GEODESICS ON THE EXTENDED SIEGEL–JACOBI UPPER HALF-PLANE

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Abstract. The semidirect product of the real Heisenberg group $H_1(\mathbb{R})$ with $SL(2,\mathbb{R})$, called the real Jacobi group $G_1^J(\mathbb{R})$, admits a four-parameter invariant metric expressed in the $S$-coordinates. We determine the geodesic equations on the extended Siegel–Jacobi upper half-plane $\tilde{X}_1^J = G_1^J(\mathbb{R}) \approx X_1^J \times \mathbb{R} \approx X_1 \times \mathbb{R}^3$, where $X_1^J$ ($X_1$) denotes the Siegel-Jacobi upper half-plane (respectively Siegel upper half-plane). Equating successively with zero the values of the three parameters in the geodesic equations on $\tilde{X}_1^J$, we get the geodesic equations on $X_1^J$, $X_1$ and $H_1(\mathbb{R})$.

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1. Introduction

The real Jacobi group of degree $n$ is defined as $G_n^J(\mathbb{R}) := Sp(n, \mathbb{R}) \ltimes H_n(\mathbb{R})$, where $H_n(\mathbb{R})$ denotes the $(2n + 1)$-dimensional real Heisenberg group, while the semidirect product $H_n \ltimes Sp(n, \mathbb{R})_C$, $Sp(n, \mathbb{R})_C := Sp(n, \mathbb{C}) \cap U(n, n)$, is denoted by $G_n^J$. Both Jacobi groups $G_n^J(\mathbb{R})$ and $G_n^J$ are intensively investigated in Mathematics, Mathematical Physics and Theoretical Physics.

The Siegel-Jacobi upper half space is the $G_n^J(\mathbb{R})$-homogeneous manifold $X_n^J := \frac{G_n^J(\mathbb{R})}{U(n) \times \mathbb{R}} \approx X_n \times \mathbb{R}^{2n}$, where $X_n$ denotes the Siegel upper half space realized as the noncompact Hermitian symmetric space $Sp(n, \mathbb{R})_C / U(n)$. The Siegel-Jacobi ball...
is $\mathcal{D}_n^J := \frac{G_n^J}{U(n) \times \mathbb{R}} \approx \mathbb{C}^n \times D_n$ [10], where $D_n \approx \text{Sp}(n, \mathbb{R})/U(n)$ denotes the Siegel (open) ball of degree $n$ [37].

The Jacobi group $G_n^J$ is a unimodular, non-reductive, algebraic group of Harish-Chandra type [21, 45, 57, 58], and $\mathcal{D}_n^J$ is a reductive, non-symmetric manifold [15, 17]. The holomorphic irreducible unitary representations of $G_n^J$ based on $D_n^J$ constructed in [21, 28, 59, 60] are relevant in several areas of mathematics such as Jacobi forms, automorphic forms, L-functions and modular forms, spherical functions, the ring of invariant differential operators, theta functions, Hecke operators, Shimura varieties and Kuga fiber varieties.

The groups $G_n^J(\mathbb{R})$, $\text{Sp}(n, \mathbb{R})$, $H_n(\mathbb{R})$, and the Kählerian homogenous manifolds $X_n^J$ and $X_n$ are isomorphic with $G_n^J$, $\text{Sp}(n, \mathbb{R})_C$, $H_n$, $D_n^J$, respectively [10, 12, 21, 28, 34, 63, 64]. We denote with the same symbol $H_n$ both isomorphic groups $H_n$ and $H_n(\mathbb{R})$.

The Jacobi group was investigated [47, 54, 55] via coherent states (CS) [52, 53, 56]. CS-systems based on the Siegel-Jacobi ball have applications in quantum mechanics, geometric quantization, dequantization, quantum optics, squeezed states, quantum teleportation, quantum tomography, nuclear structure, signal processing, Vlasov kinetic equation, see references in [3, 16, 17].

Applying Berezin’s procedure [22]-[25] to obtain balanced metric on homogenous Kähler manifolds [2, 33, 49], we have determined the two-parameter invariant metric on the $G_n^J(\mathbb{R})$ (respectively $X_n^J$), firstly for $n = 1$ in [8, 9] and then for any $n \in \mathbb{N}$ in [10, 11, 13], see also the papers of Yang [64, 65, 67] and [11, Section 5.1]. In order to determine three-parameter invariant metric on the five-dimensional manifold $X_1^J$, called extended Siegel-Jacobi half-plane [17], we have abandoned the CS-method in the favor of Cartan’s moving frame method [30, 31, 36], considering $G_1^J(\mathbb{R})$ embedded in $\text{Sp}(2, \mathbb{R})$ [28, 34]. In [18] we have determined the three-parameter invariant metric on the extended Siegel-Jacobi upper half space $\tilde{X}_n^J$, $n \in \mathbb{N}$, a generalization of the metric on $X_1^J$ obtained in [17].

In our studies on the geometry of the Siegel-Jacobi disk by the CS-method, in [8, 12, 13] we paid special attention to the determination of the geodesic equations on $\mathcal{D}_1^J$, while in [16] we calculated geodesics on the Siegel-Jacobi ball $\mathcal{D}_n^J$. Equating successively with zero parameters in the four-parameter invariant metrics on $G_1^J(\mathbb{R})$ expressed in the S-coordinates $(x, y, \theta, p, q, \kappa)$ [28, 34], we obtained in [17] the invariant metric on $\tilde{X}_1^J$, $X_1^J$, $X_1$.

The present paper, devoted to geodesic equations on homogeneous manifolds associated with the real Jacobi group $G_1^J(\mathbb{R})$, is a continuation of [3, 17]. Using the results obtained in [17] and applied in [3], we determine the geodesic equations on homogeneous manifolds associated to the real Jacobi group $G_1^J(\mathbb{R})$ in the spirit of [8, 12, 13, 16]. Although we are not able to integrate the geodesic equations, we make important remarks on geometric and physical meaning of the variables used. Taking successively the values of the four parameters of the invariant metric on $G_1^J(\mathbb{R})$ equal with zero in the geodesic equations on the extended Siegel-Jacobi upper half-plane, we get the geodesic equations on the Siegel-Jacobi upper half-plane, Siegel upper half-plane and Heisenberg group.
The paper is laid out as follows. Section 2 recalls known formulas on coherent states, balanced metric, geodesics, and describes the parametrization of $G^1_J(\mathbb{R})$ with the S-coordinates. Proposition 1 recalls the four-parameter invariant metric of $G^1_J(\mathbb{R})$ obtained in [17, Theorem 5.7]. Section 3 summarizes our previous results on the invariant Kähler two-form $\omega_{D}^{J}(w,z)$, where $(w,z) \in (D^1_J, \mathbb{C})$. Proposition 2, taken from [3, 17], expresses the effect of the change of coordinates $FC: (w,z) \rightarrow (v,\eta)$, partial Cayley transform $\Phi: (w,z) \rightarrow (v,u)$, and $(v,u) \rightarrow (x,y,p,q)$ applied to the invariant Kähler two-form on $D^1_J$. Proposition 3, a completion of Proposition 2, gives the change of coordinates $\Phi_1 := FC_1 \circ \Phi: D^1_J \ni (w,z) \rightarrow (v,\eta) = (x+i y,q+i p)$, called the second partial Cayley transform. The geodesic equations on $D^1_J$ in the complex variables $(w,z)$ are given in Proposition 4 and Remark 4, while Proposition 5 gives the geodesic equations in the real and imaginary parts of the complex variables $w,z$. In Proposition 6 and Remark 5 in Section 4 the information on geodesics on $D^1_J$ in Section 3 is carried in the information on geodesics on $X^1_J$ via the partial Cayley transform, which is proved to be a geodesic mapping [51, Definition 5.1] p 127. Geodesics on $X^1_J$ expressed in the variables $(x,y,p,q)$ are given in Proposition 7. Proposition 8 proves by brute force calculation that the second partial Cayley transform $\Phi_1$ is a geodesic mapping. Section 5 is devoted to the geodesics on the extended Siegel-Jacobi upper half-plane $\tilde{X}^1_J$ corresponding to the three-parameter invariant metric in the S-variables $(x,y,p,q,\kappa)$ determined in [17]. To help the reader, in Section 6 are recalled some facts on geodesic mappings extracted from [11, 33, 37, 51].

The new results obtained in this paper are contained in the Remarks 1, 5, item 6 of Proposition 1, Propositions 3–8 and Theorem 1, which is the main result of the present work.

**Notation**

We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. We denote the imaginary unit $\sqrt{-1}$ by $i$, the real and imaginary parts of a complex number $z \in \mathbb{C}$ by $\text{Re}z$ and $\text{Im}z$ respectively, and the complex conjugate of $z \in \mathbb{C}$ by $\bar{z}$. We denote by $\det(M)$ the determinant of the matrix $M$. $M(n,m,F)$ denotes the set of $n \times m$ matrices with entries in the field $F$. We denote by $M(n,F)$ the set $M(n,n,F)$. $1_n$ denotes the unit matrix in $M(n,F)$. We denote by $d$ the differential. We use Einstein convention i.e. repeated indices are implicitly summed over. The scalar product of vectors in the Hilbert space $\mathcal{H}$ is denoted $(\cdot,\cdot)$. If $f$ is a real or complex function of $t \in \mathbb{R}$, then sometimes we use the standard abbreviations $\dot{f} := \frac{df}{dt}$, $\ddot{f} := \frac{d^2f}{dt^2}$. We also denote $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$.

**2. Preliminaries**

**2.1. Balanced metric and coherent states.** In Perelomov’s approach to CS [56] it is supposed that there exists a continuous, unitary, irreducible representation $\pi$ of a Lie group $G$ on a separable complex Hilbert space $\mathcal{H}$. If $H$ is the isotropy group of the representation $\pi$, then two types of CS-vectors belonging to $\mathcal{H}$ are locally defined on
\[ M = G/H: \text{ the normalized (un-normalized) CS-vector } e_x \text{ (respectively, } e_z) \]
\[ e_x = \exp(\sum_{\phi \in \Delta^+} x_{\phi} X_{\phi}^+ - \bar{x}_{\phi} X_{\phi}^-) e_0, \quad e_z = \exp(\sum_{\phi \in \Delta^+} z_{\phi} X_{\phi}^+ e_0, \]

where \( e_0 \) is the extremal weight vector of the representation \( \pi \), \( \Delta^+ \) is the set of positive roots of the Lie algebra \( g \), and \( X_{\phi}^+ \) (\( X_{\phi}^- \)) are the positive (respectively, negative) generators. For \( X \in g \) we denoted in (2.1) \( X := d\pi(x) \) such that

\[ \omega_M(z) = i \sum_{\alpha, \beta=1}^{n} h_{\alpha \bar{\beta}}(z) \, d z_{\alpha} \wedge d \bar{z}_{\beta}, \quad h_{\alpha \bar{\beta}} = \bar{h}_{\beta \bar{\alpha}} = h_{\bar{\beta} \alpha}, \]

derived from the Kähler potential \( f(z, \bar{z}) \) \[ (2.1) \]

\[ \frac{\partial h_{\alpha \bar{\beta}}}{\partial z_{\gamma}} = \frac{\partial h_{\alpha \bar{\beta}}}{\partial z_{\alpha}}, \quad \alpha, \beta, \gamma = 1, \ldots, n. \]

The condition for the Hermitian metric to be a Kählerian one is, cf. \[ (2.2) \]

As was pointed out in \[ (2.3) \]

2.2. \textbf{Geodesics on Riemannian and Kähler manifolds.} Let \( (M, g) \) be a \( n \)-dimensional Riemannian manifold. In a local coordinate system \( x^1, \ldots, x^n \) the geodesic equations on a manifold \( M \) with components of the linear connection \( \Gamma \) are, see e.g. \[ (2.4) \]

\[ \frac{d^2 x^i}{dt^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad i = 1, \ldots, n. \]

The components \( \Gamma^i_{jk} \) (Christoffell’s symbols of second kind) of a Riemannian (Levi-Civita) connection \( \nabla \) are obtained from the Christoffell’s symbols of first kind \([ij, k] \) by
solving the linear system of algebraic equations, see e.g. [42 p 160]

\[ g_{lk} \Gamma^l_{jk} = [ij, k] =: \frac{1}{2} \left( \frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{kj}}{\partial x_i} \right), \ i, j, k, l = 1, \ldots, n. \]

The \( \frac{n^2(n+1)}{2} \) distinct \( \Gamma \)-symbols of an \( n \)-dimensional manifold are given by the formulas

\[ \Gamma^m_{ij} = [ij, k]g^{km}, \quad \text{where} \quad g^{mk}g_{kl} = \delta^m_l, \ i, j, k, m = 1, \ldots, n. \]

In the convention \( \alpha, \beta, \gamma, \ldots \) run from 1 to \( n \), while \( A, B, C, \ldots \) run through 1, \ldots, \( n \), \( 1, \ldots, \bar{n} \), [43 p 155], for an almost complex connection without torsion we have the relations

\[ \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}; \quad \bar{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\bar{\beta}\bar{\gamma}}, \]

and all other \( \Gamma_A^B \) are zero. For a complex manifold of complex dimension \( n \) there are \( \frac{n^2(n+1)}{2} \) distinct \( \bar{\Gamma} \)-s.

If we take into account the hermiticity condition in (2.3) of the metric and the Kählerian restrictions (2.4), the non-zero Christoffel’s symbols \( \Gamma \) of the Chern connection (cf. e.g. [4, §3.2]) also Levi-Civita connection, cf. e.g. [4 Theorem 4.17]) which appear in (2.6) are determined by the equations, see also e.g. [43 (12) at p 156]

\[ h_{\alpha\bar{\beta}} \Gamma^\alpha_{\beta\gamma} = \frac{\partial h_{\bar{\epsilon}\bar{\beta}}}{\partial z_{\gamma}}, \quad \alpha, \beta, \gamma, \epsilon = 1, \ldots, n, \]

and

\[ \Gamma^\gamma_{\alpha\beta} = \bar{h}^\gamma_{\epsilon\alpha} \frac{\partial h_{\bar{\epsilon}\bar{\beta}}}{\partial z_{\gamma}} = h^\gamma_{\epsilon\alpha} \frac{\partial h_{\bar{\epsilon}\bar{\beta}}}{\partial z_{\gamma}}, \quad \text{where} \quad h_{\alpha\bar{\epsilon}}h_{\bar{\epsilon}\beta} = \delta_{\alpha\beta}. \]

It is easy to prove

**Remark 1.** Let \( M \) be a Kähler manifold with local complex coordinates \((z^1, \ldots, z^n)\). Let \( \Gamma^i_{jk}(z) \) be the holomorphic Christoffel’s symbols in the formula (2.5) of geodesics

\[ \frac{d^2 z^i}{dt^2} + \Gamma^i_{jk} \frac{d z^j}{dt} \frac{d z^k}{dt} = 0, \quad i = 1, \ldots, n. \]

Let us make in formula (2.9) the change of variables \( z^j = \xi^j + i \eta^j \), and let us introduce the notation \( \xi^j := \eta^j, \ j' := j + n, \ j = 1, \ldots, n. \)

Then the geodesic equations (2.9) in \((\xi^1, \ldots, z^n) \in \mathbb{C}^n \) became geodesic equations in the variables \((\xi^1, \ldots, \xi^n, \xi^{n'}, \ldots, \xi^{n'n'}) \in \mathbb{R}^{2n}\)

\[
\begin{align*}
\frac{d^2 \xi^i}{dt^2} + \tilde{\Gamma}^i_{jk} \frac{d \xi^j}{dt} \frac{d \xi^k}{dt} + 2 \bar{\Gamma}^i_{jk} \frac{d \xi^j}{dt} \frac{d \xi^k}{dt} + \bar{\Gamma}^i_{jk'} \frac{d \xi^j}{dt} \frac{d \xi^{k'}}{dt} = 0, \\
\frac{d^2 \xi^{j'}}{dt^2} + \tilde{\Gamma}^j_{jk} \frac{d \xi^j}{dt} \frac{d \xi^k}{dt} + 2 \bar{\Gamma}^j_{jk} \frac{d \xi^j}{dt} \frac{d \xi^k}{dt} + \bar{\Gamma}^{j'}_{jk} \frac{d \xi^{j'}}{dt} \frac{d \xi^k}{dt} = 0,
\end{align*}
\]

where \( \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} = -\bar{\Gamma}^{j'}_{j'k'} = \text{Re} \Gamma^i_{jk}; \quad -\bar{\Gamma}^i_{jk} = \bar{\Gamma}^i_{jk} = -\Gamma^{j'}_{j'k'} = \text{Im} \Gamma^i_{jk}, \)

and the real and imaginary parts of \( \Gamma^i_{jk} \) are functions of \((\xi, \xi') \in \mathbb{R}^{2n}\).
2.3. **The Jacobi group** $G_1^J(\mathbb{R})$ **embedded in** $\text{Sp}(2, \mathbb{R})$. We have adopted in [3, 17] the notation from [28, 34] for the real Jacobi group $G_1^J(\mathbb{R})$, realized as submatrices of $\text{Sp}(2, \mathbb{R})$ of the form

\[
(2.10) \quad G_1^J(\mathbb{R}) \ni g = \begin{pmatrix} \lambda & \mu & \kappa \\ 0 & d & -p \\ 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det M = 1,
\]

where

\[
Y := (p, q) = XM^{-1} = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (\lambda d - \mu c, -\lambda b + \mu a)
\]
is related to the Heisenberg group $H_1$ described by the coordinates $(\lambda, \mu, \kappa)$. For coordinateization of the real Jacobi group we use the so-called $S$-coordinates $(x, y, \theta, p, q, \kappa)$ [28, Section 1.4].

$M \in \text{SL}(2, \mathbb{R})$ is realized as an element in $\text{Sp}(2, \mathbb{R})$ by the relation

\[
(2.11) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow g = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_1^J(\mathbb{R}), \quad g^{-1} = \begin{pmatrix} d & 0 & -b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

The Iwasawa decomposition $M = NAK$ of $M$ as in (2.11) reads, see [28, p 10]

\[
(2.12) \quad M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad y > 0.
\]

Comparing (2.12) with (2.11), we find, see [17, p 4]

\[
a = y^{1/2} \cos \theta - x y^{-1/2} \sin \theta, \quad b = y^{1/2} \sin \theta + x y^{-1/2} \cos \theta,
\]

\[
c = -y^{-1/2} \sin \theta, \quad d = y^{-1/2} \cos \theta,
\]

and

\[
x = \frac{ac + bd}{d^2 + c^2}, \quad y = \frac{1}{d^2 + c^2}, \quad \sin \theta = -\frac{c}{\sqrt{c^2 + d^2}}, \quad \cos \theta = \frac{d}{\sqrt{c^2 + d^2}}.
\]

2.3.1. **The Heisenberg group embedded in** $\text{Sp}(2, \mathbb{R})$. In this section, extracted from [3, 17], we summarize the parametrization of the Heisenberg group used in [28].

The composition law of the 3-dimensional Heisenberg group $H_1$ in (2.10) is

\[
(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda \mu' - \lambda' \mu).
\]

It is easy to observe that

**Remark 2.** For an element of $H_1$ as element of $G_1^J(\mathbb{R})$ we have in (2.10)

\[
(2.13) \quad M = \mathbb{1}_2, \quad (p, q) = (\lambda, \mu), \quad (x, y, \theta) = (0, 1, 0),
\]
and we denote an element of $H_1$ embedded in $\text{Sp}(2, \mathbb{R})$ by

$$H_1 \ni g = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ -\lambda & 1 & -\mu & -\kappa \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

In [17, p 7] we determined the left-invariant one-forms

$$\lambda^\alpha = d\lambda, \quad \lambda^\beta = d\mu, \quad \lambda^\gamma = d\kappa - \lambda d\mu + \mu d\lambda,$$

and the left-invariant vector fields on $H_1$

$$(2.14) \quad L^\alpha = \partial_\alpha - \mu \partial_\kappa, \quad L^\beta = \partial_\mu + \lambda \partial_\kappa, \quad L^\gamma = \partial_\kappa.$$ 

The left-invariant action of $H_1$ on itself is given by

$$\exp(\lambda P + \mu Q + \kappa R)(\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda \mu_0 - \mu \lambda_0),$$

where the generators $P, Q, R$ of $H_1$ verify the commutation relations [17, (3.2) in the first reference]

$$[P, Q] = 2R, \quad [P, R] = [Q, R] = 0.$$

We get the three-parameter left-invariant metric on $H_1$

$$(2.15a) \quad g^L_{H_1}(\lambda, \mu, \kappa) = a_1(\lambda^P)^2 + a_2(\lambda^Q)^2 + a_3(\lambda^R)^2$$

$$(2.15b) \quad = a_1 d\lambda^2 + a_2 d\mu^2 + a_3 (d\kappa - \lambda d\mu + \mu d\lambda)^2, \quad a_1, a_2, a_3 > 0.$$ 

2.3.2. Metrics. In [17] first reference, Proposition 5.4, Proposition 5.6, Theorem 5.7 we have introduced 6 invariant one-forms $\lambda^1, \ldots, \lambda^6$ associated with the real Jacobi group [17, first reference, (4.10), (5.15), (5.17)] and we have expressed the invariant metric on several homogenous spaces associated with $G_1^J(\mathbb{R})$ in the $S$-coordinates

**Proposition 1.** The four-parameter left-invariant metric on the real Jacobi group $G_1^J(\mathbb{R})$ in the $S$-coordinates $(x, y, \theta, p, q, \kappa)$ is

$$ds^2_{G_1^J(\mathbb{R})} = \sum_{i=1}^{6} \lambda_i^2 = \alpha \frac{dx^2 + dy^2}{y^2} + \beta \left( \frac{dx}{y} + 2d\theta \right)^2 + \gamma \frac{d(q^2 + Sdp^2 + 2xdpdq \delta d\kappa - pdq + qdp)^2, \quad S := x^2 + y^2.}$$

Depending on the parameter values $\alpha, \beta, \gamma, \delta > 0$, in the metric $2.16$, we have invariant metric on the following $G_1^J(\mathbb{R})$-homogeneous manifolds:

1) the Siegel upper half-plane $X_1$ if $\beta, \gamma, \delta = 0$,
2) the group $\text{SL}(2, \mathbb{R})$ if $\gamma, \delta = 0, \alpha \beta \neq 0$,
3) the Siegel–Jacobi half-plane $X_1^J$ if $\beta = 0$,
4) the extended Siegel–Jacobi half-plane $\tilde{X}_1^J$ if $\beta = 0$,
5) the Jacobi group $G_1^J$ if $\alpha \beta \gamma \delta \neq 0$,
6) the Heisenberg group $H_1$ if $2.13$ is verified and $a_1 = a_2 = \gamma, a_3 = \delta$ in $2.15$. 


3. Geodesics on the Siegel–Jacobi disk

In Proposition 2 below, extracted from [17, Proposition 2.1 in the first reference] and [3, Proposition 1] \((w, z) \in (D_1, \mathbb{C}), (v, u) \in (X_1, \mathbb{C})\) and the parameters \(k\) and \(\nu\) come from representation theory of the Jacobi group: \(k\) indexes the positive discrete series of \(SU(1,1)\), \(2k \in \mathbb{N}\), while \(\nu > 0\) indexes the representations of the Heisenberg group. See also [8, 15]. Below by the Berndt-Kähler two-form we mean the invariant two-parameter Kähler two-form on \(X^J_1\) determined in [26, 27, 40, 41], see also [3, 17].

**Proposition 2.** a) The Kähler two-form on \(D^J_1\), invariant to the action of \(G^J_0 = SU(1, 1) \ltimes \mathbb{C}\), is

\[
- i \omega_{D^J_1}(w, z) = \frac{2k}{P^2} \, d w \wedge d \bar{w} + \nu \frac{A \wedge \bar{A}}{P}, \quad P := 1 - |w|^2, \quad A = A(w, z) := d z + \bar{\eta} \, d w.
\]

We have the change of variables \(FC : (w, z) \to (w, \eta)\)

\[
FC: \quad z = \eta - w \bar{\eta}, \quad FC^{-1}: \quad \eta = \frac{z + \bar{z}w}{P},
\]

and

\[
FC: \quad A(w, z) \to d \eta - w d \bar{\eta}.
\]

The matrix corresponding to the balanced metric associated with the Kähler two-form \((3.1)\) reads

\[
h(w, z) = \begin{pmatrix} h_{zz} & h_{z \bar{w}} \\ h_{\bar{z}w} & h_{\bar{w} \bar{w}} \end{pmatrix} = \begin{pmatrix} \frac{\nu}{P} & \frac{\nu |\eta|^2}{P} \\ \nu \frac{2k}{P} + \nu |\eta|^2 \end{pmatrix}.
\]

b) Using the partial Cayley transform \(\Phi : (w, z) \to (v, u)\) and its inverse

\[
(3.4a) \quad \Phi: \quad w = \frac{v - i}{v + i}, \quad z = 2i \frac{u}{v + i}, \quad v, u \in \mathbb{C}, \quad \text{Im} \, v > 0,
\]
\[
(3.4b) \quad \Phi^{-1}: \quad v = i \frac{1 + w}{1 - w}, \quad u = \frac{z}{1 - w}, \quad w, z \in \mathbb{C}, \quad |w| < 1,
\]

we obtain

\[
A \left( \frac{v - i}{v + i}, \frac{2i u}{v + i} \right) = \frac{2i}{v + i} B(v, u),
\]

where

\[
B(v, u) := d u - r d v, \quad r := \frac{u - \bar{u}}{v - \bar{v}}.
\]

The Kähler two-form of Berndt–Kähler, invariant to the action of \(G^J(\mathbb{R})_0 = SL(2, \mathbb{R}) \ltimes \mathbb{C}\), is

\[
- i \omega_{X^J_1}(v, u) = - \frac{2k}{(\bar{v} - v)^2} \, d v \wedge d \bar{v} + \frac{2\nu}{i(\bar{v} - v)} B \wedge \bar{B}.
\]

We have the change of variables \(FC_1 : (v, u) \to (v, \eta)\)

\[
(3.8) \quad FC_1: \quad 2 i u = (v + i) \eta - (v - i) \bar{\eta}, \quad FC_1^{-1}: \quad \eta = \frac{u \bar{v} - \bar{u} v}{\bar{v} - v} + i r.
\]
The matrix corresponding to the balanced metric associated with the Kähler two-form reads \[ (3.7) \]
\[ (3.9) \]
\[ h(u, v) = \left( \begin{array}{cc} h_{u\bar{u}} & h_{u\bar{v}} \\ h_{u\bar{v}} & h_{v\bar{v}} \end{array} \right) = \left( \begin{array}{cc} \frac{\nu}{y} & -\frac{\nu}{y} \\ -\frac{\nu}{y} & \frac{k}{2y^2} + \frac{\nu^2}{y} \end{array} \right), \quad y := \frac{v - \bar{v}}{2i}. \]

c) If we apply the change of coordinates \( D^J_1 \ni (v, u) \rightarrow (x, y, p, q) \in X^J_1 \)
\[ (3.10) \]
then
\[ r = p, \quad B(v, u) = d u - p d v, \]
\[ (3.11) \]
\[ B(v, u) = B(x, y, p, q) := F d t = v d p + d q = (x + i y) d p + d q, \quad F := \dot{p} v + \dot{q}. \]

d) Given \[ (3.10) \]
we obtain the change of variables \( \mathbb{C} \ni u := \xi + i \rho, \quad \xi, \rho \in \mathbb{R} \),
\[ (3.12) \]
\[ (x, y, \xi, \rho) \rightarrow (x, y, p, q) : \quad \xi = px + q, \quad \rho = py, \]
and
\[ B(v, u) = B(x, y, \xi, \rho) = d u - \frac{\rho}{y} d v = d(x + i y). \]

Based on \[ 3, \text{Remark 1} \], we complete Proposition 2 with the relation between the parameter \( \eta \) introduced in \[ (3.2) \] and the \( S \)-variables \( p, q \), and we write down explicitly the change of variables \( (w, z) \rightarrow (v, \eta) \)

**Proposition 3.** The FC-transform \[ (3.2) \] relates Perelomov’s un-normalized CS-vector \( e_{wz} \) with the normalized one \( e_{\eta \eta} \)
\[ e_{\eta \eta} = (e_{wz}, e_{wz})^{-\frac{1}{2}} e_{wz}, \quad w \in D_1, \quad z, \eta \in \mathbb{C}, \]
and the \( S \)-variables \( p, q \) introduced by \[ (3.10) \] are related to the parameter \( \eta \) \[ (3.2) \] by the simple relation
\[ (3.13) \]
\[ \eta = q + i p. \]

The second partial Cayley transform
\[ \Phi_1 := FC_1 \circ \Phi : (w, z) \rightarrow (v = x + i y, \eta = q + i p) \]
from the Siegel-Jacobi disk \( D^J_1 \) to the Siegel-Jacobi upper half-plane \( X^J_1 \) (respectively its inverse) is given by \[ (3.14a) \] (respectively, \[ (3.14b) \])
\[ (3.14a) \]
\[ \Phi_1 : w = \frac{v - i}{v + i}, \quad z = 2i \frac{pv + q}{v + 1}, \]
\[ (3.14b) \]
\[ \Phi_1^{-1} : v = i \frac{1 + w}{1 - w}, \quad \eta = \frac{(1 + i \bar{v})(z - \bar{z}) + v(\bar{v} - i)(z + \bar{z})}{2i(\bar{v} - v)} = \frac{z + \bar{z}w}{P}. \]
Proof. Relation (3.13) was proved in [3, Remark 1], based on [8, Comment 6.12], (2.1), (2.2) and [13, Lemma 2]. Here we give another very simple proof.

Equating the values of $u$ in (3.10) and (3.8), we get the formula
\begin{equation}
(3.15) \quad v(\eta - \bar{\eta} - 2ip) + i(\eta + \bar{\eta} - 2q) = 0,
\end{equation}
If in (3.15) we recall that in $v = x + iy$ we have $y > 0$, then (3.13) follows.

The proof of formula (3.14b) is also very easy. □

We emphasize that the new relations (3.14) are a more precise formulation of relations of the change of coordinates which appear in [13, Remark 8] and [16, Proposition 6].

We also make the

\textbf{Remark 3.} If in formula (3.1) of $A(z,w)$ we introduce the differential of $z$ given in (3.14a) we get
\begin{equation}
(3.16) \quad \dot{z} = \frac{2i}{v + i} \left( F - \frac{\bar{\eta} \dot{\bar{w}}}{v + i} \right).
\end{equation}
With (3.13) and (3.5) equation (3.11) is proved.

Proposition 4 below is extracted from [8, Remark 7.3], [12, Remark 3.3], [13, Remark 7], [16, Proposition 1].

\begin{proposition}
The geodesic equations on the Siegel-Jacobi disk in the $(w,z)$-variables corresponding to the metric defined by the Kähler two form (3.1) are
\begin{equation}
(3.17a) \quad \bar{\eta}G_2 = 2\tau G_3, \quad G_1 := \frac{dz}{dt} + \bar{\eta} \frac{dw}{dt}, \quad G_3 := \frac{d^2 z}{dt^2} + 2 \bar{w} \frac{dz}{dt} \frac{dw}{dt}, \quad \tau := \kappa \nu;
\end{equation}
\begin{equation}
(3.17b) \quad G_1^2 = -2\tau G_2, \quad G_2 := \frac{d^2 w}{dt^2} + 2 \frac{\bar{w}}{P} \left( \frac{dw}{dt} \right)^2.
\end{equation}
\end{proposition}

A particular solution of (3.17) is given in

\textbf{Remark 4.} With $G_2$ defined in (3.17b), the geodesic equation $G_2 = 0$ on $D_1$ has the solution
\begin{equation}
(3.18) \quad w(t) = \frac{B}{|B|} \tanh |B|t, \quad w(0) = 0, \quad \dot{w}(0) = B.
\end{equation}
\eta = \eta_0 = ct gives a particular solution of equations of geodesics (3.17) on $D_1^J$
\begin{equation}
(3.19) \quad \dot{z} = \bar{\eta} \dot{w}.
\end{equation}

Differentiating $z$ in (3.2), we find with (3.19)
\begin{equation}
\dot{\eta} - w \dot{\bar{\eta}} = 0,
\end{equation}
Proof. In [5] we proved that a $w(t)$ of the type (3.18) is a solution of $G_2 = 0$ for a complex Grassmann manifold $G_n(C^{n+m})$ and its noncompact dual. If $G_2 = 0$, then $G_1 = 0$ and also
\begin{equation}
(3.19) \quad \dot{z} = -\bar{\eta} \dot{w}.
\end{equation}
which combined with its complex conjugate implies the condition
\[ P\dot{\eta} = 0, \]

i. e. \( \eta = \eta_0. \)

Conversely, if \( \eta \) is constant, then \( G_1 = 0 \), which also implies \( G_2 = 0 \) and also \( G_3 = 0. \)

Now we pass from the geodesic equations (3.17) on \( D_1^J \) in the complex variables \((w, z)\) to the real variables \((m, n, \alpha, \beta)\), where

\[ C \ni w = \alpha + i\beta, \ |w| < 1, \ C \ni z = m + i n, \ \alpha, \beta, m, n \in \mathbb{R}. \]

If we introduce (3.20) into the complex variable \( \eta \) defined in (3.22), we get

\[ P\eta = C + i D, \] where \( C := (1 + \alpha)m + n\beta, \ D := (1 - \alpha)n + m\beta. \)

We introduce also the notation

\[(3.22a) \quad W := C^2 - D^2, \ T := 2CD, \ U := 2\alpha + \frac{j}{P} W, \ V := \beta + \frac{jT}{2P}, \ j := \frac{1}{2t}; \]
\[(3.22b) \quad Z_1 := \alpha - j\frac{W}{P}, \ Z_2 := \beta - j\frac{T}{P}, \ X := \frac{3C^2 - D^2}{P^2}, \ Y := \frac{3D^2 - C^2}{P^2}; \]
\[(3.22c) \quad L := \dot{m}\dot{\beta} + \dot{n}\dot{\alpha}, \ M := \dot{m}\dot{\alpha} - \dot{n}\dot{\beta}. \]

If we express \( W, T, X, Y \) defined in (3.22) as a function of the variables \( m, n, \alpha, \beta \), we get

\[(3.23a) \quad W = [(\alpha + 1)^2 - \beta^2]m^2 + [\beta^2 - (\alpha - 1)^2]n^2 + 4mn\alpha\beta, \]
\[(3.23b) \quad \frac{T}{2} = \beta[(\alpha + 1)m^2 + (-\alpha + 1)n^2] + mn(1 - \alpha^2 + \beta^2), \]
\[(3.23c) \quad PX = [3(\alpha + 1)^2 - \beta^2]m^2 + [3\beta^2 - (\alpha - 1)^2]n^2 + 4(2\alpha + 1)\beta mn, \]
\[(3.23d) \quad PY = [3\beta^2 - (\alpha + 1)^2]m^2 + [3(\alpha - 1)^2 - \beta^2]n^2 + 4(1 - 2\alpha)\beta mn. \]

We now prove

**Proposition 5.** In the notation (3.21), (3.22), (3.23), the geodesic equations (3.17) on the Siegel-Jacobi disk corresponding to the Kähler two-form (3.1) expressed in the real variables (3.20) are

\[(3.24a) \quad \dot{\alpha} + \frac{U}{P}(\dot{\alpha}^2 - \dot{\beta}^2) + j(\dot{m}^2 - \dot{n}^2) + \frac{4V}{P}\dot{\alpha}\dot{\beta} + \frac{2j}{P}(DL + CM) = 0, \]
\[(3.24b) \quad \dot{\beta} - \frac{2V}{P}(\dot{\alpha}^2 - \dot{\beta}^2) + \frac{2U}{P}\dot{\alpha}\dot{\beta} + 2j(\dot{m}\dot{n} + \frac{W}{P}) = 0, \]
\[(3.24c) \quad \dot{m} + \frac{jC}{P} \left[ \frac{Y}{P^2}(\dot{\alpha}^2 - \dot{\beta}^2) - \dot{m}^2 + \dot{n}^2 \right] + \frac{2jD}{P}\left( \frac{X}{P^2}\dot{\alpha}\dot{\beta} + \dot{m}\dot{n} \right) + \frac{2}{P}(Z_1M + Z_2L) = 0, \]
\[(3.24d) \quad \dot{\bar{n}} + \frac{jD}{P^2} \left[ \frac{X}{P^2}(\dot{\alpha}^2 - \dot{\beta}^2) + \dot{m}^2 - \dot{n}^2 \right] + \frac{2}{P}\left( -jC\dot{m}\dot{n} + Z_1L + Z_2M \right) = 0. \]
Proof. The balanced metric associated to the Kähler two-form (3.1) on $\mathcal{D}_1$ reads
$$d s^2_{\mathcal{D}_1}(w, z) = h_{zz} d z d \bar{z} + h_{w\bar{w}} d w d \bar{w} + h_{z\bar{w}} d z d \bar{w},$$
where the matrix $h$ is defined in (3.9). With the notation (3.20), the metric matrix attached to the metric $d s^2_{\mathcal{D}_1}(m, n, \alpha, \beta)$ is
\begin{equation}
(3.25)
g_{\mathcal{D}_1}(m, n, \alpha, \beta) = \begin{pmatrix}
g_{mm} & 0 & g_{m\beta} \\
0 & g_{nn} & g_{n\beta} \\
g_{\beta m} & g_{\beta n} & 0
\end{pmatrix},
\end{equation}
where
\begin{align*}
g_{mm} &= g_{nn} = \frac{\nu^2}{P^2} + \frac{\nu^2}{P^2} (C^2 + D^2), \\
g_{\alpha\alpha} &= g_{\beta\beta} = \frac{\nu^2}{P^2} C, \\
g_{\alpha} &= g_{\beta} = -\frac{\nu^2}{P^2} D\end{align*}
Now we calculate the determinant of the metric matrix (3.25). We recall, cf. [38, Lemma 1, p 176]:
Let us consider the analytic mappings
\begin{equation}
(3.26)
w_k := f_k(z_1, \ldots, z_k), \quad z_k = x_k + i y_k, \quad w_k = u_k + i v_k, \quad k = 1, \ldots, n.
\end{equation}
The mapping (3.26) induces a transformation of the 2n-dimensional real space. We have
\begin{equation}
(3.27)
\frac{\partial (w_1, \ldots, w_n, \bar{w}_1, \ldots, \bar{w}_n)}{\partial (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)} = \left| \frac{\partial w}{\partial z} \right|^2,
\end{equation}
where
\begin{align*}
d w_i &= \sum_{j=1}^n d z_j \frac{\partial f_i}{\partial z_j}, \\
d \bar{w}_i &= \sum_{j=1}^n d \bar{z}_j \frac{\partial \bar{f}_i}{\partial \bar{z}_j}.
\end{align*}
From the relation
\begin{equation}
(3.28)
\det h(w, z) = \frac{2k\nu}{P^3}
\end{equation}
with (3.27) we get
\begin{equation}
(3.29)
\det g(m, n, \alpha, \beta) = \left( \frac{2k\nu}{P^3} \right)^2.
\end{equation}

We make use of the change of variables (3.26), (3.27) in the equations (3.17) and we express $G_i$ as
\begin{equation}
(3.30)
\mathbb{C} \ni G_i := A_i + i B_i, \quad A_i, B_i \in \mathbb{R}, \quad i = 1, 2, 3,
\end{equation}
where
\begin{align*}
A_1 &= \dot{\alpha} + \frac{i}{P}(C \dot{\alpha} + D \dot{\beta}) \\
A_2 &= \dot{\beta} + \frac{i}{P}[\alpha(\dot{\alpha}^2 - \dot{\beta}^2) + 2\beta \dot{\alpha} \dot{\beta}] \\
A_3 &= \dot{\mu} + \frac{i}{P}(\alpha M + \beta L) \\
B_1 &= \dot{\mu} + \frac{i}{P}(C \dot{\alpha} + D \dot{\beta}) \\
B_2 &= \dot{\beta} + \frac{i}{P}[2\alpha \dot{\alpha} \dot{\beta} + \beta (-\dot{\alpha}^2 + \dot{\beta}^2)] \\
B_3 &= \dot{\mu} + \frac{i}{P}(\alpha L - \beta M)
\end{align*}
Introducing the expressions (3.30) into (3.17), the geodesic equations become
\begin{align}
(3.32a) & \quad A_2 + j(A_1^2 - B_1^2) = 0, \\
(3.32b) & \quad B_2 + 2j A_1 B_1 = 0, \\
(3.32c) & \quad A_3 - 2j \Im \eta A_1 B_1 - j \Re \eta (A_1^2 - B_1^2) = 0, \\
(3.32d) & \quad B_3 - 2j \Re \eta A_1 B_1 + j \Im \eta (A_1^2 - B_1^2) = 0.
\end{align}
Introducing (3.31) in (3.32), with
\[ A_1^2 - B_1^2 = m^2 - n^2 + \frac{W}{P^2}(\alpha^2 - \beta^2) + \frac{2T}{P^2} \alpha \beta + \frac{2}{P}(CM + DL), \]
\[ A_1 B_1 = -\frac{1}{2P^2}(\alpha^2 - \beta^2) + \dot{m}n + \frac{W}{P} \alpha \beta + \frac{1}{P}(CL - DM), \]
we get (3.24).

4. GEODESICS ON THE SIEGEL–JACOBI UPPER HALF-PLANE

The geodesic equations on \( X^I \) expressed in the variables \((v, u)\) are given in Proposition 6.

a) The differential equations on \( X^I \) obtained by the partial Cayley transform (3.14) from the geodesic equations on \( D^I \) (3.17) are
\[
(4.1a) \quad rH_1^2 = \iota H_3, \quad H_1 := \ddot{u} - r \dot{v}, \quad H_3 := i \ddot{u} - \frac{\dot{u} \dot{v}}{y};
\]
\[
(4.1b) \quad H_2^2 = \iota H_2, \quad H_2 := i V_1 = i \ddot{v} - \frac{\dot{v}^2}{y}, \quad V_1 := \ddot{v} - \frac{\dot{v}^2}{iy} = \ddot{v} + 2 \frac{\dot{v}^2}{\ddot{v} - \dot{v}},
\]
where \( r \) was defined in (3.6). Equations (4.1) can be written as
\[
(4.2a) \quad \ddot{v} + i \iota \left[ \ddot{u}^2 - 2r \ddot{u} \dot{v} + \left( \frac{\dot{v}}{y} + r^2 \right) \dot{v}^2 \right] = 0,
\]
\[
(4.2b) \quad \ddot{u} + i \iota \left[ -r \ddot{u}^2 + \frac{\dot{u}}{y} (1 - 2r^2) \ddot{u} \dot{v} + \dot{v}^3 \ddot{v} \right] = 0.
\]

b) The differential equations (4.1) or (4.2) are geodesic equations in the variables \((v, u)\) on \( X^I \) attached to the metric corresponding to the Kähler two-form (3.7).

c) The partial Cayley transform \( \Phi : (w, z) \rightarrow (v, u) \) given in (3.4) is a geodesic mapping \( D^I \rightarrow X^I \).

Proof. a) In (3.17) we make the change of coordinates \((w, z) \rightarrow (v, u)\) given by the partial Cayley transform (3.4).

With (3.4a), we get successively
\[
(4.3a) \quad P = \frac{4y}{(v + i) (\ddot{v} - i)};
\]
\[
(4.3b) \quad \dot{w} = 2i \frac{\ddot{v}}{(v + i)^2};
\]
\[
(4.3c) \quad 2 \frac{\ddot{w}}{P} \dot{w}^2 = -2 \frac{\dot{v} + i}{y (v + i)} [(v + i) \ddot{u} - u \dot{v}] \dot{v};
\]
\[
(4.3d) \quad \ddot{w} = 2i \frac{(v + i) \ddot{v} - 2 \dot{v}^2}{(v + i)^3};
\]
\[
(4.3e) \quad \dot{z} = \frac{2i}{(v + i)^2} [(v + i) \ddot{u} - u \dot{v}];
\]
\[
(4.3f) \quad \ddot{z} = \frac{2i}{(v + i)^3} [(v + i)^2 \ddot{u} - u (v + i) \ddot{v} - 2 (v + i) \dot{v} \dot{u} + 2 \dot{u} \dot{v}^2].
\]
In the expression (3.17b) of \(G_2(w, z)\) we introduce (4.3b), (4.3c), (4.3d) and we obtain
\[
G_2(v, u) = \frac{2iV_1}{(v + i)^2}.
\]

In the expression (3.17a) of \(G_1(w, z)\) we introduce (4.3b) and we get
\[
G_1(v, u) = \frac{2}{(v + i)^2}[(v + i)\dot{u} - (\bar{\eta} - u)\dot{\bar{v}}].
\]

But expression (3.8) of \(\eta\) implies
\[
\bar{\eta} - u = -r(v + i),
\]
and we find
\[
G_1(v, u) = \frac{2i}{v + i}((\dot{u} - r\dot{v}).
\]

If we introduce (4.6) and (4.4) in (3.17b) we get (4.1b).

To also prove (4.1a), we introduce in expression (3.17a) of \(G_3(w, z)\) relations (4.3a), (4.3b), (4.3e), (4.3f) and we get
\[
G_3(v, u) = \frac{2}{(v + i)^3}[i((v + i)^2\bar{u} - 2(v + i)\dot{v} + (2\dot{v}^2 - u(v + i)\dot{v})) - (v + i)(\bar{v} + i)\dot{v} + (v + i)\omega^2 y].
\]

But
\[
2iuv^2 + uv^2\frac{\bar{v} + i}{y} = -(v + i)\frac{i}{y}; \quad (-2i\frac{v + i}{y})\dot{v} = -(v + i)\frac{i}{y},
\]
and we get for \(G_3(v, u)\)
\[
G_3(v, u) = \frac{2}{v + i}(i\frac{\dot{u} - i\dot{\bar{v}}}{y}) - uG_2.
\]

Now the expressions (4.6), (4.4) and (4.7) are introduced in (3.17) and we get
\[
\bar{H}_1^2 = -i[(v + i)H_3 - uH_2],
\]
(4.8b)
\[
H_1^2 = iH_2.
\]

If we introduce (4.8b) into (4.8a), we have
\[
(\bar{\eta} - u)H_1^2 = -(v + i)H_3,
\]
and, with (4.5), we get (4.1a).

b) We calculate the geodesic equations on the Siegel-Jacobi disk \(\mathcal{X}_1^J\) in the variables \((v, u)\) corresponding to the balanced metric (3.9).

In the variables \((u, v) \in (\mathbb{C}, \mathcal{X}_1)\) the geodesic equations (2.5) for the metric (3.9) read
\[
\begin{align*}
\frac{d^2 u}{dt^2} + \Gamma^u_{uu} \left(\frac{du}{dt}\right)^2 + 2\Gamma^u_{uv} \frac{du}{dt} \frac{dv}{dt} + \Gamma^u_{v} \left(\frac{dv}{dt}\right)^2 &= 0; \\
\frac{d^2 v}{dt^2} + \Gamma^v_{uu} \left(\frac{du}{dt}\right)^2 + 2\Gamma^v_{uv} \frac{du}{dt} \frac{dv}{dt} + \Gamma^v_{v} \left(\frac{dv}{dt}\right)^2 &= 0.
\end{align*}
\]

The equations (2.8) which determine the \(\Gamma\)-symbols for the Siegel-Jacobi disk are
\[
\begin{align*}
\frac{\partial h_{uu}}{\partial u} \Gamma^u_{uu} + h_{uv} \Gamma^u_{uv} &= \frac{\partial h_{uu}}{\partial u}; \\
\frac{\partial h_{uv}}{\partial u} \Gamma^u_{uv} + h_{vv} \Gamma^v_{uu} &= \frac{\partial h_{uv}}{\partial u}.
\end{align*}
\]

(4.10)
\[ \begin{align*}
\{ h_{u\bar{u}} \Gamma^u_{u\bar{u}} + h_{v\bar{u}} \Gamma^v_{u\bar{u}} &= \frac{\partial h_{u\bar{u}}}{\partial u}, \\
&= \frac{\partial h_{v\bar{u}}}{\partial u}, \\
\{ h_{u\bar{v}} \Gamma^u_{u\bar{v}} + h_{v\bar{v}} \Gamma^v_{u\bar{v}} &= \frac{\partial h_{u\bar{v}}}{\partial u}, \\
&= \frac{\partial h_{v\bar{v}}}{\partial u}.
\end{align*} \]

We calculate easily the partial derivatives of the elements of the metric \((3.3)\)
\[ \begin{align*}
\frac{\partial h_{u\bar{u}}}{\partial u} &= 0, \\
&= \frac{\partial h_{u\bar{v}}}{\partial u}, \\
&=- \frac{i \nu r}{y^2}, \\
&= \frac{3}{2} i \nu \left( \frac{r}{y} \right)^2.
\end{align*} \]

Introducing \((4.13)\) into \((4.10)-(4.12)\), we find for the Christoffel's symbols \(\Gamma\)-s the following expressions
\[ \begin{align*}
\Gamma^u_{uu} &= -i \frac{r}{y}, \\
&= i \frac{r}{y}, \\
\Gamma^v_{uv} &= i \nu \left( \frac{1}{y} - \frac{2 r^2}{y^2} \right); \\
\Gamma^v_{vu} &= -i \frac{r}{y}, \\
&= i r, \\
\Gamma^v_{vv} &= i \nu \frac{r^2}{y}, \\
&= i \nu \frac{r}{y} + \frac{3}{2} i \nu \left( \frac{r}{y} \right)^2.
\end{align*} \]

Introducing \((4.14)\) in \((4.9)\), we get the equations \((4.2)\) on \(X_1^I\) in the variables \((u, v) \in (C, X_1)\).

c) is proved because the equations \((4.1)\) represent geodesic equations \((4.2)\) for Christofell symbols \((4.14)\) associated to metric corresponding to the Kähler two-form \((3.7)\).

A particular solution of \((4.1)\) on \(X_1^I\) in the variables \((v, u)\) is given in

**Remark 5.** The Cayley transform \((3.4)\) \(D_1 \ni w \mapsto v \in X_1\) is a geodesic mapping \(\Phi^{-1} : D_1 \to X_1\).

If \(w(t)\) is the solution \((3.18)\) of the equation \(G_2 = 0\) of geodesics on the Siegel disk \(D_1\), then geodesic equations \(H_2 = 0\) on the Siegel upper half-plane \(X_1\) with \(H_3\) defined in \((4.1)\) have the solution
\[ \begin{align*}
\eta &= \eta_0 \text{ gives the particular solution of equations of geodesics } (4.1) \text{ on } X_1^I \\
v &= v(t), \\
u(t) &= \frac{\eta_0 - \bar{\eta}_0 w(t)}{1 - w(t)}, \\
(4.16) \quad u(0) &= \eta_0.
\end{align*} \]

**Proof.** For the first assertion it is observed that
\[ H_2 = -2 \frac{P}{(1 - w)^2} Q G_2, \]
where
\[ Q := |1 - w|^2. \]

\(G_2 = 0\) implies \(H_2 = 0\) and \(v\) given in \((4.16)\) is a solution of \(H_2 = 0\) if \(w\) is a solution of \(G_2 = 0\).
Now we are interested in the geodesic equations on the Siegel-Jacobi upper half-plane expressed in the variables \((v, \eta) = (x + iy, q + ip)\).

**Proposition 7.** a) The two-parameter balanced metric on the Siegel-Jacobi upper half-plane \(X_1^J\), the particular case of (2.16) corresponding to item 3) in Proposition 1 associated to the Kähler two-form (3.7), (3.11), is

\[
(4.18) \quad ds^2_{X_1^J}(x, y, p, q) = \alpha \frac{dx^2 + dy^2}{y^2} + \frac{\gamma}{y} (S \, dp^2 + dq^2 + 2x \, dp \, dq),
\]

where \(S\) was defined in (2.16) and

\[
(4.19) \quad \alpha := k/2, \quad \gamma := \nu.
\]

The geodesic equations on \(X_1^J\) corresponding to the metric (4.18) are

\[
(4.20a) \quad E_1 := \ddot{x} - \frac{2}{y} \dot{x} \dot{y} - \epsilon Ry\dot{p} = 0, \quad R := \text{Re } F = x\dot{p} + \dot{q}, \quad \epsilon := \frac{\gamma}{\alpha},
\]

\[
(4.20b) \quad E_2 := \ddot{y} + \frac{1}{y} (\dot{x}^2 - \dot{y}^2) + \frac{\epsilon}{2} (R^2 - y^2 \dot{p}^2) = 0,
\]

\[
(4.20c) \quad E_3 := \ddot{p} + R\frac{\dot{x}}{y^2} + \frac{1}{y} \dot{y} \dot{p} = 0,
\]

\[
(4.20d) \quad E_4 := \ddot{q} + \frac{\dot{x}}{y^2} (y^2 \dot{p} - xR) - \frac{\dot{y}}{y} (R + x\dot{p}) = 0.
\]

If in (4.20) we take \(\epsilon = 0\) \((\gamma = 0)\) we get the expression of geodesic equations \(H_2 = 0\) in (4.1b) in the variables \((x, y)\) on \(X_1\) corresponding to the Christoffel's symbols (4.21)

\[
(4.21a) \quad \Gamma^x_{xx} = 0, \quad \Gamma^x_{xy} = -\frac{1}{y}, \quad \Gamma^x_{yy} = 0,
\]

\[
(4.21b) \quad \Gamma^y_{xx} = \frac{1}{y}, \quad \Gamma^y_{xy} = 0, \quad \Gamma^y_{yy} = -\frac{1}{y},
\]

associated with the metric in case 1) in Proposition 1

\[
\ddot{x} - \frac{2}{y} \dot{x} \dot{y} = 0, \quad \ddot{y} + \frac{1}{y} (\dot{x}^2 - \dot{y}^2) = 0,
\]

with the solution (4.15).

b) With the change of coordinates \(D_1^J \ni (v, u) \mapsto (x, y, p, q) \in X_1^J\) (3.10), the geodesic equations (4.2) on \(D_1^J\) become on \(X_1^J\) the system of differential equations

\[
(4.22a) \quad pK_1^2 = iK_3, \quad K_1 := F, \quad K_3 := i(\dot{p} + \dot{q}) - \frac{\dot{F}}{y} + pH_2;
\]

\[
(4.22b) \quad K_1^2 = iK_2, \quad K_2 := H_2, \quad v = x + iy,
\]

where \(F\) was defined in (3.11).

c) The transform \((v, u) \mapsto (x, y, p, q)\) (3.10) is a geodesic mapping \(D_1^J \to X_1^J\).

\(\eta = q + ip = ct\) is a particular solution of (4.22).
**Proof. a)** The matrix associated with the metric (4.18) is

\[
g_{X_1'} = \begin{pmatrix} g_{xx} & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 \\ 0 & 0 & g_{pp} & g_{pq} \\ 0 & 0 & g_{qp} & g_{qq} \end{pmatrix}, \quad g_{xx} = \frac{\alpha}{y^2}, \quad g_{yy} = \frac{\alpha}{y^2},
\]

and

\[
\det(g_{X_1'}(x, y, q, p)) = \left(\frac{\alpha \gamma}{y^2}\right)^2.
\]

The inverse of the metric matrix (4.23) is

\[
g_{X_1'}^{-1} = \begin{pmatrix} g^{xx} & 0 & 0 & 0 \\ 0 & g^{yy} & 0 & 0 \\ 0 & 0 & g^{pp} & g^{pq} \\ 0 & 0 & g^{qp} & g^{qq} \end{pmatrix}, \quad g^{xx} = g^{yy} = \frac{\alpha}{y^2}, \quad g^{pq} = \frac{\gamma}{y^2}, \quad g^{pp} = \frac{1}{\gamma y}, \quad g^{qq} = \frac{\gamma}{y^2}.
\]

With formula (2.7) we determine the non-zero Christoffel’s symbols corresponding to the Riemannian metric (4.18) of the Siegel-Jacobi upper half-plane

\[
\begin{align*}
\Gamma_{x y}^x &= -\frac{1}{y}, & \Gamma_{x y}^x &= -\epsilon x y, & \Gamma_{x y}^x &= -\frac{1}{2} \epsilon y, \\
\Gamma_{x y}^{x} &= \frac{1}{y}, & \Gamma_{y y}^{y} &= -\frac{1}{y}, & \Gamma_{y y}^{p q} &= \frac{1}{2} (x^2 - y^2), & \Gamma_{y y}^{p q} &= \frac{1}{2} x, & \Gamma_{y y}^{q p} &= \frac{1}{2} y, \\
\Gamma_{x y}^{p q} &= \frac{1}{y}, & \Gamma_{x y}^{q p} &= -\frac{1}{y}, & \Gamma_{y y}^{q p} &= \frac{1}{y}, & \Gamma_{y y}^{q p} &= -\frac{1}{y}.
\end{align*}
\]

To get (4.21), we apply Remark 1 for $X_1$ and we get from the holomorphic $\Gamma$ symbols the associated real Christoffel’s symbols, see also e.g. [29, Exercise 8 p 58].

**b)** With (4.1), in $\nu H_1^2 = k H_2$ we introduce $H_1 = \dot{u} - \dot{p}v$, $H_2 = i \ddot{u} + \frac{\ddot{w}}{y}$ and taking the real and imaginary part, we get (4.20a) and (4.20b).

Taking the derivative of (3.10), we get successively

\[
\dot{u} = \dot{p}v + \dot{p}v + \ddot{q}, \quad \ddot{u} = \ddot{p}v + 2\dot{p}v + \ddot{p}v + \ddot{q}
\]

which introduced in expression (4.1b) gives the expression of $K_3$ in (4.22a). Now in the first equation (4.22a) we introduce the expression of $K_3$ and taking into consideration first equation in (4.22b), we get

\[
2i(\dot{p}v + \ddot{q}) - \dddot{F} \frac{\dddot{u}}{y} = 0.
\]

Taking the real and imaginary part of (4.26), we get (4.20c), (4.20d).

**c)** Assertion c) is a consequence of a) and b).  

We make now

**Remark 6.** The expression (4.24) of the determinant of the metric matrix of $X_1'$ in the variables $(x, y, q, p)$ can be obtained from the expression (3.29) of the metric matrix of $D_1'$ in the variables $(m, n, \alpha, \beta)$.  

\[\square\]
Proof. Let \((M, g(x)) \rightarrow (M', g'(x'))\) be an isometry of Riemannian manifolds. Then we have the relation, see e.g. [35, (9.3) p 23]

\[
\det g'(x') = \det g(x) J^2,
\]

where \(J := \det \left| \frac{\partial x^i}{\partial x'^j} \right|\).

In our case of \((D^1_J, g(m, n, \alpha, \beta)) \rightarrow (X^1_J, g(x, y, q, p))\), with the first relation (3.4a) and (3.21), we have the change of coordinates

\[
(x, y, q, p) = (−\frac{2\beta}{Q}, \frac{P}{Q}, \frac{C}{P}, \frac{D}{P}),
\]

where \(P\) (\(Q\)) was defined in (3.1) (respectively (4.17)).

With formula (4.27) applied to the change of coordinates (4.28) from the Siegel-Jacobi disk to the Siegel-Jacobi upper half-plane we have to calculate the Jacobian

\[
I := \frac{\partial(x, y, q, p)}{\partial(m, n, \alpha, \beta)} = I_1 I_2, \quad I_1 := \frac{\partial(x, y)}{\partial(\alpha, \beta)}, \quad I_2 := \frac{\partial(q, p)}{\partial(m, n)}.
\]

Using the Cauchy-Riemann equations

\[
\frac{\partial x}{\partial \alpha} = \frac{\partial y}{\partial \beta}, \quad \frac{\partial x}{\partial \beta} = -\frac{\partial y}{\partial \alpha},
\]

we have to calculate

\[
I_1 = \frac{\partial x}{\partial \alpha} \frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta} \frac{\partial y}{\partial \alpha} = \left( \frac{\partial x}{\partial \alpha} \right)^2 + \left( \frac{\partial x}{\partial \beta} \right)^2,
\]

and with (4.28)

\[
\frac{\partial x}{\partial \alpha} = \frac{4(\alpha - 1)\beta}{Q^2}, \quad \frac{\partial x}{\partial \beta} = -\frac{2(\alpha - 1)^2 - \beta^2}{Q^2},
\]

introduced in (4.31), we get

\[
I_1 = \frac{4}{Q^2}, \quad I_2 = \frac{1}{P}, \quad I = \frac{4}{Q^2} \frac{1}{P}.
\]

With (4.27) and (4.29), we get for \(J = I^{-1}\)

\[
\det g(x, y, q, p) = \left( \frac{k\nu}{2} \frac{Q^2}{P^2} \right)^2,
\]

i.e. relation (4.24) because of the notation (4.19), (4.28).

We show below that the geodesic equations (4.20) on \(X^1_J\) are found making in (3.17) the change of variables given in (3.14) with the consequence that

Proposition 8. The second partial Cayley transform (3.14) \(\Phi_1 : (D^1_J, ds^2_{2^1_J}(w, z)) \rightarrow (X^1_J, ds^2_{X^1_J}(x, y, p, q))\) is a geodesic mapping.

Proof. From (3.16), we get

\[
\ddot{z} = \frac{2i}{v + i} \left\{ Z - 2\frac{\dot{v}\dot{\eta}}{v + i} - \frac{\eta[\dot{v}(v + i) - 2\dot{v}^2]}{(v + i)^2} \right\}, \quad Z := \nu\dot{p} + \dot{q}.
\]
With (4.3a), (4.3b) and (4.32), we get for \( G_3 \) in (3.17a) the value
\[
G'_3 := G'_3(v, \eta) := F C_1 \circ \Phi(G_3(w, z))
\]
The expression
\[
(4.33) \quad G'_3 = 2i \frac{v}{v + i} \left( Z \frac{-\bar{\eta}}{v + i} V_1 + 2 \frac{v}{v - v} \right),
\]
where \( F \) was defined in (3.11) and \( V_1 \) in (4.1b).

We also find for \( G_i = G_i(v, \eta) = \Phi_1 \circ \Phi(G_i(w, z)), i = 1, 2 \) the expressions
\[
(4.34) \quad G'_1 = 2i \frac{F}{v + i}, \quad G'_2 = \frac{2i}{(v + i)^2} V_1.
\]

Introducing (4.34) into (3.17b), we obtain
\[
(4.35) \quad V_1 = -i \epsilon^{-1} F^2.
\]
The real (imaginary) part of (4.35) expresses equation (4.20a) (respectively (4.20b)) because \( \epsilon^{-1} = \frac{i}{\epsilon} \).

Introducing (4.35) into (4.33), we get
\[
(4.36) \quad G'_3 = 2i \frac{v}{v + i} \left( Z + 2i \frac{F}{v - v} + i \epsilon^{-1} \frac{\bar{\eta} F^2}{v + i} \right).
\]

Now we introduce (4.36) into (3.17a) and we get
\[
Z + 2i \frac{F}{v - v} = 0
\]
whose real (imaginary part) gives (4.20c) (respectively (4.20d)). \( \square \)

Proposition \( 8 \) is a more precise formulation of \[13\, \text{Remark} 8 \] for the \( FC \)-transform (3.2) on \( D^J_1 \) and \[16\, \text{Proposition} 6 \] for \( D^J_n, \ n \in \mathbb{N} \).

5. Geodesics on the extended Siegel–Jacobi upper half-plane

In order to get geodesic equations on the extended Siegel-Jacobi upper half-plane in the \( S \)-variables \((x, y, p, q, \kappa)\), we use the expression of the metric on \( \tilde{X}^J_1 \) given in Proposition 4. The solution (5.3) of geodesic equations (5.3) on the Heisenberg group \( H_1 \) in Theorem 1 is taken form \[50\, (11) \] in Theorem 1.

\textbf{Theorem 1.} The three-parameter metric on the extended Siegel-Jacobi upper half-plane \( \tilde{X}^J_1 \) expressed in the \( S \)-coordinates \((x, y, p, q, \kappa)\), left-invariant with respect to the action of the Jacobi group \( G^J_1(\mathbb{R}) \), is given by item 4) in Proposition 4 as
\[
\begin{align*}
\text{d}s^2_{\tilde{X}_i^J}(x, y, p, q, \kappa) &= \text{d}s^2_{\tilde{X}_i^J}(x, y, p, q) + \lambda_0^2(p, q, \kappa) \\
&= \frac{\alpha}{y^2}(\text{d}x^2 + \text{d}y^2) + \left( \frac{\gamma}{y} S + \delta q^2 \right) \text{d}p^2 + \left( \frac{\gamma}{y} + \delta p^2 \right) \text{d}q^2 + \delta \kappa^2 \\
&+ 2 \left( \frac{\gamma x}{y} - \delta pq \right) \text{d}p \text{d}q + 2\delta(q \text{d}p \kappa - p \text{d}q \kappa).
\end{align*}
\]

\textbf{a)} The geodesic equations on \( \tilde{X}^J_1 \) associated to the metric (5.1) are
\[
(5.2a) \quad E'_1 := E_1 = 0,
\]
(5.2b) \( E'_4 = E_2 = 0, \)
(5.2c) \( E'_3 = E_3 + \frac{2\tau}{y} [xq^2 + (q - px)\dot{p}q - pq^2 + R\kappa] = 0, \quad \tau := \frac{\delta}{\gamma}, \quad \xi := px + q, \)
(5.2d) \( E'_4 = E_4 + \frac{2\tau}{y} [-qS\dot{p} + (pS - xq)\dot{p}q - S\dot{p}\kappa + xq(p\dot{q} - \kappa)] = 0, \)
\[
E'_5 := \dot{\kappa} + \frac{py^2 - \xi x}{y^2} \dot{x} - \frac{\xi y^2}{y} \dot{y} - \frac{2px + q}{y} \ddot{p} - \frac{p}{y} \ddot{q} \]
(5.2e) \[+ \frac{2\tau}{y}[\dot{p}(pS + qx)(\dot{p}q - \ddot{\kappa}) + (p^2S - q^2)\dot{p}q + \xi q(\dot{p}q - \ddot{\kappa})] = 0. \]

If \( \tau = 0, \) then the first four equations (5.2) are the equations (4.20) of geodesics on the Siegel-Jacobi upper half-plane with invariant metric (4.18).

b) In particular, the geodesic equations on the Heisenberg group \( H_1 \) corresponding to the metric in case 6) in Proposition [1] are obtained as the particular cases 5.3 of (5.2c) - (5.2d), where

\[
E''_3 := \dot{p} + 2\tau(-pq^2 + \dot{q}\dot{p}q + \ddot{q} - \kappa) ,
\]
(5.3b) \( E''_4 := \dot{q} + 2\tau(-q^2p + pq^2 + \ddot{p}q) + \dot{p} \)
(5.3c) \( E''_5 := \dot{\kappa} + 2\tau[pq(-\dot{p}^2 + \ddot{q}) + (p^2 - q^2)\dot{p}q - (p\dot{p} + q\dot{q})\kappa]. \)

Geodesic lines issuing from \((0,0,0)\) such that \((x(0), \dot{y}(0), z(0)) = (r \cos \phi, r \sin \phi, \sigma), \sigma \neq 0 \) in \( H_1 \) are given by

\[
\begin{align*}
  x(t) &= \frac{\tau}{2\sigma} (\sin(2\sigma t + \phi) - \sin \phi), \\
  y(t) &= \frac{\tau}{2\sigma} (\cos \phi - \cos(2\sigma t + \phi)), \\
  z(t) &= \frac{1}{2\sigma^2} t - \frac{1}{4\sigma^2} \sin 2\sigma t,
\end{align*}
\]
while if \( \sigma = 0, \)
\[
(x(t), y(t), z(t)) = (\alpha_1 t, \beta_1 t, 0), \quad \alpha_1^2 + \beta_1^2 = 1.
\]

Proof. a) The metric matrix associated to the metric (5.1) is

\[
g_{\chi_1} = \begin{pmatrix}
g_{xx} & 0 & 0 & 0 & 0 \\
0 & g_{yy} & 0 & 0 & 0 \\
0 & 0 & g_{pp} & g_{pq} & g_{pp} \\
0 & 0 & g_{qp} & g_{qq} & g_{qk} \\
0 & 0 & g_{kp} & g_{qk} & g_{kk}
\end{pmatrix}, \quad g_{xx} = \frac{\alpha}{y^2}, \quad g_{yy} = \frac{\alpha}{y^2},
\]
\[
g_{pp} = g_{pq} = g_{qp} = g_{qq} = g_{qk} = g_{kk} = \delta.
\]

and

\[
det(g_{\chi_1}) = \delta \left( \frac{\alpha \gamma}{y^2} \right)^2.
\]

The inverse of the metric matrix (5.5) is

\[
g_{\chi_1}^{-1} = \begin{pmatrix}
g_{xx} & 0 & 0 & 0 & 0 \\
0 & g_{yy} & 0 & 0 & 0 \\
0 & 0 & g_{pp} & g_{pq} & g_{pp} \\
0 & 0 & g_{qp} & g_{qq} & g_{qk} \\
0 & 0 & g_{kp} & g_{qk} & g_{kk}
\end{pmatrix}, \quad g_{xx} = g_{yy} = \frac{\beta^2}{\alpha}, \quad g_{pq} = -\frac{x}{\gamma y},
\]
\[
g_{pp} = 1 + \frac{\alpha}{\gamma y}, \quad g_{pp} = \frac{S + pq}{\gamma y}, \quad g_{qk} = \frac{1}{\delta} + \frac{\epsilon^2(\gamma y)^2}{\gamma y}.
\]
With formula (2.7) we determine the Christoffel’s symbols corresponding to the Riemannian metric (5.1) of the extended Siegel-Jacobi upper half-plane. In formulas below we have included only the $\Gamma$-s which are not given in (4.25)

\[
\begin{align*}
\Gamma^p_{pp} &= 2\tau\frac{xq}{y}, & \Gamma^p_{pq} &= \tau\frac{yq}{2y}, & \Gamma^p_{qp} &= \tau\frac{xq}{y}, & \Gamma^p_{q\kappa} &= -2\tau\frac{2y}{q}, & \Gamma^p_{p\kappa} &= \tau\frac{1}{y}, \\
\Gamma^q_{pp} &= -2\tau\frac{yS}{y}, & \Gamma^q_{pq} &= -\tau\frac{xpS}{y}, & \Gamma^q_{qp} &= -\tau\frac{xq}{y}, & \Gamma^q_{q\kappa} &= 2\tau\frac{2y}{q}, & \Gamma^q_{p\kappa} &= -\tau\frac{2y}{y}, \\
\Gamma^\kappa_{xp} &= \frac{yp^2 - x^2}{2y}, & \Gamma^\kappa_{xq} &= -\tau\frac{2y}{2y}, & \Gamma^\kappa_{yp} &= 2\tau\frac{xq}{q}, & \Gamma^\kappa_{yq} &= -\frac{2y}{2y}, & \Gamma^\kappa_{pp} &= -2\tau\frac{2y(pS + qx)}{y}.
\end{align*}
\]

(5.6) implies the geodesic equations on $\tilde{X}_1^I$ given in (5.3).

b) According to (2.13), the Heisenberg group $H_1$ embedded into $\text{Sp}(2, \mathbb{R})$ corresponds to $(x, y, \tau) = (0, 1, 0)$ in (5.2).

Indeed, the matrix associated to the fundamental two-form (2.15) is

\[
(5.7) \quad g_{H1} = \begin{pmatrix}
g_{\lambda\lambda} & g_{\lambda\mu} & g_{\lambda\kappa} \\
g_{\mu\lambda} & g_{\mu\mu} & g_{\mu\kappa} \\
g_{\kappa\lambda} & g_{\kappa\mu} & g_{\kappa\kappa}
\end{pmatrix}, \quad g_{\lambda\lambda} = a_1 + a_3\mu^2, \quad g_{\mu\mu} = a_2 + a_3\lambda^2, \quad g_{\kappa\kappa} = a_3, \\
g_{\lambda\mu} = -a_3\lambda\mu, \quad g_{\lambda\kappa} = a_3\mu, \quad g_{\mu\kappa} = -a_3\lambda,
\]

and

\[
\det(g_{H1}) = a_1a_2a_3.
\]

The inverse of the metric matrix $g_{H1}$ (5.7) is

\[
(5.8) \quad g_{H1}^{-1} = \begin{pmatrix}
g^{\lambda\lambda} & g^{\lambda\mu} & g^{\lambda\kappa} \\
g^{\mu\lambda} & g^{\mu\mu} & g^{\mu\kappa} \\
g^{\kappa\lambda} & g^{\kappa\mu} & g^{\kappa\kappa}
\end{pmatrix}, \quad g^{\lambda\lambda} = \frac{1}{a_1}, \quad g^{\mu\mu} = \frac{1}{a_2}, \quad g^{\kappa\kappa} = \frac{1}{a_3} + \left(\frac{\lambda^2}{a_1} - \frac{\mu^2}{a_2}\right),
\]

The Christoffel’s symbols associated to the metric matrix (5.7) are

\[
\begin{align*}
\Gamma_{\lambda\lambda}^\lambda &= 0, & \Gamma_{\lambda\mu}^\lambda &= \frac{a_3}{a_1}\mu, & \Gamma_{\lambda\kappa}^\lambda &= 0, & \Gamma_{\mu\lambda}^\lambda &= -\frac{2a_3}{a_1}\lambda, & \Gamma_{\mu\kappa}^\lambda &= \frac{a_3}{a_1}, & \Gamma_{\kappa\kappa}^\lambda &= 0, \\
\Gamma_{\mu\lambda}^\mu &= -2\frac{a_3}{a_2}\mu, & \Gamma_{\mu\mu}^\mu &= \frac{a_3}{a_2}\lambda, & \Gamma_{\mu\kappa}^\mu &= -\frac{a_3}{a_2}, & \Gamma_{\mu\kappa}^\mu &= 0, & \Gamma_{\mu\kappa}^\mu &= 0, & \Gamma_{\kappa\kappa}^\mu &= 0, \\
\Gamma_{\lambda\lambda}^\kappa &= -2\frac{a_3}{a_2}\mu, & \Gamma_{\lambda\mu}^\kappa &= \frac{a_3}{a_2}\lambda, & \Gamma_{\lambda\kappa}^\kappa &= -\frac{a_3}{a_2}, & \Gamma_{\mu\lambda}^\kappa &= \frac{2a_3}{a_1}\lambda, & \Gamma_{\mu\kappa}^\kappa &= -\frac{a_3}{a_1}, & \Gamma_{\mu\kappa}^\kappa &= 0, & \Gamma_{\kappa\kappa}^\kappa &= 0.
\end{align*}
\]

The geodesic equations on $H_1$ corresponding to the three-parameter invariant metric (2.15) are

\[
\begin{align*}
(5.9a) \quad \ddot{p} + 2\frac{a_3}{a_1}(-pq^2 + q\dot{p}\dot{q} + \dot{q}\dot{\kappa}) &= 0, \\
(5.9b) \quad \ddot{q} + 2\frac{a_3}{a_2}(-qp^2 + p\dot{q}\dot{p} - \dot{p}\dot{\kappa}) &= 0, \\
(5.9c) \quad \ddot{\kappa} + 2a_3[p\dot{q}(q^2 - \dot{p}^2) + \left(\frac{p^2}{a_2} - \frac{q^2}{a_1}\right)\dot{p}\dot{q} - \frac{1}{a_2}(p\dot{p} + q\dot{q})] &= 0.
\end{align*}
\]

We have the following covariant derivatives associated to the 3-parameter invariant metric (2.15) of $H_1$

\[
\begin{align*}
(5.10a) \quad \nabla_p \partial_p &= -2\frac{a_3}{a_2}qL, & \nabla_p \partial_\kappa &= \nabla_\kappa \partial_\kappa = -\frac{a_3}{a_2}L, & \nabla_\kappa \partial_q = \nabla_q \partial_\kappa = \frac{a_3}{a_1}L, \\
(5.10b) \quad \nabla_\kappa \partial_q &= \nabla_q \partial_\kappa = a_3[q(\frac{1}{a_1}\partial_\partial_\partial + \frac{1}{a_2}\partial_\partial) + \left(\frac{p^2}{a_2} - \frac{q^2}{a_1}\right)\partial_\partial].
\end{align*}
\]
With (2.14) and (5.10), we get the covariant derivatives of the Riemannian connection of the left-invariant metric (2.15) with the orthonormal frame (2.14)

\[ \nabla_{L^p}L^q = 0, \quad \nabla_{L^p}L^q = L^r, \]

(5.11a)

\[ \nabla_{L^p}L^r = -\frac{a_3}{a_2}L^q \quad \nabla_{L^q}L^r = \frac{a_3}{a_1}L^p. \]

(5.11b)

Equations (5.10) for \( a_1 = a_2 = a_3 = 1 \) is Lemma 1 in [50] and we can apply Theorem 1 in [50], which is the content of last assertion in Theorem 1.

We take \( a_1 = a_2 = \gamma, \quad a_3 = \delta \) in (5.9) or in (5.11) we get the geodesic equations on \( H_1 \) given in (5.3).

\[ \square \]

6. Appendix: Geodesic mappings

In several places of this paper, namely in: Proposition 6, item c) for the partial Cayley transform \( \Phi \) (3.4), Remark 5, item c) for the Cayley transform, Proposition 7 for the change of coordinates (3.10) and in Proposition 8 for the second partial Cayley transform (3.14) \( \Phi_1 \), we proved that the mentioned applications are geodesic mappings.

Below we collect several well known facts about geodesic mappings.

Firstly we recall the notion of isometry [37, p 60] (or motion [1, Section 1.2]) between Riemannian manifolds.

Let us suppose that two Riemannian spaces \( (M_n, g), (\tilde{M}_n, \tilde{g}) \) have the fundamental forms [35, (40.2)]

\[ ds^2_{M_n} = g_{ij} \, dx_i \, dx_j, \quad ds^2_{\tilde{M}_n} = \tilde{g}_{ij} \, dx_i \, dx_j, \]

and \( f \) is a function

\[ f : (M_n, g) \rightarrow (\tilde{M}_n, \tilde{g}), \quad y = f(x), \]

such that the Jacobian

\[ J = \det (\frac{\partial y_i}{\partial x_j})_{i,j=1,...,n} \neq 0. \]

With [37, pages 22-24], [42, p 161], [35, (7.10) p 18] and [51, (2.15) p 59]

Proposition 9. Let (6.2) be an isometry, or motion, between the Riemannian spaces (6.2), i.e. \( f \) is a diffeomorphism of \( M_n \) onto \( \tilde{M}_n \) and

\[ f^\ast \tilde{g} = g, \]

or

\[ g_p(X,Y) = \tilde{g}_{f(p)}(d\,f_p(X), d\,f_p(Y)), \quad \forall \, p \in M_n, \quad \forall X,Y \in (TM_n)_p. \]

If

\[ X_x := X^i(x) \frac{\partial}{\partial x_i}, \quad Y_x := Y^i(x) \frac{\partial}{\partial x_i}, \quad X^{\ast}_{f(p)} := d\,f_p(X_p), \]

then

\[ X^i_x = (X^\ast_{f(p)})^a \frac{\partial x^i}{\partial y^a}. \]
and (6.5) becomes

\[
\begin{align*}
(6.6) \quad g_{ij}(x) &= \frac{\partial y^\alpha}{\partial x^i} \tilde{g}_{\alpha\beta}(f(x)) \frac{\partial y^\beta}{\partial x^j}, \\
\tilde{g}_{\alpha\beta}(f(x)) &= \frac{\partial x^i}{\partial y^\alpha} g_{ij}(x) \frac{\partial x^j}{\partial y^\beta},
\end{align*}
\]

i.e. \( g_{ij} \) is a (0,2)-contravariant tensor.

Now we recall [51, Definition 5.1 p 127]

**Definition 1.** If \( f \) \( (6.2) \) is a diffeomorphism between manifolds with affine connections, then \( f \) is called a geodesic mapping if it maps geodesic curves on \( M_n \) into geodesic curves on \( \tilde{M}_n \).

In fact, there are three methods for proving that a mapping \( (6.2) \) is a geodesic one.

a) By brute force calculation, i.e.:
1) suppose that we have geodesic equations \( G(x) \) on the manifold \( M_n \) in the variables \( x \);
2) we make the change of coordinates \( (6.2) \) in \( G(x) \) and we get a system of differential equations \( H(y) \) on \( \tilde{M}_n \);
3) we verify that \( H(y) \) are geodesic equations on \( \tilde{M}_n \) corresponding to the Christoffel symbols associate to \( \nabla_{\tilde{g}} \).

In the present paper we have applied this method.

b) Any motion group takes geodesic into a geodesic \([1, Section 1.2 p 9]\). We make some more comments.

- If \( f \) is an isometry of a Riemannian manifold, then \( f \) preserves distances \([37, p 60]\). It is proved in \([37, Theorem 11.1]\) that: Let \( M \) be a Riemannian manifold and \( f \) a distance-preserving mapping onto itself. Then \( f \) is an isometry.

- One can argue \([37, p 60], [39]\) that the isometry \( f \) also preserves the induced distances \( d_1, d_2 \) on \( M, \tilde{M} \) from \( g, \tilde{g} \) respectively, that is \( d_1(x,y) = d_2(f(x), f(y)) \) for \( x, y \in M \). It is easy to show \([39]\) that \( f \) sends geodesics on \( M \) to geodesics on \( \tilde{M} \), using the length minimizing property of geodesics and that \( f \) is distance-preserving.

c) the Levi-Civita \([46]\) equations \((6.7)\).

In \([35, Section 40 \text{"Spaces with corresponding geodesics"}, pages 131-133]\), Eisenhart considers the two Riemannian spaces \((M_n, g), (\tilde{M}_n, \tilde{g})\) with the fundamental forms \((6.1)\) and Christoffel’s symbols \( \Gamma^l_{ij} \), (respectively \( \tilde{\Gamma}^l_{ij} \)).

Based on \([35, (40.6), (40.8), (11.4)]\)

**Proposition 10.** The \( \frac{n^2(n+1)}{2} \) equivalent conditions for the spaces \( M_n, \tilde{M}_n \) to have the same geodesics are the Levi-Civita equations

\[
\begin{align*}
(6.7a) \quad \tilde{\Gamma}^k_{ij} &= \Gamma^k_{ij} + \psi^k_j \delta^i_j + \psi^k_i \delta^j_j, \\
(6.7b) \quad \tilde{g}_{ij,k} &= 2\psi^k_i \tilde{g}_{ij} + \psi^k_j \tilde{g}_{jk} + \psi^k_x \tilde{g}_{ik},
\end{align*}
\]

where \( \tilde{g}_{ij,k} \) is the covariant derivative of \( \tilde{g}_{ij} \) with respect to \( x^k \) and the fundamental tensor \( g_{ij} \) of type \((0,2)\), i.e.

\[
(6.8) \quad \tilde{g}_{ij,k} = \partial_k \tilde{g}_{ij} - \Gamma^l_{ki} \tilde{g}_{lj} - \Gamma^l_{kj} \tilde{g}_{il},
\]
\begin{equation}
\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+1)} \ln \frac{\det \tilde{g}}{\det g}.
\end{equation}

- For equations (6.7a) see also [61, (12) p 157, (72) p 270] and for (6.9) see [61, p 270]. Note that the Levi-Civita condition (6.7b) appears in [44, Theorem 94.1 p 290] in the case of geodesic mappings of constant Gaussian curvature between a portion of a surface $S$ onto a portion of another surface $S^\ast$.

- With Proposition 9 we get also the well-known relation (4.27) which allows us write down (6.9) as

\begin{equation}
\Psi = -\frac{1}{n+1} \ln |J|.
\end{equation}

- In [51, Section 7.1, p 167] it is presented the particular case considered in Proposition 10 of two Riemannian manifolds $V_n = (M, g)$ and $\tilde{V}_n = (\tilde{M}, \tilde{g})$ with corresponding Riemannian connections $\nabla$ and $\tilde{\nabla}$, respectively. It is supposed that there exists a geodesic map $f : M \to \tilde{M}$ and $\tilde{M}$ is identified via $f$ with $M$ as in [51, Section 3.1]. Then $V_n$ admits a geodesic mapping onto $\tilde{V}_n$ if and only if the equivalent Levi-Civita equations (6.7) hold.

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