Luttinger stripes in antiferromagnets

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We propose a model for the physics of stripes in antiferromagnets in which the stripes are described by Luttinger liquids hybridized with antiferromagnetic domains. Using bosonization techniques we study the model in the limit where the magnetic correlation length is larger than the inter-stripe distance and propose an explanation for the commensurate-incommensurate phase transition seen in neutron scattering in the underdoped regime of La$_{2-x}$Sr$_x$CuO$_4$. The explanation is based on a phase to anti-phase domain transition in the spin configuration which is associated with the transverse motion of the stripes. Using a non-linear $\sigma$ model to describe the antiferromagnetic regions we conjecture the crystalization of the stripes in the magnetically ordered phase.

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I. INTRODUCTION

In recent years we have seen tremendous development in the production of new materials which have a phase diagram where an insulating magnetic state is very close to a metallic or superconducting state. The most famous examples are the high temperature superconductors such as La$_{2-x}$Sr$_x$CuO$_4$, Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and Y Ba$_2$CuO$_{6+x}$ which are formed by layers of Cu O$_2$. Their phase diagram is well-known. At zero doping the ground state is insulating and antiferromagnetic. Antiferromagnetism is suppressed very rapidly with doping and it is followed by a spin-glass phase and a superconducting state with high critical temperature. There are many anomalies in these systems when compared to a normal Fermi liquid, which have been a source of a lot of controversy in the last few years.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{stripes.png}
\caption{Geometry of the problem represented by vertical stripes (shown with black and white circles) inside of an antiferromagnet (which is represented by $\uparrow$ and $\downarrow$). In the figure on the top we show a phase domain and on the bottom we show an anti-phase domain. Observe the orientation of the spins.}
\end{figure}

In this paper we will introduce a model for the discussion of these systems in the presence of segregation of charge into one-dimensional domain walls or stripes. The geometry of the problem is shown in Fig. The problem of formation of domain walls and stripes has been discussed for quite some time in terms of the proximity of these materials to phase separation. Macroscopic phase separation has been observed in La$_2$CuO$_{4+\delta}$, and stripes have been seen experimentally in La$_{2-x}$Sr$_x$NiO$_{4+y}$ and with neutrons and muons.

More recently a direct evidence for stripe formation was given in neutron scattering by Tranquada et al. in La$_{1.6-\delta}$Nd$_{0.4}$Sr$_x$CuO$_4$. In these experiments Nd was used to stabilize the low temperature tetragonal phase (LTT) which has a tilt of the oxygen octahedra which is favorable for stripe pinning. These experiments seem to be indicative of an inhomogeneous ground state. Moreover, the existence of stripes in cuprates and ladders have been confirmed in many different computer simulations of $t-J$ and Hubbard models. In these models the formation of domains is associated with the formation of pairs of holes and segregation from the magnetic regions. In the simulations only short range interactions are included and it is not clear whether long range forces are important in order to stabilize the stripe structures, especially because the materials under consideration are poor metals and screening is not guaranteed. Actually the idea of a frustrated phase separation due to long range forces as discussed by Emery and Kivelson has been very successful in explaining many of the important features of high temperature cuprates. Their theory should be contrasted with theories where it is Fermi surface effects which drive the formation of inhomogeneous states.

Here we look at this problem from a more phenomenological point of view and do not ask for the origin of the instability towards stripe formation but rather try to develop the consequences which come from it. One of the main features is that the charge of the dopants goes to specific quasi-one dimensional regions of the system which are surrounded by undoped quasi-two dimensional regions. In this way the problem splits into two independent parts which hybridize with each other: a quasi-one-dimensional interacting electron gas and a quasi-two
dimensional magnetic electron system. A lot is known about these two systems separately. The one-dimensional system has been studied via bosonization and renormalization group techniques, and it is a very mature field [11]. The field of itinerant magnetism has seen a lot of development especially from the point of view of renormalization group [12] and effective spin models [13]. The objective of this work is to try to link these two approaches in order to gain insight into the physics of these materials.

In section II we propose a model with short range forces for the physics of the stripes. In section III we discuss the antiferromagnetically ordered region and focus on the effect of the stripes on the antiferromagnet and vice-versa. We also compare our results with recent neutron scattering data. In section IV we study the magnetically disordered phase and suggest a possible phase diagram for the stripes. Section V contains our conclusions.

II. THE MODEL

Although the basic model for the problem of itinerant holes in magnets can be quite complicated we are going to simplify it by making assumptions about the geometric structure of the charge distribution in the planes of Cu O₂. Instead of starting from a microscopic model such as a t − J or Hubbard model with possible long-range interactions we choose to write an effective model which has already incorporated into it the minimal physical requirements for the presence of stripes. In this model one assumes that the regions rich in carriers (which we call stripes) are separated from the regions with no carriers (which we call magnetic regions). This segregation can be introduced if we assume that the energy of the electrons is locally raised relative to the magnetic regions. In this picture we look at the problem as a set of coupled chains where the chains that “contain” the stripes have the bottom of the band raised in energy relative to the magnetic regions (see Fig. 2).

In this way the holes tend to segregate into these chains — that is, form stripes. The electrons in the stripes can hybridize with the magnetic regions, and this hybridization gives rise to charge fluctuations. The amount of fluctuations is controlled by the shift in energy and the hybridization.

For the stripes and the magnetic regions we use a simple Hubbard model. Our complete model reads,

\[ H = H_I + H_S + H_t \]

where the magnetic region is described by the Hubbard model

\[ H_I = -t \sum_{\langle \vec{r}, \vec{r}' \rangle, \alpha = \uparrow, \downarrow} c^\dagger_{\alpha}(\vec{r}) c_{\alpha}(\vec{r}') + \text{c.c.} \]

\[ + U \sum_{\vec{r}} n_{c,\uparrow}(\vec{r}) n_{c,\downarrow}(\vec{r}) \]

where \( \vec{r} = a(n, m) \) labels the sites in a two-dimensional square lattice (\( a \) is the lattice spacing), \( n \) labels the \( x \) direction (perpendicular to the stripe), \( m \) labels the \( y \) direction (parallel to the stripe) and the brackets \( \langle \cdot \rangle \) imply sum over nearest neighbor sites. \( c^\dagger_{\alpha}(\vec{r}) \) creates a fermion with spin projection \( \alpha \) at the site \( \vec{r} \) in the magnetic region. \( n_{c,\alpha} = n_{c,\alpha}(\vec{r}) \) is the occupation number at site \( \vec{r} \). The stripes are assumed to be periodically located at \( x = n \ell \) where \( n \) is an integer and \( \ell = Na \) is the interstripe distance (the superscript in \( H_I \) means that the chains at \( n = 0, \pm N, \pm 2N, \ldots \) are excluded from the sums). The stripe Hamiltonian is also a Hubbard model that reads,

\[ H_S = -\hat{\ell} \sum_{m, n, \alpha = \uparrow, \downarrow} a^\dagger_{\alpha}(nN, m) a_{\alpha}(nN, m + 1) + \text{c.c.} \]

\[ + \epsilon_0 \sum_{m, n, \alpha = \uparrow, \downarrow} n_{a,\alpha}(nN, m) \]

\[ + \hat{U} \sum_{m, n} n_{a,\uparrow}(nN, m) n_{a,\downarrow}(nN, m) \]

where we are assuming the possibility of mass and vertex renormalization (\( \hat{\ell} \neq \ell, \hat{U} \neq U \)) and \( \epsilon_0 \) is the shift in energy as it is shown in Fig. 2. \( a^\dagger_{\alpha}(nN, m) \) creates a fermion with spin projection \( \alpha \) on the site \( m \) along the stripe located at \( n \). We assume that the operators \( c_{\alpha}(n, m) \) and \( a_{\alpha}(n, m) \) anti-commute with each other. Moreover we assume that after the stripes are formed the long range Coulomb interaction within the stripes is screened by the other stripes in the material. The interaction between the stripes and the magnetic region is given by the hybridization term,

\[ H_t = \sum_{m, n, \alpha = \uparrow, \downarrow} V_{m, n} a^\dagger_{\alpha}(nN, m) (c_{\alpha}(nN - 1, m) \]

\[ + c_{\alpha}(nN + 1, m)) + \text{c.c.} \]

where \( V_{n, m} \) is the local hybridization of the stripe with the magnet region. Observe that when \( \hat{U} = t = 0 \) the problem is equivalent to the one in the Anderson model. The antiferromagnetic regions behave like \( f \) electrons while the stripes behave like a conduction band. In this
sense the problem becomes very similar to the problem of heavy fermions in rare earth alloys. Moreover, observe that if \( \epsilon_0 > E_F \) where \( E_F \) is the Fermi energy of the magnetic region then the stripes are going to be completely empty, that is, full of holes (see Fig. 1). This seems to be the case in La\(_{2-x}\)Sr\(_x\)Ni O\(_{4+y}\) [4].

The main advantage of this Hamiltonian is that it has only short range interactions and one assumes that all the strong interaction effects are already renormalized. That is, it is an effective Hamiltonian for quasi-electrons in the Cu O\(_2\) planes. Notice that the stripes are present even for small values of the effective parameters which also makes the problem easier to treat. Moreover, the phenomenologic parameters \( t, \tilde{t}, \tilde{U}, V, \epsilon_0, \ell \) are functions of doping (and possibly temperature) and they depend intrinsically on the microscopic details, which are not available. In particular in the absence of doping (half-filling) one has \( \epsilon_0 = 0, t = \tilde{t} = V \) and \( U = \tilde{U} \) which leads to a two-dimensional half-filled Hubbard model. Naturally the full solution of this problem is of major complexity. It turns out, however, that this model simplifies considerably if one considers the physics close to the antiferromagnetic ordered region.

The closeness to the magnetically ordered region is measured by the magnetic correlation length, \( \xi \). \( \xi \) must be compared with the other characteristic scales in the problem, which are Fermi wavelength which is proportional to the lattice spacing, \( a \), and the distance between the stripes, \( \ell \). We will assume that the Fermi wavelength of the quasi-electrons is the smallest scale in the problem so that we are dealing with a large bandwidth. The change in the physics of the problem, from our point of view, comes from the comparison of \( \xi \) with \( \ell \). Close to the antiferromagnetic region one expects \( \xi > \ell \) and, in particular, in the magnetically ordered phase \( \xi \) diverges. In this case large regions of the antiferromagnet are locked together in a Neel order. In particular, on the scale of \( \xi \) many stripes are found, and it costs a large amount of energy for the holes to move around. Part of this energy is compensated by the shift of the bottom of the bands, that is, \( \epsilon_0 \) (this is the equivalent of the gain of Coulomb energy with condensation into stripes). Therefore, in this region, the holes are going to be confined to the stripes. When \( \xi \) becomes comparable to or smaller than \( \ell \) one expects the holes to make excursions into the antiferromagnetic regions, because the order is not robust enough to segregate the holes into the stripes. In the next section we discuss the physics of the problem close to the magnetically ordered state.

### III. \( \xi > \ell \), MAGNETICALLY ORDERED PHASE.

In this section we assume that the charge fluctuations into the magnetic regions are very rare because the system is close to magnetic order and large regions of the magnet are locked into an almost ordered state. As is well known, the holes disrupt the order by overturning spins and therefore are expelled from the magnetic regions into the stripes. Thus, charge fluctuations are only allowed within the stripe. In terms of our model this implies that \( U >> t \) and \( V << \epsilon_0 \). This allows us to do perturbation theory in \( t/U \) and \( V/\epsilon_0 \).

When \( U >> t \) the magnetic regions (which are half-filled) can be mapped into the Heisenberg model using degenerate perturbation theory,

\[
\mathcal{H}_H = J \sum_{(n',n)} \vec{S}(n') \cdot \vec{S}(n) \tag{3.1}
\]

where \( J = 2t^2/U \) is the exchange constant. In this limit the electrons in the stripes can only hop virtually into the magnetic regions. It is easy to show that in this case the allowed interaction at low energies is an exchange coupling which can be derived via a Schrieffer-Wolff transformation for small \( V \) [4]. The hybridization term has the form of a Kondo coupling,

\[
\mathcal{H}_I = J' \sum_{m,n} \vec{S}(n, m) \cdot \left( \vec{S}(n + 1, m) + \vec{S}(n - 1, m) \right) \tag{3.2}
\]

where \( \vec{S}(n, m) = \sum_{\alpha, \alpha'} a^\dagger_{\alpha}(n, m) \tilde{\sigma}_{\alpha, \alpha'} a_{\alpha'}(n, m) \) is the electronic spin in the stripe and \( J' \propto \frac{|V|^2}{\epsilon_0} \) is the stripe exchange (\( \tilde{\sigma}_{\alpha, \alpha'} \) represents the Pauli matrices). We have obtained a major simplification now because the charge fluctuations in the magnetic regions are frozen; only magnetic fluctuations are allowed. Thus, we can focus only on the spin dynamics.

Since we are assuming that the magnetic correlation length is larger than the interstripe distance, we can work with just one stripe at a time, because each stripe sees an effective magnetic environment. Moreover, in this region, because the levels of doping are small, the inter-stripe distance is much larger than the lattice spacing and the connection between the stripes is weak. Therefore in what follows we look at the dynamics of one isolated stripe and assume that all the other stripes essentially undergo the same dynamics. This also leads to further simplification in the calculations. In the magnetic region the stripes do not fluctuate transversally. Therefore their physics is going to be described in terms of a Luttinger liquid, that is, an interacting one-dimensional Fermi gas. We can therefore use the powerful machinery of bosonization in order to study their physics [4]. The fermion operator is written in terms of right, \( R \), and left, \( L \), moving electrons as,

\[
a_{\alpha}(y) = \psi_{R, \alpha}(y)e^{ik_Fy} + \psi_{L, \alpha}(y)e^{-ik_Fy}
\]

and these can be bosonized via the transformation

\[
\psi_{R,L, \alpha}(y) = \frac{1}{\sqrt{2\pi \eta}} e^{\pm i\pi/4} e^{\pm i\pi/4} \phi_{R,L, \alpha}(y)
\]
where $\eta$ is a lattice cut-off for the theory in the continuum. The bosonic modes $\phi$ can now be described in terms of amplitude, $\phi_\alpha$, and phase, $\theta_\alpha$, bosonic modes as $\phi_{\eta \alpha} (y) = \phi_\alpha (y) \mp \theta_\alpha (y)$. In turn these bosonic fields can be written in terms of charge and spin bosonic modes, $\phi_{\eta s} = \sqrt{2} (\phi_\uparrow \pm \phi_\downarrow)$ and $\theta_{\eta s} = \sqrt{2} (\theta_\uparrow \mp \theta_\downarrow)$, and it is easy to show that the Euclidean Lagrangean density of the system can be written as,

$$L_S = \sum_{i=\eta s} \left\{ \frac{g_s}{2u_j} \left[ (\partial_i \phi_i)^2 + v^2 (\partial_y \phi_i)^2 \right] \right\} \quad (3.3)$$

$g_s$ and $g_a$ are the Luttinger parameters for spin and charge respectively and $v_s$ and $v_p$ their velocities of propagation. Observe that this model exhibits the phenomenon of charge and spin separation, as we can clearly see from $\phi_{\eta s}$.

First let us assume that we are inside the antiferromagnetic region and there is long range antiferromagnetic order ($\xi \to \infty$). Let us consider the effect of the stripe on the antiferromagnet. In this case the interaction between the stripe and the antiferromagnet can be simplified to a Ising-like form because the spin symmetry is broken (say in the z direction). That is, let us write

$$H_I = J' \sum_m S_\xi^z (S^z (+1, m) + S^z (-1, m)). \quad (3.4)$$

In the bosonized language $S_\xi^z = \partial_y \phi_s / \sqrt{\pi}$ which shows that only $\phi_s$ is coupled to the antiferromagnet. It is easy now to trace the Luttinger liquid out of the problem and calculate the effect of the Luttinger liquid on the spins around the stripe. The final action reads,

$$S = -\frac{(J')^2 v_s}{2\pi g_s} \sum_n \int \frac{dk}{2\pi \omega_n^2 + v_s^2 k^2} \left| \Delta S^z (k, \omega_n) \right|^2$$

where $\Delta S^z (0, y) = S^z (0^+, y) + S^z (0^-, y)$ is the total local spin around the stripe. Observe that this action is retarded and non-local and the limits of zero frequency and zero wavevector do not commute in the integrand. If we impose the condition that the momentum is strictly conserved and allow energy fluctuation (that is, we take $k \to 0$ first), the interaction vanishes. In the opposite limit ($\omega \to 0$ first), if we require energy to be conserved while the momentum can fluctuate, we obtain the physical interaction between the spins. In the static case one obtains,

$$S = -\frac{(J')^2 v_s}{2\pi g_s} \int dx \int dy \left[ (S^z (0^+, y))^2 + (S^z (0^-, y))^2 + 2 S^z (0^+, y) S^z (0^-, y) \right] . \quad (3.5)$$

Therefore we have generated two kinds of terms: the first is a single ion anisotropy at the border of the stripe and the second is a ferromagnetic coupling between the opposite sides of the stripe. This type of ferromagnetic coupling has been already proposed in the literature in the context of magnetism in the cuprates [15]. The exchange energy is given by

$$J_{eff} = \frac{(J')^2 a}{\pi g_s v_s}$$

In order to estimate its value let us assume that the parameters in the stripe are exactly the same as in the magnetic regions. In this case $J' \approx J = 2t^2 / U$, $\tilde{t} \approx t$ and $\tilde{U} \approx U << \pi v_F$ and one finds,

$$J_{eff} \approx \frac{J}{\pi \sin \left( \frac{n}{2} \right)}$$

where $n = 2(k_F a) / \pi$ is the electronic density on the stripe. We see that $J_{eff} < J$ if $\frac{U}{t} > (\pi \sin \left( \frac{n}{2} \right))^{-1}$ which is easy to attain in the antiferromagnetic phase. This result implies that the coupling across the stripes is weakened by the Luttinger liquid.

In a recent paper we have proposed an explanation for the destruction of antiferromagnetism with hole doping in terms of the weakening of the antiferromagnetic correlations around the stripe [16]. This weakening introduces a spacial anisotropy in the propagation of the spin waves parallel and perpendicular to the stripes. The physical picture is that, in the presence of stripes, the quantum fluctuations in the system grow and at some critical value of the anisotropy parameter a quantum phase transition occurs.

Moreover, our results are consistent with the neutron scattering data, as we explain now. It has been known for quite some time that in La$_{2-x}$Sr$_x$CuO$_4$ and related compounds that instead of the magnetic peak at $(\pi/a, \pi/a)$ which is seen in the antiferromagnetic compound ($x \leq 0.02$) one sees four incommensurate peaks at $(\pi/a \pm \epsilon, \pi/a)$ and $(\pi/a, \pi/a \pm \epsilon)$ where $\epsilon$ depends on doping [17]. In some recent experiments by Yamada et al. [18] this incommensurability effect was measured with great precision. It was found that below $x = 0.05$ only one peak at $(\pi/a, \pi/a)$ is seen. Above $x = 0.05$ the four
It is easy to conclude that, the problem which is given by the interstripe distance of the stripes. This change in sign implies a new scale in the sign of the staggered magnetization at the position of the stripes. This change in sign implies a new scale in the problem which is given by the interstripe distance \( \xi \). That is, there is a commensurate-incommensurate phase transition as a function of doping. Our results indicate the nature of the phase transition. When \( \xi \geq \ell \) our picture is the one of a static stripe with ferromagnetic coupling given by (3.5) between the spins in the neighborhood of the stripe. Therefore, in this regime one expects a phase domain (see Fig. (3)). It is clear that in this case the staggered magnetization does not change sign at the stripe (see Fig. (3)), and one should find just one peak at \( (\pi/a, \pi/a) \) as is seen in the experiment. Therefore we are led to the conclusion that charge fluctuations must be the source of the commensurate-incommensurate phase transition. That is, the energy of the system is lowered by transverse motion of the stripes. This result makes a lot of sense. In the presence of a phase domain the transverse motion of the stripe is frustrated (see Fig. (1)) because of the increase in the antiferromagnetic energy due the hopping of electrons with the same spin. This effect does not happen for the anti-phase domain, because the hopping of a hole into the antiferromagnet does not frustrate the spins (see Fig. (1)). Therefore in an anti-phase domain the stripes can readily fluctuate transversally.

Thus our conclusion is that when the stripes start to fluctuate (which in our theory happens when \( \xi \approx \ell \)) one should see this transition. Indeed, in Fig. (4), one shows the experimental data for \( \xi \) and \( \ell \) as a function of doping from two different experiments. \( \xi \) is obtained from Birgeneau et al. \[16\] and Keimer et al. \[20\] from the measurement of the width of the magnetic peaks in the doped samples. \( \ell \) is obtained from Yamada et al. \[18\] from equation (3.7) for \( x > 0.06 \) and by extrapolating for \( x < 0.06 \) using equation (3.6). One clearly sees that the two sets of data cross around \( x = 0.03 - 0.04 \). We concluded therefore that the commensurate-incommensurate phase transition can be associated with a phase to antiphase domain transition driven by transverse fluctuations.

Thus equation (3.6) predicts that the interstripe distance is decreasing with doping. This conclusion is reasonable if one assumes that the doping does not change the filling factor of the stripes. There are some recent calculations using the exact solution of the Hubbard model that seem to agree with the assumption \[14\]. It turns out, however, that the four peaks disappear below \( x = 0.05 \). That is, there is a commensurate-incommensurate phase transition as a function of doping. Our results indicate the nature of the phase transition. When \( \xi \geq \ell \) our picture is the one of a static stripe with ferromagnetic coupling given by (3.5) between the spins in the neighborhood of the stripe. Therefore, in this regime one expects a phase domain (see Fig. (3)). It is clear that in this case the staggered magnetization does not change sign at the stripe (see Fig. (3)), and one should find just one peak at \( (\pi/a, \pi/a) \) as is seen in the experiment. Therefore we are led to the conclusion that charge fluctuations must be the source of the commensurate-incommensurate phase transition. That is, the energy of the system is lowered by transverse motion of the stripes. This result makes a lot of sense. In the presence of a phase domain the transverse motion of the stripe is frustrated (see Fig. (1)) because of the increase in the antiferromagnetic energy due the hopping of electrons with the same spin. This effect does not happen for the anti-phase domain, because the hopping of a hole into the antiferromagnet does not frustrate the spins (see Fig. (1)). Therefore in an anti-phase domain the stripes can readily fluctuate transversally.

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Let us look now at the effect of the antiferromagnet on the stripe. We concluded that the coupling between the spins of the antiferromagnet on opposite sides of the stripe is ferromagnetic. Therefore, in the ordered phase, one must have \( \langle S^z(+1,m) \rangle = \langle S^z(-1,m) \rangle = (-1)^m M_s \) where \( M_s \) is the staggered magnetization. Substituting this into (3.4) one gets

\[
\mathcal{H}_L = 2J M_s \sum_m (-1)^m S^z(m) \tag{3.8}
\]

which shows that the stripe sees a staggered magnetic field in the \( z \) direction proportional to \( B(m) \approx -2J M_s (-1)^m \). One can easily show that the effect of this field is to magnetize the Luttinger liquid (a paramagnetic effect). The relevant part of the Lagrangean can be rewritten in terms of the bosons as

\[
\mathcal{L} = \frac{g_s}{2v_s} \left[ (\partial_x \phi_s)^2 + v_s^2 (\partial_y \phi_s)^2 \right] + \kappa \cos(\pi y/a) \partial_y \phi_s
\]

where \( \kappa = \frac{2J M_s}{\sqrt{\pi}} \). Notice that this Lagrangean can be brought to a quadratic form by a simple shift in the field \( \phi_s \) as, \( \Phi_s(y) = \phi_s(y) + \frac{g_s}{\pi v_s} \sin(\pi y/a) \) where the action for \( \Phi_s \) is gaussian. It implies that, \( \langle \phi_s(y) \rangle = -\frac{g_s}{\pi v_s} \sin(\pi y/a) \). It is easy to show that the mean value of magnetization on the stripe is

\[
\langle S^z(y) \rangle = \frac{a}{\sqrt{\pi}} (\partial_y \phi_s) = -\frac{2aJ M_s}{\pi g_s v_s} \cos(\pi y/a).
\]

This gives a polarization of the stripe and now we understand the origin of the ferromagnetic coupling between the spins: an up spin in the magnetic region polarizes a down spin cloud in the stripe which propagates and couples with an up spin on the other side of the stripe. This
is just the zeroth order effect, because the transverse degrees of freedom in the antiferromagnet are also coupled to the Luttinger liquid, and from this argument it is not clear what is their effect. For that we need a better study of the transverse modes of the antiferromagnet. In the next section we look at the problem coming from the disordered phase and show that these modes actually play an important role in the phase diagram of the stripe.

**IV. \( \xi > \ell, \) MAGNETICALLY DISORDERED PHASE**

First let us turn our attention to the antiferromagnet. The low energy physics of the antiferromagnet can be described in terms of the non-linear \( \sigma \) model. In order to treat this problem we go to a path integral formulation and treat the Heisenberg spins in a spin coherent state representation. The Euclidean action associated with the Heisenberg Hamiltonian, \((3.1)\), is the usual non-linear \( \sigma \) model action that reads \((\hbar = 1)\) \cite{23},

\[
S_H = \frac{1}{2g} \sum_{i=R,L} \int d^2r \int c^\beta \, d\tau \left( \partial_\tau \hat{n}_i \right)^2
\]

where \(g = \sqrt{\chi_\perp/\rho_s} \) is the coupling constant, \(\chi_\perp = 4Ja^2\) the transverse spin susceptibility, \(\rho_s = JS^2\) the spin stiffness and \(c = 2JSa\) is the spin wave velocity. \(\hat{n}_R\) and \(\hat{n}_L\) describe the field \(\hat{n}\) on the right and left sides of the stripe respectively. Observe that the imaginary time has been scaled by \(c\).

We have concluded that the Luttinger liquid produces a ferromagnetic coupling between the two sides of the antiferromagnet. This implies that \((4.4)\) can be written as,

\[
\mathcal{H}_I = J'S \sum_m (-1)^m \vec{S}_e(m) \cdot (\hat{n}_R(+1,m) + \hat{n}_L(-1,m)).
\]

From \((4.2)\) one sees that the only field of relevance for the electronic dynamics is the field \(\vec{u}(m,\tau) = \hat{n}_R(0^+,m,\tau) + \hat{n}_L(0^-,m,\tau)\).

In order to proceed one has to understand the dynamics of \(\vec{u}\). Here we use the well known results for the non-linear \( \sigma \) model. The partition function of this model is given by,

\[
Z = \int D\vec{u} e^{-S[\vec{u}]} \delta(\vec{u}^2 - 1)
\]

where \(S_H\) is given in \((4.1)\) and the Dirac delta function insures the norm of \(\vec{u}\). This constraint can be introduced via a Lagrange multiplier, \(\lambda\) in the usual way,

\[
Z = \int D\vec{u} D\lambda e^{-S[\vec{u},\lambda]}
\]

where

\[
S[\vec{u}, \lambda] = S_H + i \int d^2r \int_0^{c^\beta} d\tau \lambda (\vec{u}^2 - 1).
\]

The constraint introduces non-linearities in this problem. Here we treat this problem in a large \( N \) expansion (where \( N \) is the number of components of the vector \(\vec{u}\)) \cite{22}. Since the symmetry is broken in the \(z\) direction we write \(\vec{u} = (\sigma, \vec{v})\) where \(\sigma\) is the component of \(\vec{u}\) in the \(z\) direction and \(\vec{v}\) are the \(N - 1\) transverse components. We can trace the transverse modes out of the problem and the saddle point equation when \( N \to \infty \) can be obtained if we set \(\sigma = M_s\) and \(\lambda = -i m^2/(2g)\) in order to find

\[
M_s m = 0
\]

\[
M_s^2 = 1 - (N - 1)g \sum_m \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \omega^2_n + m^2}.
\]

The physical interpretation is straightforward: \(M_s\) is the order parameter (the staggered magnetization) and \(m\) is the inverse of the magnetic correlation length \(\xi\) \((m = 1/\xi)\). Therefore we have two phases: an ordered phase with \(M_s \neq 0\) and \(m = 0\) and a disordered phase \(M_s = 0\) and \(m \neq 0\) where \(M_s\) and \(M_s\) are given in \((4.4)\). A major simplification happens in the disordered phase. In this phase all the modes are massive \((m \neq 0)\) and therefore the physics of the problem is equivalent to a massive linear \(\sigma\) model \cite{22}. That is, the theory is essentially gaussian. In the disordered regime the action reads,

\[
S[\vec{n}] = \frac{1}{2g} \int d^2r \int c^\beta \, d\tau \left\{ \left( \partial_\tau \vec{n} \right)^2 + m^2 \vec{n}^2 \right\}
\]

where now the constraint is gone and \(\vec{n}\) can vary arbitrarily. This result leads to a major simplification.

The action \((4.5)\) for the \(\sigma\) model can be traced out exactly because of its quadratic nature. We proceed by introducing an identity on the partition function as follows,

\[
1 = \int D\vec{u} \int D\vec{v} \exp \left\{ -i \int dy \int_0^{c^\beta} d\tau \vec{v}(y,\tau) \cdot (\vec{u}(y,\tau) - \vec{n}_R(0,y,\tau) - \vec{n}_L(0,y,\tau)) \right\}
\]

where \(\vec{v}\) is a Lagrange multiplier. Observe that because the action in \((4.5)\) is completely quadratic we can integrate the fields \(\vec{n}\) and \(\vec{v}\) out and get an action for the field \(\vec{u}\) alone. We just quote here the final result for the partition function for the fermions on the stripe which has the form,

\[
Z = \int Da Da^* e^{-S_{\sigma}[a,a^*] - S_I[a,a^*]}
\]

where

\[
S_I = \frac{3g}{4} \left( \frac{J'S}{ac} \right)^2 \sum_n \int \frac{dk}{2\pi} \frac{\left|\tilde{S}_e(k + \pi/a, \omega_n)\right|^2}{\sqrt{k^2 + \omega^2_n + m^2}}
\]
and $S_S$ is the action associated with (3.3). Observe that the interaction is a retarded, temperature dependent and long range spin-spin interaction which in real space becomes

$$S_I = -\int dy\,dy’\,d\tau’\,F(y - y’, \tau - \tau’)|\vec{S}_c(y, \tau)\cdot\vec{S}_c(y’, \tau’)|$$

where,

$$F(y, \tau) = \frac{3g\nu\cos(\pi y/a)}{16\pi a^4} \sum_n e^{-i\omega_n\tau} K_0(|y|\sqrt{\omega_n^2 + m^2})$$

is the effective spin-spin interaction ($K_0$ is a modified Bessel function). At zero temperature it becomes

$$F(y, \tau) = \frac{3g\nu\cos(\pi y/a)}{24\pi a^4} \frac{e^{-m\sqrt{\tau^2 + y^2}}}{\sqrt{\tau^2 + y^2}}$$

where $\nu = J’/J$. Since we are interested in the propagator in real time we have to Wick rotate from Euclidean space to Minkowski space ($\tau \rightarrow -i\epsilon t$). The result is

$$F(y, t) = \frac{3g\nu\cos(\pi y/a)}{24\pi a^4} \frac{e^{-m\sqrt{y^2 - c^2t^2}}}{\sqrt{y^2 - c^2t^2}}.$$ 

Thus for space-like separations ($y^2 > c^2t^2$) the propagator decays exponentially at large space-like separations. But, for time-like separations ($y^2 < c^2t^2$) we get an oscillatory decaying function with two periods $\xi$ and $\alpha$.

In the special case where the correlation length goes to zero ($m \rightarrow \infty$, $\xi \rightarrow 0$) one gets,

$$F(y, \tau) = \frac{3g\nu\xi}{16a^4} \delta(y)$$

and it becomes purely local. Taking (4.7) and rewriting in Hamiltonian form one gets a simple renormalization of the original Hubbard $U$.

At the other extreme case where the correlation length diverges ($m = 0$, $\xi \rightarrow \infty$) the interaction reads,

$$F(y, \tau) = \frac{3g\nu\cos(\pi y/a)}{24\pi a^4} \frac{1}{\sqrt{\tau^2 + y^2}}.$$ 

The interaction is oscillatory and decays like $1/R$. It was shown by Schulz that a $1/y$ interaction leads to Wigner crystalization in a one-dimensional Fermi gas [24]. This is exactly the form of the instantaneous part of (1.8) apart from the oscillatory term. Moreover, if we take the zero frequency part of the interaction one finds that $F(y, \tau) = \delta(\tau) U(y)$ where

$$U(y) = \frac{3g\nu\cos(\pi y/a)}{16\pi a^4} K_0(m|y|)$$

which for $\xi >> y >> a$ becomes,

$$U(y) = \frac{3g\nu\cos(\pi y/a)}{16\pi a^4} \cos(\pi y/a) \ln(|y|/|y|).$$

which is divergent in the antiferromagnetic phase $\xi \rightarrow \infty$. Our conjecture is therefore that the system is crystalized in the ordered phase. This result is consistent with the experimental fact that the system is insulating in this phase. We have reason to believe, therefore, that the stripe undergoes an insulator-metal transition with doping as the system goes through the antiferromagnetic-paramagnetic phase transition. However, in order to prove this conjecture we have to study how the interaction renormalizes as we go to lower frequencies. This can be done using renormalization group calculations with two frequencies: the Fermi energy of the electrons on the stripe and $c/\xi$ which is the characteristic energy of the spin waves. Although calculations with two frequencies have been done for electron-phonon interactions [23], they have not been done for electron-magnon interactions. One can show that this type of interaction leads to new diagrams that are not considered in the electron-phonon problem which are related to spin flip processes [26].

V. CONCLUSIONS

We have discussed here the limit of $\xi > \ell$ where magnetism dominates over charge effects. In the opposite limit, $\xi < \ell$, the physics is dominated by stripe fluctuations and formation of anti-phase domains. In this phase the stripes move transversally and clearly one expects the transverse motion to be related to the existence of superconductivity in cuprates [26,28]. Indeed, it is shown experimentally in La$_{2−x}$Ba$_x$CuO$_4$ [29] and in La$_{1.6−x}$Nd$_{0.4}$Sr$_x$CuO$_4$ [30] that the pinning of stripes induces drastically the superconducting temperature of the material. In the case of the model presented in this paper it implies that the hybridization $V$ of the stripe with the magnetic region becomes large and holes can hop inside the magnetic regions. Moreover, one expects in this region that interaction between the stripes will be important. In particular, the stripes can exchange electrons (and Cooper pairs) over the magnetic regions and the approximation of an isolated stripe is not a sensible one. We believe however that the main ingredients for the physics in this region are already present in [24].

It is worth noticing, however, that most of the methods used in this paper can still be used for $\xi < \ell$ if we make the appropriate changes [26].

In this paper, therefore, we focused on the limit where $\xi > \ell$. We showed that in the ordered phase the effect of the Luttinger liquid on the antiferromagnet is to generate a ferromagnetic coupling between the spins around the stripe and lead therefore to the presence of phase domains in contrast to the anti-phase domains which are seen in the samples with doping $x > 0.05$. The physical reason for this ferromagnetic ordering is a spin polarization...
of the stripes. We conjecture that the commensurate-incommensurate transition seen in neutron scattering is associated with the phase to anti-phase domain transition in the coupling between the spins. This transition is driven by the gain in the kinetic energy of the transverse motion of the stripes over the antiferromagnetic energy that tends to suppress the transverse motion. Moreover, these results are consistent with earlier work on the destruction of antiferromagnetism in the striped antiferromagnet due to the growth of quantum fluctuations \[16\]. Furthermore, in the disordered phase we have shown, using a \(\sigma\) model calculation, that close to the antiferromagnetic transition the interaction between the electronic spins on the stripe becomes singular, which leads to conjecture that the Luttinger liquid condenses into a Wigner crystal. In this case the system is completely insulating, in agreement with the experiments.

In summary, in this paper we propose a model for the physics of stripes in antiferromagnets in which the stripes are described by a Luttinger liquid hybridized with a magnetic host. Using bosonization and \(O(N)\)-\(\sigma\) models we discuss the physics of this problem in the limit where the physics is dominated by the magnetic interactions, that is, \(\xi > \ell\). We show that it is possible with very general arguments to explain a series of experiments in cuprates and to predict a transition in the spin orientation with doping in these materials.

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