GENERALIZED JACOBIANS AND EXPLICIT DESCENTS

BRENDAN CREUTZ

Abstract. We develop a cohomological description of various explicit descents in terms of generalized Jacobians, generalizing the known description for hyperelliptic curves. Specifically, given an integer \( n \) dividing the degree of some reduced effective divisor \( m \) on a curve \( C \), we show that multiplication by \( n \) on the generalized Jacobian \( J_m \) factors through an isogeny \( \varphi : A_m \to J_m \) whose kernel is naturally the dual of the Galois module \((\text{Pic}(C_k)/m)[n]\). By geometric class field theory, this corresponds to an abelian covering of \( C_k := C \times_{\text{Spec}k} \text{Spec}(k) \) of exponent \( n \) unramified outside \( m \). The \( n \)-coverings of \( C \) parameterized by explicit descents are the maximal unramified subcoverings of the \( k \)-forms of this ramified covering.

1. Introduction

Suppose \( f(x, y) \) is a binary form of degree \( d \) over a field \( k \) of characteristic not equal to 2. Pencils of quadrics with discriminant form \( f(x, y) \) have been studied in \cite{BSD63, Cas62, Cre01, BG, Wan, BGW, BGWb}. When \( d \) is even, the \( \text{SL}_d(k)/\mu_2 \)-orbits of pairs \((A, B)\) with discriminant form \( f(x, y) \) correspond to a collection of 2-coverings of the hyperelliptic curve \( C : z^2 = f(x, y) \). When \( k = \mathbb{Q} \) these coverings are used in \cite{Bha} and \cite{BGWb} to compute the average size of the 2-Selmer set of \( C \), and of the torsor \( J^1 \) parameterizing divisor classes of degree 1, respectively, from which they deduce the fantastic result that most hyperelliptic curves over \( \mathbb{Q} \) have no rational points.

The same collection of coverings can also be described in terms of the \( k \)-algebra \( L := k[x]/f(x, 1) \). This description was used in \cite{BS09} and \cite{Cre13} to compute 2-Selmer sets of \( C \) and \( J^1 \), respectively, for individual hyperelliptic curves. A key step in both \cite{Cre13} and \cite{BGWb} is to check that this collection of coverings is large enough to contain the locally soluble 2-coverings (under suitable hypotheses on \( C \)). In \cite{BGWb} this is achieved by identifying the collection of coverings as the unramified subcoverings of \( k \)-forms of the maximal abelian covering of exponent 2 unramified outside the pair of points at infinity on the affine model of \( z^2 = f(x, y) \), a characterization that is quite natural in light of the use of generalized Jacobians in \cite{PS97}.

Meanwhile the theory of explicit descents has expanded to incorporate descriptions of Selmer sets for all curves (the case of non-hyperelliptic curves of genus at least 2 in \cite{BPS} and curves of genus 1 in \cite{Cre14}). In this paper we develop a cohomological description of these descents in terms of generalized Jacobians, generalizing the description for hyperelliptic curves given in \cite{PS97}. Specifically, given an integer \( n \) dividing the degree of some reduced effective divisor \( m \) on a curve \( C \), we show that multiplication by \( n \) on the generalized Jacobian \( J_m \) factors through an isogeny \( \varphi : A_m \to J_m \) whose kernel is naturally the dual of the Galois module \((\text{Pic}(C_k)/m)[n]\). By geometric class field theory, this corresponds to an abelian covering of \( C_k := C \times_{\text{Spec}k} \text{Spec}(k) \) of exponent \( n \) unramified outside \( m \). The \( n \)-coverings of \( C \) parameterized by the explicit descents mentioned above are the maximal unramified subcoverings of the \( k \)-forms of this ramified covering.
This description unifies the methods of explicit descent on curves (and/or their \( J^1 \)) described in \[MSS96, Sta05, BS09, Cre13, Cre14, BPS\], and provides new insights into the descents on the corresponding Jacobians (For example, Lemma 2.9, Corollary 2.11 and Remark 3.4). It also allows us to show that the corresponding collection of coverings of \( J^1 \) contains the locally solvable coverings and, in particular, that the ‘descent map’ in \[BPS\] could be used to compute 2-Selmer sets of \( J^1 \) for non-hyperelliptic curves of genus \( \geq 2 \).

We expect this will be of relevance to future efforts to compute these Selmer sets on average. Namely, it should be possible to identify this collection of coverings with the orbits in some coregular representation (as is done in \[BGWD\] for the hyperelliptic case). The results in Propositions 3.10 and 3.12 would then have implications for the structure of the space of orbits. Thorne has recently made progress understanding the situation for non-hyperelliptic genus 3 curves with a marked rational point \[Tho15, Tho\]. It is our hope that the results of this paper may shed light on the corresponding situation when there are no rational points.

1.1. Notation. Throughout this paper \( n \) is an integer and \( k \) is a field of characteristic not divisible by \( n \), with separable closure \( \overline{k} \) and absolute Galois group \( \Gal_k = \Gal(\overline{k}/k) \). We will use \( C \) to denote a nice curve over \( k \), i.e. a smooth, projective and geometrically integral \( k \)-variety of dimension 1. For a nonempty finite étale \( k \)-scheme \( \Delta = \Spec(L) \) we use \( \Res_{\Delta} = \Res_{L/k} \) to denote the restriction of scalars functor taking \( L \)-schemes to \( k \)-schemes.

## 2. The modulus setup

**Definition 2.1.** Let \( C \) be a nice curve over \( k \). A **modulus setup** for \( C \) is a pair \((n, m)\) consisting of a positive integer \( n \) not divisible by the characteristic of \( k \), and a reduced effective divisor \( m \in \text{Div}(C) \) of degree \( m \), with \( n \) dividing \( m \).

Given a modulus setup \((n, m)\) we define \( \ell := \deg(m)/n \). We are primarily interested in the following examples.

- **M.1** Suppose \( \pi : C \to \mathbb{P}^1 \) is a double cover which is not ramified over \( \infty \in \mathbb{P}^1 \). Let \( n = 2 \) and \( m = \pi^*\infty \).
- **M.2** Suppose \( C \) is a plane cubic curve. Let \( n = 3 \) and let \( m \) be any triple of distinct collinear points.
- **M.3** Generalizing the previous example, suppose \( C \) is a genus one curve of degree \( m \) in \( \mathbb{P}^{n-1} \). Take \( m \) to be any reduced hyperplane section and take \( n \) to be a divisor of \( m \).
- **M.4** Suppose \( C \) is a plane quartic curve. Let \( n = 2 \) and let \( m \) be any quadruple of distinct collinear points.
- **M.5** Generalizing the previous example, suppose \( C \) is any nice curve, \( n = 2 \) and \( m \) is a canonical divisor. Then \( m = 2g - 2 \) and \( \ell = g - 1 \).

We may view \( m \) as a finite étale subscheme \( m = \Spec M \subset C \), or as a modulus in the sense of geometric class field theory (see \[Ser88\]). Let \( C_m \) denote the singular curve associated to \( m \) as in \[Ser88\] IV.4. Let \( \Pic_C \) and \( \Pic_{C_m} \) be the commutative group schemes over \( k \) representing the Picard functors of \( C \) and \( C_m \). There is an exact sequence of commutative group schemes over \( k \),

\[
0 \to T \to \Pic_{C_m} \to \Pic_C \to 0,
\]
where $T$ is an algebraic torus. The restriction of (2.1) to the identity components is an exact sequence of semiabelian varieties,

\begin{equation}
1 \to T \to J_m \to J \to 0,
\end{equation}

where $J_m$ is the generalized Jacobian of $C$ associated to the modulus $m$ and $J$ is the usual Jacobian of $C$.

**Lemma 2.2.** $T = \text{Res}_m \mathbb{G}_m / \mathbb{G}_m$ is the quotient by the diagonal embedding of $\mathbb{G}_m$, and there is an exact sequence of finite group schemes

\begin{equation}
1 \to \text{Res}_m^1 \mu_n \to T[n] \to \mu_n \to 1,
\end{equation}

where the map $N$ is induced by the norm map $\text{Res}_m \mathbb{G}_m \to \mathbb{G}_m$ and $\text{Res}_m^1 \mu_n$ is the kernel of $N : \text{Res}_m \mu_n \to \mu_n$.

**Proof.** The first statement, that $T = \text{Res}_m \mathbb{G}_m / \mathbb{G}_m$, follows from well known results on the structure of generalized Jacobians (see [Ser88 §V Prop. 7]). The inclusion map $\text{Res}_1^1 \mu_n \mathbb{G}_m \to \text{Res}_m \mathbb{G}_m$ induces a surjective map onto $\text{Res}_m \mathbb{G}_m / \mathbb{G}_m$ with kernel $\mu_n$. This gives the middle rows of the following commutative and exact diagram.

\begin{center}
\begin{tikzcd}
\mu_n \arrow{r} \arrow{d} & \text{Res}_m^1 \mu_n \arrow{r} \arrow{d} & T[n] \arrow{d} \\
1 \arrow{r} \arrow{d} & \mu_m \arrow{r} \arrow{d}{n} & \text{Res}_m \mathbb{G}_m \arrow{r} \arrow{d}{n} & \text{Res}_m \mathbb{G}_m \arrow{r} \arrow{d}{n} & 1 \\
1 \arrow{r} \arrow{d} & \mu_m \arrow{r} \arrow{d}{m/n} & \text{Res}_m \mathbb{G}_m \arrow{r} \arrow{d}{n} & \text{Res}_m \mathbb{G}_m \arrow{r} \arrow{d}{n} & 1 \\
\mu_n \arrow{r} & 1
\end{tikzcd}
\end{center}

The exact sequence in the statement of the lemma follows by applying the snake lemma. \qed

2.1. The isogeny associated to a modulus setup.

**Lemma 2.3.** Given a modulus setup $(n, m)$ there is a commutative group scheme $\mathfrak{A}$ over $k$ and isogenies $\psi : \text{Pic}_{C_m} \to \mathfrak{A}$ and $\varphi : \mathfrak{A} \to \text{Pic}_{C_m}$ such that $\text{ker}(\psi) = \text{Res}_m^1 \mu_n \mathbb{G}_m \subset T[n]$ and $\varphi \circ \psi = [n]$. Moreover, we have a commutative and exact diagram

\begin{equation}
1 \to T' \to \mathfrak{A} \to \text{Pic}_{C_m} \to 0 \\
1 \to T \to \text{Pic}_{C_m} \to \text{Pic}_{C_m} \to 0
\end{equation}

where $T'$ is a torus and $T'[\varphi] \simeq \mu_n$.

**Proof.** By Lemma 2.2 Pic$_{C_m}$ contains a finite group scheme isomorphic to $\text{Res}_m^1 \mu_n$. The quotient of Pic$_{C_m}$ by this subgroup scheme yields an isogeny $\psi : \text{Pic}_{C_m} \to \mathfrak{A}$. The existence of $\varphi$ follows from the fact that ker$(\psi)$ is contained in the kernel of multiplication by $n$. Since
ker(ψ) ⊂ T, A is an extension of PicC. The rest of the assertions follow from the diagram in Lemma 2.3.

\[ \text{Remark 2.4. When } n = m = \text{deg}(m) = 2, \text{ we have that } T[n] \cong \mu_n. \text{ Hence } \psi \text{ is the identity map on } A = \text{Pic}_{C_m} \text{ and } \varphi \text{ is multiplication by } 2. \]

2.2. Description using divisor classes. A function \( f \in k(C) \setminus \{ 0 \} \) that is regular away from \( m \) gives, by evaluation, an element \( f(m) \in M = \text{Spec } m \). We use \( \text{Div}_m(C) \) to denote the divisors of \( C \) that have support disjoint from \( m \).

\[ \text{Lemma 2.5. Let } A \text{ as defined in Lemma 2.3. Then} \]
\[ \text{Pic}_C(k) = \text{Div}(C)/(\text{div}(f) : f \in k(C) \setminus \{ 0 \}), \]
\[ \text{Pic}_{C_m}(k) = \text{Div}_m(C)/(\text{div}(f) : f \in k(C) \setminus \{ 0 \}, f(m) = 1), \]
\[ A(k) = \text{Div}_m(C)/(\text{div}(f) : f \in k(C) \setminus \{ 0 \}, f(m) \in \text{Res}_m^1 \mu_n). \]

Moreover, \( \varphi : A \rightarrow \text{Pic}_{C_m} \) is induced by multiplication by \( n \) on \( \text{Div}_m(C) \).

\[ \text{Proof. The first two statements are well known (see [Ser88]; note that } f(m) = 1 \text{ if and only if } f \equiv 1 \mod m, \text{ since } m \text{ is reduced). The } k \text{-points of the subgroup } T = \text{Res}_m G_m G_m \subset \text{Pic}_{C_m} \text{ are represented by divisors of functions which do not vanish on } m, \]
\[ T(k) = \{ \text{div}(f) : f \in k(C), f(m) \in \overline{M} \setminus \{ 0 \}, f(m) = 1 \}. \]

The description of \( A(k) \) in the statement then follows from the fact that \( A \) is the quotient of \( \text{Pic}_{C_m} \) by the image of \( \text{Res}_m^1 \mu_n \) in \( T \). The final statement follows easily from the fact that \( \varphi \circ \psi \) is multiplication by \( n \) on \( \text{Pic}_{C_m} \). \qed

2.3. Component groups. The component groups of \( \text{Pic}_{C_m}, \text{Pic}_C \) and \( A \) are all isomorphic to \( \mathbb{Z} \), the isomorphism being given by the degree map on divisor classes. The degree 0 component of \( A \) is a semiabelian variety \( A_m \) fitting into an exact sequence,
\[ 1 \rightarrow T' \rightarrow A_m \rightarrow J \rightarrow 0. \]

In particular, \( A_m \) is geometrically connected.

We label the components
\[ \text{Pic}_C = \bigsqcup_{i \in \mathbb{Z}} J^i, \quad \text{Pic}_{C_m} = \bigsqcup_{i \in \mathbb{Z}} J^i_m, \quad A = \bigsqcup_{i \in \mathbb{Z}} A^i_m, \]
so that the superscripts denote the image under the degree map. To ease notation we also denote the degree 0 components by \( J = J^0, J_m = J^0_m \) and \( A_m = A^0_m \). For any \( i \in \mathbb{Z} \), \( J^i \) and \( J^i_m \) are torsors under \( J \) and \( J_m \), respectively.

Let \( m' \in \text{Div}_m(C) \) be an effective reduced \( k \)-rational divisor linearly equivalent to and with disjoint support from \( m \) (which exists by Bertini’s theorem, provided \( k \) has sufficiently many elements). Each of the quotient groups
\[ J := \frac{\text{Pic}_C}{\mathbb{Z}m'}, \quad J_m := \frac{\text{Pic}_{C_m}}{\mathbb{Z}m'}, \quad A_m := \frac{A}{\mathbb{Z}m'} = \bigsqcup_{i = 0}^{m-1} A^i_m. \]
Remark 2.8. The pairing in Lemma 2.6 induces nondegenerate Galois equivariant pairings.

Proof. We will show that the orthogonal complements of $\text{Res}_m\mu_n$ and $T[n]$ with respect to $e$ are $\mathcal{J}_m[n]$ and $J_m[n]$, respectively. This is enough to ensure that $e$ induces the pairings stated. The pairing induced on $J_m[n]$ is evidently the Weil pairing (see Appendix A.4 in the appendix), which is known to be nondegenerate. Nondegeneracy of the other pairings follows.

Let $D_1 \in T[n]$. Then $\mathcal{D}_1$ is represented by $D_1 = \text{div}(f)$ for some $f \in k(C_T)^\times$ with $f(m) \in \text{Res}_m\mu_n$. Since $nD_1 = \text{div}(f^n)$ and $f^n(m) = 1$ we must use $h_1 = f^n$ in the definition.

2.4. Extended Weil pairings. We now define a Galois equivariant and nondegenerate bilinear pairing $e : \mathcal{A}_m[\varphi] \times \mathcal{A}_m[\varphi] \to \mu_n$ that induces nondegenerate pairings on $\mathcal{A}_m[\varphi] \times \mathcal{J}_m[n]$ and $\mathcal{J}_m[n] \times \mathcal{J}_m[n]$ via the maps in (2.6).

To begin, define a pairing on $\mathcal{J}_m[n]$ as follows. Fix $f \in k(C)^\times$ such that $\text{div}(f) = m' - m$. Given $D_1, D_2 \in \mathcal{J}_m[n]$, choose representative divisors $D_1, D_2 \in \text{Div}_m(C_T)$, and let $d_i = \deg(D_i)/\ell$. There exist unique functions $h'_i \in k(C_T)^\times$ such that $nD_i = \text{div}(h_i) + d_i m'$ and $h'(m) = 1$. Set $h_i = f^{d_i}h'_i$, so that $nD_i = \text{div}(h_i) + d_i m$. Define:

$$e(D_1, D_2) = (-1)^{d_1d_2} \prod_{P \in C_T} (-1)^{n(\text{ord}_P D_1)(\text{ord}_P D_2)} \frac{h_2^{\text{ord}_P D_1}}{h_1^{\text{ord}_P D_2}}(P) \in k^\times.$$

Lemma 2.6. This gives a Galois equivariant bilinear pairing $e : \mathcal{J}_m[n] \times \mathcal{J}_m[n] \to \mu_n$.

Proof. This may be checked exactly as in [PS97, Section 7] (one need only replace the function $x$ there with the function $f$ in the definition above).

Lemma 2.7. The pairing in Lemma 2.6 induces nondegenerate Galois equivariant pairings,

$$e : \mathcal{A}_m[\varphi] \times \mathcal{A}_m[\varphi] \to \mu_n,$$

$$e : \mathcal{A}_m[\varphi] \times \mathcal{J}_m[n] \to \mu_n,$$

$$e : \mathcal{J}_m[n] \times \mathcal{J}_m[n] \to \mu_n.$$

The induced pairing on $\mathcal{J}_m[n] \times \mathcal{J}_m[n]$ coincides with the Weil pairing.

Remark 2.8. The definition of $e$ given above depends on the choice of $m'$ in (2.3) and the function $f$ with $\text{div}(f) = m' - m$. However, as shown in the proof below, the induced pairings on $\mathcal{A}_m[\varphi] \times \mathcal{J}_m[n]$ and $\mathcal{J}_m[n] \times \mathcal{J}_m[n]$ do not depend on these choices.

Proof. We will show that the orthogonal complements of $\text{Res}_m\mu_n$ and $T[n]$ with respect to $e$ are $\mathcal{J}_m[n]$ and $J_m[n]$, respectively. This is enough to ensure that $e$ induces the pairings stated. The pairing induced on $J_m[n]$ is evidently the Weil pairing (see Appendix A.4 in the appendix), which is known to be nondegenerate. Nondegeneracy of the other pairings follows.
of the pairing. Suppose $D_2 \in \mathcal{J}[n]$ and let $D_2, h_2, d_2$ be as in the definition of the pairing. Then we have

$$
eq \prod_{P \in C(\mathbb{K})} (-1)^{(\text{ord}_P f)(\text{ord}_P m)} \frac{h_2^{\text{ord}_P f}}{h_2^{\text{ord}_P h_2 + d_2 \text{ord}_P m}} (P) \quad \text{(since } nD_2 = \text{div}(h_2) + d_2m.)$$

where $N$ denotes the induced norm $\text{Res}_m \mathbb{G}_m \to \mathbb{G}_m$. From this one easily sees that $\text{Res}_m^1 \mu_n$ lies in the kernel of the pairing and that $T[n]$ pairs trivially with the degree 0 subgroup, $J_m[n] \subset \mathcal{J}[n]$. 

Taking Galois cohomology of (2.6) yields a commutative and exact diagram

\begin{equation}
\begin{array}{ccc}
\mathbb{Z}/n\mathbb{Z} & \xrightarrow{\delta'} & H^1(A_m[\varphi]) \\
\downarrow & & \downarrow \downarrow \\
\mathbb{Z}/n\mathbb{Z} & \xrightarrow{\delta} & H^1(J[n]) \\
\downarrow & & \downarrow \downarrow \\
\text{Br}(k)[n] & \xrightarrow{\vartheta'} & \text{Br}(k)[n]
\end{array}
\end{equation}

\textbf{Lemma 2.9.} The images of $\delta(1)$ and $\delta'(1)$ in $H^1(J)$ and $H^1(A_m)$ are the classes of $J^\ell$ and $A_m^\ell$, respectively. The maps $\vartheta$ and $\vartheta'$ are given by

$$\vartheta(\xi) = \xi \cup_{e} \delta(1)$$

$$\vartheta'(\xi) = \xi \cup_{e} \delta'(1),$$

where $\cup_{e}$ denotes the cup product induced by the pairing of Lemma 2.7.

\textbf{Proof.} The extensions

$$J_m^\ell : \quad 0 \to J[n] \to \mathcal{J}[n] \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

$$J_{m,n}^\ell : \quad 0 \to A_m[\varphi] \to A_m[\varphi] \to \mathbb{Z}/n\mathbb{Z} \to 0$$

represent classes

$$[J_m] \in \text{Ext}^1_{\mathbb{Z}/n\mathbb{Z}[\text{Gal}_k]}(\mathbb{Z}/n\mathbb{Z}, J[n]) \cong H^1(J[n])$$

$$[J_{m,n}] \in \text{Ext}^1_{\mathbb{Z}/n\mathbb{Z}[\text{Gal}_k]}(\mathbb{Z}/n\mathbb{Z}, A_m[\varphi]) \cong H^1(A_m[\varphi]).$$
Let $\delta$ and $\delta'$ denote the coboundary maps in the Galois cohomology of $J_n^\ell$ and $J_{m,n}^\ell$, respectively. Then $\delta(1) = [J_n^\ell]$ and $\delta'(1) = [J_{m,n}^\ell]$. Let $\delta^\vee$ and $\delta'^\vee$ denote the coboundaries of the extensions $(J_n^\ell)^\vee$ and $(J_{m,n}^\ell)^\vee$ obtained by dualizing and let $\epsilon : J[n] \cong J[n]^\vee$ and $\epsilon' : J[n] \cong A_m[\varphi]^\vee$ be the isomorphisms induced by the $\epsilon$-pairings of Lemma 2.7.

By [NSW08 Corollary 1.4.6] the following diagrams are commutative.

\[
\begin{array}{cccc}
H^1(J[n]) & \xrightarrow{\epsilon^*} & H^1(J[n]^\vee) & \xrightarrow{\delta^\vee(\bullet)\cup 1} & H^2(\mu_m) \\
\downarrow \circ \delta(1) & & \downarrow \circ \delta(1) & & \downarrow (-1)^2 \\
H^2(J[n] \otimes J[n]) & \xrightarrow{(\epsilon \otimes \text{id})^*} & H^2(J[n]^\vee \otimes J[n]) & \xrightarrow{\text{eval}^*} & H^2(\mu_m) \\
H^1(J[n]) & \xrightarrow{\epsilon'^*} & H^1(A_m[\varphi]^\vee) & \xrightarrow{\delta'^\vee(\bullet)\cup 1} & H^2(\mu_m) \\
\downarrow \circ \delta'(1) & & \downarrow \circ \delta'(1) & & \downarrow (-1)^2 \\
H^2(J[n] \otimes A_m[\varphi]) & \xrightarrow{(\epsilon' \otimes \text{id})^*} & H^2(A_m[\varphi]^\vee \otimes A_m[\varphi]) & \xrightarrow{\text{eval}^*} & H^2(\mu_m)
\end{array}
\]

The composition along the top row is the map $\Upsilon$ (resp. $\Upsilon'$), while the path from the top-left to the bottom-right along the bottom row agrees with the description given in the statement of the lemma.

2.5. Brauer class of a $k$-rational divisor class. Given a nice curve $X$, there is a well known exact sequence

\[(2.9) 0 \to \text{Pic}(X) \longrightarrow \text{Pic}_X(k) \xrightarrow{\Theta_X} \text{Br}(k) \]

(see [Lic69]). The map $\Theta_X$ gives the obstruction to a $k$-rational divisor class being represented by a $k$-rational divisor.

Lemma 2.10. Let $d : J(k) \rightarrow H^1(k, J[n])$ denote the connecting homomorphism in the Kummer sequence. For any $x \in J(k)$ we have $\Upsilon \circ d(x) = \ell \cdot \Theta_C(x)$.

Proof. The image of $d$ is isotropic with respect to the Weil-pairing cup product $\cup_e$. This gives a commutative diagram of pairings

\[
\begin{array}{ccc}
\cup_e : & H^1(J[n]) & \times H^1(J[n]) \rightarrow \text{Br}(k) \\
d^\dagger & \downarrow \downarrow \| & \\
\langle , , \rangle : & J(k) & \times H^1(J) \rightarrow \text{Br}(k)
\end{array}
\]

By a result of Lichtenbaum (see the proof of [Lic69 Corollary 1]) we have that $\langle x, [J^1] \rangle = \Theta_C(x)$. By the previous lemma we have

$\Upsilon \circ d(x) = d(x) \cup \delta(1) = \langle x, [J_n^\ell] \rangle = \ell \cdot \langle x, [J^1] \rangle = \ell \cdot \Theta_C(x)$.

\[\square\]

Corollary 2.11. If

(1) the period of $C$ divides $\ell$, or

(2) \( k \) is a local or global field and \( \gcd(m, g - 1) \) divides \( \ell \), then \( \Upsilon \circ d = 0 \).

Proof. The image of \( \Theta_C : J(k) \to \text{Br}(k) \) is isomorphic to the cokernel of \( \text{Pic}^0(C) \to J(k) \), which is annihilated by the period of \( C \) \([PS97\text{, Prop. 3.2}]\). Over a local field, the period of \( C \) divides \( g - 1 \) \([PS97\text{, Prop. 3.4}]\). Since the period also divides \( m = \deg(m) \), (2) implies that \( \ell \) is divisible by the period locally. Hence \( \Upsilon \circ d = 0 \) locally. This must also be true globally by the local-global principle for \( \text{Br}(k) \). \( \square \)

3. The descent setup

We recall the following definition from \([BPS]\).

Defintion 3.1. A descent setup for \( C \) for a nice curve \( C \) is a triple \((n, \Delta, \beta)\) consisting of a positive integer \( n \) not divisible by the characteristic of \( k \), a nonempty finite étale \( k \)-scheme \( \Delta = \text{Spec} \, L \), and a divisor \( \beta \in \text{Div}(C \times \Delta) \) such that \( n\beta = m \times \Delta + \text{div}(f_m) \) for some \( m \in \text{Div}(C) \) and \( f_m \in k(C \times \Delta) \times \).

Suppose \((n, \Delta, \beta)\) is a descent setup for \( C \). If the divisor \( m \) appearing in the definition is reduced, then \((n, m)\) is a modulus setup, which we say is associated to \((n, \Delta, \beta)\). For each \( \delta \in \Delta(k) \), \( \beta_{\delta} \in \text{Div}(C_k) \) is a divisor such that \( n\beta_{\delta} - m \) principal. So the class of \( \beta_{\delta} \) in \( J \) lies in \( J[n] \). This gives rise to a commutative and exact diagram,

\[
\begin{array}{cccc}
\text{Res}_\Delta^0 \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \text{Res}_\Delta \mathbb{Z}/n\mathbb{Z} & \text{deg} \\
\downarrow & & \downarrow & \\
J[n] & \longrightarrow & J[n] & \text{deg} \\
\text{deg} & & \text{deg} & \\
\end{array}
\]

Defintion 3.2. We say that \((n, \Delta, \beta)\) is an \( n \)-descent setup if the vertical maps in (3.1) are surjective.

The following examples show that all of the modulus setups in setups considered in Section 2 are associated to an \( n \)-descent setup. Details for Example D.1 and Example D.2 may be found in \([BPS\text{, Examples 6.9}]\), while Example D.3 is considered in \([Cre14]\).

**D.1** Suppose \( C \) is a double cover of \( \mathbb{P}^1 \) which is not ramified over \( \infty \). Let \( \Delta(k) \) be the set of ramification points and take \( \beta \) to be the diagonal embedding of \( \Delta \) in \( C \times \Delta \). Then \((2, \Delta, \beta)\) is a 2-descent setup. Taking \( m \) be the pullback of \( \infty \in \text{Div}(\mathbb{P}^1) \) we recover the modulus setup in Example M.1.

**D.2** Suppose \( C \) is any curve of genus \( \geq 2 \). We obtain a 2-descent setup for \( C \) by taking \( \Delta \) to be the \( \text{Gal}_k \)-set of odd theta characteristics. By \([BPS\text{, Proposition 5.8}]\) there is some \( \beta \in \text{Div}(C \times \Delta) \) such that \( [eta_{\delta}] = \delta \) for \( \delta \in \Delta(k) \). We can take \( m \) to be a canonical divisor and thus recover the modulus setup in Example M.5.

**D.3** Suppose \( C \) is a genus one curve of degree \( n \) in \( \mathbb{P}^{n-1} \) (or equivalently, a genus one curve together with the linear equivalence class of \( k \)-rational divisor of degree \( n \)). We obtain an \( n \)-descent setup by taking \( \Delta \) to be the set of \( n^2 \) flex points (i.e. points \( x \in C \) such that \( n.x \) is a hyperplane section) and \( \beta \) to be the diagonal embedding of \( \Delta \) in \( C \times \Delta \). Taking \( m \) to be a generic hyperplane section recovers the modulus setup in Example M.3.
D.4 More generally, suppose \( C \) is a genus one curve of degree \( m \) in \( \mathbb{P}^{m-1} \) and \( n \mid m \). There is a \( \text{Gal}_k \)-invariant subset of flexes of size \( n \ell^2 \) whose differences represent the \( n \)-torsion points. For any such flex \( x \), \( \ell x \) is linearly equivalent to a divisor \( D_x := P_1 + \cdots + P_\ell \) with \( P_i \in C \) such that \( nD_x \) is a hyperplane section. Furthermore, these \( D_x \) may be chosen so as to give a divisor \( \beta \in \text{Div}(C \times \Delta) \). This gives an \( n \)-descent setup with corresponding modulus setup \((n, m)\), where \( m \) is a generic hyperplane section. In the case \( n = 2, m = 4 \) this is done explicitly in \cite{Sta05}.

Remark 3.3. In \cite{BPS} Section 6 it is shown that a choice of \( m \) and \( f_m \in k(C \times \Delta)^{\times} \) yields a homomorphism

\[
f_m : \text{Pic}(C) \longrightarrow \frac{L^\times}{k^\times L^\times / n}
\]

related to the connecting homomorphism \( d : J(k) \rightarrow H^1(J[\mu_n]) \) by a commutative diagram

\[
\begin{array}{ccc}
\text{Pic}^0(C) & \xrightarrow{f_m} & \text{Pic}(C) \\
\downarrow & & \downarrow \\
J(k) & \xrightarrow{d} & H^1(J[\mu_n]) \\
\end{array}
\]

The injective map on the right comes from dualizing \eqref{3.1} and taking Galois cohomology to obtain a commutative and exact diagram,

\[
\begin{array}{cccccc}
H^1(T'[\varphi]) & \longrightarrow & H^1(A_m[\varphi]) & \longrightarrow & H^1(J[\mu_n]) & \longrightarrow & H^2(T'[\varphi]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
k^\times / k^{\times n} & \longrightarrow & L^\times / L^{\times n} & \longrightarrow & H^1\left(\frac{\text{Res}_\Delta \mu_n}{\mu_n}\right) & \longrightarrow & \text{Br}(k)[n]
\end{array}
\]

From this we see that the isogeny \( \varphi : A_m \rightarrow J_m \) naturally arises in the context of explicit \( n \)-descent as its kernel is the Cartier dual of \( J[\mu_n] \).

Remark 3.4. Let \( J(k)^\bullet \subset J(k) \) be the subgroup which maps under \( \epsilon \circ d \) into the subgroup \( L^\times / k^\times L^{\times n} \). This is the largest subgroup of \( J(k) \) to which one can extend the \( f_m \) map to a map which is compatible with the connecting homomorphism and takes values in \( L^\times / k^\times L^{\times n} \). We can determine \( J(k)^\bullet \) using Lemma 2.10. For example, when \( C \) is a non-hyperelliptic curve of genus 3 over a global field with descent setup as in Example D.2, we see that \( J(k)^\bullet = J(k) \).

3.1. \( \varphi \) and \( n \)-coverings.

Definition 3.5. Suppose \( \phi : A \rightarrow B \) is an isogeny of semiabelian varieties over \( k \) and \( T \) is a \( B \)-torsor. We say \( \pi : T' \rightarrow T \) is a \( \phi \)-covering of \( T \) if there exist isomorphisms \( a, b \) of \( k \)-varieties fitting into a commutative diagram

\[
\begin{array}{ccc}
T' & \xrightarrow{b} & A \\
\downarrow{\pi} & & \downarrow{\phi} \\
T & \xrightarrow{a} & B
\end{array}
\]
Two \( \phi \)-coverings of \( T \) are isomorphic if they are isomorphic in the category of \( T \)-schemes.

Suppose \((n, m)\) is a modulus setup for a nice curve \( C \) over \( k \). The isogenies \( \varphi : A_m \to J_m \) and \( n : J \to J \) give rise to the notions of \( \varphi \)-coverings of \( J_m^1 \) and \( n \)-coverings of \( J^1 \). The pullback of a \( \varphi \)-covering \( T \to J_m^1 \) along the canonical map \((C - m) \to J_m^1\) sending a geometric point \( x \) to the class of the divisor \( x \) in \( J_m^1(k) \subset \text{Pic}_{C_m}(k) \) yields an unramified covering of \((C - m)\). Corresponding to this is a unique (up to isomorphism) morphism \( \pi : Y \to C \) of smooth projective curves over \( k \) which is unramified outside \( m \).

**Definition 3.6.** Suppose \((n, m)\) is a modulus setup for a nice curve \( C \) over \( k \). A morphism \( \pi : Y \to C \) of nice curves is a \( \varphi \)-\textbf{covering} of \( C \) if it is the unique extension of the pullback of a \( \varphi \)-covering of \( J_m^1 \) along the canonical map \((C - m) \to J_m^1\). A morphism \( \pi : X \to C \) is an \( n \)-\textbf{covering} of \( C \) if it is the pullback of an \( n \)-covering of \( J^1 \) along the canonical map \( C \to J^1 \).

By Galois theory, the field extension of \( \overline{k}(C_T) \) corresponding to a \( \varphi \)-covering is the compositum of the extensions corresponding to the index \( n \) subgroups of \( A_m[\varphi] \), or equivalently, to the points of order \( n \) in the Cartier dual \( J[n] \). If \( D \in \text{Div}(C_T) \) represents a point of order \( n \) in \( J[n] \), then there exists a function \( h_D \in \overline{k}(C_T)^\times \) such that \( \text{div}(h_D) = nD - dm \) for some \( d \in \mathbb{Z} \). The corresponding extension of \( \overline{k}(C_T) \) is obtained by adjoining an \( n \)-th root of \( h_D \). In particular, \( n \)-coverings of \( C \) are the \( k \)-forms of the maximal unramified abelian covering of \( C \) of exponent \( n \), while \( \varphi \)-coverings of \( C \) are (examples of) abelian coverings of exponent \( n \) and conductor \( m \).

For an isogeny, let \( \text{Cov}^\phi(V) \) denote the set of isomorphism classes of \( \phi \)-coverings of \( V \). When nonempty, \( \text{Cov}^\phi(V) \) is a principal homogeneous space for the group \( H^1(k, \ker(\phi)) \) acting by twisting. By geometric class field theory the canonical maps \( \text{Cov}^n(J^1) \to \text{Cov}^n(C) \) and \( \text{Cov}^\varphi(J_m^1) \to \text{Cov}^\varphi(C) \) are bijections that respect this action. There is also a canonical map \( \text{Cov}^\varphi(C) \to \text{Cov}^a(C) \), which associates to a \( \varphi \)-covering of \( C \) the maximal unramified intermediate covering of \( C \). Let \( \text{Cov}^a_0(C) \) denote the image of this map, and \( \text{Cov}^a_0(J^1) \) the corresponding subset of \( \text{Cov}^a(J^1) \). Thus, \( \text{Cov}^a_0 \) consists of isomorphism classes of \( n \)-coverings that may be lifted to a \( \varphi \)-covering.

**Remark 3.7.** Suppose \((2, m)\) is a modulus setup for \( C : z^2 = f(x, y) \), a double cover of \( \mathbb{P}^1 \) as in [Example D.1].

1. Given a pair of symmetric bilinear forms \((A, B)\) such that \( \text{disc}(Ax - By) = f(x, y) \) the Fano variety of maximal linear subspaces contained in the base locus of the pencil of quadrics generated by \((A, B)\) may be given the structure of a 2-covering of \( J^1 \). Theorem 22 and the discussion of Section 5 in [BGWb] shows that the isomorphism classes of 2-coverings of \( J^1 \) that arise in this way are precisely those in \( \text{Cov}^a_0(J^1) \).

2. Section 3 of [BS09] gives an explicit construction of a collection of 2-coverings of \( C \) from the set \( H_k \) (notation as in [BS09]). Comparing Lemma 3.4 below with the proof of [BS09] Theorem 3.4 shows that the collection of coverings they produce is precisely \( \text{Cov}^a_0(C) \). It follows from this that \( \text{Cov}^a_0(J^1) \) also coincides with the set \( \text{Cov}^a_{\text{good}}(J^1/k) \) defined in [Cre13] Section 6].

**Remark 3.8.** Suppose \((n, m)\) is a modulus setup for a genus one curve as in [Example D.3]. In Section 4 we show that the set \( \text{Cov}^a_0(C) \) defined in this paper coincides with that in [Cre14] Definition 3.3].
Lemma 3.9. Suppose \((n, m)\) is a modulus setup associated to an \(n\)-descent setup \((n, \Delta, \beta)\) and that \(\pi : X \to C\) is an \(n\)-covering. The class of \((X, \pi)\) in \(\text{Cov}^n(C)\) lies in \(\text{Cov}^0_n(C)\) if and only if \(\pi^* \beta_\delta\) is linearly equivalent to a \(k\)-rational divisor, for some \(\delta \in \Delta(k)\).

Proof. Suppose \(\pi : X \to C\) lifts to a \(\varphi\)-covering \(Y \to C\). The subfield \(k(X) \subset k(Y)\) corresponds to the subgroup \(\mu_n = T^n[\varphi] \subset A_m[\varphi]\). The extension \(k(X) \subset k(Y)\) is therefore obtained by adjoining to \(k(X)\) an \(n\)-th root of a function \(f\) such that \(\text{div}(f) = nD - \pi^* \delta m\), for some \(d \in \mathbb{Z}\) and \(f \in k(X)^\times\). Furthermore, we can arrange that \(d = 1\). Indeed, we must have \(\gcd(n, d) = 1\), otherwise there would be a proper unramified intermediate extension of \(k(X) \subset k(Y)\). Hence \(\pi^* m = nD + \text{div}(f)\) for some \(D \in \text{Div}(X)\) and \(f \in k(X)^\times\).

For the other direction, suppose \(D \in \text{Div}(X)\) is a \(k\)-rational divisor linearly equivalent to \(\pi^* \beta_\delta\). Then \(\text{div}(\pi^* f_{m, \delta}) = n \pi^* \beta_\delta - \pi^* m = nD - \pi^* m + \text{div}(f)\), for some \(f \in \overline{k}(X)^\times\). Thus, the divisor \(nD - \pi^* m \in \text{Div}(X)\) is principal and \(k\)-rational. By Hilbert’s Theorem 90 it is the divisor of some \(k\)-rational function \(g \in k(X)^\times\). The covering obtained by adjoining an \(n\)-th root of \(g\) to \(k(X)\). Over \(k\) we see that \(k(Y)\) is the compositum of \(k(X)^\times\) and \(\overline{k}(C^\gamma_{m, \delta})\), so \(Y \to C\) is a \(\varphi\)-covering of \(C\). \qed

Proposition 3.10. Suppose \((n, m)\) is a modulus setup for \(C\) associated to an \(n\)-descent setup \((n, \Delta, \beta)\). The following are equivalent.

1. The class of \(J^1_m\) in \(H^1(k, J^1_m)\) is divisible by \(\varphi\).
2. There exists a \(\varphi\)-covering of \(J^1_m\).
3. There exists a \(\varphi\)-covering of \(C\).
4. \(\text{Cov}^1_{\varphi}(C) \neq \emptyset\).
5. \(\text{Cov}^0_{\varphi}(C) \neq \emptyset\).
6. There exists an \(n\)-covering \(\pi : X \to C\) with the property that \(\pi^* \beta_\delta\) is linearly equivalent to a \(k\)-rational divisor, for some \(\delta \in \Delta(k)\).
7. The maximal unramified abelian covering of \(C^\gamma_{m, \delta}\) of exponent \(n\) descends to \(k\) and the image of the \(k\)-rational divisor class \(\pi^* \beta_\delta\) in \(\text{Br}(k)\) under the map \(\Theta_X\) of (2.9) lies in the image of the map \(\Upsilon\) of (2.8), for every maximal unramified abelian covering \(\pi : X \to C\) of exponent \(n\) and every \(\delta \in \Delta(k)\).

Proof. There exists a \(\varphi\)-covering of \((J^1_m)^\gamma_{m, \delta}\). The Galois descent obstruction to defining this over \(k\) is the image in \(H^2(k, A_m[\varphi])\) of the class of this covering under the map

\[
H^0\left(k, H^1\left((J^1_m)^\gamma_{m, \delta}, A_m[\varphi]\right)\right) \to H^2(k, A_m[\varphi])
\]

from the Hochschild-Serre spectral sequence (cf. [Sko01 Section 2.2]). This class coincides with the image of \([J^1_m]\) under the coboundary arising from the exact sequence

\[0 \to A_m[\varphi] \to A_m \to J^1_m \to 0\]

(see [Sko01, Lemma 2.4.5]). This proves the equivalence of (1) and (2), while the equivalence of (2) and (3) follows from geometric class field theory. The equivalences (3) \iff (4) \iff (5) follow immediately from the definitions, and (3) \iff (6) is given by Lemma 3.9.
It remains to prove (6) \iff (7). An \( n \)-covering \( \pi : X \to C \) is a \( k \)-form of the maximal unramified abelian covering of exponent \( n \), which we may assume exists. Then, for any \( \delta, \delta' \in \Delta(k) \) the divisors \( \pi^* \delta \) and \( \pi^* \delta' \) are linearly equivalent. Indeed \( \delta - \delta' \) represents a class in \( J[n] \). It follows that the class of \( \pi^* \delta \) in \( \text{Pic}(X) \) is fixed by \( \text{Gal}_k \). The image of this class in \( \text{Br}(k) \) is trivial if and only if the class can be represented by a \( k \)-rational divisor. Since the set of all isomorphism classes of \( n \)-coverings of \( C \) is a principal homogeneous space for \( H^1(k, J[n]) \) under the action of twisting, the equivalence of (6) and (7) follows from the next lemma.

Lemma 3.11. Suppose \( \pi : X \to C \) is an \( n \)-covering and \( \pi_\xi : X_\xi \to C \) is the twist by the cocycle \( \xi \in Z^1(J[n]) \). Then for any \( \delta \in \Delta(k) \),

\[ \Upsilon([\xi]) = \Theta_{X_\xi}(\pi^*_\delta \beta) - \Theta_X(\pi^* \beta). \]

Proof. There is an isomorphism of coverings \( \rho : \overline{X}_\xi \to \overline{X} \) with the property that \( \sigma \circ \rho^{-1} = T_{\xi} \in \text{Aut}(X/C_\pi) \) is translation by \( \xi_\sigma \in J[n] \), for every \( \sigma \in \text{Gal}_k \). Let \( W = \pi^*_\xi \beta \) and \( W' := \rho^*(W) = \pi^* \beta \). These represent Galois invariant divisor classes, hence, for any \( \sigma \in \text{Gal}_k \) there are functions \( f_\sigma \in \overline{k}(X_\xi)^\times \) and \( g_\sigma \in \overline{k}(X)^\times \) with \( \text{div}(f_\sigma) = \sigma W - W \) and \( \text{div}(g_\sigma) = \sigma W' - W' \). The classes in \( \text{Br}(k) \) of \( W \) and \( W' \) are given by the 2-cocycles

\[ a_{(\sigma, \tau)} = \frac{\sigma f_\tau \cdot f_\sigma}{f_{\sigma \tau}} \quad \text{and} \quad a'_{(\sigma, \tau)} = \frac{\sigma g_\tau \cdot g_\sigma}{g_{\sigma \tau}}, \]

both of which take values in \( \overline{k}^\times \). Since \( f_\sigma / \rho^* g_\sigma \in \overline{k}^\times \), the computation

\[ \frac{a_{(\sigma, \tau)}}{a'_{(\sigma, \tau)}} = \frac{a_{(\sigma, \tau)}}{\rho^*(a'_{(\sigma, \tau)})} = \frac{\sigma f_{\tau}}{\rho^* g_{\sigma \tau}} \cdot \frac{\sigma g_{\tau}}{\rho^* g_{\sigma \tau}} \cdot \frac{\sigma (\rho^* g_\tau)}{\rho^* (\sigma g_\tau)} \]

shows that \( \Theta_{X_\xi}(W) - \Theta_X(W') \) is represented by the 2-cocycle \( \eta \in Z^2(\text{Gal}_k, \overline{k}^\times) \) defined by

\[ \eta_{(\sigma, \tau)} = \frac{\sigma (\rho^* g_\tau)}{\rho^* (\sigma g_\tau)} = \frac{\sigma g_\tau \circ \sigma \rho}{\rho^* (\sigma g_\tau)} \cdot \rho^* \circ \rho. \]

Using that \( (\rho^{-1})^* \) is the identity on \( \overline{k} \subset \overline{k}(Y) \) and that \( \sigma \circ \rho^{-1} = T_{\xi} \), we have \( \eta_{(\sigma, \tau)} = \frac{\sigma g_\tau \circ T_{\xi}}{\rho^* (\sigma g_\tau)} \).

We recognize this as the Weil pairing \( \eta_{(\sigma, \tau)} = e_n(\sigma P_\tau, \xi_\sigma) \), where \( P_\tau \in J[n] \) is the class represented by the divisor \( \tau \beta - \beta \) (see (A.3)). The cocycle \( P_\tau \in Z^1(\text{Gal}_k, J[n]) \) represents \([J_n^\ell] \). So \( \eta_{(\sigma, \tau)} \) represents the \( e \)-pairing cup product \([J_n^\ell] \cup_e [\xi] = [\xi] \cup_e [J_n^\ell] \) which is equal to \( \Upsilon(\xi) \) by Lemma 2.3.

3.2. Soluble coverings. For an isogeny \( \phi \), let \( \text{Cov}_\phi^u(U) \) denote the set of isomorphism classes of \( \phi \)-coverings \( U \to V \) with \( U(k) \neq \emptyset \). When \( k \) is a global field, let \( \text{Sel}_\phi^u(V) \) denote the set of isomorphism classes of \( \phi \)-coverings of \( V \) that are soluble everywhere locally.

Proposition 3.12. The group \( H^1(k, J[n]) \) acts on the set \( \text{Cov}_\phi^u(J^1/k) \) by twisting. This gives rise to simply transitive actions of:

1. \( H^1(k, J[n]) \) on \( \text{Cov}_\phi^u(J^1) \), when \([J^1]\) is divisible by \( n \);
2. \( H^1(k, J[n]) \) on \( \text{Cov}_\phi^u(C) \), when \([J^1]\) is divisible by \( n \);
3. \( \ker(\Upsilon) \) on \( \text{Cov}_\phi^u(J^1) \), when \([J_m^1]\) is divisible by \( \varphi \);
4. \( \ker(\Upsilon) \) on \( \text{Cov}_\phi^u(C) \), when \([J_m^1]\) is divisible by \( \varphi \);
Corollary 3.13. Suppose $C(k) \neq \emptyset$, then $[J^1_m]$ is divisible by $\varphi$.

Proof. This follows from (6) since under our assumption $\text{Cov}_{\text{sol}}^n(J^1) \neq \emptyset$. □

Corollary 3.14. Suppose that $k$ is a global field and $C$ is everywhere locally solvable. Then $\text{Sel}^n(J^1) \subset \text{Cov}_{\text{sol}}^n(J^1)$.

Proof. If $\text{Sel}^n(J^1) = \emptyset$ there is nothing to prove. Otherwise, $[J^1] \in n\text{III}(J)$ in which the result follows from (7) and (8). □

Proof of Proposition 3.12

1. First note that $n$-coverings are $J[n]$-torsors. As in [Sko01, Section 2.2], the low degree terms of Hochschild-Serre spectral sequence give an exact sequence

$$0 \to H^1(k, J[n]) \to H^1_{\text{et}}(J^1, J[n]) \to H^0(k, H^1_{\text{et}}(k, J[n])) \xrightarrow{\partial} H^2(k, J[n]).$$

There exists an $n$-covering of $J^1_k$ and the image of its class under $\partial$ is the obstruction to the existence of an $n$-covering of $J^1$. This obstruction coincides with the coboundary of $[J^1]$ arising from the exact sequence $0 \to J[n] \to J \to J \to 0$ (see [Sko01, Lemma 2.4.5]). In particular, if $[J^1]$ is divisible by $n$, then $\text{Cov}_{\text{sol}}^n(J^1) \neq \emptyset$. In this case $H^1(k, J[n])$ acts simply transitively on $\text{Cov}_{\text{sol}}^n(J^1)$ by exactness of the sequence above.

2. It follows from geometric class field theory that the map $\text{Cov}_{\text{sol}}^n(J^1) \to \text{Cov}_{\text{sol}}^n(C)$ given by pullback is a bijection which respects the action of $H^1(k, J[n])$, so (1) $\Rightarrow$ (2).

3. This follows from Proposition 3.10 and Lemma 3.11.

4. This follows from (3) by pullback.

5. If $J^1(k) \neq \emptyset$, then $\text{Cov}_{\text{sol}}^n(J^1) \neq \emptyset$ (since in this case $[J^1] = 0$ in $H^1(k, J)$ is divisible by $n$). The difference of any two soluble $n$-coverings has trivial image in $H^1(k, J)$, hence must lie in the image of the Kummer map $J(k)/nJ(k) \to H^1(k, J[n])$.

6. This follows from (3) once we show that $\text{Cov}_{\text{sol}}^n(J^1)$ contains $\text{Cov}_{\text{sol}}^n(J^1)$. By assumption there is some $x \in C(k)$, and hence a lift of $x$ to a point $x' \in X(k)$ on some $n$-covering $\pi : X \to C$. This is the pullback of some $n$-covering $T \to J^1$, which is necessarily soluble. Since $X(k) \neq \emptyset$ we have $\text{Pic}(X) = \text{Pic}_X(k)$. Proposition 3.10 shows that $[T \to J^1] \in \text{Cov}_{\text{sol}}^n(J^1) \neq \emptyset$ and that $[J^1_m]$ is divisible by $\varphi$. By (3), $\text{Cov}_{\text{sol}}^n(J^1)$ is the orbit of $[T \to J^1]$ under $\ker(\Upsilon)$, and by (3) $\text{Cov}_{\text{sol}}^n(J^1)$ is the orbit of $[T \to J^1]$ under $J(k)/nJ(k)$. It thus suffices to prove that the image of $J(k)$ under the Kummer map is contained in the kernel of $\Upsilon$. This follows from Lemma 2.10 since our assumption that $C(k) \neq \emptyset$ implies that $\Theta_C$ is the zero map.

7. Since $[J^1] \in n\text{III}(J)$, we have that $\text{Sel}^n(J^1) \neq \emptyset$. One then argues as in (5) (everywhere locally) to see that the difference of two locally soluble $n$-coverings of $J^1$ gives an element of $\text{Sel}^n(J)$.

8. This follows from (7) once we show that $\text{Sel}^n(J^1) \subset \text{Cov}_{\text{sol}}^n(J^1)$. Suppose $T \to J^1$ is a locally soluble $n$-covering and $\pi : X \to C$ is the pullback. Applying (6) over the completions $k_v$ of $k$ we see that $\Theta_X(\pi^*\beta_0)$ has trivial image in $\text{Br}(k_v)$. But then
\( \Theta_X(\pi)/\beta_i \) must be trivial in \( \text{Br}(k) \) by the local-global principle for the Brauer group. Hence \( (X, \pi) \in \text{Cov}_0^n(C) \). Then \( T \to J^1 \) lies in \( \text{Cov}_0^n(J^1) \) as desired.

3.3. A descent map. Suppose \((n, m)\) is a modulus setup associated to a descent setup \((n, \Delta, \beta)\) for \(C\). Recall from Remark 3.3 that a choice for \( f_m \in k(C \times \Delta)^x \) induces a map \( f_m : \text{Pic}^n(C) \to L^x/k^xL^x_n \). The condition in Lemma 3.9 allows us to define a ‘descent map’ \( f_m : \text{Cov}_0^n(C) \to L^x/k^xL^x_n \) with the property that for any extension \( K/k \) and \( Q \in C(K) \),

\[
(3.3) \quad \tilde{f}_m(C', \pi) = f_m(\pi(Q)) \quad \text{in} \quad \frac{(L \otimes K)^x}{K^x(L \otimes K)^x}.
\]

This map is compatible with the action of \( \ker(\Upsilon) \) on \( \text{Cov}_0^n(C) \) and the image of \( \ker(\Upsilon) \) in \( L^x/k^xL^x_n \) under the map \( \epsilon \) in (3.2). In particular, the action of \( \text{Pic}^0(C) \) on \( \text{Cov}_0^n(C) \) via \( \text{Pic}^0(C) \subset J(k) \stackrel{d}{\to} \ker(\Upsilon) \) is compatible with the action of \( f_m (\text{Pic}^0(C)) \) on the image of \( \tilde{f}_m \) inside \( L^x/k^xL^x_n \). In the situations of Example D.1 and Example D.3, details of this construction can be found in [Cre13 Prop. 5.4] and [Cre14 Theorem 5.2], respectively. The general case can be carried out in the same way.

Composing \( f_m \) with the pullback map \( \text{Cov}_0^n(J^1) \to \text{Cov}_0^n(C) \) one obtains a map \( \tilde{g}_m : \text{Cov}_0^n(J^1) \to L^x/k^xL^x_n \).

If \( Q \in C(k) \neq \emptyset \), then Proposition 3.12(6) shows that \( \text{Cov}_0^\text{sol}(J^1) \subset \text{Cov}_0^n(J^1) \), and compatibility with the action of \( \ker(\Upsilon) \) shows that the image of \( \text{Cov}_0^\text{sol}(J^1) \) under \( \tilde{g}_m \) is the orbit under \( f_m \) of \( (J(k)/nJ(k)) \) of \( f_m(Q) \). Similarly if \( k \) is a global field and \( C \) is everywhere locally soluble, then Proposition 3.12(viii) shows \( \text{Sel}^n(J^1) \subset \text{Cov}_0^n(J^1) \); its image under \( \tilde{g}_m \) is contained in a set which can be computed from the local images of \( \text{Pic}^1(C_k) \). In [Cre13] one finds examples (in the case of hyperelliptic curves) where this is used to prove that \( \text{Sel}^n(J^1) = \emptyset \) thus concluding that \( |J^1| \neq n\text{III}(J) \). This illustrates the value of Proposition 3.12 from the perspective of explicit descents; namely it shows that the set of coverings \( \text{Cov}_0^n(J^1) \) is large enough to be of interest for such arithmetic applications.

In general, the map \( \text{Cov}_0^n(J^1) \to L^x/k^xL^x_n \) will not be injective, so the subset cut out by the local conditions may be larger than the image of \( \text{Sel}^n(J^1) \) (One says that we are computing a ‘fake Selmer set’ for \( J^1 \)). There are ways to deal with this (at least in principle; see Remark 3.20 below).

3.4. Norm conditions. When \( C : z^2 = f(x) \) is a hyperelliptic curve with \( m \) the divisor above \( \infty \in \text{Div}(\mathbb{P}^1) \), the 2-divisibility of \( |J_m^1| \) is equivalent to the vanishing of the class of the leading coefficient of \( f(x) \) in \( k^x/k^x\mathbb{N}_{L/k}(L^x) \). In this section we consider generalizations of this condition.

If the diagonal embedding \( \mathbb{Z}/n\mathbb{Z} \hookrightarrow \text{Res}_\Delta \mathbb{Z}/n\mathbb{Z} \) is contained in the kernel of the map \( \text{Res}_\Delta \mathbb{Z}/n\mathbb{Z} \to J[n] \), then \#\( \Delta = nr \) for some \( r \) and there exists a function \( g \in k(C)^x \) such that \( \text{div}(g) = \sum_{\delta \in \Delta} \delta - rm \). From the descent setup \((n, \Delta, \beta)\), we have a function \( f_m \in k(C \times \Delta)^x \) such that \( \text{div}(f_m) = n\beta - m \times \Delta \). Then \( \text{div}(N_{L/k}(f_m)) = n \sum_{\delta \in \Delta} -nr m = \text{div}(g^n) \). Hence \( N_{L/k}(f_m) = c g^n \) for some \( c \in k^x \). Different choices for \( f_m \) and \( g \) would modify \( c \) by an element of \( k^xN_{L/k}(L^x) \). Hence the class of \( c \) in \( k^x/k^x\mathbb{N}_{L/k}(L^x) \) is determined by the descent setup \((n, \Delta, \beta)\).
Proposition 3.15. Suppose that the diagonal embedding $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \text{Res}_{\Delta} \mathbb{Z}/n\mathbb{Z}$ is contained in the kernel of the map $\text{Res}_{\Delta} \mathbb{Z}/n\mathbb{Z} \to \mathcal{J}[n]$ and let $c \in k^\times$ be as defined above. If $[J_{m}]^1$ is divisible by $\varphi$, then the class of $c$ in $k^\times/k^\times\text{N}_{L/k}(L^\times)$ is trivial.

Proof. Suppose $[J_{m}]^1$ is divisible by $\varphi$. By Proposition 3.10 there exists a $\varphi$-covering $\pi : C'' \to C$. There exists $\alpha \in L^\times$ and $f \in k(C'' \times \Delta)^\times$ such that $\alpha f^n = \pi^* f_m$. Taking norms we find $N_{L/k}(\alpha) N_{L/k}(f^n) = \pi^* g^n$, where $g \in k(C)^\times$ is the function involved in the definition of $c$. From this we see that $c \in k^\times N_{L/k}(L^\times)$. \hfill $\square$

When the diagonal embedding of $\mathbb{Z}/n\mathbb{Z}$ is equal to the kernel of the map $\text{Res}_{\Delta} \mathbb{Z}/n\mathbb{Z} \to \mathcal{J}[n]$, the norm condition in the proposition is also sufficient for $\varphi$-divisibility of $[J_{m}]^1$. Indeed, in this case, dualizing (3.1) one finds an exact sequence $0 \to A_m[\varphi] \to \text{Res}_{\Delta} \mu_n \to \mu_n \to 1$ which leads to an injective map $k^\times/N_{L/k}(L^\times) k^\times_n \hookrightarrow H^2(A_m[\varphi])$ and one can show that the class of $c$ maps to $\partial[J_{m}]^1$ [BGWB, Theorem 24].

In general, the kernel of $\text{Res}_{\Delta} \mathbb{Z}/n\mathbb{Z} \to \mathcal{J}[n]$ may be larger than (or fail to contain) the diagonal embedding of $\mathbb{Z}/n\mathbb{Z}$. As described in [BPS, Appendix] one can always find a finite étale $k$-scheme $\Delta' = \text{Spec}(L')$ and a correspondence $\tau : \Delta \dashrightarrow \Delta'$ which induces a surjection of $\text{Res}_{\Delta'} \mathbb{Z}/n\mathbb{Z}$ onto the kernel. We recall that a correspondence $\tau : E \dashrightarrow E'$ between finite $k$-schemes is a homomorphism $\text{Res}_E \mathbb{Z} \to \text{Res}_{E'} \mathbb{Z}$.

Lemma 3.16. There exist finite étale $k$-schemes $\Delta' = \text{Spec}(L')$, $\Delta'' = \text{Spec}(L'')$ and correspondences $\tau : \Delta \dashrightarrow \Delta'$, $\tau' : \Delta' \dashrightarrow \Delta''$ and $\tau'' : \Delta \dashrightarrow \Delta''$ with $\tau' \circ \tau = n \tau''$ that induce an exact sequence $0 \to A_m[\varphi] \to \text{Res}_{\Delta} \mu_n \xrightarrow{\tau} \text{Res}_{\Delta'} \mu_n \xrightarrow{\tau'} \text{Res}_{\Delta''} \mu_n$.

Proof. The map $A_m[\varphi] \to \text{Res}_{\Delta} \mu_n$ comes from dualizing $\text{Res}_{\Delta} \mathbb{Z}/n\mathbb{Z} \to \mathcal{J}[n]$. For the existence of $\tau$, $\tau'$, $\tau''$ we refer the reader to [BPS, §A.1, §A.2]. \hfill $\square$

Example 3.17. If $C$ is a hyperelliptic curve with $\Delta$ its set of Weierstrass points, we can take $\Delta' = \text{Spec}(k)$, $\Delta'' = \emptyset$ and $\tau$ to be the correspondence inducing the norm $N_{L/k} : \text{Res}_{\Delta} \mu_n \to \mu_n$.

Example 3.18. Suppose $C$ is a cubic curve with $\Delta$ the set of flex points. Take $\Delta'' = \Delta$ and take $\Delta'$ to be the Galois set consisting of the twelve lines passing through three distinct flexes. The incidence relations between flexes and lines define correspondences $\tau : \Delta \dashrightarrow \Delta'$ and $\sigma : \Delta' \dashrightarrow \Delta$. The compositions of the norm maps from $\text{Res}_{\Delta} \mu_n$ and $\text{Res}_{\Delta'} \mu_n$ to $\mu_n$ with the diagonal embedding $\mu_n \hookrightarrow \text{Res}_{\Delta} \mu_n$ are induced by correspondences $\rho : \Delta \dashrightarrow \Delta$ and $\rho' : \Delta' \dashrightarrow \Delta$. Set $\tau' = \sigma - \rho'$ and $\tau'' = 1 - \rho$. The proof of [Cre10, Lemma 5.3] shows that these correspondences satisfy the conditions of Lemma 3.16.

Let $\tau, \tau', \tau''$ be as in Lemma 3.16 and define maps

\[ v : \text{Res}_{\Delta} \mathbb{G}_m \to \text{Res}_{\Delta} \mathbb{G}_m \times \text{Res}_{\Delta'} \mathbb{G}_m ; \quad v(\ell) = (\ell^n, \tau_*(\ell)) , \]

\[ v' : \text{Res}_{\Delta} \mathbb{G}_m \times \text{Res}_{\Delta'} \mathbb{G}_m \to \text{Res}_{\Delta'} \mathbb{G}_m \times \text{Res}_{\Delta''} \mathbb{G}_m ; \quad v'(\ell, \ell') = \left( \frac{\ell^m}{\tau_*(\ell)}, \frac{\tau'_*(\ell')}{\tau''_*(\ell')} \right) . \]

By Lemma 3.16 there is an exact sequence

\[ 0 \to A_m[\varphi] \to \text{Res}_{\Delta} \mathbb{G}_m \xrightarrow{v} \text{Res}_{\Delta} \mathbb{G}_m \times \text{Res}_{\Delta'} \mathbb{G}_m \xrightarrow{v'} \text{Res}_{\Delta'} \mathbb{G}_m \times \text{Res}_{\Delta''} \mathbb{G}_m . \]

Proposition 3.19. With notation as above, let $U = \text{image}(v)$ and $U' = \text{image}(v')$.

1. $H^1(U) \simeq U'(k)/U'(L^\times \times L'^\times)$.
(2) There exists \( g \in k(C \times \Delta')^\times \) such that the functions \( c' = \tau_*(f_m)/g^n \) and \( c'' = \tau'_*(f_m)/\tau'_*(g) \) are constant and \( c = (c', c'') \in U'(k) \subset L'^\times \times L''^\times \).

(3) The class of \( c \) in \( H^1(U) \) depends only on \((n, \Delta, \beta)\).

(4) The class of \( c \) in \( H^1(U) \) is trivial if \([J_n^m]\) is divisible by \( \varphi \).

**Proof.** (1) This follows immediately from the definitions.

(2) One can check that \( \tau_*(f_m) \) is \( n \) times a principal divisor (see [BPS, Lemma A.9]). Hence there exist \( c' \in L'^\times \) and \( g \in k(C \times \Delta')^\times \) such that \( c' = g^n/\tau_*(f_m) \). Using that \( \tau' \circ \tau = n\tau \) we see that \( \tau''(f_m) \) and \( \tau'_*(g) \) have the same divisor, whence the existence of \( c'' = \tau'_*(g)/\tau'_*(f_m) \in L''^\times \). To see that \( c \in U'(k) \) simply note that \( c = \varphi(f_m(x), g(x)) \), for any \( x \in C(k) \).

(3) Modifying \( f_m \) or \( g \) by a scalar leaves the class of \( c \) modulo \( \varphi(L'^\times \times L''^\times) \) unchanged.

(4) This proved in the same way as in Proposition 3.15.

\[ \square \]

**Remark 3.20.**

(1) The exact sequence (3.4) yields an injective map \( H^1(U) \hookrightarrow H^2(A_m[\varphi]) \). We expect that this sends the class of \( c \) to \( \partial[J_n^m] \), where \( \partial \) denotes the coboundary coming from the exact sequence \( 0 \rightarrow A_m[\varphi] \rightarrow A_m \rightarrow J_m \rightarrow 0 \). In particular we expect the converse of Proposition 3.19(iv) holds.

(2) As described in the appendix to [BPS], one can extend the map \( f_m : \text{Pic}(C) \rightarrow L'^\times/k^\times L''^\times \) to a map \( \text{Pic}(C) \rightarrow U(k)/\nu(L'^\times) \). Using this one can extend the descent map described in Section 3.3 to an injective map \( \text{Cov}^n(J^1) \rightarrow U(k)/\nu(L'^\times) \), thus also allowing one to compute the (full) \( n \)-Selmer set of \( J^1 \). In the case of genus one curves this is described in detail in [Cre14].

### 4. INDEX AND DIVISIBILITY OF TORSORS UNDER ELLIPTIC CURVES

Let \( T \) be a torsor under an elliptic curve \( E \). We define the index of \( T \) to be the least positive degree of a \( k \)-rational divisor on \( T \). The index \( I \) of \( T \) and the order \( P \) of \( T \) in \( H^1(k, E) \) are known to satisfy \( P \mid I \mid P^2 \), and over number fields all pairs of integers \((P, I)\) satisfying these relations are known to occur [CSI10].

**Proposition 4.1.** Let \([C] \) be a torsor under an elliptic curve \( E \) with underlying curve \( C \). The following are equivalent.

(1) There exists a torsor \([C'] \) of index dividing \( n^2 \) such that \( n[C'] = [C] \).

(2) The curve \( C \) admits a modulus setup \((n, m)\) with \( n = \deg(m) \) such that \([J_n^m]\) is divisible by \( \varphi \) in \( H^1(k, J_m) \).

**Remark 4.2.** In [Cre] it is shown that condition (2) is satisfied when \( C \) is a locally solvable curve over a global field \( k \) and the action of \( \text{Gal}_k \) on \( J[n] \) is sufficiently generic. In particular, when \( k = \mathbb{Q} \), it holds when \( n = p^r \) is any prime power with \( p > 7 \).

Our proof of Proposition 4.1 will make use of the following interpretation of the elements of \( H^1(k, E[n]) \) taken from [CFO*08].

**Definition 4.3.** A **torsor divisor class pair** \((T, Z)\) consists of a \( J \)-torsor \( T \) and a \( k \)-rational divisor class \( Z \in \text{Pic}_T(k) \). Two torsor divisor class pairs \((T, Z)\) and \((T', Z')\) are isomorphic if there is an isomorphism of torsors \( s : T \rightarrow T' \) such that \( s^*Z' = Z \).
The automorphism group of the pair \((E, n.0_E)\) can be identified with \(E[n]\), and every pair \((T, Z)\) with \(\deg(Z) = n\) can be viewed as a twist of \((E, n.0_E)\) \([\text{CFO}+08\text{ Lemmas 1.7 and 1.8}]\). It follows that the torsor divisor class pairs of degree \(n\), viewed as twists of \((E, n.0_E)\), are parameterized by the group \(H^1(k, E[n])\).

**Lemma 4.4.** Suppose \((T', Z')\) is a torsor divisor class pair representing a lift of the class of \((T, Z)\) under the map \(n_+ : H^1(k, E[n^2]) \to H^1(k, E[n])\). The Brauer classes associated to the \(k\)-rational divisor classes \(Z'\) and \(Z\) satisfy \(n[Z'] = [Z]\) in \(Br(k)\). In particular, \(Z\) is represented by a \(k\)-rational divisor if \(Z'\) is.

**Proof.** Suppose the class of \((T', Z')\) is represented by a 1-cocycle \(\xi_\sigma \in Z^1(E[n^2])\). Let \(f_\sigma, g_\sigma \in \overline{k}(E)^\times\) be functions such that \(\text{div}(f_\sigma) = \tau_{\xi_\sigma}^* [n]^0_{E} - [n]^0_{E}\) and \(\text{div}(g_\sigma) = \tau_{\xi_\sigma}^* n.0_E - n.0_E\). Comparing divisors we see that we may scale by a constant to arrange that \(f_\sigma^n = g_\sigma \circ [n]\). Moreover, using that \(\xi_\sigma\) is a cocycle, we see that the coboundaries of the 1-cochains \((\sigma \mapsto f_\sigma)\) and \((\sigma \mapsto g_\sigma)\) give 2-cocycles \(F, G \in Z^2(\kappa)\) satisfying \(F^n = G\).

To prove the lemma one shows that \(F\) and \(G\) represent the Brauer classes corresponding to \(Z'\) and \(Z\), respectively. By \([\text{CFO}+08\text{ Prop. 1.32}]\), the pair \((g_\sigma, n\xi_\sigma)\) denotes a lift of \(n\xi_\sigma\) to the theta group corresponding to the torsor divisor class pair \((E, n.0_E)\). Then \([\text{CFO}+08\text{ Prop. 2.2}]\) shows that \([G] = [Z]\). In the same way we see that \((f_\sigma, \xi_\sigma)\) gives a lift of \(\xi_\sigma\) to the theta group corresponding to \((E, [n]^0_{E})\) and so \([F] = [Z']\). \(\square\)

**Proof of Proposition 4.4.** \(\Rightarrow\ 1\). Suppose \(\Rightarrow 2\) holds and let \((n, \Delta, \beta)\) be the \(n\)-descent setup corresponding to \(m\). By Proposition 3.10 there is an \(n\)-covering \(\pi : C' \to C\) such that \(\pi^* \beta_\delta\) is linearly equivalent to a \(k\)-rational divisor for some \(\delta \in \Delta(\overline{k})\). The genus one curve \(C'\) can be endowed with a torsor structure so that \(n[C'] = [C]\) in \(H^1(k, E)\). Moreover, the index of \([C']\) divides \(\deg(\pi^* \beta_\delta) = n^2\).

\(1\Rightarrow 2\). Suppose \(1\) holds and let \(Z' \in \text{Pic}^n(C')\). Consider the torsor divisor class pair \([(C'), Z']\). The image of this class under \(n_+ : H^1(k, E[n^2]) \to H^1(k, E[n])\) is represented by a pair \([(C), Z]\). By Lemma 1.4 \(Z \in \text{Pic}^n(C)\). By Riemann-Roch the divisor class \(Z\) contains a reduced and effective divisor \(m\) of degree \(n\). Then \((n, m)\) is a modulus setup for \(C\) with \(n = \deg(m)\). The divisor \(m\) determines a map \(C \to \mathbb{P}^{n-1}\) (which is an embedding for \(n > 2\)). Let \(\Delta := \{x \in C(\overline{k}) : n.x \sim m\}\) and take \(\beta\) to be the diagonal embedding of \(\Delta\) in \(C \times \Delta\). Then \((n, m)\) is associated to the \(n\)-descent setup \((n, \Delta, \beta)\).

The pair \((C', Z')\) corresponds to an \(n^2\)-covering of \(E\), which we may assume factors through the \(n\)-covering of \(E\) determined by \((C, Z)\). In particular, there is a commutative diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{n'} & C & \xrightarrow{\pi} & E \\
\downarrow{s'} & & \downarrow{s} & & \downarrow{\;} \\
E & \xrightarrow{n} & E & \xrightarrow{n} & E
\end{array}
\]

where \(s\) and \(s'\) are isomorphisms defined over \(\overline{k}\) which determine the \(E\)-torsor structures on \(C\) and \(C'\). Now \([m] = Z = [s^* n.0_E]\), so we must have \(s^* 0_E = \beta_\delta\) for some \(\delta \in \Delta(\overline{k})\). On the other hand, \(Z'\) is the class of \(s^* n^2.0_E = s^* [n]^0_{E} = \pi^* s^* 0_E = \pi^* \beta_\delta\). As this class is represented by a \(k\)-rational divisor, Proposition 3.10 shows that \([J_m]\) is divisible by \(\varphi\). \(\square\)
A. Weil Pairings

In this appendix we recall the three equivalent definitions of the Weil pairing on the \(n\)-torsion of a Jacobian variety that have been used in the paper.

A.1. The Weil pairing via the principal polarization. Suppose \(\phi : A \to B\) is an isogeny of abelian varieties with dual isogeny \(\hat{\phi} : \hat{B} \to \hat{A}\). There is a canonical isomorphism \(\beta : \hat{B} \simeq A[\phi] : \Hom(A[\phi], \mathbb{G}_m)\), and thus a nondegenerate pairing \(e_\phi : A[\phi] \times \hat{B}[\hat{\phi}] \to \mathbb{G}_m\) defined by \(e_\phi(x, y) = \beta(y)(x)\) (see [Mum70, §15 Theorem 1]). Applying this in the case \(\phi = [n] : A \to A\) yields a pairing

\[
\tilde{e}_n : A[n] \times \hat{A}[n] \to \mathbb{G}_m.
\]

Now suppose \(C\) is a nice curve and let \(\lambda : J \to \hat{J}\) be the canonical principal polarization of its Jacobian \(J\). Using this one defines,

\[
e_n : J[n] \times J[n] \to \mathbb{G}_m ; \quad e_n(x, y) = \tilde{e}_n(x, \lambda(y)).
\]

A.2. The Weil pairing via Kummer theory. Let \(M\) be the maximal unramified abelian extension of \(K = \overline{K}(C_K)\) of exponent \(n\). Kummer theory gives a perfect pairing \(\kappa : \Gal(M/K) \times (K^\times \cap M^{\times n})/K^{\times n} \to \mu_n\).

There is an isomorphism \(r : J[n] \simeq \Gal(M/K)\) of \(\Gal_k\)-modules. Specifically, \(M\) is the function field of an étale covering \(\pi : Y \to C_K\) fitting into a cartesian diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & J_K \\
\downarrow \pi & & \downarrow n \\
C_K & \longrightarrow & J_K.
\end{array}
\]

Translation by a point of \(J[n]\) on \(J_K\) pulls back to give an automorphism of \(Y\) over \(C_K\), and hence an element of \(\Gal(M/K)\). There is also an isomorphism \(s : J[n] \simeq (K^\times \cap M^{\times n})/K^{\times n}\).

Explicitly, if \(P \in J[n]\) is represented by a divisor \(D \in \Div(C_K)\) such that \(nD = \text{div}(f)\), then there exist \(g \in M^\times\) such that \(g^n = f\) and \(r(P) = gK^{\times n}\). From [Mil86b, Section 16] and [Mil86 Remark 6.10] one deduces that the pairing in (A.2) can be written as,

\[
e_n(x, y) = \kappa(r(x), s(y)).
\]

A.3. Weil’s definition of the pairing. Let \(D_1, D_2 \in \Div(C_K)\) be divisors representing \(P_1, P_2 \in J[n]\) and let \(h_1, h_2 \in \overline{K}(C_K)^\times\) be functions such that \(\text{div}(h_i) = nD_i\). Then

\[
e_n(P_1, P_2) = \prod_{x \in C_K(k)} (-1)^{m_{\ord_x(D)}(\ord_x(E))} \frac{g^{\ord_x(D)}}{\ord_x(E)}(x).
\]

In [How96] it is proved that this agrees with (A.2).
References

[Bha] Manjul Bhargava, Most hyperelliptic curves over \(Q\) have no rational points, available at \texttt{arXiv:1308.0395}.

[BG] Manjul Bhargava and Benedict Gross, The average size of the \(2\)-Selmer group of Jacobians of hyperelliptic curves having a rational Weierstrass point, available at \texttt{arXiv:1208.1007}.

[BGWa] Manjul Bhargava, Benedict Gross, and Xiaoheng Wang, Arithmetic invariant theory II: Pure inner forms and obstructions to the existence of orbits, available at \texttt{arXiv:1310.7689}.

[BGWb] Pencils of quadrics and the arithmetic of hyperelliptic curves, available at \texttt{arXiv:1310.7692}.

[BSD63] B. J. Birch and H. P. F. Swinnerton-Dyer, Notes on elliptic curves. I, J. Reine Angew. Math. \textbf{212} (1963), 7–25.

[BPS] Nils Bruin, Bjorn Poonen, and Michael Stoll, Generalized explicit descent and its application to curves of genus 3, available at \texttt{arXiv:1205.4456}.

[BS09] Nils Bruin and Michael Stoll, Two-cover descent on hyperelliptic curves, Math. Comp. \textbf{78} (2009), no. 268, 2347–2370.

[Cas62] J. W. S. Cassels, Arithmetic on curves of genus 1. IV. Proof of the Hauptvermutung, J. Reine Angew. Math. \textbf{211} (1962), 95–112.

[CS10] Pete L. Clark and Shahed Sharif, Period, index and potential. III, Algebra Number Theory \textbf{4} (2010), no. 2, 151–174.

[Cre01] J. E. Cremona, Classical invariants and 2-descent on elliptic curves, J. Symbolic Comput. \textbf{31} (2001), no. 1-2, 71–87. Computational algebra and number theory (Milwaukee, WI, 1996).

[CFO+08] J. E. Cremona, T. A. Fisher, C. O’Neil, D. Simon, and M. Stoll, Explicit \(n\)-descent on elliptic curves. I, Algebra, J. Reine Angew. Math. \textbf{615} (2008), 121–155.

[CRe10] Brendan Creutz, Explicit second \(p\)-descent on elliptic curves (2010). Ph.D. thesis, Jacobs University.

[CRe14] Second \(p\)-descents on elliptic curves, Math. Comp. \textbf{83} (2014), no. 285, 365–409.

[CRe13] Explicit descent in the Picard group of a cyclic cover of the projective line, Algorithmic number theory: Proceedings of the 10th Biennial International Symposium (ANTS-X) held in San Diego, July 9–13, 2012 (Everett W. Howe and Kiran S. Kedlaya, eds.), Open Book Series, vol. 1, Mathematical Science Publishers, 2013, pp. 295–315.

[CRe13] Most binary forms come from a pencil of quadrics, available at \texttt{arXiv:1601.03451}.

[How96] Everett W. Howe, The Weil pairing and the Hilbert symbol, Math. Ann. \textbf{305} (1996), no. 2, 387–392.

[Lic69] Stephen Lichtenbaum, Duality theorems for curves over \(p\)-adic fields, Invent. Math. \textbf{7} (1969), 120–136.

[MSS96] J. R. Merriman, S. Siksek, and N. P. Smart, Explicit 4-descents on an elliptic curve, Acta Arith. \textbf{77} (1996), no. 4, 385–404.

[Mil86a] J. S. Milne, Jacobian varieties, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 167–212.

[Mil86b] Abelian varieties, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 103–150.

[Mum70] David Mumford, Abelian varieties, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, Cohomology of number fields, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008.

[PS97] Bjorn Poonen and Edward F. Schaefer, Explicit descent for Jacobians of cyclic covers of the projective line, J. Reine Angew. Math. \textbf{488} (1997), 141–188.

[Ser88] Jean-Pierre Serre, Algebraic groups and class fields, Graduate Texts in Mathematics, vol. 117, Springer-Verlag, New York, 1988. Translated from the French.
[Sko01] Alexei Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001.

[Sta05] Sebastian Stamminger, *Explicit 8-descent on elliptic curves* (2005). PhD Thesis, International University Bremen.

[Tho15] Jack A. Thorne, *$E_6$ and the arithmetic of a family of non-hyperelliptic curves of genus 3*, Forum Math. Pi 3 (2015), e1, 41.

[Tho] ______, *On the 2-Selmer groups of plane quartic curves with a marked point*. (preprint).

[Wan] Xiaoheng Wang, *Maximal linear spaces contained in the base loci of pencils of quadrics*, available at [arXiv:1302.2385](http://arXiv.org/1302.2385)