POSITIVE ARBOREALIZATION OF POLARIZED WEINSTEIN MANIFOLDS

DANIEL ÁLVAREZ-GAVELA, YAKOV ELIASHBERG, AND DAVID NADLER

Abstract. Let $X$ be a Weinstein manifold. We show that the existence of a global field of Lagrangian planes in $TX$ is equivalent to the existence of a positive arboreal skeleton for the Weinstein homotopy class of $X$.

Contents

1. Introduction 1
2. $\mathcal{Wc}$-manifolds and cotangent buildings 15
3. Positivity of Lagrangian planes 44
4. Arboreal models 50
5. Arboreal Lagrangians and their stability 64
6. Symplectic neighborhoods of arboreal Lagrangians 83
7. Positivity of cotangent buildings 98
8. Ridgification of Lagrangians 104
9. Arborealization of skeleta 110
References 124

1. Introduction

1.1. Arborealization program. The symplectic topology of the cotangent bundle $T^*M$ of a smooth manifold is determined by the smooth topology of its Lagrangian zero-section $M$. A general Weinstein symplectic manifold $X$ (see precise definitions below), which is the symplectic counterpart of a Stein complex manifold, can be viewed as a symplectic thickening of its skeleton, which is a singular isotropic subcomplex $\text{Skel}X \subset X$. One would like to view $X$ as the “cotangent bundle” of $\text{Skel}X$, and characterize the smooth symplectic topology of $X$ in terms of the differential topology of $\text{Skel}X$. However, in general the singularities of $\text{Skel}X$ are too complicated to be amenable to a differential topological treatment.

In the paper [N13], the third author introduced a class of Lagrangian singularities, called arboreal. For arboreal singularities, the smooth topology of the singularity determines the...
symplectic topology of its neighborhood. In particular, it is possible to calculate the local symplectic invariants of a neighborhood of an arboreal singularity in terms of the singularity. This was shown in [N13] for microlocal sheaves; for Fukaya categories, one can apply Lefschetz fibration calculations [S08] to the plumbing characterization of [S18]. Going further, if a Weinstein manifold \( W \) has a skeleton with arboreal singularities, then its global symplectic invariants can be effectively computed knowing the smooth topology of the skeleton, see [N16] for microlocal sheaves and [GPS17] for Fukaya categories.

The paper [N13] initiated a program to determine which Weinstein manifolds admit arboreal skeleta, or as we say, \textit{which Weinstein manifolds could be arborealized}. It was shown in [N15] that germs of Whitney stratified Lagrangians can always be deformed to arboreal Lagrangians in a non-characteristic fashion, i.e. without changing their microlocal invariants. The question of whether a Weinstein manifold can be arborealized via a homotopy of its Weinstein structure is more subtle. In two dimensions, the story is classical: generic ribbon graphs provide arboreal skeleta. In four dimensions, Starkston proved that arboreal skeleta always exist [S17]. In this paper, we establish that any Weinstein manifold admitting a polarization, i.e. a Lagrangian plane field, or equivalently a reduction of structure group of the tangent bundle from \( Sp(2n) \) to \( GL(n) \), can be arborealized, see Theorem 1.5 below. Moreover, our proof yields skeleta with the more specific class of \textit{positive} arboreal singularities. Conversely, Weinstein manifolds with positive arboreal skeleta admit polarizations. On the other hand it turns out not all Weinstein manifolds can be arborealized, see the discussion in Section 1.3 below. Perhaps this is not surprising: there are homotopical obstructions to defining many symplectic invariants, and so any combinatorial route to realizing them must also encounter these obstructions.

1.2. Main results. Let \((W, \lambda)\) be a \(2n\)-dimensional Liouville domain: \( W \) is a compact \(2n\)-manifold with boundary; \( \lambda \) is a 1-form, called the Liouville form, such that \( \omega = d\lambda \) is a symplectic form; and the corresponding vector field \( Z = \omega^{-1}(\lambda) \), called the Liouville vector field, is outward transverse to \( \partial W \).

By definition, the skeleton of a Liouville domain \((W, \lambda)\) is the attractor of the negative flow of the Liouville vector field:

\[
\text{Skel}(W, \lambda) = \bigcap_{t>0} Z^{-t}(W).
\]

While the \(2n\)-dimensional Lebesgue measure of \( \text{Skel}(W, \lambda) \) is equal to 0, in general \( \text{Skel}(W, \lambda) \) can be quite large if no additional conditions are imposed.

A Weinstein domain is a Liouville domain \((W, \lambda)\) which admits a Morse Lyapunov function \( \phi : W \to \mathbb{R} \), i.e. the critical points of \( \phi \) are non-degenerate and \( Z \) is gradient-like for \( \phi \). Sometimes it is convenient to relax the Morse condition to generalized Morse or Morse-Bott. We do not consider the Lyapunov function as part of the defining data of a Weinstein domain, we merely require its existence.

The skeleton of a Weinstein domain \((W, \lambda)\) is known to be the union of stable manifolds

\[
\text{Skel}(W, \lambda) = \bigcup_{\lambda_p = 0} S_p, \quad S_p = \{ x \in W \mid \lim_{t \to \infty} Z^t(x) = p \}.
\]
Each stable manifold is $\lambda$-isotropic, hence $\omega$-isotropic and so at most half-dimensional. A result of F. Laudenbach states that if $Z$ is Morse-Smale, i.e. its stable and unstable manifolds intersect transversely, and if moreover $Z$ is the gradient with respect to a Euclidean metric near critical points, then $\text{Skel}(W, \lambda)$ can be Whitney stratified. However, even in the Morse-Smale case, in general the singularities of $\text{Skel}(W, \lambda)$ are quite complicated, and their smooth topology does not determine the symplectic topology of their neighborhoods. For example, in the simplest case when $W$ is obtained by attaching a handle to a ball along a Legendrian sphere, the skeleton has a unique conical singular point and the symplectic topology depends on the Legendrian isotopy class of the link, not its smooth isotopy class (which is always trivial for manifolds of dimension $> 4$). However, for the class of arboreal singularities described in Definition 1.1 below the situation is different.

First, we introduce some auxiliary notions. A closed subset of a symplectic or contact manifold is called isotropic if it is stratified by isotropic submanifolds. It is called Lagrangian or Legendrian if it is isotropic and purely of the maximal possible dimension. The germ at the origin of a locally simply-connected isotropic subset $L \subset T^*\mathbb{R}^n$ of the cotangent bundle with its standard Liouville structure $\lambda = pdq$ admits a unique lift to an isotropic germ at the origin $\hat{L} \subset J^1\mathbb{R}^n = T^*\mathbb{R}^n \times \mathbb{R}$ of the 1-jet bundle. Given an isotropic subset $\Lambda \subset S^*\mathbb{R}^n$ of the cosphere bundle, its Liouville cone $C(\Lambda) \subset T^*\mathbb{R}$, i.e. the closure of its saturation by trajectories of the Liouville vector field $Z = p\frac{\partial}{\partial p}$, is an isotropic subset.

**Definition 1.1.** Arboreal Lagrangian (resp. Legendrian) singularities form the smallest class $\text{Arb}_{n}^{\text{symp}}$ (resp. $\text{Arb}_{n}^{\text{cont}}$) of germs of closed isotropic subsets in $2n$-dimensional symplectic (resp. $(2n+1)$-dimensional contact) manifolds such that the following properties are satisfied:

(i) (Invariance) $\text{Arb}_{n}^{\text{symp}}$ is invariant with respect to symplectomorphisms and $\text{Arb}_{n}^{\text{cont}}$ is invariant with respect to contactomorphisms.

(ii) (Base case) $\text{Arb}_{0}^{\text{symp}}$ contains $pt = \mathbb{R}^0 \subset T^*\mathbb{R}^0 = pt$.

(iii) (Stabilizations) If $L \subset (X, \omega)$ is in $\text{Arb}_{n}^{\text{symp}}$, then the product $L \times \mathbb{R} \subset (X \times T^*\mathbb{R}, \omega + dp \wedge dq)$ is in $\text{Arb}_{n+1}^{\text{symp}}$.

(iv) (Legendrian lifts) If $L \subset T^*\mathbb{R}^n$ is in $\text{Arb}_{n}^{\text{symp}}$, then its Legendrian lift $\hat{L} \subset J^1\mathbb{R}^n$ is in $\text{Arb}_{n}^{\text{cont}}$.

(v) (Liouville cones) Let $\Lambda_1, \ldots, \Lambda_k \subset S^*\mathbb{R}^n$ be a finite disjoint union of arboreal Legendrian germs from $\text{Arb}_{n-1}^{\text{cont}}$ centered at points $z_1, \ldots, z_k \in S^*\mathbb{R}^n$. Let $\pi : S^*\mathbb{R}^n \to \mathbb{R}^n$ be the front projection. Suppose
- $\pi(z_1) = \cdots = \pi(z_k)$;
- for any $i$, and smooth submanifold $Y \subset \Lambda_i$, the restriction $\pi|_Y : Y \to \mathbb{R}^n$ is an embedding (or equivalently, an immersion, since we only consider germs).
- for any distinct $i_1, \ldots, i_\ell$, and any smooth submanifolds $Y_{i_1} \subset \Lambda_{i_1}, \ldots, Y_{i_\ell} \subset \Lambda_{i_\ell}$, the restriction $\pi|_{Y_{i_1} \cup \cdots \cup Y_{i_\ell}} : Y_{i_1} \cup \cdots \cup Y_{i_\ell} \to \mathbb{R}^n$ is self-transverse.

Then the union $\mathbb{R}^n \cup C(\Lambda_1) \cup \cdots \cup C(\Lambda_k)$ of the Liouville cones with the zero-section form an arboreal Lagrangian germ from $\text{Arb}_{n}^{\text{symp}}$. 
With the above classes defined, we can also allow boundary by additionally taking the product $L \times \mathbb{R}_{\geq 0} \subset (X \times T^*\mathbb{R}, \omega + dp \wedge dq)$ for any arboreal Lagrangian $L \subset (X, \omega)$, and similarly for arboreal Legendrians.

**Figure 1.1.** The simplest non-smooth arboreal singularity is the zero section union the positive conormal of a smooth co-oriented hypersurface.

**Figure 1.2.** This arboreal singularity consists of the zero section union the positive conormal of two smoothly intersecting co-oriented hypersurfaces.

We prove in Section 5 that for fixed dimension $n$ the class of arboreal singularities contains only finitely many local models up to ambient symplectomorphism or contactomorphism. More precisely, to each member of the class $\text{Arb}_{\text{sym}}^n$ one can assign a signed rooted tree $T = (T, \rho, \varepsilon)$ with at most $n + 1$ vertices. Here $T$ is a finite acyclic graph, $\rho$ is a distinguished root vertex, and $\varepsilon$ is a sign function on the edges of $T$ not adjacent to $\rho$. This discrete data completely determines the germ:
Figure 1.3. This arboreal singularity consists of the zero section union the positive conormal of a singular co-oriented hypersurface (namely the front of the arboreal singularity of Figure 1.1).

**Theorem 1.2.** If two arboreal singularities \( L \subset (X,\omega) \), \( L' \subset (X',\omega') \) have the same dimension and signed rooted tree \( T \), then there is (the germ of) a symplectomorphism \((X,\omega) \cong (X',\omega')\) identifying \( L \) and \( L' \).

Within all arboreal singularities, there is the distinguished class with positive sign \( \varepsilon \equiv +1 \).

Figure 1.4. One can obtain a new arboreal singularity from the arboreal singularity of Figure 1.1 in two different ways depending on how it is embedded in \( S^*M \) relative to the vertical distribution \( \nu = \ker(d\pi) \). One yields a positive arboreal singularity and the other does not.
Definition 1.3. An arboreal Lagrangian $L$ (with boundary) in a symplectic manifold $(X, \omega)$ is a piecewise smooth Lagrangian with arboreal singularities (with boundary), i.e. locally modeled on the above class (with boundary). When $L$ is an arboreal Lagrangian whose boundary $\partial L$ is a smooth manifold, we will say that $L$ has smooth boundary.

A positive arboreal Lagrangian $L$ (with boundary) in a symplectic manifold $(X, \omega)$ is a Lagrangian with positive arboreal singularities (with boundary), i.e. locally modeled on the distinguished class with positive sign $\varepsilon \equiv +1$.

Given any positive arboreal Lagrangian $L \subset (X, \omega)$, possibly with boundary, its neighborhood $U \subset X$ admits a canonical, up to homotopy, Lagrangian plane field $\xi \subset TU$. This homotopy class admits a representative $\xi$ which is transverse to $L$, i.e. transverse to the closure of each smooth piece.

Definition 1.4. A polarization of a symplectic manifold $(X, \omega)$ is the choice of a Lagrangian plane field $\xi \subset TX$.

If $(W, \lambda)$ is a Weinstein domain with positive arboreal skeleton, then $(W, \lambda)$, which retracts onto an arbitrarily small neighborhood of its skeleton, admits a polarization. The main result of this paper is that the converse also holds:

![Figure 1.5](image_url)

**Figure 1.5.** The canonical polarization in the neighborhood of an $A_2$ singularity, where the positivity condition is vacuous. One could also turn the tangent field to the zero section counter-clockwise instead of clockwise. The choice is determined by what we call an orientation structure, which in this case is equivalent to a co-orientation of the origin inside the zero-section.

Theorem 1.5. A Weinstein domain $(W, \lambda)$ is homotopic to a Weinstein domain whose skeleton is positive arboreal with smooth boundary if and only if $(W, \lambda)$ admits a polarization.

**Remark 1.6.** The conclusion of Theorem 1.5 can be refined in several ways, some of which we sketch below. For precise formulations see Theorem 9.16 and its corollaries, which use the language of $Wc$-manifolds introduced in Section 2.24.

(i) In fact, for $\xi$ a polarization of $(W, \lambda)$ we can arrange it so that $\xi$ is the canonical polarization in the neighborhood of the positive arboreal skeleton, so in particular $\xi$ is transverse to the skeleton.
Figure 1.6. The canonical polarization in the neighborhood of a positive $A_3$ singularity, where the orientation structure is given by the co-orientation of the $A_3$ front.

(ii) If so desired, the smooth boundary of the positive arboreal skeleton can be pushed out to the boundary of $W$, in which case it serves as a skeleton for a Weinstein pair, see Section 2.4. The Weinstein hypersurface in the pair is thus the ribbon of a smooth Legendrian, and can also be thought of as a stop.

(iii) Theorem 1.5 also holds with input a Weinstein pair $(W, A)$ instead of a Weinstein domain. In this case the conclusion is that the Weinstein hypersurface $A$ can be enlarged by the ribbon of a smooth Legendrian so that the resulting Weinstein pair has a positive arboreal skeleton.

(iv) Moreover, the version of Theorem 1.5 for Weinstein pairs also holds in relative form if the Weinstein hypersurface in the Weinstein pair already has a positive arboreal skeleton and the polarization restricts on it to the canonical polarization. In this case the Weinstein structure can be kept fixed near the Weinstein hypersurface.

(v) Theorem 1.5 and its refinements hold more generally for arbitrary Weinstein manifolds, not necessarily of finite type (i.e. not completions of Weinstein domains), and the result is proved in the same way since the inductive argument stabilizes. We restrict the discussion to the finite type case for simplicity of exposition.

Existence of a polarization for a symplectic manifold $(X^{2n}, \omega)$ is equivalent to asking that the classifying map $X \to BU_n$ of the tangent bundle lifts to $BO_n$, where we endow $TX$ with the homotopically unique almost complex structure compatible with $\omega$.

Definition 1.7. A stable polarization of a symplectic manifold $(X, \omega)$ is the choice of a Lagrangian plane field $\xi \subset TX \oplus \mathbb{C}^d$ for some $d \geq 1$.

The existence of a stable polarization is equivalent to asking that the classifying map $W \to BU$ of the stable tangent bundle lifts to $BO$. Recall that a Weinstein domain $(W, \lambda)$ has the homotopy type of a half-dimensional CW complex. Recall also that $BO_k \to BO_{k+1}$ is
$k$-connected and $BU_k \to BU_{k+1}$ is $(2k+1)$-connected. Therefore, it follows that for Weinstein domains the existence of a polarization is equivalent to the existence of a stable polarization. From the above, one can check that many Weinstein domains admit polarizations, notably smooth complete intersections in complex affine space. Indeed, if $X \subset \mathbb{C}^N$ is a smooth complete intersection, then its normal bundle is trivial, hence its tangent bundle is stably trivial and in particular $X$ admits a stable polarization. This proves:

**Corollary 1.8.** Let $X \subset \mathbb{C}^N$ be a complete intersection. Then $X$ is Weinstein homotopic to a Weinstein manifold whose skeleton is positive arboreal with smooth boundary.

An arboreal skeleton with boundary comes with a natural additional orientation structure, induced from the ambient symplectic structure. We formalize this in the notion of an arboreal space and prove the following counterpart to Theorem 1.5.

**Theorem 1.9.** Any compact arboreal space with boundary arises as the skeleton of a Weinstein domain $(W, \lambda)$, unique up to Weinstein homotopy.

![Figure 1.7](image.png)

**Figure 1.7.** The simplest example of two non-equivalent arboreal spaces with diffeomorphic underlying arboreal Lagrangians. These are skeleta for the pair of pants and the once-punctured torus respectively. The arboreal spaces are distinguished by the orientation structure, which is additional data, and in this dimension reduces to the usual cyclic structure on ribbon graphs. We thank A. Oancea for pointing out this example.

Hence, up to Weinstein homotopy, those Weinstein domains which admit polarizations are precisely the Weinstein thickenings of positive arboreal spaces. Moreover, the relative form of Theorem 1.5 implies a uniqueness result for positive arboreal skeleta. To state it we need the notion of a Weinstein concordance, which is best defined in the language of Wc-manifolds introduced in Section 2.24. Briefly, if $\lambda_1$ and $\lambda_2$ are two Weinstein structures on the same symplectic domain $(W, \omega)$ with Liouville fields $Z_1, Z_2$, respectively, then a Weinstein
such that the Liouville field $Z$ of $\lambda$ is tangent to the boundary $W \times T^*_0[0,1] \cup W \times T^*_1[0,1]$ and restricts to $Z_1 + u\partial/\partial u$ and $Z_2 + u\partial/\partial u$ over $W \times T_0^*[0,1]$ and $W \times T_1^*[0,1]$ respectively.

**Definition 1.10.** A positive arboreal concordance is a Weinstein concordance $(W \times T^*[0,1], \lambda)$ whose skeleton is a positive arboreal Lagrangian with boundary. We assume that the boundary of the skeleton is smooth in the interior of $W \times T^*[0,1]$ but we allow simple corners with one face contained in $W \times T_0^*[0,1]$ or $W \times T_1^*[0,1]$ and the other face transverse to it.

The uniqueness statement then reads as follows:

**Theorem 1.11.** Suppose that $\lambda_1$ and $\lambda_2$ are two homotopic Weinstein structures on $W$ whose skeleta are positive arboreal Lagrangians with smooth boundary, transverse to polarizations $\xi_1$ and $\xi_2$ respectively. Then $\xi_1$ and $\xi_2$ are stably homotopic if and only if there is a positive arboreal concordance between $\lambda_1$ and $\lambda_2$.

We remark that the classification of polarizations on a Weinstein manifold up to stable homotopy is in general weaker than the classification up to homotopy.

1.3. **Further development of the arborealization program.** Work in progress of the authors extends the results of this paper and the broader arborealization program in two directions. First, it considers a 1-parametric version of the problem, by introducing a suitable notion of “arboreal homotopy of positive arboreal spaces”, i.e. the minimal sequence of combinatorial Reidemeister type moves necessary to connect two homotopic Weinstein structures with positive arboreal skeleta. Note that substantial progress in this direction appeared in Zorn’s thesis [Z18], where local mutations of arboreal skeleta are classified.

The second direction is an extension of the arborealization program to a more general class of Weinstein manifolds. Existence of a polarization is a rather restrictive condition, for example a cohomological obstruction is that all odd Chern classes are 2-torsion. But even allowing for non-positive arboreal skeleta, there are still homotopical obstructions to their existence. While existence of a polarization is not a necessary condition for existence of a non-positive arboreal skeleton, there is a closely related necessary condition: existence of an $(n,n-1)$-polarization. This is, roughly speaking, a polarization which is allowed to degenerate to an $(n-1)$-dimensional isotropic subspace. More precisely, it is defined as follows. Given a symplectic vector space $(E, \omega)$ let $L_n(E)$ denote its Lagrangian Grassmanian of non-oriented Lagrangian planes, and $I_{n-1}(E)$ denote the Grassmanian of non-oriented isotropic $(n-1)$-dimensional subspaces. We further denote by $F_{n,n-1}(E)$ the flag manifold $\{(\lambda, \mu) : \lambda \in L_n(E), \mu \in I_{n-1}(E), \mu \subset \lambda\}$. Let $\pi_L$ and $\pi_I$ be the tautological projections $\pi_L : F_{n,n-1}(E) \to L_n(E)$ and $\pi_I : F_{n,n-1}(E) \to I_{n-1}(E)$. Consider the double cone $C_n(E) := (F_{n,n-1} \times [0,1] \cup L_n(E) \cup I_{n-1}(E))/\{f \times 0 \sim \pi_L(f), f \times 1 \sim \pi_I(f)\}$. Given any symplectic vector bundle $E$, we will use the notation $C_n(E)$ for the associated cone bundle. An $(n,n-1)$-polarization on a symplectic manifold $V$ is by definition a section of the cone bundle $C_n(T(V))$. 
One can show that for \( n \leq 5 \) any \( 2n \)-dimensional Weinstein manifold admits a \((n, n - 1)\)-polarization, while for \( n = 6 \) there are obstructions (e.g. one must have \( c_3(V) = c_1(V)c_2(V) \)).\(^1\) It is possible that this condition is also sufficient for existence of an arboreal skeleton.

For completely arbitrary Weinstein manifolds it is not clear whether one should expect there to be a simple class of singularities (which by necessity would have to be more general than arboreal) which the skeleton can always be arranged to have. In any case, even the most optimistic hopes of constructing a reasonable skeleton for a Weinstein manifold will in all likelihood require the existence of “Maslov data”, i.e. the homotopical trivialization required to define Fukaya categories or microlocal sheaves.

### 1.4. Flexibility of caustics and the ridgification theorem

We discuss a toy example to illustrate our strategy of proof. Consider a Weinstein domain \((W^{2n}, \lambda)\) which is obtained from the standard Darboux ball \( B \subset (\mathbb{R}^{2n}, pdq - qdp) \) by attaching an index \( n \) handle, i.e. the neighborhood of a Lagrangian disk, along a Legendrian sphere \( \Lambda \subset \partial B = (S^{2n-1}, \xi_{\text{std}}) \).

The skeleton of \( W \) is homeomorphic to the \( n \)-sphere: it consists of the Lagrangian disk union the Liouville cone on \( \Lambda \) with respect to the radial Liouville field on \( \mathbb{R}^{2n} \), union the origin.

Replace the radial Liouville structure \( pdq - qdp \) on \( \mathbb{R}^{2n} \) with the Morse-Bott canonical Liouville structure \( pdq \) on the cotangent bundle \( T^*\mathbb{R}^n \), or rather on \( T^*D \) for \( D \subset B \) a Lagrangian \( n \)-disk bounding a Legendrian unknot \( \partial D \subset S^{2n-1} \) which we assume disjoint from \( \Lambda \). Then the radial Liouville cone on \( \Lambda \subset S^{2n-1} \) in \( B \) gets replaced with the fibrewise Liouville cone on \( \Lambda \subset S^*D \) in \( T^*D \), where \( S^*D \) denotes the cosphere bundle, see Figure 1.8.

In this way the conical singularity gets spread out over the disk. The skeleton of the new Weinstein structure on \( W \) consists of the Lagrangian handle, union the Liouville cone on \( \Lambda \) with respect to the canonical Liouville structure on \( T^*D \), union the zero section \( D \). The singularities of this new Lagrangian skeleton are thus related to the singularities of the map \( \pi|_{\Lambda} : \Lambda \to D \), where \( \pi : S^*D \to D \) is the front projection. These singularities correspond to the tangencies of \( \Lambda \) with the distribution \( \nu = \ker(d\pi) \subset T(S^*D) \) and are also known as caustics in the literature. For example, if \( \pi|_{\Lambda} : \Lambda \to D \) has no caustics, then it is an immersion and so by axiom (v) of Definition 1.1, after a generic perturbation (to ensure self-transversality), the skeleton of \( W \) is arboreal.

The idea of blowing up conical Lagrangian singularities was already present at the inception of the arborealization program, but for our purposes it is necessary to perform the blowing up procedure globally, at the level of Weinstein structures. This strategy was successfully implemented by Starkston in [S17] for the 4-dimensional case. Starkston moreover showed how to explicitly arborealize the semi-cubical cusp singularities which generically appear in the front projections of 1-dimensional Legendrians, thus leading to her result that 4-dimensional

\(^1\)We thank John Pardon for suggesting to us the notion of a \((n, n - 1)\)-polarization, and to Søren Galatius for computations of obstructions.
Weinstein manifolds always admit arboreal skeleta. In higher dimensions this strategy encounters the difficulty that there is no classification theorem for the generic singularities of front projections, nor can anything remotely close to a satisfactory classification be hoped for.

Nevertheless, there holds an h-principle for the simplification of caustics. The h-principle states that if there is no homotopy theoretic obstruction to the simplification of caustics, then the simplification can be achieved by means of a Legendrian isotopy (or for Lagrangians, by means of a Hamiltonian isotopy). Results in this direction first appeared in work of Entov [E99], and the full h-principle was established in work of the first author [AG18a], [AG18b].

Returning to our toy example, if we know that $W$ admits a polarization as in the hypothesis of our main Theorem 1.5 and we further assume that the dimension of $W$ is congruent to 2 modulo 4, then it follows from the h-principle that there exists a Legendrian isotopy which deforms $\Lambda$ to a Legendrian whose front projection $\pi|_{\Lambda} : \Lambda \to D$ only has cusp singularities. In this case we can proceed as in [S17] to conclude that the skeleton of $W$ can be arborealized by a Weinstein homotopy. When the dimension of $W$ is congruent to 0 modulo 4 there is an additional homotopical subtlety to consider, even in this toy example, but we do not pursue this further since one also encounters additional and more serious difficulties when attempting to implement the above strategy on a Weinstein manifold with a more complicated handlebody presentation. For example, one would need to know that the relevant homotopical obstruction to applying the h-principle vanishes for every handle, yet it is unclear how to arrange for this even under the global assumption of existence of a polarization.

So instead of attempting a direct application of the h-principle, we will exploit the fact that for the purposes of arborealization we can allow deformations which are more general than a Hamiltonian isotopy, namely we can deform the skeleton of a Weinstein manifold by any homotopy of the Weinstein structure. This allows the skeleton to develop genuine singularities.
(i.e. non-smooth points) along which its field of tangent planes jumps discontinuously. By introducing these jumps we can always avoid tangencies with the distribution $\nu = \ker(d\pi)$. This strategy was made precise in [AGEN19], where we introduced a class of singular Lagrangians, called ridgy, and proved an h-principle without pre-conditions, which allows one to deform any Lagrangian so that it becomes transverse to any given Lagrangian distribution, at the expense of developing ridgy singularities. This result, which we refer to as the ridgification theorem, uses the h-principle for the simplification of caustics as one of the key ingredients in its proof, but is better suited for the task at hand since there are no hypotheses needed for its application.

![Figure 1.9. The ridge singularity in one dimension.](image)

![Figure 1.10. The ridgification theorem in action: the zero section $M \subset T^*M$ gets deformed to a ridgy Lagrangian $L \subset T^*M$ which is transverse to the red distribution. Of course in this simple example a single ridge point would suffice, but the proof always produces a ridge locus which is the union of homologically trivial hypersurfaces, in this case two ridge points.](image)

Ridgy singularities are extremely simple: they consist of the corner $\{p = |q|\} \subset \mathbb{R}^2$ together with its products and stabilizations. Moreover, ridgy singularities can be explicitly arborealized while maintaining transversality to any given Lagrangian distribution, see Figure 1.11. So our revised strategy to arborealize a Weinstein manifold $W$ is the following: (1) blow up the singularities as before to replace Liouville cones on a point with Liouville cones on Lagrangian disks, (2) apply the ridgification theorem to get rid of all singularities of the
resulting front projections, and finally (3) arborealize the ridgy Lagrangians produced by the ridgification theorem, while maintaining transversality to the relevant distributions, to obtain an arboreal skeleton for $W$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ridge1.png}
\caption{Arborealization of an order 1 ridge.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ridge2.png}
\caption{Arborealization of an order 2 ridge.}
\end{figure}

Finally, a word on how we deal with the issue of compatibility at the interaction of three or more strata of the skeleton: this is where the notion of positivity comes up in an essential way. The key fact is that if a Lagrangian distribution $\eta$ on $T^*\mathbb{R}^n$ corresponds to a family of positive definite quadratic forms, then it is transverse to the conormal $T^*_H\mathbb{R}^n \subset T^*\mathbb{R}^n$ of any immersed hypersurface $H \subset \mathbb{R}^n$ in a neighborhood of the zero section. For the interaction of say three strata, we use the ridgification theorem to achieve transversality for two strata at a time and then conclude that transversality also holds for the third stratum \emph{for free} using the above observation on positivity. This yields the desired compatibility. The existence of a global polarization for $W$ will play a key role in guiding the positivity condition globally, thus enabling the successful implementation of the above strategy.

1.5. Structure of the article. The paper is organized as follows:

In Section 2 we give the necessary preliminaries regarding Weinstein manifolds. This includes a review of notions in the literature as well as the introduction of a new framework for working with Weinstein structures. The key notions are that of a Weinstein manifold with boundary and corners, which we call a Wc-manifold, and that of a cotangent building, which provides a way of presenting a Wc-manifold in terms of successive attachments of cotangent
bundles. The concept of a cotangent building gives an alternative to the standard Weinstein handlebody presentation for Weinstein manifolds, and is better suited for our purposes.

In Section 3 we discuss the notion of positivity, which is the key ingredient that allows us to control the interaction of three or more strata in the Lagrangian skeleta under consideration. In this preliminary discussion, to be taken up again in Section 7, we focus on the linear algebraic aspects.

In Sections 4, 5, and 6 we introduce and study arboreal singularities from the axiomatic viewpoint presented above. In particular we prove that there are only finitely many local models for arboreal singularities in each dimension, indexed by signed rooted trees, and that arboreal spaces have unique Weinstein thickenings. Of particular importance for the purposes of this paper are the positive arboreal singularities, which correspond to signed rooted trees where all the signs are positive.

In Section 7 we discuss positivity for cotangent buildings and relate it to the notion of positivity for arboreal spaces. We also show that existence of a positive arboreal skeleton for the homotopy class of a Weinstein structure implies the existence of a polarization, proving one half of our main theorem 1.5.

In Section 8.1 we recall the notion of a ridgy Lagrangian from our previous work [AGEN19] and state the main result of that paper, which we call the ridgification theorem, as well as its non-integrable counterpart, which we call the formal ridgification theorem, and which also plays an important role since it holds in extension form. The main point is that by allowing for ridgy singularities we can deform any Lagrangian submanifold so that it becomes transverse to any given Lagrangian distribution, and moreover the deformation can be achieved by means of a Weinstein homotopy.

Finally, in Section 9 we show how the ridgy singularities produced by the ridgification theorem can be further deformed into arboreal singularities. Furthermore, the deformation is realized by a Weinstein homotopy and the end result is a positive arboreal Lagrangian which serves as skeleton for the deformed Weinstein structure. This proves the other half of our main theorem 1.5.

1.6. Acknowledgements. We are very grateful to Laura Starkston who collaborated with us on the initial stages of this project. We are also grateful to John Pardon who helped us to crystalize the notion of an \((n, n-1)\)-polarization and to Søren Galatius who explained to us how to compute obstructions to its existence.

The first author thanks CRM Montreal for its hospitality. The second author thanks RIMS Kyoto and ITS ETH Zurich for their hospitality. The third author thanks MSRI for its hospitality.

Finally, we are very grateful for the support of the American Institute of Mathematics, which hosted a workshop on the arborealization program in 2018 from which this project has greatly benefited.
2. WC-mANIFOLDS AND cotangent BUILDINGS

In this section we recall the basic definitions concerning Liouville and Weinstein manifolds, and introduce the language of WC-manifolds and cotangent buildings, which will be used throughout the text.

2.1. Liouville manifolds.

2.1.1. Liouville domains. An exact symplectic manifold \((W, \lambda)\) is a compact manifold with boundary equipped with an exact symplectic form \(\omega\), together with a choice of primitive \(\lambda\), i.e. \(\omega = d\lambda\). Since \(\omega\) is non-degenerate, the 1-form \(\lambda\) corresponds to a vector field \(Z\) under the contraction \(\iota_Z \omega = \lambda\).

**Definition 2.1.** A Liouville domain is a Liouville domain \((W, \lambda)\) such that \(Z\) is outwards transverse to \(\partial W\).

Equivalently, \((W, \lambda)\) is Liouville if \(\lambda|_{\partial W}\) is a contact form and the orientation defined on \(\partial W\) by the volume form \(\lambda \wedge (d\lambda)^{n-1}|_{\partial W}\) coincides with the orientation of \(\partial W\) as the boundary of \(W\), where \(W\) is itself oriented by the volume form \(\omega \wedge \lambda\).

We call \(\lambda\) the Liouville form and \(Z\) the Liouville field. By Cartan’s formula for the Lie derivative, the condition \(\omega = d\lambda\) is equivalent to \(L_Z \omega = \omega\). This means that the vector field \(Z\) is symplectically conformally expanding, i.e. \((Z^t)_* \omega = e^t \omega\), \(t \geq 0\), for \(Z^t\) the flow of \(Z\). Equivalently, the vector field \(-Z\) is symplectically conformally contracting and this viewpoint will sometimes be more natural in what follows. Note that we also have \((Z^t)_* \lambda = e^t \lambda\). The object of interest in this paper is the following.

**Definition 2.2.** The skeleton of a Liouville domain \((W, \lambda)\) is the subset
\[
\text{Skel}(W, \lambda) = \bigcap_{t > 0} Z^{-t}(W).
\]

In other words, Skel\((W, \lambda)\) is the attractor of the positive flow of the contracting field \(-Z\).

We will be going back and forth between Liouville domains and the closely related notion of a Liouville manifold, whose definition we recall next.

2.1.2. Liouville manifolds. Recall that the symplectization \(S(Y, \xi)\) of a contact manifold \((Y, \xi)\) equipped with a contact form \(\alpha\) for \(\xi\) is the exact symplectic manifold \(Y \times \mathbb{R}\) equipped with the primitive \(\lambda = e^t \alpha\), where \(t \in \mathbb{R}\). We can equivalently consider \(Y \times \mathbb{R}^+\) with \(\lambda = s \alpha\) for \(s = e^t \in \mathbb{R}^+\) the multiplicative coordinate.

The symplectization of a contact manifold \((Y, \xi)\) can be defined invariantly without choosing a contact form. Assuming that \(Y\) is connected and \(\xi\) is co-oriented, we define \(S(Y, \xi)\) as follows. Let \(N(\xi) \subset T^* Y\) be the total space of the conormal line bundle to \(\xi\) and \(N_+ (\xi)\) the component of \(N(\xi) \setminus Y\) consisting of 1-forms defining the given co-orientation of \(\xi\). Then \(\lambda_\xi = pdq|_{N_+ (\xi)}\) is a primitive of the exact symplectic form \(\omega = d\lambda_\xi\) and we define \(S(Y, \xi)\) as \(N_+ (\xi)\) endowed with this structure. A choice of a contact form \(\alpha\) for \(\xi\) identifies \(N_+ (\xi)\) with \(Y \times \mathbb{R}^+\) and \(\lambda_\xi\)
with so. For example, the symplectization of the cosphere bundle $S^*M$ is the complement of the zero section in $T^*M$.

In the other direction, the contactization of an exact symplectic manifold $(N,\mu)$ is the manifold $M \times \mathbb{R}$ equipped with the contact form $\mu - du$, where $u \in \mathbb{R}$. For example, the contactization of the cotangent bundle $T^*M$ is the 1-jet bundle $J^1M$.

**Definition 2.3.** A finite type Liouville manifold $(X,\lambda)$ consists of a (non-compact) boundaryless exact symplectic manifold such that the following conditions hold.

(L1) The vector field $Z$ which is $\omega$-dual to $\lambda$ is complete for $t \to \pm \infty$.

(L2) There exists a compact domain $W \subset X$ such that $Z$ is outward transverse to the boundary $\partial W$, i.e. $(W,\lambda|_W)$ is a Liouville domain, and such that the union of forward trajectories of $Z$ starting at $\partial W$ is equal to $X \setminus \text{Int} W$.

We call Liouville domains as in (L2) defining.

**Definition 2.4.** The skeleton of a finite type Liouville manifold $(X,\lambda)$ is defined to be

$$\text{Skel}(X,\lambda) = \bigcup_{C} \bigcap_{t>0} Z^{-1}(C), \quad C \subset X \text{ is compact}.$$

Clearly, $\text{Skel}(X,\lambda) = \text{Skel}(W,\lambda)$ for any defining Liouville domain $W \subset X$. Hence the skeleton of a finite type Liouville manifold is compact. Note that for any such $W$ we may use the Liouville flow starting at $\partial W$ to parametrize $X \setminus \text{int} W$ as $\partial W \times [0,\infty)$. Since the Liouville field conformally expands $\lambda$ we have $\lambda|_{\partial W \times [0,\infty)} = e^t(\lambda|_{\partial W})$ for $t \in [0,\infty)$ the coordinate parametrizing the Liouville flow, so $X \setminus \text{int} W$ is the positive part of the symplectization of the contact manifold $(\partial W,\lambda|_{\partial W})$.

Conversely, any Liouville domain $(W,\lambda)$ can be completed to a Liouville manifold by attaching to it the positive part of the symplectization of $(\partial W,\lambda|_{\partial W})$. Explicitly:

$$X = W \bigcup_{\partial W \sim \partial W \times 0} \partial W \times [0,\infty),$$

where the Liouville form $\lambda$ on $W$ extends to $\partial W \times [0,\infty)$ as $e^t(\lambda|_{\partial W})$.

**Definition 2.5.** The ideal boundary of a finite type Liouville manifold $(X,\lambda)$ is defined to be

$$\partial \infty X = (X \setminus \text{Skel}(X,\lambda))/\mathbb{R}^+,$$

where the $\mathbb{R}^+$ action is given by the positive flow of $Z$.

Since $\lambda(Z) = 0$, the form $\lambda$ descends to $\partial \infty X$ as a contact structure $\xi_\infty$. A choice of a contact form $\alpha$ for $\xi_\infty$ yields an isomorphism $j_\alpha: S(\partial \infty X,\xi_\infty) \to (\partial \infty X \times \mathbb{R}, e^\alpha) \to (X \setminus \text{Skel}(X,\lambda),\lambda)$ which maps $\partial \infty X \times 0$ contactomorphically onto the boundary of a defining Liouville domain. We denote by $\pi_\infty: X \setminus \text{Skel}(X,\lambda) \to \partial \infty X$ the projection along the trajectories of $Z$.

**2.1.3. Deformations of Liouville structures.**

**Definition 2.6.** A homotopy of finite type Liouville structures $(X,\lambda_t)$ is a family $\lambda_t$, $t \in [0,1]$, of finite type Liouville structures on a manifold $X$ admitting a smooth family $W_t \subset X$ of defining Liouville domains.
Given such a homotopy \((X, \lambda_t)\) there exists an isotopy \(\phi_t : X \to X\) such that \(\phi_t^* \omega_t = \omega_0\), where \(\omega_t = d\lambda_t\). Moreover, one can always arrange that \(\phi_t^* \lambda_t = \lambda_0 + dH_t\) for uniformly compactly supported functions \(H_t\), see [CE12], Sections 11.1 and 11.2. In particular, it is always sufficient to consider homotopies fixing the symplectic form, and moreover changing the Liouville form by adding a compactly supported exact form. Note that for any fixed \(\omega\) the space of Liouville structures on \((X, \omega)\) which are fixed at infinity is convex, hence any two such Liouville structures are canonically homotopic.

Remark 2.7. Unless explicitly stated otherwise, all Liouville structures under consideration in this paper are of finite type and hence we will stop mentioning this below.

We will often abuse terminology and say that two Liouville manifolds \((X_1, \lambda_1)\) and \((X_2, \lambda_2)\) are Liouville homotopic if there exists a symplectomorphism \(F : (X_1, d\lambda_1) \to (X_2, d\lambda_2)\) such that \((X_1, F^* \lambda_2)\) is a Liouville manifold which is Liouville homotopic to \((X_1, \lambda_1)\). In all cases where such terminology will be abused, the symplectomorphism will be obvious and unique up to contractible choice.

If \(F^* \lambda_2 = \lambda_1\) on the nose, we say that \((X_1, \lambda_1)\) and \((X_2, \lambda_2)\) are Liouville isomorphic. For \((X, \lambda)\) and \((Y, \mu)\) Liouville domains or manifolds we call a smooth, proper embedding \(F : (X, \lambda) \to (Y, \mu)\) a Liouville embedding if \(F^* \mu = \lambda\).

2.1.4. Liouville germs. It will often be convenient to express our constructions in the language of germs.

Definition 2.8. A Liouville germ \((\mathcal{X}, \lambda)\) is the equivalence class of defining Liouville domains for a fixed Liouville manifold \((X, \lambda)\).

Definition 2.9. A Liouville embedding of a Liouville germ \((\mathcal{X}, \lambda)\) into a Liouville domain or manifold \((Y, \mu)\) is an equivalence class of Liouville embeddings \(F_W : (W, \lambda|_W) \to (Y, \mu)\), where \(W \subset X\) is a defining domain for \((X, \lambda)\) and we identify \(F_{W_0}\) with \(F_{W_1}\) if they agree on a defining domain \(W_2\) for \((X, \lambda)\) contained in both \(W_0\) and \(W_1\).

Example 2.10. The cotangent bundle \(X = T^*M\) of a closed manifold \(M\) is a Liouville manifold with \(\lambda = pdq\) the canonical 1-form. The Liouville field is the fibrewise radial vector field \(Z = p\partial_p\) and the skeleton is the zero section \(M \subset T^*M\). As a Liouville domain we can take the unit disk bundle \(W = \{||p|| \leq 1\}\) relative to any Riemannian metric on \(M\). The ideal boundary is the cosphere bundle \(S^*M\), i.e the (positively) projectivized cotangent bundle, which is contactomorphic to the unit sphere bundle \(\partial W = \{||p|| = 1\}\) for any choice of Riemannian metric. We will use the notation \(\mathcal{X}^*M\) for the germ of the cotangent bundle \(T^*M\).

2.2. Weinstein manifolds. The notion of Weinstein manifold was first introduced in [EG91] building on [E90] and [W91].
2.2.1. **Lyapunov functions.** Let $(X, \lambda)$ be a Liouville manifold. While Skel$(X, \lambda)$ always has its $2n$-dimensional Lebesgue measure equal to 0, its dimension can nonetheless be quite large if no extra conditions are imposed on the Liouville structure. For example, McDuff constructed in [MD91] a Liouville structure on $T^*\Sigma_g \setminus \Sigma_g$ for $\Sigma_g$ a closed surface of genus $g > 1$ whose skeleton has codimension 1. To tame the dynamics of the Liouville flow we require existence of a Lyapunov function.

**Definition 2.11.** We say that $\phi : X \to \mathbb{R}$ is a **Lyapunov function** for the Liouville manifold $(X, \lambda)$ if $Z$ is gradient-like for $\phi$, i.e. if there holds

\[(W1) \quad d\phi(Z) \geq \delta(\|Z\|^2 + \|d\phi\|^2)\]

for some Riemannian metric on $X$ and some constant $\delta > 0$.

Note that the space of Lyapunov functions for a given Liouville structure is convex, hence contractible as soon as it is nonempty. Consider the set of critical points Crit$(\phi) = \{d\phi = 0\}$ of the Lyapunov function, which by (W1) is the same as the zero set $\{\lambda = 0\}$ of the Liouville form. The conditions (L1) and (W1) imply that Skel$(X, \lambda)$ is the union of the $Z$-stable manifolds of the critical points of $\phi$, i.e. points converging to Crit$(\phi)$ in forward time. However, as far as we know, we have to add some further constraints on the Lyapunov function $\phi$ in order to deduce any meaningful properties.

**Definition 2.12.** A **Morse-Weinstein manifold** is a Liouville manifold $(X, \lambda)$ for which there exists a Morse Lyapunov function $\phi : X \to \mathbb{R}$.

Under this assumption it was shown in [CE12], see also [EG91, E95], that

\[(W2) \quad \text{Skel}(X, \lambda) \text{ is the union of submanifolds which are isotropic for } \lambda, \text{ and hence for } \omega.\]

![Figure 2.1](image.png)

**Figure 2.1.** The skeleton of an open Riemann surface with Morse Lyapunov function.

**Remark 2.13.** Note that $\lambda$-isotropic is equivalent to $\omega$-isotropic and $Z$-conic.
Without any assumptions on $\phi$, it is not known whether a triple $(X, \lambda, \phi)$ which satisfies (L1), (L2) and (W1) is homotopic in the class of such triples to one with a Morse Lyapunov function. However, condition (W2) holds for a larger class of taming functions, for example when $\phi$ is generalized Morse, see [CE12]. Moreover, it is often most natural to consider a Lyapunov function which is not Morse or even generalized Morse, such as the Morse-Bott function $\phi = ||p||^2$ on $(T^*M, pdq)$.

2.2.2. Morse-Bott Lyapunov functions. In this paper we will consider a version of Morse-Bott Weinstein structures which allows the critical set of the Lyapnuov function to have nonempty boundary, and even corners. Our definition is a slight variation of Starkston’s definition in [S17] of a Weinstein structure which is Morse-Bott with boundary.

Let $E^+ \oplus E^0 \oplus E^-$ be the decomposition of the tangent space to $X$ at a zero of $Z$ into generalized eigenspaces of the differential $dZ$ with eigenvalue having positive, zero or negative real part respectively. Suppose first that a connected component $C$ of $\text{Crit}(\phi)$ is a smooth, proper submanifold of $X$ without boundary. Then the Morse-Bott condition on a triple $(X, \lambda, \phi)$ along the component $C$ is the following:

(MB1) $Z$ is non-degenerate in the directions normal to $C$, i.e. $TC = E^0|_C$.

When $\partial C \neq \emptyset$ we moreover demand that in a neighborhood of $\partial C$ the triple $(X, \lambda, \phi)$ takes a special form, which we now describe. Recall the Weinstein normal form for Morse critical points in a Weinstein manifold of dimension 2. There are two possibilities depending on the index: $k = 0$ or $k = 1$.

$(k = 1)$ $\lambda_1 = 2pdq + qdp$, $\phi_1 = p^2 - \frac{1}{2}q^2$.

$(k = 0)$ $\lambda_0 = \frac{1}{2}(pdq - qdp)$, $\phi_0 = \frac{1}{4}(p^2 + q^2)$.

We also recall the standard Morse-Bott normal form corresponding to the canonical Liouville structure on $T^*\mathbb{R}$.

(std) $\lambda_{\text{std}} = pdq$, $\phi = \frac{1}{2}p^2$.

We construct local models for Morse-Bott with boundary Weinstein structures on $T^*\mathbb{R}$ with the half line $\{p = 0, q \geq 0\}$ as the critical set. The construction consists of patching up the above Morse normal forms on the half plane $\{q \leq 0\}$ with the standard Morse-Bott normal form on the half plane $\{q \geq 0\}$. Take any smooth function $\psi : \mathbb{R} \to [0, 1]$ satisfying the following properties:

(i) $\psi = 0$ on $[0, \infty)$
(ii) $\psi = 1$ on $(-\infty, -\varepsilon]$ for some $\varepsilon > 0$.
(iii) $\psi < 1$ and $\psi' < 0$ on $(-\varepsilon, 0)$ for that same $\varepsilon > 0$.

In the case $k = 1$ we set $f_1(q, p) = \psi(q)pq$ and in the case $k = 0$ we set $f_0(q, p) = -\frac{1}{2}\psi(q)pq$. In both cases the patched up model is then given by $\lambda_k^{\text{MB}} = \lambda_{\text{std}} + df_k$, which has Liouville field $Z_k^{\text{MB}} = Z_{\text{std}} + X_{f_k}$ for $X_{f_k}$ the Hamiltonian vector field of $f_k$. Explicitly, when $k = 1$ we have

$$Z_1^{\text{MB}} = (1 + \psi(q) + \psi'(q)q)p\partial_p - \psi(q)q\partial_q.$$
Since the coefficient which multiplies $p\partial_p$ is positive, we deduce that $Z_{1}^{MB}$ admits a Lyapunov function. For example, if we set $\phi = p^2 - h(q)$ for

$$h(q) = \int_0^q \psi(t)tdt,$$

then $Z_{1}^{MB}$ is the Euclidean gradient of $\phi$ along $p = 0$ and we have $d\phi(Z_{1}^{MB}) > 0$ on all of $T^*\mathbb{R}$, from which the claim follows. Similarly, one verifies that $Z_{0}^{MB}$ also admits a Lyapunov function.

**Definition 2.14.** We call $(T^*\mathbb{R}, \lambda_{k}^{MB})$ the *Weinstein normal form for Morse-Bott boundary* of index $k = 0, 1$.

![Figure 2.2](image_url)

**Figure 2.2.** The Weinstein normal form for Morse-Bott boundary of index 1 and 0 respectively.

The space of permissible $\psi$ is convex, hence the normal form is well defined up to a contractible choice. Note that the behavior of the Liouville flow along the half line $\{p = 0, q \leq 0\}$, which shares its boundary point $\{p = 0, q = 0\}$ with the critical set $\{p = 0, q \geq 0\}$, can be either repelling or attracting depending on whether $k = 0$ or $k = 1$. We also consider products of these normal forms. Thus we obtain Weinstein structures on $T^*\mathbb{R}^n$ where the critical set is the positive quadrant $\{p_j = 0, q_j \geq 0\}$, which not only has boundary but also has corners of all orders $\leq n$.

**Definition 2.15.** The *Weinstein normal form for Morse-Bott n-fold corners* $(T^*\mathbb{R}^n, \lambda_{r}^{MB})$ of index $0 \leq r \leq n$ is the product of $n - r$ copies of $(T^*\mathbb{R}, \lambda_{0}^{MB})$ and $r$ copies of $(T^*\mathbb{R}, \lambda_{1}^{MB})$.

Going back to a triple $(X, \lambda, \phi)$ as before, consider a component $C \subset X$ of the critical set of $\phi$ such that $C$ is a smooth, proper submanifold of $X$ with nonempty boundary $\partial C$ and possibly with higher order corners. By definition, if $x \in \partial_k C \subset \partial C$ is a corner of order $k$, then there is a collar neighborhood $U \times \mathcal{I}^k$ of $x$ in $C$, where $\mathcal{I}$ is the germ of the interval $[0, 1)$ at 0 and $U \subset \partial_k C$ is a neighborhood of $x$ in the order $k$ corner locus. For such a collar neighborhood we have a local symplectomorphism near $x$ between $X$ and $T^*U \times T^*\mathbb{R}^k \times \mathbb{C}^m$, with $C$ corresponding to $U \times \mathcal{I}^k \times 0$. The condition we require is that for each such $k$-fold point $x \in \partial_k C$ there exists a collar neighborhood $U \times \mathcal{I}^k$ as above such that with respect to the splitting $T^*U \times T^*\mathbb{R}^k \times \mathbb{C}^m$ there holds:
\(\lambda\) splits as a direct sum with the summand corresponding to \(T^*U\) given by the canonical Liouville form and the summand corresponding to \(T^*\mathbb{R}^k\) given by the Weinstein normal form for Morse-Bott \(k\)-fold corners \(\lambda_{MB}^r\) for some \(r \leq k\).

**Definition 2.16.** A generalized Morse-Bott Weinstein manifold is a Liouville manifold \((X, \lambda)\) which admits a Lyapunov function \(\phi\) whose critical set is a union of smooth, proper submanifolds with corners \(C \subset X\) satisfying (MB1) and (MB2).

**Remark 2.17.** It was shown by Starkston [S17] that in the boundary repelling Morse-Bott case the intersection of the \(Z\)-stable manifold of a component \(C\) of the critical set of \(\phi\) with a neighborhood of \(C\) is a smoothly embedded isotropic submanifold of \(X\), even without requiring standard local models. To check that this property also holds near the boundary attracting or mixed behavior points of the Morse-Bott structures considered above is straightforward, since it suffices to consider the explicit local model.

**Lemma 2.18.** Suppose that \((X, \lambda_0, \phi_0)\) is a generalized Morse-Bott Weinstein manifold. Then there exists a homotopy \((\lambda_t, \phi_t)\), fixed outside of a compact subset, such that \((\lambda_t, \phi_t)\) is Morse-Weinstein for \(t > 0\).

**Proof.** We give the proof in the case where each component \(C \subset \Sigma\) of the critical set of \(\phi\) is of the critical dimension \(n = \frac{1}{2} \dim(X)\), the general case is similar and in any case will not be needed for our purposes. Suppose first that \(C \subset \Sigma\) has empty boundary. Since \(C\) is Lagrangian, by the Weinstein neighborhood theorem it has a neighborhood in \(X\) symplectomorphic to a neighborhood of the zero section in \((T^*C, \lambda_{std} = pdq)\). Therefore we may from the onset replace \(X\) by \(T^*C\) with the pulled back Liouville form which we still denote by \(\lambda_0\).

Let \(f : C \to \mathbb{R}\) be a Morse function and let \(\eta : T^*C \to [0, 1]\) be a cutoff function such that \(\eta = 1\) on \(Op(C)\) and \(\eta = 0\) outside of a slightly bigger neighborhood. Fix an auxiliary Riemannian metric on \(C\) and put \(h = \eta \lambda_{std}(\nabla f)\), a function on \(T^*C\). The restriction of the Hamiltonian vector field of the function \(h\) to \(C\) is \(\nabla f\). Hence \(\phi_0 + hf\) is a Lyapunov function for \(\lambda_0 + dh\) in a neighborhood of \(C\) and therefore on all of \(T^*C\) if \(f\) is taken sufficiently \(C^1\) small. Moreover, for such \(h\) the Liouville form \(\lambda_0 + tdh\) is Morse-Weinstein for any \(t > 0\).

If \(\partial C\) is non-empty, then we need to modify the construction somewhat. Consider a slight enlargement \(\tilde{C}\) of \(C\) in \(X\) obtained by attaching an isotropic collar along \(\partial C\) so that the Liouville field \(Z_0\) is tangent to \(\tilde{C}\). Then we should require that our function \(f : \tilde{C} \to \mathbb{R}\) is Morse on \(C\), compactly supported in \(\tilde{C}\), has no critical points on \(\partial C\) and moreover has the following behavior on \(\partial C\). Let \(P\) be the closure of a component of the order 1 corner locus \(\partial_1 C = \partial C \setminus \bigcup_{k>1} \partial_k C\). If the Liouville structure is boundary attracting \((k = 1)\) along \(P\), then we demand that \(df(v) > 0\) for any inwards pointing vector \(v \in T\partial C\). If the Liouville structure is boundary repelling \((k = 0)\) along \(P\), then we demand that \(df(v) > 0\) for any outwards pointing vector \(v \in T\partial C\). With this restriction on \(f\) the proof proceeds just as before.

\qed
Remark 2.19. As we will see, if a Morse-Bott Weinstein manifold has arboreal skeleton, then one can choose the Morsifying homotopy such that $\text{Skel}(X, \lambda_t) = \text{Skel}(X, \lambda_0)$ for all $t \geq 0$.

2.2.3. Weinstein manifolds. It is easy to check that Lemma 2.18 also holds for the generalized Morse condition instead of Morse-Bott, see Proposition 9.13 in [CE12]. Therefore we will take the following as our working definition, which includes Morse, generalized Morse and Morse-Bott Weinstein structures.

Definition 2.20. A Liouville manifold $(X, \lambda_0)$ is called Weinstein if there exists a Liouville homotopy $\lambda_t$ and a family of Lyapunov functions $\phi_t$ for $Z_t$ such that $(X, \lambda_t, \phi_t)$ is Morse-Weinstein for $t > 0$.

The notions of a Weinstein domain, defining Weinstein domain for a Weinstein manifold and a Weinstein germ are entirely analogous to the corresponding notions in the Liouville case. We emphasize that the Lyapunov function $\phi$ is not part of the structure of a Weinstein manifold, we merely require its existence. For example, Weinstein embeddings are just Liouville embeddings of Weinstein type Liouville manifolds (or domains, or germs). We do not require the embedding to pull back one Lyapunov function to the other. Similarly, a Weinstein isomorphism (resp. homotopy) is a Liouville isomorphism (resp. homotopy) of Liouville manifolds of Weinstein type.

Example 2.21. The standard Liouville structure $\lambda = pdq$ on the cotangent bundle $T^*M$ of a closed manifold $M$ is Weinstein. Given a Riemannian metric on $M$, the function $\phi = ||p||^2$ is a Morse-Bott Lyapunov function for $\lambda$. By the procedure explained in Lemma 2.18, we can Morsify the Lyapunov function $\phi$ using a Morse function on $M$. By inspection, this procedure does not change the skeleton of $T^*M$, which is the zero section.

Remark 2.22. Although Definition 2.20 allows for rather degenerate Weinstein structures, in the Weinstein homotopies constructed in this paper for the purposes of arborealization of a Morse-Weinstein manifold the Weinstein structures are always of Morse-Bott type except for a finite number of times at which there is a birth/death of Morse-Bott critical components.

2.3. $Wc$-manifolds.

2.3.1. Manifolds with boundary and corners. Recall that an $n$-dimensional smooth manifold $M$ has a corner of order $k \leq n$ at $x \in M$ if there is a neighborhood of $x$ in $M$ diffeomorphic to a neighborhood of the origin in $I^k \times \mathbb{R}^{n-k}$, where $I$ denotes the germ of the interval $[0, 1)$ at 0. We denote the locus of order $k$ corners by $\partial_k M$. The closure $P$ of a connected component of $\partial_k M$ is called a boundary $k$-face. For $k = 1$ we will more simply call $P$ a boundary face. Sometimes we will call a manifold with boundary and corners (of any order) a bc-manifold for short.

We always assume that each component of $\partial_k M$ is regularly embedded, so that each $k$-face is itself a manifold with corners. Under this assumption, if $Q$ is a $k$-face, then there is an
embedded collar neighborhood $Q \times I^k \subset M$. We consider the germs of these collars as part of the structure. In particular, near each point $x \in \partial_k M$ we have canonical collar coordinates $x = (y, t)$, where $y \in \partial_k M$ and $t = (t_1, \ldots, t_k) \in I^k$. Note that in a neighborhood of $x$ we have $\partial_k M$ cut out by $t_1 = \cdots = t_k = 0$. More generally, for $j \leq k$ the components of $\partial_j M$ whose closure contains $x$ are given by setting exactly $j$ of the coordinates $t_i$ equal to zero.

We demand compatibility of these collars in the sense that the remaining $n - j$ coordinates $t_i$ give the collar structure for $\partial_j M \times I^{n-j}$ near $x$. We call such a compatible system of collar germs a corner structure on $M$. Henceforth all manifolds with corners will be implicitly equipped with a fixed but otherwise arbitrary corner structure.

![Figure 2.3. A corner structure on $M$.](image)

Let us choose a partial ordering of the boundary faces of a manifold $M$ with corners such that the faces at all corners are ordered in a compatible way, i.e. the order is that induced by the order of the coordinates $t_i$ in the collar structure. We will assume this order to be part of the structure of a manifold with corners.

Given manifolds with corners $W$ and $X$, we consider embeddings of $W$ into $X$ which satisfy the property that each $j$-face of $W$ is properly embedded into some $k$-face of $X$, where $k \geq j$.

### 2.3.2. Liouville manifolds with boundary and corners.

We now give an inductive definition of the notion of a Liouville manifold with corners of order $k \leq \frac{1}{2} \dim X$, starting with the base case $k = 0$ which is just our previous definition.

**Definition 2.23.** A Liouville manifold with corners of order $k$ consists of an exact symplectic manifold $(X^{2n}, \lambda)$ with smooth corners of order $k \leq n$ such that the following properties hold.

1. **(L1)** The vector field $Z$ which is $\omega$-dual to $\lambda$ is complete for $t \to \pm \infty$.
2. **(L2)** There exists a compact domain with corners $W \subset X$ such that $Z$ is outwards transverse to its vertical boundary $\partial_v W$, which is defined to be the closure of $\partial W \cap \text{int} X$, and such that the union of forward trajectories of $Z$ starting at $\partial_v W$ is equal to $\partial_v W \cup (X \setminus W)$.
3. **(L3)** For $r = 1, \ldots, k$, each $r$-face $P$ of the horizontal boundary $\partial_h W$, which is defined to be $\partial W \cap \partial X$, contains a Liouville submanifold with corners $N \subset P$, called the nucleus.
of $P$, such that the collar neighborhood of $P$ in $X$ is of the form $N \times T^*I^r$, where $P$ is the product of $N$ with the cotangent fibre over $0 \in I^r$, and in this neighborhood the form $\lambda$ is the direct sum of a Liouville form on $N$ and the canonical 1-form on $T^*I^r$.

It follows from the definition that the Liouville field $Z$ is tangent to every component of $\partial_kX$, $k = 1, \ldots, n$. A Liouville manifold with corners has a horizontal boundary $\partial_hX = \partial X$ and an ideal vertical boundary $\partial_\infty X = (X \setminus \text{Skel}(X, \lambda))/\mathbb{R}^+$, where just as before the action of $\mathbb{R}^+$ is defined by the flow of $Z$ and the skeleton $\text{Skel}(X, \lambda)$ is defined to be the attractor of the negative flow of $Z$.

We call $(W, \lambda|_W)$ as in (L2) a defining Liouville domain with corners for the Liouville manifold with corners $(X, \lambda)$. Note that the horizontal boundary of $W$ is $\partial_hW = W \cap \partial_hX$ and its vertical boundary $\partial_vW$, the closure of $\partial W \setminus \partial_hW$, is a contact manifold with corners which is naturally contactomorphic to $\partial_\infty X$ under the projection $\pi_\infty : \partial_vW \to \partial_\infty X$. Just as before, the equivalence class of defining domains for Liouville manifold with corners is called the Liouville germ of $(X, \lambda)$ and is denoted by $(\mathcal{X}, \lambda)$. Notions of isomorphism and homotopy of Liouville manifolds with corners are defined exactly the same way as without corners.

2.3.3. $W_c$-manifolds. We now introduce Weinstein structures for Liouville manifolds with corners, which we call $W_c$-manifolds. We restrict our discussion to the Morse-Bott category since this is the framework in which we will work throughout this paper.

**Definition 2.24.** A $W_c$-manifold is a Liouville manifold with corners $(X, \lambda)$ whose Liouville vector field admits a Morse-Bott Lyapunov function $\phi : X \to \mathbb{R}$, i.e. satisfying (MB1) and (MB2) in the interior of $X$, and such that on the collars $N \times T^*I^k$ of (L3) $\phi$ has the form $\phi_N + \sum_{i=1}^k u_i^2$ for $\phi_N$ a Morse-Bott Lyapunov function on $(N, \lambda|_N)$ and $u_1, \ldots, u_k$ the momentum coordinates on $T^*I^k$.

Defining Liouville domains for $W_c$-manifolds will be called defining $W_c$-domains, and the equivalence class of such will be called a $W_c$-germ. Notions of $W_c$-isomorphism, embedding.
and homotopy are the same as the corresponding notions for Liouville manifolds with corners, but demanding that all objects in sight are of Weinstein type.

Example 2.25. Let $M$ be a smooth $n$-dimensional manifold with corners. Then $T^*M$ is a Liouville manifold with corners, where the structure $(L3)$ is induced by a corner structure on $M$. We call the germ $\mathcal{F}^*M$ of $T^*M$ a cotangent block. The corresponding Lagrangian distribution tangent to cotangent fibers will be denoted by $\nu_M$. We denote by $\lambda_M$ and $Z_M$ the corresponding Liouville form and the Liouville vector field. The notation $S^*M$ will be used for the positively projectivized cotangent bundle of $M$, which is the ideal boundary of $\mathcal{F}^*M$. For $Q$ a $k$-face of $M$ we have $P = T^*M|_Q \subset \partial_k T^*M$ a $k$-face of $T^*M$ and $N = T^*Q$ is the nucleus of $P$. The collar coordinates near $Q$ yield the decomposition $O_P P = N \times T^*I_k$. A choice of Riemannian metric on $M$ yields defining Liouville domain with corners $W = \{\|p\| \leq \varepsilon\}$ and a Morse-Bott Lyapunov function $\phi = \|p\|^2$.

The rest of this section can be equally discussed in the Liouville and Weinstein categories. We restrict the discussion only to the Weinstein case as this is the one needed for our further applications.

2.4. Wc-hypersurfaces.

2.4.1. Weinstein hypersurfaces. We now turn our attention to Weinstein hypersurfaces with corners, or Wc-hypersurfaces. They were introduced (without corners) in [E18]. These are special cases of the Liouville hypersurfaces introduced by Avdek in [A12]. This notion is similar to the “stops” of Sylvan, [S16] and the Liouville sectors of Ganatra-Pardon-Shende, [GPS17]. Related constructions are also considered in Ekholm-Lekili’s paper [EL17]. We first recall the definition without corners.

Definition 2.26. A Weinstein hypersurface in a Weinstein manifold $(X^{2n}, \lambda)$ is a Weinstein embedding $(A^{2n-2}, \lambda'|_A) \hookrightarrow (X \setminus \text{Skel}(X, \lambda), \lambda)$ such that $\pi_\infty|_A : A \rightarrow \partial_\infty X$ is an embedding.

Note that any Weinstein hypersurface is contained in a boundary $\partial W$ of some defining domain $W$. Note also that the Reeb vector field of the contact form $\lambda|_{\partial W}$ is transverse to $A$. Note also that the boundary $\partial A$ is a codimension 2 contact submanifold of $(\partial W, \lambda|_{\partial W})$. By
the assumption that \((A, \lambda|_A)\) is Weinstein, the skeleton \(\text{Skel}(A, \lambda|_A)\) can be decomposed as a union of isotropic submanifolds, which in the top dimension \(n - 1\) are Legendrian for the contact structure \(\xi = \ker(\lambda|_{\partial W})\).

**Lemma 2.27.** The skeleton \(\text{Skel}(A, \lambda|_A)\) depends only on the projection \(\pi_\infty(A) \subset \partial_\infty X\).

**Proof.** Indeed, the Liouville fields corresponding to the Liouville structures \(\lambda_A\) and \(f\lambda|_A\) for a positive function \(f\) are proportional. In fact, as computed in Lemma 12.1 of [CE12] the form \(d(f\lambda)|_A\) is symplectic if and only if \(k := \inf(f + df(Z)) > 0\), where \(Z\) is the expanding field for \(\lambda|_A\), and in that case the restriction of the form \(f\lambda\) to \(A\) is automatically Liouville, with the expanding vector field for \(f\lambda|_A\) equal to \(\frac{1}{k}Z\). Therefore \(\text{Skel}(A, \lambda|_A)\) only depends on the contact structure \(\xi_\infty\). \(\square\)

Note that the space of functions \(f > 0\) for which \(f\lambda\) is Liouville (and hence in the considered case Weinstein) is convex, therefore contractible.

**Definition 2.28.** We say that two Weinstein hypersurfaces \(A_0, A_1\) in a Weinstein manifold \((X, \lambda)\) are related by a **translation** if \(A_1 = Zf(A_0)\) for some function \(f : A_0 \to \mathbb{R}\).

Note that given a defining domain \(W \subset X\), any Weinstein hypersurface \(A\) of \(X\) can be translated to a Weinstein hypersurface contained in \(W\). However, it may not be possible to translate it to a Weinstein hypersurface contained in \(\partial W\). Indeed, this is equivalent to the condition \(\inf(f + df(Z)) > 0\) for \(f\) the conformal factor between the contact forms corresponding to \(A\) and \(\partial W\).

**Definition 2.29.** A Weinstein hypersurface germ \(\mathcal{A} \subset X \setminus \text{Skel}(X, \lambda)\) is an equivalence class of Weinstein hypersurfaces \((A, \lambda' = \lambda|_A) \subset (X \setminus \text{Skel}(X, \lambda), \lambda)\) for \((A, \lambda')\) defining domains for a fixed Liouville manifold, where two embeddings are equivalent if they agree on a smaller defining domain, possibly after translation.

One can think of \(\mathcal{A}\) as living in the ideal boundary \(\partial_\infty X\), or one can think of \(\mathcal{A}\) as the germ of a Weinstein embedding \(\mathcal{A} \to \mathcal{X} \setminus \text{Skel}(\mathcal{X}, \lambda)\). The two viewpoints are equivalent and we will go back and forth between them.

**Example 2.30.** An important example of a Weinstein hypersurface is the **ribbon** of a Legendrian submanifold. Let \(\Lambda\) be a Legendrian submanifold in the contact boundary of a Liouville domain \((W, \lambda)\). Then it admits a Darboux neighborhood \(U(\Lambda) \subset \partial W\) contactomorphic to \(\left\{\|p\|^2 + z^2 \leq \varepsilon^2\right\} \subset J^1(\Lambda)\) for some Riemannian metric on \(\Lambda\). The strip \(\Sigma(\Lambda) := U(\Lambda) \cap \{z = 0\}\) is a Weinstein hypersurface symplectomorphic to the cotangent disk bundle of \(\Lambda\), whose skeleton is precisely \(\Lambda\). The space of all ribbon extensions of a given Legendrian \(\Lambda\) is contractible. An important example to keep in mind is the case where \(\mathcal{X} = T^*M\) for \(M\) a smooth manifold, \(W = \{\|p\| \leq 1\}\) with respect to any Riemannian metric on \(M\) and \(\Lambda\) is the unit conormal to a co-oriented hypersurface \(H \subset M\).
2.4.2. Weinstein pairs.

**Definition 2.31.** A Weinstein pair \((X, A, \lambda)\) consists of a Weinstein manifold \((X, \lambda)\) together a Weinstein hypersurface \(A \subset X \setminus \text{Skel}(X, \lambda)\).

We will also call \((X, \mathcal{A}, \lambda)\) a Weinstein pair when \(\mathcal{A}\) is the germ of a Weinstein hypersurface \(\mathcal{A} \subset X \setminus \text{Skel}(X, \lambda)\).

**Definition 2.32.** Let \((X, \lambda)\) be a Liouville manifold and \(Y \subset X \setminus \text{Skel}(X, \lambda)\) any subset. The Liouville cone on \(Y\) is the saturation of \(A\) by the Liouville trajectories:

\[
\text{Cone}(Y, \lambda) = \bigcup_{t \in \mathbb{R}} Z^{-t}(Y) \subset X.
\]

The positive Liouville cone on \(A\) is the saturation of \(A\) by the backwards Liouville trajectories:

\[
\text{Cone}^+(Y, \lambda) = \bigcup_{t \geq 0} Z^{-t}(Y) \subset X.
\]

Note that if \(Y\) is compact, then \(\text{Cone}^+(Y, \lambda)\) has compact closure while \(\text{Cone}(Y, \lambda)\) does not. The skeleton of a Weinstein pair \((X, A, \lambda)\) is defined as the compact subset:

\[
\text{Skel}(X, A, \lambda) = \text{Skel}(X, \lambda) \cup \text{Cone}^+(\text{Skel}(A, \lambda|_A), \lambda).
\]

Note that \(\text{Skel}(X, A, \lambda) \subset W\) for any defining domain \(W\) such that \(A \subset \partial W\), moreover \(\partial W \cap \text{Skel}(X, A, \lambda) = \text{Skel}(A, \lambda|_A)\). The notion of a pair at the level of germs is defined as follows.

**Definition 2.33.** A Weinstein (germ) pair \((\mathcal{X}, \mathcal{A}, \lambda)\) consists of a Weinstein germ \((\mathcal{X}, \lambda)\) together with a Weinstein hypersurface germ \(\mathcal{A} \subset \mathcal{X} \setminus \text{Skel}(\mathcal{X}, \lambda)\).

Note that for any representative \((W, \lambda)\) of \((\mathcal{X}, \lambda)\) we may take a representative \(A\) for \(\mathcal{A}\) contained in \(W\). The skeleton of a Weinstein pair \((\mathcal{X}, \mathcal{A}, \lambda)\) is defined as the non-compact subset:

\[
\text{Skel}(X, \mathcal{A}, \lambda) = \text{Skel}(X, \lambda) \cup \text{Cone}^+(\text{Skel}(\mathcal{A}, \lambda|_{\mathcal{A}}), \lambda).
\]
Equivalently in terms of germs, we can think of \( \text{Skel}(X, A, \lambda) \) as the equivalence class of the compact subsets \( \text{Skel}(X, A, \lambda) \subset W \) for all defining domains \( (W, \lambda|_W) \) and representatives \( A \) for \( \mathcal{A} \) such that \( A \subset W \).

2.4.3. \textit{Wc-hypersurfaces.} Finally, we extend the above to include boundary and corners.

\textbf{Definition 2.34.} A Wc-hypersurface in a Wc-manifold \( (X^{2n}, \lambda) \) is an equivalence class of embeddings of Wc-germs \( (\mathcal{A}^{2n-2}, \lambda'|_\mathcal{A}) \hookrightarrow (X \setminus \text{Skel}(X, \lambda), \lambda) \) such that \( \pi_\infty|_\mathcal{A} : \mathcal{A} \to \partial X_\infty \) is an embedding, with the equivalence given by translation.

All the previous discussion carried over word by word to the Wc-setting. The notion of a Weinstein pair in the Wc-category is called a Wc-pair.

\textbf{Definition 2.35.} A Wc-pair \( (X, A, \lambda) \) consists of a Wc-germ \( (X, \lambda) \) together with a Wc-hypersurface germ \( A \subset X \setminus \text{Skel}(X, \lambda) \).

2.5. \textit{Conversion between Wc-hypersurfaces and face nuclei.}

2.5.1. \textit{Nucleus-to-hypersurface conversion.} We start with a Wc-manifold \( (X, \lambda) \). Let \( P \) be a boundary face of \( X \) and \( N \subset P \) its nucleus. Let \( U \) be a collar neighborhood of \( P \) in \( X \) as in the definition of a Wc-manifold. So \( U \) is of the form \( N \times T^*\mathcal{I} \), with \( \lambda = \lambda|_N + u dt \) where \( t \) is the defining coordinate for \( P \). Attach to \( N \times T^*\mathcal{I} \) the half-infinite collar \( N \times T^*(-\infty, 0] \) along their common intersection and on the union consider the Weinstein structure given by the direct sum of \( \lambda|_N \) and the Weinstein normal form for Morse-Bott boundary of index \( k = 0 \), which agrees with \( \lambda_N + u dt \) on \( N \times T^*\mathcal{I} \). The existence of a Morse-Bott Lyapunov function is immediate from the local model. Hence the result is a Wc-manifold \( (X_N^N, \lambda_N) \). Let \( \mathcal{N} \) be the germ of \( N \). For \( \varepsilon > 0 \) note that \( \mathcal{N} \times \{ t = -\varepsilon \} \subset \mathcal{N} \times T^*(-\infty, 0] \) is a Wc-hypersurface for \( \lambda^N \). Any two differ by a translation and all are canonically identified with \( \mathcal{N} \). We will denote this Wc-hypersurface by \( \mathcal{A}^N \subset X^N \setminus \text{Skel}(X^N, \lambda^N) \).

Note that by construction we have \( \text{Skel}(X^N, \mathcal{A}^N, \lambda^N) = \text{Skel}(X, \lambda) \). The operation \( (X, N) \mapsto (X^N, \mathcal{A}^N) \) only depends on the contractible choice of a collar neighborhood and the contractible choice of the Weinstein normal form for Morse-Bott boundary. Hence it produces a Wc-germ \( (\mathcal{X}^N, \lambda^N) \) which is well defined up to Wc-homotopy.

\textbf{Definition 2.36.} The Wc-pair \( (\mathcal{X}^N, \mathcal{A}^N, \lambda^N) \) is called the \textit{nucleus-to-hypersurface conversion} of the Wc-germ \( (\mathcal{X}, \lambda) \) and the boundary nucleus \( N \).

2.5.2. \textit{Underlying Weinstein manifold of a Wc-manifold.} Let \( X \) be a Wc-manifold and let \( P \) be a boundary face. The resulting of converting the nucleus \( N \) of \( P \) to a Wc-hypersurface is a Wc-manifold which has one less boundary face than \( X \). We can continue this process inductively over all faces. After forgetting the resulting Wc-hypersurfaces, we end up with a Weinstein manifold without boundary which we denote \( \hat{X} \). It is clear that the Weinstein homotopy class of the Weinstein manifold \( \hat{X} \) does not depend on the order in which the
boundary faces are chosen. If one wishes to, one can further deform the Weinstein structure on \( \hat{X} \), which at this point is generalized Morse-Bott, so that it becomes Morse-Weinstein.

**Definition 2.37.** The Weinstein manifold \( \hat{X} \) obtained from a Wc-manifold \( X \) by converting all of its boundary nuclei to Wc-hypersurfaces is called the *underlying Weinstein manifold*.

**Example 2.38.** The underlying Weinstein manifold of the Wc-manifold \((T^*[0,1]^n, pdq)\) is Weinstein homotopic to \((\mathbb{R}^{2n}, pdq - qdp)\).

**Remark 2.39.** A Wc-manifold may be thought of as a Weinstein manifold with ‘stops’. From this viewpoint, taking the underlying Weinstein manifold of a Wc-manifold is simply removing the stops.

### 2.5.3. Hypersurface-to-nucleus conversion

The converse operation to the nucleus-to-hypersurface conversion is the **hypersurface-to-nucleus** conversion, which we discuss next. Here and below we denote \( U_{a,b} = \{0 < u \leq a, -b < t < b\} \subset T^*\mathbb{R}^n \).

**Lemma 2.40.** The inclusion \((A, \lambda|_A) \hookrightarrow (X, \lambda)\) of a Wc-hypersurface extends, uniquely up to contractible choice, to a Liouville embedding \((A \times U_{1,\varepsilon}, \lambda' + u dt) \hookrightarrow (X, \lambda)\), where \( A \) is identified with \( A \times \{u = 1, t = 0\} \subset A \times U_{1,\varepsilon} \).

**Proof.** Let \( W \subset X \) be a defining domain such that \( A \subset \partial W \) and let \( A \times (-\varepsilon, \varepsilon) \subset \partial W \) be its contact surrounding, as in [E18], so that the restriction of \( \lambda \) to this neighborhood is of the form \( \pi^*\lambda|_A + dt \), where \( \pi : A \times (-\varepsilon, \varepsilon) \rightarrow A \) is the projection and \( t \in (-\varepsilon, \varepsilon) \). For every \( x \in X \setminus \text{Skel}(X, \lambda) \), let \( x^s \) denote the backwards flow of \( \mathcal{Z}_X \) starting at \( x \), where \( s \in (-\infty, 0] \). For every \( y \in A \times (-\varepsilon, \varepsilon) \), let \( y^s \in U_{1,\varepsilon} \) denote the backwards flow of \( \mathcal{Z}_A + u \frac{\partial}{\partial u} \) starting at \( y \), where \( s \in (-\infty, 0] \). We extend the contact embedding \( A \times (-\varepsilon, \varepsilon) \rightarrow \partial W \) which sends \( y \mapsto x \) to a Liouville embedding \( A \times U_{1,\varepsilon} \rightarrow W \) by sending \( y^s \mapsto x^s \), so that by construction \( \mathcal{Z}_A + u \frac{\partial}{\partial u} \) is identified with \( \mathcal{Z}_X \). It is clear that the embedding is determined by the contact surrounding and its parametrization, which are unique up to contractible choice. \( \square \)

We now explain the **hypersurface-to-nucleus** conversion. Let \((X, \lambda)\) be a Wc-manifold and let \( \mathcal{A} \subset X \setminus \text{Skel}(X, \lambda) \) be a Wc-hypersurface. Fix defining Wc-domains \( A \) and \( W \) for \( \mathcal{A} \).
and $X$ respectively so that we have an embedding $A \to \partial W$ which extends to an embedding $A \times U_{1,\varepsilon} \to W$ as in Lemma 2.40. Consider the exact symplectic manifold which is the result of gluing $W$ and $A \times U_{3,\varepsilon}$ along $A \times U_{1,\varepsilon}$. Consider the Weinstein structure on $U_{3,\varepsilon}$, thought of as the $\varepsilon$-neighborhood of the zero section in $T^* (0, 3)$, which is the restriction of the Weinstein normal form for Morse-Bott boundary of index $k = 1$ translated two units in the $u$ direction, so that the critical set is $\{ t = 0, u \geq 2 \}$. The restriction of this structure to $U_{1,\varepsilon}$ is $u dt$, hence we obtain a patched up structure on the union of $W$ and $A \times U_{3,\varepsilon}$. Moreover, by inspection the new Liouville field points outwards at the boundary. So we may replace $U_{3,\varepsilon}$ with a smoothing $U_{3,\mu} = \{-\varepsilon < t < \varepsilon, 0 < u < \min(3, \mu(|t|))\}$ for $\mu : (0, \varepsilon) \to [1, \infty)$ a function satisfying

(i) $\mu(x) \to \infty$ as $x \to 0$
(ii) $\mu = 1$ on $(\varepsilon/2, \varepsilon)$
(iii) $\mu' < 0$ on $(0, \varepsilon/2)$

and the Liouville field is still outwards pointing for the smoothed boundary. Thus we obtain a defining domain for a Wc-manifold $(X^A, \lambda^A)$. Indeed the only think that needs to be checked is the existence of a Morse-Bott Lyapunov function, which is clear from the local models since without loss of generality we can start with a defining domain $W$ for $X$ such that $\partial W$ is a regular level set of the Morse-Bott Lyapunov function $\phi$ for $X$. Note that $A \times \{ t = 0, u = 3 \}$ is the nucleus of the new boundary face of $X^A$.

By construction we have $\text{Skel}(X^A, \lambda^A) = \text{Skel}(X, A, \lambda)$, or more precisely

$$\text{Skel}(X^A, \lambda^A) = \text{Skel}(X, A, \lambda) \cup (\text{Skel}(A, \lambda|_A) \times [1, 2]).$$

In other words, the backwards $\lambda$-Liouville cone on $A$ has a trivial collar attached to it, which does not change the topology of $\text{Skel}(X, A, \lambda)$. Note that the space of possible smoothings $\mu$ is convex, hence contractible. The choice of Morse-Bott Weinstein normal form is also contractible. Similarly the choice of collars used in the construction is contractible. Hence we get a Wc-germ $(\mathcal{X}^{A}, \lambda^{A})$ which is well defined up to Wc-homotopy. Finally, note that different representatives $A$ for $\mathcal{A}$ will also result in homotopic Wc-manifolds $(X^{A}, \lambda^{A})$. Hence the conversion operation only depends on the Wc-hypersurface germ $\mathcal{A}$ up to Wc-homotopy.

**Definition 2.41.** The Wc-germ $(\mathcal{X}^{A}, \lambda^{A})$ is called the **hypersurface-to-nucleus conversion** of the Wc-germ $X$ and the Wc-hypersurface $\mathcal{A}$.

**2.5.4. Cancellation of conversions.** The hypersurface-to-nucleus conversion is inverse to the nucleus-to-hypersurface conversion at the level of Weinstein homotopy classes, as follows from a parametric Smale cancellation of Morse-Bott index 0 and 1 critical points, the existence of which is immediate from the local models. This implies the following obvious but important lemma.

**Lemma 2.42.** Let $(X^{\mathcal{A}}, \lambda^{\mathcal{A}})$ be the result of hypersurface-to-nucleus conversion of a Wc-hypersurface $\mathcal{A} \subset X \setminus \text{Skel}(X, \lambda)$. Then the underlying Weinstein manifolds of $(X, \lambda)$ and $(X^{\mathcal{A}}, \lambda^{\mathcal{A}})$ are Wc-homotopic.
2.6. Gluing and splitting.

2.6.1. Horizontal gluing. We will now describe the basic gluing operation for Wc-manifolds. Given two Wc-manifolds \((X_0, \lambda_0)\) and \((X_1, \lambda_1)\), we define their horizontal gluing as follows. Pick some non-adjacent boundary faces \(P_1, \ldots, P_l\) of \(X_1\) and \(Q_1, \ldots, Q_l\) of \(X_0\), and for each \(j\) suppose we are given a Liouville isomorphism \(\phi_j : N_j \rightarrow M_j\), where \(N_j\) and \(M_j\) are the nuclei or \(P_j\) and \(Q_j\) respectively. We have collar neighborhoods \(N_j \times \mathbb{T}^* \subset X_1\) and \(M_j \times \mathbb{T}^* \subset X_2\) and we extend \(\phi_j\) to \(\Phi_j : P_j \rightarrow Q_j\) as \(\Phi_j = \phi_j \times \text{id}_\mathbb{R} : N_j \times \mathbb{R} \rightarrow M_j \times \mathbb{R}\).

Definition 2.43. The horizontal gluing of \((X_1, \lambda_1)\) and \((X_2, \lambda_2)\) along \(\{\phi_j\}_j\) is defined to be the Wc-germ of the Wc-manifold \(X_1 \cup X_2\) where we identify \(P_j \sim Q_j\) via \(\Phi_j\).

Example 2.44. Suppose \(M, N\) are two manifolds with boundary and we have a diffeomorphism \(\psi : \partial N \rightarrow \partial M\). Then \(\psi\) lifts to a Liouville isomorphism \(\phi\) between \(T^*(\partial N)\) and \(T^*(\partial M)\) and we can perform the horizontal gluing \(T^* N \cup_\phi T^* M\) which is the same as \(T^*(N \cup_\psi M)\) for \(N \cup_\psi M\) the smooth gluing of \(N\) and \(M\) given by \(\psi\).

Note that the skeleton of the horizontal gluing of \(X_1\) and \(X_2\) is equal to the union of the skeleta of \(X_1\) and \(X_2\).

2.6.2. Vertical gluing. We can also glue two Wc-manifolds \(X_1\) and \(X_2\) in a different way.

Definition 2.45. Suppose we have a Wc-hypersurface \(\mathcal{A}\) in \(\mathcal{X}_1\) and a Liouville isomorphism \(\phi : \mathcal{A} \rightarrow \mathcal{N}\) for \(\mathcal{N}\) the germ of the nucleus \(N\) of a boundary face of \(X_2\). The vertical gluing of \(\mathcal{X}_1\) and \(\mathcal{X}_2\) is defined to be the composition of two steps: a conversion a Wc-hypersurface into a boundary hypersurface, see Section 2.5 above, followed by a horizontal gluing.

Indeed, if we first convert \(\mathcal{A}\) into a boundary nucleus, we can then apply the horizontal gluing determined by \(\phi\) as described above. The resulting Wc-germ is well defined up to Wc-isomorphism. Note that the skeleton of the vertical gluing is equal to the union of the skeleton of the pair \((\mathcal{X}_1, \mathcal{A})\) and the skeleton of \(\mathcal{X}_2\). We record for future reference the following slight strengthening of this observation, where we denote by \(\lambda^P\) the Liouville form \(\lambda - d(ut)\) for \(t\) the defining coordinate of a boundary face \(P\).
Lemma 2.46. Let $(\mathcal{X}_i, \lambda_i), \ i = 1, 2$ be two Wc-germs, $\mathcal{A} \subset \mathcal{X}_2 \setminus \text{Skel}(\mathcal{X}_2, \lambda_2)$ a Wc-hypersurface, $N$ the nucleus of a boundary face $P$ of $X_1$ and $\mathcal{N}$ its germ, $\phi : \mathcal{A} \to \mathcal{N}$
a Liouville isomorphism and $(\mathcal{Y}, \lambda)$ the result of vertically gluing $\mathcal{X}_1$ to $\mathcal{X}_2$ along $\phi$. Then there exists a symplectic embedding of germs $\Psi : \text{int} \mathcal{X}_1 \rightarrow \mathcal{Y}$ such that the following properties hold, where we denote by $\iota : \text{int} \mathcal{X}_2 \rightarrow \mathcal{Y}$ the restriction of the inclusion to the interior of $\mathcal{X}_2$.

(i) $\iota(\text{Skel}(\mathcal{X}_2, \lambda_2)) \cup \Psi(\text{Skel}(\mathcal{X}_1, \lambda_1) \cap \text{int} \mathcal{X}_1) = \text{Skel}(\mathcal{Y}, \lambda)$

(ii) $\iota(\text{Skel}(\mathcal{X}_2, \lambda_2)) \cap \Psi(\text{int} \mathcal{X}_1) = \emptyset$.

(iii) $\Psi^{-1}(\iota(\text{int} \mathcal{X}_2)) = \mathcal{O}p\mathcal{N} \setminus \partial \mathcal{X}_1 \subset \mathcal{X}_1$ and $(\Psi)^*(\iota)_* \lambda_2 = \lambda_1^P$.

**Proof.** We recall from the gluing construction that we attached $\mathcal{N} \times (-\varepsilon, \varepsilon)$ to the top of a cylinder $\mathcal{A} \times U_{3,\varepsilon} \subset Y$. Since a neighborhood $\mathcal{O}p\mathcal{N}$ of $N$ in $X_1$ is symplectomorphic to $N \times T^*_\varepsilon(0, \delta)$, we can map $\mathcal{N} \times T^*_\varepsilon(0, \delta)$ symplectomorphically to $\mathcal{A} \times U_{3,\varepsilon} \cup \mathcal{N} \times T^*_\varepsilon(0, \delta)$, relative to $\mathcal{N} \times T^*_\varepsilon(\delta/2, \delta)$ say, so that the symplectomorphism extends to the rest of $\text{int} \mathcal{X}_1$ as the identity. Moreover, the symplectomorphism can be obtained from a reparametrization of the $t$ coordinate, hence the Liouville forms pull back as desired. $\square$

### 2.6.3. Splitting

The converse operation to gluing is splitting.

**Definition 2.47.** Given a $\text{Wc}$-manifold $(X, \lambda)$, a regularly embedded hypersurface $(H, \partial H) \subset (X, \partial X)$ with corners called splitting for $X$ if it satisfies the following conditions:

(i) $H$ divides $X$ into two parts, $X = X_+ \cup X_-$ and $X_- \cap X_+ = H$, which are manifolds with corners.

(ii) The Liouville vector field $Z$ of $X$ is tangent to $H$.

(iii) There exists a hypersurface with corners $(S, \partial S) \subset (H, \partial H)$ tangent to $Z$.

(iv) $(S, \lambda|_S)$ is a $\text{Wc}$-manifold and there exists a retraction $\pi : H \rightarrow S$ such that $\lambda|_H = \pi^*(\lambda|_S)$.

**Remark 2.48.** That $H$ is regularly embedded implies in particular that $\mathcal{O}p\mathcal{H} = H \times (-\varepsilon, \varepsilon)$.

We will refer to $S$ as the soul of the splitting hypersurface $H$ and denote it by $\text{Soul}(H)$. Note that $\text{Skel}(S, \lambda|_S) = \text{Skel}(X, \lambda) \cap H$.

**Example 2.49.** Let $M$ be a manifold with corners and $N$ a regularly embedded codimension 1 submanifold with corners, so that $\mathcal{O}p\mathcal{N} = N \times (-\varepsilon, \varepsilon)$. Assume that $N$ divides $M$ into two parts: $M_+$ and $M_-$, so that $M_+ \cup M_- = M$. Then $H = T^*M|_N$ is a splitting hypersurface. Given a decomposition of the tubular neighborhood of $N$, $\mathcal{O}p\mathcal{N} = N \times (-\varepsilon, \varepsilon)$, we have $S := T^*N \times 0 \subset T^*M = T^*N \times T^*(-\varepsilon, \varepsilon)$ the soul of $H$.

The splitting hypersurface $S$ can be used to split $X$ into two $\text{Wc}$-manifolds in two ways, horizontal and vertical. For the horizontal splitting we view $X_\pm$ as $\text{Wc}$-manifolds with an additional component $H_\partial X_\pm$. The $\text{Wc}$-manifold $S$ serves as the nucleus of the additional boundary component $H$. The vertical splitting consists of a horizontal splitting followed by a nucleus-to-hypersurface conversion operation, transforming the nucleus $S$ into a $\text{Wc}$-hypersurface.
Partial horizontal gluing. One more basic operation that we will need is the creation of corners and its associated partial horizontal gluing. Let $X$ be a Wc-manifold, $P$ a boundary face of $X$ and $N$ the nucleus of $P$. Let $\Sigma \subset N$ be a splitting hypersurface with soul $T$. Then there is a deformation of the Wc-structure on $X$ which creates an additional corner along $\Sigma$ with its nucleus $T$.

More precisely, take a collar neighborhood $O_pP = N \times T^*\mathcal{L}$, so that $\lambda = \lambda|_N + udt$ and extend it to $N \times T^*(-\varepsilon, \varepsilon)$ with the same Liouville form. Let $\Sigma \times (-\delta, \delta) \subset N$ be a tubular neighborhood of $\Sigma$ in $N$ and consider a smooth function $\mu : (0, \delta) \to [0, \infty)$ satisfying the following properties:

(i) $\mu(x) \to \infty$ as $x \to 0$.
(ii) $\mu(x) = 0$ on $[\delta/2, \delta)$
(iii) $\mu'(x) < 0$ on $(0, \delta/2)$.

Set $\eta(x) = \min(\mu(x), \epsilon/2)$ and consider the subset $\Omega \subset (\Sigma \times (-\delta, \delta)) \times T^*(-\varepsilon, \varepsilon)$ consisting of points $(\sigma, s, t, u)$ such that $\eta(s) + t \geq 0$. Then the union of $\Omega$ and $X$ along $P$ is a new Wc-manifold which has a new corner along a parallel copy of $\Sigma$.

![Figure 2.11](image-url) A creation of additional corners on $\mathcal{F}^*M$ for $M$ a closed manifold with corners is induced from a creation of additional corners on $M$.

Next, consider two Wc-manifolds $X_0, X_1$ with boundary faces $P_0, P_1$ and respective nuclei $N_0, N_1$. Let $\Sigma$ be a dividing hypersurface for $N_0$ with its soul $T$, which splits $N_0$ into Wc-manifolds $N_0'$ and $N_0''$, which share a common face $\Sigma$ with the nucleus $T$. Suppose there is a Liouville isomorphism $\phi : N_1 \to N_0'$.

**Definition 2.50.** The partial horizontal gluing of $\mathcal{F}_1$ to $\mathcal{F}_0$ using $\phi$ is the Wc-germ obtained from the following two operations:

(a) Creating a corner along $\Sigma$, i.e. adding $\Sigma$ to $\partial_2X_0$, and replacing the face $N_0$ with two faces $N'_0, N''_0$.
(b) The horizontal gluing $X_0$ and $X_1$ along $\phi$.

As before, the partial horizontal gluing is well defined up to Wc-homotopy.

2.7. Wc-buildings.
2.7.1. Iterated gluing. We now introduce the iterated vertical gluing of Wc-manifolds, which
will be a convenient notion for the constructions in this paper. Consider Wc-germs
\((X_1, \lambda_1), \ldots, (X_k, \lambda_k)\). We inductively define a k-level Wc-building, which we will denote
by \((X_k \to X_{k-1} \to \cdots \to X_1)\), as the following presentation of a Wc-germ.

**Definition 2.51.** A 1-level Wc-building is \((X_1, \lambda_1)\) itself. Suppose that the Wc-building
\((Y, \mu) := (X_k \to X_{k-1} \cdots \to X_2)\) is already defined. Consider a nucleus \((N, \mu') := \mu|_N\) of one of the boundary faces \(P\) of
\((Y, \mu)\) and let \(\phi\) be an embedding \(N\) as a Wc-hypersurface in \(X_1 \setminus \text{Skel}(X_1, \lambda_1)\). The building
\((X_k \to X_{k-1} \cdots X_2 \to X_1) := (X_k \to X_{k-1} \cdots \to X_2) \to X_1\)
is by definition the result of a vertical gluing of \((Y, \mu)\) and \((X_1, \lambda_1)\) along \(\phi\).

![Figure 2.12. A Wc-building.](image)

So a Wc-building is a Wc-germ \(Y\) equipped with a choice of presentation of its Wc-
deformation class as the iterated vertical gluing \(Y = (X_1 \to \cdots \to X_k)\).

2.7.2. Distinguished cover of a Wc-building. A Wc-building come equipped with a symplectic
cover by the Wc-manifolds it is built out of. The relevant properties of this cover are sum-
marized in the following lemma. Recall that given a Wc-manifold \((X, \lambda)\) the Liouville form \(\lambda\) restricted to a neighborhood of a face \(P\) has the form \(\lambda := \lambda|_N + u dt\), where \(N\) is the nucleus of \(P\) and \(t\) is the defining coordinate for the face \(P\). We recall the notation \(\lambda^P\) for the form \(\lambda|_N - t du = \lambda - d(ut)\) on \(Op N\).

**Lemma 2.52.** Suppose \((X, \lambda)\) is presented as a k-level building \((X_k \to X_{k-1} \to \cdots \to X_1)\).
Then there exist symplectic embeddings \(\Phi_j : intX_j \to X, j = 1, \ldots, k, \) such that

(i) \(\bigcup_{j=1}^k (\Phi_j(Skel(X_j, \lambda_j) \cap intX_j) = \text{Skel}(X, \lambda)\);

(ii) \(\Phi_j(intX_j) \cap \Phi_i(Skel(X_i, \lambda_i) \cap intX_i) = \emptyset\) if \(j > i\);

(iii) If \(\Phi_j(intX_j) \cap \Phi_i(intX_i) \neq \emptyset\) for \(j > i\) then there exists a nucleus \(N\) of a boundary
face of \(X_j\) such that \(\Phi_j^{-1}(\Phi_i(intX_i)) = Op N \cap intX_j\) and \(\Phi_j^*(\Phi_i)_*\lambda_i = \lambda^P_j\).
Proof. We inductively construct the maps $\Phi_{i,j} : X_i \to (X_k \to \cdots \to X_j)$, $i \geq j$, as follows, at the end of the inductive process we will set $\Phi_i = \Phi_{i,1}$. The map $\Phi_{k,k}$ is just the inclusion on $X_1$ into $X$. Suppose we have defined maps $\Phi_{i,j}$ for $i \geq j$. Recall that $Y = (X_1 \to \cdots \to X_j \to X_{j-1})$ is vertically glued to $X_{j-1}$ by a Liouville isomorphism between the nucleus $N$ of a boundary face $P$ of $Y$ and a Wc-hypersurface $A$ in $X_{j-1}$. Let $\Psi : Y \to (X_1 \to \cdots \to X_j \to X_{j-1})$ be the symplectic embedding from Lemma 2.46, set $\Phi_{i,j-1} = \Psi \circ \Phi_{i,j}$ and set $\Phi_{j-1,j-1}$ to be the restriction of the inclusion $X_{j-1} \hookrightarrow (X_1 \to \cdots X_j \to X_{j-1})$ to the interior of $X_{j-1}$. The desired properties for the resulting $\Phi_i$ follow from the properties stated in Lemma 2.46.

Remark 2.53. Note that on the overlap $\Phi_{i_1}(X_{i_1}) \cap \cdots \cap \Phi_{i_k}(X_{i_k}) \subset X$ the pushed-forward Liouville fields $Z_{i_1}, \ldots, Z_{i_k}$ coming from $X_{i_1}, \ldots, X_{i_k}$ all commute with each other.

2.7.3. Cotangent buildings. Finally, we introduce a particular class of Wc-buildings which will be particularly relevant for our purposes.

Definition 2.54. A Wc-building $(X_k \to \cdots \to X_1)$ for which each Wc-germ consists of a cotangent block $T^*M_j$ is called a cotangent building.

We will later show that any finite type Weinstein manifold can be realized as the underlying Weinstein manifold of a cotangent building.

2.8. Legendrian submanifolds.

2.8.1. Legendrians in cotangent bundles. Consider first a cotangent bundle $T^*M$, where $M$ an $n$-dimensional compact manifold with corners. Let $\Lambda$ be an $(n-1)$-dimensional manifold with corners.

Definition 2.55. An embedding $\Lambda \to T^*M \setminus M$ is called Legendrian if its projection to $S^*M$ is a Legendrian embedding.

With every Legendrian $\Lambda \subset T^*M$ we associate its Lagrangian Liouville cone $L := \text{Cone}(\Lambda, pdq)$ formed by forward and backwards trajectories of the Liouville field $Z$. Note that $L$ is embedded.
The splitting $P \times I^k$ near a $k$-face $P$ of $M$ provide canonical splittings

$$T^*M|_{O_p P} = T^*P \times T^*I^k$$

We will call a Legendrian embedding $\phi : L \to \mathcal{F}^*M \setminus M$ adapted to the block $T^*M$ if $f(O_p \partial L) \subset O_p \partial M$ and for each stratum $P \subset \partial_k M$ we have $L \cap O_p P = L_k \times I^k$ for a Legendrian $L_k \subset \mathcal{F}^*P$.

By conical Lagrangians $L \subset T^*M \setminus M$ we will always mean Lagrangian Liouville cones over Legendrians.

Denote by $\pi : T^*M \to M$ the cotangent bundle projection.

**Definition 2.56.** An adapted Legendrian $\Lambda$ is called regular if $\pi|_{\Lambda} : \Lambda \to M$ is an immersion with transverse self-intersections.

**Figure 2.14.** The Legendrian on the left is regular, while the one on the right is not.

2.8.2. Legendrians in $Wc$-manifolds. Let $(X, \lambda)$ be a $2n$-dimensional $Wc$-manifold.

**Definition 2.57.** An $(n-1)$-dimensional submanifold $\Lambda \subset X \setminus \text{Skel}(X, \lambda)$ is called Legendrian if it projects to $\partial_\infty X$ as an embedded Legendrian submanifold of $\partial_\infty X$. Its Liouville cone $L = \text{Cone}(\Lambda, \lambda)$ is a called a conical Lagrangian.

We say that a Legendrian $\Lambda$ is adapted to the $Wc$-structure of $(X, \lambda)$ if $\partial_k \Lambda \subset \partial_k X$ and moreover in a neighborhood of any corner $P \subset \partial_k X$ we have $\Lambda = \Lambda_N \times I^k \subset P \times T^*I^k$, where $\Lambda_N$ is a Legendrian in the nucleus $N$ of the corner $P$.

**Example 2.58.** Let $M$ be a manifold and $N \subset M$ a co-oriented hypersurface. Then any submanifold $\Lambda \subset T^*M \setminus M$ which 1-1 projects to the conormal lift of $N$ in $S^*M$ (and is never tangent to the Liouville flow) is Legendrian. If $N$ is adapted for the corner structure of $M$, then one can construct a Legendrian lift $\Lambda$ which is adapted for the $Wc$-structure of $T^*M$.

We can also think of Legendrians in $T^*M$ as equivalence classes of Legendrian embeddings into defining domains for $T^*M$, where the equivalence is given by translation along the Liouville flow. This gives us the notion of a Legendrian in a cotangent block $\mathcal{F}^*M$. Similarly, we
have the notion of a Legendrian in a Wc-germ \( \mathcal{X} \). For a Legendrian \( \Lambda \) in a defining domain \((W, \lambda|_W)\) for \((\mathcal{X}, \lambda)\), we only take its positive Liouville cone \(\text{Cone}^+(\Lambda, \lambda)\), which is contained in \(W \setminus \text{Skel}(W, \lambda)\). One can think of the equivalence class of all positive Liouville cones over all Legendrians equivalent to \(\Lambda\) as the whole Liouville cone \(\text{Cone}(\Lambda, \lambda) \subset X \setminus \text{Skel}(X, \lambda)\).

### 2.9. Gluing and splitting of Legendrians.

#### 2.9.1. Gluing Legendrians.

**Lemma 2.59.** Let \((\mathcal{X}, \lambda)\) be a Wc-germ, \(\Lambda \subset \mathcal{X} \setminus (\text{Skel}(\mathcal{X}, \lambda)\) an adapted Legendrian and \(L\) the \(\lambda\)-Liouville cone over \(\Lambda\). Let \((\mathcal{X}^N, \lambda^N)\) be the result of a nucleus-to-hypersurface conversion of the nucleus \(N\) of one of the boundary faces of \(X\). Then \(\Lambda \subset \mathcal{X}^N\) remains Legendrian and its \(\lambda^N\)-Liouville cone coincides with \(L\).

**Proof.** In a neighborhood of the face \(P\) of \(X\) whose nucleus \(N\) we convert to a boundary face we have \(\mathcal{O}_P P = T^*N \times T^*I\) and \(\Lambda = \Lambda_N \times I\). The conversion changes \(\lambda = \lambda_N + udt\) only by the differential of a function of \((u, t)\), moreover the resulting Liouville field on \(T^*I\) is tangent to \(I\) (and outwards pointing at the boundary). Hence \(\Lambda\) remains Legendrian and its Liouville cone is unchanged. \(\square\)

**Lemma 2.60.** Let \(\mathcal{X}_0, \mathcal{X}_1\) be two Wc-germs, \(P_0, P_1\) two of their boundary faces and \(N_0, N_1\) the nuclei of \(P_0, P_1\). Let \(\Lambda_0 \subset \mathcal{X}_0 \setminus \text{Skel}(\mathcal{X}_0, \lambda_0)\) and \(\Lambda_1 \subset \mathcal{X}_1 \setminus \text{Skel}(\mathcal{X}_1, \lambda_1)\) be two adapted Legendrians. Denote \(\Lambda'_i := \Lambda_i \cap N_i, i = 0, 1\). Let \(\phi : N_0 \to N_1\) be a Liouville isomorphism such that \(\phi(\Lambda'_0) = \Lambda'_1\). Then the horizontally glued Wc-manifold \(\mathcal{X}_{01} = \mathcal{X}_0 \cup_\phi \mathcal{X}_1\) contains a glued Legendrian

\[
\Lambda_{01} = \Lambda_0 \cup_{\Lambda'_i = \phi|_{\Lambda'_0}} \Lambda_1.
\]

**Proof.** This is obvious by definition of adapted. \(\square\)
2.9.2. Splitting Legendrians. Let $X$ be a Wc-manifold and $P \subset X$ a splitting hypersurface with the soul $S$. A Legendrian $\Lambda \subset X \subset \text{Skel}(X, \lambda)$ is called in in splitting position if $\Lambda$ is transverse to $P$ and $\Lambda \cap P \subset S$.

Lemma 2.61. Let $X$ be a Wc-manifold, $P \subset X$ a splitting hypersurface and consider a Legendrian $\Lambda \subset X \setminus \text{Skel}(X, \lambda)$ which is in splitting position. Then under a horizontal splitting of $X$ we can arrange it so that $\Lambda$ splits into Legendrians with corners $\Lambda_{\pm} \subset X_{\pm} \setminus \text{Skel}(X_{\pm}, \lambda_{\pm})$.

Proof. It suffices to take a neighborhood of $P$ in $\mathcal{X}$ so that $\Lambda$ is of product form with respect to the neighborhood, which is clearly possible. □

Let us recall that given a contact manifold $(M, \xi)$ a hypersurface $N \subset M$ is called convex if it admits a transverse to it contact vector field $v$. Consider a split a tubular neighborhood $U_{\varepsilon} := N \times (-\varepsilon, \varepsilon)$ of a contact hypersurface $N$ such that the contact vector field $v$ coincides with $\frac{\partial}{\partial t}$, where $t$ is the coordinate corresponding to the second factor. The contact structure $\xi$ on $U_{\varepsilon}$ can be then defined by a contact form $\alpha = f dt + \lambda$, where $f : N \to \mathbb{R}$ is a smooth function and $\lambda$ is a 1-form on $N$. The set $Q = \{f = 0\} = \{x \in N; \; v(x) \in \xi\}$ is called the dividing set. It is always a regular level set of the function $f$. Denote $N_{+} := \{f > 0\}, N_{-} := \{f < 0\}$. We also denote

$$\tilde{Q} := Q \times (-\varepsilon, \varepsilon) = \bigcup_{t \in (-\varepsilon, \varepsilon)} v^t(Q)$$

and call it the dividing wall. The contact condition implies that $\beta := \lambda|_{\tilde{Q}}$ is a contact form on $Q$ and that the form $\lambda_{f} := \frac{1}{f} \lambda$ is Liouville on $N \setminus Q = N_{+} \cup N_{-}$. The corresponding Liouville fields $Z_{\pm}$ are complete of both halves $N_{+}$ and $N_{-}$ and for any point $x \in N_{\pm} \setminus \text{Skel}(N_{\pm})$ there exists a limit $\lim_{s \to \infty} Z_{\pm}^{s}(x) \in Q$. We say that the pair $(N, v)$, where $N$ is a convex hypersurface and $v$ a transverse vector field, is of Weinstein type, if $(N_{\pm}, \lambda_{f})$ is Weinstein.

Lemma 2.62. Let $(M, \xi)$ be a contact manifold, $N \subset (M, \xi)$ a Weinstein type convex hypersurface and $\Lambda \subset (M, \xi)$ a Legendrian transverse to $N$. Then there exists a contact isotopy $\phi_{s} : M \to M$, $t \in [0, 1]$ such that

(i) $\phi_{s}$ fixes $Q$;
(ii) $\phi_{s}$ leaves $N$ invariant;
(iii) there exists $\varepsilon > 0$ such that $\phi_{1}(\Lambda \cap U_{\varepsilon}) \subset Q_{\varepsilon}$;
(iv) $\phi_{s}$ is supported in a neighborhood $U' \supset U_{\varepsilon}$.

Proof. Note that a contact vector field $Y := t\frac{\partial}{\partial t} + Z_{\pm}$ on $N_{\pm} \times (-\varepsilon, \varepsilon)$ smoothly extends to $N \times (-\varepsilon, \varepsilon)$ as equal to $t\frac{\partial}{\partial t}$ on $Q \times (-\varepsilon, \varepsilon)$. Let us choose $\varepsilon$ sufficiently small so that the coordinate $t|_{\Lambda \cap U_{\varepsilon}}$ has no critical points. Denote $\Gamma := \Lambda \cap N$. A general position argument allows us to $C^{\infty}$-perturb $\Lambda$ so that $\Gamma \cap (\text{Skel}(N_{+}) \cup \text{Skel}(N_{-})) = \emptyset$ and the projection of $\pi : \Gamma \to Q$ along Liouville trajectories of $N_{\pm}$ is an embedding. Note that $\lambda|_{\Gamma} = 0$, and hence $\pi(\Gamma) \subset Q$ is Legendrian for the contact structure $\text{ker} \beta$ on $Q$. Let us parameterize $\Lambda \cap U_{\varepsilon}$ by an embedding $h_{0} : \Gamma \times (-\varepsilon, \varepsilon) \to \Lambda \cap U_{\varepsilon}$ such that $h_{0}(\Gamma \times t) \subset N \times t$. Define a Legendrian
isotopy \( h_s : \Gamma \times (-\varepsilon, \varepsilon) \to \Lambda \cap U, \ s \in [0, \infty) \) by the formula
\[
h_s(x, t) = Y^s(h_0(x, te^{-s})).
\]

Let us observe that \( h_s(x, t) = (g_s(x), t) \), with \( g_s(x) \) flowing a \( \lambda \)-isotropic submanifold \( \Gamma \)
to \( \pi(\Gamma) \) along trajectories of \( Z_\Lambda \). The flow is fixed on \( Q \). Let us analyze the isotopy in
a neighborhood of \( Q \). The form \( \lambda_f \) on \( \mathcal{O}p Q \) can be written as \( \frac{1}{u^1} \beta + dt \) for the coordinate
\( u = f \) on \( \mathcal{O}p Q \), and hence, \( Y = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} \). The flow \( Y^t \) is given by \( Y^s(x, u, t) = (x, e^{-s}u, e^s t) \).

Assuming that \( T \) is large enough so that \( h_T(\Gamma \times (-\varepsilon, \varepsilon)) \subset \mathcal{O}p Q \times (-\varepsilon, \varepsilon) \) we get
\( h_{T+s}(x, u, t) = (x, e^{-s}x, t) \to (x, 0, t) \) as \( s \to \infty \). Hence by reparameterizing the isotopy to the interval \([0, 1]\)
and extending it to the end-point 1 as \((x, 0, 1)\). we get a smooth Legendrian isotopy \( \phi_s : \Gamma \times (-\varepsilon, \varepsilon) \) such that \( \phi_1(\Gamma \times (-\varepsilon, \varepsilon)) \subset Q \times (-\varepsilon, \varepsilon) \).
The isotopy \( \phi_s \) can be realized by an ambient contact isotopy which keeps \( N \) invariant. By cutting off this isotopy outside a smaller
neighborhood of \( N \) we get the required Legendrian isotopy of \( \Lambda \).

\[\square\]

Lemma 2.63. Consider a Wc-manifold \((X, \lambda)\) with a dividing hypersurface \( P \subset X \) and its
soul \( S \). Let \( \Lambda_0 \subset X \setminus \text{Skel}(X, \lambda) \) be any Legendrian. Then there exists a Legendrian isotopy
\( \Lambda_t \subset X \setminus \text{Skel}(X, \lambda), \ t \in [0, 1] \), such that \( \Lambda_1 \) is in a splitting position.

\textbf{Proof.} We can realize \( \partial_\infty X \) as a smooth boundary of a neighborhood of \( \text{Skel}(X, \lambda) \) and view
\( \Lambda, \partial_\infty S \) and \( \partial_\infty P \) as submanifolds of \( \partial_\infty X \). Then the boundary \( \partial_\infty P \subset \partial_\infty X \) is a convex
hypersurface in the contact manifold \( (\partial_\infty X, \lambda) \), and its dividing set \( \partial_\infty S \) divides \( \partial_\infty P \) into
two copies \( S_\pm \) of \( S \). Therefore the conclusion follows from the previous lemma.

\[\square\]

2.9.3. Legendrians in Wc-buildings. Let \((\mathcal{X}, \lambda)\) be a Wc-germ with a structure of a \( k \)-block
building \( \mathcal{X}_k \to \cdots \to \mathcal{X}_1 \). We say that a Legendrian \( \Lambda \subset \mathcal{X} \setminus \text{Skel}(\mathcal{X}, \lambda) \) is compatible with
the building structure if its Liouville cone \( L \) is invariant with respect to all Liouville fields \( Z_i \),
\( i = 1, \ldots, k \), which we can think of as living in \( \mathcal{X} \) via Lemma 2.52.

\textbf{Lemma 2.64.} Let \((X, \lambda)\) be presented as a Weinstein building \( \mathcal{X}_k \to \cdots \to \mathcal{X}_1 \). Let \( \Lambda \subset \mathcal{X} \setminus \text{Skel}(\mathcal{X}, \lambda) \) be a Legendrian. Then there exists a Legendrian isotopy \( \Lambda_t \) starting from
\( \Lambda_0 = \Lambda \) such that \( \Lambda_1 \) is compatible with the building, i.e. its \( Z \)-Liouville cone is invariant with respect to all Liouville fields \( Z_i \) where they are defined.

\textbf{Proof.} By definition we have filtration by Wc-manifolds \( \mathcal{X} \supset \mathcal{X}_{\geq 2}, \ldots \supset \mathcal{X}_{\geq k-1} \supset \mathcal{X}_k \),
where \( \mathcal{X}_{\geq j} = \mathcal{X}_k \to \cdots \to X_j \). Note that we equivalently view \( \mathcal{X} \) as a 2-block building
\( \mathcal{X}_{\geq 2} \to \mathcal{X}_1 \). By definition the building \( \mathcal{X}_{\geq 2} \to \mathcal{X}_1 \) can be split into the blocks \( \mathcal{X}_1 \) and \( \mathcal{X}_{\geq 2} \)
along a splitting hypersurface \( H \) which can be defined as follows. Recall that \( \mathcal{X}_{\geq 2} \to \mathcal{X}_1 \) is
obtained from \( \mathcal{X}_1 \) and \( \mathcal{X}_{\geq 2} \) by taking the nucleus \( N \) of one of the boundary faces of \( \mathcal{X}_{\geq 2} \),
realizing it as Wc-hypersurface in \( \mathcal{X}_1 \), converting it to a nucleus of a boundary face and then
performing the horizontal gluing along the face \( H \) of \( \mathcal{X}_1 \) and the converted face of \( \mathcal{X}_{\geq 2} \).
The collar neighborhood \( \mathcal{O}p H \) of the face \( H \) in \( \mathcal{X}_1 \) is split as \( \mathcal{O}p H = H \times \mathcal{I} \). The inverse splitting
can be done by using the \( H \times \varepsilon \) as a splitting surface. Note that making the Legendrian \( \Lambda \) compatible with the building \( \mathcal{X}_{\geq 2} \to \mathcal{X}_1 \) is equivalent to moving it to a splitting position.
with respect to the splitting hypersurface $H \times \varepsilon$. Hence, using Lemma 2.64 we can move $\Lambda$ by a Legendrian isotopy to make it compatible with the building $\mathcal{X}_{\geq 2} \to \mathcal{X}_1$. The splitting defines a Legendrian $\Lambda_{\geq 2} \subset \mathcal{X}_{\geq 2} \setminus \text{Skel}(\mathcal{X}_{\geq 2}, \lambda)$. We continue by applying again 2.64 we move $\Lambda_{\geq 2}$ by a Legendrian isotopy to make it compatible with the building $\mathcal{X}_{\geq 2} \to \mathcal{X}_2$. Continuing the process we prove the lemma.

2.10. Weinstein handlebodies revisited.

2.10.1. Rounded handles. Consider an $n$-ball $D^n \subset S^{n-k} \times D^k$ and denote

$$K^n_k := (S^{n-k} \times D^k) \setminus \text{Int}^n D.$$ 

Thus, $\partial K^n_k$ is manifold with two boundary components $\partial_- K^n_k = S^{n-k} \times \partial D^{k-1}$ and $\partial_+ K^n_k := S^{n-1}$.

Consider a Weinstein manifold with boundary $G^n_k := T^*K^n_k$. It has two boundary faces $M_-$ and $M_+$ with nuclei $P_- := T^*(S^{n-k} \times \partial D^k)$ and $P_+ := T^*S^{n-1}$.

Definition 2.65. We call $G^n_k$ a rounded subcritical Weinstein handle.

Remark 2.66. Our rounded handles $G^n_k$ have nothing to do with round handles of index $k$.

Lemma 2.67. Let $W^{2n}$ be Weinstein domain and $\phi : \partial D^k \times D^{n-k} \to \partial W$ a Legendrian embedding, $k \leq n$. Let $W'$ be the Weinstein domain obtained by a subcritical index $k$ Weinstein handle attachment along the isotropic sphere $\phi(\partial D^k \times 0)$ with the framing given by the Legendrian extension. Suppose that the embedding $\phi$ is extended to a Legendrian embedding $\Phi : \partial D^k \times S^{n-k} \to \partial W$. Let $\Sigma = T^*(\partial D^k \times S^{n-k})$ be the Weinstein ribbon of this embedding. Then the Weinstein structure obtained by the vertical gluing of the rounded handle $G^n_k$ along $\Sigma$ is homotopic to the Weinstein structure on $W'$.

Proof. The vertical gluing consists of the conversion of $\Sigma$ to a boundary nucleus and the horizontal gluing to $G^n_k$ along the new nucleus. The conversion is achieved by taking the union of $W$ with a collar $T^*(\partial D^k \times S^{n-k} \times [0,1])$ and deforming the Liouville field of $W$ near $\Sigma$ so that the result is a Weinstein manifold with boundary. Hence the vertical gluing is the union of $W$ with the cotangent bundle of $\tilde{K}_n^k = (\partial D^k \times S^{n-k} \times [0,1]) \cup (K^n_k \times 1)$. Let $L := D^k \times D^{n-k}$ be attached to $\partial D^k \times S^{n-k}$ along $\partial D^k \times D^{n-k}$, i.e. the image of $\phi$. Then $\tilde{K}_n^k$ is diffeomorphic to $T^n_k := (\partial D^k \times S^{n-k} \times [0,1]) \cup (L \times 1)$ after smoothing corners. The underlying Weinstein structure of the vertically glued Weinstein manifold is given along the boundary of $T^*\tilde{K}_k^n$ by the Weinstein normal form for Morse-Bott boundary of index 0. Before smoothing, on $T^*T^n_k$ this corresponds to the Weinstein normal form for Morse-Bott boundary, also of index 0. By a Weinstein homotopy we can deform the structure to cancel the critical points on $T^*(\partial D^k \times S^{n-k} \times [0,1])$ along the $[0,1]$ direction, similar to a parametric Smale cancellation, where we think of $\partial D^k \times S^{n-k}$ as the parameter space. We are left with the Weinstein normal form for Morse-Bott boundary and corners on $T^*L \simeq T^*(D^k \times D^{n-k})$ which has index 1 on the boundary face $\partial D^k \times D^{n-k}$ and index 0 on the boundary face $D^k \times \partial D^{n-k}$. 
This Weinstein structure is evidently homotopic to the standard Morse-Weinstein normal form of index $k$ on $T^*(D^k \times D^{n-k})$, which is by definition the Weinstein surgery of index $k$. □

2.10.2. Weinstein manifolds as cotangent buildings. Recall that a cotangent building is a $W_c$-building $W = (B_k \to \cdots \to B_1)$ consisting of cotangent blocks $B_i := \mathcal{T}^*M_i$.

**Proposition 2.68.** Any Weinstein manifold germ $(\mathcal{X}, \lambda)$ admits the structure of a cotangent building with blocks $B_j = \mathcal{T}^*M_j$, where each $M_j$ is diffeomorphic to either a disc $D^n$ or one of the manifolds $K^n_j$, $j = 1, \ldots, n-1$ with corners added to its boundary.

**Proof.** Consider a handle decomposition $H_k \to H_{k-1} \to \cdots \to H_1$ of a Weinstein germ $(\mathcal{X}, \lambda)$. If $k = 0$ then we just replace the first index 0 handle by $T^*D^n$. Suppose that the Weinstein structure on the handlebody $W_{<k} := H_{k-1} \to \cdots \to H_1$ is already compatible with a structure of a cotangent building. If the index of the handle $H_k$ is equal $m$ for $0 < m < n$ we use Lemma 2.67 to replace the handle $H_k$ by the rounded handle $G^m_n = T^*K^m_n$. If $m = n$ we keep it as is equal to $T^*D^n$. In both cases we denote by $T^*M_k$ the block we need to attach with the nucleus of the attaching face $N$ equal to $T^*\partial D^n$ in the latter case and $\partial_- G^m_n = T^*(S^m \times S^{n-m})$ in the former one. For the attaching, $N$ is realized as a Weinstein hypersurface in $W_{<k} \setminus \text{Skel}(W_{<k})$. We denote by $\Lambda$ the core Legendrian of that hypersurface. Let $L$ be the $(Z_{<k})$-cone of $\Lambda$. Using Lemma 2.64 we deform $\Lambda$ via a Legendrian isotopy to make the cone $L$ invariant with respect to the contracting fields on the blocks $B_1, \ldots, B_k$. Note that this automatically gives $\tilde{M}_k := L \cup M^k$ a new structure of a $W_c$-manifold with additional corners. The neighborhood
of $\hat{M}_k$ then gets the structure of a cotangent block $B_k = \mathcal{F}^*(\hat{M}_k)$ and $W$ can be obtained from the vertical gluing of $B_k$ to $W_{\leq k}$.  

![Figure 2.17](image)

**Figure 2.17.** If $B_2 \to B_1$ already has the structure of a cotangent building and we wish to extend this structure to a third block $B_3$, we need to make sure the Legendrian $\Lambda$ along whose ribbon the block $B_3$ will be attached is adapted, i.e. its Liouville cone is invariant under the Liouville flows of the blocks $B_1$ and $B_2$.

More generally, we have the following result:

**Proposition 2.69.** Any $Wc$-manifold admits the structure of a cotangent building.

To prove this we will inductively apply the following lemma.

**Lemma 2.70.** Given any $Wc$-pair $(\mathcal{X}, \mathcal{A}, \lambda)$, where $\mathcal{X}$ and $\mathcal{A}$ are $Wc$-germs which admit the structure of a cotangent building, the hypersurface-to-nucleus conversion $(\mathcal{X}^{cf}, \lambda^{cf})$ also admits the structure of a cotangent building.

**Proof.** Suppose that $(\mathcal{X}, \lambda)$ is a $Wc$-manifold which admits the structure of a cotangent building $B_k \to \cdots \to B_1$ and $\mathcal{A} \subset \mathcal{X} \setminus \operatorname{Skel}(\mathcal{X}, \lambda)$ is a $Wc$-hypersurface which also admits the structure of a cotangent building $B'_k \to \cdots \to B'_1$. Then it is straightforward to verify that the hypersurface-to-nucleus conversion $(\mathcal{X}^{cf}, \lambda^{cf})$ inherits an induced structure of a cotangent building $(B'_k \times \mathcal{F}^*[0,1]) \to \cdots \to (B'_1 \times \mathcal{F}^*[0,1]) \to B_k \to \cdots \to B_1$.  

**Proof of Proposition 2.69.** Given any $Wc$-manifold $(\mathcal{X}, \lambda)$, we may convert all of its boundary nuclei to $Wc$-hypersurfaces to obtain its underlying Weinstein manifold $\hat{X}$, and $\mathcal{X}$ can be recovered from $\hat{X}$ by converting back the $Wc$-hypersurfaces to face nuclei. By Proposition 2.68 we know that $\hat{X}$ admits the structure of a cotangent building, and we may assume by induction on dimension that the $Wc$-hypersurfaces also admit the structure of a cotangent building. Hence it follows from Lemma 2.70 that $\mathcal{X}$ admits the structure of a cotangent building.
3. Positivity of Lagrangian planes

In this section we begin our discussion of the positivity relation for Lagrangian planes. Positivity will be revisited later in the context of cotangent buildings, for now we focus on the linear algebraic aspects.

3.1. The positivity relation.

3.1.1. Polarizations. Let \((V, \omega)\) be a symplectic vector space and let \(\tau, \nu \subset V\) be transverse linear Lagrangian subspaces. We will refer to the pair \((\tau, \nu)\) as a polarization of \(V\), and call \(\tau\) and \(\nu\) the respective horizontal and vertical spaces of the polarization. A polarization \((\tau, \nu)\) provides a canonical linear isomorphism \(\nu \cong \tau^*\) given by \(\nu \ni v \mapsto \iota(v)\omega|_\tau \in \tau^*\), and linear symplectomorphisms \(V \cong \tau \oplus \nu \cong T^*\tau\), where we define \(T^*\tau = \tau \oplus \tau^*\).

Let \(L \subset V\) be a Lagrangian subspace transverse to \(\nu\). Via the identification \(V \cong T^*\tau\), we can regard \(L\) as the graph of the differential of a quadratic form denoted by \(L^{(\tau, \nu)} : \tau \to \mathbb{R}\). By construction, this differential is the composition of the canonical maps

\[
dL^{(\tau, \nu)} : \tau \cong V / \nu \cong L \to V \to V / \tau \cong \nu.
\]

**Definition 3.1.** Let \((V, \omega)\) be a symplectic vector space with polarization \((\tau, \nu)\). Let \(L \subset V\) be a Lagrangian subspace transverse to the vertical space \(\nu\). We write

\[L \succ_{\nu} \tau\] (resp. \(L \prec_{\nu} \tau\))

when the quadratic form \(L^{(\tau, \nu)} : \tau \to \mathbb{R}\) is positive (resp. negative) definite.

![Figure 3.1](image-url)  
**Figure 3.1.** The above Lagrangians \(L_3, L_4\) in \(\mathbb{R}^2\) are given by positive-definite and negative-definite quadratic forms respectively, hence we have \(L_3 \succ_{L_2} L_1\) and \(L_4 \prec_{L_2} L_1\).

Note that \(L \succ_{\nu} \tau\) or \(L \prec_{\nu} \tau\) implies that \(L\) and \(\tau\) are transverse.

One can check positivity by Hamiltonian reduction to smaller symplectic vector spaces, in particular two-dimensional symplectic planes. Given a coisotropic \(W \subset V\), and any subspace \(L \subset V\), denote the symplectic reduction by \([L]^W = (L \cap W) / W^\perp \subset W / W^\perp = [V]^W\). Note
that if $L$ is Lagrangian then $[L]^W$ is Lagrangian in $[V]^W$ even when the intersection $L \cap W$ is not transverse.

**Lemma 3.2.** Let $\Lambda(V)$ denote the Lagrangian Grassmanian of a symplectic space $V$. Then the map $\pi_W : \Lambda(V) \to \Lambda([V]^W)$ defined by the formula $\pi_W(L) = [L]^W$, $L \in \Lambda(V)$, is continuous.

*Proof.* This is clear. $\square$

**Lemma 3.3.** Let $V$ be a symplectic vector space of dimension $2n > 0$, $(\tau, \nu)$ a polarization of $V$ and $L \subset V$ a linear Lagrangian subspace. The following conditions are equivalent:

(i) Positivity:

$$L \succ_{_{\nu}} \tau.$$

(ii) Positivity restricted to subspaces:

$$[L]^W \succ_{_{[\nu]_W}} [\tau]^W,$$

for all coisotropics $W \subset V$ containing $\nu$ with $\dim W > n$.

(iii) Positivity restricted to lines:

$$[L]^W \succ_{_{[\nu]_W}} [\tau]^W,$$

for all coisotropics $W \subset V$ containing $\nu$ with $\dim W = n + 1$.

(iv) Positivity of all reductions:

$$[L]^W \succ_{_{[\nu]_W}} [\tau]^W,$$

for all coisotropics $W \subset V$.

(v) Sylvester’s criterion:

$$\wedge^{\text{top}}[L]_{W_i} \succ_{_{\wedge^{\text{top}}[\nu]_{W_i}}} \wedge^{\text{top}}[\tau]_{W_i},$$

for any flag of coisotropics

$$\nu \subset W_{n+1} \subset \cdots \subset W_{2n-1} \subset W_{2n} = V$$

with $\dim W_i = i$.

*Proof.* Equivalence of (i) with (ii) (resp. (iii)) means that a quadratic form is positive definite if and only if it is positive definite on all subspaces (resp. on all lines), which is clear. Clearly (iv) implies (iii), and the converse implication follows from Lemma 3.2 together with the two fact that the positivity condition $L \succ_{_{\nu}} \tau$ implies that $[\nu]^W$, $[L]^W$ and $[\tau]^W$ are pairwise transverse. Finally, (v) is a reformulation of Sylvester’s criterion that a quadratic form is positive definite if and only if its leading principal minors have positive determinant. $\square$

3.1.2. *Cyclic symmetries.* Note that given a polarization $(\tau, \nu)$ of $V$, we also have the polarization $(\nu, \tau)$. Additionally, if $L \subset V$ is transverse to $\nu$, we can regard $(L, \nu)$ as a polarization.
Lemma 3.4. The following properties hold.

(i) Duality of polarization: \( L \succ \nu \Longleftrightarrow L \prec \nu \).

(ii) Exchange of graphs: \( L \succ \nu \Longleftrightarrow \nu \prec L \).

(iii) Transitivity: suppose \( K \subset V \) is another Lagrangian subspace transverse to \( \nu \). Then

\[
K \succ L, \quad L \succ \nu \quad \Longrightarrow \quad K \succ \nu
\]

Proof. Let \( \omega \) denote the symplectic form of \( V \). For (i), by construction, \( L(\tau,\nu)(X) = \omega(dL(\tau,\nu)^{-1}Y,Y) \), for \( Y \in \nu \) we have

\[
L(\nu,\tau)(Y) = \omega((dL(\tau,\nu)^{-1}Y,Y) = -\omega(Y,(dL(\tau,\nu)^{-1}Y) = -L(\tau,\nu)(dL(\tau,\nu)^{-1}Y).
\]

Hence if \( L(\tau,\nu) \) is positive definite, then \( L(\nu,\tau) \) is negative definite and vice versa. Next, let \( K \subset V \) be a Lagrangian transverse to \( \nu \). Note that \( K(\tau,\nu) = K(L,\nu) + L(\tau,\nu) \) under the isomorphism \( \tau \simeq V/\nu \simeq L \). Both (ii) and (iii) follow immediately (for (ii) take \( K = \tau \) so that \( K(\tau,\nu) = 0 \) and hence \( \tau(L,\nu) = -L(\tau,\nu) \)).

Lemma 3.5. Suppose \( L_1, L_2, L_3 \subset V \) are Lagrangian subspaces transverse to \( \nu \). Then for any permutation \( \sigma \in \Sigma_3 \) we have

\[
L_1 \succ L_2 \Longleftrightarrow L_{\sigma(1)} \succ L_{\sigma(2)}
\]

where \( \succ \) denotes \( \succ \) for \( \sigma \) an even composition of transpositions, and \( \prec \) for \( \sigma \) an odd composition of transpositions.

Proof. The case of \( \sigma = (23) \) (resp. \( \sigma = (12) \)) is part (i) (resp. (ii)) of Lemma 3.7. These permutations generate.

One can interpret Lemma 3.7, and in turn Lemma 3.5, as saying the ternary relation \( \succ \) provides a partial cyclic order on triples of Lagrangian subspaces.

Definition 3.6. We say an ordered list \( L_1, \ldots, L_m \subset V \) of Lagrangian planes is \( \succ \)-cyclically ordered (resp. \( \prec \)-cyclically ordered) when they are pairwise transverse and satisfy

\[
L_{i_2} \succ L_{i_3} \quad (\text{resp. } L_{i_2} \prec L_{i_3})
\]

whenever \( i_1, i_2, i_3 \in \{1, \ldots, m\} \) are cyclically ordered in \( \mathbb{Z}/m \). We will also write \( L_1 \succ L_2 \succ \cdots \succ L_m \) (resp. \( L_1 \prec L_2 \prec \cdots \prec L_m \)).

We record the following useful assertions for future reference. The assertions and their proofs hold in any partial cyclic order.

Lemma 3.7. The positivity relation satisfies the following properties.

(i) Suppose \( L_1 \prec L_2 \) and either \( \tau \prec L_1 \) or \( L_2 \prec \tau \). Then \( L_1 \prec L_2 \).

(ii) Suppose \( L_3 \succ L_2 \succ L_1 \). Then \( L_2 \succ L_1 \). If additionally \( L \succ L_2 \) then \( L \succ L_1 \).
Figure 3.2. A cyclic ordering on a tuple \((L_1, \ldots, L_n)\) can be thought of as an embedding into \(S^1\) preserving the cyclic order of all triples.

(iii) If \(L_- \succ L_1\) and \(L_+ \succ L_2\), then \(L_- \succ L_+\).

Proof. (i) By cyclic symmetry applied to \(L_1 \prec \tau L_2\), we have \(L_2 \prec L_1 \tau\) and \(\tau \prec L_2 L_1\). Suppose \(\tau \prec L_3\). Then by cyclic symmetry, we have \(L_1 \prec L_2 \tau\) and hence by transitivity \(L_2 \prec L_1 \tau\).

Similarly, suppose \(L_2 \prec \tau L_3\). Then by cyclic symmetry, we have \(\tau \prec L_2 L_3\) and hence by transitivity \(L_1 \prec L_2 \tau\).

(ii) For the first assertion, \(L_3 \succ L_4\) (resp. \(L_4 \succ L_2\)) implies \(L_3 \prec L_4\) (resp. \(L_4 \prec L_2\)) by transpositional symmetry. Hence by transitivity we have \(L_3 \prec L_1\), and so by transpositional symmetry \(L_2 \succ L_1\).

For the second, suppose \(L\) satisfies \(L \succ L_2\). By the previous part, we have \(L_2 \succ L_1\), hence \(L \succ L_1\) by cyclic symmetry. By transpositional symmetry, we then have \(L \prec L_3\). By assumption, we have \(L_3 \succ L_1\) and hence by transpositional symmetry \(L_3 \prec L_4\) so by transitivity \(L \prec L_4\). Finally, by transpositional symmetry, we obtain \(L \succ L_1\).

(iii) By cyclic symmetry, we have \(L_1 \succ L_+\), and hence by transitivity \(L_- \succ L_+\). \(\square\)

3.1.3. Positive zones. It is useful to reformulate positivity more geometrically.

Definition 3.8. Let \((V, \omega)\) be a symplectic vector space with polarization \((\tau, \nu)\). Let \(\Lambda(V)\) be the Lagrangian Grassmannian of \(V\). Denote by \(C(\tau, \nu) \subset \Lambda(V)\) the subset consisting of those Lagrangians \(L \subset V\) transverse to \(\nu\) and satisfying \(L \succ \tau\). We refer to \(C(\tau, \nu)\) as a positive zone, and its closure, denoted by \(\overline{C}(\tau, \nu)\), as a closed positive zone.
Remark 3.9. As justified by part (ii) of the following lemma, we will regard any single Lagrangian $L \in \Lambda(V)$ itself as a (degenerate) closed positive zone.

Lemma 3.10. The positive zone $C(\tau, \nu) \subset \text{Gr}(V)$ satisfies the following properties.

(i) $C(\tau, \nu)$ (resp. $\overline{C}(\tau, \nu)$) is non-empty, open (resp. closed), and geodesically convex for any homogenous metric on $\text{Gr}(V)$, hence contractible.

(ii) If $\tau$ approaches $\nu$ along a geodesic in $\Lambda(V)$, then $C(\tau, \nu)$ and $\overline{C}(\tau, \nu)$ limit to $\nu$ itself.

Proof. (i) is immediate from usual statements about quadratic forms. For (ii), we can put everything into a standard position so that we are contracting positive definite quadratic forms towards zero. □

Lemma 3.11. The following properties hold.

(i) Suppose $L_2 \in C(L_1, \nu)$ and either $L_1 \in C(\tau, \nu)$ or $L_2 \in C(\tau, \nu)$. Then $L_2 \in C(L_2, \tau)$.

(ii) Suppose $L_3 \in C(L_2, L_4), L_2 \in C(L_1, L_4)$. Then $L_2 \in C(L_1, L_3)$ and $C(L_2, L_3) \subset C(L_1, L_4)$.

(iii) If $L_- \in C(L_1, L_2)$ and $L_+ \in C(L_2, L_1)$, then $L_- \in C(L_+, L_2)$.

Proof. This follows immediately from Lemma 3.7 by unwinding the definition. □

Lemma 3.12. Let $\mathcal{L} \subset \Lambda(V)$ be a non-empty set of Lagrangian planes each transverse to a fixed Lagrangian plane $L \subset V$. The set $\mathcal{S}(\mathcal{L}, L) \subset \Lambda(V)$ of Lagrangian planes $L_- \subset V$ such that $\mathcal{L} \subset C(L_-, L)$ is convex. If $\mathcal{L}$ is compact, then $\mathcal{S}(\mathcal{L}, L)$ is non-empty, hence contractible.

Proof. The condition $T \in C(L_-, L)$ is equivalent to $L_- \in C(L, T)$, and thus

$$\mathcal{S}(\mathcal{L}, L) = \bigcap_{T \in \mathcal{L}} C(L, T).$$

Fix a Lagrangian plane $H \subset V$ transverse to $L$. Then we can identify Lagrangian planes $T \subset V$ transverse to $L$ with quadratic forms $Q_T$ on $H$, and specifically, the set of Lagrangian
planes $C(L,T) \subset \Lambda(V)$ with the convex set of those quadratic forms $Q$ on $H$ satisfying $Q < Q_T$. Hence $S(L,\mathcal{L})$ is an intersection of convex sets so itself convex.

Finally, when $\mathcal{L}$ is compact, the set of quadratic forms $Q_T$, for $T \in \mathcal{L}$, is bounded, hence we may choose a quadratic form $Q_-$ such that $Q_- < Q_T$, for all $T \in \mathcal{L}$. Thus $S(L,\mathcal{L})$ is non-empty. \hfill \Box

**Lemma 3.13.** Let $W \subset V$ be a coisotropic subspace. For any polarization $(\tau,\nu)$ of $V$, the reduction map $\pi_W$ projects the positive zone $C(\tau,\nu) \subset \Lambda(V)$ to the positive zone $C([\tau]^W,[\nu]^W) \subset \Lambda([V]^W)$.

**Proof.** For any $L \in \Lambda(V)$ transverse to $\nu$, the pre-composition of the quadratic form $([L]^W)([\tau]^W,[\nu]^W)$ with the reduction map $\tau \to [\tau]^W$ is equal to the quadratic form $L^{(\tau,\nu)}$. Since $\tau \to [\tau]^W$ is surjective, the lemma follows. \hfill \Box

**3.2. Positivity of polarizations.**

**3.2.1. Polarized Legendrians.** Recall that a Legendrian embedding $\Lambda \subset \mathcal{T}^*M \setminus M$ is said to be adapted to the block $\mathcal{T}^*M$ if $Op \partial \Lambda \subset Op \partial M$ and for each stratum $P \subset \partial_k M$ we have $\Lambda \cap (Op P = \mathcal{T}^*P \times \mathcal{T}^*\mathcal{T}^k) = \Lambda_k \times \mathcal{T}^k$ for a Legendrian $\Lambda_k \subset \mathcal{T}^*P$. Recall also that an adapted Legendrian $\Lambda$ is called regular if $\pi|_{\Lambda} : \Lambda \to M$ is an immersion with transverse self-intersections, where $\pi : \mathcal{T}^*M \to M$ is the cotangent projection.

A polarization of a Lagrangian submanifold $L$ of a symplectic manifold $(M,\omega)$ is a field $\eta \subset TM|_L$ of Lagrangian planes transverse to $L$. A polarization of a Legendrian $\Lambda$ in a contact manifold $(V,\xi)$ is a Lagrangian field $\tau \subset \xi|_{\Lambda}$ transverse to $\Lambda$ (recall that $\xi$ has a well-defined conformal symplectic structure; hence, it makes sense to talk about Lagrangian fields in $\xi$).

**Definition 3.14.** A polarization of a Legendrian $\Lambda \subset \mathcal{T}^*M \setminus M$ is a field of Lagrangian planes $\mu \subset \mathcal{T}^*M|_{\Lambda}$ which projects to a polarization of the projection of $\Lambda$ to $S^\infty M$.

Note that $\mu$ has to be tangent to the Liouville field $Z$.

In particular, the Lagrangian distribution $\nu_M$ which projects to the Legendrian distribution $\ell_M$ tangent to the spherical Legendrian fibration of $S^\infty M$, defines a canonical polarization of any regular Legendrian in $\mathcal{T}^*M \setminus M$. We will refer to this polarization as tautological.

The space of polarizations of a given Legendrian is contractible. Any given polarization of a Legendrian admits a ribbon, whose tangent planes to fibers along the 0-section form the given polarization. The space of germs of ribbons for a given polarization is contractible. Hence the space of germs of ribbons for a given Legendrian is contractible.

**Definition 3.15.** A polarization $\mu$ of a regular Legendrian $L \subset \mathcal{T}^*M \setminus M$ is called positive if $\ell_M \in C(TL,\mu)$ at any point of $L$, where $\ell_M$ is the tautological polarization.

**3.2.2. Positivity implies transversality of conormals.**

**Definition 3.16.** Let $(W,\lambda)$ be a Wc-manifold. A polarization of $W$ is a global field of Lagrangian planes $\eta \subset TW$. 
Let $T^*M$ be the cotangent bundle of a compact manifold with corners, $\tau = TM$ the tangent field along $M$ and $\nu = \ker(d\pi)$ the vertical field, where $\pi : T^*M \to M$ is the cotangent bundle projection.

**Lemma 3.17.** Let $L \subset T^*M \setminus M$ be the Lagrangian cone over a regular Legendrian, and $\eta$ a polarization of $T^*M$ whose restriction to the 0-section $M$ is positive with respect to the polarization $(\tau, \nu)$. Then $\eta$ is transverse to the cone $L$ on a sufficiently small neighborhood of $M \subset T^*M$.

The statement is a corollary of the following linear algebra lemma.

**Lemma 3.18.** Let $Q$ be a quadratic form and $L_Q = \{q = Ap\} \subset T^*\mathbb{R}^n$ the corresponding linear Lagrangian. If $Q$ is positive or negative definite then for any co-oriented hyperplane $H \subset \mathbb{R}^n$ through the origin, its conormal $T^*_H\mathbb{R}^n \subset T^*\mathbb{R}^n$ is transverse to $L_Q$. Conversely, if $L_Q$ is transverse to conormal $T^*_H\mathbb{R}^n$ for all co-oriented hyperplanes $H \subset \mathbb{R}^n$ through the origin, then $Q$ is positive or negative definite.

**Proof.** Indeed, the condition $(p, q) \in L_Q \cap \tau^*$ and $(p, q) \neq 0$ is equivalent to $Q(p) = 0$. □

**Proof of Lemma 3.17.** Indeed, the Liouville cone is the conormal of the immersed front projection of $\Lambda$, and hence we can apply Lemma 3.18 in a sufficiently small neighborhood of the 0-section. □

\[ \eta = x^2 - y^2 \]

**Figure 3.4.** For a non positive-definite quadratic form such as $\eta = x^2 - y^2$, the conormal $T^*_H\mathbb{R}^2$ of a smooth hypersurface $H \subset \mathbb{R}^2$ will be tangent to $\eta$ whenever $TH$ is tangent to the light-cone $\{x = y\}$.

### 4. Arboreal models

#### 4.1. Quadratic fronts.** Before we present the local models for arboreal singularities, we introduce the quadratic fronts out of which the models will be built and discuss some of their basic properties.
4.1.1. Basic constructions. For $i \geq 0$, define functions $h_i : \mathbb{R}^i \to \mathbb{R}$ by the inductive formula

$$h_0 = 0 \quad h_i = h_i(x_1, \ldots, x_i) = x_1 - h_{i-1}(x_2, \ldots, x_i)^2$$

For example, for small $i$, we have

$$h_1(x_1) = x_1 \quad h_2(x_1, x_2) = x_1 - x_2^2 \quad h_3(x_1, x_2, x_3) = x_1 - (x_2 - x_3^2)^2$$

Fix $n \geq 0$. For $i = 0, \ldots, n$, define smooth graphical hypersurfaces

$$n\Gamma_i = \{x_0 = h_i^2\} \subset \mathbb{R}^{n+1}$$

equipped with the graphical coorientation, and consider their union

$$n\Gamma = \bigcup_{i=0}^{n} n\Gamma_i$$

Note the elementary identities

$$n\Gamma_i = i\Gamma_i \times \mathbb{R}^{n-i} \quad i = 0, \ldots, n$$

$$n\Gamma_i \cap n\Gamma_0 = n^{-1}\Gamma_{i-1} \quad i = 1, \ldots, n$$

![Figure 4.1. The hypersurfaces $1\Gamma_i$](image)

Let $T^*\mathbb{R}^n$ denote the cotangent bundle with canonical 1-form $pdx = \sum_{i=1}^{n} p_i dx_i$ where $p = (p_1, \ldots, p_n)$ are dual coordinates to $x = (x_1, \ldots, x_n)$. Let $J^1\mathbb{R}^n = \mathbb{R} \times T^*\mathbb{R}^n$ denote the 1-jet bundle with contact form $dx_0 + pdx = dx_0 + \sum_{i=1}^{n} p_i dx_i$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with graph $\Gamma_f = \{x_0 = f(x)\} \subset \mathbb{R} \times \mathbb{R}^n$, we have the conormal Lagrangian of the graph $L_{\Gamma_f} = \{x_0 = f(x), p_i = -p_0 \partial f / \partial x_i\} \subset T^*\mathbb{R}^{n+1}$, and the conormal Legendrian of the graph $\Lambda_{\Gamma_f} = \{x_0 = f(x), p_0 = 1, p_i = -\partial f / \partial x_i\} \subset J^1\mathbb{R}^n$.

For $i = 0$, let $nL_0 = \mathbb{R}^n \subset T^*\mathbb{R}^n$ denote the zero-section. For $i = 1, \ldots, n$, introduce the conormal Lagrangian

$$nL_i = L_{n^{-1}\Gamma_{i-1}} \subset T^*\mathbb{R}^n$$

of the graph $n^{-1}\Gamma_{i-1} \subset \mathbb{R}^n$, and consider their union

$$nL = \bigcup_{i=0}^{n} nL_i$$
Similarly, for $i = 0, \ldots, n$, introduce the conormal Legendrian

$n\Lambda_i = \Lambda_i \cap J^1\mathbb{R}^n$

of the graph $n\Gamma_i \subset \mathbb{R}^{n+1}$, and consider their union

$n\Lambda = \bigcup_{i=0}^{n} n\Lambda_i$

Note that the Liouville form vanishes on the conical Lagrangian $nL_i \subset T^*\mathbb{R}^n$, hence its lift to $J^1\mathbb{R}^n = \mathbb{R} \times T^*\mathbb{R}^n$ with zero primitive is a Legendrian. We have the following compatibility:

**Lemma 4.1.** The contactomorphism

$$S : J^1\mathbb{R}^n \longrightarrow J^1\mathbb{R}^n$$

$$S(x_0, x, p) = (x_0 - p_1^2/4, x_1 + p_1/2, x_2, \ldots, x_n, p_1, \ldots, p_n)$$

takes the Legendrian $n\Lambda_i$ isomorphically to the Legendrian $\{0\} \times nL_i$, and thus the union $n\Lambda$ isomorphically to the union $\{0\} \times nL$.

**Proof.** Set $h_{i,1} = h_{i-1}(x_2, \ldots, x_i)$ so that $h_i = x_1 - h_{i,1}^2$. Observe $n\Lambda_i \subset J^1\mathbb{R}^n$ is given by the equations

$$x_0 = h_i^2, \quad pdx = -dh_i^2 = -2h_idh_i = -2h_i(dx_1 - 2h_{i,1}dh_{i,1})$$

so in particular $p_1 = -2h_i$ and $\sum_{i=2}^{n} p_i dx_i = 4h_i h_{i,1} dh_{i,1}$.

If we write $(\hat{x}_0, \hat{x}, p) = S(x_0, x, p)$, for $(x_0, x, p) \in n\Lambda_i$, then we have

$$\hat{x}_0 = x_0 - p_1^2/4 = \pm(x_0 - h_i^2) = 0 \quad \hat{x}_1 = x_1 + p_1/2 = x_1 - h_i = x_1 - (x_1 - h_{i,1}^2) = h_{i,1}^2$$

Now it remains to observe $nL_i \subset T^*\mathbb{R}^n$ is given by the equations

$$x_1 = h_{i,1}^2, \quad \sum_{i=2}^{n} p_i dx_i = -p_1 dh_{i,1}^2 = -2p_1 h_{i,1} dh_{i,1}$$

This completes the proof. \qed
4.1.2. *Distinguished quadrants.* We now specify some distinguished quadrants of the $^n\Gamma$ which we will use to define our arboreal models. Which of these quadrants are cut out by our sign conventions will become clearer when the arboreal models are introduced.

For $0 \leq j < i \leq n$, set

$$h_{i,j} := h_{i-j}(x_{j+1}, \ldots, x_i)$$

so in particular $h_{i,0} = h_i(x_1, \ldots, x_i)$ and $h_{i,i-1} = h_1(x_i) = x_i$.

For fixed $0 \leq i \leq n$, consider the collection of functions

$$h_{i,0}, \ldots, h_{i,i-1}$$

Note the triangular nature of the linear terms of the collection: for all $0 \leq j \leq i - 1$, the subcollection $h_{i,j} - x_{j+1}, h_{i,j+1}, \ldots, h_{i,i-1}$ is independent of $x_{j+1}$. Thus the level sets of the collection are mutually transverse.

Fix once and for all a list of signs $\delta = (\delta_0, \delta_1, \ldots, \delta_n), \delta_i \in \{\pm 1\}$. Define the domain quadrant $^nQ_i^\delta \subset \mathbb{R}^n$ to be cut out by the inequalities

$$\delta_1 h_{i,0} \leq 0, \ldots, \delta_i h_{i,i-1} \leq 0$$

By the transversality noted above, $^nQ_i^\delta$ is a submanifold with corners diffeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-i}$. Its codimension one boundary faces are given by the vanishing of one of the functions $h_{i,j}$.

Note $^nQ_i^\delta$ only depends on the truncated list $\delta_1, \ldots, \delta_i$. In particular, it is independent of $\delta_0$ which will enter the constructions next.

Define the cooriented hypersurface $^n\Gamma_i|_\delta \subset \mathbb{R}^{n+1}$ to be the restricted signed graph

$$^n\Gamma_i|_\delta = \{x_0 = \delta_0 h_i^2\}|_{^nQ_i^\delta}$$

with the graphical coorientation.

Thus $^n\Gamma_i|_\delta$ is cut out by the equations

$$x_0 = \delta_0 h_i^2, \quad \delta_1 h_{i,0} \leq 0, \ldots, \delta_i h_{i,i-1} \leq 0$$

Since $^n\Gamma_i|_\delta$ is graphical over $^nQ_i^\delta$, it is also a submanifold with corners diffeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-i}$. Likewise, its codimension one boundary faces are given by the vanishing of one of the functions $h_{i,j}$.

Consider as well the union

$$^n\Gamma|_\delta = \bigcup_{i=0}^n ^n\Gamma_i|_\delta$$

**Remark 4.2.** Note that

$$^n\Gamma_i = \bigcup_{\delta, \delta_0 = 1} ^n\Gamma_i|_\delta \quad ^n\Gamma = \bigcup_{\delta, \delta_0 = 1} ^n\Gamma_i|_\delta$$

since $x \in ^n\Gamma_i$ implies $x \in ^n\Gamma_i|_\delta$ where for $1 \leq j \leq i$, we set $\delta_j = -\text{sgn}(h_{i,j}(x))$, when $h_{i,j}(x) \neq 0$, and choose it arbitrarily otherwise.
Remark 4.3. Note if we set $\delta' = (\delta_0, \ldots, \delta_{n-1}, -\delta_n)$, then the map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $(x_0, \ldots, x_{n-1}, x_n) \mapsto (x_0, \ldots, x_{n-1}, -x_n)$, takes $^n\Gamma|_{\delta}$ isomorphically to $^n\Gamma|_{\delta'}$ as a cooriented hypersurface. Thus we could always set $\delta_n = 1$ and not miss any new geometry.

Note $^n\Gamma_i \cap \{x_0 < 0\}$, hence also $^n\Gamma_i|_{\delta} \cap \{\delta_0x_0 < 0\}$, is empty since $^n\Gamma_i$ is the graph of $h_i^2 \geq 0$.

Lemma 4.4. Fix $\delta = (\delta_0, \ldots, \delta_n)$, and set $\delta' = (\delta_0\delta_1, \delta_2, \ldots, \delta_n)$. The homeomorphism

$$s : \delta_0\mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \delta_0\mathbb{R}_{\geq 0} \times \mathbb{R}^n$$

$$s(x_0, x_1, x_2, \ldots, x_n) = (x_0, \delta_0\delta_1(x_1 + \delta_1\sqrt{\delta_0x_0}), x_2, \ldots, x_n)$$

gives a cooriented identification

$$s(^n\Gamma_i|_{\delta} \cap \{\delta_0x_0 \geq 0\}) = \delta_0\mathbb{R}_{\geq 0} \times ^{n-1}\Gamma_i|_{\delta'} \quad 0 < i \leq n$$

Proof. Recall $^n\Gamma_i|_{\delta}$ is defined by

$$x_0 = \delta_0 h_i^2 \quad \delta_1 h_{i,0} \leq 0, \ldots, \delta_i h_{i,i-1} \leq 0$$

in particular

$$x_0 = \delta_0 h_i^2 \quad \delta_1 h_{i,0} = \delta_1 h_i \leq 0$$

Note the functions $h_{i,1}, \ldots, h_{i,i-1}$ are independent of the coordinates $x_0, x_1$.

When $\delta_0x_0 \geq 0$ and $\delta_1 h_i \leq 0$, the equation $x_0 = \delta_0 h_i^2$ is equivalent to $\sqrt{\delta_0x_0} = -\delta_1 h_i$.

Expanding this in terms of the definitions, we can rewrite this in the form

$$x_1 + \delta_1\sqrt{\delta_0x_0} = h_{i-1}(x_2, \ldots, x_i)^2$$

Thus since $\delta'_0 = \delta_0\delta_1$, we see $s$ takes $^n\Gamma_i|_{\delta} \cap \{\delta_0x_0 \geq 0\}$ into $\delta_0\mathbb{R}_{\geq 0} \times \{x_1 = \delta'_0 h_{i-1}^2\}$.

Moreover, the additional functions $h_{i,1}, \ldots, h_{i,i-1}$ cutting out $^{n-1}\Gamma_i|_{\delta'} \subset \{x_1 = \delta'_0 h_{i-1}^2\}$ pull back to the same functions $h_{i,1}, \ldots, h_{i,i-1}$ cutting out $^n\Gamma_i|_{\delta}$.

Finally, the coorientations of $^n\Gamma_i|_{\delta}, ^{n-1}\Gamma_i|_{\delta'}$ are positive on respectively $\partial x_0, \partial x_1$. Observe the $\partial x_1$-component of $s_{\ast}\partial x_0$ is in the direction of $\partial x_1$, and hence $s$ gives a cooriented identification. \hfill \square

4.1.3. Alternative presentation. For compatibility with inductive arguments, it is useful to introduce an alternative sign convention and alternative presentation of the local models.

Fix signs $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$. Consider the involution $\sigma_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\sigma_\varepsilon(x_1, \ldots, x_n) = (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$.

Define the domain quadrant $^n\mathbb{R}_i^\varepsilon \subset \mathbb{R}^n$ cut out by the inequalities

$$\varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0, \ldots, \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0$$

Define the cooriented hypersurface $^n\Gamma_i^\varepsilon \subset \mathbb{R}^{n+1}$ to be the restricted signed graph

$$^n\Gamma_i^\varepsilon = \{x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon\} \cap \mathbb{R}_{i}^\varepsilon$$
with the graphical coorientation. Thus \( n\Gamma_1^\varepsilon \) is cut out by the equations
\[
x_0 = \varepsilon_0 h_i^2 \circ \sigma_{\varepsilon} \quad \varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_{\varepsilon} \leq 0, \ldots, \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_{\varepsilon} \leq 0
\]
Consider as well the union
\[
n\Gamma_1^\varepsilon = \bigcup_{i=0}^n n\Gamma_i^\varepsilon
\]

**Remark 4.5.** A simple but important observation: \( n\Gamma_i^\varepsilon \) in fact only depends on \( \varepsilon_0, \ldots, \varepsilon_{i-1} \) and not \( \varepsilon_i \). This is because \( h_{i,i-1} = x_i \) and so \( \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_{\varepsilon} = \varepsilon_{i-1} x_i \). In particular, the union \( n\Gamma^\varepsilon \) is independent of \( \varepsilon_n \).

We have the following adaption of Lemma 4.4.

**Lemma 4.6.** Fix \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_n) \), and set \( \varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n) \). The homeomorphism
\[
s : \varepsilon_0 \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \varepsilon_0 \mathbb{R}_{\geq 0} \times \mathbb{R}^n
\]
\[
s(x_0, x_1, x_2, \ldots, x_n) = (x_0, x_1 + \varepsilon_0 \sqrt{\varepsilon_0 x_0}, x_2, \ldots, x_n)
\]
gives a cooriented identification
\[
s(n\Gamma_i^\varepsilon \cap \{ \varepsilon_0 x_0 \geq 0 \}) = \varepsilon_0 \mathbb{R}_{\geq 0} \times n-1\Gamma_i^{\varepsilon'} \quad 0 < i \leq n
\]

**Proof.** Recall \( n\Gamma_i^\varepsilon \) is defined by
\[
x_0 = \varepsilon_0 h_i^2 \circ \sigma_{\varepsilon} \quad \varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_{\varepsilon} \leq 0, \ldots, \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_{\varepsilon} \leq 0
\]
in particular
\[
x_0 = \varepsilon_0 h_i^2 \circ \sigma_{\varepsilon} \quad \varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_{\varepsilon} = \varepsilon_0 \varepsilon_1 h_i \circ \sigma_{\varepsilon} \leq 0
\]
Note the functions \( h_{i,1}, \ldots, h_{i,i-1} \) are independent of the coordinates \( x_0, x_1 \).

When \( \varepsilon_0 x_0 \geq 0 \) and \( \varepsilon_0 \varepsilon_1 h_{i} \circ \sigma_{\varepsilon} \leq 0 \), the equation \( x_0 = \varepsilon_0 h_i^2 \circ \sigma_{\varepsilon} \) is equivalent to \( \sqrt{\varepsilon_0 x_0} = -\varepsilon_0 \varepsilon_1 h_i \circ \sigma_{\varepsilon} \). Expanding this in terms of the definitions, we can rewrite this in the form
\[
x_1 + \varepsilon_0 \sqrt{\varepsilon_0 x_0} = \varepsilon_1 h_{i-1,1} \circ \sigma_{\varepsilon'}
\]
Thus we see \( s \) takes \( n\Gamma_i^\varepsilon \cap \{ \varepsilon_0 x_0 \geq 0 \} \) into \( \varepsilon_0 \mathbb{R}_{\geq 0} \times \{ x_1 = \varepsilon_1 h_{i-1,1} \circ \sigma_{\varepsilon'} \} \).

Moreover, the additional functions \( h_{i,1}, \ldots, h_{i,i-1} \) cutting out \( n-1\Gamma_i^{\varepsilon'} \subset \{ x_1 = \varepsilon_1 h_{i-1,1} \circ \sigma_{\varepsilon'} \} \) pull back to the same functions \( h_{i,1}, \ldots, h_{i,i-1} \) cutting out \( n\Gamma_i^\varepsilon \).

Finally, the coorientations of \( n\Gamma_i^\varepsilon, n-1\Gamma_i^{\varepsilon'} \) are positive on respectively \( \partial x_0, \partial x_1 \). Observe the \( \partial x_1 \)-component of \( s \circ \partial x_0 \) is in the direction of \( \partial x_1 \), and hence \( s \) gives a cooriented identification.

Here is a useful corollary that “explains” the geometric meaning of the signs \( \varepsilon \).

**Corollary 4.7.** Fix \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_n) \).

For \( i = 0, \ldots, n-1 \), we have \( \varepsilon_i = \pm 1 \) if and only if \( n\Gamma_{i+1} \) is on the \( \pm \)-side of \( n\Gamma_i \) with respect to the graphical \( dx_0 \)-coorientation.
Moreover, for \(i = 1, \ldots, n - 1\), we have \(\varepsilon_i = \pm 1\) if and only if \(n\Gamma_{i+1} \cap n\Gamma_0\) is on the ±-side of \(n\Gamma_i \cap n\Gamma_0\) with respect to the graphical \(dx_1\)-coorientation.

**Proof.** For \(i = 0\), the first assertion is immediate from the definitions \(n\Gamma_0 = \{x_0 = 0\}\) and \(n\Gamma_1 = \{x_0 = \varepsilon_0(\varepsilon_1 x_1)^2 = \varepsilon_0 x_1^2, \varepsilon_0 \varepsilon_1 (\varepsilon_1 x_1) = \varepsilon_0 x_1 \leq 0\}\).

For \(i > 0\), both assertions follow by induction from Lemma 4.6. \(\square\)

Fix signs \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1})\). For \(i = 0\), let \(nL^0_i = \mathbb{R}^n \subset T^n\mathbb{R}^n\) denote the zero-section. For \(i = 1, \ldots, n\), introduce the positive conormal bundles

\[nL^\varepsilon_i = T^+_{n-1}\Gamma^\varepsilon_{i-1} \mathbb{R}^n \subset T^*\mathbb{R}^n\]

determined by the graphical coorientation, and consider their union

\[nL^\varepsilon = \bigcup_{i=0}^n nL^\varepsilon_i\]

Fix signs \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)\). For \(i = 0, \ldots, n\), introduce the Legendrian

\[n\Lambda^\varepsilon_i \subset J^1\mathbb{R}^n\]

projecting diffeomorphically to the front \(n\Gamma^\varepsilon_i \subset \mathbb{R}^{n+1}\), and consider their union

\[n\Lambda^\varepsilon = \bigcup_{i=0}^n n\Lambda^\varepsilon_i\]

We have the following compatibility of the above Lagrangians and Legendrians analogous to Lemma 4.1.

**Lemma 4.8.** Fix signs \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)\), and set \(\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n)\). The contactomorphism

\[S_{\varepsilon_0} : J^1\mathbb{R}^n \longrightarrow J^1\mathbb{R}^n\]

\[S_{\varepsilon_0}(x_0, x, p) = (x_0 - \varepsilon_0 p_0^2/4, x + \varepsilon_0 p_1/2, x_2, \ldots, x_n, p_1, \ldots, p_n)\]

takes the Legendrian \(n\Lambda^\varepsilon_i\) isomorphically to the Legendrian \(\{0\} \times nL^\varepsilon_i\), and thus the union \(n\Lambda^\varepsilon\) isomorphically to the union \(\{0\} \times nL^\varepsilon\).

**Proof.** The proof is the same as that of Lemma 4.1 with the following observations. Consider the additional equations

\[\varepsilon_0 \varepsilon_1 \delta_1 h_{i,0} \circ \sigma_\delta \leq 0, \ldots, \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0\]

First, over \(\varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0\), when \(p_1 = -2\varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon\), we then have \(p_1 = -2\varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \geq 0\), so we obtain the positive conormal direction. Second, the remaining functions \(h_{i,1}, \ldots, h_{i,i-1}\) are independent of \(x_0, x_1\). Thus \(S_{\varepsilon_0}\) indeed takes \(n\Lambda^\varepsilon_i\) to \(\{0\} \times nL^\varepsilon_i\). \(\square\)

**Remark 4.9.** By the lemma, we see the Legendrian \(n\Lambda^\varepsilon_i \subset J^1\mathbb{R}^n\) is independent of the initial sign \(\varepsilon_0\) so only depends on \(\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n)\).

It is also useful to record the following relationship of \(n\Gamma^\varepsilon\) with the extended model \(n\Gamma\).
Lemma 4.10. Fix signs $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$.

Given a contactomorphism $J^1\mathbb{R}^n \to J^1\mathbb{R}^n$ restricting to a closed embedding $n\Lambda^\varepsilon \subset \varepsilon_0 \cdot n\Lambda$ with $n\Lambda_i^\varepsilon \subset \varepsilon_0 \cdot n\Lambda_i$, for all $i$, consider the front $\Upsilon = \pi(n\Lambda^\varepsilon) \subset \varepsilon_0 \cdot n\Gamma$.

Then either the involution $\sigma^\varepsilon$ or its composition with $x_n \mapsto \pm x_n$ takes $\Upsilon$ to $n^{-1}\Gamma^\varepsilon$.

Proof. Note we have $n\Lambda_0^\varepsilon = \varepsilon_0 \cdot n\Lambda_0 = n\Lambda_0$. Consider the intersection $\Upsilon' = \pi((n\Lambda^\varepsilon \setminus n\Lambda_0) \cap n\Lambda_0)$ as a front inside of $\pi(n\Lambda_0) = n\Gamma_0 = \{x_0 = 0\}$. By induction, either the involution $\sigma^\varepsilon$ or its composition with $x_n \mapsto \pm x_n$ takes $\Upsilon'$ to $n^{-1}\Gamma^\varepsilon'$ where $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n)$. So we may assume $\Upsilon' = n^{-1}\Gamma^\varepsilon'$. Now observe $n\Gamma^\varepsilon$ is the unique way to extend $n^{-1}\Gamma^\varepsilon'$ within $\sigma^\varepsilon(\varepsilon_0 \cdot n\Gamma)$ compatible with coorientations.

We also have the following observation about signs.

Lemma 4.11. Let $\nu_0$ be the vertical polarization of $T^*\mathbb{R}^n \to \mathbb{R}^n$.

Then we have $\varepsilon(nL_1, \nu_0, nL_2^\varepsilon) = \varepsilon_0$.

Proof. Recall $nL_1^\varepsilon$ is the positive conormal to the graph $n^{-1}\Gamma_0^\varepsilon = \{x_0 = 0\}$, and $nL_1^\varepsilon$ is the positive conormal to the graph $n^{-1}\Gamma_1^\varepsilon = \{x_0 = \varepsilon_0 x_1^2\}$. Since $\varepsilon_0 x_1^2$ is an $\varepsilon_0$-definite quadratic form in $x_1$, the assertion follows.

\[\square\]

4.2. Arboreal models. We now present the local models for arboreal singularities.

4.2.1. Signed rooted trees.

Definition 4.12. We will use the following terminology throughout:

(i) A tree $T$ is a nonempty, finite, connected acyclic graph.

(ii) A rooted tree $\mathcal{T} = (T, \rho)$ is a pair of a tree $T$ and a distinguished vertex $\rho$ called the root.

(iii) A signed rooted tree $\mathcal{T} = (T, \rho, \varepsilon)$ is a rooted tree $(T, \rho)$ and a decoration $\varepsilon$ of a sign $\pm 1$ on each edge of $T$ not adjacent to the root $\rho$.

\[\text{Figure 4.3. A signed rooted tree.}\]
Given a signed rooted tree $T = (T, \rho, \varepsilon)$, we write $v(T)$ for the set of vertices, $e(T)$ for the set of edges, and $n(T) = v(T) \setminus \rho$ for the set of non-root vertices. We regard $v(T)$ as a poset with unique minimum $\rho$, and in general $\alpha \leq \beta \in v(T)$ when the shortest path connecting $\beta$ and $\rho$ contains $\alpha$. We call a non-root vertex $\beta$ a leaf if exactly one edge of $T$ is adjacent to $\beta$, and write $\ell(T) \subset v(T)$ for the set of leaf vertices.

**Remark 4.13.** Throughout what follows, for a finite set $S$, we write $\mathbb{R}^S$ for the Euclidean space of $S$-tuples of real numbers. One may always fix a bijection $S \simeq \{1, 2, \ldots, n\}$, for some $n \geq 0$, and hence an isomorphism $\mathbb{R}^S \simeq \mathbb{R}^n$, but it will be convenient to avoid choosing such identifications when awkward. We will most often consider $S = n(T)$ the non-root vertices for some rooted tree $T = (T, \rho)$. Here if one prefers to fix a bijection $b : n(T) \sim \rightarrow \{1, 2, \ldots, |n(T)|\}$, we recommend choosing $b$ to be order-preserving: if $\alpha \leq \beta$, then one should ensure $b(\alpha) \leq b(\beta)$.

This will allow for a clear translation of our constructions.

**Definition 4.14.** A signed rooted tree $T = (T, \rho, \varepsilon)$ is called positive if the decoration $\varepsilon$ consists of signs $+1$.

We will associate to any signed rooted tree $T = (T, \rho, \varepsilon)$, a multi-cooriented hypersurface, conic Lagrangian, and Legendrian $H_T \subset \mathbb{R}^{n(T)} \quad L_T \subset T^* \mathbb{R}^{n(T)} \quad \Lambda_T \subset J^1 \mathbb{R}^{n(T)}$

where as usual we write $n(T) = v(T) \setminus \rho$ for the set of non-root vertices.

By definition, the latter two will be determined by the first as follows:

(i) $L_T$ is the union of the zero-section $\mathbb{R}^{n(T)}$ and the positive conormal to $H_T$.

(ii) $\Lambda_T$ is the Legendrian lift of $L_T$ with zero primitive.

4.2.2. **Type A trees.** Let us first consider the distinguished case of $A_{n+1}$-trees with extremal root.

**Definition 4.15.** For $n \geq 0$, a linear signed $A_{n+1}$-rooted tree is a signed rooted tree $A_{n+1} = (A_{n+1}, \rho, a)$ with vertices $v(A_{n+1}) = \{0, 1, \ldots, n\}$, edges $v(A_{n+1}) = \{[i, i+1] \mid i = 0, \ldots, n-1\}$, and root $\rho = 0$.

By definition, the sign $a$ is a length $n-1$ list of signs $(a[1,2], \ldots, a[n-1,n])$. Let us set $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1}) = (a[1,2], \ldots, a[n-1,n], 1)$ to be the length $n$ list of signs where we pad $a$ by adding a single 1 at the end.

**Definition 4.16.** The models for $A_n$-type arboreal singularities are given as follows:

(i) The arboreal $A_1$-front is the empty set $H_{A_1} = \emptyset$ inside the point $\mathbb{R}^0$.

For $n \geq 1$, the arboreal $A_{n+1}$-front is the cooriented hypersurface

$$H_{A_{n+1}} = n-1 \Gamma^\varepsilon \subset \mathbb{R}^n$$

introduced in Section 4.1.3.
(ii) For $n \geq 0$, the arboreal $A_{n+1}$-Lagrangian is the union of the zero-section and positive conormal

$$L_{A_{n+1}} = \mathbb{R}^n \cup T^*_\mathbb{R}^n H_{A_{n+1}} \subset T^* \mathbb{R}^n$$

(iii) For $n \geq 0$, the arboreal $A_{n+1}$-Legendrian is the lift

$$\Lambda_{A_{n+1}} = \{0\} \times L_{A_{n+1}} \subset J^1 \mathbb{R}^n$$

Figure 4.4. The two $A_3$ fronts with positive and negative sign.

Remark 4.17. Following Remark 4.5, the arbitrary choice of the last sign $\varepsilon_{n-1} = 1$ does not affect the arboreal $A_{n+1}$-models.

Recall the linear signed $A_{n+1}$-rooted tree $A_{n+1} = (A_{n+1}, \rho, a)$ has vertices $v(A_{n+1}) = \{0, 1, \ldots, n\}$ with root $\rho = 0$, and so the non-root vertices form the set $n(A_{n+1}) = \{1, \ldots, n\}$. In the above definition, we should more invariantly view the ambient Euclidean space $\mathbb{R}^n$ in the form $\mathbb{R}^{n(A_{n+1})}$ where the ordering of the coordinates matches that of $n(A_{n+1})$.

With this viewpoint, we rename the smooth pieces of the $A_{n+1}$-front, indexing them by non-root vertices

$$H_i = n^{-1} P_{i-1}^e \subset H_{A_{n+1}} \quad i \in n(A_{n+1}) = \{1, \ldots, n\}$$

Likewise, we rename the smooth pieces of the $A_{n+1}$-Lagrangian, indexing them by vertices

$$L_0 = \mathbb{R}^n \subset L_{A_{n+1}}$$

$$L_i = T^*_\mathbb{R}^n H_i \subset L_{A_{n+1}} \quad i \in n(A_{n+1}) = \{1, \ldots, n\}$$

and similarly, we rename the smooth pieces of the $A_{n+1}$-Legendrian, indexing them by vertices

$$\Lambda_i = \{0\} \times L_{A_{n+1}, i} \subset \Lambda_{A_{n+1}} \quad i \in v(A_{n+1}) = \{0, 1, \ldots, n\}$$
Figure 4.5. Two $A_4$ fronts with different choices of signs. The other two fronts can be obtained from these two by reflections.

**Lemma 4.18.** For $n \geq 1$, and $n \in v(A_{n+1}) = \{0, 1, \ldots, n\}$ the unique leaf vertex, and $H_n \subset H_{A_{n+1}}$ the interior of the corresponding smooth piece, we have

$$H_{A_{n+1}} \setminus \bar{H}_n = H_{A_n} \times \mathbb{R}$$

inside of $\mathbb{R}^{n(A_{n+1})} = \mathbb{R}^{n(A_n)} \times \mathbb{R}$.

**Proof.** Recall the other smooth pieces $H_i = n^{-1} P_i^{-1}$, for $i = 1, \ldots, n - 1$, are independent of the last coordinate $x_n$. $\square$

### 4.2.3. General trees.

Now we consider a general signed rooted tree $T = (T, \rho, \varepsilon)$.

To each leaf $\beta \in \ell(T)$, we associate the linear signed $A_{n+1}$-rooted tree $A_\beta = (A_\beta, \rho, a)$ where $A_\beta$ is the full subtree of $T$ on the vertices $v(A_\beta) = \{\alpha \leq \beta \in v(T)\}$, and $a$ is the restricted sign decoration.

Consider the Euclidean space $\mathbb{R}^{n(T)}$. For each $\beta \in \ell(T)$, the inclusion $n(A_\beta) \subset n(T)$ induces a natural projection

$$\pi_\beta : \mathbb{R}^{n(T)} \longrightarrow \mathbb{R}^{n(A_\beta)}$$

**Definition 4.19.** Let $T = (T, \rho, \varepsilon)$ be a signed rooted tree.

(i) The arboreal model $T$-front is the multi-cooriented hypersurface given by the union

$$H_T = \bigcup_{\beta \in \ell(T)} \pi_\beta^{-1}(H_{A_\beta}) \subset \mathbb{R}^{n(T)}$$
where \( H_{A_\beta} \subseteq \mathbb{R}^{n(A_\beta)} \) is the arboreal \( A_\beta \)-front.

(ii) The arboreal model \( \mathcal{T} \)-Lagrangian is the union of the zero-section and positive conormal
\[
L_{\mathcal{T}} = \mathbb{R}^{n(\mathcal{T})} \cup T^+_{\mathbb{R}^{n(\mathcal{T})}} H_{\mathcal{T}} \subset T^*\mathbb{R}^{n(\mathcal{T})}
\]

(iii) The arboreal model \( \mathcal{T} \)-Legendrian is the lift
\[
\Lambda_{\mathcal{T}} = \{0\} \times L_{\mathcal{T}} \subset J^1\mathbb{R}^{n(\mathcal{T})}
\]

Arboreal models \( H_{\mathcal{T}}, L_{\mathcal{T}} \) and \( \Lambda_{\mathcal{T}} \) corresponding to positive \( \mathcal{T} \) are called positive.

**Figure 4.6.** Two non \( A_n \)-type fronts with different choices of signs.

*Remark 4.20.* When \( \mathcal{T} = A_{n+1} \), the above definition recovers Definition 4.16 verbatim.

Transporting from the case of \( A_{n+1} \), we may naturally index the smooth pieces of the \( \mathcal{T} \)-front by non-root vertices
\[
H_\alpha = \pi_\beta^{-1}(H_{A_{\beta},\alpha}) \subseteq H_{\mathcal{T}} \quad \alpha \in n(\mathcal{T})
\]
where \( \beta \in \ell(\mathcal{T}) \) is any leaf with \( \alpha \leq \beta \), and \( H_{A_{\beta},\alpha} \subseteq H_{A_{\beta}} \) is the corresponding smooth piece. Likewise, we may index the smooth pieces of the \( \mathcal{T} \)-Lagrangian by vertices
\[
L_\rho = \mathbb{R}^{n(\mathcal{T})} \subseteq L_{\mathcal{T}}
\]
and the smooth pieces of the \( \mathcal{T} \)-Legendrian by vertices
\[
\Lambda_\alpha = \{0\} \times L_\alpha \subseteq \Lambda_{\mathcal{T}} \quad \alpha \in v(\mathcal{T})
\]

Let us record a basic compatibility of the above Lagrangians and Legendrians.
Fix a signed rooted tree $T = (T, \rho, \varepsilon)$. Let us first consider the situation when there is a single vertex $\rho' \in T$ adjacent to $\rho$. Let $T' = T \setminus \rho$ be the signed rooted tree with root $\rho'$ and restricted signs.

Let $\alpha_1, \ldots, \alpha_k \in T'$ be the vertices adjacent to $\rho'$, and $\varepsilon_1, \ldots, \varepsilon_k$ the signs of $T$ assigned to the respective edges from $\rho'$ to $\alpha_1, \ldots, \alpha_k$.

Let $L_\infty^T \subset S^*\mathbb{R}^n(T)$ be the ideal Legendrian boundary of $L_T \subset T^*\mathbb{R}^n(T)$. Note that $L_\infty^T$ lies in the open subspace $J^1\mathbb{R}^n(T') \simeq \{p_{\rho'} = 1\} \subset S^*\mathbb{R}^n(T)$.

**Lemma 4.21.** The contactomorphism

$$S : J^1\mathbb{R}^n(T') \longrightarrow J^1\mathbb{R}^n(T')$$

$$S(x_{\rho'}, x, p) = (x_{\rho'} - \sum_{i=1}^k \varepsilon_i p_{\alpha_i}^2 / 4, \hat{x}, p)$$

$$\hat{x}_{\alpha_i} = x_{\alpha_i} + \varepsilon_i p_{1}/2, \text{ for } i = 1, \ldots, k, \quad \hat{x}_\beta = x_\beta \text{ else}$$

takes the Legendrian $L_\infty^T$ isomorphically to the Legendrian $\{0\} \times L_{T'}$.

Thus $L_\infty^T$ itself is a model arboreal Legendrian of type $T' = T \setminus \rho$.

**Proof.** For each leaf vertex of $T$, we have a linear signed type $\mathcal{A}$ subtree of $T$ given by the vertices running from $\rho$ to the leaf. By Definition 4.19, $L_T$ is the union of the corresponding linear signed type $\mathcal{A}$ subcomplexes $L_\mathcal{A}$. Each such subcomplex is independent of the coordinate $x_\beta$ indexed by vertices $\beta$ not in the subtree, hence lies in the zero locus of the dual coordinate $p_\beta$. Thus transport of each $L_\mathcal{A}^\infty$ under the contactomorphism of the lemma reduces to that of Lemma 4.8. \hfill \Box

More generally, suppose $\rho_1, \ldots, \rho_\ell$ are the vertices adjacent to $\rho$. Observe that $T \setminus \rho$ is a disjoint union of signed rooted subtrees $T_j \subset T \setminus \rho$, for $j = 1, \ldots, \ell$, with $\rho_j$ as root and restricted signs. Let $T_j^+ = T_j \cup \rho \subset T$ be the signed rooted subtree with $\rho$ readjoined as root and with restricted signs. Set $c_j = n(T) \setminus n(T_j)$.

Let $L_\infty^T \subset S^*\mathbb{R}^n(T)$ be the ideal Legendrian boundary of $L_T \subset T^*\mathbb{R}^n(T)$. We similarly have $L_\infty^{T_j^+} \subset S^*\mathbb{R}^{n(T_j^+)}$ the ideal Legendrian boundary of $L_{T_j^+} \subset T^*\mathbb{R}^{n(T_j^+)}$.

Since $\rho_j$ is the unique vertex adjacent to $\rho$ within $T_j^+$, observe that $L_{T_j^+}$ is connected and in fact lies in

$$J^1\mathbb{R}^{n(T_j)} = \{p_{\rho_j} = 1\} \subset S^*\mathbb{R}^{n(T_j)}.$$ 

Moreover, observe that $L_\infty^T$ is the disjoint union of the connected components

$$\Lambda_j = L_\infty^{T_j^+} \times \mathbb{R}^{c_j} \subset J^1\mathbb{R}^{n(T_j)} \times T^*\mathbb{R}^{c_j} = \{p_{\rho_j} = 1\} \subset S^*\mathbb{R}^n(T)$$

By Lemma 4.21, $L_\infty^{T_j^+} \subset J^1\mathbb{R}^{n(T_j)}$ is a model arboreal Legendrian of type $T_j$, so $\Lambda_j = L_\infty^{T_j^+} \times \mathbb{R}^{c_j} \subset J^1\mathbb{R}^{n(T_j)} \times T^*\mathbb{R}^{c_j}$ is a stabilized model arboreal Legendrian of type $T_j$. This proves:

**Lemma 4.22.** Fix a signed rooted tree $T = (T, \rho, \varepsilon)$. 

Let $\rho_1, \ldots, \rho_k$ be the vertices adjacent to $\rho$. Let $T_j \subset T \setminus \rho$ be the signed rooted subtree with $\rho_j$ as root and restricted signs, and $T^+_j = T_j \cup \rho \subset T$ the signed rooted subtree with $\rho$ readjoined as root and with restricted signs. Set $c_j = n(T) \setminus n(T_j)$.

Then the ideal Legendrian boundary $L^\infty_T \subset S^*\mathbb{R}^n(T)$ of the model arboreal Lagrangian $L_T \subset T^*\mathbb{R}^n(T)$ of type $T$ is the disjoint union of the Legendrians

$$\Lambda_j = L^\infty_{T^+_j} \times \mathbb{R}^{c_j} \subset S^*\mathbb{R}^n(T),$$

which are stabilized model arboreal Legendrians of type $T_j$.

By Lemma 4.18, we also have the following.

**Corollary 4.23.** For $\beta \in \ell(T)$ a leaf vertex, and $\check{H}_\beta \subset H_T$ the interior of the corresponding smooth piece, we have

$$H_T \setminus \check{H}_\beta = H_{T \setminus \beta} \times \mathbb{R}^\beta$$

inside of $\mathbb{R}^n(T) = \mathbb{R}^n(T \setminus \beta) \times \mathbb{R}^\beta$.

**4.2.4. Extended arboreal models.** It will be useful for us also define extended arboreal models associated with rooted, but not signed trees $T = (T, \rho)$.

For the unsigned rooted tree $A_{n+1} = (A_{n+1}, \rho)$ we define

$$H_{A_{n+1}} := n^{-1}\Gamma \subset \mathbb{R}^n,$$

$$L_{A_{n+1}} := \mathbb{R}^n \cup T^\infty_{\mathbb{R}^n} H_{A_{n+1}} \subset T^*\mathbb{R}^n,$$

$$\Lambda_{A_{n+1}} := \{0\} \times L_{A_{n+1}} \subset J^1\mathbb{R}^n.$$

Similarly, for a general rooted tree $\mathcal{F} = (T, \rho)$ we define

$$H_{\mathcal{F}} = \bigcup_{\beta \in \ell(\mathcal{F})} \pi^{-1}_\beta(H_{\mathcal{A}_\beta}) \subset \mathbb{R}^n(T)$$

where $H_{\mathcal{A}_\beta} \subset \mathbb{R}^n(\mathcal{A}_\beta)$ is the arboreal $\mathcal{A}_\beta$-front. Furthermore, we define

$$L_{\mathcal{F}} = \mathbb{R}^n(\mathcal{F}) \cup T^+_{\mathbb{R}^n(\mathcal{F})} H_T \subset T^*\mathbb{R}^n(\mathcal{F})$$

and

$$\Lambda_{\mathcal{F}} = \{0\} \times \Lambda_{\mathcal{F}} \subset J^1\mathbb{R}^n(\mathcal{F})$$

Clearly, for any signed version $\mathcal{T}$ of the tree $\mathcal{F}$ we have $H_T \subset H_{\mathcal{F}}, L_T \subset L_{\mathcal{F}}, \Lambda_T \subset \Lambda_{\mathcal{F}}$.

**Lemma 4.24.** Given a closed embedding $\Lambda^\infty_{\mathcal{F}} \subset \Lambda^\infty_{\mathcal{T}}$ with $\Lambda^\infty_{\mathcal{T}, \alpha} \subset \Lambda^\infty_{\mathcal{F}, \alpha}$, for all $\alpha$, the front $\pi(\Lambda^\infty_{\mathcal{T}}) \subset H_{\mathcal{F}}$ is an embedding of $H_T$.

**Proof.** For each leaf vertex of $\mathcal{F}$, we have a linear signed type $A$ subtree of $\mathcal{T}$ given by the vertices running from $\rho$ to the leaf. By construction, $\Lambda^\infty_{\mathcal{F}}$ and $\Lambda^\infty_{\mathcal{T}}$ are the union of the corresponding type $A$ subcomplexes $L^\infty_A$ and $L^\infty_{\mathcal{A}_{\beta}}$. Each such subcomplex is independent of the coordinates $x_\beta$ indexed by vertices $\beta$ not in the subtree. Now Lemma 4.10 confirms $\pi(L^\infty_A)$ is the standard embedding of $H_A$ after a change of coordinates $x_\alpha$ indexed by vertices $\alpha$ in
the subtree. Moreover, the change of coordinates agrees for $x_\alpha$ indexed by vertices $\alpha$ in the intersection of such subtrees. By definition, $H_T$ is the union of the $H_A$. □

5. ARBOREAL LAGRANGIANS AND THEIR STABILITY

In this section we define arboreal Lagrangian and Legendrian subsets and prove their stability under symplectic reduction and Liouville cone operations.

5.1. Arboreal Lagrangians and Legendrians.

**Definition 5.1.** Arboreal Lagrangians and Legendrians are defined as follows:

(a) A closed subset $L \subset X$ of a $2m$-dimensional symplectic manifold $(X, \omega)$ is called an arboreal Lagrangian if the germ of $(X, L)$ at any point $\lambda \in L$ is symplectomorphic to the germ of the pair $(T^*\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, L_T \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $T$ with $n := n(T) \leq m$.

(b) A closed subset $\Lambda \subset Y$ of a $(2m+1)$-dimensional contact manifold $(Y, \xi)$ is called an arboreal Legendrian if the germ of $(Y, \Lambda)$ at any point $\lambda \in \Lambda$ is contactomorphic to the germ of $(J^1(\mathbb{R}^n \times \mathbb{R}^{m-n}) = J^1\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, \Lambda_T \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $T$ with $n := n(T) \leq m$.

(c) A closed subset $H \subset M$ of an $(m+1)$-dimensional manifold $M$ is called an arboreal front if the germ of $(M, H)$ at any point $m \in M$ is diffeomorphic to the germ of $(\mathbb{R}^{n+1} \times \mathbb{R}^{m-n}, H_T \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $T$ with $n := n(T) \leq m$.

The pair $(T, m)$ is called the arboreal type of the germ of $L$, $\Lambda$, or $H$ at the given point. We say $L$, $\Lambda$, or $H$ is positive if it is locally modeled on positive arboreal models at all points.

**Remark 5.2.** Later we will also allow arboreal Lagrangians to have boundary and even corners, but throughout the present discussion we restrict to the above definition for simplicity.

Given an arboreal Lagrangian we call $\sup_{\lambda \in L}\{n(T(\lambda))\}$ the maximal order of $L$, where $T(\lambda)$ is a the signed rooted tree describing the germ of $L$ at the point $\lambda$. Similarly, we define the maximal order of arboreal Legendrians and fronts.

Every arboreal Lagrangian or Legendrian is naturally stratified by isotropic strata indexed by the corresponding tree type. A Lagrangian distribution $\eta$ in $X$ is called transverse to an arboreal Lagrangian $L$ if it is transverse to all top-dimensional strata of $L$. Similarly a Legendrian distribution $\eta \subset \xi$ in a contact $(Y, \xi)$ is called transverse to an arboreal Legendrian $\Lambda$ if it has trivial intersection with tangent planes to all top-dimensional strata of $\Lambda$.

**Definition 5.3.** A polarization of $L$ or $\Lambda$ is a transverse Lagrangian distribution.

**Remark 5.4.** We emphasize the transversality to an arboreal Lagrangian means transversality to its closed smooth pieces, and not just to open strata.
Let \( L \) be an arboreal Lagrangian whose germ at a point \( \lambda \in L \) has the type \( (T, \rho, \varepsilon, m) \). Let \( L_\rho \subset T_\lambda X \) the tangent plane to the root Lagrangian corresponding to the root \( \rho \). For each vertex \( \alpha \) connected by an edge with \( \rho \) let \( L_\alpha \subset T_\lambda X \) the tangent plane to the root Lagrangian corresponding to the root \( \rho \). For each vertex \( \alpha \) connected by an edge with \( \rho \) let \( L_\alpha \subset T_\lambda X \) denote the Lagrangian plane tangent to the Lagrangian corresponding to the vertex \( \alpha \). We recall that \( L_\rho \) and \( L_\alpha \) cleanly intersect along a codimension 1 subspace. Consider a coistropic subspace \( C_\alpha := \text{Span}(L_\rho, L_\alpha) \subset T_\lambda X \). Let \( \eta \) be a Lagrangian distribution in \( X \) transverse to \( L \). Define the sign

\[
\varepsilon(\eta, L, \alpha) = \begin{cases} 
+1, & \text{if } [L_\rho]^{C_\alpha} \prec [L_\alpha]^{C_\alpha} \prec [\eta]^{C_\alpha}; \\
-1, & \text{if } [L_\rho]^{C_\alpha} \prec [\eta]^{C_\alpha} \prec [L_\alpha]^{C_\alpha}.
\end{cases}
\]

Figure 5.1. The notion of sign for the \( A_2 \) singularity.

Similarly, if \( \Lambda \) is an arboreal Legendrian in a contact manifold \( (Y, \xi) \), and \( \eta \) a Legendrian distribution transverse to \( \Lambda \), then for any point \( \lambda \in \Lambda \) of type \( T = (T, \rho, \varepsilon) \) we assign a sign \( \varepsilon(\eta, \Lambda, \alpha) \) for every vertex \( \alpha \) adjacent to the root \( \rho \) as equal to \( \pm 1 \) depending on the \( \prec \)-order of the triple \( [L_\rho]^{C_\alpha}, [L_\alpha]^{C_\alpha}, [\eta]^{C_\alpha} \) in \( [\xi_\lambda]^{C_\alpha} \).

5.2. Stability of arboreal Lagrangians and Legendrians. The following is the main result of Section 5.

**Theorem 5.5.** Let \( T \) be a signed rooted tree. Let \( \rho_1, \ldots, \rho_k \) be vertices adjacent to the root \( \rho \) and \( T_j \) be subtrees with roots \( \rho_j \) (where we removed the decoration of edges \([\rho_j \alpha]\)). Let \( \phi_j : T^* \mathbb{R}^m \to J^1 \mathbb{R}^m, m \geq n = n(T) \), be germs of Weinstein hypersurface embeddings with disjoint images. Denote \( z_j := \phi_j(0), \Lambda^j = \phi_j(L_{T_j} \times \mathbb{R}^{m-n(T_j)}) \), \( j = 1, \ldots, k \). Suppose that

(i) \( \pi(z_j) = 0 \);

(ii) the arboreal Legendrian \( \Lambda := \bigcup_{j=1}^k \Lambda^j \) projects transversely under the front projection \( J^1 \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \);

(iii) for each edge \([\rho_j \alpha]\) we have \( \varepsilon(\nu, \Lambda^j, \alpha) = \varepsilon_{[\rho_j \alpha]} \).
Then $\mathbb{R}^m \cup C(\Lambda)$ is an arboreal Lagrangian of type $(\mathcal{T}, m)$ or equivalently, the germ of the front $\pi(\Lambda)$ is diffeomorphic to $H_\mathcal{T} \times \mathbb{R}^{m-n(\mathcal{T})}$.

Theorem 5.5 is a corollary of its unsigned version which is the content of the following proposition.

**Proposition 5.6.** Let $\mathcal{T}$ be a rooted tree. Let $\rho_1, \ldots, \rho_k$ be vertices adjacent to the root $\rho$ and $\mathcal{T}_j$ be subtrees with roots $\rho_j$. Let $\phi_j : \mathcal{T} \times \mathbb{R}^m \to J^1 \mathbb{R}^m$, $m \geq n = n(\mathcal{T})$, be germs of Weinstein hypersurface embeddings. Denote $z_j := \phi_j(0)$, $\tilde{\Lambda}^j = \phi_j(\tilde{L}_{\mathcal{T}_j} \times \mathbb{R}^{m-n(\mathcal{T}_j)})$, $j = 1, \ldots, k$. Suppose that

(i) $\pi(z_j) = 0$;

(ii) the extended arboreal Legendrian $\Lambda := \bigcup_{j=1}^k \tilde{\Lambda}^j$ projects transversely under the front projection $J^1 \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$;

Then $\mathbb{R}^m \cup C(\Lambda)$ is an extended arboreal Lagrangian of type $(\mathcal{T}, m)$, or equivalently, the germ of the front $\pi(\Lambda)$ is diffeomorphic to $H_\mathcal{T} \times \mathbb{R}^{m-n(\mathcal{T})}$.

**Proof of Theorem 5.5 using Proposition 5.6.** Consider the arboreal Legendrian as a closed subcomplex of the extended model. Apply Proposition 5.6 to assume the extended front is in canonical form. Then Lemma 4.24 implies the front of the original arboreal Legendrian is a canonical model. □

Proposition 5.6 will be proven below in this section (see Section 5.6) below, but first we discuss some corollaries of Theorem 5.5.

**Corollary 5.7.** Let $\Lambda \subset \partial_\infty T^* M$ be an arboreal Legendrian. Suppose that the front projection $\pi : \Lambda \to M$ is a transverse immersion. Then $L := C(\Lambda) \cup M$ is an arboreal Lagrangian.

![Figure 5.2](image-url)  

**Figure 5.2.** In particular, the zero section union the Liouville cone on a regular Legendrian is arboreal with $A_2$ singularities along its front.

**Proof.** The intersection $H := M \cap \overline{C(\Lambda)}$ is the front of the Legendrian $\Lambda$. Each point $a \in H$ has finitely many pre-images $z_1, \ldots, z_k \in \Lambda$. The germs $\Lambda^j$ of $\Lambda$ at $z_j$ by our assumption
are images of arboreal Lagrangian models under Weinstein embeddings of their symplectic neighborhoods. Hence, by Theorem 5.5 the germ of $L$ at $z$ is of arboreal type. 

It is not a priori clear that even the standard Lagrangian (resp. Legendrian) arboreal models are arboreal Lagrangians (resp. Legendrians). However, the following corollary shows that they are.

**Corollary 5.8.** Consider a model Lagrangian $L_T \subset T^*\mathbb{R}^n$, $n = n(T)$. Then for any point $\lambda \in L_T$ the germ of $L_T$ at $\lambda$ is an $(\mathcal{T}', n)$-Lagrangian for a signed rooted tree $\mathcal{T}'$.

**Proof.** We argue by induction in $n$. The base of the induction is trivial. Assuming the claim for $n - 1$ we recall that $L_T$ can be presented as $L_\rho \cup C(\Lambda)$, where $L_\rho$ is the smooth piece corresponding to the root $\rho$ of $\mathcal{T}$ and $\Lambda$ is a union of model Legendrians of dimension $n - 1$ in $\partial_\mathcal{T} T^*(\mathbb{R}^n)$. By the induction hypothesis $\Lambda$ is an arboreal Legendrian, and hence applying Corollary 5.7 we conclude that $L_T$ is an arboreal Lagrangian. 

**Remark 5.9.** We will not need it in what follows, so only briefly comment here that it is possible to specify precisely the type $(\mathcal{T}', n)$ of the germ of $L_T$ at each point $\lambda \in L_T$. Following [N13] the underlying tree $T'$ is a canonically defined subquotient of $T$, in other words, a diagram $T' \leftarrow S \rightarrow T$, where $S \rightarrow T$ is a full subtree, and $S \rightarrow T'$ contracts some edges; conversely, any such subquotient can occur. Furthermore, if we partially order $T$ with the root $\rho \in T$ as minimum, then the root $\rho' \in T'$ is the unique minimum of the natural induced partial order on $T'$. Finally, to equip $T'$ with signs, we restrict the signs of $T$ to the subtree $S$, then push them forward to $T'$ using that each edge of $T'$ is the image of a unique edge of $S$.

**Corollary 5.10.** Let $L_T \subset T^*\mathbb{R}^n$ be a model Lagrangian associated with a signed rooted tree $(T, \rho, \varepsilon)$. Let $\eta_0, \eta_1$ be two polarizations transverse to $L_T$. Suppose that for any vertex $\alpha$ of $T$ adjacent to $\rho$ we have

$$\varepsilon(\eta_0, L, \alpha) = \varepsilon(\eta_1, L, \alpha).$$

Then there is a (germ at the origin of) a symplectomorphism $\psi : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ such that $\psi(L) = L$ and $d\psi(\eta_0) = \eta_1$ along $L$.

**Proof.** There exists embeddings $h_0, h_1 : T^*\mathbb{R}^n \rightarrow J^1\mathbb{R}^n$ as Weinstein hypersurfaces, such that $h_j(\eta_j) = \nu_0$, $j = 0, 1$, where $\nu_0$ is the canonical Legendrian foliation of $J^1\mathbb{R}^n$ by fibers of the front projection to $\mathbb{R}^n \times \mathbb{R}$. Consider the arboreal Lagrangians $L_j := C(h_j(L_T)) \cup (\mathbb{R}^n \times \mathbb{R})$, $j = 0, 1$, and note that their arboreal types are described by the same signed rooted tree $\mathcal{T}$ obtained from $T$ by adding a new root, connecting it by an edge to the old one, and assigning to edges $[\rho \alpha]$ of $T \subset \mathcal{T}$ adjacent to the old root $\rho$ the sign $\varepsilon(\eta_j, L, \alpha) = \varepsilon(\eta_1, L, \alpha)$. Applying Theorem 5.5 we find the required symplectomorphism $\psi$. 

**Corollary 5.11.** Let $H \subset M$ be an arboreal front. Then for any submanifold $\Sigma \subset M$ transverse to (all strata of) $H$ the intersection $\Sigma \cap H$ is an arboreal front in $\Sigma$. 
Proof. We can assume that $H$ is an arboreal front germ at a point $x \in H$, and hence the germ of $(M,H)$ at $x$ is diffeomorphic to the germ of $(\mathbb{R}^{n(T)}+1 \times \mathbb{R}^k, H_T \times \mathbb{R}^k)$ for some rooted signed arboreal tree $T$ and $k = n - n(T)$. Note that the transversality of $\Sigma$ to $H$ implies that $\text{codim} \Sigma \leq k$ and that the projection of $p : \Sigma \subset \mathbb{R}^{n(T)+1} \times \mathbb{R}^k \rightarrow \mathbb{R}^{n(T)+1}$ to the first factor is a submersion, and because we are dealing with germs, it is a trivial fibration. On the other hand, the projection $p|_{\Sigma \cap H} : \Sigma \cap H \rightarrow H_T$ is the restriction of this fibration to $H_T \subset \mathbb{R}^N(T)$. □

Figure 5.3. Illustration that $\Sigma \cap H$ is an arboreal front in $\Sigma$.

5.3. Parametric version. The following is the parametric version of Theorem 5.5.

Theorem 5.12. Let $T$ be a signed rooted tree. Let $\rho_1, \ldots, \rho_k$ be vertices adjacent to the root $\rho$ and $T_j$ be subtrees with roots $\rho_j$ (where we removed the decoration of edges $[\rho_j \alpha]$). Let $\phi_j^y : \mathcal{T} \times \mathbb{R}^m \rightarrow J^1 \mathbb{R}^m$, $m \geq n = n(T)$, be families of germs of Weinstein hypersurface embeddings with disjoint images, parametrized by a manifold $Y$. Denote $z_j^y := \phi_j^y(0)$, $\Lambda_j^y = \phi_j^y(L_{T_j} \times \mathbb{R}^{m-n(T_j)})$, $j = 1, \ldots, k$. Suppose that

(i) $\pi(z_j^y) = 0$;

(ii) the arboreal Legendrian $\Lambda_j^y := \bigcup_{j=1}^k \Lambda_j^y$ projects transversely under the front projection $J^1 \mathbb{R}^m \rightarrow \mathbb{R} \times \mathbb{R}^n$;

(iii) for each edge $[\rho_j \alpha]$ we have $\varepsilon(\nu, \Lambda_j^y, \alpha) = \varepsilon_{[\rho_j \alpha]}$.

Then there exists a family of diffeomorphisms $\phi_y$ between $H_T \times \mathbb{R}^{m-n(T)}$ and the front $\pi(\Lambda_y)$. If $K \subset Y$ is a closed subset and the $\phi_j^y$ are the standard embeddings of the local model for $y \in \mathcal{O}_p(K)$, then we may further assume $\phi_y = \text{Id}$ for $y \in \mathcal{O}_p(K)$.

The parametric version of Proposition 5.6 is formulated similarly. As a consequence of Theorem 5.12 we get the following result:

Corollary 5.13. Fix a signed rooted tree $T = (T, \rho, \varepsilon)$, set $n = |n(T)|$ and consider the arboreal $T$-front $H_T \subset \mathbb{R}^n$. Let $D(\mathbb{R}^n, H_T)$ be the group of germs at 0 of diffeomorphisms of $\mathbb{R}^n$ preserving $H_T$ as a front, i.e. as a subset along with its coorientation.

Then the fibers of the natural map $D(\mathbb{R}^n, H_T) \rightarrow \text{Aut}(T)$ are weakly contractible.
Proof. We deduce Corollary 5.13 from Theorem 5.12. We will argue for $T = A_{n+1}$ when $H_{A_{n+1}} = n^{-1} \Gamma$; the case of general $T$ is similar.

Since $\text{Aut}(A_{n+1})$ is trivial, we seek to show $D(\mathbb{R}^n, n^{-1} \Gamma)$ is weakly contractible. Note any $\varphi \in D(\mathbb{R}^n, n^{-1} \Gamma)$ preserves $0$, and moreover, preserves the canonical flag in $T_0 \mathbb{R}^n$ given by the tangents to the intersections $\bigcap_{i<j} n^{-1} \Gamma_i$.

Let $D(\mathbb{R}^n)$ denote the group of germs at $0$ of diffeomorphisms of $\mathbb{R}^n$. Consider a $k$-sphere of maps $f_t \in D(\mathbb{R}^n, n^{-1} \Gamma)$, $t \in S^k$. Since all $f_t$ preserve $0$ and the canonical flag in $T_0 \mathbb{R}^n$, there exists a $k + 1$-ball of diffeomorphisms $g_t \in D(\mathbb{R}^n)$, $t \in B^{k+1}$, extending $f_t$. Applying Theorem 5.12 to the Weinstein hypersurface embeddings induced by $g_t$, we can find diffeomorphisms $h_t$ such that $h_t$ takes $g_t(n^{-1} \Gamma)$ back to $n^{-1} \Gamma$ and such that $h_t$ is the identity for $t \in S^k$. Then $h_t \circ g_t \in D(\mathbb{R}^n, n^{-1} \Gamma)$, $t \in B^{k+1}$, gives an extension of $f_t$ to the $k + 1$-ball. □

We also formulate the parametric version of Corollary 5.10, which will be needed later.

Corollary 5.14. Let $L_T \subset T^* \mathbb{R}^n$ be a model Lagrangian associated with a signed rooted tree $(T, \rho, \varepsilon)$. Let $\eta_0^y, \eta_1^y$ be two families of polarizations transverse to $L_T$ parametrized by a manifold $Y$. Suppose that for any vertex $\alpha$ of $T$ adjacent to $\rho$ we have

$$\varepsilon(\eta_0^y, L, \alpha) = \varepsilon(\eta_1^y, L, \alpha).$$

Then there is a family of (germ at the origin of) symplectomorphisms $\psi^y : T^* \mathbb{R}^n \to T^* \mathbb{R}^n$ such that $\psi^y(L) = L$ and $d\psi^y(\eta_0^y) = \eta_1^y$ along $L$. Moreover, if $\eta_0^y = \eta_1^y$ for $y \in \mathcal{O}(K)$ for $K \subset Y$ a closed subset, then we can take $\psi^y = \text{Id}$ for $y \in \mathcal{O}(K)$.

The proof is just like in the non-parametric case, but applying Theorem 5.12 instead of Theorem 5.5.

5.4. Tangency loci. Before proving Proposition 5.6 and its parametric analogue we need to analyze more closely the geometry of hypersurfaces forming arboreal fronts.

Definition 5.15. Given smooth hypersurfaces $X_1, X_2 \subset \mathbb{R}^{n+1}$, we denote by $T(X_1, X_2) \subset \mathbb{R}^{n+1}$ their tangency locus, i.e. the subset of points $x \in X_1 \cap X_2$ such that $T_x X_1 = T_x X_2$.

Remark 5.16. Given smooth Legendrians $L_1, L_2 \subset J^1 \mathbb{R}^n$ whose fronts $X_1 = \pi(L_1), X_2 = \pi(L_2) \subset \mathbb{R}^{n+1}$ are smooth hypersurfaces, note that $T(X_1, X_2) = \pi(L_1 \cap L_2)$.

For $0 \leq j < i \leq n$, recall the nontation

$$h_{i,j} := h_{i-j}(x_{j+1}, \ldots, x_i)$$

so in particular $h_{i,0} = h_i(x_1, \ldots, x_i)$ and $h_{i,i-1} = h_1(x_i) = x_i$. Set

$$T_{i,j} = \{h_{i,j} = 0\} \subset \mathbb{R}^{n+1}$$

Note $h_{i,j}$ is independent of $x_0, \ldots, x_{j}$, and we have

$$T_{i,j} = \mathbb{R}^{j+1} \times n^{-j-1} \Gamma_{i-j-1}$$
Lemma 5.17. For $0 \leq j < i \leq n$, the tangency locus $T(\Gamma_i, \Gamma_j) \subset \mathbb{R}^{n+1}$ is the intersection of either $\Gamma_i$ or $\Gamma_j$ with the union

$$\{h_{i,j} = 0\} \cup \bigcup_{k=0}^{j-1} \{h_{i,k} = h_{j,k} = 0\} = T_{i,j} = \bigcup_{k=0}^{j-1} (T_{i,k} \cap T_{j,k})$$

Proof. Since $\Gamma_i, \Gamma_j$ are the graphs of $h_i^2, h_j^2$, the projection of $T(\Gamma_i, \Gamma_j)$ to the domain $\mathbb{R}^n$ is cut out by

$$h_i^2 = h_j^2 \quad \text{and} \quad dh_i^2 = dh_j^2$$

Note $h_i = h_{i,0} = x_1 - h_{i,1}^2$, $h_j = h_{j,0} = x_1 - h_{j,1}^2$. By examining the $dx_1$-component of $dh_i^2 = dh_j^2$, we see it implies $h_i = h_j$. Thus the projection of $T(\Gamma_i, \Gamma_j)$ is cut out by the single equation $dh_i^2 = dh_j^2$ which in turn implies $h_i = h_j$.

To satisfy $dh_i^2 = dh_j^2$, so in particular $h_i = h_j$, there are two possibilities: (i) $h_i = h_j = 0$; or (ii) $h_i = h_j \neq 0$. In case (i), we find the subset $\{h_{i,0} = h_{j,0} = 0\}$ appearing in the union of the assertion of the lemma. In case (ii), we observe $dh_i^2 = dh_j^2$ is then equivalent to $dh_{i,1}^2 = dh_{j,1}^2$ which in turn implies $h_{i,1} = h_{j,1}$.

Now we repeat the argument. To satisfy $dh_{i,1}^2 = dh_{j,1}^2$, so in particular $h_{i,1} = h_{j,1}$, there are two possibilities: (i) $h_{i,1} = h_{j,1} = 0$; or (ii) $h_{i,1} = h_{j,1} \neq 0$. In case (i), we find the subset $\{h_{i,1} = h_{j,1} = 0\}$ appearing in the union of the assertion of the lemma. In case (ii), we observe $dh_{i,1}^2 = dh_{j,1}^2$ is then equivalent to $dh_{i,2}^2 = dh_{j,2}^2$ which in turn implies $h_{i,2} = h_{j,2}$.

Iterating this argument, we obtain the subset $\bigcup_{k=0}^{j-1} \{h_{i,k} = h_{j,k} = 0\}$, and arrive at the final equation $dh_{i,j}^2 = 0$. By examining the $dx_{j+1}$-term, we see $dh_{i,j}^2 = 0$ holds if and only if $h_{i,j} = 0$, which gives the remaining subset of the assertion of the lemma. \qed

Remark 5.18. The only evident redundancy in the description of the lemma is $T_{i,j-1} \cap T_{j,j-1} \subset T_{i,j}$ since $h_{i,j-1} = x_j - h_{i,j}^2$, $h_{j,j-1} = x_j$, so their vanishing implies the vanishing of $h_{i,j}$.

We will be particularly interested in the locus $T_{i,j} \subset T(\Gamma_i, \Gamma_j)$ and formalize its structure in the following definition.

Definition 5.19. Given smooth hypersurfaces $X_1, X_2 \subset \mathbb{R}^{n+1}$, we denote by $\tau^o(X_1, X_2) \subset T(X_1, X_2)$ the subset of points $x \in X_1 \cap X_2$ where in some local coordinates we have $X_1 = \{x_0 = 0\}$, $X_2 = \{x_0 = x_1^2\}$. We write $\tau(X_1, X_2) \subset T(X_1, X_2)$ for the closure of $\tau^o(X_1, X_2)$, and refer to it as the primary tangency of $X_1, X_2$.

Remark 5.20. Given smooth Legendrians $L_1, L_2 \subset J^1 \mathbb{R}^n$ whose fronts $X_1 = \pi(L_1), X_2 = \pi(L_2) \subset \mathbb{R}^{n+1}$ are smooth hypersurfaces, note that $\tau^o(X_1, X_2)$ is the front projection of where $L_1, L_2$ intersect cleanly in codimension one.

We have the following consequence of Lemma 5.17.

Corollary 5.21. For $0 \leq j < i \leq n$, the primary tangency $\tau(\Gamma_i, \Gamma_j) \subset \mathbb{R}^{n+1}$ is the intersection of either $\Gamma_i$ or $\Gamma_j$ with $T_{i,j}$.  

Before continuing, let us record the following for future use.

**Lemma 5.22.** Fix $0 \leq k < j \leq n - 1$.

We have

$$\tau(\tau(\Gamma_n, n\Gamma_k), \tau(n\Gamma_j, n\Gamma_k)) = \tau(n\Gamma_n, n\Gamma_j) \cap \tau(n\Gamma_j, n\Gamma_k)$$

where the primary tangency of $\tau(\Gamma_n, n\Gamma_k)$, $\tau(n\Gamma_j, n\Gamma_k)$ of the left hand side is calculated in $n\Gamma_k \cong \mathbb{R}^n$.

![Figure 5.4](image_url)

**Figure 5.4.** Verification of the conclusion of Lemma 5.22 for $n = 2$, in this case both the right and left hand sides of the equality $\tau(\tau(2\Gamma_2, 2\Gamma_0), \tau(2\Gamma_2, 2\Gamma_0)) = \tau(2\Gamma_2, 2\Gamma_1) \cap \tau(2\Gamma_1, 2\Gamma_0)$ consist of the origin.

**Proof.** By the preceding corollary, the left hand side is the intersection $n\Gamma_k \cap \tau(T_{n,k}, T_{j,k})$.

Note $n\Gamma_k \cap T_{j,k} = \tau(n\Gamma_j, n\Gamma_k) = n\Gamma_j \cap T_{j,k}$. Hence

$$n\Gamma_k \cap \tau(T_{n,k}, T_{j,k}) = n\Gamma_j \cap \tau(T_{n,k}, T_{j,k})$$

since $y \in n\Gamma_k \cap \tau(T_{n,k}, T_{j,k}) \iff y \in n\Gamma_k \cap T_{j,k}$, $y \in \tau(T_{n,k}, T_{j,k}) \iff y \in n\Gamma_j \cap T_{j,k}$, $y \in \tau(T_{n,k}, T_{j,k}) \iff y \in n\Gamma_j \cap \tau(T_{n,k}, T_{j,k})$.

Next, recall

$$T_{n,k} = \mathbb{R}^{k+1} \times n^{-k-1}\Gamma_{n-k-1} \quad T_{j,k} = \mathbb{R}^{k+1} \times n^{-k-1}\Gamma_{j-k-1}$$

Hence by the preceding corollary, we have

$$\tau(T_{n,k}, T_{j,k}) = T_{j,k} \cap \{h_{n,j} = 0\}$$

Thus the left hand side is given by $n\Gamma_j \cap T_{j,k} \cap T_{n,j}$.

On the other hand, by the preceding corollary, the right hand side is also given by $n\Gamma_j \cap T_{n,j} \cap T_{j,k}$. \qed
5.4.1. More on distinguished quadrants.

**Corollary 5.23.** For \(0 \leq j < i \leq n\), we have

\[
^n\Gamma^\varepsilon_i \cap \n^\varepsilon_j = T(^n\Gamma^\varepsilon_i, \n^\varepsilon_j) = \tau(^n\Gamma^\varepsilon_i, \n^\varepsilon_j)
\]

and they coincide with the closed boundary face of \(^n\Gamma^\varepsilon_i\) cut out by \(h_{i,j} = 0\).

**Proof.** For \(j = 0\), we have \(^n\Gamma^\varepsilon_0 = \n^\varepsilon_0 = \{x_0 = 0\}\). From the definitions, we have

\[
^n\Gamma^\varepsilon_i \cap \n^\varepsilon_0 = T(^n\Gamma^\varepsilon_i, \n^\varepsilon_0) = \tau(^n\Gamma^\varepsilon_i, \n^\varepsilon_0)
\]

which is cut out of \(^nP^\varepsilon_i\) by \(h_{i,0} = h_i = 0\).

For \(j > 0\), the assertions follow from Lemma 4.4 by induction on \(n\). \(\square\)

**Remark 5.24.** Note for any \(0 \leq j < i \leq n\), we have

\[
\tau(^n\Gamma^\varepsilon_i, \n^\varepsilon_j) = \bigcup_{\varepsilon} \tau(^n\Gamma^\varepsilon_i, \n^\varepsilon_j)
\]

To see this, consider \(x \in \tau(^n\Gamma^\varepsilon_i, \n^\varepsilon_j)\), so that \(h_{i,j}(x) = 0\) by Corollary 5.21. Choose \(\varepsilon\) so that \(x \in ^n\Gamma^\varepsilon_i\). Then by Corollary 5.23, we have \(x \in \tau(^n\Gamma^\varepsilon_i, \n^\varepsilon_j)\).

For \(i = 0\), let \(^nL^\varepsilon_0 = \mathbb{R}^n \subset T^*\mathbb{R}^n\) denote the zero-section. For \(i = 1, \ldots, n\), consider the conormal bundles

\[
^nL^\varepsilon_i = T^{*^n\Gamma^\varepsilon_i - 1}_i \mathbb{R}^n \subset T^*\mathbb{R}^n
\]

and their union

\[
^nL^\varepsilon = \bigcup_{i=0}^n \n^nL^\varepsilon_i
\]

Similarly, for \(i = 0, \ldots, n\), consider the smooth Legendrian

\[
^n\Lambda^\varepsilon_i \subset J^1\mathbb{R}^n
\]

that maps diffeomorphically to \(^n\Gamma^\varepsilon_i \subset \mathbb{R}^{n+1}\) under the front projection \(\pi : J^1\mathbb{R}^n \to \mathbb{R}^{n+1}\), and their union

\[
^n\Lambda^\varepsilon = \bigcup_{i=0}^n \n^n\Lambda^\varepsilon_i
\]

Note the contactomorphism of Lemma 4.1 takes \(^n\Lambda^\varepsilon_i \subset J^1\mathbb{R}^n\) isomorphically to \(\{0\} \times ^nL^\varepsilon_i \subset \{0\} \times T^*\mathbb{R}^n\), and thus \(^n\Lambda^\varepsilon \subset J^1\mathbb{R}^n\) isomorphically to \(\{0\} \times ^nL^\varepsilon \subset \{0\} \times T^*\mathbb{R}^n\).

We have the following topological consequence of Lemma 4.4.

**Corollary 5.25.** As a union of smooth manifolds with corners, \(^n\Gamma^\varepsilon \subset \mathbb{R}^{n+1}\) is given by the gluing

\[
^n\Gamma^\varepsilon = (^{n-1}\Gamma^\varepsilon' \times \mathbb{R}_{\geq 0}) \coprod_{(n-1)\Gamma^\varepsilon' \times \{0\}}}^{(\mathbb{R}^n \times \{0\})}
\]

where \(\varepsilon' = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\). The front projection takes \(^nL^\varepsilon \subset J^1\mathbb{R}^n\) homeomorphically to \(^n\Gamma^\varepsilon \subset \mathbb{R}^{n+1}\).

Before continuing, let us record the following for future use.
Corollary 5.26. For $0 < j < i \leq n$, the closure of the codimension one clean intersection of $nL_i$, $nL_j$ is precisely $nL_i \cap nL_j$.

Proof. The closure of the codimension one clean intersection of $nL_i$, $nL_j$ is conic and projects to the primary tangency of $n^{-1}\Gamma_{i-1}^{-\varepsilon}$, $n^{-1}\Gamma_{j-1}^{-\varepsilon}$. By Corollary 5.21, the primary tangency of $n^{-1}\Gamma_{i-1}^{-\varepsilon}$, $n^{-1}\Gamma_{j-1}^{-\varepsilon}$ is cut out by $h_{i-1,j-1} = 0$. By Corollary 5.23, this is precisely the tangency $T(n^{-1}\Gamma_{i-1}^{-\varepsilon}, n^{-1}\Gamma_{j-1}^{-\varepsilon})$ and hence lifts precisely to the conic intersection $nL_i \cap nL_j$. □

5.5. The case of $\mathcal{A}_{n+1}$-tree. The following Theorem 5.27 will play a key role in proving Proposition 5.6.

Theorem 5.27. Let $\varphi : T^*\mathbb{R}^n \rightarrow J^1\mathbb{R}^n$ be an embedding as a Weinstein hypersurface. Assume that the image of $nL$ under $\varphi$ is transverse to the fibers of the projection $J^1\mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $\Upsilon = \pi(\varphi(nL)) \subset \mathbb{R} \times \mathbb{R}^n$ be (the germ of) the front at the central point.

Then there exists a diffeomorphism $\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^n$ taking $\Upsilon$ to the germ at the origin of $n\Gamma \subset \mathbb{R} \times \mathbb{R}^n$.

The proof of Theorem 5.27 will proceed by induction on the dimension $n$. At each stage, we will prove the fully parametric version:

Theorem 5.28. Let $\varphi^y : T^*\mathbb{R}^n \rightarrow J^1\mathbb{R}^n$ be a family of Weinstein hypersurface embeddings parametrized by a manifold $Y$. Assume that the image of $nL$ under $\varphi^y$ is transverse to the fibers of the projection $J^1\mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $\Upsilon^y = \pi(\varphi^y(nL)) \subset \mathbb{R} \times \mathbb{R}^n$ be (the germs of) the fronts at the central points.

Then there exists a family of diffeomorphisms $\psi^y : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^n$ taking $\Upsilon^y$ to the germ at the origin of $n\Gamma \subset \mathbb{R} \times \mathbb{R}^n$. If $\varphi^y = \text{Id}$ for $y \in \mathcal{O}_p(K)$, where $K \subset Y$ is a closed subset, then we may assume $\psi^y = \text{Id}$ for $y \in \mathcal{O}_p(K)$.

As usual the case of general pairs $(Y, K)$ follows from the case $Y = D^k$ and $K = S^{k-1}$.

Base case $n = 0$. The $k$-parametric version states: the germ of any graphical hypersurface $\Upsilon \subset \mathbb{R} \times \mathbb{R}^k$ is diffeomorphic to the germ of the zero-graph $0\Gamma \times \mathbb{R}^k = \{0\} \times \mathbb{R}^k$. This can be achieved by an isotopy generated by a time-dependent vector field of the form $h_t\partial_{x_0}$. This vector field is zero at infinity if $\Upsilon$ is standard at infinity.

Case $n = 1$. The next case of the induction $n = 1$ is elementary but slightly different from the others, so it is more convenient to treat separately.

With the setup of the theorem, consider the front $\Upsilon = \pi(1\Lambda) \subset \mathbb{R}^2$, and assume without loss of generality that the origin is the central point. By induction, we may assume, the front takes the form $\Upsilon = \Gamma_0 \cup \Upsilon_1 \subset \mathbb{R}^2$ where $\Gamma_0 = \{x_0 = 0\}$. Near the origin, the intersection $\Gamma_0 \cap \Upsilon_1$ and tangency locus $T(\Gamma_0, \Upsilon_1)$ coincide and consist of the origin alone. Moreover, by construction, the origin is a simple tangency, and so $\Upsilon_1 = \{x_0 = \alpha x_1^2\}$ with $\alpha(0) \neq 0$. Now it is elementary to find a time-dependent vector field of the form $h_t x_1 \partial_{x_1}$, hence vanishing on $\Gamma_0$, generating an isotopy taking $\Upsilon_1$ to either $\Gamma_1 = \{x_0 = x_1^2\}$ or $-\Gamma_1 = \{x_0 = -x_1^2\}$. In the former
case, we are done; in the latter case, we may apply the diffeomorphism \((x_0, x_1) \mapsto (-x_0, x_1)\) to arrive at the configuration \(\Gamma_0 \cup \Gamma_1\). Finally, it is evident the prior constructions can be performed parametrically, with the vector field zero at infinity if \(\Upsilon\) is standard at infinity.

**Inductive step.** The inductive step takes the following form. Suppose the fully parametric assertion has been established for dimension \(n-1\). Starting from \(^n\Lambda \subset T^*\mathbb{R}^n\), remove the last smooth piece to obtain \(^n\Lambda' = ^n\Lambda \setminus ^n\Lambda_n\), and consider the corresponding front \(\Upsilon' = \pi(^n\Lambda')\). Note that \(^n\Lambda' = ^{n-1}\Lambda \times \mathbb{R} \subset T^*(\mathbb{R}^{n-1} \times \mathbb{R})\), and so by an inductive application of the 1-parametric version of the theorem, we may assume

\[
\Upsilon' = ^{n-1}\Gamma \times \mathbb{R}
\]

Set \(\Upsilon_n = \pi(^n\Lambda_n)\). We will find a diffeomorphism \(\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\) that preserves \(\Upsilon'\) (as a subset, not pointwise), and takes \(\Upsilon_n\) to \(^n\Gamma_n\). Moreover, it will be evident the diffeomorphism can be constructed in parametric form, including the relative parametric form. This will complete the inductive step and prove the theorem.

5.5.1. **Two propositions.** The proof of the inductive step is based on the following 2 propositions.

**Proposition 5.29.** Fix \(n \geq 2\).

*With the setup of Theorem 5.27, suppose \(\Upsilon = \bigcup_{i=0}^{n-1} ^n\Gamma_i \cup \Upsilon_n\) where \(\Upsilon_n = \pi(^n\Lambda_n)\). Suppose in addition \(\Upsilon_n\) has primary tangency loci satisfying*

\[
\tau(\Upsilon_n, ^n\Gamma_i) \supset \tau(^n\Gamma_n, ^n\Gamma_i) \quad i = 0, \ldots, n-1
\]

*Then \(\Upsilon_n = \{x_0 = \alpha h_n^2\}\) where

\[
\alpha = 1 + \beta \prod_{j=1}^{n-1} h_{n,j}^2 = 1 + \beta h_{n,1}^2 \cdots h_{n,n-1}^2
\]

*Moreover, the same holds in parametric form.*

**Proof.** We have \(\Upsilon_n = \{x_0 = g\}\) for some \(g\). Since \(\tau(\Upsilon_n, ^n\Gamma_0) \supset \tau(^n\Gamma_n, ^n\Gamma_0) = \{h_n = 0\}\), we must have \(g\) is divisible by \(h_n^2\), hence \(g = \alpha h_n^2\), for some \(\alpha\). Next, for any \(j \neq 0, n\), by Lemma 5.17, \(\tau(^n\Gamma_n, ^n\Gamma_j)\) is cut out by \(h_{n,j} = 0\). Since \(\tau(\Upsilon_n, ^n\Gamma_j) \supset \tau(^n\Gamma_n, ^n\Gamma_j)\), and \(h_n \neq 0\) along a dense subset of \(\{h_{n,j} = 0\}\), taking the ratio \(g/h_n^2\) shows we must have \(\alpha = 1 + h_{n,j}^2 \beta_j\), for some \(\beta_j\). Repeating this argument, and using the transversality of the level-sets of the collection \(h_{n,j}\), we arrive at the assertion. \(\square\)

**Proposition 5.30.** Fix \(n \geq 2\).

*With the setup of Theorem 5.27, suppose \(\Upsilon = \bigcup_{i=0}^{n-1} ^n\Gamma_i \cup \Upsilon_n\) where \(\Upsilon_n = \pi(^n\Lambda_n)\). Suppose in addition \(\Upsilon_n = \{x_0 = \alpha h_n^2\}\) where

\[
\alpha = 1 + \beta \prod_{j=1}^{n-1} h_{n,j}^2 = 1 + \beta h_{n,1}^2 \cdots h_{n,n-1}^2
\]
Consider the family $\Upsilon'_n = \{x_0 = (1 - t + t\alpha)h^2_n\}$ so that $\Upsilon'_0 = ^n\Gamma_n$, $\Upsilon'_1 = \Upsilon_n$. Then there exist functions $h_t : \mathbb{R}^{n+1} \to \mathbb{R}$ such that the vector fields
\[
h_t v_{n-1} = h_t \sum_{i=0}^{n-1} x_i \frac{1}{2t} \partial x_i = h_t x_0 \partial x_0 + \frac{1}{2} h_t x_1 \partial x_1 + \cdots + \frac{1}{2^n} h_t x_{n-1} \partial x_{n-1}
\]
generate an isotopy $\varphi_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $\varphi_t(\Upsilon_n,0) = \Upsilon_{n,t}$. In addition, the functions $h_t$, hence vector fields $h_t v_{n-1}$, are divisible by the product $\prod_{j=1}^{n-1} h_{n,j}$.
Moreover, all of the above holds in parametric form.

The following lemmas are needed for the proof of Proposition 5.30.

**Lemma 5.31.** For all $0 \leq j \leq n$, the vector field
\[
v_i = \sum_{j=0}^n x_j \frac{1}{2t} \partial x_j = x_0 \partial x_0 + \frac{1}{2} x_1 \partial x_1 + \cdots + \frac{1}{2^n} x_i \partial x_i
\]
preserves each $^n\Gamma_j \subset \mathbb{R}^{n+1}$, for $j = 0, \ldots, i$.

**Proof.** Since $^n\Gamma_j \subset \mathbb{R}^{n+1}$ is independent of $x_{j+1}, \ldots, x_n$, it suffices to prove the case $i = j = n$. Recall $^n\Gamma_n$ is the zero-locus of $f = x_0 - h^2_n$. We will show $v(h_n) = \frac{1}{2} h_n$ and so $v(f) = f$. Recall $h_n = h_{n,0} = x_1 - h^2_{n,1}$, and in general $h_{n,j} = x_{j+1} - h^2_{n,j+1}$ with $h_{n,n-1} = x_n$. Thus $v_n(h_{n,n-1}) = \frac{1}{2^n} h_{n,n-1}$, and by induction, $v_n(h_{n,j}) = \frac{1}{2^n} h_{n,j}$, so in particular $v_n(h_{n,0}) = v_n(h_n) = \frac{1}{2} h_n$. □

**Remark 5.32.** In the context of the inductive step outlined above, we will use Lemma 5.31 in particular the vector field
\[
v_{n-1} = \sum_{i=0}^{n-1} x_i \frac{1}{2t} \partial x_i = x_0 \partial x_0 + \frac{1}{2} x_1 \partial x_1 + \cdots + \frac{1}{2^{n-1}} x_{n-1} \partial x_{n-1}
\]
to move $\Upsilon_n$ to $^n\Gamma_n$. The lemma confirms we will preserve $\Upsilon' = ^{n-1}\Gamma \times \mathbb{R} = \bigcup_{i=1}^{n-1} ^n\Gamma_i$.

**Lemma 5.33.** For any $0 \leq j < i \leq n$, and $1 \leq k \leq i$, we have
\[
\frac{\partial h^2_i}{\partial x_k} = -(2)^k \prod_{j=0}^{k-1} h_{i,j} = -(2)^k h_{i,0} h_{i,1} \cdots h_{i,k-1}
\]

**Proof.** Recall $h_i = h_{i,0}$ and the inductive formulas $h_{i,j} = x_{j+1} - h^2_{i,j+1}$ with $h_{i,i-1} = x_i$. Thus we have
\[
\frac{\partial h^2_{i,j}}{\partial x_{j+1}} = 2h_{i,j} \quad \frac{\partial h^2_{i,j}}{\partial x_k} = -2 h_{i,j} \frac{\partial h^2_{i,j+1}}{\partial x_k} \quad k > j + 1
\]
and the assertion follows. □

**Proof of Proposition 5.30.** Suppose $\Upsilon = \bigcup_{i=0}^{n-1} ^n\Gamma_i \cup \Upsilon_n$ where $\Upsilon_n$ is the graph of
\[
h_n = (1 + \beta \prod_{j=1}^{n-1} h^2_{n,j}) h^2_n = (1 + \beta h^2_{n,1} \cdots h^2_{n,n-1}) h^2_n
\]
Our aim is to find a normalizing isotopy, generated by a time-dependent vector field \( v_t \), taking the graph \( \Upsilon_n = \{ x_0 = h_\beta \} \) to the standard graph \( \#n \Gamma_n = \{ x_0 = h_n^2 \} \), i.e. to the graph where \( \beta = 0 \), while preserving \( \bigcup_{i=0}^{n-1} \#i \Gamma_i \). Thus for any infinitesimal deformation in the class of functions \( h_\beta \), we seek a vector field \( v \) realizing the deformation and preserving the functions \( h_0, \ldots, h_{n-1} \), i.e. we seek to solve the system

\[
\dot{h}_i = 0, \quad i = 0, \ldots, n - 1
\]

\[
\dot{h}_\beta = \gamma \prod_{j=0}^{n-1} h_{n,j}^2 = \gamma h_{n,0}^2 \cdots h_{n,n-1}^2
\]

where \( \dot{h}_\beta \) denotes the derivative of \( h_\beta \) with respect to \( v \), and \( \gamma \) is any given smooth function.

Let \( \Lambda_\beta \subset T^*\mathbb{R}^{n+1} \) denote the conormal to the graph of \( h_\beta \). Any vector field \( v = \sum_{j=0}^{n} v_j \partial/\partial x_j \) on \( \mathbb{R}^{n+1} \) extends to a Hamiltonian vector field \( v_H \) on \( T^*\mathbb{R}^{n+1} \) with Hamiltonian \( H = \sum_{j=0}^{n} p_j v_j \). We will find \( v \) deforming the graph of \( h_\beta \) by finding \( H \) so that \( v_H \) deforms the conormal to the graph \( \Lambda_\beta \).

In general, for a function \( f : \mathbb{R}^n \to \mathbb{R} \), with graph \( \Gamma_f = \{ x_0 = f \} \subset \mathbb{R}^{n+1} \), denote the conormal to the graph by \( T^*\Gamma_f \subset T^*\mathbb{R}^{n+1} \). With respect to the contact form \( p_1 dx_1 + \cdots + p_n dx_n - x_0 dp_0 \), the conormal \( T^*\Gamma_f \) is given by the generating function \( F(x_1, \ldots, x_n) = -p_0 f(x_1, \ldots, x_n) \), i.e. it is cut out by the equations

\[
p_i = -p_0 \frac{\partial f}{\partial x_i}, \quad i = 1, \ldots, n
\]

\[
x_0 = f(x_1, \ldots, x_n)
\]

Hence given a Hamiltonian \( H = \sum_{j=0}^{n} p_j v_j \), its restriction to the conormal \( T^*_f \) is given by

\[
H|_{T^*_f} = p_0 v_0 |_{x_0 = f} - p_0 \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} v_j |_{x_0 = f}
\]

and so further restricting to \( p_0 = 1 \), we find the Hamilton-Jacobi equation

\[
H|_{T^*_f \cap \{ p_0 = 1 \}} = v_0 |_{x_0 = f} - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i |_{x_0 = f} = v_0 |_{x_0 = f} - \dot{f}
\]

Let us apply the above to \( h_\beta \) and \( h_i \), for \( i = 0, \ldots, n-1 \). It allows us to transform system (3) into the system

\[
v_0(x_1, \ldots, x_n, h_i) - \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} v_j = 0, \quad i = 0, \ldots, n - 1
\]

\[
v_0(x_1, \ldots, x_n, h_\beta) - \sum_{j=1}^{n} \frac{\partial h_\beta}{\partial x_j} v_j = \gamma \prod_{j=0}^{n-1} h_{n,j}^2
\]

(4)
Note we can reformulate Lemma 5.31 from this viewpoint: when $\beta = \gamma = 0$, given any function $h = h(x_1, \ldots, x_n)$, the functions

$$v_0 = x_0 h, v_1 = \frac{x_1}{2} h, v_2 = \frac{x_2}{4} h, \ldots, v_n = \frac{x_n}{2^n} h$$

satisfy system (4).

Now let us choose $v_0, v_1, \ldots, v_{n-1}$ as in (5) but set $v_n = 0$. This will satisfy the first $n$ equations of system (4), independently of $\beta, \gamma$. From hereon, we will restrict to this class of vector fields and focus on the last equation of system (4).

Let us first set $\beta = 0$, so that $h = h^2$, and solve system (4) in this case. Using Lemma 5.33, we can then rewrite the left-hand side of the last equation of system (4) in the form

$$v_0(x_1, \ldots, x_n, h^2) - \sum_{j=1}^{n-1} \frac{\partial h^2}{\partial x_j} v_j = h \left( h^2 - \sum_{j=1}^{n-1} \frac{\partial h^2}{\partial x_j} x_j \right) = h \left( h^2 + \sum_{j=1}^{n-1} (-1)^j x_j \prod_{k=0}^{j-1} h_{n,k} \right)$$

Using $h_n = h_{n,0}, h_{n,k} - x_{k+1} = -h^2_{n,k-1}$, we can inductively simplify the term in parentheses

$$h^2 + \sum_{j=1}^{n-1} (-1)^j x_j \prod_{k=0}^{j-1} h_{n,k} = h_n(h_n - x_1 + \sum_{j=2}^{n-1} (-1)^j x_j \prod_{k=1}^{j-1} h_{n,k})$$

$$= h_n(-h^2_{n,1} + \sum_{j=2}^{n-1} (-1)^j x_j \prod_{k=1}^{j-1} h_{n,k})$$

$$= h_n h_{n,1}(-h_{n,1} + x_2 + \sum_{j=3}^{n-1} (-1)^j x_j \prod_{k=2}^{j-1} h_{n,k})$$

$$\cdots$$

$$= (-1)^n h_n h_{n,1} h_{n,2} \cdots h_{n,n-1} = (-1)^{n-1} \prod_{j=0}^{n-1} h_{n,j}$$

Thus for $\beta = 0$, the last equation of system (4) reduces to

$$(-1)^n h \prod_{j=0}^{n-1} h_{n,j} = \gamma \prod_{j=0}^{n-1} h_{n,j}^2$$

and hence can be solved by

$$h = (-1)^{n-1} \gamma \prod_{j=0}^{n-1} h_{n,j}$$

Now for general $\beta$, we will similarly calculate the left-hand side of the last equation of system (4). To simplify the formulas, set

$$F = \prod_{j=0}^{n-1} h_{n,j} \quad \theta = \beta F^2$$
Thus we have $h_\beta = (1 + \theta)h_n^2$, and our prior calculation showed when $\beta = 0$, the last equation of system (4) took the form

$$(-1)^{n-1}hF = \gamma F^2$$

so was solved by $h = (-1)^{n-1}\gamma F$.

For general $\beta$, after factoring out the function $h$ to be solved for, the left-hand side of the last equation of system (4) takes the form

$$(-1)^{n-1}(1 + \theta)F - h_n^2\sum_{j=1}^{n-1} \frac{1}{2j} \frac{\partial \theta}{\partial x_j} x_j$$

Thus the equation itself takes the form

$$((-1)^{n-1}(1 + \theta)F - h_n^2\sum_{j=1}^{n-1} \frac{1}{2j} \frac{\partial \theta}{\partial x_j} x_j)h = \gamma F^2$$

Since $\theta = \beta F^2$, we have

$$\frac{\partial \theta}{\partial x_j} = F^2 \frac{\partial \beta}{\partial q_j} + \beta \frac{\partial F^2}{\partial q_j} = F^2 \frac{\partial \beta}{\partial q_j} + 2F\beta \frac{\partial F}{\partial q_j}$$

and hence $\frac{\partial \theta}{\partial x_j}$ is divisible by $F$. Thus we can divide equation (6) by $F$, and after renaming $\gamma$, write equation (6) in the form

$$(1 + O(x))h = \gamma F$$

where $O(x)$ vanishes at the origin. We conclude we can solve the equation by $h = (1 + O(x))^{-1}\gamma F$.

This completes the proof of Proposition 5.30.

\[\square\]

5.5.2. Proof of Theorem 5.27. In this section, we use Propositions 5.29 and Proposition 5.30 to complete the inductive step outlined in 5.5, and thus, complete the proof of Theorem 5.27. Let us assume $n \geq 2$.

Then $\Upsilon = \Upsilon' \cup \Upsilon_n$ where $\Upsilon' = \bigcup_{i=0}^{n-1} \Upsilon_i$, $\Upsilon_n = \pi(\Lambda_n)$. We will implement the following strategy. Suppose for some $0 < k \leq n - 1$, we have moved $\Upsilon_n$, while preserving $\Upsilon'$, so that we have the relation of primary tangencies

$$\tau(\Upsilon_n, {}^n\Gamma_j) \supset \tau(\Gamma_n, {}^n\Gamma_j) \quad j > k$$

Then using Proposition 5.29 and Proposition 5.30, or alternatively, the cases $n = 0, 1$ when respectively $k = n - 1, n - 2$, we will move $\Upsilon_n$, while preserving $\Upsilon'$, so that we have the relation of primary tangencies

$$\tau(\Upsilon_n, {}^n\Gamma_j) \supset \tau(\Gamma_n, {}^n\Gamma_j) \quad j \geq k$$

Proceeding in this way, we will arrive at $k = 0$, where all primary tangencies have been normalized. Then a final application of Proposition 5.29 and Proposition 5.30 will complete the proof.
Figure 5.5. The strategy of the proof: inductively normalize tangencies.

To pursue this argument, we need the following control over primary tangencies.

**Lemma 5.34.** Fix $0 \leq k < j \leq n - 1$.

We have

$$\tau(\tau(Y_n, n\Gamma_k), \tau(n\Gamma_j, n\Gamma_k)) \supset \tau(Y_n, n\Gamma_j) \cap \tau(n\Gamma_j, n\Gamma_k)$$

Moreover, when $k = n - 2$, the tangency of $\tau(Y_n, n\Gamma_{n-2})$ and $\tau(n\Gamma_{n-1}, n\Gamma_{n-2})$ is nondegenerate.

**Proof.** We will assume $k > 0$ and leave the case $k = 0$ as an exercise.

Fix a point

$$y \in \tau(Y_n, n\Gamma_j) \cap \tau(n\Gamma_j, n\Gamma_k)$$

In particular $y \in Y_n$ and so $y = \pi(\tilde{y})$ for some $\tilde{y} \in n\Lambda_n$. Recall $n\Lambda_n = \bigcup_{\varepsilon} n\Lambda_n^\varepsilon$ and so $\tilde{y} \in n\Lambda_n^\varepsilon$, for some $\varepsilon$.

Note $y \in \tau(Y_n, n\Gamma_j)$ implies $\tilde{y}$ is in the closure of the clean codimension one intersection of $n\Lambda_n, n\Lambda_j$.

By Corollary 5.26, this locus intersects $n\Lambda_n^\varepsilon$ precisely along $n\Lambda_n^\varepsilon \cap n\Lambda_j^\varepsilon$ and so $\tilde{y} \in n\Lambda_j^\varepsilon$.

Similarly, note $y \in \tau(n\Gamma_j, n\Gamma_k)$ implies $\tilde{y}$ is in the closure of the clean codimension one intersection of $n\Lambda_j, n\Lambda_k$. By Corollary 5.26, this locus intersects $n\Lambda_j^\varepsilon$ precisely along $n\Lambda_j^\varepsilon \cap n\Lambda_k^\varepsilon$ and so $\tilde{y} \in n\Lambda_k^\varepsilon$.

Thus altogether $\tilde{y} \in n\Lambda_n^\varepsilon \cap n\Lambda_j^\varepsilon \cap n\Lambda_k^\varepsilon = (n\Lambda_n^\varepsilon \cap n\Lambda_k^\varepsilon) \cap (n\Lambda_j^\varepsilon \cap n\Lambda_k^\varepsilon)$.

By Corollary 5.26, the intersections $n\Lambda_n^\varepsilon \cap n\Lambda_k^\varepsilon$ and $n\Lambda_j^\varepsilon \cap n\Lambda_k^\varepsilon$ are closures of clean codimension one intersections, hence their projections lie in the primary tangencies $\tau(Y_n, n\Gamma_k)$ and
\[ \tau(\mathcal{N}_j, \mathcal{N}_k). \] Moreover, \( \mathcal{N}^\varepsilon_n \cap \mathcal{N}^\varepsilon_k \) and \( \mathcal{N}^\varepsilon_j \cap \mathcal{N}^\varepsilon_k \) intersect along their primary tangency. Since \( \pi \) restricted to \( \mathcal{N}_k \) has no critical points, the projection of this primary tangency is again a primary tangency. Hence \( y \in \pi(\tau(\mathcal{N}_n, \mathcal{N}_k)), \tau(\mathcal{N}_j, \mathcal{N}_k)) \), proving the asserted containment.

We leave the nondegeneracy of the case \( k = n - 2 \) to the reader. \( \square \)

Now we are ready to inductively normalize the primary tangencies.

Lemma 5.35. Fix \( 0 \leq k < n - 1 \).

Suppose

\[ \tau(\mathcal{Y}_n, \mathcal{N}_j) = \tau(\mathcal{N}_n, \mathcal{N}_j) \quad j > k \]

Then there exists a diffeomorphism \( \psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) preserving \( \mathcal{Y}' = \bigcup_{i=0}^{n-1} \mathcal{N}_i \) such that

\[ \tau(\psi(\mathcal{Y}_n), \mathcal{N}_j) = \tau(\mathcal{N}_n, \mathcal{N}_j) \quad j \geq k \]

Moreover, when \( k \neq n - 2 \), the diffeomorphism is an isotopy.

Proof. We will assume \( k < n - 3 \). We leave the elementary cases \( k = n - 2, n - 3 \) to the reader. They can be deduced from the parametric versions of the cases \( n = 0, 1 \) presented in 5.5, 5.5 respectively.

Throughout what follows, we use the projection \( \mathbb{R}^{n+1} \to \mathbb{R}^n \) to identify \( \mathcal{N}_k = \mathbb{R}^n \).

On the one hand, we have

\[ \tau(\mathcal{N}_j, \mathcal{N}_k) = \mathbb{R}^k \times n-k-1 \mathcal{N}_{j-k-1} \quad k < j < n \]

On the other hand, by Lemma 5.34 and assumption, we have

\[ \tau(\tau(\mathcal{Y}_n, \mathcal{N}_k), \mathcal{N}_j) = \tau(\mathcal{Y}_n, \mathcal{N}_j) \cap \mathcal{N}_k = \tau(\mathcal{N}_n, \mathcal{N}_j) \cap \mathcal{N}_k \quad k < j < n \]

Hence within \( \mathcal{N}_k = \mathbb{R}^n \), the loci \( \tau(\mathcal{Y}_n, \mathcal{N}_k) \) and \( \tau(\mathcal{N}_n, \mathcal{N}_k) \) have the same tangencies with

\[ \tau(\mathcal{N}_j, \mathcal{N}_k) = \mathbb{R}^k \times n-k-1 \mathcal{N}_{j-k-1} \quad k < j < n \]

Thus Proposition 5.29 and Proposition 5.30 provide a time-dependent vector field of the form

\[ v_t = h_t \sum_{i=k+1}^{n-1} \frac{1}{2^i} x_i \partial x_i \]

generating an isotopy \( \varphi : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k} \) satisfying

\[ \varphi(\tau(\mathcal{Y}_n, \mathcal{N}_k)) = \tau(\mathcal{N}_n, \mathcal{N}_k) \]

In addition, the function \( h_t \), hence vector field \( v_t \), is divisible by the product \( \prod_{j=k+1}^{n-1} h_{n,j} \), and thus \( \varphi \) preserves its zero-locus.

Let us complete \( v_t \) to the vector field

\[ V_t = h_t \sum_{i=0}^{n-1} \frac{1}{2^i} x_i \partial x_i \]
and consider the isotopy \( \psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) generated by \( V_t \).

Then \( \psi \) satisfies
\[
\psi(\tau(\mathcal{Y}_n, {}^n\Gamma_k)) = \tau(\mathcal{Y}_n, {}^n\Gamma_k)
\]
It also preserves \( {}^n\Gamma_i \), for \( 0 \leq i \leq n-1 \), as well as \( \tau(\mathcal{Y}_n, {}^n\Gamma_j) = \tau(\mathcal{Y}_n, {}^n\Gamma_j) \), for \( j > k \). In addition, it preserves
\[
\tau( {}^n\Gamma_j, {}^n\Gamma_k) = \mathbb{R}^k \times {}^{n-k-1}\Gamma_{j-k-1} \quad k < j < n
\]
since this is the zero-locus of \( h_{n,j} \).

Finally, let us use the lemma to complete the inductive step of the proof of Theorem 5.27 as outlined above. Suppose for some \( 0 < k \leq n-1 \), we have moved \( \mathcal{Y}_n \), while preserving \( \mathcal{Y}' \), so that we have the sought-after primary tangencies
\[
\tau(\mathcal{Y}_n, {}^n\Gamma_j) = \tau(\mathcal{Y}_n, {}^n\Gamma_j) \quad j > k
\]
Then using Lemma 5.35, we can move \( \mathcal{Y}_n \), while preserving \( \mathcal{Y}' \), so that we have the sought-after primary tangencies
\[
\tau(\mathcal{Y}_n, {}^n\Gamma_j) = \tau(\mathcal{Y}_n, {}^n\Gamma_j) \quad j \geq k
\]
Proceeding in this way, we arrive at \( k = 0 \), where all primary tangencies have been normalized.

Now a final application of Proposition 5.29 and Proposition 5.30 move \( \mathcal{Y}_n \) to \( {}^n\Gamma_n \), while preserving \( \mathcal{Y}' \), and thus complete the proof of Theorem 5.27.

5.6. Conclusion of the proof. We are now ready to prove Proposition 5.6. As a consequence we establish Theorem 5.5, and since all the above also holds parametrically this also establishes the parametric version Theorem 5.12.

Proof of Proposition 5.6. Take any point \( \lambda \) in the front \( H := \pi(\Lambda) \) and let \( \pi^{-1}(\lambda) = \{\lambda_1, \ldots, \lambda_k\} \). Let \( \Lambda_1, \ldots, \Lambda_k \) be germs of \( \Lambda \) at these points of arboreal types \((\mathcal{T}_j, n)_j\), \( n(\mathcal{T}_j) = n_j \). We need to show that the germ of the front \( H \) at \( \lambda \) is diffeomorphic to the germ of a model front \( H_{\mathcal{T}} \), where \( \mathcal{T} \) is a signed rooted tree obtained from \( \bigsqcup T_j \) by adding the root \( \rho \) and adjoining it to the roots \( \rho_j \) of the trees \( T_j \) by edges \( [\rho \rho_j] \). The signs of all edges of the trees \( T_j \) are preserved, while previously unsigned edges \( \rho_j \alpha \) get a sign \( \varepsilon(\nu, L, \alpha) \), see (2).

We proceed by induction on the number of vertices in the signed rooted tree \( \mathcal{T} = (T, \rho, \varepsilon) \).

The base case of a \((\mathscr{A}_1, m)\)-front \( H \subset \mathbb{R}^m \) is the same geometry as appearing in 5.5: any graphical hypersurface \( H \subset \mathbb{R} \times \mathbb{R}^{m-1} \) is isotopic to the germ of the zero-graph \( \{0\} \times \mathbb{R}^{m-1} \).

For the inductive step, fix a rooted tree \( \mathcal{T} = (T, \rho, \varepsilon) \), and as usual set \( n = |n(\mathcal{T})| \).
Consider a \((\mathcal{T}, m)\)-front \( H \subset \mathbb{R}^m \), with by necessity \( m \geq n \).

Fix a leaf vertex \( \beta \in \ell(\mathcal{T}) \), which always exists as long as \( \mathcal{T} \neq \mathscr{A}_1 \). Consider the smaller signed rooted tree \( \mathcal{T}' = \mathcal{T} \setminus \beta \), and the corresponding \((\mathcal{T}', m)\)-front \( H' = H \setminus H[\beta] \subset \mathbb{R}^m \), where \( H[\beta] \subset H \) is the interior of the smooth piece indexed by \( \beta \). By induction, we may
assume

\[ H' = H_{\mathcal{F}} \times \mathbb{R}^{m-n+1} \subset \mathbb{R}^m \]

Thus it remains to normalize the smooth piece \( H[\beta] \).

Let \( \mathcal{A}_\beta = (A_\beta, \rho, \varepsilon_\beta) \) be the linear signed rooted subtree of \( \mathcal{T} = (T, \rho, \varepsilon) \) with vertices \( v(A_\beta) = \{ \alpha \in v(T) \mid \alpha \leq \beta \} \). Set \( d = v(\mathcal{T}) \setminus v(\mathcal{A}_\beta) = n(\mathcal{T}) \setminus n(\mathcal{A}_\beta) \) to be the complementary vertices.

Consider the \((\mathcal{A}_\beta, m)\)-front \( K \subset H \) given by the union \( K = \bigcup_{\alpha \in n(\mathcal{A}_\beta)} K[\alpha] \) of the smooth pieces of \( H \subset \mathbb{R}^m \) indexed by \( \alpha \in n(\mathcal{A}_\beta) \). Note for \( \mathcal{A}_\beta' = \mathcal{A}_\beta \cap \mathcal{T}' \), and \( K' = K \cap H' \), we already have

\[ K' = H_{\mathcal{A}_\beta'} \times \mathbb{R}^{m-n+1+d} \subset \mathbb{R}^m \]

and seek to normalize the smooth piece \( K[\beta] = H[\beta] \).

Now we can apply Theorem 5.27 to normalize \( K[\beta] \) viewed as the final smooth piece of \( K \). More specifically, we can apply Theorem 5.27 to normalize \( K[\beta] \) while preserving \( K' \) and viewing the complementary directions \( \mathbb{R}^{m-n+1+d} \) as parameters, see Figure 5.6. This insures we preserve \( H' \) and hence do not disturb its already arranged normalization.

This concludes the proof of Proposition 5.6. \( \square \)

\[ \text{Figure 5.6. Treating the complementary directions as parameters.} \]
6. Symplectic neighborhoods of arboreal Lagrangians

In this section we introduce the notion of an arboreal space and prove that any arboreal space has a unique symplectic thickening.

6.1. Arboreal spaces.

6.1.1. Pre-Ac spaces. Recall that we associated to each signed rooted tree \( T = (T, \rho, \varepsilon) \), an arboreal Lagrangian \( L_T \subset T^*\mathbb{R}^{n(T)} \) which is a union of smooth pieces, \( L_T = \bigcup_{\alpha \in v(T)} L_\alpha \).

For fixed \( c \geq 0 \), and \( m \geq n(T) + c \), set \( d = m - n(T) + c \), and consider the product

\[
L(T, m, c) := L_T \times \mathbb{R}^d \times \mathbb{R}^c_{\geq 0}
\]

We also have the smooth pieces \( L(T, m, c)_\alpha := L_\alpha \times \mathbb{R}^d \times \mathbb{R}^c_{\geq 0} \).

We equip \( L(T, m, c) \) with the structure sheaf \( \mathcal{O}(T, m, c) \) of functions restricted from smooth functions on \( T^*\mathbb{R}^{n(T)} \times \mathbb{R}^d \times \mathbb{R}^c_{\geq 0} \) by the inclusion \( L(T, m, c) \hookrightarrow T^*\mathbb{R}^{n(T)} \times \mathbb{R}^d \times \mathbb{R}^c_{\geq 0} \).

**Definition 6.1.** An \( m \)-dimensional pre-arboreal space with corners (pre-Ac space) \((X, \mathcal{O})\) is a locally ringed Hausdorff topological space locally modeled on

\[
(L(T, m, c), \mathcal{O}(T, m, c))
\]

for varying signed rooted trees \( T \), and \( c \leq m - n(T) \).

A smooth sheet \( Y \to X \) of a pre-Ac space \((X, \mathcal{O})\) is a finite proper map from a manifold with corners such that the map is an embedding on the interior of \( Y \), and locally the map identifies the local connected components of \( Y \) with smooth pieces

\[
L(T, m, c)_\alpha \subset L(T, m, c)
\]

for varying \( \alpha \in v(T) \).

When \( c, d = 0 \), we write \((L_T, \mathcal{O}_T)\) in place of \((L(T, m, c), \mathcal{O}(T, m, c))\).

![Figure 6.1](image)

**Figure 6.1.** We can take \( X \) to be the interval \([-1, 1]\) with 1 glued to 0 as an \( A_2 \)-singularity. Then \([-1, 1] \to X \) is a smooth sheet. We will introduce the notion of Ac-building to avoid the unnecessary complications of this kind of gluing.

**Remark 6.2.** Given a pre-Ac space \((X, \mathcal{O})\), we can speak about a closed embedding into a smooth manifold \((M, \mathcal{C}^\infty_M)\). These are embeddings with closed image \( X \to M \) with a surjective map of sheaves of local rings \( \mathcal{C}^\infty_M|_X \to \mathcal{O} \).
Remark 6.3. One can relax the definition of a pre-arboreal space by choosing the alternative structure sheaf of continuous real-valued functions on \( L(\mathcal{T}, m, c) \) whose restrictions to each smooth piece \( L(\mathcal{T}, m, c)_\alpha \), for \( \alpha \in v(\mathcal{T}) \), are smooth. One can show that every such “weak” pre-arboreal space is diffeomorphic to a strong one, and any weak diffeomorphism between strong pre-arboreals is isotopic to a strong one.

6.1.2. \( \text{Ac} \) spaces. Given a local model \((L_T, \mathcal{O}_T)\), it is not possible to reconstruct the symplectic geometry of \( L_T \subset T^*\mathbb{R}^n(T) \) unless we keep track of additional structure. For a signed rooted tree \( T = (T, \rho, \varepsilon) \), recall if we remove the root \( T \setminus \rho \) what remains is a disjoint union of rooted trees with signs along their edges. Let us write \( T' = (T, \rho, \varepsilon') \) for the signed rooted tree where we negate all of the signs along the edges in any collection of components of \( T \setminus \rho \). Then negating corresponding coordinates in our local models provides an isomorphism of pre-Ac spaces \((L_T, \mathcal{O}_T) \simeq (L_{T'}, \mathcal{O}_{T'})\) that does not extend to an ambient symplectomorphism unless \( T = T' \). Additionally, in the case of the tree \( A_2 \) with a single unsigned edge, the nontrivial involution of the pre-Ac space \((L_{A_2}, \mathcal{O}_{A_2})\) that reverses the orientation on the smooth base piece \( L_\rho \subset L_{A_2} \) can only be extended to an ambient anti-symplectomorphism. We will see in Lemma 6.10 below these are the only ambiguities.

We introduce here additional structure to overcome the above ambiguities and hence enable us to reconstruct \( L_T \subset T^*\mathbb{R}^n(T) \) from \((L_T, \mathcal{O}_T)\). Let us first mention some simple data we could add. Given \((L_T, \mathcal{O}_T)\), consider the base smooth piece \( L_\rho \subset L_T \), and the hypersurface \( H = \bigcup_{\alpha \neq \rho} L_\alpha \cap L_\rho \subset L_\rho \). Then a coorientation of \( H \) provides an embedding \( L_T \subset T^*\mathbb{R}^n(T) \) as the corresponding positive conormal of \( H \). In particular, there is the specific choice of coorientation that gives the canonical embedding \( L_T \subset T^*\mathbb{R}^n(T) \) of the local model.

To go forward, we will introduce a version of the above coorientation data that can be easily glued in a global setting. For a signed rooted tree \( \mathcal{T} = (T, \rho, \varepsilon) \), and vertex \( \alpha \in v(\mathcal{T}) \), consider the orientation bundle \( \kappa(\mathcal{T}, m, c)_\alpha := \wedge^m TL(\mathcal{T}, m, c)_\alpha \).

Let \( K(\mathcal{T}, m, c) \), denote the space of real line bundles \( \kappa \to L(\mathcal{T}, m, c) \) equipped with identifications
\[
\kappa|_{L(\mathcal{T}, m, c)_\alpha} \simeq \kappa(\mathcal{T}, m, c)_\alpha \quad \alpha \in v(\mathcal{T})
\]
Note that given such a \( \kappa \), we can extract from it a coorientation of the hypersurface \( H = \bigcup_{\alpha \neq \rho} L(\mathcal{T}, m, c)_\alpha \cap L(\mathcal{T}, m, c)_\rho \) in the base smooth piece \( L(\mathcal{T}, m, c)_\rho \). Namely, we can take the coorientation of \( H \) to be that glued under \( \kappa \) with the positive Liouville direction.

**Lemma 6.4.** Each connected component of \( K(\mathcal{T}, m, c) \) is contractible, and \( \pi_0(K(\mathcal{T}, m, c)) \) is a \((\mathbb{Z}/2\mathbb{Z})^{|v(\mathcal{T})}|\)-torsor.

**Proof.** Suppose the lemma is proved for signed rooted trees with fewer vertices than \( \mathcal{T} \).
Fix $r > 0$, let $B_r = \{ |p| = r \} \subset T^*(\mathbb{R}^{m-c} \times \mathbb{R}_r^c)$ denote the radius $r$ open ball bundle, and $N_r = B_r \cap L(T, m, c)$ the corresponding open neighborhood of the zero-section $L(T, m, c)_\rho \simeq \mathbb{R}^{m-c} \times \mathbb{R}_r^c$. Note the Liouville flow of $T^*\mathbb{R}^m$ deformation retracts $N_r$ back to the zero-section.

Consider the Čech cover of $L(T, m, c)$ given by the open $N_r$ along with the disjoint opens

$$L(T_i, m - 1, c) \times \mathbb{R}_{>0} \quad i \in I$$

where $I \subset v(T)$ denotes the set of vertices adjacent to the root $\rho$, or equivalently the set of edges adjacent to $\rho$, $T_i$ is the signed rooted tree given by the connected component of $T \setminus \rho$ containing $i \in I$, and the last factor $\mathbb{R}_{>0}$ of each open is given by Liouville flow. Note each intersection $N_r \cap L(T_i, m - 1, c) \times \mathbb{R}_{>0}$, for $i \in I$, is homeomorphic to $L(T_i, m - 1, c) \times (0, r)$, in particular connected.

We see that to give a line bundle on $L(T, m, c)$, given its restriction to the above opens, is to give a sign $\pm 1$ on each intersection $N_r \cap L(T_i, m - 1, c) \times \mathbb{R}_{>0}$, for $i \in I$. This inductively establishes the lemma.

We will now use the symplectic geometry to specify a distinguished element of $\kappa(T, m, c) \in K(T, m, c)$, i.e. a trivialization of the $(\mathbb{Z}/2\mathbb{Z})^{e(T)}$-torsor of the lemma

$$K(T, m, c) \simeq (\mathbb{Z}/2\mathbb{Z})^{e(T)}$$

Given the canonical model $L(T, m, c) \subset T^*\mathbb{R}^m$, we have a canonical coorientation of the hypersurface $H = \bigcup_{\alpha \neq \rho} L(T, m, c)_\alpha \cap L(T, m, c)_\rho$ in the base smooth piece $L(T, m, c)_\rho$. To identify the orientation bundles of the smooth pieces along this intersection it suffices to specify that the positive codirection to the co-oriented front is glued to the positive Liouville direction. Note this choice is invariant under symplectomorphism.

**Definition 6.5.** The neutral element $\kappa(T, m, c) \in K(T, m, c)$ is given by the above inductively specified gluing.

**Remark 6.6.** Let us mention a characterization (we will not use) of when an arboreal model is positive. Assume for simplicity $T$ has a single vertex adjacent to the root. Then the Legendrian at infinity $L(T, m, c)_{\infty} \subset S^*\mathbb{R}^m$ projects homeomorphically to its front $H \subset \mathbb{R}^m$, and the front has a natural “tangent” bundle given by the orthogonal to the coorientation. In the case of positive $T$, the restriction of $\kappa(T, m, c)$ to $L(T, m, c)_{\infty}$ coincides with the top exterior power of the “tangent” bundle of the front under the above homeomorphism.

**Definition 6.7.** An orientation structure $\kappa$ for a pre-Ac space $(X, \mathcal{O})$ is a real line bundle $\kappa \to X$ equipped with identifications $\kappa|_Y \simeq \kappa_Y$ for each smooth sheet $Y \to X$, locally modeled on the neutral elements $\kappa(T, m, c) \in K(T, m, c)$.

**Definition 6.8.** An $m$-dimensional arboreal space with corners (Ac space) $(X, \mathcal{O}, \kappa)$ is an $m$-dimensional pre-arboreal space $(X, \mathcal{O})$ equipped with an orientation structure $\kappa$.

**Remark 6.9.** When $m = 1$, one can compare the notion of orientation structure with ribbon structure for a trivalent graph. At each trivalent vertex we have a “T” singularity and in
this case an orientation structure is equivalent to a cyclic orientation of the three edges. In particular, the orientation structure rigidifies the pre-Ac space \((L_{A_2}, O_{A_2})\).

Recall that an arboreal Lagrangian \(L\) in a symplectic manifold \((M, \omega)\) is locally symplectomorphic to a standard model. Since this model has an orientation structure given by its neutral element, and this element is invariant under symplectomorphism, we conclude that any arboreal Lagrangian has an orientation structure induced by the ambient symplectic structure. Hence every arboreal Lagrangian has the structure of an arboreal space.

Finally, let us return to the discussion at the beginning of this section, and observe that, locally, orientation structures are nothing more than choices of coorientations.

**Lemma 6.10.** Let \(T = (T, \rho, \varepsilon)\) be a signed rooted tree.

There are precisely \(2^k\) isomorphism classes of orientation structures on the pre-Ac space \((L_T, O_T)\), where \(k\) is the number of edges adjacent to the root \(\rho\).

**Proof.** By construction, an orientation structure on the local model \((L_T, O_T)\) must result by pullback of the neutral orientation structure under an isomorphism \((L_T, O_T) \simeq (L_{T'}, O_{T'})\) for some \(T'\). Moreover, note that \((L_T, O_T)\) is determined by the hypersurface \(H = \bigcup_{\alpha \neq \rho} L_{\alpha} \cap L_{\rho} \subset L_{\rho} \simeq \mathbb{R}^n(T)\), and so we must have an isomorphism \((\mathbb{R}^n(T), H) \simeq (\mathbb{R}^n(T'), H')\) of hypersurfaces. So we must have \(T' = (T, \rho, \varepsilon')\) where \(\varepsilon'\) results from \(\varepsilon\) by negating all of the signs along the edges in some collection of components of \(T \setminus \rho\). These choices correspond to the \(2^k\) choices of coorientations of the hypersurface \(H\). Each such choice provides a distinct Lagrangian realization \(L_T \subset T^*\mathbb{R}^n(T)\) with corresponding orientation structure.

**Remark 6.11.** Note therefore that the data of an orientation structure on an arboreal singularity is equivalent to a fully signed rooted tree, i.e. a signed rooted tree in the previous sense together with additional signs on each of the edges of \(T\) adjacent to the root, so that now
every edge in $T$ has a sign. We will however not use this notion of a fully signed rooted tree and will always use the term signed rooted tree for rooted trees with signs on those edges not adjacent to the root.

6.2. Arboreal Darboux-Weinstein neighborhood theorem.

6.2.1. Lagrangian subsets. Let $L$ be a path connected and locally simply-connected closed subset in a symplectic manifold $(M,\omega)$. We call $L$ a Lagrangian subset if each point of $L$ has a Darboux neighborhood $U$ such that $\int_\gamma pdq = 0$ for every loop $\gamma$ in $L \cap U$. Clearly, any arboreal Lagrangian $L \subset M$ is a Lagrangian subset. If $f$ is (the germ at a point of) a diffeomorphism $M \to M$, then $f(L)$ is (the germ of) a Lagrangian subset if and only if the image under $f$ of each smooth piece of $L$ is Lagrangian.

If $L \subset M$ is a pre-Ac space embedded as a Lagrangian subset in a symplectic manifold $(M,\omega)$, then $L$ inherits an orientation structure as follows. Returning to the inductive description of orientation structures given in Lemma 6.4, along a smooth Lagrangian piece $P \subset L$, given a Lagrangian half-plane $H$ with codimension 1 clean intersection with $P$, we specify that the inward normal direction $\nu$ to $P \cap H$ in $H$ is glued to the normal direction to $P \cap H$ in $P$ which pairs positively with $\nu$ under $\omega$. With this structure, any pre-Ac space embedded as a Lagrangian subset gets an induced structure of an Ac space.

Note the above orientation structure on $L$ as a Lagrangian subset agrees with the previously defined orientation structure on $L$ as an arboreal Lagrangian, i.e. the one defined using local models.

6.2.2. Darboux-Weinstein theorem. The main result of this section is the following:

**Theorem 6.12.** Let $V$ be a manifold with two symplectic forms $\omega_0, \omega_1$. Suppose $L \subset V$ is a compact arboreal space embedded as a Lagrangian subset for both symplectic forms. Let $\kappa_0, \kappa_1$ be the induced orientation structures on $L$.

If the orientation structures are isomorphic $\kappa_0 \simeq \kappa_1$, then there exists a diffeomorphism $\varphi : V \to V$, preserving $L$ as a subset, such that $\varphi^* \omega_1 = \omega_0$. If $\omega_0$ and $\omega_1$ coincide on $O_p A$ for a closed subset $A \subset L$ then $\varphi$ can be chosen to be the identity on $O_p A$.

**Corollary 6.13.** Any arboreal space embeds into a symplectic manifold as an arboreal Lagrangian such that the orientation structure induced from the ambient symplectic structure is isomorphic to the given orientation structure.

**Proof.** By definition, any point of an arboreal space has a model neighborhood. On overlaps, these local symplectic structures have the same orientation structure, and hence according to Theorem 6.12 they are diffeomorphic via a diffeomorphism preserving the arboreal. Moreover, the relative form of that theorem allows us to construct a symplectic neighborhood inductively extending it over neighborhoods of strata of certain stratification. □
In Theorem 6.27, proved below in Section 6.27 below, we find a Weinstein structure on a symplectic neighborhood of an arboreal space $L$ for which $L$ serves as the skeleton. Moreover, Theorem 6.27 ensures uniqueness of this structure up to Weinstein homotopy fixing the skeleton.

Theorem 6.12 is a corollary of the following two propositions.

**Proposition 6.14.** Let $V$ be a manifold with two symplectic forms $\omega_0, \omega_1$. Suppose $L \subset V$ is a compact arboreal space which is a Lagrangian subset for both symplectic forms. Let $\kappa_0, \kappa_1$ be the induced orientation structures on $L$. If the orientation structures are isomorphic $\kappa_0 \simeq \kappa_1$, then there exists a diffeomorphism $\phi$ of $\mathcal{O}(L)$ preserving $L$ pointwise such that the linear interpolation $\omega_t = (1-t)\omega_0 + t\phi^*\omega_1$ is a family of symplectic forms on $\mathcal{O}(L)$ so that $L$ is a Lagrangian subset for each $\omega_t$. If $\omega_0$ and $\omega_1$ coincides on $\mathcal{O}A$ for a closed subset $A \subset X$ then $\phi$ can be chosen to be the identity on $\mathcal{O}A$.

**Proposition 6.15.** Let $L \subset V$ be an arboreal subspace which is a Lagrangian subset for a family $\omega_t, t \in [0, 1]$, of symplectic forms on $L$. Then there exists a diffeotopy $h_t : \mathcal{O}(L) \to \mathcal{O}(L), t \in [0, 1]$, such that $h_t(L) = L$ and $h_t^*\omega_t = \omega_0$. If $\omega_t = \omega_0$ on $\mathcal{O}A$ for a closed subset $A \subset L$ then the isotopy $h_t$ can be chosen fixed on $\mathcal{O}A$.

Indeed, to prove Theorem 6.12 we can apply Proposition 6.15 to the output of Proposition 6.14. We will prove Propositions 6.14 and 6.15 in the next two subsections.

**Remark 6.16.** As the proof of Proposition 6.14 will show, the conclusion holds more generally for two symplectic manifolds $(V_i, \omega_i), i = 1, 2$, together with embeddings of a compact arboreal space $L$ onto a Lagrangian subset of $V_i, i = 1, 2$. In particular it follows that the smooth topology of a neighborhood of a Lagrangian compact arboreal space in a symplectic manifold is uniquely determined by the compact arboreal space.

6.2.3. **Proof of Proposition 6.14.** Let us revisit the notion of an orientation structure. Consider a full flag $F = (0 = V_0 \subset V_1 \subset \cdots \subset V_{2n} = V)$ of subspaces in a $2n$-dimensional vector space $V$, $\dim V_j = j$. Let $\Omega(F)$ be the space of all symplectic forms $\omega$ on $V$ such that $V_j$ is isotropic for $j \leq n$, $V_j$ is coisotropic for $j \geq n$, and $V_j^\perp \omega = V_{2n-j}$ for $j \leq n$.

We begin with following elementary assertion.

**Lemma 6.17.** The connected components of $\Omega(F)$ are determined by orientations of the symplectic quotients $V_{2n-j}/V_j$, for $j = 0, \ldots, n-1$. Any sequence of orientations can occur and each connected component is convex.

**Proof.** Represent $\omega$ as a matrix with respect to a basis $u_1, \ldots, u_{2n}$ of $V$ such that $V_j = \text{span}(u_1, \ldots, u_j)$. It has a block form with four $n$ by $n$ blocks, the two diagonal blocks being zero and the two off-diagonal blocks being $A$ and $-A$. Moreover, $A$ is a triangular matrix and its diagonal elements must be non-zero. Their signs naturally correspond to orientations of the symplectic quotients. \qed
Recall the explicit construction of the extended local model

\[ nL = \bigcup_{i=0}^{n} nL_i \subset T^*\mathbb{R}^n \]

in terms of equations in the canonical coordinates \( x_0, \ldots, x_{n-1}, p_0, \ldots, p_{n-1} \). Introduce the flag \( F(nL) = (0 = V_0 \subset V_1 \subset \cdots \subset V_{2n} = T^*\mathbb{R}^n) \) given by the assignments

\[
V_i = T_0(\bigcap_{j=0}^{n-i} nL_j) = \text{Span}(\partial_{x_{n-1}}, \ldots, \partial_{x_{n-i}}) \quad i \leq n
\]

\[
V_{2n-i} = \Theta_0(\bigcup_{j=0}^{n-i} nL_j) = \text{Span}(\partial_{x_{n-1}}, \ldots, \partial_{x_0}, \partial_{p_0}, \ldots, \partial_{p_{2n-i}}) \quad i \leq n
\]

Here \( \bigcap_{j=0}^{n-i} nL_j \) is a submanifold of \( T^*\mathbb{R}^n \), in fact a linear subspace of \( nL_0 = \mathbb{R}^n \), and \( T_0(\bigcap_{j=0}^{n-i} nL_j) \) denotes its tangent space at the origin; \( \bigcup_{j=0}^{n-i} nL_j \) is not a submanifold, but has a tangent sheaf in the sense of algebraic geometry, and \( \Theta_0(\bigcup_{j=0}^{n-i} nL_j) = (T_0/T^*_0)^{\vee} \) denotes its fiber at the origin. Note these characterizations imply \( F(nL) \) is preserved by diffeomorphisms of \( T^*\mathbb{R}^n \) preserving \( nL \) as a subset hence preserving each \( nL_i \) as a subset, i.e. \( F(nL) \) only depends on the sheaf of smooth functions on \( nL \). Note also \( V_i \) is isotropic for \( i \leq n \), \( V_i \) is coisotropic for \( i \geq n \), and \( V_i^{\perp, \omega} = V_{2n-i} \) for \( j \leq n \) for any symplectic form on \( \mathbb{R}^{2n} \) such that \( nL \subset \mathbb{R}^{2n} \) is Lagrangian.

Recall that the local model for an \( \mathcal{A}_{n+1} \)-arboreal Lagrangian is a subset of the extended local model: \( L_{\mathcal{A}_{n+1}} \subset nL \). We have the following reformulation of orientation structures for \( \mathcal{A} \)-space structures on the germ of \( L_{\mathcal{A}_{n+1}} \) at its central point:

**Lemma 6.18.** Let \( \omega_0 \) be a symplectic form on \( \mathbb{R}^{2n} \) for which \( L_{\mathcal{A}_{n+1}} \) is a Lagrangian subset.

If \( \omega_1 \) is another such form, then the orientation structures \( \kappa_0, \kappa_1 \) induced by \( \omega_0, \omega_1 \) respectively on \( L_{\mathcal{A}_{n+1}} \) are isomorphic if and only \( \omega_0 \) and \( \omega_1 \) lie in the same connected component of \( \Omega(F(nL)) \).

**Proof.** Let \( \omega \) be a symplectic structure on \( \mathbb{R}^{2n} \) such that \( nL \) is Lagrangian. The sign of the leading diagonal element of the matrix \( A \) in the proof of Lemma 6.17 applied to \( \omega \) is precisely the pairing used to define the orientation structure on \( nL \) as a Lagrangian subset of \( (\mathbb{R}^{2n}, \omega) \). The other signs are uniquely determined by quadratic information. For example, for \( L_{\mathcal{A}_3} \subset T^*\mathbb{R}^2 \) the third stratum is the positive conormal to \( \{ x_0 = x_1^2 x_1 \geq 0 \} \). Therefore, the vectors \( 2x_1 \partial/\partial x_1 + \partial/\partial x_0 \) and \( \partial/\partial p_1 + 2x_1 \partial/\partial p_0 \) should span a Lagrangian subspace. Their symplectic product is \( 2x_1(\lambda - \mu) \) where \( \lambda \) is the first diagonal term of the matrix \( A \) and \( \mu \) is the second diagonal term of the matrix \( A \). So in fact \( \lambda = \mu \). The general case can be proved similarly by induction. The conclusion is that the connected component of \( \Omega(F(nL)) \) is uniquely determined by the orientation structure, which in this case is just a single sign. \( \square \)

For more general trees \( \mathcal{T} \), one can apply the above argument to each factor \( T^*\mathbb{R}^n \) in \( T^*\mathbb{R}^{n(\mathcal{T})} \) corresponding to each \( \mathcal{A}_n \)-subtree of \( \mathcal{T} \) with the same root \( \rho \) and deduce the general version:

**Lemma 6.19.** Let \( \mathcal{T} \) be a signed rooted tree and \( \omega_0 \) a symplectic form on \( \mathbb{R}^{2n(\mathcal{T})} \) for which \( L_{\mathcal{T}} \) is Lagrangian.
If \( \omega_1 \) is another such form, then the orientation structures \( \kappa_0, \kappa_1 \) induced by \( \omega_0, \omega_1 \) respectively on \( L_T \) are isomorphic if and only if for any \( \mathcal{A}_{m+1} \)-type subtree \( \mathcal{A}_{m+1} \subset T \) with the same root \( \rho \), the restrictions of \( \omega_0 \) and \( \omega_1 \) to the factor \( \mathbb{R}^{2m} \) of \( \mathbb{R}^{2n(T)} \) corresponding to the \( \mathcal{A}_{m+1} \)-subtree lie in the same connected component of \( \Omega(F(m)L)) \).

So we have proved Proposition 6.14 for the germ of the central point of the pre-Ac space \( L(T, m) \) with \( m = n(T) \), i.e. not stabilized. In the case \( d = m - n(T) > 0 \), the situation changes as follows. Given a symplectic form \( \omega_0 \) on \( \mathbb{R}^{2m} \) for which the standard model \( L_T \times \mathbb{R}^d \subset \mathbb{R}^{2n(T)} \times \mathbb{R}^{2d} \) is Lagrangian, consider \( E \subset \mathbb{R}^{2m} \), the \((2m - d)\)-dimensional coisotropic subspace of \( \mathbb{R}^{2m} \) which is symplectically orthogonal to the isotropic \( F = 0 \times \mathbb{R}^d \). Since \( L_T \times \mathbb{R}^d \) is assumed to be Lagrangian, note that \( F \) is spanned by those directions tangent to the smooth part of \( L_T \times \mathbb{R}^d \) and \( E \) is spanned by those directions tangent to \( L_T \times \mathbb{R}^d \) at 0, hence both are independent of the symplectic form \( \omega_0 \).

The symplectic reduction of \((E, \omega_0|_E)\) is a \((2m - d)\)-dimensional symplectic vector space \( V_0 \). The quotient \( \mathbb{R}^{2m}/E \) is canonically identified via the fixed symplectic form \( \omega_0 \) with the dual of \( 0 \times \mathbb{R}^d \). Therefore there exists a canonical symplectic isomorphism \( \Phi \) between \( \mathbb{R}^{2m} \) and \( V_0 \oplus (F \oplus F^*) \).

Now suppose that we have two such symplectic forms \( \omega_0 \) and \( \omega_1 \) on \( \mathbb{R}^{2m} \) for which \( L_T \times \mathbb{R}^d \) is Lagrangian.

**Lemma 6.20.** Consider \( S = S(\omega_0, \omega_1) \), the space of linear isomorphisms \( \Psi : \mathbb{R}^{2m} \to \mathbb{R}^{2m} \) such that:

1. \( \Psi|_{L_T \times \mathbb{R}^d} = \text{Id}_{L_T \times \mathbb{R}^d} \).
2. \( \Psi^* \omega_1 \) and \( \omega_0 \) agree on \( F \oplus (\mathbb{R}^{2m}/E) \).

Then \( S \) is nonempty and contractible.

**Example 6.21.** For the smooth arboreal singularity, \( S \) consists of a single element.

**Proof.** For the existence, write \( \mathbb{R}^{2m} = \mathbb{R}^{2n(T)} \times \mathbb{R}^d \times \mathbb{R}^d \). Consider a linear isomorphism \( \Psi : \mathbb{R}^{2m} \to \mathbb{R}^{2m} \) which is the identity on the first two factors. This induces a linear isomorphism between the \( V_0 \) and \( V_1 \) corresponding to \( \omega_0 \) and \( \omega_1 \). The further condition that \( \Psi \) respects the identification \( V_0 \oplus (F \oplus F^*) \simeq V_1 \oplus (F \oplus F^*) \) uniquely determines a linear isomorphism \( \Psi \) satisfying (i) and (ii).

For the contractibility, fix \( \Psi_0 \in S \). For any other \( \Psi_1 \in S \), we will write the matrix for \( \Psi_1^{-1} \Psi_0 \) in block form with respect to the decomposition \( \mathbb{R}^{2m} = \mathbb{R}^{2n(T)} \times \mathbb{R}^d \times \mathbb{R}^d \), with the canonical basis. The matrix will have block form, with the diagonal factors the identity, and with all off-diagonal blocks zero except for the \((3, 2)\) block, which is an arbitrary symmetric matrix \( A \). The space of such is contractible, so this completes the proof. \( \square \)

**Lemma 6.22.** Let \( \omega_0, \omega_1 \) be symplectic forms on \( \mathbb{R}^{2m} \) such that \( L_T \times \mathbb{R}^d \) is Lagrangian and which induce isomorphic orientation structures. Let \( \Psi \in S \). Then \((1 - t)\omega_0 + t\Psi^* \omega_1 \) is symplectic for all \( 0 \leq t \leq 1 \).
Proof. On $\mathbb{R}^{2m}/V_0$, the symplectic forms induced by $\omega_0$ and $\Psi^*\omega_1$ agree. On $V_0$ the problem is reduced to the non-stabilized case as discussed above: the homotopy $(1 - t)\omega_0 + t\Psi^*\omega_1$ will be through symplectic forms whenever the induced orientation structures agree. Indeed, the argument is unaffected by the symplectic isomorphism $\Psi$ since it is the identity on $L_T \times \mathbb{R}^d$. □

Now we are ready to prove Proposition 6.14. Let $L \subseteq V$ be a compact arboreal space which is Lagrangian for two symplectic forms $\omega_0$ and $\omega_1$. By Lemmas 6.20 and 6.22 we may construct a family of linear isomorphisms $\Psi_x : T_x V \to T_x V$, $x \in L$, such that $\Psi_x$ is the identity on directions tangent to $L$ at $x$, and such that $(1 - t)\omega_0(x) + t\Psi^*_x\omega_1(x)$ is symplectic for all $x \in X$ and $0 \leq t \leq 1$. We may then integrate the family $\Psi_x$ to a diffeomorphism $\Psi : \mathcal{O}_p(L) \to \mathcal{O}_p(L)$ such that $\Psi|_L = Id_L$ and such that $(1 - t)\omega_0 + t\Psi^*\omega_1$ is a homotopy of symplectic forms on $\mathcal{O}_p(L)$. This completes the proof.

6.2.4. Proof of Proposition 6.15.

Lemma 6.23. Let $L \subseteq T^* M$ be a Lagrangian subset which is the union $M \cup C(\Lambda)$ where $C(\Lambda)$ is the Liouville cone over a Legendrian $\Lambda$. Let $\lambda$ be a 1-form on $T^* M$ such that $\lambda|_M = 0$ and $d\lambda = 0$ on $L$. Then there exists a smooth function $H : T^* M \to \mathbb{R}$ vanishing on $M$ together with its differential $dH$, and such that $\lambda$ coincides with $dH$ on vectors tangent to $L$.

Proof. For any point $p \in T^*_p M$ denote by $\gamma_{p,q}$ the path $t \mapsto (tp, q)$, $t \in [0,1]$. Define the function $H$ by the formula

$$H(p, q) = \int_{\gamma_{p,q}} \lambda.$$ 

For $(p, q) \in L$ the path $\gamma_{p,q}$ is contained in $L$, and hence on tangent vectors to $L$ the 1 forms $dH$ and $\lambda$ coincide. □

Lemma 6.24. Let $L \subseteq T^* M$ be a Lagrangian subset which is the union $M \cup C(\Lambda)$. Let $A \subseteq M$ be a closed subset and $\omega_t$ be a family of symplectic forms on $T^* M$ such that $\omega_0 = d(pdq)$, $\omega_t|_{\mathcal{O}_p A} = \omega_0$, and $L$ is $\omega_t$-Lagrangian for all $t$. Then there exists an isotopy $f_t : T^* M \to T^* M$ which is fixed on $M \cup \mathcal{O}_p A$, leaves $L$ invariant and such that $f_t^* \omega_t = \omega_0$.

Proof. There exists a family of Liouville forms $\lambda_t$ for $\omega_t$ such that $\lambda_0 = pdq$, $\lambda_t|_{T^* M|_A} = 0$ and $\lambda_t|_{\mathcal{O}_p A} = pdq|_{\mathcal{O}_p A}$. By Lemma 6.23 there exists a family of functions $H_t$ on $T^* M$ such that $dH_t|_L = \lambda_t|_L H_t|_{M \cup \mathcal{O}_p A} = 0$ and $dH_t|_{T^* M|_A} = 0$. Define $\mu_t := \lambda_t - dH_t$. Then $\mu_0 = pdq$, $\mu_t|_{T^* M|_A} = 0$, $\mu_t|_{\mathcal{O}_p A} = pdq|_{\mathcal{O}_p A}$ and $\mu_t|_L = 0$. Then Moser’s homotopical method yields an isotopy $f_t : T^* M \to T^* M$ which is fixed on $M \cup \mathcal{O}_p A$, leaves $L$ invariant and such that $f_t^* \omega_t = \omega_0 = d(pdq)$. □

Note that any point of an arboreal space which is realized as a Lagrangian subset of a symplectic manifold $X$ has a neighborhood which has the conical form of Lemma 6.24. Hence, we can take a finite covering of $X$ by such neighborhoods and successively apply Lemma 6.24 to extend the isotopy. This concludes the proof of Proposition 6.15, and with it the proof of Theorem 6.12.
6.3. Weinstein thickening of an Ac-building.

6.3.1. Ac-buildings. Similar to the notion of a Liouville or a Weinstein manifold with corners, by a symplectic manifold with corners we mean a possibly non-compact manifold with corners, such that near each corner point it is symplectomorphic to $\mathbb{R}^{2m} \times T^*\mathbb{R}^c_+$. An arboreal Lagrangian with boundary with corners, or ac-Lagrangian for short, is a closed subset of a symplectic manifold $W$ with corners such that the germ of $(W,L)$ at a point $x \in L$ is symplectomorphic to the germ of $(\mathbb{R}^{2n-2c} \times T^*\mathbb{R}^c_+, L_T \times \mathbb{R}^c_+)$ at the origin for some signed rooted tree $T$ and an integer $c \geq 0$. The $c$ is called the multiplicity of the corner. It follows from the definition that corners of $L$ are contained in the corresponding corners of $W$. As in the case of manifolds with boundary with corners, we denote by $\partial_j L$ the strata of corner points of order $j \geq 1$, and write $\partial L = \bigcup_j \partial_j L$. The closure of each component of $\partial_j L$ is itself an ac-Lagrangian in a symplectic submanifold of $W$ of codimension $2k$. The closures of components of $\partial_1 L$ are called boundary faces of $L$.

It will also be useful to consider the notion of an Ac-building, a special kind of Ac-space whose definition is analogous to that of a cotangent building as a special kind of $Wc$-manifold. Recall that an Ac-space $L$ is the union $L = \bigcup_j L_j$ of smooth pieces $L_j$, and each point of $L$ belongs to the interior of exactly one $L_j$.

**Definition 6.25.** An Ac-building is an Ac-space $L$ whose smooth pieces can be ordered $L_1, \ldots, L_k$ so that:

(i) Each $L_{>j} = \bigcup_{i>j} L_j$ is an Ac-space with boundary and corners.

(ii) $L_{>j-1}$ is obtained from $L_{>j}$ by gluing a boundary component $P_j$ of $L_{>j}$ to $L_j$.

**Lemma 6.26.** Let $K = \bigcup_j L_j$ be an Ac-building. Then the gluing maps $P_j \to L_j$ can be canonically lifted to embeddings $P_j \to T^*L_j \setminus L_j$ onto arboreal Legendrians.

**Proof.** This follows from the characterization of orientation structures in terms of co-orientations of fronts, as discussed above. \hfill \Box

**Theorem 6.27.** Suppose $(W,\omega)$ is a $2m$-dimensional symplectic manifold, and let $L \subset W$ be an Ac-building. Then

(i) a neighborhood of $L$ admits a Weinstein structure with $L$ as its skeleton,

(ii) the germ near $L$ of such a structure is unique up to Weinstein homotopy with fixed skeleton $L$.

An assumption that $L$ is an Ac-building rather than a general Ac-space simplifies the proof, but is not necessary. We will not need the more general result and will not prove it in this paper.

6.3.2. Existence.
Proof of (i). Consider the building presentation \( L_k \to L_{k-1} \to \cdots \to L_1 \) of \( L \). We will inductively construct a cotangent building structure with \( L \) as its skeleton. Using Theorem 6.12 this structure can be transported to \( \mathcal{O}_p L \subset W \).

We argue by induction in the number \( k \) of blocks. Suppose that we already constructed a cotangent building \( W_{>1} := B_k \to \cdots \to B_2 \) with the skeleton \( L_{>1} \) which consists of blocks \( B_j = \mathcal{T}^*L_j \). By Lemma 6.26 there exists a boundary face \( Y \) of \( L_{>1} \) such that \( L \) is obtained by attaching \( L_{>1} \) to \( L_1 \) using a front embedding \( Y \to L_1 \), i.e. an embedding which factors as \( Y \to T^*L_1 \setminus L_1 \), where \( \phi \) as an embedding onto a Legendrian arboreal. The Ac-space \( Y \) serves as the nucleus \( Q \) of a boundary face of \( W_{>1} \). The embedding \( \phi \) extends to an embedding \( \Phi : Q \to T^*L_1 \setminus L_1 \) as a Weinstein hypersurface. The required building \( L \) can be now obtained by the vertical gluing of \( W_{>1} \) to \( \mathcal{T}^*L_1 \). □

6.3.3. Liouville fields on cotangent bundles. Before proving the second part of Theorem 6.27 we need to review some facts about Liouville fields on cotangent bundles, see also [CE12, Sect. 12.3].

Let \( Y \) be an \( n \)-dimensional bc-manifold, \( T^*Y \) its cotangent bundle with the standard Liouville form \( \lambda_{st} = pdq \) and symplectic form \( \omega_{st} = d(pdq) \), and Liouville field \( Z_{st} = p \frac{\partial}{\partial p} \). Any other Liouville form \( \lambda \) has the form \( \lambda_{st} - dH \) for a function \( H : T^*M \to \mathbb{R} \). Respectively, the Liouville field \( Z \) of \( \lambda \) is given by \( Z = Z_{st} + X_H \), where \( X_H \) is the Hamiltonian field of \( H \).

We will always suppose in this section that \( Z \) is tangent to \( Y \). This is equivalent to the condition \( H|_Y = 0 \). Denote \( Z' := Z|_Y \), \( H' = H|_Y \).

Lemma 6.28. Let \( \lambda = pdq - dH \) a Liouville form on \( T^*Y \) with \( H|_Y = 0 \). Let \( a \in Y \) be a zero of \( Z' = Z|_Y \). Choose local canonical coordinates \( q_1, \ldots, q_n, p_1, \ldots, p_n \) centered at \( a \). Write \( H = \frac{1}{2} \sum_{i,j=1}^n a_{ij} p_i p_j + \sum_{i,j=1}^n b_{ij} p_i q_j + o(|p|^2 + |q|^2) \). Then

\[
Z' = \sum_{i,j=1}^n b_{ij} q_j \frac{\partial}{\partial q_j} + O(|q|^2), \quad Z = \left( \sum_{i,j=1}^n b_{ij} q_j + \sum_{i,j=1}^n a_{ij} p_j \right) \frac{\partial}{\partial q_i} + \sum_{i,j=1}^n (\delta_{ij} - b_{ij}) p_i \frac{\partial}{\partial p_j} + O(|p|^2 + |q|^2).
\]

The matrix of \( d_a Z' \) is equal to \( B \) and the matrix of \( d_a Z \) in the basis \( \frac{\partial}{\partial q}, \frac{\partial}{\partial p} \) has the form

\[
\begin{pmatrix}
B & A \\
0 & I - B^T
\end{pmatrix}
\]

where \( B = (b_{ij}) \), \( A = (a_{ij}) \).

Proof. Direct computation. □

Definition 6.29. A zero \( a \in Y \) of \( Z' \) is called transversely non-degenerate if the matrix \( I - B^T \) has no pure imaginary eigenvalues.

Lemma 6.30. Let \((W, Z)\) be a Liouville structure with an arboreal skeleton \( X \). Let \( a \in X \) be a zero of \( Z \) and \( Y \) the smooth piece corresponding to the root of the arboreal singularity at
the point $a$. Suppose that $a$ is transversely non-degenerate with respect to $Y$. Denote by $Z'$ the restriction $Z|_Y$. Then the eigenvalues $\lambda_j$ of the differential $d_a Z'$ have real parts $< 1$.

**Proof.** By assumption $Z$ is tangent to $Y$, and hence $Z$ can be written near the zero $a$ as

$$Z = \left( \sum_{i,j=1}^{n} b_{ij} q_j + \sum_{i,j=1}^{n} a_{ij} p_j \right) \frac{\partial}{\partial q_i} + \sum_{i,j=1}^{n} (\delta_{ij} - b_{ij}) p_i \frac{\partial}{\partial p_j} + O(|p|^2 + |q|^2).$$

The transverse non-degeneracy condition means the matrix $C := I - B^T = (\delta_{ij} - b_{ji})$ does not have pure imaginary eigenvalues. If $\text{Re} \lambda < 0$ for one of eigenvalues of $C$, then the stable manifold of $a$ must contain a curve tangent to the corresponding eigenvector. But the arboreal $X$ cannot contain any (two-sided) curves transverse to the root smooth piece $Y$. \hfill $\square$

**Lemma 6.31.** Let $Y$ be a compact manifold, $v$ a vector field, and $\ell : Y \to \mathbb{R}_+$ a positive Lyapunov function for $v$. Let $A$ be the set of critical points of $\ell$. Then for any neighborhood $U \supset A$ and any constant $C > 0$ there exists a function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta' > 0$ such that on $Y \setminus U$ there holds:

$$(7) \quad d(\theta \circ \ell)(v) > C \theta \circ \ell$$

**Proof.** Let $c_0 < \cdots < c_k$ be critical values of $Y$. Choose a sufficiently small $\sigma > 0$ and denote $B_j := \{ c_j - \sigma \leq \ell \leq c_j + \sigma \}$, $C_j := \{ c_j + \sigma \leq \ell \leq c_{j+1} - \sigma \}$. Note that each $C_j$ is a trivial cobordism bounded by regular level sets $\partial_- C_j = \{ \ell = c_j + \sigma \}$ and $\partial_+ C_j = \{ \ell = c_{j+1} - \sigma \}$. Denote $\delta_0 := [c_0, c_0 + \sigma]$, $\Delta_0 := [c_0 + \sigma, c_1 - \sigma]$, $\delta_1 := [c_1 - \sigma, c_1 + \sigma]$, $\Delta_1 := [c_1 + \sigma, c_2 - \sigma]$, $\cdots$, $\Delta_{k-1} := [c_{k-1} + \sigma, c_k - \sigma]$, $\delta_k := [c_k - \sigma, c_k]$. Define a diffeomorphism $\psi_j : \partial_- C_j \times \Delta_j \to C_j$ by sending vertical intervals to the corresponding trajectories of $v$ parameterized by $\ell$, so that we have $\ell(x,u) = u$, $x \in \partial_- C_j, u \in \Delta_j$. Denote $h(x,u) := d_{x,u} \ell(v)$. We have $d(\theta \circ \ell)(v) = \theta'(u) h(x,u)$. We will successively define $\theta$ on intervals. Assuming $\sigma$ so small that $B_0 \subset U$ we define $\theta(t) = t$ on $\delta_0$. Identifying $C_0$ with $\partial_- C_0 \times \Delta_0$ via $\psi_0$, the inequality (7) takes the form

$$\theta'(u) h(x,u) > C \theta(u)$$

or

$$\frac{d \ln \theta(u)}{du} > \frac{C}{h(x,u)}.$$ 

Choose a smooth positive function $\tilde{h}(u) < \min_x h(x,u)$. Solving the equation

$$\frac{d \ln \theta(u)}{du} = \frac{C}{\tilde{h}(u)},$$

we get $\theta(s) = \theta(c_0 + \sigma) e^{H(u)}$, where $H(u) = \int_0^u \tilde{h}(s) ds$. Note that we can assume that all the trajectories of $v$ in $B_j \setminus U$ begin at $\{ \ell = c_j - \varepsilon \}$ and end at $\{ \ell = c_j + \varepsilon \}$, i.e. there exists a diffeomorphism $(\partial_- B_j \setminus U) \times \delta_j \to B_j \setminus U$. Hence we can define $\theta$ on $[c_0, c_k]$, by repeating the same argument successively to $\delta_0, \Delta_0, \delta_1, \Delta_1, \ldots, \delta_k$. \hfill $\square$
Lemma 6.32. Let \( Z = p\frac{\partial}{\partial p} + X_H \) be a Liouville field on \( T^*Y \). Suppose the vector field \( Z' = Z|_Y \) is gradient like for some function \( \ell : Y \to \mathbb{R} \). Suppose for all zeroes \( a \in Y \) of the vector field \( Z' \) the eigenvalues of the differential \( d_a Z' \) have real part \( < 1 \). Then there exists a Riemannian metric on \( Y \) such that \( Z'|p|^2 \geq \varepsilon|p|^2 \) on a neighborhood of \( Y \).

Proof. Let us start with any background metric on \( Y \). Recall that the vector field \( Z' \) is gradient like for a function \( \ell : Y \to \mathbb{R} \), which can be assumed positive. According to Lemma 6.32, near each zero we have

\[
Z = \left( \sum_{i,j=1}^n b_{ij} q_j + \sum_{i,j=1}^n a_{ij} p_j \right) \frac{\partial}{\partial q_i} + \sum_{i,j=1}^n (\delta_{ij} - b_{ij}) p_i \frac{\partial}{\partial p_j} + O(|p|^2 + |q|^2).
\]

By assumption, all eigenvalues of the matrix \( B \) have real parts \( < 1 \), and hence the matrix \( I - B^T \) has eigenvalues with the real part \( > 0 \), which implies

\[
Z|p|^2 > \varepsilon|p|^2
\]

on a neighborhood \( U \) of the compact set of all zeroes of \( Z \).

Let \( \tilde{Z} \) be the Hamiltonian extension of \( Z' \) to \( T^*Y \) with the Hamiltonian \( p(Z'(q)) \). Note that

\[
Z = \tilde{Z} + \bar{Z}, \text{ where } \bar{Z} = O(|p|).
\]

Hence \( Z \cdot |p|^2 = O(|p|^2) \). We also note that \( \tilde{Z} \cdot |p|^2 = O(|p|^2) \).

Hence, \( Z \cdot |p|^2 = O(|p|^2) \). Denote \( \Delta_1 := \max\left(\frac{|Z\cdot |p|^2}{|p|^2}\right) \). Given any point \( a \in Y \setminus U \) we choose local coordinates centered at \( a \) in a neighborhood \( U_a \) and write \( |p|^2 = \sum c_{ij}(q)p_ip_j \).

Denote

\[
\Delta_2 := \max_{a \in Y \setminus U} \max_{i,j \in U_a} |Z' \cdot c_{ij}(q)|.
\]

Set \( K = \max(\Delta_1, \Delta_2) \). According to Lemma 6.31 there exists a function \( \theta : \mathbb{R} \to [1, \infty) \) such that the function \( \theta \circ \ell \) satisfies on \( Y \setminus U \) the condition

\[
Z' \cdot (\theta \circ \ell) > 3K \theta \circ \ell.
\]

Let us pull-back the function \( \ell \) to \( T^*Y \) via the projection \( T^*Y \to Y \). We will keep the notation \( \ell \) for the extended function. Note that \( \tilde{Z}(p,q) \cdot \ell = Z'(q) \cdot \ell \) and \( \tilde{Z} \cdot \ell = O(|p|) \).

Then we have

\[
|Z \cdot (\theta \circ \ell(q))|p|^2 \geq (3K \theta \circ \ell(q) - 2K \theta \circ \ell(q)) |p|^2 + \frac{\partial}{\partial p} \cdot |p|^2 + o(|p|^2) \geq \sigma|p|^2, \quad \sigma > 0,
\]

if \( |p| \) is small enough.

We need to check that the conformal scaling did not destroy the estimate on \( U \).

\[
Z(\theta \circ \ell)|p|^2 \geq \varepsilon \theta \circ \ell|p|^2 + (Z \cdot \theta \circ \ell)|p|^2
\]

\[
= \varepsilon \theta \circ \ell|p|^2 + (\tilde{Z} \cdot \theta \circ \ell)|p|^2 + (\bar{Z} \cdot \theta \circ \ell)|p|^2.
\]

But

\[
\tilde{Z} \cdot \theta \circ \ell(p,q) = \tilde{Z}' \cdot \ell(q) \geq 0,
\]
and for $|p| < \frac{1}{KC}$ we get $Z \cdot (\theta \circ \ell |p|^2) \geq \varepsilon \theta \circ \ell |p|^2 - C|p|^3$.

Hence, we get

$$Z \cdot (\theta \circ \ell |p|^2) \geq \varepsilon \theta \circ \ell |p|^2 - C|p|^3,$$

and for $|p| < \frac{1}{KC}$ we get $Z \cdot (\theta \circ \ell |p|^2) \geq \varepsilon \theta \circ \ell |p|^2$ for a reduced $\varepsilon$. Hence, rescaling the metric $|p|^2$ with the conformal factor $\theta \circ \ell$ and possibly reducing the size of the neighborhood of $Y$ we get the required metric which satisfies the inequality $Z|p|^2 \geq \varepsilon|p|^2$. 

$\square$

**Lemma 6.33.** Let $Z, \ell$ and the norm $|\cdot|$ be as in Lemma 6.32. We pull-back $\ell$ to $T^*Y$ via the projection $T^*Y \to Y$ and will keep the notation $\ell$ for the extended function. Then there exists a constant $K > 0$ such that the function $\hat{\phi} := \ell + K|p|^2$ is a Lyapunov function for $Z$ on $O\ell Y$.

**Proof.** It is sufficient to check the Lyapunov condition only in s neighborhood of the zero locus $A$ of $Z$. Indeed, elsewhere it is true by assumption along $Y$, and hence, by openness of the taming condition in its neighborhood. As in the proof of Lemma 6.32 we extend $Z'$ to $T^*Y$ as a Hamiltonian vector field $\hat{Z}$ with the Hamiltonian function $\zeta(p, q) = p(Z'(q))$. Denote $\tilde{Z} := Z - \hat{Z} - \frac{\partial}{\partial p}$. We also assume that the function $\ell$ is extended to $T^*Y$ by pulling it back from $Y$ by the projection $T^*Y \to Y$. We have

$$Z \cdot \hat{\phi} = KZ \cdot |p|^2 + Z \cdot \ell \geq K\varepsilon|p|^2 + \tilde{Z} \cdot \ell + \tilde{Z} \cdot \ell.$$ 

Furthermore,

$$\tilde{Z}(p, q) \cdot \ell = Z'(q) \cdot \ell \geq \varepsilon_1(|d\ell|^2 + |Z'(q)|^2),$$

as $\ell$ is Lyapunov for $Z'$.

We also have

$$|\tilde{Z} \cdot \ell| \leq C_1|d\ell||p|, \quad |\tilde{Z}(p, q) - Z'(q)| \leq C_2|d\ell||q||p|.$$

Thus,

$$d\hat{\phi}(Z) \geq K\varepsilon|p|^2 + \varepsilon_2(|Z'(q)|^2 + |d\ell(q)|^2) - C_3|d\ell||p|$$

$$\geq \varepsilon_3(|p|^2 + |d\ell(q)|^2 + |Z'(q)|^2),$$

if $K$ is chosen large enough. We also have

$$||Z||^2 \leq C_2(|p|^2 + |Z'(q)|^2) \quad \text{and} \quad |d\hat{\phi}|^2 \leq C_5(K^2|p|^2 + |d\ell(q)|^2).$$

Hence,

$$d\hat{\phi}(Z) \geq \varepsilon_4(||Z||^2 + |d\phi|^2).$$

$\square$

6.3.4. **Deformations.**

**Proof of Theorem 6.27(ii).** We will construct a Weinstein homotopy between the given Weinstein structure $(W, \omega, \lambda, Z, \phi)$ and the cotangent building structure constructed in the proof
of (i). Again, we will argue by induction in the number $k$ of blocks. The inductive argument will simultaneously establish the base case of the induction, which is simpler.

According to Lemmas 6.30 and 6.32 we have $\dot{Z}|p|^2 > \varepsilon|p|^2$ on $U \setminus L_1$ for a neighborhood $U \supset L_1$. Choose $\delta > 0$ to have $\{|p| \leq 6\delta\} \subset U$. Denote $U_\delta := \{|\delta \leq \phi \leq 6\delta\}$.

Assuming $\phi > 0$ and choosing a sufficiently large $C$ we can find a Lyapunov function $\psi$ for $Z$ such that

- $\psi = C_1|p|^2$ on $\{2\delta < |p| < 5\delta\}$;
- $\psi = C_2\phi$ on $W \setminus \{|p| > 5\delta\}$;
- $\psi = \phi$ on $\{|p| < \delta\}$;
- $d\psi(Z) > 0$ on $U_\delta \setminus \{2\delta < |p| < 5\delta\}$.

We can apply the same argument to the function $\hat{\phi} = \ell + K|p|^2$ constructed in Lemma 6.33 to get a function $\hat{\psi}$ which coincide with $\hat{\phi}$ on $\{|p| < \delta\}$ and with $C_1|p|^2$ on $\{2\delta < |p| < 3\delta\}$.

A convex combination of Lyapunov functions for $Z$ is again a Lyapunov function for $Z$, and hence there is a fixed on $\{|p| \geq 2\delta\}$ deformation of $\psi$ to a Lyapunov function which is equal to $\hat{\psi}$ on $\{|p| < \delta\}$. We will keep the notation $\psi$ for the deformed function.

Recall that $\lambda|\mathcal{O}_p L_1 = pdq - dH$ for a function $H$ vanishing on $L \cap \mathcal{O}_p L_1$. Take a function $\theta : [0,6\delta] \to \mathbb{R}_+$ which is equal to 0 on $[0,2\delta] \cup [5\delta,6\delta]$ and equal to 1 on $[3\delta,4\delta]$. Define a family of functions $H_t : U_\delta \to \mathbb{R}$ by the formula $H_t := (1 - t\theta(|p|))H$. Thus, $H_0 = H$, $H_1 = 0$ on $\{3\delta < |p| < 4\delta\}$ and $H_t = H$ on $\{0 \leq |p| \leq 2\delta\} \cup \{5\delta \leq |p| \leq 6\delta\}$ for all $t \in [0,1]$. Denote by $v$ the Hamiltonian field of $H$, by $v_t$ the Hamiltonian field of $H_t$ and by $Z_t$ the Liouville field $v_t + p\frac{\partial}{\partial p}$. Then $v_t = v$ on $\{0 \leq |p| \leq 2\delta\} \cup \{5\delta \leq |p| \leq 6\delta\}$ and $v_t = (1 - t\theta(|p|))v + H^\prime(|p|)w$ on $\{3\delta < |p| < 4\delta\}$, where we denoted by $w$ the Hamiltonian vector field of the Hamiltonian $|p|^2$. Let us observe that the Liouville field $Z_t$ is tangent to the arboreal $L$. Then we have $v_t \cdot \psi = v \cdot \psi$ on $\{0 \leq |p| \leq 2\delta\} \cup \{5\delta \leq |p| \leq 6\delta\}$ and

$$v_t \cdot |p|^2 = (1 - t\theta(|p|))v \cdot |p|^2 + H^\prime(|p|)w \cdot |p|^2 = (1 - t\theta(|p|))v \cdot |p|^2$$

on $\{3\delta < |p| < 4\delta\}$. On the other hand, on $\{3\delta < |p| < 4\delta\}$ we have

$$Z_t \cdot |p|^2 = (1 - t\theta(|p|))Z \cdot |p|^2 + 2t\theta(|p|)|p|^2 > 0.$$ 

Thus, $Z_t$ is gradient like for $\psi$ for all $t \in [0,1]$.

According to Lemma 6.32 the vector field $Z$ also satisfies the condition $Z \cdot |p|^2 \geq \varepsilon|p|^2$. Hence, the vector field $p\frac{\partial}{\partial p} + v_1$ is gradient like for the function $|p|^2 + \psi$. Define $Z_t$ for $t \in [1,2]$ on $\{|p| \leq 4\delta\}$ by the formula

$$Z_t := p\frac{\partial}{\partial p} + (2 - t)v_1.$$ 

We claim that the Liouville field $Z_t$ is gradient like for the function $\psi_t := |p|^2 + (2 - t)\psi$.

Away from the 0-section both vector fields, $(2 - t)Z_1$ and $(t - 1)p\frac{\partial}{\partial p}$ are gradient like for $|p|^2$, and hence, so is $Z_t = (2 - t)Z_1 + (t - 1)p\frac{\partial}{\partial p}$. Hence, it is sufficient to check the Lyapunov condition in an arbitrary small neighborhood of the 0-section.
Note that we have $p \frac{\partial}{\partial p} \cdot \ell = 0$. Hence,

$$Z_t \cdot \psi_t = ((t-1)p \frac{\partial}{\partial p} + (2-t)Z_1) \cdot (|p|^2 + (2-t)\psi)$$
$$= (t-1)K(t)|p|^2 + (2-t)^2 Z_1 \cdot \psi \geq 2(t-1)K(t)|p|^2 + (2-t)^2 \epsilon(|Z_1|^2 + |d\psi|^2).$$

Furthermore, $|Z_t|^2 \leq 2((2-t)^2|Z_1|^2 + (t-1)^2|p|^2)$ and $|d\psi_t|^2 \leq 2(|p|^2 + (2-t)^2|d\psi_1|^2)$. Therefore,

$$Z_t \cdot \psi_t \geq \epsilon_1(|Z_t|^2 + |d\psi_t|^2).$$

We also note that the Liouville field $Z_t$ is tangent to the arboreal $L$ for all $t$.

All functions $\psi_t$ are proportional to $|p|^2$ on $\{3\delta < |p| < 4\delta\}$, and hence can be extended as proportional to $\psi$ Lyapunov functions for $Z_t$ for all $t \in [0,2]$.

Finally, we observe that $Z_2$ is equal to $Z_{at} = p \frac{\partial}{\partial p}$ on $\{|p| \leq 4\delta\}$, and therefore one can use splitting procedure, as it is described in Section 2.6 to split the Weinstein manifold along the hypersurface $\{|p| = 4\}$ into the block $T^*L_1$ and a $W_c$-manifold with the skeleton $L_{\geq 1}$. Hence the induction hypothesis completes the proof. 

\[\square\]

7. Positivity of cotangent buildings

In this section we introduce the notion of a positive cotangent building and relate their skeleta to positive arboreal spaces.

7.1. Positive cotangent buildings.

7.1.1. Reductions. Let $W$ be a cotangent building with blocks $B_j = \mathcal{F}^*M_j, j \in J$. The notions we introduce in this section will depend on sizes of blocks, so we deviate here from the point of view of germs and take $B_j$ to be a fixed defining domain for $T^*M_j$. Set $B_j^0 = B_j \setminus M_j$, for $j \in J$. For each $i \in J$, consider the line bundle $\operatorname{Span}(Z_i) \subset TB_i^0$ generated by $Z_i$, let $\zeta_i = \operatorname{Span}(Z_i)^\perp / \operatorname{Span}(Z_i) \rightarrow B_i^0$ denote the corresponding symplectic normal bundle, and consider the natural reduction diagram

$$TB_i^0 \xrightarrow{\iota} \operatorname{Span}(Z_i)^\perp \xrightarrow{p} \zeta_i$$

Given a linear subspace $\nu \subset TB_i^0$, let $[\nu]^i := [\nu]^{\operatorname{Span}(Z_i)^\perp} \subset \zeta_i$ be the reduction along the above correspondence

$$[\nu]^i = p(i^{-1}(\nu)) = (\nu + \operatorname{Span}(Z_i)) \cap \operatorname{Span}(Z_i)^\perp / \operatorname{Span}(Z_i).$$

We will refer to $[\nu]^i$ as the $i$-reduction of $\nu$. More generally, given a multi-index $I = (i_1 < \cdots < i_m) \subset J$ (which could be empty), consider the natural reduction diagram

$$T(B_{i_1}^0 \cap \cdots \cap B_{i_m}^0) \xrightarrow{\iota} \operatorname{Span}(Z_{i_1}, \ldots, Z_{i_m})^\perp \xrightarrow{p} \zeta_I$$

$$\zeta_I = \operatorname{Span}(Z_{i_1}, \ldots, Z_{i_t})^\perp / \operatorname{Span}(Z_{i_1}, \ldots, Z_{i_m}).$$
Given a linear subspace \( \nu \subset TB^i \), we refer to \([\nu]^I := [\nu]^\text{Span}(Z_{i_1}, \ldots, Z_{i_\ell}) \perp = p(i^{-1}(\nu)) \subset \zeta_I\) as the \( I \)-reduction of \( \nu \). If \( I = \emptyset \), then \([\nu]^I = \nu\).

7.1.2. **Positivity.** Let \( W = \bigcup_j B_j \) be a cotangent building. Since the sizes of the blocks \( B_j \) are fixed, we can assign a type to any point of \( \text{Skel}(W) = \bigcup_j \hat{M}_j \) as follows:

**Definition 7.1.** We say that a point \( a \in \hat{M}_i \) is of type \( I = (i_1 < \cdots < i_m) \), where \( i_m < i \), if \( a \in B_{i_1} \cap \cdots \cap B_{i_m} \) and \( a \notin B_j \), for \( j \notin I \) and \( j < i \).

Note that we allow the type \( I \) to be empty. For any multi-index \( I = (i_1 < \cdots < i_m) \subset J \) and integer \( s \), where \( 1 \leq s \leq m \), denote \( I(s) = (i_s < \cdots < i_t) \subset I \).

**Definition 7.2.** A cotangent building \( W = \bigcup_{j=1}^k B_j \) is positive if for any point \( a \in \hat{M}_j \) of type \( I = (i_1 < \cdots < i_m) \), \( I \neq \emptyset \) and any \( s \leq m \) the tuple of Lagrangian planes

\[
[T_a M_j]^I(s), [\nu_j(a)]^I(s), [\nu_{i_m}(a)]^I(s), \ldots, [\nu_{i_s}(a)]^I(s)
\]

is \( \prec \)-cyclically ordered.

**Definition 7.3.** Let \( W \) be a cotangent building \( W \) with the notation as above. A Lagrangian distribution \( \eta \) on \( W \) is called positive if it is transverse to all the 0-sections \( M_i \) and at any point \( a \in \hat{M}_i \) of type \( I \) (which could be empty), we have \([\nu_i(a)]^I \in C([T_a M_i]^I, [\eta(a)]^I)\).

7.1.3. **Existence and uniqueness of positive distributions.** Positive distributions exist and are unique up to a contractible choice on any positive cotangent building. Before we prove this we need two lemmas.

**Lemma 7.4.** Let \( B = \mathcal{T}^* M \) be a cotangent block and \( \tau \) any Lagrangian distribution extending \( TM \). Let \( \Sigma \subset \mathcal{T}^* M \) be a conormal of a co-oriented hypersurface \( H \subset M \). For any Lagrangian
plane $T \in T_aB$, $a \in B \setminus M$ denote by $\overline{T} := [T]^{\operatorname{Span}(Z)^\perp}$ the reduction with respect to the conormal of the Liouville field $Z$. Then the angle between $\overline{T\Sigma}(a)$ and $\overline{\tau}(a)$ converges to $0$ as $a \to M$.

Proof. The statement is local so we can assume that $M = \mathbb{R}^n$, $H = \{q_n = 0\}$, $a = 0$ and $\Sigma = \{q_n = 0, p_j = 0, j < n\}$. Moreover, it is sufficient to prove the lemma for a single particular extension $\tau$, and hence we can choose $\tau = \operatorname{Span}(\partial_{q_1}, \ldots, \partial_{q_n})$. The Liouville field is $Z = \sum_i p_i \partial_{p_i}$, hence $Z|_{\Sigma} = p_n \partial_{p_n}$. Therefore $\operatorname{Span}(Z)^\perp|_{\Sigma} = \operatorname{Span}(\partial_{q_1}, \ldots, \partial_{q_{n-1}}, \partial_{p_1}, \ldots, \partial_{p_n})$, $T\Sigma \cap Z^\perp|_{\Sigma} = \operatorname{Span}(\partial_{q_1}, \ldots, \partial_{q_{n-1}}, \partial_{p_n})$ and $\tau \cap Z^\perp|_{\Sigma} = \operatorname{Span}(\partial_{q_1}, \ldots, \partial_{q_{n-1}})$ so in fact $\overline{T\Sigma}$ and $\overline{\tau}$ are identical along $\Sigma$. \hfill \Box

Lemma 7.5. Let $B_1 = \mathcal{F}^* M_1$ be the base block of a positive cotangent building $W$ and $\eta$ a distribution on $\mathcal{O}p M_1$ such that at any point $a \in M_1$ the triple $T_a M_1, \nu_1(a), \eta(a)$ is cyclically $\prec$-ordered. Then $\eta$ is positive for $W$ on $\mathcal{O}p M_1$.

Proof. Take any extension $\tau$ of the distribution $TM_1$ to $B_1 = \mathcal{F}^* M_1$. For any point $a \in B_1 \cap \dot{M}_j$ of type $I = (i_1 = 1 < i_2 < \cdots < i_k), i_k < j$, the tuple

$$[T_a M_j]^I, [\nu_j(a)]^I, \ldots, [\nu_1]^I$$

is cyclically $\prec$-ordered. On the other hand, if the block $B_1$ is sufficiently thin then the triple

$$[\tau(a)]^I, [\nu_1(a)]^I, [\eta(a)]^I$$

is also cyclically $\prec$-ordered. Hence Lemma 7.4 implies that

$$[T_a M_j]^I, [\nu_1(a)]^I, [\eta(a)]^I$$

is cyclically $\prec$-ordered as well. Therefore,

$$[T_a M_j]^I, [\nu_j(a)]^I, \ldots, [\nu_1]^I, [\eta(a)]^I$$

is cyclically $\prec$-ordered, i.e. $\eta$ is positive for $W$ on $B_1$. \hfill \Box

Let us denote by $\text{Pos}(W)$ the space of positive distributions on a positive cotangent building $W = \bigcup_j B_j$.

Proposition 7.6. For any positive cotangent building $W$ the space $\text{Pos}(W)$ is non-empty and (weakly) contractible.

Proof. We will show that $\text{Pos}(W) \neq \emptyset$ using the convexity of the space of positive definite quadratic forms. The weak contractibility claim is a parametric version of the same argument. We argue by induction on the number of blocks.

For $W_{\leq 1} = \mathcal{F}^* M_1$ a positive distribution from $\text{Pos}(W)$ restricted to $M_1$ can be viewed as a field of positive definite quadratic forms on the canonical polarization $\nu_1$. Take any such field $\eta$ and its arbitrary extension to $\mathcal{F}^* M_1$. According to Lemma 7.5 the distribution $\eta$ satisfies the positivity condition for $W$ if the block $B_1$ is chosen sufficiently thin.
Suppose that we have already constructed \( \eta \) on \( W_{<m} = \bigcup_{j<m} B_j \) and let us consider the next block \( B_m = \mathcal{F}^* M_m \). Recall that \( M_m \) is a manifold with corners. Its corners are enumerated by the types of points in their neighborhood, i.e. if \( b \in \partial_k M_m \) then points \( a \in \tilde{M}_m \) in a sufficiently small neighborhood of \( b \) have a fixed type \( I = (i_1 < \cdots < i_k) \), where \( i_k < m \). The Liouville fields \( Z_{ij} \) of blocks \( B_{ij}, j = 1, \ldots, k \), are tangent to \( \tilde{M}_m \) and yield the canonical splitting \( \mathcal{O}_p \tilde{M}_m b = \mathcal{O}_p \partial_k M_m \times \mathcal{I}^k \). Consider the restriction of \( \eta \) to \( \tilde{M}_m \cap W_{<m} \). It is transverse to \( M_m \) and can therefore be viewed as a field of quadratic forms on \( \nu_m \). The positivity condition means that:

\[(\dagger) \text{ for any point } a \in \tilde{M}_m \text{ of type } I, \text{ the quadratic form } \eta(a) \text{ is positive definite on the subspace of } \nu_m(a) \text{ dual to } \operatorname{Span}(Z_{i_1}(a), \ldots, Z_{i_k}(a)).\]

Recall that the notion of type depends on the thickness of the blocks \( B_1, \ldots, B_{m-1} \). Let us take slightly thinner blocks \( B'_i \subset B_i \) and choose a Lagrangian field along \( \tilde{M}_m \) that

\[\begin{align*}
(i) & \text{ coincides on } \tilde{M}_m \cap \left( W'_{<m} := \bigcup_{i=1}^{m-1} B'_i \right) \text{ with the restriction of } \eta \text{ from } W'_{<m}, \\
(ii) & \text{ is given by a field of positive definite quadratic forms on } \nu_m.
\end{align*}\]

Note in particular that (ii) implies that condition \((\dagger)\) holds on \( W_{<m} \cap \tilde{M}_m \). Then choosing any extension to \( \mathcal{O}_p \tilde{M}_m \), and choosing the block \( B_m \) sufficiently thin, we get the required positive distribution on \( W_{\leq m} \). \( \square \)

### 7.2. Positive cotangent buildings and positive arboreals.

We prove in this section that the skeleton of a positive cotangent building is generically a positive arboreal, where throughout this section generically means after a \( C^\infty \)-small perturbation of the building structure. This corresponds to a \( C^\infty \)-small perturbation of the underlying Weinstein structure. Conversely, if a cotangent building has a positive arboreal skeleton, we show the building structure can be adjusted to be made positive without changing the skeleton.

#### 7.2.1. From positive buildings to positive arboreals.

The key definition is the following:

**Definition 7.7.** Given an arboreal Lagrangian \( L \) in a symplectic manifold \( X \), a Lagrangian distribution \( \eta \) along \( L \) is called positive with respect to \( L \) if the following condition is satisfied. Take any singular point \( a \in L \). Let \( T \) be the tangent space to the root Lagrangian at \( a \), and \( T' \) a tangent plane to any other smooth piece adjacent to \( a \). Then \( T' \in C(T, \eta(a)) \).

We will later have use for the following reduced variant:

**Definition 7.8.** Let \( L \subset X \) be a positive arboreal Lagrangian and \( Z \) a non-zero Liouville vector field tangent to \( L \). A Lagrangian distribution \( \mu \) along \( L \) is called reduced positive with respect to \( (L, Z) \) if for any singular point \( a \in L \) the following condition is satisfied. Let \( T \) be the tangent space to the root Lagrangian at \( a \), and \( T' \) a tangent plane to any other smooth piece adjacent to \( a \). Then \( [T']^\zeta \in C([T]^\zeta, [\eta(a)]^\zeta) \), where \( \zeta = \operatorname{Span}(Z)^{1/2} \).

Theorem 5.5 implies:
Lemma 7.9. Let $W$ be a $W$-complex $B_k \rightarrow \cdots \rightarrow B_0$ and denote by $\nu_j$ the polarization of the block $B_j$, $j = 0, \ldots, k$.

(i) If the skeleton $\text{Skel}(W)$ is arboreal, then for each point $a \in \hat{B}_j \cap M_i$, $0 \leq j < i \leq k$, the Lagrangian distribution $[\nu_j]^i$ is transverse to $[TM_j]^i$. Conversely, if for each point $a \in \hat{B}_j \cap M_i$, $0 \leq j < i \leq k$, the Lagrangian distribution $[\nu_j]^i$ is transverse to $[TM_j]^i$ then $\text{Skel}(W)$ is generically arboreal.

(ii) If the skeleton $\text{Skel}(W)$ is positive arboreal, then for each point $a \in \hat{B}_j \cap M_i \cap M_\ell$, $0 \leq j < i < \ell \leq k$, the triple $[T_a M_i]^N, [T_a M_\ell]^N, [\nu_j]^N$, where $N := \text{Span}(T_a M_i, T_a M_\ell)$, is $\prec$-cyclically ordered. Conversely, if for each point $a \in \hat{B}_j \cap M_i \cap M_\ell$, $0 \leq j < i < \ell \leq k$, the triple $[T_a M_i]^N, [T_a M_\ell]^N, [\nu_j]^N$ is $\prec$-cyclically ordered, then $\text{Skel}(W)$ is generically positive arboreal.

Proof. The top block $B_k$ is attached to $B_{k-1}$ along a ribbon of a smooth Legendrian, and the transversality condition implies that the front projection to $M_{k-1}$ is an immersion. Generically it has transverse self-intersections, and hence $\text{Skel}(W_{\geq k-1})$ is arboreal in the case (i) and positive arboreal in the case (ii). Continuing by induction we will assume that $\text{Skel}(W_{\geq j})$ is arboreal. Then the skeleton of the attaching hypersurface $V_j$ of $W_{\geq j}$ to $B_j$ is arboreal as well. By Theorem 5.5, the transversality condition then implies that generically $\text{Skel}(W_{\geq j})$ is arboreal, and in the case (ii) is positive arboreal. The converse direction is straightforward. □

Corollary 7.10. The skeleton of a positive cotangent building is generically a positive arboreal.

7.2.2. More on polarizations. We will need the following lemma from linear algebra.

Lemma 7.11. Let $\tau, \nu$ denote 0-section and the cotangent fiber in $T^*\mathbb{R}^n$. Given a collection of hyperplanes $\Pi = \{\Pi_1, \ldots, \Pi_k\} \subset \mathbb{R}^n$ denote by $\nu_1, \ldots, \nu_k$ their Lagrangian conormals, and set $P_j := \text{Span}(\tau, \nu_j)$. Denote by $\mathcal{P}(\Pi)$ the space of Lagrangian planes $\eta$ transverse to $\tau$ such that the triples of lines $([\tau]^j, [\eta]^j, [\nu]^j)$ are $\prec$-cyclically ordered for all $j = 1, \ldots, k$. Denote by $\mathcal{P}_+$ the subspace of $\mathcal{P}(\Pi)$ consisting of those Lagrangian planes such that the triple $(\tau, \nu, \eta)$ is $\prec$-cyclically ordered. Then $\mathcal{P}(\Pi)$ and $\mathcal{P}_+ \subset \mathcal{P}(\Pi)$ are non-empty convex subsets of the affine space of Lagrangian planes transverse to $\tau$.

Proof. Any Lagrangian plane transverse to $\tau$ can be viewed as a quadratic form on $\nu$. The subspace $\mathcal{P}_+$ consists of positive definite quadratic forms, while $\mathcal{P}(\Pi)$ consists of quadratic forms which take positive values on the covectors $A_j \in \nu = \tau^*$ which define the hyperplanes $\Pi_j$, $j = 1, \ldots, k$. Both spaces are non-empty and convex. □

Below in this section we view polarizations of a Lagrangian as transverse Lagrangian foliations on its neighborhood. Similarly, we assume that polarizations of a Legendrian $\Lambda$ are extended to its ribbon, i.e. a given embedding of $\mathcal{T}^*\Lambda$ as a Weinstein hypersurface.

Lemma 7.12. Let us fix a reference polarization $\nu$ of a Lagrangian $L_0 \subset (X, \omega)$. Then the space $\text{Pol}(L_0)$ of all polarizations on $L_0$ can be viewed as the space of germs along $L_0$ of
symplectomorphisms \( T^*L_0 \to T^*L_0 \) which fix \( L_0 \), or alternatively as the space of germs along \( L_0 \) of functions on \( T^*L_0 \) which vanish on \( L_0 \) together with their differential.

**Proof.** For any polarization \( \eta \in \text{Pol}(L_0) \) there exists a unique symplectomorphism of \( T^*L_0 \) which fixes \( L_0 \) and which sends \( \nu \) onto \( \eta \) as foliations. Each function on \( H \) on \( T^*L_0 \) vanishing on \( L_0 \) together with its differential generates a symplectomorphism fixing \( L_0 \) as the time 1 map of its Hamiltonian flow. Conversely, each symplectomorphism germ \( \phi \) fixing \( L_0 \) defines the required function \( H \) by the conditions \( \phi^*pdq = pdq + dH, \, H|_{L_0} = 0 \).

Let \( L = C(\Lambda) \cup L_0 \) be an arboreal Lagrangian, where \( \Lambda \subset \partial_{\infty} T^*L_0 \) is an arboreal Legendrian. A polarization \( \eta \) is called **positive** for \( L \) if for any point \( \lambda \in L \cap \overline{C(\Lambda)} \) and any Lagrangian plane \( T \in T_0(T^*L_0) \) tangent to one of smooth pieces of \( L \) different from \( L_0 \) the triple \( ([T\lambda L_0]^N, [T]^N, [\eta(\lambda)]^N) \) is \( \prec \)-cyclically ordered, where \( N = \text{Span}(T\lambda L_0, T) \). Such a polarization is automatically transverse to \( L \) on \( \text{Op} \, L_0 \) (comp. Lemma 3.17).

**Lemma 7.13.** Let \( L = C(\Lambda) \cup L_0 \) be an arboreal Lagrangian, where \( \Lambda \subset \partial_{\infty} T^*L_0 \) is an arboreal Legendrian and \( \eta \) a positive polarization of \( L_0 \). Denote by \( \text{Pol}(L, \eta) \) the space of polarizations \( \mu \) of \( L_0 \) for which the corresponding Liouville field is tangent to the cone \( C(\Lambda) \) and the triple \( (T\lambda L_0, \mu(\lambda), \eta(\lambda)) \) is \( \prec \)-cyclically ordered for any \( \lambda \in L_0 \). Then \( \text{Pol}(L, \eta) \) is non-empty and convex.

**Proof.** Viewing \( \mu \in \text{Pol}(L, \eta) \) as a function on \( T^*L_0 \) the condition \( \mu \in \text{Pol}(L, \eta) \) means that \( \mu \) vanishes on \( L \) and that the quadratic part of \( \eta - \mu \) along \( L_0 \) is positive definite on cotangent fibers. Hence, the convexity claim for \( \text{Pol}(L, \eta) \) is straightforward. Let us show that the space \( \text{Pol}(L, \eta) \) is non-empty. Take any polarization \( \eta' \) such that the triple \( (T\lambda L_0, \nu(\lambda), \eta'(\lambda)) \) is \( \prec \)-cyclically ordered for all \( \lambda \in L_0 \). In particular, \( \zeta' \) is transverse to \( L \) on \( \text{Op} \, L_0 \).

The tangent cone to \( L \setminus L_0 \) at a point \( \lambda \in L_0 \) is the union of Lagrangian conormals \( \nu_{j,\lambda} \) to a collection \( T_{j,\lambda} \) of transverse hyperplanes \( \Pi_{j,\lambda} \subset T\lambda L_0, \, j = 1, \ldots, k \). Using Lemma 7.11 we can find a homotopy \( \eta_t \) connecting \( \eta_0 = \eta \) and \( \eta_1 = \eta' \) such that the triple \( ([T\lambda L_0]^N, [\nu]_j^{N_{j,\lambda}}, [\eta_t]^{N_{j,\lambda}}) \) is \( \prec \)-cyclically ordered for all \( 0 \leq t \leq 1 \). Here we denoted \( N_{j,\lambda} := \text{Span}(T\lambda L_0, \nu_{j,\lambda}) \).

In particular, \( \eta_t \) is transverse to the germ of \( L \) along \( L_0 \) for all \( t \in [0, 1] \). There exists a Hamiltonian isotopy \( \psi_t \) defined on \( \text{Op} \, L_0 \) such that \( \psi_t|_{L_0} = L_0 \) and \( \psi_t(\eta_t) = \eta, \, t \in [0, 1] \). Denote \( L^t := \psi_t(L) \). Using Corollary 5.14 we can construct for each point \( \lambda \in L \) a Hamiltonian isotopy \( \phi_{t,\lambda} \) on \( \text{Op} \lambda \) such that

(i) \( \phi_{t,\lambda}|_{L_0 \cap \text{Op} \lambda} = \text{Id} \);
(ii) \( \phi_{t}(\eta|_{\text{Op} \lambda}) = \eta|_{\text{Op} \lambda} \);
(iii) \( \phi_{t,\lambda}(L^t \cap \text{Op} \lambda) = L \cap \text{Op} \lambda \).

Then the polarization \( \mu_{\lambda} := \phi_{1,\lambda} \circ \psi_1(\mu) \) belongs to the space \( \text{Pol}(L \cap \text{Op} \lambda, \eta|_{\text{Op} \lambda}) \). In view of compactness of \( L_0 \) we can cover \( L_0 \) by finitely many balls \( U_j, \, j = 1, \ldots, N \), centered at the points \( \lambda_j \) such that \( \mu_{\lambda_j} \in \text{Pol}(L \cap U_j, \eta|_{U_j}) \). Given a partition of unity \( \sum_1^N \theta_j = 1 \) subordinate to the covering, we define the required element of \( \text{Pol}(L, \eta) \) by the formula \( \mu = \sum_1^N \theta_j \mu_{\lambda_j} \).
7.2.3. From positive arboreals to positive buildings.

**Proposition 7.14.** If the skeleton of a cotangent building is positive arboreal, then the cotangent building structure can be deformed to be positive without changing the skeleton. Similarly, if \( \eta \) is a Lagrangian distribution which is positive for the skeleton, then it can be made positive for the building without changing the skeleton.

**Proof.** We argue by induction on blocks beginning with the bottom block \( B_1 \). For a cotangent building consisting of a single block \( B_1 = \mathcal{T}^* M_1 \) the positivity condition is vacuous and to make \( \eta \) positive we just need to choose the polarization \( \nu_1 \), so that along \( M_1 \) we have \( \nu_1 \in C(TM_1, \eta) \).

Suppose we have already constructed a positive building structure on \( W_{<j} = \bigcup_{i<j} B_i \) such that \( \eta|_{W_{<j}} \) is positive. Let \( L = \text{Skel}(W) \) and denote \( L_j := L \cap (B_j = T^* M_j) \). Suppose the block \( B_j \) intersects with blocks \( B_{j_1}, \ldots, B_{j_m}, 1 \leq j_1 < \cdots < j_m < j \), and let \( P_i \) be the face of \( M_j \) such that \( B_j \) is attached to \( B_{j_i} \) along a ribbon \( V_i = \mathcal{T}^* P_i \subset B_{j_i} \setminus M_{j_i} \). The arboreal Lagrangian \( L_j \) intersects the ribbon \( V_i \) along an arboreal Legendrian \( \Lambda_i \) containing \( P_i \). Denote by \( \mu_i \) the polarization of \( P_i \) induced by the polarization \( \nu_j \) of \( M_j \) and by \( \nu_j \) the polarization obtained by reducing the polarization \( \nu_{j_i} \) of \( B_{j_i} \) with respect to the symplectic conormal of the Liouville field \( Z_j \).

Arguing by induction beginning with \( i = m \), we first use Lemma 7.13 to deform \( \mu_m \) to a polarization \( \mu'_m \) of \( P_m \) such that its Liouville field is tangent to \( \Lambda_j \) and the triple \((TP_m, \mu'_m, \nu_j)\) is \( \prec \)-cyclically ordered. We extend \( \mu'_m \) to \( B_j \) as a polarization such that its Liouville field is tangent to \( L_j \), which we will still denote by \( \nu_j \), forgetting the previous polarization of \( M_j \). Note that on \( Op P_m \cap P_{m-1} \) the reduced triple \( ([TP_{m-1}]_{\xi_{m-1}}, [\mu_{m-1}]_{\xi_{m-1}}, [\nu_{j_{m-1}}]_{\xi_{m-1}}) \) is \( \prec \)-ordered, thanks to the positivity of the building \( W_{<j} \). Applying Lemma 7.13 again, we can deform the polarization \( \mu_{m-1} \) to a polarization \( \mu'_{m-1} \) whose Liouville field is tangent to \( \Lambda_{j-1} \), which coincides with \( \nu_j := [\nu_j]_{\xi_{m-1}} \) on a neighborhood \( U \supset P_m \cap P_{m-1} \), and such that the triple \((TP_{m-1}, \mu'_{m-1}, \nu_{j_{m-1}})\) is \( \prec \)-cyclically ordered outside \( U \). The extension claim follows from convexity: one first constructs \( \mu'_{m-1} \) satisfying the positivity condition in the complement of a neighborhood \( U' \subset U \). Then, using a cut-off function \( \theta \) equal to 1 on \( U' \) and 0 outside \( U \), one redefines \( \mu_{m-1} \) as \( (1-\theta)\mu'_{m-1} + \theta \nu_j \). Again we extend to a polarization of \( B_j \) such that its Liouville field is tangent to \( L_j \) and we still call the new polarization \( \nu_j \). Continuing this process, we obtain a polarization \( \nu_j \) on \( B_j \) making the building \( W_{<j} \) positive. Finally, to ensure that \( \eta \) is positive along \( M_j \) we can further deform \( \nu_j \) via a homotopy fixed on \( W_{<j} \cap B_j \) by final application of Lemma 7.13. \( \square \)

8. Ridgification of Lagrangians

In this section we recall the ridification theorem and its formal analogue, and describe the canonical \( Wc \)-structure in the neighborhood of a ridgy Lagrangian.

8.1. Geomorphology of Lagrangian ridges.
8.1.1. **Ridgy Lagrangians.** As a stepping stone towards arboreal skeleta it will be useful to consider a class of singular Lagrangian and Legendrian submanifolds, called *ridgy*, which were introduced in [AGEN19]. Let us recall their definition. In the standard symplectic \((\mathbb{R}^2, dx \wedge dy)\) consider the subset \(R = R_{1,2} = \{xy = 0, x \geq 0, y \geq 0\}\).

**Definition 8.1.** The *model ridge* \(R_{k,n} \subset \mathbb{R}^{2n}\) of order \(k\) is the product \(R_{k,n} = R^k \times \mathbb{R}^{n-k} \subset (T^*\mathbb{R})^k \times T^*\mathbb{R}^{n-k} = T^*\mathbb{R}^n\).

**Example 8.2.** The order \(n\) model ridge \(R_{n,n} \subset T^*\mathbb{R}^n\) is the union to all the inner conormals of the faces of a quadrant in \(\mathbb{R}^n\), hence is the union of the \(2^n\) linear Lagrangians \(\{p_j = q_k = 0, q_j, p_k \geq 0, j \in I, k \notin I\}\), where \(I \subset \{1, \ldots, n\}\).

**Definition 8.3.** An \(n\)-dimensional *ridgy Lagrangian* in a symplectic manifold \((X, \omega)\) is a closed subset \(L \subset X\) which is covered by open neighborhoods \(U_i\) such that each \((U_i, U_i \cap L)\) is symplectomorphic to some \((B, B \cap R_{k,n})\), \(0 \leq k \leq n\).

![Figure 8.1.](image-url) A 2-dimensional ridgy Lagrangian has order 1 ridges along a union of separating curves, which intersect in order 2 ridges along a discrete set of points.

A ridgy Lagrangian \(L\) admits a natural stratification \(L = \bigsqcup_{j=0}^{n} L_j\) according to the order of ridges, so that \(L_0\) is the smooth part of the ridgy Lagrangian. Any ridgy Lagrangian can be viewed as the limit as \(\varepsilon \to 0\) of a family of smooth Lagrangians \(L(\varepsilon)\) homeomorphic to \(L = L(0)\); they are obtained by using the model \(R(\varepsilon) = \{xy = \varepsilon, x \geq 0, y \geq 0\}\) instead of \(R\).

**Definition 8.4.** Given a contact manifold \((Y, \xi)\), a ridgy Legendrian \(\Lambda \subset \xi\) is defined as a ridgy Lagrangian in a Weinstein hypersurface \(\Sigma \subset Y\).

Note that \(R_{k,n} \subset \mathbb{R}^{2n}\) is invariant with respect to the radial contracting vector field \(Z_n = \sum_{j=1}^{n} q_j \partial/\partial q_j - p_j \partial/\partial p_j\). Hence its link \(\partial R_{k,n} := R_{k,n} \cap S^{2n-1}\) in the standard contact sphere is an \((n-1)\)-dimensional ridgy Legendrian. Similarly, \(R_{k,n} \subset \mathbb{R}^{2k} \times T^*\mathbb{R}^{n-k}\) is invariant with respect to the contracting field \(Z_k = \sum_{j=1}^{n-k} p_j \partial/\partial p_j\), where \(\sum_{j=1}^{n-k} p_j \partial/\partial p_j\) is the canonical Liouville field for the factor \(T^*\mathbb{R}^{n-k}\).
Figure 8.2. The 2-fold model ridge $R^2 = R \times R \subset T^*R \times T^*R$ is symplectomorphic to the union in $T^*R^2$ of the first quadrant in $R^2$, the inner conormals to the positive $x$ and $y$ axes and the quarter-conormal to the origin lying in the first quadrant.

8.1.2. The formal ridgification theorem. First, we recall from [AGEN19] the notion of a tectonic field, which is the formal (i.e. non-integrable) analogue of a ridgy Lagrangian. Let $M$ be a manifold with corners and $B = T^*M$ the corresponding cotangent block. Recall that a Lagrangian plane field $\eta$ along $M$ which is transverse to the vertical distribution $\nu$ can be viewed as a field of quadratic forms on $TM$. To simplify the notation we will use the same symbol to denote the Lagrangian distribution and the corresponding field of quadratic forms.

By a dividing hypersurface $N \subset M$ we mean a properly embedded, co-oriented codimension 1 submanifold with corners such that $(N, \partial N) \subset (M, \partial M)$ is homologically trivial. A collection of dividing hypersurfaces $\{N_j\}_j$ is said to be in general position if each $N_j$ is transverse to the intersection $N_i \cap \cdots \cap N_k$ of any subset of the other hypersurfaces.

Given a collection of dividing hypersurfaces $N_1, \ldots, N_k \subset M$ in general position, a tectonic field $\lambda$ with faults along $\{N_j\}_j$ is a collection of smooth graphical Lagrangian plane fields $\{\sigma_Q\}_Q$ defined over the closures $\overline{Q}$ of the connected components $Q$ of $M \setminus \bigcup_{j=1}^k N_j$, such that for every $j = 1, \ldots, k$ there exists a field $\ell_j$ of of non-vanishing 1-forms along $N_j$ for which the following conditions are satisfied:

(i) For each component $P$ of $N_i \setminus \bigcup_{j \neq i} N_j$ adjacent to components $Q_\pm$ of $M \setminus \bigcup_j N_j$ we have that the difference $\lambda_{Q_+} - \lambda_{Q_-}$ is the rank 1 quadratic form $\eta_i = \ell_i^2$, where the co-orientation of $N_j$ points into $Q_+$;

(ii) Along each intersection $N_{j_1} \cap \cdots \cap N_{j_m}$ the hyperplane fields $\tau_{j_s} = \ker(\eta_{j_s})$, $s = 1, \ldots, m$, are transverse to all possible intersections of the $\tau_{js}$, $r \neq s$.

A tectonic field $\eta$ is called $\varepsilon$-small if it deviates from $TM$ by an angle $< \varepsilon$, where a fixed but arbitrary Riemannian metric is understood. A tectonic field $\lambda$ is called collared if for each $k$-face $F$ of $M$ the tectonic field $\lambda$ splits as a product of a tectonic field on $F$ and the trivial (tangent) field in the collar directions. In terms of quadratic forms, this means that the quadratic form is the sum of a quadratic form on $TF$ and the zero form in the collar directions.
directions. In particular this gives the notion of a smooth Lagrangian plane field which is collared with respect to a corner structure. We can now state the formal version of the ridgification theorem.

**Theorem 8.5.** Let $M$ be a manifold with corners and $\eta$ any collared Lagrangian plane field in $T^*M$. Then for any $\varepsilon > 0$ there exists a collared $\varepsilon$-small tectonic field $\lambda$ transverse to $\eta$. If for a compact set $A \subset M$ we are already given an $\varepsilon$-small tectonic field $\lambda_0$ over $O_p A$ transverse to $\eta$, then $\lambda$ can be chosen equal to $\lambda_0$ over $O_p A$.

8.1.3. **The ridgification theorem.** Given a tectonic field $\lambda$ in $T^*M$, a ridgy Lagrangian $L \subset T^*M$ is called $\delta$-close to $\lambda$ if for any point $a \in L$ and any tangent plane $T$ to $L$ at $a$ there exists a non-fault point $b \in M$, $\delta$-close to $a$ such that the angle between $\lambda(b)$ and the plane $T$ parallel transported to $b$ along a geodesic is $< \delta$. As before, a fixed but otherwise arbitrary Riemannian metric on $M$ is understood.

A ridgy Lagrangian in $L \subset T^*M$ is called **collared** if for each collar $Q \times I^k \subset M$, where $Q$ is a $k$-face, we have $L \cap T^*(Q \times I^k) = L_Q \times I^k$ for $L_Q \subset T^*Q$ a ridgy Lagrangian and $I^k \subset T^*I^k$ the zero section.

**Remark 8.6.** If $L \subset T^*M$ is a collared ridgy Lagrangian, then it is adapted for the $Wc$-structure of $T^*M$ inherited from the collar structure of $M$.

The ridgy Lagrangians we will consider are of a special type: they are all obtained from the zero section $M \subset T^*M$ by means of a ridgy isotopy, which is defined as follows.

(i) Let $N_1, \ldots, N_m \subset M$ be co-oriented separating hypersurfaces defined by equations $\phi_j = 0$ for some $C^\infty$-functions $\phi_j : M \to \mathbb{R}$ without critical points on $N_j$. We assume that the $N_j$ are co-oriented by the outward transversals to the domains $\{ \phi_j \leq 0 \}$. Denote $\phi_j^+ = \max(\phi_j, 0)$ and choose a cut-off function $\theta_j$ which is equal to 1 on $N_j$ and to 0 outside a neighborhood of $N_j$. Define a function $\Phi : M \to \mathbb{R}$ (which is $C^1$ and piecewise $C^\infty$) by the formula

$$\Phi := \sum_{j=1}^m \theta_j \left( \phi_j^+ \right)^2.$$

An earthquake isotopy with faults $N_j$ is defined as a family of Lagrangians $L_t \subset T^*M$ given by the homotopy of generating functions $t\Phi$, i.e. $L_t = \{ p = td\Phi \}$, $t \geq 0$.

(ii) A **ridgy isotopy** is an earthquake isotopy followed by an ambient Hamiltonian isotopy.

Note that $L_0 = M$ and the earthquake isotopy can be realized by an ambient Hamiltonian homotopy beginning from any $t > 0$. We can now state the integrable version of the ridgification theorem.

**Theorem 8.7.** For any collared plane field $\eta$ in $T^*M$ and any $\delta > 0$ there exists a collared ridgy Lagrangian $L$ such that

(i) $L$ is ridgy isotopic to $M$ with the Hamiltonian isotopy $\delta$-close to the identity in the $C^0$-norm;
(ii) \( L \) is transverse to \( \eta \).

If \( \eta \) is transverse to \( M \) on \( \mathcal{O}p\ A \) for \( A \subset M \) a closed subset, then we may assume that the ridgy isotopy is constant on \( \mathcal{O}p\ A \).

Theorems 8.5 and Theorem 8.7 are proved in [AGEN19].

8.2. \textit{Wc-structures for ridgy Lagrangians.}

8.2.1. \textit{Wc-building structure associated with a ridgy Lagrangian.} Next we discuss a Darboux/Weinstein type theorem for the symplectic structure in the neighborhood of a ridgy Lagrangian. Denote by \( L_k \) the locus of \( k \)-dimensional locus of \((n-k)\)-fold ridges in \( L \). This stratifies \( L \) as a union of smooth, relatively open submanifolds \( L = L_0 \cup \cdots \cup L_n \).

Given a manifold with corners \( M \) we denote by \( M^{\Delta,j} \) the manifold with corners obtained from \( M \) by truncating all corners of dimension \( \leq j \). Thus, each \( i \)-face \( P \) from \( \partial_iM \) for \( i < j \) yields a 1-face \( P^{\bullet,i} := P^{\Delta,i} \times \Delta^{j-i} \subset \partial_1M^{\Delta,j} \), where we denote by \( \Delta^{j-i} \) an open \((j-i)\)-dimensional simplex. Denote by \( \mathbb{R}^{2k} \) the germ of the standard symplectic \( \mathbb{R}^{2k} \) at the origin.

**Lemma 8.8.** Let \( L \) be a ridgy Lagrangian in a symplectic manifold \((X,\omega)\) such that for each \( k \geq 1 \) the symplectic normal bundle to the order \( k \) ridge locus is trivial. Then a neighborhood of \( L \) admits a structure of a Wc-building \( W = (U_0 \rightarrow \cdots \rightarrow U_n) \), where

\[
U_j := \bigcup_Q \mathcal{T}^*Q^{\Delta,j-1} \times \mathbb{R}^{2j}
\]

and the union is taken over all components \( Q \) of \( L_j \), such that the following properties hold:

(i) The inclusion of \( W \) into \( X \) is a symplectic embedding.

(ii) The skeleton of \( W \) is \( L \).

(iii) The Weinstein structure underlying \( W \) is Weinstein homotopic to the cotangent bundle structure \( \mathcal{T}^*L(\varepsilon) \) for a smoothing \( L(\varepsilon) \) of \( L \).

**Remark 8.9.** The hypothesis on the symplectic normal bundles is included only for the sake of simplicity. It is obviously satisfied by the ridgy Lagrangians produced by the ridgification theorem and these are the only ridgy Lagrangians we care about for our applications. In general one should replace \( \mathcal{T}^*Q^{\Delta,j-1} \times \mathbb{R}^{2j} \) by the appropriate symplectic bundle over \( \mathcal{T}^*Q^{\Delta,j-1} \).
and the result is proved in the same way. Note that in either case the Wc-building structure is not a cotangent building.

**Proof.** Recall that $L_k$ denotes the locus of $k$-dimensional locus of $(n-k)$-fold ridges in $L$. Denote by $U_0$ the union of so small closed disjoint balls $B_j$ centered at the points of $L_0$ that $L \cap B_j$ is invariant with respect to the radial contracting vector field inherited from the local model. Denote $L_j^> := L_j \setminus U_n$. Choose a small (in a similar sense) closed tubular neighborhood $U_1$ of $L_j^>$. Set $L_j^{>1} = L_j^> \setminus U_1$ for $j > 1$.

Continuing this process we define $U_j \supset L_j^{>j-1}$ for $j = 0, 1, \ldots, n$. Note also that $U_j$ can be presented as a symplectic fibration $\pi_j : U_j \to T^* \mathcal{T}_j^{>j-1}$. The restriction $\pi_j |_{\mathcal{T}_j^{>j-1}}$ contains a subfibration $L \cap U_j \to \mathcal{T}_j^{>j-1}$ with the fiber $R^{n-j,n-j} = R^{n-j}$.

The structural group of the fibration $\pi_j$ reduces to the discrete group of symmetries of the model ridge $R^{n-j}$. These symmetries commute with the action of the radial contracting field $Z_j$, and hence $U_j$ admits a global contracting field $Z_j^{>j-1}$ which restricts to the radial field $Z_j$ on fibers over $\mathcal{T}_j^{>j-1}$ and which vanishes along $\mathcal{T}_j^{>j-1}$. This makes $(U_j, Z_j^{>j-1})$ a Wc-manifold.

The required Wc-structure on $\mathcal{O}p L$ can now be constructed by successive vertical gluing. Starting with $(U_0, Z_0)$ we vertically attach $(U_1, Z_1^{>0})$ using a Wc-hypersurface $U_0$, namely the ribbon of $\partial_\infty U_0 \cap L$, and the similar hypersurface in $\partial U_1$ which plays the role of the nucleus of one of the boundaries of $\partial U_1$. Continuing this process we obtained the required Wc-structure.

Properties (i) and (ii) are immediate from the construction and for (iii) the homotopy between the constructed Wc-structure and $T^* L(\varepsilon)$ can be obtained by repeating the whole construction for the smoothed $L(\varepsilon)$ with the stratification inherited from the order of ridges in $L$. Finally the description of the $U_j$ given in the statement of the lemma follows from the usual Darboux/Weinstein theorem for isotropic submanifolds with trivial symplectic normal bundles. □

**8.2.2. Ridgy Lagrangians in Wc-buildings.** Let $W = \bigcup_j B_j$ be a cotangent building, with $\nu_j$ the vertical distribution for $B_j$. Given a Liouville cone $L$ over an adapted Legendrian in $W \setminus \text{Skel}(W)$ we say that $L$ is reduced transverse to all $\nu_i$ defined near its corners if for each

**Figure 8.4.** The Wc-building structure in the neighborhood of a ridgy Lagrangian
tangent plane $T$ to $L \cap B_{i_1} \cap \cdots \cap B_{i_m}$ at a point $a \in B_{i_1} \cap \cdots \cap B_{i_m}$ the reduction $[T]_{\zeta_I} \subset \zeta_I$ is transverse to the reductions $[\nu_j]_{\zeta_I}$, where $I = (i_1 < \cdots < i_m)$.

**Proposition 8.10.** Let $\hat{W}$ be a symplectic manifold and $W \subset \hat{W}$ a symplectic embedding of a positive cotangent building. Let $L \subset \hat{W} \setminus \text{Skel}(W)$ be a Lagrangian with corners such that $L \cap W$ is a Liouville cone over an adapted Legendrian in $W \setminus \text{Skel}(W)$. Let $\nu_{-1}$ be a positive distribution which is extended to $\hat{W}$. Then there exists a $C^0$-small ridgy isotopy which keeps $L \cap W$ adapted and deforms $L$ to a ridgy Lagrangian $\hat{L}$ which is transverse to $\nu_{-1}$ and reduced transverse to all the corresponding distributions $\nu_i$ defined near its corners.

**Proof.** Consider $L$ as the 0-section of a block $T^*L$. Let $P_1$ be a face of $L$ which is a Legendrian in one of the blocks $B_k$. The distribution $\nu_k$ is adapted for this block structure and is invariant (and tangent) to the contracting field $Z_{P_1}$ adapted to this block. We apply the formal ridgification theorem 8.5 to deform $TP_1$ to an $\varepsilon$-small collared tectonic field $\sigma'_1$ in $T^*P_1$, achieving transversality to $[\nu_k]_{\zeta_{P_1}}$. By taking $\sigma_1 := \text{Span}(\sigma'_1, Z_{P_1})$ we lift it to a tectonic field in $T^*L$ given on $Op P_1$. Thus $\sigma_1$ is reduced transverse to $\nu_k$. The tectonic field $\sigma_1$ on $Op P_1$ can be extended as a product to a neighborhood $Op P_1$. We will keep the notation $\sigma_1$ for this extension, which is collared by construction. Note that according to Lemma 3.17, $[\sigma_1]_{\zeta_{P_1}}$ is transverse to $[\nu_j]_{\zeta_{P_1}}$, or in other words, that $\sigma_1$ is reduced transverse to $\nu_j$. Using the extension form of the formal ridgification theorem 8.5 we can further inductively deform the tectonic field $\sigma_1$ into $\sigma_2, \ldots, \sigma_k$ which is split transverse to $\nu_{k-1}, \ldots, \nu_1$, and finally construct a tectonic field $\sigma$ which is in addition is transverse to $\nu_{-1}$. We conclude the proof by using Theorem 8.7 to deform $L$ into a ridgy collared $\hat{L}$ which is $\delta$-close to $\sigma$. If $\delta$ is small enough then $\hat{L}$ is automatically transverse to $\nu_{-1}$ and split transverse to all $\nu_j$. □

9. **Arborealization of skeleta**

In this final section we complete the proof of our main result: a Weinstein manifold admitting a polarization can be deformed to have a positive arboreal skeleton.

9.1. **Immersions into arboreals.** In this section we will present an inductive scheme to arborealize a ridgy Lagrangian. The most direct way of doing so produces an arboreal Lagrangian with boundary, i.e. whose boundary is itself arboreal. However, we can do slightly better by inductively capping off the new boundary components that arise so that the end result is an arboreal Lagrangian with smooth boundary. We therefore begin with a preliminary discussion which describes the capping process.

9.1.1. **Capping arboreal Legendrians.**

**Lemma 9.1.** Let $M$ be a smooth manifold, possibly with boundary and corners, and $\Lambda \subset (T^*M \setminus M) \cap T^*\hat{M}$ an arboreal Legendrian with smooth boundary $\partial \Lambda$. Suppose that $\Lambda$ is positive and that it is also positive with respect to the vertical polarization $\nu$ of $T^*M$. Let $\hat{\Lambda} \supset \Lambda$ be an arboreal space without boundary such that $\hat{\Lambda} \setminus \Lambda$ is a manifold. Suppose that
there exists an immersion \( h : \hat{\Lambda} \to \hat{M} \), transverse to \( \pi(\Lambda) \), extending the co-oriented front projection \( \pi : \mathcal{O}p \partial \Lambda \to \hat{M} \). Let \( H \subset \mathcal{T}^*M \setminus M \) be the Legendrian positive conormal lift of the immersion \( h \). Then the closed arboreal Legendrian \( \tilde{\Lambda} := \Lambda \cup H \subset \mathcal{T}^*M \setminus M \) is positive arboreal and is positive with respect to \( \nu \).

**Proof.** Indeed, we just added to \( \Lambda \) a smooth Legendrian transverse to \( \nu \). \( \square \)

**Remark 9.2.** To be more precise, we should say that there exists a Legendrian lift \( H \subset \mathcal{T}^*M \setminus M \) of \( h \) such that the conclusion of the lemma holds, or equivalently we could work in the idealized boundary \( S^*M \) from the onset, where the lift \( H \) is unique.

9.1.2. **Genuine transversality.** Let \( \eta \) be a Lagrangian distribution in a symplectic manifold \( W \). We introduce a notion of transversality for piecewise smooth Lagrangians which can be intuitively thought of as “transversality even after smoothing”.

**Definition 9.3.** A piecewise smooth Lagrangian \( L \subset W \) is called **genuinely transverse** to \( \eta \) if for each point \( a \in L \) there exist Darboux coordinates \( (p,q) \) near \( a \), with \( \eta(a) \) is tangent to the cotangent fiber \( q = 0 \) at the origin \( 0 \), such that the Lagrangian \( \mathcal{O}p a \subset L \) is generated by a \( C^1 \)-function \( H(q) \).

**Remark 9.4.** The ridgy Lagrangians produced by the ridgification theorem are transverse to the distribution used as input in the theorem, but in general are not genuinely transverse.

Similarly, a piece-wise smooth Lagrangian map \( f : L \to W \), i.e. not necessarily an embedding, is said to be genuinely transverse to a Lagrangian distribution \( \eta \) if for every point \( a \in L \) there exists a neighborhood \( \mathcal{O}p a \subset L \) such that \( f|\mathcal{O}p L \) is a piecewise smooth Lagrangian embedding and \( f(\mathcal{O}p L) \) is genuinely transverse to \( \eta \). Note that any piece-wise smooth Lagrangian which is genuinely transverse to \( \eta \) can be approximated by a family of Lagrangians \( L_t \), \( t > 0 \), which are transverse to \( \eta \) and such that \( L_t \overset{C^0}{\to} L \) as \( t \to 0 \), where convergence is smooth where \( L \) is smooth. Indeed, this follows from the corresponding approximation property for \( C^1 \)-smooth generating functions.

**Definition 9.5.** Let \( L \) be an arboreal space and \( M \) a smooth manifold of the same dimension \( n \). A map \( f : M \to L \) is called an **immersion** if there exists a stratification \( M = M_0 \cup M_1 \cup \cdots \cup M_n \) by manifolds \( M_j \) of dimension \( (n-j) \) such that \( f \) is an immersion on each stratum.

Recall that as an arboreal space, \( L \) is equipped with an orientation structure \( \kappa \), which is a line bundle equipped with identifications with the orientation line bundles of the smooth pieces and compatibility at the singularities determined by symplectic geometry. The pullback \( f^*\kappa \) is a line bundle on \( M \), with fixed identifications with the orientation bundle \( \wedge^nTM_0 \) for \( P \) on \( M_0 \).

**Definition 9.6.** An immersion \( f : M \to L \) is **oriented** if \( f^*\kappa \) is isomorphic to the orientation bundle of \( M \) relative to the fixed identification on \( M_0 \).
The condition that \( f : M \to L \) is oriented can be understood as follows. Let \( A \) be a component of \( M_1 \) and \( P, Q \) be components of \( M_0 \) adjacent to \( A \). At a point \( f(x) \in L \) in the closure of both \( f(P) \) and \( f(Q) \), the singularity of \( L \) is modeled on signed rooted tree \( \mathcal{T} \). If \( f(P) \) is contained in a smooth piece of \( L \) closer to the root \( \rho \) of \( \mathcal{T} \) than \( f(Q) \) then the outward boundary coorientation of \( A \subset \partial P \) coincides with the coorientation of \( f(A) \) in \( P \). See also Remark 6.6.

Let \( L \subset W \) be a positive arboreal Lagrangian and \( \eta \) a positive distribution for \( L \). Then the symplectic vector bundle \( TW|_L \to L \) is isomorphic to \( \eta \oplus \eta^\ast \). We can realize \( \eta^\ast \) as a Lagrangian plane field on \( TW|_L \) which is transverse to \( \eta \) and the space of such is contractible.

**Lemma 9.7.** Let \( L \subset W \) be a positive arboreal Lagrangian and \( \eta \) a positive distribution for \( L \). For any oriented immersion \( f : N \to L \) and any point \( x \in N \), the germ of \( f(L) \) at \( f(x) \) is graphical with respect to the polarization \((\eta^\ast, \eta)\).

*Proof.* We first check the condition along \( M_1 \). Let \( x \in M_1 \), so \( f(x) \in L \) is an \( A_2 \) type singularity. There are two possibilities for the germ of \( f : M \to L \) near \( x \), excluding the trivial case where \( f(\partial P(x)) \) stays in the smooth part of \( L \), i.e. the zero section piece corresponding to the root \( \rho \) of the signed rooted tree \( \mathcal{T} \) which classified the singularity of \( L \) at \( f(x) \). The first possibility is that \( f(\partial P(x)) \) is the conormal of a cooriented hyperplane together with the half-plane lying in the direction of the coorientation. The second possibility is to take the other half-plane. By inspection of the local model, the former is graphical with respect to \((\eta^\ast, \eta)\) and the latter is not. On the other hand, the former gives an oriented immersion while the latter does not. The condition on \( M_k, k > 1 \) can be verified inductively using the above argument, or more directly one can inspect the explicit local model, in which the polarization \( \eta \) can also be taken to be canonical by virtue of Corollary 5.10. \( \square \)

**Proposition 9.8.** Let \( L \subset W \) be a \( n \)-dimensional positive arboreal Lagrangian and \( \eta \) a positive distribution for \( L \). Then for any oriented immersion \( f : N \to L \) of an \( n \)-dimensional manifold \( N \) the piecewise smooth immersion \( N \to L \hookrightarrow W \) is genuinely transverse to \( \eta \).

*Proof.* Near each point \( x \in N \) the germ of \( f(L) \) is graphical with respect to \((\eta^\ast, \eta)\) so near \( x \) we can generate \( f(L) \) by a piecewise smooth function on \( \eta^\ast \). Moreover, the generating function can be smoothed and the result is still transverse to \( \eta \) since it remains graphical with respect to \((\eta^\ast, \eta)\). To achieve a global smoothing of \( L \) one can patch up the local smoothings with a partition of unity. \( \square \)

### 9.1.3. Capping arboreal Lagrangians

Let \( L \) be an arboreal space and \( \Lambda \) a smooth component of its boundary \( \partial L \). We call the component \( \Lambda \) bounding if there exists a manifold \( M \) with \( \partial M = \Lambda \) and an oriented immersion \( M \to L \) extending the inclusion \( \Lambda \hookrightarrow L \).

**Lemma 9.9.** Let \( \Lambda \subset \mathcal{T}^* M \setminus M \) be a positive arboreal Legendrian which is positive with respect to the vertical distribution \( \nu \). Suppose that the boundary of \( \Lambda \) is smooth and bounding. Then there exists a closed arboreal Legendrian \( \hat{\Lambda} \subset \mathcal{T}^* M \setminus M \) which is transverse to \( \nu \) and
satisfies the following properties, where we denote by \( \hat{L} \subset T^*M \) the arboreal Lagrangian formed by the union of the Liouville cone of \( \hat{\Lambda} \) and the 0-section.

(i) \( \Lambda' := \hat{\Lambda} \setminus \Lambda \) is smooth;

(ii) \( \hat{\Lambda} \) bounds in \( \hat{L} \) an immersed submanifold which is genuinely transverse to any Lagrangian distribution \( \eta \) in \( T^*M \) such that \( TM, \nu, \eta \) are \(<\)-ordered.

Proof. Let \( f : N \to \Lambda \) be an immersion bounding \( \partial \Lambda \) in \( \Lambda \). Denote by \( \pi : T^*M \to M \) the front projection. Then the image \( \Sigma := \pi \circ f(N) \subset M \) is a \( C^1 \)-smooth immersed cooriented hypersurface. Hence it has a \( C^\infty \)-smooth, fixed on \( \partial \Sigma \) push-off \( \Sigma' \) in the direction of the co-orientation, which together with \( \Sigma \) bounds an immersed domain \( \Omega \subset M \) with a \( C^1 \)-smooth boundary \( \partial \Omega = \Sigma \cup \Sigma' \). Let \( \Lambda' \) be the conormal lift of \( \Sigma' \). Then \( \hat{\Lambda} := \Lambda \cup \Lambda' \) is the required Legendrian, where property (ii) follows from Proposition 9.8.

9.2. Cones over arboreals. We now discuss the key lemma which will be used for the inductive arborealization of ridgy Lagrangians.
9.2.1. Arborealization of radial cones on arboreals.

**Definition 9.10.** An arboreal Lagrangian in a Wc-manifold $W$ is called *asymptotically conical* if it coincides outside of a neighborhood of $\text{Skel}(W)$ with a Liouville cone over a Legendrian.

Note that the boundary of an asymptotically conical arboreal Lagrangian $L$ is also asymptotically conical. We call a smooth component $C \subset \partial L$ *bounding* if there exists a possibly non-compact manifold $N$ with $\partial N = C$ and an asymptotically conical immersion $f : N \to L$ bounding $C$.

**Lemma 9.11.** Let $\Lambda$ be a positive arboreal Legendrian in $S^{2n-1}$ endowed with the standard contact structure. Let $L \subset \mathbb{R}^{2n}$ be the Lagrangian cone over $\Lambda$ centered at 0, i.e. with respect to the radial Liouville structure on the unit ball $B \subset \mathbb{R}^{2n}$. Let $(\tau, \nu)$ be a polarization of $\mathbb{R}^{2n}$ such that every Lagrangian plane $T$ tangent to $L$ at 0 satisfies the condition $T^{\tau,\nu} > 0$. Let $L_0 = \tau(0)$ be the Lagrangian plane through the origin, and $\Lambda_0 \subset S^{2n-1}$ its Legendrian link. Suppose that each boundary component of $\Lambda$ is bounding. Then there exist:

(i) a closed arboreal Legendrian $\tilde{\Lambda} \supset \Lambda$ such that $\Lambda' := \tilde{\Lambda} \setminus \Lambda$ is smooth and such that every Lagrangian plane $T$ tangent to the Liouville cone $L'$ over $\Lambda'$ satisfies the condition $T^{\tau,\nu} > 0$;

(ii) an asymptotically conical arboreal Lagrangian $\hat{L}$ in $\mathbb{R}^{2n}$ endowed with the standard Liouville structure such that:

- $\hat{L}$ coincides at infinity with the Lagrangian cone over $\Lambda \cup \Lambda_0$;
- $\hat{L} \supset \tilde{\Lambda}$ and $\hat{L} \supset L_0$;
- $\hat{L}$ has smooth boundary and every boundary component is bounding;
- $\hat{L}$ is transverse to any Lagrangian distribution $\eta \in C(\nu, \tau)$, i.e negative with respect to $(\tau, \nu)$.

(iii) a compactly supported homotopy $\lambda_t = \lambda_0 + df_t$ of $\lambda_0 = pdq$ such that the Wc-manifold obtained from $(\mathbb{R}^{2n}, \lambda_1)$ by converting the ribbons of the Legendrians $\Lambda \cup \Lambda_0$ into nuclei of boundary faces has the structure of a positive cotangent building with skeleton $\hat{L}$.

![Figure 9.3](image-url)  

**Figure 9.3.** The goal of Lemma 9.11 when $n = 1$. 
Figure 9.4. The goal of Lemma 9.11 when \( n = 2 \).

\textbf{Proof.} (i) Let \( p, q \) be canonical coordinates for the polarization \( (\tau, \nu) \) Note that the function \( H = pq \) restricts to \( L \setminus 0 \) as a positive function which satisfies the condition \( H|_L \geq c|p|^2 \) for some \( c > 0 \). Take the function \( G = \max(H, \frac{1}{2}c|p|^2) \). Then \( \{H = \varepsilon\} \cap L = \{G = \varepsilon\} \cap L \) for any \( \varepsilon > 0 \). The function \( G \) has star-shaped level sets with corners away from \( L \). Hence, it can be smoothed to a function with the same property keeping the restriction to \( L \) equal to \( pq|_L \). We abuse notation and denote the smoothed function by the same symbol \( G \). Note that in particular we have

\[ (pdq = \frac{1}{2}(pdq - qdp) + d(pq))|_{L \cap \{G=\varepsilon\}} = 0, \]

i.e. \( L \cap \{G = \varepsilon\} \) is Legendrian in the level set \( G = \varepsilon \) for both contact structures, defined by the Liouville forms \( pdq \) and \( \frac{1}{2}(pdq - qdp) \). Consider the Liouville form

\[ \lambda = \frac{1}{2}(pdq - qdp) + \frac{1}{2}d(\alpha(G)) = \frac{1}{2}(pdq - qdp) + \frac{1}{2}\alpha'(G)d(pq), \]

where \( \alpha : \mathbb{R}_+ \to [0, 1] \) a monotone cut-off function which is equal to the identity on \( [0, \varepsilon] \) and to 0 on \( [\varepsilon := \varepsilon + \delta, \infty) \) for \( \delta \ll \varepsilon \). Then \( \lambda = pdq \) in \( \{G < \varepsilon\} \) and \( \lambda = \frac{1}{2}d(pdq - qdp) \) in \( \{G \geq \varepsilon\} \).

Let \( \tilde{L} = \tilde{L}_{\varepsilon, \delta} \) be the saturation of \( \Lambda_{\varepsilon} = L \cap \{G = \varepsilon\} \) by the forward and backward Liouville trajectories of the Liouville field of the form \( \lambda \).

As \( \delta \to 0 \), note that \( \tilde{L}_{\varepsilon, \delta} \) converges to the union of the forward \( pdq \)-cone and the backwards \( \frac{1}{2}(pdq - qdp) \) cone over \( \Lambda_{\varepsilon} \). Hence for small \( \delta \) a plane \( \tilde{T} \) tangent to \( \tilde{L} \) near \( \Lambda_{\varepsilon} \) is close to a...
plane spanned by a tangent plane η to \( \tilde{\Lambda}_e \) and a convex linear combination \((1 - t)R + tV\), \(0 \leq t \leq 1\) of the contracting vector fields \(R = -\frac{1}{2}(p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q})\) and \(V = -p\frac{\partial}{\partial p}\). When \(t = 0\) the plane \(\tilde{T}\) coincides with a plane \(T\) tangent to \(L\), and hence corresponds to a positive definite quadratic form \(Q\). Increasing \(t\) corresponds to adding to \(Q\) the quadratic form \(\frac{1}{1-t}\ell^2\), where the hyperplane \(\{\ell = 0\}\) is the front projection of \(\eta\). On the other hand, the distribution \(\nu\) is given by our assumption by a negative definite quadratic form \(P\). The intersection of \(\tilde{T} \cap \nu\) corresponds to critical points of the quadratic form \(Q + \frac{1}{1-t}\ell^2 - P\) which is positive definite. This assures the transversality of \(\nu\) to \(\tilde{L}\).

Recall that each boundary component \(C = \partial \Lambda\) is bounding. Let \(f : N \to \Lambda\), where \(\partial N = C\), be the bounding immersion. Let \(Z^t\) be the Liouville flow of the Liouville vector field dual to \(\lambda\). Consider the conical immersion \(F : N \times \mathbb{R} \to \mathbb{R}^{2n}\) given by

\[
F(x,t) = Z^t(x), \quad x \in N, t \in \mathbb{R}.
\]

Then the limit

\[
F_\infty(\cdot) = \lim_{t \to -\infty} F(\cdot, t)
\]

defines a cooriented \(C^1\)-smooth immersion \(F_\infty : N \to L_0\). Let \(F'_\infty : N \to L_0\) be a \(C^\infty\)-smooth normal push-off of \(N\) in the direction of its co-orientation. By modifying \(F'_\infty\) near \(\partial N = C\) we can arrange that the immersion \(F'_\infty\) together with \(F\) define a \(C^1\)-smooth immersion \(\tilde{F} := F_\infty \cup F'_\infty : \tilde{N} := N \cup C N \to L_0\) of the closed manifold \(\tilde{N}\) obtained by gluing two copies of \(N\) along the boundary \(C\). The manifold \(\tilde{N}\) bounds a manifold \(M\), namely the product \(N \times [0,1]\) with a smoothed boundary. Note that we can arrange that the immersion \(\tilde{F}\) extends to \(M\). There exists a Legendrian embedding \(\phi : N \to S^{2n-1}\) such that \(\phi(\partial N) = C\) and such that

\[
\lim_{t \to -\infty} Z^{-t} \circ \phi = F'_\infty.
\]

Set \(\tilde{N} := \phi(N)\) and denote by \(\tilde{L}\) the backward \(\lambda\)-Liouville cone of \(\tilde{N}\). Then \(\tilde{L} := \tilde{L} \cup \tilde{L} \cup L_0\) is the required arboreal Lagrangian with boundary. Its boundary (which can be smoothed) is the union of \(\Lambda_0\) and the forward Liouville cone of \(C\) glued along \(C\) to \(\tilde{N}\).

The required Liouville form \(\lambda_1\) can then be constructed as follows: starting with \(\lambda\), convert the ribbon of \(\tilde{\Lambda}\) to a boundary nucleus and then back to a \(W_c\)-hypersurface. This has the effect of adding the backwards Liouville cone of \(\tilde{\Lambda}\) to the skeleton. Then, up to smoothing, further conversion of the ribbons of \(\Lambda \cup \Lambda_0\) to boundary nucleus will produce a \(W_c\)-structure with skeleton equal to \(\tilde{L}\). The positive cotangent building structure is obtained by stabilizing the positive cotangent building structure for the ribbon of \(\Lambda\) and adding the block \(\mathcal{F}^*\Lambda_0\).

9.2.2. Parametric version. Lemma 9.11 also holds in a parametric form. Namely:

**Lemma 9.12.** Let \(\Lambda^z \subset S^{2n-1}\) be a a family of positive arboreal Legendrians in \(S^{2n-1}\) endowed with the standard contact structure, parametrized by \(z \in Z\) for \(Z\) a compact manifold. Let \(L^z \subset \mathbb{R}^{2n}\) be the Lagrangian cone over \(\Lambda^z\) centered at \(0\), i.e. with respect to the radial Liouville structure on the unit ball \(B \subset \mathbb{R}^{2n}\). Let \((\tau_z, \nu_z)\) be a family of polarizations of \(\mathbb{R}^{2n}\) such that
every Lagrangian plane $T_z$ tangent to $L^z$ at 0 satisfies the condition $T_z^z,\nu > 0$. Let $L_0^z = \tau_z(0)$ be the Lagrangian plane through the origin, and $\Lambda_0^z \subset S^{2n-1}$ its Legendrian link. Suppose that each boundary component of $\Lambda^z$ is bounding and moreover that the bounding manifolds $N^z$ form a fibre bundle over $Z$. Then there exist:

(i) a family of closed arboreal Legendrians $\hat{\Lambda}^z \supset \Lambda^z$ such that $\Lambda' = \hat{\Lambda}^z \setminus \Lambda^z$ is smooth and such that every Lagrangian plane $T_z$ tangent to the Liouville cone $L_z'$ over $\Lambda'$ satisfies the condition $T_z,\nu > 0$;

(ii) a family of asymptotically conical arboreal Lagrangians $\hat{L}^z$ in $\mathbb{R}^{2n}$ endowed with the standard Liouville structure such that:
- $\hat{L}^z$ coincides at infinity with the Lagrangian cone over $\Lambda^z \cup \Lambda_0^z$;
- $\hat{L}^z \supset \hat{\Lambda}^z$ and $\hat{L}^z \supset L_0^z$;
- $\hat{L}^z$ has smooth boundary and every boundary component is bounding, moreover with bounding manifolds forming a fibre bundle over $Z$;
- $\hat{L}^z$ is transverse to any family of Lagrangian distributions $\eta_z \in C(\nu_z, \tau_z)$, i.e negative with respect to $(\tau_z, \nu_z)$.

(iii) a family of compactly supported homotopies $\lambda_z^t = \lambda_0^t + df_z^t$ of $\lambda_0 = pdq$ such that the $W_c$-manifold obtained from $(\mathbb{R}^{2n}, \lambda_0^t)$ by converting the ribbons of the Legendrians $\Lambda^z \cup \Lambda_0^z$ into nuclei of boundary faces is a positive cotangent building with skeleton $\hat{L}^z$.

**Proof.** The proof proceeds exactly as in the non-parametric case by adding a parameter everywhere. \qed

9.3. From ridgy to arboreal.

9.3.1. Recap on reduced positivity. We recall the Definition 7.8 of reduced positivity. Let $L \subset X$ be a positive arboreal Lagrangian and $Z$ a non-zero Liouville vector field tangent to $L$. A Lagrangian distribution $\mu$ along $L$ is called reduced positive with respect to $(L, Z)$ if for any singular point $a \in L$ the following condition is satisfied. Let $T$ be the tangent space to the root Lagrangian at $a$, and $T'$ a tangent plane to any other smooth piece adjacent to $a$. Then $[T']^z \in C([T]^z, [\eta(a)]^z)$, where $\zeta = \text{Span}(Z)^{\perp, \nu}$.

9.3.2. The distribution $\nu_\infty$. We consider in this section the following setup. Let $M$ be a bc-manifold and $X = T^*\mathcal{M}$. Suppose the boundary faces of $M$ are ordered: $\partial_1 M = P_1 \cup \cdots \cup P_k$. Let $u_j$ be defining coordinates near faces $P_j$. Suppose we are given the following objects for each $j = 1, \ldots, k$:

- a neighborhood $V_j \supset P_j$;
- the canonical Liouville fields $Z_j$ on $\mathcal{T}^*P_j$ and the Liouville field $\hat{Z}_j := u_j \partial_{\hat{u}_j} + Z_j$, on $V_j$;
- a Lagrangian distribution $\nu_j$ on $V_j$, which is tangent to $\hat{Z}_j$ and invariant with respect to its negative flow.
Suppose, in addition, we are given
- a ridgy Lagrangian $L \subset T^*M$ which on $V_j \cap L$ is tangent to $\hat{Z}_j$;
- a Lagrangian distribution $\nu_{-1}$ on $T^*M$ which is transverse to $L$ and all distributions $\nu_j$.

Suppose that the following conditions are satisfied:
- for any multi-index $I = \{i_1 < \cdots < i_\ell\}$ we have
  
  \[
  [\nu_{i_\ell}(a)]^{I'} < \cdots < [\nu_{i_1}(a)]^{I'} < [\nu_{-1}(a)]^{I'} \text{ on } \bigcap_{1}^{\ell} V_{i_j} \setminus \bigcup_{j \notin I} V_j;
  \]
  
  where $I' = \{i_1 < \cdots < i_{\ell-1}\}$.
- for any point $a \in L \cap V_j$ and any tangent plane $T$ to $L$ at $a$ we have
  
  \[
  [T]^j < [\nu_j]^j < [\nu_{-1}]^j.
  \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9_5}
\caption{The condition on $[\nu_{-1}]^1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9_6}
\caption{The condition on a Lagrangian plane $T$ tangent to $L$.}
\end{figure}

**Lemma 9.13.** There exist neighborhoods $V_j \ni V'_j \ni P_j$, $j = 1, \ldots, k$, and a distribution $\nu_\infty$ on $T^*M$ which satisfies the following conditions:
(i) $\nu_\infty = \nu_m$ on $V'_m \setminus \bigcup_{j > m} V_j$, $m = 1, \ldots, k$;

(ii) for any $1 \leq i < j \leq k$ and any point $a \in V'_i \cap (V_j \setminus V'_j)$ we have

$$[\nu_\infty(a)]^I \in C([\nu_i(a)]^I, [\nu_j(a)]^I),$$

where $I$ is the multi-index $\{i_0 = i < i_1 < \cdots < i_\ell, i_\ell < j\}$, such that

$$a \in \bigcap_{1 \leq s \leq \ell} V_{i_s} \setminus \bigcup_{m \not\in I, m < j} V_m.$$

(iii) $\nu_\infty$ is transverse to $\nu_{-1}$ and for any point $a \in L \setminus \bigcup_{1 \leq j \leq k} V_j$ and any tangent plane $T$ to $L$ at $a$ we have $T \prec \nu_\infty(a) \prec \nu_{-1}(a)$.

**Figure 9.7.** The conclusion on $[\nu_\infty]^1$.

**Figure 9.8.** The conclusion on a Lagrangian plane $T$ tangent to $L$.

**Proof.** Define the required distribution as equal to $\nu_m$ on $V'_m \setminus \bigcup_{j > m} V_j$, $m = 1, \ldots, k$, and any distribution on $\mathcal{T}^*M \setminus \bigcup_{1 \leq j \leq k} V_j$ which is transverse to $\nu_{-1}$ and satisfies condition (iii) along $L$. Next, we successively extend $\nu_\infty$ to $V_k$: first to $(V_k \setminus V'_k) \cap V'_{k-1}$ to satisfy the condition
[ν∞]^{k-1} \in C([ν_k]^{k-1}, [ν_{k-1}]^{k-1})$, then continuing the process by extending to $V_k \setminus (V'_k \cup V'_{k-1}) \cap V'_{k-2}$ to satisfy the condition $[ν∞]^{k-1} \in C([ν_k]^{k-2,k-1}, [ν_{k-1}]^{k-2,k-1})$, etc. Next, we similarly successively extend $ν∞$ to $V_k \setminus (V'_k \cap V_1), V_k \setminus (V'_1 \cup V_k), ..., V_k \setminus (V'_1 \cup V_2 \cup ... \cup V_k)$. Each extension is possible because it amounts to an extension of a section of a fibration with open convex (and therefore contractible) fiber. □

9.3.3. Arborealization of ridges. Recall that for an isotropic submanifold $C$ of a symplectic manifold, its symplectic normal bundle, is defined as $(TC)^{\perp}/TC$. Denote by $π_C = []C$ the reduction map. We consider a ridgy Lagrangian $L \subset \mathcal{F}^*M$ with the notation as in Section 9.3.2 above. Consider the stratification $L = L_0 \cup L_1 \cup ... \cup L_n$, where $L_j$ is the $j$-dimensional locus of order $(n - j)$ ridges. Thus the top-dimensional stratum $L_n$ is the smooth locus of $L$.

For each component $C_j$ of $L_j$, we denote by $N(C_j)$ the $2j$-dimensional symplectic normal bundle to $C_j$. A Lagrangian tangent plane $T$ to $L$ at a point of $C_j$ reduces to a Lagrangian plane $[T]^C_j := π_{C_j}(T) \subset N(C_j)$. If moreover $a ∈ C_{j-1} \subset C_j$ for $C_{j-1}$ a component of $L_{j-1}$ and $T$ a Lagrangian plane at $a$, then the plane $[T]^C_{j-1}$ further reduces to $[T]^C_{j-1}$.

Note that if $L$ is tangent to $C_j$ on $V_j$, $j = 1, ..., k$, then so are all $L_i, i = 1, ..., k$. In particular, if $i < ℓ$ we have $L_i ∩ V_{i_1} ∩ ... ∩ V_{i_ℓ} = ∅$.

Let $U_n → ... → U_0$ be the canonical $W_{C-}$structure associated to a ridgy Lagrangian $L$.

Definition 9.14. An arborealization of the ridgy Lagrangian $L$ is the structure of a cotangent building $B_m → ... → B_0$ on a neighborhood of $L$ whose skeleton is arboreal and whose underlying Weinstein manifold is homotopic to that of $U_n → ... → U_0$.

We say that the arborealization is positive when the resulting arboreal Lagrangian is positive, which implies $B_m → ... → B_0$ is a positive complex.

In the next proposition we continue using the notation and assumptions introduced in Section 9.3.2.

Proposition 9.15. The ridgy Lagrangian $L$ can be arborealized to a positive arboreal Lagrangian $\tilde{L}$ that is positive with respect to $ν_{-1}$, and moreover, over $V'_j$ is invariant with respect to the negative flow of $\tilde{Z}_j$ and reduced positive with respect to $ν_j$, for $j = 1, ..., k$.

Proof. Any Lagrangian distribution can be viewed as a field of quadratic forms with respect to the polarization $(ν∞, ν_{-1})$. Fix a field of positive definite quadratic forms $Q$. Denote by $η_τ$ the Lagrangian distribution generated by $-tQ$, and by $ζ_τ$ the distribution generated by $\frac{1}{t}Q$. Choose $ε > 0$ so small that the following conditions are satisfied:

- $ν_j < η_ε < ν_{-1}$ on $V_j, j = 1, ..., k$;
- for any point $a ∈ L ∩ V_j$ and any tangent plane $T$ to $L$ at $a$ we have $[T]^j_⊥ < [ηε]^j_⊥ < [ν∞]^j_⊥$. 


For each component $C_j$ of $L_j$ define Lagrangian distributions in the bundle $N(C_j)$ over $C_j$, $j = 0, \ldots, n$.

$$s_j := \left[ \eta_{n+1} | C_j \right]^{C_j},$$
$$\bar{s}_j = \left[ \zeta_{n-j+1} | C_j \right]^{C_j}.$$

Note that if $C_{j-1} \subset C_j$ then

$$[s_{j-1}]^{C_j} \prec s_j | C_{j-1} \prec s_j | C_j \prec \bar{s}_j | [s_{j-1}]^{C_j}.$$

We will refer to the system of distributions $s_j, \bar{s}_j$ as a frame of arborealization for $L$.

Consider the $Wc$-building structure $U_n \to \cdots \to U_0$ on a neighborhood of $L$ where the germ $\mathcal{U}_n$ is decomposed as $\bigcup Q \mathcal{T}^* Q^{\Delta,j-1} \times \mathcal{A}^{2(n-j)}$. The $Wc$-manifold $W$ is obtained by successively gluing $U_n$ to $U_{n-1}$, the result of that to $U_{n-2}$, the result of that to $U_{n-3}$, etc. In this process we get a sequence of $Wc$-manifolds $W^{>j} = \bigcup_{i>j} U_i$ with skeletons $L^{>j} = L \setminus \bigcup_{i\geq j} U_i$. We will be proving the proposition by induction for $W^{>n-j}, j = 1, \ldots, n+1$. We will also include into the induction hypothesis the following additional property. Recall that $W^{>j}$ is attached vertically to $U_j$ along a $Wc$-hypersurface $\Sigma$ (which is the nucleus of the attaching face) whose skeleton is fibered over $L^{\Delta,j-1}$ with the fiber which is the link $\partial R^{n-j}$ of the ridgy singularity $R^{n-j}$. We will require that the arborealization $\hat{L}^j$ intersects the attaching face in an arboreal skeleton of $\Sigma$ which is fibered over $L^{\Delta,j-1}$, whose fiber is the arborealization of the link $\partial R^{n-j}$.

The base of induction $j = 0$ is trivial because $L_n$ in this case is smooth. Suppose that we already arboREALIZED the ridgy Lagrangian $L^{>j}$ to $\hat{L}^{>j}$ and deformed correspondingly the Weinstein structure on $W^{>j}$. i.e. made $\hat{L}^{>j}$ the skeleton of the new structure so that, by inductive assumption, (i) the arboreal $\hat{L}^{>j}$ has a smooth boundary $\partial \hat{L}^{>j}$ which bounds an immersed $n$-dimensional manifold $A \subset \hat{L}^{>j}$, (ii) $\hat{L}^{>j}$ intersects the nucleus $\Sigma$ in an arborealization of the skeleton of $\Sigma$ and (iii) the skeleton is Legendrian for $U_j$. We also observe
that the condition that each boundary component is bounding is inherited by the skeleton of $\Sigma$. Hence, using the fiberwise polarization $(\tau = \delta_j, \nu = s_j)$ we can apply Lemma 9.12 to arborealize the fiberwise Lagrangian cone over the arboreal Legendrian.

Note that by our construction, the distribution $\nu_{-1}$ is positive for the constructed arboreal $\hat{L}$, and $\nu_j$ are reduced positive on $V_j'$.

\[\square\]

Figure 9.10. In the case $n = 2$ the inductive procedure has 2 steps. The figure illustrates the moment between the two steps: one has arborealized the order 1 ridges and it remains to arborealize a radial Liouville cone over an arboreal Legendrian with smooth boundary in a sphere around point corresponding to a order 2-ridge.

9.4. Conclusion of the proof. We are now ready to prove our main result.

9.4.1. Proof of the main theorem. Let $(X, \lambda)$ be a $Wc$-manifold and $\nu \subset TW$ a Lagrangian plane field. Then there exists a Weinstein homotopy $\lambda_t$ of $\lambda_0 = \lambda$ and a $Wc$-hypersurface $A \subset X \setminus \text{Skel}(X, \lambda_1)$ such that the result of converting $A$ to a boundary nucleus yields a $Wc$-manifold $(X, \lambda_1^*)$ which admits a structure of a positive cotangent building with a minimal distribution $\nu_{-1}$ equal to $\nu$. In particular, $\text{Skel}(X, \lambda_1^*)$ is a positive arboreal Lagrangian (with boundary) transverse to $\nu$. Moreover, we can arrange it so that $\text{Skel}(A, \lambda)$ is smooth, i.e. $\text{Skel}(X, \lambda_1^*)$ has smooth boundary.

Proof. We choose a presentation of $(X, \lambda)$ as a cotangent building $(B_k \to \cdots \to B_0)$, $B_i = \mathcal{F}^*M_i$, see Proposition 2.69. Without loss of generality we may assume that the restriction of $\nu$ to $B_0$ serves as $\nu_{-1}$, i.e. satisfies the condition $\nu_0 \in C(TM_0, \nu_{-1})$ over $M_0$. Indeed, $M_0$ is a disk, $B_0 = T^*M$ is a ball and we are free to change polarization if desired. The cotangent block $B_1$ is attached to $B_0$ along $\mathcal{F}^*P$, where $P$ is a boundary face of $M_1$.

Consider the block $B_1$ and the restrictions of the distributions $\nu_0$ and $\nu$ to $B_1$. Applying Theorem 8.7 we can find a ridgy isotopy of $M_1$ to a ridgy Lagrangian $M_1'$ transverse to $\nu$, reduced transverse to $\nu_0$, invariant with respect to the negative flow of $Z_0$, and such that at each point $a \in M_1'$ and any tangent plane $T$ at $a$ we have $[T]^0 \prec [\nu_0]^0 \prec [\nu_{-1}]^0$. 


Moreover, this deformation can be realized by a deformation of skeleta of Weinstein structures of $B_1$. Next, we apply to the ridgy Lagrangian $M'_1$ Proposition 9.15 to further deform the Weinstein structure on $B_1$ to a structure of a cotangent building, which is positive with respect to $\nu_{-1}$ and reduced positive with respect to $\nu_0$.

This yields a building structure on $B_0 \cup B_1$ with positive skeleton, positive with respect to $\nu_{-1}$. Using Proposition 7.14 we can deform the building structure on this building to make it positive and positive for $\nu_{-1}$.

Before we proceed with the next block, we note that the Lagrangian $\tilde{M}_2 \cap W^{(1)}$ is no longer conical for the constructed Wc-structure of $W^{(1)}$. So we apply Lemma 2.64 to arrange for this property to hold. Once this is achieved we can proceed as before to produce a Wc-manifold $(W^{(2)}, \lambda^{(2)})$ with a positive cotangent building which is positive with respect to $\nu_{-1}$.

### 9.4.2. Variants of the main theorem

Next we state our main theorem for Weinstein pairs:

**Theorem 9.17.** Let $(X, \lambda)$ be a Wc-manifold, $A_0 \subset X \setminus \text{Skel}(X, \lambda)$ a Wc-hypersurface and $\nu \subset TW$ a Lagrangian plane field. Then there exists a Weinstein homotopy $\lambda_t$ of $\lambda_0 = \lambda$ and a Wc-hypersurface $A_1 \subset X \setminus \text{Skel}(X, \lambda_1)$ disjoint from $A_0$ such that the result of converting $A = A_0 \cup A_1$ to a boundary nucleus yields a Wc-manifold $(X, \lambda^A_1)$ which admits a structure of a positive cotangent building with a minimal distribution $\nu_{-1}$ equal to $\nu$. In particular, $\text{Skel}(X, \lambda^A_1)$ is a positive arboreal Lagrangian (with boundary) transverse to $\nu$. Moreover, we can arrange it so that $\text{Skel}(A_1, \lambda)$ is smooth, i.e. $\text{Skel}(X, \lambda^A_1)$ has smooth boundary away from $A_0$.

**Proof.** The same proof works applied to the cotangent building structure associated to a Weinstein pair. □

Finally, we state our main theorem in relative version:

**Theorem 9.18.** Let $(X, \lambda)$ be a Wc-manifold, $A_0 \subset X \setminus \text{Skel}(X, \lambda)$ a Wc-hypersurface and $\nu \subset TW$ a Lagrangian plane field. Suppose that $A_0$ is endowed with the structure of a positive cotangent building with minimal distribution equal to the reduction of $\nu|_{A_0}$. Assume moreover that $\nu$ is transverse to the Liouville field $Z$ of $X$ near $A_0$. Then there exists a Weinstein homotopy $\lambda_t$ of $\lambda_0 = \lambda$, fixed near $A_0$, and a Wc-hypersurface $A_1 \subset X \setminus \text{Skel}(X, \lambda_1)$ disjoint from $A_0$ such that the result of converting $A = A_0 \cup A_1$ to a boundary nucleus yields a Wc-manifold $(X, \lambda^A_1)$ which admits a structure of a positive cotangent building with a minimal distribution $\nu_{-1}$ equal to $\nu$. In particular, $\text{Skel}(X, \lambda^A_1)$ is a positive arboreal Lagrangian (with boundary) transverse to $\nu$ and $\text{Skel}(X, \lambda^A_1) \cap A_0 = \text{Skel}(A_0, \lambda)$. Moreover, we can arrange it so that $\text{Skel}(A_1, \lambda)$ is smooth, i.e. $\text{Skel}(X, \lambda^A_1)$ has smooth boundary away from $A_0$. □
\textbf{Proof.} All the intermediary steps hold in relative form, so one may apply the same proof to the cotangent building structure associated to the Weinstein pair keeping everything fixed near $A_0$ at every stage of the argument. \hfill \Box

Theorem 9.16 implies our main Theorem 1.5 as stated in the Section 1.2, simply by taking the underlying Weinstein manifold of the Wc-manifold $(X, \lambda^A_1)$, i.e. converting $A$ back to a Wc-hypersurface. Similarly, Theorems 9.17 and 9.18 imply the variants outlined in the remarks below the statement. For the concordance statement of Theorem 1.11 simply apply Theorem 9.18 to the Wc-pair obtained from $W \times T^*[0,1]$ by converting the face nuclei $A_0 = (W \times 0) \cup (W \times 1)$ to Wc-hypersurfaces, where the Liouville form is $\lambda = \lambda_t + u dt$ for $\lambda_t$ the homotopy between $\lambda_0$ and $\lambda_1$.

\textbf{References}

[AG18a] D. Álvarez-Gavela, \textit{Refinements of the holonomic approximation lemma}, Algebraic & Geometric Topology 18 (2018) 2265–2303. 11

[AG18b] D. Álvarez-Gavela, \textit{The simplification of singularities of Lagrangian and Legendrian fronts}, Inventiones Mathematicae, 214(2) (2018) 641–737. 11

[AGEN19] D. Alvarez-Gavela, Y. Eliashberg, and D. Nadler, \textit{Geomorphology of Lagrangian ridges} arXiv:1912.03439 12, 14, 105, 106, 108

[A12] R. Avdek, \textit{Liouville hypersurfaces and connect sum cobordisms}, arXiv:1204.3145. 25

[CE12] K. Cieliebak and Y. Eliashberg, \textit{From Stein to Weinstein and Back – Symplectic Geometry of Affine Complex Manifolds}, Colloquium Publications Vol. 59, Amer. Math. Soc. (2012). 17, 18, 19, 22, 26, 93

[EL17] T. Ekholm and Y. Lekili, \textit{Duality between Lagrangian and Legendrian invariants}, arXiv:1701.01284. 25

[E95] Y. Eliashberg, \textit{Symplectic geometry of plurisubharmonic functions}, in “Gauge Theory and Symplectic Geometry”, Vol. 488 of the series NATO ASI Series, 49–67. 18

[E90] Y. Eliashberg, \textit{Topological characterization of Stein manifolds of dimension > 2}, Internat. J. Math. 1(1990), no. 1, 29-46. 17

[E18] Y. Eliashberg, \textit{Weinstein manifolds revisited}, Proc. of Symp. in Pure Math., Amer. Math. Soc., 99(2018), 59–82. 25, 29

[EGL15] Y. Eliashberg, S. Ganatra and O. Lazarev, \textit{Flexible Lagrangians}, Int. Math. Rec. Notices, //https://doi.org/10.1093/imrn/rny078.

[EG91] Y. Eliashberg, M Gromov, \textit{Convex symplectic manifolds}, Proc. Symp. Pure Math., 52(1991), Amer. Math. Soc., Providence, RI, 135–162. 17, 18

[E99] M. Entov, \textit{Surgery on Lagrangian and Legendrian Singularities}, Geometric and Functional Analysis, 9(2) (1999) 298–352. 5 11

[GPS17] S. Ganatra, J. Pardon and V. Shende, \textit{Covariantly functorial Floer theory on Liouville sectors}, arXiv:1706.03152. 2, 25

[MD91] D. McDuff, \textit{Symplectic manifolds with contact type boundaries}, Invent. Math. 103(1991), 651–671. 18

[N13] D. Nadler, \textit{Arboreal Singularities}, arXiv:1309.4122. 1, 2, 67

[N15] D. Nadler, \textit{Non-characteristic expansion of Legendrian singularities}, arXiv:1507.01513. 2

[N16] D. Nadler, \textit{Wrapped microlocal sheaves on pairs of pants}, arXiv:1604.00114. 2

[S08] P. Seidel \textit{Fukaya categories and Picard-Lefschetz theory}, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. 2

[S18] V. Shende, \textit{Arboreal singularities from Lefschetz fibrations}, arXiv:1809.10359. 2

[S16] Z. Sylvan, \textit{On partially wrapped Fukaya categories}, arXiv: 1604.02540v2. 25
[S17] L. Starkston, *Arboreal Singularities in Weinstein Skeleta*, arXiv:1707.03446. 2, 10, 11, 19, 21

[W91] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. 20(1991), 241–251. 17

[Z18] A. Zorn, *A Combinatorial Model of Lagrangian Skeleta*, Ph.D. thesis, UC Berkeley, 2018. 9

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, 02139
*Email address*: da21@math.princeton.edu

Department of Mathematics, Stanford University, Stanford, CA 94305
*Email address*: eliash@stanford.edu

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840
*Email address*: nadler@math.berkeley.edu