COHOMOGENEITY-ONE SOLITONS IN LAPLACIAN FLOW:
LOCAL, SMOOTHLY-CLOSING AND STEADY SOLITONS

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Abstract. We initiate a systematic study of cohomogeneity-one solitons in Bryant’s Laplacian flow of closed $G_2$-structures on a 7-manifold, motivated by the problem of understanding finite-time singularities of that flow. Here we focus on solitons with symmetry groups $Sp(2)$ and $SU(3)$; in both cases we prove the existence of continuous families of local cohomogeneity-one gradient Laplacian solitons and characterise which of these local solutions extend smoothly over their unique singular orbits. The main questions are then to determine which of these smoothly-closing solutions extend to complete solitons and furthermore to understand the asymptotic geometry of these complete solitons.

We provide complete answers to both questions in the case of steady solitons. Up to the actions of scaling and discrete symmetries, we show that the set of all smoothly-closing $SU(3)$-invariant steady Laplacian solitons defined on a neighbourhood of the zero-section of $\Lambda^2 \mathbb{C}P^2$ is parametrised by $\mathbb{R}_{\geq 0}$, the set of nonnegative reals. We then determine precisely which of these solutions extend to a complete soliton defined on the whole of $\Lambda^2 \mathbb{C}P^2$. An open interval $I = (0, c_* \cap \mathbb{R}_{\geq 0}$ corresponds to complete nontrivial gradient solitons that are asymptotic to the unique $SU(3)$-invariant torsion-free $G_2$-cone. The point $0 \in \partial I$ corresponds to the well-known Bryant–Salamon asymptotically conical torsion-free structure on $\Lambda^2 \mathbb{C}P^2$ viewed as a trivial steady soliton, while the other point $c_* \in \partial I$ corresponds to an explicit complete gradient steady soliton with exponential volume growth and novel asymptotic geometry. The open interval $(c_*, \infty)$ consists entirely of incomplete solutions.

In addition, we find an explicit complete gradient shrinking soliton on $\Lambda^2 S^4$ and $\Lambda^2 \mathbb{C}P^2$. Both these shrinkers are asymptotic to closed but non-torsion-free $G_2$-cones. Like the nontrivial AC gradient steady solitons on $\Lambda^2 \mathbb{C}P^2$, these shrinkers appear to be potential singularity models for finite-time singularities of Laplacian flow. We also compare the behaviour of the Laplacian solitons we construct to solitons in Ricci flow.

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1. Introduction and main results

1.1. Laplacian flow and torsion-free $G_2$-structures. Bryant’s Laplacian flow is a weakly parabolic geometric flow of closed positive 3-forms (closed $G_2$-structures) on a 7-manifold which evolves a closed 3-form in the direction of its Hodge-Laplacian. Laplacian flow arises as an upward gradient flow for Hitchin’s volume functional: its stationary points are torsion-free $G_2$-structures and these are necessarily local maxima of the volume functional. Some further basic geometric and analytic features of Laplacian flow and its solitons will be reviewed in Section 2.

Any torsion-free $G_2$-structure induces a Ricci-flat Riemannian metric whose holonomy group reduces to a subgroup of the compact exceptional simple Lie group $G_2$. Currently the only known source of Ricci-flat metrics on compact simply connected odd-dimensional manifolds are such $G_2$-holonomy metrics. Several methods are now known for constructing compact $G_2$-holonomy manifolds [27,39,40,43], all based on methods of nonlinear elliptic PDEs and, more specifically, on variations on the degeneration/gluing/perturbation method pioneered by Joyce in his construction of the first compact $G_2$-holonomy manifolds [41]. Although several obstructions to the existence of $G_2$-holonomy metrics are known, currently we know rather little about which compact orientable spin 7-manifolds can admit such metrics.

A long-term goal in Laplacian flow would be to use the flow to give a parabolic approach to construct torsion-free $G_2$-structures on compact 7-manifolds, potentially shedding some light on which manifolds can admit such structures. To produce torsion-free $G_2$-structures via Laplacian flow one needs to establish long-time existence and convergence results for the flow. A major obstruction to proving long-time existence results for the flow is that in general finite-time singularities are expected to develop. However, the study of singularity formation in Laplacian flow is still in its infancy, especially compared to the high level of development now achieved in understanding

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singularities in various better-known geometric flows, *e.g.* Ricci flow and Kähler–Ricci flow, mean curvature flow (especially in the codimension one and Lagrangian settings), Yamabe flow, Yang–Mills flow and harmonic map heat flow.

1.2. **Laplacian solitons.** Given a smooth 7-manifold $M$ (compact or noncompact), a triple $(\phi, X, \lambda)$ consisting of a $G_2$–structure $\phi$, a vector field $X$ and a constant $\lambda \in \mathbb{R}$ is called a *Laplacian soliton* if the triple satisfies the following system of partial differential equations

\[
\begin{aligned}
\{ d\phi &= 0, \\
\Delta_{\phi} \phi &= \lambda \phi + \mathcal{L}_X \phi.
\end{aligned}
\]

The second equation is nonlinear in $\phi$ because $\Delta_{\phi}$ depends on the Hodge-star operator of the metric $g_{\phi}$ induced by $\phi$ and $g_{\phi}$ depends nonlinearly on $\phi$. Any Laplacian soliton $(\phi, X, \lambda)$ gives rise to a self-similar solution to the Laplacian flow. We call the soliton shrinking if $\lambda < 0$, steady if $\lambda = 0$ and expanding if $\lambda > 0$; we call $\lambda$ the dilation constant of the soliton.

The Laplacian soliton equations constitute a diffeomorphism-invariant overdetermined system of nonlinear PDEs for the pair $(\phi, X)$. Given its overdetermined nature a priori it is unclear how plentiful solutions of (LSE) are even locally. However, Bryant [12] has used the methods of overdetermined PDEs to prove that locally there are in fact many solutions of (LSE) (though questions remain about the local generality of solutions when the vector field $X$ is assumed to be gradient). The question therefore is how to produce global solutions to (LSE), by which we mean solitons on compact manifolds or on complete noncompact manifolds (in the compact case the only nontrivial solitons could be expanders, though no such examples are currently known).

There are no general analytic methods currently available that can produce global solutions to (LSE), a problem that one also faces in the study of Ricci solitons (except in the Kähler setting where recently complex Monge–Ampère methods have enabled general analytic constructions of Kähler–Ricci solitons [24–26]). Therefore we are led to study solutions of (LSE) on which we impose additional geometric structures that makes their construction more tractable. For many nonlinear geometric PDEs a natural approach is to impose a continuous group of symmetries on the problem. In particular, in Ricci flow many of the known solitons arise by considering cohomogeneity-one group actions, *i.e.* where the generic orbit of the symmetry group has codimension one. Imposing such a group of symmetries reduces the system of nonlinear PDEs governing solitons to a system of nonlinear ODEs. Some of the best-known examples include: Hamilton’s cigar soliton [37]; Bryant’s rotationally-invariant steady soliton and his 1-parameter family of expanders [13]; Cao’s $U(n)$-invariant Kähler expanders and steady solitons on $\mathbb{C}^n$ and the Feldman–Ilmanen–Knopf $U(n)$-invariant Kähler shrinkers [29]; the noncollapsed steady non-Kähler solitons found recently by Appleton [4], and the infinite discrete family of complete gradient asymptotically conical shrinkers and expanders on $\mathbb{R}^p \times S^{q-1}$ (with $p, q \geq 3$ and $p + q \leq 10$) found recently by Angenent–Knopf [2]. General features of cohomogeneity-one Ricci solitons were studied in a whole series of papers by Dancer and Wang.

1.3. **Cohomogeneity-one Laplacian solitons.** The problem of understanding cohomogeneity-one Laplacian solitons can naturally be divided up into two distinct steps, each of which has a different character. The first step is to understand which Lie groups can possibly act with cohomogeneity one on closed $G_2$-manifolds: this problem has already been solved by Cleyton–Swann [23] (see also Cleyton’s thesis). The methods needed here are (not surprisingly) mainly of a Lie-theoretic nature. The second step is to derive the equations for $G$-invariant Laplacian solitons for each possible group action $G$ and then to study the resulting nonlinear system of ODEs.

The second step is much more involved than the first. The fundamental difficulty is that for a typical group $G$ the general solution of the nonlinear ODEs governing $G$-invariant Laplacian solitons does not correspond to a complete soliton and therefore one has to find a way to recognise which (if any) of the local solutions represent complete solutions. The difficulties are further compounded by two factors: in most cases the general solutions to the ODE systems are not explicitly available;
also we do not know in advance what types of asymptotic geometry complete noncompact solitons may exhibit. As a result of these issues there are no general systematic approaches to determining which of the local solutions extend to complete ones.

In this paper we begin a detailed study of the geometry of cohomogeneity-one Laplacian solitons with principal orbit type \( \mathbb{CP}^3 \) and \( F_{1,2} := SU(3)/T^2 \); these have symmetry groups \( Sp(2) \) and \( SU(3) \) respectively. It will turn out that the ODE system governing \( Sp(2) \)-invariant Laplacian solitons can be regarded as a special case of the system governing \( SU(3) \)-invariant solitons. We have several reasons for singling out these two among the seven types of simply connected principal orbit that can occur for cohomogeneity-one \( G_2 \)-structures [23, Theorem 3.1]. We explain those reasons in detail in our review of the basics of cohomogeneity-one \( G_2 \)-structures in Section 3.4. However, the ultimate justification for our choice of principal orbit types is that they do indeed lead to the existence of interesting complete shrinking, steady and expanding gradient Laplacian solitons.

1.4. Main results of the paper. We now describe the main results of this paper.

The ODE systems for \( G \)-invariant Laplacian solitons. On the face of it the nonlinear ODE system governing \( SU(3) \)-invariant Laplacian solitons consists of five mixed-order differential equations for four unknown functions \((f_1, f_2, f_3, u)\) defined on some interval \( I \subset \mathbb{R} \). The triple \( f = (f_1, f_2, f_3) \) determines an \( SU(3) \)-invariant \( G_2 \)-structure while \( u \) determines the invariant vector field \( X = u \partial_t \).

At a given value of \( t \in I \), the triple \( f \) determines an \( SU(3) \)-invariant Riemannian metric \( g_f \) on \( SU(3)/T^2 \). There are three distinct homogeneous \( S^2 \)-fibrations of \( SU(3)/T^2 \)

\[
\mathbb{S}_j^2 = U(2)_j/T^2 \to SU(3)/T^2 \to SU(3)/U(2)_j = \mathbb{CP}^2
\]

that arise from three different \( U(2)_j \) subgroups of \( SU(3) \) all of which contain the diagonal subgroup \( T^2 \subset SU(3) \); the three fibres \( \mathbb{S}_j^2 \) are mutually orthogonal with respect to any homogeneous metric \( g \) on \( SU(3)/T^2 \), and the restriction of \( g \) to the \( j \)th fibre determines a homogeneous metric \( g_j \) on \( \mathbb{S}_j^2 \). Moreover, \( g \) is determined by the size of these three fibres at a single point, \( i.e. \) by three positive parameters. With appropriate normalisations each of the invariant metrics \( g_j \) is just the standard round metric of sectional curvature 1. In other words, the geometric interpretation of the components of the triple \( f \) is that the \( j \)th component \( f \) determines the “size” of the \( j \)th spherical fibre \( \mathbb{S}_j^2 \) with respect to the homogeneous metric \( g_f \).

For most purposes it is more convenient not to work directly with the mixed-order system (4.6) but instead to work with an equivalent real-analytic first-order system. The variables in this first-order reformulation are the previous triple of functions \( f = (f_1, f_2, f_3) \) along with a further triple of functions \((\tau_1, \tau_2, \tau_3)\) that describes the torsion 2-form \( \tau \) of the \( SU(3) \)-invariant closed \( G_2 \)-structure determined by the triple \( f \). The algebraic constraints on the torsion of a closed \( G_2 \)-structure, \( i.e. \) that the torsion \( \tau \) is a 2-form of type 14, imply a single algebraic constraint on \((f, \tau)\).

An \( SU(3) \)-invariant Laplacian soliton can therefore be interpreted as an integral curve of an explicit vector field (depending on the dilation constant \( \lambda \)) on a 5-dimensional smooth noncompact phase space \( \mathcal{P}^5 \subset \mathbb{R}^3 \times \mathbb{R}^3 \). An \( Sp(2) \)-invariant Laplacian soliton can be interpreted as an integral curve of the restriction of this vector field to the 3-dimensional invariant submanifold \( \mathcal{P}^3 \subset \mathcal{P}^5 \) obtained by imposing the conditions \( f_2 = f_3 \) and \( \tau_2 = \tau_3 \). These first-order reformulations of the ODE system for \( G \)-invariant Laplacian solitons immediately imply the following result about the space of all local \( G \)-invariant Laplacian solitons.

Theorem A (Local \( G \)-invariant Laplacian solitons).

(i) the space of local \( G \)-invariant nonsteady solitons up to scale is 4-dimensional for \( G = SU(3) \) and 2-dimensional for \( G = Sp(2) \);

(ii) the space of local \( G \)-invariant steady solitons up to scale is 3-dimensional for \( G = SU(3) \) and 1-dimensional for \( G = Sp(2) \).
Here a local cohomogeneity-one soliton is a solution of the soliton ODE system that exists on $I \times P$ for some interval $I$ where $P$ denotes the principal orbit type.

The reduction by one in the parameter count in the steady cases arises because of their invariance under rescaling (while rescaling a nonsteady soliton also changes the value of $\lambda$).

Next we want to attempt to understand which of these local $G$-invariant Laplacian solitons extends to a complete Laplacian soliton. While there is no systematic way to address this completeness question there is however one aspect of the completeness issue that can be studied systematically: the problem of extending a $G$-invariant Laplacian soliton smoothly over a so-called singular orbit.

$G$-invariant Laplacian solitons that extend smoothly over the singular orbit. For the actions we are considering Cleyn–Swann’s work on the structure of cohomogeneity-one $G_2$–structures [23] implies that any complete $G$-invariant Laplacian soliton must possess a unique singular orbit, i.e. a nongeneric $G$-orbit, which in our cases is necessarily of lower dimension. Moreover, the structure of this lower-dimensional singular orbit is determined by $G$: $\mathbb{C}P^2 \cong SU(3)/U(2)$ for $G = SU(3)$ and $S^4 \cong Sp(2)/Sp(1) \times Sp(1)$ for $G = Sp(2)$. To understand which of the local $G$-invariant Laplacian solitons extend to complete Laplacian solitons (which would necessarily be defined on $\Lambda^2 \mathbb{C}P^2$ or on $\Lambda^2 S^4$ respectively) our first task is therefore to understand which $G$-invariant Laplacian solitons extend smoothly over the (unique) singular orbit. There is a relatively systematic way to understand such so-called smooth closure problems and applying these methods leads to the following result.

**Theorem B** (Smoothly-closing $G$-invariant Laplacian solitons). Fix any $\lambda \in \mathbb{R}$.

(i) For $G = Sp(2)$, among the 2-dimensional space of local $G$-invariant Laplacian solitons with dilation constant $\lambda$ defined on $I \times \mathbb{C}P^3$ for some interval $I \subset \mathbb{R}$, there is a 1-parameter family of smoothly-closing $G$-invariant Laplacian solitons, i.e. a $G$-invariant Laplacian soliton that extends smoothly over the zero-section $S^4 \subset \Lambda^2 S^4$.

(ii) For $G = SU(3)$, among the 4-dimensional space of local $G$-invariant Laplacian solitons with dilation constant $\lambda$ defined on $I \times SU(3)/T^2$ for some interval $I \subset \mathbb{R}$, there is a 2-parameter family of smoothly-closing $G$-invariant Laplacian solitons, i.e. a $G$-invariant Laplacian soliton that extends smoothly over the zero-section $\mathbb{C}P^2 \subset \Lambda^2 \mathbb{C}P^2$.

Theorem B holds for shrinkers, expanders and steady solitons (and the proof turns out to be insensitive to the sign or the dilation constant $\lambda$). However, because of the scaling invariance of steady solitons Theorem B implies:

Up to rescaling there is unique smoothly-closing $Sp(2)$-invariant steady soliton and a 1-parameter family of distinct smoothly-closing $SU(3)$-invariant steady solitons.

In the $Sp(2)$-invariant case the Bryant–Salamon asymptotically conical $G_2$–holonomy metric on $\Lambda^2 S^4$ already provides a smoothly-closing (trivial) steady soliton. It follows that any smoothly-closing $Sp(2)$-invariant steady soliton must be trivial, i.e. torsion-free with vanishing soliton vector field. Therefore to find complete nontrivial steady solitons we must look at the $SU(3)$-invariant setting.

**Complete Laplacian solitons.** The next tasks are to try to understand which of the smoothly-closing solitons constructed in Theorem B give rise to complete solitons and then to determine the asymptotic geometry of these complete solutions. In full generality this is a highly nontrivial task for the reasons already explained. In this paper we find an explicit complete $G$-invariant shrinking soliton for both $G = SU(3)$ and $G = Sp(2)$ and achieve a full understanding of all the complete $SU(3)$-invariant steady Laplacian solitons. In the sequel we study complete $G$-invariant Laplacian expanders and shrinkers more generally.

**Theorem C.** There exists an explicit complete noncompact $Sp(2)$-invariant gradient shrinking soliton on $\Lambda^2 S^4$ with principal orbit $\mathbb{C}P^3$ and an explicit complete noncompact $SU(3)$-invariant gradient shrinking soliton on $\Lambda^2 \mathbb{C}P^2$ with principal orbit $SU(3)/T^2$. Both shrinking solitons are asymptotic to closed but non-torsion-free $G$-invariant $G_2$-cones.
In terms of the triple $f$ of functions described in (1.1) above (see also Lemmas 3.20 and 3.46 for more precise definitions) and the function $u$ determining the soliton vector field $X = u \partial_t$, the precise statement is that for any $b > 0$ the quadruple $(f_1, f_2, f_3, u)$

$$f_1 = t, \quad f_2 = f_3 = \sqrt{b^2 + \frac{1}{4} t^2}, \quad u = \frac{3t}{4b^2} + \frac{4t}{4b^2 + t^2},$$

is a complete AC shrinker with dilation constant $\lambda = -\frac{9}{4b^2}$. (The AC end behaviour is encoded by each of the $f_i$ being asymptotically linear.)

Shrinking solitons are usually the rarest type of soliton, reflecting the hope/expectation that finite-time singularities of a ‘nice’ geometric flow beginning with smooth initial data on a compact manifold cannot be arbitrarily bad.

The asymptotically conical nature of these explicit complete Laplacian shrinkers motivates the further study of asymptotically conical Laplacian solitons. Indeed, it will turn out that there are complete $G$-invariant asymptotically conical Laplacian solitons of all three types: shrinkers, expanders and steady solitons. However, there is a fundamental difference between the steady case and the shrinker/expander cases: the asymptotic cones of AC ends of steady $G$-invariant Laplacian solitons must be torsion-free. In this paper we deal only with the case of complete steady solitons; to understand AC ends of $G$-invariant expanders or shrinkers requires the development of substantial additional theory and will be described in the sequel.

**Theorem D.** Among the 1-parameter subfamily $S_{1,c}$ of smoothly-closing SU(3)-invariant steady gradient solitons on $\Lambda^2 \mathbb{C}P^2$ the 1-parameter subfamily with $c^2 \leq \frac{9}{2}$ consists of complete solitons, while the 1-parameter subfamily with $c^2 > \frac{9}{2}$ consists entirely of incomplete solitons. Moreover we have the following additional properties

(i) Any complete SU(3)-invariant steady soliton with principal orbit $\text{SU}(3)/\mathbb{T}^2$ belongs to this family (up to scaling and discrete symmetries).

(ii) $S_{1,0}$ is a trivial steady soliton, arising from the Bryant–Salamon $G_2$-holonomy metric on $\Lambda^2 \mathbb{C}P^2$. It is asymptotic with rate $-4$ to the cone $C_{1t}$, the unique SU(3)-invariant torsion-free $G_2$-cone over $\text{SU}(3)/\mathbb{T}^2$.

(iii) For $0 < c^2 < \frac{9}{2}$, $S_{1,c}$ is a nontrivial steady soliton asymptotic with rate $-1$ to the cone $C_{1t}$.

(iv) For $c^2 = \frac{9}{2}$, $S_{1,c}$ is a complete steady soliton on $\Lambda^2 \mathbb{C}P^2$ which has exponential volume growth, and asymptotically constant negative scalar curvature.

For $c = 3/\sqrt{2}$ the solution in (iv) has the explicit expression

$$f_1 = 2 \sinh \frac{t}{2}, \quad f_2 = \sqrt{1 + e^t}, \quad f_3 = \sqrt{1 + e^{-t}}, \quad u = \tanh \frac{t}{2}.$$  

(The Bryant–Salamon torsion-free $G_2$-structure in (ii) is described in Example 3.43.)

Steady solitons, being eternal solutions have features of both ancient solutions (like shrinking solitons) and of immortal solutions (like expanding solitons). Compared to the well-known steady solitons in Ricci flow and Kähler–Ricci flow, *e.g.* Hamilton’s cigar soliton [37], Bryant’s unique rotationally-invariant steady soliton in each dimension $n \geq 3$ [13] and Cao’s $U(n)$-invariant steady Kähler solitons on $\mathbb{C}^n$ [19], the existence of such asymptotically conical steady solitons is a distinctive
feature of Laplacian flow. In fact, it is impossible to have a nontrivial steady soliton in Ricci flow that is asymptotic to a Ricci-flat cone whose cross-section is smooth (there are of course various well-known AC shrinking Ricci solitons and expanding Ricci solitons asymptotic to regular cones).

The asymptotic geometry of the explicit steady soliton (1.2), with its exponential volume growth and asymptotically constant negative scalar curvature, is further removed yet from the asymptotic behaviour of Ricci solitons. We can describe some basic features of the asymptotic geometry of the explicit steady soliton relatively easily.

Recall that the sinh-cone over a compact Einstein manifold $E$ with positive Einstein constant is the product $\mathbb{R}_{\geq 0} \times E$ endowed with the Riemannian metric $g = dr^2 + \sinh^2 r \, g_E$. The sinh-cone over the round $n - 1$ sphere of radius 1 yields the standard warped product description of the hyperbolic metric on $\mathbb{R}^n$. In general, the sinh-cone over $E$ is a mildly-singular Einstein space with negative Einstein constant; it has a single isolated conical singularity at $r = 0$ modelled on the Ricci-flat cone $g_C = dr^2 + r^2 g_E$, and a complete end with exponential volume growth as $r \to \infty$. The asymptotic geometry of the explicit steady soliton is modelled by a nontrivial 2-sphere bundle over the complete end of the sinh-cone over $\mathbb{CP}^2$ endowed with the Fubini-Study metric. This 2-sphere fibration structure arises from viewing the flag variety $F_{1, 2} = SU(3)/T^2$ as a homogeneous 2-sphere fibration over $\mathbb{CP}^2 = SU(3)/U(2)$, cf. (1.1): the base metric on $\mathbb{CP}^2$ is controlled by the coefficients $f_1$ and $f_2$ of the triple $f$, whereas $f_3$ controls the scale of the round 2-sphere fibre. That the base metric is well approximated by the complete end of the sinh-cone over $\mathbb{CP}^2$ is due to the fact that for $t$ sufficiently large, (1.2) has $f_1 \simeq f_2 \simeq \sinh t$; in contrast to the exponential growth of $f_1$ and $f_2$, the coefficient $f_3$ is bounded and converges asymptotically to 1, i.e. the $S^2$-fibre asymptotically has unit size.

1.5. Organisation of the paper. Here we describe in some detail the structure of the paper, explain some of its overall logic and roughly how the proofs of our main theorems proceed.

Section 2 gives a rapid account of Laplacian flow and some of its basic features; it describes some results that have been proven about it and also other aspects of Laplacian flow that remain open. This material is intended in part for readers familiar with other geometric flows, like Ricci flow, but not with Laplacian flow or with solitons in Laplacian flow, and also in part to motivate our study of solitons in Laplacian flow.

Section 3 summarises the facts that we will need in the rest of the paper about cohomogeneity-one closed $G_2$-structures. The material of this section is used mainly to derive the fundamental ODEs governing $G$-invariant Laplacian solitons in Section 4. Much of this material appears in some form already in Cleyton’s thesis and in Cleyton–Swann [23], but the perspective adopted in Section 3.3 gives us a slightly different viewpoint. The main point is that we have chosen to make systematic use of the description of a cohomogeneity-one closed $G_2$-structure in terms of a 1-parameter family of $SU(3)$-structures (with some constraints on their intrinsic torsion). We have also chosen to describe in detail some aspects related to the discrete symmetries of these structures: these play an important role in a couple of places later in the paper. However, our main reason for treating this material in detail, rather than simply quoting more extensively from Cleyton–Swann, is our desire to make the paper more accessible (and self-contained) for those familiar with Ricci solitons, but who are perhaps much less familiar with $G_2$-geometry.

Section 4 derives the systems of nonlinear ODEs (4.6) and (4.7) governing $SU(3)$-invariant and $Sp(2)$-invariant Laplacian solitons. As previously mentioned, rather than work directly with the mixed-order systems (4.6) and (4.7), we instead prefer to work with equivalent real-analytic first-order systems that we derive in Section 4.3. The variables in this reformulation are a pair of triples $f = (f_i)$ and $\tau = (\tau_i)$ defined on some interval $I \subset \mathbb{R}$. Any such triple $f$ (subject to satisfying one scalar differential equation) determines a closed $SU(3)$-invariant $G_2$-structure and $\tau$ represents its torsion 2-form. In this first-order reformulation the soliton vector field $X = u \partial_x$ is no longer explicit.
in the ODE problem, but is recovered algebraically from \( f, \tau \) and the dilation constant \( \lambda \). Theorem A follows easily from our first-order reformulation of the soliton ODEs. Finally Section 4.4 describes a reformulation of the first-order ODE system in terms of one explicitly scale-dependent variable \( g := (f_1 f_2 f_3)^{1/3} \) and scale-normalised versions of the triples \( f \) and \( \tau \). This scale-normalised ODE system (4.30) will play a key role in our understanding of the behaviour (and construction) of AC solitonic ends, both in this paper and in the sequel. In this paper we will only need to study (4.30) in the steady case where it has additional special features that we will describe in Section 6.3.

Section 5 deals with the so-called smooth closure problem: understanding which of the local \( G \)-invariant solitons extend smoothly over a singular orbit. There is a slightly involved, but by now well-understood, method for systematically addressing such smooth extension questions and then constructing all smoothly-closing solutions. The general approach originates in the setting of cohomogeneity-one Einstein metrics in the work of Eschenburg–Wang [28]; see also [31, 33] where this approach was adapted to solve the same problem for cohomogeneity-one torsion-free \( G_2 \)-structures and nearly Kähler structures respectively and Maria Buzano’s work [18] in the setting of cohomogeneity-one gradient Ricci solitons. We work out the details of the smooth closure problem in our particular setting in Section 5 using the first-order reformulations of the \( G \)-invariant Laplacian soliton equations. The initial conditions guaranteeing that an \( SU(3) \)-invariant closed \( G_2 \)-structure extend smoothly over \( \mathbb{CP}^2 \) are described in Section 5.1. These initial conditions lead to a singular initial value problem for the first-order reformulation (4.12) of the \( SU(3) \)-invariant Laplacian soliton ODE system. The singularities of this singular initial value problem are of so-called regular type: this means that any formal power series solution has a nonzero radius of convergence and hence defines a real-analytic solution to the system. Analysing this initial value problem leads to the statements claimed in Theorem B.

The power series solutions for the smoothly-closing \( SU(3) \)-invariant Laplacian solitons can in theory be computed algorithmically in terms of the dilation constant \( \lambda \) and the two parameters \( b > 0 \) and \( c \in \mathbb{R} \) that determine the admissible initial data for the singular initial value problem. The first several terms of these power series solutions are detailed in Appendix A. The power series solutions for smoothly-closing \( Sp(2) \)-invariant Laplacian solitons are obtained by setting \( c = 0 \) in the \( SU(3) \)-invariant power series. The specific forms of the first several nonzero terms of these power series solutions turn out to play important roles at a couple of key points in this paper.

One immediate application is the following: by careful inspection of the power series solutions in the \( Sp(2) \)-invariant case we notice that significant simplifications occur when \( \lambda \) and \( b \) satisfy \(-4b^2 \lambda = 9\). This leads us to Theorem 5.20 in which we find explicitly the smoothly-closing shrinkers on \( A_2 \mathbb{S}^4 \) satisfying \(-4b^2 \lambda = 9\). This shrinker is complete and is asymptotic to a closed \( Sp(2) \)-invariant non-torsion-free \( G_2 \)-cone. The same solution also represents an \( SU(3) \)-invariant complete asymptotically conical shrinker on \( A_2 \mathbb{CP}^2 \) which possesses an additional free isometric \( \mathbb{Z}_2 \) action. Acting with the discrete symmetries available produces two further variants of this AC shrinker on \( A_2 \mathbb{CP}^2 \); these differ from each other by their singular orbit structure and the closed \( SU(3) \)-invariant \( G_2 \)-cone to which they are asymptotic.

Section 6 develops the theory of complete \( SU(3) \)-invariant steady solitons in its entirety, culminating in the proof of Theorem D; in the next subsection we describe some of the main ideas that go into its proof. Finally in Section 7 we compare some of the Laplacian solitons we have constructed with analogous known (or conjectured) Ricci solitons, describing some key differences.

1.6. Some key ideas from Section 6. Understanding \( SU(3) \)-invariant steady Laplacian solitons turns out to be significantly simpler than shrinkers or expanders; in large part this is due to the fact that the three quantities \( C_i := \tau_i - u f_i^2 \) are conserved for any steady soliton (but not on shrinkers or expanders). The torsion 2-form \( \tau \) of a general closed \( SU(3) \)-invariant \( G_2 \)-structure involves both the triple \( f \) and its first derivatives: see (3.38a). Using the three conserved quantities \( C_i \) we can
express the torsion $\tau$ of a steady soliton in terms of the values of the three conserved quantities and rational functions of the triple $f$ alone, i.e. with no first derivatives of $f$ involved. This allows us to derive a reduced (self-contained) first-order ODE system for the triple $f$ whose coefficients depend on the values $c_i$ of the three conserved quantities $C_i$. For smoothly-closing steady solitons, the $c_i$ are determined from a single real number $c$ by $(c_1, c_2, c_3) = (0, c, -c)$, and we obtain the first-order rational ODEs

$$\frac{d}{dt} \ln f_i^2 = u + \frac{c_i}{f_i^2} + \frac{1}{f_1 f_2 f_3} (\overline{f}^2 - 2 f_i^2) = \frac{c}{3} \left( \frac{1}{f_3^2} - \frac{1}{f_i^2} \right) + \frac{c_i}{f_i^2} + \frac{1}{f_1 f_2 f_3} (\overline{f}^2 - 2 f_i^2),$$

where $\overline{f}^2 := \sum f_i^2$. Furthermore, the variable change

$$F_i := \frac{f_i^2}{f_1 f_2 f_3}$$

leads to another positive triple $F$ which now satisfies the polynomial ODE system

$$(1.3) \quad \frac{d}{dt} \ln F_i^2 = -\frac{c}{3} F_1 (F_2 - F_3) + (c_i F_j F_k - c_j F_i F_k - c_k F_i F_j) + (F_j + F_k - 3 F_i).$$

The first step is to derive some qualitative properties of solutions to these smoothly-closing steady ODE systems. From now on we use the scaling symmetry of steady solitons (and the action of the discrete symmetries available) to normalise so that $c = 3$; this is convenient because then all the solutions we are considering have the same values of the conserved quantities $C_i$. (This is different from the normalisation used in the statement of Theorem D, where we found it more convenient to use the scaling symmetry to fix $b = 1$ within the 2-parameter family of smoothly-closing steady solitons $S_{b,c}$.)

The first important observation is that by considering the sign of $(f_i - f_j)'$ at points where $f_i = f_j$ one can prove that certain orderings of the coefficients $f_i$ are preserved. Using this observation and the power series solutions for smoothly-closing steady solitons to establish an initial ordering of the three coefficients $f_i$, one finds that the coefficient $f_2$ is dominant, i.e. $f_2 > \max (f_1, f_3)$, throughout the lifetime of any smoothly-closing steady soliton. That helps to show that for any complete solution, $f_1$ and $f_2$ have comparable growth rates and must grow at least linearly in the arclength parameter $t$, whereas $f_3$ grows at most linearly in $t$. Developing these ideas further, we show that if a smoothly-closing steady soliton is defined on the finite interval $[0, T)$ then $T$ is its maximal existence time if and only if $\lim_{t \to T} f_3 = 0$; in the opposite direction, we show that $f_3$ is bounded away from 0 for all $t > 0$ if the solution is complete.

It turns out that there are two fundamentally different types of asymptotic behaviour that a complete steady soliton can exhibit. We now explain one possible way to understand this fact. To understand possible asymptotic behaviour of complete steady solitons it is natural to look at the polynomial version of the steady ODE system, because asymptotic behaviour that occurs as some of the $f_i$ tend to infinity can instead occur at finite values of the $F_i$. It is therefore natural to consider all the fixed points of (1.3) and understand their local stable manifolds.

The polynomial ODE system (1.3) has no fixed points in the positive octant, but fixed points do occur on the boundary of the positive octant: the origin and the point $(1, 1, 0)$. The fixed point $(1, 1, 0)$ turns out to be a hyperbolic fixed point with a 2-dimensional local stable manifold. This immediately implies the existence of a 1-parameter family of forward-complete steady ends with exponential volume growth.

The origin, on the other hand, is a totally degenerate fixed point of (1.3). To understand the behaviour of solutions asymptotic to the origin it is better to consider another variation of the steady ODE system: the scale-invariant version of the system. The latter system turns out to have a unique fixed point that corresponds to $C_{15}$, the unique SU(3)-invariant torsion-free $G_2$-structure on the cone over SU(3)/$\mathbb{T}^2$. This fixed point is strictly stable and hence there is an open subset of the
3-dimensional space of local steady solitons consisting of forward-complete steady ends that are all asymptotic to the cone $C^\text{tf}$ (generically with rate $-1$).

Part of the content of Theorem D is that the asymptotic behaviour of any complete steady soliton is modelled by one of these two types of steady end and moreover that an exponential volume growth end occurs only at transitions between an open subset consisting of complete AC solutions and an open subset consisting of incomplete smoothly-closing solutions. To prove Theorem D, we focus on the explicit solution (1.2) that occurs at the transition. When rescaled to account for the change of normalisation from $b=1$ to $c=3$, this critical solution occurs at $b^2=2$. In the $F_i$ coordinates it is asymptotic to the fixed point $(1,1,0)$, and it traces out the curve in the positive octant defined by the algebraic relations

$$F_1(F_2 + F_3) = 1, \quad F_2 - F_3 = 1.$$ 

For the other smoothly-closing solutions, we can get a handle on the completeness by considering the evolution of the quantities

$$\Lambda := F_1(F_2 + F_3) = \frac{1}{f_2^2} + \frac{1}{f_3^2} \quad \text{and} \quad D := F_2 - F_3 = \frac{f_2^2 - f_3^2}{f_1f_2f_3}.$$ 

In particular, the previously described arguments mean that completeness is equivalent to boundedness of $\Lambda$. With the normalisation $c=3$, the initial value of $\Lambda$ is $\Lambda(0) = 2/b^2$. If $b^2 > 2$ then $\Lambda(0) < 1$, and completeness follows from $\Lambda < 1$ being preserved forward in time, while showing $\Lambda \to 0$ as $t \to \infty$ is key to proving that the component $f_3$ grows at the same linear rate as $f_1$ and $f_2$, leading to the AC asymptotics. For $b^2 < 2$ we instead use that the condition $\Lambda > D > 1$ is preserved forward in time, show that $\Lambda$ is unbounded, and deduce from this that the solution cannot be complete.

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2. The Laplacian flow and Laplacian solitons

2.1. Bryant’s closed Laplacian flow. For a parabolic approach to the problem of finding torsion-free $G_2$–structures on a compact oriented spin 7-manifold it is natural to seek a geometric flow on positive 3-forms. Although a number of different flows on positive 3-forms have been considered, in this paper we discuss only what is widely considered to be the most promising of these flows with the nicest geometric and analytic features: Bryant’s closed $G_2$–Laplacian flow [14]. A 1-parameter family of closed $G_2$–structures $\varphi(t)$ evolves according to the Laplacian flow if it satisfies

$$\frac{\partial \varphi}{\partial t} = \Delta_\varphi \varphi$$

where $\Delta_\varphi$ is the Hodge Laplacian on 3-forms determined by the evolving metric $g_{\varphi(t)}$. Clearly any torsion-free $G_2$–structure gives rise to a fixed point of Laplacian flow and on a compact manifold integration by parts shows that these are the only fixed points. For any closed $G_2$–structure $\varphi$ there is a unique 2-form $\tau$ of type 14 with the property that $d(*\varphi) = \tau \wedge \varphi$. $\tau$ is called the torsion of $\varphi$ and it encodes all the first-order local invariants of a closed $G_2$–structure. Using the algebraic properties of 2-forms of type 14 it is readily seen that $\Delta_\varphi \varphi = d\tau$ and so in particular under Laplacian flow the cohomology class of $\varphi(t)$ remains constant. The flow of $\varphi(t)$ induces a flow of metrics $g_t := g_{\varphi(t)}$ which has the form

$$\frac{\partial g}{\partial t} = -2 \text{Ric}(g) + \frac{8}{21} |\tau|^2 g + \frac{1}{4} j_\varphi(*\varphi(\tau \wedge \tau)),$$

where the map $j_\varphi : \Omega^3(M) \to S^2(T^*M)$ sends a 3-form $\alpha$ to the symmetric covariant 2-tensor defined by $j_\varphi(V,W)(\alpha) = *(V \wedge_\varphi W \wedge_\varphi \alpha)$. In particular, the metric $g_t$ evolves by Ricci flow with the addition of two quadratic correction terms involving the torsion $\tau$. Since the Ricci curvature
of any closed $G_2$–structure is determined by $\varphi$, $\tau$ and $d\tau$ \cite[(4.37)]{14} one can view these additional terms quadratic in $\tau$ as ‘lower-order’ corrections.

However, these ‘correction terms’ have a profound impact on certain geometric features of the flow. For instance, a standard computation \cite[(6.14)]{14} shows that the induced volume form $\text{Vol}_\varphi$ evolves via

$$\frac{\partial}{\partial t} \text{Vol}_\varphi = \frac{1}{3} |\tau_\varphi|^2 \text{Vol}_\varphi,$$

i.e. the induced volume form is pointwise increasing in $t$. This implies that for a non-torsion-free $G_2$–structure on a compact manifold the total volume $\text{Vol}_\varphi(M)$ is increasing in $t$. This is clearly very different from Ricci flow where for instance every compact Einstein manifold with positive scalar curvature shrinks homothetically to a point in finite time. Hitchin provided a more geometric understanding of why the total volume should be increasing by exhibiting a gradient structure for Laplacian flow. More specifically, Hitchin considered the volume functional

$$\mathcal{V}(\varphi) := \frac{1}{7} \int_M \varphi \wedge \ast \varphi = \int_M \text{Vol}_\varphi = \text{Vol}(M, g_\varphi)$$

and proved that for an appropriate Riemannian metric on the space of all closed $G_2$–structures in a fixed cohomology class the Laplacian flow is the (upward) gradient flow of $\mathcal{V}$. Moreover, Hitchin proved that any critical point of $\mathcal{V}$ on $[\varphi]$ is a strict maximum (modulo the action of diffeomorphisms). This suggests that if one could prove long-time existence for solutions of Laplacian flow and if the volume is bounded above for all $t$ that perhaps the solution should converge as $t \to \infty$ to a torsion-free $G_2$–structure $\varphi_\infty$ on $M$ in the original cohomology class.

Like Ricci flow, because of its diffeomorphism invariance the Laplacian flow is not strictly parabolic. However, by making a suitable gauge-fixing in the spirit of DeTurck and appealing to some of Hamilton’s Nash–Moser-type methods, Bryant and Xu \cite{16}, proved short-time existence (and uniqueness) for Laplacian flow on any compact manifold with any smooth closed $G_2$–structure as initial data. The extra technical difficulties arise because the linearisation is parabolic only in the direction of exact forms: see also the recent note by Bedulli and Vezzoni \cite{6} observing that short-time existence can also be proven using that the fact that solutions to the gauged Laplacian flow fit into a general framework introduced by Hamilton in his original 1982 paper on Ricci flow \cite{36}.

More recently, Laplacian flow analogues of various analytic results well known in Ricci flow were proven by Lotay and Wei in a series of three papers \cite{45–47}. These include long-time existence criteria based on curvature and torsion estimates along the flow, Shi-type estimates, uniqueness and compactness theory (the analogue of Hamilton’s compactness theorem for Ricci flows), real analyticity of the the flow, and stability of critical points, i.e. when the initial data $\varphi(0)$ is sufficiently close to a torsion-free $G_2$–structure $\varphi_{tf}$ in the same cohomology class then the solution to Laplacian flow exists for all time and converges modulo diffeomorphisms to $\varphi_{tf}$. For short introductions to many of these analytic results we refer the reader to the recent Fine–Yao survey article \cite{30} on hypersymplectic flow (which can be viewed as a dimensional reduction of Laplacian flow when $M^7 = \mathbb{T}^3 \times N^4$).

2.2. Singularity models, ancient solutions and solitons. Recall that an ancient solution to Ricci flow is a smooth solution that exists on a time interval $(-\infty, b)$ (where $b$ could be finite or infinite; in the latter case the solution is said to be eternal). Ancient solutions to Ricci flow are fundamental to finite-time singularity analysis because performing the natural parabolic blow-up procedure for such a singularity produces an ancient solution. Any ancient solution of Ricci flow that arises as the blow-up limit of a finite-time singularity of smooth Ricci flow is called a singularity model. One also knows that any singularity model is necessarily $\kappa$-noncollapsed. The $\kappa$-noncollapsed condition already enables one to prove that many ancient solutions are not (finite-time) singularity models (e.g. the product of Hamilton’s cigar soliton with a Euclidean space).
A special class of ancient solutions to Ricci flow are provided by shrinking and steady Ricci solitons. By now there is a very extensive literature on (complete gradient) Ricci solitons. Broadly speaking one can divide these results into three categories:

(a) the construction of various gradient Ricci solitons;
(b) general structural results about gradient Ricci solitons (of both topological and geometric natures);
(c) classification results about gradient Ricci solitons.

Even though gradient Ricci solitons are easier to study directly than general ancient solutions there are still many challenges on the construction side. Ricci solitons satisfy an elliptic (modulo diffeomorphism) system of PDEs that generalises the Einstein equations; since we lack any general analytic methods to produce Einstein metrics (except in the setting of special holonomy) it is not too surprising that it has proven difficult to construct solitons by analytic methods. Rather, as with Einstein metrics, many constructions are based on a symmetry assumption or other special metric ansatz (e.g. a warped product structure or bundle structure) that reduces the system of PDEs to ODEs or even to algebraic equations (as for homogeneous Einstein metrics or solitons).

One can then try to leverage results about steady or shrinking solitons either to construct or prove structural or classification results for more general ancient solutions: Perelman proved that any complete nonflat 3-dimensional ancient $\kappa$-solution has a rescaled backward time limit which is a nonflat gradient shrinking Ricci soliton; he also constructed a compact rotationally-invariant ancient solution to 3-dimensional Ricci flow that at large negative times resembles two steady Bryant solitons glued together to obtain a 3-sphere with a long neck and which close to its extinction time approaches a shrinking round 3-sphere; Brendle’s recent proof of Perelman’s conjecture on the complete classification of noncompact ancient $\kappa$-solutions in three dimensions [11] builds on his earlier classification result for 3-dimensional $\kappa$-noncollapsed steady gradient solitons with positive curvature [10].

### 2.3. Finite-time singularity formation in Laplacian flow

Currently significantly less is known about finite-time singularity formation in Laplacian flow than in Ricci flow (or various other better-studied geometric flows like mean curvature flow or harmonic map heat flow). The first difficulty is that as in Hamilton’s compactness theorem for Ricci flows, the Lotay–Wei compactness results [45, Theorems 7.1 and 7.2] assume that a uniform lower bound on the injectivity radius holds. In Ricci flow, Perelman’s $\kappa$-noncollapsing theorem guarantees this holds at any finite-time singularity. Gao Chen extended Perelman’s $\kappa$-noncollapsing theorem to perturbations of Ricci flow by a symmetric two-tensor $h$ under some boundedness assumptions on $h$ along the flow [20, Theorem 4.2]. Since by (2.2) under Laplacian flow the induced metric indeed evolves by such a perturbation of Ricci flow, under some assumptions on the behaviour of the torsion (need to give the required control of the perturbation term $h$) one can find a singularity model that is a complete nonflat torsion-free $G_2$-structure with Euclidean volume growth. But without making such a priori assumptions on the behaviour of the torsion along Laplacian flow it is not yet known how to pass to a singularity model at finite-time singularities.

An analogous result in Ricci flow to Chen’s result in Laplacian flow is that under uniform (upper and lower) bounds on the scalar curvature any finite-time singularity model is a smooth nonflat complete Ricci-flat manifold with Euclidean volume growth. Finite-time singularities of this kind are now known to occur in U(2)-invariant non-Kähler Ricci flow (on certain simple asymptotically cylindrical 4-manifolds and where the singularity model is the Eguchi–Hansen metric) [5]. However such a uniform bound on scalar curvature is not always satisfied along a Ricci flow: for Kähler-Ricci flow on closed manifolds it is known that the scalar curvature must blow up at any finite-time singularity. In some cases one can prove that finite-time singularities must occur and even identify the singularity model that appears: in a compact U(2)-invariant Kähler setting Maximo [50] proved that an embedded $(-1)$-sphere can collapse to a point in finite time and that the associated singularity
model is the $U(2)$-invariant Feldman–Ilmanen–Knopf Kähler shrinker on the one-point blowup of $\mathbb{C}^2$ [29].

2.4. Ancient and eternal solutions to Laplacian flow. Rather little work has been done so far to understand ancient solutions of the Laplacian flow. Homogeneous solitons in Laplacian flow are by far the simplest solitons to study and the resulting problems have a Lie-theoretic flavour. There is now a growing literature on homogeneous Laplacian solitons and more generally on the evolution of homogeneous metrics under Laplacian flow [44] using some of the techniques developed for the study of Ricci solitons and Ricci flows of homogeneous metrics. Note that there are no nontrivial gradient homogeneous Ricci solitons [53, Theorem 2.3]. We refer the reader to [44] for further references on homogeneous Laplacian solitons.

Outside the homogeneous setting Ball has found complete nontrivial steady gradient Laplacian solitons on topological cylinders $\mathbb{R} \times N$, where $N^6$ is either the twistor space of an anti-self-dual Ricci-flat $4$-manifold $B$ or a particular $T^2$-bundle over certain hyperKähler $4$-manifolds. Note that a nontrivial complete noncompact steady gradient Ricci soliton must be connected at infinity [52, Corollary 1.1] so these nontrivial steady gradient Laplacian solitons on topological cylinders are a new feature of Laplacian flow. In the first case Ball exhibits a $2$-parameter family of explicit solutions and when $B$ is compact (a $K3$ surface or a $4$-torus for instance) on one end the $G_2$–structure is asymptotic to the product torsion-free $G_2$–structure on $\mathbb{R}^3 \times B$ while the other end has finite volume. (Note that a finite volume end is not possible for a complete steady gradient Ricci soliton because by [51, Theorem 5.1] any end must have at least linear volume growth). In the second case the solutions have linear volume growth at one end and cubic volume growth at the other end.

2.5. Laplacian solitons. Given a smooth $7$-manifold $M$ (compact or noncompact), a triple $(\varphi, X, \lambda)$ consisting of a $G_2$–structure $\varphi$, a vector field $X$ and a constant $\lambda \in \mathbb{R}$ is called a Laplacian soliton if the triple satisfies the following system of equations

$$\begin{aligned}
\text{(LSE)} & \quad \left\{ 
\begin{array}{l}
\varphi = 0, \\
\Delta_{\varphi} \varphi = \lambda \varphi + \mathcal{L}_X \varphi.
\end{array}
\right.
\end{aligned}
$$

Basic facts about the torsion of a closed $G_2$–structure $\varphi$ imply that $\Delta_{\varphi} \varphi = d\tau_2$ where $\tau_2$ is the torsion $2$-form of type $14$ determined by $d(\ast \varphi) = \tau_2 \wedge \varphi$. Hence an alternative formulation of the Laplacian soliton system is

$$\begin{aligned}
\text{(2.3)} & \quad \left\{ 
\begin{array}{l}
d\varphi = 0, \\
(d\tau_2 - i_X \varphi) = \lambda \varphi.
\end{array}
\right.
\end{aligned}
$$

Remark 2.4. A simple but important observation is that for any nonsteady Laplacian soliton the closed $3$-form $\varphi$ must actually be exact.

Any Laplacian soliton $(\varphi, X, \lambda)$ gives rise to a self-similar solution to the Laplacian flow as follows: for any time $t \in \mathbb{R}$ satisfying $2\lambda t + 3 > 0$ we define a $1$-parameter family of closed $G_2$–structures $\varphi_t$ with $\varphi_0 = \varphi$ that evolves by Laplacian flow by defining

$$\varphi_t = \left(\frac{3 + 2\lambda t}{3}\right)^{\frac{3}{2}} \phi_t^* \varphi,$$

where $\phi_t$ is the $1$-parameter family of diffeomorphisms of $M$ generated by the time-dependent vector field $X(t) = \left(\frac{3}{3 + 2\lambda t}\right) X$ such that $\phi_0$ is the identity, e.g. see [45, Section 9]. The proof that $\varphi_t$ evolves via Laplacian flow is an elementary calculation.

Based on the behaviour of the scaling factor that appears in the definition of $\varphi_t$ one says that a Laplacian soliton is steady if $\lambda = 0$, expanding if $\lambda > 0$ and shrinking if $\lambda < 0$. For a shrinking soliton with $\lambda$ normalised to be $-1$ we therefore have an ancient solution to Laplacian flow defined
on the time interval \((-\infty, \frac{3}{2})\), whereas for an expanding soliton with \(\lambda\) normalised to be 1 we have an immortal solution to Laplacian flow defined on the time interval \((-\frac{3}{2}, \infty)\). Steady solitons give rise to eternal solutions to Laplacian flow defined for all \(t \in \mathbb{R}\).

3. Cohomogeneity-one closed \(G_2\)-structures

Section 3.1 recalls key facts about closed \(G_2\)-structures and Section 3.2 collects some basic facts about the automorphism group of a closed \(G_2\)-structure. Section 3.3 explains how to pass from certain 1-parameter families of \(SU(3)\)-structures to a closed \(G_2\)-structure; we use this construction in our discussion of cohomogeneity-one closed \(G_2\)-structures later in this section. Section 3.4 recalls the facts that we will need from the general theory of cohomogeneity-one spaces and the work of Cleyton and Swann on cohomogeneity-one closed \(G_2\)-structures. The most important parts of this section are Sections 3.5 and 3.6 in which we write down the most general cohomogeneity-one \(SU(3)\)-invariant (respectively \(Sp(2)\)-invariant) closed \(G_2\)-structures. These results are used to derive the ODE system satisfied by invariant Laplacian solitons in Section 4; these soliton ODE systems are central role to the rest of the paper.

3.1. Closed \(G_2\)-structures. We recall some basic facts about closed \(G_2\)-structures following Bryant \[14\] (to which we refer the reader for further details and proofs). For readers looking for a thorough introduction to the linear algebra associated with \(G_2\) we also recommend the notes of Salamon–Walpuski \[55\].

The first-order local invariants of a \(G_2\)-structure \(\varphi\) are all encoded in a terms of a quadruple of differential forms called the torsion forms of \(\varphi\). These torsion forms arise as components of the decomposition of the exterior derivatives of \(\varphi\) and \(\ast \varphi\) into their \(G_2\)-irreducible components. Recall that the exterior powers of the standard 7-dimensional representation \(V\) of \(G_2\) decompose as

\[(3.1a)\quad \Lambda^2(V^*) = \Lambda^2_{14} \oplus \Lambda^2_7,\]
\[(3.1b)\quad \Lambda^3(V^*) = \Lambda^3_{27} \oplus \Lambda^3_7 \oplus \Lambda^3_1,\]
\[(3.1c)\quad \Lambda^4(V^*) = \Lambda^4_{27} \oplus \Lambda^4_7 \oplus \Lambda^4_1,\]
\[(3.1d)\quad \Lambda^5(V^*) = \Lambda^5_{14} \oplus \Lambda^5_7,\]

where the subscript denotes the dimension of the irreducible module and

\[(3.2a)\quad \Lambda^2_7 = \{X \ast \varphi \mid X \in V\} = \{\omega \in \Lambda^2(V^*) \mid \ast (\varphi \wedge \omega) = 2\omega\} \cong V,\]
\[(3.2b)\quad \Lambda^2_{14} = \{\omega \in \Lambda^2(V^*) \mid \omega \wedge \ast \varphi = 0\} = \{\omega \in \Lambda^2(V^*) \mid \ast (\varphi \wedge \omega) = -\omega\} \cong \mathfrak{g}_2,\]
\[(3.2c)\quad \Lambda^3_1 = \{r \varphi \mid r \in \mathbb{R}\} \cong \mathbb{R},\]
\[(3.2d)\quad \Lambda^3_7 = \{X \ast \varphi \mid X \in V\} \cong V,\]
\[(3.2e)\quad \Lambda^3_{27} = \{\gamma \in \Lambda^3(V^*) \mid \gamma \wedge \varphi = 0, \gamma \wedge \ast \varphi = 0\} \cong \text{Sym}^2_0(V).\]

The Hodge star gives isomorphisms \(\Lambda^p_j \cong \Lambda^{7-p}_j\). In particular this gives us the irreducible decomposition of \(\Lambda^4\) from that of \(\Lambda^3\).

**Lemma 3.3.** For any \(G_2\)-structure \(\varphi\) on \(M^7\) there exist unique differential forms \(\tau_0 \in \Omega^0(M), \tau_1 \in \Omega^1(M), \tau_2 \in \Omega^2_{14}(M, \varphi)\) and \(\tau_3 \in \Omega^3_{27}(M, \varphi)\) such that

\[d \varphi = \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast \tau_3,\]
\[d(\ast \varphi) = 4\tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi.\]

The quadruple \((\tau_0, \tau_1, \tau_2, \tau_3)\) defined above can be identified with the intrinsic torsion of \(\varphi\). We will only be interested in closed \(G_2\)-structures, i.e. \(d \varphi = 0\), which by the previous lemma is equivalent to
the vanishing of the torsion forms \( \tau_0, \tau_1, \) and \( \tau_2. \) Hence the intrinsic torsion of a closed \( G_2 \)-structure \( \varphi \) can be identified with \( \tau_2 \in \Omega^2_{14}(M, \varphi) \) and it satisfies

\[
(3.4) \quad d(* \varphi) = \tau_2 \wedge \varphi.
\]

Since \( \tau_2 \) is of type 14, \(* (\tau_2 \wedge \varphi) = -\tau_2 \) and so \( d^* \varphi = -* d\varphi = \tau_2. \) Hence

\[
(3.5) \quad \Delta \varphi \varphi = dd^* \varphi = d \tau_2.
\]

Since \( \tau_2 \) is of type 14 it also satisfies \( \tau_2 \wedge * \varphi = 0. \) Taking the exterior derivative of both sides of this equation and using again the characterisation of \( \Lambda^2_{14} \) as the \(-1\) eigenspace of \(* (\varphi \wedge \cdot) \) yields

\[
\tau_2 \wedge * \varphi = |\tau_2|^2 \varphi + 1.
\]

Taking the exterior derivative of \((3.4)\) implies that

\[
(3.6) \quad d \tau_2 \wedge \varphi = 0
\]

and therefore the 3-form \( d \tau_2 \) has no type 7 component. Hence we can write

\[
(3.7) \quad \Delta \varphi \varphi = d \tau_2 = \frac{1}{7} |\tau_2|^2 \varphi + \gamma_{27}
\]

for some 3-form \( \gamma_{27} \in \Omega^3_{27}(M, \varphi). \)

The scalar curvature of \( g_\varphi \) for a closed \( G_2 \)-structure \( \varphi \) is given by

\[
(3.8) \quad S(g_\varphi) = -\frac{1}{2} |\tau_2|^2.
\]

In particular its scalar curvature is nonpositive and vanishes if and only if \( \varphi \) is torsion free.

### 3.2. Symmetries of closed \( G_2 \)-structures.

For any \( G_2 \)-structure \( \varphi \) on a 7-manifold \( M \) we define its automorphism group to be

\[
\text{Aut}_{\varphi}(M) := \{ f \in \text{Diff}(M) | f^* \varphi = \varphi \}.
\]

\( \text{Aut}_{\varphi}(M) \) is a closed subgroup of \( \text{Iso}_{g_\varphi}(M) \) and therefore it is compact whenever \( \text{Iso}_{g_\varphi}(M) \) is, \( \text{e.g.} \) when \( M \) is compact. The Lie algebra to the identity component \( \text{Aut}^0_{\varphi}(M) \) of \( \text{Aut}_{\varphi}(M) \) defined by

\[
\text{aut}_{\varphi}(M) := \{ X \in \mathfrak{X}(M) | L_X \varphi = 0 \}
\]

is then a Lie subalgebra of the algebra of Killing fields \( \mathcal{K}_{g_{\varphi}} \) of \( g_{\varphi}. \) If \( M \) is compact and \( \varphi \) is closed the Lie algebra \( \text{aut}_{\varphi}(M) \) must be abelian and if \( \varphi \) is exact then \( \text{aut}_{\varphi}(M) = (0). \)

#### Lemma 3.9.

The Lie algebra \( \text{aut}_{\varphi}(M) \subseteq \mathcal{K}_{g_{\varphi}} \) of a closed \( G_2 \)-structure on a compact manifold \( M \) is abelian and satisfies \( \dim \text{aut}_{\varphi}(M) \leq b^2(M). \) If \( \varphi \) is exact then \( \text{aut}_{\varphi}(M) = (0). \)

**Proof.** When \( \varphi \) is closed the image of \( \text{aut}_{\varphi} \) under the isomorphism between \( \mathfrak{X}(M) \) and \( \Omega^2_{14}(M) \) given by \( X \mapsto X_{\varphi} \) consists of closed and closed 2-forms: \( X_{\varphi} \) is closed because, \( 0 = L_X \varphi = d(X_{\varphi}) \) (the latter equality holding because \( \varphi \) is closed) for any \( X \in \text{aut}_{\varphi}(M). \) Then any closed 2-form \( \alpha \) of type 7 is also closed, because \(* \alpha \wedge \varphi \) and \( d(\alpha \wedge \varphi) = d\alpha \wedge \varphi + \alpha \wedge d\varphi = 0. \) Hence the 2-form \( X_{\varphi} \) is \( \Delta_{\varphi}\)-harmonic.

For any harmonic form \( \alpha \) and \( X \in \text{aut}_{\varphi}(M), \) the Lie derivative \( L_X \alpha \) is also harmonic (since the Laplacian commutes with isometries). But since \( \alpha \) is closed the harmonic form \( L_X \alpha = d(X_{\varphi}) \) is also exact. Hence if \( M \) is compact then by the Hodge decomposition \( L_X \alpha = 0. \) In particular for any \( X, Y \in \text{aut}_{\varphi}(M) \) we have \( L_X(Y_{\varphi}) = 0 \) and therefore also

\[
[X, Y]_{\varphi} = L_X(Y_{\varphi}) - Y_{\varphi}(L_X \varphi) = 0.
\]

Suppose now that \( \varphi = d\vartheta. \) Then if \( M \) is compact without boundary for any \( X \in \text{aut}_{\varphi}(M) \) we have

\[
6 \|X\|^2 = 6 \int_M g_{\varphi}(X, X) \text{Vol}_{\varphi} = \int_M (X_{\varphi}) \wedge (X_{\varphi}) \wedge \varphi = \int_M d((X_{\varphi}) \wedge (X_{\varphi}) \wedge \varphi) = 0
\]

and hence the vector field \( X \) must vanish identically. \( \square \)
The first statement in Lemma 3.9 was first observed in [54, §2], where further details about the possible dimensions of $\text{aut}_\varphi(M)$ are given. The second statement was observed by Fowdar [34, Prop. 4.13].

**Remark 3.10.** Lemma 3.9 implies (a) that a 7-manifold that admits a *closed* $G_2$–structure with a nonabelian symmetry algebra $\text{aut}_\varphi(M)$ must be noncompact and (b) that Laplacian expanders on a compact manifold must have $\text{aut}_\varphi(M) = (0)$. The hypothesis that $M$ be compact in both (a) and (b) is necessary, e.g. there are noncompact complete cohomogeneity-one torsion-free examples with nonabelian symmetries and there are noncompact homogeneous expanders [44]. Since our interest is in constructing highly-symmetric Laplacian solitons we are therefore forced to consider noncompact manifolds $M$. (Recall also that there are no non-torsion free steady and no shrinking Laplacian solitons when $M$ is compact, regardless of the symmetry question.) Clearly the assumption that $\varphi$ is closed is also necessary, since the 7-sphere admits homogeneous $G_2$–structures.

### 3.3. Closed $G_2$–structures from 1-parameter families of SU(3)-structures

Any smooth oriented hypersurface in a 7-manifold $M$ with a $G_2$–structure $\varphi$ inherits an SU(3)-structure. Recall that an SU(3)-structure on a 6-manifold is a pair of smooth differential forms $(\omega, \Omega)$, where $\omega$ is a nondegenerate 2-form and $\Omega$ is a complex volume form (i.e. a locally decomposable complex 3-form such that $\Omega \wedge \Omega$ is nowhere zero) satisfying the algebraic constraints

\begin{align}
\omega \wedge \Omega &= 0, \\
\frac{1}{16} \omega^3 &= \frac{1}{4} \Re \Omega \wedge \Im \Omega.
\end{align}

If we choose a family of equidistant hypersurfaces $P_t$ in $(M, \varphi)$ then for $t$ sufficiently small we can view this as giving us a 1-parameter family of SU(3)-structures on a fixed 6-manifold $P$. We can also reverse this procedure and recover a $G_2$–structure from a 1-parameter family of SU(3)-structures on $P$. If we start with a 1-parameter family of homogeneous SU(3)-structures on $P$ we will obtain a cohomogeneity-one $G_2$–structure on $I \times P$, but the method applies more generally. If additionally we impose some conditions on the torsion of the $G_2$–structure then the torsion of the 1-parameter family of SU(3)-structures induced on its equidistant hypersurfaces will also satisfy some constraints on its torsion. This idea was popularised by Hitchin [38, Theorem 8] in the setting of torsion-free $G_2$–structures, in which case the induced SU(3)-structure is a so-called half-flat structure. Hitchin viewed his equations as the Hamiltonian flow of a certain functional. It has proven to be a useful formalism for understanding some properties of the systems of ODEs governing cohomogeneity-one torsion-free $G_2$–structures [8, 33, 48] and a similar idea proved to be useful in the study of cohomogeneity-one nearly Kähler 6-manifolds [31]. We will be interested in the case that $\varphi$ is closed.

Let $P$ be a fixed 6-manifold and suppose that $(\omega, \Omega)$ is a 1-parameter family of SU(3)-structures on $P$ depending on $t \in I \subset \mathbb{R}$. Consider the $G_2$–structure on $I \times P$ defined by

\begin{align}
\varphi &= dt \wedge \omega + \Re \Omega, \\
\ast \varphi &= \frac{1}{2} \omega^2 - dt \wedge \Im \Omega.
\end{align}

The exterior derivatives of $\varphi$ and $\ast \varphi$ are given by

\begin{align}
d \varphi &= d \Re \Omega + (d \omega - \partial_t \Re \Omega) \wedge dt, \\
d (\ast \varphi) &= \omega \wedge d \omega + (d \Im \Omega + \omega \wedge \partial_t \omega) \wedge dt.
\end{align}

Hence the condition that $\varphi$ be closed is equivalent to

\begin{align}
d \Re \Omega &= 0, \\
\partial_t \Re \Omega &= d \omega.
\end{align}
We call (3.14a) the static closure condition: it imposes a restriction on the torsion of the SU(3)-structure \((\omega, \Omega)\) that holds for every \(t \in I\). We call (3.14b) the dynamic closure condition: it imposes a condition on how \((\omega, \Omega)\) evolves with \(t\).

Remark. The torsion of a general SU(3)-structure \((\omega, \Omega)\) takes values in a 42-dimensional space. The static closure condition imposes 15 conditions on this torsion: in the notation of [32, Prop. 2.10] it implies that the function \(\bar{w}_1 = 0\), the 1-form \(w_5 = 0\) and the primitive \((1, 1)\)-form \(w_2 = 0\). Hence the exterior derivatives of \(\omega\) and \(\Omega\) satisfy

\[
d\omega = 3w_1 \operatorname{Re} \Omega + w_3 + w_4 \wedge \omega,
\]

\[
d\operatorname{Im} \Omega = -2w_1\omega^2 + \bar{w}_2 \wedge \omega,
\]

where \(w_1\) is a function, \(\bar{w}_2\) is a primitive \((1, 1)\)-form, \(w_3\) is a 3-form of type 12 and \(w_4\) is a 1-form. Note that an SU(3)-structure which in addition satisfies \(d\omega^2 = 0\) is called \textit{half-flat}: this imposes a further 6 conditions on the torsion, namely the 1-form \(w_4\) also vanishes.

Remark 3.15. An important point to notice is that the Klein four-group \(\mathbb{Z}_2 \times \mathbb{Z}_2\) acts naturally on the space of SU(3)-structures satisfying the static closure condition (3.14a). For any SU(3)-structure \((\omega, \Omega)\) define a pair of involutions \(\overline{T}\) and \(T_\pi\) by

\[
T(\omega, \Omega) = (-\omega, \overline{\Omega}), \quad T_\pi(\omega, \Omega) = (\omega, -\Omega).
\]

Both \(\overline{T}\) and \(T_\pi\) preserve the set of SU(3)-structures; \(\overline{T}\) and \(T_\pi\) commute and therefore \(\overline{T} \circ T_\pi\) defines another involution on the space of SU(3)-structures. The group generated by \(\overline{T}\) and \(T_\pi\) is therefore isomorphic to the Klein four-group \(\mathbb{Z}_2 \times \mathbb{Z}_2\). \(T_\pi(\omega, \Omega)\) induces the same orientation as \((\omega, \Omega)\) whereas \(\overline{T}(\omega, \Omega)\) induces the opposite orientation. All three involutions preserve the set of SU(3)-structures satisfying the static closure condition (3.14a).

Note that the 3-form \(\varphi\) defined in (3.12a) is invariant under \((t, \omega, \Omega) \mapsto (-t, \overline{T}(\omega, \Omega))\). In other words, if we reverse the sense of the interval \(I\) then we get the same \(G_2\)-structure \(\varphi\) but parameterised in the opposite sense. Under \((t, \omega, \Omega) \mapsto (-t, T_\pi(\omega, \Omega))\), the \(G_2\)-structure satisfies \(\varphi \mapsto -\varphi\); but clearly this still sends a closed \(G_2\)-structure to another closed \(G_2\)-structure.

3.4. \textbf{Cohomogeneity-one closed \(G_2\)-structures.} In preparation for our review of the work of Cleyton and Cleyton–Swann [23] on cohomogeneity-one (closed) \(G_2\)-structures we now present a brief summary of the requisite background from the theory of cohomogeneity-one manifolds. For general references on cohomogeneity-one theory we refer the reader to [9, Chapter IV] for the smooth aspects of the theory and to [1] for its more Riemannian aspects.

\textit{Basic cohomogeneity-one theory.} Recall that the orbit space \(M/G\) of a cohomogeneity-one isometric action of a compact Lie group \(G\) is a connected Riemannian 1-manifold (potentially) with boundary, \(i.e\). it is either \(S^1\) or an interval \(I\) and there are 3 kinds of intervals: \(\mathbb{R}\), \([0, \infty)\) or \([0, \ell]\). Interior points of \(M/G\) correspond to principal orbits and any boundary points of \(I\) correspond to singular orbits. The isotropy group of any principal orbit is conjugate to a fixed Lie subgroup \(K \subset G\), and the isotropy subgroup \(H\) of any singular orbit has the properties that \(K \subset H \subset G\) and that \(H/K\) is diffeomorphic to a sphere. Moreover, there is an orthogonal representation \(\rho: H \to O(V)\) such that a neighbourhood of the singular orbit \(G/H\) in \(M\) is \(G\)-equivariantly diffeomorphic to a neighbourhood of the zero section of the vector bundle \(G \times_H V \to G/H\). There are at most two singular orbits \(G/H_1\) and \(G/H_2\). When \(I = [0, \ell]\), so that there are two singular orbits, then \(M\) is necessarily compact: it is obtained by identifying the two disc bundles \(G \times_H D_i, D_i \subset V_i\) over the singular orbits along their common boundary \(G/K\). When the orbit space is the circle \(S^1\) there are no singular orbits and \(M\) is compact with infinite fundamental group. In the remaining two cases \(I = \mathbb{R}\) or \(I = [0, \infty)\), \(M\) is noncompact.

Given any point \(p \in M^o\), the open dense set of principal points in \(M\), there exists a unique unit-speed geodesic \(\gamma\) through \(p\) that is orthogonal to every principal orbit \(G/K\). When \(M/G\) is an
interval \( I \) the map \( I \times G/K \to M \) given by \( (t, gK) \mapsto g \cdot \gamma(t) \) is surjective and the restriction of this map to the interior \( I^0 \) of \( I \) is a diffeomorphism onto \( M^0 \) with the property that the composition \( \pi \circ \gamma : I^0 \to M^0/G \) is an isometry with respect to the standard metric \( dt^2 \) on \( I^0 \) and the quotient metric on \( M^0/G \) (here \( \pi : M \to M/G \) denotes the orbit projection). Therefore we can identify smooth \( G \)-invariant tensors on \( M^0 \) with smooth \( t \)-dependent \( G \)-invariant tensors on \( G/K \). The latter can be determined by standard methods in representation theory.

It is a separate matter to analyse when a smooth \( G \)-invariant tensor on \( M^0 \) extends to a smooth \( G \)-invariant tensor on \( M \). Since this extension question depends only on the geometry of \( M \) in a neighbourhood of its (at most two) singular orbits and a neighbourhood of any singular orbit \( G/H \) is diffeomorphic to a neighbourhood of the zero section of the vector bundle \( G \times_H V \to G/H \) one can again reduce to problems of a representation-theoretic nature: see Eschenburg–Wang [28, §1] for details. In the cases of interest to us for a given principal orbit type \( G/K \) there will be a unique singular orbit type \( G/H \) and also there must be precisely one singular orbit, i.e. the orbit space \( M/G \) is the interval \( I = [0, \infty) \) and \( M \) is diffeomorphic to a homogeneous vector bundle \( G \times_H V \to G/H \) over the unique singular orbit.

**Principal orbits of cohomogeneity-one closed \( G_2 \)-structures.** Cleyton in his thesis [22] and Cleyton–Swann [23, Theorem 3.1] analysed the possible principal orbits for \( G_2 \)-structures (not necessarily closed) that admit an isometric cohomogeneity-one action of a compact connected Lie group \( G \). The requirement that \( G \) preserve the 3-form \( \varphi \) implies that the representation of the isotropy group \( K \subset G \) on the tangent space of a principal orbit \( G/K \) must occur as a subgroup of \( \text{SU}(3) \) on its standard 6-dimensional representation on \( \mathbb{C}^3 \), i.e. \( \mathfrak{k} \) must be \( \text{su}(3), \text{u}(2), \text{su}(2), \text{u}(1) \oplus \text{u}(1), \text{u}(1) \) or \( \{0\} \). For each of these six possible \( \mathfrak{k} \) they determine what the possible \( \text{Ad}_G(K) \)-invariant complements to the isotropy representation of a principal orbit. They prove that (up to finite quotients) there are seven possibilities for the topology of a principal orbit: \( S^6, \text{CP}^3, F_{1,2}, S^3 \times S^3, S^5 \times S^1, S^3 \times T^3 \) and \( T^6 \). Except for \( S^3 \times S^3 \), which admits three homogeneous space structures with different isotropy groups, the homogeneous space structure \( G/K \) on each topological orbit type is unique. Moreover, any cohomogeneity-one 7-manifold with such principal orbits admits some cohomogeneity-one \( G_2 \)-structure.

In the first three cases the group \( G \) is simple—\( G_2, \text{Sp}(2) \) or \( \text{SU}(3) \) respectively—with principal isotropy subgroup \( K \) being \( \text{SU}(3), \text{Sp}(1) \times \text{U}(1) \) or \( \text{U}(1) \times \text{U}(1) \) respectively. (For the remaining four cases see [23, Theorem 3.1] for the list of \( G \) and \( K \) that arise.) The case with isometry group \( G = G_2 \) and principal isotropy \( K = \text{SU}(3) \) (which acts irreducibly on the tangent space of a principal orbit) is easy to analyse: any cohomogeneity-one closed \( G_2 \)-structure is necessarily torsion-free and the associated metric must be flat [23, Theorem 8.1]). We will discuss in detail the two remaining cases where \( G \) is simple: \( \text{Sp}(2) \) or \( \text{SU}(3) \). Some of our motivations for considering these two cases are the following:

(i) The two symmetry groups \( \text{Sp}(2) \) and \( \text{SU}(3) \) should naturally be considered together: as pointed out by Cleyton–Swann [23] the ODEs governing \( \text{Sp}(2) \)-invariant closed \( G_2 \)-structures can be regarded as specialisations of the ODEs governing \( \text{SU}(3) \)-invariant closed \( G_2 \)-structures when some of the coefficients are set equal. An equivalent way of saying the same thing is that the ODEs satisfied by \( \text{SU}(3) \)-invariant closed \( G_2 \)-structures whose induced Riemannian metrics possess a certain additional free isometric \( \mathbb{Z}_2 \)-action are the same ODEs satisfied by \( \text{Sp}(2) \)-invariant closed \( G_2 \)-structures.

(ii) Four of the seven principal orbit types arise as the principal orbit of a complete cohomogeneity-one torsion-free \( G_2 \)-structure. The standard constant \( G_2 \)-structure on \( \mathbb{R}^7 \) is cohomogeneity one with respect to the action of \( G_2 \subset \text{SO}(7) \) and clearly has \( S^6 \) as its principal orbit type. However the induced metric is the Euclidean metric on \( \mathbb{R}^7 \) and so has trivial holonomy; this is the only way \( S^6 \) arises as the principal orbit of a closed cohomogeneity-one \( G_2 \)-structure. \( \text{CP}^3, \text{SU}(3)/\mathbb{T}^2 \).
and $S^3 \times S^3$ arise as the principal orbits of the (irreducible holonomy) Bryant–Salamon metrics on $\Lambda^2 S^4$, $\Lambda^2 CP^2$ and on the spinor bundle of $S^3$ respectively. (iii) The 3-form underlying a nonsteady closed Laplacian soliton is necessarily exact and in general it is expected that finite-time singularity models for Laplacian flow must be exact. For topological reasons complete cohomogeneity-one $G_2$–structures with our principal orbits are necessarily exact, whereas the AC torsion-free $G_2$–structure on the spinor bundle of $S^3$ (with principal orbit $S^3 \times S^3$) fails to be exact (the zero-section is a nontrivial compact associative 3-fold).

(iv) Atiyah and Witten’s physics-inspired description of the potential relations between well-known complete noncompact torsion-free $G_2$–structures and certain noncompact special Lagrangian 3-folds in $\mathbb{C}^3$ admits a natural extension that suggests the existence of some links between Laplacian solitons and solitons of Lagrangian mean curvature flow (LMCF) in $\mathbb{C}^3$. Consideration of some of the known cohomogeneity-one solitons in LMCF naturally leads one to the study of our two principal orbit types. Some of the Laplacian solitons we construct can therefore be viewed as 7-dimensional (M-theory uplifts in physics terms) analogues of known Lagrangian MCF solitons in $\mathbb{C}^3$.

Singular orbits of cohomogeneity-one $G_2$–structures. Since $G$ is nonabelian, Lemma 3.9 implies that we cannot obtain compact $G$-invariant manifolds which admit closed $G_2$–structures. This implies that the orbit space cannot be $S^1$ or $[0, 1]$ and hence there can be at most one singular orbit. In fact in both of our cases there must be a singular orbit (see for instance Lemma 3.31), i.e. we must have $I = [0, \infty)$. $M$ is therefore diffeomorphic to some $G$-equivariant vector bundle over the singular orbit $G/H$.

Cleyton–Swann analysed the possible singular orbits that can appear. It follows from their analysis [23, Tables 2 &3] that the only spaces admitting a cohomogeneity-one action of $G = Sp(2)$ or $G = SU(3)$ on which there exist any (metrically complete) closed cohomogeneity-one $G_2$–structures are the total spaces of the vector bundles $\Lambda^2 S^4$ or $\Lambda^2 CP^2$ respectively, with the zero section of the vector bundle being the unique singular orbit of the $G$-action. Cleyton–Swann [23, §9] understood the conditions under which a smooth closed $G$-invariant $G_2$–structure on the open dense set of principal orbits $M^0$ extends to a smooth $G$-invariant $G_2$–structure on $M$. We will use these smooth extension conditions later in the paper, where their results will be recalled. The asymptotically conical $G_2$–holonomy metrics on $\Lambda^2 S^4$ and $\Lambda^2 CP^2$ constructed by Bryant–Salamon [15] provide instances of such closed $G_2$–structures. In this paper we will also construct complete cohomogeneity-one Laplacian shrinkers and steady solitons on both these vector bundles (and in the sequel we also construct complete expanders). For most of the examples we construct, like the well-known $G_2$–holonomy examples, the metrics underlying these solitons will be asymptotically conical; unlike for AC torsion-free $G_2$–structures, however, the asymptotic cones of these solitons need not be torsion-free $G_2$-cones. Indeed the complete shrinkers we construct are asymptotic to $G$-invariant closed but non-torsion-free $G_2$-cones (with cross-section $CP^3$ or $F_{1,2}$ respectively). We also construct complete steady solitons on $\Lambda^2 CP^2$ that are not asymptotically conical: instead they have exponential volume growth.

To proceed further we need to recall the description from Cleyton–Swann [23] of the invariant forms of interest on the principal orbit $G/K$ in the cases $G/K = SU(3)/T^2$ and $G/K = Sp(2)/Sp(1) \times U(1)$. We refer the reader to their paper and Cleyton’s thesis for further details if needed.

3.5. Closed $SU(3)$-invariant $G_2$–structures. We begin with the description of the space of $SU(3)$-invariant tensors on the principal orbit $G/K = SU(3)/T^2$. When $G = SU(3)$ the principal isotropy group $K = T^2 = S^1 \times S^1$ acts on the standard representation on $\mathbb{C}^3$ as $L_1 + L_2 + \bar{L}_1 \bar{L}_2$ where $L_i$ are the standard (complex) representations of $S^1 \simeq U(1)$. The isotropy representation is $[L_1 \bar{L}_2] + [L_1 L_2] + [\bar{L}_1 \bar{L}_2]$, where $[L]$ denotes the real representation such that $[L] \otimes \mathbb{C} = L \oplus \bar{L}$. 

Each of the three irreducible submodule of the isotropy representation carries an invariant metric $g_i$ and an invariant 2-form $\omega_i$ for $i = 1, 2, 3$, while the space of invariant 3-forms is 2-dimensional. To be more concrete we identify $\mathbb{T}^2$ with the diagonal matrices in $SU(3)$ and fix the following basis $E_1, \ldots, E_6$ of the tangent space at the origin
\[
E_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{2i} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_4 = \frac{1}{2i} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_5 = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_6 = \frac{1}{2i} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then the invariant bilinear forms and 2-forms are spanned by
\[
g_1 = e_1^2 + e_2^2, \quad g_2 = e_3^2 + e_4^2, \quad g_3 = e_5^2 + e_6^2,
\]
and
\[
\omega_1 = e_{12}, \quad \omega_2 = e_{34}, \quad \omega_3 = e_{56}
\]
respectively where $\{e_1, \ldots, e_6\}$ denotes the dual basis to $\{E_i\}$. In particular an arbitrary $SU(3)$-invariant Riemannian metric on $SU(3)/\mathbb{T}^2$ takes the form
\[
(3.17) \quad g_f = f_1^2 g_1 + f_2^2 g_2 + f_3^2 g_3
\]
for some triple $f = (f_1, f_2, f_3)$ with $f_1 f_2 f_3 \neq 0$. The 2-dimensional space of invariant 3-forms is spanned by
\[
\alpha = e_{246} - e_{235} - e_{145} - e_{136}, \quad \beta = e_{135} - e_{146} - e_{236} - e_{245}.
\]
There are no nontrivial invariant 1-forms or 5-forms; in particular the wedge product of any invariant 2-form with any invariant 3-form vanishes (as one can also check directly). We set $\Vol_0 = e_{123456}$.

The exterior derivatives of these invariant forms satisfy the structure equations
\[
(3.18a) \quad d\omega_1 = d\omega_2 = d\omega_3 = \frac{1}{2} \alpha, \quad d\alpha = 0, \quad d\beta = -2(\omega_1 \wedge \omega_2 + \omega_2 \wedge \omega_3 + \omega_3 \wedge \omega_1).
\]

**Invariant $SU(3)$-structures on $SU(3)/\mathbb{T}^2$.** We want to describe the $SU(3)$-invariant $SU(3)$-structures on $SU(3)/\mathbb{T}^2$. In fact, since our main interest in them is as a tool to describe cohomogeneity-one closed $G_2$-structures, (3.14a) implies that we really want to understand $SU(3)$-structures $(\omega, \Omega)$ that satisfy the additional condition that $d \Re \Omega = 0$. (Invariant $SU(3)$-structures on $SU(3)/\mathbb{T}^2$ were not discussed explicitly in Cleyton’s thesis or by Cleyton–Swann.)

Define invariant forms by
\[
(3.19) \quad \omega_1 + \omega_2 + \omega_3 \quad \text{and} \quad \Omega := \alpha + i\beta.
\]

**Lemma 3.20.**

(i) Any $SU(3)$-invariant $SU(3)$-structure on $SU(3)/\mathbb{T}^2$ can be written in the form
\[
(3.21a) \quad \begin{cases} \omega_f, \Omega_f, \theta \\ (-\omega_f, \Omega_f, \theta) = \overline{\Theta}(\omega_f, \Omega_f, \theta) \end{cases}
\]

where $\Theta$ is the involution defined in Remark 3.15 and
\[
(3.21b) \quad (\omega_f, \Omega_f, \theta) := \left( f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3, (f_1 f_2 f_3) e^{-i\theta} \Omega \right)
\]

for some $e^{i\theta} \in S^1$ and some triple of real numbers $(f_1, f_2, f_3)$ satisfying $f_1 f_2 f_3 \neq 0$.

(ii) In the first case in (3.21a) the induced orientation is $\Vol_0$, while the second case induces the opposite orientation.

(iii) The induced invariant metric is $g_f$ as defined in (3.17).
(iv) \( \Omega_{f,\theta} \) as in (3.21b) satisfies \( d \Re \Omega_{f,\theta} = 0 \) if and only if \( \sin \theta = 0 \).

**Proof.** Note that the two cases in (3.21a) are clearly exchanged by the involution \( \mathbf{T} \) defined in Remark 3.15. They are never in the same connected component of the space of \( \SU(3) \)-structures because they induce opposite orientations. Since there are no invariant 5-forms on \( \SU(3)/\mathbb{T}^2 \) the condition (3.11a) holds for any invariant 2-form and 3-form. Clearly \( \omega \) as defined in (3.19) is a nondegenerate 2-form and it is easy to check that \( \omega \) and \( \Omega \) satisfy the second condition in (3.11) and hence the pair defines an invariant \( \SU(3) \)-structure. More generally, up to an overall sign, any nondegenerate invariant 2-form can be written in the form \( \omega_f \) as defined in (3.21b) for some real triple \( (f_1, f_2, f_3) \) satisfying \( f_1 f_2 f_3 \neq 0 \). Similarly any invariant complex volume form can be written as a nonzero real multiple \( \mu \) of either \( e^{-i\theta} \Omega \) or \( e^{i\theta} \overline{\Omega} \) for some \( e^{i\theta} \in \mathbb{S}^1 \). Since in the first case of (3.21a) we have

\[
4\omega_f^3 = 4f_1^2 f_2^2 f_3^2 \omega^3 = 6f_1^2 f_2^2 f_3^2 \Re \Omega \wedge \Im \Omega = 6\mu^2 \Re \Omega \wedge \Im \Omega
\]

(3.11b) holds if and only if \( \mu^2 = (f_1 f_2 f_3)^2 \). In the second case we have the same condition on \( \mu \) because the signs in \( -\omega_f^3 \) and \( \Re \overline{\Omega} \wedge \Im \Omega \) cancel. The condition on \( \theta \) required for \( d \Re \Omega_{f,\theta} = 0 \) follows immediately from the structure equations (3.18), more specifically that \( d\alpha = d \Re \Omega = 0 \) and \( d\beta = d \Im \Omega \neq 0 \).

\( \square \)

**Remark (The invariant nearly Kähler structure on \( \SU(3)/\mathbb{T}^2 \).)** The invariant \( \SU(3) \)-structure

\[
\omega_{nK} := \frac{1}{4} \omega \quad \text{and} \quad \Omega_{nK} := \frac{1}{8} \Omega
\]

satisfies

\[
d\omega_{nK} = 3 \Re \Omega_{nK}, \quad d \Im \Omega_{nK} = -2 \omega_{nK}^2.
\]

Hence \( (\omega_{nK}, \Omega_{nK}) \) is the unique \( \SU(3) \)-invariant nearly Kähler structure on \( \SU(3)/\mathbb{T}^2 \). It corresponds to taking \( f_1 = f_2 = f_3 = \frac{1}{2} \) and \( \theta = 0 \) in (3.21b).

The next result tells us that the Klein four-group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) action defined in Remark 3.15 acts simply transitively on the connected components of the space of invariant \( \SU(3) \)-structures on \( \SU(3)/\mathbb{T}^2 \) satisfying the static constraint \( d \Re \Omega = 0 \).

**Corollary 3.23 (The connected components of the space of invariant \( \SU(3) \)-structures satisfying the static constraint).**

(i) *The space of \( \SU(3) \)-invariant \( \SU(3) \)-structures on \( \SU(3)/\mathbb{T}^2 \) satisfying the static constraint \( d \Re \Omega = 0 \) is a smooth noncompact 3-manifold with four connected components each diffeomorphic to the positive octant in \( \mathbb{R}^3 \). The Klein four-group described in Remark 3.15 acts simply transitively on these connected components.

(ii) *The connected component containing the nearly Kähler \( \SU(3) \)-structure \( (\omega_{nK}, \Omega_{nK}) \) defined in (3.22) consists of all \( \SU(3) \)-structures of the form \( (\omega_f, \Omega_{f,\theta}) \) where \( (\theta, f) \) is any positive triple and we use the notation of (3.21b).

(iii) *The smooth map that sends an invariant \( \SU(3) \)-structure on \( \SU(3)/\mathbb{T}^2 \) satisfying \( d \Re \Omega = 0 \) to its induced invariant metric is a smooth covering map of degree 4.*

**Proof.** By Lemma 3.20 we know that any invariant \( \SU(3) \)-structure can be written in the form (3.21a).

We already observed that the two cases in (3.21a) are exchanged by \( \mathbf{T} \), induce different orientations and so are necessarily in different connected components of the space of \( \SU(3) \)-structures. Hence it suffices to consider the case that \( \omega_f = \omega_{f'} \) and \( \Omega_{f,\theta} = \Omega_{f',\theta'} \) for some \( (\theta, f) \) and \( (\theta', f') \). (The second case in (3.21a) can be analysed the same way). These two equalities hold if and only if there exist \( k_1, k_2, k_3 \in \mathbb{Z}_2 \) such that

\[
f_i = (-1)^{k_i} f'_i \quad \text{and} \quad \text{sgn} \left( \frac{f'_1 f'_2 f'_3}{f_1 f_2 f_3} \right) = (-1)^{k_1+k_2+k_3} = e^{i(\theta-\theta')}.
\]
Hence $\theta' = \theta \mod 2\pi$ when $k_1 + k_2 + k_3$ is even and $\theta' - \theta = \pi \mod 2\pi$ when $k_1 + k_2 + k_3$ is odd. In the space of all invariant SU(3)-structures where we are free to vary the parameter $\theta$ continuously, by shifting $\theta$ by $\pi$ we could change the sign of $f_1f_2f_3$ remaining within the same connected component. However, once we impose the condition that $\sin \theta = 0$ this is no longer possible; hence the sign of $f_1f_2f_3$ is well-defined on each connected component of invariant SU(3)-structures satisfying the static constraint $d\text{Re}\Omega = 0$. The involution $T_\pi$ exchanges the two connected components that share the same orientation but on which $f_1f_2f_3$ has different signs. Hence there are four connected components of the space of invariant SU(3)-structures satisfying the static constraint $d\text{Re}\Omega = 0$ and they correspond to the four elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. \hfill \Box

Three homogeneous fibrations of $SU(3)/T^2$. We now describe a more geometric interpretation of the parameters $f_1^3, f_2^3$ and $f_3^3$ that specify an SU(3)-invariant Riemannian metric $g_f$ on $SU(3)/T^2$. Consider the following three diagonal SU(3) matrices

$$(3.24) \quad I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Since all three matrices have two coincident eigenvalues their centralisers in SU(3), which we denote by $U(2)_j$, are each subgroups containing the (diagonal) maximal torus $T^2 \subset SU(3)$ and isomorphic to $U(2)$. These three different $U(2)$ subgroups give rise to three distinct homogeneous $S^2$-fibrations of $SU(3)/T^2$

$$S_j^2 = U(2)_j/T^2 \rightarrow SU(3)/T^2 \rightarrow SU(3)/U(2)_j = CP^2_j.$$ 

For any homogeneous metric $g$ on $SU(3)/T^2$, the three fibres $S_1^2$, $S_2^2$ and $S_3^2$ are mutually orthogonal and the restriction of $g$ to the $j$th fibre determines a homogeneous metric on $S_j^2$. Moreover, $g$ is determined by the size of these three fibres at a single point, i.e. by three positive parameters. Each of the invariant metrics $g_j$ can therefore be regarded as a homogeneous metric on $S_j^2$ which with our conventions is the standard round metric of sectional curvature 1. In other words, the parameter $f_j^3$ determines the “size” of the $j$th spherical fibre $S_j^2$ with respect to the homogeneous metric $g_f$.

The Weyl group action on invariant metrics and SU(3)-structures. Let $W$ be the Weyl group of SU(3), i.e. $W = N(T^2)/T^2$ where $N(T^2)$ denotes the normaliser of $T^2$ in SU(3). By standard Lie theory $W$ is isomorphic to the symmetric group on 3 letters: we can take the cosets that contain

$$a_{231} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad a_{132} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$ 

as order 3 and order 2 generators respectively of $W$.

The natural action of the normaliser $N(T^2)$ on $SU(3)/T^2$, i.e. $(n,gT^2) \mapsto gT^2 \cdot n^{-1} = gn^{-1}T^2$, induces a free action of $W$ on $SU(3)/T^2$ by diffeomorphisms and hence also on all invariant tensors on $SU(3)/T^2$. We are interested in this action on the cone of left-invariant metrics and on the invariant 2-forms and 3-forms. Using the explicit generators of $W \cong S_3$ specified above one can verify (see also [23, p. 214]) the following:

**Lemma 3.25.** The Weyl group $W \cong S_3$ acts on SU(3)-invariant tensors on SU(3)/T^2 as follows.

1. $S_3$ acts as the standard representation on our chosen basis of invariant metrics $(g_1, g_2, g_3)$;
2. $S_3$ leaves the 3-form $\beta$ invariant and acts as the sign representation on $\alpha$;
3. A transposition $(ij) \in S_3$ acts on invariant 2-forms by sending $(\omega_i, \omega_j, \omega_k) \mapsto - (\omega_j, \omega_i, \omega_k)$; hence the subgroup $A_3$ acts via cyclic permutations of $(\omega_1, \omega_2, \omega_3)$;
4. $S_3$ acts on invariant SU(3)-structures: any transposition $\sigma \in S_3$ acts via

$$(\omega_f, \Omega_{f,\theta}) \mapsto (-\omega_{\sigma(f)}, -e^{-2i\theta} \Omega_{f,\theta}).$$
the action of the Weyl group
any invariant 
SU
invariant

Corollary 3.26. Up to the actions of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) described in Remark 3.15 and the Weyl group \( W \cong S_3 \) any invariant \( SU(3) \)-structure on \( SU(3)/\mathbb{T}^2 \) satisfying the static constraint can be written uniquely in the form

\[
(\omega_f, \Omega_f) = \left( f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3, f_1 f_2 f_3 \Omega \right)
\]

for some positive real triple \((f_1, f_2, f_3)\) satisfying \( f_1^2 \leq f_2^2 \leq f_3^2 \).

Proof. By Corollary 3.23, after acting with some element of the group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) described in Remark 3.15 we can assume that our invariant \( SU(3) \)-structure satisfying the static constraint belongs to the connected component containing \((\omega_{nK}, \Omega_{nK})\); this component is parameterised by

\[
(\omega_f, \Omega_f) = \left( f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3, f_1 f_2 f_3 \Omega \right)
\]

where the real triple \((f_1, f_2, f_3)\) is positive. By acting with \(A_3 \subset W\) we can further arrange that \( f_1^2 \leq f_2^2 \leq f_3^2 \). \(\square\)

Remark 3.28. Finally we remark that \( SU(3) \)-invariant objects on \( SU(3)/\mathbb{T}^2 \) invariant under a nontrivial subgroup of \( W \) enjoy extra discrete symmetries.

- Any invariant metric \( g_f \) for which \( f_i^2 = f_j^2 \) for unique \( i \neq j \) admits an additional free isometric action of \( \mathbb{Z}_2 \); however this \( \mathbb{Z}_2 \)-action does not preserve the \( SU(3) \)-structure \((\omega_f, \Omega_f, g_{12})\).
- The invariant metrics with \( f_1^2 = f_2^2 = f_3^2 \) possess an additional free isometric action of \( S_3 \); these are precisely the metrics that arise from multiples of the Cartan–Killing form on \( SU(3) \). The corresponding invariant \( SU(3) \)-structures possess an additional \( A_3 \) symmetry, but not the full \( S_3 \) symmetry. Note that the nearly Kähler structure \( SU(3) \)-structure \((\omega_{NK}, \Omega_{NK}) \) defined in (3.22) corresponds to \( f_1 = f_2 = f_3 = \frac{1}{2} \) and hence possesses this additional \( A_3 \) symmetry.

3.5.1. Closed invariant \( G_2 \)-structures from 1-parameter families of invariant \( SU(3) \)-structures. We are ready now to apply the general method described in Section 3.3 to understand \( SU(3) \)-invariant closed \( G_2 \)-structures on \( I \times SU(3)/\mathbb{T}^2 \).

Proposition 3.29. Up to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-action defined in Remark 3.15 any smooth closed \( SU(3) \)-invariant \( G_2 \)-structure on \( I \times SU(3)/\mathbb{T}^2 = I \times F_{1,2} \) can be written uniquely in the form

\[
(3.30a) \quad \varphi_f = \omega_f \wedge dt + \text{Re} \Omega_f = \left( f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3 \right) \wedge dt + f_1 f_2 f_3 \text{Re} \Omega,
\]

\[
(3.30b) \quad \ast \varphi_f = \frac{1}{2} \omega_f^2 - dt \wedge \text{Im} \Omega_f = f_1^2 f_3^2 f_2 \omega_3 + f_3^2 f_1^2 f_2 \omega_1 + f_1^2 f_2^2 f_3 \omega_2 - dt \wedge f_1 f_2 f_3 \text{Im} \Omega,
\]

\[
(3.30c) \quad g_{\varphi_f} = dt^2 + g_f = dt^2 + f_1^2 g_1 + f_2^2 g_2 + f_3^2 g_3,
\]

\[
(3.30d) \quad V_0 \varphi_f = f_1^2 f_2^2 f_3^2 V_0 \wedge dt,
\]

where \( t \in I \subset \mathbb{R} \) is the arclength parameter of an orthogonal geodesic, and \( f = (f_1, f_2, f_3) : I \to \mathbb{R}^3 \) is a triple of positive smooth real functions satisfying the ODE

\[
(3.30e) \quad 2(f_1 f_2 f_3)' = f_1^2 + f_2^2 + f_3^2,
\]

and where \((\omega_f, \Omega_f)\) denotes the invariant \( SU(3) \)-structure defined in (3.27). Furthermore, by using the action of the Weyl group \( W \cong S_3 \) we can take the triple \( f \) to satisfy \( f_1 \leq f_2 \leq f_3 \).
Proof. By the discussion in Section 3.3 (and by (3.14) in particular) for any closed SU(3)-invariant \( G_2 \)-structure on \( I \times SU(3)/\mathbb{T}^2 \) we can assume that its restriction to any constant \( t \)-slice is an invariant SU(3)-structure on \( SU(3)/\mathbb{T}^2 \) satisfying the static constraint \( d \text{Re} \Omega = 0 \). Therefore by Corollary 3.26, up to the action of the Weyl group \( W \) and the Klein four-group action defined in Remark 3.15, any smooth closed SU(3)-invariant \( G_2 \)-structure on \( I \times SU(3)/\mathbb{T}^2 \) can be written in the form given in (3.30a) where the 1-parameter family of SU(3)-structures \( (\omega_f, \Omega_f) \) is defined by (3.27) and must satisfy the system (3.14). We have imposed the static closure condition (3.14a) throughout, so it remains only to understand the dynamic closure condition (3.14b). Using the structure equations (3.18) we see easily that (3.14b) is equivalent to (3.30e). \( \square \)

Equation (3.30e) has the following elementary but important consequences.

**Lemma 3.31** (cf. [23, equation (9.2)]). Assume that \( \varphi_f \) is a closed SU(3)-invariant \( G_2 \)-structure on \( I \times SU(3)/\mathbb{T}^2 \) in the form specified in Proposition 3.29, i.e. \( \varphi_f = dt \wedge \omega_f + \text{Re} \Omega_f \) and \( f : I \to \mathbb{R}^3 \) is a smooth positive triple satisfying (3.30e).

(i) The triple \( f = (f_1, f_2, f_3) \) satisfies

\[
\frac{d}{dt} (f_1 f_2 f_3)^{1/3} \geq \frac{1}{2}
\]

with equality if and only if \( f_1 = f_2 = f_3 \).

(ii) No 6-dimensional orbit of \( \varphi_f \) is a critical point of the orbital volume. Hence there can be no exceptional orbits and there is at most one singular orbit.

(iii) If \( g_\varphi \) is complete then there is a unique singular orbit; this singular orbit must be of the form \( \mathbb{CP}^2 = SU(3)/U(2) \) and \( g_\varphi \) defines a complete Riemannian metric on \( \Lambda^2 \mathbb{CP}^2 \) with at least Euclidean volume growth and with nonpositive scalar curvature.

Proof. Recall that for \( p > 0 \) the \( p \)-th power mean \( \mathcal{M}_p \) of a nonnegative triple \( f = (f_1, f_2, f_3) \) is defined by \( \mathcal{M}_p(f) := \left( \frac{1}{3}(f_1^p + f_2^p + f_3^p) \right)^{1/p} \) and \( \mathcal{M}_0(f) \) is defined to be the geometric mean \( (f_1 f_2 f_3)^{1/3} \). The power means inequality states that for any real numbers \( r < s \)

\[
\mathcal{M}_r(f) \leq \mathcal{M}_s(f)
\]

with equality if and only if all the \( f_i \) are equal (or \( s \leq 0 \) and \( f_i = 0 \) for some \( i \)). (3.30e) written in terms of power means is equivalent to \( 2(\mathcal{M}_3^3)'/3 = 3\mathcal{M}_3^3 \). Since we are assuming that \( f_1 f_2 f_3 \neq 0 \) (3.33) therefore implies that

\[
(\mathcal{M}_0(f))' = \frac{1}{2} \left( \frac{\mathcal{M}_2(f)}{\mathcal{M}_0(f)} \right)^2 \geq \frac{1}{2}
\]

with equality if and only if \( f_1 = f_2 = f_3 \). Exceptional orbits are immediately ruled out because they are necessarily critical points of the orbital volume. If there were two singular orbits then since the orbital volume goes to zero for both singular orbits then there would also have to be an orbit of maximal volume. Suppose for a contradiction that the solution is complete but contains no singular orbit, then the orbit space must be \( \mathbb{R} \). Then at \( t = 0 \) say, \( (f_1 f_2 f_3)^{1/3} \) is some finite positive number, but \( (f_1 f_2 f_3)^{1/3} \) decreases backwards in \( t \) at least as fast as \( \frac{1}{2}t \) and hence in finite backwards time it reaches zero, which contradicts our assumptions. Once we know that the orbit space is \( [0, \infty) \) it follows by Cleyton–Swann’s classification of singular orbits that the singular isotropy group is \( H = U(2) \), so that \( G/H = \mathbb{CP}^2 \) and that \( M \) is \( G \)-equivariantly diffeomorphic to \( \Lambda^2 \mathbb{CP}^2 \). \( \square \)

*The type decomposition on invariant forms for closed SU(3)-invariant \( G_2 \)-structures.* We need the following straightforward result about the \( G_2 \)-type decomposition when restricted to invariant 2-forms and 3-forms.
Lemma 3.34. Assume that \( \varphi_f \) is a closed \( SU(3) \)-invariant \( G_2 \)-structure on \( I \times SU(3)/\mathbb{T}^2 \) in the form specified in Proposition 3.29, i.e. \( \varphi_f = dt \wedge \omega_f + Re \Omega_f \) and \( f : I \rightarrow \mathbb{R}^3 \) is a smooth positive triple satisfying (3.30e).

(i) The invariant 2-forms of type 7 are generated by \( \omega_f \); the invariant 2-form

\[ \beta = \beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3 \]

is of type 14 if and only if its coefficients satisfy the constraint

\[ \sum \frac{\beta_i}{f_i^2} = 0. \]

Therefore the invariant forms of type 14 are generated by \( f_1^2 \omega_1 - f_2^2 \omega_2 \) and \( f_1^2 \omega_1 - f_3^2 \omega_3 \).

(ii) The invariant 3-forms of type 7 are generated by the invariant 3-form \( \beta \) and the invariant 3-forms of type 27 are generated by the triple

\[ \omega_i \wedge dt - \frac{f_if_k}{4f_i} \alpha \]

for \( i = 1, 2, 3 \) and \( (ijk) \) a permutation of \((123)\).

Proof. Recall that the decomposition for 2-forms takes the form

\[ \Lambda_i^2 = \{ X \varphi_f \} = \{ \beta \mid \beta \wedge \varphi_f = 2*\beta \}, \quad \Lambda_i^2_{14} = \{ \beta \mid \beta \wedge \varphi_f = 0 \} = \{ \beta \mid \beta \wedge \varphi_f = -*\beta \}. \]

The first characterisation of type 7 shows that the invariant 2-form \( \omega_f \) generates the invariant 2-forms of type 7. For an invariant 2-form \( \beta = \beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3 \) the first characterisation of type 14 shows that \( \beta \) is of type 14 if and only if \( \sum \beta_i f_i^2 = 0 \) which is clearly equivalent to (3.35).

Recall that for 3-forms the type decomposition takes the form

\[ \Lambda_1^3 = \langle \varphi_f \rangle, \quad \Lambda_7^3 = \{ X \wedge \varphi_f \}, \quad \Lambda_7^2 = \{ \eta \in \Lambda_7^3 \mid \varphi_f \wedge \eta = 0, *\varphi_f \wedge \eta = 0 \}. \]

If \( X \) is an invariant vector field, i.e. \( X = u(t) \partial_t \) for some function \( u \), then \( X \wedge \varphi_f = (uf_1f_2f_3)\beta \), i.e. the invariant 3-forms of type 7 are generated by the invariant 3-form \( \beta \). If \( \eta \) is an arbitrary invariant 3-form with coefficients

\[ \eta = (\eta_1 \omega_1 + \eta_2 \omega_2 + \eta_3 \omega_3) \wedge dt + \eta_\alpha \alpha + \eta_3 \beta \]

then \( \eta \wedge \varphi = 0 \) if and only if \( \eta_\beta = 0 \) and \( \eta \wedge *\varphi = 0 \) if and only if

\[ \eta_\alpha = -\frac{f_1f_2f_3}{4} \sum \eta_i f_i^2. \]

Therefore the invariant 3-forms of type 27 are generated by the triple of invariant 3-forms claimed. Note also that \( \eta_\alpha = 0 \) if and only if the invariant 2-form \( \eta_1 \omega_1 + \eta_2 \omega_2 + \eta_3 \omega_3 \) is of type 14.

3.5.2. The intrinsic torsion of \( SU(3) \)-invariant closed \( G_2 \)-structures. Recall that the intrinsic torsion of any closed \( G_2 \)-structure \( \varphi \) is the unique 2-form \( \tau \) of type 14 that satisfies (3.4), i.e. \( d(*\varphi) = \tau \wedge \varphi \).

If \( \varphi \) is also \( SU(3) \)-invariant then so is its torsion \( \tau \) and so by the results of the previous section we can write the torsion 2-form as

\[ \tau = \tau_1 \omega_1 + \tau_2 \omega_2 + \tau_3 \omega_3 \]

for a triple of functions \( (\tau_1, \tau_2, \tau_3) \) satisfying the type 14 constraint (3.35). The following lemma determines these torsion coefficients \( \tau_i \) for \( \varphi_f \) in terms of the triple \( f = (f_1, f_2, f_3) \) and its first derivatives.
Lemma 3.37. Let $\varphi_f$ be a SU(3)-invariant closed G$\text{\textregistered}$-structure on $I \times SU(3)/\mathbb{T}^2$ written in the normal form described in Proposition 3.29. The intrinsic 2-form $\tau = \tau_1 \omega_1 + \tau_2 \omega_2 + \tau_3 \omega_3$ of $\varphi_f$ is the unique invariant 2-form of type 14 whose coefficients are the triple of real-valued functions $(\tau_1, \tau_2, \tau_3) : I \to \mathbb{R}^3$ satisfying

$$\tau_i = -\frac{f_i^2}{f_j^2 f_k^2} \left( \tau_j f_k^2 + \tau_k f_j^2 \right) = -\frac{f_i^2}{f_j^2 f_k^2} \left( (f_j^2 f_k^2)' - 2 f_1 f_2 f_3 \right),$$

for $(ijk)$ any permutation of $(123)$ or equivalently

$$\tau_i = (f_i^2)' + \frac{f_i^2}{f_1 f_2 f_3} \left( 2 f_i^2 - f_1^2 \right),$$

where for a more compact notation we define

$$\overline{f_i^2} := f_i^2 + f_1^2 + f_2^2.$$

Proof. It follows from the structure equations (3.18) that

$$d(*\varphi_f) = \sum_{(ij)} \left( (f_i^2 f_j^2)' - 2 f_1 f_2 f_3 \right) \omega_i \omega_j \wedge dt,$$

where $(ij)$ is summed over $(12), (23)$ and $(31)$. For $(ijk)$ any permutation of $(123)$ we define

$$A_i := (f_i^2 f_j^2)' - 2 f_1 f_2 f_3.$$

Then using (3.30a) and (3.40) it is straightforward to verify that the condition (3.4) is equivalent to

$$A_i = \tau_k f_j^2 + \tau_j f_k^2, \quad \text{for} (ijk) \text{ any permutation of} (123).$$

Multiplying this by $f_i^2$ and using (3.35) and (3.41) we obtain (3.38a). To see the equivalence of (3.38a) and (3.38b) note that from the equality

$$(f_i^2)' f_j^2 f_k^2 = (f_i^2 f_j^2 f_k^2)' - f_i^2 (f_j^2 f_k^2)'$$

using the closure of $\varphi_f$, i.e. (3.30e), to rewrite the first term of the right-hand side, the definition of $A_i$ and (3.38a) we obtain

$$(f_i^2)' f_j^2 f_k^2 = f_1 f_2 f_3 \overline{f_i^2} - f_i^2 (A_i + 2 f_1 f_2 f_3) = f_1 f_2 f_3 (\overline{f_i^2} - 2 f_i^2) + f_i^2 f_j^2 f_k^2 \tau_i.$$

Rearranging this gives (3.38b). \qed

Example 3.43. By (3.30e) and (3.38a), the SU(3)-invariant G$\text{\textregistered}$-structure defined by $f_1, f_2, f_3$ is torsion-free if and only if $2(f_1 f_2 f_3)' = f_1^2 + f_2^2 + f_3^2$ and each $(f_j^2 f_k^2)' = 2 f_1 f_2 f_3$. The simplest solution to these equations is

$$f_1 = f_2 = f_3 = \frac{1}{2} t.$$

This incomplete solution describes a conical SU(3)-invariant torsion-free G$\text{\textregistered}$-cone with cross-section $F_{1,2} = SU(3)/\mathbb{T}^2$ arising from the unique SU(3)-invariant nearly Kähler structure on $F_{1,2}$ that we described in (3.22). To instead obtain a complete metric, the solution must satisfy the conditions for closing smoothly over a nonprincipal orbit (described in detail in section 5.1, see also Cleyton–Swann [23, p. 217]). The smoothly-closing conditions force two of the $f_i$ to coincide and (up to symmetry) any complete solution can be written in terms of the parameter $r = f_1 f_3$ and a constant $\mu \in \mathbb{R}$ as

$$f_1 = r (r^2 - \mu^2)^{-\frac{1}{2}}, \quad f_2 = f_3 = (r^2 - \mu^2)^{\frac{1}{4}}.$$

These solutions define the asymptotically conical holonomy G$\text{\textregistered}$ metrics on $\Lambda^2_+ \mathbb{CP}^2$ first found by Bryant and Salamon [15]; they are asymptotic to the torsion-free G$\text{\textregistered}$-cone just described. Permuting the $f_i$ gives three AC G$\text{\textregistered}$-manifolds asymptotic to the same cone, but topologically different in that the special $\mathbb{CP}^2$ orbit fits in differently. In fact, one can show that any complete torsion-free G$\text{\textregistered}$-structure asymptotic to the torsion-free G$\text{\textregistered}$-cone described above must be one of these Bryant–Salamon solutions [42, Corollary 6.10].
3.6. **Closed Sp(2)-invariant G2-structures.** When \( G = \text{Sp}(2) \), the principal isotropy group \( K = \text{U}(1) \times \text{Sp}(1) \) acts on the standard representation \( \mathbb{H}^2 \cong \mathbb{C}^4 \) as \( H + L + \overline{L} \) where \( H \cong \mathbb{H} \) is the standard representation of \( \text{Sp}(1) \) and \( L \cong \mathbb{C} \) is the standard representation of \( \text{U}(1) \). It follows that the isotropy representation is \([L^2] + [HL] \). Both of these irreducible modules admit an invariant metric \( g \) and invariant 2-form \( \omega_i \) and the space of invariant 3-forms on their sum is 2-dimensional.

To be more concrete we equip the isotropy representation with the basis

\[
E_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad E_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & -k \end{pmatrix}, \quad E_3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

\[
E_4 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_5 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad E_6 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},
\]

and denote the corresponding dual basis by \( \{e_1, \ldots, e_6\} \). Then \( \{e_1, e_2\} \) is a basis for \([L^2]^* \) and \( \{e_3, \ldots, e_6\} \) is a basis for \([HL]^* \). We can scale \( g_i \) and \( \omega_i \) so that

\[g_1 = e_1^2 + e_2^2, \quad g_2 = e_3^2 + e_4^2 + e_5^2 + e_6^2,\]

\[\omega_1 = e_{12}, \quad \omega_2 = e_{34} + e_{56}.\]

\( g_1 \) may be identified with an \( \text{Sp}(1) \)-invariant metric on the 2-sphere \( \text{Sp}(1) \times \text{Sp}(1)/\text{Sp}(1) \times \text{U}(1) \). By computing its sectional curvature one can check that \( g_1 \) is the standard round metric with constant sectional curvature \( \frac{1}{2} \). Similarly \( g_2 \) can be identified with an \( \text{Sp}(2) \)-invariant metric on \( \text{Sp}(2)/\text{Sp}(1) \times \text{Sp}(1) \cong \mathbb{H}P^1 \cong S^4 \). Again by computing its sectional curvature one can check that \( g_2 \) is the standard round metric with constant sectional curvature \( \frac{1}{2} \). The 2-dimensional space of invariant 3-forms is then spanned by

\[\alpha = e_{246} - e_{235} - e_{145} - e_{136}, \quad \beta = e_{135} - e_{146} - e_{236} - e_{245},\]

and we set \( \text{Vol}_0 = e_{123456} \). The exterior derivatives of these invariant forms satisfy

\[
d\omega_1 = \frac{1}{2} \alpha, \quad d\omega_2 = \alpha, \quad d\alpha = 0, \quad d\beta = -2\omega_1 \wedge \omega_2 - \omega_1 \wedge \omega_2.\]

Using the same method as in the \( SU(3) \)-invariant case we obtain the following description of \( \text{Sp}(2) \)-invariant \( SU(3) \)-structures on \( \mathbb{CP}^3 \) that satisfy the static closure condition \( d\text{Re} \Omega = 0 \).

**Lemma 3.46.**

(i) Up to the action of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) described in Remark 3.15 a general \( \text{Sp}(2) \)-invariant \( SU(3) \)-structure on \( \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) \) \( = \mathbb{CP}^3 \) can be written in the form

\[(\omega_f, \Omega_{f,\theta}) = (f_1^2 \omega_1 + f_2^2 \omega_2, f_1 f_2^* e^{-i\theta} \Omega)\]

for \( e^{i\theta} \in \mathbb{S}^1 \), \( \Omega := \alpha + i \beta \) and \( f = (f_1, f_2) \in \mathbb{R}_+ \subset \mathbb{R}^2 \). The induced metric is \( g_f = f_1^2 g_1 + f_2^2 g_2 \).

(ii) \( (\omega_f, \Omega_{f,\theta}) \) satisfies the static closure condition \( d\text{Re} \Omega_{f,\theta} = 0 \) if and only if \( \sin \theta = 0 \).

(iii) The \( \text{Sp}(2) \)-invariant \( SU(3) \)-structure with \( f_1 = f_2 = \frac{1}{2} \) and \( \theta = 0 \) is the standard \( \text{Sp}(2) \)-invariant nearly Kähler structure on \( \mathbb{CP}^3 \).

(iv) The space of \( \text{Sp}(2) \)-invariant \( SU(3) \)-structures on \( \mathbb{CP}^3 \) satisfying the static closure condition has four connected components each diffeomorphic to the positive quadrant \( \mathbb{R}_+^2 \) in \( \mathbb{R}^2 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts simply transitively on these four components. The connected component containing the nearly Kähler structure on \( \mathbb{CP}^3 \) is parameterised by

\[(\omega_f, \Omega_f) = (f_1^2 \omega_1 + f_2^2 \omega_2, f_1 f_2^* \Omega)\]

with \( f = (f_1, f_2) \in \mathbb{R}^2 \).

Similarly we have the following result about the ‘Weyl group’ \( W \) and its action on invariant \( SU(3) \)-structures in the \( \text{Sp}(2) \)-invariant setting.

**Lemma 3.47.** Let \( K \) denote the principal isotropy group \( \text{Sp}(1) \times \text{U}(1) \).
Remark. An important observation (due to Cleyton–Swann) is that if we set
specified in Proposition 3.48.

(ii) The generator of \( W \) fixes the invariant bilinear forms \( g_1, g_2 \), and the 3-form \( \beta \) and acts as
\((-1)\) on \( \omega_1, \omega_2 \) and \( \alpha \). In particular any invariant metric \( g_f \) on \( \mathbb{CP}^3 \) possesses an additional
free isometric \( \mathbb{Z}_2 \)-symmetry, but this \( \mathbb{Z}_2 \) does not preserve any invariant \( SU(3) \)-structure.

(iii) \( W \) acts on invariant \( SU(3) \)-structures and preserves the subset of structures satisfying the
static constraint \( d\text{Re}\Omega = 0. \)

Proof. Parts (i) and (ii): The normaliser and the Weyl group \( W \) are described explicitly in [23, p.
216] (but note that there is a sign error there in describing its action on invariant 3-forms); see also
[22, pp. 120–121], but notice that Cleyton’s choice of \( \alpha \) and \( \beta \) is the opposite to ours and that in
[23], namely his form \( \beta \) is closed. Part (iii) follows easily from part (ii) and the previous lemma. □

By appealing to the method we used in the \( SU(3) \)-invariant case but using Lemmas 3.46 and 3.47
we deduce the following result.

**Proposition 3.48.** Up to the action of the discrete symmetries \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) of Remark 3.15 any smooth
closed \( Sp(2) \)-invariant \( G_2 \)-structure on \( I \times Sp(2)/Sp(1) \times U(1) = I \times \mathbb{CP}^3 \) can be written in the
form

\begin{align}
(3.49a) \quad \varphi &= \omega_f \wedge dt + \text{Re}\, \Omega_f = (f_1^2 \omega_1 + f_2^2 \omega_2) \wedge dt + f_1 f_2^2 \alpha \\
(3.49b) \quad *\varphi &= \frac{1}{2} \omega_f^2 + \text{Im}\, \Omega_f \wedge dt = f_1^2 f_2^2 \omega_1 \wedge \omega_2 + f_2^4 \omega_1 \wedge \omega_2 + f_1 f_2^2 \beta \wedge dt, \\
(3.49c) \quad g_\varphi &= dt^2 + g_f = dt^2 + f_1^2 g_1 + f_2^2 g_2, \\
(3.49d) \quad \text{Vol}_\varphi &= f_1^2 f_2^2 dt \wedge \text{Vol}_0,
\end{align}

where \( t \in I \subset \mathbb{R} \) is the arclength parameter of an orthogonal geodesic and \( f = (f_1, f_2) : I \to \mathbb{R}^2 \) is a
pair of positive smooth real functions satisfying

\begin{equation}
(3.49e) \quad 2(f_1 f_2^2)' = f_1^2 + 2f_2^2.
\end{equation}

**Remark.** An important observation (due to Cleyton–Swann) is that if we set \( f_2 = f_3 \) in Proposition
3.29 then all the statements there reduce to those in Proposition 3.48 above. Hence we can treat
the \( Sp(2) \)-invariant system as if it were a special case of the \( SU(3) \)-invariant one. The one difference
to note is that because in the \( Sp(2) \)-invariant setting the Weyl group \( W \cong \mathbb{Z}_2 \) preserves all invariant
bilinear forms (whereas in the \( SU(3) \) setting \( W \cong S_3 \) permutes \( g_1, g_2 \) and \( g_3 \)) we cannot a priori
assume that \( f_1 \leq f_2 \).

As in the \( SU(3) \)-invariant setting the closed condition (3.49e) already implies significant restrictions
on the possible orbit structure of a closed \( Sp(2) \)-invariant \( G_2 \)-structure.

**Lemma 3.50.** Assume that \( \varphi_f \) is a closed \( Sp(2) \)-invariant \( G_2 \)-structure on \( I \times \mathbb{CP}^3 \) in the form
specified in Proposition 3.48, i.e. \( \varphi_f = dt \wedge \omega_f + \text{Re}\, \Omega_f \) and \( f = (f_1, f_2) : I \to \mathbb{R}^2 \) is a smooth positive
pair satisfying (3.49e).

- (i) \( f = (f_1, f_2) \) satisfies \( \frac{d}{dt}(f_1 f_2^2)^{1/3} \geq \frac{1}{2} \) with equality if and only if \( f_1 = f_2 \).
- (ii) No 6-dimensional orbit is a critical point of the orbital volume. Hence there are no exceptional
orbits and there is at most one singular orbit.
- (iii) If \( g_\varphi \) is complete then there is a unique singular orbit; this singular orbit must be of the form
\( S^4 = Sp(2)/Sp(1) \times Sp(1) \) and \( g_\varphi \) defines a complete Riemannian metric on \( \Lambda^2 S^4 \) with at
least Euclidean volume growth and with nonpositive scalar curvature.

The \( G_2 \)-type decomposition for \( Sp(2) \)-invariant forms on \( \varphi_f \) takes the following form.

**Lemma 3.51.** Assume that \( \varphi_f \) is a closed \( Sp(2) \)-invariant \( G_2 \)-structure on \( I \times \mathbb{CP}^3 \) in the form
specified in Proposition 3.48.
(i) The invariant 2-forms of type 7 are generated by $\omega_1$; the invariant 2-form $\beta = \beta_1 \omega_1 + \beta_2 \omega_2$ is of type 14 if and only if its coefficients satisfy the constraint

$$\frac{1}{f_1^2} \beta_1 + \frac{2}{f_2^2} \beta_2 = 0.$$

Therefore the invariant 2-forms of type 14 are generated by $2f_1^2 \omega_1 - f_2^2 \omega_2$.  

(ii) The invariant 3-forms of type 7 are generated by the invariant 3-form $\beta$.

(iii) The torsion 2-form $\tau = \tau_1 \omega_1 + \tau_2 \omega_2$ of $\varphi$ is given in terms of $f_1$ and $f_2$ by

$$\tau_1 = (f_1^2)' - 2f_1 + \frac{f_3^2}{f_2^2}, \quad \tau_2 = (f_2^2)' - f_1,$$

and satisfies the type 14 constraint

$$\frac{1}{f_1^2} \tau_1 + \frac{2}{f_2^2} \tau_2 = 0.$$

In particular the torsion coefficients $\tau_1$ and $\tau_2$ have opposite signs whenever they are nonzero.

**Remark 3.54.** Since both types of torsion-free $SU(3)$-invariant solution given in Example 3.43 had $f_2 = f_3$, they also give rise to solutions to the torsion-free $Sp(2)$-invariant ODE system. They correspond to the unique $Sp(2)$-invariant torsion-free cone over $\mathbb{CP}^3$ arising from the $Sp(2)$-invariant nearly Kähler structure on $\mathbb{CP}^3$, and to the complete $Sp(2)$-invariant solutions on $\mathbb{A}^2 \mathbb{S}^4$ asymptotic to that cone respectively; these solutions were also found by Bryant and Salamon [15].

4. THE ODE SYSTEMS FOR COHOMOGENEITY-ONE LAPLACIAN SOLITONS

In this section we derive the ODE systems satisfied by $G$-invariant Laplacian solitons for the cohomogeneity-one actions of $G = SU(3)$ and $G = Sp(2)$ described in the previous section.

Section 4.1 makes some general remarks about closed Laplacian solitons that we use later. In Section 4.2 we derive the $G$-invariant Laplacian soliton equations. Section 4.3 gives a reformulation of these equations as a real-analytic first-order system; for most purposes the first-order reformulations turn out to be more convenient than the form of the systems derived in Section 4.2.

Recall from (2.3) that the system for a closed Laplacian soliton may be recast in the form

\[
\text{(LSE)} \quad d\varphi = 0, \quad d(\tau - X_\varphi) = \lambda \varphi,
\]

where $\tau$ is the intrinsic torsion of $\varphi$, i.e. the unique 2-form of type 14 satisfying

$$d(*\varphi) = \tau \wedge \varphi.$$

For the cohomogeneity-one actions of the groups $Sp(2)$ and $SU(3)$ considered in the previous section we now seek cohomogeneity-one Laplacian solitons, i.e. the 3-form $\varphi$ has the form specified in Proposition 3.29 or Proposition 3.48 respectively and the vector field $X = u \partial_t$ for some function $u = u(t) : I \to \mathbb{R}$. In both cases the system of PDEs (LSE) reduces to a system of nonlinear ODEs that we will write down explicitly. Moreover, the ODE system for $Sp(2)$-invariant solitons can be obtained from the system for $SU(3)$-invariant solitons by setting $f_2 = f_3$, $\tau_2 = \tau_3$ and considering the resulting ODE system in a smaller number of variables.

4.1. General remarks on closed Laplacian solitons. It is useful to decompose by type the 3-forms appearing on both sides of the Laplacian soliton equation (LSE) and to consider the resulting equations on the components of type 1, 7 and 27 separately. Recall from (3.7) that $d\tau$ has no type 7 component and its type 1 component is $\frac{1}{|t|^2} \tau^2 \varphi$. Lotay–Wei [45, §9] show that the type decomposition of $\mathcal{L}_X \varphi$ for any closed $G_2$-structure $\varphi$ has components of type 1, 7 and 27 respectively given by

$$\frac{3}{2} \text{div}(X) \varphi, \quad \frac{1}{2} (d^* (\iota_X \varphi))^2 \ast^\varphi, \quad \iota \left( \frac{1}{2} \mathcal{L}_X g \varphi - \frac{1}{4} \text{div}(X)g \varphi \right).$$
Hence the 1 and 7 components of the Laplacian soliton equation (LSE) read

\[(LS_1)\]
\[\text{div} X = \frac{1}{3} |\tau|^2 - \frac{7}{3} \lambda,\]
\[\text{(LS}_7)\]
\[d^*(X \lrcorner \varphi) = 0\]

respectively. The 27 component of (LSE) can be derived from the expression given in [45, (9.12)].

Remark 4.2. Integrating both sides of \((LS_1)\) over a compact manifold and applying the Divergence Theorem implies that compact shrinking Laplacian solitons cannot exist and that for compact steady Laplacian solitons the underlying \(G_2\)-structure must be torsion free and therefore the vector field \(X\) must be an automorphism of \(\varphi\), i.e. \(L_X \varphi = 0\). Hence also \(X\) must also be a Killing field of \(g_\varphi\); a torsion-free \(G_2\)-structure admits nontrivial Killing fields only when the holonomy of \(g_\varphi\) is reducible. If we have a gradient Laplacian soliton, i.e. if \(X\) is the gradient of a potential \(f\), and \(\lambda \leq 0\), then we have \(\Delta f \geq \frac{1}{4} |\tau|^2 \geq 0\), i.e. the potential function \(f\) is subharmonic on any steady or shrinking gradient Laplacian soliton. This fact can be useful in the complete noncompact setting.

In the SU(3)-invariant setting note that for any invariant closed 3-form \(\varphi_f\) and invariant vector field \(X = u \partial_t\), we have \(\ast(X \lrcorner \varphi_f) = u \left( \sum f_i^2 f_3^2 \omega_j \wedge \omega_k \wedge dt \right)\). Since \(d \omega_i = \frac{1}{2} \alpha\) and \(\omega_i \wedge \alpha = 0\), this implies that \(d \ast(X \lrcorner \varphi) = 0\). Hence the 7 component of the Laplacian soliton equation is automatically satisfied in this case. The 1 component of the Laplacian soliton equation however yields useful information. Since \(X\) is assumed to be invariant we have

\[\text{div} X = \text{div} (u \partial_t) = \frac{1}{(f_1 f_2 f_3)^2} (u f_2^2 f_3^2)’ = u’ + 2u (\ln f_1 f_2 f_3)’ = u’ + u \left( \frac{f_2^2}{f_1 f_2 f_3} \right)\]

where in the final equality we use \((3.30e)\). Hence in this setting \((LS_1)\) reads

\[u’ + u \left( \frac{f_2^2}{f_1 f_2 f_3} \right) = \frac{1}{3} |\tau|^2 - \frac{7}{3} \lambda.\]

As a immediate consequence of \((LS_1)\) we deduce the following result about SU(3)-invariant shrinkers.

**Lemma 4.4.** Let \((\varphi_f, X = u \partial_t, \lambda)\) be any SU(3)-invariant shrinking Laplacian soliton. If the vector field \(X = u \partial_t\) is positive at some \(t_+\) then \(X\) remains positive for all \(t \geq t_+\) (within the lifetime of the soliton). Moreover if soliton is forward-complete, i.e. the solution exists for all \(t\) sufficiently large, then \(X\) is eventually positive.

**Proof.** \((4.3)\) implies that for any shrinker \(u’ \geq -\frac{7}{3} \lambda > 0\) whenever \(u \leq 0\). \(\square\)

**Remark.** In the same spirit, but with a little more work to understand the torsion term, one can also make deductions about the positivity of \(u\) being preserved for steady solitons.

### 4.2. The ODE system for SU(3)-invariant closed Laplacian solitons on \(M^0\).

In this section we derive the ODEs satisfied by SU(3)-invariant closed Laplacian solitons on \(M^0\), the open dense set of principal points for the SU(3)-action, and derive some of the basic properties of this ODE system. The ODEs satisfied by Sp\(_2\)-invariant Laplacian solitons arise by specialisation of this system.

**Lemma 4.5.** The triple \((\varphi_f, X, \lambda)\) is an SU(3)-invariant closed Laplacian soliton on \(I \times \mathbb{F}_{1,2}\) if and only if up to the action of the Klein four-group defined in \((3.15)\) it can be written in the form

\[\varphi_f = \omega_f \wedge dt + Re \Omega_f, \quad X = u(t) \partial_t\]

where \((\omega_f, \Omega_f)\) is the invariant SU(3)-structure defined in \((3.27)\), \(u\) is a real function on the interval \(I\) and \((f_1, f_2, f_3) : I \rightarrow \mathbb{R}^3\) is a positive triple satisfying the equations

\[2(f_1^2 f_2^2 + f_3^2),\]
\[(\tau_i - u f_i^2)’ = \lambda f_i^2, \quad \text{for} \quad i = 1, 2, 3,\]
\[\tau_1 + \tau_2 + \tau_3 = u(f_1^2 + f_2^2 + f_3^2) + 2\lambda f_1 f_2 f_3,\]
where \( \tau_i \) are the components of the torsion 2-form \( \tau = \sum \tau_i \omega_i \) of \( \varphi_f \) as determined by (3.38a).

**Proof.** Proposition 3.29 already showed that up to the action of the Klein four-group any closed SU(3)-invariant \( G_2 \)-structure on \( I \times \mathbb{F}_{1,2} \) can be written in the form above and must satisfy (4.6a). So it remains to establish (4.6b) and (4.6c). Since \( dw_i = \frac{1}{2} \alpha \) we calculate that

\[
d\tau = \frac{1}{2} \left( \sum \tau_i \right) \alpha + \sum \tau_i' dt \land \omega_i
\]

and

\[
d(X \varphi_f) = \left( \frac{1}{2} u \sum f_i^2 \right) \alpha + \sum (uf_i^2)' dt \land \omega_i.
\]

(4.6b) and (4.6c) now follow by equating the coefficients of \( \omega_i \land dt \) and of \( \alpha \) respectively on both sides of the second equation in (LSE).

\[ \square \]

4.2.1. The ODE system for Sp(2)-invariant closed Laplacian solitons. Recall from (3.49) that up to the action of discrete symmetries any closed Sp(2)-invariant \( G_2 \)-structure on \( I \times \mathbb{CP}^3 \) can be written in the form

\[
\varphi_f = \omega_f \land dt + \text{Re } \Omega_f = (f_1^2 \omega_1 + f_2^2 \omega_2) \land dt + f_1 f_2^2 \alpha,
\]

where \( (f_1, f_2) : I \to \mathbb{R}^2 \) is a positive pair satisfying the first-order ODE (3.49e). The Laplacian soliton system for such a closed Sp(2)-invariant \( G_2 \)-structure reduces to

\[
\begin{align*}
(4.7a) & \quad (f_1 f_2^2)' = \frac{1}{2} f_1^2 + f_2^2; \\
(4.7b) & \quad (\tau_1 - uf_1^2)' = \lambda f_1; \\
(4.7c) & \quad (\tau_2 - uf_2^2)' = \lambda f_2; \\
(4.7d) & \quad \tau_1 + 2 \tau_2 = u f_1^2 + 2 f_2^2 + 2\lambda f_1 f_2^2,
\end{align*}
\]

where the components \( \tau_1 \) and \( \tau_2 \) of the torsion 2-form \( \tau = \tau_1 \omega_1 + \tau_2 \omega_2 \) are given by (3.52). This ODE system can be obtained from (4.6) by setting \( f_2 = f_3 \) and \( \tau_2 = \tau_3 \), and as a result many of its properties are inherited from that larger ODE system.

4.2.2. Basic properties of the SU(3)-invariant soliton system. We make several observations about the structure of the ODE system (4.6) governing SU(3)-invariant Laplacian solitons. Define

\[
H := H(f_1, f_2, f_3, f_1', f_2', f_3', u, \lambda) = \sum_{i=1}^{3} (\tau_i - uf_i^2) - 2\lambda f_1 f_2 f_3 = \bar{\tau} - u \bar{f}^2 - 2\lambda f_1 f_2 f_3,
\]

where as above we use (3.38a) to express the torsion components \( \tau_i \) in terms of \( f_i \) and \( f_i' \) and where for a more compact expressions we introduce the notation

\[
(4.8) \quad \bar{\tau} := \tau_1 + \tau_2 + \tau_3, \quad \bar{f}^2 := f_1^2 + f_2^2 + f_3^2.
\]

Then the four equations in (4.6a) and (4.6b) imply that \( H' = 0 \) on any SU(3)-invariant Laplacian soliton. However, the final equation (4.6c) implies that in fact \( H = 0 \). For this reason we will sometimes refer to equation (4.6c) as the conservation law for the system (4.6), since it fixes the value of the conserved quantity \( H \) to be zero.

An immediate consequence of this observation is that if \( f_1, \tau_1 \) and \( u \) satisfy equations (4.6a), (4.6c) and any two out of the three equations (4.6b) then they necessarily satisfy the full system of ODEs. Also if \( \lambda \neq 0 \) then differentiating equation (4.6c) and subtracting the sum of the three equations in (4.6b) shows that equation (4.6a) is a consequence of the other equations in (4.6).

Using (3.35) and the fact that \( \tau \) is of type 14 we can also rewrite the left-hand side of the conservation law as

\[
\bar{\tau} = \tau_1 + \tau_2 + \tau_3 = \frac{f_1^2 - f_2^2}{f_1^2} \tau_1 + \frac{f_2^2 - f_3^2}{f_2^2} \tau_2.
\]
Equations (4.6a) and (4.6c) involve at most first derivatives of the $f_i$ and no derivatives of $u$. Moreover each term $f_i'$ appears linearly in both equations, $u$ does not appear in (4.6a) and appears linearly in (4.6c). The $i$th equation in (4.6b) depends linearly on $f_i''$, does not depend on the other 2nd derivatives of the $f$, and also depends on all the first derivatives $f_1', f_2', f_3'$ and $u'$.

Remark 4.9. It is often useful to keep in mind how solutions of the ODE system (4.6) behave under rescaling: any solution $(f_1, f_2, f_3, u)(t)$ of (4.6) for a given value of $\lambda \in \mathbb{R}$ transforms into a solution of (4.6) with $\hat{\lambda} = \mu^{-2}\lambda$ provided we make the following rescalings

$$(4.10) \quad (t, f_1, u, \lambda, \tau, \varphi, g_{\varphi}, *\varphi) \mapsto (\mu t, \mu f_1, \mu^{-2}u, \mu^{-2}\lambda, \mu\tau, \mu^3\varphi, \mu^2 g_{\varphi}, \mu^4 *\varphi).$$

We will call the powers of $\mu$ that appear above the scaling weights. Since we also have $\partial_t \mapsto \mu^{-1}\partial_t$, each successive time-derivative of $f_i$ or $u$ decreases its scaling weight by one. Note the elementary but important fact that re-scalings of steady solitons are steady solitons.

4.3. A first-order reformulation of the SU(3)-invariant Laplacian soliton system. The goal of this section is to reformulate the mixed-order ODE system (4.6) for SU(3)-invariant solitons on $\mathcal{M}^0$ as a nonsingular real analytic first-order system in an alternative system of variables. We achieve this by exploiting the conservation law (4.6c) of this system, the equation (LS$_1$) satisfied by the 1 component of a soliton and the type 14 constraint on the torsion $\tau$ (3.35). This first-order reformulation of the soliton ODE system will make it clear that for any $\lambda \neq 0$ there is a 4-parameter family of local real analytic SU(3)-invariant Laplacian solitons and that there is a 3-parameter family of local SU(3)-invariant steady Laplacian solitons. This reformulation will also enable us to apply a known technique for proving the existence of solutions that extend smoothly over the singular orbit $\mathbb{C}P^2$: see Section 5.

Proposition 4.11. Let $f = (f_1, f_2, f_3) : I \to \mathbb{R}^3$ be a triple of positive functions and let $\tau_1, \tau_2, \tau_3 : I \to \mathbb{R}$ be functions defined on the connected interval $I$ satisfying the first-order ODE system

$$(4.12a) \quad (f_i^2)' = \tau_i - \frac{f_i^2}{f_1 f_2 f_3} \left(2f_i^2 - \overline{f_i}^2\right),$$

$$(4.12b) \quad \overline{f_i}^2 \tau_i' = \frac{4\lambda}{3} f_i^2 S_i + \tau \left(\tau_i - \frac{2f_i^4}{f_1 f_2 f_3}\right) + \frac{1}{3} |\tau|^2 \overline{f_i}^2 f_i,$$

where the quantity $S_i$ is defined by

$$(4.13) \quad S_i := 3f_i^2 - \overline{f_i}^2 - \frac{3f_1 f_2 f_3 \tau_i}{2f_i^2},$$

and the initial data satisfies $\sum \frac{\tau_i}{f_i^2} = 0$ for some given $t_0 \in I$. Then for $\lambda \in \mathbb{R}$ the triple $(\varphi_f, X, \lambda)$

$$\varphi_f := \omega_f \wedge dt + \text{Re} \Omega_f = (f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3) \wedge dt + f_1 f_2 f_3 \alpha,$$

$$X := u \partial_t \quad \text{with} \quad u \text{ defined by} \quad \overline{u} f^2 := -2\lambda f_1 f_2 f_3 + \tau$$

is an SU(3)-invariant closed Laplacian soliton on $I \times \mathbb{F}_{1,2}$, i.e. $(f_1, f_2, f_3, u)$ satisfies the ODE system (4.6). In particular $\sum \frac{\tau_i}{f_i^2} = 0$ holds on $I$. Conversely, (up to discrete symmetries) any SU(3)-invariant closed Laplacian soliton on $I \times \mathbb{F}_{1,2}$ arises from a solution of (4.12) that satisfies $\sum \frac{\tau_i}{f_i^2} = 0$ throughout its lifetime.

Before proving this result we make a related remark.

Remark 4.14. Define a 5-dimensional smooth (in fact real analytic) noncompact manifold $\mathcal{P}$ by

$$\mathcal{P} := \{(f, \tau) \in (\mathbb{R}_+^3 \times \mathbb{R}_+^3) | \sum \tau_i f_i^{-2} = 0\}.$$
an integral curve of (4.12) that remains in $\mathcal{P}$ and conversely that any integral curve of (4.12) that starts in $\mathcal{P}$ gives rise to an SU(3)-invariant Laplacian soliton. In other words, we can view $\mathcal{P}$ as the phase space that parameterises all possible principal orbits of SU(3)-invariant Laplacian solitons.

Note also that $\mathcal{P}$ has some additional structure: it is naturally a real analytic 2-plane subbundle $\xi$ of the trivial real 3-plane bundle $\mathbb{R}^3$ over the positive octant $\mathbb{R}^3_+$ with projection map $\pi : (f, \tau) \mapsto f$ and fibre $\xi_f := \{(\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3 | \sum \tau_i f_i^{-2} = 0\}$. We note also that the fibre $\xi_f$ of $\xi$ depends only on the homothety class $[f_1^2, f_2^2, f_3^2] \in \mathbb{S}^2_+$. In other words, $\mathcal{P}$ is the radial extension of a real 2-plane bundle over $\mathbb{S}^2_+$.

**Proof of Proposition 4.11.** First we show that any solution of (4.6) gives rise to a solution of the first-order system (4.12). So suppose we have a solution $(f_1, f_2, f_3, u_i)$ of the SU(3)-invariant Laplacian soliton ODE system (4.6) for some $\lambda \in \mathbb{R}$ and let $(\tau_1, \tau_2, \tau_3)$ denote the coefficients of its torsion 2-form $\tau = \sum \tau_i \omega_i$. Since

$$\varphi_f := (f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3) \wedge dt + f_1 f_2 f_3 \alpha$$

is assumed to be a closed SU(3)-invariant $G_2$-structure its torsion coefficients satisfy (4.12a) thanks to (3.38b). It remains therefore to derive (4.12b). To this end we now seek to eliminate $u$ from the ODE system (4.6) and hence obtain expressions for $\tau_i'$ in terms of $\lambda$, the coefficients $f_i$ (but not their derivatives) and the $\tau_i$. Recall the conservation law (4.6c) for the ODE system has the form

(CL) \quad \quad \quad \quad \quad \quad u f^2 = \tau - 2\lambda f_1 f_2 f_3.

Substituting this expression for $u$ into (4.3), which recall was simply the 1 component of the Laplacian soliton equation, yields

$$3(u' + \lambda) = 2\lambda - \frac{3\tau}{f_1 f_2 f_3} + \sum \frac{\tau_i^2}{f_i^3} = 2\lambda + |\tau|^2 - \frac{3\tau}{f_1 f_2 f_3}.$$  

Since we are assuming all $f_i$ to be positive the system of equations (4.6b) is equivalent to

$$\frac{\tau_i'}{f_i^3} = \frac{u (f_i^2)'}{f_i^2} + (u' + \lambda) \quad \text{for } i = 1, 2, 3.$$  

Substituting the expressions for $(f_i^2)'$, $u$ and $u' + \lambda$ given in (4.12a), (CL) and (4.15) respectively into the right-hand side of these equations yields

$$\tau_i' = \left(\frac{\tau_i}{f_i^3} - \frac{2\lambda f_i f_2 f_3}{f_i^3} \right) + \frac{1}{3} \left(2\lambda + |\tau|^2 - \frac{3\tau}{f_1 f_2 f_3}\right).$$

Multiplying both sides by $f_1 f_2 f_3 \frac{f_i^2}{f_i}$ and rearranging yields

$$\frac{f_i f_j f_k}{f_i} \tau_i' = (\tau - 2\lambda f_1 f_2 f_3) \left(f_i f_j f_k \tau_i - (2\lambda - \tau_i^2)\right) + \frac{1}{3} \left(2\lambda + |\tau|^2 - \frac{3\tau}{f_1 f_2 f_3}\right).$$

In the last line we have first grouped the terms not involving the coefficients of $\tau$, then those linear in its coefficients and finally those quadratic in its coefficients. The previous equation is equivalent to

$$\tau_i' = \frac{4\lambda}{3} f_i f_2 f_3 (3 f_i^2 - \tau_i^2) + \frac{1}{2} \frac{f_i}{f_i f_2 f_3} \left(2\lambda + |\tau|^2 - \frac{3\tau}{f_1 f_2 f_3}\right).$$

Grouping the terms containing $\lambda$ together gives the form of the equation stated in the Proposition.

Conversely suppose that $(f_1, f_2, f_3, \tau_1, \tau_2, \tau_3)$ satisfy (4.12) and that initially the type 14 constraint is satisfied, i.e. initially $\sum \tau_i^2 = 0$ at some $t_0 \in I$. Now define a function $u$ in terms of $f_i$ and $\tau_i$ via the conservation law (CL). First we want to prove that the resulting SU(3)-invariant $G_2$-structure $\varphi$ is closed and has the invariant 2-form $\tau := \sum \tau_i \omega_i$ as its torsion 2-form (and therefore satisfies
$\sum \frac{\tau_i}{f_i^2} = 0$ throughout its lifetime). It is straightforward to verify that the invariant 3-form $\varphi$ is closed if and only if the condition $\sum \frac{\tau_i}{f_i^2} = 0$ holds for all $t \in I$. Since we have defined $u$ via the conservation law (CL) we can rewrite (4.12a) as

$$\frac{\tau_i}{f_i^2} = \frac{4\lambda E_i}{3f_i^2} + \frac{u\tau_i}{f_i} - \frac{2f_i^2}{f_1f_2f_3f_i^2} + \frac{1}{3}|\tau|^2,$$

where $E_i := 3f_i^2 - \overline{f_i}$. Summing these equations over $i$ yields

$$\left(\sum \frac{\tau_i}{f_i^2}\right) = -\frac{2\overline{f_i}}{f_1f_2f_3} + |\tau|^2 + u\left(\sum \frac{\tau_i}{f_i^2}\right).$$

Using (4.12a) yields

$$\frac{\tau_i}{f_i^2} \frac{(f_i^2)'}{f_i^2} = \frac{\tau_i^2}{f_i^2} - \frac{2\tau_i}{f_1f_2f_3} + \frac{\overline{f_i}}{f_1f_2f_3} \frac{2\tau_i}{f_i},$$

and therefore summing over $i$ gives

$$\sum \frac{\tau_i}{f_i^2} \frac{(f_i^2)'}{f_i^2} = |\tau|^2 - \frac{2\overline{f_i}}{f_1f_2f_3} + \frac{\overline{f_i}}{f_1f_2f_3} \left(\sum \frac{\tau_i}{f_i^2}\right).$$

Hence we find that

$$\frac{d}{dt} \left(\sum \frac{\tau_i}{f_i^2}\right) = \left(\sum \frac{\tau_i}{f_i^2}\right) - \sum \frac{\tau_i}{f_i^2} \frac{(f_i^2)'}{f_i^2} = \left(\sum \frac{\tau_i}{f_i^2}\right) \left(u - \frac{\overline{f_i}}{f_1f_2f_3}\right).$$

In particular, this equation implies that if a solution to (4.12) has $\sum \frac{\tau_i}{f_i^2} = 0$ at some $t_0 \in I$ then it continues to vanish for all $t \in I$. Hence $\varphi$ defines a closed SU(3)-invariant G$_2$-structure whose torsion 2-form $\tau \in \Omega^2_{14}$ is equal to the invariant 2-form $\tau := \sum \tau_i\omega_i$.

It remains to prove that $(f_1, f_2, f_3, u)$ satisfies the SU(3)-invariant Laplacian soliton equations (4.6). Clearly (4.6a) is satisfied (because we just proved that $\varphi$ is closed) and since we are defining $u$ to satisfy the conservation law our solution automatically satisfies (4.6c). Therefore it remains only to check that $u$ satisfies (4.6b) for all $i$. Earlier in the proof we saw that (4.6b) holds if and only if (4.16) does. Differentiating the conservation law (CL) that defines $u$ yields

$$u'\overline{f_i} = \overline{(f_i^2)'} - 2\lambda(f_1f_2f_3)' = \overline{(f_i^2)'} - \lambda\overline{f_i} - 2\frac{f_1f_2f_3}{f_i^2} \left(\sum \frac{\tau_i}{f_i^2}\right)$$

and hence

$$(u' + \lambda) = \frac{\overline{(f_i^2)'}}{f_i} - \frac{u\overline{(f_i^2)'}}{f_i} - 2\frac{f_1f_2f_3}{f_i^2} \left(\sum \frac{\tau_i}{f_i^2}\right) = \frac{\overline{(f_i^2)'}}{f_i} - \frac{u\overline{(f_i^2)'}}{f_i},$$

where the final equality holds because by our assumptions the term $\sum \frac{\tau_i}{f_i^2}$ vanishes (initially and therefore throughout the lifetime of the solution). Using (4.12b) and summing over $i$ yields

$$\frac{\overline{(f_i^2)'} f_i}{f_i^2} = \left(\frac{4\lambda D(f)}{3(f_i^2)^2} - \frac{2\overline{f_i}^4}{f_1f_2f_3(f_i^2)^2}\right) + \frac{1}{3}|\tau|^2 + \frac{u\overline{f_i}^2}{f_i^2} = \left(-\frac{4\lambda}{3} - \frac{2u\overline{f_i}^4}{f_1f_2f_3f_i^2}\right) + \frac{1}{3}|\tau|^2 + \frac{u\overline{f_i}^2}{f_i^2},$$

while using (4.12a) and summing over $i$ yields

$$\frac{u\overline{(f_i^2)'}}{f_i^2} = \frac{u\overline{f_i}^2}{f_1f_2f_3} - \frac{2uf_i^4}{f_1f_2f_3f_i^2}.$$
Therefore the right-hand side of (4.16) is equal to
\[
\left(\frac{u\tau_i}{f_i^2} - \frac{2uf_i^2}{f_1f_2f_3} + \frac{u\tau_i^2}{f_1f_2f_3}\right) + \left(-\frac{4\lambda}{3} + \frac{1}{3}|\tau|^2 - \frac{u\tau_i^2}{f_1f_2f_3}\right) = \frac{u\tau_i}{f_i^2} - \frac{2uf_i^2}{f_1f_2f_3} - \frac{4\lambda}{3} + \frac{1}{3}|\tau|^2.
\]
Hence the difference between the two sides of (4.16) is
\[
\frac{\tau_i'}{f_i^2} - \left(\frac{u(f_i^2)'}{f_i^2} + (u' + \lambda)\right) = \frac{4\lambda(E_i + \tau_i^2)}{3f_i^2} - \frac{2f_i^2\tau_i}{f_1f_2f_3f_i^2} + \frac{2uf_i^2}{f_1f_2f_3} = 2\left(\frac{2\lambda f_1f_2f_3 + u\tau_i^2 - \tau_i}{f_1f_2f_3f_i^2}\right)f_i^2 = 0
\]
as required (the final equality follows because of the conservation law (CL) used to define \(u\).

**Corollary 4.17.** For any \(\lambda \neq 0\) there is a 4-parameter family of distinct local real analytic \(\text{SU}(3)\)-invariant Laplacian solitons; there is a 3-parameter family of distinct local real analytic \(\text{SU}(3)\)-invariant steady Laplacian solitons.

**Proof.** On any principal orbit the system (4.12) is a nonsingular real analytic first-order system of ODEs. Hence standard ODE theory gives the existence of unique (local in \(t\)) real analytic solutions depending real analytically on the initial data, i.e., on the choice of principal orbit or equivalently of a point in the phase space \(P\) described in Remark 4.14. Since \(P\) is 5-dimensional and an \(\text{SU}(3)\)-invariant Laplacian soliton corresponds to an integral curve of (4.12) that remains in \(P\) there is a 4-parameter family of local real analytic \(\text{SU}(3)\)-invariant Laplacian solitons for any \(\lambda \in \mathbb{R}\). However, since rescalings take steady solitons to other steady solitons, geometrically this implies the existence of a 3-parameter family of distinct local \(\text{SU}(3)\)-invariant steady Laplacian solitons, but a 4-parameter family of distinct local \(\text{SU}(3)\)-invariant Laplacian solitons for any \(\lambda \neq 0\).

**4.3.1. Specialisation to the \(\text{Sp}(2)\)-invariant case.** For future purposes we record the form of the first-order system that arises from specialising the system (4.12) to the \(\text{Sp}(2)\)-invariant case, i.e., by setting \(f_2 = f_3\) and \(\tau_2 = \tau_3\).

**Proposition 4.18.** If \((f_1, f_2, \tau_1, \tau_2, u)\) is any solution to the \(\text{Sp}(2)\)-invariant soliton system (4.7) then \((f_1, f_2, \tau_2)\) satisfies the first-order ODE system
\[
\begin{align*}
(f_1^2)' &= 2f_1 - \frac{f_1^2}{f_2^2}(f_1 + 2\tau_2), \quad (f_2^2)' = f_1 + \tau_2, \quad \tau_2' = \frac{4R_1S}{3f_1(f_1^2 + 2f_2^2)},
\end{align*}
\]
where
\[
R_1 := \lambda f_1f_2^2 - 3\tau_2, \quad S := f_2^2 - f_1^2 - \frac{3}{2}f_1\tau_2.
\]
Conversely, if \((f_1, f_2, \tau_2)\) is a solution to the first-order system (4.19) with \(f_1, f_2 > 0\) and if we define \(\tau_1\) and \(u\) in terms of \((f_1, f_2, \tau_2)\) by
\[
\begin{align*}
\tau_1 &= -\frac{2f_1^2}{f_2^2}\tau_2, \quad u := \frac{2(f_2^2 - f_1^2)\tau_2 - 2\lambda f_1f_2^4}{f_2^2(f_1^2 + 2f_2^2)},
\end{align*}
\]
then \((f_1, f_2, \tau_1, \tau_2, u)\) solves the \(\text{Sp}(2)\)-invariant soliton system (4.7).

**Proof.** The proof is a routine calculation obtained by specialising the formulae from the \(\text{SU}(3)\)-invariant case to the \(\text{Sp}(2)\)-invariant case where \(f_2 = f_3\) and \(\tau_2 = \tau_3\).

**Corollary 4.21.** For any \(\lambda \neq 0\) there is a 2-parameter family of distinct local real analytic \(\text{Sp}(2)\)-invariant Laplacian solitons; there is a 1-parameter family of distinct local real analytic \(\text{Sp}(2)\)-invariant steady Laplacian solitons.
4.4. **A scale-normalised version of the first-order soliton system.** To analyse the behaviour of asymptotically conical invariant Laplacian solitons it will prove useful to consider the equations satisfied by the following set of variables:

\[(4.22)\]

\[g^3 := f_1 f_2 f_3, \quad F_i := \frac{f_i}{g}, \quad \tau_i := \frac{\tau_i}{g} \quad i = 1, 2, 3.\]

Note that \(g\) the geometric mean of the positive triple \(f = (f_1, f_2, f_3)\) has scaling weight one under dilations of \(f\) whereas all the variables \(F_i\) and \(\tau_i\) are scale invariant. Notice also that by definition the \(F_i\) satisfy \(F_1 F_2 F_3 = 1\). It also convenient to introduce scale-invariant versions of the quantities \(S_i\) introduced in (4.13). More specifically we define

\[(4.23)\]

\[S_i := \frac{S_i}{g^2} = \frac{(2F_i^2 - F_j^2 - F_k^2) - 3\tau_i}{2F_i^2} = E_i - \frac{3\tau_i}{2F_i^2},\]

where we define the *excesses* \(E_i\) and their scale-invariant analogues \(E_i\) by

\[(4.24a)\]

\[E_i := 3f_i^2 - F_i^2,\]

\[(4.24b)\]

\[E_i := \frac{E_i}{g^2} = 3F_i^2 - \bar{F}^2.\]

The excess \(E_i\) gives a scale-invariant measure of the deviation (or excess) of \(f_i^2\) from the average \(\frac{1}{3} \sum_j f_j^2\). We note some elementary properties of various sums related to these excesses:

\[\sum_i E_i = 0, \quad \sum_i E_i^2 = 3D(f), \quad \sum_i E_i F_i^2 = D(f).\]

Obvious analogues also hold for sums of their scale-invariant versions. The scale-invariant version of the final equality also yields

\[(4.25)\]

\[\sum_i F_i^2 S_i = D(F) - \frac{3\tau}{2}.\]

Since

\[(4.26)\]

\[3F_i^2 = E_i + \bar{F}^2 > 0\]

and \(F_1 F_2 F_3 = 1\) we also have

\[(4.27)\]

\[\prod_{i=1}^3 (E_i + \bar{F}^2) = 27.\]

Rearranging (4.23) gives

\[\tau_i = \frac{2}{3} F_i^2 (3 \frac{E_i^2}{F_i^2} - S_i) = \frac{2}{3} F_i^2 (E_i - S_i).\]

Hence we can also express \(\tau_i^2\) in terms of \(g\) and scale-invariant variables as

\[(4.28)\]

\[\tau_i^2 = \sum_i \tau_i^2 \frac{F_i^2}{f_i^2} = \frac{1}{g^2} \sum_i \left(\frac{\tau_i}{F_i^2}\right)^2 = \frac{4}{9g^2} \sum (E_i - S_i)^2 = \frac{4}{9g^2} \left(3D(F) + \sum (S_i^2 - 2E_i S_i)\right).\]

Similarly using the conservation law (CL) yields the following expression for \(u\) in terms of \(g\) and scale-invariant variables

\[(4.29)\]

\[u \bar{F}^2 = -2 \lambda g + \frac{\tau}{g}.\]

A direct calculation using the previous first-order system (4.12) shows that if \((f, \tau)\) is a solution of the first-order version of the invariant Laplacian soliton system then the ODE system satisfied by
the variables \((g, \mathcal{T}_i, \mathcal{J}_i)\) defined in (4.22) is

\[ g' = \frac{1}{6} \mathcal{J}_i, \]

\[ g_{\mathcal{T}_i} = \frac{1}{2 \mathcal{T}_i} \left( \mathcal{J}_i - 2 \mathcal{T}_i^4 + \frac{2}{3} \mathcal{J}_i^2 \mathcal{J}_i^2 \right) = -\frac{1}{3} \mathcal{T}_i \mathcal{S}_i, \]

\[ g_{\mathcal{J}_i} = \frac{4 \lambda g^2}{3} \sum_{i} \mathcal{T}_i \mathcal{S}_i + \frac{\mathcal{T}}{2 \mathcal{T}_i} (\mathcal{J}_i - 2 \mathcal{T}_i^4) + \frac{1}{3} \left( \sum \frac{\mathcal{T}_i^2}{\mathcal{J}_i^2} \right) \mathcal{J}_i^2 - \frac{1}{6} \mathcal{T}_i \mathcal{T}_i^2. \]

In particular (4.30b) implies that \(\mathcal{T}_i\) is monotone in \(t\) if and only if \(\mathcal{S}_i\) has a definite sign (since \(\sum \mathcal{S}_i = 0\) the \(\mathcal{S}_i\) cannot all have the same sign).

**Remark 4.31.** Using (4.23) and (4.25) the right-hand side of (4.30c) can be rewritten as

\[ \frac{4 \lambda g^2}{9 g^2} \left( 3 \lambda g^2 - D(\mathcal{F}) + \left( \sum \mathcal{T}_i^2 \mathcal{S}_i \right) \right) + \frac{4}{9} \mathcal{J}_i^2 \sum \mathcal{T}_i^2 \mathcal{S}_i + \frac{4}{9} \left( \sum \left( \mathcal{S}_i^2 - 2 \mathcal{E}_i \mathcal{S}_i \right) \right) - \frac{1}{6} \mathcal{T}_i \mathcal{T}_i^2. \]

Note that when all \(\mathcal{S}_i\) vanish the only (potentially) nonvanishing term in the expression above is \(-\frac{1}{6} \mathcal{T}_i \mathcal{T}_i^2\).

**Remark 4.32.** The evolution of other scale-invariant variables can be computed from (4.30). For instance we have

\[ g(\mathcal{J}_i)' = -\frac{2}{3} \sum \mathcal{T}_i^2 \mathcal{S}_i = \mathcal{F} - \frac{2}{3} D(\mathcal{F}), \]

\[ g(\mathcal{F})' = \frac{2 \lambda g^2}{3 \mathcal{F}^2} \left( 2D(\mathcal{F}) - 3 \mathcal{F} \right) + \frac{\mathcal{F}}{2 \mathcal{F}_i} \left( \mathcal{F} - 2 \mathcal{F}_i - \frac{1}{6} (\mathcal{F}_i^2)^2 \right) + \frac{1}{3} \left( \frac{\sum \mathcal{T}_i^2}{\mathcal{F}_i^2} \right) \mathcal{F}_i^2, \]

\[ g(\mathcal{E}_i)' = -2 \mathcal{T}_i^2 \mathcal{S}_i - \mathcal{F} + \frac{2}{3} D(\mathcal{F}) = (3 \mathcal{T}_i - \mathcal{F}) - \frac{2}{3} \left( 3 \mathcal{T}_i^2 \mathcal{E}_i - (\sum \mathcal{E}_j \mathcal{T}_j^2) \right). \]

**Characterisations of the Gaussian solitons.** Using the scale-normalised version of the Laplacian soliton ODE system (4.30) we now prove two simple results that provide characterisations of the Gaussian solitons over the unique torsion-free SU(3)-invariant \(G_2\)-structure on the cone over SU(3)/\(\mathbb{T}^2\). The first result says that these Gaussian solitons are the only conical SU(3)-invariant Laplacian solitons; the second says that they are the only nontrivial invariant solitons for which the underlying \(G_2\)-structure is torsion-free.

**Lemma 4.36.** For any value of \(\lambda \in \mathbb{R}\) let \((\varphi, X, \lambda)\) be an SU(3)-invariant closed Laplacian soliton for which the induced metric \(g_f\) is conical, then \(\varphi\) coincides with a portion of the Gaussian soliton over the unique torsion-free SU(3)-invariant \(G_2\)-structure on the cone over SU(3)/\(\mathbb{T}^2\), i.e.

\[ f_1 = f_2 = f_3 = \frac{1}{2} t \quad \text{and} \quad X = -\frac{1}{3} \lambda t \partial_t. \]

**Proof.** Since \(g_f\) is conical the scale-invariant variables \(\mathcal{T}_i\) must all be constant and therefore \(\mathcal{S}_i = 0\) for all \(i\) by (4.30b). Similarly the scale-invariant variables \(\mathcal{J}_i\) must also all be constant and therefore \(\mathcal{J}_i = 0\) for all \(i\). Since \(\mathcal{S}_i = 0\) for all \(i\) then by Remark 4.31 the vanishing of the right-hand side of (4.30c) implies that \(\mathcal{T}_i = 0\) for all \(i\). Therefore \(\mathcal{T}\) vanishes and hence by (4.25) so does \(D(\mathcal{F})\) and this implies that \(\mathcal{F}_i = 1\) for all \(i\). Therefore by (4.30a) \(g' = \frac{1}{2}\) and hence (up to a time-translation) \(f_i = \frac{1}{2} t\) for all \(i\).

Another easy consequence of the equations derived in Remark 4.32 is the following

**Lemma 4.37.** An SU(3)-invariant Laplacian soliton \((\varphi, X = u \partial_t, \lambda)\) whose underlying closed \(G_2\)-structure \(\varphi\) is torsion free is either a trivial soliton, i.e. \(\lambda = 0\), \(u \equiv 0\), or a Gaussian soliton over the torsion-free cone over \(\mathbb{F}_{1,2}\) for any \(\lambda \in \mathbb{R}\).
Proof. If \( \lambda = 0 \) and the torsion 2-form \( \tau \) vanishes, then (4.29) implies that \( u \equiv 0 \). Now suppose that \( \lambda \neq 0 \) and the torsion 2-form \( \tau \) vanishes. Then the only potentially nonvanishing term on the right-hand side of (4.34) is

\[
\frac{4\lambda g^2 D(f)}{3f^2}.
\]

Since we must have \( f' \equiv 0 \) this implies that \( D(f) \equiv 0 \) and hence that \( f_i \equiv 1 \) for all \( i \). \( \square \)

5. Cohomogeneity-one \( G_2 \)-solitons extending smoothly over a singular orbit

The main result of this section is the existence for any \( \lambda \in \mathbb{R} \) of a 2-parameter family of local \( SU(3) \)-invariant \( G_2 \)-solitons that extend smoothly over some neighbourhood of the singular orbit \( CP^2 \subset \Lambda^2 CP^2 \); the same method also yields the existence of a 1-parameter family of \( Sp(2) \)-invariant \( G_2 \)-solitons that extend smoothly over the singular orbit \( S^4 \subset \Lambda^2 S^4 \).

These existence proofs are relatively standard given the first-order reformulation of the \( SU(3) \)-invariant \( G_2 \)-soliton equations derived in the previous section: the conditions that a smooth invariant soliton defined on the principal set \( M^0 \) extends smoothly over the singular orbit lead to a singular initial value problem for the first-order real analytic ODE system (4.12). The singularities turn out to be of regular type: for suitable initial data we prove the existence of a unique formal power series solution to this system; then a general result guarantees the existence of a real analytic solution whose Taylor series is equal to the formal series solution (which is therefore convergent).

The formal power series solutions can be in principle be computed symbolically (in terms of \( \lambda \) and two additional real parameters) using a computer algebra package: a number of terms for these power series solutions are detailed in Appendix A. Explicit knowledge of some of the lowest-order terms proves to be important in several later arguments and some general features of the series also turn out to be suggestive when trying to understand limiting behaviour of solutions in certain regimes. In this section consideration of these power series solutions lead to Theorem 5.20 which details explicit complete (asymptotically conical) shrinkers on \( \Lambda^2 S^4 \) and \( \Lambda^2 CP^2 \).

5.1. The initial conditions for closing smoothly over a \( CP^2 \) singular orbit. Table 2 in Cleyton–Swann [23] tells us that for \( G/K = SU(3)/T^2 \), complete closed \( SU(3) \)-invariant \( G_2 \)-structures exist only when the singular orbit is \( G/H = SU(3)/U(2) \cong CP^2 \). Without loss of generality (recall we can always arrange this by acting with the Weyl group \( W = S_3 \)) we will take the isotropy group of the singular orbit to be the subgroup \( U(2)_1 \) defined as the centraliser of the matrix \( I_1 \) given in (3.24), i.e. as we approach the singular orbit the size \( f_1^2 \) of the fibre \( S^2_1 \) goes to zero.

The conditions for a smooth \( SU(3) \)-invariant \( G_2 \)-structure defined on on \( (0, \epsilon) \times CP^3 \) to extend smoothly across a singular orbit \( CP^2 = SU(3)/U(2) \) at \( t = 0 \) were already determined in [23, §9]. Here we recall their result without proof, referring the reader to [23] or Cleyton’s thesis for further details as needed. Alternatively some readers may prefer the approach taken by Chi to prove [21, Prop. 2.8]: Chi applies the representation-theoretic approach of Eschenburg–Wang [28] to study the smooth extension problem for an \( SU(3) \)-invariant Riemannian metric to extend smoothly over the zero section of \( \Lambda^2 CP^2 \). (Clearly if the \( SU(3) \)-invariant \( G_2 \)-structure \( \varphi \) extends smoothly over \( CP^2 \) then the invariant Riemannian metric \( g_{\varphi} \) extends smoothly over \( CP^2 \), so that Chi’s stated conditions are certainly necessary conditions for smooth extension. See [23, §10] for a converse.)

For a closed \( SU(3) \)-invariant \( G_2 \)-structure expressed in the form

\[
\varphi_f = \omega_f \wedge dt + \text{Re} \Omega_f = (f_1^2 \omega_1 + f_2^2 \omega_2 + f_3^2 \omega_3) \wedge dt + f_1f_2f_3 \alpha
\]

the conditions that \( \varphi_f \) extend smoothly over the singular orbit \( CP^2_1 = SU(3)/U(2)_1 \) at \( t = 0 \) are:

- \( f_1 \) is odd in \( t \) and \( |f_1(0)| = 1 \),
- \( f_2(t) = f_3(-t) \) and \( f_2(0) \neq 0 \).
In particular these conditions force that the product \( f_2 f_3 \) is even in \( t \). Recall also that by acting with Klein four-group we are able to assume that \((f_1, f_2, f_3)\) is a positive triple for all \( t > 0 \). In particular we must have that \( f_1(0) = +1 \) and \( f_2(0) = f_3(0) > 0 \). With these sign assumptions understood then the smooth extension conditions above then imply that \( f_2 + f_3 \) is even and \( f_2 - f_3 \) is odd. In particular, we can write the coefficients \( f_i \) as
\begin{equation}
(5.1) \quad f_1 = t + t^3 \hat{f}_1, \quad f_2 = b + t \hat{f}_2, \quad f_3 = b + t \hat{f}_3,
\end{equation}
for some \( b > 0 \) and new functions \( \hat{f}_i \) with the following symmetries:
- \( \hat{f}_1, \hat{f}_2 \hat{f}_3 \) and \( \hat{f}_2 - \hat{f}_3 \) are even in \( t \);
- \( \hat{f}_2 + \hat{f}_3 \) is odd in \( t \).
In particular, \( \hat{f}_2(0) + \hat{f}_3(0) = 0 \), so if \( \hat{f}_2(0) \neq 0 \) then the solutions will not satisfy \( f_2 = f_3 \).

### 5.2. SU(3)-invariant G\(_2\)-solitons extending smoothly over \( \mathbb{CP}^2 \).

The main result for smoothly-closing SU(3)-invariant solitons is the following:

**Theorem 5.2.** Fix any \( \lambda \in \mathbb{R} \) and \( b > 0 \) and \( c \in \mathbb{R} \). Then there exists a unique local SU(3)-invariant \( G_2 \)-soliton which closes smoothly on the singular orbit \( \mathbb{CP}^2 \subset \Lambda^2 \mathbb{CP}^2 \) and satisfies
\begin{equation}
(5.3) \quad f_1 = t + t^3 \hat{f}_1, \quad f_2 = b + t \hat{f}_2, \quad f_3 = b + t \hat{f}_3, \quad \tau_1 = t^3 \hat{\tau}_1, \quad \tau_2 = c + t \hat{\tau}_2, \quad \tau_3 = -c + t \hat{\tau}_3,
\end{equation}
where the leading-order terms of the \( \hat{f}_i \) and \( \hat{\tau}_i \) satisfy
- \( \hat{f}_1(0) = -\frac{1}{6b^2} + \frac{2 \lambda}{27} \frac{c^2}{18b^4} \), \( \hat{f}_2(0) = -\hat{f}_3(0) = \frac{c}{6b} \),
- \( \hat{\tau}_1(0) = \frac{2(2 \lambda b^4 - c^2)}{9b^4} \), \( \hat{\tau}_2(0) = \hat{\tau}_3(0) = \frac{2}{9} \left( \frac{\lambda b^2 + c^2}{b^2} \right) \),

respectively. The solution is real analytic on \([0, \epsilon) \times \text{SU}(3)/\mathbb{T}^2\) for some \( \epsilon > 0 \).

Moreover, up to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-action, any SU(3)-invariant \( G_2 \)-soliton which closes smoothly on the singular orbit \( \mathbb{CP}^2 \subset \Lambda^2 \mathbb{CP}^2 \) belongs to this 2-parameter family of solutions.

We have performed a computer-assisted symbolic computation of the terms of the power series solutions of the 2-parameter family of smoothly-closing solitons constructed in Theorem 5.2 in terms of the real parameters \( b, c \) and \( \lambda \). The first several terms of these solutions are listed in Appendix A.

**Remark 5.4.** The mean curvature of the singular orbit is zero, because by the smooth closure conditions \( \lim_{t \to 0} \frac{\hat{f}_2}{\hat{f}_2} + \frac{\hat{f}_3}{\hat{f}_3} = 0 \). However, when \( c \neq 0 \) the singular orbit is not totally geodesic.

The main technical tool for proving the previous theorem is the following general result about regular first-order singular initial value problems. (Here regular refers to the fact that the singular term is of order \( t^{-1} \) rather than \( t^{-k} \) for some \( k > 1 \).)

**Theorem 5.5.** Consider the singular initial value problem
\begin{equation}
(5.6) \quad y = \frac{1}{t} M_{-1}(y) + M(t, y), \quad y(0) = y_0,
\end{equation}
where \( y \) takes values in \( \mathbb{R}^k \), \( M_{-1}: \mathbb{R}^k \to \mathbb{R}^k \) is a (real) analytic function of \( y \) in a neighbourhood of \( y_0 \) and \( M: \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k \) is analytic in \( t, y \) in a neighbourhood of \((0, y_0)\). Assume that
- \( M_{-1}(y_0) = 0 \);
- \( h \text{Id} - d_{y_0} M_{-1} \) is invertible for all \( h \in \mathbb{N}, h \geq 1 \).

Then there exists a unique solution \( y(t) \) of (5.6) which moreover is analytic on \([0, \epsilon)\) for some \( \epsilon > 0 \). Furthermore \( y \) depends continuously on \( y_0 \) satisfying (i) and (ii).
The condition (ii) guarantees the existence of a unique formal power series solution \( y(t) \) to (5.6). The fact that any formal power series solution to (5.6) converges is due to Malgrange [49]. As for the continuous dependence on the initial conditions: the coefficients of the formal power series solution \( y(t) \) depend differentiably on \( y_0 \) satisfying (i) and (ii) and the operator used in the fixed point argument is uniformly contracting with respect to the initial conditions.

Proof. The structure of the proof is to rewrite the (autonomous) first-order ODE system (4.12) for the pair of triples \((f, \tau)\) as an \(t\)-dependent ODE system for the pair of triples \((\hat{f}, \hat{\tau})\) and to show that the hypotheses of Theorem 5.5 apply to the latter ODE system. It is therefore particularly important to identify the leading-order term \(M_{-1}\) in order to be able to understand condition (i) and verify that the hypothesis required in (ii) is satisfied.

Warmup: the torsion-free case. For SU\((3)\)-invariant torsion-free \(G_2\)-structures we already know from Cleyton–Swann that the only solutions (described in Example 3.43) that extend smoothly across the \(\mathbb{CP}^2\) singular orbit have \(\hat{f}_2(0) = \hat{f}_3(0) = 0\) and so must have \(f_2 \equiv f_3\). Cleyton–Swann proved this by finding an explicit parametrisation of any local SU\((3)\)-invariant torsion-free \(G_2\)-structure [23, p. 217] and observing that the conditions on the singular orbit force \(f_2^2 \equiv f_3^2\). As a warmup for the general SU\((3)\)-invariant soliton case we want to obtain this result in the torsion-free by an application of Theorem 5.5, since that method will apply even when we are not able to find explicit parametrisations of all the local SU\((3)\)-invariant \(G_2\)-solitons. Moreover, all the calculations presented here are also needed in the more general soliton analysis.

If we assume that all the torsion coefficients \(\tau_i\) vanish identically then the first-order ODE system (4.12) reduces to the following first-order ODE system for the triple \(f\)

\[
(f_i^2)' = \frac{f_j^2 + f_k^2 - f_i^2}{f_if_jf_k}, \quad (ijk) \text{ a cyclic permutation of } (123).
\]

Now we rewrite (5.7) as a (singular) ODE system for \(\hat{f}_1, \hat{f}_2, \hat{f}_3\) as defined above in (5.1). It will also be useful to record the ODE system satisfied by the sums and differences of pairs of the components \(f_i\). A straightforward calculation shows that

\[
(f_j + f_k)' = \left(\frac{f_j^2 - (f_j - f_k)^2}{2f_1f_2f_3}\right)(f_j + f_k), \quad (f_j - f_k)' = \left(\frac{f_j^2 - (f_j + f_k)^2}{2f_1f_2f_3}\right)(f_j - f_k),
\]

where as above \((ijk)\) is any cyclic permutation of \((123)\). In particular, if on some principal orbit we have \(f_j = f_k\) then we will have \(f_j \equiv f_k\) up to the maximal existence time of the solution.

For later purposes we record some intermediate calculations used to calculate the recast ODE system:

\[
\begin{align*}
2f_2f_3 &= 2 \left(b^2 + t(\hat{f}_2 + \hat{f}_3) + t^2\hat{f}_2\hat{f}_3\right), \\
2f_1f_3 &= 2t(b + tf_3)(1 + t^2\hat{f}_1), \\
2f_1f_2 &= 2t(b + tf_2)(1 + t^2\hat{f}_1).
\end{align*}
\]
For \( i = 1 \) rewriting (5.7) yields
\[
t^3 \dot{f}_1' + 1 + 3t^2 \ddot{f}_1 = \frac{2b^2 + 2bt(\dot{f}_2 + \dot{f}_3) - t^2(1 - f_2^2 - f_3^2 + 2t^2 \ddot{f}_1 + t^4 \dddot{f}_1)}{2 (b^2 + t(\dot{f}_2 + \dot{f}_3) + t^2 \ddot{f}_2 \ddot{f}_3)}
\]
\[
= 1 - t^2 \left( \frac{1 - (\dot{f}_2 - \dot{f}_3)^2 + 2t^2 \ddot{f}_1 + t^4 \dddot{f}_1}{2 (b^2 + t(\dot{f}_2 + \dot{f}_3) + t^2 \ddot{f}_2 \ddot{f}_3)} \right).
\]
Hence we have
\[
(5.8a) \quad t\dddot{f}_1' = -3\dddot{f}_1 + \frac{1}{2b^2}(\dot{f}_2 - \dot{f}_3)^2 - \frac{1}{2b^2} + tM_1(t, \hat{f}).
\]
Similarly for \( i = 2 \) and \( i = 3 \) (5.7) is equivalent to
\[
(5.8b) \quad t\dddot{f}_2' = -\dddot{f}_2 + \frac{2b(\dot{f}_3 - \dot{f}_2) + t(1 + \dot{f}_2^2 - \dot{f}_3^2 + 2t^2 \ddot{f}_1 + t^4 \dddot{f}_1)}{2(b + t\ddot{f}_3)(1 + t^2 \ddot{f}_1)} = -2\dddot{f}_2 + \dddot{f}_3 + tM_2(t, \hat{f}),
\]
and
\[
(5.8c) \quad t\dddot{f}_3' = \dddot{f}_2 - 2\dddot{f}_3 + tM_3(t, \hat{f}),
\]
respectively. Note also that the sum and difference of \( \dddot{f}_2 \) and \( \dddot{f}_3 \) satisfy the ODEs
\[
(5.8d) \quad t(\dddot{f}_2 + \dddot{f}_3)' = -(\dddot{f}_2 + \dddot{f}_3) + O(t), \quad t(\dddot{f}_2 - \dddot{f}_3) = -3(\dddot{f}_2 - \dddot{f}_3) + O(t).
\]
The ODE system (5.8) is now in the form (5.6) and so Theorem (5.5) tells us that to have a real analytic solution around \( t = 0 \) first we must impose that the singular term \( M_{-1} \) given by
\[
M_{-1}(\dddot{f}_1, \dddot{f}_2, \dddot{f}_3) = \left(-3\dddot{f}_1 + \frac{1}{2b^2}(\dot{f}_2 - \dot{f}_3)^2 - \frac{1}{2b^2}, 2\dddot{f}_2 + \dddot{f}_3, \dddot{f}_2 - 2\dddot{f}_3 \right)
\]
vanishes initially. In our case this means that the initial conditions for \( \dddot{f}_1, \dddot{f}_2, \) and \( \dddot{f}_3 \) at \( t = 0 \) must satisfy
\[
-3\dddot{f}_1 + \frac{(\dddot{f}_2 - \dddot{f}_3)^2 - 1}{2b^2} = 0, \quad 2\dddot{f}_2 + \dddot{f}_3 = 0, \quad \dddot{f}_2 - 2\dddot{f}_3 = 0.
\]
These equations force the initial conditions to satisfy
\[
(5.10) \quad \dddot{f}_1(0) = -\frac{1}{6b^2}, \quad \dddot{f}_2(0) = \dddot{f}_3(0) = 0.
\]
To guarantee a unique formal series solution for all \( b \neq 0 \) it remains to verify that \( n\text{Id} - dM_{-1} \) evaluated at the initial conditions (5.10) is invertible for all positive integers \( n \). Evaluating the differential of \( M_{-1} \) at the initial conditions (5.10) yields
\[
n\text{Id} - dM_{-1} = \begin{pmatrix} n + 3 & 0 & 0 \\
0 & n + 2 & -1 \\
0 & -1 & n + 2 \end{pmatrix},
\]
which having determinant \( (n + 3)^2(n + 1) \) is indeed invertible for all positive integers \( n \). Hence Theorem 5.5 applies. However we already know that there are solutions of the ODE system with any such initial values that have \( f_2 \equiv f_3 \), namely the classical AC Bryant–Salamon solutions on \( \Lambda^2 \mathbb{CP}^2 \) from Example 3.43. So by local uniqueness these solutions must coincide.

The general case. First we must determine what constraints the initial values of an invariant soliton must satisfy to extend smoothly over a singular orbit \( \mathbb{CP}^2 \) at \( t = 0 \). The conditions that the 3-form \( \varphi \) must satisfy have already been described. The coefficient \( u \) determining the vector field \( X = u \partial_t \) must be odd in \( t \), and in particular \( u(0) = 0 \). We now use these constraints to understand the behaviour of the components of the torsion \( \tau \) close to the singular orbit.
Since \( u \) is odd, using the initial conditions for \( f_i \) given in (5.1) and the conservation law (4.6c) we find that \( \tau \) must be odd and in particular \( \tau(0) = 0 \). Since \( f_1 \) is odd and \( f_2 f_3 \) is even, (3.38a) implies that \( \tau_1 \) is odd and therefore also \( \tau_2 + \tau_3 \) is odd. In particular \( \tau_1 = 0 \) and \( \tau_2 + \tau_3 = 0 \) at \( t = 0 \).

To proceed further we choose to single out the 5 variables \((f_1, f_2, f_3, \tau_2, \tau_3)\) and to recover \( \tau_1 \) by defining

\[
\tau_1 := -f_1^2 \left( \frac{\tau_2}{f_2} + \frac{\tau_3}{f_3} \right). 
\]

Then we write

\[
\tau_2 = c + t\hat{\tau}_2, \quad \tau_3 = -c + t\hat{\tau}_3,
\]

where \( \hat{\tau}_2 + \hat{\tau}_3 \) is even (since \( \tau_2 + \tau_3 \) was odd).

The key point is to determine the potentially singular terms \( M_{-1} \) in the first-order ODE system rewritten in terms of the variables \((f_1, \hat{f}_2, \hat{f}_3, \hat{\tau}_2, \hat{\tau}_3)\). For \( \hat{f}_1 \) we can do this by understanding what corrections to the singular terms \( M_{-1} \) that we calculated in the torsion-free case in (5.9) will appear due to the presence of the torsion terms \( \hat{f}_1 f_i \).

For \( i = 2, 3 \) this is straightforward: the terms in \( M_{-1} \) change only by the addition/subtraction respectively of the term \( \frac{c}{2b} \). In other words we have

\[
t f_2' = -2\hat{f}_2 + \hat{f}_3 + \frac{c}{2b} + O(t), \quad t f_3' = \hat{f}_2 - 2\hat{f}_3 - \frac{c}{2b} + O(t).
\]

The ODEs satisfied by the sum \( \hat{f}_2 + \hat{f}_3 \) and the difference \( \hat{f}_2 - \hat{f}_3 \) are

\[
t(\hat{f}_2 + \hat{f}_3)' = -(\hat{f}_2 + \hat{f}_3) + O(t),
\]

\[
t(\hat{f}_2 - \hat{f}_3)' = -3(\hat{f}_2 - \hat{f}_3) + \frac{c}{b} + O(t).
\]

The vanishing of these two components of \( M_{-1} \) on the initial data therefore forces

\[
(5.12) \quad \hat{f}_2(0) = -\hat{f}_3(0) = \frac{c}{6b}.
\]

The computation for \( i = 1 \) requires a little more effort and we return to this at the end.

Instead next we consider the equations for \( \tau_2 \) and \( \tau_3 \), i.e. (4.12b) rewritten in terms of \( \hat{\tau}_2 \) and \( \hat{\tau}_3 \). For \( i = 2, 3 \) the coefficients \( \overline{f_i^2} f_j f_k f_i^{-1} \) appearing on the left-hand side of (4.12b) are both equal to

\[
2tb^2 + O(t^2).
\]

Hence to determine the terms that will contribute to \( M_{-1} \) we need only look at terms on the right-hand side of (4.12b) that give rise to terms linear in \( t \). Two of the five terms, namely \(-2\lambda f_j f_k^2 \tau_i \) and \( \tau f_j f_k f_i^{-1} \), contain only terms quadratic in \( t \) and higher: for the first term this is immediate from the fact that it contains \( f_i^2 \) and for the second recall that \( \tau \) is odd in \( t \). So we have to consider the remaining three terms: one not involving the coefficients of \( \tau \), one linear and one quadratic in those coefficients. The first term \( \frac{4}{3} \lambda f_1 f_2 f_3 (2f_i^2 - f_j^2 - f_k^2) \) is easily seen to contribute

\[
\frac{4\lambda b^4}{3} t.
\]

To see the \( t \) contribution made by the second term first note that \( \tau = t(\hat{\tau}_2 + \hat{\tau}_3) + O(t^2) \). This is because \( \tau \) and \( \tau_1 \) are both odd and (5.11) combined with the leading-order behaviour of \( f_2, f_3, \hat{\tau}_2 \) and \( \hat{\tau}_3 \) force that the leading-order behaviour of \( \tau_1 \) is \( t^3 \). (Below we will need to determine this term). It is therefore clear that the term linear in the coefficients of \( \tau \), namely \(-2\tau f_i^2 \), contributes

\[
-2tb^2(\hat{\tau}_2 + \hat{\tau}_3).
\]
It remains to compute the $t$ contribution made by the term $\frac{1}{2} f_1^2 f_2 f_3 |\tau|^2$. Using (5.11) again, we can replace the term $\tau^2 f_1^{-4}$ and obtain

$$|\tau|^2 = \left( \sum \tau_i^2 f_i^{-4} \right) = 2(\tau_2^2 f_2^{-4} + \tau_3^2 f_3^{-4} + \tau_2 \tau_3 f_2^{-2} f_3^{-2}).$$

Using the leading-order behaviour of those coefficients we find that $\left( \sum \tau_i^2 f_i^{-4} \right) = 2c^2 b^{-4} + O(t)$. Therefore the $t$ contribution of this term is

$$\frac{4}{3} t c^2.$$

Combining all this we find

$$t \dot{\tau}_2 = -2 \dot{\tau}_2 - \dot{\tau}_3 + \frac{2}{3} \lambda b^2 + \frac{2 c^2}{3 b^2} + O(t),$$

$$t \dot{\tau}_3 = -2 \dot{\tau}_2 - 2 \dot{\tau}_3 + \frac{2}{3} \lambda b^2 + \frac{2 c^2}{3 b^2} + O(t).$$

We also note that the ODEs satisfied by the sum $\dot{\tau}_2 + \dot{\tau}_3$ and the difference $\dot{\tau}_2 - \dot{\tau}_3$ are

$$t(\dot{\tau}_2 + \dot{\tau}_3)' = -3(\dot{\tau}_2 + \dot{\tau}_3) + \frac{4}{3} \left( \lambda b^2 + \frac{c^2}{b^2} \right) + O(t),$$

$$t(\dot{\tau}_2 - \dot{\tau}_3)' = -(\dot{\tau}_2 - \dot{\tau}_3) + O(t).$$

Requiring these two components of $M_{-1}$ to vanish on the initial data therefore forces

$$\dot{\tau}_2(0) = \dot{\tau}_3(0) = \frac{2}{9} \left( \lambda b^2 + \frac{c^2}{b^2} \right).$$

Finally we return to the change in the contribution to (5.8a), the equation for $f_1$ in the torsion-free case, due to the presence of the additional torsion term $\tau_2 f_2^{-2} + \tau_3 f_3^{-2}$. To evaluate this we need to find the term linear in $t$ in the expansion of $\tau_2 f_2^{-2} + \tau_3 f_3^{-2}$. A short calculation shows

$$\tau_1 \frac{\tau_1}{f_1^2} := \left( \frac{\tau_2}{f_2^2} + \frac{\tau_3}{f_3^2} \right) = -\left( \frac{2c}{b^2} (f_2 - \hat{f}_2) + \frac{1}{b^2} (\dot{\tau}_2 + \dot{\tau}_3) \right) t + O(t^2).$$

Substituting for the initial values given in (5.12) and (5.13) we find

$$\tau_1 \frac{\tau_1}{f_1^2} = \frac{2}{9} \left( -2 \lambda + \frac{c^2}{b^4} \right) t + O(t^2).$$

Combining (5.8a) and (5.14) we conclude that

$$t \dot{f}_1 = \left( -3 \dot{f}_1 + \frac{1}{2 b^2} (\hat{f}_2 - \hat{f}_3)^2 - \frac{1}{2 b^2} \right) + \frac{c}{b^3} (\hat{f}_2 - \hat{f}_3) - \frac{1}{2 b^2} (\dot{\tau}_2 + \dot{\tau}_3) + O(t).$$

Hence the constraint that the corresponding component of $M_{-1}$ evaluated at the initial condition must vanish also yields

$$\dot{f}_1(0) = -\frac{1}{6 b^2} - \frac{2 \lambda}{27} + \frac{c^2}{18 b^4},$$

where to obtain this we also used the previously determined constraints on the initial conditions given by (5.12) and (5.13). Evaluating the differential of $M_{-1}$ at the initial conditions determined
by (5.12) at (5.15) yields

\[
 n \text{Id} - dM_{-1} = \begin{pmatrix}
 n + 3 & -\frac{4c}{3b^2} & \frac{4c}{3b^2} & \frac{1}{2b^2} & \frac{1}{2b^2} \\
 0 & n + 2 & -1 & 0 & 0 \\
 0 & -1 & n + 2 & 0 & 0 \\
 0 & 0 & 0 & n + 2 & 1 \\
 0 & 0 & 0 & 1 & n + 2
\end{pmatrix},
\]

which having determinant \((n + 3)^3(n + 1)^2\) is indeed invertible for all positive integers \(n\). Hence Theorem (5.5) applies and our result follows. \(\square\)

5.3. \textbf{Sp}(2)-invariant \(G_2\)-solitons extending smoothly over \(S^4\). We can also deduce results about \(\text{Sp}(2)\)-invariant \(G_2\)-solitons that close smoothly on the singular orbit \(S^4 \subset \Lambda^2 S^4\) by specialising to the case where \(f_2 = f_3\) and \(\tau_2 = \tau_3\). We now given further details of the results obtained this way.

For a closed \(\text{Sp}(2)\)-invariant \(G_2\)-structure \(\varphi_f\) expressed in the form

\[
 \varphi_f = \omega_f \wedge dt + \Re \Omega_f = (f_1^2 \omega_1 + f_2^2 \omega_2) \wedge dt + f_1 f_2^2 \alpha
\]

the conditions that \(\varphi_f\) extend smoothly over the singular orbit \(S^4 = \text{Sp}(2)/\text{Sp}(1) \times \text{Sp}(1)\) at \(t = 0\) are: \(f_1\) is odd in \(t\) and \(|f_1(0)| = 1\); \(f_2\) is even in \(t\).

Recall also that by acting with Klein four-group we may assume that \((f_1, f_2)\) is a positive pair for all \(t > 0\). In particular, we must have that \(f_1'(0) = +1\) and \(f_2(0) > 0\). Hence we can write the coefficients \(f_1\) and \(f_2\) as

\[
 (5.16) \quad f_1 = t + t^3 \hat{f}_1, \quad f_2 = b + t^2 \hat{f}_2,
\]

for some \(b > 0\) and new functions \(\hat{f}_1, \hat{f}_2\) that are both even in \(t\). Similarly, we write the torsion coefficients \(\tau_1\) and \(\tau_2\) as

\[
 \tau_1 = t^3 \hat{\tau}_1, \quad \tau_2 = t \hat{\tau}_2,
\]

for functions \(\hat{\tau}_1\) and \(\hat{\tau}_2\) that are both even in \(t\).

The main theorem for smoothly-closing solitons in the \(\text{Sp}(2)\)-invariant setting is the following:

**Theorem 5.17.** Fix any \(\lambda \in \mathbb{R}\) and \(b > 0\). Then there exists a unique local \(\text{Sp}(2)\)-invariant \(G_2\)-soliton which closes smoothly on the singular orbit \(S^4 \subset \Lambda^2 S^4\) and which satisfies

\[
 f_1 = t + t^3 \hat{f}_1, \quad f_2 = b + t^2 \hat{f}_2, \quad \tau_1 = t^3 \hat{\tau}_1, \quad \tau_2 = t \hat{\tau}_2,
\]

where the terms \(\hat{f}_1\) and \(\hat{\tau}_1\) satisfy

\[
 \hat{f}_1(0) = -\frac{1}{6b^2}, \quad \hat{f}_2(0) = \frac{(2\lambda b^2 + 9)}{36b}, \quad \hat{\tau}_1(0) = -\frac{4\lambda}{9}, \quad \hat{\tau}_2(0) = \frac{2}{9} \lambda b^2.
\]

The solution is real analytic on \([0, \epsilon) \times \mathbb{C}P^3\) for some \(\epsilon > 0\) (depending on \(b\) and \(\lambda\)). Moreover, up to the \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-action, any \(\text{Sp}(2)\)-invariant \(G_2\)-soliton which closes smoothly on the singular orbit \(S^4 \subset \Lambda^2 S^4\) belongs to this 1-parameter family of solutions.

Taking into account the scaling behaviour of steady solitons immediately gives us the following:

**Corollary 5.18.** Any local \(\text{Sp}(2)\)-invariant steady \(G_2\)-soliton which closes smoothly on the singular orbit \(S^4 \subset \Lambda^2 S^4\) is torsion-free and has vector field \(X \equiv 0\). Hence the underlying \(G_2\)-structure is the (unique up to scale) standard asymptotically conical torsion-free \(G_2\)-structure on \(\Lambda^2 S^4\) constructed by Bryant–Salamon. In particular, its asymptotic cone is the cone over the unique \(\text{Sp}(2)\)-invariant nearly Kähler structure on \(\mathbb{C}P^3\).
5.4. Explicit complete asymptotically conical shrinkers. In this section we observe that properties of the power series expansions of $\text{Sp}(2)$-invariant smoothly-closing Laplacian solitons close to the singular orbit lead us naturally to a 1-parameter family of explicit complete asymptotically conical shrinking solitons. Up to the action of scaling this yields a unique explicit asymptotically conical shrinker on $\Lambda^2 S^4$ and also on $\Lambda^2 \mathbb{C}P^2$. For notational simplicity it will be convenient to set $x = f_1$ and $y = f_2 = f_3$ in the discussion below.

The starting point is to notice the following property of the power series expansions close to the singular orbit for the smoothly-closing shrinkers constructed in Theorem 5.17 and described in Appendix A: when we write

$$x = t \left(1 + \sum_{i=0} \lambda x i \tau^2_i\right), \quad \tau_2 = t \left(\sum_{i=0} T_i \tau^2_i\right)$$

then the coefficients $x_1, x_2, T_1$ and $T_2$ all contain $x_0$ as a factor. Moreover, $x_0$ vanishes if and only if $\lambda b^2 = -\frac{9}{4}$. In other words, when the latter equality holds we have

$$x = t + O(t^9), \quad \tau_2 = -\frac{1}{2} t + O(t^9).$$

This suggests that $x + 2\tau_2 \equiv 0$ and $x = t$ when $\lambda b^2 = -\frac{9}{4}$.

We will see that this is indeed the case and then find explicit solutions also for $y$ and $u$. First note that the ODE for $(x^2)'$ can be written as $(x^2)' = 2x - \frac{x^2}{4}(x + 2\tau_2)$. So if we impose the condition $x + 2\tau_2 \equiv 0$ we are forced to have $x' = 1$ everywhere and hence by the initial conditions $x = t$. Then for $y$ we have $(y^2)' = x + \tau_2 = \frac{1}{2} x = \frac{1}{2} t$, and therefore $y^2 = b^2 + \frac{1}{4} t^2$. The quantity $S = y^2 - x^2 - \frac{3}{2} x \tau_2$ also has a particularly simple behaviour: $S = b^2 + \frac{1}{4} t^2 - \frac{3}{4} t^2 \equiv b^2$. Now we compute that $4R_1 S = 4(\lambda xy^2 - 3\tau_2 S) = -\frac{32}{3} (4b^2 + 3t^2)$ and $3x(x^2 + y^2) = \frac{32}{3} (4b^2 + 3t^2)$ and therefore the right-hand side of the ODE for $\tau_2$ is identically equal to $-\frac{1}{2}$. Hence we do have a consistent solution to the ODE system (4.19). Finally using the second equation in (4.20) we calculate that

$$u = \frac{3t}{4b^2} + \frac{4t}{4b^2 + t^2}.$$

In other words, the unique solution of the initial value problem for the $\text{Sp}(2)$-invariant shrinker equation with initial conditions as in the statement of Theorem 5.17 and $\lambda = -\frac{9}{4b^2} < 0$ is

$$x = t, \quad y^2 = b^2 + \frac{1}{4} t^2, \quad \tau_2 = -\frac{1}{2} t, \quad u = \frac{3t}{4b^2} + \frac{4t}{4b^2 + t^2}.$$

Note that $u + \frac{3}{4} t = \frac{4t}{4b^2 + t^2} \geq 0$ and that the latter tends to zero as $t \to \infty$. For $t \gg 1$ this shrinker satisfies $y \approx \frac{1}{2} t$, so that it is asymptotic to the closed non-torsion-free $\text{Sp}(2)$-invariant $G_2$-cone $x = c_1 t, y = c_2 t$ with $c_1 = 1, c_2 = \frac{1}{2}$.

Remark 5.19. Note also that in this case asymptotically we have $x > y$, whereas initially $y > x$, and moreover there is a unique time $t_0 = \sqrt{2b^2/3}$ at which $x = y$.

We summarise the situation in the following Theorem.

**Theorem 5.20.** For any $b > 0$ let $\lambda = -\frac{9}{4b^2} < 0$ and define the following functions

$$x = t, \quad y^2 = b^2 + \frac{1}{4} t^2, \quad \tau_2 = -\frac{1}{2} t, \quad u = \frac{3t}{4b^2} + \frac{4t}{4b^2 + t^2}, \quad t \geq 0.$$

(i) The closed $\text{Sp}(2)$-invariant $G_2$-structure $\varphi = xy^2 \alpha + (x^2 \omega_1 + y^2 \omega_2) \wedge dt$, together with the vector field $X = u \partial_t$ determines a complete asymptotically conical Laplacian shrinker on $\Lambda^2 S^4$ asymptotic to the closed but non-torsion-free $\text{Sp}(2)$-invariant $G_2$-cone defined by $x = t, y = \frac{1}{2} t$. 
(ii) The closed SU(3)-invariant G\textsubscript{2}-structure \( \varphi = xy^2\alpha + (x^2\omega_1 + y^2\omega_2 + y^2\omega_3) \wedge dt \) together with the vector field \( X = u \partial_t \) determines a complete asymptotically conical Laplacian shrinker on \( \Lambda^2 \mathbb{CP}^2 \) that closes smoothly on \( \mathbb{CP}^1 \) and is asymptotic to the closed but non-torsion-free SU(3)-invariant G\textsubscript{2}-cone defined by \( f_1 = t, f_2 = f_3 = \frac{1}{2}t \).

**Remark.** By Remark 3.28 the metric induced by the AC shrinking SU(3)-invariant soliton on \( \Lambda^2 \mathbb{CP}^2 \) has an additional free isometric \( \mathbb{Z}_2 \)-action which does not however preserve the G\textsubscript{2}-structure. By acting with the nontrivial elements of \( A_3 \subset W = S_3 \) we also get two other variants of this AC shrinking soliton; these variants have different free \( \mathbb{Z}_2 \) actions, different asymptotic cones and close smoothly not on the singular orbit \( \mathbb{CP}^2 \) but on \( \mathbb{CP}^2_2 = SU(3)/U(2)_2 \) or \( \mathbb{CP}^2_3 = SU(3)/U(2)_3 \) instead.

**Remark.** Fowdar [34, Section 5.2] found explicit complete inhomogeneous gradient shrinking Laplacian solitons on certain \( T^2 \)-bundles over hyperKähler 4-manifolds. The induced metric on the base \( B \) and the fibre \( T^2 \) both change by homotheties along the cylinder direction \( \mathbb{R} \) but the base and the fibre scale differently. In the simplest case where \( B = T^4 \) then resulting soliton is of cohomogeneity one and the principal orbit \( N^6 \) is a compact nilmanifold, the 3-dimensional Iwasawa manifold, i.e. the compact quotient of the complex 3-dimensional Heisenberg group by the lattice generated by the Gaussian integers. Apostolov–Salamon [3, Example 1 p55] and Gibbons et al [35] explain how the analogous construction of a cohomogeneity-one torsion-free G\textsubscript{2}-structure with the same principal orbit \( N \) arises from the Bryant–Salamon Sp(2)-invariant AC torsion-free G\textsubscript{2}-structure on \( \Lambda^2 \mathbb{S}^4 \) by a ‘contraction’ of the isometry group. Based on this it is tempting to speculate that Fowdar’s example arises from contraction of the isometry group of our explicit Sp(2)-invariant AC shrinker on \( \Lambda^2 \mathbb{S}^4 \).

### 6. Complete SU(3)-invariant steady solitons

In this section we characterise precisely which of the smoothly-closing SU(3)-invariant steady solitons constructed in Theorem 5.2 are complete and which are incomplete; for the complete ones we achieve a detailed understanding of their asymptotic geometry.

We will denote the smoothly-closing steady solitons by \( S_{b,c} \) where \( b > 0 \) and \( c \in \mathbb{R} \). Recall that by the invariance of the steady soliton system under rescaling this yields only a 1-parameter family of distinct smoothly-closing steady solitons. By acting with the element of the Weyl group \( W \) that exchanges \( f_2 \) and \( f_3 \), recall Lemma 3.25, we can also transform a solution with \( c < 0 \) into one with \( c > 0 \). Therefore for simplicity of exposition in most of this section we will make the assumption that \( c > 0 \); results about solutions with \( c < 0 \) follow easily by minor modifications of those obtained for solutions with \( c > 0 \).

Recall also that by Corollary 5.18, any smoothly-closing Sp(2)-invariant steady soliton is necessarily a trivial soliton, i.e. has vanishing vector field and the underlying G\textsubscript{2}-structure is torsion free. So throughout the rest of this section we will only consider SU(3)-invariant steady solitons.

#### 6.1. Special features of the steady ODE system.

The steady case has many special features which we now detail; arguably the single most important feature is the existence of three conserved quantities that are not present for shrinkers or expanders.

First observe from (4.6b) that when \( \lambda = 0 \) the three quantities \( \tau_i - u f_i^2 \) are conserved. Using (5.3) we therefore have that any smoothly-closing steady soliton satisfies

\[
\tau_1 = u f_1^2, \quad \tau_2 = u f_2^2 + c, \quad \tau_3 = u f_3^2 - c,
\]

for some \( c \in \mathbb{R} \). Using (6.1) one can verify that the type 14 condition on \( \tau \) is equivalent to

\[
u = \frac{c}{3} \left( \frac{1}{f_2^2} - \frac{1}{f_3^2} \right) = \frac{c}{3} \left( \frac{f_2^2 - f_3^2}{f_2^2 f_3^2} \right).
\]
Substituting the expression for $u$ from (6.2) into (6.1) we obtain the following expressions for the components of $\tau$ in terms of the $f_i^2$

$$\tau_1 = \frac{c f_1^3 (f_2^2 - f_3^2)}{3 f_2 f_3^2}, \quad \tau_2 = \frac{c}{3} \left(2 + \frac{f_2^3}{f_3^2}\right), \quad \tau_3 = -\frac{c}{3} \left(2 + \frac{f_2^3}{f_3^2}\right).$$

Similarly, substituting (6.1) into the expression $|\tau|^2 = \sum \tau_i^2 f_i^{-4}$ yields

$$|\tau|^2 = u^2 + (u + cf_2^{-2})^2 + (u - cf_3^{-2})^2 = \frac{2c^2}{3} \left(\frac{1}{f_3^4} + \frac{1}{f_2^4} + \frac{1}{f_2^2 f_3^2}\right),$$

where the second equality uses (6.2).

An immediate consequence of formulae (6.1) and (6.2) is that in the steady case the first-order ODE system (4.12) for $(f_i, \tau_i)$ can be rewritten as a self-contained first-order system involving only the triple $(f_i)$ and the constant triple $(c_1, c_2, c_3)$ that specifies the values of the three conserved quantities $\tau_i - uf_i^2$ (which in the smoothly-closing case has the form $(0, c, -c)$ for some $c \in \mathbb{R}$).

**Lemma 6.5.** Any smoothly-closing steady soliton satisfies the first-order ODE system

$$\frac{d}{dt} \ln f_i^2 = u + \frac{c_i}{f_i^2} + \frac{1}{f_1 f_2 f_3} (f_i^2 - 2 f_j^2) = \frac{c}{3} \left(\frac{1}{f_3^2} - \frac{1}{f_2^2}\right) + \frac{1}{f_1 f_2 f_3} (f_j^2 - 2 f_i^2)$$

where $(c_1, c_2, c_3) = (0, c, -c) \in \mathbb{R}^3$ are the values of the three conserved quantities $\tau_i - uf_i^2$ and $(ijk)$ is any permutation of $(123)$.

Conversely, for any $c \in \mathbb{R}$, given any solution $f = (f_1, f_2, f_3)$ to the ODE system (6.6), then defining $u$ via (6.2) and $\tau_i$ via (6.1) makes $(f_1, f_2, f_3, u)$ a solution of the mixed-order soliton ODE system (4.6) and $(f_1, f_2, f_3, \tau_1, \tau_2, \tau_3)$ a solution of the first-order soliton ODE system (4.12).

**Remark.** Note that when $c = 0$ the system (6.6) reduces to the torsion-free ODE system (5.7).

We note that (6.6) implies that the ratio $f_i/f_j$ satisfies the differential equation

$$\frac{d}{dt} \log \left(\frac{f_i^2}{f_j^2}\right) = \frac{c_i}{f_i^2} - \frac{c_j}{f_j^2} + \frac{2(f_j^2 - f_i^2)}{f_1 f_2 f_3}.$$

These differential equations for the ratios $f_i/f_j$ will play an important role in controlling the behaviour of the smoothly-closing steady solitons. The first application is the following result establishing that a certain ordering of the coefficients of the triple $(f_i)$ persists throughout the lifetime of any smoothly-closing steady soliton $S_{b,c}$.

**Lemma 6.8.** For any smoothly-closing steady soliton $S_{b,c}$ with $c > 0$ the component $f_2$ is dominant, i.e. for all positive times within the lifetime of the solution the triple $f$ satisfies the ordering properties

$$f_2 > f_3, \quad f_2 > f_1.$$

Hence

$$u > 0, \quad f_2 > \frac{1}{2}t$$

also hold for all positive times within the lifetime of the solution.

**Proof.** The small-$t$ power series for the smoothly-closing solutions given in Appendix A imply that

$$f_2 - f_1 = b + \frac{c - 6b}{6b} t + \text{h.o.t.}, \quad f_2 - f_3 = \frac{c}{3b} t + \text{h.o.t.}$$

Hence $f_2 > f_1$ and $f_2 > f_3$ (since we assumed $c > 0$) hold for $t > 0$ sufficiently small. At any instant when $f_i = f_j$, (6.7) implies that

$$f_i' - f_j' = \frac{1}{2f_i} (c_i - c_j).$$
Hence \((f_2 - f_1)' = \frac{c}{f_2} > 0\) whenever \(f_1 = f_2\) and \((f_2 - f_3)' = \frac{c}{f_2} > 0\) whenever \(f_2 = f_3\). It follows that the conditions \(f_2 > f_1\) and \(f_2 > f_3\) are preserved under the (forward) evolution of (6.6). Positivity of \(u\) now follows immediately from the dominance of \(f_2\) and (6.2). Recall from (3.32) that the closure condition already gave that \(8f_1f_2f_3 > t^3\) and hence the dominance of \(f_2\) implies that \(2f_2 > t\). (Finally we remark that \((f_1 - f_3)' = \frac{c}{2f_1} > 0\) whenever \(f_1 = f_3\) so that the condition \(f_1 > f_3\) is also preserved under the evolution of (6.6). However, since the solution \(S_{b,c}\) satisfies \((f_1 - f_3)(0) = -b < 0\) the condition \(f_1 > f_3\) is not initially satisfied.)

By exploiting the dominance of \(f_2\) just proven in Lemma 6.8 we can obtain useful qualitative information about smoothly-closing steady solitons by deriving certain differential inequalities from (6.6) and (6.7).

**Lemma 6.9.** For any smoothly-closing steady soliton \(S_{b,c}\) with \(c > 0\) the following all hold.

(i) \(f_1\) is an increasing function of \(t\) throughout the lifetime of the solution.

(ii) Away from a neighbourhood of \(t = 0\) there exists \(C_{12} > 0\) so that ratio \(f_1 / f_2\) satisfies

\[
\frac{f_1}{f_2} > C_{12} > 0
\]

throughout the lifetime of the solution. In particular, \(f_1\) and \(f_2\) have comparable growth and both must grow at least linearly if the solution is complete.

(iii) There exists a constant \(C > 0\) so that \(f_3 < C\) holds throughout the lifetime of the solution. In particular, \(f_3\) has at most linear growth on any complete solution.

(iv) Let \(C_{12} > 0\) be the constant appearing in part (ii). Suppose there exists an instant \(t_0\) at which

\[
f_3 < A \left(\frac{2C_{12}}{C_{12} + 1}\right)
\]

holds for some \(A \in (0, 1)\). Then for \(t > t_0\), \(f_3\) decreases monotonically toward 0 in finite time and so, in particular, the solution is incomplete. Hence for any complete solution there exists a constant \(C_3 > 0\) so that

\[
f_3 > C_3 > 0
\]

holds for all \(t \geq 0\).

**Proof.** (i) The component \(f_1^2\) satisfies

\[
\frac{d}{dt} \ln f_1^2 = u + \frac{f_2}{f_1 f_3} + \frac{f_3 - f_1^2}{f_1 f_2 f_3} = u + \frac{f_3}{f_1 f_2} + \frac{f_3^2 - f_1^2}{f_1 f_2 f_3} > u + \frac{f_3}{f_1 f_2} > u > 0.
\]

The first equality is simply (6.6) in the case \(i = 1\) (using the fact that \(c_1 = 0\)). The two inequalities claimed follow immediately from the dominance of \(f_2\) and the positivity of \(u\), as proven in Lemma 6.8.

(ii) Now consider the ratio \(r_{12} = \frac{f_1^2}{f_2}\). Using (6.7) and the dominance of \(f_2\) we obtain the following differential inequality

\[
(6.10) \quad \frac{d}{dt} \log r_{12} = \frac{2}{f_3} \left(\frac{f_2}{f_1} - \frac{f_1}{f_2}\right) - \frac{3}{f_2} > \frac{2}{f_2} \left(\frac{f_2}{f_1} - \frac{f_1}{f_2}\right) - \frac{3}{f_2} = \frac{2}{f_1} \left(1 - \left(1 + \frac{3}{2f_1}\right) r_{12}\right).
\]

In particular, whenever \(r_{12}\) is sufficiently close to zero the right-hand side of this inequality is positive. In other words, \(\frac{d}{dt} \log r_{12} > 0\) whenever \(r_{12}\) is sufficiently close to zero and therefore \(r_{12}\) must be bounded away from zero as \(t \to \infty\), say \(r_{12} > C_{12}^2 > 0\) for all \(t\) sufficiently large (clearly the constant \(C_{12}\) will depend on \(b\) but that dependence is not important to us here).

(iii) From (6.6) we obtain

\[
(6.11) \quad \frac{d}{dt} \log f_3^2 = \frac{1}{f_3} \left(\frac{f_1^2 + f_2^2}{f_1 f_2}\right) - \left(\frac{2}{f_3^2} + \frac{1}{f_1^2} + \frac{f_3}{f_1 f_2}\right).
\]
Hence using part (ii) and the fact that \( f_2 \) dominates we obtain that
\[
2f_3' = \left( \frac{f_1}{f_2} + \frac{f_2}{f_1} - \frac{2}{f_3} \right) - f_3 \left( \frac{1}{f_2} + \frac{f_3}{f_1 f_2} \right) < \left( 1 + \frac{1}{C_{12}} \right) - \frac{2}{f_3}
\]
holds away from a neighbourhood of 0. But from the small-\( t \) expansions for \( S_{k,c} \) we already have control of \( f_3' \) close to \( t = 0 \).

(iv) From (6.11) and part (ii) we see that
\[
f_3' < \frac{1}{2} \left( \frac{f_1}{f_2} + \frac{f_2}{f_1} \right) < \frac{1}{2} \left( 1 + C_{12}^{-1} \right)
\]
holds away from a neighbourhood of 0. Hence at any instant \( t_0 \) where \( f_3 < A \left( \frac{2C_{12}}{C_{12} + 1} \right) \) holds we have
\[
2f_3' < - \left( \frac{1 + C_{12}}{C_{12}} \right) A^{-1}(1 - A).
\]
Therefore \( f_3 \) is decreasing at \( t_0 \) and so the condition \( f_3 < A \left( \frac{2C_{12}}{C_{12} + 1} \right) \) persists during the remainder of the lifetime of the solution and therefore so does the inequality 
\[
2f_3' < - C \left( \frac{1 + C_{12}}{C_{12}} \right) A^{-1}(1 - A).
\]
Hence for \( t > t_0 \), \( f_3 \) decreases monotonically to 0 in finite time. \( \square \)

The ODE system (6.6) that the triple \( f \) satisfies is a rational ODE system. The next result shows that a suitable change of variables transforms it into a polynomial ODE system. As we will shortly, this polynomial reformulation gives us some important insights into possible asymptotic behaviour of steady solitons not readily apparent from (6.6).

**Lemma 6.12.** Given a smoothly-closing solution \( f \) of the steady ODE system (6.6) the triple \( F = (F_i) \) defined by
\[
F_i := \frac{f_i}{f_j f_k} = \frac{f_i^2}{g^3}, \quad (ij) = (123),
\]
where as usual \( (ijk) \) is a cyclic permutation of \((123)\), satisfies the polynomial ODE system
\[
\begin{align*}
F_1' &= \frac{c}{3} F_1^2 (F_2 - F_3) + \frac{1}{2} F_1 (F_2 + F_3 - 3F_1) \\
F_2' &= \frac{c}{3} F_1 F_2 (F_2 + 2F_3) + \frac{1}{2} F_2 (F_1 + F_3 - 3F_2) \\
F_3' &= -\frac{c}{3} F_1 F_3 (2F_2 + F_3) + \frac{1}{2} F_3 (F_1 + F_2 - 3F_3),
\end{align*}
\]
where \( c \in \mathbb{R} \) is the constant determined by (6.1).

**Proof.** First we note that \( F_i F_j = f_k^{-2} \) and therefore by (6.2) we can write
\[
3u = cF_1(F_2 - F_3).
\]
Therefore using (6.1) and (6.6) we obtain
\[
\frac{d}{dt} \ln F_i^2 = \left( \frac{\tau_i}{f_i^2} - \frac{\tau_j}{f_j^2} - \frac{\tau_k}{f_k^2} \right) + (F_j + F_k - 3F_i) = -u + \left( \frac{c_i}{f_i^2} - \frac{c_j}{f_j^2} - \frac{c_k}{f_k^2} \right) + (F_j + F_k - 3F_i)
\]
where \( (c_1, c_2, c_3) = (0, c, -c) \). Therefore we have
\[
\frac{d}{dt} \ln F_i^2 = -\frac{c}{3} F_1(F_2 - F_3) + (c_i F_j F_k - c_j F_i F_k - c_k F_i F_j) + (F_j + F_k - 3F_i),
\]
where \( (ijk) \) is a permutation of \((123)\) from which the claimed result now follows. \( \square \)

Note that (6.14) is an inhomogeneous cubic (for \( c \neq 0 \)) system with no constant or linear terms.
Lemma 6.16. Fix any \( c > 0 \), the constant appearing in (6.14).

(i) Any solution of (6.14) defined for all positive \( t \) satisfies

\[
\lim_{t \to \infty} F_1 F_2 F_3 = 0. \tag{6.17}
\]

(ii) There are no critical points of (6.14) within the positive octant.

(iii) The only critical points of (6.14) within the closure of the positive octant are the origin and the point

\[
(F_1, F_2, F_3) = (3c^{-1}, 3c^{-1}, 0). \tag{6.18}
\]

(iv) The origin is a totally degenerate critical point, i.e. the linearisation of (6.14) at the origin is the zero map.

(v) The critical point \((F_1, F_2, F_3) = (3c^{-1}, 3c^{-1}, 0)\) is a hyperbolic fixed point of the system, with two negative eigenvalues and a single positive one. It therefore has a 2-dimensional local stable manifold which at the fixed point has normal vector \((1, 1, 1)\).

Proof. (i) The system of equations (6.14) implies that

\[
\frac{d}{dt} \ln F_1 F_2 F_3 = -\frac{1}{2}(F_1 + F_2 + F_3). \tag{6.19}
\]

In fact this equation is equivalent to the closure condition \(2(f_1 f_2 f_3)' = \overline{f_2}^2\) and moreover the inequality (3.32) is equivalent to

\[
\frac{d}{dt} \log (F_1 F_2 F_3)^{1/3} \leq -\frac{1}{2}.
\]

In particular, \(F_1 F_2 F_3\) is a strictly decreasing function of \(t\) and the previous inequality implies that any solution of (6.14) defined for all positive \( t \) satisfies

\[
\lim_{t \to \infty} F_1 F_2 F_3 = 0.
\]

(Indeed this has to be the case since \(F_1 F_2 F_3 = (f_1 f_2 f_3)^{-1}\) and \(f_1 f_2 f_3\) is increasing and in fact \(F_1 F_2 F_3 < 8t^{-3}\) because \(8f_1 f_2 f_3 > t^3\).)

(ii) A critical point of (6.14) within the positive octant would have \(\frac{d}{dt} \ln F_1 F_2 F_3 = 0\) and hence \(F_1 + F_2 + F_3 = 0\) by (6.19).

(iii) Each of the hyperplanes \(F_i = 0\) is an invariant subset of (6.14). It is easy to check that the only critical point contained in the hyperplanes \(F_1 = 0\) or \(F_2 = 0\) is the origin. Clearly, the linearisation of the system at the origin is the zero map, since all terms are at least quadratic in \(F\). The hyperplane \(F_3 = 0\) contains the additional critical point given in (6.18) that (for any \(c > 0\)) lies on the boundary of the positive octant. Moreover, if \(F\) is a solution of (6.14) asymptotic to a critical point (6.18) as \(t \to \infty\), then by (6.19) \(F_1 F_2 F_3\) decays exponentially to 0 with rate at least \(3c^{-1}\). This is equivalent to the fact that \(f_1 f_2 f_3\) grows exponentially in \(t\) with rate at least \(3c^{-1}\), i.e. that the corresponding steady soliton has a forward-complete end with exponential volume growth.

We leave the computation of the linearisation of (6.14) at the fixed point (6.18) to the reader; the proofs of our main results about completeness and incompleteness of smoothly-closing steady solitons will not rely on the properties of this critical point. \(\square\)

As an immediate consequence of the existence of a 2-dimensional stable manifold for the fixed point (6.18) we have the following:

Corollary 6.20. For each \( c > 0 \) there is a 1-parameter family of distinct forward-complete steady soliton ends all with exponential volume growth.
Given this result it is natural to ask:

Which if any of the smoothly-closing steady solutions $S_{b,c}$ has a complete end for which the associated triple $F$ is asymptotic to the critical point (6.18)?

Since the origin is a totally degenerate fixed point of (6.14) we cannot immediately derive any information about the behaviour of solutions close to the origin. We will shortly introduce another reformulation of the steady ODEs that will shed light on the character of this critical point.

6.2. Blow-up analysis for smoothly-closing steady solitons. Here we give a simple criterion under which a smoothly-closing steady soliton defined on an interval $[0,T)$ can be extended past $t = T$; we use this extension criterion to prove that for any smoothly-closing steady soliton with $c > 0$ there exists a unique $t_*$ within its lifetime at which $f_1 = f_3$. The extension criterion will also play an important role later in proving completeness results for smoothly-closing steady solitons.

Proposition 6.21. If a smoothly-closing steady soliton $S_{b,c}$ with $c > 0$ exists on the finite interval $[0,T)$, then $T$ is the maximal existence time for the solution if and only if $\lim_{t \to T} f_3(t) = 0$.

Proof. First we observe by part (iv) of Lemma 6.9 that once $f_3$ is sufficiently close to 0 then it decreases monotonically to 0. Clearly if $\lim_{t \to T} f_3(t) = 0$ then the solution cannot be extended past $T$, so we need only prove that if $T < \infty$ is the maximal existence time, then we must have $\lim_{t \to T} f_3(t) = 0$. Assume therefore that $T < \infty$ is the maximal existence time of the solution $f$ and consider the corresponding solution $F$ of the polynomial ODE system (6.14). Recall that for $t \approx 0$ we have $F \approx (b^{-1}t, b^{-2}t^{-1}, b^{-2}t^{-1})$. Suppose first for a contradiction that the positive triple $F$ remains bounded above as $t \to T$. If we can also show that each component of $F$ remains bounded away from 0 as $t \to T$ then by appealing to the local existence theory we may extend the solution $F$ of (6.14) beyond $t = T$ maintaining the positivity of the triple $F$; therefore via $f_3^2 = 1/(F_jF_k)$, we also obtain a positive triple $f$ that solves (6.6) and exists past $t = T$, thereby contradicting the maximality of $T$. Away from a sufficiently small neighbourhood of $t = 0$, $F$ takes values in a bounded subset of $\mathbb{R}^3_{\geq 0}$. Hence by (6.15), $\frac{d}{dt} \log F_i^2$ has a uniform lower bound (away from a neighbourhood of $t = 0$). Integrating this implies that away from a neighbourhood of $t = 0$, $\log F_i^2(t)$ is bounded below for all $t < T$. Therefore each $F_i$ is bounded away from 0 as $t \to T$, as we required for our contradiction, and hence $F$ must be unbounded as $t \to T$, the maximal existence time. Since the triple $F$ has the same ordering as the triple $f$ this implies that $F_2$ is unbounded as $t \to T$. But since away from a neighbourhood of $t = 0$, part (ii) of Lemma 6.9 and the dominance of $f_2$ imply that

$$\frac{1}{f_3} < F_2 = \frac{f_2}{f_1 f_3} < \frac{1}{C_{12} f_3},$$

this forces $f_3$ to accumulate to 0 as $t \to T$. Hence, by our first observation, $\lim_{t \to T} f_3 = 0$. □

The smoothly-closing initial conditions (5.3) imply that the inequality $f_3 > f_1$ holds for $t$ sufficiently small. If this condition were preserved throughout the lifetime of the solution then the previous result and the fact that $f_1$ is increasing would prove that its lifetime must be infinite. However, as remarked at the end of the proof of Lemma 6.8, instead it is the condition $f_1 > f_3$ that is preserved.

Lemma 6.22. For any smoothly-closing steady soliton $S_{b,c}$ with $c > 0$, if the equality $f_1 = f_3$ holds at some instant then for all subsequent times $f_1 > f_3$ holds.

As an immediate consequence of Proposition 6.21 and the previous lemma we have the following

Corollary 6.23. If a smoothly-closing steady soliton $S_{b,c}$ with $c > 0$ has finite lifetime $T$ then there exists a unique $T' \in (0,T)$ such that $f_1(T') = f_3(T')$ and $(f_1 - f_3)(t) > 0$ for $t > T'$, i.e. it cannot reach its maximal existence time while the condition $f_3 \geq f_1$ is maintained.
Remark. An immediate consequence of Lemmas 6.8 and 6.25 and (6.17) is that the triple such that which for sufficiently large contradicts 2f2 > t. Hence there must exist some finite T' > 0 with f1(T') = f3(T'). □

Remark. An immediate consequence of Lemmas 6.8 and 6.25 and (6.17) is that the triple F associated via (6.13) to any complete smoothly-closing steady soliton $S_{b,c}$ with c > 0 satisfies

$$\lim_{t \to \infty} F_3 = 0.$$ 

6.3. The purely scale-normalised steady ODE system. The use of the polynomial reformulation (6.14) of the steady ODE system allowed us to identify and describe in detail one possible type of asymptotic behaviour of complete steady solitons: namely those solutions that belong to the 2-dimensional local stable manifold of the hyperbolic fixed point of that system. The other fixed point of the system was the origin; however, since the origin was a totally degenerate fixed point its character was unclear. To analyse the behaviour of steady solitons for which F is asymptotic to the origin it is convenient to use (yet) another reformulation of the steady ODE system.

To this end we return to the scale-normalised ODE system (4.30) and observe that in the steady case, after a suitable reparametrisation of t, the equations for $T_i$ and $T'_i$ can be rewritten entirely in terms of the scale-invariant variables $F_i$ and $F'_i$, i.e. with no appearance of the scale-factor $g$. For any smoothly-closing steady soliton $S_{b,c}$ with c > 0 there exists unique $T' > 0$ such that $f_1(T') = f_3(T')$ and $(f_1 - f_3)(t) > 0$ for $t > T'$. Proof. By Corollary 6.23 it suffices to prove the result for a soliton with infinite maximal existence time $T = +\infty$. So assume for a contradiction that $f_1 < f_3$ holds for all $t \geq 0$. It therefore follows from (6.24) that $f_1/f_3$ is a strictly increasing function of t for all $t \geq 0$. In particular, since by Lemma 6.9 $f_1$ grows at least linearly in t so must $f_3$. This linear lower bound for $f_3$ together with (6.2) yields the upper bound $3u < cf_3^{-2} = Ct^{-2}$ for some $C > 0$.

We now seek to bound the growth of $f_2$ in order to derive a contradiction to the fact that $f_2 > \frac{1}{2}t$ holds for all $t > 0$. For any $t > 0$ we have the upper bound

$$\frac{d}{dt} \ln f_2^2 = u + \frac{c}{f_2} + f_1 \frac{1}{f_3 f_2} + \frac{f_3^2 - f_2^2}{f_1 f_2 f_3} < u + \frac{c}{f_2^2} + \frac{1}{f_2} < \frac{1}{3} Ct^{-2} + 4ct^{-2} + 2t^{-1}$$

using $f_1 < f_3$ and $f_3 < f_2$, our upper bound for $u$ and the linear lower bound for $f_2$. So for $t \geq 1$ we obtain

$$\left(\ln f_2^2\right)' < C't^{-1}.$$ 

For any $t > 1$, integrating from 1 to $t$, yields the upper bound

$$f_2^2(t) < \left( f_2^2(1)e^{C't} \right) t,$$

which for $t$ sufficiently large contradicts $2f_2 > t$. Hence there must exist some finite $T' > 0$ with $f_1(T') = f_3(T')$. □

Proof. We know that $f_3 > f_1$ holds for $t \geq 0$ sufficiently small. Suppose that $T < \infty$ is the maximal existence time of the solution and that $f_3 - f_1 > 0$ holds on the whole interval $[0,T)$. Since by Lemma 6.9 $f_1$ is increasing, we have $\lim_{t \to T} f_1(T) > 0$. By continuity, $\lim_{t \to T} f_3(t) \geq \lim_{t \to T} f_1(t) > 0$ and hence by Proposition 6.21 the solution can be extended past $t = T$, contradicting the maximality of $T$. The remainder of the statement now follows immediately from Lemma 6.22. □

Rewriting (6.7) yields the differential equation

$$\left( \frac{f_3}{f_1} \right)' = -\frac{1}{f_2} \left( \left( \frac{f_3}{f_1} \right)^2 - 1 \right) - \frac{c}{2f_1 f_3}.$$ 

In particular for $c > 0$ the function $f_3/f_1$ is strictly decreasing whenever its value is at least 1.

Lemma 6.25. For any smoothly-closing steady soliton $S_{b,c}$ with $c > 0$ there exists unique $T' > 0$ such that $f_1(T') = f_3(T')$ and $(f_1 - f_3)(t) > 0$ for $t > T'$.
More specifically, using the fact that when $\lambda = 0$ the terms on the right-hand side of the ODE system (4.30) involving the scale-factor $g$ vanish, the ODE system reduces to

\begin{align}
(6.26a) & \quad \dot{\mathcal{F}}_i = \frac{1}{2\mathcal{F}_i} \left( \mathcal{F}_i - \frac{2}{3} \mathcal{F}_i^3 + \frac{2}{3} \mathcal{F}_i^3 \right) = -\frac{1}{3} \mathcal{F}_i S_i, \\
(6.26b) & \quad \mathcal{F}_i = \frac{\mathcal{F}_i}{\mathcal{F}^2} (\mathcal{F}_i - 2 \mathcal{F}_i^4) + \frac{1}{3} \left( \sum \mathcal{F}_i^4 \mathcal{F}_i^2 - \frac{1}{6} \mathcal{F}_i^3 \mathcal{F}_i^2, \right)
\end{align}

where $\dot{}$ denotes differentiation with respect to a new variable $s$ defined (up to a constant) by

\begin{equation}
(6.27) \quad \frac{dt}{ds} = g.
\end{equation}

Note also that an immediate consequence of (4.30a) and (6.27) is that $g(s)$ satisfies

\begin{equation}
(6.28) \quad \frac{d}{ds} \ln g = \frac{1}{6} \mathcal{F}^2.
\end{equation}

Equation (6.28) clearly implies that given a solution $({\mathcal{F}}, {\mathcal{T}})$ to the purely scale-invariant steady soliton system (6.26) we can recover the scale-factor $g$ via quadrature.

**Remark.** The fact that in the steady case the scale-invariant variables $\mathcal{F}_i$ and $\mathcal{T}_i$ satisfy a self-contained ODE system and that the scale-factor $g$ is then determined by the scale-invariant variables is strongly reminiscent of an analogous decoupling that occurs in Böhm’s work on noncompact cohomogeneity-one Ricci-flat metrics [7, Section 3].

A simple but key observation about the ODE system (6.26) is that it has a unique critical point $c_{\text{tf}}$; this critical point corresponds to the torsion-free $G_2$–structure on the cone over the flag variety $F_{1,2} = SU(3)/T^2$; it will also be crucial that $c_{\text{tf}}$ is a stable critical point of (6.26).

**Lemma 6.29.**

(i) **The purely scale-invariant steady soliton ODE system (6.26) has a unique critical point given by $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = 1$, $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0$; this critical point corresponds to the scale-invariant description of the SU(3)-invariant torsion-free $G_2$–structure on the cone over the flag variety $F_{1,2} = SU(3)/T^2$. We therefore we denote this critical point by $c_{\text{tf}}$.**

(ii) **The linearisation of (6.26) at $c_{\text{tf}}$ has eigenvalues $-\frac{1}{2}$ and $-2$, each with multiplicity two. Therefore $c_{\text{tf}}$ is an exponentially stable critical point of (6.26) with (exponential) rate of convergence at worst $\frac{1}{2}$.**

**Proof.** Let $({\mathcal{F}_1}, {\mathcal{F}_2}, {\mathcal{F}_3}, {\mathcal{T}_1}, {\mathcal{T}_2}, {\mathcal{T}_3})$ be any critical point of (6.26). Equation (6.26a) implies that $S_i = 0$ for all $i$. Hence by Remark 4.31 the condition $\dot{\mathcal{T}}_i = 0$ for all $i$ forces $\mathcal{T}_i = 0$ for all $i$. As we observed previously in the proof of Lemma 4.36, the vanishing of all $S_i$ and all $\mathcal{T}_i$ implies that $\mathcal{T}_i = 1$ for all $i$. Denote this unique critical point $(1,1,1,0,0,0)$ by $c_{\text{tf}}$. The phase space of (6.26) is the smooth 4-manifold $\mathcal{P} \subset \mathbb{R}^3 \times \mathbb{R}^3$ defined by

\[ \mathcal{P} := \left\{ (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \mid \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 = 1, \frac{\mathcal{F}_1}{\mathcal{F}^2} + \frac{\mathcal{F}_2}{\mathcal{F}^2} + \frac{\mathcal{F}_3}{\mathcal{F}^2} = 0 \right\}. \]

$\mathcal{P}$ has the structure of a real 2-plane bundle over the smooth connected surface of $\mathbb{R}^3$, cut out by the equation $\mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 = 1$. Radial projection shows that the latter surface is diffeomorphic to $S^2_+ := S^2 \cap \mathbb{R}^3_+$. The tangent space to $\mathcal{P}$ at $(\mathcal{F}, \mathcal{T})$ is the kernel of the constant rank 2 linear map $L(\mathcal{F}, \mathcal{T}) : \mathbb{R}^6 \to \mathbb{R}^2$

\[ L(\mathcal{F}, \mathcal{T}) := \begin{pmatrix} \mathcal{F}_2 \mathcal{F}_3 & -2 \mathcal{F}_1 \mathcal{F}_3 & \mathcal{F}_1 \mathcal{F}_2 & \mathcal{F}_1 \mathcal{F}_2 & 0 & 0 \\ -2 \mathcal{F}_1 \mathcal{F}_3 & -2 \mathcal{F}_2 \mathcal{F}_3 & -2 \mathcal{F}_3 \mathcal{F}_2 & 0 & \mathcal{F}_3^{-2} & \mathcal{F}_2^{-2} \end{pmatrix}. \]
The linearisation of (6.26) about its unique critical point \( c_{ctf} \) is given by the restriction of the linear vector field \((\zeta, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3\)

\[
\begin{align*}
\frac{d\zeta_i}{ds} &= -2\zeta_i + \frac{1}{2}\eta, \\
\frac{d\eta_i}{ds} &= -\frac{1}{2}\eta,
\end{align*}
\]

(6.30a) to the tangent space to \( P \) at \( c_{ctf} \), i.e. the 4-dimensional linear subspace defined by \( \zeta = 0 = \eta \) (here \( \zeta \) means the sum of its three coefficients), which clearly is an invariant subspace of the vector field on \( \mathbb{R}^6 \). This vector field has eigenvalues \(-2\) and \(-\frac{1}{2}\) each with multiplicity two and

\[
E_{-2} = \{(\zeta, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 | \zeta = 0, \eta = 0\}, \quad E_{-\frac{1}{2}} = \{(\zeta, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 | 3\zeta = \eta, \zeta = \eta = 0\},
\]

where \( E_{\mu} \) denotes the eigenspace of eigenvalue \( \mu \).

**Remark.** Note that a deformation in the direction of any vector in the eigenspace \( E_{-2} \) does not change the torsion to first order, because any element of \( E_{-2} \) has \( \eta = 0 \). In other words, elements of the 2-dimensional space \( E_{-2} \) can be identified with decaying SU(3)-invariant infinitesimal deformations of the torsion-free \( G_2 \)-structure on the cone over SU(3)/\( T^2 \). The three 1-parameter families of asymptotically conical torsion-free \( G_2 \)-structures on \( \Lambda^2 \mathbb{C} \mathbb{P}^2 \) constructed by Bryant–Salamon provide torsion-free deformations that integrate some of these infinitesimal deformations: the three families correspond to the three different choices for which coefficient \( f_i \) should degenerate on the singular orbit \( \mathbb{C} \mathbb{P}^2 \). Note however in all three 1-parameter families that if the coefficient \( f_i \to 0 \) on the singular orbit then we have \( f_j = f_k \) (globally), where as usual \( (ijk) \) is a permutation of \( (123) \). In other words, the obviously integrable decaying infinitesimal torsion-free deformations of the torsion-free \( G_2 \)-cone belong to the three 1-dimensional subspaces \( E_{-2} = \{(\zeta, \eta) \in E_{-2} | \zeta_j = \zeta_k\} \).

Note that all of the AC Bryant–Salamon torsion-free \( G_2 \)-structures approach the asymptotic cone with rate \(-4\); rate \(-4\) is consistent with the exponential decay rate \(-2\) with respect to the variable \( s \) because asymptotically \( t \) and \( s \) are related by \( t^{-2} \sim e^{-s} \).

In the opposite direction, to obtain a deformation of the torsion-free \( G_2 \)-cone into a genuine steady soliton whose torsion is nonzero to first order then that deformation must have a nontrivial projection onto the eigenspace \( E_{-\frac{1}{2}} \), and so must decay to its asymptotic cone only at rate \(-1\).

By combining the stability of the critical point \( c_{ctf} \) of the purely scale-invariant steady ODE system (6.26) with the continuous dependence of the smoothly-closing steady solitons \( S_{b,c} \) of Theorem 5.2 on the initial data \((b, c)\) we can prove the following result about the nature of solutions with \( c \) sufficiently close to 0.

**Proposition 6.31.**

(i) For any fixed \( b > 0 \) there exists \( \epsilon_b > 0 \) so that for \( c \in (-\epsilon_b, \epsilon_b) \) the smoothly-closing steady soliton \( S_{b,c} \) of Theorem 5.2 extends to a complete nontrivial steady gradient soliton on \( \Lambda^2 \mathbb{C} \mathbb{P}^2 \) with nonzero torsion \( \tau \).

(ii) For any \( c \neq 0 \in (-\epsilon_b, \epsilon_b) \) \( S_{b,c} \) is asymptotic to the unique torsion-free SU(3)-invariant \( G_2 \)-cone over \( \text{SU}(3)/\mathbb{T}^2 \) with rate \(-1\); all the torsion components \((\tau_1, \tau_2, \tau_3)\) are bounded functions and as \( c \to 0 \) they converge uniformly on compact subsets to 0.

We leave the details of the proof to the interested reader. Later in Theorem 6.37 we will give a complete characterisation of which members of the 1-parameter family of smoothly-closing steady solitons \( S_{b,c} \) are complete and of those which are asymptotically conical. Our proof does not rely on Proposition 6.31, nor is the result used elsewhere in the paper; however our proof of Theorem 6.37, among other things, uses the stability of the critical point \( c_{ctf} \) of the purely scale-invariant steady ODE system (6.26).
6.4. Complete and incomplete steady solitons. Proposition 6.31 showed that for any $c \neq 0$ sufficiently small the smoothly-closing steady soliton $S_{b,c}$ is complete and is asymptotic with rate $-1$ to the torsion-free $\text{SU}(3)$-invariant $G_2$-cone over $\text{SU}(3)/\mathbb{T}^2$. This naturally raises the question of whether the smoothly-closing steady solitons $S_{b,c}$ remain complete and asymptotically conical for all values of $c$. Numerical simulation strongly suggests that this is not the case and in fact suggests that the smoothly-closing steady solitons are:

- asymptotically conical when $\frac{c^2}{b^2} < \frac{9}{2}$;
- incomplete when $\frac{c^2}{b^2} > \frac{9}{2}$.

Based on this numerically-observed transition we were led to investigate steady solitons at the critical ratio $\frac{c^2}{b^2} = \frac{9}{2}$. By scaling and use of the discrete symmetries we can suppose that $b = \sqrt{2}$ and $c = 3$. In this case the small-$t$ power series expansion for the coefficient $f_1$ given in Appendix A specialises to

$$f_1 = t + \frac{t^3}{24} + \frac{t^5}{1920} + \frac{t^7}{322560} + \frac{t^9}{92897280} + \frac{t^{13}}{25505877196800} + \cdots$$

which we recognise as the opening terms in the Taylor series for the function $2 \sinh \frac{1}{2} t$ centred at $t = 0$. Inspection of the expansions for $f_2$ and $f_3$ also reveal expansions that are consistent with being the Taylor series for other hyperbolic trigonometric functions. This leads us to the following result.

**Theorem 6.32.** The smoothly-closing steady soliton $S_{\sqrt{2},3}$ is given explicitly by

$$(6.33) \quad f_1^2 = 2(\cosh t - 1) = 4 \sinh^2 \frac{t}{2}, \quad f_2^2 = 1 + e^t, \quad f_3^2 = 1 + e^{-t}.$$ 

It then follows from (6.2), (6.3) and (6.4) that $u$ and $f_1 f_2 f_3$ are given by

$$f_1 f_2 f_3 = 2 \sinh t, \quad u = \frac{e^t - 1}{e^t + 1} = \tanh \frac{t}{2},$$

that the $\tau_i$ are given by

$$\tau_1 = \frac{(e^t - 1)^3}{e^t (e^t + 1)} = 4 \tanh \frac{t}{2} \sinh^2 \frac{t}{2}, \quad \tau_2 = 2 + e^t, \quad \tau_3 = -(2 + e^{-t}),$$

and that

$$|\tau|^2 = \frac{6(e^{2t} + e^t + 1)}{(e^t + 1)^2} = 6 - \frac{3}{2} \text{sech}^2 \frac{t}{2}.$$ 

In particular, $S_{\sqrt{2},3}$ is complete and has exponential volume growth. Its scalar curvature decays exponentially fast and is asymptotic to $-3$. The vector field $X = u \partial_t$ decays exponentially fast to the constant vector field $\partial_s$.

**Proof.** By uniqueness of solutions to the initial value problem it suffices to verify by direct computation that the explicit triple $f$ given in (6.33) satisfies both the ODE system (6.6) and the initial conditions needed for $S_{\sqrt{2},3}$. The statement about scalar curvature follows immediately from (3.8).

**Remark 6.34.** The solution of the polynomial version of the ODE system (6.14) that corresponds to the explicit steady soliton given in (6.33) is

$$(6.35) \quad F_1 = \tanh \frac{t}{2}, \quad F_2 = \frac{e^t}{e^t - 1}, \quad F_3 = \frac{1}{e^t - 1}.$$ 

Note that $(F_1, F_2, F_3) \to (1, 1, 0)$ as $t \to +\infty$ and that the point $(1, 1, 0)$ is precisely the hyperbolic fixed point of (6.14) that we identified in Lemma 6.16 when $c = 3$. In other words, the triple $F$ given in (6.35) that corresponds to the solution $S_{\sqrt{2},3}$ yields a curve that belongs to the 2-dimensional local stable manifold of $(1, 1, 0)$. 
We now seek to establish that the condition (6.36b) on a nonnegative triple $F$ defines a smooth surface within the nonnegative octant; this surface contains the hyperbolic fixed point $p = (1, 1, 0)$ and its tangent space at $p$ coincides with the stable eigenspace of $p$.

The following theorem proves that smoothly-closing steady solitons do indeed behave exactly as the numerics suggested.

**Theorem 6.37.**

(i) For any $b \geq \sqrt{2}$ the smoothly-closing steady soliton $S_{b,3}$ is complete.

(ii) For any $b > \sqrt{2}$, $S_{b,3}$ is asymptotic with rate $-1$ to the torsion-free $G_2$-cone on $SU(3)/T^2$.

(iii) For any $b < \sqrt{2}$ the smoothly-closing steady soliton $S_{b,3}$ is incomplete.

**Proof.** (i) Motivated by the relation (6.36b), for any smoothly-closing steady soliton we define the following nonnegative function of $t$

\[(6.38) \quad \Lambda := F_1(F_2 + F_3) = \frac{1}{f_2^2} + \frac{1}{f_3^2}.
\]

Using the polynomial form of the steady ODE system (6.14) we compute (using the normalisation $c = 3$) that $\Lambda$ satisfies the differential equation

\[(6.39) \quad \Lambda' = -F_1 \left( \Lambda (1 - (F_2 - F_3)^2) + (1 - \Lambda)(F_2 - F_3)^2 \right).
\]

In particular, (6.39) implies that $\Lambda' \leq 0$ holds whenever $\Lambda \leq 1$ and moreover that if $\Lambda \leq 1$ holds then $\Lambda' = 0$ if and only if $F_2 - F_3 = 1$ and $\Lambda = 1$. (Note that these two conditions are the algebraic relations (6.36) satisfied by the explicit complete steady soliton). Hence the conditions $\Lambda \leq 1$ and $\Lambda < 1$ are both preserved forward in $t$. Next notice that any upper bound on $\Lambda$ at a given $t$, implies lower bounds for both $f_2$ and $f_3$ at that instant. In particular, by our extension result Proposition 6.21, to prove completeness it will suffice to prove that the quantity $\Lambda$ remains bounded above throughout the lifetime of the solution. To see this simply notice that the smoothly-closing steady soliton $S_{b,3}$ satisfies

\[\Lambda(0) = \frac{2}{b^2}\]

and hence for $b \geq \sqrt{2}$ the inequality $\Lambda(0) \leq 1$ holds. Therefore by our previous observations $\Lambda(t) \leq 1$ holds for all $t \geq 0$.

(ii) We now seek to establish that $S_{b,3}$ is asymptotically conical for any $b > \sqrt{2}$. (In fact, our argument will show the following more general result: any complete smoothly-closing steady soliton for which $f_3$ is unbounded is asymptotic to $C_{ct}$ with rate $-1$. In other words, the assumption $b > \sqrt{2}$ will be used only to derive completeness and the fact that $f_3$ is unbounded. Note in contrast that for the explicit steady soliton $S_{\sqrt{2},3}$ the function $f_3$ is strictly decreasing with values in $(1, \sqrt{2})$).

The argument proceeds in several steps which we now outline. First we establish that the coefficient $f_3$ is unbounded above as $t \to \infty$. This implies that all the coefficients $f_i \to +\infty$ as $t \to \infty$, because by Lemma 6.9 we already know that $f_1$ and $f_2$ both grow at least linearly. Next we want to show that the growth of $f_3$ is comparable to that of $f_1$ and $f_2$: for this it will suffice to show that the ratio $f_3/f_1$ (which by Lemma 6.25 is eventually bounded above by 1) is bounded away from 0 as $t \to \infty$. We achieve this by studying the ODE satisfied by the ratio $f_1^2/f_3^2$. Because by Lemma 6.9 $f_3$ has at most linear growth this forces all three components $f_i$ to grow linearly. Once we have shown linear growth of all the $f_i$ we can then prove that the solution must become asymptotic to the torsion-free cone $C_{ct}$ by showing that the corresponding solution of the purely scale-invariant

Observe also that these functions satisfy the following algebraic relations

\[(6.36a) \quad F_2 - F_3 = 1,
\]

\[(6.36b) \quad F_1(F_2 + F_3) = 1.
\]
ODE system (6.26) must eventually enter the basin of attraction of \( C_{ct} \).

We now provide the details of this argument. For \( b = \sqrt{2} \) we know that \( \Lambda \equiv 1 \). For any \( b > \sqrt{2} \) we claim that instead \( \lim_{t \to \infty} \Lambda(t) = 0 \). Since \( \Lambda(0) = 2/b^2 < 1 \) holds for any \( b > \sqrt{2} \) we know from (6.39) that \( \Lambda' < 0 \) holds for all \( t \geq 0 \). Since \( \Lambda \) is nonnegative and decreasing the limit \( \lim_{t \to \infty} \Lambda(t) = \Lambda_{\infty} \) exists and is nonnegative; we want to prove that \( \Lambda_{\infty} = 0 \) and (since \( f_2 > \frac{1}{2} t \)) proving this is equivalent to establishing that \( f_3 \) is unbounded. Moreover, knowing \( \lim_{t \to \infty} \Lambda(t) = \Lambda_{\infty} = 0 \) then implies that \( \lim_{t \to \infty} f_3^{−1} \) exists and is zero.

First we observe that \( F_1 \) cannot be bounded away from 0 as \( t \to \infty \). So suppose that for a contradiction that \( F_1 > \epsilon > 0 \) for all \( t \) sufficiently large. Notice that (6.39) can be rewritten as

\[
\Lambda' = -F_1 \left( \Lambda(1 - \Lambda) + (F_2 - F_3 - \Lambda)^2 \right),
\]

and therefore since \( 0 < \Lambda < 1 \) holds for all \( t \), the following differential inequality holds

\[
\frac{\Lambda'}{\Lambda(1 - \Lambda)} < -F_1 < -\epsilon.
\]

This inequality clearly implies that \( \Lambda \to 0 \) as \( t \to \infty \), but this yields a contradiction because \( \Lambda > F_1 F_2 > F_1^2 > \epsilon^2 \). Since \( F_1 \) is not bounded away from zero and

\[
F_1^2 = \frac{f_1^2}{f_2 f_3} \frac{1}{f_3} = r_{12} \frac{1}{f_3},
\]

the fact that \( r_{12} \) is bounded away from zero for \( t \) sufficiently large (Lemma 6.9) implies that \( f_3 \) must be unbounded as claimed. Hence as observed above actually \( \lim_{t \to \infty} f_3^{−1} \) exists and is zero.

Next we show that all of the coefficients \( f_i \) have comparable large-\( t \) growth and that in fact this common growth rate is linear. As explained in the proof sketch above this will follow from our previous results as soon as we establish a lower bound for the ratio \( f_3/f_1 \) as \( t \to \infty \). To this end, we consider \( r_{31} = \frac{f_2^3}{f_1} \) and find using (6.7) and the dominance of \( f_2 \) that the following differential inequality holds for all \( t \) sufficiently large

\[
(6.40) \quad \frac{d}{dt} \log r_{31} = 2 \left( \frac{f_1^2 - f_3^2}{f_1 f_2 f_3} \right) - \frac{3}{f_3} \left( \frac{f_1^2 - f_3^2}{f_2 f_3} \right) > \frac{1}{f_3} \left( -\frac{3}{f_3} + 2 \frac{f_1}{f_2} - 2 r_{31} \right) > \frac{1}{f_3} \left( -\frac{3}{f_3} + \frac{2}{C} - 2 r_{31} \right),
\]

where to obtain the second inequality we used the asymptotic lower bound \( r_{12} > C^2 \) for some \( C > 0 \). Since we proved that \( f_3^{−1} \to 0 \) as \( t \to \infty \) it follows that the right-hand side of this inequality is positive for any \( r_{31} \) sufficiently close to zero. Therefore \( \frac{d}{dt} \log r_{31} > 0 \) holds whenever \( r_{31} \) is sufficiently close to zero and hence \( r_{31} \) is bounded away from zero as \( t \to \infty \).

It remains to prove that in fact the solution must be asymptotic to the unique SU(3)-invariant torsion-free \( G_2 \)-cone. To prove this it suffices to show that the corresponding solution of the purely scale-invariant ODE system (6.26) must eventually enter the basin of attraction of the stable fixed point \( c_{ct} \). To see this, first we prove that all the scale-invariant torsion components \( T_i \) have vanishing limits as \( t \to \infty \). Since all the \( f_i \) grow linearly in \( t \) so does the scale-factor \( g = (f_1 f_2 f_3)^{1/3} \). Hence the vanishing of \( T_i = \tau_i/g \) will follow from boundedness of the (un-normalised) torsion components \( \tau_i \).

The latter follows immediately from (6.3) and the boundedness of all the ratios \( f_i/f_j \) already proven. Note also that since \( g \geq \frac{1}{2} t \) it follows from (6.27) and the infinite-\( t \) lifespan of the solutions that their \( s \)-lifetimes are also infinite. In terms of the variable \( s \) the equation satisfied by the scale-invariant average \( \overline{T^2} \) is

\[
\frac{d}{ds} \overline{T^2} = -\frac{2}{3} D(\mathcal{F}) + \mathcal{F}.
\]

Suppose that for a contradiction that \( \overline{T^2} \) remains bounded away from 3 as \( s \to \infty \). This is equivalent to supposing that there exists a constant \( \tilde{C} > 0 \) so that \( D(\mathcal{F}) > \tilde{C} \) holds for all \( s \) sufficiently large. The term \( \mathcal{F} \) on the right-hand side of the previous equation tends to 0 as \( s \to \infty \) (because
lim_{t \to \infty} J_i = 0 \text{ for all } i). Hence for } s \text{ sufficiently large we have } \frac{d}{ds} \sqrt{F} < -\frac{1}{2} \hat{C}, \text{ but since the solution has infinite } s\text{-lifetime this contradicts the fact that (by its definition) } \frac{d}{ds} \sqrt{F} \text{ is bounded below by } 3. \text{ The fact that as } s \to \infty, \sqrt{F} \text{ has } 3 \text{ as a limit point and that } J_i \to 0 \text{ for all } i \text{ implies that the solution must eventually enter the basin of attraction of the stable fixed point } c_{\text{st}}.

(iii) Recall that if } \Lambda \text{ remains bounded above throughout the lifetime of a smoothly-closing solution then that solution must be complete. To instead prove incompleteness of } S_{b,3} \text{ for } b < \sqrt{2} \text{ we want to establish a suitable growth estimate for } \Lambda \text{ valid throughout the lifetime of the solution. If we assume the solution to be complete then our growth estimate for } \Lambda \text{ will force } \Lambda \to \infty \text{ as } t \to \infty \text{ and hence that } f_3 \to 0 \text{ as } t \to \infty. \text{ But this contradicts part (iv) of Lemma 6.9, which guaranteed that } f_3 \text{ is bounded away from } 0 \text{ on any complete smoothly-closing steady soliton. More specifically, we claim that for any } b < \sqrt{2} \text{ the solution } S_{b,3} \text{ satisfies}

(6.41) \quad \frac{d}{dt} \log (\Lambda - 1) > \tanh \frac{1}{2} t,

throughout its lifetime. Clearly this forces } \Lambda \text{ to grow exponentially in } t.

To derive (6.41) first we rewrite the polynomial ODE system (6.14) entirely in terms of the quantities } \Lambda, F_2 - F_3 \text{ and } F_1. \text{ A calculation shows that (6.14) is equivalent to}

(6.42a) \quad D' = -\frac{1}{2} F_1 D (D - 1) + \frac{3\Lambda}{2 F_1} (\Lambda - D),

(6.42b) \quad \Lambda' = F_1 \left( (\Lambda - 1) D^2 - \Lambda (D - 1)^2 \right),

(6.42c) \quad F_1' = F_1^2 \left( D - \frac{3}{2} \right) + \frac{1}{2} \Lambda,

where to achieve a more compact presentation we have introduced the notation } D := F_2 - F_3. \text{ Note in particular that the first two equations of (6.42) are satisfied automatically when } D = F_2 - F_3 = 1 = \Lambda \text{ and that the third equation then reduces to}

F_1' = \frac{1}{2} (1 - F_1^2)

whose unique solution with } F_1(0) = 0 \text{ is } F_1 = \tanh \frac{1}{2} t. \text{ (This gives an alternative derivation of the solution } F \text{ of (6.14) corresponding to the explicit steady solution } S_{\sqrt{2},3}. \text{ However, without first having discovered the explicit steady soliton it seems unlikely we would have thought to recast (6.14) in terms of } \Lambda, D \text{ and } F_1). \text{ The small-}t \text{ expansions for } S_{b,3} \text{ given in Appendix A imply that}

\Lambda(0) = D(0) = \frac{2}{b^2}, \quad \Lambda - D = \frac{3(2 - b^2)}{5 b^6} t^2 + \text{h.o.t.}

Hence for any } b^2 < 2 \text{ we have the following inequalities for } t > 0 \text{ sufficiently small}

(6.43) \quad \Lambda > D > 1.

We claim that the condition (6.43) persists throughout the lifetime of the solution. Granted this claim (proven below in Lemma 6.45) we can establish (6.41) as follows. First rewrite (6.42b) as

\Lambda' = \left( (\Lambda - 1) D^2 - \Lambda (D - 1)^2 \right) F_1 = (-D^2 + 2D\Lambda - \Lambda) F_1.

Since the quadratic } q(D) = -D^2 + 2D\Lambda - \Lambda \text{ is an increasing function of } D \in [1, \Lambda], \text{ (6.43) implies}

(6.44) \quad \Lambda' > q(1) F_1 = (-1 + 2\Lambda - \Lambda) F_1 = (\Lambda - 1) F_1.

(In particular, this implies that } \Lambda \text{ is strictly increasing on any connected interval } I \text{ on which (6.43) holds). Furthermore, from (6.42c) we observe that}

F_1' = F_1^2 \left( D - \frac{3}{2} \right) + \frac{1}{2} \Lambda > \frac{1}{2} (1 - F_1^2)
where the inequality holds because of (6.43). Hence $F_1 > \tanh \frac{1}{2}t$ holds throughout the lifetime of the solution; combining this with (6.44) yields (6.41).

To complete the proof it remains only to prove our claim about the persistence of the inequalities (6.43). We state and prove this in the following separate lemma. □

**Lemma 6.45.** Suppose that a solution of (6.42) satisfies the inequality
\begin{equation}
\Lambda > D > 1
\end{equation}
at some instant $t_0$, then it must satisfy (6.46) for the remainder of its lifetime. In particular, any solution of (6.42) that arises from a smoothly-closing steady soliton $S_{b,3}$ with $b < \sqrt{2}$ satisfies (6.46) throughout its lifetime.

**Proof.** Suppose for a contradiction that (6.46) eventually fails and that $t > t_0$ is the first instant at which it fails. As we observed immediately after (6.44), $\Lambda$ must be increasing on the interval $[t_0, t)$ and hence $\Lambda(t) > 1$. So $t$ must satisfy one of the following two conditions:

(i) $D(t) = 1$, or
(ii) $\Lambda(t) = D(t)$.

In each case we will derive a contradiction. In the first case we have $D > 1$ on $[t_0, t)$ and $D(t) = 1$, so $D'(t) \leq 0$. On the other hand, it follows from (6.42) that at any point where $D = 1$ we have
\begin{equation}
D' = \frac{3}{2F_1} \Lambda(\Lambda - 1).
\end{equation}
Hence $D'(t) > 0$ since $\Lambda(t) > 1$.

In the second case we have $\Lambda - D > 0$ on $[t_0, t)$ and $(\Lambda - D)(t) = 0$, so $(\Lambda - D)'(t) \leq 0$. Again using (6.42) we find that at any point where $D = \Lambda$
\begin{equation}
(\Lambda - D)' = \frac{3}{2} \Lambda(\Lambda - 1)F_1.
\end{equation}
Hence $(\Lambda - D)'(t) > 0$ because $\Lambda(t) > 1$. □

**Remark 6.47.** The same type of argument proves the following analogue of Lemma 6.45:

*If a solution of (6.42) satisfies $\Lambda < D < 1$ at some instant $t_0$ then it continues to satisfy the same inequalities for the remainder of its lifetime. In particular, any solution of (6.42) that arises from a smoothly-closing steady soliton $S_{b,3}$ with $b > \sqrt{2}$ satisfies $\Lambda < D < 1$ for all $t > 0$."

Our proof that smoothly-closing steady solitons $S_{b,3}$ with $b > \sqrt{2}$ are all asymptotically conical already crucially used the fact that the condition $\Lambda < 1$ is preserved; we did not directly use that the remaining inequalities $D < 1$ and $\Lambda < D$ hold for all $t$.

### 7. Comparisons to Ricci solitons

**Cohomogeneity-one steady Ricci solitons on $\Lambda^2 S^4$ and $\Lambda^2 CP^2$.** In order to compare the behaviour of Laplacian flow with Ricci flow in dimension 7 it is natural to ask what one can say about cohomogeneity-one Ricci solitons in dimension 7. Given the examples we have studied in this paper it is particularly natural to compare $G$-invariant Ricci solitons on $\Lambda^2 S^4$ and $\Lambda^2 CP^2$ for $G = Sp(2)$ and $G = SU(3)$ respectively to the complete Laplacian solitons we have found on those spaces. We describe known results in this direction.

We begin with results in the $Sp(2)$-invariant setting. We are not aware of any results or numerical evidence either for or against the existence of complete $Sp(2)$-invariant gradient AC Ricci shrinkers on $\Lambda^2 S^4$. However for complete $Sp(2)$-invariant steady solitons one has the following result.

**Theorem 7.1.** [57, Theorem 3.1] *There exists a 1-parameter family of complete $Sp(2)$-invariant gradient steady Ricci solitons on $\Lambda^2 S^4$. The metric coefficients $f_1$ and $f_2$ both grow asymptotically...*
like \( \sqrt{t} \) where \( t \) is the arclength parameter along a unit-speed geodesic normal to every principal orbit and the potential for the soliton vector field has linear growth with asymptotic slope \(-1\).

This statement was conjectured by Buzano–Dancer–Gallaugher–Wang [17] based on numerical investigations they conducted. In fact, they considered a higher-dimensional generalisation, namely to cohomogeneity-one manifolds of dimension \( 4m + 3 \) that are 3-dimensional disc bundles over \( \mathbb{HP}^m \) for \( m \geq 1 \). In this case the defining triple of groups \( K \subset H \subset G \) is \( G = \text{Sp}(m + 1) \), \( H = \text{Sp}(m) \times \text{Sp}(1) \) and \( K = \text{Sp}(m) \times \text{U}(1) \). The principal orbit is therefore diffeomorphic to \( \mathbb{CP}^{2m+1} \) and the singular orbit is \( \mathbb{HP}^m \). They conjectured that the same results as stated above hold for any \( m \geq 1 \). Wink proved this conjecture for \( m \geq 3 \) [56, Theorem A], and subsequently he proved the conjecture for all \( m \geq 1 \) [57, Theorem 3.1] by a different method.

In the \( SU(3) \)-invariant case we are aware only of the following recent result of H. Chi for Ricci-flat metrics (with generic holonomy). In particular, nontrivial \( SU(3) \)-invariant steady Ricci solitons do not yet seem to have been studied.

**Theorem 7.2.** [21, Theorems 1.2 & 1.5] For any \( c \in \mathbb{R} \) there is a unique (up to scale) smoothly-closing \( SU(3) \)-invariant Ricci-flat metric \( \mathcal{RF}_c \) defined on a neighbourhood of the zero-section of \( \Lambda^2 \mathbb{CP}^2 \) satisfying the initial conditions

\[
    f_1 = t + O(t^3), \quad f_2 = 1 + ct + O(t^2), \quad f_3 = 1 - ct + O(t^2).
\]

There exists \( \epsilon > 0 \) so that for \( |c| < \epsilon \) the smoothly-closing metric \( \mathcal{RF}_c \) extends to a complete \( SU(3) \)-invariant AC Ricci-flat metric on \( \Lambda^2 \mathbb{CP}^2 \) asymptotic to the unique \( SU(3) \)-invariant torsion-free \( G_2 \)-cone. The metric \( \mathcal{RF}_0 \) is the Bryant–Salamon \( G_2 \)-metric on \( \Lambda^2 \mathbb{CP}^2 \), while for any \( c \neq 0 \) the Ricci-flat metric \( \mathcal{RF}_c \) has generic holonomy.

We note that Chi’s proof of Theorem 7.2 does not give any information about the completeness (or otherwise) of \( \mathcal{RF}_c \) for \( |c| \) large; nor are any numerical results presented in [21] that shed any light on this question. In this sense his result is closely analogous to our initial stability result, Proposition 6.31, for nontrivial steady solitons close to the Bryant–Salamon torsion-free AC \( G_2 \)-structure. Recall however that in our case Theorem 6.37 provided a complete understanding of for which parameter values our \( SU(3) \)-invariant smoothly-closing steady solitons lead to asymptotically conical solitons and also what type of degeneration occurs at the boundary of the space of smoothly-closing AC solutions. Since Ricci-flat metrics have at most Euclidean volume growth, any degeneration that could occur in Chi’s family of smoothly-closing \( SU(3) \)-invariant Ricci-flat metrics would necessarily be quite different from what we encountered in the steady soliton case.

**Appendix A. The formal power series solutions for smoothly-closing invariant solitons**

**SU(3)-invariant solitons.** A computer-assisted symbolic computation of the power series expansions around \( t = 0 \) of the 2-parameter family of smoothly-closing solitons constructed in Theorem 5.2 in terms of the real parameters \( b, c \) and \( \lambda \) was performed up to high order. The first several terms
are listed below

\[ f_1 = t - \frac{t^3}{54b^4} (4\lambda b^4 + 9b^2 - 3c^2) + \frac{t^5}{48600b^8} (464b^8 \lambda^2 + 2844b^6 \lambda - 972b^4 c^2 \lambda + 4050b^4 - 2322b^2 c^2 + 321c^4) + \ldots \]

\[ f_2 = b + \frac{c}{6b} t + \frac{t^2}{72b^3} (4\lambda b^4 + 18b^2 - c^2) - \frac{ct^3}{6480b^5} (152\lambda b^4 + 126b^2 - 63c^2) + \ldots \]

\[ f_3 = b - \frac{c}{6b} t + \frac{t^2}{72b^3} (4\lambda b^4 + 18b^2 - c^2) + \frac{ct^3}{6480b^5} (152\lambda b^4 + 126b^2 - 63c^2) + \ldots \]

\[ \tau_1 = -\frac{2(2\lambda b^4 - c^2)}{9b^4} t^3 + \frac{2c}{405b^5} (26b^8 \lambda^2 + 81b^6 \lambda - 40b^2 c^2 \lambda - 54b^2 c^2 + 12c^4) + \ldots \]

\[ \tau_2 = c + \frac{2(\lambda b^4 + c^2)}{9b^2} t - \frac{ct^2}{54b^4} (5\lambda b^4 - 4c^2) - \frac{t^3}{1215b^6} (8b^8 \lambda^2 + 18b^6 \lambda + 92b^4 c^2 \lambda + 99b^2 c^2 - 42c^4) + \ldots \]

\[ \tau_3 = -c + \frac{2(\lambda b^4 + c^2)}{9b^2} t + \frac{ct^2}{54b^4} (5\lambda b^4 - 4c^2) - \frac{t^3}{1215b^6} (8b^8 \lambda^2 + 18b^6 \lambda + 92b^4 c^2 \lambda + 99b^2 c^2 - 42c^4) + \ldots \]

\[ \tau = \frac{4(\lambda b^4 + c^2)}{9b^2} t - \frac{4c^3}{1215b^6} (4b^8 \lambda^2 + 144b^6 \lambda + 46b^4 c^2 \lambda - 18b^2 c^2 - 21c^4) + \ldots \]

\[ u = -\frac{7(\lambda b^4 - 2c^2)}{9b^4} t + \frac{2c^3}{1215b^6} (26b^8 \lambda^2 + 126b^6 \lambda - 61b^4 c^2 \lambda - 117b^2 c^2 + 21c^4) + \ldots \]

Note the following hold for \( t > 0 \) sufficiently small

- If \( b \) and \( c \) are assumed to be positive then we have the ordering \( f_1 < f_3 < f_2 \).
- If \( \lambda \leq 0 \) then \( u \) and \( \tau_1 \) are positive.
- If \( \lambda \geq 0 \) then \( \tau \) is positive.

**Steady solitons.** In the steady case \( \lambda = 0 \) these power series specialise to

\[ f_1 = t - \frac{t^3}{18b^4} (3b^2 - c^2) + \frac{(1350b^4 - 774b^2 c^2 + 107c^4)}{16200b^8} t^5 + \ldots \]

\[ f_2 = b + \frac{c}{6b} t + \frac{t^2}{72b^3} (18b^2 - c^2) - \frac{7ct^3}{720b^5} (2b^2 - c^2) + \frac{(-2700b^4 + 636b^2 c^2 + 7c^4)}{51840b^7} t^4 + \ldots \]

\[ f_3 = b - \frac{c}{6b} t + \frac{t^2}{72b^3} (18b^2 - c^2) + \frac{7ct^3}{720b^5} (2b^2 - c^2) + \frac{(-2700b^4 + 636b^2 c^2 + 7c^4)}{51840b^7} t^4 + \ldots \]

\[ \tau_1 = \frac{2c^2}{9b^4} t^3 + \frac{4c^2 t^3}{135b^8} (11b^2 - 3c^2) + \ldots \]

\[ \tau_2 = c + \frac{2c^2}{9b^2} t + \frac{2c^3 t^2}{27b^4} + \frac{c^3 t^3}{405b^6} (-33b^2 + 14c^2) - \frac{2c^3 t^4}{405b^8} (11b^2 - 3c^2) + \ldots \]

\[ \tau_3 = -c + \frac{2c^2}{9b^2} t - \frac{2c^3 t^2}{27b^4} + \frac{c^3 t^3}{405b^6} (-33b^2 + 14c^2) + \frac{2c^3 t^4}{405b^8} (11b^2 - 3c^2) + \ldots \]

\[ \tau = \frac{4c^2}{9b^2} t^2 + \frac{4c^2 t^3}{405b^6} (6b^2 + 7c^4) + \ldots \]

\[ u = -\frac{2c^2}{9b^4} t + \frac{2c^2 t^3}{405b^8} (-39b^2 + 7c^2) + \ldots \]

Note that the coefficient of \( t^k \) in any of the \( f_i \) or \( \tau_i \) has the general structure \( t^k b^{1-k} P_k(b/c) \) for some 1-variable polynomial \( P_k \) in the variable \( b/c \) with rational coefficients. This is consistent with the scaling behaviour of solitons and the fact that rescaling a steady soliton yields another steady soliton.
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