Nonlinear resonance in Hořava–Lifshitz bouncing cosmologies

Rodrigo Maier

Institute of Cosmology and Gravitation, University of Portsmouth, Dennis Sciama Building, Portsmouth, PO1 3FX, UK

E-mail: rodrigo.maier@port.ac.uk

Received 15 January 2013, in final form 7 April 2013
Published 7 May 2013
Online at stacks.iop.org/CQG/30/115011

Abstract
In this paper, I examine the phase space dynamics in the framework of non-projectable Hořava–Lifshitz bouncing cosmologies. By considering a closed Friedmann–Lemaître–Robertson–Walker geometry, the first integral contains a correction term that leads to nonsingular metastable bounces in the early evolution of the Universe. The matter content of the model is a massive conformally coupled scalar field, dust and radiation. A nonvanishing cosmological constant connected to a de Sitter attractor in the phase space is also assumed. In narrow windows of the parameter space, labeled by an integer $n \geq 2$, nonlinear resonance phenomena may destroy the Kolmogorov–Arnold–Moser (KAM) tori that trap the scalar field, leading to an exit to the de Sitter attractor. As a consequence, nonlinear resonance imposes constraints on the parameters and in the initial configurations of the models so that an accelerated expansion may be realized.

PACS numbers: 98.80.Jk, 98.80.Qc, 05.45.–a

1. Introduction

Although General Relativity is the most successful theory that currently describes gravitation, it presents some intrinsic crucial problems when one tries to construct a cosmological model in accordance with observational data. In cosmology, the $\Lambda$CDM model gives us important predictions concerning the evolution of the Universe and about its current state [1]. However, let us assume that the initial conditions of our Universe were fixed when the early Universe emerged from the semi-Planckian regime and started its classical expansion. Evolving back such initial conditions using the Einstein field equations, we see that our Universe is driven toward an initial singularity where the classical regime is no longer valid [2].

Notwithstanding the cosmic censorship conjecture [3], there is no doubt that General Relativity must properly be corrected or even replaced by a completely new theory, let us say a quantum theory of gravity. This demand is in order to solve the issue of the presence of the initial singularity predicted by classical General Relativity in the beginning of the Universe.
One of the most important characteristics of our Universe supported by observational data is its large scale of homogeneity and isotropy. However, when we consider a homogeneous and isotropic model filled with baryonic matter, we find several difficulties by taking into account the primordial state of our Universe. Among such difficulties, we can mention the horizon and flatness problems [1]. Although the inflationary paradigm [4] allows one to solve problems like these, inflationary cosmology does not solve the problem of the initial singularity.

On the other hand, since 1998 [5] observational data have been giving support to the highly unexpected assumption that our Universe is currently in a state of accelerated expansion. In order to explain this state of late-time acceleration, cosmologists have been considering the existence of some field—known as dark energy—that violates the strong energy condition. Although it poses a problem to quantum field theory on how to accommodate its observed value with vacuum energy calculations [6], the cosmological constant seems to be the simplest and most appealing candidate for dark energy. Therefore, nonsingular models which provide late-time acceleration should be strongly considered.

Over the last decades, bouncing models [7, 8] have been considered in order to solve the problem of initial singularity predicted by General Relativity. Such models (as [9]) might provide attractive alternatives to the inflationary paradigm once they can solve the horizon and flatness problems, and justify the power spectrum of primordial cosmological perturbations inferred by observations.

In 2009, Hořava proposed a modified gravity theory by considering a Lifshitz-type anisotropic scaling between space and time at high energies [10]. In this context, it has been shown [11, 12] that higher spatial curvature terms can lead to regular bounce solutions in the early Universe. Since its proposal, several versions of Hořava–Lifshitz gravity have emerged.

In the case of a four-dimensional \((1 + 3)\) spacetime, the basic assumption which is required by all the versions of Hořava–Lifshitz theories is that a preferred foliation of spacetime is \emph{a priori} imposed. Therefore, it is natural to work with the Arnowitt–Deser–Misner decomposition of spacetime
\[
d s^2 = N^2 \, dt^2 - g_{ij}(N^i \, dt + dx^i)(N^j \, dt + dx^j),
\]
where \(N = N(t, x^i)\) is the lapse function, \(N' = N'(t, x')\) is the shift and \(g_{ij} = g_{ij}(t, x')\) is the spatial geometry. In this case, the final action of the theory will not be invariant under diffeomorphisms as in General Relativity. Nevertheless, an invariant foliation preserving diffeomorphisms can be assumed. This is achieved if the action is invariant under the symmetry of time reparametrization together with time-dependent spatial diffeomorphisms. That is,
\[
t \to \bar{t}(t), \quad x^i \to \bar{x}(t, x^i).
\]
It turns out that the only covariant object under spatial diffeomorphisms that contain one time derivative of the spatial metric is the extrinsic curvature \(K_{ij}\):
\[
K_{ij} = \frac{1}{2N} \left[ \frac{\partial g_{ij}}{\partial t} - \nabla_j N_i - \nabla_i N_j \right],
\]
where \(\nabla_i\) is the covariant derivative built with the spatial metric \(g_{ij}\). Thus, to construct the general theory which is of second order in time derivatives, one needs to consider the quadratic terms \(K_{ij} K^{ij}\) and \(K^2\)—where \(K\) is the trace of \(K_{ij}\)—in the extrinsic curvature. By taking these terms into account, we obtain the following general action:
\[
S \propto \int N \sqrt{-g} \left[ K_{ij} K^{ij} - \lambda K^2 - U(g_{ij}, N) \right] \, dx^3 \, dt,
\]
where \(g\) is the determinant of the spatial metric and \(\lambda\) is a constant which corresponds to a dimensionless running coupling. As in General Relativity the term \(K_{ij} K^{ij} - K^2\) is invariant
under four-dimensional diffeomorphisms, we expect to recover the classical regime for $\lambda \to 1$. That is why it is a consensus that $\lambda$ must be a parameter sufficiently close to 1. In general, $U(g_{ij}, N)$ can depend on the spatial metric and the lapse function because of the symmetry of the theory. It is obvious that there are several invariant terms that one could include in $U$. Particular choices resulted in different versions of Hořava–Lifshitz gravity.

Motivated by condensed matter systems, Hořava proposed a symmetry on $U$ that substantially reduces the number of invariants [10]. In this case, $U$ depends on a superpotential $W$ given by the Chern–Simons term, the curvature scalar and a term which mimics the cosmological constant. It has been shown [13] that this original assumption has to be broken if one intends to build a theory in agreement to current observations.

The simplification $N = N(t)$ was also originally proposed by Hořava [10]. This condition defines a version of Hořava–Lifshitz gravity called projectable. As $\partial N/\partial x^i \equiv 0$, the projectable version also reduces the number of invariants that one can include in $U$. The linearization of this version assuming a Minkowski background provides an extra scalar degree of freedom which is classically unstable in the IR when $\lambda > 1$ or $\lambda < 1/3$, and is a ghost when $1/3 < \lambda < 1$ [14]. Although some physicists argue that higher order derivatives can cut off these instabilities, it has been shown [13, 15–17] that a perturbative analysis is not consistent when $\lambda \to 1$ and the scalar mode gets strongly coupled. This is because the strongly coupled scale is unacceptably low. In this case, higher order operators would modify the graviton dynamics at very low energies, being in conflict with current observations.

Besides pure curvature invariants of $g_{ij}$, one may also include invariant contractions of $\partial (\ln N)/\partial x^i$ in $U$. This assumption defines the so-called non-projectable version of Hořava–Lifshitz gravity. Connected to the lowest order invariant $\partial (\ln N)/\partial x^i$, there is a parameter $\sigma$ which defines a ‘safe’ domain of the theory [14, 18]. In fact, in this case there is also an extra scalar degree of freedom when one linearizes the theory in a Minkowski background. However, when $0 < \sigma < 2$ and $\lambda > 1$, this mode is neither a ghost nor classically unstable (as long as detailed balance is not imposed). Although the non-projectable version also exhibits a strong coupling [13, 18, 19], it has been argued that its scale is too high to be phenomenologically accessible from gravitational experiments [14].

In this paper, I adhere to the so-called non-projectable Hořava–Lifshitz gravity in which I consider a nonsingular Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological model [18]. The matter content is given by dust, radiation and a conformally coupled scalar field. I also assume a nonvanishing cosmological constant in order to obtain a de Sitter attractor in the phase space. In this context, I show how an alternative exit to late-time acceleration (connected to the de Sitter attractor) may be realized.

In the following section, I present a nonsingular homogeneous and isotropic cosmological model—sourced with perfect fluids, a cosmological constant and a conformally coupled scalar field—which arises from non-projectable Hořava–Lifshitz gravity. In section 3, I analyze the structure of the phase space. In section 4, I restrict myself to the case of dust and radiation in order to construct a simple model. In section 5, I show how nonlinear resonance can provide an exit to the de Sitter attractor. In section 6, I exhibit the pattern of the resonance windows and show in which regions in the parametric space late-time acceleration may be realized. Final remarks are given in section 7.

2. The model

Let us consider a model in which the matter content is given by a non-minimally coupled massive scalar field $\phi$ and $N$ noninteracting perfect fluids with equation of state $p_i = \omega_i \rho_i$ ($i = 1, \ldots, N$). In this context, the 4D covariant Lagrangian $\mathcal{L}_m$ for the matter content can be
written as
\[ L_m = \sum_{i=1}^{N} L_i - L_\phi, \]  
where \( L_i \) are the Lagrangians for the noninteracting perfect fluids and
\[ L_\phi = \frac{1}{4} \left[ (\phi, \alpha \phi, \beta \phi^\alpha + m^2 \phi^2) + \xi R \phi^2 \right], \]  
with \( R \) being the 4D Ricci scalar. That is, \( L_m \) is the Lagrangian density of the massive scalar field plus perfect fluids whose dynamics interact only with the metric \( g_{\alpha\beta} \). We further assume that the scalar field is non-minimally coupled with \( g_{\alpha\beta} \), with the coupling parameter \( \xi \).

The fundamental symmetry assumed in Hořava–Lifshitz gravity (invariant under diffeomorphisms that preserve the foliation) provides enough gauge freedom to choose
\[ N = 1, \quad N^i = 0, \quad g_{ij} = -a(t)^2 \gamma_{ij}, \]  
This puts the geometry (1) into the FLRW form
\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right], \]  
where \( k \) is the spatial curvature, \( a(t) \) is the scale factor, \( t \) is the cosmological time and \( (r, \theta, \chi) \) are the comoving coordinates. It is straightforward to show that the energy density connected to \( L_m \) is given by
\[ \rho_m = \sum_i \rho_i + \frac{1}{2} (\dot{\phi}^2 + m^2 \phi^2) + 3\xi \left[ H + \frac{k}{a^2} \right] \phi^2 + 6\xi H \phi \dot{\phi}, \]  
with the \( \dot{a} \equiv da/dt \) and \( H \equiv \dot{a}/a \).

By considering the local Hamiltonian constraint in the non-projectable version of Hořava–Lifshitz gravity [18], we obtain the following first integral:
\[ \frac{\ddot{a}}{2} + 2 \left[ \frac{2}{3} \pi^2 + \frac{\alpha_2 k^3}{6a^4} + \frac{\alpha_3 k^2}{2a^2} + \frac{k}{2} - \frac{\Lambda}{6} a^2 - \frac{4\pi G}{3} \rho_m a^2 \right] = 0, \]  
where \( \alpha_2 \) and \( \alpha_3 \) are constant coupling coefficients. From (10), we note that the correction term proportional to \( \alpha_2 \) behaves just like a radiation component.

Let \( T^{\mu \beta}_{m} \) be the energy momentum tensor connected to the matter content connected to \( L_m \). As in this case the conservation equations \( \nabla_{\alpha} T^{\alpha \beta}_{m} = 0 \) still apply, we obtain
\[ \dot{\rho}_i + 3H (\rho_i + p_i) = 0 \rightarrow \dot{\rho}_i = \frac{E_i}{a^{3(1+\omega_i)}} \]  
for the noninteracting perfect fluids, where \( E_i \) are constants of motion. On the other hand, for the non-minimal coupled scalar field, we obtain the following equation of motion:
\[ \ddot{\phi} + 3H \dot{\phi} + m^2 \phi + 6\xi \left[ \frac{\ddot{a}}{a} + H^2 + \frac{1}{a^2} \right] \phi = 0. \]  
If \( \omega_i < 1 \) for all \( i \), the conditions for the bounce are given by
\[ \lambda > 1/3, \quad \alpha_3 k^3 > 0. \]  
From now on, I will restrict myself to the case of closed geometries, that is, \( k = 1 \). It is worth mentioning that this model is not the only possibility of generating a bounce in non-relativistic theories. In fact, it has been shown—for the case closed of FLRW geometries—that the Universe can undergo through a bounce as long as the terms which violate relativity lead to a dark radiation component with negative energy density [20].

In order to simplify the numerical analysis, I will also fix \( \lambda = 1 \) and \( \alpha_3 > 0 \). In the framework of Hořava–Lifshitz gravity, \( \lambda \) must be close to unity in order to ensure that severe
Lorentz violation do not occur. Thus, the assumption of $\lambda = 1$ enables the following model to be a fair approximation derived from of a healthy non-projectable Hořava–Lifshitz theory. On the other hand, $\alpha_3 > 0$ is the necessary and sufficient condition for the bounce.

By choosing the so-called conformal coupling ($\xi = 1/6$), equations (10) and (12) may be rewritten using the conformal time $\eta = \int \frac{1}{a} dt$ as

$$3a^2 + V(a) - \frac{1}{2}(\phi'^2 + (1 + m^2 a^2)\phi^2) = 8\pi G(E_{\text{rad}} - \alpha_4),$$

and

$$\phi'' + (1 + m^2 a^2)\phi = 0,$$

where $E_{\text{rad}}$ corresponds to a constant connected to $\omega_{\text{rad}} = \frac{1}{3}$, the primes denote derivatives with respect to the conformal time, $\phi \equiv a \phi \sqrt{\frac{8}{3\pi G}}$, $\alpha_4 \equiv \frac{3}{8\pi G}$ and

$$V(a) = \frac{\alpha_3}{a^2} + 3a^2 - \Lambda a^4 - 8\pi G \sum_{i \neq \text{rad}} \frac{E_i}{a^{1 - 3\omega_i}}.$$  

(16)

It is worth noting that for $m = 0$ the system of equations (14) and (15) is separable and, therefore, integrable. That is, in this case, equation (15) has a first integral $E_{\psi} = \frac{1}{2}(\phi'^2 + \phi^2)$ which is a constant of motion and, from (14), we obtain

$$\eta \equiv \int \sqrt{\frac{3}{8\pi G(E_{\text{rad}} - \alpha_4) + E_{\psi} - V(a)}} da.$$  

(17)

3. The structure of the phase space

By considering equations (14) and (15), the following dynamical system can be defined:

$$\phi' = p_{\phi},$$

$$a' = \frac{p_a}{6},$$

$$p_{\phi}' = -(k + m^2 a^2)\phi,$$

$$p_a' = -6a + 4\Lambda a^3 + \frac{2\alpha_3}{a} + 8\pi G \sum_{i \neq \text{rad}} \frac{E_i}{a^{1 - 3\omega_i}}.$$  

(21)

Equations (18) and (19) are mere redefinitions. On the other hand, $(a, p_a)$ can be shown to be canonically conjugated by considering the first integral (14) as a Hamiltonian constraint. Now, we focus on three basic structures that organize the dynamics in the phase space of the above dynamical system.

3.1. Invariant plane

Let us consider the arbitrary dynamical system with $n$ degrees of freedom $(\psi_1, \ldots, \psi_n) \in \mathbb{R}^n$, whose differential equations are given by

$$\frac{d\psi_i}{d\xi} = F_i(\psi_1, \ldots, \psi_n),$$

(22)

where $i = 1, \ldots, n$. Let $\psi_i(\xi)$ be its solution for given initial conditions $\psi_i(\xi_0) \in M \subset \mathbb{R}^n$. Thus, $M$ is defined as an invariant manifold if the solution $\psi_i(\xi) \in M$, for each $\psi_i(\xi_0) \in M$.

If we fix the initial conditions $p_{\phi_0} = 0 = \phi_0$, we see from equations (18)–(21) that the dynamics is integrable and does not evolve in the $\phi$ and $\phi'$ directions. That is, orbits with these
initial conditions remain contained in the plane \((a, p_\phi)\) during all the evolution of the system. Therefore, the invariant plane is defined by

\[
\varphi = 0 = p_\psi.
\]

(23)

It is worth noting that the dynamics in this plane is analogous to that of the dynamics in the separable case \(m = 0\). In fact, in both cases, the dynamics is separable and integrable, and its description is given by similar orbits which differ by a constant \(E_0\) in the \((a, p_\phi)\) sector. As we shall see, in order to furnish an exit to a de Sitter attractor due to parametric resonance, I will always assume initial conditions sufficiently close to the invariant plane.

3.2. Critical points

In the phase space \((\psi_1, \ldots, \psi_n)\) of an arbitrary dynamical system like (22), a critical point \(\psi^{cr}_i\) is defined as a solution of the equations \(F_i(\psi^{cr}_1, \ldots, \psi^{cr}_n) = 0\). That is, it is a stationary solution of (22). If one takes the initial condition \(\psi_i(\xi_0) = \psi^{cr}_i\), then \(\psi_i(\xi) = \psi^{cr}_i\) for all \(\xi\).

The structure of the phase space of (18)–(21) allows the presence of critical points \(P = (\varphi = 0, a = a^{cr}, p_\phi = 0, p_\psi = 0)\), where \(a^{cr}\) satisfies the relation

\[
V'(a^{cr}) \equiv \left. \frac{dV}{da} \right|_{a=a^{cr}} = 0.
\]

(24)

It is easy to see that, by definition, the critical points are contained in the invariant plane. Furthermore, according to (24), the critical points are associated with potential extrema. For specific numerical values of \(\Lambda, E_i\) and \(\omega_i\), we may obtain one or many extrema for \(V(a)\) (characterized by one or many values of \(a^{cr}\)). In fact, that is the case for a fixed value of \(\Lambda\) and suitable domains of \(E_i\). For \(\Lambda = 0\), the dynamical system (18)–(21) has only one critical point connected to a global minimum of the potential \(V(a)\). As an exit to the de Sitter attractor cannot be performed in this case, I will not consider such configurations.

Linearizing the dynamical system (22) around the critical points, we obtain

\[
\frac{d\psi_i}{d\xi} \simeq \sum_{j=1}^{n} dF_{ij}(\psi_j - \psi^{cr}_j),
\]

(25)

where

\[
dF_{ij} = \begin{pmatrix}
\frac{\partial F_1}{\partial \psi_i} & \cdots & \frac{\partial F_i}{\partial \psi_i} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial \psi_i} & \cdots & \frac{\partial F_n}{\partial \psi_i}
\end{pmatrix}.
\]

(26)

Let us then assume that the matrix \(dF_{ij}\) has \(\mu_l = i\theta_i\) pure imaginary eigenvalues (where \(l = 1, \ldots, 2p\), with \(p\) being an integer smaller than \(n/2\)) and \(v_i = \lambda_{s} (\text{where } s = 2p+1, \ldots, n)\) real eigenvalues. The respective eigenvectors are given by \(x_l = w_l + iq_l\) and \(v_i\). Therefore, we define the following local subspaces (in a small neighborhood of the critical point) in the phase space:

\[
W^s = \text{Span}\{v_s | \lambda_s < 0\},
\]

(27)

\[
W^u = \text{Span}\{v_s | \lambda_s > 0\},
\]

(28)

\[
W^c = \text{Span}\{w_l, q_l | \lambda_l = 0\}.
\]

(29)

The superscripts \(s\), \(u\) and \(c\) denote stable, unstable and center manifolds, respectively. If \(dF_{ij}\) has only pure imaginary eigenvalues, the critical point is called a center. If \(dF_{ij}\) has \(n - 2\) pure
imaginary eigenvalues and two real eigenvalues (one positive and one negative), the critical point is called a *saddle-center*.

In the case of the dynamical system (18)–(21), we obtain

\[
\begin{pmatrix}
\varphi' \\
p_{\varphi}' \\
p_a'
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
-(1 + m^2 a_{\text{cr}}^2) & 0 & 0 \\
0 & 0 & 1/6 \\
0 & 0 & -\frac{\partial^2 V(a)}{\partial a^2} \bigg|_{a=a_{\text{cr}}} \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\varphi \\
p_\varphi \\
a - a_{\text{cr}} \\
p_a
\end{pmatrix}.
\]

(30)

It can be shown that the above matrix has the following eigenvalues:

\[
\mu_{1,2} = \pm \sqrt{1 + m^2 a_{\text{cr}}^2}, \quad \mu_{3,4} = \pm \sqrt{-\frac{V''(a_{\text{cr}})}{6}}.
\]

(31)

By considering the plane \((\varphi, p_\varphi)\), the corresponding eigenvectors of \(\mu_{1,2}\) and \(\mu_{3,4}\) engender the local topology of \(S^1\) around the origin \((\varphi = 0, p_\varphi = 0)\). Thus, the local topology of the critical points is determined by the second derivative of the potential \(V(a)\).

When \(\left.\frac{\partial^2 V(a)}{\partial a^2}\right|_{a=a_{\text{cr}}}>0\), we obtain a local minimum for the potential \(V(a)\) and one more pair of pure imaginary eigenvalues. In this case, by considering the plane \((a, p_a)\), the corresponding eigenvectors of \(\mu_3\) and \(\mu_4\) engender the local topology of \(S^1\) around the point \((a = a_{\text{cr}}, p_a = 0)\). As a consequence, the topology around the critical points with \(\left.\frac{\partial^2 V(a)}{\partial a^2}\right|_{a=a_{\text{cr}}}>0\) is given by \(S^1 \times S^1\) (or 2-torus). We denote by \(P_0\) a critical point with this property. It is called a *center* by definition.

When \(\left.\frac{\partial^2 V(a)}{\partial a^2}\right|_{a=a_{\text{cr}}}<0\), we obtain a local maximum for the potential \(V(a)\) and two real eigenvalues (one positive and one negative). In this case, by considering the plane \((a, p_a)\), the corresponding eigenvectors of \(\mu_3\) and \(\mu_4\) engender the local topology of a *saddle* around the point \((a = a_{\text{cr}}, p_a = 0)\). As a consequence, the topology around the critical points with \(\left.\frac{\partial^2 V(a)}{\partial a^2}\right|_{a=a_{\text{cr}}}<0\) is given by \(R \times S^1\). We denote by \(P_1\) a critical point with this property. It is called a *saddle-center* by definition.

The expansion of the first integral (14) around the critical points reads

\[
H = \frac{1}{12} p_a^2 + \frac{1}{2} V'(a_{\text{crit}})(a - a_{\text{crit}})^2 - \frac{1}{2}\left[p_\varphi^2 + (1 + m^2 a_{\text{crit}}^2)\varphi^2\right] + E_{\text{crit}}
\]

\[= -8\pi G(E_{\text{rad}} - a_4) + \mathcal{O}(3) = 0,\]

(32)

where \(\mathcal{O}(3)\) denote terms of higher order in the expansion and \(E_{\text{crit}} \equiv V(a_{\text{crit}})\) is the energy of the respective critical point. In a small neighborhood of the critical points, these higher order terms can be neglected and the dynamics is nearly separable in the sectors \((a, p_a)\) and \((\varphi, p_\varphi)\) with constants of motion given by

\[
E_a = \frac{1}{12} p_a^2 + \frac{1}{2} V'(a_{\text{crit}})(a - a_{\text{crit}})^2,
\]

(33)

\[
E_{\varphi} = \frac{1}{2}\left[p_\varphi^2 + (1 + m^2 a_{\text{crit}}^2)\varphi^2\right],
\]

(34)

with \(E_a - E_{\varphi} = E_{\text{crit}} - 8\pi G(E_{\text{rad}} - a_4) \sim 0\) and \(|E_{\text{crit}} - 8\pi G(E_{\text{rad}} - a_4)|\) sufficiently small. While in the sector \((\varphi, p_\varphi)\) we have a rotational motion with energy \(E_{\varphi}\) around the critical points, in the sector \((a, p_a)\) we have two possibilities. If \(V''(a_{\text{cr}}) > 0\), we have a rotational motion with energy \(E_a\) in a small neighborhood of the critical point \(P_0\). Otherwise \((V''(a_{\text{cr}}) < 0)\), we obtain a hyperbolic motion around the critical point \(P_1\). The critical point \(P_1\) defines an universe analogous to that of the unstable Einstein static universe \([2]\). On the other hand, the configuration of stable Einstein static universe, corresponding to the critical point \(P_0\), does not possess any classical analogue.
3.3. Separatrices

According to the definition above, from the saddle-center emerges a special structure consisting in two local subspaces. While one of them is generated by the unstable manifold $W^u$, the other is generated by the stable manifold $W^s$. Being transversal to each other, they define the separatrices of the saddle-center critical point.

Let us then assume that one of the initial conditions is given by a point in the phase space which lies on $W^u$ in a neighborhood of the saddle-center critical point. Although this manifold is locally unstable, in general this does not mean that the final stage of the dynamics differs from the saddle-center critical point. In fact, sometimes the nonlinearities of the dynamics may induce an orbit to join the saddle-center point to itself. In this case, the final stage of the dynamics is the very same saddle-center critical point. Orbits with such a property are called homoclinic orbits.

From the saddle-center critical point $P$ (when present) emerges a structure of separatrices $S$ contained in the invariant plane. One of them tends to a de Sitter attractor at infinity, defining an escape of orbits to an accelerated phase regime. In fact, a straightforward analysis of the infinity of the phase space shows the presence of a pair of critical points in this region, one acting as an attractor (stable de Sitter configuration) and the other as a repeller (unstable de Sitter configuration). The scale factor approaches the de Sitter attractor as $a(\eta) \sim (C_0 - \eta)^{-1}$ for $\eta \to C_0$, or $a(t) \sim \exp(t\sqrt{\Lambda/3})$.

4. A simple model

Let us now consider that the noninteracting perfect fluids are given by dust and radiation. In this case, it can be shown that the potential

$$V(a) = 3a^2 - \Lambda a^3 - 8\pi G E_{\text{dust}} a + \frac{\alpha_3}{a^2},$$

will have two extrema (one local minimum and one local maximum) as long as the following conditions hold:

$$27(8\pi GE_{\text{dust}})^2 > \frac{256}{\Lambda},$$

and

$$6\gamma - 4\gamma^3\Lambda - 8\pi GE_{\text{dust}} > \frac{2\alpha_3}{\gamma},$$

where

$$\gamma = \frac{\rho}{12\Lambda} + \frac{4}{\rho},$$

$$\rho^3 = \left[ -108(8\pi GE_{\text{dust}}) + 12\sqrt[3]{27(8\pi GE_{\text{dust}})^2 - \frac{256}{\Lambda}} \right] \Lambda^2.$$

It can be numerically shown (cf figure 1) that the increase $E_{\text{dust}}$ has the effect of spoiling the potential configuration with two extrema.

If $m = 0$, the dynamics is integrable and thus separable. In this case, the first integral (14) reads

$$\frac{p_u^2}{12} + V(a) = 8\pi G(E_{\text{rad}} - \alpha_4) + \mathcal{E}_0 \equiv E_{\text{a}}^0,$$

and the scalar field behaves just like a radiation component in the dynamics of the scale factor.
Figure 1. The potential $V(\alpha)$ for several values $\delta \equiv \varepsilon_{\text{dust}}$. For higher values of $\delta$, the minimum of the potential is no longer present. For $\delta \approx 2.3$, the two extrema of the potential (connected to the stable and unstable Einstein static universes) vanish. Here, we fixed $\Lambda_1 = 1.5$, $\alpha_3 = 10^{-3}$ and $8\pi G = 1$.

Figure 2. The phase portrait of invariant plane dynamics with the critical points $P_0$ (center) and $P_1$ (saddle-center) corresponding to stable and unstable Einstein universes. The periodic orbits of region I describe perpetually bouncing universes. Orbits in regions II and III are solutions of one-bounce universes. A separatrix $S_1$ emerges from $P_1$ defining an escape to the de Sitter attractor. Here, we fixed $\Lambda_1 = 1.5$, $\alpha_3 = 10^{-3} = \varepsilon_{\text{dust}}$, $E_{\text{rad}} = 0.5$ and $8\pi G = 1$.

In figure 2, I exhibit the phase portrait in the invariant plane $\varphi = 0 = p_{\varphi}$. With suitable values for the parameters, the two extrema of $V(\alpha)$ are connected to $P_0$ and $P_1$. This model furnishes us with perpetually bouncing universes (periodic orbits in region I). The two separatrices $S_1$ and $S_2$ that emerge from $P_1$ coalesce generating a boundary for region I.
This boundary defines an homoclinic orbit by definition. Orbits in regions II and III correspond to universes with one bounce only.

Now, I focus on some structural differences between the integrable dynamics in the invariant plane and the integrable dynamics given by $m = 0$. If $\phi(0)$ and/or $p\phi(0)$ do not vanish (in the integrable case $m = 0$), the phase portrait in the plane $(a, p_a)$ is analogous to that of the invariant plane (cf figure 2). Let $X_0$ and $X_1$ be the analogous to that of $P_0$ and $P_1$ (connected to the potential extrema). In this case, $X_0$ and $X_1$ would no longer be critical points. Instead, they define stable and unstable periodic orbits respectively. Due to the system’s periodicity in a neighborhood of $P_0$, orbits are confined on two-dimensional surfaces in the phase space which topologically coincide with 2-tori. Therefore, the integrable dynamics ($m = 0$) is not constrained in the invariant plane but, in a neighborhood of $P_0$, evolves on 2-tori which are the direct product of closed curves (analogous to that of in region I) with periodic orbits in the sector $(\phi, p\phi)$.

Another topological feature to be considered—in the integrable case $m = 0$—is the dynamics of orbits in a neighborhood of $P_1$. Given the initial conditions $p\phi(0) = 0$ and $a_0 = a|_{X_1}$, the motion of the system corresponds to an unstable periodic orbit which generates a circle in the sector $(\phi, p\phi)$. Let $\Gamma$ denotes one of these orbits. Therefore, the direct product of periodic orbits in the sector $(\phi, p\phi)$ with the stable and unstable manifolds (analogous to that of $S_1$ and $S_2$) generates semi-cylinders which coalesce in the very same orbit $\Gamma$. When we consider a nonlocal description of the topology, the nonlinearities induce such cylinders to close onto themselves. As the projection of such cylinders in the sector $(a, p_a)$ defines a homoclinic orbit, we call these structures as homoclinic cylinders. On the other hand, the direct product of periodic orbits in the sector $(\phi, p\phi)$ with the analogous to that of separatrices $S_3$ and $S_4$ generates semi-infinite cylinders which coalesce in the orbit $\Gamma$. A summary of structural differences between the dynamics in the invariant plane and the integrable case $m = 0$ is given in the table below.

| Dynamics in the invariant plane: | Integrable dynamics $m = 0$: |
|--------------------------------|-------------------------------|
| No motion in the sector $(\phi, p\phi)$ | Separable motion in the sectors $(a, p_a)$ and $(\phi, p\phi)$ |
| Critical points $P_0$ and $P_1$ | Stable and unstable periodic orbits |
| Periodic orbits (region I) | Integrable tori |
| Separatrices between regions I and II | Homoclinic cylinders |
| Separatrices between regions II and III | Semi-infinity integrable cylinders |

5. Nonlinear resonance of KAM tori

According to the Liouville–Arnold theorem [21], if the motion of a Hamiltonian system with $n$ degrees of freedom is integrable and bounded, then the orbits of such a system are confined on $n$-dimensional hyper-surfaces in the phase space which topologically coincide with $n$-tori. That is exactly what occurs with our system when we consider the integrable case $m = 0$ in a neighborhood of $P_0$.

In order to give a more precise analysis, let us consider the surfaces with energy $E_0^\phi = 8\pi G(E_{\text{rad}} - a_4) + E_0^{\phi} < E_{\text{crit}}(P_1) \equiv V(P_1)$. This region of the phase space is foliated by 2-tori $S^1 \times S^1$ which are the topological product of periodic orbits of the separable sectors $(\phi, p\phi)$ and $(a, p_a)$. $E_0^{\phi}$ and $E_0^a$ are conserved quantities for those orbits. The frequency $\nu_a$ of the periodic orbit in the sector $(a, p_a)$ is given by the integral

$$\frac{1}{\nu_a} = \sqrt{\frac{12}{\Lambda}} \int_{P_1}^\beta \frac{a \, da}{\sqrt{\prod_{i=1}^6 (a - \beta_i)}},$$

(39)
where \( \beta_i \) \((i = 1, \ldots, 6)\) are the real roots of \( V(a) = 8\pi G(E_{\text{rad}} - \alpha_s) + \varepsilon_0^\alpha \). Here, \( I \) denote \( \beta_i \) \((i = 1, \ldots, 3)\) as the three positive real roots with \( \beta_1 < \beta_2 < \beta_3 \). On the other hand, the periodic orbits in the sector \((\varphi, p_\varphi)\), parameterized by \( \varepsilon_0^\alpha \), have the frequency \( \nu_\varphi = 1/2\pi \).

The importance of \( n \)-tori in integrable Hamiltonian systems comes from the fact that these surfaces trap the dynamics in a finite region of the phase space. In our case, such tori in a neighborhood of \( P_0 \) avoid an exit to the de Sitter attractor. However, a relevant question which arises is whether such tori ‘survive’ when one introduces a small perturbation connected to the mass \( m \) of the scalar field. Assuming initial conditions \((\varphi_0, p_\varphi 0)\) sufficiently close to the invariant plane, equation (15) may be rewritten as

\[
\varphi'' + \left( 1 + m^2 \alpha_0^2(\tau) \right) \varphi = 0,
\]

(40)

where \( \alpha_0(\eta) \) is the background solution for the scale factor of the integrable dynamics (17). Equation (40) is a Lamé equation. Defining \( \nu_\varphi \) as the frequency in the \((\varphi, p_\varphi)\) given by (40), the resonance phenomena \( [21] \) will occur when the ratio

\[
R \simeq \frac{v_\varphi}{\nu_\varphi}
\]

(41)

is a rational number. Expanding \( \alpha_0(\eta) \) in the Lamé equation, one can show that

\[
\tilde{\nu}_\varphi \simeq \frac{1}{2\pi} \sqrt{1 + \frac{(0.9m)^2}{2} \left( \beta_1^2 + \beta_2^2 \right)}.\]

(42)

However, as the system evolves, the amplitude of the scalar field may grow, so that the solution of the integrable case \( \alpha_0(\eta) \) is no longer a good approximation to be introduced in (15). As we shall see in the following section, this process may lead the dynamics to a more unstable behavior, with the amplification of the resonance mechanism and the break of the KAM tori \([22]\). To analytically show this behavior, let us now consider the following approximation of constraint (14):

\[
\mathcal{H} \equiv E_a^0 - \varepsilon_0^\alpha - \frac{1}{2} m^2 \alpha_0^2(\eta) \varphi^2(\eta) \simeq 8\pi G(E_{\text{rad}} - \alpha_s),
\]

(43)

where \( \varphi(\eta) \) is an approximate solution of the Lamé equation. Now, I introduce the action–angle variables \( [21] \) \((\Theta_\varphi, J_\varphi, \Theta_\alpha, J_\alpha)\). The angle variables are defined by \((\Theta_\varphi = \tilde{\nu}_\varphi \eta, \Theta_\alpha = \varphi_\eta)\) in such a way that they span the interval \([0, 1]\) during a complete period of the system. Taking into account that the function \( \alpha_0(\eta) \) is periodic with period \( T_a = \nu_\varphi^{-1} \), the expansion of the non-integrable term of (43) is given by \( [23] \)

\[
- \frac{1}{2} m^2 \alpha_0^2(\eta) \varphi^2(\eta) = - \frac{1}{2} m^2 J_\alpha(0) J_\varphi(0) \sum_n (A_n \cos 2n\pi \Theta_\alpha) \cos 4\pi \Theta_\varphi,
\]

(44)

where \( A_n \) are constant coefficients. The superior index in \( J_\varphi \) and \( J_\alpha \) denotes that these are the action variables for the integrable case. The Hamilton equation for \( J_\alpha \), derived from (43), can then be integrated furnishing us in the first approximation with

\[
J_\alpha \sim \frac{1}{2} m^2 J_\alpha^{(0)} J_\varphi^{(0)} \sum_n \frac{A_n}{2\pi n \nu_\varphi} \left[ \frac{\cos(2\pi n(\Theta_\alpha - 4\pi \Theta_\varphi))}{R - 2/n} + \frac{\cos(2\pi n(\Theta_\alpha + 4\pi \Theta_\varphi))}{R + 2/n} \right].
\]

(45)

From (45), we see that the dominant resonance terms are those for which \( R \simeq 2/n \). It can be shown that the condition \( n \geq 2 \) must hold in order to obtain a real positive numerical value for the mass \( m \). Therefore,

\[
R \simeq \frac{v_\varphi}{\nu_\varphi} \simeq \frac{2}{n}, \quad (n \geq 2)
\]

(46)

is a good approximation in order to determine the resonances of the dynamical system (18)–(21) in the presence of dust and radiation. When such resonances occur, one can eventually
Figure 3. Poincaré map with section $p_\alpha = 0$ for several initial conditions with $m = 9.6$ in the domain of parametric resonance $n = 3$. Here, we see that the resonance of the exact dynamics is connected to the bifurcation of the stable periodic orbit at the origin ($\phi = 0, p_\phi = 0$). That is, when the resonance occurs this stable periodic orbit turns into an unstable periodic orbit. In this case, no KAM tori are present around the origin of the map so orbits with initial conditions in a neighborhood of the origin may escape to the de Sitter attractor. The presence of such bifurcation is a crucial feature in order to allow the dynamics to an exit to the de Sitter attractor.

obtain a loss of stability with the break of the KAM tori [22], allowing the dynamics to an exit to the de Sitter attractor.

Let us now consider a Poincaré map in the variables ($\phi, p_\phi$) with section $p_\alpha = 0$. As a convention, I will assume that this map is unidirectional. That is, a chosen orbit of the system crosses the plane of section $p_\alpha = 0$ only once after a period of time $T_\alpha$. If $E_\phi^0 = 0$, the periodic orbits in the sector ($a, p_\alpha$) correspond to the point ($\phi = 0, p_\phi = 0$). For $E_\phi^0 \neq 0$ and $m = 0$, the tori are characterized by closed curves around the origin of this map. For a small value of $m$, this configuration is maintained around the origin ($\phi = 0, p_\phi = 0$). In fact, according to the KAM theorem [24], if $\nu_a$ and $\nu_\phi$ are sufficiently irrationals (that is, satisfy the diophantine condition), then the solutions of the perturbed system are quasi-periodic for a sufficiently small value of $m$.

In order to compare the approximation (46) with the exact dynamics, I will perform a numerical analysis of the evolution of the system in the Poincaré map with section $p_\alpha = 0$. For computational simplicity, I will fix $\lambda = 1.5, E_{\text{rad}} = 0.5, \alpha_3 = 10^{-3} = E_{\text{dust}}$ and $8\pi G = 1$. In this way, one can define the parametric space ($\alpha_4, m$) where the resonances may occur. For several initial conditions around ($p_\alpha = 0, \phi_0 = 0$), I construct the Poincaré map with $m = 9.6$ (figure 3) and $m = 11.5$ (figure 4). According to approximation (46), the first map shows the resonant behavior of the system for $n = 3$. The second map shows the pattern of parametric stability in a region between the resonances $n = 3$ and 4. The structure of the stochastic sea [25] in figure 3 shows that initial conditions near the invariant plane can generate orbits with a long time of diffusion before escaping to the de Sitter attractor.

In figure 5, I numerically construct the resonance chart using the exact dynamics. Taking the initial conditions $p_{\alpha_0} = 0 = p_{\phi_0}$ and $\phi_0 = 10^{-3}$, the value of $a_0$ is obtained by substituting the numerical values of $\alpha_4$ and $m$ in the constraint (14). For a suitable value of $\alpha_4$, the approximate expression (46) is an accurate guide in order to localize the respective values of $m$ in which the resonances in the parametric space ($\alpha_4, m$) occur. The dashed curves (cf figure 5) in the parametric space ($\alpha_4, m$) are constructed using the approximation (46) and they allow us to localize a given domain of resonance. As I previously pointed out, the dominant resonances
Figure 4. Poincaré map of section $p_\varphi = 0$ for several initial conditions with $m = 11.5$ in the domain of parametric stability between the resonances $n = 3$ and $n = 4$. The topology of the system around the origin (which defines the stable periodic orbit) corresponds to a 2-torus. In this case, the remaining KAM tori from the integrable case trap the orbits with initial conditions in a neighborhood of the origin avoiding an escape to the de Sitter attractor. Therefore, the region of parametric stability of the system does not favor late-time accelerating scenarios.

Figure 5. Resonance chart in the plane $(\alpha_4, m)$. The dashed lines are solutions of the conditions of resonance (46). The hatched areas correspond to resonance domains of the exact dynamics. These domains correspond to regions where the bifurcation of periodic orbits at the origin (in the Poincaré map of section $p_\varphi = 0$) occurs. The white remaining region corresponds to the domain of parametric stability where the dynamics is trapped by the KAM tori. For computational simplicity, I am still fixing the parameters $\Lambda = 1.5$, $E_{\text{rad}} = 0.5$, $\alpha_3 = 10^{-3} = E_{\text{dust}}$ and $8\pi G = 1$.

of the system are connected to the bifurcation of stable periodic orbits at the origin. Although the approximation (46) allows us to identify the curves in the parametric space $(\alpha_4, m)$ where the resonances occur, the effect of the exact dynamics tends to stretch these domains. In fact, as one may numerically verify, for a fixed value of $\alpha_4$, there is a continuum domain of values of $m$ (where the bifurcation of stable periodic orbits at the origin occurs) for each resonance. The windows of exact resonance are shown as hatched regions in figure 5.
6. The resonance pattern

The resonance regions in the parametric space \((\alpha_4, m)\) possess a substructure which I now examine. In order to simplify this analysis, I will restrict myself to the resonance domain \(n = 3\) (cf figure 5) with \(\alpha_4 = 0\). For these fixed values, the resonance occurs in the interval \(\Delta m \cong [8.7, 9.8]\). In this interval, one can notice three distinct regions.

(i) For \(m < 9.3\), the dynamics is highly unstable and the resonances provide a rapid escape to the de Sitter attractor. In figure 6, the behavior of \(a\) and \(\phi\) (with respect to the conformal time) is shown by taking \(m = 8.8\). In this figure, one can observe that a rapid escape to the de Sitter attractor occurs when \(\eta \simeq 70\), so there is no enough recurrence in order to construct a Poincaré map.

(ii) When \(9.7 < m < 9.8\), the motion of orbits is resonant and chaotic, although stable. In figure 7, the behavior of an orbit in this region is exhibited for \(m = 9.8\). This is the pattern close to the right edge of resonance \(n = 3\). Due to its stability, these orbits are not interesting from the late-time accelerating point of view. It is worth mentioning that this behavior is in agreement to the analytical description regarding cyclic universes given in [20].

(iii) A region of transition occurs when \(9.3 < m \leq 9.7\). In this case, orbits go through a long time of diffusion before escaping to the de Sitter attractor. In figure 8, a Poincaré map (with \(p_a = 0\)) of an orbit with \(m = 9.7\) is shown. This map illustrates what happens in the above interval. This is an example of how orbits can go through a long time of diffusion before escaping to the de Sitter attractor.

The above substructure is a pattern which is maintained for every value of \(\alpha_4\) in the resonance zone \(n = 3\). Furthermore, it can be shown that this pattern is qualitatively equivalent for every value of \(n\). Throughout this analysis, one notices that nonsingular perpetually bouncing models from Hořava–Lifshitz possess a restrict domain in the parametric space where late-time acceleration (connected to the de Sitter attractor) may be realized. For typical variations of the parameters \(E_{\text{dust}}\) and/or \(\alpha_3\), the domains of the parametric space \((\alpha_4, m)\)—where the system is resonant—can be stretched or shrunken. Nevertheless, the pattern in
resonance windows and its substructure are maintained as one may numerically verify. In this sense the pattern is said to be structurally stable.

7. Conclusions

In this paper, I examine the effect of parametric resonance in bouncing cosmologies originating from Hořava–Lifshitz gravity. In this context, terms arising from foliation preserving diffeomorphism invariance—which breaks 4D covariance—implement nonsingular bounces in the early evolution of the Universe. The matter content of the model is given by perfect fluids,
namely dust and radiation. Furthermore, I also assume a nonvanishing cosmological constant (connected to a de Sitter attractor in the phase space which provides late-time acceleration) and a massive conformally coupled scalar field.

By considering the case of closed geometries, I obtain a potential well with a local minimum and a local maximum (cf figure 1) respectively connected to the critical points $P_0$ (center) and $P_1$ (saddle-center) in the phase space. Assuming a conformally coupled scalar field, the oscillatory behavior of the dynamical system around $P_0$ might become metastable when the system is driven into a resonance window of the parameter space—labeled by an integer $n \geq 2$. In this case, I determine the physical domain of the parameters (cf figure 5) in which the breakup of KAM tori may occur, leading the Universe to a late-time acceleration regime.

It is worth mentioning that, as examined in [26], a chaotic exit to accelerated expansion can also be realized—in the dynamical system (18)–(21)—if one assumes initial condition sets taken in a small neighborhood of the stable separatrix $S_1$. These sets possess fractal basin boundaries connected to a code recollapse/escape leading to a chaotic exit to an accelerated regime.

Although the cosmological constant poses a crucial problem to quantum field theory on how to match its observed value with vacuum energy calculations, the cosmological constant is by far the simplest explanation for the present acceleration of the Universe. Indeed, the $\Lambda$CDM standard model assumes that there exists a cosmological constant which becomes dynamically important when the typical scale of the Universe has the size of the present Hubble radius. In this sense, the model of this paper does not exhibit an alternative explanation for late-time acceleration. Instead, the core of this paper is to examine the dynamics in the phase space of the above model, showing how to provide an alternative exit to late-time acceleration.

**Acknowledgments**

I acknowledge financial support of CNPq/MCTI-Brazil through a post-doctoral grant no. 201907/2011-9. I would like to thank David Wands for his useful comments and suggestions. I would also like to acknowledge the Institute of Cosmology and Gravitation for their hospitality. Figures were generated using the Wolfram Mathematica 7 and DYNAMICS SOLVER packet [27].

**References**

[1] Mukhanov V 2005 *Physical Foundations of Cosmology* (Cambridge: Cambridge University Press)
[2] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)
[3] Penrose R 1965 *Phys. Rev. Lett.* 14 57
[4] Abbott L F and Pi S-Y 1986 *Inflationary Cosmology* (Singapore: World Scientific)
[5] Riess A G et al 1998 *Astron. J.* 116 1009
   Perlmutter S et al 1999 *Astrophys. J.* 517 565
   Tonry J L et al 2003 *Astrophys. J.* 594 1
   John M V 2004 *Astrophys. J.* 614 1
   Astier P et al 2006 *Astron. Astrophys.* 447 31
   Riess A G et al 2007 *Astrophys. J.* 659 98
   Rubin D et al 2009 *Astrophys. J.* 695 591
   Hicken M et al 2009 *Astrophys. J.* 700 1097
[6] Weinberg S 1989 *Rev. Mod. Phys.* 61 1–23
[7] Novello M and Perez Bergliaffa S E 2008 *Phys. Rep.* 463 127
[8] Tolman R C 1931 *Phys. Rev.* 38 1758
   Murphy G 1973 *Phys. Rev. D* 8 4231
Novello M and Salim J M 1979 Phys. Rev. D 20 377
Melnikov V and Orlov S 1979 Phys. Lett. A 70 263
Acacio de Barros J, Pinto-Neto N and Sagioro-Leal M A 1998 Phys. Lett. A 241 229
Colistete R Jr, Fabris J C and Pinto-Neto N 2000 Phys. Rev. D 62 083507
Khouri J, Ovrut B A, Steinhardt P J and Turok N 2001 Phys. Rev. D 64 123522
De Lorenzi V A, Klippert R, Novello M and Salim J M 2002 Phys. Rev. D 65 063501
Alvarenga F G, Fabris J C, Lemos N A and Monerat G A 2002 Gen. Rel. Grav. 34 651
Fabris J C, Furtado R G, Peter P and Pinto-Neto N 2003 Phys. Rev. D 67 124003
Ashketar A, Bojowald M and Lewandowski J 2003 Adv. Theor. Math. Phys. 7 233
Biswas T, Brandenberger R, Mazumdar A and Siegel W 2007 J. Cosmol. Astropart. Phys. JCAP12(2007)011
Abramo L R, Peter P and Yasuda I 2010 Phys. Rev. D 81 023511
Peter P and Pinto-Neto N 2008 Phys. Rev. D 78 063506
Cai Y F, Brandenberger R and Zhang X 2011 J. Cosmol. Astropart. Phys. JCAP03(2011)003

[9] Maier R, Pereira S, Pinto-Neto N and Siffert B B 2012 Phys. Rev. D 85 023508
[10] Hoˇrava P 2009 Phys. Rev. D 79 084008
[11] Calcagni G 2009 J. High Energy Phys. JHEP09(2009)112
[12] Brandenberger R 2009 Phys. Rev. D 80 043516
[13] Charmousis C, Niz G, Padilla A and Saffin P M 2009 J. High Energy Phys. JHEP08(2009)070
[14] Sotiriou T P 2011 J. Phys. Conf. Ser. 283 012034
[15] Blas D, Pujolas O and Sibiryakov S 2009 J. High Energy Phys. JHEP10(2009)029
[16] Bogdanos C and Saridakis E N 2010 Class. Quantum Grav. 27 075005
[17] Koyama K and Arroja F 2010 J. High Energy Phys. JHEP03(2010)061
[18] Sotiriou T P, Visser M and Weinfurtner S 2009 J. High Energy Phys. JHEP10(2009)033
[19] Papazoglou A and Sotiriou T P 2010 Phys. Lett. B 685 197–200
[20] Cai Y F and Saridakis E N 2009 J. Cosmol. Astropart. Phys. JCAP10(2009)020
[21] Arnold V I 1989 Mathematical Methods of Classical Mechanics (Berlin: Springer)
[22] Berry M 1978 AIP Conf. Proc. 46 16–120
[23] Abramowitz M and Stegun I 1964 Handbook of Mathematical Functions (NBS Applied Mathematical Series vol 55) (Washington, DC: National Bureau of Standards)
[24] Kolmogorov A N 1979 Stochastic Behaviour in Classical and in Quantum Hamiltonian Systems (Lecture Notes in Physics vol 93) ed G Casati and J Ford (Berlin: Springer)
Arnold V I 1963 Russ. Math. Surv. 18 9
Moser J 1962 Nachr. Akad. Wiss. Goett.II Math.-Phys. Kl. 1 1–20
[25] Zaslavsky G M, Sagdeev R Z, Usikov D A and Chernikov A A 1991 Weak Chaos and Quasi-Regular Patterns (Cambridge: Cambridge University Press)
[26] Maier R, Soares I D and Tonini E V 2009 Phys. Rev. D 79 023522
[27] Aguirregabiria J M Dynamics solver http://tp.lc.ehu.es/jma.html