H-SPACE STRUCTURES ON SPACES OF METRICS OF
POSITIVE SCALAR CURVATURE

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Abstract. We construct and study an H-space multiplication on \( R^+(M) \) for manifolds \( M \) which are nullcobordant in their own tangential 2-type. This is applied to give a rigidity criterion for the action of the diffeomorphism group on \( R^+(M) \) via pullback. We also compare this to other known multiplicative structures on \( R^+(M) \).

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1. Introduction

Let \( R^+(M) \) denote the space of metrics of positive scalar curvature (hereafter: psc-metrics) on a given compact manifold \( M \), equipped with the Whitney \( C^\infty \)-topology. In this paper we will examine multiplicative structures on \( R^+(M) \). In order to state our results with the least amount of technicalities we confine ourselves to the case of Spin-manifolds. A Spin-manifold \( M \) is called Spin \( \times B\pi_1(M) \)-nullcobordant if for a classifying map \( f: M \rightarrow B\pi_1(M) \) of the universal cover, the element \([f: M \rightarrow B\pi_1(M)]\) vanishes in the cobordism group \( \Omega^d_{\text{Spin}}(B\pi_1(M)) \). The following is our main theorem (see Theorem 3.1 for the general version).

**Theorem A.** Let \( M \) be a Spin-manifold of dimension at least 6, which is Spin \( \times B\pi_1(M) \)-nullcobordant. Then \( R^+(M) \) is a homotopy-associative, homotopy-commutative H-space.

**Remark.** Note that any Spin \( \times B\pi_1(M) \)-nullcobordant manifold of dimension at least 5 admits a psc-metric as a consequence of the famous Gromov–Lawson–Schoen–Yau surgery theorem (see [GL80] and [SY79]).
Our main result applies in particular to high-dimensional spheres, generalizing a result of Walsh [Wal14], and products of arbitrary Spin-manifolds with $S^n$ for $n \geq 2$. The key feature of this $H$-space structure is that the multiplication map is given “geometrically”. To explain what this means, let us recall the main result from [Fre19a] (see also [Fre19b]): Let $(M_0, f_0), (M_1, f_1)$ be $(d-1)$-dimensional Spin-manifolds with maps $f_i : M_i \to B\pi$ where $\pi := \pi_1(M_i)$. We define $\Omega^\Spin_\pi(M_0, M_1)$ to be the set of equivalence classes of pairs $(W, F)$ of $d$-dimensional Spin-manifolds $W$ together with maps $F : W \to B\pi$ such that $\partial W = M_0 \amalg M_1$ and $F$ extends $f_0$ and $f_1$. The relation is given as follows: $(W, F) \sim (W', F')$ if there exists a $(d+1)$-dimensional relative Spin $\times B\pi$-cobordism connecting $(W, F)$ and $(W', F')$, i.e. $\Omega^\Spin_\pi(M_0, M_1)$ is the set of (relative) cobordism classes of cobordisms from $M_0$ to $M_1$. For spaces $X, Y$ let $[X, Y]$ denote the set of homotopy classes of maps $X \to Y$. In [Fre19a] we constructed a map
\[ \Omega^\Spin_\pi(M_0, M_1) \to [\mathcal{R}^+(M_0), \mathcal{R}^+(M_1)], \]
providing that $d \geq 7$ and $f_1$ is a classifying map for the universal cover of $M_1$. We will omit the maps $f, F$. Now let $M$ be a Spin-manifold with fundamental group $\pi$ and let us assume that $M$ is Spin $\times B\pi$-nullcobordant via $W : \emptyset \sim M$. This gives a homotopy class of a map $S_W := S(W) : \mathcal{R}^+(\emptyset) \to \mathcal{R}^+(M)$ and since $\mathcal{R}^+(\emptyset) = \{g_\emptyset\}$ is a point, we get a base point component of $\mathcal{R}^+(M)$. Furthermore let $X_W := W^\op \amalg W^\op : M \amalg M \sim M$, where $W^\op$ denotes the flipped cobordism. Then the homotopy class of the map
\[ \mu_W := S(X_W) : \mathcal{R}^+(M) \times \mathcal{R}^+(M) \to \mathcal{R}^+(M) \]
gives the $H$-space structure in Theorem A with the neutral element given by $e_W := S_W(g_\emptyset)$. Since $\mu_W$ only depends on the class of $X_W$ in $\Omega^\Spin_\pi(M \amalg M, M)$, it is possible to prove Theorem A by doing computations in this cobordism set. This leads to a form of computation which we call graphical calculus. Since the definition of $\mu_W$ required the choice of a null-cobordism $W$, it is natural to ask whether $\mu_W$ is independent of this choice. This is answered by the following lemma.

**Lemma B (Lemma 3.5).** Let $M$ and $N$ be Spin-manifolds of dimension at least 6 with the same fundamental group $\pi$. Let $W : \emptyset \sim M$ and $W' : \emptyset \sim N$ be respective Spin $\times B\pi$-nullcobordisms. Then the map
\[ S(W^\op \amalg W') : (\mathcal{R}^+(M), \mu_W) \to (\mathcal{R}^+(N), \mu_W) \]
is an equivalence of $H$-spaces. If $W' = W \amalg B$ for $B$ a closed Spin-manifold with non-vanishing $\alpha$-invariant, then $S(W^\op \amalg W')$ does not fix any path component and is in particular not homotopic to the identity.

We also show that the components of invertible elements are independent of the nullcobordism $W$ (see Proposition 3.6). If furthermore $N$ is a (not necessarily nullcobordant) Spin-manifold with the same fundamental group $\pi$, then we define a map
\[ \rho_W := S(N \times [0, 1] \amalg W^\op) : \mathcal{R}^+(M) \times \mathcal{R}^+(N) \to \mathcal{R}^+(N) \]
which gives an action of $\mathcal{R}^+(M)$ on $\mathcal{R}^+(N)$ in the homotopy category (see Proposition 3.8). Using graphical calculus we obtain a triviality criterion for the action of the oriented diffeomorphism group $\text{Diff}(N)$ on $\mathcal{R}^+(N)$ in the case $\pi = 1$.

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1See also [Fre19b, Section 3.1].
Note that for an orientation preserving diffeomorphism \( f : N \to N \) of a simply connected Spin-manifold \( N \) there exist 2 Spin-structures on the mapping torus \( T_f := N \times [0,1]/(f(x),1) \sim (x,0) \).

**Theorem C** (Theorem 3.9). Let \( N, M \) be simply connected Spin-manifolds of dimension at least 6, let \( W : \emptyset \sim M \) be a Spin-cobordism and let \( f : N \to N \) be an orientation preserving diffeomorphism. Then \( f^*: \mathcal{R}^+(N) \to \mathcal{R}^+(N) \) is homotopic to the identity if there exists a Spin-structure on \( T_f \) such that \( e_W \) is isotopic to \( \mathcal{S}(M \times [0,1]\Pi T_f)(e_W) \). If \( N \) is Spin-nullcobordant equivalence holds.

**Remark.** Considering the special case that \( M = N \) we get that \( f^* \) is homotopic to the identity if and only if \( f^*e_W \sim e_W \). This extends [Fre19a, Proposition D].

In the final Section 4 we compare \( \mu_W \) to other multiplicative structures on \( \mathcal{R}^+(M) \). We show that Walsh’s multiplication from [Wal14] agrees with \( \mu_D \) for the disk \( D : \emptyset \sim S^{d-1} \) provided that \( d \geq 7 \). We then examine the multiplicative structure on concordance classes introduced by Stolz in [Sto91] and further studied in [WY15] and [XYZ19]. We show that this is induced by a map of spaces and if the manifold is Spin \( \times B\pi \)-nullcobordant it is induced by \( \mu_W \). Finally we examine the \( H \)-multiplication \( \mu_{cyl} \) given by concatenation of metrics on cylinders. It is shown in [ERW19b] that for a certain class of manifolds this yields an infinite loop space structure on the subspace of so-called stable metrics. In the special case of the cylinder over a sphere we show that gluing in the torpedo metric on both sides yields an equivalence of \( H \)-spaces

\[
(R^+(S^{d-2} \times [0,1])_{g_*g_\ast}, \mu_{cyl}) \to (R^+(S^{d-1}), \mu_D).
\]

As a corollary we get the following.

**Corollary D.** The underlying \( H \)-space structures of the \( (d-1) \)-fold loop space structure from [Wal14] and the infinite loop space structure from [ERW19b] on \( \mathcal{R}^+(S^{d-1})^{st} \) agree for \( d \geq 7 \).

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2. **Tangential structures and the surgery map**

For \( d \geq 0 \) let \( BO(d+1) \) be the classifying space of the \( (d+1) \)-dimensional orthogonal group and let \( U_{d+1} \) be the universal vector bundle over \( BO(d+1) \). Let \( \theta : B \to BO(d+1) \) be a fibration. We call \( \theta \) a tangential structure.

**Definition 2.1.** A \( \theta \)-structure on a real rank \( (d+1) \)-vector bundle \( V \to X \) is a bundle map \( \tilde{l} : V \to \theta^*U_{d+1} \). A \( \theta \)-structure on a manifold \( W^{d+1} \) is a \( \theta \)-structure on \( TW \) and a \( \theta \)-manifold is a pair \((W,\tilde{l})\) consisting of a manifold \( W \) and a \( \theta \)-structure \( \tilde{l} \) on \( W \). For \( 0 \leq k \leq d \) a stabilized \( \theta \)-structure on \( M^k \) is a \( \theta \)-structure on \( TM \oplus \mathbb{R}^{d+1-k} \).

**Definition 2.2.** Let \( \theta : B \to BO(d+1) \) be a tangential structure. We call \( \theta \) the (stabilized) tangential 2-type of a \( (d-1) \)-dimensional manifold \( M \) if the map \( \theta \) is 2-connected and there exists a (stabilized) \( \theta \)-structure \( \tilde{l} \) on \( M \) such that the underlying map \( l : M \to B \) is 2-connected.
Example 2.3 ([Fre19a, Example 3.3], [Fre19b, Example 1.1.6]).

(1) The (stabilized) tangential 2-type of a connected spin manifold $M$ of dimension at least 3 is $B\text{Spin}(d+1) \times B\pi_1(M)$.

(2) The (stabilized) tangential 2-type of a simply connected, non-spinnable manifold $M$ of dimension at least 3 is $BSO(d+1)$.

Definition 2.4. Let $M_0^{d-1}, M_1^{d-1}$ be closed manifolds with (stabilized) $\theta$-structures $\hat{l}_0, \hat{l}_1$. We define the cobordism set of manifolds with $\theta$-structure and fixed boundary by

$$\Omega^\theta_d((M_0, \hat{l}_0), (M_1, \hat{l}_1)) := \{(W, \hat{\ell})\}/\sim.$$

Here, $W$ is a $d$-manifold with boundary $\partial W = M_0 \sqcup M_1$ and $\hat{\ell}$ is a stabilized $\theta$-structure on $W$ such that $(-1)^i\hat{\ell}_i = \hat{\ell} |_{M_i}$. We call $M_0$ the incoming boundary and $M_1$ the outgoing boundary (see Figure 1).

![Figure 1. A representative of an element in $\Omega^{\text{Spin}}_d(M_0, M_1)$.](image)

The equivalence relation is given by the relative cobordism relation: We say that $(W, \ell)$ and $(W', \ell')$ are $\theta$-cobordant if there exists a $(d+1)$-dimensional $\theta$-manifold $(X, \ell_X)$ with corners such that there exists a partition of

$$\partial X = M_0 \times I \cup W \cup M_1 \times I \cup W'$$

such that the $\theta$-structures fit together (see Figure 2).

![Figure 2. The cobordism relation.](image)
Then there is a map
\[ S : \Omega^B_d((M_0, \hat{l}_0), (M_1, \hat{l}_1)) \rightarrow [R^+(M_0), R^+(M_1)] \]
such that \( S(M_0 \times [0, 1]) = \text{id} \) and \( S \) is compatible with compositions, i.e. \( S(W \cup W') = S(W') \circ S(W) \).

\( S \) is called the surgery map and we will sometimes write \( S(W) = S_W \). Note that if \( B \) is not connected, say \( B = B' \sqcup B'' \), and \( M_i = M_i' \sqcup M_i'' \) for \( i = 0, 1 \) and we have
\[ \Omega^B_d(M_0, M_1) = \Omega^B_d(M_0', M_1') \times \Omega^B_d(M_0'', M_1'') . \]

The following proposition is one of the key features of the cobordism relation.

**Proposition 2.6** ([Fre19a, Proposition 3.25], see also [Fre19b, Proposition 1.3.3]). Let \( W^d : M_0 \sim M_1 \) be a \( \theta \)-cobordism. Then there exists a \( \theta \)-structure on \( W^{op} : M_1 \sim M_0 \) such that \( W \cup W^{op} \sim M_0 \times [0, 1] \) relative to \( M_0 \times \{0, 1\} \). In particular, if \( W: \emptyset \sim M \) is a \( \theta \)-nullcobordism, the double \( dW := W \cup W^{op} \) is \( \theta \)-nullcobordant and \( W^{op} \sqcup W \) is \( \theta \)-cobordant to the cylinder \( M \times [0, 1] \).

Let us close this section by recalling the definition \( H \)-spaces. From now on the symbol “\( = \)” will denote equality in the homotopy category of spaces, i.e. \( f = f' \) means \( f \) and \( f' \) are homotopic. Let us start by recalling the notion of an \( H \)-space.

**Definition 2.7.** An \( H \)-space is a triple \( (X, \mu, e) \) where \( X \) is a space, \( e \in X \) and \( \mu : X \times X \rightarrow X \) is a homotopy class of a map, such that \( \mu(e, \cdot) = \mu(\cdot, e) = \text{id} \). An \( H \)-space is called homotopy-commutative if \( \mu \circ \tau = \mu \), for \( \tau : X \times X \rightarrow X \times X \) the switch map and it is called homotopy-associative if \( \mu \circ (\mu, \text{id}) = \mu \circ (\text{id}, \mu) \). An equivalence of \( H \)-spaces \( (X, \mu, e) \) and \( (X', \mu', e') \) is a (homotopy class of a) homotopy equivalence \( \varphi : X \rightarrow X' \) such that \( \mu \circ (\varphi, \varphi) = \varphi \circ \mu \) and \( \varphi(e) \sim e' \).

**Remark 2.8.** Usually the definition of an \( H \)-space involves the choice of an actual map \( X \times X \rightarrow X \). The definition given here is more in spirit of an \( H \)-space being a unital magma object in the homotopy category of spaces. Furthermore, since the neutral element of an \( H \)-space is well-defined and unique up to homotopy it suffices to specify the component of \( e \).

**Definition 2.9.** Let \( Y \) be a space and let \( X = (X, \mu, e) \) be an \( H \)-space. An action of \( X \) on \( Y \) in the homotopy category is a homotopy class of a map
\[ \rho : X \times Y \rightarrow Y , \]
such that \( \rho(e, \cdot) = \text{id}_Y \) and \( \rho(\mu, \text{id}) = \rho(\text{id}, \rho) \).

3. **Graphical calculus**

Let \( d \geq 7 \), let \( M^{d-1} \) be a manifold and let \( \theta \) be its tangential 2-type. Let \( \hat{l} \) be a 2-connected \( \theta \)-structure and let \( W : \emptyset \sim M \) be a \( \theta \)-nullcobordism of \( (M, \hat{l}) \). We get a map \( S(W) : R^+(\emptyset) = \{ \theta \} \rightarrow R^+(M) \) which gives a base-point component \( e_W \) of \( R^+(M) \). Furthermore, let \( X_W := W^{op} \sqcup W^{op} \sqcup W : M \sqcup M \sim_M M \) (see Figure 3). We define
\[ \mu_W := S(X_W) : R^+(M) \times R^+(M) \rightarrow R^+(M) \]
Theorem 3.1. \((\mathcal{R}^+(M), \mu_W, e_W)\) is a homotopy-commutative, homotopy-associative \(H\)-space.

Proof. First we show that \(e_W\) really is the neutral element. We need to show that \(\mu_W \circ (\text{id}, S(W))\) is homotopic to the identity. Now \((\text{id}, S(W)) = S((M \times I) \cup W)\) and so \(\mu_W \circ (\text{id}, S(W)) = SX_W \circ S((M \times I) \cup W) = S((M \times I) \cup W) \sim \text{id}\) as the double of \(W\) is nullcobordant by Proposition 2.6. This computation relies on the cobordism relation and is depicted in Figure 4.

For commutativity, the composition \(\mu_W \circ \tau\), where \(\tau\) is the map switching the factors, has to be homotopic to \(\mu_W\). The map \(\tau\) however is given by the surgery map \(S\) for the cobordism in Figure 5 and the composition of this cobordism with \(X_W\) is cobordant to \(X_W\) relative to the boundary.

For associativity we need to show that \(\mu \circ (\mu, \text{id}) = \mu \circ (\text{id}, \mu)\). Again, all maps are given by surgery maps and the proof is finished by Figure 6.

Corollary 3.2. The set \(\pi_0(\mathcal{R}^+(M))\) carries the structure of an abelian monoid induced by \(\mu_W\), \(\pi_1(\mathcal{R}^+(M), e_W)\) is an abelian group and \(H^*(\mathcal{R}^+(M); \mathbb{F})\) is a graded Hopf algebra for any field \(\mathbb{F}\).
Remark 3.3. A word of warning is appropriate here: Using pictures to do computations can be dangerous as illustrated by the following example: consider the cobordism $X := W^{op} II W^{op} II W II W : M II M \rightsquigarrow M II M$ (see Figure 7).

We then have two decompositions $(W^{op} II W) II(W^{op} II W) = X = X_W II W$ of $X$. One might be tempted to think that $(\mu_W, e) = S_{X_W II W} = S_{(W^{op} II W) II(W^{op} II W)} = (id, id)$ implying that $R^+(M)$ is contractible. This computation is wrong, as one needs to consider the tangential 2-type of the outgoing boundary which is not connected in this case. Hence the corresponding tangential 2-type $\theta : B \to BO(d + 1)$ is not connected (in the sense that $B$ is not connected) and a $\theta$-structure on $W$ is a map into a disconnected space that respects the given one on the outgoing boundary. Therefore one has to specify which component of $W$ is mapped to which component of $B$. In particular, the components of the incoming boundary are already coupled with components of the outgoing boundary. The manifolds $(W^{op} II W) II(W^{op} II W)$ and $X_W II W$ are different as $\theta$-manifolds, even though the underlying manifolds are equal.

However, when the outgoing boundary is connected so is the corresponding tangential 2-type and one does not need to be as careful. This is the case in the computations in the proof of Theorem 3.1 and will be in every computation in this section.

Example 3.4. By the definition of $S$ we get $e_D = g_{d-1}$ for $D = D^d : \emptyset \rightsquigarrow S^{d-1}$.

The next lemma explains the dependence of $\mu_W$ on $W$ and on $M$.

Lemma 3.5. Let $W : \emptyset \rightsquigarrow M$ and $V : \emptyset \rightsquigarrow N$ be to $\theta$-nullcobordisms. Then

$$\varphi := S(W^{op} II V) : (R^+(M), \mu_W, e_W) \to (R^+(N), \mu_V, e_V)$$

is an equivalence of $H$-spaces. If $M = N$ is simply connected and Spin and $V = W II B$ for a closed Spin-manifold $B$ with non-vanishing $\alpha$-invariant (cf. [LM89, p. 92]), then $\varphi$ does not fix any path component and in particular is not homotopic to the identity.
Proof. An inverse is given by $S_{V \op \Pi W}$, so $\varphi$ is a homotopy equivalence. We have $\varphi \circ \mu_W = \mu_V \circ (\varphi, \varphi)$ because of Figure 8 and $e_V = \varphi(e_W)$ because of Figure 9.

$\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}$

Figure 8. $\varphi \circ \mu_W = \mu_V \circ (\varphi, \varphi)$.

$\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}$

Figure 9. $e_V = \varphi(e_W)$.

The final part follows from Proposition 2.6 and [Fre19a, Proposition 3.35]$^2$.

Even though $\mu_W$ and $\mu_V$ might be different maps, the path components of invertible elements are the same. Let $G_W$ denote the components of invertible elements with respect to $\mu_W$.

Proposition 3.6. Let $V, W: \emptyset \to M$ be two $\theta$-nullcobordisms. Then $G_W = \varphi(G_W) = G_V$.

This follows from the following, more general lemma.

Lemma 3.7. Let $U: M \to M$ be a $\theta$-cobordism. Then

$$\mu_W \circ (S(U), \id) = \mu_W \circ (\id, S(U)) = S(U) \circ \mu_W$$

and in particular $S(U)(G_W) = G_W$.

Proof. Since $W \op \Pi W$ is cobordant to $M \times I$, the formula follows from Figure 11. For the second part let $g, g' \in G_W$ such that $\mu_W(g, g') \sim e_W$ and let $S(U \op)(g') \sim g'' \in \pi_0(\mathcal{R}^+(M))$. Then $\mu_W(S(U)(g), g'') \sim \mu_W(g, S(U)(g'')) \sim \mu_W(g, g') \sim e$ and so we have $S(U)(G_W) \subset G_W$. The other inclusion follows by the same argument for $U \op$.

Now, let $M$ be as before and let $N$ be a manifold with the same tangential 2-type but not necessarily $\theta$-nullcobordant. We get a $\theta$-cobordism $Y_W := W \op \Pi N \times [0, 1]: M \Pi N \to N$ (see Figure 10) and a surgery map

$$\rho_W := S(Y_W): \mathcal{R}^+(M) \times \mathcal{R}^+(N) \to \mathcal{R}^+(N).$$

Proposition 3.8. $\rho_W$ defines an action of $\mathcal{R}^+(M)$ on $\mathcal{R}^+(N)$ in the homotopy category of spaces.

Proof. We need to show that $\rho_W(e_W, \cdot) = \id$ and $\rho_W \circ (\id, \rho_W) = \rho_W \circ (\mu_W, \id)$ which follows from Figure 12 and Figure 13.

$^2$see also [Fre19b, Proposition 4.4.3]
For the final result of this section recall that for a $\theta$-diffeomorphism\(^3\) $f: (N, \hat{l}_N) \to (N, \hat{l}_N)$ the main result \cite[Theorem A resp. Corollary 3.32]{Fre19a} states that the pullback map $f^*: \mathcal{R}^+(N) \to \mathcal{R}^+(N)$ is homotopic to $\mathcal{S}(N \times [0,1] \amalg T_f)$, where $T_f$ denotes the $\theta$-structured mapping torus.

\(^3\)If $\theta: B\text{Spin}(d+1) \to B\text{O}(d+1)$, a $\theta$-diffeomorphism is an orientation preserving diffeomorphism $f: N \xrightarrow{\cong} N$ together with an isomorphism of Spin-structures $f^*\hat{l}_N \cong \hat{l}_N$. For more on general $\theta$-diffeomorphisms see \cite[Section 3.3]{Fre19a} or \cite[Section 1.2]{Fre19b}.

\(^4\)see also \cite[Corollary B]{Fre19b}
\textbf{Theorem 3.9.} Let \( f : N \to N \) be a \( \theta \)-diffeomorphism of \( N \). If \( S(M \times [0, 1] \amalg T_f)(e_W) \) and \( e_W \) lie in the same component of \( \mathcal{R}^+(M) \), then \( f^* : \mathcal{R}^+(N) \to \mathcal{R}^+(N) \) is homotopic to the identity. If furthermore \( N \) is \( \theta \)-nullcobordant, equivalence holds.

\textbf{Remark 3.10.} In particular this shows the following for \( N = M \): If \( f^* e_V \) and \( e_V \) lie in the same path component of \( \mathcal{R}^+(N) \), then \( f^* \) is homotopic to the identity.

\textit{Proof of Theorem 3.9.} The first part is implied by
\[ f^* = S(N \times [0, 1] \amalg T_f) = \rho_W(e_W, S(N \times [0, 1] \amalg T_f)) = \rho_W(S(M \times [0, 1] \amalg T_f)(e_W), \text{id}) \]
where the last equality follows from Figure 14.

\textbf{Figure 13.} \( \rho_W \circ (\text{id}, \rho_W) = \rho_W \circ (\mu_W, \text{id}) \).

\textbf{Figure 14.} \( \rho_W(\text{id}, S(N \times [0, 1] \amalg T_f)) = \rho_W(S(M \times [0, 1] \amalg T_f), \text{id}) \).

If \( N \) is \( \theta \)-nullcobordant as well, say via \( V : \emptyset \sim N \), then \( \rho_W = \mu_V(S(W^\text{op} \amalg V), \text{id}) \) (see Figure 15) and we compute
\[ \rho_W(S(M \times [0, 1] \amalg T_f)(e_W), \text{id}) = \mu_V \left( S(W^\text{op} \amalg V)(S(M \times [0, 1] \amalg T_f)(e_W)), \text{id} \right) \].

This is homotopic to the identity if and only if \( S(M \times [0, 1] \amalg T_f)(e_W) \sim e_W \) since \( S(W^\text{op} \amalg V) \) is an equivalence of \( H \)-spaces. \( \square \)

Since every orientation preserving diffeomorphism of a simply connected Spin-manifold \( N \) lifts to a Spin-diffeomorphism, \textbf{Theorem C} follows immediately from \textbf{Theorem 3.9}.

As a corollary of the the computation in Figure 15 we get:
where the last equivalence holds because $S$.

Proof. Let again $V$ be a $\theta$-nullcobordism of $N$. By Figure 15 we have $\rho_W(g) = \mu_V(S(W^{op} \Pi V))(g, \text{id})$. It follows that

$$\rho_W(g) = \text{id} \iff S(W^{op} \Pi V)(g) \sim e_V \iff g = e_W,$$

where the last equivalence holds because $S(W^{op} \Pi V)$ is an equivalence of $H$-spaces.

Remark 3.12. The computations from this section rely on the maps being given geometrically via cobordisms. This is reminiscent of quantum field theories which are functors from cobordism categories. Note however, that we also make frequent use of the cobordism relation which allows us to introduce and cancel doubles.

4. COMPARISON TO OTHER MULTIPlicative STRUCTURES ON $\mathcal{R}^+(M)$

4.1. Walsh’s construction. Let us start by recalling the construction from [Wal14].

Let $d - 1 \geq 4$ and let $\varphi_i : D^{d-1} \hookrightarrow S^{d-1}$ be disjoint embeddings for $i = 1, 2, 3$. Let $g_{\text{tor}}$ be a torpedo metric on $D^{d-1}$, i.e. a rotationally symmetric metric of positive scalar curvature that restricts to the cylinder over the round metric near the boundary (cf. [EF21, Definition 2.9] for a more precise definition). By the parametrized version of the Gromov–Lawson–Schoen–Yau surgery theorem (cf. [Che04], see also [EF21]) there exists a metric $u'$ on $S^{d-1} \setminus \text{im } (\varphi_1 \Pi \varphi_2 \Pi \varphi_3)$ such that $u := u' \cup (\varphi_1)_{*}g_{\text{tor}} \cup (\varphi_2)_{*}g_{\text{tor}} \cup (\varphi_3)_{*}g_{\text{tor}} \in \mathcal{R}^+(S^{d-1}, \varphi_1 \Pi \varphi_2 \Pi \varphi_3)^5$ lies in the component of the round metric in $\mathcal{R}^+(S^{d-1})$. For clarity let us from now on index the spheres: $S_0^{d-1}$ and $S_1^{d-1}$ will denote the spheres on which we multiply and $S_2^{d-1}$ is the remaining “reference” sphere. A multiplication map

$$\mu_{\text{tor}} : \mathcal{R}^+(S_0^{d-1}, \varphi_1) \times \mathcal{R}^+(S_1^{d-1}, \varphi_1) \to \mathcal{R}^+(S_2^{d-1}, \varphi_1)$$

is given as follows: For $i = 0, 1$, let $g_i \in \mathcal{R}^+(S_i^{d-1}, \varphi_1)$, say $g_i = g_i' \cup (\varphi_1)_{*}g_{\text{tor}}$. We define $\mu_{\text{tor}}(g_0, g_1) := f^*(g_0' \cup u' \cup g_1 \cup (\varphi_1)_{*}(g_{\text{tor}}))$ for a fixed diffeomorphism

$$f : S^{d-1} \to (S_0^{d-1} \setminus \text{im } \varphi_1) \Pi (S_2^{d-1} \setminus \text{im } \varphi_3) / \sim$$

For an embedding $\varphi : D^{d-1} \to S^{d-1}$ the space $\mathcal{R}^+(S^{d-1}, \varphi)$ is defined to be the subspace of those metrics, which have restrict to $\varphi_{*}g_{\text{tor}}$ on the image of $\varphi$. If there are several disjoint such embeddings $\varphi_1 \Pi \cdots \Pi \varphi_n$ the analogous space is denoted by $\mathcal{R}^+(S^{d-1}, \varphi_1 \Pi \cdots \Pi \varphi_n)$.
The relation identifies $\partial(\text{im } \varphi_1)$ from $S^d_{0-1}$ with $\partial(\text{im } \varphi_2)$ from $S^d_{2-1}$ and $\partial(\text{im } \varphi_1)$ from $S^d_{1-1}$ with $\partial(\text{im } \varphi_3)$ from $S^d_{2-1}$ (see Figure 16). Furthermore we may choose $f$ so that $f \circ \varphi_1 = \varphi_1$ and $\varphi_1$ for $\varphi_1 : D^{d-1} \hookrightarrow S^d_1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16}
\caption{The multiplication $\mu^{\text{tor}}$.}
\end{figure}

The obtained metric $\mu^{\text{tor}}(g_0, g_1)$ restricts to $(\varphi_1)^*(g_{\text{tor}})$ on $\text{im } \varphi$ and hence lies in $\mathcal{R}^+(S^d_{d-1}, \varphi_1)$. Since the inclusion $\mathcal{R}^+(S^d_{d-1}, \varphi_1) \hookrightarrow \mathcal{R}^+(S^d_{d-1})$ is a weak equivalence, this defines an $H$-space multiplication $\mu^{\text{tor}}$ with neutral element given by the round metric on $\mathcal{R}^+(S^d_{d-1})$ (cf. [Wal14, Theorem 5.1]). It turns out that the component of the round metric $g_0$ on $S^d_{d-1}$ is a $(d-1)$-fold loop space (cf. [Wal14, Theorem 9.6]).

Now let $\varphi_{12} : S^0 \times D^{d-1} \hookrightarrow S^d_{0-1} \amalg S^d_{2-1}$ and $\varphi_{13} : S^0 \times D^{d-1} \hookrightarrow S^d_{1-1} \amalg S^d_{2-1}$ be the disjoint union of $\varphi_1$ with $\varphi_2$ or $\varphi_3$ respectively. Since $u = g_0 = e_D$, the map $\mu^{\text{tor}}$ is given by the surgery map for the cobordism (see Figure 17 for a visualization)

$$W = \left( (S^d_{0-1} \amalg S^d_{1-1}) \times [0,1] \amalg D^d \right) \cup \left( \text{tr} \ (\varphi_{12}) \ amalg (S^d_{1-1} \times [0,1]) \cup \text{tr} \ (\varphi_{13}) \right)$$

where $\text{tr}$ denotes the trace of a surgery.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17}
\caption{}
\end{figure}

Let $D = D^d : \emptyset \hookrightarrow S^d_{d-1}$ denote the $d$-dimensional disk.
Proposition 4.1. $W$ is Spin-cobordant to $D^{op} \amalg D^{op} \amalg D$.

Proof. The respective cobordisms $tr(\varphi_{12})$ and $tr(\varphi_{13})$ are both Spin-cobordant to $D^{op} \amalg D^{op} \amalg D$ via connected sum on the interior. So $W$ is cobordant to $D^{op} \amalg D^{op} \amalg D \amalg (2D \cup D^{op})$.

\hfill $\square$

Corollary 4.2. If $d \geq 7$, then $\mu^{tor}$ and $\mu_{D}$ are homotopic.

4.2. Stolz’s construction. Let $M$ be a manifold of dimension $d - 1 \geq 5$ of positive scalar curvature. In [Sto91] Stolz proved the existence of a group structure on concordance classes of psc-metrics on $M$ which was further analysed by Weinberger–Yu in [WY15] and Xie–Yu–Zeidler [XYZ19]. For this and the succeeding subsection we need to consider spaces of metrics on manifolds with boundaries. Let $W$ be a manifold with boundary $M$ and let $\mathcal{R}^{+}(W)$ denote the space of those psc-metrics on $W$ that restrict to a cylinder $g + dt^{2}$ in some neighbourhood of the boundary. Since $\text{scal}(g + dt^{2}) = \text{scal}(g)$, we have a well-defined restriction map

$$\text{res}: \mathcal{R}^{+}(W) \longrightarrow \mathcal{R}^{+}(M)$$

and for $g \in \mathcal{R}^{+}(M)$ we define the space $\mathcal{R}^{+}(W)_{g} := \text{res}^{-1}(g)$ to consist of those metrics that restrict to $g$ on the boundary. In this situation, we will sometimes call $g$ a boundary condition.

Definition 4.3. Two metrics $g_{0}, g_{1} \in \mathcal{R}^{+}(M)$ are called \textit{concordant} if there exists a metric $G \in \mathcal{R}^{+}(M \times [0, 1])_{g_{0}, g_{1}}$. The metric $G$ is called a \textit{concordance}. Being concordant is an equivalence relation and we denote the \textit{set of concordance classes of psc-metrics on $M$} by $\tilde{\pi}_{0}(\mathcal{R}^{+}(M))$.

As a convention we denote concordance classes of metrics by $[g]_{c}$, and isotopy classes by $[g]$. Since isotopy implies concordance, we get a canonical map $\pi_{0}(\mathcal{R}^{+}(M)) \twoheadrightarrow \tilde{\pi}_{0}(\mathcal{R}^{+}(M))$. We have the following result:

Proposition 4.4 ([Fre19a, Proposition 3.16 and Remark 3.17]). Let $\theta$ be the tangential 2-type of $M_{1}$ and let $W: M_{0} \sim M_{1}$ be a $\theta$-cobordism. Then $S(W)$ induces a map $\tilde{\pi}_{0}(\mathcal{R}^{+}(M_{0})) \rightarrow \tilde{\pi}_{0}(\mathcal{R}^{+}(M_{1}))$. Furthermore, if there exists a $G \in \mathcal{R}^{+}(W)_{g,h}$, then $S(W)([g]_{c}) = [h]_{c}$.

Proof. Let $G \in \mathcal{R}^{+}(W)_{g,h}$ and $S(W)([g]_{c}) = [h]_{c}$. By [Wal11, Theorem 3.1] there exists $G' \in \mathcal{R}^{+}(W)_{g',h'}$ and hence $G^{op} \cup G' \in \mathcal{R}^{+}(W^{op} \cup W)_{h,h'}$ where $G^{op} \in \mathcal{R}^{+}(W^{op})_{h,g}$ denotes the flipped metric. Now $W^{op} \cup W$ is $\theta$-cobordant to $M_{1} \times [0, 1]$ relative to the boundary and by the surgery theorem, there exists a metric $H \in \mathcal{R}^{+}(M_{1} \times [0, 1])_{h,h'}$, hence $[h']_{c} = [h]_{c}$. The rest has been proven in [Fre19a, Proposition 3.16].

The multiplication of Stolz on $\tilde{\pi}_{0}\mathcal{R}^{+}(M)$ is defined as follows. We take the disjoint union of two cylinders over $M$ and consider them as a $\theta$-cobordism from $M \amalg M \amalg M \sim M$ as in Figure 18. Here $-M$ denotes the same underlying manifold with the opposite $\theta$-structure.

After performing surgery on this we obtain a cobordism $X_{C}: M \amalg M \amalg M \sim M$ such that the inclusion of the outgoing boundary $M \hookrightarrow X_{C}$ is 2-connected. Let $u \in \mathcal{R}^{+}(M)$ be fixed. The multiplication $\mu_{\text{conc},u}$ of Stolz is then defined by $\mu_{\text{conc},u}([g_{0}]_{c}, [g_{1}]_{c}) = [g]$, if there exists a psc-metric $G$ on $X_{C}$ restricting to $(g_{0} \amalg u \amalg g_{1}) \amalg g$ on the boundary. We have the following result relating this multiplication to the surgery map and the $H$-space structure from Theorem 3.1.
Proposition 4.5.

1. The map $\mu_{\text{conc},u}$ is associative, commutative and induced by a map $\mathcal{R}^+(M) \times \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ of spaces.

2. If $M$ is nullcobordant in its own tangential 2-type via a nullcobordism $W: \emptyset \sim M$, then $\mu_{\text{conc},eW} = \mu_W$.

Proof.

1. It follows directly that from Proposition 4.4 that $\mu_{\text{conc},u}(g_0, g_1) = [\mathcal{S}_{XC}(g_0, u, g_1)]_c$ and so the multiplication $\mu_{\text{conc},u}$ is induced by the map $\mathcal{S}_{XC}$. Associativity and commutativity of $\mu_{\text{conc},u}$ can then be proven using graphical calculus, where we mark the part incoming boundary that does not belong to the multiplication by $u$ (see Figure 19 and Figure 20).

![Figure 19. Commutativity of $\mu_{\text{conc},u}$](image1)

![Figure 20. Associativity of $\mu_{\text{conc},u}$](image2)
(2) Let $M$ be nullcobordant in its own tangential 2-type via a nullcobordism $W: \emptyset \sim M$. Since $X_W \sim (M \times [0, 1] \amalg -W \amalg M \times [0, 1]) \cup X_C$ (see Figure 21), we have:

$$\mu_{\text{cone},e_W} = S_{X_C}(-,e_W,\_\_) = \mu_W.$$
Question 4.7.

1. Is there a boundary condition \( g \in \mathcal{R}^+ (M) \) and a \( \theta \)-nullcobordism \( W : \emptyset \sim dN \) such that there exists an equivalence of \( H \)-spaces

\[
(\mathcal{R}^+(M \times [0, 1])_{g|\partial g}, \mu_{\text{cyl}}) \to (\mathcal{R}^+(dN), \mu_W)\]

2. If so, can one choose \( g \) such that there exists a right-stable metric \( G_{\text{rest}} \in \mathcal{R}^+(N) \) for which the map \( \text{cl}_{G_{\text{rest}}} \) is an equivalence?

The natural starting point for investigating this question is the case that \( M = S^{d-2} \), \( g = g^d_{\text{tor}} \) is the round metric, \( N = D^{d-1} \), \( G_{\text{rest}} = g_{\text{tor}} \) is the torpedo metric and \( W = D := D^d \). We identify \( dD^{d-1} = S^{d-1} = \partial W \). In this case it is possible to get a more explicit form of the multiplication map \( \mu_W \):

Let \( \varphi : S^0 \times D^{d-1} \hookrightarrow S^{d-1} \amalg S^{d-1} \) be the inclusion of the lower hemisphere into the first and the upper hemisphere into the second factor. We define the map \( \overline{\mathcal{S}}_\varphi : \mathcal{R}^+(S^{d-1} \amalg S^{d-1}, \varphi) \to \mathcal{R}^+(S^{d-1}) \) by

\[
\overline{\mathcal{S}}_\varphi (\text{cl} \ (g \cup g^\text{tor}_{\text{op}}) \amalg (g^\text{tor} \cup g')) = g \cup (g^d_{\text{tor}} + dt^2) \cup g'.
\]

By the parametrized version of the Gromov–Lawson–Schoen–Yau surgery theorem ([Che04], see also [EF21]) the inclusion map \( \mathcal{R}^+(S^{d-1} \amalg S^{d-1}, \varphi) \hookrightarrow \mathcal{R}^+(S^{d-1} \amalg S^{d-1}) \) is a weak homotopy equivalence and we denote the composition of its homotopy inverse with \( \overline{\mathcal{S}}_\varphi \) by \( \mathcal{S}_\varphi \). By definition (see [Fre19a, Definition 2.23 (3)]\(^6\)) this agrees with \( \mathcal{S}(X_W) \) and the map \( \mu_W \) is therefore homotopic to \( \mathcal{S}_\varphi \). Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{R}^+(S^{d-1}) \times \mathcal{R}^+(S^{d-1}) & \xrightarrow{\mu_W} & \mathcal{R}^+(S^{d-1}) \\
\mathcal{R}^+(S^{d-1} \amalg S^{d-1}) & \xrightarrow{\mathcal{S}_\varphi} & \mathcal{R}^+(S^{d-1}) \\
\mathcal{R}^+(S^{d-1} \amalg S^{d-1}, \varphi) & \xrightarrow{\text{cl}_{g_{\text{tor}}}} & \mathcal{R}^+(S^{d-2} \times [0, 1]_{g_{\text{tor}}, g_{\text{tor}}}) \\
\mathcal{R}^+(S^{d-2} \times [0, 1]_{g_{\text{tor}}, g_{\text{tor}}}) & \xrightarrow{\mu_{\text{cyl}}} & \mathcal{R}^+(S^{d-2} \times [0, 1]_{g_{\text{tor}}, g_{\text{tor}}})
\end{array}
\]

where the triangles commute up to homotopy by the definition and the cobordism invariance of \( \mathcal{S} \) and the lower square commutes up to homotopy by Equation 4.8 after appropriately rescaling the cylinders. We therefore can affirmatively answer Question 4.7 in this special case:

**Theorem 4.9.** The map \( \text{cl}_{g_{\text{tor}}} : (\mathcal{R}^+(S^{d-2} \times [0, 1]_{g_{\text{tor}}, g_{\text{tor}}}, \mu_{\text{cyl}}) \to (\mathcal{R}^+(S^{d-1}), \mu_D) \) is an equivalence of \( H \)-spaces provided \( d \geq 7 \).

**Corollary D.** Now follows from Corollary 4.2 and Theorem 4.9.

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\(^6\)see also [Fre19b, Definition 3.1.1 (3)]
References

[Che04] V. Chernysh. On the homotopy type of the space $\mathcal{R}^+(M)$. 2004, math/0405235.

[EF21] J. Ebert and G. Frenck. The Gromov-Lawson-Chernysh surgery theorem. *Bol. Soc. Mat. Mex.* (3), 27(2):37, 2021. doi:10.1007/s40590-021-00310-w.

[ERW19a] J. Ebert and O. Randal-Williams. Infinite loop spaces and positive scalar curvature in the presence of a fundamental group. *Geom. Topol.*, 23(3):1549–1610, 2019. doi:10.2140/gt.2019.23.1549.

[ERW19b] J. Ebert and O. Randal-Williams. The positive scalar curvature cobordism category. 2019, 1904.12951.

[Fre19a] G. Frenck. The action of the mapping class group on metrics of positive scalar curvature, 2019, 1912.08613.

[Fre19b] G. Frenck. *The Action of the mapping class group on spaces of metrics of positive scalar curvature*. PhD thesis, WWU Münster, Available through the author’s website, July 2019. URL http://frenck.net/Math/articles/thesis.pdf.

[GL80] M. Gromov and H. B. Lawson, Jr. The classification of simply connected manifolds of positive scalar curvature. *Ann. of Math.* (2), 111(3):423–434, 1980. doi:10.2307/1971103.

[LM89] H. B. Lawson, Jr. and M.-L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[Sto91] S. Stolz. Concordance classes of positive scalar curvature metrics. 1991. URL https://www3.nd.edu/~stolz/preprint.html.

[SY79] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979. doi:10.1007/BF01647970.

[Wal11] M. Walsh. Metrics of positive scalar curvature and generalised Morse functions, Part I. *Mem. Amer. Math. Soc.*, 209(983):xviii+80, 2011. doi:10.1090/S0065-9266-10-00622-8.

[Wal14] M. Walsh. $H$-spaces, loop spaces and the space of positive scalar curvature metrics on the sphere. *Geom. Topol.*, 18(4):2189–2243, 2014. doi:10.2140/gt.2014.18.2189.

[WY15] S. Weinberger and G. Yu. Finite part of operator $K$-theory for groups finitely embeddable into Hilbert space and the degree of nonrigidity of manifolds. *Geom. Topol.*, 19(5):2767–2799, 2015. doi:10.2140/gt.2015.19.2767.

[XYZ19] Z. Xie, G. Yu, and R. Zeidler. On the range of the relative higher index and the higher rho-invariant for positive scalar curvature, 2019, 1712.03722.

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