Quantum cloning and the capacity of the Pauli channel

Nicolas J. Cerf

1 W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125
2 Information and Computing Technologies Research Section, Jet Propulsion Laboratory, Pasadena, California 91109
3 Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, 1050 Bruxelles, Belgium

March 1998

A family of quantum cloning machines is introduced that produce two approximate copies from a single quantum bit, while the overall input-to-output operation for each copy is a Pauli channel. A no-cloning inequality is derived, describing the balance between the quality of the two copies. This also provides an upper bound on the quantum capacity of the Pauli channel with probabilities \( p_x, p_y \) and \( p_z \). The capacity is shown to be vanishing if \( \sqrt{p_x}, \sqrt{p_y}, \sqrt{p_z} \) lies outside an ellipsoid whose pole coincides with the depolarizing channel that underlies the universal cloning machine.

PACS numbers: 03.67.Hk, 03.65.Bz, 89.70.+c

KRL preprint MAP-220

A remarkable consequence of the linearity of quantum mechanics is that an unknown quantum state cannot be duplicated, as recognized after the seminal papers by Dieks [1] and Wootters and Zurek [2]. Specifically, a universal cloning machine (UCM) can be defined that creates two copies characterized each by the same density operator \( \rho \). Recently, it has been shown by Buzek and Hillery [3] that it is nevertheless possible to construct a cloning machine that yields two approximate copies of an input qubit. Specifically, a universal cloning machine (UCM) can be defined that creates two copies characterized each by the same density operator \( \rho \), the fidelity of cloning being \( f = \frac{\langle \psi | \rho | \psi \rangle}{\langle \psi | \psi \rangle} = 5/6 \). The UCM was later proved to be optimal by Bruss et al. [4], and Gisin and Massar [5]. This cloning machine is universal in the sense that the copies are state-independent (both output qubits emerge from a depolarizing channel of probability 1/4, that is, the Bloch vector characterizing the input qubit is shrunk by a factor 2/3 regardless of its orientation). A great deal of effort has been devoted recently to quantum cloners because of their use in the context of quantum communication and cryptography (see, e.g., [6,7]). For example, an interesting application of the UCM is that it can be used to establish an upper bound on the quantum capacity \( C \) of a depolarizing channel, namely \( C = 0 \) at \( p = 1/4 \) [4].

In this paper, I introduce a family of asymmetric cloning machines that produce two (not necessarily identical) output qubits, each emerging from a Pauli channel. This family of cloners, which I call Pauli cloning machines (PCM), relies on a parametrization of 4-qubit wave functions for which all qubit pairs are in a mixture of Bell states. Using these PCMs, I derive a no-cloning inequality governing the tradeoff between the quality of both copies of a single qubit imposed by quantum mechanics. It is, by construction, a tight inequality for qubits which is saturated using a Pauli channel. I then consider a subclass of symmetric PCMs in order to express an upper limit on the quantum capacity of a Pauli channel, generalizing the considerations of Ref. [4] for a depolarizing channel.

When processed by a Pauli channel, a qubit is rotated by one of the Pauli matrices or remains unchanged: it undergoes a phase-flip \( (x) \), a bit-flip \( (z) \), or their combination \( (y) \) with respective probabilities \( p_x, p_z, p_y \). (A depolarizing channel corresponds to the special case \( p_x = p_y = p_z \).) If the input qubit \( x \) of the Pauli channel is initially in a fully entangled state with a reference qubit \( r \), say in the Bell state \( | \Phi^+ \rangle \), then the joint state of \( r \) and the output \( y \) is a mixture of the four Bell states \( | \Phi^\pm \rangle = 2^{-1/2}(|00\rangle \pm |11\rangle) \) and \( | \Psi^\pm \rangle = 2^{-1/2}(|01\rangle \pm |10\rangle) \).

\[
\rho_{ry} = (1 - p)| \Phi^+ \rangle \langle \Phi^+ | + p_z | \Phi^- \rangle \langle \Phi^- | + p_x | \Psi^+ \rangle \langle \Psi^+ | + p_y | \Psi^- \rangle \langle \Psi^- |
\]

with \( p = p_x + p_y + p_z \). For an asymmetric cloning machine whose two outputs, \( y_1 \) and \( y_2 \), emerge from (distinct) Pauli channels, the density operators \( \rho_{ry_1} \) and \( \rho_{ry_2} \) must then be mixtures of Bell states. Focusing on the first output \( y_1 \), we see that a 4-dimensional Hilbert space is necessary in general to purify \( \rho_{ry_1} \) since we need to accommodate its four (nonzero) eigenvalues. The 2-dimensional space of second output qubit \( y_2 \) is thus insufficient for this purpose, so that we must introduce an additional qubit \( y_3 \), which may be viewed as an ancilla or the cloning machine itself [4].

Thus, instead of specifying a PCM by a particular unitary operation acting on the state \( | \psi \rangle = \alpha |0\rangle + \beta |1\rangle \) of the input qubit \( x \) (together with the two auxiliary qubits in a fixed state \(|0\rangle \)), it is more convenient to characterize it by the wave function \( \langle \psi | r_{y_1 y_2 y_3} \rangle \) underlying the entanglement of the three outputs with \( r \). So, the question here is to find in general the 4-qubit wave functions \( | \psi \rangle_{abcd} \) that satisfy the requirement that the state of every pair of two qubits is...
a mixture of the four Bell states. Making use of the Schmidt decomposition for the bipartite partition $ab$ vs $cd$, it is clear that $|\psi\rangle_{abcd}$ can be written as a superposition of double Bell states

$$
|\psi\rangle_{abcd} = \left\{ v |\Phi^+\rangle |\Phi^+\rangle + z |\Phi^+\rangle |\Phi^-\rangle + x |\Psi^+\rangle |\Psi^+\rangle + y |\Psi^-\rangle |\Psi^-\rangle \right\}_{abcd},
$$

where $x$, $y$, $z$, and $v$ are parameters ($|x|^2 + |y|^2 + |z|^2 + |v|^2 = 1$). The above requirement is then satisfied for the qubit pairs $ab$ and $cd$, that is, $\rho_{ab} = \rho_{cd}$ is of the form of Eq. (1) with $p_x = |x|^2$, $p_y = |y|^2$, $p_z = |z|^2$, and $1 - p = |v|^2$. It is worth noting that these double Bell states for the partition $ab$ vs $cd$ transform into superpositions of double Bell states for the two other possible partitions ($ac$ vs $bd$, $ad$ vs $bc$), e.g.,

$$
|\Phi^+\rangle_{ab} |\Phi^+\rangle_{cd} = \frac{1}{2} \left\{ (|\Phi^+\rangle |\Phi^+\rangle + |\Phi^-\rangle |\Phi^-\rangle + |\Psi^+\rangle |\Psi^+\rangle + |\Psi^-\rangle |\Psi^-\rangle) \right\}_{ac:bd}.
$$

This implies that $|\psi\rangle_{abcd}$ is also a superposition of double Bell states (albeit with different amplitudes) for these two other partitions, which, therefore, also yield mixtures of Bell states when tracing over half of the system. Table I summarizes the amplitudes of $|\psi\rangle_{abcd}$ for the three partitions of $abcd$ into two pairs, starting from Eq. (2). Thus, identifying $a$ with the reference qubit (initially entangled with the input), $b$ and $c$ with the two outputs, and $d$ with an idle qubit (or the cloning machine), Table I defines the desired family of asymmetric Pauli cloning machines.

FIG. 1. Pauli cloning machine of input $x$ and outputs $y_1$ and $y_2$. The third output $y_3$ refers to an idle qubit (or the cloning machine). The three outputs emerge in general from distinct Pauli channels.

Let us consider a PCM whose first output $b$ emerges from a depolarizing channel of probability $p = 3|x|^2$, i.e.,

$$
\rho_{ab} = |v|^2 |\Phi^+\rangle \langle \Phi^+| + |x|^2 \left( |\Phi^-\rangle \langle \Phi^-| + |\Psi^+\rangle \langle \Psi^+| + |\Psi^-\rangle \langle \Psi^-| \right) .
$$

From Table I, the second output $c$ necessarily emerges from a depolarizing channel of probability $p' = \frac{3}{4} |v - x|^2$, or, more precisely,

$$
\rho_{ac} = \frac{|v + 3x|^2}{4} |\Phi^+\rangle \langle \Phi^+| + \frac{|v - x|^2}{4} \left( |\Phi^-\rangle \langle \Phi^-| + |\Psi^+\rangle \langle \Psi^+| + |\Psi^-\rangle \langle \Psi^-| \right) .
$$

(The idle qubit $d$ emerges in general from a different Pauli channel.) Clearly, both outputs of this asymmetric PCM are state-independent when they simply correspond to a (different) shrinking of the vector characterizing the input qubit in the Bloch sphere. Combining Eqs. (4) and (5), the tradeoff between the quality of the two copies can be described by the no-cloning inequality

$$
x^2 + x'^2 + xx' \geq \frac{1}{4}, \quad \text{(with } x, x' \geq 0) \tag{6}
$$

where the copying error is measured by the probability of the depolarizing channel underlying each output, i.e., $p = 3x^2$ and $p' = 3x'^2$. (I assume here that that the amplitudes of $|\psi\rangle_{abcd}$ are real and positive.) Equation (5) corresponds to the domain in the $(x, x')$-space located outside an ellipse whose semimajor axis $b = 1/\sqrt{2}$ is oriented in the direction $(1, 1)$, as shown in Fig. 2. (The semimajor axis is $a = 1/\sqrt{2}$.) The origin in this space corresponds to a (nonexisting) cloner whose two outputs would be perfect $p = p' = 0$. The ellipse characterizes the ensemble of values for $p$ and $p'$ that can actually be achieved with a PCM. It intercepts its minor axis at $(1/\sqrt{12}, 1/\sqrt{12})$, which corresponds to the universal cloning machine (UCM), i.e., $p = p' = 1/4$. This point is the closest to the origin (i.e., the cloner with minimum $p + p'$), and characterizes in this sense the best possible copying, as expected. Note that the underlying wave function

$$
|\psi\rangle_{abcd} = \sqrt{\frac{3}{4}} |\Phi^+\rangle_{ab} |\Phi^+\rangle_{cd} + \sqrt{\frac{1}{12}} \left\{ (|\Phi^-\rangle |\Phi^-\rangle + |\Psi^+\rangle |\Psi^+\rangle + |\Psi^-\rangle |\Psi^-\rangle) \right\}_{abcd}
$$

$$
= \sqrt{\frac{1}{3}} \left\{ (|\Phi^+\rangle |\Phi^+\rangle + |\Phi^-\rangle |\Phi^-\rangle + |\Psi^+\rangle |\Psi^+\rangle) \right\}_{abcd} .
$$

2
is symmetric under the interchange of $a$ and $d$ (or $b$ and $c$). The ellipse crosses the $x$-axis at $(1/2, 0)$, which describes the situation where the first output emerges from a 100%-depolarizing channel ($p = 3/4$) while the second emerges from a perfect channel ($p' = 0$). Of course, $(0, 1/2)$ corresponds to the symmetric situation. The argument used in \[7\] strongly suggests that the imperfect cloning achieved by such an asymmetric PCM is optimal: a single additional qubit $d$ is sufficient to perform optimal cloning, i.e., to achieve the minimum $p$ and $p'$ for a fixed ratio $p/p'$. The domain inside the ellipse corresponds then to the values for $p$ and $p'$ that cannot be achieved simultaneously, reflecting the impossibility of close-to-perfect cloning, and Eq. \[6\] is the tightest no-cloning bound that can be written for a qubit.

FIG. 2. Ellipse delimiting the best quality of the two outputs of an asymmetric PCM that can be achieved simultaneously (only the quadrant $x, x' \geq 0$ is of interest here). The outputs emerge from depolarizing channels of probability $p = 3x^2$ and $p' = 3x'^2$. Any close-to-perfect cloning characterized by a point inside the ellipse is forbidden.

Consider now the class of symmetric PCMs that have both outputs merging from a same Pauli channel, i.e., $\rho_{ab} = \rho_{ac}$. Using Table \[\ref{tab:1}\] we obtain the conditions $4|v|^2 = |v+z+x+y|^2$, $4|z|^2 = |v+z-x-y|^2$, $4|x|^2 = |v-z+x-y|^2$, and $4|y|^2 = |v-z-x+y|^2$, which yields

$$v = x + y + z,$$

(9)

where $x$, $y$, $z$, and $v$ are assumed to be real. Equation \[9\], together with the normalization condition, describes a 2D surface in a space where each point $(x, y, z)$ represents a Pauli channel of parameters $p_x = x^2$, $p_y = y^2$, and $p_z = z^2$ (I only consider here the first octant $x, y, z \geq 0$). This surface,

$$x^2 + y^2 + z^2 + xy + xz + yz = \frac{1}{2},$$

(10)

is an oblate ellipsoid $E$ with symmetry axis along the direction $(1, 1, 1)$, as shown in Fig. \[\ref{fig:3}\]. The semiminor axis (or polar radius) is $a = 1/2$ while the semimajor axis (or equatorial radius) is $b = 1$. In this representation, the distance to the origin is $p_x + p_y + p_z$, so that the pole $(1/\sqrt{12}, 1/\sqrt{12}, 1/\sqrt{12})$ of this ellipsoid—the closest point to the origin—corresponds to the special case of a depolarizing channel of probability $p = 1/4$. Thus, this particular PCM coincides with the UCM. This simply illustrates that the requirement of having an optimal cloning (minimum $p_x + p_y + p_z$) implies that the cloner is state-independent ($p_x = p_y = p_z$).

FIG. 3. Oblate ellipsoid representing the class of symmetric PCMs whose two outputs emerge from the same Pauli channel of parameters $p_x = x^2$, $p_y = y^2$, and $p_z = z^2$ (only the octant $x, y, z \geq 0$ is considered here). The pole of this ellipsoid corresponds to the UCM. The capacity of a Pauli channel that lies outside this ellipsoid must be vanishing.
The class of symmetric PCMs characterized by Eq. (10) can be used in order to put a limit on the quantum capacity of a Pauli channel, thereby extending the result of Bruss et al. [4]. Indeed, applying an error-correcting scheme separately on each output of the cloning machine (obviously of the other output) would lead to a violation of the no-cloning theorem if the capacity $C$ was nonzero. Since $C$ is a nonincreasing function of $p_x$, $p_y$, and $p_z$, for $p_x, p_y, p_z \leq 1/2$ (i.e., adding noise to a channel cannot increase its capacity), I conclude that $C(p_x, p_y, p_z) = 0$ for $(x, y, z) \not\in E$, that is, the quantum capacity is vanishing for any Pauli channel that lies outside the ellipsoid $E$. In particular, Eq. (10) implies that the quantum capacity vanishes for (i) a depolarizing channel with $p = 1/4$ ($p_x = p_y = p_z = 1/12$); (ii) a "2-Pauli" channel with $p = 1/3$ ($p_x = p_z = 1/6, p_y = 0$); and (iii) a dephasing channel with $p = 1/2$ ($p_x = p_y = 0, p_z = 1/2$). Furthermore, using the fact that $C$ cannot be superadditive for a convex combination of a perfect and a noisy channel [6], an upper bound on $C$ can be written using a linear interpolation between the perfect channel $(0, 0, 0)$ and any Pauli channel lying on $E$: \[ C \leq 1 - 2(x^2 + y^2 + z^2 + xy + xz + yz) \]. (11)

Note that another class of symmetric PCMs can be found by requiring $\rho_{ab} = \rho_{ad}$, which implies $v = x - y + z$ rather than Eq. (10). This requirement gives rise to the reflection of $E$ with respect to the $xz$-plane, i.e., $y \rightarrow -y$. It does not change the above bound on $C$ because this class of PCMs has noisier outputs in the octant $x, y, z \geq 0$.

Let us now turn to the fully symmetric PCMs that have three outputs emerging from the same Pauli channel, which corresponds to a family of (non-optimal) quantum triplicating machines. The requirement $\rho_{ab} = \rho_{ac} = \rho_{ad}$ implies $(v = x + z) \land (y = 0)$. Incidentally, we notice that if all pairs are required to be in the same mixture of Bell states, this mixture cannot have a singlet $|\Psi^-\rangle$ component. The outputs of the corresponding triplicators emerge therefore from a "2-Pauli" channel ($p_y = 0$), so that these triplicators are state-dependent, in contrast with the one considered in Ref. [3]. (For describing a state-independent triplicator, a 6-qubit wave function should be used.) These triplicators are represented by the intersection of $E$ with the $xz$-plane, that is, the ellipse \[ x^2 + z^2 + xz = \frac{1}{2}, \] (12)

with seminor axis $b = 1/\sqrt{3}$ [oriented along the direction $(1, 1, 0)$] and semimajor axis $a = 1$. The intersection of this ellipse with its seminor axis $(x = z = 1/\sqrt{6})$ corresponds to the 4-qubit wave function \[ |\psi\rangle_{abcd} = \frac{2}{\sqrt{6}}|\Phi^+\rangle|\Phi^+\rangle + \frac{1}{\sqrt{6}}|\Phi^-\rangle|\Phi^-\rangle + \frac{1}{\sqrt{6}}|\Psi^+\rangle|\Psi^+\rangle, \] (13)

which is symmetric under the interchange of any two qubits and maximizes the 2-bit entropy (or minimizes the mutual entropy between any two outputs of the triplicator). Equation (13) thus characterizes the best triplicator of this ensemble, whose three outputs emerge from a "2-Pauli" channel with $p = 1/3$ ($p_x = p_z = 1/6$). Equivalently, the (state-dependent) operation of this triplicator on an arbitrary qubit can be written as \[ |\psi\rangle \rightarrow \frac{1}{2} |\psi\rangle\langle\psi| + \frac{1}{6} |\psi^*\rangle\langle\psi^*| + \frac{1}{3} (\mathbb{I}/2). \] (14)

If $|\psi\rangle$ is real, Eq. (14) reduces to the triplicator that is considered in Ref. [10]. The fidelity of cloning is then the same as for the UCM.

I have generalized the Buzek-Hillery cloning machine and introduced an asymmetric Pauli cloning machine, whose outputs emerge from distinct Pauli channels. This allowed me to derive a tight no-cloning inequality for quantum bits, quantifying the impossibility of copying due to quantum mechanics. Furthermore, I have established an upper bound on the quantum capacity of the Pauli channel relying on a class of symmetric PCMs.
ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant Nos. PHY 94-12818 and PHY 94-20470, and by a grant from DARPA/ARO through the QUIC Program (#DAAH04-96-1-3086).

[1] D. Dieks, Phys. Lett. 92A, 271 (1982).
[2] W. K. Wootters and W. H. Zurek, Nature (London) 299, 802 (1982).
[3] V. Buzek and M. Hillery, Phys. Rev. A 54, 1844 (1996).
[4] D. Bruss, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. A. Smolin, Report No. quant-ph/9705038.
[5] N. Gisin and S. Massar, Phys. Rev. Lett. 79, 2153 (1997).
[6] N. Gisin and B. Huttner, Phys. Lett. A 228, 13 (1997).
[7] It is shown in Refs. [3,4] that a 2-dimensional ancilla space is sufficient, so we only have to consider a single qubit $y_3$ for the cloner.
[8] Permutations of the Bell states in Eq. (3) are not considered for simplicity.
[9] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[10] V. Buzek, S. L. Braunstein, M. Hillery, and D. Bruss, Phys. Rev. A 56, 3446 (1997).

| $|\psi\rangle_{abcd}$ | $|\Phi^+\rangle|\Phi^+\rangle$ | $|\Phi^-\rangle|\Phi^-\rangle$ | $|\Psi^+\rangle|\Psi^+\rangle$ | $|\Psi^-\rangle|\Psi^-\rangle$ |
|----------------|----------------|----------------|----------------|----------------|
| $ab$ vs $cd$ | $v$ | $z$ | $x$ | $y$ |
| $ac$ vs $bd$ | $\frac{1}{2}(v + z + x + y)$ | $\frac{1}{2}(v + z - x - y)$ | $\frac{1}{2}(v - z + x - y)$ | $\frac{1}{2}(v - z - x + y)$ |
| $ad$ vs $bc$ | $\frac{1}{2}(v + z + x - y)$ | $\frac{1}{2}(v + z - x + y)$ | $\frac{1}{2}(v - z + x + y)$ | $\frac{1}{2}(v - z - x - y)$ |

TABLE I. Amplitudes of $|\psi\rangle_{abcd}$ in terms of the double Bell states for the three possible partitions of the four qubits $abcd$ into two pairs.