Computer Simulation of Transition Regimes of Solitons in Four-Photon Resonant Parametric Processes in Case of Two-Photon Resonance

Vladimir Feshchenko¹, Galina Feshchenko²

¹Dawson College, Montreal, Canada
²Vanier College, Montreal, Canada

Email: vfeshchenko@place.dawsoncollege.qc.ca, feshche@vaniercollege.qc.ca

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Abstract

The transition regimes of solitons in four-photon resonant processes in the case of two-photon absorption of the fundamental radiation are numerically investigated. The standard system of equations for the amplitudes of probability of finding the system in state with certain energy is used to derive the expression for the induced polarization in the nonlinear medium. As for the equations for the amplitudes of the optical pulses, the general case is considered in which both the amplitudes and phases are space-time dependent. We focus on the finite difference methods and the case of simultaneously propagating solitons at all frequencies of the interacting waves (simultons). The obtained results indicate that upon certain threshold conditions all interacting pulses become the solitons of Lorentzian shape. The numerical analysis has also shown that the soliton amplitudes significantly depend on the ratio between the nonlinear polarizability at the fundamental frequency $\omega_0$ and that of combination of $\omega_0$ and the trigger-field frequency $\omega_1 \left( 2 \omega_0 + \omega_1 \right)$. In the second part of the paper, we apply the method of phase planes to show that at typical values of parameters, the solitons are stable.

Keywords

Solitons, Transition Regime, Stability

1. Introduction

Solitons or self-reinforcing solitary waves can emerge spontaneously in a physical system in which some energy
is fed in, for instance as thermal energy or by an excitation with an electromagnetic wave, even if the excitation
does not match exactly the soliton solution. Therefore, if a system possesses the necessary properties to allow
the existence of solitons, it is highly likely that any large excitation will indeed lead to their formation [1]-[3].
The field of solitons and related nonlinear phenomena has been substantially advanced and enriched by research
and discoveries in nonlinear optics [4]-[7].

In our previous research [8], we established the possibility of the existence of simultons (simultaneously
propagating solitons at different frequencies) in the case of nonstationary Raman scattering with excitation of
polar optical phonons under the conditions of the interaction of ultrashort pulses of exciting and Stokes radiation
in nonlinear crystals. The relevance of this study is connected both with the fact that one can extract additional
information on the optical characteristics of matter, and with the possibility of obtaining of ultrashort pulses.

The second topical problem in modern nonlinear optics is the production of coherent and frequency-tunable
radiation in the far ultraviolet (UV) and infrared (IR). In these spectral areas, solid materials have broad absorp-
tion bands and this narrows down the application of nonlinear crystals for the generation of electromagnetic
radiation. Possible ways of overcoming those difficulties are related with the utilization of nonlinear phenomena
in gases and metal vapors. The resonant four-photon interaction (RFPI) in the case of two-photon resonance is
one of them. Among the advantages of gases are the presence of narrow resonances and possibility of conti-
nuous variation of density, width of spectral line, length of the medium, etc. [9]-[11]. Ultrashort pulse propaga-
tion in the case of two-photon resonance was first examined in [12] where the two-photon self-induced transpa-
rency effect was predicted. This prediction was subsequently confirmed experimentally [13] and by numerical
studies [14]. Third-harmonic generation (THG) in media exhibiting resonance behaviour has also attracted con-
siderable attention [15]-[21]. However, RFPIs that are not frequency degenerate are of no less interest; they can
be used to transfer the tuning of radiation from one range to another [22] [23].

The present paper is devoted to the computer simulation of transition regimes of RFPI solitons in the case of
two-photon resonance. The basic equations describing this process are given in Section 2 [12] [24]. The results
of computer simulation are shown in Section 3. The stability of solitons is considered in Section 4.

2. Fundamental Principle

Let us assume that two optical pulses with frequencies \( \omega_{0,1} \) propagate in the nonlinear medium at the angles
\( \theta_{0,1} \) with respect to the z-axis. The value of \( 2\omega_0 \) is close to the frequency of resonant transition between levels
2 and 1 in the medium \( (\omega_2 = 2\omega_0 - \Delta\omega) \). The nonlinear interaction between \( \omega_{0,1} \) and the medium results in
parametric generation of \( \omega_2 = 2\omega_0 + \omega_1 \) and \( \omega_3 = 2\omega_0 - \omega_1 \). The values of \( \omega_{0,1,2,3} \) are considered to be in the
transparent range of frequencies. We also assume that all electromagnetic waves have the same polarizations.

To find the system of equations that governs the processes of propagation of optical pulses with frequencies
\( \omega_{0,1,2,3} \) in the medium we take the standard system of equations for the amplitudes of probability
\( a_k \) of finding the system in state with energy \( E_k \) [25]

\[
\dot{a}_k = \frac{1}{\hbar} \sum_{\ell} e^{i\omega_{\ell} t} V_{k\ell} a_{\ell},
\]

where \( V_{k\ell} = -\frac{1}{2} \mu_{k\ell} \sum_m E_m (e^{i\phi_m + e^{-i\phi_m}}), \Phi_m = \omega_m t - k_m z + \phi_m, (\Phi_m = -\Phi_m, E_m = E_{-m}) \),

\[
(2)
\]

\( \mu_{k\ell} \) is the dipole moment of the transition \( k \rightarrow \ell \); \( \omega_k, k_m, E_m, \phi_m \) are the frequencies, wave vectors, real
“slowly-varying amplitudes”, and phases of the interacting waves, respectively.

We next use (1) and the theory of perturbations [26] to find \( a_l \) \( (l \neq 1, 2) \) (the perturbation coefficient is of
order \( \mu_{02} E_{-2} / \hbar \omega_{02} \))

\[
a_l = \frac{1}{2\hbar} \sum_{m, p=\pm 2} \frac{\mu_{02} E_{-2}}{(\omega_0 + \omega_m)} e^{i(\phi_m + \phi_2)} a_p, \quad (l \neq 1, 2)
\]

(3)

To obtain the system of equations for \( a_{1,2} \) we introduce the expression (3) into the Equation (1), which be-


The system of Equations (4) and (5) can now be rewritten in terms of \( r_{11}^{(m)} \) and \( r_{12}^{(m)} \):

\[
i \frac{\partial a_i}{\partial t} = \left( \sum_{m=0}^{3} \frac{i r_{11}^{(m)} E_m^2}{4h} \right) a_i - \frac{1}{4h} \left( r_{12}^{(1)} E_0^2 + r_{12}^{(2)} E_2 E_0 e^{i\Delta} + r_{12}^{(3)} E_3 E_0 e^{i\Delta} \right) e^{i\Delta} a_2,
\]

\[
i \frac{\partial a_2}{\partial t} = -\frac{1}{4h} \left( r_{12}^{(1)} E_0^2 + r_{12}^{(2)} E_2 E_0 e^{i\Delta} + r_{12}^{(3)} E_3 E_0 e^{i\Delta} \right) e^{i\Delta} a_1 - \left( \sum_{m=0}^{3} \frac{r_{22}^{(m)} E_m^2}{4h} \right) a_2,
\]

where

\[
r_{11}^{(m)} = \frac{1}{\hbar \omega_m^2} \sum_{l=1,2} \frac{\mu_{l1} \mu_{l2} \omega_{l1,2}}{(\omega_{l1,2} - \omega_m^2)}, \quad r_{12}^{(1)} = \frac{1}{\hbar \omega_1 \omega_2} \sum_{l=1,2} \frac{\mu_{l1} \mu_{l2}}{\omega_{l1,2}}, \quad r_{12}^{(2)} = \frac{1}{\hbar \omega_2} \sum_{l=1,2} \frac{\mu_{l1} \mu_{l2} \omega_{l1,2}}{\omega_{l1,2} - \omega_m^2},
\]

\[
r_{12}^{(3)} = \frac{1}{\hbar \omega_{12}} \sum \mu_{l1} \mu_{l2} \left( \frac{1}{\omega_{l1} + \omega_{l2}} + \frac{1}{\omega_{12} + \omega_m} \right),
\]

\[
r_{22}^{(m)} = \frac{2}{\hbar} \sum \frac{\mu_{l1} \mu_{l2} \omega_{l1,2}}{(\omega_{l1,2} - \omega_m^2)}, \quad \Delta = \Delta_0 t - 2k_z^2 \bar{z} + 2\bar{q}_0,
\]

\[
\Delta_1 = \left( \frac{2k_0^2}{2} + k_1^2 - k_2^1 \right) \bar{z} + \bar{q}_1 - 2\bar{q}_0.
\]

The expression for the polarization induced by the superposition of nonlinear waves is defined by

\[
P = \sum_{i=1,2} \left( a_i^* \mu_i a_i e^{i\omega m t} + \alpha_i^* \mu_i a_i e^{i\omega m t} + c.c. \right).
\]

We introduce (3) into (7) and find that the expression for the induced polarization becomes

\[
P = \sum_{m=0}^{3} \left( \Delta r^{(m)} n + x^{(m)} \right) E_m \cos \Phi_m + 2r_{12}^{(1)} E_0 P_1 \cos \Phi_0 + 2r_{12}^{(2)} E_0 P_2 \sin \Phi_0
\]

\[
+ \left[ r_{12}^{(1)} E_2 \left( P_1 \cos \Delta_2 - P_2 \sin \Delta_2 \right) + r_{12}^{(2)} E_2 \left( P_1 \cos \Delta_2 + P_2 \sin \Delta_2 \right) \right] \sin \Phi_1
\]

\[
+ \left[ r_{12}^{(1)} E_0 \left( P_1 \cos \Delta_0 - P_2 \sin \Delta_0 \right) + r_{12}^{(2)} E_0 \left( P_1 \cos \Delta_0 + P_2 \sin \Delta_0 \right) \right] \cos \Phi_1
\]

\[
+ r_{12}^{(2)} E_1 \left( P_1 \cos \Delta_2 - P_2 \sin \Delta_2 \right) \cos \Phi_2 + r_{12}^{(2)} E_1 \left( P_1 \cos \Delta_2 + P_2 \sin \Delta_2 \right) \sin \Phi_2
\]

\[
+ r_{12}^{(2)} E_1 \left( P_1 \cos \Delta_1 - P_2 \sin \Delta_1 \right) \sin \Phi_3 + r_{12}^{(2)} E_1 \left( P_1 \cos \Delta_1 + P_2 \sin \Delta_1 \right) \cos \Phi_3,
\]

where \( P_i = \text{Re} \left( a_i^* a_i e^{i\omega m t} \right), \quad P_2 = \text{Im} \left( a_1^* a_2 e^{i\omega m t} \right), \quad n = |a_1|^2 - |a_2|^2, \quad \Delta r^{(m)}, \kappa^{(m)} = 0.5 \left( r_{22}^{(m)} + r_{11}^{(m)} \right).

The system of Equations (4) and (5) can now be rewritten in terms of \( P_{1,2} \) and \( n \) as follows

\[
\frac{\partial P_1}{\partial t} = -\Delta \Omega \tau_0 P_2 - \frac{n}{4} \left( r_{12}^{(1)} E_2 \cos \Delta_2 + r_{12}^{(2)} E_2 \sin \Delta_2 \right),
\]

\[
\frac{\partial P_2}{\partial t} = \Delta \Omega \tau_0 P_1 - \frac{n}{4} \left( r_{12}^{(1)} E_2 \sin \Delta_2 + r_{12}^{(2)} E_2 \cos \Delta_2 \right),
\]

\[
\frac{\partial n}{\partial t} = \left( r_{12}^{(1)} E_0 + r_{12}^{(2)} E_1 \sin \Delta_2 + r_{12}^{(3)} E_1 \cos \Delta_2 \right) P_2 + \left( r_{12}^{(1)} E_2 + r_{12}^{(2)} E_3 \cos \Delta_2 + r_{12}^{(3)} E_3 \sin \Delta_2 \right) P_1,
\]

where \( \Delta = \sum_{m=0}^{3} \frac{\Delta r^{(m)} + \Delta m + 2\Delta \varphi_0}{2h} E_m \cos \Phi_m + 2 T \frac{\partial \varphi_0}{\partial t} \), \( \tilde{E}_m = \frac{E_m}{A_0}, \quad \tilde{r}_{12}^{(1)} = \frac{r_{12}^{(1)} \tau_0 A_0^2}{h}, \quad \tilde{r}_{12}^{(2)} = \frac{r_{12}^{(2)} \tau_0 A_0^2}{h}, \quad \tau_0 \) is the maximum pulse amplitude; \( \varphi_0 \) is the pulse width.

To make the system (8) - (10) complete we add Maxwell’s equations for the all real “slowly-varying amplitudes” \( \tilde{E}_{0,1,2,3} \) and their phases \( \varphi_{0,1,2,3} \). We obtain

\[
\frac{\partial \tilde{E}_0}{\partial z} + \frac{1}{\tilde{v}_{E_0}} \frac{\partial \tilde{E}_0}{\partial t} = -2\alpha_0 \tilde{E}_0 P_2,
\]

\[
\frac{\partial \varphi_0}{\partial t} + \frac{1}{\tilde{v}_{\varphi_0}} \frac{\partial \varphi_0}{\partial t} = -\beta_0 n - \tilde{\gamma}_0 - 2\alpha_0 \varphi_0,
\]

\[
\frac{\partial \tilde{E}_1}{\partial z} + \frac{1}{\tilde{v}_{\tilde{E}_1}} \frac{\partial \tilde{E}_1}{\partial t} = -\alpha_1 \tilde{E}_1 \left( P_0 \cos \Delta_2 - P_2 \sin \Delta_2 \right) - \tilde{v}_{\tilde{E}_1} \left( P_1 \cos \Delta_2 + P_2 \sin \Delta_2 \right),
\]
The optical pulses on the pump and trigger frequencies \( n_1 \) and \( n_2 \) are locked to \( \omega_0 \). We assume that both processes \( \phi_0, \phi_1 \) were chosen to be of Gaussian shape. The accuracy of numerical results was based upon monitoring the conservation of energy of the system at every cross-sectional area in the medium. We considered the following trends: the phase locking and confinement; the length of medium needed for soliton formation; the relationship between the soliton speed and characteristics of nonlinear medium; the conditions leading to formation solitary wave at one frequency (instead of generation of sequence of them); the connection between the amplitude of soliton and two frequencies \( n_1, n_2 \). This condition is usually satisfied in gases [29]. Moreover, we also suggest that the phase differences \( \Delta \omega_{01} \) due to the nonlinear effects [30]. In this case \( P_0 = 0 \) (\( \Delta \Omega \tau = \pm 1 \)). Let \( \Delta \omega_1 \) be \( \Delta \omega_1 = 2\pi n + \Delta \omega_2 \), or \( \Delta \omega_1 = 2\pi n + \Delta \omega_2 \), where \( \Delta \omega_2 \) are some small phase fluctuations. Finally, the modified system of (12) (16) (18) and (18) can be written in terms of \( \Delta \omega_2 \) as follows

\[
\frac{d\Delta \omega_2}{d\xi} = \left( f_1, \sin \Delta \omega_2 \right) \sin \Phi, \\
\frac{d\Delta \omega_2}{d\xi} = \left( f_2, \sin \Delta \omega_2 \right) \sin \Phi,
\]
The behaviour of the latter system is analyzed in terms of phase planes. As an example, Figure 5 shows the phase plane of the following system

$$\frac{d\tilde{\Delta}_{i}}{d\tilde{\xi}} = \left(\sin \tilde{\Delta}_{i} + \sin \tilde{\Delta}_{j}\right) \sin \Phi,$$  \hspace{1cm} (21)
Figure 3. The generation of the soliton at frequency $\omega_2$.

Figure 4. The generation of the soliton at frequency $\omega_3$.

at 11 different initial conditions for $\tilde{\Lambda}_{1,2}$.

5. Conclusion

The space-time evolution of the optical pulses by using the computer simulation of transition regimes of four-photon resonant parametric processes in case of two-photon resonance is investigated. The computer simulation was based on application of the finite difference methods to the system of nonlinear equations modeling the foregoing interactions. It is shown that at certain boundary conditions (those result from the “area theorem” (see, e.g. [30])) the incoming laser pulses at frequencies $\omega_{0,1}$ first generate new waves at $\omega_{2,3}$, and then all become simultons of Lorentzian shape. It has also been shown that upon the conditions of phase locking ($\Delta_{1,2} = 2\pi n + \tilde{\Lambda}_{1,2}$)
Figure 5. The phase plane of the system (21) and (22) \((x = \Delta_1, \quad y = \Delta_2)\) for 11 consecutive initial conditions for \(\Delta_{1,2} = 2.0, 1.9, 1.8, 1.0\).

or \(\Delta_{1,2} = (2n+1)\pi + \Delta_{1,2}\) and synchronism \((k_1^7 \approx 2k_0^7 - k_1^7, \quad k_2^7 \approx 2k_0^7 + k_1^7)\) in wide range of typical values of polarizabilities, the simultons are stable. These results could be useful for the applications related with designing the lossless communication systems using the tunable frequencies ranging from IR \((\omega_h)\) to UV \((2\omega_h + \omega_k)\).

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