AN EFFICIENT NUMERICAL APPROACH FOR STOCHASTIC EVOLUTION PDES DRIVEN BY RANDOM DIFFUSION COEFFICIENTS AND MULTIPLICATIVE NOISE*

XIAO QI$^1$ MEJDI AZAIEZ$^{1,2}$ CAN HUANG$^1$ CHUANJU XU$^{1,3}$

Abstract. In this paper, we investigate the stochastic evolution equations (SEEs) driven by log-Whittle-Matérn (W-M) random diffusion coefficient field and $Q$-Wiener multiplicative force noise. First, the well-posedness of the underlying equations is established by proving the existence, uniqueness, and stability of the mild solution. A sampling approach called approximation circulant embedding with padding is proposed to sample the random coefficient field. Then a spatio-temporal discretization method based on semi-implicit Euler-Maruyama scheme and finite element method is constructed and analyzed. An estimate for the strong convergence rate is derived. Numerical experiments are finally reported to confirm the theoretical result.

1. Introduction

Stochastic partial differential equations (SPDEs) appears in many fields of science and engineering, and have been subject of many theoretical and numerical investigations. It is commonly believed that incorporating noise and/or uncertainty into models is closer to reality in mathematical modeling, due to the existence of uncertainty stemming from various sources such as thermal fluctuation, impurities of materials and so on. As an active area of research, numerical study of stochastic evolution equations (SEEs) has attracted increasing attention in the past decades; see, e.g., monographs [32, 41, 43, 37, 26, 56] and references therein. Although much progress has been made, it is still far from being satisfactory due to the numerical approximations to SEEs encounter all the difficulties that may arise in solving deterministic differential equations on one hand, and caused by the infinite dimensional nature of the driving noise processes on the other hand. The present work focus on the SEEs perturbed by a smooth random diffusion coefficient field as well as multiplicative force noise, and aims to propose and analyze an efficient numerical method for this equation.

When considering the numerical approaches for SEEs with various noises, two categories of convergence errors may be involved, namely weak error and strong error. The former is related

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$^1$School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High Performance Scientific Computing, Xiamen University, 361005 Xiamen, China.

$^2$Bordeaux INP, Laboratoire I2M UMR 5295, 33607 Pessac, France.

$^3$Corresponding author. Email: cjxu@xmu.edu.cn.
to the approximation of the probability law of the solution. Concerning weak convergence error of numerical methods for SEEs, we refer to, for instance, [45, 25, 12, 14, 20, 13, 33, 34, 38, 53, 6, 2, 9, 7] and references therein for a list of literature in this direction. Unlike weak convergence error, the strong convergence error measures the deviation from the trajectory of an exact solution. It has been extensively investigated in various types of SPDEs, see, e.g., [1, 11, 17, 55, 50, 28, 23, 31, 8, 51, 52, 35, 27, 36, 46, 22, 19, 5, 29, 39, 40] and references therein. We mention here some works on strong convergence of the numerical schemes for linear SEEs with additive or multiplicative noise. For example, Allen et al. [1] described, analyzed and compared the finite element and difference methods for parabolic SPDEs driven by additive white noise. Du et al. [17] investigated numerical solutions of linear SEEs perturbed by special additive noises, ranging from the space time white noise to colored noises generated by some infinite dimensional Brownian motions with a prescribed covariance operator. Yan [55] studied the finite element method for linear SEEs with multiplicative noise in multidimensional case. The case of strong convergence of nonlinear SEEs is generally more subtle and challenging, and has received widely attention in the research community in recent years. For instance, Kloeden et al. [28, 31] proposed a discretization based on the Galerkin method in space and exponential integrator in time for the nonlinear SEEs with cylindrical additive noise. Kruse [36] analysed the strong convergence error for a finite element method/linear implicit Euler spatio-temporal discretization of semilinear SEEs with multiplicative noise and Lipschitz continuous nonlinearities, and deduced the optimal error estimates. Wang [51] derived strong convergence results for a spatio-temporal discretization of the semilinear SEEs with additive noise, where the approximation in space was performed by a standard finite element method and in time by a linear implicit Euler method. Moreover it was shown how exactly the strong convergence rate of the full discretization relies on the regularity of the driven process. Kovács et al. [35] used Euler type splitstep method to study the semidiscretisation in time of the stochastic Allen-Cahn equation perturbed by smooth additive Gaussian noise, and showed that the strong convergence rate is 1/2 with respect to the step size. Liu et al. [40] proposed a general theory of optimal strong error estimation for some drift-implicit Euler schemes of a second-order nonlinear SPDE with monotone drift driven by a multiplicative infinite-dimensional Wiener process.

In this paper, we consider the SEEs with both multiplicative force noise and random diffusion coefficient field, which has not yet been addressed in the literature to the best of our knowledge. The main contributions/novelties of this paper are as follows:

- The well-posedness of the considered stochastic equation is established. That is, the existence, uniqueness, and stability of the mild solution is proved.
- The diffusion coefficient considered in the current work is a log-Whittle-Matérn Gaussian random field with a parametrized covariance function whose regularity can be controlled by a parameter. Therefore different cases can be tested and compared in a convenient way.
- A sampling approach called approximation circulant embedding with padding [16, 54, 44] is employed to render the equation solvable. Then for each sample diffusion coefficient, a time-stepping scheme based on a semi-implicit Euler-Maruyama approach is constructed for the resulting equation. The standard piecewise linear finite element method is employed for the
Sobolev spaces, γ \| \cdot \|_2 where the norm \( L^2 \) throughout the paper we use \( \gamma \) orthonormal eigenfunctions \( \{ \phi_j(x) \in H_0^2(D) : j \in \mathbb{N} \} \) and corresponding positive eigenvalues \( \{ q_j \} \); see, e.g., [55, 36, 51] for more details.

We start by defining our problem. Let \( T > 0, D := (0,1), L^2(D) \) and \( H_0^2(D) \) are classical Sobolev spaces, \( \gamma \geq 0 \). \( L(L^2(D)) \) represents the space of bounded linear operators \( A : L^2(D) \to L^2(D) \) equipped with operator norm \( \| A \|_{L(L^2(D))} = \sup_{u \neq 0} \frac{\| Au \|_{L^2(D)}}{\| u \|_{L^2(D)}} \). \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) is a filtered probability space with a normal filtration. Let \( W(t,x) \) be a \( \mathcal{F}_t \)-adapted \( H_0^2(D) \)-valued Wiener process with covariance operator \( Q \), where \( Q \) is a positive definite and symmetric operator with orthonormal eigenfunctions \( \{ \phi_j(x) \in H_0^2(D) : j \in \mathbb{N} \} \) and corresponding positive eigenvalues \( \{ q_j \} \); see, e.g., [55, 36, 51] for more details.

Let \( Q^\frac{1}{2}(H_0^2(D)) := \{ Q^\frac{1}{2}v : v \in H_0^2(D) \} \). Let \( \mathcal{L}_Q \) be the set of linear operators \( B : Q^\frac{1}{2}(H_0^2(D)) \to L^2(D) \), which satisfies

\[
\left( \sum_{j=1}^\infty \| BQ^\frac{1}{2}\phi_j \|_{L^2(D)}^2 \right)^{\frac{1}{2}} < +\infty.
\]

\( \mathcal{L}_Q \) endowed with the norm \( \| B \|_{\mathcal{L}_Q} := \left( \sum_{j=1}^\infty \| BQ^\frac{1}{2}\phi_j \|_{L^2(D)}^2 \right)^{\frac{1}{2}} \) is actually the space of Hilbert-Schmidt operators [21]. We will also use the space \( L^2(\Omega, \mathcal{L}_Q) \) of all random Hilbert-Schmidt operators \( B : \Omega \to \mathcal{L}_Q \), equipped with the norm

\[
\| B(\omega) \|_{L^2(\Omega, \mathcal{L}_Q)} := \mathbb{E}[\| B(\omega) \|_{\mathcal{L}_Q}^2]^{\frac{1}{2}}.
\]

Throughout the paper we use \( c \), with or without subscripts, to mean generic positive constants (independent of \( \omega \) in particular), which may not be the same at different occurrences.

2. Problem and its well-posedness

We start by defining our problem. Let \( T > 0, D := (0,1), L^2(D) \) and \( H_0^2(D) \) are classical Sobolev spaces, \( \gamma \geq 0 \). \( L(L^2(D)) \) represents the space of bounded linear operators \( A : L^2(D) \to L^2(D) \) equipped with operator norm \( \| A \|_{L(L^2(D))} = \sup_{u \neq 0} \frac{\| Au \|_{L^2(D)}}{\| u \|_{L^2(D)}} \). \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) is a filtered probability space with a normal filtration. Let \( W(t,x) \) be a \( \mathcal{F}_t \)-adapted \( H_0^2(D) \)-valued Wiener process with covariance operator \( Q \), where \( Q \) is a positive definite and symmetric operator with orthonormal eigenfunctions \( \{ \phi_j(x) \in H_0^2(D) : j \in \mathbb{N} \} \) and corresponding positive eigenvalues \( \{ q_j \} \); see, e.g., [55, 36, 51] for more details.

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\]

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\[
\| B(\omega) \|_{L^2(\Omega, \mathcal{L}_Q)} := \mathbb{E}[\| B(\omega) \|_{\mathcal{L}_Q}^2]^{\frac{1}{2}}.
\]

Throughout the paper we use \( c \), with or without subscripts, to mean generic positive constants (independent of \( \omega \) in particular), which may not be the same at different occurrences.
Our point of interest is the SEE with random diffusion coefficient and multiplicative noise, written in the abstract form:

\[ du(x, t) = (-Lu + f(u))dt + G(u)dW(x, t), \quad 0 < t < T, \quad x \in D, \]
(2.2)

\[ u(x, t) = 0, \quad 0 \leq t \leq T, \quad x \in \partial D, \]
\[ u(x, 0) = u_0(x), \quad x \in \bar{D}, \]

where \( L \) is the elliptic operator \(-\partial_x(a(x, \omega)\partial_x)\) with the coefficient \( a(x, \omega) \) being a log-Gaussian random field with the scale parameter \( \varepsilon \), i.e.,

\[ a(x, \omega) = \varepsilon e^{z(x, \omega)}. \]

This type of random diffusion coefficient field has received a lot of attention in the study of uncertainty quantification (UQ) problems \([3, 41]\), and appeared in some applications, e.g., geostatistical modelling \([19, 30]\). We consider the random field \( z(x, \omega) \) in (2.3) to be a mean-zero Whittle-Matérn Gaussian random field, which is a stationary random field with the covariance function

\[ c_q(x) := \frac{\sqrt{2\Gamma(q + 1/2)}}{\Gamma(q)} \int_0^\infty \left(\frac{2}{\pi}\right)^{1/2} \cos(\lambda x) \frac{1}{(1 + \lambda^2)^{q/2}} d\lambda, \quad x \in [0, 1], \quad q > 2, \]
(2.4)

where \( \Gamma(\cdot) \) is the Gamma function.

The theoretical result established in this paper depends on the following assumption on the nonlinear drift term \( f(\cdot) \):

\[ \|f(v)\|_{L^2(D)} \leq c(1 + \|v\|_{L^2(D)}), \quad \forall v \in L^2(D), \]
(2.5)

\[ \|f(v_1) - f(v_2)\|_{L^2(D)} \leq c(\|v_1 - v_2\|_{L^2(D)}), \quad \forall v_1, v_2 \in L^2(D). \]
(2.6)

These assumptions are often used to prove the existence and uniqueness of the solution for SPDEs, see, e.g., \([36, 41]\).

We are interested in the mild solution of problem (2.2) in the Itô sense \([10]\), defined by

\[ u(t) = S(t)u_0 + \int_0^t S(t - \tau)f(u(\tau))d\tau + \int_0^t S(t - \tau)G(u(\tau))dW(\tau), \]
(2.7)

where \( S(t) := e^{-tL} \) is a semigroup generated by the operator \( L \) \([15]\). The well-posedness of the problem (2.2) thus consists in verifying that the integrals in (2.7) are well defined and a function \( u \) satisfying the integral equation (2.7) uniquely exists. We first notice that the realization of the random field \( a(x, \omega) \) given in (2.3) is \( 2 \) times mean-square differentiable due to \( q > 2 \) \([41]\). Thus, almost surely (\( \mathbb{P}\)-a.s.), \( a(x, \omega) \in C^1(\bar{D}) \) and \( 0 < a_{\min}(\omega) \leq a(x, \omega) \leq a_{\max}(\omega) < \infty \), where \( a_{\min}(\omega) \) and \( a_{\max}(\omega) \) represent respectively the essential infimum and supremum of \( a(x, \omega) \).

In order to well define the integral \( \int_0^t S(t - s)G(u(s))dW(s) \) and prove the existence and uniqueness of mild solution (2.7), we assume that there exists \( a_{\min} \) and \( a_{\max} \) such that

\[ 0 < a_{\min} \leq a_{\min}(\omega) \leq a_{\max}(\omega) \leq a_{\max} < +\infty, \quad \mathbb{P}\)-a.s.
(2.8)

One verifies readily that \( \mathcal{D}(L) = H^2(D) \cap H_0^1(D) \) almost surely \([4]\), where \( \mathcal{D}(L) \) is the domain of the operator \( L \).

We also need some assumptions on the nonlinear term \( G \), which are collected below:
- $L^sG(\cdot), 0 \leq s \leq \frac{1}{2}$, is a mapping from $L^2(D)$ to $L_Q$ such that:

$$
\|L^sG(v)\|_{L_Q} \leq c(1 + \|v\|_{L^2(D)}), \quad \forall v \in L^2(D),
$$

(2.9)

$$
\|L^s(G(v_1) - G(v_2))\|_{L_Q} \leq c\|v_1 - v_2\|_{L^2(D)}, \quad \forall v_1, v_2 \in L^2(D).
$$

(2.10)

- $\{G(v(\tau)) : \tau \in [0,T]\}$ is a predictable $L_Q$-valued process, such that

$$
\int_0^T \mathbb{E}[\|G(v)\|_{L_Q}^2] d\tau < +\infty, \quad \forall v \in L^2(D).
$$

(2.11)

**Remark 2.1.** The assumptions (2.9) and (2.10) impose some restrictive conditions on the nonlinear term $G(\cdot)$, which include a combination of the nonlinear term $G(\cdot)$, the elliptic operator $L$, and the covariance operator $Q$. Notice that the similar or more general assumptions have been considered in [21, 53, 2].

We define the space $L^1_2$ for $t \in [0, T]$, which is the Banach space of $L^2(D)$-valued predictable processes $\{v(\tau) : \tau \in [0, t]\}$, equipped with the norm

$$
\|v\|_{L^1_2} := \sup_{\tau \in [0, t]} \|v(\tau)\|_{L^2(\Omega, L^2(D))} < +\infty.
$$

Now we are in a position to state and prove the well-posedness of the mild solution to (2.2).

**Theorem 2.1.** Suppose that the initial data $u_0 \in L^2(\Omega, L^2(D))$. Then, there exists a unique mild solution $u \in L^1_2$ to (2.2). Furthermore, the following stability inequality holds

$$
\sup_{t \in [0, T]} \|u(t)\|_{L^2(\Omega, L^2(D))} \leq c_T(1 + \|u_0\|_{L^2(\Omega, L^2(D))}).
$$

(2.12)

**Proof.** We define the integral operator $M$ by: for all $v \in L^1_2$, $0 \leq t \leq T$,

$$
(Mv)(t) := S(t)u_0 + \int_0^t S(t)\mathbb{E}[G(v(\tau))dW(\tau).
$$

(2.13)

Obviously if there is a fixed point $u \in L^1_2$ for the operator $M$, then this fixed point is a mild solution defined by (2.7). The proof basically consists of two steps: 1) prove that the integral operator $M$ is well-defined under the assumptions given above; 2) use the Fixed Point Theorem [11, Theorem 1.10] to establish the existence of a unique mild solution. This can be done by following the same lines as in [11, Theorem 10.26], using the imposed assumptions and a number of known results including the Karhunen-Loève (K-L) expansion of $Q$-Wiener process $W(s)$, Itô isometry, and the inequality

$$
\|S(\tau)\|_{L(\Omega, L^2(D))} \leq 1, \quad \forall \tau \in (0, T)
$$

(2.14)

for the semigroup $S(\tau)$. We emphasize here that $S(\tau)$ involves the random diffusion coefficient, thus the inequality (2.14) must be understood in the sense of almost surely. This, compared to the case of deterministic diffusion coefficient (see, e.g., [11, Theorem 10.26] for details), causes no essential difficulty in establishing the desired results. □
3. Random field sampling and fully discrete scheme

Our first goal in this section is to employ a method called approximation circulant embedding with padding to uniformly sample the random diffusion coefficient \( a(x, \omega) \). It is notable that some other sampling methods are available, such as turning bands method \([42, 15]\) and quadrature sampling method \([47, 48]\). However the turning bands method is only applicable to isotropic Gaussian random fields, and the quadrature sampling method needs to know the spectral density function of the covariance function of random fields. One of the advantages of the sampling method we employ here is its applicability to stationary Gaussian random fields including isotropic random fields, and does not require prior knowledge of the spectral density function of the covariance function.

It is obvious from (2.3) that if we want to sample \( a(x, \omega) \), we only need to sample \( z(x, \omega) \). The crucial ingredient of the circulant embedding sampling is that the target covariance matrix can be embedded into a large circulant matrix, which can be decomposed by discrete Fourier transform. Then a new random field based on the combination of decomposition factors is constructed, which will be used to obtain the approximations of \( z(x, \omega) \) for \( x \in \bar{D} \).

Another purpose in the section is to present semi-implicit Euler-Maruyama scheme and finite element method to discrete problem (2.2) in time and space, respectively. We start by random field sampling.

3.1. Approximation circulant embedding with padding. Consider uniform sampling of random field \( z(x, \omega) \) in \( \bar{D} := [0, 1] \). We set

\[
0 = x_1 \leq ... \leq x_P = 1, \quad \Delta x = \frac{1}{P - 1} = x_{p+1} - x_p, \quad p = 1, ..., P - 1.
\]

Let \( C := (c_{ij}) \) denote the \( P \times P \) covariance matrix with respect to \( z(x_p, \omega) \) for \( p = 1, ..., P \), where \( c_{ij} := \text{cov}(z(x_i, \omega), z(x_j, \omega)) = c_q(|x_i - x_j|) \) for \( i, j = 1, ..., P \). If we set \( c_{i-j} := c_{ij} \), then

\[
C = \begin{pmatrix}
c_0 & c_{-1} & \cdots & c_{1-P} \\
c_1 & c_0 & \cdots & c_{2-P} \\
\vdots & \ddots & \ddots & \vdots \\
c_{P-1} & \cdots & c_1 & c_0
\end{pmatrix}.
\]

One verifies readily that \( C \) is a symmetric Toeplitz matrix, and it can be well defined by its first column \( c_1 = (c_0, ..., c_{P-1})^T \in \mathbb{R}^P \). If we define \( \tilde{c}_1 := (\tilde{c}_1) \in \mathbb{R}^{P+M} \) with \( 0 \in \mathbb{R}^M \) be a zero padding vector, a new symmetric Toeplitz matrix denoted by \( \tilde{C} \in \mathbb{R}^{(P+M) \times (P+M)} \) can be generated from \( \tilde{c}_1 \). Next, we carry out the minimal circulant extension \([41, Definition 6.48]\) to \( \tilde{C} \) such that it can be embedded into a bigger circulant matrix denoted by \( \tilde{\tilde{C}} \in \mathbb{R}^{2P \times 2P} \) for \( \tilde{P} := P + M - 1 \). Let \( \tilde{c}_1 \) be the first column of \( \tilde{\tilde{C}} \), \( W^* \) represent the conjugate transpose of discrete Fourier matrix \( W \in \mathbb{C}^{2P \times 2P} \), and \( d_j \) be the \( j \)-th entry of \( \sqrt{2P}W^*\tilde{c}_1 \). Then by Fourier representation, the circulant matrix \( \tilde{\tilde{C}} \) can be decomposed as follows:

\[
\tilde{\tilde{C}} = W(\Lambda_+ - \Lambda_-)W^*.
\]
where $\Lambda_\pm$ represents the diagonal matrix whose $j$-th diagonal element is $\pm \lambda_j := \max\{0, \pm d_j\}$, i.e.,

\begin{equation}
\Lambda_\pm = \text{diag}(\pm \lambda_1, \ldots, \pm \lambda_{2\hat{p}}).
\end{equation}

Let $z := (z(x_1, \omega), \ldots, z(x_P, \omega))^T$. Our main goal is to take the sample approximations to the random vector $z$. To this end, we construct a new random field vector $Z$, defined by

\begin{equation}
Z := W\Lambda_+^{1/2}\xi, \quad \xi \sim \text{CN}(0, 2I_{2\hat{p}}),
\end{equation}

where $\text{CN}(\cdot, \cdot)$ denotes the complex Gaussian distribution [11, Definition 6.15]. It's readily to deduce that $Z \sim \text{CN}(0, 2(\tilde{C} + \tilde{C}_-))$ with $\tilde{C}_- := W\Lambda_- W^*$, which means both real and imaginary parts of $Z$ obey real Gaussian distribution $N(0, (\tilde{C} + \tilde{C}_-))$. Notice that $\|\tilde{C}_-\|_2 \leq \rho(\Lambda_-)$ with $\rho(\Lambda_-)$ representing the spectral radius of $\Lambda_-$. It is known that $\rho(\Lambda_-)$ can be small enough by increasing the dimension $M$ of zero padding vector [54]. Therefore $\tilde{C}$ can be approximately treated as a non-negative definite matrix when the dimension $M$ is large enough, which is crucial for obtaining a good approximation of the random vector $z$. Then the sample approximations of the random vector $z$ can be provided by truncating the real or imaginary part of $Z$.

The sampling procedure is summarized as follows:

i) Embed $C$ shown in (3.1) into the padded circulant matrix $\tilde{C} \in \mathbb{R}^{2(P+M-1) \times 2(P+M-1)}$ with dimension $M$ large enough;

ii) Compute $\Lambda_+$ by (3.2);

iii) Construct a new random field vector $Z$ by (3.3) and take its real or imaginary part, denoted by $Z_1 \in \mathbb{R}^{2(P+M-1)}$;

iv) Truncate the first $P$ terms of $Z_1$ and use it as an approximation to the random vector $z$.

It is worthwhile to point out that the sampling method described above is convenient in the sense that it can simultaneously produce two sets of independent and identically distributed (i.i.d) samples in one sampling.

For each of the sampling data of the random diffusion coefficient, the problem (2.2) becomes a SEE with randomness only on the $G$-term.

3.2. Spatio-temporal discretization. In this subsection we propose and analyze a discretization method for the problem (2.2). The proposed method is based on a finite element discretization in space and semi-implicit Euler-Maruyama approach in time.

We first describe the $P_1$ finite element method for the spatial discretization. Let $K > 0, h = \frac{1}{K + 1}, x_0 = 0, x_k = kh, I_k = [x_{k-1}, x_k], k = 1, \ldots, K + 1$. Define the finite element space $V_h$ by

$$V_h := \{v \in C^0(\bar{D}) : v|_{I_k} \in P_1(I_k), k = 1, \ldots, K + 1; v(0) = v(1) = 0\},$$

where $P_1(I_k)$ denotes the space of the polynomials of degree $\leq 1$ defined in $I_k$. Let $\varphi_i(x)$ be the nodal basis functions satisfying $\varphi_i(x_j) = \delta_{ij}, i, j = 0, 1, \ldots, K + 1$. Then $V_h = \text{span}\{\varphi_1(x), \ldots, \varphi_K(x)\}$.

Let $P_h$ be the orthogonal projection from $L^2(D)$ to $V_h$, and $P^T_h$ be the projection from $H^1_0(D)$ to the finite-dimensional space $\text{span}\{\phi_1, \ldots, \phi_J\}$. The spatial semi-discrete scheme of the problem
In actual calculation, we will use discretization (3.5) to the mild solution (2.7). Here, strong convergence is understood in the
\[ (L_h w, v) := (a(x, \omega) \partial_x w, \partial_x v), \quad \forall w, v \in V_h \]
with \((\cdot, \cdot)\) be the \(L^2\)-inner product.

We now describe the temporal discretization. Let \(N\) be a positive integer, \(\Delta t := T/N\) be the uniform time step. Then the spatio-temporal full discretization of the problem (2.2), called hereafter the finite element method/semi-implicit Euler Maruyama scheme, reads:
\[
(I + \Delta t L_h)u_h^{n+1} = u_h^n + \Delta t P_h f(u_h^n) + P_h (G(u_h^n) P^w_j \Delta W^n), \quad n = 0, ..., N - 1,
\]
\[
u_h^0 = P_h u_0,
\]
where \(P^w_j \Delta W^n := \sum_{j=1}^J (\beta_j(t_{n+1}) - \beta_j(t_n)) \phi_j\) with \(\beta_j(t)\) be the i.i.d \(\mathcal{F}_t\)-Brownian motions.

Before carrying out the error analysis, we briefly discuss the implementation of the above scheme. The weak formulation of (3.5) is:
\[
(u_h^{n+1}, v_h) + \Delta t (a(x, \omega) \partial_x u_h^{n+1}, \partial_x v_h) = (g_h^n, v_h), \quad v_h \in V_h,
\]
where \(g_h^n := u_h^n + \Delta t f(u_h^n) + G(u_h^n) P^w_j \Delta W^n\). Expressing the solution \(u_h^{n+1}\) under the basis \(\{\varphi_k\}_{k=1}^K\),
\[
u_h^{n+1}(x) = \sum_{k=1}^K \hat{u}_h^{n+1} \varphi_k(x), \quad n = 0, ..., N - 1,
\]
and taking the test function \(v_h\) in (3.6) to be each of the basis functions, we arrive at the following linear system:
\[
(M + \Delta t S) \hat{u}_h^{n+1} = M \hat{g}_h^n, \quad n = 0, ..., N - 1,
\]
where \(\hat{u}_h^{n+1} := (\hat{u}_1^{n+1}, ..., \hat{u}_K^{n+1})^T, \hat{g}_h^n\) is the expansion coefficient vector of \(g_h^n\) under the basis \(\{\varphi_k\}_{k=1}^K\). \(M\) and \(S\) are respectively the mass and stiffness matrix defined by
\[
M = (m_{ij}), \quad m_{ij} := (\varphi_i, \varphi_j), \quad \forall i, j = 1, ..., K,
\]
\[
S = (s_{ij}), \quad s_{ij} := (a(x, \omega) \partial_x \varphi_i, \partial_x \varphi_j), \quad \forall i, j = 1, ..., K.
\]
In actual calculation, we will use \(\frac{1}{2}(a(x_{k-1}, \omega) + a(x_k, \omega))\) to approximate \(a(x, \omega)\) for \(x \in I_k\). Therefore the overall cost of the scheme is roughly equal to solving a linear system with random variable coefficients at each time step.

4. Error estimate

This section is devoted to analyzing the strong convergence error of the spatio-temporal full discretization (3.5) to the mild solution (2.7). Here, strong convergence is understood in the
sense of convergence with respect to the norm $\|\cdot\|_{L^2(\Omega, L^2(D))}$. We first note that the full-discrete scheme (3.5) can be rewritten under form:

$$(4.1) \quad u_{n+1}^h = (I + \Delta t L_h)^{-1} \left( u_n^h + \Delta t \mathcal{P}_h f(u_n^h) + \mathcal{P}_h G(u_n^h) \mathcal{P}_f^w \Delta W_n \right), \quad n = 0, \ldots, N-1.$$  

It is readily seen that $L_h$ is reversible in $V_h$, i.e., $L_h^{-1} v_h$ is well defined for all $v_h \in V_h$. We now extend the definition of $L_h^{-1}$ to all $v \in L^2(D)$ by $L_h^{-1} v = L_h^{-1} \mathcal{P}_h v$. By the assumption on $a(x, \omega)$, we know that for almost every $\omega \in \Omega$, $L_h^{-1}$ is a non-negative definite operator from $L^2(D)$ to $V_h$. In fact, for all $v \in L^2(D)$, there exists $w_h \in V_h$ such that $L_h w_h = \mathcal{P}_h v$, and thus

$$(L_h^{-1} v, v) = (L_h^{-1} \mathcal{P}_h v, v) = (L_h^{-1} \mathcal{P}_h v, \mathcal{P}_h v) = (L_h^{-1} L_h w_h, L_h w_h) = (w_h, L_h w_h) = (a(x, \omega) \partial_x w_h, \partial_x w_h) \geq 0.$$  

Let $S^n_{h, \Delta t} := (I + \Delta t L_h)^{-n}$. The fully discrete approximation can be expressed under the form:

$$(4.2) \quad u^n_h = S^n_{h, \Delta t} \mathcal{P}_h u_0 + \sum_{k=0}^{n-1} \Delta t S^{n-k}_{h, \Delta t} \mathcal{P}_h f(u^k_h) + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S^{n-k}_{h, \Delta t} \mathcal{P}_h G(u^k_h) \mathcal{P}_f^w dW(\tau).$$  

Subtracting (4.2) from the mild solution (2.7) gives

$$(4.3) \quad u(t_n) - u^n_h = \theta_1 + \theta_2 + \theta_3$$  

with $\theta_i, i = 1, 2, 3,$ representing

$$(4.4) \quad \theta_1 := S(t_n) u_0 - S^n_{h, \Delta t} \mathcal{P}_h u_0,$$

$$(4.5) \quad \theta_2 := \sum_{k=0}^{n-1} \left( \int_{t_k}^{t_{k+1}} S(t_n - \tau) f(u(\tau)) d\tau - \Delta t S^{n-k}_{h, \Delta t} \mathcal{P}_h f(u^k_h) \right),$$

$$(4.6) \quad \theta_3 := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (S(t_n - \tau) G(u(\tau)) - S^{n-k}_{h, \Delta t} \mathcal{P}_h G(u^k_h) \mathcal{P}_f^w) dW(\tau).$$

Our goal in the following is to estimate $\theta_1$, $\theta_2$, $\theta_3$ separately in the sense of strong convergence. To this end, we first give some preliminaries that will be used in subsequent analysis.

- If the initial value $u_0 \in L^2(\Omega, \mathcal{D}(L))$, then there exists a constant $c$ depended on $u_0$ such that the mild solution $u$ defined in (2.7) satisfies the following temporal Hölder regularity:

$$(4.7) \quad \|u(\tau_2) - u(\tau_1)\|_{L^2(\Omega, L^2(D))} \leq c(\tau_2 - \tau_1)^{1/2}, \quad \forall 0 \leq \tau_1 \leq \tau_2 \leq T.$$  

The proof of (4.7) can be done by following the same lines as in [11] Lemma 10.27], which is omitted here. Basically, it makes use of the properties of the operator $L$ and its associated semigroup $S(t)$, satisfied in the sense of almost surely.

- The operator $L$ and the induced semigroup $S(t)$ satisfy the following estimates, which is a straightforward extension of the classical results (see, e.g., [36, 51]) to the sense of almost surely:

$$(4.8) \quad \|L^n S(t)\|_{L(L^2(D))} \leq c t^{-\alpha}, \quad \forall t > 0.$$
4.1. Strong error estimate. We first focus on strong error estimate for the term $\theta_1$. It is worth pointing out that, although our analysis is inspired by the work [36] based on the rational function approach, our proof makes full use of the standard framework of the finite element approximation to the linear parabolic equation as well as the fact that operator $L_h^{-1}$ is non-negative definite from $L^2(D)$ to $V_h$. Let

$$T_n := (e^{-t_n L} - (I + \Delta t L_h)^{-n} P_h).$$

Then it follows from the definition (4.4) that $\theta_1 = T_n u_0$.

**Lemma 4.1 (Error estimate of $\theta_1$).** Suppose $u_0 \in L^2(\Omega, D(L))$. Then there exists a constant $c$ independent of $h$ and $\Delta t$ (but depends on $u_0$), such that

$$\|\theta_1\|_{L^2(\Omega, L^2(D))} \leq c(\Delta t + h^2),$$

where $\theta_1$ is given in (4.4).

**Proof.** Obviously, $\theta_1$ characterizes the error between the exact solution $e^{-t_n L}u_0$ and the full discrete solution $(I + \Delta t L_h)^{-n}P_h u_0$, which can be split into two parts:

$$\theta_1 = e_1^n + e_2^n,$$

where

$$e_1^n := e^{-t_n L} u_0 - e^{-t_n L} P_h u_0$$

is the spatial discretization error, while

$$e_2^n := (e^{-t_n L} - (I + \Delta t L_h)^{-n}) P_h u_0$$

is the temporal discretization error. Clearly, we have $e_1^n = y(t_n) - y_h(t_n)$, where $y(t) \in H^1_0(D)$ and $y_h(t) \in V_h$ are the solutions of the parabolic equation

$$\frac{\partial y(t)}{\partial t} + Ly(t) = 0, \quad y(0) = u_0.$$
and its finite element semi-discrete equation

\[
\frac{\partial y_h(t)}{\partial t} + L_h y_h(t) = 0, \quad y_h(0) = \mathcal{P}_h u_0
\]

respectively. Let

\[
e_1(t) := y(t) - y_h(t) , \quad \rho(t) := L_h^{-1} \frac{\partial e_1(t)}{\partial t} + e_1(t).
\]

It can be verified that

\[
\rho(t) = (L^{-1} - L_h^{-1}) Ly(t).
\]

Using the non-negative definite of the operator \( L_h^{-1} \) as well as the standard error analysis of the finite element approximation to the parabolic equation [41, Lemma 3.51] gives: for almost every \( \omega \in \Omega \),

\[
\| e_1(t) \|_{L^2(D)} \leq c \sup_{0 \leq \tau \leq t} \left( \| \rho(\tau) \|_{L^2(D)} + \| \frac{\partial \rho(\tau)}{\partial \tau} \|_{L^2(D)} \right).
\]

The terms in the right-hand side can be bounded by:

\[
\| \rho(\tau) \|_{L^2(D)} = \| (L^{-1} - L_h^{-1}) Ly(\tau) \|_{L^2(D)} \leq c h^2 \| e^{-L\tau} u_0 \|_{L^2(D)} \leq c h^2,
\]

\[
\| \frac{\partial \rho(\tau)}{\partial \tau} \|_{L^2(D)} = \tau \| (L^{-1} - L_h^{-1}) L^2 y(\tau) \|_{L^2(D)} \leq c \tau h^2 \| L^2 e^{-L\tau} u_0 \|_{L^2(D)} \leq c h^2 \| Lu_0 \|_{L^2(D)} \leq c h^2.
\]

Thus

\[
(4.13) \quad \| e_1^n \|_{L^2(D)} = \| e_1(t_n) \|_{L^2(D)} \leq c h^2, \quad n = 0, 1, \ldots, N.
\]

We now turn to estimate the temporal discretization error \( e_2^n \). A direct calculation gives

\[
\| e_2^n \|_{L^2(D)} = \| (e^{-n \Delta t L_h} - (I + \Delta t L_h)^{-n}) L_h^{-1} \mathcal{P}_h u_0 \|_{L^2(D)}
\]

\[
\leq \| (e^{-n \Delta t L_h} - (I + \Delta t L_h)^{-n}) L_h^{-1} \|_{\mathcal{L}(L^2(D))} \| L_h \mathcal{P}_h u_0 \|_{L^2(D)}.
\]

Noticing that the operator \( L_h \) is symmetric, and the \( \mathcal{L}(L^2(D)) \)-norm of the operator \( (e^{-n \Delta t L_h} - (I + \Delta t L_h)^{-n}) L_h^{-1} \) is equal to its spectral radius, i.e.,

\[
\sup_{j = 1, \ldots, K} \left| \frac{(e^{-n \Delta t \lambda_j^h} - (1 + \Delta t \lambda_j^h)^{-n})}{\lambda_j^h} \right|,
\]

where \( \lambda_j^h > 0, j = 1, \ldots, K \), are the eigenvalues of \( L_h \). Note that \(|(e^{-nx} - (1+x)^{-n})/x|\) is bounded for \( x > 0 \), therefore taking \( x = \lambda_j^h \Delta t \) gives

\[
\sup_{j = 1, \ldots, K} \left| \frac{(e^{-n \Delta t \lambda_j^h} - (1 + \Delta t \lambda_j^h)^{-n})}{\lambda_j^h} \right| \leq c \Delta t.
\]

This proves

\[
(4.14) \quad \| e_2^n \|_{L^2(D)} \leq c \Delta t, \quad n = 0, 1, \ldots, N.
\]

Combining (4.12), (4.13), and (4.14) gives

\[
\| \theta_1 \|_{L^2(D)} \leq c(\Delta t + h^2).
\]

The above estimate holds for almost all \( \omega \in \Omega \). Therefore

\[
\| \theta_1 \|_{L^2(\Omega, L^2(D))} \leq c(\Delta t + h^2).
\]
Remark 4.1. If \( u_0 \in L^2(\Omega, L^2(D)) \). Then we have only \([36, 41]\): for almost every \( \omega \in \Omega \),

\[
\|\theta_1\|_{L^2(D)} = \|T_n u_0\|_{L^2(D)} \leq c \|u_0\|_{L^2(D)} \frac{\Delta t + h^2}{t_n}, ~ n = 1, \ldots, N.
\]

We next derive the error estimate for the term \( \theta_2 \), which is based on the standard error analysis for the deterministic semilinear evolution equation, the semigroup property, and the temporal regularity of the mild solution.

**Lemma 4.2** (Error estimate of \( \theta_2 \)). Suppose \( u_0 \in L^2(\Omega, \mathcal{D}(L)) \). Then there exists a constant \( c \) independent of \( h \) and \( \Delta t \) (but depends on \( \|u_0\|_{L^2(\Omega, \mathcal{D}(L))} \)), such that

\[
\|\theta_2\|_{L^2(\Omega, L^2(D))} \leq c \left[ \Delta t^2 + (\Delta t + h^2) \ln(\Delta t^{-1}) + \sum_{k=0}^{n-1} \|u(t_k) - u_h^k\|_{L^2(\Omega, L^2(D))} \Delta t \right], ~ n = 1, \ldots, N,
\]

where \( \theta_2 \) is given by (4.5).

**Proof.** The term to be bounded can be expressed by

\[
\theta_2 = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (S(t_n - \tau) f(u(\tau)) - S_{h,\Delta t}^n P_h f(u_h^k)) d\tau,
\]

which can be decomposed into

\[
\theta_2 = \theta_2^1 + \theta_2^2 + \theta_2^3 + \theta_2^4
\]

with

\[
\begin{align*}
\theta_2^1 & := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (S(t_n - \tau) - S(t_n - t_k)) f(u(\tau)) d\tau, \\
\theta_2^2 & := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (S(t_n - t_k) - S_{h,\Delta t}^{n-k} P_h) f(u(\tau)) d\tau, \\
\theta_2^3 & := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S_{h,\Delta t}^{n-k} P_h (f(u(\tau)) - f(u(t_k))) d\tau, \\
\theta_2^4 & := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S_{h,\Delta t}^{n-k} P_h (f(u(t_k)) - f(u_h^k)) d\tau.
\end{align*}
\]

For the part \( \theta_2^1 \), it follows from the norm definition (2.1):

\[
\|\theta_2^1\|_{L^2(\Omega, L^2(D))} \leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \|S(t_n - \tau) - S(t_n - t_k)\|_{L^2(L^2(D))}^2 \|f(u(\tau))\|_{L^2(D)}^2 \right] d\tau.
\]
According to (4.8) and (4.9), the operator norm $\|S(t_n - \tau) - S(t_n - t_{n-1})\|_{\mathcal{L}(L^2(D))}$ is bounded $\mathbb{P}$-a.s. by:

$$
\|S(t_n - \tau) - S(t_n - t_{n-1})\|_{\mathcal{L}(L^2(D))} \leq c,
$$

and

$$
\|S(t_n - \tau) - S(t_n - t_k)\|_{\mathcal{L}(L^2(D))} \leq \|LS(t_n - \tau)\|_{\mathcal{L}(L^2(D))} \|L^{-1}(I - S(\tau - t_k))\|_{\mathcal{L}(L^2(D))} \leq c\frac{\tau - t_k}{t_n - \tau}, \quad \tau \in (t_k, t_{k+1}), \quad k = 0, \ldots, n - 2.
$$

We further use (2.5) and (2.12) to derive

$$
\|\theta_1^2\|_{L^2(\Omega, L^2(D))} \leq c\Delta t + \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \frac{\Delta t}{t_n - t_{k+1}} d\tau \leq c\Delta t \left(1 + \sum_{k=1}^{n}\frac{1}{k}\right) \leq c\Delta t (1 + \ln n) \leq c\Delta t (1 + \ln(\Delta t^{-1})).
$$

The estimate of $\theta_2^2$ follows from (4.11), (4.15), (2.5), and (2.12):

$$
\|\theta_2^2\|_{L^2(\Omega, L^2(D))} \leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|T_{n-k} f(u(\tau))\|_{L^2(\Omega, L^2(D))} d\tau
\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\|T_{n-k}\|^2_{\mathcal{L}(L^2(D))} \|f(u(\tau))\|^2_{L^2(D)}\right]^{\frac{1}{2}} d\tau
\leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\Delta t + h^2}{t_n - t_{n-k}} d\tau = c(\Delta t + h^2) \sum_{k=0}^{n-1} \frac{1}{n - k} \leq c(\Delta t + h^2) \ln(\Delta t^{-1}).
$$

For the part $\theta_3^2$, by $\|S_{h, \Delta t}^{n-k}\|_{\mathcal{L}(L^2(D))} \leq 1$ ($\mathbb{P}$-a.s.), $\|P_h\|_{\mathcal{L}(L^2(D))} \leq 1$, (2.6), and (4.7), we have

$$
\|\theta_3^2\|_{L^2(\Omega, L^2(D))} \leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|f(u(\tau)) - f(u(t_k))\|_{L^2(\Omega, L^2(D))} d\tau
\leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|u(\tau) - u(t_k)\|_{L^2(\Omega, L^2(D))} d\tau
\leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (\tau - t_k)^{\frac{1}{2}} d\tau \leq c\Delta t^2.
$$

The part $\theta_3^3$ can be estimated similarly:

$$
\|\theta_3^3\|_{L^2(\Omega, L^2(D))} \leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|f(u(t_k)) - f(u^k)\|_{L^2(\Omega, L^2(D))} d\tau
\leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|u(t_k) - u^k\|_{L^2(\Omega, L^2(D))} d\tau
\leq c \sum_{k=0}^{n-1} \|u(t_k) - u_k\|_{L^2(\Omega, L^2(D))} \Delta t.
$$

Finally, we conclude by combining all above estimates with the triangle inequality. □
In order to estimate the error contribution term $\theta_3$, we need to derive an estimate related to the nonlinear term $G$.

**Lemma 4.3.** Suppose $u_0 \in L^2(\Omega, D(L))$, and the eigenvalues of $Q$ satisfy $q_j = O(j^{-(2\gamma+1+\epsilon)})$ for some $\gamma \geq 0$ and $\epsilon > 0$. Then it holds: for $0 \leq \tau_1 \leq \tau_2 \leq T$,

\[
\|P_h(G(u(\tau_2)) - G(u(\tau_1))P_J^u)\|_{L^2(0,\tau_2,\mathcal{L}Q)} \leq c(\tau_2 - \tau_1)^{\frac{1}{2}} + J^{-\gamma}).
\]

Proof. Using the triangle inequality:

\[
\|P_h(G(u(\tau_2)) - G(u(\tau_1)))\|_{L^2(0,\tau_2,\mathcal{L}Q)} \leq \|P_h(G(u(\tau_2)) - G(u(\tau_1)))\|_{L^2(0,\tau_2,\mathcal{L}Q)} + \|P_h(G(u(\tau_1)) - G(u(\tau_1))P_J^u)\|_{L^2(0,\tau_2,\mathcal{L}Q)},
\]

we are led to estimate the two terms in the right-hand side. First, employing (2.10) and (4.7) gives:

\[
\|P_h(G(u(\tau_2)) - G(u(\tau_1)))\|_{L^2(0,\tau_2,\mathcal{L}Q)} \leq c\|u(\tau_2) - u(\tau_1)\|_{L^2(0,\tau_2,\mathcal{L}Q)} \leq c(\tau_2 - \tau_1)^{\frac{1}{2}}.
\]

Then under the assumptions (2.9) and (2.12), we have

\[
\|P_h(G(u(\tau_1)) - G(u(\tau_1))P_J^u)\|_{L^2(0,\tau_2,\mathcal{L}Q)} \leq c\|P_h(G(u(\tau_1)) - G(u(\tau_1))P_J^u)\|_{L^2(0,\tau_2,\mathcal{L}Q)} \leq cJ^{-\gamma}.
\]

This proves (4.16). \qed

**Lemma 4.4** (Error estimate of $\theta_3$). Under the assumptions of Lemma 4.3, further assume $\Delta t = O(h^2) = O(J^{-\gamma})$. Then there exists a constant $c$ independent of $\Delta t$ and $h$, such that

\[
\|\theta_3\|_{L^2(0,T,J^2)}^2 \leq c\left((\Delta t^2 + h^2)^2 + \sum_{k=0}^{n-1}\|u(t_k) - u_h^k\|_{L^2(0,T,J^2)}^2 \right), \quad n = 1, \ldots, N,
\]

where $\theta_3$ is given by (4.6).

Proof. Split $\theta_3$ as $\theta_3 = \sum_{i=1}^{4} \theta_3^i$, where

\[
\theta_3^i := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} X_i dW(\tau)
\]

with

\[
X_1 := (S(t_n - \tau) - S(t_n - t_k))G(u(\tau)), \quad X_2 := (S(t_n - t_k) - S_{h,\Delta t}^{n-k}P_h)G(u(\tau)), \quad X_3 := S_{h,\Delta t}^{n-k}P_h(G(u(\tau)) - G(u(t_k)))P_J^w, \quad X_4 := S_{h,\Delta t}^{n-k}P_h(G(u(t_k))P_J^w - G(u_h^k))P_J^w).
\]
For the part \( \theta^1_3 \), it follows from the Itô isometry:

\[
\| \theta^1_3 \|^2_{L^2(Ω,L^2(D))} = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[\|X_1\|^2_{L^2}] d\tau
\]

\[
\leq \int_{t_{n-1}}^{t_n} \mathbb{E}\left[ \|S(t_n - \tau) - S(t_n - t_{n-1})\|_{L^2(Ω)}^2 \| G(u(\tau)) \|_{L^2(Ω)}^2 \right] d\tau
\]

\[
+ \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ \|S(t_n - \tau) - S(t_n - t_k)\|_{L^2(Ω)} \| G(u(\tau)) \|_{L^2(Ω)}^2 \right] d\tau.
\]

We are led to estimate the two terms on the right-hand side of the inequality. First using

\[
\| S(t_n - \tau) - S(t_n - t_{n-1}) \|_{L^2(Ω)}^2 \leq c \ (\mathbb{P}-\text{a.s.}),
\]

(2.9), and (2.12) yields

\[
\int_{t_{n-1}}^{t_n} \mathbb{E}\left[ \|S(t_n - \tau) - S(t_n - t_{n-1})\|_{L^2(Ω)}^2 \| G(u(\tau)) \|_{L^2(Ω)}^2 \right] d\tau \leq c\Delta t.
\]

Then employing (4.8) and (4.9) gives

\[
\| (S(t_n - \tau) - S(t_n - t_k)) L^{-\frac{1}{2}} \|_{L^2(Ω)}^2 = \| L^{\frac{1}{2}} S(t_n - \tau) L^{-1} (I - S(\tau - t_k)) \|_{L^2(Ω)}^2
\]

\[
\leq \| L^{\frac{1}{2}} S(t_n - \tau) \|_{L^2(Ω)}^2 \| L^{-1} (I - S(\tau - t_k)) \|_{L^2(Ω)}^2
\]

\[
\leq c \frac{(\tau - t_k)^2}{t_n - \tau}.
\]

Making use of (2.9), (2.12) gives

\[
\sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ \|S(t_n - \tau) - S(t_n - t_k)\|_{L^2(Ω)}^2 \| L^{\frac{1}{2}} G(u(\tau)) \|_{L^2(Ω)}^2 \right] d\tau
\]

\[
\leq c \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \frac{\Delta t^2}{t_n - t_{k+1}} d\tau \leq c\Delta t^2 \ln(\Delta t^{-1}).
\]

Therefore

\[
\| \theta^1_3 \|^2_{L^2(Ω,L^2(D))} \leq c(\Delta t + \Delta t^2 \ln(\Delta t^{-1})).
\]

The estimate for \( \theta^2_3 \) follows from Itô isometry, (4.11), and (4.15):

\[
\| \theta^2_3 \|^2_{L^2(Ω,L^2(D))} = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[\|X_2\|^2_{L^2}] d\tau = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[\|T_{n-k} G(u(\tau))\|^2_{L^2}] d\tau
\]

\[
\leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ \left( \frac{\Delta t + h^2}{t_n - t_k} \right)^2 \| G(u(\tau)) \|_{L^2}^2 \right] d\tau.
\]

We further use (2.9), (2.12), and \( \Delta t = O(h^2) \) to get

\[
\| \theta^2_3 \|^2_{L^2(Ω,L^2(D))} \leq c \left( \frac{\Delta t + h^2}{\Delta t} \right)^2 \frac{1}{(n-k)^2} \leq c \left( \frac{(\Delta t + h^2)^2}{\Delta t} \right) \leq c\Delta t.
\]
Finally we combine all above estimates and keep only the leading order to conclude.

$$\|\theta_3^2\|_{L^2(\Omega;L^2(D))} = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[|X_3|^2] d\tau \leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (|\tau - t_k|^\frac{3}{2} + J^{-\gamma})^2 d\tau$$

$$\leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (|\tau - t_k|^\frac{3}{2} + h^2)^2 d\tau$$

$$\leq c(\Delta t^\frac{3}{2} + h^2)^2.$$ 

The last part $\theta_3^2$ can be estimated by using Itô isometry, $\|S_{h,\Delta t}^{n-k}\|_{\mathcal{L}(L^2(D))} \leq 1$ (P-a.s.), and (4.10):

$$\|\theta_3^2\|_{L^2(\Omega;L^2(D))} = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[|X_4|^2] d\tau = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[\|S_{h,\Delta t}^{n-k} P_h(G(u(t_k))P_w^w - G(u_k^k)P_w^w)\|^2] d\tau$$

$$\leq c \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[\|u(t_k) - u_h^k\|^2_{L^2(D)}] d\tau$$

$$= c \sum_{k=0}^{n-1} \|u(t_k) - u_h^k\|^2_{L^2(\Omega;L^2(D))} \Delta t.$$ 

Finally we combine all above estimates and keep only the leading order to conclude. \(\square\)

Thanks to the results established in the previous lemmas, we are now in a position to derive the full discretization error bound, which is stated in the following theorem.

**Theorem 4.1.** Let $u$ be the mild solution defined in (2.7), and $u_h^n$ be the numerical solution of (3.5). Then under the assumptions stated in Lemmas 4.1–4.4 there exists a constant $c$ independent of $\Delta t$ and $h$, such that

$$\|u(t_n) - u_h^n\|_{L^2(\Omega;L^2(D))} \leq c(\Delta t^\frac{3}{2} + (\Delta t + h^2) \ln(\Delta t^{-1})),$$  

for $n = 1, \ldots, N$.

**Proof.** It follows from (4.3), Lemmas 4.1–4.4 and the triangle inequality:

$$\|u(t_n) - u_h^n\|^2_{L^2(\Omega;L^2(D))} \leq c \left[ (\Delta t^\frac{3}{2} + (\Delta t + h^2) \ln(\Delta t^{-1}))^2 + \sum_{k=0}^{n-1} \|u(t_k) - u_h^k\|^2_{L^2(\Omega;L^2(D))} \Delta t \right],$$

for $n = 1, \ldots, N$.

Then the discrete Gronwall inequality yields

$$\|u(t_n) - u_h^n\|^2_{L^2(\Omega;L^2(D))} \leq c(\Delta t^\frac{3}{2} + (\Delta t + h^2) \ln(\Delta t^{-1}))^2,$$  

for $n = 1, \ldots, N$.

This ends the proof. \(\square\)

**Remark 4.2.** Notice that the term $\Delta t^\frac{3}{2}$ dominates the term $\Delta t \ln(\Delta t^{-1})$, the estimate given in Theorem 4.1 can be simplified by

$$\|u(t_n) - u_h^n\|_{L^2(\Omega;L^2(D))} \leq c(\Delta t^\frac{3}{2} + h^2 \ln(\Delta t^{-1})).$$

Also notice that $\Delta t^{-\varepsilon_0}$ dominates $\ln(\Delta t^{-1})$ for arbitrarily small $\varepsilon_0 > 0$, we have

$$\|u(t_n) - u_h^n\|_{L^2(\Omega;L^2(D))} \leq c(\Delta t^\frac{3}{2} + h^2 \Delta t^{-\varepsilon_0})$$
or, since $\Delta t = O(h^2)$,
\[
\|u(t_n) - u_h^n\|_{L^2(\Omega, L^2(D))} \leq c(\Delta t^{1/2} + h^{2-\varepsilon_0}).
\]

5. Numerical results

Several numerical experiments are presented in this section to validate our theoretical estimates and show the effect of stochastic factors on numerical solutions. We start by testing the convergence orders of time and space.

Example 5.1 (Accuracy Test). We take the stochastic Allen-Cahn (AC) equation with random diffusion coefficient field and multiplicative force noise as a numerical example to test the temporal and spatial convergence orders. The underlying equation is expressed as:
\[
\begin{align*}
    du(x,t) &= \varepsilon \partial_x(e^{z(x,\omega)}\partial_x u)dt + (u - u^3)dt + G(u)dW(x,t), \quad 0 < t < T, \ x \in D, \\
    u(x,t) &= 0, \quad 0 \leq t \leq T, \ x \in \partial D, \\
    u(x,0) &= u_0(x), \ x \in \bar{D},
\end{align*}
\]
(5.1)

where $z(x,\omega)$ is a W-M Gaussian random field with mean-zero and covariance function $c_q(x)$, and $W(x,t)$ is a $H_0^\gamma$-valued Wiener process defined by
\[
W(t,x) = \sum_{j=1}^{\infty} \sqrt{q_j} \sin(j\pi x) \beta_j(t),
\]
(5.2)

where $q_j = \mathcal{O}(j^{-(2\gamma+1+\epsilon)})$ with arbitrary small positive $\epsilon$.

The strong convergence rate in space and time is measured in terms of mean-square approximation errors at the endpoint $T = 0.1$, caused by the spatial and temporal discretizations. The expected value of error is approximated by computing the mean of 100 samples. Note that the exact solution of the problem (5.1) is unknown, and we will use the reference solution computed in the fine space-time mesh size as an approximation to the exact solution. If we denote by $u_{j,\text{ref}}$ the reference solution of the $j$-th sample of the exact solution $u(T)$, and denote by $u_{j,h}$ the value of the $j$-th sample of the fully discrete numerical solution $u_h^N$. Then the mean-square error $\|u(T) - u_h^N\|_{L^2(\Omega, L^2(D))}$ is approximately calculated by
\[
\|u(T) - u_h^N\|_{L^2(\Omega, L^2(D))} \approx \left( \frac{1}{100} \sum_{j=1}^{100} \|u_{j,\text{ref}} - u_{j,h}\|_{L^2(D)}^2 \right)^{1/2} =: u_{\text{error}}.
\]

We first test the time accuracy with different nonlinear terms $G$. Take the numerical solution computed by spatial mesh $h = 1/128$ and time step size $\Delta t = 10^{-6}$ as the reference solution for every sample. The approximation error $u_{\text{error}}$ under different time step size is calculated by taking $u_0(x) = \sin(2\pi x)$, $\varepsilon = 10^{-3}$, $\gamma = 1$ and $q = 2$. Table 1 and Table 2 respectively show the results for the cases where $G(u) = (1 - u^2)/2$ and $G(u) = u/2$, from which we observe that this is as predicted by the theory.

Next, we test the spatial accuracy. Now take the numerical solution computed by $h = 1/512$ and $\Delta t = 10^{-6}$ as the reference solution for every sample. We compute the approximation error $u_{\text{error}}$ under different spatial mesh size by taking $u_0(x) = \sin(2\pi x)$, $\varepsilon = 10^{-3}$, $\gamma = 1$ and $q = 2$. The results for the cases where $G(u) = (1 - u^2)/2$ and $G(u) = u/2$, from which we observe that this is as predicted by the theory.
again. Table 3 and Table 4 separately shows the relevant error and spatial convergence order for $G(u) = (1 - u^2)/2$ and $G(u) = u/2$, which is also consistent with the theoretical result.

**Table 1. Time accuracy test**

| $\Delta t$  | $u_{error}$ | Order |
|-------------|-------------|-------|
| 1.00E-2     | 3.75E-3     | -     |
| 5.00E-3     | 2.66E-3     | 0.49  |
| 2.50E-3     | 1.81E-3     | 0.55  |
| 1.25E-3     | 1.34E-3     | 0.43  |
| 6.25E-4     | 9.32E-4     | 0.52  |

**Table 2. Time accuracy test**

| $\Delta t$  | $u_{error}$ | Order |
|-------------|-------------|-------|
| 1.00E-2     | 5.23E-3     | -     |
| 5.00E-3     | 3.90E-3     | 0.42  |
| 2.50E-3     | 2.67E-3     | 0.55  |
| 1.25E-3     | 1.83E-3     | 0.55  |
| 6.25E-4     | 1.33E-3     | 0.46  |

**Table 3. Spatial accuracy test**

| $h$  | $u_{error}$ | Order |
|------|-------------|-------|
| 1/16 | 1.28E-3     | -     |
| 1/32 | 3.77E-3     | 1.76  |
| 1/64 | 1.17E-3     | 1.69  |
| 1/128| 3.61E-4     | 1.70  |
| 1/256| 9.74E-5     | 1.89  |

**Table 4. Spatial accuracy test**

| $h$  | $u_{error}$ | Order |
|------|-------------|-------|
| 1/16 | 1.36E-2     | -     |
| 1/32 | 4.09E-3     | 1.73  |
| 1/64 | 1.33E-3     | 1.62  |
| 1/128| 4.20E-4     | 1.67  |
| 1/256| 1.11E-4     | 1.91  |

**Example 5.2 (Phenomenon comparison).** In this example, the time evolution of the numerical solution of the stochastic AC equation shown in (5.1) is compared to that of the deterministic AC equation to show the effect of random perturbations, where the deterministic version is expressed by:

\[
\begin{align*}
    u_t(x,t) &= \varepsilon \partial_{xx} u + u - u^3, \quad 0 < t < T, \ x \in D, \\
    u(x,t) &= 0, \quad 0 \leq t \leq T, \ x \in \partial D, \\
    u(x,0) &= u_0(x), \ x \in \bar{D}.
\end{align*}
\]

We first show the effect of random field $a(x,w) = \varepsilon e^{z(x,w)}$ on the numerical solution in the absence of the nonlinear term $G$ (i.e., $G(u) = 0$), where $z(x,\omega)$ is a mean-zero Gaussian random field with covariance function $c_q(x)$. Given a sample point, by taking $u_0 = \sin(4\pi x)$, $T = 0.1$, $\Delta t = 10^{-5}$, $h = 1/128$ and $\varepsilon = 10^{-2}$, we plot in Figure 5.1 the contour maps of the numerical solution under different cases, where figure (a) represents the deterministic case and figures (b) and (c) denote the random case with $q = 0.1$ and $q = 2$, respectively. Compared to the deterministic model, it is seen from figures (b) and (c) that the random diffusion coefficient makes the diffusion process uncertain. Moreover, it’s known that the larger the parameter $q$, the more regular the random field $z(x,\omega)$ in, which results in the diffusion process shown in figure (c) is more uniform than that in figure (b).

Then we give a demonstration of the case with both random diffusion coefficients as well as multiplicative force noise. Given a sample point, by taking $u_0 = \sin(4\pi x)$, $T = 4$, $\Delta t = 10^{-4}$,
$h = 1/128$, $q = 2$, $\varepsilon = 10^{-5}$ and $G(u) = \frac{1}{2}(1 - u^2)$, we plot the time evolution of the numerical solution in Figure 5.2, where figure (a) denotes the deterministic model and figures (b) and (c) represent the case with $\gamma = 0.5$ and $\gamma = 1$, respectively. Compared to figure (a), it can be seen from figures (b) and (c) that there are small-scale structures resulted from noise, which are not present in the deterministic model. Noise plays a significant role, it changes the properties of the solutions. Notably, the static kink corresponding to the deterministic model varies greatly after the incorporation of noise and random diffusion coefficient fields. The kinks can interact, even annihilate each other, and some new kinks may arise. One more thing to point out that the larger the regularity parameter $\gamma$, the smoother the noise and the smaller the kink variation, which seems to be observed between figures (b) and (c).

![Figure 5.1](image1.png)

**Figure 5.1.** Time evolution of numerical solution with different diffusion coefficients. (a): deterministic case, (b): random case with $q = 0.1$, (c): random case with $q = 2$.

![Figure 5.2](image2.png)

**Figure 5.2.** Time evolution of numerical solution with different noise. (a): deterministic case, (b): random case with $\gamma = 0.5$, (c): random case with $\gamma = 1$.

**REFERENCES**

[1] E. J. Allen, S. J. Novosel, and Z. Zhang. Finite element and difference approximation of some linear stochastic partial differential equations. *An International Journal of Probability and Stochastic Processes*, 64(1-2):117–142, 1998.
[2] A. Andersson and S. Larsson. Weak convergence for a spatial approximation of the nonlinear stochastic heat equation. *Mathematics of Computation*, 85(299):1335–1358, 2016.

[3] I. Babuška, F. Nobile, and R. Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 45(3):1005–1034, 2007.

[4] I. Babuška, R. Tempone, and G. E. Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM Journal on Numerical Analysis*, 42(2):800–825, 2004.

[5] M. Beccari, M. Hutzenthaler, A. Jentzen, R. Kurniawan, F. Lindner, and D. Salimova. Strong and weak divergence of exponential and linear-implicit euler approximations for stochastic partial differential equations with superlinearly growing nonlinearities. *arXiv preprint arXiv:1903.06066*, 2019.

[6] C.-E. Bréhier. Approximation of the invariant measure with an Euler scheme for stochastic PDEs driven by space-time white noise. *Potential Analysis*, 40(1):1–40, 2014.

[7] M. Cai, S. Gan, and X. Wang. Weak Convergence Rates for an Explicit Full-Discretization of Stochastic Allen-Cahn Equation with Additive Noise. *Journal of Scientific Computing*, 86(3):1–30, 2021.

[8] Y. Cao, J. Hong, and Z. Liu. Approximating stochastic evolution equations with additive white and rough noises. *SIAM Journal on Numerical Analysis*, 55(4):1958–1981, 2017.

[9] J. Cui and J. Hong. Strong and weak convergence rates of a spatial approximation for stochastic partial differential equation with one-sided Lipschitz coefficient. *SIAM Journal on Numerical Analysis*, 57(4):1815–1841, 2019.

[10] G. Da Prato and J. Zabczyk. Stochastic equations in infinite dimensions. *Cambridge university press*, 2014.

[11] A. Davie and J. Gaines. Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations. *Mathematics of Computation*, 70(233):121–134, 2001.

[12] A. De Bouard and A. Debussche. Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation. *Applied Mathematics and Optimization*, 54(3):369–399, 2006.

[13] A. Debussche. Weak approximation of stochastic partial differential equations: the nonlinear case. *Mathematics of Computation*, 80(273):89–117, 2011.

[14] A. Debussche and J. Printems. Weak order for the discretization of the stochastic heat equation. *Mathematics of computation*, 78(266):845–863, 2009.

[15] C. R. Dietrich. A simple and efficient space domain implementation of the turning bands method. *Water Resources Research*, 31(1):147–156, 1995.

[16] C. R. Dietrich and G. N. Newsam. Fast and exact simulation of stationary Gaussian processes through circulant embedding of the covariance matrix. *SIAM Journal on Scientific Computing*, 18(4):1088–1107, 1997.

[17] Q. Du and T. Zhang. Numerical approximation of some linear stochastic partial differential equations driven by special additive noises. *SIAM Journal on Numerical Analysis*, 40(4):1421–1445, 2002.

[18] K. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. *Semigroup Forum*, 63:278–280, 1999.

[19] X. Feng, Y. Li, and Y. Zhang. Finite element methods for the stochastic Allen-Cahn equation with gradient-type multiplicative noise. *SIAM Journal on Numerical Analysis*, 55(1):194–216, 2017.

[20] M. Geißen, M. Kovács, and S. Larsson. Rate of weak convergence of the finite element method for the stochastic heat equation with additive noise. *BIT Numerical Mathematics*, 49(2):343–356, 2009.

[21] I. Gohberg, S. Goldberg, and M. A. Kaashoek. Hilbert-schmidt operators. In *Classes of Linear Operators Vol. I*, pages 138–147. Springer, 1990.

[22] I. Gyöngy, S. Sabanis, and D. Šiška. Convergence of tamed Euler schemes for a class of stochastic evolution equations. *Stochastics and Partial Differential Equations: Analysis and Computations*, 4(2):225–245, 2016.

[23] I. Gyöngy and A. Millet. Rate of convergence of space time approximations for stochastic evolution equations. *Potential analysis*, 30(1):29–64, 2009.

[24] E. Hausenblas. Approximation for semilinear stochastic evolution equations. *Potential Analysis*, 18(2):141–186, 2003.
[25] E. Hausenblas. Weak approximation for semilinear stochastic evolution equations. In *Stochastic analysis and related topics VIII*, pages 111–128. Springer, 2003.

[26] M. Hutzenthaler and A. Jentzen. *Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients*, volume 236. American Mathematical Society, 2015.

[27] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong and weak divergence in finite time of Euler’s method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2130):1563–1576, 2011.

[28] A. Jentzen and P. E. Kloeden. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 465(2102):649–667, 2008.

[29] A. Jentzen and P. Pušnik. Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities. *IMA Journal of Numerical Analysis*, 40(2):1005–1050, 2020.

[30] Y. Kazashi. Quasi-monte carlo integration with product weights for elliptic PDEs with log-normal coefficients. *IMA J. Numer. Anal.*, 39(3):1563–1593, 2019.

[31] P. E. Kloeden, G. J. Lord, and A. et al Neuenkirch. The exponential integrator scheme for stochastic partial differential equations: Pathwise error bounds. *Journal of Computational and Applied Mathematics*, 235(5):1245–1260, 2011.

[32] P. E. Kloeden and E. Platen. Numerical solution of stochastic differential equations. *Springer Science and Business Media*, 2013.

[33] M. Kovács, S. Larsson, and F. Lindgren. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise. *BIT Numerical Mathematics*, 52(1):85–108, 2012.

[34] M. Kovács, S. Larsson, and F. Lindgren. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise II. Fully discrete schemes. *BIT Numerical Mathematics*, 53(2):497–525, 2013.

[35] M. Kovács, S. Larsson, and F. Lindgren. On the discretisation in time of the stochastic Allen–Cahn equation. *Mathematische Nachrichten*, 291(5-6):966–995, 2018.

[36] R. Kruse. Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise. *IMA Journal of Numerical Analysis*, 34(1):217–251, 2014.

[37] R. Kruse. *Strong and weak approximation of semilinear stochastic evolution equations*. Springer, 2014.

[38] F. Lindner and R. Schilling. Weak order for the discretization of the stochastic heat equation driven by impulsive noise. *Potential Analysis*, 38(2):345–379, 2013.

[39] Z. Liu and Z. Qiao. Strong approximation of monotone stochastic partial differential equations driven by white noise. *IMA Journal of Numerical Analysis*, 40(2):1074–1093, 2020.

[40] Z. Liu and Z. Qiao. Strong approximation of monotone stochastic partial differential equations driven by multiplicative noise. *Stochastics and Partial Differential Equations: Analysis and Computations*, 9(3):559–602, 2021.

[41] G. J. Lord, C. E. Powell, and T. Shardlow. An introduction to computational stochastic PDEs. *Cambridge University Press*, 2014.

[42] A. Mantoglou and J. L. Wilson. The turning bands method for simulation of random fields using line generation by a spectral method. *Water Resources Research*, 18(5):1379–1394, 1982.

[43] G. N. Milstein and M. V. Tretyakov. Stochastic numerics for mathematical physics. *Springer Science and Business Media*, 2013.

[44] G. N. Newsam and C. R. Dietrich. Bounds on the size of nonnegative definite circulant embeddings of positive definite Toeplitz matrices. *IEEE Transactions on Information Theory*, 40(4):1218–1220, 1994.

[45] J. Printems. On the discretization in time of parabolic stochastic partial differential equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 35(6):1055–1078, 2001.
[46] M. Sauer and W. Stannat. Lattice approximation for stochastic reaction diffusion equations with one-sided Lipschitz condition. *Mathematics of Computation*, 84(292):743–766, 2015.

[47] M. Shinozuka. Simulation of multivariate and multidimensional random processes. *The Journal of the Acoustical Society of America*, 49(1B):357–368, 1971.

[48] M. Shinozuka and C. M. Jan. Digital simulation of random processes and its applications. *Journal of sound and vibration*, 25(1):111–128, 1972.

[49] J. L. Wadsworth and J. A. Tawn. Efficient inference for spatial extreme value processes associated to log-Gaussian random functions. *Biometrika*, 101(1):1–15, 2014.

[50] J. B. Walsh. Finite element methods for parabolic stochastic PDEs. *Potential Analysis*, 23(1):1–43, 2005.

[51] X. Wang. Strong convergence rates of the linear implicit Euler method for the finite element discretization of SPDEs with additive noise. *IMA Journal of Numerical Analysis*, 37(2):965–984, 2017.

[52] X. Wang. An efficient explicit full-discrete scheme for strong approximation of stochastic Allen–Cahn equation. *Stochastic Processes and their Applications*, 130(10):6271–6299, 2020.

[53] X. Wang and S. Gan. Weak convergence analysis of the linear implicit Euler method for semilinear stochastic partial differential equations with additive noise. *Journal of Mathematical Analysis and Applications*, 398(1):151–169, 2013.

[54] A. TA. Wood and G. Chan. Simulation of stationary Gaussian processes in $[0,1]^d$. *Journal of Computational and Graphical Statistics*, 3(4):409–432, 1994.

[55] Y. Yan. Galerkin finite element methods for stochastic parabolic partial differential equations. *SIAM Journal on Numerical Analysis*, 43(4):1363–1384, 2005.

[56] Z. Zhang and G. Karniadakis. *Numerical methods for stochastic partial differential equations with white noise*. Springer, 2017.