Null controllability for the parabolic equation with a complex principal part∗

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Abstract
This paper is addressed to a study of the null controllability for the semilinear parabolic equation with a complex principal part. For this purpose, we establish a key weighted identity for partial differential operators \((\alpha + i\beta)\partial_t + \sum_{j,k=1}^n \partial_k(a^{jk}\partial_j)\) (with real functions \(\alpha\) and \(\beta\)), by which we develop a universal approach, based on global Carleman estimate, to deduce not only the desired explicit observability estimate for the linearized complex Ginzburg-Landau equation, but also all the known controllability/observability results for the parabolic, hyperbolic, Schrödinger and plate equations that are derived via Carleman estimates.

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1 Introduction and main results
Given \(T > 0\) and a bounded domain \(\Omega\) of \(\mathbb{R}^n\) \((n \in \mathbb{N})\) with \(C^2\) boundary \(\Gamma\). Fix an open non-empty subset \(\omega\) of \(\Omega\) and denote by \(\chi_\omega\) the characteristic function of \(\omega\). Let \(\omega_0\) be another non-empty open subset of \(\Omega\) such that \(\overline{\omega_0} \subset \omega\). Put
\[
Q = (0, T) \times \Omega, \quad \Sigma = (0, T) \times \Gamma, \quad Q_0 = (0, T) \times \omega_0.
\]
In the sequel, we will use the notation \(y_j = y_{x_j}\), where \(x_j\) is the \(j\)-th coordinate of a generic point \(x = (x_1, \ldots, x_n)\) in \(\mathbb{R}^n\). In a similar manner, we use the notation \(z_j, v_j\), etc. for the partial derivatives of \(z\) and \(v\) with respect to \(x_j\). Throughout this paper, we will use \(C = C(T, \Omega, \omega)\) to denote generic positive constants which may vary from line to line (unless otherwise stated). For any \(c \in \mathbb{C}\), we denote its complex conjugate by \(\overline{c}\).

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Fix \( a^{jk}(\cdot) \in C^{1,2}(\overline{Q}; \mathbb{R}) \) satisfying
\[
a^{jk}(t, x) = a^{kj}(t, x), \quad (t, x) \in \overline{Q}, \quad j, k = 1, 2, \ldots, n, \tag{1.1}
\]
and for some constant \( s_0 > 0 \),
\[
\sum_{j,k} a^{jk} \xi_j \xi_k \geq s_0 |\xi|^2, \quad (t, x, \xi) \equiv (t, x, \xi_1, \ldots, \xi_n) \in \overline{Q} \times \mathbb{C}^n. \tag{1.2}
\]
Next, we fix a function \( f(\cdot) \in C^1(\mathbb{C}) \) satisfying \( f(0) = 0 \) and the following condition:
\[
\lim_{s \to \infty} \frac{|f(s)|}{|s| \ln^{1/2} |s|} = 0. \tag{1.3}
\]
Note that \( f(\cdot) \) in the above can have a superlinear growth. We are interested in the following semilinear parabolic equation with a complex principal part:
\[
\begin{cases}
(1 + ib)y_t - \sum_{j,k=1}^n (a^{jk} y_j)_k = \chi \omega u + f(y) & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } \Omega,
\end{cases} \tag{1.4}
\]
where \( i = \sqrt{-1}, \; b \in \mathbb{R} \). In (1.4), \( y = y(t, x) \) is the state and \( u = u(t, x) \) is the control.

One of our main objects in this paper is to study the null controllability of system (1.4), by which we mean that, for any given initial state \( y_0 \), find (if possible) a control \( u \) such that the weak solution of (1.4) satisfies \( y(T) = 0 \).

In order to derive the null controllability of (1.4), by means of the well-known duality argument (see [16, p.282, Lemma 2.4], for example), one needs to consider the following dual system of the linearized system of (1.4) (which can be regarded as a linearized complex Ginzburg-Landau equation):
\[
\begin{cases}
Gz = q(t, x)z & \text{in } Q, \\
z = 0 & \text{on } \Sigma, \\
z(T) = z_T & \text{in } \Omega,
\end{cases} \tag{1.5}
\]
where \( q(\cdot) \in L^\infty(0, T; L^n(\Omega)) \) is a potential and
\[
Gz \triangleq (1 + ib)z_t + \sum_{j,k=1}^n (a^{jk} z_j)_k. \tag{1.6}
\]

By means of global Carleman inequality, we shall establish the following explicit observability estimate for solutions of system (1.5).

**Theorem 1.1** Let \( a^{jk}(\cdot) \in C^{1,2}(\overline{Q}; \mathbb{R}) \) satisfy (1.1)–(1.2), \( q(\cdot) \in L^\infty(0, T; L^n(\Omega)) \) and \( b \in \mathbb{R} \). Then there is a constant \( C > 0 \) such that for all solutions of system (1.5), it holds
\[
|z(0)|_{L^2(\Omega)} \leq C(r)|z|_{L^2((0,T) \times \omega)}, \quad \forall \; z_T \in L^2(\Omega) \tag{1.7}
\]
where
\[
C(r) \triangleq C_0 e^{C_0 r^2}, \quad C_0 = C(1 + b^2), \quad r \triangleq |q|_{L^\infty(0,T;L^n(\Omega))}. \tag{1.8}
\]
Thanks to the dual argument and the fixed point technique, Theorem 1.1 implies the following controllability result for system (1.4).

**Theorem 1.2** Let \( a^{jk}(\cdot) \in C^{1,2}(\overline{Q}; \mathbb{R}) \) satisfy (1.1)–(1.2), \( f(\cdot) \in C^1(\mathbb{C}) \) satisfy \( f(0) = 0 \) and (1.3), and \( b \in \mathbb{R} \). Then for any given \( y_0 \in L^2(\Omega) \), there is a control \( u \in L^2((0,T) \times \omega) \) such that the weak solution \( y(\cdot) \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)) \) of system (1.4) satisfies \( y(T) = 0 \) in \( \Omega \).

The controllability problem for system (1.4) with \( b = 0 \) (i.e., linear and semilinear parabolic equations) has been studied by many authors and it is now well-understood. Among them, let us mention [4, 5, 13, 9] on what concerns null controllability, [7, 8, 9, 28] for approximate controllability, and especially [29] for recent survey in this respect. However, for the case \( b \neq 0 \), very little is known in the previous literature. To the best of our knowledge, [10] is the only paper addressing the global controllability for multidimensional system (1.4). We refer to [20] for a recent interesting result on local controllability of semilinear complex Ginzburg-Landau equation.

We remark that condition (1.3) is not sharp. Indeed, following [4], one can establish the null controllability of system (1.4) when the nonlinearity \( f(y) \) is replaced by a more general form of \( f(y, \nabla y) \) under the assumptions that \( f(\cdot, \cdot) \in C^1(\mathbb{C}^{1+n}) \), \( f(0,0) = 0 \) and

\[
\lim_{|(s,p)| \to \infty} \frac{\left| \int_0^1 f_s(\tau s, \tau p) d\tau \right|}{\ln^{3/2}(1 + |s| + |p|)} = 0, \tag{1.9}
\]

where \( p = (p_1, \ldots, p_n) \). Moreover, following [5], one can show that the assumptions on the growth of the nonlinearity \( f(y, \nabla y) \) in (1.9) are sharp in some sense. Since the techniques are very similar to [4, 5], we shall not give the details here.

Instead, as a byproduct of the fundamental identity established in this work (to show the observability inequality (1.7)), we shall develop a universal approach for controllability/observability problems governed by partial differential equations (PDEs for short), which is the second main object of this paper. The study of controllability/observability problem for PDEs began in the 1960s, for which various techniques have been developed in the last decades ([1, 3, 13, 17, 29]). It is well-known that the controllability of PDEs depends strongly on the nature of the system, say time reversibility or not. Typical examples are the wave and heat equations. It is clear now that there exists essential differences between the controllability/observability theories for these two equations. Naturally, one expects to know whether there are some relationship between these two systems of different nature. Especially, it would be quite interesting to establish a unified controllability/observability theory for parabolic and hyperbolic equations. This problem was posed by D.L. Russell in [21], where one can also find some preliminary result; further results are available in [18, 26]. In [15], the authors analyzed the controllability/observability problems for PDEs from the point of view of methodology. It is well known that these problems may be reduced to the obtention of suitable observability inequalities for the underlying homogeneous systems.
However, the techniques that have been developed to obtain such estimates depend heavily on the nature of the equations. In the context of the wave equation one may use multipliers ([17]) or microlocal analysis ([1]); while, in the context of heat equations, one uses Carleman estimates ([8, 13]). Carleman estimates can also be used to obtain observability inequalities for the wave equation ([25]). However, the Carleman estimate that has been developed up to now to establish observability inequalities of PDEs depend heavily on the nature of the equations, and therefore a unified Carleman estimate for those two equations has not been developed before. In this paper, we present a point-wise weighted identity for partial differential operators $(\alpha + i\beta)\partial_t + \sum_{j,k=1}^{n} \partial_k(a^{jk}\partial_j)$ (with real functions $\alpha$ and $\beta$), by which we develop a unified approach, based on global Carleman estimate, to deduce not only the controllability/observability results for systems (1.4) and (1.5), but also all the known controllability/observability results for the parabolic, hyperbolic, Schrödinger and plate equations that are derived via Carleman estimates (see Section 2 for more details). We point out that this identity has other interesting applications, say, in [19] it is applied to derive an asymptotic formula of reconstructing the initial state for a Kirchhoff plate equation with a logarithmic convergence rate for smooth data; while in [11] it is applied to establish sharp logarithmic decay rate for general hyperbolic equations with damping localized in arbitrarily small set by means of an approach which is different from that in [2].

The rest of this paper is organized as follows. In Section 2, we establish a fundamental point-wise weighted identity for partial differential operators of second order and give some of its applications. In Section 3, we derive a modified point-wise inequality for the parabolic operator. This estimate will play a key role when we establish in Section 4 a global Carleman estimate for the parabolic equation with a complex principal part. Finally, we will prove our main results in Section 5.

2 A weighted identity for partial differential operators and its applications

In this section, we will establish a point-wise weighted identity for partial differential operators of second order with a complex principal part, which has an independent interest. First, we introduce the following second order operator:

$$\mathcal{P}_z \triangleq (\alpha + i\beta)z_t + \sum_{j,k=1}^{m} (a^{jk}z_j)_k, \quad m \in \mathbb{N}.$$  \hspace{1cm} (2.1)

We have the following fundamental identity.

**Theorem 2.1** Let $\alpha, \beta \in C^2(\mathbb{R}^{1+m}; \mathbb{R})$ and $a^{jk} \in C^{1,2}(\mathbb{R}^{1+m}; \mathbb{R})$ satisfy $a^{jk} = a^{kj}$ ($j, k = 1, 2, \cdots, m$). Let $z, v \in C^2(\mathbb{R}^{1+m}; \mathbb{C})$ and $\Psi, \ell \in C^2(\mathbb{R}^{1+m}; \mathbb{R})$. Set $\theta = e^\ell$ and $v = \theta z$. 

Then,
\[
\theta(Pz I_1 + \overline{P}z I_1) + M_t + \text{div} \, V \\
= 2|I_1|^2 + \sum_{j,k,j',k'}^m \left[ 2(a^{jk} \ell_j)_{kj'}a^{jk'} - (a^{jk} a^{jk'} \ell_{j'})_{k'} + \frac{1}{2}(\alpha a^{jk})_t - a^{jk} \Psi \right] (v_k \overline{v}_j + \overline{v}_k v_j) \\
+i \sum_{j,k=1}^m \left[ (\beta a^{jk} \ell_j) + a^{jk}(\beta \ell_j) \right](\overline{v}_k v - v_k \overline{v}) - \sum_{j,k=1}^m a^{jk} \alpha_k (v_j \overline{v}_t + \overline{v}_j v_t) \\
+i \left[ \beta \Psi + \sum_{j,k=1}^m (\beta a^{jk} \ell_j)_k \right] (\overline{v} v_t - v \overline{v}_t) + B|v|^2,
\]
where
\[
A \triangleq \sum_{j,k=1}^m a^{jk} \ell_j \ell_k - \sum_{j,k=1}^m (a^{jk} \ell_j)_k - \Psi, \\
I_1 \triangleq i \beta v_t - \alpha \ell_t v + \sum_{j,k=1}^m (a^{jk} v_j)_k + Av,
\]
and
\[
M \triangleq [(\alpha^2 + \beta^2) \ell_t - \alpha A] |v|^2 + \alpha \sum_{j,k=1}^m a^{jk} v_j \overline{v}_k + i \beta \sum_{j,k=1}^m a^{jk} \ell_j (\overline{v}_k v - v_k \overline{v}), \\
V \triangleq [V^1, \ldots, V^k, \ldots, V^m], \\
V^k \triangleq \sum_{j,j',k'=1}^m \left\{ -i \beta \left[a^{jk} \ell_j (\overline{v}_v - v \overline{v}_v) + a^{jk} \ell_t (v_j \overline{v} - v \overline{v}_v) \right] - \alpha a^{jk} (v_j \overline{v}_t + \overline{v}_j v_t) \\
+ (2a^{jk'} a^{jk} - a^{jk} a^{jk'}) \ell_j (v_j \overline{v}_k + \overline{v}_j v_k) - \Psi a^{jk} (v_j \overline{v} + v \overline{v}_v) \\
+ a^{jk} (2A \ell_j + \Psi v - \alpha \ell_j \ell_t) |v|^2 \right\}, \\
B \triangleq (\alpha^2 \ell_t) + (\beta^2 \ell_t) - (\alpha A) - 2 \sum_{j,k=1}^m (\alpha a^{jk} \ell_j \ell_t)_k + \alpha \Psi \ell_t \\
+ \sum_{j,k=1}^m (a^{jk} \Psi_k)_j + 2 \sum_{j,k=1}^m (a^{jk} \ell_j A)_k + A \Psi, \]

Several remarks are in order.

**Remark 2.1** We see that only the symmetry condition of $a^{jk}_{m \times m}$ is assumed in the above. Therefore, Theorem 2.1 is applicable to ultra-hyperbolic or ultra-parabolic differential operators.

**Remark 2.2** Note that when $\Psi = -\sum_{j,k=1}^m (a^{jk} \ell_j)_k$ and $\alpha(t, x) \equiv a, \beta(t, x) \equiv b$ with $a, b \in \mathbb{R}$, Theorem 2.1 is reduced to [10, Theorem 1.1]. Here, we add an auxiliary function $\Psi$ in the right-hand side of the multiplier $I_1$ so that the corresponding identity coincides with [12, Theorem 4.1] for the case of hyperbolic operators. Moreover, we will see that the modified identity is better than [10] in some sense.
Remark 2.3 By choosing \( \alpha(t, x) \equiv 1, \beta(t, x) \equiv 0 \) and \( \Psi = -2 \sum_{j,k=1}^{m} (a^j k \ell_j)_k \) in Theorem 2.1, one obtains a weighted identity for the parabolic operator. By this and following [22], one may recover all the controllability/observability results for the parabolic equations in [4] and [13].

Remark 2.4 By choosing \( a^j k(t, x) \equiv a^j k(x) \) and \( \alpha(t, x) = \beta(t, x) \equiv 0 \) in Theorem 2.1, one obtains the key identity derived in [12] for the controllability/observability results on the general hyperbolic equations.

Remark 2.5 By choosing \( (a^j k)_{1 \leq j,k \leq m} \) to be the identity matrix, \( \alpha(t, x) \equiv 0, \beta(t, x) \equiv 1 \) and \( \Psi = -\Delta \ell \) in Theorem 2.1, one obtains the pointwise identity derived in [14] for the observability results for the nonconservative Schrödinger equations. Also, this yields the controllability/observability results in [27] for the plate equations and the results for inverse problem for the Schrödinger equation in [6].

Remark 2.6 By choosing \( (a^j k)_{1 \leq j,k \leq m} \) to be the identity matrix, \( \alpha(t, x) \equiv 0, \beta(t, x) \equiv p(x) \) and \( \Psi = -\Delta \ell \) in Theorem 2.1, one obtains the pointwise identity for the Schrödinger operator: \( \mathrm{i} p(x) \partial_t + \Delta \). Further, by choosing

\[
\ell(t, x) = s \varphi, \quad \varphi = e^{\gamma (|x-x_0|^2-c|t-t_0|^2)}
\]

with \( \gamma > 0, c > 0, x_0 \in \mathbb{R}^n \setminus \Omega \) and assuming \( \nabla \log p \cdot (x - x_0) > -2 \). One can recover the fundamental Carleman estimate for Schrödinger operator \( \mathrm{i} p(x) \partial_t + \Delta \) derived in [24, Lemma 2.1].

Proof of Theorem 2.1. The proof is divided into several steps.

Step 1. By \( \theta = e^\ell, v = \theta z \), we have (recalling (2.4) for the definition of \( I_1 \))

\[
\theta \mathcal{P} z = (\alpha + i \beta) v_t - (\alpha + i \beta) \ell_t v + \sum_{j,k=1}^{m} (a^j k \ell_j)_k
\]

\[
+ \sum_{j,k=1}^{m} a^j k \ell_j k v - 2 \sum_{j,k=1}^{m} a^j k \ell_j k v - \sum_{j,k=1}^{m} (a^j k \ell_j)_k v
\]

\[
= I_1 + I_2,
\]

where

\[
I_2 \equiv \alpha v_t - i \beta \ell_t v - 2 \sum_{j,k=1}^{m} a^j k \ell_j v_k + \Psi v.
\]

Hence, by recalling (2.3) for the definition of \( I_1 \), we have

\[
\theta(\mathcal{P} z T_1 + \overline{\mathcal{P} z I_1}) = 2|I_1|^2 + (I_1 T_2 + I_2 \overline{T}_1).
\]

Step 2. Let us compute \( I_1 \overline{T}_2 + I_2 \overline{T}_1 \). Denote the four terms in the right hand side of \( I_1 \) and \( I_2 \) by \( I^1_1 \) and \( I^1_2 \), respectively, \( j = 1, 2, 3, 4 \). Then

\[
I^1_1 (\overline{T}_1^2 + \overline{T}_2^2) + \overline{T}_1^1 (I_2^1 + I_2^2) = - (\beta^2 \ell_t |v|^2)_t + (\beta^2 \ell_t t |v|^2).
\]
Note that
\[
\begin{align*}
2\nu_t &= (|v|^2)_t - (\nu v_t - v\nu_t), \\
2\nu_k &= (|v|^2)_k - (\nu v_k - v\nu_k).
\end{align*}
\]

Hence, we get
\[
I_1^2(I_2^2 + I_3^2) + I_1^3(I_2^3 + I_3^3)
= -2i \sum_{j,k=1}^m \left[ (\beta a^{jk} \ell_j v\nu_k)_t - (\beta a^{jk} \ell_j)_t v\nu_k \right] + 2i \sum_{j,k=1}^m \left[ (\beta a^{jk} \ell_j v\nu_t)_k - (\beta a^{jk} \ell_j)_k v\nu_t \right] - i\beta \Psi (v\nu_t - v_t\nu) - i \sum_{j,k=1}^m (\beta a^{jk} \ell_j) \nu_t - v\nu_t \right]_k
- i \sum_{j,k=1}^m (\beta a^{jk} \ell_j)(v\nu_k - v\nu_k) + i \left[ \beta \Psi + \sum_{j,k=1}^m (\beta a^{jk} \ell_j) \right] (v\nu_t - v\nu_t).
\]

Next,
\[
I_1^2(T_2^2 + T_2^3 + \overline{T}_2^3) + I_1^3(T_2^3 + I_2^3 + I_2^3 + I_2^4)
= -(\alpha^2 \ell_t |v|^2)_t + 2 \sum_{j,k=1}^m (\alpha a^{jk} \ell_j \ell_t |v|^2)_k
+ (\alpha^2 \ell_t |v|^2) - 2 \left[ \sum_{j,k=1}^m (\alpha a^{jk} \ell_j \ell_t)_k + \alpha \Psi \ell_t \right] |v|^2.
\]

Noting that \(a^{jk} = a^{kj}\), we have
\[
I_1^2(T_2^2 + T_2^3 + \overline{T}_2^3) + I_1^3(I_2^3 + I_2^3)
= \sum_{j,k=1}^m \left[ \alpha a^{jk} (v_j \nu_t + \nu_j v_t) \right] - \sum_{j,k=1}^m a^{jk} \alpha_k (v_j \nu_t + \nu_j v_t) - \sum_{j,k=1}^m (\alpha a^{jk} v_j \nu_k) + \sum_{j,k=1}^m (\alpha a^{jk} v_j \nu_t) + i \sum_{j,k=1}^m \left[ \beta a^{jk} \ell_t (v_j \nu - \nu_j v) \right]_k + i \sum_{j,k=1}^m a^{jk} (\beta \ell_t)_k (v_j v - v_j \nu)
= \sum_{j,k=1}^m \left[ \alpha a^{jk} (v_j \nu_t + \nu_j v_t) + i \beta a^{jk} \ell_t (v_j \nu - \nu_j v) \right]_k
- \sum_{j,k=1}^m (\alpha a^{jk} \nu_j \nu_k)_t - \sum_{j,k=1}^m a^{jk} \alpha_k (v_j \nu_t + \nu_j v_t)
+ \frac{1}{2} \sum_{j,k=1}^m (\alpha a^{jk})_t (v_j \nu_k + v_k \nu_j) + i \sum_{j,k=1}^m a^{jk} (\beta \ell_t)_k (v_j v - v_j \nu).
\]
Using the symmetry condition of $a^{jk}$ again, we obtain

$$2 \sum_{j,k,j',k'=1}^{m} a^{jk} a^{j'k'} \ell_j (v_j \overline{\nu}_{kk'} + \overline{\nu}_{j'} v_{k'})$$

$$= \sum_{j,k,j',k'=1}^{m} \left[ a^{jk} a^{j'k'} \ell_j (v_j \overline{\nu}_{k'} + \overline{\nu}_{j'} v_{k'}) \right] - \sum_{j,k,j',k'=1}^{m} (a^{jk} a^{j'k'}) \ell_k (v_j \overline{\nu}_{k'} + \overline{\nu}_{j'} v_{k'}).$$

(2.12)

By (2.12), we get

$$I_1^3 \overline{I_2} + I_1^4 I_2^3$$

$$= -2 \sum_{j,k,j',k'=1}^{m} \left[ a^{jk} \ell_j a^{j'k'} (v_j \overline{\nu}_k + \overline{\nu}_j v_k) \right] + 2 \sum_{j,k,j',k'=1}^{m} a^{j'k'} (a^{jk} \ell_j) (v_j \overline{\nu}_k + \overline{\nu}_j v_k)$$

$$+ \sum_{j,k,j',k'=1}^{m} \left[ a^{jk} a^{j'k'} \ell_j (v_j \overline{\nu}_{k'} + \overline{\nu}_{j'} v_{k'}) \right] - \sum_{j,k,j',k'=1}^{m} (a^{jk} a^{j'k'}) \ell_k (v_j \overline{\nu}_{k'} + \overline{\nu}_{j'} v_{k'}).$$

(2.13)

Further,

$$I_1^3 \overline{I_2} + I_1^4 I_2^3 = \sum_{j,k=1}^{m} \left[ \Psi a^{jk} (v_j \overline{\nu} + \overline{\nu}_j v) \right] - \sum_{j,k=1}^{m} a^{jk} \ell_j (v_j \overline{\nu}_k + \overline{\nu}_j v_k)$$

$$- \sum_{j,k=1}^{m} \left[ a^{jk} \ell_j |v|^2 \right] + \sum_{j,k=1}^{m} (a^{jk} \ell_j A |v|^2)$$

(2.14)

Finally,

$$I_1^4 \overline{T_2} + T_2^3 + T_2^3 + T_1^4 + I_2^4 + I_2^3 + I_2^3 + I_2^3 + I_2^3$$

$$= (\alpha A |v|^2) - (\alpha A) t |v|^2 - 2 \sum_{j,k=1}^{m} (a^{jk} \ell_j A |v|^2)$$

$$+ 2 \left[ \sum_{j,k=1}^{m} (a^{jk} \ell_j A) + A \Psi \right] |v|^2.$$

(2.15)

**Step 3.** By (2.8)-(2.15), combining all $\frac{\partial}{\partial v}$-terms, all $\frac{\partial}{\partial \nu_k}$-terms, and (2.7), we arrive at the desired identity (2.2).

We have the following pointwise estimate for the complex parabolic operator $Gz$.

**Corollary 2.1** Let $b \in \mathbb{R}$ and $a^{jk}(t, x) \in C^{1,2}(\mathbb{R}^{1+n}; \mathbb{R})$ satisfy condition (1.1). Let $z, v \in C^2(\mathbb{R}^{1+m}; \mathbb{C})$ and $\Psi, \ell \in C^2(\mathbb{R}^{1+m}; \mathbb{R})$. Set $\theta = e^t$ and $v = \theta z$. Put

$$\Psi = -2 \sum_{j,k=1}^{n} (a^{jk} \ell_j)_k.$$

(2.16)

Then

$$\theta^2 |Gz|^2 + M_t + \text{div } V$$

$$\geq \sum_{j,k,j',k'=1}^{n} \left[ 2(a^{jk} \ell_j)_{k'} a^{j'k'} - a^{jk} a^{j'k'} \ell_{j'} + \frac{1}{2} a^{jk} + a^{jk} (a^{j'k'} \ell_{j'})_{k'} \right] (v_k \overline{\nu}_j + \overline{\nu}_j v_k)$$

$$+ ib \sum_{j,k=1}^{n} (a^{jk} \ell_j + 2a^{jk} \ell_j)(\overline{\nu}_k v - v_k \overline{\nu}_j) - ib \sum_{j,k=1}^{n} (a^{jk} \ell_j)_{k}(\overline{\nu}_k v - v_k \overline{\nu}_j) + B |v|^2,$$

(2.17)
where

\[
M = [(1 + b^2)\ell_t - A]v^2 + \sum_{j,k=1}^n a_{jk}v_j\overline{v}_k + ib \sum_{j,k=1}^n a_{jk}\ell_j(\overline{v}_k v - v_k\overline{v}),
\]

\[
V^k = \sum_{j,j',k'=1}^n \left\{-ib\left[a_{jk}\ell_j(\overline{v}_t v - \overline{v}v_t) + a_{jk}\ell_t(v_j\overline{v} - \overline{v}_j v)\right] - a_{jk}(v_j\overline{v}_t + \overline{v}_j v_t)
\right. \\
\left. + \left(2a_{jk'}a_{j'k} - a_{jk}a_{j'k'}\right)\ell_j(v_{j'}\overline{v}_{k'} + \overline{v}_{j'}v_{k'}) - \Psi a_{jk}(v_j\overline{v} + \overline{v}_j v)
\right.
\left. + a_{jk}(2\ell\ell_t + \Psi_j - 2\ell_j\ell_t)|v|^2 \right\}, (2.18)
\]

\[
B = (1 + b^2)\ell_{tt} - A_t - 2\sum_{j,k=1}^n (a_{jk}\ell_j\ell_t) + \Psi \ell_t
\]

\[
+ \sum_{j,k=1}^n (a_{jk}\Psi) + 2\sum_{j,k=1}^n (a_{jk}\ell_j A) + A\Psi.
\]

Proof. Recalling (1.6) for the definition of \( G_z \), taking \( m = n, \alpha(x) \equiv 1, \beta(x) \equiv b \) in Theorem 2.1, by using Hölder inequality and simple computation, we immediately obtain the desired result.

\( \square \)

3 A modified point-wise estimate

Note that the term \( ib \sum_{j,k=1}^n (a_{jk}\ell_j)(\overline{v}_t v - v\overline{v}) \) in the right-hand side of (2.17) is not good. In this section, we make some modification on this term and derive the following modified point-wise inequality for the parabolic operator with a complex principal.

Theorem 3.1 Let \( b \in \mathbb{R} \) and \( a^{jk}(t, x) \in C^{1,2}(\mathbb{R}^{1+n}; \mathbb{R}) \) satisfy (1.1)–(1.2). Let \( z, v \in C^2(\mathbb{R}^{1+m}; \mathbb{C}) \) and \( \Psi, \ell \in C^2(\mathbb{R}^{1+m}; \mathbb{R}) \) satisfy (2.16). Put

\[
\theta = e^\ell, \quad v = \theta z.
\] (3.1)

Then

\[
2\theta^2|Gz|^2 + M_t + \text{div} \tilde{V}
\]

\[
\geq \sum_{j,k=1}^n c^{jk}(v_k\overline{v}_j + \overline{v}_k v_j) + \tilde{B}|v|^2
\]

\[
+ ib \sum_{j,k=1}^n \left[a_{jk}\ell_j + 2a_{jk}\ell_{jt} + \frac{1}{1 + b^2} \sum_{j',k'=1}^n (a_{j'k'}\ell_{j'k'}) a_{jk}\right](\overline{v}_k v - v_k\overline{v}), (3.2)
\]
where $M$, $V^k$, $B$ is given by (2.18) and

$$
\begin{align*}
\tilde{V}^k &= V^k + \sum_{j,j',k'=1}^n \left\{ \frac{ib}{1+b^2} \sum_{j,k,j',k=1}^n \left[ (a^{j'k'}\ell_{j'})_{k'}a^{jk}(\overline{v}_j v - v_j \overline{v}) \right] \\
&\quad - \frac{b^2}{1+b^2} \theta^2 (a^{j'k'}\ell_{j'})_{k'}a^{jk}(\overline{z}_j z + z_j \overline{z}) + \frac{b^2}{1+b^2} \left[ \theta^2 (a^{j'k'}\ell_{j'})_{k'} \right] a^{jk}|z|^2 \\
&\quad - \frac{2b^2}{1+b^2} (a^{j'k'}\ell_{j'})_{k'}a^{jk}|v|^2 \right\}, \\
c^{jk} &= \sum_{j',k'=1}^n \left[ 2(a^{j'k'}\ell_{j'})_{k'}a^{jk} - a^{jk}a^{j'k'}\ell_{j'} + \frac{1}{2}a^{jk} + \frac{1}{2(1+b^2)}a^{jk}(a^{j'k'}\ell_{j'})_{k'} \right], \\
\tilde{B} &= B - \frac{2b^2}{1+b^2} \sum_{j,k,j',k'=1}^n (a^{j'k'}\ell_{j'})_{k'}a^{jk}\ell_{k'} - \frac{b^2}{1+b^2} \sum_{j,k=1}^n (a^{jk}\ell_{j})_k^2 \\
&\quad + \frac{b^2}{1+b^2} \sum_{j,k,j',k'=1}^n \left\{ 2a^{jk}\ell_k(a^{j'k'}\ell_{j'})_{k'} + (a^{jk}(a^{j'k'}\ell_{j'})_{k'})_k \right\}.
\end{align*}
$$

(3.3)

Proof. We divided the proof into several steps.

**Step 1.** Note that $v = \theta z$, it is easy to check that

$$
\overline{v}_t - v\overline{t} = \theta^2 (z\overline{z}_t - z\overline{z}_t), \quad \overline{v}_k - v\overline{k} = \theta^2 (z_k\overline{z} - z_k\overline{z}).
$$

(3.4)

Recalling (1.6) for the definition of $Gz$ and by (3.4), we have

$$
-ib \sum_{j,k=1}^n (a^{jk}\ell_j)_k(\overline{v}_t - v\overline{t}) = -ib \sum_{j,k=1}^n (a^{jk}\ell_j)_k \theta^2(\overline{z}_t - z\overline{z}_t)
$$

$$
= -\frac{ib\theta^2}{1+b^2} \sum_{j,k=1}^n (a^{jk}\ell_j)_k \left[ (1-ib)Gz\overline{z} - (1-ib)Gz\overline{z} \right]
$$

$$
= -\frac{ib\theta^2}{1+b^2} \sum_{j,k=1}^n (a^{jk}\ell_j)_k \sum_{j,k=1}^n \left[ (a^{jk}\overline{z}_j)z - (a^{jk}z_j)\overline{z} \right] - \frac{b^2\theta^2}{1+b^2} \sum_{j,k=1}^n (a^{jk}\ell_j)_k \sum_{j,k=1}^n \left[ (a^{jk}\overline{z}_j)z + (a^{jk}z_j)\overline{z} \right]
$$

$$
\equiv \sum_{k=1}^3 J_k.
$$

(3.5)

**Step 2.** Let us estimate $J_k$ ($k = 1, 2, 3$) respectively.

First, note that $v = \theta z$, we have

$$
J_1 = -\frac{ib\theta^2}{1+b^2} \sum_{j,k=1}^n (a^{jk}\ell_j)_k \left[ (1-ib)Gz\overline{z} - (1-ib)Gz\overline{z} \right]
$$

$$
\geq -\frac{(1-ib)\theta Gz}{\sqrt{1+b^2}} - \frac{ib\theta z}{\sqrt{1+b^2}} \sum_{j,k=1}^n (a^{jk}\ell_j)_k^2
$$

$$
= -\theta^2 |Gz|^2 - \frac{b^2}{1+b^2} \sum_{j,k=1}^n (a^{jk}\ell_j)_k^2 |v|^2.
$$

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Next, by using (3.4) again, we have

\[ J_2 = -\frac{ib\theta^2}{1 + b^2} \sum_{j',k'=1}^n (a^{j'k'} \ell_{j'})_{k'} \sum_{j,k=1}^n \left[ (a^{jk} z_j)_{kz} - (a^{jk} z_j)_{k\bar{z}} \right] \]

\[ = -\frac{ib}{1 + b^2} \sum_{j,k,j',k'=1}^n \left[ \theta^2 (a^{jk'k'} \ell_{j'})_{k'} a^{jk}(\overline{\sigma}_j z - z_j \overline{\sigma}) \right]_k \]

\[ + \frac{ib}{1 + b^2} \sum_{j,k,j',k'=1}^n \theta^2 \left[ 2\ell_k(a^{jk'k'} \ell_{j'})_{k'} + (a^{jk'k'} \ell_{j'})_{k'k} \right] a^{jk}(\overline{\sigma}_j z - z_j \overline{\sigma}) \] (3.7)

\[ = -\frac{ib}{1 + b^2} \sum_{j,k,j',k'=1}^n \left[ (a^{jk'k'} \ell_{j'})_{k'} a^{jk}(\overline{\sigma}_j v - v_j \overline{\sigma}) \right]_k \]

\[ + \frac{ib}{1 + b^2} \sum_{j,k,j',k'=1}^n \left[ 2\ell_k(a^{jk'k'} \ell_{j'})_{k'} + (a^{jk'k'} \ell_{j'})_{k'k} \right] a^{jk}(\overline{\sigma}_j v - v_j \overline{\sigma}). \]

Next, noting that \( a^{jk} \) satisfy (1.1), and recalling \( v = \theta z \), it follows

\[ J_3 = b^2 \frac{\theta^2}{1 + b^2} \sum_{j',k'=1}^n (a^{j'k'} \ell_{j'})_{k'} \sum_{j,k=1}^n \left[ (a^{jk} z_j)_{kz} + (a^{jk} z_j)_{k\bar{z}} \right] \]

\[ = b^2 \frac{1}{1 + b^2} \sum_{j,k,j',k'=1}^n \left[ \theta^2 (a^{j'k'k'} \ell_{j'})_{k'} a^{jk}(\overline{\sigma}_j z - z_j \overline{\sigma}) \right]_k \]

\[ - b^2 \frac{\theta^2}{1 + b^2} \sum_{j,k,j',k'=1}^n (a^{j'k'} \ell_{j'})_{k'} a^{jk}(\overline{z}_j z_k + z_j \overline{z}_k) \]

\[ + b^2 \frac{1}{1 + b^2} \sum_{j,k,j',k'=1}^n \left[ \theta^2 (a^{j'k'k'} \ell_{j'})_{k'} a^{jk} \right]_k |z|^2 \] (3.8)

\[ = b^2 \frac{1}{1 + b^2} \sum_{j,k,j',k'=1}^n \left[ \theta^2 (a^{j'k'k'} \ell_{j'})_{k'} a^{jk}(\overline{\sigma}_j z - z_j \overline{\sigma}) \right]_k \]

\[ - b^2 \frac{\theta^2}{1 + b^2} \sum_{j,k,j',k'=1}^n (a^{j'k'k'} \ell_{j'})_{k'} a^{jk}(\overline{\sigma}_j v - v_j \overline{\sigma}) + \left\{ \sum_{j,k,j',k'=1}^n a^{jk} \ell_j \ell_k (a^{jk'k'} \ell_{j'})_{k'} \right\} |v|^2 \]

\[ + 2 \sum_{j,k,j',k'=1}^n a^{jk} \ell_k (a^{jk'k'} \ell_{j'})_{k'} + \sum_{j,k,j',k'=1}^n (a^{jk} (a^{jk'k'} \ell_{j'})_{k'}) |v|^2 \]
where we have used the following fact:

\[
\theta^2 \sum_{j,k,j',k'=1}^n (a^{jk'} \ell_{j'}) k' a^{j k}(\overline{z}_j z_k + z_j \overline{z}_k) \]

\[
= \sum_{j,k,j',k'=1}^n (a^{jk'} \ell_{j'}) k' a^{j k}(\overline{\n}_j v_k + v_j \overline{\n}_k) + 2 \sum_{j,k,j',k'=1}^n (a^{jk'} \ell_{j'}) k' a^{j k} \ell_j \ell_k |v|^2 \tag{3.9}
\]

\[
-2 \sum_{j,k,j',k'=1}^n [(a^{jk'} \ell_{j'}) k' a^{j k} \ell_k |v|^2] + 2 \sum_{j,k,j',k'=1}^n [(a^{jk'} \ell_{j'}) k' a^{j k} \ell_k ] |v|^2.
\]

and

\[
\sum_{j,k,j',k'=1}^n \left\{ \left[ \theta^2 (a^{jk'} \ell_{j'})_{k'} \right] a^{j k} \right\} |z|^2
\]

\[
= \sum_{j,k,j',k'=1}^n \left\{ \theta^2 \left[ 2 a^{j k} \ell_j (a^{jk'} \ell_{j'})_{k'} + a^{jk} (a^{jk'} \ell_{j'})_{k' j} \right] \right\} |z|^2
\]

\[
= \left\{ 4 \sum_{j,k,j',k'=1}^n a^{jk} \ell_j \ell_k (a^{jk'} \ell_{j'})_{k'} + 4 \sum_{j,k,j',k'=1}^n a^{jk} \ell_k (a^{jk'} \ell_{j'})_{k' j} \right\} |v|^2 + 2 \sum_{j,k,j',k'=1}^n (a^{jk} \ell_j) |v|^2 + \sum_{j,k,j',k'=1}^n \left( a^{jk} (a^{jk'} \ell_{j'})_{k' j} \right) |v|^2.
\]

Step 3. Noting that \(a^{jk}\) satisfying (1.1)--(1.2), we have the following fact.

\[
\frac{1}{2(1+b^2)} \sum_{j,k=1}^n a^{jk} (\overline{\n}_j v_k + v_j \overline{\n}_k) - \frac{2i}{1+b^2} \sum_{j,k=1}^n a^{jk} \ell_k (\overline{\n}_j v - v_j \overline{\n})
\]

\[
= \frac{1}{1+b^2} \sum_{j,k=1}^n a^{jk} (v_j \overline{\n}_k - 2ib \ell_k (\overline{\n}_j v - v_j \overline{\n}) + 2b^2 \ell_j \ell_k |v|^2 - \frac{4b^2}{1+b^2} \sum_{j,k=1}^n a^{jk} \ell_j \ell_k |v|^2
\]

\[
= \frac{1}{1+b^2} \sum_{j,k=1}^n a^{jk} (v_j - 2ib \ell_j v) (v_k - 2ib \ell_k v) - \frac{4b^2}{1+b^2} \sum_{j,k=1}^n a^{jk} \ell_j \ell_k |v|^2
\]

\[
\geq -\frac{4b^2}{1+b^2} \sum_{j,k=1}^n a^{jk} \ell_j \ell_k |v|^2.
\]

Finally, combining (2.17), (3.4)--(3.8) and (3.11), we arrive at the desired result (3.2). \(\square\)

4 Global Carleman estimate for parabolic operators with complex principal part

We begin with the following known result.

Lemma 4.1 ([13], [23]) There is a real function \(\psi \in C^2(\overline{\Omega})\) such that \(\psi > 0\) in \(\Omega\) and \(\psi = 0\) on \(\partial \Omega\) and \(|\nabla \psi(x)| > 0\) for all \(x \in \overline{\Omega} \setminus \omega_0\).
For any (large) parameters \( \lambda > 1 \) and \( \mu > 1 \), put
\[
\ell = \lambda \rho, \quad \varphi(t, x) = \frac{e^{\mu \psi(x)}}{t(T - t)}, \quad \rho(t, x) = \frac{e^{\mu \psi(x)} - e^{2\mu |\psi|_{C(\Omega, R)}}}{t(T - t)}.
\] (4.1)

For \( j, k = 1, \ldots, n \), it is easy to check that
\[
\ell_t = \lambda \rho_t, \quad \ell_j = \lambda \mu \varphi_j, \quad \ell_{jk} = \lambda \mu^2 \varphi_j \psi_k + \lambda \mu \varphi_j \psi_k
\] (4.2)
and
\[
|\rho_t| \leq C e^{2\mu |\psi|_{C(\Omega)}} \varphi^2, \quad |\varphi_t| \leq C \varphi^2.
\] (4.3)

In the sequel, for \( k \in \mathbb{N} \), we denote by \( O(\mu^k) \) a function of order \( \mu \) for large \( \mu \) (which is independent of \( \lambda \)); by \( O_\mu(\lambda^k) \) a function of order \( \lambda^k \) for fixed \( \mu \) and for large \( \lambda \).

We have the following Carleman estimate for the differential operator \( G \) defined in (1.6):

**Theorem 4.1** Let \( b \in \mathbb{R} \) and \( a^{jk} \) satisfy (1.1)–(1.2). Then there is a \( \mu_0 > 0 \) such that for all \( \mu > \mu_0 \), one can find two constants \( C = C(\mu) > 0 \) and \( \lambda_1 = \lambda_1(\mu) \), such that for all \( z \in C((0, T]; L^2(\Omega)) \cap C((0, T]; H^1_0(\Omega)) \) and for all \( \lambda \geq \lambda_1 \), it holds
\[
\lambda^3 \mu^4 \int_\Omega \varphi^2 |z|^2 \, dt \, dx + \lambda \mu^2 \int_\Omega \varphi^2 |\nabla z|^2 \, dt \, dx \leq C(1 + b^2) \left[ \int_\Omega \theta^2 |G z|^2 \, dt \, dx + \lambda^3 \mu^4 \int_{(0, T) \times \omega} \varphi^2 |z|^2 \, dt \, dx \right].
\] (4.4)

**Proof.** The proof is long, we divided it into several steps.

**Step 1.** Recalling (3.3) for the definition of \( c^{jk} \), by (4.2)–(4.3), we have
\[
\sum_{j, k=1}^n c^{jk}(v_k \overline{\psi}_j + \overline{\psi}_k v_j)
= \sum_{j, k, j', k'=1}^n \left[ 2(a^{j'k'} \ell_{j'}) \alpha^{jk'} - a^{jk'} a^{j'k'} \ell_{j'} \right] v_k \overline{\alpha}^{j'k'} v_j
+ \frac{1}{2} a^{jk} + \frac{1}{2(1 + b^2)} a^{jk} (a^{j'k'} \ell_{j'}) v_k \overline{v}_j
= 4 \lambda^3 \mu^2 \varphi \left[ \sum_{j, k=1}^n a^{jk} \overline{\psi}_j \overline{\psi}_k \right]^2
+ \frac{1}{2(1 + b^2)} \lambda^3 \mu^2 \varphi \sum_{j, k, j', k'=1}^n a^{jk} a^{j'k'} \overline{\psi}_j \overline{\psi}_k (v_k \overline{\psi}_j + \overline{\psi}_k v_j) + \lambda \varphi O(\mu) \overline{\psi}.$
\] (4.5)

On the other hand, by (4.2)–(4.3), recalling (2.16) and (2.3) for the definitions of \( \Psi \) and \( A \), it is easy to check that
\[
\Psi = -2 \lambda \mu^2 \varphi \sum_{j, k=1}^n a^{jk} \overline{\psi}_j \overline{\psi}_k + \lambda \varphi O(\mu), \quad A = \lambda^2 \mu^2 \varphi^2 \sum_{j, k=1}^n a^{jk} \overline{\psi}_j \overline{\psi}_k + \varphi^2 O_\mu(\lambda).
\] (4.6)
Next, recalling (2.3) and (2.18) for the definition of $A$ and $B$, respectively. By (4.2)–(4.3) and combining (4.6), we have

\begin{equation}
B = 2 \sum_{j,k=1}^{n} a^{jk} \ell_j A_k - 2A \sum_{j,k=1}^{n} (a^{jk} \ell_j)_k \\
+ (1 + b^2) \ell_{tt} - A_t - 2 \sum_{j,k=1}^{n} a^{jk} \ell_j \ell_{tk} + 2 \ell_t \sum_{j,k=1}^{n} (a^{jk} \ell_j)_k + \sum_{j,k=1}^{n} (a^{jk} \Psi_k)_j
\end{equation}

(4.7)

\begin{equation}
= 2 \lambda^3 \mu^4 \varphi^3 \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \leq \lambda^3 \varphi^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2). 
\end{equation}

Hence, by recalling (3.3) for the definition of $\tilde{B}$, we have

\begin{equation}
\tilde{B} = \frac{2}{1 + b^2} \lambda^3 \mu^4 \varphi^3 \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \leq \lambda^3 \varphi^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2). 
\end{equation}

(4.8)

\textbf{Step 2.} By (4.2)–(4.3), we have

\begin{equation}
\left| \frac{ib}{1 + b^2} \sum_{j,k=1}^{n} (a^{jk} \ell_j) (\overline{v} k v - v_k \overline{v}) \right| \\
\leq C \lambda \mu | \varphi^2 \sum_{j,k=1}^{n} a^{jk} \psi_j (\overline{v} k v - v_k \overline{v}) | \leq C \lambda \mu | \sum_{j,k=1}^{n} a^{jk} \psi_j \overline{v} k v |^2 + C \lambda \mu \varphi^3 | v |^2.
\end{equation}

(4.9)

It is easy to see that (4.9) can be absorbed by (4.5) and (4.8).

Similarly, by using (4.2)–(4.3) again, we have

\begin{equation}
\left| \frac{ib}{1 + b^2} \sum_{j,k,j',k'=1}^{n} (a^{j'k'}) \ell_{j'} \ell_{j'} a^{jk} (\overline{v} k v - v_k \overline{v}) \right| \\
\leq \frac{1}{1 + b^2} \left| b \left[ \lambda \mu^3 \varphi^2 \sum_{j',k'=1}^{n} a^{j'k'} \overline{v} j' v - \varphi^2 O(\mu^2) \right] a^{jk} \psi_j (\overline{v} k v - v_k \overline{v}) \right| \\
\leq \frac{1}{1 + b^2} \left| b \lambda \mu^3 \varphi^2 \sum_{j',k'=1}^{n} a^{j'k'} \overline{v} j' v - \varphi^2 O(\mu^2) \right| a^{jk} \psi_j (\overline{v} k v - v_k \overline{v}) \right| \\
+ \frac{C}{1 + b^2} \left| b \lambda \mu^2 \varphi^2 \sum_{j,k=1}^{n} a^{jk} \psi_j (\overline{v} k v - v_k \overline{v}) \right| \\
\leq \frac{1}{1 + b^2} \lambda \mu^2 \varphi \left| \sum_{j,k=1}^{n} a^{jk} \psi_j \overline{v} k v \right|^2 + \frac{b^2}{1 + b^2} \lambda \mu^4 \varphi^3 \left| \sum_{j,k=1}^{n} a^{jk} \psi_j \psi_k \right|^2 | v |^2 \\
+ \frac{C}{1 + b^2} \lambda \mu \varphi \left| \sum_{j,k=1}^{n} a^{jk} \psi_j \overline{v} k v \right|^2 + \frac{Cb^2}{1 + b^2} \lambda \mu^3 \varphi^3 | v |^2.
\end{equation}

Therefore (4.10) also can be absorbed by (4.5) and (4.8).
Combining (4.5), (4.8)–(4.10), by (1.2), we end up with

The right-hand side of (3.2)

\[
\geq \frac{1}{1 + b^2} \lambda \mu^2 \varphi \sum_{j',k'=1}^n a^{j'k'} \psi_{j'} \psi_{k'} \sum_{j,k=1}^n a^{jk} v_j \overline{\psi}_j + \lambda \varphi O(\mu)|\nabla v|^2 \\
+ \left[ \frac{2}{1 + b^2} \lambda^3 \mu^4 \varphi^3 \right] \sum_{j,k=1}^n a^{jk} \psi_j \psi_k^2 + \lambda^3 \varphi^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2) |v|^2 \tag{4.11}
\]

\[
\geq \frac{1}{1 + b^2} \left[ s_0^2 \lambda \mu^2 |\nabla \psi|^2 + \lambda \varphi O(\mu) \right] |\nabla v|^2 \\
+ \frac{2}{1 + b^2} \left[ s_0^2 \lambda^3 \mu^4 \varphi^3 |\nabla \psi|^4 + \lambda^3 \varphi^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2) \right] |v|^2.
\]

Step 3. Integrating (3.2) on \(Q\), by (4.11), noting that \(\theta(0) = \theta(T) \equiv 0\), we have

\[
\int_Q \varphi \left[ \lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu) \right] |\nabla v|^2 dx dt \\
+ \int_Q \varphi^3 \left[ \lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) + O_\mu(\lambda^2) \right] |v|^2 dx dt \\
\leq C \left[ \left| \theta G_z \right|^2_{L^2(Q)} + \int_Q \text{div} \tilde{V} \cdot \nu dx dt \right].
\tag{4.12}
\]

Next, recalling (2.4) and (3.3) for the definition of \(V\) and \(\tilde{V}\). Noting that \(z|_\Sigma = 0\) and \(v_i = \frac{\partial v}{\partial \nu} \nu_i\) (which follows from (1.5) and \(v|_\Sigma = 0\), respectively), by (4.2) and Lemma 4.1, we have

\[
\int_Q \text{div} \tilde{V} \cdot \nu dx dt = \int_Q \text{div} V \cdot \nu dx dt \\
= \lambda \mu \int_{\Sigma} \frac{\partial \psi}{\partial \nu} \left( \sum_{j,k=1}^n a^{jk} \nu_j \nu_k \right)^2 dt dx \leq 0.
\tag{4.13}
\]

On the other hand,

Left-side of \(4.12\) \(= \int_0^T \left( \int_{\Omega \setminus \omega_0} + \int_{\omega_0} \right) \left[ \varphi \left( \lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu) \right) |\nabla v|^2 \\
+ \varphi^3 \left( \lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2) \right) |v|^2 \right] dt dx \\
\geq \int_0^T \int_{\Omega \setminus \omega_0} \left[ \varphi \left( \lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu) \right) |\nabla v|^2 \\
+ \varphi^3 \left( \lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2) \right) |v|^2 \right] dt dx \\
- C \lambda \mu^2 \int_{Q_0} \varphi (|\nabla v|^2 + \lambda^2 \mu^2 \varphi^2 |v|^2) dt dx.
\tag{4.14}
\]
By the choice of $\psi$, we know that $\min_{x \in \Omega \setminus \omega_0} |\nabla \psi| > 0$. Hence, there is a $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, one can find a constant $\lambda_1 = \lambda_1(\mu)$ so that for any $\lambda \geq \lambda_1$, it holds

$$\int_0^T \int_{\Omega \setminus \omega_0} \varphi [\lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu)] |\nabla v|^2 dtdx$$

$$+ \int_Q \varphi^3 [\lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2)] |v|^2 dtdx$$

$$\geq c_0 \lambda \mu^2 \int_0^T \int_{\Omega \setminus \omega_0} \varphi (|\nabla v|^2 + \lambda^2 \mu^2 \varphi^2 |v|^2) dtdx,$$

where $c_0 = \min \left( \min_{x \in \Omega \setminus \omega_0} |\nabla \psi|^2, \min_{x \in \Omega \setminus \omega_0} |\nabla \psi|^4 \right)$ is a positive constant.

Next, note that $v = \theta z_j$ by (4.2), we have

$$z_j = \theta^{-1} (v_j - \ell_j v) = \theta^{-1} (v_j - \lambda \mu \varphi \psi_j v), \quad v_j = \theta (z_j + \ell_j z) = \theta (z_j + \lambda \mu \varphi \psi_j z).$$

(4.16)

By (4.16), we get

$$\frac{1}{C} \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 |z|^2) \leq |\nabla v|^2 + \lambda^2 \mu^2 \varphi^2 |v|^2 \leq C \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 |z|^2).$$

(4.17)

Therefore, it follows from (4.14)–(4.15) and (4.17) that

$$\lambda \mu^2 \int_Q \varphi \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 |z|^2) dtdx$$

$$= \lambda \mu^2 \int_0^T \left( \int_{\Omega \setminus \omega_0} + \int_{\omega_0} \right) \varphi \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 |z|^2) dtdx$$

$$\leq C \left\{ \int_Q \varphi [\lambda \mu^2 |\nabla \psi|^2 + \lambda O(\mu)] |\nabla v|^2 dtdxight.$$

$$+ \int_Q \varphi^3 [\lambda^3 \mu^4 |\nabla \psi|^4 + \lambda^3 O(\mu^3) + \varphi^3 O_\mu(\lambda^2)] |v|^2 dtdx$$

$$+ \lambda \mu^2 \int_{Q_0} \varphi \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 |z|^2) dtdx \right\}.$$  

(4.18)

Now, combining (4.12)–(4.13) and (4.18), we end up with

$$\lambda \mu^2 \int_Q \varphi \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 |z|^2) dtdx$$

$$\leq C \left[ \int_Q \theta^2 |Gz|^2 dtdx + \lambda \mu^2 \int_{Q_0} \varphi \theta^2 (|\nabla z|^2 + \lambda^2 \mu^2 \varphi^2 |z|^2) dtdx \right].$$

(4.19)

\textit{Step 4.} Finally, we choose a cut-off function $\zeta \in C_0^\infty (\omega; [0, 1])$ so that $\zeta \equiv 1$ on $\omega_0$. Then

$$\left[ \zeta^2 \varphi \theta^2 (1 + ib) \right](z_t^2) = \zeta^2 (1 + ib) z^2 (\varphi \theta^2) t + \zeta^2 \varphi \theta^2 (1 + ib) (1 - ib)(\bar{z} z_t + z \bar{z}_t).$$

(4.20)
By (1.6), (4.20) and noting \( \theta(0, x) = \theta(T, x) \equiv 0 \), we find

\[
0 = \int_{Q_0} \zeta^2 \left[ |(1 + ib)z|^2 (\varphi \theta^2)_t + \varphi \theta^2 (1 + ib)(\overline{\varphi z_t + z \overline{\varphi}}) \right] dtdx
\]

\[
= \int_{Q_0} \zeta^2 \theta^2 \left\{ |(1 + ib)z|^2 (\varphi_t + 2\lambda \varphi \rho_t) + \varphi (1 - ib)\overline{\varphi} \left[ -\sum_{j,k} (a^{jk} z_j)_k + Gz \right] + \varphi (1 + ib)z \left[ -\sum_{j,k} (a^{jk} \overline{z_j})_k + \overline{Gz} \right] \right\} dtdx
\]

\[
= \int_{Q_0} \theta^2 \left\{ \zeta^2 |(1 + ib)z|^2 (\varphi_t + 2\lambda \varphi \rho_t)ight.
\]

\[
+ \zeta^2 \varphi \sum_{j,k} a^{jk} [(1 - ib)z_j \overline{z_k} + (1 + ib)\overline{z_j} z_k]
\]

\[
+ \mu \zeta^2 \varphi \sum_{j,k} a^{jk} [(1 - ib)\overline{z_j} \psi_k + (1 + ib)z \overline{z_j} \psi_k]
\]

\[
+ 2\lambda \mu \zeta^2 \varphi^2 \sum_{j,k} a^{jk} [(1 - ib)\overline{z_j} \psi_k + (1 + ib)z \overline{z_j} \psi_k]
\]

\[
+ 2\zeta \varphi \sum_{j,k} a^{jk} [(1 - ib)\overline{z_j} \zeta_k + (1 + ib)z \overline{z_j} \zeta_k]
\]

\[
+ \zeta \varphi \left[ (1 - ib)\overline{z} Gz + (1 + ib)z \overline{Gz} \right] dtdx.
\]

Hence, by (4.21) and (1.2), we conclude that, for some \( \delta > 0 \),

\[
2 \int_{Q_0} \zeta^2 \varphi \theta^2 |\nabla z|^2 dx dt
\]

\[
= \int_{Q_0} \theta^2 \left\{ \zeta^2 |(1 + ib)z|^2 (\varphi_t + 2\lambda \varphi \rho_t)ight.
\]

\[
+ \mu \zeta^2 \varphi \sum_{j,k} a^{jk} [(1 - ib)\overline{z_j} \psi_k + (1 + ib)z \overline{z_j} \psi_k]
\]

\[
+ 2\lambda \mu \zeta^2 \varphi^2 \sum_{j,k} a^{jk} [(1 - ib)\overline{z_j} \psi_k + (1 + ib)z \overline{z_j} \psi_k]
\]

\[
+ 2\zeta \varphi \sum_{j,k} a^{jk} [(1 - ib)\overline{z_j} \zeta_k + (1 + ib)z \overline{z_j} \zeta_k]
\]

\[
+ \zeta \varphi \left[ (1 - ib)\overline{z} Gz + (1 + ib)z \overline{Gz} \right] dtdx
\]

\[
\leq \delta \int_{Q_0} \zeta^2 \varphi \theta^2 |\nabla z|^2 dtdx
\]

\[
+ \frac{C}{\delta} \left[ \frac{(1 + b^2)}{\lambda^2 \mu^2} \int_{Q_0} \theta^2 |Gz|^2 dtdx + \lambda^2 \mu^2 \int_{Q_0} \varphi^3 \theta^2 |z|^2 dtdx \right].
\]

Now, we choose \( \delta = 1 \). By (4.22), we conclude that

\[
\int_{Q_0} \varphi \theta^2 |\nabla z|^2 dtdx \leq C(1 + b^2) \left[ \frac{1}{\lambda^2 \mu^2} \int_{Q_0} \theta^2 |Gz|^2 dtdx + \lambda^2 \mu^2 \int_{Q_0} \varphi^3 \theta^2 |z|^2 dtdx \right].
\]
Finally, combining (4.19) and (4.23), we arrive
\[
\lambda^3\mu^4 \int_Q \varphi^3 \theta^2 |z|^2 dtdx + \lambda \mu^2 \int_Q \varphi \theta^2 |\nabla z|^2 dtdx \\
\leq C(1 + b^2) \left[ \int_Q \theta^2 |Gz|^2 dtdx + \lambda^3 \mu^4 \int_{(0,T) \times \omega} \varphi^3 \theta^2 |z|^2 dtdx \right],
\]
which gives the proof of Theorem 4.1. \( \square \)

5 Proof of the main results

In this section, we will give the proofs of Theorem 1.1 and 1.2. Thanks to the classical dual argument and the fixed point technique, proceeding as [4], Theorem 1.2 is a consequence of the observability result (1.7). Therefore, we only give here a brief proof of Theorem 1.1.

Proof of Theorem 1.1. We apply Theorem 4.1 to system (1.5). Recalling that \( q(\cdot) \in L^\infty(0, T; L^n(\Omega)) \) and using (4.2), we obtain
\[
\lambda^3\mu^4 \int_Q \varphi^3 \theta^2 |z|^2 dtdx + \lambda \mu^2 \int_Q \varphi \theta^2 |\nabla z|^2 dtdx \\
\leq C(1 + b^2) \left[ \int_Q \theta^2 |qz|^2 dtdx + \lambda^3 \mu^4 \int_{(0,T) \times \omega} \varphi^3 \theta^2 |z|^2 dtdx \right] \tag{5.1}
\]
Choosing \( \lambda \geq C(1 + b^2)(1 + |r|^2) \) and \( \mu \) large enough, from (5.1), one deduces that
\[
\int_Q \varphi^3 \theta^2 |z|^2 dtdx \leq C \int_{(0,T) \times \omega} \varphi^3 \theta^2 |z|^2 dtdx. \tag{5.2}
\]
Finally, by (5.2) and applying the usual energy estimate to system (1.5), we conclude that inequality (1.7) holds, with the observability constant \( C \) given by (1.8), which completes the proof of Theorem 1.1. \( \square \)

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