Clebsch-Gordan coefficients of discrete groups in subgroup bases

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We express each Clebsch-Gordan (CG) coefficient of a discrete group as a product of a CG coefficient of its subgroup and a factor, which we call an embedding factor. With an appropriate definition, such factors are fixed up to phase ambiguities. Particularly, they are invariant under basis transformations of irreducible representations of both the group and its subgroup. We then impose on the embedding factors constraints, which relate them to their counterparts under complex conjugate and therefore restrict the phases of embedding factors. In some cases, the phase ambiguities are reduced to sign ambiguities. We describe the procedure of obtaining embedding factors and then calculate CG coefficients of the group \( \mathcal{E} \) in terms of embedding factors of its subgroups \( S_4 \) and \( T_7 \).

I. INTRODUCTION

Discrete subgroups of \( SU(3) \) are widely used in flavor model building in particle physics, where one needs to study the mathematical properties of the selected group, e.g., its Clebsch-Gordan (CG) coefficients, which determine how fields are coupled in the model.

For some continuous groups, CG coefficients are usually expressed in terms of those of their subgroups. For example, \( SU(3) \) CG coefficients can be factored into \( SU(2) \) CG coefficients and so-called isoscalar factors. With the notations of Ref. [8], the \( SU(3) \) CG coefficients for the tensor product \( \mu_1 \otimes \mu_2 = \mu_3 \oplus \cdots \), with \( \mu_i \) being irreducible representations (irreps), can be written as

\[
\begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 \\
\nu_1 & \nu_2 & \nu_3
\end{pmatrix}
= C_{I_1 \otimes I_2 \otimes I_3}^{I_1 I_2 I_3}
\begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 \\
I_1 Y_1 & I_2 Y_2 & I_3 Y_3
\end{pmatrix},
\]

where \( I_1 \) and \( I_2 \) are collections of indices carrying the number of isoscalar factors is smaller than the number of \( SU(3) \) CG coefficients.

In this paper, we discuss the relation between CG coefficients of a discrete group and those of its subgroup. To our knowledge, a general study of this idea for discrete groups has not been done in the literature.

For a discrete group \( G \) with subgroup \( H \), we want to find a set of factors that relate CG coefficients of \( G \) and \( H \), denoted as \( M^{(G)} \) and \( M^{(H)} \),

\[
M_{AB}^{(G)} = \mathcal{E}_A M_B^{(H)},
\]

where \( B \) is a collection of indices to specify the subgroup CG coefficients and \( A \) a collection of indices carrying the information of how irreps of \( H \) are embedded in irreps of \( G \). In analogy to the isoscalar factors, we call the coefficients \( \mathcal{E} \) the embedding factors. Since CG coefficients are basis-dependent, Eq. (2) is also basis-dependent. Then one may ask the following questions: 1) how can we define \( \mathcal{E} \) independent of bases? 2) are the coefficients in \( \mathcal{E} \) unique for all CG coefficients \( M^{(H)} \) in all bases? To answer these questions, we show that embedding factors \( \mathcal{E} \) can be defined in a way that is invariant under basis transformations of irreps of \( G \) and \( H \). It implies that there does exist a set of embedding factors for all bases of irreps of \( G \) and \( H \). This does not exhaust all the ambiguities of the embedding factors because the coefficients of \( \mathcal{E} \) still have phase ambiguities, which stem from those of the subgroup CG coefficients.

We analyze the phase ambiguities of the embedding factors and propose a way to reduce them, with some ambiguities remained to be eliminated by other conventions. The advantage of our convention is that it only depends on general properties of groups and irreps, and hence it can apply to any groups. We impose on embedding factors constraints which require that a contraction of two irreps, in the form \( (X \otimes Y)_Z \), behaves the same as the corresponding irrep \( Z \) under the action of complex conjugate. Such constraints lead to the following consequences:

- Case I: If all of \( X \), \( Y \), and \( Z \) are real or pseudoreal irreps, the overall phase of embedding factors is fixed up to a sign factor. In particular, if the irreps can be decomposed into real or pseudoreal subgroup irreps, then the corresponding embedding factors are real numbers fixed up to sign factors.
- Case II: If \( Z \) is real or pseudoreal and \( X \) is complex conjugate of \( Y \), the overall phase of embedding factors is also fixed up to a sign factor.
- Case III: In other cases, the embedding factors for \( X \otimes Y \rightarrow Z \) and \( X \otimes Y \rightarrow Z \) are complex conjugate to each other. Here, \( X \) represents conjugate of \( X \) if it is complex, or \( X \) itself otherwise. This statement looks trivial and one may think that CG coefficients always have such a property. But actually such relation not always holds for CG coefficients (See Proposition 2 of Section [11]). This is another advantage of embedding factors compared with CG coefficients.

We introduce a procedure to calculate the embedding factors. The calculation involves complicated cyclotomic numbers, which are polynomials of roots of unity \( e^{2k\pi/n} \).
and hence it is difficult to obtain simplified results. We therefore implement an algorithm to perform arithmetic calculations of cyclotomic numbers.

We apply our technique to the group $PSL_2(7)$ and its subgroups, the groups $S_4$ and $T_7$, and obtain representation matrices of $PSL_2(7)$ in its subgroup bases, in which subgroup elements are block diagonal matrices. For both subgroups, we acquire complete lists of embedding factors of the group $PSL_2(7)$. We also find the embedding factors for $S_4$ and its subgroup $A_4$. We automate much of the procedure in the Mathematica code in Ref. [5], which can be easily adjusted for calculating embedding factors of other discrete groups.

The complete list of $PSL_2(7)$ CG coefficients and the presentation matrices of $PSL_2(7)$ in $S_4$ basis are new results, which could be helpful for flavor model building based on $PSL_2(7)$, or studying connections among $PSL_2(7)$ models [6, 7], models [13, 18, 21, 22], and $S_4$ models (for a review of $S_4$ models, see Ref. [2]). A subgroup tree of $SU(3)$ discrete subgroups can be found in Ref. [23]. For systematic analysis of discrete groups used in flavor model buildings, see Refs. [17] and [19].

The remainder of the paper is organized as follows. In Section II we analyze the ambiguities of CG coefficients. In Section III we define the embedding factors and then show that they are basis independent but still have phase ambiguities. In Section IV phase conventions are introduce to reduce the phases ambiguities of embedding factors. In Section V we describe the procedure of calculating the embedding factors. The procedure is then applied to the group $PSL_2(7)$ and its subgroups $S_4$ and $T_7$. Specifically, in Section VI the representation matrices of the $PSL_2(7)$ group are obtained in its subgroup bases. In Section VII we calculate embedding factors of the tensor product $6 \otimes 6 \rightarrow 6$ of the group $PSL_2(7)$ as an example. The group theory properties of relevant groups are given in Appendix A. In Appendix B we describe the algorithm for arithmetic calculations of cyclotomic numbers. Appendix C and D list complete sets of CG coefficients of $PSL_2(7)$ in $S_4$ and $T_7$ bases.

### Conventions

Some of the conventions used in the main text are as follows. $G$ represents a discrete group and $H$ is its subgroup. The boldface and capitalized letters $X, Y$, and $Z$ are irreps of the group $G$ and $X, Y, Z$ are corresponding vectors in Hilbert spaces of these irreps. The boldface and lowercase letters $x, y, z$ are irreps of the subgroup $H$, and $x, y, z$ are corresponding vectors in Hilbert space of these irreps. Contraction of irreps of $G$ are denoted as $[X \otimes Y]_Z$, which means a contraction of $X$ and $Y$ to $Z$. Similarly, $\{x \otimes y\}$ represents a contraction of $H$ irreps. The letters $i, j, k$ label a single component of a vector while $a, b, c$ label an irrep of the subgroup. For example, $x_i$ means the $i$-th component of $x$ while $x_a$ is a vector of the irreps $x_a$. A matrix realization (representation matrix) of group element $g$ in $X$ irrep is denoted as $\rho_X (g)$. Representation matrices are always unitary in this paper.

### II. CLEBSCH-GORDAN COEFFICIENTS AND THEIR AMBIGUITIES

The tensor product of two irreps of a group is in general reducible. Let $G$ be a discrete group with irreps $X, Y, Z$, and $X, Y, Z$ be vectors of the corresponding Hilbert spaces on which the group elements act. If $Z$ is contained in the tensor product $X \otimes Y$, then given $X$ and $Y$, there exist a $Z$ and a set of coefficients $M^{(XY \rightarrow Z)}_{i\alpha}$ such that, for all $g \in G$,

$$ (\rho_Z (g) Z)_k = \sum_{\alpha} M^{(XY \rightarrow Z)}_{i\alpha} (\rho_X (g) X \otimes \rho_Y (g) Y)_\alpha, $$

where $\rho_Z$ is a matrix realization of irrep $Z$ and elements of the tensor product $\rho_X (g) X \otimes \rho_Y (g) Y$, a column vector with dimension $(\dim X \times \dim Y)$, are of the form $\rho_X (g) X|_i (\rho_Y (g) Y)_\alpha$. The coefficients $M^{(XY \rightarrow Z)}_{i\alpha}$ are the Clebsch-Gordan coefficients. Note that CG coefficients can also be defined as a unitary transformation between the tensor products of group elements and the direct sum of their irreps, e.g. Ref. [2]. In this paper, it is more convenient to define CG coefficients with vector spaces. In the remainder of this paper, we may suppress the superscript $(XY \rightarrow Z)$ if there is no chance of confusion.

If $Z$ and $M^{(XY \rightarrow Z)}$ satisfy eq. (3), then for any nonzero c-number $\lambda$, $\lambda Z$ and $\lambda M^{(XY \rightarrow Z)}$ also satisfy the equation. So it is then conventional to impose on CG coefficients, in addition to orthogonality, normalization constraints, which give rise to unitary CG coefficients

$$ \sum_{\alpha} M^\alpha_{i\alpha} M^{\alpha\beta}_{\beta\beta} = \delta_{kl}. $$

There are, however, other ambiguities. Firstly, CG coefficients are basis-dependent. Under basis transformations

$$ X \rightarrow X' = U_X X, \quad \rho_X (g) \rightarrow U_X \rho_X (g) U_X^{-1}, $$

and similarly for $Y$ and $Z$, eqs. (3) and (4) are invariant if the matrices $M$ simultaneously transform as

$$ M \rightarrow M' = U_Z M (U_X^{-1} \otimes U_Y^{-1}) . $$

We shall call this basis ambiguity in the remainder of the paper. The second ambiguity is phase ambiguity, meaning that eqs. (3) and (4) are invariant under the transformation $M \rightarrow e^{i\theta} M$, $Z \rightarrow e^{i\theta} Z$. The last ambiguity exists when there are nontrivial multiplicities in a tensor product, i.e., $X \otimes Y \rightarrow Z^{(1)} \oplus \cdots \oplus Z^{(p)}$, then a linear combination of the CG coefficients $\lambda_1 M^{(1)} + \cdots + \lambda_p M^{(p)}$ with $\sum_i \lambda_i^* \lambda_i = 1$ is also a set of unitary CG coefficients.
III. EMBEDDING FACTORS

Let $H$ be a subgroup of $G$ and assume that the irreps $X, Y, Z$ can be decomposed into irreps of $H$ as

$$X = \bigoplus_a x_a, \quad Y = \bigoplus_b y_b, \quad Z = \bigoplus_c z_c, \quad (6)$$

which reduces to eq. (7) when $g = c$.

In the following, we may write $\mathcal{E}_{c,ab}^{(XY \rightarrow Z)}$ simply as $\mathcal{E}_{c,ab}$ for convenience. The orthonormalization constraints (11) now become

$$\sum_{ab} \mathcal{E}_{c,ab}^* \mathcal{E}_{d,ab} = \delta_{cd}. \quad (10)$$

We remark that the rhs of each equations of (6) may contain duplicated irreps. Such a case can be avoided by choosing a large enough subgroup $H$. Therefore, for simplicity, we only consider the case that no irrep is contained twice in an irrep of the large group.

**Proposition 1.** The coefficients $\mathcal{E}_{c,ab}^{(XY \rightarrow Z)}$ defined as eq. (4) are invariant under basis transformations of irreps of both the group and its subgroup.

**Proof.** Under a basis transformation of the group $G$, the vector $X$, projection matrix $P_{X \rightarrow a}$, and matrices $\rho_X (g)$ transform as $X' = U_X X$, $P'_{X \rightarrow a} = P_{X \rightarrow a} U_X^{-1}$, and $\rho'_{X} (g) = U_X \rho_{X} (g) U_X^{-1}$ respectively. It follows that $(P_{X \rightarrow a} \rho_{X} (g) X)$ is invariant, so are $(P_{Y \rightarrow b} \rho_{Y} (g) Y)$ and $(P_{Z \rightarrow c} \rho_{Z} (g) Z)$. Hence, both sides of eq. (9) are invariant under the transformation.

Now consider basis transformations of the subgroup irreps. For simplicity, we can now fix projection matrices to special forms, since we can always perform basis transformations to bring projection matrices to desired forms without changing $\mathcal{E}_{c,ab}$ coefficients. So we choose bases such that elements of $H$ are block diagonal matrices, which means that the projection matrices have the form

$$P_{X \rightarrow a} = \begin{pmatrix} O_1 & \cdots & O_{a-1} & P^{(u)}_{X \rightarrow a} & O_{a+1} & \cdots \end{pmatrix}, \quad (11)$$

where $x_a, y_b, z_c$ are irreps of $H$. For the contraction $X \otimes Y \rightarrow Z$, we can write

$$P_{Z \rightarrow c} Z = \sum_{a,b} \mathcal{E}_{c,ab}^{(XY \rightarrow Z)} \{(P_{X \rightarrow a} X) \otimes (P_{Y \rightarrow b} Y)\} z_c, \quad (7)$$

where $P_{X \rightarrow a}$ is a matrix of dimension $\text{dim} x_a \times \text{dim} X$ to project the $x_a$ components from $X$ and $\mathcal{E}_{c,ab}^{(XY \rightarrow Z)}$ are the embedding factors. The projection matrices act like similarity transformations on $H$ elements between representation $X$ and $x_a$, i.e.,

$$(P_{X \rightarrow a})_{ik} [\rho_{X} (h)]_{kl} \left( P'_{X \rightarrow b}\right)_{lj} = \delta_{ab} [\rho_{x_a} (h)]_{ij}, \quad \forall h \in H. \quad (8)$$

In analogy to eq. (3), the coefficients $\mathcal{E}_{c,ab}^{(XY \rightarrow Z)}$ should actually satisfy a stronger constraint

$$\mathcal{E}_{c,ab}^{(XY \rightarrow Z)} = \mathcal{E}_{c,ab}^{(YX \rightarrow Z)} \quad \forall g \in G, \quad (9)$$

where $P^{(u)}_{X \rightarrow a}$ is a unitary matrix of dimension $\text{dim} x_a \times \text{dim} x_a$ and $O_b$ are zero matrices of dimension $\text{dim} x_a \times \text{dim} x_b$. For basis transformations of $H$

$$(x_a, y_b, z_c) \rightarrow (x'_a, y'_b, z'_c) = (U_a x_a, U_b y_b, U_c z_c), \quad (12a)$$

the subgroup contraction $\{x_a \otimes y_b\} z_c$ should transform as

$$z_c = \{x_a \otimes y_b\} z_c \rightarrow z'_c = \{U_a x_a \otimes U_b y_b\} z'_c, \quad (12b)$$

where we have used the primed brackets to indicate the new CG coefficients in the new basis. It then implies

$$\{U_a x_a \otimes U_b y_b\} z'_c = U_c \{x_a \otimes y_b\} z_c. \quad (13)$$

Under these transformations, $\rho_{X} (g) X$ transforms as

$$\rho_{X} (g) X \rightarrow U_X \rho_{X} (g) X, \quad (14)$$

where

$$U_X = \bigoplus_{x_a \text{ in } X} \left( P^{(u)}_{X \rightarrow a} \right)^{-1} U_a P^{(u)}_{X \rightarrow a}. \quad (15)$$

Since $P_{X \rightarrow a}$ is in the form of (11), it is easy to see that

$$P_{X \rightarrow a} U_X = U_a P_{X \rightarrow a}. \quad (16)$$

(14) and (10) imply that $P_{X \rightarrow a} \rho_{X} (g) X$ transforms as

$$P_{X \rightarrow a} \rho_{X} (g) X \rightarrow U_a P_{X \rightarrow a} \rho_{X} (g) X. \quad (17)$$

Now consider the both sides of eq. (9) under the basis transformations. The lhs becomes $U_c P_{Z \rightarrow c} \rho_{Z} (g) Z$ and
the rhs becomes

\[ \text{rhs} = \sum_{a,b} \mathcal{E}_{c,ab} \{ U_a P_{X \rightarrow a} \rho_X (g) X \otimes U_b P_{Y \rightarrow b} \rho_Y (g) Y \} \xi_c^{(\Gamma)} \]

\[ = U_c \sum_{a,b} \mathcal{E}_{c,ab} \{ P_{X \rightarrow a} \rho_X (g) X \otimes P_{Y \rightarrow b} \rho_Y (g) Y \} \xi_c \]

\[ = U_c P_{Z \rightarrow c} \rho_Z (g) Z = \text{lhs}, \]

where we have used eq. (13) in the second equality. Eq. (9) is therefore invariant under the basis transformations (12). 

We have showed that embedding factors are independent of bases of both the group and its subgroup. There are, however, still ambiguities in embedding factors due to phase ambiguities of the projection matrices and subgroup CG coefficients. Consider the $U (1)$ transformations on projection matrices and subgroup CG coefficients:

\[ P_{X \rightarrow a} \rightarrow e^{i \theta_a} P_{X \rightarrow a}, \]

\[ \{ x_a \otimes y_b \} \rightarrow e^{i \theta_{a+b-c}} \{ x_a \otimes y_b \}. \]

Under these $U (1)$ transformations, the embedding factors transform as

\[ \mathcal{E}_{c,ab}^{(XY \rightarrow Z)} \rightarrow e^{(\theta_a - \theta_b - \theta_c) + (\theta_{a+b-c})} \mathcal{E}_{c,ab}^{(XY \rightarrow Z)}. \]

We see that there are in general four phase ambiguities for each embedding factor. They can be removed or reduced by appropriate phase conventions, which are usually basis-dependent. For example, for $SU (2)$ CG coefficients, it is conventional to choose a particular CG coefficient to be real and positive. In the following section, we introduce a basis-independent convention, which can reduce the number of $U (1)$ ambiguities and, in some cases, reduce the ambiguities to $Z_2$ ambiguities, i.e., ambiguities of sign factors.

### IV. Reducing Phase Ambiguities

To introduce the convention, we first discuss real and complex representations. A real or pseudoreal representation is a representation whose complex conjugate is equivalent to itself while a complex representation is a representation that is inequivalent to its complex conjugate. So we can define the complex conjugate of a real (or pseudoreal) representation to be itself. Now if $\rho_X$ is a matrix realization of $X$ and $\rho_{\tilde{X}}$ the one of $\tilde{X}$, then there exists a unitary matrix $\Gamma_X$ such that

\[ \Gamma_X \rho_X (g)^* \Gamma_X^\dagger = \rho_{\tilde{X}} (g), \quad \forall g \in G. \]

When $X$ is complex, we can always choose $\rho_{\tilde{X}} (g) = \rho_X (g)^*$ so that $\Gamma_X$ can be the identity matrix. When $X$ is real or pseudoreal, then $\rho_{\tilde{X}}$ is identical to $\rho_X$ and $\Gamma_X$ is in general a nontrivial unitary matrix depending on the basis of the representation. For real $X$, $\Gamma_X$ is symmetric; for pseudoreal $X$, $\Gamma_X$ is antisymmetric.

Eq. (20) implies that $\Gamma_X X^*$ should transform as a vector in the representation space of $\tilde{X}$. We therefore can define a vector $\tilde{X}$ to be

\[ \tilde{X} \equiv \Gamma_X X^*. \]

With such a definition, it is natural to impose the following constraints

\[ \Gamma_Z ([X \otimes Y] Z)_z = [\tilde{X} \otimes \tilde{Y}]_z, \]

\[ \Gamma_c \left( (x_a \otimes y_b)_z \right)_z = (\bar{x}_a \otimes \bar{y}_b)_{\bar{z}_c}, \]

where the matrix $\Gamma_Z$ transforms $Z$ to $\tilde{Z}$, and $\Gamma_c$ transforms $Z$ to $\bar{Z}$. The constraints (22a) and (22b) imply that the contractions $[X \otimes Y]_Z$ and $(x_a \otimes y_b)_z$ should behave the same as $Z$ and $\bar{z}_c$ under the complex conjugate operation. It is also natural to impose similar constraints on subgroup irreps embedded in a large group irrep, meaning that $\Gamma_c$ should transform the complex conjugate $Z_c$ components of $Z$ to $\bar{z}_c$ components of $\bar{Z}$.

\[ \Gamma_c (P_{Z \rightarrow c} Z)_z = P_{Z \rightarrow c} \bar{Z} = P_{Z \rightarrow c} \Gamma_Z Z^*, \]

from which it follows that

\[ \Gamma_c P_{Z \rightarrow c} = P_{Z \rightarrow c} \Gamma_Z. \]

These constraints lead to the following consequences.

**Proposition 2.** The CG coefficients $M^{(x_a y_b \rightarrow z_a)}$ and $M^{(x_a y_b \rightarrow \bar{z}_c)}$ are related by

\[ M^{(\bar{x}_a \bar{y}_b \rightarrow \bar{z}_c)} = \Gamma_c \left( M^{(x_a y_b \rightarrow z_a)} \right)^* (\Gamma_a^{-1} \otimes \Gamma_b^{-1}). \]

**Proof.** The CG coefficients $M^{(x_a y_b \rightarrow z_a)}$ and $M^{(\bar{x}_a \bar{y}_b \rightarrow \bar{z}_c)}$ are defined as

\[ z_a = M^{(x_a y_b \rightarrow z_a)} (x_a \otimes y_b), \quad \bar{z}_c = M^{(\bar{x}_a \bar{y}_b \rightarrow \bar{z}_c)} (\bar{x}_a \otimes \bar{y}_b). \]

The constraint (22b) then implies

\[ \Gamma_c M^{(x_a y_b \rightarrow z_a)} (x_a^* \otimes \bar{y}_b^*) = M^{(\bar{x}_a \bar{y}_b \rightarrow \bar{z}_c)} (\bar{x}_a \otimes \bar{y}_b) \]

\[ = \Gamma_a (\bar{x}_a \otimes \bar{y}_b) \Gamma^{-1} \Gamma_b (x_a^* \otimes y_b^*), \]

where we used $\bar{x}_a = \Gamma_c x_a^*$ and $\bar{y}_b = \Gamma_b y_b^*$ in the second equality. Comparing the coefficients of both sides yields eq. (23). 

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1 According to [3], projection matrices can also transformation as

\[ P_{X \rightarrow a} \rightarrow P_{X \rightarrow a} \rho_X (g'), \]

where $g'$ is an element of center of $G$. But we can see that it is equivalent to replacing $g$ with $g' g$ in eq. (3) and hence does not change embedding factors.
We see that the relation between $M^{(x_yyightarrow z_c)}$ and $M^{(x_yyightarrow z_c)}$ depends on basis of irreps. $M^{(x_yyightarrow z_c)}$ is in general not simply the complex conjugate of $M^{(x_yyightarrow z_c)}$ unless that $\Gamma_{a,b,c}$ matrices are all identity matrices. The matrix $\Gamma_{a}$ is the identity matrix in two cases: 1) $x_a$ is complex; 2) $x_a$ is real (not pseudoreal) irrep and is in a basis that the matrices of its generators are all real. The latter implies that, if all the three irreps are real, there exist bases that the CG coefficients $M^{(x_yyightarrow z_c)}$ are all real. The overall phase of $M^{(x_yyightarrow z_c)}$ is fixed up to a sign factor in two cases: i) $x_a, y_b, z_c$ are all real or pseudoreal irreps; ii) $z_a$ is real or pseudoreal and $x_a$ is the complex conjugate of $y_b$. In these two cases, the phase $\phi^{(a,b,c)}$ in eq. (19) can only be 0 or $\pi$.

**Proposition 3.** When $\Gamma_a$ and $\Gamma_X$ are fixed, the phase ambiguity of the projection matrix $P_{Xightarrow a}$, denoted as $\theta_a^{(X)}$ in eq. (17), are constrained by $e^{i\theta_a^{(X)}} = e^{-i\theta_a^{(X)}}$.

**Proof.** This follows directly from (22c). \qed

If both $X$ and $x_a$ are real or pseudoreal, we have $\theta_a^{(X)}$ is fixed to 0 or $\pi$ and the $U(1)$ ambiguity is reduced to a $Z_2$ ambiguity; if any of the irreps is complex, we have $\theta_a^{(X)} = -\theta_a^{(X)}$, which implies that two $U(1)$ ambiguities are reduced to one $U(1)$ ambiguity.

**Proposition 4.** The embedding factors $\epsilon_c^{(XYightarrow Z)}$ and $\epsilon_{c,ab}^{(XYightarrow Z)}$ satisfy

$$\epsilon_{c,ab}^{(XYightarrow Z)} = \left(\epsilon_{c,ab}^{(XYightarrow Z)}\right)^*.$$  

**Proof.** The coefficients $\epsilon_{c,ab}^{(XYightarrow Z)}$ and $\epsilon_{c,ab}^{(XYightarrow Z)}$ are defined as

$$P_{Zightarrow c} = \sum_{a,b} \epsilon_{c,ab}^{(XYightarrow Z)} \left\{ P_{Xightarrow a} \times P_{Yightarrow b} \right\}_{Z_c},$$  

$$P_{Zightarrow c} = \sum_{a,b} \epsilon_{c,ab}^{(XYightarrow Z)} \left\{ P_{Xightarrow a} \times P_{Yightarrow b} \right\}_{z_c}.$$  

Applying eq. (22b) to eq. (26) yields

$$P_{Zightarrow c} = \sum_{a,b} \epsilon_{c,ab}^{(XYightarrow Z)} \Gamma_c \left( \left\{ P_{Xightarrow a} \times P_{Yightarrow b} \right\}_{Z_c} \right)^*.$$  

On the other hand, we can also write $P_{Zightarrow c}$ as, using eqs. (22c) and (23)

$$P_{Zightarrow c} = \sum_{a,b} \left( M^{(XYightarrow Z)} \right)^* \Gamma_c \left( \left\{ P_{Xightarrow a} \times P_{Yightarrow b} \right\}_{z_c} \right)^*.$$  

Comparing rhs of eqs. (27) and (28) gives rise to eq. (24).

Note that eq. (24) holds only when the CG coefficients of the subgroup obey the constraint (22c). The constraint (24) on $\epsilon_{c,ab}$ is much simpler than the constraint (23) on $M$. It is basis-independent and $\epsilon_c^{(XYightarrow Z)}$ is simply the complex conjugate of $\epsilon_c^{(XYightarrow Z)}$. The overall phase of $\epsilon_c^{(XYightarrow Z)}$ is fixed up to a sign factor when $[X \otimes Y]_{Z}$ and $[\bar{X} \otimes \bar{Y}]_{Z}$ represent the same contraction, which occurs in the following two cases:

- **Case I:** all of $X, Y,$ and $Z$ are real or pseudoreal. Particularly, if subgroup irreps $z_c, x_a, y_b$ are also real or pseudo real, then the coefficient $\epsilon_c^{(XYightarrow Z)}$ is a real number and fixed up to a sign factor.

- **Case II:** $Z$ is real or pseudoreal and $X = Y$ are complex.

For contractions of more than two vectors, we have similar constraints as (22c). Consider a contraction of three vectors $[X \otimes [V \otimes W]]_{Y}$ and its counterpart under complex conjugate $[\bar{X} \otimes \bar{V} \otimes \bar{W}]_{Y}$. The relation between these two is

$$\Gamma_Z ( [X \otimes [V \otimes W]]_{Y} )^{*} = [\bar{X} \otimes [\bar{V} \otimes \bar{W}]_{Y}]_{Z}.$$  

It can be shown as follows:

$$\text{lhs} = \Gamma_Z M^{(XYightarrow Z)} \ast (X^{*} \otimes ([V \otimes W]_{Y})^{*})$$

$$= \Gamma_Z M^{(XYightarrow Z)} (\Gamma_X^{-1} \otimes \Gamma_Y^{-1}) (X^{*} \otimes ([V \otimes W]_{Y}))$$

$$= M^{(XYightarrow Z)} (\bar{X} \otimes ([\bar{V} \otimes \bar{W}]_{Y}))$$

$$= [\bar{X} \otimes [\bar{V} \otimes \bar{W}]_{Y}]_{Z}.$$  

where in the third equality we used eq. (23). We can generalize eq. (23) to contractions of arbitrary number of vectors

$$\Gamma_Z ([X \otimes Y \otimes \cdots \otimes W]_{Z})^{*} = [\bar{X} \otimes \bar{Y} \otimes \cdots \otimes \bar{W}]_{Z},$$

where the rhs is the complex conjugate counterpart of the term inside the round parentheses of the lhs and we have suppressed all the nesting structures and intermediate irreps. When $Z$ is the trivial singlet representation, $\Gamma_Z$ is the one-dimensional identity matrix and $[X \otimes Y \otimes \cdots \otimes W]_{Z}$ is a c-number complex conjugate to $[X \otimes Y \otimes \cdots \otimes W]_{Z}$. Particularly, if the contraction $[X \otimes Y \otimes \cdots \otimes W]_{1}$, with 1 being the trivial singlet, is invariant under complex conjugate operation of irreps, i.e. $[X \otimes Y \otimes \cdots \otimes W]_{1} = [\bar{X} \otimes \bar{Y} \otimes \cdots \otimes \bar{W}]_{1}$, then the contraction is a real number.

**A. Remarks**

The above property has an implication in flavor physics models with discrete flavor symmetries. To make the Lagrangian Hermitian, one needs to add its complex conjugate to for each term of the Lagrangian. For example,
if a term like $\lambda [X \otimes [V \otimes W]_X]_1$ is contained in the Lagrangian, then a counterpart term $\lambda^* [X \otimes [V \otimes W]_X^*]_1$ is presumably contained in the Lagrangian as well. However, for general CG coefficients, $[X \otimes [V \otimes W]_X]_1$ and $[X \otimes [V \otimes W]_X^*]_1$ are not necessary complex conjugate to each other. Therefore, the coupling constant $\lambda$ in general has to be a complex number with certain phase to make the Lagrangian real. With the embedding factors defined under constraints (22), the coefficient $\lambda$ can always be a real number.

One should not confuse the $\Gamma$ matrices with the unitary matrix $U$ of generalized charge-parity (CP) transformations,

$$\phi_i \xrightarrow{CP} U_i \phi_i^*. \quad (30)$$

where $u$ is an automorphism of the group. For a physical CP transformation, $u$ should be class-inverting and involutory. If such an automorphism exists, a model employing the group $G$ as flavor symmetry can be invariant under transformation (30). The matrices $U_i$ are in general different with the $\Gamma$ matrices defined as (20). In fact, under the transformation like $X \rightarrow \Gamma_X X^*$, we have

$$X \rightarrow \tilde{X}, \quad \tilde{X} \rightarrow \begin{cases} -X & \text{pseudoreal } X \\ X & \text{complex or real } X \end{cases} \quad (32)$$

The Lagrangian is not invariant under (22) if it contains contractions with odd number of pseudoreal irreps. But there exist groups with pseudoreal irreps admitting a CP symmetry. For example, the group $Q_8$, which has one 2-dimensional pseudoreal irrep, admits a CP symmetry. On the other hand, even if a group does not have pseudoreal irreps, the transformation $X \rightarrow \Gamma_X X^*$ is not necessary a CP transformation, since the class-inverting involutory automorphism for such a transformation might not exist. For example, the group $T_7$, of which all irreps are complex except the real trivial singlet, does not admit a physical CP transformation but the Lagrangian is invariant under the transformation $X \rightarrow \Gamma_X X^*$.

V. PROCEDURE TO CALCULATE EMBEDDING FACTORS

In this section, we will describe the procedure to calculate embedding factors. First, we remark that there are some existing methods to calculate CG coefficients, for example, the Mathematica package Discrete, which implements the algorithm of Ref. [20], and the method introduced by Ref. [19]. These methods work well for low-dimensional irreps and groups with small order. However, when it comes to CG coefficients of large discrete group or those involving high-dimensional irreps, they are usually not effective. Furthermore, our goal is to calculate the embedding factors, we therefore introduce the following procedure.

The step zero is to find representation matrices and CG coefficients of its subgroup. Here, we assume that the subgroup is relatively small and its representation matrices and CG coefficients are known or easy to find. Moreover, the CG coefficients of the subgroup should satisfy eq. (29).

- Step I

Find representation matrices of $G$ in the subgroup basis. For simplicity, we choose a basis that projection matrices are in the simplest form, meaning that $P_{\Gamma_{\rightarrow \Gamma}}^{(n)}$, in (11) are identity matrices. One can first find the representation matrices of low-dimensional irreps then build the high-dimensional irreps from tensor products. Usually the low-dimensional representation matrices, in a certain basis, are already known in the literature or can be obtained from the GAP[14]. We therefore focus on finding a similarity transformation that transforms the matrices to the desired basis.

To find the similarity transformation, we need to diagonalize the representation matrices, see Section VI. Entries of these matrices are usually cyclotomic numbers, which are polynomials of $n$th roots of unity for certain fixed $n$. It is difficult to find the eigenvectors of these matrices directly by Mathematica. We developed a algorithm to perform arithmetic operation of cyclotomic numbers. The details are discussed in Appendix B. With the algorithm, we can find the eigenvector of a matrix for a given eigenvalue, which, for low-dimensional irreps, usually can be calculated directly by Mathematica.

- Step II

Write down the most general expression of a contraction $[X \otimes Y]_Z$ in terms of subgroup contractions, as eq. (8), with undetermined embedding factors $E_{\epsilon, \alpha, \beta}$ and then setup equations for these coefficients. With eq. (7), we obtain the expression of $Z$ in terms of $E_{\epsilon, \alpha, \beta}$ and bilinear forms of $X$ and $Y$. We then substitute the expression

2 We do not know the order of groups or the dimension of irreps, beyond which these methods become ineffective. Based on our testing, for the group $PSL_2(7)$ of order 168, the Discrete package did not give any result for the calculation of CG coefficients of two six-dimensional irreps, even after days of computation. The method of Ref. [19] requires diagonalization of representation matrices by Mathematica, which, based on our testing, failed to give any result for the six-dimensional representation matrices of the group $PSL_2(7)$.

3 There is a special case that the projection matrix cannot be in the trivial form. This occurs when two irreps of the group are identified to the same irrep of the subgroup. For example, for the $PSL_2(7)$ group, both 3 and 3 are identified to the $3^2$ of $S_3$. If we choose $P_{\Gamma_{\rightarrow \Gamma}}^{(n)}$ to be a identity matrix, then $P_{\Gamma_{\rightarrow \Gamma}}^{(n)}$ cannot be identity matrix simultaneously. The key point here is to make projection matrices as simple as possible.
of $Z$ into eq. (9) with $g$ being generators of $G$. If a generator is a member of the subgroup, then eq. (9) is automatically satisfied. So we only need to substitute $g$ with generators that are in the subgroup. By matching of the coefficients of bilinear forms of $X$ and $Y$, we obtain homogeneous equations with respect to the unknown variables $E_{c,ab}$. In this way, the number of equations we obtained are usually much more than the number of unknown variables. Many of the equations are dependent on others and hence redundant.

Alternatively, instead of matching coefficients of bilinear forms, we can generate the equations by replacing $X$ and $Y$ with some constant vectors

$$\begin{align*}
X &= V(p),
Y &= W(p),
\end{align*}$$

where $p = 1, 2, \ldots$, with each $p$ corresponding to one set of inputs. There are different choices of the constant vectors $V(p)$ and $W(p)$. A simple choice is that each vector has only one nonzero component, i.e.,

$$Y'(p) = \delta_{k,i_p}, W'(p) = \delta_{k,j_p},$$

where $\{i_p\}$ and $\{j_p\}$ are two sets of appropriately chosen positive integers. In this way, we can reduce the number of equations. Of course, we need to choose enough number of $i_p$ and $j_p$ and there could still be redundant equations and some of the equations are trivially 0 = 0. If $X \otimes Y \rightarrow Z$ has multiplicity $\mu_Z$, then there will be at most $N_c - \mu_Z$ independent equations, where $N_c$ is the number of the unknown variables $M_{c,ab}$.

- **Step III**

  The third step is to solve for the unknown variables $E_{c,ab}$. In the solution of the homogeneous linear equations, there will be $\mu_Z$ free variables and the other $N_c - \mu_Z$ variables be expressed as linear combinations of these free variables. In principle, we could solve these linear equations using standard methods. However, as the coefficients of these linear equations are cyclotomic numbers, which come from the matrices of group generators, the exact solutions are usually involved. If we use Mathematica to solve the equations directly, it usually cannot simplify the solution to appropriate forms.

  There are two ways to solve the issue. The first way is to use the calculation technique of cyclotomic numbers. We can use the Gaussian elimination algorithm with arithmetic operation of cyclotomic numbers to solve the equations. To be efficient, the Gaussian elimination procedure should apply to a set of independent equations, which can be found by converting the coefficients of equations to floating numbers and apply regular Gaussian elimination algorithm with certain error tolerance.

  The second way to solve the issue is to use a Mathematica programming trick. Instead of solving the equations directly, we convert all the coefficients to floating numbers and solve the equation numerically. We then convert the float numbers back into exact numbers using the Mathematica function RootApproximant. Because of numerical instability, the dependency relations of the equations are broken when the coefficients are converted to float numbers. Hence, it is crucial to pick out the maximal set of independent equations before solving the equations. Again, this can be done by the Gaussian elimination algorithm with an appropriate error tolerance. The method is less rigorous comparing to the first one. But we found it very effective in the calculation of $PSL_2(7)$ CG coefficients.

- **Step IV**

  The last step is to solve the constraint (24) and orthonormalize the embedding factors. The constraint (24) simply means that the embedding factors of $[X \otimes Y]_Z$ are complex conjugate of those of $[X \otimes Y]_Z$. Thus, care should be taken when $[X \otimes Y]_Z$ is invariant under the complex conjugate transformation of irreps, i.e., the Case I and II of section II. The constraint under these two cases can be solved as follows. Let $\{E_A\}$ be the sets of free parameters in the solution of step III, where $A$ denotes subscripts of the form $(c,ab)$, then the constraint (24) is translated into equations of the form

$$\sum_A \alpha_A E_A = \sum_A \beta_A E_A^*,$$

where $\{\alpha_A\}$ and $\{\beta_A\}$ are constant c-numbers. These equations can be solved by expressing $E_A$ in terms of real and imaginary parts, i.e., $E_A = E_A^r + iE_A^i$. Finally, if $\mu_Z > 1$, we can use the Gram-Schmidt process to build $\mu_Z$ sets of orthonormal embedding factors.

We have shown the procedure to find embedding factors. Let us now count the number of degrees of freedom (DOF) of embedding factors. There are $\mu_Z$ free complex coefficients $E_{c,ab}$ after solving the homogeneous linear equations in step III. For Case I and II, the constraint (24) generates $\mu_Z$ independent equations as (33), which reduce the $\mu_Z$ complex DOF to $\mu_Z$ real DOF. The normalization condition reduces one more real DOF. Therefore, there are $\mu_Z$ sets of independent embedding factors with $\mu_Z - 1$ real DOF. For other cases, the constraint (24) relates embedding factors of $[X \otimes Y]_Z$ to those of $[X \otimes Y]_Z$. Then there are $2\mu_Z$ sets of embedding factors with $\mu_Z$ complex DOF. Again, the normalization condition reduces one more real DOF. The $2\mu_Z$ sets of embedding factors therefore have $2\mu_Z - 1$ real DOF.

In the following sections, we will demonstrate the above procedure with the group $PSL_2(7)$ and its subgroups $S_6$, and $T_7$. We automate much of the procedures in Mathematica package files, which can be found in Ref. [3]. We note that the source code can be adapted for different groups.

VI. REPRESENTATION MATRICES OF $PSL_2(7)$

In this section, we will find representation matrices of $PSL_2(7)$ in its subgroup bases. To begin with, let us first
give a brief introduction of the group and its subgroups. Much of the group theories can be found in Appendix A.

$\mathcal{PSL}_2(7)$, the largest discrete subgroup of $SU(3)$ of order 168, is the projective special linear group of $(2 \times 2)$ matrices over $\mathbb{F}_7$, the finite Galois field of seven elements. The generators of the group are defined as

$$\left( A, B \right)A^2 = B^3 = (AB)^7 = [A, B]^4 = e,$$

where $[A, B] = A^{-1}B^{-1}AB$. It has six irreps: the complex 3 and its conjugate, $\bar{3}$, as well as four reals, 1, 6, 7, and 8. Two of its subgroups are $S_4$, generated by

$$a^4 = b^2 = (ab)^3 = e. \quad (36)$$

and $T_7$, generated by

$$c^7 = d^3 = 1, \quad d^{-1}cd = c^4. \quad (37)$$

The $S_4$ and $T_7$ generators can be expressed in terms of $\mathcal{PSL}_2(7)$ generators as

$$a = [A, B], \quad b = \left( AB^2 (AB)^2 \right)^2, \quad (38)$$

and

$$c = AB, \quad d = AB \left( AB^2 \right)^2 (AB)^2 (AB^2)^2. \quad (39)$$

We note that these expressions are not unique and they can be found by GAP. An example GAP code to find such relations can be found in Ref. 3.

In the following subsection, we will choose bases of $\mathcal{PSL}_2(7)$ irreps so that the projection matrices $P_{X \rightarrow \eta}$ are in the form of eq. (11) with $F_{X \rightarrow \eta}$ being identity matrices, except for a special case that we will see shortly. In such bases, the subgroup irreps are contained in $\mathcal{PSL}_2(7)$ irreps following their orders in the embedding relations shown as table A.4. For example, irreps of the subgroup $T_7$ are embedded in the 6 irrep as $6 = 3 \oplus \bar{3}$, then the first three components of a sextet form a $T_7$ triplet and the last three components form a $T_7$ anti-triplet.

We will first find the representation matrices in the $S_4$ basis and then the $T_7$ basis. For both subgroups, we will first seek for the representation matrices of 3 and $\bar{3}$, then build high-dimensional irreps from the tensor product of low-dimensional irreps. We will denote contractions of $\mathcal{PSL}_2(7)$ by square brackets and those of $S_4$ and $T_7$ by curly brackets.

### A. In the $S_4$ basis

To find representation matrices of $\mathcal{PSL}_2(7)$ in the $S_4$ basis, we first need to find representation matrices of $S_4$, which usually can be obtained from the literature or the GAP, and then the CG coefficients of $S_4$. But since $S_4$ has the subgroup $A_4$, which is also a group popular in model building of flavor physics, we seek for a matrix realization of $S_4$ in $A_4$ basis. $A_4$ has only one non-singlet irrep, 3, whose matrix realization is given as eq. (A.2). The procedures to find $S_4$ representation matrices in $A_4$ basis are similar to what we will talk in this section but the calculation is kind of trivial. So we simply give the results in Appendix A and focus on finding representation matrices of $\mathcal{PSL}_2(7)$.

Although we require the projection matrices to be the simplest form, there are still ambiguities in the representation matrices of $\mathcal{PSL}_2(7)$ because of the phase ambiguity of subgroup representation spaces. For complex representations $X$ and $\bar{X}$, if $x$ and $\bar{x}$ are vectors in their representation spaces satisfying $S_{X,x} s = x$, then $e^{i\theta} x$ and $e^{-i\theta} \bar{x}$ are also vectors of the representation spaces and satisfy the same constraint; for real or pseudoreal representation $X$, the phase ambiguity becomes a sign ambiguity. Since $S_4$ is an ambivalent group, whose irreps all are real or pseudoreal, the vectors of $S_4$ representation spaces have sign ambiguities. It then implies that, consulting the embedding of $S_4$ irreps in $PSL_2(7)$ irreps as table A.4, representation matrices of $PSL_2(7)$ irreps 6, 7, and 8 are fixed up to similarity transformation of diagonal sign-factor matrices,

$$O^{[R]} \rightarrow S^{[R]} \cdot O^{[R]} \left( S^{[R]} \right)^{\dagger}, \quad O = A, B, \quad R = 6, 7, 8, \quad (40)$$

where

$$S^{[6]} = \text{diag} (\pm 1, \pm I_2, \pm I_3),$$

$$S^{[7]} = \text{diag} (\pm 1, \pm I_3, \pm I_3),$$

$$S^{[8]} = \text{diag} (\pm I_2, \pm I_3, \pm I_3),$$

with $I_n$ being the $n \times n$ identity matrix. Since the triplet and anti-triplet decomposition are 3 = 3_2 and $\bar{3}$ = 3_2, their representation matrices are fixed because a similarity transformation of the above form does not change the matrices. Under above transformations, the embedding factors transform as eq. (19) with $g^{(ab,ac)} = 0$ and $g^{(a,y,z)} = 0$ or $\pi$. In the following results, the $S$ matrices are chosen to be identity matrices for simplicity.

#### 1. The triplet representation

Representation matrices of $\mathcal{PSL}_2(7)$ generators already exist in the literature. We will use the existing results and transform them to the $S_4$ basis. In Ref. 20 the representation matrices of $\mathcal{PSL}_2(7)$ in triplet irrep are

$$\tilde{A}^{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta - \eta^6 & \eta - \eta^5 & \eta - \eta^4 \\ \eta^4 - \eta^5 & \eta^2 - \eta^6 & \eta^3 - \eta^4 \\ \eta^3 - \eta^5 & \eta^3 - \eta^4 & \eta^2 - \eta^6 \end{pmatrix}, \quad \eta = \exp (i2\pi/7), \quad (41a)$$

$$\tilde{B}^{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta - \eta^7 & \eta - \eta^6 & \eta - \eta^5 \\ \eta^2 - 1 & \eta^6 - \eta^5 & \eta^6 - \eta^4 \\ \eta^3 - \eta^5 & \eta^4 - 1 & \eta^3 - \eta^6 \end{pmatrix}. \quad (41b)$$
Since $3$ of $\mathcal{PSL}_2(7)$ is identified to $3_z$ of $S_4$, the $S_4$ generators $a$ and $b$ in $3_z$ irrep can be generated by $\bar{A}^{[3]}$ and $\bar{B}^{[3]}$. Using eq. (38), we have

$$\bar{a} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^5 - \eta & \eta^3 - \eta^3 & \eta^3 - \eta^3 & \eta^3 - \eta^2 \\ \eta^5 - \eta^5 & \eta^4 - \eta^4 & \eta^4 - \eta^4 & \eta^4 - \eta^3 \\ \eta^5 - \eta^5 & \eta^4 - \eta^4 & \eta^4 - \eta^3 & \eta^3 - \eta^3 \\ \eta^5 - \eta^5 & \eta^4 - \eta^3 & \eta^3 - \eta & \eta^3 - \eta \\ \eta^6 - \eta^2 & \eta^6 - \eta^2 & \eta^6 - \eta^2 & \eta^6 - \eta^2 \end{pmatrix}.$$

$$\bar{b} = -\frac{i}{\sqrt{7}} \begin{pmatrix} \eta^6 - \eta & \eta - \eta & \eta - \eta & \eta - \eta \\ \eta^6 - \eta^4 & \eta^4 - \eta^4 & \eta^4 - \eta^3 & \eta^4 - \eta^2 \\ \eta^6 - \eta^4 & \eta^4 - \eta^3 & \eta^3 - \eta & \eta^3 - \eta \\ \eta^6 - \eta^4 & \eta^4 - \eta^3 & \eta^3 - \eta & \eta^3 - \eta \\ \eta^6 - \eta^4 & \eta^4 - \eta^3 & \eta^3 - \eta & \eta^3 - \eta \\ \eta^6 - \eta^4 & \eta^4 - \eta^3 & \eta^3 - \eta & \eta^3 - \eta \end{pmatrix}^2.$$

We now want to find a unitary matrix $U$ that simultaneously transforms $\bar{a}$ to $a^{[3z]}$ and $\bar{b}$ to $b^{[3z]}$, where $a^{[3z]}$ and $b^{[3z]}$ are given as eqs. (43).6

$$U^\dagger \bar{a} U = a^{[3z]},$$

$$U^\dagger \bar{b} U = b^{[3z]}.$$

The matrix $U$ can be found as follows. Since $\bar{a}$ and $a^{[3z]}$ have the same eigenvalues $\{1, i, -i\}$, there exist unitary matrices $U_1$ and $U_2$ such that

$$U_1^\dagger \bar{a} U_1 = U_2^\dagger a^{[3z]} U_2 = \text{diag}(1, i, -i).$$

Now the matrix $U$ can be written as

$$U = U_1 \begin{pmatrix} e^{i\theta_1} & e^{i\theta_2} & e^{i\theta_3} \end{pmatrix} U_2^\dagger.$$ (44)

Substituting above into eq. (42b), we can solve for $\theta_2$ and $\theta_3$ in terms of $\theta_1$ and determine the matrix $U$ up to an irrelevant overall phase. To diagonalize $\bar{a}$, we can use the algorithm of cyclotomic number calculation to find its eigenvectors.

Alternatively, an easier way to find $U$ is using the $A_4$ generators $s = a^2$ and $t = ab$. In the desired basis, $\bar{t} = \text{diag}(1, \omega, \omega^2)$ is diagonal and, in the basis of (11), $\bar{t}$ has a simple form

$$\bar{t} = \bar{a} \bar{b} = \begin{pmatrix} 0 & \eta^4 & 0 \\ \eta^4 & 0 & \eta \\ 0 & \eta & 0 \end{pmatrix}.$$ (45a)

Replacing $\bar{a}$ and $a^{[3z]}$ with $\bar{t}$ and $t$ in eq. (43) and repeating the calculation, we find that $U_2$ is the identity matrix and

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ \eta^5 & \omega \eta^3 & \omega^2 \eta^3 \\ \eta^5 & \omega^2 \eta^2 & \omega \eta^2 \end{pmatrix}, \quad \omega = \exp \left(\frac{2\pi i}{3}\right).$$ (45a)

Requiring that $U$ transforms $\tilde{s} = \bar{a}^2$ to the matrix $s$ of eq. (42a), we obtain

$$e^{i(\theta_2 - \theta_1)} = \frac{1}{28} \left[\left(-9 - \sqrt{3}i\right) \eta^5 - 2 \left(9 + \sqrt{3}i\right) \eta^4 \right] + \frac{1}{28} \left[\left(-6 + 4i\sqrt{3}\right) \eta^5 + \left(6 - 4i\sqrt{3}\right) \eta^4 \right] + \frac{1}{28} \left[\left(-3 + 9i\sqrt{3}\right) \eta + \sqrt{3}i - 5 \right].$$ (45b)

$$e^{i(\theta_3 - \theta_1)} = \frac{1}{28} \left[\left(-9 + \sqrt{3}i\right) \eta^5 - 2 \left(9 - \sqrt{3}i\right) \eta^4 \right] + \frac{1}{28} \left[\left(-6 - 4i\sqrt{3}\right) \eta^5 + \left(6 + 4i\sqrt{3}\right) \eta^4 \right] + \frac{1}{28} \left[\left(-3 - 9i\sqrt{3}\right) \eta - \sqrt{3}i - 5 \right].$$ (45c)

Substituting eqs. (45) and $U_2 = I$ into eq. (44), we obtain the matrix $U$ in a complicated expression. Fortunately, applying the unitary transformation with $U$ to $A^{[3]}$ and $B^{[3]}$, we get simple expressions of $A$ and $B$ in the desired basis

$$A^{[3]} = \begin{pmatrix} -\frac{4}{3} & \frac{2}{3} \omega & \frac{2}{3} \omega^2 \\ \frac{2}{3} \omega & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} \omega^2 & \frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

$$B^{[3]} = \begin{pmatrix} \frac{i(\sqrt{3}+\sqrt{7})}{6} \omega b_7^2 & \frac{i(\sqrt{3}-\sqrt{7})}{6} \omega b_7^2 \\ \frac{i(\sqrt{3}+\sqrt{7})}{6} \omega^2 b_7^2 & \frac{i(\sqrt{3}-\sqrt{7})}{6} \omega^2 b_7^2 \\ \frac{1+i\sqrt{7}}{2\sqrt{2}} \omega & -\frac{1}{3} \end{pmatrix},$$ (46b)

where $b_7$ and $\bar{b}_7$ are pure phases

$$b_7 = \frac{\eta + \eta^2 + \eta^4}{\sqrt{2}} = -1 + \frac{i\sqrt{7}}{\sqrt{2}}, \quad \bar{b}_7 = b_7^*.$$ (47)

The $\bar{3}$ matrix realization is the complex conjugate of $3$. However, the projection matrix $P_{\bar{3} \rightarrow 3z}$ is not the identity matrix but equals the matrix $\Gamma_{3z}$,

$$P_{\bar{3} \rightarrow 3z} = \Gamma_{3z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ (48)

It can be explained as follows. If $3_z$ were a complex representation, there would exist its complex conjugate $\bar{3}_z$. The decomposition of $3$ of $\mathcal{PSL}_2(7)$ to $S_4$ irreps would be $3 = 3_z \oplus \bar{3}_z$. But both of the projection matrices be the identity matrix. Now $\bar{3}_z$ and $3_z$ are equivalent and related by a similarity transformation $\Gamma_{3z}$. Therefore $3_z$ and $3$ should also be related by the same similarity transformation, and hence, $P_{\bar{3} \rightarrow 3z} = \Gamma_{3z}$.

2. Sextet, Octet, and Septet Representations

We now build representation matrices of high-dimensional irreps with those of $3$ and $\bar{3}$. The generators in $6$ irrep can be obtained from the tensor product
$3 \otimes 3 \rightarrow 6$. The decompositions of $\mathcal{PSL}_2(7)$ irreps into $S_4$ irreps

$$6 = 1_0 \oplus 2 \oplus 3_1, \quad 3 = 3_2,$$  
(48)

and the tensor product of $S_4$ irreps

$$3_2 \otimes 3_2 \rightarrow 1_0 \oplus 2 \oplus 3_1$$
determine the $\mathcal{PSL}_2(7)$ contraction $[3 \otimes 3]_6$ to be

$$[3 \otimes 3]_6 = \begin{pmatrix} e^{i\theta_1} \{3_2 \otimes 3_2\}_{1_0} \\ e^{i\theta_2} \{3_5 \otimes 3_2\}_{2} \\ e^{i\theta_3} \{3_2 \otimes 3_2\}_{3_1} \end{pmatrix},$$

where the phases are to be determined. The generators of $6$ can be extracted from the equations

$$O^{[6]}_6 = \left[ O^{[3]}_3 \otimes O^{[3]}_3 \right]_6, \quad O = A, B.$$  

We then obtain matrices $A^{[6]}$ and $B^{[6]}$ with unknown phases $\theta_i$. The phases $\theta_i$ can be determined by the constraints

$$\Gamma_6 O^{[6]}_6 \Gamma_6^\dagger = O^{[6]}_6, \quad O = A, B,$$

(50)

where $\Gamma_6$ can be determined by eq. (22a) with the projection matrices in the simplest form. It turns out that

$$\Gamma_6 = (1) \oplus \Gamma^{(S_4)}_2 \oplus \Gamma^{(S_4)}_{3_1}$$

with $\Gamma^{(S_4)}$ given as eq. (22a). With above constraints, we find that $e^{i\theta_2} = -i\bar{b} e^{i\theta_1}$ and $e^{i\theta_3} = i\bar{b} e^{i\theta_1}$ with $\theta_1$ being a free unphysical phase. Choosing $\theta_1 = 0$, we obtain the contraction

$$[3 \otimes 3]_6 = \begin{pmatrix} \{3_2 \otimes 3_2\}_{1} \\ -i\bar{b} \{3_5 \otimes 3_2\}_{2} \\ i\bar{b} \{3_2 \otimes 3_2\}_{3_1} \end{pmatrix},$$

and the generators

$$A^{[6]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B^{[6]} = \begin{pmatrix} -\frac{1}{6} & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} \\ -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix}.$$

The representation matrices of $8$ can be calculated with the tensor product $3 \otimes \bar{3} \rightarrow 8$ and the embedding relations

$$8 = 2 \oplus 3_1 \oplus 3_2, \quad 3 = 3_2, \quad \bar{3} = 3_2.$$  

Since the complex conjugate of $3 \otimes \bar{3} \rightarrow 8$ is itself, the overall phase of the CG coefficients is fixed. By a little algebra, we find that

$$[3 \otimes \bar{3}]_8 = \begin{pmatrix} \{3_2 \otimes 3_2\}_{2} \\ \{3_2 \otimes 3_2\}_{3_1} \\ i \{3_2 \otimes 3_2\}_{3_1} \end{pmatrix}.$$  

The generator $A^{[8]}$ has a simple form

$$A^{[8]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{3} & \frac{3}{3} & \frac{3}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$  

The generator $B^{[8]}$ is given by
Finally, let us consider the 7 irrep. We can obtain the 7 irrep from $3 \otimes 6 \rightarrow 7$. However, unlike the 6 case, in which the contraction $[3 \otimes 3]_6$ is determined up to phases solely by embedding relations and subgroup tensor products, the absolute values of the embedding factors of $[3 \otimes 6]_6$ cannot be determined now. Instead, we have to first determine the embedding factors of $[3 \otimes 6]_3$ and $[3 \otimes 6]_8$, and then obtain those of $[3 \otimes 6]_7$ by orthogonality. By a straightforward calculation, we obtain

\[
P_{3 \rightarrow 3_2} [3 \otimes 6]_3 = \frac{1}{\sqrt{6}} \left( 3_2 \otimes 1_0 \right)_{3_2} - \frac{i B_7^3}{\sqrt{3}} \left( 3_2 \otimes 2 \right)_{3_2} + \frac{i B_7}{\sqrt{2}} \left( 3_2 \otimes 3_1 \right)_{3_2},
\]

\[
[3 \otimes 6]_8 = \left( \begin{array}{c}
\{3_2 \otimes 3_2\} \\
\{3_2 \otimes 3_2\} \\
\{3_2 \otimes 3_2\} \\
\end{array} \right) + \frac{1}{\sqrt{6}} \left( \begin{array}{c}
\{3_2 \otimes 3_1\} \\
\{3_2 \otimes 3_1\} \\
\{3_2 \otimes 3_1\} \\
\end{array} \right) + \frac{1}{\sqrt{3}} \left( \begin{array}{c}
\{3_2 \otimes 3_2\} \\
\{3_2 \otimes 3_1\} \\
\{3_2 \otimes 3_1\} \\
\end{array} \right),
\]

up to overall phases. The orthogonality of embedding factors determines $[3 \otimes 6]_7$ to be

\[
[3 \otimes 6]_7 = \left( e^{i \theta_1} \{3_2 \otimes 3_1\} + e^{i \theta_2} \{3_2 \otimes 3_1\},
\right)
\]

with $\theta_i$ being unknown phases. Applying a constraint in analogy to [30] with $\Gamma_7 = (1) \oplus \Gamma_3 \oplus \Gamma_3$, we arrive at $e^{i (\theta_2 - \theta_1)} = \frac{b_7^3}{b_7}$ and $e^{i (\theta_3 - \theta_1)} = - \frac{i b_7^3}{\sqrt{3}}$, which then give rise to

\[
A[7] = \left( \begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 \omega & 2 \omega^2 & 0 & 0 & 0 \\
0 & 0 & 2 \omega & 2 \omega^2 & 0 & 0 & 0 \\
0 & 2 \omega & 2 \omega^2 & 0 & 0 & 0 & 0 \\
0 & 2 \omega & 2 \omega^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right),
\]

\[
B[7] = \left( \begin{array}{ccccccc}
- \frac{\sqrt{3}}{2} & \frac{1}{2} i \omega_3 & - \frac{1}{2} \omega_3 & 0 & 0 & \frac{1}{2} \omega_3 & - \frac{1}{2} \omega_3 \\
0 & \frac{1}{2} i \omega_3 & \frac{1}{2} \omega_3 & 0 & 0 & \frac{1}{2} \omega_3 & - \frac{1}{2} \omega_3 \\
0 & \frac{1}{2} \omega_3 & - \frac{1}{2} i \omega_3 & 0 & 0 & \frac{1}{2} \omega_3 & - \frac{1}{2} \omega_3 \\
0 & \frac{1}{2} i \omega_3 & \frac{1}{2} \omega_3 & 0 & 0 & \frac{1}{2} \omega_3 & - \frac{1}{2} \omega_3 \\
0 & \frac{1}{2} \omega_3 & - \frac{1}{2} i \omega_3 & 0 & 0 & \frac{1}{2} \omega_3 & - \frac{1}{2} \omega_3 \\
\end{array} \right)
\]

B. In the $T_7$ basis

We now find the representation matrices of $\mathcal{P} \mathcal{S} \mathcal{L}_2 (7)$ in the $T_7$ basis. In fact, Ref. [20] has built representation matrices of $\mathcal{P} \mathcal{S} \mathcal{L}_2 (7)$ with those of $T_7$. Our calculation is
similar to the one of Ref. 20, but with slightly different bases. We will take care of the phase ambiguities of the matrices which is not covered in Ref. 20.

Since all the $T_7$ irreps are complex except the trivial singlet, according to the same arguments leading to eq. (18), the representation matrices of $\mathcal{PSL}_2(7)$ irreps have phase ambiguities. We can obtain new representation matrices by performing similarity transformations with diagonal pure phase matrices,

$$O[R] \rightarrow U[R]O[R](U[R]^{-1})^T, \quad O = A, B$$

where

$$U[6] = \text{diag}(e^{i\theta_0}I_3, e^{-i\theta_3}I_3),$$
$$U[7] = \text{diag}(\pm 1, e^{i\theta_0}I_3, e^{-i\theta_3}I_3),$$
$$U[8] = \text{diag}(e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_3}I_3, e^{-i\theta_3}I_3).$$

Under these transformations, the embedding factors change according to eq. (19) with $\phi^{(a\text{b}\text{c})} = 0$ and $\phi^{(X\text{Y}Z)}$ being the angles in the $U$ matrices. In the following, we will set all the $U$ matrices to be identity matrix for simplicity.

The triplet representation is given by eq. (41), in which the $T_7$ generator $e = AB$ is a diagonal matrix

$$(AB)[3] = \text{diag} (\eta, \eta^2, \eta^4) \equiv \tilde{T}^{[3]}.$$ 

Since $3$ and $\bar{3}$ are the only two non-singlet irreps of $T_7$, and in all singlets irreps, $AB$ are diagonal matrices in all $\mathcal{PSL}_2(7)$ irreps in the $T_7$ basis

$$(AB)[6] = \text{diag} (\tilde{p}[3], \tilde{p}^{[3]}),$$
$$(AB)[7] = \text{diag} (1, \tilde{p}[3], \tilde{p}^{[3]}),$$
$$(AB)[8] = \text{diag} (I_2, \tilde{p}[3], \tilde{p}^{[3]}),$$

where $\tilde{p}[3] = (p^{[3]})^\dagger$. Given since $A^2 = e$, the matrices of $B$ can be expressed as

$$B[R] = A[R](AB)[R], \quad R = 3, \bar{3}, 6, 7, 8.$$ 

We therefore only need to find the matrices of $A$ in the following.

The $6$ irrep can be extract from the tensor product $3 \otimes 3 \rightarrow 6$. Repeating the calculation for the $S_4$ case, up to a similarity transformation of $U[6]$, we obtain the matrix of $A$

$$A[6] = \frac{2}{7} \left( \begin{array}{ccc} R_1 & \sqrt{2}R_2 & \sqrt{2}R_2 \\
\sqrt{2} & 2c_1 + c_2 & c_1 + c_2 + c_3 \\
c_3 - c_2 & c_1 + c_2 & c_1 + c_3 \\
1 & 1 & 1 \end{array} \right),$$

where $R_{1,2}$ are real $3 \times 3$ symmetric matrices

$$R_1 = \begin{pmatrix} 1 - c_2 & 1 - c_1 & 1 - c_3 \\
1 - c_1 & 1 - c_3 & 1 - c_2 \\
1 - c_3 & 1 - c_2 & 1 - c_1 \end{pmatrix},$$
$$R_2 = \begin{pmatrix} c_1 - c_3 & c_3 - c_2 & c_2 - c_1 \\
c_3 - c_2 & c_2 - c_1 & c_1 - c_3 \\
c_2 - c_1 & c_1 - c_3 & c_3 - c_2 \end{pmatrix}.$$ 

with $c_k = \cos \frac{2\pi k}{21}$. Similarly, the $8$ irrep can be obtained from the tensor product $3 \otimes \bar{3} \rightarrow 8$,

$$A[8] = \frac{2}{7} \left( \begin{array}{ccc} C_1 & C_2 & C_2 \\
C_1^\dagger & R_3 & R_3 \\
C_2 & R_3 & R_3 \end{array} \right),$$

where $C_1$ and $C_2$ are complex matrices

$$C_1 = \frac{1}{2} \begin{pmatrix} 0 & \omega & 0 \\
\omega^2 & 0 & -c_2 + c_3\omega^2 \\
0 & \omega & 0 \end{pmatrix} + i\sqrt{3} \begin{pmatrix} 0 & -c_2 + c_3\omega^2 & 0 \\
-c_2 + c_3\omega^2 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix},$$
$$C_2 = \frac{i}{2} \begin{pmatrix} -\omega^2 & 1 - \omega^2 \\
-\omega & 1 & \omega^2 \\
-\omega & \omega & 1 \end{pmatrix} + \sqrt{3} \begin{pmatrix} c_2\omega - c_3 & c_2\omega^2 - c_3\omega & c_2 - \omega^2c_3 \\
c_2\omega^2 - c_3 & c_2\omega - c_3\omega^2 & (c_2 - \omega)c_3 \end{pmatrix},$$

and $R_3$ is a real $3 \times 3$ symmetric matrix

$$R_3 = \begin{pmatrix} c_2 - c_1 & c_1 - c_3 & c_3 - c_2 \\
c_3 - c_2 & c_1 - c_3 & c_3 - c_2 \\
c_3 - c_2 & c_1 - c_3 & c_3 - c_2 \end{pmatrix}.$$ 

Note that $R_3$ can be obtained by applying the cyclic permutation $c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow c_1$ on $R_2$.

The $7$ irrep can be extracted from the tensor product $3 \otimes 6 \rightarrow 3 \oplus 8 \oplus 7$. We first find the embedding factors of $3 \otimes 6 \rightarrow 3$ and $3 \otimes 6 \rightarrow 8$, then embedding factors of $3 \otimes 6 \rightarrow 7$ are fixed up to phases. The generator $A[7]$ is

\begin{align*}
A[7] &= \frac{2}{7} \left( \begin{array}{ccccccc}
\frac{1}{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
\sqrt{2} & 2c_1 + c_2 & c_1 + c_2 + c_3 & 2c_2 + c_3 & 2c_2 + 1 & 2c_1 + 1 & 2c_3 + 1 \\
\sqrt{2} & c_1 + c_2 & c_2 + c_3 & 2c_1 + c_2 & c_1 + 1 & 2c_3 + 1 & 2c_2 + 1 \\
\sqrt{2} & 2c_2 + c_3 & c_1 + c_2 + c_3 & 2c_3 + 1 & 2c_1 + 1 & 2c_3 + 1 & 2c_2 + 1 \\
\sqrt{2} & 2c_1 + 1 & 2c_2 + c_3 & 2c_1 + c_2 & 1 & c_1 + 1 & 2c_2 + c_3 \\
\sqrt{2} & 2c_2 + c_3 & 2c_2 + 1 & c_1 + 1 & 2c_1 + 1 & 2c_1 + 2c_3 & 2c_1 + c_2 \\
\sqrt{2} & 2c_1 + 1 & 2c_3 + 1 & 2c_2 + 1 & c_1 + 1 & 2c_2 + c_3 & 2c_1 + c_2 \\
\sqrt{2} & 2c_3 + 1 & 2c_2 + 1 & c_1 + 1 & 2c_2 + c_3 & 2c_1 + c_2 & 2c_1 + c_3 \\
\sqrt{2} & 2c_1 + 1 & 2c_3 + 1 & 2c_2 + 1 & c_1 + 1 & 2c_2 + c_3 & 2c_1 + c_2 \end{array} \right).
\end{align*}

To conclude this section, let us discuss the relation between our bases with those of Ref. 20. The triplet
and anti-triplet are the same. For the sextet, octet and septet irrep’s, the similarity transformations between our bases and those of Ref. [20] are

$$\hat{O}^{[R]} = U^{[R]} \hat{O}^{[R]} U^{[R]^\dagger}, \quad R = 6, \ 7, \ 8, \ O = A, \ B,$$

where $\hat{O}$ denotes the generators in the basis of Ref. [20] and the $U$ matrices are

$$U^{[6]} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},$$

$$U^{[7]} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},$$

$$U^{[8]} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & i & 0 & 0 & -i & 0 & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\omega^2 & -\omega & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

VII. $\mathcal{P}SL_2(7)$ EMBEDDING FACTORS

With the generator matrices, we can now find the embedding factors of $\mathcal{P}SL_2(7)$. To demonstrate the procedure, we will take the tensor product $6 \otimes 6 \to 6_s^{(1)} \oplus 6_s^{(2)}$ as an example. Here the superscripts $(1)$ and $(2)$ indicates the multiplicity of irrep’s in the tensor product and the subscript $s$ ($a$) indicates the symmetric (antisymmetric) part.

We first consider the embedding factors for $\mathcal{P}SL_2(7)$ and the $S_4$ subgroup. Using the embedding relation $6 = 1_0 \oplus 2 \oplus 3_1$, we can write down the most general expression of the $[6 \otimes 6]_{6_s}$ built out of $S_4$ contractions,

$$1_0 = \mathcal{E}_{1,11} \{ 1_0 \otimes 1_0 \}^{(1)}_{1_0} + \mathcal{E}_{1,22} \{ 2 \otimes 2 \}^{(2)}_{1_0} + \mathcal{E}_{1,33} \{ 3_1 \otimes 3_1 \}^{(3)}_{1_0}, \quad (53a)$$

$$2 = \mathcal{E}_{2,12} \{ 1_0 \otimes 2 \}^{(1)}_{2} + \mathcal{E}_{2,22} \{ 2 \otimes 2 \}^{(2)}_{2} + \mathcal{E}_{2,33} \{ 3_1 \otimes 3_1 \}^{(3)}_{2}, \quad (53b)$$

$$3_1 = \mathcal{E}_{3,13} \{ 1_0 \otimes 3_1 \}^{(1)}_{3_1} + \mathcal{E}_{3,23} \{ 2 \otimes 3_1 \}^{(2)}_{3_1} + \mathcal{E}_{3,33} \{ 3_1 \otimes 3_1 \}^{(3)}_{3_1}, \quad (53c)$$

where $\mathcal{E}_{c,ab}$ are embedding factors to be determined. In the above expressions, the notation $\{ x \otimes y \}^s_z$ denotes the contraction of two $S_4$ irrep’s, where $x$ components is embedded in the first $6$ and $y$ embedded in the second $6$.

Since $6 \otimes 6 \to 6_s$ has multiplicity 2 and there are nine unknowns, we can setup seven homogeneous linear equations by feeding values into $X$, $Y$, and $g$ of eq. (53a). To do this, we choose $g$ to be the generator $B$ and $X$, $Y$ to be constant vectors, whose only one nonzero components are specified by integer lists $\{ i_p \}$ and $\{ j_p \}$, see eqs. (53a) and (53b). It turns out that choosing $\{ i_p \} = \{ 1, 3 \}$ and $\{ j_p \} = \{ 1, 4 \}$ for $X$ and $Y$ is enough to generate the equations. By solving these equations, we express all coefficients in terms of $\mathcal{E}_{1,11}$ and $M_{1,22}$:

$$\mathcal{E}_{1,33} = -\frac{\mathcal{E}_{1,11}}{\sqrt{3}} - \frac{2 \mathcal{E}_{2,12}}{\sqrt{3}}, \quad (54a)$$

$$\mathcal{E}_{2,22} = -\frac{2 \mathcal{E}_{2,12}}{\sqrt{3}} - \frac{3 \mathcal{E}_{2,22}}{\sqrt{14}}, \quad (54b)$$

$$\mathcal{E}_{2,33} = \frac{\mathcal{E}_{1,22}}{\sqrt{42}} - \frac{2 \mathcal{E}_{1,11}}{\sqrt{21}}, \quad (54c)$$

$$\mathcal{E}_{3,13} = -\frac{\sqrt{2}}{3} \mathcal{E}_{1,11} - \frac{2 \mathcal{E}_{1,22}}{3}, \quad (54d)$$

$$\mathcal{E}_{3,23} = \frac{1}{3} \sqrt{7} \mathcal{E}_{1,22} - \frac{8 \mathcal{E}_{1,11}}{3 \sqrt{7}}, \quad (54e)$$

$$\mathcal{E}_{3,33} = \sqrt{\frac{2}{21}} \mathcal{E}_{1,11} + \frac{5 \mathcal{E}_{1,22}}{\sqrt{21}}. \quad (54f)$$

Since all the irrep’s of $S_4$ are real, the constraints $\mathcal{E}_{c,ab} = \mathcal{E}_{c,ba}$ are satisfied if $\mathcal{E}_{1,11}$ and $\mathcal{E}_{1,22}$ are real. To build two orthonormal sets of embedding factors, we consider the embedding factors of the $1_0$ component. We need to find two orthonormal vectors $\{ \mathcal{E}_{1,11}, \mathcal{E}_{1,22}, \mathcal{E}_{1,33} \}$ corresponding to two independent solutions. By the Gram–Schmidt process, we find the solutions $(\mathcal{E}_{1,11}, \mathcal{E}_{1,22}) = \left( \frac{\sqrt{2}}{\sqrt{3}}, 0 \right)$ and $(\mathcal{E}_{1,11}, \mathcal{E}_{1,22}) = \left( -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}} \right)$ can generate such two vectors. Substituting the two solutions into $(54a)$ and $(53b)$, we obtain the embedding factors of $6 \otimes 6 \to 6_s^{(1)} \oplus 6_s^{(2)}$ shown in Appendix [C].
The calculation of embedding factors in $T_7$ basis is similar. According to the decomposition $6 = 3 \oplus \bar{3}$ and tensor products of $T_7$, $[6 \otimes 6]_{9a}$ can be written as

$$3 = \mathcal{E}_{3,33}\{3 \otimes 3\}_3 + \mathcal{E}_{3,33}\{\bar{3} \otimes \bar{3}\}_{\bar{3}} + \frac{\mathcal{E}_{3,33}}{\sqrt{2}}\{(3 \otimes \bar{3}\}_3 + \{(\bar{3} \otimes 3\}_3$$

where the curly brackets denote $T_7$ contractions. The homogeneous linear equations can be generated in the same way as $S_4$ case with input $\{i_p\} = \{1, 3\}$ and $\{j_p\} = \{2, 4\}$ and generator $A$. Solving the linear equations yields

$$\mathcal{E}_{3,33} = -\sqrt{2}\mathcal{E}_{3,33} - \frac{\mathcal{E}_{3,33}}{\sqrt{2}}$$

$$\mathcal{E}_{3,33} = -\mathcal{E}_{3,33} - \mathcal{E}_{3,33}$$

$$\mathcal{E}_{3,33} = \sqrt{2}\mathcal{E}_{3,33}, \quad \mathcal{E}_{3,33} = \frac{1}{\sqrt{2}}\mathcal{E}_{3,33}$$

Now the constraints $\mathcal{E}_{c,ab} = \mathcal{E}_{c,ab}$ can be solved by expressing $\mathcal{E}_{3,33}$ and $\mathcal{E}_{3,33}$ in terms of real and imaginary parts. It turns out that the solution is $\mathcal{E}_{3,33} = \mathcal{E}_{3,33}$. The Gram–Schmidt process then give rise to two sets of independent embedding factors shown in Appendix D.

VIII. CONCLUSION

We have introduced the embedding factors, which express CG coefficients of a discrete group in terms of CG coefficients of its subgroup. Embedding factors are fixed up to phase ambiguities and invariant under basis transformations of irreps of the group and the subgroup. Their phase ambiguities are reduced by a phase convention defined as (22). Particularly, the phase ambiguities are reduced to sign ambiguities in the Case I and II of Section IV. We also obtained complete sets of embedding factors of the group $PSL(2|7)$ in the bases of its subgroup $S_4$ and $T_7$.

The work can be extended in several directions. One direction is to apply the method to other discrete groups. To give a few examples, the group $\Sigma (360\phi)$ of order 1080 has subgroups $\Sigma (60) \simeq A_5$ of order 60 and $\Sigma (36\phi)$ of order 108; the group $\Sigma (216\phi)$ of order 648 has the subgroup tree $\Sigma (216\phi) \supset \Sigma (72\phi) \supset \Sigma (36\phi)$, where $\Sigma (72\phi)$ is of order 216 and $\Sigma (36\phi)$ of order 108. More subgroup examples can be found in Ref. 22.

Another possible direction is to simplify the procedure to find embedding factors. As we have shown, embedding factors are basis independent and fixed up to possible phases. It seems unnecessary to find representation matrices of the group and subgroups in order to calculate these coefficients, at least for their absolute values. A natural guess is that embedding factors have something to do with the coset structure of the group. The procedure to calculate these coefficients would be much simpler if there exists a method to determine them without knowing the representation matrices and CG coefficients of subgroups.

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Appendix A: Group Theory

1. The $A_4$ group

The $A_4$ group is generated by two elements $s$ and $t$ which fulfill

$$s^2 = t^3 = (st)^3 = e. \quad (A.1)$$

The group has four irreps: one trivial singlet representation 1, two nontrivial singlets 1' and 1'' that are complex conjugate to each other, and one triplet 3. The character table of $A_4$ is shown as table A.1. The 3 representation of the group can be written as

$$s = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & \omega \\ \omega & \omega^2 \end{pmatrix}, \quad \omega = e^{2\pi i/3}. \quad (A.2)$$

Let $x$, $y$ be two triplets, $s'$ and $\bar{s}'$ be 1' and 1'' respectively. The the CG coefficients of $A_4$ in $\{A.2\}$ represen-
The non-singlet irreps of $S_4$ can be expressed as

\[ 2 : \quad a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad (A.5) \]

\[ 3_1 : \quad a^{[3_1]} = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2\omega & -1 \\ 2\omega & -1 & 2\omega^2 \end{pmatrix}, \quad (A.6a) \]

\[ b^{[3_1]} = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad (A.6b) \]

\[ a^{[3_2]} = -a^{[3_1]}, \quad b^{[3_2]} = -b^{[3_1]}, \quad (A.6c) \]

Their $\Gamma$ matrices, see eq. (20) for the definition, are

\[ \Gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_{3_1} = \Gamma_{3_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (A.7) \]

We remark that the matrices of (A.5) and (A.6) are in the $A_4$ basis, meaning that, in the $2 = 1^\prime \oplus 1^\prime$ representation (A.6), $A_4$ elements are diagonal matrices, and in the $3_1 = 3$ and $3_2 = 3$ representations, $A_4$ elements are generated by the same matrices as the ones in (A.2).

To obtain these matrices, we use GAP to obtain an arbitrary matrix realization of $S_4$ and then perform similarity transformations to transform them to the desired bases. The procedures are described in Section VI.

Using the CG coefficients of $A_4$, we obtain the CG coefficients of two triplets of $S_4$. In the following CG coefficients, $x$ and $y$ are triplets, $z = (s'_w, s''_w)$ and $w = (s'_w, s''_w)$ are nontrivial singlets of $A_4$, and $s$ is a trivial singlet of $S_4$ and $A_4$.

- $3_1 \otimes 3_1 \rightarrow (1_0 + 2 + 3_1)_s + 3_2$: 
  \[ \{ x \otimes y \}_1 = (x \otimes y)_{1_0} \]
  \[ \{ x \otimes y \}_2 = (x \otimes y)_{1^\prime} \]
  \[ \{ x \otimes y \}_3 = (x \otimes y)_{3_s} \]
  \[ \{ x \otimes y \}_3 = i (x \otimes y)_{3_a} \]

- $3_2 \otimes 3_2 \rightarrow (1_0 + 2 + 3_1)_s + 3_2$: 
  \[ \{ x \otimes y \}_1 = (x \otimes y)_{1_0} \]
  \[ \{ x \otimes y \}_2 = (x \otimes y)_{1^\prime} \]
  \[ \{ x \otimes y \}_3 = (x \otimes y)_{3_s} \]
  \[ \{ x \otimes y \}_3 = i (x \otimes y)_{3_a} \]
• $3_1 \otimes 3_2 \rightarrow 1_1 + 2 + 3_1 + 3_2$:

$$\{x \otimes y\}_{1_1} = (x \otimes y)_1$$
$$\{x \otimes y\}_{2_1} = \left( \begin{array}{c}
i(x \otimes y)_{1'} \\
-i(x \otimes y)_{1'}
\end{array} \right)$$
$$\{x \otimes y\}_{3_1} = i(x \otimes y)_{3_a}$$
$$\{x \otimes y\}_{3_2} = (x \otimes y)_{3_s}$$

• $3_1 \otimes 2 \rightarrow 3_1 + 3_2$:

$$\{x \otimes z\}_{3_1} = \frac{1}{\sqrt{2}}(x \otimes s'_3) + \frac{1}{\sqrt{2}}(x \otimes s'_3)$$
$$\{x \otimes z\}_{3_2} = \frac{i}{\sqrt{2}}(x \otimes s'_3) - \frac{i}{\sqrt{2}}(x \otimes \bar{s}'_3)$$

• $3_2 \otimes 2 \rightarrow 3_1 + 3_2$:

$$\{x \otimes z\}_{3_1} = \frac{i}{\sqrt{2}}(x \otimes s'_3) - \frac{i}{\sqrt{2}}(x \otimes \bar{s}'_3)$$
$$\{x \otimes z\}_{3_2} = \frac{1}{\sqrt{2}}(x \otimes s'_3) + \frac{1}{\sqrt{2}}(x \otimes \bar{s}'_3)$$

• $2 \otimes 2 \rightarrow (1_0 + 2_1) + 1_1$:

$$\{z \otimes w\}_{1_0} = \frac{1}{\sqrt{2}}(s'_3 \otimes s'_3) + \frac{1}{\sqrt{2}}(s'_3 \otimes s'_3)$$
$$\{z \otimes w\}_{1_1} = \frac{i}{\sqrt{2}}(s'_3 \otimes \bar{s}'_3) - \frac{i}{\sqrt{2}}(s'_3 \otimes \bar{s}'_3)$$
$$\{z \otimes w\}_{2_1} = \left( \begin{array}{c}
\bar{s}'_3 \\
s'_3
\end{array} \right)_{1'}$$

• $2 \otimes 1_1 \rightarrow 2$:

$$\{z \otimes s\}_{2} = \left( \begin{array}{c}
i(s'_3 \otimes s)_{1'} \\
-i(s'_3 \otimes s)_{1'}
\end{array} \right).$$

Note that the above CG coefficients in $A_4$ basis satisfy the constraints of eqs. 22.

3. The $T_7$ group

The Frobenius group of order $21$ is the smallest finite non-Abelian subgroup of $SU(3)$. It contains elements of order three and seven, with the presentation

$$\langle c, d | c^7 = d^3 = 1, d^{-1}cd = c^4 \rangle. \quad (A.8)$$

Its irreps are, a real singlet, one complex triplet $3$, a complex singlet, $1'$, and their inequivalent conjugates, $\bar{3}$, and $1'$. Their Kronecker products are

$$1' \otimes 1' = \bar{1}', \quad 1' \otimes \bar{1}' = 1$$
$$3 \otimes 1' = 3, \quad 3 \otimes \bar{1}' = \bar{3}$$
$$3 \otimes 3 = (3 + \bar{3})_a + \bar{3}_a, \quad 3 \otimes \bar{3} = 1 + 1' + \bar{1}' + 3 + \bar{3}.$$
The general form of an element in the field is

\[
\text{field} = \exp(2\pi i/n) = \sum_{k=0}^{n-1} q_k \exp(2k\pi i/n), \quad q_k \in \mathbb{Q}. \quad \text{(B.1)}
\]

Discrete group characters are usually cyclotomic numbers for fixed \(n\). We therefore need to perform arithmetic operators over cyclotomic numbers in order to calculate CG coefficients of discrete groups. Here we introduce the algorithm we used for calculations involving cyclotomic numbers.

Cyclotomic fields are closed in the arithmetic of addition, multiplication, and division. The addition and multiplication operators are trivially the operators of polynomials. So we only need to discuss the division. If we can find the reciprocal of a cyclotomic number (B.1), then the division operator becomes a multiplication operator and the problem is solved.

Let us now consider how to find reciprocal of (B.1). For convenience, we now write \(\exp(2\pi i/n) = r_n\). Let \(g = \sum p_k r_n^k\) be the reciprocal of \(f\). Collecting \(r_n^k\) terms in the product of \(f\) and \(g\), we have

\[
\begin{align*}
0 &= \sum_i q_{[k]} - i p_i, \quad k > 0, \quad [k] = k, \text{ or } n + k, \quad \text{(B.2a)} \\
1 &= q_0 p_0 + \sum_i q_{n-i} p_i. \quad \text{(B.2b)}
\end{align*}
\]

We therefore have \(n\) linear equations for \(n\) unknown variables \(p_k\). It seems enough to solve the equations. However, these equations are in general not independent and therefore have no solution. The root cause is that the expression (B.1) is not unique.

One of the ambiguities comes from that \(r_n^k\) are not independent, e.g.,

\[
\sum_{k=0}^{n-1} r_n^k = 0.
\]

We can always eliminate \(r_n^{n-1}\) from the expression (B.1). Is it the only dependent relation among \(r_n^k\)? The answer is no. If \(n\) has a factor \(1 < p < n\), then

\[
\sum_{k=0}^{n/p} r_n^{kp} = 0 \quad \text{(B.3)}
\]

is also a dependent relation. If \(n\) has \(s\) positive factors (not including \(n\) itself), we can setup \(s\) equations in the form of (B.3) and solve for \(r_n^{k_1}, \ldots, r_n^{k_s}\) in terms of the remaining \(n - s\) nth root of unities. By doing this, there are at most \(n - s\) terms in (B.1) and the number of equations in (B.2) is \(n - s\). It turns out that, after doing this, the equations are always solvable and has only one solution.

### Appendix C: CG Coefficients of \(PSL_2(7)\) in \(S_4\) Basis

The notations used in this appendix are as follows. If \(X, Y, Z\) are decomposed to subgroup irreps as

\[
X = \bigoplus x_a, \quad Y = \bigoplus y_b, \quad Z = \bigoplus z_c
\]

then the CG coefficients for \(X \otimes Y \rightarrow Z\) will be written as a list of expressions of the form

\[
z_c = \sum_{a,b} \mathcal{E}_{c,ab} \{x_a \otimes y_b\} z_c, \quad \text{(C.1)}
\]

where \(\mathcal{E}_{c,ab}\) are embedding factors and \(\{x_a \otimes y_b\} z_c\) are contractions of subgroup irreps. Note that, for each term \(\{x_a \otimes y_b\} z_c\) on the rhs of eq. (C.1), the first subgroup irrep \(x_a\) always comes from \(X\) and second subgroup irrep \(y_b\) always comes from \(Y\).

In the following, \(b_7\) is a pure phase complex constant

\[
b_7 = \frac{\eta + \eta^2 + \eta^4}{\sqrt{2}} = -1 + i\sqrt{7}/2\sqrt{2}, \quad \eta = e^{2\pi i/7}.
\]

\[
\begin{align*}
3 \otimes 3 &= 6_s + \bar{3}_a \\
3 \otimes 3 &= 3_a; \\
3_2 &= \{3_2 \otimes 3_2\} 3_2 \\
3 \otimes 3 &= 6_s; \\
1_0 &= \{3_2 \otimes 3_2\} 1_0 \\
2 &= -i b_7 \{3_2 \otimes 3_2\} 2 \\
3_1 &= i b_7 \{3_2 \otimes 3_2\} 3_1
\end{align*}
\]
\[3 \otimes 3 \rightarrow 1 + 8\]

- \(3 \otimes 3 \rightarrow 1:\)
  \[1_0 = \{3_2 \otimes 3_2\}_1\]

- \(3 \otimes 3 \rightarrow 8:\)
  \[2 = \{3_2 \otimes 3_2\}_2\]
  \[3_1 = \{3_2 \otimes 3_2\}_3\]
  \[3_2 = i\{3_2 \otimes 3_2\}_3\]

\[3 \otimes 6 \rightarrow 3 + 7 + 8\]

- \(3 \otimes 6 \rightarrow 3:\)
  \[3_2 = \frac{1}{\sqrt{6}} \{3_2 \otimes 1_0\}_3 - \frac{i}{\sqrt{36}} \{3_2 \otimes 2\}_3 + \frac{i}{\sqrt{2b_7}} \{3_2 \otimes 3_1\}_3\]

- \(3 \otimes 6 \rightarrow 7:\)
  \[1_1 = \{3_2 \otimes 3_1\}_1\]
  \[3_1 = \frac{1}{\sqrt{3b_7}} \{3_2 \otimes 2\}_3 + \frac{\sqrt{2}}{b_7} \{3_2 \otimes 3_1\}_3\]
  \[3_2 = \frac{\sqrt{2}}{b_7} \{3_2 \otimes 1_0\}_3 + \frac{2}{3b_7} \{3_2 \otimes 2\}_3 - \frac{1}{\sqrt{6b_7}} \{3_2 \otimes 3_1\}_3\]

- \(3 \otimes 6 \rightarrow 8:\)
  \[2 = \{3_2 \otimes 3_1\}_2\]
  \[3_1 = -\frac{\sqrt{3}}{3} b_7 \{3_2 \otimes 2\}_3 + \frac{1}{\sqrt{3}} \{3_2 \otimes 3_1\}_3\]
  \[3_2 = \frac{2}{3b_7} \{3_2 \otimes 1_0\}_3 - \frac{1}{3} i \sqrt{2b_7} \{3_2 \otimes 2\}_3 + \frac{i}{\sqrt{3}} \{3_2 \otimes 3_1\}_3\]

\[3 \otimes 7 \rightarrow 6 + 7 + 8\]

- \(3 \otimes 7 \rightarrow 6:\)
  \[1_0 = \{3_2 \otimes 3_2\}_1\]
  \[2 = -\frac{\sqrt{2}}{b_7^2} \{3_2 \otimes 3_1\}_2 + \frac{2}{\sqrt{7}b_7^2} \{3_2 \otimes 3_2\}_2\]
  \[3_1 = \frac{\sqrt{2}}{b_7} \{3_2 \otimes 1_1\}_3 - \frac{2}{\sqrt{7}b_7^2} \{3_2 \otimes 3_1\}_3\]
  \[-\frac{1}{\sqrt{7}b_7} \{3_2 \otimes 3_2\}_3\]

- \(3 \otimes 7 \rightarrow 7:\)
  \[1_1 = \{3_2 \otimes 3_1\}_1\]
  \[3_1 = -\frac{1}{\sqrt{3}} \{3_2 \otimes 1_1\}_3 - \frac{1}{\sqrt{3b_7}} \{3_2 \otimes 3_1\}_3\]
  \[-\frac{1}{\sqrt{3b_7}} \{3_2 \otimes 3_2\}_3\]
  \[3_2 = \frac{1}{\sqrt{3b_7}} \{3_2 \otimes 3_1\}_3 + \frac{\sqrt{2}}{b_7^2} \{3_2 \otimes 3_2\}_3\]

- \(3 \otimes 7 \rightarrow 8:\)
  \[2 = \frac{2}{\sqrt{7}} \{3_2 \otimes 3_1\}_2 + \frac{\sqrt{2}}{b_7} \{3_2 \otimes 3_2\}_2\]
  \[3_1 = \frac{2}{\sqrt{3}b_7} \{3_2 \otimes 1_1\}_3 + \frac{\sqrt{2}}{b_7^2} \{3_2 \otimes 3_1\}_3\]
  \[+ \frac{13\sqrt{7} + 7i}{28\sqrt{3}} \{3_2 \otimes 3_2\}_3\]
  \[3_2 = \frac{\sqrt{2}}{b_7} \{3_2 \otimes 1_0\}_3 - \frac{1}{\sqrt{3b_7}} \{3_2 \otimes 3_2\}_3\]

\[3 \otimes 8 \rightarrow 3 + 6 + 7 + 8\]

- \(3 \otimes 8 \rightarrow 3:\)
  \[3_2 = \frac{1}{2} \{3_2 \otimes 2\}_3 + \frac{\sqrt{2}}{2} \{3_2 \otimes 3_1\}_3\]
  \[\frac{1}{2} i \sqrt{2} \{3_2 \otimes 3_2\}_3\]
\[ \bullet \ 3 \otimes 8 \rightarrow 6: \]
\[ 1_0 = \{3_2 \otimes 3_2\}_{1_0} \]
\[ 2 = \frac{1}{2} \sqrt{3b_7^3} \{3_2 \otimes 3_1\}_2 + \frac{1}{2} b_7^2 \{3_2 \otimes 3_2\}_2 \]
\[ 3_1 = -\frac{b_7}{\sqrt{2}} (3_2 \otimes 2)_{3_1} - \frac{b_7}{2} \{3_2 \otimes 3_1\}_{3_1} + \frac{i b_7}{2} \{3_2 \otimes 3_2\}_{3_1} \]

\[ \bullet \ 3 \otimes 8 \rightarrow 7: \]
\[ 1_1 = \{3_2 \otimes 3_1\}_{1_1} \]
\[ 3_2 = \frac{b_7^2}{2} \{3_2 \otimes 2\}_{3_2} + \frac{i (5\sqrt{7} + i)}{8\sqrt{6}} \{3_2 \otimes 3_1\}_{3_2} + \frac{1}{2} \sqrt{\frac{7}{6} b_7^3} \{3_2 \otimes 3_2\}_{3_2} \]

\[ \bullet \ 3 \otimes 8 \rightarrow 8: \]
\[ 2 = \frac{1}{2} \{3_2 \otimes 3_1\}_2 + \frac{i \sqrt{3}}{2} \{3_2 \otimes 3_2\}_2 \]
\[ 3_1 = \frac{1}{\sqrt{6}} \{3_2 \otimes 2\}_{3_1} - \sqrt{\frac{2}{3}} b_7^2 \{3_2 \otimes 3_1\}_{3_1} + \frac{i b_7}{\sqrt{6}} \{3_2 \otimes 3_2\}_{3_1} \]
\[ 3_2 = \frac{-i}{\sqrt{2}} \{3_2 \otimes 2\}_{3_2} - \sqrt{\frac{2}{3}} b_7 \{3_2 \otimes 3_1\}_{3_2} - \frac{b_7^2}{\sqrt{3}} \{3_2 \otimes 3_2\}_{3_2} \]

\[ \bullet 6 \otimes 6 \rightarrow 6^{(1)}: \]
\[ 1_0 = \frac{\sqrt{3}}{2} \{1_0 \otimes 1_0\}_{1_0} - \frac{1}{2} \{3_1 \otimes 3_1\}_{1_0} \]
\[ 2 = -\frac{\sqrt{3}}{7} \{2 \otimes 2\}_2 - \frac{2}{\sqrt{7}} \{3_1 \otimes 3_1\}_2 \]
\[ 3_1 = -\frac{1}{2\sqrt{3}} (\{1_0 \otimes 3_1\}_{3_1} + \{3_1 \otimes 1_0\}_{3_1}) - \frac{2}{3} \sqrt{\frac{2}{21}} (\{2 \otimes 3_1\}_{3_1} + \{3_1 \otimes 2\}_{3_1}) + \frac{1}{\sqrt{14}} \{3_1 \otimes 3_1\}_{3_1} \]

\[ \bullet 6 \otimes 6 \rightarrow 6^{(2)}: \]
\[ 1_0 = -\frac{1}{2\sqrt{3}} \{1_0 \otimes 1_0\}_{1_0} + \frac{\sqrt{2}}{3} \{2 \otimes 2\}_1 \]
\[ -\frac{1}{2} \{3_1 \otimes 3_1\}_{1_0} \]
\[ 2 = \frac{1}{\sqrt{3}} ((\{1_0 \otimes 2\}_2 + \{2 \otimes 1_0\}_2) - \frac{2}{\sqrt{21}} \{2 \otimes 2\}_2 + \frac{1}{\sqrt{7}} \{3_1 \otimes 3_1\}_2 \]
\[ 3_1 = -\frac{1}{2\sqrt{3}} (\{1_0 \otimes 3_1\}_{3_1} + \{3_1 \otimes 1_0\}_{3_1}) + \sqrt{\frac{2}{21}} (\{2 \otimes 3_1\}_{3_1} + \{3_1 \otimes 2\}_{3_1}) + \frac{3}{\sqrt{14}} \{3_1 \otimes 3_1\}_{3_1} \]

\[ \bullet 6 \otimes 6 \rightarrow 8: \]
\[ 2 = \frac{1}{\sqrt{6}} ((\{1_0 \otimes 2\}_2 + \{2 \otimes 1_0\}_2) + 2 \sqrt{\frac{2}{21}} \{2 \otimes 2\}_2 - \sqrt{\frac{7}{21}} \{3_1 \otimes 3_1\}_2 \]
\[ 3_1 = \frac{1}{\sqrt{3}} (\{1_0 \otimes 3_1\}_{3_1} + \{3_1 \otimes 1_0\}_{3_1}) - \frac{1}{\sqrt{42}} (\{2 \otimes 3_1\}_{3_1} + \{3_1 \otimes 2\}_{3_1}) + \sqrt{\frac{2}{7}} \{3_1 \otimes 3_1\}_{3_1} \]
\[ 3_2 = -\frac{1}{\sqrt{2}} (\{2 \otimes 3_1\}_{3_2} + \{3_1 \otimes 2\}_{3_2}) \]

\[ \bullet 6 \otimes 6 \rightarrow 6 + 8 + 7 \]
\[ \bullet 6 \otimes 6 \rightarrow 1: \]
\[ 1_0 = \frac{1}{\sqrt{6}} \{1_0 \otimes 1_0\}_{1_0} + \frac{1}{\sqrt{3}} \{2 \otimes 2\}_1 \]
\[ + \frac{1}{\sqrt{2}} \{3_1 \otimes 3_1\}_{1_0} \]
$6 \otimes 6 \to 7_a$:

$1_1 = \{2 \otimes 2\}_{1_1}$

$3_1 = \frac{\sqrt{3}}{3} (\{1_0 \otimes 3_1\}_{3_1} - \{3_1 \otimes 1_0\}_{3_1})$
$+ \frac{1}{3} (\{2 \otimes 3_1\}_{3_1} - \{3_1 \otimes 2\}_{3_1})$

$3_2 = \frac{1}{\sqrt{3}} (\{2 \otimes 3_1\}_{3_2} - \{3_1 \otimes 2\}_{3_2})$
$- \frac{1}{\sqrt{3}} (3_1 \otimes 3_1)_{3_2}$

$6 \otimes 6 \to 8_a$:

$2 = \frac{1}{\sqrt{2}} (\{1_0 \otimes 2\}_2 - \{2 \otimes 1_0\}_2)$

$3_1 = -\frac{1}{\sqrt{3}} (\{1_0 \otimes 3_1\}_{3_1} - \{3_1 \otimes 1_0\}_{3_1})$
$+ \frac{\sqrt{2}}{3} (\{2 \otimes 3_1\}_{3_1} - \{3_1 \otimes 2\}_{3_1})$

$3_2 = -\frac{1}{\sqrt{6}} (\{2 \otimes 3_1\}_{3_2} - \{3_1 \otimes 2\}_{3_2})$
$- \frac{\sqrt{2}}{3} (3_1 \otimes 3_1)_{3_2}$

$6 \otimes 7 \to 3 + \overline{3} + 6 + 7^{(1)} + 7^{(2)} + 8^{(1)} + 8^{(2)}$

$6 \otimes 7 \to 3$:

$3_2 = \frac{1}{\sqrt{6}} (1_0 \otimes 3_2)_{3_2} + \frac{b_7^2}{\sqrt{7}} (2 \otimes 3_1)_{3_2}$
$+ \frac{2b_7^2}{\sqrt{21}} (2 \otimes 3_2)_{3_2} + \frac{1}{\sqrt{7b_7}} (3_1 \otimes 1_1)_{3_2}$
$+ \frac{\sqrt{2}}{b_7} (3_1 \otimes 3_1)_{3_2} - \frac{b_7^2}{\sqrt{14b_7}} (3_1 \otimes 3_2)_{3_2}$

$6 \otimes 7 \to \overline{3}$:

$3_2 = \frac{1}{\sqrt{6}} (1_0 \otimes 3_2)_{3_2} + \frac{1}{\sqrt{7b_7^2}} (2 \otimes 3_1)_{3_2}$
$+ \frac{2}{\sqrt{21b_7}} (2 \otimes 3_2)_{3_2} + \frac{b_7^2}{\sqrt{7}} (3_1 \otimes 1_1)_{3_2}$
$+ \frac{\sqrt{2}}{b_7^2} (3_1 \otimes 3_1)_{3_2} - \frac{1}{\sqrt{14b_7}} (3_1 \otimes 3_2)_{3_2}$

$6 \otimes 7 \to 6$:

$1_0 = \{3_1 \otimes 3_1\}_{1_0}$

$2 = -\frac{\sqrt{7}}{7} (2 \otimes 1_1)_{3_2} + \frac{1}{\sqrt{7}} (3_1 \otimes 3_1)_{3_2}$
$- \frac{\sqrt{2}}{7} (3_1 \otimes 3_2)_{3_2}$

$3_1 = -\frac{1}{\sqrt{6}} (1_0 \otimes 3_1)_{3_1} - \frac{2}{21} (2 \otimes 3_1)_{3_1}$
$- \frac{\sqrt{2}}{7} (2 \otimes 3_2)_{3_1} - \frac{2}{7} (3_1 \otimes 3_2)_{3_1}$

$6 \otimes 7 \to 7^{(1)}$:

$1_1 = \{1_0 \otimes 1_1\}_{1_1}$

$3_1 = -\frac{1}{6} (1_0 \otimes 3_1)_{3_1} + \frac{7}{6} (2 \otimes 3_1)_{3_1}$
$+ \frac{\sqrt{2}}{2} (2 \otimes 3_2)_{3_1} + \frac{\sqrt{2}}{2} (3_1 \otimes 3_1)_{3_1}$
$- \frac{\sqrt{2}}{2} (3_1 \otimes 3_2)_{3_1}$

$3_2 = -\frac{1}{6} (1_0 \otimes 3_2)_{3_2} + \frac{\sqrt{2}}{2} (2 \otimes 3_1)_{3_2}$
$+ \frac{\sqrt{2}}{6} (2 \otimes 3_2)_{3_2} + \frac{\sqrt{2}}{2} (3_1 \otimes 3_1)_{3_2}$
$- \frac{\sqrt{2}}{2} (3_1 \otimes 3_2)_{3_2}$

$6 \otimes 7 \to 7^{(2)}$:

$1_1 = \{3_1 \otimes 3_2\}_{1_1}$

$3_1 = -\frac{\sqrt{7}}{6} (1_0 \otimes 3_1)_{3_1} + \frac{7}{6\sqrt{2}} (2 \otimes 3_1)_{3_1}$
$- \frac{1}{2\sqrt{6}} (2 \otimes 3_2)_{3_1} - \frac{1}{2\sqrt{6}} (3_1 \otimes 3_1)_{3_1}$
$+ \frac{1}{2\sqrt{6}} (3_1 \otimes 3_2)_{3_1}$

$3_2 = \frac{\sqrt{7}}{6} (1_0 \otimes 3_2)_{3_2} - \frac{1}{2\sqrt{6}} (2 \otimes 3_1)_{3_2}$
$- \frac{1}{6\sqrt{2}} (2 \otimes 3_2)_{3_2} + \frac{1}{\sqrt{3}} (3_1 \otimes 1_1)_{3_2}$
$- \frac{1}{2\sqrt{6}} (3_1 \otimes 3_1)_{3_2} - \frac{\sqrt{2}}{2} (3_1 \otimes 3_2)_{3_2}$
\[ \begin{align*}
\text{6} \otimes 7 & \rightarrow 8^{(1)}: \\
2 & = \frac{1}{\sqrt{2}} \{2 \otimes 1\}_2 - \frac{1}{\sqrt{2}} \{3 \otimes 3\}_2 \\
3_1 & = \frac{1}{\sqrt{6}} \{2 \otimes 2\}_3 - \sqrt{\frac{2}{3}} \{3 \otimes 3\}_3 \\
3_2 & = -\frac{2}{3} \{1 \otimes 3\}_2 - 2 \sqrt{\frac{2}{21}} \{2 \otimes 3\}_2 \\
3_3 & = \frac{1}{\sqrt{6}} \{3 \otimes 2\}_3 - \sqrt{\frac{2}{3}} \{2 \otimes 3\}_3 \\
\text{6} \otimes 7 & \rightarrow 8^{(2)}: \\
2 & = \frac{1}{\sqrt{14}} \{2 \otimes 1\}_2 + \sqrt{\frac{6}{7}} \{3 \otimes 3\}_2 \\
3_1 & = \frac{2}{3} \{1 \otimes 3\}_3 + \frac{2}{3} \{2 \otimes 3\}_3 \\
3_2 & = \frac{1}{\sqrt{2}} \{2 \otimes 3\}_2 - \frac{1}{\sqrt{3}} \{3 \otimes 1\}_2 \\
3_3 & = \frac{1}{\sqrt{6}} \{1 \otimes 3\}_3 + \frac{1}{\sqrt{3}} \{2 \otimes 3\}_3 \\
\text{6} \otimes 8 & \rightarrow 3 + 3 + 6^{(1)} + 6^{(2)} + 7^{(1)} + 7^{(2)} + 8^{(1)} + 8^{(2)} \\
\text{6} \otimes 8 & \rightarrow 3: \\
3_2 & = \frac{1}{\sqrt{6}} \{1 \otimes 3\}_2 - \frac{1}{2\sqrt{2}} \{2 \otimes 3\}_2 \\
+ \frac{i}{\sqrt{2}} \{2 \otimes 3\}_3 - \frac{1}{2\sqrt{2}} \{3 \otimes 1\}_3 \\
\text{6} \otimes 8 & \rightarrow 3: \\
3_2 & = \frac{1}{\sqrt{6}} \{1 \otimes 3\}_2 - \frac{b^2}{2} \{2 \otimes 3\}_2 \\
- \frac{ib^2}{2\sqrt{3}} \{2 \otimes 3\}_3 - \frac{b_2}{2} \{3 \otimes 1\}_2 \\
+ \frac{b_2}{2\sqrt{2}} \{3 \otimes 1\}_2 + \frac{ib_2}{2\sqrt{2}} \{3 \otimes 2\}_2 \\
\text{6} \otimes 8 & \rightarrow 6^{(1)}: \\
1_0 & = \{2 \otimes 2\}_1 \\
2 & = \frac{1}{\sqrt{2}} \{1 \otimes 2\}_2 + \frac{1}{\sqrt{2}} \{1 \otimes 2\}_2 \\
3_1 & = \frac{1}{\sqrt{14}} \{3 \otimes 3\}_2 + \frac{1}{\sqrt{42}} \{3 \otimes 2\}_3 \\
3_2 & = \frac{1}{\sqrt{2}} \{3 \otimes 2\}_3 + \frac{1}{\sqrt{7}} \{3 \otimes 2\}_3 \\
\text{6} \otimes 8 & \rightarrow 6^{(2)}: \\
1_0 & = \{1 \otimes 3\}_1 \\
2 & = \sqrt{\frac{3}{2}} \{2 \otimes 2\}_2 + \frac{3}{4} \{2 \otimes 2\}_3 \\
3_1 & = \frac{1}{\sqrt{14}} \{3 \otimes 3\}_2 + \frac{1}{\sqrt{42}} \{3 \otimes 2\}_3 \\
3_2 & = \frac{1}{\sqrt{2}} \{3 \otimes 2\}_3 + \frac{1}{\sqrt{7}} \{3 \otimes 2\}_3 \\
\text{6} \otimes 8 & \rightarrow 7^{(1)}: \\
1_1 & = \{2 \otimes 2\}_1 \\
3_1 & = \frac{\sqrt{7}}{12} \{1 \otimes 3\}_3 + \frac{1}{6\sqrt{2}} \{2 \otimes 3\}_3 \\
- \frac{\sqrt{7}}{2} \{2 \otimes 3\}_3 + \frac{1}{4} \{3 \otimes 2\}_3 \\
- \frac{7}{4\sqrt{6}} \{3 \otimes 3\}_3 + \frac{\sqrt{7}}{4} \{3 \otimes 2\}_3 \\
3_2 & = -\frac{7}{12} \{1 \otimes 3\}_3 + \frac{1}{2\sqrt{6}} \{2 \otimes 3\}_3 \\
- \frac{\sqrt{7}}{6} \{2 \otimes 3\}_3 + \frac{\sqrt{7}}{4} \{3 \otimes 2\}_3 \\
+ \frac{5}{4\sqrt{6}} \{3 \otimes 3\}_3 - \frac{\sqrt{7}}{4} \{3 \otimes 2\}_3 \\
\end{align*} \]
\[ \cdot 6 \otimes 8 \rightarrow 7^{(2)}: \]
\[ 1_1 = \{3_1 \otimes 3_2\}_{1_1} \]
\[ 3_1 = -\frac{7}{12}\{1_0 \otimes 3_1\}_{3_1} - \frac{\sqrt{7}}{6}\{2 \otimes 3_1\}_{3_1} - \frac{1}{2\sqrt{6}}\{2 \otimes 3_2\}_{3_1} - \frac{\sqrt{7}}{4}\{3_1 \otimes 3_1\}_{3_1} - \frac{1}{4\sqrt{6}}\{3_1 \otimes 3_2\}_{3_1} \]
\[ 3_2 = -\frac{\sqrt{7}}{12}\{1_0 \otimes 3_2\}_{3_2} + \frac{\sqrt{7}}{2}\{2 \otimes 3_1\}_{3_2} + \frac{5}{6\sqrt{2}}\{2 \otimes 3_2\}_{3_2} + \frac{\sqrt{7}}{4}\{3_1 \otimes 2\}_{3_2} - \frac{\sqrt{7}}{4}\{3_1 \otimes 3_1\}_{3_2} + \frac{\sqrt{3}}{4}\{3_1 \otimes 3_2\}_{3_2} \]
\[ 7 \otimes 7 \rightarrow (1 + 6^{(1)} + 6^{(2)} + 7 + 8)_s + (3 + 3 + 7 + 8)_a \]
\[ \cdot 7 \otimes 7 \rightarrow 1_a: \]
\[ 1_0 = -\frac{1}{\sqrt{7}}\{1_1 \otimes 1_1\}_{1_0} + \sqrt{3}\{3_1 \otimes 3_1\}_{1_0} + \frac{\sqrt{3}}{7}\{3_2 \otimes 3_2\}_{1_0} \]
\[ 7 \otimes 7 \rightarrow 6_a^{(1)}: \]
\[ 1_0 = \frac{\sqrt{3}}{2}\{1_1 \otimes 1_1\}_{1_0} - \frac{1}{2}\{3_2 \otimes 3_2\}_{1_0} \]
\[ 2 = -\frac{\sqrt{3}}{7}\{3_1 \otimes 3_2\}_{2} - \frac{\sqrt{3}}{7}\{3_2 \otimes 3_1\}_{2} + \frac{1}{\sqrt{7}}\{3_2 \otimes 3_2\}_{2} \]
\[ 3_1 = -\frac{1}{2\sqrt{7}}\left(\{1_1 \otimes 3_2\}_{3_1} + \{3_2 \otimes 1_1\}_{3_1}\right) + \sqrt{7}\{3_1 \otimes 3_1\}_{3_1} + \frac{\sqrt{7}}{4}\{3_2 \otimes 3_1\}_{3_1} - \frac{1}{\sqrt{14}}\{3_2 \otimes 3_2\}_{3_1} \]
\[ 3_2 = -\frac{\sqrt{3}}{2}\{3_2 \otimes 3_2\}_{3_2} - \frac{\sqrt{7}}{4}\{3_1 \otimes 2\}_{3_2} + \frac{1}{2\sqrt{7}}\{3_1 \otimes 3_1\}_{3_2} - \frac{1}{\sqrt{14}}\{3_2 \otimes 3_2\}_{3_2} \]
\[ 7 \otimes 7 \rightarrow 6_a^{(2)}: \]
\[ 1_0 = -\frac{\sqrt{4}}{2}\{1_1 \otimes 1_1\}_{1_0} + \frac{2}{\sqrt{7}}\{3_1 \otimes 3_1\}_{1_0} - \frac{3}{2\sqrt{7}}\{3_2 \otimes 3_2\}_{1_0} \]
\[ 2 = -\{3_1 \otimes 3_1\}_{2} \]
$3_1 = -\frac{1}{2} \left( \{1_1 \otimes 3_2\}_{3_1} + \{3_2 \otimes 1_1\}_{3_1} \right) + \frac{1}{\sqrt{2}} \{3_2 \otimes 3_2\}_{3_1}$

**$7 \otimes 7 \rightarrow 7_4$:**

$1_1 = \frac{1}{\sqrt{2}} \{3_1 \otimes 3_2\}_{1_1} + \frac{1}{\sqrt{2}} \{3_2 \otimes 3_1\}_{1_1}$

$3_1 = \frac{1}{6} \left( \{1_1 \otimes 3_2\}_{3_1} + \{3_2 \otimes 1_1\}_{3_1} \right) + \frac{1}{\sqrt{3}} \{3_1 \otimes 3_1\}_{3_1} + \frac{1}{\sqrt{3}} \{3_2 \otimes 3_2\}_{3_1}$

$3_2 = \frac{1}{6} \left( \{1_1 \otimes 3_1\}_{3_2} + \{3_1 \otimes 1_1\}_{3_2} \right) + \frac{1}{\sqrt{3}} \{3_1 \otimes 3_2\}_{3_2} + \frac{1}{\sqrt{3}} \{3_2 \otimes 3_1\}_{3_2}$

**$7 \otimes 7 \rightarrow 8_5$:**

$2 = \frac{1}{\sqrt{2}} \{3_1 \otimes 3_2\}_2 + \frac{1}{\sqrt{2}} \{3_2 \otimes 3_1\}_2 + \frac{6}{7} \{3_2 \otimes 3_2\}_2$

$3_1 = \frac{1}{\sqrt{21}} \left( \{1_1 \otimes 3_2\}_{3_1} + \{3_2 \otimes 1_1\}_{3_1} \right) - 2 \frac{2}{\sqrt{21}} \{3_1 \otimes 3_1\}_{3_1} + \frac{3}{14} \{3_1 \otimes 3_2\}_{3_1} - \frac{3}{14} \{3_2 \otimes 3_1\}_{3_1} + \frac{2}{21} \{3_2 \otimes 3_2\}_{3_1}$

$3_2 = -\frac{1}{\sqrt{3}} \left( \{1_1 \otimes 3_1\}_{3_2} + \{3_1 \otimes 1_1\}_{3_2} \right) + \frac{1}{\sqrt{6}} \{3_1 \otimes 3_2\}_{3_2} + \frac{1}{\sqrt{6}} \{3_2 \otimes 3_1\}_{3_2}$

**$7 \otimes 7 \rightarrow 3_6$:**

$3_2 = \frac{1}{7} \left( \{1_1 \otimes 3_1\}_{3_2} - \{3_1 \otimes 1_1\}_{3_2} \right) + \frac{b_7}{\sqrt{2}} \{3_1 \otimes 3_1\}_{3_2} - \frac{b_7}{\sqrt{2}} \{3_1 \otimes 3_2\}_{3_2}$

$3_2 = \frac{1}{\sqrt{7}} \left( \{1_1 \otimes 3_1\}_{3_2} - \{3_1 \otimes 1_1\}_{3_2} \right) + \frac{1}{\sqrt{7b_7}} \{3_1 \otimes 3_1\}_{3_2} - \frac{1}{\sqrt{7b_7}} \{3_1 \otimes 3_2\}_{3_2}$

**$7 \otimes 7 \rightarrow 7_5$:**

$1_1 = \frac{1}{\sqrt{2}} \{3_1 \otimes 3_2\}_{1_1} - \frac{1}{\sqrt{2}} \{3_2 \otimes 3_1\}_{1_1}$

$3_1 = -\frac{1}{6} \left( \{1_1 \otimes 3_2\}_{3_1} - \{3_2 \otimes 1_1\}_{3_1} \right) + \frac{1}{\sqrt{3}} \{3_1 \otimes 3_2\}_{3_1} + \frac{1}{\sqrt{3}} \{3_1 \otimes 3_2\}_{3_1}$

$3_2 = \frac{1}{6} \left( \{1_1 \otimes 3_1\}_{3_2} - \{3_1 \otimes 1_1\}_{3_2} \right) + \frac{1}{\sqrt{3}} \{3_1 \otimes 3_2\}_{3_2} - \frac{1}{\sqrt{3}} \{3_2 \otimes 3_2\}_{3_2}$

**$7 \otimes 7 \rightarrow 8_6$:**

$2 = \frac{1}{\sqrt{2}} \{3_1 \otimes 3_2\}_2 - \frac{1}{\sqrt{2}} \{3_2 \otimes 3_1\}_2$

$3_1 = -\frac{1}{\sqrt{3}} \left( \{1_1 \otimes 3_2\}_{3_1} - \{3_2 \otimes 1_1\}_{3_1} \right) - \frac{1}{\sqrt{6}} \{3_1 \otimes 3_2\}_{3_1} - \frac{1}{\sqrt{6}} \{3_2 \otimes 3_1\}_{3_1}$

$3_2 = -\frac{1}{\sqrt{21}} \left( \{1_1 \otimes 3_1\}_{3_2} - \{3_1 \otimes 1_1\}_{3_2} \right) + 2 \frac{2}{\sqrt{21}} \{3_1 \otimes 3_1\}_{3_2} + \frac{3}{14} \{3_1 \otimes 3_2\}_{3_2} - \frac{3}{14} \{3_2 \otimes 3_1\}_{3_2} + \frac{2}{21} \{3_2 \otimes 3_2\}_{3_2}$

$7 \otimes 8 \rightarrow 3 + 3 + 6^{(1)} + 6^{(2)} + 7^{(1)} + 7^{(2)} + 8^{(1)} + 8^{(2)} + 8^{(3)}$

**$7 \otimes 8 \rightarrow 3$:**

$3_2 = \frac{1}{\sqrt{7}} \{1_1 \otimes 3_1\}_{3_2} - \frac{b_7}{\sqrt{7}} \{3_1 \otimes 2\}_{3_2}$

$3_2 = \frac{1}{\sqrt{7b_7}} \{3_1 \otimes 3_1\}_{3_2} + \frac{b_7}{2} \{3_1 \otimes 3_2\}_{3_2}$

**$7 \otimes 8 \rightarrow 3$:**

$3_2 = \frac{1}{\sqrt{7}} \{1_1 \otimes 3_1\}_{3_2} - \frac{b_7}{\sqrt{7}} \{3_1 \otimes 2\}_{3_2}$

$3_2 = \frac{1}{\sqrt{7b_7}} \{3_1 \otimes 3_1\}_{3_2} + \frac{b_7}{2} \{3_1 \otimes 3_2\}_{3_2}$

$3_2 = \frac{1}{\sqrt{7}} \{1_1 \otimes 3_1\}_{3_2} - \frac{b_7}{\sqrt{7}} \{3_1 \otimes 2\}_{3_2}$

$3_2 = \frac{1}{\sqrt{7b_7}} \{3_1 \otimes 3_1\}_{3_2} + \frac{b_7}{2} \{3_1 \otimes 3_2\}_{3_2}$
• $7 \otimes 8 \rightarrow 6^{(1)}$

$$1_0 = \{3_1 \otimes 3_1\}_{1_0}$$

$$2 = \frac{\sqrt{7}}{2} (1_1 \otimes 2)_2 + \frac{1}{\sqrt{7}} (3_1 \otimes 3_1)_2$$
$$+ \frac{3}{4} \sqrt{4} \{3_2 \otimes 3_1\}_2 - \frac{3}{4} \{3_2 \otimes 3_2\}_2$$

$$3_1 = -\frac{1}{2} (1_1 \otimes 3_2)_{3_1} + \sqrt{3} (3_1 \otimes 2)_{3_1}$$
$$\quad - \frac{1}{2\sqrt{7}} \{3_2 \otimes 3_2\}_3 - \frac{3}{2\sqrt{14}} \{3_2 \otimes 3_1\}_3$$
$$\quad - \frac{1}{2\sqrt{2}} \{3_2 \otimes 3_2\}_3$$

• $7 \otimes 8 \rightarrow 6^{(2)}$

$$1_0 = \{3_2 \otimes 3_2\}_{1_0}$$

$$2 = \frac{\sqrt{7}}{2} (1_1 \otimes 2)_2 - \frac{3}{4} \sqrt{4} \{3_2 \otimes 3_1\}_2$$
$$\quad + \frac{1}{4\sqrt{7}} \{3_2 \otimes 3_2\}_2$$

$$3_1 = \frac{1}{2} (1_1 \otimes 3_2)_{3_1} + \frac{1}{\sqrt{7}} (3_1 \otimes 3_1)_{3_1}$$
$$\quad - \frac{1}{\sqrt{14}} \{3_2 \otimes 3_2\}_3 - \frac{1}{2} (3_2 \otimes 2)_{3_1}$$
$$\quad + \frac{1}{2\sqrt{2}} \{3_2 \otimes 3_1\}_3 + \frac{1}{2\sqrt{14}} \{3_2 \otimes 3_2\}_3$$

• $7 \otimes 8 \rightarrow 7^{(1)}$

$$1_1 = \{3_1 \otimes 3_2\}_{1_1}$$

$$3_1 = \frac{\sqrt{3}}{4} (1_1 \otimes 3_2)_{3_1} + \frac{\sqrt{7}}{2} (3_1 \otimes 3_1)_{3_1}$$
$$\quad - \frac{1}{2\sqrt{6}} \{3_1 \otimes 3_2\}_3 + \frac{\sqrt{7}}{4} \{3_2 \otimes 2\}_3$$
$$\quad - \frac{\sqrt{7}}{4} \{3_2 \otimes 3_1\}_3 - \frac{5}{4\sqrt{6}} \{3_2 \otimes 3_2\}_3$$

$$3_2 = -\frac{\sqrt{3}}{4} (1_1 \otimes 3_1)_{3_2} + \frac{\sqrt{7}}{2} (3_1 \otimes 3_1)_{3_2}$$
$$\quad - \frac{1}{2\sqrt{6}} \{3_1 \otimes 3_2\}_3 - \frac{\sqrt{7}}{4} \{3_2 \otimes 2\}_3$$
$$\quad - \frac{\sqrt{7}}{4} \{3_2 \otimes 3_1\}_3 - \frac{1}{4\sqrt{6}} \{3_2 \otimes 3_2\}_3$$

• $7 \otimes 8 \rightarrow 7^{(2)}$

$$1_1 = \{3_2 \otimes 3_1\}_{1_1}$$

$$3_1 = -\frac{\sqrt{3}}{4} (1_1 \otimes 3_2)_{3_1} - \frac{1}{2\sqrt{6}} (3_1 \otimes 3_1)_{3_1}$$
$$\quad - \frac{\sqrt{7}}{2} (3_1 \otimes 3_2)_{3_1} + \frac{\sqrt{3}}{4} (3_2 \otimes 2)_{3_1}$$
$$\quad + \frac{5}{4\sqrt{6}} (3_2 \otimes 3_1)_{3_1} - \frac{\sqrt{7}}{4} (3_2 \otimes 3_2)_{3_1}$$

$$3_2 = -\frac{\sqrt{3}}{4} (1_1 \otimes 3_1)_{3_2} - \frac{1}{2\sqrt{6}} (3_1 \otimes 3_1)_{3_2}$$
$$\quad + \frac{\sqrt{7}}{2} (3_1 \otimes 3_2)_{3_2} + \frac{3}{4\sqrt{6}} (3_1 \otimes 3_2)_{3_2}$$
$$\quad + \frac{1}{4\sqrt{14}} (3_2 \otimes 3_1)_{3_2} + \frac{23}{8\sqrt{21}} (3_2 \otimes 3_1)_{3_2}$$
$$\quad + \frac{1}{8\sqrt{3}} (3_2 \otimes 3_2)_{3_2}$$

• $7 \otimes 8 \rightarrow 8^{(1)}$

$$2 = \frac{3}{4\sqrt{2}} (1_1 \otimes 2)_2 - \frac{5}{8} (3_2 \otimes 3_1)_2$$
$$\quad - \frac{\sqrt{21}}{8} \{3_2 \otimes 3_2\}_2$$

$$3_1 = -\frac{\sqrt{3}}{4} (1_1 \otimes 3_2)_{3_1} + \frac{1}{2\sqrt{3}} (3_1 \otimes 3_1)_{3_1}$$
$$\quad + \frac{\sqrt{3}}{2} (3_1 \otimes 3_2)_{3_1} + \frac{3}{4} (3_2 \otimes 2)_{3_1}$$
$$\quad + \frac{5}{8\sqrt{3}} (3_2 \otimes 3_1)_{3_1} - \frac{\sqrt{3}}{8} \{3_2 \otimes 3_2\}_3$$

$$3_2 = -\frac{3\sqrt{3}}{4} (1_1 \otimes 3_1)_{3_2} + \frac{2}{21} (3_1 \otimes 2)_{3_2}$$
$$\quad + \frac{1}{2\sqrt{21}} (3_1 \otimes 3_1)_{3_2} + \frac{1}{2\sqrt{3}} (3_1 \otimes 3_2)_{3_2}$$
$$\quad - \frac{1}{4\sqrt{14}} (3_2 \otimes 2)_{3_2} + \frac{23}{8\sqrt{21}} (3_2 \otimes 3_1)_{3_2}$$
$$\quad + \frac{1}{8\sqrt{3}} (3_2 \otimes 3_2)_{3_2}$$

• $7 \otimes 8 \rightarrow 8^{(2)}$

$$2 = \frac{1}{4\sqrt{7}} (1_1 \otimes 2)_2 + \frac{6}{\sqrt{3}} (3_1 \otimes 3_1)_2$$
$$\quad - \frac{3}{4\sqrt{14}} (3_2 \otimes 3_1)_{3_2} + \frac{\sqrt{7}}{4} (3_2 \otimes 3_2)_{3_2}$$
\[ 3_1 = -\frac{\sqrt{3}}{4} \{1_1 \otimes 3_2\}_{3_1} - \frac{2}{\sqrt{7}} \{3_1 \otimes 2\}_{3_1} - \frac{3\sqrt{3}}{14} \{3_2 \otimes 3_1\}_{3_1} - \frac{\sqrt{2}}{4} \{3_2 \otimes 3_2\}_{3_1} \]
\[ 3_2 = \frac{\sqrt{3}}{4} \{1_1 \otimes 3_1\}_{3_2} - \frac{1}{4} \{3_2 \otimes 2\}_{3_2} + \frac{\sqrt{2}}{4} \{3_2 \otimes 3_1\}_{3_2} - \frac{\sqrt{2}}{4} \{3_2 \otimes 3_2\}_{3_2} \]

\[ 7 \otimes 8 \rightarrow 8^{(3)}: \]
\[ 2 = \frac{3}{4\sqrt{2}} \{1_1 \otimes 2\}_{2} + \frac{2}{\sqrt{7}} \{3_1 \otimes 3_2\}_{2} + \frac{3}{8} \{3_2 \otimes 3_1\}_{2} + \frac{\sqrt{3}}{8} \{3_2 \otimes 3_2\}_{2} \]

\[ 8 \otimes 8 \rightarrow (1 + 6^{(1)} + 6^{(2)} + 7 + 8^{(1)} + 8^{(2)})_a + (3 + 3 + 7^{(1)} + 7^{(2)} + 8)_a \]

\[ 8 \otimes 8 \rightarrow 1_1: \]
\[ 1_0 = \frac{1}{2} \{2 \otimes 2\}_{1_0} + \frac{\sqrt{3}}{2} \{3_1 \otimes 3_1\}_{1_0} + \frac{\sqrt{3}}{2} \{3_2 \otimes 3_2\}_{1_0} \]

\[ 8 \otimes 8 \rightarrow 6^{(1)}: \]
\[ 1_0 = \frac{\sqrt{3}}{5} \{2 \otimes 2\}_{1_0} - \frac{2}{5} \{3_2 \otimes 3_2\}_{1_0} \]

\[ 2 = -\frac{\sqrt{4}}{2} \{2 \otimes 2\}_{2} - \frac{27}{4\sqrt{70}} \{3_1 \otimes 3_1\}_{2} + \frac{\sqrt{3}}{4} \{3_1 \otimes 3_2\}_{2} + \frac{\sqrt{3}}{4} \{3_2 \otimes 3_1\}_{2} + \frac{\sqrt{7}}{4} \{3_2 \otimes 3_2\}_{2} \]

\[ 8 \otimes 8 \rightarrow 6^{(2)}: \]
\[ 1_0 = -\frac{\sqrt{3}}{2} \{2 \otimes 2\}_{1_0} + \frac{\sqrt{3}}{2} \{3_1 \otimes 3_1\}_{1_0} - \frac{3}{2\sqrt{10}} \{3_2 \otimes 3_2\}_{1_0} \]
\[ 2 = -\frac{\sqrt{10}}{2} \{2 \otimes 2\}_{2} + \frac{1}{4\sqrt{70}} \{3_1 \otimes 3_1\}_{2} + \frac{3\sqrt{3}}{4} \{3_1 \otimes 3_2\}_{2} + \frac{3\sqrt{3}}{4} \{3_2 \otimes 3_1\}_{2} + \frac{3}{4\sqrt{10}} \{3_2 \otimes 3_2\}_{2} \]
\[ 3_1 = -\frac{\sqrt{6}}{35} \{2 \otimes 3_1\}_{3_1} + \{3_1 \otimes 2\}_{3_1} - \frac{1}{\sqrt{10}} \{2 \otimes 3_2\}_{3_1} + \{3_2 \otimes 2\}_{3_1} - \frac{9}{4\sqrt{35}} \{3_1 \otimes 3_1\}_{3_1} - \frac{3}{4\sqrt{5}} \{3_1 \otimes 3_2\}_{3_1} + \frac{3}{4\sqrt{5}} \{3_2 \otimes 3_1\}_{3_1} + \frac{\sqrt{3}}{4} \{3_2 \otimes 3_2\}_{3_1} \]

\[ 8 \otimes 8 \rightarrow 7: \]
\[ 1_1 = \frac{1}{\sqrt{2}} \{3_1 \otimes 3_2\}_{1_1} + \frac{1}{\sqrt{2}} \{3_2 \otimes 3_1\}_{1_1} \]
\[3_1 = -\frac{1}{\sqrt{6}} \left( \{2 \otimes 3_2\}_{3_1} + \{3_2 \otimes 2\}_{3_1} \right) - \sqrt{\frac{7}{4}} \left\{ 3_1 \otimes 3_1 \right\}_{3_1} + \sqrt{\frac{3}{4}} \left\{ 3_1 \otimes 3_2 \right\}_{3_1} - \frac{\sqrt{3}}{4} \left\{ 3_2 \otimes 3_1 \right\}_{3_1} - \frac{\sqrt{7}}{4} \left\{ 3_2 \otimes 3_2 \right\}_{3_1}\]

\[3_2 = \frac{\sqrt{7}}{2} \left( \left\{ 2 \otimes 3_1 \right\}_{3_2} + \left\{ 3_1 \otimes 2 \right\}_{3_2} \right) - \sqrt{\frac{7}{16}} \left( \left\{ 2 \otimes 3_2 \right\}_{3_2} + \left\{ 3_2 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{7}}{12} \left\{ 3_1 \otimes 3_2 \right\}_{3_2} + \sqrt{\frac{7}{12}} \left\{ 3_2 \otimes 3_1 \right\}_{3_2}\]

- **8 \otimes 8 \rightarrow 3_8^{(1)}:**

\[2 = \frac{\sqrt{7}}{5} \left\{ 2 \otimes 2 \right\}_2 - \sqrt{\frac{3}{10}} \left\{ 3_1 \otimes 3_1 \right\}_2 - \sqrt{\frac{3}{10}} \left\{ 3_2 \otimes 3_2 \right\}_2\]

\[3_1 = -\frac{1}{\sqrt{6}} \left( \{2 \otimes 3_1\}_{3_1} + \{3_1 \otimes 2\}_{3_1} \right) + \frac{\sqrt{3}}{4} \left\{ 3_1 \otimes 3_1 \right\}_{3_1} - \frac{\sqrt{3}}{4} \left\{ 3_1 \otimes 3_2 \right\}_{3_1} - \frac{1}{\sqrt{3}} \left\{ 3_2 \otimes 3_2 \right\}_{3_1} - \frac{\sqrt{7}}{4} \left\{ 3_2 \otimes 3_1 \right\}_{3_1}\]

\[3_2 = \frac{\sqrt{7}}{2} \left( \left\{ 2 \otimes 3_1 \right\}_{3_2} + \left\{ 3_1 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{7}}{12} \left\{ 3_1 \otimes 3_2 \right\}_{3_2} + \sqrt{\frac{7}{12}} \left\{ 3_2 \otimes 3_1 \right\}_{3_2}\]

- **8 \otimes 8 \rightarrow 3_8^{(2)}:**

\[2 = \frac{3}{2\sqrt{35}} \left\{ 2 \otimes 2 \right\}_2 + \frac{3}{\sqrt{45}} \left\{ 3_1 \otimes 3_1 \right\}_2 + \sqrt{\frac{5}{4}} \left\{ 3_1 \otimes 3_2 \right\}_2 + \sqrt{\frac{5}{4}} \left\{ 3_2 \otimes 3_1 \right\}_2 - \frac{\sqrt{21}}{4} \left\{ 3_2 \otimes 3_2 \right\}_2\]

\[3_1 = \frac{3}{2\sqrt{70}} \left( \{2 \otimes 3_1\}_{3_1} + \{3_1 \otimes 2\}_{3_1} \right) - \frac{\sqrt{3}}{2} \left\{ \{2 \otimes 3_2\}_{3_1} + \{3_2 \otimes 2\}_{3_1} \right\} + \frac{13}{2\sqrt{105}} \left\{ 3_1 \otimes 3_1 \right\}_{3_1} + \frac{\sqrt{7}}{2} \left\{ 3_2 \otimes 3_2 \right\}_{3_1}\]

\[3_2 = \frac{\sqrt{3}}{2} \left( \left\{ 2 \otimes 3_1 \right\}_{3_2} - \left\{ 3_1 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{7}}{8} \left( \left\{ 2 \otimes 3_2 \right\}_{3_2} - \left\{ 3_2 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{3}}{4} \left\{ 3_1 \otimes 3_1 \right\}_{3_2} + \frac{\sqrt{21}}{8} \left\{ 3_1 \otimes 3_2 \right\}_{3_2} - \frac{\sqrt{21}}{8} \left\{ 3_2 \otimes 3_1 \right\}_{3_2}\]

- **8 \otimes 8 \rightarrow 3_{a}^{(1)}:**

\[3_1 = \frac{1}{4} \left( \left\{ 2 \otimes 3_1 \right\}_{3_2} - \left\{ 3_1 \otimes 2 \right\}_{3_2} \right) - \frac{\sqrt{3}}{4} \left( \left\{ 2 \otimes 3_2 \right\}_{3_2} - \left\{ 3_2 \otimes 2 \right\}_{3_2} \right) - \frac{1}{2\sqrt{7}} \left\{ 3_1 \otimes 3_1 \right\}_{3_2} + \frac{i}{4\sqrt{7}} \left\{ 3_1 \otimes 3_2 \right\}_{3_2} - \frac{i}{407} \left\{ 3_2 \otimes 3_1 \right\}_{3_2} - \frac{1}{2\sqrt{208}} \left\{ 3_2 \otimes 3_2 \right\}_{3_2}\]

- **8 \otimes 8 \rightarrow 3_{a}^{(2)}:**

\[3_1 = \frac{1}{2\sqrt{2}} \left( \left\{ 2 \otimes 3_1 \right\}_{3_2} - \left\{ 3_1 \otimes 2 \right\}_{3_2} \right) - \frac{\sqrt{7}}{2} \left( \left\{ 2 \otimes 3_2 \right\}_{3_2} - \left\{ 3_2 \otimes 2 \right\}_{3_2} \right) - \frac{1}{\sqrt{2}} \left\{ 3_1 \otimes 3_1 \right\}_{3_2} + \frac{ib_7}{4} \left\{ 3_1 \otimes 3_2 \right\}_{3_2} + \frac{ib_7}{4} \left\{ 3_2 \otimes 3_1 \right\}_{3_2} - \frac{b_7^2}{2\sqrt{2}} \left\{ 3_2 \otimes 3_2 \right\}_{3_2}\]

- **8 \otimes 8 \rightarrow 7_{a}^{(1)}:**

\[3_1 = \frac{1}{8\sqrt{2}} \left( \left\{ 2 \otimes 3_1 \right\}_{3_2} - \left\{ 3_1 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{21}}{8} \left( \left\{ 2 \otimes 3_2 \right\}_{3_2} - \left\{ 3_2 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{21}}{8} \left\{ 3_1 \otimes 3_2 \right\}_{3_2} + \frac{\sqrt{21}}{8} \left\{ 3_2 \otimes 3_1 \right\}_{3_2}\]

\[3_2 = -\frac{\sqrt{3}}{8} \left( \left\{ 2 \otimes 3_1 \right\}_{3_2} - \left\{ 3_1 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{7}}{8} \left( \left\{ 2 \otimes 3_2 \right\}_{3_2} - \left\{ 3_2 \otimes 2 \right\}_{3_2} \right) + \frac{\sqrt{3}}{4} \left\{ 3_1 \otimes 3_1 \right\}_{3_2} + \frac{\sqrt{21}}{8} \left\{ 3_1 \otimes 3_2 \right\}_{3_2} - \frac{\sqrt{21}}{8} \left\{ 3_2 \otimes 3_1 \right\}_{3_2}\]
Appendix D: CG Coefficients of $\mathcal{P}S\mathcal{L}_2 (7)$ in $T_7$ Basis

The notations used in this appendix are the same as those of Appendix C. Two constant angles will be used in the following results

$$\alpha = \arctan \frac{\sqrt{5}}{2}, \quad \beta = \arctan \left(3\sqrt{3} + 2\sqrt{6}\right).$$

$3 \otimes 3 \rightarrow 6_s + \bar{3}_a$

- $3 \otimes 3 \rightarrow 3_a$:
  \[
  \bar{3}_a = \{3 \otimes 3\}_3
  \]
- $3 \otimes 3 \rightarrow 6_s$:
  \[
  3 = \{3 \otimes 3\}_3
  \]
  \[
  \bar{3} = \{3 \otimes 3\}_\bar{3}
  \]

$3 \otimes \bar{3} \rightarrow 1 + 8$

- $3 \otimes \bar{3} \rightarrow 1$:
  \[
  1 = \{3 \otimes \bar{3}\}_1
  \]
- $3 \otimes \bar{3} \rightarrow 8$:
  \[
  1' = \{3 \otimes \bar{3}\}_1
  \]
  \[
  \bar{1}' = \{3 \otimes \bar{3}\}_\bar{1}
  \]
  \[
  3 = \{3 \otimes \bar{3}\}_3
  \]
  \[
  \bar{3} = \{3 \otimes \bar{3}\}_\bar{3}
  \]

$3 \otimes 6 \rightarrow \bar{3} + 7 + 8$

- $3 \otimes 6 \rightarrow \bar{3}$:
  \[
  \bar{3} = \frac{1}{\sqrt{2}} (3 \otimes 3)_\bar{3} + \frac{1}{\sqrt{2}} (3 \otimes \bar{3})_\bar{3}
  \]
- $3 \otimes 6 \rightarrow 7$:
  \[
  1 = \{3 \otimes 6\}_1
  \]
  \[
  3 = \frac{1}{\sqrt{3}} (3 \otimes 3)_3 + \frac{2}{\sqrt{3}} (3 \otimes \bar{3})_3
  \]
  \[
  \bar{3} = -\frac{1}{\sqrt{6}} (3 \otimes 3)_\bar{3} + \frac{2}{\sqrt{3}} (3 \otimes \bar{3})_\bar{3}
  \]
  \[
  + \frac{1}{\sqrt{6}} (3 \otimes \bar{3})_\bar{3}
  \]
- $3 \otimes 6 \rightarrow 8$:
  \[
  1' = \{3 \otimes 6\}_1
  \]
  \[
  \bar{1}' = -\{3 \otimes 6\}_1
  \]
  \[
  3 = i\sqrt{\frac{2}{3}} (3 \otimes 3)_3 - i\sqrt{3} (3 \otimes \bar{3})_3
  \]
  \[
  \bar{3} = -i\sqrt{\frac{2}{3}} (3 \otimes 3)_\bar{3} - i\sqrt{\frac{3}{3}} (3 \otimes \bar{3})_3
  \]
  \[
  + i\sqrt{\frac{3}{3}} (3 \otimes \bar{3})_\bar{3}
  \]
\[ 3 \otimes 7 \rightarrow 6 + 7 + 8 \]

- **3 \otimes 7 \rightarrow 6:**
  \[ 3 = \frac{2}{7} \{3 \otimes 1\}_3 + \frac{1}{\sqrt{7}} \{3 \otimes 3\}_3 \]
  \[ + \frac{2}{\sqrt{7}} \{3 \otimes 3\}_3 \]
  \[ \bar{3} = -\frac{1}{\sqrt{3}} \{3 \otimes 3\}_3 - \frac{2}{\sqrt{3}} \{3 \otimes 3\}_3 \]
  \[ + \frac{2}{\sqrt{3}} \{3 \otimes 3\}_3 \]

- **3 \otimes 7 \rightarrow 7:**
  \[ 1 = \{3 \otimes 3\}_1 \]
  \[ 3 = -\frac{1}{\sqrt{3}} \{3 \otimes 1\}_3 + \sqrt{\frac{2}{3}} \{3 \otimes 3\}_3 \]
  \[ 3 = -\frac{1}{\sqrt{3}} \{3 \otimes 3\}_3 - \sqrt{\frac{2}{3}} \{3 \otimes 3\}_3 \]

- **3 \otimes 7 \rightarrow 8:**
  \[ 1' = \{3 \otimes 3\}_1 \]
  \[ \bar{1}' = -e^{2i\alpha} \{3 \otimes 3\}_1 \]
  \[ 3 = -2i \sqrt{\frac{2}{21}} e^{i\alpha} \{3 \otimes 1\}_3 - \frac{2ie^{i\alpha}}{\sqrt{21}} \{3 \otimes 3\}_3 \]
  \[ + i \sqrt{\frac{6}{7}} e^{i\alpha} \{3 \otimes 3\}_3 \]
  \[ \bar{3} = -i \sqrt{\frac{6}{7}} e^{i\alpha} \{3 \otimes 3\}_3 + i \sqrt{\frac{2}{21}} e^{i\alpha} \{3 \otimes 3\}_3 \]
  \[ - \frac{ie^{i\alpha}}{\sqrt{21}} \{3 \otimes 3\}_3 \]

- **3 \otimes 8 \rightarrow 3 + 6 + 7 + 8**
  \[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 1'\}_3 + \frac{1}{2\sqrt{2}} \{3 \otimes \bar{1}'\}_3 \]
  \[ + \sqrt{\frac{2}{2}} \{3 \otimes 3\}_3 + \sqrt{\frac{2}{2}} \{3 \otimes 3\}_3 \]

- **3 \otimes 8 \rightarrow 6:**
  \[ 3 = \frac{1}{2} \{3 \otimes 1'\}_3 - \frac{1}{2} \{3 \otimes \bar{1}'\}_3 \]
  \[ - \frac{i}{2} \{3 \otimes 3\}_3 + \frac{i}{2} \{3 \otimes 3\}_3 \]
  \[ \bar{3} = \frac{i}{2} \{3 \otimes 3\}_3 - \frac{i}{2} \{3 \otimes 3\}_3 \]
  \[ + \frac{i}{\sqrt{2}} \{3 \otimes 3\}_3 \]

- **3 \otimes 8 \rightarrow 7:**
  \[ 1 = \{3 \otimes 3\}_1 \]
  \[ 3 = \frac{1}{2} i \sqrt{\frac{7}{6}} e^{i\alpha} \{3 \otimes 1'\}_3 + \frac{1}{2} i \sqrt{\frac{7}{6}} e^{-i\alpha} \{3 \otimes \bar{1}'\}_3 \]
  \[ + \frac{1}{2\sqrt{6}} \{3 \otimes 3\}_3 - \sqrt{\frac{2}{2}} \{3 \otimes 3\}_3 \]
  \[ \bar{3} = \frac{\sqrt{3}}{2} \{3 \otimes 3\}_3 + \frac{1}{\sqrt{3}} \{3 \otimes 3\}_3 \]
  \[ + \frac{1}{\sqrt{6}} \{3 \otimes 3\}_3 \]

- **3 \otimes 8 \rightarrow 8:**
  \[ 1' = \{3 \otimes 3\}_1 \]
  \[ \bar{1}' = -\omega^2 \{3 \otimes 3\}_1 \]
  \[ 3 = \frac{\omega^2}{\sqrt{3}} \{3 \otimes 1'\}_3 - \frac{1}{\sqrt{3}} \{3 \otimes \bar{1}'\}_3 \]
  \[ - \frac{i\omega}{\sqrt{3}} \{3 \otimes 3\}_3 \]
  \[ \bar{3} = -i \sqrt{\frac{2}{3}} e^{i\alpha} \{3 \otimes 3\}_3 + \frac{i\omega}{\sqrt{3}} \{3 \otimes 3\}_3 \]

- **6 \otimes 6 \rightarrow (1 + 6^{(1)} + 6^{(2)} + 8) + (7 + 8)_a**
  \[ 6 \otimes 6 \rightarrow 1_1:**
  \[ 1 = \frac{1}{\sqrt{2}} (\{3 \otimes 3\}_1 + \{\bar{3} \otimes 3\}_1) \]

- **6 \otimes 6 \rightarrow 6^{(1)}:**
  \[ 3 = \frac{1}{\sqrt{14}} (3 - \sqrt{2}) \{3 \otimes 3\}_3 \]
  \[ + \frac{1}{\sqrt{14}} (3 - \sqrt{2}) \{3 \otimes 3\}_3 + \{\bar{3} \otimes 3\}_3 \]
  \[ - \frac{1}{\sqrt{14}} (5 + 3\sqrt{2}) \{3 \otimes 3\}_3 \]

- **6 \otimes 6 \rightarrow 6^{(1)}:**

- **6 \otimes 6 \rightarrow 6^{(1)}:**
\[
3 = -\sqrt{\frac{1}{14}} \left( 5 + 3\sqrt{2} \right) \{3 \otimes 3\}_{3s} \\
+ \sqrt{\frac{1}{14}} \left( 3 - \sqrt{2} \right) (\{3 \otimes \bar{3}\}_{3s} + \{\bar{3} \otimes 3\}_{3s}) \\
+ \sqrt{\frac{1}{14}} \left( 3 - \sqrt{2} \right) \{3 \otimes 3\}_{3s}
\]

- **6 \otimes 6 \rightarrow 6_{s}^{(2)}:**

\[
3 = i \sqrt{\frac{1}{14}} \left( 3 + \sqrt{2} \right) \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{1}{14}} \left( 3 + \sqrt{2} \right) (\{3 \otimes \bar{3}\}_{3s} + \{\bar{3} \otimes 3\}_{3s}) \\
- i \sqrt{\frac{1}{14}} \left( 5 - 3\sqrt{2} \right) \{3 \otimes 3\}_{3s} \\
3 = i \sqrt{\frac{1}{14}} \left( 5 - 3\sqrt{2} \right) \{3 \otimes 3\}_{3s} \\
+ i \sqrt{\frac{1}{14}} \left( 3 + \sqrt{2} \right) (\{3 \otimes \bar{3}\}_{3s} + \{\bar{3} \otimes 3\}_{3s}) \\
- i \sqrt{\frac{1}{14}} \left( 3 + \sqrt{2} \right) \{3 \otimes 3\}_{3s}
\]

- **6 \otimes 6 \rightarrow 8_{s}:**

1' = e^{-i\alpha} \left( \{3 \otimes \bar{3}\}_{1'} + \{\bar{3} \otimes 3\}_{1'} \right) \\
\bar{1}' = e^{i\alpha} \left( \{3 \otimes \bar{3}\}_{1} + \{\bar{3} \otimes 3\}_{1} \right) \\
3 = -\frac{2}{\sqrt{7}} \{3 \otimes 3\}_{3s} \\
- \frac{1}{\sqrt{14}} (\{3 \otimes 3\}_{3s} + \{\bar{3} \otimes 3\}_{3s}) \\
- \sqrt{\frac{2}{7}} \{3 \otimes \bar{3}\}_{3s} \\
\bar{3} = -\sqrt{\frac{2}{7}} \{3 \otimes 3\}_{\bar{3}s} \\
\bar{3} = -\sqrt{\frac{2}{7}} \{3 \otimes 3\}_{\bar{3}s} \\
\bar{3} = -\frac{1}{\sqrt{14}} (\{3 \otimes 3\}_{\bar{3}s} + \{\bar{3} \otimes 3\}_{\bar{3}s}) \\
\bar{3} = -\frac{2}{\sqrt{7}} \{3 \otimes \bar{3}\}_{\bar{3}s}
\]

- **6 \otimes 6 \rightarrow 7_{s}:**

1 = \frac{i}{\sqrt{2}} \left( \{3 \otimes \bar{3}\}_{1} - \{\bar{3} \otimes 3\}_{1} \right)

3 = -\frac{i}{\sqrt{3}} \left( \{3 \otimes \bar{3}\}_{3s} - \{\bar{3} \otimes 3\}_{3s} \right) \\
- \frac{i}{\sqrt{3}} \{3 \otimes 3\}_{3s} \\
\bar{3} = i \frac{1}{\sqrt{3}} \{3 \otimes 3\}_{\bar{3}s} \\
\bar{3} = i \frac{1}{\sqrt{3}} \{3 \otimes 3\}_{\bar{3}s} \\
3 = i \frac{1}{\sqrt{3}} \left( \{3 \otimes 3\}_{3s} - \{\bar{3} \otimes 3\}_{3s} \right) \\
- i \frac{1}{\sqrt{3}} \left( \{3 \otimes 3\}_{3s} - \{\bar{3} \otimes 3\}_{3s} \right) \\
3 = \frac{1}{\sqrt{7}} \{3 \otimes 3\}_{3s} + \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
+ \frac{1}{\sqrt{14}} \{3 \otimes \bar{3}\}_{3s} + \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- \frac{1}{\sqrt{14}} \{3 \otimes \bar{3}\}_{\bar{3}s} + \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{\bar{3}s}
\]

- **6 \otimes 7 \rightarrow 3:**

3 = \frac{1}{\sqrt{7}} \{3 \otimes 1\}_{3} + \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
+ \frac{1}{\sqrt{14}} \{3 \otimes \bar{3}\}_{3s} + \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- \frac{1}{\sqrt{14}} \{3 \otimes \bar{3}\}_{\bar{3}s} + \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{\bar{3}s}
\]

- **6 \otimes 7 \rightarrow \bar{3}:**

\bar{3} = \frac{1}{\sqrt{14}} \{3 \otimes 3\}_{\bar{3}s} - \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{\bar{3}s} \\
- \frac{1}{\sqrt{14}} \{3 \otimes \bar{3}\}_{\bar{3}s} - \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{\bar{3}s} \\
- \frac{1}{\sqrt{14}} \{3 \otimes 3\}_{\bar{3}s} - \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{\bar{3}s}
\]

- **6 \otimes 7 \rightarrow 6:**

3 = \frac{i}{\sqrt{7}} \{3 \otimes 1\}_{3} - i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} + i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
3 = \frac{i}{\sqrt{7}} \{3 \otimes 1\}_{3} - i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} + i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
3 = \frac{i}{\sqrt{7}} \{3 \otimes 1\}_{3} - i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} + i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
3 = \frac{i}{\sqrt{7}} \{3 \otimes 1\}_{3} - i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} + i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
3 = \frac{i}{\sqrt{7}} \{3 \otimes 1\}_{3} - i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} + i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
3 = \frac{i}{\sqrt{7}} \{3 \otimes 1\}_{3} - i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} + i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
3 = \frac{i}{\sqrt{7}} \{3 \otimes 1\}_{3} - i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
- i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} + i \sqrt{\frac{2}{7}} \{3 \otimes 3\}_{3s} \\
\[ 3 = -i \sqrt{\frac{2}{7}} (3 \otimes 3)_3 + \frac{i}{\sqrt{7}} (\bar{3} \otimes 1)_3 + i \sqrt{\frac{2}{7}} (\bar{3} \otimes 3)_3 \]

\[ 3 = \frac{1}{\sqrt{6}} (3 \otimes 1)_3 - \frac{i}{\sqrt{6}} (3 \otimes 3)_3 + \frac{1}{6} (\sqrt{3} + \sqrt{6}) (3 \otimes 3)_3 - \frac{1}{\sqrt{6}} (3 \otimes 3)_3 \]

\[ 3 = \frac{1}{6} (\sqrt{3} - \sqrt{6}) (3 \otimes 3)_3 - \frac{1}{\sqrt{6}} (3 \otimes 3)_3 + \frac{1}{6} (\sqrt{3} + \sqrt{6}) (3 \otimes 3)_3 - \frac{1}{\sqrt{6}} (3 \otimes 3)_3 \]

\[ 3 = \frac{1}{\sqrt{21}} (3 \otimes 3)_1 + \frac{i}{\sqrt{21}} (3 \otimes 3)_3 \]

\[ 1' = -i \omega^{2i\alpha - 2i\beta} \sqrt{\frac{2}{7}} (3 \otimes 3)_1 - i \omega^{2i\alpha - 2i\beta} (3 \otimes 3)_3 + i \sqrt{2} (\bar{3} \otimes 3)_3 \]

\[ 1' = \frac{i}{\sqrt{21}} (3 \otimes 3)_1 - \frac{i}{\sqrt{21}} (3 \otimes 3)_3 + i \sqrt{2} (\bar{3} \otimes 3)_3 \]

\[ 3 = -i \omega^{2i\alpha - 2i\beta} \sqrt{\frac{2}{7}} (3 \otimes 3)_1 - i \omega^{2i\alpha - 2i\beta} (3 \otimes 3)_3 + i (\sqrt{2} - 2) (3 \otimes 3)_3 + \frac{i}{\sqrt{21}} (3 \otimes 3)_3 - \frac{i (\sqrt{2} - 2)}{2 \sqrt{21}} (\bar{3} \otimes 3)_3 \]

\[ 3 = \frac{1}{\sqrt{21}} (3 \otimes 3)_1 - \frac{i}{\sqrt{21}} (3 \otimes 3)_3 + i \sqrt{2} (\bar{3} \otimes 3)_3 \]

\[ 6 \otimes 7 \rightarrow 7^{(1)}: \]

\[ 6 \otimes 7 \rightarrow 7^{(2)}: \]

\[ 6 \otimes 8 \rightarrow 3 + 3 + 6^{(1)} + 6^{(2)} + 7^{(1)} + 7^{(2)} + 8^{(1)} + 8^{(2)}: \]

\[ 6 \otimes 8 \rightarrow 3: \]
\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]

\[ 3 = \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3a} + \frac{1}{2\sqrt{2}} \{3 \otimes 3\}_{3b} + \frac{i}{2\sqrt{2}} \{3 \otimes 3\}_{3c} \]
\[ 3 = \frac{ie^{i\beta}}{\sqrt{6}} (3 \otimes 1)^{1} + \frac{ie^{-i\beta}}{\sqrt{6}} (3 \otimes \bar{1})^{3} \]
\[ - \sqrt{\frac{5 + 3\sqrt{2}}{42}} (3 \otimes 3)^{3} + \sqrt{\frac{6 - 2\sqrt{2}}{21}} (3 \otimes 3)^{3} \]
\[ - \sqrt{\frac{5 + 3\sqrt{2}}{42}} (\bar{3} \otimes 3)^{3} + \sqrt{\frac{3 - \sqrt{2}}{21}} (\bar{3} \otimes \bar{3})^{3} \]
\[ \bar{3} = \sqrt{\frac{3 - \sqrt{2}}{21}} (3 \otimes 3)^{3} - i \frac{e^{i\beta}}{\sqrt{6}} (3 \otimes \bar{1})^{3} \]
\[ + \frac{ie^{i\beta}}{\sqrt{6}} (3 \otimes \bar{1})^{3} + \frac{ie^{-i\beta}}{\sqrt{6}} (3 \otimes 1)^{3} \]
\[ + \sqrt{\frac{6 - 2\sqrt{2}}{21}} (3 \otimes 3)^{3} - \sqrt{\frac{5 + 3\sqrt{2}}{42}} (3 \otimes 3)^{3} \]
\[ 6 \otimes 8 \rightarrow 8^{(2)}: \]
\[ 1' = \frac{\omega e^{i\alpha - i\beta}}{\sqrt{2}} (3 \otimes 3)^{1} - \frac{\omega e^{i\alpha - i\beta}}{\sqrt{2}} (3 \otimes 3)^{1} \]
\[ \bar{1}' = -\frac{\omega e^{i\alpha - i\beta}}{\sqrt{2}} (3 \otimes \bar{3})^{1} + \frac{\omega e^{i\alpha - i\beta}}{\sqrt{2}} (3 \otimes \bar{3})^{1} \]
\[ 3 = \frac{\omega e^{i\alpha - i\beta}}{\sqrt{6}} (3 \otimes 3)^{3} + \frac{\omega e^{i\alpha - i\beta}}{\sqrt{6}} (3 \otimes \bar{1})^{3} \]
\[ + i \sqrt{\frac{5 - 3\sqrt{2}}{42}} (3 \otimes 3)^{3} + i \sqrt{\frac{6 + 2\sqrt{2}}{21}} (3 \otimes \bar{3})^{3} \]
\[ - i \sqrt{\frac{5 - 3\sqrt{2}}{42}} (\bar{3} \otimes 3)^{3} - i \sqrt{\frac{3 + \sqrt{2}}{21}} (3 \otimes 3)^{3} \]
\[ 3 = i \sqrt{\frac{3 + \sqrt{2}}{21}} (3 \otimes 3)^{3} + i \sqrt{\frac{5 - 3\sqrt{2}}{42}} (3 \otimes \bar{3})^{3} \]
\[ + \frac{\omega e^{i\alpha - i\beta}}{\sqrt{6}} (3 \otimes \bar{1})^{3} - \frac{\omega e^{i\alpha - i\beta}}{\sqrt{6}} (3 \otimes 1)^{3} \]
\[ - i \sqrt{\frac{6 + 2\sqrt{2}}{21}} (3 \otimes 3)^{3} - i \sqrt{\frac{5 - 3\sqrt{2}}{42}} (3 \otimes 3)^{3} \]
\[ 7 \otimes 7 \rightarrow (1 + 6^{(1)} + 6^{(2)} + 7 + 8) \]
\[ 7 \otimes 7 \rightarrow 6^{(2)}: \]
\[ 3 = \frac{1}{\sqrt{7}} ((1 \otimes 3)^{3} + (3 \otimes 1)^{3}) \]
\[ + \frac{1}{14} (2\sqrt{7} + \sqrt{14}) (3 \otimes 3)^{3} \]
\[ - \frac{1}{\sqrt{7}} ((3 \otimes 3)^{3} + (3 \otimes 3)^{3}) \]
\[ + \frac{1}{14} (\sqrt{14} - 2\sqrt{7}) (3 \otimes 3)^{3} \]
\[ 3 = \frac{1}{\sqrt{7}} ((1 \otimes 3)^{3} + (3 \otimes 1)^{3}) \]
\[ + \frac{1}{14} (\sqrt{14} - 2\sqrt{7}) (3 \otimes 3)^{3} \]
\[ - \frac{1}{\sqrt{7}} ((3 \otimes 3)^{3} + (3 \otimes 3)^{3}) \]
\[ + \frac{1}{14} (2\sqrt{7} + \sqrt{14}) (3 \otimes 3)^{3} \]
\[ 7 \otimes 7 \rightarrow 6^{(1)}: \]
\[ 3 = \frac{1}{\sqrt{7}} ((1 \otimes 3)^{3} + (3 \otimes 1)^{3}) \]
\[ + \frac{1}{14} (2\sqrt{7} + \sqrt{14}) (3 \otimes 3)^{3} \]
\[ - \frac{1}{\sqrt{7}} ((3 \otimes 3)^{3} + (3 \otimes 3)^{3}) \]
\[ + \frac{1}{14} (\sqrt{14} - 2\sqrt{7}) (3 \otimes 3)^{3} \]
\[ 3 = \frac{1}{\sqrt{7}} ((1 \otimes 3)^{3} + (3 \otimes 1)^{3}) \]
\[ + \frac{1}{14} (\sqrt{14} - 2\sqrt{7}) (3 \otimes 3)^{3} \]
\[ - \frac{1}{\sqrt{7}} ((3 \otimes 3)^{3} + (3 \otimes 3)^{3}) \]
\[ + \frac{1}{14} (2\sqrt{7} + \sqrt{14}) (3 \otimes 3)^{3} \]
\[ 7 \otimes 7 \rightarrow 7: \]
\[ 1 = \sqrt{\frac{6}{7}} (1 \otimes 1)_{1} \]
\[ - \frac{1}{\sqrt{14}} ((3 \otimes 3)^{3} + (3 \otimes 3)^{3}) \]
\[ 3 = -\frac{1}{\sqrt{42}} ((1 \otimes 3)^{3} + (3 \otimes 1)^{3}) \]
\[ - \frac{2}{\sqrt{21}} (3 \otimes 3)^{3} \]
\[ - \frac{2}{\sqrt{21}} ((3 \otimes 3)^{3} + (3 \otimes 3)^{3}) \]
\[ - \frac{2}{\sqrt{21}} (3 \otimes 3)^{3} \]
\[ \bar{3} = -\frac{1}{\sqrt{42}} (\{1 \otimes \bar{3}\}_{\bar{3}} + \{\bar{3} \otimes 1\}_{\bar{3}}) \]
\[ -2\frac{\sqrt{2}}{21} \{3 \otimes 3\}_{a} \]
\[ -\frac{2}{\sqrt{21}} (\{3 \otimes 3\}_{3} + \{\bar{3} \otimes 3\}_{3}) \]
\[ -\frac{2}{\sqrt{21}} \{\bar{3} \otimes \bar{3}\}_{\bar{3}} \]

**7 \otimes 7 \rightarrow 8_{s}:**
\[ 1' = \frac{ie^{-i\alpha}}{\sqrt{2}} (\{3 \otimes \bar{3}\}_{1'} + \{\bar{3} \otimes 3\}_{1'}) \]
\[ 1 = \frac{i}{\sqrt{2}} (\{3 \otimes \bar{3}\}_{1} - \{\bar{3} \otimes 3\}_{1}) \]
\[ 3 = \frac{i}{\sqrt{6}} (\{1 \otimes \bar{3}\}_{3} - \{\bar{3} \otimes 1\}_{3}) \]
\[ -i\frac{2}{\sqrt{3}} \{3 \otimes \bar{3}\}_{a} \]
\[ 3 = -i\frac{2}{\sqrt{3}} (\{1 \otimes \bar{3}\}_{3} - \{\bar{3} \otimes 1\}_{3}) \]
\[ + i\frac{2}{\sqrt{3}} \{3 \otimes \bar{3}\}_{a} \]

**7 \otimes 7 \rightarrow 8_{a}:**
\[ 1' = \frac{e^{i\alpha}}{\sqrt{2}} (\{3 \otimes \bar{3}\}_{1} - \{\bar{3} \otimes 3\}_{1}) \]
\[ 1' = \frac{-e^{-i\alpha}}{\sqrt{2}} (\{3 \otimes \bar{3}\}_{1} - \{\bar{3} \otimes 3\}_{1}') \]
\[ 3 = \frac{2i}{\sqrt{21}} (\{1 \otimes \bar{3}\}_{3} - \{\bar{3} \otimes 1\}_{3}) \]
\[ -i\sqrt{\frac{3}{14}} (\{3 \otimes \bar{3}\}_{a} - \{\bar{3} \otimes 3\}_{a}) \]
\[ + \frac{2i}{\sqrt{21}} \{3 \otimes \bar{3}\}_{a} \]
\[ 3 = -\frac{2i}{\sqrt{21}} (\{1 \otimes \bar{3}\}_{3} - \{\bar{3} \otimes 1\}_{3}) \]
\[ -\frac{2i}{\sqrt{21}} (\{3 \otimes \bar{3}\}_{a} - \{\bar{3} \otimes 3\}_{a}) \]
\[ -i\sqrt{\frac{3}{14}} (\{3 \otimes \bar{3}\}_{a} - \{\bar{3} \otimes 3\}_{a}) \]
\[ 7 \otimes 8 \rightarrow 3 + \bar{3} + 6^{(1)} + 6^{(2)} + 7^{(1)} + 7^{(2)} + 8^{(2)} + 8^{(3)} \]

**7 \otimes 8 \rightarrow 3:**
\[ 3 = \frac{1}{\sqrt{7}} (\{1 \otimes \bar{3}\}_{3} - \{\bar{3} \otimes 1\}_{\bar{3}}) \]
\[ + \frac{ie^{-i\alpha}}{2\sqrt{2}} \{3 \otimes 1'\}_{\bar{3}} \]
\[ + \frac{ie^{-i\alpha}}{2\sqrt{2}} \{3 \otimes \bar{1}'\}_{\bar{3}} - \frac{3}{2\sqrt{14}} \{3 \otimes 3\}_{3} \]
\[ + \frac{1}{2\sqrt{14}} \{3 \otimes 3\}_{3} + \frac{1}{2\sqrt{14}} \{\bar{3} \otimes 3\}_{3} \]
\[ + \frac{3}{2\sqrt{14}} \{\bar{3} \otimes \bar{3}\}_{a} - \frac{1}{2\sqrt{7}} \{3 \otimes \bar{3}\}_{3} \]
\[7 \otimes 8 \rightarrow 3:\]
\[
\mathbf{3} = \frac{1}{\sqrt{7}} \{1 \otimes \mathbf{3}\}_3 + \frac{3}{2\sqrt{7}} \{3 \otimes \mathbf{3}\}_3a
- \frac{i\omega e^{\text{i}\alpha - 2i\beta}}{2\sqrt{2}} \{3 \otimes \mathbf{1}'\}_3
- \frac{3\sqrt{2} - 2}{4\sqrt{7}} \{3 \otimes \mathbf{3}\}_3
- \frac{i\omega e^{\text{i}2\beta - \text{i}2\beta}}{2\sqrt{2}} \{3 \otimes \mathbf{1}'\}_3
- \frac{3\sqrt{2} - 2}{4\sqrt{7}} \{3 \otimes \mathbf{3}\}_3
+ \frac{1}{2\sqrt{14}} \{3 \otimes \mathbf{3}\}_3a - \frac{3}{2\sqrt{14}} \{3 \otimes \mathbf{3}\}_3a
\]

\[7 \otimes 8 \rightarrow 6^{(1)}:\]
\[
\mathbf{3} = \frac{1}{\sqrt{7}} \{1 \otimes \mathbf{3}\}_3 + \frac{3}{2\sqrt{7}} \{3 \otimes \mathbf{3}\}_3a
- \frac{i\omega e^{\text{i}2\beta - \text{i}2\beta}}{2\sqrt{2}} \{3 \otimes \mathbf{1}'\}_3
- \frac{3\sqrt{2} - 2}{4\sqrt{7}} \{3 \otimes \mathbf{3}\}_3
- \frac{i\omega e^{\text{i}2\beta - \text{i}2\beta}}{2\sqrt{2}} \{3 \otimes \mathbf{1}'\}_3
- \frac{3\sqrt{2} - 2}{4\sqrt{7}} \{3 \otimes \mathbf{3}\}_3
+ \frac{1}{2\sqrt{14}} \{3 \otimes \mathbf{3}\}_3a - \frac{3}{2\sqrt{14}} \{3 \otimes \mathbf{3}\}_3a
\]

\[7 \otimes 8 \rightarrow 6^{(2)}:\]
\[
\mathbf{3} = \frac{i}{\sqrt{7}} \{1 \otimes \mathbf{3}\}_3 - \frac{\omega e^{\text{i}2\beta - \text{i}2\beta}}{2\sqrt{2}} \{3 \otimes \mathbf{1}'\}_3
+ \frac{\omega e^{\text{i}2\beta - \text{i}2\beta}}{2\sqrt{2}} \{3 \otimes \mathbf{1}'\}_3 - \frac{i(\sqrt{2} - 2)}{4\sqrt{7}} \{3 \otimes \mathbf{3}\}_3
- \frac{i(2 + 3\sqrt{2})}{4\sqrt{7}} \{3 \otimes \mathbf{3}\}_3
+ \frac{i(\sqrt{2} - 4)}{4\sqrt{7}} \{3 \otimes \mathbf{3}\}_3
+ \frac{3i}{2\sqrt{14}} \{3 \otimes \mathbf{3}\}_3a
\]

\[7 \otimes 8 \rightarrow 7^{(2)}:\]
\[
1 = \frac{i}{\sqrt{2}} \{3 \otimes \mathbf{3}\}_1 - \frac{i}{\sqrt{2}} \{\bar{3} \otimes \mathbf{3}\}_1
\]
\[
3 = \frac{i}{\sqrt{6}} \{1 \otimes \mathbf{3}\}_3 + \frac{1}{\sqrt{6}} \{3 \otimes \mathbf{3}\}_3
- \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3 + \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3
- \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3 - \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3
+ \frac{i\sqrt{3}}{4} \{3 \otimes \mathbf{3}\}_3a
- \frac{i\sqrt{3}}{4} \{3 \otimes \mathbf{3}\}_3a
\]

\[7 \otimes 8 \rightarrow 7^{(1)}:\]
\[
1 = \frac{1}{\sqrt{2}} \{3 \otimes \mathbf{3}\}_1 + \frac{1}{\sqrt{2}} \{\bar{3} \otimes \mathbf{3}\}_1
\]
\[
3 = \frac{1}{\sqrt{6}} \{1 \otimes \mathbf{3}\}_3 + \frac{1}{\sqrt{6}} \{3 \otimes \mathbf{3}\}_3
- \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3 + \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3
- \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3 - \frac{1}{\sqrt{3}} \{3 \otimes \mathbf{1}'\}_3
+ \frac{1}{\sqrt{6}} \{3 \otimes \mathbf{3}\}_3a
\]

\[7 \otimes 8 \rightarrow 8^{(1)}:\]
\[
1' = \frac{3\sqrt{5} - 4i}{\sqrt{91}} \{1 \otimes \mathbf{1}'\}_1 - 2\frac{6}{\sqrt{91}} \{3 \otimes \mathbf{3}\}_1
- 2\frac{6}{\sqrt{91}} \{3 \otimes \mathbf{3}\}_1
\]
\[ i' = \frac{3\sqrt{3} + 4i}{\sqrt{91}} \{1 \otimes i'\}_1 - 2\frac{\sqrt{6}}{91} \{3 \otimes 3\}_1, \]
\[ - 2\frac{6}{91} \{3 \otimes 3\}_1. \]
\[ 3 = - \frac{\sqrt{3}}{91} \{1 \otimes 3\}_3 + 2\frac{\sqrt{2}}{91} \omega^2 \{3 \otimes 1'\}_3 + 2\frac{6}{91} \{3 \otimes 3\}_3, \]
\[ - 2\frac{6}{91} \{3 \otimes 3\}_3 + 2\frac{3}{91} \{3 \otimes 3\}_3, \]
\[ - 2\frac{3}{91} \{3 \otimes 3\}_3. \]

- \[ 7 \otimes 8 \rightarrow 8^{(3)}; \]
\[ 1' = - e^{-i\alpha} \omega \{3 \otimes 3\}_1 + e^{-i\alpha} \omega^2 \{3 \otimes 3\}_1, \]
\[ i' = \frac{e^{i\alpha} \omega}{\sqrt{2}} \{3 \otimes 3\}_1 - \frac{e^{i\alpha} \omega^2}{\sqrt{2}} \{3 \otimes 3\}_1, \]
\[ 3 = - \frac{2i}{\sqrt{21}} \{1 \otimes 3\}_3 - \frac{e^{i\alpha} \omega}{\sqrt{6}} \{3 \otimes 1'\}_3 + \frac{e^{-i\alpha} \omega^2}{\sqrt{6}} \{3 \otimes 1'\}_3 - \frac{3}{14} \{3 \otimes 3\}_3, \]
\[ + \frac{3}{14} \{3 \otimes 3\}_3 + \frac{e^{i\alpha} \omega}{\sqrt{6}} \{3 \otimes 3\}_3, \]
\[ - \frac{3}{14} \{3 \otimes 3\}_3. \]
\[ 8 \otimes 8 \rightarrow (1 + 6^{(4)} + 6^{(2)} + 7 + 8^{(1)} + 8^{(2)})_s + (3 + \bar{3} + 7^{(1)} + 7^{(2)} + \bar{8})_a, \]

- \[ 8 \otimes 8 \rightarrow 1; \]
\[ 1 = \frac{1}{2\sqrt{2}} \{(1' \otimes 1')_1 + (1' \otimes 1')_1 \}
\[ + \frac{\sqrt{3}}{2} \{(3 \otimes 3)_1 + (3 \otimes 3)_1 \}, \]

- \[ 8 \otimes 8 \rightarrow 6^{(1)}; \]
\[ 3 = \frac{i e^{i\theta}}{2\sqrt{2}} \{(\{1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \}
\[ - \frac{i e^{-i\theta}}{2\sqrt{2}} \{(1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \}
\[ + \sqrt{\frac{1}{14}} (3 - \sqrt{7}) \{3 \otimes 3\}_3, \]
\[ - \frac{1}{2} \sqrt{\frac{1}{14}} (5 + 3\sqrt{2}) \{(3 \otimes 3)_3 + \{3 \otimes 3\}_3 \}
\[ + \frac{1}{2} \sqrt{\frac{1}{7}} (3 - \sqrt{2}) \{3 \otimes 3\}_3, \]}
\[ 3 = \frac{ie^{i\beta}}{2\sqrt{2}} \left( \{1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \right) \]
\[ - \frac{ie^{-i\beta}}{2\sqrt{2}} \left( \{1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \right) \]
\[ + \frac{1}{2} \sqrt{\frac{1}{7} \left( 3 - \sqrt{2} \right)} \{3 \otimes 3\}_3 \]
\[ - \frac{1}{2} \sqrt{\frac{1}{14} \left( 5 + 3\sqrt{2} \right)} \left( \{3 \otimes 3\}_3 + \{3 \otimes 3\}_3 \right) \]
\[ + \sqrt{\frac{1}{14} \left( 3 - \sqrt{2} \right)} \{3 \otimes 3\}_3 \]

\[ \bullet 8 \otimes 8 \rightarrow 6_s^{(2)}:\]
\[ 3 = \frac{\omega^2 e^{i\alpha - i\beta}}{2\sqrt{2}} \left( \{1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \right) \]
\[ + \frac{\omega e^{i\beta - i\alpha}}{2\sqrt{2}} \left( \{1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \right) \]
\[ + i \sqrt{\frac{1}{14} \left( 3 + \sqrt{2} \right)} \{3 \otimes 3\}_3 \]
\[ + \frac{1}{2} \sqrt{\frac{1}{14} \left( 5 - 3\sqrt{2} \right)} \left( \{3 \otimes 3\}_3 + \{3 \otimes 3\}_3 \right) \]
\[ + \frac{1}{2} \sqrt{\frac{1}{7} \left( 3 + \sqrt{2} \right)} \{3 \otimes 3\}_3 \]

\[ \bullet 8 \otimes 8 \rightarrow 8_s^{(1)}:\]
\[ 1' = -\frac{\omega^2}{\sqrt{3}} \left( \overline{1'} \otimes \overline{1'} \right)_1' \]
\[ - \frac{i}{\sqrt{3}} \left( \{3 \otimes 3\}_1' + \{3 \otimes 3\}_1' \right) \]
\[ \overline{1'} = -\frac{\omega}{\sqrt{3}} \left( 1' \otimes 1' \right)_1' \]
\[ + \frac{i}{\sqrt{3}} \left( \{3 \otimes 3\}_1' + \{3 \otimes 3\}_1' \right) \]

\[ \bullet 8 \otimes 8 \rightarrow 7_s:\]
\[ 1 = \sqrt{\frac{2}{7}} \left( \{1' \otimes 1'\}_1 + \{1' \otimes 1'\}_1 \right) \]
\[ - \frac{1}{2\sqrt{2}} \left( \{3 \otimes 3\}_1 + \{3 \otimes 3\}_1 \right) \]

\[ \bullet 8 \otimes 8 \rightarrow 7_s:\]
\[ \overline{3} = \frac{ie^{i\beta}}{2\sqrt{3}} \left( \{1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \right) \]
\[ - \frac{ie^{-i\beta}}{2\sqrt{3}} \left( \{1' \otimes 3\}_3 + \{3 \otimes 1'\}_3 \right) \]
\[ + \frac{1}{2} \sqrt{\frac{1}{7} \left( 3 - \sqrt{2} \right)} \{3 \otimes 3\}_3 \]
\[ - \frac{1}{2} \sqrt{\frac{1}{2\sqrt{3}} \left( 5 + 3\sqrt{2} \right)} \left( \{3 \otimes 3\}_3 + \{3 \otimes 3\}_3 \right) \]
\[ + \sqrt{\frac{1}{14} \left( 3 - \sqrt{2} \right)} \{3 \otimes 3\}_3 \]

\[ \bullet 8 \otimes 8 \rightarrow 8_s^{(1)}:\]
\[ 1' = -\frac{\omega^2}{\sqrt{3}} \left( \overline{1'} \otimes \overline{1'} \right)_1' \]
\[ - \frac{i}{\sqrt{3}} \left( \{3 \otimes 3\}_1' + \{3 \otimes 3\}_1' \right) \]
\[ \overline{1'} = -\frac{\omega}{\sqrt{3}} \left( 1' \otimes 1' \right)_1' \]
\[ + \frac{i}{\sqrt{3}} \left( \{3 \otimes 3\}_1' + \{3 \otimes 3\}_1' \right) \]

\[ \bullet 8 \otimes 8 \rightarrow 7_s:\]
\[ 1 = \sqrt{\frac{2}{7}} \left( \{1' \otimes 1'\}_1 + \{1' \otimes 1'\}_1 \right) \]
\[ - \frac{1}{2\sqrt{2}} \left( \{3 \otimes 3\}_1 + \{3 \otimes 3\}_1 \right) \]

\[ \bullet 8 \otimes 8 \rightarrow 7_s:\]
\(8 \otimes s \rightarrow s^{(2)}_a:\)

\[
1' = \sqrt{\frac{2}{3}} e^{i\alpha} \{I' \otimes I\}
- \frac{i e^{-i\alpha} \omega}{\sqrt{6}} \left((3 \otimes \bar{3})_{1a} + (\bar{3} \otimes 3)_{1a}\right)
\]

\[
\bar{I}' = \sqrt{\frac{2}{3}} e^{i\alpha} \{I' \otimes I'\}
+ \frac{i e^{i\alpha} \omega}{\sqrt{6}} \left((3 \otimes 3)_{1a} + (\bar{3} \otimes \bar{3})_{1a}\right)
\]

\[
3 = \frac{i e^{i\alpha} \omega^2}{3\sqrt{2}} \left\{\{1' \otimes 3\}_{3a} + \{3 \otimes 1'\}_{3a}\right\}
- \frac{i e^{-i\alpha} \omega}{3\sqrt{2}} \left\{\{\bar{I}' \otimes 3\}_{3a} + \{3 \otimes \bar{I}'\}_{3a}\right\}
- \frac{2\sqrt{3}}{3} \{3 \otimes 3\}_{3a}
- \frac{2\sqrt{3}}{3} \left\{\{3 \otimes \bar{3}\}_{3a} + \{\bar{3} \otimes 3\}_{3a}\right\}
+ \frac{5}{3\sqrt{7}} \{\bar{3} \otimes \bar{3}\}_{3a}
\]

\(8 \otimes s \rightarrow s_{3a}:\)

\[
3 = \frac{1}{2\sqrt{2}} \left\{\{1' \otimes 3\}_{3a} - \{3 \otimes 1'\}_{3a}\right\}
- \frac{\omega}{2\sqrt{2}} \left\{\{\bar{I}' \otimes 3\}_{3a} - \{3 \otimes \bar{I}'\}_{3a}\right\}
+ \frac{i \omega^2}{2} \left\{\{3 \otimes 3\}_{3a} - \{\bar{3} \otimes \bar{3}\}_{3a}\right\}
- \frac{i \omega^2}{2\sqrt{6}} \{3 \otimes 3\}_{3a}
\]

\(8 \otimes s \rightarrow \bar{r}_{a}^{(1)}:\)

\[
1 = \frac{i}{\sqrt{2}} \left\{\{1' \otimes \bar{I}'\}_{1a} - \{\bar{I}' \otimes 1'\}_{1a}\right\}
\]

\[
3 = \frac{\omega}{4} \left\{\{1' \otimes 3\}_{3a} - \{3 \otimes 1'\}_{3a}\right\}
+ \frac{\omega^2}{4} \left\{\{\bar{I}' \otimes 3\}_{3a} - \{3 \otimes \bar{I}'\}_{3a}\right\}
+ \frac{\sqrt{3}}{4} \left\{\{3 \otimes 3\}_{3a} - \{\bar{3} \otimes \bar{3}\}_{3a}\right\}
- \frac{\sqrt{3}}{2} \{3 \otimes \bar{3}\}_{3a}
\]

\(8 \otimes s \rightarrow \bar{r}_{a}^{(2)}:\)

\[
1 = \frac{i}{\sqrt{2}} \left\{\{3 \otimes 3\}_{1a} - \{\bar{3} \otimes 3\}_{1a}\right\}
\]

\[
3 = -\frac{1}{4} \sqrt{\frac{7}{3}} e^{i\alpha} \omega \left\{\{1' \otimes 3\}_{3a} - \{3 \otimes 1'\}_{3a}\right\}
+ \frac{1}{4} \sqrt{3} e^{-i\alpha} \omega^2 \left\{\{\bar{I}' \otimes 3\}_{3a} - \{3 \otimes \bar{I}'\}_{3a}\right\}
+ \frac{i \sqrt{3}}{4} \left\{\{3 \otimes 3\}_{3a} - \{\bar{3} \otimes \bar{3}\}_{3a}\right\}
- \frac{i}{2\sqrt{6}} \{3 \otimes 3\}_{3a}
\]
\[3 = \frac{1}{4} \sqrt{\frac{7}{3}} e^{i\omega} \left( \{1' \otimes 3\}_3 - \{3 \otimes 1'\}_3 \right) \]
\[ - \frac{1}{4} \sqrt{\frac{7}{3}} e^{-i\omega} \left( \{\bar{1}' \otimes \bar{3}\}_3 - \{\bar{3} \otimes \bar{1}'\}_3 \right) \]
\[ + \frac{i}{2\sqrt{6}} \left( \{3 \otimes 3\}_3 \right) \]
\[ + \frac{i}{\sqrt{3}} \left( \{\bar{3} \otimes \bar{3}\}_3 - \{\bar{3} \otimes \bar{3}\}_3 \right) \]

\[ \bullet \ 8 \otimes 8 \rightarrow 8_a: \]
\[1' = -\frac{\omega}{\sqrt{2}} \left( \{3 \otimes \bar{3}\}_1' - \{\bar{3} \otimes 3\}_1' \right) \]
\[\bar{1}' = \frac{\omega^2}{\sqrt{2}} \left( \{3 \otimes \bar{3}\}_{\bar{1}}' - \{\bar{3} \otimes 3\}_{\bar{1}}' \right) \]
\[\bar{3} = \frac{\omega^2}{\sqrt{6}} \left( \{1' \otimes \bar{3}\}_3 - \{\bar{3} \otimes 1'\}_3 \right) \]
\[ - \frac{\omega}{\sqrt{6}} \left( \{1' \otimes \bar{3}\}_3 - \{\bar{3} \otimes 1'\}_3 \right) \]
\[ - \frac{i}{\sqrt{3}} \left( \{3 \otimes 3\}_3 \right) \]
\[+ \frac{i}{\sqrt{3}} \left( \{\bar{3} \otimes \bar{3}\}_3 \right) \]

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