Abstract. Given an \( m \)-component link \( L \) in \( S^3 \) \((m \geq 2)\), we construct a family of links which are link homotopic, but not link isotopic, to \( L \). Every proper sublink of such a link is link isotopic to the corresponding sublink of \( L \). Moreover, if \( L \) is an unlink then there exist links that in addition to the above properties have all Milnor invariants zero.

1. Introduction

An \( n \)-component link \( L \) in \( S^3 \) is a collection of piecewise linear maps \((l_1, \ldots, l_n) : S^1 \to S^3\), where the images \( l_1(S^1), \ldots, l_n(S^1) \) are pairwise disjoint. A link with one component is a knot.

Two links in \( S^3 \), \( L_1 \) and \( L_2 \), are said to be isotopic if there is an orientation preserving homeomorphism \( h : S^3 \to S^3 \) such that \( h(L_1) = L_2 \) and \( h \) is isotopic to the identity map.

The notion of link homotopy was introduced by Milnor in \([4]\). Two links \( L \) and \( L' \) are said to be link homotopic if there exist homotopies \( h_{i,t}, \) between the maps \( l_i \) and the maps \( l'_i \) so that the sets \( h_{1,t}(S^1), \ldots, h_{n,t}(S^1) \) are disjoint for each value of \( t \). In particular, a link is said to be link homotopically trivial if it is link homotopic to the unlink. Notice that this equivalence allows self-crossings, that is, crossing changes involving two strands of the same component.

The question that arises now is: how different are these two notions of link equivalence? From the definition it is clear that a link isotopy is a link homotopy as well. Link homotopy allows self-crossings, which dramatically simplifies the equivalence between links. Under link homotopy, all knots become trivial knots, despite the fact that they are non-trivial under isotopy. Link homotopy in some sense measures the linking between different components of a link. So the question reduces to whether two link homotopic links are also isotopic. We have the example of the Whitehead link, which is a non-trivial link that becomes trivial under link homotopy. This gives us an example of a 2-component link that is link homotopic, but not isotopic, to the unlink. But there are no known examples of such links with more than two components.

The main theorem in this article provides a method to generate examples of links with more than two components that are link homotopic but not isotopic to any given link. In particular, we prove the following.

**Theorem 1.1.** Let \( L \) be any \( m \)-component link in \( S^3 \) for \( m \geq 2 \). There exists a family of \( m \)-component links, \( L_m \), with the following properties:

1. Each \( L_m \) is link homotopic to \( L \)
2. Each \( L_m \) is not link isotopic to \( L \)
3. Every proper sublink of \( L_m \) is link isotopic to the corresponding sublink of \( L \). In particular, if \( L \) is an unlink then every proper sublink of \( L_m \) is isotopic to the unlink.

The proof of this theorem constitutes Sections 4 and 5. In the last section we show how to generate examples of links with an additional property about Milnor invariants. We prove the following.
**Theorem 1.2.** There exist links in $S^3$ with the following properties.

1. It is link homotopic to the unlink.
2. It is not link isotopic to the unlink.
3. Every proper sublink is link isotopic to the unlink.
4. All Milnor’s $\mu$-invariants, $\overline{\mu}(i_1 \ldots i_s)$, are 0.

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2. **Jones polynomial**

The Jones polynomial gives an invariant of an oriented link, that assigns to each link a Laurent polynomial in the variable $t^{1/2}$ with integer coefficients. If two links are link isotopic then they have the same Jones polynomial.

The Jones polynomial is a function

$$V : \{\text{Oriented links in } S^3\} \rightarrow \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

with the following properties.

1. $V(\text{unknot}) = 1$.
2. $V(L \cup \emptyset) = (-t^{-1/2} - t^{1/2})V(L)$.
3. $V$ satisfies the Skein relation: whenever three oriented links $L_+, L_-$ and $L_0$ are the same, except in the neighbourhood of a point where they are as shown in Figure 1, then

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0.$$

![Figure 1. Skein relation](image)

The Jones polynomial is discussed in detail in the book [2].

3. **Whitehead link**

The Whitehead link is given in Figure 2. It can be seen to be link homotopic to the unlink by making a crossing change at the central crossing, $A$. However, its Jones polynomial is computed to be $t^{-3/2}(-1 + t - 2t^2 + t^3 - 2t^4 + t^5)$, while the Jones polynomial of the unlink is $(-t^{-1/2} - t^{1/2})$.

This shows that the Whitehead link is not link isotopic to the unlink. Moreover, removing any component from the Whitehead link leaves us with the unknot. Thus, it satisfies the conclusion of Theorem 1.1 for $m = 2$.

![Figure 2. Whitehead link](image)
4. Unlink

In this section, we construct examples of links as in Theorem 1.1 where $L$ is an unlink.

4.1. Construction. A Brunnian link is a non-trivial link such that each sublink is link isotopic to an unlink. An example of a Brunnian link with 3 components are the Borromean rings (Figure 3). For $m \geq 3$, a family of Brunnian links, $B_m$, was given by Milnor [4] as shown in Figure 4. Notice that these links satisfy properties (2) and (3) of our theorem, but not property (1).

![Figure 3. Borromean rings](image)

**Figure 3.** Borromean rings

![Figure 4. Brunnian link with $m$ components, $B_m$](image)

**Figure 4.** Brunnian link with $m$ components, $B_m$

**Definition 4.1** (Whitehead double). Let $K$ be a component of a link $L$ in $S^3$, regarded as $h(\{0\} \times S^1)$ for some embedding $h : D^2 \times S^1 \rightarrow S^3 \setminus (L \setminus K)$, such that $K$ and $h((0,1) \times S^1)$ have linking number zero. Let $n$ be a non-zero integer. Consider in the solid torus $T = D^2 \times S^1$ the knot $W_n$ with $n$ crossings, as depicted in Figure 5. The knot $h(W_n)$ is called the Whitehead $n$-double of $K$, and is denoted by $W_n(K)$.

![Figure 5. Whitehead $n$ twists, $W_n$](image)

(A) $n$ negative twists  
(B) $n$ positive twists

**Figure 5.** Whitehead $n$ twists, $W_n$

If the components of the link $L$ are numbered, $L = K_1 \cup \ldots \cup K_m$, then the link obtained by considering the Whitehead $n$-double on the $i$-th component of $L$ will be denoted by $W_n^i(L)$. We are now ready to prove our theorem.
4.2. **Case** \( m = 3 \). We first consider the case of 3-component links. Let \( B_3 \) be the Brunnian link with 3 components, with its components numbered as shown in Figure 6. Consider the Whitehead \( n \)-double (with negative twists) of the third component of \( B_3 \) to obtain the new link, \( W_n^3(B_3) \). To simplify notation, we will denote this by \( W_n(B_3) \) hereafter (Figure 7).

Observe that when \( n \) is even, this link is link homotopic to the unlink. Even a single crossing change in \( W_n(B_3) \) unwinds the link to give an unlink. Note that \( W_n(B_3) \) satisfies properties (1) and (3) of our theorem.

We claim that there is an integer \( n > 0 \) for which \( W_n(B_3) \) is not link isotopic to the unlink, that is, it satisfies property (2). In fact, such an \( n \) occurs in every pair of consecutive even integers. We show this by showing that the Jones polynomial of \( W_n(B_3) \) for some \( n \) is not the same as that of the unlink.

Consider the skein relation at the crossing indicated in Figure 8.

\[
t^{-1}V(W_{n-2}(B_3)) - tV(W_n(B_3)) + (t^{-1/2} - t^{1/2})V(L_3) = 0. \tag{1}
\]
If our claim is false, then both $W_{n-2}(B_3)$ and $W_n(B_3)$ must be link isotopic to the unlink and must have the same Jones polynomial as the unlink with 3 components, namely, $(-t^{-1/2} - t^{1/2})^2$.

In that case, the above equation becomes

$$t^{-1}(-t^{-1/2} - t^{1/2})^2 - t(-t^{-1/2} - t^{1/2})^2 + (t^{-1/2} - t^{1/2})V(L_3) = 0. \tag{2}$$

Therefore,

$$(t^{-1/2} - t^{1/2})V(L_3) = (t - 1)(-t^{-1/2} - t^{1/2})^2. \tag{3}$$

That is,

$$V(L_3) = (-t^{-1/2} - t^{1/2})^3. \tag{4}$$

Now we compute the Jones polynomial for $L_3$ through multiple applications of the skein relations. Consider the link $L_3$ and focus on the part of the link indicated in Figure 9.

We apply the skein relation at the crossings indicated to get the following equations.

$$t^{-1}V(L_3) = tV(L^1) - (t^{-1/2} - t^{1/2})V(L^2) \tag{5}$$

$$tV(L^1) = t^{-1}V(L^3) + (t^{-1/2} - t^{1/2})V(L^4) \tag{6}$$
\[ tV(L^2) = t^{-1}V(L^5) + (t^{-1/2} - t^{1/2})V(L^6) \] (7)

Under isotopy the links \( L^3, L^4, L^5 \) and \( L^6 \) can be drawn as shown in Figure 10.

Figure 10

Now define the link \( A \) as shown.

\[
A = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{DiagramA.png}
\end{array}
\]

Notice that the links \( L^3, L^4, L^5 \) and \( L^6 \) can be written, up to isotopy, using the links \( A \) and the Brunnian link, \( B_3 \). Indeed, we have \( L^3 = B_3 \cup \bigcirc \), \( L^4 = B_3 \), \( L^5 = B_3 \), \( L^6 = A \cup \bigcirc \).

So now our computation is reduced to finding the Jones polynomial for \( B_3 \) and \( A \). After applying further Skein relations, we can calculate:

\[
V(B_3) = -t^3 + 3t^2 - 2t + 4 - 2t^{-1} + 3t^{-2} - t^{-3} \\
V(A) = t^2 + 2 + t^{-2}.
\]

Using the Skein relations above, this allows us the calculate:

\[
V(L_3) = t^{9/2} - 2t^{7/2} + t^{5/2} - 2t^{3/2} - 2t^{1/2} - 2t^{-1/2} - 2t^{-3/2} + t^{-5/2} - 2t^{-7/2} + t^{-9/2}.
\] (8)

However, this is different from the expression we obtained in (4) for \( V(L_3) \), proving our assumption to be incorrect. Hence, both \( W_n \) and \( W_{n-2} \) cannot be unlinks. This shows that at least one of any pair of consecutive even numbers gives a link with properties as required in Theorem 1.1. We have thus proved our theorem for links with \( m = 3 \) components.
4.3. Case $m > 3$. Now consider the case when the number of components is greater than 3. Consider the $m$-component Brunnian link, $B_m$ ($m > 3$), and the Whitehead $n$-double of the $m$-th component to obtain the link $W_n(B_m)$. Using the Skein relations, similar to the ones in Figure 8, we obtain the link $L_m$ as shown in Figure 11.

If $W_n(B_m)$ and $W_{n-2}(B_m)$ were both isotopic to the unlink, then computations similar to equations (1)–(4) show that the Jones polynomial for $L_m$ will be

$$V(L_m) = (-t^{-1/2} - t^{1/2})^m. \tag{9}$$

On the other hand, we compute the Jones polynomial of $L_m$ by application of Skein relations at the crossing (A) and then at (B). This reduces the link $L_m$ to the links $L_m^1, \ldots, L_m^6$, similar to the ones in equations (5)–(7). Here $L_m^3 = B_m \cup \circ$, $L_m^4 = B_m$, $L_m^5 = B_m$ and $L_m^6 = L_{m-1} \cup \circ$.

Computations similar to the ones in the case $m = 3$ give us the following recurrence relation for the Jones polynomial for $L_m$, for every $m > 3$:

$$V(L_m) = V(L_{m-1})(t^{5/2} - t^{3/2} - t^{-3/2} + t^{-5/2}) + (-t^{1/2} - t^{-1/2})^m(t^2 - 2t + 3 - 2t^{-1} + t^{-2}) \tag{10}$$

We claim that $\deg(V(L_m)) > m/2$ for every $m \geq 3$. From equation (8), we know that $\deg(V(L_3)) = \frac{9}{2} > \frac{3}{2}$, hence the result is true for $m = 3$.

Assume that this is true for $m - 1$. From the relation (10) we see that the highest power of $t$ in $V(L_m)$ comes from one of the terms, $t^{5/2}V(L_{m-1})$ and $(-t^{1/2} - t^{-1/2})^m t^2$. By the induction hypothesis, we know that $\deg(V(L_{m-1})) \geq \frac{m-1}{2}$. Hence, $\deg((t^{5/2}V(L_{m-1}))) > \frac{5}{2} + \frac{m-1}{2} = \frac{m+4}{2}$. This shows that degree of $V(L_m)$ must be the degree of the term $t^{5/2}V(L_{m-1})$, that is, $\deg(V(L_m)) = \deg(V(L_{m-1}))+ \frac{5}{2}$. Also, $\deg(V(L_m)) > \frac{m+4}{2} > \frac{m}{2}$, which proves our claim.

It also gives us a recurrence relation for the degree of $V(L_m)$, namely,

$$\deg(V(L_m)) = \deg(V(L_{m-1})) + \frac{5}{2}.$$ 

This in turn gives us that, for $m \geq 3$, $\deg(V(L_m)) = \frac{5}{2}(m-3) + \frac{9}{2}$.

This shows that

degree of $V(L_m) >$ degree of $(-t^{1/2} - t^{-1/2})^m$, for $m \geq 3$,

showing that the expression obtained in (9) must be different from the one in the recurrence relation (10).

Thus one of the links, $W_n(B_m)$ or $W_{n-2}(B_m)$, must be non-isotopic to the unlink. This completes the proof of Theorem 1.4 for $m \geq 3$. 

![Figure 11](image-url)
Remark. J.-B. Meilhan pointed out that a result of Meilhan-Yasuhara (Theorem 1, [3]) can also be used to show that $W_n(B_m)$ are non-trivial links for every $n > 0$. Their result shows that these links have non-zero Milnor isotopy invariants [5], showing that they are link isotopically non-trivial. And hence they also fail to satisfy property (4) of Theorem 1.2. In Section 6, we give examples satisfying properties (1) – (4).

4.4. Non-isotopic links. Here we point out that the method described in this section is useful to further show that each of the links, $W_n(B_3)$, are in fact non-isotopic links. To see this, observe that one can obtain the Jones polynomial of $W_n(B_3)$ as a function of $W_{n-2}(B_3)$, using the Skein relation in equation (1). We will do that here.

Define the function $\Phi$ as follows.

$$\Phi : \mathbb{Z}[t^{-1/2}, t^{1/2}] \to \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

$$\Phi(p) = t^{-2}p + t^{-1}(t^{-1/2} - t^{1/2})V(L_3),$$

where $V(L_3)$ is the Jones polynomial of the link $L_3$ as given in equation (8). Then $V(W_n(B_3)) = \Phi(V(W_{n-2}(B_3)))$, for every $n \geq 2$. Hence we only need to analyse the function $\Phi$ in order to understand the Jones polynomials of the Whitehead doubles.

One can see that the degree of $\Phi(p)$ depends on that of the polynomial $p$. Let us denote by $\deg_L(p)$ the degree of the lowest degree term of $p$. Then,

$$\deg_L(\Phi(p)) = \min \{\deg_L(p) - 2, \deg_L(V(L_3)) - 3/2\}.$$

We know from our previous computations that $\deg_L(V(L_3)) = -9/2$. Hence,

$$\deg_L(\Phi(p)) = \min\{\deg_L(p) - 2, -6\}. \quad (11)$$

For $n = 2$, we have that $V(W_2(B_3)) = \Phi(V(W_0(B_3)))$. Since $W_0(B_3)$ is nothing but the unlink with 3 components, $V(W_0(B_3)) = (-t^{-1/2} - t^{1/2})^3$, and $\deg_L(V(W_0(B_3))) = -1$. Hence $\deg_L(V(W_2(B_3))) = \deg_L(\Phi(V(W_0(B_3)))) = -6$.

Continuing in a similar fashion, we get that $\deg_L(V(W_4(B_3))) = -8$, $\deg_L(V(W_6(B_3))) = -10$ and so on.

We can do similar computations for odd $n$. Using skein relations, we get $V(W_1(B_3)) = -t^4 + 2t^3 - t^2 + 2t + 1 + t^{-2} - t^{-3} + 2t^{-4} - t^{-5}$. So, $\deg_L(V(W_1(B_3))) = -5$. Using the above relation (11) we get $\deg_L(V(W_3(B_3))) = \deg_L(\Phi(V(W_1(B_3)))) = -7$, $\deg_L(V(W_5(B_3))) = -9$ and so on.

Together, we get that $\deg_L(V(W_{n}(B_3))) = -n - 4$ for $n \geq 1$. This shows that each of these links are neither isotopic to the unlink, nor to each other.

Further, we can generalize this for $m > 3$ to show that

$$\deg_L(V(W_n(B_m))) = \frac{5 - 5m}{2} - (n - 1) \text{ for } n \geq 1 \text{ and } m > 3.$$

This shows that the links $W_n(B_m)$ are all non-isotopic links.

5. General links

In this section we will prove Theorem 1.1 for a general link. We first define what we mean by connect sum of links.

Definition 5.1 (Connect sum of links) Let $L = (l_1, \ldots, l_n)$ and $L' = (l'_1, \ldots, l'_n)$ be two $n$-component links in $S^3$. The links can be considered to be in $\mathbb{R}^3$. 
Let \( H_+ = \{(x,y,z) \in \mathbb{R}^3 : z > 1\} \) and \( H_- = \{(x,y,z) \in \mathbb{R}^3 : z < -1\} \) be two half-spaces in \( \mathbb{R}^3 \). Consider the set of \( n \) points \( \{x_i \in \mathbb{R}^3 : x_i = (i,0,1), i = 1,2,\ldots,n\} \) in \( \partial H_+ \) as shown in Figure 12 and similarly another set \( \{x_i' \in \mathbb{R}^3 : x_i' = (i,0,-1), i = 1,2,\ldots,n\} \) in \( \partial H_- \). Let \( I_i \) be line segments in \( \mathbb{R}^3 \) joining \( x_i \) and \( x_i' \) for \( i = 1,2,\ldots,n \) (Figure 12).

Now we can assume that \( L \) lies in \( H_+ \) and \( L' \) in \( H_- \). Let \( \phi_+ \) be an isotopy of \( \mathbb{R}^3 \) supported in the upper half space \( \{(x,y,z) : z > 0\} \) such that \( \phi_+(L) \) lies in \( \mathbb{R}_+ \) and that each component \( \phi(l_i) \) of \( \phi(L) \) intersects \( \partial H_+ \) in exactly one point, \( x_i \). Similarly, let \( \phi_- \) be an isotopy of \( \mathbb{R}^3 \) supported in the lower half space such that \( \phi_-(L') \) lies in \( \mathbb{R}_- \) and that each component \( \phi(l_i') \) intersects \( \partial H_- \) in the point \( x_i' \). See Figure 13.

Since \( \phi_+ \) and \( \phi_- \) are isotopies, we can call the new links \( L \) and \( L' \). A connect sum of each pair of components \( l_i \) and \( l_i' \) along the segment \( I_i \) is formed in the following way: small unknotted arcs of \( l_i \) and \( l_i' \) are removed in small neighbourhoods of \( x_i \) and \( x_i' \). The endpoints of these arcs are connected by strands parallel to \( I_i \) as shown.

The resulting \( n \)-component link is called a connect sum of \( L \) and \( L' \) and denoted by \( L \# L' \), as shown in Figure 14.

If the links \( L \) and \( L' \) are oriented links, we can define \( L \# L' \) to be an oriented link by having \( h \) preserve the orientation of each component.

Note that the definition of connect sum of links is not well-defined in general. It depends on
- the ordering of the components,
- the isotopies \( \phi_+ \) and \( \phi_- \).

For example, Figure 15 shows two different connect sums of the Whitehead link and the Hopf link.

We first consider the case of 2-component links. Let \( L \) be any given 2-component link in \( S^3 \). Let \( H \) be the Hopf link. Then let \( L \# H \) be a connect
Figure 15. Connect sums of the Whitehead link and Hopf link

sum of $L$ and $H$. Recall that, by definition of connect sums, this involves making choices for the isotopies of half spaces containing $L$ and $H$. We will call this choice the type of the connect sum. Now replace one of the components of $H$ by its Whitehead $n$-double as defined in Def 4.1 and call the resulting link $W_n(H)$. Construct a connect sum of $L$ and $W_n(H)$ using the same choice of isotopies, and hence of the same type. We will denote this particular choice of the connect sum by $L_n(H)$.

For example, let $L$ be the link with the Figure-8 knot and trefoil as its components as shown in Figure 16. Let $L\#H$ be its connect sum with the Hopf link as shown in Figure 17a. Then the connect sum of a Whitehead $n$-double of $H$ with $L$ corresponding to the fixed isotopies will be as shown in Figure 17b. This gives us a family of connect sums of the same type.

Let $L_n(H)$ be the connect sum shown in Figure 17b.

Notice that when $n$ is even a single (self) crossing change in the Whitehead twists will reduce the link $L_n(H)$ to the original link $L$, showing that it is link homotopic to $L$. Also, if one component from $W_n(H)$ is removed then we get an unknot, and hence removing one component from the connect sum will give us the corresponding component of the link $L$. We will show that $L_n(H)$ is
not isotopic to \( L \) for large enough even integers \( n \). Before we do that let us consider the case of links with more than two components.

Let \( L \) be any given \( m \)-component link in \( S^3 \) for \( m \geq 3 \). Let \( B_m \) be the Brunnian link with \( m \) components, and \( L \# B_m \) be a connect sum. Just like in the case of 2-component links, we fix the isotopies for this connect sum. Consider the Whitehead \( n \)-double of one of the components of \( B_m \) and call the resulting link \( W_n(B_m) \). Now consider the connect sum \( L \# W_n(B_m) \) with the same choices of isotopies and call it \( L_n(B_m) \).

For example let \( L \) be the 3-component link with pairwise linking numbers = 1 as shown in Figure 18a and let \( L_n(B_3) \) be a family of connect sums of the same type as shown in Figure 18b. Just as in the case of 2-component links we can easily see that for \( n \) even, \( L_n(B_3) \) is link homotopic to \( L \) and any proper sublink will be isotopic to the corresponding sublink of \( L \).

Now it remains to show that these links are indeed isotopically different from \( L \). To do this we consider the case when the number of components \( m = 3 \).

We use Skein relations on the indicated crossing in Figure 19 to get

\[
t^{-1}V(L_n(B_3)) - tV(L_{n-2}(B_3)) + (t^{-1/2} - t^{1/2})V(A) = 0,
\]

where \( A \) is the link as shown in Figure 19. We can compute the Jones polynomial of \( A \) using Skein relations multiple times to get

\[
V(A) = t^{33/2} - 3t^{21/2} + 5t^{19/2} - 2t^{17/2} - 11t^{15/2} + 15t^{13/2} - t^6 - 17t^{11/2} + 8t^5 + 20t^{9/2} - 15t^4 - 28t^{7/2} + 12t^5 + 21t^{5/2} - 7t^2 - 8t^{3/2} + 8t - 4t^{1/2} - 7 + 4t^{-1/2} + 4t^{-1} - 3t^{-3/2} - t^{-2} + t^{-5/2}.
\]

As before, consider the function

\[
\Phi : \mathbb{Z}[t^{-1/2}, t^{1/2}] \to \mathbb{Z}[t^{-1/2}, t^{1/2}]
\]

\[
\Phi(p) = t^{-2}p + t^{-1}(t^{-1/2} - t^{1/2})V(A),
\]

where \( V(A) \) is the Jones polynomial of the link \( A \) as given in equation (12). Then \( V(L_n(B_3)) = \Phi(V(L_{n-2}(B_3))) \), for every \( n \geq 2 \). Hence we only need to analyse the function \( \Phi \) in order to understand the Jones polynomials of the links \( L_n \). Notice that the degree of the lowest degree term of \( \Phi(p) \) can be given by

\[
\deg_L(\Phi(p)) = \min \{ \deg_L(p) - 2, \deg_L(V(A)) - 3/2 \}.
\]

We know from our computations that \( \deg_L(V(A)) = -5/2 \) and hence,

\[
\deg_L(\Phi(p)) = \min \{ \deg_L(p) - 2, -4 \}.
\]
For \( n = 2 \), we have that \( V(L_2(B_3)) = \Phi(V(L_0(B_3))) \). But \( L_0(B_3) \) is nothing but the link \( L \). We can compute the Jones polynomial of \( L \), which comes out to be \( 2t^6 + t^4 - t^2 \).

Hence \( \deg_L(V(L_0(B_3))) = 2 \), and \( \deg_L(V(L_2(B_3))) = \deg_L(\Phi(V(L))) = -4 \). Continuing in a similar fashion, we get that \( \deg_L(V(L_4(B_3))) = -6 \), \( \deg_L(V(L_0(B_3))) = -8 \) and so on.

We can generalize this to obtain that \( \deg_L(V(L_n(B_3))) = -n - 2 \) for \( n \geq 2, n \) even. This shows that each of the links \( L_n \) are neither isotopic to \( L \), nor to each other.

Notice that it is crucial that we chose the connect sums \( L_n(B_m) \) to be of the same type, otherwise the link \( A \) might change at every stage of the iteration giving us different Jones polynomials \( V(A) \).

In general, the link \( A \) depends on the link \( L \) and the type of connect sum, and may have a different Jones polynomial. However, we still have that \( \deg_L(V(L_n(B_3))) = \min \{ \deg_L(V(L_{n-2}(B_3))) - 2, \deg_L(V(A)) - 3/2 \} \) for \( n \geq 2 \). Hence, we will get a positive integer \( N \) such that for every even \( n > N \) \( \deg_L(V(L_n(B_3))) = \deg_L(V(L_{n-2}(B_3))) - 2 \), showing that eventually all the Jones polynomials \( V(L_n(B_3)) \) will be distinct and hence the links will be non-isotopic.

6. Milnor invariants

In this section we give a proof of Theorem 1.2. Let \( L \) be an \( m \)-component link in \( S^3 \). Let \( F(L) \) be the fundamental group, \( \pi_1(S^3 - L) \), and \( F_q(L) \) be its \( q \)-th lower central subgroup. We now define Milnor’s \( \mu \)-invariants [5].

We have a presentation of the group \( F(L)/F_q(L) \) with \( m \) generators, \( \alpha_1, \ldots, \alpha_m \), where \( \alpha_i \) represents the \( i \)-th meridian of the link \( L \). For each \( i = 1, \ldots, m \), the longitude \( l_i \) of the \( i \)-th component of \( L \) is represented by a word in \( \alpha_1, \ldots, \alpha_m \).

On substituting \( \alpha_j = 1 + \kappa_j \), and \( \alpha_j^{-1} = 1 - \kappa_j + \kappa_j^2 - \kappa_j^3 + \ldots \) in the expression for \( l_i \), we can represent \( l_i \) by a formal power series in \( \kappa_1, \ldots, \kappa_m \). Let \( \mu(j_1 \ldots j_s) \) denote the coefficient of \( \kappa_{j_1} \ldots \kappa_{j_s} \) in this series, for every \( s \geq 1 \).

Let \( \Delta(i_1 \ldots i_r) \) denote the greatest common divisor of \( \mu(j_1 \ldots j_s) \), where \( j_1 \ldots j_s(2 \leq s < r) \) ranges over all sequences obtained by cancelling at least one of the indices \( i_1 \ldots i_r \), and permuting the remaining indices cyclically. And now \( \overline{\mu}(i_1 \ldots i_r) \) is defined as the residue class of \( \mu(i_1 \ldots i_r) \) modulo \( \Delta(i_1 \ldots i_r) \). These are Milnor’s isotopy invariants of links.

Consider the link obtained from the Brunnian link \( B_3 \) by taking the Whitehead double twice, on the first and last component (Figure 20). We denote this link by \( W_{n,m}^{1,3}(B_3) \), where \( k \) and \( n \) are the number of twists on the first and third components respectively.
By Corollary 4.2 in [3], we know that all the Milnor invariants, \( \mu \), of \( W_{k,n}^{1,3}(B_3) \) are 0. Hence these links satisfy property (4) of Theorem 1.2. However, we can no longer use Milnor invariants to show that the links are not isotopic to unlink. But we can use the Jones polynomials to show existence of links that satisfy all four properties.

It is clear that the links will satisfy properties (1) and (3). We show as before that the Jones polynomial will come out to be non-trivial showing that the links satisfy (2).

On considering the Skein relation at the crossing (A) (Figure 20) we get the Skein relation

\[
t^{-1}V(W_{k,n-2}(B_3)) - tV(W_{k,n}(B_3)) + (t^{-1/2} - t^{1/2})V(L_{k,3}) = 0,
\]

where \( L_{k,3} \) is the link in Figure 21.

If both \( W_{k,n-2}(B_3) \) and \( W_{k,n}(B_3) \) are unlinks then we will have \( V(L_{k,3}) = (-t^{-1/2} - t^{1/2})^3 \).

Now consider the Skein relation at crossing (B) in Figure 21. We get the following relation.

\[
t^{-1}V(L_{k-2,3}) - tV(L_{k,3}) + (t^{-1/2} - t^{1/2})V(L_{3}') = 0,
\]

where \( L_3' \) is the following link.
We compute \( V(L'_3) \) in similar fashion as we computed \( V(L_3) \) in Section 4.2. We first apply Skein relations at the crossings shown in the box (a) in Figure 22 and then on the crossings in box (b).

This gives us the Jones polynomial
\[
V(L'_3) = -t^6 + t^5 + t^3 + 2t^2 + 2t + 6 + 2t^{-1} + 2t^{-2} + t^{-3} + t^{-5} - t^{-6}.
\]

This shows that either \( V(L_{k-2,3}) \) or \( V(L_{k,3}) \) is different from \((-t^{-1/2} - t^{1/2})^3\). This in turn shows that at least one of \( W_{k,n}(B_3) \), \( W_{k,n-2}(B_3) \), \( W_{k-2,n-2}(B_3) \) and \( W_{k-2,n-2}(B_3) \) is isotopically non-trivial, showing the existence of links satisfying (1) – (4) in Theorem 1.2.

We can extend these computations to links \( L'_m \) for \( m > 3 \) to get the recurrence relation
\[
V(L'_m) = V(L'_{m-1})(t^{5/2} - t^{3/2} - t^{-3/2} + t^{-5/2}) - (-t^{1/2} - t^{-1/2})m(t^{5/2} - t^{3/2} - t^{1/2} - t^{-1/2} - t^{-3/2} + t^{-5/2}).
\]

Analysis similar to that in the previous section shows that \( V(L'_m) \) is not equal to \((-t^{-1/2} - t^{1/2})^m\) (the degree of \( V(L'_m) \) being strictly larger than \( m \)), which in turn proves Theorem 1.2 for \( m \)-component links with \( m > 3 \).

7. CONCLUDING REMARKS

7.1. Other invariants. Consider the Seifert matrix of the link \( W_n^-(B_3) \), the Whitehead double of \( B_3 \) with \( n \) negative twists, where \( n \geq 4 \) is even.

\[
S_n = \begin{pmatrix}
& & & 0 \\
& & & \\
& & & \\
& & & \\
0 & \cdots & 0 & -1 & 0 & -1 & \ddots & \\
& & & & 1 & -1 & & \\
\end{pmatrix}_{n \times n},
\]

where \( S \) is nothing but the Seifert matrix for the link \( L_3 \) from Figure 9.

On looking at the form of the matrix, one is tempted to compare the signatures of the two links.

To do this, consider the symmetric bilinear form given by \( A_n = S_n + S_n^T \). The signature of the link \( W_n^- (B_3) \), \( \sigma(W_n^- (B_3)) \), by definition, the signature of \( A_n \).

\[
A_n = \begin{pmatrix}
& & & 0 \\
& & & \\
& & & \\
& & & \\
0 & \cdots & 0 & -1 & 0 & -1 & \ddots & \\
& & & & 1 & -1 & & \\
\end{pmatrix}_{n \times n},
\]

where \( A \) gives the bilinear form corresponding to the link \( L_3 \). (Detailed discussion on Seifert form and signature of links can be found in [2], Chapters 6 and 8.)

Suppose \( A \) defines a bilinear form on the vector space \( V \), and \( A_n \) on \( V \oplus \mathbb{R} \). Let \( w_n \) be the eigenvector for the largest eigenvalue, \( \lambda_n \), of \( A_n \), and \( H_n \) be the subspace orthogonal to \( w_n \). The form of the matrix \( A_n \) allows us to conclude that \( w_n \to (0, \ldots, 0, \lambda) \) and \( \lambda_n \to \infty \), as \( n \to \infty \). This in turn gives us that subspace \( H_n \) approaches \( V \) as \( n \to \infty \). Hence, the form \( A_n|_{H_n} \) approaches \( A_n|_V \). However, \( A_n|_V \) is same as the form \( A \).

This shows that the eigenvalues of \( A_n|_{H_n} \) must also approach those of \( A \). Thus, \textit{as long as none of the eigenvalues of} \( A \) \textit{are zero}, for a large enough value of \( n \), we would obtain \( \sigma(A_n|_{H_n}) = \sigma(A) \),
which would force $\sigma(A_n) = \sigma(A) + 1$. Alternatively, if $W^+_n(B_3)$ is a link having $n$ positive twists with $n \geq 4$ being even, then a similar analysis shows that for large $n$, $\sigma(A_n) = \sigma(A) - 1$.

Thus, for large enough $n$ even, $\sigma(W^+_n(B_3)) = \sigma(L_3) - 1$ and $\sigma(W^-_n(B_3)) = \sigma(L_3) + 1$. Hence at least one of the links, $W^+_n(B_3)$ or $W^-_n(B_3)$, must have signature different from that of the unlink, proving the existence of a link as in Theorem 1.1 for $m = 3$. The same method would work for links $W_n(B_m)$ for $m > 3$ as well.

Unfortunately, the following observation shows that $\det(A) = 0$, so $A$ will have zero as an eigenvalue, showing that the above method will not work for these examples of links.

A coloring (modulo $n$) of a link $L$ is a function $c$ from the set of arcs in its link diagram to $\mathbb{Z}/n\mathbb{Z}$ if at every crossing we have $2c(x_i) \equiv c(x_j) + c(x_k) \mod n$, where $x_i, x_j, x_k$ are arcs at a crossing as shown.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure23.png}
\caption{Figure 23}
\end{figure}

A system of these equations gives us the coloring matrix. Deleting any one column and one row yields a new matrix. We know that $\det(A)$ is equal to the determinant of this new matrix for link $L_m$. And this determinant is 0 if and only if the link has a coloring mod $n$ for every $n \in \mathbb{Z}$, and in particular, if it has a coloring over $\mathbb{Z}$ ([1], Chapter 3).

For every $m \geq 3$, we have the following coloring of $L_m$ over $\mathbb{Z}$, which forces $\det(A) = 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure24.png}
\caption{Figure 24}
\end{figure}

7.2. 3-manifold theory. As per the comments by an anonymous referee, one can use 3-manifold theory to show that the Whitehead double of (any non-split) link is isotopically non-trivial, and in fact, non-split. The complement of a Brunnian link in $S^3$ gives an irreducible 3-manifold with incompressible boundary, and so does the complement of the Whitehead $n$-twists in the solid torus.

Let $W_n$ be the Whitehead $n$ twists in the solid torus $D^2 \times S^1$ (Figure 5). Let $M_1 := S^3 \setminus N(B_m)$ and $M_2 := (D^2 \times S^1) \setminus N(W_n)$, where $N(B_m)$ and $N(W_n)$ denote the tubular neighbourhoods. Let $T^2$ be the torus boundary of the tubular neighborhood of the $m$-th component of $B_m$, $N(B_m^m)$. Then we have $S^3 \setminus N(W_n(B_m)) = M_1 \cup_{T^2} M_2$. Since $M_1$ and $M_2$ are irreducible manifolds, gluing
them along their incompressible boundary gives us another irreducible manifold. This shows that $W_n(B_m)$ cannot be a trivial link, since otherwise we get a 2-sphere that separates $S^3 \setminus N(W_n(B_m))$.

To see that gluing $M_1$ and $M_2$ along their incompressible boundary gives us another irreducible manifold, say $M$, let us first assume that $M$ is not irreducible. Then there must be a copy of $\alpha: S^2 \hookrightarrow M$ which separates $M$. Also, $\alpha(S^2)$ must intersect $T^2$, since otherwise it would lie completely in either $M_1$ or $M_2$, which are irreducible. We may assume that $\alpha$ is transverse to $T^2$. Then $\alpha(S^2) \cap T^2$ is a codimension 1 submanifold in $T^2$. Then $\alpha^{-1}(T^2) \hookrightarrow S^2$ also has codimension 1, and hence must be a union of copies of $S^1$ in $S^2$.

Consider a copy of $S^1$, say $S^1_{\beta}$, in $\alpha(S^2) \cap T^2$ that bounds a disk in $M_1$. Since $T^2$ is incompressible in $M$, it must bound a disk in $T^2$. This gives a copy of $S^2$ in $M_1$. Being the complement of the link $B_m$, $M_1$ is aspherical. Hence the 2-sphere must bound a 2-ball in $M_1$. Then the map $\alpha$ can be perturbed so that $\alpha(S^2) \cap T^2$ no longer contains $S^1_{\beta}$. Doing this for all the copies of $S^1$ in $S^2 \cap T^2$ that bound a disk in $M_1$ will eventually show that there is a copy of $S^2$ in $M_2$ that separates $M_2$ which contradicts the irreducibility of $M_2$.

We note here that a similar 3-manifold theory technique does not help in proving the result for a general link $L$. In the case of a general link, we need to show that the connect sum is not isotopic to the link $L$, which is different from showing that there is a separating 2-sphere in $S^3$. Further, the decomposition of $S^3 \setminus N(L \# W_n(B_m))$ is a bit more complicated than the unlink case and similar techniques don’t work anymore.

We further note that the Jones polynomial methods help us to show that all the links $L \# W_n(B_m)$ are isotopically distinct giving us an infinite family of such links.

References Cited

[1] C. Livingston Knot Theory, Mathematical Association of America, Washington DC, 1993.
[2] W.B.R. Lickorish An Introduction to Knot Theory, Springer-Verlag New York, Inc. 1997.
[3] J.-B. Meilhan, A. Yasuhara Whitehead double and Milnor invariants, Osaka Journal of Mathematics. Vol. 48, No. 2 (2011), pp. 371-381
[4] J. Milnor Link Groups, The Annals of Mathematics, Second Series, Vol. 59, No. 2 (Mar., 1954), pp. 177-195
[5] J. Milnor Isotopy of links, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 280-306. Princeton University Press, Princeton, N. J., 1957.