The shape of surfaces that contain a great and a small circle through each point

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Abstract

We classify the possible shapes of surfaces in the 3-dimensional unit-sphere that contain both a great and a small circle through each point. A corollary is that such a surface either is the pointwise product of circles in the unit quaternions or contains six concurrent circles.

Keywords: real surfaces, topology, pencils of circles, singular locus, Möbius geometry, elliptic geometry, Clifford translations, unit quaternions

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1 Introduction

Can we recover the shape of an embedded surface from the knowledge that this surface contains at least two algebraic curves of minimal degree through each point? Surfaces in this article can be defined as the zero-set of algebraic equations and therefore with surface is meant a real irreducible reduced algebraic surface.

For example, a surface in $\mathbb{R}^3$ that contains $\lambda \geq 3$ lines through each point must be a plane. A surface that contains $\lambda = 2$ lines through each point, must be a doubly ruled quadric and is shaped as either a horse saddle or a cooling tower. Such hyperboloid structures are of interest to architects [13, Chapter 9].

Now let us consider the analogue of lines in elliptic geometry, namely great circles in the unit-sphere $S^3 \subset \mathbb{R}^4$. If a surface in $S^3$ contains $\lambda \geq 3$ great circles through each point, then it must be a sphere, and if this surface contains $\lambda = 2$ great circles through each point, then it is homeomorphic to a torus by [10, Theorem 2]. If a smooth surface in $S^3$ is covered by great circles that are fibres of a single Hopf fibration, then this surface has degree $d \leq 8$ by [12, Corollary 1].

In this article we consider the shape of surfaces that contain both a great circle and a small circle through each point. Here we call a circle small if it is not great. In Figure 1 we see possible shapes of stereographic projections of such surfaces. It was already known to Hipparchus (190–120 BCE) that a stereographic projection $\pi: S^3 \rightarrow \mathbb{R}^3$ sends circles to either circles or lines.

![Figure 1: Stereographic projections of surfaces in $S^3$ that contain a great (red) and a small (blue) circle through each point, together with their great types.](image)

A surface $Z \subset S^3$ is $\lambda$-circled if $Z$ contains at least $\lambda \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ circles through a general point. If $\lambda \in \mathbb{Z}_{\geq 0}$, then we assume that $Z$ is not $(\lambda + 1)$-circled. We call $Z$ great if it contains a great circle through a general point. We say that a surface $Z \subset S^3$ is of great type I, II or III if $Z$ is great and 2-circled, and if one of the
following holds:

I. $Z$ is homeomorphic to two linked ring tori that touch along a common Villarceau circle. Both topological tori components are a union of great circles in $Z$. A small circle in $Z$ meets the great double circle in $Z$ in two points. A great circle in $Z$ is either disjoint from or coincides with the great double circle.

II. $Z$ is homeomorphic to a torus. A small circle in $Z$ meets the great double circle in $Z$ tangentially in one point. A great circle in $Z$ is either disjoint from or coincides with the great double circle. The double circle is cuspidal as a general hyperplane section of the surface has a cusp at this circle.

III. $Z$ is homeomorphic to the disjoint union of a torus and a circle such that the circle is linked with the torus.

We remark that surfaces in $S^3$ are homeomorphic if and only if their stereographic projections in $\mathbb{R}^3$ are equivalent up to homeomorphisms and inversions (see Definition 4). For example, the two surfaces in Figure 2 are related by an inversion that turns the orange torus component of the surface inside out.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure_2.png}
\caption{Different stereographic projections of the same surface in $S^3$.}
\end{figure}

In order to construct the surfaces in Figure 1 we apply an idea of William Kingdon Clifford (1845–1879) and identify $S^3$ with the unit-quaternions. The unit-quaternions form a Lie group with respect to the Hamiltonian product $\star$. We define $A \star B$ for curves $A, B \subset S^3$ as the Zariski closure of $\{a \star b \in S^3 \mid a \in A \text{ and } b \in B\}$. For example, if $A$ and $B$ are distinct great circles, then $A \star B$ is called a Clifford torus. Let consider the great circle $A_0 \subset S^3$ and the small circles $B_1, B_2, B_3 \subset S^3$ that are parametrized as follows:

$A_0 := \{(\cos \alpha, \sin \alpha, 0, 0) \mid 0 \leq \alpha < 2\pi\}$,

$B_1 := \left\{ \left( \frac{12+8 \cos \beta}{17+12 \cos \beta}, \frac{8 \sin \beta}{17+12 \cos \beta}, 0, \frac{9+12 \cos \beta}{17+12 \cos \beta} \right) \mid 0 \leq \beta < 2\pi \right\}$,

$B_2 := \left\{ \left( \frac{2+2 \cos \beta}{3+2 \cos \beta}, \frac{\sin \beta}{3+2 \cos \beta}, 0, \frac{2+2 \cos \beta}{3+2 \cos \beta} \right) \mid 0 \leq \beta < 2\pi \right\}$, and

$B_3 := \left\{ \left( \frac{6+2 \cos \beta}{11+6 \cos \beta}, \frac{2 \sin \beta}{11+6 \cos \beta}, 0, \frac{9+6 \cos \beta}{11+6 \cos \beta} \right) \mid 0 \leq \beta < 2\pi \right\}$. 

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The normal forms for surfaces of great types I, II and III in Figure 1 are constructed as $\pi(A_0 \ast B_1)$, $\pi(A_0 \ast B_2)$ and $\pi(A_0 \ast B_3)$, respectively, where the center of stereographic projection $\pi$ is assumed to be $(0, 0, 0, 1)$.

Let us summarize the state of the art concerning the topology of an analytic $\lambda$-circled surface $Z \subset S^3$ of degree $d$ such that $\lambda \geq 2$. We know from [15, Theorem 1] that $Z$ must be algebraic and we have $d \in \{2, 4, 8\}$ by [9, Theorem 1]. If $d = 4$ and $Z$ is smooth, then $Z$ is homeomorphic to either a torus, a sphere or the disjoint union of two spheres by [21, Proposition 4.1] (see also [14, Figure 12]).

The following theorem is the main result of this article, but depends on [10, Theorem 2] for the degree four case. We call surface $Z \subset S^3$ a CO cyclide or EO cyclide if $Z$ stereographically projects to a circular cone and elliptic cone, respectively. A CO cyclide is also known as a spindle cyclide. A 6-circled surface $Z \subset S^3$ is called a Blum cyclide and is smooth [2, 9].

**Theorem 1.** Suppose that $Z \subset S^3$ is a great $\lambda$-circled surface such that $\lambda \geq 2$. The name or great type of $Z$ and a homeomorphic normal form for $Z$ together with its degree $d$ and the number $\lambda$ is characterized by a row in Table 2. If $d = 8$, then $Z = A \ast B$ such that $\{A, B\}$ consists of a great circle and a small circle in $S^3$.

| $d$ | $\lambda$ | name or great type | homeomorphic normal form |
|-----|-----------|---------------------|--------------------------|
| 2   | $\infty$ | sphere              | sphere                   |
| 4   | 6         | Blum cyclide        | torus                    |
| 4   | 4         | Clifford torus      | torus                    |
| 4   | 3         | EO cyclide          | doubly pinched torus     |
| 4   | 2         | spindle cyclide     | doubly pinched torus     |
| 8   | 2         | I                   | $A_0 \ast B_1$           |
| 8   | 2         | II                  | $A_0 \ast B_2$           |
| 8   | 2         | III                 | $A_0 \ast B_3$           |

In [18, Main Theorem 1.1] it is shown using quaternion factorization that a $\lambda$-circled surface $Z \subset S^3$ such that $\lambda \geq 2$ and $\text{deg } Z \neq 4$ is Möbius equivalent to either $A \ast B$, or the Zariski closure of $\pi^{-1}(\{a + b \mid a \in A, b \in B\})$ for some circles $A$ and $B$. The following corollary confirms and strengthens [18] under the assumption that $Z$ is great. The alternative proof in this article is complementary in that it exposes the self-intersections of such surfaces and does not depend on the quaternion algebra.
Corollary 3. If a surface \( Z \subset S^3 \) contains great and small circles through a general point, then either \( Z = A \star B \) for some circles \( A, B \subset S^3 \), or \( Z \) contains six concurrent circles.

Overview. Suppose that \( Z \subset S^3 \) is a great 2-circled surface of degree eight. In §2 we setup a projective model for real elliptic geometry and in §3 we characterize the incidences between circles and complex double lines in \( Z \) using divisor classes. In §4 we introduce an invariant for curve components in the singular loci of projective surfaces. In §5 we use this invariant in combination with the central projection to obtain a characterization of the incidences between circles and double curves in \( Z \). This characterization is used in §6 to construct non-vanishing vector fields and this enables us to classify \( Z \) up to homeomorphisms.

2 Projective model for elliptic geometry

In order to prove Theorem 1 we investigate curves at complex infinity. To uncover these hidden curves we define a real variety \( X \) to be a complex variety together with an antiholomorphic involution \( \sigma: X \to X \) (see [17, Section I.1]) and we denote its real points by \( X^R := \{ p \in X \mid \sigma(p) = p \} \). Such varieties can always be defined by polynomials with real coefficients [16, Section 6.1].

Points, curves, surfaces and projective spaces \( \mathbb{P}^n \) are real algebraic varieties and maps between such varieties are compatible with the real structure \( \sigma \) unless explicitly stated otherwise. Moreover, a variety is irreducible, and subvarieties such as hyperplane sections inherits the real structure unless explicitly stated otherwise. By default we assume that the real structure \( \sigma: \mathbb{P}^n \to \mathbb{P}^n \) sends \( x \) to \((x_0: \ldots : x_n)\).

As circles play a central role, it is natural to consider the Möbius quadric for our space: \( S^3 := \{ x \in \mathbb{P}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \} \). The Möbius transformations of \( S^3 \) are defined as biregular automorphisms \( \text{Aut} S^3 \) and are linear so that \( \text{Aut} S^3 \subset \text{Aut} \mathbb{P}^4 \). If \( V \subset S^3 \) is a variety, then we define \( V(\mathbb{R}) := \gamma(V_R) \), where the isomorphism \( \gamma: S^3_R \to S^n \) sends \( x \) to \((x_1/x_0, \ldots , x_4/x_0)\). Notice that \( S^3(\mathbb{R}) = S^3 \).

Definition 4. An inversion with respect to a sphere \( O \subset \mathbb{R}^3 \) with center \( c \) and radius \( r \) is the map \( f: \mathbb{R}^3 \setminus \{ c \} \to \mathbb{R}^3 \setminus \{ c \} \) such that \( ||c - x|| \cdot ||c - f(x)|| = r^2 \) for all \( x \in \mathbb{R}^3 \setminus \{ c \} \). Such a map exchanges the interior and exterior of \( O \) and takes circles to circles and/or lines (see also [22, Definition 2.2.1]).

Notice that the image \( \pi^{-1}(\mathbb{R}^3) \) of an inverse stereographic projection \( \pi^{-1} \) defines an
isomorphic copy of $\mathbb{R}^3$ inside $S^3 = S^3(\mathbb{R})$. Thus inversions of $\mathbb{R}^3$ commute via $\pi \circ \gamma$ with Möbius transformations of $S^3$.

The elliptic absolute is defined as the hyperplane section $E := \{x \in S^3 \mid x_0 = 0\}$. This doubly ruled quadric has no real points and we refer to its two families of non-real lines as left generators and right generators. Notice that the complex conjugate of a left (right) generator is again a left (right) generator. The elliptic transformations are defined as $\{\varphi \in \text{Aut} S^3 \mid \varphi(E) = E\}$ and induce rotations of $S^3 = S^3(\mathbb{R})$. The left Clifford translations are defined as

$$\{\varphi \in \text{Aut} S^3 \mid \exists a \in S^3 \forall p \in S^3 : (\gamma \circ \varphi)(p) = a \ast \gamma(p)\}.$$  

Similarly, we define the right Clifford translations as

$$\{\varphi \in \text{Aut} S^3 \mid \exists b \in S^3 \forall p \in S^3 : (\gamma \circ \varphi)(p) = \gamma(p) \ast b\}.$$  

The left/right Clifford translations induce isoclinic rotations of $S^3$, namely rotations where each point in $S^3$ rotates with the same angle.

**Proposition 5.** The left or right Clifford translations of $S^3$ leave the left and right generators of the elliptic absolute $E$ invariant, respectively.

**Proof.** See [10, Proposition 1] or [3, 7.93]. \hfill $\square$

**Definition 6.** We call a conic $C \subset S^3$ a great/small circle, if $C(\mathbb{R}) \subset S^3$ is a great/small circle. A surface $X \subset S^3$ is called $\lambda$-circled, great, CO cyclide, EO cyclide, Clifford torus, or Blum cyclide if $X(\mathbb{R}) \subset S^3$ is defined as such in §1. \hfill $\triangle$

### 3 Divisor classes of curves

We recall some results from [9] about divisor classes of complex double lines and circles in 2-circled surfaces of degree eight in the Möbius quadric (recall Definition 6).

A smooth model of a surface $X \subset \mathbb{P}^n$ is a birational morphism $\varphi: Y \rightarrow X$ from a nonsingular surface $Y$, that does not contract complex $(-1)$-curves. See [8, Theorem 2.16] for the existence and uniqueness of the smooth model up to biregular isomorphisms.

We denote the singular locus of a surface $X \subset \mathbb{P}^n$ by $\text{sng} \ X$.

The Néron-Severi lattice $N(X)$ is an additive group defined by the divisor classes on $Y$ up to numerical equivalence. This group comes with an unimodular intersection product $\cdot$ and a unimodular involution $\sigma_*: N(X) \rightarrow N(X)$ induced by the real structure $\sigma: X \rightarrow X$. 
Suppose that $C \subset X$ is a complex and possibly reducible curve. Its class $[C] \in N(X)$ is defined as either the divisor class of the Zariski closure of $\varphi^{-1}(C \setminus \text{sng } X)$ if $C \not\subset \text{sng } X$, or the divisor class of $\varphi^{-1}(C)$ if $C \subset \text{sng } X$. The class of hyperplane sections $h \in N(X)$ is defined as the class of any hyperplane section of $X$. We will denote the canonical class of $X$ in $N(X)$ by $k$.

**Remark 7.** If $C,D \subset X$ are complex irreducible curves such that $[C] \cdot [D] > 0$, then $|C \cap D| > 0$. However, we cannot assume that $|C \cap D| = [C] \cdot [D]$. In particular, if $\varphi^{-1}(C)$ and $\varphi^{-1}(D)$ are disjoint in the smooth model $Y$, then $C$ and $D$ may still meet at the singular locus of $X$.

If $N(X)$ is generated by $\langle \ell_0, \ell_1 \rangle_Z$, then we assume that $\ell_0^2 = \ell_1^2 = 0$ and $\ell_0 \cdot \ell_1 = 1$.

**Proposition 8.** If $X \subset S^3$ is a 2-circled surface of degree eight, then $N(X) \cong \langle \ell_0, \ell_1 \rangle_Z$, $\sigma_* = \text{id}$, the class of hyperplane sections $h$ is equal to the anticanonical class $-k = 2\ell_0 + 2\ell_1$ and the class of a circle is in $\{\ell_0, \ell_1\}$.

*Proof.* Direct consequence of [9, Lemma 1 and Proposition 2].

**Lemma 9.** If $X \subset \mathbb{P}^n$ is a surface with canonical class $k$ and if $C \subset X$ is an irreducible and reduced curve, then $\frac{1}{2}([C]^2 + [C] \cdot k) + 1 \geq 0$.

*Proof.* Direct consequence of the geometric genus formula [6, Remark IV.1.1.1 and Exercise IV.1.8.a].

Let $X \subset \mathbb{P}^n$ be a surface with smooth model $\varphi: Y \rightarrow X$. Its linear normalization $X_N \subset \mathbb{P}^m$ is defined as the image of $Y$ via the map $\varphi_h$ associated to the linear equivalence class $h$ of the pullback to $Y$ of a hyperplane section of $X$.

**Remark 10.** The associated map $\varphi_h: Y \rightarrow X_N \subset \mathbb{P}^m$ is compatible with the real structures and $X_N$ is unique up to $\text{Aut } \mathbb{P}^m$ as a direct consequence of the definitions (see [6, Remark II.7.8.1]). We have that $m \geq n$ and there exists a degree-preserving linear projection $\eta: \mathbb{P}^m \dashrightarrow \mathbb{P}^n$ such that $\eta(X_N) = X$.

**Proposition 11.** If $X \subset S^3$ is a 2-circled surface of degree eight, then its linear normalization $X_N \subset \mathbb{P}^8$ is isomorphic to its smooth model $\mathbb{P}^1 \times \mathbb{P}^1$.

*Proof.* See [9, Theorem 3].
A pencil on a surface $X \subset \mathbb{P}^n$ is defined as an irreducible hypersurface $P \subset X \times \mathbb{P}^1$ such that $\pi_1(P) = X$ and $\pi_2(P) = \mathbb{P}^1$, where $\pi_1$ and $\pi_2$ are the projections of $P$ to its two factors $X$ and $\mathbb{P}^1$, respectively. A member $P_i$ of $P$ is defined as $\pi_1(P \cap X \times \{i\}) \subset X$ for all $i \in \mathbb{P}^1$. The complex points in the intersection $\cap_{i \in \mathbb{P}^1} P_i$ are called base points of $P$. We call $P$ a pencil of circles/conics if $P_i$ is a circle/conic for a general point $i \in \mathbb{P}^1$.

Proposition 12. Suppose that $X \subset S^3$ is a 2-circled surface of degree eight.

a) $X$ is covered by two base point free pencils of circles.

b) If $L \subset X$ is a complex line, then $L \subset \text{sng} X$ and $L$ is not real.

c) The singular locus of $X$ does not contain isolated singularities.

Proof. Assertion a) follows from [9, Theorem 3] and assertion b) follows from [9, Lemma 1d, Lemma 2 and Proposition 2].

c) Suppose by contradiction that complex $p \in \text{sng} X$ is an isolated singularity and let $\rho: \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ be the complex linear projection with center $p$. A general hyperplane section of $\rho(X)$ is smooth by Bertini’s theorem and pulls back to a hyperplane section $H \subset X$ such that $p \in H$. It follows from the proof of [9, Lemma 7] that $p \in \text{sng} H$. Hence the geometric genus of $H$ is strictly less than the geometric genus of a general hyperplane section of $X$. We deduce from a) that $\rho(X)$ is covered by two pencils of conics and not covered by lines. It follows from [15, Theorem 5] that a general hyperplane section of $X$ and $\rho(X)$ is one and zero, respectively. We arrived at a contradiction as $\rho(X)$ is by [15, Theorem 8] either ruled or contains infinitely many conics through a general point. We conclude that $X$ does not have isolated singularities. \hfill \Box

4 Sectional delta invariant

In this section, we introduce an invariant for curve components in the singular locus of a surface. This invariant measures how singular such a component is.

Definition 13. The delta invariant of a point $p$ in the curve $C \subset \mathbb{P}^n$ with structure sheaf $\mathcal{O}$ is defined as $\delta_p(C) := \text{length}(\widetilde{\mathcal{O}}_p/\mathcal{O}_p)$, where $\widetilde{\mathcal{O}}_p$ denotes the integral closure of the stalk $\mathcal{O}_p$ (see [6, Exercise IV.1.8] or [19, Tag 0C3Q]). \hfill ◄
Notice that the delta invariant of a singular point in a curve is a non-zero positive integer. Informally, we may think of \( \delta_p(C) \) as the number of double points that are concentrated at \( p \) (see [11, page 85]).

**Lemma 14.** Let \( X \) be a complex surface and suppose that \( H \subset X \) is a general hyperplane section with arithmetic genus \( p_a(H) \) and geometric genus \( p_g(H) \).

a) The complex curve \( H \) is irreducible and \( \text{sng} \ H = H \cap \text{sng} \ X \).

b) \( p_g(H) = \frac{1}{2}(h^2 + h \cdot k) + 1 \), where \( h \) and \( k \) are the class of hyperplane sections and the canonical class, respectively.

c) \( p_a(H) - p_g(H) = \sum_{p \in \text{sng} \ H} \delta_p(H) \).

d) \( p_a(H) = p_a(H') \) and \( p_g(H) \geq p_g(H') \) for all irreducible hyperplane sections \( H' \subset X \).

**Proof.** a) This assertion follows from the Bertini theorems at [19, Tag 0G4C] and [5, Theorem 17.16].

b) Suppose that \( \varphi : Y \to X \) is a smooth model and let \( D \) be the proper transform of \( H \) along \( \varphi \). We have \( p_a(D) = \frac{1}{2}(h^2 + h \cdot k) + 1 \) by the arithmetic genus formula [6, Exercise V.1.3]. It follows from the Bertini theorem at [6, Corollary 10.9] that the general curve \( D \) in the linear series associated to the morphism \( \varphi \) is smooth, which implies that \( p_a(D) = p_g(D) \). As the geometric genus is a birational invariant, it follows that \( p_g(D) = p_g(H) \) as asserted.

c) This is the genus formula at [6, Exercise IV.1.8a] (see also [4, Section 2.4.6]).

d) It follows from [6, Exercise V.1.3] that \( p_a(H) \) only depends on the projective equivalence class of \( H \). A non-general hyperplane section \( H' \subset X \) maybe more singular than \( H \) and thus \( \sum_{p \in \text{sng} \ H'} \delta_p(H') \geq \sum_{p \in \text{sng} \ H} \delta_p(H) \). Hence this assertion follows from c). \( \square \)

Let \( X \subset \mathbb{P}^n \) be a complex surface. The *sectional arithmetic genus* \( a(X) \) and the *sectional geometric genus* \( g(X) \) are defined as the arithmetic and geometric genus of a general hyperplane section of \( X \), respectively. The *total delta invariant* of \( X \) is defined as \( \delta(X) := a(X) - g(X) \). These invariants are well-defined by Lemma 14.

**Lemma 15.** Let \( X \subset \mathbb{P}^n \) be a complex degree \( d \) surface, with canonical class \( k \) and class of hyperplane sections \( h \).

a) If \( X \subset \mathbb{P}^3 \), then \( \delta(X) = \frac{d}{2} (d - 4) - \frac{1}{2} h \cdot k \).
b) If $X \subset S^3$, then $\delta(X) = \frac{d}{2} \left( \frac{d}{2} - 3 \right) - \frac{1}{2} h \cdot k$.

**Proof.** Suppose $H \subset X$ is a general hyperplane section so that $h = [H]$ and $d = h^2$.

a) Since $H$ is a planar curve so that $a(X) = p_a(H) = \frac{1}{2}(d - 1)(d - 2)$ by [4, Example 2.17] and we conclude from Lemma 14b that $\delta(X)$ is as asserted.

b) We observe that $H$ is a complete intersection curve of degree $d$ that is contained in a two-sphere $Q \subset S^3$. We have $N(Q) \cong (\ell_0, \ell_1)_Z$, $h_Q = \ell_0 + \ell_1$ is the class of a hyperplane section of $Q$, and $k_Q = -2h_Q$ is the canonical class of $Q$. Suppose that $[H]_Q$ is the class of $H$ in $N(Q)$ so that $h_Q \cdot [H]_Q = (\ell_0 + \ell_1) \cdot (\ell_0 + \alpha \ell_1) = 2\alpha = d$. We find that $\alpha = \frac{1}{2} d$ so that $a(X) = p_a(H) = \frac{1}{2}(d^2 - 1) + k_Q \cdot [H]_Q + 1 = \frac{1}{4}d^2 - d + 1$ by the arithmetic genus formula. We conclude from Lemma 14b that $\delta(X)$ is as asserted. \qed

**Proposition 16.** If $X \subset S^3$ is a 2-circled surface of degree eight, then $\delta(X) = 8$.

**Proof.** Direct consequence of Proposition 8 and Lemma 15. \qed

**Definition 17.** Let $X \subset \mathbb{P}^n$ be a complex surface such that the 1-dimensional part of the singular locus of $X$ admits the following decomposition into complex irreducible curve components: $C_1 \cup \cdots \cup C_r$. A **sectional delta invariant** for $X$ is a function $\Delta_X : \{ C_i | 1 \leq i \leq r \} \to \mathbb{Z}_{>0}$ that satisfies the following axioms for all $1 \leq i \leq r$, real structures $\sigma : X \to X$ and $\alpha \in \text{Aut} \mathbb{P}^n$:

1. $\Delta_X(C_1) + \cdots + \Delta_X(C_r) = \delta(X)$.
2. $\Delta_X(C_i) \geq \deg C_i$.
3. $\Delta_X(C_i) = \Delta_X(\sigma(C_i))$ and $\Delta_X(C_i) = \Delta_{\sigma(X)}(\sigma(C_i))$.
4. If $\rho : X \dasharrow Z \subset \mathbb{P}^m$ is a complex birational linear map such that $g(X) = g(Z)$ and $\rho|_{C_i} : C_i \dasharrow \rho(C_i)$ is birational, then $\Delta_Z(\rho(C_i)) \cdot \deg C_i = \Delta_X(C_i) \cdot \deg \rho(C_i)$.
5. If $\rho : X \to Z \subset \mathbb{P}^m$ is a finite $q : 1$ linear map such that $\deg \rho(C_i) = (\deg C_i)/q$, then $\Delta_Z(\rho(C_i)) \cdot \deg C_i \geq \Delta_X(C_i) \cdot \deg \rho(C_i)$.

We write $\Delta(C)$ instead of $\Delta_X(C)$ if it is clear from the context that $C \subset X$. \quad \Box

**Proposition 18.** If $X \subset \mathbb{P}^n$ is a complex surface, then there exists a sectional delta invariant $\Delta_X$.

**Proof.** Let $\mathcal{H}$ denote the set of irreducible and reduced hyperplane sections of $X$. Suppose that $H \subset X$ is a general hyperplane section of $X$. We consider the following
functions for all \(1 \leq i \leq r\) and \(W \in \mathcal{H}\):
\[
\Delta_X(C_i, W) := \sum_{p \in W \cap C_i} \delta_p(W) \quad \text{and} \quad \Delta_X(C_i) := \Delta_X(C_i, H).
\]
It follows from Lemma 14a that \(\text{sgn}\; H = (C_1 \cup \cdots \cup C_r) \cap H\) and thus axiom 1 is a direct consequence of Lemma 14c. Axiom 2 is a direct consequence of Bézout’s theorem.

**Claim 1.** \(\delta_p(H) = \delta_q(H) = \delta_{\sigma(p)}(\sigma(H)) = \delta_{\alpha(p)}(\alpha(H))\) for all \(p, q \in C_i \cap H, 1 \leq i \leq r\), real structures \(\sigma : X \to X\) and \(\alpha \in \text{Aut} \mathbb{P}^n\).

Let \(t_0 x_0 + \cdots + t_n x_n\) with \(t \in \mathbb{P}^n\) be the defining polynomial of the complex hyperplane \(H\). Notice that the ideal of \(C_i \subset \mathbb{P}^n\) is generated by polynomial forms in \(\mathbb{C}[x]\). Instead over the coefficient field \(\mathbb{C}\), let us consider the defining polynomials of the complex curve \(C_i\) and the hyperplane over the function field \(K := \mathbb{C}(t_0, \ldots, t_n)\). As we do not choose any particular value for \(t \in \mathbb{P}^n\), we ensure that \(H\) is general. In the algebraic closure \(\overline{K}\) the Galois group acts transitively on the roots corresponding to the complex points in \(H \cap C_i\). The delta invariant is an algebraic invariant in the sense that it can be computed in the ring \(\overline{K}[x]\) from the defining polynomials. Hence, all algebraic invariants (and in particular the delta invariant) are the same for all complex points in \(C_i \cap H\). Since both \(\sigma\) and \(\alpha\) act on \(\overline{K}[x]\) we find that an algebraic invariant of \(p\) is equal to \(\sigma(p)\) and \(\alpha(p)\) for all \(p \in X\).

Axiom 3 is a direct consequence of claim 1. It follows from claim 1 and Lemma 14d that \(\Delta_X(C_i) \leq \Delta_X(C_i, W)\) for all \(W \in \mathcal{H}\) and \(1 \leq i \leq r\). Hence, by axiom 1, \(\Delta_X(C_i)\) does not depend on the choice of a general hyperplane section \(H\) and is therefore well-defined.

We now proceed with the proof for axiom 4. Let \(H' \subset Z\) be a general hyperplane section and let \(W := \rho^{-1}(H')\). We already established that \(H'\) (like \(H \subset X\)) must be irreducible and thus \(W \in \mathcal{H}\). However, \(W\) is not necessarily general as the hyperplane spanned by \(W\) passes through the center of the linear projection \(\rho\). Let \(U_p \subset X\) be an arbitrary small complex analytic neighborhood of \(p \in W \cap C_i\) for any \(1 \leq i \leq r\). Since \(\rho\) is birational, it is defined at \(W \cap C_i\) and thus restricts to a complex analytic isomorphism \(U_p \to \rho(U_p)\). The delta invariant \(\delta_p(W)\) is a complex analytic invariant by [6, Exercise IV.1.8c] and thus \(\delta_p(W) = \delta_{\rho(p)}(H')\). We deduce from claim 1 applied to \(H' \subset Z\) and Bézout’s theorem that \(\Delta_Z(\rho(C_i)) = \deg(\rho(C_i)) \cdot \delta_{\rho(p)}(H')\). Again by Bézout’s theorem it follows that \(\Delta_X(C_i, W) = \deg(C_i) \cdot \delta_p(W)\).
Recall that $\Delta_X(C_i, W) \geq \Delta_X(C_i)$ and thus we established that
\[
\frac{\Delta_Z(\rho(C_i))}{\deg \rho(C_i)} = \frac{\Delta_X(C_i, W)}{\deg C_i} \geq \frac{\Delta_X(C_i)}{\deg C_i}.
\]
However, since $g(X) = g(Z)$ by assumption, we must by Lemma 14d have the equality $\Delta_X(C_i, W) = \Delta_X(C_i)$ and thus we conclude that axiom 4 holds.

The proof of axiom 5 follows the proof of axiom 4. Again the finite morphism $\rho$ is defined at $p \in W \cap C_i$. In this case however, $\rho$ defines a complex analytic isomorphism $U_{p'} \rightarrow \rho(U_{p'})$ for each point $p'$ in the fibre $(\rho^{-1} \circ \rho)(p)$. In other words, the $q : 1$ covering $\rho$ defines locally a complex analytic isomorphism on each of its $q$ sheets. The remaining arguments are the same as for axiom 4, except we do not need to prove the equality. This concludes the proof for the only remaining axiom 5. □

**Remark 19.** We will not use axiom 4 in this article, but we believe that notion of sectional delta invariant in Definition 17 is of interest outside the scope of this article. We conjecture that the $g(X) = g(Z)$ assumption in axiom 4 can be omitted and that the inequality in axiom 5 can be replaced by an equality. ◁

## 5 Singular components via central projection

The central projection of a great 2-circled surface $X \subset S^3$ of degree eight is a surface in $\mathbb{P}^3$ of degree four. We will show that the intersection of this quartic surface with the branching locus consist of four complex double lines. We then argue that the ramification locus $X \cap E$ must be a union of two double left generators and two double right generators. This will allow us to recover the complete singular locus of $X$ by using the sectional delta invariant. Since the right Clifford translations leave the two double right generators in $X$ invariant, we then obtain an essential ingredient for the proof of Corollary 3, namely, Proposition 31.

The central projection $\tau: S^3 \rightarrow \mathbb{P}^3$ sends $(x_0 : \ldots : x_4)$ to $(x_1 : x_2 : x_3 : x_4)$. Thus $\tau$ is a 2:1 linear projection with ramification locus $E$ and branching locus $\tau(E) = \{y \in \mathbb{P}^3 | y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0\}$. We will call the central projection of a left generator or right generator into the branching locus $\tau(E)$, also left generator and right generator, respectively.

Notice that a fiber of the central projection $S^3 \rightarrow \mathbb{R}^3$ induced by $\tau$ consists of antipodal points of $S^3$ and that great circles are send to lines.

**Notation 20.** We denote by $\mathcal{X}$ a great 2-circled surface of degree 8 in $S^3$. ◁
Lemma 21. The central projection $\tau(\mathcal{X})$ is of degree 4 and covered by a pencil of lines and a pencil of conics. Moreover, $\mathcal{X}$ is covered by a pencil of great circles and a pencil of small circles.

Proof. The degree of a $q : 1$ linear projection of a surface $X \subset \mathbb{P}^n$ is equal to $(1/q) \cdot \deg X$. Since $X$ is covered by great circles that are centrally projected to lines it follows that $\deg \tau(\mathcal{X}) = 4$. Recall from Proposition 8 that $\mathcal{X}$ is covered by two pencils of circles. This concludes the proof as $\mathcal{X}$ is not covered by two pencils of great circles, because $\tau(\mathcal{X})$ would in this case be a doubly ruled quadric. \hfill $\square$

Lemma 22. We have $N(\tau(\mathcal{X})) \cong \langle \ell_0, \ell_1 \rangle \mathbb{Z}$, where $\sigma_* = \text{id}$, $h = 2 \ell_0 + \ell_1$ is the class of a hyperplane section, $k = -2(\ell_0 + \ell_1)$ is the canonical class and $\delta(\tau(\mathcal{X})) = 3$. The class of a line or irreducible conic in $\tau(\mathcal{X})$ that is not contained in $\text{sng} \tau(X)$ is equal to $\ell_0$ and $\ell_1$, respectively.

Proof. Recall from Lemma 21 that $\tau(\mathcal{X})$ is covered by pencils of lines and conics and thus a rational surface by Noether’s theorem. More precisely, $\tau(\mathcal{X})$ is a geometrically ruled surface over $\mathbb{P}^1$. If follows from [1, Proposition III.18] that $N(\tau(X)) \cong \langle h, \ell_0 \rangle \mathbb{Z}$, where $h^2 = \deg \tau(\mathcal{X}) = 4$, $h \cdot \ell_0 = 1$, $k = -2 h + 2 \ell_0$ and $\ell_0$ is the class of the fiber. We set $\ell_1 := h - 2 \ell_0$ so that $\ell_1$ is the class of a conic and thus $N(\tau(X))$ is as asserted. We conclude from Lemma 15 that $\delta(\tau(\mathcal{X})) = 3$. \hfill $\square$

Lemma 23. There exists a possibly reducible curve $F \subset \tau(\mathcal{X}) \cap \tau(\mathbb{E})$ such that the complex tangent plane of $\tau(\mathcal{X})$ at a general complex point $p \in F$ is spanned by the complex tangent vector of $F$ at $p$ and the complex tangent vector of an irreducible complex conic in $\tau(\mathcal{X})$ that passes through $p$.

Proof. We know from Lemma 21 that $\mathcal{X}$ is covered by a pencil of small circles. Suppose that $B \subset \mathcal{X}$ is a general small circle in this pencil and let $q \in \mathbb{E} \cap B$. Since the elliptic absolute $\mathbb{E}$ does not contain real points, we find that $B$ meets $\mathbb{E}$ in two complex conjugate points by Bézout’s theorem. Thus $B$ meets the possibly reducible curve $\mathbb{E} \cap \mathcal{X}$ transversally at the complex point $q$. We claim that the central projection $\tau(B)$ meets $\tau(\mathcal{X}) \cap \tau(\mathbb{E})$ transversally at $\tau(q)$ as well. Notice that the central projection identifies antipodal circles in $S^3$ via the map $\gamma : \mathbb{S}^2_\mathbb{R} \to S^3$ that was defined in §2. From this observation we deduce that the restricted central projection $\tau|_{\mathcal{X}}$ is a 2:1 covering that defines locally a complex analytic isomorphism between each of the two complex sheets that intersect at the complex ramification locus $\mathbb{E} \cap \mathcal{X}$,
and a neighborhood of $\tau(X)$ around the branching point $\tau(q)$. Thus the central projection $\tau(B)$ meets the branching locus $\tau(E) \cap \tau(X)$ transversally at $\tau(q)$ as was claimed. Because $\tau(B)$ is a conic that meets the hyperquadric $\tau(E)$ in two complex points, it follows from Bézout’s theorem that this conic must meet $\tau(E)$ tangentially. We deduce that the complex tangent lines at $\tau(q)$ of $\tau(X) \cap \tau(E)$ and $\tau(B)$ span a complex tangent plane of both $\tau(X)$ and $\tau(E)$. Recall from Proposition 12a that the pencil of small circles in $X$ is base point free and thus the pencil of conics in $\tau(X)$, that is defined by the central projections of small circles in $X$, is base point free as well. We concluded the proof as $\tau(X)$ must meet $\tau(E)$ tangentially along the (not necessarily irreducible) component $F$ that is traced out by irreducible complex conics in $\tau(X)$.

**Notation 24.** By default we will consider $\tau(X) \cap \tau(E)$ as a scheme-theoretic intersection of $\tau(X)$ with the quadric surface $\tau(E)$. With “scheme-theoretic” we mean that the underlying ring structure is taken into account [6, Exercise 7.12].

**Lemma 25.** The intersection $\tau(X) \cap \tau(E)$ is as a set not bijective to a reducible or irreducible quartic curve $G$ with the property that $[G]$ is equal to the class of hyperplane sections $h \in N(\tau(X))$.

**Proof.** Since $E := \tau(X) \cap \tau(E)$ is a scheme-theoretic intersection and $\tau(E)$ is a hyperquadric, it follows from Lemma 22 that $[E] = 2h$. Now suppose by contradiction that $E$ is as a set bijective to $G$ such that $[G] = h = 2\ell_0 + \ell_1$. The degree of a complex curve $C \subseteq G$ is equal to $h \cdot [C]$, and if $[C] = 2\ell_0$, then $C$ is by Lemma 9 either a complex double line, or the union of two complex lines. Since $G$ does not contain real points, it cannot have a single line as component. It follows that the irreducible components of $G$ consist of either

1. a conic with class $\ell_1$ and two complex conjugate lines with class $\ell_0$,
2. a conic with class $2\ell_0 + \ell_1$ so that $G \subseteq sng \tau(X)$, or
3. an irreducible quartic curve with class $2\ell_0 + \ell_1$.

Suppose by contradiction that case 1 or 2 holds so that $G$ either contains or is equal to a real conic without real points. A general line in $\tau(X)$ has class $\ell_0$ and thus intersects the conic component with class in $\{\ell_1, 2\ell_0 + \ell_1\}$ in a complex point. This line must also pass through the complex conjugate of this point. We arrived at a contradiction as $\tau(X)$ is not a plane.
Finally, suppose by contradiction that case 3 holds. We recall from Lemma 22 that 
$3 = \delta(\tau(\mathcal{X})) < \deg G = 4$ and thus it follows from axioms 1 and 2 at Definition 17
that $G$ is not contained in the singular locus of $\tau(\mathcal{X})$. A general line in $\tau(\mathcal{X})$ meets $G$
in complex conjugate points $p, \overline{p} \in G$. It follows from Lemma 23 that through $p$
passes a complex irreducible conic that intersects $\tau(E)$ tangentially. The line meets $\tau(E)$ at $p$
transversally and thus this line meets the tangent plane of $\tau(\mathcal{X})$ transversally as well. We arrived at a contradiction, since $G \subseteq \text{sng} \tau(\mathcal{X})$.

We concluded the proof as we arrived at a contradiction for each possible case. \qed

**Lemma 26.** The intersection $\tau(\mathcal{X}) \cap \tau(E)$ consists of complex conjugate generators $\tau(L)$ and $\tau(\overline{L})$ such that $[\tau(L)] = [\tau(\overline{L})] = \ell_0$ in $N(\tau(\mathcal{X}))$ and a reducible curve with class $2 \ell_1 \in N(\tau(\mathcal{X}))$.

**Proof.** Let $E := \tau(\mathcal{X}) \cap \tau(E)$ be the scheme-theoretic intersection so that $[E] = 2h = 4 \ell_0 + 2 \ell_1$ in $N(\tau(\mathcal{X}))$ by Lemma 22. It follows from Lemma 23 that $E = F \cup F'$, where $F$ is a possibly reducible curve along which conics in $\tau(X)$ intersect $\tau(E)$ tangentially, and $F'$ is either the remaining component or the empty-set. Let $[F] = \alpha \ell_0 + \beta \ell_1$ for some $\alpha, \beta \in \mathbb{Z}$. Recall that the class of a line and conic is $\ell_0$ and $\ell_1$ in $N(\tau(\mathcal{X}))$, respectively, and thus $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, since a general line and conic in $\tau(X)$ has positive intersection product with $E$ so that $[F] \cdot \ell_0 \geq 0$ and $[F] \cdot \ell_1 \geq 0$. Moreover, as $[F'] \cdot \ell_1 = 0$ and $[F'] \cdot \ell_0 \geq 0$ we find that $[F'] = \gamma \ell_1$ for some $\gamma \in \mathbb{Z}_{\geq 0}$. As $F$ contributes with multiplicity two to $E$ we find that $[E] = 2[F] + [F'] = 2(\alpha \ell_0 + \beta \ell_1) + \gamma \ell_1 = 4 \ell_0 + 2 \ell_1$ and thus $(\alpha, \beta, \gamma) \in \{(2, 1, 0), (2, 0, 2)\}$. It follows from Lemma 25 that $[F] \neq h$ so that $[F] = 2 \ell_0$ and $[F'] = 2 \ell_1$. We find that $F$ is reducible by Lemma 9 and therefore consists of generators $\tau(L)$ and $\tau(\overline{L})$. The remaining component $F'$ is by Lemma 9 reducible as well. \qed

**Lemma 27.** The intersection $\tau(\mathcal{X}) \cap \tau(E)$ consist of generators $\tau(L)$, $\tau(\overline{L})$, $\tau(R)$ and $\tau(\overline{R})$ such that $[\tau(L)] = [\tau(\overline{L})] = \ell_0$ and $[\tau(R)] = [\tau(\overline{R})] = \ell_1$ in $N(\tau(\mathcal{X}))$.

**Proof.** Let $Q := \tau(E)$ so that $N(Q) = \langle \ell_0, \ell_1 \rangle_{\mathbb{Z}}$, where $\sigma_{\ast} = \text{id}$. Notice that $h_Q = \ell_0 + \ell_1$ is the class of hyperplane sections of $Q$, and that $k_Q = -2(\ell_0 + \ell_1)$ is the canonical class of $Q$. A generator in $Q$ has class either $\ell_0$ or $\ell_1$ in $N(Q)$. Let us consider $E := Q \cap \tau(\mathcal{X})$ as the scheme-theoretic intersection so that $[E] = 4h_Q$.
Recall from Lemma 23 that $E = F \cup F'$ is the reducible curve such that irreducible conics in $\tau(\mathcal{X})$ intersect $Q$ tangentially at $F$, and $F'$ is the remaining component. Therefore, $[E] = 2[F] + [F']$ and we know from Lemma 26 that $[F]$ consist of two
generators that without loss of generality each have class $\ell_0$ in $N(Q)$. We find that $[E] = 2[F] + [F'] = 4\ell_0 + [F']$ so that $[F'] = 4\ell_1$ in $N(Q)$. Notice that $F'$ is reducible by Lemma 9. However, $F'$ does not have an irreducible conic as component, since such a conic in $Q$ has class $h_Q = \ell_0 + \ell_1$.

Now let us consider $[F']$ as a class in $N(\tau(\mathcal{X}))$ instead of $N(Q)$. Recall from Lemma 26 that $[F'] = 2\ell_1$, where $\ell_1 \in N(\tau(\mathcal{X}))$ is the class of an irreducible conic in the linear normalization $\tau(\mathcal{X})_N$. Since $F'$ does not have an irreducible conic as component, we conclude that $F'$ consist of two double lines that are projections of two disjoint irreducible conics in $\tau(\mathcal{X})_N$. This concludes the proof, as these double lines must be generators of $Q$ as asserted.

Lemma 28. We have $\text{sng} \tau(\mathcal{X}) = \tau(R) \cup \tau(\overline{R}) \cup \tau(V)$ and

$$\tau(\mathcal{X}) \cap \tau(\mathcal{E}) = \tau(L) \cup \tau(\overline{L}) \cup \tau(R) \cup \tau(\overline{R})$$

such that the incidences between the components are as in Figure 3 and where

- $\tau(L)$ and $\tau(\overline{L})$ are generators with class $[\tau(L)] = [\tau(\overline{L})] = \ell_0$,
- $\tau(R)$ and $\tau(\overline{R})$ are double generators such that $[\tau(R)] = [\tau(\overline{R})] = \ell_1$ and $\Delta(\tau(R)) = \Delta(\tau(\overline{R})) = 1$,
- $\tau(V)$ is a double line such that $[\tau(V)] \in \{2\ell_0, \ell_0\}$ and $\Delta(\tau(V)) = 1$, and
- if $C \subset \tau(\mathcal{X})$ is a line or irreducible conic such that $C \notin \text{sng} \tau(\mathcal{X})$, then $[C]$ equals $\ell_0$ and $\ell_1$, respectively.

![Figure 3: See Lemma 28.](image)

Proof. The last assertion follows from Lemma 22. Recall from Lemma 27 that $\tau(\mathcal{X}) \cap \tau(\mathcal{E})$ consists of generators $\tau(L), \tau(\overline{L})$ and double generators $\tau(R), \tau(\overline{R})$, such that $[\tau(L)] = [\tau(\overline{L})] = \ell_0$ and $[\tau(R)] = [\tau(\overline{R})] = \ell_1$. Since $\delta(\tau(\mathcal{X})) = 3$ by Lemma 22, it follows from axioms 1 and 3 in Definition 17 that $\Delta(\tau(R)) = \Delta(\tau(\overline{R})) = 1$. Hence, by axiom 2, the remaining component of $\text{sng} \mathcal{X}$ consists of a double line $\tau(V)$ such that $\Delta(\tau(V)) = 1$. 

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Recall from Lemma 22 that $h := 2 \ell_0 + \ell_1$ is the class of any hyperplane section $H \subset \tau(X)$. Thus if $\tau(V) \subset H$, then $[H] = \alpha[\tau(V)] + \beta[H \setminus \tau(V)] = 2\ell_0 + \ell_1$ for some $\alpha, \beta > 0$. Therefore we deduce that $[\tau(V)] \in \{2\ell_0, \ell_0, \ell_1, \ell_0 + \ell_1\}$.

Suppose by contradiction that $[\tau(V)] = \ell_1$. If $\tau(V) \cap \tau(R) \neq \emptyset$, then $\tau(V) \cup \tau(R)$ forms a hyperplane section by Bézout’s theorem. Since $[\tau(V) \cup \tau(R)] = \alpha \ell_1 \neq h$ for any $\alpha > 0$, it follows that $\tau(V) \cap \tau(R) = \emptyset$. Moreover, $\tau(R)$, $\tau(\overline{R})$ and $\tau(V)$ form three skew double lines each with class $\ell_1$. Recall that each line in $\tau(\mathcal{X})$ has class $\ell_0$ and is a member of the pencil of lines that covers $\tau(\mathcal{X})$. Each line in this pencil meets the three skew lines with class $\ell_1$. We arrived at a contradiction, since $\tau(\mathcal{X})$ is a not a doubly ruled quadric.

Suppose by contradiction that $[\tau(V)] = \ell_0 + \ell_1$. The double line $\tau(V)$ meets in this case the double line $\tau(R)$, because $[\tau(R)] = \ell_1$. Thus $\tau(V) \cup \tau(R)$ forms a hyperplane section by Bézout’s theorem so that $[\tau(V) \cup \tau(R)] = h$. This is a contradiction, since $[\tau(V) \cup \tau(R)] = \alpha(\ell_0 + \ell_1) + \beta \ell_1 \neq h$ for all $\alpha, \beta > 0$.

Suppose by contradiction $\tau(V) \cap \tau(L) \neq \emptyset$. In this case $\tau(V) \cup \tau(L) \cup \tau(L')$ for some line $\tau(L')$ is a hyperplane section of $\tau(\mathcal{X})$ by Bézout’s theorem. Thus $[\tau(V) \cup \tau(L) \cup \tau(L')] = \alpha[\tau(V)] + \beta[\tau(L)] + \gamma[\tau(L')] = h$ for some $\alpha, \beta, \gamma > 0$, where $[\tau(L)] = [\tau(L')] = \ell_0$ by assumption. We arrived at a contradiction, since we already established that $[\tau(V)] \neq \ell_1$.

This concludes the proof as the main assertion is the only remaining possibility. 

**Proposition 29.** We have $\text{sng} \mathcal{X} = L \cup \overline{L} \cup R \cup \overline{R} \cup V$ such that the incidences between the components are as in Figure 4 and where

- $L, \overline{L} \subset \mathbb{E}$ are complex conjugate double generators such that $[L] = [\overline{L}] = \ell_0$ and $\Delta(L) = \Delta(\overline{L}) = 1$,

- $R, \overline{R} \subset \mathbb{E}$ are complex conjugate double generators such that $[R] = [\overline{R}] = \ell_1$ and $\Delta(R) = \Delta(\overline{R}) = 2$,

- $V$ is a great double circle such that $[V] \in \{2\ell_0, \ell_0\}$ and $\Delta(V) = 2$, and

- the great circles and small circles in $\mathcal{X} \setminus \text{sng} \mathcal{X}$ have class $\ell_0$ and $\ell_1$, respectively.

Moreover, if $H \subset \mathbb{P}^4$ is a hyperplane such that $V \subset H$, then $H \cap \mathcal{X} = V \cup C \cup C'$, where $C(\mathbb{R})$ and $C'(\mathbb{R})$ are antipodal small circles in the sphere $H(\mathbb{R}) \cap S^3$. 

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Proof. Recall from Proposition 8 that $N(\mathcal{X}) \cong \langle \ell_0, \ell_1 \rangle_Z$, $h = -k = 2(\ell_0 + \ell_1)$ and the class of a circle is in $\{\ell_0, \ell_1\}$. It follows from Lemma 28 that $\mathcal{X} \cap \mathcal{E}$ consist of left (or right) generators $L, \overline{L}$ and right (or left) generators $R, \overline{R}$. By Proposition 12 these generators must be non-real lines in $\text{sng} \mathcal{X}$ and thus projections of complex irreducible conics in the linear normalization $X_N \subset \mathbb{P}^8$. Since the central projection $\tau(V)$ is a line it follows that $V$ is a great double circle.

We may assume without loss of generality that $\ell_0$ and $\ell_1$ are the classes of a great circle and small circle, respectively. It follows from Lemma 22 that the planar sections of $\tau(\mathcal{X})$ that contain the double line $\tau(V)$ defines a pencil of conics. The preimage $H \subset \mathbb{S}^3$ of such a plane is a 2-sphere that contains $V$ and two little circles that are centrally projected 2:1 to a conic in $\tau(\mathcal{X})$. Since $|V \cap L| = |V \cap \overline{L}| = 0$ and $|V \cap R| = |V \cap \overline{R}| = 1$ as a direct consequence of Lemma 28.

Remark 30. Notice that in Proposition 29, either $L, \overline{L} \subset \mathcal{E}$ are both left generators and $R, \overline{R} \subset \mathcal{E}$ are both right generators, or vice versa.

Proposition 31. If $\mathcal{X}(\mathbb{R})$ contains the identity quaternion $1 \in \mathbb{S}^3$, then $\mathcal{X}(\mathbb{R}) \in \{A(\mathbb{R}) \star B(\mathbb{R}), B(\mathbb{R}) \star A(\mathbb{R})\}$, where $A \subset \mathcal{X}$ is a great circle and $B \subset \mathcal{X}$ is a small circle such that $A(\mathbb{R}) \cap B(\mathbb{R}) = \{1\}$.

Proof. We know from Proposition 29 that the great circles in $\mathcal{X}$ meet the elliptic
absolute $\mathbb{E}$ at the double generators $R$ and $\overline{R}$. Recall from Remark 30 that these generators are either both left or both right.

First suppose that $R$ and $\overline{R}$ are both right generators. Let $F$ be the family of great circles $F_b$ indexed by $b \in B(\mathbb{R})$ such that $F_b(\mathbb{R}) := A(\mathbb{R}) \star \{b\}$. It follows from Proposition 5 that $F_b \cap R \neq \emptyset$ and $F_b \cap \overline{R} \neq \emptyset$ for all $b \in B(\mathbb{R})$. Suppose by contradiction that the family $F$ does not cover $\mathcal{X}$. In this case there exists $b \in B(\mathbb{R})$ such that the great circle $F_b$ is not contained in $\mathcal{X}$. Let $C \subset \mathcal{X}$ be the great circle such that $b \in C(\mathbb{R})$. Observe that $b \in F_b(\mathbb{R}) \cap B(\mathbb{R})$, since $1 \in A(\mathbb{R})$ by assumption. The incidence relations for the current scenario are schematically depicted in Figure 5, where $\gamma(\beta) = b$ and $\gamma(\varepsilon) = 1$ (see §2 for $\gamma: S^3(\mathbb{R}) \to S^3$). The central projections $\tau(F_b)$ and $\tau(C)$ are lines that meet the right generators $\tau(R)$ and $\tau(\overline{R})$ in the complex doubly ruled quadric $\tau(\mathbb{E})$ such that $\tau(F_b) \cap \tau(C) = \{\tau(\beta)\}$. We arrived at a contradiction as $\tau(F_b)$ and $\tau(C)$ span a plane so that $\tau(R)$ and $\tau(\overline{R})$ cannot be skew. We conclude that $F$ covers $\mathcal{X}(\mathbb{R})$ so that $\mathcal{X}(\mathbb{R}) = A(\mathbb{R}) \star B(\mathbb{R})$.

Finally, suppose that $R$ and $\overline{R}$ are both left generators. In this case the proof is the same except that we define $F_b(\mathbb{R})$ to be $\{b\} \star A(\mathbb{R})$ so that $\mathcal{X}(\mathbb{R}) = B(\mathbb{R}) \star A(\mathbb{R})$. □

Figure 5: See the proof of Proposition 31.

6 Shapes

In this section we classify the possible shapes of $\mathcal{X}(\mathbb{R})$, where $\mathcal{X} \subset S^3$ is defined at Notation 20. The real points $(\mathcal{X}_N)_R$ of the linear normalization $\mathcal{X}_N \subset \mathbb{P}^8$ is homeomorphic to a topological torus by the Poincaré-Hopf theorem, as a family of conics in $\mathcal{X}_N(\mathbb{R})$ defines a non-vanishing vector field. The shape of $\mathcal{X}(\mathbb{R})$ can be recovered as a linear projection of this topological torus in $\mathbb{P}^8(\mathbb{R})$.

Notation 32. In this section we shall denote by $V$ the great double circle in $\mathcal{X}$ (see Proposition 29) and we suppose that $P \subset \mathcal{X} \times \mathbb{P}^1$ is the pencil of small circles that
covers $\mathcal{X}$ (see Lemma 21). Recall that points, lines and curves (such as conics) are real by default.

Lemma 33. There exists a birational morphism $\xi: Q \to \mathcal{X}$ from a doubly ruled quadric $Q \subset \mathbb{P}^3$. If $L \subset Q$ is a line, then $\xi(L)$ is a circle and $L$ is bijective to $\xi(L)$.

Proof. It follows from Proposition 11 that the linear normalization $\mathcal{X}_N \subset \mathbb{P}^8$ of $\mathcal{X}$ is biregular isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Recall from Remark 10 that there exists a degree preserving linear projection $\eta: \mathcal{X}_N \to \mathcal{X}$. We can identify $\mathcal{X}_N$ with a doubly ruled quadric $Q \subset \mathbb{P}^3$ such that lines in $Q$ correspond to irreducible conics in $\mathcal{X}_N$. The composition of the isomorphism $Q \cong \mathcal{X}_N$ with $\eta$ defines a birational morphism $\xi: Q \to \mathcal{X}$. If $L \subset Q$ is a line, then $\xi(L)$ is the projection of an irreducible conic in $\mathcal{X}_N$ and thus either a circle, a double line or an isolated singularity. It follows from Proposition 12 that $\xi(L)$ can only be a circle. We concluded the proof as no two points in $L$ are send via $\xi$ to the same point in $\mathcal{X}$.

Lemma 34.

a) There exists $q \in \{0, 1, 2\}$ such that $|P_i(\mathbb{R}) \cap V(\mathbb{R})| = q$ for all $i \in \mathbb{P}^1_{\mathbb{R}}$.

b) If $G \subset \mathcal{X}$ is a great circle, then either $G = V$ or $G \cap V = \emptyset$.

Proof. Let $\xi: Q \to \mathcal{X}$ be defined as in Lemma 33 and let us call a complex line $L \subset Q$ great or small if $\xi(L)$ is a great and small circle, respectively. We deduce from Lemma 33 that $Q_\mathbb{R}$ is covered by a pencil of small lines, and a pencil of great lines. It follows from Proposition 29 that the $\xi$-preimage of $V$ is either one great line or two complex great lines. Hence, if $L \subset Q$ is a small line, then $|\xi^{-1}(V_\mathbb{R}) \cap L| \leq 2$, which implies that $q := |V_\mathbb{R} \cap \xi(L)| \leq 2$ by Lemma 33. In particular, we deduce that $q$ does not depend on the choice of $L$. We concluded the proof for assertion a) as $q = |V(\mathbb{R}) \cap \xi(L)| = |V(\mathbb{R}) \cap P_i(\mathbb{R})|$ for all $i \in \mathbb{P}^1_{\mathbb{R}}$. Assertion b) is a direct consequence of Lemma 33 and the fact that the complex great lines form a single pencil of lines on $Q$.

Remark 35. If $C \subset \mathcal{X}$ is any small circle, then $\llbracket C \rrbracket = \ell_1$ and $\llbracket V \rrbracket \in \{\ell_0, 2\ell_0\}$ by Proposition 29. Let $\xi: Q \to \mathcal{X}$ be the birational morphism as defined at Lemma 33. It follows from Proposition 11 that $\xi$ defines a smooth model, and thus $C \cap V = \llbracket C \rrbracket \cdot \llbracket V \rrbracket$ by Lemma 33. If $|C(\mathbb{R}) \cap V(\mathbb{R})| \in \{0, 2\}$, then $|C \cap V| = 2$ so that $\llbracket V \rrbracket = 2\ell_0$. If $|C(\mathbb{R}) \cap V(\mathbb{R})| = 1$, then $|C \cap V| = 1$ and thus $\llbracket V \rrbracket = \ell_0$. \hspace{1cm} $\triangle$
Example 36. Suppose that the center of stereographic projection $\pi : S^3 \rightarrow \mathbb{R}^3$ lies in $V(\mathbb{R})$. In Figure 6 (left/middle/right) we depict $\pi(\mathcal{X}(\mathbb{R}))$ in case $|P_i(\mathbb{R}) \cap V(\mathbb{R})|$ is equal to 2, 1 and 0, respectively, for all $i \in \mathbb{P}^1_{\mathbb{R}}$. It follows from Proposition 29 that if $\pi(H(\mathbb{R})) \subset \mathbb{R}^3$ is a hyperplane that contains the double line $\pi(V(\mathbb{R}))$, then $\pi(H(\mathbb{R})) \cap \pi(\mathcal{X}(\mathbb{R})) = \pi(V(\mathbb{R})) \cup \pi(C(\mathbb{R})) \cap \pi(C'(\mathbb{R}))$ where $C(\mathbb{R})$ and $C'(\mathbb{R})$ are small circles in $\mathcal{X}(\mathbb{R})$. The circles $\pi(C(\mathbb{R}))$ and $\pi(C'(\mathbb{R}))$ as depicted in Figure 6 meet the double line either in two points, tangentially at one point, or in zero points. Notice the analogy of these three types of great 2-circled surfaces with the spindle, horn and ring cyclide, respectively.

Figure 6

Proposition 37. If $|P_i(\mathbb{R}) \cap V(\mathbb{R})| = 0$ for all $i \in \mathbb{P}^1_{\mathbb{R}}$, then $\mathcal{X}(\mathbb{R})$ is of great type III.

Proof. Since $P_i \cap V$ consists of two complex conjugate points for all $i \in \mathbb{P}^1_{\mathbb{R}}$, we find that the great double circle $V(\mathbb{R})$ is isolated. The great circles define a non-vanishing vector field and thus $T := \mathcal{X}(\mathbb{R}) \setminus V(\mathbb{R})$ has Euler characteristic 0 by the Poincaré-Hopf theorem. We deduce that $T$ must be a topological torus. Recall from Proposition 29 that the hyperplane sections of $\mathcal{X}(\mathbb{R})$ containing $V(\mathbb{R})$ consist additionally of two small circles. Thus if the center of the stereographic projection $\pi$ lies on $V(\mathbb{R})$, then the line $\pi(V(\mathbb{R}))$ passes through the hole of the topological torus $\pi(T)$ as is illustrated in Figure 6 (right). Hence, $V(\mathbb{R})$ and $T$ are linked and thus $\mathcal{X}(\mathbb{R})$ is of great type III by Lemma 34b.

Proposition 38. If $|P_i(\mathbb{R}) \cap V(\mathbb{R})| = 1$ for all $i \in \mathbb{P}^1_{\mathbb{R}}$, then $\mathcal{X}(\mathbb{R})$ is of great type II.

Proof. We know from Proposition 29 that the hyperplane sections of $\mathcal{X}(\mathbb{R})$ containing the great double circle $V(\mathbb{R})$ consist additionally of two antipodal small circles. Thus if the center of the stereographic projection $\pi$ lies on $V(\mathbb{R})$, then a hyperplane in $\mathbb{R}^3$ containing the line $L = \pi(V(\mathbb{R}))$ contains additionally two circles $C := \pi(P_u(\mathbb{R}))$ and $C' := \pi(P_v(\mathbb{R}))$ for some $u, v \in \mathbb{P}^1_{\mathbb{R}}$. Hence $|L \cap C| = |L \cap C'| = 1$ as is illustrated in Figure 6 (middle) so that $P_i(\mathbb{R})$ must intersect $V(\mathbb{R})$ tangentially.
for all $i \in \mathbb{P}_R^1$. We know from Remark 10 and Proposition 11 that there exists a linear projection $\eta: \mathbb{P}^8 \rightarrow \mathbb{P}^4$ such that $\eta(\mathcal{X}_N) = \mathcal{X}$. Recall from Remark 35 that $[V] = \ell_0$ and thus the double circle $V$ is the 1:1 linear projection of a conic $V_N \subset \mathcal{X}_N$. This implies that the $\eta$-preimage of a hyperplane in $\mathbb{P}^4$ is a hyperplane in $\mathbb{P}^8$ that meets the linear normalization $\mathcal{X}_N$ tangentially at $V_N$. Therefore, the hyperplane sections of $\mathcal{X}$ have a cuspidal singularity at $V$. The small circles define a non-vanishing vector field and thus $\mathcal{X}(\mathbb{R})$ has Euler characteristic 0 by the Poincaré-Hopf theorem, which implies that $\mathcal{X}(\mathbb{R})$ must be a topological torus. We conclude from Lemma 34b that $\mathcal{X}(\mathbb{R})$ is of great type II.

**Proposition 39.** If $|P_i(\mathbb{R}) \cap V(\mathbb{R})| = 2$ for all $i \in \mathbb{P}_R^1$, then $\mathcal{X}(\mathbb{R})$ is of great type I.

**Proof.** Let $Z \subset \mathbb{R}^3$ denote the stereographic projection of $\mathcal{X}(\mathbb{R}) \subset S^3$. We may assume without loss of generality that the center of stereographic projection does not lie on $\mathcal{X}(\mathbb{R})$ so that $\deg Z = 8$. We shall refer to the stereographic projections of great circles and small circles in $\mathcal{X}(\mathbb{R})$ as red circles and blue circles, respectively. Figure 7 illustrates an example of how red and blue circles cover $Z$. We know from Proposition 29 and Remark 35 that the great double circle $V \subset \mathcal{X}$ has class $[V] = 2 \ell_0$. The great and small circles have class $\ell_0$ and $\ell_1$, respectively. By Lemma 34b a general red circle does not meet the double red circle $C := \pi(V(\mathbb{R}))$. By assumption each blue circle meets $C$ in two points.

![Figure 7](image)

Let us consider a red circle $A \subset S^3$ that moves in its pencil. First let $A$ coincide with $C$. As we move $A$ we trace out first a manifold $T$, then $A$ coincides again with $C$ and finally $A$ traces out a second manifold $T'$ until $A$ coincides again with its starting position $C$. Notice that $Z = T \cup T'$ and $C = T \cap T'$. The tangent vectors of the red circles define a non-vanishing vector field on both $T$ and $T'$. It now follows
from the Poincaré-hopf theorem that $T$ and $T'$ each have Euler characteristic 0. From this we deduce that $T$ and $T'$ must be topological tori.

A loop on a topological torus in $\mathbb{R}^3$ has *winding pair* $(w_1, w_2) \in \mathbb{Z}_2^2$, if the loop winds $w_1$ times around the “hole” and $w_2$ times around the “tube” of this torus. For example, the latitudinal, longitudinal and Villarceau circles on a ring torus have winding pair $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively.

Suppose that $C$ has winding pair $(\alpha, \beta)$ with respect to the torus $T$. Notice that each red circle in the torus $T$ is homeotopic to $C$ and thus has winding pair $(\alpha, \beta)$. We established before that the pencil of red circles defines a non-vanishing vector field on $T$ and thus $(\alpha, \beta) \neq (0, 0)$. The topological torus $T$ is covered by a family of blue arcs such that each arc meets $C$ in two different points. Let us consider a blue circle $B$ that moves in its pencil as is illustrated in Figure 8 (left) so that the blue arc $B \cap T$ traces out the torus $T$. As each the trajectory of each point on the arc $B \cap T$ traces out a red circle we deduce that $\alpha, \beta \leq 1$. Therefore we find that $(\alpha, \beta) \in \{(1, 0), (0, 1), (1, 1)\}$.

![Figure 8](image)

Suppose by contradiction that $(\alpha, \beta) = (0, 1)$. We illustrated in Figure 8 (right) the red double circle $C \subset T$ and a blue arc $B \cap T$ that is the intersection of any blue circle $B$ with the torus $T$. Each red circle on $T$ has also winding pair $(0, 1)$ and thus does not go around the hole of the torus. From this we deduce that the blue arc $B \cap T$ must go around the hole of the torus. Hence, the blue arc meets the spanning plane $H$ of $C$ at the points $p \in C$ and $q \notin C$. By Bézout’s theorem $|B \cap H| \leq 2$ and thus $B \cap C = \{p\}$. We arrived at a contradiction as the two end points of the blue arc must lie in $C$.

Suppose by contradiction that $(\alpha, \beta) = (1, 0)$. Stereographic projections of $\mathcal{X}(\mathbb{R})$ into $\mathbb{R}^3$ from different centers in $S^3$ are related by inversions. By placing the center
of an inversion inside $T$ we can turn this torus inside out while sending longitudinal
circles to latitudinal circles and vice versa. An animation of this fact can be found
on the Wikipedia page for “Torus”. Thus $(\alpha, \beta) = (0, 1)$ up to inversions so that we
arrive at a contradiction as before.

We conclude that $T \cap T'$ has winding pair $(1, 1)$ on $T$ and thus also winding pair
$(1, 1)$ on $T'$. Moreover, we established that $\mathcal{X}(\mathbb{R})$ is of great type I.

\textbf{Proof of Theorem 1.} We know that $d \in \{2, 4, 8\}$ by [9, Theorem 1]. If $d = 4$, then
the assertion follows from [10, Theorem 2]. Finally, suppose that $d = 8$ so that $Z =
\mathcal{X}(\mathbb{R})$ by Proposition 12. In this case Lemma 34a reduces the proof to three cases
that are treated by Proposition 37, Proposition 38 and Proposition 39. For proving
the remaining assertion of Theorem 1, we identify $S^3$ with the unit-quaternions.

\textbf{Proof of Corollary 3.} By assumption, $Z \subset S^3$ is a great $\lambda$-circled surface such that
$\lambda \geq 2$. If $Z$ is a sphere, Blum cyclide, EO cyclide or spindle cyclide, then $Z$
contains six concurrent circles. If $\deg Z = 8$, then $Z = A \star B$ for some circles $A, B \subset S^3$
by Theorem 1. Now suppose that $Z = X(\mathbb{R})$, where $X \subset S^3$ is a great ring cyclide
(recall Definition 6). We know from [10, Example 1 and Figure 4] that $|\text{sng } X| = 4$
and $|\text{sng } X_{\mathbb{R}}| = 0$. Moreover, $X$ contains two pairs of complex conjugate lines,
and these lines meet at the four non-real isolated singularities. Recall that the
central projection $\tau: S^3 \to \mathbb{P}^3$ is a 2:1 covering with branching locus $\mathbb{E}$ so that
great circles are send to lines. Hence, $\tau(X)$ is a ruled quadric and $|\text{sng } \tau(X)| \leq 1$.
Suppose by contradiction that $p \in \text{sng } \tau(X)$. In this case $p \in \tau(X)_{\mathbb{R}}$ and thus the
fiber $\tau^{-1}(p) = \{q, r\}$ spans a real line $U \subset \mathbb{P}^4$ that contains $(1 : 0 : 0 : 0 : 0)$. But
this implies that $\gamma(U \cap S^3_{\mathbb{R}})$ consist of two antipodal points in $S^3$. These points
must be real and thus $q, r \in \text{sng } X_{\mathbb{R}}$. We arrived at a contradiction and thus $\text{sng } X \subset \mathbb{E}$,
which implies that $\tau(X)$ is smooth. Moreover, it follows from Bézout’s theorem that
$X \cap \mathbb{E}$ consist of two left generators and two right generators. Since $\tau(X)$ is a doubly
ruled quadric, we deduce that $X$ is covered by great circles that meet each of the
complex conjugate right generators $R, \overline{R} \subset X \cap \mathbb{E}$ in complex conjugate points. We
may assume without loss of generality that $X(\mathbb{R})$ contains the identity quaternion (see the proof of Theorem 1). We now apply the proof of Proposition 31, but with $X$ instead of $\mathcal{X}$, and establish that $Z = A \star B$ for some circles $A, B \subset S^3$. We concluded the proof of the main assertion as it follows from Theorem 1 that we considered all possible cases for $Z$. 

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