Interpolation and Sampling for Generalized Bergman spaces on finite Riemann Surfaces

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1 Introduction

In [Seip-92, SW-92, Seip-93] Seip et al. characterized the sampling and interpolating sequences in the Bargmann-Fock space of entire functions that are square integrable with respect to the weight function $e^{-|z|^2}$, and in the Bergman space of square integrable holomorphic functions on the unit disk $D$.

In both cases, the results are given in terms of densities. In the complex plane $\mathbb{C}$, let $\Gamma$ be a discrete, uniformly separated sequence, and define

$$ D^+(\Gamma) := \limsup_{r \to \infty} \sup_{z \in \mathbb{C}} \frac{\#(\Gamma \cap D(z, r))}{r^2} $$

and

$$ D^-(\Gamma) := \liminf_{r \to \infty} \inf_{z \in \mathbb{C}} \frac{\#(\Gamma \cap D(z, r))}{r^2}. $$

In the unit disk $D$, the corresponding densities are defined in an analogous manner, which is nevertheless slightly different. Let $\Gamma$ be a sequence that is uniformly separated in the pseudo-hyperbolic distance. We then set

$$ D^+(\Gamma) := \limsup_{r \to 1} \sup_{z \in D} \frac{2 \int_0^r \#(\Gamma \cap D(z, s)) ds}{\log \frac{1}{1-r}} $$

and

$$ D^-(\Gamma) := \liminf_{r \to 1} \inf_{z \in D} \frac{2 \int_0^r \#(\Gamma \cap D(z, s)) ds}{\log \frac{1}{1-r}}, $$

where $D(z, s)$ is the pseudohyperbolic disk of center $z$ and radius $s$. The numbers $D^\pm(\Gamma)$ are often called the upper and lower densities of $\Gamma$. Seip et al. proved the following, now celebrated theorem.

**Theorem:** A uniformly separated sequence $\Gamma$ is an interpolating sequence for the Bargmann-Fock space or the Bergman space if and only if $D^+(\Gamma) < 1$. It is a sampling sequence if and only if $D^-(\Gamma) > 1$.

The goal of the present article is to generalize the sufficiency part of the Theorem of Seip et al. to the case of open Riemann surfaces. We fall somewhat short of this goal, but it is not clear how short. Indeed, we establish generalizations for the case of finite Riemann surfaces. However, the methods used do generalize to other Riemann surfaces, and may even generalize to all Riemann surfaces; we were unable to decide.

We now present our main results. To this end, every open Riemann surface admits a metric, locally denoted $e^{-2\nu |dz|^2}$, that we call the fundamental metric. (See definition 2.1.) Moreover, if the Riemann surface is hyperbolic, then this metric is unique. With the fundamental metric at hand, we can associate to a smooth function $\varphi : X \to \mathbb{R}$ and a discrete subset $\Gamma \subset X$ two Hilbert spaces

$$ B^2 = B^2(X, \varphi) := \left\{ h \in \mathcal{O}(X) : ||h||^2 := \int_X |h|^2 e^{-2\varphi} dA \nu < +\infty \right\} $$

and

$$ b^2 = b^2(\Gamma, \varphi) := \left\{ (s_\gamma)_{\gamma \in \Gamma} : \sum_{\gamma \in \Gamma} |s_\gamma|^2 e^{-2\varphi(\gamma)} < +\infty \right\}. $$

**Definition.** A discrete set $\Gamma$ is said to be...
1. an interpolation set if for every \( (s, \gamma) \in b^2 \) there exists \( F \in B^2 \) such that for all \( \gamma \in \Gamma \), \( F(\gamma) = s, \gamma, \) and
2. a sampling set if there is a constant \( M \) such that for all \( F \in B^2 \),
\[
\frac{1}{M} ||F||^2 \leq \sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-2\varphi(\gamma)} \leq M ||F||^2.
\]

For each locally integrable function \( f : [0, R_X) \to [0, \infty) \) and each \( r \in (0, R_X) \), let \( c_r := 2\pi \int_0^r tf(t)dt \) and
\[
\xi_r(z, \zeta) = \frac{1}{c_r} f(\rho_2(\zeta)) e^{2\nu |d\rho_2(\zeta)|^2} \mathbf{1}_{A}(z, \zeta),
\]
where \( \mathbf{1}_A \) denotes the characteristic function of a set \( A \). To every uniformly separated sequence (see section \ref{A.1.4} for the definition) we associate the upper and lower densities
\[
D^+_f(\Gamma) := \limsup_{r \to R_X} \sup_{z \in X} \sum_{\gamma \in \Gamma} \frac{\xi_r(\gamma, z)}{e^{2\nu |\Delta \varphi(z)|}}
\]
and
\[
D^-_f(\Gamma) := \liminf_{r \to R_X} \inf_{z \in X} \sum_{\gamma \in \Gamma} \frac{\xi_r(\gamma, z)}{e^{2\nu |\Delta \varphi(z)|}}
\]
where \( \Delta \) is the Laplace operator. Our main theorem can now be stated as follows.

**Theorem 1.** Let \( X \) be a finite Riemann surface, \( \Gamma \subset X \) a uniformly separated sequence, and \( \varphi : X \to R \) a subharmonic function such that for some constant \( C, \frac{1}{C} \leq e^{2\nu \Delta \varphi} \leq C \). If \( D^+_f(\Gamma) < 1 \) the \( \Gamma \) is an interpolation set, while if \( D^-_f(\Gamma) > 1 \), then \( \Gamma \) is a sampling set.

Partial results covering our theorem have been proved by others. For the case of the plane but with more general subharmonic weights, Theorem 1 was proved by Berndtsson-Ortega Cerdà \cite{BO-95}.

Ohsawa \cite{O-94} has proved results on interpolation only, and in the much more general context extension of \( L^2 \) holomorphic functions from closed submanifolds of Stein manifolds. His approach is somewhat different than ours; he uses a method, pioneered by himself and Takegoshi, of using a twisted \( \overline{\partial} \) theorem at (an earlier stage of the construction) to do the extension directly, rather than use \( \overline{\partial} \), as is done in \cite{BO-95}. (Ohsawa argued later \cite{O-01} that his approach is more conducive to generalization than the \( \overline{\partial} \) approach used here. We believe the methods of this paper show that both approaches are equally generalizable.)

In the same paper \cite{BO-95}, Berndtsson and Ortega Cerdà also treat the case of functions that are square integrable with respect to a subharmonic weight on the unit disk. A more careful analysis shows that because of the curvature of the fundamental metric, Theorem 1 does not cover all the cases those authors treat. A way to compensate is to prove a second theorem in the case of the disk, which allows some relaxation of the condition on \( \Delta \varphi \). This is indeed what was done in \cite{BO-95}. Aesthetically, this approach has the disadvantage of making the results of the disk and the plane appear distinct. Instead, in section \ref{S.3} we use conformal metrics on the Riemann surface to obtain more general norms on our generalized Bergman spaces with metrics. We then prove results (Theorems \ref{3.2} and \ref{3.3}) which encompass Theorem 1.

One additional novelty in our work is the introduction of a family of densities, parameterized by locally integrable functions \( f \) on the positive real line. This feature of our densities is likely to be useful in applications, and gives new results even in the classical cases of the plane and the disk. We demonstrate some useful consequences in the short Section \ref{S.2} at the end of the paper.

It is worth mentioning two additional things.

(i) As of right now we have not addressed the question of necessity. While we know that some of our density conditions are not necessary for interpolation or sampling, we do not know which, if any, are necessary.
Though we prove our theorems for finite Riemann surfaces, our work applies to a much broader class of open Riemann surfaces. In fact, we know of no example of a Riemann surface where our methods cannot be used to prove the corresponding version of Theorem 1.

The organization of our paper is as follows. In section 2 we discuss potential theory and the resulting analytic geometry of finite Riemann surfaces. In section 3 we introduce metrics into our scheme, and state Theorems 3.2 and 3.3 which are the main results of paper. The hypotheses in those theorems appear rather rigid, and thus it is not clear if or when they are satisfied. Thus in the same section we show that in fact there is always a non-trivial case in which the hypotheses are satisfied. In section 4 we discuss a $\partial$ theorem, due to Ohsawa [O-01], that will be used in the proof of the interpolation theorem 3.2. Since Ohsawa’s theorem is more general, we give a short, ad hoc proof of the case we need here. In section 5 we take a brief detour and establish interpolation and sampling results for compact Riemann surfaces. There are no non-constant holomorphic functions on compact Riemann surfaces, so we must look at sections of line bundles. These spaces are always finite dimensional, and sheaf theoretic methods give a complete answer to the interpolation and sampling question. Nevertheless we prove a special case of the interpolation theorem in this setting, to demonstrate how the $\partial$ theorem is later used. In section 6 we prove Theorems 3.2 and 3.3. Finally, in section 7 we present a collection of examples of our main results in some special cases.

2 Analytic Geometry of Finite Riemann surfaces

2.1 Potential theoretic preliminaries

We recall some basic, well known facts about fundamental solutions of the Laplacian and about harmonic functions on Riemann surfaces.

2.1.1 Extremal fundamental solutions and the fundamental metric

We write

$$\Delta := \sqrt{-1} \partial\bar{\partial}$$

for the Laplace operator. Note that this is the complex analytic convention, which is 1/4 of the usual Laplace operator one encounters in electromagnetism. Let $\delta_\zeta$ denote the Dirac mass at $\zeta$. The following definition is standard.

**Definition.** The Green’s function on a Riemann surface $X$ is the function $G : X \times X \to [-\infty, 0)$ with the following properties.

(a) For each $\zeta \in X$, $\Delta_\zeta G(z, \zeta) = \frac{\pi}{2} \delta_\zeta(z)$.

(b) If $H : X \times X \to [-\infty, 0)$ is another function with property (a), then $H(z, w) \leq G(z, w)$ whenever $z \neq w$.

It can easily be deduced that the Green’s function is symmetric.

Recall that a Riemann surface is said to be hyperbolic if it admits a bounded subharmonic function, elliptic if it is compact and parabolic otherwise. It is well known that a Riemann surface has a Green’s function if and only if it is hyperbolic. Property (b) guarantees that the Green’s function is unique.

On the other hand, a Riemann surface admits an Evans kernel if and only if it is parabolic (see page 352 of [NS-70]). Moreover, after prescribing (with somewhat limited possibility) the logarithmic singularity at infinity, the Evans kernel is unique up to an additive constant.

**Definition.** An Evans kernel on a Riemann surface $X$ is a symmetric function $S : X \times X \to [-\infty, +\infty)$ with the following properties.

(a) For each $\zeta \in X$, $\Delta_\zeta S(z, \zeta) = \frac{\pi}{2} \delta_\zeta(z)$.

(b) For each $r \in \mathbb{R}$ and $p \in X$, the level set \{ $\zeta \in X ; S(\zeta, p) = r$ \} is compact and non-empty.
We shall use the notation \( E : X \times X \to [-\infty, R_X] \) to denote either the Green’s function or some chosen Evans kernel, depending on whether the Riemann surface is hyperbolic or parabolic, respectively. Define

\[
R_X := \begin{cases} 
1 & \text{X is hyperbolic} \\
+\infty & \text{X is parabolic}
\end{cases}
\]

Using the extremal fundamental solution we next define a notion of distance on our Riemann surface, a distance that in general fails to satisfy the triangle inequality.

**Definition 2.1.** Let

\[
\rho_\zeta(z) := e^{E(z, \zeta)}, \quad D_\zeta(z) := \{ \zeta \in X : \rho_\zeta(\zeta) < \varepsilon \} \quad \text{and} \quad S_\zeta(z) = \partial D_\zeta(z).
\]

The fundamental metric \( e^{-2\nu} \) is given by the formula

\[
e^{-2\nu(z)}|dz|^2 = \lim_{\zeta \to z} |\partial \rho_\zeta(\zeta)|^2.
\]

**2.1.2 Green’s Formula and mean values**

Recall that on a Riemann surface with a conformal metric, the Hodge star operator simplifies somewhat when expressed in analytic coordinates \( z = x + \sqrt{-1}y \): if \( f \) is a real-valued function, \( \alpha = \alpha_1 dx + \alpha_2 dy \) is a real 1-form and \( \varphi dx \wedge dy \) is a real 2-form, then one has

\[
\ast f = f dA_g = e^{-2\psi} f dx \wedge dy \\
\ast \alpha = -\alpha_2 dx + \alpha_1 dy \\
\ast(\varphi dx \wedge dy) = e^{2\psi} \varphi.
\]

Using this, we have \( 4\Delta = d \ast d \) (recall that \( \Delta = \partial \partial = \frac{1}{4}(\partial_x^2 + \partial_y^2) \) in our convention), and Green’s formula can be written

\[
4 \int_D f \Delta h - h \Delta f = \int_{\partial D} f \ast dh - h \ast df.
\] (1)

Let \( X \) be an open Riemann surface and \( Y \subset X \) an open connected subset whose boundary consists of finitely many smooth Jordan curves. It is well known that the Green’s function \( G_Y \) for \( Y \) exists and is continuous up to the boundary. Moreover, the exterior derivative \( d(G_Y(\zeta, \cdot)) \) is also continuous up to the boundary.

**Remark.** One can construct the Green’s function \( G_Y \) from the extremal fundamental solution \( E \) of \( X \) as follows. Since \( Y \) has smooth boundary, the Dirichlet Problem of harmonic extension from the boundary can be solved on \( Y \). We then take

\[
G_Y(\zeta, z) := E(\zeta, z) - h_{\zeta}(z),
\]

where \( h_\zeta \) is the harmonic function in \( Y \) that agrees with \( E(\zeta, \cdot) \) on the boundary of \( Y \).

We write

\[
H_{r,\zeta}(z) := G_{D_r(\zeta)}(\zeta, z).
\]

In fact, the function \( H_{r,\zeta} \) has a particularly simple form in terms of the extremal fundamental solution \( E \):

\[
H_{r,\zeta}(z) = E(z, \zeta) - \log r, \quad z \in D_r(\zeta).
\] (2)

Moreover, in this case we don’t need to assume that \( r \) is a regular value of \( \rho_\zeta \).

Putting \( D = D_r(z) \) and \( h = H_{r, z} \) in (1) and using the definition of Green’s function, we obtain the following lemma.
Lemma 2.2. Let $r < R_E$ and $\zeta \in X$. Then
\[
2\pi f(z) = \int_{S_r(z)} f \ast dE_z + \int_{D_r(z)} H_{r,z} \Delta f.
\]
(3)

In particular, if $f$ is subharmonic, then
\[
f(z) \leq \frac{1}{2\pi} \int_{S_r(z)} f \ast dE_z
\]
(4)
with equality when $f$ is harmonic.

2.2 Finite Riemann surfaces

2.2.1 Definition and construction of finite Riemann surfaces

Recall that a finite Riemann surface is a two dimensional compact manifold with boundary, possibly with a finite number of points removed. Thus the topological data determining the Riemann surface is finite, hence the name.

There are two types of finite Riemann surfaces. One type has only punctures and no one dimensional components, while the other type does have smooth boundary components. The first type of is always parabolic (unless it has no punctures, in which case it is elliptic) while the second type is always hyperbolic.

An alternate description of a finite Riemann surface $X$ can be given as follows: $X$ is a (not necessarily compact) manifold with compact boundary, and in addition $X$ can be decomposed as
\[
X = X_{\text{core}} \cup \bigcup_{j=1}^{N} U_j,
\]
where $X_{\text{core}}$ is a compact manifold with smooth boundary, and each $U_j$ is biholomorphic to a punctured disk whose outer boundary is one of the smooth boundary curves of $X_{\text{core}}$. (Of course, $X_{\text{core}}$ may have some other boundary components that do not meet one of the $U_j$.) The $U_j$ correspond to the punctures.

While every finite Riemann surface with no one dimensional boundary is obtained from a compact Riemann surface by removal of a finite number of points, there is an almost equally simple way to construct hyperbolic finite Riemann surfaces; simply take a compact Riemann surface and remove a finite number of smooth Jordan curves so that the resulting surface as two components. Then either component is a finite Riemann surface, and one can further remove any finite number of points.

In fact, all finite Riemann surfaces are of this type. Indeed, we can fill in the punctures complex analytically (since they are just punctured disks) to obtain a compact Riemann surface with boundary
\[
\tilde{X} = X_{\text{core}} \cup \bigcup_{j=1}^{N} \overline{U_j},
\]
and then form the so-called double of $\tilde{X}$. For more on this well-known construction see, for example, [SS-54].

2.2.2 Analytic-Geometric properties of finite Riemann surfaces

We shall now derive certain analytic-geometric properties of finite Riemann surfaces that are useful in the proofs of our main theorems.

Theorem 2.3. Let $X$ be a finite Riemann surface with extremal fundamental solution $E$. Then for each sufficiently small $\sigma \in (0, R_X)$ there is a constant $C = C_\sigma$ such that for all $z \in X$ and all $\zeta \in D_\sigma(z)$ the following estimate holds.
\[
\frac{1}{C} \leq e^{2\nu} |\partial \rho_z(\zeta)|^2 \leq C.
\]
(5)
Before getting to the proof, we make a few observations. Suppose that we are given an extremal fundamental solution $E$ on our Riemann surface $X$. By the definition of a fundamental solution of the Laplacian, if $z$ is a local coordinate on $U \subset X$ then there exists a function $h_U(\zeta, \eta)$, harmonic in each variable separately, such that $h_U(\zeta, \eta) = h_U(\eta, \zeta)$ and

$$E(p, q) = \log |z(p) - z(q)| + h_U(z(p), z(q)).$$

The dependence of $h_U$ on $z$ is determined by the fact that $E$ is globally defined.

For simplicity of exposition, we abusively write

$$E(\zeta, \zeta) = \log |\zeta - z| + h(z, \zeta).$$

We then have that $\rho_z(\zeta) = |z - \zeta| e^{h(z, \zeta)}$ and, differentiating, we obtain

$$\partial \rho_z(\zeta) = \frac{\zeta - z}{|\zeta - z|} e^{h(z, \zeta)} \left( \frac{1}{2} + (z - \zeta) \partial_\zeta h(z, \zeta) \right).$$

It follows that

$$e^{-2\nu(\zeta)} = e^{2h(\zeta, \zeta)},$$

and

$$e^{2\nu(\zeta)} |\partial \rho_z(\zeta)|^2 = e^{2(h(z, \zeta) - h(\zeta, \zeta))} \left[ 1 + 2(z - \zeta) \frac{\partial h(z, \zeta)}{\partial \zeta} \right]^2. \quad (6)$$

We point out that this implies in particular that the right hand side of (6) is well defined, since this is the case for the left hand side.

**Proof of theorem 2.3.** We shall break up the proof into the hyperbolic and parabolic case.

**The case of bordered Riemann surfaces.**

We realize $X$ as an open subset of its double $Y$. Since $\overline{X} = X \cup \partial X$ is compact, it suffices to bound the right hand side of (6) in a set $U \cap X$, where $U$ is a coordinate chart in $Y$. For coordinate charts whose closure lies in the interior $X$, it is clear that this can be done. Indeed, if $U \subset X$ and $z, \zeta \in U$, then $h$ is a smooth function that is harmonic in each variable separately, and $\rho_z(\zeta) \asymp |\zeta - z|$ uniformly on $U$. Thus by taking $\sigma$ sufficiently small, we obtain the estimate (5) for all $z \in U$ and $\zeta \in D_\sigma(z)$. We thus restrict our attention to the boundary.

There are two types of boundary points; zero dimensional and one dimensional. However, the Green’s function ignores isolated zero dimensional boundary components, since they have capacity zero. (In particular, the distance $\rho_z$ fails to be proper when there are punctures.) Thus we may assume that there are no punctures.

Let $U \subset Y$ be a coordinate neighborhood of a boundary point $x \in \partial X$. By taking $U$ sufficiently small, we may assume that $U$ is the unit disk in the plane, that $U \cap X$ lies in the upper half plane and that $\partial X$ lies on the real line. It follows that the Green’s function is given by

$$E(z, \zeta) = \log |z - \zeta| - \log |\bar{z} - \zeta| + F(z, \zeta),$$

where $F(z, \zeta)$ is smooth and harmonic in each variable on a large open set containing the closure of $U$. Indeed, the Green’s function for the upper half plane is $\log |z - \zeta| - \log |\bar{z} - \zeta|$. The regularity of $F$ then follows from the construction of Green’s functions on finite Riemann surfaces using harmonic differentials on the double. (See [SS-54], §4.2.) It follows that in $U$,

$$2 \frac{\partial h(z, \zeta)}{\partial \zeta} = \frac{1}{z - \zeta} + 2 \frac{\partial F(z, \zeta)}{\partial \zeta} \quad \text{and} \quad \rho_z(\zeta) \geq C \frac{|z - \zeta|}{|\bar{z} - \zeta|}.$$

Thus

$$|2(z - \zeta) \frac{\partial h(z, \zeta)}{\partial \zeta}| \leq \frac{|z - \zeta|}{|\bar{z} - \zeta|} + 2 |z - \zeta| \left| \frac{\partial F(z, \zeta)}{\partial \zeta} \right| \leq C \frac{|z - \zeta|}{|\bar{z} - \zeta|} \leq C' \rho_z(\zeta),$$

6
where the constant $C'$ depends only on the neighborhood $U$. The proof in the hyperbolic case is thus complete.

**The case of compact Riemann surfaces with punctures.** Let $E$ be the Evans kernel of $X$. Fix $p \in X$ and choose $r$ so large that the set $X - D_r(z)$ is a union of punctured disks $U_1, \ldots, U_N$. We may think of each $U_j$ as sitting in $\mathbb{C}$, with the puncture at the origin.

Since $D_r(z) \subset X$, each $x \in D_r(z)$ has a neighborhood $U$ for which the expression (6) is bounded above and below by positive constants, depending only on $U$, whenever $\rho_z(\zeta) < \sigma$ for some sufficiently small $\sigma$ again depending only on $U$. Indeed, in any such neighborhood the function $h$ is very regular, and $\rho_z(\zeta)$ is uniformly comparable to $|z - \zeta|$.

For $z, \zeta \in U_j$, the Evans kernel has the form

$$E(z, \zeta) = \log|z - \zeta| - \lambda_j \log|\zeta| + F(z, \zeta),$$

where $\lambda_j > 0$ with $\lambda_1 + \ldots + \lambda_N = 1$, and $F(z, \zeta)$ is smooth across the origin (see [NS-70]). Indeed, using the method of constructing harmonic differentials with prescribed singularities (see [SS-54] §2.7) we can construct a function with the right singularities, defined everywhere on $X$. Such a function clearly can be written in the form (7) near the puncture. Thus by the uniqueness of the Evans kernel with prescribed singularities at the punctures, this function must differ from $E$ by a constant.

It follows that in $U$,

$$2 \frac{\partial h(z, \zeta)}{\partial \zeta} = -\frac{\lambda_j}{\zeta} + \frac{\partial F(z, \zeta)}{\partial \zeta}$$

and

$$\rho_z(\zeta) \geq C \frac{|z - \zeta|}{|\zeta|}.$$

Thus

$$2(|z - \zeta|) \frac{\partial h(z, \zeta)}{\partial \zeta} \leq \lambda_j \frac{|z - \zeta|}{|\zeta|} + 2|z - \zeta| \frac{\partial F(z, \zeta)}{\partial \zeta} \leq C \frac{|z - \zeta|}{|\zeta|} \leq C' \rho_z(\zeta),$$

where again the constant $C'$ depends only on the neighborhood $U$. The proof of Theorem 2.3 is thus complete. □

**Proposition 2.4.** Let $X$ be a finite Riemann surface. Then there exists a constant $C$ such that, for sufficiently small $\sigma > 0$ and all $z \in X$,

$$\sup_{w \in D_\sigma(z)} \exp \left( \frac{4}{\pi} \int_{D_{2\sigma}(z)} -G(w, \zeta)e^{-2\nu(\zeta)} \right) \leq C \inf_{w \in D_\sigma(z)} \exp \left( \frac{4}{\pi} \int_{D_{2\sigma}(z)} -G(w, \zeta)e^{-2\nu(\zeta)} \right) < +\infty,$$

where $G$ is the Green’s function for the domain $D_{2\sigma}(z)$.

**Sketch of proof.** Once again we can use compactness properties of finite surfaces. The finiteness of the integrals in question is easy, since extremal fundamental solutions have only a logarithmic singularity, and are thus locally integrable. Thus we restrict ourselves to estimating near the boundary.

The local analysis used in the proof of Theorem 2.3 shows that, near the boundary, the disks $D_\sigma(z)$ are simply connected and that the metric $e^{-2\nu}$ is equivalent to the Poincaré metric of the disk in the hyperbolic case, and the metric $|z|^{-2}dz^2$ in the parabolic case.

The hyperbolic case follows from the fact that the Green’s function $G(w, \zeta)$ is comparable to the Green’s function of the disk. In the parabolic case it is easier to work with the complement of the unit disk rather than the punctured
disk. Then the metric $e^{-2\nu}$ is comparable to the Euclidean metric, the Green’s function $G(w, \zeta)$ is comparable to the Green’s function of the plane, and the necessary estimate follows as in the Euclidean case. This completes the sketch of proof. \hfill \Box

We shall also have use for the following lemma.

**Lemma 2.5.** Let $X$ be a finite Riemann surface. Let $\sigma > 0$ be a fixed, sufficiently small constant. If $\phi$ is a function for which $e^{2\nu} \Delta \phi$ is bounded above and below by positive constants, then there is a constant $C = C_\sigma$ such that, for all $z \in X$ and all $w \in D_\sigma(z)$,

$$\exp \left( \frac{4}{\pi} \int_{D_{2\sigma}(z)} - G(w, \zeta) \Delta \phi(\zeta) \right) \leq C$$

(9)

**Proof.** By Theorem 2.3, Proposition 2.4 and the boundedness of $\Delta \phi$, it suffices to prove the result when $\Delta \phi(\zeta) = |d\rho_z(\zeta)|^2$ and $w = z$. In this case, it is easy to show that the integral is equal to $8\sigma^2$. \hfill \Box

The next result we will need is a global version of the Cauchy estimates on a Riemann surface with Riemannian metric.

**Proposition 2.6.** Let $X$ be a finite Riemann surface and let $g$ be a conformal metric for $X$. Then for every $\sigma \in [0, R_X)$ and $\varepsilon > 0$ there exists a constant $C_{\varepsilon, \sigma}$ such that for any $x \in X$ the following Cauchy estimates hold.

$$\sup_{D_\varepsilon(x)} |h|^2 \leq C_{\varepsilon, \sigma} \int_{D_\varepsilon(x)} |h|^2 dA_g,$$

(10)

and

$$\sup_{D_\varepsilon(x)} |\partial \rho|^2 |h'|^2 \leq C_{\varepsilon, \sigma} \int_{D_\varepsilon(x)} |h|^2 dA_g.$$

(11)

**Proof.** We begin with the following lemma.

**Lemma 2.7.** Let $X$ be a finite Riemann surface. Then for every $x \in X$ there exists a function $K^\varepsilon : X \times X \to \mathbb{R}$ such that the following hold for any $\sigma \in [0, R_X)$:

1. In the sense of distributions, $\Delta_z K^\varepsilon(z, \zeta) = \frac{\varepsilon^2}{2} \delta_z(\zeta)$ for all $z, \zeta \in D_\sigma(x)$.

2. For every $\varepsilon < \sigma/4$ there exists a constant $C_{\varepsilon, \sigma}$ such that for any $x \in X$ the following estimates hold:

$$\sup_{z \in D_\varepsilon(x)} \int_{V_{\sigma}(x)} e^{2\psi} \left| \frac{\partial \rho_z}{\partial \zeta} \frac{\partial K^\varepsilon(z, \zeta)}{\partial \zeta} \right|^2 \leq C_{\varepsilon, \sigma}$$

(12)

$$\sup_{z \in D_\varepsilon(x)} |\partial \rho| \left| e^{2\psi} \frac{\partial \rho_z}{\partial \zeta} \right|^2 \leq C_{\varepsilon, \sigma}$$

(13)

Here $V_\sigma(x) := D_\sigma(x) - D_{\sigma/2}(x)$.

**Sketch of proof.** In the case of a bordered Riemann surface with a finite number of punctures, one can find a function $K^\varepsilon$ that does not depend on the point $x$. This is done as follows. Let $Y$ be the double of $X$, and fix any smooth distance function on $Y$. We let $X_\varepsilon$ be the set of all $x \in Y$ that are a distance less than $\varepsilon$ from $X$. For $\varepsilon$ sufficiently small, $X_\varepsilon - X$ is a finite collection of annuli whose inner boundaries form the boundary of $X$. We may take for our Cauchy-Green kernel the Green’s function of $X_\varepsilon$. We leave it to the reader to check that the relevant estimates hold.

In the case of an $N$-punctured compact Riemann surface, one decomposes $X$ as

$$X = X_{\text{core}} \cup \bigcup_{j=1}^N U_j,$$

8
where $X_{\text{core}}$ is a bordered Riemann surface, and each $U_j$ is a neighborhood of a puncture biholomorphic to the punctured disk. Each surface in the union has a Cauchy-Green kernel by the construction in the bordered Riemann surface case, and thus we are done.

Completion of the proof of Proposition 2.6. Let $f \in C^\infty_0(D_\sigma(x))$ and write $K^x_\cdot(z) = K^x(z, \zeta)$. Applying formula (1) with $h(\zeta) = K^x(z,\zeta)$, we obtain

$$\pi^2 f(z) = \int_{D_\sigma(x)} K^x_z \partial \bar{f} = \int_{D_\sigma(x)} \partial f \wedge \partial K^x_z.$$  

Now let $\varepsilon < \sigma/4$ and let $\chi \in C^\infty_0([0, 3\sigma/4))$ be such that

$$\chi|[0, \sigma/2] \equiv 1 \quad \text{and} \quad \sup |\chi'| \leq \frac{5}{\sigma}.$$  

If $h \in \mathcal{O}(D_\sigma(x))$, then with $z \in D_\varepsilon(x)$ we have

$$h(z) = \int_{D_\sigma(x)} h\chi'(\rho_x) \overline{\partial \rho_x} \wedge \partial K^x_z.$$  

(14)

An application of the Cauchy-Schwarz inequality and the estimate (12) gives the inequality (10), while differentiation of (14) followed by an application of the Cauchy-Schwarz inequality and the estimate (13) gives inequality (11).

Remark. Note that were it not for the requirement that $C^\infty_{\varepsilon,\sigma}$ be independent of $x$, Proposition 2.6 would follow without (12) and (13).

2.2.3 Discrete subsets in finite Riemann surfaces

Let $X$ be an open Riemann surface. Our work on sampling and interpolation sequences requires the notion of the separation of a sequence. For a measurable subset $A \subset X$, let

$$D_r(A) = \{ w \in X : w \in D_r(a) \text{ for some } a \in A \}.$$  

We define two separation conditions on a sequence $\Gamma$, both of which are given in terms of the distance induced by the extremal fundamental solution.

**Definition 2.8.** Let $\Gamma \subset X$ be a discrete set.

1. The separation constant of $\Gamma$ is the number

$$\sigma(\Gamma) := \sup \{ r : D_r(\gamma) \cap D_r(\gamma') = \emptyset \},$$  

and say that $\Gamma$ is uniformly separated if $\sigma(\Gamma) > 0$.

2. We say $\Gamma$ is sparse if there is a positive constant $N_{r,\varepsilon}$, depending only on $0 < r, \varepsilon < R_X$, such that the number of points of $\Gamma$ lying in the set $D_r(D_\varepsilon(z))$ is at most $N_{r,\varepsilon}$ for all $z \in X$.

In both the complex plane and the unit disk, the triangle inequality allows one to easily show that a uniformly separated sequence is sparse.

In both of these situations, the triangle inequality allows one to estimate the diameter of a set $D_\varepsilon(D_r(a))$ in terms of $\varepsilon$ and $r$, uniformly for all $a$.

Such an estimate can always be found if it is allowed to depend on the base point $a$. This situation can be made uniform when $X$ is a finite Riemann surface. As in the proofs of Theorem 2.3 and Propositions 2.4 and 2.6, we can take advantage of the compactness in the picture. In particular, we have uniform estimates if we have them in neighborhoods of the boundary. But on the boundary, the potential theory of $X$ is either like that (near the boundary) in the upper half plane or (near infinity) in the plane, where we know, from triangle inequalities in those cases, that the needed estimates hold. We thus have the following proposition.

**Proposition 2.9.** In a finite Riemann surface $X$ every uniformly separated sequence is sparse.

We do not know whether Proposition 2.9 holds if one removes the finiteness condition.
3 Bergman spaces with metrics

It is also interesting to introduce, in addition to the weight in question, a metric. Thus, suppose in addition to an open Riemann surface \( X \), a discrete subset \( \Gamma \) and a weight function \( \varphi \), we are also given a conformal metric \( g \). Thus we modify our Hilbert spaces as follows:

\[
\mathcal{B}^2 = \mathcal{B}^2(X, g, \varphi) := \{ h \in \mathcal{O}(X) ; \| h \|^2 := \int_X |h|^2 e^{-2\varphi} dA_g < +\infty \},
\]

and

\[
b^2 = b^2(\Gamma, g, \varphi) := \left\{ (s_\gamma)_{\gamma \in \Gamma} ; \sum_{\gamma \in \Gamma} |s_\gamma|^2 e^{-2\varphi(\gamma)} A_g(D_{\sigma}(\gamma)) < +\infty \right\},
\]

where \( A_g(B) = \int_B dA_g \).

As before, we say that a uniformly separated sequence \( \Gamma \) is interpolating if for any \((s_\gamma) \in b^2\) there exists \( F \in \mathcal{B}^2 \) such that for all \( \gamma \in \Gamma \), \( F(\gamma) = s_\gamma \). On the other hand, the sequence \( \Gamma \) is sampling if there exists a constant \( M \) such that for all \( F \in \mathcal{B}^2 \),

\[
\frac{1}{M} \| F \|^2 \leq \sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-2\varphi(\gamma)} A_g(D_{\sigma}(\gamma)) \leq M \| F \|^2.
\]

(15)

To obtain sufficient conditions for interpolation and sampling sequences, the definition of the densities must be changed slightly.

3.1 The definition of the upper and lower densities

We associate to our metric \( g = e^{-2\varphi} \) the two functions

\[
u := \psi - \nu \quad \text{and} \quad \tau := e^{2\nu} \left( \Delta \psi + 2\Delta u_\psi - 2|\partial u_\psi|^2 \right).
\]

For each locally integrable function \( f : [0, R_X) \to [0, \infty) \) and each \( r \in (0, R_X) \), we associate to every uniformly separated sequence \( \Gamma \) upper and lower densities, defined by

\[
D_f^{+}(\Gamma) := \limsup_{r \to R_X} \sup_{z \in X} \sum_{\gamma \in \Gamma} \frac{\xi_r(\gamma, z)}{e^{2\nu(z)} \Delta \varphi(z) + \tau_\psi(z)},
\]

(16)

and

\[
D_f^{-}(\Gamma) := \liminf_{r \to R_X} \inf_{z \in X} \sum_{\gamma \in \Gamma} \frac{\xi_r(\gamma, z)}{e^{2\nu(z)} \Delta \varphi(z)},
\]

(17)

where \( \xi_r \) is defined as in section II.

3.2 A sub-mean value lemma

We will have occasion to use the following result.

Lemma 3.1. For any function \( F \),

\[
\int_X F(w) \xi_r(z, w) e^{-2\nu(w)} \sqrt{-1} dw \wedge d\bar{w} = \frac{1}{c_r} \int_{D_r(z)} F \chi (\rho_z) d\rho_z \wedge * d\rho_z
\]

\[
= \frac{1}{c_r} \int_0^r t f(t) \left( \int_{S_t(z)} F * dE_z \right) dt.
\]
Thus, in view of (4) of Lemma 2.2, if \( h \) is subharmonic then
\[
h(z) \leq \int_X \xi_r(z, w)h(w)e^{-2\nu(w)}\sqrt{-1}dw \land d\bar{w}
\]
(18)
with equality if \( h \) is harmonic.

3.3 Interpolation and sampling theorems

Theorem 3.2. Let \( X \) be a finite open Riemann surface with metric \( g = e^{-2\psi}|dz|^2 \) and let \( \varphi \) be a weight function on \( X \) such that, for some \( c > 1, \frac{1}{c} \leq e^{2\nu}\Delta \varphi \leq c \),
\[
e^{2\nu}\Delta(\varphi) + \tau_\varphi \geq \frac{1}{c} \quad \text{and} \quad e^{2\nu}|\partial u_\varphi|^2 \leq c.
\]
(19)
Then every uniformly separated sequence \( \Gamma \) satisfying \( D_f^+(\Gamma) < 1 \) is an interpolation sequence.

Theorem 3.3. Let \( X \) be a finite open Riemann surface with metric \( g = e^{-2\psi}|dz|^2 \) and \( \Gamma \subset X \) a uniformly separated sequence. Suppose \( \varphi \) is a weight function on \( X \) such that, for some \( C > 1, \frac{1}{C} \leq e^{2\nu}\Delta \varphi \leq C \). Assume also that the metric \( g \) is bounded above by the fundamental metric \( e^{-2\nu} \) (i.e., \( u_\psi \geq 0 \)) and moreover satisfies the differential inequality
\[
2e^{2\nu}|\partial u_\varphi|^2 \leq e^{2\nu}\Delta u_\varphi.
\]
(20)
Then \( \Gamma \) is a sampling sequence whenever \( D_f^-(\Gamma) > 1 \).

Theorems 3.2 and 3.3 imply Theorem 1.

3.4 Existence of metrics satisfying (20)

Let \( X \) be an open Riemann surface. Observe that a function \( u \) on \( X \) satisfies (20) if and only if
\[
\Delta(-e^{-2u}) \geq 0.
\]
Since the function \(-e^{-2u}\) is bounded above, we immediately obtain the following proposition.

Proposition 3.4. Let \( X \) be a compact or parabolic Riemann surface. Then any function satisfying the inequality \( (20) \) is constant.

In particular, we may assume that when \( X \) is parabolic, the metric \( g \) in Theorem 3.3 is just the fundamental metric.

Let us turn now to the hyperbolic case and suppose that \( o \in X \). Then
\[
\rho_o \Delta \rho_o = \rho_o^2 \Delta E(o, \cdot) + \rho_o^2 |\partial E(o, \cdot)|^2 = \rho_o^2 |\partial E(o, \cdot)|^2 = |\partial \rho_o|^2,
\]
and thus
\[
\Delta \rho_o^2 = 4|\partial \rho_o|^2 = |d\rho_o|^2.
\]
We let
\[
u = -\frac{1}{2}\log(1 - \rho_o^2).
\]
The reader versed in Several Complex Variables will recognize this function as the negative log-distance-to-the-boundary. Calculating, we have
\[
\Delta u - 2|\partial u|^2 = \left(\frac{\Delta \rho_o^2}{2(1 - \rho_o^2)} + \frac{|\partial \rho_o|^2}{2(1 - \rho_o^2)^2}\right) - 2 \cdot \frac{|\partial \rho_o|^2}{4(1 - \rho_o^2)^2} = \frac{|d\rho_o|^2}{2(1 - \rho_o^2)} \geq 0.
\]

11
Moreover, observe that
\[ |\partial u|^2 = \frac{\rho_o^2 |\partial \rho_o|^2}{(1 - \rho_o^2)^2}. \]

**Proposition 3.5.** Let \( X \) be a hyperbolic Riemann surface. Then there is always a non-trivial function \( u \geq 0 \) satisfying the differential inequality \( \Delta u \leq 0 \). Moreover, if \( X \) is a finite hyperbolic surface, then one can choose \( u \) such that \( e^{2\nu} |\partial u|^2 \) is uniformly bounded.

**Sketch of proof.** It remains only to verify the last assertion. By compactness, it suffices to prove the desired estimate in a neighborhood of the form \( \{ z \in \mathbb{C} : |z| < 1, \text{Im} z > 0 \} \) in the upper half plane. As above, one can take \( \nu = -\log \text{Im} z \) the Poincaré potential. Moreover, one can show that
\[ 1 - \rho_o(z) = \text{Im} z + \text{higher order terms}. \]

The proposition now follows. \( \square \)

## 4 A Theorem of Ohsawa on the solution of \( \overline{\partial} \)

In our proof of the interpolation theorem, we require a theorem for solving \( \overline{\partial} \) with certain \( L^2 \) estimates. Such a theorem has been stated by Ohsawa [O-94] in a very general situation, but there seem to be counterexamples at this level of generality (see [Siu-02]). However, in the case of Riemann surfaces there is a short proof of Ohsawa’s theorem. Since it is not easily accessible in the literature, we shall give a proof here using methods adapted from [Siu-02].

Let \( X \) be a Riemann surface with conformal metric \( g = e^{-2\psi} \) and let \( V \to X \) be a holomorphic line bundle with Hermitian metric \( h = e^{-2\xi} \) that is allowed to be singular, i.e., \( \xi \) may be in \( L^1_{\text{loc}} \). One has the Bochner-Kodaira identity
\[ ||\overline{\partial} \beta||^2 = ||\nabla \beta||^2 + (2e^{2\psi} \Delta (\xi + \psi) \beta, \beta), \] (21)

where
\[ \nabla (f d\bar{z}) := (f \bar{z} + 2\psi \bar{z} f)d\bar{z} \otimes \bar{z}. \]

Indeed, straight-forward calculations show that the formal adjoints \( \overline{\partial}^* \) of \( \partial \) and \( \nabla^* \) of \( \nabla \) are given by
\[ \overline{\partial}^* (hd\bar{z}) = -e^{2\psi} \left( \frac{\partial h}{\partial z} - 2\frac{\partial \xi}{\partial z} h \right) \text{ and } \nabla^* (hd\bar{z} \otimes \bar{z}) = -e^{2\psi} \left( \frac{\partial h}{\partial z} - 2\frac{\partial \xi}{\partial z} h \right). \] (22)

Using these, another calculation shows that \( \overline{\partial} \beta - \nabla^* \nabla \beta = 2e^{2\psi} \Delta (\xi + \psi) \beta \), which gives (21).

We shall now make a simple but far-reaching modification of the identity (21). To this end, let \( e^{-2\xi} \) and \( e^{-2(\xi-u)} \) be two metrics of the same line bundle. (Thus \( u \) is a globally defined function.) We assume moreover that \( e^{2(\psi-u)} |\partial u|^2 \) is uniformly bounded.

Formula (22) implies that
\[ \overline{\partial}_{\xi-u} \beta = \overline{\partial}_\xi \beta - 2e^{2\psi} \partial u \wedge \beta. \] (23)

Substituting (22) into the Bochner-Kodaira identity (21), we obtain
\[ ||\overline{\partial}_{\xi-u} \beta||^2_\xi = ||\nabla \beta||^2_\xi \]
\[ + (2e^{2\psi} \left\{ \Delta (\xi + \psi) - 2|\partial u|^2 \right\} \beta, \beta) \]
\[ - 2\text{Re}(\overline{\partial}^* \beta, 2e^{2\psi} \partial u \wedge \beta)_\xi. \] (24)

Identity (24) is sometimes called the Bochner-Kodaira identity with two weights. The Cauchy-Schwarz inequality then shows that for any \( \varepsilon > 0 \) we have
\[ (1 + \varepsilon^{-1}) ||\overline{\partial}_{\xi-u} \beta||^2_\xi \geq (2e^{2\psi} \left\{ \Delta (\xi + \psi) - 2(1+\varepsilon)|\partial u|^2 \right\} \beta, \beta)_\xi. \] (25)
Letting $Tf := \overline{\partial}(e^{-u}f)$, we can rewrite (15) as

$$||T^*\beta||^2_{\xi-u} \geq C\varepsilon \left( 2e^{2(\psi-u)} \{ \Delta(\xi + \psi) - 2(1 + \varepsilon)|\partial u|^2 \} \beta, \beta \right)_{\xi-u}. \tag{26}$$

Suppose now that, for some $\delta > 0$, one has the estimate

$$e^{2(\psi-u)} \{ \Delta(\xi + \psi) - 2|\partial u|^2 \} \geq \delta.$$ 

Since $e^{2(\psi-u)}|\partial u|^2$ is bounded, we may choose $\varepsilon > 0$ sufficiently small in (26) to obtain

$$||T^*\beta||^2_{\xi-u} \geq C||\beta||_{\xi-u}. \tag{27}$$

A standard Hilbert space argument yields a function $f$ such that

$$Tf = \alpha$$

with the estimate

$$\int_X |f|^2 e^{2u} e^{-2\xi} dA_g \leq C||\alpha||_{\xi-u}. \tag{28}$$

Finally, choosing $u = u_\psi = \psi - \nu$, $\varphi := \xi - 2u$ and $U = e^{-u}f$ gives the following.

**Theorem 4.1.** (O-94) Suppose that for some $\delta > 0$,

$$e^{2\nu} \Delta \varphi + \tau_\psi \geq \delta \quad \text{and} \quad e^{2\nu}|\partial \psi|^2 < \frac{1}{\delta}.$$ 

Then there exists a constant $C = C_\delta$ such that for any $\alpha$ satisfying

$$\int_X \sqrt{-1}\alpha \wedge \bar{\alpha} e^{-2u_\psi} e^{-2\varphi} < +\infty,$$

the equation $\overline{\partial}U = \alpha$ has a solution satisfying

$$\int_X |U|^2 e^{-2\varphi} dA_g \leq C \int_X \alpha \wedge \bar{\alpha} e^{-2u_\psi} e^{-2\varphi}.$$ 

5 Compact Riemann surfaces

At this point we take a short detour to consider the problems of sampling and interpolation on elliptic Riemann surfaces. While the essence of this situation is different from that of open Riemann surfaces, we note that the estimates on the solution of the $\overline{\partial}$ problem discussed in the previous section are applicable.

Let $X$ then be a compact Riemann surface and let $V \to X$ be a holomorphic line bundle. We denote by $V_x$ the fiber of $V$ over $x \in X$. Then $\Gamma$ is interpolating if and only if the evaluation map

$$H^0(X, L) \ni s \mapsto \sum_{\gamma \in \Gamma} s(\gamma) \in \bigoplus_{\gamma \in \Gamma} V_\gamma \tag{29}$$

is surjective, and sampling if and only if (29) is injective.

Let $\Lambda$ be the line bundle corresponding to the effective divisor $\Gamma$. One can understand the situation completely using the short exact sequence of sheaves

$$0 \to \mathcal{O}_X(L \otimes \Lambda^*) \to \mathcal{O}_X(L) \to \bigoplus_{\gamma \in \Gamma} V_\gamma \to 0.$$
where $V_{\gamma}(U) = V_{\gamma}$ if $\gamma \in U$ and $V_{\gamma}(U) = 0$ if $\gamma \notin U$. Passing to the long exact sequence, we have that

$$0 \rightarrow H^0(X, L \otimes \Lambda^*) \xrightarrow{\Theta} H^0(X, L) \xrightarrow{\Theta} \bigoplus_{\gamma \in \Gamma} V_{\gamma} \xrightarrow{\delta_0} H^1(X, L \otimes \Lambda^*) \xrightarrow{\iota_1} H^1(X, L) \rightarrow \ldots$$

We see that $\Theta$ is injective if and only if $\text{Image}(i_0) = \{0\}$ and surjective if and only if $i_1$ is injective, i.e., $\text{Image}(\delta_0) = \{0\}$. We then have the following proposition.

**Proposition 5.1.** Let $X$ be a compact Riemann surface of genus $g$, $\Gamma \subset X$ a finite subset and $L \rightarrow X$ a holomorphic line bundle.

1. If $\# \Gamma < \deg(L) + 2 - 2g$, then $\Gamma$ is interpolating.

2. If $\# \Gamma > \deg(L)$, then $\Gamma$ is sampling.

**Proof.** To establish 1, note that by Serre duality, $h^1(X, L \otimes \Lambda^*) = h^0(X, K_X \otimes \Lambda \otimes L^*)$, and the latter vanishes if

$$\# \Gamma + 2g - 2 - \deg(L) = \deg(K_X \otimes \Lambda \otimes L^*) < 0.$$ 

Similarly, if $\deg(L) - \# \Gamma = \deg(L \otimes \Lambda^*) < 0$, then $h^0(X, L \otimes \Lambda^*) = 0$.  

Part 1 of Proposition 5.1 can also be proved using Theorem 4.1. Because it is similar to the proof of our main interpolation theorem, we sketch this method here.

**Analytic proof of Proposition 5.1.** Let $\sum v_\gamma \in \bigoplus V_{\gamma}$. First, observe that there is a smooth section $\eta$ of $L$ such that $\eta(\gamma) = v_\gamma$ for all $\gamma \in \Gamma$. In fact, by the usual cutoff method, we can take $\eta$ supported near $\Gamma$ and holomorphic in a neighborhood of $\Gamma$.

Fix a conformal metric $e^{-2\varphi}$ on $X$. Let $\tau$ be the canonical section of $\Lambda$ corresponding to the divisor $\Gamma$. By the degree hypothesis, there is a metric $e^{-2\varphi}$ for the line bundle $L \otimes \Lambda^*$ such that the curvature current $\sqrt{-1} \Theta d(\varphi + \psi)$ of $L \otimes \Lambda^* \otimes K_X$ is strictly positive on $X$. Then $e^{-2\varphi}/|\tau|^2$ is a singular metric for $L$ such that the curvature current of $e^{-2(\varphi + \psi)}/|\tau|^2$ is still strictly positive on $X$. Moreover, since $\eta$ is holomorphic in a neighborhood of $\Gamma$, we have

$$\int_X |\overline{\partial}\eta|^2 |\tau|^2 e^{-2\varphi} < +\infty.$$ 

By Theorem 4.1 (with $u_\psi \equiv 0$; c.f. Proposition 3.4) there is a section $u$ of $L$ such that $\overline{\partial} u = \overline{\partial}\eta$ and $\int_X |u|^2 |\tau|^2 e^{-2(\varphi + \psi)} < +\infty$. But since $\tau$ vanishes on $\Gamma$, so does $u$. Thus $\sigma = \eta - u$ is holomorphic and solves the interpolation problem.

**Remark.** We note that if $e^{-2\varphi}$ is a metric for a holomorphic line bundle $L$, then we have

$$\deg(L) = \frac{1}{2\pi} \int_X \Delta \varphi,$$

showing the resemblance between Proposition 5.1 and our main theorems.

### 6 Proofs of Theorems 3.2 and 3.3

#### 6.1 Functions and singular weights

In this paragraph we define certain functions that play important roles in the proofs of Theorems 3.2 and 3.3.

**A local construction of a holomorphic function**

In the proofs of Theorems 3.2 and 3.3, we will need, for each $\gamma \in \Gamma$, a holomorphic function defined in a neighborhood of $\gamma$ and satisfying certain global estimates. For reasons that will become clear later, the size of this neighborhood cannot be taken too small. As a consequence, we must overcome certain difficulties presented by the topology of the neighborhood.
Lemma 6.1. Let $X$ be a finite open Riemann surface. Assume $e^{2\varphi} \Delta \varphi$ is bounded above and below by positive constants. Let $\Gamma$ be a uniformly separated sequence and $\sigma = \sigma(\Gamma)$. There exists a constant $C = C_{\Gamma} > 0$ and, for each $\gamma \in \Gamma$, a holomorphic function $F_{\gamma} \in \mathcal{O}(D_{\sigma}(\gamma))$ such that $F_{\gamma}(\gamma) = 0$ and for all $z \in D_{\sigma}(\gamma)$,

$$
\frac{1}{C} e^{-2\varphi(\gamma)} \leq |e^{-2\varphi + 2F_{\gamma}}| \leq C e^{-2\varphi(\gamma)}.
$$

Proof. Let $G$ be the Green’s function for the domain $D_{2\sigma}(\gamma)$. Consider the function

$$
T_{\gamma}(z) := \frac{2}{\pi} \int_{D_{2\sigma}(\gamma)} -G(z, \zeta) \Delta \varphi(\zeta).
$$

By Green’s formula, we have that

$$
T_{\gamma}(z) = -\varphi(z) + \frac{1}{2\pi} \int_{S_{2\sigma}(\gamma)} \varphi(\zeta) * d\zeta G(z, \zeta).
$$

We claim that the harmonic function

$$
h_{\gamma} := \frac{1}{2\pi} \int_{S_{2\sigma}(\gamma)} \varphi(\zeta) * d\zeta G(z, \zeta)
$$

has a harmonic conjugate, i.e., it is the real part of a holomorphic function. Indeed, if $C$ is a Jordan curve in $D_{r}(\gamma)$, then

$$
\int_{C} *dh_{\gamma}(z) = \frac{1}{2\pi} \int_{S_{2\sigma}(\gamma)} \varphi(\zeta) * d\zeta \left( \int_{C} *dE(z, \zeta) \right).
$$

Since $S_{2\sigma}(\gamma) \cap C = \emptyset$, the function $z \mapsto G(z, \zeta)$ is harmonic and thus $*dE(z, \zeta)$ is a closed form. It follows that the term in the parentheses on the right hand side of (31) depends only on the homology class $[C] \in H_{1}(X, \mathbb{Z})$. Since $H_{1}(X, \mathbb{Z})$ is discrete and $*dE(z, \zeta)$ is continuous in $\zeta$, we see that the right hand side of (31) vanishes, as claimed.

Let

$$
H_{\gamma} := h_{\gamma} + \sqrt{-1} \int_{\gamma} *dh_{\gamma}
$$

be the holomorphic function whose real part is $h_{\gamma}$, and let $F_{\gamma} := H_{\gamma} - H_{\gamma}(\gamma)$. We have

$$
|2\varphi(\gamma) - 2\varphi(z) + 2\text{Re} F_{\gamma}(z)| = 2 |T_{\gamma}(\gamma) - T_{\gamma}(z)| \leq 2|T_{\gamma}(\gamma)| + 2|T_{\gamma}(z)|.
$$

Taking exponentials and applying Lemma 2.3 completes the proof. \qed

A function with poles along $\Gamma$

For $z, \zeta \in X$ and $r < R_{X}$, let

$$
I(\zeta, z) = \int_{X} \xi_{r}(\zeta, w) E(w, z) e^{-2\nu(w)} \sqrt{-1} dw \wedge d\bar{w}.
$$

$$
= \frac{1}{c_{r}} \int_{a}^{r} tf(t) \left( \int_{S_{t}(\zeta)} E(w, z) * dE_{\zeta}(w) \right) dt.
$$

Since $E$ is a fundamental solution to the Laplacian,

$$
e^{2\nu(z)} \Delta I(\zeta, z) = \frac{\pi}{2} \int_{X} \xi_{r}(\zeta, w) \delta_{z}(w) = \frac{\pi}{2} \xi_{r}(\zeta, z).
$$

Next it follows from (18) that, since $E(\cdot, z)$ is subharmonic, $E(\zeta, z) \leq I(\zeta, z)$ and, since $E(\cdot, z)$ is harmonic in the region $\{w \in X : \rho_{\zeta}(w) > r\}$, $E(\zeta, z) = I(\zeta, z)$ if $\rho_{\zeta}(\zeta) > r$. Moreover, in view of (2), an application of (3) shows that

$$
\frac{1}{2\pi} \int_{S_{t}(\zeta)} E(w, z) * dE_{\zeta}(w) = E(z, \zeta) - 1_{D_{t}(\zeta)}(\zeta) (E(z, \zeta) - \log t).
$$

15
We see that
\[
I(\zeta, z) = \frac{2\pi}{c_r} \left( \log(p_\zeta(\zeta)) \int_0^{p_\zeta(\zeta)} tf(t)dt + \int_{p_\zeta(\zeta)}^r tf(t) \log t dt \right)
\]
if \( p_\zeta(\zeta) < r \). Note that
\[
\left| \frac{1}{c_r} \int_{p_\zeta(\zeta)}^r tf(t) \log(t) dt \right| \leq D_r,
\]
where \( D_r \) depends only on \( r \). We then have
\[
|I(\zeta, z)| \leq K_r p_\zeta(\zeta) |\log(p_\zeta(\zeta))| + D_r
\]
for all \( z, \zeta \in X \) satisfying \( p_\zeta(\zeta) < r \). Since the expression on the right hand side is bounded by a constant that depends only on \( r \), we have
\[
|I(\zeta, z)| \leq C_r
\]
whenever \( p_\zeta(\zeta) < r \).

Let \( \Gamma \) be a discrete sequence. We define the function
\[
v_\Gamma(z) = \sum_{\gamma \in \Gamma} (E(\gamma, z) - I(\gamma, z)).
\]
By the preceding remarks, \( v_\Gamma(z) \leq 0 \) and
\[
v_\Gamma(z) = \sum_{\gamma \in \Gamma \cap D_r(z)} (E(\gamma, z) - I(\gamma, z)).
\]
Moreover,
\[
e^{2\nu} \Delta v_\Gamma = \frac{\pi}{2} \sum_{\gamma \in \Gamma} (e^{2\nu} \delta_\gamma - \xi_\Gamma(\gamma, \cdot)).
\]
Writing
\[
X_{\Gamma, \varepsilon} := \left\{ z \in X : \min_{\gamma \in \Gamma} \rho_\gamma(z) > \varepsilon \right\},
\]
we have the following lemma.

**Lemma 6.2.** Let \( \Gamma \) be a sparse, uniformly separated sequence and let \( \varepsilon \leq \sigma(\Gamma) \). The function \( v_\Gamma \) is uniformly bounded on \( X_{\Gamma, \varepsilon} \). Moreover, \( v_\Gamma \) satisfies the following estimate: if \( \gamma \in \Gamma \) and \( \rho_\gamma(z) < \sigma(\Gamma) \), then
\[
|v_\Gamma(z) - \log \rho_\gamma(z)| \leq C_{r, \varepsilon}.
\]

**Proof.** Let \( z \in X_{\Gamma, \varepsilon} \). Since \( \Gamma \) is sparse, there are at most \( N = N_{r, \varepsilon} \) members of \( \Gamma \), say \( \gamma_1, \ldots, \gamma_N \), lying in \( D_r(z) \), and so
\[
|v_\Gamma(z)| \leq \sum_{j=1}^N (|E(\gamma_j, z)| + |I(\gamma_j, z)|) \leq \sum_{j=1}^N (|\log(\rho_\gamma(\gamma_j))| + C_r).
\]
Note that the number \( N \) does not depend on \( z \). Since \( \varepsilon < \rho_\gamma(\gamma_j) < r \), the term involving the logarithm has a bound that depends only on \( \varepsilon \) and \( r \). We thus see that \( v_\Gamma \) is uniformly bounded on \( X_{\Gamma, \varepsilon} \).

Let \( \gamma \in \Gamma \). Since \( \Gamma \) is sparse, there are at most \( N = N_{r, \varepsilon} \) elements of \( \Gamma \) that lie in \( D_r(D_\varepsilon(\gamma)) \). We write \( \Gamma \cap D_r(D_\varepsilon(\gamma)) = \{ \gamma_1, \ldots, \gamma_N \} \), where \( \gamma_1 = \gamma \). Again, \( N \) does not depend on \( z \). Then
\[
|v_\Gamma(z) - \log \rho_\gamma(\gamma)| \leq \left( \sum_{j=2}^N |E(\gamma_j, z)| + \sum_{j=1}^N |I(\gamma_j, z)| \right) + |E(\gamma, z) - \log \rho_\gamma(\gamma)|.
\]
The first sum is bounded because \( \sigma(\Gamma) < \rho_\gamma(\gamma_j) < r \) for \( j = 2, \ldots, N \). The second sum is bounded by \( (32) \), and the third term vanishes. This completes the proof of the lemma.
A function with bumps along $\Gamma$

In this paragraph, we shall use area forms associated to the points of $\Gamma$. We define

$$dA_{E,\gamma}(\zeta) := d\rho_{\gamma}(\zeta) \wedge \ast d\rho_{\gamma}(\zeta).$$

Let

$$A_{E,\gamma}(D) := \int_D dA_{E,\gamma}.$$ 

Given a distribution $f$, we consider its regularization

$$\frac{1}{A_{E,\gamma}(D_\varepsilon(z))} \int_{D_\varepsilon(z)} f dA_{E,\gamma}$$

using the area element $dA_{E,\gamma}$, where $\gamma \in \Gamma$.

Observe that

$$A_{E,\gamma}(D_\varepsilon(\gamma)) = \int_{D_\varepsilon(\gamma)} d\rho_{\gamma} \wedge \ast d\rho_{\gamma} = \int_{D_\varepsilon(\gamma)} \rho_{\gamma} d\rho_{\gamma} \wedge \ast dE(\gamma, \cdot)$$

$$= \int_0^\varepsilon t \left( \int_{S_\varepsilon(\gamma)} \ast dE(\gamma, \cdot) \right) dt = 2\pi \int_0^\varepsilon t dt = \pi \varepsilon^2. \quad (35)$$

Consider the function

$$v_{r,\varepsilon}(z) = t \sum_{\gamma \in \Gamma} \frac{1}{\pi \varepsilon^2} \int_{D_\varepsilon(\gamma)} (E(\zeta, z) - I(\zeta, z)) dA_{E,\gamma}(\zeta)$$

where $0 << t < 1$.

**Lemma 6.3.** The function $v_{r,\varepsilon}$ has the following properties.

1. 

$$e^{2\nu(z)} \Delta v_{r,\varepsilon}(z) = t \sum_{\gamma \in \Gamma} \frac{1}{2\varepsilon^2} e^{2\nu(z)} |d\rho_{\gamma}(z)|^2 1_{D_\varepsilon(z)}$$

$$- t \sum_{\gamma \in \Gamma} \frac{1}{2\varepsilon^2} \int_{D_\varepsilon(\gamma)} \xi_r(\gamma, \cdot) dA_{E,\gamma},$$

In particular,

$$\lim_{\varepsilon \to 0} e^{2\nu} \Delta v_{r,\varepsilon} = \frac{\pi}{2} t \sum_{\gamma \in \Gamma} (e^{2\nu} \delta_\gamma - \xi_r(\gamma, \cdot))$$

in the sense of distributions.

2. There exists a positive constant $C_{r,\varepsilon}$ such that

$$z \in X \Rightarrow -C_{r,\varepsilon} \leq v_{r,\varepsilon}(z) \leq 0 \quad (36)$$

and for any $\gamma \in \Gamma$,

$$\rho_{\gamma}(z) \Rightarrow v_{r,\varepsilon}(z) = \frac{t}{\pi \varepsilon^2} \int_{D_\varepsilon(\gamma)} E(\zeta, z) dA_{E,\gamma}(\zeta) \leq C_{r,\varepsilon} \quad (37)$$
Proof. 1. The formula for the Laplacian is a straightforward calculation, and the limit is a standard consequence of the regularization of currents.

2. Since $E(\zeta, z) = I(\zeta, z)$ whenever $\rho_z(\zeta) > r$, we have, in view of formula (35),

$$v_{r, \varepsilon}(z) = \sum_{\gamma \in D_\varepsilon(D_r(z))} \frac{t}{\varepsilon^2} \int_{D_\varepsilon(\gamma)} (E(\zeta, z) - I(\zeta, z)) dA_{E, \gamma}(\zeta).$$

Choose $\gamma \in \Gamma$. Since $\Gamma$ is sparse, there exist $\gamma_1, \ldots, \gamma_N \in \Gamma - \{\gamma\}$ such that for all $z \in D_\varepsilon(\gamma)$

$$v_{r, \varepsilon}(z) = \frac{t}{\varepsilon^2} \int_{D_\varepsilon(\gamma)} (E(\zeta, z) - I(\zeta, z)) dA_{E, \gamma}$$

Moreover, $N$ is independent of $\gamma$, and depends only on $r$ and $\varepsilon$. It follows that

$$\left|v_{r, \varepsilon}(z) - \frac{t}{\varepsilon^2} \int_{D_\varepsilon(\gamma)} E(\cdot, z) dA_{E, \gamma}\right|$$

$$\leq \frac{t}{\varepsilon^2} \int_{D_\varepsilon(\gamma)} |I(\cdot, z)| dA_{E, \gamma} + \sum_{j=1}^N \frac{t}{\varepsilon^2} \int_{D_\varepsilon(\gamma_j)} (|E(\cdot, z)| + |I(\cdot, z)|) dA_{E, \gamma_j}.$$

We have estimates for $I(\zeta, z)$ as in the proof of Lemma 6.2, and since, by uniform separation, $\rho_z(\zeta) > \sigma$ for any $\zeta \in D_\varepsilon(\gamma_j)$, we can estimate the right hand side by a constant that depends only on $r$. This proves (37), and (36) follows from (35). Lemma 6.2 and the fact that $v_r \leq 0$. \hfill \Box

Finally, we shall have use for the following lemma.

Lemma 6.4. For any $z \in D_\varepsilon(\gamma)$,

$$\frac{1}{A_{E, \gamma}(D_\varepsilon(\gamma))} \int_{D_\varepsilon(\gamma)} E(z, \zeta) dA_{E, \gamma}(\zeta) \leq \log \frac{1}{\varepsilon} + \frac{1}{2}$$

(38)

Proof. Observe that if $z \in D_\varepsilon(\gamma)$ and $t \in (0, \varepsilon]$, then

$$\int_{S_{t}(\gamma)} d\zeta E(z, \zeta) = \int_{D_\varepsilon(\gamma)} d\zeta * d\zeta E(z, \zeta) = 2\pi \mathbf{1}_{D_\varepsilon(\gamma)}(z) \leq 2\pi.$$

Applying Green’s formula (11) with $f = E(z, \cdot)$ and $h = E(\gamma, \cdot)$, we obtain

$$\int_{S_{t}(\gamma)} E(z, \zeta) * d\zeta E(\gamma, \zeta) = \int_{S_{t}(\gamma)} E(\gamma, \zeta) * d\zeta E(z, \zeta).$$

We thus have

$$- \int_{D_\varepsilon(\gamma)} \log \rho_z d\rho_{\gamma} * d\rho_{\gamma} = - \int_0^\varepsilon t \left( \int_{S_{t}(\gamma)} E(z, \zeta) * d\zeta E(\gamma, \zeta) \right) dt$$

$$= - \int_0^\varepsilon t \left( \int_{S_{t}(\gamma)} E(\gamma, \zeta) * d\zeta E(z, \zeta) \right) dt$$

$$= - \int_0^\varepsilon t \log t \left( \int_{S_{t}(\gamma)} d\zeta E(z, \zeta) \right) dt$$

$$\leq -2\pi \int_0^\varepsilon t \log t dt = \pi \varepsilon^2 \left( \frac{1}{2} - \log \varepsilon \right).$$

The lemma now follows from (35). \hfill \Box
6.2 Proof of Theorem 3.2

Let \((s\gamma) \in b^2(\Gamma, g, \varphi)\). We begin by constructing a smooth function \(\eta \in L^2(X, g, \varphi)\) that interpolates \((s\gamma)\). To this end, let \(\chi \in C_\infty^0([0, \sigma))\) satisfy

\[
0 \leq \chi \leq 1, \quad \chi[0, \sigma/2] \equiv 1 \quad \text{and} \quad |\chi'| \leq \frac{3}{\sigma}.
\]

We define

\[
\eta(z) := \sum_{\gamma \in \Gamma} \chi \circ \rho_\gamma(z) s\gamma e^{F_\gamma(z)},
\]

where \(F_\gamma\) is as in Lemma 6.1. Observe that \(\eta(\gamma) = s\gamma\) for all \(\gamma \in \Gamma\), and that

\[
\int_X |\eta|^2 e^{-2\varphi} dA_g = \sum_{\gamma \in \Gamma} |s\gamma|^2 \int_{D_\sigma(\gamma)} |\chi \circ \rho_\gamma|^2 \ e^{2(F_\gamma - 2\varphi)} \ dA_g
\]

\[
\leq C \sum_{\gamma \in \Gamma} |s\gamma|^2 e^{-2\varphi(\gamma)} A_g(D_\sigma(\gamma)) < +\infty.
\]

Next we wish to correct \(\eta\) by adding to it a function \(U\) that lies in \(L^2(X, g, \varphi)\) and vanishes along \(\Gamma\). The standard approach is to solve the equation \(\partial U = \partial \eta\) with singular weights, using Ohsawa’s \(\partial\) Theorem 4.1. The singular weight we will use is the weight \(\tilde{\varphi} := \varphi + vr\), and one computes that

\[
\partial \eta = \sum_{\gamma \in \Gamma} \chi'(\rho_\gamma) \partial \rho_\gamma s\gamma e^{F_\gamma}.
\]

(39)

Since \(D_f^J(\Gamma) < 1\), there exist \(r < R_X\) and \(\delta > 0\) such that

\[
e^{2\nu} \Delta \tilde{\varphi} + \tau_\psi = e^{2\nu} \Delta \varphi + \tau_\psi + e^{2\nu} \Delta vr
\]

\[
\geq (e^{2\nu} \Delta \varphi + \tau_\psi) \left(1 - \sum_{\gamma \in \Gamma} e^{2\nu} \Delta \tilde{\varphi} + \tau_\psi\right)
\]

\[
> \delta (e^{2\nu} \Delta \varphi + \tau_\psi),
\]

where the first inequality follows from (33). It follows from hypothesis (19) in Theorem 3.2 that

\[
e^{2\nu} \Delta \tilde{\varphi} + \tau_\psi \geq C\delta > 0.
\]

Next, (39) and Lemma 6.2 imply that \(\tilde{\varphi}\) is comparable to \(\varphi\) on the support of \(\partial \eta\), which lies in \(V_\sigma(\gamma) := D_\sigma(\gamma) - D_\varphi(\gamma)\). We then have the estimate

\[
\int_X |\partial \eta|^2 e^{-2u_\psi} e^{-2\tilde{\varphi}} \leq \frac{C}{\sigma^2} \sum_{\gamma \in \Gamma} |s\gamma|^2 e^{-2\varphi(\gamma)} \int_{V_\sigma(\gamma)} |\partial \rho_\gamma|^2 e^{-2u_\psi}
\]

\[
\leq \frac{C}{\sigma^2} \sum_{\gamma \in \Gamma} |s\gamma|^2 e^{-2\varphi(\gamma)} \int_{D_\sigma(\gamma)} |\partial \rho_\gamma|^2 e^{-2u_\psi}
\]

\[
\leq C' \sum_{\gamma \in \Gamma} |s\gamma|^2 e^{-2\varphi(\gamma)} \int_{D_\sigma(\gamma)} e^{-2(u_\psi + u_\varphi)}
\]

\[
< +\infty,
\]

19
where the first inequality follows from Lemma 6.1 and the last inequality follows from 5. Applying Theorem 4.1, we obtain a function \( U \in L^2(X, g, \tilde{\varphi}) \subset L^2(X, g, \varphi) \) such that \( \nabla U = \partial \eta \). Moreover, since \( e^{-2\tilde{\varphi}} \sim \frac{1}{|z-\gamma|^s} \) for \( z \) sufficiently close to \( \gamma \), we see that \( U(\gamma) = 0 \) for all \( \gamma \in \Gamma \). Thus the function
\[
 f := \eta - U \in \mathcal{B}^2(X, g, \varphi)
\]
interpolates \((s, \gamma)\), and the proof of Theorem 5.2 is complete. \( \square \)

### 6.3 Proof of Theorem 3.3

Let \( \tilde{\varphi} := \varphi + \nu_{r, \epsilon} \). The main idea behind the proof of Theorem 3.3 is the following sampling type lemma.

**Lemma 6.5.** Suppose the metric \( e^{-2\psi} \) satisfies the differential inequalities (20). For each \( h \in \mathcal{B}^2(X, g, \tilde{\varphi}) \),
\[
\int_X |h|^2 e^{-2\tilde{\varphi}} e^{2\nu} \Delta \tilde{\varphi} dA_g \geq 0.
\]

**Proof.** Consider the function \( S = |h|^2 e^{-2\tilde{\varphi}} \). Then
\[
\frac{\Delta S}{S} = \Delta \log S + \frac{1}{S^2} |\partial S|^2 = \frac{1}{S^2} |\partial S|^2 + \Delta \log |h|^2 - 2\Delta \tilde{\varphi}
\]
and thus
\[
e^{2\nu} \Delta S \geq -2S \left( e^{2\nu} \Delta \tilde{\varphi} \right) \geq -2S \left( e^{2\nu} \Delta \tilde{\varphi} \right).
\]

We claim that
\[
\int_X e^{2\nu} \Delta S \, dA_g \leq 0.
\]

To prove the claim, let \( z_0 \in X \). Take \( \lambda \in C^\infty_0([0, 1/2]) \) such that \( \lambda(t) \equiv 1 \) for \( 0 \leq t \leq 1/4 \), and put
\[
\chi_a(r) := \lambda(r^2(1-a)).
\]

Then
\[
\int_X e^{-2(\psi - \nu)} \Delta S = \int_X e^{-2u} \Delta S
\]
\[
= \lim_{a \to 1} \int_X e^{-2u} \chi_a \circ \rho_{z_0} \Delta S
\]
\[
= \lim_{a \to 1} \int_X S \Delta (e^{-2u} \cdot (\chi_a \circ \rho_{z_0}))
\]
\[
= \lim_{a \to 1} \int_X S \left( (\Delta(e^{-2u}) \chi_a \circ \rho_{z_0} + \partial(e^{-2u}) \wedge \nabla(\chi_a \circ \rho_{z_0})
\right.
\]
\[
\left. + \partial(e^{-2u}) \wedge \nabla(\chi_a \circ \rho_{z_0}) + e^{-2u} \Delta(\chi_a \circ \rho_{z_0}) \right)
\]
\[
= \lim_{a \to 1} \int_X S \left( (\Delta(e^{-2u}) \chi_a \circ \rho_{z_0} + e^{-2u} \Delta(\chi_a \circ \rho_{z_0})) \right),
\]

where the third equality follows from Stokes' Theorem. Now,
\[
\lim_{a \to 1} \int_X S e^{-2u} \Delta(\chi_a \circ \rho_{z_0})
\]
\[
= \lim_{a \to 1} \int_X S e^{-2u} \left( \chi_a''(\rho_{z_0}) |\partial \rho_{z_0}|^2 + \chi_a'(\rho_{z_0}) \Delta \rho_{z_0} \right)
\]
\[
= \lim_{a \to 1} \int_X S e^{-2u} \left( \chi_a''(\rho_{z_0}) + \frac{\chi_a'(\rho_{z_0})}{\rho_{z_0}} \right) |\partial \rho_{z_0}|^2
\]
\[
= \lim_{a \to 1} \int_X |h|^2 e^{-2\tilde{\varphi}} \left( \chi_a''(\rho_{z_0}) + \frac{\chi_a'(\rho_{z_0})}{\rho_{z_0}} \right) e^{2\nu} |\partial \rho_{z_0}|^2 dA_g = 0,
\]

20
where the last equality follows from (5) and the definition of $\chi_a$. It follows that
\[
\int_X e^{-2(\psi-\nu)} \Delta S = \int_X S \Delta e^{-2u_\psi}.
\]

Since
\[
\Delta e^{-2u_\psi} = 2e^{-2u_\psi} (2|\partial u_\psi|^2 - \Delta u_\psi) = 2e^{-2\psi} e^{2\nu} (2|\partial u_\psi|^2 - \Delta u_\psi),
\]
the lemma now follows from (20).

**Conclusion of the proof of Theorem 3.3.** Let $h \in \mathcal{H}^2(X, g, \varphi)$. By Lemma 6.3 we calculate that
\[
e^{2\nu(z)} \Delta \hat{\varphi}(z) = e^{2\nu(z)} \Delta \varphi(z) + e^{2\nu(z)} \Delta v_r,\varepsilon(z)
\]
\[
= e^{2\nu(z)} \Delta \varphi(z) \left( 1 - t \sum_{\gamma \in \Gamma} \frac{1}{2\varepsilon^2} \int_{D_\varepsilon(\gamma)} \frac{\xi_\varepsilon(\zeta, z)}{e^{2\nu(z)} \Delta \varphi(z)} dA_{E, \gamma}(\zeta)
\right.
\]
\[
+ t \sum_{\gamma \in \Gamma} e^{2\psi(z)} \frac{1}{\varepsilon^2} \frac{e^{2\nu(z)} |\partial \rho_\gamma(z)|^2}{e^{2\nu(z)} \Delta \varphi(z)} 1_{D_\varepsilon(\gamma)}(z) \right).
\]

Applying the hypotheses $D^*_f(\Gamma) > 1$ and (5), we see therefore that, for $t$ sufficiently close to 1, there exist $\rho, \delta, C > 0$ such that
\[
e^{2\nu} \Delta \hat{\varphi} \leq -t e^{2\nu} \Delta \varphi \left( 1 - C \sum_{\gamma \in \Gamma} e^{2\psi} \frac{2}{\varepsilon^2} 1_{D_\varepsilon(\gamma)} \right).
\]

(41)

We then apply Lemma 6.3 to get
\[
\int_X |h|^2 e^{-2\varphi} dA_g \leq \int_X |h|^2 e^{-2\hat{\varphi}} dA_g
\]
\[
\leq C \int_X e^{2\nu} \Delta(\varphi)|h|^2 dA_g
\]
\[
\leq C' \sum_{\gamma \in \Gamma} \frac{2}{\varepsilon^2} \int_{D_\varepsilon(\gamma)} e^{2\nu} \Delta(\varphi)|h|^2 e^{-2\hat{\varphi}} dA_g
\]
\[
\leq C'' \sum_{\gamma \in \Gamma} \frac{2}{\varepsilon^2} \int_{D_\varepsilon(\gamma)} |h|^2 e^{-2\hat{\varphi}} dA_g
\]
\[
\leq C''' \sum_{\gamma \in \Gamma} \frac{2}{\varepsilon^2 + \varepsilon^2} \int_{D_\varepsilon(\gamma)} |h|^2 e^{-2\varphi} dA_g,
\]

where the first inequality follows from Lemma 6.3, the third inequality follows from integration of (41) together with
Lemma 6.5 and the last inequality follows from Lemmas 6.3 and 6.4. Now,
\[
\int_{D_\gamma(x)} |h|^2 e^{-2\varphi} dA_g = \int_{D_\gamma(x)} |he^{-F_\gamma}|^2 e^{-2\varphi + 2ReF_\gamma} dA_g
\]
\[
\leq Ce^{-2\varphi(x)} \int_{D_\gamma(x)} |he^{-F_\gamma}|^2 dA_g
\]
\[
\leq C' A_g(D_\gamma(x)) e^{-2\varphi(x)} \left( |h(\gamma)|^2 + \varepsilon^2 \sup_{D_\gamma(x)} \frac{|(he^{-F_\gamma})|^2}{|\partial \rho_\gamma|^2} \right)
\]
\[
\leq C' A_g(D_\gamma(x)) e^{-2\varphi(x)} |h(\gamma)|^2 + \varepsilon^2 C'' A_g(D_\gamma(x)) \int_{D_\gamma(x)} |h|^2 e^{-2\varphi} dA_g,
\]
where the first and last inequalities follow from Lemma 6.1, the second inequality follows from Taylor’s theorem, and the third inequality from the Cauchy estimate (11). Next, since \( X \) is fundamentally finite and \( e^{-2\varphi} \leq e^{-2\nu} \), we see that
\[
A_g(D_\gamma(x)) \leq \int_{D_\gamma(x)} e^{-2\nu} \leq C \int_{D_\gamma(x)} |\partial \rho_\gamma|^2 = \pi C \varepsilon^2
\]
for all sufficiently small \( \varepsilon \) and some \( C \) independent of \( \gamma \), where the last equality follows from (14). We thus obtain
\[
\int_X |h|^2 e^{-2\varphi} dA_g \leq \sum_{\gamma \in \Gamma} \left( \frac{C_1}{\varepsilon^{2+2t}} |h(\gamma)|^2 e^{-2\varphi(\gamma)} A_g(D_\gamma(x)) + C_2 \varepsilon^{2-2t} \int_{D_\gamma(x)} |h|^2 e^{-2\varphi} dA_g \right)
\]
\[
\leq \sum_{\gamma \in \Gamma} \left( \frac{C_1}{\varepsilon^{2+2t}} |h(\gamma)|^2 e^{-2\varphi(\gamma)} A_g(D_\gamma(x)) \right) + C_2 \varepsilon^{2-2t} \int_X |h|^2 e^{-2\varphi} dA_g.
\]
By taking \( \varepsilon \) sufficiently small, we obtain the left hand side of (15). For the right hand side of (15), we argue as follows.
\[
\sum_{\gamma \in \Gamma} |h(\gamma)|^2 e^{-2\varphi(\gamma)} A_g(D_\gamma(x)) = \sum_{\gamma \in \Gamma} |h(\gamma)e^{-F_\gamma(\gamma)}|^2 e^{-2\varphi(\gamma)} A_g(D_\gamma(x))
\]
\[
\leq C \sigma^2 \sum_{\gamma \in \Gamma} e^{-2\varphi(\gamma)} \int_{D_\gamma(x)} |he^{-F_\gamma}|^2 dA_g
\]
\[
\leq C' \sigma \int_{\Gamma} |h|^2 e^{-2\varphi} dA_g
\]
\[
\leq C'' \int_X |h|^2 e^{-2\varphi} dA_g,
\]
where the first inequality follows from (10), the second from Lemma 6.1 and the third from the definition of the separation constant. The proof of Theorem 3.3 is thus complete. \( \square \)
7 Examples

7.1 The Euclidean plane

In this paragraph, we consider the case of the Euclidean complex plane \((X, g) = (\mathbb{C}, |dz|^2)\). The generalized Bergman space in this situation is

\[
\mathcal{BF}^2 = \left\{ h \in \mathcal{O}(\mathbb{C}) : ||h||^2_\varphi := \int_{\mathbb{C}} |h|^2 e^{-2\varphi} \, dm < +\infty \right\},
\]

where \(dm\) is Lebesgue measure in the plane, and

\[
\mathfrak{b}f^2 = \left\{ (s_\gamma) \subset \mathbb{C} : ||(s_\gamma)||^2_\varphi := \sum_{\gamma \in \Gamma} |s_\gamma|^2 e^{-2\varphi(\gamma)} < +\infty \right\}.
\]

The space \(\mathcal{BF}^2\) is sometimes called generalized Bargmann-Fock space. When \(\varphi(z) = |z|^2/2\) we obtain the classical Bargmann-Fock space.

The plane is the main example of a parabolic Riemann surface. The Evans kernel in \(\mathbb{C}\) is unique and is given by

\[
E(z, \zeta) = \log |z - \zeta|.
\]

Thus \(\rho_z(\zeta) = |z - \zeta|\) and the disks \(D_\sigma(z)\) are simply the Euclidean disks \(|z - \zeta| < \sigma\). A simple calculation shows that

\[
|d\rho_z(\zeta)|^2 = 4|\partial\rho_z(\zeta)|^2 = 1,
\]

and thus the fundamental metric is just a multiple of the Euclidean metric.

The upper and lower densities are given by

\[
D_f^+(\Gamma) = \limsup_{r \to \infty} \sup_{z \in \mathbb{C}} \frac{\sum_{\Gamma \cap D_r(z)} f(|z - \gamma|)}{4\Delta \varphi \int_0^r t f(t) \, dt}
\]

and

\[
D_f^-(\Gamma) = \liminf_{r \to \infty} \inf_{z \in \mathbb{C}} \frac{\sum_{\Gamma \cap D_r(z)} f(|z - \gamma|)}{4\Delta \varphi \int_0^r t f(t) \, dt}.
\]

If we choose as our locally integrable function \(f\) the constant function, we recover the results of [BO-95]. However, by making other choices, we can get other sufficient conditions that, although not necessary, might be of use in some applications.

For the sake of simplicity, we will consider in the following examples only the classical Bargmann-Fock space.

Example 7.1. (i) Let \(f(t) = e^{-t}\). Then \(\Gamma\) is interpolating if

\[
\sup_{z \in \mathbb{C}} \sum_{\Gamma \cap (D_r(z))} e^{-|z - \gamma|} < 2
\]

and sampling if

\[
\inf_{z \in \mathbb{C}} \sum_{\Gamma \cap (D_r(z))} e^{-|z - \gamma|} > 2.
\]

Integration by parts, together with a standard argument shows that \(\Gamma\) is interpolating if

\[
\sup_{z \in \mathbb{C}} \int_0^{\infty} \frac{\#(\Gamma \cap D_s(z)) \, ds}{e^s} < 2
\]

and sampling if

\[
\inf_{z \in \mathbb{C}} \int_0^{\infty} \frac{\#(\Gamma \cap D_s(z)) \, ds}{e^s} > 2.
\]

(ii) Let \(f_a := 1_{[0,a]}\). We then obtain:

If \(a > 1/\sqrt{2}\) and every disk of radius \(a\) contains at most one member of \(\Gamma\), then \(\Gamma\) is interpolating.

If \(a < 1/\sqrt{2}\) and every disk of radius \(a\) contains at least one member of \(\Gamma\), then \(\Gamma\) is sampling.
7.2 The disk

In this paragraph we consider the case of the Poincaré unit disk \((X, g) = (\mathbb{D}, \frac{|dz|^2}{1-|z|^2})\). The disk is the main example of a regular hyperbolic Riemann surface. Its Green’s function is

\[ E(z, \zeta) = \log |\phi_z(\zeta)|, \quad \text{where} \quad \phi_z(\zeta) = \frac{z - \zeta}{1 - \overline{z}\zeta}, \]

is the standard involution. Thus \(\rho_z(\zeta) = |\phi_z(\zeta)|\) and the disks \(D_\sigma(z)\) are the well-known pseudo-hyperbolic disks; they are geometrically Euclidean disks, but their Euclidean centers and radii are different.

Standard calculations show that

\[ |d\rho_z(\zeta)|^2 = \frac{1 - |z|^2}{(1 - z\overline{\zeta})^2} \]

so we have \(\nu(\zeta) = \log 2 + \log(1 - |\zeta|^2)\), and it is clear that (5) holds.

As suggested by the proof of Proposition 3.5, we take

\[ u_\psi(z) = -\frac{1}{2} \log(1 - |z|^2), \]

and thus we have

\[ e^{2\nu(z)}(\Delta u_\psi(z) - 2|\partial u_\psi(z)|^2) = \frac{1}{2}(1 - |z|^2) \geq 0 \quad \text{and} \quad \tau_\psi(z) = \frac{1}{2(1 - |z|^2)}. \]

We also have

\[ A_g(D_\sigma(\gamma)) = C_\sigma(1 - |\gamma|^2). \]

Thus our Hilbert spaces are

\[ B_\varphi^2 := \left\{ h \in \mathcal{O}(\mathbb{D}) : \int_\mathbb{D} |h|^2 e^{-2\varphi} \frac{dm}{1 - |z|^2} < +\infty \right\} \]

and

\[ b_\varphi^2 := \left\{ (s_\gamma) : \sum_{\gamma \in \Gamma} |s_\gamma|^2 e^{-2\varphi(\gamma)}(1 - |\gamma|^2) < +\infty \right\}. \]

The densities are given by

\[ D^+_\varphi(\Gamma) = \lim \sup \sup_{r \to 1} \sum_{z \in \mathbb{D}} \frac{f(\rho_z(\gamma))(1 - \rho_z(\gamma))^2}{4 \left(1 - |z|^2\right)^2 \Delta \varphi(z) + \frac{1}{2} \left(1 - |\gamma|^2\right) \int_0^t tf(t)dt}, \]

and

\[ D^-_\varphi(\Gamma) = \lim \inf \inf_{r \to 1} \sum_{z \in \mathbb{D}} \frac{f(\rho_z(\gamma))(1 - \rho_z(\gamma))^2}{4(1 - |z|^2)^2 \Delta \varphi(z) \int_0^t tf(t)dt}. \]

If we take

\[ f(t) = \frac{-\log t}{(1 - t^2)^2} \frac{1}{1 - \frac{1}{2}}, \]

Theorems 3.2 and 3.3 recover the results from [BO-95].

Again for the sake of illustration we will consider below only the classical unweighted Bergman space, which is obtained by setting \(\varphi = \frac{1}{2} \log(1 - |z|^2)\).
Example 7.2.  (i) Letting $f = 1$, we see that $\Gamma$ is interpolating if
\[
\sup_{z \in D} \sum (1 - \rho_z(\gamma))^2 < 1
\]
and sampling if
\[
\inf_{z \in D} \sum (1 - \rho_z(\gamma))^2 > 1.
\]
(ii) Letting $f(t) = (1 - t^2)^{-2}$, we see that $\Gamma$ is interpolating if
\[
\lim_{r \to 1} \sup_{z \in D} \#(\Gamma \cap D_r(z)) \frac{A_{\text{hyp}}(D_r(z))}{A_{\text{hyp}}(D_r(z))} < 1
\]
and sampling if
\[
\lim_{r \to 1} \inf_{z \in D} \#(\Gamma \cap D_r(z)) \frac{A_{\text{hyp}}(D_r(z))}{A_{\text{hyp}}(D_r(z))} > 1,
\]
where
\[
A_{\text{hyp}}(D_r(z)) = \frac{1}{2\pi} \int_{D_r(z)} \frac{dm(z)}{(1 - |z|^2)^2}
\]
denotes hyperbolic area of $D_r(z)$.
(iii) Let $f_a := 1_{[0,a]}$. We then obtain:
If $\delta > \frac{1}{\sqrt{2}}$ and $\Gamma$ has at most one point in every disk of radius $\delta$, then $\Gamma$ is interpolating.
If $\delta < \frac{1}{\sqrt{2}}$ and every disk of radius $\delta$ contains at least one member of $\Gamma$, then $\Gamma$ is sampling.

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