THE MORSE INDEX OF REDUTIBLE SOLUTIONS
OF THE $SW$-EQUATIONS

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Abstract. The $2^{nd}$ variation formula of the Seiberg-Witten functional is obtained in order to estimate the Morse index of reduitible solutions $(A, 0)$. It is shown that their Morse index is given by the dimension of the largest negative eigenspace of the operator $\triangle_A + \frac{k}{g}$, hence it is finite.

1. Introduction

Let $(X, g)$ be a closed riemannian 4-manifold and

$$Spin^c(X) = \{\beta + \gamma \in H^2(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}_2) \mid w_2(X) = \alpha (mod \ 2)\}.$$ the space of $Spin^c$ structures on $X$.

Originally, the $SW_\alpha$-monopole equations in 2 were not obtained through a variational principle, but they can be described as the stable solutions of the functional in 3, named $SW_\alpha$-functional. Since this functional satisfies the Palais-Smale condition it is allowed to related its critical points with the topology of its configuration space $A_\alpha \times G_\alpha \Gamma(S^+_\alpha)$. As shown in [3], by considering the embedding of the Jacobian Torus

$$i : T^{b_1(X)} = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})} \hookrightarrow A_\alpha \times G_\alpha \Gamma(S^+_\alpha), \ b_1(X) = \dim_\mathbb{R}H^1(X, \mathbb{R})$$

into the configuration space, the variational formulation of the $SW_\alpha$-equations gives us an interpretation for the topology of $A_\alpha \times G_\alpha \Gamma(S^+_\alpha)$:

Theorem 1.0.1. Let $(X, g)$ be a closed riemannian 4-manifold with scalar curvature $k_g$.

Thus,

(1) if $k_g \geq 0$, then the gradient flow of the $SW_\alpha$-functional defines an homotopy equivalence among $A_\alpha \times G_\alpha \Gamma(S^+_\alpha)$ and $i(T^{b_1(X)})$.

(2) if $k_g < 0$, then $A_\alpha \times G_\alpha \Gamma(S^+_\alpha)$ has the same homotopy type of $T^{b_1(X)}$.

Therefore, the existence of $SW_\alpha$-monopoles are not a consequence of the homotopy type of $A_\alpha \times G_\alpha \Gamma(S^+_\alpha)$. It is also known that they may exist for at most a finite number of classes $\alpha$ and, whenever $X$ has simple type, they satisfy $\alpha^2 = 2\chi(X) + 3\sigma(X)$ \footnote{$\chi(X)$-euler characteristic of $X$, $\sigma(X)$=signature of $X.$} corresponding, each one, to a almost complex structure on $X$. 

\[\chi(X)-euler\ \text{characteristic\ of\ } X, \ \sigma(X)=\text{signature\ of\ } X.\]
An application of theorem 1.0.1 is that the space $A_\alpha \times G_\alpha \Gamma(S^+_{\alpha}) - \{(A,0) \mid A \in A_\alpha\}$ has the homotopy type of $T^h(X) \times CP^\infty$.

I am grateful to Professor Clifford Taubes for sharing his knowledge about the spectral properties of the operator $\triangle + \frac{\ell b}{4}$.

2. Background

In order to define the $SW_\alpha$ functional, we fix a $Spin^c$ structure on $X$ and also describe the Sobolev Space structure defined on the configuration space. For each $\alpha \in Spin^c(X)$, there is a representation $\rho_\alpha : SO_4 \rightarrow Cl_4$ inducing a pair of vector bundles $(S^\alpha, L_\alpha)$ over $X$, where

(i): $S_\alpha = P_{SO_4} \times_{\rho_\alpha} V = S^\alpha_+ \oplus S^\alpha_-$. The bundle $S^\alpha_\pm$ are the positive and the negative complex spinors bundle (fibers are $Spin^c_\alpha$-modules isomorphic to $\mathbb{C}^2$).

(ii): $L_\alpha = P_{SO_4} \times_{det(\alpha)} \mathbb{C}$. It is called the determinant line bundle associated to $\alpha$. $(c_1(L_\alpha) = \alpha)$

Thus, given $\alpha \in Spin^c(X)$ we associate a pair of bundles

$$\alpha \in Spin^c(X) \quad \mapsto \quad (L_\alpha, S_\alpha^+)$$

From now on, we considered on $X$ a Riemannian metric $g$ and on $S_\alpha$ an hermitian structure $h$.

Let $P_\alpha$ be the $U_1$-principal bundle over $X$ obtained as the frame bundle of $L_\alpha$ ($c_1(P_\alpha) = \alpha$). Also, we consider the associated bundles

$$Ad(U_1) = P_{U_1} \times_{Ad} U_1 \quad ad(u_1) = P_{U_1} \times_{ad} u_1,$$

where $Ad(U_1)$ is a fiber bundle with fiber $U_1$, and $ad(u_1)$ is a vector bundle with fiber isomorphic to the Lie Algebra $u_1$.

Let $A_\alpha$ be (formally) the space of connections (covariant derivative) on $L_\alpha$, $\Gamma(S^\alpha_+)$ the space of sections of $S^\alpha_+$ and $G_\alpha = \Gamma(Ad(U_1))$ the gauge group acting on $A_\alpha \times \Gamma(S^\alpha_+)$ as follows:

$$g.(A, \phi) = (A + g^{-1}dg, g^{-1}\phi).$$

So, the action is free whenever $\phi \neq 0$, otherwise the pair $(A,0)$ has isotropy subgroup isomorphic to $U_1$. $A_\alpha$ is an affine space which vector space structure, after fixing an origin, is isomorphic to the space $\Omega^1(ad(u_1))$ of $ad(u_1)$-valued 1-forms. Once a connection $\nabla^0 \in A_\alpha$ is fixed, a bijection $A_\alpha \leftrightarrow \Omega^1(ad(u_1))$ is explicated by $\nabla^A \leftrightarrow A$, where $\nabla^A = \nabla^0 + A$. $G_\alpha = Map(X, U_1)$, since $Ad(U_1) \simeq X \times U_1$. The curvature of a 1-connection form $A \in \Omega^1(ad(u_1))$ is the 2-form $F_A = dA \in \Omega^2(ad(u_1))$.

The configuration space is $C_\alpha$. A topology is fixed on $C_\alpha$ by considering the following Sobolev structures:

- $A_\alpha = L^{1,2}(\Omega^1(ad(u_1)))$;
- $\Gamma(S^\alpha_+) = L^{1,2}(\Omega^0(X, S^\alpha_+))$;
- $C_\alpha = A_\alpha \times \Gamma(S^\alpha_+)$;
- $G_\alpha = L^{2,2}(X, U_1) = L^{2,2}(Map(X, U_1))$.

($G_\alpha$ is an $\infty$-dimensional Lie Group which Lie algebra is $g = L^{1,2}(X, u_1)$).
2.1. **Seiberg-Witten Monopole Equation.** In dimension 4, the vector bundle \( \Omega^2(ad(u_1)) \) splits as
\[
\Omega^2_+(ad(u_1)) \oplus \Omega^2_-(ad(u_1)),
\]
where (+) is the seld-dual component and (-) the anti-self-dual.

Fixed \( \alpha \in Spin^c(X) \), the 1st-order SW_\alpha-monopole equations, defined over \( \mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma(S_\alpha^+) \), are
\[
\begin{aligned}
D_A^+(\phi) &= 0, \\
F_A^+ &= \sigma(\phi),
\end{aligned}
\]
where
(i): \( D_A^+ \) is the Spin^c-Dirac operator defined on \( \Gamma(S_\alpha^+) \),
(ii): The quadratic form \( \sigma : \Gamma(S_\alpha^+) \to \text{End}^0(S_\alpha^+) \) is defined as
\[
\sigma(\phi) = \phi \otimes \phi^* - \frac{|\phi|^2}{2}.I
\]
performs the coupling of the ASD-equation with the \( \text{Dirac}^c \) operator.

A solution of equations 2 is named as SW_\alpha-monopole. It can be described as the space \( F^{-1}(0) \), where \( F_\alpha : \mathcal{C}_\alpha \to \Omega^2_+(X) \oplus \Gamma(S_\alpha^-) \) is the map
\[
F_\alpha(A, \phi) = (F^+_A - \sigma(\phi), D^+_A(\phi)).
\]
The SW_\alpha-equations are \( g_\alpha \)-invariant and the map \( F \) is a Fredholm map up to gauge equivalence.

**Definition 2.1.1.** \((A, \phi)\) is named a SW_\alpha-monopole if it satisfies the equation 2 and \( \phi \neq 0 \).

2.2. **SW_\alpha Lagrangean.**

**Definition 2.2.1.** For each \( \alpha \in Spin^c(X) \), the Seiberg-Witten Functional is the functional \( SW_\alpha : \mathcal{C}_\alpha \to \mathbb{R} \) given by
\[
SW(A, \phi) = \int_X \left\{ \frac{1}{4} | F_A |^2 + \frac{1}{8} | \nabla^4 \phi |^2 + \frac{k_g}{4} | \phi |^4 - \frac{k_g}{4} | \phi |^2 \right\} dv_g + \pi^2 \alpha^2,
\]
where \( k_g = \text{scalar curvature of } (X, g) \) and \( \alpha^2 = Q_X(\alpha, \alpha) \) (Q_X is the intersection form of \( X \)).

**Remark 2.2.2.**
(i): The functional is well defined on \( \mathcal{C}_\alpha \), since in dimension \( n=4 \) there is the Sobolev embeddings \( L^4(\Omega^1(ad(u_1))) \subset L^{1,2}(\Omega^1(ad(u_1))) \) and \( L^4(\Omega^0(X, S_\alpha^+)) \subset L^{1,2}(\Omega^0(X, S_\alpha^+)) \).
(ii): the functional \( SW_\alpha : \mathcal{C}_\alpha \to \mathbb{R} \) is gauge invariant.
(iii): Whenever \( k_g \geq 0 \), \( SW_\alpha(A, \phi) \geq \int_X | F_A |^2 dx + \pi^2 \alpha^2 \). Therefore, the stable solutions are \((A, 0)\).
(iv): \((A, 0)\) is a SW_\alpha-monopole if and only if \( A \) is anti-self-dual connection. Whenever \( b^+_2 \geq 3 \), it is known [?] that such solutions do not exist for a dense set of metrics on \( X \).
The tangent bundle $TC_{\alpha}$ is trivial because $C_{\alpha}$ is contractible. The fiber over $(A, \phi)$ is

$$T_{(A, \phi)}C_{\alpha} = \Omega^1(ad(u_1)) \oplus \Gamma(S^+_\alpha).$$

A Riemannian structure on $TC_{\alpha}$ is defined by the inner product

$$<,>: T_{(A, \phi)}C_{\alpha} \times T_{(A, \phi)}C_{\alpha} \to \mathbb{R}$$

$$(\Theta + V, \Lambda + W) \to < \Theta + V, \Lambda + W >= < \Theta, \Lambda > + < V, W > .$$

where

$$< \Theta, \Lambda >= \int_X < \Theta(x), \Lambda(x) > dx,$$

$$< V, W >= \int_X [< V(x), W(x) > dx.$$  

The derivative of $SW$-functional defines a 1-form $dSW \in \Omega^1(C_{\alpha}).$ For each point $(A, \phi)$, the functional $dSW_{(A, \phi)} : \Omega^1(ad(u_1)) \oplus \Gamma(S^+_\alpha) \to \mathbb{R}$ is decomposed in components as

$$dSW_{(A, \phi)}(\Theta + \Lambda) = d_1SW_{(A, \phi)}(\Theta) + d_2SW_{(A, \phi)}(\Lambda),$$

where

(i): $d_1$

$$d_1SW_{(A, \phi)}(\Theta) = \lim_{t \to 0} \frac{SW(A+t\Theta, \phi) - SW(A, \phi)}{t} =$$

$$= \frac{1}{2} \int_X Re \{ < F_A, d\Theta > + 4 < \nabla^A(\phi), \Theta(\phi) > \} dx =$$

$$= \frac{1}{2} Re \left( \int_X < d^*F_A + 4\Phi^*(\nabla^A(\phi)), \Theta > \right) dx =$$

$$= \frac{1}{2} Re \left( < d^*F_A + 4\Phi^*(\nabla^A(\phi)), \Theta > \right).$$

where $\Phi : \Omega^1(u_1) \to \Omega^1(S^+_\alpha)$ is the linear operator $\Phi(\Theta) = \Theta(\phi),$ and

$$\Phi^*(\nabla^A(\phi)) = \frac{1}{2} d(| \phi |^2).$$

(ii): $d_2$

$$d_2SW_{(A, \phi)}(V) = \lim_{t \to 0} \frac{SW(A, \phi + tV) - SW(A, \phi)}{t} =$$

$$= 2 \int_X Re \{ < \nabla^A\phi, \nabla^AV > + < \frac{| \phi |^2 + k\phi}{4}, V > \} dx =$$

$$= 2 \int_X Re \{ < \triangle_A\phi + \frac{| \phi |^2 + k\phi}{4}, V > \} dx =$$

$$= 2 Re \left( < \triangle_A\phi + \frac{| \phi |^2 + k\phi}{4}, V > \right).$$
where $\Delta_A = (\nabla^A)^* \nabla^A$.

Thus, the Euler-Lagrange equations of $SW_\alpha$-functional are

\begin{align*}
&d^* F_A + 4\Phi^*(\nabla^A \phi) = 0, \\
&\Delta_A \phi + \frac{|\phi|^2}{4} \phi + \frac{k_g}{4} \phi = 0.
\end{align*}

### 2.3. Hodge Solutions

It follows from the identity

$$SW_\alpha(A, \phi) = \int_X \{ |DA \phi|^2 + |F_A^+ - \sigma(\phi)|^2 \} dx,$$

(see in [4]) that the $SW_\alpha$ monopoles are the stable solutions. From [11], it is known that such $SW$-monopoles exist only for a finite number of classes $\alpha \in Spin^c(X)$.

Thus, it arise the question about a sufficient condition to guarantee their existence. The easiest solutions of the Euler-Lagrange equations above are the ones of type $(A, 0)$, where the curvature $F_A$ is a harmonic 2-form. Hodge theory guarantees the existence of such solution. The gauge isotropy subgroup of $(A, 0)$ is isomorphic to $U_1$, justifying their nickname reducible. Furthermore,

**Proposition 2.3.1.** The space of solutions of $d^* F_A = 0$, module the $G_\alpha$-action, is diffeomorphic to the Jacobian Torus

$$T^{b_1}(X) = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})}, \quad b_1(X) = \text{dim}_\mathbb{R}H^1(X, \mathbb{Z}).$$

**Proof.** The equation $d^* F_A = 0$ implies that $F_A$ is an harmonic 2-form and, by Hodge theory, it is the only one. Let $A$ and $B$ be solutions and consider $B = A + b$; so,

$$d^* F_B + d^* F_A + d^* db = 0 \quad \Rightarrow \quad db = 0.$$

By fixing the origin at $A$, we associate $B \leadsto b \in H^1(X, \mathbb{R})$; note that $F_B = F_A$.

Let $B_1$ be a solution gauge equivalent to $B_2$, so there exists $g \in G_\alpha$ such that $B = A + g^{-1} dg$ and $F_{B_1} = F_{B_2}$. However, the 1-form $g^{-1} dg \in H^1(X, \mathbb{Z})$. Consequently, if $B_1 = A + b_1$ and $B_2 = A + b_2$, then $[b_2] = [b_1]$ in $H^1(X, \mathbb{Z})$. \hfill \Box

### 3. 2nd - Variation Formula

The 2nd - variation formula is the symmetric bilinear form

$$H^{SW}_{(A, \phi)} : T_{(A, \phi)} \mathcal{C}_\alpha \times T_{(A, \phi)} \mathcal{C}_\alpha \rightarrow \mathbb{R},$$

obtained by computing the hessian of $SW$. A neat exposition may achieved by considering the operator $H^{SW}_{(A, \phi)}$ as a linear functional $H^{SW}_{(A, \phi)} : T_{(A, \phi)} \mathcal{C}_\alpha \otimes T_{(A, \phi)} \mathcal{C}_\alpha \rightarrow \mathbb{R}$. By taking $\mathcal{U} = \Omega^1(ad(u_1))$ and $\mathcal{V} = \Gamma(S^+_1)$, we have the functional

$$H^{SW}_{(A, \phi)} : (\mathcal{U} \otimes \mathcal{U}) \bigoplus (\mathcal{U} \otimes \mathcal{V}) \bigoplus (\mathcal{V} \otimes \mathcal{V}) \bigoplus (\mathcal{V} \otimes \mathcal{V}) \rightarrow \mathbb{R}$$

de decomposing into
\[ H_{(A,\phi)}^{\text{SW}}((\Theta, V), (\Lambda, W)) = H_{(A,\phi)}^{\text{SW}}(\Theta \otimes \Lambda + \Theta \otimes W + V \otimes \Lambda + V \otimes W) = d_{11}(\Theta \otimes \Lambda) + d_{12}(\Theta \otimes W) + d_{21}(V \otimes \Lambda) + d_{22}(V \otimes W). \]

Thus, \( H_{(A,\phi)}^{\text{SW}} = d_{11} + d_{12} + d_{21} + d_{22}, \) where

1. \( d_{11} : \Omega^2(\text{ad}(u_1)) \otimes \Omega^1(\text{ad}(u_1)) \to \mathbb{R} \)

\[
d_{11}^{\text{SW}}(\Theta, \Lambda) = \lim_{t \to 0} \frac{1}{t} \left\{ d_{11}^{\text{SW}}_{(A+t\Lambda,\phi)}(\Theta - d_{1}^{\text{SW}}_{(A,\phi)}(\Theta) = \right.
\]

\[
= \frac{1}{2} \int_X \text{Re}\{ <d\Theta, d\Lambda + 4 + \Theta(\phi), \Lambda(\phi) > \} dx = \text{Re}\{ <d\Theta, d\Lambda + 4 + \Theta(\phi), \Lambda(\phi) > \} = \frac{1}{2} \text{Re}\{ <\Theta, (d^*d + 4\Phi^*\Phi)(\Lambda) > \}.
\]

2. \( d_{12} : \Omega^2(\text{ad}(u_1)) \otimes \Gamma(S^+_\alpha) \to \mathbb{R} \)

\[
d_{12}^{\text{SW}}(\Theta, W) = \lim_{t \to 0} \frac{1}{t} \left\{ d_{12}^{\text{SW}}_{(A,\phi+tW)}(\Theta - d_{1}^{\text{SW}}_{(A,\phi)}(\Theta) = \right.
\]

\[
= 2 \int_X \text{Re}\{ <\nabla^A \phi, \Theta(W) > + <\nabla^A W, \Theta(\phi) > \} dx = 2 \text{Re}\{ <\nabla^A \phi, \Theta(W) > + <\nabla^A W, \Phi(\Theta) > \} = 2 \text{Re}\{ <\nabla^A \phi, \Theta(W) > + <\Theta, \Phi^*\nabla^A W > \}
\]

Introducing the operator \( W : \Gamma(S^+_\alpha) \to \Omega^1(\text{ad}(u_1)), \) as \( W(\Theta) = \Theta(W), \)

\[
d_{12}^{\text{SW}}(\Theta, W) = 2 \text{Re}\{ <\Theta, P(W) > \}
\]

where \( P : \Gamma(S^+_\alpha) \to \Omega^1(\text{ad}(u_1)) \) is \( P(W) = W^*(\nabla^A \phi) + \Phi^*(\nabla^A W). \)

3. \( d_{21} : \Omega^1(\text{ad}(u_1)) \otimes \Omega^1(\text{ad}(u_1)) \to \mathbb{R} \)

\[
d_{21}^{\text{SW}}(V, \Lambda) = \lim_{t \to 0} \frac{1}{t} \left\{ d_{21}^{\text{SW}}_{(A+t\Lambda,\phi)}(V - d_{2}^{\text{SW}}_{(A,\phi)}(V) = \right.
\]

\[
= 2 \int_X \text{Re}\{ <\nabla^A \phi, \Lambda(V) > + <\Lambda(\phi), \nabla^A V > \} dx = 2 \text{Re}\{ <\nabla^A \phi, \Lambda(V) > + <\Phi(\Lambda), \nabla^A V > \} = 2 \text{Re}\{ <\nabla^A \phi, \Lambda(V) > + <\Lambda(\phi), \Phi^*(\nabla^A \phi) > \}
\]

Analogously, by considering the operator \( Q : \Omega^1(\text{ad}(u_1)) \to \Gamma(S^+_\alpha), \)
\( Q(\Lambda) = \Lambda^*\nabla^A \phi + (\nabla^A)^*\Phi(\Lambda), \)

\[
d_{21}^{\text{SW}}(V, \Lambda) = 2 \text{Re}\{ <Q(\Lambda), V > \}.
\]

It is straightforward that \( Q = P^*. \)
Furthermore, is also the tangent space to the Jacobian torus $T\Omega$, the tangent space of $A$. Since $\Omega(6)$ which matrix representation is

\[ H = \left( \begin{array}{c} d^*d + 4\Phi^*\Phi \\ Q \end{array} \right) \left( \begin{array}{c} P \\ (\nabla^A)^*\nabla^A + \frac{1}{2} < \phi, \cdot > + \frac{k^2|\phi|^2}{4}I \end{array} \right). \]

Therefore, the induced quadratic form $\tilde{H}^{SW}_{(A,\phi)} : \Omega^1(ad(u_1)) \oplus \Gamma(S^+_\alpha) \to \mathbb{R}$ is

\[ H^{SW}_{(A,\phi)}(\Theta, V) = |d\Theta|^2 + |\Phi(\Theta)|^2 + 2 < \nabla^A\phi, \Theta(V) > + 2 < \Phi(\Theta), \nabla^A V > + \]

\[ + |\nabla^A V|^2 + \frac{k^2|\phi|^2}{4}, |V|^2 + \frac{1}{2} < \phi, V >^2. \]

Hence, $H^{SW}_{(A,\phi)} : T_{(A,\phi)}C_\alpha \otimes T_{(A,\phi)}C_\alpha \to \mathbb{R}$ is a bounded operator.

### 3.1. Morse Index of Reducible Solutions

From Hodge theory, we have the complex

\[ \Omega^\theta(ad(u_1)) \xrightarrow{d} \Omega^1(ad(u_1)) \xrightarrow{d} \Omega^2(ad(u_1)) \xrightarrow{d} \ldots \Omega^4(ad(u_1)). \]

Since $\Omega^1(ad(u_1)) = d(\Omega^0) \oplus \ker(d^*(\Omega^1))$, and $d(\Omega^0)$ is the tangent space to the orbits, the tangent space of $A_\alpha \times \phi_\alpha \Gamma(S^+_\alpha)$ at $(A, \phi)$ can be decomposed into $\mathfrak{W}_1 \oplus \mathfrak{W}_2$, where

\[ \mathfrak{W}_1 = \{ \Theta \in \Omega^1(ad(u_1)); \ d^*\Theta = 0 \}, \]

\[ \mathfrak{W}_2 = \{ W \in \Gamma(S^+_\alpha); < W, A(\phi) >= 0, \ \forall A \in \Omega^\theta(ad(u_1)) \}. \]

Furthermore, $\mathfrak{W}_1 = d^*(\Omega^2(ad(u_1))) \oplus \mathcal{H}_1$, where $\mathcal{H}_1$ is space of harmonic forms and is also the tangent space to the Jacobian torus $T^{b_1}(X)$ at $(A, 0)$.

The quadratic form in 7, when evaluated at a reducible solution $(A, 0)$, is given by
\begin{align}
H(\Theta, V) &= \| d\Theta \|_{L^2}^2 + \| \nabla^A V \|_{L^2}^2 + \int_X \frac{k_g}{4} | V |^2 \, dx.
\end{align}

Indeed, at \((A, 0)\), it follows from expression 7

\begin{align}
H &= \left( \begin{array}{cc}
d^*d & 0 \\
0 & \triangle_A + \frac{k_g}{4} \end{array} \right), \\
H^{Sw}_{(A, \phi)}(\Theta, V) &= \langle (\Theta, V), H(\Theta, V) \rangle.
\end{align}

The Morse index of \((A, 0)\) is equal to the dimension of the largest negative eigenspace of the operator \(L_A = \triangle_A + \frac{k_g}{4} : \Gamma(S_\alpha) \to \Gamma(S_\alpha)\). Let \(V_\lambda \subset T_{(A, 0)} A_\alpha \times _{\tilde{\alpha}} \Gamma(S_\alpha^+)\) be the eigenspace associated to the eigenvalue \(\lambda\). Since \(L_A\) is an elliptic operator, \(V_0\) has finite dimension;

**Proposition 3.1.1.** Let \((A, 0)\) be a reducible solution. So,

1. If \(k_g > 0\), then \(V_0 = T_{(A, 0)} b_1(X)\) and so \(SW_\alpha : A_\alpha \times _{\tilde{\alpha}} \Gamma(S_\alpha^+) \to \mathbb{R}\) is a Morse-Bott function at \((A, 0)\).
2. If \(k_g = 0\), then \(V_0 = T_{(A, 0)} b_1(X) \oplus \{ V \in \Gamma(S_\alpha^+) \mid \nabla^A V = 0 \}\).

**Proof.** It is straightforward from equation 10. \(\square\)

**Proposition 3.1.2.** The Morse index of a reducible solution \((A, 0)\) is finite.

**Proof.** It follows from the spectral theory applied to the elliptic, self-adjoint operator \(L_A\) ([1], [12]) that the spectrum of \(L_A\)

(i): is discrete,

(ii): each eigenvalue has finite multiplicity,

(iii): there are no points of accumulation,

(iv): there are but a finite number of eigenvalues below any given number.

Therefore, the spectrum of \(L_A\) is bounded from below. \(\square\)

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