Dispersion properties of transverse waves in electrically polarized BECs

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Abstract

Further development of the method of quantum hydrodynamics in applications for Bose–Einstein condensates (BECs) is presented. To consider the evolution of polarization direction along with particle movement, we have developed a corresponding set of quantum hydrodynamic equations. It includes equations of the polarization evolution and the polarization-current evolution along with the continuity equation and the Euler equation (the momentum-balance equation). Dispersion properties of the transverse waves, including the electromagnetic waves propagating through the BECs, are considered. To this end, we consider a full set of the Maxwell equations for the description of electromagnetic field dynamics. This approximation gives us the possibility of considering the electromagnetic waves along with the matter waves. We find a splitting of the electromagnetic-wave dispersion on two branches. As a result, we have four solutions, two for the electromagnetic waves and two for the matter waves; the last two are the concentration-polarization waves appearing as a generalization of the Bogoliubov mode. We also find that if the matter wave propagates perpendicular to the external electric field then the dipolar contribution does not disappear (as it follows from our generalization of the Bogoliubov spectrum). A small dipolar frequency shift exists in this case due to the transverse electric field of perturbation.

Keywords: Bose–Einstein condensate, collective excitations, polarization, transverse waves, quantum hydrodynamic model

1. Introduction

At studying of electrically polarized Bose–Einstein condensates (BECs) the generalization of the Gross-Pitaevskii (GP) equation [1–5] and some its further generalizations [6–8] have been used, see also review papers [10–15]. This nonlinear Schrödinger equation defines the complex scalar wave function $\Phi(r, t)$, which describes the evolution of the particle concentration $n(r, t) = \Phi^*(r, t)\Phi(r, t)$. When the direction of polarization of each particle $d$ is not changing during the considering processes and all of them have the same direction (the aligned dipoles) we can write a formula for the density of polarization $P = \Phi^*(r, t)d\Phi(r, t)$. Thus, the polarization density changes due to particle movement, and consequently, when the concentration changes. The waves of polarization can easily propagate even in the ferroelectric materials as well as spin waves in ferromagnetic materials. Furthermore, the more we can expect the existence of such waves in the polarized BECs. To consider the evolution of polarization direction along with particle movement, we have developed a set of quantum hydrodynamic equations that includes equations of the polarization evolution and the polarization current evolution along with the continuity equation and the Euler equation (the momentum balance equation) [16–19]. Earlier, we developed a similar approach for the electric dipolar ultracold fermions [20].

There is a field in the study of dipolar BECs where the evolution of dipole directions is under consideration. This is research of spinor BECs, which was suggested by T Ohmi and Machida [21] and Ho [22] in 1998 and was recently reviewed in [23, 24]. We should mention that this field is dedicated to magnetic dipole BECs, but the dipole–dipole (spin–spin) interaction is neglected there.
It was noted in [16] that the full and correct form of the Hamiltonian of the electric dipole interaction, which also corresponds to the Maxwell equations, is

$$H_{dd} = -\alpha^2 \delta^2 \frac{1}{r} \cdot d_i^a d_i^b.$$  \hspace{1cm} (1)

This formula is in accordance with classical electrodynamics [25] (see also [34]).

Using well-known identity (see, for instance, [26, 27])

$$-\alpha^2 \delta^2 \frac{1}{r} = \frac{\delta^{\alpha \rho} - 3 r^\rho r^\alpha / r^2}{r^3} + \frac{4\pi}{3} \delta^{\alpha \rho} \delta(r),$$  \hspace{1cm} (2)

we can see that the Hamiltonian (1) differs from usually used one

$$H_{dd} = \frac{\delta^{\alpha \rho} - 3 r^\rho r^\alpha / r^2}{r^3} - d_i^a d_i^b.$$  \hspace{1cm} (3)

Moreover, the Hamiltonian (1) is the original form of the potential energy of electric dipole interaction. The Hamiltonian of electric dipole interaction must be considered because it must be in accordance with the Maxwell equations. This connection exists since Maxwell equations describe the electric field and its connection with the sources. In our case, the source of the field is the density of electric polarization. Action of the electric field on the density of polarization comes in to equations of motion via the force field. As a result, it describes the interaction of the polarization (the electric dipole moments).

It has been shown in [28–30] that using the non-linear Schrödinger equation, which generalized the GP equation for the electrically polarized BECs, and using the formula (3) for description of the dipole–dipole interaction lead to the appearance of the contribution of the equilibrium polarization in the dispersion dependence of the Bogoliubov mode. This contribution leads to the anisotropy of the dispersion dependence. The anisotropy of the sound velocity in dipolar Bose gases has recently been confirmed experimentally (see [31]).

In [17, 18], we considered the evolution of electric dipole directions and got dispersion dependencies for two waves. It means that the account of the polarization evolution leads to the appearance of a new wave in the BECs. This is a polarization wave, which is an analog of spin waves in systems of spinning particles.

As a first step in studying the polarization direction evolution, we considered propagation of waves parallel to the direction of the electric field, so their dispersion reveals no angle dependence [17, 18]. As an intermediate step in studying the polarization evolution in BECs, we considered fully polarized BECs. Usually, the theory of fully polarized BECs is based on a formula (3) (see, for instance, [10, 14, 15]). The formula has the following consequences. The dipolar part of $\alpha^2$ is proportional to $(\cos^2 \theta - 1/3)$, so it can change its sign, where $\theta$ is the angle between the direction of wave propagation $k$ and the direction of an external field. If the polarization part is negative or positive and is small enough to occur in the short-range part of the spectrum, one finds the roton instability [10, 14, 15]. Our consideration of fully polarized electric dipolar BECs is based on formula (1). Consequently, the dipolar part of $\alpha^2$ is proportional $\cos^2 \theta$. So it is positive for all angles $\theta$ and shows no instability of three-dimensional (3D) BECs with the repulsive short-range interaction [32]. Moreover, full electric dipole interaction (1) leads to the stabilization of the Bogoliubov spectrum with attractive short-range interaction [32], since the positive dipolar part can exceed negative contribution of the short-range interaction. We obtained that consideration of the full Hamiltonian (1) is essential for fully polarized electric BECs, the full Hamiltonian of spin–spin interaction was considered for the fully polarized magnetic BECs as well. In both cases, we have one wave solution only, as it was presented in [32]. However the dipolar part of the dispersion dependence changes due to the account of the delta-function part of the full potential. As a result, we find that the dipolar part in the electric dipolar BECs becomes positive for all angles (all directions of wave propagation). We have a different picture for the magnetic dipolar BECs, where we have another coefficient in front of the delta-function (see formula (83.9) in [33]). Consequently, the dipolar part of the spectrum is also different—it is negative for all angles. Hence, there is no roton instability for the electrically polarized BECs, but the roton instability exists in the magnetically polarized BECs at all angles for a small enough constant of the short-range interaction.

Consideration of the full electric dipolar BEC theory, including the dipole direction evolution, was presented in [16], where we applied our model to the calculation of dispersion of waves propagating at different angles $\theta$. Thus, [16] contains generalization of results obtained in [17]. The waves considered in [16–18] and [18] were longitudinal; that means perturbations of the electric field are parallel to the direction of wave propagation $k||E$. Many features of the quantum hydrodynamics of dipolar BECs and meaning of considered approximations were described in [16] as well.

In this paper, we suggest a more general model in comparison with our recent papers. We consider the full set of the Maxwell equations instead of the pair of quasi-electrostatic equations describing the electric field considered earlier. This approximation gives us the possibility to consider the electromagnetic waves along with the matter waves. Below, we will show that in the electrically polarized BECs we have a splitting of the electromagnetic waves on two branches. So, we have four solutions, two for the electromagnetic waves and two for the matter waves, the last two of which are waves of concentration–polarization. It is important and interesting to see that in this approximation we find anisotropy in the dispersion dependencies for all four waves.

This paper is organized as follows. In section 2, we present the quantum hydrodynamics model for description of the electrically polarized BECs. In section 3, we present dispersion equation and describe the method to get it. In section 4, we present an analysis of the dispersion equation. In section 5, a brief discussion of the obtained results is presented.
2. Basic equations

In our previous papers [16, 34] we have presented the derivation of the QHD equation for the electrically polarized BECs. These equations were directly derived from the microscopic many-particle Schrödinger equation by means of the QHD method. The method of quantum hydrodynamics was suggested in 1999 for the derivation of hydrodynamic equations for quantum plasmas [35]. In 2008, this method was applied to ultracold quantum gases [36]. Later, it was used for a system of particles possessing electric dipole moment [16, 18, 19]. Let us now present the set of the quantum hydrodynamics equations.

The first equation of the QHD equation system is the continuity equation

$$\partial n + V \cdot (n v) = 0,$$ (4)

showing conservation of the particle number, and giving time evolution of the particle concentration.

The momentum balance equation (Euler equation) for the polarized BECs has the form

$$mn \left( \partial_t + v \cdot \nabla \right) v^n + \partial \rho \sigma^{\theta \phi}$$

$$= \frac{\hbar^2}{m} \partial^2 \nabla^2 n + \frac{\hbar^2}{4m} \partial \rho \left( \frac{\sigma^n \cdot \sigma^n}{n} \right)$$

$$= Y n \delta \nabla^2 n + \frac{1}{2} Y_2 \sigma n \nabla^2 n + P \rho \sigma E^\beta,$$ (5)

where

$$Y = \frac{4\pi}{3} \int dr(r) \frac{\partial U(r)}{\partial r},$$ (6)

and

$$Y_2 \equiv \frac{\pi}{30} \int dr(r) \frac{\partial U(r)}{\partial r},$$ (7)

are the numerical coefficients. In equation (5), we define a parameter $Y_2$ as (7). This definition differs from the one in the work [36]. Here, we put multiplier 1/8 within the definition of $Y_2$ to make the equations look better. Terms proportional to $h^2$ appear as a result of using of the quantum kinematics. They are usually called the quantum Bohm potential. The first two terms at the right-hand side of the equation (5) are first terms of expansion of the quantum stress tensor. They occur due to the short-range interaction potential $U_{pp}$. The interaction potential $U_{pp}$ determines the macroscopic interaction constants $Y$ and $Y_2$. The term proportional to $Y_2$ contains higher spatial derivative compared with the Gross–Pitaevskii term $\sqrt{\sigma} / 2$. This is why it is called a non-local interaction. Contributions of the third order by the radius of the short-range interaction (terms proportional $Y_2$) in non-linear properties of quantum gases were considered in [37, 38] for non-polarized quantum gases. The influence of $Y_2$ on dispersion of waves in dipolar fermions was described in [20]. A non-local interaction for BECs being used as an analogue gravity system is described in [39]. The last term of the equation (5) describes a force field that affects the dipole moment in a unit of volume as the effect of the external electrical field and the field produced by other dipoles. It is written using the self-consistent field approximation [16, 19, 35]. The tensor of the kinetic pressure is $P^{\theta \phi}(r, t)$, which depends on particle thermal velocities and makes no contribution into the BEC dynamics at near zero temperatures. In equation (5) and below $E$ is the sum of external and internal electric fields, where an internal electric field is created by dipoles, $P$ is the density of the electric dipole moment (the polarization of medium).

The first order by the interaction radius constant for dilute gases may be presented in the form

$$Y = -\frac{4\pi \hbar^2 a}{m},$$ (8)

where $a$ is the scattering length, so we see $Y = -g$, where $g$ is the Gross-Pitaevskii interaction constant [36, 40]. We have also the field equations

$$\nabla \cdot E(r, t) = -4\pi \nabla \cdot P(r, t),$$ (9)

and

$$\nabla \times E(r, t) = 0.$$ (10)

These equations show a relation between the polarization $P$ and the electric field $E$ caused by the polarization $P$. Equation (9) allows us to consider the propagation of waves parallel to the equilibrium polarization only. For the consideration of waves propagating in an arbitrary direction, we have to consider the couple of equations (9) and (10).

Having a short-range interaction in a microscopic many-particle Schrödinger equation we obtain the corresponding force field $F_{SR}$ in the Euler equation as a result of applying of the QHD method for the derivation of equations for collective evolution description [36]. Using the fact that interaction is short-range, we find the force field $F_{SR}$ as a series. The force field also appears as the divergence of a second rank tensor, which is the quantum stress tensor $\sigma^{\theta \phi}$. Considering the first two non-zero terms in the series, existing for spherically symmetric potential of the short-range interaction for particles in the BEC state, we get the Gross–Pitaevskii term and a non-local term containing the third spatial derivative of the square of the particle concentration $F_{SR} = -\partial \rho \sigma^{\theta \phi} = -\frac{1}{2} g V^\theta V^\phi \nabla^2 n^2 + \frac{1}{2} Y_2 V^\theta V^\phi \nabla^2 n^2$, with $\sigma^{\theta \phi} = \frac{1}{2} g \delta^{\theta \phi} n^2 - \frac{1}{2} Y_2 \delta^{\theta \phi} \nabla^2 n^2$. Consequently we see that there is no corresponding non-linear Schrödinger equation, since there is no Cauchy integral of the Euler equation. Nevertheless, a formal representation of the hydrodynamic equations in the form of a non-linear Schrödinger equation was made in [36].

One may try to include non-local terms in the GP equation or the corresponding Lagrangian, including spatial derivatives of the particle concentration. For instance, one may consider $gn = Y_2 \nabla^2 n$ instead of $gn$. However, a microscopic derivation of QHD equations shows that the Euler equation has a more complex form. We found $F = -\frac{1}{2} g V^\theta V^\phi \nabla^2 n^2 + \frac{1}{2} Y_2 V^\theta V^\phi \nabla^2 n^2$ instead of $F = -n V (g - Y_2 \nabla) n$ corresponding to a non-linear Schrödinger equation.

If particles do not contain the dipole moment, the continuity equation and the momentum balance equation form a
closed set of equations. If the dipole moment is taken into account in the momentum balance equation when a new physical quantity emerges. This is the polarization vector field $\mathbf{P}(r, t)$. It causes the set of equations to become incomplete.

If we have a system of fully polarized particles as pictured on figure 1 then polarization changes due to the motion of particles in space. Consequently we have $\mathbf{P}(r, t) = d\mathbf{n}(r, t)$, where $d$ is the electric dipole moment of particles, and $\mathbf{n}$ is the direction of all dipoles. A non-linear Schrödinger equation can be derived for a system of fully polarized dipoles from the first principal derived QHD equations [36]. In an integral form, it can be written as

$$i\hbar\partial_t \Phi(r, t) = \left(-\frac{\hbar^2}{2m} \nabla^2 + \mu(r, t) + V_{\text{ext}}(r, t)\right)\Phi(r, t) + g|\Phi(r, t)|^2 \Phi(r, t),$$

where $\Phi(r, t)$ is the macroscopic wave function, $\mu$ is the chemical potential, its explicit definition is presented in appendix along with other hydrodynamic variables, $V_{\text{ext}}$ is the potential of an external field, $g$ is the constant of short-range interaction, $d$ is the dipole moment of a single particle, $\theta$ is the angle between $r - r'$ and the direction of external electric field, $m$ is the mass of a particle and $\hbar$ is the Planck constant divided by $2\pi$. This is a generalized GP equation, where we included the Dirac delta function as part of the full potential of electric dipole interaction, as we did earlier in the appendix of [16]. Let us admit that the macroscopic wave function $\Phi(r, t)$ appears to be defined in hydrodynamic variables as follows $\Phi(r, t) = \sqrt{n(r, t)} \exp \left(i m \psi(r, t) / \hbar \right)$ with $\psi(r, t)$ is the potential of the velocity field $\mathbf{v}(r, t) = \nabla \psi(r, t)$.

The Delta function term in equation (11) can be combined with the short-range interaction term presented by the term with cubic non-linearity. So, we might obtain shifted constant of the short-range interaction $\tilde{g} = g + 4\pi d^2/3$. But this would be an unwise move. We can go another way, allowing us to simplify equation (11) without extra approximations. We can explicitly introduce the electric field created by dipoles (see appendix B formula (52)) and rewrite the non-linear Schrödinger equation in a non-integral form [34]

$$i\hbar\partial_t \Phi(r, t) = \left(-\frac{\hbar^2}{2m} \nabla^2 + g|\Phi(r, t)|^2 \right. - \mathbf{d} \cdot \mathbf{E}$$

$$\left. \times \Phi(r, t), \right)$$

where $\mathbf{d} \cdot \mathbf{E}$ is the scalar product of vectors $\mathbf{d}$ and $\mathbf{E}$. We do not include the chemical potential in equation (12). Equation (12) contains an extra function $\mathbf{E}(r, t)$, which obeys the quasi-electrostatic Maxwell equations (9) and (10). So, we have a set of coupled equations for medium motion (12) and fields (9) and (10). The non-integral form (12) of the generalized GP equation (11) was introduced in [34]. See also the discussion of this equation in [32]. Let us acknowledge that in this paper we discuss the theory of dipolar BECs in a model of point-like particles. Finite-size particles were considered in [32], where a radius of molecules explicitly comes in the model.

The short-range interaction constant $g$ in equation (12) does not depend on the electric dipole moment $d$. Hence, the corresponding scattering length does not depend on $d$ either.

The dipole-dependent scattering length for the short-range part of the interaction in dipolar BECs is discussed in [41] and [42].

Getting back to our analysis, we separated the short-range interaction and the dipole–dipole interaction with the beginning of our paper. So we have the scattering length $a$, which does not depend on $d$. All dependencies on the electric dipole moment are included in the dipole–dipole interaction.

A generalized integral GP equation, but with the shorted potential of dipole–dipole interaction (3), still finds a lot of applications. Frequencies of collective excitations in trapped dipolar BECs were calculated in the Thomas–Fermi regime in [43]. One-dimensional solitons on a weak two-dimensional (2D) square and triangular optical lattice potentials placed perpendicular to the polarization direction were considered in [44]. Order and chaos in the dipolar BECs are investigated in [45] considering the behavior of bifurcations. A dipolar boson–fermion mixture and its stability was studied in [46]. The stability of dipolar Bose and Fermi gases was investigated in [47]. Solitons in dipolar boson–fermion mixtures were described in [48]. The influence of dipole–dipole interaction on the formation of vortices in a rotating dipolar BECs atoms in quasi-2D geometry was considered in [49]. Solitons in dipolar BECs trapped in the 2D plane were studied in [50]. Modification of the properties of super-fluid vortices by dipole–dipole interactions within the context of a 2D atomic Bose gas of fully polarized dipoles was considered in [51]. A discussion of linear properties of dipolar BEC based on an integral GP equation with shorted dipole–dipole potential (3) is also presented in [51]. Judging upon formula (2) in [52], we assume that potential (3) was used for the consideration of collective modes in fermionic dipolar liquid.
We can also find explicit examples using of identity (2) when authors work with the shorted potential of dipole–dipole interaction (3); see text before formula (418) in review [23], and see formula (36) in [53], formula (8) in [54], formula (8) and text before this formula in [26], and formula (4) in [55].

Other approaches to dipolar BECs were developed in recent years ([9, 56, 57]). Some of them we briefly describe below. Various beyond-mean-field effects on Bose gases at zero temperature were considered within the realm of the Bogoliubov–de Gennes theory [9]. For the homogeneous system, the condensate depletion, the ground-state energy, the equation of state, and the speed of sound are discussed in detail. Authors analyze the influence of quantum fluctuations on properties of quantum gases. During calculations the condensate density is replaced by the total number density, which includes distribution of particles on states with non-minimal energy. The fluctuation Hamiltonian density is included. The corrected sound velocity due to beyond mean-field equation of state, which is a generalization of the Lee–Huang–Yang quantum corrected equation of state, is presented by formula (34). Corresponding hydrodynamic equations are presented by formulas (32) and (33) of [9]. Applying the many-particle Schrödinger equation for ultracold dipolar Fermi gases in anisotropic traps, which are polarized in the z direction, the couple of quantum hydrodynamic equations, i.e. the continuity and Euler equations, was obtained via the equation of motion for the one-body density matrix [56]. Collective motion of polarized dipolar Fermi gases in the hydrodynamic regime was considered in [58] applying a variational time-dependent Hartree–Fock approach.

Collective excitations of quasi-2D trapped dipolar fermions in a framework of the collisional Boltzmann–Vlasov equation obtained by reducing the evolution equations of the non-equilibrium Green functions to quantum kinetic equations were considered in [57]. This analysis is performed at finite temperatures. Transition from a collisionless to a hydrodynamic regime was considered.

In this paper, we are interested in dipole direction evolution instead of the consideration of fully polarized BECs. So, let us make a step towards the main topic. The variation of dipole direction can be different in different areas of space. Consequently, we have a different magnitude of polarization in different points of space. These variations can cause a turning of the polarization vector \( \mathbf{P} \) relative to the direction of the external electric field \( \mathbf{E}_{\text{ex}} \) (see figure 2(a)) or it can change the module of \( \mathbf{P} \) without a change of direction (see figure 2(b)).

If particles possessing an electric-dipole moment are in different cells of an optical lattice in the Mott insulator regime, then their ability to get close to each other is limited. However, there is no such limitation in the superfluid phase or when particles are in the absence of a lattice. So, particles can get close to each other. In this case, the magnitude of the dipole–dipole interaction increases and the direction of the dipoles can change due to the dipole–dipole interaction. It provides the possibility of polarization wave propagation, which are waves related to the evolution of dipole direction (see figure 3).

As a result, we included dipole direction evolution in our model. Consequently, we derived the next equation of the chain of QHD equations, which is the equation for the time evolution of the polarization (density of the electric dipoles) \( \mathbf{P} \). Explicit definitions of the QHD variables allowed us to derive the set of QHD equations that are presented in the appendix.

The next equation we need for studying of the collective excitation dispersion is the equation of polarization evolution:

\[
\partial_t P_{\alpha\beta}(\mathbf{r}, t) + \partial^\mathbf{r} R_{\alpha\beta}^{\text{pol}}(\mathbf{r}, t) = 0,
\]

where \( R_{\alpha\beta}^{\text{pol}}(\mathbf{r}, t) \) is the current of polarization.

Equation (13) does not contain information about the effect of the interaction on the polarization evolution. So, we should consider the equation for \( R_{\alpha\beta}^{\text{pol}}(\mathbf{r}, t) \) evolution, which can be derived by means of the quantum hydrodynamic method [19, 36]. Using the self-consistent field approximation of the dipole–dipole interaction we obtain the equation for the

![Figure 2](image2.png)

**Figure 2.** This figure shows formation of full polarization in a vicinity of a point at dipole direction evolution. Figure 2a illustrates possibility of turn of the polarization vector \( \mathbf{P} \) due to asymmetrical distribution of dipole directions, which can appear at propagation of the polarization wave. Figure 2b shows decreasing of module of the polarization \( \mathbf{P} \) without change of direction. It happens due to symmetric distribution of dipole direction in the vicinity.

![Figure 3](image3.png)

**Figure 3.** This figure shows motion of dipoles including evolution of dipole direction. Here we have two mechanisms of polarization change. The first of them is the same as in systems of fully polarized particles. It is motion of each particle in space, so the particle concentration changes and causes change of the polarization. The second mechanism is related to variation of direction of dipoles from direction of external field due to interparticle interaction.
polarization current \( R^{\alpha \beta}(r, t) \) evolution

\[
\partial_t R^{\alpha \beta} + \partial_t \left( R^{\alpha \beta} \partial_t + R^{\alpha \beta} \partial_{\sigma} - P^{\alpha \beta} \partial_{\sigma} \right) \\
+ \frac{1}{m} \partial_t \partial_t R^{\alpha \beta} - \frac{\hbar^2}{4m^2} \partial_{\beta} \delta R^{\alpha \beta} \\
+ \frac{\hbar^2}{8m^2} \partial_{\beta} \left( \frac{\partial_j P^{\alpha \sigma} \partial_n}{n} + \frac{\partial_j P^{\alpha \sigma} \partial_n}{n} \right) \\
= \frac{1}{2m} \mathcal{Y} \partial_t (n P^{\alpha \beta}) + \frac{\sigma}{m} \frac{P^{\alpha \beta} P^\gamma}{n} - \partial_{\beta} E^\gamma,
\]

(14)

where \( r^{\alpha \beta}(r, t) \) presents the contribution of the thermal movement of the polarized particles into the dynamics of \( R^{\alpha \beta}(r, t) \). As we deal with the BECs below, the contribution of \( r^{\alpha \beta}(r, t) \) may be neglected. The last term of the formula (14) includes both an external electrical field and the self-consistent field that particle dipoles create. This term contains numerical constant \( \sigma \). The second group of terms in the left-hand side of equation (14) has a kinematic nature. It is an analog of \( m \mathcal{Y} \partial_t (n P^{\alpha \beta}) \) in the Euler equation (14). The fourth and fifth groups of terms are the contribution of the quantum Bohm potential in the polarization evolution. Their appearance is related to the de-Broglie wave nature of quantum particles. The \( \partial_t \partial_t R^{\alpha \beta} \) term (14) includes both an external electrical field and the self-consistent field that particle dipoles create. This term contains numerical constant \( \sigma \).

3. Collective excitations in the electric polarized BEC: Dispersion equation

We can analyze the linear dynamics of collective excitations in the polarized BEC using the QHD equations (4), (5), (13), (14) and the Maxwell equations (15)–(18). Let us assume the system is placed in an external electrical field \( E_0 = E_0 \mathbf{e}_z \). The values of concentration \( n_0 \) and polarization \( P_0 = \varepsilon E_0 \) for the system in an equilibrium state are constant and uniform, and the equilibrium field strength is assumed to have a non-zero value. Expressing all the quantities entering the system of equations in terms of the electric field, we come to the equation

\[
\Lambda^\theta(\omega, \mathbf{k}) E^\beta(\omega, \mathbf{k}) = 0,
\]

(20)

where

\[
\Lambda^\theta = \left( \frac{\omega^2}{c^2} - k^2 \right)^2 \delta^\theta + k^2 \mathbf{k}^\beta \\
- 4\pi \frac{\omega^2}{c^2} k^2 \mathbf{k}^\beta \\
= \frac{p_0^\alpha p_0^\beta}{4m^2} - \frac{\hbar^2 k^4}{4m^2} + \frac{\n_0 k^2}{m} \\
\times \left( \frac{\sigma}{mn_0} \mathbf{k}^\gamma \mathbf{P}^\beta \right) + \frac{k^2}{m^2} - \frac{\n_0 \mathbf{k}^\gamma \mathbf{P}^\beta}{m}
\]

(21)

with \( k^2 = k_x^2 + k_y^2 + k_z^2 \). The dielectric permittivity tensor of electric dipolar BECs is \( \Lambda^\theta \).

We suppose that the external electric field parallel to the \( z \)-axis. Thereby, the equilibrium polarization \( P_0 \) also is parallel to the \( z \)-axis \( \mathbf{P}_0 \parallel \mathbf{k}_z \). In this case, the tensor (21) becomes simpler:

\[
\Lambda(\omega, \mathbf{k})
\]
4. Dispersion dependencies

4.1. Matter waves

Equation (24) simplifies in the low-frequency limit ($\omega \ll k c$)

$$k^2 c^2 + \beta(\omega)(k^2 c^2 \cos^2 \theta - \omega^2) = 0.$$  \hfill (25)

If $\cos \theta$ is not close to zero, when the first term in the large bracket is much larger than the second one. Consequently, equation (25) simplifies to

$$1 + \cos^2 \theta \beta(\omega) = 0,$$  \hfill (26)

whereas at $\cos^2 \theta = 1$ we have

$$1 + \beta(\omega) = 0.$$  \hfill (27)

Equation (27) was obtained in [16–18]. Equation (26) was also obtained in [16], and its properties were considered under the condition of maximal polarization contribution, which corresponds to $\cos \theta = 1$. So, the properties of equation (27) were studied in full detail in [16].

Equation (26) has the following solution

$$\omega^2 = \frac{1}{2m} \left( -\frac{3}{2} \frac{Yn_0 k^2}{n_0} + \frac{\hbar^2 k^4}{2m} + \frac{\hbar^2 k^4}{n_0} \right) + 4\pi \sigma \cos^2 \theta \frac{P_0^2 k^2}{n_0} \left( \frac{1}{2} Yn_0 k^2 - \frac{Y_2 n_0 k^4}{n_0} \right) \frac{1}{2}.$$ \hfill (28)

In the case of wave propagation parallel to the equilibrium polarization $\cos \theta = 1$, we get the solution obtained in [16–18], as well as one considered at great length in [16]. The contribution of equilibrium polarization becomes smaller when angle $\theta$ is increased from 0 to $\pi/2$, and it vanishes at $\theta = \pi/2$. However, equation (26) does not describe the area around $\theta = \pi/2$. This area is described by equation (32) and below.

An analysis of formula (28) was presented in [16] for $k \parallel E_{ext}$. This analysis includes the contribution of the short-range interaction up to the third order by the interaction radius terms proportional to $Y_2$. Let us discuss some properties of formula (28) in the first order by the interaction radius of the short-range interaction (neglecting terms proportional to $Y_2$). We also assume $\sigma = 1$ and $g = -1$. Applying all these approximations, we obtain two separate formulas from formula (28):

$$\omega^2 = \frac{1}{2m} \frac{gn_0 k^2}{m^2} + \frac{\hbar^2 k^4}{4m^2},$$ \hfill (29)

and

$$\omega^2 = \frac{1}{m} gn_0 k^2 + \frac{4\pi \sigma P_0^2 k^2}{m n_0} \cos^2 \theta + \frac{\hbar^2 k^4}{4m^2}.$$ \hfill (30)

Assuming $P_0 = d m_0$, we can rewrite the dipolar part of the spectrum as $4\pi d n_0 k^2 \cos^2 \theta$. Formula (30) is a generalization of the Bogoliubov spectrum for dipolar particles. Extra solution (29) appears as a consequence of polarization evolution. Solutions (29) and (30) are presented in figure 4.

Solution (30) is rather different from the well-known result containing the signature of the roton instability (see, for instance, formula (5.1) in [10], the formula next to formula (6) in [14], formula (11) in [15], and the formula (416) in [23]). Experimental results in [31] were obtained for the magnetic

Figure 4. This figure shows dispersion of waves in electric dipolar BECs with evolution of dipole direction. Black (upper) line presents the dispersion dependence of the Bogoliubov mode (30). Green (lower) line shows dispersion of wave caused by evolution of the polarization $P$ (29). Dashed line shows the Bogoliubov mode (30) at zero equilibrium polarization $P_0 = 0$. This figure is obtained for $n_0 = 10^{19}$ cm$^{-3}$, $d = 1D = 1 \times 10^{-8}$ CGS units, $m = 10^{-24}$ g.

$\theta = \pi/4$.

$$\left( \frac{\alpha^2}{c^2} - k^2 c^2 \right) \begin{pmatrix} k, k, k, k \cr k, k, k, k \cr k, k, k, k \cr k, k, k, k \end{pmatrix} = \begin{pmatrix} 0 & \frac{\alpha^2}{c^2} & 0 & 0 \cr 0 & 0 & \frac{\alpha^2}{c^2} & 0 \cr 0 & 0 & 0 & \frac{\alpha^2}{c^2} \end{pmatrix} \begin{pmatrix} 1 + \beta(\omega) \cr 1 + \beta(\omega) \cr 1 + \beta(\omega) \cr 1 + \beta(\omega) \end{pmatrix}.$$ \hfill (22)

where

$$\beta(\omega) \equiv -4\pi \frac{P_0^2}{n_0} \frac{k^2}{\alpha^2} \frac{\hbar^2 k^4}{4m^2} + \frac{\hbar^2 k^4}{2m}$$

$$\times \left( \frac{\alpha}{m} + \frac{k^2 Y_0}{2m^2} \right).$$ \hfill (23)

The dispersion equation emerges as a condition of the existence of non-zero perturbations of the electric field. That means the determinant of matrix (22) equals zero: $\det \Lambda = 0$. Using formula (22), we find an explicit form of the dispersion equation

$$\left( \alpha^2 - k^2 c^2 \right)^2 + \beta(\omega) \left( \alpha^2 - \left( 1 + \cos^2 \theta \right) \alpha^2 \right) k^2 c^2 + \cos^2 \theta k^4 c^4 = 0.$$ \hfill (24)
dipolar BECs. Consequently, our analysis does not relate to this experiment. However, we should admit that our previous results for the fully polarized BECs based on the full potential of magnetic dipole interaction gives an angle dependence of the spectrum corresponding to the one obtained experimentally in [31]. Hence, our result (30) does not contradict to the presence of the roton instability in the spectrum of magnetic dipolar BECs. The Bogoliubov spectrum for the dipolar BECs presented in [10, 14, 15], has the following form

\[ \omega^2 = k^2 \left( \frac{n_0}{m} + \frac{C_{dd}}{3} \left( 3 \cos^2 \theta - 1 \right) \right) + \frac{\hbar^2 k^2}{4 m^2} \]  

(31)

\( C_{dd} \) is the dipolar coupling constant \( C_{dd} = d^2/\varepsilon_0 \) in SI units, where \( \varepsilon_0 \) is the vacuum permittivity, or \( C_{dd} = 4\pi d^2 \) in the CGS units. It has been applied to both the magnetic and the electric dipolar BECs. In our recent papers, we have shown that the generalization of the Bogoliubov spectrum is different for the electric and the magnetic dipolar BECs, even for the fully polarized BECs [32, 34]. These generalizations differ from formula (31). These differences appear due to the consideration of the full potentials of dipole–dipole interactions, which contain the Dirac delta function term [see formulas (1) and (2) for the electric dipoles]. Coefficients before the delta functions are different for the electric and magnetic dipole interactions. The delta function term in the electric dipole interaction gives an additional repulsion, when the delta interactions. The delta function term in the electric dipole interaction gives an angle dependence of wave function. At \( \phi = 0 \), the GP equation has symmetry-terminated conserved current related to the rotational symmetry of dipoles.

4.2. Transverse-longitudinal matter waves

Consideration of the transverse component of waves in dipolar BECs manifests in a special regime of matter-wave dispersion. At \( \cos \theta = 0 \), equation (25) appears as

\[ k^2 c^2 - \beta(\omega) \omega^2 = 0. \]  

(32)

We did not have such a limited case as equation (32) when we considered longitudinal waves [16–18]. The appearance of equation (32) is related to consideration of the full set of Maxwell equations (15)–(18) and presence of the perturbation of the electric field \( E_L \) perpendicular to the direction of wave propagation. As a signature of the presence of the transverse electric field \( E_L \), we have given the light speed in equation (32) and its solutions below.

The explicit form of equation (32) is

\[ \left( 1 + \frac{4\pi \sigma P_0^2}{m n_0 c^2} \right) \omega^4 - \left( \frac{\hbar^2 k^2}{2 m^2} + \frac{3 g n_0 k^2}{m} + \frac{Y_1 n_0 k^4}{m} + \frac{1}{2} \frac{4\pi P_0^2}{m n_0 c^2} g n_0 k^2 \right) \omega^2 \right) + \left( \frac{\hbar^2 k^2}{4 m^2} + \frac{g n_0 k^2}{4 m^2} + \frac{Y_2 n_0 k^4}{m} \right) \omega^2 = 0. \]  

(33)

Neglecting contributions of the quantum Bohm potential (terms proportional to \( \hbar^2 \)) and the third order by the interaction radius (terms proportional to \( Y_2 \)), we find the following solution

\[ \omega^2 = \frac{g n_0 k^2}{4 m} \left( 1 + \frac{4\pi \sigma P_0^2}{m n_0 c^2} \right) \times \left( 3 + \left( 2\sigma + 1 \right) \sigma \right) \frac{4\pi P_0^2}{m n_0 c^2} \]  

\[ \pm \sqrt{\left( 1 + \left( 2\sigma + 1 \right) \left( 3 + \frac{16\pi P_0^2}{m n_0 c^2} \right) \right)^2 + \frac{16\pi P_0^2}{m n_0 c^2}}. \]  

(34)

Let us acknowledge that solution (34) is obtained for \( \cos \theta = 0 \). If \( R_t \approx 0 \) in formula (34) simplifies to \( \omega^2 = \frac{gn_0k^2}{m} \) and \( \omega^2 = \frac{gn_0k^2}{2m} \), the first of them corresponds to the Bogoliubov spectrum. Getting the first order amendment on \( \frac{4\pi P_0^2}{m n_0 c^2} \) as in the case of small polarization, we obtain

\[ \omega^2 = \frac{g n_0 k^2}{m} \left( 1 + \frac{4\pi \sigma P_0^2}{m n_0 c^2} \right). \]  

(35)

and

\[ \omega^2 = \frac{g n_0 k^2}{2 m} \left( 1 - \left( 1 + \sigma \right) \frac{4\pi P_0^2}{m n_0 c^2} \right). \]  

(36)

In the opposite limit, when \( \frac{4\pi P_0^2}{m n_0 c^2} \gg 1 \), we find
\[ \omega^2 = \frac{gn_0 k^2}{m} \left( 1 + \frac{1}{2a} \right) \] (modified Bogoliubov mode) and
\[ \omega^2 = -\frac{gn_0 k^2}{2m} \left( \frac{4\pi\sigma n_0^2}{m\omega^2} \right)^{-1}. \] (37)

The last solution has a module of frequency much smaller than the frequencies of the Bogoliubov mode. Solution (37) shows an aperiodic damping and no wave solution, since \( \omega^2 < 0 \). On the other hand, it might cause the instability of dipolar BEC cloud to collapse.

**4.3. Electromagnetic waves**

In this subsection, we do not assume small frequencies. On the contrary, we focus our attention on properties of high-frequency excitations. If \( R_0 = 0 \), formula (24) gives one electromagnetic wave solution only: \( \omega = kc \). In this case, we do not have influence of the medium on properties of the electromagnetic waves. Let us acknowledge that the model under consideration does not account for the resonance interaction of the electromagnetic waves with the medium when the frequencies of the waves cause electron transitions within atoms.

Under the condition \( \theta = 0 \) or \( \theta = \pi \), equation (24) assumes the form
\[ \left( \omega^2 - k^2c^2 \right) \left[ \omega^2 - k^2c^2 + \beta(\omega)\omega^2 \right] = 0. \] (38)

From this equation, we find the usual dispersion dependence of the light \( \omega = kc \), while equation \( 1 + \beta(\omega) = 0 \) describes the two matter waves, which appear instead of the Bogoliubov mode in the electrically polarized BECs; see formulas (28)–(30) of this paper and \( [16–18] \).

In the opposite case, when \( \theta = \pi/2 \), equation (24) simplifies to
\[ \left( \omega^2 - k^2c^2 \right) \left( \omega^2 - k^2c^2 + \beta(\omega)\omega^2 \right) = 0. \] (39)

In this case we have two independent equations.

One of them, \( \omega^2 - k^2c^2 = 0 \), describes changeless dispersion of the light.

The second equation is a hybrid equation describing interacting light and matter
\[ \omega^2 \left( 1 + \beta(\omega) \right) - k^2c^2 = 0. \] (40)

This is an equation of the third degree relative to \( \omega^2 \). So, it contains three wave branches. We can acknowledge that it contains both, which are modified by the BEC dispersion of the light wave and the two matter waves. This equation has no low-frequency limitation as (26), thus, we have here a high frequency ‘tail’ in the dispersion of the matter waves.

Characteristic frequencies of the matter waves in dipolar BECs are \( \omega_1 = \sqrt{\frac{\omega_{m0}}{m}} k \) and \( \omega_2 = \sqrt{k^3/\omega_{m0}} k \). Each of them is considerably less than the frequency of light \( kc \). Thus the hybrid matter-light dispersion equation (40) contains a light-wave solution with matter-modified dispersion dependence, and this modification is rather small compared with \( kc \). We can use the iterative procedure to find the high-frequency solution. We present its frequency as \( \omega = kc + \delta\omega \), assuming \( \delta\omega \ll kc \). We substitute this representation of frequency in equation (40) and obtain
\[ 2kc\delta\omega + \delta\omega^2 + \beta(kc + \delta\omega)(kc + \delta\omega)^2 = 0. \] (41)

Here, we neglect \( \delta\omega \) in comparison with \( kc \). Hence, we find
\[ \omega \approx kc \left( 1 - \frac{1}{2}\beta(kc) \right). \] (42)

where \( \beta(kc) = -4\pi\sigma n_0^2d^2 (\frac{2m}{\omega_{m0}^2}) \). Expanding formula (42) in a series by \( \beta(kc) \), we get
\[ \omega = kc \left[ 1 + 2\pi\sigma n_0 d^2 \left( \sigma - \frac{8\pi n_0}{2mc^2} \right) \right]. \] (43)

with \( R_0 = n_0d \). We can see that polarized BECs cause splitting of the light on two waves at light propagation perpendicular to the direction of the equilibrium polarization. The magnitude of splitting \( M_i \) is equal to
\[ M_i \approx kc\frac{2\pi\sigma n_0 d^2}{mc^2}. \] (44)

From this formula, we find \( M_i > 0 \), so the frequency of the second wave is increased compared to the expression \( kc \): \( \omega > kc \). The second wave has a phase velocity greater than speed of light, which agrees with the general properties of waves (see, for instance, the Feynman lectures vol I \([59]\), the comment after formula 48.13). We consider the reasons for the appearance of superluminal phase velocity of the second wave in our model as a future problem. However, this shift is comment after formula 48.13). We consider the reasons for the appearance of superluminal phase velocity of the second wave in our model as a future problem. However, this shift is significant in comparison with \( kc \).

In the large frequency limit, we can put \( \beta = \beta(\omega = kc) \) in formula (44). Then, formula (24) becomes an equation of the second degree for \( \omega^2 \). Solving this approximate equation for all \( \theta \) we get two solutions
\[ \omega^2 = k^2c^2, \] (45)
and
\[ \omega^2 = \frac{1 + \beta \cos^2\theta}{1 + \beta} k^2c^2, \] (46)
accumulating all results described in this subsection.

At the parallel propagation of the light to the external electric field \( \mathbf{E} \parallel \mathbf{k} \) and \( \theta = 0 \), the electric field of the light wave is perpendicular to the external field and the equilibrium polarization of the medium (see figure 5a). We find it for both linear polarization of the light wave. Hence, we have equal dispersion dependencies for both polarizations of the light. These dispersion dependencies are given by formula (38). This formula shows that dispersion is not affected by the medium. We have found it for \( \delta\mathbf{E}_i, \mathbf{E}_0 \), where \( \delta\mathbf{E}_i \) is the electric field in the electromagnetic (light) wave, and sub-index \( i = 1,2 \) corresponds to two linear polarizations of the light wave.

If the light propagates perpendicular to the external electric field, we have that one polarization of the light is...
parallel to the external field and the other one is perpendicular. This case is depicted in figure 5b. We can expect that the light wave described by $\delta E_1$ is not affected by the medium, since $\delta E_1 \neq 0$. It corresponds to the first solution of equation (39), which coincides with the dispersion of the light in a vacuum. We also have the light wave with another polarization described by vector $\delta E_2$ (see figure 5b). Its electric field is parallel to the external field $\delta E_2 \parallel E_0$. In this case, we find solution (42). We see the contribution of medium in the light dispersion leading to non-trivial permittivity.

At oblique propagation of light relative to the external field, which is shown in figure 5c, we find that one linearly polarized light wave has the electric field vector perpendicular to the vector of the external electric field (see vector $\delta E_1$ on figure 5a). Thus, even at oblique propagation of the light wave, one of its dispersion branches is not affected by the medium. It has a dispersion dependence coinciding with the dispersion of the light in a vacuum [see formula (45)]. Electric field in the light wave with another polarization is directed at angle of to direction of the external field (see vector $\delta E_2$ in figure 5c). Hence, it has non-zero projection on the direction of the external field. Consequently, it is affected by the medium. Dispersion of this light wave is described by formula (46).

At light propagation through a medium, the speed of light depends on the permittivity of the medium. Formulas (45) and (46) show these dependencies. Moreover, equation (24), as its particular case (39), manifests even more than dependence of the light speed on medium permittivity. It shows splitting of the light wave dispersion into two branches (45) and (46). Each of these branches has its own permittivity. Similar effects take place in magnetized plasmas, where one finds the O wave, the X wave, the R wave and the L wave instead of the light wave (see, for instance, [60]). For instance, the phase velocity of the X wave in magnetised plasmas rises from c until the cut-off at $\omega_K = (\omega_L^2 + \omega_0^2/4 + \omega_e/2)$ is reached, where $\omega_0^2 = 4\pi e^2 n_0/m$ is the square of Langmuir frequency of plasmas, $\omega_e = eB_0/(mc)$ is the cyclotron frequency, $e$ is the charge of electron, and $B_0$ is the external magnetic field.

5. Conclusion

We have considered transverse waves in electrically polarized BECs. To do so, we have used an entire set of the Maxwell equations for the description of electromagnetic fields caused by the dynamic of the electric dipoles of the medium. In comparison to the case of longitudinal waves considered in our previous papers, we have found an advantage, which is a possibility of consideration of the light propagating through the electrically polarized BECs. Thus, we have considered dispersion of the light propagating through the medium and we have found the splitting of the light into two waves when the light propagates at an angle to the direction of the equilibrium polarization. We have calculated the magnitude of frequencies splitting for the case of the light propagation perpendicular to the direction of equilibrium polarization. There is no splitting of the light propagation along the direction of the equilibrium polarization, so there is also no contribution of the medium in the light dispersion dependence in this case.

As in the case of longitudinal waves, we have two matter waves that exist instead of the Bogoliubov mode existing in the unpolarized BECs and fully polarized BECs. We have found analytical solution for the plane-matter waves propagating in the 3D BECs and showing the influence of the anisotropy on the value of the frequencies.

As a result, we have developed a generalization of the quantum hydrodynamic equations for the electrically polarized BECs, including transverse wave propagation. Using it, we have shown that in the electrically polarized BEC situated in the external electric field, there are four waves: two of them are high-frequency waves, and are associated with the light. Two more waves are the anisotropic matter waves.

Appendix: Definition of quantum hydrodynamic variables

The concentration of particles is defined as the quantum average of the concentration operator in the coordinate representation $n = \sum_i \delta r - \tau$, with the sum over all particles in the system

$$n(r, t) = \int dR \sum_i \Psi^*(R, t) \delta r \Psi(R, t),$$

where $R = \{r_1, r_2, ..., r_N\}$ is a vector in $3N$ dimensional configurational space containing a set of coordinates of all $N$ particles in the system under consideration, $dR = \prod_{p=1}^{N} d\mathbf{r}_p$ is the volume element in $3N$ dimensional space, $\Psi(R, t) = a(R, t)e^{i\mathbf{S}(R)/\hbar}$ is the wave function describing the
exact evolution of the $N$ particle quantum system. The expression $\Psi(R, t)$ obeys the many-particle Schrödinger equation. The operator of the particle concentration contains the Dirac delta functions, which provide the projection of $3N$ dimensional quantum dynamics in 3D physical space, where real particles move.

The particle flux or the momentum density appears as

$$ j^a (r, t) = \int dR \sum_i \delta (r - r_i) \frac{1}{2m_i} \left( p^a_i \Psi (R, t) + \text{c.c.} \right), $$

where $p_i = -i\hbar \nabla_i$ is the momentum operator of the $i$th particle, and c. c. stands for the complex conjugation. The particle flux allows us to derive the Euler equation by applying $j = nv$.

Polarization appears in the Euler equation as follows:

$$ P^a (r, t) = \int dR \sum_i \delta (r - r_i) \Psi^* (R, t) \tilde{d}_i^a \Psi (R, t), $$

where $\tilde{d}_i^a$ is the operator of the electric dipole moment of a neutral particle.

The polarization current or polarization flux is

$$ R^{a\beta} (r, t) = \int dR \sum_i \delta (r - r_i) \frac{\tilde{d}_i^a}{2m_i} \times \left( \Psi^* (R, t) p^\beta \Psi (R, t) + \text{c.c.} \right). $$

The internal electric field caused by dipoles has the following explicit form

$$ E^a (r, t) = \int d^3r' G^{a\beta} (r, r') P^\beta (r', t). $$

The chemical potential in equation (11) of this paper is defined as follows

$$ \nu \mu (r, t) = \frac{\nabla p (r, t)}{mn (r, t)}, $$

where $p$ is the isotropic thermal pressure $p^{a\beta} = p\delta^{a\beta}$; see also formula (36) of [36].

The thermal pressure $p^{a\beta}$ and its analog in the the polarization current evolution equation $r^{a\beta}$ have the following explicit definitions

$$ p^{a\beta} (r, t) = \int dR \sum_{i=1}^N \delta (r - r_i) \alpha^2 (R, t) m_i u_i^a u_i^\beta, $$

and

$$ r^{a\beta} (r, t) = \int dR \sum_{i=1}^N \delta (r - r_i) \alpha^2 (R, t) m_i d_i^a u_i^\beta u_i^\gamma, $$

where $u_i (r, R, t) = v_i (R, t) - v (r, t)$ is a quantum equivalent of the thermal speed, and $v_i (R, t) \equiv \frac{1}{m_i} V_i S (R, t)$ is the velocity of $i$th particle proportional to gradient of phase of the many-particle wave function $\Psi (R, t)$. Definitions (53) and (54) appear at the derivation of many-particle QHD equations.

Formula (53) can be found in [35] [see the first term in formula (35)] and in [36] [see formula (15)], and in [19] [see formula (9)]. Formula (54) can be found in [19] [see formula (26)].

To derive QHD equations, we need the Schrödinger equation $i\hbar \partial_t \Psi = \hat{H} \Psi$, which governs the evolution of the many-particle wave function. We also need the explicit form of the Hamiltonian of the system under consideration, which within the quasi-static approximation is

$$ \hat{H} = \sum_i \left( \frac{1}{2m_i} \hat{p}_i^2 - \hat{d}_i \cdot \hat{E}_{ext} + V_{trap} (r_i, t) \right) + \frac{1}{2} \sum_{i,j\neq i} \left( U_{ij} - \hat{d}_i^a \hat{d}_j^\beta G^{a\beta}_{ij} \right). $$

The first term in the Hamiltonian is the operator of the kinetic energy. The second term represents the interaction between the dipole moment $\hat{d}_i^a$ and the external electric field. The subsequent terms represent the short-range $U_{ij}$ and the long range dipole–dipole $d_i^a d_j^\beta G^{a\beta}_{ij}$ interactions between neutral particles possessing the electric dipole moment. The Green function for the dipole–dipole interaction reads as $G^{a\beta}_{ij} = V_i^a V_j^\beta (1/\eta_i)$.

For the derivation of the QHD equations, we use quasi-static approximations for the dipole–dipole interaction. This means that at each moment of time, we consider dipole–dipole interaction between motionless dipoles. This interaction changes the state of the translational motion of molecules and the direction of their dipoles. At the next moment of time, particles are in new positions and have a new direction of dipoles. In this new state, particles interact as motionless, etc. Quasi-static interaction is an interaction via the electric field created by motionless dipoles. The slow motion of dipoles creates a small magnetic field and electromagnetic radiation. This can be accounted for by considering of the full set of Maxwell equations, as we do at the end of section II.

More details on the derivation of quantum collective observable evolution by quantum hydrodynamic methods can be found in the following References: [19, 35] (focused on dipole evolution) and [36] (focused on BECs evolution).

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