The Geometry of Dyonic Instantons in 5-dimensional Supergravity

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Abstract

We systematically construct and study smooth supersymmetric solutions in 5-dimensional $\mathcal{N}=1$ Yang-Mills-Einstein supergravity. Our solution is based on the ADHM construction of (dyonic) multi-instantons in Yang-Mills theory, which extends to the gravity-coupled system. In a simple supergravity model obtained from $\mathcal{N}=2$ theory, our solutions are regular ring-like configurations, which can also be interpreted as supertubes. By studying the $SU(2)$ 2-instanton example in detail, we find that angular momentum is maximized, with fixed electric charge, for circular rings. This feature is qualitatively same as that of supertubes. Related to the existence of this upper bound of angular momentum, we also check the absence of closed timelike curves for the circular rings. Finally, in supergravity and gauge theory models with non-Abelian Chern-Simons terms, we point out that the solution in the symmetric phase carries electric charge which does not contribute to the energy. A possible explanation from the dynamics on the instanton moduli space is briefly discussed.

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1 Introduction

Remarkable progress has been made recently in our understanding of the supersymmetric solutions in supergravity theories in various dimensions. The general consequence of the existence of a Killing spinor has been analyzed in 5 dimensional minimal supergravity [1], and then in gauged and/or matter-coupled supergravity theories [2, 3, 4] in 5 dimension. Similar studies in higher dimensions have also been carried out: just to mention a few of them, 6-dimensional minimal supergravity [5], 11-dimensional supergravity [6, 7] and type IIB supergravity [7].

The general properties of supersymmetric solutions have proven to be useful in finding new explicit solutions. For instance, in 5-dimensional supergravity theories mentioned above, new black rings [8, 9, 10, 11, 12] and $AdS_5$ black holes [13, 3, 14] are discovered, fully utilizing these structures.
A purpose of this paper is to broaden our understanding to the 5-dimensional supergravity coupled to the vector multiplets with non-Abelian gauge groups. Technically, this Yang-Mills-Einstein supergravity is obtained by a procedure called gauging. The gauging relevant to this theory is that of a global non-Abelian isometry of the scalar manifold in the vector multiplet, as we review below.

5-dimensional Yang-Mills-Einstein supergravity should have a large class of supersymmetric solutions, which we expect from our knowledge of 5-dimensional supersymmetric Yang-Mills field theory in flat space. Firstly, it is well-known that there are supersymmetric instanton particles in the latter theory, which are finite energy solutions of the self-duality equation for Yang-Mills field strength in spatial $\mathbb{R}^4$, carrying topological charge which we call the instanton number. The general solution of this non-linear partial differential equation with finite topological charge is known, called the ADHM construction [15, 16], which we shall review and heavily use in this paper. This construction has a remarkable property of completely solving the self-duality differential equation, up to an algebraic constraint on the parameters appearing in the ansatz of the solution. Even if the latter constraint is notorious as a general closed-form solution is not available, all the differential equation is completely solved.

A dyonic version of this instanton particle is also known [17]. This configuration carries electric charge as well as topological one. It is an instanton particle in the Coulomb phase of the theory. Ordinary instantons tend to collapse in this phase, while nonzero electric charge stabilizes this collapse to a finite size. This ‘dyonic instanton’ has been studied in various directions, with its interpretation as supertubes [18, 20] (ending on D4 branes) [21, 22, 23, 24]. The equations for supersymmetric solutions can again be solved modulo a set of algebraic constraints, using the ADHM construction [17, 25].

In this paper, firstly, we present the set of general conditions for the bosonic supersymmetric solutions in 5-dimensional Yang-Mills-Einstein supergravity, preserving time-like supersymmetry. This is a simple generalization of [3, 4] obtained in Maxwell-Einstein supergravity theories. This condition also generalizes the equations for the dyonic instanton in the field theory to the gravity-coupled case. A more general analysis of such conditions is presented in [26], but we shall explain the derivation to be self-consistent. Secondly, we show that this set of equations determining the gauge fields, scalars and the metric can be ‘solved’ in a way which naturally generalizes the ADHM construction. Namely, we solve all differential equations leaving a set of algebraic conditions. The solution that we obtain in this manner is manifestly regular at the generic point of the instanton moduli space.

From our solution for the metric, one can easily read-off the ADM angular momentum of the configuration. In models with ‘rigid’ limits, in which 5-dimensional gauge theory description of [27] would become relevant, one naturally expects that same result could also have been obtained from the Noether angular momentum in the field theory, which is an integral of angular momentum.
density over spatial $\mathbb{R}^4$. The latter integral could not be evaluated yet. We show that one of the differential conditions we solve in this paper can be used to make this Noether integrand into a surface term, giving the same answer as the above ADM value.

Having the expression for angular momentum and electric charge at hand, we investigate the $\mathcal{N} = 1$ truncated model of $\mathcal{N} = 2$ supergravity with $SU(2)$ gauge group in detail. We find for 2-instanton configurations that various components of the angular momentum have upper bounds given by the electric charges, where the maximum is attained when the configuration becomes a ‘round circle’ on a 2-plane with $U(1)^2$ symmetry like a ring. This is a feature which also happens for the supertubes $[28, 29, 24]$. Our analysis provides another evidence for the supertube interpretation of our solutions. We also study the geometry of this $U(1)^2$ symmetric configuration in detail, where the radius of the ‘ring’ is one of the free parameters of the solution. In particular, we show that this geometry has no closed timelike curves (CTC). This should be naturally related to the above fact that the angular momentum has an upper bound, since it is over-rotation which usually causes the naked CTC to appear. The general solution we find does not admit such a source for over-rotation, which leads us to a conjecture that CTC would be absent in the general solution we found. We do not attempt to check it in this paper.

The interpretation of our solution becomes subtler, but interesting, when there is a non-Abelian Chern-Simons term in the theory for $SU(N)$ gauge group with $N \geq 3$. For example, such gauge theories have been obtained from M-theory on singular Calabi-Yau 3-folds $[27]$, where the non-Abelian Chern-Simons coupling arises either classically or by integrating out massive Dirac fermions. Since our solution is new even in the gauge theory case, we present our ADHM solution in the context of both supergravity and gauge theory. The instanton carries electric charges even in the symmetric phase, namely, with zero asymptotic VEV for adjoint scalars. The structure of our general solution suggests a natural model for its moduli space dynamics, on which we only comment briefly in this paper.

The rest of this paper is organized as follows. In section 2 we summarize the necessary backgrounds on 5 dimensional supergravity coupled to vector multiplets. Special geometry, gauging and several models are explained. In section 3 we analyze the general structure of supersymmetric solutions in this theory and derive a set of differential conditions, generalizing the analysis in the literature. We also systematically construct regular solutions of these equations using the ADHM construction. The physical charges, some of which have been unknown, are computed as well. In section 4 we consider examples. We first consider the properties of gauge theory solitons, especially in the theory with Chern-Simons coupling. We also consider the $SU(2)$ 2-instantons in detail: we find various bounds on physical charges, and identify the structure of regular ring. Section 5 concludes the paper with discussions. Derivation of our ADHM solution is given in detail in appendix A. Properties of Killing spinor bilinears are summarized in appendix B.
2 Special geometry and gauging

In this section we summarize some aspects of 5 dimensional $\mathcal{N}=1$ Maxwell-Einstein supergravity (preserving 8 real supersymmetries), and explain the gauging of this theory to obtain the Yang-Mills-Einstein supergravity. We also explain some models of our interest, including the related supersymmetric Yang-Mills-Chern-Simons gauge theory.

The 5 dimensional $\mathcal{N}=1$ supergravity coupled to $n_V$ Abelian vector multiplets contains the following fields: (1) metric $g_{\mu\nu}$, (2) gravitino $\psi^i_{\mu}$ ($i=1,2$), (3) a graviphoton plus $n_V$ vector fields which are put together and written as $A^I_{\mu}$ ($I=1,2,\cdots,n_V+1$), (4) $n_V$ gauginos $\lambda^x$ and (5) $n_V$ real scalars $\varphi^x$ ($x=1,2,\cdots,n_V$). The coupling of gravity to the vector multiplets is conveniently described by the real special geometry \cite{30}. One introduces $n_V+1$ real scalars $X^I$ together with the vector fields $A^I_{\mu}$. The scalars $X^I$ have one more degree than is needed to parameterize $n_V$ dimensional moduli space of $\varphi^x$, which we call $\mathcal{M}_{n_V}$. $X^I$ is constrained as

$$\mathcal{V}(X) \equiv \frac{1}{6} C_{IJK} X^I X^J X^K = 1 ,$$

where $C_{IJK}$ is a set of parameters of the theory, totally symmetric in its indices. When we write $X^I(\varphi^x)$, it is understood that the above constraint is solved by $\varphi^x$. The above constraint can be written as

$$X^I X_I = 1 \quad \text{where} \quad X_I \equiv \frac{1}{6} C_{IJK} X^J X^K .$$

The bosonic part of the action of this theory is

$$S = \frac{1}{16\pi G} \int \left( \ast R - Q_{IJ} F^I \wedge \ast F^J - Q_{IJ} \ast dX^I \wedge \ast dX^J - \frac{1}{6} C_{IJK} A^I \wedge F^J \wedge F^K \right)$$

where we use the metric with mostly plus signature and

$$Q_{IJ} \equiv \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K$$

is the coupling matrix of $U(1)^{n_V+1}$ gauge fields. This matrix satisfies $Q_{IJ} X^J = \frac{3}{2} X_I$.

In some theories, including one that we consider in this paper, the constant $C_{IJK}$ satisfies the so-called symmetric space condition:

$$C^{IJK} C_{(LM)C_{NP)K}} = \frac{4}{3} \delta^I_L C_{MNP} \quad (C^{IJK} \equiv C_{IJK}) .$$

In this case, the following relations hold:

$$\mathcal{V} = \frac{9}{2} C^{IJK} X_I X_J X_K , \quad X^I = \frac{9}{2} C^{IJK} X_J X_K .$$

The properties of symmetric space are not used when we derive the supersymmetry conditions or our regular solutions in section 3, but are used to analyze specific examples in section 4.2.
Now we turn to the gauging of the above theory \cite{31,32}. To this end, we explain the global symmetry of this theory. The theory has a global $SU(2)_R$ R-symmetry, which rotates $\psi^i_\mu$ and $\lambda^x_i$ as doublets. Apart from this, there can be a symmetry $G$ which leaves the cubic polynomial $\mathcal{V}(X)$ in (2.1) invariant. The infinitesimal $G$-transformation is given as

$$
\delta X^I = M^I_{\, J} X^J, \quad \delta A^I_\mu = M^I_{\, J} A^J_\mu \tag{2.7}
$$

$$
M^I_{\, (JC)KL}_I = 0. \tag{2.8}
$$

Leaving the polynomial $\mathcal{V}(X)$ invariant, this transformation becomes a global symmetry of the Lagrangian, and especially generates an isometry on $\mathcal{M}_{n_V}$ with the metric

$$
g_{xy} \equiv Q^I_{\, J} \frac{\partial X^I}{\partial \varphi^x} \frac{\partial X^J}{\partial \varphi^y}. \tag{2.9}
$$

Among the global symmetry group $SU(2)_R \times G$, we want to gauge a subgroup $K \subset G$ to obtain Yang-Mills-Einstein supergravity. We summarize some aspects of this gauging, referring the readers to \cite{32} and references therein for details. The gauge field $A^I_\mu$ and the scalar $X^I$, which are both in an $n_V+1$ dimensional (generally reducible) representation of $G$, decompose to

$$(n_V+1)_G \rightarrow \text{adj}_K \oplus (\text{singlets})_K \oplus (\text{other non singlets})_K \tag{2.10}
$$

under $K$, where $\text{adj}_K$ denotes the adjoint representation. The last part consists of the non-singlets apart from the adjoint we picked out. We label the gauge fields belonging to the adjoint representation as

$$
A^a_\mu, \quad a = 1, \cdots, k \equiv \text{dim(adj}_K). \tag{2.11}
$$

To gauge the theory with group $K$, one should appropriately insert the $K$-connection $A^a_\mu$ in the action and supersymmetry transformations to make this symmetry $K$ a local one: covariantize the derivatives acting on all non-singlet components of the fields $X^I(\varphi^x)$ and $\lambda^x_i$, change the field strength $F^a = dA^a$ into a non-Abelian one, and change the Chern-Simons term into a non-Abelian one. This modification of the action containing adjoint and other non-singlet fields, if any, breaks the modified supersymmetry transformation in general. If there are no non-singlet fields in (2.10), the only thing one should do to restore supersymmetry is to add a suitable Yukawa interaction for fermions without further deforming supersymmetry transformation rule \cite{31,32}. If there exist non-singlet non-adjoint fields, one has to work harder to restore supersymmetry \cite{33}.

In this paper, we only consider the case in which the decomposition (2.10) consists of one adjoint and arbitrary number of singlets. We label the $n_V+1-k$ singlet fields as $A^a_\mu$ and $X^a$. The constants $C_{IJK}$ are constrained from the symmetry $K$ as

$$
C_{abc} = cd_{abc}, \quad C_{aab} = C_a \delta_{ab}, \quad C_{a\beta a} = 0 \quad \left( d_{abc} \equiv \frac{1}{2} \text{tr}(T^a\{T^b, T^c\}), \quad \text{tr}(T^a T^b) = \frac{1}{2} \delta_{ab} \right) \tag{2.12}
$$

\footnote{Another possibility which we do not consider here is the gauging which includes a subgroup of $SU(2)_R$. In this case one has to introduce a scalar potential.}
where $T^a$’s are the generators of $K$. $C_{\alpha\beta\gamma}$ is not constrained. Below we present models of this type derived from string theory.

The gauging of the subgroup $K \subset G$ outlined above can be done as follows. The isometry of $M_{\nu\nu}$ is generated by a set of Killing vectors. The $k$ Killing vectors $K^x_a(\varphi^x)$ are given as

$$K^x_a(\varphi) = \frac{3}{2} f^c_{ab} X_c(\varphi) \left( g^{xy} \partial_y X^b(\varphi) \right) = \frac{3}{2} f^c_{ab} \left( g^{xy} \partial_y X_c(\varphi) \right) X^b(\varphi),$$

where the second expression is equal to the first one since $(f^a_{bc})$ is the structure constant of $K$.

$$M^L(p_{JK})L = 0 \rightarrow f^L_{a(1C_{JK})L} = 0 \rightarrow f^c_{ab} X_c X^b = 0. \quad (2.14)$$

Firstly the derivatives and field strengths have to be covariantized. Since our main interest in this paper is to analyze bosonic solutions, here we only record the bosonic part of the covariantization:

$$\partial_\mu \varphi^x \rightarrow D_\mu \varphi^x = \partial_\mu \varphi^x + g K^x_a A^a_\mu, \quad (2.15)$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu, \quad (2.16)$$

where $g$ is the coupling constant. Other singlet quantities, like $F^a_{\mu\nu}$, are unchanged. If $d_{abc} \neq 0$, which is possible only for $SU(N)$ with $N \geq 3$ among simple groups, the Chern-Simons term is covariantized to the non-Abelian one:

$$d_{abc} A^a \wedge dA^b \wedge dA^c \rightarrow tr_{SU(N)} \left( A \wedge F \wedge F + \frac{i}{2} A \wedge A \wedge A \wedge F - \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A \right), \quad (2.17)$$

where $F = dA - iA \wedge A$ and $A = A^a T^a$.

Actually, the isometry of our model with (2.12) has a simpler realization as follows. The Killing vector $K^x_a$ transforms $X^I$ as

$$\delta_a X^I = f^I_{aJ} X^J = \begin{cases} 
  f^b_{ac} X^c & (I = b) \\
  0 & (I = \alpha)
\end{cases} \quad (2.18)$$

which is basically the reason why we required $f^c_{a(b} C_{cd)I} = 0$ for the polynomial $V(X)$ to be invariant under $K$. This (rather obvious) statement can also be checked directly from the above definition of $K^x_a$.

Therefore one finds

$$D_\mu X^I = \partial_\mu X^I + g A^a_\mu \delta_a X^I = \begin{cases} 
  \partial_\mu X^a + g f^a_{bc} A^b_\mu X^c & (I = a) \\
  \partial_\mu X^\alpha & (I = \alpha)
\end{cases}, \quad (2.20)$$

where structure constants other than $f^c_{ab}$ are all zero, and we used $f^I_{JK} X^J X^K = 0$. From this we confirm $D_\mu X^I = \partial_\mu X^I + f^I_{ab} A^b_\mu X^J$ is given as (2.19). Similarly, one finds $\delta_a X_I = -f^J_{aI} X_J$.\footnote{From the definition of Killing vector and special geometry, one finds

$$\delta_a X^I = \partial_a X^I K_a^x = -\frac{3}{2} f^c_{aK} X^K \left( g^{xy} \partial_x X^I \partial_y X_J \right) = f^c_{aK} X^K \left( \delta^I_J - X^J X^I \right) = f^c_{aK} X^K \quad (2.19)$$

where structure constants other than $f^c_{ab}$ are all zero, and we used $f^I_{JK} X^J X^K = 0$. From this we confirm $D_\mu X^I = \partial_\mu X^I + f^I_{ab} A^b_\mu X^J$ is given as (2.19). Similarly, one finds $\delta_a X_I = -f^J_{aI} X_J$.}
and similarly $D_\mu X_a = \partial_\mu X_a + gf_{ab}A_\mu^b X^c$.

We will sometimes consider the above supergravity together with a related 5 dimensional Yang-Mills gauge theory model presented in [27]. Firstly, we normalize the gauge fields and scalars in the adjoint representation $(A_\mu^a, X^a)$ such that the covariant derivatives do not contain the coupling $g$. We define

$$(A_\mu^a, X^a)_{SYM} = g(A_\mu^a, X^a)_{SUGRA}. \quad (2.21)$$

We write $\phi^a \equiv (X^a)_{SYM}$. The first limit we consider is the one in which the scalars $\phi^a$ and the gauge fields $A_\mu^a$ are ‘small’. Let us write $\phi^a \sim M$ and $\partial_\mu \phi^a \sim M$, where $M$ is the scale of the gauge theory, or more specifically of the classical solutions, which we are interested in. Taking $M \ll g$, we can regard the singlet scalars $X^\alpha$ as constants (of $\sim O(1)$). The metric $g_{\mu \nu}$ can also be taken to be approximately constant ($\approx \eta_{\mu \nu}$), while other singlet fields like $F^\alpha_{\mu \nu}$ are set to be nearly zero. One also finds

$$Q_{ab} \approx - \frac{1}{2} C_{abI} (X^I)_{SUGRA} = \frac{1}{2} (-C_{\alpha X^\alpha}) \delta_{ab} - \frac{c}{2g} d_{abc} \phi^c. \quad (2.22)$$

If $C_{\alpha X^\alpha} < 0$, one introduces the following Yang-Mills and Chern-Simons coupling ‘constants’

$$\frac{1}{g_{YM}^2} = \frac{(-C_{\alpha X^\alpha})}{16\pi G g^2}, \quad c_{YM} = - \frac{c}{16\pi G g^3}. \quad (2.23)$$

The bosonic part of the resulting gauge theory action is given as

$$S = \int d^5 x \left[ - \left( \frac{1}{g_{YM}} \delta_{ab} + c_{YM} d_{abc} \phi^c \right) \left( \frac{1}{4} F_{ab}^a F^{ab} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^b \right) \right] + S_{CS} \quad (2.24)$$

where

$$S_{CS} = + \frac{c_{YM}}{6} \int \text{tr} \left( A \wedge F \wedge F + \frac{i}{2} A \wedge A \wedge A \wedge F - \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A \right). \quad (2.25)$$

This theory can be obtained from the prepotential $F(\phi) = \frac{1}{2g_{YM}^2} \phi^a \phi^a + \frac{c_{YM}}{6} d_{abc} \phi^a \phi^b \phi^c$ [27], analogous to $\mathcal{V}(X)$ appearing in (2.1). Demanding that the exponential of the Chern-Simons term be invariant under large gauge transformations, $c_{YM}$ should be $\frac{1}{(2\pi)^2}$ times an integer [27], which can be checked from $\frac{1}{m!} \int_{R^{2n}} \text{tr}(F \wedge \cdots \wedge F) \in (2\pi)^n \mathbb{Z}$.

We close this section by explaining some supergravity models that will be considered in this paper.

We shall consider in some detail a supergravity model obtained by an $\mathcal{N} = 1$ truncation of the $\mathcal{N} = 2$ supergravity. The latter can be obtained as a low-energy theory of type II string theory on $K3 \times S^1$ or its various U-duals like heterotic string theory on $T^5$. We start from the $\mathcal{N} = 2$ supergravity coupled to $n$ vector multiplets. The latter vector multiplets contain $n$ gauge fields, $2n$ symplectic Majorana fermions and $5n$ real scalars. Especially, the scalar manifold is given as

$$SO(1,1) \times \frac{SO(5,n)}{SO(5) \times SO(n)} \quad (2.26)$$
up to discrete quotients, where the first factor comes from the dilaton in the $\mathcal{N} = 2$ gravity multiplet. We consider an $\mathcal{N} = 1$ truncation of this $\mathcal{N} = 2$ theory, keeping only the $\mathcal{N} = 1$ gravity and vector multiplets while setting the hypermultiplets and gravitino multiplet to zero. The scalars $\varphi^x$ in the truncated model live on the $n_V = n + 1$ dimensional space

$$M_{n_V} = SO(1,1) \times \frac{SO(1, n_V-1)}{SO(n_V-1)}, \quad (2.27)$$

whose special geometry is determined by the polynomial

$$V(X) = \frac{1}{2} X^1 (\eta_{ab} X^a X^b), \quad (2.28)$$

where $a, b = 2, 3, \cdots, n_V + 1$ and $\eta_{ab} = \text{diag}(+, -, -, \cdots, -)$. One can easily show that this set of cubic coefficients, $C_{1ab} = \eta_{ab}$ and others zero, satisfies the symmetric space condition (2.5). There is an obvious global symmetry $SO(1, n_V - 1)$ on $M_{n_V}$. The group $K$ we would like to gauge is in its compact subgroup, $K \subset SO(n_V - 1)$.

For the above string theory compactification, the massless scalar moduli is generically given by (2.26) or (2.27) with $n = 21$. Near certain points of the moduli space, namely the fixed points of the isometry $K$, the $U(1)^{21}$ gauge symmetry enhances to non-Abelian symmetry which technically is realized as the supergravity gauging. A simple example, among many others, is $SO(32) \times U(1)^5$ or $E_8 \times E_8 \times U(1)^5$ where the non-Abelian factors may be regarded as being inherited from 10 dimensional heterotic gauge symmetry for certain values of the moduli. At the level of supergravity, the gauging of the theory described by (2.28) with respect to any Lie group $K$ can be done by first enlarging the scalar manifolds as

$$SO(1,1) \times \frac{SO(1,n)}{SO(n)} \rightarrow SO(1,1) \times \frac{SO(1,k-r+n)}{SO(k-r+n)}, \quad (2.29)$$

where $k$ and $r$ are the dimension and rank of $K$, respectively. The cubic polynomial is (2.28) with $a, b = 2, \cdots, k-r+n+2 \equiv n_V + 1$. The matrix $\eta_{ab}$ becomes $-\delta_{ab} \propto \text{tr}(T_a T_b)$ in the $k$-dimensional subspace with negative signature, proportional to the quadratic Casimir of any group $K$ of dimension $k$. $V(X)$ is therefore invariant under the action of $K$, which can can be gauged. Under $K$, the $n_V + 1 = n + k - r + 2$ dimensional representation decomposes as $(\text{adj})_k \oplus (n-r+2 \ \text{singlets})$, which is the class of theory we discussed. For instance, taking $k = 496$ and $r = 16$, one can gauge either subgroup $SO(32)$ or $E_8 \times E_8$ of $SO(496)$.

Another interesting example is obtained from M-theory on $K3$-fibred Calabi-Yau 3-folds [32]. In order to correctly gauge these models, one has to take care of the 1-loop effect of massive Dirac fermions in the adjoint representation of $K$, renormalizing the Chern-Simons coupling $C_{1JK}$. This model is not treated in this paper. We just mention that there is no such renormalization in the above $\mathcal{N} = 2$ theory due to the underlying 16 supersymmetry.
3 Supersymmetric regular solutions

3.1 General properties of supersymmetric solutions

In this section we investigate the general supersymmetric solutions in the Yang-Mills-Einstein supergravity explained in the previous section. The strategy is closely related to the ones in, e.g., [1, 3, 4]. Conventions on geometry and spinors follows [1]. Especially we use mostly negative metric $\eta_{\mu\nu} = (+ - - -)$ only in this subsection and Appendix B, to parallel our results with the similar ones in [3, 4]. To go to the latter convention, changing sign in front of the Einstein-Hilbert term and the scalar kinetic term would suffice in the bosonic action (2.3).

We start by assuming the existence of a Killing spinor $\epsilon^i (i = 1, 2)$ in a purely bosonic background, satisfying the following equations coming from the supersymmetry transformations of gravitino and gaugino:

\begin{equation}
0 = \delta \psi^i_\mu = \left( \nabla_\mu + \frac{1}{8} X_I (\gamma_\mu^\rho - 4 \delta_\mu^\nu \gamma_\rho) F^I_{\nu\rho} \right) \epsilon^i \tag{3.1}
\end{equation}

and

\begin{equation}
0 = \delta \lambda^i_\alpha = \left( \frac{1}{4} Q_{IJ} \gamma^{\mu\nu} F^J_{\mu\nu} + \frac{3}{4} \gamma_\mu D_\mu X_I \right) \epsilon^i \frac{\partial X^I}{\partial \phi^\alpha} . \tag{3.2}
\end{equation}

Here $\nabla_\mu$ denotes the spacetime-covariant derivative, while $D_\mu$ (acting on $X_I$) is used to emphasize that it is $K$-covariantized. Its action on $X_I$ is given as

\begin{equation}
D_\mu X_I = \partial_\mu X_I + f_{IJ}^K A^K_\mu X_I \quad (\text{where } f^\alpha_{**} = f^{*\alpha} = 0) , \tag{3.3}
\end{equation}

while its action on $F^I_{\mu\nu}$ should also include Christoffel connection in curved spaces. Using the property $X_I \partial_\nu X^I = 0$ of special geometry, the gaugino equation (3.2) can be written as [3, 4]

\begin{equation}
0 = \left( \left( \frac{1}{4} Q_{IJ} - \frac{3}{8} X_I X_J \right) F^J_{\mu\nu} \gamma^{\mu\nu} + \frac{3}{4} \gamma_\mu D_\mu X_I \right) \epsilon^a . \tag{3.4}
\end{equation}

A bosonic configuration solving the above equation, should additionally satisfy the equation of motion for the gauge fields (including the Gauss’ law) to be a solution. This equation is

\begin{equation}
D(Q_{IJ} \ast F^J) = - \frac{1}{4} C_{JKL} F^J \wedge F^K + Q_{JK} f^J_{IL} X^L_\ast (\ast DX^K) . \tag{3.5}
\end{equation}

Assuming this equation, other equations of motion will turn out to be guaranteed from the integrability of Killing spinor equation, in the case we consider (in which timelike supersymmetry is preserved, to be explained below).

Having a solution of the equations (3.1) and (3.2), it is helpful to study the various spinor
bilinear following, for instance, [1, 3, 4]:

\[
\bar{\epsilon}^i \epsilon^j = f \epsilon^{ij} \quad (3.6)
\]

\[
\bar{\epsilon}^i \gamma_\mu \epsilon^j = V_\mu \epsilon^{ij} \quad (3.7)
\]

\[
\bar{\epsilon}^i \gamma_\mu \epsilon^j = \Phi^{ij}_{\mu} \quad (i \leftrightarrow j \text{ symmetric}), \quad (3.8)
\]

real 2 forms $J^a_{\mu \nu}$: $\Phi^{11} = J^1 + iJ^2$, $\Phi^{12} = -iJ^3$. \( (3.9) \)

They satisfy a set of algebraic relations due to Fierz identity, and differential conditions obtained by using the Killing spinor equation. The structure of these conditions are similar to the ones presented in [1, 3, 4] and are summarized in appendix B. Firstly, all algebraic conditions and differential condition obtained from gravitino equation (3.1) are same as the results [4] for the Maxwell-Einstein theory. There are minor difference in conditions obtained from the gaugino equation (3.2) and the equation of motion (3.5), modified by the gauging.

Equations (B.7) shows that $V$ is a Killing vector. From (B.1), it may be either timelike or null. In this paper we consider the timelike case, which is what we meant by timelike supersymmetry. Introducing coordinates $(t, x^m)$ \((m = 1, \cdots, 4)\) such that $V = \frac{\partial}{\partial t}$, the metric can be written as

\[
-ds^2 = -f^2(dt + \omega)^2 + f^{-1}h_{mn}dx^m dx^n \quad (3.10)
\]

where $f$, $\omega$ and $h_{mn}$ are independent of $t$. $h_{mn}$ is a metric on 4 dimensional base space, which we call $\mathcal{B}$. Following [4], we set $e^0 = f(dt + \omega)$, choose the volume form $(\text{vol})_4$ of $\mathcal{B}$ and take $e^0 \wedge (\text{vol})_4$ to be the 5 dimensional volume form. With $(\text{vol})_4$, we can decompose $d\omega$ as

\[
fd\omega = G^++G^- , \quad (3.11)
\]

namely into self-dual and anti-self-dual 2-forms on $\mathcal{B}$, again following the above references. One can see from (B.3) and (B.4) that $J^i$ can all be regarded as anti-self-dual 2-forms on $\mathcal{B}$, while from (B.5) and (B.9) that they provide an integrable hyper-Kähler structure on $\mathcal{B}$ [1].

Now we turn to the gauge fields. $A^I$ can be written as $A^I = A^I_0 e^0 + A^I$ where $A^I$ is a 1-form on $\mathcal{B}$. We choose the gauge $A^I_0 = X^I$, which is not essential but convenient:

\[
A^I = X^I e^0 + A^I . \quad (3.12)
\]

Using (B.12), one can follow [4] and write

\[
F^I = -f^{-1} e^0 \wedge D(f X^I) + \Psi^I + \Theta^I + X^I G^+ \quad (3.13)
\]

where $\Theta^I$ and $\Psi^I$ are self-dual and anti-self-dual on $\mathcal{B}$, respectively. Inserting this expression into (B.6) and (B.8), one obtains

\[
X_I \Psi^I = G^- , \quad X_I \Theta^I = -\frac{2}{3} G^+ . \quad (3.14)
\]
However, since (B.11) requires $\Psi^I$ to be proportional to $X^I$, one finds

$$\Psi^I = X^IG^-.$$  \hspace{1cm} (3.15)

Inserting this back to (3.13), one obtains

$$F^I = D(X^Ie^0) + \Theta^I,$$  \hspace{1cm} (3.16)

where $\Theta^I$ is related to $G^+$ as (3.14). Since this field strength is related to the potential (3.12) as $F^I = dA^I + \frac{1}{2}f^I_{JK}A^J \wedge A^K$, which is

$$dA^I + \frac{1}{2}f^I_{JK}A^J \wedge A^K = D(X^Ie^0) + \left( dA^I + \frac{1}{2}f^I_{JK}A^J \wedge A^K \right),$$  \hspace{1cm} (3.17)

one concludes that the self-dual component $\Theta^I$ is given by the 1-form $A^I$ on $B$ as

$$\Theta^I = dA^I + \frac{1}{2}f^I_{JK}A^J \wedge A^K,$$  \hspace{1cm} (3.18)

which is exactly the Yang-Mills field strength of $A^I$ on the space $B$. The set of constraints on $\Theta^I$ is (3.18), self-duality on $B$, and (3.14).

Following the Maxwell-Einstein supergravity, one can also show that the above conditions are sufficient to show the Killing spinor equations. Firstly, imposing the projection $\gamma^0 \epsilon^i = \epsilon^i$, the gaugino equation follows from (3.16) and the fact $\Theta^I = *_4 \Theta^I$. The gravitino equation reduces to

$$\partial_t \epsilon^i = 0, \quad \nabla_m (f^{-\frac{1}{2}} \epsilon^i) = 0.$$  \hspace{1cm} (3.19)

As in the Maxwell-Einstein theory, there exist 4 real independent components solving these equations and $\gamma^0 \epsilon^i = \epsilon^i$ on the hyper-Kähler space $B$.

Apart from the conditions for supersymmetry, one also has to impose the equation of motion for the gauge fields. After imposing the supersymmetry conditions, it turns out that the only nontrivial component of this equation is the Gauss’ law:

$$\mathcal{D}^m \mathcal{D}_m (f^{-1}X_I) = \frac{1}{6}C_{IJK} *_4 \left( \Theta^J \wedge \Theta^K \right).$$  \hspace{1cm} (3.20)

As mentioned above, the supersymmetry conditions and this Gauss’ law guarantee other equations of motion also hold in our timelike case.

To summarize, one obtains the following set of equations to be solved:

$$\Theta^I = *_4 \Theta^I \quad (\Theta^I = dA^I + f^I_{JK}A^J \wedge A^K) \hspace{1cm} (3.21)$$

$$\mathcal{D}^m \mathcal{D}_m (f^{-1}X_I) = \frac{1}{6}C_{IJK} *_4 \left( \Theta^J \wedge \Theta^K \right) \hspace{1cm} (3.22)$$

$$(1 + *_4)d\omega = -3 f^{-1}X_I \Theta^I.$$  \hspace{1cm} (3.23)
where $D_m$ is the covariant derivative on $\mathcal{B}$ with the connection $A^I$. These equations should be solved to give the fields $A^I$, $X^I$, $f$ and $\omega$. The basic fields are given by (3.10) and (3.12). The above three equations are similar to those in the Maxwell theory [10, 12]. There, if one tries to solve them in the order listed above, they can be regarded as linear equations with source. The situation is nearly the same here. The first equation is non-linear to start with. However, the latter two can be solved linearly, regarding the right hand sides as external source terms once the previous equations are solved. Even the first non-linear equation is has been studied in depth, since it is the famous equation describing self-dual instantons in Yang-Mills theory. In the next subsection, we present a large class of (semi-)explicit solutions of this set of equations.

### 3.2 ADHM instantons and regular solutions

From now we assume the base space $\mathbb{R}^4$ and systematically find a class of configurations solving (3.21), (3.22), (3.23). The self-dual Yang-Mills gauge field configurations on $\mathbb{R}^4$ can be found by the so-called ADHM construction [15, 16]. We base our analysis on the ADHM construction to find solutions of the other equations we listed in the previous subsection.

Before starting the analysis, we would like to clarify the different normalizations in supergravity and Yang-Mills theory. So far we naturally normalized the scalars $X^I$ and gauge fields $A^I_{\mu}$ to have mass dimension 0. The gauge coupling $g$ has dimension 1. A convenient normalization for the analysis of solitons in gauge theory is to set this coupling to 1 by rescaling $X^I_{YM} = gX^I_{SUGRA}$ and $A^I_{YM} = gA^I_{SUGRA}$, where the prefactor in front of the kinetic terms of vector multiplet fields becomes $\frac{1}{16\pi G g^2}$. We assume the latter normalization in this subsection and Appendix A. In this normalization, scalars satisfy $\frac{1}{6} C_{IJK}X^I X^J X^K = g^3$. The equations (3.21) and (3.22) takes the same form replacing $X_I \equiv \frac{1}{6} C_{IJK}X^J X^K$ and $\Theta^I$ into the new ones, while (3.23) becomes

$$(1 + \star_4) d \omega = -3g^{-3}(f^{-1} X_I) \Theta^I$$

(3.24)

with the new normalization.

As mentioned above, we choose the 4 dimensional base space to be $\mathbb{R}^4$ with the flat metric $h_{mn} = \delta_{mn}$, even though there are more general possibilities of base space. With this choice of the base space, the general solution to the self-dual field equation (3.21) is given by the ADHM construction which we explain now. We will exclusively consider the case with $SU(N)$ gauge group in this paper, even if we expect the cases with $SO(N)$ and $Sp(N)$ gauge groups can be treated in a similar way. Following [25], one starts the construction of $SU(N)$ $k$-instantons by writing down an $(N + 2k) \times 2k$ matrix $\Delta_{\dot{a}}(x)$

$$\Delta_{\dot{a}} \equiv a_{\dot{a}} + b^a x_{a\dot{a}}$$

(3.25)

where

$$x_{a\dot{a}} \equiv x^m \sigma^m_{a\dot{a}} , \quad x^m \in \mathbb{R}^4 , \quad \sigma^m_{a\dot{a}} = (1, i\vec{\sigma}) , \quad \bar{\sigma}^{\dot{a}\alpha} = (1, -i\vec{\sigma})$$

(3.26)
and
\[ a_\dot{\alpha} \equiv \begin{pmatrix} \omega_{\dot{\alpha}} \\ a_{\dot{\alpha}}' \end{pmatrix}, \quad b^\alpha \equiv \begin{pmatrix} 0_{N \times 2k} \\ 1_{2k \times 2k} \end{pmatrix}. \] (3.27)

The constant matrices \( \omega_{\dot{\alpha}} \) and \( a_{\dot{\alpha}}' \equiv a_n \sigma_n^\alpha \), \( a_n \) are \( N \times 2k \), \( 2k \times 2k \) and \( k \times k \) matrices, respectively, and we suppressed all matrix indices except for the 2-component \( SO(4) \) spinor indices \( \alpha \) and \( \dot{\alpha} \). We refer the readers to [25] for more details on notations.

The self-dual field strength \( \Theta_{mn} \), or the connection \( A_m \), is given by an \( (N + 2k) \times N \) matrix \( U(x) \) satisfying the following conditions
\[ \bar{\Delta}^{\dot{\alpha}}(x)U(x) = 0, \quad \bar{U}U = 1_{N \times N}. \] (3.28)

The gauge field \( A_m \) is given as
\[ A_m = i\bar{U}(x)\partial_m U(x), \] (3.29)
whose field strength is guaranteed to be self-dual if \( \omega_{\dot{\alpha}} \) and Hermitian matrices \( a_n \) satisfy the following algebraic equation (\( \bar{\sigma}_{mn} \equiv \bar{\sigma}_{[m} \sigma_{n]} \) and \( \sigma_{mn} \equiv \sigma_{[m} \bar{\sigma}_{n]} \)):
\[ \bar{\omega}^{\dot{\alpha}} \omega_j^{\beta} (\bar{\sigma}_{mn})^{\dot{\beta}}_{\dot{\alpha}} = 2(1 - \ast_4)[a_m, a_n]. \] (3.30)

With (3.30) satisfied, one can show that the \( 2k \times 2k \) matrix \( \bar{\Delta}^{\dot{\alpha}} \Delta_\beta \) takes the form
\[ \bar{\Delta}^{\dot{\alpha}} \Delta_\beta = F^{-1}(x)\delta^{\dot{\alpha}}_\beta \] (3.31)
with an invertible \( k \times k \) Hermitian matrix \( F(x) \). The field strength \( \Theta^\alpha T^a \), where \( T^a \)'s are \( SU(N) \) generators with the normalization in section 2, is given as
\[ \Theta_{mn} \equiv \Theta^\alpha_{mn} T^a = 2i\bar{U}b^\alpha (\sigma_{mn})^\beta_\beta F\bar{b}_\beta U. \] (3.32)

The general solution to the \( k \times k \) matrix equation (3.30) is not known, but we will say that one ‘solved’ the equation (3.21) in the sense that partial differential equation is reduced to an algebraic one. The number of unconstrained real degrees in the matrices are \( 4Nk \): from the original \( 4Nk + 4k^2 \) degrees in \( \omega_{\dot{\alpha}} \) and \( a_n \), one subtracts the number of equations in (3.30), \( 3k^2 \), as well as the \( U(k) \) gauge transformation degree \( k^2 \) [25]. This actually is the general self-dual configuration with given topological charge \( k \), deduced from a suitable index theorem.

Having this general solution parameterized by \( 4Nk \) data, one has to solve the covariant Laplace equation with sources (3.22). We first consider the scalars in the adjoint representation, \( I = a \). The Laplace equation without source is solved in [25], see their Appendix C. In Appendix A.1, we generalize this construction to the case with sources provided by the non-Abelian Chern-Simons term. The equation and our solution in matrix notation are
\[ \mathcal{D}^2(f^{-1}X_a T^a) = \frac{c}{24} \left( \Theta_{mn} \Theta_{mn} - \frac{1}{N} \text{tr}(\Theta_{mn} \Theta_{mn}) 1_N \right). \] (3.33)
and
\[ f^{-1}X_a T^a = \bar{U}(x)J_0 U(x) - \frac{c}{24N} \partial^2 \log(\det F(x)) \mathbf{1}_N \]  
where the \((N+2k) \times (N+2k)\) matrix \(J_0\) is given as
\[ J_0 = \begin{pmatrix} v_{N \times N} & (\varphi_{k \times k} - \frac{c}{12} F(x) \otimes \mathbf{1}_2) \end{pmatrix}. \]

We hope using \(\varphi\) will not cause confusion with scalars \(\varphi^x\) in section 2. Here the \(k \times k\) matrix \(\varphi\) should satisfy
\[ L\varphi \equiv \frac{1}{2} \{ \tilde{\omega}^\dot{\alpha} \omega_\dot{\alpha}, \varphi \} + [a_n, [a_n, \varphi]] = \tilde{\omega}^\dot{\alpha} v_{\omega_\dot{\alpha}} - \frac{c}{6} \mathbf{1}_k \]
for (3.34) to solve (3.33). The \(N \times N\) matrix \(v = v_a T^a\) is the asymptotic value of \(X_a T^a\) at infinity. Equation (3.36) is linear in \(\varphi\). We will present the explicit 2-instanton solutions in the next section. Anyhow, the differential equation is solved modulo the algebraic equation (3.36).

Now we turn to the Laplace equation for the singlet scalars with source terms,
\[ \partial^2 (f^{-1}X_\alpha) = \frac{C_\alpha}{6} \ast_4 (\Theta^a \wedge \Theta^a) = \frac{C_\alpha}{6} \text{tr} (\Theta_{mn} \Theta_{mn}) . \]
It can be solved using the Osborn’s formula [34] for the topological charge density:
\[ \text{tr}_N (\Theta_{mn} \Theta_{mn}) = (\partial^2)^2 \log (\det F_{k \times k}(x)) \].
From this one obtains
\[ f^{-1}X_\alpha = + \frac{1}{6} C_\alpha \partial^2 \log (\det F(x)) + h_\alpha , \]
where \(h_\alpha\) are constants. One might have inserted any harmonic function \(H_\alpha(x)\) on \(\mathbb{R}^4\) instead of \(h_\alpha\), which is a homogeneous solution of this equation. However, in foresight, we do not insert any nontrivial homogeneous solution, which in \(\mathbb{R}^4\) is associated with singular sources, in order to obtain regular solutions.

Finally, we turn to the differential equation (3.23) for the 1-form \(\omega_m\). In Appendix A.2, we derive the following solution in general ADHM instanton background:
\[ \omega_m = -\frac{3i}{g^3} \text{tr} \left( J_0 \frac{\mathcal{P} \partial_m \mathcal{P} - \partial_m \mathcal{P} \mathcal{P}}{2} + 2[\varphi, a_m] F - \frac{c}{72} \epsilon_{mnpq} \partial_n F^{-1} F \partial_p F^{-1} F \partial_q F^{-1} F \right) , \]
where \(\mathcal{P}(x) \equiv U \bar{U}\). Again, one might add arbitrary homogeneous solution \(\Delta \omega_m\) to the equation (3.23), where \(d(\Delta \omega)\) is anti-self-dual. For nonzero \(\Delta \omega_m\) to vanish at asymptotic infinity, it should also be associated with a singular source since the Maxwell equation \(d^\dagger d(\Delta \omega) = 0\) is satisfied for anti-self-dual \(d(\Delta \omega)\). For instance, adding
\[ \Delta \omega_m = \frac{j}{r^4} (\delta_m^{[1} x^{2]} + \delta_m^{[3} x^{4]}) \]
to a spherically symmetric black hole would change the solution into the BMPV black hole with the self-dual angular momentum \((J_L)_{12} = (J_L)_{34} \sim j\). Adding it to our solution would result in
closed timelike curves. Anyhow we again do not add such homogeneous solutions. This completes
the construction of our solution of (3.21)-(3.23).

We emphasize that the solution we obtained is manifestly smooth ‘generically’: namely all
components of the fields \((g_{\mu\nu}, F^I_{\mu\nu}, X^I)\) are finite and smooth in space-time coordinates \((t,x^m)\),
at a generic point on the instanton moduli space. This is guaranteed from the construction itself,
once the matrix \(F(x)\) introduced in (3.31) is invertible. This assumption is not true on a certain
point of the instanton moduli space. For example, there are parameters which can be identified
as the sizes of instantons. When any of these sizes is taken to zero, the configuration \(\Theta_{a m n}\)
starts to be singular at the ‘location’ of this small instanton. This singularity propagates to the other
fields at this point. Just to mention one phenomenon, let us consider the Chern-Simons coupling
\(C_\alpha A^\alpha \wedge (F^a \wedge F^a)\) which induces \(U(1)\) electric charges of
\(A^\alpha_{\mu}\) to an instanton. As the instanton becomes small, the source for \(F^\alpha_{\mu\nu}\) becomes point-like, which has an effect of replacing \(h^\alpha\) in (3.39)
by a harmonic function sourced by a point charge. Away from such ‘singular’ points on the
instanton moduli space, our configuration is smooth.

In [35], regular solutions for the gravitating single monopoles and instantons in 4- and 5-
dimensional (super-)gravity saturating BPS energy bounds are constructed. Moreover, in the
’t Hooft (dyonic) multi-instanton background, the regular solutions in 10 dimensional heterotic
supergravity is obtained in [36, 37]. Our solution is a generalization of these works in the 5
dimensional setting.

We close this section by computing the physical charges of our solutions.

The \(U(1)^{N-1} \subset SU(N)\) electric charge \(q\) is given as (choosing the orientation \(dt \wedge dr \wedge \text{vol}(S^3)\))
\[
q_a \sim \frac{1}{8\pi G} \int_{S^3} Q_{a I} \wedge F^I = \frac{3}{16\pi G} \int_{S^3} *4\mathcal{D}(f^{-1}X_a) \tag{3.42}
\]
where the integral is over the asymptotic 3-sphere. We multiply \(\frac{1}{g^3}\) on the right hand side, which
will turn out to be the most natural normalization. Expanding the integrand, the electric charge
is given by the \(N\) diagonal entries of the following \(N \times N\) matrix,
\[
q = \frac{3}{16\pi G g^3} 4\pi^2 \left( \frac{1}{2} \{v, \omega_\alpha \bar{\omega}^\alpha\} - \omega_\alpha \bar{\omega}^\alpha \right) - \frac{c k}{6N} 1_N \equiv 4\pi^2 \left( \frac{1}{2} \{\bar{v}, \omega_\alpha \bar{\omega}^\alpha\} - \omega_\alpha \bar{\omega}^\alpha \right) - \frac{c_{YM} k}{2N} 1_N \tag{3.43}
\]
where we introduce new variables \(\bar{v} \equiv -\frac{3}{16\pi G g^3} v\), \(\bar{\varphi} \equiv -\frac{3}{16\pi G g}\varphi\) in foresight. In section 4.1 we
show this is the natural normalization in Yang-Mills field theory. \(c_{YM}\) is already introduced as
(2.23). The matrix \(q\) is traceless if \(\varphi\) satisfies (3.36). This is a simple generalization of the result
in [17] to the case \(c \neq 0\).

We now compute the ADM angular momentum associated with the Killing vector \(-\xi_{ab}\), where
\(\xi_{ab} \equiv x_a \partial_b - x_b \partial_a\):
\[
J_{ab} = -\frac{1}{16\pi G g^3} \int_{S^3} *\nabla \xi_{ab} = -\frac{4\pi^2 k_{ab}}{16\pi G g^3} \tag{3.44}
\]
where $\omega_m \approx \frac{k m a^n}{g^4}$ as $r \to \infty$. Expanding the matrices $F$ and $P$, one obtains from (3.40) the following:

$$\omega_m \approx 3i g^3 x_n r^4 \{ \text{tr}_k (\bar{\omega}^\alpha v \omega^\beta) (\bar{\sigma}_{mn})^\beta_\alpha + 4 \text{tr}_k \left( \varphi [a_m, a_n] \right) \}$$

$$= 3i g^3 x_n r^4 \{ \text{tr}_k (\bar{\omega}^\alpha v \omega^\beta) (\bar{\sigma}_{mn})^\beta_\alpha + 2(1 + *_4) \text{tr}_k \left( \varphi [a_m, a_n] \right) \},$$

(3.45)

where we used the ADHM constraint (3.30) on the second line. Therefore, one finally obtains

$$J_{mn} = + 4 \pi^2 \left\{ \text{tr}_k (\bar{\omega}^\alpha v \omega^\beta) (\bar{\sigma}_{mn})^\beta_\alpha + 2(1 + *_4) \text{tr}_k \left( \varphi [a_m, a_n] \right) \right\},$$

(3.46)

where again the new variables $\tilde{v}$ and $\tilde{\varphi}$ are introduced as shown in the previous paragraph, to compare the result (3.46) with the one from field theory in section 4.1.

The ADM mass is associated with the Killing vector $\xi = \partial_t$:

$$M = + \frac{3 \pi \alpha}{4G},$$

(3.47)

if $f \approx 1 - \frac{\alpha}{r^2}$ as $r \to \infty$. With the asymptotic behavior

$$f^{-1} X^I \approx h_I + \frac{\mu_I}{r^2}, \quad f^{-1} X^I \approx h_I + \frac{\mu_I}{r^2},$$

(3.48)

where $h^I = X^I(\infty)$, one can easily find $\alpha = \mu_I h^I$. From (3.34) and (3.39) one finds

$$M = q_a \phi^a(\infty) + \frac{8 \pi^2 k}{g^4_{YM}},$$

(3.49)

where $\phi^a(\infty)$ is the expectation value of $\phi^a(= gX^a_{\text{SUGRA}})$ at infinity, and $g^2_{YM}$ is given by (2.23). This saturates the BPS bound given in [17].

4 Examples and applications

4.1 The Yang-Mills(-Chern-Simons) gauge theory

When the Yang-Mills gauge fields and scalars are taken to be ‘small’, as explained in section 2, our solution reduces to that of the gauge theory of [27]. The dyonic instanton configuration in the gauge theory without non-Abelian Chern-Simons term has been first studied in [17]. The general ADHM solution in the presence of the non-Abelian Chern-Simons term has been unknown in the gauge theory, so we shall take a more detailed look at our new solution in this context. Another problem in the gauge theory which has not been answered is the computation of the Noether angular momentum of the configuration. In the previous section we obtained the ADM angular

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3The Killing vector for mass always picks up a minus sign relative to those for spatial momenta [13].
momentum, but it seems that the same answer should be obtained on the gauge theory side as the Noether charge. We also explain this point in this subsection.

In the Yang-Mills-Chern-Simons theory, the differential conditions for the supersymmetric solutions are

\[
F_{mn} = \star_4 F_{mn} \\
\mathcal{D}^2 \tilde{\phi}_a = + \frac{c_{YM}}{4} d_{abc} F_{mn}^b F_{mn}^c
\]

where

\[
\tilde{\phi}_a \equiv \frac{\partial F(\phi)}{\partial \phi^a} = \frac{1}{g_{YM}^2} \phi^a + \frac{c_{YM}}{2} d_{abc} \phi^b \phi^c
\]

\[
\mathcal{F}(\phi) \equiv \frac{1}{2 g_{YM}^2} \phi^a \phi^a + \frac{c_{YM}}{6} d_{abc} \phi^a \phi^b \phi^c
\]

Furthermore, even if the metric degree considered in the previous section is irrelevant in the gauge theory, we would still like to consider the regular solution of the following differential equation:

\[
(1 + \star_4)(d\alpha)_{mn} = -6 \text{tr}_N(\tilde{\phi} F_{mn})
\]

where the regular solution for the 1-form \(\alpha_m\) can be obtained as we got \(\omega_m\) before. If one takes the scaling of fields \(F_{mn}^{YM} = g F_{mn}^{SUGRA}\) and \(\tilde{\phi}_a = -\frac{3}{16\pi G g^3} \phi^{a}\) into account, one obtains

\[
\alpha_m = \omega_m(\tilde{v}) = -\frac{3}{16\pi G g^3} \omega_m(v)
\]

with \(\omega_m\) given as (A.36). This differential equation and the solution \(\alpha_m\) will still play interesting roles as we explain below.

Firstly, let us re-consider the electric charge computed in the previous section. The expression (3.43) is exactly the same as that in [17] in the case \(c = 0\) (correcting a factor 2 typo there). The quantization of this electric charge was studied from the moduli space dynamics of Yang-Mills instantons [17], where the electric charge is understood as a momentum conjugate to the coordinate on the moduli space parameterizing the global gauge zero mode. See [38] also. A potential of the schematic form \(\propto v^2 |\omega_a|^2\) is generated on the moduli space, which holds the motion in the moduli space in a finite \(\omega_a\) region. Since \(\varphi\) is also proportional to \(v\), one finds that the electric charge depends linearly on the asymptotic value \(v^a\) of \(\tilde{\phi}^a\).

When there is a non-zero Chern-Simons term, \(c \neq 0\), the physics becomes different. In this case one finds that the configuration carries nonzero electric charge even when \(\phi^a(\infty) = 0\) (or \(v = 0\)), as the second and third terms of (3.43) are still nonzero. From the dynamics on the instanton moduli space, this quantity should also be understood as the momentum conjugate to the global gauge zero mode. The Lagrangian should acquire modifications other than the potential to explain this charge. From (3.49), the electric charge, or momentum, does not contribute to the BPS mass if \(\phi^a(\infty) = 0\). It is likely that the states with electric charges should provide a sort of lowest
Landau level degeneracy from the viewpoint of moduli space dynamics, by an addition of external magnetic field on the moduli space.

For simplicity, let us briefly comment on the single instantons in the unbroken phase ($\phi^a(\infty) = 0$) when $c_{YM} \neq 0$. The magnetic field $\Theta^a_{\mu\nu}$ is given by the $SU(N)$ embedding of single $SU(2)$ 't Hooft solution. In this background, one finds nonzero scalar and electric field. However, the electric charge contribution to the energy is zero since $v = 0$. One finds $\tilde{\phi} = -\frac{c_{YM}}{4\lambda^2} \rightarrow q = \pi^2 c_{YM} \left( P - \frac{2}{N} 1^N \right)$, (4.5)

where $\lambda$ is the size of the instanton, $P$ is the projector to the 2 dimensional subspace of the $N$ dimensional space in which $SU(2)$ 't Hooft solution is embedded. With $\tilde{v} = 0$, since the potential $\propto \lambda^2 \tilde{v}^2$ confining $\lambda$ is absent, the nature of the corresponding motion on the moduli space should be quite different. What we expect from (4.5) is a motion on the moduli space with appropriate 'magnetic field.' Just for convenience, let us assume that $\lambda$ is much larger than $c_{YM} g^2_{YM}$, the only scale of this system. Then we can trust the moduli space metric for single instantons with $c_{YM} = 0$, which is a cone over $SU(N)/U(N-2)$ with homogeneous metric on the base. Upon coupling the system to a suitable 1-form $A \sim c_{YM} \theta$, where $d\theta$ gives the Kahler 2-form of the space $SU(N)/SU(2) \times U(N-2)$, one finds that the rest particle solution carries an angular momentum of the form (4.5). More comment is in order in the conclusion section.

Now we consider the angular momentum of the configuration. The Noether angular momentum is given by the following 4 dimensional integral:

$$J_{mn} = - \int d^4x (x^m T_{0n} - x^n T_{0m}) \, ,$$

(4.6)

where

$$T_{0m} = \left( \frac{1}{g_{YM}} \delta_{ab} + c_{YM} d_{abc} \phi^c \right) F_{0a}^a F_{0b}^b = -2 \partial_n tr (\tilde{\phi} F_{mn}) \, .$$

(4.7)

The integral (4.6) can be written as

$$J_{mn} = -2 \int_{S^3} r^3 d\Omega^k \left( x^m tr (\tilde{\phi} F_{nk}) - x^n tr (\tilde{\phi} F_{mk}) \right) + 4 \int_{\mathbb{R}^4} d^4x \ tr (\tilde{\phi} F_{mn}) \, ,$$

(4.8)

where $d\Omega^k$ is the vector normal to the unit 3-sphere whose length is the volume element of $S^3$. In [37], the second term is shown to be zero for the 't Hooft multi-instanton background. The first surface term is easily evaluated to give an expression for $J_{mn}$ in this case. For general ADHM instanton, the second term is nonzero and the general expression of $J_{mn}$ has not been available yet. However, one can also change the second term of (4.8) into a surface term, using the differential condition (4.3):

$$\int d^4x tr (\tilde{\phi} F_{mn}) = -\frac{1}{6} \int_{S^3} r^3 \left( d\Omega^m \alpha_n - d\Omega^n \alpha_m + \epsilon_{mnpq} d\Omega^p \alpha_q \right)$$

(4.9)

\footnote{We thank David Tong for pointing it out to us.}

Overall minus sign is inserted since positive energy is given by $\int d^4x T_{00}$, while spatial momentum has a relative minus sign in its definition.
where we used the following fact for an integral over a region $\Sigma$ in $\mathbb{R}^4$:

$$
\int_{\Sigma} d^4x \partial_a f = \int_{\partial \Sigma} dS^a f .
$$

(4.10)

Evaluating the two surface integrals, one finds

$$
-2 \int_{S^3} r^3 d\Omega^k \left( x^a \text{tr} (\phi F_{nk}) - x^m \text{tr} (\phi F_{nk}) \right) = +4\pi^2 i \text{tr} \left( \bar{\omega}^\alpha \bar{\omega}_\beta (\sigma_{mn})^\beta_\alpha \right)
$$

(4.11)

$$
-\frac{2}{3} \int_{S^3} r^3 (d\Omega^m \alpha_n - d\Omega^n \alpha_m + \epsilon_{mpq} d\Omega^p \alpha_q) = +\frac{2\pi^2}{3} (1 + \star_4) k_{mn}(\tilde{v})
$$

(4.12)

where $\alpha_m(\tilde{v}) \approx \frac{k_{mn}(\tilde{v}) x^n}{r}$ near $r \to \infty$. Adding the above two, what we get is exactly same as the ADM angular momentum $\mathbf{3.40}$.

### 4.2 SU(2) 2-instantons: closed timelike curves and charge bounds

In this subsection we investigate the the case with $SU(2)$ gauge group in detail. Since $d_{abc} = 0$ for $SU(2)$, there is no non-Abelian Chern-Simons term here. Since the single instanton is basically given by the ’t Hooft solution, which is quite special rather than being generic, we concentrate on the case in which instanton number $k$ is 2. (For simplicity, we set $g = 1$.)

The Yang-Mills 2-instanton for $SU(2)$ gauge group is completely given by the so-called Jackiw-Nohl-Rebbi (JNR) solution [39]. For $k = 2$, it is parameterized by three positions $a_i$ ($i = 0, 1, 2$) in $\mathbb{R}^4$, and associated scales $\lambda_i$. The solution is given as

$$
A_m^a = -\tilde{\eta}^a_{mn} \partial_n \log H(x), \quad H \equiv \sum_{i=0}^{2} \frac{\lambda_i^2}{|x - a_i|^2},
$$

(4.13)

where the anti-self-dual ’t Hooft tensor $\tilde{\eta}^a_{mn}$ is defined as $\tilde{\sigma}_{mn} \equiv i\tilde{\eta}^a_{mn} \sigma^a$ (or $\tilde{\eta}^a_{bc} = \epsilon_{abc}$ and $\tilde{\eta}^a_{b4} = -\delta^a_{b4}$). One of the three scales $\lambda_i$ is unphysical, since overall scaling of $H(x)$ does not affect the gauge field $A_m^a$. Furthermore, as shown in [39, 40], one of the twelve real parameters in $a_i$ is unphysical. To be more precise, there is a unique circle in $\mathbb{R}^4$ passing through the three points $a_i$. It is shown that moving the three points along this circle with relative ‘speed’ $\lambda_i^2$ can be undone by a local gauge transformation. Thus one is left with $15 - 1 - 1 = 13$ independent parameters. Together with the 3 degrees in global gauge orientation, they provide the complete parameterization of the moduli space of $SU(2)$ 2-instantons.

For convenience, we assume the scalars in vector multiplet live on the coset, which is a symmetric space, explained in section 2. The neutral and charged [22] scalars are given as ($C_\alpha = -1$ with $\alpha = 1$ only, for this symmetric space example)

$$
-f^{-1} X_\alpha = h_\alpha + C_\alpha \partial^2 \left( \log \left( \frac{s_0}{|x_0|^2 |x_1|^2 |x_2|^2} \right) - \log H \right) = h_\alpha + \frac{C_\alpha}{6} \left( \frac{\partial_m H \partial_m H}{H^2} - \sum_{i} \frac{4}{|x_i|^2} \right)
$$

(4.14)

\footnote{From the general ADHM solution, the above JNR solution can be obtained by appropriate singular gauge transformation. See, for instance, [22] for details.}
and
\[ \phi_a \sigma_a^2 = \frac{1}{2 \Sigma H(x)} \left( \frac{ZvZ}{s^2} + \frac{C\sigma_a^2}{2} \right) \left( \frac{(x_0)^m (x_1)^n}{|x_0|^2 |x_1|^2} + \frac{(x_1)^m (x_2)^n}{|x_1|^2 |x_2|^2} + \frac{(x_2)^m (x_0)^n}{|x_2|^2 |x_0|^2} \right) \] (4.15)

where \( Z = \sigma_m \frac{s(x)}{|x|^m} \equiv \sigma_m Z_m, \) \( v = v_a \sigma_a^2, \) \( s_i \equiv (\lambda_i)^2, \) \( s_S = s_0 + s_1 + s_2, \) \( x_i = x - a_i \) and
\[ C \equiv \frac{4v_a \eta_{mn}}{(s_0s_1)^{-1} |a_0 - a_1|^2 + (s_1s_2)^{-1} |a_1 - a_2|^2 + (s_2s_0)^{-1} |a_2 - a_0|^2}. \] (4.16)

From this expression one can obtain the function \( f. \) Assuming the above symmetric space with \( V(x) = \frac{1}{2} X^1((X^2)^2 - X^0X^0), \) one finds
\[ f^{-3} = \frac{27}{2} (f^{-3} X^1) \left( h_2^2 - \phi^a \phi^a \right) \quad (> 0 \ everywhere) \quad \left( \frac{27}{2} h_1 \left( h_2^2 - v^a v^a \right) = 1 \right). \] (4.17)

Otherwise, we just understand that \( f \) is given by the algebraic equation \( \frac{1}{2} C_{IJK} X^I X^J X^K = 1. \)

Now we turn to the 1-form \( \omega_m. \) Firstly, one can write
\[ -i \mathrm{tr} (J_0 (\mathcal{P} \partial_m \mathcal{P} - \partial_m \mathcal{P} \mathcal{P})) = -2 \mathrm{tr} (\phi A_m) + i \mathrm{tr} (U J_0 \partial_m U - \partial_m \bar{U} J_0 U). \] (4.18)

After some computation, the second term can be written as
\[ i \mathrm{tr} (U J_0 \partial_m U - \partial_m \bar{U} J_0 U) = \frac{2v_a \eta_{np}^a}{s\Sigma H(x)} \left( \frac{s_i (x_i)^n}{|x|^2} \right) \partial_m \left( \frac{s_j (x_j)^p}{|x|^2} \right) \] (4.19)
\[ + \frac{C}{s\Sigma H(x)} \sum_{i=1}^3 \epsilon^{ijk} \left( \frac{(x_j)^n}{|x|^2} \right) \partial_m \left( \frac{(x_k)^n}{|x|^2} \right). \]

With the following gauge field
\[ A_m = -\eta_{mn} \frac{\sigma_a^2}{2} \frac{\partial_m H}{H} = + \frac{i}{2} \bar{\eta}_{mn} \frac{\partial_m H}{H}, \] (4.20)

and charged scalar solution, the first term becomes
\[ -2 \mathrm{tr} (\phi A_m) = -\frac{1}{s\Sigma} \partial_n \left( \frac{1}{H} \right) \left( 2v_a (Z_m \eta_{np}^a - Z_n \eta_{mp}^a) Z_p + v_a Z_p Z_p \eta_{mn}^a \right) \]
\[ -\frac{C}{s\Sigma} \partial_n \left( \frac{1}{H} \right) \left( \sum_i \epsilon^{ijk} \frac{(x_j)^m}{|x|^2} \frac{(x_k)^n}{|x|^2} - \frac{1}{2} \epsilon_{mpq} \sum_i \epsilon^{ijk} \frac{(x_j)^p}{|x|^2} \frac{(x_k)^q}{|x|^2} \right) \]
where we used
\[ \bar{\eta}_{mn} \eta_{pq} = \delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np} - \epsilon_{mpq} \] (4.21)
\[ \mathrm{tr} (\bar{\sigma}_p \eta_{q}^a \bar{\sigma}_m \sigma_n) = 2iv_a (\delta_{mp} \eta_{qm}^a - \delta_{mp} \eta_{qm}^a) + iv_a \delta_{pq} \eta_{mn}^a. \] (4.22)

Adding the two contributions, (4.18) becomes
\[ -\frac{4v_a}{s\Sigma} \eta_{mn}^a Z_n - \frac{2C}{s\Sigma H} \left( \frac{(a_0 - a_1)^m}{|x_0|^2 |x_1|^2} + \frac{(a_1 - a_2)^m}{|x_1|^2 |x_2|^2} + \frac{(a_2 - a_0)^m}{|x_2|^2 |x_0|^2} \right) \eta_{mn}^a \partial_n \phi^a (x) \] (4.23)
where we used
\[ Z_m = \partial_m \left( \sum_i s_i \log |x_i| \right) \rightarrow \partial_m Z_n - \partial_n Z_m = 0 . \] (4.24)

The second term appearing in (3.40) is
\[ -i \text{tr} \left( \bar{b} \beta_{\alpha} a_\alpha \bar{\sigma}_m \right) + \text{c.c.} = + \frac{\mathcal{C}}{s_\Sigma} \frac{|x_0|^2(a_1 - a_2)^m + |x_1|^2(a_2 - a_0)^m + |x_2|^2(a_0 - a_1)^m}{s_0|x_1|^2|x_2|^2 + s_1|x_2|^2|x_0|^2 + s_2|x_0|^2|x_1|^2} , \] (4.25)
so that \( \omega_m \) itself simply becomes
\[ \omega_m = -\frac{3}{2} \bar{\eta}^a \partial_n \phi^a(x) - \frac{v_a}{s_\Sigma} \bar{\eta}^a mn Z_n , \] (4.26)
where the scalar is given as (4.15). One can explicitly check that (4.26) is regular everywhere including \( x = a_i \), even if each term is not. This is just re-confirming the regularity of our general solution.

To be concrete, let us consider the case where the three points \( a_i \) form an equilateral triangle on, say \( x^1 \times x^2 \) plane, with scales \( \lambda_i \) being all equal:
\[ a_0 = (R, 0, 0, 0) , \ a_1 = (-\frac{R}{2}, \frac{\sqrt{3}R}{2}, 0, 0) , \ a_2 = (-\frac{R}{2}, -\frac{\sqrt{3}R}{2}, 0, 0) , \ s_0 = s_1 = s_2 = 1 . \] (4.27)
Then one obtains
\[ \mathcal{C} = \frac{2v_3}{\sqrt{3}} \] (4.28)
The function \( \det F(x) \) in this case has \( U(1)^2 \) symmetry, rotations on two 2-planes:
\[ \det F^{-1}(x) = |x_0|^2|x_1|^2|x_2|^2H = 3 \left( r^2 + \rho^2 + R^2 \right) , \] (4.29)
where \( r^2 \equiv (x^1)^2 + (x^2)^2, \rho^2 \equiv (x^3)^2 + (x^4)^2 \). If we take the scalar expectation to be \( v_1 = v_2 = 0 \), which we do, this symmetry of the gauge field becomes the symmetry of the full solution. To see this, we first find that the gauge-invariant combination \( \phi^a \phi^a \) has this symmetry:
\[ \phi^a \phi^a = v^2 - \frac{4v^2R^2}{9} \frac{3(r^2 + \rho^2) + 5}{(r^2 + \rho^2 + R^2)^2 - R^2r^2} + \frac{4v^2R^4}{3} \frac{\rho^4 + \rho^2(r^2 + 2R^2) + R^2r^2 + R^4}{((r^2 + \rho^2 + R^2)^2 - R^2r^2)^2} . \] (4.30)
One can also obtain the 1-form \( \omega_m \): defining \( z \equiv x^1 + ix^2 \) and \( z' \equiv x^3 + ix^4 \), one obtains after some algebra the following,
\[ \omega_1 - i\omega_2 = \frac{2ivR^2 \bar{z}(2(r^2 + \rho^2 + R^2)^2 + R^2r^2)}{((r^2 + \rho^2 + R^2)^2 - R^2r^2)^2} \] (4.31)
\[ \omega_3 - i\omega_4 = -\frac{2ivR^2 \bar{z'}((r^2 + \rho^2 + R^2)^2 + 2R^2r^2)}{((r^2 + \rho^2 + R^2)^2 - R^2r^2)^2} , \] (4.32)
which also has symmetry under \( U(1)^2 \) rotations. The full geometry is smooth everywhere.
Now we investigate if there is any closed timelike curves (CTC) in the above geometry. We would check that there are no timelike directions on the constant $t$ hyperspace. Pick up any unit vector $N^m(x)$ in $\mathbb{R}^4$, that is $N^t = 0$ and $N^m N^m = 1$. The norm of this vector is
\begin{equation}
\left| g_{\mu\nu} N^\mu N^\nu \right| = f^{-1} \left( 1 - f^3 (\omega_m N^m)^2 \right) \geq f^{-1} \left( 1 - f^3 |\omega_m|^2 \right). \tag{4.33} \end{equation}
Showing that the last expression never becomes negative will be sufficient for proving the absence of CTC. To be precise, there exists an ambiguity $\omega \rightarrow \omega + d\lambda$ associated with shifting $t$ by $\lambda(x)$. However, we work with (4.31) which will turn out to be enough to show $f^3 |\omega_m|^2 < 1$ everywhere. In fact we find
\begin{equation}
f^3 |\omega_m|^2 = \frac{(2 v R^2)^2 [(4 r^2 + \rho^2)(r^2 + \rho^2 + R^2)^4 + 4 R^2 r^2 r^2 (r^2 + \rho^2)^2 + R^4 r^4 (r^2 + 4 \rho^2)]}{[(r^2 + \rho^2 + R^2)^2 - R^2 r^2]}^{-1} \times \left( h_1 + \frac{1}{6} \partial^2 \log \left[ (r^2 + \rho^2 + R^2)^2 - R^2 r^2 \right] \right)^{-1} \times \left( \frac{1}{h_1} + 6 v^2 R^2 \left( \frac{3 (r^2 + \rho^2) + 5 R^2}{(r^2 + \rho^2 + R^2)^2 - R^2 r^2} - \frac{3 R^2 r^2 (r^2 + \rho^2 + R^2)}{[(r^2 + \rho^2 + R^2)^2 - R^2 r^2]^2} \right)\right)^{-1} < \frac{1}{4}. \tag{4.34} \end{equation}
In particular, the upper bound 1 is never attained. This confirms that there are no CTC’s in this $U(1)^2$-symmetric solutions. The upper bound $\frac{1}{4}$ is asymptotically attained when $v^2 \rightarrow \infty$ and $h_1 \rightarrow 0^+$, at $r = R$ and $\rho = 0$.

This absence of CTC in the above example may not be very surprising since CTC appears as one tries to obtain an over-rotating solution. For instance, one obtains the over-rotating BMPV black hole as one takes to coefficient of the homogeneous solution for (3.23) to be too large. Since we only keep in $\omega_m$ the terms which are not associated with singular sources, there seems to be no degree in our solution to cause such an over-rotation.

Even if we believe that the absence of CTC can be true for our general regular solutions, this seems to be hard for us to show without symmetry, like the $U(1)^2$ isometry in the above example. However, we shall provide an indirect evidence for this conjecture for more general configurations. We show in the general 2-instanton sector that the angular momentum has an upper bound given by other charges. Especially, given the instanton number $k = 2$ and electric charge $q$, one finds that certain components of angular momenta are maximized for the above $U(1)^2$ symmetric configurations.

For general 2-instantons, one obtains the following self-dual angular momentum
\begin{equation}
j_{mn} = 8 \pi^2 (1 + \kappa_4) \operatorname{tr} (\varphi [a_m, a_n]) = \frac{8 \pi^2 i}{s_2^2} (1 + \kappa_4) \left( \sum a_i \wedge a_{i+1} \right) \frac{v^b p^b \sum_i (a_i \wedge a_{i+1})_{pq}}{s_2^2 \sum_i (s_i s_{i+1})^{-1} |a_i - a_{i+1}|^2}. \tag{4.35} \end{equation}
Note that, for 2-instantons, one can locate the three positions $a_i$ on the 12 plane without losing
generality. Defining $j_{mn} \equiv \eta_{mn}^a j^a$, one finds that only $j^3$ is nonzero and

\[
j^3 = \frac{1}{4} \eta_{mn}^a j_{mn} = \frac{16\pi^2 v^3 \text{vol}(\Delta(a_0 a_1 a_2))}{s_\Sigma^2} \left( v^a \eta_{mn} \sum_i (a_i \wedge a_{i+1})_{mn} \right) = \frac{4\pi^2 v^3}{s_\Sigma^2} \left( 4\text{vol}(\Delta(a_0 a_1 a_2)) \right)^2 \left( \sum_i (s_i s_{i+1})^{-1} |a_i - a_{i+1}|^2 \right)
\]

where $\text{vol}(\Delta(a_0 a_1 a_2))$ is the area of the triangle made by three vectors $a_0$, $a_1$ and $a_2$. The electric charge is given as [22]

\[
v^aq^a = \frac{4\pi^2}{s_\Sigma^2} \left( v^2 \sum_i s_i s_{i+1} |a_i - a_{i+1}|^2 - \frac{\left( v^a \eta_{mn} \sum_i (a_i \wedge a_{i+1})_{mn} \right)^2}{\sum_i (s_i s_{i+1})^{-1} |a_i - a_{i+1}|^2} \right)
\]

\[
\geq \frac{4\pi^2}{s_\Sigma^2} v^2 \left( \sum_i |a_i - a_{i+1}|^2 \right)^2 - \frac{\left( v^a \eta_{mn} \sum_i (a_i \wedge a_{i+1})_{mn} \right)^2}{\sum_i (s_i s_{i+1})^{-1} |a_i - a_{i+1}|^2}
\]

\[
= 2v^2 \left( |a_{01}|^2 - |a_{12}|^2 \right)^2 + \left( |a_{12}|^2 - |a_{20}|^2 \right)^2 + \left( |a_{20}|^2 - |a_{01}|^2 \right)^2 \geq 0.
\]

The two inequalities are saturated in the following cases, respectively: (1) the first one if $s_0 = s_1 = s_2$ and $v^1 = v^2 = 0$, and (2) the second one if $|a_{01}|^2 = |a_{12}|^2 = |a_{20}|^2$. Therefore we find $|j| \leq \frac{1}{2} v^aq^a$, which is saturated by $U(1)^2$ invariant rings.

The anti-self-dual part of the angular momentum is given as

\[
\tilde{j}_{mn} = 4\pi^2 i \text{tr}(\tilde{\omega}^a \nu a_\beta)(\tilde{\sigma}_{mn})^\dagger \tilde{\alpha} = \frac{4\pi^2}{s_\Sigma^2} \left( s_{00} s_1 \tilde{a}_{01} v^a a_{01} + s_{12} \tilde{a}_{12} v^a a_{12} + s_{20} \tilde{a}_{20} v^a a_{20} \right)^\dagger \tilde{\alpha},
\]

(4.39)

Here $a_{ij} \equiv (a_i - a_j)_m \sigma^m$. We define $\tilde{j}_{mn} \equiv \tilde{\eta}^a_{mn} \tilde{j}^a$, and again align the vectors $a_{ij}$ on the 12 plane, $a_{ij} = a^1_{ij} \sigma^1 + a^2_{ij} \sigma^2$. Decomposing $v = v_\| + v_\perp = v^3 \sigma^3 + (v^1 \sigma^1 + v^2 \sigma^2)$, one finds

\[
\tilde{j}_3 = \frac{4\pi^2 v_3}{s_\Sigma^2} \left( \sum_i s_i s_{i+1} |a_i - a_{i+1}|^2 \right)
\]

(4.40)

\[
\tilde{j}_1 \frac{\sigma^1}{2} + \tilde{j}_2 \frac{\sigma^2}{2} = -\frac{4\pi^2}{s_\Sigma^2} \left( s_{00} s_1 \tilde{a}_{01} v^a a_{01} + s_{12} \tilde{a}_{12} v^a a_{12} + s_{20} \tilde{a}_{20} v^a a_{20} \right)
\]

\[
= -\frac{4\pi^2}{s_\Sigma^2} \left( s_{00} |a_{01}|^2 v^0_\perp + s_{12} |a_{12}|^2 v^1_\perp + s_{20} |a_{20}|^2 v^2_\perp \right)
\]

(4.41)

where $v_{ij} \equiv \frac{\tilde{a}_{ij}}{|a_{ij}|} v^a_{ij} |a_{ij}|$. Regarding $v_\perp$ as a 2-dimensional vector spanned by $\sigma^1$ and $\sigma^2$, it is a rotation of $v_\perp$. From the structure of (4.41), one finds that $(\tilde{j}_1)^2 + (\tilde{j}_2)^2$ is maximized when all $v_{ij}$
are parallel, which is possible when all $a_i$ lie on the same line in $\mathbb{R}^4$. One finds

$$\sqrt{(\tilde{j}_1)^2 + (\tilde{j}_2)^2} \leq \frac{4\pi^2 |v_1|}{s_\Sigma^2} \left( \sum_i s_i s_{i+1} |a_i - a_{i+1}|^2 \right). \quad (4.42)$$

Therefore one obtains

$$|j| = \sqrt{\sum_a j_a^2} \leq \frac{4\pi^2 |v|}{s_\Sigma^2} \left( \sum_i s_i s_{i+1} |a_i - a_{i+1}|^2 \right). \quad (4.43)$$

This inequality is saturated if (i) $v_3 = 0$ and $a_{ij}$ all parallel, or (ii) $v_\perp = 0$. With this result, one finds that the anti-self-dual angular momentum $\tilde{j}$ is also has an upper bound given by the electric charge:

$$\frac{s_\Sigma^2}{4\pi^2} \left( v^a q^a - \frac{2}{3} |v||\tilde{j}| \right) \geq \frac{|v|^2}{3} \sum_i s_i s_{i+1} |a_i - a_{i+1}|^2 - \frac{\left( v^a \eta^a \sum_i (a_i \wedge a_{i+1})_{mn} \right)^2}{\sum_i (s_i s_{i+1})^{-1} |a_i - a_{i+1}|^2} \geq 0, \quad (4.44)$$

where we applied the same inequalities used in (4.38). We therefore find $|\tilde{j}| \leq \frac{3}{2} \frac{v^a q^a}{|v|}$. For all inequalities used in the intermediate steps to be saturated, the configuration should again satisfy $s_0 = s_1 = s_2$, $v_1 = v_2 = 0$ and $|a_{01}| = |a_{12}| = |a_{20}|$. Especially, both $|j|$ and $|\tilde{j}|$ are bound by $\frac{a^a q^a}{|v|}$.\footnote{This question was raised in [24], where a similar conclusion in a slightly different setting was obtained.}

We suspect there could exist similar upper bound for general $SU(2)$ instantons with topological charge $k \geq 3$: perhaps similar to what we found here, like $\frac{a^a q^a}{|v|} \geq c_k |j|$ and $\frac{a^a q^a}{|v|} \geq \tilde{c}_k |\tilde{j}|$. We do not attempted to explore it here, partly because we have not solved (3.36) for $\varphi$ with general $k$, and also because we cannot solve the ADHM constraint completely. For $k = 1$, it is known [37] that $j_{mn} = 0$ while $|\tilde{j}|$ is proportional to $\frac{a^a q^a}{|v|}$. For $k = 2$, one finds $j \neq 0$ in general, but the upper bound for anti-self-dual part $|\tilde{j}|$ is still larger than that for the self-dual part. The large $k$ expectation is that the two bounds would be the same, namely $\frac{c_k}{\tilde{c}_k} \to 1$ for $k \to \infty$ [19, 28, 24]. To see how such bounds behave for $k \geq 3$, if they exist at all, one could restrict one’s interest to the multi JNR instanton of [39], where the ADHM data is also known [22]. The matrix $\varphi$ is also obtained recently for some values of $k \geq 3$ [11].

5 Concluding remarks

In this paper we studied supersymmetric solutions of 5 dimensional $\mathcal{N} = 1$ Yang-Mills-Einstein supergravity. We systematically obtained explicit solutions to the differential equations imposed on supersymmetric configurations based on ADHM construction, modulo a set of algebraic conditions on the parameters of the solutions. The solution carries topological charge, electric charge and angular momentum. This gravitating dyonic instanton solution is regular on the generic point of the instanton moduli space.
We also checked the absence of CTC in the $U(1)^2$-invariant solution carrying instanton charge $2$, and conjectured the absence for our general solution. It is indirectly supported in the general 2-instanton sector by showing the existence of an upper bound for angular momenta in $\mathbb{R}^4$. It will be interesting to further explore it.

In the truncated $\mathcal{N} = 2$ model, the dyonic instantons in 5 dimensional super-Yang-Mills theory have been argued to be supertubes, configurations carrying suitable dipole charges and expanding into ‘tubular’ or ‘ring-like’ shapes in space. We find further evidence for this interpretation in the theory with $SU(2)$ gauge group, by showing that both self-dual and anti-self-dual components of the angular momentum are maximized for circular configurations with $U(1)^2$ symmetry in the 2-instanton sector.

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In the theory with non-Abelian Chern-Simons term, even the gauge theory soliton needs further study. There we find that our configuration has non-zero electric charge even if the adjoint scalars take zero VEV, leaving $SU(N)$ gauge symmetry unbroken. The equation (3.36) for $\varphi$ appearing in the scalar solution has a natural interpretation as a non-dynamical auxiliary degree in the matrix quantum mechanics describing the dynamics of $k$-instanton moduli $\omega_\alpha$ and $a_\alpha$: the latter model arises either as the moduli space approximation or as describing open strings degrees attached to $D0$-$D4$ branes. When $c = 0$, from the latter viewpoint, since there is a $U(k)$ gauge symmetry on $k$ stacks of $D0$ branes, one introduces a gauge field $A_0$ and its superpartner scalar, which we call $\varphi$, living on the worldline. The equation of motion for $\varphi$ is exactly (3.36) with $c = 0$. We managed to find a deformation of this matrix model with the parameter $c \neq 0$, preserving 8 supersymmetries, which yields (3.36) as the equation of motion for $\varphi$, and further reproduces (4.5) in the single instanton sector.

It should be interesting to understand this finding more physically.

In a broader perspective, one could extend the study of non-Abelian supersymmetric solutions to other gauged supergravity theories. For example, if one gauges both $U(1) \subset SU(2)_R$ as well as an isometry on scalar manifolds, the resulting theory has nonzero scalar potential. Gauged supergravity with $\mathcal{N} = 2$ (16 real) or $\mathcal{N} = 4$ (32 real) supersymmetry is another direction. In a theory where a subgroup of $SU(2)_R$ is gauged, the global $SU(2)_R$ symmetry is broken by picking up a $U(1)$ subgroup. Related to this, the hyper-Kahler structure on the base space that we got should be relaxed [2], which could render the system richer and/or more complicated.

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This is in progress in collaboration with Ki-Myeong Lee.
A Derivation of the ADHM solutions

In this appendix, we solve the differential conditions (3.22) and (3.23) using ADHM technique. For convenience, we set the gauging parameter $g = 1$ here, which can be recovered easily.

A.1 Adjoint scalar solution

In this subsection, we derive the solution of the covariant Laplace equation with a source term coming from non-Abelian Chern-Simons coupling:

$$\mathcal{D}^2(f^{-1}X_a) = \frac{c}{12}d_{abc}\Theta^b_{mn}\Theta^c_{mn}. \tag{A.1}$$

Alternatively, in $N \times N$ matrix notation, one may first solve an auxiliary equation

$$\mathcal{D}^2\Phi = \frac{c}{24}\Theta_{mn}\Theta_{mn}(\{T^a, T^b\} = \frac{1}{N}\delta^{ab}1_N + 4d^{abc}T^c). \tag{A.2}$$

Since there is an overall $U(1)$ part, whose solution is given by the Osborn’s formula

$$\text{tr}\Phi = \frac{c}{24}\partial^2\log(\det F(x)), \tag{A.3}$$

$f^{-1}X_a$ is obtained from $\Phi$ as

$$f^{-1}X_aT^a = \Phi - \frac{1}{N}(\text{tr}\Phi)1_N = \Phi - \frac{c}{24N}\partial^2\log(\det F(x))1_N. \tag{A.4}$$

Using $(\sigma_{mn})^\beta_{\gamma}(\sigma_{mn})^\delta_{\gamma} = -4(\delta^\beta_{\gamma}\delta^\delta_{\gamma} + \epsilon_{\alpha\gamma}\epsilon^{\beta\delta})$, one obtains from $\Theta_{mn} = 2i\bar{U}b(\sigma_{mn}F)bU$ the following:

$$\Theta_{mn}\Theta_{mn} = +16(\bar{U}b^aF\bar{b}_bPb^b\bar{b}_aU + \bar{U}b^aF\bar{b}_bPb^b\bar{b}_aU). \tag{A.5}$$

As a first trial, we compute $\mathcal{D}^2(\bar{U}J_1U)$ with $J_1 \equiv b^a\bar{b}_a$. The general expression in [25] is

$$\mathcal{D}^2(\bar{U}J_1U) = -4\bar{U}\{b^aF\bar{b}_a, J_1\}U + 4\bar{U}b^aF\bar{\Delta}^\beta_{\gamma}J_1\Delta_\alpha F\bar{b}_aU$$

$$+ \bar{U}\partial^2J_1U - 2\bar{U}b^aF\sigma_{\alpha\beta\gamma}\bar{\Delta}^\alpha\partial_\alpha J_1U - 2\bar{U}\partial_\alpha J_1\Delta_\alpha\sigma_{\alpha\beta}\partial_\beta F\bar{b}_aU. \tag{A.6}$$

Inserting $J_1 = b^a\bar{b}_a$, one obtains

$$\mathcal{D}^2(\bar{U}J_1U) = -8\bar{U}(b^aF\bar{b}_a)^2U + 4\bar{U}b^aF\bar{\Delta}^\beta_{\gamma}b^\beta\bar{b}_a\Delta_\alpha F\bar{b}_aU$$

$$- 4\bar{U}b^aF\bar{b}_bPb^b\bar{b}_aU + 8\bar{U}b^aF\bar{\Delta}^\beta_{\gamma}b^\beta\bar{b}_a\Delta_\alpha F\bar{b}_aU \tag{A.7}$$

where $\mathcal{P} = \bar{U}U$. Here we used

$$\partial^2J_1 = -4b^aF\bar{b}_bPb^b\bar{b}_a \tag{A.8}$$

$$\partial_\alpha J_1 = \begin{cases} -b^aF\sigma_{\alpha\beta\gamma}\bar{b}_b\Delta_\alpha F\bar{b}_a \\ -b^aF\bar{\Delta}^\beta_{\gamma}b^\beta\sigma_{\alpha\beta}\partial_\alpha F\bar{b}_a \end{cases} \tag{A.9}$$
and $\sigma_{\alpha\beta\dot{\alpha}\dot{\beta}} = 2\delta_{\alpha}^\beta\delta_{\dot{\alpha}}^\dot{\beta}$. We try to massage the second and fourth terms:

$$4\bar{U} b^\alpha F\bar{\Delta}^\dot{\alpha} b^\beta \bar{F}_\beta \Delta_\dot{\alpha} F_{\bar{b}_\alpha} U = 4\bar{U} b^\alpha F\bar{F}_{\bar{b}_\beta}(1 - \mathcal{P}) b^\beta \bar{F}_{\bar{b}_\alpha} U = 8\bar{U} (b^\alpha F\bar{b}_\alpha)^2 U - 4\bar{U} b^\alpha F\bar{b}_\beta \mathcal{P} b^\beta \bar{F}_{\bar{b}_\alpha} U$$

$$8\bar{U} b^\alpha F\bar{\Delta}^\dot{\alpha} b^\beta \bar{F}_{\beta} \Delta_\dot{\alpha} F_{\bar{b}_\beta} U = 8\bar{U} b^\alpha F\bar{b}_\beta(1 - \mathcal{P}) b^\beta \bar{F}_\alpha U = 8\bar{U} (b^\alpha F\bar{b}_\alpha)^2 U - 8\bar{U} b^\alpha F\bar{b}_\beta \mathcal{P} b_\alpha F_{\bar{b}_\beta} U$$

where we used $\bar{\Delta}_\dot{\alpha} b_\alpha = \bar{b}_\alpha \Delta_\dot{\alpha}$. Inserting these back to (A.7), one obtains

$$D^2(\bar{U} J_\alpha U) = 8\bar{U} (b^\alpha F\bar{b}_\alpha)^2 U - 8\bar{U} b^\alpha F\bar{b}_\beta \mathcal{P} b^\beta F_{\bar{b}_\alpha} U - 8\bar{U} b^\alpha F\bar{b}_\beta F_{\bar{b}_\alpha} F_{\bar{b}_\beta} U$$

$$= 8\bar{U} (b^\alpha F\bar{b}_\alpha)^2 U - \frac{1}{2} \Theta_{mn} \Theta_{mn} . \quad (A.10)$$

Therefore, $\Phi$ has to satisfy

$$D^2 \left( \Phi + \frac{c}{12} \bar{U} b^\alpha F\bar{b}_\alpha U \right) = \frac{2c}{3} \bar{U} b^\alpha F^2 \bar{b}_\alpha U . \quad (A.11)$$

The last equation can be solved by generalizing the ansatz taken in [25] to solve the covariant Laplace equation. We try

$$\Phi + \frac{c}{12} \bar{U} b^\alpha F\bar{b}_\alpha U = \bar{U} \begin{pmatrix} v & 0 \\ 0 & \varphi \otimes 1_2 \end{pmatrix} U \equiv \bar{U} J_\alpha U , \quad (A.12)$$

where $v$ is the asymptotic value of $X_\alpha T^\alpha$, and $\varphi$ is a constant matrix to be determined. Plugging this ansatz in (A.11) and following the computation (C.31) of [25], one obtains

$$4\bar{U} b^\alpha F \left( -L \varphi + \bar{\omega}^\alpha v \omega_\alpha \right) F_{\bar{b}_\alpha} U = \frac{2c}{3} \bar{U} b^\alpha F^2 \bar{b}_\alpha U \quad (A.13)$$

where $L \varphi = \frac{1}{2} \left( \bar{\omega}^\alpha \omega_\alpha, \varphi \right) + [a_n, [a_n, \varphi]]$. This equation is solved if one demands

$$L \varphi = \bar{\omega}^\alpha v \omega_\alpha - \frac{c}{6} 1_k , \quad (A.14)$$

which is solvable since $L$ is generically invertible. The final answer is

$$f^{-1} X_\alpha T^\alpha = \bar{U} \begin{pmatrix} v & 0 \\ 0 & \varphi - \frac{c}{12} F(x) \end{pmatrix} U - \frac{c}{24N} \partial^2 \log(\det F(x)) 1_N \quad (A.15)$$

with (A.14).

### A.2 The 1-form $\omega_m$

Here we derive the solution of (3.23), where the scalar on the right hand side is given by (A.15).

Again as a first trial, we would like to compute the action of $\frac{1 + \alpha}{2} d$ on the 1-form $\text{tr} (J_{\alpha}\mathcal{P} \partial_m \mathcal{P})$. Using the following identities [25],

$$\partial_m F = \begin{cases} -F \sigma_{\alpha m} \bar{b}_\alpha \Delta_\dot{\alpha} F \\ -F \bar{\Delta}^\dot{\alpha} b^\alpha \sigma_{\alpha m} F \\ -F \bar{\Delta}^\dot{\alpha} b^\alpha \sigma_{\alpha m} F \end{cases} \quad (A.16)$$

$$\partial_m \mathcal{P} = -\Delta_\dot{\alpha} F \sigma_{\alpha m} \bar{b}_\alpha \mathcal{P} - \mathcal{P} b^\alpha \sigma_{\alpha n} \Delta_\dot{\alpha} F \quad , \quad (A.17)$$
one obtains
\[ \frac{1 + *_4}{2} \text{tr} (\mathcal{J}_0 \partial_{[m} \mathcal{P} \partial_{n]} \mathcal{P}) = \frac{1 + *_4}{2} \text{tr} \left( \mathcal{J} \left( \mathcal{P} b^\alpha (\sigma_{mn})_\alpha^\beta F \bar{b}_\beta \mathcal{P} + \Delta_\alpha F \bar{\sigma}_{[m} \bar{a} \mathcal{P} b^\beta \sigma_{n] \beta} F \bar{\Delta}^\beta \right) \right). \]  

(A.18)

To treat the second term, one needs
\[ (\sigma_m)_{\alpha \beta} (\bar{\sigma}_n) \bar{\gamma}^\delta = \frac{1}{2} \bar{\delta}_m \delta_\alpha^\beta \delta_\beta^\delta - \frac{1}{2} \delta_\alpha^\beta \bar{\delta}_m (\bar{\sigma}_{mn}) \bar{\gamma}^\beta + \frac{1}{2} (\sigma_{mn})_\alpha^\beta \delta_\beta^\bar{\gamma} - \frac{1}{2} (\bar{\sigma}_{mn})_{\bar{\alpha} \bar{\beta}} \bar{\gamma} (\sigma^m)_{\bar{\alpha} \bar{\beta}}. \]  

(A.19)

The last term on the right hand side of (A.19) is zero after anti-symmetrizing \( mn \) indices. We thus find a useful identity
\[ \frac{1 + *_4}{2} (\sigma_m)_{\alpha \beta} (\bar{\sigma}_n) \bar{\gamma}^\delta = \frac{1}{2} (\sigma_{mn})_\alpha^\beta \delta_\beta^\bar{\gamma}. \]  

(A.20)

We also need the following property [25],
\[ \bar{\Delta}^\bar{\alpha} \mathcal{J}_0 \Delta_{\bar{\alpha}} = \omega^\alpha \nu \omega_{\bar{\alpha}} - \mathbf{L} \varphi + \{ \varphi, F^{-1} \} = \frac{c}{6} \mathbf{1}_k + \{ \varphi, F^{-1} \}. \]  

(A.21)

Using these, the quantity inside the \( \frac{1 + *_4}{2} \) projector of (A.18) can be written as
\[ \text{tr} \left( \mathcal{J}_0 \mathcal{P} b^\alpha (\sigma_{mn})_\alpha^\beta F \bar{b}_\beta \mathcal{P} \right) + \frac{1}{2} \text{tr} \left( (\bar{\Delta}^\bar{\alpha} \mathcal{J}_0 \Delta_{\bar{\alpha}}) F \bar{b}_\alpha \mathcal{P} (\sigma_{mn})_\alpha^\beta b^\beta F \right) \]  

(A.22)

\[ = \text{tr} \left( (\bar{\bar{U}} \mathcal{J}_0 \mathcal{U}) \left( \bar{U} b^\alpha (\sigma_{mn})_\alpha^\beta F \bar{b}_\beta \mathcal{U} \right) \right) + \frac{1}{2} \text{tr} \left( \{ \varphi, F \} \bar{b}_\alpha \mathcal{P} (\sigma_{mn})_\alpha^\beta b^\beta \right) - \frac{c}{12} \text{tr} \left( b^\alpha (\sigma_{mn})_\alpha^\beta F \bar{b}_\beta (1 - \mathcal{P}) \bar{\mathcal{J}_1} \mathcal{P} \right) \]  

where \( f^{-1} X \equiv f^{-1} X_a T^a \) is given as (A.15). The first term is what we need on the right hand side of (3.23). We shall explain how to deal with the other two terms below.

First we show that the second term on the last line of (A.22) can be arranged to take the form \( \frac{1 + *_4}{2} d(\cdots) \). First, \( \mathcal{P} \) appearing in this term can be replaced by \( -(1 - \mathcal{P}) = -\Delta_\alpha F \Delta^\alpha \), since there is \( \bar{b}_\alpha \mathcal{J} b^\beta = \varphi \delta_\alpha^\beta \) in the subtracted term, from which one finds \( (\sigma_{mn})_\alpha^\alpha = 0 \). Thus we consider
\[ -\frac{1}{2} \text{tr} \left( \{ \Delta_\alpha F \Delta^\alpha, \mathcal{J} \} b^\beta (\sigma_{mn})_\alpha^\beta F \bar{b}_\alpha \right). \]  

(A.23)

We use (A.20) to rewrite this term as
\[ -\frac{1 + *_4}{2} \text{tr} \left( \{ \Delta_\alpha F \Delta^\alpha, \mathcal{J} \} b^\beta (\sigma_{[n})_{\beta \bar{\beta}} F (\bar{\sigma}_{m]}_\bar{\alpha} \bar{b}_\alpha \bar{b}_\bar{\beta} \right) = -\frac{1 + *_4}{2} \text{tr} \left( \{ \Delta_\alpha F \Delta^\alpha, \mathcal{J} \} b^\beta (\sigma_{mn})_\alpha^\beta F \bar{b}_\alpha \right). \]  

(A.24)

Using the fact \( (\partial_m \bar{\Delta}^\bar{\alpha}) \Delta_{\bar{\alpha}} = \bar{\Delta}^\bar{\alpha}(\partial_m \Delta_{\bar{\alpha}}) = \partial_m F^{-1} \), this can be written as
\[ + \frac{1 + *_4}{2} \text{tr} \left( \mathcal{J} \Delta_{\bar{\alpha}} \partial_\alpha F \partial_\beta \mathcal{J} \Delta_{\bar{\alpha}} \partial_\alpha F \Delta^\beta \right). \]  

(A.25)

Each term inside the \( \frac{1 + *_4}{2} \) projector is exact. The first term is
\[ \text{tr} \left( \mathcal{J} \Delta_{\bar{\alpha}} \partial_\alpha F \partial_\beta \mathcal{J} \Delta_{\bar{\alpha}} \partial_\alpha F \Delta^\beta \right) = \text{tr} \left( \mathcal{J} (a_{\bar{\alpha}} + b^\alpha \sigma_{[m} b^\alpha x^p \partial_\alpha F \bar{\sigma}_{n]} \bar{b}_\beta \right) \]  

(A.26)

\[ = -\partial_{[n} \text{tr} \left( \bar{b}_\beta J a_{\bar{\alpha}} F \bar{\sigma}_{m]} \bar{\alpha} \bar{\beta} + 2x_{[m} F \right), \]  

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and similarly the second term is
\[
\text{tr} \left( \mathcal{J} \partial_n \Delta_{\alpha} \partial_m F \tilde{\Delta} \hat{\alpha} \right) = + \partial_m \text{tr} \left( \tilde{\alpha}^\alpha \mathcal{J} b^\beta F \sigma_{n|\beta} + 2r_n \varphi F \right) .
\] (A.27)

Collecting all, one obtains the following expression
\[
\frac{1}{2} \text{tr} \left( \{ \mathcal{P}, \mathcal{J} \} b^\beta (\sigma_{nm})_m^\alpha \tilde{F} b_\alpha \right) = \frac{1 + \ast_4}{2} \partial_m \text{tr} \left( \left( \tilde{\alpha}^\alpha \mathcal{J} b^\beta \sigma_{n|\beta} \tilde{a}^\hat{\alpha} - \tilde{b}_\beta \mathcal{J} a_\alpha \tilde{\sigma}_{n|\hat{\alpha}} \right) F \right) .
\] (A.28)

The expression inside the derivative can simply be rewritten as
\[
\text{tr} \left( \left( \tilde{b}_\beta \mathcal{J} a_\alpha \tilde{\sigma}_{n|\hat{\alpha}} - \tilde{\alpha}^\alpha \mathcal{J} b^\beta \sigma_{n|\beta} \tilde{a}^\hat{\alpha} \right) F \right) = 2 \text{tr}_k \left( [\varphi, a_m] F \right) ,
\] (A.29)

where the lower $2k \times 2k$ block of $a_\hat{\alpha}$ is written as $a_n \sigma_{n\hat{\alpha}}^a$. This completes the analysis of the second term of [A.22].

As the final step, we try to write the third term of [A.22] in the form $\frac{1 + \ast_4}{2} d(\cdots)$. We first note that this last term can be written in either of the following ways:
\[
\mathcal{O}_{mn} \equiv \text{tr} \left( b^\alpha (\sigma_{mn})_m^\beta F \tilde{b}_\beta (1 - \mathcal{P}) \mathcal{J}_1 \mathcal{P} \right) = \text{tr} \left( b^\alpha (\sigma_{mn})_m^\beta F \tilde{b}_\beta \mathcal{P} \mathcal{J}_1 (1 - \mathcal{P}) \right) .
\] (A.30)

One can write this term in another way by using the following identity,
\[
(\sigma_{mn})_m^\beta \delta^\gamma_\delta = \frac{1}{2} (\sigma_{mn})_m^\beta \delta^\gamma_\beta + \frac{1}{2} (\sigma_{mn})_m^\gamma \delta^\beta_\delta - \frac{1}{4} \left( (\sigma_{mp})_m^\alpha (\sigma_{np})_n^\beta - (\sigma_{np})_n^\delta (\sigma_{mp})_m^\gamma \right) .
\] (A.31)

Applying this identity to the latter form in [A.30], one obtains
\[
\mathcal{O}_{mn} = \text{tr} \left( b^\alpha (\sigma_{mn})_m^\beta F^2 \tilde{b}_\beta (\Delta_{\alpha} F \tilde{\Delta} \hat{\alpha}) \right) - \text{tr} \left( b^\alpha (\sigma_{mn})_m^\beta F \tilde{b}_\beta (\Delta_{\alpha} F \tilde{\Delta} \hat{\alpha}) b^\gamma F \tilde{b}_\gamma (\Delta_{\beta} F \tilde{\Delta} \hat{\beta}) \right)
= \frac{1 + \ast_4}{2} \text{tr} \left( 2 \partial_m F^{-1} F^2 \partial_n F^{-1} F \right) + \text{tr} \left( \Delta_{\alpha} F \tilde{\Delta} \hat{\alpha} b^\alpha (\sigma_{mn})_m^\beta F \tilde{b}_\beta \Delta_{\beta} F \tilde{\Delta} \hat{\beta} b^\gamma F \tilde{b}_\gamma \right)
= \frac{1 + \ast_4}{2} \text{tr} \left( \partial_m F^{-1} F \partial_n F^{-1} F \right) + \frac{1 + \ast_4}{2} \text{tr} \left( (F \partial_m F^{-1} F \partial_n F^{-1} F) \tilde{b}_\alpha \Delta_{\alpha} F \tilde{\Delta} \hat{\alpha} b^\alpha \right)
= 2 \frac{1 + \ast_4}{2} \text{tr} \left( (F \partial_m F^{-1} F \partial_n F^{-1} F) (1 - \tilde{b}_\alpha \mathcal{P} b^\alpha) \right)
= 2 \frac{1 + \ast_4}{2} \left( (\partial_m F^{-1} F \partial_n F^{-1} F + \tilde{F}^2 F + 4 F^2) \right) \equiv \frac{1 + \ast_4}{2} \varrho_{mn} .
\] (A.32)

We applied the above identity [A.31] to the second term on the first line, and also used $\partial^2 F = -4 F \tilde{b}_\alpha \mathcal{P} b^\alpha F$ [25] one the 4th line. Here we note that, inside the projector $\frac{1 + \ast_4}{2}$, proving that the 2-form $\varrho_{mn}$ is co-exact is also fine for our purpose. Namely we try to write $\varrho = d^t \omega^{(3)}$ with certain 2-form $\omega^{(3)}$, where $d^t \equiv \ast_4 d \ast_4$. The following 3-form
\[
\Lambda_{mnp} = \text{tr} \left( \partial_{[m} F^{-1} F \partial_{n]} F^{-1} \partial_{p} F \right) = -\text{tr} \left( \partial_{[m} F^{-1} F \partial_{n]} F^{-1} F \partial_{p} F^{-1} F \right)
\] (A.33)

turns out to be helpful. Since the three indices $mnp$ are symmetric under cyclic permutations, anti-symmetrizing $mn$ guarantees that the indices are totally anti-symmetric. Acting $\partial_p$ on this 3-form, and using $\partial_n \partial_n F^{-1} = 2 \delta_{mn} 1_k$, one obtains
\[
\partial_p \Lambda_{mnp} = 2 \varrho_{mn} .
\] (A.34)

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Inside the projector \( \frac{1 + 4}{2} \), one can write
\[
\frac{1 + 4}{2} \partial_p \Lambda_{mnp} = \frac{1 + 4}{2} (\star_4 d\lambda^{(1)})_{mn} = \frac{1 + 4}{2} (d\lambda^{(1)})_{mn} \tag{A.35}
\]
where \( \lambda^{(1)} \equiv \star_4 \Lambda \).

Collecting all, the 1-form \( \omega_m \) is given as
\[
\omega_m = -3i \text{tr} \left( J_0 \frac{\partial_m P - \partial_m P \partial_m}{2} + 2[\varphi, a_m]F - \frac{c}{72} \epsilon_{mnpq} \partial_n F^{-1} F \partial_p F^{-1} F \partial_q F^{-1} F \right) \tag{A.36}
\]
where the traces are either over \( N + 2k \) or \( k \) dimensional matrices, and \( \varphi \) appearing in \( J_0 \) satisfies (A.14).

## B Summary of the properties of spinor bilinear

In this appendix we summarize the algebraic and differential conditions satisfied by the differential forms constructed from the Killing spinor bilinear. These are nearly the same as the conditions in Maxwell-Einstein supergravity. We follow the notations of [4].

The algebraic conditions following from the Fierz identity are
\[
V_\mu V^\mu = f^2 \tag{B.1}
\]
\[
J^i \wedge J^j = -2 \delta_{ij} f \star V \tag{B.2}
\]
\[
i_V J^i = 0 \tag{B.3}
\]
\[
i_V \star J^i = -f J^i \tag{B.4}
\]
\[
J_{\mu \nu} J^i_{\rho \sigma} = \delta_{ij} (f^2 \eta_{\mu \nu} - V_\mu V_\nu) + \epsilon_{ijk} f J^k_{\mu \nu} \tag{B.5}
\]
The differential conditions that one obtains from the gravitino Killing spinor equation are
\[
df = -i_V \left( X_1 F^I \right) \tag{B.6}
\]
\[
\nabla_{(\mu} V_{\nu)} = 0 \tag{B.7}
\]
\[
dV = 2f X_1 F^I + X_1 \star (F^I \wedge V) \tag{B.8}
\]
\[
\nabla_{\mu} J^i_{\nu \rho} = -\frac{1}{2} X_1 \left( 2 F^I \sigma \left( \star J^i \right)_{\sigma \nu \rho} - 2 F^I [\nu \sigma \left( \star J^i \right)_{\rho] \mu \sigma} + \eta_{\mu [\nu F^I \sigma \tau \left( \star J^i \right)_{\rho] \sigma \tau} \right) \tag{B.9}
\]
These conditions are same as the results in [4], except that we are setting \( \chi = 0 \) (a parameter in their scalar potential) in their formulae. The conditions coming from the gaugino Killing spinor equation is slightly different to [4]. Contracting this equation with \( \bar{\epsilon}^j \), one obtains
\[
V^\mu D_\mu X_I = 0 \tag{B.10}
\]
\[
\left( \frac{1}{4} Q_{IJ} - \frac{3}{8} X_I X_J \right) F_{\mu \nu} J^{i \mu \nu} = 0 \tag{B.11}
\]
Contracting it with $\bar{\epsilon}^j \gamma_{\mu}$, one obtains

$$i_V F^I = -D(fX^I) \quad \text{(B.12)}$$

$$- \left( \frac{1}{4} Q_{IJ} - \frac{3}{8} X_I X_J \right) F^J_{\mu \nu} (\ast J^I)_\rho^{\mu \nu} = -\frac{3}{4} (J^I)_\rho^{\mu \nu} D_{\mu} X_I \quad \text{(B.13)}$$

where $D$ without subscript denotes exterior $K$-gauge covariant derivatives. Finally, contracting it with $\bar{\epsilon}^j \gamma_{\mu \nu}$, one obtains equations similar to those in [4]. We do not record them as they will not be used in this paper.

References

[1] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, “All supersymmetric solutions of minimal supergravity in five-dimensions,” Class. Quant. Grav. 20, 4587 (2003) [arXiv:hep-th/0209114].

[2] J.P. Gauntlett and J.B. Gutowski, “All supersymmetric solutions of minimal gauged supergravity in five-dimensions,” Phys. Rev. D68, 105009 (2003) [arXiv:hep-th/0304064].

[3] J.B. Gutowski and H.S. Reall, “General supersymmetric AdS_5 black holes,” JHEP 0404, 048 (2004) [arXiv:hep-th/0401129].

[4] J.B. Gutowski and W. Sabra, “General supersymmetric solutions of five-dimensional supergravity,” JHEP 0510, 039 (2005) [arXiv:hep-th/0505185].

[5] J.B. Gutowski, D. Martelli and H.S. Reall, “All supersymmetric solutions of minimal supergravity in six-dimensions,” Class. Quant. Grav. 20, 5049 (2003) [arXiv:hep-th/0306235].

[6] J.P. Gauntlett and S. Pakis, JHEP 0304, 039 (2003) [arXiv:hep-th/0212008]; J.P. Gauntlett, J.B. Gutowski and S. Pakis, JHEP 0312, 049 (2003) [arXiv:hep-th/0311112].

[7] J. Gillard, U. Gran and G. Papadopoulos, Class. Quant. Grav. 22, 1033 (2005) [arXiv:hep-th/0410155]; U. Gran, J.B. Gutowski and G. Papadopoulos, Class. Quant. Grav. 22, 2453 (2005) [arXiv:hep-th/0501177].

[8] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, “A supersymmetric black ring,” Phys. Rev. Lett. 93, 211302 (2004) [arXiv:hep-th/0407065].

[9] J.P. Gauntlett and J.B. Gutowski, “Concentric black rings,” Phys. Rev. D71, 025013 (2005) [arXiv:hep-th/0408010].

31
[10] I. Bena and N.P. Warner, “One ring to rule them all... and in the darkness bind them?” Adv. Theor. Math. Phys. 9, 667 (2005) [arXiv:hep-th/0408106].

[11] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, “Supersymmetric black rings and three-charge supertubes,” Phys. Rev. D71, 024033 (2005) [arXiv:hep-th/0408120].

[12] J.P. Gauntlett and J.B. Gutowski, “General concentric black rings,” Phys. Rev. D71, 045002 (2005) [arXiv:hep-th/0408122].

[13] J.B. Gutowski and H.S. Reall, “Supersymmetric AdS5 black holes,” JHEP 0402, 006 (2004) [arXiv:hep-th/0401042].

[14] H.K. Kunduri, J. Lucietti and H.S. Reall, “Supersymmetric multi-charge AdS5 black holes,” JHEP 0604, 036 (2006) [arXiv:hep-th/0601156].

[15] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, “Construction of instantons,” Phys. Lett. A65, 185 (1978).

[16] N.H. Christ, E.J. Weinberg and N.K. Stanton, “General selfdual Yang-Mills solutions,” Phys. Rev. D18, 2013 (1978).

[17] N.D. Lambert and D. Tong, “Dyonic instantons in five dimensional gauge theories,” Phys. Lett. B462, 89 (1999) [arXiv:hep-th/9907014].

[18] D. Mateos and P.K. Townsend, “Supertubes,” Phys. Rev. Lett. 87, 011602 (2001) [arXiv:hep-th/0103030].

[19] D. Mateos, S. Ng and P.K. Townsend, “Tachyons, supertubes and brane/anti-brane systems,” JHEP 0203, 016 (2002) [arXiv:hep-th/0112054].

[20] M. Kruczenski, R.C. Myers, A.W. Peet and D.J. Winters, “Aspects of supertubes,” JHEP 0205, 017 (2002) [arXiv:hep-th/0204103].

[21] D. Bak and K. Lee, “Supertubes connecting D4 branes,” Phys.Lett. B 544, 329 (2002) [arXiv:hep-th/0206185].

[22] S. Kim and K. Lee, “Dyonic instanton as supertube between D4 branes,” JHEP 0309, 035 (2003) [arXiv:hep-th/0307048].

[23] P.K. Townsend, “Field theory supertubes,” Comptes Rendus Physique 6, 271 (2005) [arXiv:hep-th/0411206].

[24] H.-Y. Chen, M. Eto and K. Hashimoto, “The shape of instantons: cross-section of supertubes and dyonic instantons,” JHEP 0701, 017 (2007) [arXiv:hep-th/0609142].

32
[25] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, “The calculus of many instants,” Phys. Rept. 371, 231 (2002) [arXiv:hep-th/0206063].

[26] J. Bellorin and T. Ortin, “Characterization of all the supersymmetric solutions of gauged N=1,d=5 supergravity,” JHEP 0708, 096 (2007) [arXiv:0705.2567].

[27] K. Intriligator, D.R. Morrison and N. Seiberg, “Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces,” Nucl. Phys. B497, 56 (1997) [arXiv:hep-th/9702198].

[28] D. Bak, Y. Hyakutake and N. Ohta, “Phase moduli space of supertubes,” Nucl. Phys. B696, 251 (2004) [arXiv:hep-th/0404104].

[29] D. Bak, Y. Hyakutake, S. Kim and N. Ohta, “A geometric look on the microstates of supertubes,” Nucl. Phys. B712, 115 (2005) [arXiv:hep-th/0407253].

[30] M. Gunaydin, G. Sierra and P.K. Townsend, “The geometry of N=2 Maxwell-Einstein supergravity and Jordan algebras,” Nucl. Phys. B242, 244 (1984).

[31] M. Gunaydin, G. Sierra and P.K. Townsend, “Gauging the d=5 Maxwell-Einstein supergravity theories: more on Jordan algebras,” Nucl. Phys. B253, 573 (1985).

[32] T. Mohaupt and M. Zagermann, “Gauged supergravity and singular Calabi-Yau manifolds,” JHEP 0112, 026 (2001) [arXiv:hep-th/0109055].

[33] M. Gunaydin and M. Zagermann, “The gauging of five-dimensional, N=2 Maxwell-Einstein supergravity theories coupled to tensor multiplets,” Nucl. Phys. B572, 131 (2000) [arXiv:hep-th/9912027].

[34] H. Osborn, “Semiclassical functional integrals for selfdual gauge fields,” Ann. Phys. 135, 373 (1981).

[35] G.W. Gibbons, D. Kastor, L.A.J. London, P.K. Townsend and J.H. Traschen, “Supersymmetric selfgravitating solitons,” Nucl. Phys. B416, 850 (1994) [arXiv:hep-th/9310118].

[36] A. Strominger, “Heterotic solitons,” Nucl. Phys. B343, 167 (1990); Erratum-ibid. B353, 565 (1991).

[37] E. Eyras, P.K. Townsend and M. Zamaklar, “The heterotic dyonic instanton,” JHEP 0105, 046 (2001) [arXiv:hep-th/0012016].

[38] K. Peeters and M. Zamaklar, “Motion on moduli spaces with potentials,” JHEP 0112, 032 (2001) [arXiv:hep-th/0107164].

[39] R. Jackiw, C. Nohl and C. Rebbi, “Conformal properties of pseudoparticle configurations,” Phys. Rev. D15, 1642 (1977).
[40] M.F. Atiyah and N.S. Manton, “Geometry and kinematics of two skyrmions,” Comm. Math. Phys. 153, 391 (1993).

[41] M.Y. Choi, K.K. Kim, C. Lee and K. Lee, *to appear.*