The Dirichlet problem for degenerate fully nonlinear elliptic equations on Riemannian manifolds

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Abstract
We derive the existence of $C^{1,1}$-solutions to the Dirichlet problem for degenerate fully nonlinear elliptic equations on Riemannian manifolds under appropriate assumptions.

1 Introduction

Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary $\partial M$, $\bar{M} := M \cup \partial M$, with the Levi-Civita connection $\nabla$. Let $\chi$ be a smooth symmetric $(0, 2)$-tensor. Let $f$ be a symmetric function defined in an open symmetric convex cone $\Gamma \subset \mathbb{R}^n$ containing positive cone, i.e.

$$\Gamma_n := \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \} \subseteq \Gamma$$

and with vertex at the origin and with boundary $\partial \Gamma \neq \emptyset$.

In this paper, we consider the Dirichlet problem

$$f(\lambda(\nabla^2 u + \chi)) = \psi \text{ in } M, \quad u = \varphi \text{ on } \partial M$$

with degenerate right-hand side

$$\inf_M \psi = \sup_{\partial \Gamma} f, \quad (1.2)$$

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where \( \lambda(\nabla^2 u + \chi) \) denote the eigenvalues of \( \nabla^2 u + \chi \) with respect to \( g \), and
\[
\sup_{\partial \Gamma} f = \sup_{\lambda_0 \in \partial \Gamma} \lim_{\lambda \to \lambda_0} f(\lambda).
\]

In addition, some standard assumptions are imposed as follows:
\[
f \in C^\infty(\Gamma) \cap C(\overline{\Gamma}), \quad \text{where } \overline{\Gamma} = \Gamma \cup \partial \Gamma, \tag{1.3}
\]
\[
f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n, \tag{1.4}
\]
\[
f \text{ is concave in } \Gamma. \tag{1.5}
\]

For \( M = \Omega \subset \mathbb{R}^n \) and \( \chi = 0 \), the Dirichlet problem for degenerate Monge-Ampère equation has been studied by Guan [7] (with homogeneous boundary data) and Guan-Trudinger-Wang [9], which was later extended by Ivochkina-Trudinger-Wang [14] to more general degenerate Hessian equations. In a series of papers [10, 11, 13], Harvey-Lawson proposed the concepts of strict \( F \) and \( \tilde{F} \) convexity assumptions on the boundary then derived with Perron method the existence and uniqueness of continuous solutions to the Dirichlet problem which have broader forms than that of (1.1). See also the Survey [12]. For more progress and open problems on degenerate nonlinear elliptic equations, we also refer the readers to [8].

This paper is a complement to these works. We derive \( C^{1,1} \) weak solutions to the Dirichlet problem for degenerate equations on Riemannian manifolds subject to
\[
(-\kappa_1, \ldots, -\kappa_{n-1}) \in \overline{\Gamma}_\infty \text{ in } \partial M. \tag{1.6}
\]
Here \( \kappa_1, \ldots, \kappa_{n-1} \) denote the principal curvatures of the boundary, and \( \overline{\Gamma}_\infty \) is the closure of
\[
\Gamma_\infty = \left\{ \lambda' \in \mathbb{R}^{n-1} : (\lambda', \lambda_n) \in \Gamma \right\}.
\]

First we present some notion. We say \( u \in C^2(\bar{M}) \) is admissible if
\[
\lambda(\nabla^2 u + \chi) \in \Gamma \text{ in } \bar{M}.
\]
We say \( u \) is a subsolution of the Dirichlet problem (1.1) if
\[
f(\lambda(\nabla^2 u + \chi)) \geq \psi \text{ in } \bar{M}, \quad u = \varphi \text{ on } \partial M. \tag{1.7}
\]
Furthermore, it is called a strictly subsolution if
\[ f(\lambda(\nabla^2 u + \chi)) > \psi \text{ in } \bar{M}, \quad u = \varphi \text{ on } \partial M. \] (1.8)

**Theorem 1.1.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\) with smooth boundary satisfying (1.6). Suppose (1.3)-(1.5) hold. For the \(\varphi \in C^{2,1}(\partial M), \psi \in C^{1,1}(\bar{M})\) satisfying (1.2), we assume that there exists a \(C^{2,1}\) admissible strictly subsolution to the Dirichlet problem (1.1). Then the Dirichlet problem admits a weak solution \(u \in C^{1,1}(\bar{M})\) with
\[ \lambda(\nabla^2 u + \chi) \in \bar{\Gamma} \text{ in } \bar{M}, \quad \Delta u \in L^\infty(\bar{M}). \]

Condition (1.4) ensures (1.1) to be elliptic at any admissible solutions, while (1.5) implies that the operator \(F(A) = f(\lambda(A))\) is concave with respect to \(A\) when \(\lambda(A) \in \Gamma\). Consequently, according to Evans-Krylov theorem [3, 15] and classical Schauder theory, higher order estimates for admissible solutions follow from
\[ |u|_{C^2(\bar{M})} \leq C. \] (1.9)

When \(\chi = 0\) and \(M = \Omega\) is a smooth bounded domain in \(\mathbb{R}^n\), the estimate (1.9) was established by Caffarelli-Nirenberg-Spruck [1] for the Dirichlet problem under assumptions that the principal curvatures \(\kappa_1, \cdots, \kappa_{n-1}\) of \(\partial \Omega\) satisfy
\[ (\kappa_1, \cdots, \kappa_{n-1}) \in \Gamma_\infty \]
and \(f\) satisfies proper assumptions including a unbounded condition. The bounded case has been further studied by Trudinger [18] on \(\Omega \subset \mathbb{R}^n\). On general closed Riemannian manifolds, the estimate (1.9) for equation (1.1) with \(\chi = g\) has been obtained by Li [16] when the underlying manifold admits nonnegative sectional curvature, by Urbas [19] when replacing nonnegative sectional curvature assumption by certain extra assumptions on \(f\) including
\[ f_j(\lambda) \geq \delta \sum_{i=1}^n f_i(\lambda) \text{ if } \lambda_j \leq 0. \] (1.10)

The Dirichlet problem (1.1) has been studied by Guan [4], where he imposed either one of such two assumptions, beyond the subsolution assumption. We shall remark that in [16, 19, 4], condition (1.10) and the assumption of nonnegative
sectional curvature are only used to prove gradient estimate. The author [21] proposed partial uniform ellipticity then confirm the key condition (1.10) when
\[
\lim_{t \to +\infty} f(t, \lambda) > f(\mu) \text{ for any } \lambda, \mu \in \Gamma,
\]
thereby extending the a priori estimates to a large amount of Hessian type equations on Riemannian manifolds. (See [22] for more extension). However, in the literature cited above, the right-hand sides of equations are assumed to satisfy a nondegenerate condition
\[
\inf_M \psi > \sup_{\partial M} f.
\]
Consequently, the estimates established there don’t apply to degenerate equations.

In order to prove Theorem 1.1, besides the gradient estimate mentioned above, the other key issue is to bound the second order derivatives at the boundary in terms of a constant depending not on \((\delta_{\psi, f})^{-1}\), where
\[
\delta_{\psi, f} = \inf_M \psi - \sup_{\partial M} f.
\]
Such a boundary estimate is fairly delicate. To this end, following some idea from [20], we achieve such two ingredients simultaneity through deriving a quantitative boundary estimate. It is very different from previous works [7, 9, 14].

**Theorem 1.2.** Assume that \((f, \Gamma)\) satisfies (1.4), (1.5) and (1.11). Let \((M, g)\) be a compact Riemannian manifold with smooth boundary subject to (1.6). Let \(\varphi \in C^\infty(\partial M)\) and \(\psi \in C^\infty(\bar{M})\) satisfy (1.12). Suppose in addition that the Dirichlet problem (1.1) admits a \(C^2\)-admissible subsolution \(u \in C^2(\bar{M})\) satisfying (1.7). Then for any admissible solution \(u \in C^3(M) \cap C^2(\bar{M})\) to (1.1), we have
\[
\sup_{\partial M} \Delta u \leq C(1 + \sup_M |\nabla u|^2)
\]
where \(C\) is a constant depending not on \((\delta_{\psi, f})^{-1}\).

The paper is organized as follows. In Section 2 we summarize some lemmas. They are key ingredients in proof of Theorem 1.2. In Sections 3 and 4 we derive the quantitative boundary estimate. Finally, we complete the proof of Theorem 1.1 in Section 5.

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*The paper [21] is essentially extracted from [arXiv:2011.08580] and [arXiv:2101.04947].

†The paper [20] is essentially extracted from [arXiv:2203.03439] and the first parts of [arXiv:2001.09238; arXiv:2106.14837].
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2 Preliminaries

The lemma below plays an important role in the proof of Proposition 3.1.

Lemma 2.1 ([5, Lemma 2.2]). Suppose (1.4) and (1.5) hold. Let $K$ be a compact subset of $\Gamma$ and $\beta_0 > 0$. There is a constant $\epsilon > 0$ such that, for $\mu \in K$ and $\lambda \in \Gamma$, when $|\nu_\mu - \nu_\lambda| \geq \beta_0$,

$$\sum_{i=1}^{n} f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \epsilon(1 + \sum_{i=1}^{n} f_i(\lambda)). \quad (2.1)$$

Here $\nu_\lambda = Df(\lambda)/|Df(\lambda)|$ denotes the unit normal vector to the level set, $\{\lambda' \in \Gamma : f(\lambda') = f(\lambda)\}$, passing through $\lambda$, where $Df(\lambda) = (f_1(\lambda), \cdots, f_n(\lambda))$.

The following lemmas are key ingredients for quantitative boundary estimate for pure normal derivatives.

Lemma 2.2 ([20]). Let $A$ be an $n \times n$ Hermitian matrix

$$\begin{pmatrix}
  d_1 & a_1 \\
  d_2 & a_2 \\
  \vdots & \vdots \\
  d_{n-1} & a_{n-1} \\
  \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & a \\
\end{pmatrix}$$

with $d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1}$ fixed, and with $a$ variable. Denote the eigenvalues of $A$ by $\lambda = (\lambda_1, \cdots, \lambda_n)$. Let $\epsilon > 0$ be a fixed constant. Suppose that the parameter $a$ in $A$ satisfies the quadratic growth condition

$$a \geq \frac{2n - 3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + \frac{(n - 2)\epsilon}{2n - 3}. \quad (2.3)$$

Then the eigenvalues (possibly with a proper permutation) behave like

$$|d_\alpha - \lambda_\alpha| < \epsilon, \ \forall 1 \leq \alpha \leq n - 1; \quad 0 \leq \lambda_n - a < (n - 1)\epsilon.$$

†The lemma was proposed in [arXiv:2203.03439] that was reorganized as a part of [20].
The concavity of $f$ yields that

$$\text{For any } \lambda, \mu \in \Gamma, \quad \sum_{i=1}^{n} f_i(\lambda)\mu_i \geq \limsup_{t \to +\infty} f(t\mu)/t.$$  

Inspired by this observation, the author [20] introduced the following conditions:

$$\text{For any } \lambda \in \Gamma, \quad \lim_{t \to +\infty} f(t\lambda) > -\infty, \quad (2.4)$$

$$\text{For any } \lambda \in \Gamma, \quad \limsup_{t \to +\infty} f(t\lambda)/t \geq 0. \quad (2.5)$$

Obviously, it leads to

Lemma 2.3 ([20]). Suppose $f$ satisfies (1.5) and (2.5). Then

$$\sum_{i=1}^{n} f_i(\lambda)\mu_i \geq 0 \text{ for any } \lambda, \mu \in \Gamma. \quad (2.6)$$

In particular

$$\sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq 0, \quad \forall \lambda \in \Gamma. \quad (2.7)$$

If, in addition, $\sum_{i=1}^{n} f_i(\lambda) > 0$ then

$$\sum_{i=1}^{n} f_i(\lambda)\mu_i > 0 \text{ for any } \lambda, \mu \in \Gamma. \quad (2.8)$$

We now give characterizations of concave functions satisfying (1.11).

Lemma 2.4 ([20]). In the presence of (1.4) and (1.5), the following statements are equivalent to each other.

(1) $f$ satisfies (1.11).

(2) $f$ satisfies (2.4).

(3) $f$ satisfies (2.5).

(4) $f$ satisfies (2.6).

(5) $f$ satisfies (2.7).

(6) $f$ satisfies (2.8).
3 Quantitative boundary estimate for mixed derivatives

For a point \( x_0 \in \partial M \), we shall choose local coordinates

\[ x = (x_1, \ldots, x_n) \tag{3.1} \]

with origin at \( x_0 \) such that, when restricted to \( \partial M \), \( \frac{\partial}{\partial x_n} \) is normal to \( \partial M \); moreover, we assume \( g_{ij}(x_0) = \delta_{ij} \).

We will carry out the computations in such local coordinates, and set

\[ \nabla_i = \nabla_{\frac{\partial}{\partial x_i}}, \quad \nabla_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \]

with a similar convention for higher derivatives, where \( \Gamma_{ij}^k \) are the Christoffel symbols

\[ \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}. \]

**Proposition 3.1.** Suppose, in addition to (1.4), (1.5) and (1.11), that the data \( \varphi \in C^3(\partial M), \psi \in C^1(\bar{M}) \) satisfies (1.12), and that there is an admissible subsolution \( u \in C^2(\bar{M}) \). Then for any \( x_0 \in \partial M \), under local coordinate system (3.1), the admissible solution \( u \) of Dirichlet problem (1.1) must satisfy

\[ |\nabla_{\alpha n} u(x_0)| \leq C(1 + \sup_M |\nabla u|), \quad \forall 1 \leq \alpha \leq n-1 \tag{3.2} \]

where \( C \) is a uniform positive constant depending not on \((\delta_{\psi,f})^{-1}\).

3.1 Useful formulas and notation

Throughout this paper we use the notation

\[ g = \nabla^2 u + \chi, \quad \bar{g} = \nabla^2 \bar{u} + \chi. \]

By direct computation one has

\[ \nabla_{ij} \nabla_k u = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} - \Gamma_{ij}^l \frac{\partial^2 u}{\partial x_k \partial x_l}, \tag{3.3} \]
\[ \nabla_{ijk} u = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} - \frac{\partial \Gamma^l_{ij}}{\partial x_k} \frac{\partial u}{\partial x^l} - \frac{\partial \Gamma^l_{ij}}{\partial x_j} \frac{\partial u}{\partial x^l} - \frac{\partial \Gamma^l_{ij}}{\partial x_i} \frac{\partial u}{\partial x^l} - \Gamma^l_{kij} \nabla_i u - \Gamma^l_{kij} \nabla_j u - \Gamma^l_{kij} \nabla_k u. \] (3.4)

Let \( \sigma(x) \) be the distance function from \( x \in M \) to \( \partial M \), \( \rho(x) \) be the distance from \( x \) to \( x_0 \). We denote
\[ \Omega_\delta = \{ x \in M : \rho(x) < \delta \}. \]

The boundary value condition, \( u = u = \varphi \) on \( \partial M \), implies that
\[ \nabla_\alpha (u - \varphi) = \nabla_n \sigma \nabla_n (u - \varphi) \] on \( \partial M \cap \overline{\Omega_\delta} \),
(3.5)
\[ \nabla_{\alpha \beta} u = \nabla_{\alpha \beta} u + \nabla_{\alpha \beta} \sigma \nabla_n (u - u) \text{ at } x_0. \]
(3.6)

Let \( h \) be the solution to
\[ \Delta h + \text{tr}_g \chi = 0 \text{ in } M, \quad h = \varphi \text{ on } \partial M. \]
(3.7)

The maximum principle also yields
\[ u \leq u \leq h \text{ in } \partial M, \]
(3.8)

then at \( x_0 \)
\[ \nabla_n u \leq \nabla_n u \leq \nabla_n h. \]
(3.9)

In particular, we have \( C^0 \)-bound and boundary gradient estimate
\[ \sup_M |u| + \sup_{\partial M} |\nabla u| \leq C. \]
(3.10)

Under local coordinates (3.1), the tangential operator on boundary is given by
\[ \mathcal{T} = \pm \left( \frac{\partial}{\partial x_\alpha} - \eta \frac{\partial}{\partial x_n} \right), \text{ for } 1 \leq \alpha \leq n - 1, \]
(3.11)

where \( \eta = \nabla_\alpha \sigma / \nabla_n \sigma \). The (3.5) means that
\[ \mathcal{T} (u - \varphi) = 0 \text{ on } \partial M \cap \overline{\Omega_\delta}. \]
(3.12)
3.2 Completion of the proof of Proposition 3.1

Let $L$ be the linearized operator of equation (1.1) at $u$. Locally, it is given by

$$Lv = F^{ij} \nabla_{ij}v$$

for $v \in C^2(M)$, where $F^{ij} = \frac{\partial F}{\partial a_{ij}}(g)$. First, we have the following lemma.

**Lemma 3.2.** Let $u \in C^3(M) \cap C^1(\bar{M})$ be an admissible solution to equation (1.1). For some small $\delta > 0$, we have

$$|L(T(u - \varphi))| \leq C \left( 1 + (1 + \sup_M |\nabla u|) \sum_{i=1}^{n} f_i + \sum_{i=1}^{n} |\lambda_i| \right), \text{ in } \Omega_{\delta} \quad (3.13)$$

where $C$ depends on $|\varphi|_{C^3(M), \xi|_{C^1(M)}, \psi|_{C^1(M)}}$ and other known data (but not on $(\delta_{\phi, \psi})^{-1}$).

**Proof.** Differentiating the equation one has

$$F^{ij} \nabla_{ij} u = \nabla_{\alpha} \psi.$$

Combining with (3.3)-(3.4) one derives (3.13). □

Proposition 3.1 can be proved by constructing barrier functions similar to that used in [20] in complex variables. The construction of this type of barriers follows [6]. Let’s take

$$\bar{\Psi} = A_1 \sqrt{b_1(u - u)} - A_2 \sqrt{b_1 \rho^2} + A_3 \sqrt{b_1(N \sigma^2 - t \sigma)} + \frac{1}{\sqrt{b_1}} \sum_{\tau < \phi} |\nabla_{\tau}(u - \varphi)|^2 + T(u - \varphi),$$

where $b_1 = 1 + \sup_M |\nabla u|^2$.

Let $\delta > 0$ and $t > 0$ be sufficiently small such that $N \delta - t \leq 0$ (where $N$ is a positive constant sufficiently large to be determined), $\sigma$ is $C^2$ in $\Omega_{\delta}$ and

$$\frac{1}{2} \leq |\nabla \sigma| \leq 2, \quad |L \sigma| \leq C_2 \sum_{i=1}^{n} f_i, \quad \frac{|L \rho|^2}{C_2} \leq C_2 \sum_{i=1}^{n} f_i \text{ in } \Omega_{\delta} \quad (3.14)$$

and

$$L(N \sigma^2 - t \sigma) = (2N \sigma - t) L \sigma + 2NF^{ij} \nabla_{i} \sigma \nabla_{j} \sigma \text{ in } \Omega_{\delta}. \quad (3.15)$$

Furthermore, we can choose $\delta$ and $t$ small enough such that $|2N \delta - t|$ is small.
By straightforward calculation and \(|a - b|^2 \geq \frac{1}{2}|a|^2 - |b|^2\), one derives

\[
L(\sum_{r < n} |\nabla_r (u - \varphi)|^2) \geq \sum_{r < n} F^{ij}_{\mathfrak{g}_{rj}} g_{ij} - C'_1 \sqrt{b_1} \sum_{i=1}^{n} f_i |\lambda_i| - C'_1 b_1 \sum_{i=1}^{n} f_i - C'_1 \sqrt{b_1}. \tag{3.16}
\]

By [4, Proposition 2.19], there is an index \(r\) such that

\[
\sum_{r < n} F^{ij}_{\mathfrak{g}_{rj}} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2.
\]

Using Lemma 2.4 and \(\sum_{i=1}^{n} f_i (\lambda_i - \lambda_i) \geq 0\) respectively, we obtain

\[
\sum_{i=1}^{n} f_i |\lambda_i| = 2 \sum_{\lambda_i \geq 0} f_i \lambda_i - \sum_{\lambda_i \leq 0} f_i \lambda_i < 2 \sum_{\lambda_i \geq 0} f_i \lambda_i,
\]

\[
\sum_{i=1}^{n} f_i |\lambda_i| = \sum_{i=1}^{n} f_i \lambda_i - 2 \sum_{\lambda_i < 0} f_i \lambda_i < \sum_{i=1}^{n} f_i \lambda_i - 2 \sum_{\lambda_i < 0} f_i \lambda_i.
\]

In conclusion, we have

\[
\sum_{i=1}^{n} f_i |\lambda_i| \leq \frac{\epsilon}{4 \sqrt{b_1}} \sum_{i \neq r} f_i \lambda_i^2 + \frac{C \sqrt{b_1}}{\epsilon} \sum_{i=1}^{n} f_i. \tag{3.17}
\]

By (3.14), (3.15), (3.16), Lemma 3.2 and (3.17), we obtain

\[
L(\tilde{\Psi}) \geq A_1 \sqrt{b_1} L(u - u) + \frac{1}{2 \sqrt{b_1}} \sum_{i \neq r} f_i \lambda_i^2 + A_3 \sqrt{b_1} L(N \sigma^2 - t \sigma) - C_1 - C_1 \sum_{i=1}^{n} f_i |\lambda_i| - (A_2 C_2 + C_1) \sqrt{b_1} \sum_{i=1}^{n} f_i
\]

\[
\geq A_1 \sqrt{b_1} L(u - u) + 2 N A_3 \sqrt{b_1} F^{ij}_{\nabla_i \sigma \nabla_j \sigma}
\]

\[
+ A_3 (2 N \sigma - t) \sqrt{b_1} L \sigma - C \sqrt{b_1} \sum_{i=1}^{n} f_i - C. \tag{3.18}
\]

Next we will prove

\[
L \tilde{\Psi} \geq 0, \text{ in } \Omega_{\delta}
\]

for \(0 < \delta \ll 1\), if we appropriately choose \(A_1 \gg A_2 \gg A_3 > 1, N \gg 1\) and \(0 < t \ll 1\).
Case I: If $|\nu_1 - \nu_2| \geq \beta_0$, then by Lemma 2.1 we have
\[
\sum_{i=1}^{n} f_i(\lambda_i - \lambda_i) \geq \varepsilon \left(1 + \sum_{i=1}^{n} f_i\right),
\]
where we take $\beta_0 = \frac{1}{2} \min_{M} \text{dist}(\nu, \partial \Gamma_n)$ as above, $\varepsilon$ is the positive constant in Lemma 2.1. Taking $A_1 \gg A_3 \gg 1$ we obtain
\[
\mathcal{L}\tilde{\Psi} \geq 0 \text{ on } \Omega_\delta.
\]
Case II: Suppose that $|\nu_1 - \nu_2| < \beta_0$. Then $\nu_1 - \beta_0 \tilde{\mathbf{i}} \in \Gamma_n$ and
\[
f_i \geq \frac{\beta_0}{\sqrt{n}} \sum_{j=1}^{n} f_j.
\]
By (3.14), we have $|\nabla \sigma| \geq \frac{1}{2}$ in $\Omega_\delta$, then
\[
2A_1 N \sqrt{b_1} F_{ij} \sigma_i \sigma_j \geq \frac{A_1 N \beta_0}{2 \sqrt{n}} \sum_{i=1}^{n} f_i \text{ on } \Omega_\delta.
\]
On the other hand, $\mathcal{L}(u - u) \geq 0$. Thus
\[
\mathcal{L}(\tilde{\Psi}) \geq 0 \text{ on } \Omega_\delta, \text{ if } A_1 N \gg 1.
\]
The boundary value condition $u - \varphi = 0$ on $\partial M$ implies $\mathcal{T}(u - \varphi) = 0$ and $|\nabla \tau(u - \varphi)| \leq C \rho$ on $\partial M \cap \overline{\Omega}_\delta$. Thus
\[
\tilde{\Psi} = A_1 \sqrt{b_1} (u - u) - A_2 \sqrt{b_1} \rho^2 + A_3 \sqrt{b_1} (N \sigma^2 - t \tau) + \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |\nabla \tau(u - \varphi)|^2 + \mathcal{T}(u - \varphi) \leq 0, \text{ on } \partial M \cap \overline{\Omega}_\delta.
\]
Note that $\rho = \delta$ and $u - u \leq 0$ on $M \cap \partial \Omega_\delta$. Hence, if $A_2 \gg 1$ then $\tilde{\Psi} \leq 0$ on $M \cap \partial \Omega_\delta$, where we use $N \delta - t \leq 0$. Therefore $\tilde{\Psi} \leq 0$ in $\Omega_\delta$ by applying maximum principle. Together with $\tilde{\Psi}(0) = 0$, one has $\nabla \tau \tilde{\Psi}(0) \leq 0$. Thus
\[
\nabla \tau \mathcal{T}(u - \varphi)(0) \leq C'(1 + \sup_{M} |\nabla u|).
\]
Here we use (3.10). Therefore
\[
\pm \nabla_{\text{on}} u \leq C(1 + \sup_{M} |\nabla u|), \text{ at } x_0,
\]
where $C$ depends only on $|\varphi|_{C^1(M)}$, $|\mathcal{M}|_{C^2(M)}$, $|\psi|_{C^1(M)}$ and other known data (but not on $\sup_{M} |\nabla u|$).
Moreover, the constant $C$ in (3.22) does not depend on $(\delta_{\varphi,f})^{-1}$. \[11\]
4 Quantitative boundary estimate for pure normal derivatives

Given $x_0 \in \partial M$. As in (3.1), we choose local coordinate $x = (x_1, \cdots, x_n)$ with origin at $x_0$ such that, when restricted to $\partial M$, $\frac{\partial}{\partial x_n}$ is an inner normal vector to $\partial M$, and $g_{ij}(x_0) = \delta_{ij}$. Furthermore, we assume $\{g_{\alpha\beta}\}$ is diagonal at $x_0$.

When imposing assumption (1.6) on boundary we can derive

**Proposition 4.1.** Let $(M, g)$ be a compact Riemannian manifold with smooth boundary satisfying (1.6). Suppose, in addition to (1.4), (1.5), (1.11) and (1.12), that there exists an admissible subsolution $u \in C^2(\bar{M})$. Then any admissible solution $u \in C^2(\bar{M})$ of the Dirichlet problem (1.1) satisfies

$$\nabla_{nn} u(x_0) \leq C \left(1 + \sum_{\alpha=1}^{n-1} |g_{\alpha n}|^2\right), \quad \forall x_0 \in \partial M,$$

where $C$ does not depend on $(\delta_{\psi, f})^{-1}$.

**Proof.** In what follows the discussion is done at $x_0$, and the Greek letters $\alpha, \beta$ range from 1 to $n - 1$. By (3.6)

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \nabla_n (u - u) \nabla_{\alpha\beta}\sigma$$

(4.1)

The proof consists of two steps.

**Step 1.** As in [20] there exist two uniform positive constants $\varepsilon_0, R_0$ depending on $\bar{g}$ and $f$, such that

$$f(\bar{g}_{11} - \varepsilon_0, \cdots, \bar{g}_{n-1}(n-1) - \varepsilon_0, R_0) \geq \psi,$$

(4.2)

$$\{\bar{g}_{11} - \varepsilon_0, \cdots, \bar{g}_{n-1}(n-1) - \varepsilon_0, R_0\} \in \Gamma.$$

**Step 2.** Next, we apply Lemmas 2.2 and 2.4 derive the quantitative boundary estimates for double normal derivative. Let’s denote

$$I(r) = \begin{pmatrix} 1 & 0 \\ \vdots & \ddots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & r \end{pmatrix}$$

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We fix $\varepsilon_0 \in \mathbb{R}^+$ given above, there exists a uniform positive constant $r_0$ such that

$$B + \varepsilon_1 I(r_0/\varepsilon_1) \in \Gamma.$$  \hspace{1cm} (4.3)

We fix $\varepsilon_0$, $\varepsilon_1$, and $r_0$ that we have chosen. From (4.2) we see that

$$\lambda(A(R) - \varepsilon_1 \nabla_n(u - \underline{u})I(r_0/\varepsilon_1)) \in \Gamma, \quad R \geq R_0 + r_0 \nabla_n(u - \underline{u}).$$  \hspace{1cm} (4.4)

On the other hand, the identity (4.1) means that

$$A(R) = \left[A(R) - \varepsilon_1 \nabla_n(u - \underline{u})I(r_0/\varepsilon_1)\right] + \nabla_n(u - \underline{u})[B + \varepsilon_1 I(r_0/\varepsilon_1)].$$

For simplicity, we denote

$$H(R) = A(R) - \varepsilon_1 \nabla_n(u - \underline{u})I(r_0/\varepsilon_1).$$
Notice that $\nabla_n(u - u) \geq 0$. According to Lemma 2.4 we obtain

$$f(A(R)) \geq f(H(R)). \quad (4.5)$$

Let’s pick the parameter $\epsilon_0 = \epsilon_0$ in Lemma 2.2, and we assume

$$R_c = \frac{8(2n - 3)}{\epsilon_0} \sum_{a=1}^{n-1} |g_{an}|^2 + (n - 1) \sum_{a=1}^{n-1} \left( |g_{an}| + \frac{\epsilon_0}{8} \right) + \frac{(n - 2)\epsilon_0}{8(2n - 3)} + R_0 + r_0 \nabla_n(u - u),$$

where $\epsilon_0$ and $R_0$ are the constants from (4.2).

Lemma 2.2 applies to $H(R_c)$ and the eigenvalues of $H(R_c)$ (possibly with an order) shall behave like

$$\lambda(H(R_c)) \in \left( g_{11} - \frac{\epsilon_0}{4}, \cdots, g_{n-1,n-1} - \frac{\epsilon_0}{4}, R_c \right) + \Gamma_n \subset \Gamma. \quad (4.6)$$

Applying (1.4), (4.2), (4.5) and (4.6), we hence derive

$$F(A(R_c)) \geq F(H(R_c)) \geq f(g_{11} - \frac{\epsilon_0}{4}, \cdots, g_{n-1,n-1} - \frac{\epsilon_0}{4}, R_c) > \psi$$

which then gives $g_{\omega}(x_0) \leq R_c$.

□

5 Proof of existence results

The condition (1.11) is necessary for the blow-up argument in [17].

Lemma 5.1 ([20]). Suppose $f$ satisfies (1.3)-(1.5). Then it obeys (1.11).

Proof. Since $f$ satisfies (1.3), we have

$$\lim_{t \to 0^+} f(t \mathbf{1}) = f(\mathbf{0}) > -\infty. \quad (5.1)$$

For $\lambda \in \Gamma$ there is $t_\lambda > 0$ such that $f(t_\lambda \mathbf{1}) = f(\lambda)$. Thus

$$f(\lambda) \geq f(\mathbf{0}).$$

Consequently, (1.11) holds according to Lemma 2.4.

□
As shown in (3.10), we have $C^0$-estimate and boundary gradient estimate

$$\sup_M |u| + \sup_{\partial M} |\nabla u| \leq C.$$

From Propositions 3.1 and 4.1 we deduce Theorem 1.2. Namely,

$$\sup_{\partial M} \Delta u \leq C(1 + \sup_M |\nabla u|^2).$$

By $\Gamma \subset \Gamma_1$, $\Delta u > -\text{tr} g \chi$. Together with the second estimate,

$$\sup_M \Delta u \leq C(1 + \sup_M |\nabla u|^2 + \sup_{\partial M} |\Delta u|)$$

announced in [17, Section 8], we obtain

$$\sup_M \Delta u \leq C(1 + \sup_M |\nabla u|^2).$$  (5.2)

With such a second order estimate at hand, we can derive the gradient estimate via a blow-up argument (see [2, 17]), thereby establishing (1.9). Notice furthermore that the obtained estimates are independent of $(\delta_{\psi,f})^{-1}$, we can prove Theorem 1.1 by standard approximate method.

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