FRIEDLANDER'S EIGENVALUE INEQUALITIES AND THE DIRICHLET-TO-NEUMANN SEMIGROUP

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Abstract. If Ω is any compact Lipschitz domain, possibly in a Riemannian manifold, with boundary Γ = ∂Ω, the Dirichlet-to-Neumann operator D_λ is defined on L^2(Γ) for any real λ. We prove a close relationship between the eigenvalues of D_λ and those of the Robin Laplacian ∆_µ, i.e. the Laplacian with Robin boundary conditions ∂_ν u = µu. This is used to give another proof of the Friedlander inequalities between Neumann and Dirichlet eigenvalues, λ_N^{k+1} ≤ λ_D^k, k ∈ ℕ, and to sharpen the inequality to be strict, whenever Ω is a Lipschitz domain in ℝ^d. We give new counterexamples to these inequalities in the general Riemannian setting. Finally, we prove that the semigroup generated by −D_λ, for λ sufficiently small or negative, is irreducible.

1. Introduction. Let Ω ⊂ ℝ^d be a bounded domain with ∂Ω = Γ. Let λ_D^1 < λ_D^2 ≤ λ_D^3 ≤ ··· and λ_N^1 < λ_N^2 ≤ λ_N^3 ≤ ··· be the eigenvalues of the Dirichlet and Neumann Laplacians on Ω, respectively. There is a beautiful set of inequalities discovered by Friedlander [9] which compares the elements of these two lists, namely

λ_N^{k+1} ≤ λ_D^k for all k. (1.1)

The fundamental tool in his proof is the Dirichlet-to-Neumann operator associated to ∆ − λ; his methods require that ∂Ω be at least C^1. Friedlander’s inequalities have attracted substantial attention since then, starting from a geometric recasting of his argument by the second author [19]. More recently, Filonov [8] discovered a substantially simpler proof of (1.1) based on the minimax characterization of eigenvalues, assuming only that Ω has finite measure and that the inclusion H^1(Ω) ⊂ L^2(Ω) be compact. An extension of Filonov’s ideas by Gesztesy and Mitrea [10] provides a comparison between generalized Robin and Dirichlet eigenvalues, while Safarov [24] showed how to describe all of this in a purely abstract setting involving only quadratic forms on Hilbert spaces.

The present paper is a substantially shortened version of the preprint [3], which apparently provided some motivation for [10], and hence should be placed before that paper in the chronology. We have decided to revise it for publication since we believe that the point of view espoused here is still of interest and should lead to further progress on some of the questions we consider. We return to the use
of the Dirichlet-to-Neumann operator, formulated weakly so that our argument applies on Lipschitz domains. (This is still less general than the domains considered by Filonov.) Our starting point is the folklore observation that if \( \lambda \) and \( \mu \) are real numbers, then \( \mu \) is an eigenvalue of the Dirichlet-to-Neumann operator \( \mathcal{D}_\lambda \) associated to \( \Delta - \lambda \) if and only if \( \lambda \) is an eigenvalue of the Robin Laplacian \( \Delta_\mu \), i.e. the operator \( \Delta \) on \( \Omega \) with boundary condition \( \partial_\nu u = \mu u \). We prove that \( \lambda \) depends strictly monotonically on \( \mu \), and vice versa. This has been rediscovered several times before our proof of it in [3]; it is equivalent to the monotonicity for \( \mathcal{D}_\lambda \) used by Friedlander [9], see also [19], but traces back at least as far as the paper of Grégoire, Nédélec and Planchard [11] in the mid ’70’s, though they in turn attribute the idea to earlier unpublished work of Caseau. This relationship and monotonicity was known to S.T. Yau in the ’70’s as well. In any case, this is a lovely set of ideas which deserves to be more widely appreciated and utilized. We show here that it leads directly to yet another proof of (1.1). We also show that (1.1) need not be true for general manifolds with boundary. This was already discussed in [19], and the counterexample given there is any spherical cap larger than a hemisphere. We prove here that (1.1) also fails if \( \Omega \) is the complement of a sufficiently small set in any closed manifold \( M \).

Our second goal in this paper is to present some facts about the semigroup associated to the Dirichlet-to-Neumann operator \( \mathcal{D}_\lambda \) (for any \( \lambda < \lambda_1 \)). Specifically, we prove that it is positive and irreducible. While this is somewhat disjoint from the question of eigenvalue inequalities, the proof is yet another illustration of the close link between the Robin Laplacian and \( \mathcal{D}_\lambda \). A consequence of this is that the first eigenvalue of \( \mathcal{D}_\lambda \) is simple and has a strictly positive eigenfunction. Note that this irreducibility of \( T \) requires only that \( \Omega \) be connected, though its boundary may have several components. This reflects the non-local nature of \( \mathcal{D}_\lambda \).

We mention also the recent paper [4] which considers a number of issues related to the ones here. For general information about eigenvalue problems we refer to [14] and [15].

We shall be brief since various of the papers cited above contain good expositions of all the background material needed here, as well as the history of eigenvalue inequalities preceding (1.1). The next section contains a short review of the correspondence between coercive symmetric forms and self-adjoint operators and the weak formulation of normal derivatives on Lipschitz domains, and then records the quadratic forms underlying the various operators we study in this paper. §3 describes the eigenvalue monotonicity and its application to the proof of the eigenvalue inequalities. The Dirichlet-to-Neumann semigroup is the subject of §4.

2. The Robin Laplacian and Dirichlet-to-Neumann operator. Let \( H \) be an infinite dimensional separable Hilbert space and \( V \) another Hilbert space which is embedded as a dense subspace in \( H \), so that \( V \subset H \subset V^* \). Suppose that \( a \) is a closed, symmetric, real-valued, coercive quadratic form, i.e.
\[
a(u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad \text{for all } u \in V
\]
for some \( \omega \in \mathbb{R} \) and \( \alpha > 0 \). Associated to \( a \) is a bounded operator \( A_1 : V \to V^* \). Also associated to \( a \) is an unbounded self-adjoint operator \( A_2 \) on \( H \) with domain \( D(A_2) \subset V \subset H \). Thus \( x \in D(A_1) \) and \( A_1 x = y \in V^* \) if and only if \( a(x,v) = \langle y,v \rangle \) for all \( v \in V \). The operator \( A_2 \) is the part of \( A_1 \) in \( D(A_2) \), and hence we simply write either operator as \( A \) and drop the subscript. The form \( a \) is accretive (i.e. \( a(u) \geq 0 \) for all \( u \in V \)) if and only if \( A \) is nonnegative (i.e. \( \langle Au,u \rangle_H \geq 0 \) for all
These definitions are set so that Green’s formula still holds:

\[ \lambda_n = \sup_{V \in \mathcal{G}_{n-1}(V)} \inf \{ a(u) : u \in V_{n-1}, \|u\| = 1 \} \tag{2.1} \]

where \( \mathcal{G}_{n-1}(V) \) denotes the set of all subspaces of \( V \) of codimension \( n - 1 \).

Let \( (\Omega, g) \) be a compact Riemannian manifold with Lipschitz boundary. In other words, we assume that \( \Omega \) is a connected, compact subset in a larger smooth manifold \( M \), that the metric \( g \) on \( \Omega \) is the restriction of a smooth metric on \( M \), and that \( \Gamma = \partial \Omega \) is locally a Lipschitz graph such that \( \Omega \) lies locally on one side of \( \Gamma \). (The results below extend in a straightforward manner if we only assume that \( M \) has a \( C^{1,1} \) structure and that the metric \( g \) is Lipschitz.) We refer to [12], [13], [17], [18] for more about the (straightforward) generalizations of the analytic facts used in this paper from the setting of Lipschitz domains in \( \mathbb{R}^d \) to domains in manifolds.

The volume form and gradient for \( g \) lead naturally to the Hilbert spaces \( L^2(\Omega) \) and \( H^1(\Omega) \), as well as the space \( L^2(\Gamma) \). As usual, \( H^1_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in \( H^1(\Omega) \). The boundary restriction map \( u \mapsto u|_\Gamma := \text{Tr} u \) is well-defined for any \( u \in H^1(\Omega) \cap C^0(\Omega) \), and this map extends to a bounded operator \( \text{Tr} : H^1(\Omega) \to L^2(\Gamma) \), with nullspace \( H^0(\Omega) \). We write \( u|_\Gamma \) or \( \text{Tr} u \) interchangeably.

We next recall the weak formulations of well-known operators and identities.

a) If \( u \in H^1(\Omega) \), we say that \( \Delta u \in L^2(\Omega) \) if there exists \( f \in L^2(\Omega) \) such that

\[ \int_\Omega \nabla u \cdot \nabla v \, dV_g = \int_\Omega f v \, dV_g \quad \text{for all } v \in H^1_0(\Omega). \]

b) Suppose that \( u \in H^1(\Omega) \) and \( \Delta u \in L^2(\Omega) \). We say that \( \partial_\nu u \in L^2(\Gamma) \) if there exists \( b \in L^2(\Gamma) \) such that

\[ \int_\Omega (\nabla u \cdot \nabla v - \Delta u \nu) \, dV_g = \int_\Gamma b v \, d\sigma_g \quad \text{for all } v \in H^1(\Omega), \]

and we then write \( \partial_\nu u = b \).

To be explicit, our conventions are that \( \Delta = -\text{div} \nabla \) and \( \nu \) is the outer unit normal; also, \( dV_g \) and \( d\sigma_g \) are the volume forms on \( \Omega \) and \( \Gamma \) associated to \( g \). Here and later we often omit the trace signs under the integral, e.g. simply write \( \int_\Gamma b v = \int_\Gamma b v|_\Gamma \).

These definitions are set so that Green’s formula still holds:

\[ \int_\Omega (\nabla u \cdot \nabla v - \Delta u \nu) \, dV_g \Rightarrow \int_\Gamma \partial_\nu u v \, d\sigma_g \]

for all \( v \in H^1(\Omega) \) whenever \( u \in H^1(\Omega) \), \( \Delta u \in L^2(\Omega) \) and \( \partial_\nu u \in L^2(\Gamma) \).

Consider the form, for any \( \mu \in \mathbb{R} \),

\[ b_\mu(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dV_g - \mu \int_\Gamma u v \, d\sigma_g, \tag{2.2} \]

for \( u, v \in H^1(\Omega) \). It is not hard to show that \( b_\mu \) is coercive, and hence determines an operator \( \Delta_\mu \). Letting \( v \in H^1_0(\Omega) \) shows that \( \Delta_\mu \) is just the standard Laplacian in the interior, and we then deduce that \( u \in \mathcal{D}(\Delta_\mu) \) implies \( \partial_\nu u = \mu u \), at least in the weak sense. Thus, altogether,

\[ \mathcal{D}(\Delta_\mu) = \{ u \in L^2(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u \text{ exists and } \partial_\nu u = \mu u|_\Gamma \}. \]
The special case $\mu = 0$ corresponds to the Neumann Laplacian $\Delta^N$.

We next consider the form
\[
b_{-\infty}(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dv,
\]
for $u, v \in H^1_0(\Omega)$. The discussion in the next section motivates why the moniker $b_{-\infty}$ is reasonable. The coercivity of this form is obvious, and its corresponding operator is the Dirichlet Laplacian $\Delta^D$.

Since $H^1(\Omega)$ is compactly included in $L^2(\Omega)$, each of these operators has discrete spectrum. We write
\[
\sigma(\Delta_\mu) = \{ \lambda_j(\mu) \}_{j=1}^\infty, \quad \sigma(\Delta^D) = \{ \lambda_j^D \}_{j=1}^\infty, \quad \text{and} \quad \sigma(\Delta^N) = \{ \lambda_j^N \}_{j=1}^\infty.
\]
Thus, $\lambda_j(0) = \lambda_j^N$, whereas $\lim_{\mu \to -\infty} \lambda_j(\mu) = \lambda_j^D$ (see Proposition 3.2 below). Hence the Robin eigenvalues interpolate between the Dirichlet and Neumann eigenvalues.

We now define, for each $\lambda \in \mathbb{R}$, the Dirichlet-to-Neumann operator $D_\lambda$. If $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$, then the classical definition is that if $g$ is a (sufficiently smooth) function on $\Gamma$ and $u$ is the unique function on $\Omega$ such that $(\Delta - \lambda)u = 0$, $\text{Tr}u = g$, then
\[
D_\lambda g = \partial_{\nu} u|_{\Gamma}.
\]
Note that $u$ is indeed uniquely defined if and only if $\lambda \notin \sigma(\Delta^D)$. There are several equivalent ways to circumvent this apparent need to avoid the Dirichlet eigenvalues. The first and most classical is simply to consider the Cauchy data subspace, sometimes also called the Calderon subspace, which is defined for any $\lambda \in \mathbb{R}$ by
\[
C_\lambda = \{ (g, h) \in L^2(\Gamma) \times L^2(\Gamma) : \exists u \in H^1(\Omega) \text{ such that } \Delta u = \lambda u, u|_{\Gamma} = g, \partial_{\nu} u = h \}.
\]
It follows from Proposition 1 below that $C_\lambda$ is a closed subspace of $L^2(\Gamma) \times L^2(\Gamma)$. If $\lambda \notin \sigma(\Delta^D)$, then $C_\lambda$ intersects $\{0\} \times L^2(\Gamma)$ only at the origin, and hence there is a densely defined closed operator $D_\lambda$ on $L^2(\Gamma)$ for which $C_\lambda$ is the graph.

We may also consider $C_\lambda$ as a multi-valued selfadjoint operator when $\lambda \in \sigma(\Delta^D)$.

In order to avoid this, we define $D_\lambda$ as follows. Let $\lambda \in \sigma(\Delta^D)$ and define $K(\lambda) := \{ h \in L^2(\Gamma) : (0, h) \in C_\lambda \}$; clearly
\[
K(\lambda) = \{ \partial_{\nu} w : w \in \ker(\lambda - \Delta^D), \partial_{\nu} w \in L^2(\Gamma) \}.
\]
Let $L^2_\perp(\Gamma) := K(\lambda)^\perp$ (the orthogonal taken in $L^2(\Gamma)$). Since $\text{dim} K(\lambda) < \infty$, $L^2_\perp(\Gamma)$ is an infinite-dimensional closed subspace of $L^2(\Gamma)$. We now let $D_\lambda$ be the unique operator on $L^2_\perp(\Gamma)$ whose graph is $C_\lambda \cap (L^2_\perp(\Gamma) \times L^2_\perp(\Gamma))$. In this way, the operator $D_\lambda$ is defined for all $\lambda \in \mathbb{R}$. It follows from our definition of the normal derivative that $D_\lambda$ is symmetric. In order to show that $D_\lambda$ is self-adjoint (i.e., that $(is - D_\lambda)$ is invertible for $s \in \mathbb{R} \setminus \{0\}$), we use the following result by Grégoire, Nédelec and Planchard [11, Proposition 1].

Proposition 1. Fix $\lambda \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$. Then given any $h \in L^2(\Gamma)$, there exists a unique $u \in H^1(\Omega)$ which satisfies
\[
\Delta u = \lambda u, \quad isu|_{\Gamma} - \partial_{\nu} u = h.
\]
This solution $u$ is uniquely determined by the condition that
\[
\int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega uv = \int_\Gamma ((isu|_{\Gamma} - h)v|_{\Gamma}.
\]
for all $v \in H^1(\Omega)$.

For $h \in L^2(\Gamma)$, let $R(is)h = u_{|\Gamma}$ where $u$ is the solution above. By the Closed Graph Theorem, $L^2(\Gamma) \ni h \mapsto u \in H^1(\Omega)$ is bounded, so by the compactness of $\text{Tr}: H^1(\Omega) \to L^2(\Gamma)$, we see that $R(is)$ is compact on $L^2(\Gamma)$. We let $L^2_\Delta(\Gamma) = L^2(\Gamma)$ if $\lambda \notin \sigma(\Delta^D)$. The relationship with $D_\lambda$ is as follows.

**Proposition 2.** The operator $D_\lambda$ is selfadjoint for every $\lambda \in \mathbb{R}$. In fact, for $s \in \mathbb{R} \setminus \{0\}$ the resolvent is given by

$$(is - D_\lambda)^{-1} h = R(is)h \quad (h \in L^2_\lambda(\Gamma)).$$

In particular, $D_\lambda$ has compact resolvent.

**Proof.** Let $h \in L^2_\lambda(\Gamma)$, $R(is)h = g$, and define the corresponding $u \in H^1(\Omega)$ as in Proposition 1 with $u_{|\Gamma} = g$. We claim that $g \in K(\lambda)^\perp$. In fact, let $\partial_\nu w \in K(\lambda)$, where $w \in \ker(\lambda - \Delta^D)$. Since $w \in H^1_0(\Omega)$, it follows from the last identity in Proposition 1 for $v := w$

$$0 = \int_\Omega \nabla u \cdot \nabla w - \lambda \int_\Omega uv = \int_\Omega \nabla u \cdot \nabla w - \int_\Omega u \Delta w$$

$$= \int_\Gamma u_{|\Gamma} \partial_\nu w = \langle g, \partial_\nu w \rangle_{L^2(\Gamma)}.$$  

Thus $g \in K(\lambda)^\perp = L^2_\Delta(\Gamma)$.

Since $isg - \partial_\nu u = h \in L^2_\Delta(\Gamma)$, it follows that $\partial_\nu u \in L^2_\Delta(\Gamma)$ as well. Moreover, since $\partial_\nu u = isg - h$ one has $(g, isg - h) \in C_\lambda \cap (L^2_\Delta(\Gamma) \times L^2_\Delta(\Gamma))$. Thus $g \in D(D_\lambda)$ and $D_\lambda g = isg - h$. 

**Lemma 2.1.** Let $(g, h) \in C_\lambda$. Then $g \in D(D_\lambda)$. If $(g, h) = 0$, then $\langle D_\lambda g, g \rangle = 0$.

**Proof.** Let $(g, h) \in C_\lambda$. We show that $g \in L^2_\lambda(\Gamma)$. By definition, there exists $u \in H^1(\Omega)$ such that $\Delta u = \lambda u, u_{|\Gamma} = g, \partial_\nu u = h$. Let $k \in K(\lambda)$, we have to show that $(k, g) = 0$. There exists $w \in \ker(\lambda - \Delta^D)$ such that $k = \partial_\nu w$. Thus $(k, g) = \int_\Gamma \partial_\nu w u_{|\Gamma} - \int_\Omega \lambda w \Delta u = 0$ since $w \in H^1_0(\Omega)$. This proves the claim. Since $C_\lambda$ is closed, also $K(\lambda)$ is a closed subspace of $L^2(\Gamma)$. Thus we can write $h = h_0 + h_1$ with $h_0 \in L^2_\lambda(\Gamma), h_1 \in K(\lambda)$. Hence $g \in D(D_\lambda)$ and $D_\lambda g = h_0$. Now assume that $(g, h) = 0$. Since $g \in L^2_\lambda(\Gamma)$ one has $(g, h_1) = 0$. Consequently, $(g, D_\lambda g) = (g, h_0) = (g, h - h_1) = 0$. 

We will see later, in Theorem 3.1, that the operator $D_\lambda$ is bounded below. Thus its spectrum consists of eigenvalues $\alpha_k(\lambda), k = 1, 2, \ldots$, which we arrange in increasing order repeated according to multiplicity.

**Remark 1.** Using Proposition 1 Grégoire et al. [11] define the unitary operator $B(s)$ on $L^2(\Gamma)$:

$$B(s) = (is - C_\lambda)(is + C_\lambda)^{-1}$$

for $\lambda = s^2$ (where $C_\lambda$ is considered as a multi-valued operator, which is such that its resolvent is single-valued). Hence as $\lambda$ increases, the poles of $D_\lambda$ as $\lambda$ crosses a Dirichlet eigenvalue transform to a more innocuous spectral flow across the value 1.

We conclude this discussion by an alternative form definition of $D_\lambda$ in the case where $\lambda \notin \sigma(\Delta^D)$. 


Lemma 2.2. For any \( \lambda \in \mathbb{R} \setminus \sigma(\Delta^D) \), define \( H^1(\lambda) = \{ u \in H^1(\Omega) : \Delta u = \lambda u \} \). Then
\[
H^1(\Omega) = H^1(\Omega) \oplus H^1(\lambda).
\]

Proof. If \( \lambda \notin \sigma(\Delta^D) \), then \( \Delta^D - \lambda : H^1_0(\Omega) \to (H^1_0(\Omega))^* \) is an isomorphism. Let \( u \in H^1(\Omega) \) and consider the element \( F \in H^1_0(\Omega)^* \) given by \( F(v) = \int_\Omega (\nabla u \cdot \nabla v - \lambda uv) \). Since \( \lambda \notin \sigma(\Delta^D) \), there exists \( u_0 \in H^1_0(\Omega) \) such that \( (\Delta^D - \lambda)u_0 = F \). Thus \( u_1 := u - u_0 \in H^1(\lambda) \), and hence \( u = u_0 + u_1 \in H^1_0(\Omega) + H^1(\lambda) \). We have now shown that \( H^1(\Omega) = H^1_0(\Omega) + H^1(\lambda) \). Since \( \lambda \notin \sigma(\Delta^D) \) one has \( H^1_0(\Omega) \cap H^1(\lambda) = \{0\} \). The fact that \( H^1(\Omega) \) is the topological direct sum of these two spaces follows from the open mapping theorem.

Since \( \text{Tr} : H^1(\lambda) \to L^2(\Gamma) \) is injective and \( H^1(\lambda) \hookrightarrow L^2(\Omega) \) is compact, it is not difficult to show that there exist \( \alpha > 0, \omega \geq 0 \) such that
\[
\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 + \omega \int_\Gamma |u|^2 \geq \alpha \|u\|_{H^1}^2
\]
for every \( u \in H^1(\lambda) \). Define
\[
a_\lambda(u|_\Gamma, v|_\Gamma) = \int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega uv.
\]
Then \( a_\lambda \) is a closed, symmetric form on \( L^2(\Gamma) \) and \( D_\lambda \) is the associated self-adjoint operator. We refer to [3] for more details.

3. Eigenvalue comparison. We now recall the relationship between the eigenvalues \( \lambda_k(\mu) \) of \( \Delta_\mu \) and \( \alpha_k(\lambda) \) of \( D_\lambda \).

Theorem 3.1. Let \( \lambda, \mu \in \mathbb{R} \). Then
\(a\) \ \( \mu \in \sigma(D_\lambda) \iff \lambda \in \sigma(\Delta_\mu) \);  
\(b\) \ \( \dim \ker(\mu - D_\lambda) = \dim \ker(\lambda - \Delta_\mu) \).

Proof. Both assertions follow from the fact that the mapping
\[
S : \ker(\Delta_\mu - \lambda) \longrightarrow \ker(\Delta_\lambda - \mu), \quad u \mapsto \text{Tr} u
\]
is an isomorphism.

To prove this, let \( u \in \ker(\Delta_\mu - \lambda) \). Then \( b_\mu(u, v) = \lambda \int_\Omega uv \) for all \( v \in H^1(\Omega) \), i.e.
\[
\int_\Omega (\nabla u \cdot \nabla v - \lambda uv) = \mu \int_\Gamma uv
\]
for all \( v \in H^1(\Omega) \). This implies that \( (\text{Tr} u, \mu \text{Tr} u) \in C_\lambda \). Let \( \partial_\nu w \in K(\lambda) \) where \( w \in \ker(\lambda - \Delta^D) \). Then
\[
\int_\Gamma \partial_\nu w \text{Tr} u = \int_\Omega \nabla w \cdot \nabla u - \int_\Omega \Delta w \overline{v} = \int_\Omega \nabla w \cdot \nabla u - \lambda \int_\Omega w \overline{v} = \int_\Omega w \Delta u - \lambda \int_\Omega w \overline{v} = 0
\]
since \( w \in H^1_0(\Omega) \) and \( \Delta u = \lambda u \). Thus \( \text{Tr} u \in K(\lambda)^\perp = L^2(\Gamma) \). Hence \( \text{Tr} u \in D(D_\lambda) \) and \( D_\lambda \text{Tr} u = \mu \text{Tr} u \). Thus \( S \) is well-defined.

Next, \( S \) is injective, since if \( u \in \ker(\Delta_\mu - \lambda) \) is such that \( \text{Tr} u = 0 \), then \( u = 0 \) by Proposition 1. To show surjectivity, let \( \varphi \in \ker(D_\lambda - \mu) \). Then there exists
u ∈ H^1(Ω) such that ∆u = λu, ∂_ν u = μTr u, φ = Tr u. Thus u ∈ D(Δ_μ) and Δ_μ u = λu.

We now describe how the Robin eigenvalues λ_k(μ) vary with μ.

**Proposition 3.** For each k, the function λ_k(μ) is strictly decreasing and satisfies

\[
\lim_{μ \to -∞} λ_k(μ) = λ_k^D, \quad \lim_{μ \to ∞} λ_k(μ) = -∞.
\]

**Proof.** It follows from the definition of b_μ and the max-min definition of eigenvalues that λ_k is at least nonincreasing. To see that it decreases strictly, suppose that λ_k(μ_1) = λ_k(μ_2) for some μ_1 < μ_2. Then setting λ := λ_k(μ_1), it follows from Theorem 3.1 that μ ∈ σ(Δ_λ) for all μ ∈ [μ_1, μ_2]. But this is impossible since σ(Δ_λ) is discrete.

Standard eigenvalue perturbation theory shows that each λ_k is continuous in μ, and is even analytic if one follows the eigenvalue branches correctly across their crossings, see [16]). We refer to [3, Theorem 2.4] for the proof that \( \lim_{μ \to -∞} λ_k(μ) = λ_k^D \). On the other hand, if there exist k ∈ N and λ ∈ R such that λ_k(μ) > λ > −∞ for all μ ∈ R, then by Theorem 3.1, for that value of λ, σ(Δ_λ) ⊂ \{μ ∈ R, λ_j(μ) = λ, j = 1, ..., k − 1\}, which is a finite set. This is impossible. □

We are now almost in a position to reprove the Friedlander eigenvalue inequalities.

**Lemma 3.2.** If Ω ⊂ R^d is a bounded Lipschitz domain, then σ(Δ_λ) ∩ (−∞, 0) ≠ ∅ for any λ > 0.

**Proof.** Fix λ > 0 and define W := \{ω ∈ R^d : |ω|^2 = λ\}. For ω ∈ W, set u_ω(x) = e^{iωx}. Then u_ω ∈ H^1(Ω), ∆u_ω = λu_ω and (∂_ν u_ω)(x) = i(ω, ν(x))e^{iωx} on Γ. Thus (u_ω, ∂_ν u_ω) ∈ C_λ. It follows from the divergence theorem that

\[
\int_Γ g_ω ∆_ν u_ω = -i \int_Γ ⟨ω, ν(z)⟩ = 0.
\]

Now it follows from Lemma 2.1 that g_ω := u_ω_{1∗} ∈ D(Δ_λ) and

\[
⟨Δ_λ g_ω, g_ω⟩ = 0
\]

for all ω ∈ W. Suppose that D_λ is a nonnegative operator. Then for any h ∈ D(Δ_λ),

\[
⟨Δ_λ g_ω, h⟩ ≤ ⟨Δ_λ g_ω, g_ω⟩^{1/2} (Δ_λ h, h)^{1/2} = 0,
\]

which implies that D_λ g_ω = 0 for every ω ∈ W. This is a contradiction since it would mean that ker Δ_λ is infinite-dimensional. □

This same set of test functions was used in [9] and later in [8] for the same purpose.

Using this Lemma and Theorem 3.1, we now obtain the strict Friedlander inequalities.

**Theorem 3.3.** If Ω ⊂ R^d is a bounded Lipschitz domain, then

\[
λ_{k+1}^N < λ_k^D \quad \text{for all } k ∈ N.
\]

**Proof.** Suppose that λ_{k+1}^N ≥ λ_k^D for some k. Choose any λ ∈ [λ_k^D, λ_{k+1}^N]; then for any μ < 0,

\[
j ≤ k \quad \Rightarrow \quad λ_j(μ) ≤ λ_k(μ) < λ_k^D ≤ λ,
\]

\[
j ≥ k + 1 \quad \Rightarrow \quad λ_j(μ) ≥ λ_{k+1}(μ) > λ_{k+1}(0) = λ_{k+1}^N ≥ λ.
\]
Hence $\lambda_j(\mu) \neq \lambda$ for any $j \in \mathbb{N}$. It follows from Theorem 3.1 that $\mu \notin \sigma(D)$, which contradicts Lemma 3.2.

\[
\begin{array}{c}
\lambda_{k+1}(\mu) \\
\lambda_k \\
\lambda_N \\
\lambda_{k}(\mu) \\
\mu
\end{array}
\]

A quantitative version of this inequality which appears in [9] for $\lambda \notin \sigma(\Delta^D)$ can be proved by similar considerations.

**Proposition 4.** For any $\lambda \in \mathbb{R}$, define
\[
N^N(\lambda) = \text{card}\{k \in \mathbb{N} : \lambda_k^N \leq \lambda\}, \\
N^D(\lambda) = \text{card}\{k \in \mathbb{N} : \lambda_k^D \leq \lambda\}.
\]

Then $N^N(\lambda) - N^D(\lambda)$ is the number of all eigenvalues of $D$ which are $\leq 0$.

**Proof.** Let $\lambda \in \mathbb{R}$. Then
\[
\{k \in \mathbb{N} : \lambda_k^N \leq \lambda < \lambda_k^D\} = \{k \in \mathbb{N} : \exists \mu \leq 0 \text{ such that } \lambda_k(\mu) = \lambda\}.
\]

Since $N^N(\lambda) - N^D(\lambda) = \text{card}\{k \in \mathbb{N} : \lambda_k^N \leq \lambda < \lambda_k^D\}$ the claim follows from Theorem 3.1. \hfill \Box

We now consider the functions $\mu_k : (-\infty, \lambda_k^D) \to \mathbb{R}$ which are the inverses of the $\lambda_k(\mu)$, of course, and each $\mu_k$ is continuous (see [3]), strictly decreasing and satisfies
\[
\lim_{\lambda \to -\infty} \mu_k(\lambda) = \infty, \quad \lim_{\lambda \to \lambda_k^D} \mu_k(\lambda) = -\infty.
\]

Theorem 3.1 now gives the following description of the spectrum of $D$, where we use the convention $\lambda_k^D = -\infty$.

**Proposition 5.** For any $\lambda \in \mathbb{R}$, choose $n \in \mathbb{N}$ such that $\lambda_{n-1}^D \leq \lambda < \lambda_n^D$. Then
\[
\sigma(D) = \{\mu_j(\lambda) : j \geq n\}.
\]

We conclude this section with a broader class of counterexamples of (1.1) when $\Omega$ is no longer Euclidean than those presented in [19]. We first quote an old result by Rauch and Taylor which is a special case of Theorem 2.3 in [23]:

**Lemma 3.4.** Let $(M^d, g)$ be a compact Riemannian manifold, $d \geq 2$. Let $K_1 \supset K_2 \supset \ldots \supset \{a\}$ be a decreasing sequence of closed subsets with Lipschitz boundary which decrease to a point $a \in M$. Set $\Omega_n = M \setminus K_n$, and denote by $\Delta_M$ the Laplace operator on all of $M$. Then for all $k \in \mathbb{N}$,
\[
\lim_{n \to \infty} \lambda_k^D(\Omega_n) = \lim_{n \to \infty} \lambda_k^N(\Omega_n) = \lambda_k(M)
\]

**Remark 2.** The precise criterion in [23] is that the capacities of the sets $K_n$ tend to 0.
Proposition 6. Choose any $k \in \mathbb{N}$ such that $\lambda_k(M) < \lambda_{k+1}(M)$. Then for $n$ sufficiently large, $\lambda_k^N(\Omega_n) < \lambda_{k+1}^N(\Omega_n)$.

Proof. This is immediate from
\[
\lim_{n \to \infty} \lambda_{k+1}^N(\Omega_n) = \lambda_{k+1}(M) > \lambda_k(M) = \lim_{n \to \infty} \lambda_k^D(\Omega_n).
\]
\[\square\]

On the other hand, a straightforward perturbation result using the variational characterization of the eigenvalues also proves the following.

Proposition 7. Let $(M^d, g)$ be any compact Riemannian manifold and let $k \in \mathbb{N}$. Then for any $\lambda > 0$ there exists an $r_0$ which depends on $\lambda$ and $g$ such that
\[
\lambda_{k+1}^N(\Omega) < \lambda_k^D(\Omega)
\]
for any Lipschitz domain $\Omega$ in $M$ which is contained in a geodesic ball $B_{r_0}(p)$, and for all $k$ such that $\lambda_k^D(\Omega) \leq \lambda$.

Note that from this sort of perturbation argument, it is impossible to discern whether these inequalities hold for all $k$ independent of the size of $\Omega$.

4. Positivity. We now turn to a study of the semigroup generated by $-D_\lambda$ on $L^2(\Gamma)$. Some of the facts established here appear also in the paper [7].

A $C_0$-semigroup $T = (T(t))_{t \geq 0}$ on a space $L^p$ is called positive if $T(t)f \geq 0$ for all $t \geq 0$ whenever $0 \leq f \in L^p$.

Theorem 4.1. If $\lambda < \lambda_k^D$, then the semigroup generated by $-D_\lambda$ on $L^2(\Gamma)$ is positive.

Proof. If $w$ is any function, then we recall the standard notation $w^+ = \max\{w, 0\}$ and $w^- = -\min\{w, 0\}$. If $u \in H^1(\Omega)$, then both $u^\pm \in H^1(\Omega)$ as well. Let $\varphi = \text{Tr} u$, which is an element of the space $V$ consisting of all boundary traces of elements of $H^1(\Omega)$. Then the terms $\varphi^\pm$ in its decomposition are precisely the boundary traces $\text{Tr} u^\pm$, and in particular both $\varphi^\pm \in V$ as well.

By the Beurling-Deny criterion (see [6] or [22, Theorem 2.6]), the semigroup $T(t)$ is positive if and only if $a_\lambda(\varphi^+, \varphi^-) \leq 0$ for all $\varphi \in V$. Now suppose that $u \in H^1(\lambda)$, and write $u^\pm = u_0^\pm + u_1^\pm \in H_0^1(\Gamma) \oplus H^1(\lambda)$. We use this short notation here even though $u_0^\pm$ is not the positive part $u_0$ (and similarly for $u_0^+, u_1^+$). Since $u = (u_0^+ - u_0^-) + (u_1^+ - u_1^-) \in H^1(\lambda)$, i.e. $u$ has no component in $H_0^1(\Gamma)$, it follows that $u_0^+ = u_0^-$. We then compute
\[
a_\lambda(\varphi^+, \varphi^-) &= \int_\Omega (\nabla u_1^+ \cdot \nabla u_1^- - \lambda u_1^+ u_1^-) \\
&= \int_\Omega \nabla(u_1^+ + u_0) \nabla(u_1^- + u_0) \\
&\quad - \int_\Omega (\nabla u_1^+ \nabla u_0 + \nabla u_0 \nabla u_1^- + |\nabla u_0|^2) \\
&\quad - \lambda \int_\Omega ((u_1^+ + u_0)(u_1^- + u_0) - u_1^+ u_0 - u_0 u_1^- - (u_0)^2) \\
&= \int_\Omega \nabla u^+ \nabla u^- - \lambda \int_\Omega u^+ u^- - \int_\Omega |\nabla u_0|^2 + \lambda \int_\Omega (u_0)^2 \leq 0,
\]
by the Poincaré inequality. In the last identity we used that fact that
\[ \int_{\Omega} \nabla u^+_T \nabla u_0 = \lambda \int_{\Omega} u^+_T u_0 \]
since \( u^+_T \in H^1(\Omega) \).

Let \((Y, \Sigma, \nu)\) be a measure space endowed with a positive \(C_0\)-semigroup \( T \) acting on \( L^p(Y) \) for some \( 1 \leq p < \infty \). A subspace \( J \subset L^p(Y) \) is called a \textbf{closed ideal} if and only if there exists \( S \subset \Sigma \) such that \( J = \{ f \in L^p(Y) : f = 0 \ \text{a.e. on} \ \ Y \setminus S \} =: L^p(S) \). A closed ideal \( J \) is said to be \textbf{invariant} (with respect to \( T \)) if \( T(t)J \subset J \) for all \( t > 0 \). The semigroup \( T \) is called \textbf{irreducible} if the only invariant closed ideals are \( J = \{ 0 \} \) and \( J = L^p(Y) \). For any \( f \in L^p(Y) \) we write \( f > 0 \) if \( f(y) \geq 0 \ \text{a.e. and} \ \nu(\{ y \in Y : f(y) > 0 \}) > 0 \), while \( f \gg 0 \) if \( f(y) > 0 \) \ a.e. If \( T \) is holomorphic then irreducibility implies that \( T(t)f \gg 0 \) for all \( t > 0 \) and \( f > 0 \) (see \cite{20}, C-III. Theorem 3.2, (b)).

Irreducible semigroups have interesting spectral properties. Assume that \( T \) is positive and irreducible (and hence that its generator \( -B \) has compact resolvent). Denote by \( \lambda_1(B) \) the first eigenvalue of \( B \). Then the eigenspace \( \ker(\lambda_1(B) - B) \) has dimension 1. Moreover, there exists a strictly positive eigenvector \( u \); i.e. \( u \in D(B), Bu = \lambda_1(B)u \) and \( u \gg 0 \). This actually characterizes the first eigenvalue: whenever \( \lambda \in \mathbb{R} \) is an eigenvalue with positive eigenvector, then \( \lambda = \lambda_1(B) \). This set of results is frequently referred to as the Krein-Rutman Theorem, see \cite{20} for more information. The following comparison result will be used below. Let \( T \) be another \( C_0 \)-semigroup on \( L^p(Y) \) whose generator \( \bar{B} \) has compact resolvent. If
\[ T(t)f \leq \bar{T}(t)f \]
for all \( t \geq 0 \) and \( f \geq 0 \) then \( \lambda_1(B) \leq \lambda_1(\bar{B}) \). Moreover, \( \lambda_1(B) = \lambda_1(\bar{B}) \) if and only if \( B = \bar{B} \) (see \cite[Theorem 1.3]{1}).

An example of this is the semigroup generated by the Robin Laplacian, or slightly more generally, the Laplacian \( \Delta_{\beta} \) with boundary conditions \( \partial_{\nu} u = \beta u_{\nu} \) for some fixed \( \beta \in L^\infty(\Gamma) \). To make this precise, let \( \Omega \) be a compact manifold with Lipschitz boundary \( \Gamma \), as before, and fix \( \beta \in L^\infty(\Gamma) \). Then define the form
\[ a_\beta(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dV_g - \int_\Gamma \beta uv \, d\sigma_g \]  
with domain \( H^1(\Omega) \). The associated self-adjoint operator is \( \Delta_\beta \), and has domain
\[ D(\Delta_\beta) = \{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \ \partial_{\nu} u \in L^2(\Gamma), \ \partial_{\nu} u = \beta u \}. \]
Moreover, \( -\Delta_\beta \) generates a positive irreducible \( C_0 \)-semigroup \( T_\beta \) on \( L^2(\Omega) \) and if \( \bar{\beta} \leq \beta \) then
\[ 0 \leq T_\beta(t) \leq T_{\bar{\beta}}(t). \]
We refer to \cite{5} for this and further information. The Krein-Rutman Theorem shows that if \( \tilde{\beta} \leq \beta \), then
\[ \lambda_1(\Delta_\beta) = \lambda_1(\Delta_{\tilde{\beta}}) \] if and only if \( \Delta_\beta = \Delta_{\tilde{\beta}} \).

Let us return to our primary goal, which is to prove the

\textbf{Theorem 4.2.} Suppose that \( \Omega \) is connected, and let \( \lambda < \lambda_1^D \). Then the semigroup \( T \) generated by \( -D_\lambda \) on \( L^2(\Gamma) \) is irreducible.
Proof. Let Γ₁ ⊂ Γ be a Borel set and assume that the closed ideal \( L^2(Γ_1) := \{ b ∈ L^2(Γ) : b|_{Γ_2} = 0 \text{ a.e.} \} \) is invariant under \( T \), where we have set \( Γ_2 = Γ \setminus Γ_1 \). Then \( T_1(t) := T(t)|_{L^2(Γ_1)} \) is a positive \( C_0 \)-semigroup on \( L^2(Γ_1) \) and \( T_1(t) \) is compact for all \( t > 0 \). Consequently its generator \(-A_1\) has compact resolvent. Let \( μ \) be the first eigenvalue of \( A_1 \). By the Krein-Rutman Theorem there exists \( 0 < b ∈ L^2(Γ_1) \) such that \( T_1(t)b = e^{-μ t}b \ (t > 0) \). Consequently, \( T(t)b = e^{-μ t}b \) for all \( t > 0 \), and hence \( b ∈ D(A_μ) \) and \( D_μ b = μ b \). By the definition of \( D_μ \) there exists \( u ∈ H^1(Γ) \) such that \( Tr u = b \) and \( ∂_ν u = μ b \). We show that \( u ≥ 0 \). In fact,

\[
\int_Ω (\nabla u \cdot \nabla v - λ uv) = μ \int_Γ uv \tag{4.4}
\]

for all \( v ∈ H^1(Ω) \). Since \( Tr u ≥ 0 \), one has \( u^− ∈ H^1_0(Ω) \). Thus inserting \( v = u^− \) into this equation gives

\[
−\int_Ω |\nabla u^−|^2 + λ \int_Ω |u^−|^2 = 0 .
\]

Combined with the Poincaré inequality, \( ∫_Ω |\nabla u^−|^2 ≥ λ_1^D ∫_Ω |u^−|^2 \) we obtain \( λ ∫_Ω |u^−|^2 ≥ λ_1^D ∫_Ω |u^−|^2 \). Since \( λ < λ_1^D \) we deduce that \( ∫_Ω |u^−|^2 = 0 \), and hence \( u^− = 0 \), i.e. \( u ≥ 0 \). It follows that \( u ∈ D(Δ_μ) \) and \( Δ_μ u = λ u \). Since \( u ≥ 0 \) but \( u ≠ 0 \), the Krein-Rutman Theorem implies that \( λ = λ_1(μ) \) is the first eigenvalue of \( Δ_μ \). Now define \( β ∈ L^∞(Γ) \) by

\[
β(z) = \begin{cases} 
μ & z ∈ Γ_1 \\
μ_1 & z ∈ Γ_2,
\end{cases}
\]

where \( μ_1 ≠ 0 \) is chosen so that \( μ < μ_1 \). Since \( Tr u = 0 \) on \( Γ_2 \), it follows from (4.4) that

\[
a_β(u, v) = ∫_Ω ∇ u ∇ v − ∫_Γ β uv = ∫_Ω ∇ u ∇ v − μ ∫_Γ uv = λ ∫_Ω uv
\]

for all \( v ∈ H^1(Ω) \). This implies that \( u ∈ D(Δ_β) \) and \( Δ_β u = λ u \). Since \( u ≥ 0 \) (and \( u \) is nontrivial), applying the Krein-Rutman Theorem once again gives that \( λ = λ_1(Δ_β) \). We have shown that \( λ_1(Δ_β) = λ_1(Δ_μ) \). Since \( μ ≤ β \), the semigroup \( T_β \) generated by \(-Δ_β \) satisfies \( 0 ≤ T_β(t) ≤ T_μ(t) \). Now it follows from (3.3) that \( Δ_μ = Δ_β \). This implies that \( a_β = a_μ \). In particular

\[
∫_{Γ_1} μ u^2 + ∫_{Γ_2} (μ_1) u^2 = ∫_Γ μ u^2
\]

for all \( u ∈ H^1(Ω) \). Hence \( ∫_{Γ_2} u^2 dσ = 0 \) for all \( u ∈ H^1(Ω) \). In particular \( ∫_Γ u^2 1_{Γ_2} dσ = 0 \) for all \( u ∈ D(Δ_β) \). Since \( \{ Tr u : u ∈ C_0^∞(R^d) \} \) is dense in \( C(Γ) \), we see that \( ∫_Γ φ^2 1_{Γ_2} dσ = 0 \) for all \( φ ∈ C(Γ) \). This implies that the Borel measure \( 1_{Γ_2} dσ \) is 0. Hence \( σ(Γ_2) = 0 \).
Acknowledgement. The first author is most grateful to the Department of Mathematics of Stanford University for its hospitality and inspiring atmosphere during the time this work was carried out. The second author is grateful to Jean-Claude Nédélec for pointing out the paper [11]; he was supported by the NSF grant DMS-0805529.

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