MULTIDIMENSIONAL CONTINUED FRACTIONS
AND A MINKOWSKI FUNCTION

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Abstract. The Minkowski Question Mark function can be characterized as the unique homeomorphism of the real unit interval that conjugates the Farey map with the tent map. We construct an \( n \)-dimensional analogue of the Minkowski function as the only homeomorphism of an \( n \)-simplex that conjugates the piecewise-fractional map associated to the Mönkemeyer continued fraction algorithm with an appropriate tent map.

1. Preliminaries

The \( n \)th order Farey set \( F_n \) in the real unit interval \([0, 1]\) is defined by recursion: one starts with \( F_0 = \{0/1, 1/1\} \) and obtains \( F_n \) by adding to \( F_{n-1} \) all the Farey sums \( v_1 \oplus v_2 = (a_1 + a_2)/(b_1 + b_2) \) of two consecutive elements \( v_i = a_i/b_i \) of \( F_{n-1} \). The union of all the \( F_n \)'s is the set of all rational numbers in \([0, 1]\). Analogously, by starting with \( B_0 = F_0 \) and replacing the Farey sum with the barycentric sum \( v_1 \oplus v_2 = (v_1 + v_2)/2 \), we obtain an increasing sequence \( B_0 \subset B_1 \subset B_2 \subset \cdots \), whose union is the set of all dyadic rationals in \([0, 1]\). For every \( n \geq 0 \), there exists a unique order-preserving bijection from \( F_n \) to \( B_n \). The union of these bijections is a bijection from \( \bigcup_{n \geq 0} F_n \) to \( \bigcup_{n \geq 0} B_n \), which extends uniquely by continuity to an order-preserving bijection \( \Phi : [0, 1] \to [0, 1] \). This last map is the Minkowski Question Mark function \([7], [15], [13], [20]\). Among others, \( \Phi \) has the following properties:

1. it is an order-preserving homeomorphism of \([0, 1]\);
2. it maps bijectively the rational numbers to the dyadic rationals, and the real algebraic numbers of degree \( \leq 2 \) to the rationals (all these sets restricted to \([0, 1]\), of course);
3. it is singular w.r.t. the Lebesgue measure \( \lambda \) (i.e., there exists a measurable set \( A \subseteq [0, 1] \) such that \( \lambda(A) = 1 \) and \( \lambda(\Phi[A]) = 0 \));
4. it has a fractal structure —which is apparent in the following approximate sketch—

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(5) it conjugates the Farey map

\[ F(x) = \begin{cases} 
  x/(1-x), & \text{if } 0 \leq x < 1/2; \\
  (1-x)/x, & \text{if } 1/2 \leq x \leq 1; 
\end{cases} \]

with the tent map

\[ T(x) = \begin{cases} 
  2x, & \text{if } 0 \leq x < 1/2; \\
  2-2x, & \text{if } 1/2 \leq x \leq 1. 
\end{cases} \]

Property (4) means, more precisely, the following: let \( v_1 < v_2 \) be consecutive elements of some \( \mathcal{F}_n \). Then there exists a unique element \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of \( \text{PSL}_2 \mathbb{Z} \) such that the corresponding fractional-linear transformation \( G(x) = (ax+b)/(cx+d) \) maps \( v_1 \) to 0 and \( v_2 \) to 1. Analogously, there is a unique affine transformation \( H(x) = rx+s \) such that \( H(\Phi(v_1)) = 0 \) and \( H(\Phi(v_2)) = 1 \). One then checks easily that \( \Phi = H \circ (\Phi \mid [v_1,v_2]) \circ G^{-1} \).

We note the following for future reference.

**Proposition 1.1.** Property (5) characterizes \( \Phi \).

**Proof.** Let \( \Psi \) be a homeomorphism of \([0,1]\) such that \( T = \Psi \circ F \circ \Psi^{-1} \). The only point which is fixed by \( F \) (respectively, \( T \)), and whose removal does not disconnect \([0,1]\) is 0; therefore \( \Psi(0) = 0 \) and \( \Psi \) is order-preserving. For every \( n \geq 0 \) we have \( \mathcal{F}_n = F^{-(n+1)}(\{0\}) \) and \( B_n = T^{-(n+1)}(\{0\}) \). Hence, for every \( n \), \( \Psi \) restricts to a bijection between \( \mathcal{F}_n \) and \( B_n \). Since these bijections are order-preserving, \( \Psi \) must coincide with \( \Phi \). \( \square \)

In [2] a generalization of the Minkowski function to a selfmap \( \delta \) of a 2-dimensional triangle is proposed. The construction of \( \delta \) proceeds in stages, and parallels that for \( \Phi \): assume that \( \langle v_1, v_2, v_3 \rangle \) and \( \langle v'_2, v'_3, v'_4 \rangle \) are paired triangles that appear at the \( (n-1) \)th stage of the construction “on the Farey side” and “on the barycentric side”, respectively. Then, at the \( n \)th stage, \( \langle v_1, v_2, v_3 \rangle \) is subdivided into three subtriangles \( \langle v'_1, v_2, w' \rangle, \langle v'_1, w', v'_3 \rangle, \langle w', v'_2, v'_3 \rangle \), where \( w' \) is the Farey sum of \( v'_1, v'_2, v'_3 \), and \( w^2 \) is the barycentric sum of \( v'_2, v'_3, v'_4 \). The new triangles are paired in the obvious way, and the function \( \delta \) is defined using an appropriate limiting process.

This \( \delta \) function is not injective, nor is continuous at all points [2 p. 117]. This is essentially due to the fact that not every sequence of nested Farey triangles \( \langle v_1(0), v_2(0), v_3(0) \rangle \supset \langle v_1(1), v_2(1), v_3(1) \rangle \supset \langle v_1(2), v_2(2), v_3(2) \rangle \supset \cdots \) intersects in a single point (here, for every \( n \), \( \langle v_1(n), v_2(n), v_3(n) \rangle \) is one of the three triangles resulting from the subdivision of \( \langle v_1(n-1), v_2(n-1), v_3(n-1) \rangle \) at stage \( n \)). In terms of multidimensional continued fractions [4 17], this amounts to saying that
the continued fraction algorithm naturally associated with the 2-dimensional Farey partition is not topologically convergent [17, Definition 9].

In this paper we will construct another generalization of the Minkowski function, by replacing the 2-dimensional Farey continued fraction algorithm with the \( n \)-dimensional Mönkemeyer algorithm. The latter algorithm is topologically convergent, and this fact allows us to construct, for every \( n \geq 2 \), an \( n \)-dimensional Minkowski function \( \Phi \) which is a homeomorphism. We will show that appropriate analogs of the properties (1)–(5) continue to hold, with the exception of (2), for which we have partial results.

A remark on terminology is in order here: the multidimensional continued fraction algorithm we are going to use has been rediscovered over and over again. We call it the Mönkemeyer algorithm—and we call Mönkemeyer map the associated piecewise-fractional map—since the first reference we are aware of is [14]. The name Selmer algorithm is more widely used; as a matter of fact, the Mönkemeyer algorithm is just the restriction of the Selmer one [18] to the absorbing simplex \( D \) of [17, Theorem 22]. In [1] the same algorithm is called the GMA (generalized mediant algorithm), and is defined on a simplex obtainable from \( D \) via a permutation of the coordinates. All these versions are easily shown to be equivalent to each other.

2. AN \( n \)-DIMENSIONAL MINKOWSKI FUNCTION

We will define our generalization \( \Phi \) of the Minkowski function as the only homeomorphism of a certain \( n \)-dimensional simplex \( \Delta \) that conjugates the Mönkemeyer map \( M \) with a version of the tent map \( T \), both maps to be defined shortly. In order to streamline the presentation, we fix some notation. First of all, we fix an integer \( n \geq 1 \), and we identify \( \mathbb{R}^n \) with the plane \( \pi = \{ x_{n+1} = 1 \} \) in \( \mathbb{R}^{n+1} \). If \( \mathbf{v} = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in \mathbb{R}^{n+1} \) and \( \alpha_{n+1} > 0 \), we denote the projection of \( \mathbf{v} \) on \( \pi \) by \( \mathbf{v} = (\alpha_1/\alpha_{n+1}, \ldots, \alpha_n/\alpha_{n+1}) \). Conversely, if \( \mathbf{v} \in \mathbb{Q}^n \), then we denote by \( \mathbf{v} \) the unique point \( \mathbf{v} = (l_1, \ldots, l_n, l_{n+1}) \in \mathbb{Z}^{n+1} \) such that \( l_1, \ldots, l_{n+1} \) are relatively prime, \( l_{n+1} > 0 \), and \( \mathbf{v} \) projects to \( \mathbf{v} \). In this case, we say that \( \mathbf{v} \) is a rational point and that the coordinates of \( \mathbf{v} \) are the primitive projective coordinates of \( \mathbf{v} \). Note that this convention differs from the one used in [17], where projective coordinates range from 0 to \( n \), and \( \pi = \{ x_0 = 1 \} \).

An \( n \)-dimensional simplex in \( \mathbb{R}^n \) is unimodular if its vertices \( v_1, \ldots, v_{n+1} \) are rational and \( v_1, \ldots, v_{n+1} \) constitute a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^{n+1} \). In all this paper, \( \Delta \) will denote the simplex whose vertices \( v_1, \ldots, v_{n+1} \) are given, in primitive projective coordinates, by the columns of the following matrix:

\[
V = \begin{pmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 1 & 1 & \cdots & 1 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
1 & 1 & 1 & \cdots & 1 & 1
\end{pmatrix}
\]

More precisely, all entries of \( V \) are 0, except those in position \( ij \), with either \( i = n+1 \) or \( j \geq 2 \) and \( i + j \leq n + 2 \), that have value 1. Clearly \( \Delta \) is unimodular.
Consider now the following $(n + 1) \times (n + 1)$ matrices:

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Here, all entries of $A_0$ and $A_1$ are 0, except $(A_0)_{11}, (A_1)_{21},$ and all elements in position $1(n + 1)$, $2(n + 1)$, $(j + 1)j$, for $2 \leq j \leq n$, that have value 1. Let $\Delta_0, \Delta_1$ be the unimodular simplices whose vertices are given, in projective coordinates, by the columns of $VA_0$ and $VA_1$, respectively. For $a = 0, 1$, the matrix $M_a = VA_a^{-1}V^{-1} \in \text{GL}_{n+1}(\mathbb{Z})$ expresses, in projective coordinates, a fractional-linear homeomorphism—with a slight abuse of notation, again denoted by $M_a$—from $\Delta_a$ to $\Delta$ as follows: if $x = (x_1 \cdots x_n)^{tr} \in \Delta_a$, then the projective coordinates of $M_a(x)$ are $M_a(x_1 \cdots x_n)^{tr}$. Note that $\Delta = \Delta_0 \cup \Delta_1$ and $M_0 = M_1$ on $\Delta_0 \cap \Delta_1$. Indeed, the $(n-1)$-simplex $\Delta_0 \cap \Delta_1$ has vertices given by the columns of $VA'$ (where $A'$ is the $(n + 1) \times n$ matrix obtained from either $A_0$ or $A_1$ by removing the first column) and $M_aVA' = M_1VA'$. We remark here, for future reference in the proof of Proposition 2.2 that $M_0[\Delta_0 \cap \Delta_1] = M_1[\Delta_0 \cap \Delta_1]$ is the $(n-1)$-dimensional face of $\Delta$ whose vertices are $v_2, \ldots, v_{n+1}$ (just consider the columns of $VA_1^{-1}A'$).

The continuous piecewise-fractional map $M : \Delta \to \Delta$ defined by $M(x) = M_a(x)$, for $x \in \Delta_a$, is the Mönkemeyer map. A simple matrix computation shows that $\Delta_0 = \{x \in \Delta : x_1 + x_n \leq 1\}$ and that, in affine coordinates,

$$M(x_1, x_2, \ldots, x_n) = \begin{cases} \left( \frac{x_1}{1 - x_n}, \frac{x_1 - x_n}{1 - x_n}, \ldots, \frac{x_{n-1} - x_n}{1 - x_n} \right), & \text{if } x_1 + x_n \leq 1; \\ \left( \frac{1 - x_n}{x_1}, \frac{x_1 - x_n}{x_1}, \ldots, \frac{x_{n-1} - x_n}{x_1} \right), & \text{if } x_1 + x_n \geq 1. \end{cases}$$

For $a = 0, 1$, let now $B_a$ be the $(n + 1) \times (n + 1)$ matrix which is identical to $A_a$ except for the last column, where the two 1’s are replaced by 1/2. The matrices $V$ and $VB_a$ agree in the last row $(1 \cdots 1 1)$. Therefore, the product of the first one with the inverse of the second, i.e., the matrix $T_a = VB_a^{-1}V^{-1}$, has last row $(0 \cdots 0 1)$ and determines an affine map $T_a : \Delta_a \to \Delta$ as follows: if $x = (x_1 \cdots x_n)^{tr} \in \Delta_a$ and $y = (y_1 \cdots y_n)^{tr} = T_a(x)$, then $T_a(x_1 \cdots x_n)^{tr} = (y_1 \cdots y_n)^{tr}$. As above, $T_0 = T_1$ on $\Delta_0 \cap \Delta_1$. The continuous piecewise-affine map $T : \Delta \to \Delta$ defined by $T(x) = T_a(x)$, for $x \in \Delta_a$, is the tent map. In affine coordinates, $T$ is expressed by

$$T(x_1, x_2, \ldots, x_n) = \begin{cases} (x_1 + x_n, x_1 - x_n, \ldots, x_{n-1} - x_n), & \text{if } x_1 + x_n \leq 1; \\ (2 - x_1 - x_n, x_1 - x_n, \ldots, x_{n-1} - x_n), & \text{if } x_1 + x_n \geq 1. \end{cases}$$

Note that, for $n = 1$, the Mönkemeyer map and the tent map defined above coincide with the Farey map and the tent map of [1]. The following theorem is then an $n$-dimensional generalization of Proposition 1.1.

**Theorem 2.1.** There exists a unique homeomorphism $\Phi : \Delta \to \Delta$ such that $T = \Phi \circ M \circ \Phi^{-1}$.

The rest of this section is devoted to the proof of Theorem 2.1, we first prove the existence of $\Phi$, then its uniqueness. Recall that a [rational] simplicial complex
in \( \mathbb{R}^n \) is a finite set \( C \) of simplexes in \( \mathbb{R}^n \) such that: (1) all vertices of all simplexes in \( C \) are rational; (2) if \( \Gamma \in C \) and \( \Sigma \) is a face of \( \Gamma \), then \( \Sigma \in C \); (3) every two simplexes intersect in a common face. The support of \( C \) is the set-theoretic union \( |C| \) of all simplexes in \( C \). A complex \( C \) refines a complex \( D \), written \( C \geq D \), if \( |C| = |D| \) and every simplex of \( C \) is contained in some simplex of \( D \). The mesh of \( C \), written \( \text{mesh}(C) \), is the maximum diameter of the simplexes in \( C \) or, equivalently [12, Corollary 5.18], the maximum length of the 1-simplexes in \( C \).

The set \( F_1 \) of all faces of \( \Delta_0 \) and \( \Delta_1 \) is a simplicial complex supported in \( \Delta \). For short, we list only the maximal (w.r.t. the relation of being a face) simplexes, thus writing \( F_1 = \{ \Delta_0, \Delta_1 \} \); we also write \( F_0 = \{ \Delta \} \). For every finite sequence \( a_0, \ldots, a_{t-1} \in \{0,1\} \), we define by recursion

\[
\Delta_{a_0 \ldots a_{t-1}} = \Delta_{a_0} \cap M^{-1} \Delta_{a_1 \ldots a_{t-1}}
\]

\[
= \{ x : x \in \Delta_{a_0} \land M(x) \in \Delta_{a_1} \land M^2(x) \in \Delta_{a_2} \land \cdots \land M^{t-1}(x) \in \Delta_{a_{t-1}} \},
\]

and we call \( F_t = \{ \Delta_{a_0 \ldots a_{t-1}} : a_0, \ldots, a_{t-1} \in \{0,1\} \} \) the time-\( t \) partition for \( M \).

**Proposition 2.2.** Every \( F_t \) is a simplicial complex, whose maximal elements are the \( 2^t \) \( n \)-simplexes \( \Delta_{a_0 \ldots a_{t-1}} \). For every \( t \geq 0 \), the complex \( F_{t+1} \) refines \( F_t \).

**Proof.** Let \( \psi_a = M_a^{-1} : \Delta \to \Delta_a \) be the two inverse branches of \( M_a \) for \( a = 0, 1 \). Note that \( \Delta_{a_0 \ldots a_{t-1}} = \psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_{t-1}}[\Delta] \). Indeed, this is true for \( t = 1 \), and follows by induction otherwise, since

\[
\Delta_{a_0 \ldots a_{t-1}} = \Delta_{a_0} \cap M^{-1} \Delta_{a_1 \ldots a_{t-1}}
\]

\[
= \Delta_{a_0} \cap [\psi_0[\Delta_{a_1 \ldots a_{t-1}}] \cup \psi_1[\Delta_{a_1 \ldots a_{t-1}}]]
\]

\[
= \psi_{a_0}[\Delta_{a_1 \ldots a_{t-1}}].
\]

We now proceed by induction: \( F_0 \) and \( F_1 \) are simplicial complexes supported on \( \Delta \), with \( F_0 \supseteq F_1 \). Assuming that \( F_t \) is such a complex, then the elements of \( F_{t+1} \) are given by \( \psi_0[F_t] \cup \psi_1[F_t] \), where \( \psi_a[F_t] \) is the set of all \( \psi_a \)-images of the elements of \( F_t \). Since \( \psi_a \) is fractional-linear, \( \psi_a[F_t] \) is a simplicial complex supported in \( \Delta_a \).

It is therefore sufficient to show that \( \psi_0[F_t] \) and \( \psi_1[F_t] \) agree (i.e., induce the same complex) on the intersection \( \Delta_0 \cap \Delta_1 \). This fact is true because, as we remarked in the course of the definition of the Mönkemeier map, \( M_0 \) and \( M_1 \) agree on \( \Delta_0 \cap \Delta_1 \), and provide a fractional-linear homeomorphism between \( \Delta_0 \cap \Delta_1 \) and the \((n-1)\)-dimensional face \( \Lambda \) of \( \Delta \) whose vertices are \( v_2, \ldots, v_{n+1} \). Therefore \( \psi_0 \) and \( \psi_1 \) agree on \( \Lambda \). This implies that \( \psi_0[F_t] \) and \( \psi_1[F_t] \) induce the same complex on \( \Delta_0 \cap \Delta_1 \), namely the \( \psi_0 \)-image, which is also the \( \psi_1 \)-image, of the complex induced by \( F_t \) on \( \Lambda \). The fact that \( F_{t+1} \) refines \( F_t \) is immediate, since every maximal simplex \( \Delta_{a_0 \ldots a_{t-1}} \) is contained in \( \Delta_{a_0 \ldots a_{t-1}} \). \( \square \)

We construct analogously the time-\( t \) partition \( B_t \) for \( T \). Namely, we let \( B_0 = F_0 \), \( \Gamma_0 = \Delta_0 \), \( \Gamma_1 = \Delta_1 \), and \( \Gamma_{a_0 \ldots a_{t-1}} = \Gamma_{a_0} \cap T^{-1} \Gamma_{a_1 \ldots a_{t-1}} \). The analogue of Proposition 2.2 holds verbatim, and we have simplicial complexes \( B_t = \{ \Gamma_{a_0 \ldots a_{t-1}} : a_0, \ldots, a_{t-1} \in \{0,1\} \} \), with \( B_{t+1} \) refining \( B_t \). An obvious induction on \( t \) shows that there exists a unique combinatorial isomorphism from \( F_t \) to \( B_t \) that fixes the vertices of \( \Delta \). At the level of maximal simplexes, the isomorphism is given by \( \Delta_{a_0 \ldots a_{t-1}} \mapsto \Gamma_{a_0 \ldots a_{t-1}} \). We draw a picture of \( F_1 \) and \( B_1 \), for \( n = 2 \), labeling the 2-simplex \( \Gamma_{a_0 a_1 a_2} \in B_1 \) by \( a_0 a_1 a_2 a_3 \).
Let \( \{0,1\}^\mathbb{N} \) be the Cantor space, i.e., the set of all infinite sequences \( \bar{a} = a_0a_1a_2 \ldots \) of elements of \( \{0,1\} \), endowed with the product topology. For \( t \geq 1 \), we write \( \bar{a} \upharpoonright t \) for \( a_0a_1 \ldots a_{t-1} \), and we let \( [a_0 \ldots a_{t-1}] \) be the cylinder \( \{ \bar{b} : \bar{a} \upharpoonright t = \bar{b} \upharpoonright t \} \); we extend this convention by setting \( \Delta_{\bar{a}|0} = \Gamma_{\bar{a}|0} = \Delta \) and \( [a_0 \ldots a_{-1}] = \{0,1\}^\mathbb{N} \).

**Lemma 2.3.** For every \( \bar{a} \in \{0,1\}^\mathbb{N} \), the intersection \( \bigcap_{t \geq 0} \Delta_{\bar{a}|t} \) is a singleton, and the intersection \( \bigcap_{t \geq 0} \Gamma_{\bar{a}|t} \) is a singleton as well.

**Proof.** The first statement amounts to saying that the Mönkemeyer algorithm is topologically convergent [17, Definition 9]; this fact is proved in [14, Satz 10] as well as in [17, Lemma 19]. Note that [14, Satz 10] assumes that \( \bigcap_{t \geq 0} \Delta_{\bar{a}|t} \) contains a point whose coordinates are not all rational. But this is not a restriction since, if \( \bigcap_{t \geq 0} \Delta_{\bar{a}|t} \) contained two distinct points \( p, q \), then, by convexity, it would contain all the points in the line segment connecting \( p \) with \( q \), and hence a point whose coordinates are not all rational.

In order to prove the second statement note that the vertices of \( \Gamma_{\bar{a}|t} \) are given, in projective coordinates, by the columns of \( VB_{a_0} \cdots B_{a_{t-1}} \). Let \( \mathcal{R} \) be the set of all \( (n+1) \times (n+1) \) column-stochastic matrices (i.e., all nonnegative matrices having the property that the entries in each column sum up to 1). Observe that \( \mathcal{R} \) is a compact submonoid of \( (\text{Mat}_{(n+1) \times (n+1)} \mathbb{R}) \cdot \{\text{Id}\} \). Let \( B_{\bar{a}|t} = B_{a_0} \cdots B_{a_{t-1}} \in \mathcal{R} \). We will apply [6, Theorem 6.1] to show that, for every \( \bar{a} \in \{0,1\}^\mathbb{N} \), the limit \( \hat{B} = \lim_{t \to \infty} B_{\bar{a}|t} \) exists (necessarily in \( \mathcal{R} \), since the latter is closed), and all columns of \( \hat{B} \) are equal. Recall that a column-stochastic matrix \( C = C_{ij} \) is \( (j_1, j_2) \)-scrambling if there exists a row index \( i \) such that \( C_{ij_1} \) and \( C_{ij_2} \) are both \( > 0 \); \( C \) is scrambling if it is \( (j_1, j_2) \)-scrambling for every pair \( (j_1, j_2) \) of columns indices [10]. By [6, Theorem 6.1], it will be sufficient to prove the following:

(A) there exists \( s > 0 \) such that all products of \( s \) matrices from \( \{B_0, B_1\} \) (repetitions allowed) are scrambling.

It is simpler to argue on the incidence graphs \( G(B_0) \) and \( G(B_1) \) associated to \( B_0 \) and \( B_1 \). The graph \( G(B_0) \) has \( n+1 \) vertices and there is a directed edge connecting the \( j \)-th vertex to the \( i \)-th if \( (B_0)_{ij} > 0 \); similarly for \( G(B_1) \). We combine \( G(B_0) \) and \( G(B_1) \) in a single graph \( G \) as in the following picture, with the understanding that in \( G(B_0) \) the edge \( e_0 \) is activated and the edge \( e_1 \) is discarded, and conversely in \( G(B_1) \).
We will deduce property (A) from the existence of a winning strategy for a certain game on $G$. The game starts with two Lovers sitting in distinct vertices. A move of the game consists of the following: first, the Enemy chooses which of the two edges $e_0$ and $e_1$ is to be active at that move, and then each Lover moves one step along an edge departing from his/her current vertex. The Lovers win the game if after finitely many moves they end up in the same vertex.

Claim. Regardless of the initial position, the Lovers win in at most $(n + 1)n/2$ moves.

Assuming the Claim, let us prove (A). We take $s = (n + 1)n/2$, and we fix a product $B = B_{a_0} \cdots B_{a_{n-1}}$ of $s$ matrices from $\{B_0, B_1\}$. No column in $B_0$ or in $B_1$ is identically 0; therefore, for every $(j_1, j_2)$-scrambling matrix $C$, both $B_0C$ and $B_1C$ are $(j_1, j_2)$-scrambling. Choose column indices $j_1, j_2$; by the above, we can assume $j_1 \neq j_2$. Consider the game in which the Lovers apply the winning strategy and start in position $j_1$ and $j_2$, while the Enemy activates the edge $e_{a_{s-r}}$ at the $r$th move ($r \geq 1$). By the Claim, this game ends after $1 \leq t \leq s$ moves, leaving the Lovers in the same vertex $i$. This implies that there exists a path in $G$ connecting $j_1$ to $i$, and such that the $r$th edge in the path is an edge of $G(B_{a_{s-r}})$. By the elementary properties of the incidence graphs of nonnegative square matrices, the $ij_1$th entry of $B_{a_0} \cdots B_{a_{s-r}}B_{a_{s-1}}$ is $> 0$. Analogously, the $ij_2$th entry is $> 0$; hence $B_{a_0} \cdots B_{a_{s-r}}B_{a_{s-1}}$ is $(j_1, j_2)$-scrambling, and so is $B$. Since $j_1$ and $j_2$ are arbitrary, $B$ is scrambling.

Proof of Claim. Given any vertex $p$ of $G$, there exists a unique vertex path $p = p_0, p_1, p_2, \ldots$ such that, for every $i \geq 1$, there is an oriented edge in $G$, different from both $e_0$ and $e_1$, which connects $p_i$ to $p_{i-1}$. Let us call such a path a backward path. The length of a finite backward path $p_0, p_1, p_2, \ldots, p_r$ is $r$, its origin is $p_0$, and its endpoint is $p_r$. Let $\mathcal{V}$ be the set, of cardinality $(n + 1)n/2$, whose elements are all unordered pairs of distinct vertices of $G$. If $\{p, q\} \in \mathcal{V}$, then the gap $g(p, q)$ of $\{p, q\}$ is the minimal length of a backward path whose origin is one of the vertices $p, q$, and whose endpoint is the other vertex. The origin of such a path is the leading vertex of $\{p, q\}$, and the defect $d(p, q)$ of $\{p, q\}$ is 0 if the leading vertex is 1, and is $g(1, \text{leading vertex})$ otherwise. The leading vertex, and the numbers $1 \leq g(p, q) \leq n$ and $0 \leq d(p, q) \leq n$, are uniquely determined by the pair, with the exception of the case in which $n$ is even, $p$ and $q$ are both $\neq 1$, and $g(p, q) = n/2$. In this case, we define the leading vertex to be the vertex whose gap from 1 is minimal, and we define the defect accordingly. Let $T$ be the set $\{1, \ldots, n\} \times \{0, \ldots, n\}$, ordered lexicographically: $(g, d) < (g', d')$ iff $g < g'$ or $(g = g'$ and $d < d')$. The
map $\chi : V \to T$ defined by $\chi \{p, q\} = (g(p, q), d(p, q))$ is injective: indeed $d(p, q)$ determines uniquely the leading vertex of the pair (start from 1 and go backwards $d(p, q)$ steps in $G$, never using the edges $e_0$ and $e_1$), and then $g(p, q)$ determines the other vertex.

Assume now that at a certain stage of the game the Lovers are in distinct vertices $p, q$, and consider the following strategy.

(a) If $\{p, q\} = \{1, n + 1\}$, then the Lovers win at the next move, either by meeting at 1 (if the Enemy chooses to activate $e_0$) or by meeting at 2 (if the Enemy activates $e_1$).

(b) Otherwise, if $n + 1 \notin \{p, q\}$, then each Lover follows the unique edge at his disposal.

(b) Otherwise, without loss of generality $p = n + 1$ and $q \notin \{1, n + 1\}$. Then the Lover at $q$ follows the unique edge at his disposal, while the Lover at $p$ moves to 1 provided that $p$ is the leading vertex of the pair $\{p, q\}$, otherwise moves to 2.

Let $p', q'$ be the vertices occupied by the Lovers at the next step, and assume that the game is not finished yet (hence case (a) did not apply, and $p' \neq q'$). An easy case analysis, distinguishing the three cases (i) the leading vertex is 1, (ii) the leading vertex is $n + 1$, and (iii) the leading vertex is in $\{2, \ldots, n\}$, shows that $\chi(p', q') < \chi(p, q)$. In other words, at each step either the gap of the pair decreases, or the gap stays the same and the defect decreases. Since $\chi[Y]$ has finite size and is totally ordered by $\prec$, the length of the longest possible game, final winning move included, coincides with this size, namely $(n + 1)n/2$.

Note that the bound in the proof of Lemma 2.3 is sharp: for $n = 2$ the matrix $A_0A_1$ is not scrambling, and for $n = 3$ the matrix $A_0A_0A_1A_0$ is not scrambling either.

**Corollary 2.4.** Both $\lim_{t \to \infty} \text{mesh}(F_t)$ and $\lim_{t \to \infty} \text{mesh}(B_t)$ exist and have value 0.

**Proof.** Suppose that statement is false for, say, the Farey complexes. Then there exists $\varepsilon > 0$ such that, for every $t$, the set

$$S_t = \{a_0a_1 \ldots a_{t-1} : \text{the diameter of } \Delta_{a_0 \ldots a_{t-1}} \text{ is } \geq \varepsilon\}$$

is not empty. If $a_0a_1 \ldots a_{t-1} \in S_t$ and $0 < k \leq t$, then $a_0a_1 \ldots a_{k-1} \in S_k$, so the union of all $S_t$'s is an infinite subtree of the full binary tree. By König’s Lemma 19 Lemma 3.3.19] this subtree has an infinite branch $\bar{a} \in \{0, 1\}^{\mathbb{N}}$. This contradicts Lemma 2.3.

As a side remark note that, for $n = 2$, all the $2^t$ triangles in $B_t$ are congruent, because the $2 \times 2$ matrices obtained from $T_0^{-1}$ and $T_1^{-1}$ by removing the third row and the third column are both of the form $1/\sqrt{2} \cdot$ (an orthogonal matrix). This is no longer true for $n \geq 3$.

Given $\bar{a} \in \{0, 1\}^{\mathbb{N}}$, let $\varphi(\bar{a})$ be the unique point in $\cap_{t \geq 0} \Delta_{\bar{a}[t]}$, and let $\nu(\bar{a})$ be the unique point in $\cap_{t \geq 0} \Gamma_{\bar{a}[t]}$.

**Lemma 2.5.** The mappings $\varphi, \nu : \{0, 1\}^{\mathbb{N}} \to \Delta$ are continuous, surjective, and have the same fibers (i.e., $\varphi(\bar{a}) = \varphi(\bar{b})$ iff $\nu(\bar{a}) = \nu(\bar{b})$).
Proof. Clearly $\varphi$ is surjective and we have

$$\Delta_{a_0 \ldots a_{t-1}} \subseteq \varphi[[a_0 \ldots a_{t-1}]]; \quad \text{(*)}$$

$$[a_0 \ldots a_{t-1}] \subseteq \varphi^{-1}\Delta_{a_0 \ldots a_{t-1}}. \quad \text{(**)}$$

If $U \subseteq \Delta$ is open, then we have

$$\varphi^{-1}U = \bigcup\{[a_0 \ldots a_{t-1}] : \Delta_{a_0 \ldots a_{t-1}} \subseteq U\}. \quad \text{(***)}$$

Indeed, the $\supseteq$ inclusion is immediate from (**). On the other hand, let $\varphi(\bar{a}) \in U$. Then $(\Delta \setminus U) \cap \bigcap_{t \geq 0} \Delta_{\bar{a}|_t} = \emptyset$, and hence by compactness there exists $t \geq 0$ such that $\Delta_{\bar{a}|_t} \subseteq U$. Therefore $\bar{a}$ belongs to the right-hand side of (**), and equality follows. Since the right-hand side of (***) is open in the Cantor space, $\varphi$ is continuous. Exactly the same proof shows that $\psi$ is surjective and continuous as well.

We assume now that $\bar{a}, \bar{b}$ are such that $\varphi(\bar{a}) = \varphi(\bar{b})$, and prove $\psi(\bar{a}) = \psi(\bar{b})$. By hypothesis, for every $t \geq 0$ we have $\Delta_{\bar{a}|_t} \cap \Delta_{\bar{b}|_t} \neq \emptyset$, and hence $\Delta_{\bar{a}|_t}$ and $\Delta_{\bar{b}|_t}$ intersect in a common nonempty face. As observed above, $\mathcal{F}_t$ and $\mathcal{B}_t$ are combinatorially isomorphic; therefore, for every $t$, $\Gamma_{\bar{a}|_t}$ and $\Gamma_{\bar{b}|_t}$ intersect in a common nonempty face as well. Again by compactness, $\bigcap_{t \geq 0} (\Gamma_{\bar{a}|_t} \cap \Gamma_{\bar{b}|_t}) \neq \emptyset$. Since by definition $\bigcap_{t \geq 0} \Gamma_{\bar{a}|_t} = \{\psi(\bar{a})\}$ and $\bigcap_{t \geq 0} \Gamma_{\bar{b}|_t} = \{\psi(\bar{b})\}$, we have $\psi(\bar{a}) = \psi(\bar{b})$. Clearly the rôle of $\bar{a}$ and $\bar{b}$ can be reversed, and it follows that $\varphi$ and $\psi$ have the same fibers. $\square$

Let $\equiv$ be the equivalence relation on the Cantor space defined by $\bar{a} \equiv \bar{b}$ iff $\varphi(\bar{a}) = \varphi(\bar{b})$ iff $\psi(\bar{a}) = \psi(\bar{b})$. Let $\chi : \{0,1\}^N \to \{0,1\}^N / \equiv$ be the quotient mapping, and endow $\{0,1\}^N / \equiv$ with the quotient topology: $V$ is open in $\{0,1\}^N / \equiv$ iff $\chi^{-1}V$ is open in $\{0,1\}^N$. We have an obvious factorization in continuous mappings

$$\begin{array}{ccc}
\{0,1\}^N & \xrightarrow{\varphi} & \Delta \\
\downarrow{\chi} & & \downarrow{\psi} \\
\{0,1\}^N / \equiv & \xrightarrow{\psi} & \Delta
\end{array}$$

The quotient space $\{0,1\}^N / \equiv$ is compact, and $\Delta$ is Hausdorff. Hence the continuous bijections $\varphi$ and $\psi$ are both homeomorphisms.

**Definition 2.6.** We define $\Phi : \Delta \to \Delta$ as the homeomorphism $\Phi = \psi \circ \varphi^{-1}$. Equivalently, $\Phi(p) = \psi(\varphi(p))$, for any $\bar{a}$ such that $\varphi(\bar{a}) = p$.

For every $a_0, \ldots, a_{t-1}$, the homeomorphism $\Phi$ restricts to a bijection from $\Delta_{a_0 \ldots a_{t-1}}$ to $\Gamma_{a_0 \ldots a_{t-1}}$; this follows from (*) and (**) in the proof of Lemma 2.5 and the corresponding inclusions for $\psi$. For every $t > 1$ we have $M[\Delta_{a_0 \ldots a_{t-1}}] = M \circ \psi \circ \varphi \circ [\Delta_{a_0 \ldots a_{t-1}}] = \Delta_{a_1 \ldots a_{t-1}}$; this makes sense also for $t = 1$, since $M[\Delta_{a_0}] = \Delta$. Analogously we have $T[\Gamma_{a_0 \ldots a_{t-1}}] = \Gamma_{a_1 \ldots a_{t-1}}$. Denoting by $S$ the shift map on
By chasing the diagram, one sees immediately that \( T = \Phi \circ M \circ \Phi^{-1} \), as required.

We now show the uniqueness of \( \Phi \), by assuming that \( \Psi \) is a homeomorphism of \( \Delta \) such that \( T = \Psi \circ M \circ \Psi^{-1} \) and proving that \( \Psi = \Phi \). Observe that the boundary \( \partial \Delta \) of \( \Delta \) is characterized—in purely topological terms, with no reference to the immersion of \( \Delta \) in \( \mathbb{R}^n \)—as the set of points \( p \in \Delta \) whose removal leaves \( \Delta \setminus \{p\} \) contractible. Therefore, \( \Phi[\partial \Delta] = \Psi[\partial \Delta] = \partial \Delta \). Let \( \Sigma \) be the set-theoretic union of the proper faces of \( \Delta_0 \) and \( \Delta_1 \); we have \( \Sigma = M^{-1} \partial \Delta = T^{-1} \partial \Delta \) and, as a consequence, \( \Sigma = \Phi^{-1} \Sigma = \Psi^{-1} \Sigma \). Indeed, e.g., \( p \in \Sigma \) iff \( M(p) \in \partial \Delta \) iff \( \Psi \circ M(p) \in \partial \Delta \) iff \( \Psi(p) \in T^{-1} \partial \Delta = \Sigma \) iff \( p \in \Psi^{-1} \Sigma \).

Note that \( \Phi(v_1) = \Psi(v_1) = v_1 \). Indeed, \( v_1 \) is a point in \( \partial \Delta \) which is fixed by \( M \), and therefore both \( \Phi(v_1) \) and \( \Psi(v_1) \) must be points in \( \partial \Delta \) which are fixed by \( T \). But there is only one such point, namely \( v_1 \) itself. Observe now that \( \Delta_0 \) can be characterized as the set of points in \( \Delta \) that can be connected to \( v_1 \) by a continuous path whose relative interior does not intersect \( \Sigma \). Since \( \Phi \) and \( \Psi \) are homeomorphisms, both fixing \( \Sigma \) globally, we can safely conclude that \( \Phi[\Delta_a] = \Psi[\Delta_a] \), for \( a = 0, 1 \).

Let now \( p \) be any point of \( \Delta \), and choose \( \bar{a} \) such that \( \varphi(\bar{a}) = p \) and \( v(\bar{a}) = \Phi(p) \). For every \( t \geq 0 \) we have \( M^t(p) \in \Delta_{a_t} = \Gamma_{a_t} \) and therefore \( \Psi^{-1} \circ T^t(\Psi(p)) \in \Gamma_{a_t} \), i.e., \( T^t(\Psi(p)) \in \Psi[\Gamma_{a_t}] = \Gamma_{a_t} \). Hence, by definition of \( v \), \( \Psi(p) = v(\bar{a}) = \Phi(p) \). This concludes the proof of Theorem 2.1.

### 3. Fractal structure, periodicity and singularity

In this section we will discuss how properties §1(1)–(5) of the classical Minkowski function generalize to our \( n \)-dimensional setting. Basically, all properties continue to hold, with the exception of §1(2), whose full validity turns out to be an open problem. Let us first treat §1(4).

**Proposition 3.1.** Let \( t \geq 0 \), let \( \Xi \in \mathcal{F}_t \), and let \( \Lambda \) be the simplex in \( \mathcal{B}_t \) corresponding to \( \Xi \) under the combinatorial isomorphism defined before Lemma 2.3. Then \( \Phi \) restricts to a homeomorphism between \( \Xi \) and \( \Lambda \). Moreover, for every \( \Delta_{a_0...a_{t-1}} \in \mathcal{F}_t \) we have

\[
\Phi = (T_{a_{t-1}} \circ \cdots \circ T_{a_0}) \circ (\Phi \mid \Delta_{a_0...a_{t-1}}) \circ (M_{a_{t-1}} \circ \cdots \circ M_{a_0})^{-1}.
\]

**Proof.** We can give an equivalent definition of \( \Phi \) and \( \Phi^{-1} \) as follows. For each \( t \), we define a simplicial homeomorphism \( \Phi_t : \Delta \rightarrow \Delta \) by first mapping the vertices of \( \mathcal{F}_t \) to the vertices of \( \mathcal{B}_t \) according to the combinatorial isomorphism, and then using barycentric coordinates to extend \( \Phi_t \) to all of \( \Delta \). More precisely, if \( \Delta_{a_0...a_{t-1}} \) has vertices \( w_1, \ldots, w_{n+1} \) and \( \Delta_{a_0...a_{t-1}} \ni p = \sum \alpha_i w_i \) in barycentric coordinates, then
Hence $\Phi[\Xi] \subseteq k$ and every $\Phi$ main diagonal are those in the last row of $VA\Phi$. Let $\Delta$ Choose Proof.

The mappings $M_{a_{t-1}} \circ \cdots \circ M_{a_0} = M \mid \Delta_{a_0 \ldots a_{t-1}} : \Delta_{a_0 \ldots a_{t-1}} \to \Delta,$

and $T_{a_{t-1}} \circ \cdots \circ T_{a_0} = T \mid \Gamma_{a_0 \ldots a_{t-1}} : \Gamma_{a_0 \ldots a_{t-1}} \to \Delta,$

are both homeomorphisms, the former fractional-linear and the latter affine. Our second statement is then immediate, since it amounts to the restriction of the identity $\Phi \circ M = T \circ \Phi$ to $\Delta_{a_0 \ldots a_{t-1}}.$

Next, §1(1) generalizes to the following proposition.

**Proposition 3.2.** $\Phi$ is an orientation-preserving homeomorphism.

**Proof.** Choose $t$ such that $F_t$ contains a vertex $p$ in the topological interior $\Delta^o$ of $\Delta$. Let $\Delta_{a_0 \ldots a_{t-1}} \in F_t$, and let $D$ be the diagonal matrix whose entries along the main diagonal are those in the last row of $VA_{a_0} \cdots A_{a_{t-1}}$. Then the affine homeomorphism $\Phi_t : \Delta_{a_0 \ldots a_{t-1}}$ defined in the proof of Proposition 3.1 is expressed by the matrix $(V B_{a_0} \cdots B_{a_{t-1}})(V A_{a_0} \cdots A_{a_{t-1}} D^{-1})^{-1}$, which has last row $(0 \cdots 0 1)$ and determinant $> 0$ (because $D$ has positive determinant, and $A_a, B_a$ have determinant of the same sign, for $a \in \{0, 1\}$). This holds for every maximal simplex $\Delta_{a_0 \ldots a_{t-1}} \in F_t$, and it follows that $\Phi_t$ is orientation-preserving.

Let now $q$ be the vertex in $B_t$ corresponding to $p$. Again $q$ is in $\Delta^o$, and $\Phi(p) = \Phi_t(p) = q$. Let $X = \Delta \setminus \{p\}$ and $Y = \Delta \setminus \{q\}$. Then $\Phi \mid X$ and $\Phi_t \mid X$ are homeomorphisms from $X$ to $Y$, and we claim that they are homotopic. Indeed, a homotopy $F : X \times [0, 1] \to Y$ is given by $(x, r) = (1 - r)\Phi(x) + r\Phi_t(x).$ This works because, assuming $x \in \Delta_{a_0 \ldots a_{t-1}}$, we have by Proposition 3.1 that $\Phi(x)$ and $\Phi_t(x)$ are both in $\Gamma_{a_0 \ldots a_{t-1}} \setminus \{q\}.$ Since $\Gamma_{a_0 \ldots a_{t-1}} \setminus \{q\}$ is convex, the image of $F$ is $Y.$ One checks easily that $F$ is continuous, and this establishes our claim.

Note that, given any points $p', q' \in \Delta^o$, the homology groups $H_{n-1}(\Delta \setminus \{p'\})$ and $H_{n-1}(\Delta \setminus \{q'\})$ (coefficients in $\mathbb{Z}$) are canonically identifiable, since they are both canonically isomorphic to the relative homology group $H_n(\Delta, \Delta \setminus B)$, where $B$ is any ball in $\Delta^o$ containing $p'$ and $q'$. By [10] p. 233], we have that $\Phi$ (respectively, $\Phi_t$) is orientation-preserving if $\Phi \mid X$ (respectively, $\Phi_t \mid X$) induces in homology the identity mapping between $H_{n-1}(X)$ and $H_{n-1}(Y)$ (these two infinite cyclic groups canonically identified as above). Since $\Phi \mid X$ and $\Phi_t \mid X$ are homotopic, they induce the same isomorphism in homology, and we conclude that $\Phi$ is orientation-preserving if and only if $\Phi_t$.

As remarked at the beginning of this section, a proper generalization of §1(2) is critical. Indeed, the periodicity properties of the various multidimensional continued fraction algorithms are a long-standing open problem. Even for the most studied algorithm, the Jacobi-Perron one [10], [17], it is still unknown whether points $p = (\alpha_1, \ldots, \alpha_n)$ such that $[\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}] \leq n + 1$ are always preperiodic under the piecewise-fractional map associated to the algorithm. The situation for the Mönkemeyer algorithm is no better; we list a few simple facts in order to describe the problem.
Let $p \in \Delta$: the grand orbit of $p$ under $M$ is
\[ \text{GO}_M(p) = \{ q \in \Delta : M^t(p) = M^s(q) \text{ for some } t, s \geq 0 \}, \]
and its eventual periodic orbit is
\[ \text{EPO}_M(p) = \{ q \in \Delta : q = M^t(p) = M^s(p) \text{ for some } 0 \leq t < s \}. \]

EPO$_M(p)$ is always a finite set, possibly empty; if it is nonempty, then $p$ is preperiodic under $M$. One defines GO$_T(p)$ and EPO$_T(p)$ similarly; of course
\[ \Phi[\text{GO}_M(p)] = \text{GO}_T(\Phi(p)) \text{ and } \Phi[\text{EPO}_M(p)] = \text{EPO}_T(\Phi(p)). \]

Let $\mathbb{Z}[1/2] = \{ a/2^m \in \mathbb{Q} : a, m \in \mathbb{Z} \text{ and } m \geq 0 \}$ be the ring of dyadic rationals. It is a p.i.d., since it is a localization of the p.i.d. $\mathbb{Z}$. For $p = (\alpha_1, \ldots, \alpha_n) \in \Delta$, we write $\mathbb{Q}(p)$ for the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, and we write $\mathbb{Z}[1/2](p)$ for the $\mathbb{Z}[1/2]$-module $\mathbb{Z}[1/2] \alpha_1 + \cdots + \mathbb{Z}[1/2] \alpha_n + \mathbb{Z}[1/2]$, which is free of rank $\leq n + 1$. Since the matrices $M_0, M_1$ determining $M$ are in $\text{GL}_{n+1} \mathbb{Z}$, we have clearly $\mathbb{Q}(p) = \mathbb{Q}(q)$, for any $q \in \text{GO}_M(p)$. Analogously, the matrices $T_0, T_1 \in \text{Mat}_{n+1} \mathbb{Z}$ determining $T$ are in $\text{GL}_{n+1} \mathbb{Z}[1/2]$, and hence $\mathbb{Z}[1/2](p) = \mathbb{Z}[1/2](q)$, for any $q \in \text{GO}_T(p)$. We call a point $p = (\alpha_1, \ldots, \alpha_n) \in \Delta$ a rational point (respectively, a dyadic point) if $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$ (respectively, $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[1/2]$). In order to prove that $\Phi$ determines a 1-1 correspondence between the rational points and the dyadic ones, we need two technical lemmas.

Remember that a nonsingular matrix $H = H_{ij} \in \text{Mat}_{n \times n} \mathbb{Z}$ is in row Hermite Normal Form (HNF) if it is upper triangular, $H_{ij} > 0$ for every $j$, and $0 \leq H_{ij} < H_{jj}$ for every $1 \leq i < j$. Every nonsingular $A \in \text{Mat}_{n \times n} \mathbb{Z}$ has a unique HNF (i.e., there exists a unique $H$ in HNF and a —unique— $X \in \text{GL}_n \mathbb{Z}$ such that $H = XA$) \[ \text{[2.4.2]} \]

In particular, two nonsingular matrices $A, B \in \text{Mat}_{n \times n} \mathbb{Z}$ have the same HNF iff there exists $X \in \text{GL}_n \mathbb{Z}$ such that $B = XA$; in this case we write $A \sim B$.

**Lemma 3.3.** Let $t \geq 1$, and let $a_0, \ldots, a_{t-1} \in \{0, 1\}$. The matrices $T_{a_{t-1}} \cdots T_{a_0}$ and $T_{a_0}^t$ have the same HNF.

**Proof.** The last row of $T_0$ and $T_1$, and hence of all products $T_{a_{t-1}} \cdots T_{a_0}$, is $(0 \cdots 0 1)$. Hence we can safely replace $T_a$ with the $n \times n$ matrix $Q_a$ obtained from $T_a$ by removing the last row and the last column. It now suffices to show that $Q_{a_{t-1}} \cdots Q_{a_0}$ and $Q_{a_0}^t$ have the same HNF. Direct computation shows that the entries of $Q_0$ are as follows:

\[ (Q_0)_{ij} = \begin{cases} 1, & \text{if } ij = 11, \text{ or } ij = 1n, \text{ or } i = j + 1; \\
-1, & \text{if } i \geq 2 \text{ and } j = n; \\
0, & \text{otherwise.} \end{cases} \]

We have $Q_0^n = 2E_n$, where $E_n$ is the $n \times n$ identity matrix; in particular, all powers of $Q_0^n$ commute with everything. For $1 \leq r \leq n - 1$, denote the HNF of $Q_0^n$ by $E_r$; we have explicitly:

\[ (E_r)_{ij} = \begin{cases} 2, & \text{if } i = j \geq n - r + 1; \\
1, & \text{if } i = j < n - r + 1, \text{ or } i < j = n - r + 1; \\
0, & \text{otherwise.} \end{cases} \]

We work by induction on $t$. Denote by $D$ the $n \times n$ diagonal matrix whose entries along the diagonal are $-1, 1, \ldots, 1$. Since $Q_1 = DQ_0$, we always have $Q_{a_0} \sim Q_0$, and the case $t = 1$ is settled. By inductive hypothesis, assume $Q_{a_{t-1}} \cdots Q_{a_1} \sim Q_0^{t-1}$.
Then $Q_{\gamma t-1} \cdots Q_{\alpha t}Q_{\alpha 0} \sim Q_{\gamma t-1}^t Q_{\alpha 0}$, and we claim that $Q_{\gamma t-1}^t Q_{\alpha 0} \sim Q_{\alpha 0}^t$. This is immediate if $a_0 = 0$, so we assume $a_0 = 1$. Let $t - 1 = mn + r$, for some $m \geq 0$ and $0 \leq r < n$. Note that $E_D, D \sim E_r$; indeed, $E_D$ is obtainable from $E_r$ by row operations, namely by first forming $DE_r$ and then, if $0 < r$, summing to the first row of $DE_r$ the $(n-r+1)$th row. Therefore we have

$$Q_{\gamma t-1}^t Q_{\alpha 0} = Q_{\gamma t}^t Q_{\alpha 0} = Q_{\alpha 0}^t Q_{\gamma t}^{m+1} \sim E_r DQ_0^{mn+1} \sim E_r Q_{\gamma t}^{m+1} \sim Q_{\gamma t}^t Q_{\alpha 0}^m = Q_0^t,$$

as claimed. \(\square\)

**Lemma 3.4.** Let $s > 0$, let $a_0, \ldots, a_{s-1} \in \{0, 1\}$, and let $M_* = M_{a_{s-1}} \cdots M_{a_0}$, $T_* = T_{a_{s-1}} \cdots T_{a_0}$. Then:

(i) there exists a unique point $q = (\alpha_1, \ldots, \alpha_n) \in \Delta$ such that $(\alpha_1, \ldots, \alpha_n) \in \Delta_0$ is a right eigenvector for $M_*$ whose corresponding eigenvalue $\xi$ is positive; we then have $Q(\xi) = Q(\alpha_1, \ldots, \alpha_n)$;

(ii) an analogous statement holds for $T_*; in this case $\xi = 1$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$.

**Proof.** Recall that we are identifying $\mathbb{R}^n$ with the plane $\{x_{n+1} = 1\}$ in $\mathbb{R}^{n+1}$, the latter viewed as a space of column vectors. Accordingly, given a simplex $\Sigma$ in $\mathbb{R}^n$, we write $\mathbb{R}_{\geq 0} \Sigma$ for the polyhedral cone $\{r(\alpha_1, \ldots, \alpha_n) : r \geq 0 \text{ and } (\alpha_1, \ldots, \alpha_n) \in \Sigma\}$. Let $\tilde{a} \in \{0, 1\}^n$ be defined by $a_k = a_k \text{ (mod } s)$. Then, for every $k \geq 0$, we have $M_*^k [\mathbb{R}_{\geq 0} \Delta] = \mathbb{R}_{\geq 0} \Delta_{\tilde{a}^t k}$. By Lemma 2.3 the intersection $\bigcap_{k \geq 0} \Delta_{\tilde{a}^t k}$ is the singleton of a point $q = (\alpha_1, \ldots, \alpha_n)$. This immediately implies that $M_*^{-1}$ has (up to scalar multiples) a unique eigenvector $\mathbb{R}_{\geq 0} \Delta$, namely $(\alpha_1, \ldots, \alpha_n)^{tr}$, whose corresponding eigenvalue $\xi^{-1}$ is positive, and the first statement in (i) follows.

Observe now that $V^{-1} M_*^{-1} V = A_{a_0} \cdots A_{a_{s-1}}$ is a nonnegative matrix. By the Perron-Frobenius theory (see, e.g., [3, Chapter III]), there exists a permutation matrix $P$ such that $P^{-1} A_{a_0} \cdots A_{a_{s-1}} P$ has the block form

$$
\begin{pmatrix}
E_1 & 0 & 0 & 0 \\
* & E_2 & 0 & 0 \\
* & * & \ddots & 0 \\
* & * & * & E_r
\end{pmatrix}
$$

with each $E_i$ a nonsingular primitive matrix. The $m \times m$ matrix $E_r$ has a dominant simple eigenvalue $\rho > 0$ whose corresponding one-dimensional right eigenspace is spanned by a strictly positive column vector $(\beta_1 \cdots \beta_m)^{tr} \in Q(\rho)^m$. Since $M_*^{-1}$ and $A_{a_0} \cdots A_{a_{s-1}}$ are conjugate by $V$, and the $V$-image of the positive orthant $\mathbb{R}_{\geq 0}^{n+1}$ of $\mathbb{R}^{n+1}$ is $\mathbb{R}_{\geq 0} \Delta$, we have from the first part of the proof that $A_{a_0} \cdots A_{a_{s-1}}$ has exactly (up to scalar multiples) one eigenvector in the positive orthant whose corresponding eigenvalue is positive. This eigenvector is necessarily $P(0 \cdots 0 \beta_1 \cdots \beta_m)^{tr}$, and $\rho = \xi^{-1}$. Going back to $\mathbb{R}_{\geq 0} \Delta$, we have that $(\alpha_1, \ldots, \alpha_n)^{tr}$ is a real multiple of $V P(0 \cdots 0 \beta_1 \cdots \beta_m)^{tr}$. Hence $(\alpha_1, \ldots, \alpha_n)^{tr}$ is a real multiple of a vector in $Q(\rho)^{n+1}$, and therefore $\alpha_1, \ldots, \alpha_n \in Q(\rho) = Q(\xi)$: since $M_*$ has integer entries, $\xi \in Q(\alpha_1, \ldots, \alpha_n)$. The same proof shows (ii); in this case $\xi = 1$ because the last row of $T_*$ is $(0 \cdots 0 1)$. \(\square\)

Lemma 3.4(i) should be compared with [4, Theorem 3.1] and [17, Theorem 42]. In both cases it is proved that a purely periodic point has coordinates in a field of the form $Q(\xi)$, for $\xi$ an eigenvalue of an appropriate periodicity matrix. However,
Theorem 3.5. Let \( p \in \Delta \). Then:

(i) \( p \) is rational iff \( \text{EPO}_M(p) = \{v_1\} \) iff \( p \) is a vertex of some \( F_t \);
(ii) \( p \) is dyadic iff \( \text{EPO}_T(p) = \{v_1\} \) iff \( p \) is a vertex of some \( B_t \).

In particular, the set of rational points \( \text{GO}_M(v_1) \) is mapped bijectively by \( \Phi \) to the set of dyadic points \( \text{GO}_T(v_1) \). Moreover, we have:

(iii) \( |\mathbb{Q}(p) : \mathbb{Q}| \leq n + 1 \) if \( p \) is \( M \)-preperiodic;
(iv) \( p \) is rational iff \( p \) is \( T \)-preperiodic.

Proof. By construction, the \( M \)-counterimage of the set of vertices of \( F_t \) is the set of vertices of \( F_{t+1} \). Moreover, the vertices \( v_1, \ldots, v_{n+1} \) of \( \Delta \) are such that \( M(v_1) = v_1 \) and \( M(v_j) = v_{j-1} \), for \( 2 \leq j \leq n + 1 \). Analogous statements hold for \( T \), so in (i) and (ii) the equivalence of the second condition with the third is clear.

(i) If \( \text{EPO}_M(p) = \{v_1\} \), then \( \mathbb{Q}(p) = \mathbb{Q}(v_1) = \mathbb{Q} \), and \( p \) is rational. Let \( p \) be rational, and let \( l_1, \ldots, l_{n+1} \in \mathbb{Z} \) be its primitive projective coordinates. Let \( p = \varphi(\bar{a}) \). Then, for every \( t \geq 0 \), \( p \in \Delta_{\bar{a}|t} \) and, since \( \Delta_{\bar{a}|t} \) is unimodular, there exist \( 0 \leq k_1(t), \ldots, k_{n+1}(t) \in \mathbb{Z} \) such that

\[
\begin{pmatrix}
  l_1 \\
  \vdots \\
  l_{n+1}
\end{pmatrix} = VA_{a_0} \cdots A_{a_{t-1}} \begin{pmatrix}
  k_1(t) \\
  \vdots \\
  k_{n+1}(t)
\end{pmatrix}.
\]

Let \( (c_1(t) \cdots c_{n+1}(t)) \) be the last row of \( VA_{a_0} \cdots A_{a_{t-1}} \). The reader can easily prove (compare with [9, pp. 40-41]) that \( 1 \leq c_1(t) \leq \cdots \leq c_{n+1}(t) \) and that the sequence \( \{c_2(t)\}_{t \geq 0} \) is nondecreasing, with limit \( \infty \). Let \( t \) be such that \( l_{n+1} < c_2(t) \). Then we must have \( k_1(t) = 1 \) and \( k_2(t) = \cdots = k_{n+1}(t) = 0 \). In other words, \( (l_1 \cdots l_{n+1})^{tr} \) is the first column of \( VA_{a_0} \cdots A_{a_{t-1}} \), and hence \( p \) is a vertex of \( F_t \).

(ii) If \( \text{EPO}_T(p) = \{v_1\} \), then \( \mathbb{Z}[1/2](p) = \mathbb{Z}[1/2](v_1) = \mathbb{Z}[1/2] \), and \( p \) is dyadic. Conversely, let \( p = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}[1/2]^n \) be dyadic, \( p = v(\bar{a}) \). Choose \( m \geq 0 \) such that \( 2^m p \in \mathbb{Z}^n \). Then, in projective coordinates, we have

\[
T^{mn}(p) = T_{a_{m-1}} \cdots T_{a_0} \begin{pmatrix}
  \alpha_1 \\
  \vdots \\
  \alpha_n \\
  1
\end{pmatrix}.
\]

By Lemma 3.3, there exists \( X \in \text{GL}_{n+1} \mathbb{Z} \) such that

\[
T^{mn}(p) = XT_{0}^{mn} \begin{pmatrix}
  \alpha_1 \\
  \vdots \\
  \alpha_n \\
  1
\end{pmatrix} = X \begin{pmatrix}
  2^m & \cdots & 2^m \\
  \vdots & \ddots & \vdots \\
  2^m & \cdots & 2^m
\end{pmatrix} \begin{pmatrix}
  \alpha_1 \\
  \vdots \\
  \alpha_n \\
  1
\end{pmatrix} \in \mathbb{Z}^{n+1}.
\]

Hence \( T^{mn}(p) \) is one of the vertices of \( \Delta \) and \( T^{mn+n}(p) = v_1 \).

(iii) Let \( (\alpha_1, \ldots, \alpha_n) = q = M^s(q) = \varphi(\bar{a}) \in \text{EPO}_M(p) \) for some \( s > 0 \). Then \( (\alpha_1 \cdots \alpha_n)^{tr} \) is a right eigenvector for the matrix \( M_{a_{s-1}} \cdots M_{a_0} \in \text{GL}_{n+1} \mathbb{Z} \). The
corresponding eigenvalue $\xi$ is a real algebraic unit of degree $\leq n + 1$, and by Lemma 3.4(i) we have $Q(p) = Q(q) = Q(\xi)$.

(iv) Let $p$ be rational, and let $0 < k \in \mathbb{Z}$ be such that $kp \in \mathbb{Z}^n$. Since $T_0$ and $T_1$ have both integer entries, the forward $T$-orbit of $p$ is contained in $\Delta \cap (k^{-1}\mathbb{Z})^n$, which is finite set; hence $p$ is preperiodic. The reverse implication is analogous to (iii), using Lemma 3.4(ii). $\square$

Finally, we discuss $\S 1(3)$, i.e., the singularity of $\Phi$ w.r.t. the Lebesgue measure $\lambda$. We normalize $\lambda$ so that $\lambda(\Delta) = 1$. Let $h(x_1, \ldots, x_n) \in L_1(\Delta, \lambda)$ be defined by

$$h(x_1, \ldots, x_n) = \frac{1}{x_1(x_1 - x_2 + 1)(x_1 - x_3 + 1) \cdots (x_1 - x_n + 1)};$$

and let $\mu$ be the probability measure on $\Delta$ induced by the density $h$, properly normalized (for the rest of this paper we are assuming $n \geq 2$, since otherwise $\mu$ is infinite):

$$\mu(A) = \int_A h \, d\lambda / \int_\Delta h \, d\lambda.$$  

The Mönkemeyer map $M$ preserves $\mu$ and is ergodic w.r.t. it [17, Theorems 23–24]. Note that in the above reference $M$ appears as the restriction of the Schmeil map to the absorbing $n$-simplex $D$ in [17, Theorem 22], and the invariant density is $h'(x_1, \ldots, x_n) = \prod_i x_i^{-1}$. We leave to the reader—as a simple exercise in the calculus of Jacobians—to check that our $h$ on $\Delta$ is the density corresponding to $h'$ on $D$.

Given an $n$-simplex $\Lambda$ in \{ $x_{n+1} = 1$ \} $\subset \mathbb{R}^{n+1}$, let $L$ be an $(n+1) \times (n+1)$ real matrix whose columns express the vertices of $\Lambda$ in projective coordinates, and such that the entries $L_{(n+1)1}, \ldots, L_{(n+1)(n+1)}$ in the last row are all $> 0$. Then one easily computes that

$$\lambda(\Lambda) = \frac{|\text{det}(L)|}{L_{(n+1)1} \cdots L_{(n+1)(n+1)}}.$$  

Applying this fact to $L = VB_{a_0} \cdots B_{a_{t-1}}$, we obtain

$$\lambda(\Gamma_{a_0 \ldots a_{t-1}}) = 2^{-t}. \tag{*}$$  

Remember that if $\rho : X \to Y$ is a Borel map and $\sigma$ is a Borel probability measure on $X$, then the push-forward of $\sigma$ by $\rho$ is the measure $\rho_* \sigma$ on $Y$ defined by $(\rho_* \sigma)(A) = \sigma(\rho^{-1}(A))$. If $\beta$ denotes the Bernoulli measure on $\{0,1\}^\mathbb{N}$ obtained by giving 0 and 1 equal weight $1/2$, the formula \( (*) \) implies that $\nu_* \beta = \lambda$ (because such an identity holds on the simplexes $\Delta_{\bar{a}[t]}$, for $\bar{a} \in \{0,1\}^\mathbb{N}$ and $t \geq 0$, and these simplexes generate the Borel sets in $\Delta$). Since $\nu$ induces a conjugation between the shift map $S$ on $\{0,1\}^\mathbb{N}$ and the tent map $T$ on $\Delta$, it follows that $T$ is ergodic w.r.t. $\lambda$, and hence $M$ is ergodic w.r.t. $\Phi^{-1} \lambda$. Now, $\mu$ and $\Phi^{-1} \lambda$ are different (e.g., $(\Phi^{-1}_* \lambda)(\Delta_0) = 1/2 \neq \mu(\Delta_0)$), and are both ergodic w.r.t. the same transformation $M$. Therefore they are mutually singular [21, Theorem 6.10(iv)], and there exists a measurable set $A \subset \Delta$ such that $\mu(A) = 1$ and $\lambda(\Phi[A]) = 0$. Since $h \geq 1$ on $\Delta$, we have $\mu \geq C\lambda$ for some constant $C > 0$. It follows that each of $\mu$ and $\lambda$ is absolutely continuous w.r.t. the other, and in particular they have the same sets of full measure. We conclude that $\lambda(A) = 1$, and $\Phi$ is singular w.r.t. $\lambda$.

If $p = \phi(\bar{a}) \in \Delta$, it is natural to look at the limit

$$\lim_{t \to \infty} \frac{\lambda(\Phi[\Delta_{\bar{a}[t]}])}{\lambda(\Delta_{\bar{a}[t]})}. \tag{**}$$
as an index of the singularity of $\Phi$ at $p$. As we already observed, $\lambda(\Phi|\Delta_{\bar{a}|t}) = 2^{-t}$. By the Shannon-McMillan-Breiman Theorem [3 §13] we have, for $\mu$-all $p$ (and hence for $\lambda$-all $p$, since $\mu$ and $\lambda$ have the same nullsets), that

$$\lim_{t \to \infty} -\frac{\log \mu(\Delta_{\bar{a}|t})}{t} = h_\mu,$$

where $h_\mu$ is the metrical entropy of $M$ w.r.t. $\mu$. Without loss of generality, we can assume that $p$ is in the topological interior of $\Delta$. For such a $p$, there exist $t_0$ and a constant $C > 0$ such that $C\mu(\Delta_{\bar{a}|t}) \leq \lambda(\Delta_{\bar{a}|t}) \leq C^{-1}\mu(\Delta_{\bar{a}|t})$, for all $t \geq t_0$. It follows that in the identity $(***)$ we can safely substitute $\mu$ with $\lambda$. The value $h_\mu$ is explicitly computed in [1 §5.2] as follows: if

$$G(n) = \int_0^1 \frac{\log(1 + s^n)}{s} ds,$$

then

$$h_\mu = \frac{(n + 1) G(n)}{n G(n - 1)}.$$

Taking logarithms in $(*)$ we have

$$\lim_{t \to \infty} \left[ \log \lambda(\Gamma_{\bar{a}|t}) - \log \lambda(\Delta_{\bar{a}|t}) \right] = \lim_{t \to \infty} \left( -\log 2 - \frac{\log \lambda(\Delta_{\bar{a}|t})}{t} \right) t = \lim_{t \to \infty} (-\log 2 - h_\mu)t.$$

For $n = 2$ we have $h_\mu \sim 0.54807 \ldots$ and, as shown in [1 §5.2], $h_\mu$ is monotonically increasing with $n$, tending to the limit $\log 2 \sim 0.69314 \ldots$ — which is the topological entropy of $M$ in every dimension — as $n$ goes to infinity. We conclude that, for $\lambda$-all $p$ and every $n \geq 2$, the limit $(**)$ approaches 0 exponentially fast. On the other hand, since $\lim_{n \to \infty} (\log 2 - h_\mu) = 0$, we might loosely say that the singularity of $\Phi$ decreases with the dimension.

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