On the dynamics of Navier-Stokes-Fourier equations

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Abstract

In this paper we are concerned with a non-isothermal compressible Navier-Stokes-Fourier model with density dependent viscosity that vanish on the vacuum. We prove the global existence of weak solutions with large data in the three-dimensional torus \(\Omega = T^3\). The main point is that the pressure is given by \(P = R\rho\theta\) without additional cold pressure assumption.

Keywords: weak solutions; compressible non-isothermal model; global existence.

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1 Introduction

A compressible and heat-conducting fluid governed by the Navier-Stokes-Fourier equations satisfies the following system in \(R^+ \times \Omega\):

\[\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}\mathcal{S}, \\
\partial_t (\rho E) + \text{div}(\rho Eu) + \text{div}(q) + \text{div}(Pu) &= \text{div}(\mathcal{S}u),
\end{align*}\]

where the functions \(\rho, u, \theta\) represent the density, the velocity field, the absolute temperature. \(P\) stands for the pressure, \(\mathcal{S}\) denotes the viscous stress tensor. \(\rho E = \rho e + \frac{\rho|u|^2}{2}\) the total energy, \(e\) the internal energy. \(q\) the heat flux. Eqs. (1.1), (1.2), (1.3) respectively express the conservation of mass, momentum and total energy.

Our analysis is based on the following physically grounded assumptions:

- The viscosity stress tensor \(\mathcal{S}\) is determined by the Newton’s rheological law

\[\mathcal{S} = 2\mu(\rho)D(u) + \lambda(\rho)\text{div}_x u\mathbb{I},\]

where \(3\lambda + 2\mu \geq 0\) and \(D(u) = \frac{1}{2} \left( \nabla u + \nabla^T u \right)\) denotes the strain rate tensor, we require \(\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho))\). For simplicity, we only consider a particular case \(\mu(\rho) = \rho, \lambda(\rho) = 0\).

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• A key element of the system (1.1)-(1.3) is pressure $P$, which obeys the following equation of state:

$$P(\rho, \theta) = R\rho\theta,$$

where $R$ is the perfect gas constant, for simplicity, we set $R = 1$. This assumption means ideal gas given by Boyle’s law.

• In accordance with the second thermodynamics law, the form of the internal energy reads:

$$e = C_\nu\theta,$$

where $C_\nu$ is termed the specific heat at constant volume, for simplicity, we set $C_\nu = 1$.

• The heat flux $q$ is expressed through the classical Fourier’s law:

$$q = -\kappa \nabla \theta,$$

where the heat conducting coefficient $\kappa$ is assumed to satisfy:

$$\kappa(\rho, \theta) = \kappa_0(\rho, \theta)(1 + \theta^a),$$

where $a \geq 2$, $\kappa_0$ is a continuous function of temperature and density satisfying: $C_1 \leq \kappa_0(\rho, \theta) \leq \frac{1}{C_1}$, for some positive $C_1$.

Assuming smoothness of the flow the total energy equation (1.3) can be written using the equation for the thermal energy equation

$$\partial_t (\rho e) + \text{div}(\rho eu) + \text{div}q = S : \nabla u - P\text{div}u.$$  

Finally, to complete the system (1.1)-(1.3), the initial conditions are given by

$$\rho(0, \cdot) = \rho_0, (\rho u)(0, \cdot) = m_0, \theta(0, \cdot) = \theta_0,$$

(1.10)

together with the compatibility condition:

$$m_0 = 0 \text{ on the set } \{x \in \Omega | \rho_0(x) = 0\}. \quad (1.11)$$

Now we give the definition of a variational solution to (1.1)-(1.10).

**Definition 1.1.** We call $(\rho, u, \theta)$ is as a variational weak solution to the problem (1.1)-(1.10), if the following is satisfied.

(1)the density $\rho$ is a non-negative function satisfying the internal identity

$$\int_0^T \int_\Omega \rho \partial_t \phi + \rho u \cdot \nabla \phi dx dt + \int_\Omega \rho_0 \phi(0) dx = 0,$$

for any test function $\phi \in \mathcal{D}([0, T] \times \overline{\Omega})$.

(2) The momentum equation holds in $D'((0, T) \times \Omega)$, that means,

$$\int_\Omega m_0 \phi(0) dx + \int_0^T \int_\Omega \rho u \cdot \partial_t \phi + \rho(u \otimes u) \cdot \nabla \phi + P\text{div}\phi dx dt$$

$$= \int_0^T \int_\Omega S : \nabla \phi dx dt \text{ for any } \phi \in \mathcal{D}([0, T] \times \overline{\Omega}),$$

(1.13)
(3) The temperature $\theta$ is a non-negative function satisfying

$$
\int_0^T \int_\Omega \rho \partial_t \phi + \rho \theta \cdot \phi + K(\theta) \Delta \phi \, dx \, dt \leq \int_0^T \int_\Omega (R\rho \theta - S : \nabla u) \phi \, dx \, dt + \int_\Omega \rho_0 \theta_0 \, dx = 0,
$$

for any $\phi \in C^\infty([0,T] \times \Omega)$, $\phi \geq 0$, $\phi(T) = 0$, where

$$K = \int_0^T \kappa(z) \, dz;$$

(4) The total energy inequality holds for a.a. $\tau \in (0,T)$ with

$$\rho E(\tau) \leq \rho E(0),$$

where

$$\rho E(0) = \int_\Omega \frac{1}{2} \frac{|m_0|^2}{\rho_0} + \rho_0 \theta_0;$$

Now, we are ready to formulate the main result of this paper.

**Theorem 1.2.** Let $\Omega$ be the periodic box $T^3$. Assume that the pressure $P$, the conductivity coefficient $\kappa$ and the viscosity coefficient $\mu$ satisfy the condition (1.4)-(1.8). Assume the initial data $\rho_0, u_0, \theta_0$ satisfy

$$\rho_0 \geq 0, \quad \nabla \sqrt{\rho_0} \in L^2(\Omega),$$

$$\rho_0 |u_0|^2 \in L^1(\Omega), \quad \rho_0 (1 + |u_0|^2) \ln(1 + |u_0|^2) < \infty,$$

$$\theta_0 \in L^\infty(\Omega), \quad 0 < \theta_0 \leq \theta \quad \text{for a.e. } x \in \Omega.$$

Then, for any given $T > 0$, there exists a variational weak solution of (1.1)-(1.3) on the set $(0,T) \times \Omega$.

**Remark 1.1.** Compared with the constant viscosity and viscosity depending temperature case stated in [5], [6] gives global weak solutions to the nonlinear problem (1.1)-(1.3). Here the viscosity is depending viscosity.

**Remark 1.2.** Compared with the viscosity depending density case stated in [3] gives global weak solutions to the nonlinear problem (1.1)-(1.3) with additional pressure. Here the pressure is only ideal gas condition, i.e. $P = R\rho \theta$.

There is a large amount work on the global existence of weak solutions for the compressible Navier-Stokes equation, in the constant viscosity coefficients case, one of the main result of the nineties is due to P.L. Lions [8], who proved the global existence of weak solutions for the compressible Navier-Stokes system in the case of barotropic equations of state. Later, this result has been extended to the somehow optimal case $\gamma > n/2$ in [4] using oscillation defect measures on density sequences associated with suitable approximation solutions. For the full compressible Navier-Stokes equation, i.e., including the temperature equation, Feireisl [9] firstly prove the global existence of so-called variational solutions for the full compressible Navier-Stokes and heat-conducting system.
Later on, he also extended this result to the temperature depending viscosity case\cite{6}. Such an existence result is obtained for specific pressure laws, given by general pressure equation

\[ P(\rho, \theta) = P_b(\rho) + \theta P_\theta(\rho). \]

Unfortunately, the perfect gas equation of state is not covered by this result. Namely the dominant role of the first, barotropic pressure \( P_b \) is one of the key argument to obtain such an existence result.

Recently Bresch-Desjardins \cite{1} have made important progress in the case of viscosity coefficients depending on the density \( \rho \), under some structure constraint on the viscosity coefficients, they discover a new entropy inequality (called BD entropy) which can yield global in time integrability properties on density gradients. This new structure was first applied in \cite{2} in the framework of capillary fluid. Later on, they founded that this BD entropy inequality also can applied in the compressible Navier-Stokes equation without capillarity. By this new BD entropy inequality, they succeeded in obtaining global existence of weak solutions in the barotropic fluids with some additional drag terms. However, there are some difficulties without any additional drag term, as lack of estimates for the velocity. By obtaining a new apriori estimate on smooth approximation solutions, Mellet-Vasseur \cite{10} study the stability of barotropic compressible Navier-Stokes equations. Unfortunately, they cannot construct smooth approximation solutions. Li and Xin \cite{9} recently have been constructed some suitable approximate system which has smooth solutions satisfying the energy inequality, the BD entropy inequality, and the Mellet-Vasseur type estimate, therefore they completely solved an open problem.

As for the density depending viscosities case, the existence of global weal solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids was firstly proved by D. Bresch and B. Desjardins \cite{3}. The equation of state is ideal polytropic gas type:

\[ P = R\rho\theta + P_c(\rho), \]

However, they still need additional cold pressure assumption \( P_c \). Therefore, Our aim in this work is to remove additional assumption on the equation of state \( P_c \). In order to prove the global existence of variational weak solutions, we need to construct an adapted approximation scheme and have enough compactness to pass the limit. Suppose we can construct a sequence of approximate solutions \( \{\rho_n\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty}, \{\theta_n\}_{n=1}^{\infty} \) we come accross two major stumbling blocks when we passing the limit: (1) the lack of the strong convergence for \( \sqrt{\rho_n}u_n \) in \( L^2 \). (2) concentrations in \( \{\theta_n\}_{n=1}^{\infty} \), more specifically, the lack a priori bounds on \( K(\theta_n) \).

The problem of strong convergence for \( \sqrt{\rho_n}u_n \) in \( L^2 \) can be solved by establishing a Mellet-Vasseur inequality. The inequality was discovered by Mellet-Vasseur in the baratropic case, providing a \( L^\infty(0,T;L \log L(\Omega)) \) estimate of \( \rho_n|u_n|^2 \). However, it is difficult to construct a adapted approximate scheme verifying the B-D entropy inequality and the Mellet-Vasseur inequality. To deal with this issue, we follow the idea in A.F.Vasseur and C.Yu \cite{12}, \cite{11}. Additional damping terms \( r_0u, r_1\rho|u|^2u \) and quantum term \( \kappa \rho \nabla (\nabla \sqrt{\rho} \cdot \nabla \sqrt{\rho}) \) were introduced in their paper.

The problem of temperature concentration lies in the fact that there are only poor a priori bounds on \( K(\theta_n) \), specifically,

\[ K(\theta_n) \text{ are bounded in } L^1((0,T) \times \Omega), \]
we adopted an technique from Feireisl where the limit in the sense of renormalized limit of \( K(\theta_n) \).
i.e, the thermal energy inequality is stable with respect to the topology induced by the renormalized limit.

This paper is organized as follows. In section 2, we construct approximate system by adding some term to the origin system and using the Faedo-Galerkin approximation, also we establish the uniform estimates which is independent of \( N \) and pass the limit \( N \to \infty \). In section 3, we deduce the BD entropy energy estimates and pass the limit \( \varepsilon \to 0 \). In section 4, follow the idea in [11], we will get the approximate Mellet-Vasseur inequality for the weak solution. In section 5, 6 and 7, we recover the original system by vanishing these parameter \( m \to \infty, K \to \infty, \kappa \to 0, n \to \infty, r_0 \to 0, r_1 \to 0 \), therefore our main theorem is proved.

2 Faedo-Galerkin approximation

In this section we introduce a approximating scheme which involves a system of regularized equations and the Faedo-Galerkin method. More specifically, we follow the idea in [Feireisl]. In begin with, we fix \( u_N \) in the space \( C([0,T];X_N) \) and use it to find a unique smooth solution to (2.1) \( \rho = \rho(u_N) \), then we solve a regularized thermal equation to (2.3) \( \theta = \theta(\rho,u_N) \), in the following we find a local solution to the momentum equation by Schauder fixed theorem. Finally, in according with the uniform estimates, we can extend the local solutions for the whole time interval.

We define a finite-dimensional space \( X_N = \text{span}\{e_1,e_2,...,e_N\} \), where \( N \in \mathbb{N} \), each \( e_i \) is an orthonormal basic of \( L^2(\Omega) \) which is also an orthogonal basis of \( H^2(\Omega) \). We notice that \( u \in C^0([0,T];X_N) \) is given by

\[
  u_N(t,x) = \sum_{i=1}^{N} \lambda_i(t)e_i(x), \quad (t,x) \in [0,T] \times \Omega,
\]

for some functions \( \lambda_i(t) \), and because of all the norms are equivalence on \( X_N \), hence, \( u \) can be bound in \( C^0([0,T];C^k(\Omega)) \) for any \( k \geq 0 \), thus

\[
  \|u_N\|_{C^0([0,T];C^k(\Omega))} \leq \|u_N\|_{C^0([0,T];L^2(\Omega))}.
\]

2.1. Continuity equation

For any given \( u_N \in C^0([0,T];X_N) \), by the classical theory of parabolic equations, there exists a smooth solution \( \rho \) to the following approximated system

\[
  \rho_t + \text{div}(\rho u_N) = \varepsilon \Delta \rho, \quad \text{in} \quad (0,T) \times \Omega,
\]

with the initial data

\[
  \rho(0,x) = \rho_0 \geq \nu > 0 \quad \text{and} \quad \rho_0(x) \in C^{\infty}(\Omega),
\]

where \( \nu > 0 \) is a constant. The following lemma can be seen in [3].

Lemma 2.1. Let \( u_N \in C([0,T];X_N) \) for \( N \) fixed and \( \rho_0 \) be as above. Then there exists the unique classical solution to (2.1), i.e., \( \rho \in V^\rho_{[0,T]} \), where

\[
  V^\rho_{[0,T]} = \left\{ \begin{array}{ll} 
  \rho \in C([0,T];C^{2+\nu}(\Omega)), \\
  \partial_t \rho \in C([0,T];C^{\nu}(\Omega)),
\end{array} \right. 
\]
Moreover, the mapping \( u_N \mapsto \rho(u_N) \) maps bounded sets in \( C([0,T];X_N) \) into bounded sets in \( V^\rho_{[0,T]} \) and is continuous with values in \( C([0,T];C^{2+\nu'}(\Omega)) \), \( 0 < \nu' < \nu < 1 \),
\[
\inf_{x \in \Omega} \rho_0(x)e^{\int_0^T \|\text{div}u_N\|_{L^\infty(\Omega)} ds} \leq \rho(t,x) \leq \sup_{x \in \Omega} \rho_0(x)e^{\int_0^T \|\text{div}u_N\|_{L^\infty(\Omega)} ds}
\]

Finally, for fixed \( N \in \mathbb{N} \), the function \( \rho \) is smooth in the space variable.

### 2.2. Temperature equation

Next, given \( \rho, u_N \), the temperature will be looked for as a solution of the approximate thermal energy equation:
\[
\partial_t((\varepsilon + \rho)\theta) + \text{div}(\rho \theta u) - \Delta K(\theta) + \varepsilon \theta^{\alpha+1} = S : \nabla u - \rho \text{div}u,
\]
with
\[
(\varepsilon + \rho)\theta(0,x) = (\varepsilon + \rho_0)\theta_0,
\]
is fulfilled pointwisely on \((0,T) \times \Omega\). Note that we need to regularize the coefficient of \( \theta^{\alpha+1} \) with respect to time. A standard approach yields the following result:

**Lemma 2.2.** Let \( u_N \in C([0,T];X_N) \) be a given vector field and let \( \rho(u) \) be the unique solution of \((2.1)\). Then \((2.3)\) with the initial condition defined as above admits a unique strong solution \( \theta = \theta(u_N) \) which belong to
\[
V^{\theta}_{[0,T]} = \left\{ \theta \in L^\infty(0,T;W^{1,2}(\Omega)), \quad \theta, \theta^{-1} \in L^\infty((0,T) \times \Omega), \quad \Delta \theta \in L^2((0,T) \times \Omega) \right\}
\]
Moreover, the mapping \( u_N \) to \( \theta(u_N) \) maps bound sets in \( C([0,T];X_N) \) into bound sets in \( V^{\theta}_{[0,T]} \) and the mapping is continuous with values in \( L^2(0,T;W^{1,2}(\Omega)) \).

### 2.3. Momentum equation

The Faedo-Galerkin approximation for the weak formulation of the momentum balance is given by
\[
\begin{align*}
\int_\Omega \rho u_N(T) \psi dx - \int_\Omega m_0 \psi dx + \varepsilon \int_0^T \int_\Omega \Delta u_N \cdot \Delta \psi dx dt & - \int_0^T \int_\Omega (\rho u_N \otimes u_N) : \nabla \psi dx dt + \int_0^T \int_\Omega 2\rho \text{div} u_N : \nabla \psi dx dt \\
& - \int_0^T \int_\Omega P \nabla \psi dx dt + \varepsilon \int_0^T \int_\Omega \rho^{-1} \nabla \psi dx dt + \varepsilon \int_0^T \int_\Omega \nabla \rho \cdot \nabla u_N \psi dx dt \\
& = -\mu_0 \int_0^T \int_\Omega u_N \psi dx dt - \mu_1 \int_0^T \int_\Omega |u_N|^2 u_N \psi dx dt + \varepsilon \int_0^T \int_\Omega \Delta \sqrt{\rho} \nabla \sqrt{\rho} \psi dx dt \\
& - \kappa \int_0^T \int_\Omega \sqrt{\rho} \nabla \sqrt{\rho} \text{div} \psi dx dt + \varepsilon \int_0^T \int_\Omega \rho \Delta^4 \rho \psi dx dt
\end{align*}
\]
for any test function \( \psi \in X_N \). The extra term \( \varepsilon \Delta^2 u_N \) is not only necessary to extend the local solution obtained by the fixed point theorem to a global one at the Gerlakin level but also to make
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sure \( \partial_t (\frac{\nabla \rho}{\rho}) \in L^2((0, T) \times \Omega) \) so that it can be as a test function when we compute the B-D entropy at next level, the extra term \( \varepsilon \nabla \rho^{-10} \) and \( \varepsilon \rho \Delta^9 \rho \) are necessary to keep the density bounded, and bounded away from zero for all time. This enables us to take \( \frac{\nabla \rho}{\rho} \) as a test function to derive the BD entropy.

Following the same arguments in [6,7,11], we can solve (2.5) by the fixed point argument. To that purpose, we introduce an operator on the set \( \{ \rho \in L^1(\Omega), \rho > \rho > 0 \} \), where \( \rho = \xi_0 \):

\[
\mathcal{M}[\rho](t) : X_N \to X_N^*, \quad < \mathcal{M}[\rho]u, w > = \int_{\Omega} \rho u \cdot w dx \text{ for } u, w \in X_N,
\]

We can show that \( \Xi[\rho] \) is invertible,

\[
\|\mathcal{M}^{-1}(\rho)\|_{L(X_N^*, X_N)} \leq \rho^{-1},
\]

where \( L(X_N^*, X_N) \) is the set of all bounded linear mappings from \( X_N^* \) to \( X_N \). It is Lipschitz continuous in the following sense,

\[
\|\mathcal{M}^{-1}(\rho_1) - \mathcal{M}^{-1}(\rho_2)\|_{L(X_N^*, X_N)} \leq C(N, \rho)\|\rho_1 - \rho_2\|_{L^1(\Omega)},
\]

for any \( \rho_1 \) and \( \rho_2 \) from the following set

\[
N_\nu = \{ \rho \in L^1(\Omega) | \inf_{x \in \Omega} \rho \geq \nu > 0 \},
\]

We also define a mapping

\[
\mathcal{T} : C([0, \tau]; X_N) \to C([0, \tau]; X_N), \mathcal{T}(v_N) = u_N,
\]

them, can rewrite (2.5) as the following problem:

\[
u_N(t) = \mathcal{M}^{-1}[\rho(v_N)](m^0 + \int_0^T P_{X_N}N(v_N)ds),
\]

where

\[
< \mathcal{N}(v_N), \phi > = \int_{\Omega} (\rho v_N \otimes v_N) : \nabla \phi dx - \int_{\Omega} 2\rho \nabla v_N : \nabla \phi dx + \int_{\Omega} P\nabla \phi dx + \varepsilon \int_{\Omega} \Delta v_N \cdot \Delta \phi dx + \varepsilon \int_{\Omega} \rho^{-10} \nabla \phi dx + \varepsilon \int_{\Omega} \nabla \rho \cdot \nabla v_N \phi dx
\]

\[
+ \varepsilon \int_{\Omega} \rho \Delta^9 \rho \phi dx - r_0 \int_{\Omega} \rho |v_N|^2 v_N \phi dx - 2\kappa \int_{\Omega} \Delta \sqrt{\rho} \nabla \sqrt{\rho} \phi dx - \kappa \int_{\Omega} \Delta \sqrt{\rho} \div \phi dx.
\]

Next, we consider a ball \( \mathcal{B} \) in the space \( C([0, T]; X_N) \):

\[
\mathcal{B}_{R, \tau} = \{ v \in C([0, T]; X_N) : \|v\|_{C([0, T]; X_N)} \leq R \}
\]

It is easier to show that the operator \( \mathcal{T} \) is continuous and maps \( \mathcal{B}_{R, \tau} \) into itself, provided \( \tau \) is sufficiently small. Moreover, thanks to lemma 2.1 and 2.2, \( \mathcal{T} \) is a continuous mapping and its image consists of Lipschitz functions, thus it is compact in \( \mathcal{B}_{R, \tau} \). It allows us to apply the Schauder theorem to infer that there exists at least one fixed point \( u \) solving (2.5) on \([0, \tau] \).
2.4. Uniform estimates and global-in-time solvability

In order to extend this solution for the whole time interval \([0,T]\), we need uniform estimates of the solution with \(N\). Taking \(\psi = N\) in (2.3) and using the approximate continuity equation, we obtain the kinetic energy balance

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho_N |u_N|^2 + \frac{\eta}{10} \rho_N^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho_N}|^2 + \frac{\varepsilon}{2} |\nabla \Delta^4 \rho_N|^2 + \varepsilon \int_{\Omega} |\Delta u_N|^2 dx + \int_{\Omega} \rho_N |\nabla u_N|^2 dx \\
+ \varepsilon^2 \int_{\Omega} |\Delta^5 \rho_N|^2 dx + \varepsilon^2 \int_{\Omega} |\nabla \rho_N^{-5}|^2 dx + r_0 \int_{\Omega} |u_N|^2 dx + r_1 \int_{\Omega} \rho_N |u_N|^4 dx \\
+ \kappa \varepsilon \int_{\Omega} \rho_N |\nabla^2 \log \rho_N|^2 dx = \int_{\Omega} P(\rho_N, \theta_N) \text{div} u_N dx,
\]

Adding, to this, equality (2.3) integrated with respect to space and integrating the resulting sum with respect to time we obtain the total energy balance

\[
\int_{0}^{T^*} \||\Delta u_N||_{L^2}^2 dt < \infty.
\] (2.7)

Due to the equivalence of norms on the finite dimensional of \(X_N\), we deduce the uniform bound for \(u\) in \(C([0,T];X_N)\). Thus, we can extend local time \(\tau\) to global time \(T\), i.e. there exists a solution \((\rho, u, \theta)\) to (2.1), (2.3), (2.5) for any \(T > 0\).

2.5. Estimates independent of \(N\)

Our goal now is to identify a limit \(N \to \infty\) of the approximate solutions \(\rho_N, u_N, \theta_N\) as a solution of the problem (2.1), (2.3), (2.5). In order to achieve this, additional estimates are needed. In the following compactness analysis, we will always need a lemma proved by Jüngel [7].

**Proposition 2.3.**

\[
\int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx \geq \frac{1}{4} \int_{\Omega} |\nabla \sqrt{\rho}|^2 dx, \tag{2.8}
\]

and

\[
\int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx \geq \frac{1}{8} \int_{\Omega} |\nabla \rho^\frac{1}{4}|^4 dx, \tag{2.9}
\]

By energy equality (2.6), we have

\[
\kappa \varepsilon \int_{\Omega} \rho_N |\nabla^2 \log \rho_N|^2 dx < \infty, \tag{2.10}
\]

By Prop 2.3, we have the following uniform estimates:

\[
(\kappa \varepsilon)^\frac{1}{2} \|\sqrt{\rho_N}\|_{L^2(0,T;H^2(\Omega))} + (\kappa \varepsilon)^\frac{1}{2} \|\nabla^\frac{1}{4} \rho_N\|_{L^4(0,T;L^4(\Omega))} \leq C, \tag{2.11}
\]
where the constant $C > 0$ is independent of $N$.

To conclude this part, we have the following lemma on the approximate solutions $(\rho_N, u_N, \theta_N)$.

**Proposition 2.4.** Let $(\rho_N, u_N, \theta_N)$ be the solution of (2.1), (2.3), (2.5) on $(0, T) \times \Omega$ constructed above, then we have the following energy inequality

\[
\sup_{t \in (0,T)} \int_{\Omega} E(\rho_N, u_N, \theta_N) + \varepsilon \int_{\Omega} |\Delta u_N|^2 dx + \varepsilon^2 \int_{\Omega} |\Delta^5 \rho_N|^2 dx + \varepsilon^2 \int_{\Omega} \nabla \rho_N^{-5} dx + r_0 \int_{\Omega} |u_N|^2 dx + r_1 \int_{\Omega} \rho_N |u_N|^4 dx + \varepsilon \int_{\Omega} \theta_N^{\alpha+1} dx + \kappa \varepsilon \int_{\Omega} \rho_N \nabla^2 \log \rho_N^2 dx \leq E_0(\rho_N, u_N, \theta_N),
\]

(2.12)

where

\[
E(\rho_N, u_N, \theta_N) = \int_{\Omega} \left( \frac{1}{2} \rho_N |u_N|^2 + \frac{\varepsilon}{10} \rho_N^{-10} + \frac{\kappa}{2} \nabla \sqrt{\rho_N}^2 + \varepsilon \nabla \Delta \rho_N^2 + (\varepsilon + \rho_N) \theta_N dx, \right)
\]

(2.13)

Moreover, we have the following uniform estimates

\[
(\kappa \varepsilon)^{\frac{1}{2}} \| \sqrt{\rho_N} \|_{L^2(0,TH^2(\Omega))} + (\kappa \varepsilon)^{\frac{1}{2}} \| \nabla \rho_N^\frac{1}{2} \|_{L^4(0,TL^4(\Omega))} \leq C,
\]

(2.14)

where the constant $C > 0$ is independent of $N$.

In particular, we have the following estimates,

\[
\sqrt{\rho_N} u_N \in L^\infty(0, T; L^2(\Omega)), \sqrt{\varepsilon} \Delta u_N \in L^2((0, T) \times \Omega),
\]

(2.15)

\[
\varepsilon \Delta^5 \rho_N \in L^2((0, T) \times \Omega), \sqrt{\varepsilon} \rho_N \in L^\infty(0, T; H^9(\Omega)), \sqrt{\kappa} \sqrt{\rho_N} \in L^\infty(0, T; H^1(\Omega)),
\]

(2.16)

\[
\varepsilon \frac{1}{10} \rho_N^{-1} \in L^\infty(0, T; L^{10}(\Omega)), \varepsilon \nabla \rho_N^{-5} \in L^2((0, T) \times \Omega),
\]

(2.17)

\[
u_N \in L^2((0, T) \times \Omega), \rho_N^\frac{1}{2} u_N \in L^4((0, T) \times \Omega),
\]

(2.18)

\[
\rho_N \theta_N \in L^\infty(0, T; L^1(\Omega)), \theta_N^{\alpha+1} \in L^\infty(0, T; L^1(\Omega)),
\]

(2.19)

At this stage of approximation, We multiply (2.3) by $h(\theta_N)$, where $h$ enjoys the properties such that

\[
h \in C^2[0, \infty), \ h(0) = 1, \ h \ non-increasing \ on \ [0, \infty), \ \lim_{z \to \infty} h(z) = 0,
\]

(2.20)

\[
h'' \geq 2(h'(z))^2 \ for \ all \ z \geq 0.
\]

Accordingly, we obtain

\[
\partial_t ((\varepsilon + \rho_N)Q_h(\theta_N)) + \text{div}(\rho_N Q_h(\theta_N) u_N) - \Delta K_h(\theta_N) + \varepsilon \theta_N^{\alpha+1} h(\theta_N)
\]

(2.21)

\[
\begin{align*}
&= h(\theta_N) [\nabla u_N - \kappa(\theta_N) h'(\theta_N) \nabla \theta_N^2] - h(\theta_N) \rho_N \theta_N \text{div} u_N \\
&\quad + \varepsilon \Delta \rho_N (Q_h(\theta_N) - \theta_N h(\theta_N)),
\end{align*}
\]

where $Q_h, K_h$ are determined by

\[
Q_h = \int_0^{\theta_N} h(z) dz, \quad K_h = \int_0^{\theta_N} \kappa(z) h(z) dz,
\]

(2.22)
Integrating (2.21) over $\Omega$ yields

$$
\frac{d}{dt} \int_{\Omega} (\varepsilon + \rho_N) Q h(\theta_N) dx + \varepsilon \int_{\Omega} \theta_N^{\alpha+1} h(\theta_N) dx = \int_{\Omega} h(\theta_N) S : \nabla u_N - \kappa(\theta_N) h'(\theta_N)|\nabla \theta_N|^2 dx
$$

$$
+ \int_{\Omega} \varepsilon (\nabla \rho_N \cdot \nabla \theta_N) h'(\theta_N) - \theta_N h(\theta_N)\rho_N \theta_N \text{div} u_N dx.
$$

(2.23)

In particular, the choice $h(\theta) = (1 + \theta)^{-1}$ leads to relations

$$
-\int_{\Omega} \kappa(\theta_N) h'(\theta_N)|\nabla \theta_N|^2 dx \geq C \int_{\Omega} |\nabla \theta_N^{\alpha/2}|^2 dx,
$$

while

$$
\varepsilon |\int_{\Omega} (\nabla \rho_N \cdot \nabla \theta_N) h'(\theta_N)| \leq \varepsilon \|\nabla \rho_N\|_{L^2(\Omega)} \|\nabla \theta_N^2\|_{L^2(\Omega)},
$$

and

$$
\varepsilon |\int_{\Omega} \theta_N h(\theta_N) \rho_N \theta_N \text{div} u_N| \leq C \|\rho_N \theta_N\|_{L^2(\Omega)} \|\text{div} u_N\|_{L^2(\Omega)},
$$

It follows from hypothesis (1.8) and the energy estimates (2.12) that the right-hand side of the last inequality is bounded in $L^1(0,T)$ by a constant that depends only on $\delta$.

Consequently, (2.23) integrated with respect to $t$ together with the energy estimates (2.12) yield a bound

$$
\|\nabla \log \theta_N\|_{L^2((0,T) \times \Omega)} \leq C(\varepsilon), \quad \|\nabla \theta_N^{\alpha/2}\|_{L^2((0,T) \times \Omega)} \leq C(\varepsilon),
$$

(2.24)

which is independent of $N$.

We note that both the energy estimates and entropy estimates are independent of $N, \varepsilon$.

### 2.6. The first level approximate solutions

At this stage we are ready to pass to the limit for $N \to \infty$ in the sequence of approximate solutions $\{\rho_N, u_N, \theta_N\}$ in order to obtain a solution to the system (2.1), (2.3), (2.5). As for uniform estimates of the sequence $\{\theta_N\}$, we need an auxiliary result.

**Proposition 2.5.** Let $\Lambda \geq 1$ a given constant. Let $\rho \geq 0$ be a measurable function satisfying

$$
0 < M \leq \int_{\Omega} \rho dx, \quad \int_{\Omega} \rho^\chi dx \leq K,
$$

for

$$
\chi > \frac{6}{5}.
$$

Then there exists a constant $C = C(M, K)$ such that

$$
\|v\|_{L^2(\Omega)} \leq C(M, K)(\|\nabla v\|_{L^2(\Omega)} + [\int_{\Omega} \rho |v|^{\frac{2}{\chi}}]^\Lambda),
$$

for any $v \in W^{1,2}(\Omega)$.

Based on the previous estimates, we have the following estimates uniform in $N$. 

Lemma 2.6. The following estimates hold for any fixed positive constants $\varepsilon, r_0, r_1$ and $\kappa$:

\[
\|\sqrt{\rho_N}\|_{L^2((0,T)\times\Omega)} + \|\rho_N\|_{L^2(0,T;H^2(\Omega))} \leq C
\]  \hspace{1cm} (2.25)

\[
\|\rho_N\|_{L^2((0,T)\times\Omega)} + \|\rho_N\|_{L^2(0,T;H^{10}(\Omega))} \leq C
\]  \hspace{1cm} (2.26)

\[
\|\rho_N u_N\|_{L^2(0,T;H^{-9}(\Omega))} + \|\rho_N u_N\|_{L^2((0,T)\times\Omega)} \leq C
\]  \hspace{1cm} (2.27)

\[
\nabla(\rho_N u_N) \text{ is uniformly bounded in } L^4(0,T;L^\frac{6}{5}(\Omega)) + L^2(0,T;L^\frac{3}{2}(\Omega)).
\]  \hspace{1cm} (2.28)

\[
\|\rho_N^{-1}\|_{L^\frac{10}{9}(0,T)\times\Omega} \leq C
\]  \hspace{1cm} (2.29)

\[
\|\log \theta_N\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\theta_N^{\frac{3}{2}}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C
\]  \hspace{1cm} (2.30)

where $C$ is independent of $N$ and depends on $\varepsilon, r_0, r_1, \kappa$.

Proof. The proof of (2.25)-(2.29) is same as the Lemma 2.2 in [11].

The estimate (2.24) together with (2.19) make it possible to apply Proposition 2.5 such that (2.30) hold.

Applying the Aubin-Lions lemma and Lemma 2.6, we conclude

\[
\rho_N \to \rho \text{ strongly in } L^2(0,T;H^9(\Omega)), \text{ weakly in } L^2(0,T;H^{10}(\Omega)),
\]  \hspace{1cm} (2.31)

\[
\sqrt{\rho_N} \to \sqrt{\rho} \text{ strongly in } L^2(0,T;H^1(\Omega)), \text{ weakly in } L^2(0,T;H^2(\Omega)),
\]  \hspace{1cm} (2.32)

and

\[
\rho_N u_N \to \rho u \text{ strongly in } L^2((0,T)\times\Omega),
\]  \hspace{1cm} (2.33)

we notice that $u_N \in L^2((0,T)\times\Omega)$, thus

\[
u_N \to \nu \text{ weakly in } L^2((0,T)\times\Omega),
\]

Thus we can pass to the limits for the term $\rho_N u_N \otimes u_N$ as follows,

\[
\rho_N u_N \otimes u_N \to \rho u \otimes u
\]

in the distribution sense.

Here we state the following lemma on the strong convergence of $\rho_N|u_N|^2 u_N$, which will be used later again. The proof is essentially the same as Lemma 2.3 in [11].

Lemma 2.7. When $N \to \infty$, we have

\[
\rho_N|u_N|^2 u_N \to \rho|u|^2 u, \text{ strongly in } L^1(0,T;L^1(\Omega)).
\]

Meanwhile, we have to mention the following Sobolev inequality

\[
\|\rho^{-1}\|_{L^\infty(\Omega)} \leq C(1 + \|\rho\|_{H^{k+2}(\Omega)})^2(1 + \|\rho^{-1}\|_{L^2})^3
\]

for $k \geq \frac{3}{2}$. Thus the estimates on density from (2.16)-(2.17) enable us to use the above inequality to have

\[
\|\rho\|_{L^\infty((0,T)\times\Omega)} \geq C(\delta, \eta) > 0 \text{ a.e. in } (0,T) \times \Omega.
\]  \hspace{1cm} (2.34)
allow us to have $\rho_{N}^{-10}$ converges almost everywhere to $\rho^{-10}$. Thanks to (2.29), we deduce

$$\rho_{N}^{-10} \to \rho^{-10} \text{ strongly in } L^1((0,T) \times \Omega), \quad (2.35)$$

In order to continue, we have to show pointwise convergence of the sequence $\{\theta_N\}$. To this end, we use the fact that the time derivatives $\partial_t \theta_N$ satisfy the thermal energy inequality.

**Lemma 2.8.** Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of functions such that

$v_n$ are bounded in $L^2(0,T; L^q(\Omega)) \cap L^\infty(0,T; L^1(\Omega))$, with $q > \frac{2N}{N+2}$.

Furthermore, assume that

$$\partial_t v_n \geq g_n \text{ in } D'((0,T) \times \Omega)$$

where

$g_n$ are bounded in $L^1(0,T; W^{-m,r}(\Omega))$

for a certain $m \geq 1$, $r > 1$.

Then $\{v_n\}_{n=1}^{\infty}$ contains a subsequence such that

$$v_n \to v \text{ in } L^2(0,T; W^{-1,2}(\Omega)).$$

Now we want to apply Lemma 2.7 to the sequence $(\varepsilon + \rho_N)\theta_N$ appearing in the thermal equation (2.3). Note that, in accordance with the estimate (2.31) for the temperature, we have

$$\rho_N \log \theta_N \text{ bounded in } L^2(0,T; L^q(\Omega)) \cap L^\infty(0,T; L^1(\Omega)), \text{ with } q > \frac{2N}{N+2}. \quad (2.36)$$

Thus we can use Lemma 2.7 together with (2.30) and thermal energy inequality (2.3) to obtain

$$(\varepsilon + \rho_N)\theta_N \to (\varepsilon + \rho)\theta(\text{strongly}) \text{ in } L^2(0,T; W^{-1,2}(\Omega)).$$

Consequently, in view of $\theta_N \in L^2(0,T; W^{1,2}(\Omega))$

$$(\varepsilon + \rho_N)|\theta_N|^2 \to (\varepsilon + \rho)|\theta|^2 \text{ in } [D'((0,T) \times \Omega)]^N.$$  

As the function $z \mapsto \varepsilon z^2 + \rho z^2$ is non-decreasing, this relation allow us to conclude that strong convergence

$$\theta_N \to \theta \text{ strongly in } L^1((0,T) \times \Omega), \quad (2.37)$$

Now, a simple interpolation argument can be used to deduce form (2.37), (2.19), (2.30) that

$$\theta_N \to \theta \text{ strongly in } L^p((0,T) \times \Omega), \text{ for a certain } p > \alpha, \quad (2.38)$$

Thus we know that

$\theta$ is strictly positive a.e. on $(0,T) \times \Omega$, $\log \theta = \log \theta$, $\theta^3 = \theta^3$, \quad (2.39)

Here we state the following lemma on the convergence of $\rho_N |u_N|^2 u_N$ which is proved in Lemma 2.3 ([11]).
Lemma 2.9. When \( N \to \infty \), we have
\[
|\varepsilon| u_N^2 u_N \to \varepsilon |u|^2 u \quad \text{strongly in } L^1((0, T) \times \Omega),
\]
(2.40)

By the above compactness, we are ready to pass to the limits as \( N \to \infty \) in the approximation system. Thus we have shown that \((\rho, u)\) solves
\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) = \varepsilon \Delta \rho, \quad \text{pointwise in } (0, T) \times \Omega.
\]
(2.41)

and for any test function \( \psi \) such that the following integral hold:
\[
\int_{\Omega} \rho u(T) \psi dx - \int_{\Omega} m_0 \psi dx + \mu \int_{0}^{T} \Delta u \cdot \Delta \psi dx dt
\]
\[
- \int_{0}^{T} \int_{\Omega} (\rho \otimes u) : \nabla \psi dx dt + \int_{0}^{T} \int_{\Omega} 2\rho \nabla u : \nabla \psi dx dt
\]
\[
- \int_{0}^{T} \int_{\Omega} R \rho \theta \nabla \psi dx dt + \eta \int_{0}^{T} \int_{\Omega} \rho^{-10} \nabla \psi dx dt + \varepsilon \int_{0}^{T} \int_{\Omega} \nabla \rho \cdot \nabla u \psi dx dt
\]
\[
= -r_0 \int_{0}^{T} \int_{\Omega} u \psi dx dt - r_1 \int_{0}^{T} \int_{\Omega} |u|^2 u \psi dx dt - 2\kappa \int_{0}^{T} \int_{\Omega} \Delta \sqrt{\rho} \nabla \sqrt{\rho} \psi dx dt
\]
\[
- \kappa \int_{0}^{T} \int_{\Omega} \Delta \sqrt{\rho} \nabla \psi dx dt + \delta \int_{0}^{T} \int_{\Omega} \rho \Delta^9 \rho \psi dx dt
\]
(2.42)

Thanks to the weak lower semicontinuity of convex functions, we are able to pass to the limits in the energy inequality (2.12); by the strong convergence of the density and temperature, we have the following energy inequality in the sense of distributions on \((0, T)\):
\[
\sup_{t \in (0, T)} \int_{\Omega} E(\rho, u, \theta) + \varepsilon \int_{\Omega} |\nabla u|^2 dx + \varepsilon^2 \int_{\Omega} |\nabla^5 \rho|^2 dx + \varepsilon^2 \int_{\Omega} |\nabla \rho^{-5}|^2 dx + r_0 \int_{\Omega} |u|^2 dx
\]
\[
+ r_1 \int_{\Omega} |\rho|^4 dx + \kappa \varepsilon \int_{\Omega} \nabla^2 \log \rho |^2 dx + \varepsilon \int_{\Omega} \theta^{\alpha+1} dx \leq E_0(\rho, u, \theta),
\]
(2.43)

where
\[
E(\rho, u, \theta) = \int_{\Omega} \left( \frac{1}{2} |\rho|^2 + \frac{n}{10} \rho^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho|^2 + (\varepsilon + \rho \theta) \right) dx,
\]
(2.44)

Finally, we will pass to the limit for \( N \to \infty \) in (2.21) to obtain (2.47). Note that it is enough to show that one can pass to the limit in all non-linear terms contained in (2.47). To this end, we have used weak lower-continuity of the dissipative estimate: To begin with, we can use (2.42) together with estimates (2.33), (2.31), (2.19), (2.36) to deduce
\[
(\varepsilon + \rho N) Q_h(\theta_N) \to (\varepsilon + \rho) Q_h(\theta) \quad \text{in } L^1((0, T) \times \Omega)
\]
(2.45)

and
\[
\rho N Q_h(\theta_N) u_N \to \rho Q_h(\theta) u \quad \text{weakly in } L^r((0, T) \times \Omega)
\]
(2.46)

and
\[
\rho N \theta_N h(\theta_N) \text{div} u_N \to \rho \theta h(\theta) \text{div} u \quad \text{weakly in } L^r((0, T) \times \Omega)
\]
(2.47)

for a certain \( r > 1 \).
Moreover, because of convexity of the function 

\[
[M, \theta] \mapsto \begin{cases} 
  h(\theta) \left( \frac{1}{2} M : M + \lambda(\text{tr}[M])^2 \right), & \text{if } \theta \geq 0, \ M \in \mathbb{R}^{N^2}, \\
  \infty, & \text{if } \theta < 0,
\end{cases}
\]

we get

\[
\int_0^T \int_{\Omega} h(\theta)S : \nabla u \psi \, dx \, dt \leq \lim \inf_{N \to \infty} \int_0^T \int_{\Omega} h(\theta_N)S : \nabla u_N \psi \, dx \, dt,
\]

for any non-negative test function \( \psi \). Similarly,

\[
-\int_0^T \int_{\Omega} \psi \kappa(\theta)\theta' h'(\theta) |\nabla \theta|^2 \, dx \, dt \leq \lim \inf_{N \to \infty} \int_0^T \int_{\Omega} \psi \kappa(\theta_N)\theta'_N h'(\theta_N) |\nabla \theta_N|^2 \, dx \, dt,
\]

Now, because of strong convergence of \( \nabla \rho_N \) established in (2.33), we

\[
\int_0^T \int_{\Omega} \varepsilon \nabla (\psi (\log \theta_N - 1)) \cdot \nabla \rho_N + \psi \rho_N \text{div} u_N \, dx \, dt \\
\to \int_0^T \int_{\Omega} \varepsilon \nabla (\psi (\log \theta - 1)) \cdot \nabla \rho + \psi \text{div} u \, dx \, dt
\]

Finally, by virtue of (2.50), (2.51), (2.33), (2.31), (2.19), (2.36)

\[
K_h(\theta_N) \to K_h(\theta) \quad \text{in } L^1((0,T) \times \Omega),
\]

\[
h(\theta_N)\theta_N^{\alpha+1} \to h(\theta)\theta^{\alpha+1} \quad \text{in } L^1((0,T) \times \Omega),
\]

Making use of these estimates (2.45)-(2.52) we are able to let \( N \to \infty \) in (2.21) in order to obtain a renormalized thermal energy inequality:

\[
\int_0^T \int_{\Omega} ((\varepsilon + \rho)Q_h(\theta)) \partial_t \psi + (\rho Q_h(\theta)u) \cdot \nabla \psi + \Delta K_h(\theta) \Delta \psi - \varepsilon \theta^{\alpha+1} h(\theta) \psi \, dx \, dt \\
\leq \int_0^T \int_{\Omega} (\kappa(\theta_N)h'(\theta_N) |\nabla \theta_N|^2 - h(\theta_N)S : \nabla u_N) \, dx \, dt + \int_0^T \int_{\Omega} h(\theta_N)\rho_N \text{div} u_N \, dx \, dt
\]

\[
+ \varepsilon \int_0^T \int_{\Omega} \Delta \rho_N (Q_h(\theta_N) - \theta_N h(\theta_N)) \, dx \, dt - \int_{\Omega} \varepsilon + \rho_{0,N})Q_h(\theta_{0,N}) \, dx,
\]

to be satisfied for any test function

\[
\psi \in C^\infty([0,T] \times \Omega), \quad \psi \geq 0, \quad \psi(0) = 1, \quad \psi(T) = 0,
\]

### 3 BD entropy and vanishing limits \( \varepsilon \to 0 \)

The goal of this section is to pass into the limits for \( \varepsilon \to 0 \) in the family of approximate solutions \{\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon\} constructed in Section 2. In order to achieve this task, we will deduce the BD entropy for the approximation system in Section 2. By (2.45) and (2.46), we have

\[
\rho_\varepsilon \geq C(\varepsilon) > 0, \quad \text{and} \quad \rho_\varepsilon \in L^2(0,T;H^{10}(\Omega)) \cap L^\infty(0,T;H^9(\Omega)).
\]
### 3.1. BD entropy

Thanks to (3.1), we can use $\psi = \nabla (\log \rho_\varepsilon)$ to test the momentum equation to derive the BD entropy. Thus we have the following lemma.

**Lemma 3.1.**

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u_\varepsilon + \frac{\nabla \rho_\varepsilon}{\rho_\varepsilon}|^2 + \frac{\varepsilon}{10} \rho_\varepsilon^{-10} + \frac{\kappa}{2} \left| \nabla \sqrt{\rho_\varepsilon} \right|^2 + \frac{\delta}{2} \left| \nabla^4 \rho_\varepsilon \right|^2 dx + \eta \int_{\Omega} \left| \nabla \rho_\varepsilon^{-5} \right|^2 dx + \kappa \int_{\Omega} \rho_\varepsilon \left| \nabla \log \rho_\varepsilon \right|^2 dx
\]

\[
+ \frac{\kappa}{2} \left| \nabla \right| \rho_\varepsilon \left( \Delta^2 \rho_\varepsilon \right) dx + 2 \varepsilon \int_{\Omega} \left| \Delta \rho_\varepsilon \right|^2 dx + 2 \varepsilon \int_{\Omega} \rho_\varepsilon \left| \nabla^4 \rho_\varepsilon \right|^2 dx + \varepsilon \int_{\Omega} \left| \Delta \rho_\varepsilon \right|^2 dx
\]

\[
+ \int_{\Omega} \left| \nabla \rho_\varepsilon \right|^2 \theta_\varepsilon = \varepsilon \int_{\Omega} \nabla \rho_\varepsilon \cdot \nabla u_\varepsilon \cdot \nabla \rho_\varepsilon + \varepsilon \int_{\Omega} \Delta \rho_\varepsilon \left| \nabla \log \rho_\varepsilon \right|^2 dx
\]

\[
- \varepsilon \int_{\Omega} \Delta u_\varepsilon \cdot \nabla \Delta \log \rho_\varepsilon dx - \varepsilon \int_{\Omega} \Delta \rho_\varepsilon \left| \nabla \rho_\varepsilon \right|^2 dx
\]

\[= R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8,
\]

(3.2)

We follow the same arguments in [16] to control terms $R_i$ for $i = 1, 2, 3, 4, 5, 6$, and they approach to zero as $\varepsilon \to 0$ or $\mu \to 0$ or $r_0 \to 0$ or $r_1 \to 0$. We estimate $R_7$ as follows:

\[
|R_7| \leq \varepsilon \int_{\Omega} \rho_\varepsilon |\nabla u_\varepsilon|^2 dx + C(\varepsilon) \int_{\Omega} \rho_\varepsilon \theta_\varepsilon^2 dx
\]

\[
\leq \varepsilon \int_{\Omega} \rho_\varepsilon |\nabla u_\varepsilon|^2 dx + C(\varepsilon) \|\theta_\varepsilon\|_{L^2}^2 \|\nabla \rho_\varepsilon\|_{L^2}^2,
\]

(3.3)

and for $R_8$, we have

\[
|R_8| \leq C \int_{\Omega} \rho_\varepsilon \theta_\varepsilon^2 dx + C \int_{\Omega} \frac{\kappa |\nabla \theta_\varepsilon|^2}{\varepsilon^2} dx
\]

\[
\leq C \int_{\Omega} |\nabla \sqrt{\rho_\varepsilon}|^2 dx + C
\]

(3.4)

Thus, by taking $\varepsilon$ small enough, (3.2) and Sobolev inequality, $\theta_\varepsilon \in L^2([0, T]; L^6(\Omega))$, it is possible to get some a priori estimates via Gronwall’s inequality. Therefore, we have the following inequality

**Lemma 3.2.**

\[
\int_{\Omega} \left( \frac{1}{2} \rho_\varepsilon |u_\varepsilon + \frac{\nabla \rho_\varepsilon}{\rho_\varepsilon}|^2 + \frac{\varepsilon}{10} \rho_\varepsilon^{-10} + \frac{\kappa}{2} \left| \nabla \sqrt{\rho_\varepsilon} \right|^2 + \frac{\delta}{2} \left| \nabla^4 \rho_\varepsilon \right|^2 \right) dx + \eta \int_{0}^{T} \int_{\Omega} \left| \nabla \rho_\varepsilon^{-5} \right|^2 dx + \kappa \int_{0}^{T} \int_{\Omega} \rho_\varepsilon \left| \nabla \log \rho_\varepsilon \right|^2 dx
\]

\[
+ \frac{\kappa}{2} \left| \nabla \right| \rho_\varepsilon \left( \Delta^2 \rho_\varepsilon \right) dx + 2 \varepsilon \int_{0}^{T} \int_{\Omega} \left| \Delta \rho_\varepsilon \right|^2 dx + 2 \varepsilon \int_{0}^{T} \int_{\Omega} \rho_\varepsilon \left| \nabla^4 \rho_\varepsilon \right|^2 dx + \varepsilon \int_{0}^{T} \int_{\Omega} \left| \Delta \rho_\varepsilon \right|^2 dx
\]

\[
+ \int_{0}^{T} \int_{\Omega} \left| \nabla \rho_\varepsilon \right|^2 \theta_\varepsilon \leq \int_{0}^{T} \int_{\Omega} \frac{1}{2} \rho_0 \varepsilon |u_{0, \varepsilon} + \frac{\nabla \rho_{0, \varepsilon}}{\rho_{0, \varepsilon}}|^2 + \frac{\varepsilon}{10} \rho_{0, \varepsilon}^{-10} + \frac{\kappa}{2} \left| \nabla \sqrt{\rho_{0, \varepsilon}} \right|^2 + \frac{\delta}{2} \left| \nabla^4 \rho_{0, \varepsilon} \right|^2 dx + 2E_0,
\]

(3.5)

Then, we infer the following estimate from the BD entropy:

\[
\kappa \int_{0}^{T} \int_{\Omega} \rho_\varepsilon \left| \nabla \log \rho_\varepsilon \right|^2 dx \leq C
\]
where $C$ is independent of $\varepsilon$.

Applying Lemma 2.1, we have the following uniform estimate
\[
\frac{1}{2} \| \sqrt{\rho_\varepsilon} \|_{L^2(0,T;H^2(\Omega))} + \frac{1}{2} \| \nabla \rho_\varepsilon \|_{L^4(0,T;L^4(\Omega))} \leq C,
\]
(3.6)
where $C$ is independent of $\varepsilon$.

### 3.2. Uniform estimates with $\varepsilon$

From the energy estimate (3.6), we have the following uniform estimates on $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$:
\[
\sqrt{\rho_\varepsilon} u_\varepsilon \in L^\infty(0,T;L^2(\Omega)), \quad \sqrt{\rho_\varepsilon} \nabla u_\varepsilon \in L^2((0,T) \times \Omega), \quad \sqrt{\varepsilon} \Delta u_\varepsilon \in L^2((0,T) \times \Omega),
\]
(3.7)
\[
\varepsilon \Delta^5 \rho_\varepsilon \in L^2((0,T) \times \Omega), \quad \sqrt{\varepsilon} \rho_\varepsilon \in L^\infty(0,T;H^9(\Omega)), \quad \sqrt{\varepsilon} \rho_\varepsilon \in L^\infty(0,T;H^1(\Omega)),
\]
(3.8)
\[
\varepsilon \Delta^{\frac{1}{2}} \rho_\varepsilon^{-1} \in L^\infty(0,T;L^{10}(\Omega)), \quad \varepsilon \nabla \rho_\varepsilon^{-\frac{5}{2}} \in L^2((0,T) \times \Omega),
\]
(3.9)
\[
u_\varepsilon \in L^2((0,T) \times \Omega), \quad \rho_\varepsilon^2 u_\varepsilon \in L^4((0,T) \times \Omega),
\]
(3.10)
\[
\rho_\varepsilon \theta_\varepsilon \in L^\infty(0,T;L^1(\Omega)), \quad \varepsilon \theta_\varepsilon^{\frac{2}{3}+1} \in L^\infty(0,T;L^1(\Omega)),
\]
(3.11)

Moreover, by the BD entropy, we have
\[
\nabla \sqrt{\rho_\varepsilon} \in L^\infty(0,T;L^2(\Omega)), \quad \sqrt{\varepsilon} \Delta^5 \rho_\varepsilon \in L^2((0,T) \times \Omega),
\]
(3.12)
\[
\nabla \rho_\varepsilon^\frac{3}{2} \in L^2((0,T) \times \Omega), \quad \sqrt{\varepsilon} \nabla \rho_\varepsilon^{-\frac{5}{2}} \in L^2((0,T) \times \Omega).
\]
(3.13)

Also, we have
\[
\frac{1}{2} \| \sqrt{\rho_\varepsilon} \|_{L^2(0,T;H^2(\Omega))} + \frac{1}{2} \| \nabla \rho_\varepsilon \|_{L^4(0,T;L^4(\Omega))} \leq C,
\]
(3.14)
where $C$ is independent of $\varepsilon$.

In according with Lemma 3.2, one deduces
\[
\int_0^T \int_{\Omega} \rho_\varepsilon |\nabla u_\varepsilon - \nabla^T u_\varepsilon|^2 \leq C
\]
(3.15)
which together with (3.7), yields
\[
\int_0^T \int_{\Omega} \rho_\varepsilon |\nabla u_\varepsilon|^2 \leq C,
\]
(3.16)
where $C$ is independent of $\varepsilon$. Based on the above estimates, we have the following lemma.

**Lemma 3.3.** The following further uniform estimates independent of $\varepsilon$ hold:
\[
\| (\sqrt{\rho_\varepsilon})^t \|_{L^2((0,T) \times \Omega)} + \| \sqrt{\rho_\varepsilon} \|_{L^2(0,T;H^2(\Omega))} \leq C
\]
(3.17)
\[
\| (\rho_\varepsilon^2 u_\varepsilon) \|_{L^2(0,T;H^{-5}(\Omega))} + \| \rho_\varepsilon u_\varepsilon \|_{L^2((0,T) \times \Omega)} \leq C
\]
(3.18)
\[
\nabla (\rho_\varepsilon u_\varepsilon) \text{ is uniformly bounded in } L^4(0,T;L^6(\Omega)) + L^2(0,T;L^\frac{3}{2}(\Omega)),
\]
(3.19)
\[
\| \rho_\varepsilon^{-10} \|_{L^\frac{2}{3}(0,T) \times \Omega)} \leq C
\]
(3.20)
where $C$ is independent of $\varepsilon$ and depends on $r_0, r_1, \kappa$.

**Proof.** By (3.7)-(3.16), following the same path as in the proof of Lemma 2.2, we can prove the above estimates. \qed
3.3. Temperature estimate

Now taking
\[ \psi(t, x) = \varphi(t), \quad 0 \leq \varphi \leq 1, \quad \varphi \in \mathcal{D}(0, T), \quad h(\theta) = \frac{\omega}{\omega + \theta}, \quad \omega > 0, \]
in (2.53), we deduce
\[
\int_0^T \int_\Omega \left( \frac{1}{\omega + \theta_\varepsilon} S_\varepsilon : \nabla u_\varepsilon + \frac{\kappa(\theta_\varepsilon)}{(\omega + \theta_\varepsilon)^2} |\nabla \theta_\varepsilon|^2 \right) dx dt \\
\leq \int_0^T \int_\Omega \left( \frac{\theta_\varepsilon}{\omega + \theta_\varepsilon} \rho_\varepsilon \operatorname{div} u_\varepsilon dx dt + \varepsilon \int_0^T \int_\Omega \theta_\varepsilon dx dt \\
- \int_\Omega (\rho_0 + \varepsilon) Q_{h, \omega}(\theta_{0, \varepsilon}) dx + \int_\Omega (\rho_\varepsilon + \varepsilon) Q_{h, \omega}(\theta_\varepsilon) dx, \tag{3.21}
\]
where
\[ Q_{h, \omega}(\theta) = \int_1^\theta \frac{1}{\omega + z} dz, \]
Letting \( \omega \to 0 \) and taking hypothesis (1.38) together with the estimate (3.16) into account, we have
\[
\int_0^T \int_\Omega \left( \frac{1}{1 + \theta_\varepsilon} S_\varepsilon : \nabla u_\varepsilon + |\nabla \theta|^2 + |\nabla \theta_\varepsilon^{\alpha/2}|^2 \right) dx dt \\
\leq C(1 + \int_0^T \int_\Omega \rho_\varepsilon \operatorname{div} u_\varepsilon dx dt) \leq C, \tag{3.22}
\]
By virtue of Lemma 3.2 and above estimate, we know that
\[
\theta_\varepsilon \text{ bounded in } L^2(0, T; W^{1,2}(\Omega)), \\
\theta_\varepsilon^{\alpha/2} \text{ bounded in } L^2(0, T; W^{1,2}(\Omega)), \tag{3.23}
\]
Similarly, taking
\[ \psi(t, x) = \varphi(t), \quad 0 \leq \varphi \leq 1, \quad \varphi \in \mathcal{D}(0, T), \quad h(\theta) = \frac{1}{(1 + \theta)^\omega}, \quad 0 < \omega < 1, \]
in (2.53), we can also deduce that
\[
\int_0^T \int_\Omega \left( \frac{1}{1 + \theta_\varepsilon} S_\varepsilon : \nabla u_\varepsilon + \omega \frac{\kappa(\theta_\varepsilon)}{(1 + \theta_\varepsilon)^{1+\omega}} |\nabla \theta_\varepsilon|^2 \right) dx dt \\
\leq C(1 + \int_0^T \int_\Omega \rho_\varepsilon \theta_\varepsilon \operatorname{div} u_\varepsilon dx dt) \leq C, \tag{3.24}
\]
where \( C \) is independent of both \( \delta \) and \( \omega \), which yields
\[
\|\theta_\varepsilon^{(\alpha+1-\omega)/2}\|_{L^2(0, T; W^{1,2}(\Omega))} \leq C(\omega) \quad \text{for any } \omega > 0. \tag{3.25}
\]
Finally, using Holder’s inequality as in Section 5.2 [5] we establish that
\[
\int_{\{\rho_\varepsilon > \omega\}} \theta_\varepsilon^{\alpha+1} dx dt \leq C(\omega) \quad \text{for any } \omega > 0. \tag{3.25}
\]
Since the density \( \rho_\varepsilon \) solves the mass equation in \( D'((0,T)) \), the total mass \( M_\varepsilon \) is a constant of motion, and we have
\[
\int_{\{\rho_\varepsilon > \omega\}} \rho_\varepsilon dx \geq M_\varepsilon - \omega |\Omega| \geq \frac{M}{2} - \omega |\Omega|, \tag{3.26}
\]

On the other hand, a straightforward application of Holder inequality gives rise to
\[
\int_{\{\rho_\varepsilon > \omega\}} \rho_\varepsilon dx \leq |\{\rho_\varepsilon \geq \omega\}|^{2/3} \|\rho_\varepsilon\|_{L^3(\Omega)}. \tag{3.27}
\]
Consequently, by virtue of \( (3.26), (3.27), (3.12) \) there exists a function \( d = d(\omega) \), which is independent of \( \varepsilon \), such that
\[
|\{\rho_\varepsilon > \omega\}| \geq d(\omega) > 0 \text{ for all } t \in [0,T] \text{ provided } 0 \leq \omega < \frac{M}{2|\Omega|}. \tag{3.28}
\]
Fix \( 0 < \omega < M/4|\Omega| \) and find a function \( B \in C^\infty(R) \) such that
\[
B : R \to \text{non-increasing}, B(z) = 0 \text{ for } z \leq \omega, \quad B(z) = -1 \text{ for } z \geq 2\omega.
\]
For each \( t \in [0,T] \), let \( \eta = \eta_\varepsilon \) be the unique strong solution of the Neumann problem
\[
\Delta \eta_\varepsilon = B(\rho_\varepsilon(t)) - \frac{1}{|\Omega|} \int_{\Omega} B(\rho_\varepsilon(t))dx \text{ in } \Omega,
\]
\[
\nabla \eta_\varepsilon \cdot n = 0 \text{ on } \partial \Omega,
\]
\[
\int_{\Omega} \eta_\varepsilon dx = 0,
\]
Since the right-hand side of \( (3.29) \) is uniformly bounded independently of \( \varepsilon \), there is a constant \( \underline{\eta} \) such that
\[
\eta_\varepsilon \geq \underline{\eta} \text{ for all } t \in [0,T], x \in \Omega, \delta > 0,
\]
Accordingly, we can take a test function
\[
\varphi(t,x) \equiv \psi(t)(\eta_\varepsilon(t,x) - \underline{\eta}), \psi \in D(0,T), \quad 0 \leq \psi \leq 1
\]
in \( (2.53) \) to deduce
\[
\int_0^T \int_{\Omega} \psi K_h(\theta_\varepsilon)(B(\rho_\varepsilon) - \frac{1}{|\Omega|} \int_{\Omega} B(\rho_\varepsilon(t))dx)dxdt
\]
\[
\leq 2\|\eta_\varepsilon\|_{L^\infty((0,T) \times \Omega)}(\int_0^T \int_{\Omega} \varepsilon \theta_\varepsilon^{\alpha+1} + \theta_\varepsilon \rho_\varepsilon |\text{div}\, u_\varepsilon|dxdt)
\]
\[
+ \|\nabla \eta_\varepsilon\|_{L^\infty((0,T) \times \Omega)}(\int_0^T \int_{\Omega} \rho_\varepsilon Q_h(\theta_\varepsilon)|u_\varepsilon|dxdt)
\]
\[
+ \int_0^T \int_{\Omega} (\rho_\varepsilon + \varepsilon)Q_h(\theta_\varepsilon)(\underline{\eta} - \eta_\varepsilon)\partial_t \psi - (\rho_\varepsilon + \varepsilon)Q_h(\theta_\varepsilon)\partial_t \eta_\varepsilon \psi dxdt,
\]
Now we can take a sequence of function \( h = h_n \nearrow 1 \) so that \( (3.30) \) gives rise to
\[
\int_0^T \int_{\Omega} \psi K(\theta_\varepsilon)(B(\rho_\varepsilon) - \frac{1}{|\Omega|} \int_{\Omega} B(\rho_\varepsilon(t))dx)dxdt
\]
\[
\leq C(1 + \int_0^T \int_{\Omega} (\rho_\varepsilon + \varepsilon)\theta_\varepsilon |\partial \eta|dxdt).
\]

Moreover,\[
\int_0^T \int_\Omega \psi K(\theta_\varepsilon)(B(\rho_\varepsilon)) - \frac{1}{|\Omega|} \int_\Omega B(\rho_\varepsilon(t))dx dt \\
= \int_{\{\rho_\varepsilon < \omega\}} \psi K(\theta_\varepsilon)(B(\rho_\varepsilon)) - \frac{1}{|\Omega|} \int_\Omega B(\rho_\varepsilon(t))dx dt \\
+ \int_{\{\rho_\varepsilon \geq \omega\}} \psi K(\theta_\varepsilon)(B(\rho_\varepsilon)) - \frac{1}{|\Omega|} \int_\Omega B(\rho_\varepsilon(t))dx dt,
\]
where, by virtue of (3.25), the second integral on the right-hand side is bounded in dependent of $\varepsilon > 0$.

On the other hand,\[
-\frac{1}{|\Omega|} \int_\Omega B(\rho_\varepsilon(t))dx \geq \frac{1}{|\Omega|} \int_{\{\rho_\varepsilon \geq 2\omega\}} B(\rho_\varepsilon(t))dx = \frac{|\rho_\varepsilon \geq 2\omega|}{|\Omega|} \geq \frac{d(2\omega)}{|\Omega|},
\]
where we have used (3.28). Thus we get\[
\int_{\{\rho_\varepsilon < \omega\}} \psi K(\theta_\varepsilon)(B(\rho_\varepsilon)) - \frac{1}{|\Omega|} \int_\Omega B(\rho_\varepsilon(t))dx dt \geq \frac{d(2\omega)}{|\Omega|} \int_{\{\rho_\varepsilon < \omega\}} \psi K(\theta_\varepsilon)dx dt,
\]
(3.32)

This inequality, together with (3.31), yields\[
\int_0^T \int_{\{\rho_\varepsilon < \omega\}} \psi K(\theta_\varepsilon)dx dt \leq C(1 + \int_0^T \int \{\rho_\varepsilon + \varepsilon\} \theta_\varepsilon |\partial_\eta| dx dt).
\]
Thus, the desired estimates on $\theta_\varepsilon$ in the space $L^{a+1}((0, T) \times \Omega)$ provided we show that the integrals on the right hand side of (3.33) are bounded.

To this end, we use the fact that $\rho_\delta$ is a solution of the renormalized continuity equation and, consequently,\[
\Delta \partial_\eta_\varepsilon = \partial_t (\Delta \eta_\varepsilon) = \partial_t B(\rho_\varepsilon) - \frac{1}{|\Omega|} \int_\Omega \partial_t B(\rho_\varepsilon)dx \\
= -\text{div}(B(\rho_\varepsilon)u_\varepsilon) - b(\rho_\varepsilon)\text{div}u_\varepsilon + \frac{1}{|\Omega|} b(\rho_\varepsilon)\text{div}u_\varepsilon dx,
\]
whence\[
\partial_\eta \text{ in } L^2(0,T;W^{1,2}(\Omega)),
\]
which, together with (3.12), (3.23), yields boundedness of the integrals on the right hand side of (3.33).

Thus, we have shown that\[
\theta_\varepsilon \text{ is bounded in } L^{a+1}((0, T) \times \Omega),
\]
by a constant which is independent of $\varepsilon > 0$.

3.4. Strict positivity of the temperature

It is easy to see that inequality (2.53) holds also for functions\[
h(\theta) = \frac{1}{\omega + \theta}, \quad \omega > 0,
\]
\[
\varphi(t, x) = \psi(t), \quad 0 \leq \psi \leq 1, \quad \psi(0) = 1, \quad \psi(T) = 1, \quad \psi \in C^\infty[0, T].
\]

\[
(3.35)
\]
According, in view of the estimates obtained above, we have
\[
\int_0^T \int_\Omega (\varepsilon + \rho_\varepsilon) Q_{h,\varepsilon}(\theta_\varepsilon) \partial_t \psi + \frac{k_1}{(\omega + \theta_\varepsilon)^2} |\nabla \theta_\varepsilon|^2 \psi dx dt \\
\leq C - \int_\Omega (\varepsilon + \rho_{0,\varepsilon}) Q_{h,\varepsilon}(\theta_{0,\varepsilon}) dx,
\]
where
\[
Q_{h,\varepsilon}(\theta_\varepsilon) \equiv \int_1^\theta C_v(z) (\omega + z) dz,
\]
Letting \(\omega \to 0\) we can conclude that
\[
\log(\theta_\varepsilon) \text{ is bounded in } L^2((0, T) \times \Omega)
\]
by a constant independent of \(\varepsilon > 0\).

3.5. Passing to the limits as \(\varepsilon \to 0\).

Applying the Aubin-Lions lemma and Lemma 3.3, we conclude
\[
\sqrt{\rho_\varepsilon} \to \sqrt{\rho} \text{ strongly in } L^2(0, T; H^1(\Omega)), \text{ weakly in } L^2(0, T; H^2(\Omega)),
\]
and
\[
\rho_\varepsilon u_\varepsilon \to \rho u \text{ strongly in } L^2((0, T) \times \Omega),
\]
we notice that \(u_\varepsilon \in L^2((0, T) \times \Omega)\), thus
\[
u_\varepsilon \to u \text{ weakly in } L^2((0, T) \times \Omega),
\]
Thus we can pass to the limits for the term \(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon\) as follows,
\[
\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \to \rho u \otimes u
\]
in the distribution sense.

We can show
\[
\rho_\varepsilon |u_\varepsilon|^2 u_\varepsilon \to \rho |u|^2 u \text{ strongly in } L^1((0, T) \times \Omega),
\]
similarly to Lemma 2.7.

By the previous estimates, for any test function \(\psi \in L^\infty(0, T; L^\infty(\Omega))\) we can deduce that
\[
\varepsilon \int_0^T \int_\Omega \Delta \rho_\varepsilon \psi \leq \varepsilon \|\Delta \rho_\varepsilon\|_{L^2(0, T; L^2(\Omega))} \|\psi\|_{L^2(0, T; L^2(\Omega))} \to 0 \text{ as in } \varepsilon \to 0,
\]
and
\[
\varepsilon \int_0^T \int_\Omega \nabla \rho_\varepsilon \nabla u_\varepsilon \psi \leq \varepsilon \|\nabla \sqrt{\rho_\varepsilon}\|_{L^2(0, T; L^2(\Omega))} \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^2(0, T; L^2(\Omega))} \\|\psi\|_{L^\infty(0, T; L^\infty(\Omega))} \to 0 \text{ as in } \varepsilon \to 0,
\]
For the convergence of term \(\varepsilon \Delta^2 u_\varepsilon\), for any test function \(\psi \in L^2(0, T; H^2(\Omega))\), thanks to (3.7), we have
\[
| \int_0^T \int_\Omega \varepsilon \Delta^2 u_\varepsilon \psi dx dt | \leq \sqrt{\varepsilon} \|\varepsilon \Delta u_\varepsilon\|_{L^2((0, T) \times \Omega)} \|\Delta \psi\|_{L^2((0, T) \times \Omega)} \to 0 \text{ as } \varepsilon \to 0,
\]
For the convergence of terms \( \varepsilon \rho^{-10} \) and \( \varepsilon \rho \nabla \Delta^3 \rho \), we refer to the lemma 3.6 and 3.7 in [11].

Thus, by the compactness argument, we can pass to the limits as \( \varepsilon \to 0 \), yield that the limit function \((\rho, u, \theta)\) satisfy the continuity equation as well as the momentum equation:

\[
\partial_t \rho + \text{div}(\rho u) = 0, \quad \text{pointwise in } (0, T) \times \Omega. \tag{3.45}
\]

and

\[
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P - 2\text{div}(\rho \mathbb{D} u) + r_0 u + r_1 \rho |u|^2 = \kappa \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \tag{3.46}
\]

holds in the sense of distribution on \((0, T) \times \Omega\),

Furthermore, thanks to the weak lower semicontinuity of convex functions, we are able to pass to the limits in the energy inequality (2.12) and B-D entropy inequality (3.5); by the strong convergence of the density and temperature, we have the following energy inequality in the sense of distributions on \((0, T)\):

\[
\sup_{t \in (0, T)} \int_{\Omega} E(\rho, u, \theta) + r_0 \int_{\Omega} |u|^2 dx + r_1 \int_{\Omega} \rho |u|^4 dx + \kappa \int_{\Omega} \rho |\nabla \log \rho|^2 dx \leq E_0(\rho, u, \theta), \tag{3.47}
\]

where

\[
E(\rho, u, \theta) = \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\eta}{10} \rho^{-10} + \frac{\kappa}{2} \nabla \sqrt{\rho}^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho|^2 + \rho \theta + \beta \theta^4 \right) dx, \tag{3.48}
\]

and

\[
\int_{\Omega} \left( \frac{1}{2} \rho |u + \nabla \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 - r_0 \log \rho \varepsilon \right) dx + \kappa \int_{0}^{T} \int_{\Omega} \rho |\nabla \log \rho|^2 dx \\
+ \frac{1}{2} \int_{0}^{T} \int_{\Omega} \rho |\nabla u - \nabla^T u|^2 dx + \int_{\Omega} \frac{|\nabla \rho|^2}{\rho} \theta \leq \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0 + \nabla \rho_0|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 + 2E_0, \tag{3.49}
\right)
\]

3.6. Thermal energy equation

In order to complete the limit passage \( \varepsilon \to 0 \), we have to show that \( \rho, u \) and \( \theta \) represent a variational solution of the thermal energy equation (1.36) in the sense of Definition 1.1.

Because of the uniform estimate (3.22), we know that

\[
Q(\theta \varepsilon) \to Q(\theta) \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)). \tag{3.50}
\]

which, together with (3.37), yields

\[
\rho \varepsilon Q(\theta \varepsilon) \to \rho Q(\theta) \quad \text{weakly in } L^2(0, T; L^2(\Omega)).
\]

Thus, we are allowed to apply to Lemma 6.3 to deduce

\[
\rho \varepsilon Q(\theta \varepsilon) \to \rho F(\theta) \quad \text{in } L^2(0, T; W^{-1,2}(\Omega)).
\]

In accordance with (3.50), we have

\[
\rho \varepsilon Q(\theta \varepsilon) \to \rho F(\theta) \quad \text{in } D'(0, T \times \Omega).
\]
Since $Q$ is sublinear, we can infer that
\[
\theta_\varepsilon \to \overline{\theta} \quad \text{(strongly) in } L^r(\{\rho > 0\}) \text{ for a certain } r > 1.
\]

Now we can pass to limits for $\varepsilon \to 0$ in (2.53) to obtain
\[
\int_0^T \int_{\Omega} \rho Q_h(\overline{\theta}) \partial_t \varphi + \rho_\varepsilon \rho Q_h(\overline{\theta}) u \cdot \nabla \varphi + K_h(\overline{\theta}) \Delta \varphi \\
\int_0^T \int_{\Omega} (h(\overline{\theta}) S : \nabla u + h(\overline{\theta}) \rho \text{div} u) \varphi dx dt - \int_{\Omega} \rho_0 Q_h(\theta_0) \varphi_0 dx,
\]
where
\[
\rho K_h(\theta) = \rho K_h(\overline{\theta}),
\]
and
\[
\log(K_h(\theta)) \in L^2((0,T) \times \Omega).
\]

Take
\[
h(\theta) = \frac{1}{(1+\theta)^\omega}, \quad 0 < \omega < 1,
\]
in (3.51) and let $\omega \to 0$ in order to deduce
\[
\int_0^T \int_{\Omega} \rho \overline{\theta} \partial_t \varphi + \rho_\varepsilon \rho \overline{\theta} u \cdot \nabla \varphi + K(\overline{\theta}) \Delta \varphi \leq \\
\int_0^T \int_{\Omega} (S : \nabla u + \overline{\theta} \text{div} u) \varphi dx dt - \int_{\Omega} \rho_0 \theta_0 \varphi_0 dx,
\]
Finally, we set
\[
\theta \equiv K^{-1}(K(\overline{\theta})).
\]
Obviously, the new function $\theta$ is non-negative, specifically,
\[
\theta \in L^{a+1}((0,T) \times \Omega), \quad \log(\theta) \in L^2((0,T) \times \Omega),
\]
Therefore we obtain a variational form of the thermal energy inequality:
\[
\int_0^T \int_{\Omega} \rho \theta \partial_t \phi + \rho \theta \cdot \phi + K(\theta) \Delta \phi dx dt \leq \\
\int_0^T \int_{\Omega} (R \rho \theta - S : \nabla u) \phi dx dt - \int_{\Omega} \rho_0 \theta_0 \phi dx,
\]
to be satisfied for any test function
\[
\phi \in C^\infty([0,T] \times \Omega), \quad \phi \geq 0, \quad \phi(0) = 1 \quad \phi(T) = 0,
\]
4 Approximation of the Mellet-Vasseur type inequality

As seen before, we can deduce the strong compactness of the density and temperature from the B-D energy estimate and entropy estimate. Note that estimates are independent of all approximation parameter. Unfortunately, the primary obstacle to prove the compactness of the solution to (1.1) is the lack of strong convergence for $\sqrt{\rho}u$ in $L^2$. To solve this problem, a new estimate as established in Mellet and Vasseur [10], providing a $L^\infty(0,T;L\log L(\Omega))$ control on $\rho|u|^2$. This new estimate enable us to pass to the limit $r_0 \to 0, r_1 \to 0$ and $\kappa \to 0$.

In this section, we construct an approximation of the Mellet-Vasseur type inequality for any weak solutions to the following level of approximate system:

$$\partial_t \rho + \text{div}(\rho u) = 0, \quad (4.1)$$

$$\rho u_t + \text{div}(\rho u \otimes u) + \nabla P - 2\text{div}(\rho \nabla u) + r_0 u + r_1 \rho|u|^2 = \kappa \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \quad (4.2)$$

$$\partial_t (\rho \theta + \beta \theta^4) + \text{div}(u(\rho \theta + \beta \theta^4)) + \text{div} q = \rho |\nabla u|^2 - \text{Pdiv} u, \quad (4.3)$$

Following the idea in [12], we define two $C^\infty,$ nonnegative cut-off function $\phi_m$ and $\phi_K$ as follows:

$$\phi_m(\rho) = 1 \quad \text{for any } \rho > \frac{1}{m}, \quad \phi_m(\rho) = 0 \quad \text{for any } \rho < \frac{1}{2m}, \quad (4.4)$$

where $m > 0$ is any real number, and $|\phi'| \leq 2m$; and $\phi_K(\rho) \in C^\infty(\mathbb{R})$ is a nonnegative function such that

$$\phi_K(\rho) = 1 \quad \text{for any } \rho < K, \quad \phi_K(\rho) = 0 \quad \text{for any } \rho > 2K, \quad (4.5)$$

where $K > 0$ is any real number, and $|\phi'_K| \leq \frac{2}{K}.$

We define $v = \phi(\rho)u$, and $\phi(\rho) = \phi_m(\rho)\phi_K(\rho).$ The following lemma will be useful to construct the approximation of the Mellet-Vasseur type inequality. The structure of the $\kappa$ quantum term is essential to get this lemma in 3D.

**Lemma 4.1.** For any fixed $\kappa > 0$, we have

$$\|\nabla v\|_{L^2(0,T;L^2(\Omega))} \leq C \quad (4.6)$$

where the constant $C$ depend on $\kappa > 0, r_1, K$ and $m$; and

$$\rho_t \in L^4(0,T;L^6(\Omega)) + L^2(0,T;L^4(\Omega)) \quad \text{uniform in } \kappa. \quad (4.7)$$

We introduce a new nonnegative cut-off function $\varphi_n$ which is in $C^1(\mathbb{R}^3)$:

$$\varphi_n(u) = \tilde{\varphi}_n(|u|^2), \quad (4.8)$$

where $\tilde{\varphi}$ is given on $\mathbb{R}^+$ by

$$\varphi''_n(y) = \begin{cases} \frac{1}{1+y} & \text{if } 0 \leq y \leq n, \\ \frac{1}{1+y} & \text{if } n \leq y \leq C_n, \\ 0 & \text{if } y \geq C_n, \end{cases}$$

with $\varphi'_n(0) = 0, \varphi_n(0) = 0$, and $C_n = e(1+n)^2 - 1.$

Here we gather the properties of the function $\varphi'_n$ in the following lemma:
Lemma 4.2. Let $\varphi_n$ and $\tilde{\varphi}_n$ be defined as above. Then they verify

- (a) For any $u \in \mathbb{R}^3$, we have
  \[ \varphi_n''(u) = 2(2\tilde{\varphi}_n'(|u|^2)u \otimes u + I\tilde{\varphi}_n'(|u|^2)), \]  
  \[ (4.9) \]

  where $I$ is $3 \times 3$ identity matrix.

- (b) $\varphi_n''(y) \leq \frac{1}{1+y}$ for any $n > 0$ and $y \geq 0$.

- (c) $\varphi_n'(y) \begin{cases} 
  1 + \ln(1+y) & \text{if } 0 \leq y \leq n, \\
  0 & \text{if } y \geq C_n, \\
  \geq 0, \text{ and } & \leq 1 + \ln(1+y) \text{ if } n \leq y \leq C_n, 
\end{cases}$

In one word, $0 \leq \varphi_n' \leq 1 + \ln(1+y)$ for any $y \geq 0$, and it is compactly supported.

- (d) For any given $n > 0$, we have
  \[ |\varphi_n''(u)| \leq 6 + 2\ln(1+n) \]  
  \[ (4.10) \]

  for any $u \in \mathbb{R}^3$.

- (e) \[ \tilde{\varphi}_n(y) \begin{cases} 
  (1+y)\ln(1+y) & \text{if } 0 \leq y \leq n, \\
  2(1+\ln(1+n))y - (1+y)\ln(1+y) + 2(\ln(1+n) - n) & \text{if } n \leq y \leq C_n, \\
  e(1+n)^2 - 2n - 2 & \text{if } y \geq C_n, 
\end{cases} \]

$\tilde{\varphi}_n(y)$ is a nondecreasing function with respect to $y$ for any fixed $n$, and it is a nondecreasing function with respect to $n$ for any fixed $y$, and

\[ \tilde{\varphi}_n(y) \rightarrow (1+y)\ln(1+y) \text{ a.e.} \]  
  \[ (4.11) \]

as $n \rightarrow \infty$.

By the mollifier method, we can construct the approximation of the Mellet-Vasseur type inequality which is shown in the following lemma:

Lemma 4.3. For any weak solution to \[ (4.1) - (4.3) \], and any $\psi \in \mathcal{D}(-1, +\infty)$, we have

\[ -\int_0^T \int_\Omega \psi_1 \rho \varphi_n(v) dx dt + \int_0^T \int_\Omega \psi(t)\varphi_n'(v) F dx dt \]
\[ + \int_0^T \int_\Omega \psi(t) S : \nabla(\varphi_n'(v)) dx dt = \int_\Omega \rho_0 \varphi_n(v_0) \psi(0) dx, \]  
  \[ (4.12) \]

where

\[ S = \rho \varphi(\phi)(Du + \kappa \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}), \]  
  \[ (4.13) \]

\[ F = \rho^2 u\phi_K'(\rho) \text{div} u + \nabla \rho \rho \phi_K'(\rho) + r_0 u \phi_K'(\rho) + r_1 \rho |u|^2 u \phi_K'(\rho) + \kappa \sqrt{\rho} \nabla \phi_K'(\rho) \Delta \sqrt{\rho} + 2 \kappa \phi_K'(\rho) \nabla \sqrt{\rho} \Delta \sqrt{\rho}, \]  
  \[ (4.14) \]

where $I$ is an identical matrix.
5 Recover the limits as $m \to \infty$ and $K \to \infty$

In this section, we want to recover the limits from (4.12) as $m \to \infty$ and $K \to \infty$. Firstly, we will pass to the limit $m \to 0$. For the $K \to \infty$ case, it is similar to the $m \to 0$ process. For any fixed weak solution $(\rho, u)$, $\phi_m(\rho)$ converges to 1 almost everywhere for $(t, x)$, and it is uniform bounded in $L^\infty(0, T; \Omega)$, and

$$r_0\phi_K(\rho)u \in L^2(0, T; L^2(\Omega)).$$

(5.1)

Thus, we find $v_m = \phi_m \phi_K(\rho)u \in L^2(0, T; L^2(\Omega)) \to \phi_K u$ almost everywhere for $(t, x)$. (5.2) as $m \to \infty$. The dominated convergence theorem allows us to have $v_m \to \phi_K u$ in $L^2(0, T; L^2(\Omega))$ (5.3) as $m \to \infty$, and hence

$$\varphi_n(v_m) \to \varphi_n(\phi_K u) \quad \text{in} \quad L^p((0, T) \times \Omega) \quad \text{for any} \quad 1 \leq p < \infty.$$ (5.4)

for any $1 \leq p < \infty$. Thus, we can show that

$$\int_0^T \int_\Omega \psi'(t)(\rho \varphi_n(v_m)) dx dt \to \int_0^T \int_\Omega \psi'(t)(\rho \varphi_n(\phi_K u)) dx dt,$$ (5.5)

and

$$\int_0^T \int_\Omega \rho \varphi_n(v_m0) dx dt \to \int_0^T \int_\Omega \rho_0 \varphi_n(\phi_K(\rho_0)u_0) dx dt,$$ (5.6)

as $m \to \infty$.

Meanwhile, for any fixed $\rho$, we have

$$\phi'_m(\rho) \to 0 \quad \text{almost everywhere for} \quad (t, x)$$ (5.7)

as $m \to \infty$.

Calculating $|\phi'_m(\rho)| \leq 2m$ as $\frac{1}{2m} \leq \rho \leq \frac{1}{m}$, and otherwise, $\phi'_m(\rho) = 0$, thus

$$|\rho \phi'_m(\rho)| \leq 1 \quad \text{for all} \quad \rho.$$ (5.8)

To pass into the limits in (4.12) as $m \to \infty$, we rely on the following Lemma:

**Lemma 5.1.** If

$$\|a_m\|_{L^\infty(0, T; \Omega)} \leq C, \quad a_m \to a \quad a.e. \quad \text{for} \quad (t, x)$$

and in $L^p((0, T) \times \Omega)$ for any $1 \leq p < \infty$.

$f \in L^1((0, T) \times \Omega)$, then we have

$$\int_0^T \int_\Omega \phi_m(\rho)a_m f dx dt \to \int_0^T \int_\Omega a f dx dt \quad \text{as} \quad m \to \infty.$$ (5.10)
Calculating

\[
\int_0^T \int_{\Omega} \psi(t) S_m : \nabla (\varphi'(v_m)) dx dt
\]
\[
\int_0^T \int_{\Omega} \psi(t) S_m \varphi''(v_m)(\nabla \phi_m \phi_K u + \phi_m \nabla \phi_K u + \phi_m \phi_K \nabla u) dx dt
\]
\[
\int_0^T \int_{\Omega} \phi_m a_{m1} f_1 dx dt + \int_0^T \int_{\Omega} \phi_m a_{m2} f_2 dx dt,
\]

where

\[a_{m1} = \phi_m(\rho) \varphi''(v_m)\]
\[f_1 = \psi(t) \rho \phi_K(\rho)(\nabla \rho + \kappa \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}) \nabla \phi_K + \phi_K(\rho) \nabla u,\]

and

\[a_{m2} = \phi_m(\rho) \rho \varphi''(v_m) = \varphi''(v_m) v_m,\]
\[f_2 = \psi(t) \phi_K(\rho)(\nabla \rho + \kappa \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}) \nabla \phi_K + \phi_K(\rho) \nabla u,\]
\[= 2 \psi(t) \phi_K(\rho)(\kappa \Delta \rho \nabla \sqrt{\rho} + \sqrt{\rho} \nabla \rho \nabla \sqrt{\rho}).\]

So applying Lemma 3.1 to (5.11), one obtains

\[
\int_0^T \int_{\Omega} \psi(t) S_m : \nabla (\varphi'(v_m)) dx dt \rightarrow \int_0^T \int_{\Omega} \psi(t) S_m : \nabla (\varphi'(\phi_K(\rho) u)) dx dt
\]

as \( m \to \infty \), where \( S = \phi_K(\rho)(\nabla \rho + \kappa \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}) \).

Letting \( F_m = F_{m1} + F_{m2} \), where

\[F_{m1} = \rho^2 u \varphi(\rho) \text{div} u + \rho \nabla \phi(\rho) \nabla u + \kappa \sqrt{\rho} \nabla \phi(\rho) \Delta \sqrt{\rho}\]
\[= \rho(\varphi'(\rho) \phi_K(\rho) + \phi_m(\rho) \phi'_K(\rho))(\rho u \text{div} u + \nabla \rho \cdot \nabla u + \kappa \nabla \rho \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}),\]

where

\[\phi_K(\rho)(\rho u \text{div} u + \nabla \rho \cdot \nabla u + \kappa \nabla \rho \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}) \in L^1((0, T) \times \Omega),\]

and

\[\rho \varphi'(\rho) \phi_K(\rho) \rho u \text{div} u + \nabla \rho \cdot \nabla u + \kappa \nabla \rho \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \in L^1((0, T) \times \Omega),\]

and

\[F_{m2} = \phi_m(\rho) \phi_K(\rho)(2 \rho^2 \nabla \rho \nabla u + r_0 u + r_1 |u|^2 u + 2 \kappa \sqrt{\rho} \Delta \sqrt{\rho}),\]

where

\[\phi_K(\rho)(2 \rho^2 \nabla \rho \nabla u + r_0 u + r_1 |u|^2 u + 2 \kappa \sqrt{\rho} \Delta \sqrt{\rho}) \in L^1((0, T) \times \Omega).\]

Using Lemma 3.1, we obtain

\[
\int_0^T \int_{\Omega} \psi(t) \varphi'(v_m) F_m dx dt \rightarrow \int_0^T \int_{\Omega} \psi(t) \varphi'(\phi_K(\rho) u) F_m dx dt,
\]
Lemma 6.1. In this section, our aim is to recover the limits in (5.22) as \( \kappa \to 0 \) and \( S \to \infty \), which in turn gives us the following lemma:

Thus, letting \( m \to \infty \) in (4.12), and using the above convergence in this section, we find

\[
\begin{align*}
- \int_0^T \int_\Omega \psi_t \rho \varphi_n(\phi(\rho)u) \, dxdt + \int_0^T \int_\Omega \psi_r(t) \varphi_n'(\phi(\rho)u) F \, dxdt \\
+ \int_0^T \int_\Omega \psi_r(t) \mathbb{S} : \nabla(\varphi_n'(\phi(\rho)u)) \, dxdt = \int_\Omega \rho_0 \varphi_n(\phi(\rho)u_0) \, dxdt,
\end{align*}
\]  

(5.19)

which in turn gives us the following lemma:

**Lemma 5.2.** For any weak solution to (4.1) - (4.3), we have

\[
\begin{align*}
- \int_0^T \int_\Omega \psi_t \rho \varphi_n(\phi(\rho)u) \, dxdt + \int_0^T \int_\Omega \psi_r(t) \varphi_n'(\phi(\rho)u) F \, dxdt \\
+ \int_0^T \int_\Omega \psi_r(t) \mathbb{S} : \nabla(\varphi_n'(\phi(\rho)u)) \, dxdt = \int_\Omega \rho_0 \varphi_n(\phi(\rho)u_0) \, dxdt,
\end{align*}
\]  

(5.20)

where \( \mathbb{S} = \phi(\rho)\rho(\mathbb{D}u + \kappa \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}) \), and

\[
F = \rho^2 u \phi'_K(\rho) \text{div} u + \nabla P \phi(\rho) + r_0 u \phi(\rho) + r_1 |u|^2 u \phi(\rho) + \kappa \sqrt{\rho} \nabla \phi(\rho) \Delta \sqrt{\rho} + 2 \kappa \phi(\rho) \nabla \sqrt{\rho} \Delta \sqrt{\rho},
\]  

(5.21)

where \( \mathbb{I} \) is an identical matrix.

Similar to the passage \( m \to 0 \), letting \( K \to \infty \), we deduce the following lemma:

**Lemma 5.3.** For any weak solution to (4.1) - (4.3), we have

\[
\begin{align*}
- \int_0^T \int_\Omega \psi_t \rho \varphi_n(u) \, dxdt + \int_0^T \int_\Omega \psi_r(t) \varphi_n'(u) F \, dxdt \\
+ \int_0^T \int_\Omega \psi_r(t) \mathbb{S} : \nabla(\varphi_n'(u)) \, dxdt = \int_\Omega \rho_0 \varphi_n(u_0) \, dxdt,
\end{align*}
\]  

(5.22)

where \( \mathbb{S} = \rho(\mathbb{D}u + \kappa \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \mathbb{I}) \), and

\[
F = \nabla P + r_0 u + r_1 |u|^2 u + 2 \kappa \nabla \sqrt{\rho} \Delta \sqrt{\rho},
\]  

(5.23)

where \( \mathbb{I} \) is an identical matrix.

### 6 Recover the limits as \( \kappa \to 0 \).

In this section, our aim is to recover the limits in (5.2) as \( \kappa \to 0 \). First, we have the following lemma.

**Lemma 6.1.** Let \( \kappa \to 0 \), we have for any fixed \( n \),

\[
\rho_n \varphi_n(u_n) \to \rho \varphi_n(u) \quad \text{strongly in } L^1((0, T) \times \Omega),
\]  

(6.1)

and

\[
\rho_n \theta_n^2 (1 + \varphi'_n(|u_n|^2)) \to \rho \theta^2 (1 + \varphi'_n(|u|^2)) \quad \text{strongly in } L^1((0, T) \times \Omega),
\]  

(6.2)
Lemma 6.2. Let $\kappa \to 0$, for any $\psi \geq 0$ and $\psi' \leq 0$, we have

\[-\int_0^T \int_\Omega \psi' \rho \varphi_n(u) dx dt\]
\[
\leq 8\|\psi\|_{L^\infty} \left( \int_\Omega (|\rho_0| u_0|^2) + \frac{\rho^3_0}{\gamma - 1} \right) + |\nabla \sqrt{\rho_0}|^2 - r_0 \log_+ \rho_0 dx + 2E_0 \tag{6.3}\]
\[
+ C(\|\psi\|_{L^\infty}) \int_0^T \left( \int_\Omega (\rho^2)^{\frac{2}{n-1}} dx \right)^{\frac{3}{2}} \times \left( \int_\Omega (1 + \varphi_n'(|u|^2)) \frac{\chi^2}{2} dx \right)^{\frac{4}{2}} dt,\]

Proof. By use of Lemma 6.1, we can handle the first and forth term in (5.22) as follows, that is,

\[
\int_0^T \int_\Omega \psi(t)(\rho \varphi_n(u_\kappa)) dx dt \to \int_0^T \int_\Omega \psi(t)(\rho \varphi_n(u)) dx dt \tag{6.4}\]

and

\[
\psi(0) \int_\Omega \rho_0 \varphi_0^\prime(u_{\kappa,0}) dx \to \psi(0) \int_\Omega \rho_0 \varphi'(u_0) dx \tag{6.5}\]
as $\kappa \to 0$.

On the other hand, for the second term in (5.22)

\[
\int_0^T \int_\Omega \psi(t) \varphi_n' (u_\kappa) \cdot \nabla (\rho \theta_n) dx dt = - \int_0^T \int_\Omega \psi(t) \rho \theta_n \varphi_n'' : \nabla u_\kappa dx dt = P, \tag{6.6}\]

Thanks to Part b of Lemma 4.2, we have

\[
\varphi''(u_\kappa) : \nabla u_\kappa = 4\varphi''(\rho \theta_n |u_\kappa|^2) \nabla u_\kappa : (u_\kappa \otimes u_\kappa) + 2\text{div}_a \varphi_n'(|u_\kappa|^2). \tag{6.7}\]

Using Part b of Lemma 4.2, we find that

\[
|\varphi_n''(u_\kappa^2) \nabla u_\kappa : (u_\kappa \otimes u_\kappa)| \leq |\varphi_n''(\rho \theta_n |u_\kappa|^2)| \nabla u_\kappa |u_\kappa|^2 \\
\leq |\nabla u_\kappa| \frac{|u_\kappa|^2}{1 + |u_\kappa|^2} \leq |\nabla u_\kappa|, \tag{6.8}\]

where we denote $|\nabla u_\kappa|^2 = \sum_{i,j} |\partial_i u_j|^2$. Hence

\[
|P| \leq 4 \int_0^T \int_\Omega \psi(t)|\rho \theta_n| |\nabla u_\kappa| dx dt \]
\[
+ 2 \int_0^T \int_\Omega \psi(t) \int_\Omega |\varphi_n'(u_\kappa^2)| |\rho \theta_n| \|\text{div}_a u_\kappa\| dx dt \leq 4\|\psi\|_{L^\infty} \int_0^T \int_\Omega \rho \theta_n |\nabla u_\kappa|^2 dx dt + C(\|\psi\|_{L^\infty}) \int_0^T \int_\Omega \rho \theta_n^2 dx dt \tag{6.9}\]
\[
+ 2 \int_0^T \int_\Omega \psi(t) \int_\Omega |\varphi_n'(u_\kappa^2)| |\rho \theta_n| \|\text{div}_a u_\kappa\| dx dt,\]

With this lemma in hand, we are ready to recover the limits in (5.22) as $\kappa \to 0$. We have the following lemma.

**Lemma 6.2.** Let $\kappa \to 0$, for any $\psi \geq 0$ and $\psi' \leq 0$, we have
Lemma 6.1, we have

Thus,

as $\kappa \to 0$.

Finally

$$
2 \int_0^T \int_\Omega \psi(t) \int_\Omega |\varphi'_n(|u_\kappa|^2)| \rho_\kappa \theta_\kappa |\text{div} u_\kappa| dxdt \\
\leq 2 \int_0^T \int_\Omega \psi(t) |\varphi'_n(u_\kappa)| \rho_\kappa |\mathbb{D} u_\kappa|^2 dxdt \\
+ C(\|\psi\|_{L^\infty}) \int_0^T \int_\Omega |\varphi'_n(|u_\kappa|^2)| \rho_\kappa \theta^2_\kappa dxdt.
$$

The first right hand side term will be controlled by

$$
4\|\psi\|_{L^\infty}(\int_\Omega (\rho_0|u_0|^2 + \frac{\rho_0^2}{\gamma - 1}) + |\nabla \sqrt{\rho_0}|^2 - r_0 \log \rho_0) dx + 2E_0
$$

and the second right hand side term will be absorbed by the dispersion term $A_1$ in (6.17). By Lemma 6.1, we have

$$
\int_0^T \int_\Omega (1 + \varphi'_n(|u_\kappa|^2)) \rho_\kappa \theta^2_\kappa dxdt \to \int_0^T \int_\Omega (1 + \varphi'_n(|u|^2)) \rho \theta^2 dxdt
$$

as $\kappa \to 0$.

Note that

$$
\int_0^T \int_\Omega \psi(t) \varphi'(u_\kappa)(r_\kappa u_\kappa + r_1 \rho_\kappa |u_\kappa|^2 u_\kappa) dxdt \geq 0,
$$

so this term can be dropped directly.

For the term $S_\kappa = \phi_K(\rho_\kappa) \rho_\kappa (\mathbb{D} u_\kappa + \kappa \nabla \sqrt{\rho_\kappa}) = S_1 + S_2$, we calculate as follows

$$
\kappa \int_0^T \int_\Omega \psi(t) \mathbb{S}_1 : \nabla (\varphi'(u_\kappa)) dxdt = \int_0^T \int_\Omega \psi(t) \mathbb{D} u_\kappa : \nabla (\varphi'_n(u_\kappa)) dxdt \\
= \int_0^T \int_\Omega \psi(t) |\nabla u_\kappa \varphi''(u_\kappa)| \rho_\kappa : \mathbb{D} u_\kappa dxdt = 2 \int_0^T \int_\Omega \psi(t) \varphi'(u_\kappa) \rho_\kappa \mathbb{D} u_\kappa : \nabla u_\kappa dxdt \\
+ 4 \int_0^T \int_\Omega \psi(t) \varphi''(u_\kappa) \rho_\kappa (\nabla u_\kappa u_\kappa \otimes u_\kappa) : \mathbb{D} u_\kappa dxdt
$$

$$
= A_1 + A_2.
$$
Notice that
\[ \mathbb{D} u_\kappa : \nabla u_\kappa = |\mathbb{D} u_\kappa|^2, \]
thus
\[ A_1 \geq 2 \int_0^T \int_\Omega \psi(t) \varphi'(v_\kappa)(\phi_K(\rho_\kappa))^2 \rho_\kappa |\mathbb{D} u_\kappa|^2 dx dt \]
\[ - 4\|\psi\|_{L^\infty} \int_0^T \int_\Omega \rho_\kappa |\nabla u_\kappa|^2 dx dt, \]
where we control \( A_2 \)
\[ A_2 \leq 4 \int_0^T \int_\Omega |\psi(t)| \frac{|v_\kappa|^2}{1 + |v_\kappa|^2} \rho_\kappa |\nabla u_\kappa|^2 dx dt \]
\[ \leq 4 \|\psi\|_{L^\infty} \int_0^T \int_\Omega \rho_\kappa |\nabla u_\kappa|^2 dx dt \]
\[ \leq 4 \|\psi\|_{L^\infty} \left( \int_\Omega (\rho_0 |u_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1}) + |\nabla \sqrt{\rho_0}|^2 - r_0 \log_+ \rho_0 \right) dx + 2E_0, \]
We need to treat the term related to \( S_2 \),
\[ \kappa \int_0^T \int_\Omega \psi S_2 : \nabla (\varphi_n'(u_\kappa)) dx dt = \kappa \int_0^T \int_\Omega \psi \nabla u_\kappa \varphi_n'(u_\kappa) : \sqrt{\rho_\kappa} \Delta \sqrt{\rho_\kappa} dx dt = B, \]
we control \( B \) as follows
\[ |B| \leq C(n, \psi) \|\sqrt{\rho_\kappa} \nabla u_\kappa\|_{L^2(0,T;L^2(\Omega))} \|\sqrt{\kappa} \Delta \sqrt{\rho_\kappa}\|_{L^2(0,T;L^2(\Omega))} \sqrt{\kappa} \]
\[ \leq C_\kappa \frac{1}{\kappa} \to 0 \]
as \( \kappa \to 0 \).

With (6.4) – (6.20), in particularly, letting \( \kappa \to 0 \) in (6.22), dropping the positive terms on the left side, we hav the following inequality
\[ - \int_0^T \int_\Omega \psi'(t) \rho \varphi_n(u)dx dt \]
\[ \leq 4 \|\psi\|_{L^\infty} \left( \int_\Omega (\rho_0 |u_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1}) + |\nabla \sqrt{\rho_0}|^2 - r_0 \log_+ \rho_0 \right) dx + 2E_0 \] \[ + \psi(0) \int_\Omega \rho_0 \varphi_n(u_0) dx + C(\|\psi\|_{L^\infty}) \int_0^T \int_\Omega (1 + \varphi_n'(|u|^2)) \rho \theta^2 dx dt, \]
and
\[ C(\|\psi\|_{L^\infty}) \int_0^T \int_\Omega (1 + \varphi_n'(|u|^2)) \rho \theta^2 dx dt \]
\[ \leq C(\|\psi\|_{L^\infty}) \int_0^T \left( \int_\Omega (\rho \theta^2)^{\frac{2}{2-s}} dx \right)^{\frac{s}{2-s}} \times \left( \int_\Omega (1 + \varphi_n'(|u|^2))^{\frac{2}{s}} dx \right)^{\frac{s}{2}} dt, \]
which in turn gives us Lemma 6.2. \( \square \)

7 Limit when \( n \to \infty, r_0 \to 0 \) and \( r_1 \to 0 \)

Thanks to the total energy estimate, thermal energy estimate, B-D entropy energy estimate, Mellet-Vasseur estimate, we have enough compactness to pass the final three parameter limit to get the weka solutions foe the Navier-Stokes-Fouier equations. The passage limit process is similar to the one in Section 5,6, so omit the details, here. Therefore we complete the proof of Theorem 1.2.
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