New class of g-modes and unexpected convection in neutron stars

Mikhail E. Gusakov$^{1}$ and Elena M. Kantor$^{1,2}$

$^1$Ioffe Physical Technical Institute, Polytekhnicheskaya 26, 194021 St.-Petersburg, Russia
$^2$St.-Petersburg State Polytechnical University, Polytekhnicheskaya 29, 19521 St.-Petersburg, Russia

We suggest a specific new class of low-frequency g-modes in superfluid neutron stars. We determine the Brunt-Väisälä frequency for these modes and demonstrate that they can be unstable with respect to convection. The criterion for the instability onset (analogue of the well known Schwarzschild criterion) is derived. It is very sensitive to equation of state and a model of nucleon superfluidity. In particular, convection may occur for both positive and negative temperature gradients. Our results have interesting implications for neutron star cooling and seismology.

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Introduction. — This Letter is devoted to gravity oscillation modes (g-modes) and the related phenomenon of convection in neutron stars (NSs). The restoring force for g-modes is buoyancy that originates from the dependence of the pressure on at least two quantities (e.g., density and temperature or density and chemical composition). g-modes and convection are actively studied in laboratory experiments (e.g., [1–3]) and are widespread in nature. For instance, g-modes are observed in Earth atmosphere and ocean, in white dwarfs [4], in slowly pulsating B-stars [5], and in other objects [6], while convection is typical for most of the stars (including the Sun).

In application to NSs, g-modes were studied, e.g., in Refs. [7–12]. In all these works the NS matter was assumed to be nonsuperfluid (normal). However, according to microscopic calculations [13, 14], baryons in the internal layers of NSs become superfluid (SFL) at temperatures $T \lesssim 10^8 \pm 10^{10}$ K which has a drastic impact on stellar dynamics and evolution [14, 15]. Recent real-time observations [16] of the cooling NS in Cas A supernova remnant indicate that this NS has an SFL core [17, 18].

A number of attempts [19–21] have been made to theoretically predict g-modes in cold SFL NSs, but they have failed. This led to a general belief that g-modes do not exist in SFL interiors of NSs or, more precisely, their frequencies $\omega$ are all degenerate at zero. In this Letter we show that proper account of finite temperature effects extracts g-modes from the zero frequency domain. Moreover, these modes can be unstable with respect to convection. Possible applications of these results are outlined. Below the Planck constant, the speed of light, and the Boltzmann constant equal unity, $\hbar = c = k_B = 1$.

Convection in NSs and SFL g-modes. — For simplicity, consider npe NS cores, composed of neutrons ($n$), protons ($p$), and electrons ($e$). To start with, assume that all particles are nonsuperfluid. Any thermodynamic quantity in npe-matter (e.g., the heat function $w = e + P$, where $e$ is the energy density and $P$ is the pressure) can be presented as a function of 3 variables, say, $P$, $x_n \equiv n_n/n_b$, and $x_S \equiv S/n_b$. Here $n_i$ is the number density for particles $i = n, p, e$; $n_b$ is the baryon number density; $S$ is the entropy density. What is the local criterion for the absence of convection in npe-matter? It is easy to derive it in the same manner as it was done in Refs. [22, 23] (see also [11]). Assume that a spherically symmetric star is in hydrostatic equilibrium (but not necessarily in thermal or beta-equilibrium), that is $\nabla P = -w \nabla \phi$, where $\phi(r)$ is the gravitational potential and $r$ is the radial coordinate. Here and below $\nabla \equiv d/dr$ because all quantities of interest depend on $r$ only. Consider two close points 1 and 2 with $r = r_1$ and $r_2$. Let $A_1$ and $A_2$ be the values of some thermodynamic quantity $A$ at points 1 and 2, respectively, and $\Delta A \equiv A_2 - A_1$. Let us displace adiabatically a small fluid element upward from point 1 to point 2. At point 2 the pressure of the fluid element adjusts itself to the surrounding pressure $P_2 = P_1 + \Delta P$, while $x_e$ and $x_S$ remain unchanged and equal to $x_{e1}$ and $x_{S1}$ (we assume that beta-processes are slow). The matter is stable against convection if the inertial mass density of the lifted element is larger than the equilibrium density at point 2. For relativistic matter the role of the inertial mass density of the lifted element is much smaller than the first one and can be neglected. The stability requires that $w(P_2, x_{e2}, x_{S2}) < w(P_2, x_{e1}, x_{S1})$. Expanding $w$ in Taylor series near point 1, we obtain

$$\partial_{x_e} w(P, x_e, x_S) \nabla x_e + \partial_A w(P, x_e, x_S) \nabla x_S < 0,$$

(1)

where $\partial_A \equiv \partial/\partial A$. When $w$ is a function of $P$ and $x_S$ only, Eq. (1) immediately reproduces the textbook criterion for the absence of convection (see, e.g., [22]). In a strongly degenerate matter the second term in Eq. (1) is much smaller than the first one and can be neglected. Similarly, to calculate the first term in Eq. (1) it is sufficient to set $T = 0$ and $x_S = 0$. Then, Eq. (1) reduces to $\partial_{x_e} w(P, x_e) \nabla x_e < 0$. This Ledoux-type criterion is always satisfied in beta-equilibrated NSs, i.e., they are stable against convection. Oscillations of such a matter near equilibrium correspond to temperature-independent composition g-modes, first studied in Ref. [10].

Assume now that neutrons (and possibly protons) are SFL. What will be the analogue of criterion (1) in that case? Nucleon SFL leads to the appearance of two independent velocity fields: SFL neutron velocity $V_{sn}$ and velocity of normal liquid component $V_{ql}$, composed of
neutron Bogoliubov excitations, protons and electrons \[23\]. The presence of extra velocity field \(V_{\text{ex}}\) results in additional (besides equation \(\nabla P = -w \nabla \phi\)) condition of hydrostatic equilibrium in SFL matter \[26, 27\]:

\[
\nabla \left( \mu_n e^\phi \right) = 0,
\]

where \(\mu_n\) is the relativistic neutron chemical potential. As a result, when we displace the fluid element, ‘attached’ to the normal liquid component, from point 1 to point 2, both \(P\) and \(\mu_n\) adjust themselves to their equilibrium values \(P_2\) and \(\mu_{n2}\) at point 2. The pressure adjusts by contraction/expansion of the fluid element, while \(\mu_n\) adjusts by the variation in the number of ‘SFL neutrons’ in this element. Note that, since SFL neutrons can freely escape from the fluid element attached to the normal particles, the total number of neutrons in the element is not conserved, and neither are the quantities \(x_e = n_e/n_b\) and \(x_S = S/n_b\). In this situation the conserved quantity is \(x_S = S/n_e\), because both the entropy and electrons flow with the same velocity \(V_{\text{ex}}\) (e.g., \[26, 28\]). Bearing this in mind, it is convenient to consider \(w\) as a function of \(P\), \(\mu_n\), and \(x_S\). Then the condition for stability against convection can be written as

\[
\partial_{x_S} w(P, \mu_n, x_S) \nabla x_e < 0.
\]

A similar condition was derived in a different way in Ref. \[29\], where internal gravity waves were analyzed in a mixture of SFL He-4 and a normal fluid (see also \[30, 31\]). Note that the left-hand side of Eq. (2) depends on \(T\) and vanishes at \(T = 0\). Then the system is marginally stable, since there is no restoring force acting on a displaced fluid element. Thus, it is not surprising that the authors of Refs. \[10, 21\], who assumed \(T = 0\), did not find g-modes in SFL NSs. In contrast, consistent treatment of the temperature effects should reveal g-modes.

To check it we perform a local analysis of SFL hydrodynamic equations (see, e.g., \[26, 27\]), describing oscillations of an NS in the weak-field limit (\(\phi \ll 1\)) at \(T \neq 0\). We analyze short-wave perturbations, proportional to \(\exp(i\omega t)\) \(\exp[i f' dr' k(r')]Y_{lm}\), where the wave number \(k\) of a perturbation weakly depends on \(r\) \((k \gg |d\ln k/dr|\), WKB approximation), and \(Y_{lm}\) is a spherical harmonic. Solving oscillation equations in the Cowling approximation (in which \(\phi\) is not perturbed \[32\]), we find the standard \[10\] short-wave dispersion relation for the SFL g-modes:

\[
\omega^2 = N^2 \left[ (l+1)/l(l+1) + k^2 r^2 \right],
\]

is the corresponding Brunt-Väisälä frequency squared; \(g = \nabla \phi\); \(n_b = n_b Y_{np}/[\mu_n (Y_{nn} Y_{np} - Y_{np}^2)]\); \(1 > 0\); \(Y_{ik}\) being the relativistic entrainment matrix (see, e.g., \[33, 34\] and comment \[35\]). The stability condition for these g-modes, \(N^2 > 0\), coincides with Eq. (2).

For numerical evaluation of Eq. (3) it is convenient to introduce a new set of independent variables \(n_b, n_e\), and \(T\) instead of \(P, \mu_n\), and \(x_S\). Then Eq. (3) is approximately rewritten as

\[
N^2 \approx \alpha C_Y F \left[ -\nabla T/(T gT) + F - 1 \right],
\]

where we neglect small terms of the second and higher orders in \(T/\epsilon_F\) (\(\epsilon_F\) is the typical particle Fermi energy). In Eq. (4) \(\alpha = g^2/2(1+y)/(y \mu_n n_b) > 0\) and \(C_Y = -\partial T/\partial S > 0\). Finally, \(F = 1 + \mu_n G_2/(G_1 C_Y)\), where \(G_1 = \partial n_b P \partial n_b \mu_n - \partial n_b P \partial n_b \mu_n\) and \(G_2 = \partial n_b P \partial n_b S + \partial n_b \mu_n (S - n_b \delta_{nn} S - n_b \delta_{nn} S).\) For an NS in thermal equilibrium, the red-shifted temperature \(T^\infty = T e^\phi\) is constant throughout the core, \(\nabla T^\infty = (\nabla T + gT) e^\phi = 0\). In that case \(N^2\) in Eq. (4) is positive and reduces to

\[
N^2 \approx \alpha C_Y F^2.
\]

If NS is not in thermal equilibrium, \(N^2\) can be negative for certain \(\nabla T^\infty\). These gradients follow from Eq. (4) (or Eq. 2) and are defined by the inequality \(\nabla^2 T^\infty > gT^\infty F^2\), which is the analogue of the ordinary Schwarzschild criterion for convection \[23\]. This inequality, as well as Eq. (2), is valid not only in the weak-field limit \(\phi \ll 1\), but also in the full general relativity. When it is satisfied, convective instability occurs \[36\]. Thus, the critical gradient for the instability onset is given by

\[
\nabla T^\infty_{\text{crit}} = gT^\infty F.
\]

Clearly, \(F\) determines the sign of \(\nabla T^\infty_{\text{crit}}\). If \(F < 0\) one gets the instability while heating the matter from below; if \(F > 0\) the instability occurs when it is heated from above. Note that both signs of \(\nabla T^\infty\) can be realized in cooling NSs \[37\].

Results. — First, consider the limit \(T \ll T_{cn}, T_{cp}\) \((T_{ci}\) is the critical temperature for particles \(i = n, p\)), in which the nucleon entropy is exponentially suppressed and the entropy density \(S\) is provided by electrons, \(S = S_e = T(3\pi^2 n_e^3)^{2/3}/3\). In this limit both \(N\) and \(N^2\) are \(\propto T\). Fig. (1a) presents the Brunt-Väisälä frequency \(N\) (see Eq. (5)) as a function of \(n_b\) for npe-matter in thermodynamic equilibrium for 5 equations of state (EOSs) and \(T = 10^7\) \(K\). One sees that for any EOS \(N\) vanishes at a certain \(n_b\) that corresponds to \(F = 0\). Fig. (1b) shows the critical gradient \(\nabla T^\infty_{\text{crit}}\) (see Eq. (6)) versus \(n_b\) for the APR EOS \[38\]. As expected, \(\nabla T^\infty_{\text{crit}}\) changes sign when \(F = 0\). The region of parameters, where convection occurs, is filled with gray.

When \(T\) is not too low \((0.1 T_{ci} \lesssim T \lesssim T_{ci}, i = n, p)\), nucleonic contribution to \(S\) is non-negligible. The results in that case strongly differ from those obtained in the limit \(T \ll T_{cn}, T_{cp}\) and are presented in Fig. (2). For illustration, we adopt the APR EOS and take \(T = 1.5 \times 10^8\) \(K\). Some realistic profiles of singlet proton \(T_{cp}(n_b)\) and triplet neutron \(T_{cn}(n_b)\) critical temperatures, employed in our numerical calculations, are shown in Fig. (2a). Figs. (2b, c) demonstrate \(N(n_b)\) and...
$$T = 10^7 \text{K}$$

$$\nabla T^\infty_{\text{crit}}(n_b)$$, respectively. Solid lines in Figs. 2(b,c) are obtained for \(T_{\text{cn}}(n_b)\) and \(T_{\text{cp}}(n_b)\) from Fig. 2(a). Dot-dashed lines are plotted for \(T_{\text{cn}}(n_b)\) from Fig. 2(a), but for \(T_{\text{cp}} \to \infty\). Finally, dashed lines in Figs. 2(b,c) correspond to the limit \(T_{\text{cn}}, T_{\text{cp}} \to \infty\) of Fig. 1 (note, however, that Fig. 1 is plotted for different \(T\)). For simplicity, we assume that the density entropy \(S_i\) of particles \(i = n, p\) and \(e\) depends only on \(n_i\) and \(T\), so that \(S = S_n(n_n, T) + S_p(n_p, T) + S_e(n_e, T)\). Then \(F\), which enters Eqs. 5 and 6, can be rewritten as

$$F = 1 + \frac{\mu_n}{G_1 C V} \left[ (\partial_{n_n} P - n_n \partial_{n_n} \mu_n) \partial_{n_n} S_n - n_e \partial_{n_e} P - n_e \partial_{n_e} \mu_e - n_e \partial_{n_e} S_e + S \partial_{n_n} \mu_n \right].$$

Clearly, the behaviour of \(N\) and \(\nabla T^\infty_{\text{crit}}\) results from interplay between various derivatives of \(S_i\). The key role is played by the nucleon derivatives \(\partial_{n_i} S_i\) (i.e. \(n, p\)). Because \(S_i\) strongly depends on \(T_{\text{cr}}\), which is in turn a very strong function of \(n_i\) (especially, on the slopes of \(T_{\text{cr}}(n_i)\), see Fig. 2(c)), \(\partial_{n_i} S_i\) can be very large and thus determine the dependence of \(N\) and \(\nabla T^\infty_{\text{crit}}\) on \(n_b\).

To illustrate this point let us compare dot-dashed and dashed curves in Fig. 2(c). Dashed curve is plotted assuming \(S_n = S_p = 0\) \((T_{\text{cn}}, T_{\text{cp}} \to \infty)\) while for dot-dashed curve only \(S_p = 0\). The terms in \(F\) related to \(\partial_{n_n} S_n\) are negative for \(n_b \lesssim 0.2 \text{ fm}^{-3}\) and positive for \(n_b \gtrsim 0.2 \text{ fm}^{-3}\). This can be easily understood if one bears in mind that for densities of interest \((i)\) \(G_1 > 0\) and \(\partial_{n_n} P - n_n \partial_{n_n} \mu_n > 0\); \((ii)\) \(S_n\) decreases with increasing \(T_{\text{cn}}\); and \((iii)\) \(T_{\text{cn}}\) reaches maximum at \(n_b \sim 0.2 \text{ fm}^{-3}\) (Fig. 2a). As a result, \(\nabla T^\infty_{\text{crit}}\) given by the dot-dashed curve vanishes at lower \(n_b\) than for dashed curve (Fig. 2c).

Now let us consider the effects related to the proton entropy density \(S_p\) (solid line in Fig. 2b). When \(T \ll T_{\text{cp}}\) the solid and dot-dashed curves coincide, because in that case \(S_p\) is negligible. When \(T\) approaches \(T_{\text{cp}}\) with growing \(n_b\), \(\nabla T^\infty_{\text{crit}}\) rapidly increases because \(dT_{\text{cp}}/dn_b\) is large and negative (i.e. \(\partial_{n_b} S_p > 0\) and \(\partial_{n_b} \mu_p < 0\) [see the corresponding term in Eq. 7]). At

FIG. 1: Panel (a): \(N\) versus \(n_b\) for EOSs of Armani et al. [34], APR [38], and PAL [40]. We adopt the model I of PAL family with three values of the compression modulus, 120, 180 and 240 MeV. Panel (b): \(\nabla T^\infty_{\text{crit}}\) versus \(n_b\) for APR EOS. The instability region is filled with gray. Vertical dot-dashed line corresponds to the crust-core interface. Both panels are plotted for \(T = 10^7 \text{K}\). Here and in Fig. 2, \(g = 10^{14} \text{ cm s}^{-2}\).

FIG. 2: Panel (a): \(T_{\text{cn}}\) and \(T_{\text{cp}}\) versus \(n_b\). Panels (b) and (c): \(N\) and \(\nabla T^\infty_{\text{crit}}\) versus \(n_b\) for EOS APR and \(T = 1.5 \times 10^6 \text{K}\) [see the horizontal dotted line in panel (a)]. Solid lines in panels (b) and (c) are obtained for \(T_{\text{cn}}(n_b)\) and \(T_{\text{cp}}(n_b)\) from panel (a); dot-dashed lines: \(T_{\text{cn}}(n_b)\) is from panel (a), \(T_{\text{cp}} \to \infty\); dashed lines: \(T_{\text{cn}}, T_{\text{cp}} \to \infty\). The right vertical dot-dashed line indicates the boundary between the SFL and normal neutron matter (in the latter SFL g-modes are absent). The vertical dotted line shows a similar boundary for protons. Other notations are the same as in Fig. 1.
T = T_{cp} (vertical dotted line in Fig. 2), \( \partial_{n_b} \mathcal{S}_p \) and \( \partial_{T} \mathcal{S}_p \) are discontinuous that results in discontinuities of \( \mathcal{N} \) and \( \nabla T^\infty_{\text{crit}} \). At \( n_b \gtrsim 0.26 \text{ fm}^{-3} \) all protons are normal and \( \mathcal{S}_p = T m_p^2 p_{Fp} / 3 \), where \( m_p^* \) and \( p_{Fp} \) are the proton effective mass and Fermi momentum, respectively. At such \( n_b \) the proton contribution to \( \mathcal{F} \) is negative, i.e. the solid curve in Fig. 2(c) goes lower than the dot-dashed curve. The most important conclusion drawn from the analysis of Fig. 2 is that SFL g-modes and convection in the internal layers of NSs are extremely sensitive to the EOS and the model of nucleon SFL. An account for the single neutron SFL at lower densities \( n_b \lesssim 0.08 \text{ fm}^{-3} \) may additionally affect \( \mathcal{N} \) and \( \nabla T^\infty_{\text{crit}} \) near the crust-core interface.

Discussion and conclusion. — Our results indicate that NSs can have convective internal layers. This could affect the thermal evolution of young NSs (such as in Cas A), for which \( \nabla T^\infty \) is not completely smoothed out by the thermal conductivity, as well as the thermal relaxation of quasi-persistent X-ray transients [41, 42]. Note that in this Letter we only consider SFL g-modes and convection in the NS cores (but not in the crust). If the NS crust is elastic, as it is usually assumed, then it is most likely that core SFL g-modes do not penetrate the crust, while the crustal SFL g-modes are ‘mixed’ with the shear modes [9] (for which the restoring force is elasticity), and pushed to frequencies \( \omega \gg \mathcal{N} \) (but see comment [43]). In that case convection is absent. However, if the inner crust (especially, mantle [15]) is plastic [44] then the existence of crustal SFL g-modes and convection cannot be excluded, that can have even more interesting implications for NS cooling.

We have shown that g-modes can propagate in SFL NS matter. But how can they be excited? Among the potential scenarios is the excitation of stable SFL g-modes by unstable ones (i.e., by convective motions). Another possibility was considered in Refs. [11, 45] in application to composition g-modes of normal NSs. It consists in resonant excitations of SFL g-modes by tidal interaction in coalescing binary systems, when the frequency of the tidal driving force equals one of the g-mode frequencies. Finally, SFL g-modes in rotating NSs could be excited due to gravitational driven (CFS) instability, as it is usually assumed, then it is most likely that core SFL g-modes do not penetrate the crust, while the crustal SFL g-modes are ‘mixed’ with the shear modes [9] (for which the restoring force is elasticity), and pushed to frequencies \( \omega \gg \mathcal{N} \) (but see comment [43]). In that case convection is absent. However, if the inner crust (especially, mantle [15]) is plastic [44] then the existence of crustal SFL g-modes and convection cannot be excluded, that can have even more interesting implications for NS cooling.

To conclude, we have predicted a new class of g-modes in SFL NSs. We have calculated their Brunt-Väisälä frequency \( \mathcal{N} \), which strongly depends on \( T \) and vanishes at \( T = 0 \). The SFL g-modes appear to be unstable for certain temperature gradients (that correspond to \( \mathcal{N}^2 < 0 \)). We have derived the criterion for convective instability (analogue of the Schwarzschild criterion) in SFL NS cores. We have shown that convection in the NS core may occur for both positive and negative temperature gradients and is extremely sensitive to the model EOS and nucleon SFL. We have only outlined the properties of SFL g-modes. In particular, we have not calculated their frequency spectrum and damping times. We plan to fill these gaps in the future publication. Though we have only considered npe-matter of NSs, our analysis can be easily extended to SFL hyperon and quark stars, for which we also predict the existence of global, low-frequency SFL g-modes.

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