POISSON STRUCTURES ON CLOSED MANIFOLDS

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Abstract. We prove an $h$-principle for poisson structures on closed manifolds.

1. INTRODUCTION

In this paper we prove an $h$-principle for poisson structures on closed manifolds. Similar results on open manifolds has been proved by Fernandes and Frejlich in [6]. We state their result below.

Let $M^{2n+q}$ be a $C^\infty$-manifold equipped with a co-dimension-$q$ foliation $\mathcal{F}_0$ and a 2-form $\omega_0$ such that $(\omega_0^n)|_{T\mathcal{F}_0} \neq 0$. Denote by $\text{Fol}_q(M)$ the space of co-dimension-$q$ foliations on $M$ identified as a subspace of of $\Gamma(\text{Gr}_2^n(M))$, where $\text{Gr}_2^n(M) \xrightarrow{pr} M$ be the grassmann bundle, i.e, $pr^{-1}(x) = \text{Gr}_2^n(T_xM)$ and $\Gamma(\text{Gr}_2^n(M))$ is the space of sections of $\text{Gr}_2^n(M) \xrightarrow{pr} M$ with compact open topology. Define

$$\Delta_q(M) \subset \text{Fol}_q(M) \times \Omega^2(M)$$

$$\Delta_q(M) := \{(\mathcal{F}, \omega_0^n) : \omega_0^n|_{T\mathcal{F}} \neq 0\} \neq 0$$

Obviously $(\mathcal{F}_0, \omega_0) \in \Delta_q(M)$. In this setting Fernandes and Frejlich has proved the following

Theorem 1.1. ([6]) Let $M^{2n+q}$ be an open manifold with $(\mathcal{F}_0, \omega_0) \in \Delta_q(M)$ be given. Then there exists a homotopy $(\mathcal{F}_t, \omega_t) \in \Delta_q(M)$ such that $\omega_1$ is $d\mathcal{F}_1$-closed (actually exact).

In the language of poisson geometry the above result [13] takes the following form. Let $\pi \in \Gamma(\wedge^2 TM)$ be a bi-vectorfield on $M$, define $\#\pi : T^*M \rightarrow TM$ as $\#\pi(\eta) = \pi(\eta, -)$. If $\text{Im}(\#\pi)$ is a regular distribution then $\pi$ is called a regular bi-vectorfield.

Theorem 1.2. Let $M^{2n+q}$ be an open manifold with a regular bi-vectorfield $\pi_0$ on it such that $\text{Im}(\#\pi)$ is an integrable distribution then $\pi_0$ can be homotoped through such bi-vectorfields to a poisson bi-vectorfield $\pi_1$.

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In [1] above $d_F$ is the tangential exterior derivative, i.e., for $\eta \in \Gamma(\wedge^k T^* F)$, $d_F \eta$ is defined by the following formula:

$$d_F \eta(X_0, X_1, ..., X_k) = \Sigma_i (-1)^i X_i(\eta(X_0, ..., \hat{X}_i, ..., X_k)) + \Sigma_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k)$$

where $X_i \in \Gamma(T F)$. So if we extend a $F$-leafwise closed $k$-form $\eta$, i.e., $d_F \eta = 0$, to a form $\eta'$ by the requirement that $\ker(\eta') = \nu F$, where $\nu F$ is the normal bundle to $F$, then $d\eta' = 0$.

In order to fix the foliation in [1] one needs to impose an openness condition on the foliation, we refer the readers to [1] for precise definition of this openness condition. Under this hypothesis Bertelson proved the following

**Theorem 1.3.** ([1]) If $(M, F)$ be an open foliated manifold with $F$ satisfies some openness condition and let $\omega_0$ be a $F$-leafwise 2-form then $\omega_0$ can be homotoped through $F$-leafwise 2-forms to a $F$-leafwise symplectic form.

She also constructed counter examples in [2] that without this openness condition the above theorem fails. A contact analogue of Bertelson’s result on any manifold (open or closed) has recently been proved in [3] by Borman, Eliashberg and Murphy. We have used this theorem in our argument. So let us state the theorem.

**Theorem 1.4.** ([3]) Let $M^{2n+q+1}$ be any manifold equipped with a co-dimension-$q$ foliation $F$ on it and let $(\alpha_0, \beta_0) \in \Gamma(T^* F \oplus \wedge^2 T^* F)$ be given such that $\alpha_0 \wedge \beta_0^n$ is nowhere vanishing, then there exists a homotopy $(\alpha_t, \beta_t) \in \Gamma(T^* F \oplus \wedge^2 T^* F)$ such that $\alpha_t \wedge \beta_t^n$ nowhere vanishing and $\beta_1 = d_F \alpha_1$.

Now we state the main theorem of this paper.

**Theorem 1.5.** Let $M^{2n+q}$ be a closed manifold with $q = 2$ and $(F_0, \omega_0) \in \Delta_q(M)$ be given. Then there exists a homotopy $F_t$ of singular foliations on $M$ with singular locus $\Sigma_t$ and a homotopy of two forms $\omega_t$ such that the restriction of $\omega_t$ to $T F_t$ is non-degenerate and $\omega_1$ is closed.

In terms of poisson geometry [3] states

**Theorem 1.6.** Let $M^{2n+q}$ be a closed manifold with $q = 2$ and $\pi_0$ be a regular bi-vectorfield of rank $2n$ on it such $\text{Im}(\# \pi_0)$ is integrable distribution. Then there exists a homotopy of
bi-vectorfields $\pi_t$, $t \in I$ (not regular) such that $\text{Im}(\#\pi_t)$ integrable and $\pi_1$ is a poisson bi-vectorfield.

We organize the paper as follows. In section-2 we shall explain the preliminaries of the theory of $h$-principle and of wrinkle maps which are needed in the proof of 1.5 which we present in section-3.

2. Preliminaries

We begin with the theory of $h$-principle. Let $X \to M$ be any fiber bundle and let $X^{(r)}$ be the space of $r$-jets of jerms of sections of $X \to M$ and $j^r f : M \to X^{(r)}$ be the $r$-jet extension map of the section $f : M \to X$. A section $F : M \to X^{(r)}$ is called holonomic if there exists a section $f : M \to X$ such that $F = j^r f$. In the following we use the notation $Op(A)$ to denote a small open neighborhood of $A \subset M$ which is unspecified.

Theorem 2.1. (4) Let $A \subset M$ be a polyhedron of positive co-dimension and $F_z : Op(A) \to X^{(r)}$ be a family of sections parametrized by a cube $I^m$, $m = 0, 1, 2, \ldots$ such that $F_z$ is holonomic for $z \in Op(\partial I^m)$. Then for given small $\varepsilon, \delta > 0$ there exists a family of $\delta$-small (in the $C^0$-sense) diffeotopies $h_z : M \to M$, $z \in I^m$ such that

1. $h_z = \text{id}_M$ and $\tilde{F}_z = F_z$ for all $z \in Op(\partial I^m)$

2. $\text{dist}(\tilde{F}_z(x), (F_z)_{Op h_z^1(A)}(x)) < \varepsilon$ for all $x \in Op(h_z^1(A))$

Remark 2.2. Relative version of 2.1 is also true. More precisely let the sections $F_z$ be already holonomic on $Op(B)$ for a sub-polyhedron $B$ of $A$, then the diffeotopies $h_z$ can be made to be fixed on $Op(B)$ and $\tilde{F}_z = F_z$ on $Op(B)$.

Now we briefly recall preliminaries of wrinkled maps following [5]. Consider the following map

$$w : \mathbb{R}^{q-1} \times \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{q-1} \times \mathbb{R}$$

$$w_s(y, x, z) = (y, z^3 + 3(|y|^2 - 1)z - \Sigma_{i=1}^2 x^2_i + \Sigma_{i=1}^{2n} x^2_i)$$

where $y \in \mathbb{R}^{q-1}$, $z \in \mathbb{R}$ and $x \in \mathbb{R}^{2n}$. Observe that the singular locus of $w_s$ is

$$\Sigma(w_s) = \{x = 0, z = 1, |y|^2 = 1\}$$
Let \( D \) be the disc enclosed by \( \Sigma(w_s) \), i.e.,
\[
D = \{ x = 0, \ z^2 + |y|^2 \leq 1 \}
\]

**Definition 2.3.** ([5]) A map \( f : M \to Q \) is called a wrinkled map if there exists a disjoint union of open subsets \( U_1, \ldots, U_l \subset M \) such that \( f|_{M-U} \) is a submersion, where \( U = \bigcup U_i \) and \( f|_{U_i} \) is equivalent to \( w_s \), for some \( s \).

A fibered map over \( B \) is given by a map \( f : U \to V \), where \( U \subset M \) and \( V \subset Q \) with submersions \( a : U \to B \) and \( b : V \to B \) such that \( b \circ f = a \). Denote by \( TBQ \) and \( ker(a) \subset TM \) and \( ker(b) \subset TQ \) respectively. The fibered differential \( df|_{TBQ} \) is denoted by \( d_B f \).

If we consider the projection on first \( k \)-factors, where \( k < q-1 \), then \( w_s \) is a fibered map. So we can define fibered version of a wrinkled map. We refer the reader [5] for more details. By combining Lemma-2.1B and Lemma-2.2B of [5] we get the following

**Theorem 2.4.** ([5]) Let \( g : I^n \to I^q \) be a fibered submersion over \( I^k \) and \( \theta : I^n \to I^n \) be a fibered wrinkled map over \( I^k \) with one wrinkle. Then there exists a fibered wrinkled map \( \psi \) with very small wrinkles and which agrees with \( \theta \) near \( \partial I^n \) such that \( g \circ \psi \) is a fibered wrinkled map.

3. Main Theorem

In this section we prove [1,5]

Consider \( \hat{M} = M \times \mathbb{R} \) and let us denote the co-dimension-\( q \) foliation \( F_0 \times \mathbb{R} \) on \( \hat{M} \) by \( \hat{F} \) with a \( \hat{F} \)-leaf wise one form \( \alpha_0 \) such that \( \alpha_0(\partial_s) = 1 \) and \( ker(\alpha_0)|_{(x,s)} = T_x F_0 \). Observe that if we extend \( \omega_0 \) to \( \hat{M} \) by the requirement that \( \omega_0(\partial_s,-) = 0 \), then \( (\alpha_0 \wedge \omega_0^0)|_{\hat{F}} \neq 0 \). Let \( (\omega_0)|_{\hat{F}} = \beta_0 \). Then \( (\alpha_0, \beta_0) \) is a \( \hat{F} \)-leaf wise almost contact structure. Then according [1,4] there exists a homotopy of pairs \((\alpha_t, \beta_t)\) defining a homotopy of \( \hat{F} \)-leaf-wise almost contact structures consisting of a \( \hat{F} \)-leaf-wise one form \( \alpha_t \) and a \( \hat{F} \)-leaf-wise two form \( \beta_t \) such that \( \beta_1 = d_F \alpha_1 \), i.e., \( (\alpha_1, \beta_1) \) is a \( \hat{F} \)-leaf-wise contact structure. Now let \( L_t = ker(\alpha_t) \subset T\hat{F} \) and \( G_t^1 = L_t \oplus \nu\hat{F} \oplus \mathbb{R} \subset \hat{M} \times \mathbb{R} \), where \( \nu\hat{F} \) is the normal bundle.

Now observe that the embedding \( f_0 : M \to M \times \{0\} \to \hat{M} \times \mathbb{R} \) is \( \ominus \) to \( \hat{F} \times \mathbb{R} \) and \( Im(df_0) \cap (T\hat{F} \times \mathbb{R}) = L_0 \). First extend \( \beta_t \) to \( \hat{M} \) and call it \( \hat{\beta}_t \) in such a way that \( ker(\hat{\beta}_t) = \nu\hat{F} \). Let \( X_t = ker(\hat{\beta}_t) \) be the vector field on \( \hat{M} \) and consider the family of 2-dimensional foliation \( G_t \) generated by \( X_t \) and \( \partial_w \), where \( w \) is the \( \mathbb{R} \)-variable in \( \hat{M} \times \mathbb{R} \). Observe that \( \alpha_t \wedge dw \) is a
$G_t$-leaf-wise symplectic form.

Now we shall perturb $f_0$ by a homotopy of immersions $f_t$ such that $f_t$ will be tangent to $\tilde{\mathcal{F}} \times \mathbb{R}$ only on $\Sigma_t$ and on $M - \Sigma_t$, $f_t \cap \tilde{\mathcal{F}} \times \mathbb{R}$, i.e., $\text{Im}(df_t) \cap (T\tilde{\mathcal{F}} \times \mathbb{R})$ is of dimension $2n$ and $\text{Im}(df_t) \cap (T\tilde{\mathcal{F}} \times \mathbb{R})$ is close to $L_i$. As $\tilde{\beta}_t^n + \alpha_t \wedge dw$ is non-degenerate on $\text{Im}(df_t) \cap T\tilde{\mathcal{F}} \times \mathbb{R}$. Hence we only need to set $F_t = f_t^{-1}(\tilde{\mathcal{F}} \times \mathbb{R})$ and $\omega_t = f_t^*(\tilde{\beta}_t + \alpha_t \wedge dw)$.

First divide the interval $I$ as

$$I = \bigcup_N^N [(i - 1)/N, i/N]$$

and assume that $f_t$ is defined on $[0, (i - 1)/N]$. Observe that the limit

$$\lim_{x \to \Sigma_{(i-1)/N}} \text{Im}(df_{(i-1)/N}) \cap (T\tilde{\mathcal{F}} \times \mathbb{R})$$

exists and is of dimension $2n$ and is close to $L_{(i-1)/N}$. Let $\tilde{L}_{(i-1)/N} \subset T\tilde{\mathcal{F}} \times \mathbb{R}$ be the $2n$-dimensional distribution which equals $\text{Im}(df_{(i-1)/N}) \cap T\tilde{\mathcal{F}} \times \mathbb{R}$ on $M - \Sigma_{(i-1)/N}$ and on $\Sigma_{(i-1)/N}$ it is the limit. Set $\nu_{(i-1)/N} = \text{Im}(df_{(i-1)/N})/\tilde{L}_{(i-1)/N}$ and $G_t^i$, $t \in [(i - 1)/N, i/N]$ as

$$G_t^i = L_t \oplus \nu_{(i-1)/N}$$

Observe that $\text{Im}(df_{(i-1)/N})$ approximates $G_t^i_{(i-1)/N}$. So if $N$ is large then there exists a family of monomorphisms $F_t$, $t \in [(i - 1)/N, i/N]$ such that $F_{(i-1)/N} = df_{(i-1)/N}$ and $\text{Im}(F_t)$ approximates $G_t^i$ and hence $F_t$ tangent to $T\tilde{\mathcal{F}} \times \mathbb{R}$ only on a slightly perturbed $\Sigma_{(i-1)/N}$.

Choose a triangulation of $M$ which is fine and $\Sigma_{(i-1)/N} \subset A$, where $A$ is the $(2n + q - 1)$-skeleton of the triangulation. As the triangulation is fine all $(2n + q)$-simplices under the image of $f_{(i-1)/N}$ is contained in a neighborhood diffeomorphic to $I^{2n+q+2}$ and on it $\tilde{\mathcal{F}} \times \mathbb{R}$ is given by the projection $\pi : I^{2n+q+2} \to I^q$ (projection on the first $q$-factors).

Without loss of generalization let us assume $F_t$ is defined for $t \in I$ instead of $t \in [(i - 1)/N, i/N]$. Let

$$\tilde{F}_t = F_{\sigma(t)}$$
Theorem 3.1. Let $f : I \to I$ be a smooth map such that $\sigma = 0$ on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$ and $\sigma = 1$ on a neighborhood of $1/2$.

Use (2.1) for $\tilde{F}_i$ to get a family of immersions $\tilde{f}_i$ defined on $Op(h_1^i(A))$ and approximating $\tilde{F}_i$ on $Op(h_1^i(A))$, where $h_1^i$ is $\delta$ small with $h_1^i = id$ for $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$. The $\delta$ above will be used later so the reader needs to keep note of this fact. We approximate $\tilde{F}_i$ by $F'_i$ such that $F'_i = df_i$ on $Op(h_1^i(A))$.

It is enough to consider one simplex $\Delta$. Let $\Delta' \subset \Delta$ be a $2\delta$-smaller simplex so that $h_1^i(\Delta)$ does not intersect $\Delta'$, $\delta$ is produced by applying (2.1) to $\tilde{F}_i$ above.

Define monomorphisms $\tilde{F}_i^\delta$ depending on $\delta$ as follows. On $Op(\partial \Delta)$, set $\tilde{F}_i^\delta = d(f_i \circ h_1^i)$. Now observe there exists an isotopy of embeddings $\tilde{g}_r : \Delta - Op(\partial \Delta) \to \Delta - Op(\partial \Delta)$ such that $\tilde{g}_0 = id$ and $\tilde{g}_1(\Delta - Op(\partial \Delta)) = \Delta'$. Any element of $(\Delta - Op(\partial \Delta)) - \Delta'$ is of the form $\tilde{g}_r(x)$, $x \in \partial(\Delta - Op(\partial \Delta))$.

Let $\gamma_i^\tau : I \to M$ be the path

$$
\gamma_i^\tau(\tau) = \begin{cases} 
  h_i^{1-2\tau}(x), & \tau \in [0, 1/2] \\
  \tilde{g}_{2\tau-1}(x), & \tau \in [1/2, 1]
\end{cases}
$$

Set $(\tilde{F}_i^\delta)_\gamma(x) = (F'_i)_\gamma(x)$. Observe that $\gamma_i^\tau(1) = \tilde{g}_1(x) \in \partial \Delta'$. As $\tilde{F}_i^\delta$-agrees with $F'_i$ along $\partial \Delta'$, we can extend $\tilde{F}_i^\delta$ on $\delta$ by defining it to be $F'_i$ on $\Delta'$. Observe that

$$
\Sigma_i^\delta = \{ \tilde{F}_i^\delta \text{ tangent to } T\tilde{F} \times \mathbb{R} \} \subset \Delta - \Delta'
$$

The next theorem (3.1) extends $f_i$ from $t \in [0, (i - 1)/N]$ to $t \in [0, i/N]$. To start the process i.e., to extend $f_0$ to $f_i$, $t \in [0, 1/N]$ we take a fine triangulation of $M$ so that image under $f_0$ of all top dimensional simplices lies in a neighborhood diffeomorphic to $I^{2n+q+2}$ and on it $\tilde{F} \times \mathbb{R}$ is given by the projection on the first $q$ factors $\pi : I^{2n+q+2} \to I^q$.

Theorem 3.1. Let $I_\delta = [\delta, 1 - \delta]$, $I_\varepsilon = [\varepsilon, 1 - \varepsilon]$ with $\varepsilon = \varepsilon(\delta) < \delta$ and $(F_i^\delta, h_i^\delta) : TI^{2n+q+2} \to TI^{2n+q+2}$ be a family of monomorphisms such that

1. $F_i^\delta = dh_i^\delta$ on $I^{2n+q} - I^{2n+q}_{\varepsilon(\delta)}$
Proof. Let $\sigma : I \to I$ be a smooth map such that $\sigma = 0$ on $I - I_{\varepsilon(\delta)}$ and $\sigma = 1$ on a neighborhood of $1/2$. Let

$$F^\delta : T(I \times I^{2n+q}) \to T(I \times I^{2n+q+2})$$

be monomorphisms given by the matrix

$$F^\delta_{(t,x)} = \begin{pmatrix} 1 & 0 \\ \partial_t b^\delta_{\sigma(t)}(x) & F^\delta_{\sigma(t)}(x) \end{pmatrix}$$

Which covers $b^\delta(t,x) = (t, b^\delta_{\sigma(t)}(x))$. So $F^\delta = db^\delta$ on $I \times (I^{2n+q} - I^{2n+q+q})$. Let $\chi^\delta : I^{2n+q+1} \to I$ be a smooth map such that $\chi^\delta = 0$ on $I^{2n+q+1} - I^{2n+q+q+1}$ and $\chi^\delta = 1$ on $I^{2n+q+1}$, $\delta' < \delta$. Set $\Xi_\tau : I^{2n+q} \to I^{2n+q}$, $\tau \in I$ as

$$\Xi_\tau(x_1, ..., x_{2n+q}) = (x_1, ..., x_{q-1}, (1 - \chi^\delta)x_q + \chi^\delta(\tau - \gamma'(\tau), x_q), x_{q+1}, ..., x_{2n+q})$$

where $\gamma' : I \to [-\varepsilon(\delta)/2, \varepsilon(\delta)/2]$ be linear homeomorphism such that $\gamma'(0) = -\varepsilon(\delta)/2$ and $\gamma'(1) = \varepsilon(\delta)/2$. Now set $(F^\delta_{\tau})_{(t,x)} = F^\delta_{(t,\Xi_\tau(x))}$ which covers $b^\delta_{\tau}(t,x) = b^\delta(t, \Xi_\tau(x))$. Observe that

(1) $F^\delta_{\tau} = db^\delta = db^\delta_{\tau}$ on $I_{\varepsilon(\delta)} \times I_{\varepsilon(\delta)}^{-1} \times I \times I^{2n}$. 

(2) $F^\delta \cap L$ on $I^{2n+q}$ for all $t$ and $Im(F^\delta_t) \cap T\mathcal{L}$ is of dimension $2n$ and is close to $L_t$ for all $t$ on $I^{2n+q}$

(3) $I^2 \{ F^\delta \text{ tangent to } T\mathcal{L} \} \subset (I^{2n+q} - I^{2n+q})$

where $\mathcal{L}$ is the foliation on $I^{2n+q+2}$ induced by the projection $\pi : I^{2n+q+2} \to I^q$ (projection on the first $q$-factors), $\hat{\mathcal{L}}$ is such that $\mathcal{L} = \hat{\mathcal{L}} \times I$ and $L_t \subset T\hat{\mathcal{L}}$ is a family of $2n$-dimensional distribution. Then there is a $\delta''$ and a family of immersions $f_t : I^{2n+q} \to I^{2n+q+2}$ such that

(1) $f_t = b^{\delta''}_t$ on $I^{2n+q} - I^{2n+q+q}/2$

(2) $(\pi \circ f_t)|_{I^{2n+q}}$ is a wrinkle map

(3) If $\Sigma_t(I^{2n+q} - I^{2n+q}) = \{ x \in I^{2n+q} - I^{2n+q} : f_t(x) \text{ tangent to } \mathcal{L} \}$, then on $(I^{2n+q} - I^{2n+q}) - \Sigma_t(I^{2n+q} - I^{2n+q}), \text{Im}(df_t) \cap T\mathcal{L}$ is of dimension $2n$ and is close to $L_t$. 

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(2) \( F_0^\delta = db^\delta = db^\delta_0 \) on \( I \times I^{q-1} \times [0, \varepsilon(\delta)] \times I^{2n} \)

(3) \( F_1^\delta = db^\delta = db^\delta_1 \) on \( I \times I^{q-1} \times [1 - \varepsilon(\delta), 1] \times I^{2n} \)

Moreover observe that \( F_0^\delta \) and \( F_1^\delta \) are holonomic and for \( \tau \in I_\delta \)

\[
\Sigma^\delta_\tau = \{ F^\delta_\tau \not\pitchfork to T\mathcal{L} \times \mathbb{R}^2 \} \subset I^{2n+q+1} - I^{2n+q+1}_\delta
\]

Using the 2.4 we can approximate \( F^\delta_\tau \) on \( h^1_\tau(I \times I^{q-1} \times \{1/2\} \times I^{2n}) \) by \( df^\delta_\tau \), where \( f^\delta_\tau \) is a family of immersions defined on \( h^1_\tau(I \times I^{q-1} \times \{1/2\} \times I^{2n}) \). Now consider three smooth functions \( \chi^i, i = 1, 2, 3 \) defined as follows

\( \chi^1 : [0, \delta'] \rightarrow [0, 1], \chi^1(0) = 0 \) and \( \chi^1(\delta') = 1. \chi^2 : [1 - \delta', 1] \rightarrow [0, 1], \chi^2(1 - \delta') = 1 \) and \( \chi^2(1) = 0. \chi^3 : [\delta', 1 - \delta'] \rightarrow [0, 1], \chi^3(\delta') = 0 \) and \( \chi^3(1 - \delta') = 1 \) also \( \chi^3(\delta) = \delta \) and \( \chi^3(1 - \delta) = 1 - \delta. \) Now define \( g^\delta_\tau \) as follows

\[
\begin{align*}
g^\delta_\tau &= b^\delta_0 \circ g^\tau_{\chi^1(\tau)}, \quad \tau \in [0, \delta'] \\
&= f^\delta_{\chi^2(\tau)} \circ h^1_{\chi^2(\tau)} \circ g_1, \quad \tau \in [\delta', 1 - \delta'] \\
&= b^\delta_1 \circ g^\tau_{\chi^3(\tau)}, \quad \tau \in [1 - \delta', 1]
\end{align*}
\]

Where \( g_s : I \times I^{2n+q} \rightarrow I \times I^{2n+q+2}, s \in I \) is an isotopy of embeddings defined as follows

\[
I \times I^{2n+q} = g_0(I \times I^{2n+q})
\]
Let \( g_s : I \to I \) be such that \( g_0 = \text{id} \) and \( g_1(I) \subset Op(1/2) \). Then we set \( g_s = \text{id}_{I_{\varepsilon(\delta)}} \times \text{id}_{I_{\varepsilon(\delta)}} \times g_s \times \text{id}_{I_{2n}} \) on \( I_{\varepsilon(\delta)} \times I_{\varepsilon(\delta)} \times I \times I_{2n} \). This is shown in the central shaded region in the above pictures.

In the non-shaded region in the third picture i.e, in the picture of \( g_1(I^{2n+q+1}) \),

\[
f_0^\delta = b_0^\delta = b_1^\delta = b_1^\delta = f_1^\delta
\]

and hence \( g_s^\delta \) is well defined.

Now observe that for \( \tau \in I_\delta \), \{\( g_s^\delta \text{ not } \cap \text{ to } \mathcal{L} \times \mathbb{R} \} \subset (I^{2n+q+1} - I_\delta^{2n+q+1})$. 
For an integer $l > 0$ take a function $\phi_l : I \rightarrow I$ such that

\[
\phi_l = 1, \text{ on } I_{1/(8l)} \\
0, \text{ outside } I_{1/(16l)}
\]

which is increasing on $[1/(16l), 1/(8l)]$ and decreasing on $[1 - 1/(8l), 1 - 1/(16l)]$. Set

\[
\gamma_l(t) = t + \phi_l(t)\sin(2\pi lt), \ t \in I
\]

Let $J_i$ be the interval of length $9/(16l)$ centered at $(2i - 1)/2l$. Observe that $\gamma_l$ is non-singular outside $\bigcup J_i$ and $(\gamma_l)_{J_i}$ is a wrinkle. Also

\[
\partial_t \gamma_l(t) \geq l, \ t \in I - \bigcup J_i
\]

Let $\bar{\chi}_\delta : I^{2n+q+1} \rightarrow I$ be such that

\[
\bar{\chi}_\delta = 0, \text{ near } \partial(I^{2n+q+1}) \\
1, \text{ on } I^{2n+q+1}
\]

Now we take $\delta = \delta(l) << 1/(16l)$. Set $\tilde{\gamma}_l(x) = (1 - \bar{\chi}_\delta(x))x_q + \bar{\chi}_\delta(x)\gamma_l(x_q)$. Let $\lambda : I \rightarrow I$, be such that $\lambda(0) = 0, \lambda(1) = 1$

1. $\lambda = (2i - 1)/2l$, on $J_i$
2. $0 < \partial_t \lambda < 3$, on $I - \bigcup J_i$

Set $\tilde{g}_\delta = g_{\lambda(\tau)}^{\delta}, \ \tau \in I$. Now consider

\[
(t, x_1, ..., x_{2n+q}) \overset{\delta}{\mapsto} \tilde{g}_{x_q}^{\delta}(t, x_1, ..., x_{q-1}, \tilde{\gamma}_l(x), x_{q+1}, ..., x_{2n+q})
\]

Let $\theta$ be the function $\theta(t, x) = (t, x_1, ..., x_{q-1}, \tilde{\gamma}_l(x), x_{q+1}, ..., x_{2n+q})$. Then $\theta$ is a wrinkle map and as $\delta = \delta(l) << 1/(16l)$, the wrinkles of $\theta$ do not intersect $\{g^2 \text{ not } \in L \times \mathbb{R}\}$, for $\tau \in I_\delta$. On $I \times I^{n-1} \times J_i \times I^{2n}$, $\rho_t$ is of the form $g^\delta_i \circ \theta_i$, where $\theta_i = \theta_{i\times I^{n-1} \times J_i \times I^{2n}}$. Using (3.1) we can replace $\theta_i$ by another wrinkle map $\psi_i$ such that $\pi \circ \tilde{g}_i^{\delta(l)} \circ \psi_i$ turns out to be a fibered wrinkle map, fibered over the first factor $I$. But observe that $\tilde{g}_i^{\delta(l)} \circ \psi_i$ is not an immersion. So we need to regularize it.

For all $i, \pi \circ \tilde{g}_i^{\delta(l)} \circ \psi_i$ has many wrinkles and near each wrinkle it is of the form

\[
w_s(t, y, z, x) = (t, y, z^3 + 3((t, y)^2) - 1)z - \Sigma_1^4 x_i^2 + \Sigma_2^{2n} x_i^2
\]
and hence $g_i^{(l)} \circ \psi_i$ is of the form
\[(t, y, z, x) \mapsto (t, y, z^3 + 3(|(t, y)|^2 - 1)z - \Sigma_{j=1}^{2n} a_j^2, a_1(t, y, z, x), ..., a_{2n+2}(t, y, z, x))\]

Its derivative is given by the matrix
\[
\begin{pmatrix}
I_q & 0 & 0 \\
* & 3(|(t, y)|^2 - 1) & (\pm 2x_i)^{2n}_1 \\
* & (\partial_x a_j)^{2n+2} & (\partial_x, a_j)_{i=1, j=1}^{2n+2}
\end{pmatrix}
\]

and from the proof of [3] it follows that $\partial_x a_j = 0$ for all $j$ along $\{z^2 + |(t, y)|^2 - 1 = 0\}$.

So in order to regularize it one needs to $C^1$-approximate $a_j$'s by $a_j'$'s so that not all of $\partial_x a_j'$ vanish simultaneously along $\{z^2 + |(t, y)|^2 = 1\}$. But we shall moreover want the $\partial_x a_{2n+1}'s$ to be different from 0 along $\{z^2 + |(t, y)|^2 - 1 = 0\}$, where $a_{2n+1}'$ corresponds to the $R$-factor of $\tilde{M} = M \times \mathbb{R}$.

Now let us set $\varphi : I^{2n+q+2} \to [0, 1]$ be a smooth function such that $\varphi = 1$ outside a neighborhood of $D$, where $D$ is the disc which encloses $\{z^2 + |(t, y)|^2 - 1 = 0\}$ and on $\{z^2 + |(t, y)|^2 - 1 = 0\}$, $\varphi = 0$ and $\partial_{(n, \varphi)} = 0$, moreover $\phi + y_1 \partial_{y_1} \varphi$ is non-vanishing outside $\{z^2 + |(t, y)|^2 - 1 = 0\}$. Now let $y = (y_1, ..., y_q)$ in the above. Now replace the resulting map by
\[(t, y, z, x) \mapsto (t, \varphi(t, y, z, x)y_1, y_2, ..., y_q, z^3 + 3(|(t, y)|^2 - 1)z - \Sigma_{i=1}^{2n} x_i^2 + \Sigma_{i=1}^{2n+2} a_1'(t, y, z, x), ..., a_{2n+1}'(t, y, z, x) + y_1 - y_1 \varphi(t, y, z, x))\]

Where in the above the last component corresponds to the $R$-component of $\tilde{M} \times \mathbb{R}$, i.e. the $w$-variable. Its derivative is given by
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\partial_{(y_1 \varphi)} & \varphi + y_1 \partial_{y_1} \varphi & * & * & * \\
0 & 0 & I_{q-2} & 0 & 0 \\
* & * & * & 3(|(t, y)|^2 - 1) & (\pm 2x_i)^{2n}_1 \\
* & * & * & (\partial_x a_j')^{2n+2}_1 & (\partial_x, a_j')_{i=1, j=1}^{2n+2} \\
* - \partial_t(y_1 \varphi) & * + 1 - \partial_{y_1}(y_1 \varphi) & * - \partial_t(y_1 \varphi) & (\partial_x a_2' - \partial_x(y_1 \varphi)) & (\partial_x, a_2')_{i=1, j=1}^{2n+2}
\end{pmatrix}
\]

Now observe that the projections of the column vectors
\[\begin{pmatrix}
(0, *, 0, 3(|(t, y)|^2 - 1), (\partial_x a_j')^{2n+1}_1, (\partial_x a_{2n+2}' - \partial_x(y_1 \varphi))
\end{pmatrix}
\]

and
\[\begin{pmatrix}
(0, *, 0, (\pm 2x_i)^{2n}_1, (\partial_x, a_j')_{i=1, j=1}^{2n+1}, (\partial_x a_{2n+2}' - \partial_x(y_1 \varphi))_{i=1}^{2n})^T
\end{pmatrix}
\]
onto $T\tilde{F} \times \mathbb{R}$ are
\[(\partial z_a^j)^{2n+1}_1, (\partial z_a^j(y_1\varphi))T\]
and
\[(\partial x_i^a)^i=2n,j=2n+1, (\partial x_i^a(y_1\varphi))i=1_{2n})^T\]
and their projection on $T\tilde{F}$ are
\[(\partial z_a^j)^{2n+1}_1)^T\]
and
\[(\partial x_i^a)^i=2n,j=2n+1, (\partial x_i^a(y_1\varphi))i=1_{2n})^T\]

Whose span was already close to $\mathbb{R} \times L_t$.

Along \(\{q \geq 2\}_{\mathbb{N}} \ni \text{ to } L \times \mathbb{R}\} \subset I_2^{2n+q+1} - I_2^{2n+q+1} , \tau \in I_2\), we can apply the same technique as above. For this we decompose \(\{q \geq 2\}_{\mathbb{N}} \ni \text{ to } L \times \mathbb{R}\} \subset I_2^{2n+q+1} - I_2^{2n+q+1} , \tau \in I_2\) as
\[\{q \geq 2\}_{\mathbb{N}} \ni \text{ to } L \times \mathbb{R}\} = \{\partial_{y_1} \text{ tangent to } L \times \mathbb{R}\} \cup \{\partial_{y_2} \text{ tangent to } L \times \mathbb{R}\}\]

Now we use the same technique as above along \(\{\partial_{y_1} \text{ tangent to } L \times \mathbb{R}\} \cup \{\partial_{y_2} \text{ tangent to } L \times \mathbb{R}\}, i.e, rotating $y_1$-component to be tangent to $L \times \mathbb{R}$ along \(\{\partial_{y_2} \text{ tangent to } L \times \mathbb{R}\} \cup \{\partial_{y_1} \text{ tangent to } L \times \mathbb{R}\}\). This way we make \(\{q \geq 2\}_{\mathbb{N}} \ni \text{ to } L \times \mathbb{R}\} to \{g_\tau^r \text{ tangent to } L \times \mathbb{R}\}.

Note that if $q > 2$, then along intersection of three sets \(\cup_{i=1}^3 \{\partial_{y_i} \text{ tangent to } L \times \mathbb{R}\}\), we can not make \(\{q \geq 2\}_{\mathbb{N}} \ni \text{ to } L \times \mathbb{R}\} to \{g_\tau^r \text{ tangent to } L \times \mathbb{R}\}, otherwise the rank will drop and it will no longer be regular.

Let $\bar{p}_l$ be the regularized map, then $\bar{p}_l$ is of the form $\bar{p}_l(t,x) = (t,x(t))$, where $x(t)$ are functions of $t$. So the required family of immersions is given by
\[f_l(x) = x(\sigma^{-1}(t)), t \in [0,1/2]\]

with reparametrization. Clearly $f_l$ has the property (1) and (2). Condition (3) follows from the fact that for large $l$, $d_I\rho_l$ approximates $d_I\tilde{g}_t^r$ on $I \times I^{q-1} \times (I - \cup_i J_i) \times I^{2n}$ and on $I^{2n+q+1} - I^{2n+q+1}_l$ whose proof is same as in 2.3A of [5] and we refer the readers to [5]. As $\delta(l)$ depends on $l$ and $\varepsilon(\delta)$ depends on $\delta$, we are done. □
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