $A, V$ are identical to the ones in radiative pion decays $\pi^+ \to l^+\nu\gamma$ [3], as a result of which one has

$$
\gamma_\pi = \gamma_K + O(\mathcal{M}) ,
$$

We display in Table 4 the branching ratios $\text{BR}_I$ (3.21) which follow from the prediction (3.37). These predictions satisfy of course the inequalities found from experimental data (see Table 3).

B) $W^2$ dependence of the form factors

The chiral prediction gives constant form factors at order $p^4$. Terms of order $p^6$ have not yet been calculated. They would, however, generate a non-trivial $W^2$ dependence both in $V$ and $A$. In order to estimate the magnitude of these corrections, we consider one class of $p^6$ contributions: terms which are generated...
Figure 1: The rate $dP(x)/dx$ in (3.40), evaluated with the form factors (3.39) and $N_{\text{tot}} = 9 \cdot 10^9$. The solid line corresponds to $M_{K^*} = 890$ MeV, $M_{K_1} = 1.3$ GeV. The dashed line is evaluated with $M_{K^*} = 890$ MeV, $M_{K_1} = \infty$, and the dotted line corresponds to $M_{K^*} = M_{K_1} = \infty$. The total number of events is also indicated in each case.

by vector and axial vector resonance exchange with strangeness [17, 18],

$$V(W^2) = \frac{V}{1 - W^2/M_{K^*}^2}, \quad A(W^2) = \frac{A}{1 - W^2/M_{K_1}^2},$$

(3.39)

where $V, A$ are given in (3.36). We now examine the effect of the denominators in (3.39) in the region $y \geq 0.95, x \geq 0.2$ which has been explored in $K^+ \to e^+\nu_e\gamma$ [11]. We put $m_e = 0$ and evaluate the rate

$$\frac{dP(x)}{dx} = \frac{N_{\text{tot}}}{\Gamma_{\text{tot}}} \int_{y=0.95}^{1} \rho_{SD^+}(x, y) dy,$$

(3.40)

where $N_{\text{tot}}$ denotes the total number of $K^+$ decays considered, and $\Gamma_{\text{tot}} = 1.24 \cdot 10^{-8}$ s.

The function $\frac{dP(x)}{dx}$ is displayed in Fig. 1 for three different values of $M_{K^*}$ and $M_{K_1}$, with $N_{\text{tot}} = 9 \cdot 10^9$. The total number of events

$$N_p = \int_{x=0.2}^{x=1} dP(x)$$

(3.41)

is also indicated in each case. The difference between the dashed and the dotted lines shows that the nearby singularity in the anomaly form factor influences the
decay rate substantially at low photon energies. The effect disappears at \( x \to 1 \), where \( W^2 = M_K^2(1 - x) \to 0 \). To minimize the effect of resonance exchange, the large-\( x \) region should thus be considered. The low-\( x \) region, on the other hand, may be used to explore the \( W^2 \) dependence of \( V \) and of \( A \). For a rather exhaustive discussion of the relevance of this \( W^2 \) dependence for the analysis of \( K_{l2} \) decays we refer the reader to Ref. [17].

3.6 Outlook

Previous experiments have used various cuts in phase space in order (i) to identify the individual contributions IB, SD\( ^\pm \), INT\( ^\pm \) as far as possible, and (ii) to reduce the background from \( K_{l3} \) decays. This background has in fact forced such severe cuts that only the upper end of the lepton spectrum remained.

The experimental possibilities to reduce the background from \( K_{l3} \) decays are presumably more favourable with today’s techniques. Furthermore, the annual yield of \( 9 \cdot 10^9 K^+ \) decays at e.g. DAFNE is more than two orders of magnitude higher than the samples which were available until now [11, 12, 13, 16]. This allows for a big improvement in the determination of the amplitudes \( A \) and \( V \), in particular in \( K_{\mu2\gamma} \) decays. It would be very interesting to pin down the combination \( L_9 + L_{10} \) of the low-energy constants which occurs in the chiral representation of the amplitude \( A \), and to investigate the \( W^2 \) dependence of the form factors.

4 The decays \( K^\pm, \pi^\pm \to l^\pm \nu_l l'^\pm \bar{l}'^- \)

Here we consider decays where the photon turns into a lepton–antilepton pair,

\[
\begin{align*}
K^+ & \to e^+ \nu_e \mu^+ \mu^- & (4.1) \\
K^+ & \to \mu^+ \nu_\mu e^+ e^- & (4.2) \\
\pi^+ & \to \mu^+ \nu_\mu e^+ e^- & (4.3) \\
K^+ & \to e^+ \nu_e e^+ e^- & (4.4) \\
K^+ & \to \mu^+ \nu_\mu \mu^+ \mu^- & (4.5) \\
\pi^+ & \to e^+ \nu_e e^+ e^- & (4.6)
\end{align*}
\]

4.1 Matrix elements

We start with the processes (4.1) and (4.2),

\[
K^+(p) \to l^+(p_1)\nu_l(p_\nu)l'^+(p_1)\bar{l}'^-(p_2)
\]
\( (l, l') = (e, \mu) \) or \((\mu, e)\). \hspace{1cm} (4.7)

The matrix element is

\[
T = -iG_F e V_{us}^\ast \tau^\prime \left\{ F_K \mathcal{T}^\mu - \mathcal{T}^{\mu\nu} l_\mu \right\}
\]

where

\[
\mathcal{T}^\mu = m_l \overline{u}(p_\nu) (1 + \gamma_5) \left\{ \frac{2p_\mu - q_\mu}{2pq - q^2} - \frac{2p_\mu + q\gamma_\mu}{2pq + q^2} \right\} v(p_l)
\]

\[
l^\mu = \overline{u}(p_\nu) \gamma^\mu (1 - \gamma_5) v(p_l)
\]

\[
\mathcal{T}^{\mu\nu} = iV_1 e^\rho_{\mu\alpha\beta} q_\alpha p_\beta - A_1 (q W g^{\rho\mu} - W^{\rho} q_\mu)
\]

\[-A_2 (q^2 g^{\rho\mu} - q^\rho q^\mu) - A_4 (q W q^\rho - q^2 W^\rho) W^\mu , \]

with

\[
A_4 = \frac{2F_K}{M_K^2 - W^2} \frac{F_V^K(q^2) - 1}{q^2} + A_3 . \hspace{1cm} (4.10)
\]

The form factors \(A_i(q^2, W^2)\), \(V_1(q^2, W^2)\) are the ones defined in Appendix B, and \(F_V^K(q^2)\) is the electromagnetic form factor of the \(K^+\). Finally the quantity \(\overline{\epsilon}^\mu\) stands for

\[
\overline{\epsilon}^\mu = \frac{e}{q^2} \overline{u}(p_2) \gamma^\mu v(p_1) , \hspace{1cm} (4.11)
\]

and the four-momenta are

\[
q = p_1 + p_2 , \quad W = p_t + p_\nu = p - q , \hspace{1cm} (4.12)
\]

such that \(q_\mu \overline{\epsilon}^\mu = 0\). The first term in Eq. (4.8) contains the part where the off-shell photon radiates off the final-state lepton and those corrections that are reabsorbed in the definition of \(F_K\). This can always be done independently of the model used, because of gauge invariance [13].

In order to obtain the matrix element for (4.4) and (4.5),

\[
K^+(p) \rightarrow l^+(p_l) \nu(p_\nu) l^+(p_1) l^-(p_2) , \hspace{1cm} (4.13)
\]

one identifies \(m_l\) and \(m_l'\) in (4.8) and subtracts the contribution obtained from interchanging \(p_1 \leftrightarrow p_l\):

\[
(p_1, p_l) \rightarrow (p_l, p_1)
\]

\[
q \rightarrow p_l + p_2
\]

\[
W \rightarrow p - q = p_\nu + p_l . \hspace{1cm} (4.14)
\]

For the decays of a pion, the same formulas with the following replacements apply:

\[
M_K \rightarrow M_\pi , \quad \quad V_{us} \rightarrow V_{ud} ,
\]

\[
F_K \rightarrow F_\pi , \quad \quad F_V^K(q^2) \rightarrow F_V^\pi(q^2) . \hspace{1cm} (4.15)
\]
4.2 Decay distributions

The decay width is given by

$$d\Gamma = \frac{1}{2M_K(2\pi)^8} \sum_{\text{spins}} |T|^2 d_{LIPS}(p; p_1, p_\nu, p_1, p_2)$$

(4.16)

and the total rate is the integral over this for the case $l \neq l'$. For the case $l = l'$ the integral has to be divided by a factor of 2 for two identical particles in the final state.

We first consider the case where $l \neq l'$ and introduce the dimensionless variables

$$x = \frac{2pq}{M_K^2}, \quad y = \frac{2p_l p_\nu}{M_K^2}, \quad z = \frac{q^2}{M_K^2},$$

$$r_l = \frac{m_l^2}{M_K^2}, \quad r'_l = \frac{m^2_{l'}}{M_K^2}.$$  

(4.17)

Then one obtains, after integrating over $p_1$ and $p_2$ at fixed $q^2$ [20],

$$d\Gamma_{K^+ \to l^+ \nu_l l'^- l'^-} = \alpha^2 G_F^2 |V_{us}|^2 M_K^5 F(z, r'_l) \left\{- \sum_{\text{spins}} T^\mu T^\mu \right\} dx dy dz$$

$$F(z, r'_l) = \frac{1}{192\pi^3 z} \left\{1 + \frac{2r'_l}{z}\right\} \sqrt{1 - \frac{4r'_l}{z}}$$

$$T^\mu = M_K^{-2} \left\{F_K T^\mu - \overline{F}^{\mu
u} l_\nu\right\}.$$  

(4.18)

The quantity $\left\{- \sum_{\text{spins}} T^\mu T^\mu\right\}$ can be found in Ref. [14]. The expression or a Monte Carlo program containing this trace in FORTRAN can be obtained from the authors. This result allows one to evaluate, e.g. the distribution $d\Gamma/dz$ of produced $l^+ l'^-$ pairs, rather easily. The kinematically allowed region is

$$4r'_l \leq z \leq 1 + r_l - 2\sqrt{r_l}$$

$$2\sqrt{z} \leq x \leq 1 + z - r_l$$

$$A - B \leq y \leq A + B,$$

(4.19)

with

$$A = \frac{(2 - x)(1 + z + r_l - x)}{2(1 + z - x)},$$

$$B = \frac{(1 + z - x - r_l)\sqrt{x^2 - 4z}}{2(1 + z - x)}.$$  

(4.20)

The case $l = l'$ is slightly more elaborate since the integration over part of the phase space cannot be done analytically as before. The expression for $\sum_{\text{spins}} |T|^2$,
Figure 2: The two classes of contributions to the decays $K^+ \rightarrow l^+ \nu_l l'^+ l'^−$.

together with the Monte Carlo program to do the phase-space integrals, is available from the authors.

All formulas in this section are also valid with the replacements of Eq. (4.15) for the pion-decay case.

4.3 Theory

The form factors $A_i$, $V_1$ and $F^K_V$ have been discussed in all kinds of models, Vector Meson Dominance, hard meson, etc. (see Ref. [17] for a discussion). Here, we will work rigorously within the framework of CHPT.

There are two classes of contributions to these decays. They are depicted in Fig. 2. The first class of diagrams is the well understood Bremsstrahlung off the final-state lepton. This contribution is entirely contained in the first term in Eq. (4.18). The second class of diagrams, which also contains the radiation off the kaon line, contributes to both terms in Eq. (4.18). The one-loop diagrams contributing to this class are shown in Fig. 3.

To leading order we have

$$V_1 = 0$$
$$A_1 = A_2 = A_3 = 0 .$$

We also have $F^K_V = 1$ and $F_K$ is replaced by $F$. The rate here is entirely given by the inner Bremsstrahlung contribution.

\textsuperscript{6}The momentum dependence of the $W$ propagator is neglected.
The effects of the next-to-leading order CHPT corrections manifest themselves in the replacement $F \to F_K$ and in non-zero values for several of the form factors:

\[
\begin{align*}
V_1 &= -\frac{1}{8\pi^2 F} \\
A_1 &= -\frac{4}{F} (L_9^r + L_{10}^r) \\
A_2 &= -\frac{2F_K (F_V^K(q^2) - 1)}{q^2} \\
A_3 &= 0 \\
F_V^K(q^2) &= 1 + H_{\pi\pi}(q^2) + 2H_{KK}(q^2).
\end{align*}
\]

These results obey the current algebra relation of Ref. [17]. The function $F_V^K(q^2)$ does, however, deviate somewhat from the linear parametrization that is often used. The function $H(t)$ is defined in Appendix A. It can be easily seen from (4.8), (4.9) and (4.23) that the one-loop contributions to the decay matrix elements vanish for $q^2 \to 0$ and only the contributions from the Wess–Zumino term (to the form factor $V_1$) and from the $p^4$ Lagrangian remain.

The fact that the form factors at next-to-leading order could be written in terms of the kaon electromagnetic form factor in a simple way is no longer true at the $p^6$ level. The Lagrangian at order $p^6$ contains a term of the form

\[
\text{tr}\left\{D_\alpha F_L^{\alpha\mu} U^\dagger D^\beta F_{R\beta\mu} U\right\}
\]

that contributes to $A_2$ and $A_3$, but not to the kaon electromagnetic form factor, $F_V^K(q^2)$.
Table 5: Theoretical values for the branching ratios for the decay $K^+ \to \mu^+\nu_\mu e^+e^-$ for various cuts.

| Cut Condition | Tree Level Value | CHPT Form Factors Value |
|---------------|------------------|------------------------|
| Full phase space | $2.49 \cdot 10^{-5}$ | $2.49 \cdot 10^{-5}$ |
| $z \leq 10^{-3}$ | $2.07 \cdot 10^{-5}$ | $2.07 \cdot 10^{-5}$ |
| $z \geq 10^{-3}$ | $4.12 \cdot 10^{-6}$ | $4.20 \cdot 10^{-6}$ |
| $z \geq (20 \text{ MeV}/M_{K} )^2$ | $3.15 \cdot 10^{-6}$ | $3.23 \cdot 10^{-6}$ |
| $z \geq (140 \text{ MeV}/M_{K} )^2$ | $4.98 \cdot 10^{-8}$ | $8.51 \cdot 10^{-8}$ |
| $x \geq 40 \text{ MeV}/M_{K}$ | $1.58 \cdot 10^{-5}$ | $1.58 \cdot 10^{-5}$ |

Essentially the same result holds for the decay $\pi^+ \to l^+\nu_l e^+e^-$ after doing the replacements (1.15) in Eqs. (4.8), (4.9) and (1.23). The pion electromagnetic form factor to one loop is given by [9, 21]

$$F_V^\pi(q^2) = 1 + 2H_{\pi\pi}(q^2) + H_{KK}(q^2) .$$

(4.24)

Our result for the $\pi^+ \to l^+\nu_l l^+ l^-$ form factors agrees in the limit of two flavours with the result of Ref. [3].

4.4 Numerical results

Using the formulas of the previous subsections, we have calculated the rates for a few cuts, including those given in the literature. The values for the masses used are those of $K^+$ and $\pi^+$. For $L_9$ and $L_{10}$ we used the values given in Section 2.

For the case of unequal leptons, the results are given in Table 5 for the decay $K^+ \to \mu^+\nu_\mu e^+e^-$. These include the cuts used in Refs. [20] and [22], $x \geq 40 \text{ MeV}/M_{K}$ and $z \geq (140 \text{ MeV}/M_{K} )^2$, respectively. It can be seen that for this decay most of the branching ratio is generated at very low electron–positron invariant masses. As can be seen from the result for the cuts used in Ref. [22], the effect of the structure-dependent terms is most visible at high invariant electron–positron mass. Our calculation, including the effect of the form factors, agrees well with their data. We disagree, however, with the numerical result obtained in Ref. [24] by about an order of magnitude.

For the decay $K^+ \to e^+\nu_e \mu^+\mu^-$, we obtain for the tree level or IB contribution a branching ratio

$$BR_{\text{IB}}(K^+ \to e^+\nu_e \mu^+\mu^-) = 3.06 \cdot 10^{-12}$$

(4.25)

and, including the form factors,

$$BR_{\text{total}}(K^+ \to e^+\nu_e \mu^+\mu^-) = 1.12 \cdot 10^{-8} .$$

(4.26)
Table 6: Theoretical values for the branching ratios for the decay $\pi^+ \rightarrow \mu^+\nu_\mu e^+e^-$ for various cuts. The effect of the form factors is not visible to the accuracy quoted.

| full phase space | tree level |
|------------------|------------|
| z $\geq 10^{-5}$ | $3.3 \cdot 10^{-7}$ |
| z $\geq 10^{-2}$ | $2.8 \cdot 10^{-9}$ |

Here the structure-dependent terms are the leading contribution since the inner Bremsstrahlung contribution is helicity-suppressed, as can be seen from the factor $m_\ell$ in $L_\mu$.

The decay $\pi^+ \rightarrow \mu^+\nu_\mu e^+e^-$ is entirely given by the inner Bremsstrahlung contribution. The effects of the form factors are very small. The IB amplitude is not helicity-suppressed here since $M_\mu^2/M_\pi^2 \approx 1$. The effects of the form factors are partly suppressed since only very small $e^+e^-$ masses are allowed by phase space. The rates for full phase space and various cuts on $z$, defined as in Eq. (4.29), but with $M_\pi$ instead of $M_K$, are given in Table 6.

For the decays with identical leptons we obtain for the muon case a branching ratio of

$$BR_{\text{total}}(K^+ \rightarrow \mu^+\nu_\mu \mu^+\mu^-) = 1.35 \cdot 10^{-8}$$  \hspace{1cm} (4.27)

for the full phase space, including the effects of the form factors. The inner Bremsstrahlung or the tree level branching ratio for this decay is

$$BR_{\text{IB}}(K^+ \rightarrow \mu^+\nu_\mu \mu^+\mu^-) = 3.79 \cdot 10^{-9}.$$  \hspace{1cm} (4.28)

For the decay with two positrons and one electron, the integration over phase space for the tree-level results is very sensitive to the behaviour for small pair masses. We have given the tree level and the full prediction including form-factor effects in Table 7. The cuts used for the last row are similar to those of Ref. 23. We require a minimum energy of 15 MeV for the three charged particles and an opening angle $\theta_{1,2}$ for each electron–positron pair with $\cos \theta_{1,2} \leq 0.95$.  

For the decay $\pi^+ \rightarrow e^+\nu_e e^+e^-$, the integration over phase space is again very sensitive to the region for small invariant masses of the pairs. Tree level and full results are given for several cuts in Table 8; $z$ and $z_1$ are defined as in Eq. (4.29) with $M_K$ replaced by $M_\pi$. The cuts used for the last row are similar to those of Ref. 23.
Table 7: Theoretical values for the branching ratios for the decay \( K^+ \rightarrow e^+\nu_ee^+e^- \) for various cuts.

| Cut Conditions | Tree Level | Form Factors as Given by CHPT |
|---------------|------------|-------------------------------|
| Full phase space | \( \approx 4 \times 10^{-9} \) | \( 1.8 \times 10^{-7} \) |
| \( z, z_1 \geq 10^{-3} \) | \( 3.0 \times 10^{-10} \) | \( 1.22 \times 10^{-7} \) |
| \( z, z_1 \geq (50 \text{ MeV}/M_K)^2 \) | \( 5.2 \times 10^{-11} \) | \( 8.88 \times 10^{-8} \) |
| \( z, z_1 \geq (140 \text{ MeV}/M_K)^2 \) | \( 2.1 \times 10^{-12} \) | \( 3.39 \times 10^{-8} \) |

Table 8: Theoretical values for the branching ratios for the decay \( \pi^+ \rightarrow e^+\nu_ee^+e^- \) for various cuts.

| Cut Conditions | Tree Level | Form Factors as Given by CHPT |
|---------------|------------|-------------------------------|
| \( z, z_1 \geq 10^{-3} \) | \( 2.4 \times 10^{-9} \) | \( 3.0 \times 10^{-9} \) |
| \( z, z_1 \geq 10^{-2} \) | \( 4.2 \times 10^{-10} \) | \( 8.7 \times 10^{-10} \) |
| Cuts similar to [23] | \( 2.1 \times 10^{-12} \) | \( 5.7 \times 10^{-10} \) |

4.5 Experimental status

For kaon decays only those with an electron positron pair in the final state, decays (4.2) and (4.4), have been observed.

Both were measured in the same experiment [22]. The decay \( K^+ \rightarrow \mu^+\nu_\mu e^+e^- \) was measured with a branching ratio of \( (1.23 \pm 0.32) \times 10^{-7} \) with a lower cut on the electron–positron invariant mass of 140 MeV. The measurement is compatible with our calculation including the form-factor effects for the relevant region of phase space. This measurement was then extrapolated [22] using the result of [20] to the full phase space. Since we disagree with that calculation, we also disagree with the extrapolation.

In the same experiment, four events of the type \( K^+ \rightarrow e^+\nu_ee^+e^- \) were observed where both electron–positron pair invariant masses were above 140 MeV. This corresponds to a branching ratio for this region of phase space of \( (2.8^{+2.8}_{-1.4}) \times 10^{-8} \). This result is compatible within errors with our calculation, see Table 6. The matrix element of Ref. [20] was again used for the extrapolation to full phase space [22]. Apart from our numerical disagreement, the calculation of Ref. [24] was for the case of non-identical leptons and cannot be applied here.

For the decay \( K^+ \rightarrow \mu^+\nu_\mu\mu^+\mu^- \) an upper limit of \( 4.1 \times 10^{-7} \) exists [24]. This upper limit is compatible with our theoretical result, Eq. (4.27).

The decay \( K^+ \rightarrow e^+\nu_e\mu^+\mu^- \) has not been looked for so far but should be within the capabilities of a \( \Phi \) factory, given the branching ratio predicted in the
previous subsection. This decay proceeds almost entirely through the structure-dependent terms and, as such, is a good test of our calculation.

The decay $\pi^+ \rightarrow \mu^+ \nu_\mu e^+ e^-$ has not yet been looked for. It is almost entirely given by the inner Bremsstrahlung contribution so it is not very interesting as a test of the form factors. The decay $\pi^+ \rightarrow e^+ \nu_e e^+ e^-$ has been measured in [25]. We do not make a direct comparison with our result, because the acceptance of the apparatus is very difficult to implement, see Ref. [26]. We do agree with the form factors as determined there.

### 4.6 Outlook

The decays discussed in this section, $K^+ \rightarrow l^+ \nu_l l'^- l'^-$, are complementary to the decays $K^+ \rightarrow l^+ \nu_l \gamma$. As was the case for the analogous decay $\pi^+ \rightarrow e^+ \nu_e e^+ e^-$ [25], it may be possible to explore phase space more easily with this process than with $K^+ \rightarrow l^+ \nu_l \gamma$ to resolve ambiguities in the form factors.

As can be seen from our predictions, Tables 3 and 4, all the decays considered in this section should be observable at a $\Phi$ factory like DAFNE or in fixed-target experiments at future kaon factories. Large improvements in statistics are possible since less severe cuts than those used in the past experiments should be possible. In the decays with a $\mu^+ \mu^-$ pair and the decay $K^+ \rightarrow e^+ \nu_e e^+ e^-$, the effects of the form factors are already large in the total rates. In the decay $K^+ \rightarrow \mu^+ \nu_\mu e^+ e^-$ most of the total rate is for a small invariant mass of the pair and is given by the inner Bremsstrahlung contribution. There are, however, regions of phase space where the form-factor effects are large and where branching ratios are still large enough that sufficient statistics can be accumulated.

### 5 Radiative $K_{l3}$ decays

The decay channels considered in this section are

$$K^+(p) \rightarrow \pi^0(p') l^+(p_l) \nu_l(p_{\nu}) \gamma(q) \quad [K^+_{l3\gamma}]$$
$$K^0(p) \rightarrow \pi^-(p') l^+(p_l) \nu_l(p_{\nu}) \gamma(q) \quad [K^0_{l3\gamma}]$$

and the charge conjugate modes. We only consider real photons ($q^2 = 0$).
5.1 Kinematics and invariant amplitudes

The matrix element for $K^+ \ell_3 \gamma$ has the general structure

$$T = \frac{G_F}{\sqrt{2}} \epsilon^\mu(p_\ell) \gamma^\nu (1 - \gamma_5)v(p_l) \left\{ (V_{\mu\nu}^+ - A_{\mu\nu}^+) \pi(p_\nu) \gamma^\nu (1 - \gamma_5)u(p_\mu) \right\} \equiv \epsilon^\mu A_\mu^+ .$$

The diagram of Fig. 4a, corresponding to the first part of Eq. (5.1), includes Bremsstrahlung off the $K^+$. The lepton Bremsstrahlung diagram of Fig. 4b is represented by the second part of Eq. (5.1). The hadronic tensors $V_{\mu\nu}^+, A_{\mu\nu}^+$ are defined as

$$I_{\mu\nu}^+ = i \int d^4 x e^{iqx} \langle \pi^0(p') | T \{ V_{\mu\nu}^+ (x) I_{\nu}^{+ - i5}(0) \} | K^+(p) \rangle , \quad I = V, A ; \quad (5.2)$$

$F_{\nu}^+$ is the $K_{l3\gamma}^+$ matrix element

$$F_{\nu}^+ = \langle \pi^0(p') | V_{\nu}^{+ - i5}(0) | K^+(p) \rangle . \quad (5.3)$$

The tensors $V_{\mu\nu}^+$ and $A_{\mu\nu}^+$ satisfy the Ward identities (cf. Appendix C)

$$q^\mu V_{\mu\nu}^+ = F_{\nu}^+ , \quad (5.4)$$

$$q^\mu A_{\mu\nu}^+ = 0 , \quad (5.5)$$

leading in turn to

$$q^\mu A_\mu^+ = 0 , \quad (5.5)$$

as is required by gauge invariance.

For $K_{l3\gamma}$, one obtains the corresponding amplitudes and hadronic tensors by making the replacements

$$K^+ \rightarrow K^0 , \quad \pi^0 \rightarrow \pi^- ,$$

$$V_{\mu\nu}^+ \rightarrow V_{\mu\nu}^0 , \quad A_{\mu\nu}^+ \rightarrow A_{\mu\nu}^0 ,$$

$$F_{\nu}^+ \rightarrow F_{\nu}^0 , \quad A_{\mu}^+ \rightarrow A_{\mu}^0 . \quad (5.6)$$

To make the infrared behaviour transparent, it is convenient to separate the tensors $V_{\mu\nu}^+, V_{\mu\nu}^0$ into two parts:

$$V_{\mu\nu}^+ = \hat{V}_{\mu\nu}^+ + \frac{p_\mu}{pq} F_{\nu}^+ , \quad (5.7)$$

$$V_{\mu\nu}^0 = \hat{V}_{\mu\nu}^0 + \frac{p_\nu}{pq} F_{\nu}^0 .$$

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Due to Low’s theorem [19], the amplitudes \( \hat{V}_{\mu\nu}^{+,0} \) are finite for \( q \to 0 \). The axial amplitudes \( A_{\mu\nu}^{+,0} \) are automatically infrared-finite. The Ward identity (5.4) implies that the vector amplitudes \( \hat{V}_{\mu\nu}^{+,0} \) are transverse:

\[
q^\mu \hat{V}_{\mu\nu}^{+,0} = 0 .
\]

(5.8)

For on-shell photons, Lorentz and parity invariance together with gauge invariance allow the general decomposition (dropping the superscripts +, 0 and terms that vanish upon contraction with the photon polarization vector)

\[
\hat{V}_{\mu\nu} = V_1 \left( g_{\mu\nu} - \frac{W_{\mu\nu} q_{\nu}}{q W} \right) + V_2 \left( p'_\mu q_{\nu} - \frac{p' q}{q W} W_{\mu q} \right) \\
+ V_3 \left( p'_\mu W_{\nu} - \frac{p' q}{q W} W_{\mu W} \right) + V_4 \left( p'_\mu p_{\nu} - \frac{p' q}{q W} W_{\mu q} \right)
\]

For the decomposition of the axial tensor amplitude, use was made of Schouten’s identity

\[
\sum_{(\lambda\mu\nu\rho\sigma)} v_\lambda \varepsilon_{\mu\nu\rho\sigma} = 0 ,
\]

(5.10)

where the sum extends over all cyclic permutations of the five indices and \( v \) is an arbitrary four-vector. With the decomposition (5.7) we can write the matrix element for \( K_{1\gamma}^+ \) in (5.1) in a form analogous to Eqs. (3.2,3.3) for \( K_{1\gamma} \):

\[
T = \frac{G_F}{\sqrt{2}} e V_{us}^* \bar{e}^\nu(p)(q)^* \left\{ \hat{V}_{\mu\nu}^{+} - A^{+}_{\mu\nu} \right\} \overline{\pi}(p_\nu) \gamma^\nu (1 - \gamma_5) v(p_l) \\
+ F_\nu^+ \overline{\pi}(p_\nu) \gamma^\nu (1 - \gamma_5) \left[ \frac{p_\mu}{p q} - \frac{2 p_\mu + q r_\mu}{2 p q} \right] v(p_l) \right\} .
\]

(5.11)

The four invariant vector amplitudes \( V_1, \ldots, V_4 \) and the four axial amplitudes \( A_1, \ldots, A_4 \) are functions of three scalar variables. A convenient choice for these variables is

\[
E_\gamma = p q / M_K , \quad E_\pi = p p' / M_K , \quad W = \sqrt{W^2} ,
\]

(5.12)

where \( W \) is the invariant mass of the lepton pair. The amplitudes \( C_1, C_2 \) can be expressed in terms of the \( K_{1\gamma}^+ \) form factors and depend only on the variable \( (p - p')^2 = M_K^2 + M_\pi^2 - 2 M_K E_\pi \). For the full kinematics of \( K_{1\gamma} \) two more variables are needed, e.g.

\[
E_1 = p p_l / M_K , \quad x = p_l q / M_K^2 .
\]

(5.13)
Figure 4: Diagrammatic representation of the $K_{l3\gamma}^+$ amplitude.

The variable $x$ is related to the angle $\theta_{l\gamma}$ between the photon and the charged lepton in the $K$ rest frame:

$$xM_K^2 = E_{\gamma}(E_l - \sqrt{E_l^2 - m_l^2 \cos \theta_{l\gamma}}).$$

(5.14)

$T$ invariance implies that the vector amplitudes $V_1, \ldots, V_4$, the axial amplitudes $A_1, \ldots, A_4$ and the $K_{l3}$ form factors $C_1, C_2$ are (separately) relatively real in the physical region. We choose the standard phase convention in which all amplitudes are real.

For $\theta_{l\gamma} \to 0$ (collinear lepton and photon), there is a lepton-mass singularity in (5.1), which is numerically relevant for $l = e$ [27]. The region of small $E_{\gamma}, \theta_{l\gamma}$ is dominated by the $K_{l3}$ matrix elements. The new theoretical information of $K_{l3\gamma}$ decays resides in the infrared-finite tensor amplitudes $\hat{V}_{\mu\nu}$ and $A_{\mu\nu}$. The relative importance of these contributions can be enhanced by cutting away the region of low $E_{\gamma}, \theta_{l\gamma}$.

5.2 CHPT to $O(p^4)$

Prior to CHPT, the most detailed calculations of $K_{l3\gamma}$ amplitudes were performed by Fearing, Fischbach and Smith [28] using current-algebra techniques.

To lowest order in the chiral expansion [$O(p^2)$], the invariant amplitudes for the two charge modes are given by

$K_{l3\gamma}^+$:
Figure 5: Loop diagrams (without tadpoles) for $K_{l3}$. For $K_{l3\gamma}$, the photon must be appended on all charged lines and on all vertices.

\[
V_1^+ = \frac{1}{\sqrt{2}} \\
V_2^+ = -\frac{1}{\sqrt{2}pq} \\
C_1^+ = 2C_2^+ = \sqrt{2}
\] (5.15)

\[
K_{l3\gamma}^0:
\]

\[
V_1^0 = -1 \\
V_2^0 = \frac{1}{pq} \\
C_1^0 = 2C_2^0 = 2
\] (5.16)

All other invariant amplitudes vanish to $O(p^2)$.

At next-to-leading order in CHPT, there are in general three types of contributions [4]: anomaly, local contributions due to $\mathcal{L}_4$, and loop amplitudes. The chiral anomaly encoded in the WZW functional yields the axial amplitudes

\[
A_{\mu\nu}^+ = \frac{i\sqrt{2}}{16\pi^2F^2} \left\{ \varepsilon_{\mu\rho\sigma}(4p' + W)^\sigma + \frac{4}{W^2 - M_K^2}\varepsilon_{\mu\lambda\rho\sigma}W_\nu p'^\lambda q^\rho W^\sigma \right\} 
\] (5.17)

\[
A_{\mu\nu}^0 = -\frac{i}{8\pi^2F^2}\varepsilon_{\mu\rho\sigma}q^\rho W^\sigma.
\]
Table 9: Coefficients for the $K^{+}_{i3\gamma}$ loop amplitudes corresponding to the diagrams $I = 1, 2, 3$ in Fig. [3] All coefficients $c^I_j$ must be divided by $6\sqrt{2}F^2$.

| $I$ | $\mathcal{M}_I$ | $m_I$ | $c^I_1$ | $c^I_2$ | $c^I_3$ |
|-----|----------------|------|---------|---------|---------|
| 1   | $M_K$         | $M_\pi$ | 1       | $-2$    | $-M^2_K - 2M^2_\pi$ |
| 2   | $M_K$         | $M_\eta$ | 3       | $-6$    | $-M^2_K - 2M^2_\pi$ |
| 3   | $M_\pi$      | $M_K$ | 0       | $-6$    | $-6M^2_\pi$ |

The local parts of these amplitudes can be read off the Lagrangian (2.18). The non-local kaon pole term in $A^+_{\mu\nu}$ is due to the anomalous $K^+K^-\pi^0\gamma$ vertex contained in (2.19) with the outgoing $K^+$ turning into a $W^+$. Its contribution to the matrix element is helicity-suppressed [28, 8] and can be neglected for $l = e$. However, for $K^+_{i3\gamma}$ it is a normal-size $O(p^4)$ contribution to the axial amplitude.

The loop diagrams for $K^+_{i3\gamma}$ are shown in Fig. [3]. We first write the $K^+_{i3\gamma}$ matrix element in terms of three functions $f^+_1, f^+_2, f^+_3$ which will also appear in the invariant amplitudes $V^+_i$. Including the contributions from the low-energy constants $L_5, L_9$ in $\mathcal{L}_4$, the $K^+_{i3}$ matrix element $F^+_\nu$ is given by

\[
C^+_1 = f^+_1(t) \\
C^+_2 = \frac{1}{2}(M^2_K - M^2_\pi - t)f^+_2(t) + f^+_3(t) \\
f^+_1(t) = \sqrt{2} + \frac{4L_9}{\sqrt{2}F^2}t + 2 \sum_{i=1}^3 (c^+_2 - c^+_1)B^+_2(t) \\
f^+_2(t) = -\frac{4L_9}{\sqrt{2}F^2} + \frac{1}{2} \sum_{i=1}^3 \left\{ (c^+_1 - c^+_2) \left[ 2B^+_2(t) - \frac{(t + \Delta_I)\Delta_I J_I(t)}{2t} \right] - c^+_2\Delta_I J_I(t) \right\} \\
f^+_3(t) = \frac{F_K}{\sqrt{2}F_\pi} + \frac{1}{2t} \sum_{i=1}^3 \left\{ (c^+_1 + c^+_3)(t + \Delta_I) - 2c^+_3\right\} \Delta_I J_I(t) \\
\mathcal{L}_9 = L_5(\mu) - \frac{1}{256\pi^2} \ln \frac{M_\pi M^2_K M_\eta}{\mu^4} \\
\Delta_I = M^2_I - m^2_I, \ t = (p - p')^2.
\]

$\mathcal{L}_9$ is a scale-independent coupling constant and we have traded the tadpole contribution together with $L_5$ for $F_K/F_\pi$ in $f^+_3(t)$ [cf. Eq. (2.17)]. The sum over $I$ corresponds to the three one-loop diagrams of Fig. [3] with the coefficients $c^+_1, c^+_2, c^+_3$ displayed in Table [3]. We use the Gell-Mann–Okubo mass formula throughout to express $M^2_\eta$ in terms of $M^2_K, M^2_\pi$. The functions $J_I(t)$ and $B^+_2(t)$ can be found in Appendix [A].
The standard $K_{l3}$ form factors $f_+(t), f_-(t)$ are

\[
\begin{align*}
  f_+(t) &= \frac{1}{\sqrt{2}} f_1^+(t) \\
  f_-(t) &= \frac{1}{\sqrt{2}} \left[ (M_K^2 - M_\pi^2 - t)f_2^+(t) + 2f_3^+(t) - f_1^+(t) \right].
\end{align*}
\] (5.19)

It remains to calculate the infrared-finite tensor amplitude $\hat{V}_1^+$. The invariant amplitudes $V_i^+$ can be expressed in terms of the previously defined functions $f_i^+$ and of additional amplitudes $I_1, I_2, I_3$. Diagrammatically, the latter amplitudes arise from those diagrams in Fig. 3 where the photon is not appended on the incoming $K^+$ (non-Bremsstrahlung diagrams). The final expressions are

\[
\begin{align*}
  V_1^+ &= I_1 + p' W_q f_2^+(W_q^2) + f_3^+(W_q^2) \\
  V_2^+ &= I_2 - \frac{1}{pq} \left[ p' W_q f_2^+(W_q^2) + f_3^+(W_q^2) \right] \\
  V_3^+ &= I_3 + \frac{1}{pq} \left[ p' W f_2^+(W^2) + f_3^+(W^2) - p' W_q f_2^+(W_q^2) - f_3^+(W_q^2) \right] \\
  V_4^+ &= \frac{f_1^+(W^2) - f_1^+(W_q^2)}{pq} \\
  W_q &= W + q = p - p'.
\end{align*}
\] (5.20)

The amplitudes $I_1, I_2, I_3$ in Eq. (5.20) are given by

\[
\begin{align*}
  I_1 &= \frac{4qW}{\sqrt{2}F^2} (L_9 + L_{10}) + \frac{8p'q}{\sqrt{2}F^2} L_9 \\
  &\quad + \frac{3}{2} \sum_{i=1} \left\{ [(W_q^2 + \Delta_i)(c_2^i + c_3^i) - 2(c_2^i p' W_q + c_3^i)] \left[ \frac{(W_q^2 - \Delta_i) \hat{J}_I}{2W_q^2} - 2G_I \right] \\
  &\quad + \frac{1}{2} \left[ p' W_q \left( \frac{(W_q^4 - \Delta_i^2) \hat{J}_I}{W_q^2} + 4\hat{B}_2^I \right) + p' W (W_q - q) L_m^I \right] \\
  &\quad + \frac{2(c_2^i - c_1^i)}{qW} \left[ p' q(\hat{B}_2^I - (W_q^2 + \Delta_i)G_I) + p' W (\hat{B}_2^I - B_2^I) \right] \} \\
  I_2 &= -\frac{8L_9}{\sqrt{2}F^2} + \frac{2}{qW} \sum_{i=1} \left( c_2^i - c_1^i \right) \left[ F_I - (W^2 + \Delta_i) G_I \right] \\
  I_3 &= -\frac{4L_9}{\sqrt{2}F^2} + \frac{3}{2} \sum_{i=1} \left\{ 2(c_2^i - c_1^i) \left[ G_I + \frac{L_m^I}{4} + \frac{\hat{B}_2^I - B_2^I}{qW} \right] - c_1^i \Delta_i J_I \right\} \\
  L_{10} &= L_{10}^*(\mu) + \frac{1}{256\pi^2} \ln \left( \frac{M_\eta M_K^2}{\mu^4} \right) \\
  L_m^I &= \frac{\Sigma_I}{32\pi^2 \Delta_I} \ln \left( \frac{m_I^2}{\mu^4} \right).
\end{align*}
\] (5.21)
Table 10: Coefficients for the $K_{l3\gamma}^0$ loop amplitudes corresponding to the diagrams $I = 1, 2, 3$ in Fig. 5. All coefficients $c^I_i$ must be divided by $6\sqrt{2}F^2$.

\[
\begin{array}{cccccc}
I & M_I & m_I & c^I_1 & c^I_2 & c^I_3 \\
1 & M_K & M_\pi & 0 & -3 & -3M_K^2 \\
2 & M_K & M_\eta & 6 & -3 & M_K^2 + 2M_\pi^2 \\
3 & M_\pi & M_K & 4 & -2 & -2M_K^2 + 2M_\pi^2 \\
\end{array}
\]

\[
F_I = \hat{B}_2^I - \frac{W^2}{4}L^I_m + \frac{1}{qW} \left( W^2B_2^I - W^2_q\hat{B}_2^I \right)
\]

\[
G_I = \frac{M_I^2}{2}C(W_q^2, W^2, M_I^2, m_I^2) + \frac{1}{8qW} \left[ (W_q^2 + \Delta_I)\hat{J}_I - (W^2 + \Delta_I)J_I \right] + \frac{1}{64\pi^2}
\]

\[
J_I \equiv J_I(W^2), \quad \hat{J}_I \equiv J_I(W_q^2)
\]

\[
B_2^I \equiv B_2^I(W^2), \quad \hat{B}_2^I \equiv B_2^I(W_q^2).
\]

The function $C(W_q^2, W^2, M_I^2, m_I^2)$ is given in Appendix A. All the invariant amplitudes $V_1^+, \ldots, V_4^+$ are real in the physical region. Of course, the same is true for the $K_{l3}^0$ form factors $C_1^+, C_2^+$.

The $K_{l3\gamma}^0$ amplitude has a very similar structure. Both the $K_{l3}^0$ matrix element $F_{\nu}^0$ and the infrared-finite vector amplitude $\hat{V}_{\mu\nu}^0$ can be obtained from the corresponding quantities $F_{\nu}^+$ and $\hat{V}_{\mu\nu}^+$ by the following steps:

- interchange $p'$ and $-p$;
- replace $\frac{F_K}{F_\pi}$ by $\frac{F_\pi}{F_K}$ in $f_3^+$;
- insert the appropriate coefficients $c^I_i$ for $K_{l3\gamma}^0$ listed in Table 10;
- multiply $F_{\nu}^+$ and $\hat{V}_{\mu\nu}^+$ by a factor $-\sqrt{2}$.

### 5.3 Numerical results

The total decay rate is given by

\[
\Gamma(K \rightarrow \pi l\nu\gamma) = \frac{1}{2M_K(2\pi)^8} \int dLIPS(p; p', p_\nu, q) \sum_{\text{spins}} | T |^2 \quad (5.22)
\]

in terms of the amplitude $T$ in (5.1). The square of the matrix element, summed over photon and lepton polarizations, is a bilinear form in the invariant amplitudes $V_1, \ldots, V_4, A_1, \ldots, A_4, C_1, C_2$. The explicit expression can be found in Ref. [14], which also contains a summary of the presently available data sample.
Table 11: Branching ratios and expected number of events at DAFNE for $K^{\pm}_{\ell 3 \gamma}$.

| $K^{\pm}_{\ell 3 \gamma}$ | BR  | #events/yr |
|---------------------------|-----|------------|
| full $O(p^4)$ amplitude   | $3.0 \times 10^{-4}$ | $2.7 \times 10^6$ |
| tree level                | $2.8 \times 10^{-4}$ | $2.5 \times 10^6$ |
| $O(p^4)$ without loops    | $3.2 \times 10^{-4}$ | $2.9 \times 10^6$ |

Detailed experimental studies of $K_{\ell 3 \gamma}$ decays will become feasible with the high kaon fluxes of various factories, which are either under construction or still in the planning stage. The following number of events corresponds to the kaon fluxes expected in the proposed detector KLOE [29] for the $\Phi$ factory DAFNE in Frascati. These fluxes are based on a luminosity of $5 \times 10^{32}$ cm$^{-2}$ s$^{-1}$ equivalent to an annual rate of $9 \times 10^9 (1.1 \times 10^9)$ tagged $K^\pm (K_L)$ assuming a year of $10^7$ s.

For the calculation of decay rates we have used the following cuts:

$$E_\gamma \geq 30 \text{ MeV} \quad (5.23)$$

$$\theta_{l\gamma} \geq 20^\circ.$$  

The physical values of $M_\pi$ and $M_K$ are used in the amplitudes; $M_\pi$ is calculated from the Gell-Mann–Okubo mass formula. The values of the other parameters can be found in Sect. 2. For $K^0_{\ell 3 \gamma}$, the rates are to be understood as $\Gamma(K_L \to \pi^\pm l^\mp \nu \gamma)$.

The results for $K^{\pm}_{\ell 3 \gamma}$ and $K^0_{\ell 3 \gamma}$ are displayed in Tables [11] and [12] respectively. For comparison, the tree-level branching ratios and the rates for the amplitudes without the loop contributions are also shown. The separation between loop and counterterm contributions is of course scale-dependent. This scale dependence is absorbed in the scale-invariant constants $L_9, L_{10}$ defined in Eqs. (5.18), (5.21). In other words, the entries in Tables [11], [12] for the amplitudes without loops correspond to setting all coefficients $c^i_l$ in Tables [8], [10] equal to zero.

These numerical results show very clearly that the non-trivial CHPT effects of $O(p^4)$ can be detected at dedicated machines such as DAFNE, in all four channels, without any problem of statistics. Of course, the rates are bigger for the electronic modes. On the other hand, the relative size of the structure-dependent terms is somewhat bigger in the muonic channels (around 8% for the chosen cuts). We observe that there is negative interference between the loop and counterterm amplitudes. With the chosen distinction between loop and counterterm contribu-
Table 12: Branching ratios and expected number of events at DAFNE for $K_{l3\gamma}^0$.

|                | BR            | #events/yr |
|----------------|---------------|------------|
| $K_{e3\gamma}^0$ |               |            |
| full $O(p^4)$ amplitude | $3.8 \times 10^{-3}$ | $4.2 \times 10^6$ |
| tree level     | $3.6 \times 10^{-3}$ | $4.0 \times 10^6$ |
| $O(p^4)$ without loops | $4.0 \times 10^{-3}$ | $4.4 \times 10^6$ |
| $K_{\mu3\gamma}^0$ |               |            |
| full $O(p^4)$ amplitude | $5.6 \times 10^{-4}$ | $6.1 \times 10^5$ |
| tree level     | $5.2 \times 10^{-4}$ | $5.7 \times 10^5$ |
| $O(p^4)$ without loops | $5.9 \times 10^{-4}$ | $6.5 \times 10^5$ |

The loops amount to approximately 50% of the rate due to the local amplitudes of $O(p^4)$ (including the anomalous ones).

The sensitivity to the counterterm coupling constants $L_9, L_{10}$ and to the chiral anomaly can be expressed as the difference in the number of events between the tree level and the $O(p^4)$ amplitudes (without loops). In the optimal case of $K_{e3\gamma}^0$, this amounts to more than $4 \times 10^5$ events/yr at DAFNE. Most of this difference is due to $L_9$.

The chiral anomaly is more important for $K_{l3\gamma}^+$, but even there it influences the total rates rather little. A dedicated study of differential rates is necessary to locate the chiral anomaly in $K_{l3\gamma}$ amplitudes, if this is at all possible.

However, taking into account that both $L_9$ and $L_{10}$ are already known to rather good accuracy (see Sect. 2), $K_{l3\gamma}$ decays will certainly allow for precise tests of CHPT to $O(p^4)$.

6 Conclusions

1. In this paper we have calculated the matrix elements for the radiative semileptonic kaon decays $K^+ \rightarrow l^+\nu_l\gamma, l^+\nu_l l'^+ l'^-$ and $K \rightarrow \pi l\nu_l\gamma(l, l' = e, \mu)$ to next-to-leading order in CHPT. All measured $[10]$ semileptonic $K$ decay amplitudes are now known $[9, 14, 30, 31]$ to order $p^4$ in CHPT, except for one form factor in $K_{\mu4}$ decays $[32]$.

2. The $K_{l2\gamma}$ decay does not receive any contribution from the one-loop corrections, except for those that can be absorbed into $F_K$. This decay is then completely predicted, including effects of $SU(3)$ breaking, from the knowledge of the $\gamma$ parameter as measured in $\pi^+ \rightarrow e^+\nu_e\gamma$. This prediction was
successfully confronted with the present experimental data, and the relevance of more detailed measurements at future kaon facilities was discussed.

3. For the decays with a lepton–antilepton pair in the final state \((K^+(\pi^+) \rightarrow l^+\nu_ll^+l^-)\), the additional effects of one-loop contributions can be described by the electromagnetic form factor. It was shown that this relation breaks down at order \(p^6\). We noted that decays with a \(\mu^+\mu^-\) pair are especially useful probes of the form factors. All these decays should be measurable in the near future with sufficient statistics to measure the form factors.

4. The last considered decay, \(K_{l3\gamma}\), is a useful test of the non-trivial aspects of CHPT since the one-loop effects are of the same magnitude as the effects of the tree-level \(O(p^4)\) action. All form factors vary rather significantly over the available kinematical region, so that an analysis of experiments with a constant form factor will not be able to uncover these effects. We therefore urge future experiments to perform the analysis of their data directly with the formulas given here.

5. Overall, the agreement with the presently available data is quite satisfactory. On the other hand, future kaon facilities like DAFNE will have the opportunity to test the predictions of CHPT at next–to–leading order in much more detail than is possible with present data.

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APPENDICES

A  Notation and loop integrals

The notation for phase space is the one without the factors of $2\pi$. For the decay rate of a particle with four-momentum $p$ into $n$ particles with momenta $p_1, \ldots, p_n$, this is

$$d_{LIPS}(p; p_1, \ldots, p_n) = \delta^4 \left( p - \sum_{i=1}^n p_i \right) \prod_{i=1}^n \frac{d^3 p_i}{2 p_i^0}. \quad (A.1)$$

We use a covariant normalization of one-particle states,

$$\langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 2^0 \delta^3 (\vec{p}' - \vec{p}), \quad (A.2)$$

together with the spinor normalization

$$\bar{u}(p, r) u(p, s) = 2m \delta_{rs}. \quad (A.3)$$

The kinematical function $\lambda(x, y, z)$ is defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx). \quad (A.4)$$

We take the standard model in the current × current form, i.e. we neglect the momentum dependence of the $W$ propagator. The currents used in the text are:

$$V_{\mu}^{4-15} = \bar{q} \gamma_\mu \frac{1}{2} (\lambda_4 - i \lambda_5) q = \bar{s} \gamma_\mu u$$

$$A_{\mu}^{4-15} = \bar{q} \gamma_\mu \gamma_5 \frac{1}{2} (\lambda_4 - i \lambda_5) q = \bar{s} \gamma_\mu \gamma_5 u$$

$$V_{\mu}^{em} = \bar{q} \gamma_\mu Q q$$

$$Q = \text{diag}(2/3, -1/3, -1/3). \quad (A.5)$$

The numerical values used in the programs are the physical masses for the particles as given by the Particle Data Group [10]. In addition we have used the values for the decay constants derived from the most recent measured charged pion and kaon semileptonic decay rates [10, 33]:

$$F_\pi = 93.2 \text{ MeV}$$

$$F_K = 113.6 \text{ MeV}. \quad (A.6)$$

We do not need values for the quark masses. For the processes considered in this paper we can always use the lowest-order relations to rewrite them in terms of the pseudoscalar meson masses (see Section 2). For the KM matrix element $|V_{us}|$ we used the central value, 0.220, of Ref. [10]. The numerical values for the $L'_i(M_\rho)$ are those given in Section 2.
Whenever we quote a branching ratio for a semileptonic \( K^0 \) decay, it stands for the branching ratio of the corresponding \( K_L \) decay, e.g.

\[
BR(K^0 \to \pi^- l^+ \nu_l) \equiv BR(K_L \to \pi^\pm l^\mp \nu) .
\]  

We use the Condon–Shortley phase conventions throughout.

Next we define the functions appearing in the loop integrals used in the text. First we define those needed for loops with two propagators, mainly in the form given in Ref. [4]. We consider a loop with two masses, \( M \) and \( m \). All needed functions can be given in terms of the subtracted scalar integral \( \bar{J}(t) = J(t) - J(0) \),

\[
J(t) = -i \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + M^2)(p^2 - m^2)}
\]  

with \( t = k^2 \). The functions used in the text are then:

\[
\bar{J}(t) = \frac{1}{16\pi^2} \int_0^1 dx \log \frac{M^2 - tx(1 - x) - \Delta x}{M^2 - \Delta x} = \frac{1}{32\pi^2} \left\{ 2 + \frac{\Delta}{t} \log \frac{m^2}{M^2} - \frac{\Sigma}{\Delta} \log \frac{m^2}{M^2} - \sqrt{\lambda} \frac{t}{t} \log \frac{(t + \sqrt{\lambda})^2 - \Delta^2}{(t - \sqrt{\lambda})^2 - \Delta^2} \right\} .
\]

\[
J'(t) = \bar{J}(t) - 2k ,
\]

\[
M'(t) = \frac{1}{2t} \left\{ t - 2\Sigma \right\} \bar{J}(t) + \frac{\Delta^2}{3t^2} \bar{J}(t) + \frac{1}{288\pi^2} - \frac{k}{6} \frac{1}{96\pi^2} \left\{ \Sigma + 2 \frac{M^2m^2}{\Delta} \log \frac{m^2}{M^2} \right\} ,
\]

\[
L(t) = \frac{\Delta^2}{4t} \bar{J}(t) ,
\]

\[
K(t) = \frac{\Delta}{2t} \bar{J}(t) ,
\]

\[
H(t) = \frac{2}{3F^2} \left[ tM'(t) - L(t) \right] ,
\]

\[
\Delta = M^2 - m^2 ,
\]

\[
\Sigma = M^2 + m^2 ,
\]

\[
\lambda = \lambda(t, M^2, m^2) = (t + \Delta)^2 - 4tM^2 .
\]  

(A.9)

In the text these are used with subscripts,

\[
\bar{J}_{ij}(t) = \bar{J}(t) \quad \text{with} \quad M = M_i, m = M_j
\]  

and similarly for the other symbols. The subtraction-point-dependent part is contained in the constant \( k \):

\[
k = \frac{1}{32\pi^2} \frac{M^2 \log \left( \frac{M^2}{\mu^2} \right) - m^2 \log \left( \frac{m^2}{\mu^2} \right)}{M^2 - m^2} ,
\]  

(A.11)
where $\mu$ is the subtraction scale.

In addition, in Section 5 these functions and symbols appear in a summation over loops $I$ with

\[
\begin{align*}
J_I(t) &= \bar{J}(t) \quad \text{with} \quad M = M_I, m = m_I, \\
\Sigma_I &= M_I^2 + m_I^2
\end{align*}
\]  

(A.12)

and again similarly for the others. There the combination $B_2$ appears as well:

\[
B_2(t, M^2, m^2) = B_2(t, m^2, M^2) = \frac{1}{288\pi^2 (3\Sigma - t)} - \frac{\lambda(t, M^2, m^2)\bar{J}(t)}{12t} + \frac{t\Sigma - 8M^2m^2}{384\pi^2\Delta} \log \frac{M^2}{m^2}.
\]  

(A.13)

The last formula to be defined is the three-propagator-loop integral function $C(t_1, t_2, M^2, m^2)$, where one of the three external momenta has zero mass and two of the propagators have the same mass $M$. Here $t_1 = (q_1 + q_2)^2$, $t_2 = q_2^2$ and $q_1^2 = 0$:

\[
\begin{align*}
C(t_1, t_2, M^2, m^2) &= -i \int \frac{d^4p}{(2\pi)^d} \frac{1}{(p^2 - M^2)((p + q_1)^2 - M^2)((p + q_1 + q_2)^2 - m^2)} \\
&= -\frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{1}{M^2 - y(\Delta + t_1) + xy(t_1 - t_2) + y^2t_1} \\
&= \frac{1}{(4\pi^2(t_1 - t_2))} \left\{ Li_2 \left( \frac{1}{y_+(t_2)} \right) + Li_2 \left( \frac{1}{y_-(t_2)} \right) \\
&\quad - Li_2 \left( \frac{1}{y_+(t_1)} \right) - Li_2 \left( \frac{1}{y_-(t_1)} \right) \right\},
\end{align*}
\]  

(A.14)

where $Li_2$ is the dilogarithm

\[
Li_2(x) = -\int_0^1 \frac{dy}{y} \log(1 - xy).
\]  

(A.15)

\section*{B \ Decomposition of the hadronic tensors $I^{\mu\nu}$}

Here we consider the tensors

\[
I^{\mu\nu} = \int dx e^{ixq + iWy} \langle 0 | TV_{em}^{\mu}(x) I_{4-i5}^{\nu}(y) | K^+(p) \rangle, \quad I = V, A
\]  

(B.1)
and detail its connection with the matrix element (3.2).

The general decomposition of $A_{\mu\nu}$, $V_{\mu\nu}$ in terms of Lorentz-invariant amplitudes reads [17], for $q^2 \neq 0$:

$$\frac{1}{\sqrt{2}} A^{\mu\nu} = -F_K \left\{ \frac{(2W^\mu + q^\mu)W^\nu}{M_K^2 - W^2} + g^{\mu\nu} \right\}
+ A_1(qWg^{\mu\nu} - W^{\mu}q^{\nu}) + A_2(q^2g^{\mu\nu} - q^{\mu}q^{\nu})
+ \left\{ \frac{2F_K(F_K^V(q^2) - 1)}{(M_K^2 - W^2)q^2} + A_3 \right\} (qWq^{\mu} - q^2W^{\mu})W^{\nu} \quad (B.2)$$

and

$$\frac{1}{\sqrt{2}} V^{\mu\nu} = iV_1 \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta , \quad (B.3)$$

where the form factors $A_i(q^2, W^2)$ and $V_1(q^2, W^2)$ are analytic functions of $q^2$ and $W^2$; $F_K^V(q^2)$ denotes the electromagnetic form factor of the kaon ($F_K^V(0) = 1$). The amplitudes $I^{\mu\nu}$ satisfy the Ward identities (cf. Appendix C)

$$q_{\mu} A^{\mu\nu}_{IB} = -\sqrt{2} F_K p^{\nu} \quad (B.4)$$

The decomposition in Eq. (B.2) may be derived as follows [17]. One first isolates the contribution from Bremsstrahlung off the kaon,

$$A^{\mu\nu} = A^{\mu\nu}_{IB} + \bar{A}^{\mu\nu}$$

$$A^{\mu\nu}_{IB} = -\sqrt{2} F_K \frac{F_K^V(q^2)}{M_K^2 - W^2} (2W^\mu + q^\mu)W^{\nu} \quad (B.5)$$

$$q_{\mu} A^{\mu\nu}_{IB} = -\sqrt{2} F_K F_K^V(q^2)W^{\nu}.$$

The general structure of the remainder $\bar{A}^{\mu\nu}$ is

$$- \bar{A}^{\mu\nu} = A_1 W^{\mu}q^{\nu} + A_2 q^{\mu}q^{\nu} + A_3 q^2W^{\mu}W^{\nu} + A_4 g^{\mu\nu} + A_5 q^{\mu}W^{\nu} \quad (B.6)$$

The factor $q^2$, which multiplies $A_3$, is dictated by the Ward identity Eq. (B.4) which implies

$$qW A_1 + q^2 A_2 + A_4 = \sqrt{2} F_K$$

$$q^2 qW A_3 + q^2 A_5 = \sqrt{2} F_K \left[ 1 - F_K^V(q^2) \right] . \quad (B.7)$$

Eliminating $A_4$ and $A_5$ gives (B.2).

In the process (3.1) the photon is real. As a consequence, only the two form factors $A_1(0, W^2)$ and $V_1(0, W^2)$ contribute. We set

$$A(W^2) = A_1(0, W^2)$$
$$V(W^2) = V_1(0, W^2) \quad (B.8)$$
and obtain, for the matrix element (3.2):

\[ T = -iG_F/\sqrt{2}e V_{us}^* c_\mu \left\{ \sqrt{2}F_K l_1^\mu - (V^{\mu\nu} - A^{\mu\nu}) l_\nu \right\}_{q^2=0}, \tag{B.9} \]

with

\[ l_\mu = \bar{u}(p_\nu) \gamma^\mu (1 - \gamma_5) v(p_l) \]

\[ l_1^\mu = l_\mu + m_l \bar{u}(p_\nu)(1 + \gamma_5) \frac{2p_l^\mu + q^\mu}{m_l^2 - (p_l + q)^2} v(p_l). \tag{B.10} \]

Grouping terms into an IB and a SD piece gives (3.2), (3.3). As a consequence of (B.4), \( T \) is invariant under the gauge transformation \( \epsilon_\mu \rightarrow \epsilon_\mu + q_\mu \).

The amplitudes \( A_1, A_2 \) and \( V_1 \) are related to the corresponding quantities \( F_A, R \) and \( F_V \) used by the PDG [10] by

\[ -\sqrt{2}M_K(A_1, A_2, V_1) = (F_A, R, F_V). \tag{B.11} \]

The last term in (B.2) is omitted in [10]. It contributes to processes with a virtual photon, \( K^\pm \rightarrow l^\pm \nu l^+ l^- \).

Finally, the relation to the notation used in [11, 12] is

\[ 2(A \pm V)^2 = (a_k \pm v_k)^2 \quad [11] \]

\[ \sqrt{2}(A, V) = (F_A, F_V) \quad [12]. \tag{B.12} \]

### C Ward identities

Here we derive the Ward identity

\[ A^{\mu\nu}(q, p) = \int dx e^{iqx + iWy} \langle 0 | TV^\mu_{em}(x) A^\nu_{A+5}(y) | K^+(p) \rangle \quad \tag{C.1} \]

\[ q_\mu A^{\mu\nu} = -\sqrt{2}F_K p^\nu, \quad W = p - q \quad \tag{C.2} \]

without using formal manipulations with \( T \)-products and with current commutators. Our treatment is similar to the considerations in Ref. [34] and parallels the ones in Ref. [35].

The reason we wish to discuss this issue is the following. Owing to the short-distance singularities of the operator product of two currents, integrals such as the one in Eq. (C.1) are \textit{a priori} not well defined. As a result of this, the Ward identities for \( A^{\mu\nu} \) depend on the convention chosen for \( \langle 0 | TV^\mu(x) A^\nu(y) | K \rangle \). In
CHPT, the Green functions are obtained through (functional derivatives of) the generating functional $Z(v, a, s, p)$. This functional is constructed such that it is invariant under transformations generated by the vector currents (see Section 2),

$$Z(v', a', s', p') = Z(v, a, s, p), \quad (C.3)$$

where

$$v'_\mu \pm a'_\mu = g(v_\mu \pm a_\mu)g^+ + ig\partial_\mu g^+$$
$$s' + ip' = g(s + ip)g^+$$
$$g(x) \in SU(3). \quad (C.4)$$

The relation (C.3) embodies all the Ward identities associated with the transformations (C.4). In particular, seagulls and Schwinger terms are automatically taken into account [34, 35]. For an infinitesimal transformation

$$g = 1 + i\alpha + O(\alpha^2),$$
$$\delta v_\mu = \partial_\mu \alpha + i[a, v_\mu] \cong D_\mu \alpha,$$
$$\delta I = i[a, I]; I = a_\mu, s, p, \quad (C.5)$$

one obtains from (C.3)

$$\left< \alpha D_\mu \frac{\delta Z}{\delta v_\mu(x)} \right> = i \sum_{I=a_\mu,s,p} \left< [a, I(x)] \frac{\delta Z}{\delta I(x)} \right>, \quad (C.6)$$

where

$$I = \frac{\lambda^a}{2} I^a, \quad \frac{\delta Z}{\delta I} = \frac{\lambda^a}{2} \frac{\delta Z}{\delta I^a}. \quad (C.7)$$

The Ward identities for current correlation functions are obtained from this general formula by taking additional functional derivatives, evaluated at

$$v_\mu = a_\mu = p = 0, \quad s = \text{diag}(m_u, m_d, m_s). \quad (C.8)$$

In the following we consider the isospin symmetry limit $m_u = m_d$. In order to derive Eq. (C.2), we consider the quantity

$$\Gamma^{\mu\nu\rho}(x, y, z) = \langle 0|TV_{em}(x)A_{4-i5}^\nu(y)A_{4+i5}^\rho(z)|0 \rangle, \quad (C.9)$$

which provides an off-shell extension of $A^{\mu\nu}$, and translate the Ward identity for $\Gamma^{\mu\nu\rho}$ into the corresponding relation for $A^{\mu\nu}$. Differentiating (C.6) with respect to $A_{4-i5}^\nu(y)$ and $A_{4+i5}^\rho(z)$ at (C.8), one obtains with $\alpha = \frac{1}{3} \text{diag}(2, -1, -1)$

$$\partial_\mu^{\alpha} \Gamma^{\mu\nu\rho}(x, y, z) = -[\delta^4(x - y) - \delta^4(x - z)] \langle 0|TA_{4-i5}^\nu(y)A_{4+i5}^\rho(z)|0 \rangle. \quad (C.10)$$
Next we introduce the Fourier transforms

\[
\begin{align*}
&i \int dz e^{-ip^* z} \langle 0|TA^\nu_{4-i5}(z)A^\rho_{4+i5}(0)|0 \rangle = \frac{R^{\nu \rho}_2(p^*)}{M^2_K - p^{*2}} \\
&\int dx dz e^{iqx+iW^* y-ip^* z} \Gamma^{\mu \nu \rho}(x,y,z) = \frac{R^{\mu \nu \rho}_3(q,W^*)}{M^2_K - p^{*2}}
\end{align*}
\] (C.11)

where

\[W^* = p^* - q, \ p^{*2} \neq M^2_K .\] (C.12)

By use of \footnote{Again note that we work with the Condon–Shortley–de Swart phase convention.}

\[
\lim_{p^* \to p} \begin{cases}
R^{\nu \rho}_2 = 2F^2_K p^\nu p^\rho \\
R^{\mu \nu \rho}_3 = \sqrt{2}F_K A^{\mu \nu} p^\rho
\end{cases}
\] (C.13)

one finds from (C.10)

\[q_\mu A^{\mu \nu} = -\sqrt{2}F_K p^\nu .\] (C.14)

The other Ward identities considered in this article may be derived in an analogous manner.
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