EQUIVARIANT OPEN GROMOV-WITTEN THEORY OF
\[ \mathbb{RP}^{2m} \hookrightarrow \mathbb{CP}^{2m} \]

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ABSTRACT. We define equivariant open Gromov-Witten invariants for \( \mathbb{RP}^{2m} \hookrightarrow \mathbb{CP}^{2m} \) as sums of integrals of equivariant forms over resolution spaces, which are blowups of products of moduli spaces of stable disc-maps modeled on trees. These invariants encode the quantum deformation of the equivariant cohomology of \( \mathbb{RP}^{2m} \) by holomorphic discs in \( \mathbb{CP}^{2m} \) and, for \( m = 1 \), specialize to give Welschinger’s signed count of real rational planar curves in the non-equivariant limit.

This paper prepares the ground for \cite{18} in which we prove a fixed-point formula computing these invariants.

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1. Introduction

This paper is concerned with equivariant invariants obtained from stable maps of holomorphic discs to \( \mathbb{CP}^2 \) with boundary on the Lagrangian \( \mathbb{RP}^{2m} \). The pair \((X, L) = (\mathbb{CP}^{2m}, \mathbb{RP}^{2m})\) is a real homogeneous variety (see [20, Definition 2], and Example 3 ibid.). This alleviates many of the difficulties involved with the construction of the virtual fundamental chain. In fact, the moduli space of stable disc maps \( \mathcal{M}_{0,k,l}(X, L, \beta) \) is an orbifold with corners, constructed from the moduli of closed genus zero maps, \( \mathcal{M}_{0,k+2l}(X, \beta) \). It carries a relative orientation by a result of Solomon [14]. Roughly speaking, this means \( \mathcal{M}_{0,k,l}(X, L, \beta) \) defines a smooth singular chain \( M \) in \( L^k \times X \).

Of course in general, the chain \( M \) is not a cycle, \( \partial M \neq 0 \), so capping cohomology classes against \( M \) will not produce invariants. In particular, if we try to count disc-map configurations subject to constraints, as we modify the constraints through a cobordism some configurations may “fall off the boundary”, making the count ill-defined.

This type of problem can be overcome in various ways. Open invariants are defined and computed in the works of Katz and Liu [8], Pandharipande, Solomon and Walcher [10], Georgieva and Zinger [5] and Tehrani and Zinger [16].

We will focus on an approach taken by Fukaya [3], making use of the recursive structure of the boundary and extracting invariants from the \( A_\infty \) algebra associated with \( L \hookrightarrow X \). In a similar vein, Solomon and Tukachinsky [15] consider the potential \( \Phi(w) \) of a weak bounding cochain \( w \) for the Fukaya \( A_\infty \) algebra of \( L \), satisfying certain conditions. In case \( \dim L \) is odd and certain obstructions vanish, they prove that such a \( w \) exists and \( \Phi(w) \) is independent of all choices.

The key result in this paper is the construction of a complex \( (\Omega_b, D) \) of extended equivariant forms. These extended forms behave as if they’re equivariant differential forms on a closed manifold equipped with a torus action: there’s an integration map

\[
\int_b : \Omega_b \to \mathbb{R}[\bar{\lambda}]
\]

satisfying Stokes’ theorem, \( \int_b D\omega = 0 \). This allows us to define equivariant open Gromov-Witten invariants by

\[
I(k, \bar{l}, \beta) = \int_b \omega(k, \bar{l}, \beta)
\]

for some \( \omega(k, \bar{l}, \beta) \in \Omega_b \). Stokes’ theorem also plays a key role in [18] where we obtain a fixed-point formula, simplifying considerably the computation of \( \int_b \omega \) for any \( \omega \in \Omega_b \) and specializing to give a closed formula for \( \int_b \omega(k, \bar{l}, \beta) \).

Although seemingly quite different, this integration approach to open Gromov-Witten theory is in fact based on [15]. The open Gromov-Witten invariants \( \int_b \omega(k, \bar{l}, \beta) \) are also the coefficients of the potential \( \Phi(w) \) of a kind of weak bounding cochain \( w \). We can use the homological perturbation lemma (see [19]) to reduce the choices involved in the construction of \( w \) to selecting a single homotopy retraction operator. Moreover, we can interpret \( \Phi(w) \) as encoding the quantum deformation class of the equivariant cohomology of \( \mathbb{RP}^{2m} \). The relationship between the \( A_\infty \) perspective and the integration perspective is explained in [18] §1.6.

\[1\] The last two works study real invariants, but these are closely related to disc invariants using the reflection principle, cf. Remark [3].
It is worth noting that for \( m = 1 \), the \( (\mathbb{CP}^2, \mathbb{RP}^2) \) invariants we define are an equivariant extension of Welschinger’s signed count \([17]\) of real rational planar curves passing through \( k \) points in \( \mathbb{RP}^2 \) and \( l \) conjugation-invariant pairs of points in \( \mathbb{CP}^2 \), see Remark \([4]\).

Let us now provide a more detailed overview of the contents of this paper.

1.1. **Resolutions and Stokes’ theorem.** Fix a positive integer \( m \) and let \((X, L) = (\mathbb{CP}^{2m}, \mathbb{RP}^{2m})\). Let \( T = U(1)^m \) denote the rank \( m \) torus group. Let \( b = (k, l, \beta) \) be a 3-tuple of non-negative integers such that \( k + 2l + 3\beta \geq 3 \) and \( k + \beta = 1 \mod 2 \), and denote

\[
\mathcal{T}_b^r = T \times \text{Sym}(k) \times \text{Sym}(l) \times \text{Sym}(r),
\]

where \( \text{Sym}(n) \) is the symmetric group on \( n \) elements. In Sections \([2]\) and \([3]\) we construct, for \( r \geq 0 \) a diagram of \( T_b \)-orbi-folds with corners

\[
\mathcal{M}_b^r \xrightarrow{\text{bu}_b^r} \mathcal{M}_b^r \xrightarrow{\text{For}_b^r} \mathcal{M}_b^r,
\]

where \( \text{For}_b^r \) is a kind of forgetful map and \( \text{bu}_b^r \) is obtained from a spherical blow up construction. Let us first explain the construction of \( \mathcal{M}_b^r \) and the forgetful map. We set \( \mathcal{M}_b^0 = \mathcal{M}_{0,k,l}^b(X, L, \beta) \). We construct \( \mathcal{M}_b^1 \) from \( \mathcal{M}_b^0 \) by replacing some of the fiber products in

\[
\partial \mathcal{M}_b^0 = \bigsqcup_{i=1}^{16} \mathcal{M}_{0,k_1,l_1}^b(X, L, \beta_i) \times L \mathcal{M}_{0,k_2,l_2}^b(X, L, \beta_2)
\]

by products. This can be thought of as a kind of resolution, allowing the two components of the stable disc to move independently of one another. By Leibniz’ rule, \( \partial \mathcal{M}_b^1 \) is also a disjoint union of fibered products, and \( \mathcal{M}_b^2 \) is obtained by replacing some of these fiber products by products. The spaces \( \mathcal{M}_b^r \) for \( r \geq 3 \) are defined in a similar, recursive, fashion, so that \( \mathcal{M}_b^r \) resolves a clopen component of \( \partial^r \mathcal{M}_{0,k,l}(X, L, \beta) \), see \([11]\), and is a disjoint union of products of moduli spaces indexed by certain labeled trees (see Lemma \([15]\)). The group \( T_b \) acts by translating maps and relabeling markings, making \( \partial^r \mathcal{M}_{0,k,l}(X, L, \beta) \to \mathcal{M}_b^r \) a \( \text{Sym}(r) \)-equivariant map. The spaces \( \mathcal{M}_b^r \) are obtained from \( \mathcal{M}_b^0 \) by forgetting one of each of the \( r \) pairs of marked points corresponding to the nodes. We denote the forgetful map by \( \text{For}_b^r : \mathcal{M}_b^r \to \mathcal{M}_b^r \).

Let us now sketch the blow up construction, \( \mathcal{M}_b^r \to \mathcal{M}_b^r \). Let \( \text{bu}_\Delta : \Delta \times L \to L \times L \) denote the spherical blow up of \( L \times L \) along the diagonal \( \Delta \subset L \times L \) (cf. \([8, 2]\)). Evaluation at the node markings produces a map \( \text{ev}_b^r : \mathcal{M}_b^r \to (L \times L)^r \), which is transverse to \( \Delta^r \subset (L \times L)^r \), so we can define the blow up of \( \mathcal{M}_b^r \) using the cartesian square

\[
\begin{array}{ccc}
\mathcal{M}_b^r & \xrightarrow{\text{bu}_b^r} & \mathcal{M}_b^r \\
\text{ev}_b^r \downarrow & & \text{ev}_b^r \downarrow \\
(L \times L)^r & \xrightarrow{\text{bu}_\Delta} & (L \times L)^r
\end{array}
\]

In Definition\([50]\) we define an extended form \( \omega \) for \( b \) as a collection of \( T \)-equivariant forms

\[
\omega = \left\{ \tilde{\omega}_r \in \Omega^*(\mathcal{M}_b^r; \mathcal{E}_b^r \otimes_\mathbb{R} \mathbb{R} \langle \lambda \rangle)^{\mathbb{C}^*} \right\}_{r \geq 0},
\]

for \( \mathcal{E}_b^r := \bigotimes_{i=1}^{k} (\text{ev}_b^r)_i^* \text{Or}(TL) \).
satisfying a boundary compatibility condition and Sym(r)-invariance. The set of extended forms is a complex, \((\Omega_b, D)\), with \(D\) acting level-wisely by the Cartan-Weil differential \(D = d - \sum_{j=1}^{m} \lambda_j I_{\xi_j}\). Here \(\xi_j\) are vector fields generating the \(T\)-action and \(I_{\xi_j}\) denotes contraction with \(\xi_j\). In Definition 34 we introduce an \(\mathbb{R} \{\bar{\lambda}\}\)-linear, Sym(k)×Sym(l)-invariant integration map

\[
\int_b : \Omega_b \to \mathbb{R} \{\bar{\lambda}\},
\]

by the finite sum of integrals

\[
\int_b \omega = \sum_{r \geq 0} \frac{1}{r!} \int_{\bar{\mathcal{M}}^r_b} (\text{For}^r \circ \text{b} \mathcal{H}_b)^* \bar{\omega}_r \cdot \bar{\mathcal{E}}^{B^r}_b \Lambda^{B^r}.
\]

Here \(\Lambda \in \Omega \{(\bar{L} \times L; \bar{\mathfrak{p}}_2) (\text{Or} (TL)) \otimes \mathbb{R} \{\bar{\lambda}\}\}^T\) is an equivariant homotopy kernel, see 3.5. The integrals are defined using local system isomorphisms

\[
\tilde{F}^r_b : \text{Or} (T\bar{\mathcal{M}}^r_b) \to \bar{\mathcal{E}}^{\ast \beta \circ \tilde{\mathcal{H}}^b} \otimes \left((\bar{\mathfrak{p}}_2)^{\ast \beta \circ \tilde{\mathcal{H}}^b}\right)^{-1} \text{Or} (TL)^{\otimes r},
\]

see (72).

Clearly, for \(r = 0\) we have \(\bar{\mathcal{M}}^0_b = M^0_b = \bar{M}^0_b\), so the first summand in (1) is just

\[
\int_{\bar{M}^0_{0, k, l}(X, L, \beta)} \omega_0.
\]

One may think of the summands for \(r \geq 1\) as corrections accounting for the boundary and corners of \(\bar{M}^0_{0, k, l}(X, L, \beta)\).

A central result of this paper is Stokes’ theorem, Theorem 31, which states that

\[
\int_b Dv = 0
\]

for any \(v \in \Omega_b\). In the next subsection we will use this version of Stokes’ theorem to define equivariant open Gromov-Witten invariants. Equation (3) also plays an important role in the proof of the fixed-point formula in 15.

1.2. Equivariant open Gromov-Witten invariants. For any pair of non-negative integers \(k, \beta\) and tuple of non-negative integers \(\bar{l} = (l_0, \ldots, l_{2m}) \in \mathbb{Z}_{\geq 0}^{2m+1}\), the equivariant open Gromov-Witten invariant

\[
I(k, \bar{l}, \beta) \in \mathbb{R} \{\bar{\lambda}\}
\]

is defined as follows. Let \(l = l_0 + \cdots + l_{2m}\). If \(k + 2l + 3\beta < 3\) or \(k + \beta = 0 \mod 2\), we set \(I(k, \bar{l}, \beta) = 0\), so suppose this is not the case.

We define

\[
I(k, \bar{l}, \beta) = \int_b \omega,
\]

where the extended form \(\omega\) is constructed as follows. Fix an \(l\)-tuple \((d_1, \ldots, d_l) \in \{0, \ldots, 2m\}^l\) so that for all \(0 \leq d \leq 2m\) we have

\[
\# \{1 \leq i \leq l | d_i = d\} = l_d.
\]

We define an extended form \(\omega = \{\bar{\omega}_r\} \in \Omega_b\) by

\[
\bar{\omega}_r = \prod_{1 \leq j \leq l} (\text{ev}_{d_j}^r)^* \eta^{d_j} \cdot \prod_{1 \leq k \leq k} (\text{ev}_{b_k}^r)^* \rho_0.
\]
where \( \rho_0 \in \Omega \left( L; \text{Or} \left( TL \right) \otimes \mathbb{R} \left[ \lambda \right] \right)^T \) is a \( D \)-closed form Poincaré dual to the (unique) fixed point \( p_0 \in L \), and \( \eta \in \Omega \left( X; \mathbb{R} \left[ \lambda \right] \right)^T \) is a \( D \)-closed form representing the equivariant hyperplane class \( H \) (cf. \[18\]). Clearly, \( D\omega = 0 \), and by \[3\], this is independent of the representatives \( \eta \) and \( \rho \). Independence of the tuple \((d_1, \ldots, d_i)\) representing \( l \) follows from \( \text{Sym}(l) \)-equivariance of the evaluation maps.

**Remark 1.** The same argument shows that, in fact,

\[
I(k, \bar{l}, \beta) = \int_b \prod_{1 \leq j \leq l} \left( \text{ev}_j^* \right)^* \eta_j \cdot \prod_{1 \leq i \leq k} \left( \text{evb}_i^* \right)^* \rho_i
\]

for any \( \{ \rho_i \}, \{ \eta_j \} \) with \( [\rho_i] = \text{pt} \) for \( 1 \leq i \leq k \) and \( [\eta_j] = H^{d_j} \) for \( 1 \leq j \leq l \).

**Remark 2.** \[18\] Theorem 27] can be restated as saying that for every \( \omega \in \Omega_b \) with \( D\omega = 0 \) we have

\[
\int_b \omega = \sum_{r,C} \text{Cont}_C (\omega)
\]

where for \( r \geq 0 \), \( C \subset E^T \) ranges over the connected components of \( E^T \), the \( \mathbb{T} \)-fixed-points of a clopen component (or disjoint union of connected components) \( E \in \mathcal{M}_b \), and where \( \text{Cont}_C (\omega) \) is given by a certain integral over \( C \) which depends only on the cohomology class of \( \omega \in \Omega_b \). In this sense, the invariants can be refined. Moreover, the explicit form of \( \text{Cont}_C (\omega) \) shows that it does not depend on the choice of equivariant homotopy kernel \( \Lambda \). It may be useful to find a direct proof of the invariance of \( \int_b \omega \) on the choice of \( \Lambda \) that does not require the fixed-point localization argument.

We have

\[
\deg I(k, \bar{l}, \beta) = \deg \omega - \dim \mathcal{M}_{0,k,l} (\beta) = \\
= \sum_{j=0}^{2m} (2j - 1) l_j + (2m - 1) \cdot k - (2m + 1) (\beta + 1) + 4.
\]

**Remark 3.** In Remark \[32\] we note that the same construction can be used to define non-equivariant extended forms and their integrals, which satisfy \( \int d\upsilon = 0 \). In particular, open Gromov-Witten invariants with \( \deg I(k, \bar{l}, \beta) = 0 \) can be computed using any \( d \)-closed forms \( \eta_i, \rho_j \) representing their cohomology classes. This can be used to offer a more geometric interpretation of the invariants, as follows.

Choose for \( 1 \leq i \leq l \) a submanifold \( W_i \subset X \) Poincaré dual to \( H^{d_i} \) and \( k \) points \( \{z_j\}_{j=1}^{k} \subset L \) such that \( (\text{evb}^* \times \text{ev}_i^*) \circ \text{For}_b^* \circ \text{bu}_b^* \) is \( b \)-transverse to \( W_1 \times \cdots \times W_i \times \{z_1\} \times \cdots \times \{z_k\} \subset X^l \times L^k \) for all \( r \geq 0 \), so the inverse image \( W^r \subset \mathcal{M}_b^r \) is an embedded suborbifold. We can then take \( \eta_i \) and \( \rho_j \) in \( \[4\] \) to be Thom forms for \( W_i, z_j \). Shrinking the support of these forms we obtain, in the limit,

\[
I(k, \bar{l}, \beta) = \sum_{r \geq 0} \frac{1}{r!} \int_{W^r} \left( \text{evb}^* \right)^* \Lambda^{\text{b}r}.
\]

In particular, the \( r = 0 \) term is just \( \int_{W^0} 1 \): a signed, isotropy-weighted count of the discs satisfying the constraints.

---

\[2\] See Definition \[17\] This means that \( (\text{evb}^* \times \text{ev}_i^*) \circ \text{For}_b^* \circ \text{bu}_b^* \circ \partial^x_{\mathcal{M}_b^r} \) is transverse to \( W_1 \times \cdots \times W_i \times \{z_1\} \times \cdots \times \{z_k\} \) for all \( c \geq 0 \), where \( \partial^x_{\mathcal{M}_b^r} : \partial^x \mathcal{M}_b^r \to \mathcal{M}_b^r \) is the structure map.
Remark 4. When $m = 1$, the invariants with deg $I(k, l, \beta) = 0$ for $(\mathbb{CP}^2, \mathbb{RP}^2)$ overlap with those defined in [14]. Indeed, the sign-of-conjugation argument used there to cancel the contributions of the boundary, can be used to show that the $r > 0$ terms in (1) vanish and $I(k, l, \beta)$ reduces to a disc count (see Remark 3). As in [14], Schwarz reflection associates to every stable disc-map a real rational planar curve passing through $k$ real points and $l$ conjugation-invariant pairs of complex points, and this can be used to identify disc counts with twice Welschinger’s signed count of such curves [17]. This was verified with a computer for all invariants of degree $\leq 6$, using the fixed-point formula [18].

Remark 5. Invariants with deg $I(k, l, \beta) < 0$ must vanish, so the fixed-point formula [18] produces non-trivial relations involving descendent integrals on discs. See also Remark 29 for a simple example of this.

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2. Resolutions of Moduli Spaces

Throughout the paper, $m$ will be some fixed positive integer, and we will consider the open Gromov-Witten theory of $(X, L) = (\mathbb{CP}^{2m}, \mathbb{RP}^{2m})$.

2.1. The pair $(\mathbb{CP}^{2m}, \mathbb{RP}^{2m})$, torus actions and moduli spaces. We begin by introducing notation and reviewing some results. We assume the reader is familiar with the notion of a real homogeneous pair [20, Definition 1].

Let $G_X = U(2m + 1)$ denote the unitary group, with $T^C = U(1)^{2m+1} < U(2m + 1)$ the subgroup of diagonal matrices. Projectivizing the standard action on $V = \mathbb{C}^{2m+1}$ defines a transitive group action

$$\alpha_{U,X} : U(2m + 1) \times X \to X.$$

Restricting to $T^C$ and decomposing $V = V_0 \oplus \cdots \oplus V_{2m}$ into complex irreducible representations we find

$$X = \mathbb{CP}^{2m} = \mathbb{CP}(V_0 \oplus V_1 \oplus \cdots \oplus V_{2m}).$$

where $(u_0, ..., u_{2m}) \in T^C$ acts on $V_i$ by $z \mapsto u_i \cdot z$.

$T^C$-equivariant cohomology is defined over the ring

$$H^*_{T^C} = H^*_{T^C}(pt) = \mathbb{R}[\alpha_0, ..., \alpha_{2m}], \quad \text{deg} \alpha_i = 2.$$

The total space of the tautological line bundle $\tau$ on $\mathbb{CP}(V)$ is the blow up of $V$, so there is a natural lift of the $T^C$ action to $\tau$. Let $H = -c_1^{\mathbb{CP}}(\tau) \in H^2_{T^C}(X)$ denote minus the equivariant Chern class of $\tau$. $H$ is an equivariant extension of the hyperplane class. We have

$$H^*_{T^C}(X) = \mathbb{R}[H, \alpha_0, ..., \alpha_{2m}] / \left( \prod_{i=0}^{2m} (H - \alpha_i) \right).$$

In particular, $H^*_{T^C}(X)$ is generated as an $H^*_{T^C}$-module by $H^0, H^1, ..., H^{2m}$. 

We now introduce a conjugation action. Let \( \gamma : V \to V \) be the involution given by
\[
(7) \quad (z_0, \ldots, z_{2m}) \mapsto (\overline{z}_0, \overline{z}_{2m}, \overline{z}_{2m-1}, \ldots, \overline{z}_1),
\]
let \( c_X : X \to X \) denote its projectivization, and let \( c_G : U (2m + 1) \to U (2m + 1) \) be defined by
\[
(8) \quad g \mapsto c_V \circ g \circ c_V^{-1}.
\]
Clearly \( c_G (\mathbb{T}^C) = \mathbb{T}^C \) and \( O (2m + 1) \) and \( \mathbb{T} \) are the \( c_G \)-fixed subgroups of \( U (2m + 1) \) and \( \mathbb{T}^C \), respectively, where \( \mathbb{T} \cong U (1)^m \) is the image of
\[
U (1)^m \ni (u_1, \ldots, u_m) \mapsto \text{diag} (1, u_1, \ldots, u_m, \overline{u}_m, \ldots, \overline{u}_1) \in \mathbb{T}^C < U (2m + 1).
\]
The monomorphism \( \mathbb{T} \to \mathbb{T}^C \) corresponds to the map
\[
(9) \quad H_{c_G}^* = \mathbb{R} [\alpha_0, \ldots, \alpha_{2m}] \xrightarrow{\rho_T} H_T^* = \mathbb{R} [\lambda_1, \ldots, \lambda_m]
\]
given by \( \alpha_0 \to 0 \) and, for \( 1 \leq i \leq m \), \( \alpha_i \to \lambda_i \) and \( \alpha_{2m+1-i} \to -\lambda_i \). Let \( W_0 = \mathbb{R} \) denote the trivial representation of \( \mathbb{T} \), and for \( 1 \leq i \leq m \) let \( W_i = \mathbb{R}^2 \) denote the real representations of \( \mathbb{T} \), on which \( (e^{-\sqrt{-1} t_1}, \ldots, e^{-\sqrt{-1} t_m}) \) acts by
\[
\begin{pmatrix}
\cos t_i & -\sin t_i \\
\sin t_i & \cos t_i
\end{pmatrix}.
\]
As \( \mathbb{Z}/2 \times \mathbb{T} \) representations we have
\[
V_0 = \mathbb{C} \otimes W_0
\]
and for \( 1 \leq i \leq m \),
\[
V_i \oplus V_{2m+1-i} = \mathbb{C} \otimes W_i,
\]
where the \( \mathbb{Z}/2 \) action is the usual conjugation action on the \( \mathbb{C} \) factor.

We find that
\[
(X, \omega, J, G_X = U (2m + 1), \alpha_{U,X}, c_G, c_X)
\]
is a real homogeneous variety (cf. [20] Definition 1]), where \( \omega \) is the Fubini-Study symplectic form and \( J \) is the standard complex structure. Let \( (X, L = \mathbb{R}^{2m}) \) be the associated real homogeneous pair. As usual, the induced action of \( G_X^{\mathbb{Z}/2} = O (2m + 1) \) on \( L \) is transitive. It restricts to a \( \mathbb{T} \)-action on \( L \) specified by the equivariant identification
\[
L = \mathbb{P}_R \left( W_0 \oplus W_1 \oplus \cdots \oplus W_m \right).
\]
Recall the map
\[
(10) \quad H_2 (X, L) \to H_2 (X)
\]
sends a singular chain \( \sigma \in C_2 (X) \) with \( \partial \sigma \in C_1 (L) \) to the class represented by the cycle \( c_X \ast \sigma + \sigma \). We identify \( H_2 (X) = \mathbb{Z} \) using the complex structure and fix an isomorphism \( H_2 (X, L) \cong \mathbb{Z} \), so that \( H_2 (X) \to H_2 (X, L) \) corresponds to multiplication by \( +2 \). The map \( \text{Id}_\mathbb{Z} \) then becomes \( \text{Id}_\mathbb{Z} \). We see that an integer \( \beta \in \mathbb{Z} \) can either represent an element of \( H_2 (X, L) \) or its image in \( H_2 (X) \) under \( \text{Id}_\mathbb{Z} \), depending on the context.

Let \((k, l, \beta)\) be non-negative integers with \( k + 2l + 3\beta \geq 3 \), and write
\[
G_{k,l} = O (2m + 1) \times \text{Sym} (k) \times \text{Sym} (l)
\]
for the product of the orthogonal group with the symmetric groups on $k$ and $l$
eq0 elements. By [20, Theorem 2] the moduli space
\[ \mathcal{M}_{0,k,l}(X, L, \beta) \]
is a $G_{k,l}$-orbifold with corners admitting a $G_{k,l}$-equivariant map to the moduli space $\mathcal{M}_{0,k+2l}(X, \beta)$, and there's a $G_{k,l}$-equivariant evaluation map
\[ \mathcal{M}_{0,k,l}(X, L, \beta) \to L^k \times X^l \]
induced from the closed evaluation map $\mathcal{M}_{0,k+2l}(\beta) \to X^{k+2l}$.

We will often use general sets as labels. For instance, we may consider the moduli spaces $\mathcal{M}_{0,k,l}(\beta)$ where $k,l$ are finite disjoint sets.

2.2. Moduli specifications and an overview. Let $\mathbb{N} = \{1, 2, \ldots\}$. For any $S \subseteq \mathbb{N}$, we denote $s_S' = \{s_i^l\}_{i \in S}$ and similarly for $s_S''$, $s_S'''$. Hollow stars will be used to denote markings related to boundary nodes, and solid stars will be used for interior nodes.

Definition 6. A \emph{pre-moduli specification} $b$ is a 3-tuple $(k, l, \beta)$ where
- $k \in \mathbb{N} \bigcup s_{n}''$ and $l \in \mathbb{N}$ are finite subsets. Elements of $k$ and of $l$ are called \emph{orienting labels} and \emph{interior labels}, respectively.
- $\beta$ is a non-negative integer, \emph{the degree}, which we think of as an element of $H_2(X, L)$.

A \emph{basic moduli specification} is a pre-moduli specification $b = (k, l, \beta)$ that is
- \emph{stable}, meaning $k + 2l + 3\beta \geq 3$; henceforth we use standard Roman letters to denote the sizes of sets labeled by the corresponding Serif letters, so $k = |k|$ and $l = |l|$.
- \emph{orientable}, meaning
\[ k + \beta = 1 \mod 2. \]

A \emph{moduli specification} $s$ is a pair $s = (b, \sigma)$ where $b = (k, l, \beta)$ is an orientable pre-moduli specification and $\sigma \subseteq s_{n}''$ is a finite subset such that $k + |\sigma| + 2l + 3\beta \geq 3$. We call $\sigma$ the \emph{superfluous} (boundary) labels, and $k = k \bigcup \sigma$ the \emph{boundary labels}.

A moduli specification $s = (b, \sigma)$ is called \emph{sturdy} if $b$ is stable and \emph{wobbly} otherwise.

Let $s = (b, \sigma)$ be a moduli specification. If $s$ is sturdy, then $b$ is a basic moduli specification; if it is wobbly, it is necessarily of the form
\[ ((k, \emptyset, 0), \sigma) \] with $|k|=1$ and $|\sigma| \geq 2$.

Either way, the \emph{combined moduli specification} $\bar{s} := (\bar{k}, l, \beta)$ is a basic moduli specification.

A 3-tuple of non-negative integers $b = (k, l, \beta)$ with $k + 2l + 3\beta \geq 3$ and $k + \beta = 1 \mod 2$ may be used in place of a basic moduli specification, taking $b = ([k], [l], \beta)$ where we denote $[k] = \{1, 2, \ldots, k\}$.

The following proposition collects the main properties of resolutions that are proved in this section, and which we will need when we discuss extended forms and integration in Section 3. We refer the reader to the Appendix, §4, for the definition of the category of orbifolds with corners and related notions.

Let $b = (k, l, \beta)$ be a basic moduli specification. Let $T_b = T \times \text{Sym}(k) \times \text{Sym}(l)$ and $T'_b = T_b \times \text{Sym}(r)$. 

Proposition 7. For every non-negative integer \( r \geq 0 \) (the number of resolved nodes) and subset \( S \subset \mathbb{N} \) (the subset of node tails we do not forget) there exist

- \( T_b^r \)-orbifolds with corners \( \hat{\mathcal{M}}_b^{r,S} \). We set \( \check{\mathcal{M}}_b^r = \hat{\mathcal{M}}_b^{r,\emptyset} \) and \( \mathcal{M}_b^r = \hat{\mathcal{M}}_b^{r,N} \).
- \( T_b^r \)-equivariant b-fibrations (the forgetful maps)

\[
\begin{align*}
\text{For}_b^{r,S} &: \mathcal{M}_b^r \to \hat{\mathcal{M}}_b^{r,S} \\
\text{For}_b^r &: \hat{\mathcal{M}}_b^{r,S} \to \hat{\mathcal{M}}_b^r.
\end{align*}
\]

We set

\[
\text{For}_b^r = \text{For}_b^{r,\emptyset} : \mathcal{M}_b^r \to \hat{\mathcal{M}}_b^r.
\]

For \( b \) induces a \( T_b^r \)-equivariant decomposition of the boundary, denoted

\[
\partial \mathcal{M}_b^r = \partial_+ \mathcal{M}_b^r \coprod \partial_- \mathcal{M}_b^r,
\]

see \([4]\),

- boundary evaluation maps

\[
e_{\beta}^{b,r,S} : \hat{\mathcal{M}}_b^{r,S} \to L \quad \text{for} \quad x \in k \coprod \{ \beta' \}_{i=1}^r \coprod \{ \beta_i \}_{i=\alpha[r] \cap S}, \quad \text{and}
\]

- interior evaluation maps

\[
e_{\beta}^{b,r,S} : \hat{\mathcal{M}}_b^{r,S} \to X \quad \text{for} \quad x \in l.
\]

- a \( T_b^r \)-equivariant involution \( \text{inv}_b^r \) \( \partial_+ \mathcal{M}_b^r \to \partial_- \mathcal{M}_b^r \), and

- local system maps

\[
\mathcal{F}_b^r : \text{Or}(T\mathcal{M}_b^r) \to \text{Or}(T\hat{\mathcal{M}}_b^r)
\]

lying over \( \text{For}_b^r \) and

\[
\check{\mathcal{J}}_b^r : \text{Or}(T\hat{\mathcal{M}}_b^r) \to \text{Or}(TL)^{\#(k+r)}
\]

lying over \( \prod_{x \in k} \{ \beta_i \}_{i=1}^r \cdot e_{\beta}^{b,r} : \hat{\mathcal{M}}_b^r \to L^{k+r} \). We set

\[
\check{\mathcal{J}}_b^r := \check{\mathcal{J}}_b^r \circ \mathcal{F}_b^r.
\]

These satisfy the following properties.

(a) \( \mathcal{M}_b^0 = \overline{\mathcal{M}}_{0,k,1}(\beta) \) and \( \mathcal{M}_b^r = \emptyset \) for sufficiently large \( r \).
(b) there’s a cartesian square

\[
\begin{array}{ccc}
\partial_- \mathcal{M}_b^r & \xrightarrow{\partial_- \mathcal{M}_b^r + 1} & \mathcal{M}_b^{r+1} \\
\downarrow & & \downarrow \\
L & \xrightarrow{\Delta_L} & L \times L
\end{array}
\]

where the right and bottom maps are b-transverse (see Definition [47]). This induces a local system map

\[
\mathcal{G}_b^{r+1} : \text{Or}(T\partial_- \mathcal{M}_b^r) \to \text{Or}(T\mathcal{M}_b^{r+1}) \ominus \left( e_{\beta}^{b,r+1} \right)^{-1} \text{Or}(TL)
\]

lying over \( g_b^{r+1} \).

(c)

\[
\check{\mathcal{J}}_b^r \circ \left( \mathcal{I}_{\mathcal{M}_b^r} \circ \partial_- \mathcal{M}_b^r \right) \circ \text{Or}(d \text{inv}_b^r) = (-1) \check{\mathcal{J}}_b^r \circ \left( \mathcal{I}_{\mathcal{M}_b^r} \circ \partial_- \mathcal{M}_b^r \right)
\]
(d) For all \( x \in k \cup \{ \ast' \} \) we have
\[
ev^b_\ast x \circ \partial^r \circ \ev^b_\ast = \ev^b_\ast \circ \partial^r \circ \ev^b_\ast
\]
(e) Let \( \mathcal{J}_0 \) be the local system map derived from \( \mathcal{J}_0 \) be the local system map derived from \( \mathcal{J}_0 \) be the local system map derived from \( \mathcal{J}_0 \) be the local system map derived from \( \mathcal{J}_0 \) be the local system map derived from \( \mathcal{J}_0 \). Then
\[
(\mathcal{J}_0) = \mathcal{J}_0 \circ \partial^r = \mathcal{J}_0 \circ \partial^r \circ \partial^r.
\]
(f) The maps
\[
\prod_{x \in k \cup \{ \ast' \}} \ev^x : \mathcal{M}_b^r \to L^k \times X^r
\]
and
\[
\prod_{x \in k \cup \{ \ast' \}} \ev^x : \mathcal{M}_b^r \to L^k \times X^r
\]
are \( T^r \)-equivariant (see Remark 3). In both cases, \( T \) acts diagonally on the codomain and \( \text{Sym}(k) \times \text{Sym}(l) \times \text{Sym}(r) \) acts by shuffling factors.

(g) \( \mathcal{J}_0 \) is \( T \times \text{Sym}(k) \times \text{Sym}(l) \times \text{Sym}(r) \) invariant, but the action of \( \tau \in \text{Sym}(r) \times T^r \) involves a sign: if we let \( \tau : \mathcal{M}_b^r \to \mathcal{M}_b^r \) denote the diffeomorphism induced by the action of \( \tau \) then
\[
\mathcal{J}_0 \circ \partial \cdot \mathcal{M}_b^r = \text{sgn} (\tau) \cdot \mathcal{J}_0,
\]
where \( \text{sgn} (\tau) \in \{ \pm 1 \} \) is the sign of \( \tau \).

(h) We have \( \ev^b_\ast \circ \partial^r \circ \ev^b_\ast = \ev^b_\ast \circ \partial^r \circ \ev^b_\ast \) and \( \ev^b_\ast \circ \partial^r \circ \ev^b_\ast = \ev^b_\ast \circ \partial^r \circ \ev^b_\ast \) whenever both sides are defined.

Remark 8. Let \( X \) be an orbifold and \( M \) be a manifold. The maps \( X \to M \) are equivalent to a set, so it makes sense to say that two maps \( f_1, f_2 : X \to M \) are equal. If \( X, M \) are equipped with an action of a group \( H \), then the \( H \)-equivariant maps form a subset of the set of all maps, so we may treat \( H \)-equivariance as a property (in contrast, if \( Y \) is a general \( H \)-orbifold the forgetful functor between \( H \)-equivariant maps \( X \to Y \) and ordinary maps is not full and faithful). Similar remarks hold for maps of local systems whose codomain is a local system over a manifold.

2.3. Wobbly boundary involution.

2.3.1. Boundary decomposition by a b-normal map. Following Joyce [6], we can use a b-normal map \( X \to Y \) to decompose the boundary \( \partial X \) into a horizontal \( \partial^b X \) and a vertical \( \partial^v X \) part.

As we discuss in the appendix, if \( X \to Y \) is a smooth map of orbifolds with corners, we have an induced interior map of l-orbifolds
\[
C (X) = \bigcup_{k \geq 0} C_k (X) \overset{C(f)}{\longrightarrow} C (Y) = \bigcup_{l \geq 0} C_l (Y).
\]
It follows from [6 Proposition 2.11] that a smooth map \( f : X \to Y \) is b-normal if and only if
\[
C (X) \cap C_l (Y) \subset \bigcup_{r \geq 0} C_r (Y)
\]
for all \( r \geq 0 \).
Proof. This follows from maps. Then so the boundary of $X$ is a disjoint union

$$\partial X = C_1(X) = \partial f X \coprod \partial f Y,$$

and $C(f)|_{C_1(X)}$ is given by the b-normal maps

$$f_+ : \partial f X \to Y$$

and

$$f_- : \partial f X \to \partial Y.$$

**Lemma 10.** Let $h : X \to Z$ denote the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of two b-normal maps. Then

(a) $h$ is b-normal.

(b) we have

$$\partial^h X = \partial f X \coprod f^{-1} (\partial^g Y)$$

and

$$\partial^h X = f^{-1} (\partial^f X).$$

(c) we have

$$h_- = g_- \circ f_- \quad \text{and} \quad h_+ |_{\partial f X} = g \circ f_+ \quad \text{and} \quad h_+ |_{f^{-1}(\partial^g Y)} = (g_+ \circ f_-).$$

**Proof.** This follows from $C(h) = C(g) \circ C(f)$ (see [6, Definition 2.10]).

More generally, given a chain

$$\xymatrix{X_r & \cdots & X_j & \cdots & X_0 \ar[l]_{f_{j+1}} \ar[l]_{f_{j}} \ar[l]_{f_{j-1}} \ar[l]_{f_{j-2}} \ar[l]_{f_{j-3}} \ar[l]_{f_{j-4}} \ar[l]_{f_{j-5}} \ar[l]_{f_{j-6}} \ar[l]_{f_{j-7}} \ar[l]_{f_{j-8}} \ar[l]_{f_{j-9}} \ar[l]_{f_{j-10}}}$$

of b-normal maps we define $f_{sj} : X_r \to X_j$ and $f_{sj} : X_j \to X_0$ for $j = 1, \ldots, r$ by the indicated compositions, and we have

$$\partial^{f_{sr}} X_r = \coprod_{j=1}^{r} \partial f_{sj} X_r = \coprod_{j=1}^{r} (f_{sj})^{-1} \left( \partial f_{j} X_j \right).$$

2.3.2. The forgetful map boundary decomposition. For a sturdy moduli specification $s = (b, \sigma) = ((k, l, \beta), \sigma)$ and $S \subset \mathbb{N}$, we define

$$\tilde{\mathcal{M}}_S^S = \mathcal{M}(k \cup (\sigma \cap s_S^S), l, \beta).$$

We will mostly be interested in $\mathcal{M}_S : = \tilde{\mathcal{M}}_S^N$ and $\tilde{\mathcal{M}}_S : = \tilde{\mathcal{M}}_S^O$.

Let $\text{For}^S_\sigma : \mathcal{M}_S \to \tilde{\mathcal{M}}_S$ be the map that forgets the markings $s_\sigma^S$. For any $S \subset \mathbb{N}$ we have a decomposition

$$\xymatrix{\mathcal{M}_S \ar[r]^<<<<<<<<<{\text{For}^S_\sigma} & \tilde{\mathcal{M}}_S^S \ar[r]^<<<<<<<<<{\text{For}^S_\sigma} & \tilde{\mathcal{M}}_S}$$

where $\text{For}^S_\sigma$ (respectively, $\tilde{\text{For}}^S_\sigma$) is the map that forgets the markings $\sigma \setminus s_\sigma^S$ (resp. $\sigma \cap s_\sigma^S$).

**Lemma 11.** The maps $\text{For}^S_\sigma, \tilde{\text{For}}^S_\sigma, \text{For}^S_\sigma$ are well-defined b-fibrations.
Proof. [20] Lemma 8] says that the map forgetting a single boundary marked point is a b-fibration, whenever its codomain does not contain any E-type nodes. Since each of \( E\), \( E_f \), \( E^S \) can be written as a composition of maps forgetting a single boundary marking, and b-fibrations are closed under composition, we reduce to showing that for any moduli specification \( s = ((k, l, \beta), \sigma) \) and set \( S \) the configuration parameters parameterized by the space \( \hat{M}^S_0 \) do not have any E-type nodes.

To show this, note that the map \( D := \text{id}_{H_2(X)} + (c_X)_* : H_2(X) \to H_2(X) \) is just multiplication by 2, so (by the orientability assumption, \(|k| + \beta = 1 \text{ mod } 2\)), we either have \( k \bigcup (\sigma \cap s'_\beta) \neq \emptyset \) or \( \beta \) is not in the image of \( D \), which implies there are no E-nodes by [20, Remark 7].

Remark 12. The forgetful map is not a submersion; it is not even strongly smooth.

We write

\[
\partial M_0 = \partial_M \bigcup \partial_\ast M_0
\]

with \( \partial_M \bigcup \partial_\ast M_0 \). We call \( \partial_M \bigcup \partial_\ast M_0 \) the sturdy boundary and \( \partial_\ast M_0 \) the wobbly boundary. The boundary also decomposes as

\[
\partial M_0 = \bigcup M_0 \times \bigcup M_0
\]

where the disjoint union is taken over all pairs of moduli specifications

\[
s' = ((k', l', \beta'), \sigma') \bigcup s'_\ast \quad \text{and} \quad s'' = ((k'', l'', \beta''), \sigma'')
\]

for

\[
\sigma = \sigma' \bigcup \sigma'', \quad k = k' \bigcup k'', \quad l = l' \bigcup l'', \quad \text{and} \quad \beta = \beta' + \beta'',
\]

and where \( r \) is any sufficiently large integer, so \( s'_\ast, s''_\ast \) denote two new boundary markings (representing the special boundary points identified by the node). We emphasize that the orientability condition specifies the order of the fiber factors. Using the \( O(2m + 1) \) action, we see that the restriction of \( d\text{ev}_{s'_\ast} \) to the codimension \( k \) strata,

\[
d\text{ev}_{s'_\ast} : TS^k (M_{s'_\ast}) \to TZ,
\]

is surjective for every \( k \geq 0 \), so the evaluation maps \( \text{ev}_{s'_\ast}, \text{ev}_{s''_\ast} \) are b-transverse (see the proof of Lemma [16] below, which generalizes this).

In terms of (17), the sturdy boundary \( \partial_M \bigcup \partial_\ast M_0 \) consists of those components where \( s', s'' \) are both sturdy. The wobbly boundary \( \partial_\ast M_0 \bigcup \partial_\ast M_0 \) consists of those components where precisely one of \( s' \) or \( s'' \) is wobbly.

2.3.3. Wobbly boundary involution. We construct a fixed-point free involution

\[
\text{inv}_0 : \partial_\ast M_0 \to \partial_\ast M_0
\]

as follows. If \( s_0 = ((k_0, l_0, \beta_0), \sigma_0) \) is any moduli specification and \( S \subset \sigma_0 \) is a two element subset, we abuse notation and denote by \( S' \in \text{Sym}(\sigma_0) \) the permutation that swaps the elements in \( S \), and by \( S' : M_{s_0} \to M_{s_0} \) the induced diffeomorphism. We let \( \text{inv}_0 \) acts on \( B = M_{s'} \times M_{s''} \subset \partial_\ast M_0 \) as follows. Precisely one of \( s', s'' \) is a wobbly boundary specification, and we denote it by \( s_0 = ((k_0, l_0, \beta_0), \sigma_0) \).
It follows that $|\sigma_0| \geq 2$. We take $S_0$ to be the first two elements of $\sigma_0$, where $\sigma' \sqcup s_0', \sigma''$ are ordered as subsets of $s_0'$. Let

$$\text{inv}_s|_B = \begin{cases} (S_0) \times \text{id} & \text{if } s_0 = s' \\
\text{id} \times (S_0) & \text{otherwise} \end{cases}.$$  

Clearly,

$$(\text{For}_s)_+ \circ \text{inv}_s = (\text{For}_s)_+.$$  

In the next subsection we will see that the wobbly boundary is inessential, in the sense that $\text{inv}_s$ defines an orientation-reversing involution on $\partial_s \mathcal{M}_g$.

2.4. Orienting moduli spaces. Our goal in this subsection is to prove the following.

**Proposition 13.** Let $s = ((k, l, \beta), \sigma)$ be a sturdy moduli specification

(a) There exists an isomorphism of local systems

$$\mathcal{J}_s : \text{Or} (T \mathcal{M}_s) \to \text{Or} (TL)^{\oplus k},$$  

lying over $\prod_{z \in k} ev_z : \mathcal{M}_s \to L^{\times k}$.

(b) The local system isomorphism

$$\text{Or} (d \text{inv}_s) : \text{Or} (T \partial_s \mathcal{M}_s) \to \text{Or} (T \partial_s \mathcal{M}_s)$$  

satisfies

$$\mathcal{J}_s \circ i^\beta_{\mathcal{M}_s} \circ \text{Or} (d \text{inv}_s) = (-1) \mathcal{J}_s \circ i^\beta_{\mathcal{M}_s},$$

where $i^\beta_{\mathcal{M}_s} = i^\beta_{\mathcal{M}_s}|_{\partial_s \mathcal{M}_s}$ is the boundary local system map (see [14]).

The proof of this proposition will occupy the remainder of this subsection. By Lemma [52], it suffices to construct local systems and maps of local systems on interior (depth zero) points of orbifolds with corners. We will make repeated use of this result.

2.4.1. Construction of $\mathcal{J}_s$. If $m$ is even (respectively odd), there are two $\text{Pin}^+$ (resp., $\text{Pin}^-$) structures on $\mathbb{R}^2$. Provisionally, let $p$ denote one of them. Consider some sturdy moduli specification $s = (b, \sigma) = ((k, l, \beta), \sigma)$. We will shortly define $\mathcal{J}_s$ to be the composition

$$\text{Or} (T \mathcal{M}_s) \xrightarrow{\mathcal{F}_s} \text{Or} (T \hat{\mathcal{M}}_s) \xrightarrow{\check{\mathcal{J}}_s} \text{Or} (TL)^{\oplus k},$$

where $\check{\mathcal{J}}_s$ is the local system map constructed by Solomon [14] (we will review the definition below) and $\mathcal{F}_s : \text{Or} (T \mathcal{M}_s) \to \text{Or} (T \hat{\mathcal{M}}_s)$ is defined using the natural orientation of the fibers of $\text{For}_s$. Reversing the choice of $p$ corresponds to replacing $\check{\mathcal{J}}_s$ by $-\check{\mathcal{J}}_s$ (see [14] Lemma 2.10), and so we may fix $p$ by requiring that $\check{\mathcal{J}}_{s_1, ((1, 2), 0, 1), \sigma}$ gives positive orientation to both points of any generic fiber of $ev_1^{\times 1} \times ev_2^{\times 1}$.

We set

$$\mathcal{F}_s : \text{Or} (T \mathcal{M}_s) \to \text{Or} (T \hat{\mathcal{M}}_s)$$

to be the local system map over $\text{For}_s$, which is defined over the interior of $(\hat{\mathcal{M}}_s)^o$ using the ordered direct sum decomposition

$$TM_s \cong TM_s \oplus \ker (d \text{For}_s).$$
The fiber of For$_S$|$_{(M_s)^g}$ over an interior point of $(M_s)^g$, represented by $(\Sigma, \nu, \kappa, \lambda, u)$, is naturally identified with an open subset of $(\partial \Sigma)^g$ which is oriented; here we use the order on $\sigma \in S'_g$ to order the product.

For any $S \subset \mathbb{N}$ we have a decomposition

\begin{equation}
\mathcal{F}_\Sigma = \mathcal{F}_S^S \circ \mathcal{F}_\Sigma^S
\end{equation}

lying over For$_S = \text{For}_S^S \circ \text{For}_S^S$; the maps $\mathcal{F}_S^S, \mathcal{F}_\Sigma^S$ are defined similarly to $\mathcal{F}_e$, using the orientation on $(\partial \Sigma)^{g_1} S$ and $(\partial \Sigma)^{g_2} S$ respectively, except we twist $\mathcal{F}_S^S$ by a suitable shuffle sign so that (23) holds.

Next we review the construction of $\mathcal{J}_S$ following [14], beginning with $\mathcal{J}_S^{\text{main}} = \mathcal{J}_S|_{M_\text{main}^g}$ where $M_\text{main}^g \subset M_s$ is the clopen component which is the closure of points represented by $(\Sigma = D^2, \nu, \kappa, \lambda, u)$ in which $\kappa : k \to S^1$ preserves the cyclic order; henceforth we consider $k \in \mathbb{N} \bigcup s''_n$ as ordered by putting the elements of $\mathbb{N}$ first, in order, then the elements of $s''_n$ in order.

Let $\mathcal{M}_b^{\text{reg}}(\beta)$ denote the space of holomorphic maps $(D^2, \partial D^2) \to (X, L)$ of degree $\beta$, where $D^2 \subset \mathbb{C}$ is the standard unit disc. Let $\delta$ be a tuple of max $(0, 2 - |k|)$ dummy markings, so that $\delta \bigcup k$ has length $k_+ \geq 2$, and set $\tilde{k}$ to be the tuple obtained by omitting the first two elements of $\delta \bigcup k$. Consider the subspace

$$\tilde{\mathcal{M}}_b \subset \mathcal{M}_b^{\text{reg}}(\beta) \times (\partial D^2)^{\tilde{k}} \times (D^2)^{1}$$

in which the components $(z_{1}, \ldots, z_{k_+}) \in (\partial D^2)^{\tilde{k}}$ are such that

$$(z_{1}, z_{2}, \ldots, z_{k_+}) = (+1, -1, z_{3}, \ldots, z_{k_+})$$

is cyclically ordered. Using similar notation to [4, Chapter 8], our orientation convention is summarized by the following equality of oriented bases for $T_p \mathcal{M}_b$:

\begin{equation}
\mathcal{M}_b \times \mathbb{R}_\beta = \mathcal{M}_b^{\text{reg}}(\beta) \times (\partial D^2)^{\tilde{k}} \times (D^2)^{1}.
\end{equation}

Here $D^2$ and $\partial D^2$ are oriented using the complex structure. Letting $\mathcal{B}_b = ((k, l, \beta), \delta)$, $\mathcal{M}_b \times \mathbb{R}_\beta$ stands for the pullback of a local oriented base for $T \mathcal{M}_b$, under the map

$$q_{1, 2} : \mathcal{M}_b \to \mathcal{M}_b$$

$$q_{1, 2}(u, z_{3}, \ldots, z_{k_+}, w) \mapsto [\Sigma = D^2, \nu = \emptyset, \kappa, \lambda, u]$$

for $k_+ \in \delta \bigcup k = (+1, -1, z_{3}, \ldots, z_{k_+})$ and $\lambda (1) = w$. $\mathbb{R}_\beta$ is an oriented real line bundle on $\mathcal{M}_b^{\text{reg}}(\beta)$ representing the action of the subgroup of $PSL_2(\mathbb{R})$ fixing the points $\pm 1$, with the positive direction corresponding to the flow from $+1$ to $-1$ (cf. [4, Convention 8.3.1]).

Now assume without loss of generality that $k = (1, 2, \ldots, k)$. Let $u : \mathcal{M}_b \to \mathcal{M}_b^{\text{reg}}(\beta)$ and $z (j) : \mathcal{M}_b \to \partial D^2$, for $1 \leq j \leq k$, denote the projections. Let $u (\cdot) : \mathcal{M}_b \times D^2 \to X$ be the corresponding evaluation map. By (24), to construct $\mathcal{J}_S$ it suffices to produce a section $u^{-1} \text{Or} (T \mathcal{M}_b^{\text{reg}}(\beta)) \otimes \bigotimes_{j=1}^{k} u (z(j))^{-1} \text{Or}(T L)^{\nu}$. We choose an arbitrary orientation for $(u (z(j)))^{-1} T L$ and transport it along $u|_{\partial D^2}$, obtaining orientations for $(u (z(j)))^{-1} T L$ for $2 \leq j \leq k$ and an orientation for $u|_{\partial D^2}^{-1} T L$ if orientable (that is, if $\beta = 0 \mod 2$). Using the Pin$^+$ structure $p$ and the orientation for $u|_{\partial D^2}^{-1} T L$ if $\beta = 0 \mod 2$ we obtain an orientation for $\text{Or}(T \mathcal{M}_b^{\text{reg}}(\beta))$ using [14, Proposition 2.8]. Reversing the initial choice of orientation for $(u (z(j)))^{-1} T L$ reverses the orientations $(u (z(j)))^{-1} T L$ for all $1 \leq j \leq k$ and, if $\beta = 0$, also the orientation.
for \(\text{Or}(T\mathcal{M}^\text{tau}_g(\beta))\) (see \[14\] Lemma 2.9). Since \(k + \beta = 1 \mod 2\) it follows that the section of the tensor product is independent of the initial choice of orientation. This completes the definition of \(\mathcal{J}_g^{\text{main}} : \text{Or}(T\mathcal{M}^\text{main}_g) \to \text{Or}(TL)^g\).

Let \(\tau : \mathcal{M}_g \to \mathcal{M}_g\) denote the action of the permutation \(\tau \in \text{Sym}(k)\). Let \(\tau_0\) denote the cyclic shift, which generates the stabilizer subgroup \(\{\tau |\tau.\mathcal{M}^\text{main}_g \subset \mathcal{M}^\text{main}_g\}\).

We have
\[
\mathcal{J}_g^{\text{main}} \circ \text{Or}(d\tau_0) = \mathcal{J}_g^{\text{main}}.
\]
Indeed, the shift introduces a sign of \((-1)^{k-1} = (-1)^\beta\). On the other hand, we need to compare the orientation transport \(OT(1)\) beginning at \(z(1)\), with the orientation transport \(OT(2)\) beginning at \(z(2)\). We can choose the initial orientation for \(OT(2)\) so it agrees with \(OT(1)\) at all of the points \(\tau(2), \tau(3), \ldots, \tau(k)\). With this choice, \(OT(1)\) and \(OT(2)\) also agree at \(z(1)\) if and only if \(\beta = 0 \mod 2\). Eq (25) follows. We extend the definition of \(\mathcal{J}_g\) to \(\mathcal{M}_g\) by setting
\[
\mathcal{J}_g|_{\tau^{-1}(\mathcal{M}^\text{main}_g)} := \mathcal{J}_g^{\text{main}} \circ \text{Or}(d\tau).
\]
This is well defined by (24), and shows that
\[
\mathcal{J}_g \circ \text{Or}(d\tau.) = \mathcal{J}_g
\]
for all \(\tau \in \text{Sym}(k)\). Using this it is straightforward to check that
\[
\mathcal{J}_g \circ \text{Or}(d\tau.) = \mathcal{J}_g
\]
for \(\tau \in \text{Sym}(k)\), whereas for \(\tau \in \text{Sym}(\sigma)\) we have
\[
\mathcal{F}_g \circ \text{Or}(d\tau.) = \text{sgn}(\tau) \cdot \mathcal{F}_g
\]
and hence
\[
\mathcal{J}_g \circ \text{Or}(d\tau.) = \text{sgn}(\tau) \cdot \mathcal{J}_g.
\]

2.4.2. Checking \(\text{inv}_g\) reverses the orientation. We turn to proving part (b) of Proposition 13. We will prove that
\[
\mathcal{F}_g \circ \iota^{\delta_{\mathcal{M}_g}} \circ \text{Or}(d\text{inv}_g) = (-1) \mathcal{F}_g \circ \iota^{\delta_{\mathcal{M}_g}}.
\]
which clearly implies (21).

Consider a wobbly boundary component
\[
\mathcal{B} = M_{s'}_{ev} = M_{s''} \subset \partial_+ \mathcal{M}_g.
\]
Recall \(\text{inv}_g|_{\mathcal{B}}\) was defined by swapping \(S_0\), the first two elements of \(\sigma' \bigcup \{s'_{s''}\}\) or the first two elements of \(\sigma''\), depending on which side is wobbly. If \(\sigma_0 \leq \sigma = \sigma' \bigcup \sigma''\), \(\text{inv}_g|_{\mathcal{B}}\) is the restriction of \((S_0) : \mathcal{M}_g \to \mathcal{M}_g\),
\[
\iota^{\delta_{\mathcal{M}_g}} \circ \text{inv}_g|_{\mathcal{B}} = (S_0) \cdot \iota^{\delta_{\mathcal{M}_g}}|_{\mathcal{B}}
\]
so (30) holds over \(\mathcal{B}\) by (28).

In case \(s'_{s''} \in S_0\) we have
\[
M_{s''} = M_{((x), s, s')}(s', x, s) = \{(s', x, s), (s', s, x)\}
\]
for some \(x \in k, s = s'_{i} \in \sigma', i < r\). On the right, each 3-tuple specifies the cyclic order of the markings on the boundary of the disc. In this case,
\[
\text{inv}_g|_{\mathcal{B}} = sw \times \text{id}_{\mathcal{M}^\sigma_{s''}} : \mathcal{B} \to \mathcal{B}
\]
where sw swaps the two tuples. By \((24)\) we can write
\[
F_s = \hat{F}_s \circ \hat{F}_s^S
\]
for \(S = \sigma \setminus \{i\}\). In other words, we forget \(s = s'\) first, then the rest of \(\sigma\).

Since the local system map \(i_{\mathcal{M}_s}^{\partial} : \text{Or}(T\partial \mathcal{M}_s) \rightarrow \text{Or}(T\mathcal{M}_s)\) was defined using the outward normal orientation convention, we have
\[
\hat{F}_s^S \circ i_{\mathcal{M}_s}^{\partial} \circ \text{Or}(d\text{inv}_s)|_B = -\hat{F}_s^S \circ i_{\mathcal{M}_s}^{\partial}
\]
(over each fiber of \(\text{For}_s^S\), \(\text{inv}_s|_B\) swaps two interval endpoints which are identified by \(\text{For}_s^S \circ i_{\mathcal{M}_s}^{\partial}|_B\)). So \((30)\) holds over \(B\) by \((32)\) and \((33)\).

2.5. Resolutions. In this subsection we construct resolutions of moduli spaces, which are orbifolds with corners \(\mathcal{M}_s^\rho\) where \(s\) is a sturdy moduli specification and \(\rho \subset \mathbb{N}\) is a finite subset. We will see that these are simply products of moduli spaces modeled on trees. Ultimately, we will be interested in \(\rho = [r]\) and \(s = (b, \varnothing)\) for some basic moduli specification \(b\), setting
\[
\mathcal{M}_b^r = \mathcal{M}_b^{[r]} = \mathcal{M}_{(b, \varnothing)}^{[r]}.
\]
The use of sets is needed to write cleaner recursive definitions. The discussion in this section may seem pedantic; the added detail is needed only for precise orientation computations.

**Definition 14.** Given a sturdy moduli specification
\[
s = (b, \sigma) = ((k, 1, \beta), \sigma) \text{ and } S \subset \mathbb{N}
\]
we define recursively for every finite subset \(\rho \subset \mathbb{N}\) such that
\[
s'_{\rho} \cap k = s'_{\rho} \cap \sigma = \varnothing
\]

- An orbifold with corners \(\hat{\mathcal{M}}_{s}^{\rho, S}\),
- a pair of forgetful maps
\[
\mathcal{M}_s^{\rho} : \hat{\mathcal{M}}_{s}^{\rho, N} \xrightarrow{\text{For}_{s}^{\rho, S}} \hat{\mathcal{M}}_{s}^{\rho, S} \xrightarrow{\text{For}_{s}^{\rho, S}} \hat{\mathcal{M}}_{s}^{\rho, \varnothing} = \hat{\mathcal{M}}_{s}^{\rho, \varnothing},
\]

- a finite set of sturdy \((\rho, s)\)-labeled trees, \(\mathcal{T}_s^{\rho}\), and
- a locally constant map \(\hat{\pi}_s^{\rho} : \hat{\mathcal{M}}_{s}^{\rho} \rightarrow \mathcal{T}_s^{\rho}\),

as follows. For \(\rho = \varnothing\) we set \(\hat{\mathcal{M}}_{s}^{\rho, S} = \mathcal{M}_{(b, s^{s'})}, \text{For}_{s}^{\rho, S} = \text{For}_{s}^{S}\) and \(\text{For}_{s}^{\rho, S} = \text{For}_{s}^{S}\) (cf. \((2.3.2)\)). We take \(\mathcal{T}_s^{\varnothing}\) to be a set with one element, which should be thought of as representing a tree with a single vertex labeled \(s\). We set \(\hat{\pi}_s^{\varnothing}\) to be the unique map to a point.
If $\rho \neq \emptyset$, let $r$ denote the maximal element of $\rho$ and let $\tilde{\rho} = \rho \setminus \{r\}$. We then define

\begin{equation}
\hat{M}^{\rho,S}_x = \bigcup_{s,\rho} \hat{M}^{\rho,S}_x \times \hat{M}^{\rho''}_x,
\end{equation}

\begin{equation}
\hat{F}^{\rho,S}_x = \bigcup_{s,\rho} \hat{F}^{\rho,S}_x \times \hat{F}^{\rho''}_x.
\end{equation}

Henceforth, $\bigcup_{s,\rho}''$ denotes a disjoint union over all partitions $\rho = \rho' \bigcup \rho''$ and pairs of sturdy moduli specifications $s = ((k', l', \beta'), \sigma' \bigcup \{s'_e\})$ and $s'' = ((k'', l'', \beta''), \sigma'')$ such that

$$\sigma = \sigma' \bigcup \sigma'', \quad k = k' \bigcup k'', \quad l = l' \bigcup l'', \quad \beta = \beta' + \beta''.\$$

We introduce some more notation. Define $\hat{F}^\rho_x = \bigcup_{s,\rho} \hat{F}^{\rho,S}_x$, $\pi^\rho_x : \hat{M}^\rho_x \to \hat{F}^\rho_x$ by

$$\pi^\rho_x = \tilde{\pi}^\rho_x \circ \hat{F}^\rho_x,$$

and set $\hat{M}_T = (\tilde{\pi}^\rho_x)^{-1}(T)$ and $\hat{M}_T = (\pi^\rho_x)^{-1}(T)$.

For $x \in k \bigcup s'_\rho \bigcup \{(\sigma \bigcup s'_\rho) \cap s'_l\}$ we have boundary evaluation maps

$$e^\rho_{x} : \hat{M}^\rho_x \to L$$

defined in the obvious way. Similarly, for $x \in l$ there are interior evaluation maps $e^\rho_{x} : \hat{M}^\rho_x \to X$.

For $j \in \rho \cap S$ we define $e^\rho_j : \hat{M}^\rho_x \to L \times L$ by

$$e^\rho_{x} = e^\rho_{x} \times e^\rho_{y},$$

and $e^\rho_{y} := \prod_{j \in \rho} e^\rho_{y} : \hat{M}^\rho_x \to (L \times L)$. We will mostly be interested in $e^\rho_{x} = e^\rho_{x} \ast N$ and $e^\rho_{x} = e^\rho_{x} \ast S$. We have

$$e^\rho_{x} = e^\rho_{x} \circ e^\rho_{x} \ast S.$$

for any $x$ such that both sides of the equality make sense, and similarly

$$e^\rho_{x} = e^\rho_{x} \ast S \circ e^\rho_{x} \ast S.$$
Lemma 15. Let $\mathfrak{s} = ((k, l, \beta), \sigma)$ be a sturdy moduli specification and $\rho$ be a tuple.

(a) The set $\mathcal{P}_0^\rho$ is in natural bijection with isomorphism types of $(\mathfrak{s}, \rho)$-labeled trees $T$. These are trees with set of oriented edges $T_1 = \rho$, set of vertices $T_0$, and maps $s_T$ assigning to each vertex a sturdy moduli specification, 

$$s_T(v) = ((k_T(v), l_T(v), \beta_T(v)), \sigma_T(v)),$$

such that:

(i) The head (respectively, the tail) of the edge $j \in T_1$ is the vertex $v$ if and only if $s''_j \in k_T(v)$ (resp. $s'_j \in \sigma_T(v)$)

(ii) We have

$$\prod_{v \in T_0} k_T(v) = k \prod_{j \in \rho} \{s''_j\}, \prod_{v \in T_0} l_T(v) = l, \sum \beta_T(v) = \beta$$

and $\prod_{v \in T_0} \sigma_T(v) = \sigma \prod_{j \in \rho} \{s'_j\}$

(b) There’s a natural order on the vertex set $T_0$ of each $T \in \mathcal{P}_0^\rho$ and we have, for any $S \subset \mathbb{N}$,

$$\hat{M}_S^r := \prod_{T \in \mathcal{P}_0^\rho} \hat{M}_T^S$$

for $\hat{M}_S^r = \prod_{v \in T_0} \hat{M}_S(v)$;

and

$$\text{For}_{S}^r := \prod_{T \in \mathcal{P}_0^\rho} \text{For}_{S}^T(v), \text{For}_{S}^r := \prod_{T \in \mathcal{P}_0^\rho} \text{For}_{S}^T(v)$$

and the map $\hat{\pi}_r$ is defined by $\hat{M}_T := (\hat{\pi}_r)^{-1}(T) = \prod_{v \in T_0} \hat{M}(v)$.

(c) Sym $(\rho)$ acts on $\mathcal{M}_0^\rho, \mathcal{M}_S^r, \mathcal{P}_0^\rho$ in a way which commutes with the $O_b$ action and makes all the maps $O_b$-equivariant in the obvious sense.

Proof. We prove (a) by induction on $\rho$. For $\rho = \emptyset$ the claim is trivial. If $r = \max \rho$ and $\rho' = \rho \setminus r$, it suffices to give a bijection of isomorphism types

$$\{(\mathfrak{s}, \rho) \text{ trees}\} \simeq \prod_{\rho' \in \rho}\{\{(\mathfrak{s}', \rho') \text{ trees}\} \times \{(\mathfrak{s}'', \rho'') \text{ trees}\}.$$

One direction is immediate: given a sturdy $(\mathfrak{s}', \rho')$-labeled tree $T'$ and a sturdy $(\mathfrak{s}'', \rho'')$-labeled tree $T''$, we construct $T$ by connecting the unique vertex $v' \in T'_0$ such that $s''_j \in s_T(v)$ with the unique vertex $v'' \in T''_0$ such that $s''_j \in s_T(v)$.

For the other direction, given a sturdy $(\mathfrak{s}, \rho)$-labeled tree $T$, let $T_0$ denote $T$ after we remove the interior of the edge $e_r$ labeled by $r$. We obtain an ordered pair of trees $(T', T'')$ with $T'$ (respectively, $T''$) corresponding to the connected component of $T$ containing the tail $v'$ (resp. the head $v''$) of the edge $e_r$. Let $\rho' \subset \rho$ and $T'_0 \subset T_0$ be the subset of edges and vertices belonging to $T'$, and set $s_{T'} = \sigma_{T'}|T'_0$. We claim

$$s' = \left(\prod_{v \in T'_0} k_{T'}(v) \setminus \{s''_j\}_{j \in \rho'}, \prod_{v \in T'_0} l_{T'}(v), \sum_{v \in T'_0} \beta_{T'}(v), \prod_{v \in T''_0} \sigma_{T''}(v) \setminus \{s'_j\}_{j \in \rho'}\right).$$

is a sturdy moduli specification. To see this, write $s' = ((k', l', \beta'), \sigma')$. We check stability: any oriented tree has a vertex with no incoming edges, and we let $u$ be such a vertex for $T'$, so $k_{T'}(u) \subset k'$. Since we always have $l_{T'}(u) \subset l'$ and $\beta_{T'}(u) \subset \beta'$, stability of $(k', l', \beta')$ follows from stability of $(k_{T'}(u), l_{T'}(u), \beta_{T'}(u))$. Orientability
follows from the following computation mod 2:

\[ \left| \bigoplus_{v \in T_0'} k_{T'} (v) \setminus \{ s'' \} \right| = \sum |k_{T'} (v)| - |\rho'| = \sum |\beta_{T'} (v) + |T_0| - |\rho'| = \beta' + 1. \]

Clearly, \( T' \) is an \((s', \rho')\)-labeled tree. Similarly, labeling \( T'' \) by \( s'_{T''}|T_0'' \) we obtain a \((s'', \rho'')\)-labeled tree for the appropriate \((s', \rho', s'', \rho'') \) and one checks that \( s', \rho', s'', \rho'' \) satisfy (17). The result follows.

The proof of (b) is straightforward; the order on \( T_0 \) is defined recursively, so

\[ T_0 = T'_0 \bigcup T''_0 \]

(that is, the vertices in \( T_0' \) appear before those in \( T''_0 \)).

We prove part (c). It is clear that \( \text{Sym} (\rho) \) acts on \((s, \rho)\)-labeled trees, and this defines an action on \( T_0'' \) using part (a). Suppose \( \tau \in \text{Sym} (\rho) \) sends \( T^1 \in T_0'' \) to \( T' \in T_0'' \). We define a diffeomorphism \( \tau : M_{T'} \to M_{T''} \) by first relabeling and permuting the markings so \( s'_j \) maps to \( s''_{\tau (j)} \) and \( s''_{\tau (j)} \) by \( s''_{\tau (j)} \), and then permuting factors so that the order of the factors agrees with the order specified in part (b). Consider for example the following component \( M_{T'} \subset M_{s''} \) for \( \rho = (1, 2) \):

\[ M_{T'} = M_{\{(k_1, l_1, \beta_1), (s'_j)\}} \times \left( M_{\{(k_2, l_2, \beta_2), (s'_j)\}} \times M_{\{(k_3 \cup (s''_j, \beta_j)), \beta_3\}} \right) \]

the nontrivial element \( \tau \in \text{Sym} (\rho) \) acts by first applying a product of three diffeomorphisms, with codomain

\[ M_{\{(k_1, l_1, \beta_1), (s'_j)\}} \times \left( M_{\{(k_2, l_2, \beta_2), (s'_j)\}} \times M_{\{(k_3 \cup (s''_j, \beta_j)), \beta_3\}} \right) \]

where the diffeomorphism on the right involves a permutation of the tuple \( (\partial D^2)^{k_3} \cup (s''_j, \beta_j) \)

(by the left and middle diffeomorphism are just relabeling of values in this case). Then we apply the associator and commutator for the product to identify this with \( M_{T''} \subset M_{s''} \) for

\[ M_{T''} = M_{\{(k_2, l_2, \beta_2), (s'_j)\}} \times \left( M_{\{(k_1, l_1, \beta_1), (s'_j)\}} \times M_{\{(k_3 \cup (s''_j, \beta_j)), \beta_3\}} \right) \]

We define a diffeomorphism \( \tilde{M}_{T'} \to \tilde{M}_{T''} \) similarly by mapping \( s''_{\tau (j)} \) to \( s''_{\tau (j)} \) and permuting factors. Clearly these maps commute with the \( O_b \) action and the other maps.

This lemma gives a manifestly \( \text{Sym} (\rho) \)-symmetric alternative to Definition 14

Lemma 16. For every \( S \subset \rho \) the product \( \text{ed}^\rho_S : \prod_{j \in S} \text{ed}_j^\rho : (L \times L)^S \to (L \times L)^S \) is \( b \)-transverse to the diagonal

\[ \Delta^S_L : L^S \to (L \times L)^S, \]

so the fiber product

\[ \left( \text{ed}^\rho_S \right)^{-1} (\Delta) = L^S \Delta^S_L \times \prod_{j \in S} \text{ed}_j^\rho : M^\rho_S \]

exists, and its corners are described by Eq (80).

Proof. \( O \left( 2m + 1 \right)^T_0 \) acts on \( M_{T'} \subset T_0 \) \( M_{T''} \), and the maps \( \text{ev}^\rho_{x,T'} : \tilde{M}_{T'} \to L \) are naturally \( O \left( 2m + 1 \right)^T_0 \to O \left( 2m + 1 \right)^T_0 \) equivariant with respect to the appropriate projection. The map linearization of the action

\[ \phi \left( 2m + 1 \right) \to T_y L \]
is surjective at every \( y \in \mathcal{L} \).

Let \( p \in S^k (\mathcal{M}_s^r) \) be a point of depth \( k \) with \( \text{ed}_S^*(p) = \Delta_l^S (y) \) for \( y = (y_1, \ldots, y_s) \in L^S \). Let \( v = (v_1, \ldots, v_s) \), \( v_i \in T_{y_i}L \) represent a normal vector to \( \Delta^S \) at \( y \). More precisely, we use the isomorphism \( N_\Delta T \to \text{pr}_2^* T \mathcal{L} \) where \( \text{pr}_2 \) denotes projection on the head of the edge \( i \). Let \( T \setminus \mathcal{L} \) denote the tree with the edges corresponding to \( S \) removed. For each connected component \( C_j \subset T \setminus \mathcal{L} \) we fix an \( O(2m + 1) \)-fundamental vector field \( \theta_j \) on \( L \) in such a way that \( \theta_j|_{y_i} - \theta_j|_{y_{i+1}} = v_i \) whenever the tail and head of \( i \) are incident to \( C_j \) and \( C_{j+1} \) respectively. The corresponding lie algebra elements define a lift of \( v \) to \( w \in T_p S^k (\mathcal{M}_s^r) \). Now use Remark 15 and Lemma 15.

\[ \square \]

**Remark 17.** For \( |S| = 1 \), \( \text{ed}_S^* \) is in fact a b-submersion.

2.5.1. **Resolutions and the boundary.** As \( \rho \) varies, the spaces \( \mathcal{M}_s^\rho \) form a resolution of the sturdy boundary in the sense that for \( r > \max \rho \) there's a map \( g^\rho : \partial \mathcal{L} \to \mathcal{L}_s \), which we shall now define.

Comparing (17) and (34) we see that for any \( r \in \mathbb{N} \) there's a map

\[ g^r_\phi : \partial \mathcal{M}_s^\phi \to \mathcal{M}_s^\phi \]

sitting in a cartesian square

\[
\begin{array}{ccc}
\partial \mathcal{M}_s^\phi & \xrightarrow{g^r_\phi} & \mathcal{M}_s^r \\
\downarrow & & \downarrow \\
L & \xrightarrow{\Delta} & L \times L
\end{array}
\]

More generally for \( r > \max \rho \) we define an \( O^\rho \)-equivariant map

\[ g^\rho_{\phi} : \partial \mathcal{M}_s^\rho \to \mathcal{M}_s^\rho \]

by recursion on \( \rho \). For \( \rho = \emptyset \) we've already defined it. For \( |\rho| \geq 1 \) we set \( g^\rho_{\phi} \) to be the composition of the following maps

\[
\partial \mathcal{M}_s^\rho = \bigcap_{s, \rho} \left( \left( \partial \mathcal{M}_s^\rho \times \mathcal{M}_s^\rho \right) \bigcap \left( \mathcal{M}_s^\rho \times \partial \mathcal{M}_s^\rho \right) \right) \xrightarrow{(1)}
\]

\[
\bigcup_{s, \rho} \left( \left( \mathcal{M}_s^\rho \times \mathcal{M}_s^\rho \right) \bigcap \left( \mathcal{M}_s^\rho \times \partial \mathcal{M}_s^\rho \right) \right) \xrightarrow{(1)}
\]

\[
\bigcup_{s, \rho} \left( \left( \mathcal{M}_s^\rho \times \mathcal{M}_s^\rho \right) \bigcap \left( \mathcal{M}_s^\rho \times \partial \mathcal{M}_s^\rho \right) \right) \xrightarrow{(1)}
\]

\[
\mathcal{M}_s^\rho \to \mathcal{M}_s^\rho \]

The map (1) is \( \bigcup_{s, \rho} \left( \left( \mathcal{M}_s^\rho \times \mathcal{M}_s^\rho \right) \bigcap \left( \mathcal{M}_s^\rho \times \partial \mathcal{M}_s^\rho \right) \right) \).

We define the map \( g^\rho_{\phi} \) as the composition

\[ (39) \quad g^\rho_{\phi} : \partial \mathcal{M}_s^\rho \xrightarrow{\text{sym} \rho} \mathcal{M}_s^\rho \xrightarrow{\tau} \mathcal{M}_s^\rho \]

where \( \tau \in \text{Sym} (\rho \bigcup \{ r \}) \) is the cyclic permutation that sends \( r \) to \( \min \rho \). \( g^\rho_{\phi} \) is an \( O^\rho = O (2m + 1) \times \text{Sym} (k) \times \text{Sym} (l) \times \text{Sym} (\sigma) \times \text{Sym} (\rho) \) equivariant map, which
sits in a cartesian square

\[ \partial_- M^\varrho_s \xrightarrow{\partial^\varrho_s \cup r} M^\varrho_s \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ L \quad \Delta L \quad L \times L \]

Since \( \text{For}_s^\varrho \) is \( \text{Sym}(\rho) \) equivariant, the decomposition \( \partial M^\varrho_s = \partial_- M^\varrho_s \coprod \partial_+ M^\varrho_s \) is \( \text{Sym}(\rho) \)-equivariant. The following lemma will allow us to compute the sign of the \( \text{Sym}(\rho) \) action on local systems more easily.

**Lemma 18.** Let \( s \) be a sturdy moduli specification, \( \rho \subset \mathbb{N} \) finite, and \( r \) a non-negative integer. We denote \( \rho_+=\rho\{\max \rho+1,...,\max \rho+r\} \).

(a) There exists a clopen component \( \partial_-^r M^\varrho_s \subset \partial^r M^\varrho_s \) and an \( \text{O}_s^\varrho \)-equivariant map

\[ g^\varrho_{s+r^r} : \partial^r_- M^\varrho_s \rightarrow M^{r^r}_s \]

which sits in a cartesian square

\[ \partial^r_- M^\varrho_s \xrightarrow{g^\varrho_{s+r^r}} M^{r^r}_s \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ L^r \quad \Delta L^r \quad (L \times L)^r \]

where \( \text{ed}^\varrho_{(1,...,r)} := \text{ed}^\varrho_{r^r} \times ... \times \text{ed}^\varrho_{r^r} \) is \( b \)-transverse to \( \Delta L^r \).

(b) As a subgroup of \( \text{Sym}(\rho_+) \), \( \text{Sym}(r) \) acts on \( M^{r^r}_s \). \( \text{Sym}(r) \) also acts on \( \Delta L^r \) and \( (L \times L)^r \) by permuting factors. The induced action on the fibered product \( \partial_-^r M^\varrho_s \), is the restriction of the \( \text{Sym}(r) \) action on \( \partial^r_- M^\varrho_s \) permuting the local boundary components.

**Proof.** We prove part (a), by induction on \( r \). For \( r = 0 \) we take \( \partial^0_- M^\varrho_s = \partial^0_- M^\varrho_s = M^\varrho_s \) and \( g^\varrho_{s,0} = \text{id} \) so the claim is trivial.

For \( r \geq 1 \), use the inductive hypothesis to obtain a cartesian square

\[ \partial_- (r-1) M^\varrho_s \xrightarrow{\partial^\varrho_s \cup (r-1)} M^\varrho_{s%(r-1)} \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \Delta^{r-1} \quad (L \times L)^{r-1} \]

We take

\[ \partial^r_- M^\varrho_s := (\partial^\varrho_s \cup (r-1))^{-1} (\partial_- M^\varrho_{s%(r-1)}) \]

where \( \partial_- M^\varrho_{s%(r-1)} := \partial^\varrho_{s%(r-1)} M^\varrho_{s%(r-1)} \) is the horizontal clopen component as in \([2.3.1 \text{ We obtain a pair of cartesian squares}]

\[ \partial^r_- M^\varrho_s \xrightarrow{\partial_- M^\varrho_{s%(r-1)}} \partial_- M^\varrho_{s%(r-1)} \quad \partial_- M^\varrho_{s%(r-1)} \xrightarrow{\partial_- M^\varrho_{s%(r-1)}} M^\varrho_{s%(r-1) \cup \{r\}} \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \Delta^{r-1} \quad (L \times L)^{r-1} \quad \Delta \quad L \times L \]
Note that $e[r-1] = e[r+(r-1)]$ factors through $\partial \mathcal{M}_{s+1}^{r} \to \mathcal{M}_{s}^{r+1(r-1)} = \mathcal{M}_{s}^{r}$ and in each of the squares the bottom and right maps are b-transverse to one another by Lemma 16, so the b-transverse cartesian square (41) is obtained from (43). More precisely, a simple diagram chase shows that given two cartesian squares

\[
\begin{array}{ccc}
P & \rightarrow & Q \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array}
\]

with $df \oplus dg : TQ \oplus TA \to TB$ and $da \oplus db : TE \oplus TC \to TD$ surjective, the square

\[
\begin{array}{ccc}
P & \rightarrow & E \\
\downarrow & & \downarrow \\
A \times C & \rightarrow & B \times D
\end{array}
\]

is cartesian too and $d(r \times a) \oplus d(g \times b)$ is surjective. The result follows from this by Remark 36.

Part (b) is easy to see. □

We now want discuss an analogous relationship between the spaces $\mathcal{M}_{s}^{T}$ for various $\rho$, but first we introduce some more notation. Let $r > \max \rho$. We define a map

\[
(44) \quad \text{cnt} = \text{cnt}_{s}^{\rho} \mathcal{U}(r) : \mathcal{F}_{s}^{\rho} \mathcal{U}(r) \to \mathcal{F}_{s}^{\rho}
\]

which one may think of as contracting the edge $e_{r}$ to some vertex $v_{i} \in \text{cnt} (T_{0})$ which carries the sum of the degrees and the disjoint union of the labels of the two incident vertices, except for $s'_{i}, s''_{i}$ which are discarded. More precisely, the map sends a sturdy tree $T_{r} \in \mathcal{F}_{s}^{\rho} \mathcal{U}(r)$ to the unique $T \in \mathcal{F}_{s}^{\rho}$ such that

\[
\left(\gamma_{s}^{\rho} \mathcal{U}(r)\right)^{-1} (\mathcal{M}_{T_{r}}) \subset \partial \mathcal{M}_{T}.
\]

Setting

\[
\partial^{T_{r}} \mathcal{M}_{T} := \left(\gamma_{s}^{\rho} \mathcal{U}(r)\right)^{-1} (\mathcal{M}_{T_{r}}),
\]

we have

\[
\partial \mathcal{M}_{T} = \bigsqcup_{\{T_{r} : \text{cnt} (T_{r}) = T\}} \partial^{T_{r}} \mathcal{M}_{T}.
\]

Now consider some $S \subset \mathbb{N}$, $r \notin S$. We define

\[
(45) \quad \partial^{T_{r}} \mathcal{M}_{T}^{S} = \left(\Gamma_{r}^{S} \right) \left(\partial^{T_{r}} \mathcal{M}_{T}\right) \subset \partial \mathcal{M}_{T}^{S}.
\]

If $T_{1}, T_{2} \in \mathcal{F}_{s}^{\rho}$ differ by moving $s'_{i}, i \notin S$ from the head of $e_{r}$ to the tail of $e_{r}$ or vice-versa, then $\partial^{T_{1}} \mathcal{M}_{T}^{S} = \partial^{T_{2}} \mathcal{M}_{T}^{S}$. There’s a map

\[
(46) \quad \gamma_{T_{r}}^{S} : \partial^{T_{r}} \mathcal{M}_{T}^{S} \to \mathcal{M}_{T}^{S} \mathcal{U}(r)
\]
sitting in cartesian squares

\[
\begin{array}{c}
F \rightarrow T \xrightarrow{\partial T_{\mathcal{M}}} \mathcal{M}_{\mathcal{M}}^T \\
\left(F_{\mathcal{M}_{\mathcal{M}}^T}^{\rho, (r)}\right) \downarrow \downarrow \\
\left(F_{\mathcal{M}_{\mathcal{M}}^T}^{\rho, (r)}\right) \\
\end{array}
\]

whose composition is the restriction of (40) to \(\mathcal{M}_{\mathcal{M}}^T\).

We define

\[
\begin{array}{c}
\mathcal{M}_{\mathcal{M}}^T \rightarrow \mathcal{M}_{\mathcal{M}}^T \\
\mathcal{M}_{\mathcal{M}}^T \rightarrow \mathcal{M}_{\mathcal{M}}^T \\
L \node{\Delta} L \times L
\end{array}
\]

by \(\hat{g}_{\mathcal{M}_{\mathcal{M}}^T} \circ \partial T_{\mathcal{M}}\).

2.6. Orienting resolutions.

2.6.1. Overview. Our goal in this section is to construct local system maps

\[
\mathcal{F}_\rho^0 : \text{Or} (T \mathcal{M}_\rho^0) \rightarrow \text{Or} (T \mathcal{M}_\rho^0)
\]

and

\[
\mathcal{J}_\rho^0 : \text{Or} (T \mathcal{M}_\rho^0) \rightarrow \text{Or} (T \mathcal{M}_\rho^0) \text{ over } \prod_{x \in v^0_\mathcal{M}_\rho} \text{ev}_{x}^0
\]

extending \(\mathcal{F}_\rho = \mathcal{F}_\rho^0\) and \(\mathcal{J}_\rho = \mathcal{J}_\rho^0\) so that \(\mathcal{J}_\rho^0 := \mathcal{J}_\rho^0 \circ \mathcal{F}_\rho^0\) can be used to integrate forms on \(\mathcal{M}_\rho^0\). We will show that the following two properties, essential to the proof of Stokes’ theorem, hold:

\[
\begin{align*}
\text{(coherence)} & \quad \left(\mathcal{J}_\rho^0 \text{ev}_{x}^0\right)^{-1} \circ \mathcal{G}_\rho^0 \text{ev}_{x}^0 = \mathcal{J}_\rho^0 \circ \mathcal{G}_\rho^0, \\
\text{(Sym} (\rho)\text{-equivariance)} & \quad \mathcal{J}_\rho^0 \circ \text{Or} (d\tau) = \text{sgn} (\tau) \cdot \mathcal{J}_\rho^0.
\end{align*}
\]

Here the local system map

\[
\left(\mathcal{J}_\rho^0 \text{ev}_{x}^0\right)^{-1} : \text{Or} (T \mathcal{M}_\rho^0) \otimes \left(\text{ev}_{x}^0\right)^{-1} \text{Or} (T \mathcal{M}_\rho^0) \rightarrow \text{Or} (T \mathcal{M}_\rho^0)
\]

is obtained from \(\mathcal{J}_\rho^0\) by the tensor-hom adjunction, and

\[
\mathcal{G}_\rho^0 (r) : \text{Or} (T \mathcal{M}_\rho^0) \rightarrow \text{Or} (T \mathcal{M}_\rho^0) \otimes \left(\text{ev}_{x}^0\right)^{-1} \text{Or} (T \mathcal{M}_\rho^0)
\]

is induced from the cartesian square (40). Orienting fiber products involves some computation in \([10]\). To this end, we use the short exact sequences

\[
\begin{align*}
0 \rightarrow \text{Or} (T \mathcal{M}_\rho^0) & \rightarrow \left(\mathcal{J}_\rho^0 \text{ev}_{x}^0\right)^{-1} T \mathcal{M}_\rho^0 \oplus \text{ev}_{x}^{-1} \rightarrow T (L \times L) \rightarrow 0 \tag{49}
\end{align*}
\]

and

\[
0 \rightarrow \text{ev}_{x}^{-1} T L \rightarrow \text{ev}_{x}^{-1} T (L \times L) \rightarrow \text{ev}_{x}^{-1} T L \rightarrow 0,
\]

in conjunction with (42).

We will conclude this subsection with an explicit computation of \(\mathcal{J}_\rho^0 I_{\mathcal{M}_{\mathcal{M}}^T}\) for \(\mathcal{T} \in \mathcal{F}_\rho\) a special kind of tree, which will come up in the fixed point localization computation in [15].
2.6.2. Construction of $\mathcal{F}_s^\rho, J_s^\rho$. Let $s$ be a sturdy moduli specification, $\rho \subset \mathbb{N}$ a finite subset and $S \subset \mathbb{N}$ any subset. Using the order on vertices in Lemma 19(b) we obtain an isomorphism of local systems

\[
\text{Or} \left( T\mathcal{M}_s^\rho, S \right) \simeq \coprod_{T \in \mathcal{F}_s^\rho} \text{Or} \left( T\mathcal{M}_{s(T)}^S \right)
\]

Using this and

\[
\coprod_{T \in \mathcal{F}_s^\rho} \mathcal{F}_s^\rho(T), \coprod_{T \in \mathcal{F}_s^\rho} \mathcal{F}_s^S(T), \coprod_{T \in \mathcal{F}_s^\rho} \mathcal{J}_s^S(T), \coprod_{T \in \mathcal{F}_s^\rho} \mathcal{J}_s(T)
\]

we obtain local system maps

\[
\left( \prod \mathcal{F} \right)_s^\rho : \text{Or} \left( T\mathcal{M}_s^\rho \right) \to \text{Or} \left( T\mathcal{M}_s^\rho, S \right)
\]

\[
\left( \prod \mathcal{J} \right)_s^\rho : \text{Or} \left( T\mathcal{M}_s^\rho, S \right) \to \text{Or} \left( T\mathcal{M}_s^\rho, S \right)
\]

\[
\left( \prod \mathcal{J} \right)_s^\rho : \text{Or} \left( T\mathcal{M}_s^\rho \right) \to \text{Or} \left( T\mathcal{M}_s^\rho, S \right)
\]

\[
\left( \prod \mathcal{J} \right)_s^\rho : \text{Or} \left( T\mathcal{M}_s^\rho, S \right) \to \text{Or} \left( T\mathcal{M}_s^\rho, S \right)
\]

and

\[
\left( \prod \mathcal{J} \right)_s^\rho : \text{Or} \left( T\mathcal{M}_s^\rho \right) \to \text{Or} \left( T\mathcal{M}_s^\rho, S \right)
\]

with \((\prod \mathcal{J})_s^\rho = (\prod \mathcal{J})_s^\rho \circ (\prod \mathcal{F})_s^\rho\) and \((\prod \mathcal{J})_s^\rho = (\prod \mathcal{J})_s^\rho \circ (\prod \mathcal{F})_s^\rho\).

However, \((\prod \mathcal{J})_s^\rho\) does not satisfy the coherence condition above. The plan is as follows. We will twist \((\prod \mathcal{F})_s^\rho, (\prod \mathcal{J})_s^\rho\) by certain signs to make them coherent (in a sense which we will make precise shortly), and prove that this makes \((\prod \mathcal{J})_s^\rho\) coherent. We will see \((\prod \mathcal{J})_s^\rho\) equivariant follows from coherence.

**Lemma 19.** Let $\mathcal{F} : \mathcal{L}_1 \to \mathcal{L}_2$ be a local system map lying over a b-normal map of orbifolds $f : \mathcal{X} \to \mathcal{Y}$. There’s an induced local system map $\mathcal{F}_- : \left( i^{\partial _\mathcal{X}}_\mathcal{X} \right)^{-1} \mathcal{L}_1 \to \left( i^{\partial _\mathcal{Y}}_\mathcal{Y} \right)^{-1} \mathcal{L}_2$ over $f_-$. In case $\mathcal{L}_1 = \text{Or} \left( T\mathcal{X} \right)$ and $\mathcal{L}_2 = \text{Or} \left( T\mathcal{Y} \right)$ we can use $i^{\partial _\mathcal{X}}_\mathcal{X}, i^{\partial _\mathcal{Y}}_\mathcal{Y}$ to define a local system map

\[
\partial \mathcal{F} : \text{Or} \left( T\partial _\mathcal{X} \mathcal{X} \right) \to \text{Or} \left( T\partial _\mathcal{Y} \mathcal{Y} \right)
\]

over $f_-$, which satisfies $\mathcal{F} \circ i^{\partial _\mathcal{Y}}_\mathcal{Y} = i^{\partial _\mathcal{X}}_\mathcal{X} \circ \partial \mathcal{F}$.

**Proof.** straightforward. \(\square\)

Fix some $r > \max \rho$. For every $T_s \in \mathcal{F}_s^\rho \mathcal{L}(r)$ we define

\[
\mathcal{G}_{T_s} : \text{Or} \left( \partial \mathcal{F}_s^\rho \mathcal{L}(r) \right) \to \text{Or} \left( \partial \mathcal{F}_s^\rho \mathcal{L}(r) \right) \circ \left( \mathcal{F}_s^\rho \mathcal{L}(r) \right)^{-1} \text{Or} \left( T\mathcal{L} \right)
\]

over $\mathcal{G}_{T_s}$ by the bottom cartesian square in (47), using the same convention as in (49). We define a local system map over $\mathcal{G}_{T_s} = \mathcal{F}_s^\rho \mathcal{L}(r) \circ \mathcal{G}_{T_s}$ by

\[
\mathcal{G}_{T_s} = \left( (\prod \mathcal{J})_s^\rho \mathcal{L}(r) \right) \circ \alpha \circ \mathcal{G}_{T_s}.
\]
Here
\[ \alpha : (e_{s_t}^{a_s, \rho})_{s_t}^{\rho, r} \rightarrow (e_{s_t}^{a_s, \rho})_{s_t}^{\rho, r} \rightarrow (T L) \]

is the associator local system map lying over \( F_{\rho, s}^{\rho, r} \), associated with the factorization of the evaluation map \( e_{s_t}^{a_s, \rho} = e_{s_t}^{a_s, \rho} \circ F_{\rho, s}^{\rho, r} \). In what follows we will abuse notation and use \( \alpha \) to denote any such associator. We will also avoid excessive decorations and write just \( e_{s_t}^{a_s, \rho} \) if the domain is clear from the context.

The following proposition defines \( \mathcal{F}_s^{\rho} \) and the coherence condition that it satisfies.

**Proposition 20.** There exists some function \( \delta_s^\rho : \mathcal{F}_s^{\rho} \rightarrow \{\pm 1\} \) such that if we define
\[ \mathcal{F}_s^{\rho} = (\delta_s^\rho \circ \pi_s^\rho) \cdot (\prod \mathcal{F})^\rho_s, \]
then we have for any \( r > \max \rho \) and \( T_r \in \mathcal{F}_s^{\rho, r} \),
\[ [\mathcal{F}_s^{\rho, r} \otimes \alpha] \circ G_s^{\rho, r} |_{\partial T_r} = \mathcal{G}_{T_r} \circ \partial T_r \mathcal{F}_s^{\rho, r} \]
where we denote \( T = \text{cnt } T_r \) and \( \partial T_r \mathcal{F}_s^{\rho, r} := \partial \mathcal{F}_s^{\rho, r} |_{\partial T_r} \).

The proof relies on the following lemma, which says that the failure of \( \mathcal{F} \) to satisfy coherence is measured by a function \( \epsilon \) which is constant on components of the form \( \partial T_r \mathcal{M}_{\text{cnt } T_r} \). We will then define the correction function \( \delta \) recursively by a kind of difference equation which depends on \( \epsilon \).

**Lemma 21.** There’s some function \( \epsilon_s^{\rho, r} : \mathcal{F}_s^{\rho, r} \rightarrow \{\pm 1\} \) such that for every \( T_r \in \mathcal{F}_s^{\rho, r} \), \( T = \text{cnt } T_r \), we have
\[ \mathcal{G}_{T_r} \circ \partial T_r \mathcal{F}_s^{\rho, r} = \epsilon_s^{\rho, r} \mathcal{F}_s^{\rho, r} |_{\partial T_r} \mathcal{M}_{\text{cnt } T_r} \]

**Proof.** Consider some \( T_r \in \mathcal{F}_s^{\rho, r} \). Let \( T_r^! = \tau^{-1} (T_r) \) where \( \tau \in \text{Sym } (\rho \mathcal{U}^ r) \) is the cyclic shift that sends \( r \) to \( \min \rho \). The four local system maps in \( \vdash \) form the four outer edges of the following diagram
\[ \begin{align*}
\text{Or} (T \mathcal{M}_{\text{cnt } T_r^{\rho, r}}) & \xrightarrow{} \text{Or} (T \mathcal{M}_{T_r^{\rho, r}}) \xrightarrow{} \text{Or} (T \mathcal{M}_{T_r^{\rho, r}}) \\
\text{Or} (T \mathcal{M}_{\text{cnt } T_r^{\rho, r}}) & \xrightarrow{} \text{Or} (T \mathcal{M}_{T_r^{\rho, r}}) \xrightarrow{} \text{Or} (T \mathcal{M}_{T_r^{\rho, r}})
\end{align*} \]
\[ \begin{array}{c}
\text{Or} (T \mathcal{M}_{\text{cnt } T_r^{\rho, r}}) \\
\text{Or} (T \mathcal{M}_{\text{cnt } T_r^{\rho, r}})
\end{array} \]
Here \( \mathcal{G}_{T_r^{\rho, r}}, \mathcal{G}_{T_r^{\rho, r}} \) are induced from the cartesian squares for \( g_r^{T_r^{\rho, r}} \) and \( g_r^{T_r^{\rho, r}} \), respectively, and we set
\[ \text{Or} (T \mathcal{M}_{\text{cnt } T_r^{\rho, r}}) = \text{Or} (T \mathcal{M}_{\text{cnt } T_r^{\rho, r}}) \otimes \text{ev}_{\rho, r}^{-1} \text{Or} (T L), \]
and similarly for the other occurrences of subscript \( \otimes \). The map \( (\prod \mathcal{F})^{\rho, r}_s \) is defined similarly to \( (\prod \mathcal{F})^{\rho, r}_s \), except we now treat \( s'_t \) as the first element of any \( \sigma_T(v) \) containing it, so there’s no sign twist in this case (compare to \( \vdash \)). The compositions of the top (respectively, bottom) row of arrows in this diagram give \( \mathcal{G}_{T_r} \) (resp., \( \mathcal{G}_{T_r} \)), and we reduce to showing the two squares commute up to signs which depend only on \( T_r \).
Consider the left square. We have
\begin{align}
\partial (\prod \mathcal{F})_\partial^\partial |_{\mathcal{M}_s (T_1)} \times \cdots \times \partial \mathcal{M}_s (T_{n}) & = (-1)^{\dim \mathcal{M}_s (T_1)} \cdot \text{id} \otimes \partial \mathcal{M}_s (T_{n}) \otimes \text{id} \\
\end{align}
and similarly for \( \mathcal{M}_s^\partial \) in place of \( \mathcal{M}_s \), so that
\begin{align}
\partial (\prod \mathcal{F})_\partial^\partial |_{\mathcal{M}_s (T_1)} \times \cdots \times \partial \mathcal{M}_s (T_{n}) & = (-1)^{\cdot} \mathcal{F}_{s} (T_1) \times \cdots \times \partial \mathcal{F}_{s} (T_{n}) \times \cdots \times \mathcal{F}_{s} (T_{n+1}) \\
\end{align}
for
\begin{align}
\bullet & = \sum_{j < i} \dim \mathcal{M}_s (T_{j}) = \sum_{j < i} |\sigma_T (v_j)|, \\
\end{align}
which depends only on \( T_\star \). Thus, we reduce to computing the sign for \( \rho = \emptyset \). So let \( T_\star \) be some tree with a single edge labeled by \( r \), and \( T \) be the unique \((s, \emptyset)\)-labeled tree. Then the oriented base for \( T \partial T_\star \mathcal{M}_T \) picked out by going up then left starting from \( \text{Or}(T \mathcal{M}_T^{(\cdot)}) \) at the bottom center in (53) is defined by the equation
\begin{align}
\partial T_\star \mathcal{M}_T \times L & = \mathcal{M}((k', \nu', \beta'), \cdot) \times (\partial D^2)^{\sigma' \setminus \nu' , T} \times \mathcal{M}_s \times (\partial D^2)^{\sigma''} \\
\end{align}
Where we let \( \mathcal{X} \) stand for a local oriented base for \( T \mathcal{X} \). On the other hand, if we go left then up, we obtain an oriented base defined by the equation
\begin{align}
\partial \mathcal{M}_s^\partial \times L & = \mathcal{M}((k', \nu', \beta'), \cdot) \times \mathcal{M}_s \times (\partial D^2)^{\sigma}. \\
\end{align}
Since \( k'' + \beta'' \equiv 1 \mod 2 \) we find that \( \dim \mathcal{M}_s \equiv 0 \mod 2 \) so the sign discrepancy for the square is simply
\begin{align}
\text{sgn} (\sigma' \setminus \nu' , T) & = \text{sgn} (\sigma' \setminus \nu' , T), \\
\end{align}
where \((\sigma' \setminus \nu' , T)\) denotes the shuffle permutation \((\sigma' \setminus \nu' , T) \rightarrow \sigma\), which depends only on \( T_\star \).

It is not hard to check that the right square of (53) also depends only on \( T_\star \). We will not need the explicit formula for this sign.

Proof of Proposition 20. We define \( \delta_s^\partial \) recursively. We set \( \delta_s^\partial \equiv 1 \). Suppose \( \delta_s^\partial \) is given. Using (44) we can write
\begin{align}
\delta_s^\partial \circ \pi_s^\partial \circ i_{\mathcal{M}_s^\partial} = \nu \circ \pi_s^\partial \circ g_s^\partial \mathcal{U}_r \\
\end{align}
for some \( \nu : \mathcal{F}_s^{\mathcal{U}_r} \rightarrow \{ \pm 1 \} \). We then compute, using Lemma 21 for every \( T_\star \in \mathcal{F}_s^{\mathcal{U}_r} \) and \( T = \text{cnt} T_\star \),
\begin{align}
\mathcal{G}_{T_\star} \circ \partial \mathcal{F}_s^\partial = \left( \delta_s^\partial (\mathcal{T}) \right) \cdot \mathcal{G}_{T_\star} \circ \partial \left( \prod \mathcal{F} \right)_{s}^\partial = \\
= \left( \nu (T_\star) \right) \cdot \mathcal{G}_{T_\star} \circ \partial \left( \prod \mathcal{F} \right)_{s}^\partial = \\
= \left( \left( v \cdot \nu^\partial \mathcal{U}_r \right) (T_\star) \right) \cdot \left( \left( \prod \mathcal{F} \right)_{s}^\partial \mathcal{U}_r \otimes \alpha \right) \circ \left( \left( \prod \mathcal{F} \right)_{s}^\partial \mathcal{U}_r \otimes \alpha \right) \circ \mathcal{G}_{s}^{\mathcal{U}_r} = \\
= \left[ \left( v \cdot \nu^\partial \mathcal{U}_r \right) (T_\star) \right] \cdot \left( \left( \prod \mathcal{F} \right)_{s}^\partial \mathcal{U}_r \otimes \alpha \right) \circ \mathcal{G}_{s}^{\mathcal{U}_r}, \\
\end{align}
so if we take \( \delta_s^{\mathcal{U}_r} = \nu \cdot \nu^\partial \mathcal{U}_r \). Eq (51) holds for \( \mathcal{U}_r \).
we have, for \( T_s \in T_s^\rho \cup^r, T = ctn T_s \),

\[
(\tilde{\mathcal{J}}^r_s \circ \hat{\mathcal{G}}_{T_s}) = \mathcal{J}_s^\rho \circ \iota_{\mathcal{M}^r_s}.
\]

**Proof.** This is similar to the proof of Proposition 20 above, except we use Lemma 21 below, which computes the failure of coherence for \( \prod \tilde{\mathcal{J}} \).

**Corollary 23.** \( \mathcal{J}^r_s := \tilde{\mathcal{J}}^r_s \circ \mathcal{F}^r_s \) satisfies (coherence):

\[
(\mathcal{J}_s^\rho \cup^r) = \mathcal{J}_s^\rho \circ \iota_{\mathcal{M}^r_s}.
\]

**Proof.** Consider some \( T_s \in T_s^\rho \cup^r \) and \( T = ctn T_s \). Restricting our attention to \( \partial T_s := \partial T_s \cdot \mathcal{M} \) we have

\[
(\mathcal{J}_s^\rho \cup^r)^{-1} \circ G^0_s \cup^r \mid_{\partial T_s = \partial T_s \cdot \mathcal{M}^r_s} = (\mathcal{J}_s^\rho \cup^r)^{-1} \circ F^0_s \cup^r \circ G^0_s \cup^r \mid_{\partial T_s} =
\]

\[
= (\mathcal{J}_s^\rho)^{-1} \circ \big( \tilde{\mathcal{J}}^r_s \cup^r \circ G^0_s \cup^r \big) \circ T_{\hat{\partial} T_s} = \tilde{\mathcal{J}}^r_s \circ \iota_{\mathcal{M}^r_s} \circ \partial F^0_s \mid_{\partial T_s} = \tilde{\mathcal{J}}^r_s \circ \iota_{\mathcal{M}^r_s} = \mathcal{J}_s^\rho \circ \iota_{\mathcal{M}^r_s}.
\]

**Lemma 24.** There exists a function \( \eta_s^\rho \cup^r : T_s^\rho \cup^r \rightarrow \{\pm 1\} \) such that

\[
(\prod \tilde{\mathcal{J}})^r_s \circ \iota_{\mathcal{M}^r_s} \mid_{\partial T_s} = (\eta_s^\rho \cup^r (T_s)) \big( \prod \tilde{\mathcal{J}}^r_s \cup^r \big)^{-1} \circ \hat{\mathcal{G}}_{T_s}
\]

for every \( T_s \in T_s^\rho \cup^r, T = ctn T_s \).

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
\text{Or}(T \partial T_s \cdot \mathcal{M}^r_s) & \xrightarrow{\tilde{\mathcal{G}}_{T_s}} & \text{Or}(T \mathcal{M}^r_s) \\
\xrightarrow{\iota_{\mathcal{M}^r_s}} & & \xrightarrow{\iota_{\mathcal{M}^r_s}} \\
\text{Or}(T \mathcal{M}^r_s) & \xrightarrow{\text{Or}(d_T)} & \text{Or}(T \mathcal{M}^r_s) \\
\xrightarrow{\text{Or}(d_T)} & & \xrightarrow{\text{Or}(d_T)} \\
\text{Or}(T \mathcal{M}^r_s) & \xrightarrow{\text{Or}(d_T)} & \text{Or}(T \mathcal{M}^r_s) \\
\end{array}
\]

where \( (\prod \tilde{\mathcal{J}})^r_s \cup^r \) is defined similarly to \( (\prod \tilde{\mathcal{J}})^r_s \cup^r \), with no sign twist, and we have

\[
(\prod \tilde{\mathcal{J}})^r_s \cup^r = (\prod \tilde{\mathcal{J}})^r_s \cup^r \circ (\prod \tilde{\mathcal{J}})^r_s \cup^r.
\]

Using (57), we reduce the computation of the sign of the pentagon on the left to the case \( \rho = \emptyset \); we may assume \( s = ((k, l, \beta), \emptyset) \) for \( k = (1, \ldots, k) \), so we’re working over \( \partial \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M} = (\prod \mathcal{J}^r_s) \). Using 20 for both the domain and codomain of the map we may further assume (without effecting the sign) that \( k' = (1, \ldots, k') \) and \( k'' = (s_k', k + 1, \ldots, k) \) and that the image in \( \mathcal{M} \times \mathcal{M} \) is represented by

\[
(\Sigma' = D^2, \nu' = \text{id}, \kappa', \lambda', u'), (\Sigma'' = D^2, \nu'' = \text{id}, \kappa'', \lambda'', u'').
\]
so that \( \kappa' \) and \( \kappa'' \) both preserve the cyclic order. We can also use the same initial orientation for \( \mathcal{M}_{s'} \) and \( \mathcal{M}_s \) (both of the orientation transports begin at \( 1 \in k' \subset k \) see \( \text{(2.4.1)} \)), and continue the orientation transport of \( \mathcal{M}_{s'} \) to \( s'_1 \) and use the orientation of \( (u_{s'_1}')^{-1} TL = (u_{s''})^{-1} TL \) as the initial orientation for the orientation transport of \( \mathcal{M}_{s''} \). In particular, the orientations for the real boundary conditions \( u_{\partial_3}^1, TL, u_{\partial_3}^1, TL \) and \( u_{\partial_2}^1, TL \), if orientable, are compatible, so that by \( \text{(13)} \) the corresponding orientations for

\[
T(\tilde{\mathcal{M}}(\beta'), u^{(+1)}_\varsigma\times u^{(-1)}_\varsigma\tilde{\mathcal{M}}(\beta''), T\partial\tilde{\mathcal{M}}(\beta))
\]

differ by the sign

\[
W_m(\beta', \beta'') = \begin{cases} 
-1 & m = 1 \mod 2 \\
+1 & m = 0 \mod 2 
\end{cases}
\]

(recall the parity of \( m \) affects whether \( L = \mathbb{RP}^{2m} \) is \( Pin^k \)).

From here the computation is a variation on the proof of \( \text{(4)} \) Proposition 8.3.3], and we try to use similar notation. We assume that \( \delta' = \delta'' = \emptyset \) and \( l = \emptyset \), the computation in the other cases being similar and giving the same result. We use subscript \( L \) to denote the place where we \emph{remove} (the pullback along \( ev_{s'_1} = ev_{s'''} \) of) an oriented base for \( TL \) (this is picked out by the orientation transport as discussed above; since \( \dim L = 2m \) we can shift this without an additional sign).

We let \( \mathbb{R}_{\beta'}, \mathbb{R}_{\beta''}, \mathbb{R}_{\beta'+\beta''} \) denote copies of the 1-parameter subgroup of \( PSL_2(\mathbb{R}) \) fixing \( \pm 1 \in \partial D^2 \). \( \mathbb{R}_{out} \) stands for the outward normal boundary vector, whose positive generator is the image of

\[
(1, -1) \in \text{Lie}(\mathbb{R}_{\beta'} \times \mathbb{R}_{\beta''}) = \mathbb{R}_{\beta'} \oplus \mathbb{R}_{\beta''},
\]

see \( \text{(4)} \). With this in place, the computation runs as follows.

\[
\mathbb{R}_{out} \times \mathcal{M}_{s'} \times_L \mathcal{M}_{s''} \times \mathbb{R}_{\beta'+\beta''} = \\
= (-1)^{d'} (\mathcal{M}_{s'} \times \mathbb{R}_{\beta'}) \times_L (\mathcal{M}_{s''} \times \mathbb{R}_{\beta''}) \\
= (-1)^{d'} \left[ \tilde{\mathcal{M}}(\beta') \times (\partial D^2)^{(k+1),1,2,3,...,k',s'} \times \tilde{\mathcal{M}}(\beta'') \times (\partial D^2)^{k'} \right]_L \\
= (-1)^{d'+k'} \left[ \tilde{\mathcal{M}}(\beta') \times (\partial D^2)^{(k+1),1,2,3,...,k',s'} \times \tilde{\mathcal{M}}(\beta'') \times (\partial D^2)^{k'} \right]_L \\
= (-1)^{d'+k'}(k'-1) \left[ \tilde{\mathcal{M}}(\beta') \times \tilde{\mathcal{M}}(\beta'') \times (\partial D^2)^{(k+1),1,2,3,...,k} \right]_L \\
= (-1)^{d'+k'}(k'-1) \cdot W_m(\beta', \beta'', \tilde{\mathcal{M}}(\beta' + \beta'')) \times (\partial D^2)^{(1,2,3,...,k)} \\
= (-1)^{d'+k'}(k'-1) \cdot W_m(\beta', \beta''). \mathcal{M}_s \times \mathbb{R}_\beta
\]

This should be read as a string of equalities of oriented bases for the tangent space at a point of \( \tilde{\mathcal{M}}(\beta') \times_L \tilde{\mathcal{M}}(\beta') \times (\partial D^2)^{k-2} \). For instance, \( \mathcal{M}_{s'} \) denotes the pullback of an oriented base for \( \mathcal{M}_{s'} \) along the map

\[
q_{1,2} : \tilde{\mathcal{M}}(\beta') \times (\partial D^2)^{(k'+1)-2} \to \mathcal{M}_{s'} \\
(\beta', z_1, ..., z_{k'-1}) \mapsto [D^2, \emptyset, \kappa' = (+1, -1, z_1, ..., z_{k'-1}), \lambda' = \emptyset, \sigma']
\]
On the fourth line, we switch to a different point of $\mathcal{N}(\beta') \times \partial D^2$, mapping to the same point of the moduli space but now using the map

$$g_{1,k+1} : (\tilde{u}', \tilde{z}_1, ..., \tilde{z}_k) \mapsto [D^2, \varnothing, k' = (-1, \tilde{z}_1, ..., \tilde{z}_k, +1), \chi' = \varnothing, \tilde{u}'].$$

Consider $g \in PSL(2,\mathbb{R})$ which takes $(+1,-1,\tilde{z}_1, ..., \tilde{z}_k)$ to $(-1,\tilde{z}_1, ..., \tilde{z}_k, +1)$. The action of $g$ preserves the orientation of $\mathcal{N}(\beta')$ by [14, Proposition 2.8]. It is easy to see that the differential of the action of $g$ is in the same connected component of $GL(k' - 1, \mathbb{R})$ as the map

$$\begin{pmatrix}
1 & -1 \\
\vdots & \vdots \\
1 & -1
\end{pmatrix}$$

which has determinant $(-1)^{k'}$. A similar change occurs on the sixth line, as indicated by the shifting of hats. Finally, we have

$$d' := \dim \mathcal{M}_{g'} = 2m + (2m + 1) \cdot \beta' - 3 + (k' + 1) + 2t' \equiv 2 (k' + 1) + \beta' + 1 \equiv 1$$

$$d'' := \dim \mathcal{N}(\beta'') = (2m + 1) \beta'' \equiv 2 \beta'',$$

so in sum, for $\hat{\rho} = \emptyset$ the left pentagon in (58) commutes up to

$$(-1)^{k' + \beta''(k' - 1)} W_m(\beta', \beta'') = (-1)^{k' + \beta' - \beta''} W_m(\beta', \beta'') = (-1)^{k' + (1 + m) \beta'} \beta'',$$

which clearly factors through $\pi_s^{(1)}$.

The square in (58) commutes (with +1 sign), since the moduli factors appearing in $\mathcal{M}_{g'} \sqcup r$ are all even dimensional, so there’s at most one odd dimensional moduli factor in the spaces lying above them, $\mathcal{M}_{g'} \sqcup r$ and $\hat{\mathcal{M}}_{g'}$. The triangle commutes up to a sign which factors through $\pi_s^r \circ g_s^{[r]}$ (we will not need an explicit formula for it).

2.6.3. Sym$(\rho)$ equivariance.

**Proposition 25.** For $\tau \in$ Sym$(\rho)$ we have

$$J_{s}^\rho \circ \text{Or}(d\tau) = \text{sgn}(\tau) \cdot J_{s}^\rho,$$

$$\hat{J}_{s}^\rho \circ \text{Or}(d\tau) = \text{sgn}(\tau) \cdot \hat{J}_{s}^\rho,$$

$$F_{s}^\rho \circ \text{Or}(d\tau) = \text{Or}(d\tau) \circ F_{s}^\rho,$$

**Proof.** We prove (60). Assume without loss of generality $\rho = \{1, ..., r\}$. Let

$$G_{s}^{[r],r} : \text{Or}(T\partial C\mathcal{M}_{s}^g) \rightarrow \text{Or}(T \mathcal{M}_{s}^{1, ..., r}) \otimes_{\oplus_{i=1}^r \text{ev}_i} \text{Or}(TL)$$

be the local system map over $g_{s}^{[r],r}$ associated with the cartesian square (11). If we equip the domain with the Sym$(r)$-action that permutes boundary faces, we have

$$G_{s}^{[r],r} \circ \text{Or}(d\tau) = \text{Or}(d\tau) \circ G_{s}^{[r],r}.$$

Iteratively applying coherence we find that

$$\left(J_{s}^{(1, ..., r)} \right)^{-1} \circ G_{s}^{[r],r} = (-1)^{\ell} J_{s}^\rho \circ \text{Or}(\mathcal{M}_{s}^g).$$
Here the sign \((-1)^{\zeta(T)}\) comes from
\[
G^{[r],r}_x = (-1)^{\zeta(T)} G^{(1,\ldots,r)}_x \circ \partial G^{(1,\ldots,r-1)}_x \circ \ldots \circ \partial^{r-1} G^{(1)}_x
\]
(the cartesian natural local system maps between pullbacks of \(\text{Or}(TL)\) are suppressed in this equation), which reflects a reversal of the order of the outward normal vectors; at any rate all we need is that this sign factor is constant on \(\mathcal{M}^{[r]}_x\).

Clearly, \(J^{\#}_x\) is \(\text{Sym}(r)\) invariant and \(\iota^{\partial r} \circ \text{Or} (d\tau) = \text{sgn} (\tau) \iota^{\partial r} \). Combining this with (64) and (63) we deduce that (60) holds on the image of \(g^{[r],r}_x\). By the long exact sequence of a fibration, \(g^{[r],r}_x\) visits every connected component of \(\mathcal{M}^{[r+1,\ldots,r]}_x\), so (60) holds everywhere and the proof of (60) is complete. The proof of (61) is similar, and (62) follows directly from (60) and (61). \(\square\)

2.6.4. Explicit signs for odd-even trees. We conclude the discussion of orientations with a result that will be used in deriving the explicit formula for the fixed point contributions in [15].

First we state a general lemma which aids the computation of \(J\). For a labeled tree \(T \in \mathcal{J}^{[r]}_x = \mathcal{J}^{(1,\ldots,r)}_x\), we have
\[
J^{[r]}_x \big|_{\mathcal{M}^T} = \theta (\mathcal{T}) \cdot (\prod_j J)_x^{[r]}
\]
for \(\theta (\mathcal{T}) = \delta (\mathcal{T}) \cdot \zeta (\mathcal{T})\) (see Propositions 20 and 22). The smoothing sequence of \(\mathcal{T}\) is
\[
\mathcal{T} = \mathcal{T}^{(0)}, \mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(r)}
\]
where, for \(1 \leq j \leq r\), \(\mathcal{T}^{(j)} \in \mathcal{J}^{(j+1,\ldots,r)}_x\) is obtained from \(\mathcal{T}^{(j-1)}\) by contracting the edge \(j\); more precisely it is uniquely specified by requiring
\[
\partial M_{\mathcal{T}^{(j)}} \subset \left( g^{(j,\ldots,r-1)}_x \right)^{-1} (M_{\mathcal{T}^{(j-1)}}).
\]

Lemma 26. We have
\[
\theta (\mathcal{T}) = \prod_{a=1}^r \xi (\mathcal{T}^{(a-1)})
\]
and
\[
\zeta (\mathcal{T}) = \prod_{a=1}^r \xi (\mathcal{T}^{(a-1)})
\]
where for \(1 \leq a \leq r\), \(\xi (\mathcal{T}^{(a-1)}), \xi (\mathcal{T}^{(a-1)})\) are computed as follows. We denote by \(v', v''\) the tail and head, respectively, of the edge \(a\) in \(\mathcal{T}^{(a-1)}\), and let
\[
k' = |k_{\mathcal{T}^{(a-1)}} (v')|, \beta' = \beta_{\mathcal{T}^{(a-1)}} (v'), \beta'' = \beta_{\mathcal{T}^{(a-1)}} (v'')
\]
Let \(\mathcal{T}_0^{(a)} = (v_1, \ldots, v_{r+1-a})\) and let \(v_1\) be the vertex obtained from contracting the edge \(a\). We have
\[
\xi_T (\mathcal{T}^{(a-1)}) = (-1)^{r-a} \cdot (-1)^{k' \times (1+m) \beta' \beta''} \cdot (-1)^{\sum_{s \in \mathcal{T}_0^{(a)}} |(v_s)\cdot \text{sgn} (\sigma') \in \mathcal{M}^{[r],r}_x\}
\]
\[
\xi_T (\mathcal{T}^{(a-1)}) = (-1)^{r-a} \cdot (-1)^{k' \times (1+m) \beta' \beta''}
\]

Proof. We consider \(\xi_T\) first. We have \(\theta (\mathcal{T}^{(r)}) = 1\), so it suffices to prove that \(\theta (\mathcal{T}^{(a-1)}) = \xi (\mathcal{T}^{(a)}) \cdot \theta (\mathcal{T}^{(a)})\). Without loss of generality, assume \(a = 1\). Let
\( \tau : (1, \ldots, r) \mapsto (r, 1, \ldots, r - 1) \) be the label-preserving bijection. We have
\[
(-1)^{r-1} \left( \mathcal{J}_b^{[r]} \right)^{-} \circ \mathcal{G}^i_{\tau(0)} = \left( \mathcal{J}_b^{[r]} \right)^{-} \circ \text{Or} (d\tau) \circ \mathcal{G}^i_{\tau(0)} = \\
= \left( \mathcal{J}_b^{[r]} \right)^{-} \circ \mathcal{G}^i_{\tau(0)} = \mathcal{J}_b^{[r-1]} \circ \iota_{\mathcal{T}_b^{[r-1]}},
\]
where the sign \((-1)^{i} = (-1)^{k'(1+m)} \beta' \beta'' \cdot (-1)^{\Sigma_{v \in c_1} \sigma(\ell)} \cdot \text{sgn} (\sigma') \cdot \sigma'' \) is given by \[\text{(38)}\] (this is the product of signs of the left square in \[\text{(38)}\] and the left trapezoid in \[\text{(38)}\]). The result follows.

The computation of \( \xi_{\tau} \) is similar:
\[
(-1)^{r-1} \left( \tilde{\mathcal{J}}_b^{[r]} \right)^{-} \circ \mathcal{G}_{\tau(0)}^{i} = \left( \tilde{\mathcal{J}}_b^{[r]} \right)^{-} \circ \text{Or} (d\tau) \circ \mathcal{G}_{\tau(0)}^{i} = \\
= \left( \tilde{\mathcal{J}}_b^{[r]} \right)^{-} \circ \mathcal{G}_{\tau(0)}^{i} = \tilde{\mathcal{J}}_b^{[r-1]} \circ \iota_{\mathcal{T}_b^{[r-1]}},
\]

Here we have \( \mathcal{G}_{\tau(0)}^{i} := \text{Or} (d\tau)^{-1} \circ \mathcal{G}_{\tau(0)}^{i} \) (justifying the equality between the first and second line above). The sign \((-1)^{i} \) is the sign of the parallelogram on the bottom left of the following diagram (compare \[\text{(38)}\]):

\[
\begin{array}{ccc}
\text{Or} \left( T \partial \mathcal{M}_\mathcal{T} \right) & \xrightarrow{\mathcal{G}_{\tau(0)}^i} & \text{Or} \left( T \mathcal{M}_\mathcal{T}^{\{r\}} \right) \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\text{Or} \left( T \hat{\mathcal{M}}_b^{\{r\}} \right) & \xrightarrow{\theta} & \text{Or} \left( T \hat{\mathcal{M}}_b^{\{r\}} \right)
\end{array}
\]

The dashed arrow is \( \mathcal{G}_{\tau(0)}^i \), which by definition is the map that makes the triangle directly above it commute. By the definition of \( \mathcal{G}_{\tau(0)}^i \), the triangle directly below this map also commutes. We’ve argued above that the square commutes, so \( \mathcal{G}_{\tau(0)}^i \circ \mathcal{G}_{\tau(0)}^i = \left( \prod \mathcal{J}_b^{[r]} \right)^{\rho} \circ \mathcal{G}_{\tau(0)}^i \), so that
\[
(-1)^{i} = (-1)^{k'(1+m)} \beta' \beta''
\]
by \[\text{(38)}\].

\[\text{Definition 27.} \quad \text{(a) A labeled tree } \mathcal{T} \in \mathcal{J}_b^{[r]} \text{ will be called an \textit{odd-even tree} if all the moduli specifications } ((k, \ell, \beta, \sigma) = \ell_{\mathcal{T}}(v) \text{ for a vertex } v \in \mathcal{T}_0 \text{ satisfy the following condition: if } \beta = 0 \mod 2 \text{ then } \sigma = \emptyset \text{ and if } \beta = 1 \mod 2 \text{ then } k = \emptyset \text{ (in particular this means the tree is bipartite with respect to the partition into odd degree and even degree vertices, which explains the name).}\]

An odd-even tree $T \in \mathcal{T}^{[r]}_b$ will be called sorted if the graph spanned by the edges $1, \ldots, a$ is connected for every $1 \leq a \leq r$, and such that for any $v \in \mathcal{T}_0$, if $\{s'_i, s'_j\} \subset \sigma_T(v)$ for some $i < j$, then $\sigma'_a \in \sigma_T(v)$ for all $1 \leq a \leq j$.

For every odd-even tree $T \in \mathcal{T}^{[r]}_b$ there exists at least one $\tau \in \text{Sym}(r)$ such that $\tau \cdot T$ is sorted. The following facts are readily verified:

- Ordering the vertices of $T$, $\mathcal{T}_0 = (v_1, \ldots, v_r+1)$ as in Lemma 15(b), if $\beta_T(v_i) = 1 \mod 2$ and $\beta_T(v_j) = 0 \mod 2$ then $i < j$, so the odd vertices appear before the even vertices.
- Let $n_{\text{odd}}$ denote the number of odd vertices. There are integers $r \geq s_1 \geq s_2 \geq \cdots \geq s_{n_{\text{odd}}} = 0$ such that for $1 \leq i \leq n_{\text{odd}}$ we have
  $$\sigma_T(v_i) = \left(s'_{s_i+1}, \ldots, s'_{s_{i-1}}\right)$$
  and $\sigma_T(v_{n_{\text{odd}}}) = \emptyset$ for $n_{\text{odd}} + 1 \leq i \leq r + 1$.
- $T(\tau)$ is sorted odd-even for every $T^{(\tau)}$ in the smoothing sequence of $T$.

**Proposition 28.** Let $b = (k, 1, \beta)$ be a basic moduli specification. For a sorted odd-even tree $T \in \mathcal{T}^{[r]}_b := \mathcal{J}^{[r]}_b(\emptyset)$ we have

\[
\mathcal{J}^{[r]}_b|_{\mathcal{M}_T} = (-1)^{(1+m)\xi} (\prod \mathcal{J})^{[r]}_b
\]

\[
\mathcal{J}^{[r]}_b|_{\mathcal{M}_T} = (-1)^{(\xi \cdot \beta)} (-1)^{(1+m)\xi} (\prod \mathcal{J})^{[r]}_b
\]

\[
\mathcal{J}^{[r]}_b|_{\mathcal{M}_T} = (-1)^{(\xi \cdot \beta)} (\prod \mathcal{J})^{[r]}_b
\]

where $\xi$ is the number of odd vertices.

**Proof.** We prove (65). Using the notation of Lemma 26 and the properties of sorted odd-even trees listed above, we have

$$\theta(T) = \prod_{a=1}^r \xi(T^{(a-1)})$$

for

$$\xi(T^{(a-1)}) = (-1)^{(1+m)\beta'_{\beta''}} (-1)^{(r-a)} (-1)^{\sum_{v_i} |\sigma_T(v_i)|} (\sigma'/\sigma'').$$

It is easy to see that

$$\prod_{a=1}^r f_1(a) = (-1)^{(1+m)\beta|\sigma_T(v_i)|(\beta'(v_j))} = (-1)^{(1+m)\xi}.$$ 

Fix some $1 \leq a \leq r$. Let $1 \leq j \leq a$. be such that $s'_a \in \left(s'_{s_{a+1}}, \ldots, s'_{s_{a-1}}\right) = \sigma_T(v_j)$. For every $1 \leq b < s_{a+1}$, $s'_a$ contributes to $f_3(b)$. For $b = s_{a+1}$, $s'_a$ contributes to the sign of the shuffle $\left(\sigma'/\sigma_{s_{a+1}}, \sigma''\right)$, and for $b > s_{a+1}$ it contributes nothing (with the convention that an element contributes to the shuffle sign when it is commuted past elements preceding it, but not when elements succeeding it are commuted past it). This shows that the total contribution of $s'_{a}$ to $\prod_{b=1}^r \left(f_3(b) \cdot f_4(b)\right)$ is $(-1)^{(r-a)}$, which cancels $f_2(a)$.

The proof of (66) is simpler, since $f_3(a)$ and $f_4(a)$ terms are absent. Equation (67) follows from the previous two equations since $\mathcal{J}^{[r]}_b = \mathcal{J}^{[r]}_b \circ \mathcal{J}^{[r]}_b$ and $(\prod \mathcal{J})^{[r]}_b = (\prod \mathcal{J})^{[r]}_b \circ (\prod \mathcal{J})^{[r]}_b$. □
Remark 29. The factor \((-1)^{(1+m)-(l)}\) was verified using relations derived from fixed-point localization, see [18] Example 35.

2.7. Proof of Proposition 7 Most of the theorem is obtained directly from the results of the previous subsections, by specializing to \(\rho = [r]\) and \(s = (b, \emptyset)\). We now fill in the few remaining gaps.

By Lemma 15 and the stability condition on the specification \(s_T(v)\) of the vertices, \(\mathcal{F}_b^r = \emptyset\) for sufficiently large \(r\), and then \(\mathcal{M}_b^r = \emptyset\) too.

We construct the involution \(\text{inv}^r_s : \partial_+, \mathcal{M}_b^r \to \partial_+, \mathcal{M}_b^r\) recursively. We have

\[
\partial^r_{\text{for}^r} \mathcal{M}_b^r = \prod \left( \left( \partial^r_{\text{for}^r} \mathcal{M}_b^r \times \mathcal{M}_b^r \right) \prod \left( \mathcal{M}_b^r \times \partial^r_{\text{for}^r} \mathcal{M}_b^r \right) \right),
\]

so we can define recursively \(\text{inv}^r_s\) by setting \(\text{inv}^r_s = \text{inv}^r_s\) and

\[
\text{inv}^r_s = \prod \left( \left( \text{inv}^r_s \times \text{id} \right) \prod \left( \text{id} \times \text{inv}^r_s \right) \right).
\]

A simple inductive proof based on (30) shows that \((\prod \mathcal{F})^r_s \circ \iota_{\mathcal{M}_b^r} \circ \text{Or} (d \text{inv}^r_s) = (-1)^{(\prod \mathcal{F})^r_s} \circ \iota_{\mathcal{M}_b^r} \text{ and that } \pi^r_s \circ \text{inv}^r_s = \pi^r_s\). Since \((\prod \mathcal{F})^r_s\) and \(\mathcal{F}^r_s\) differ by a sign which factors through \(\pi^r_s\) we find that

\[\mathcal{F}^r_s \circ \iota_{\mathcal{M}_b^r} \circ \text{Or} (d \text{inv}^r_s) = (-1)^{(\prod \mathcal{F})^r_s} \circ \iota_{\mathcal{M}_b^r}.\]

This completes the proof of Proposition 7.

3. Extended forms and Stokes' Theorem

In this section we define an integration map and prove Stokes' theorem. Throughout this section, we fix some stable basic moduli specification \(b = (k, l, \beta)\), and we may omit it from the notation to avoid clutter.

The internal and external local systems on \(\mathcal{M}_b^r = \mathcal{M}_b^r = \mathcal{M}_b^r\) are defined, respectively, by

\[
\mathcal{I}_b^r = \bigotimes_{x \in \mathcal{I}_b^r} (ev_{x, r})^{-1} \text{Or} (TL) \quad \text{and} \quad \mathcal{E}_b^r = \bigotimes_{x \in \mathcal{E}_b^r} (ev_{x, r})^{-1} \text{Or} (TL).
\]

The internal and external local systems on \(\mathcal{M}_b^r\) are given by \(\mathcal{I}_b^r = (\text{For}^r_b)^{-1} \mathcal{I}_b^r\) and \(\mathcal{E}_b^r = (\text{For}^r_b)^{-1} \mathcal{E}_b^r\), respectively.

3.1. Statement of Stokes' theorem.

Definition 30. An extended form \(\omega\) for \(b\) is a sequence

\[
\left\{ \tilde{\omega}_r \in \Omega \left( \mathcal{M}_b^r, \mathcal{E}_b^r \otimes \mathbb{R} [\alpha] \right) \right\}_{r \geq 0},
\]

such that

(a) the following coherence property holds for all \(r \geq 0\) and \(r \in \mathcal{F}_b^{r+1}\),

\[
\left( \left( \iota_{\mathcal{M}_b^r} \right)^* \tilde{\omega}_r \right) \mid_{\partial^r_{\mathcal{M}_b^r}} = \left( \tilde{\omega}_{r+1} \right)^* \left( \iota_{\mathcal{M}_b^r} \right)
\]

(see [15, 43]).

(b) \(\tilde{\omega}_r\) is \(\text{Sym} (r)\)-invariant.

We denote by \(\Omega_b\) the differential graded \(\mathbb{R} [\lambda_1, ..., \lambda_m]\)-module of all extended forms; the degree \((+1)\) differential \(D : \Omega_b \to \Omega_b\) is given by

\[
D \{ \tilde{\omega}_r \}_{r \geq 0} = \{ D \tilde{\omega}_r \}.
\]
Our goal in this section is to define an $\mathbb{R}[\tilde{\alpha}]$-module map
\[
\int_b : \Omega_b \to \mathbb{R}[\tilde{\alpha}] = \mathbb{R}[\lambda_1, \ldots, \lambda_m]
\]
of degree
\[-[(2m + 1)\beta + 2m + (k - 3) + 2l] = -\dim M^b_i
\]
and prove

**Theorem 31. (Stokes’ Theorem)** We have $\int_b D\omega = 0$ for any $\omega \in \Omega_b$.

**Remark 32.** One can also consider the complex of non-equivariant extended forms $\Omega^\text{ne}_b$, consisting of sequences of forms $\tilde{\omega}_r \in \Omega(M^b_0, E_0^r \otimes \mathbb{R})$ satisfying the same two conditions as in Definition 30 above; the differential is given by the level-wise action of the exterior derivative $d$, integration is defined using the same formula as in Definition 34 below, except we replace $\Lambda$ by
\[
\Lambda^\text{ne} := \Lambda \mod (\lambda_1, \ldots, \lambda_m)
\]
The proof of Stokes’ theorem carries through and we find
\[
(69) \quad \int_b d\omega = 0.
\]
The map $\mathbb{R}[\tilde{\lambda}] \to \mathbb{R}$ lifts to a map $R_b : \Omega_b \to \Omega^\text{ne}_b$ commuting with integration.

We mention this for two reasons. First, we do not know whether the induced map $H(R_b) : H(\Omega_b) \to H(\Omega^\text{ne}_b)$, is surjective, so there may be non-equivariant invariants which do not admit an equivariant extension. Note, however, that $X^l \times L^k$ has cohomology in even degrees so the map
\[
H^\bullet(X^l \times L^k) \to H^\bullet(X^l \times L^k)
\]
is surjective (cf. [19, Proposition 32]). This means the equivariant open Gromov-Witten invariants we considered in §1.2 exhaust both equivariant and non-equivariant invariants defined by pull back along the evaluation maps.

The second reason to consider non-equivariant invariants is that (69) represents a stronger invariance property, which can be used to offer a geometric interpretation of the invariants with deg $I(k,l,\beta) = 0$, as we discussed in Remark 3.

### 3.2. Resolution blow ups.

In this subsection we write $G$ for the group $O(2m + 1)$. Fix some $G$-invariant Riemannian metric on $L \times L$, and construct a $G$-equivariant tubular neighborhood
\[
(70) \quad N_\Delta \overset{i_\Delta}{\longrightarrow} V_\Delta \overset{\gamma_\Delta}{\longrightarrow} L \times L.
\]
Let $\overline{L \times L} \overset{\text{bun}}{\longrightarrow} L \times L$ denote the blow up of the diagonal $\Delta L \subset L \times L$ (cf. [19, Definition 37]). It is a $G$-equivariant map of manifolds with corners. Explicitly,
\[
\overline{L \times L} = S(N_\Delta) \times (0,\epsilon) \bigcup_{S(N_\Delta) \times (0,\epsilon)} (L \times L \setminus \Delta L).
\]

For each $T \in \mathcal{B}^r$, $G^{r+1}$ acts on $\mathcal{M}_T$ where the $i$'th factor of $G$ acts on the $i$'th factor of $\mathcal{M}_T = \prod_{\nu \in \mathcal{G}_0} \mathcal{M}_{T^{(\nu)}}$, making $ev^i_\nu |_{\mathcal{M}_T}$, $ev^i_\nu |_{\mathcal{M}_{T^{(\nu)}}}$ into $(G^{r+1} \to G)$-equivariant maps for each $i$, with respect to a suitable projection $(G^{r+1} \to G)$. This implies that $ed_b : \mathcal{M}^b_0 \to (L \times L)^r$ is $b$-transverse to $\text{bun}_b ^r : \overline{L \times L}^r \to (L \times L)^r$ (recall that this means that the restrictions of these two maps to corner strata are also transverse, see Remark 35), and we use Lemma 48 to construct the cartesian square
By a slight abuse of notation we denote this map by $\tilde{\mathrm{comb.alt1}}$. Since the right and bottom map are $G \times \text{Sym}(r)$ equivariant, there’s a natural $G \times \text{Sym}(r)$ action on $\tilde{\mathcal{M}}_b^r$ making the entire square equivariant.

By Lemma 48(c), since $\tilde{\mathcal{M}}_b^r$ is $b$-normal so is $\tilde{\mathcal{M}}_b^r$. In particular we have a decomposition $\partial \tilde{\mathcal{M}}_b^r = \partial_+ \tilde{\mathcal{M}}_b^r \amalg \partial_\tilde{\mathcal{M}}_b^r \amalg \partial \tilde{\mathcal{M}}_b^r$, where $\partial_+ \tilde{\mathcal{M}}_b^r = (\tilde{\mathcal{M}}_b^r)_{-1} \big( \partial_+ \tilde{\mathcal{M}}_b^r \big)$.

Writing $\tilde{\mathcal{M}}_b^r$ as a composition of maps blowing up a single edge at a time, using 15 to further break down $\partial_+ \tilde{\mathcal{M}}_b^r$, we obtain the following decomposition of the blow up boundary:

$$
\partial \tilde{\mathcal{M}}_b^r = \partial_+ \tilde{\mathcal{M}}_b^r \amalg \partial_\tilde{\mathcal{M}}_b^r \amalg \prod_{j=1}^r \big( \tilde{\mathcal{M}}_b^r \big)^{-1} S(N_\Delta).
$$

We turn to discuss orientations. Clearly $d\tilde{\mathcal{M}}_b^r$ is a diffeomorphism away from the boundary of $\tilde{\mathcal{M}}_b^r$, so it induces (see Lemma 52(a)) a $G \times \text{Sym}(r)$ equivariant local system map

$$
\text{Or}(T\tilde{\mathcal{M}}_b^r) \to \text{Or}(T\mathcal{M}_b^r).
$$

By a slight abuse of notation we denote this map by $\text{Or}(d\tilde{\mathcal{M}}_b^r)$.

We set $\mathcal{J}_b^r : \text{Or}(T\tilde{\mathcal{M}}_b^r) \to \text{Or}(TL)\amalg(k \amalg s^r)$ to be the composition

$$
(72) \quad \mathcal{J}_b^r = \mathcal{J}_b^r \circ \text{Or}(d\tilde{\mathcal{M}}_b^r);
$$

it follows that

$$
(73) \quad \mathcal{J}_b^r \circ \text{Or}(d\tau) = \text{sgn}(\tau) \mathcal{J}_b^r.
$$

Since $\partial_+ \mathcal{M}_b^r$ is the fiber product of $\partial \mathcal{M}_b^r$ with $\tilde{\mathcal{M}}_b^r$, the pair of maps

$$
\big( \text{id}(L \times L), \text{inv}^r \big)
$$

define an involution

$$
\text{inv}^r : \partial_+ \mathcal{M}_b^r \to \partial_+ \mathcal{M}_b^r
$$

such that $\tilde{\mathcal{M}}_b^r \circ \partial_+ \mathcal{M}_b^r \circ \text{inv}^r = \text{inv}^r \circ \tilde{\mathcal{M}}_b^r \circ \partial_+ \mathcal{M}_b^r$. It follows that

$$
(74) \quad \mathcal{J} \circ \partial_+ \mathcal{M}_b^r \circ \text{Or}(d\text{inv}^r) = (-1) \mathcal{J} \circ \partial_+ \mathcal{M}_b^r.
$$
3.3. Equivariant homotopy kernel. We fix an equivariant homotopy kernel $\Lambda \in \Omega(\tilde{L} \times \tilde{L}; \tilde{pr}_2^*(Or(TL)) \oplus \mathbb{R} \frac{\Delta}{\partial})$ for $L$. Namely,

$$\Lambda = \sigma \left( \tilde{pr}_S(N_D) \right)^* \phi + b\sigma^r \Upsilon$$

for $\phi$ an equivariant angular form for $S(N_D)$, $\sigma : [0, \infty) \to [0, 1]$ a smooth, compactly supported cutoff function with $\sigma(0) = +1$, and $\Upsilon$ chosen so that

$$DA = \tilde{pr}_2^* \rho_0 \in \text{Im}(\tilde{pr}_2^*) \subset \text{Im}(b\sigma^r),$$

where $\rho_0$ is an equivariant form representing the point class. It follows that

$$\left(\frac{\partial}{\partial L \times \tilde{L}}\right)^* \Lambda = \phi + \left(\pi^{S(N_D)}\right)^* \Upsilon|\Delta.$$

Remark 33. Compare this to Definition 55 and Proposition 56 in [19]. First, there is a minus sign introduced in $\sigma$ for convenience. More importantly, here we require only conditions (75) and (76), whereas in [19] $\Lambda$ (denoted $\Lambda'$ there) depended on a particular choice of form $\rho$ representing the point class, and the associated homotopy operator $h'$ was modified further in order to satisfy the side conditions (see Definition 23 ibid.). If one can construct a unital cyclic retraction that is represented by a smooth kernel $\Lambda$ as above, then the open Gromov-Witten invariants we define here also encode the unital cyclic homotopy type of the twisted equivariant Fukaya $A_\infty$ algebra of $\mathbb{R}P^{2m} \to \mathbb{C}P^{2m}$. See [18] §1.6 for a detailed discussion.

3.4. Integration of extended forms.

Definition 34. Let $\omega = \{\omega_r\} \in \Omega_b$ be an extended form. We define

$$\int_b \omega = \sum_{r \geq 0} \frac{1}{r!} \int_{\tilde{M}_b^r} (b\omega_b^r)^* \omega_r \cdot (\tilde{c}\tilde{d}_b^r)^* \Lambda^{gr},$$

where $\omega_r := (\text{For}_b^r \omega_r)$ and $(\tilde{c}\tilde{d}_b^r)^* \Lambda^{gr}$ are forms with values in $(b\omega_b^r)^{-1} \mathcal{E}_b^r \otimes \mathbb{R} [\lambda_1, \ldots, \lambda_m]$ and in $(b\omega_b^r)^{-1} \mathcal{I}_b^r \otimes \mathbb{R} [\lambda_1, \ldots, \lambda_m]$, respectively, so that the integrand takes values in

$$(b\omega_b^r)^{-1} (\mathcal{E}_b^r \otimes \mathcal{I}_b^r) \otimes \mathbb{R} [\tilde{\alpha}] = \bigotimes_{x \in k} \bigoplus_{e' \in \mathcal{F}_x^r} \left(\tilde{c}\tilde{d}_b^r(\tilde{\alpha}) \right)^{-1} Or(TL)$$

and the integral is computed using $\tilde{F}_b^r$. More precisely, integration of real-valued forms is defined as pushforward along the horizontally-submersive map $\tilde{M}_b^r \to \text{pt}$, see [20] Eq (24), which becomes an oriented map using $\tilde{F}_b^r$. The integral is then extended $\mathbb{R} [\tilde{\alpha}]$-linearly to define integration of equivariant forms.
Proof of Theorem 31 (Stokes’ Theorem). The computation goes as follows.

\[
\int_{\mathcal{M}} D\omega = \sum_{r \geq 0} \frac{1}{r!} \int_{\mathcal{M}} (bu_b^r)^* D(\omega) \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} \tag{1}
\]

\[
\begin{aligned}
\sum_{r \geq 0} \frac{1}{r!} \int_{\partial \mathcal{M}_b} (bu_b^r)^* \omega \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} = & \sum_{r \geq 0} \frac{1}{r!} \sum_{i=0}^{r-1} \int_{\partial \mathcal{M}_b} \omega_i \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} + \\
\sum_{r \geq 0} \frac{1}{(r+1)!} \int_{\partial \mathcal{M}_b} (bu_b^r)^* \omega_1 \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} + \\
= & \sum_{r \geq 0} \frac{1}{(r+1)!} \int_{\partial \mathcal{M}_b} (bu_b^r)^* \omega_1 \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} + \\
\end{aligned}
\]

\[
\int_{\partial \mathcal{M}_b} (bu_b^r)^* \omega \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} = 0.
\]

The equality marked (1) is justified by Lemma 35 below. The equality marked (2) is the usual Stokes’ theorem. To justify the equality (3), we argue that

\[
\int_{\partial \mathcal{M}_b} (bu_b^r)^* \omega \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} = 0.
\]

For this use (4) and observe that the integrand is \( \tilde{\tilde{\omega}} \)-invariant since

\[
\text{For} \circ \omega_b \circ \partial \mathcal{M}_b \circ \tilde{\tilde{\omega}} = \text{For} \circ \omega_b \circ \partial \mathcal{M}_b
\]

and

\[
\tilde{\tilde{w}} \circ \partial \mathcal{M}_b \circ \tilde{\tilde{\omega}} = \tilde{\tilde{w}} \circ \partial \mathcal{M}_b.
\]

We justify the equality (4). Write the expression on the fourth line as

\[
\sum_{i=0}^{r-1} \int_{\partial \mathcal{M}_b} \omega_i \cdot (\widetilde{ed}_{r})^* \Lambda^{gr}.
\]

We claim that

\[
I_{r+1}(i) = I_{r+1}(r+1),
\]

so \( \sum_{i=0}^{r+1} \int_{\partial \mathcal{M}_b} \omega_i \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} \), which immediately gives the equality (4). To prove (77), pullback the integrand by some \( \tau \in \operatorname{Sym}(r+1) \) with \( \tau(i) = r+1 \). \( \tilde{\tilde{w}} \) picks up a sign, \( \text{sgn}(\tau) \), by (73), which cancels the sign of permuting the odd-degree \( \Lambda \)'s.

Eq (45) implies \( \omega_{r+1} = (g_0^{r+1})^* \omega_r \). Using this and Corollary 23 we obtain equality (5) by integrating out \( \phi \).

\[
\text{Lemma 35.} \quad \int_{\partial \mathcal{M}_b} (bu_b^r)^* \omega \cdot (\widetilde{ed}_{r})^* \Lambda^{gr} = 0.
\]

Proof. By (73), the integrand is pulled back from the orbifold \( \mathcal{M}' \), obtained from \( \mathcal{M}_b \) by forgetting \( \ast_i \) and blowing up \( \mathcal{M}_b \) by \( \Delta \) for \( i \neq j \). Since

\[
\dim \mathcal{M}' = \dim \mathcal{M}_b - 1,
\]
the integral vanishes. □

4. Appendix: Orbifolds with Corners

This appendix summarizes briefly some definitions and results from [20, §3]. The reader should consult that reference for the proofs and more detail.

4.1. Manifolds with corners. We refer the reader to Joyce [6, §2] for the terminology we use regarding manifolds with corners. The manifolds we’ll consider have “ordinary” corners (as opposed to generalized corners), which are modeled on \( \mathbb{R}^n \) equipped with a maximal \(-\)-dimensional atlas of charts \((U, \phi)\) where \( U \subset X \) is open and \( \phi : U \to \mathbb{R}^n \) is a homeomorphism \( (n \text{ is fixed, } k \text{ may vary}) \), with weakly smooth transitions. A weakly smooth map \( f : X \to Y \) between manifolds with corners is a continuous map which is of this form in every coordinate patch. A weakly smooth map \( f : X \to Y \) is said to be smooth, strongly smooth, interior, b-normal, simple, or a b-fibrations as in [6] Definitions 2.1, 4.3. “A map” between manifolds with corners will always be assumed to be smooth unless specifically stated otherwise, and we denote by \( \text{Man}^c \) the category of manifolds with corners with smooth maps.

The depth of a point \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) is defined by depth \( (x) = \# \{1 \leq i \leq k | x_i = 0 \} \). It is easy to see that the transitions preserve the depth, so we can speak of the depth of a point \( x \in X \). We define \( S^k(X) = \{ x \in X | \text{depth} (x) = k \} \). A local k-corner component \( \gamma \) of \( X \) at \( x \) is a local choice of connected component of \( S^k(X) \) near \( x \) (cf. [6] Definition 2.7); a local 1-corner component is also called a local boundary component.

We have manifolds with corners
\[
\partial X = C_1(X) = \{(x, \beta) | x \in X, \beta \text{ is a local boundary component of } X \text{ at } x \}
\]
and, for every \( k \geq 0 \),
\[
C_k(X) = \{(x, \gamma) | x \in X, \gamma \text{ is a local } k \text{-corner component of } X \text{ at } x \}.
\]

Letting \( \partial^k X \) denote the iterated boundary, we find that \( C_k(X) \simeq \partial^k X / \text{Sym}(k) \) where \( \text{Sym}(k) \) acts by permuting the local boundary components.

We can consider \( C(X) = \bigsqcup_{k \geq 0} C_k(X) \) as a local manifold with corners (or “manifold with corners of mixed dimension”, in Joyce’s terms). These form a category and the various properties of maps can be used to describe maps between local manifolds with corners. If \( f : X \to Y \) is a smooth map of manifolds with corners, there’s an induced interior map
\[
C(f) : C(X) \to C(Y)
\]
We denote by \( i^\partial_X : \partial X \to X \) the map defined by \( i^\partial_X ((x, \beta)) = x \). Even if \( X \) is connected, \( \partial X \) may be disconnected and \( i^\partial_X \) may not be injective. Sometimes we abbreviate \( i^\partial = i^\partial_X \).

A strongly smooth map \( f : X \to Y \) between manifolds with corners is a submersion if, whenever \( x \) of depth \( k \) maps to \( y = f(x) \) of depth \( l \), both \( df|_x : T_x X \to T_y Y \)
and \( df|_x : T_x S^k(X) \rightarrow T_y S^l(Y) \) are surjective (see [7, Definition 3.2]; beware that a “smooth map” there is what we call a strongly smooth map, see [6, Remark 2.4,(iii)]). We say a map \( f : X \rightarrow Y \) is perfectly simple if it is simple and maps points of depth \( k \) to points of depth \( k \), and is étale if it is a local diffeomorphism.

If \( X \) is a manifold with corners its tangent bundle \( TX \) is defined in the obvious way. In addition, one can consider the \( b \)-tangent bundle \( bTX \). It is a vector bundle on \( X \) whose sections can be identified with sections \( v \in C^\infty(X) \) such that \( v|_{S^k(X)} \) is tangent to \( S^k(X) \) for all \( k \) (cf. [6, Definition 2.15]). If \( f : X \rightarrow Y \) is an interior map of orbifolds with corners, there’s an induced map \( bdf : bTX \rightarrow bTY \). Two interior maps \( f : X \rightarrow Z \) and \( g : Y \rightarrow Z \) are called \( b \)-transverse if for any \( x \in S^j(X), y \in S^k(Y) \) such that \( f(x) = g(y) = z \), the map

\[
 bdf \oplus bg : bT_x X \oplus bT_y Y \rightarrow bT_z Z
\]

is surjective.

**Remark 36.** In case \( \partial Z = \emptyset \), \( f, g \) are \( b \)-transverse if and only if for every \( x \in S^j(X), y \in S^k(Y) \) with \( f(x) = g(y) = z \) the map

\[
 df|_{TS^k(X)} \oplus dg|_{TS^l(Y)} : TS^k(X) \oplus TS^l(Y) \rightarrow T_z Z
\]

is surjective.

**Lemma 37.** Let \( X, Y, Z \) be manifolds with corners and let \( f : X \rightarrow Z \) and \( g : Y \rightarrow Z \) be continuous. Consider the topological fiber product

\[
 P = X \times_Y Z = \{(x, y) \in X \times Y | f(x) = g(y)\}.
\]

Suppose at least one of the following conditions holds.

(i) \( f \) is a \( b \)-normal submersion and \( g \) is strongly smooth and interior,

(ii) \( f \) is étale, \( g \) is a smooth map,

(iii) \( f \) is a \( b \)-submersion, \( g \) is perfectly simple, or

(iv) \( \partial Z = \emptyset \), \( f, g \) are \( b \)-transverse and smooth.

Then \( P \) admits a unique structure of a manifold with corners making it the fiber product in \( \text{Man}^b \), and we have

\[
 C_i(W) = \coprod_{j,k,l \geq 0; i = j + k - l} C^l_j(X) \times_{C^i_l(Z)} C^k_j(Y)
\]

where \( C^l_j(X) = C_j(X) \cap C(f)^{-1}(C^i_l(Z)) \) and \( C^k_j(Y) = C_k(Y) \cap C(g)^{-1}(C^i_l(Z)) \), and the fiber product is taken over \( C(f), C(g) \).

Moreover, if \( X \xrightarrow{f} Z \) (respectively, \( Y \xrightarrow{g} Z \)) is \( b \)-normal then so is \( P \xrightarrow{f'} Y \) (resp., \( P \xrightarrow{g'} X \)).

In what follows the discussion diverges from [9] (see more specifically §4.2 there). More precisely we introduce a stronger notion of a closed immersion, that has the implicit function theorem built into it. This is the only kind of closed immersion that we need to consider, and makes the discussion considerably simpler.

**Definition 38.** A map \( f : X \rightarrow Y \) of manifolds with corners is called a **closed immersion** if for every \( p \in X \) there exists an open neighborhood \( p \in U \subset X \), an open neighborhood \( f(U) \subset V \subset Y \), and a strongly smooth submersion \( h : V \rightarrow \mathbb{R}^N \) for
some integer $N \geq 0$ such that the following square is cartesian

\[
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow & & \downarrow h \\
0 & \rightarrow & \mathbb{R}^N
\end{array}
\]

(it follows that $N = \dim Y - \dim X$). The fiber product exists by Lemma 37 since $h$ is (vacuously) b-normal, and $0 \rightarrow \mathbb{R}^N$ is strongly smooth and interior.

**Remark 39.** Any b-submersion to a manifold without boundary is automatically a strongly smooth submersion, so in Definition 38 it suffices to assume that $h$ is a b-submersion.

**Definition 40.** A map $f : X \rightarrow Y$ of manifolds with corners is called a closed embedding if it is a closed immersion, has a closed image, and induces a homeomorphism on its image.

**Definition 41.** A map $f : X \rightarrow Y$ of manifolds with corners is an open embedding if it is étale and injective.

**Definition 42.** (a) Let $f : X \rightarrow Y$ be a map of manifolds with corners. We say $f$ is horizontally submersive if for every $\tilde{x} \in X$ the germ $f_{\tilde{x}}$ is isomorphic to the projection $\mathbb{R}^k \rightarrow \mathbb{R}^k'$,

\[(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{k'}, x_{k+1}, \ldots, x_{k+n'}) .\]

(b) Let $f : X \rightarrow Y$ be a b-normal map. We call

\[C^\text{hor}_k (X) := (C(f)^{-1}(C_0(Y)) \cap C_k(X))\]

the horizontal $k$-corners of $X$ with respect to $f$.

**Lemma 43.** A map $f : X \rightarrow Y$ is horizontally submersive if and only if it is b-normal and the induced map $C^\text{hor}_k (X) \xrightarrow{C(f)} Y$ is a submersion for every $k$; that is,

\[T_x C^\text{hor}_k (X) \xrightarrow{dC(f)} T_y Y\]

is surjective for all $x \in C^\text{hor}_k(X)$.

Suppose now $X, Y$ are manifolds with corners, $f$ is horizontally submersive with oriented fibers, and let $\omega$ be a compactly supported differential form on $X$. In this case we can define $f_\ast \omega$ by integration along the fiber.

### 4.2. Orbifolds with corners.

**Definition 44.** A groupoid $(G_0, G_1, s, t, e, i, m)$ is a category where every arrow is invertible. Namely, $G_0$ is a class of points and $G_1$ is a class of arrows. The maps $s, t : G_1 \rightarrow G_0$ take an arrow to its source and target objects, respectively. The composition map $m : \{(f, g) \in G_1 \times G_1 | t(f) = s(g)\} \rightarrow G_1$ takes a pair of composable arrows to their composition. The identity map $e : G_0 \rightarrow G_1$ takes an object to the identity arrow and the inverse map $i : G_1 \rightarrow G_1$ takes an arrow to its inverse.
The equivalence classes of the equivalence relation $\text{Im}(s \times t) \subseteq G_0 \times G_0$ are called the orbits of the groupoid; the class of all orbits is denoted $G_0/G_1$. We will use different notations for groupoids, depending on how much of the structure we want to label:

$$\left( G_0, G_1, s, t, e, i, m \right) = G_s = G_1 \Rightarrow G_0.$$ 

**Definition 45.** A groupoid $(X_0, X_1, s, t, e, i, m)$ will be called étale if $X_0, X_1$ are objects of $\text{Man}^i$, and the maps $s, t, e, i, m$ are all étale (in fact, it suffices to require that $s : X_1 \to X_0$ is étale). An étale groupoid will be called proper if the map $s \times t : X_1 \to X_0 \times X_0$ is proper. We will mostly be interested in proper étale groupoids, or PEG’s for short.

Let $X_\bullet$ be a PEG. The set of orbits $X_0/X_1$, taken with the quotient topology, forms a locally compact Hausdorff space. $X_\bullet$ is called compact if $X_0/X_1$ is compact.

Let $X_\bullet, Y_\bullet$ be two PEG’s. A smooth functor $X_\bullet \xrightarrow{F} Y_\bullet$ consists of a pair of smooth maps $F_0 : X_0 \to Y_0$ and $F_1 : X_1 \to Y_1$ which is a functor between the underlying categories. If $F_\bullet, G_\bullet : X_\bullet \to Y_\bullet$ are two functors a smooth transformation $\alpha : F_\bullet \Rightarrow G_\bullet$ is a smooth map $X_0 \to Y_1$ which is a natural transformation between the underlying functors. In this way we obtain a bicategory (see [2]) PEG, whose objects, or 0-cells, are proper étale groupoids, morphisms (or 1-cells) are smooth functors, and 2-cells are natural transformations. A refinement $R_\bullet : X_\bullet \to X'_\bullet$ is a smooth functor which is an equivalence of categories and such that $R_0$ (hence also $R_1$) is an étale map.

**Lemma 46.** As a subset of the 1-cells of PEG the refinements admit a right calculus of fractions, in the sense of [11] §2.1.

We define the category $\text{Orb}$ of orbifolds (always with corners, unless specifically mentioned otherwise) to be the 2-localization of PEG by the refinements. We usually denote orbifolds by calligraphic letters $\mathcal{X}, \mathcal{Y}, \mathcal{M}$... They are given by proper étale groupoids. Maps $\mathcal{X} \to \mathcal{Y}$ are given by fractions $F_\bullet|R_\bullet$ with $X_\bullet \xleftarrow{r} X'_\bullet$ a refinement and $X'_\bullet \xrightarrow{f} Y_\bullet$ a smooth functor. We refer the reader to [11] for further details, including the definition of the 2-cells, the composition operations, etc.

**Definition 47.** We say $f$ is strongly-smooth, étale, interior, b-normal, submersive, b-submersive, horizontally submersive, simple or perfectly simple if $F_0$ has the corresponding property as a map of manifolds with corners. It is easy to check that these properties are preserved by 2-cells (and thus are properties of the homotopy class of $f$). The map $f$ is called a b-fibration if it is b-normal and b-submersive (cf. [2] Definition 4.3).

For $i = 1, 2$ let $f^i = F^i|R^i : \mathcal{X}^i \to \mathcal{Y}$ be an interior map. We say $f^1$ and $f^2$ are $b$-transverse if $F^1_0, F^2_0$ are $b$-transverse (as maps of manifolds with corners).

An equivalence in $\text{Orb}$ is called a diffeomorphism. We say $f = F|R : \mathcal{X} \to \mathcal{Y}$ is full, essentially surjective, or faithful if $F$ is full, essentially surjective, or faithful, respectively.

If $\mathcal{X} \xrightarrow{X_\bullet} X_0$ is an orbifold with corners, $\partial \mathcal{X} = \partial X_1 \Rightarrow \partial X_0$ is naturally an orbifold with corners and the smooth functor $(i_{X_0}^{\partial}, i_{X_0}^{\partial}, \ldots, i_{X_0}^{\partial})$ induces a map $i_X^\partial : \partial \mathcal{X} \to \mathcal{X}$. We denote

$$i_X^\partial := i_X^\partial \circ i_{\partial \mathcal{X}}^\partial \circ \cdots \circ i_{\partial ^{n-1} \mathcal{X}}^\partial : \partial \mathcal{X} \to \mathcal{X}.$$
Since the maps $s, t, e, i, m$ are étale, they preserve the depth and we obtain orbifolds with corners

$$C_k(X) = C_k(X_1) \Rightarrow C_k(X_0)$$

for all $k$. A local orbifold with corners $X = \bigsqcup X_n$ (or just an $l$-orbifold) is a disjoint union of orbifolds with corners with $\dim X_n = n$. It is obvious how to turn this into a category and extend the definitions of various types of maps to this situation. If $X$ is an orbifold with corners, we can consider $C(X) = \bigsqcup_{k \geq 0} C_k(X)$ as an $l$-orbifold.

A smooth map $f : X \to Y$ induces an interior map $C(f) : C(X) \to C(Y)$.

We turn to a discussion of the weak fibered product in $\text{Orb}$.

**Lemma 48.** Let

$$f : X \xleftarrow{R} X' \xrightarrow{F} Z \text{ and } g : Y \xleftarrow{S} Y' \xrightarrow{G} Z$$

be two 1-cells in $\text{Orb}$. Suppose at least one of the following conditions holds.

(i) $F$ is a $b$-normal submersion and $G$ is strongly smooth and interior,

(ii) $F$ is étale, $G$ is a smooth map,

(iii) $F$ is a $b$-submersion, $G$ is perfectly simple, or

(iv) $\partial Z = \emptyset$, $F$ and $G$ are $b$-transverse (see Remark 36 for an equivalent condition) and smooth.

Then

(a) The weak fiber product $\mathcal{P} = X \xleftarrow{f} \xrightarrow{g} Y$ exists in $\text{Orb}$. In fact, we can take

$$\mathcal{P} = X' \xleftarrow{F} \xrightarrow{G} Y'$$

the weak fiber product in $\text{PEG}$, given by the groupoid $P_1 \Rightarrow P_0$ where

$$P_0 = X_0' F_0 \times_S Z_1 t^* G_0 Y_0',$$

$$P_1 = X_1' s t F_1 \times_S Z_1 t G_1 Y_1'.$$

Here an element of $P_1$ specifies the three solid arrows in the diagram below,

$$\begin{array}{ccc}
  x^1 & F_0(x^1) & G_0(y^1) \\
  a | & \downarrow \quad | \downarrow G_1(b) & b \\
  x^2 & F_0(x^2) & G_0(y^2)
\end{array}$$

The horizontal dashed arrow is uniquely determined by requiring the square to be commutative; $s, t : P_1 \to P_0$ are the projections on the top and bottom rows of the diagram, respectively, and the other structure maps are computed similarly.

(b) We have

$$C_i(\mathcal{P}) = \bigsqcup_{j, k, l \geq 0, i = j, k - l} C_j(X) \times_{C_l(Z)} C_k(Y)$$

where $C_j(X) = C_j(X) \cap C(f)^{-1}(C_l(Z))$ and $C_k(Y) = C_k(Y) \cap C(g)^{-1}(C_l(Z))$, and the weak fiber product is taken over $C(f), C(g)$.

(c) If we assume, in addition, that $X \xleftarrow{f} Z$ (respectively, $Y \xrightarrow{g} Z$) is $b$-normal then so is $\mathcal{P} \xleftarrow{f'} \xrightarrow{g'} Y$ (resp., $\mathcal{P} \xrightarrow{g'} \xleftarrow{f'} X$).

**Definition 49.** A map $F|R : X \to Y$ of orbifolds with corners is a closed immersion if $F_0$ is a closed immersion. In this case, the same holds for any map homotopic to $F|R$. 

A manifold with corners $M$ specifies an orbifold $\underline{M} = M \to \mathcal{X}$ with only identity morphisms, and this extends to a 2-fully-faithful pseudofunctor $\text{Man}^c \to \text{Orb}$ (namely, it restricts to an equivalence $\text{Man}^c(X, Y) \simeq \text{Orb}(\underline{X}, \underline{Y})$ for any pair $X, Y$ of objects of $\text{Man}^c$). We say an orbifold “is” a manifold with corners if it is in the essential image of this functor.

**Definition 50.** Let $\mathcal{X}$ be an orbifold with corners. An atlas for $\mathcal{X}$ is a map $p : \underline{M} \to \mathcal{X}$ where $M$ is some manifold with corners, such that for any other map $f : \underline{N} \to \mathcal{X}$ from a manifold with corners, $\underline{M} \times_{\mathcal{X}} \underline{N}$ is a manifold with corners and the projection $\underline{M} \times_{\mathcal{X}} \underline{N} \to \underline{N}$ is étale and surjective (as a map of $\text{Man}^c$).

The obvious map $X_0 \to (X_1 \to X_0)$ is an atlas. Conversely, any atlas $\underline{M} \to \mathcal{X}$ defines an orbifold equivalent to $\mathcal{X}$, whose objects are $\underline{M}$ and morphisms are $\underline{M} \times_{\mathcal{X}} \underline{M}$.

**Definition 51.** A map $f : \mathcal{X} \to \mathcal{Y}$ of orbifolds with corners is a closed (respectively, open) embedding if for some (hence any) atlas $p : \underline{M} \to \mathcal{Y}$, the 2-pullback $\underline{M} \times_{\mathcal{Y}} \mathcal{X}$ is a manifold with corners and the map $\underline{M} \times_{\mathcal{Y}} \mathcal{X} \to \underline{M}$ is a closed (resp. open) embedding of manifolds with corners.

If $f : \mathcal{X} \to \mathcal{Y}$ is a closed embedding we may refer to $\mathcal{X}$ as a suborbifold of $\mathcal{Y}$.

The notion of a sheaf on an orbifold $\mathcal{X}$ is the same as the notion of a sheaf on the underlying topological orbifold (see [9 11]). A vector bundle $E$ on an orbifold with corners $\mathcal{X} = X_1 \to X_0$ is given by $(E_0, \phi)$ where $E_0$ is a smooth vector bundle on $X_0$ and

$$\phi : s^* E_0 \to t^* E_0$$

is an isomorphism satisfying some obvious compatibility requirements with the groupoid structure. The sections of $(E_0, \phi)$ form a sheaf over $\mathcal{X}$. A local system on an orbifold $\mathcal{X}$ is a sheaf which is locally isomorphic to the constant sheaf $\mathbb{Z}$. We extend the conventions set forth in [19 §1.1, §6.1] to proper étale groupoids with corners in the obvious way.

In particular, for every vector bundle $E$ on $X_*$ there’s a local system $\text{Or}(E)$ on $X_*$. The orientation local system of $X_*$ is $\text{Or}(TX_*)$. We have a local system isomorphism

$$(81) \quad i_\partial^* : \text{Or}(T\partial X_*) \to \text{Or}(TX_*)$$

lying over $i_\partial^* : \partial X_* \to X_*$, defined by appending the outward normal vector at the beginning of the oriented base. Given a short exact sequence of vector bundles

$$0 \to E_1 \xrightarrow{f} E \xrightarrow{q} E_2 \to 0$$
on $\mathcal{X}$, we obtain a local system isomorphism

$$(82) \quad \text{Or}(E_1) \otimes \text{Or}(E_2) \to \text{Or}(E),$$

which, using oriented bases to represent orientation, can be expressed by

$$\left[ e_1, \ldots, e_{n_1} \right] \otimes \left[ e_2, \ldots, e_{n_2} \right] \mapsto \left[ f(e_1), \ldots, f(e_{n_1}), g(e_1), \ldots, g(e_{n_2}) \right]$$

where $g : E_2 \to E$ is any section of $q$.

---

Note there we had to work with $\mathbb{C}$-valued local systems, but for the purposes of this paper we can work with $\mathbb{Z}$-valued local systems.
Maps of local systems are always assumed to be cartesian, so to specify a local system map \( \mathcal{L}_1 \xrightarrow{\mathcal{F}} \mathcal{L}_2 \) over \( \mathcal{X}_1 \xrightarrow{f} \mathcal{X}_2 \) is equivalent to giving an isomorphism \( \mathcal{L}_1 \to f^{-1}\mathcal{L}_2 \). In this case we may say that \( \mathcal{F} \) lies over \( f \).

**Lemma 52.** Let \( \mathcal{X} \) be an orbifold with corners. We denote by \( \hat{\mathcal{X}} := S^0(\mathcal{X}) \) the orbifold (without boundary or corners) consisting of points of depth zero, and by \( j : \hat{\mathcal{X}} \to \mathcal{X} \) be the inclusion.

(a) The pushforward and inverse image functors \( j_* \), \( j^{-1} \) form an adjoint equivalence of groupoids between local systems on \( \hat{\mathcal{X}} \) and local systems on \( \mathcal{X} \).

(b) \( \text{Or} (dj) : \text{Or} (T\hat{\mathcal{X}}) \to j^{-1}\text{Or} (T\mathcal{X}) \) is an isomorphism.

Let \( f : \mathcal{X} \to \mathcal{Y} \) be a b-normal map of orbifolds with corners.

(c) There exists a unique map \( \hat{f} : \hat{\mathcal{X}} \to \hat{\mathcal{Y}} \) with \( f \circ j_{\hat{\mathcal{X}}} = j_{\hat{\mathcal{Y}}} \circ \hat{f} \).

Let \( \mathcal{L} \) be a local system on \( \mathcal{X} \) and let \( \mathcal{L}' \) be a local system on \( \mathcal{Y} \), and denote by \( \hat{\mathcal{L}} = j_{\hat{\mathcal{X}}}^{-1}\mathcal{L} \), \( \hat{\mathcal{L}}' = j_{\hat{\mathcal{Y}}}^{-1}\mathcal{L}' \) their restrictions to \( \hat{\mathcal{X}}, \hat{\mathcal{Y}} \), respectively. Define a map taking a map of sheaves \( \mathcal{F} : \mathcal{L} \to \mathcal{L}' \) over \( f \) to the map \( \hat{\mathcal{F}} : \hat{\mathcal{L}} \to \hat{\mathcal{L}}' \) over \( \hat{f} \) given by the composition

\[
j_{\hat{\mathcal{X}}}^{-1}\mathcal{L} \xrightarrow{j_{\hat{\mathcal{X}}}^{-1}\mathcal{F}} j_{\hat{\mathcal{Y}}}^{-1}\mathcal{L}' = \hat{f}^{-1}j_{\hat{\mathcal{Y}}}^{-1}\mathcal{L}'.
\]

(d) \( \mathcal{F} \mapsto \hat{\mathcal{F}} \) is a bijection

\[
\{ \text{maps } \mathcal{F} : \mathcal{L} \to \mathcal{L}' \text{ over } f \} \cong \{ \text{maps } \hat{\mathcal{F}} : \hat{\mathcal{L}} \to \hat{\mathcal{L}}' \text{ over } \hat{f} \}.
\]

and together with \( \mathcal{L} \mapsto \hat{\mathcal{L}} \) forms a functor from the category of sheaves (respectively, local systems) over orbifolds with corners with b-normal maps to the category of sheaves (resp. local systems) over orbifolds.

Let \( \mathcal{X} \) be an orbifold with corners and \( \mathcal{L} \) a local system on \( \mathcal{X} \). We define the complex of differential forms on \( \mathcal{X} \) with values in \( \mathcal{L} \)

\[
\Omega (\mathcal{X} ; \mathcal{L}) = \Gamma (C^\infty (\wedge T\mathcal{X}) \otimes_\mathbb{Z} \mathcal{L})
\]
as the global sections of the sheaf of sections of the vector bundle \( \wedge T\mathcal{X} \), twisted by \( \mathcal{L} \).

Suppose \( \mathcal{X}, \mathcal{Y} \) are compact orbifolds with corners, \( \mathcal{K}, \mathcal{L} \) are local systems on \( \mathcal{X} \) and on \( \mathcal{L} \), respectively, and \( f : (\mathcal{X}, \mathcal{K}) \to (\mathcal{Y}, \mathcal{L}) \) is an oriented map, which means it is a local system map \( \mathcal{K} \to \mathcal{L} \) lying over a smooth map of orbifolds with corners \( \mathcal{X} \to \mathcal{Y} \). We have a pullback operation

\[
\Omega (\mathcal{Y} ; \mathcal{L}) \xrightarrow{f^\ast} \Omega (\mathcal{X} ; \mathcal{K}).
\]

If, in addition, we assume that \( f \) is horizontally submersive, then there’s a push-forward operation

\[
\Omega (\mathcal{X} ; \mathcal{K} \otimes \text{Or} (T\mathcal{X})^\vee) \xrightarrow{f_\ast} \Omega (\mathcal{Y} ; \mathcal{L} \otimes \text{Or} (T\mathcal{Y})^\vee).
\]

We now sketch how these operations are constructed. Define the complex of compactly supported differential forms on \( \mathcal{X} \) by

\[
\Omega_c (\mathcal{X} ; \mathcal{L}) := \text{coker} (t_\ast - s_\ast : \Omega_c (X_1; s^* \mathcal{L}_0) \to \Omega_c (X_0; \mathcal{L}_0)),
\]

where on the right hand side, \( \Omega_c \) denotes the usual complex of compactly supported forms on a manifold with corners. In case \( f = (F_0, F_1) : \mathcal{X} \to \mathcal{Y} \) is a smooth functor,
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$F_0^*$ induces a pullback map (83) and (if $f$ is horizontally submersive) $(F_0)_*$ induces a pushforward map of compactly supported forms,

$$\Omega_c(\mathcal{X}; \mathcal{K} \otimes \text{Or}(T\mathcal{X})) \to \Omega_c(\mathcal{Y}; \mathcal{L} \otimes \text{Or}(T\mathcal{Y})).$$

In defining the operations $F_0^*$, $(F_0)_*$ (for forms on manifolds with corners) we follow the conventions in [19]. A partition of unity for $\mathcal{X}$ is a smooth map $\rho : X_0 \to [0, 1]$ such that $\text{supp}(s^*\rho) \cap t^{-1}(K)$ is compact for every compact subset $K \subset X_0$ and $t_*s^*\rho \equiv 1$ (the fiber of $t$ is discrete, hence canonically oriented). Partitions of unity always exist; since $\mathcal{X}$ is assumed to be compact we can require that $\rho$ has compact support in $X_0$, and use this to construct an isomorphism

$$\Omega(\mathcal{X}; \mathcal{L}) \simeq \Omega_c(\mathcal{X}; \mathcal{L}),$$

see Behrend [1]. The isomorphism (86) allows us to define (84) using (85). Now if $f = \mathcal{X} \leftarrow \mathcal{X}' \xrightarrow{F} \mathcal{Y}$ is a general oriented map, we define (83) by

$$f^* = R_*F^*,$$

pulling back along the smooth functor $F$ and then pushing forward along the refinement $R$ ($R$ is a horizontally submersive since it is étale; moreover, any refinement defines an equivalence between the categories of local systems on $\mathcal{X}$ and on $\mathcal{X}'$, so orientations for $f$ are in natural bijection with orientations for $F$). If $f$ is oriented and horizontally submersive we define the pushforward (84) by

$$f_* = F_*R_*.$$

By construction, the operations (83) (84) extend the operations defined in [19] for the case $\mathcal{X}, \mathcal{Y}$ are manifolds, and they satisfy the same relations.

To make the paper more readable, outside of this appendix we will sometimes abuse notation and refer to maps which have a specified isomorphism as being equal. For example, if $G$ acts on $\mathcal{X}$ (see §4.4 below) we may write

$$g \cdot h = (gh).$$

even though in general the two sides differ by a (specified) 2-cell. The same goes for orbifolds which are canonically equivalent (that is, with a given equivalence, or with an equivalence which is specified up to a unique 2-cell). For example we may write

$$(\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3 = \mathcal{M}_1 \times (\mathcal{M}_2 \times \mathcal{M}_3).$$

When we write $p \in \mathcal{X}$ we mean $p \in X_0$, where $\mathcal{X} = X_1 \rightrightarrows X_0$.

4.3. Hyperplane Blowup. An important step in the construction of the moduli spaces of discs from the moduli spaces of curves, is the notion of a hyperplane blowup, which we now discuss.

4.3.1. Hyperplane blowup of manifolds.

**Definition 53.** (a) Let $h : W \to X$ be a proper closed immersion between manifolds without boundary. Write $h^{-1}(x) = \{w_1, ..., w_r\}$ (this is finite since $h$ is proper), and let $N^\vee_{w_i} = \ker\left(T^\vee_x W \xrightarrow{\partial h^\vee_{w_i}} T^\vee_{w_i} W\right)$ denote the conormal bundle to $h$. We say $h$ has **transversal self-intersection at** $x \in \mathcal{X}$ if the induced map

$$\bigoplus_{i=1}^r N^\vee_{w_i} \to T^\vee_x \mathcal{X}$$

is a pullback map (83) and (if $f$ is horizontally submersive) $(F_0)_*$ induces a pushforward map of compactly supported forms,
is injective. We say $h$ has transversal self-intersection if it has transversal self intersection at every $x \in X$.

(b) Let $h : W \to X$ be a proper closed immersion which has transversal self intersection. Suppose further that $h$ is codimension one, i.e. $\dim X - \dim W = 1$. In this case we call $E = \operatorname{Im} h$ a hyper subset, and call $h$ a hyper map. Since the conditions on $h$ can be checked locally on the codomain $X$, being a hyper subset is a local property. Moreover, it follows from Proposition ?? below that the map $h$ is essentially unique: if $W \xrightarrow{h} X, W' \xrightarrow{h'} X$ are two hyper maps with $\operatorname{Im} h = \operatorname{Im} h'$ then there’s a unique diffeomorphism $W \xrightarrow{\phi} W'$ such that $h = h' \circ \phi$.

(c) Let $Y \to X$ be a smooth map of manifolds without boundary, and let $E \subset X$ be a hyper subset. We say $f$ is multi-transverse to $E$ if for some (hence any) hyper map $h$ such that $E = \operatorname{Im} h$, $f$ is transverse to $h$ and the pullback $f^{-1}W \xrightarrow{f^{-1}h} Y$ has transversal self intersection (so in fact, since $f^{-1}h$ is necessarily a codimension one proper closed immersion, $f^{-1}E \subset Y$ is a hyper subset).

**Definition 54.** (a) Let $X$ be a manifold without boundary, let $U \subset X$ be an open subset. Consider the set of germs of connected components,

$$I(X,U) = \bigcup_{x \in X} \{x\} \times \varprojlim_{x \in V \subset X} \pi_0^X(V \cap U)$$

where for $V$ an open neighborhood of $x \in X$, $\pi_0^X(V \cap U)$ denotes the set of connected components $C \subset V \cap U$ with $x \in C$ in the closure. If $V_1 \subset V_2$ are two such neighborhoods, there’s an induced map $\pi_0^X(V_1 \cap U) \to \pi_0^X(V_2 \cap U)$, and $\varprojlim_{x \in V \subset X} \pi_0^X(V \cap U)$ denotes the inverse limit of this system of sets.

(b) If $(X_1,U_1) \to (X_2,U_2)$ is a map of pairs there’s an induced map $I(X_1,U_1) \to I(X_2,U_2)$ making $I$ a functor; there’s an obvious natural transformation $I(X,U) \to X$.

(c) Let $E \subset X$ be a hyper subset. As a set, the blow up of $X$ along $E$ is given by

$$B(X,E) = I(X,X \setminus E).$$

The associated natural transformation is denoted $B(X,E) \xrightarrow{\beta(X,E)} X$, and if $Y \xrightarrow{f} X$ is multi-transverse to $E$ write

$$B(Y,f^{-1}E) \xrightarrow{B(f)} B(X,E)$$

for the induced map.

**Proposition 55.** Let $\text{Man}^*$ denote the category of marked manifolds, whose objects are pairs $(X,E)$ where $X$ is a manifold without boundary and $E$ is a hyper subset of $X$, and where an arrow $(X_1,E_1) \to (X_2,E_2)$ is given by a map $X_1 \xrightarrow{f} X_2$ which is multi-transverse to $E_2$ and such that $f^{-1}E_2 = E_1$. Let $\text{Man}^*_\text{ps}$ denote the category of manifolds with corners with perfectly simple maps. Then blowing up gives a faithful functor

$$B : \text{Man}^* \to \text{Man}^*_\text{ps}$$

together with a natural transformation $B(X,E) \xrightarrow{\beta(X,E)} X$.

Moreover, if $(X_1,E_1), (X_2,E_2)$ are any two objects of $\text{Man}^*$, any étale map $f : X_1 \to X_2$ is a morphism of $\text{Man}^*$ and $B(f)$ is also étale in this case.
The following definition characterizes the manifold with corners structure on the blow up. More precisely, \( B(X,E) \) will be equipped with the unique manifold with corners structure on the set \( B(X,E) \) making the map \( \beta_{(X,E)} \) rectilinear:

**Definition 56.** Let \( C \) be a manifold with corners, \( M \) a manifold without boundary. A map \( f : C \to M \) will be called \emph{rectilinear} if the restriction of \( f \) to interior points is an injective map \( \bar{C} \to M \), and for every \( c \in C \) there exist a non-negative integer \( k \) and coordinate charts \( U \xrightarrow{\varphi} \mathbb{R}^n, c \in U, \varphi(c) = 0 \) and \( V \xrightarrow{\psi} \mathbb{R}^n, f(c) \in V, \psi(f(c)) = 0 \) such that \( f(U) \subset V \) and \( \psi \circ f \circ \varphi^{-1} \) is the standard embedding of \( \mathbb{R}^n \) to \( \mathbb{R}^n \), restricted to \( \varphi(U) \).

4.3.2. *Hyperplane blowup of orbifolds.* We consider the bicategory \( \text{PEG}^+ \) of marked proper étale groupoids. The objects of \( \text{PEG}^+ \) are pairs \( (X_1 \xrightarrow{s,t} X_0) \) where \( X_1 \xrightarrow{s,t} X_0 \) is a proper étale groupoid without boundary, and \( E \subset X_0 \) is a hyper subset which is a union of orbits, \( s^{-1}E = t^{-1}E \).

If \( (X^{(1)}_s, E^{(1)}), (X^{(2)}_t, E^{(2)}) \) are two objects, a 1-cell of \( \text{PEG}^+ \) consists of a smooth functor

\[
X^{(1)}_s \xrightarrow{F_0,F_1} X^{(2)}_t
\]

such that \( F_0 \) is multi-transverse to \( E^{(2)} \) and \( E^{(1)} = F_0^{-1}E^{(2)} \). Every étale map, and in particular every refinement, satisfies this condition. The 2-cells in \( \text{PEG}^+ \) are all the 2-cells of \( \text{PEG} \) spanned by the 1-cells specified above.

For emphasis, in this subsection we denote the bicategory of proper étale groupoids and orbifolds in the category \( \text{Man}^+ \) by \( \text{PEG}^+ \) and \( \text{Orb}^+ \), respectively. We denote by \( \text{PEG}_{ps}^+ \), \( \text{Orb}_{ps}^+ \) the subcategories whose maps are perfectly simple maps.

If \( X_1 \xrightarrow{s,t} X_0 \) is a groupoid, we write

\[
X_2 = X_1 \times_s X_1
\]

for the manifold with corners parameterizing composable arrows

\[
x_1 \xrightarrow{a} x_2 \xrightarrow{b} x_3
\]

and, for \( i = 1, 2, 3 \), we denote by \( p_i : X_2 \to X_0 \) the map sending a composable arrow as above to \( x_i \).

**Theorem 57.** The functor \( B \) extends to a strict 2-functor

\[
B : \text{PEG}^+ \to \text{PEG}_{ps}^+
\]

which takes

\[
\left( X = X_2 \xrightarrow{m} X_1 \xrightarrow{s,t} X_0, E \right)
\]

to

\[
B(X) = B(X_2, p_1^{-1}E) \xrightarrow{B(m)} B(X_1, s^{-1}E) \xrightarrow{\beta_{(X,E)}} B(X_0, E)
\]

together with the obvious strict natural transformation \( B(X) \xrightarrow{\beta_{(X,E)}} X \).

This functor takes refinements to refinements, and thus there’s an induced functor between the 2-localization of these categories

\[
B : \text{Orb}^+ \to \text{Orb}_{ps}^+
\]
4.3.3. A hyper map between orbifolds. There’s a natural way to construct objects and arrows in \( \text{Orb}^\ast \). Let \( h : \mathcal{W} \rightarrow \mathcal{X} \) be a map of orbifolds without boundary, given by a pair of smooth functors

\[
\mathcal{W} \xrightarrow{\mathcal{W}} (\mathcal{W} = \mathcal{W}_1 \rightrightarrows \mathcal{W}_0) \xrightarrow{H = (H_1, H_0)} (\mathcal{X} = \mathcal{X}_1 \rightrightarrows \mathcal{X}_0)
\]

with \( S \) a refinement.

Let \( W_0 = X_1 \times H_0 \mathcal{W}_0 \). Since \( t \) is étale this fiber product exists. We let \( H_0' : W_0' \rightarrow X_0 \) denote the composition \( X_1 \times H_0 \mathcal{W}_0 \rightarrow X_1 \rightrightarrows X_0 \). We call the image of \( H_0' \) the essential image of \( h \), and denote it \( \text{Im} h \). Fix some point \( x \in X_0 \). The essential fiber of \( h \) over \( x \) is a topological groupoid, with object space \( (H_0')^{-1}(x) \) and with arrows between \( (x \xrightarrow{\alpha} H_0(w), w) \) and \( (x \xrightarrow{\alpha'} H_0(w'), w') \) consisting of the arrows in \( \mathcal{W}_1 \) between \( w \) and \( w' \) (this is a special case of the weak fiber product, see Lemma 48). If \( R \) is a refinement, the essential fiber of \( h \) over \( x \) and of \( R \circ h \) over \( R(x) \) are equivalent. The essential image and, up to bijection the essential fiber, depend only on the homotopy class of \( H \) (in particular, they do not depend on \( S \)).

**Definition 58.** We say that \( h \) is **hyper** if the following five conditions are met (cf. Definition 53)

- \( h \) is **faithful**, which means \( H_0 \) is faithful. This implies the essential fiber over every point is equivalent to a set (with a topology).
- \( h \) is a **closed immersion**, which means \( H_0 \) is a closed immersion. This implies the orbit space of each essential fiber has the discrete topology.
- \( h \) is **proper**, which means the essential fibers have compact orbit spaces.
- Given our previous assumptions this means the essential fiber is equivalent to a finite set, and and we fix representatives \( \{ q_i = x \xrightarrow{\alpha_i} H_0(w_i) \}_{i=1}^r \).
- We require that \( h \) has **transversal self-intersection**, that is, we require the map

\[
\bigoplus_{i=1}^r N^\gamma_{w_i} \rightarrow T^\gamma_x X_0
\]

be injective, where \( N^\gamma_{w_i} = \ker \left( T^\gamma_{w_i} W_0 \rightarrow T^\gamma_x X_0 \right) \); this is independent of the choice of representatives.
- \( h \) has **codimension one**, meaning \( \dim \mathcal{X} - \dim \mathcal{W} = 1 \).

If \( \mathcal{Y} \) is another orbifold without boundary, we say a map \( \mathcal{Y} \xrightarrow{f} \mathcal{X} \) is **multi-transverse** to \( h \) if \( h \) is \((b-)\text{transverse to } f \) and the 2-pullback \( f^{-1}h \) has transversal self-intersection (it is automatically a proper, faithful closed immersion).

**Lemma 59.** (a) Let \( \mathcal{W} \xrightarrow{h} \mathcal{X} \) be a hyper map. Then \( (\mathcal{X}, \text{Im } h) \) is an object of \( \text{Orb}^\ast \).

(b) If \( f \) is multi-transverse to \( h \), \( (\mathcal{Y}, \text{Im } f^{-1}h) \xrightarrow{f} (\mathcal{X}, \text{Im } h) \) is an arrow in \( \text{Orb}^\ast \).

4.4. **Group actions.** Let \( G \) be a compact lie group, with multiplication \( m : G \times G \rightarrow G \) and identity \( e : \text{pt} \rightarrow G \). Given a bicategory of spaces \( \mathcal{C} \) such as \( \text{Man}^\ast, \text{Orb}, \text{Orb}^\ast \), we construct a category \( G-\mathcal{C} \) of \( G \)-equivariant objects following
Romagny [12]. We briefly explain how to translate his definitions to our setup, and refer the reader to [12] for more details. A 0-cell of $G - C$ is given by a 4-tuple $(X, \mu, \alpha, a)$ where $X$ is a 0-cell of $C$, $\mu : G \times X \to X$ is a 1-cell, and $\alpha$ and $a$ are 2-cells filling in, respectively, the following square and triangle:

$$
\begin{array}{c}
G \times G \times X \xrightarrow{\mu \times \text{id}_X} G \times X \\
\downarrow \quad \downarrow \\
G \times X \xrightarrow{\mu} X \\
\end{array}
\quad
\begin{array}{c}
G \times X \xrightarrow{\mu} X \\
\downarrow \quad \downarrow \\
X \xrightarrow{\text{id}_X} X
\end{array}
$$

A 1-cell (or $G$-equivariant map)

$$(X, \mu, \alpha, a) \to (X', \mu', \alpha', a')$$

is given by a pair $\left(X \xrightarrow{f} X', \sigma\right)$ where $\sigma$ is a 2-cell filling in the square

$$
\begin{array}{c}
G \times X \xrightarrow{\mu} X \\
\downarrow \quad \downarrow \\
G \times X' \xrightarrow{\mu'} X'
\end{array}
$$

A 2-cell $(f, \sigma) \Rightarrow (f', \sigma')$ is given by a 2-cell $f \Rightarrow f'$. As usual, the 2-cells $\alpha, a, \sigma, \beta$ are required to satisfy some coherence conditions, cf. [12, Definition 2.1].

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