Finite $n$ Largest Eigenvalue Probability Distribution Function of Gaussian Ensembles

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Abstract

In this paper we focus on the finite $n$ probability distribution function of the largest eigenvalue in the classical Gaussian Ensemble of $n \times n$ matrices (GE$_n$). We derive the finite $n$ largest eigenvalue probability distribution function for the Gaussian Orthogonal and Symplectic Ensembles and also prove an Edgeworth type Theorem for the largest eigenvalue probability distribution function of Gaussian Symplectic Ensemble. The correction terms to the limiting probability distribution are expressed in terms of the same Painlevé II functions appearing in the Tracy-Widom distribution.

1 Introduction

In applications of the limiting probability distributions laws from Random Matrix Theory (e.g. [1], [15]) it is important to have an estimate on the convergence rates, if possible have a control on this rate of convergence. For recent reviews of applications of these distributions we refer the reader to [3, 7, 8, 14, 25]. In our desire to control the rate of convergence of the probability distribution of the largest eigenvalue from the Gaussian Orthogonal Ensemble $GOE_n$, Gaussian Unitary Ensemble $GUE_n$ and Gaussian Symplectic Ensembles $GSE_n$, we introduce a fine tuning constant $c$ in the scaling of the desired eigenvalue. The finite large $n$ expansion is therefore a function of $c$. We use this constant to fine-tune the convergence rate. This work was done for the $GUE_n$ and $GOE_n$. In completing the same work for the $GSE_n$, we decided to find a closed formula (opposed to a large $n$ asymptotic formula) whenever possible for each function appearing in the the final expression for these probabilities distribution functions. This approach has the advantages that the final results is a finite $n$ representation of the distributions functions, we only need to perform the large $n$
expansion once for each function to have an Edgeworth type expansion for the desired probability.

The Gaussian $\beta$-ensembles are probability spaces on $n$-tuples of random variables \( \{\lambda_1, \ldots, \lambda_n\} \) (think of them as eigenvalues of a randomly chosen matrix from the ensemble.) with probability density that the variables lie in an infinitesimal intervals about the points $x_1, \ldots, x_n$ is

\[
P_{n,\beta}(x_1, \ldots, x_n) = C_{n,\beta} \exp \left( -\frac{\beta}{2} \sum_{i=1}^{n} x_i^2 \right) \prod_{j<k} |x_j - x_k|^\beta, \tag{1.1}\]

with

\[-\infty < x_i < \infty, \quad \text{for } i = 1, \ldots, n. \tag{1.2}\]

Here $C_{n,\beta}$ is the normalizing constant such that the total integral over the $x_i$'s is one. When $\beta = 1$ we have the GOE$_n$, when $\beta = 2$ we have the GUE$_n$ and when $\beta = 4$ we have the GSE$_n$. We denote the largest eigenvalue by $\lambda_{\text{max}}^\beta$, and by

\[F_{n,\beta}(t) = P(\lambda_{\text{max}}^\beta \leq t) \tag{1.3}\]

his probability distribution function.

When $\beta = 2$, the harmonic oscillator wave functions (see [13], or [18] for a complete definition)

\[\varphi_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x) e^{-x^2/2} \quad k = 0, 1, 2, \ldots\]

play an important role. We also have the Hermite kernel

\[K_{n,2}(x, y) = \sum_{k=0}^{n-1} \varphi_k(x) \varphi_k(y) = \sqrt{\frac{n}{2}} \frac{\varphi_n(x) \varphi_{n-1}(y) - \varphi_n(y) \varphi_{n-1}(x)}{x - y}, \tag{1.4}\]

which is the kernel of the integral operator $K_{n,2}$ acting on $L^2(t, \infty)$, with resolvent kernel

\[R_{n,2}(x, y; t) = (I - K_{n,2})^{-1} \cdot K_{n,2}(x, y). \tag{1.5}\]

Note here that the dot denotes operator multiplication. We have the following representation of $(1.1)$, (see for example, [17] for a derivation of this result)

\[P_{n,2}(x_1, \ldots, x_n) = \det(K_{n,2}(x_i, x_j))_{1 \leq i,j \leq n}. \]

Following Tracy and Widom in [20, 21, 22, 23, 24, 25, 26], we define

\[\varphi(x) = \left( \frac{n}{2} \right)^{\frac{1}{2}} \varphi_n(x), \quad \psi(x) = \left( \frac{n}{2} \right)^{\frac{1}{2}} \varphi_{n-1}(x), \tag{1.6}\]

by $\varepsilon$ the integral operator with kernel

\[\varepsilon_t(x) = \frac{1}{2} \text{sgn}(x - t), \tag{1.7}\]
If \( Ai \) is the Airy function, the kernel

\[ Q_{n,i}(x; t) = ((I - K_{n,2})^{-1}, x^i \varphi) \]  

and

\[ P_{n,i}(x; t) = ((I - K_n)^{-1}, x^i \psi). \]  

We introduce the following quantities

\[ q_{n,i}(t) = Q_{n,i}(t; t), \quad p_{n,i}(t) = P_{n,i}(t; t) \]  

\[ u_{n,i}(t) = (Q_{n,i}, \varphi), \quad v_{n,i}(t) = (P_{n,i}, \varphi), \]  

\[ \tilde{v}_{n,i}(t) = (Q_{n,i}, \psi), \quad \text{and} \quad w_{n,i}(t) = (P_{n,i}, \psi). \]  

Here \((\cdot, \cdot)\) denotes the inner product on \( L^2(t, \infty) \). In our notation, the subscript without the \( n \) represents the scaled limit of that quantity when \( n \) goes to infinity, and we dropped the second subscript \( i \) when it is zero.

If \( Ai \) is the Airy function, the kernel \( K_{n,2}(x, y) \) then scales\(^1\) to the Airy kernel

\[ K_{Ai}(X, Y) = \frac{Ai(X) Ai'(Y) - Ai(Y) Ai'(X)}{X - Y}. \]

Our conventions are as follows:

\[ Q_i(x; s) = ((I - K_{Ai})^{-1}, x^i Ai), \quad Q_0(x; s) = Q(x; s), \]  

\[ P_i(x; s) = ((I - K_{Ai})^{-1}, x^i Ai'), \quad P_0(x; s) = P(x; s), \]  

\[ q_i(s) = Q_i(s; s), \quad q_0(s) = q(s), \quad p_i(s) = P_i(s; s), \quad p_0(s) = p(s), \]  

\[ u_i(s) = (Q_i, Ai), \quad u_0(s) = u(s), \quad v_i(s) = (P_i, Ai), \quad v_0(s) = v(s), \]  

\[ \tilde{v}_i(s) = (Q_i, Ai'), \quad \tilde{v}_0(s) = \tilde{v}(s), \quad w_i(t) = (P_i, Ai'), \quad \text{and} \quad w_0(t) = w(s). \]

Here \((\cdot, \cdot)\) denotes the inner product on \( L^2(s, \infty) \) and \( i = 0, 1, 2, \cdots \).

We also note that \( q(s) \) is the Haskins-Macleod solution to the Pailevé II equation \( q''(s) = sq(s) + 2q^3(s) \) with the boundary condition \( q(s) \sim Ai(s) \) as \( s \to \infty \).

We use the subscript \( n \) for unscaled quantities only.

\[ R_{n,1} := \int_{-\infty}^{t} R_n(x, t; t)dx, \quad P_{n,1} := \int_{-\infty}^{t} P_n(x; t)dx, \quad Q_{n,1} := \int_{-\infty}^{t} Q_n(x; t)dx, \]  

and

\[ R_{n,A}(t) := \int_{-\infty}^{\infty} \varepsilon_i(x) R_n(x, t; t)dx, \quad P_{n,A}(t) := \int_{-\infty}^{\infty} \varepsilon_i(x) P_n(x; t)dx, \]  

\[ Q_{n,A}(t) := \int_{-\infty}^{\infty} \varepsilon_i(x) Q_n(x; t)dx. \]  

\(^1\)as \( n \to \infty \) in the change \( x = \sqrt{2n+\epsilon} + 2^{-\frac{1}{2}}n^{-\frac{1}{4}}X \) and \( y = \sqrt{(n+\epsilon)} + 2^{-\frac{1}{2}}n^{-\frac{1}{4}}Y \).
The epsilon quantities are
\[ Q_{n,\varepsilon}(x; t) = (I - K_n)^{-1}(x, y), \varepsilon \varphi(y), \quad q_{n,\varepsilon}(t) = Q_{n,\varepsilon}(t; t) \]  
(1.21)

\[ u_{n,\varepsilon}(t) = (Q_{n,\varepsilon}(x; t), \varphi(x)), \quad \tilde{v}_{n,\varepsilon}(t) = (Q_{n,\varepsilon}(x; t), \psi(x)), \]  
(1.22)

where \((\cdot, \cdot)\) denotes the inner product on \(L^2(t, \infty)\).

The GOE and GSE analogue of (1.27) in Theorem 1.1 below will follow from representations (equations (40) and (41) of [22]).

\[ F_{n,1}(t) = F_{n,2}(t) \cdot \left( (1 - \tilde{v}_{n,\varepsilon}(t)) (1 - \frac{1}{2} R_{n,1}(t)) - \frac{1}{2} (q_{n,\varepsilon}(t) - c\varphi) P_{n,1}(t) \right) \]  
(1.23)

and

\[ F_{n,4}(t/\sqrt{2})^2 = F_{n,2}(t) \cdot \left( (1 - \tilde{v}_{n,\varepsilon}(t)) (1 + \frac{1}{2} R_{n,4}(t)) + \frac{1}{2} (q_{n,\varepsilon}(t) P_{n,4}(t) \right. \]  
(1.24)

To complete the work, we will use the close form of \(R_{n,1}, P_{n,1}, R_{n,4}, P_{n,4}, \tilde{v}_{n,\varepsilon}, \) and \(q_{n,\varepsilon}\) derived in [5] to give a finite \(n\) representation of \(F_{n,1}\) and \(F_{n,4}\), then the corresponding large \(n\) expansion to find an Edgeworth type expansion for \(F_{n,4}\) as outlined in [4]. We will also need the following results,

**Theorem 1.1.** [2] If we set
\[ \tau(s) = (2(n + c))^\frac{1}{2} + 2^{-\frac{1}{4}} n^{-\frac{1}{4}} s \quad \text{and} \]
\[ E_{c,2} := E_{c,2}(s) = 2w_1 - 3w_2 + (-20c^2 + 3)v_0 + u_1 v_0 - u_0 v_1 + u_0 v_2^2 - u_0^2 w_0. \]  
(1.25)

Then as \(n \to \infty\)
\[ F_{n,2}(\tau(s)) = F_2(s) \left\{ 1 + c u_0(s) n^{-\frac{1}{3}} - \frac{1}{20} E_{c,2}(s) n^{-\frac{2}{3}} \right\} + O(n^{-1}) \]  
(1.27)

uniformly in \(s\), and
\[ F_2(s) = \lim_{n \to \infty} F_{n,2}(t) = \exp \left( - \int_s^\infty (x - s) q(x)^2 \, dx \right) \]  
(1.28)

is the Tracy-Widom distribution.

To state the next result we need the following definitions
\[ \alpha := \alpha(s) = \int_s^\infty q(x) u(x) \, dx, \]  
(1.29)

\[ \mu := \mu(s) = \int_s^\infty q(x) \, dx, \]  
(1.30)
\[ \nu := \nu(s) = \int_s^\infty p(x)dx = \alpha(s) - q(s), \quad (1.31) \]

\[ \eta := \eta(s) = \frac{1}{20\sqrt{2}} \int_s^\infty (6qv + 3pu + 2p_2 + 2p_1v + 2pv - 2q_2u - 2q_1u_1 - 2qu_2)(x)dx - \]

\[ \frac{20c^2q'(s) + 3p(s)}{20\sqrt{2}} \]

\[ E_{c,1}(s) = -\frac{1}{20}E_{c,2}(s) e^{-\mu} - \frac{c\alpha}{2\mu^2} + \frac{cp}{2\mu} + \frac{(2c - 1)\nu^2}{4\mu^2} + c u \left( c q e^{-\mu} - \frac{\nu}{2\mu} (1 - e^{-\mu}) \right) + \]

\[ e^{-2\mu} \left( \frac{\nu (\nu + 8c q)}{32\mu} - \frac{\eta}{4\sqrt{2}} \right) + e^{-\mu} \left( \frac{2\sqrt{2} c^2 q^2 - 3\eta}{4\sqrt{2}} + \frac{\nu^2 - 8(2cp + c^2 q_2^2) - 4c^2 \alpha^2}{32\mu} \right) \]

\[ - \frac{c^2 q^2}{8\mu^2} + \frac{2 - \mu}{2\mu^2} \left( c q \alpha + \frac{1}{4} \nu^2 + (c^2 - c) q^2 \right) - (4c^2 \alpha^2 + 3c^2 q_2^2 - \nu^2) \frac{\cosh(\mu)}{8\mu^2}. \quad (1.32) \]

**Theorem 1.2.** [4] As \( n \to \infty \)

\[ F_{n,1}(\tau(s))^2 = F_2(s) \left\{ e^{-\mu(s)} + \left[ c(q(s) + u(s))e^{-\mu(s)} - \frac{\nu(s)}{2\mu(s)} (1 - e^{-\mu(s)}) \right] n^{-\frac{1}{2}} + \]

\[ E_{c,1}(s)n^{-\frac{3}{4}} \right\} + O(n^{-1}) \quad (1.34) \]

uniformly for bounded \( s \).

### 1.1 Statement of our results

If we set \( a(t) := \int_t^\infty q_n(x)dx \) and \( b(t) = \int_t^\infty p_n(x)dx \),

then for the Gaussian Orthogonal Ensemble,

**Theorem 1.3.** For \( n \) even,

\[ F_{n,1}(t)^2 = F_{n,2}(t) \left\{ \frac{1}{2} - c_\phi^2 \frac{b(t)}{a(t)} - 2\sqrt{\frac{b(t)}{2a(t)}c_\phi \sinh \sqrt{2a(t)b(t)}} \right\} \]

\[ \left( \frac{1}{2} + c_\phi^2 \frac{b(t)}{a(t)} \right) \cosh \sqrt{2a(t)b(t)} \}

and for the Gaussian Simplectic Ensemble,
Theorem 1.4. For $n$ odd,

$$F_{n,4}(t/\sqrt{2})^2 = F_{n,2}(t)\frac{1}{2} \left(1 + \cosh \sqrt{2a(t)b(t)}\right).$$

Theorem (1.4) leads to the following Edgeworth Expansion of $F_{n,4}$.

Theorem 1.5. Then as $n \to \infty$

$$F_{n,4}\left(\frac{\tau(s)}{\sqrt{2}}\right)^2 = F_2(s) \cdot \left\{ \cosh^2\left(\frac{\mu}{2}\right) + \frac{c}{2}u_0(1 + \cosh(\mu)) - q \sinh(\mu) \right\} n^{-\frac{1}{3}}$$

$$\frac{1}{4} \left[ \frac{\nu}{2\sqrt{2}\mu} \sinh(\mu) + c^2q^2 \cosh(\mu) + \frac{\cosh(\mu) - 1}{10} E_{c,2} + \sqrt{2}(\eta - \sqrt{2}c^2u_0) \sinh(\mu) \right] n^{-2/3} \} + \frac{\sinh(\mu)}{\mu} O(n^{-1})$$

uniformly for bounded $s$.

In §2 we complete the derivation of (1.24) as outlined in [22]. In §3 Theorems 1.3 and 1.4 are derived. In §4 we justify Theorem 1.5 followed in §5 by a brief discussion on the rate of convergence of these distributions as a function of the fine-tuning constant $c$ for large $n$.

2 Derivation of $F_{n,4}$

We treat here the case $n$ odd. (Most of the algebra used here can be found in [10], [11], [12], [16], and [27].) We note here that

$$F_{n,4}(t/\sqrt{2})^2 = \det(I - K_{n,4}),$$

(2.1)

with

$$2K_{n,4} = \chi \left( \begin{array}{cc} K_{n,2} + \psi \otimes \varphi & K_{n,2} D - \psi \otimes \varphi \\ \varepsilon K_{n,2} + \varepsilon \psi \otimes \varepsilon \varphi & K_{n,2} + \varepsilon \varphi \otimes \psi \end{array} \right) \chi.$$  

(2.2)

We set $J = (t, \infty)$, and $\chi$ represents the multiplication by the function $\chi_J(x)$. This notation allows us to think of the operator with kernel $K_{n,4}$ as an operator acting on $\mathbb{R}$ instead of acting on $(t, \infty)$. We will like to also note that this section is identical to the derivation made in [22]; we only provide here details intentionally left out by the authors of [22] (we suppose to shorten the length of the paper, even if they provided the necessary steps to complete the derivation.)

Using the following commutators,

$$[K_{n,2}, D] = \varphi \otimes \psi + \psi \otimes \varphi, \quad [\varepsilon, K_{n,2}] = -\varepsilon \varphi \otimes \varepsilon \psi - \varepsilon \psi \otimes \varepsilon \varphi$$

(2.3)
We therefore have,

\[ K_{n,2} + \psi \otimes \varphi = D \varepsilon K_{n,2} + D \varepsilon \psi \otimes \varepsilon \varphi = D(\varepsilon K_{n,2} + \varepsilon \psi \otimes \varepsilon \varphi) = D(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi), \]

\[ K_{n,2} D - \psi \otimes \varphi = D K_{n,2} + \varphi \otimes \psi = DK_{n,2} + D \varepsilon \varphi \otimes \psi = D(K_{n,2} + \varepsilon \varphi \otimes \psi) \]

and

\[ \varepsilon K_{n,2} + \varepsilon \psi \otimes \varepsilon \varphi = K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi \]

as \( D\varepsilon = I \). Our kernel is now

\[ 2K_{n,4} = \chi \left( \begin{array}{cc} D(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) & D(K_{n,2} + \varepsilon \varphi \otimes \psi) \\
K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi & K_{n,2} + \varepsilon \varphi \otimes \psi \end{array} \right) \chi \]  

(2.4)

\[ = \left( \begin{array}{cc} \chi & 0 \\
0 & \chi \end{array} \right) \cdot \left( \begin{array}{cc} (K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi & (K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \\
(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi & (K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \end{array} \right) \]  

(2.5)

Since \( K_{n,4} \) is of the form \( AB \), we can use the fact that \( \det(I - AB) = \det(I - BA) \) and deduce that the Fredholm determinant of \( K_{n,4} \) is unchanged if instead we take \( 2K_{n,4} \) to be

\[ \left( \begin{array}{cc} (K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi & (K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \\
(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi & (K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \end{array} \right) \cdot \left( \begin{array}{cc} \chi & 0 \\
0 & \chi \end{array} \right) \]  

(2.6)

\[ = \left( \begin{array}{cc} (K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi D & (K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \\
(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi D & (K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \end{array} \right). \]  

(2.7)

Thus

\[ \det(I - K_{n,4}) = \det \left( I - \frac{1}{2}(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi D - \frac{1}{2}(K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \right). \]  

(2.8)

Performing row and column operations on the matrix\(^2\) does not change the Fredholm determinant. We first subtract row 1 from row 2, next we add column 2 to column 1 to have the following matrix

\[ \left( \begin{array}{cc} I - \frac{1}{2}(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi D - \frac{1}{2}(K_{n,2} + \varepsilon \varphi \otimes \psi) \chi & -\frac{1}{2}(K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \\
0 & I \end{array} \right). \]  

(2.9)

We therefore have,

\[ \det(I - K_{n,4}) = \det \left( I - \frac{1}{2}(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi D - \frac{1}{2}(K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \right) \]  

(2.10)

\[ = \det \left( I - K_{n,2} \chi - \frac{1}{2}(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi D + \frac{1}{2}(K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \right) \]  

(2.11)

\[ = \det \left( I - K_{n,2} \chi - \frac{1}{2}(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi D + \frac{1}{2}K_{n,2} \chi + \frac{1}{2} \varepsilon \varphi \otimes \psi \chi - \varepsilon \varphi \otimes \psi \chi \right) \]

\(^2\)This does not change the determinant, for more details see [22].
εrank operator. To this end we introduce in this section the following notation, $I$ factor of (2.12). We will represent this factor in the form $(I - K_{n,2} + R_{n,2}K_{n,2})ε[χ D] - \frac{1}{2}Q_{n,ε}⊗ ψ[χ D] - Q_{n,ε}⊗ ψχ).$ We used the fact that $ε D = I$ to write $χ = ε D χ$ and the fact that $ε$ is antisymmetric to have $εφ⊗ ψχ D = εφ⊗ ψε^t χ D = -εφ⊗ ψεχ D.$

Next we factor out $I - K_{n,2}$ and note that $(I - K_{n,2})^{-1} = I + R_{n,2}$, where $R_{n,2}$ was defined as the resolvent of $K_{n,2}$, and $(I - K_{n,2})^{-1}εφ = Q_{n,ε}.$ We are interested on the determinant of the following operator

$$(I - K_{n,2}) (I - \frac{1}{2}(K_{n,2} + R_{n,2}K_{n,2})ε[χ D] - \frac{1}{2}Q_{n,ε}⊗ ψ[χ D] - Q_{n,ε}⊗ ψχ).$$ (2.12)

The determinant of the first factor, $det(I - K_{n,2}χ) = F_{n,2}$ is a familiar object that was studied in our work in GUE$_n$ see [2] or [3]. Attention will be given to the second factor of (2.12). We will represent this factor in the form $(I - \sum_{j=1}^{k} α_j ⊗ β_j)$ and use the well known formula $det(I - \sum_{j=1}^{k} α_j ⊗ β_j) = det(δ_{i,j} - (α_i, β_j))_{i,j=1,...,k}$ to expand the Fredholm determinant. First we need to find a representation of $ε[χ D]$ as a finite rank operator. To this end we introduce in this section the following notation,

$ε_k(x) = ε(x - a_k), \quad R_k(x) = R_{n,2}(x, a_k), \quad δ_k(x) = δ(x - a_k), \quad a_1 = t, \quad$ and $a_2 = ∞.$

With the new notation $J = (t, ∞) = (a_1, a_2)$, and the commutator $[χ D] = -δ_1 ⊗ δ_1 + δ_2 ⊗ δ_2,$ gives $ε[χ D] = -ε_1 ⊗ δ_1 + ε_2 ⊗ δ_2.$

With this representation, we have

$$(K_{n,2} + R_{n,2}K_{n,2})ε[χ D] = \sum_{k=1,2} (-1)^k(K_{n,2} + R_{n,2}K_{n,2})ε_k ⊗ δ_k$$

and

$$Q_{n,ε}⊗ ψ[χ D] = \sum_{k=1,2} (-1)^k(Q_{n,ε}⊗ ψ) · (ε_k ⊗ δ_k) = \sum_{k=1,2} (-1)^k(ψ, ε_k)Q_{n,ε}⊗ δ_k.$$ (2.12)

In the last equation we used the formula $(α ⊗ β) · (γ ⊗ δ) = (β, γ)α ⊗ δ$. If we substitute this in the second factor of (2.12), it becomes

$$I - \frac{1}{2} \sum_{k=1,2} (-1)^k(K_{n,2} + R_{n,2}K_{n,2})ε_k ⊗ δ_k - \frac{1}{2} \sum_{k=1,2} (-1)^k(ψ, ε_k)Q_{n,ε}⊗ δ_k - Q_{n,ε}⊗ ψχ.$$ (2.13)
We have
\[ \varepsilon_2(x) = \varepsilon_\infty(x) = \frac{1}{2} \text{sgn}(x - \infty) = -\frac{1}{2}, \quad \text{and} \quad R_2 = R_{n,2}(x, \infty) = 0. \]

If we substitute these values in (2.13), it then becomes,
\[ I - Q_{n,\varepsilon} \otimes \chi \psi - \frac{1}{2} [(K_{n,2} + R_{n,2} K_{n,2}) \varepsilon_t + (\psi, \varepsilon_t) Q_{n,\varepsilon}] \otimes \delta_t + \frac{1}{4} [(K_{n,2} + R_{n,2} K_{n,2}) + (\psi, 1) Q_{n,\varepsilon}] \otimes \delta_\infty. \]  

This operator is of the desired form
\[ I - \sum_{k=1,2,3} \alpha_k \otimes \beta_k \quad \text{and} \quad \det(I - \sum_{j=1}^{3} \alpha_j \otimes \beta_j) = \det(\delta_{i,j} - (\alpha_i, \beta_j))_{i,j=1,...,3} \]

with
\[ \alpha_1 = Q_{n,\varepsilon}, \]
\[ \alpha_2 = \frac{1}{2} [(K_{n,2} + R_{n,2} K_{n,2}) + (\psi, \varepsilon_t) Q_{n,\varepsilon}], \]
\[ \alpha_3 = -\frac{1}{4} [(K_{n,2} + R_{n,2} K_{n,2}) + (\psi, 1) Q_{n,\varepsilon}] \]

and
\[ \beta_1 = \chi \psi, \quad \beta_2 = \delta_t \quad \text{and} \quad \beta_3 = \delta_\infty. \] (2.15)

As pointed out in [21], the contribution of \( \beta_3 \) is zero
\[(\alpha_1, \beta_3) = (\alpha_2, \beta_3) = (\alpha_3, \beta_3) = 0, \]

thus the determinant reduces to the contribution of \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \). The corresponding inner product are;
\[(\alpha_1, \beta_1) = \tilde{v}_{n,\varepsilon}, \quad (\alpha_1, \beta_2) = q_{n,\varepsilon} + c_\varphi = q_{n,\varepsilon} \quad \text{as} \quad c_\varphi = 0 \quad \text{for} \ n \ \text{odd} \] (2.16)

where we have
\[ c_\varphi = \varepsilon \varphi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi(x) \, dx = 0 \quad \text{as} \quad \varphi \quad \text{is odd} \] (2.17)

we also note that
\[ c_\psi = \varepsilon \psi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(x) \, dx, \] (2.18)

and for \( n \) odd
\[ c_\psi = (\pi(n - 1))^{1/4} 2^{-3/4-(n-1)/2} ((n - 1)!)^{1/2} / ((n-1)/2)! \] (2.19)

and
\[(\alpha_2, \beta_1) = -\frac{1}{2} (P_{n,4} - a_4 + a_4 \tilde{v}_{n,\varepsilon}), \quad (\alpha_2, \beta_2) = -\frac{1}{2} (R_{n,4} + a_4 q_{n,\varepsilon}). \] (2.20)

with \( a_4 = (\psi, \varepsilon_t) \). The determinant of (2.13) is therefore
\[ (1 - \tilde{v}_{n,\varepsilon}(t)) \left( 1 + \frac{1}{2} R_{n,4}(t) \right) + \frac{1}{2} q_{n,\varepsilon}(t) P_{n,4}(t). \] (2.21)

We will make use of large \( n \) expansion of the functions appearing (2.21) (see [5]) combine with the large \( n \) expansion of \( F_{n,2} \) to give an edgeworth type expansion of \( F_{n,4} \).
3 Finite $n$ Expression of $F_{n,1}$ and $F_{n,4}$

In [5] we derived the following results; if we recall that

$$a(t) = \int_t^\infty q_n(x) \, dx \quad \text{and} \quad b(t) = \int_t^\infty p_n(x) \, dx$$

then for the GSE$_n$,

$$\tilde{v}_{n,\epsilon}(t) = 1 - \frac{1}{2}[1 + \cosh \sqrt{2a(t)b(t)}] \quad (3.1)$$

and

$$q_{n,\epsilon}(t) = -\sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)} \quad (3.2)$$

for the epsilon independent functions, and the epsilon dependent functions are

$$\mathcal{P}_{n,4}(t) = -c_\varphi \frac{1}{2}[1 + \cosh \sqrt{2a(t)b(t)}] - \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)}, \quad (3.3)$$

and

$$\mathcal{R}_{n,4}(t) = -c_\varphi \sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)} + \cosh \sqrt{2a(t)b(t)} - 1. \quad (3.4)$$

If we recall that

$$F_{n,2}(t) = \det(I - K_{n,2}) = \exp \left( -2 \int_t^\infty (x-t)q_n(x)p_n(x) \, dx \right), \quad (3.5)$$

then have

**Theorem 3.1.** For $n$ odd,

$$F_{n,4}(t/\sqrt{2})^2 = F_{n,2}(t) \frac{1}{2} \left( 1 + \cosh \sqrt{2a(t)b(t)} \right)$$

Or what is the same as

$$F_{n,4}(t/\sqrt{2}) = \cosh \sqrt{\frac{a(t)b(t)}{2}} \exp \left( - \int_t^\infty (x-t)q_n(x)p_n(x) \, dx \right) \quad (3.6)$$

The surprising fact here is that $c_\varphi$ drops out and the formula is as simple as possible. We couldn’t wish for a better result.

In a similar way we can also use the following representations

$$F_{n,1}(t)^2 = F_{n,2}(t) \left\{ (1 - \tilde{v}_{n,\epsilon})(1 - \frac{1}{2}\mathcal{R}_{n,1}) - \frac{1}{2}(q_{n,\epsilon} - c_\varphi)\mathcal{P}_{n,1} \right\},$$

together with the following representations (see [5] for a derivation of these results),
\[ \tilde{v}_{n,\epsilon}(t) = 1 - \frac{1}{2}[1 + \cosh \sqrt{2a(t)b(t)}] + c_\varphi \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)} \] (3.7)

and

\[ q_{n,\epsilon}(t) = -\sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)} + c_\varphi \cosh \sqrt{2a(t)b(t)}. \] (3.8)

We also have

\[ \mathcal{P}_{n,1}(t) = c_\varphi \frac{b(t)}{a(t)} [\cosh \sqrt{2a(t)b(t)} - 1] - \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)} \] (3.9)

and

\[ \mathcal{R}_{n,1}(t) = 1 - 2c_\varphi \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)} - \cosh \sqrt{2a(t)b(t)}, \] (3.10)

to have for the GOE

**Theorem 3.2.** For \( n \) even,

\[ F_{n,1}^2(t) = F_{n,2}(t) \left\{ \frac{1}{2} - c_\varphi^2 \frac{b(t)}{a(t)} - 2\sqrt{\frac{b(t)}{2a(t)}} c_\varphi \sinh \sqrt{2a(t)b(t)} \right\} \]

\[ \left( \frac{1}{2} + c_\varphi^2 \frac{b(t)}{a(t)} \right) \cosh \sqrt{2a(t)b(t)} \] (3.11)

Unlike the GSE \( n \) case, the finite \( n \) GOE \( n \) does not simplify as well, but it is still a very applicable formula.

Theorems 3.1 and 3.6 are the main results in this section, they provide a general \( n \) formula for the probability distribution function of the largest eigenvalue. In the next section, we will find a large \( n \) expansion of these two functions.

### 4 Edgeworth Expansion for GSE\(_n\)

The previous section tells us that these three probabilities distribution functions \( F_{n,1} \), \( F_{n,2} \) and \( F_{n,4} \) are all functions of \( q_n \) and \( p_n \), thus to find a large \( n \) expansion of these functions, we need large \( n \) expansion for these functions \( q_n \) and \( p_n \). Upon substituting these expansions into the sought after functions (3.11) and (3.6), we recover an Edgeworth type expansion of these probability distribution functions. This work was carried out in [2, 3] for the GUE\(_n\), and in [4] for the GOE\(_n\). In this paper we follow the same technique to find the corresponding expansion for GSE\(_n\). This presentation is different from the previous results since it gives a closed form for \( F_{n,1} \) and \( F_{n,4} \) that can be used to compute these expansions.
We quickly see that after substitution of the large $n$ expansion of $q_n$ in $a(t)$ and the large $n$ expansion of $p_n$ in $b(t)$, the result is

$$F_{n,1}(\tau(s)) = F_{n,2}(\tau(s)) \cdot \left\{ e^{-\mu(s)} + \left( \frac{\nu(s)}{2\mu(s)}(e^{-\mu(s)} - 1) + cq(s)e^{-\mu(s)} \right) n^{-\frac{1}{3}} + \frac{1}{16\mu(s)^2} \left( e^{-\mu(s)} \left( \nu(s)^2 \left( -1 + e^{\mu(s)} \right) - 3 + 8c + e^{\mu(s)} \right) + 2\mu(s) \left( 1 - 4c + 4c^2\mu(s) \right) \right) - 8cv(s) \left( -1 + e^{\mu(s)} + \mu(s)(-1 + 2c\mu(s)) \right) \int_{s}^{\infty} q[x]u[x] \, dx + 8\mu(s) \left( 20c \left( -1 + e^{\mu(s)} \right) \left( \int_{s}^{\infty} q[x]v[x] \, dx - \int_{s}^{\infty} q_1[x] \, dx \right) + \mu(s) \left( -3 + 20c^2 \right) \right) \int_{s}^{\infty} p[x]u[x] \, dx - 3 \int_{s}^{\infty} q[x]v[x] \, dx + 2 \int_{s}^{\infty} q[x]p_1[x] \, dx + 3 \int_{s}^{\infty} q_1[x] \, dx + 2 \int_{s}^{\infty} v[x]p_1[x] \, dx + 3 \int_{s}^{\infty} q_1[x] \, dx + 2 \int_{s}^{\infty} q[x]v[x] \, dx + 2 \int_{s}^{\infty} q[x]p_2[x] \, dx + 3 \int_{s}^{\infty} q[x]v[x] \, dx + 20c \left( 2 \int_{s}^{\infty} q[x]u[x]^2 \, dx - 3 \int_{s}^{\infty} q[x]v[x] \, dx + \int_{s}^{\infty} q_1[x] \, dx \right) + 2 \left( \int_{s}^{\infty} q_1[x] \, dx - \int_{s}^{\infty} p[x]v_1[x] \, dx \right) \right) n^{-\frac{2}{3}} + O\left( \frac{1}{n} \right) \right\}\right.$$

$$= F_{n,2}(\tau(s)) \cdot \left\{ e^{-\mu(s)} + \left( \frac{\nu(s)}{2\mu(s)}(e^{-\mu(s)} - 1) + cq(s)e^{-\mu(s)} \right) n^{-\frac{1}{3}} + f(s)n^{-\frac{2}{3}} + O\left( \frac{1}{n} \right) \right\}\right.$$

Using the Edgeworth Expansion of $F_{n,2}$ given in Theorem 1.1 leads to Theorem 1.2. For the GSE we have the following,

$$F_{n,4}(\frac{\tau(s)}{\sqrt{2}})^2 = F_{n,2}(\tau(s)) \cdot \left\{ \cosh^2\left( \frac{\mu(s)}{2} \right) - \frac{1}{2}cq(s)\sinh\mu(s) n^{-\frac{3}{4}} + \frac{1}{16\mu(s)} \left( \left( 4c^2\cosh[\mu(s)]\mu(s)(\nu(s) - \int_{s}^{\infty} q[x]u[x] \, dx)^2 + \left( -\nu(s)^2 + 4\mu(s) \right) \left( (3 - 20c^2) \int_{s}^{\infty} p[x]u[x] \, dx + 3 \int_{s}^{\infty} q[x]v[x] \, dx + 2 \int_{s}^{\infty} v[x]p_1[x] \, dx + 2 \int_{s}^{\infty} q[x]v[x] \, dx + \int_{s}^{\infty} q[x]v[x] \, dx + 2 \int_{s}^{\infty} q[x]p_2[x] \, dx + 3 \int_{s}^{\infty} q[x]v[x] \, dx + 20c \left( 2 \int_{s}^{\infty} q[x]u[x]^2 \, dx - 3 \int_{s}^{\infty} q[x]v[x] \, dx + \int_{s}^{\infty} q_1[x] \, dx \right) + 2 \left( \int_{s}^{\infty} q[x]u[x] \, dx - \int_{s}^{\infty} p[x]v_1[x] \, dx \right) \right) \right) n^{-\frac{3}{4}} + O\left( n^{-1} \right) \right\}\right.$$
5 Conclusion

The major finding about this studies of the probability distribution function of the largest eigenvalues of the classical Gaussian Random Matrix Theory Ensemble, is their dependence on just two quantities, $q_n$ and $p_n$. Indeed, comparing the representation for these three ensembles, we see that a complete statistical study would only require the knowledge of of these two functions. The simplicity of representation (3.5) and (3.6) allows us to have a simple form for the probability density function of the largest eigenvalues, and the somewhat not so simple representation (3.11) will do the similar work for the GOE$_n$.

The second finding is the similarity between the unitary and the Symplectic ensembles, the probability distribution functions are very simple to represent, the fine-tuning constant $c$ does the work, as in these two cases, we can eliminate the first correction term to the Tracy-Widom limit by setting $c$ to zero. But there are some numerical evidences showing that there is a way to fine-tune the result for the GOE$_n$ to speed the convergence rate to the Tracy-Widom distribution. This gives us hope to find the right way to capture this phenomenon for the Orthogonal Ensemble.

It would also be convenient to find a simplify version of the functions $f(s)$ appearing in (1.1) and the function $g(s)$ appearing in (1.2).
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