REDUCTION mod $p$ OF CUSPIDAL REPRESENTATIONS OF $GL_2(F_{p^n})$ AND SYMMETRIC POWERS

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Abstract. We show the existence of integral models for cuspidal representations of $GL_2(F_{p^n})$, whose reduction modulo $p$ can be identified with the cokernel of a differential operator on $F_q[X,Y]$ defined by J-P. Serre. These integral models come from the crystalline cohomology of the projective curve $XY^q - X^qY - Z^{q+1} = 0$.

As an application, we can extend a construction of C. Khare and B. Edixhoven (\cite{5}) giving a cohomological analogue of the Hasse invariant operator acting on spaces of mod $p$ modular forms for $GL_2$.

1. Introduction

Let $q = p^n$ be a power of a prime number $p$; the irreducible complex representations of $G = GL_2(F_q)$ (of dimension greater than one) that are not twists of the Steinberg representation are of two types: the principal series representations, having dimension $q+1$ and obtained by inducing to $G$ characters of the Borel subgroup of $G$, and the cuspidal representations, having dimension $q-1$ and characterized by the property that they do not occur as a factor of a principal series.

The dimensions of the cuspidal representations and of the principal series representations of $G$ appear in the study of the modular representations of the group: for $k \geq 0$ let $V_k = \text{Sym}^k F_q^2$ be the $k$th symmetric power of the standard left representation of $G$ over the field $F_q$; in \cite{13}, J-P. Serre noted the following identity in the Grothendieck group of $G$ (cf. Proposition 2.3):

$$V_k - e \cdot V_{k-(q+1)} = V_{k-(q-1)} - e \cdot V_{k-2q} \quad (k \in \mathbb{Z}),$$

where the definition of the $V_k$'s (as elements of the Grothendieck group of $G$) has been extended in a suitable way for $k < 0$ (cf. Definition 2.2), and $e$ denotes the character determinant.

A look at the left hand side of the above relation leads naturally to consider the $F_q[G]$-equivariant map:

$$\theta_q : e \otimes V_{k-(q+1)} \rightarrow V_k$$

defined, for any $k > q$, as multiplication by the Dickson invariant $X^qY - XY^q \in F_q[X,Y]$ (cf. \cite{8}) and realizing $e \otimes V_{k-(q+1)}$ as a subobject of $V_k$. The cokernel of this map has dimension $q+1$ and turns out to be the reduction mod $p$ of a principal series representation of $G$: $\text{coker} \theta_q \simeq \text{Ind}^G_B (\eta^k)$, where $B \leq G$ is the subgroup of upper triangular matrices, and $\eta$ is the character of $B$ given by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a$. In particular, one deduces the following periodicity result (cf. Proposition 2.7):
Proposition. For any integer $k > q$ and any $\lambda \geq 0$ there are isomorphisms of $G$-modules:

$$\frac{V_k}{e \otimes V_{k-(q+1)}} \simeq \text{Ind}_\mathcal{B}^G(\eta^k) \simeq \frac{V_{k+\lambda(q-1)}}{e \otimes V_{k+\lambda(q-1)-(q+1)}}.$$ 

It seems natural to expect that the other ”period” $q-1$ of the group $G$ is associated to reduction mod $p$ of cuspidal representations of $G$. At this purpose, in response to a message of C. Khare, Serre defined a $G$-map $D : V_k \to V_{k+(q-1)}$ that raises the power of the symmetric module $V_k$ by $q-1$. This function is defined as a derivation map of the algebra $\mathbb{F}_q[X,Y]$, as follows:

$$D : \mathbb{F}_q[X,Y] \to \mathbb{F}_q[X,Y] : \quad f \mapsto X^q \frac{\partial f}{\partial X} + Y^q \frac{\partial f}{\partial Y}.$$ 

The derivation map $D$ in not always injective (cf. Proposition 3.3) and seems to capture essential properties related to the existence or non-existence of embeddings of $G$-modules of the form $V_k \to V_{k+(q-1)}$ (cf. Proposition 3.5).

In this paper we show how the cokernel of the $D$-map can be identified with the reduction of a cuspidal representation of $G$ over the field $\overline{\mathbb{F}}_p$, more precisely, in sections 3 and 4 we consider, for $2 \leq k \leq p-1$, the exact sequence of $G$-modules (Proposition 4.1):

$$0 \to e \otimes V_{k-2} \xrightarrow{\overline{\eta_k}} \frac{V_{k+(q-1)}}{D(V_k)} \to \text{coker} \overline{\theta} \to 0$$

and we identify it with an exact sequence constructed by B. Haastert and J.C. Jantzen in [19], and coming from the crystalline cohomology of the Deligne-Lusztig variety of the group $SL_2/\mathbb{F}_q$, i.e. the smooth projective curve $C/\mathbb{F}_q : XY^q - X^qY - Z^{q+1} = 0$. We then deduce the main result of the paper, analogous to the periodicity result for reduction of principal series representations of $G$ (cf. Theorem 4.2):

**Theorem.** Let $q > 2$, $2 \leq k \leq p-1$ with $k \neq \frac{2p-1}{2}$ and let us denote by $\Theta(\chi^k)$ the cuspidal $\overline{\mathbb{F}}_p$-representation of $G$ associated to the $k$th-power of the Teichmüller character $\chi$. Then there exists a natural integral model $\widetilde{\Theta}(\chi^k)$ of $\Theta(\chi^k)$ coming from the $-k$-eigenspace of the first crystalline cohomology group of the projective curve $C/\mathbb{F}_q$, such that there is an isomorphism of $\mathbb{F}_q[G]$-modules:

$$\frac{V_{k+(q-1)}}{D(V_k)} \simeq \widetilde{\Theta}(\chi^k) \pmod{p}.$$ 

In the last section of the paper, we switch to modular forms mod $p$ for $GL_2$: in this context, thanks to the Eichler-Shimura isomorphism, the modules $V_k$'s appear in the study of the systems of eigenforms for the action of the Hecke algebra on the space of modular forms $M_k(\Gamma_1(N), \mathbb{F}_p)$. As a consequence, the maps $\theta_p$ and $D$ considered above play a role too: in [11], A. Ash and G. Stevens identify a group theoretical analogue of the theta operator in the map induced in cohomology by the Dickson homomorphism $\theta_p$; in [15], B. Edixhoven and C. Khare build a cohomological analogue of the Hasse invariant operator as a map $\alpha : H^1(\Gamma_1(N), V_0) \to H^1(\Gamma_1(N), V_{p-1})$. The existence of the map $D$ might lead to further results in this direction.
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2. Reduction of principal series representations

We introduce some notation that we will be using throughout the paper: let $q = p^n > 1$ be an integral power of a prime number $p$ and let $\mathbb{F} = \mathbb{F}_q$ be a field of cardinality $q$, with algebraic closure $\mathbb{F}$. $G = GL_2(\mathbb{F})$ will denote the group of $2 \times 2$ non-singular matrices with coefficients in $\mathbb{F}$, and $V = \mathbb{F}^2$ the standard left $G$-representation.

2.1. Definition of the $G$-modules $V_k$. For any non-negative integer $k$ let $V_k = \text{Sym}^k V$ be the symmetric $k$th-power of the $G$-representation $V$: $V_k$ will be identified with the $\mathbb{F}$-vector space $\mathbb{F}[X,Y]_k$ of homogeneous polynomials over $\mathbb{F}$ in two variables and of degree $k$, endowed with the left action of $G$ induced by:

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot X = aX + cY, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot Y = bX + dY.$$

It is possible to extend the definition of the $V_k’s$ as elements of the Grothendieck group of $G$ - to the negative integers: this is done by considering (a variant of) the Euler-Poincaré characteristic of the twists $\mathcal{O}(k)$ of the structure sheaf on $\mathbb{P}^1_{\mathbb{F}}$ (cf. Serre, [18]). In order to justify the presence of this sheaf, we will start by considering the case of the algebraic group $GL_2$.

Let $G = GL_2$ as an algebraic group over $\mathbb{F}$, and let $T \subset G$ be the maximal split torus of diagonal matrices. We identify the character group $X(T)$ of $T$ with $\mathbb{Z}^2$ in the usual way, so that the roots associated to the pair $(G,T)$ are $(1,-1)$ and $(-1,1)$; we fix a choice of positive root $\alpha = (1,-1)$. The corresponding Borel subgroup $B$ is the group of upper triangular matrices in $G$; we denote by $B^-$ the opposite Borel subgroup.

For a fixed $\lambda \in X(T)$, let $M_\lambda$ be the one dimensional left $B^-$-module on which $B^-$ acts (through its quotient $T$) via the character $\lambda$. Denote by $\text{ind}_{B^-}^G M_\lambda$ the left $G$-module given by algebraic induction from $B^-$ to $G$ of $M_\lambda$:

$$\text{ind}_{B^-}^G M_\lambda := \{ f \in \text{Mor}(G,G_a) : f(bg) = \lambda(b)f(g), \text{ for all } g \in G, b \in B^- \},$$

where $\text{Mor}(B^-, G_a)$ denotes the set of morphisms of functors from $B^-$ to $G_a$, and the left action of $G$ on the above space is given by letting $(gf)(x) = f(xg)$ for all $x,g \in G$ and $f \in \text{ind}_{B^-}^G M_\lambda$. 
Define the following generalization of the dual Weyl module for $\lambda$ (cf. [14], II.5):

$$W(\lambda) := \sum_{i \geq 0} (-1)^i \cdot R^i \text{ind}_{\mathbb{B}^-}^G (M_\lambda),$$

where $R^i \text{ind}_{\mathbb{B}^-}^G$ denote the $i^{th}$ right derived functor of the functor $\text{ind}_{\mathbb{B}^-}^G$.

Notice that $W(\lambda)$ is a well defined element of the Grothendieck group $K_0(\mathbb{G})$ of $\mathbb{G}$, because each $R^i \text{ind}_{\mathbb{B}^-}^G (M_\lambda)$ is a finite dimensional $\mathbb{G}$-module, and $R^i \text{ind}_{\mathbb{B}^-}^G (M_\lambda)$ is zero for $i > 1$ ([14], II.4.2).

By [14], I.5.12, we have for each $i \geq 0$ a canonical isomorphism of $\mathbb{G}$-modules:

$$R^i \text{ind}_{\mathbb{B}^-}^G (M_\lambda) \simeq H^i(\mathbb{B}^- \setminus \mathbb{G}, \mathcal{L}(M_\lambda)),$$

where $\mathcal{L}(M_\lambda)$ is the sheaf on the projective scheme $\mathbb{B}^- \setminus \mathbb{G} \simeq \mathbb{P}^1$ associated to $M_\lambda$, i.e. for an open subset $U \subseteq \mathbb{P}^1$ we have:

$$\mathcal{L}(M_\lambda)(U) := \{ f \in \text{Mor}(\pi^{-1}U, \mathbb{G}_a) : f(bx) = \lambda(b)f(x), \text{ for all } x \in \pi^{-1}U, b \in \mathbb{B}^- \},$$

where $\pi : \mathbb{G} \to \mathbb{B}^- \setminus \mathbb{G} \simeq \mathbb{P}^1$ is the canonical projection (the $\mathbb{G}$-module structure on $H^i(\mathbb{B}^- \setminus \mathbb{G}, \mathcal{L}(M_\lambda))$ comes from the fact that $\mathcal{L}(M_\lambda)$ is a $\mathbb{G}$-linearized sheaf, cf. Remark in [14], I.5.12).

Let us fix $\lambda_k \in X(\mathbb{T})$ corresponding to the pair of integers $(k, 0)$ ($k \in \mathbb{Z}$); we have an isomorphism of sheaves on $\mathbb{P}^1$ (cf. [14] II.4.3.):

$$\mathcal{L}(M_{\lambda_k}) \simeq \mathcal{O}(k).$$

To see this, let us work for simplicity over $\mathbb{F}$; the canonical projection $\pi$ is given by:

$$(x_1 \ x_2 \ x_3 \ x_4) \mapsto (x_1 : x_2) \in \mathbb{P}^1.$$

Denote by $\omega : \mathbb{B}^- \to \mathbb{F}^\times$ the character associated to the weight $\lambda = (1, 0) \in X(\mathbb{T})$, and let $N := \text{Ker} \omega$. Fix an open subset $U$ of $\mathbb{P}^1$; by definition of the sheaf $\mathcal{L}(M_\lambda)$, if $f \in \mathcal{L}(M_\lambda)(U)$ then $f(b \cdot x) = \omega(b)f(x)$ for all $x \in \pi^{-1}U, b \in \mathbb{B}^-$, and $f$ is left $N$-invariant. We make the identification of sets $N \backslash \pi^{-1}U \simeq U$ through the map $Nx \mapsto e_1 \cdot x$, where $e_1 = (1, 0)$ and $x \in \pi^{-1}U$. We deduce the existence of an isomorphism of $\mathcal{L}(M_\lambda)(U)$ with the group:

$$\{ f \in \text{Mor}(U, \mathbb{F}) : f(e_1 \cdot b \cdot x) = \omega(b)^k f(e_1 \cdot x), \text{ for all } x \in \pi^{-1}U, b \in \mathbb{B}^- \}.$$

Since $e_1 \cdot b = \omega(b) e_1$, we deduce:

$$\mathcal{L}(M_\lambda)(U) \simeq \{ f \in \text{Mor}(U, \mathbb{F}) : f(av) = a^k f(v), \text{ for all } v \in U, a \in \mathbb{F} \} \simeq \mathcal{O}(k)(U).$$

We conclude the existence of isomorphisms of $\mathbb{G}$-modules $H^i(\mathbb{B}^- \setminus \mathbb{G}, \mathcal{L}(M_{\lambda_k})) \simeq H^i(\mathbb{P}^1, \mathcal{O}(k))$ for every $i \geq 0$.

If $k \geq 0$, $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$ so that $W(\lambda_k) = H^0(\mathbb{P}^1, \mathcal{O}(k)) = \text{Sym}^k \mathbb{F}^2$ in $K_0(\mathbb{G})$; if $k < 0$ we have $H^0(\mathbb{P}^1, \mathcal{O}(k)) = 0$ and $W(\lambda_k) = -H^1(\mathbb{P}^1, \mathcal{O}(k))$; the canonical perfect pairing of $\mathbb{G}$-modules:

$$H^0(\mathbb{P}^1, \mathcal{O}(-k - 2)) \times H^1(\mathbb{P}^1, \mathcal{O}(k)) \to H^1(\mathbb{P}^1, \mathcal{O}(-2)) \simeq \text{det}^{-1} \otimes \mathbb{G}_a.$$
(cf. [10], III.5) gives the following equalities in $K_0(G)$:

$$W(\lambda_k) = -\text{Hom}_G(H^0(\mathbb{P}^1, \mathcal{O}(-k-2)), \det^{-1} \otimes G_a) =$$

$$= -\text{Hom}_G((\det \otimes H^0(\mathbb{P}^1, \mathcal{O}(-k-2)), G_a) =$$

$$= - (\det \otimes H^0(\mathbb{P}^1, \mathcal{O}(-k-2)))^* =$$

$$= -e^{1+k} \cdot \text{Sym}^{-k-2} \mathbb{F}^2,$$

where $\cdot$ denotes the product induced in $K_0(G)$ by the tensor product of $G$-modules, and $e$ is the determinant character.

**Remark 2.1.** In more general settings, let $G$ be a connected split reductive group over $\mathbb{F}$; the vanishing of the modules $R^i \text{Ind}_{B}^{G}(\mathbb{M}_{\lambda})$ for $i > 0$ whenever $\lambda$ is a dominant weight for $(G, T)$ is the content of Kempf’s vanishing Theorem (cf. [14], II.4); on the other side $\lambda \in X(T)$ is dominant if and only if $\text{Ind}_{B}^{G}(\mathbb{M}_{\lambda}) \neq 0$ (cf. [14], II.2.6). In our case, with the choice of positive root $\alpha$, $\lambda_k$ is dominant if and only if $k \geq 0$.

The above considerations on the generalized Weyl modules $W(\lambda_k)$ for $G$ lead to the following definition for the finite group $G$:

**Definition 2.2.** For any integer $k$, define the element $V_k$ of the Grothendieck group $K_0(G)$ of $G$ over $\mathbb{F}$ by:

$$V_k := \begin{cases} 
\text{Sym}^k \mathbb{F}^2 & \text{if } k \geq 0 \\
0 & \text{if } k = -1 \\
e^{1+k} \cdot V_{-k-2} & \text{if } k \leq -2
\end{cases}$$

Notice that the definition of $V_k$ for $k < 0$ is also suggested by the expression of the Brauer character of the representation $V_{-k}$ (cf. next paragraph).

We remark that in the rest of the paper, for any non-negative integer $k$, the symbol $V_k$ will refer sometimes to the $G$-module $\text{Sym}^k \mathbb{F}^2$ and sometimes to its image in $K_0(G)$: the meaning of the symbol $V_k$ will be clear from the context.

**2.2. An identity in $K_0(G)$.** Let us fix an embedding $\iota : \mathbb{F}_q^2 \hookrightarrow M_2(\mathbb{F})$, corresponding to a choice of $\mathbb{F}$-basis for the degree 2 extension of $\mathbb{F}$ inside $\overline{\mathbb{F}}$. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of the $p$-adic field $\mathbb{Q}_p$ and let us fix an isomorphism between $\overline{\mathbb{F}}$ and the residue field of the ring of integers $\mathbb{Z}_p$ of $\overline{\mathbb{Q}}_p$; denoting by $\chi : \mathbb{F}_q^\times \rightarrow \mu(\mathbb{Z}_p)$ the corresponding Teichmüller lifting, one can compute the Brauer character $G_{\text{reg}} \rightarrow \overline{\mathbb{Q}}_p^\times$ of the representations $V_k$ ($k \geq 1$) and get:

$$\begin{pmatrix} a & 0 \\
0 & a \end{pmatrix} \mapsto (k+1)\chi(a)^k, \quad a \in \mathbb{F}_q^\times$$

$$\begin{pmatrix} a & 0 \\
0 & b \end{pmatrix} \mapsto \frac{\chi(a)^{k+1} - \chi(b)^{k+1}}{\chi(a) - \chi(b)}, \quad a, b \in \mathbb{F}_q^\times, a \neq b$$

$$\iota(c) \mapsto \frac{\chi(c)^{q(k+1)} - \chi(c)^{k+1}}{\chi(c)^q - \chi(c)} , \quad c \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^\times.$$
Proposition 2.3. The following relation holds in $K_0(G)$, for any integer $k$:

\[ V_k - e \cdot V_{k-(q+1)} = V_{k-(q-1)} - e \cdot V_{k-2q}. \]

More can be shown: let us assume $k > q$ and let $\theta_q = X^qY - XY^q \in \mathbb{F}[X,Y]_{q+1}$. (Notice this polynomial naturally appears in the classical theory of the $SL_2(\mathbb{F})$-invariants of the symmetric algebra $\text{Sym}^* \mathbb{F}^2$; by [4], the ring of $SL_2(\mathbb{F})$-invariants in $\text{Sym}^* \mathbb{F}^2$ is generated by the two polynomials

\[ \theta_q = \det \begin{pmatrix} X^q & Y^q \\ X & Y \end{pmatrix} \text{ and } \det \begin{pmatrix} X^{q^2} & Y^{q^2} \\ X & Y \end{pmatrix} / \theta_q. \]

Let us denote by $\theta_q$ also the $G$-equivariant map $e \otimes V_{k-(q+1)} \to V_k$ given by multiplication by the above polynomial. This map is monic and its cokernel is isomorphic to the induced representation $\text{Ind}_G^H(\eta^k)$, where $B \leq G$ is the subgroup of upper triangular matrices, and $\eta$ is the character of $B$ given by \((a \ b \ c) \mapsto a. \) Since $\eta$ has order $q - 1$ one deduces the following result (cf. 2):

Proposition 2.4. For $k > q$, there is an exact sequence of $G$-modules $0 \to e \otimes V_{k-(q+1)} \to V_k \to \text{Ind}_G^H(\eta^k) \to 0$; in particular, for $k \geq 2q$ we have a $G$-isomorphism:

\[ \frac{V_k}{e \otimes V_{k-(q+1)}} \simeq \frac{V_{k-(q-1)}}{e \otimes V_{k-2q}} \]

We can obtain the isomorphism [3] also as follows: as in [1], we let $I_k$ be, for any $k \geq 0$, the space of functions $\mathbb{F}^2 \to \mathbb{F}$ that are homogeneous of degree $k$ and vanish at $(0,0)$, endowed with the $G$-action given by $(g \cdot f)(x,y) = f((g^{-1}(x,y))^t)$ for $g \in G, f \in I_k$ and $(x,y) \in \mathbb{F}^2$. If $\rho : G \to \text{GL}(V)$ is any representation of $G$ over some field, we denote by $\rho^t$ the representation defined by $\rho^t(g) := \rho((g)^t)^{-1}$, where $g^t$ is the transpose of the matrix $g \in G$. Denote then by $\tau_k : V_k \to I_k^t$ the $G$-map sending a polynomial in $V_k$ to the associated polynomial function in $I_k^t$.

Assume now $k > q$ and let $T_0 := X^{q-(q-1)} \prod_{a \in \mathbb{F}^t(A - aY)} \in V_k$: then $\tau_k(T_0)$ is non-zero only at $[1 : 0] \in \mathbb{P}^1(\mathbb{F})$: since $G$ act transitively on $\mathbb{P}^1(\mathbb{F})$, for any $P \in \mathbb{P}^1(\mathbb{F})$ we can find a polynomial $T_P \in V_k$ such that $\tau_k(T_P)$ is non zero only at $P$: $\tau_k$ is therefore onto. Since $\theta_q(e \otimes V_{k-(q+1)}) \subset \ker \tau_k$ and $\dim I_k^t = q + 1$ we conclude that $\text{coker}(\theta_q : e \otimes V_{k-(q+1)} \to V_k) \simeq I_k^t \simeq I_{k+(q-1)}$.

Notice that for any $k \geq 0$ we have $I_k \simeq e^{-k} \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F})]$, where $\mathbb{F}[\mathbb{P}^1(\mathbb{F})]$ is the $G$-module of $\mathbb{F}$-valued functions on $\mathbb{P}^1(\mathbb{F})$ with the usual left $G$-action; denoting by $M_0$ (resp $M_1$) the $G$-submodule of $\mathbb{F}[\mathbb{P}^1(\mathbb{F})]$ consisting of the constant (resp. average zero) functions, we have $\mathbb{F}[\mathbb{P}^1(\mathbb{F})] = M_0 \oplus M_1$. We now need two lemmas:

Lemma 2.5. In $\mathbb{F}[X,Y]$ one has the identity $X^q - 1 + \sum_{a \in \mathbb{F}} (aX + Y)^{q-1} = 0$.

Proof. Writing $\sum_{a \in \mathbb{F}} (aX + Y)^{q-1} = \sum_{a \in \mathbb{F}} \sum_{j=0}^{q-1} \binom{q-1}{j} a^j X^j Y^{q-1-j}$, we see that the coefficient of $X^q$ in this polynomial is $q - 1$ (resp. $Y^{q-1}$), hence it’s enough we prove that for any $0 < j < q - 1$ we have $\sum_{a \in \mathbb{F}} a^j = 0$. Let us chose a generator $e$ of the multiplicative group of $\mathbb{F}$ and let $\lambda$ be the order of $e$ in this group; if $q - 1 = \lambda d$ in $\mathbb{Z}$, we have:

\[ \sum_{a \in \mathbb{F}} a^j = \sum_{n=0}^{\lambda-1} e^{nj} = d \sum_{n=0}^{\lambda-1} e^{nj}. \]
Since $0 = \varepsilon^{\lambda j} - 1 = (\varepsilon^j - 1) \sum_{n=0}^{\lambda - 1} \varepsilon^{nj}$ and $\varepsilon^j \neq 1$, we conclude that the above summation is zero. 

\[ \square \]

**Lemma 2.6.** There exists a $G$-module epimorphism $\vartheta : \mathbb{F} \left[ \mathbb{P}^1(\mathbb{F}) \right] \to V_{q-1}$ whose kernel is $M_0$. In particular $M_1 \simeq V_{q-1}$ as $G$-modules.

**Proof.** Let $\vartheta$ be the $G$-map defined as follows:

\[ f \in \mathbb{F} \left[ \mathbb{P}^1(\mathbb{F}) \right] \mapsto \sum_{[a:b] \in \mathbb{P}^1(\mathbb{F})} f([a:b]) (aX + bY)^{q-1}. \]

For any point $P$ of the $\mathbb{F}$-projective line, let $f_P \in \mathbb{F} \left[ \mathbb{P}^1(\mathbb{F}) \right]$ be the function equal to $1$ at $P$ and zero everywhere else. Notice that $X^{q-1} = \vartheta \left( f_{[1:0]} \right)$; let then $\alpha$ be an integer, $0 \leq \alpha < q - 1$ and let $f = \sum_{P \in \mathbb{P}^1(\mathbb{F})} \lambda_P f_P$, where $\lambda_P \in \mathbb{F}$.

We have $\vartheta (f) = X^\alpha Y^{q-1-\alpha}$ if, for example, $\lambda_{[1:0]} = 0$ and the following $q$ equations in the variables $\{\lambda_{[a:1]}\}_{a \in \mathbb{F}}$ have a common solution:

\[
\begin{align*}
\sum_{a \in \mathbb{F}} \lambda_{[a:1]} a^\alpha &= \left(\binom{q-1}{\alpha}\right)^{-1} \\
\sum_{a \in \mathbb{F}} \lambda_{[a:1]} a^j &= 0 \quad (0 \leq j < q - 1, j \neq \alpha).
\end{align*}
\]

(Notice that the identity $(X + Y)^q = (X + Y) \sum_{i=0}^{q-1} (-1)^i X^{q-1-i} Y^i$ in $\mathbb{F}[X,Y]$ implies that $\binom{q-1}{i} \equiv (-1)^i (\bmod p)$ for any $0 \leq i \leq q - 1$, hence $\binom{q-1}{\alpha}$ is invertible in $\mathbb{F}$).

Since the matrix of this $q \times q$ system is of Vandermonde type over the elements of $\mathbb{F}$, the system has a unique solution, so that $\vartheta^{-1} \left( X^\alpha Y^{q-1-\alpha} \right) \neq \varnothing$ and $\vartheta$ is onto.

Since the kernel of $\vartheta$ has dimension $1$, the previous lemma implies $\ker \vartheta = M_0$. 

As a consequence of our computations we conclude:

**Proposition 2.7.** For any integer $k > q$ and any $\lambda \geq 0$ there are isomorphisms of $G$-modules:

\[
\frac{V_k}{e \otimes V_{k-(q+1)}} \simeq e^k \otimes (\mathbb{F} \oplus V_{q-1}^t) \simeq \frac{V_{k+\lambda(q-1)}}{e \otimes V_{k+\lambda(q-1)-(q+1)}},
\]

where the inclusion $e \otimes V_{k+\lambda(q-1)-(q+1)} \hookrightarrow V_{k+\lambda(q-1)}$ is induced by the multiplication by $\vartheta_q$.

**Proof.** Just notice that $I_{k}^t \simeq (e^{-k} \otimes (\mathbb{F} \oplus V_{q-1}))^t \simeq e^k \otimes (\mathbb{F}^t \oplus V_{q-1}^t) = e^k \otimes (\mathbb{F}^t \oplus V_{q-1}^t)$. 

\[ \square \]

3. **Reduction of cuspidal representations**

Let us rewrite the identity 2 as:

\[ (4) \quad V_k - V_{k-(q-1)} = e \cdot \left( V_{k-(q+1)} - V_{k-2q} \right), \quad (k \in \mathbb{Z}). \]

One is naturally led to ask when $V_k - V_{k-(q-1)}$ is positive in $K_0(GL_2(\mathbb{F}))$; we would like to have, at least for some values of $k$, a monic map of $GL_2(\mathbb{F})$-modules $V_{k-(q-1)} \to V_k$ that "raises the weight by $q - 1$". The Serre $D$-map will play this role, if $1 \leq k \leq p - 1$. 
3.1. **The D-map of Serre.** We assume from now on that $k \geq 0$ and we will impose some other restrictions on it later.

Let $\chi : \mathbb{F}_{q^2}^\times \to \mathbb{G}_p^\times$ be the (restriction of the) Teichmüller character fixed in the previous section, and let $k$ be an integer such that $k \not\equiv 0 \pmod{q+1}$, so that $\chi^k$ is indecomposable (i.e. it does not factor through the norm map $\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$). Under these assumptions, there is a unique $\mathbb{Q}_p$-representation $\Theta(\chi^k)$ of $G$ characterized by the property that $\Theta(\chi^k) \otimes \text{sp} \simeq \text{Ind}_G^{\mathbb{F}_{q^2}^\times} (\chi^k)$, where $\mathbb{F}_{q^2}^\times$ is embedded in $G$ by the map $\iota$ fixed in the previous section, and $\text{sp}$ is the $G$-subrepresentation of $\mathbb{Q}_p[[\mathbb{P}^1(\mathbb{F})]$ consisting of functions of average zero. $\Theta(\chi^k)$ is the cuspidal representation associated to $\chi^k$.

The character of $\Theta(\chi^k)$ is given by:

\[
\begin{align*}
\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} & \mapsto \quad \left(q - 1\right) \chi(a)^k, \quad a \in \mathbb{F}_q^\times \\
\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} & \mapsto \quad -\chi(a)^k, \quad a \in \mathbb{F}_q^\times \\
\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} & \mapsto \quad 0, \quad a, b \in \mathbb{F}_q^\times, a \neq b \\
\iota(c) & \mapsto \quad -\chi(c)^k - \chi(cq)^k, \quad c \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times.
\end{align*}
\]

A computation of Brauer characters gives:

**Proposition 3.1.** Let $k$ be an integer such that $k \not\equiv 0 \pmod{q+1}$; the following identity holds in $K_0(G)$:

$$V_{k+(q-1)} - V_k = \overline{\Theta(\chi^k)},$$

where the bar on the left hand side denotes the reduction mod $p$ of any fixed integral model $\overline{\Theta(\chi^k)}$ of $\Theta(\chi^k)$ (the choice of integral model is not relevant since the identity is written in $K_0(G)$).

**Definition 3.2.** (J-P. Serre) The map $D : \mathbb{F}[X,Y] \to \mathbb{F}[X,Y]$ is the $\mathbb{F}$-linear map defined by:

$$Df(X,Y) := X^q \cdot \frac{\partial f}{\partial X} + Y^q \cdot \frac{\partial f}{\partial Y},$$

for any polynomial $f \in \mathbb{F}[X,Y]$ (notice here $q = \#\mathbb{F}$).

The map $D$ is $G$-equivariant: if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $\alpha, \beta \geq 0$ the polynomial $D(g \cdot X^{a}Y^{b})$ equals:

$$\alpha (aX + cY)^\alpha (aX + cY)^{\alpha-1} (bX + dY)^\beta + \beta (bX + dY)^\beta (aX + cY)^\alpha (bX + dY)^{\beta-1}.$$  

One easily sees that this is exactly $(gX)^\alpha (g \cdot X^{a-1}Y^{b}) + (gY)^\alpha (g \cdot Y^{\beta-1})$. Furthermore, N. Fakhruddin proved the following:

\footnote{An analogue of the map $D$ in characteristic zero appears in the theory of $G$-invariants, in particular cf. the definition of the polarization map in [9], Appendix F, §1.}
Proposition 3.3. The kernel of the map \( D : \mathbb{F}[X,Y] \rightarrow \mathbb{F}[X,Y] \) is given by 
\[ \ker D = \mathbb{F}[X^p, Y^p, \theta_q]. \]

Proof. Let \( A = \mathbb{F}[X^p, Y^p, \theta_q] \) and \( B = \ker D \); notice that \( B \) is a ring and we have the inclusions \( \mathbb{F}[X^p, Y^p] \subseteq A \subseteq B \subseteq \mathbb{F}[X,Y] \). The polynomial \( t^p - (X^pY^p - X^pY^p) \in \mathbb{F}(X^p, Y^p)[t] \) is irreducible in \( \mathbb{F}(X^p, Y^p)[t] \) since \( X^pY^p - X^pY^p \) does not have a \( p^{th} \)-root in \( \mathbb{F}(X^p, Y^p) \), so that we have \( [Q(A) : \mathbb{F}(X^p, Y^p)] = p \), where we denote by \( Q(R) \) the field of fractions of an integral domain \( R \) inside some extension of \( \mathbb{F} \).

Now observe that \( Q(B) \) is properly contained inside \( \mathbb{F}(X,Y) \): if not, we could write \( X = \frac{f}{g} \) with \( f, g \in B, g \neq 0 \) and \( 1 = af + bg \) for some \( a, b \in B \); this would imply \( X = f \cdot (aX + b) \) so that \( f \in B \) would be an associate of \( X \) in \( \mathbb{F}[X,Y] \). Since \( [\mathbb{F}(X,Y) : \mathbb{F}(X^p, Y^p)] = p^2 \) we have therefore \( Q(A) = Q(B) \). Notice that \( \mathbb{F}[X,Y]/A \) is an integral extension, so that \( B/A \) is too.

Now observe that the domain \( A \) is normal, since the corresponding variety has equation \( X_q^2X_2 - X_1X_2^2 - X_3^2 = 0 \), and then it defines an hypersurface of \( h_{2/2}^{1/2} \) that is non-singular in codimension one (cf. \cite{10}, II.8.23). We conclude \( A = B \).

We will need the following:

Lemma 3.4. If \( q > p > 3 \) is a prime number, then \( V_p \) is a non-trivial extension of \( V_{p-2} \) via \( V_1 \) in the category of \( \mathbb{F}_p[SL_2(\mathbb{F}_p)] \)-modules. In particular, for \( p > 3 \), the one dimensional \( \mathbb{F}_p \)-space \( Ext^1_{\mathbb{F}_p SL_2(\mathbb{F}_p)}(V_{p-2}, V_1) \cong Ext^1_{\mathbb{F}_p SL_2(\mathbb{F}_p)}(V_1, V_{p-2}) \) is generated by the class of \( V_p \).

Proof. A computation of Brauer characters tells us that \( V_p - V_1 = V_{p-2} \) in \( K_0(SL_2(\mathbb{F}_p)) \); the existence of the monic map \( D : V_1 \rightarrow V_p \) and the fact that \( V_{p-2} \) is irreducible give therefore an exact sequence of the form:

\[
0 \rightarrow V_1 \xrightarrow{D} V_p \xrightarrow{\pi'} V_{p-2} \rightarrow 0.
\]

We can choose \( \pi' \) so that it sends \( X^p \) and \( Y^p \) to zero and \( \pi'(X^{2r}Y^{p-r}) := k(r) \cdot X^{r-1}Y^{p-1-r} \) for \( 0 < r < p \), where \( k(r) := \frac{(p-2)}{(r-1)} \neq 0 \) (cf. the proof of Proposition 3.3). With this choice of \( \pi' \), the sequence:

\[
0 \rightarrow V_1 \xrightarrow{D} V_p \xrightarrow{\pi'} V_{p-2} \rightarrow 0
\]

is exact in the category of \( GL_2(\mathbb{F}_p) \)-modules; since \( SL_2(\mathbb{F}_p) \) is normal in \( GL_2(\mathbb{F}_p) \) of index prime to \( p \), we have that \( \Box \) splits as a sequence of \( SL_2(\mathbb{F}_p) \)-modules if and only if \( \Box \) splits as a sequence of \( GL_2(\mathbb{F}_p) \)-modules. We will show that \( \Box \) does not split.

A splitting \( GL_2(\mathbb{F}_p) \)-homomorphism \( \sigma : e \otimes V_{p-2} \rightarrow V_p \) for \( \pi' \) has to send \( k(r) \cdot X^{r-1}Y^{p-1-r} \rightarrow X^{r}Y^{p-r} + h_1(r) Y^p + h_2(r) Yp \) for some \( h_1(r), h_2(r) \in \mathbb{F}_p \) (0 < \( r < p \)); imposing the equivariance of the map \( \sigma \) with respect to the maximal split torus of diagonal matrices in \( GL_2(\mathbb{F}_p) \), we find the conditions:

\[
h_1(r) = a^{-1}b^{1-r} \cdot h_1(r),
h_2(r) = a^r b^{r-1} \cdot h_2(r),
\]

that have to be satisfied for every \( a, b \in \mathbb{F}_p^\times \) and every \( 0 < r < p \). We therefore have \( h_1(r) = 0 \) if \( 2 \leq r \leq p-1 \) and \( h_2(r) = 0 \) if \( 1 \leq r \leq p-2 \).
This implies a contradiction, since then - being \( p > 3 \) - we would get:

\[
\sigma \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot k(2)XY^{p-3} \right) \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left( k(2)XY^{p-3} \right).
\]

The second assertion in the Lemma follows from the Propositions in §§12.1,12.2 of [12].

We are now able to prove the following result, due to G. Savin:

**Proposition 3.5.** If \( q = p > 3 \), then there are no \( \mathbb{F}_p[GL_2(\mathbb{F}_p)] \)-embeddings \( V_p \to V_{2p-1} \).

**Proof.** The \( \mathbb{F}_p[SL_2(\mathbb{F}_p)] \)-module \( (V_{p-1} \otimes_{\mathbb{F}_p} \mathbb{F}_p) \otimes_{\mathbb{F}_p} (V_1 \otimes_{\mathbb{F}_p} \mathbb{F}_p) \)^{Fr} \) is irreducible by Steinberg’s tensor product theorem, and it is isomorphic to \( V_{2p-1} \otimes_{\mathbb{F}_p} \mathbb{F}_p \) via the map induced by \( X^iY^{p-1-i} \otimes X^jY^{1-j} \to X^{i+p}Y^{2p-1-i-pj} \) (0 \( \leq \) \( i \leq p-1 \), 0 \( \leq \) \( j \leq 1 \)). We deduce the existence of an isomorphism of \( \mathbb{F}_p[SL_2(\mathbb{F}_p)] \)-modules \( V_{2p-1} \cong V_1 \otimes_{\mathbb{F}_p} V_{p-1} \).

Since \( V_1 \) is an \( \mathbb{F}_p[SL_2(\mathbb{F}_p)] \)-submodule of \( V_p \), to prove the proposition it suffices to show:

\[
\text{Hom}_{\mathbb{F}_p[SL_2(\mathbb{F}_p)]}(V_1, V_{2p-1}) = 0.
\]

We have isomorphisms of \( \mathbb{F}_p \)-vector spaces:

\[
\text{Hom}_{\mathbb{F}_p[SL_2(\mathbb{F}_p)]}(V_1, V_{2p-1}) \cong \text{Hom}_{\mathbb{F}_p[SL_2(\mathbb{F}_p)]}(V_1, V_1 \otimes_{\mathbb{F}_p} V_{p-1}) \\
\cong \text{Hom}_{\mathbb{F}_p[SL_2(\mathbb{F}_p)]}(V_1 \otimes_{\mathbb{F}_p} V_1, V_{p-1}),
\]

where the second isomorphism is a consequence of the self-duality of \( V_1 \) as \( \mathbb{F}_p[SL_2(\mathbb{F}_p)] \)-module. Now notice that the vector space \( \text{Hom}_{\mathbb{F}_p[SL_2(\mathbb{F}_p)]}(V_1 \otimes_{\mathbb{F}_p} V_1, V_{p-1}) \) is trivial, since \( V_{p-1} \) is a simple \( \mathbb{F}_p[SL_2(\mathbb{F}_p)] \)-module of dimension \( p > 3 \).

The last proposition shows that when \( q = p > 3 \) and the Serre \( D \)-map fails to be injective as map \( V_p \to V_{2p-1} \), then there are actually no subobjects of \( V_{2p-1} \) isomorphic to \( V_p \). Notice that an analogous result is clearly true for maps \( V_0 \to V_{p-1} \), since \( V_{p-1} = St \) is irreducible over \( \mathbb{F}_p[GL_2(\mathbb{F}_p)] \).

We also have:

**Proposition 3.6.** The element \( V_k - V_{k-(q-1)} \) of \( K_0(G) \) is positive if and only if \( k \neq -2 \) (mod \( q + 1 \)).

**Proof.** By [4] we can restrict ourselves to the case \(-2 \leq k \leq q-2\). If \( 0 \leq k \leq q-2 \) then \( k-(q-1) < 0 \), so that \( V_k - V_{k-(q-1)} = V_k - V_{-(k-q+1)} = V_k - (-e^{1-(q-1-k)} \cdot V_{q-1-k-3}) = V_k + e^{1+k} \cdot V_{q-k-3} \), and this element is positive in \( K_0(G) \) (notice that here both \( k \) and \( q-k-3 \) are non-negative integers). If \( k = -1 \), then \( V_{-1} - V_{-1-(q-1)} = 0 + V_{q-2} = V_{q-2} \) is again positive in the Grothendieck group of \( G \).

If \( k = -2 \), we need to consider the element \( V_{-2} - V_{-(q+1)} \) of \( K_0(G) \): this is equal to \( e^{-1} \cdot (V_{q-1} - V_0) \) and is not positive, since \( V_{q-1} = St \) and \( V_0 \) are distinct non-zero irreducible representations of \( G \).
Assuming now that $1 \leq k \leq p - 1$, the map of $G$-modules $D : V_k \to V_{k+q-1}$ is monic and we deduce, via a computation of Brauer characters, the existence of a $G$-module isomorphism:

\[(V_{k+q-1})^{ss} \cong (\Phi(\chi^k))^{ss},\]

where for a $G$-module $M$, $M^{ss}$ denotes the $G$-module obtained by semi-simplifying $M$.

We will show that this isomorphism comes from an isomorphism of modules without the semi-simplification, for $2 \leq k \leq p - 1$, $k \neq \frac{q+1}{2}$ and some integral model $\Phi(\chi^k)$ of $\Theta(\chi^k)$.

3.2. An exact sequence from crystalline cohomology. We recall some constructions from [9]. Let $R$ be either a field extension $K$ of $F_{q^2}$ inside $\mathbb{F}$ or the ring of Witt vectors $W = W(K)$ of such a field, together with the natural embedding $\mathbb{F}_{q^2} \hookrightarrow R^\times$; let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(R)$ and:

\[H(X,Y,Z) = (X,Y,Z) \cdot \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \cdot (X,Y,Z) \in R[X,Y,Z],\]

where the upper bar denotes the $q^{th}$-power map. The homogeneous polynomial $H$ is irreducible over $R$ and defines a projective, smooth and irreducible curve $C_{/R}$ having equation $XY^q - X^qY - Z^{q+1} = 0$.

We let $U_2(\mathbb{F}_{q^2}) = \{g \in GL_2(\mathbb{F}_{q^2}) : g \cdot A = A\}$ and we consider $\mathbb{F}_{q^2}$ embedded in $U_2(\mathbb{F}_{q^2})$ via $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-q} \end{pmatrix}$, so that $U_2(\mathbb{F}_{q^2}) = \mathbb{F}_{q^2}^\times \cdot SL_2(\mathbb{F})$. By construction, $U_2(\mathbb{F}_{q^2})$ acts on $C_{/K}$ via the embedding:

\[U_2(\mathbb{F}_{q^2}) \hookrightarrow GL_3(\mathbb{F}_{q^2}) : g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix},\]

and the induced action of $\mathbb{F}_{q^2}^\times \hookrightarrow U_2(\mathbb{F}_{q^2})$ lifts to an action on $C_{/W}$. In particular the group $\mu = \{t \in \mathbb{F}_{q^2}^\times : t^{q+1} = 1\} \subset \mathbb{F}_{q^2}^\times$ acts on $C_{/R}$ and its cohomology groups.

Let $i \geq 0$ be any integer; we denote by $H^i_{cris}(C_{/K})$ the $i$th group of the crystalline cohomology of $C_{/K}$ (cf. [16]), and we let $H^i_{dR}(C_{/R})$ be the $i$th group of the de Rham cohomology of $C_{/R}$. Since $C_{/K}$ admits a smooth lifting to characteristic zero (i.e. $C_{/W}$), we have canonical isomorphisms of $W$-modules $H^i_{cris}(C_{/K}) \cong H^i_{dR}(C_{/W})$.

The de Rham cohomology groups of $C_{/R}$ are computed as hypercohomology of the de Rham complex for $C_{/R}$, so that we have the corresponding (Hodge-de Rham) spectral sequence $E^{p,q}_{1} = H^q(C_{/R}, \Omega^p_{C_{/R}}) \Rightarrow H_{dR}^{p+q}(C_{/R})$ ($p, q \geq 0$) that degenerates at the first page $E_1$, since $C_{/R}$ is a curve (cf. [2], II.6.). We deduce from this that $H_{dR}^i(C_{/R}) \cong R$ for $i = 0, 2$ and $H_{dR}^i(C_{/R}) = 0$ for $i > 2$.

For $i = 1$ the degeneracy of $E^{p,q}_{1}$ at the first page gives:

**Proposition 3.7.** There is a natural exact sequence of free $R$-modules:

\[0 \to H^0(C_{/R}, \Omega^1_{C_{/R}}) \to H^1_{dR}(C_{/R}) \to H^1(C_{/R}, \mathcal{O}_{C_{/R}}) \to 0.\]
For $1 \leq k \leq q$ let $k : \mu \rightarrow \mu$ denote the character $t \mapsto t^k \ (t \in \mu)$. Since $|\mu| = q+1$ is invertible in $R$, the proposition gives:

**Corollary 3.8.** For every $1 \leq k \leq q$, there is a natural exact sequence of $R$-modules,:

$$0 \rightarrow H^0(C/R, \Omega^1_{C/R})_{-k} \rightarrow H^1_{dR}(C/R)_{-k} \rightarrow H^1(C/R, \mathcal{O}_{C/R})_{-k} \rightarrow 0,$$

where the subscript $-k$ designates the corresponding $\mu$-eigenspace.

In the particular case $R = \mathbb{K}$, we have an action of $U_2 \left( \mathbb{F}_q^2 \right)$ on $C_\mathbb{K}$ and its cohomology. We have:

**Proposition 3.9.** There is a natural exact sequence of $\mathbb{K} \left[ U_2 \left( \mathbb{F}_q^2 \right) \right]$-modules, for every $1 \leq k \leq q$:

$$0 \rightarrow H^0(C/\mathbb{K}, \Omega^1_{C/\mathbb{K}})_{-k} \rightarrow H^1_{dR}(C/\mathbb{K})_{-k} \rightarrow H^1(C/\mathbb{K}, \mathcal{O}_{C/\mathbb{K}})_{-k} \rightarrow 0.$$

We will interested in the sequel in writing down explicitly the maps occurring in the above sequences.

**Remark 3.10.** For any $i \geq 0$ there is an exact sequence of "universal coefficients":

$$0 \rightarrow H^i_{\text{cris}}(C/\mathbb{K}) \otimes_W \mathbb{K} \rightarrow H^i_{dR}(C/\mathbb{K}) \rightarrow \text{Tor}_1^W(H^{i+1}_{\text{cris}}(C/\mathbb{K}), \mathbb{K}) \rightarrow 0.$$

(cf. [13]). Taking $i = 1$ and using the identification $H^1_{\text{cris}}(C/\mathbb{K}) \simeq H^1_{dR}(C/W)$ and the fact that $\text{Tor}_1^W(W, \mathbb{K}) = 0$, we deduce the existence of a natural isomorphism:

$$H^1_{dR}(C/\mathbb{K}) \simeq H^1_{dR}(C/W) \otimes_W \mathbb{K}.$$

### 3.3. An exact sequence of representations of $U_2 \left( \mathbb{F}_q^2 \right)$

We work over the field $\mathbb{F}_q^2$, the quadratic extension of $\mathbb{F}$ inside $\overline{\mathbb{F}}$. We assume from now on that $p$ is an odd prime and we put as usual $G = GL_2(\mathbb{F})$; let $k$ be an integer such that $2 \leq k \leq p-1$, and let $V_k = \text{Sym}^k \mathbb{F}_q^2$ be the usual left representation of $U_2 \left( \mathbb{F}_q^2 \right) \subseteq GL_2(\mathbb{F}_q^2)$ over $\mathbb{F}_q^2$. Notice that we changed the notation from section 2: in this section we will be denoting by $V_k$ the $G$-representation $\text{Sym}^k \mathbb{F}^2$, so that $V_k = V_k \otimes_{\mathbb{F}} \mathbb{F}_q^2$, as $G$-representations.

The multiplication by $\theta_q$ (not by $\theta_q^2$!) induces a $\mathbb{F}_q^2 \left[ U_2 \left( \mathbb{F}_q^2 \right) \right]$-monomorphism $\theta_q : V_{k-2} \hookrightarrow V_{k+q-1}$. Let $D' : V_k' \rightarrow V_{k+(q-1)}$ be the Serre $D$-map that increases the degree of $q-1$, as in Definition 3.2 the monic $\mathbb{F}_q^2$-linear map $D := D' \otimes \mathbb{F}_q^2 : V_k \rightarrow V_{k+(q-1)}$ is $SL_2(\mathbb{F})$-equivariant, but not $\mathbb{F}_q^2$-equivariant; nevertheless $D \left( V_k \right)$ is an $\mathbb{F}_q^2 \left[ U_2 \left( \mathbb{F}_q^2 \right) \right]$-submodule of $V_{k+(q-1)}$ and hence we get the well defined $U_2 \left( \mathbb{F}_q^2 \right)$-map of $\mathbb{F}_q^2$-spaces:

$$\theta_q : V_{k-2} \rightarrow V_{k+(q-1)} \frac{D \left( V_k \right)}{D \left( V_k \right)}.$$

Notice that this map is non-zero, then monic, i.e. $\text{Im} \theta_q \cap D \left( V_k \right) = 0$, because our restriction on the range of $k$ implies that $V_k'$ is an irreducible representation of $G$.

By dimension considerations, we have the $\mathbb{F}_q^2$-vector space decomposition $V_{k+(q-1)} = D \left( V_k \right) \oplus \theta_q \left( V_{k-2} \right) \oplus W$, where

$$W := \bigoplus_{r=k}^{q-1} \mathbb{F}_q^2 X^r Y^{k+q-1-r}.$$

Notice that the action of $\mathbb{F}_q^\times$ upon $W$ decomposes the space into the direct sum of eigenspaces of dimension one: if $k \leq r \leq q-1$, $\mathbb{F}_q X^r Y^{k+r-1}Y$ is the subspace of $W$ on which $\mathbb{F}_q^\times$ acts by the character $t \mapsto t^{r(q+1)+(q-1)-qk}$.

**Proposition 3.11.** There exist $U_2 (\mathbb{F}_q^2)$-epimorphisms of $\mathbb{F}_q^2$-spaces $\frac{V_{k+q-1}}{D(V_k)} \to e^k \otimes V_{q-1-k}$ whose kernel is equal to $\text{Im} \overline{\theta}_q$, if and only if $q = p$ is prime; in this case they are given by the maps $\{ \omega_s \}_{s \in \mathbb{F}_q^\times}$, where for any $s \in \mathbb{F}_q^\times$ the homomorphism $\omega_s$ is defined by $\omega_s (\text{Im} \overline{\theta}_q) = 0$ and:

$$X^r Y^{k+r-1} \mod D(V_k) \mapsto s \cdot \frac{(q-1-k)}{(r-k)} \cdot X^{r-k} Y^{q-1-r}, \quad (k \leq r \leq q-1).$$

**Proof.** We need to construct the $U_2 (\mathbb{F}_q^2)$-equivariant onto maps $\omega : V_{k+q-1} \to e^k \otimes V_{q-1-k}$ whose kernel is $D(V_k) \oplus \theta_q (V_{k-2})$, therefore we need to define $\omega$ only on $W$. Since $t \in \mathbb{F}_q^\times$ acts on the subspace $\mathbb{F}_q X^r Y^{q-1-k-s} \subset e^k \otimes V_{q-1-k}$ as multiplication by $t^{s(q+1)+(q-1)-kq}$ ($0 \leq s \leq q-1-k$), and since $\omega$ has to respect the decomposition of the spaces in $\mathbb{F}_q^2$-eigenspaces, we have $\omega : X^r Y^{k+r-1} \mapsto \alpha (r) X^{r-k} Y^{q-1-r}$ for some $\alpha (r) \in \mathbb{F}_q^\times$ ($k \leq r \leq q-1$). For any $k \leq r \leq q-1$ we find the following conditions to be satisfied:

$$\alpha (r) = (-1)^k \alpha (q+k-1-r)$$

$$\alpha (r) = \alpha (q-1) \cdot \frac{(q-1-k)}{(r-k)},$$

where the first equation is equivalent to the equivariance of $\omega$ with respect to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the second is equivalent to the equivariance with respect to $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ for any $u \in \mathbb{F}$. Notice that $(q-1-k)$ is not divisible by $p$ for every $k \leq r \leq q-1$ if and only if $q = p$, in fact if $q > p$ we have $(q-1-k) \equiv 0 (\text{mod} \, p)$.

If $q = p$ we can choose $\alpha (q-1) \neq 0$ arbitrarily, so that we get the maps in the statement; all of them are acceptable since:

$$U_2 (\mathbb{F}_q^2) = \mathbb{F}_q^\times \cdot \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} : u \in \mathbb{F} \right\}.$$

\[ \square \]

Note that the above proposition gives, for $q > p$, equivariant non-surjective maps $\frac{V_{k+q-1}}{D(V_k)} \to e^k \otimes V_{q-1-k}$ whose kernel properly contains $\text{Im} \overline{\theta}_q$.

**Corollary 3.12.** Let $q = p$ be prime. For any $s \in \mathbb{F}_q^\times$ we have an exact sequence of $\mathbb{F}_q^2 [U_2 (\mathbb{F}_q^2)]$-modules:

$$0 \to V_{k-2} \overset{\overline{\theta}_q}{\to} \frac{V_{k+q-1}}{D(V_k)} \overset{\omega_s}{\to} e^k \otimes V_{q-1-k} \to 0,$$

coming, if $s \in \mathbb{F}_q^\times$, from the exact sequence of $\mathbb{F} [\text{SL}_2 (\mathbb{F})]$-modules:

$$0 \to V'_{k-2} \overset{\overline{\theta}_q'}{\to} \frac{V'_{k+q-1}}{D(V'_k)} \overset{\omega_s'}{\to} V'_{q-1-k} \to 0.$$
Remark 3.13. If $q$ is any power of the prime $p$ and $2 \leq k \leq p-1$, one sees (formula 10) that there is an exact sequence of $\mathbb{F} [GL_2 (\mathbb{F})]$-modules:

$$0 \to e \otimes V'_{k-2} \xrightarrow{\theta_q} \frac{V'_{k+(q-1)}}{D (V'_k)} \to \text{coker } \theta_q \to 0.$$ 

The same reasoning of the proof of Proposition 3.11 gives that there exist $GL_2 (\mathbb{F})$-epimorphisms of $\mathbb{F}$-spaces $\frac{V'_{k+(q-1)}}{D (V'_k)} \to e^k \otimes V'_{q-1-k}$ whose kernel is equal to $\text{Im } \theta_q$ if and only if $q = p$ is prime (one just needs to consider the equivariance of the desired maps with respect to the characters of the split torus of $GL_2 (\mathbb{F})$, and proceed as in the proof of the proposition).

Furthermore, a computation of Brauer characters gives the following identity in the proof of the proposition):

$$\text{coker } \theta_q = (V'_{q-1-k})^* = e^k \cdot V'_{q-1-k},$$

(notice that $(V'_k)^* = e^k \cdot V'_k$ in $K_0 (G, \mathbb{F})$ for any $k \in \mathbb{Z}$). In the case $q = p$, $e^k \otimes V'_{q-1-k}$ is an irreducible $GL_2 (\mathbb{F})$-module, so that we find (again) that $\text{coker } \theta_q \simeq e^k \otimes V'_{q-1-k}$; if $q > p$ we cannot guarantee the irreducibility of $e^k \otimes V'_{q-1-k}$ and we can only deduce the existence of isomorphisms:

$$(\text{coker } \theta_q)^{ss} \simeq ((V'_{q-1-k})^*)^{ss} \simeq (e^k \otimes V'_{q-1-k})^{ss}.$$ 

We conclude this paragraph with the following result:

**Proposition 3.14.** For any $2 \leq k \leq p-1$, the exact sequence of $\mathbb{F} [SL_2 (\mathbb{F})]$-modules $0 \to V'_{k-2} \to \frac{V'_{k+(q-1)}}{D (V'_k)} \to \text{coker } \theta_q \to 0$ is non-split.

**Proof.** If the sequence of $SL_2 (\mathbb{F})$-modules $0 \to V'_{k-2} \to \frac{V'_{k+(q-1)}}{D (V'_k)} \to \text{coker } \theta_q \to 0$ splits, then by tensoring with $\mathbb{F}_q^\times$ we would get a split sequence of $U_2 (\mathbb{F}_q^\times)$-modules, since $SL_2 (\mathbb{F}) \triangleleft U_2 (\mathbb{F}_q^\times)$ is of index prime to $p$.

Let us assume that $0 \to V_{k-2} \xrightarrow{\theta_q} \frac{V_{k+(q-1)}}{D (V_k)} \to \text{coker } \theta_q \to 0$ is a split exact sequence of $\mathbb{F}_q^\times [U_2 (\mathbb{F}_q^\times)]$-modules. A basis of $\frac{V_{k+(q-1)}}{D (V_k)}$ is given by the set:

$$\mathcal{S} = \left\{ X^i Y^{k+q-1-r} \mod D (V_k), X^i Y^{k-2-s} \theta_q \mod D (V_k) \right\}_{k \leq r \leq q-1, 0 \leq s \leq k-2, i \leq k},$$

and the group $\mathbb{F}_q^\times$ acts on the 1-dimensional space generated by each of the vectors in $\mathcal{S}$ as multiplication by a character, and this action is via different characters on different spaces (in fact, $\mathbb{F}_q^\times$ acts on $X^i Y^{k-2-s} \theta_q \mod D (V_k)$ via the character $t \mapsto t^{(q+1) - kq + 2s}$ for any $0 \leq s \leq k-2$; for the action on the other basis vectors see the computation before Proposition 3.11). This implies that if $M$ is a complement of $\text{Im } \theta_q$ inside $\frac{V_{k+(q-1)}}{D (V_k)}$, there is basis of $M$ contained in $\mathcal{S}$; since $M \cap \text{Im } \theta_q = 0$ we have $M = (W \oplus D (V_k))/D (V_k)$ (see 8 for the definition of $W$). This is not possible, since this space is not closed under the action of $U_2 (\mathbb{F}_q^\times)$. In fact we have:

$$\left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right) \cdot X^k Y^{q-1} \mod D (V_k) = \sum_{i=0}^{k} t^{k \choose i} X^i Y^{k+q-1-i} \mod D (V_k),$$

and $\sum_{i=0}^{k} t^{k \choose i} X^i Y^{k+q-1-i} = X^k Y^{q-1} + \sum_{i=1}^{k-1} t^{k \choose i} X^i Y^{k+q-1-i}$, where the first term is in $W$ and the second is not in $W \oplus D (V_k)$. \[\square\]
Corollary 3.15. If $2 \leq k \leq p - 1$ and $q = p$ is prime, the one dimensional $F$-vector space $\text{Ext}^1_{SL_2(F)}(V'_{k-2}, V'_{q-1-k}) \simeq \text{Ext}^1_{SL_2(F)}(V'_{q-1-k}, V'_{k-2})$ is generated by the class of $\frac{V_{k+q-1}}{D(V_k)}$.

Proof. This follows from the previous result and from the Propositions in §§12.1,12.2 of [12]. □

3.4. Connecting the two exact sequences. We would like to be able to relate the exact sequences coming from Corollary 3.13 (for $K = \mathbb{F}_q^2$) and Corollary 3.12 (with the additional assumption $q = p$), by constructing - for some integers $\alpha, \beta, \gamma$ - a commutative diagram of the following form:

$$
\begin{array}{c}
0 \rightarrow H^0(C, \Omega^1_{C}) \rightarrow H^1_{dR}(C) \rightarrow H^1(C, \mathcal{O}_C) \rightarrow 0 \\
\downarrow \simeq \downarrow \simeq \\
0 \rightarrow e^\alpha \otimes V_{k-2} \rightarrow e^\beta \otimes \frac{V_{k+q-1}}{D(V_k)} \rightarrow e^\gamma \otimes V_{q-1-k} \rightarrow 0.
\end{array}
$$

In the sequel, we will construct the vertical isomorphisms; we will also treat the case $q > p$.

3.4.1. The left vertical maps. We assume from now on $K = \mathbb{F}_q^2$ and $C$ will denote the smooth projective curve $C_{/\mathbb{F}_q^2}$; as usual we fix the integer $k$ such that $2 \leq k \leq p - 1$ ($p > 2$). We will always allow $q$ to be any positive power of $p$, unless otherwise stated.

There is a natural isomorphism of $U_2(\mathbb{F}_q^2)$-modules $e^{-1} \otimes H^0(\mathbb{P}^2(\mathbb{F}_q^2), \mathcal{O}(q - 2)) \simeq e^{-1} \otimes \text{Sym}^{q-2}(\mathbb{F}_q^2)$; furthermore by [9], Prop. 2.1. the map:

$$
e^{-1} \otimes H^0(\mathbb{P}^2(\mathbb{F}_q^2), \mathcal{O}(q - 2)) \rightarrow H^0(C, \Omega^1_C) \quad X^b Y^a Z^{q-2-a-b} \rightarrow -t_1^a t_2^{-2-a-b} dt_2,
$$

where $a, b \geq 0, a + b \leq q - 2$ and $t_1 = Y/X, t_2 = Z/X$, is a natural $U_2(\mathbb{F}_q^2)$-isomorphism. Here the action of $U_2(\mathbb{F}_q^2)$ on the codomain is given by embedding $U_2(\mathbb{F}_q^2)$ in $GL_3(\mathbb{F}_q^2)$ via $g \mapsto \begin{pmatrix} (g^{-1})^t & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and letting the latter group act on $\mathbb{F}_q^2[X, Y, Z]$ in the standard way (see formula (1) in Section 2). For example we have, for $a, b \geq 0$ and $a + b \leq q - 2$:}

$$
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t_1^a t_2^b dt_2 = \begin{pmatrix} Y^a \\ Z^b \end{pmatrix} X^{-1} d\left(\frac{X}{Y}\right) = \begin{pmatrix} X^a \\ Z^b \end{pmatrix} Y^{-1} d\left(\frac{X}{Y}\right) = \begin{pmatrix} (X^{-1})^a \\ (Z^{-1})^b \end{pmatrix} Y d\left(\frac{Z}{-Y}\right),
$$

(9)

then we use the identity $t_2^{q+1} = t_1^q - t_1$ to get $dt_1 = -t_2^q dt_2, dt_1 t_2 = t_2^{q-2} dt_2$ and conclude that:

$$
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t_1^a t_2^b dt_2 = (-1)^{a+b+1} t_1^a t_2^{q-2-a-b} dt_2.
$$
We recall for the sequel the action of the group $U_2 \left( \mathbb{F}_q^2 \right)$ on every element (for a proof, cf. [9]): for any $a, b \geq 0$ and $a + b \leq q - 2$:

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} t_1^q t_2^b dt_2 = (-1)^{a+b+1} t_1^{q-2-a-b} t_2^b dt_2
\]

(10)

\[
\begin{pmatrix}
t & 0 \\
0 & t^{-q}
\end{pmatrix} t_1^q t_2^b dt_2 = \sum_{i=0}^{a} \binom{a}{i} (-u)^{a-i} t_1^i t_2^b dt_2, \quad (u \in \mathbb{F})
\]

(11)

\[
\begin{pmatrix}
t & 0 \\
0 & t^{-q}
\end{pmatrix} t_1^q t_2^b dt_2 = t^{a(q+1)+b+1} t_1^q t_2^b dt_2, \quad (t \in \mathbb{F}_q^*)
\]

(12)

Therefore we can make the identification $H^0 \left( C, \Omega^1_C \right)_{-k} = \bigoplus_{n=-k}^{k-2} \mathbb{F}_q \cdot t_1^n t_2^{q-k} dt_2$.

**Proposition 3.16.** The isomorphisms of $\mathbb{F}_q^*[U_2 \left( \mathbb{F}_q^2 \right)]$-modules $H^0 \left( C, \Omega^1_C \right)_{-k} \to e^{1-k} \otimes V_{k-2}$ are the maps $\{ \varphi_s \}_{s \in \mathbb{F}_q^*}$, where for any $s \in \mathbb{F}_q^*$, $\varphi_s$ is defined by:

\[
t_1^q t_2^{q-k} dt_2 \mapsto (-1)^a \cdot s \cdot X^a Y^{k-2-a}, \quad (0 \leq a \leq k - 2).
\]

**Proof.** The isomorphisms $\varphi$ we are looking for preserve the $\mathbb{F}_q^*$-eigenspace decomposition of the modules: for any fixed $0 \leq i \leq k-2$, $\mathbb{F}_q^* X^i Y^{k-2-i} \subset e^{1-k} \otimes V_{k-2}$ is the eigenspace relative to the character of $\mathbb{F}_q^*$ represented by the mod $(q^2 - 1)$ integer $(i+1)(q+1) - k$; $t_1^q t_2^{q-k} dt_2 \subset H^0 \left( C, \Omega^1_C \right)_{-k}$ is the eigenspace for $a(q+1)+q-k+1$ $(0 \leq a \leq k - 2)$. We deduce that $\varphi$ has to send $t_1^q t_2^{q-k} dt_2$ to $\beta(a) X^a Y^{k-2-a}$ for any $0 \leq a \leq k - 2$; by imposing the condition of equivariance with respect to the matrices:

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
t & 0 \\
0 & t^{-q}
\end{pmatrix} (u \in \mathbb{F}),
\]

we find respectively, for any $0 \leq a \leq k - 2$:

\[
\beta(a) = (-1)^k \beta(k-2) - a
\]

\[
\beta(a) = (-1)^k \beta(k-2) + a.
\]

Any fixed value for $\beta(k-2) \in \mathbb{F}_q^*$ makes these equations satisfied. \hfill \square

3.4.2. The central vertical maps. We recall a computation from [9, §4]: let $U_0 = \text{Spec} \left( \mathbb{F}_q[Y/X, Z/X] \right)$, $U_1 = \text{Spec} \left( \mathbb{F}_q[Z/Y, Z/Y] \right)$ be open in $\mathbb{P}^2 \left( \mathbb{F}_q \right)$ and let $\mathcal{U} = \{ U_0 \cap C, U_1 \cap C \}$. Using Čech cohomology with respect to open affine covering $\mathcal{U}$ of $C$, we recover $H^1_{dR}(C)$ as the quotient of the space:

\[
\{ (\omega_0, \omega_1, f_{01}) : \omega_i \in H^0 \left( U_i \cap C, \Omega^1_C \right), f_{01} \in H^0 \left( U_i \cap C, \mathcal{O}_C \right), df_{01} = \omega_0 - \omega_1 \},
\]

by the subspace $\{ (df_0, df_1, f_0 - f_1) : f_i \in H^0 \left( U_i \cap C, \mathcal{O}_C \right) \}$. Furthermore, the non-trivial maps in the exact sequence in Proposition 8.1 are induced by $\omega \mapsto \left[ (\omega|_{U_i \cap C}, \omega|_{U_i \cap C}) \right]$ and by $\left[ (\omega_0, \omega_1, f_{01}) \right] \mapsto f_{01}$.

Once we fix the integer $2 \leq k \leq p - 1$ and we pass to the $-k$ eigenspaces of the above cohomology groups, we see that a basis for $H^1_{dR}(C)_{-k}$ is built up by two sets: $\mathcal{A} = \{ e_{a-k} \}_{a=0}^{k-2}$ and $\mathcal{B} = \{ e_{a,q+1-k} \}_{a=1}^{q-k}$, where:

\[
e_{a,q-k} := [(t_1^q t_2^{q-k} dt_2, t_1^q t_2^{q-k} dt_2, 0)]
\]

\[\text{The basis constructed in [9] works in general for any } 1 \leq k \leq q.\]
\[ e_{a,q+1-k} := [-(at_1^{q+1} + at_2^{q+1}) dt_2, -(a + 1 - k)t_1^q t_2^q dt_2, t_1^q t_2^{q+1-k}] \]

This follows from the previous observations and from Proposition 2.3. in [9], that gives explicitly a basis for \( H^1(C, O_C) \).

Notice that the action of \( U_2(F_q^2) \) on \( A \) is clear (see formulae [10, 11 and 12]), while for \( B \) we refer to [9] for the following computation: for any integer \( a \) such that \( 1 \leq -a \leq q - k \) and any \( u \in F_t \), \( t \in F_q^\times \), one has:

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} e_{a,q+1-k} = (-1)^{q+k+1} e_{-(a+q+1-k),q+1-k},
\]

\[
\begin{pmatrix}
1 & 0 \\
u & 1
\end{pmatrix} e_{a,q+1-k} = \sum_{i=0}^{a+q-k} \binom{q+a}{i} (-u)^i e_{a-i,q+1-k+1} + \sum_{i=a+q+1-k}^{a+q-1} \binom{q+a-1}{i} (-u)^i ae_{q+a-1-i,q-k},
\]

\[
\begin{pmatrix}
t & 0 \\
0 & t-q
\end{pmatrix} e_{a,q+1-k} = t^{a(q+1)+q-k+1} e_{a,a+q+1-k}.
\]

Observe that the canonical projection \( \pi : H^1_{dR}(C)_k \rightarrow H^1(C, O)_k \) coming from Corollary 3.9 is given by sending the elements of \( A \) to zero and by \( e_{a,q+1-k} \mapsto t_1^q t_2^{q+1-k} \), for any \( 1 \leq -a \leq q - k \).

We can now prove:

**Proposition 3.17.** The \( F_q^2[U_2(F_q^2)] \) modules \( H^1_{dR}(C)_k \) and \( e^{1-k} \otimes \frac{V_k(1-k)}{D(V_k)} \) are isomorphic. A family of \( U_2(F_q^2) \)-equivariant isomorphisms is given by the maps \( \{f_s\}_{s \in F_q^\times} \), where, for any \( s \in F_q^\times \), \( f_s \) is defined by:

\[
f_s(e_{a,q-k}) = (-1)^{a} \cdot s \cdot X^a Y^{k-2-a} \theta_q \bmod D(V_k), \quad \text{for } 0 \leq a \leq k - 2,
\]

\[
f_s(e_{a,q+1-k}) = (-1)^{a} \cdot k \cdot s \cdot X^a Y^{k-1-a} \bmod D(V_k), \quad \text{for } 1 \leq -a \leq q - k.
\]

**Proof.** Let \( f \) be an isomorphism as above: the action of \( F_q^\times \) decomposes both \( H^1_{dR}(C)_k \) and \( e^{1-k} \otimes \frac{V_k(1-k)}{D(V_k)} \) in the direct sum of one dimensional eigenspaces, therefore an easy computation shows that:

\[
f(e_{a,q-k}) = \gamma(a) \cdot X^a Y^{k-2-a} \theta_q \bmod D(V_k), \quad \text{for } 0 \leq a \leq k - 2,
\]

\[
f(e_{a,q+1-k}) = \delta(a) \cdot X^a Y^{k-1-a} \bmod D(V_k), \quad \text{for } 1 \leq -a \leq q - k.
\]

where \( \gamma(a), \delta(a) \in F_q^\times \). Since the span of the \( e_{a,q-k} \)'s for \( 0 \leq a \leq k - 2 \) is just \( H^0(C, O_C)_k \) and it has to be mapped by \( f \) isomorphically onto \( \theta_q \left( e^{1-k} \otimes V_{k-2} \right) \cong e^{1-k} \otimes V_{k-2} \), Proposition 3.16 gives \( \gamma(a) = (-1)^{a} s \) for some \( s \in F_q^\times \) that we consider now fixed.

Let \( 1 \leq -a \leq q - k \); the relation:

\[
f\left( \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} e_{a,q+1-k} \right) = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \cdot \delta(a) X^a Y^{k-1-a} \bmod D(V_k)
\]

gives \( (d_{a,q} \delta(a) - (-1)^{a+k+1} \delta(k - a - q - 1)) X^{k-1-a} Y^{a+q} \in D(V_k) \); by a degree consideration this can only happen when the coefficient of the monomial is zero,
that is when:
\[ \delta(a) = (-1)^k \delta(k - a - q - 1). \]
For any fixed \( u \in \mathbb{F} \) we also require:
\[
f \left( \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \right) e_{a,q+1-k} = \left( \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \right) \cdot \delta(a) X^{a+q} Y^{k-1-a} \mod D(V_k). \]
By making the action of the matrix explicit, the condition we find after some simplifications is:
\[
\sum_{i=0}^{a+q-k} \binom{q+a}{i} (-u)^i \delta(a - i) X^{a-i+q} Y^{k-1-a+i} + 
\sum_{i=a+q-k+1}^{a+q-1} \binom{q+a}{i} (a - i) (-u)^i \gamma(q + a - 1 - i) X^{a-i+2q-1} Y^{k-q-a+i} + 
\sum_{i=a+q-k+1}^{a+q-1} \binom{q+a}{i} (a - i) (-u)^i \gamma(q + a - 1 - i) X^{a-i+q} Y^{k-1-a+i} = 
\sum_{i=0}^{a+q-k} \binom{q+a}{i} u^i \delta(a) X^{a-i+q} Y^{k-1-a+i} \mod D(V_k). 
\]
This is equivalent to:
\[
\sum_{i=0}^{a+q-k} \binom{q+a}{i} u^i \left[ (-1)^i \delta(a - i) - \delta(a) \right] X^{a-i+q} Y^{k-1-a+i} + 
\sum_{i=a+q-k+1}^{a+q-1} \binom{q+a}{i} (a - i) u^i (-1)^a s X^{a-i+2q-1} Y^{k-q-a+i} + 
\sum_{i=a+q-k+1}^{a+q-1} \binom{q+a}{i} u^i [(a - i) (-1)^a s + \delta(a)] X^{a-i+q} Y^{k-1-a+i} \equiv 0 \mod D(V_k). 
\]
Notice that for \( 0 \leq i \leq a + q - k \), the monomial \( X^{a-i+q} Y^{k-1-a+i} \) does not belong to \( D(V_k) \). The above congruence holds if \( \delta(a) = (-1)^i \delta(a - i) \) (for any \( i = 0, \ldots, a + q - k \) and any \( -a = 1, \ldots, q - k \)) and the polynomial \( \sum_{i=a+q-k+1}^{a+q-1} \binom{q+a}{i} u^i \), \( g_{a,i,s}(X,Y) \) is in \( D(V_k) \), where:
\[
g_{a,i,s}(X,Y) = (a - i) (-1)^a s X^{(a-i+q)+(q-1)} Y^{k-q-a+i} + 
\left[ (a - i) (-1)^a s + \delta(a) \right] X^{a-i+q} Y^{(k-q-a+i)+(q-1)}. 
\]
In our range, we have \( 1 \leq a + q - i \leq k - 1 \); since a basis for \( D(V_k) \) is given by:
\[
\left\{ j X^{j+q-1} Y^{k-j} + (k-j) X^j Y^{k-j+q-1} \right\}_{j=0}^k, 
\]
we deduce that the above polynomial is in \( D(V_k) \) if for any \( a \) such that \( 1 \leq a \leq q - k \) we can find an element \( \sigma_a \in \mathbb{F}_q^k \) such that \( \sigma_a \cdot (a - i) = (-1)^a (a - i) s \) and \( \sigma_a \cdot (k + i - a) = (-1)^a (a - i) s - \delta(a) \). This is equivalent to \( \delta(a) = ks (-1)^{a+1} \) for every \( 1 \leq a \leq q - k \). Thus this expression for \( \delta(\cdot) \) is compatible with all the previous conditions we imposed on it. \( \square \)

3.4.3. The right vertical maps. We now complete the construction of the main diagram. Recall that \( 2 \leq k \leq p - 1 \) and denote by \( \pi' \) (resp. \( \pi'' \)) the canonical projection \( H^1_{\text{dR}}(C)_-k \to \text{coker} \iota \) (resp. \( e^{1-k} \otimes \frac{V_{k}^{(k+1)}}{D(V_k)} \to \text{coker} \overline{\theta}_q \)), where \( \iota : H^0(C, \Omega^1_C)_-k \to H^1_{\text{dR}}(C)_-k \). We have:

**Corollary 3.18.** For any \( s \in \mathbb{F}_q^* \) there is a unique \( U_2(\mathbb{F}_q^*) \)-equivariant isomorphism of \( \mathbb{F}_q \)-spaces \( \psi_s : \text{coker} \iota \to \text{coker} \overline{\theta}_q \) such that \( \psi_s \circ \pi' = \pi'' \circ f_s \). In particular we have \( \text{coker} \overline{\theta}_q \cong H^1(C, \mathcal{O}_C)_-k \).
Proof. By Proposition 3.16 and Proposition 3.17 we have the commutative diagram:
\[
\begin{array}{ccc}
0 & \rightarrow & H^0(C, \Omega^1_C)_{-k} \\ \downarrow \varphi_s & & \downarrow f_s \\ 0 & \rightarrow & e^{1-k} \otimes V_{k-2}
\end{array}
\rightarrow
\begin{array}{ccc}
H^1_{dR}(C)_{-k} & \rightarrow & H^1_{dR}(C)_{-k} \\ f_s & \rightarrow & e^{1-k} \otimes \frac{V_{k+(q-1)}}{D(V_k)}
\end{array}
\]
whose rows are exact. The statement just follows from this and the identification \( \text{coker } \psi_q \simeq H^1(C, \mathcal{O}_C)_{-k} \).
\[\square\]

Remark 3.19. If \( q = p \), by Corollary 3.12, we have an isomorphism of \( U_2 (\mathbb{F}_q^r) \)-modules \( H^1(C, \mathcal{O}_C)_{-k} \simeq e \otimes V_{q-1-k} \). By [9], Prop. 2.8, we also have, for any \( q \), the isomorphisms of \( \text{SL}_2 (\mathbb{F}) \)-modules:
\[\text{coker } \overline{\theta}_q \simeq H^1(C, \mathcal{O}_C)_{-k} \simeq V_{q-1-k}.\]

Notice that if \( q = p \) then \( V_{q-1-k} \) is irreducible as representation of \( \text{SL}_2 (\mathbb{F}) \), and self-dual.

3.4.4. Back to the base field. Now we go back to our original base field \( \mathbb{F} = \mathbb{F}_q \), \( q = p^n \). At this purpose, we consider the projective curve \( C \) as defined over \( \bar{\mathbb{F}} \), so that its various cohomology groups are \( \mathbb{F} \)-spaces. Notice that the elements of the basis we considered for the cohomology groups of \( C/\mathbb{F}_q \) are defined over the prime field of \( \mathbb{F} \), so that we will write \( H^1_{dR} (C/\mathbb{F}_q)_{-k} \) to denote the \( \mathbb{F} \)-space with basis \( \{ e_{a,q-1-k} \}_{a=0}^{k-2} \cup \{ e_{a,q+1-k} \}_{a=1}^{q-k} \) and similarly for the other cohomology (sub)spaces we have been treating. In particular the sequence of \( \mathbb{F} [\text{SL}_2 (\mathbb{F})] \)-modules:
\[
0 \rightarrow H^0(C/\mathbb{F}_q, \mathcal{O}_C^1)_{-k} \rightarrow H^1_{dR}(C/\mathbb{F}_q)_{-k} \rightarrow H^1(C/\mathbb{F}_q, \mathcal{O}_{C/\mathbb{F}_q})_{-k} \rightarrow 0
\]
remains exact. (Notice that we do not change the names of the maps involved with respect to the case of modules over \( \mathbb{F}_q^2 \) if the behavior of the map is clear as above).

We can summarize some of the results we obtained as follows:

Proposition 3.20. Let \( q > 2 \) be a power of a rational prime \( p \), \( \mathbb{F} \) a field with \( q \) elements, and let \( k \) be an integer such that \( 2 \leq k \leq p - 1 \). Let \( \text{SL}_2 (\mathbb{F}) \) act on \( \mathbb{F}^2 \) in the standard way, and define the left \( \text{SL}_2 (\mathbb{F}) \)-representation \( V'_k := \text{Sym}^k \mathbb{F}^2 \).

Define furthermore \( C \) to be the projective curve associated to the polynomial \( X^q - Y^q - Z^q+1 \in \mathbb{F}[X,Y,Z] \). For any \( s \in \mathbb{F}^\times \) we have the following commutative diagram of \( \mathbb{F} [\text{SL}_2 (\mathbb{F})] \)-modules, in which the rows are exact and the vertical arrows are isomorphisms:
\[
\begin{array}{ccc}
0 & \rightarrow & H^0(C/\mathbb{F}_q, \mathcal{O}_{C/\mathbb{F}_q}^1)_{-k} \\ \downarrow \varphi_s & & \downarrow f_s \\ 0 & \rightarrow & e^{1-k} \otimes V'_{k-2}
\end{array}
\rightarrow
\begin{array}{ccc}
H^1_{dR}(C/\mathbb{F}_q)_{-k} & \rightarrow & H^1_{dR}(C/\mathbb{F}_q)_{-k} \\ f_s & \rightarrow & e^{1-k} \otimes \frac{V'_{k+(q-1)}}{D(V'_k)}
\end{array}
\rightarrow
\begin{array}{ccc}
H^1(C/\mathbb{F}_q, \mathcal{O}_{C/\mathbb{F}_q})_{-k} & \rightarrow & 0 \\ \psi_s & \rightarrow & \text{coker } \theta_q \\
\end{array}
\rightarrow
0.

4. Integral models

From now on we work over the field \( \mathbb{F} \); we then go back to the notation of Section 2 and write \( V_h \) for the left \( G = \text{GL}_2 (\mathbb{F}) \)-representation \( \text{Sym}^h \mathbb{F}^2 \) (\( h \geq 0 \)).
4.1. Extending the action to $GL_2$. Let $k$ be an integer such that $2 \leq k \leq p - 1$; the natural action of $G$ on $V_{k-2}$ and on $\frac{V_{k+(q-1)}}{D(V_k)}$ gives an exact sequence of $F[G]$-modules:

$\theta$ 

(16) \hspace{1cm} 0 \to e \otimes V_{k-2} \xrightarrow{\pi_q} \frac{V_{k+(q-1)}}{D(V_k)} \xrightarrow{\pi''_q} \text{coker} \Theta_q \to 0.$

For any $s \in F^\times$, there is a unique extension of the action of $SL_2(F)$ on $H^0(C/F, \Omega^1_{C/F})$ and on $H^1_{dR}(C/F) - k$ to an action of $G$ making $\varphi_s$ and $f_s$ isomorphisms of $G$-modules. By definition of $\varphi_s$ and $f_s$ we see that this extension does not depend upon $s$ and is given by defining, for any $x_1, x_2 \in F^\times$:

\begin{align*}
\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \cdot e_{a,q-k} & : = x_1^{a+1} x_2^{k-a-1} e_{a,q-k}, \quad (0 \leq a \leq k - 2) \\
\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \cdot e_{a,q+1-k} & : = x_1^{a+1} x_2^{k-a-1} e_{a,q+1-k}, \quad (1 \leq -a \leq q - k).
\end{align*}

Note that this definition of the action of the standard maximal torus of $G$ on $H^1_{dR}(C/F) - k$ is forced if we want it to be compatible with any $U_2(F_{q^2})$-isomorphism $H^1_{dR}(C/F) - k \to \frac{V_{k+(q-1)}}{D(V_k)} \otimes_F F_{q^2}$ (see proof of Proposition 3.17). Furthermore, the above definition can also be deduced from Proposition 4.10. in [9]. We deduce:

**Proposition 4.1.** For any $q > 2$ and any $2 \leq k \leq p - 1$ we have an isomorphism of exact sequences of $F[G]$-modules:

\[
\begin{array}{c}
0 \to H^0(C/F, \Omega^1_{C/F}) \to H^1_{dR}(C/F) \to H^1(C/F, \mathcal{O}_{C/F}) \to 0 \\
\downarrow \cong \hspace{1cm} \downarrow \cong \hspace{1cm} \downarrow \cong \hspace{1cm} \downarrow \cong \\
0 \to e \otimes V_{k-2} \xrightarrow{\pi_q} \frac{V_{k+(q-1)}}{D(V_k)} \xrightarrow{\pi''_q} \text{coker} \Theta_q \to 0.
\end{array}
\]

Furthermore, if $q = p$ we have $\text{coker} \Theta_q \simeq e^k \otimes V_{q-1-k}$.

4.2. Integral models. For a finite field $F'$, let us denote by $W(F')$ the ring of Witt vectors of $F'$. We fix a rational prime $l \neq p$, field isomorphisms $\mathbb{F}_l \simeq \mathbb{C}$ and $\mathbb{Q}_p \simeq \mathbb{C}$, and ring embeddings $W(F) \hookrightarrow W(F_{q^2}) \hookrightarrow \mathbb{Q}_p \simeq \mathbb{C}$. We also denote by $\chi : F^\times_{q^2} \to \mathbb{Q}_p^\times$ the Teichmüller lifting we considered at the beginning of section 2, and we let $\Theta(\chi^k)$ be the cuspidal $\mathbb{Q}_p^\times$-representation of $G = GL_2(F)$ associated to the indecomposable character $\chi^k$ ($2 \leq k \leq p - 1$). We can now go back to formula (7) and prove the following:

**Theorem 4.2.** Let $q > 2$ and $2 \leq k \leq p - 1$ with $k \neq \frac{q+1}{2}$; there exists an integral model $\tilde{\Theta}(\chi^k)$ of the $\mathbb{Q}_p[G]$-module $\Theta(\chi^k)$ coming from the $-k$-eigenspace of the first crystalline cohomology group of the projective curve $C : XY^q - X^qY - Z^{q+1} = 0$ such that there is an isomorphism of $F[G]$-modules:

\[
\frac{V_{k+(q-1)}}{D(V_k)} \simeq \Theta(\chi^k).
\]
Proof. As observed in [9], §2.10, we have for any \( g \in U_2(F_q) \):
\[
\sum_{i \geq 0} (-1)^i \text{tr} \left( g, H^i_{\text{cris}}(C/F_q) \otimes W(F_q) \otimes \mathcal{Q}_p \right) = \sum_{i \geq 0} (-1)^i \text{tr} \left( g, H^i(C, \mathcal{Q}_p) \right),
\]
where \( H^i \) denotes étale cohomology over \( \mathcal{Q}_p \) (for more details, see [3]). This gives an isomorphisms of \( U_2(F_q) \)-modules:
\[
H^1_{dR}(C/W(F_q)) \otimes W(F_q) \mathbb{C} \simeq H^1_{\text{cris}}(C/F_q) \otimes W(F_q) \mathbb{C} \simeq H^1(C, \mathcal{Q}_p).
\]
If \( 1 \leq k \leq q \) we deduce \( H^1_{dR}(C/W(F_q))_{-k} \otimes W(F_q) \mathbb{C} \simeq H^1(C, \mathcal{Q}_p)_{-k} \), hence also an isomorphism of \( SL_2(F) \)-modules:
\[
H^1_{dR}(C/W(F_q))_{-k} \otimes W(F_q) \mathbb{C} \simeq H^1(C, \mathcal{Q}_p)_{-k}.
\]
Since \( H^1(C, \mathcal{Q}_p)_{-k} \) is the subspace of \( H^1(C, \mathcal{Q}_p) \) on which \( \mu \) acts via the character \( \vartheta_{-k} : \mu \to \mathcal{Q}_p^\times : t \mapsto t^{-k} \), by [10], Example 2.20, the \( \mathcal{Q}_p \)-representation of \( SL_2(F) \) afforded by \( H^1(C, \mathcal{Q}_p)_{-k} \) is of the following type: if \( \vartheta_{-k} \) is in general position (i.e. \( \vartheta_{-k}^2 \neq 1 \) or equivalently \( k \neq (q + 1)/2 \) \( H^1(C, \mathcal{Q}_p)_{-k} \) is an (irreducible) cuspidal representation of \( SL_2(F) \) over \( \mathcal{Q}_p \). If \( k = (q + 1)/2 \), then \( H^1(C, \mathcal{Q}_p)_{-k} = V \oplus V^* \), with \( V \) cuspidal.

From the character theory of \( SL_2(F) \) and \( GL_2(F) \) over an algebraically closed field of characteristic zero (cf. [4]), we know that, if \( \vartheta_{-k} \) is in general position, there is an indecomposable character \( \varsigma : F_q^\times \to \mathbb{C}^\times \simeq \mathcal{Q}_p^\times \) for which there is an isomorphism of \( SL_2(F) \)-modules:
\begin{equation}
H^1(C, \mathcal{Q}_p)_{-k} \simeq \operatorname{Res}^{SL_2(F)}_{SL_2(F_q)}(\Theta(\varsigma)).
\end{equation}
(Notice that \( \varsigma \) is not unique and can be changed by \( \varsigma^{-1} \) or any other indecomposable character that equals \( \varsigma \) on \( \mu \).)
If now we take any \( p \)-adic integral model of each side of the above isomorphism (e.g. we can take \( H^1_{dR}(C/W(F_q))_{-k} \) for the étale cohomology group) and we reduce mod \( p \), we find the \( SL_2(F) \)-module isomorphism \( H^1_{dR}(C/W(F_q))^\text{ss}_{-k} \simeq (\Theta(\varsigma))^\text{ss} \) (notice that by Remark 3.10 we have a natural identification \( H^1_{dR}(C/W(F_q))_{-k} \simeq H^1_{dR}(C/W(F_q))_{-k} \otimes W(F_q) \)). If furthermore \( 2 \leq k \leq p - 1 \), the first module is isomorphic to \( (V_{\chi^2 + (q + 1)}) \text{ss} \) and hence to \( (\Theta^j(\chi^k))^\text{ss} \) for any choice of integral model \( \Theta^j(\chi^k) \) of \( \Theta(\chi^k) \), by formula [7] Therefore the (reductions of the) Brauer characters of \( \Theta(\varsigma) \) and \( \Theta^j(\chi^k) \) need to coincide.

In the notation of §2.2., let \( \iota : F_q^\times \to M_2(F) \) be given by \( c = x + y\sqrt{\varepsilon} \mapsto \begin{pmatrix} x & y \\ y & x \end{pmatrix} \), where \( x, y \in F \) and \( \varepsilon \) is a generator of \( F^\times \) (recall \( p > 2 \)). If \( \iota(c) \in SL_2(F) \) we have \( c \in \mu \); the formulae giving the Brauer characters of the cuspidal representations of \( GL_2(F) \) imply that, if \( \varsigma_{\mu} = \chi^h_{\mu} \) \((0 \leq h \leq q)\), we have \( \chi(c)^h + \chi(c)^{-h} = \chi(c)^h + \chi(c)^{-h} \) for any \( c \in \mu \), so that \( (\chi(c)^k + h - 1)(\chi(c)^k - \chi(c)^h) = 0 \). We conclude that \( k \equiv \pm h \text{ (mod } q + 1) \) and \( \varsigma_{\mu} = \chi^h_{\mu} \). We can assume without loss of generality \( \varsigma = \chi^k \); this implies that the \( SL_2(F) \)-action on \( H^1(C, \mathcal{Q}_p)_{-k} \) extends to a \( GL_2(F) \)-action giving an isomorphism \( H^1(C, \mathcal{Q}_p)_{-k} \simeq \Theta(\chi^k) \).

If \( \tilde{\Theta}(\chi^k) \) is the \( W(F) \)-model of \( \Theta(\chi^k) \) corresponding to \( H^1_{dR}(C/W(F))_{-k} \) in the above isomorphism, we have \( \tilde{\Theta}(\chi^k) \simeq H^1_{dR}(C_{/F})_{-k} \). \( \square \)
5. Modular forms

In this section we assume that \( q = p \) is a prime number larger than 3 and, for any non-negative integer \( k \), we denote by \( V_k \) the \( \overline{\mathbb{F}}_p[GL_2(\mathbb{F}_p)] \)-module \( \text{Sym}^k \overline{\mathbb{F}}_p^2 \).

Let us fix an integer \( N \geq 5 \) not divisible by the prime \( p \); let \( E \to X_1(N) \) be the universal generalized elliptic curve over the modular curve \( X_1(N) \) (all the schemes here are over \( \overline{\mathbb{F}}_p \)) and let \( \omega_{X_1(N)} = 0^* \Omega^1_{E/X_1(N)} \) be the coherent sheaf obtained by pulling-back via the zero section \( 0 : X_1(N) \to E \) the sheaf of Kähler differentials of \( E \) over \( X_1(N) \). For any integer \( k \geq 0 \) we let \( M_k (\Gamma, \overline{\mathbb{F}}_p) := H^0(X_1(N), \omega_{X_1(N)}^\otimes k) \) be the space of mod \( p \) modular forms of weight \( k \) and level \( \Gamma := \Gamma_1(N) \).

In view of the Eichler-Shimura isomorphism and of \([1]\), Proposition 2.5., the study of Hecke eigensystems of mod \( p \) modular forms of weight \( k \geq 2 \) and level \( N \) leads naturally to the study of the eigenvalues of the Hecke algebra \( \mathcal{H}_N \) acting on the cohomology group \( H^1(\Gamma, V_{k-2}) \), where \( \Gamma \) acts on \( V_{k-2} \) via its reduction mod \( p \), and the action of \( \mathcal{H}_N \) is defined as in \([1]\) (here \( \mathcal{H}_N \) is generated over \( \overline{\mathbb{F}}_p \) by the Hecke operators \( T_r \) where \( p \not| r \)).

As for the group \( GL_2(\mathbb{F}_p) \) we have two operators raising the levels of the symmetric power representations \( V_k \)'s by \( p + 1 \) and \( p - 1 \) respectively, the theory of modular forms mod \( p \) offers two maps that increase weights by \( p + 1 \) and \( p - 1 \): the theta operator \( \Theta \) and the Hasse invariant \( A \) respectively (cf. \([17]\) for their definitions). One might expect a connection between the representation theoretical side discussed in the previous sections and the modular forms side.

Indeed, in \([1]\), A. Ash and G. Stevens identified a group-theoretical analogue of the \( \Theta \)-operator \((\text{[}\text{[1]}\text{]}, \text{Theorem }3.4.)\) in the map induced in cohomology by the operator \( \theta_p \), that is by the multiplication by the Dickson polynomial \( X^p Y - XY^p \) : for any \( k \geq 2 \) the map \( \theta_p \) induces an Hecke-equivariant map:

\[
\theta_{p,*} : H^1(\Gamma, V_{k-2}) \to H^1(\Gamma, V_{k+p-1})
\]

that corresponds, on the space of modular forms, to raising the weight by \( p + 1 \).

5.1. Group cohomology and the Hasse invariant. In \([5]\), Edixhoven and Khare construct a cohomological analogue of the Hasse invariant: by studying the degeneracy map \( H^1(\Gamma, V_0)^2 \to H^1(\Gamma \cap \Gamma_0(p), V_{p-1}) \), they determine a monic Hecke-equivariant homomorphism:

\[
\alpha : H^1(\Gamma, V_0) \hookrightarrow H^1(\Gamma, V_{p-1}).
\]

Notice that \( \alpha \) is not defined in \([5]\) as coming from a morphism on coefficients.

The \( D \)-map allows us to extend the existence of an injection as the map \( \alpha \) above:

**Proposition 5.1.** Let \( \mathfrak{M} \) be a non-Eisenstein maximal ideal of \( \mathcal{H}_N \):

1. if \( k \geq 0 \) and \( H^1(\Gamma, V_k)_{2\mathfrak{M}} \neq 0 \), then also \( H^1(\Gamma, V_{k+(p-1)})_{2\mathfrak{M}} \neq 0 \);
2. if \( 0 \leq k \leq p - 1 \), there is an embedding of Hecke modules:

\[
H^1(\Gamma, V_k)_{2\mathfrak{M}} \hookrightarrow H^1(\Gamma, V_{k+(p-1)})_{2\mathfrak{M}}
\]

that is induced by \( D \) if \( 0 < k \leq p - 1 \), and is the above map \( \alpha \) for \( k = 0 \).

**Proof.** If \( k \geq 0 \) and \( k \neq 0 \mod (p+1) \), Proposition 5.1 applied with \( q = p \) (and a suitable re-indexing) says that \( V_{k+(p-1)} - V_k \) is positive in \( K_0(GL_2(\mathbb{F}_p)) \), giving the first assertion for \( k \neq 0 \mod (p+1) \).
If $1 \leq k \leq p - 1$ we have the exact sequence of $GL_2(\mathbb{F}_p)$-modules:

$$0 \to V_k \overset{D}{\to} V_{k+(p-1)} \to \text{coker} \ D \to 0.$$  

By passing to the long exact sequence in cohomology and localizing with respect to the non-Eisenstein maximal ideal $\mathfrak{M}$ we get the second statement for $1 \leq k \leq p - 1$ (cf. [15]). If $k = 0$, the existence of a monic map $\alpha : H^1(\Gamma, \mathbb{F}_p)_{\mathfrak{M}} \to H^1(\Gamma, V_{p-1})_{\mathfrak{M}}$ is the cited above result of Edixhoven and Khare ([5]).

The existence of $\alpha$ also implies the first statement for $k \equiv 0(\text{mod } p+1)$: if $k = s(p+1)$ for some $s \geq 0$, formula [4] gives the following identity in $K_0(GL_2(\mathbb{F}_p))$:

$$V_{s(p+1)+p-1} = e^s \cdot V_{p-1} + (V_{s(p+1)} - e^s \cdot V_0). \quad (18)$$

Notice that $V_{s(p+1)} - e^s \cdot V_0 > 0$ because of the existence of the monic map $\theta_p : e \otimes V_0 \to V_{p+1}$. If $H^1(\Gamma, V_{s(p+1)})_{\mathfrak{M}} \neq 0$ then $H^1(\Gamma, e^s \otimes V_0)_{\mathfrak{M}} \neq 0$ or $H^1(\Gamma, V_{s(p+1)}) / e^s \otimes V_0)_{\mathfrak{M}} \neq 0$; in the first case, by applying $\alpha$ to the twisted module $H^1(\Gamma, e^s \otimes V_0)_{\mathfrak{M}}$, we deduce $H^1(\Gamma, e^s \otimes V_{p-1})_{\mathfrak{M}} \neq 0$ and hence, by [18] above, $H^1(\Gamma, V_{s(p+1)+p-1})_{\mathfrak{M}} \neq 0$. If it is $H^1(\Gamma, V_{s(p+1)}/e^s \otimes V_0)_{\mathfrak{M}} \neq 0$, we conclude again $H^1(\Gamma, V_{s(p+1)+p-1})_{\mathfrak{M}} \neq 0$. \hfill $\Box$

**Remark 5.2.** The above result cannot be deduced only by the existence of the map $D$ and by Proposition 5.1, since if $k = 0$ then $V_{p-1} - V_0$ is not positive in $K_0(GL_2(\mathbb{F}_p))$.

### 5.2. Construction of the maps $\alpha_k$ for $k > 0$.

The definition of $\alpha$ is given in [5] for $k = 0$; however starting from this case it is not hard to generalize the construction to any integer $k \geq 0$. Recall that we fixed a prime $p > 3$ and an integer $N \geq 5$ prime to $p$.

Let $\Gamma_0 := \Gamma \cap \Gamma_0(p)$, $g := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $g^* = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$; denote by $\beta' : H^1(\Gamma, V_k) \to H^1(\Gamma_0, V_k)$ the restriction map twisted by $g$, i.e. the homomorphism induced by assigning to the $1$-cocycle $\zeta \in Z^1(\Gamma, V_k)$ the cocycle:

$$x \mapsto g^* \cdot \zeta(gxg^{-1}), \quad x \in \Gamma_0.$$  

(For $k = 0$ the action of $g^*$ is trivial, and $\beta'$ is the twisted restriction considered in [5]; if $k > 0$, $g^*$ acts as multiplication by the elementary matrix $E_{11} \in M_2(\mathbb{F}_p)$). Notice that by Shapiro’s lemma, we have a natural isomorphism of Hecke modules $\beta'' : H^1(\Gamma_0, V_k) \to H^1(\Gamma, \text{Ind}_{\Gamma_0}^\Gamma V_k)$.

Let us now fix a left transversal $\{x_1, \ldots, x_{p+1}\}$ of $\Gamma_0$ inside $\Gamma$; e.g. we can take:

$$x_i := \begin{pmatrix} 1 \\ N(i-1) \end{pmatrix}, \quad x_{p+1} = \begin{pmatrix} Bp \\ AN \end{pmatrix}, \quad (1 \leq i \leq p),$$

$$x_{i+1} := \begin{pmatrix} 1 \\ N(i-1) \end{pmatrix}, \quad (1 \leq i \leq p+1),$$

where $AN + Bp = 1$ and $A, B \in \mathbb{Z}$. Denote by $\mathbb{F}_p[\Gamma/\Gamma_0]$ the $\mathbb{F}_p$-vector space of functions $f : \Gamma/\Gamma_0 \to \mathbb{F}_p$ endowed with the left $\Gamma$-action defined by $(x \cdot f)(x\Gamma_0) = f(x^{-1}x\Gamma_0)$, for every $x \in \Gamma, f \in \mathbb{F}_p[\Gamma/\Gamma_0]$ and $1 \leq i \leq p+1$. There is a $\Gamma$-isomorphism $b : \text{Ind}_{\Gamma_0}^\Gamma V_k \to \mathbb{F}_p[\Gamma/\Gamma_0] \otimes_{\mathbb{F}_p} V_k$ (where the codomain is endowed with the diagonal $\Gamma$-action) given by sending an element $f : \Gamma/\Gamma_0 \to V_k$ of $\text{Ind}_{\Gamma_0}^\Gamma V_k$ to:

$$\sum_{i=1}^{p+1} x_i \Gamma_0 \otimes x_i f(x\Gamma_0).$$
Fixing the choice of transversal given by [19], we can identify $\mathbb{F}_p [\Gamma/\Gamma_0]$ and $\mathbb{F}_p [\mathbb{P}^1(\mathbb{F}_p)]$ through the bijection $\Gamma/\Gamma_0 \to \mathbb{P}^1(\mathbb{F}_p)$ defined by $x_i \mapsto (1 : N(i-1))$ for $1 \leq i \leq p$, and $x_{p+1} \mapsto (0 : 1)$. Using Lemma 2.6 we find:

$$\mathbb{F}_p [\Gamma/\Gamma_0] \otimes_{\mathbb{F}_p} V_k \simeq \mathbb{F}_p [\mathbb{P}^1(\mathbb{F}_p)] \otimes_{\mathbb{F}_p} V_k \simeq V_k \oplus (V_{p-1} \otimes_{\mathbb{F}_p} V_k).$$

The composition of $\beta'' \circ \beta'$ with the map induced in cohomology by $b$, and the above isomorphisms yield an Hecke equivariant map:

$$\beta : H^1(\Gamma, V_k) \to H^1(\Gamma, V_k) \oplus H^1(\Gamma, V_{p-1} \otimes_{\mathbb{F}_p} V_k).$$

By composing $\beta$ with the projection onto the second factor and then with the map induced in cohomology by polynomial multiplication $V_{p-1} \otimes_{\mathbb{F}_p} V_k \to V_{k+(p-1)}$, we finally obtain:

**Proposition 5.3.** For every integer $k \geq 0$ there is an Hecke-equivariant homomorphism:

$$\alpha_k : H^1(\Gamma, V_k) \to H^1(\Gamma, V_{k+(p-1)})$$

such that $\alpha_0 = \alpha$ is the map defined in [5].

Some questions arise naturally. For example, we know that $\alpha_0$ is monic, and one might expect that also $\alpha_k$ is monic in the range $1 \leq k \leq p-1$, i.e. when $V_k \neq \mathbb{F}_p$ is irreducible. In this range, by Proposition 5.1, we also have - for any non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathcal{H}_N$ - a monic Hecke-map $H^1(\Gamma, V_k)_{\mathfrak{m}} \to H^1(\Gamma, V_{k+(p-1)})_{\mathfrak{m}}$ induced by $D$, that one would like to compare with $\alpha_k$.

To this purpose, it could be useful to see the maps $\alpha_k$’s as homomorphisms between groups of modular symbols, using the Hecke-equivariant isomorphism

$$\text{Hom}_V(\mathcal{D}_0, V_k) \xrightarrow{\sim} H^1_c(\Gamma, V_k),$$

where $\mathcal{D}_0$ is the group of divisors of degree zero of $\mathbb{P}^1(\mathbb{Q})$, and $H^1_c(\Gamma, \cdot)$ denotes compactly supported cohomology (cf. [1], §4).

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