A SIMPLE INDUCTIVE PROOF OF LEVY–STEINITZ THEOREM

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Abstract. We present a relatively simple inductive proof of the classical Levy–Steinitz Theorem saying that for a sequence \((x_n)_{n=1}^{\infty}\) in a finite-dimensional Banach space \(X\) the set \(\Sigma\) of all sums of rearranged series \(\sum_{n=1}^{\infty} x_{\sigma(n)}\) is an affine subspace of \(X\) such that \(\Sigma = \Sigma + \Gamma^\perp\) where \(\Gamma = \{f \in X^* : \sum_{n=1}^{\infty} |f(x_n)| < \infty\}\) and \(\Gamma^\perp = \bigcap_{f \in \Gamma} f^{-1}(0)\). The affine subspace \(\Sigma\) is not empty if and only if for any linear functional \(f : X \to \mathbb{R}\) the series \(\sum_{n=1}^{\infty} f(x_{\sigma(n)})\) is convergent for some permutation \(\sigma\) of \(\mathbb{N}\). This answers a problem of Vaja Tarieladze, posed in Lviv Scottish Book in September, 2017.

Also we construct a sequence \((x_n)_{n=1}^{\infty}\) in the torus \(\mathbb{T} \times \mathbb{T}\) such that the series \(\sum_{n=1}^{\infty} x_{\sigma(n)}\) is divergent for all permutations \(\sigma\) of \(\mathbb{N}\) but for any continuous homomorphism \(f : \mathbb{T}^2 \to \mathbb{T}\) to the circle group \(\mathbb{T} := \mathbb{R}/\mathbb{Z}\) the series \(\sum_{n=1}^{\infty} f(x_{\sigma(n)})\) is convergent for some permutation \(\sigma_f\) of \(\mathbb{N}\). This example shows that the second part of Levy–Steinitz Theorem (characterizing sequences with non-empty set of potential sums) does not extend to locally compact Abelian groups.

1. Introduction

In this paper we present a simple inductive proof of the famous Levy–Steinitz theorem on sums of rearranged series in finite-dimensional Banach spaces.

We shall say that a series \(\sum_{n=1}^{\infty} x_n\) in a Banach space \(X\) is \emph{potentially convergent} to a point \(x \in X\) if for some permutation \(\sigma\) of \(\mathbb{N}\) the sequence of partial sums \((\sum_{n=1}^{m} x_{\sigma(n)})_m^{\infty}\) converges to \(x\). In this case we shall say that \(x\) is a \emph{potential sum} of the series \(\sum_{n=1}^{\infty} x_n\).

A series \(\sum_{n=1}^{\infty} x_n\) in a Banach space \(X\) is called \emph{\(\ast\)-potentially convergent} if for any linear continuous functional \(f \in X^*\) the series \(\sum_{n=1}^{\infty} f(x_n)\) is potentially convergent to some real number. Here by \(X^*\) we denote the dual Banach space to \(X\).

A subset \(A\) of a linear space \(X\) is called \emph{affine} if \(\sum_{i=1}^{n} a_i \in A\) for any \(n \in \mathbb{N}\), points \(a_1, \ldots, a_n \in A\) and real numbers \(t_1, \ldots, t_n\) with \(\sum_{i=1}^{n} t_i = 1\). According to this definition, the empty subset of \(X\) is affine.

Theorem 1 (Levy, Steinitz). For a sequence \((x_n)_{n \in \omega}\) of points of a finite-dimensional Banach space \(X\) the set \(\Sigma\) of potential sums of the series \(\sum_{n=1}^{\infty} x_n\) is an affine subspace of \(X\) such that \(\Sigma = \Sigma + \Gamma^\perp\) where \(\Gamma = \{f \in X^* : \sum_{n=1}^{\infty} |f(x_n)| < \infty\}\) and \(\Gamma^\perp = \bigcap_{f \in \Gamma} f^{-1}(0)\). The set \(\Sigma\) is not empty if and only if the series \(\sum_{n=1}^{\infty} x_n\) is \(\ast\)-potentially convergent.

Levy–Steinitz Theorem was initially proved by Levy [5] in 1905. But the proof was very complicated and contained a gap in the higher-dimensional case, which was filled by Steinitz [7] in 1913. The proof of Steinitz also was very long and it was eventually clarified and simplified by Gross [2], Halperin [3], Rosenthal [6], Kadets and Kadets [4, §2.1]. But even after simplifications, the proof remained too complicated (8 pages in [6]). This situation motivated Vaja Tarieladze to pose a problem in Lviv Scottish Book [8] of finding a simple and transparent proof of Levy–Steinitz Theorem or at least of its second part (characterizing series with non-empty set of potential sums). Inspired by this question of Tarieladze, in this paper we present a simple inductive proof of Levy–Steinitz Theorem.

2. Proof of Levy–Steinitz Theorem

The proof of the theorem is by induction on the dimension of the Banach space \(X\). For zero-dimensional Banach spaces the theorem is trivial. Assume that the theorem is true for all Banach spaces of dimension less than some number \(d \in \mathbb{N}\). Let \(X\) be a Banach space of dimension \(d\), \(\sum_{n=1}^{\infty} x_n\) be a series in \(X\), and \(\Sigma\) be the set of all potential sums of this series in \(X\). If the series \(\sum_{n=1}^{\infty} x_n\) is not \(\ast\)-potentially convergent, then it not potentially convergent and the set \(\Sigma\) is empty.

So, we assume that the series \(\sum_{n=1}^{\infty} x_n\) is \(\ast\)-potentially convergent. In this case we shall prove that \(\Sigma\) is a non-empty affine subset of \(X\) and \(\Sigma = \Sigma + \Gamma^\perp\) where \(\Gamma = \{f \in X^* : \sum_{n=1}^{\infty} |f(x_n)| < \infty\}\).
Without loss of generality, each \( x_n \) is not equal to zero. Also we can assume that the norm of \( X \) is generated by a scalar product \( \langle \cdot, \cdot \rangle \), so \( X \) is a finite-dimensional Hilbert space, whose dual \( X^* \) can be identified with \( X \). Under such identification, the set \( \Gamma \) coincides with the set \( \{ y \in X : \sum_{n=1}^{\infty} |\langle y, x_n \rangle| < \infty \} \). Let \( S = \{ x \in X : \|x\| = 1 \} \) denote the unit sphere of the Banach space \( X \).

**Claim 1.** The sequence \( (x_n)_{n=1}^{\infty} \) tends to zero.

**Proof.** Assuming that \( (x_n)_{n=1}^{\infty} \) does not tend to zero, we can find an \( \varepsilon > 0 \) such that the set \( E = \{ n \in \mathbb{N} : \|x_n\| \geq \varepsilon \} \) is infinite. By the compactness of the unit sphere \( S = \{ x \in X : \|x\| = 1 \} \) the sequence \( \{ x_n/\|x_n\| \}_{n \in E} \subset S \) has an accumulating point \( x_\infty \in S \). Then for the linear functional \( f : X \to \mathbb{R}, f : x \mapsto \langle x, x_\infty \rangle \), the series \( \sum_{n=1}^{\infty} f(x_n) \) is not potentially convergent as its terms do no tend to zero. But this contradicts our assumption (that \( \sum_{n=1}^{\infty} x_n \) is \( * \)-potentially convergent).

A point \( x \) of the sphere \( S \) is defined to be a divergence direction of the series \( \sum_{n=1}^{\infty} x_n \) if for every neighborhood \( U \subset S \) of \( x \) the set

\[
\mathbb{N}_U := \{ n \in \mathbb{N} : \frac{x_n}{\|x_n\|} \in U \}
\]

is infinite and the series \( \sum_{n \in \mathbb{N}_U} \|x_n\| \) is divergent.

Let \( D \subset S \) be the (closed) set of all divergence directions of the series \( \sum_{n=1}^{\infty} x_n \). If \( D \) is empty, then the (compact) sphere \( S \) admits a finite cover \( \mathcal{U} \) by open subsets \( U \subset S \) for which the series \( \sum_{n \in \mathbb{N}_U} \|x_n\| \) is convergent. Then \( \sum_{n=1}^{\infty} \|x_n\| \leq \sum_{U \in \mathcal{U}} \sum_{n \in \mathbb{N}_U} \|x_n\| < \infty \), so the series \( \sum_{n=1}^{\infty} x_n \) is absolutely convergent, \( \Gamma = X \) and the set \( \Sigma \) is a singleton, equal to \( \Sigma = \Sigma + \{0\} = \Sigma + \Gamma \).

So, assume that the (compact) set \( D \) is not empty. We claim that the convex hull of \( D \) contains zero. In the opposite case we can apply the Hahn-Banach Theorem and find a linear functional \( f : X \to \mathbb{R} \) such that \( f(D) \subset (0, +\infty) \) hence the series \( \sum_{n=1}^{\infty} f(x_n) \) fails to converge potentially, which contradicts our assumption.

This contradicts shows that zero belongs to the convex hull of the set \( D \). Let \( D_0 \) be a subset of smallest (finite) cardinality containing zero in its convex hull \( \text{conv}(D_0) \) and let \( X_0 \) be the linear hull of \( D_0 \) in \( X \). The minimality of \( D_0 \) ensures that the set \( D_0 \) is affinely independent and zero is contained in the interior of the convex hull \( \text{conv}(D_0) \) in \( X_0 \). Let \( X_1 := \bigcap_{y \in X_0} \{ x \in X : \langle x, y \rangle = 0 \} \) be the orthogonal complement of \( X_0 \) in \( X \), and let \( \text{pr}_0 : X \to X_0 \) and \( \text{pr}_1 : X \to X_1 \) be the orthogonal projections.

**Claim 2.** For every \( z \in D_0 \) there exists a subset \( \Omega_z \subset \mathbb{N} \) such that

1. \( \sum_{n \in \Omega_z} \|\text{pr}_1(x_n)\| < \infty \);
2. for each neighborhood \( U \subset S \) of \( z \) the series \( \sum_{n \in \Omega_z \cap \mathbb{N}_U} \|x_n\| \) is divergent.

**Proof.** For every \( k \in \omega \) consider the neighborhood \( O_k = \{ x \in S : \|x - z\| < \frac{1}{2k} \} \) of \( z \) in \( S \) and observe that each point \( x \in O_k \) has

\[
\|\text{pr}_1(x)\| = \|\text{pr}_1(x - z) + \text{pr}_1(z)\| = \|\text{pr}_1(x - z) + 0\| \leq \|x - z\| < \frac{1}{2k}.
\]

Since \( z \in D_0 \subset D \), the set \( \mathbb{N}_{O_k} = \{ n \in \mathbb{N} : \frac{x_n}{\|x_n\|} \in O_k \} \) is infinite and the series \( \sum_{n \in \mathbb{N}_{O_k}} \|x_n\| \) is divergent. Using Claim 1 we can find a finite subset \( F_k \subset \mathbb{N}_{O_k} \) such that

\[
k < \sum_{n \in F_k} \|x_n\| < k + 1.
\]

We claim that the set \( \Omega_z = \bigcup_{k \in \omega} F_k \) has the required properties. Indeed,

\[
\sum_{n \in \Omega_z} \|\text{pr}_1(x_n)\| \leq \sum_{k=0}^{\infty} \sum_{n \in F_k} \|\text{pr}_1(x_n)\| = \sum_{k=0}^{\infty} \sum_{n \in F_k} \|\text{pr}_1(\frac{x_n}{\|x_n\|})\| \cdot \|x_n\| < \sum_{k=0}^{\infty} \sum_{n \in F_k} \frac{1}{2k} \|x_n\| = \sum_{k=0}^{\infty} \sum_{n \in F_k} \|x_k\| < \sum_{k=0}^{\infty} \frac{k+1}{2k} < \infty.
\]

So, the first condition is satisfied. To check the second condition, take any neighborhood \( U \subset S \) of \( z \) and find \( k \in \omega \) such that \( O_k \subset U \). The latter inclusion implies \( F_m \subset \mathbb{N}_{O_k} \subset \mathbb{N}_U \) for all \( m \geq k \) and hence

\[
\sum_{n \in \mathbb{N}_U} \|x_n\| \geq \sup_{m \geq k} \sum_{n \in F_m} \|x_n\| \geq \sup_{m \geq k} m = \infty,
\]

so the series \( \sum_{n \in \mathbb{N}_U} \|x_n\| \) is divergent. □

Claim 2 implies that the set \( \Omega = \bigcup_{z \in D_0} \Omega_z \) has the following properties:
exists a finite set \( n \in \Omega \) such that \( \|x\| < \infty\); (Ω2) for each \( x \in D_0 \) and each neighborhood \( U \subset S \) of \( x \) the series \( \sum_{n \in \Omega \cap D_0} \|x\| \) is divergent.

**Claim 3.** There exists a positive constant \( C \) such that for every \( x \in X_0, \varepsilon > 0 \) and finite set \( F \subset \Omega \), there exists a finite set \( E \subset \Omega \setminus F \) such that

\[
(1) \quad \|x - \sum_{n \in F} x_n\| < \varepsilon \quad \text{and} \quad (2) \quad \|\sum_{n \in \Omega \setminus F} x_n\| \leq C \cdot \max\{\|x\|, \varepsilon\} \quad \text{for any subset} \ E' \subset E.
\]

**Proof.** Since the interior of the convex hull of the set \( D_0 \) contains zero, it also contains some closed ball \( \{x \in X_0 : \|x\| \leq \delta\} \) of positive radius \( \delta < \frac{1}{2} \). Replacing \( \delta \) by a smaller number, we can assume that \( \|x - y\| > 2\delta \) for any distinct points \( x, y \in D_0 \). We claim that the constant \( C = \frac{C}{\delta} \) where

\[
C_\delta = \sup \left\{ \left\| \sum_{z \in D_0} t_z x_z \right\| : \forall z \in D_0 \left( t_z \in [0, 1], \|x_z - z\| < \delta \right) \}
\]

satisfies our requirements.

Indeed, fix any \( x \in X, \varepsilon > 0 \), and a finite set \( F \subset \Omega \). Replacing \( \varepsilon \) by a smaller number, we can assume that \( \varepsilon < \delta < \frac{1}{4} \). Let

\[
c := \frac{1}{\delta} \cdot \max\{\|x\|, \varepsilon\}.
\]

It follows that \( \|\frac{x}{c}\| \leq \delta \) and hence the point \( \frac{x}{c} \) belongs to the interior of the convex hull of the affinely independent set \( D_0 \) in \( X_0 \). So, \( \frac{x}{c} = \sum_{z \in D_0} t_z z \) and hence \( x = \sum_{z \in D_0} c t_z z \) for some sequence \( \{t_z\} \subset D_0 \) of positive real numbers with \( \sum_{z \in D_0} t_z = 1 \).

For every \( z \in D_0 \) consider the spherical disk \( S_z = \{x \in S : \|x - z\| < \min\{\delta, \frac{\|x\|}{\|z\|}\} \} \) and its convex cone \( \hat{S}_z = \{t \cdot s : t > 0, s \in S_z, \} \). Since \( \delta < \frac{1}{2}\|x - z\| \) for any distinct points \( z, z' \in D_0 \), the cones \( \hat{S}_z, z \in D_0, \) are pairwise disjoint. Observe also that for any elements \( x, y \in S_z \), we have the lower bound

\[
\langle x, y \rangle = \langle x, x \rangle + \langle x, y - x \rangle \geq 1 - \|x - y\| > 1 - 2\delta \geq \frac{1}{2}.
\]

Then for any elements \( x, y \in \hat{S}_z \) we have the inequality

\[
(1) \quad \|x + y\| \geq \langle x + y, \frac{x}{\|x\|} \rangle = \|x\| + \|y\| \langle \frac{x}{\|x\|}, \frac{x}{\|x\|} \rangle \geq \|x\| + \frac{1}{2}\|y\|.
\]

By the property (Ω2) of the set \( \Omega \), for every \( z \in D_0 \) the set \( \Omega_z := \{n \in \Omega : \frac{x}{\|x\|} \in S_z \} = \{n \in \Omega : x_n \in \hat{S}_z \} \) is infinite and the series \( \sum_{n \in \Omega_z} \|x_n\| \) is divergent. Since the sequence \( (x_n)_{n \in \Omega_z} \) is contained in the cone \( \hat{S}_z \), tends to zero, and the series \( \sum_{n \in \Omega_z} \|x_n\| \) is divergent, we can use the inequality (1) and choose a finite subset \( E_z \subset \Omega \setminus F \) such that the sum \( s_z = \sum_{n \in E_z} x_n \in \hat{S}_z \) has norm in the interval

\[
c \cdot t_z - \frac{\varepsilon}{2\|D_0\|} < \|s_z\| \leq c \cdot t_z
\]

where \( |D_0| \) denotes the cardinality of the finite set \( D_0 \).

Then

\[
\|s_z - c t_z z\| \leq \left\| \frac{s_z}{\|s_z\|} - c t_z \frac{s_z}{\|s_z\|} \right\| + \left\| c t_z \frac{s_z}{\|s_z\|} - c t_z z \right\| = \left\| \frac{s_z}{\|s_z\|} - c t_z \right\| + \left\| c t_z \frac{s_z}{\|s_z\|} - z \right\| < \frac{\varepsilon}{2\|D_0\|} + c t_z \frac{\varepsilon}{2c}
\]

and for the finite set \( E = \bigcup_{z \in D_0} E_z \subset \Omega \setminus F \) we get

\[
\left\| \sum_{n \in E} x_n - x \right\| = \left\| \sum_{n \in E} x_n - \sum_{z \in D_0} c t_z z \right\| = \sum_{z \in D_0} \left\| x_n - c t_z z \right\| = \sum_{z \in D_0} \sum_{n \in E_z} \left\| s_z - c t_z z \right\| < \left( \sum_{z \in D_0} \frac{\varepsilon}{2\|D_0\|} + t_z \frac{\varepsilon}{2} \right) = \frac{\varepsilon}{2} \left( 1 + \sum_{z \in D_0} t_z \right) = \varepsilon.
\]

Now we prove that the set \( E \) satisfies the second condition of Claim 3. Take any subset \( E' \subset E \) and for every \( z \in D_0 \) let \( E'_z = E \cap E_z \). Then \( \sum_{n \in E'_z} x_n \in \hat{S}_z \) and \( \|\sum_{n \in E'_z} x_n\| \leq \|\sum_{n \in E_z} x_n\| \leq c t_z \leq c \), which implies that \( \sum_{n \in E'_z} x_n \in [0, c] \cdot S_z \) and thus

\[
\left\| \sum_{n \in E'} x_n \right\| \leq c C_\delta = C_\delta \cdot \max\left\{\frac{\varepsilon}{\delta}, \frac{\|x\|}{\delta}\right\} = C \cdot \max\{\|x\|, \varepsilon\}
\]

by definition of the numbers \( C_\delta \) and \( C = C_\delta / \delta \).

**Claim 4.** The sets \( \Gamma = \{y \in X : \sum_{n=1}^\infty \|y, x_n\| < \infty\} \) and \( \Gamma_1 = \{y \in X_1 : \sum_{n=1}^\infty \|y, pr_1(x_n)\| < \infty\} \) coincide. □
Proof. First we show that $\Gamma \subset X_1 = X_0^\perp$. In the opposite case we would find points $y \in \Gamma$ and $z \in D_0$ such that $\langle y, z \rangle \neq 0$. Then $U = \{ x \in S : \| \langle y, x \rangle \| > \frac{1}{2} \| \langle y, z \rangle \| \}$ is an open neighborhood of $z$ in the sphere $S$ and by the definition of the set $D \ni z$ the set $N_U = \{ n \in \mathbb{N} : \| x_n \| \in U \}$ is infinite and the series $\sum_{n \in N_U} \| x_n \|$ diverges. It follows that for every $n \in \mathbb{N}$ we have $\| \langle y, x_n \rangle \| > \frac{1}{2^{n+1}} \| \langle y, z \rangle \|$, which implies the divergence of the series $\sum_{n \in N_U} \| \langle y, x_n \rangle \|$ but this contradicts the choice of $y \in \Gamma$. This contradiction shows that $\Gamma \subset X_1$.

Next, observe that for any $y \in X_1$ and $n \in \mathbb{N}$ we get

$$\langle y, x \rangle = \langle y, x - pr_1(x) \rangle + \langle y, pr_1(x) \rangle = \langle y, pr_0(x) \rangle + \langle y, pr_1(x) \rangle = 0 + \langle y, pr_1(x) \rangle = \langle y, pr_1(x) \rangle$$

and thus $\sum_{n=1}^{\infty} \| \langle y, x \rangle \| = \sum_{n=1}^{\infty} \| \langle y, pr_1(x) \rangle \|$, which implies the equality $\Gamma = X_1 \cap \Gamma = \Gamma$. □

Now we are ready to complete the inductive proof of the theorem. By our assumption, the series $\sum_{n=1}^{\infty} x_n$ is $\ast$-potentially convergent. Then the series $\sum_{n=1}^{\infty} pr_1(x_n)$ is $\ast$-potentially convergent in the Hilbert space $X_1$.

Since the space $X_1$ has dimension strictly smaller than $d$, the induction hypothesis and Claim 3 guarantee that the set $\Sigma$ of potential sums of the series $\sum_{n \in \mathbb{N}} pr_1(x_n)$ is a non-empty affine subspace of $X_1$ such that $\Sigma_1 = \Sigma + \Gamma_1^\perp$, where $\Gamma_1^\perp = \{ x \in X_1 : \forall y \in \Gamma_1 \langle y, x \rangle = 0 \}$ and $\Gamma_1 = \{ y \in X : \sum_{n=1}^{\infty} \langle y, x_n \rangle \}$ according to Claim 3.

It remains to prove that the set $\Sigma$ of potential sums of the series $\sum_{n=1}^{\infty} x_n$ coincides with the non-empty affine subspace $X_0 + \Sigma_1$ of $X$.

The inclusion $\Sigma \subset X_0 + \Sigma_1$ is trivial. To prove the reverse inclusion, we need to show that for any points $x \in X_0$ and $y \in \Sigma_1$ there exists a permutation $\sigma$ of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\sigma(n)} = x + y$.

It will be more convenient instead of permutation of $\mathbb{N}$, to construct a well-order $\prec$ of order type $\omega$ on the set $\mathbb{N}$ such that $\sum_{n \in \omega} x_n = x + y$, which means that for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ with $k \leq m$ we get $\| \sum_{n \leq k} x_n - (x + y) \| < \varepsilon$.

Let $s_1 \in X_1$ be the sum of the absolutely convergent series $\sum_{n \in \Omega} pr_1(x_n)$. Since $y \in \Sigma_1$, there exists a permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that $y = \sum_{n=1}^{\infty} pr_1(x_{\sigma(n)})$. The permutation $\sigma$ determines a well-order $\preceq$ on the set $\Lambda := \mathbb{N} \setminus \Omega$ defined by $n \preceq m$ iff $\sigma(n) \leq \sigma(m)$. For this well-order on $\Lambda$ we get $\sum_{n \in \Lambda} x_n = y - s_1$. From now on, talking about minimal elements of subsets of $\Lambda$ we have in mind the well-order $\preceq$. The set $\Omega \subset \mathbb{N}$ is endowed with the standard well-order $\leq$.

By induction, we shall construct decreasing sequences of sets $(\Lambda_i)_{i \in \omega}$ and $(\Omega_i)_{i \in \omega}$ and an increasing sequence of finite sets $(F_i)_{i \in \omega}$ such that the following conditions are satisfied for every $n \in \omega$:

1. $\Lambda_{i+1} = \Lambda_i \setminus \{ \min \Lambda_i \}$;
2. $\Omega_{i+1} = \Omega_i \setminus \{ \min \Omega_i \}$;
3. $F_{i+1} = F_i \cup \{ \min \Lambda_i \} \cup (\Omega_i \setminus \Omega_{i+1})$;
4. $\| x - \sum_{n \leq F_i} pr_0(x_n) \| < \frac{1}{2^i}$;
5. for any subset $E \subset F_{i+1} \setminus F_i$ we get $\| \sum_{n \in E} x_n \| \leq C \left( \frac{1}{2^i} + \| x_{\min \Lambda_i} + x_{\min \Omega_i} \| \right)$.

We start the inductive construction, applying Claim 3 and choosing a finite set $F_0 \subset \Omega$ such that $\| x - \sum_{n \in F_0} pr_0(x_n) \| \leq \| x - \sum_{n \in F_0} x_n \| < 1$. Next, we put $\Omega_0 = \Omega \setminus F_0$ and $\Lambda_0 = \Lambda = \mathbb{N} \setminus \Omega$.

Assume that for some $i \geq 0$ the sets $F_i$, $\Omega_i$ and $\Lambda_i$ have been constructed. Consider the element $a_i := x - pr_0(x_{\min \Lambda_i} + x_{\min \Omega_i} + \sum_{n \in F_i} x_n)$, observe that the inductive assumption (4) guarantees that

$$\| a_i \| \leq \| x - \sum_{n \in F_i} pr_0(x_n) \| + \| x_{\min \Lambda_i} + x_{\min \Omega_i} \| < \frac{1}{2^i} \| x_{\min \Lambda_i} + x_{\min \Omega_i} \|.$$

Applying Claim 3 find a finite subset $E_i \subset \Omega_i \setminus (F_i \cup \{ \min \Omega_i \})$ such that $\| a_i - \sum_{n \in E_i} x_n \| < \frac{1}{2^{i+1}}$ and for every $E \subset E_i$

$$\| \sum_{n \in E} x_n \| < C \cdot \max \left( \frac{1}{2^{i+1}}, \| a_i \| \right) \leq C \left( \frac{1}{2^i} + \| x_{\min \Lambda_i} + x_{\min \Omega_i} \| \right)$$

Let $F_{i+1} = F_i \cup E_i \cup \{ \min \Omega_i, \min \Lambda_i \}$, $\Omega_{i+1} = \Omega_i \setminus (E_i \cup \{ \min \Omega_i \})$ and $\Lambda_{i+1} = \Lambda_i \setminus \{ \min \Lambda_i \}$. This completes the inductive step.

After completing the inductive construction of the sequence $(F_i)_{i \in \omega}$, on the union $\bigcup_{i \in \omega} F_i = \mathbb{N}$ fix any well-order $\preceq$ such that $n \preceq m$ for any $i \in \omega$ and numbers $n \in F_i$ and $m \notin F_i$. Observe that the well-order $\preceq$ induces the well-order $\preceq$ on the set $\Lambda$, which, combined with the absolute convergence of the series $\sum_{n \in \Omega} pr_1(x_n)$, ensures that the series $\sum_{n \in \mathbb{N}} pr_1(x_n)$ converges to $y$. On the other hand, the conditions (4), (5) of the inductive construction and the convergence of the sequence $(x_n)_{n=1}^{\infty}$ to zero imply that the series $\sum_{n \in \mathbb{N}} pr_0(x_n)$ converges to $x$. Consequently, the series $\sum_{n \in \mathbb{N}} x_n$ converges to $x + y$, which yields the desired inclusion $x + y \in \Sigma$.

Finally, observe that $\Sigma + \Gamma_1^\perp = (X_0 + \Sigma_1) + (X_0 + \Gamma_1^\perp) = X_0 + (\Sigma_1 + \Gamma_1^\perp) = X_0 + \Sigma_1 = \Sigma$. 

3. Generalizing Levý-Steinitz Theorem to Abelian topological groups

The problem of extending the Levý-Steinitz Theorem to infinite-dimensional Banach space was posed by Stefan Banach in the Scottish Book [9] Problem 106 and to Abelian topological groups by Stanislaw Ulam, who noticed (without proof) in [10] that for any convergent series \( \sum_{n=1}^{\infty} x_n \) in a compact metrizable Abelian topological group \( X \) the set \( \Sigma \) of potential sums of the series \( \sum_{n=1}^{\infty} x_n \) coincides with the shift \( H + \sum_{n=1}^{\infty} x_n \) of some subgroup \( H \subset X \). This fact (whose proof was given in Banaszczyk [11, 10.2]) can be considered as an extension of the first part of Levý-Steinitz Theorem (describing the structure of the set \( \Sigma \) of potential sums of a series). On the other hand, the second part of Levý-Steinitz Theorem (characterizing series with non-empty set \( \Sigma \) of partial sums) does not extend to locally compact Abelian topological groups as shown by the following simple example.

Example 1. Let \( T = \mathbb{R}/\mathbb{Z} \) be the circle group. The compact topological group \( X = T \times T \) contains a sequence \( (x_n)_{n \in \omega} \) such that

1. \( (x_n)_{n=1}^{\infty} \) converges to zero;
2. for every continuous homomorphism \( f : T \to T \) the series \( \sum_{n=1}^{\infty} f(x_n) \) is potentially convergent in \( T \);
3. the series \( (x_n)_{n=1}^{\infty} \) is not potentially convergent in \( X \).

Proof. Let \( q : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = T \) be the quotient homomorphism and \( q^2 : \mathbb{R}^2 \to \mathbb{T}^2 = X \) be the covering homomorphism defined by \( q^2(x,y) = (q(x),q(y)) \) for \( (x,y) \in \mathbb{R}^2 \). The compactness of the topological group \( X = \mathbb{T}^2 \) implies that the set of all non-zero continuous homomorphisms \( h : X \to \mathbb{T} \) is countable and hence can be enumerated as \( \{h_n\}_{n \in \omega} \). For every \( n \in \omega \) consider the continuous homomorphism \( f_n = h_n \circ q^2 : \mathbb{R}^2 \to \mathbb{T} \). Since the space \( \mathbb{T}^2 \) is simply-connected, the homomorphism \( f_n \) can be lifted to a unique continuous homomorphism \( \hat{f}_n : \mathbb{R}^2 \to \mathbb{R} \) such that \( q \circ \hat{f}_n = f_n \). Being continuous and additive, the homomorphism \( \hat{f}_n \) is linear. Choose any non-zero linear continuous functional \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( f \neq \{\hat{f}_n\}_{n \in \omega} \). Then for every \( n \in \omega \) the sets \( \{x \in \mathbb{R}^2 : f(x) > 0, f_n(x) > 0\} \) and \( \{x \in \mathbb{R}^2 : f(x) > 0, f_n(x) < 0\} \) are not empty. This allows us to choose a sequence \( (z_n)_{n=1}^{\infty} \) in the open half-plane \( \{z \in \mathbb{R}^2 : f(z) > 0\} \) such that

1. the sequence \( (z_n)_{n=1}^{\infty} \) converges to zero;
2. for every \( n \in \omega \) the series \( \sum_{n=1}^{\infty} f(x_n) \) diverges;
3. for every \( k \in \omega \) the series \( \sum_{n=1}^{\infty} \max\{0, f_k(z_n)\} \) and \( \sum_{n=1}^{\infty} \min\{0, f_k(z_n)\} \) are divergent.

In the group \( \mathbb{T}^2 \) consider the sequence \( (x_n)_{n \in \omega} \) of points \( x_n = q^2(z_n) \), \( n \in \mathbb{N} \). The conditions (1) and (2) imply that the series \( \sum_{n=1}^{\infty} z_n \) is not potentially convergent in \( \mathbb{R}^2 \) and hence the series \( \sum_{n=1}^{\infty} x_n \) is not potentially convergent in \( \mathbb{T}^2 \). On the other hand, the condition (3) combined with the Riemann Rearrangement Theorem imply that for every \( k \in \omega \) the series \( \sum_{n=1}^{\infty} \hat{f}_k(z_n) \) is potentially convergent in the real line and hence the series \( \sum_{n=1}^{\infty} h_n(x_n) = \sum_{n=1}^{\infty} q \circ \hat{f}_n \circ q^2(z_n) \) is potentially convergent in the circle \( T \). Therefore, the sequence \( (x_n)_{n \in \omega} \) satisfies the conditions (1)–(3) of Example [11].

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