Random weighting to approximate posterior inference in LASSO regression

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Abstract: We consider a general-purpose approximation approach to Bayesian inference in which repeated optimization of a randomized objective function provides surrogate samples from the joint posterior distribution. In the context of LASSO regression, we repeatedly assign independently-drawn standard-exponential random weights to terms in the objective function, and optimize to obtain the surrogate samples. We establish the asymptotic properties of this method under different regularization parameters $\lambda_n$. In particular, if $\lambda_n = o(\sqrt{n})$, then the random-weighting (weighted bootstrap) samples are equivalent (up to the first order) to the Bayesian posterior samples. If $\lambda_n = O(n^c)$ for some $1/2 < c < 1$, then these samples achieve conditional model selection consistency. We also establish the asymptotic properties of the random-weighting method when weights are drawn from other distributions, and also if weights are assigned to the LASSO penalty terms.

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1. Introduction

1.1. Background and Motivation

Computational and modeling considerations in contemporary Bayesian analysis have led to renewed interest in a class of weighted bootstrap algorithms for posterior inference. An important example in this class is the weighted likelihood bootstrap (WLB), which was designed to yield approximate posterior samples in parametric models (Newton and Raftery, 1994). Compared to Markov chain Monte Carlo (MCMC), for example, WLB provides computationally efficient approximate posterior samples in cases where likelihood optimization is relatively easy. Asymptotic arguments demonstrate that for sufficiently regular models WLB samples provide a valid posterior approximation as the amount of data increases. However, the utility of WLB approximation is in doubt as we consider the availability of sophisticated MCMC schemes and codes that are simulation consistent: i.e., they produce exact posterior summaries on a fixed data set as the amount of computing resources increases without bound (e.g., Carpenter et al., 2017). Contemporary models also strain the validity of regularity conditions that support existing WLB asymptotic analysis.

The decades since publication of WLB have seen dramatic improvements in algorithms and code systems for optimization, as well as the interpenetration of these techniques into statistics (e.g., Duchi et al. (2015), Bhadra et al. (2019), Tibshirani and Taylor (2011)). This period has likewise seen improvements in Bayesian analysis, but there continue to be difficulties with posterior computation in some settings, especially given the problem to assure Monte Carlo error bounds with MCMC (e.g., Mossel and Vigoda (2006)), the increased size of data sets (e.g., Welling and Teh, 2011), the increased complexity of modeling techniques (e.g., Jordan et al., 2013), and the growing emphasis on tools that are not overly sensitive to modeling assumptions. The search for scalable, accurate posterior inference tools continues to be an important challenge in computational statistics.

Framing WLB in a contemporary context, Newton, Polson and Xu (2018) extended the posterior approximation scheme to a class of penalized likelihood objective functions. They saw good performance of the proposed Weighted Bayesian Bootstrap (WBB) extension in high-dimensional regression, trend filtering, and deep-learning applications. Others have recognized the utility of weighted bootstrap computations beyond the realm of parametric posterior approximation. A critical perspective was provided by Bissiri, Holmes and Walker (2016) with the concept of generalized Bayesian inference. Rather than constructing a fully specified probabilistic model for data, as in traditional Bayesian analysis, the authors told us to focus on an objective function for a parameter of interest, sidestepping the marginal posterior inference on this parameter by creating a generalized Bayesian posterior defined directly using this objective function. Lyddon, Holmes and Walker (2019) discovered a key connection between the generalized Bayesian posterior and WLB sampling, and constructed a modification called the loss-likelihood bootstrap to leverage this connection.
Further links to nonparametric Bayesian inference were recently reported in Lyddon, Walker and Holmes (2018) and Fong, Lyddon and Holmes (2019). These works demonstrate renewed interest in the operating characteristics of weighted bootstrap computation.

A useful perspective provided by recent work on weighted bootstrapping concerns the parameter, which we denote \( \theta \) following conventional presentations, as residing in some parameter space \( \Theta \), usually a nice subset of \( p \)-dimensional Euclidean space. A typical model-based approach treats \( \theta \) as an index to probability distributions in the specified model. This is not the primary perspective taken in the recent work cited above, where the focus is more nonparametric. Whether or not the model specification is valid, we may identify the distribution within the parametric model closest to the generative distribution \( F \) as a solution to an optimization problem

\[
\theta := \theta(F) := \arg \min_{t \in \Theta} \int l(t, y) dF(y).
\]  

(1.1)

Here \( y \) denotes a data point, which is distributed \( F \), and \( l(\cdot, \cdot) \) is a loss function specified by the analyst. Denoting \( p(y|\theta) \) as the density function in a working probability model, a natural loss function is \( l(\theta, y) = -\log p(y|\theta) \). From the nonparametric perspective, \( \theta \) is a model-guided feature of \( F \).

If we place a Dirichlet prior on \( F \) and have a random sample \( y_1, y_2, \ldots, y_n \) of data points, then the posterior for \( F \) is also a Dirichlet process (e.g., Ferguson, 1973). Computationally, by sampling \( F \) from this posterior and recomputing \( \theta = \theta(F) \) each time – i.e. by repeating the optimization in (1.1) – we obtain posterior draws of \( \theta \). As Fong, Lyddon and Holmes (2019) described, the weighted log likelihood of the WLB approximates the objective function in (1.1) for any single posterior-sampled \( F \). Hence we get posterior \( \theta \) draws (and thereby Bayesian integration) by repeated optimization of a randomized objective function. Sampling \( F \) is relatively straightforward; even when optimizations are relatively difficult they may be farmed out to separate cores in a parallel computation, making the entire scheme scalable to large problems. The perspective is not completely new (e.g., Muliere and Secchi (1996), Hjort (1991), Lo (1991)), but the renewed interest certainly reflects a recognition of the general loss functions allowed in (1.1) as well as the potential utility of repeated optimizations for scalable Bayesian inference.

Whether we aim for approximate parametric Bayes, generalized Bayes, or model-guided nonparametric Bayes, it is important to understand the distributional properties of these \textit{random-weighting} procedures. Precise answers are difficult, even with simple loss functions (e.g., Hjort and Ongaro, 2005), and so asymptotic methods are helpful to study the conditional distribution of \( \theta(F) \) given data. Adopting a Dirichlet prior on \( F \), Fong, Lyddon and Holmes (2019) pointed out that WBB sampling is consistent under suitable regularity conditions, due to posterior consistency property of the Dirichlet process (e.g., Ghosal, Ghosh and Ramamoorthi (1999), Ghosal, Ghosh and van der Vaart (2000)). Newton and Raftery (1994)’s first-order analysis of the weighted bootstrap samples yields the same Gaussian limits as the standard Bernstein-von-
Mises results (e.g., van der Vaart, 1998) under a correctly-specified Bayesian parametric model. Under model misspecification setting, Lyddon, Holmes and Walker (2019) showed that the Gaussian limits of weighted bootstrap sampling do not coincide with their Bayesian counterparts in Kleijn and van der Vaart (2012). Instead, they mimic the Gaussian limits in Huber (1967) – the asymptotic covariance matrix of the weighted bootstrap sampling is in fact the well-known sandwich covariance matrix in robust statistics literature. With the work reported here, we aim to extend asymptotic analysis for weighted bootstrap distributions to high-dimensional regression models. Our work adapts frequentist-theory asymptotic arguments, notably the works of Knight and Fu (2000) and Zhao and Yu (2006), to the present context.

1.2. Problem Setup

Consider the linear regression model with fixed design

\[ Y = \beta_0 1 + X \beta + \epsilon, \]  

where \( Y = (y_1, \ldots, y_n)' \in \mathbb{R}^n \) is the response vector, \( 1 \) is a \( n \times 1 \) vector of ones, \( X \in \mathbb{R}^{n \times p} \) is the design matrix, \( \beta \) is the vector of regression coefficients, and \( \epsilon = (\epsilon_1, \ldots, \epsilon_n)' \) is the vector of independent and identically distributed (i.i.d.) random errors with mean 0 and variance \( \sigma^2 \). We switch notation from \( \theta \) to \( \beta \) in deference to regression literature. Without loss of generality, we assume that the columns of \( X \) are centered, and take \( \hat{\beta}_0 = \bar{Y} \), in which case we can replace \( Y \) in (1.2) with \( Y - \bar{Y} 1 \), and concentrate on estimating \( \beta \). Again, without loss of generality, assume \( \bar{Y} = 0 \). Let \( \beta_0 \in \mathbb{R}^p \) be the true model coefficients with \( q \) non-zero components, where \( q \leq \min(p, n) \). Note that \( Y, X \) and \( \epsilon \) are all indexed by sample size \( n \), but we omit the subscript whenever this does not cause confusion.

The linear model has been studied extensively from both Bayesian and frequentist perspectives. Typical frequentist approach for handling sparse linear model (1.2) involves maximizing a penalized likelihood, with a penalty on model complexity. Some landmark developments include the least absolute shrinkage and selection operator (LASSO) (Tibshirani, 1996), the elastic net (Zou and Hastie, 2005) algorithm, the smoothly-clipped absolute deviation (SCAD) algorithm (Fan and Li, 2001), and the Dantzig selector (Candes and Tao, 2007). See, for example, Liu and Yu (2013) for an overview of frequentist theory in linear model variable selection. Meanwhile, in a simple Bayesian setup that gives closed-form analytical solutions, normal likelihood is specified, which corresponds to normality of the error terms \( \epsilon \), along with a conjugate Normal prior for \( \beta \) and inverse Gamma prior for \( \sigma^2 \) (Robert and Casella, 2004). For Bayesian variable selection, various priors on the model space have been studied, and these affect the structure of posterior inference. George and McCulloch (1993) introduced latent variables \( z = (z_1, \ldots, z_p)' \) representing active-class membership (\( z_j = 1 \) represents non-zero \( \beta_j \), whereas \( z_j = 0 \) indicates that the \( j^{th} \) variable is dropped) and specified suitable priors for \( z \) and \( \beta \). Other notable work on
Bayesian analysis of linear regression is in Berger and Pericchi (1996) (on Bayes factors) and Liang et al. (2008) (on g-priors). Posterior model selection consistency gained traction following the works of Johnson and Rossell (2012) (on non-local priors) and Narisetty and He (2014) (on spike-slab priors).

Recall, the LASSO estimator is given by

\[
\hat{\beta}_n := \arg \min_{\beta} \sum_{i=1}^{n} (y_i - x_i' \beta)^2 + \lambda_n \sum_{j=1}^{p} |\beta_j|,
\]

(1.3)

for a scalar penalty \(\lambda_n\) (Tibshirani, 1996), where \(x_i'\) is the \(i^{th}\) row of \(X\). From a Bayesian perspective, this objective function corresponds to the negative log posterior density from a Gaussian likelihood and a double Exponential (Laplace) prior, which may be represented with a scale mixture of normals (Andrews and Mallows, 1974), and so the solution to (1.3) is also the maximum a posteriori (MAP) estimator in a certain Bayesian model. Full posterior analysis in this model is possible using the Gibbs sampler (Park and Casella, 2008).

Following Newton, Polson and Xu (2018), the weighted Bayesian bootstrap (WBB) version of (1.3) is

\[
\hat{\beta}_n^w := \arg \min_{\beta} \left\{ \sum_{i=1}^{n} W_i (y_i - x_i' \beta)^2 + \lambda_n \sum_{j=1}^{p} W_0,j |\beta_j| \right\},
\]

(1.4)

where, for \(i = 1, 2, \ldots, n\), the analyst constructs \(W_i \overset{iid}{\sim} \text{Exp}(1)\). The precise treatment of penalty-associated weights \(W_0 = (W_{0,1}, \cdots, W_{0,p})\) induces several WBB variations, the simplest of which has \(W_{0,j} = 1\) for all \(j\), and the most elaborate of which has all entries i.i.d. exponential variates.

**Algorithm 1: Weighted Bayesian Bootstrap (General)**

**Input:**
- data: \(D = (y, X)\)
- number of draws: \(B\)

**Output:** \(B\) parameter samples \(\{\hat{\beta}_n^w\}_{b=1}^{B}\)

for \(b = 1 \text{ to } B\) do

\begin{enumerate}
  \item Draw \(W_i, W_{0,j} \overset{iid}{\sim} \text{Exp}(1)\) \(\forall\ i = 1, \ldots, n\) and \(j = 1, \ldots, p\);
  \item Store \(\hat{\beta}_n^w\) obtained by optimizing (1.4);
\end{enumerate}

end

These WBB algorithms produce independent samples and are trivially parallelizable over \(b = 1, \ldots, B\). Newton, Polson and Xu (2018) compared them to MCMC-based computations via the Bayesian LASSO (Park and Casella, 2008), and demonstrated good numerical properties in terms of estimation error, prediction error, credible set construction, and agreement with the Bayesian LASSO posterior. Here we investigate asymptotic properties of (1.4), and in particular, details of its conditional distribution given data. In addition, we examine how its
Algorithm 2: Weighted Bayesian Bootstrap (Basic)

Input:
- data: \( D = (y, X) \)
- number of draws: \( B \)

Output:
- \( B \) parameter samples \( \{ \hat{\beta}_{n,b}^w \}_{b=1}^B \)

for \( b = 1 \) to \( B \) do
  Draw \( W_i \sim \text{Exp}(1) \) \( \forall i = 1, \ldots, n \);
  Set \( W_0,j = 1 \) \( \forall j = 1, \ldots, p \);
  Store \( \hat{\beta}_{n,b}^w \) obtained by optimizing (1.4);
end

Statistical properties evolve with the relaxation of the distributional assumption for the random weights.

We consider both fixed-dimensional \((p_n = p)\) and growing-dimensional \((p_n\) increases with \(n)\) settings. Throughout this paper, we assume that the number of non-zero true regression parameters \(q\) to be fixed, and

\[
\mathbb{E}(\varepsilon_i^4) < \infty \quad \forall \ i,
\]

and all predictors are bounded, i.e. \( \exists \ M_1 > 0 \) such that

\[
|x_{ij}| \leq M_1 \quad \forall \ i = 1, \ldots, n ; \ j = 1, \ldots, p_n,
\]

where \( x_{ij} \) refers to the \((i, j)^{th}\) element of \( X \). Without loss of generality, we could partition \( \beta_0 \) into

\[
\beta_0 = \begin{bmatrix} \beta_{0(1)} \\ \beta_{0(2)} \end{bmatrix},
\]

where \( \beta_{0(1)} \) refers to the \( q \times 1 \) vector of non-zero true regression parameters, and \( \beta_{0(2)} \) is a \((p - q) \times 1\) zero vector. Similarly, we could partition the columns of the design matrix \( X \) into

\[
X = \begin{bmatrix} X_{(1)} \\ X_{(2)} \end{bmatrix}
\]

which corresponds to \( \beta_{0(1)} \) and \( \beta_{0(2)} \) respectively. We further assume that for some \( M_2 > 0 \),

\[
\alpha^T \left[ \frac{X_{(1)'X_{(1)}}}{n} \right] \alpha \geq M_2 \quad \forall \ ||\alpha||_2 = 1.
\]

If \( p_n = p \) is fixed, we also assume that there exists a non-singular matrix \( C \) such that

\[
\frac{1}{n} X'X = \frac{1}{n} \sum_{i=1}^n x_i x_i' \to C \quad \text{as} \ n \to \infty,
\]
where $x_i$ is the $i^{th}$ row of the design matrix $X$.

**Comments on assumptions:** The fixed-$q$ assumption comes naturally under the fixed-$p$ setting. For growing $p_n$, the fixed-$q$ assumption is commonly found in Bayesian linear-model literature, such as Johnson and Rossell (2012) and Narisetty and He (2014). Since we intend to investigate the feasibility of the random-weighting approach in approximating posterior inference, we make the fixed-$q$ assumption to align with existing theory. The finite moment assumption (1.5) is weaker than the normality assumption commonly specified under a Bayesian approach (George and McCulloch (1993), Robert and Casella (2004), Park and Casella (2008), Johnson and Rossell (2012), Narisetty and He (2014)). Assumption (1.6) can also be found in some seminal papers, such as Zhao and Yu (2006) and Chatterjee and Lahiri (2011), and in fact, can be (trivially) achieved by standardizing the covariates. Assumption (1.7) is equivalent to providing a lower bound to the minimum eigenvalue of $\frac{1}{n}X′X$. This eigenvalue assumption is very common in both frequentist and Bayesian literature, such as Zhao and Yu (2006) and Narisetty and He (2014). Note that assumptions (1.6) and (1.7), coupled with the fact that $q$ is fixed, ensure that $\frac{1}{n}X′(1)X(1)$ is invertible $\forall n$, a fact that we rely on throughout this paper. Finally, assumption (1.8) is common in the LASSO literature under fixed $p$ setting, which can be traced back to Knight and Fu (2000) and Zhao and Yu (2006). This assumption basically explains the relationship between the predictors under a fixed design model, and can be interpreted as the direct counterpart to the variance-covariance matrix of $X$ under a random design model. For the case of growing $p_n$, assumption (1.8) is no longer appropriate since the dimension of $\frac{1}{n}X′X$ grows.

The remaining sections present our asymptotic analysis of weighted bootstrapping. In Section 2 we set the necessary notation used throughout, and then we report our main results in Section 3. Allowing different rates of growth of the regularization parameter $\lambda_n$ and under suitable regularity conditions, we prove that, conditional on data, the random-weighting method has the following properties:

- conditional consistency (for fixed $p_n = p$)
- conditional asymptotic normality (for fixed $p_n = p$)
- conditional model selection consistency (for both fixed $p$ and growing $p_n$).

Section 3 concerns with problem setup (1.4) where $W_{0,j} = 1$ for all $j$, whereas random $W_0 = (W_{0,1}, \cdots, W_{0,p})$ is explored in later sections. Section 4 discusses how these statistical properties of the random-weighting approach relate to standard Bayesian procedures. In particular, conditional consistency and conditional asymptotic normality validate the use of random-weighting approach to approximate MCMC sampling, whereas the conditional model selection consistency property is comparable to the posterior model selection consistency of certain Bayesian methods. In Section 5 we explore how these statistical properties change when the random weights are drawn from distributions other than the standard exponential distribution, and when random weights are also assigned to...
the $l_1$ penalty term in (1.4). In particular, we show that the random-weighting estimator retains the same statistical properties when the weights are drawn from certain class of distributions, or when the standard exponential weights are also assigned to the $l_1$ penalty term. We conclude in Section 6 with some open problems related to the random-weighting approach. Appendix A provides extensive details, including the common probability space for our random-weighting scheme, and proofs for all theorems, propositions, and corollaries.

2. Preliminaries

There are two sources of variation in the random-weighting setup (1.4), namely the error terms $\epsilon$ and the user-defined weights $W$. To investigate the random-weighting approximation, we focus on the conditional probabilities given data, that is, given the sigma-field generated by $Y$. We refer readers to Appendix A for measure-theoretic details. In particular, we denote the common probability measure to be $P = P_D \times P_W$, where $P_D$ is the probability measure of the observed data $Y_1, Y_2, \ldots$, and $P_W$ is the probability measure of the triangular array of random weights. The use of product measure reflects the independence of user-defined $W$ and data-associated $\epsilon$.

**Conditional Convergence Notations:** Let

$$F_n := \sigma(Y_1, \ldots, Y_n) = \sigma(\epsilon_1, \ldots, \epsilon_n),$$

and let random variables (or vectors) $U, V_1, V_2, \ldots$ be defined on $(\Omega, A)$. We say $V_n$ converges in conditional probability a.s. $P_D$ to $U$ if $\forall \epsilon > 0$

$$P(\|V_n - U\| > \epsilon|F_n) \to 0 \quad a.s. \quad P_D$$
as $n \to \infty$. The notation a.s. $P_D$ is read as *almost surely under $P_D$*, and means *for almost every infinite sequence of data $Y_1, Y_2, \ldots$*. For brevity, this convergence is denoted

$$V_n \xrightarrow{c.p.} U \quad a.s. \quad P_D.$$

Similarly, we say $V_n$ converges in conditional distribution a.s. $P_D$ to $U$ if for any Borel set $A \subset \mathbb{R}$,

$$P(V_n \in A|F_n) \to P(U \in A) \quad a.s. \quad P_D$$
as $n \to \infty$. For brevity, this convergence is denoted

$$V_n \xrightarrow{c.d.} U \quad a.s. \quad P_D.$$

**Other Notation:** Following the usual convention, denote $\Phi\{\cdot\}$ as the cumulative distribution function of the standard normal distribution. For ease of notation, let $A_i$ and $A_{j}$ be the $i^{th}$ row and $j^{th}$ column (respectively) of a matrix $A$. Besides that, for any two vectors $u$ and $v$ of the same dimension, we denote
\( u \circ v \) as the Hadamard (entry-wise) product of the two vectors. In addition, define
\[
\begin{bmatrix}
C_n(11) & C_n(12) \\
C_n(21) & C_n(22)
\end{bmatrix} := \frac{1}{n} X'X = \frac{1}{n} \begin{bmatrix}
X'(1)X(1) & X'(1)X(2) \\
X'(2)X(1) & X'(2)X(2)
\end{bmatrix}.
\]

Note that an immediate consequence of Assumption (1.8) is that
\( C_n(ij) \to C_{ij} \forall i, j = 1, 2 \), where \( C_{11} \) is invertible. Furthermore, let \( D_n = \text{diag}(W_1, \ldots, W_n) \), and define
\[
\begin{bmatrix}
C_w(n)(11) & C_w(n)(12) \\
C_w(n)(21) & C_w(n)(22)
\end{bmatrix} := \frac{1}{n} X'D_nX = \frac{1}{n} \begin{bmatrix}
X'(1)D_nX(1) & X'(1)D_nX(2) \\
X'(2)D_nX(1) & X'(2)D_nX(2)
\end{bmatrix}.
\]

Finally, an estimator \( \hat{\beta} \) is said to be equal in sign to the true parameter \( \beta_0 \), if
\[
\text{sgn}(\hat{\beta}) = \text{sgn}(\beta_0),
\]
and is denoted as
\[
\hat{\beta} \overset{=}{} \beta_0.
\]

3. Main Results

Throughout this section, we consider problem setup (1.4) where \( W_{0,j} = 1 \) for all \( j \). The first theorem tells us about conditional consistency of parameter estimation by the random-weighting method under fixed dimensional setting.

**Theorem 3.1.** Assume (1.5), (1.6), (1.8) and that \( p \) is fixed.

(a) **(Conditional Consistency)** If \( \frac{\lambda_n}{n} \to 0 \), then
\[
\frac{\hat{\beta}_n}{n} \overset{c.p.}{\to} \beta_0 \ \text{a.s.} \ P_D.
\]

(b) If \( \frac{\lambda_n}{n} \to \lambda_0 \in (0, \infty) \), then
\[
\left( \frac{\hat{\beta}_n}{n} - \beta_0 \right) \overset{c.p.}{\to} \arg \min_u g(u) \ \text{a.s.} \ P_D,
\]
where
\[
g(u) = u'C'u + \lambda_0 \| \beta_0 + u \|_1.
\]

If \( \lambda_n \) vanishes in the limit, then the random-weighting estimator is consistent (conditional on data) in estimating \( \beta_0 \). Otherwise, the estimation is asymptotically biased as shown in part (b) of the theorem.

Next, we establish the conditional asymptotic normality of the random-weighting estimator under fixed-dimensional setting.
Theorem 3.2. (Asymptotic Conditional Distribution) Assume (1.5), (1.6), (1.8) and that $p$ is fixed. Let $\hat{\beta}_n^{OLS}$ be the ordinary least squares estimator of $\beta$ in the linear model (1.2). If $\frac{\lambda_n}{\sqrt{n}} \to 0$, then

$$\sqrt{n} \left( \hat{\beta}_n^w - \hat{\beta}_n^{OLS} \right) \xrightarrow{c.d.} N \left( 0, \sigma^2 \epsilon C^{-1} \right) \quad a.s. \ P_D. \tag{3.1}$$

Due to Assumption (1.8), the ordinary least squares estimator $\hat{\beta}_n^{OLS}$ is strongly consistent (Lai, Robbins and Wei, 1978). In fact, $\hat{\beta}_n^{OLS}$ can be replaced with any strongly consistent estimator of $\beta$ that satisfies

$$\frac{1}{\sqrt{n}} X' e_n \to 0 \quad a.s. \ P_D,$$

where $e_n$ is the residual of the strongly consistent estimator. Meanwhile, a more sophisticated argument is needed to establish the asymptotic conditional distribution for the case of $\left( \frac{\lambda_n}{\sqrt{n}} \right) \to \lambda_0 \in (0, \infty)$. First, note that for $j \in \{ j : \beta_{0,j} = 0 \}$, $\sqrt{n} \hat{\beta}_n^{OLS}$ has an asymptotic normal distribution, denoted $Z_j$. By the Skorokhod representation theorem, there exists random variables $U_{n,j}$ and $U_j$ on $(\Omega, \mathcal{A}, P)$ such that $U_{n,j} \overset{d}{=} \sqrt{n} \hat{\beta}_n^{OLS}$, $U_j \overset{d}{=} Z_j$, and $U_{n,j} \to U_j$ a.s. $P_D$. Then, for $\left( \frac{\lambda_n}{\sqrt{n}} \right) \to \lambda_0 \in (0, \infty)$,

$$\sqrt{n} \left( \hat{\beta}_n^w - \hat{\beta}_n^{OLS} \right) \xrightarrow{c.d.} \arg \min_u V^*(u) \quad a.s. \ P_D, \tag{3.1}$$

where

$$V^*(u) = -2u' \Psi + u'C u$$

$$+ \lambda_0 \sum_{j=1}^p \left[ u_j \text{sgn}(\beta_{0,j}) \mathbb{1}_{\{\beta_{0,j} \neq 0\}} + (|U_j + u_j| - |U_j|) \mathbb{1}_{\{\beta_{0,j} = 0\}} \right] ,$$

for $\Psi \sim N(0, \sigma^2 \epsilon C)$. Interestingly, this mimics the asymptotic distribution of the LASSO parametric residual bootstrap (Knight and Fu, 2000).

The next two results concern with the property of conditional model selection consistency given data. In particular, we are interested in the conditional probability of the random-weighting estimator matching the signs of $\beta_0$. Notably, sign consistency is stronger than variable selection consistency which requires only matching of zeroes. Nevertheless, we agree with Zhao and Yu (2006)'s argument of considering sign consistency – it allows us to avoid situations where models have matching zeroes but reversed signs, which hardly qualify as correct models. First, we establish the lower bound for this conditional probability.

Proposition 3.1. Assume (1.5), (1.6) and (1.7). Here, $p$ can either be fixed or grow with $n$. Furthermore, assume the strong irrerepresentable condition (Zhao and Yu, 2006): there exists a positive constant vector $\eta$ such that

$$\left| C_n^{(21)} \left( C_n^{(11)} \right)^{-1} \text{sgn} (\beta_{0(1)}) \right| \leq 1 - \eta, \tag{3.2}$$
where $\mathbf{1}$ is a $(p - q) \times 1$ vector of ones, $0 < \eta_j \leq 1 \forall j = 1, \ldots, p - q$, and the inequality holds element-wise. Then

$$P \left( \hat{\beta}_n^w(\lambda_n) = \beta_0 | \mathcal{F}_n \right) \geq P \left( A_n^w \cap B_n^w | \mathcal{F}_n \right) \quad \text{a.s. } P_D,$$

where

$$A_n^w = \left\{ \left[ C_n(1) - 1 \right] Z_n^{w(1)} + \rho_{1,n}^w Z_n^{w(1)} - \frac{\lambda_n}{2\sqrt{n}} \rho_{1,n}^w \text{sgn} [\beta_0(1)] \right\} \leq \sqrt{n} \left( |\beta_0(1)| + \frac{\lambda_n}{2n} \left[ C_n(1) - 1 \right] \text{sgn} [\beta_0(1)] \right) \text{ element-wise},$$

$$B_n^w = \left\{ \left[ C_n(21) C_n(1) - 1 \right] Z_n^{w(2)} + \rho_{2,n}^w Z_n^{w(1)} - \frac{\lambda_n}{2\sqrt{n}} \rho_{2,n}^w \text{sgn} [\beta_0(1)] \right\} \leq \frac{\lambda_n}{2\sqrt{n}} \eta \text{ element-wise},$$

and

$$Z_n^{w(1)} = \frac{1}{\sqrt{n}} X_n^{(1)} D_n \epsilon,$$

$$Z_n^{w(2)} = \frac{1}{\sqrt{n}} X_n^{(2)} D_n \epsilon,$$

$$\rho_{1,n}^w = \left( C_n(1) - 1 \right)^{-1} - C_n^{11},$$

$$\rho_{2,n}^w = C_n^{21} \left( C_n^{11} \right)^{-1} - C_n(21) C_n^{11}.$$
Again, we would like to highlight the fact that conditional on data, the randomness of $A^w_n$ and $B^w_n$ derives from the random weights in $Z^w_{n(1)}$, $Z^w_{n(2)}$, $\rho^w_{1,n}$ and $\rho^w_{2,n}$. Due to the random-weighting setup in (1.4), we end up with additional terms $\rho^w_{j,n} Z^w_{n(j)}$ and $\frac{\lambda_n}{2\sqrt{n}} \rho^w_{j,n} \text{sgn} [\beta_{0(1)}]$, for $j = 1, 2$. As seen in the proofs in the Appendix, these terms vanish in the limit with a mild additional constraint on the regularization parameter $\lambda_n$ as we see in the following:

**Theorem 3.3. (Conditional Model Selection Consistency – $p$ fixed)**
Assume (1.5), (1.6), (1.7), (1.8), (3.2) and that $p$ is fixed. Then, for any $\frac{1}{2} < c_1, c_2 < 1$ such that $c_1 + c_2 < 1.5$, and for all $\lambda_n$ that satisfies

$$\frac{\lambda_n}{n^{1.5-c_1}} \to 0,$$

we have

$$P \left( \hat{\beta}^w_n (\lambda_n) \right) \to 1 \text{ a.s. } P_D.$$  

Finally, we revisit the conditional model selection consistency property under the growing-dimensional setting.

**Theorem 3.4. (Conditional Model Selection Consistency – $p_n$ grows)**
Assume (1.5), (1.6), (1.7) and (3.2). For any $0 < c_3 < \frac{1}{2} < c_1, c_2 < 1$ such that $c_3 < 2 \min(c_1, c_2) - 1$ and $c_1 + c_2 < 1.5$, for which $p_n = \mathcal{O}(n^{c_2})$ and $\lambda_n = \mathcal{O}(n^{c_2})$, we have

$$P \left( \hat{\beta}^w_n (\lambda_n) \right) \to 1 \text{ a.s. } P_D.$$  

We point out the fact that even under fixed-dimensional setting, the conditional model selection consistency could not be achieved at an exponential rate, due to the presence of the additional terms

$$\rho^w_{j,n} Z^w_{n(j)} \text{ and } \frac{\lambda_n}{2\sqrt{n}} \rho^w_{j,n} \text{sgn} [\beta_{0(1)}]$$

for $j = 1, 2$ that impedes the rate of convergence.

4. Connection to Bayesian Inference

We could use the asymptotic properties of the random-weighting method established in Section 3 to draw its connection to the standard Bayesian inference.

4.1. First-order Approximate Bayesian Inference

Theorems 3.1 and 3.2 describe the first order behavior of the conditional distribution of $\beta^w_n$. Under typical parametric Bayesian inference for $\beta$ in the linear
model (1.2), for any prior measure of $\beta$ that is absolutely continuous in a neighborhood of $\beta_0$ with a continuous positive density at $\beta_0$, the Bernstein-von Mises Theorem (see, for example, Theorem 10.1 of van der Vaart (1998)) ensures that for every Borel set $A \subset \Theta \subset \mathbb{R}^p$,

$$P \left[ \sqrt{n} \left( \beta - \hat{\beta}^{MLE}_n \right) \in A | F_n \right] \to P \left[ Z \in A \right]$$

along almost every sample path, where $Z \sim N(0, \sigma^2 \mathbf{C}^{-1})$, and $\hat{\beta}^{MLE}_n$ is the maximum likelihood estimator of $\beta$ in (1.2), which corresponds to $\hat{\beta}^{OLS}_n$ under normality assumption $\epsilon \sim N(0, \sigma^2 \mathbf{I})$. Hence, based on Theorem 3.2, for all $\lambda_n = o(\sqrt{n})$, the conditional distribution of the random-weighting estimator $\hat{\beta}^w_n$ converges to the same limit as in the Bernstein-von Mises Theorem, ie. The conditional distribution of $\hat{\beta}^w_n$ is the same – at least up to the first order – as the posterior distribution of $\beta$ under the regime of Bayesian inference. We remind the readers that the random-weighting method does not embed any prior information of $\beta$ and hence is not simulation-consistent.

We also like to point out that Newton and Raftery (1994)’s first-order approximation theory for the random-weighting method relies on some classical regularity assumptions which do not hold in the LASSO setting (1.3). Our work proves the affirmative in that setting, assuming the conditions laid out in Theorems 3.1 and 3.2.

4.2. Posterior Model Selection Consistency

Firstly, we note that Bayesian and frequentist notions of model selection consistency do not necessarily coincide. For instance, Zhao and Yu (2006) define “strong sign consistency” as selected model having the same signs as the true model for any pre-selected regularization parameter $\lambda_n = f(n)$ (i.e. a function of $n$ independent of data) with probability converging to one; whereas “general sign consistency” is achieved if there exists the right amount of regularization parameter $\lambda_n$ that allows selected model to match the signs of the true model with probability converging to one. On the other hand, Narisetty and He (2014), who deployed a Bayesian framework for posterior inference of $\beta$, referred to “strong model selection consistency” as posterior probability of the true model converging to one, whereas “model selection consistency” is defined as selected model equals the true model with posterior probability converging to one. Even within the Bayesian literature, the notion of variable selection consistency may not be the same. For instance, some Bayesian work (e.g., Moreno, Giron and Casella, 2010) considered pairwise Bayes factor consistency – the Bayes factor of any other fitted model with respect to the true model goes to zero.

We have shown, in Theorems 3.3 and 3.4, that the conditional probability of random-weighting estimator having the same signs as the true model converges to one for almost every data set. We find this result to be more closely related to the notion of Bayesian (strong) model selection consistency achieved by Johnson and Rossell (2012) and Narisetty and He (2014).
We also acknowledge the fact that strong model selection consistence is a “stronger” version of the posterior model selection consistency, since it requires the sum of posterior probabilities of all other models in the feature space to shrink to zero. Next, our assumptions on regularity of design (1.6) and (1.8), as well as the strong irrepresentable condition (3.2), are arguably slightly stronger than Narisetty and He (2014)’s conditions 3, 4 and 5. Furthermore, Narisetty and He (2014) could handle high-dimensional problem where \( p_n \) grows nearly exponential with sample size \( n \), whereas the random-weighting calculations here allow \( p_n \) to grow at a polynomial rate of \( n \).

### 4.3. Connection to the MAP estimate of a Bayesian model

We argue another tangential connection between the random-weighting method and a Bayesian posterior inference. Narisetty and He (2014) explained the connection between their Bayesian procedure with the \( l_0 \) penalization. In particular, they introduced a spike-slab Gaussian prior with prior variances dependent on sample size \( n \). For large \( n \), the resulting Bayesian MAP estimator for model selection is equivalent to the model selected by minimizing the following objective function

\[
m(\beta) := \|Y - X\beta\|_2^2 + \psi_n (\|\beta\|_0 - \|\beta_0\|_0),
\]

where \( \|\cdot\|_0 \) denotes the \( l_0 \) norm, \( \psi_n = M \times nd_n \) for some \( M > 0 \), and \( nd_n = o(n) \).

Minimizing the objective function (4.1) is equivalent to minimizing

\[
m(u) = -2u' \left( \frac{X' \epsilon}{\sqrt{n}} \right) + u' \left( \frac{X'X}{n} \right) u + \frac{\psi_n}{\sqrt{n}} (\|u + \sqrt{n}\beta_0\|_0 - \|\sqrt{n}\beta_0\|_0),
\]

which looks similar to the derivation of Theorem 2 in Knight and Fu (2000), except that here we have \( l_0 \) penalization instead of \( l_1 \) penalization. If we approximate the \( l_0 \) penalization with \( l_1 \) penalization, and if \( d_n = O \left( \frac{1}{\sqrt{n}} \right) \) such that \( \frac{\psi_n}{\sqrt{n}} \to \lambda_0 \),

then (4.1) will essentially have the same asymptotic distribution as highlighted in Theorem 2 of Knight and Fu (2000), which is comparable to our results in Theorem 3.2. This reveals another connection between Bayesian inference and the random-weighting method.

### 5. Other Issues

In this section, we examine how the statistical properties in Section 3 of the random-weighting method change if

1. the random weights are drawn from distributions other than the standard exponential distribution, and
2. independent standard exponential random weights are also assigned on the penalty term.
5.1. Non Standard-Exponential Random Weights

We highlight the fact that all theorems established in Section 3 only depend on the first four moments of the standard exponential distribution for the random weights. In fact, we would obtain the same asymptotic properties if the random weights are any i.i.d positive random variables with unit mean, unit variance and finite fourth moment.

Again, throughout this subsection, we consider problem setup (1.4) where $W_{0,j} = 1$ for all $j$. We could generalize all theorems in Section 3 with different random-weight distributions as follows.

**Corollary 5.1.** Adopt assumptions in Theorem 3.1. The random weights could be any i.i.d. positive random variables with any mean $\mu_W \in (0, \infty)$, any variance $\sigma^2_W \in (0, \infty)$ and finite fourth moment. Then,

(a) **(Conditional Consistency)** If $\frac{\lambda_n}{n} \to 0$, then

$$\hat{\beta}_n^w \xrightarrow{c.p.} \beta_0 \quad a.s. \ P_D.$$  

(b) If $\frac{\lambda_n}{n} \to \lambda_0 \in (0, \infty)$, then

$$\left(\hat{\beta}_n^w - \beta_0\right) \xrightarrow{c.p.} \arg \min_u g(u) \quad a.s. \ P_D,$$

where

$$g(u) = \mu_W u'Cu + \lambda_0\|\beta_0 + u\|_1.$$  

Corollary 5.1 shows that conditional consistency is not affected by moments of the random weights if $\lambda_n = o(n)$. Otherwise, the estimation is biased with an additional factor $\mu_W$.

**Corollary 5.2.** Adopt assumptions in Theorem 3.2. The random weights could be any i.i.d. positive random variables with any mean $\mu_W \in (0, \infty)$, any variance $\sigma^2_W \in (0, \infty)$ and finite fourth moment. Then,

$$\sqrt{n} \left(\hat{\beta}_n^w - \hat{\beta}_n^{OLS}\right) \xrightarrow{c.d.} N \left(0, \frac{\sigma^2_W \sigma^2_T}{\mu_W} C^{-1}\right) \quad a.s. \ P_D.$$  

Corollary 5.2 has important implication for the random-weighting approach in approximating posterior inference, since it highlights the fact that non-unitary mean or variance of the random weights causes the random-weighting samples to converge to a conditional normal distribution with an asymptotic variance that is different from the one guaranteed by the Bernstein-von-Mises Theorem under a typical Bayesian method. Note that if $\lambda_n = \mathcal{O}(\sqrt{n})$, the Skorokhod
argument after Theorem 3.2 follows accordingly here, except that now $V^*(u)$ becomes

$$V^*(u) = -2u'\Psi + \mu_W u'C u$$

$$+ \lambda_0 \sum_{j=1}^p \left[ u_j \text{sgn}(\beta_{0,j}) \mathbb{1}_{\{\beta_{0,j} \neq 0\}} + (|U_j + u_j| - |U_j|) \mathbb{1}_{\{\beta_{0,j} = 0\}} \right],$$

where $\Psi \sim N(0, \sigma_W^2 \sigma^2_C)$.

**Corollary 5.3.** Adopt assumptions in Theorem 3.3. The random weights could be any i.i.d. positive random variables with any mean $\mu_W \in (0, \infty)$, any variance $\sigma_W^2 \in (0, \infty)$ and finite fourth moment. Then,

$$P \left( \hat{\beta}_n^w(\lambda_n) = \beta_0 \mid F_n \right) \to 1 \quad \text{a.s. } P_D.$$  

**Corollary 5.4.** Adopt assumptions in Theorem 3.4. The random weights could be any i.i.d. positive random variables with any mean $\mu_W \in (0, \infty)$, any variance $\sigma_W^2 \in (0, \infty)$ and finite fourth moment. Then,

$$P \left( \hat{\beta}_n^w(\lambda_n) = \beta_0 \mid F_n \right) \to 1 \quad \text{a.s. } P_D.$$  

It is interesting to note that, both Corollaries 5.3 and 5.4 guarantee conditional model selection consistency for any choice of random weight distributions with positive support and finite fourth moment.

5.2. Random Weighting on the Penalty Term

In the weighted Bayesian bootstrap (WBB), Newton, Polson and Xu (2018) considered assigning independent standard exponential random weights to the $l_1$ penalty term in equation (1.4), in two similar fashions as follows:

$$\hat{\beta}_n^w = \arg\min_{\beta} \left\{ \sum_{i=1}^n W_i(y_i - x'_i\beta)^2 + \lambda_n \sum_{j=1}^p |\beta_j| \right\}, \quad (5.1)$$

where $W_i \overset{iid}{\sim} \text{Exp}(1) \quad \forall \quad i = 0, 1, \ldots, n$, or

$$\hat{\beta}_n^w = \arg\min_{\beta} \left\{ \sum_{i=1}^n W_i(y_i - x'_i\beta)^2 + \lambda_n \sum_{j=1}^p W_j |\beta_j| \right\}, \quad (5.2)$$

where $W_i, W_j \overset{iid}{\sim} \text{Exp}(1) \quad \forall \quad i = 1, \ldots, n$ and $j = 1, \ldots, p$. The number of covariates $p$ in (5.1) and (5.2) can either be fixed or grow with $n$. We investigate the asymptotic properties of the samples obtained based on these general versions of our problem setup (1.4).
Corollary 5.5. Adopt assumptions in Theorem 3.1, where \( p \) is fixed.

(a) **Conditional Consistency** For both weighting schemes \((5.1)\) and \((5.2)\), if \( \frac{\lambda_n}{n} \to 0 \), then

\[
\hat{\beta}_n^w \xrightarrow{c.p.} \beta_0 \quad \text{a.s. } P_D.
\]

(b) If \( \frac{\lambda_n}{n} \to \lambda_0 \in (0, \infty) \), then

\[
\left( \hat{\beta}_n^w - \beta_0 \right) \xrightarrow{c.d.} \arg \min_u g(u) \quad \text{a.s. } P_D,
\]

where

\[
g(u) = u'C u + \lambda_0 W \| \beta_0 + u \|_1
\]

for \( W \sim \text{Exp}(1) \) under weighting scheme \((5.1)\); and

\[
g(u) = u'C u + \lambda_0 \sum_{j=1}^{p} W_j | \beta_{0,j} + u_j |
\]

for \( W_j \sim \text{Exp}(1) \) under weighting scheme \((5.2)\).

Corollary 5.5 (a) shows that conditional consistency is not affected by random-weighting the penalty term if \( \lambda_n = o(n) \). On the other hand, note that in Corollary 5.5 (b), convergence in conditional distribution takes place instead of convergence in conditional probability, since the non-vanishing penalty term contains the random weights.

Corollary 5.6. Adopt assumptions in Theorem 3.2, where \( p \) is fixed. Then, for both weighting schemes \((5.1)\) and \((5.2)\),

\[
\sqrt{n} \left( \hat{\beta}_n^w - \hat{\beta}_n^{OLS} \right) \xrightarrow{c.d.} N \left( 0, \sigma^2 C^{-1} \right) \quad \text{a.s. } P_D.
\]

Corollary 5.6 shows that random-weighting the penalty terms does not affect the ability of the random-weighting samples to approximate Bayesian posterior samples if \( \lambda_n = o(n) \). Note that if \( \lambda_n = \mathcal{O}(\sqrt{n}) \), the Skorokhod argument after Theorem 3.2 follows accordingly here, except that now, for weighting scheme \((5.1)\),

\[
V^*(u) = -2u' \Psi + u'C u
+ \lambda_0 W \sum_{j=1}^{p} \left[ u_j \text{sgn}(\beta_{0,j}) \mathbb{1}_{\beta_{0,j} \neq 0} + (|U_j + u_j| - |U_j|) \mathbb{1}_{\beta_{0,j} = 0} \right],
\]
where $\Psi \sim N(0, \sigma^2_C)$ and $W \sim \text{Exp}(1)$ and $W \perp \Psi$; whereas for weighting scheme (5.2),
\[
V^*(u) = -2u'\Psi + u'Cu
+ \lambda_0 \sum_{j=1}^p W_j \left[u_j \text{sgn}(\beta_{0,j})1_{\{\beta_{0,j} \neq 0\}} + (|U_j + u_j| - |U_j|)1_{\{\beta_{0,j} = 0\}}\right],
\]
where $\Psi \sim N(0, \sigma^2_C)$ and $W_j \overset{iid}{\sim} \text{Exp}(1)$ and $W_j \perp \Psi \forall j = 1, \ldots, p$.

**Corollary 5.7.** Adopt assumptions in Theorem 3.3, where $p$ is fixed.

(a) Under weighting scheme (5.1), we immediately have
\[
P\left(\hat{\beta}_n^w(\lambda_n) = \beta_0|F_n\right) \rightarrow 1 \text{ a.s. } P_D.
\]

(b) Under weighting scheme (5.2), if $\eta = 1$, then
\[
P\left(\hat{\beta}_n^w(\lambda_n) = \beta_0|F_n\right) \rightarrow 1 \text{ a.s. } P_D.
\]

Note that for the weighting scheme (5.2), each of the penalty term has its own weight, which adversely affects/violates the strong irreprese ntable assumption (3.2), unless under a stringent condition where $\eta = 1$. One sufficient condition for $\eta = 1$ would be zero correlation between any relevant predictor and any irrelevant predictor, i.e. $C_{n(21)} = 0$.

**Corollary 5.8.** Assume (1.5), (1.6), (1.7) and (3.2).

(a) Under weighting scheme (5.1), for any $0 < c_3 < \frac{1}{2} < c_1 < c_2 < 1$ such that $c_1 + c_2 < 1.5$ and $c_3 < \min\{2(c_2 - c_1), 2c_1 - 1\}$, for which $p_n = O(n^{c_3})$ and $\lambda_n = O(n^{c_2})$, we have
\[
P\left(\hat{\beta}_n^w(\lambda_n) = \beta_0|F_n\right) \rightarrow 1 \text{ a.s. } P_D.
\]

(b) Under weighting scheme (5.2), for any $0 < c_3 < \frac{1}{2} < c_1 < c_2 < 1$ such that $c_1 + c_2 < 1.5$ and $c_3 < \min\{\frac{2}{5}(c_2 - c_1), 2c_1 - 1, c_2 - \frac{1}{2}\}$, for which $p_n = O(n^{c_3})$ and $\lambda_n = O(n^{c_2})$, if $\eta = 1$, then
\[
P\left(\hat{\beta}_n^w(\lambda_n) = \beta_0|F_n\right) \rightarrow 1 \text{ a.s. } P_D.
\]

Comparing Theorem 3.4 and Corollary 5.8, we can see that assigning random weights on the penalty term further impedes how fast $p_n$ could increase with $n$ while achieving conditional model selection consistency, especially when the penalty terms do not share a common random weight. Again for weighting scheme (5.2), conditional model selection could only be achieved under the stringent assumption of $\eta = 1$.

To emphasize, results in Corollaries 5.5 - 5.8 continue to hold if i.i.d. random weights are drawn from any distribution with positive support, unit mean, unit variance and finite fourth moment.
6. Discussion

We establish asymptotic properties of solutions to a randomized LASSO objective function in the context of high-dimensional linear regression. In particular, for fixed-\(p\) setting, if \(\lambda_n = o(\sqrt{n})\), then the randomized solutions are equivalent – up to the first order – to Bayesian posterior samples (under some mild conditions on prior distribution), whenever the random weights are drawn from any distributions with positive support, unit mean, unit variance and finite fourth moment. Assigning random weights – that are drawn from distributions with non-unit mean or variance – affects the random-weighting method’s ability to approximate Bayesian posterior samples, since the asymptotic variance of the limit distribution is scaled by the square of coefficient of variation (CoV) of the weights. Even if standard exponential random weights are also assigned to the \(l_1\) penalty term, the random-weighting method still approximates the posterior samples as long as \(\lambda_n = o(\sqrt{n})\).

We also examine the ability of random-weighting to correctly select variables conditional on data. In short, under both fixed-\(p\) and growing-\(p_n = o(\sqrt{n})\) settings, the random-weighting samples have conditional model selection consistency for \(\lambda_n = O(n^c)\) for some \(1/2 < c < 1\), even when the random weights are drawn from distributions other than the standard exponential distribution, or when a common standard exponential random weight is also assigned to the \(l_1\) penalty term. If each \(l_1\) penalty term \(|\beta_j|_1\) is assigned a different and independent random weight, then conditional model selection consistency may not be achieved unless the all relevant predictors are uncorrelated with the irrelevant predictors.

There are several interesting open problems or possible extensions pertaining to the random-weighting method. Firstly, we expect that arguments could be extended to the use of random-weighting for the generalized linear model. Besides that, one could examine the asymptotic properties of the random-weighting method under model-misspecification, that is, if the true model is not \(X\beta_0\). It would also be interesting if performance of the random-weighting method could also be compared with variational inference, which is another approximate MCMC method. Furthermore, this paper has only addressed the random-weighting method’s ability to provide first-order asymptotic (Gaussian) approximation to posterior samples, whereas Newton, Polson and Xu (2018)’s numerical simulations reveal that the random-weighting method may also be able to recover multi-modal finite-sample posterior densities. It would be interesting to examine if the random-weighting method could also provide finite-sample approximation to the posterior samples. Finally, one could also study the statistical properties of the random-weighting samples when this approach is extended to other penalized objective functions with different penalty structures to cater to different problems.
Appendix A

First, we refer to Chapter 3 of Newton (1991) for constructing a common probability space as foundation for the theoretical framework of this paper. We outline the important details as follows.

**Probability Space:** The independent data set \( Y_1, \ldots, Y_n, \ldots \), each a mapping \( Y_i : \Omega_d \rightarrow \mathbb{R} \) and each defined on a probability space \((\Omega_d, A_d, P_d)\). Given a data set \((Y_1, \ldots, Y_n)\), we draw an independent set of positive weights \((W_1, \ldots, W_n)\), each a mapping \( W_i : \Omega_w \rightarrow \mathbb{R}^+ \) and each defined on a probability space \((\Omega_w, A_w, P_w)\). Specifically, if \( W_i \overset{iid}{\sim} \text{exp}(1) \), then \( P_w \) induces a standard exponential distribution on \( \mathbb{R}^+ \).

We proceed to define two infinite product measurable spaces, namely

\[
(\Omega_D, A_D, P_D) := (\Omega_d^\infty, A_d^\infty, P_d^\infty)
\]

and

\[
(\Omega_W, A_W, P_W) := (\Omega_w^\infty, A_w^\infty, P_w^\infty),
\]

where a single point \( \omega_D \in \Omega_D \) determines an infinite sequence of data, and a single point \( \omega_W \in \Omega_W \) determines an infinite triangular array of real weights. Now, we could define the data and weights on the same probability space as the following product measurable space

\[
(\Omega, A, P) := (\Omega_D \times \Omega_W, A_D \times A_W, P_D \times P_W),
\]

since weights are drawn independent of data. Then, we can view each of \( Y_i \) or \( W_i \) as a coordinate mapping of \( \omega = (\omega_D, \omega_W) \in \Omega \). A single \( \omega \in \Omega \) determines a realization of (infinite sequence of) data and (infinite triangular array of) weights.

Our consideration focuses on the conditional probabilities given data. Formally, for each \( A \in \mathcal{A} \), such a conditional probability is a function

\[
P(A|\sigma(Y_1, \ldots, Y_n))(\omega) \quad \omega \in \Omega
\]

which is measurable with respect to

\[
\sigma(Y_1, \ldots, Y_n) \subset \mathcal{A},
\]

and which satisfies the relation

\[
\int_B P(A|\sigma(Y_1, \ldots, Y_n))dP = P(A \cap B) \quad \forall B \in \sigma(Y_1, \ldots, Y_n).
\]
This allows us to view conditional probabilities $P(Y_1, \ldots, Y_n)$ as random variables on $(\Omega_D, A_D, P_D)$.

The rest of the section deals with the proofs for all the theorems, proposition and corollaries in this paper. Many subsequent proofs rely on this following result.

**Lemma A.1.** Let $U_1, U_2, \ldots$ be any i.i.d. random variables with $E(U_i) = 0$ and $E[(U_i)^2] = \sigma^2 < \infty$. Then for any bounded sequence of real numbers $\{k_i\}$ and for any $1/2 < c_1 < 1$,

$$\frac{1}{n^{c_1}} \sum_{i=1}^{n} k_i U_i \xrightarrow{a.s.} 0.$$

**Proof.** Since $\{k_i\}$ are bounded, then $\exists M > 0$ such that $|k_i| \leq M \forall i$. Then

$$\sum_{n=1}^{\infty} Var\left(\frac{k_n U_n}{n^{c_1}}\right) = \sigma^2 \sum_{n=1}^{\infty} \frac{k_n^2}{n^{2c_1}} \leq \sigma^2 M^2 \sum_{n=1}^{\infty} \frac{1}{n^{2c_1}} < \infty.$$

Then, by Theorem 2.5.3 of Durrett (2010), with probability one,

$$\sum_{n=1}^{\infty} \frac{k_n U_n}{n^{c_1}} < \infty.$$

Finally, apply Kronecker’s Lemma to obtain the desired result.

**Lemma A.2.** Suppose that $p$ is fixed. Assume (1.6), and (1.8). Then, as $n \to \infty$,

$$\frac{1}{n} X'D_n X \xrightarrow{a.s.} C$$

**Proof.** Due to assumption (1.6), the Strong Law of Large Numbers gives

$$\frac{1}{n} X'(D_n - I)X = \frac{1}{n} \sum_{i=1}^{n} (W_i - 1)x_i x_i' \xrightarrow{a.s.} 0,$$

where $x_i$ is the $i^{th}$ row of $X$. Then, due to assumption (1.8), by Continuous Mapping Theorem,

$$\frac{1}{n} X'D_n X = \frac{1}{n} X'(D_n - I)X + \frac{1}{n} X'X \xrightarrow{a.s.} C.$$

□
An immediate consequence of Lemma A.2 is that
\[ C_{n(ij)}^w \xrightarrow{a.s.} C_{ij} \quad \forall \ i, j = 1, 2 \]
when \( p \) is fixed.

Proof of Theorem 3.1. Conditional on data,
\[
\hat{\beta}^w_n = \arg \min_{\beta} \left\{ \frac{1}{n}(Y - X\beta)'D_n(Y - X\beta) + \frac{\lambda_n}{n} \|\beta\|_1 \right\}
\]
\[ = \arg \min_{\beta} \left\{ \frac{1}{n}[\epsilon - X(\beta - \beta_0)]'D_n[\epsilon - X(\beta - \beta_0)] + \frac{\lambda_n}{n} \|\beta_0 + \beta - \beta_0\|_1 \right\}. \]

Therefore,
\[
(\hat{\beta}^w_n - \beta_0)
\]
\[ = \arg \min_{u} \left\{ \frac{1}{n}[\epsilon - Xu]'D_n[\epsilon - Xu] + \frac{\lambda_n}{n} \|\beta_0 + u\|_1 \right\}
\]
\[ = \arg \min_{u} \left\{ -2u' \left( \frac{X'D_n\epsilon}{n} \right) + u' \left( \frac{X'D_nX}{n} \right) u + \frac{\lambda_n}{n} \|\beta_0 + u\|_1 \right\}
\]
\[ := \arg \min_{u} g_n(u). \]

First note that
\[
\frac{1}{n}X'D_n\epsilon = \frac{1}{n}X'(D_n - I)\epsilon + \frac{1}{n}X'\epsilon.
\]

Conditional on data,
\[
\frac{1}{n}X'\epsilon = \frac{1}{n} \sum_{i=1}^{n} x_i \epsilon_i \rightarrow 0 \quad a.s. \ P_D,
\]
and assumption (1.6) as well as the Strong Law of Large Numbers ensure that \( \forall \ j = 1, \ldots, p, \)
\[
\frac{1}{n^2} \sum_{i=1}^{n} c_i^2 x_{ij}^2 E(W_i^2) = \frac{1}{n} \times \frac{1}{n} \sum_{i=1}^{n} x_{ij}^2 \epsilon_i^2 \rightarrow 0 \quad a.s. \ P_D.
\]

Hence, by the Weak Law of Large Numbers (see, for example, Theorem 1.14(ii) of Shao (2003)),
\[
\frac{1}{n}X'(D_n - I)\epsilon = \frac{1}{n} \sum_{i=1}^{n} c_i x_i (W_i - 1) \xrightarrow{c.p.} 0 \quad a.s. \ P_D.
\]

Finally, by the Continuous Mapping Theorem,
\[
\frac{X'D_n\epsilon}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i \epsilon_i W_i \xrightarrow{c.p.} 0 \quad a.s. \ P_D. \quad (A.1)
\]
Similarly, note that
\[
\frac{1}{n} \epsilon' D_n \epsilon = \frac{1}{n} \epsilon' (D_n - I) \epsilon + \frac{1}{n} \epsilon' \epsilon.
\]

Conditional on data,
\[
\frac{1}{n} \epsilon' = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \to \sigma^2 \epsilon \ a.s. \ P_D,
\]
and assumption (1.5) as well as the Strong Law of Large Numbers ensure that
\[
\frac{1}{n^2} \sum_{i=1}^{n} \epsilon_i^4 E(W_i^2) = \frac{1}{n} \times \frac{1}{n} \sum_{i=1}^{n} \epsilon_i^4 \to 0 \ a.s. \ P_D.
\]
Hence, by the Weak Law of Large Numbers,
\[
\frac{1}{n} \epsilon' (D_n - I) \epsilon = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i^2 (W_i - 1) \xrightarrow{c.p.} 0 \ a.s. \ P_D.
\]
Finally, by the Continuous Mapping Theorem,
\[
\frac{\epsilon' D_n \epsilon}{n} = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i^2 W_i \xrightarrow{c.p.} \sigma^2 \epsilon \ a.s. \ P_D. \quad (A.2)
\]
Therefore, by (A.1), (A.2) and Lemma A.2, for \( \lambda_n \to \lambda_0 \in [0, \infty) \),
\[
g_n(u) \xrightarrow{c.p.} g(u) + \sigma^2 \epsilon \ a.s. \ P_D.
\]
Since \( g_n(u) \) is convex, it follows from pointwise convergence of conditional probability that
\[
\hat{\beta}_n^w - \beta_0 = \mathcal{O}_p(1),
\]
and
\[
\sup_{u \in K} \left| g_n(u) - g(u) - \sigma^2 \epsilon \right| \xrightarrow{c.p.} 0 \ a.s. \ P_D
\]
for any compact set \( K \) by applying the Convexity Lemma (Pollard, 1991). Therefore,
\[
\left( \hat{\beta}_n^w - \beta_0 \right) = \arg\min_u g_n(u) \xrightarrow{c.p.} \arg\min_u g(u) \ a.s. \ P_D.
\]
If \( \lambda_0 = 0 \), \( \arg\min_u g(u) = 0 \), i.e. \( \hat{\beta}_n^w \xrightarrow{c.p.} \beta_0 \ a.s. \ P_D. \)

**Proof of Theorem 3.2.** Define
\[
Q_n(z) := \left\| D_n^\frac{1}{2} (y - X z) \right\|_2^2 + \lambda_n \| z \|_1,
\]
which leads to

\[ Q_n \left( \beta_{n, OLS} + \frac{1}{\sqrt{n}} u \right) = \left\| D_n^2 \left[ Y - X \left( \hat{\beta}_{n, OLS} + \frac{1}{\sqrt{n}} u \right) \right] \right\|_2^2 + \lambda_n \left\| \hat{\beta}_{n, OLS} + \frac{1}{\sqrt{n}} u \right\|_1, \]

and

\[ Q_n \left( \beta_{n, OLS} \right) = \left\| D_n^2 \left( Y - X \beta_{n, OLS} \right) \right\|_2^2 + \lambda_n \left\| \beta_{n, OLS} \right\|_1. \]

Now define

\[ V_n(u) := Q_n \left( \beta_{n, OLS} + \frac{1}{\sqrt{n}} u \right) - Q_n \left( \beta_{n, OLS} \right), \]

such that

\[ \arg \min_u V_n(u) = \arg \min_u Q_n \left( \beta_{n, OLS} + \frac{1}{\sqrt{n}} u \right) = \sqrt{n} \left( \hat{\beta}_n^w - \beta_{n, OLS} \right). \]

Notice that \( V_n(u) \) can be simplified into

\[ -2u' \left( \frac{X'D_n \epsilon_{n, OLS}}{\sqrt{n}} \right) + u' \left( \frac{X'D_n X}{n} \right) u + \frac{\lambda_n}{\sqrt{n}} \left\{ \left\| \sqrt{n} \beta_n^w + u \right\|_1 - \left\| \sqrt{n} \beta_{n, OLS} \right\|_1 \right\}. \]

Lemma A.2 gives

\[ \frac{X'D_n X}{n} \xrightarrow{a.s.} C, \]

while

\[ \frac{\lambda_n}{\sqrt{n}} \left\{ \left\| \sqrt{n} \beta_n^w + u \right\|_1 - \left\| \sqrt{n} \beta_{n, OLS} \right\|_1 \right\} \xrightarrow{c.p.} 0 \quad a.s. \quad P_D, \quad (A.3) \]

since \( \frac{\lambda_n}{\sqrt{n}} \to 0 \). Now, note that

\[
\begin{align*}
\frac{1}{\sqrt{n}} X' D_n \epsilon_{n, OLS} &= \left( \frac{1}{n} \sum_{i=1}^{n} e_i^2 x_i x_i' \right)^{\frac{1}{2}} \times \left( \sum_{i=1}^{n} e_i^2 x_i x_i' \right)^{-\frac{1}{2}} \left[ \sum_{i=1}^{n} e_i x_i W_i \right] \\
&= \left( \frac{1}{n} \sum_{i=1}^{n} e_i^2 x_i x_i' \right)^{\frac{1}{2}} \times \left( \sum_{i=1}^{n} e_i^2 x_i x_i' \right)^{-\frac{1}{2}} \left[ \sum_{i=1}^{n} e_i x_i (W_i - 1) \right].
\end{align*}
\]
where the last equality follows from the fact that
\[ \sum_{i=1}^{n} e_i x_i = X' e_n^{\text{OLS}} = X' Y - X' X (X' X)^{-1} X' Y = 0. \]

Since \( \hat{\beta}_n^{\text{OLS}} \) is a strongly consistent estimator due to assumption (1.8) (Lai, Robbins and Wei, 1978), we have
\[ \hat{\beta}_n^{\text{OLS}} \rightarrow \beta_0 \quad \text{a.s.} \quad P_D. \]

Then, by assumption (1.8), the Strong Law of Large Numbers and Continuous Mapping Theorem, we can easily show that conditional on data,
\[ \frac{1}{n} \sum_{i=1}^{n} e_i^2 x_i' x_i' \rightarrow \sigma_\varepsilon^2 C \quad \text{a.s.} \quad P_D, \tag{A.4} \]
because
\[ \sigma_\varepsilon^2 \frac{1}{n} \sum_{i=1}^{n} x_i' x_i' \rightarrow \sigma_\varepsilon^2 C, \]
whereas
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} (e_i^2 - \sigma_\varepsilon^2) x_i' x_i' \right\|_2^2 \\
\leq \sum_{k=1}^{p} \sum_{l=1}^{p} \left[ \frac{1}{n} \sum_{i=1}^{n} x_{i,k} x_{i,l} (e_i^2 - \sigma_\varepsilon^2) \right]^2 \\
+ \sum_{k=1}^{p} \sum_{l=1}^{p} \left[ (\hat{\beta}_n^{\text{OLS}} - \beta_0)' \left( \frac{1}{n} \sum_{i=1}^{n} x_{i,k} x_{i,l} x_i' x_i' \right) (\hat{\beta}_n^{\text{OLS}} - \beta_0) \right]^2 \\
- 2 \sum_{k=1}^{p} \sum_{l=1}^{p} \left[ (\hat{\beta}_n^{\text{OLS}} - \beta_0)' \left( \frac{1}{n} \sum_{i=1}^{n} x_{i,k} x_{i,l} e_i x_i' x_i' \right) \right]^2 \\
\rightarrow 0 \quad \text{a.s.} \quad P_D.
\]

Similarly,
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} e_i^4 x_i' x_i' x_i' x_i' \right\|_2^2 \\
\leq p^2 M_1^8 \times \left( \frac{1}{n} \sum_{i=1}^{n} \left| e_i' - x_i' \left( \hat{\beta}_n^{\text{OLS}} - \beta_0 \right) \right|^4 \right)^2 \\
\leq p^2 M_1^8 \times \left[ \frac{1}{n} \sum_{i=1}^{n} \left( |e_i| + p M_1 \left\| \hat{\beta}_n^{\text{OLS}} - \beta_0 \right\|_2 \right)^4 \right] \times \left( \frac{1}{n} \sum_{i=1}^{n} \left| e_i' - x_i' \left( \hat{\beta}_n^{\text{OLS}} - \beta_0 \right) \right|^4 \right)^2
\]

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which is bounded almost surely due to assumptions (1.5) and (1.6), as well as the fact that \( \hat{\beta}_n^{\text{OLS}} \) is a strongly consistent estimator for \( \beta \) in the linear model (1.2). This implies that

\[
\sum_{i=1}^{n} e_i x_i' x_i' = O(n) \quad \text{a.s. } P_D.
\]

Hence, the Lindeberg’s Central Limit Theorem gives us

\[
\left( \sum_{i=1}^{n} e_i^2 x_i' x_i' \right)^{-\frac{1}{2}} \left[ \sum_{i=1}^{n} e_i x_i (W_i - 1) \right] \stackrel{c.d.}{\rightarrow} N_p(0, I_p) \quad \text{a.s. } P_D,
\]

(A.5)

since the Liapounov’s sufficient condition is satisfied as follows:

\[
\begin{aligned}
&\left[ \sum_{i=1}^{n} e_i^2 x_i' x_i' \text{Var}(W_i) \right] \left[ \sum_{i=1}^{n} e_i x_i x_i' x_i' E(W_i - 1)^4 \right] \\
&= O(n^{-2}) \times O(n) \\
&= O(n^{-1}) \quad \text{a.s. } P_D.
\end{aligned}
\]

We mention in passing that, using Slutsky’s theorem, (A.5) can also be achieved if we replace \( \hat{\beta}_n^{\text{OLS}} \) with any other strongly consistent estimator of \( \beta \) which satisfies

\[
\frac{1}{\sqrt{n}} X' e_n \rightarrow 0 \quad \text{a.s. } P_D.
\]

Combining (A.3), (A.4), (A.5) and Lemma A.2, Slutsky’s Theorem gives us

\[
V_n(u) \stackrel{c.d.}{\rightarrow} V(u) := u'C u - 2u'\Psi \quad \text{a.s. } P_D,
\]

where \( \Psi \) has a \( N(0, \sigma^2_C) \) distribution. Since \( V_n(u) \) is convex and \( V(u) \) has a unique minimum, it follows from Geyer (1996) that conditional on data,

\[
\arg \min_u V_n(u) = \sqrt{n} \left( \hat{\beta}_n^w - \hat{\beta}_n^{\text{OLS}} \right) \stackrel{c.d.}{\rightarrow} \arg \min_u V(u) = C^{-1} \Psi \sim N(0, \sigma^2_C C^{-1}).
\]

We will also prove the penalty term of \( V^*(u) \) in (3.1). From the penalty term of \( V_n(u) \), we have

\[
\frac{\lambda_n}{\sqrt{n}} \left\{ \left\| \sqrt{n} \hat{\beta}_n^{\text{OLS}} + u \right\|_1 - \left\| \sqrt{n} \hat{\beta}_n^{\text{OLS}} \right\|_1 \right\}
\]

\[
= \frac{\lambda_n}{\sqrt{n}} \sum_{j=1}^{p} \left\{ \sqrt{n} \left[ \beta_{a,j} + \left( \hat{\beta}^{\text{OLS}}_{n,j} - \beta_{a,j} \right) \right] + \mu_j \right\} - \left[ \sqrt{n} \left[ \beta_{a,j} + \left( \hat{\beta}^{\text{OLS}}_{n,j} - \beta_{a,j} \right) \right] \right] \\
:= \frac{\lambda_n}{\sqrt{n}} \sum_{j=1}^{p} p_n(u_j).
\]
We focus on the case when $\frac{\lambda_n}{\sqrt{n}} \to \lambda_0 \in (0, \infty)$. When $\beta_{o,j} \neq 0$, $\hat{\beta}_{n,j}^{\text{OLS}} - \beta_{o,j}$ a.s. $\to 0$, and hence $\sqrt{n}\beta_{o,j}$ dominates $u_j$ for large $n$. Thus, it is easy to verify that for $\beta_{o,j} \neq 0$, $p_n(u_j)$ converges to $\sqrt{n}\beta_{o,j} \text{sgn}(\beta_{o,j}) \mathbb{I}_{\{\beta_{o,j} \neq 0\}}$.

For $\beta_{o,j} = 0$, $p_n(u_j)$ is back to $\left|\sqrt{n}\hat{\beta}_{n,j}^{\text{OLS}} + \mu_j\right| - \left|\sqrt{n}\hat{\beta}_{n,j}^{\text{OLS}}\right|$, which depends on the sample path of realized data. This necessitates the Skorokhod argument, thus leading to the penalty term in (3.1).

**Proof of Proposition 3.1.** First, we note that when $p < n$, the random weighting setup (1.4) is strictly convex, and thus has unique solution (Tibshirani, 2013). We do not consider the case where $p \gg n$ in this paper. Next, similar to previous proofs, note that

\[
(\hat{\beta}_n^w - \beta_0) = \arg \min_{u_n} \left\{-2u_n'(X'D_n\epsilon) + u_n'(X'D_nX)u_n + \lambda_n\|\beta_0 + u_n\|_1\right\}.
\]

Differentiating the first two terms with respect to $u_n$ yields

\[
2X'D_nXu_n - 2X'D_n\epsilon = 2\sqrt{n}\left[C_n^w\left(\sqrt{n}u_n\right) - Z_n^w\right].
\]

Note that $\hat{\beta}_n^w = \hat{u}_n + \beta_0$, which can be partitioned into

\[
\hat{\beta}_n^w = \begin{bmatrix} \hat{\beta}_n^{w(1*)} \\ \hat{\beta}_n^{w(2*)} \end{bmatrix},
\]

where $\hat{\beta}_n^{w(1*)}$ consists of non-zero elements of $\hat{\beta}_n^w$, and $\hat{\beta}_n^{w(2*)} = 0$. The asterisk here is to distinguish the empirical partition of $\hat{\beta}_n^w$ from the true partition of $\beta_0$. If both partitions are the same, then the random-weighting method selects the true model. Based on the empirical partition of $\hat{\beta}_n^w$, we have

\[
2\sqrt{n}\left[C_n^w\left(\sqrt{n}\hat{u}_n\right) - Z_n^w\right] = 2\sqrt{n}\left\{C_n^w\left[C_n^{w(1*)} C_n^{w(12*)} C_n^{w(21*)} C_n^{w(22*)}\right] \times \sqrt{n}\left[\hat{u}_n^{(1*)} \hat{u}_n^{(2*)}\right] - Z_n^{w(1*)} Z_n^{w(2*)}\right\}.
\]

Note that $\hat{u}_n^{(2*)}$ does not necessarily equal to 0 unless the random-weighting method selects the right model. As a consequence of the Karush-Kuhn-Tucker (KKT) conditions, we have

\[
C_n^{w(1*)}\left[\sqrt{n}\hat{u}_n^{(1*)}\right] + C_n^{w(12*)}\left[\sqrt{n}\hat{u}_n^{(2*)}\right] - Z_n^{w(1*)} = -\frac{\lambda_n}{2\sqrt{n}}\text{sgn}\left(\hat{\beta}_n^{w(1*)}\right),
\]

(A.6)
and
\[ C_{n(21)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] + C_{n(22)}^w \left[ \sqrt{n} \hat{u}_{n(2)} \right] - Z_{n(2')}^w \leq \frac{\lambda_n}{2\sqrt{n}} \mathbf{1} \quad (A.7) \]
element-wise. It is also easy to verify that
\[ \left\{ \text{sgn} \left( \hat{\beta}_{n(1)}^w \right) = \text{sgn} \left( \beta_{0(1)} \right) \right\} \supseteq \left\{ \left| \hat{u}_{n(1)} \right| < \left| \beta_{0(1)} \right| \right\} \text{ element-wise}, \quad (A.8) \]
because
\[ \left\{ \left| \hat{u}_{n(1)} \right| < \left| \beta_{0(1)} \right| \right\} \implies \left\{ \hat{u}_{n(1)} < \beta_{0(1)} \right\} \cap \left\{ \hat{u}_{n(1)} > -\beta_{0(1)} \right\} \implies \left\{ \hat{\beta}_{n(1)}^w < \beta_{0(1)} + \left| \beta_{0(1)} \right| \right\} \cap \left\{ \hat{\beta}_{n(1)}^w > \beta_{0(1)} - \left| \beta_{0(1)} \right| \right\}, \]
where all inequalities hold element-wise. Thus, it is clear that if \( \beta_{0(1)} < 0 \)
element-wise, then \( \hat{\beta}_{n(1)}^w < 0 \) element-wise. Vice-versa, if \( \beta_{0(1)} > 0 \)
element-wise, then \( \hat{\beta}_{n(1)}^w > 0 \) element-wise. Thus,
\[ \left\{ \left| \hat{u}_{n(1)} \right| < \left| \beta_{0(1)} \right| \right\} \text{ element-wise} \implies \left\{ \text{sgn} \left( \hat{\beta}_{n(1)}^w \right) = \text{sgn} \left( \beta_{0(1)} \right) \right\}. \]

Therefore, conditional on data, by (A.6), (A.7), (A.8), and uniqueness of solution for the random-weighting setup (1.4), if there exists \( \hat{u}_n \) such that the following equation and inequalities hold:
\[ C_{n(11)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(1)}^w = -\frac{\lambda_n}{2\sqrt{n}} \text{sgn} \left( \beta_{0(1)} \right) \quad (A.9) \]
\[ -\frac{\lambda_n}{2\sqrt{n}} \mathbf{1} \leq C_{n(21)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(2)}^w \leq \frac{\lambda_n}{2\sqrt{n}} \mathbf{1} \quad \text{element-wise} \quad (A.10) \]
\[ \left| \hat{u}_{n(1)} \right| < \left| \beta_{0(1)} \right| \quad \text{element-wise}, \quad (A.11) \]
then we have
\[ \text{sgn} \left( \hat{\beta}_{n(1)}^w \right) = \text{sgn} \left( \beta_{0(1)} \right) \quad \text{and} \quad \hat{u}_{n(2)} = \hat{\beta}_{n(2)}^w = \beta_{0(2)} = 0, \]
ie. \( \hat{\beta}_n = \beta_0 \), due to (A.8) and the fact that
\[ \left\{ \left| C_{n(21)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(2)}^w \right| \leq \frac{\lambda_n}{2\sqrt{n}} \mathbf{1} \right\} \quad \text{element-wise} \]
\[ \bigcap \left\{ \left| C_{n(11)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(1)}^w \right| = -\frac{\lambda_n}{2\sqrt{n}} \text{sgn} \left( \beta_{0(1)} \right) \right\} \bigcap \left\{ \hat{\beta}_n^w = \beta_0 \right\}^c \]
\[ = \emptyset, \]
which implies
\[ \left\{ \left| C_{n(21)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(2)}^w \right| \leq \frac{\lambda_n}{2\sqrt{n}} \mathbf{1} \right\} \quad \text{element-wise} \]
because of the following two cases:

1. If \( \hat{u}_{n(2^*)} \neq \hat{u}_{n(2)} = 0 \), i.e. zero entries of \( \hat{\beta}_n^w \) do not match with those of \( \beta_0 \), we cannot obtain (A.9) and (A.10) from (A.6) and (A.7) for almost every data set;

2. if \( \hat{u}_{n(2^*)} = \hat{u}_{n(2)} = 0 \) but \( \text{sgn} [\hat{\beta}_n^w] \neq \text{sgn} [\beta_0] \), we cannot obtain (A.9) from (A.6) for almost every data set.

Therefore,

\[
P(\hat{\beta}_n^w = \beta_0 | F_n) \geq P \left( \left| C_{n(11)} \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(1)} \right| \leq \frac{\lambda_n}{2\sqrt{n}} \text{ element-wise} \right) \]

\[
\cap \left\{ C_{n(11)} \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(1)} = -\frac{\lambda_n}{2\sqrt{n}} \text{sgn} [\beta_0] \right\} \]

\[
\cap \left\{ |\hat{u}_{n(1)}| < |\beta_0| \text{ element-wise} \right\} \bigg| F_n \bigg).
\]

Due to assumptions (1.6), (1.7), and the fact that \( q \) is fixed, \( C_{n(11)} \) is invertible for all \( n \), and from Lemma A.2,

\[
(C_{n(11)}^{-1}) = [C_{11} + o_p]^{-1} = O_p(1),
\]

where \( x_i(1) \) is the \( i^{th} \) row of \( X(1) \). Now we proceed to simplify these equation and inequalities (A.9), (A.10) and (A.11).

Equation (A.9) can be re-written as

\[
\rho_{1,1}^w Z_{n(1)} + C_{n(11)}^{-1} Z_{n(1)} = -\frac{\lambda_n}{2\sqrt{n}} \text{sgn} [\beta_0] \]

\[
= \sqrt{n} \left( \hat{u}_{n(1)} + \frac{\lambda_n}{2n} C_{n(11)}^{-1} \text{sgn} [\beta_0] \right).
\]

Substituting inequality (A.11) into the equation above leads to \( A_n^w \).

Meanwhile, equation (A.9) can also be re-expressed as

\[
\sqrt{n} \hat{u}_{n(1)} = (C_{n(11)}^{-1}) Z_{n(1)} = -\frac{\lambda_n}{2\sqrt{n}} (C_{n(11)}^{-1}) \text{sgn} (\beta_0) \].

Substitute this into inequality (A.10) and simple arithmetic yield

\[ \left\{ \left| \rho_{2,n}^w Z_{n(1)}^w + C_{n(2)}^{-1} C_{n(1)}^{-1} Z_{n(2)}^w - \frac{\lambda_n}{2\sqrt{n}} C_{n(2)}^{-1} \left( C_{n(1)}^{-1} \right)^{-1} \text{sgn} [\beta_0(1)] \right| \right. \]

\[ - \frac{\lambda_n}{2\sqrt{n}} \left| C_{n(2)}^{-1} C_{n(1)}^{-1} \text{sgn} [\beta_0(1)] \right| \]

\[ \leq \frac{\lambda_n}{2\sqrt{n}} \left( 1 - \left| C_{n(2)}^{-1} C_{n(1)}^{-1} \text{sgn} [\beta_0(1)] \right| \right) \text{ element-wise} \equiv \tilde{B}_n^w. \]

Now, observe that \( B_n^w \subseteq \tilde{B}_n^w \), since (LHS of \( B_n^w \)) \( \geq \) (LHS of \( \tilde{B}_n^w \)) element-wise, whereas (RHS of \( B_n^w \)) \( \leq \) (RHS of \( \tilde{B}_n^w \)) element-wise due to the Irrepresentable condition (3.2). Therefore

\[ P \left( \hat{\beta}_n^w = \beta_0 \left| \mathcal{F}_n \right. \right) \geq P \left( A_n^w \cap \tilde{B}_n^w \left| \mathcal{F}_n \right. \right) \geq P \left( A_n^w \cap B_n^w \left| \mathcal{F}_n \right. \right) \text{ a.s. } P_D. \]

Lemma A.3. Assume (1.5), (1.6), (1.7), (1.8) and (3.2). Here, \( q \) is fixed but \( p \) can either be fixed or grows with \( n \). Then, for any \( \frac{1}{2} < c_1 < 1 \) and \( \frac{1}{2} < c_2 < 1 \) such that \( c_1 + c_2 < 1.5 \), and for all \( \lambda_n \) that satisfies

\[ \frac{\lambda_n}{n^{c_2}} \to \infty \quad \text{but} \quad \frac{\lambda_n}{n^{1.5-c_1}} \to 0, \]

we have

\[ P \left[ (A_n^w)^c \left| \mathcal{F}_n \right. \right] = o(1) \quad \text{a.s. } P_D. \]

Proof. First, we focus on

\[ \frac{1}{\sqrt{n}} X_{(1)}^t (D_n - I) \epsilon \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i x_{i(1)} (W_i - 1) \]

\[ = \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i^2 x_{i(1)}^t x_{i(1)}^t \right]^{\frac{1}{2}} \times \left[ \sum_{i=1}^n \epsilon_i^2 x_{i(1)}^t x_{i(1)}^t \right]^{-\frac{1}{2}} \times \left( \sum_{i=1}^n \epsilon_i x_{i(1)} (W_i - 1) \right), \]

where \( x_{i(1)} \) is the \( i^{th} \) row of \( X_{(1)} \). Assumption (1.6) and the Strong Law of Large Numbers ensure that

\[ \frac{1}{n} \sum_{i=1}^n (\epsilon_i^2 - \sigma^2) x_{i(1)} x_{i(1)}^t \to 0_{q \times q} \quad \text{a.s. } P_D, \]

thus, by assumption (1.8) and Continuous Mapping Theorem,

\[ \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i^2 x_{i(1)} x_{i(1)}^t \right]^{\frac{1}{2}} \xrightarrow{c.p.} \sigma \epsilon C_{11}^\frac{1}{2} \quad \text{a.s. } P_D. \quad (A.12) \]
Meanwhile, the Lindeberg’s Central Limit Theorem gives
\[
\left[ \sum_{i=1}^{n} \epsilon_i x_{i(1)} x'_{i(1)} \right]^{-\frac{1}{2}} \left[ \sum_{i=1}^{n} \epsilon_i x_{i(1)}(W_i - 1) \right] \xrightarrow{c.d.} N_q(0, I_{q \times q}) \quad a.s. P_D, \quad (A.13)
\]
since the Liapounov’s sufficient condition is satisfied due to assumption (1.5) as follows:
\[
\left[ \sum_{i=1}^{n} \epsilon_i^2 x_{i(1)} x'_{i(1)} \right]^{-2} \left[ \sum_{i=1}^{n} \epsilon_i^4 \left( x_{i(1)} x'_{i(1)} x_{i(1)} x'_{i(1)} \right) \right] = O(n^{-2}) \times O(n) = O(n^{-1}) \quad a.s. P_D.
\]
Next, examine \( l_2 \)-norm of each term. Firstly,
\[
\| \text{sgn} [\beta_{0(1)}] \|_2 \leq \sqrt{q},
\]
whereas for all \( n \),
\[
\left\| C_{n(11)}^{-1} \right\|_2 < \infty \quad \text{and} \quad \left\| (C_{n(11)}^w)^{-1} \right\|_2 = O_p(1).
\]
By assumption (1.6) and Lemma A.1,
\[
\left\| X_{(1)}' (D_n - I) X_{(1)}' / n^{c_1} \right\|_2 \leq \left\| \frac{1}{n^{c_1}} \sum_{i=1}^{n} x_{i(1)} x'_{i(1)} (W_i - 1) \right\|_F
\]
\[
= \sqrt{\sum_{k=1}^{q} \sum_{l=1}^{q} \left[ \frac{1}{n^{c_1}} \sum_{i=1}^{n} x_{i,k(1)} x_{i,l(1)} (W_i - 1) \right]^2} \xrightarrow{a.s.} 0.
\]
Hence,
\[
- \frac{\lambda_n}{2\sqrt{n}} \rho_{1,n}^w \text{sgn} [\beta_{0(1)}] \xrightarrow{p} 0. \quad (A.14)
\]
Similarly, due to (A.12), (A.13) and Lemma A.1,
\[
\rho_{1,n}^w Z_{n(1)}^w
\]
\[
= - C^{-1}_{w(n)} \left[ \frac{X'_1(D_n - I)X'_1}{n^{c_1}} \right] \left( C^{-1}_{w(n)} \right)^{-1} \\
\times \left\{ \left[ \frac{1}{n^{1-c_1}} \frac{X'_1(D_n - I)\epsilon}{\sqrt{n}} \right] + \left[ \frac{1}{n^{1.5-c_1}} \sum_{i=1}^{n} \epsilon_i x_i(1) \right] \right\} \\
= o_p(1) \times [o_p(1) + o(1)] \\
= o_p(1) \quad \text{a.s. } P_D. \quad \text{(A.15)}
\]

Combining (A.12), (A.13), (A.14), (A.15) and assumption (1.8), Slutsky’s Theorem gives
\[
\rho^w_{1,n} Z^w_{n(1)} + \left[ C^{-1}_{w(n)} \frac{1}{\sqrt{n}} X'_1(D_n - I)\epsilon \right] - \frac{\lambda_n}{2\sqrt{n}} \rho^w_{1,n} \text{sgn} [\beta_{0(1)}] \\
\xrightarrow{c.d.} N_q \left( 0, \sigma^2 C^{-1}_{11} \right) \quad \text{a.s. } P_D.
\]

In addition, due to the fact that \( q \) is fixed, as well as assumptions (1.6) and that \( C_{n(1)} \) is invertible for all \( n \), every element of \( C^{-1}_{n(1)} x_i(1) \) is bounded. Let \( x^*_i \) be the \( j^{th} \) element of \( C^{-1}_{n(1)} x_i(1) \). Then \( \forall j = 1, \ldots, q \),
\[
\frac{1}{n} \sum_{i=1}^{n} \epsilon_i x^*_i \to 0 \quad \text{a.s. } P_D
\]

by the Strong Law of Large Numbers. Furthermore, note that
\[
\frac{\lambda_n}{2n} C^{-1}_{n(11)} \text{sgn} [\beta_{0(1)}] \to 0.
\]

For ease of notation, let
\[
\tilde{z}_n = [\tilde{z}_{n,1}, \ldots, \tilde{z}_{n,q}]' \\
\tilde{z}^w_{n(1)} := \rho^w_{1,n} Z^w_{n(1)} + \left[ C^{-1}_{w(n)} \frac{1}{\sqrt{n}} X'_1(D_n - I)\epsilon \right] - \frac{\lambda_n}{2\sqrt{n}} \rho^w_{1,n} \text{sgn} [\beta_{0(1)}],
\]
and let
\[
z_n = [z_{n,1}, \ldots, z_{n,q}]' \\
z^w_{n(1)} := \rho^w_{1,n} Z^w_{n(1)} + \left[ C^{-1}_{w(n)} \frac{1}{\sqrt{n}} X'_1(D_n - I)\epsilon \right] - \frac{\lambda_n}{2\sqrt{n}} \rho^w_{1,n} \text{sgn} [\beta_{0(1)}],
\]
and let \( s^2_j \) be the \( j^{th} \) diagonal value of \( \sigma^2 C^{-1}_{11} \), and let \( s^2_{\max} \) be the maximum of these diagonal values. Therefore, for almost every dataset,
\[
P \left[ (A^w_n)' [F_n] \right] \leq \sum_{j=1}^{q} P \left[ |\tilde{z}_{n,j}| > \sqrt{n} \left| \beta_{0,j} \right| + \frac{\lambda_n}{2n} \left| (C^{-1}_{n(11)})_{j,j} \text{sgn} [\beta_{0(1)}] \right| \right] |F_n|
\]
\[
\leq \sum_{j=1}^{q} P \left[ |z_{n,j}| > \sqrt{n} \left( |\beta_{0,j}| + \frac{\lambda_n}{2n} \left( C_{n(11)}^{-1} \right)_{j} \text{sgn} \left[ \beta_{0(1)} \right] - \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_{i}x_{ij} \right| \right) \right] F_{n}
\]
\[
\leq 2 \sum_{j=1}^{q} \left[ 1 - \Phi \left( \frac{\sqrt{n} |\beta_{0,j}| \left[ 1 + o(1) \right]}{s_{z,\text{max}}} \right) \right] + o(1)
\]
\[
\leq 2 \sum_{j=1}^{q} \left[ \frac{s_{z,\text{max}}}{\sqrt{n} |\beta_{0,j}|} \exp \left\{ -\frac{n |\beta_{0,j}|^2}{2s_{z,\text{max}}^2} \left[ 1 + o(1) \right] \right\} \right] + o(1)
\]
\[= o(1),\]

where the second last line follows from the well-known Gaussian tail bound

\[1 - \Phi(t) \leq t^{-1}e^{-t^2}. \quad (A.16)\]

**Remark A.1.** For \( P \left[ (A_n^w)^c \right] |F_n | \) to achieve an exponential rate of convergence

\[2 \left[ 1 + O(1) \right] \sum_{j=1}^{q} \left[ 1 - \Phi \left( \frac{\sqrt{n} |\beta_{0,j}| \left[ 1 + o(1) \right]}{s_{z,\text{max}}} \right) \right] = o(e^{-n}),\]

we need

\[
P \left( |z_{n,j}| > a_n |F_n | \right) \rightarrow 1
\]

for some sequence of positive real numbers \( a_n = o(n) \) as \( n \rightarrow \infty \). This requires additional conditions on the moments or distribution function of \( z_{n,j} \) (see, for example, Linnik (1961) and Nagaev (1979)), which may not be satisfied by the component terms in \( A_n^w \): Consider the simplest case where \( q = 1 \). Due to assumption (1.7), for \( W \sim \text{Exp}(1) \),

\[\left( C_{n(11)}^{-1} \right)_{w} \leq \frac{1}{M_2 \times W},\]

which does not have finite first moment.

**Lemma A.4.** Assume (1.5), (1.6), (1.7), (1.8) and (3.2). Here, both \( q \) and \( p \) are fixed. Then, for any \( \frac{1}{2} < c_1 < 1 \) and \( \frac{1}{2} < c_2 < 1 \) such that \( c_1 + c_2 < 1.5 \), and for all \( \lambda_n \) that satisfies

\[
\frac{\lambda_n}{n^{c_2}} \rightarrow \infty \quad \text{but} \quad \frac{\lambda_n}{n^{1.5-c_1}} \rightarrow 0,
\]

we have

\[P \left[ (B_n^w)^c \right] |F_n | = o(1) \quad \text{a.s. } P_D.
\]

**Proof.** First, we focus on

\[
\frac{1}{\sqrt{n}} \left\{ C_{n(21)}C_{n(11)}^{-1} X_{(1)}' - X_{(2)}' \right\} (D_n - I) \epsilon := \frac{1}{\sqrt{n}} H_n \epsilon.
\]
Let $h_i^A$ be the $i^{th}$ row of $H_A = X_1 C_{n(1)}^{-1} C_{n(2)} - X_2$. Then rewrite
\[
\frac{1}{\sqrt{n}} H_A'(D_n - I) \epsilon \\
= \left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i^2 h_i^A (h_i^A)' \right]^{1/2} \times \left[ \sum_{i=1}^{n} \epsilon_i^2 h_i^A (h_i^A)' \right]^{-1/2} \left[ \sum_{i=1}^{n} h_i^A \epsilon_i (W_i - 1) \right]
\]
Due to Assumption (1.6) and that $C_{n(1)}$ is invertible for all $n$, coupled with the fact that both $q$ and $p$ are fixed, every element of $H_A$ is bounded for all $n$. Then, the Strong Law of Large Numbers ensures that
\[
\frac{1}{n} \sum_{i=1}^{n} (\epsilon_i^2 - \sigma^2) h_i^A (h_i^A)' \rightarrow 0_{(p-q) \times (p-q)} \text{ a.s. } P_D,
\]
whereas assumption (1.8) gives
\[
\frac{1}{n} \sum_{i=1}^{n} h_i^A (h_i^A)' = \frac{1}{n} H_A H_A \\
= C_{n(22)} - C_{n(21)} C_{n(11)}^{-1} C_{n(12)} \\
\rightarrow C_{22} - C_{21} C_{11}^{-1} C_{12},
\]
thus, by Continuous Mapping Theorem,
\[
\left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i^2 h_i^A (h_i^A)' \right]^{1/2} \xrightarrow{c.p.} \sigma \left[ C_{22} - C_{21} C_{11}^{-1} C_{12} \right]^{1/2} \text{ a.s. } P_D. \tag{A.17}
\]
Meanwhile, the Lindeberg’s Central Limit Theorem gives
\[
\left[ \sum_{i=1}^{n} \epsilon_i^2 h_i^A (h_i^A)' \right]^{-1/2} \left[ \sum_{i=1}^{n} \epsilon_i h_i^A (W_i - 1) \right] \xrightarrow{c.d.} N_{p-q} (0, I_{(p-q) \times (p-q)}) \text{ a.s. } P_D, \tag{A.18}
\]
since by Assumption (1.5), the Liapounov’s sufficient condition is satisfied as follows:
\[
\left[ \sum_{i=1}^{n} \epsilon_i^2 h_i^A (h_i^A)' \right]^{-2} \left[ \sum_{i=1}^{n} \epsilon_i^4 [h_i^A (h_i^A)' h_i^A (h_i^A)'] \right] \xrightarrow{a.s.} \mathcal{O} (n^{-2}) \times \mathcal{O} (n) = \mathcal{O} (n^{-1}) \text{ a.s. } P_D.
\]
Now, re-express
\[
- \frac{\lambda_n}{2 \sqrt{n}} \rho_{2,n}^{w} \text{sgn } \beta_{0(1)}
\]
\[\begin{align*}
&= -\frac{\lambda_n}{2\sqrt{n}} \left\{ \frac{X'_{(2)}(D_n - I)X_{(1)}}{n} - C_{n(21)}C_{n(11)}^{-1} \left( \frac{X'_{(1)}(D_n - I)X_{(1)}}{n} \right) \right\} \\
&\times \left( C_{n(11)}^{w} \right)^{-1} \text{sgn} [\beta_{0(1)}] \\
&= -\frac{\lambda_n}{2n^{1.5-c_1}} \left\{ \frac{X'_{(2)}(D_n - I)X_{(1)}}{n^{c_1}} - C_{n(21)}C_{n(11)}^{-1} \left( \frac{X'_{(1)}(D_n - I)X_{(1)}}{n^{c_1}} \right) \right\} \\
&\times \left( C_{n(11)}^{w} \right)^{-1} \text{sgn} [\beta_{0(1)}]
\end{align*}\]

We examine \(l_2\) norm of each term. Recall, from Lemma A.3,
\[\|\text{sgn} [\beta_{0(1)}]\|_2 \leq \sqrt{q} \quad \text{and} \quad \left\| C_{n(11)}^{-1} \right\|_2 < \infty \quad \text{and} \quad \left\| C_{n(11)}^{w} \right\|_2^{-1} = O_p(1),\]
and
\[\left\| \frac{X'_{(1)}(D_n - I)X_{(1)}}{n^{c_1}} \right\|_2 \overset{a.s.}{\longrightarrow} 0.\]

Due to Assumption (1.6), it is clear that
\[\left\| C_{n(21)} \right\|_2 \leq \left\| C_{n(21)} \right\|_F \leq \sqrt{(p - q) \times q} \times M_1^2.\]

Now let \(x_{i(2)}\) be the \(i\)th row of \(X_{(2)}\). Then, by assumption (1.6) and Lemma A.1,
\[\left\| \frac{X'_{(2)}(D_n - I)X_{(1)}}{n^{c_1}} \right\|_2 \leq \left\| \frac{1}{n^{c_1}} \sum_{i=1}^{n} x_{i(2)}x_{i(1)}(W_i - 1) \right\|_F \]
\[= \sqrt{\sum_{k=1}^{p} \sum_{l=1}^{q} \left[ \frac{1}{n^{c_1}} \sum_{i=1}^{n} x_{i,k}^{(2)}x_{i,l}^{(1)}(W_i - 1) \right]^2} \]
\[\overset{a.s.}{\longrightarrow} 0.\]

Hence,
\[-\frac{\lambda_n}{2\sqrt{n}} \rho_{2,n}^{w} \text{sgn} [\beta_{0(1)}] \overset{p}{\longrightarrow} 0. \quad (A.19)\]

Furthermore, due to (A.17), (A.18) and Lemma A.1, we also have
\[\begin{align*}
\rho_{Z_{n(1)}}^{w}Z_{n(1)}^{w} &= \left\{ \frac{X'_{(2)}(D_n - I)X_{(1)}}{n^{c_1}} - C_{n(21)}C_{n(11)}^{-1} \left( \frac{X'_{(1)}(D_n - I)X_{(1)}}{n^{c_1}} \right) \right\} \left( C_{n(11)}^{w} \right)^{-1} \\
&\times \left\{ \frac{1}{n^{1-c_1}} \times \frac{X'_{(1)}(D_n - I)e}{\sqrt{n}} + \left[ \frac{1}{n^{1.5-c_1}} \sum_{i=1}^{n} x_{i(1)}e_i \right] \right\}
\end{align*}\]
Combining (A.17), (A.18), (A.19) and (A.20), Slutsky’s Theorem gives

\[ \rho_{w,n} Z_{n(1)} w + \sqrt{n} H_A(D_n - I) \epsilon - \frac{\lambda_n}{2\sqrt{n}} \rho_{2,n} \text{sgn} [\beta_{0(1)}] \]

c.d. \( N_{p-q} (0, \sigma^2 \sigma C_{22} - C_{21} C_{11}^{-1} C_{12}) \) a.s. \( P_D \).

Finally, note that by Lemma A.1 and the fact that every element of \( H_A \) is bounded for all \( n \),

\[ \frac{1}{n^{c_2}} \sum_{i=1}^{n} h_i^A \epsilon_i \to 0 \text{ a.s. } P_D. \]

For ease of notation, let

\[ \tilde{\zeta} = [\tilde{\zeta}_1, \ldots, \tilde{\zeta}_{(p-q)}]' := \rho_{w,n} Z_{n(1)} w + \sqrt{n} H_A(D_n - I) \epsilon - \frac{\lambda_n}{2\sqrt{n}} \rho_{2,n} \text{sgn} [\beta_{0(1)}], \]

and let

\[ \zeta = [\zeta_1, \ldots, \zeta_{(p-q)}]' := \rho_{w,n} Z_{n(1)} w + \sqrt{n} H_A(D_n - I) \epsilon - \frac{\lambda_n}{2\sqrt{n}} \rho_{2,n} \text{sgn} [\beta_{0(1)}], \]

and let \( s^2_{\zeta, \max} \) be the maximum of these diagonal values. Then, for almost every dataset,

\[
P \left[ \left( B_n^w \right)^c \mid F_n \right] \\
\leq \sum_{j=1}^{p-q} P \left( \left| \tilde{\zeta}_j \right| > n^{c_2 - \frac{1}{2}} \left( \frac{\lambda_n}{2n^{c_2}} \eta_j \right) \right) \left| F_n \right] \\
\leq \sum_{j=1}^{p-q} P \left( \left| \zeta_j \right| > n^{c_2 - \frac{1}{2}} \left( \frac{\lambda_n}{2n^{c_2}} \eta_j - \frac{1}{n^{c_2}} \sum_{i=1}^{n} h_{ij} \epsilon_i \right) \right) \left| F_n \right] \\
\leq 2 \sum_{j=1}^{p-q} \left[ 1 - \Phi \left( \frac{n^{c_2 - \frac{1}{2}}}{s_{\zeta, \max}} \left( \frac{\lambda_n}{2n^{c_2}} \eta_j \right) \left[ 1 - o(1) \right] \right) \right] + o(1) \\
= o(1),
\]

where the second last line follows from (A.16).

\[ \square \]

Again, \( P \left[ \left( B_n^w \right)^c \mid F_n \right] \) may not achieve exponential rate of convergence

\[ 2 [1 + O(1)] \sum_{j=1}^{p-q} \left[ 1 - \Phi \left( \frac{n^{c_2 - \frac{1}{2}}}{s_{\zeta, \max}} \left( \frac{\lambda_n}{2n^{c_2}} \eta_j \right) \left[ 1 - o(1) \right] \right) \right] = o \left( e^{-n^{c_2}} \right) \]

due to similar reasons highlighted in Remark A.1.
Proof of Theorem 3.3. From Lemma 3.1,
\[
P \left( \hat{\beta}_n^w(\lambda_n) \nrightarrow \beta_0 | F_n \right) \geq P \left( A_n^w \cap B_n^w | F_n \right)
\]
\[
= 1 - P \left[ (A_n^w)^c \cup (B_n^w)^c | F_n \right]
\]
\[
= 1 - \{ P \left[ (A_n^w)^c | F_n \right] + P \left[ (B_n^w)^c | F_n \right] \}
\]
= 1 - o(1) \ a.s. \ P_D,
\]
where the last line follows from Lemma A.3 and Lemma A.4.

\[ \Box \]

Proof of Theorem 3.4. Since \( q \) is fixed and with assumptions (1.6) and (1.7), we can easily use similar techniques in the proof of Lemma A.3 to show that
\[
\rho_{1,n}^w Z_{n(1)}^w - \frac{\lambda_n}{2\sqrt{n}} \rho_{1,n}^w \text{sgn} \left[ \beta_0(1) \right] \overset{\text{c.p.}}{\rightarrow} 0 \ a.s. \ P_D. \tag{A.21}
\]
Repetitive steps are omitted. Next, for ease of notation, let
\[
u_z = [v_{z,1}, \ldots, v_{z,q}]' := C_{n(1)}^{-1} Z_{n(1)}^w,
\]
and let
\[
\nu_z = [v_{z,1}, \ldots, v_{z,q}]' := \rho_{1,n}^w Z_{n(1)}^w - \frac{\lambda_n}{2\sqrt{n}} \rho_{1,n}^w \text{sgn} \left[ \beta_0(1) \right],
\]
and (following the same notation in the proof of Lemma A.3) let \( x_{ij}^* \) be the \( j \)-th element of \( C_{n(1)}^{-1} x_{i(1)} \). Then, for any \( \xi > 0 \), we have
\[
P \left[ (A_n^w)^c | F_n \right]
\]
\[
\leq \sum_{j=1}^q P \left( |u_{z,j} + v_{z,j}| > \sqrt{n} \left( |\beta_{0,j}| + \frac{\lambda_n}{2n} \left( C_{n(1)}^{-1} \right)_{j, j} \right) \left( C_{n(1)}^{-1} \right)_{j, j} \text{sgn} \left[ \beta_0(1) \right] \right) | F_n \right)
\]
\[
\leq \sum_{j=1}^q P \left( |u_{z,j}| > \sqrt{n} \left( |\beta_{0,j}| + \frac{\lambda_n}{2n} \left( C_{n(1)}^{-1} \right)_{j, j} \right) \left( C_{n(1)}^{-1} \right)_{j, j} \text{sgn} \left[ \beta_0(1) \right] \right) \right) - \xi | F_n \right)
\]
\[
+ \sum_{j=1}^q P \left( |v_{z,j}| > \xi | F_n \right).
\tag{A.22}
\]
Due to (A.21), the second term of (A.22) converges to zero for almost every dataset. For the first term of (A.22), by Chebyshev’s inequality,
\[
\sum_{j=1}^q P \left( |u_{z,j}| > \sqrt{n} \left( |\beta_{0,j}| + \frac{\lambda_n}{2n} \left( C_{n(1)}^{-1} \right)_{j, j} \right) \left( C_{n(1)}^{-1} \right)_{j, j} \text{sgn} \left[ \beta_0(1) \right] \right) \right) - \xi | F_n \right)
\]
\[
\begin{align*}
&\leq \sum_{j=1}^{q} P \left( \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_{ij}^* \varepsilon_i (W_i - 1) \right| \right. \\
&\hspace{1cm} > \sqrt{n} \left( |\beta_{0,j}| + \frac{\lambda_n}{2n} \left( C_{n(11)}^{-1} \right)_{jj} \sgn \left[ \beta_{0(1)} \right] \right) - \frac{1}{n} \sum_{i=1}^{n} \left| \varepsilon_i x_{ij}^* \right| - \frac{\xi}{\sqrt{n}} \left| \mathcal{F}_n \right| \\
&\hspace{1cm} \leq \sum_{j=1}^{q} \frac{1}{n} \sum_{i=1}^{n} \left( x_{ij}^* \right)^2 \frac{\varepsilon_i^2}{\eta_j} \\
&\hspace{1cm} \to 0 \quad \text{a.s.} \ P_D
\end{align*}
\]

since the numerator of the second last line is bounded for almost every dataset due to assumptions (1.6) and (1.7). Again, for ease of notation, let

\[
\mathbf{u}_\xi = \left[ u_{\xi,1}, \ldots, u_{\xi,(p_n-q)} \right] : = \frac{1}{\sqrt{n}} H_A D_n \varepsilon,
\]

where \( H_A = X(1) C_{n(11)}^{-1} C_{n(12)} - X(2) \) is previously defined in the proof of Lemma A.4. In addition, let

\[
\mathbf{v}_\xi = \left[ v_{\xi,1}, \ldots, v_{\xi,(p_n-q)} \right] : = \rho_{2,n}^{w} Z_{n(1)}^w - \frac{\lambda_n}{2\sqrt{n}} \rho_{2,n}^{w} \sgn \left[ \beta_{0(1)} \right],
\]

where \( \rho_{2,n}^{w} \) is previously defined in Proposition 3.1. Besides that, let

\[
\min_{1 \leq j \leq p_n-q} \eta_j = \eta_*
\]

and note that \( 0 < \eta_* \leq 1 \) from the strong irrepresentable condition (3.2). Then, for any \( \xi > 0 \),

\[
P \left( (B_n^w)^c \big| \mathcal{F}_n \right)
\]

\[
= P \left( \bigcup_{j=1}^{p_n-q} \left\{ |u_{\xi,j} + v_{\xi,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_j \right\} \big| \mathcal{F}_n \right)
\]

\[
\leq P \left( \bigcup_{j=1}^{p_n-q} \left\{ |u_{\xi,j} + v_{\xi,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_* \right\} \big| \mathcal{F}_n \right)
\]

\[
= P \left( \max_{1 \leq j \leq p_n-q} |u_{\xi,j} + v_{\xi,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_* \big| \mathcal{F}_n \right)
\]

\[
\leq P \left( \left\{ \max_{1 \leq j \leq p_n-q} |u_{\xi,j}| + \max_{1 \leq j \leq p_n-q} |v_{\xi,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_* \right\} \right)
\]

\[
\bigcap_{1 \leq j \leq p_n-q} \left\{ |v_{\xi,j}| \leq \xi \right\} \big| \mathcal{F}_n \right) + P \left( \max_{1 \leq j \leq p_n-q} |v_{\xi,j}| > \xi \big| \mathcal{F}_n \right)
\]

\[
\leq P \left( \max_{1 \leq j \leq p_n-q} |v_{\xi,j}| > \xi \big| \mathcal{F}_n \right) + P \left( \max_{1 \leq j \leq p_n-q} |u_{\xi,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_* - \xi \big| \mathcal{F}_n \right)
\]
\[
\leq P \left( \| v_\zeta \|_2 > \xi \left| \mathcal{F}_n \right\) + \sum_{j=1}^{p_n-q} P \left( |u_{\zeta,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_* - \xi \left| \mathcal{F}_n \right\) . \tag{A.23}
\]

For the first term of (A.23), note that
\[
\| v_\zeta \|_2 \leq \left\| \rho_{w,n}^w Z_{n(1)}^w \right\|_2 + \left\| \frac{\lambda_n}{2\sqrt{n}} \rho_{z,n}^w \text{sgn} \left[ \beta_{0(1)} \right] \right\|_2 .
\]

Now, using Lemma A.1 and similar reasoning in the proof of Lemma A.4, we have
\[
\left\| \rho_{w,n}^w Z_{n(1)}^w \right\|_2^2 \\
\leq \left\| \frac{1}{n^{\epsilon_1}} H_A'(D_n - I)x_{(1)}^{(1)} \right\|_2^2 \times \left\| \left( C_{n(1)}^w \right)^{-1} \right\|_2^2 \\
\times \left\| \frac{X_{(1)}'(D_n - I)\varepsilon}{n^{1.5-\epsilon_1}} + \frac{1}{n^{1.5-\epsilon_1}} \sum_{i=1}^n x_{(1)i} \varepsilon_i \right\|_2^2 \\
\leq \left\| \frac{1}{n^{\epsilon_1}} H_A'(D_n - I)x_{(1)}^{(1)} \right\|_2^2 \times \mathcal{O}_p(1) \times o_p(1) \quad \text{a.s. } P_D \\
= \left[ \sum_{i=1}^q \sum_{k=1}^{p_n-q} \left( \frac{1}{n^{\epsilon_1}} \times \frac{1}{n^{\epsilon_1}} \sum_{i=1}^n (W_i - 1) h_{i,k}^A x_{i,j}^{(1)} \right)^2 \times \mathcal{O}_p(1) \times o_p(1) \quad \text{a.s. } P_D \\
= o_p(1) \quad \text{a.s. } P_D \
\]

Similarly,
\[
\left\| \frac{\lambda_n}{2\sqrt{n}} \rho_{z,n}^w \text{sgn} \left[ \beta_{0(1)} \right] \right\|_2 \\
\leq \left\| \frac{1}{n^{\epsilon_1}} H_A'(D_n - I)x_{(1)}^{(1)} \right\|_2 \times \left\| \left( C_{n(1)}^w \right)^{-1} \right\|_2 \times \left\| \frac{\lambda_n}{2n^{1.5-\epsilon_1}} \text{sgn} \left[ \beta_{0(1)} \right] \right\|_2 \\
= o_p(1).
\]

Hence,
\[
\left\| \rho_{w,n}^w Z_{n(1)}^w \right\|_2 = o_p(1) \quad \text{a.s. } P_D,
\]
so for every \( \xi > 0 \),
\[
P \left( \| v_\zeta \|_2 > \xi \left| \mathcal{F}_n \right\) = o(1) \quad \text{a.s. } P_D . \tag{A.24}
\]

For the second term of (A.23), using Chebyshev’s inequality,
\[
\sum_{j=1}^{p_n-q} P \left( |u_{\zeta,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_* - \xi \left| \mathcal{F}_n \right\) 
\]

\[ \leq \sum_{j=1}^{p_n-q} P \left( \left| \frac{\sum_{i=1}^{n} h_{ij} \epsilon_i (W_i - 1)}{\sqrt{n}} \right| > n^{c_2 - \frac{1}{2}} \left( \frac{\lambda_n}{2n^{c_2}} \eta_j \right) - \xi \right) \frac{1}{n^{c_2}} \sum_{i=1}^{n} h_{ij} \epsilon_i \right) - \xi \left| F_n \right) \]

\[ \leq \sum_{j=1}^{p_n-q} \frac{1}{n} \sum_{i=1}^{n} (h_{ij})^2 \epsilon_i^2 \left( n^{c_2 - \frac{1}{2}} \times \frac{\lambda_n}{2n^{c_2}} \eta_j [1 + o(1)] \right)^2 \]

\[ = O( p_n) \times O \left( n^{1-2c_2} \right) \]

\[ = o(1) \text{ a.s. } P_D. \quad (A.25) \]

Combining (A.24) and (A.25), we have

\[ P \left[ (B_n^w)^c \mid F_n \right] = o(1) \text{ a.s. } P_D. \]

Finally,

\[ P \left( \beta_n^w(\lambda_n) = \beta_0 \mid F_n \right) \geq 1 - (P \left[ (A_n^w)^c \mid F_n \right] + P \left[ (B_n^w)^c \mid F_n \right] ) \geq 1 - o(1) \text{ a.s. } P_D. \]

Sketch of Proof for Corollary 5.1. Using similar techniques for the proof of Theorem 3.1, we can easily show that

\[ \frac{1}{n} X'D_n X \xrightarrow{a.s.} \mu_W C, \]

and

\[ \frac{1}{n} \epsilon' D_n \epsilon \xrightarrow{c.p.} \mu_W \sigma^2 \epsilon \text{ a.s. } P_D, \]

and we still have

\[ \frac{1}{n} X'D_n \epsilon \xrightarrow{c.p.} 0 \text{ a.s. } P_D. \]

Then

\[ g_n(u) \xrightarrow{c.p.} g(u) + \mu_W \sigma^2 \epsilon \text{ a.s. } P_D, \]

where

\[ g(u) = \mu_W u'C u + \lambda_0 || \beta_0 + u ||_1. \]

The rest of the proofs follow through. We omit the repetitive details.

Sketch of Proof for Corollary 5.2. Using similar techniques in the proof of Theorem 3.2, we can easily show that

\[ \frac{1}{\sqrt{n}} X'D_n e_n^{OLS} \]

\[ = \frac{1}{\sqrt{n}} X' (D_n - \mu_W I) e_n^{OLS} + \frac{\mu_W}{\sqrt{n}} X' e_n^{OLS} \]
\[ \frac{1}{\sqrt{n}} X' (D_n - \mu_W I) e_n \overset{\text{OLS}}{=} \sqrt{n} \sum_{i=1}^{n} c_i x_i x'_i \left( \frac{1}{\sigma_W^2} \sum_{i=1}^{n} c_i x'_i x_i \right)^{-\frac{1}{2}} \left[ \sum_{i=1}^{n} x_i e_i (W_i - \mu_W) \right] \]

\[ \overset{\text{c.d.}}{\rightarrow} N \left( 0, \sigma_W^2 \sigma_W^2 \right) \quad \text{a.s. } P_D. \]

Then,
\[ V_n(u) \overset{\text{c.d.}}{\rightarrow} V(u) \equiv u' (\mu_W C) u - 2u' \Psi \quad \text{a.s. } P_D, \]
where \( \Psi \sim N \left( 0, \sigma_W^2 \sigma_W^2 \right) \). Finally, conditional on data,
\[ \arg \min_u V_n(u) = \sqrt{n} \left( \hat{\beta}_w - \hat{\beta}_w^{\text{OLS}} \right) \overset{\text{c.d.}}{\rightarrow} \arg \min_u V(u) = \frac{C^{-1} \Psi}{\mu_W} \sim N \left( 0, \frac{\sigma_W^2 \sigma_W^2}{\mu_W^2} C^{-1} \right). \]

Sketch of Proof for Corollary 5.3. First, note that Proposition 3.1 works for any i.i.d positive random weights. Now, for \( \mu_W = 1 \) and for any \( \sigma_W^2 \in (0, \infty) \), using similar techniques in the proofs of Lemmas A.3 and A.4, we can easily show that
\[ C_{n(11)}^{-1} \frac{1}{\sqrt{n}} X'_{(1)} (D_n - I) e \overset{\text{c.d.}}{\rightarrow} N \left( 0, \sigma_W^2 \sigma_W^2 C_{11}^{-1} \right) \quad \text{a.s. } P_D, \]
and
\[ \frac{1}{\sqrt{n}} H'_A(D_n - I) e \overset{\text{c.d.}}{\rightarrow} N \left( 0, \sigma_W^2 \sigma_W^2 \left[ C_{22} - C_{21} C_{11}^{-1} C_{12} \right] \right) \quad \text{a.s. } P_D. \]

The properties for the rest of the terms in Lemmas A.3 and A.4 do not change, and we obtain conditional model selection consistency. Repetitive details are omitted here.

When \( \mu_W \neq 1 \), we still have
\[ P \left[ (B_n^w)^c \mid F_n \right] = o(1) \quad \text{a.s. } P_D, \]
because
\[ -\frac{\lambda_n}{2} \rho_{2,n} \overset{\text{c.d.}}{\rightarrow} \left[ \beta_{0(1)} \right] \]
\[ = -\frac{\lambda_n}{2n^{1.5-c_1}} \left\{ X'_{(2)} (D_n - \mu_W I) X(1) \right\} - C_{n(21)} C_{n(11)}^{-1} X'_{(1)} (D_n - \mu_W I) X(1) \]
\[ \times \left[ C_{n(11)}^{-1} \right] \overset{\text{c.d.}}{\rightarrow} \left[ \beta_{0(1)} \right] \]
\[ = o_p(1), \]
and
\[
\begin{align*}
\rho_{2,n}^w Z_{n(1)}^w &= \left[ X'_2 - C_{n(2)} C_{n(1)}^{-1} X'_1 \right] \frac{D_n - \mu W I}{n^{c_1}} X_1 \left( C_{n(1)}^{-1} \right)^{-1} \\
&\times \left\{ \frac{1}{n^{1-c_1}} \frac{X'_1 (D_n - \mu W I) \epsilon}{\sqrt{n}} + \left[ \frac{\mu W}{n^{1.5-c_1}} \sum_{i=1}^{n} x_i(1) \epsilon_i \right] \right\} \\
&= o_p(1) \quad a.s. \quad P_D,
\end{align*}
\]

and
\[
\begin{align*}
\frac{1}{\sqrt{n}} H'_w (D_n - \mu W I) \epsilon &\xrightarrow{c.d.} N (0, \sigma^2_w \sigma^2_w \left[ C_{22} - C_{21} C_{11}^{-1} C_{12} \right]) \quad a.s. \quad P_D,
\end{align*}
\]

and
\[
\frac{\mu W}{n^{c_2}} \sum_{i=1}^{n} b_{ij}^w \epsilon_i \to 0 \quad a.s. \quad P_D \quad \forall \ j = 1, \ldots, p-q.
\]

However,
\[
\begin{align*}
- \frac{\lambda_n}{\sqrt{n}} \rho_{1,n}^w \text{sgn} [\beta_{0(1)}] &= - \frac{\lambda_n}{\sqrt{n}} \left\{ I - C_{n(1)}^{-1} \frac{X'_1 (D_n - \mu W I) X_1}{n} - \mu W I \right\} \left( C_{n(1)}^{-1} \right)^{-1} \text{sgn} [\beta_{0(1)}] \\
&= - \frac{\lambda_n}{\sqrt{n}} C_{n(1)}^{-1} \frac{X'_1 (D_n - \mu W I) X_1}{n} \left( C_{n(1)}^{-1} \right)^{-1} \text{sgn} [\beta_{0(1)}] \\
&\quad - \frac{\lambda_n}{\sqrt{n}} (1 - \mu W) \left( C_{n(1)}^{-1} \right)^{-1} \text{sgn} [\beta_{0(1)}],
\end{align*}
\]

where the second term is unbounded as \( n \to \infty \), unless \( \mu W = 1 \). Instead, we could just work with
\[
\tilde{A}_n^w = \left\{ \left( C_{n(1)}^{-1} \right)^{-1} Z_{n(1)}^w - \frac{\lambda_n}{2 \sqrt{n}} \left( C_{n(1)}^{-1} \right)^{-1} \text{sgn} [\beta_{0(1)}] \leq \sqrt{n} | \beta_{0(1)} | \right\},
\]

which satisfies
\[
P \left( \tilde{\beta}_n^w = \beta_0 \cap B_n^w | F_n \right) \geq P \left( \tilde{A}_n^w \cap B_n^w | F_n \right)
\]

based on the derivations for Proposition 3.1. Now, notice that
\[
\begin{align*}
\left( C_{n(1)}^{-1} \right)^{-1} \frac{1}{\sqrt{n}} Z_{n(1)}^w - \frac{\lambda_n}{2n} \left( C_{n(1)}^{-1} \right)^{-1} \text{sgn} [\beta_{0(1)}] &= \left( C_{n(1)}^{-1} \right)^{-1} \left\{ \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} X'_1 (D_n - \mu W I) \epsilon + \frac{\mu W}{n} X'_1 \epsilon \right\} \\
&\quad - \frac{\lambda_n}{2n} \left( C_{n(1)}^{-1} \right)^{-1} \text{sgn} [\beta_{0(1)}]
\end{align*}
\]
Similarly, proof of Corollary 5.4, we immediately obtain $P \left( (A_n^w )^c \mid \mathcal{F}_n \right) = o(1) \ a.s. \ P_D$ by slightly modifying the proof of Corollary 5.3. Where $P \left( (B_n^w )^c \mid \mathcal{F}_n \right)$ is concerned, we use the same notations in the proof of Theorem 3.4. Again, for every $\xi > 0$,

$$P \left( (B_n^w )^c \mid \mathcal{F}_n \right) \leq \left( \|v_c\|_2 > \xi \right) \mathcal{F}_n + \sum_{j=1}^{p_n - q} P \left( |u_{\xi,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_s - \xi \right) \mathcal{F}_n.$$ 

By Lemma A.1,

$$\frac{1}{n^{c_2}} H_A(D_n - \mu_W I) X_1 \int \left( C_{n(1)}^{-1} \right)^2 \times \frac{\mu_W}{n^{1.5 - c_1}} \sum_{i=1}^{n} X_{(1)}^i n^{c_i} \right)^2 \times \mathcal{O}_P(1) \times o_p(1) \ a.s. \ P_D$$

$$= \mathcal{O}(p_n) \times o \left( n^{-c_3} \right) \times \mathcal{O}_P(1) \times o_p(1) \ a.s. \ P_D$$

Similarly,

$$\frac{\lambda_n}{2\sqrt{n}} \rho_{2,n}^w \mathcal{O}_P(1) \ a.s. \ P_D$$

Hence,

$$\frac{\lambda_n}{2\sqrt{n}} \rho_{2,n}^w \mathcal{O}_P(1) \ a.s. \ P_D$$

so for every $\xi > 0$,

$$P \left( \|v_c\|_2 > \xi \right) \mathcal{F}_n = o(1).$$
Next, use Chebyshev’s inequality to obtain
\[
\sum_{j=1}^{p_n-q} P \left( |u_{\zeta,j}| > \frac{\lambda_n}{2\sqrt{n}} \eta_j - \xi \right) \leq \sum_{j=1}^{p_n-q} P \left( \left| \sum_{i=1}^{n} h_{ij}^A \epsilon_i (W_i - \mu W) \right| > n^{c_2-\frac{1}{2}} \left( \frac{\lambda_n}{2n^{c_2}} \eta_j - \frac{\mu W \sum_{i=1}^{n} h_{ij}^A \epsilon_i}{n^{c_2}} - \frac{\xi}{n^{c_2-\frac{1}{2}}} \right) \right) \leq \frac{p_n-q}{n} \frac{\sigma_W^2}{\sum_{i=1}^{n} (h_{ij}^A)^2 \epsilon_i^2} \leq 0 \left( p_n \right) \times O \left( n^{1-2c_2} \right) = o(1) \text{ a.s. } P_D.
\]

Therefore,
\[
P \left[ (B'_n)^c \mid F_n \right] = o(1) \text{ a.s. } P_D.
\]

**Sketch of Proof for Corollary 5.5.** First, consider weighting scheme (5.1). Now, we have
\[
g_n(u) = -2 u' \left( \frac{X' D_n e}{n} \right) + u' \left( \frac{X' D_n X}{n} \right) u + \frac{e' D_n e}{n} + \frac{\lambda_n W_0}{n} \| \beta_0 + u \|_1,
\]
and
\[
g(u) = u' C u + \lambda_0 W \| \beta_0 + u \|_1,
\]
which is a random function except when \( \lambda_0 = 0 \), which leads to
\[
\frac{\lambda_n W_0}{n} \| \beta_0 + u \|_1 \overset{p}{\rightarrow} 0.
\]
Thus, part (a) of the corollary follows immediately. For part (b) of the corollary, since \( g_n(u) \) is convex and \( g(u) \) has a unique minimum, it follows from Geyer (1996) that conditional on data,
\[
\arg \min_u g_n(u) = \hat{\beta}_n^w - \beta_0 \overset{c.d.}{\rightarrow} \arg \min_u g(u).
\]
For weighting scheme (5.2), simply replace the term
\[
\frac{\lambda_n W_0}{n} \| \beta_0 + u \|_1
\]
with
\[ \frac{\lambda_n}{n} \sum_{j=1}^{p} W_j |\beta_{0,j} + u_j| \]
in the proof above and we will arrive at Corollary 5.5 (b). Repetitive steps are omitted.

\[ \square \]

Sketch of Proof for Corollary 5.6. First, consider weighting scheme (5.1). We have
\[ V_n(u) = -2u' \left( X'D_n e_n^{\text{OLS}} \sqrt{\frac{n}{\lambda_n}} \right) + u' \left( X'D_n X \right) u \]
\[ + \frac{\lambda_n W_0}{\sqrt{n}} \{ \sqrt{n}\beta_n^{\text{OLS}} + u \}_1 - \sqrt{n}\beta_n^{\text{OLS}} \}_1 \}

where \( W_0 \) is independent of \( \frac{1}{\sqrt{n}} X'D_n e_n^{\text{OLS}} \) and \( \frac{1}{n} X'D_n X \). Thus, if \( \frac{\lambda_n}{\sqrt{n}} \rightarrow 0 \), we have
\[ \frac{\lambda_n W_0}{\sqrt{n}} \{ \sqrt{n}\beta_n^{\text{OLS}} + u \}_1 - \sqrt{n}\beta_n^{\text{OLS}} \}_1 \} \rightarrow 0 \quad \text{a.s.} \quad P_D, \]
and the corollary follows. If \( \frac{\lambda_n}{\sqrt{n}} \rightarrow \lambda_0 \in (0, \infty) \), then again, due to the fact that \( W_0 \) is independent of \( \frac{1}{\sqrt{n}} X'D_n e_n^{\text{OLS}} \) and \( \frac{1}{n} X'D_n X \), the Skorokhod argument follows. For weighting scheme (5.2), replace the term
\[ \frac{\lambda_n W_0}{\sqrt{n}} \{ \sqrt{n}\beta_n^{\text{OLS}} + u \}_1 - \sqrt{n}\beta_n^{\text{OLS}} \}_1 \}

with
\[ \frac{\lambda_n}{\sqrt{n}} \sum_{j=1}^{p} W_j \left( \sqrt{n}\beta_n^{\text{OLS}} + u_j \right) - \sqrt{n}\beta_n^{\text{OLS}} \}_1 \}

in the proof above, and we will obtain similar result due to the fact that \( \forall j = 1, \ldots, p, W_j \perp X'D_n e_n^{\text{OLS}} \) and \( W_j \perp X'D_n X \).

\[ \square \]

Sketch of Proof for Corollary 5.7. First, consider the weighting scheme (5.1). Following the same reasoning for Proposition 3.1, we now have
\[ P \left( \tilde{\beta}_n^w(\lambda_n) \perp \beta_0 | \mathcal{F}_n \right) \geq P \left( A_n^w \cap B_n^w | \mathcal{F}_n \right) \quad \text{a.s.} \quad P_D, \]
where
\[ A_n^w = \left\{ \left( C_n^w(11) \right)^{-1} Z_n^w - \frac{\lambda_n W_0}{2\sqrt{n}} \left( C_n^w(11) \right)^{-1} \text{sgn} \left[ \beta_0(1) \right] \right\} \]
\[ B_n^w \equiv \left\{ \frac{1}{W_0} \rho_{n,2}^w Z_{n(1)}^w + \frac{1}{W_n} \left[ H_A^w \left( D_n - I \right) \epsilon + \frac{1}{\sqrt{n}} H_A^w \right] \right\} - \frac{\lambda_n}{2\sqrt{n}} \rho_{n,2}^w \text{sgn} \left[ \beta_{0(1)} \right] \]

\[ \leq \frac{\lambda_n}{2\sqrt{n}} \text{\ element-wise} \]

Note that \( W_0 \) is independent of \( Z_{n(1)}^w \) and \( C_{n(1)}^w \). Hence, we have

\[ \left( C_{n(1)}^w \right)^{-1} \frac{1}{\sqrt{n}} Z_{n(1)}^w - \frac{\lambda_n}{2n} W_0 \left( C_{n(1)}^w \right)^{-1} \text{sgn} \left[ \beta_{0(1)} \right] \]

\[ = \left( C_{n(1)}^w \right)^{-1} \left\{ \frac{1}{\sqrt{n}} X_1(D_n - I)\epsilon + \frac{1}{n} X_1' \epsilon \right\} - \frac{\lambda_n}{2n} W_0 \left( C_{n(1)}^w \right)^{-1} \text{sgn} \left[ \beta_{0(1)} \right] \]

\[ \xrightarrow{c.p.} 0 \ \text{a.s. } P_D, \]

where \( W_0 \sim \text{Exp}(1) \), whereas \( \left| \beta_{0(1)} \right|_j > 0 \ \forall \ j = 1, \ldots, q \). Therefore,

\[ P \left[ (A_n^w)^c | \mathcal{F}_n \right] = o(1) \ \text{a.s. } P_D. \]

Now, we have to work with

\[ B_n^w \equiv \left\{ \frac{1}{W_0} \rho_{n,2}^w Z_{n(1)}^w + \frac{1}{W_n} \left[ H_A^w (D_n - I) \epsilon + \frac{1}{\sqrt{n}} H_A^w \epsilon \right] \right\} - \frac{\lambda_n}{2\sqrt{n}} \rho_{n,2}^w \text{sgn} \left[ \beta_{0(1)} \right] \]

\[ \leq \frac{\lambda_n}{2\sqrt{n}} \text{\ element-wise} \]

Again, we use the same technique

\[ \frac{1}{W_0} \rho_{n,2}^w Z_{n(1)}^w \xrightarrow{c.p.} 0 \ \text{a.s. } P_D, \]

whereas \( \frac{\lambda_n}{2n^{\frac{1}{2}} H_j} \to \infty \ \forall \ j = 1, \ldots, p - q \). Hence, it follows that

\[ P \left[ (B_n^w)^c | \mathcal{F}_n \right] = o(1) \ \text{a.s. } P_D. \]

For weighting scheme (5.2), let \( W_p = (W_1, \ldots, W_p)' \), which can be trivially partitioned accordingly to \( (W_1, W_2)' \). Then, equation (A.9) becomes

\[ C_{n(11)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(1)}^w = \frac{\lambda_n}{2\sqrt{n}} W_1 \circ \text{sgn} \left[ \beta_{0(1)} \right], \]

whereas equation (A.10) becomes

\[ \left| C_{n(21)}^w \left[ \sqrt{n} \hat{u}_{n(1)} \right] - Z_{n(2)}^w \right| \leq \frac{\lambda_n}{2\sqrt{n}} W_2 \ \text{element-wise}. \]
These lead to
\[
A_w^n = \left\{ \left( C_{n(11)}^{w} \right)^{-1} Z_{n(1)}^{w} - \frac{\lambda_n}{2\sqrt{n}} \left( C_{n(11)}^{w} \right)^{-1} \left( W_{(1)} \circ \text{sgn} \left[ \beta_{0(1)} \right] \right) \right\}
\leq \sqrt{n} |\beta_{0(1)}| \text{ element-wise},
\]
and
\[
\tilde{B}_n^w = \left\{ \left( C_{n(21)}^{w} \right) \left( C_{n(11)}^{w} \right)^{-1} Z_{n(1)}^{w} - Z_{n(2)}^{w} - \frac{\lambda_n}{2\sqrt{n}} C_{n(21)}^{w} \left( C_{n(11)}^{w} \right)^{-1} \left( W_{(1)} \circ \text{sgn} \left[ \beta_{0(1)} \right] \right) \right\}
\leq \frac{\lambda_n}{2\sqrt{n}} W_{(2)} \text{ element-wise},
\]
and so we consider
\[
B_n^w = \left\{ \left( C_{n(21)}^{w} \right) \left( C_{n(11)}^{w} \right)^{-1} Z_{n(1)}^{w} - Z_{n(2)}^{w} - \lambda_n \left( W_{(1)} \circ \text{sgn} \left[ \beta_{0(1)} \right] \right) \right\}
\leq \lambda_n \sqrt{n} \left| \beta_{0(1)} \right| \text{ element-wise}.
\]

Now, we apply similar technique as we did in the proof of Corollary 5.3. For \( A_n^w \), divide both sides of the inequality by \( \sqrt{n} \), we obtain
\[
\left( C_{n(11)}^{w} \right)^{-1} \frac{1}{\sqrt{n}} Z_{n(1)}^{w} - \frac{\lambda_n}{2n} \left( C_{n(11)}^{w} \right)^{-1} \left( W_{(1)} \circ \text{sgn} \left[ \beta_{0(1)} \right] \right) \xrightarrow{c.p.} 0 \text{ a.s. } P_D,
\]
whereas \( |\beta_{0(1)}|_j > 0 \) \( \forall j = 1, \ldots, q \). Therefore,
\[
P \left[ (A_n^w)^\tau | F_n \right] = o(1) \text{ a.s. } P_D.
\]

For ease of notation, let
\[
\tilde{v}_n = \left[ \tilde{v}_{n,1}, \ldots, \tilde{v}_{n,(p-q)} \right]'
:= C_{n(21)}^{w} \left( C_{n(11)}^{w} \right)^{-1} Z_{n(1)}^{w} - Z_{n(2)}^{w} - \frac{\lambda_n}{2\sqrt{n}} W_{(1)} \circ \text{sgn} \left[ \beta_{0(1)} \right],
\]
and using similar notation as before, let
\[
\eta_\ast = \min_{1 \leq j \leq p-q} \eta_j.
\]

Using similar technique, if we divide both sides of the inequality in \( B_n^w \) by \( n^{\frac{3}{2} - \frac{1}{2}} \), LHS of the inequality becomes
\[
\frac{1}{n^{c_2 - \frac{1}{2}}} \| \tilde{v}_n \|_2 = o_p(1) \text{ a.s. } P_D.
\]
In addition, due to the strong irrepresentable condition (3.2), notice that $\forall i = 1, \ldots, q$,

$$
\left| C_{n(21)}^{-1} C_{n(11)}^{-1} W_{(1)} \circ \text{sgn} \left[ \beta_{0(1)} \right] \right|_i \\
\leq C_{n(21)}^{-1} C_{n(11)}^{-1} \text{sgn} \left[ \beta_{0(1)} \right] \times \left( \max_{1 \leq i \leq q} W_{(1)i} \right) \\
\leq (1 - \eta_i) \left( \max_{1 \leq i \leq q} W_{(1)i} \right) \\
\leq (1 - \eta_*) \left( \max_{1 \leq i \leq q} W_{(1)i} \right).
$$

Therefore,

$$
P \left[ (B^w_n)^c \mid \mathcal{F}_n \right] \\
\leq P \left( \| \tilde{\eta}_n \|_2 > \frac{\lambda_n}{2n^{\varepsilon/2}} \left( \min_{1 \leq j \leq p - q} W_{(2)j} \right) - (1 - \eta_*) \left( \max_{1 \leq i \leq q} W_{(1)i} \right) \right) \mid \mathcal{F}_n \Bigg] \\
= P \left( \| \tilde{\eta}_n \|_2 > \frac{\lambda_n}{2n^{\varepsilon/2}} \left( \min_{1 \leq j \leq p - q} W_{(2)j} \right) - (1 - \eta_*) \left( \max_{1 \leq i \leq q} W_{(1)i} \right) \right) \mid \mathcal{F}_n \Bigg] \\
\rightarrow 0 \quad \text{a.s. } P_D \quad \text{if} \quad \eta_ = 1 \Leftrightarrow \eta = 1.
$$

\[\square\]

**Sketch of Proof for Corollary 5.8.** First, consider the weighting scheme (5.1). Similar to Corollary 5.7, we have

$$
A_n^w = \left\{ \left| C_{n(11)}^{-1} Z_{n(1)}^w - \frac{\lambda_n W_0}{2n^{\varepsilon}} \right| C_{n(11)}^{-1} \text{sgn} \left[ \beta_{0(1)} \right] \right\} \\
\leq \sqrt{n} |\beta_{0(1)}| \text{ element-wise}
$$

$$
B_n^w = \left\{ \rho_{2,n}^w Z_{n(1)}^w + C_{n(21)}^{-1} Z_{n(2)}^w - Z_{n(2)}^w - \frac{\lambda_n W_0}{2n^{\varepsilon}} \rho_{2,n}^w \text{sgn} \left[ \beta_{0(1)} \right] \right\} \\
\leq \frac{\lambda_n W_0}{2n^{\varepsilon}} \eta \text{ element-wise}
$$

Due to assumption (1.7) and that $q$ is fixed, with very slight modification to the proof of Corollary 5.7, we immediately obtain

$$
P \left[ (A_n^w)^c \mid \mathcal{F}_n \right] = o(1) \quad \text{a.s. } P_D.
$$

For ease of notation, let

$$
u_n = [v_{n,1}, \ldots, v_{n,(p - q)}] := \frac{1}{\sqrt{n}} H^t_A D_n \epsilon = C_{n(21)}^{-1} Z_{n(1)}^w - Z_{n(2)}^w,
$$

$$
u_n = [v_{n,1}, \ldots, v_{n,(p - q)}] := \rho_{2,n}^w Z_{n(1)}^w - \frac{\lambda_n W_0}{2n^{\varepsilon}} \rho_{2,n}^w \text{sgn} \left[ \beta_{0(1)} \right],
$$

where $H_A = X_{(1)} C_{n(11)}^{-1} C_{n(12)} - X_{(2)}$ is previously defined. In addition, let

$$
u_n = [v_{n,1}, \ldots, v_{n,(p - q)}]^t := \rho_{2,n}^w Z_{n(1)}^w - \frac{\lambda_n W_0}{2n^{\varepsilon}} \rho_{2,n}^w \text{sgn} \left[ \beta_{0(1)} \right].$$
where $\rho_{2,n}^w$ is previously defined in Proposition 3.1. Besides that, let
\[ \min_{1 \leq j \leq p_n - q} \eta_j = \eta_* \]
and note that $0 < \eta_* \leq 1$ from the strong irrepresentable condition (3.2). Then, for any $\xi > 0$,
\[ P \left( \left\| B^w_n \right\|_2 > \xi \left| \mathcal{F}_n \right. \right) \leq P \left( \left\| u_n \right\|_2 > \frac{\lambda_n W_0}{2 \sqrt{n}} \eta_* - \xi \left| \mathcal{F}_n \right. \right). \]
From the proof of Theorem 3.4,
\[ \left\| \rho_{2,n}^w Z_{n(1)}^w \right\|_2 = o_p(1) \text{ a.s. } P_D, \]
and note that
\[ \left\| \frac{\lambda_n W_0}{2 \sqrt{n}} \rho_{2,n}^w \text{sgn} [\beta_0(1)] \right\|_2 \leq \left\| \frac{1}{n^{c_1}} H_A (D_n - I) X_{n(1)} \right\|_2 \times \left\| \left(C_{n(1)}^w \right)^{-1} \right\|_2 \times \left\| \frac{\lambda_n W_0}{2n^{1-c_1}} \text{sgn} [\beta_0(1)] \right\|_2 \]
\[ = o(1) \times O_p(1) \times o_p(1) \text{ a.s. } P_D \]
\[ = o_p(1) \text{ a.s. } P_D. \]
Hence, $\forall \xi < 0$,
\[ P \left( \left\| v_n \right\|_2 > \xi \left| \mathcal{F}_n \right. \right) = o(1) \text{ a.s. } P_D. \]
Now, observe that
\[ \frac{1}{n^{c_2 - \frac{1}{2}}} \left\| u_n \right\|_2 \leq \frac{1}{n^{c_2 - \frac{1}{2}}} \left\| u_n \right\|_2 \]
\[ = \sqrt{\frac{1}{n^{2(c_2 - \frac{1}{2})}} \sum_{j=1}^{p_n - q} \left[ \frac{1}{n} \sum_{i=1}^{n} h_{ij}^A \epsilon_i (W_i - 1) + \frac{1}{n} \sum_{i=1}^{n} h_{ij}^A \epsilon_i \right]^2} \]
\[ = \sqrt{O \left(n^{-2(c_2-c_1)}\right) \times o(n^{c_3})} \]
\[ = o(1) \text{ a.s. } P_D, \]
which leads to
\[ P \left( \frac{\lambda_n W_0}{2 \sqrt{n}} \eta_* < \left\| u_n \right\|_2 + \xi \left| \mathcal{F}_n \right. \right) \]
\[ = P \left( \frac{\lambda_n W_0}{n^{c_2}} < \frac{2}{\eta_* n^{c_2 - \frac{1}{2}}} \left\| u_n \right\|_2 + o(1) \left| \mathcal{F}_n \right. \right) \]
\[ = \text{ asymptotically negligible.} \]
→ \( P(W < 0) = 0, \)

where \( W \sim \text{Exp}(1) \). Therefore,

\[
P\left[ (B_n^w)^c \mid F_n \right] = o(1) \quad \text{a.s.} \quad P_D.
\]

Now, consider weighting scheme (5.2). Similar to Corollary 5.7, we have

\[
A_n^w
= \left\{ \left(\frac{C_n^w}{n(11)}\right)^{-1} Z_n^w - \frac{\lambda}{2\sqrt{n}} \left(\frac{C_n^w}{n(11)}\right)^{-1} \left( W(1) \circ \text{sgn} [\beta_{0(1)}] \right) \right\}
< \sqrt{n} |\beta_{0(1)}| \quad \text{element-wise}
\]

\[
B_n^w
= \left\{ \left[ \frac{1}{\sqrt{n}} H_A D_n \epsilon + \rho_{2,n}^w Z_n^w - \frac{\lambda}{2\sqrt{n}} \rho_{2,n}^w W(1) \circ \text{sgn} [\beta_{0(1)}] \right] \right\}
\leq \frac{\lambda}{2\sqrt{n}} \left[ W(2) - \left| C_n(2) C_n(1)^{-1} ( W(1) \circ \text{sgn} [\beta_{0(1)}] ) \right| \right] \quad \text{element-wise}
\]

With very slight modification to the proof of Corollary 5.7, we immediately obtain

\[
P\left[ (A_n^w)^c \mid F_n \right] = o(1) \quad \text{a.s.} \quad P_D.
\]

For ease of notation, let

\[
v_n = [v_{n,1}, \ldots, v_{n,(p_n-q)}]^t := \rho_{2,n}^w Z_n^w - \frac{\lambda}{2\sqrt{n}} \rho_{2,n}^w ( W(1) \circ \text{sgn} [\beta_{0(1)}] )
\]

where \( \rho_{2,n}^w \) is previously defined in Proposition 3.1, and let

\[
u_n = [u_{n,1}, \ldots, u_{n,(p_n-q)}]^t := \frac{1}{\sqrt{n}} H_A D_n \epsilon = C_n(2) C_n(1)^{-1} Z_n^w - Z_n^w
\]

as defined previously. Using similar technique in the proof of Corollary 5.7, we can show that for any \( \xi > 0 \),

\[
P\left[ (B_n^w)^c \mid F_n \right]
\leq P\left( \|v_n\|_2 > \xi \mid F_n \right)
+ P\left( \|u_n\|_2 > \frac{\lambda}{2\sqrt{n}} \left( \min_{1 \leq j \leq p_n-q} W(2)_{ij} \right) - (1 - \eta_n) \left( \max_{1 \leq i \leq q} W(1)_{ij} \right) - \xi \mid F_n \right).
\]

Similar as before, we can verify that \( \forall \xi > 0 \),

\[
P\left( \|v_n\|_2 > \xi \mid F_n \right) = o(1) \quad \text{a.s.} \quad P_D.
\]
Since 
\[
\left( \min_{1 \leq j \leq p_n - q} W_{(2)j} \right) \sim \exp (p_n - q),
\]
then, for \( 1 - \eta_n = 0 \Leftrightarrow \eta_n = 1 \),
\[
P \left( \frac{1}{n^{c_2 - \frac{1}{2}}} \| u_n \|_2 > \frac{\lambda_n}{2n^{c_2}} \min_{1 \leq j \leq p_n - q} W_{(2)j} - \frac{\xi}{n^{c_2 - \frac{1}{2}}} \middle| F_n \right)
\]
\[
= 1 - \exp \left\{ -\frac{2n^{c_2}}{\lambda_n} \times (p_n - q) \times \frac{\| u_n \|_2 + \xi}{n^{c_2 - \frac{1}{2}}} \right\}
\]
\[
= o(1) \quad \text{a.s. } P_D,
\]
which leads to
\[
P \left[ (B_n^w)^c \middle| F_n \right] = o(1) \quad \text{a.s. } P_D.
\]

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