Article

Anti-Intuitionistic Fuzzy Soft a-Ideals Applied to BCI-Algebras

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Abstract: The notion of anti-intuitionistic fuzzy soft a-ideals of BCI-algebras is introduced and several related properties are investigated. Furthermore, the operations, namely; AND, extended intersection, restricted intersection, and union on anti-intuitionistic fuzzy soft a-ideals are discussed. Finally, characterizations of anti-intuitionistic fuzzy soft a-ideals of BCI-algebras are given.

Keywords: BCI-algebras; soft set; fuzzy soft set; intuitionistic fuzzy soft set; anti-intuitionistic fuzzy soft ideals; anti-intuitionistic fuzzy soft a-ideals

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1. Introduction

The theory of fuzzy set, intuitionistic fuzzy sets, soft set, and more other theories were introduced to deal with uncertainty. In [1], Zadeh introduced the concept of a fuzzy subset of a set. Later on, a number of generalizations of this fundamental notion have been studied by many authors in different directions. The notion of an intuitionistic fuzzy set defined in [2] is a generalization of a fuzzy set. It gives more opportunity to be accurate when dealing with uncertain objects. Soft set theory was initially suggested by Molodstov in [3], then Maji et al. in [4] combined the soft set theory and the intuitionistic fuzzy set theory, and introduced the notion intuitionistic fuzzy soft sets.

Algebra is the language in which combinatorics are usually expressed. Combinatorics is the study of discrete structures that arise not only in areas of pure mathematics, but in other areas of science, for example, computer science, statistical physics and genetics. From ancient beginnings, this subject truly rose to prominence from the mid-20th century, when scientific discoveries (most notably of DNA) showed that combinatorics is key to understanding the world around us, whilst many of the great advances in computing were built on combinatorial foundations. These concepts were widely studied over different classes of logical algebras as the essential classes of BCK/BCI-algebras presented by Iseki [5]. The concepts intuitionistic fuzzy ideals of BCK-algebras were studied in [6]. Bej et al. [7] declared the concept of doubt intuitionistic fuzzy subalgebra and doubt intuitionistic fuzzy ideal in BCK/BCI-algebras. Muhiuddin et al. studied various concepts on fuzzy sets and applied them to BCK/BCI-algebras, and other related notions (see for e.g., [8–18]). Also, some new generalizations of fuzzy sets and other related concepts in different algebras have been studied in (see for e.g., [6,19–35]). Additionally, Balamurugan et al. [36] introduced the concepts of intuitionistic fuzzy soft subalgebras, intuitionistic fuzzy soft ideals, and intuitionistic fuzzy soft a-ideals of B-algebra and studied several properties of these notions.

In the present paper, we introduce the notion of anti-intuitionistic fuzzy soft a-ideals in BCI-algebras. The results of present paper are organized, as follows: Section 2 summarizes some
basic definitions and properties that are needed to develop our main results while in Section 3, we introduce the notion of anti-intuitionistic fuzzy soft a-ideals of BCI-algebras and investigate related properties. In Section 4, we give characterizations of anti-intuitionistic fuzzy soft a-ideals of BCI-algebras while using the concept of a soft level set.

2. Preliminaries

In this section, we recall basic definitions and results that are related to the subject of the paper.

Definition 1. [5] An algebra \((\Omega; \odot, 0)\) of type \((2, 0)\) is called a BCI-algebra if it satisfies the following conditions:

1. \(((l \odot m) \odot (l \odot n)) \odot (n \odot m) = 0,\)
2. \((l \odot (l \odot m)) \odot m = 0,\)
3. \(l \odot l = 0;\)
4. \(l \odot m = 0\) and \(m \odot l = 0 \Rightarrow l = m,\) for all \(l, m, n \in \Omega.\)

Any BCI-algebra \(\Omega,\) satisfies the following axioms:

1. \(l \odot 0 = l,\)
2. \(l \leq m \Rightarrow l \odot n \leq m \odot n\) and \(n \odot m \leq n \odot l,\)
3. \((l \odot n) \odot (m \odot n) \leq l \odot m,\)
4. \(0 \odot (0 \odot (l \odot m)) = (0 \odot m) \odot (0 \odot l),\)
5. \((l \odot m) \odot n = (l \odot n) \odot m,\)

where \(l \leq m \Rightarrow l \odot m = 0,\) for any \(l, m, n \in \Omega.\)

A non-empty subset \(\Delta\) of a BCK-algebra \(\Omega\) is called an ideal of \(\Omega\) if it satisfies

1. \(0 \in \Delta,\)
2. \(\forall l, m \in \Omega, l \ast m \in \Delta, m \in \Delta \Rightarrow l \in \Delta.\)

A non-empty subset \(\Delta\) of a BCK-algebra \(\Omega\) is called an a-ideal of \(\Omega\) if it satisfies (1) and

3. \(\forall l, m \in \Omega, (l \odot n) \odot (0 \odot m) \in \Delta, n \in \Delta \Rightarrow m \odot l \in \Delta.\)

For an initial set \(\Omega\) and a set of parameters \(\Delta,\) a pair \((Y, \Delta)\) is said to be a soft set over \(\Omega \Leftrightarrow \exists Y : \Delta \rightarrow \varphi(\Omega),\) where \(\varphi(\Omega)\) is a family of subsets of \(\Omega.\) (see [30] for more details on soft set theory).

Definition 2. [4] Let \(\Pi\) be a collection of parameters and let \(Y(\Omega)\) indicate the collection of all fuzzy sets in \(\Omega.\) Then \((Y, \Delta)\) is called a fuzzy soft set over \(\Omega,\) where \(\Delta \subseteq \Pi\) and \(Y : \Delta \rightarrow Y(\Omega).\)

Definition 3. [36] Let \((Y, \Delta)\) be a fuzzy soft set (abbr. FSS). Then \((Y, \Delta)\) is an anti-fuzzy soft ideal (abbr. AFSID) of \(\Omega\) if \(Y[\omega] = \{\xi_{Y[\omega]}(l) : l \in \Omega\) and \(\omega \in \Delta}\) is an AFID of \(\Omega\) satisfies the following assertions:

i. \(\xi_{Y[\omega]}(0) \leq \xi_{Y[\omega]}(l),\)
ii. \(\xi_{Y[\omega]}(l) \leq \xi_{Y[\omega]}(l \odot m) \lor \xi_{Y[\omega]}(m),\)

for all \(l, m, n \in \Omega\) and \(\omega \in \Delta.\)

Definition 4. [36] Let \((Y, \Delta)\) be a fuzzy soft set (abbr. FSS). Then \((Y, \Delta)\) is an anti-fuzzy soft a-ideal (abbr. AFSID) of \(\Omega\) if \(Y[\omega] = \{\xi_{Y[\omega]}(l) : l \in \Omega\) and \(\omega \in \Delta\) is an AFID of \(\Omega\) satisfies the following assertions:

i. \(\xi_{Y[\omega]}(0) \leq \xi_{Y[\omega]}(l),\)
ii. \(\xi_{Y[\omega]}(m \odot l) \leq \xi_{Y[\omega]}(l \odot n) \lor (0 \odot m) \lor \xi_{Y[\omega]}(n),\)

for all \(l, m, n \in \Omega\) and \(\omega \in \Delta.\)
**Definition 5.** [4] Let Π be a collection of parameters and let Φ(Ω) indicate the collection of all intuitionistic fuzzy sets in Ω. Subsequently, (Y, Δ) is called an intuitionistic fuzzy soft set over Ω, where Δ ⊆ Π and Y : Δ → Φ(Ω).

3. Anti-Intuitionistic Fuzzy Soft a-Ideal

In what follows, we write Ω to denote a BCI-algebra (Ω; ∪, 0) and IFSs for intuitionistic fuzzy sets and we will introduce an abbreviation for the notions in the following definitions to be used in the rest of the paper.

**Definition 6.** Let (Y, Δ) be an intuitionistic fuzzy soft set (abbr. IFSS). Afterwards, (Y, Δ) is an anti-intuitionistic fuzzy soft ideal (abbr. AIFSID) of Ω if Y[ω] = {(ξY[ω](l), ζY[ω](l)) : l ∈ Ω and ω ∈ Δ} is an AIFID of Ω satisfies the following assertions:

(i) \( ξY[ω](0) ≤ ξY[ω](l) \) and \( ζY[ω](0) ≥ ζY[ω](l) \),
(ii) \( ξY[ω](l) ≤ ξY[ω](l ∪ m) \lor ξY[ω](m) \),
(iii) \( ξY[ω](l) ≥ ξY[ω](l ∪ m) \land ξY[ω](m) \),

for all \( l, m, n \in Ω \) and \( ω \in Δ \).

**Definition 7.** An IFSS (Y, Δ) is called an anti-intuitionistic fuzzy soft a-ideal (abbr. AIFSAID) of Ω if \( Y[ω] = \{ (ξY[ω](l), ζY[ω](l)) : l ∈ Ω \) and \( ω \in Δ \} \) is an AIFAID of Ω satisfies the following assertions:

(i) \( ξY[ω](0) ≤ ξY[ω](l) \) and \( ζY[ω](0) ≥ ζY[ω](l) \),
(ii) \( ξY[ω](m ∪ l) ≤ ξY[ω](l ∪ n) \lor (0 ∩ m) \lor ξY[ω](n) \),
(iii) \( ξY[ω](m ∪ l) ≥ ξY[ω](l ∪ n) \lor (0 ∩ m) \land ξY[ω](n) \),

for all \( l, m, n \in Ω \) and \( ω \in Δ \).

**Example 1.** Suppose that there are four patients in the initial universe set \( Ω = \{ p_1, p_2, p_3, p_4 \} \) given by

| ∩ | P1 | P2 | P3 | P4 |
|---|---|---|---|---|
| P1 | P1 | P2 | P3 | P4 |
| P2 | P2 | P1 | P4 | P3 |
| P3 | P3 | P4 | P2 | P1 |
| P4 | P4 | P3 | P2 | P1 |

Afterwards, \((Ω; ∪, p_1)\) is a BCI-algebra.

Let a set of parameters, we consider \( Δ = \{ f, s, n \} \) be a status of patients, in which

- \( f \) stands for the parameter “fever” can be treated by antibiotic,
- \( s \) stands for the parameter “sneezing” can be treated by antiallergic,
- \( n \) stands for the parameter “nosal block” can be treated by nosal drops.

Subsequently, Y[\( f \)], Y[\( s \)], and Y[\( n \)] are IFSs over Ω represented by:

| Y   | p1 | p2 | p3 | p4 |
|-----|----|----|----|----|
| \( f \) | [0.1, 0.8] | [0.1, 0.8] | [0.2, 0.6] | [0.2, 0.6] |
| \( s \) | [0.0, 0.9] | [0.0, 0.9] | [0.3, 0.7] | [0.3, 0.7] |
| \( n \) | [0.2, 0.7] | [0.2, 0.7] | [0.4, 0.6] | [0.4, 0.6] |

Therefore, Y[\( f \)], Y[\( s \)], and Y[\( n \)] are an AIFSAID of Ω with respect to \( f, s, \) and \( n \), respectively. Hence, (Y, Δ) is an AIFSAID of Ω.

**Proposition 1.** For any AIFSAID (Y, Δ) of Ω, the following inequalities hold:

\( ξY[ω](m ∪ l) ≤ ξY[ω](l ∩ (0 ∩ m)) \) and \( ζY[ω](m ∪ l) ≥ ζY[ω](l ∩ (0 ∩ m)) \), for any \( ω \in Δ \) and \( l, m \in Ω \).
Theorem 1. Over $\Omega$, any AIFSAID is an AIFSID.

Proof. Let $(Y, \Delta)$ be an AIFSAID of $\Omega$. Subsequently, $Y[\omega] = \{(\xi_{Y[\omega]}(l), \xi_{Y[\omega]}(l)) : l \in \Omega$ and $\omega \in \Delta\}$ is an AIFSAID of $\Omega$.

Thus, for every $l, m, n \in \Omega$ and $\omega \in \Delta$,
\[\xi_{Y[\omega]}(m \odot l) \leq \xi_{Y[\omega]}((l \circ n) \odot (0 \circ m)) \lor \xi_{Y[\omega]}(n)\]
and
\[\xi_{Y[\omega]}(m \odot l) \geq \xi_{Y[\omega]}((l \circ n) \odot (0 \circ m)) \land \xi_{Y[\omega]}(n).\]

By substituting $n = 0$, we get,
\[\xi_{Y[\omega]}(m \odot l) \leq \xi_{Y[\omega]}((l \circ 0) \odot (0 \circ m)) \lor \xi_{Y[\omega]}(0)\]
\[= \xi_{Y[\omega]}(l \odot (0 \circ m)) \lor \xi_{Y[\omega]}(0)\]
and
\[\xi_{Y[\omega]}(m \odot l) \geq \xi_{Y[\omega]}((l \circ 0) \odot (0 \circ m)) \land \xi_{Y[\omega]}(0)\]
\[= \xi_{Y[\omega]}(l \odot (0 \circ m)) \land \xi_{Y[\omega]}(0)\]
\[\xi_{Y[\omega]}(m \odot l) \geq \xi_{Y[\omega]}(l \odot (0 \circ m)).\]

Example 2. Let $\Omega = \{0, p, q, r, s\}$ with Cayley table:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\circ & 0 & p & q & r & s \\
\hline
0 & 0 & 0 & s & r & q \\
p & p & 0 & s & r & q \\
q & q & q & 0 & s & r \\
r & r & r & q & 0 & s \\
s & s & s & r & q & 0 \\
\hline
\end{array}
\]

The converse of Theorem 1 is not true in general i.e., an AIFSID might not be an AIFSAID, as shown in the next example and we will give in the latter theorem a condition for this converse to be true.
Hence, \( \Delta = \{ \theta, \vartheta, \kappa \} \) be a set of parameters and consider the IFSS \((Y, \Delta)\) over \(\Omega\). Then \(Y[\theta], Y[\vartheta], \) and \(Y[\kappa] \) are IFSSs over \(\Omega \) represented by:

| \(Y\) | \(0\) | \(p\) | \(q\) | \(r\) | \(s\) |
|------|------|------|------|------|------|
| \(\theta\) | \([0.1, 0.9]\) | \([0.4, 0.4]\) | \([0.3, 0.6]\) | \([0.2, 0.8]\) | \([0.5, 0.1]\) |
| \(\vartheta\) | \([0.0, 0.9]\) | \([0.1, 0.7]\) | \([0.4, 0.4]\) | \([0.3, 0.5]\) | \([0.2, 0.6]\) |
| \(\kappa\) | \([0, 1]\) | \([0.2, 0.6]\) | \([0.3, 0.5]\) | \([0.4, 0.3]\) | \([0.1, 0.7]\) |

Afterwards, \((Y, \Delta)\) is an AIFSID of \(\Omega\), but since
\[
\xi_{Y[\theta]}(p \circ s) = \xi_{Y[\theta]}(q) = 0.4 \leq 0.2 = \xi_{Y[\theta]}((s \circ 0) \circ (0 \circ p)) \lor \xi_{Y[\theta]}(0)
\]
and
\[
\xi_{Y[\theta]}(p \circ s) = \xi_{Y[\theta]}(q) = 0.4 \geq 0.6 = \xi_{Y[\theta]}((s \circ 0) \circ (0 \circ p)) \land \xi_{Y[\theta]}(0),
\]
i.e., \(Y[\theta] = \{(\xi_{Y[\theta](l)}, \xi_{Y[\theta](l)}) : l \in \Omega \) and \(\theta \in \Delta \) is not an AIFAID of \(\Omega\).

Therefore \((Y, \Delta)\) is not an AIFAID of \(\Omega\) with respect to \(\theta\).
Hence \((Y, \Delta)\) is not an AIFSID of \(\Omega\).

**Theorem 2.** Let \((Y, \Delta)\) be an AIFSID over \(\Omega\). If for any \(\omega \in \Delta \) and \(l, m \in \Omega\), \(\xi_{Y[\omega]}(m \circ l) \leq \xi_{Y[\omega]}(l \circ (0 \circ m)) \) and \(\xi_{Y[\omega]}(m \circ l) \geq \xi_{Y[\omega]}(l \circ (0 \circ m))\), then \((Y, \Delta)\) is an AIFSID over \(\Omega\).

**Proof.** Let \((Y, \Delta)\) be an AIFSID over \(\Omega\).
Therefore, \(Y[\omega] = \{(\xi_{Y[\omega](l)}, \xi_{Y[\omega](l)}) : l \in \Omega \) and \(\omega \in \Delta \} \) is an AIFD of \(\Omega\).

Thus, for any \(\omega \in \Delta \) and \(l, m, n \in \Omega\),
\[
\xi_{Y[\omega]}(m \circ l) \leq \xi_{Y[\omega]}(l \circ (0 \circ m))
\]
and
\[
\xi_{Y[\omega]}(m \circ l) \geq \xi_{Y[\omega]}(l \circ (0 \circ m)) \lor \xi_{Y[\omega]}(n)
\]

Thus, \((Y, \Delta)\) is an AIFSID of \(\Omega\).

**Theorem 3.** If \((Y, \Delta)\) is an AIFSID of \(\Omega\), then for any parameter \(\omega \in \Delta \) and \(l, m, n \in \Omega\), \(\xi_{Y[\omega]}((l \circ n) \circ (0 \circ m)) \leq \xi_{Y[\omega]}(l \circ (n \circ m)) \) and \(\xi_{Y[\omega]}((l \circ n) \circ (0 \circ m)) \geq \xi_{Y[\omega]}(l \circ (0 \circ m))\).

**Proof.** Let \((Y, \Delta)\) be an AIFSID of \(\Omega\).

Because \((l \circ n) \circ (0 \circ m) = (l \circ n) \circ ((n \circ m) \circ n) \leq l \circ (n \circ m)\).

Therefore, \((l \circ n) \circ (0 \circ m) \circ (l \circ (n \circ m)) = 0\).

By Theorem 1, \((Y, \Delta)\) is an AIFSID of \(\Omega\).
Thus, \(Y[\omega] = \{(\xi_{Y[\omega](l)}, \xi_{Y[\omega](l)}) : l \in \Omega \) and \(\omega \in \Delta \} \) is an AIFD of \(\Omega\).

Thus, for every \(l, m, n \in \Omega\) and \(\omega \in \Delta\),
\[
\xi_{Y[\omega]}((l \circ n) \circ (0 \circ m)) \leq \xi_{Y[\omega]}(((l \circ n) \circ (0 \circ m)) \circ (l \circ (n \circ m))) \lor \xi_{Y[\omega]}(l \circ (n \circ m))
\]
and
\[
\xi_{Y[\omega]}((l \circ n) \circ (0 \circ m)) \geq \xi_{Y[\omega]}(((l \circ n) \circ (0 \circ m)) \circ (l \circ (n \circ m))) \land \xi_{Y[\omega]}(l \circ (n \circ m))
\]

**Definition 8.** Let \((Y, \Delta)\) and \((\Gamma, \Psi)\) be two IFSSs over \(\Omega\). Then \((Y, \Delta) \text{ “AND” } (\Gamma, \Psi)\) written as \((Y, \Delta) \land (\Gamma, \Psi)\) is \((\Pi, \Delta \times \Psi)\) of \(\Omega\), where \(\Pi[\omega, \omega] = Y[\omega] \cap \Gamma[\omega]\) for all \((\omega, \omega) \in \Delta \times \Psi\).
Theorem 4. If \((Y, \Delta)\) and \((\Gamma, \Psi)\) are two AIFS AIDs of \(\Omega\), then \((\Pi, \Delta \times \Psi)\) is also an AIFS AID of \(\Omega\).

Proof. By definition, \((Y, \Delta) \cap \Gamma = (\Pi, \Delta \times \Psi)\), where
\[
\Pi[\omega, \nu] = Y[\omega] \cap \Gamma[\nu] = \{(\xi_{Y[\omega] \cap \Gamma[\nu]}(l), \xi_{Y[\omega] \cap \Gamma[\nu]}(l)) : l \in \Omega\text{ and } (\omega, \nu) \in \Delta \times \Psi\}.
\]
For any \(l \in \Omega\) and \((\omega, \nu) \in \Delta \times \Psi\),
\[
\xi_{\Pi[\omega, \nu]}(0) = \xi_{Y[\omega] \cap \Gamma[\nu]}(0) = \xi_{Y[\omega]}(0) \vee \xi_{\Gamma[\nu]}(0) \\
\leq \xi_{Y[\omega]}(l) \vee \xi_{\Gamma[\nu]}(l) = \xi_{Y[\omega] \cap \Gamma[\nu]}(l)
\]
and
\[
\xi_{\Pi[\omega, \nu]}(l) = \xi_{Y[\omega] \cap \Gamma[\nu]}(l) = \xi_{Y[\omega]}(l) \wedge \xi_{\Gamma[\nu]}(l)
\]
\[
\xi_{\Pi[\omega, \nu]}(0) \leq \xi_{\Pi[\omega, \nu]}(l) \\
\xi_{\Pi[\omega, \nu]}(l) \geq \xi_{\Pi[\omega, \nu]}(0).
\]
For any \(l, m, n \in \Omega\) and \((\omega, \nu) \in \Delta \times \Psi\),
\[
\xi_{\Pi[\omega, \nu]}(m \circ l) = \xi_{Y[\omega] \cap \Gamma[\nu]}(m \circ l) = \xi_{Y[\omega]}(m \circ l) \vee \xi_{\Gamma[\nu]}(m \circ l) \\
\leq (\xi_{Y[\omega]}(m \circ l)) \vee (\xi_{\Gamma[\nu]}(m \circ l)) \vee (\xi_{Y[\omega]}(m \circ l) \wedge \xi_{\Gamma[\nu]}(m \circ l))
\]
\[
= (\xi_{Y[\omega]}(m \circ l)) \vee (\xi_{\Gamma[\nu]}(m \circ l)) \vee (\xi_{Y[\omega]}(m \circ l) \wedge \xi_{\Gamma[\nu]}(m \circ l))
\]
\[
= \xi_{Y[\omega] \cap \Gamma[\nu]}(m \circ l) \vee \xi_{Y[\omega] \cap \Gamma[\nu]}(m \circ l) \vee \xi_{Y[\omega] \cap \Gamma[\nu]}(m \circ l)
\]
and
\[
\xi_{\Pi[\omega, \nu]}(m \circ l) = \xi_{Y[\omega] \cap \Gamma[\nu]}(m \circ l) = \xi_{Y[\omega]}(m \circ l) \wedge \xi_{\Gamma[\nu]}(m \circ l) \\
\geq (\xi_{Y[\omega]}(m \circ l) \wedge \xi_{\Gamma[\nu]}(m \circ l)) \wedge (\xi_{Y[\omega]}(m \circ l) \wedge \xi_{\Gamma[\nu]}(m \circ l))
\]
\[
= (\xi_{Y[\omega]}(m \circ l) \wedge \xi_{\Gamma[\nu]}(m \circ l)) \wedge (\xi_{Y[\omega]}(m \circ l) \wedge \xi_{\Gamma[\nu]}(m \circ l))
\]
Thus, \(\Pi[\omega, \nu] = Y[\omega] \cap \Gamma[\nu]\) is an AIFS AID of \(\Omega\) for any \((\omega, \nu) \in \Delta \times \Psi\).

Hence, \((\Pi, \Delta \times \Psi)\) is an AIFS AID of \(\Omega\) for any \((\omega, \nu) \in \Delta \times \Psi\).

Definition 9. The "extended intersection" of two IFSSs \((Y, \Delta)\) and \((\Gamma, \Psi)\) denoted by \((Y, \Delta) \cap E (\Gamma, \Psi)\) is \((\Pi, \Theta)\), where \(\Theta = \Delta \cup \Psi\) and for every \(\omega \in \Theta\),
\[
\Pi(\omega) = \begin{cases} 
Y[\omega], & \omega \in \Delta - \Psi, \\
\Gamma[\omega], & \omega \in \Psi - \Delta, \\
Y[\omega] \cap \Gamma[\omega], & \omega \in \Delta \cap \Psi.
\end{cases}
\]

Theorem 5. If \((Y, \Delta)\) and \((\Gamma, \Psi)\) are AIFS AIDs of \(\Omega\), then \((Y, \Delta) \cap E (\Gamma, \Psi)\) is an AIFS AID of \(\Omega\).

Proof. We know that \((Y, \Delta) \cap E (\Gamma, \Psi)\) is \((\Pi, \Theta)\), where \(\Theta = \Delta \cup \Psi\) and for every \(\omega \in \Theta\),
\[
\Pi(\omega) = \begin{cases} 
Y[\omega], & \omega \in \Delta - \Psi, \\
\Gamma[\omega], & \omega \in \Psi - \Delta, \\
Y[\omega] \cap \Gamma[\omega], & \omega \in \Delta \cap \Psi.
\end{cases}
\]
For any \(\omega \in \Theta\), if \(\omega \in \Delta - \Psi\), then \(\Pi(\omega) = Y(\omega)\) is an AIFS AID of \(\Omega\).
Likewise, if \(\omega \in \Psi - \Delta\), \(\Pi(\omega) = \Gamma(\omega)\), which is an AIFS AID of \(\Omega\).
Moreover if \(\omega \in \Theta\), such that \(\omega \in \Delta \cap \Psi\), then \(\Pi(\omega) = Y[\omega] \cap \Gamma[\omega]\) is also an AIFS AID of \(\Omega\).
Therefore, \(\Pi(\omega)\) is an AIFS AID of \(\Omega\).

Hence, \((\Pi, \Theta)\) is an AIFS AID of \(\Omega\).

We deduce the following Corollary.
Corollary 1. The “restricted intersection” of two AIFSAIDs is an AIFSAID.

Definition 10. Let \((Y, \Delta)\) and \((\Gamma, \Psi)\) be two IFSSs over \(\Omega\). Subsequently, the “union” denoted by \((Y, \Delta) \cup (\Gamma, \Psi)\) is \((\Pi, \Theta)\), where \(\Theta = \Delta \cup \Psi\) and for every \(\omega \in \Theta\),

\[
\Pi(\omega) = \begin{cases} 
Y[\omega], & \omega \in \Delta - \Psi, \\
\Gamma[\omega], & \omega \in \Psi - \Delta, \\
Y[\omega] \cup \Gamma[\omega], & \omega \in \Delta \cap \Psi.
\end{cases}
\]

The union of two AIFSAIDs is not necessarily an AIFSAID, as shown in the next example.

Example 3. Let \(\Omega = \{0, p, q, r, s\}\) with Cayley table given by:

| \(\circ\) | 0 | p | q | r | s |
|---|---|---|---|---|---|
| 0 | 0 | 0 | q | r | s |
| p | p | 0 | q | r | s |
| q | q | q | 0 | s | r |
| r | r | r | s | 0 | q |
| s | s | s | r | q | 0 |

Subsequently, \((\Omega; \circ, 0)\) is a BCI-algebra.

Let \(\Delta = \{\theta, \vartheta, \kappa, \delta\}\) and \(\Psi = \{\kappa, \delta, \eta\}\) be two collections of parameters and consider the IFSS \((Y, \Delta)\) over \(\Omega\). Afterwards, \(Y[\theta], Y[\vartheta], Y[\kappa]\) and \(Y[\delta]\) are IFSSs over \(\Omega\) given by:

\[
\begin{array}{cccc}
Y & 0 & p & q & r & s \\
\theta & [0, 0.9] & [0, 0.9] & [0.3, 0.4] & [0.1, 0.4] & [0.3, 0.4] \\
\vartheta & [0.2, 0.6] & [0.2, 0.6] & [0.4, 0.3] & [0.4, 0.3] & [0.3, 0.5] \\
\kappa & [0.1, 0.8] & [0.1, 0.8] & [0.5, 0.2] & [0.5, 0.2] & [0.5, 0.5] \\
\delta & [0.2, 0.7] & [0.2, 0.7] & [0.5, 0.5] & [0.5, 0.3] & [0.5, 0.5]
\end{array}
\]

Then \(Y[\omega]\) is an AIFSAID of \(\Omega\) with respect to \(\theta, \vartheta, \kappa,\) and \(\delta\).

Thus \((Y, \Delta)\) is an AIFSAID of \(\Omega\).

Now let \((\Gamma, \Psi)\) be an IFSS over \(\Omega\). Then \(\Gamma[\kappa], \Gamma[\delta]\) and \(\Gamma[\eta]\) are IFSSs over \(\Omega\) given by:

\[
\begin{array}{cccc}
\Gamma & 0 & p & q & r & s \\
\kappa & [0, 0.7] & [0, 0.7] & [0.3, 0.5] & [0.5, 0.2] & [0.5, 0.2] \\
\delta & [0.2, 0.6] & [0.2, 0.6] & [0.5, 0.2] & [0.5, 0.2] & [0.3, 0.4] \\
\eta & [0.9] & [0.9] & [0.3, 0.4] & [0.1, 0.6] & [0.3, 0.4]
\end{array}
\]

Subsequently, \(\Gamma[\omega]\) is an AIFSAID of \(\Omega\) with respect to \(\kappa, \delta,\) and \(\eta\).

Thus, \((\Gamma, \Psi)\) is an AIFSAID of \(\Omega\).

Note that \((Y, \Delta) \cup (\Gamma, \Psi)\) is not an AIFSAID of \(\Omega\) based on \(\kappa \in \Delta \cap \Psi\). If \(\Delta \cap \Psi = \emptyset\), then the union is an AIFSAID of \(\Omega\) proved in the next theorem.

Theorem 6. Let \((Y, \Delta)\) and \((\Gamma, \Psi)\) be two AIFSAIDs of \(\Omega\). If \(\Delta \cap \Psi = \emptyset\), then \((Y, \Delta) \cup (\Gamma, \Psi) = (\Pi, \Theta)\) is an AIFSAID of \(\Omega\).

Proof. We know that \((Y, \Delta) \cup (\Gamma, \Psi) = (\Pi, \Theta)\), where \(\Theta = \Delta \cup \Psi\) and for every \(\omega \in \Theta\),

\[
\Pi(\omega) = \begin{cases} 
Y[\omega], & \omega \in \Delta - \Psi, \\
\Gamma[\omega], & \omega \in \Psi - \Delta, \\
Y[\omega] \cup \Gamma[\omega], & \omega \in \Delta \cap \Psi.
\end{cases}
\]
Because $\Delta \cap \Psi = \emptyset$, then either $\omega \in \Delta - \Psi$ or $\omega \in \Psi - \Delta$ for all $\omega \in \Theta$.

If $\omega \in \Delta - \Psi$, then $\Pi(\omega) = Y(\omega)$, which is an AIFSAID of $\Omega$.

Thus, $(\Delta, \Psi)$ is an AIFSAID of $\Omega$.

Similarly $\omega \in \Psi - \Delta$, then $\Pi(\omega) = \Gamma(\omega)$ is an AIFSAID of $\Omega$.

Thus, $(\Gamma, \Psi)$ is an AIFSAID of $\Omega$.

Hence, $(\Delta, \Psi) \cup (\Gamma, \Psi)$ is an AIFSAID of $\Omega$. □

**Definition 11.** Let $(\Delta, \Psi)$ be an anti-soft BCI-algebra (abbr. AS_{BCI}A) over $\Omega$. An IFSS $(\Gamma, \Psi)$ over $\Omega$ is an AIFSID of $(\Delta, \Psi)$, denoted by $(\Delta, \Psi)\triangleleft(\Gamma, \Psi)$, if $\Psi \subset \Delta$ and for any $\omega \in \Psi$,

$$\Gamma(\omega) = \{(\xi_{\Gamma(\omega)}(\ell), \xi_{\Delta(\omega)}(\ell)) : \ell \in \Omega\} \cup Y(\omega).$$

**Definition 12.** Let $(\Delta, \Psi)$ be an AS_{BCI}A over $\Omega$. An IFSS $(\Gamma, \Psi)$ over $\Omega$ is an AIFSID of $(\Delta, \Psi)$, denoted by $(\Gamma, \Psi)\triangleleft_{\theta}(\Delta, \Psi)$, if $\Psi \subset \Delta$ and for any $\omega \in \Psi$,

$$\Gamma(\omega) = \{(\xi_{\Gamma(\omega)}(\ell), \xi_{\Delta(\omega)}(\ell)) : \ell \in \Omega\} \cup Y(\omega).$$

**Example 4.** Let $\Omega = \{0, p, q, r, s\}$ with Cayley table:

|   | 0 | p | q | r | s |
|---|---|---|---|---|---|
| 0 | p | p | 0 | q | r |
| p | p | 0 | 0 | q | r |
| q | q | q | 0 | s | r |
| r | r | r | s | 0 | q |
| s | s | s | r | q | 0 |

Subsequently, $(\Omega; \odot, 0)$ is a BCI-algebra.

Let $\Delta = \{\theta, \theta, \kappa\}$ be a set of parameters and let $(\Delta, \Psi)$ be a soft set over $\Omega$ and so let $Y[\theta] = Y[\theta] = \{0, q, r, s\}$, $Y[k] = \{0, q\}$, that are all sub-algebras of $\Omega$.

Hence, $(\Delta, \Psi)$ is an AS_{BCI}A over $\Omega$.

Let $(\Gamma, \Psi)$ be an IFSS over $\Omega$, where $\Psi = \{\theta, \theta\} \subset \Delta$. Afterwards, $\Gamma[\theta]$ and $\Gamma[\kappa]$ are IFSs in $\Omega$ defined by:

|   | 0 | p | q | r | s |
|---|---|---|---|---|---|
| $\theta$ | [0.2, 0.7] | [0.2, 0.7] | [0.2, 0.7] | [0.4, 0.1] | [0.4, 0.1] |
| $\theta$ | [0.3, 0.7] | [0.3, 0.7] | [0.3, 0.7] | [0.5, 0.4] | [0.5, 0.4] |

Afterwards, $\Gamma[\theta] = \{(\xi_{\Gamma[\theta]}(\ell), \xi_{\Delta[\theta]}(\ell)) : \ell \in \Omega\}$ and $\Gamma[\theta] = \{(\xi_{\Gamma[\theta]}(\ell), \xi_{\Delta[\theta]}(\ell)) : \ell \in \Omega\}$ are AIFSAIDs of $\Omega$ related to $\Gamma[\theta]$ and $\Gamma[\kappa]$, respectively.

Hence, $(\Gamma, \Psi)\triangleleft_{\theta}(\Delta, \Psi)$.

Any AIFSID $(\Gamma, \Psi)$ of an AS_{BCI}A $(\Delta, \Psi)$ is an AIFSID of $(\Delta, \Psi)$, but the converse is not true, as proved by the next example.

**Example 5.** Let $\Omega = \{0, p, q, r, s\}$ with Cayley table:

|   | 0 | p | q | r | s |
|---|---|---|---|---|---|
| 0 | p | p | p | 0 | 0 |
| p | p | 0 | 0 | 0 | 0 |
| q | q | q | 0 | q | 0 |
| r | r | r | r | r | 0 |
| s | s | s | r | q | 0 |

Subsequently, $(\Omega; \odot, 0)$ is a “BCK-algebra” and, thus, a “BCI-algebra”.

Let $\Delta = \{\theta, \theta, \kappa, \delta, \eta\}$ be a set of parameters.
Let $(Y, \Delta)$ be a soft set over $\Omega$ and so we let $Y[\theta] = \Omega$, $Y[\vartheta] = Y[\kappa] = \{0, q, r, s\}$ and $Y[\delta] = Y[\eta] = \{0, q\}$, that are all subalgebras of $\Omega$.

Hence, $(Y, \Delta)$ is an $\text{AS}_{\text{BC1}}A$ over $\Omega$.

Suppose that $(\Gamma, \Psi)$ is an IFSS over $\Omega$, where $\Psi = \{\kappa, \delta, \eta\} \subset \Delta$. Afterwards, $\Gamma[\kappa], \Gamma[\delta]$ and $\Gamma[\eta]$ are an IFSs in $\Omega$ represented by:

| $\kappa$ | $\delta$ | $\eta$ |
|----------|----------|--------|
| $[0, 0.7]$ | $[0.1, 0.8]$ | $[0.1, 0.5]$ |
| $[0, 1, 0.6]$ | $[0, 2, 0.7]$ | $[0, 2, 0.4]$ |
| $[0, 2, 0.5]$ | $[0, 3, 0.6]$ | $[0, 3, 0.3]$ |
| $[0, 3, 0.3]$ | $[0, 0.4]$ | $[0, 0.1]$ |
| $[0, 0.4]$ | $[0, 0.4]$ | $[0, 0.1]$ |

Subsequently, $(\Gamma, \Psi)$ is an AIFSId of $(Y, \Delta)$, but since
\[
\zeta_{Y[\kappa]}(r \lor q) = \zeta_{Y[\kappa]}(r) = 0.3 \leq 0.2 = \zeta_{Y[\kappa]}((q \lor 0) \lor (0 \lor r)) \lor \zeta_{Y[\vartheta]}(0)
\]
and
\[
\zeta_{Y[\kappa]}(r \lor q) = \zeta_{Y[\kappa]}(r) = 0.3 \leq 0.5 = \zeta_{Y[\kappa]}((q \lor 0) \lor (0 \lor r)) \lor \zeta_{Y[\vartheta]}(0).
\]

i.e., $\Gamma[\kappa] = \{(\zeta_{\Gamma[\kappa]}(l), \zeta_{\Gamma[\kappa]}(l)) : l \in \Omega\}$ is not an AIFSId of $\Omega$ related to $Y[\kappa]$.

Therefore $(\Gamma, \Psi)$ is not an AIFSId of $\text{AS}_{\text{BC1}}A (Y, \Delta)$.

**Theorem 7.** Let $(Y, \Delta)$ be an $\text{AS}_{\text{BC1}}A$ over $\Omega$. If $(\Gamma, \Psi)$ and $(\Pi, \Lambda)$ are AIFSIds of $(Y, \Delta)$, then the “extended intersection” of $(\Gamma, \Psi)$ and $(\Pi, \Lambda)$ is an AIFSIds of $(Y, \Delta)$.

**Proof.** We know that $(\Gamma, \Psi) \cap_{E} (\Pi, \Lambda) = (\Xi, \Theta)$, where $\Theta = \Psi \cup \Lambda \subset \Delta$ and for every $\omega \in \Theta$,
\[
\Xi(\omega) = \left\{ \begin{array}{l}
\Gamma[\omega], \; \; \; \omega \in \Psi - \Lambda,
\Pi[\omega], \; \; \; \omega \in \Lambda - \Psi,
\Gamma[\omega] \cap \Pi[\omega], \; \; \; \omega \in \Psi \cap \Lambda.
\end{array} \right.
\]

For any $\omega \in \Theta$, if $\omega \in \Psi - \Lambda$, then $\Xi(\omega) = \Gamma[\omega] = \{(\zeta_{\Gamma[\omega]}(l), \zeta_{\Gamma[\omega]}(l)) : l \in \Omega\} \uparrow_{\Delta} Y[\omega]$, since $(\Gamma, \Psi) \uparrow_{\Delta} (Y, \Delta)$.

Likewise, if $\omega \in \Lambda - \Psi$, then $\Xi(\omega) = \Pi[\omega] = \{(\zeta_{\Pi[\omega]}(l), \zeta_{\Pi[\omega]}(l)) : l \in \Omega\} \uparrow_{\Delta} Y[\omega]$, since $(\Pi, \Lambda) \uparrow_{\Delta} (Y, \Delta)$.

Moreover if $\omega \in \Theta$, such that $\omega \in \Psi \cap \Lambda$, then $\Xi(\omega) = \Gamma[\omega] \cap \Pi[\omega] = \{(\zeta_{\Gamma[\omega]}(l) \lor \zeta_{\Pi[\omega]}(l)), (\zeta_{\Gamma[\omega]}(l) \lor \zeta_{\Pi[\omega]}(l))\} \uparrow_{\Delta} Y[\omega]$. Therefore, $\Xi(\omega) \uparrow_{\Delta} Y[\omega]$ for any $\omega \in \Theta$.

Hence, $(\Xi, \Theta) = (\Gamma, \Psi) \cap_{E} (\Pi, \Lambda) \uparrow_{\Delta} (Y, \Delta)$. □

Next corollary follows directly.

**Corollary 2.** Let $(\Gamma, \Psi)$ and $(\Pi, \Lambda)$ be two AIFSIds of an $\text{AS}_{\text{BC1}}A (Y, \Delta)$. If $\Psi \cap \Lambda = \emptyset$, then the “union” $(\Gamma, \Psi) \cup_{\Delta} (\Pi, \Lambda)$ is an AIFSId of $(Y, \Delta)$.

4. Characterization of Anti-Intuitionistic Fuzzy Soft a-Ideals

In this section, we give characterizations of an AIFSId $(Y, \Delta)$ over $\Omega$ while using the idea of a soft $(\gamma, \nu)$-level set, $L(Y[\omega]; \gamma; \nu) = \{l \in \Omega \mid \zeta_{Y[\omega]}(l) \leq \gamma$ and $\zeta_{Y[\omega]}(l) \geq \gamma, \nu \in [0, 1]\}$.

**Theorem 8.** An AIFS $(Y, \Delta)$ over $\Omega$ is an AIFSId over $\Omega$ if and only if the non-empty soft $(\gamma, \nu)$-level set, $L(Y[\omega]; \gamma; \nu) = \{l \in \Omega \mid \zeta_{Y[\omega]}(l) \leq \gamma$ and $\zeta_{Y[\omega]}(l) \geq \nu \}$ is an a-ideal of $\Omega$, for any $\omega \in \Delta$ and $\gamma, \nu \in [0, 1]$.

**Proof.** Let $(Y, \Delta)$ be an AIFSId over $\Omega$.

Afterwards, $Y[\omega] = \{(\zeta_{Y[\omega]}(l), \zeta_{Y[\omega]}(l)) : l \in \Omega\}$ is an AIFSId of $\Omega$, for any $\omega \in \Delta$.

Let $L(Y[\omega]; \gamma; \nu) = \{l \in \Omega \mid \zeta_{Y[\omega]}(l) \leq \gamma$ and $\zeta_{Y[\omega]}(l) \geq \nu \} \neq \emptyset$, for any $\omega \in \Delta$ and $\gamma, \nu \in [0, 1]$.


Subsequently, for any \( l \in L(Y[\omega]; \gamma; \nu) \),
\[
\xi_{Y[\omega]}(0) \leq \xi_{Y[\omega]}(l) \leq \gamma \text{ and } \xi_{Y[\omega]}(0) \geq \xi_{Y[\omega]}(l) \geq \nu,
\]
i.e., \( 0 \in L(Y[\omega]; \gamma; \nu) \).

Now, let \((l \odot n) \odot (0 \odot m) \in L(Y[\omega]; \gamma; \nu)\) and \(n \in L(Y[\omega]; \gamma; \nu)\), for any \( l, m, n \in \Omega \).

Subsequently, \( \xi_{Y[\omega]}((l \odot n) \odot (0 \odot m)) \leq \gamma, \xi_{Y[\omega]}(n) \leq \gamma \)
and
\[
\xi_{Y[\omega]}((l \odot n) \odot (0 \odot m)) \geq \nu, \xi_{Y[\omega]}(n) \geq \nu.
\]
Thus, for any \( l, m, n \in \Omega \),
\[
\xi_{Y[\omega]}(m \odot l) \leq \xi_{Y[\omega]}((l \odot n) \odot (0 \odot m)) \lor \xi_{Y[\omega]}(n) \leq \gamma.
\]
\[
\xi_{Y[\omega]}(m \odot l) \geq \xi_{Y[\omega]}((l \odot n) \odot (0 \odot m)) \land \xi_{Y[\omega]}(n) \geq \nu.
\]
i.e., \( m \odot l \in L(Y[\omega]; \gamma; \nu) \).

Hence, \( L(Y[\omega]; \gamma; \nu) \neq \emptyset \) is an a-ideal of \( \Omega \), for any \( \omega \in \Delta \) and \( \gamma, \nu \in [0, 1] \).

Conversely assume that \( L(Y[\omega]; \gamma; \nu) \) is an a-ideal of \( \Omega \), for any \( \omega \in \Delta \) and \( \gamma, \nu \in [0, 1] \).

If for some \( l_0 \in \Omega \) and \( \omega_0 \in \Delta \), \( \xi_{Y[\omega_0]}(0) > \xi_{Y[\omega_0]}(l_0) \) and \( \xi_{Y[\omega_0]}(0) < \xi_{Y[\omega_0]}(l_0) \), then \( \xi_{Y[\omega_0]}(0) > \gamma_0 \geq \xi_{Y[\omega_0]}(l_0) \) and \( \xi_{Y[\omega_0]}(0) < \nu_0 \leq \xi_{Y[\omega_0]}(l_0) \), for some \( \gamma_0, \nu_0 \in [0, 1] \).

This implies that \( l_0 \in L(Y[\omega_0]; \gamma_0; \nu_0) \) and that \( 0 \notin L(Y[\omega_0]; \gamma_0; \nu_0) \), this contradicts the hypothesis that \( L(Y[\omega_0]; \gamma_0; \nu_0) \) is an a-ideal of \( \Omega \).

Thus \( \xi_{Y[\omega]}(0) \leq \xi_{Y[\omega]}(l) \) and \( \xi_{Y[\omega]}(0) \geq \xi_{Y[\omega]}(l) \), for any \( \omega \in \Delta \) and \( l \in \Omega \).

Moreover, if there are elements \( l_0, m_0, n_0 \in \Omega \) and \( \omega_0 \in \Delta \), such that
\[
\xi_{Y[\omega_0]}(m_0 \odot l_0) > \xi_{Y[\omega_0]}((l_0 \odot n_0) \odot (0 \odot m_0)) \lor \xi_{Y[\omega_0]}(n_0)
\]
and
\[
\xi_{Y[\omega_0]}(m_0 \odot l_0) < \xi_{Y[\omega_0]}((l_0 \odot n_0) \odot (0 \odot m_0)) \land \xi_{Y[\omega_0]}(n_0).
\]

Afterwards, for some \( \gamma_0, \nu_0 \in [0, 1] \),
\[
\xi_{Y[\omega_0]}(m_0 \odot l_0) > \gamma_0 \geq \xi_{Y[\omega_0]}((l_0 \odot n_0) \odot (0 \odot m_0)) \lor \xi_{Y[\omega_0]}(n_0)
\]
and
\[
\xi_{Y[\omega_0]}(m_0 \odot l_0) < \nu_0 \leq \xi_{Y[\omega_0]}((l_0 \odot n_0) \odot (0 \odot m_0)) \land \xi_{Y[\omega_0]}(n_0).
\]
i.e., \( m_0 \odot l_0 \notin L(Y[\omega_0]; \gamma_0; \nu_0) \), again a contradiction.

Thus, for any \( l, m, n \in \Omega \) and for any \( \omega \in \Delta \),
\[
\xi_{Y[\omega]}(m \odot l) \leq \xi_{Y[\omega]}((l \odot n) \odot (0 \odot m)) \lor \xi_{Y[\omega]}(n)
\]
and
\[
\xi_{Y[\omega]}(m \odot l) \geq \xi_{Y[\omega]}((l \odot n) \odot (0 \odot m)) \land \xi_{Y[\omega]}(n)
\]
i.e., \( Y[\omega] = \{(\xi_{Y[\omega]}(l), \xi_{Y[\omega]}(l)) \mid l \in \Omega \} \) is an \textit{AIFSAID} of \( \Omega \), for any \( \omega \in \Delta \).

Hence, \((Y, \Delta)\) is an \textit{AIFSAID} over \( \Omega \).

From the above theorem we get the following corollary.

**Corollary 3.** An \textit{AIFSS} \((Y, \Delta)\) over \( \Omega \) is an \textit{AIFSAID} over \( \Omega \) \(\iff\) the non-empty soft \((\gamma, \nu)\)-level set,
\( L(Y[\omega]; \gamma; \nu) = \{ l \in \Omega \mid \xi_{Y[\omega]}(l) \leq \gamma \text{ and } \xi_{Y[\omega]}(l) \geq \nu \} \), is an a-ideal of \( \Omega \), for any \( \omega \in \Delta \) and \( \gamma, \nu \in (1/2, 1] \).

**Theorem 9.** A non-empty soft \((\gamma, \nu)\)-level set, \( L(Y[\omega]; \gamma; \nu) = \{ l \in \Omega \mid \xi_{Y[\omega]}(l) \leq \gamma \text{ and } \xi_{Y[\omega]}(l) \geq \nu \} \), is an a-ideal of \( \Omega \), for any \( \omega \in \Delta \) and \( \gamma, \nu \in (1/2, 1] \) \(\iff\) the following conditions hold:

\( l \) \( (\xi_{Y[\omega]}(0) \lor 1/2) \leq \xi_{Y[\omega]}(l) \) \(\lor \) \( (\xi_{Y[\omega]}(0) \lor 1/2) \geq \xi_{Y[\omega]}(l) \),

\( m \) \( (\xi_{Y[\omega]}(m \odot l) \lor 1/2) \leq \xi_{Y[\omega]}(l \odot n) \odot (0 \odot m) \lor \xi_{Y[\omega]}(n) \),

\( n \) \( (\xi_{Y[\omega]}(m \odot l) \lor 1/2) \leq \xi_{Y[\omega]}((l \odot n) \odot (0 \odot m)) \land \xi_{Y[\omega]}(n) \),

for any \( \omega \in \Delta \) and \( l, m, n \in \Omega \).

**Proof.** Let the non-empty soft \((\gamma, \nu)\)-level set, \( L(Y[\omega]; \gamma; \nu) = \{ l \in \Omega \mid \xi_{Y[\omega]}(l) \leq \gamma \text{ and } \xi_{Y[\omega]}(l) \geq \nu \} \), be an a-ideal of \( \Omega \), for any \( \omega \in \Delta \) and \( \gamma, \nu \in (1/2, 1] \).

If for some \( l_0 \in \Omega \) and \( \omega_0 \in \Delta \),
\( (\xi_{Y[\omega_0]}(0) \lor 1/2) > \xi_{Y[\omega_0]}(l_0) \) and \( (\xi_{Y[\omega_0]}(0) \lor 1/2) < \xi_{Y[\omega_0]}(l_0) \).
Then there are \( \gamma_0, \nu_0 \in (1/2, 1] \), such that
\[
(\xi_{[m]_0}(0) \lor 1/2) > \gamma_0 \geq \xi_{[m]_0}(l_0) \text{ and } (\xi_{[m]_0}(0) \lor 1/2) < \nu_0 \leq \xi_{[m]_0}(l_0).
\]
This implies that \( \xi_{[m]_0}(0) > \gamma_0 \geq \xi_{[m]_0}(l_0) \) and \( \xi_{[m]_0}(0) \lor 1/2 < \nu_0 \leq \xi_{[m]_0}(l_0) \).

Moreover, for some \( l_0, m_0, n_0 \in \Omega \) and \( \omega_0 \in \Delta \), such that
\[
(\xi_{[m]_0}(m_0 \lor l_0) \lor 1/2) > \gamma_0 \geq \xi_{[m]_0}((l_0 \lor n_0) \lor (0 \lor m_0)) \lor \xi_{[m]_0}(n_0)
\]
and
\[
(\xi_{[m]_0}(m_0 \lor l_0) \lor 1/2) < \gamma_0 \geq \xi_{[m]_0}((l_0 \lor n_0) \lor (0 \lor m_0)) \land \xi_{[m]_0}(n_0).
\]

Hence, (i) is valid.

Thus, (i) is valid.

Moreover, if there are elements \( l_0, m_0, n_0 \in \Omega \) and \( \omega_0 \in \Delta \), such that
\[
(\xi_{[m]_0}(m_0 \lor l_0) \lor 1/2) > \gamma_0 \geq \xi_{[m]_0}((l_0 \lor n_0) \lor (0 \lor m_0)) \lor \xi_{[m]_0}(n_0)
\]
and
\[
(\xi_{[m]_0}(m_0 \lor l_0) \lor 1/2) < \gamma_0 \geq \xi_{[m]_0}((l_0 \lor n_0) \lor (0 \lor m_0)) \land \xi_{[m]_0}(n_0).
\]

Subsequently, for some \( \gamma_0, \nu_0 \in (1/2, 1] \),
\[
(\xi_{[m]_0}(m_0 \lor l_0) \lor 1/2) > \gamma_0 \geq \xi_{[m]_0}((l_0 \lor n_0) \lor (0 \lor m_0)) \lor \xi_{[m]_0}(n_0)
\]
and
\[
(\xi_{[m]_0}(m_0 \lor l_0) \lor 1/2) < \gamma_0 \geq \xi_{[m]_0}((l_0 \lor n_0) \lor (0 \lor m_0)) \land \xi_{[m]_0}(n_0).
\]

Thus, (ii) and (iii) are valid.

Conversely, suppose that the conditions (i), (ii), and (iii) are valid.

Let \( L(Y[\omega]; \gamma; \nu) = \{ l \in \Omega \mid \xi_{[m]_0}(l) \leq \gamma \text{ and } \xi_{[m]_0}(l) \geq \nu \} \neq \emptyset \), for any \( \omega \in \Delta \) and \( \gamma, \nu \in (1/2, 1] \).

Subsequently, for any \( l \in L(Y[\omega]; \gamma; \nu) \),
\[
(\xi_{[m]_0}(0) \lor l) \leq \gamma \text{ and } \xi_{[m]_0}(0) \lor l \geq \nu.
\]

which implies \( \xi_{[m]_0}(0) \leq \gamma \) and \( \xi_{[m]_0}(0) \geq \nu. \)

Thus, \( 0 \in L(Y[\omega]; \gamma; \nu) \).

Now let \( (l \lor n) \lor (0 \lor m) \in L(Y[\omega]; \gamma; \nu) \) and \( n \in L(Y[\omega]; \gamma; \nu) \), for any \( l, m, n \in \Omega \).

Subsequently, \( \xi_{[m]_0}((l \lor n) \lor (0 \lor m)) \leq \gamma, \xi_{[m]_0}(n) \leq \gamma \)

and
\[
\xi_{[m]_0}((l \lor n) \lor (0 \lor m)) \geq \gamma, \xi_{[m]_0}(n) \geq \gamma.
\]

Thus, (ii), we get,
\[
(\xi_{[m]_0}(m \lor l) \lor 1/2) \leq \xi_{[m]_0}((l \lor n) \lor (0 \lor m)) \lor \xi_{[m]_0}(n) \leq \gamma
\]
and
\[
(\xi_{[m]_0}(m \lor l) \lor 1/2) \geq \xi_{[m]_0}((l \lor n) \lor (0 \lor m)) \land \xi_{[m]_0}(n) \geq \gamma.
\]

This implies, \( \xi_{[m]_0}(m \lor l) \leq \gamma \) and \( \xi_{[m]_0}(m \lor l) \geq \gamma. \)

Thus, \( m \lor l \in L(Y[\omega]; \gamma; \nu) \).

Therefore, \( L(Y[\omega]; \gamma; \nu) \) is an a-ideal of \( \Omega \), for any \( \omega \in \Delta \) and \( \gamma, \nu \in (1/2, 1] \).

5. Conclusions

The notion of anti-intuitionistic fuzzy soft a-ideal (abbr. AIFSAID) is introduced and studied over a BCI-algebra \( \Omega \). We proved that any AIFSAID is an anti-intuitionistic fuzzy soft ideal (abbr. AIFS) of \( \Omega \) and that the converse is not always true. We proved that the operations “AND”, “extended intersection”, and “restricted intersection” between any two AIFSAIDs of \( \Omega \), is also an AIFSAID of \( \Omega \) whereas the “union” is not necessarily an AIFSAID. Moreover, characterizations of AIFSAID using the concept of a soft level set were given.
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