Individual consistency of 2-events quantum histories

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It is argued that the property of consistency of consistent history approach to quantum physics is not an individual property, in the sense that when such consistency holds, it cannot be attributed to each single sample of the physical system. This fact is not a logical inconsistency but it is in striking contrast with the physical idea of consistency. In this letter we introduce a meaningful notion of consistency, named self-decoherence, based on the concept of mirror projection. We prove that self-decoherence satisfies our tentative criterion of individuality. Furthermore, it is proved that self-decoherence forbids contrary inferences.

In 1984 R.B. Griffiths [1] proposed a reinterpretation of quantum formalism with the aim of giving a solution to the “well-known conceptual difficulties which arise in various interpretations of quantum mechanics”. While standard quantum theory is based on the concept of *event*, represented by a projection operator $E$ of the Hilbert space $\mathcal{H}$ describing the system, the *consistent history approach* (CHA) is based on the concept of *history*, which is any finite ordered sequence
$h = (E_1, E_2, ..., E_n)$ of events. The CHA provides the framework in which it is possible to establish whether histories have physical meaning [2]. Such framework is made up of suitable families of histories. Let $E_1, E_2, ..., E_n$ be finite resolutions of the identity, i.e. $E_k = \{E_k^{(1)}, E_k^{(2)}, ..., E_k^{(i_k)}\}$, where $E_k^{(i)} \perp E_k^{(j)}$ if $i \neq j$ and $\sum_{i=1}^{i_k} E_k^{(i)} = 1$.

A family $C$ of histories is the set of all histories $h = (E_1, E_2, ..., E_n)$ such that $E_k = \sum_{i} E_k^{(i)}$ for a fixed $n$-uple $E_1, E_2, ..., E_n$ of resolutions of the identity. When every event $E_k$ constituting a history $h$ is just an event of $E_k$, i.e. if $E_k \in E_k$ for all $k = 1, 2, ..., n$, then $h$ is called elementary history. Hence the set $E$ of all elementary histories of $C$ is the cartesian product $E = E_1 \times E_2 \times \cdots \times E_n$. Two histories $h_1 = (E_1, E_2, ..., E_n), h_2 = (F_1, F_2, ..., F_n) \in C$ are summable if they differ in only one place, say $k$, hence $E_j = F_j$ for all $j \neq k$, and $E_k \perp F_k$; in such a case their sum is $h_1 + h_2 = (E_1, E_2, ..., E_k + F_k, ..., E_n) \in C$.

The histories $h_1$ and $h_2$ are said to be alternative if there is $k$ such that $E_k \perp F_k$.

Let $h = (E_1, E_2, ..., E_n)$ be a commutative history, i.e. all $E_k$ commute with each other. According to quantum theory, the statement “$h$ occurs” means that all events $E_1, E_2, ..., E_n$ occur in the given order. Therefore, $h$ is identified with the single event $E_1 \cdot E_2 \cdots E_n = E_1 \land E_2 \land \cdots \land E_n$. Though the mathematical notions of CHA are given within the standard quantum theoretical formalism, quantum theory is unable to consider and describe the occurrence of a history when it is not commutative. On the contrary, according to CHA, the histories of a family $C$ have physical meaning whenever a condition of consistency is satisfied, which allows to assign a probability of occurrence $p(h)$ to every $h \in C$. According to such idea of consistency, the occurrence of an elementary history must imply the non-occurrence of every other elementary history. Therefore, if there is a probability
$p(h)$ of occurrence of $h$, then it must satisfy the sum rule

\[(C.0) \quad p \left( \sum_j h_j \right) = \sum_j p(h_j); \quad \sum_{h \in \mathcal{E}} p(h) = 1.\]

Moreover, the empirical validity of the theory requires that such probability should be consistent with the probability assigned to single events by quantum theory. Then, another condition for $p$ is

\[(C.1) \quad \text{whenever } h = (E_1, E_2, \ldots, E_n) \text{ and } [E_j, E_k] = 0 \text{ then}
\]

\[p(h) = Tr(E_n E_{n-1} \cdots E_1 \rho),\]

where $\rho$ is the density operator such that $Tr(E \rho)$ is the quantum probability of occurrence of the event $E$.

Condition $(C.1)$ is satisfied if $p$ is the functional $p: \mathcal{C} \to [0,1], p(h) = Tr(C_h \rho C_h^*)$, where $C_h = E_n E_{n-1} \cdots E_1$. Such $p$ satisfies also $(C.0)$ if and only if $[2]$

\[Re[Tr(C_{h_1} \rho C_{h_2}^*)] = 0 \quad \text{for all summable } h_1, h_2 \in \mathcal{E}. \quad (1)\]

When (1) holds, $\mathcal{C}$ is said to be weakly decohering.

**Definition 1.** A family of histories $\mathcal{C}$ is said to be consistent with respect to $\rho$ if it is weakly decohering.

The following P1 and P2 are the basic principles of CHA.

**P1:** All predictions about the physical system are those obtained by interpreting $p(h) = Tr(C_h \rho C_h^*)$ as probability of occurrence of $h$, within a consistent family $\mathcal{C}$.

The notion of family of histories of CHA turns out to be a generalization of the notion of observable of standard quantum theory; this last can be recovered within CHA by considering families of one-event histories $h = (E)$, i.e. generated by only one resolution of the identity. As well as in standard quantum theory it is not possible to
non-contextually assign values to all observables [3], in CHA it is not possible to assign the occurring histories in all consistent families together, without giving rise to contrary inferences, i.e. to contradictions of Kochen-Specker type [4]. This is the content of the single family rule:

P2: the occurrence or the non-occurrence of a history h can be considered only within a single consistent family C, i.e. when h ∈ C and C is weakly decohering.

The correct use of the basic principles of CHA makes it possible to recover all results of standard quantum theory, avoiding important conceptual difficulties [2].

The question we face in the present paper is whether the consistency of a given family C is a property to be attributed to every single sample of the physical system or not. From the point of view of our intuition, given a consistent family C and a history h ∈ C, for each individual sample of the physical system there are two mutually exclusive alternatives: either h occurs or h does not occur. Therefore, the physical idea of consistency which is at the root of CHA suggests that consistency should be an individual property.

In quantum theory and in CHA there are properties which are individual and also properties which are not individual. For instance, the property of having a given value c of an observable C is individual. The following example shows that the consistency of C in definition 1 is not an individual property.

Example 1. – Let us consider two density operators ρ₁ = |ψ₁⟩⟨ψ₁| and ρ₂ = |ψ₂⟩⟨ψ₂|, where ψ₁ and ψ₂ are two mutually orthonormal vectors of H. Let ϕ = \frac{1}{\sqrt{2}}(ψ₁ + ψ₂) be a third unit vector. If we put E₁ = |φ⟩⟨φ|, then E₁ψ₁ = \frac{1}{2}(ψ₁ + ψ₂) = E₁ψ₂ and E₁′ψ₁ = −E₁′ψ₂,
where $E'_1 = 1 - E_1$. Therefore, taking $\rho = \frac{1}{2} [\rho_1 + \rho_2]$, we have

$$\text{Tr}(E_2 E_1 \rho E'_1 E_2) = \frac{1}{2} [\langle E'_1 \psi_1 | E_2 E_1 \psi_1 \rangle + \langle E'_1 \psi_2 | E_2 E_1 \psi_2 \rangle] = 0 \quad (2)$$

for all projections $E_2$. Since $\text{Tr}(E'_2 E_1 \rho E_1 E_2) = 0$ whatever $E_2$, the family of histories $C$ generated by the history $(E_1, E_2)$ is consistent, whatever the projection operator $E_2$. This $E_2$ can be chosen in such a way that $C$ turns out to be consistent neither with respect to $\rho_1$, nor with respect to $\rho_2$. Indeed, by representing vectors and operators of $\mathcal{H}$ with respect to any fixed orthonormal basis $(u_n)_{n \in \mathbb{N}}$ so that $u_1 = \psi_1$ and $u_2 = \psi_2$, we have $\psi_1 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\psi_2 \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $E_1 \equiv \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let us consider the histories $h_1 = (E_1, E_2)$, $h_2 = (1 - E_1, E_2)$ and $h = h_1 + h_2 = (1, E_2)$, where $E_2 \equiv \begin{bmatrix} \cos^2 \frac{\theta}{2} & -i \frac{1}{2} \sin \theta & 0 \\ i \frac{1}{2} \sin \theta & \sin^2 \frac{\theta}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, with $0 < \theta < \frac{\pi}{2}$.

Then, $\text{Tr}(C_h \rho_1 C_h^*) = \cos^2 \frac{\theta}{2}$, while $\text{Tr}(C_{h_1} \rho_1 C_{h_1}^*) = \text{Tr}(C_{h_2} \rho_1 C_{h_2}^*) = \frac{1}{4}$, and this implies $\text{Tr}(C_{h_1 + h_2} \rho_1 C_{h_1 + h_2}^*) \neq \text{Tr}(C_{h_1} \rho_1 C_{h_1}^*) + \text{Tr}(C_{h_2} \rho_1 C_{h_2}^*)$. The same argument applied to $\rho_2$ shows that $\text{Tr}(C_{h_1 + h_2} \rho_2 C_{h_1 + h_2}^*) \neq \text{Tr}(C_{h_1} \rho_2 C_{h_1}^*) + \text{Tr}(C_{h_2} \rho_2 C_{h_2}^*)$. Therefore the family $C$ generated by $h_1$ and $h_2$ is not consistent with respect to $\rho_1$ and $\rho_2$, but it becomes consistent by mixing together the two statistical ensembles represented by $\rho_1$ and $\rho_2$, i.e. with respect to the mixture $\rho = \frac{1}{2} [\rho_1 + \rho_2]$.

Example 1 suggests the following tentative definition of what is an “individual property” of the physical system.

**Definition 2 – A property $\pi$ is individual for a quantum system if the following statement holds.**

If $\pi$ does not hold when the system is described by $\rho_1$ or $\rho_2$, then $\pi$ does not hold when the system is described by any mixture $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$, with $0 < \lambda < 1$. 


It must be said that several notions of consistency other than weak decoherence have been introduced in literature to achieve a more strict adherence with the idea of consistency.

M. Gell-Mann and J.B. Hartle [5] introduced the stronger notion of medium decoherence: a family $C$ has the property of medium decoherence if $\text{Tr}(C_{h_1}\rho C_{h_2}^*) = 0$ for all alternative $h_1, h_2 \in C$. Now, from (3) it follows that the family $C$ of example 1 has the property of medium decoherence with respect to $\rho$; but with respect to $\rho_1$ and $\rho_2$ it is not weakly decohering and therefore even medium decoherence does not hold. Thus, medium decoherence is not an individual property.

The linearly positive decoherence proposed by S. Goldstein and D.N. Page [6] consists in requiring that $\text{Re}\{\text{Tr}(C_h\rho)\} \geq 0$ for all $h \in C$; it is weaker than weak decoherence. Therefore, the family $C$ of example 1 is also linearly positive with respect to $\rho$, whatever $E_2$. We can choose $E_2$ so that $C$ is not linearly positive with respect to $\rho_1$. Let us consider the projection operator

$$E_2 = \begin{bmatrix}
\cos^2 \frac{\theta}{2} & \frac{1}{2}e^{-i\alpha}\sin \theta & 0 \\
\frac{1}{2}e^{i\alpha}\sin \theta & \sin^2 \frac{\theta}{2} & 0 \\
0 & 0 & 0
\end{bmatrix},$$

and the history $h_1 = (E_1, E_2)$. We have

$$\text{Tr}(C_{h_1}\rho_1) = \langle \psi_1 | E_2 E_1 \psi_1 \rangle = \frac{1}{2} \left( \cos^2 \frac{\theta}{2} + e^{-i\alpha}\sin \frac{\theta}{2} \cos \frac{\theta}{2} \right).$$

Therefore, for $0 < \theta < \frac{\pi}{2}$ the condition $\text{Re}(\text{Tr}(C_{h_1}\rho_1)) \geq 0$ of linear positivity becomes $\cos^2 \frac{\theta}{2} + \cos \alpha \sin \frac{\theta}{2} \geq 0$ and it can be violated by a suitable choice of $\theta$ and $\alpha$. Thus, also linear positivity violates the individuality condition.

Now we consider the ordered consistency introduced by A. Kent to avoid contrary inferences [4]. Following A. Kent we define the ordering $h_1 \leq h_2$ iff $E_k \leq F_k$ for all $k$, where $h_1 = (E_1, E_2, ..., E_k, ...)$ and $h_2 = (F_1, F_2, ..., F_k, ...)$. A history $h_1$ is said ordered consistent if $h_1 \leq h_2$. 
implies $\text{Tr}(C_{h_1} \rho C_{h_1}^*) \leq \text{Tr}(C_{h_2} \rho C_{h_2}^*)$, where both $h_1$ and $h_2$ belong to two medium decohering families. When all histories of a medium decohering family $\mathcal{C}$ are ordered consistent, then $\mathcal{C}$ is said to be ordered consistent. Not even ordered consistency is individual. Indeed, if we take $\mathcal{H} = \mathbb{C}^2$ in example 1, then $\mathcal{C}$ must be ordered consistent with respect to $\rho$, but it does not with respect to $\rho_1$ and $\rho_2$ because it is not weakly decohering.

The lack of individuality exhibited by all these notions of consistency is in striking contrast with the idea of consistency of which they should be the mathematical representation. However, this is not a problem for the logical coherence of the theories, but, rather, it reflects their inability in implementing the individuality of consistency.

Furthermore, the fact that all notions of consistency so far proposed are not individual gives rise to the suspect that individual consistency is a chimera.

Now we show that on the contrary, at least for 2-events histories, a meaningful notion of individual consistency exists, which we call self-decoherence. It is stronger than medium decoherence. Furthermore, contrary inferences are forbidden by self-decoherence.

Our proposal is based on the concept of mirror projection [7]. Given a 2-event history $h = (E_1, E_2)$ and a density operator $\rho$, a projection operator $T$ is a mirror projection for $(h, \rho)$ if

M1. $[T, E_1] = [T, E_2] = 0$,

M2. $\text{Tr}(TE_1 \rho) = \text{Tr}(T \rho) = \text{Tr}(E_1 \rho)$.

To understand the physical meaning of the mirror projection, we notice that, since (by (M1)) $T$ commutes with $E_1$, we may compute the quantum conditional probabilities $p(T \mid E_1) = \frac{\text{Tr}(TE_1 \rho)}{\text{Tr}(T \rho)}$ and $p(E_1 \mid T) = \frac{\text{Tr}(TE_1 \rho)}{\text{Tr}(T \rho)}$, which are both 1 because of (M2). Therefore, the events $T$ and $E_1$ are directly correlated: $T$ occurs iff $E_1$ occurs.
Given the history \( h = (E_1, E_2) \) with \([E_1, E_2] \neq 0\), standard quantum theory is unable to describe the occurrence of \( h \). The existence of a mirror projection \( T \) for \((h, \rho)\) allows to introduce the following notion of occurrence of \( h \).

\((oc)\) The history \( h \) occurs if both events \( T \), which is directly correlated to \( E_1 \), and \( E_2 \) occur.

Then we are led to the following notion of consistency:

**Definition 3.** A family \( \mathcal{C} \) of 2-event histories is said self-decohering with respect to \( \rho \) if there is a mirror projection for \((h, \rho)\), for all \( h \in \mathcal{C} \).

Interesting physical situations may be described by self-decohering histories. The following example was suggested by some works of M.O. Scully, B-G. Englert and H. Walter [8].

**Example 2.** – Let us consider the two-slits experiment for a particle which possesses, besides the spatial degrees of freedom \((x_1, x_2, x_3) = x\), an internal degree of freedom \( s \) corresponding to a dichotomic observable \( S \) with spectrum \( \sigma(S) = \{1, 0\} \). Such a system is described in the Hilbert space \( L_2(\mathbb{R}^3) \otimes \mathbb{C}^2 \). The event “the particle goes through slit 1 (resp., 2)” is represented by the projection operator \( E_1 \) (resp., \( F_1 \)). Given any interval \( \Delta \) on the final screen, by \( E_2 \) we denote the projection operator which represents the event “the particle hits the final screen in a point within \( \Delta \)”. Therefore \( h_1 = (E_1, E_2) \) and \( h_2 = (F_1, E_2) \) are non-commutative histories which generate a family \( \mathcal{C} \). Now suppose that the state vector of the particle is \( \Psi = \frac{1}{\sqrt{2}}[\psi_1 \otimes |1\rangle + \psi_2 \otimes |0\rangle] \), where \( \psi_1 \) (resp. \( \psi_2 \)) is a spatial wave function localized in slit 1 (resp., 2) when the particle is in the two-slits’ region. Therefore

\[
E_1 \psi_1 = \psi_1, \quad F_1 \psi_2 = \psi_2, \quad E_1 \psi_2 = F_1 \psi_1 = 0.
\]

In this situation the projection operators \( T = |1\rangle \langle 1| \) and \( U = |0\rangle \langle 0| \) are mirror projections for \((h_1, |\Psi\rangle \langle \Psi|)\) and \((h_2, |\Psi\rangle \langle \Psi|)\). Therefore the family \( \mathcal{C} \) is self-decohering, so that the history \( h_1 \) (resp., \( h_2 \)) may be
interpreted as “the particle hits the final screen in $\Delta$ passing through slit 1 (resp., 2)”. Actually, the which-slit test can be performed for each individual sample of the physical system by measuring together $E_2$, $T$ and $U$. Then we assign history $h_1$ ($h_2$) to that sample if both $E_2$ and $T$ ($U$) yield a positive outcome.

Now we prove that self-decoherence is an individual property. Let us suppose that (M2) holds for $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$. From $Tr(E_1 T \rho) = Tr(T \rho)$ we get

$$\lambda Tr[(T - E_1 T) \rho_1] + (1 - \lambda) Tr[(T - E_1 T) \rho_2] = 0. \quad (3)$$

The traces in this equation are non-negative because $E_1 T \leq T$. Therefore (3) implies $Tr[(T - E_1 T) \rho_1] = Tr[(T - E_1 T) \rho_2] = 0$. In a similar way, $Tr[(E_1 - E_1 T) \rho_1] = Tr[(E_1 - E_1 T) \rho_2] = 0$ follows from $Tr(E_1 T \rho) = Tr(E_1 \rho)$. Then $T$ must be a mirror projection for both $(h, \rho_1)$ and $(h, \rho_2)$. Thus individuality condition is satisfied by self-decoherence.

Now we prove that medium decoherence, and hence weak decoherence, hold in a self-decohering family. We limit ourselves to pure density operators $\rho = |\psi\rangle\langle\psi|$: the extension to general density operators is straightforward.

**Proposition 1.** If $T$ and $U$ are mirror projections respectively for $(h_1 = (E_1, E_2), \rho)$, $(h_2 = (F_1, E_2), \rho)$, where $\rho = |\psi\rangle\langle\psi|$, then the following statement holds.

$$E_1 \perp F_1 \quad \text{implies} \quad \langle \psi | E_1 E_2 F_1 \psi \rangle = 0. \quad (4)$$

**Proof.** Let $T$ and $U$ be mirror projections for $(h_1, \rho)$ and $(h_2, \rho)$, respectively, and let $T \lor U$ denote the projection operator which is the least upper bound of $T$ and $U$. If $E_1 \perp F_1$, by (M2) we get [9]

$$T\psi \perp U\psi, \quad (T \lor U)\psi = T\psi + U\psi, \quad T\psi = (T \lor U)\psi - U\psi. \quad (5)$$
Therefore,
\[
\langle \psi \mid E_1 E_2 F_1 \psi \rangle = \langle T \psi \mid E_2 U \psi \rangle = \langle (T \lor U) \psi \mid E_2 U \psi \rangle - \langle U \psi \mid E_2 U \psi \rangle \\
= \langle \psi \mid (T \lor U) E_2 U \psi \rangle - \langle \psi \mid E_2 U \psi \rangle \\
= \langle \psi \mid E_2 (T \lor U) U \psi \rangle - \langle \psi \mid E_2 U \psi \rangle \\
= \langle \psi \mid E_2 U \psi \rangle - \langle \psi \mid E_2 U \psi \rangle = 0.
\]
In the fourth equation we have used the fact that since \(E_2\) commutes with both \(T\) and \(U\), then \(E_2\) must commute with \(T \lor U\) (see, for instance, theorem 2.24 in [10]). Thus, proposition 1 is proved.

Individuality is not sufficient to assign the meaning of consistency to self-decoherence. A sensible notion of consistency should satisfy conditions (C.0) and (C.1). Now, if \(C\) is self-decohering, the probability of occurrence of \(h = (E_1, E_2) \in C\) which agrees with \((oc)\) is \(p(E_1, E_2) = Tr(E_2 T \rho) = Tr(E_2 E_1 \rho)\). Therefore, it satisfies both (C.0) and (C.1). Furthermore, because of (M1) and (M.2) we have \(p(h) = Tr(E_2 T \rho) = Tr(E_2 T \rho T E_2) = Tr(E_2 E_1 \rho E_1 E_2) = Tr(C_h \rho C_h^*)\). Therefore we arrive at the same formula of the probability assumed by CHA, without imposing it. It turns out to be, rather, a natural consequence of the notion of occurrence of a history \((oc)\) we have introduced by means of the concept of mirror projection.

The possibility of contrary inferences is the main critique opposed to CHA. Let us briefly describe them. Suppose that \(C_1\) and \(C_2\) are two different weakly decohering families such that \(h_1 = (E_1, E_2) \in C_1\) and \(h_2 = (F_1, E_2) \in C_2\), with \(E_1 \perp F_1\). A. Kent [4] was able to find examples in which the conditional probabilities \(p_{C_1}(h_1 \mid E_2) = \frac{p_{C_1}(h_1)}{p_{C_1}(E_2)}\) and \(p_{C_2}(h_2 \mid E_2) = \frac{p_{C_2}(h_2)}{p_{C_2}(E_2)}\) are both 1. Therefore, when \(E_2\) occurs we may state, according to CHA, that also \(E_1\) occurs within the family \(C_1\), and that also \(F_1\) occurs within the family \(C_2\); on the other hand, \(E_1 \perp F_1\) means that the occurrence of \(E_1\) excludes the occurrence of \(F_1\):
then we have two inferences which are contrary to each other. They
do not entail logical inconsistency for CHA, because they take place in different consistent families. But the meaning of the occurrence
of $E_1$, or $F_1$, once $E_2$ has occurred, becomes obscure. This state of
affairs has been judged negatively by some authors [4][11], according
to whom CHA is an unsatisfactory theory.

We can easily prove that such kind of contrary inferences cannot
take place if we consider only self-decohering families. Indeed, if $C_1$
and $C_2$ are self-decohering we have

$$p_{C_1}(h_1) + p_{C_2}(h_2) = \langle \psi | E_1 E_2 E_1 \psi \rangle + \langle \psi | F_1 E_2 F_1 \psi \rangle$$

$$= \langle \psi | E_1 E_2 E_1 \psi \rangle + \langle \psi | F_1 E_2 F_1 \psi \rangle +$$

$$+ \langle \psi | E_1 E_2 F_1 \psi \rangle + \langle \psi | F_1 E_2 E_1 \psi \rangle \quad \text{by prop.1}$$

$$= \langle \psi | (E_1 + F_1) E_2 (E_1 + F_1) \psi \rangle \leq \langle \psi | E_2 \psi \rangle = p(E_2).$$

Then the sum of $p_{C_1}(h_1 | E_2) = \frac{p_{C_1}(h_1)}{p_{C_1}(E_2)}$ and $p_{C_2}(h_2 | E_2) = \frac{p_{C_2}(h_2)}{p_{C_2}(E_2)}$
cannot be greater than 1. Thus, contrary inferences are forbidden.

We end with a necessary remark. At this stage we cannot state
that self-decoherence is the ultimate consistency’s notion able to solve
all difficulties of CHA. Several questions should be seriously examined.
A problem is how to extend the notion of self-decoherence to histo-
ries made up of more than two events. Another question is whether
the following condition should be required for a property $\pi$ being an
individual property.

C) If $\pi$ holds with respect to $\rho_1$ and $\rho_2$, then $\pi$ holds with respect
$\lambda \rho_1 + (1 - \lambda) \rho_2$.

Actually, self-decoherence does not satisfy such further condition [12].
This notwithstanding, we think that self-decoherence possesses suf-
ficiently interesting features to be submitted to the attention of re-
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\{u_n\} of the Hilbert space \mathcal{H} such that
\[ u_1 = \frac{T\psi}{\|T\psi\|} \quad \text{and} \quad u_2 = \frac{U\psi}{\|U\psi\|}, \]
then by representing the vector \((T \lor U)\psi\) with respect to \{u_n\} we have
\[ (T \lor U)\psi = T\psi + U\psi + P\psi, \]
where \(P\psi \perp \{T\psi, U\psi\}\). It is obvious that
\((T \lor U) \leq (T + U)\). Therefore
\[ \langle \psi | (T + U)\psi \rangle = \|T\psi\|^2 + \|U\psi\|^2 \geq \langle (T \lor U)\psi | (T \lor U)\psi \rangle = \|T\psi\|^2 + \|U\psi\|^2 + \|P\psi\|^2. \] Thus \(P\psi = 0\) and \((T \lor U)\psi = T\psi + U\psi\).
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[12] Indeed, if (C) holds, whenever \(T_1\) and \(T_2\) are mirror projections for \(\langle h, \psi_1 \rangle \langle \psi_1 |\) and \(\langle h, \psi_2 \rangle \langle \psi_2 |\), then one mirror projection \(T\) should exist for both. This implies that such \(T\) should be a mirror projection for \(\langle h, \frac{\psi_1 + \psi_2}{\sqrt{2}} \rangle \langle \frac{\psi_1 + \psi_2}{\sqrt{2}} |\). But this is not always true, as proved in [7].