Non-helical Large-Scale Dynamo in Stellar Radiative Zones

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ABSTRACT

Our understanding of large-scale magnetic fields in stellar radiative zones remains fragmented and incomplete. Such fields, which must be produced by some form of dynamo mechanism, are thought to dominate angular-momentum transport, making them crucial to stellar evolution more generally. A major difficulty is the effect of stable stratification, which generally suppresses dynamo action. We propose a non-helical large-scale dynamo (LSD) mechanism that we find can operate in a stably stratified plasma alongside two ingredients: mean shear and non-helical magnetic fluctuations. Both ingredients are easily sourced in the presence of differential rotation. Our idealized direct numerical simulations, supported by mean-field theory, demonstrate the generation of near equipartition large-scale toroidal fields. The mechanism is robust to increasing stable stratification as long as a source of non-helical magnetic fluctuations is present, e.g. from a small-scale dynamo. Additionally, a scan over magnetic Reynolds number shows no change in the growth or saturation of the LSD, providing good numerical evidence of a quenching-free dynamo mechanism, which has been an issue for helical dynamos. These properties—the lack of quenching and robustness to stable stratification—make the mechanism a plausible candidate for generating in-situ large-scale magnetic fields in stellar radiative zones.

Keywords: magnetic fields–dynamo–stellar interiors-solar tachocline

1. INTRODUCTION

Massive stars are born rapidly rotating and quantifying their spin down throughout the course of stellar evolution is a key issue in stellar astrophysics. The angular momentum present in the radiative interior at the end of a massive star’s life has a strong impact on the dynamics of core-collapse and the spin distribution at formation of the subsequent compact remnant (MacFadyen & Woosley 1999; Heger et al. 2000; Yoon et al. 2006). Measurements of core rotation rates of red giant stars (Cantiello et al. 2014; Eggenberger et al. 2017; Ouazzani et al. 2019), white dwarf spins (Hermes et al. 2017), rotation periods of neutron stars at birth (Faucher-Giguere & Kaspi 2006; Gullón et al. 2014), and black hole spins by LIGO (Zaldarriaga et al. 2018; Roulet et al. 2021) all indicate efficient angular momentum transport in the progenitor cores that cannot be explained by most hydrodynamic processes. Instead, torques from large-scale magnetic fields generated by a dynamo are invoked as the dominant form of angular momentum transport in regions of radial differential rotation. The leading candidate is the Tayler-Spruit (TS) dynamo (Spruit 2002), whose recently updated prescription in 1D stellar evolution codes (Fuller et al. 2019) finds generally improved agreement with observations (Fuller & Lu 2022), although discrepancies remain (Eggenberger et al. 2019; Den Hartogh et al. 2020). First-principles investigations of the TS dynamo are pressingly needed to inform 1D prescriptions, similar to the feedback between mixing-length theory and 3D convection simulations in the modeling of stellar convection zones. In particular, we still lack understanding of what nonlinear dynamo mechanisms could enable the TS mechanism, remaining viable in the high magnetic Reynolds number and stably stratified conditions of a radiative zone (RZ).

In the global context, the Tayler-Spruit dynamo provides torques through the Maxwell stress of axisymmet-
ric poloidal and toroidal magnetic fields whose energy is sourced from the radial differential rotation (DR) itself. Successful operation requires closure of a dynamo loop: the axisymmetric toroidal field needs to be generated from the axisymmetric poloidal field and vice versa. The first direction is straightforward and uncontroversial: the toroidal field is generated by winding of the axisymmetric poloidal field by the radial DR. However, the opposite direction, regeneration of the axisymmetric poloidal field, remains an open question and is likely a non-linear dynamo effect (Zahn et al. 2007; Fuller et al. 2019). The original Spruit (Spruit 2002) study suggested that the amplified toroidal field goes unstable to the Tayler instability (Tayler 1973; Markey & Tayler 1973) and creates poloidal field, but this is insufficient because only non-axisymmetric modes are generated by the Tayler instability as first pointed out by Zahn et al. (Zahn et al. 2007). Follow-up studies alternatively proposed that a non-linear dynamo effect such as the \( \alpha \)-effect (driven by the helical part of the turbulence from the Tayler instability) may close the dynamo loop (Zahn et al. 2007; Fuller et al. 2019). However, the alpha effect seems to suffer catastrophic quenching at high magnetic Reynolds numbers in the presence of small-scale magnetic fields, which means that the large-scale dynamo might saturate on resistive timescales (Cattaneo & Hughes 1996; Brandenburg 2001; Rincon 2019) (see (Brandenburg 2018) and references within on possible ways to avoid quenching). These can be comparable to or longer than stellar lifetimes. An alternative dynamo mechanism that is immune to quenching and can generate equipartition large-scale magnetic fields on dynamical timescales is thus desirable.

Such a non-linear dynamo mechanism must also be able to operate in the stably stratified conditions characteristic of stellar RZs. Stable stratification generally suppresses dynamo action because it imposes a particular form of anisotropy on the velocity field: fluid motions are unrestricted horizontally while vertical motions are rapidly restored (Riley & Lelong 2000; Billant & Chomaz 2001; Lindborg 2006; Brethouwer et al. 2007; Chini et al. 2022). In the limit of arbitrarily strong stable stratification, the velocity field is approximated by a two component and three dimensional field, which is well known to be unable to support dynamo action (Zeldovich & Ruzmaikin 1980). Thus stable stratification and DR tend to play opposite roles in suppressing and supporting dynamo action. Finding a successful dynamo loop driven by DR while surviving the extreme stable stratification of RZs is a primary challenge of stellar interior physics.

One promising mechanism is the magnetic shear-current (MSC) effect, in which large-scale magnetic fields result from the combination of mean shear and non-helical magnetic fluctuations (Rogachevskii & Kleeorin 2004; Rädler & Stepanov 2006; Squire & Bhattacharjee 2016; Käpylä et al. 2020; Zhou & Blackman 2021). Originally applied in the context of accretion disks, the MSC effect helps explain large-scale magnetic field generation in magnetic turbulence driven by the magneto-rotational instability in a sheared (Keplerian) flow (Lesur & Ogilvie 2008; Squire & Bhattacharjee 2015a; Shi et al. 2016). In the stellar context, we propose that the MSC effect would operate as follows: a radial shear generates a toroidal field from the poloidal field; statistical correlations in the non-helical magnetic turbulence lead to an off-diagonal turbulent resistivity that sources poloidal field from the toroidal field, thus closing the dynamo loop. The magnetic fluctuations can in principle originate from a variety of sources including magnetic turbulence from the Tayler instability (Zahn et al. 2007; Fuller et al. 2019) or from a small-scale dynamo (SSD) operating in stably stratified turbulence (Skoutnev et al. 2021) driven by hydrodynamic instabilities, such as horizontal shear instabilities of latitudinal DR (Zahn 1974, 1992; Prat & Lignières 2013, 2014; Cope et al. 2020; Garaud 2020, 2021). Additionally, because the MSC effect is driven by the non-helical part of the magnetic turbulence, the generated large-scale fields are also non-helical and are not subject to the MHD helicity constraints that can lead to quenching of alpha-effect dynamos (Gruzinov & Diamond 1994; Rogachevskii & Kleeorin 2004; Rincon 2019). MSC driven dynamos are therefore unlikely to be affected by quenching in the same way as \( \alpha \) dynamos driven by helical turbulence. This quenching-free property, in combination with ample sources of magnetic turbulence and shear flows, make the MSC effect a promising mechanism for generating large-scale magnetic fields in differentially rotating stellar RZs.

The aim of this paper is to assess the viability of the MSC effect as a novel dynamo mechanism to explain large-scale magnetic field generation in the stellar context. The primary difficulty is the generally unknown effect of stable stratification on large-scale dynamo mechanisms, particularly in the extreme parameter regimes of high magnetic Reynolds number and strong stable stratification. Can the large-scale dynamo operate in the background of weak stable stratification? What level of stratification is needed to shut down the large-scale dynamo mechanism? Where in parameter space are stellar RZs relative to this threshold? We attempt to answer these questions with combined analytical and numerical
Figure 1. Schematic of a local shearing box embedded in a section of a differentially rotating radiative zone. Orientation of the Cartesian coordinates used in our setup are shown relative to the local unit vectors of the spherical coordinate system.

approaches. We use an available analytical framework (mean-field theory) to study perturbatively the effects of weak stratification on large-scale dynamos followed by direct numerical simulations (DNS) to additionally study the non-perturbative limit of strong stratification. A general agreement between analytic and numerical results allows us to extrapolate these results to realistic parameters for stellar RZs.

**Paper Outline**—Section 2 presents our local model of a stellar RZ and a mean-field theory framework for the MHD Boussinesq system. This allows calculation of the modifications to the MSC effect that result from perturbative stratification. However, the non-perturbative limit of strong stable stratification relevant to stars lies outside the formally valid regime of mean-field theory and therefore requires numerical exploration. Section 3 then presents sets of DNS where the effect of varying the stable stratification on the LSD is examined. Section 4 discusses two possible applications of the MSC effect in RZs and Section 5 concludes.

2. THEORETICAL CONSIDERATIONS

2.1. Model of Turbulence in a Radiative Zone

To most simply capture the physics of a local section of a differentially rotating RZ, we consider a shearing box model as shown in Figure 1 with an imposed vertical shear profile, stable stratification, and a turbulent velocity field driven with a body force at intermediate scales. The vertical (radial) shear profile represents a local section of radial DR, \( \Omega(r) \), which has a shear rate \( S = r(\partial \Omega/\partial r) \) in the rotating frame. The driven turbulence represents stably stratified turbulence that may be sourced from an aforementioned instability. We ignore mean rotation to focus on understanding the novel effect of stable stratification on the LSD, a critical step to the full problem. We discuss the possible effects of mean rotation (likely subdominant to the effects of radial shear) in Section 4.

This setup allows a study of the SSD and the LSD as well as their crucial non-linear feedback. We require that 1) the scale separation between the outer box scale and the intermediate forcing scale is large enough to allow an unambiguous definition of large scales and 2) the scale separation between the intermediate forcing scale and the smallest (viscous) scales is large enough to allow a significant turbulent cascade.

The unstratified case of this setup is actually destabilized by a large scale hydrodynamic instability known as the vorticity dynamo (VD) (Elperin et al. 2003; Käpylä et al. 2009). The VD generates large-scale vortical (shear) flows and saturates at large amplitudes that may disrupt the operation of a LSD (Teed & Proctor 2016). Complications caused by the VD are therefore important to understand and we work through the mean-field framework for both the LSD and VD in the next sections. It will turn out that the VD is strongly suppressed by stable stratification and so does not play a role in the stratified problem.

2.2. Equation Formulation

The standard hydrodynamical model of a stably stratified, collisional plasma with subsonic velocity fluctuations on vertical length scales small compared to the local scale height is the set of Spiegel–Veronis–Boussinesq equations (Spiegel & Veronis 1960). With magnetic fields included, we will call them the MHD Boussinesq equations. In the following sections, we apply the mean-field approach to the MHD Boussinesq equations for the total fields to separate the evolution of the large and small-scale fields. This allows for an analysis of the two large-scale instabilities of the system: the hydrodynamic vorticity dynamo of the velocity field and the large-scale dynamo of the magnetic field.

The equations for the total quantities (the velocity field \( \mathbf{U}_T \), magnetic field \( \mathbf{B}_T \), buoyancy \( \Theta_T \), and pressure \( P_T \)) are:

\[
\begin{align*}
\partial_t \mathbf{U}_T + \mathbf{U}_T \cdot \nabla \mathbf{U}_T &= -\nabla P_T + \Theta_T \hat{x} + \mathbf{J}_T \times \mathbf{B}_T + \nu \Delta \mathbf{U}_T + \sigma_f, \\
\partial_t \mathbf{B}_T &= \nabla \times (\mathbf{U}_T \times \mathbf{B}_T) + \eta \Delta \mathbf{B}_T, \\
\partial_t \Theta_T + \mathbf{U}_T \cdot \nabla \Theta_T &= -N^2 \mathbf{U}_T \cdot \hat{x} + \kappa \Delta \Theta_T,
\end{align*}
\]
\[ \nabla \cdot \mathbf{U}_T = 0, \quad \nabla \cdot \mathbf{B}_T = 0, \tag{4} \]

where \( \nu \) is the kinematic viscosity, \( \eta \) is the resistivity, \( \kappa \) is the thermal diffusivity, \( \sigma_f \) is non-helical forcing at small scales, and \( \mathbf{B}_T \) is normalized by \( \sqrt{4\pi \rho_0} \) (\( \rho_0 \) is the constant background plasma density). Note that the “vertical” direction is aligned with \( \hat{z} \). In this formulation (Spiegel & Veronis 1960; Kundu & Cohen 2002; Garaud 2021), the buoyancy field is related to the temperature fluctuations by \( \Theta_T = \alpha_T g T' \) and the Brunt-Väisälä frequency is given by \( N^2 = \alpha_T g (T_{0,x} - T_{ad,x}) > 0 \), where \( g > 0 \) is the local gravitational acceleration, \( \alpha_T \) is the coefficient of thermal expansion, and \( T_{0,x} \) and \( T_{ad,x} \) are the background and adiabatic temperature gradients, respectively.

The mean-field approach splits up the total fields into mean components and fluctuating components, which we will denote with upper and lower case letters. We define the mean field as a spatial average over \( x \) and \( y \) of the total field (e.g., \( \langle \mathbf{B}_T \rangle = \int \mathbf{B}_T \, dx \, dy = \mathbf{B}(z) \)). Thus the magnetic field is \( \mathbf{B}_T = \mathbf{B} + \mathbf{b} \), the velocity field is \( \mathbf{U}_T = \mathbf{U}' + \mathbf{u} \) with \( \mathbf{U}' = \mathbf{U}_0 + \mathbf{U} \), and the buoyancy field is \( \Theta_T = \Theta + \theta \). Note that the divergence free conditions require \( B_z(z) \) and \( U_z(z) \) to be constants (which we set to 0), so the mean magnetic field, for example, is of the form \( \mathbf{B}(z,t) = (B_z(z,t), B_y(z,t), 0) \). The local radial DR is modeled as an imposed linear shear flow \( \mathbf{U}_0 = -S x \hat{y} \), which varies vertically and flows in the “toroidal” \( \hat{y} \) direction. See Figure 1 for the geometry. Lastly, to aid with studying the VD, we define the vorticity \( \mathbf{W}_T = \nabla \times \mathbf{U}_T = \mathbf{W}' + \mathbf{w} \), where \( \mathbf{W}' = \mathbf{W}_0 + \mathbf{W} \) and \( \mathbf{W}_0 = \nabla \times \mathbf{U}_0 = -S \hat{z} \).

The mean-field equations are obtained by substituting the scale-separated fields into the equations for the total quantities and taking spatial averages:

\[ \partial_t \mathbf{W} = \nabla \times (\mathbf{U}' \times \mathbf{W}' + \mathbf{J} \times \mathbf{B}) + \partial_z \Theta \hat{y} + \nabla \times (\nabla \cdot ((-\mathbf{u} \mathbf{u} + \mathbf{b} \mathbf{b}))) + \nu \nabla^2 \mathbf{W}, \tag{5} \]

\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{U}' \times \mathbf{B} + \langle \mathbf{u} \times \mathbf{b} \rangle) + \eta \nabla^2 \mathbf{B}, \tag{6} \]

\[ \partial_t \Theta + \mathbf{U}' \cdot \nabla \Theta + \langle \mathbf{u} \cdot \nabla \Theta \rangle = -N^2 \mathbf{U} \cdot \hat{z} + \kappa \Delta \Theta, \tag{7} \]

\[ \nabla \cdot \mathbf{U} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{8} \]

In this form, it is clear that the mean vorticity and mean magnetic field can be driven by the Reynolds and Maxwell stresses, \( \mathcal{T} = (-\mathbf{u} \mathbf{u} + \mathbf{b} \mathbf{b}) \), and the electromagnetic force, \( \mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle \), respectively, of the fluctuating quantities. The equations for the fluctuating quantities are obtained by subtracting the equations for the mean-fields from the those of the total fields. The fluctuation equations are shown in the Appendix A where they are used to compute transport coefficients.

The small scales contain stably stratified turbulence (driven by \( \sigma_f \)) and the small-scale dynamo. We now discuss their details as well as the dimensionless parameters of the problem before continuing with mean-field theory of the large scales.

**Dimensionless parameters**—Suppose the forcing \( \sigma_f \) at length scale \( l_f = 2\pi/k_f \) leads to steady state turbulence with outer-scale velocity fluctuations \( u_{rms} \) prior to the growth of any instabilities. The system is then described by five dimensionless parameters, \( Re, Sh, Fr, Pm, \) and \( Pr \). The Reynolds number \( Re = u_{rms}/k_f \nu \) captures the ratio of the viscous timescale to the outer scale eddy turnover time. The shear number \( Sh = S l_f/u_{rms} \) captures the ratio of the outer scale eddy turnover time to the shearing time scale. The Froude number \( Fr = u_{rms}/Nl_f \) captures the ratio of the gravitational restoring timescale (Brunt-Väisälä period) to the outer scale eddy turnover time. Lastly, the magnetic Prandtl \( Pr = \nu/\kappa \) numbers measure ratios of diffusivity timescales. We set both to unity \( \nu = \kappa = 1 \) for simplicity in our DNS, but will discuss their expected effects based on theory and previous simulations in Section 4.

**Stably Stratified Hydrodynamic Turbulence**—The forcing, \( \sigma_f \), leads to stably stratified turbulence that provides the background, hydrodynamic turbulence in which the SSD, VD, and LSD may grow. We briefly review properties relevant to the dynamo. The turbulent cascade in a stably stratified fluid with energy injection \( \epsilon \approx u_{rms}^3 k_f \) at wavenumber \( k_f \) and dissipation at viscous wavenumber \( k_o \) contains two inertial ranges, one above and one below the Ozmidov wavenumber \( k_O / 2\pi = (N^3 / \epsilon)^{1/2} \) where the eddy turnover frequency matches the Brunt-Väisälä frequency (Brethouwer et al. 2007; Clini et al. 2022). At larger scales with wavenumbers less than \( k_O \), the velocity field is highly anisotropic due to the restriction of vertical motions by the stable stratification (Riley & Le-long 2000; Riley & Lindborg 2010). The scale separation between \( k_f \) and \( k_O \) is controlled by the Froude number \( k_O = Fr^{-3/2} k_f \). At smaller scales with wavenumbers greater than \( k_O \), the velocity field is nearly isotropic due to the negligible effect of buoyancy on the fast timescales of small eddies. The scale separation between \( k_O \) and \( k_{\nu} \) is set by the buoyancy Reynolds number \( k_{\nu} = Rb^{3/4} k_O \), where \( Rb = Re Fr^2 \). Rb has been found to be the primary control parameter of stably stratified turbulence.
and needs to be larger than one to avoid the viscosity-affected stratified flow regime where the isotropic inertial range disappears (Billant & Chomaz 2001; Waite & Bartello 2004; Lindborg 2006; Brethouwer et al. 2007; Maffioli & Davidson 2016; Chini et al. 2022). DNS of strong stably stratified turbulence thus simultaneously requires $Fr \ll 1$ and $Rb \gg 1$, which is computationally challenging (Bartello & Tobias 2013).

Small-Scale Dynamo—The SSD typically has a much faster growth rate than either the VD or LSD and generates magnetic fields on length scales smaller than the forcing scale. Here we briefly review the instability criterion. The SSD will operate if the turbulence is sufficiently vigorous: magnetic stretching will statistically win over magnetic diffusion and lead to amplification of any seed magnetic field to near-equipartition with the turbulent kinetic energy. In isotropic turbulence, the SSD is unstable when $Rm$ is above a critical value $Rm > Rm^c = O(10^2)$. $Rm^c$ depends on the magnetic Prandtl number $Rm^c = Rm^c(Pm)$ and is higher in the low $Pm$ limit applicable to stellar interiors (Iskakov et al. 2007). When unstable, its exponential growth rate scales as $\gamma_{SSD} \sim u_{rms} Rm^{1/2}/l_f$ (Rincon 2019). However, in the presence of stable stratification ($Fr < 1$), the largest scales in the system are anisotropic and inefficient at contributing to the dynamo, leading to a reduced effective $Rm$ known as the magnetic buoyancy Reynolds number $Rb_m = Rm Fr^2$ (Skoutnev et al. 2021). The new criterion for the SSD to operate becomes $Rb_m > Rb_m^c$, where $Rb_m^c$ has a dependence on $Pm$ (Skoutnev et al. 2021) similar to $Rm^c$ in isotropic turbulence. Thus strong enough stable stratification ($Fr \ll 1$) can shut off the SSD even at high magnetic Reynolds numbers $Rm \gg Rm^c$. In this framework, the SSD operates within the fluctuation equations (see Appendix A) and rapidly provides a source of background magnetic fluctuations that interact with the LSD.

Mean-Field Theory—At this point, the mean-field and fluctuation equations together are still exact and just as difficult to solve as the original MHD Boussinesq equations. The primary issue is finding a closure for the evolution of the mean vorticity and magnetic fields driven by $\mathcal{T}$ and $\mathcal{E}$, respectively. To work around this issue, we use the second order correlation approximation (SOCA), which works with linear fluctuation equations by neglecting the problematic third and higher-order terms (Brandenburg & Subramanian 2005; Rädler & Stepanov 2006; Squire & Bhattcharjee 2015b). This allows for solving a closed system of equations for the mean-field evolution. A background of isotropic small-scale turbulence in both the velocity and magnetic field is assumed onto which anisotropic effects such as shear and stratification are added perturbatively. Physically the small-scale magnetic field should arise from the SSD but its statistics are treated as given for this calculation. Arbitrarily small seeds of the mean-fields may then be linearly unstable.

The choice of horizontal averages leads to no direct coupling between the mean-field vorticity and induction equations (e.g. $J \times B = 0$, $\nabla \times (U \times B) = 0$) except through $\mathcal{E}$ and $\mathcal{T}$. We make the standard assumption that $\mathcal{E} = \mathcal{E}(B)$ depends only on the mean magnetic field, which decouples the mean-field induction and vorticity equations and allows the VD and LSD to be analyzed independently. While it is possible to have joint mean vorticity-magnetic field instabilities (Blackman & Chou 1997; Courvoisier et al. 2010), simulations in Section 3 support our no-coupling assumption because we always observe and LSD with no accompanying growth of the VD when it is suppressed by stratification.

We note that the drastic nature of the SOCA limits the rigorous validity of any results for the LSD to either low magnetic Reynolds numbers $Rm \ll 1$ (in the limit of low conductivity $l_f^2/\eta \tau_c \ll 1$) or small Strouhal numbers $St = u_{rms} \tau_c/l_f \ll 1$ (in the limit of high conductivity $l_f^2/\eta \tau_c \gg 1$), where $\tau_c$ is the turbulence correlation time (Brandenburg & Subramanian 2005). In realistic astrophysical turbulence, $Rm$ is extremely large and $St$ is typically order unity. Results from the SOCA can, as a consequence, be used at most to suggest what effects may be qualitatively operating at $Rm \gg 1$ and $St \sim 1$. The combination of DNS at moderate $Rm$ alongside the results from mean-field theory is therefore important to improve our confidence in understanding the dynamo mechanisms that operate in astrophysical regimes.

2.3. Vorticity Dynamo

The imposed shear flow in the unstratified case is unstable to a purely hydrodynamic instability known as the vorticity dynamo. We will show that even a small amount of stable stratification will stabilize the VD. We extend the original formulation of the VD in Elperin et al. (Elperin et al. 2003) to include stable stratification in the framework of mean-field theory. Evolution equations of the mean vorticity field $W(z,t) = (W_x(z,t), W_y(z,t), 0) = (-\partial_z U_y, \partial_z U_x, 0)$ can be written in the form (dropping magnetic field terms):

$$
\partial_t W_x = -SW_y - \nu_{xy} S l_f^2 \partial_z^3 W_x + \nu_l u_{rms} l_f \partial_z^2 W_x,
$$

$$
\partial_t W_y = -\nu_{yx} S l_f^2 \partial_z^3 W_x + \partial_z \Theta + \nu_l u_{rms} l_f \partial_z^2 W_y,
$$

$$
\partial_t \Theta = -\nu_l u_{rms} l_f \partial_z^2 \Theta.
$$
These equations are identical to that of Elperin et al. (Elperin et al. 2003) except the addition of stratification. We normalize transport coefficients by their expected scalings so that they are dimensionless. Modes of the form $e^{i k_z z + \gamma t}$ have a growth rate:

$$\gamma_{VD} = \frac{u_{rms}}{l_f} \left( \sqrt{-k^2 l_f^2 \nu_{yx} Sh^2 - Fr^{-2} - \nu_t k^2 l_f^2} \right),$$

(12)

where we use the standard assumptions that the dimensionless transport coefficients are small and there is enough scale separation ($k^2 l_f^2 |\nu_{yx}| \ll 1$). We see that there are growing solutions for small enough $k_z$, small enough $Fr^{-1}$, and $\nu_{yx} < 0$ ($\nu_{yx}$ is negative in the unstratified case (Elperin et al. 2003; Käpylä et al. 2009)). With the addition of stable stratification, it is clear that even weak stratification reduces or possibly fully stabilizes the growth of the VD since the stratification-related term $Fr^{-2}$ in Eq. 12 is not multiplied by any transport coefficients and $|k^2 l_f^2 \nu_{yx}| \ll 1$. Any modifications of the transport coefficients (e.g. $\nu_{yx}$ or $\nu_t$) by stratification are therefore unimportant because they only appear as higher order corrections to the growth rate.

For a system of arbitrarily length in the z-direction, a growing VD with maximum growth rate

$$\gamma_{VD}^{\text{max}} = \frac{u_{rms}}{l_f} \left( -Sh^2 \nu_{yx} / 4 \nu_t - \nu_t Fr^{-2} / Sh^2 \nu_{yx} \right),$$

(13)

occurs at a wavenumber

$$k_{\text{max}}^2 l_f^2 = -Sh^2 \nu_{yx} / 4 \nu_t^2 + Fr^{-2} / Sh^2 \nu_{yx}. $$

(14)

Instability requires that the stratification be weaker than $Fr^{-1} < -Sh^2 \nu_{yx} / 2 \nu_t$ or the shear stronger than $Sh > \sqrt{-2 \nu_t / Fr \nu_{yx}} \equiv Sh_{\text{V}D}^c$. With the assumption $Sh = O(1)$ and again that the dimensionless transport coefficients are small (and typically off diagonal turbulent diffusivity coefficients are smaller than the diagonals ones $|\nu_{yx}| \ll \nu_t$), even a weak stratification $Fr \approx O(1)$ can shut off the VD.

Note that in a finite system, the unstable modes must be able to fit into the domain and so the critical $Sh_{\text{V}D}^c$ for instability may instead depend on the lowest available wavenumber.

2.4. Large-Scale Dynamo

In the presence of the imposed shear ($U_0 = -S x \hat{y}$) and non-helical magnetic fluctuations, the system is unstable to a LSD due to the magnetic shear-current effect (Squire & Bhattacharjee 2016). Extending mean-field theory to include stable stratification leads to evolution equations:

$$\partial_t B_x = -\eta_{yx} S_{ij} l_f^2 \partial_j^2 B_y + \eta_t u_{rms} l_f \partial_j^2 B_x,$$

$$\partial_t B_y = -SB_x - \eta_{yx} S_{ij} l_f^2 \partial_j^2 B_x + \eta_t u_{rms} l_f \partial_j^2 B_y,$$

(15)

(16)

Modes of the form $e^{i k_z z + \gamma t}$ have a growth rate:

$$\gamma_{MSC} = \frac{u_{rms}}{l_f} \left( k_z l_f Sh \sqrt{-\eta_{yx} - \eta_t k^2 l_f^2} \right),$$

(17)

where we have used the standard assumption that transport coefficients are small and there is enough scale separation ($k^2 l_f^2 |\eta_{yx}| \ll 1$). There are growing LSD dynamo solutions when the off-diagonal turbulent resistivity is negative $\eta_{yx} < 0$, which occurs only when magnetic fluctuations are present (Squire & Bhattacharjee 2015b). A positive $\gamma_{MSC} > 0$ requires $Sh > k_{sys} l_f \eta_t / \sqrt{-\eta_{yx}} \equiv Sh_{\text{MSC}}$ for the lowest wavenumber $k_{sys}$ that fits into the domain. The maximum growth rate is

$$\gamma_{MSC}^{\text{max}} = -\frac{u_{rms}}{l_f} \frac{Sh^2 \eta_{yx}}{4 \eta_t},$$

(18)

and occurs at wavenumber

$$k_{\text{max}} l_f = \frac{Sh \sqrt{-\eta_{yx}}}{2 \eta_t}.$$ 

(19)

These predictions are consistent with our simulations in Section 3.4 where we observe that a dominant mode emerges independent of increasing the domain length $L_z$ and that there are no temporal variations in the phase of the growing LSD mode because the growth rate is purely real.

Unlike for the VD, the mean-field induction equations 15 and 16 look identical to their unstratified case in Squire et al. Squire & Bhattacharjee (2016) except now the the transport coefficients can have additional contributions from the effects of stable stratification. We carry out a calculation of the transport coefficients that incorporates Boussinesq effects into SOCA framework and report the relevant results here for consideration of space. Details of the calculation are described in the Appendix A.

We find that out of $\eta_{yx}$, $\eta_{xy}$, and $\eta_t$, only the isotropic turbulent resistivity $\eta_t$ is modified by stable stratification as indeed must be the case due to the perturbative expansion in shear and stratification. So now $\eta_t = \eta_{t,0} + \eta_{t,N2}$. The modified coefficient is further split up into contributions from the non-helical velocity and magnetic fluctuations, i.e. $\eta_{t,N2} = \eta_t^{(u)} + \eta_t^{(b)}$. The result is:
\[ \eta_{k,Nz}^{(u)} = N^2 \int dk d\omega \left( \frac{3\bar{\eta}(\bar{v}k^2 - \omega^2)W_u(k,\omega)}{10(\bar{\eta}^2 + \omega^2)(\bar{v}^2 + \omega^2)(\bar{\kappa}^2 + \omega^2)} \right), \]

\[ \eta_{k,Nz}^{(b)} = N^2 \int dk d\omega \left( \frac{(\bar{v}k^2 - (\bar{\kappa} + 2\bar{\nu})\omega^2)W_b(k,\omega)}{60(\bar{v}^2 + \omega^2)(\bar{\kappa}^2 + \omega^2)} \right), \]

where \( \bar{\nu} = \nu k^2 \), \( \bar{\eta} = \eta k^2 \), \( \bar{\kappa} = \kappa k^2 \). \( W_u(k,\omega) \) and \( W_b(k,\omega) \) are the statistics of the non-helical background velocity and magnetic fluctuations (with the magnetic component assumed to arise from the SSD). We find that for a standard Gaussian model of the fluctuation statistics (Rädler & Stepanov 2006) both contributions of stable stratification to \( \eta_{k,Nz} \) are positive and that the kinetic term is dominant over the magnetic term (see Appendix A). According to Eqs. 18 and 19, mean-field theory predicts that weak stable stratification will slightly weaken the growth rate and push the dominant modes to larger scales (lower \( k_{\text{max}} \)). In other words, unlike the VD, we expect the LSD to be slightly modified but remain unstable in the presence of stable stratification, so long as there are still sufficient small-scale magnetic fluctuations.

We note that the generated large-scale fields are non-helical (i.e. the helicity \( \mathcal{H} = \int \mathbf{B} \cdot \mathbf{A} dV = 0 \), where \( \mathbf{A} \) is the vector potential). Further discussion of catastrophic quenching and helicity, in particular why standard arguments for quenching do not apply to the MSC mechanism, is given in the Appendix B.

2.5. Summary of Theoretical Predictions

**Summary without stratification**—If we start with a seed magnetic field, and turn on forcing at \( t = 0 \), then 1) the VD will begin to grow if \( Sh > Sh_{\text{VD}} \), and 2) the SSD will begin to grow if \( Rm > Rm' \). Once the SSD saturates such that magnetic fluctuation are in equipoition with velocity fluctuations, then the LSD will begin to grow if \( Sh > Sh_{\text{LSD}} \). Once the mean vorticity and magnetic fields become strong, they can possibly interact with each other through their back reaction on the turbulent flow and its statistics. It turns out in DNS (both in this article at moderate \( Re \approx 120 \) and in Teed & Proctor 2016) at low \( Re \approx 5 \), the zonal shear flow of the VD saturates at amplitudes orders of magnitude larger than the original vertical shear and the driving small-scale turbulence. We interpret this as a destabilization of the model because such a strong zonal flow would likely redistribute its energy in the global context on dynamical timescales and destroy the steady vertical shear assumed in the local box model. Thus we argue it is unphysical to consider the LSD in the context of a local box model in the parameter regimes where the VD is unstable.

**Summary with stable stratification**—With stable stratification that is sufficiently weak, so \( Rb_m = RmFr^2 > Rb_{m}^* \) and \( Fr \sim 1 \), at \( t = 0 \) we expect at least the SSD to grow. The mean-field model predicts a regime where the VD will be stable (\( Sh < Sh_{\text{VD}} \)), but the LSD will continue to operate if \( Sh > Sh_{\text{LSD}} \). This is astrophysically interesting since it allows the in-situ generation of a mean magnetic field without destabilization of the background hydrodynamic flow. Since stable stratification is only added perturbatively in the SOCA, it cannot predict how the LSD will behave with increasingly stronger stratification (increasing \( Fr^{-1} \)). Further increasing the stable stratification will slowly suppress the SSD and also increase \( \eta_{k,Nz} \). Therefore one can speculate that the LSD should be at least slowly suppressed. As an upper bound, the LSD will stay active at most until the stable stratification is so strong that \( Rb_m < Rb_{m}^* \), which is when the SSD is shut down. Determining the robustness of the LSD for intermediate stratification requires DNS of the full equation set.

3. NUMERICAL STUDY AND RESULTS

3.1. Numerical Setup

We use SNOOPY (Lesur 2015), a 3D pseudospectral code, with low-storage third-order Runge-Kutta time stepping and 3/2 de-aliasing to carry out DNS of the MHD Boussinesq equations. Our default domain has a size \( (L_x, L_y, L_z) = (1, 1, 4) \). Periodic boundary conditions are used in the \( y \) and \( z \) direction and shear periodic boundary conditions in the \( x \) direction to model the imposed shear flow \( U_0 = -Sz\hat{y} \). The initial seed magnetic field is random at all scales and extremely weak \( (E_b(t = 0) = 10^{-16}) \) to allow self-consistent amplification by the SSD, if present. The momentum equation is driven with an isotropic, time-correlated forcing term \( \sigma_f \) and the system is integrated in time (alternative forcing types give similar results). This is our model of a local patch of stably stratified turbulence in a differentially rotating stellar RZ as sketched in Figure 1. A visualization of a representative simulation is shown in Figure 2.

The forcing is restricted to a waveband of width \( \pi/L \) centered at \( k_f = 5 \cdot 2\pi/L \), and has a correlation time \( \tau_c = 0.3 \), chosen to satisfy the relation \( u_{rms} \approx 2\pi/(k_f \tau_c) \) at early times. The forcing wavenumber \( k_f \) is chosen to allow more than an order of magnitude scale separation from the large scale at which the mean-fields are expected to grow (i.e. \( k_f L_z/2\pi = 20 \)), while still supporting a moderate turbulent cascade of the injected energy.
V. Skoutnev, J. Squire, A. Bhattacharjee

Figure 2. Annotated visualization of the y-component of the magnetic field $B_y(x,y,z)$ from a simulation with $Rm = 210$, $Sh = 1$, and $Fr^{-1} = 3$. Boussinesq stratification (blue arrow) is imposed in the x-direction with Brunt-Väisälä frequency $N$. A shear flow (green arrows) is imposed in the x-direction with profile $U_0 = -Sx\hat{y}$. Forcing of the momentum equation at length scale $l_f = 2\pi/k_f$ (black arrow) generates velocity fluctuations, which drive a SSD to generate magnetic fluctuations. The subsequent evolution of the large-scale velocity, $U(z)$, and magnetic field, $B(z)$ (light orange), is studied.

to the smallest, viscous scales. An explicit viscosity, resistivity, and thermal diffusivity (with $Pm = Pr = 1$) is used to resolve the diffusive scales in the spectral code.

Signatures of the VD and LSD are most visible in two main diagnostics: 1) the isotropic energy spectra and 2) the time evolution of the energy in the large and small scales. The isotropic energy spectrum is defined in the standard way:

$$E_B(k,t) = \sum_{|k| \in [k_s - \frac{\pi}{2}, k_s + \frac{\pi}{2}]} \frac{1}{2} |\hat{B}_k(t)|^2,$$

(22)

where $\hat{B}_k(t)$ is the Fourier transform of the magnetic field $B(x, t)$ in the simulation. We define the wavenumber $k_s = 2\pi/L$ as the separation between the large and small scales. Then, the large and small-scale magnetic energies are:

$$E_B(t) = \sum_{k \leq k_s} E_B(k, t),$$

(23)

$$E_b(t) = \sum_{k > k_s} E_B(k, t).$$

(24)

The analogous definition holds for the kinetic energy spectra, $E_u(k, t)$, and the energy in the large and small-scale velocity fields $E_U(t)$ and $E_u(t)$, respectively. We additionally denote normalized magnetic energy with a tilde, e.g. $\tilde{E}_B(t) = E_B(t)/E_u(t)$ where $E_u(t)$ is the kinetic energy averaged over the last $50\tau_c$ of a simulation.

Figure 3. Examination of how the vorticity dynamo is easily stabilized with increasing stratification (increasing $Fr^{-1}$). Kinetic energy (top) and spectra (bottom) diagnostics of hydrodynamic simulations with varying shear $Sh$ and stratification $Fr^{-1}$ at fixed $Re \approx 120$. Solid lines in the top panel are the energy in the mean field, $E_U$, while dotted lines are the energy in the small-scale velocity field, $E_u$. The spectra are computed from a snapshot at the last time point. Green and orange shaded regions in the bottom panel represent our definition of the large and small scales. The spectral resolution of the simulations is $N_x \times N_y \times N_z = 192^2 \times 768$ modes.

3.2. Hydrodynamic Vorticity Dynamo

Unstratified VD—We begin by confirming the results of Elperin et al. (2003) and Käpylä et al. (2009) for the unstratified VD. We increase the shear parameter $Sh$ from $Sh = 0$ to $Sh \approx 1$ (by increasing $S = 0$ to $S = 4$) while keeping $Re \approx 120$ fixed and $Fr^{-1} = 0$. These runs are hydrodynamic and the diagnostics are shown in gray and blue in Figure 3. For the energy evolution (top panel), the energy of the mean vorticity field $E_U(t)$ of the high $Sh \approx 1$ run (solid blue) grows exponentially and saturates at orders of magnitude larger energies than the driven small-scale velocity turbulence, whose baseline level is $E_u(t)$ of the VD-stable runs (dotted gray,
green, or orange). The normalized kinetic energy spectra at the last time point (bottom panel) also clearly reveals that the low wavenumber $k < k_s$ modes have more energy than the forcing wave numbers for the VD-unstable simulation (blue), while the same is not true for the run without shear (gray). Additionally, most of the energy in the large-scale modes is dominated by the $y$-component of the velocity field (not shown). These are the characteristic signatures of the VD that destabilizes this shearing box setup without stratification.

**Stably Stratified VD**—To test the mean-field theory prediction, we slowly increase the strength of the background stable stratification and explore the effect on the hydrodynamic VD. We increase the stratification parameter $Fr^{-1} \in \{0.04, 0.21\}$ while keeping $Sh \approx 1$ and $Re \approx 120$ fixed by increasing $N \in \{0.2, 1\}$. Figure 3 show the diagnostics in orange and green. Based on the evolution of $E_U(t)$ (top panel), the VD is already only marginally unstable at $Fr^{-1} = 0.04$ (solid orange) and becomes fully stable at and above $Fr^{-1} = 0.21$ (solid green). This is also seen in the kinetic energy spectra (bottom panel, orange and green) where the low wavenumber $k < k_s$ modes remain in subequilibration with the energy at the forcing wave number, except in the marginal case (orange) where they are modestly excited. Note that $Fr^{-1} \geq 1$ corresponds to stratification being important at and above forcing scales (as well as a range of smaller scales), while $Fr^{-1} < 1$ means stratification only affects scales larger than the forcing scale, which we call weak stratification here. Thus, our simulations qualitatively agree well with the mean-field theory prediction: weak stable stratification easily shuts down the VD.

We note that adding magnetic fields to these simulations (not shown) does produce a SSD and LSD (only in the shearing cases), but their addition does not change the above results.

**3.3. Stably Stratified Large-scale Dynamo**

**Weak Stable Stratification**—We turn to the case of the LSD in turbulence where the stratification is weak, but sufficient to stabilize the VD. The magnetic evolution of a VD-stable, weakly stratified system proceeds in three main phases as shown in the diagnostics of Figure 4. A simulation with $Rm \approx 120$, $Fr^{-1} = 0.2$ with a spectral resolution of $N_x \times N_y \times N_z = 192^3 \times 768$ modes. Top panel: solid lines are the energy in the mean field normalized by the kinetic energy, $E_B$, while dotted lines are the energy in the small-scale magnetic fields, $E_b$, similarly normalized by the kinetic energy. The inset plot shows the same diagnostics for the velocity field. Red, orange, and green shaded regions represent the SSD growth phase, LSD growth phase, and the LSD saturated phase. Bottom panel: normalized magnetic energy spectra at representative times (different colors). Green and orange shaded regions represent our definition of the large and small scales.

![Figure 4](image.png)

**Figure 4.** Comparing a simulation with shear ($Sh = 0.9$) that is unstable to the LSD with a simulation with no shear ($Sh = 0$). Both simulations have $Rm \approx 120$, $Fr^{-1} = 0.2$ with a spectral resolution of $N_x \times N_y \times N_z = 192^3 \times 768$ modes. Top panel: solid lines are the energy in the mean field normalized by the kinetic energy, $E_B$, while dotted lines are the energy in the small-scale magnetic fields, $E_b$, similarly normalized by the kinetic energy. The inset plot shows the same diagnostics for the velocity field. Red, orange, and green shaded regions represent the SSD growth phase, LSD growth phase, and the LSD saturated phase. Bottom panel: normalized magnetic energy spectra at representative times (different colors). Green and orange shaded regions represent our definition of the large and small scales.
SSD spectrum. This is a domain-size-dependent effect—if the domain size was increased, the initial seed value of the mean field would decrease and lead to a longer linear phase. However, increasing the domain size any further is currently prohibitively expensive for DNS (but see Section 3.4 for a convergence test at a lower \( Rm \)).

The majority of the growth phase ends and the LSD saturation phase (green shaded region in top panel) begins for the sheared case around \( t/\tau_c = 100 \) where the large-scale field energy undergoes a quasi-random behavior with slow oscillations on the timescale of hundreds of dynamical times (the origin of the quasi-random oscillations is not understood but is presumably related to the saturation mechanism). The difference compared to the no-shear case is striking as the LSD grows to be nearly in equipartition with the small-scale magnetic fields (\( \tilde{E}_B \approx \tilde{E}_b \)) around \( t/\tau_c \approx 400 \). In the no shear case, the random, large-scale fields remain several orders of magnitude weaker than the small-scale fields (\( \tilde{E}_B \ll \tilde{E}_b \)) for all times. The difference is also clearly visible in the time evolution of magnetic energy spectra (bottom panel) where the energy in the lowest \( k \) modes of the sheared case steadily increases throughout the linear (\( t/\tau_c = 75 \)) and saturation (\( t/\tau_c = 200 \)) phases of the LSD and are nearly four orders of magnitude larger than that of the no shear case (dashed gray) at late times (\( t/\tau_c = 400 \)). Additionally, the lowest-\( k \) modes of the sheared case are individually more than an order of magnitude larger in energy than the peak of the magnetic spectra at smaller scales near \( k_f \) (this peak is simply that of the saturated SSD). These weakly stratified simulations are in good qualitative agreement with mean-field predictions in limit of perturbative stratification: the VD is easily suppressed while the LSD remains unstable.

**Strong Stable Stratification**—The next question to address is whether the LSD can operate in stable stratification that is non-perturbative and strong enough to affect the small-scale turbulence (\( Fr^{-1} > 1 \)). Figure 5 shows two revealing cases, one where the SSD is strongly suppressed but still active (\( Fr^{-1} \approx 4 \), orange) and another where the SSD has been shut down (\( Fr^{-1} \approx 7 \), green). The case where the SSD is shut down by strong stratification shows no LSD growth since both \( \tilde{E}_b \) and \( \tilde{E}_B \) decay, which confirms the expectation that without a source of magnetic fluctuations the MSC effect does not operate. The \( Fr^{-1} \approx 4 \) case however still shows a robust but slower LSD growth from \( t/\tau_c \approx 200 \) to \( t/\tau_c \approx 600 \) when equipartition is reached \( \tilde{E}_B \approx \tilde{E}_b \). The low \( k \) modes of the magnetic spectra for the \( Fr^{-1} \approx 4 \) case (orange, bottom panel) are highly energized. The slower growth rate of the MSC effect is not surprising because the level of magnetic fluctuations is slightly lower due to the strong stratification, which can also been seen by the much lower growth rate of the SSD of the \( Fr^{-1} \approx 4 \) case (dashed orange, top panel) compared to the e.g. \( Fr^{-1} = 0.2 \) case (dashed blue, top panel). Despite this, the final saturation of the LSD is similar in both cases.

Suppression of the dynamo with increasing stratification is quantified in Figure 6 by showing the equiparition level of the large scale (blue, \( E_B/E_u \)) and small-scale (orange, \( E_b/E_u \)) magnetic energy at saturation of the LSD versus the stratification parameter \( Rm/Fr \). Figure 6 is generated by a parameter scan of \( Fr \) at fixed \( Rm \approx 120 \) and \( Sh \approx 0.9 \). Solid lines in the top panel are the energy in the mean field, \( \tilde{E}_B \), while dotted lines are the energy in the small-scale magnetic fields, \( \tilde{E}_b \). The reference simulation with no shear (\( Sh = 0 \)) is shown in gray. All spectra are computed from a snapshot at the last time point. Green and orange shaded regions in the bottom panel represent our definition of the large and small scales. The spectral resolution of the simulations is \( N_x \times N_y \times N_z = 192^2 \times 768 \) modes.

![Figure 5. Exploring the effect of stable stratification on the LSD](image-url)
The scaling of an incoherent dynamo with volume can be determined as follows. For the simplest case, consider zero mean \( \langle \alpha_{yy}(t) \rangle = 0 \) fluctuations with a variance \( \langle \alpha_{yy}(t) \alpha_{yy}(t') \rangle = D_{yy} \delta(t - t') \), which corresponds to a fluctuating EMF of the form \( E_y = \alpha_{yy}(t) B_y \) (Mitra & Brandenburg 2012; Squire & Bhattacharjee 2015c).

One can show that the maximum growth rate scales as \( \gamma_\alpha \propto D_{yy}^{1/2} \) (Visniac & Brandenburg 1997). Because increasing the volume of the domain by a factor of \( N \) decreases the variance \( D_{yy} \) of the \( \alpha_{yy} \) fluctuations by a factor of \( N \) (assuming each of the \( N \) sub-volumes are statistically independent), the stochastic dynamo growth rate must scale with the inverse square root of the domain volume \( \gamma_\alpha \propto V^{-1/2} \) (Squire & Bhattacharjee 2015c).

Therefore if our simulations are dominated by an incoherent effect, we should expect a significant decrease in the strength of the LSD when the volume is increased.

We carry out this test by progressively quadrupling the volume by changing the box aspect ratio \( (L_x, L_y, L_z) / L = (1, 1, 4) \), \( (2, 2, 4) \), and \( (4, 4, 4) \) at fixed turbulent forcing scale and fiducial parameters \( Rm \approx 60 \), \( Sh \approx 1.0 \), \( Fr^{-1} \approx 0.2 \). These parameters are stable to the VD, but unstable to the SSD and LSD. We note that the growth rate of the LSD right after the SSD saturates is difficult to interpret because the pseudo-linear phase of the LSD is short in finite size simulations, as discussed in Section 3.3.

The results are shown in Figure 7. If the LSD was driven by an incoherent effect, we would expect the growth rate to decrease by a factor of 2 and 4 for the \( (2, 2, 4) \) and \( (4, 4, 4) \) boxes compared to the \( (1, 1, 4) \) box. This does not appear to be observed. All three runs have a similar quasi-exponential growth of \( E_B(t) \) for \( 50 \lesssim t / \tau_c \lesssim 100 \) followed by non-linear growth for \( 100 \lesssim t / \tau_c \lesssim 200 \) at rates that differ by at most a factor of approximately two. The \( (2, 2, 4) \) run has a similar, if not faster, growth rate than the \( (1, 1, 4) \) run and reaches a higher \( E_B(t) \) before beginning its slow random oscillations. The \( (4, 4, 4) \) run on the other hand appears to have a slower non-linear growth rate for \( 100 \lesssim t / \tau_c \lesssim 200 \) than the \( (1, 1, 4) \) run but then has a similar growth rate for \( 200 \lesssim t / \tau_c \lesssim 300 \) before entering its fully saturated phase. All three runs at late times saturate at similar energies and with similar magnetic spectra. Overall, there appears to be no trend in the LSD growth rate with increasing volume. Since the nonlinear phase seems to be quasi-random, a more detailed investigation would require running a ensemble of simulations; however, that is not affordable at the present time.

Examining the phase variation of the mean field over time also offers an additional way to check for the pres-

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**Figure 6.** The equipartition level of the saturated magnetic field with varying stratification. Colored lines show the energy fraction of large scale (blue) and small-scale (orange) magnetic fields relative to the turbulent kinetic energy as a function of \( Rb_m = RmFr^{-2} \). The simulations with increasing stratification (decreasing \( Rb_m \)) are carried out at constant \( Rm \approx 120 \). Energies and standard deviations are calculated from the right. Long after saturation from the last 50\( \tau_c \) in each run. Hashed region denotes where the SSD is inactive due to strong stratification (for \( Rb_m < Rb^c_m = 3 \) when \( Prm = 1 \)).

3.4. Role of the Incoherent Dynamo and Aspect Ratio

**Incoherent Dynamo**—Finite size domains are susceptible to an additional LSD mechanism known as the incoherent dynamo and it is important to confirm that our observed LSD is in fact driven by the coherent MSC effect and not a finite-domain-size effect. The primary incoherent dynamo in a shear flow and non-helical turbulence is the stochastic-\( \alpha \) effect, in which zero-mean fluctuations of the \( \alpha \) transport coefficients can drive growth of the variance of the mean-field (\( B^2 \)) despite a zero ensemble average mean-field (\( B \)) = 0 (Visniac & Brandenburg 1997; Brandenburg & Subramanian 2005; Heinemann et al. 2011; Mitra & Brandenburg 2012). These statistical properties make the stochastic-\( \alpha \) effect dependent on the domain size unlike coherent dynamo mechanisms, which presents a way to test its relative contribution (Squire & Bhattacharjee 2015c).
We conclude that the stochastic-\( \alpha \) effect plays a sub-dominant role in our simulations.

**Aspect Ratio and Convergence**—While changing the aspect ratio, we can also check if our simulations are converged in the \( z \)-dimension (Yousef et al. 2008). A comparison of the magnetic spectra in the the top panel of Figure 7 of the (1, 1, 4) and (1, 1, 8) aspect ratios shows that the MSC effect is fully captured in our (1, 1, 4) runs since the dominant mode is clearly the \( kL/2\pi = 1/4 \) mode in the larger (1, 1, 8) simulation (black). Additionally, the time evolution of \( E_B(t) \) of both runs is similar until they enter the saturation regime \( t/\tau_c \gtrsim 150 \) and slowly oscillate with similar amplitudes. We conclude that our (1, 1, 4) simulations are sufficiently large for capturing the LSD.

### 3.5. Quenching of the MSC effect

An outstanding problem in dynamo theory is understanding the amplitude and timescale of non-linear saturation of various LSD mechanisms in the limit of large \( Rm \), as relevant to the astrophysical regime. LSDs based on the \( \alpha \)-effect face the well-known issue of catastrophic quenching, in which the amplitude and/or timescale of saturation scales strongly with the microscopic resistivity, suggesting an extremely weak LSD in the \( Rm \approx 1 \) regime (Brandenburg & Subramanian 2005; Rincon 2019). Although possibilities such as helicity fluxes through boundaries (Blackman & Field 2000; Vishniac & Cho 2001; Kleerolin et al. 2000; Brandenburg et al. 2002) and alternative scalings of small-scale helicity dissipation (Brandenburg et al. 2002; Brandenburg & Subramanian 2005; Blackman 2016) may resolve the problem, simulations so far have given mixed results (Brandenburg & Subramanian 2005; Rincon 2019). Because the MSC effect is nonhelical, the usual helicity constraints that cause quenching do not apply (see Appendix B). This means there is no a-priori reason that it should be quenched, making it a promising mechanism that may operate at astrophysically large \( Rm \).

To test the quenching behavior of the MSC effect, we carry out a \( Rm \) scan at \( Prm = 1 \) to computationally accessible values, with all other parameters fixed. The fiducial parameters are \( Sh \approx 1.0 \) and \( Fr^{-1} \approx 0.2 \) as before. \( Rm \) is increased by a factor of four from \( Rm \approx 50 \) up to \( Rm \approx 200 \) by decreasing \( \eta \). A sign of catastrophic quenching in this test would be to observe the growth rate or saturation amplitude progressively decrease as a near power law with increasing \( Rm \), as observed in helical dynamo simulations (Bhat et al. 2016; Rincon 2021). Hence, the factor of four difference between the \( Rm \approx 50 \) and \( Rm \approx 200 \) simulations should have an easily discernible effect. The result is shown in Figure 8.
where we find that the LSD has no systematic $Rm$ dependence. Increasing $Rm$ only increases the SSD growth rate, as expected; this is seen through the progressively earlier saturation of $\tilde{E}_B(t)$ (dotted lines). This causes the LSD ($\tilde{E}_B(t)$, solid lines) to begin growth earlier, but they subsequently grow at a similar rate (albeit with the random oscillations discussed above) and each run saturates at a similar level. Additionally, the magnetic spectra in the inset panel are independent of $Rm$ at large scales (low $k$).

Thus our test finds no signs of quenching with increasing $Rm$ up to the largest computationally accessible value of $Rm \approx 200$. These numerical results, in combination with the above theoretical arguments, strongly suggest that the MSC effect is a quenching-free dynamo mechanism that may operate in the astrophysically large $Rm$ regime applicable to RZs.

4. APPLICATION TO RADIATIVE ZONES

Our main proposal is that the MSC effect may be an important non-linear dynamo mechanism that closes the global dynamo loop in differentially rotating RZs. Axisymmetric toroidal fields may grow from the shearing (from radial DR) of an axisymmetric poloidal if there is a source of non-helical magnetic fluctuations to drive the MSC effect and regenerate the axisymmetric poloidal field, closing the dynamo loop. We discuss two avenues for producing magnetic fluctuations (1) Tayler-instabilities of the toroidal field and (2) small-scale dynamo operating in stably stratified turbulence driven by horizontal shear instabilities of latitudinal DR. We assume that the MSC effect is agnostic to the instability that sources the magnetic fluctuations and that the turbulence is predominantly non helical at small scales. We leave studies of specific instabilities for future work.

**Tayler Instabilities**—In the framework of the TS dynamo, a sufficiently strong toroidal field is unstable to kink-type modes known as Tayler instabilities (Tayler 1973), leading to magnetic turbulence. The Tayler-modes are non-axisymmetric and themselves cannot be directly sheared to regenerate the axisymmetric toroidal field, but instead are argued to contribute to a non-linear dynamo mechanism that regenerates the axisymmetric poloidal field (Zahn et al. 2007; Fuller et al. 2019). The MSC effect is a natural candidate mechanism because it is driven by magnetic fluctuations and is robust to stable stratification as shown in this study. Suggestions of an alpha based mechanism by previous studies (Zahn et al. 2007; Fuller et al. 2019) are complicated by the known issue that small-scale magnetic fields generally suppress the alpha-effect and likely cause catastrophic quenching at the high $Rm$ regime relevant to RZs, as discussed earlier. The quenching-free nature of the MSC effect makes it a promising alternative.

**Horizontal Shear Instabilities**—Another pathway way to generate magnetic fluctuations in a RZ is through the SSD operating in stably stratified turbulence driven by hydrodynamic instabilities. A likely possibility is horizontal shear instability of latitudinal DR. Vertical shear instabilities of radial DR, while generally stronger than latitudinal DR (Zahn 1992), are most likely stabilized by the strong stratification in RZs (Garaud 2021). We propose that magnetic fluctuations from the SSD combined with radial DR may generate mean toroidal and poloidal fields through the MSC effect. Extrapolating from the results of Section 3 suggests the toroidal field would reach near-equiparitition with the turbulence sourced by instability of the latitudinal DR. Note that our local shearing box setup cannot capture such instabilities directly, because we do not include a horizontal shear, but heuristically captures the resulting small-scale turbulence through the external forcing term. Here, we use dimensionless numbers estimated based on helioseismology of the solar tachocline (Hughes et al. 2007), the upper portion of the solar RZ, to examine the feasibility of our proposal.

The tachocline is approximately a thin spherical shell with radius $R = 0.7R_\odot$, thickness $\Delta R \approx 10^{-2}R$, and
differential rotation profile $\Omega(r, \theta)$. For the latitudinal DR, the differential angular velocity between the equator and the poles $(\Delta\Omega)_{\theta}$ is approximately $O(10^{-1})$ of the solar rotational frequency, i.e. $(\Delta\Omega)_{\theta} \approx 0.1\Omega_{\odot}$. For the radial DR, the differential angular velocity between the top and bottom of the tachocline at the equator is of similar strength $(\Delta\Omega)_{r} \approx 0.1\Omega_{\odot}$. Turbulence from the horizontal shear instabilities has an upper bound on the turbulent velocity $u_{\text{rms}} \sim (\Delta\Omega)_{\theta}R$ and an effective forcing number likely comparable to $k_f \sim 2\pi/R$ (Cope et al. 2020; Garaud 2020). The radial DR provides mean shear that is stable (Garaud 2021) with a shear frequency that we estimate as $S = r(\partial\Omega/\partial r) \approx R(\Delta\Omega)_{r}/\Delta R$. The associated dimensionless numbers are:

$$Re = 8 \cdot 10^{13} \left( \frac{R}{5 \cdot 10^8 m} \right)^2 \left( \frac{R \Omega_{\theta}}{3 \cdot 10^{-7} s^{-1}} \right) \left( \frac{\nu}{10^{-3} m^2 s^{-1}} \right)^{-1}$$

$$Fr = 3 \cdot 10^{-4} \left( \frac{\Delta\Omega_{\theta}}{3 \cdot 10^{-7} s^{-1}} \right) \left( \frac{N}{10^{-3} s^{-1}} \right)^{-1}$$

$$Rb_{m} = 7 \cdot 10^{4} \left( \frac{Pm}{10^{-2}} \right) \left( \frac{Re}{8 \cdot 10^{13}} \right) \left( \frac{Fr}{3 \cdot 10^{-4}} \right)^{2}$$

$$Sh = 10^2 \left( \frac{\Delta\Omega_{r}/\Delta\Omega_{\theta}}{1} \right) \left( \frac{R/\Delta R}{10^2} \right)$$

Comparing the magnetic buoyancy Reynolds number $Rb_{m} = O(10^4)$ to the critical value $Rb_{c} = O(10)$ suggests that the SSD is unstable (Skoutnev et al. 2021). The combination of a large shear number $Sh \gg 1$ and a SSD providing magnetic fluctuations satisfies two of the criteria for operation of the MSC effect. Therefore the MSC effect may generate near-equipartition magnetic fields with $B_0 = O(1)T$, where we have use the equipartition estimate $B_0^2/2\mu_0 = \rho u_{\text{rms}}^2/2$ with $\rho \approx 200 kg/m^3$. We expect these scalings to be reasonable in the interior of RZs of stars where latitudinal DR provides the dominant source of turbulence and the MSC effect is operating in isolation.

**Effect of Rotation and Low Prandtl Numbers**—We briefly describe the possible modifications to our results by additional effects in RZs not explored in our study, that of rotation and low Prandtl numbers. These are likely subdominant to effects of strong radial shear and stable stratification in RZs. Rotation modifies the MSC effect directly through an orientation-dependent contribution to $\eta_{yz}$ and will have a significant effect if $S/\Omega \lesssim 1$ (Squire & Bhattacharjee 2015c). While this is likely not important in the tachocline where $S/\Omega \approx ((\Delta\Omega)_{r}/\Omega_{\odot}) (R/\Delta R) \approx 10$, it may be important in other stars with weaker radial DR or faster rotation. Another important source of uncertainty are the effects of low Prandtl numbers typical of stellar interiors ($Pr = O(10^{-6})$ in the tachocline (Garaud 2021)). A lower $Pm$ generally makes a SSD more difficult to sustain (Iakov et al. 2007), which is captured by the dependence of $Rb_{m}(Pm)$ on $Pm$ (Skoutnev et al. 2021). However, for the LSD, calculations with the SOCA have found that the MSC effect is not sensitive to $Pm$ (Squire & Bhattacharjee 2015b). Unfortunately confirming these results for the LSD with DNS at low $Pm$ is currently impractical due to an even larger requirement for $Re$. On the other hand, we expect a low $Pr = \nu/\kappa$ to make the SSD and LSD more unstable. A higher thermal diffusivity, $\kappa$, decreases the effect of stratification and leads to more isotropic turbulence, which generally enables more efficient dynamo action.

5. CONCLUSION

We propose and test a large-scale dynamo mechanism that can operate in the differentially rotating and stably stratified plasma of a stellar radiative zone. The dynamo loop we propose is fundamentally closed by the magnetic shear-current effect (Bogachevskii & Kleoerin 2004; Rädler & Stepanov 2006; Squire & Bhattacharjee 2016; Käpylä et al. 2020; Zhou & Blackman 2021), which generates toroidal field from the shearing of a poloidal field and regenerates poloidal field from the toroidal field through statistical correlations of local, non-helical MHD turbulence. Our analysis is based on idealised theory and simulations modeling a local section of a RZ, providing good evidence to support a broader picture of dynamos in RZs. The key pieces of evidence are:

1. Perturbative mean-field dynamo theory, when extended to include stable stratification, predicts the MSC instability remains robust with a decreased growth rate compared to the unstratified case. The hydrodynamic vorticity dynamo, however, is rapidly stabilized by stratification.

2. Shearing box simulations show that a mean shear combined with an unstable small-scale dynamo in stably stratified turbulence is unstable to a LSD, qualitatively agreeing with mean-field dynamo theory. The simulations also confirm that the hydrodynamic vorticity dynamo is stabilized by the addition of weak stable stratification.

3. Simulations show that the energy in the mean magnetic field at saturation is comparable to the
energy in the small-scale magnetic fields, which are themselves in near equipartition with the kinetic energy of the turbulence.

4. A numerical scan of the magnetic Reynolds number demonstrates that the LSD does not suffer catastrophic quenching. In particular, the saturation time and amplitude are found to be independent of $Rm$. This is expected because there are no obvious constraints arising from magnetic helicity conservation on a non-helical dynamo mechanism such as the MSC effect.

Put together, these idealized results suggest that the MSC effect in a RZ requires (1) a source of non-helical magnetic fluctuations and (2) sufficient radial differential rotation (velocity shear). The resulting mean-field should saturate near equipartition with the magnetic fluctuations in a manner that is free from quenching (a significant issue for helicity-based alpha dynamo mechanisms).

Extrapolating our results from a local shearing box model to a realistic RZ, we propose two pathways to provide non-helical magnetic fluctuations for operation of a large-scale dynamo through the MSC effect in a region of radial DR. The first is Tayler instabilities (Tayler 1973; Markey & Tayler 1973) of the toroidal field, which directly result in the magnetic fluctuations that we propose may drive the MSC effect and regenerate axisymmetric poloidal field, thereby closing the Tayler-Spruit dynamo (Spruit 2002; Fuller et al. 2019). The second is the small-scale dynamo operating in stably stratified turbulence, which may be driven by instabilities of latitudinal DR (Zahn 1974, 1992; Prat & Lignières 2013, 2014; Cope et al. 2020; Garaud 2020). The reality may be a mixture of the two processes, and perhaps others. Although significant uncertainties remain (effects of spherical geometry, the helicity fraction of instability-driven magnetic turbulence, low Prandtl numbers etc.), the near-equipartition saturation, robustness to stable stratification, and quenching-free properties of the MSC effect make it worthy of further consideration in more complex global dynamo models in stellar RZs.

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**APPENDIX**

**A. ELECTROMOTIVE FORCE CALCULATION FOR THE MHD BOUSSINESQ EQUATIONS**

The general setup for calculating mean field transport coefficients using the second order correlation approximation (SOCA) supposes a bath of homogeneous and isotropic velocity and magnetic field fluctuations in the presence of anisotropic perturbations such as shear flows $\mathbf{U}_0$ of form $U_i = U_{ij}x_j$ (with no vertical component $U_x = 0$ in the Boussinesq case), rotation ($\Omega$), and stable stratification (in the $\hat{x}$ direction with Brunt-Vaisala frequency $N$). While rotation is not included in the main paper, it is simple to include it here to demonstrate the mean-field MHD Boussinesq framework for the full problem. We assume the mean velocity field does not evolve and write the total fields as $\mathbf{U}_T = \mathbf{U}_0 + \mathbf{u}$, $\mathbf{B}_T = \mathbf{B} + \mathbf{b}$, and $\Theta_T = \Theta$ ($\Theta = 0$ because there cannot be any mean vertical flows that could drive $\Theta$). The mean field induction equation then is given by:

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{U}_0 \times \mathbf{B}) + \nabla \times \mathcal{E} + \eta \Delta \mathbf{B},$$

where the EMF is:

$$\mathcal{E} = (\mathbf{u} \times \mathbf{b}) = \mathcal{E}(\mathbf{B})$$

The fluctuation equations for $\mathbf{u}$, $\mathbf{b}$, and $\Theta$ are obtained by subtracting the mean field equations from those of the total fields:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U}_0 + \mathbf{U}_0 \cdot \nabla \mathbf{u} + (\mathbf{u} \cdot \nabla \mathbf{u})' + 2\Omega \times \mathbf{u} = -\nabla p + \theta \hat{x}$$

$$+ \mathbf{B} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{B} + (\mathbf{b} \cdot \nabla \mathbf{b})' + \nu \nabla^2 \mathbf{u} + \sigma_f,$$
where we have used the notation \( (A)' = A - \langle A \rangle \). Following Radler et al. (Rädler & Stepanov 2006), the EMF \( \mathcal{E} \) can be Taylor expanded and linearly related to the mean field \( B_0 \) and its derivative \( B_{ij} \) (i.e. \( \mathcal{E}_i = a_{i,j}B_j + b_{ij,k}B_{j,k} \)) assuming sufficient scale separation. This then provides a closure of the mean field induction equation. Taking into account all symmetry properties, the most general form of transport coefficients linearly relating \( \mathcal{E} \) and \( B \) is:

\[
\mathcal{E} = \alpha^{(0)}_H B - \alpha^{(D)}_HD_{ij}B_j - \gamma^{(W)}_H \Omega \times B - \gamma^{(1)}_W B \times \Omega - \alpha^{(1)}_1 (\hat{g} \cdot \Omega) B - \alpha^{(D)}_2 [(\hat{g} \cdot B)\Omega + (B \cdot \Omega)\hat{g}] - \alpha^{(W)}_1 (\hat{g} \cdot W)B
\]

\[
- \alpha^{(W)}_2 [(\hat{g} \cdot B)W + (B \cdot W)\hat{g}] - \alpha^{(D)}(\epsilon_{ilm}D_{ij}g_m + \epsilon_{jlm}D_{ij}g_m)B_j - (\gamma^{(0)}_2 + \gamma^{(1)}_W)\hat{g} \times \Omega + \gamma^{(W)}_W \hat{g} \times W + \gamma^{(D)}D_{i,j} \hat{g}_j \times B
\]

\[
- \beta^{(0)}J - \beta^{(D)}D_{i,j}J_j - (\delta^{(W)}W + \delta^{(1)}W)\Omega \times J + (\kappa^{(W)}W + \kappa^{(1)}\Omega) \Omega \times \nabla B)_{ij} - 2\kappa^{(D)}\epsilon_{ijk}D_{kr} \nabla B)_{ij}
\]

(A7)

The calculation of \( \mathcal{E} \) at this point can be carried out with the help of the SOCA. The SOCA assumes that the higher order correlation terms (the primed terms such as \( (u \cdot \nabla u) ' \)) are small compared to terms involving the mean fields (e.g. \( u \cdot \nabla u_0 \)) and can be neglected. This will lead to linear time evolution equations for \( u, b, \) and \( \theta \) that can be solved perturbatively. The perturbation expansion is done on both \( u \) and \( b \) around their background, homogeneous fluctuation values \( u_0 \) and \( b_0 \) and then substituted into Equation A2 for the EMF. While statistics of the background fluctuations will be assumed, they physically arise from the forcing term \( \sigma_f \) of the momentum equation (sustaining \( u_0 \)) and a resulting small-scale dynamo (sustaining \( b_0 \) in equiparition). The forcing itself models some hydrodynamic instability, such as horizontal shear instabilities discussed in the main article. In general, the background fluctuations have both helical and non-helical components, which we include in the calculation. To allow the perturbative expansion, all anisotropic parameters such as \( N^2, S, \Omega \) are considered to be small. With the notation \( u = u_0 + u^{(0)} + u^{(1)} \) for the expansion of \( u \) (as well as \( b \) and \( \theta \)), \( \mathcal{E} \) to second order is given by:

\[
\mathcal{E} = \langle u_0 \times b^{(0)} \rangle + \langle u^{(0)} \times b_0 \rangle + \langle u^{(0)} \times b^{(0)} \rangle + \langle u_0 \times b^{(1)} \rangle + \langle u^{(1)} \times b_0 \rangle
\]

(A8)

where \( \langle u_0 \times b_0 \rangle = 0 \) is assumed.

The calculation of \( \mathcal{E} \) is carried out in Fourier space and explained in detail in Radler et. al. Rädler & Stepanov (2006). We only give a brief description of the approach in order to point out how we handle the new addition of the buoyancy equation and the buoyancy term in the momentum equation in the SOCA formalism. To begin, we write out the evolution equations in real space for each order by applying the expansion to the fluctuation equations and then using the SOCA where applicable. The background, zeroth order, and first order equations are shown below.

**Background Turbulence**—The homogeneous, background fluctuations satisfy:

\[
\partial_t u_0 + (u_0 \cdot \nabla u_0)' = -\nabla \rho_0 + \theta_0 \hat{x} + (b_0 \cdot \nabla b_0)' + \nu \Delta u_0 + \sigma_f
\]

(A9)

\[
\partial_t b_0 = \nabla \times (u_0 \times b_0)' + \eta \Delta b_0
\]

(A10)

\[
\partial_t \theta_0 + u_0 \cdot \nabla \theta_0 = \kappa \nabla^2 \theta_0
\]

(A11)

While the homogeneous velocity and magnetic fluctuations \( u_0 \) and \( b_0 \) are assumed to be in a steady state driven by the forcing \( \sigma_f \) and a SSD, the buoyancy equation for the buoyancy fluctuations does not have any source term. Therefore, any initial buoyancy variable fluctuations \( \theta_0(t = 0) \) will be passively advected and thermally diffuse to zero after a transient phase. These background equations themselves are not used in the calculation for \( \mathcal{E} \), but a model for the homogeneous statistics of the background turbulence will be used.
Zeroth Order — The SOCA approximation assumes terms such as \((u^{(0)} \cdot \nabla u^{(0)})'\) are much smaller than \(u_0 \cdot \nabla U_0\) so then the equations for the zeroth order fluctuations become:

\[
(\partial_t - \nu \Delta) u^{(0)} = - (U_0 \cdot \nabla u_0 + u_0 \cdot \nabla U_0) - \nabla p^{(0)} - 2\Omega \times u_0 + \theta^{(0)} \hat{x} + B \cdot \nabla b_0 + b_0 \cdot \nabla B \tag{A12}
\]

\[
(\partial_t - \eta \Delta) b^{(0)} = \nabla \times (U_0 \times b_0 + u_0 \times B) \tag{A13}
\]

\[
(\partial_t - \kappa \Delta) \theta^{(0)} = - N^2 u_{0,x} \tag{A14}
\]

where we have dropped the \(U_0 \cdot \nabla \theta_0\) term in the buoyancy Equation A14 because the \(\theta_0\) fluctuations are zero after a transient phase as discussed above.

First Order — Similarly, the equations for the first order fluctuations are:

\[
(\partial_t - \nu \Delta) u^{(1)} = - (U_0 \cdot \nabla u^{(0)} + u^{(0)} \cdot \nabla U_0) - \nabla p^{(1)} - 2\Omega \times u^{(0)} + \theta^{(1)} \hat{x} + B \cdot \nabla b^{(0)} + b^{(0)} \cdot \nabla B \tag{A15}
\]

\[
(\partial_t - \eta \Delta) b^{(1)} = \nabla \times (U_0 \times b^{(0)} + u^{(0)} \times B) \tag{A16}
\]

\[
(\partial_t - \kappa \Delta) \theta^{(1)} = - U_0 \cdot \nabla \theta^{(0)} - N^2 u_x^{(1)} \tag{A17}
\]

Calculation of \(\mathcal{E}\) — The calculation of \(\mathcal{E}\) proceeds exactly as in Squire et. al. (Squire & Bhattacharjee 2015b) except with the addition of the buoyancy terms. The zeroth and first order fluctuation equations A12 through A17 are transformed to Fourier space and substituted into the Fourier space version of EMF equation A8 (see (Squire & Bhattacharjee 2015b) and (Rädler & Stepanov 2006) for extensive details). Because \(\theta_0\) fluctuations diffuse away and are not relevant for the background turbulence, it is unnecessary to make an additional model of the turbulent statistics between \(\theta_0\) and \(u_0\) or \(b_0\). The lack of \(\theta_0\) is what allows \(\theta^{(0)}\) and \(\theta^{(1)}\) to be solved for in terms of \(u_0\) and \(u^{(0)}\) using the buoyancy equations A14 and A17 and then substituted directly into the momentum equations A12 and A15, respectively. The net result of weak stably stratified therefore is added terms proportional to \(N^2\) in the momentum equations A12 and A15.

We use the open source VEST package (Squire et al. 2014) for Mathematica to carry out the calculation, similar to Squire et. al. (Squire & Bhattacharjee 2015b), including both helical and non-helical portions of the background velocity and magnetic fluctuations. We report only the four transport coefficients that we find are modified by stable stratification, the isotropic turbulent resistivity and alpha coefficient. All other transport coefficients are identical to the result in Squire et. al. (Squire & Bhattacharjee 2015b). The modified coefficients in Fourier space are shown below:

\[
(\tilde{\beta}^{(0)})_u = \frac{u_{\text{rms}f}}{\eta} \left[ \frac{\tilde{k}^2}{3(\tilde{k}^4 + q^2 \tilde{\omega}^2)} + (N \tau_c)^2 \left[ \frac{2 \tilde{k}^2 (\tilde{k}^2 Pm^2 - q^2 \tilde{\omega}^2)}{10 (\tilde{k}^4 + q^2 \tilde{\omega}^2)} \right] \right] \tag{A18}
\]

\[
(\tilde{\beta}^{(0)})_b = (N \tau_c)^2 \left[ \frac{q^2 \tilde{k}^2 (\tilde{k}^4 Pm^2 - (\frac{Pm}{\tau_c} + 2 Pm) q^2 \tilde{\omega}^2)}{60 (\tilde{k}^4 Pm^2 + q^2 \tilde{\omega}^2)^2 (\tilde{k}^4 Pm^2 + q^2 \tilde{\omega}^2)} \right] \tag{A19}
\]

\[
(\tilde{\alpha}^{(0)}_H)_u = \frac{u_{\text{rms}f}}{\eta} \left[ \frac{2 \tilde{k}^2}{3(\tilde{k}^4 + q^2 \tilde{\omega}^2)} + (N \tau_c)^2 \left[ \frac{3 \tilde{k}^2 (\tilde{k}^2 Pm^2 - q^2 \tilde{\omega}^2)}{15 (\tilde{k}^4 + q^2 \tilde{\omega}^2) (\tilde{k}^4 Pm^2 + q^2 \tilde{\omega}^2)} \right] \right] \tag{A20}
\]

\[
(\tilde{\alpha}^{(0)}_H)_b = \frac{u_{\text{rms}f}}{\eta} \left[ \frac{2 \tilde{k}^2 Pm}{3(\tilde{k}^4 Pm^2 + q^2 \tilde{\omega}^2)} + (N \tau_c)^2 \left[ \frac{4 \tilde{k}^2 (\tilde{k}^4 Pm^3 - (\frac{Pm}{\tau_c} + 2 Pm) q^2 \tilde{\omega}^2)}{15 (\tilde{k}^4 Pm^2 + q^2 \tilde{\omega}^2)^2 (\tilde{k}^4 Pm^2 + q^2 \tilde{\omega}^2)} \right] \right] \tag{A21}
\]

where \(\tilde{k} = kl_f\), \(\tilde{\omega} = \omega \tau_c\), \(g = l_f^2 / \eta \tau_c\) is the ratio of the resistive to the correlation time (a measure conductivity), and \(\tau_c\) and \(l_f\) are the turbulence correlation time and length. We note that \((N \tau_c)^2\) is assumed small in the perturbative approach of the SOCA. The formal bound requires that the stratification term be smaller than the dominant terms in the momentum equation, which can be clearly seen by considering the zeroth order momentum and buoyancy equations.
Figure 9. The modification of the isotropic turbulent resistivity by stratification. Plots of the ratio of the stratification term to the unstratified term on a linear (left) and a log scale (right). The contribution to the stratification term from velocity fluctuations is in blue and from magnetic fluctuations is in orange.

and balancing the $\partial_t u_0^{(0)} \sim \theta^{(0)}$ and $\partial_t \theta^{(0)} \sim -N^2 u_{0x}$ terms to get $|u_z^{(1)}| \sim (N\tau_c)^2 |u_{0x}|$. Therefore, $(N\tau_c)^2$ needs to be on the order of the perturbation expansion parameter.

The physical transport coefficients are obtained by an inverse Fourier transform given by $\beta^{(0)} = (\beta^{(0)})_u + (\beta^{(0)})_b = 4\pi \int d\tilde{k} d\tilde{\omega} \tilde{k}^2 [(\tilde{\beta}^{(0)})_u W_u(\tilde{k}, \tilde{\omega}) + (\tilde{\beta}^{(0)})_b W_b(\tilde{k}, \tilde{\omega})]$, where $W_u$ and $W_b$ are the statistics of the non-helical background velocity and magnetic fluctuations, respectively. The non-helical velocity and magnetic fields are assumed to follow Gaussian statistics: $W_u(\tilde{k}, \tilde{\omega}) = W_b(\tilde{k}, \tilde{\omega}) = \frac{2\tilde{k}^2}{8(2\pi)^{5/2}} \exp(-\tilde{k}^2/2)/(-\tilde{\omega}^2)$ (Rädlér & Stepanov 2006). The same applies for $\alpha^{(0)}_H$ but using the model statistics for the helical fraction of the turbulence. The isotropic turbulent resistivity here is the same as in the main article ($\eta = \beta^{(0)}$) but with a different notation. We are interested in quantifying the relative size of the stratification term to the unstratified value of $\beta^{(0)}$ since this term affects the MSC effect. To this end, we split the unstratified and stratified contributions to $\beta^{(0)}$ and then examine their ratio. For concreteness, we set unity Prandtl numbers ($Pm = Pr = 1$) and define the separate contributions as follows:

$$
(\beta^{(0)})_u = (\beta^{(0)})_{u(N^2=0)}(q) + (N\tau_c)^2(\beta^{(0)})_{u(N^2)}(q),
(\beta^{(0)})_b = (N\tau_c)^2(\beta^{(0)})_{b(N^2)}(q)
$$

(A22)

$$
(\beta^{(0)})_{u(N^2=0)}(q) = 4\pi \int d\tilde{k} d\tilde{\omega} \tilde{k}^2 \frac{\tilde{k}^2}{3(\tilde{k}^4 + q^2\tilde{\omega}^2)} \tilde{W}_u(\tilde{k}, \tilde{\omega})
$$

(A23)

$$
(\beta^{(0)})_{u(N^2)}(q) = 4\pi \int d\tilde{k} d\tilde{\omega} \tilde{k}^2 \frac{3q^2\tilde{k}^2(\tilde{k}^4 - q^2\tilde{\omega}^2)}{10(k^4 + q^2\tilde{\omega}^2)^3} \tilde{W}_u(\tilde{k}, \tilde{\omega})
$$

(A24)

$$
(\beta^{(0)})_{b(N^2)}(q) = 4\pi \int d\tilde{k} d\tilde{\omega} \tilde{k}^2 \frac{q^2\tilde{k}^2(\tilde{k}^4 - 3q^2\tilde{\omega}^2)}{60(k^4 + q^2\tilde{\omega}^2)^3} \tilde{W}_b(\tilde{k}, \tilde{\omega})
$$

(A25)

$$
(\beta^{(0)})_{b(N^2=0)}(q) = \frac{\tilde{\beta}^{(0)}_{b(N^2=0)}}{\tilde{\beta}^{(0)}_{b(N^2)}}
$$

(A26)

Figure 9 shows the relative size of $(\beta^{(0)})_{u(N^2)}(q)$ and $(\beta^{(0)})_{b(N^2)}(q)$ with respect to the unstratified value $(\beta^{(0)})_{u(N^2=0)}(q)$ by plotting their ratios versus $q$, which can be thought of as a measure of the conductivity $q = Rm/\text{St}$. Both kinetic and magnetic contributions from stratification are positive, which is expected since stratification should intuitively reduce the dynamo efficiency. We see that the kinetic contribution is much larger than the magnetic contribution $(\beta^{(0)})_{b(N^2)}(q) \gg (\beta^{(0)})_{u(N^2)}(q)$. As $q \to \infty$, the magnetic contribution remains small $(\beta^{(0)})_{b(N^2)}(q)/(\beta^{(0)})_{u(N^2=0)} \ll 1$. The kinetic contribution may become important as $q \to \infty$ since $(\beta^{(0)})_{b(N^2)}(q)/(\beta^{(0)})_{u(N^2=0)} \gg 1$ may become important. However, the formally valid limit $(N\tau_c)^2 \ll 1$ means that the contributions from stratification are still small compared
to the unstratified isotropic turbulent resistivity. While \( q \gg 1 \) seems like the relevant limit for large \( \text{Rm} \), recall that SOCA is only formally valid for \( \text{Rm} \ll 1 \) at \( q \ll 1 \) or \( \text{St} \ll 1 \) at \( q \gg 1 \). In reality, \( q = O(1) \) probably provides the most reasonable estimate for nonlinear turbulence (see discussions in (Brandenburg & Subramanian 2005; Rädler & Stepanov 2006; Squire & Bhattacharjee 2015b)).

B. HELICITY GENERATION AND CATASTROPHIC QUENCHING

In this section, we first apply the two scale approach to the total magnetic helicity, then describe how catastrophic quenching results from helicity constraints, and finally show that the MSC effect does not produce helicity. The resulting takeaway is that the MSC effect is immune to catastrophic quenching and therefore likely remains a robust mechanism in the astrophysical limit of large \( \text{Rm} \).

The total magnetic helicity is given by \( \mathcal{H}_T = \int \mathbf{A}_T \cdot \mathbf{B}_T dV \), where \( \mathbf{A}_T \) is the vector potential \( (\mathbf{B}_T = \nabla \times \mathbf{A}_T) \). \( \mathcal{H} \) can be split into the helicity of the large scale and small-scale magnetic fields \( \mathcal{H} = \int \mathbf{A} \cdot \mathbf{B} dV \) and \( h = \int (\mathbf{a} \cdot \mathbf{b}) dV \), where brackets denote the mean field average. Using \( \partial_t \mathbf{A}_T = -\mathbf{E}_T + \nabla \phi \) from the induction equation (where \( \mathbf{E}_T = \mathbf{U}_T \times \mathbf{B}_T + \eta \nabla \times \mathbf{B}_T \) is the electric field and \( \phi \) is an arbitrary scalar field), it is straightforward to show that:

\[
\partial_t \mathcal{H} = +2 \int \mathbf{E} \cdot \mathbf{B} dV - 2\eta \int (\nabla \times \mathbf{B}) \cdot \mathbf{B} dV \\
\partial_t h = -2 \int \mathbf{E} \cdot \mathbf{B} dV - 2\eta \int ((\nabla \times \mathbf{b}) \cdot \mathbf{b}) dV
\]

(B27) (B28)

where we have assumed helicity fluxes through the volume boundaries are zero due to periodicity or perfectly conducting boundary conditions. The argument for quenching of helical dynamos stems from the observation that large scale and small scale helicities are produced through the EMF at the same rate, but with opposite signs (Rincon 2019) (i.e. the source term is \( \pm 2 \int \mathbf{E} \cdot \mathbf{B} dV \)). Consider a turbulent MHD system at large \( \text{Rm} \) with a growing, but still weak mean magnetic field. The SSD will be the first to saturate and so the small-scale helicity will remain roughly constant \( \partial_t h \approx 0 \), leaving a balance between between helicity injection and dissipation at small scales in Equation B28 (i.e. \( \int \mathbf{E} \cdot \mathbf{B} dV \approx -2\eta \int ((\nabla \times \mathbf{b}) \cdot \mathbf{b}) dV \)). As a result, the large-scale helicity in Equation B27 becomes constrained to grow at the dissipation rate of small-scale helicity, \( \partial_t \mathcal{H} \approx -2\eta \int ((\nabla \times \mathbf{b}) \cdot \mathbf{b}) dV \) (since dissipation of helicity by the small-scale fields is much faster than by the large-scale fields). The dependence of \( \partial_t \mathcal{H} \) on the microscopic resistivity \( \eta \) leads to a resistivity limited growth (catastrophic quenching) of the large-scale dynamo in the astrophysical limit of \( \eta \to 0 \) (i.e. \( \text{Rm} \to \infty \)). It may be possible to avoid catastrophic quenching if small-scale helicity can be transported out of the domain boundaries at the production rate of large-scale helicity by dropping the ideal boundary condition assumption. However, simulations have so far found mixed results (Rincon 2019).

LSDs driven by helical turbulence produce helical large-scale fields because the form of the EMF gives a non-zero source term. For example, the simplest alpha dynamo \( \mathcal{E}_i = \alpha_H^0 \mathbf{B}_i \) has \( \int \mathbf{E} \cdot \mathbf{B} dV = \alpha_H^0 \int |\mathbf{B}|^2 dV \neq 0 \) and \( \partial_t \mathcal{H} \neq 0 \). The same can be shown for all other alpha-effect based dynamos (e.g. \( \alpha$$\Omega$$ \) dynamos) whose transport coefficients (such as \( \alpha_H^0 \), \( \alpha_H^{(D)} \) etc.) are based on the helical part of the background turbulence.

On the other hand, LSDs driven by non-helical turbulence generate non-helical large scale fields because the source term is zero. The MSC effect is an example of the general class of shear-current effects that have contributions from the following terms in the EMF:

\[
\mathcal{E}_i^{\text{SC}} = -\beta^{(D)} D_{ij} \mathbf{J}_j - \delta^{(W)} \epsilon_{ijk} W_k \mathbf{J}_j - \kappa^{(W)} W_j (\nabla \mathbf{B})_{ji}^{(s)} - 2\kappa^{(D)} \epsilon_{ijk} D_{kr} (\nabla \mathbf{B})_{jr}^{(s)}
\]

(B29)

The crucial coefficient in the MSC effect, \( \eta_{\|} = -\delta^{(W)} + \frac{1}{2} (\kappa^{(W)} - \beta^{(D)} + \kappa^{(D)}) \), has contributions from each of these terms. It is straightforward to show that \( \int \mathcal{E}^{\text{SC}} \cdot \mathbf{B} dV = 0 \), with each term independently having a null contribution. Consider the term proportional to \( \kappa^{(W)} \):

\[
\int \kappa^{(W)} W_j (\nabla \mathbf{B})_{ji}^{(s)} B_i dV = \frac{1}{2} \kappa^{(W)} W_j \int \left( \frac{\partial B_j}{\partial x_i} + \frac{\partial B_i}{\partial x_j} \right) B_i dV = \frac{1}{2} \kappa^{(W)} W_j \int \left( \frac{\partial (B_j B_i)}{\partial x_i} + \frac{\partial (B_i B_j)}{\partial x_j} \right) dV = 0
\]

(B30)

where the divergence free condition has been used, \( \frac{\partial B_i}{\partial x_j} = 0 \). The calculation for every other term is similar. As a result, the MSC effect, driven by non-helical magnetic fluctuations, generates non-helical large-scale magnetic fields
and is not affected by the helicity constraints usually used to argue for the inevitability of catastrophic quenching at high $R_m$.

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