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Orthogonal polynomials
and deformed oscillators

We discuss the construction of oscillator-like systems associated with orthogonal polynomials on the example of the Fibonacci oscillator. In addition, we consider the dimension of the corresponding Lie algebras.

Key words: Generalized oscillator, orthogonal polynomials, Fibonacci numbers, algebras of oscillator-like systems.

Introduction. The basis for this work was the talk given by one of the authors (EVD) at the conference "In search of Fundamental Symmetries" dedicated to the 90th anniversary Yuri V. Novozhilov. Therefore, the writing style is not quite academic in nature.

The authors studied in the physics faculty of Leningrad State University during the period when the department of "Field Theory and Elementary Particle Physics" was headed by Yu. V. Novozhilov, who supported the education traditions formed under the influence of academicians V.I. Smirnov and V.A. Fock. High level of physical and mathematical education in the physical department was opened for its graduates wide possibilities of application of the forces both in physics and in mathematics. This circumstance explains the fact that the author who received physical education (EVD) became interested in the achievements of mathematical physics, and the author who received the mathematical education (VVB) became interested in physical problems. As a result, since 2000, the authors have combined their efforts in the study of algebraic structures of quantum physics. In this paper we present some results obtained by the authors in recent years. These results are associated with the generalization of the notion of a harmonic oscillator, which is one of the basic concepts of quantum mechanics.

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The development of quantum physics in the last century, in particular, the emergence of quantum groups and quantum algebras [1]-[4] (L.D. Faddeev school, V. Drinfeld [1983/87]), naturally led to construction of various generalizations of the notion of quantum harmonic oscillator [5]-[8], (L.C. Biedenharn, J.A. Macfarlane, P.P. Kulish and E.V. Damaskinsky [1989/90]) connected with q-deformations of the canonical commutation relations of the algebra of harmonic oscillator 3. Further researches led to the construction of generalizations of the Heisenberg algebra associated with orthogonal polynomials from the Askey-Wilson scheme and their q-analogues [14].

In the work [15] one of the authors (VVB) developed a general scheme for the construction of such oscillator-like algebras for an arbitrary family of orthogonal polynomials on the real axis (hereinafter such algebras are called the generalized oscillators). In recent years we have applied this approach to the construction of generalized oscillator algebras associated with

- some classical orthogonal polynomials of a continuous argument (Laguerre, Chebyshev (first and second kind), Legendre, Gegenbauer and Jacobi polynomials);

- some classical orthogonal polynomials of a discrete argument (Meixner, Charlier, Kravchuk polynomials);

- some q-analogues of the classical orthogonal polynomials (discrete and continuous q-Hermite polynomials, q-Charlier polynomials);

- generalized Fibonacci polynomials;

- the Chebyshev-Koornwinder polynomials of two variables.

In addition, in some cases were constructed and investigated the corresponding coherent states of Barut-Girardello and Klauder-Gaseau type.

The construction of a generalized oscillator associated with the system of orthogonal polynomials. Let we have the system of orthogonal polynomials \( \{p_k\}_{k=0}^{\infty} \) forming an orthogonal basis in Hilbert space \( H_\mu = L^2(\mathbb{R}; \mu(dx)) \). For a given measure \( \mu \), one can define oscillator-like system for which these polynomials play the same role as the Hermite polynomials

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3Although several attempts of generalization of the Heisenberg commutation relations were made before [11]-[13].
for the standard quantum harmonic oscillator. Here we briefly describe some details of this construction (see [15]). Now, let \( \mu \) be a (symmetric) probability measure on \( \mathbb{R} \) with finite moments \( \mu_n = \int_{-\infty}^{\infty} x^n \, d\mu \) \((\mu_{2k+1} = 0)\). These moments uniquely define a positive sequence \( \{b_n\}_{n=0}^{\infty} \) and a system of orthogonal polynomials with recurrent relations

\[
x p_n(x) = b_n p_{n+1}(x) + b_{n-1} p_{n-1}(x). \tag{1}
\]

These polynomials form an orthogonal basis in the Hilbert space \( \mathcal{H}_\mu \). The relations (1) determine the action of the operator coordinates \( X_\mu \)

\[
X_\mu p_n(x) = b_n p_{n+1}(x) + b_{n-1} p_{n-1}(x).
\]

Similarly, one can define the momentum operator \( P_\mu \), which is conjugate to the operator coordinates with respect to the basis

\[
P_\mu p_n(x) = i(-b_n p_{n+1}(x) + b_{n-1} p_{n-1}(x)).
\]

Then one can define creation and annihilation operators (conjugate to each other in the Hilbert space \( \mathcal{H}_\mu \)) \( a_\mu^\pm = \frac{1}{\sqrt{2}} \left( X_\mu \pm i P_\mu \right) \); operator \( N_\mu \), numbering basis states, and selfadjoint Hamiltonian \( H_\mu = X_\mu^2 + P_\mu^2 = a_\mu^+ a_\mu^- + a_\mu^- a_\mu^+ \); moreover

\[
a_\mu^+ p_n(x) = \sqrt{2b_n} p_{n+1}(x); \quad a_\mu^- p_n(x) = \sqrt{2b_{n-1}} p_{n-1}(x);
\]

\[
N_\mu p_n(x) = n p_n(x), \quad H_\mu p_n(x) = \lambda_n p_n(x),
\]

where \( \lambda_0 = 2b_0^2 \), \( \lambda_n = 2(b_{n-1}^2 + b_n^2) \). These operators satisfy the commutation relations of generalized Heisenberg algebra

\[
[a_\mu^-, a_\mu^+] = 2(B(N_\mu + I) - B(N_\mu)); \quad [N_\mu, a_\mu^\pm] = \pm a_\mu^\pm,
\]

where the operator-function \( B(N_\mu) \) is defined by the relation

\[
B(N_\mu) p_n(x) = b_{n-1}^2 p_n(x), \quad p_n(x) \in \mathcal{H}_\mu.
\]

The center of this algebra is generated by the element \( C = 2B(N_\mu) - a_\mu^+ a_\mu^- \). With appropriate changes (see [15]) the same reasoning applies to a more general case (with asymmetric measure \( \mu \))

\[
x p_n(x) = b_n p_{n+1}(x) + a_n p_{n+1}(x) + b_{n-1} p_{n-1}(x). \tag{2}
\]
Coherent states of Barut - Girardello type for such generalized oscillator are determined by the relations

\[ a^- \left| z \right> = z \left| z \right>, \quad \left| z \right> = \mathcal{N}^{-1/2}(\left| z \right|^2) \sum_{n=0}^{\infty} \frac{z^n}{(\sqrt{2b_{n-1}})!} p_n, \]

where

\[ \mathcal{N}(\left| z \right|^2) = <z|z> = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2b_{n-1})!}. \]

It is possible to prove that the so-defined coherent states form an overcomplete family of states in the Hilbert space. In addition, these states minimize the corresponding uncertainty relation.

The main difficulty in applying this approach is the solution the moment problem that arises when constructing measures of orthogonality for polynomials (not in the classical case) and in finding of measures, participating in the (over)completeness relation of the coherent states. There are also some problems with obtaining the explicit form of coherent states in terms of special (hypergeometric or basic hypergeometric) functions.

**Fibonacci Numbers and the Fibonacci oscillator.** In 1202, Italian merchant and mathematician Leonardo of Pisa (1180-1240), known as Fibonacci, published the essay "Liber Abaca". In this work were collected almost all of the mathematical information known to that time. In particular, from this book the European mathematics, using the Latin calculation system, learned about Arabic (decimal) one, which is significantly simplified arithmetic calculations. Among the many problems given in this book was widely known, "problem of rabbits"\(^4\), the solution of which gives the sequence of numbers known as the Fibonacci numbers:

\[ F_n : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \]

The elements of this sequence (Fibonacci numbers) are determined by the recurrence relation

\[ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2) \]

with initial conditions \( F_0 = 1, F_1 = 1 \).\(^5\)

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\(^4\)Some extensions of this problem, for example, the problem of mortal rabbits discussed in [16,17].

\(^5\)Elementary properties of Fibonacci numbers are given in a brochure by N.N.Vorobiev [18].
It is known \[19, 20, 21\] that $F_n$ are associated with the Chebyshev polynomials of the 2nd kind $U_n(x)$ by the relation

$$F_{n+1} = (-i)^n U_n(i \sinh(\theta_0)), \quad U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta},$$

where $\theta_0 > 0$ and $\sinh(\theta_0) = \frac{1}{2}$.

There are many generalizations of the Fibonacci numbers. The most natural of them is defined by the relation

$$F_{a,b,n+1} = aF_{a,b,n} + bF_{a,b,n-1}.$$ 

Using the relationship of the Fibonacci numbers with Chebyshev polynomials, Ismail \[21\] suggest their one-parametric generalization

$$F_{n+1}(\theta) = (-i)^n U_n(i \sinh(i\theta)), \quad F_n(\theta) = e^{(n-1)\theta} \frac{1 - Q^n}{1 - Q},$$

where $Q = -e^{-2\theta}$. These generalizations satisfy the recurrent relations

$$y_{n+1}(\theta) = 2 \sinh \theta y_n(\theta) + y_{n-1}(\theta)$$

with initial conditions $F_1(\theta) = 1$, $F_2(\theta) = 2 \sinh \theta$. Ismail showed that $F_n(\theta)$ generate the classical moment problem for measure

$$\nu(x) = (1 - q^x) \sum_{k=0}^{\infty} q^{ak} \delta(x - q^k e^{-\theta}),$$

where $\delta(x - c)$ is unique discrete measure concentrated at the point $x = c$. When $\alpha$ is even this measure is positive with unit total mass, and in other cases, $\nu$ is the unit "signed" measure. This measure correspond to orthogonal polynomials \( \{ p_n(xe^\theta; q^{\alpha-1}, 1) \} \), known as the little $q$-Jacobi polynomials, which explicit form looks as

$$p_n(x; a, b) = 2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx)$$

$$= \sum_{j=0}^{n} \frac{(q; q)_n(abq^{n+1}; q)_j}{(q; q)_j(q; q)_{n-j}} \frac{(-x)^j}{(aq; q)_j},$$

where $q$-factorials are defined by the formula

$$(\lambda; q)_s = (1 - \lambda)(1 - \lambda q) \ldots (1 - \lambda q^{s-1}).$$
Richardson [24] noted that the matrix $F_n$, with elements $\frac{1}{F_{i+j+1}}$, where $F_n$ is the $n$-th Fibonacci number, has as its inverse the matrix with integer elements. (Since the same property has the Hilbert matrix, he called this matrix as the Filbert matrix; this term was established according to the rule $F$ibonacci+$(H)$ilbert). Using this result, Berg [25] showed that the sequence $F_{n+2}$ of numbers inverse to Fibonacci numbers are a sequence of moments for discrete probability measure. He also found that this measure is an orthogonality measure for small $q$-Jacobi polynomials

$$p_n(x; a, b; q) = 2\phi_1 \left( \frac{q^{-n}, abq^{n+1}}{aq}; q, xq \right),$$

for $a = q$, $b = 1$ and $q = \frac{1 - \sqrt{5}}{1 + \sqrt{5}}$.

Applying the described above method of constructing of the generalized oscillator to the case of polynomials $p_n(x) \equiv p_n(x; a, b; q)$ we get appropriate oscillator-like system and a set of coherent states for her. Further, this system is called the Fibonacci oscillator. [6]

In our case, recurrence relations have the form

$$-xp_n(x) = A_n p_{n+1}(x) - (A_n + C_n) p_n(x) + C_n p_{n-1}(x),$$

where $p_0(x) = 1$ and

$$A_n = q^n \frac{(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n})(1 - abq^{2n+1})}, \quad C_n = aq^n \frac{(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$ 

Let us denote $p_n(x) = \gamma_n \Psi_n(x)$, where

$$\gamma_n = \sqrt{\frac{C_1 C_2 \cdots C_n}{A_0 A_1 \cdots A_{n-1}}} = \left( a^n q^n \frac{1 - ab}{1 - abq^{2n+1}} \frac{(q, q)_n (bq, q)_n}{(aq, q)_n (abq, q)_n} \right)^{1/2}.$$ 

[6]Note that in the literature (following [26]) by oscillator Fibonacci sometimes means two-parameter deformed oscillator [23], associated with the basic number $[n; q, p] = \frac{q^n - p^{-n}}{q - p^{-1}}$, which (as well as other basic numbers) satisfies the generalized variant of Fibonacci recurrent relations

$$[n + 1; q, p] = (q + p)^{-1} [n; q, p] - q^{-1} p^{-1} [n - 1; q, p], \quad [0; p, q] = 0, \quad [1; p, q] = 1.$$
Note that in the case $a = q, b = 1$ we have
\[
\gamma_n = q^n \frac{(q, q)_n}{(q^2, q)_n} \sqrt{\frac{1 - q^2}{1 - q^{2(n+1)}}}.
\]
Then for $\Psi_n(x)$ we obtain
\[
x \Psi_n(x) = -b_n \Psi_{n+1}(x) + a_n \Psi_n(x) - b_n \Psi_{n-1}(x),
\]
where $\Psi_0(x) = 1, a_n = A_n + C_n$ and $b_{n-1} = \sqrt{A_{n-1}C_n}$.
For $a = q, b = 1$ we have
\[
a_n = \frac{q^n}{1 - q^{2(n+1)}} \left( \frac{(1 - q^n)^2}{1 - q^{2n+1}} + \frac{(1 - q^{n+2})^2}{1 - q^{2n+3}} \right),
\]
\[
b_{n-1} = \frac{q^n}{1 - q^{2n+1}} \frac{(1 - q^n)(1 - q^{n+1})}{\sqrt{(1 - q^{2n})(1 - q^{2(n+1)})}}.
\]
Now let $X_\mu = \text{Re}(\tilde{X}_\mu - \tilde{P}_\mu)$, $P_\mu = (-i)\text{Im}(\tilde{X}_\mu - \tilde{P}_\mu)$, where operators $\tilde{X}_\mu$ and $\tilde{P}_\mu$ are defined as
\[
\tilde{X}_\mu \Psi_n = b_{n-1} \Psi_{n-1} + a_n \Psi_n + b_n \Psi_{n+1};
\]
\[
\tilde{P}_\mu \Psi_n = i (b_{n-1} \Psi_{n-1} + a_n \Psi_n - b_n \Psi_{n+1}).
\]
As a result, using the above relations, we define the algebra of Fibonacci oscillator.

Coherent states of Barut-Girardello type for this oscillator are defined as above by the relations
\[
a^{-|z|} |z| = |z|, \quad |z| = N^{-1/2}(|z|^2) \sum_{n=0}^\infty \frac{|z|^n}{(\sqrt{2b_{n-1}})!} \Psi_n.
\]
After some transformations we obtain the following expression for normalization factor
\[
N(|z|^2) = 6 \phi_1 \left( \begin{array}{cccc}
-q & -q^{3/2} & q^{3/2} & -q^{3/2} \\
-q^{3/2} & q^2 & -q^2 & -q^2 \\
-q^{3/2} & -q^2 & q^2 & -q^2 \\
-q^2 & -q^2 & -q^2 & q^2
\end{array} \right) \frac{|z|^2}{2}.
\]

\footnote{Note that in the papers [27]-[28] also discussed deformed oscillators associated with (generalized) Fibonacci numbers.}
where by definition
\[ e^{\phi_1} \left( \frac{a_1, a_2, a_3, a_4, a_5, a_6}{b_1} \big| q; z \right) = \sum_{k=0}^{\infty} \frac{(-1)^k q ^{-2k} (z)}{(b_1; q)_k} \]

Here we used the standard notation for the Pochhammer $q$-symbol:
\[ (a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \]
\[ (a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n. \]

In the case $a = q$, $b = 1$, the expression for the coherent states is simplified and has the form
\[ |z> = \mathcal{N}^{-1/2} |z|^2 \sum_{n=0}^{\infty} q^{-n(n+3)/2} p_n(x; q, 1|q) \left( \frac{q^3; q^2}{q; q}_n^2 \right) \left( \frac{z}{\sqrt{2}} \right)^n, \]
with the same expression for the normalizing factor. To complete our construction it would be desirable to find an explicit expression for the measure participating in the overcompleteness relation for coherent states of the Fibonacci oscillator. Currently, this task is still under development.

**The dimension of generalized oscillator algebras.** In a recently published paper [29] the authors investigated the conditions under which algebra of the generalized oscillator $\mathfrak{A}$, associated with orthogonal polynomials in the manner described above, is finite-dimensional. In [29] considered only case of orthogonal polynomials for a symmetric measure on the real axis (when the Jacobi matrix corresponding to the recurrent relations [1] has zero diagonal). In the work [30] we have clarified the formulation of sufficient conditions and extended the results to the case of the Jacobian matrix with nonzero diagonal, corresponding to recurrent relations [2]. Following property holds [30]:

**Teorema.** Let us define the sequence
\[ A_n^{(0)} = b_n^2 - b_{n-1}^2, \ldots, A_n^{(j)} = A_n^{(j-1)} - A_n^{(j-1)}, \quad j = 1, 2, \ldots, \quad n = 0, 1, \ldots. \]

Then
1. If for any fixed $j > 0$, the sequence \( \left\{ A_n^{(j)} \right\}_{n=0}^{\infty} \) is not constant, i.e. \( A_n^{(j)} \neq \text{const}, \quad n = 0, 1, \ldots \), then the generalized oscillator algebra $\mathfrak{A}$ is infinite dimensional.
2. The generalized oscillator algebra $\mathfrak{A}$ is finite dimensional if and only if
\[ b_n^2 = (\beta_0 + \beta_2 n)(1 + n), \quad \beta_0, \beta_2 \in \mathbb{R}. \tag{4} \]
and in this case dimension of $\mathfrak{A}$ is equal 4.

Let us consider some examples illustrating this theorem.

As a first example we consider the case of Chebyshev polynomials of the first kind $T_n(x)$ which was not considered in [29]. The polynomials $T_n(x)$ is defined by the relation
\[ T_n(x) = \cos(n \arccos(x)), \quad x \in [-1, 1], \]
and orthogonal in the Hilbert space $L^2_{[-1,1]}(\frac{1}{\sqrt{1-x^2}}dx)$. Normalized polynomials
\[ \Psi_n^{\frac{1}{2}}(x) = \sqrt{2}T_n(x), \quad n \geq 1, \quad \Psi_0^{\frac{1}{2}}(x) = T_0(x) = 1, \]
fulfill the symmetric recurrence relations (1) with
\[ b_n = \frac{1}{2}, \quad n \geq 1, \quad b_0 = \frac{1}{\sqrt{2}}. \]
In this case $A_n^{(j)} \neq \text{constant},$ $n = 0, 1, \ldots,$ for all fixed $j > 0$ and, consequently, the corresponding algebra $\mathfrak{A}$ is infinite dimensional.

As a second example, we consider the Laguerre polynomials satisfying nonsymmetric recurrent relations. These polynomials
\[ L_\alpha^\alpha(x) = \frac{\alpha + 1}{n!} F_1(-n, \alpha + 1; x). \]
are orthogonal in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+^1; x^\alpha \exp(-x)dx)$. Normalized polynomials
\[ \Psi_n(x) = d_n^{-1} L_\alpha^\alpha(x), \quad d_n = \sqrt{\frac{\Gamma(n+\alpha+1)}{n!}}, \quad n \geq 0 \]
fulfill the nonsymmetric recurrence relations (2) with
\[ b_n = -\sqrt{(n+1)(n+\alpha+1)}, \quad a_n = 2n + \alpha + 1. \]
In this case, $b_n^2$ have the form (4) and, hence, the corresponding algebra $\mathfrak{A}_L$ 4-dimensional but not isomorphic to the algebra of the harmonic oscillator.
Finally, the last example is the oscillator Fibonacci discussed above. Indeed, in this case, the coefficients $b_n$ are determined by the formulas (3), as in the first example, $A_n^{(j)} \neq \text{constant}, \quad n = 0, 1, \ldots$, for all fixed $j > 0$. Consequently, the corresponding algebra $\mathfrak{A}$ is infinite dimensional.

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