Random field Ising model: statistical properties of low-energy excitations and equilibrium avalanches

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Received 10 June 2011
Accepted 27 June 2011
Published 19 July 2011

Online at stacks.iop.org/JSTAT/2011/P07010
doi:10.1088/1742-5468/2011/07/P07010

Abstract. With respect to the usual thermal ferromagnetic transitions, the zero-temperature finite-disorder critical point of the random field Ising model (RFIM) has the peculiarity of involving some ‘droplet’ exponent θ that enters the generalized hyperscaling relation \( 2 - \alpha = \nu (d - \theta) \). In the present paper, to better understand the meaning of this droplet exponent θ beyond its role in the thermodynamics, we discuss the statistics of low-energy excitations generated by an imposed single spin-flip with respect to the ground state, as well as the statistics of equilibrium avalanches, i.e. the magnetization jumps that occur in the sequence of ground states as a function of the external magnetic field. The droplet scaling theory predicts that the distribution \( dl/l^{1+\theta} \) of the linear size \( l \) of low-energy excitations transforms into the distribution \( ds/s^{1+\theta/d_\ell} \) for the size \( s \) (number of spins) of excitations of fractal dimension \( d_\ell \) (\( s \sim l^{d_\ell} \)). In the non-mean-field region \( d < d_c \), droplets are compact \( d_\ell = d \), whereas in the mean-field region \( d > d_c \), droplets have a fractal dimension \( d_\ell = 2 \theta \) leading to the well-known mean-field result \( ds/s^{3/2} \). Zero-field equilibrium avalanches are expected to display the same distribution \( ds/s^{1+\theta/d_\ell} \). We also discuss the statistics of equilibrium avalanches integrated over the external field and finite-size behaviors. These expectations are checked numerically for the Dyson hierarchical version of the RFIM, where the droplet exponent \( \theta(\sigma) \) can be varied as a function of the effective long-range interaction \( J(r) \sim 1/r^{d+\sigma} \) in \( d = 1 \).

Keywords: classical phase transitions (experiment), critical exponents and amplitudes (theory), disordered systems (theory), avalanches (theory)

ArXiv ePrint: 1106.1742
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1. Introduction

Since its introduction by Imry and Ma [1], the random field Ising model where Ising spins \( S_i = \pm 1 \) interact ferromagnetically \( (J_{i,j} > 0) \) and experience random fields \( h_i \) of strength \( W \):

\[
E = - \sum_{\langle i,j \rangle} J_{i,j} S_i S_j - \sum_i h_i S_i
\]  

(1)

has remained one of the most studied disordered models over the years (see, for instance, the review [2] and references therein). In addition to the usual critical exponents \( (\alpha, \beta, \gamma, \nu) \) which govern respectively the singularities of the free energy \( f_{\text{sing}} \sim |t|^{2-\alpha} \), magnetization \( m \sim |t|^\beta \), susceptibility \( \chi \sim |t|^{-\gamma} \) and correlation length \( \xi \sim |t|^{-\nu} \) as in thermal ferromagnetic phase transitions, the zero-temperature phase transition that occurs at some finite-disorder strength \( W_c \) in the random field Ising model (RFIM) has the peculiarity of involving a new exponent called \( \theta > 0 \), which enters the generalized hyperscaling relation (see, for instance, the recent discussion [3] and references therein)

\[
2 - \alpha = \nu(d - \theta).
\]  

(2)

This specificity of the RFIM can be understood as follows. For the usual thermal transitions, the free-energy singularity \( F(\xi^d) \) associated to a correlated volume \( \xi^d \) is of the order of unity:

\[
F(\xi^d) \sim T.
\]  

(3)

This corresponds to the density singularity:

\[
f_{\text{sing}} \sim \frac{T}{\xi^d} \sim |T_c - T|^{\nu d} \sim |T_c - T|^{2-\alpha}
\]  

(4)

and leads to the usual hyperscaling relation \( 2 - \alpha = d \nu \). For the RFIM, however, the singular ground state energy \( E(\xi^d) \) of a correlated volume \( \xi^d \) is not of the order of unity as in equation (3), but instead grows as some power of the size \( \xi \):

\[
E_{\text{sing}}(\xi^d) \sim \xi^\theta
\]  

(5)

so that the singular energy density is governed instead by

\[
e_{\text{sing}} \sim \frac{\xi^\theta}{\xi^d} \sim |W - W_c|^{\nu(d-\theta)} \sim |W - W_c|^{2-\alpha}
\]  

(6)

leading to equation (2). As emphasized by Bray and Moore [4], the presence of \( \theta \) is directly related to the zero-temperature nature of the fixed point: for a thermal fixed point occurring at a finite \( T_c \), the fixed point corresponds to a fixed ratio \( J_L/T_c \) so that the renormalized coupling \( J_L \) is invariant under a change of scale \( L \); for the RFIM, however, the fixed point corresponds to a fixed ratio \( J_L/W_L \), where \( W_L \) represents the renormalized disorder strength. So even if this ratio is fixed, both the renormalized coupling \( J_L \) and the disorder strength \( W_L \) can actually both grow as \( L^\theta \) with the scale \( L \). This explains how \( \theta \) enters the singular part of the energy in equation (5). Another insight into the physical meaning of \( \theta \) comes from the following scaling picture: in a correlated volume of size \( \xi^d \),
the random fields correspond to an averaged external field of size
\[ h_{\text{eff}}(\xi^d) \simeq \frac{1}{\xi^d} \sum_{i=1}^{\xi^d} h_i \simeq \xi^{-d/2} \] (7)
which is expected to produce a magnetization of order
\[ m_{\text{eff}}(\xi^d) \simeq \chi h_{\text{eff}}(\xi^d) \simeq \chi \xi^{-d/2} \] (8)
and to lead to the following singularity in the energy density:
\[ e_{\text{sing}} \simeq m_{\text{eff}}(\xi^d) h_{\text{eff}}(\xi^d) \simeq \chi \xi^{-d} \simeq |W - W_c|^{-\gamma + d\nu}. \] (9)
If this analysis is correct, the exponent \( \theta \) of equation (6) should be directly related to the exponents \( \gamma \) and \( \nu \):
\[ \theta = \frac{\gamma}{\nu}. \] (10)
Since this analysis neglects possible correlations, equation (10) is usually not considered as true, but it can be converted into the following bound \[2,5\]:
\[ \theta \geq \frac{\gamma}{\nu}. \] (11)
Taking into account the identity \( 2 - \alpha = 2\beta + \gamma \) and the generalized hyperscaling relation of equation (2), the inequality (10) can be rewritten as
\[ \theta \geq \frac{d}{2} - \frac{\beta}{\nu}. \] (12)
Recent works based on nonperturbative functional renormalization \[6\] have concluded that the equality (10) does not hold in general. However, numerically, it seems difficult to obtain clear evidence, since the inequalities (11) or (12) are actually satisfied as equalities within numerical errors for the short-range model in various dimensions: in dimension \( d = 3 \), the value of \( \beta/\nu \) turns out to be extremely small, of the order of \( \beta/\nu \simeq 0.012 \) \[7\], and the droplet exponent \( \theta \simeq 1.49 \) \[7\] is not really optimized with respect to the non-optimized value \( d/2 = 3/2 \). In dimension \( d = 4 \), one has a ‘reasonable’ finite value \( \beta/\nu \simeq 0.19 \) \[8\] and the droplet exponent \( \theta \simeq 1.82 \) \[8\] remains close to \( d/2 - \beta/\nu \).

Besides this thermodynamic analysis, one expects that the exponent \( \theta \) has also the meaning of a ‘droplet exponent’ as a consequence of equation (5). In particular, it should govern the statistics of critical droplets, defined as low-energy excitations, as well as the statistics of ‘equilibrium avalanches’, i.e. the avalanches between ground states as a function of the external field \( H \). The aim of the present paper is to study in detail these statistical properties of droplets and avalanches. The various predictions are checked numerically for the one-dimensional Dyson hierarchical version, where large systems can be studied, and where the value of the exponent \( \theta \) can be varied as a function of the exponent \( \sigma \) of the long-range ferromagnetic interactions. Previous studies on these questions for the short-range model in dimension \( d = 3 \) can be found in \[9,10\] for the statistics of low-energy excitations, and in \[11,12\] for the statistics of equilibrium avalanches.

This paper is organized as follows. After a reminder on the RFIM in the presence of long-range interactions in section 2, we describe the thermodynamical properties of
the Dyson hierarchical model in section 3. In section 4, we discuss the statistics of critical droplets, i.e. of low-energy excitations generated by an imposed single spin-flip with respect to the true ground state. Section 5 is then devoted to the statistics of equilibrium avalanches, i.e. the magnetization jumps that occur in the ground state as the external field is varied. In section 6, we discuss the finite-size properties in the mean-field region $d > d_c$ where the usual finite-size scaling is known to break down. Our conclusions are summarized in section 7. In the appendix we describe how the sequence of ground states as a function of the external field can be constructed via an exact recursion for the Dyson hierarchical RFIM model.

2. Reminder on the RFIM with long-range interactions

For statistical physics models in finite dimensions $d$, a phase transition usually exists only above some lower critical dimension $d > d_l$. The thermodynamical critical exponents depend upon $d$ in a region $d_l < d < d_c$, below the upper critical dimension $d_c$, and then take their mean-field values for $d > d_c$. For the short-range RFIM, one expects, for instance, $d_l = 2$ and $d_c = 6$. Besides this short-range case, it is often convenient, both theoretically and numerically, to consider the case where the ferromagnetic coupling $J_{i,j}$ in equation (1) is long range and decays only as a power law in the distance $r = |i − j|$:  

$$J(r) \sim \frac{1}{r^{d+\sigma}}$$ (13)

where $\sigma > 0$ to have an extensive energy. Renormalization group analysis of this long-range random field model can be found in [13]–[15].

2.1. Imry–Ma argument for the lower critical dimension $d_l$

Let us consider the stability of the ferromagnetic ground state where all spins are (+), with respect to small random fields. If we flip a domain $v \sim l^d$ spins, the ferromagnetic cost scales as the double integral of equation (13):

$$E^{DW}(l) \propto l^{d-\sigma}$$ (14)

whereas the random fields may correspond to an energy gain of order

$$E^{RF}(l) = \sum_{i=1}^{l^d} h_i \propto W l^{d/2}.$$ (15)

This argument yields that the lower critical dimension is

$$d_l = 2\sigma.$$ (16)

Indeed for $d > d_l$, $E^{DW}(l) > E^{RF}(l)$ for sufficiently large $l$, so the ferromagnetic ground state is stable with respect to small random fields, and one needs a finite critical disorder $W_c > 0$ to destroy it. In contrast, for $d < d_l$, $E^{DW}(l) < E^{RF}(l)$ for sufficiently large $l$, so the ferromagnetic ground state is unstable with respect to small random fields, i.e. $W_c = 0$. In the short-range case, the domain-wall cost scales as the surface $E^{DW}(l) \propto l^{d-1}$, yielding the usual value $d_l = 2$. 

doi:10.1088/1742-5468/2011/07/P07010
2.2. Mean-field region $d > d_c$

The mean-field exponents for the thermodynamical observables at $H = 0$ in the thermodynamic limit $L \to +\infty$ are given by the usual values

\[ \begin{align*}
\epsilon_{\text{sing}}(W) &\sim |W - W_c|^{2-\alpha_{\text{MF}}} \quad \text{with } \alpha_{\text{MF}} = 0 \\
m(W) &\sim |W - W_c|^{\beta_{\text{MF}}} \quad \text{with } \beta_{\text{MF}} = \frac{1}{2} \\
\chi(W) &\sim |W - W_c|^{-\gamma_{\text{MF}}} \quad \text{with } \gamma_{\text{MF}} = 1.
\end{align*} \tag{17} \]

The Gaussian nature of the fixed point yields that the connected correlation (averaged over disorder)

\[ C_{\text{connected}}(r) = \left\langle S_0 S_r \right\rangle - \left\langle S_0 \right\rangle \left\langle S_r \right\rangle \tag{18} \]

is for $\sigma \leq 2$

\[ C_{\text{connected}}(r) \sim \int d^d q e^{iqr} \hat{C}_{\text{connected}}(q) \quad \text{with } \hat{C}_{\text{connected}}(q) \sim \frac{T}{|W - W_c| + |q|^\sigma} \tag{19} \]

where the term $|q|^\sigma$ represents the leading low-$q$ singularity of the Fourier transform of the coupling $J(r) \sim 1/r^{d+\sigma}$ (the usual short-range case corresponds to the usual term $q^2$ recovered for $\sigma \geq 2$). The term $|W - W_c|$ with power unity comes from the compatibility with the value $\gamma_{\text{MF}} = 1$ for the susceptibility since

\[ \chi(W) = \int d^d r C_{\text{connected}}(r) \sim \int d^d r \int d^d q e^{iqr} \hat{C}_{\text{connected}}(q) \]

\[ = \hat{C}_{\text{connected}}(q = 0) \sim |W - W_c|^{-1}. \tag{20} \]

The large-$r$ behavior of the connected correlation of equation (19) is thus the following power-law at criticality:

\[ C_{\text{connected}}^{W=W_c}(r) \sim \int d^d q e^{iqr} \frac{T}{q^\sigma} \propto r^{\sigma-d} = \frac{1}{r^{d-2+\eta_{\text{MF}}}} \quad \text{with } \eta_{\text{MF}} = 2 - \sigma \tag{21} \]

which leads to the following finite-size divergence of the susceptibility at $W_c$:

\[ \chi_L(W_c) = \int_0^L d^d r C_{\text{connected}}^{W=W_c}(r) = \int_0^L d^d r r^{\sigma-d} \sim L^\sigma. \tag{22} \]

Off criticality, the exponential decay of the connected correlation defines the correlation length $\xi$:

\[ C_{\text{connected}}(r) \sim \int dq e^{iqr} \frac{T}{|W - W_c| + |q|^\sigma} \propto e^{-r/\xi} \quad \text{with } \xi \sim |W - W_c|^{-\nu_{\text{MF}}} \]

and $\nu_{\text{MF}} = \frac{1}{\sigma}$. \tag{23}

The upper critical dimension $d_c$ is the dimension $d$ where the generalized hyperscaling relation of equation (2) is satisfied with the mean-field exponents:

\[ 2 - \alpha_{\text{MF}} = \nu_{\text{MF}}(d_c - \theta_{\text{MF}}) \tag{24} \]
where the mean-field exponent $\theta_{\text{MF}}$ is given by equation (10):

$$\theta_{\text{MF}} = \frac{\gamma_{\text{MF}}}{\nu_{\text{MF}}} = \frac{1}{\nu_{\text{MF}}} \tag{25}$$

leading to

$$d_c = \frac{3}{\nu_{\text{MF}}} \tag{26}$$

For $\sigma \leq 2$ where $\nu_{\text{MF}} = 1/\sigma$ (equation (23)), this yields

$$d_c(\sigma \leq 2) = 3\sigma \tag{27}$$

whereas the usual short-range value $d_c = 6$ is recovered for $\sigma \geq 2$ where $\nu_{\text{MF}} = 1/2$ and $\theta_{\text{MF}} = 2$.

3. Dyson hierarchical model with random fields

To better understand the notion of phase transition in statistical physics, Dyson [16] has introduced long ago a hierarchical ferromagnetic spin model, which can be studied via exact renormalization for probability distributions. In this approach, the hierarchical couplings are chosen to mimic effective long-range power-law couplings in real space, so that phase transitions are possible already in one dimension. This type of hierarchical model has thus attracted a great deal of interest in statistical physics, both among mathematicians [17]–[20] and among physicists [21]–[24]. In the field of quenched disordered models, Dyson hierarchical models have been introduced for spin systems with random fields [25] or with random couplings [26]–[28], as well as for Anderson localization [29]–[36]. In the following, we consider the Dyson hierarchical random field model [25].

3.1. Definition of the model

The Hamiltonian for $2^N$ spins in an exterior magnetic field $H$ is

$$\mathcal{H}_{2^N}(H; \{h_1, \ldots, h_{2^N}\}) \equiv - \sum_{i=1}^{2^N} (H + h_i) S_i$$

$$-J_1[(S_1 + S_2)^2 + (S_3 + S_4)^2 + (S_5 + S_6)^2 + (S_7 + S_8)^2 + \cdots]$$

$$-J_2[(S_1 + S_2 + S_3 + S_4)^2 + (S_5 + S_6 + S_7 + S_8)^2 + \cdots]$$

$$-J_3[(S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8)^2 + \cdots] - \cdots$$

$$-J_N(S_1 + S_2 + \cdots + S_{2^{N-1}} + S_{2^N})^2. \tag{28}$$

The random fields $h_i$ represent quenched disordered variables drawn with some distribution, for instance the box distribution of width $W$:

$$P_W^{\text{Box}}(h) = \frac{1}{W^\theta} \left( \frac{-W}{2} \leq h \leq \frac{W}{2} \right). \tag{29}$$
The ferromagnetic couplings $J_n$ are chosen to decay exponentially with the level $n$ of the hierarchy

$$J_n = \left(\frac{1}{2^{1+\sigma}}\right)^n. \quad \text{(30)}$$

To make the link with the physics of long-range one-dimensional models, it is convenient to consider that the sites $i$ of the Dyson model are displayed on a one-dimensional lattice, with a lattice spacing of unity. Then the site $i = 1$ is coupled via the coupling $J_n$ to the sites $2^{n-1} < i \leq 2^n$. At the scaling level, the hierarchical model is thus somewhat equivalent to the following power-law dependence in the real-space distance $L_n = 2^n$:

$$J_n = \left(\frac{1}{L_n}\right)^{1+\sigma}. \quad \text{(31)}$$

One thus expects that the Dyson hierarchical model will have the same essential properties as the long-range model discussed in the previous section for the case $d = 1$. In particular, the model is well defined with an extensive energy for $\sigma > 0$. The lower critical dimension $d_l = 2\sigma$ (equation (16)) coming from the Imry–Ma argument and the upper critical dimension $d_c = 3\sigma$ (equation (27)) coming from the analysis of the Gaussian fixed point are the same: so in $d = 1$, the mean-field region and the non-mean-field region exist respectively in the following domains of the parameter $\sigma$:

- mean-field region $d = 1 > d_c$ for $0 < \sigma < \frac{1}{3}$
- non-mean-field region $d_l < d = 1 < d_c$ for $\frac{1}{3} < \sigma < \frac{1}{2}$. \quad \text{(32)}

### 3.2. Value of the droplet exponent $\theta = \sigma$ in the non-mean-field region

Whereas for the short-range model, we are not aware of any conjecture concerning the value of $\theta$ in the non-mean-field region $d_l = 2 < d < d_c = 6$, Grinstein [13] has conjectured that, in the presence of long-range interactions, the exponent $\theta$ always takes the following simple value:

$$\theta = \sigma \quad \text{(33)}$$

which is consistent with various limits (in particular $d \to d_l = 2\sigma$ and $d \to d_c = 3\sigma$) and various perturbative expansions [13]–[15]. This conjecture was then shown to be wrong perturbatively at order $O(\epsilon^2)$ [14] for the long-ranged case discussed in section 2, but to be true in the Dyson hierarchical version [25].

Equation (33) can be understood via the following scaling analysis. Let us introduce the exponent $y$ governing the power-law decay of the magnetization per spin with the system size $L$ exactly at the critical point $W_c$:

$$m(W_c, L) \propto L^{-y}. \quad \text{(34)}$$

The effective magnetic field per spin resulting from the random fields scales as

$$h_{\text{eff}} L = \frac{1}{L^d} \sum_{i=1}^{L^d} h_i L \propto L^{-d/2}. \quad \text{(35)}$$

doi:10.1088/1742-5468/2011/07/P07010
The characteristic Zeeman energy of the $L^d$ spins of magnetization $M_L = L^d m(W_c, L)$ is thus

$$E_Z(W_c, L) \simeq h^\text{eff}_L M_L \propto L^{(d/2) - y}$$

whereas the ferromagnetic energy associated with the effective coupling at scale $L$ (see equation (13))

$$J^\text{eff}_L \sim L^{-d - \sigma}$$

behaves as

$$E_{\text{ferro}}(W_c, L) \simeq J^\text{eff}_L M^2_L \propto L^{d - \sigma - 2y}.$$ (38)

At the critical point, the two energies of equations (36) and (38) should remain in competition at all scales, i.e. they should have the same scaling:

$$y = \frac{d}{2} - \sigma.$$ (39)

Then, the two energies scale as

$$E_Z(W_c, L) \sim E_{\text{ferro}}(W_c, L) \sim L^\theta$$

with $\theta = \sigma$. (40)

So equation (33) relies on the fact that the renormalized ferromagnetic coupling $J^\text{eff}_L$ is directly given by the power law defining the model (equation (37)) as a consequence of the exact hierarchical structure. As a final remark, let us mention that, in the short-range case, the effective renormalized ferromagnetic coupling $J^\text{eff}_L$ cannot be simply estimated, and this is why there is no simple conjecture for the value of $\theta$.

### 3.3. Finite-size properties of thermodynamical observables

As a consequence of equation (33) that holds exactly for the Dyson hierarchical model, many finite-size behaviors exactly at criticality can be explicitly computed as a function of $\sigma$, for all $d > d_c$, i.e. both in the non-mean-field region $d < d_c$ and in the mean-field region $d > d_c$ (since in the mean-field region one has also $\theta_{\text{MF}} = \sigma$ (equations (23) and (25))). In particular, the singular part of the energy density scales as (equation (40))

$$e_{\text{sing}}(W_c, L) \sim \frac{L^\theta}{L^d} = L^{\sigma - d}.$$ (41)

the divergence of the susceptibility scales as (equations (34), (35) and (39)):

$$\chi(W_c, L) \sim \frac{m(W_c, L)}{h^\text{eff}_L} \sim L^{d/2 - y} = L^\eta.$$ (42)

The exponent of the two-point correlation may also be obtained as

$$S_Sr \sim m^2(W_c, r) \sim \frac{1}{r^{2y}} = \frac{1}{r^{d - 2 + \tilde{\eta}}} \quad \text{with } \tilde{\eta} = 2 - 2\sigma.$$ (43)

At low temperature $T$, the connected correlation involves another exponent

$$\langle S_0 S_r \rangle - \langle S_0 \rangle \langle S_r \rangle \simeq (T)^{-(d - 2 + \eta)}.$$ (44)
As a consequence of the scaling of equation (40), only a rare fraction $T/r^\theta$ can contribute to the connected correlation function, so that one expects the following shift:

$$\eta = \tilde{\eta} + \theta = 2 - \sigma.$$  \hfill (45)

3.4. Numerical results on the magnetization

As explained in the appendix, the hierarchical structure of the Dyson model allows us to write exact recursions to compute the ground states that occur as a function of the external field for each disordered sample. We have studied the following sizes $2^6 \leq L \leq 2^{21}$ with corresponding statistics of $4 \times 10^7 > n_s(L) \geq 45 \times 10^3$ independent samples.

3.4.1. Location of the critical point. Exactly at criticality, the magnetization $|m(W_c, L)|$ is expected to decay as $L^{-y}$ with $y = 1/2 - \sigma$ (see equations (34) and (39)), both in the mean-field region and in the non-mean-field region. With our numerical data, we indeed find that the curves $L^{1/2-\sigma} |m(W, L)|$ for various sizes $L$ cross more and more sharply as $L$ grows: this crossing allows us to locate the critical disorder strength $W_c$, as shown in figure 1 for the two cases $\sigma = 0.2$ and $\sigma = 0.4$. We also have data concerning the cases $\sigma = 0.1$ and $0.3$ (not shown). All numerical results presented below concern the critical point $W_c$, i.e. with our data

$$W_c(\sigma = 0.1) \simeq 50 \quad W_c(\sigma = 0.2) \simeq 23 \quad W_c(\sigma = 0.3) \simeq 13.5 \quad W_c(\sigma = 0.4) \simeq 8.8.$$ \hfill (46)

3.4.2. ‘Non-mean-field’ region $\frac{1}{3} < \sigma < \frac{1}{2}$ with usual finite-size scaling. In the ‘non-mean-field’ region, thermodynamical observables follow standard finite-size scaling forms in

doi:10.1088/1742-5468/2011/07/P07010
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Figure 2. Finite-size scaling of the magnetization at $W_c$ as a function of the external magnetic field $H$: $L^{1/2-\sigma}m(H, L)$ as a function of the scaling variable $HL^{1/2}$ (see equation (48)) for $\sigma = 0.4$ (non-mean-field region).

$(W - W_c)L^{1/\nu}$ involving the correlation length exponent $\nu$, so that the known behaviors in $L$ at criticality for the magnetization, susceptibility and singular energy yield

$$\frac{\beta}{\nu} = y = \frac{1}{2} - \sigma \quad \frac{\gamma}{\nu} = \sigma \quad \frac{2 - \alpha}{\nu} = 1 - \sigma. \quad (47)$$

Since the correlation length $\nu$ is not known exactly, the values of $(\alpha, \beta, \gamma)$ are not known either, but the ratios of equation (47) are exactly known.

Besides these good finite-size properties in $(W - W_c)L^{1/\nu}$ at $H = 0$, one also expects good finite-size properties in the variable $HL^{1/2}$ (see equation (35)) exactly at $W_c$. The following finite-size scaling form:

$$m(W_c, H \geq 0) \propto L^{\sigma - 1/2} \mathcal{M}(HL^{1/2}) \quad (48)$$

is indeed well satisfied as shown in figure 2 for the case $\sigma = 0.4$.

The finite-size properties in the mean-field region $d > d_c$ where usual finite-size scaling is known to break down will be discussed later in section 6.

In conclusion of this section, the Dyson hierarchical RFIM is a convenient model where large systems can be studied numerically with a good statistics, and where the droplet exponent $\theta = \sigma$ can be varied by choosing the parameter $\sigma$ of the effective long-range interactions. Now that we have located the zero-temperature finite-disorder critical point for various $\sigma$, we may study the statistics of low-energy excitations and of equilibrium avalanches in the remaining sections.

4. Statistics of low-energy excitations at criticality

In disordered systems, there can be states that have an energy very close to the ground state energy but which are very different from the ground state in configuration space. In
the droplet theory, developed initially for the spin-glass phase [37] and then for the frozen phase of the directed polymer in a random medium [38]–[40], the low-temperature physics is described in terms of rare regions with nearly degenerate excitations which appear with a probability that decays with a power law of their size. In these models of spin glasses or directed polymers, the droplet exponent $\theta$ is a property of the low-temperature disorder-dominated phase $T < T_c$. In the RFIM, however, the originality is that the droplet exponent $\theta$ is a property of the zero-temperature finite-disorder critical point. We have already recalled in the introduction how this exponent $\theta$ enters in the critical properties of thermodynamical observables, in particular in the generalized hyperscaling relation of equation (2). In the present section, we discuss how this droplet exponent $\theta$ governs the power-law distribution of low-energy excitations at $W_c$.

4.1. Low-energy excitations generated by an imposed spin-flip

To generate low-energy excitations above the ground state in disordered systems, various procedures have been followed in the literature (see, for instance, the various methods concerning spin glasses [41]–[45]). In the following, since we wish to generate an elementary local excitation, we have chosen the ‘single spin-flip method’, already used in [10] for the short-range 3D RFIM. The idea is the following: in each disordered sample, we first compute the true ground state. Then we impose the flip of a given spin $S_{i0}$ with respect to its orientation in the true ground state and we compute the new modified ground state when this constraint is taken into account. We measure the number $s$ of spins that are different in the two ground states. This excitation has a finite energy by construction: if only $S_{i0}$ flips, the cost is simply $\Delta E = 2|h_{loc}^{i0}|$ in terms of the local field $h_{loc}^{i0}$; if the system chooses to flip $s$ spins, it is because the energy cost is lower.

At criticality, the probability distribution $D_L^{W_c}(s)$ of the number $s$ of spins of this low-energy excitation is expected to decay as a power law in the thermodynamic limit $L \to +\infty$:

\[ D_L^{W_c}(s) \sim \frac{1}{L^{\tau_D}} \]

where $\tau_D$ is defined by this equation. For finite $L$, a finite cutoff

\[ s_s(L) \propto L^\rho \]

is expected to govern the far-exponential decay

\[ D_L^{W_c}(s) \sim e^{-s/s_s(L)}. \]  

Let us first discuss the relation between the exponent $\tau_D$ of equation (49) and the droplet exponent $\theta$.

4.2. Relation with the statistics of the linear size $l$ of droplets

From the definition of the droplet exponent $\theta$, a droplet of linear size $l$ has an energy cost of order

\[ E^{\text{droplet}}(l) \propto l^\theta u \]

\[ \text{doi:10.1088/1742-5468/2011/07/P07010} \]
where $u$ is a positive random variable of order $O(1)$ distributed with some law $p(u)$. The random variable $u$ is expected to have a zero weight $p(u = 0) > 0$ at the origin. The probability to have a droplet of linear size $l$ and of finite energy $E_{\text{droplet}}(l) < E_0$ then scales as $\text{Prob}(u < E_0/l^\theta) \simeq p(u = 0)(E_0/l^\theta)$. Taking into account the logarithmic measure $dl/l$ in the size $l$ to ensure independent droplets, one obtains the following distribution for the linear size $l$:

$$
dl P_{\text{droplet}}(l) \simeq \frac{dl}{l^{1+\theta}} E_0 p(u = 0)$$

(53)

as in other models like spin glasses [37] or directed polymers [38]–[40].

To obtain the statistics in the size $s$ (number of spins) of droplets, one needs to know the fractal dimension $d_f$ of droplets:

$$s \sim l^{d_f}.$$  

(54)

The probability distribution in $l$ of equation (53) transforms into the following power-law distribution of equation (49) with

$$\tau_D = 1 + \frac{\theta}{d_f}.$$  

(55)

In the context of spin glasses, a similar relation between the droplet exponent $\theta$ and the avalanche exponent $\tau$ (which coincides with the exponent $\tau_D$ of low-energy excitations of equation (55) as discussed in the next section) has been discussed in [56], where it is assumed that the density of droplets is $(dl/l^{1+\theta}) \times l^{d-d_f}$ which is different from equation (53) whenever the droplets are non-compact $d_f \neq d$. We believe that equation (53) is correct, and is consistent with our numerical results where we generate droplets by flipping an arbitrary spin, both in the case of compact or fractal droplets, as we now discuss.

4.3. Statistics of the size $s$ of low-energy excitations in the non-mean-field region $d < d_c$

In the non-mean-field region, one expects that droplets are compact, i.e. the fractal dimension of equation (54) is simply

$$d_f = d$$

(56)

so that equation (55) is

$$\tau_D = 1 + \frac{\theta}{d} \quad \text{for } d \leq d_c.$$  

(57)

By considering avalanches in the next section 5 (see equation (91)), we expect that the cutoff exponent $\rho$ is directly related to the droplet exponent $\theta$:

$$\rho = 2\theta.$$  

(58)

Our numerical data for the Dyson hierarchical model of the parameter $\sigma = 0.4$ are in agreement with these predictions. In figure 3(a), we show a log–log plot of the droplet distribution $D_{k,l}^W(s)$ for various $L$ and we measure the slope $\tau_D(\sigma = 0.4) = 1 + \theta = 1 + \sigma = 1.4$. In figure 3(b), we show that a satisfactory data collapse can be obtained in terms of the rescaled variable $s/L^\rho$ with $\rho(\sigma = 0.4) = 2\theta = 2\sigma = 0.8$. 

doi:10.1088/1742-5468/2011/07/P07010

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4.4. Statistics of the size $s$ of low-energy excitations in the mean-field region $d > d_c$

In the mean-field region $d > d_c$, one expects that ‘loops’ are not important, so that the probability distribution $D(s)$ to return $s$ spins to obtain the new ground state when one imposes the flip of one given spin $S_i$ should take the same form as the probability that a spin belongs to a cluster of size $s$ for the percolation problem of the Bethe lattice that has no loops [46]:

$$D_{\text{MF}}(s) = \frac{1}{s^{\tau_{\text{MF}}} e^{-s(W-W_c)^2}} \quad \text{with} \quad \tau_{\text{MF}} = \frac{3}{2}. \quad (59)$$

This means that the density $n(s)$ of clusters of size $s$ per spin is

$$n(s) = \frac{D_{\text{MF}}(s)}{s} = \frac{1}{s^{\tau_{\text{MF}}} + 1} e^{-s(W-W_c)^2}. \quad (60)$$

Let us now adapt the Coniglio scaling analysis [47] concerning the percolation transition for $d > d_c$ to our present case. In a volume $l^d$, the number $N_{\text{cl}}$ of clusters has the following singularity in terms of the cutoff $s_{\text{max}} \sim (W-W_c)^{-2}$ of equation (60):

$$N_{\text{cl}} = l^d \sum_s n(s) = l^d [s_{\text{max}}]^{-\tau_{\text{MF}}} = l^d (W-W_c)^{2\tau_{\text{MF}}}. \quad (61)$$

In particular, on a correlated volume $\xi^d = |W-W_c|^{-\nu_{\text{MF}}}$, the number of clusters has for singularity

$$N_{\xi^d} = \xi^d (W-W_c)^{2\tau_{\text{MF}}} = \xi^{d-d_c} \quad \text{with} \quad d_c = \frac{2\tau_{\text{MF}}}{\nu_{\text{MF}}} = \frac{3}{\nu_{\text{MF}}}. \quad (62)$$
As stressed by Coniglio [47], the meaning of the upper critical dimension $d_c$ is that it separates:

(i) the mean-field region $d > d_c$ where the number $N_{\xi^d}$ of clusters diverges with $\xi$ as $\xi^{d-d_c}$ and

(ii) the non-mean-field region $d < d_c$ where a correlated volume contains a single relevant cluster.

In our present long-range case where $\nu_{\text{MF}} = 1/\sigma$ (equation (23)) we recover $d_c = 3\sigma$ (equation (27)), whereas the short-range case where $\nu_{\text{MF}} = 1/2$ corresponds to $d_c = 6$.

In the mean-field region, the fact that a correlated volume contains a large number of clusters $N_{\xi^d} \gg 1$ is possible because each cluster has a fractal dimension $d_t$ given by

$$s_{\text{max}} \sim (W - W_c)^{-2} \sim \xi^{d_t} \quad \text{with} \quad d_t = \frac{2}{\nu_{\text{MF}}} = 2\theta_{\text{MF}}$$

in agreement with the relation of equation (55)

$$\tau_{\text{MF}} = 1 + \frac{\theta_{\text{MF}}}{d_t} = \frac{3}{2}.$$  \hspace{1cm} (64)

In our present long-range case where $\nu_{\text{MF}} = 1/\sigma$ (equation (23)), the fractal dimension of droplets is thus

$$d_t = \frac{2}{\nu_{\text{MF}}} = 2\sigma$$

whereas in the short-range case where $\nu_{\text{MF}} = 1/2$, the fractal dimension is $d_t = 4$.

In this mean-field region where droplets are non-compact, the only constraint on the maximal linear size of droplets is the system size $L$ itself: $l_{\text{max}} \sim L$, so we may expect that the maximal size in $s$ of avalanches in a finite system is

$$s^*(L) \sim L^{d_t} = L^{2\sigma}.$$  \hspace{1cm} (66)

Our conclusion is that, in the mean-field region, the exponent $\rho$ of the finite-size cutoff of equation (50) should be

$$\rho = d_t = 2\sigma.$$  \hspace{1cm} (67)

Our data for $\sigma = 0.1$, $\sigma = 0.2$ and $\sigma = 0.3$ are indeed compatible with the values $\tau_{\text{MF}} = 3/2$ and $\rho = 2\sigma$, as shown in figure 4 for $\sigma = 0.2$.

5. Statistics of equilibrium avalanches

Power-law avalanches occur in many domains of physics. In the field of disordered systems, non-equilibrium avalanches have been much studied, in particular in the context of driven elastic manifolds in random media (see the review [48] and references therein, as well as the more recent works [49, 50]) and in the random field Ising model (see [51]–[53] and references therein). To better understand the properties of these non-equilibrium avalanches, it seems useful to make the comparison with the equilibrium avalanches that occur in the same systems. The statistics of equilibrium avalanches has been studied for elastic manifolds in random media [55], for spin glasses [56] and for the RFIM [11, 12], where some universality
Figure 4. Distribution $D_{W_c}^L(s)$ of the size $s$ (number of spins) of critical droplets for $\sigma = 0.2$ (mean-field region): (a) $\ln D_{W_c}^L(s)$ as a function of $\ln s$ to measure the droplet exponent $\tau_D$ in equation (49): the slope corresponds to the mean-field exponent $\tau_D = \tau_D^{MF} = 3/2$, and (b) finite-size scaling $\ln(L^{\rho \tau_D^{MF}} D_{W_c}^L(s))$ as a function of $\ln(s/L^\rho)$ with $\rho = 2\sigma = 0.4$.

has been found between equilibrium and non-equilibrium avalanches [12]. In this section, we discuss the statistics of equilibrium avalanches in the RFIM.

5.1. Observables concerning equilibrium avalanches

Let us introduce the average number of avalanches of size $s$ that occur at $H$ in a system of size $L$:

$$N_L(s, H) = \sum_i \delta(H - h_{\text{flip}}(i))\delta\left(s - \frac{m_i+1 - m_i}{2}\right) \quad (68)$$

where the $h_{\text{flip}}(i)$ and the $m_i$ are the fields and the magnetization occurring in the sequence of ground states of a sample as a function of the external field (see more details in the appendix). The total number of avalanches for $-\infty < H < +\infty$ is then

$$N_L^{\text{tot}} \equiv \int_{-\infty}^{+\infty} dH \int_0^{+\infty} ds N_L(s, H). \quad (69)$$

Since during the history, each spin of the $L^d$ spins has to flip exactly once, one has the exact sum rule

$$\int_{-\infty}^{+\infty} dH \int_0^{+\infty} ds s N_L(s, H) = L^d. \quad (70)$$

In terms of these avalanches, the difference between the magnetization at $H$ and at $H = 0$ can be written as

$$M(H) - M(0) = 2 \int_0^H dh \int_0^{+\infty} ds s N_L(s, h). \quad (71)$$

doi:10.1088/1742-5468/2011/07/P07010
so that the susceptibility is

\[
\chi_L(H) = \frac{dm}{dH} = \frac{1}{L^d} \frac{dM}{dH} = \frac{1}{L^d} \int_0^{+\infty} ds \, s N_L(s, H).
\] 

(72)

5.2. Probability distribution of zero-field avalanches at \(W_c\)

To measure the probability distribution of the size \(s\) (number of spins) of avalanches exactly at \(H = 0\) (see the definition of equation (68))

\[
P_L^{(H=0)}(s) = \frac{N_L(s, H = 0)}{\int_0^{+\infty} ds \, N_L(s, H = 0)}
\] 

(73)

we have considered, in each disordered sample, the avalanche occurring at the smallest \(|h_{\text{flip}}(i)|\):

\[
h_{\text{flip}}^{\text{min}} = \min |h_{\text{flip}}(i)|.
\] 

(74)

The probability distribution \(R(h_{\text{loc}})\) of the local fields \(h_{\text{loc}}(i) \geq 0\) of spins \(S_i\) in the ground state is expected to have a finite weight \(R(h_{\text{loc}} = 0) > 0\) at the origin \(h_{\text{loc}} = 0\). Then the closest avalanche from \(H = 0\) occurs at a field of order

\[
h_{\text{flip}}^{\text{min}} \sim \frac{1}{L^d}.
\] 

(75)

Indeed, one draws \(L^d\) variables from this distribution, the minimal local field will scale as \(h_{\text{loc}}^{\text{min}} \propto 1/L^d\) from the estimate

\[
\frac{1}{L^d} = \int_0^{h_{\text{loc}}^{\text{min}}} dh \, R(h) = R(0) h_{\text{loc}}^{\text{min}}.
\] 

(76)

We have checked the validity of equation (75) for the Dyson hierarchical RFIM, both in the non-mean-field and in the mean-field regions.

When the external field \(H\) reaches this \(h_{\text{loc}}^{\text{min}} \propto 1/L^d\), at least one spin becomes unstable and it induces an avalanche of size \(s\) with some probability \(P_L(s)\). From this argument, it seems natural to expect that this \(P_L(s)\) exactly coincides with the droplet distribution \(D_L(s)\) discussed in the previous section 4 (the only difference is that, for the droplet distribution, we have chosen to force the flipping of an arbitrary spin, whereas here the field forces the flipping of the spin having the lowest local field)

\[
P_L^{(H=0)}(s) = D_L(s) = \frac{1}{s^\tau} e^{-s/s_*(L)} \quad \text{with } \tau = \tau_D
\] 

(77)

with the same finite-size cutoff \(s_*(L) \sim L^\rho\). As explained in section 4, the difference between the mean-field and the non-mean-field regions is that

\[
\tau (d < d_c) = \tau_D (d < d_c) = 1 + \frac{\theta}{d} \quad \tau (d > d_c) = \tau_D^{\text{MF}} = \frac{3}{2}.
\] 

(78)
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Figure 5. Probability distribution $P_L^{(H=0)}(s)$ of the size $s$ of avalanches at $W_c$ and $H = 0$ in the non-mean-field region for $\sigma = 0.4$ and the sizes ($2^6 \leq L \leq 2^{21}$): (a) $\ln P_L^{(H=0)}(s)$ as a function of $\ln s$: the slope corresponds to $\tau = 1 + \sigma = 1.4$, and (b) finite-size scaling analysis: $\ln(L^{\rho \tau} P_L^{(H=0)}(s))$ as a function of $\ln(s/L^{\rho})$ with $\tau = 1.4$ and $\rho = 2\sigma = 0.8$.

Figure 6. Probability distribution $P_L^{(H=0)}(s)$ of the size $s$ of avalanches at $W_c$ and $H = 0$ in the mean-field region for $\sigma = 0.2$ and the sizes ($2^6 \leq L \leq 2^{21}$): (a) $\ln P_L^{(H=0)}(s)$ as a function of $\ln s$: the slope corresponds to the exponent $\tau = \tau_{\text{MF}} = \frac{3}{2}$, and (b) finite-size scaling of the probability distribution $\ln P_L^{(H=0)}(s)$ of avalanches: $\ln(L^{\rho \tau} P_L^{(H=0)}(s))$ as a function of $\ln(s/L^{\rho})$ with $\tau = \frac{3}{2}$ and $\rho = 2\sigma = 0.4$.

Our numerical data for the Dyson hierarchical RFIM of the parameter $\sigma = 0.4$ (non-mean-field region) and $\sigma = 0.2$ (mean-field region) are in agreement with this picture, as shown in figures 5 and 6, respectively.

doi:10.1088/1742-5468/2011/07/P07010
5.3. Integrated probability distribution of avalanches for $-\infty < H < +\infty$ at $W_c$

We have also measured numerically the so-called ‘integrated’ distribution of the size $s$ of avalanches (where ‘integrated’ means ‘integrated over the external field $-\infty < H < +\infty$’):

$$P_{int}^L(s) = \frac{\int_{-\infty}^{+\infty} dH N_L(s, H)}{N_{tot}^L}$$  \hspace{1cm} (79)

where the notations $N_L(s, H)$ and $N_{tot}^L$ have been introduced in equations (68) and (69).

Let us now focus on the non-mean-field region $d < d_c$, where one expects that the statistics of avalanches of equation (68) will follow some finite-size scaling form:

$$dH \, ds \, N_L(s, H) \simeq dH \, L^a \frac{ds}{s^\tau} \Phi \left( \frac{s}{L^\rho}, H^\psi \right).$$  \hspace{1cm} (80)

The exponents $\tau$ and $\rho$ have already been introduced as the exponents that characterize the statistics of zero-field avalanches of equation (73):

$$P_{L(H=0)}^L(s) = \frac{N_L(s, H = 0)}{\int_0^{+\infty} ds \, N_L(s, H = 0)} = \frac{1}{s^\tau} \Phi \left( \frac{s}{L^\rho}, 0 \right).$$  \hspace{1cm} (81)

The exponent $a$ governs the total number $N_{tot}^L$ of avalanches (equation (69)) occurring in a sample of size $L$:

$$N_{tot}^L = \int dH \int_0^{+\infty} ds \, N_L(s, H) \simeq L^a.$$

The exponent $\psi$ describes the finite-size scaling in the external field $H$. In this non-mean-field region where avalanches are compact $s \sim l^d$, we expect that the appropriate scaling variable is $H l^{d/2} = H s^{1/2}$ so that $\psi$ takes the simple value

$$\psi = \frac{1}{2} \quad \text{for } d < d_c.$$

The integrated distribution of avalanches introduced in equation (79) can now be written in terms of the finite-size scaling form of equation (80):

$$P_{int}^L(s) = \int_{-\infty}^{+\infty} dH \, N_L(s, H) \frac{1}{N_{tot}^L} = \frac{1}{s^\tau} \int_{-\infty}^{+\infty} dH \, \phi \left( \frac{s}{L^\rho}, H^\psi \right) = \frac{1}{s^\tau + \psi} \int_{-\infty}^{+\infty} dh \, \phi \left( \frac{s}{L^\rho}, h \right).$$

In the thermodynamic limit $L \to +\infty$, the integrated distribution is also a power law:

$$P_{int}^{L\to+\infty}(s) \sim \frac{1}{s^\tau_{int}}$$  \hspace{1cm} (85)

where the exponent $\tau_{int}$ is shifted from the exponent $\tau = 1 + \theta/d$ of the zero-field avalanches (equation (77)) by the factor $\psi_{nonMF} = 1/2$ (equation (83)):

$$\tau_{int} = \tau + \psi = \frac{3}{2} + \frac{\theta}{d}.$$

Our numerical data for $\sigma = 0.4$ are in agreement with this value $\tau_{int} = \frac{3}{2} + \sigma = 1.9$ as shown in figure 7(a).
Let us now write the consistency equations that fix the values of $\rho$ and $a$. Equation (72) concerning the susceptibility yields

$$\chi_{L}(H=0) = \frac{1}{L^d} \int_{0}^{+\infty} ds s N_L(s, H = 0) = L^{a-d} \int_{0}^{+\infty} ds s^{1-\tau} \Phi\left(\frac{s}{L^{\rho}}, 0\right) \sim L^{a-d} (L^{\rho})^{2-\tau}. \tag{87}$$

From the divergence of the susceptibility at criticality $\chi_L \sim L^{\gamma/\nu} = L^\theta$ (equation (10)), we obtain the relation

$$\theta = a - d + \rho(2 - \tau). \tag{88}$$

The other consistency relation comes from the sum rule of equation (70) that is

$$L^d = \int_{-\infty}^{+\infty} dH \int_{0}^{+\infty} ds s N_L(s, H) = L^a \int_{-\infty}^{+\infty} dH \int_{0}^{+\infty} ds s^{1-\tau} \Phi\left(\frac{s}{L^{\rho}}, H s^\psi\right)$$

$$= L^a \int_{0}^{+\infty} ds s^{1-\tau-\psi} \int_{-\infty}^{+\infty} dh \Phi\left(\frac{s}{L^{\rho}}, h\right) \sim L^a (L^{\rho})^{2-\tau-\psi} \tag{89}$$

i.e. one obtains the following relation between exponents:

$$d = a + \rho(2 - \tau - \psi). \tag{90}$$

The difference with the previous relation of equation (88) yields

$$\rho = \frac{\theta}{\psi} = 2\theta \tag{91}$$

in agreement with the value measured for droplets (see figure 3(b)), for zero-field avalanches (see figure 5(b)), and for integrated avalanches (see figure 7(b)).

doi:10.1088/1742-5468/2011/07/P07010
The exponent $a$ governing the total number of avalanches (equation (82)) is then from equation (88), (91) and $\tau = 1 + \theta/d$:

$$a = \theta + d - \rho(2 - \tau) = d - \theta + 2\frac{\theta^2}{d}.$$  \hfill (92)

Besides our numerical checks concerning the Dyson hierarchical RFIM, we may also compare with the numerical results of [12] concerning the statistics of equilibrium avalanches in the short-range RFIM in $d = 3$: using the value of the droplet exponent $\theta \simeq 1.49$ [7], we expect that zero-field avalanches correspond to the exponent $\tau = 1 + \theta/d \simeq 1.5$ (unfortunately very close to the mean-field value!), and that the integrated avalanches correspond to the exponent $\tau_{\text{int}} = 3/2 + \theta/d \simeq 2$, in agreement with the value measured in [12].

In the non-mean-field region $d > d_c$ where the usual finite-size scaling forms breaks down, the analysis of finite-size properties requires a more subtle analysis, as discussed in the next section.

6. Finite-size properties in the ‘mean-field’ region $d > d_c$

As is well known, the usual finite-size scaling properties are not valid in the mean-field region $d > d_c$ (see, for instance, [57]–[61] and references therein). In particular, the equalities of equation (47) concerning thermodynamic observables are not valid anymore, because the correlation length $\xi \sim (W - W_c)^{-\nu}$ is not the only important divergent length scale in the system. These properties can be more clearly understood by the scaling theory developed by Coniglio [47] in the example of the percolation transition for $d > d_c$. In section 4.4 concerning the statistics of low-energy excitations in the mean-field region $d > d_c$, we have already started to explain how Coniglio’s approach could be adapted to the RFIM. In the present section, we continue this analysis and derive the consequences for the finite-size properties of various observables.

6.1. Finite-size properties at $H = 0$ as a function of $(W - W_c)$

In section 4.4 concerning the statistics of low-energy excitations in the mean-field region $d > d_c$, we have recalled how the irrelevance of loops for $d > d_c$ leads to equation (60) for the density of droplets, and to the fractal dimension $d_l = 2/\nu_{\text{MF}} = 2\theta_{\text{MF}}$ (equation (63)). We have already described how the number of clusters $N_{\xi^d}$ in a correlated volume $\xi^d$ grows as $\xi^{d-d_c}$ (equation (62)). This means that the correlation length $\xi$ is not the only important length scale (in contrast to the region $d < d_c$). In particular, it is clear that another important scale is the smaller length $\xi_1$ defined by the requirement that the number $N_{\xi_1^d}$ of clusters in a volume $\xi_1^d$ is one

$$N_{\xi_1^d} = \xi_1^d \sum_s n(s) = \xi_1^d \xi^{-d_c} = 1 \quad \text{yields } \xi_1 = \xi^{d_c/d} = (W - W_c)^{-\nu_{\text{MF}}d_c/d}.$$  \hfill (93)

For instance, in the short-range case where $\nu_{\text{MF}} = 1/2$ and $d_c = 6$, one has $\xi \sim (W - W_c)^{-1/2}$ and $\xi_1 \sim (W - W_c)^{-3/d}$. In the one-dimensional $(d = 1)$ long-range case where $\nu_{\text{MF}} = 1/\sigma$ and $d_c = 3\sigma$, one has $\xi \sim (W - W_c)^{-1/\sigma}$ and $\xi_1 \sim (W - W_c)^{-3}$. 

doi:10.1088/1742-5468/2011/07/P07010
In a finite system of linear size \( L \), one may thus expect three regimes:

(i) For \( L < \xi_1 \): there exists \( N_c(L) = 1 \) cluster of singular energy \( E_c(L) = L^\theta \).

(ii) For \( \xi_1 < L < \xi \): there exists \( N_c(L) = L^d/\xi_1^d \) clusters of singular energy \( E_c(L) = L^\theta \).

(iii) For \( \xi < L \): there exists \( N_c(L) = L^d/\xi_1^d \) clusters of singular energy \( E_c(L) = \xi^\theta \).

The singular energy density \( e_{\text{sing}}(L,W) \) will then present a complicated finite-size form involving the two ratios \( L/\xi_1 \) and \( \xi/L \):

\[
e_{\text{sing}}(L, W, H = 0) = \frac{N_c(L)E_c(L)}{L^d} = \frac{L^\theta}{\xi_1^d} \Phi\left( \frac{\xi_1}{L}, \frac{L}{\xi} \right)
\]

where the function \( \Phi(x_1, x_2) \) should describe the crossover between (i) and (ii) when \( x_2 = 0 \):

\[
\Phi(x_1 = 0, x_2 = 0) = 1 \quad \Phi(x_1 \to +\infty, x_2 = 0) \simeq x_1^d
\]
and should describe the crossover between (ii) and (iii) when \( x_1 = 0 \):

\[
\Phi(x_1 = 0, x_2 \to +\infty) \simeq \frac{1}{x_2^d}.
\]

Note that, in the usual thermal transitions where there is no droplet exponent \( \theta = 0 \), the crossover between (ii) and (iii) actually disappears, so that the finite-size behaviors are only governed by the single ratio \( L/\xi_1 \) as proposed in [57]–[61]. However, in the presence of the droplet exponent \( \theta > 0 \) for the RFIM, we expect that this simple recipe does not work anymore. In particular, the finite-size behavior exactly at criticality \( W_c \) (regime (i) since \( \xi_1 \) and \( \xi \) are infinite)

\[
e_{\text{sing}}(L, W_c, H) = \frac{L^\theta}{L^d} = L^{\theta-d}
\]

is not connected to the result for \( W \neq W_c \) in the thermodynamic limit \( L \to +\infty \) (regime (iii) since \( \xi_1 \) and \( \xi \) are finite)

\[
e_{\text{sing}}(L = \infty, W \neq W_c) = \frac{\xi^\theta}{\xi_1^d} = \xi^{\theta-dc} = \xi^{(1/\nu_{\text{MF}}) - (3/\nu_{\text{MF}})} = |W - W_c|^2
\]

via a single crossover.

Let us now translate this scaling analysis for the finite-size properties exactly at \( W_c \) as a function of the external field \( H \), to clarify the finite-size properties of the magnetization and of the susceptibility that can be obtained from the singularity of the energy density by successive derivation with respect to \( H \).

**6.2. Finite-size properties at \( W_c \) as a function of \( H \)**

In section 2.2, we have described the mean-field critical exponents for thermodynamic observables at \( H = 0 \). Let us now describe what happens as a function of the external field \( H \) exactly at \( W_c \) in the thermodynamic limit \( L \to +\infty \):

\[
e_{\text{sing}}(W_c, H) \sim |H|^{1/(\delta_{\text{MF}})+1} \quad m(W_c, H) \sim |H|^{1/\delta_{\text{MF}}} \quad \chi(W_c, H) \sim |H|^{1/(\delta_{\text{MF}})-1}
\]
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in terms of the exponent

$$\delta_{\text{MF}} = 1 + \frac{\gamma_{\text{MF}}}{\beta_{\text{MF}}} = 3. \quad (100)$$

To reproduce the correct divergence of the susceptibility $\chi(W_c, H) \sim |H|^{-2/3}$ via the analog of equation (20), the analog of equation (19) for the Fourier transform of the connected correlation is

$$\hat{C}_{\text{connected}}(q) \sim \frac{1}{|H|^{2/3} + |q|^\sigma} \quad \text{with} \quad \xi^{(H)} \sim |H|^{-\nu_{\text{MF}}}$$

so that the exponential decay for $H \neq 0$ is governed by

$$C_{\text{connected}}(r) \sim \int dq \, e^{iqr} \frac{1}{|H|^{2/3} + |q|^\sigma} \propto e^{-(r/\xi^{(H)})}$$

and $\nu_{\text{MF}}^{(H)} = \frac{2}{3\sigma}$. \quad (101)

The density $n(s)$ of clusters of size $s$ per spin of equation (60) becomes

$$n(s) = \frac{1}{s^{\delta_{\text{MF}} + 1}} e^{-s|H|^{4/3}} \quad \text{with} \quad \tau_{\text{MF}} = \frac{3}{2} \quad (103)$$

In a volume $l^d$, the number $N_{l^d}$ of clusters has the following singularity in terms of the cutoff $s_{\text{max}} \sim |H|^{-(4/3)}$ of equation (103):

$$N_{l^d} = l^d \sum s n(s) = l^d s_{\text{max}}^{\delta_{\text{MF}}} = l^d |H|^{(4/3)\tau_{\text{MF}}} = l^d |H|^2. \quad (104)$$

So the length scale $\xi_1$ introduced in equation (93) now diverges as

$$\xi_1^{(H)} = |H|^{-(2/d)}. \quad (105)$$

The analysis leading to equation (94) is now the same, provided one uses $\xi_1^{(H)}$ and $\xi^{(H)} \sim (\xi_1^{(H)})^{d/dc} = H^{-2/dc}$, i.e. we may write after changes of variables to make clearer the dependence upon $H$:

$$e_{\text{sing}}(L, W_c, H > 0) = L^d H^2 \mathcal{E} \left( \frac{1}{HL^{d/2}}; HL^{d/2} \right) \quad (106)$$

where the function $\mathcal{E}(y_1, y_2)$ satisfies

$$\mathcal{E}(y_1 = 0, y_2 = 0) = 1 \quad \mathcal{E}(y_1 \to +\infty, y_2 = 0) \simeq y_1^2 \quad \text{and} \quad \mathcal{E}(y_1 = 0, y_2 \to +\infty) \simeq y_2^{-2d/dc} \quad (107)$$

to reproduce the three regimes $L \ll \xi_1, \xi_1 \ll L \ll \xi$ and $\xi \ll L$:

$$e_{\text{sing}}(L, W_c, H > 0) \simeq L^{d-d} \quad \text{for} \quad L \ll H^{-2/d}$$
$$e_{\text{sing}}(L, W_c, H > 0) \simeq L^{d} H^2 \quad \text{for} \quad H^{-2/d} \ll L \ll \xi H^{-2/dc} \quad (108)$$
$$e_{\text{sing}}(L, W_c, H > 0) \simeq H^{2-2d/dc} = H^{4/3} \quad \text{for} \quad H^{-2/dc} \ll L.$$
By differentiation with respect to $H$, we thus expect the following finite-size behavior for the magnetization:

$$m(L, W_c, H > 0) = L^\theta H M \left( \frac{1}{HL^{d/2}}, HL^{d/2} \right)$$

where the function $M(y_1, y_2)$ satisfies

$$M(y_1 = 0, y_2 = 0) = 1, \quad M(y_1 \to +\infty, y_2 = 0) \simeq y_1, \quad M(y_1 = 0, y_2 \to +\infty) \simeq y_2^{-2\theta/d_c}$$

(110)

to reproduce the three regimes $L \ll \xi_1, \xi_1 \ll L \ll \xi$ and $\xi \ll L$:

$$m(L, W_c, H > 0) \simeq L^{\theta - d/2} \quad \text{for } L \ll H^{-2/d}$$

$$m(L, W_c, H > 0) \simeq L^\theta H \quad \text{for } H^{-2/d} \ll L \ll \xi H^{-2/d_c}$$

$$m(L, W_c, H > 0) \simeq H^{1/3} \quad \text{for } H^{-2/d_c} \ll L.$$  

(111)

To test the presence of the two crossovers of equation (111) for the magnetization, we have plotted our data for the Dyson hierarchical model for $\sigma = 0.2$ in terms of the ratios $L/\xi_1$ and $L/\xi$, respectively. In figure 8(a), we show the test of the scaling form involving $L/\xi_1$:

$$L^{1/2-\sigma} m(L, W_c, H > 0) = M_1 \left( HL^{1/2} \right)$$

(112)

whereas in figure 8(b), we show the test of the other scaling form involving $L/\xi$:

$$H^{-1/3} m(L, W_c, H > 0) = M_2 \left( HL^{3\sigma/2} \right).$$

(113)
By another differentiation of the magnetization of equation (111) with respect to \( H \), one obtains that the crossover at \( \xi_1 \) actually disappears in the finite-size behavior of the susceptibility (i.e. regimes (i) and (ii) give the same contribution)

\[
\chi(L, W_c, H > 0) = L^\theta G(HL^{d_c/2})
\]

where the function \( G(y) \) satisfies

\[
G(y = 0) = 1 \quad G(y_2 \to +\infty) \simeq y^{-2\theta/d_c}
\]

(115)

corresponding to the two regimes \( L \ll \xi \) and \( \xi \ll L \):

\[
\begin{align*}
\chi(L, W_c, H > 0) &\simeq L^\theta \quad \text{for } L \ll \xi H^{-2/d_c} \\
\chi(L, W_c, H > 0) &\simeq H^{-2/3} \quad \text{for } H^{-2/d_c} \ll L.
\end{align*}
\]

(116)

7. Conclusion

In this paper, we have discussed the statistics of low-energy excitations and of equilibrium avalanches for the zero-temperature finite-disorder critical point of the random field Ising model (RFIM). Besides its role in the thermodynamics, the droplet exponent \( \theta \) is expected to govern the distribution \( dl/l^{1+\theta} \) of the linear size \( l \) of low-energy excitations or zero-field avalanches. In terms of the number \( s \sim l^{d_f} \) of spins of these excitations of fractal dimension \( d_f \), the power-law distribution thus is \( ds/s^{1+\theta/d_f} \). In the non-mean-field region \( d < d_c \), droplets are compact, \( d_f = d \), whereas in the mean-field region \( d > d_c \), droplets have a fractal dimension \( d_f = 2\theta \) leading to the well-known mean-field result \( ds/s^{3/2} \). We have also discussed in detail finite-size effects, both in the non-mean-field region \( d < d_c \), where standard finite-size scaling is valid, and in the mean-field region \( d > d_c \), where standard finite-size scaling breaks down, because the correlation length is not the only relevant length scale. We have explained how to adapt Coniglio’s scaling approach to understand the finite-size properties of the RFIM for \( d > d_c \) in terms of the two scales \( \xi_1 \) and \( \xi \). All expectations have been checked numerically for the Dyson hierarchical version of the RFIM, where large systems can be studied with good statistics via exact recursion, and where the droplet exponent \( \theta \) can be varied as a function of the parameter \( \sigma \) of the effective power-law ferromagnetic coupling.

Appendix: Exact recursion for the Dyson hierarchical RFIM

In this appendix, we explain how the sequence of ground states that occur as a function of the external field \( H \) in any given disordered sample can be easily computed by recursion for the Dyson model introduced in equation (28). All numerical results presented in the text have been obtained by this method.

A.1. Exact recursion for the partition function

Following the method of [24] for the ferromagnetic model, we use the Gaussian identity

\[
e^{\left[ \beta J_N \left( \sum_{i=1}^{N} S_i \right) \right]^2} = \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{+\infty} dx \, e^{-\beta x^2 + 2J_N \sum_{i=1}^{N} S_i} \quad (A.1)
\]

doi:10.1088/1742-5468/2011/07/P07010
to rewrite the partition function for a system containing $2^N$ spins of the Hamiltonian given by equation (28):

$$Z_{2N}(\beta; H; \{h_1, \ldots, h_{2N}\}) \equiv \sum_{S_1=\pm 1 \ldots S_N=\pm 1} e^{-\beta H_{2N}(H;\{h_1, \ldots, h_{2N}\})} \quad (A.2)$$

as

$$Z_{2N}(\beta; H; \{h_1, \ldots, h_{2N}\}) = \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{+\infty} dx \, e^{-\beta x^2} Z_{2N-1}(\beta; H + 2x\sqrt{J_N}; \{h_1, \ldots, h_{2N-1}\}) \times Z_{2N-1}(\beta; H + 2x\sqrt{J_N}; \{h_{2N-1+1}, \ldots, h_{2N}\}). \quad (A.3)$$

On the right-hand side appear the partition functions of the two half-systems in the modified exterior magnetic field $H' = H + 2x\sqrt{J_N}$.

The initial condition corresponding to $N = 0$ and a single spin $2^0 = 1$ is

$$Z_1(\beta; H; \{h_1\}) = \sum_{S_1=\pm 1} e^{\beta(H+h_1)S_1} = 2 \cosh(\beta(H+h_1)). \quad (A.4)$$

A.2. Limit of zero temperature

In the limit of zero temperature where $\beta = 1/T \to +\infty$, the partition function of equation (A.2) becomes dominated by the ground state energy $E_{2N}^{GS}(H;\{h_1, \ldots, h_{2N}\})$:

$$Z_{2N}(\beta; H; \{h_1, \ldots, h_{2N}\}) \approx \frac{\beta}{\beta \to +\infty} e^{-\beta E_{2N}^{GS}(H;\{h_1, \ldots, h_{2N}\})}. \quad (A.5)$$

In the limit $\beta \to +\infty$, the recursion of equation (A.3) can thus be evaluated via the saddle-point approximation. This yields the following recursion for the ground state energy:

$$E_{2N}^{GS}(H;\{h_1, \ldots, h_{2N}\}) = \max_x [x^2 + E_{2N-1}^{GS}(H + 2x\sqrt{J_N}; \{h_1, \ldots, h_{2N-1}\}) \quad + E_{2N-1}^{GS}(H + 2x\sqrt{J_N}; \{h_{2N-1+1}, \ldots, h_{2N}\})]
\quad = \min_{H'} \left[ \frac{(H' - H)^2}{4J_N} + E_{2N-1}^{GS}(H'; \{h_1, \ldots, h_{2N-1}\}) \quad + E_{2N-1}^{GS}(H'; \{h_{2N-1+1}, \ldots, h_{2N}\}) \right] \quad (A.6)$$

The initial condition (equation (A.4)) is

$$E_1(H; \{h_1\}) = -|H + h_1|. \quad (A.7)$$

A.3. Sequence of ground states as a function of the exterior field $H$

It is convenient to characterize each disordered sample of $2^n$ spins by its ‘sequence’ of ground states as the exterior field $H$ is swept from $(-\infty)$ to $(+\infty)$: this sequence contains a certain number $(1 + p_{\text{max}})$ of configurations $\{C_0, C_1, \ldots, C_{p_{\text{max}}}\}$, where $C_0$ is the configuration where all spins are negative $S_i = -1$ (ground state when $H \to -\infty$),
and where \( C_{p_{\text{max}}} \) is the configuration where all spins are positive \( S_i = 1 \) (ground state when \( H \to +\infty \)).

The energy of each configuration \( C_p \) depends linearly on the exterior field \( H \), with a slope determined by the magnetization \( M_{C_p} = \sum_i S_i \) of the configuration \( C_p \):

\[
E(C_p, H) = -M_{C_p} H + a_{C_p}.
\]  
(A.8)

The value of the field \( H = H_{C_p, C_{p+1}} \), where the ground state changes from \( C_p \) to \( C_{p+1} \), is simply given by the intersection of the corresponding two lines (equation (A.8)):

\[
H_{C_p, C_{p+1}} = \frac{a_{C_{p+1}} - a_{C_p}}{(m_{C_{p+1}} - m_{C_p})}.
\]  
(A.9)

A.4. Notion of ‘no-passing rule’

The notion of ‘no-passing rule’ has been first developed for non-equilibrium dynamics concerning charge-density waves [62], driven elastic manifolds [63] and the non-equilibrium dynamics of the RFIM [64]. This notion has then been extended to the equilibrium of the RFIM [65,66], where it means that the sequence of ground states that appear as a function of the external field \( H \) are ‘ordered’ in magnetization

\[
M_{C_0} = -2^n < M_{C_1} < M_{C_2} < \cdots < M_{C_{p_{\text{max}}}} = 2^n.
\]  
(A.10)

Since the difference between the magnetizations of two consecutive configurations is bounded from below by the value 2, that corresponds to a single spin flip:

\[
m_{C_{p+1}} - m_{C_p} \geq 2
\]  
(A.11)

so that the number \( p_{\text{max}} \) is bounded by the number of spins

\[
1 \leq p_{\text{max}} \leq 2^n
\]  
(A.12)

i.e. it always remain very small with respect to the total number \( 2^{2n} \) of possible configurations.

A.5. Recursion on the sequence of ground states as a function of the exterior field \( H \)

Let us now assume that we know the sequences of ground states in \( H \) of two independent half-systems of size \( 2^{N-1} \), and we wish to construct the sequence for the whole system when these two half-systems are coupled.

We consider the set of pairs of ground states of the two subsystems that exist at a given same exterior field \( H' \): let us call \( I_{p_1, p_2} \) the interval of the exterior field \( H' \), where the first subsystem has for the ground state the configuration \( C_{p_1}^{(1)} \) of energy (equation (A.8))

\[
E_{2^{N-1}}^{(1)}(H') = -m_{C_{p_1}^{(1)}} H' + a_{p_1}^{(1)}
\]  
(A.13)

and where the second subsystem has for ground state the configuration \( C_{p_2}^{(2)} \) of energy

\[
E_{2^{N-1}}^{(2)}(H') = -m_{C_{p_2}^{(2)}} H' + a_{p_2}^{(2)}.
\]  
(A.14)
The recursion of equation (A.6) means that, in this interval $I_{p_1,p_2}$ of $H'$, the function that has to be minimized is

$$\phi_{I_{p_1,p_2}}(H') = \frac{(H' - H)^2}{4J_N} + E_{2N-1}^{(1)}(H') + E_{2N-1}^{(2)}(H')$$

$$= \frac{(H' - H)^2}{4J_N} - m_{C_p^{(1)}} H' + a_{p_1}^{(1)} - m_{C_p^{(2)}} H' + a_{p_2}^{(2)}.$$  \hspace{1cm} (A.15)

The minimization over $H'$

$$0 = \delta_{H'} \phi_{I_{p_1,p_2}}(H') = \frac{(H' - H)}{2J_N} - m_{C_p^{(1)}} - m_{C_p^{(2)}}$$  \hspace{1cm} (A.16)

yields the solution

$$H_s'(H) = H + 2J_N(m_{C_p^{(1)}} + m_{C_p^{(2)}})$$  \hspace{1cm} (A.17)

that corresponds to the value (equation (A.15))

$$\phi_{I_{p_1,p_2}}(H_s'(H)) = J_N(m_{C_p^{(1)}} + m_{C_p^{(2)}})^2$$

$$- (m_{C_p^{(1)}} + m_{C_p^{(2)}})(H + 2J_N(m_{C_p^{(1)}} + m_{C_p^{(2)}})) + a_{p_1}^{(1)} + a_{p_2}^{(2)}$$

$$= -J_N (m_{C_p^{(1)}} + m_{C_p^{(2)}})^2 - (m_{C_p^{(1)}} + m_{C_p^{(2)}})H + a_{p_1}^{(1)} + a_{p_2}^{(2)}$$

$$\equiv E_{2N}(C = (C_p^{(1)}, C_p^{(2)}), H)$$  \hspace{1cm} (A.18)

representing the energy of the global configuration $C = (C_p^{(1)}, C_p^{(2)})$ made of $C_p^{(1)}$ and $C_p^{(2)}$ for the two subsystems, when the exterior field is $H$.

We now need to minimize over all possible intervals $I_{p_1,p_2}$ for $H'$ (equation (A.6)):

$$E_{2N}^{(GS)}(H) = \min_{(C_p^{(1)}, C_p^{(2)})} \left[ E_{2N}(C = (C_p^{(1)}, C_p^{(2)}), H) \right]$$  \hspace{1cm} (A.19)

where the minimization is over all the pairs $(C_p^{(1)}, C_p^{(2)})$ of configurations that are ground states of the isolated subsystems for the same value $H'$ of the exterior field.

In practice, we have thus used the following procedure.

(i) We make the ordered list in the field $H'$ of the pairs $(C_p^{(1)}, C_p^{(2)})$ of configurations that are ground states of the two isolated subsystems. Let us call $\{C_0, C_1, \ldots, C_{q_{\text{max}}}\}$ these ‘candidate’ configurations of the whole system and compute their energies. For instance, if $C_q = (C_p^{(1)}, C_p^{(2)})$, its energy is simply

$$E_{2N}(C_q = (C_p^{(1)}, C_p^{(2)}, H)) = -M_{C_q} H + a_{C_q}$$

$$M_{C_q} \equiv M_{C_p^{(1)}} + M_{C_p^{(2)}} + a_{C_p^{(1)}}^{(1)} + a_{C_p^{(2)}}^{(2)}$$

$$a_{C_q} \equiv -J_N (m_{C_p^{(1)}} + m_{C_p^{(2)}})^2 + a_{C_p^{(1)}}^{(1)} + a_{C_p^{(2)}}^{(2)}.$$  \hspace{1cm} (A.20)

(ii) Now these energies are in competition to be the ground state of the whole system at some given exterior field $H$. Since they are ordered in magnetization, one may proceed as follows [65, 66]. We start from the two known extremal configurations: the first configuration is $C_0 = (C_0^{(1)}, C_0^{(2)})$ where all spins are negative (ground state for $H \rightarrow -\infty$) and the last configuration is $C_{\text{last}} = (C_p^{(1)}_{q_{\text{max}}}, C_p^{(2)}_{q_{\text{max}}})$ where all spins are
positive (ground state for $H \to +\infty$). We compute the crossing field $H_{0,\text{last}}$ where the two corresponding energies cross. We now compute the energies of all intermediate candidates at this crossing field $H_{0,\text{last}}$ and select the minimal value: the corresponding configuration $C_{q_a}$ is then the ground state at $H_{0,\text{last}}$. We may now iterate this procedure: we compute the crossing field $H_{0,q_a}$ and find the minimal energy at this field among the candidates $0 \leq q \leq q_a$; similarly we compute the crossing field $H_{q_a,\text{last}}$, etc. This method allows us to compute the sequence of ground states that really appear as a function of $H$ for the whole interval. To make this procedure even clearer, we now describe as an example the first step where two systems of one spin are coupled to form a system of two spins.

A.6. Example with $N = 1$ corresponding to $2^1 = 2$ spins

Each subsystem contains only one spin. So the sequence of the ground state of the first subsystem as a function of the exterior field contains only the two configurations $(C_0^{(1)}, C_1^{(1)})$ corresponding to $(S_1 = -1, S_1 = +1)$ and the energies are (equations (A.7) and (A.8))

\[
E(C_0^{(1)}, H') = -m_{C_0^{(1)}} H' + a_{C_0^{(1)}} = H' + h_1 \\
E(C_1^{(1)}, H') = -m_{C_1^{(1)}} H' + a_{C_1^{(1)}} = -H' - h_1
\]

so that the field $H' = H_{C_0^{(1)}, C_1^{(1)}}$ where the ground state changes from $C_0^{(1)}$ to $C_1^{(1)}$ is

\[
H_{C_0^{(1)}, C_1^{(1)}} = -h_1.
\]

Similarly, the sequence of the second subsystem is described by the parameters

\[
E(C_0^{(2)}, H') = -m_{C_0^{(2)}} H' + a_{C_0^{(2)}} = H' + h_2 \\
E(C_1^{(2)}, H') = -m_{C_1^{(2)}} H' + a_{C_1^{(2)}} = -H' - h_2
\]

\[
H_{C_0^{(2)}, C_1^{(2)}} = -h_2.
\]

Let us assume that $h_1 > h_2$ (otherwise exchange the labels 1 and 2). Then $H_{C_0^{(1)}, C_1^{(1)}} = -h_1 < -h_2 = H_{C_0^{(2)}, C_1^{(2)}}$ so that the ordered list of candidates for the ground states of the whole system of two spins is $(C_0 = (---); C_1 = (+-); C_2 = (++)$. The energies of these three candidates are

\[
E(C_0, H) = 2H + h_1 + h_2 - 4J_1 \\nE(C_1, H) = -h_1 + h_2 \\
E(C_2, H) = -2H - h_1 - h_2 - 4J_1.
\]

The crossing field $H_{C_0, C_2}$ where the two energies of the extremal ground states cross is

\[
H_{C_0, C_2} = \frac{h_1 + h_2}{2}
\]

and the corresponding energy is

\[
E(C_0, H_{C_0, C_2}) = E(C_2, H_{C_0, C_2}) = -4J_1.
\]
We now have to compute the energy of the intermediate candidate $C_1$ at this crossing field

$$E(C_1, H_{C_0, C_2}) = -h_1 + h_2$$  \hspace{1cm} (A.27)

to see if it is lower or higher than the crossing energy of equation (A.27).

(i) If $E(C_1, H_{C_0, C_2}) < E(C_0, H_{C_0, C_2}) = E(C_2, H_{C_0, C_2})$, then $C_1$ is indeed the true ground state at the crossing field $H_{C_0, C_2}$. So the sequence of ground states indeed contains the three configurations $(C_0, C_1, C_2)$ with the frontiers given by the crossing fields $H_{C_0, C_1}$ and $H_{C_1, C_2}$.

(ii) If $E(C_1, H_{C_0, C_2}) > E(C_0, H_{C_0, C_2}) = E(C_2, H_{C_0, C_2})$, then the candidate $C_1$ has to be eliminated. The sequence of ground states only contains the two configurations $(C_0 = (--), C_2 = ++) \text{ with a frontier given by } H_{C_0, C_2}$. Note that the condition for this direct jump between the two ferromagnetic configurations is

$$4J_1 > h_1 - h_2 (>0)$$ \hspace{1cm} (A.28)

i.e. it is more probable at low disorder as it should be.

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