Minimizing the Machine Processing Time in a Flow Shop Scheduling Problem under Piecewise Quadratic Fuzzy Numbers

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A flow shop is the most studied production setting in the literature on scheduling. In [1], one of the earliest results in flow shop scheduling is an algorithm for minimizing the completion time of all activities in a two or three-machine shop. Gupta [2] suggested a method for determining the best time to schedule a flow shop scheduling problem (FSSP) with a certain structure.

The method [2] has important considerations and developed by several scholars; see [3−7]. Narian and Bagga [9] investigated the problem of obtaining a sequence that provides the lowest possible cost of renting while minimizing the time spent. Schulz et al. [10] explored a mixture of FSSP with varying discrete production speed levels. An upgraded multi-objective algorithm was used by Gheisarihe et al. [11] to solve the flexible FSSP with sequence-based transportation time, a probable network, and setup time.

In the real-world applied scientific problems, due to the complexity of different systems and the inaccuracy of data, classical methods cannot take into account inaccuracies in discussions. Therefore, using tools such as fuzzy perspective [12] can be helpful in managing this important task (see also [13, 14]).

The theory of fuzzy sets and its applications in optimization were proposed by Zimmermann [15]. Kaufmann and Gupta [16] studied several fuzzy mathematical models with their applications to engineering and management sciences.

By using triangular fuzzy sets to describe work processing times, Petrovic and Song [17] studied the task sequence problem in a two-machine flow shop. Multi-product parallel multi-stage cell manufacturing organizations can apply Saracoglu and Suer’s methodology [18] to create items.
on time. They employed this methodology in the case study of a shoe manufacturing plant to produce products on time. Pang et al. [19] presented the FSSP and hybrid flow shop scheduling with the intention of determining the optimal scheduling approach for manufacturing facilities. Shao et al. [20] examined a distributed fuzzy blocking FSSP with processing times represented by fuzzy numbers, with the goal of minimizing the fuzzy makespan across all components. Recently, some papers are introduced to deal with real-world problems in fuzzy environments and their extensions (see [21–26]).

This study aims to investigate a particular n-job of scheduling with piecewise quadratic fuzzy number (PQFN). Given the total time elapsed, in which processing times are shown in PQFN, an innovative approach to sequencing tasks is proposed, which minimizes the cost of renting machines.

1.1. Research Gap and Motivation. The following points may lead to motivation of the proposed study.

(1) The piecewise quadratic fuzzy number (PQFN) introduced by Jain [27] is an extended concept of fuzzy set.

(2) In real-world scenarios, distinct parameters are further classified into disjoint sets having sub-parametric values. It presents the optimal selection with the help of suitable parameters. In decision making, the jury may endure some sort of tendency and proclivity while paying no attention to such parametric categorization during the decision.

(3) Inspired from the above literature, new notions of PQFN are conceptualized along with some elementary essential properties and generalized typical results. Moreover, decision-making algorithmic approaches are proposed.

1.2. Main Contributions and Advantages. The following are the main contributions of this proposed study:

(1) The existing relevant models are made adequate with the consideration of multi-argument approximate function through the development of the fuzzy set theory.

(2) The scenario where parameters are further partitioned into sub-parametric values in the form of sets is tackled by using PQFNs.

(3) Some fundamentals like elementary properties and arithmetic operations of PQFNs are characterized.

(4) Decision-making applications are discussed based on the proposal of PQFN arithmetic operations.

(5) The results of the proposed similarity are compared with relevant existing models.

(6) The proposed structure is compared with relevant models under suitable evaluating indicators.

(7) The advantageous aspects of the proposed structure are discussed. The generalization of proposed structure is presented.

1.3. Paper Organization. This paper is organized as follows. The next section introduces the preliminaries of PQFNs and some notations. A three-stage FSSP model is provided in Section 3. Section 4 provides an efficient method for determining the sequence of jobs that minimizes the cost of equipment rental. Section 5 gives a numerical example for illustration. Section 6 introduces a comparative study with the existing methods. Finally, the conclusions are drawn in Section 7.

2. Prerequisites

Here, we study some preliminaries that we need for the main sections (for more details, see [27]).

Definition 1. A PQFN is denoted by $W_{PQ} = (w_1, w_2, w_3, w_4, w_5)$, where $w_1 \leq w_3 \leq w_1 \leq w_3 \leq w_5$ are real numbers, and its membership function $\mu_{W_{PQ}}$ is given by

$$\mu_{W_{PQ}} = \begin{cases} 0, & x < w_1; \\ \frac{1}{2} \left( \frac{1}{w_2 - w_1} \right) (x - w_1)^2, & w_1 \leq x \leq w_2; \\ \frac{1}{2} \left( \frac{1}{w_3 - w_2} \right) (x - w_2)^2 + 1, & w_2 \leq x \leq w_3; \\ \frac{1}{2} \left( \frac{1}{w_4 - w_3} \right) (x - w_3)^2 + 1, & w_3 \leq x \leq w_4; \\ \frac{1}{2} \left( \frac{1}{w_5 - w_4} \right) (x - w_4)^2, & w_4 \leq x \leq w_5; \\ 0, & x > w_5. \end{cases}$$

Figure 1 shows the graphical representation of a PQFN.

Definition 2. Let $U_{PQ} = (u_1, u_2, u_3, u_4, u_5)$ and $V_{PQ} = (v_1, v_2, v_3, v_4, v_5)$ be two PQFNs. Then, we have

(i) Addition: $U_{PQ}(+)V_{PQ} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5)$.

(ii) Subtraction: $U_{PQ}(-)V_{PQ} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, u_4 - v_4, u_5 - v_5)$.

(iii) Scalar multiplication: $kU_{PQ} = \{ (ku_1, ku_2, ku_3, ku_4, ku_5), \quad k > 0, \\
(\text{or} (ku_2, ku_3, ku_4, ku_5), \quad k < 0. \}$

Definition 3. For the close interval approximation (CIA) of PQFN of $[U] = [U^*, U^*_a]$, we call $\bar{U} = U^* + U^*_a/2$ as the associated real number of $[U]$. 

Definition 4. For $[U] = [U_a^- , U_a^+]$ and $[V] = [V_a^- , V_a^+]$, we have the following properties:

1. Addition: $[U] + [V] = [U_a^- + V_a^- , U_a^+ + V_a^+]$.
2. Subtraction: $[U] - [V] = [U_a^- - V_a^- , U_a^+ - V_a^-]$.
3. Scalar multiplication: $k[U] = \begin{cases} \{kU_a^- , kU_a^+\}, & k > 0 \\ \{kU_a^+ , kU_a^-\}, & k < 0 \end{cases}$.
4. Multiplication: $[U] \cdot [V] = [U_a^- V_a^- + U_a^+ V_a^+ / 2 , U_a^- V_a^- + U_a^+ V_a^- / 2]$.
5. Division: $[U] / [V] = \begin{cases} \{2(U_a^- / V_a^- + V_a^+), 2(U_a^+ / V_a^- + V_a^-)\}, & [V] > 0, V_a^- + V_a^+ \neq 0 \\ \{2(U_a^+ / V_a^- + V_a^-), 2(U_a^- / V_a^- + V_a^+)\}, & [V] < 0, V_a^- + V_a^+ \neq 0 \end{cases}$.
6. The order relations:

\[
\begin{align*}
\text{(i) } [U] < [V] & \text{ if } U_a^- \leq V_a^- \text{ and } U_a^+ \leq V_a^+ \text{ or } U_a^- + U_a^+ \leq V_a^- + V_a^+. \\
\text{(ii) } [U] > [V] & \text{ if and only if } U_a^- \geq V_a^- , U_a^+ \geq V_a^+.
\end{align*}
\]

2.1. Symbolization. Table 1 shows the symbols of our work.

3. Methodology

Before we discuss the issue formulation, let us define the rental cost.

3.1. Cost of Renting. The machines are rented out if needed and returned if they are no longer needed. For example, the first machine is rented at the beginning of the work process, the second machine is rented when the first work is completed in the first machine, and so on.

2. Suppose that some tasks $i, j = 1, \ldots, n$ under the definite rental policy $L$ are managed on three machines $M_j$, $j = 1, 2, 3$. Let $\tilde{a}_{ij}^{PQ}$ be the PQFPT of $i$-th task on $j$-th machine (see Table 2). Let $S_{ij}$, $i = 1, \ldots, n; j = 1, 2, 3$. Determine the related processing times with crisp number on devices $M_1, M_2$, and $M_3$ in such a way that either $\tilde{a}_{ij} \leq \tilde{a}_{i1}$ or $\tilde{a}_{ij} \leq \tilde{a}_{i2}$; $\forall i, j$. Our objective is to determine $[S_k]$ of the tasks that minimizes the cost of renting the equipment.

The problem may be expressed mathematically as follows:

\[
\min R^{PQ}(S_k) = \sum_{i=1}^{n} a_{ij}^{PQ} \times C_1 + D_1^{PQ}(S_k) \times C_2 + D_2^{PQ}(S_k) \times C_3 \text{ Subject to rental policy } L.
\]

Using the CIA of PQFN, model (10) may be reformulated as follows:

\[
\min\{R_a^-(S_k), R_a^+(S_k)\} = \sum_{i=1}^{n} [\{a_{ij}^-\}, \{a_{ij}^+\}] \times C_1 + [U_a^- (S_k), U_a^+ (S_k)] \times C_2 + [U_a^- (S_k), U_a^+ (S_k)] \times C_3 \text{ Subject to rental policy } L.
\]

with PQF-based processing time while ignoring the makespan.

Step 1. Find the associated ordinary number for all tasks.

Step 2. If $\tilde{a}_{ij} \leq \tilde{a}_{i1}$ or $\tilde{a}_{ij} \leq \tilde{a}_{i2}$; $\forall i, j$, i.e., $\max[a_{ij}] \geq \min[a_{ij}]$ or $\max[a_{ij}] \geq \min[a_{ij}]$; $\forall i, j$, go to next step; otherwise, break.

4. Proposed Algorithm

In this part, we show our strategy for minimizing the time and, consequently, the cost of renting a three-stage FSSP...
Table 1: List of symbols.

| Abbreviations | Descriptions |
|---------------|-------------|
| $S$           | Arrangement of jobs, $i = 1, n$ |
| $S_k$         | Sequence obtained through the method [1], $k = 1, 2, \ldots, n$ |
| $M_k$         | Machine $h, h = 1, 2, 3$ |
| $M$           | Minimum makespan |
| $\tilde{a}_{ij}^{h}$ | PQF processing time (PQFPT) for the $i$-th task on $M_h, h = 1, 2, 3$ |
| $t_j(S_k)$    | Close interval estimate of the PQFPT of the $i$-th task in sequence $S_k$ running on $M_j$ |
| $U_j(S_k)$    | Time of $i$-th task for $S_k$ on $M_j$ |
| $CT(S_k)$     | Consumption time for $M_j$ that is necessary for $S_k$ |
| $L_i(S_k)$    | Whole completion time |
| $\tilde{a}_{ij}$ | Idle time |
| $R(S_k)$      | Corresponding normal time of the $i$-th task on $M_j$ |
| $C$           | Whole rental payment |
|               | Cost of renting |

Table 2: Description of the problem with the PQFN matrix.

| Tasks | $M_1$ | $M_2$ | $M_3$ |
|-------|-------|-------|-------|
| $i$   | $\tilde{a}_{i1}^{PQ}$ | $\tilde{a}_{i2}^{PQ}$ | $\tilde{a}_{i3}^{PQ}$ |
| 1     | $\tilde{a}_{11}^{PQ}$ | $\tilde{a}_{12}^{PQ}$ | $\tilde{a}_{13}^{PQ}$ |
| 2     | $\tilde{a}_{21}^{PQ}$ | $\tilde{a}_{22}^{PQ}$ | $\tilde{a}_{23}^{PQ}$ |
| $\ldots$ | $\tilde{a}_{n1}^{PQ}$ | $\tilde{a}_{n2}^{PQ}$ | $\tilde{a}_{n3}^{PQ}$ |

Table 3: The problem with CIA matrix.

| Tasks | $M_1$ | $M_2$ | $M_3$ |
|-------|-------|-------|-------|
| $i$   | $[(a_{i1})_0, (a_{i1})_1]$ | $[(a_{i2})_0, (a_{i2})_1]$ | $[(a_{i3})_0, (a_{i3})_1]$ |
| 1     | $[(a_{11})_0, (a_{11})_1]$ | $[(a_{12})_0, (a_{12})_1]$ | $[(a_{13})_0, (a_{13})_1]$ |
| 2     | $[(a_{21})_0, (a_{21})_1]$ | $[(a_{22})_0, (a_{22})_1]$ | $[(a_{23})_0, (a_{23})_1]$ |
| $\ldots$ | $[(a_{n1})_0, (a_{n1})_1]$ | $[(a_{n2})_0, (a_{n2})_1]$ | $[(a_{n3})_0, (a_{n3})_1]$ |

Table 4: The problem with the corresponding crisp matrix form.

| Tasks | $M_1$ | $M_2$ | $M_3$ |
|-------|-------|-------|-------|
| $i$   | $(a_{i1})_0 + (a_{i1})_1/2$ | $(a_{i2})_0 + (a_{i2})_1/2$ | $(a_{i3})_0 + (a_{i3})_1/2$ |
| 1     | $(a_{11})_0 + (a_{11})_1/2$ | $(a_{12})_0 + (a_{12})_1/2$ | $(a_{13})_0 + (a_{13})_1/2$ |
| 2     | $(a_{21})_0 + (a_{21})_1/2$ | $(a_{22})_0 + (a_{22})_1/2$ | $(a_{23})_0 + (a_{23})_1/2$ |
| $\ldots$ | $(a_{n1})_0 + (a_{n1})_1/2$ | $(a_{n2})_0 + (a_{n2})_1/2$ | $(a_{n3})_0 + (a_{n3})_1/2$ |

Step 3. Define dummy machines $H_1$ and $H_2$, and their processing times $H_1$ and $H_2$ are as follows: $H_1 = \tilde{a}_{i1} + \tilde{a}_{i2}, H_2 = \tilde{a}_{i2} + \tilde{a}_{i3} + \tilde{a}_{i1}, \forall i$.

Step 4. Use the existing algorithm [1] on $H_1$ and get $S_1$.

Step 5. Put the $2^{nd}, \ldots, n^{th}$ tasks of the $S_1$ in the first position and all other tasks of $S_1$ in the same order.

Step 6. For all possible sequences $S_k, k = 1, n$, calculate: $R(S_k) = \sum_{i=1}^{n} \tilde{a}_{ij} \times C_1 + \tilde{U}_1(S_k) \times C_2 + \tilde{U}_2(S_k) \times C_3$.

Step 7. Set $\min\{R(S_k)\}, k = 1, n$ as the optimal solution.

5. Numerical Example

Consider Table 5 as the problem. Now, we solve this problem by our model.

At first, in Tables 6 and 7, we compute the related interval and crisp numbers for each PQFPT.

Then, using Step 3 of our algorithm, the processing times can be computed as shown in Table 8.

Using procedure [1], $S_1$ : $2-4-5-1-3$.

The subsequent viable sequences correspond to the minimal rental cost: $S_2$: $4-2-5-1-3$, $S_3$: $1-2-4-5-3$, $S_4$: $3-2-4-5-1$.

Tables 9 and 10 illustrate the in-out flow for the sequence $S_i$ in the PQFNs and CIA forms.

For $S_1$, we get the following.

The completion time for $S_1$ is $\tilde{C}_{\text{T}}^{PQ}(S_1) = (40, 49, 56, 62, 76)$, $\left[CT(S_1)\right] = [49, 62]$, and $\tilde{C}_{\text{T}}^{PQ}(S_1) = 55.5$.

The consumption time for machine $M_1$ is $\tilde{U}_2^{PQ}(S_1) = (11, 23, 30, 38, 53)$, $\left[U_2(S_1)\right] = [23, 38]$, and $\tilde{U}_2(S_1) = 30.5$.

The consumption time for machine $M_2$ is $\tilde{U}_3^{PQ}(S_1) = (14, 28, 37, 45, 61)$, $\left[U_3(S_1)\right] = [28, 45]$, and $\tilde{U}_3(S_1) = 36.5$.

$$\tilde{R}^{PQ}(S_1) = \sum_{i=1}^{5} \tilde{a}_{1i} \times C_1 + \tilde{U}_2(S_1) \times C_2 + \tilde{U}_3(S_1) \times C_3$$

$$= (572, 726, 831, 935, 1145), \left[R(S_1)\right] = [726, 935] \text{and } \tilde{R}(S_1) = 830.5.$$

Similarly, we have the following.

For $S_2$:

$$\tilde{C}_{\text{T}}^{PQ}(S_2) = (37, 47, 54, 61, 73),$$

$$\left[CT(S_2)\right] = [47, 61], \text{and } \tilde{C}_{\text{T}}^{PQ}(S_2) = 54,$$

$$\tilde{U}_2^{PQ}(S_2) = (12, 23, 30, 38, 52), \left[U_2(S_2)\right] = [23, 38], \text{and } \tilde{U}_2(S_2) = 30.5,$$
Table 5: PQFN processing times for machines.

| Tasks | $M_1$ | $M_2$ | $M_3$ |
|-------|-------|-------|-------|
| $i$   | $\hat{a}_{i1}^{P}$ | $\hat{a}_{i2}^{P}$ | $\hat{a}_{i3}^{P}$ |
| 1     | (6.7,8.9,10) | (5.6,7.8,10) | (2.3,4.5,7) |
| 2     | (11.12,13,14,17) | (4.5,6.7,9) | (3.4,5.6,8) |
| 3     | (7.8,10,12,14) | (3.4,5.6,9) | (5.6,7.8,9) |
| 4     | (9.10,11,12,14) | (3.5,6.7,9) | (10.11,12,13,15) |
| 5     | (7.9,10,11,13) | (2.5,6.8,10) | (5.8,9,10,11) |

Table 6: PQFN processing times with interval format.

| Tasks | $M_1$ | $M_2$ | $M_3$ |
|-------|-------|-------|-------|
| $i$   | $[a_{i1}^{P}, a_{i2}^{P}, a_{i3}^{P}]$ | $[a_{i1}^{P}, a_{i2}^{P}, a_{i3}^{P}]$ | $[a_{i1}^{P}, a_{i2}^{P}, a_{i3}^{P}]$ |
| 1     | [7.9] | [6.8] | [3.5] |
| 2     | [12.14] | [5.7] | [4.6] |
| 3     | [8.12] | [4.6] | [6.8] |
| 4     | [10.12] | [5.7] | [11.13] |
| 5     | [9.11] | [5.8] | [8.10] |

Table 7: PQFN processing times with crisp format.

| Tasks | $M_1$ | $M_2$ | $M_3$ |
|-------|-------|-------|-------|
| $i$   | $\hat{a}_{i1}$ | $\hat{a}_{i2}$ | $\hat{a}_{i3}$ |
| 1     | 8      | 7      | 4     |
| 2     | 13     | 6      | 5     |
| 3     | 10     | 5      | 7     |
| 4     | 11     | 6      | 12    |
| 5     | 10     | 6.5    | 9     |

Table 8: The related crisp numbers of the processing times.

| Tasks | $H_1$ | $H_2$ |
|-------|-------|-------|
| $i$   |       |       |
| 1     | 15    | 11    |
| 2     | 19    | 11    |
| 3     | 15    | 12    |
| 4     | 17    | 18    |
| 5     | 16.5  | 15.5  |

Table 9: The in-out flow for $S_i$ in the PQFNs.

| Tasks | $M_1$ | $M_2$ | $M_3$ |
|-------|-------|-------|-------|
| $i$   | In-out | In-out | In-out |
| 2     | (11.12,13,14,17) | (15.17,19,21,26) | (18.21,24,27,34) |
| 4     | (19.22,24,26,31) | (18.22,25,28,35) | (28.32,36,39,49) |
| 5     | (26,31,34,37,44) | (20,27,31,36,45) | (33.40,45,49,60) |
| 1     | (32,38,42,46,54) | (25,33,38,44,55) | (35,43,49,54,67) |
| 3     | (39,46,52,58,68) | (28,37,43,50,64) | (40,49,56,62,76) |

\[
\bar{U}^{PQ}_3(S_2) = (14, 28, 37, 45, 61), [U_3(S_2)] = [28, 45], \text{and } \bar{U}_2(S_2) = 36.5,
\] (7)

Table 10: The in-out flow for $S_i$ in CIA.

| Jobs | Machine $M_1$ | Machine $M_2$ | Machine $M_3$ |
|------|----------------|----------------|----------------|
| 2    | [12,14]        | [17,21]        | [21,27]        |
| 4    | [22,26]        | [22,28]        | [32,39]        |
| 5    | [31,37]        | [27,36]        | [40,49]        |
| 1    | [38,46]        | [33,44]        | [43,54]        |
| 3    | [46,58]        | [37,50]        | [49,62]        |

\[
\bar{R}^{PQ}(S_2) = \sum_{i=1}^{3} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (8)

\[
\bar{R}^{PQ}(S_3) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (9)

\[
\bar{R}^{PQ}(S_4) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (10)

\[
\bar{R}^{PQ}(S_5) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (11)

\[
\bar{R}^{PQ}(S_6) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (12)

\[
\bar{R}^{PQ}(S_7) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (13)

\[
\bar{R}^{PQ}(S_8) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (14)

\[
\bar{R}^{PQ}(S_9) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (15)

\[
\bar{R}^{PQ}(S_{10}) = \sum_{i=1}^{2} \bar{a}_{i1}^{PQ} \times C_1 + \bar{U}^{PQ}_2(S_2) \times C_2 + \bar{U}^{PQ}_3(S_2) \times C_3
\] (16)
Table 11: Comparison of different researchers’ contributions.

| Author                          | Processing time | Piecewise quadratic fuzzy numbers | Close approximate interval | Minimum rental cost |
|---------------------------------|-----------------|-----------------------------------|-----------------------------|---------------------|
| Ruiz et al. [28]                | ↑               | ↓                                 | ↑                           | ↑                   |
| Liang et al. [29]               | ↑               | ↓                                 | ↑                           | ↑                   |
| Sanchez-Herrera et al. [30]     | ↑               | ↓                                 | ↑                           | ↑                   |
| Our proposed approach           | ↑               | ↑                                 | ↑                           | ↑                   |

For $S_3$, we have

$$\bar{R}(S_3) = 2066.6257.$$  \hspace{1cm} (17)

Thus,

$$R^{PQ}(S_3) = \sum_{i=1}^{5} a_{i1}^{PQ} \times C_1 + \bar{d}_{i2}^{PQ}(S_3) \times C_2 + \bar{d}_{i3}^{PQ}(S_3) \times C_3$$

$$= (562, 686, 791, 896, 1075),$$  \hspace{1cm} (18)

$$[R(S_3)] = [686, 896].$$  \hspace{1cm} (19)

Therefore, $S_3$: $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 3$ is the optimal sequence subject to the minimum rental cost, and $\bar{R}(S_3) = 791$ is the minimum rental cost irrespective of the total time passed.

6. Comparative Study

In this section, the proposed approach is compared with some existing studies to illustrate the advantages of the proposed approach. The results for this analysis are summarized in Table 11. The symbol “↑” or “↓” shown in the table represents whether the associated feature satisfies or not.

7. Conclusions and Future Works

In this paper, the problem of minimizing the cost of renting machines for flow shop scheduling with a specific structure is investigated. An innovative approach to solve it is then proposed in which the processing times are fragmented as piecewise quadratic fuzzy numbers. The result shows that the proposed method has its advantage in flexible decision making corresponding to favorite priorities of alternatives. This study may be extended to additional fuzzy-like structures, such as interval-valued fuzzy set, Pythagorean fuzzy set, spherical fuzzy set, intuitionistic fuzzy set, picture fuzzy set, neutrosophic set, and so on, in future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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