QCD with Adjoint Scalars in 2D: Properties in the Colourless Scalar Sector

P. Bialas\textsuperscript{1,a}, A. Morel\textsuperscript{2,b}, B. Petersson\textsuperscript{3,c}, K. Petrov\textsuperscript{3,d} and T. Reisz\textsuperscript{2,4,e}

\textsuperscript{1}Inst. of Comp. Science, Jagellonian University
33-072 Krakow, Poland

\textsuperscript{2}Service de Physique Théorique de Saclay, CE-Saclay,
F-91191 Gif-sur-Yvette Cedex, France

\textsuperscript{3}Fakultät für Physik, Universität Bielefeld
P.O.Box 100131, D-33501 Bielefeld, Germany

\textsuperscript{4}Institut für Theor. Physik, Universität Heidelberg
Philosophenweg 16, D-69120 Heidelberg, Germany

Abstract

We present a numerical study of an SU(3) gauged 2D model for adjoint scalar fields, defined by dimensional reduction of pure gauge QCD in (2+1)D at high temperature. We show that the correlations between Polyakov loops are saturated by two colourless bound states, respectively even and odd under the $\mathbb{Z}_2$ symmetry related to time reversal in the original theory. Their contributions (poles) in correlation functions of local composite operators $A_n$ respectively of degree $n = 2p$ and $2p + 1$ in the scalar fields ($p = 1, 2$) fulfill factorization. The contributions of two particle states (cuts) are detected. Their size agrees with estimates based on a meanfield-like decomposition of the $p = 2$ operators into polynomials in $p = 1$ operators. In contrast to the naive picture of Debye screening, no sizable signal in any $A_n$ correlation can be attributed to $1/n$ times a Debye screening length associated with $n$ elementary fields. These results are quantitatively consistent with the picture of scalar “matter” fields confined within colourless bound states whose residual “strong” interactions are very weak.

\textsuperscript{a}pbialas@agrest.if.uj.edu.pl
\textsuperscript{b}morel@spht.saclay.cea.fr
\textsuperscript{c}bengt@physik.uni-bielefeld.de
\textsuperscript{d}petrov@physik.uni-bielefeld.de
\textsuperscript{e}t.reisz@thphys.uni-heidelberg.de
1 Introduction

Dimensional reduction is a powerful technique to study the infrared region of field theories at high temperature[1]–[3]. Combined with a non-perturbative lattice simulation of the reduced model, it has been employed to investigate the properties of gauge theories and QCD with dynamical quarks in the plasma phase [4]–[7]. For a recent review see [10].

It is, however, still not clear, what are the limitations and the domain of validity of this approach. Therefore, in a recent work [11], we have studied in detail the reduction to two dimensions of pure gauge QCD in (2+1) dimensions at high temperature. We refer the reader to this article for a more complete discussion of our motivations, references to the related literature and details on the reduction procedure. The reduced model is a model for scalars belonging to the SU(3) algebra (formerly the electric gluons in a static gauge). They interact with the 2D gauge fields, the static parts of the original 3D spatial gauge fields, and via an effective potential whose self-couplings are computed by perturbative integration over the non-static degrees of freedom. In [11] we restricted the perturbative integration to the one loop order. The main conclusion of our investigation is that dimensional reduction works very well in this case. In particular it was shown that it works within a few percent for the correlation function of Polyakov loops (as well as for spacelike Wilson loops) down to 1.5\( T_c \), where \( T_c \) is the critical temperature in the (2 + 1)\( D \) theory. In fact for \( T \geq 1.5T_c \) numerical simulations showed that the Polyakov loops correlations measured in (2+1)D QCD and in the reduced model were very close to one another down to quite short distances, although our formalism is in principle an expansion both in high temperature and in \( p/T \), where \( p \) is the relevant momentum scale. It was further shown, by simulations at different values of the bare parameters, that our measurements are in the scaling region, and thus our results are valid in the continuum limit.

Less expected was the observation that these correlations assume a shape typical of single particle propagation, as opposed to the standard picture of a screening mass associated with two massive electric gluons. This feature, together with our previous motivations, invites us to pursue the numerical exploration of the reduced model per se, in particular the investigation of states connected with the Polyakov loop correlations.

While in [11] we only measured the correlations of Polyakov loops, here we analyze separately those of SU(3) invariant polynomials of degree \( n \) in the elementary scalars \( A \), namely \( A_n = \text{tr} A^n \). The action is invariant under a global sign reversal of all the \( A \)'s. This \( Z_2 \)-symmetry, which we denote by \( R_\tau \), following the authors of Ref. [12], corresponds to Euclidean time reversal in the (2 + 1)\( D \) theory. In this latter article it was suggested to investigate operators odd under this symmetry, to obtain a possible gauge invariant definition of the Debye mass. To investigate both operators
which are even and odd under $R_t$ we consider separately correlations involving even and odd polynomials, $n = 2p$ and $2p + 1$. The $R_t$-symmetry may be spontaneously broken in the 2D model. In fact there exists two phases, corresponding to $R_t$ being conserved or broken. Only in the symmetric phase, the model corresponds to the reduction of the high temperature (2 + 1) QCD phase [13], [14], [12], [9]. The details of the actual phase diagram will not be studied in the present article. From our data we can conclude, however, that the values of the coupling constants in the reduced 2D theory are in the unphysical broken phase. In the same way as has been done for the reduction from (3 + 1)D to 3D, we solve this problem by working in the metastable part of the symmetric phase. Using zero field initial conditions on large enough lattices, we make sure that we stay in the phase of unbroken $R_t$, where even and odd operators do not mix. Investigations of states in the full and reduced model in the case of the (3 + 1) → 3 reduction can be found in [6]–[9] and [15]–[20], although a detailed analysis of the nature of these states, as we perform in this paper, have not yet been made.

The 2D action under consideration is recalled in Section 2, together with its meaning in terms of the (2+1)D QCD model, from which it originates, and the relevant operators and correlations are defined. The simulations are performed in the temperature range $2T_c$ to $12T_c$, where $T_c$ is the deconfining temperature of the latter model. Our results are presented and discussed in Sections 3 and 4. In Section 3, we first show that the measured correlations fulfill the factorization properties expected if the lowest states in the $n$–even and $n$–odd channels are two distinct one particle states, whose masses are then extracted from the large distance decays. According to the criteria proposed in Ref. [12], the state found in the odd channel is a candidate to define a Debye screening mass, although not the only one. In Section 4, we further analyze the composite operators $A_n$, $n = 2p$ and $2p + 1$ and their correlations $A_{n,m}$, showing that all the properties observed for $p = 2$ can be deduced with a good accuracy from their knowledge for $p = 1$. This follows from the assumption that, given the SU(3) and $R_t$ symmetry constraints, the effective model for the elementary $A$'s after integration over the gauge fields is a free field model for the massive composites $A_2$ and $A_3$. In particular, we give evidence that the (small) deviations from factorization observed at short distances are mainly due to intermediate states containing two of the above particles. The summary and the conclusions can be found in a last Section 5.
2 The Reduced Action, Operators and Correlations

In this section, we write down the reduced 2D action derived in [11], and define our notations and the quantities of interest for the present work.

The lattice is an \( L_s \times L_s \) square; the spacing is \( a \), set to one unless specified otherwise. The weight in the partition function is written \( \exp(-S) \), with \( S \) a function of the SU(3) group elements \( U(x;i), i = 1,2 \) on the links and of the scalars \( A(x) \) in the adjoint representation on the sites:

\[
A(x) = \sum_{\alpha=1}^{8} A^\alpha(x) \lambda^\alpha, \quad \text{tr} \lambda^\alpha \lambda^\beta = \frac{1}{2} \delta_{\alpha\beta}. \tag{1}
\]

Greek superscripts on \( A \) will always denote colour indices, unlike integers \( n, m \) used in powers of the algebra element \( A \).

We write the 2D reduced action as follows:

\[
S = S_U + S_{U,A} + S_A, \tag{2}
\]

\[
S_U = \beta_3 L_0 \sum_x \left( 1 - \frac{1}{3} \text{Re} U(x;1)U(x+a\hat{1};2)U(x+a\hat{2};1)^{-1}U(x;2)^{-1} \right), \tag{3}
\]

\[
D_i(U)A(x) = U(x;i)A(x+a\hat{i})U(x;i)^{-1} - A(x),
\]

\[
S_A = \sum_x k_2 \text{tr} A^2(x) + k_4 (\text{tr} A^2(x))^2.
\]

In the above, \( S_U \) is the pure gauge term, \( S_{U,A} \) the gauge invariant kinetic term for the scalars and \( S_A \) the scalar potential, whose self couplings \( k_2 \) and \( k_4 \) result from the one-loop integration over the non-static components of the 3D gauge fields. All terms have the global \( R_\tau \)-symmetry \( A(x) \to -A(x) \), while the \( Z_3 \) symmetry of the original \((2 + 1)D\) SU(3) model is broken by the perturbative reduction procedure. It was found in [11] that

\[
k_2 = -\frac{3 \pi}{2} \left( c_0 \log L_0 + c_1 \right); \quad c_0 = 1, \quad c_1 = \frac{5}{2} \log 2 - 1, \tag{4}
\]

\[
k_4 = \frac{L_0^2}{64 \pi}.
\]

The values of the parameters \( \beta_3, L_0 \) follow from the original lattice regularization of 3D pure QCD at temperature \( T \) and gauge coupling squared \( g_3^2 \) in the continuum.
The latter has dimension one in energy and is used to set the scale:

\[ \beta_3 = \frac{6}{a g_3^2}, \]
\[ L_0 = \frac{1}{a T}. \]  

Accordingly, the continuum limit \( a \to 0 \) is obtained by letting \( \beta_3 \) and \( L_0 \) go to infinity with the dimensionless temperature

\[ \tau = \frac{T}{g_3^2} = \frac{\beta_3}{6L_0} \]

being kept fixed.

The original three dimensional pure \( SU(3) \) gauge theory has a global \( Z_3 \) symmetry. The corresponding order parameter is the Polyakov loop. It is a static operator. In the reduced theory it has the form

\[ L(x) = \frac{1}{3} \text{tr} \exp[i L_0 A(x)]. \]

At sufficiently high temperature the symmetry is spontaneously broken, signalling the deconfinement of static charges in the fundamental representation. The reduced theory in the above form does not have the \( Z_3 \) symmetry any more, because the perturbative reduction is made around one of the broken vacua, where \( A(x) = 0 \). The phase transition in the three dimensional theory appears at \( \tau_c \approx 0.61 \) \[23\]. The reduced model should be valid in the deconfined phase, at sufficiently high temperature and long distances. In \[11\] we employed it to investigate the correlations between Polyakov loops in this phase, where they are related to screening. We performed a detailed numerical analysis in the reduced model and a comparison with the results in the full \((2 + 1)D\) theory. Our simulations were performed for two values of the parameter \( L_0 \), namely 4 and 8. It was shown that scaling was very good, when comparing the data collected for fixed \( \tau \) at these two values of \( L_0 \). In this paper, we will show data collected for \( L_0 = 4 \). As this is in the scaling domain, we can give the results in physical units. Distances \( R \) are given by

\[ RT = r/L_0, \]

where \( r \) is the distance in lattice units, and temperatures are measured in units of the three dimensional critical temperature \( T_c \). For \( L_0 \) fixed in the scaling region one may use

\[ T/T_c = \beta_3/\beta_{3c}. \]
To discuss the effective Lagrangian of the model at fixed $\tau$ in the scaling limit, it is convenient to normalize the scalar fields differently, defining $\phi(x)$ via

$$A(x) = \phi(x) \sqrt{\frac{6}{L_0 \beta_3}}. \tag{10}$$

The corresponding Lagrangian $L_{\text{eff}}$ was derived in [11] from the small $a$ expansion of the effective action $S$. For clarity of the discussion we reproduce it here below. In $S$, we define $A_i$ by

$$U(x; i) = \exp[i a g_2 A_i(x)], \tag{11}$$

where $g_2^2 = g_3^2 T$ is the effective coupling of the 2D theory. Taking the limit $a \to 0$ (but for the UV logarithm $\log L_0 \equiv -\log a T$ in the $\phi^2$ term), one obtains

$$L_{\text{eff}} = \frac{1}{4} \sum_{c=1}^{8} F_{ij}^c F_{ij}^c + \text{tr} [D_i \phi]^2 + \frac{g_2^2}{32\pi} \left( \frac{g_2}{T} \right)^2 \text{tr} \phi^4 + L_{\text{CT}}, \tag{12}$$

$$D_i \phi = \partial_i \phi + i g_2 [A_i, \phi],$$

$$F_{ij} = \partial_i A_j - \partial_j A_i + i g_2 [A_i, A_j],$$

$$L_{\text{CT}} = -\frac{3g_2^2}{2\pi} \left[ -\log(aT) + 5/2 \log 2 - 1 \right] \text{tr} \phi^2. \tag{13}$$

This is a 2D, $SU(3)$ gauge invariant Lagrangian for an adjoint scalar $\phi$, but it is far from being the most general one. The gauge coupling $g_2$, with its canonical dimension one in energy, sets the scale. The non kinetic quadratic term is the counterterm $L_{\text{CT}}$, suited to a lattice UV regularization with spacing $a$. The appearance of this term in the context of dimensional reduction in e.g. the lattice regularization framework was first discussed in Refs.[3][5]. It is well known that such logarithmic terms also appear in general, when one wants to define the continuum limit of a 2D lattice model, see e.g.[21][22]. The reflection symmetry $\phi \to -\phi$ is related to the euclidean time reflection in the original $(2 + 1)D$ theory, noted $R_\tau$ and discussed by Arnold and Yaffe [12]. In the two dimensional model of Eq. (3), it may, however, be spontaneously broken in some subspace of the unrestricted parameter space $\beta_3 L_0, k_2, k_4$. As was discussed in Refs. [13][14][9], and as follows from the invariance of the original three dimensional theory under euclidean time reflection, the physical phase is the $R_\tau$ symmetric phase. The phase diagram in the 3D adjoint Higgs model, related to $4 \to 3$ QCD reduction has been studied in Ref. [16]. For the case of SU(2) in (3+1)D, $R_\tau$ is the center of the gauge group so that $R_\tau$ breaking is also gauge symmetry breaking, a subject previously discussed in Refs. [13], [14] and [9].

We have made a numerical investigation of the relative positions of the phase transition and of the reduction point in the reduced model, for $T/T_c = 1.97$, and for two values $L_0 = 4$ and 8. We find that the reduction point is near to the phase
transition, but in the broken phase. The phase transition is first order and strong enough, so that we can study the reduction in the metastable symmetric phase, by using appropriate starting values for the fields, and employing a large enough $(32 \times 32)$ lattice.

We now turn to the definitions of the quantities relevant for the present investigation of the 2D model. The Polyakov loop correlation $P(x)$ is defined by
\begin{equation}
P(x) = \langle L(0) L^\dagger(x) \rangle - |\langle L \rangle|^2,
\end{equation}
where $L(x)$ is the Polyakov loop operator of Eq. (7). We now expand $L(x)$ in powers of $A(x)$, and we will study the operators $A^n(x)$ and connected correlations of their traces, defining
\begin{align}
A_n(x) &\equiv \text{tr} A^n(x), \\
A_{n,m}(x) &\equiv \langle A_n(x) A_m(0) \rangle - \langle A_n \rangle \langle A_m \rangle.
\end{align}

When not ambiguous, the notation $A_n$ may also represent $\langle A_n(x) \rangle$. Any operator $A_n(x)$ is gauge invariant, even or odd under the $R_\tau$-symmetry of $S$ for $n$ even or odd respectively. Because the reduced model was derived from a small $A$-fields expansion, its properties are significant for the (2+1)D model in the unbroken $R_\tau$ phase only, where $A_{2p+1} = 0$ and $A_{2p}$ is small.

In this article we will concentrate on the study of correlations of these operators, which are directly related to the static operator $L(x)$ and therefore relevant for the high temperature properties of the original (2 + 1)D SU(3) theory. As can be easily seen from the effective Lagrangian above, the 2D model has further symmetries beside $R_\tau$, namely reflections of the 1- and 2-axis, which can be used to classify further operators, whose correlations may be studied. (In 2D there is no spin quantum number.) A corresponding analysis has been made in [16] [20] in the case of the three dimensional adjoint Higgs model. Although a full numerical analysis of the two dimensional adjoint Higgs model certainly has an interest per se, the operators other than those which we defined above are not directly related to a static (2+1)D operator. One would need a further study to acertain to what extent their correlations are related to the high temperature physics of the original theory. We therefore do not discuss them in the context of this paper.

We have performed a numerical simulation of the model defined above, with a flat measure for the $A$’s and the standard De Haar measure for the gauge fields. The algorithm and error estimate techniques used are the same as in [11] and not reproduced here. The lattice size is $L_S = 32$, and $L_0 = 4$, throughout the present work. The $\beta_3$ values are 29, 42, 84 and 173. This corresponds to values of $T/T_c$ equal approximately to 1.97, 2.85, 5.70 and 11.73 respectively. We were able to extract information from operators and correlations corresponding to $n = 2$ to 5. The cases $n = 2$ and 3 for the 3D reduced model were investigated in [10].
3 The Lowest States of the Scalar Spectrum

In this section we present our results for the $A_{n,m}$ correlations measured, and describe them for each temperature in terms of two states $S$ and $P$, respectively even and odd under $R_\tau$ and appearing for for $n, m$ both even and both odd. Their physical masses will be denoted $M_S$ and $M_P$. For simplicity we often use in this and the next section the bare parameters $r$ and $\beta_3$, related to $R$ and $T$ by Eqs. (8) and (9).

In the simulations reported here, all runs were initialized with zero $A$-fields values and, as already stated, the system stayed in the metastable $R_\tau$ unbroken phase, as desired, with $\langle A_{2p+1}(x) \rangle$ being always compatible with zero. In this respect, the situation is thus similar to that encountered in (13)-(16).

For a first look at the data obtained in even and odd channels, we show in Fig. 1 the on-axis correlations $A_{n,m}(r)$, $n \leq m \in [2, 5]$ at $T/T_c = 1.97 (\beta_3 = 29)$. They are plotted against $R_T$, that is the physical distance in units of the inverse temperature. In the even cases, the three correlations all have the same shape, and they decrease by about one order of magnitude each time two more powers of $A$ are involved. The same is true for the odd cases, with a common decay of the correlations steeper than in the former case (smaller correlation length in lattice units). The overall situation is similar for $\beta_3$ higher. For $n, m$ larger than 5, as well as for $\beta_3$ very large, the signal/noise ratio becomes very small. This can be understood at the qualitative level by noting that the rescaling Eq.(10) of the $A$-fields normalizes the kinetic term for the $\phi$-fields to the standard, parameter independent form $1/2 \text{tr} (D_i \phi)^2$. Hence if the field renormalization by the interactions is weak, the $\phi$ correlations should depend only weakly on $\beta_3$, which means that $A_{n,m}$ scales like $\beta_3^{-(n+m)/2}$. This will be illustrated more quantitatively in section 4. Due to this scale factor, the $A$-fields remain “small” in practice down to quite low values of $\beta_3$, which a posteriori explains why the perturbative reduction may still work at a temperature as low as $1.5 T_c$. In fact, we checked that the Polyakov loop correlations are actually fully reconstructed within errors by keeping $\{n, m\}$ up to $\{5, 5\}$ only in their expansion in $A_{n,m}$’s obtained from the small $A$ expansion of (14).

Now we want to analyze these $A_{n,m}$ data quantitatively in terms of the lowest states of the spectrum. Let $m_i = a M_i$ be the lowest mass with quantum number $i = S, P$, in lattice units. For particle $i$ we introduce a lattice propagator $\tilde{\Delta}_{\text{Latt}}(m_i, p)$ in momentum space:

$$\tilde{\Delta}_{\text{Latt}}^{-1}(m_i, p) = \hat{p}^2 + 4 \sinh(m_i^2/4), \quad \hat{p}^2 = 4 \sin^2(p_1/2) + 4 \sin^2(p_2/2).$$

(17)
The corresponding contribution to $A_{n,m}(r)$ then reads

$$A_{n,m}(m_i, r) = g_{n,m}^{i} \frac{1}{L_s^2} \sum_{p_1, p_2} \cos(p_1 r) \Delta_{\text{Latt}}(m_i, p),$$

where $g_{n,m}^{i}$ measures the residue of $A_{n,m}$ at the pole of (17), which on large enough lattices sits at $p^2 \sim \tilde{p}^2 \sim -m_i^2$. With our definitions, $g_{n,m}^{i}$ is non zero only for $i = S$ if $n$ and $m$ are even, and for $i = P$ if $n$ and $m$ are odd. Prior to any fit of the masses to the data, we notice that a first consequence of our expectations on the lowest part of the spectrum comes from residue factorization $g_{n,m}^{i} = \gamma_{n}^{i} \gamma_{m}^{i}$, a property which we can probe directly on the correlations since, as $r$ becomes large, it implies

$$X_n \equiv \frac{A_{n,n}(r) A_{n+2,n+2}(r)}{A_{n,n+2}^2(r)} \to 1.$$  

That it is so is demonstrated on Figs. (2-5) showing $X_2$ and $X_3$ (symbols ⋄) for $T/T_c = 1.97$ and 5.7 ($\beta_3 = 29$ and 84). The agreement is very good in all cases, although the quality of the data is poorer for $X_3$ due to the correlations involving $A_5$ getting very small. Similar results are obtained for other values of $T/T_c$. We thus conclude at this point that single particle propagation accounts very well for the largest correlation length occurring in each of the two channels. Most of the observed deviations of $X_n$ from one will be interpreted in the next section in terms of two particle state contributions (symbols ◎ in the same figures).

We now proceed to assign values to the two lowest masses $M_S$ and $M_P$ expected from the above findings. This we do by various ways, in order to further enforce the statement that the correlations do have the characteristics associated with the pole structure of Eq. (17). Down to $r \sim 1$, an excellent approximation to the on-axis correlation (18) is given by

$$A_{n,m}(m_i, r) \approx c \left( \frac{1}{m_i r^{1/2}} e^{-m_i r} + \frac{1}{m_i (L_s - r)^{1/2}} e^{-m_i (L_s - r)} \right),$$

where $c$ is constant in $r$. We performed fits of this formula to all our $A_{n,m}(r)$ data taken at $r > r_{\text{min}}$. These fits are stable with respect to $r_{\text{min}}$ provided it is larger than about 4, and the values found for $m_i$ in different correlations are always consistent with each other. The smallest errors were obtained by using fits to $A_{2,2}$ and $A_{3,3}$.

Effective masses can also be obtained without any fitting by using 0–momentum correlations, defined for a generic $x$–space correlation $C(x_1, x_2)$ by

$$C^0(r) = \frac{1}{L_s} \sum_{x_2} C(r, x_2).$$

9
If the lowest mass in $C$ is $m$, the ansatz (17) gives
\[ C_0^0(r) \propto \cosh\left(m\left(L_s/2 - r\right)\right), \]  
(22)
and $m$ can be extracted at any $r$ by inverting this relation:
\[ m^{\text{eff}}(r) = \log\left(Y(r) + \sqrt{Y^2(r) - 1}\right), \]  
(23)
\[ Y(r) = \frac{C_0^0(r + 1) + C_0^0(r - 1)}{2C_0^0(r)}. \]

As an overall consistency check, we have extracted an effective mass $m^{\text{eff}}(r)$ from 0–momentum Polyakov loops correlations (14), and compared it to the $m_S$ values obtained by our fits to $A_{2,2}$. We find that $m^{\text{eff}}(r)$ is indeed nearly constant, in fact slowly decreasing towards a value compatible with $m_S$, due to smaller and steeper contributions to (14) of the heavier particle $P$.

A contrario, we invalidate the interpretation of the largest correlation length in $A_{n,n}$ as being $n$ times shorter than the “Debye screening length”, the inverse of a mass $m_E$ associated with “electric” gluons of the initial (2+1)D model (the scalars of the reduced model). This scenario was advocated by D’Hoker in his perturbative study of $QCD_3$ at high temperature [24]. If such was the case, the on-axis correlations $A_{n,n}$ should rather look like
\[ A_{n,n}(nm_E, r) \propto \left(\frac{1}{m_E^{1/2} e^{-m_E r}} + \frac{1}{m_E(L_s - r)^{1/2} e^{-m_E(L_s - r)}}\right)^n, \]  
(24)
which differs in shape from (24), as was illustrated in [11] for Polyakov loop correlations. We nevertheless tried fits with (24), but got definitely worse agreement, in the range of temperatures, which we have investigated, i.e up to $12T_c$. Hence the above picture is ruled out by the data in this temperature range, and if a mass can be defined for the electric gluon in high temperature $QCD_3$ it is most probably larger than both $m_S/2$ and $m_P/3$. In a ”constituent gluon” picture, as advocated in Ref. [25], one would have bound states instead of a cut. One would, however, expect $m_P/m_S \approx 3/2$.

Our final results for the $S$ and $P$ masses in units of the scale $\sqrt{g_s^2 T}$ are collected in Table 1 for the values of $T/T_c$ investigated. They are taken from fits to $A_{2,2}$ and $A_{3,3}$ respectively. The values for $M_S$ agree with those obtained in [11] from the Polyakov loop correlations. As can be seen from the tables, the ratios $M_P/M_S$ vary with $T/T_c$. There is, however, no clear tendency in the region we have investigated, the ratios being 1.8, 2.0, 1.7, 1.6 in order of increasing temperature. We can, of course, not exclude that the ratio goes to 1.5 at still higher temperatures.
4 Weak “Strong” Interactions Between Colourless States

Here we will show that even at quite short distances ($r$ small compared to $m_2^{-1}$) all the condensates $A_n \equiv \langle A_n(x) \rangle$ and correlations $A_{n,m}(r)$ can be reconstructed to a good accuracy from the data for $A_2$, $A_{2,2}$ and $A_{3,3}$. The assumption is that the elementary fields $A^\alpha(x)$ (Greek superscripts are colour indices) interact only through $S$ and $P$ exchanges between the non-interacting composite $A_2(x)$ and $A_3(x)$, the scale of the fields being fixed by the size of $A_2$, while $A_3=0$. The precise way how this idea is implemented and the corresponding technicalities are detailed in the appendix.

Here we limit ourselves to the simplest applications and give the results, starting with the local condensates.

4.1 Weak Residual Interactions: The $A$-fields condensates

Since SU(3) has rank 2, any $A_n(x)$ can be reduced to a polynomial in $A_2(x)$ and $A_3(x)$. An elegant method [26] and explicit formulae are given in the Appendix. For $n$ odd $A_n$ is zero by $R_\tau$ symmetry. For $n$ even, we apply Wick contraction to all pairs of $A^\alpha$ elementary fields, followed by the meanfield-like substitution

$$A^\alpha(x)A^\beta(x) \rightarrow \frac{1}{4}\delta_{\alpha,\beta} A_2(x). \quad (25)$$

As an illustration, consider $A_4$. With the definitions of Section 2, we have

$$2A_4(x) = A_2^2(x) = \frac{1}{2^2} \sum_{\alpha,\beta=1}^8 A^\alpha(x)A^\alpha(x)A^\beta(x)A^\beta(x). \quad (26)$$

There we apply (25) and then replace $A_2(x)$ by its average $A_2$. The $A^\alpha A^\alpha$ and $A^\beta A^\beta$ contractions give $(8 \times A_2/4)^2$, and the additional contributions from $\alpha = \beta$
give $2 \times 8(A_2/4)^2$. Noting that $A_{2,2}(0) = \langle A_2^2(x) \rangle - A_2^2$ (see (16)), the net result can be put into the two equivalent forms

$$2A_4 = \frac{5}{4} A_2^2, \quad (27)$$
$$A_{2,2}(0) = \frac{1}{4} A_2^2. \quad (28)$$

This prediction is remarkably well verified in all cases. At $\beta_3 = 29$, the left and right hand sides of (28) are respectively $4.86(1) 10^{-3}$ and $4.825(10) 10^{-3}$. They are $6.093(13) 10^{-4}$ and $6.069(1) 10^{-4}$ at $\beta_3 = 84$. Similar manipulations lead to

$$A_{3,3}(0) = \frac{5}{64} A_2^3. \quad (29)$$

In this case the left and right hand sides are measured to be $2.105(7) 10^{-4}$ and $2.095(6) 10^{-4}$ for $\beta_3 = 29$, $9.365(30) 10^{-6}$ and $9.344(3) 10^{-6}$ for $\beta_3 = 84$. Hence the effects of residual interactions via non quadratic effective couplings in $A_2(x)$ and $A_3(x)$ are less than the percent in the correlations at zero distance.

Before going to the correlations at non zero distance, let us discuss their normalization, as measured by the values of $A_{2,2}(0)$ and $A_{3,3}(0)$ described just above. At the beginning of section 3, we argued that the behaviour in $\beta_3, n, m$ observed for $A_{n,m}$ could follow from the absence of a large renormalization, by the interactions, of the $\phi$-fields defined by Eq. (10). Here we note that in the confined phase the effective degrees of freedom are the massive composites $\phi_i = \text{tr} \phi^i, i = 2, 3$, so that in the limit where they are considered as free fields, one may write (see (17))

$$\langle \phi_i(0) \phi_i(0) \rangle \simeq R_i \int \frac{1}{\hat{p}^2 + 4 \sinh(m_i^2/4)}, \quad (30)$$
$$\hat{p}^2 = 4 \sin^2(p_1/2) + 4 \sin^2(p_2/2),$$

the residue $R_i$ being one if neither $\phi$ nor the composites get renormalized. We computed $R_i$ as the ratio of the l.h.s. of (30), directly measured, to the integral in the r.h.s. evaluated numerically on the lattice for the mass values fitted to the correlation data. The result is shown in Fig.6: in the whole temperature range, both residues in the even and odd channels remain uniformly very close to one.

4.2 Weak Residual Interactions: Properties of the Correlations

As we have seen in section 2 (see Fig. 1), the different $A_{n,m}(r)$’s corresponding to the same channel have very similar shapes. Their analysis in terms of one particle exchange was successful, confirmed by the agreement with residue factorization.
Nevertheless although the quantity $X_n$ of Eq. (19) does go to one at large distances, it is significantly different from one at medium and short distances (see Figs. 2-5). We will now show that two particle exchange is responsible for most of this lack of factorization.

The simplest consequence of our assumptions for correlations at finite $r$ is, using Eq. (27),

$$A_{2,4}(r) = \frac{5}{4} A_2 A_{2,2}(r),$$

which is very well verified at any distance as shown in Fig.7 for $T/T_c = 1.97$ ($\beta_3 = 29$). A similar agreement is found for the relation

$$A_{3,5}(r) = \frac{35}{24} A_2 A_{3,3}(r),$$

derived in the appendix. A new situation arises when we consider $A_{4,4}$ or $A_{5,5}$ where both the initial and final states may couple to a two particle state, $(S S)$ or $(S P)$ respectively. Then the intermediate state in a connected correlation between 0 and $r$ may consist of either one or two particles. For example, to compute $A_{4,4}(r)$, we apply the substitution rule (25) to the sum (26), and then average using the definitions of $A_{2,2}$ and $A_2$. One finds

$$4A_{4,4}(r) = \left(\frac{5}{4}\right)^2 \left[ 4A_2^2 A_{2,2}(r) + 2A_{2,2}^2(r) \right].$$

A similar treatment given in the appendix leads to the prediction

$$A_{5,5}(r) = \left(\frac{35}{24}\right)^2 \left[ A_2^2 A_{3,3}(r) + A_{2,2}(r) A_{3,3}(r) \right].$$

In the two expressions above, the second contribution, a product of two propagators in space, is that of a two-particle intermediate state, and it provides a correction to exact factorization. From the definitions Eq.(19) of $X_2$ and $X_3$, one actually gets the following estimates:

$$X_2(r) \approx \tilde{X}_2(r) \equiv 1 + \frac{A_{2,2}(r)}{2A_2^2},$$

$$X_3(r) \approx \tilde{X}_3(r) \equiv 1 + \frac{A_{2,2}(r)}{A_2^2}.$$

The estimates $\tilde{X}_2(r), \tilde{X}_3(r)$ are displayed in Figs. 2,3 (resp. 4,5), for comparison with the measured values $X_2(r), X_3(r)$ at $T/T_c = 1.97$ (resp. 5.7), i.e. $\beta_3 = 29$ (resp. 84). We see that the corrections to factorization implied by two-particle propagation provide a reasonable explanation of the behaviour observed for the
X’s at intermediate and short distances. This is especially true in the case of $X_2$, showing that there is very little room for contributions from direct non-quadratic couplings in $A_2(x)$ in the full effective action (that resulting from integration over the gauge fields). This justifies our statement that the residual interactions between the colourless boundstates of the adjoint scalars are very weak.

5 Conclusions

In this paper, we have studied properties of the two dimensional model derived in [11] by dimensional reduction of 3D QCD at high temperature. In this model, scalar fields $A$ in the adjoint representation of SU(3) interacts via SU(3) gauge fields $U$, in addition to a self-interaction generated by integration over the non-static 3D gauge degrees of freedom. Such properties are interesting since it was shown in [11] that dimensional reduction works remarkably well in this case. Also, they offer an opportunity to explore non-perturbative features in a low dimensional situation where the IR singularities are particularly severe.

By means of numerical simulations, we explored that part of phase space where the $R_\tau$-symmetry $A \rightarrow -A$ is unbroken, in accordance with the small $A$ expansion used to derive the model, known to be valid quite soon above the transition temperature of pure 3D QCD. We identified two boundstates $S$ and $P$, respectively even and odd under $R_\tau$, and thus coupled to monomials respectively of degree $2n$ and $2n + 1$ in the $A$'s. The $S$ signal coincides with that previously obtained from Polyakov loops correlations [11], where however the $P$-state contributions could not be disentangled.

These results came out from the measurement of three even-even and three odd-odd distinct correlations, as functions of the on-axis lattice distance $r$. Great care was taken in the analysis of their shape in $r$, with the result that in all cases, the signal found was that expected from the occurrence of genuine poles in momentum space. A contrario, this demonstrates that the picture where the decay with $r$ of such correlations reflects the propagation in 3D of $p = 2n$ or $2n + 1$ “electric gluons”, i.e. a correlation length equal to $1/p$ times the “Debye screening length”, is inadequate in the case under study.

By comparing the size of the three different correlations measured for each of the $S$ and $P$ sectors, we were able to show that residue factorization holds, as expected on general grounds when one particle propagates between different states. The agreement with factorization was expectedly found to be particularly good at large distances, but we could even show that deviations at shorter distances are to a large extent compatible with propagation of two particles, namely two $S$ or $S$ and $P$ respectively in the $S$ or $P$ channel. The overall picture thus is that the scalar sector of the reduced model at large distances, thought to accurately describe
static properties of 3D QCD at high temperature, consists of two weakly interacting colourless particles, respectively even and odd under the $R_{\tau}$ symmetry of the model.

There are several problems, which this study invites to investigate further. Of course, similarly detailed analysis for full QCD in (3+1)D would be interesting. Furthermore, the construction of a reduced model where the $Z_3$-symmetry of the pure gauge theory is not spoiled by the reduction process is highly desirable [27], with the hope that it exhibits a transition to a symmetric $Z_3$ phase analogous to the low temperature QCD phase.

6 Acknowledgments

We thank the DFG for support under the contract Ka 1198/4-1. K.P. was also supported by DAAD and P.B partially by KBN grant P03B01917.

Appendix: Mean Field Technique for Composite Fields Correlators

A Formulae for Traces and Determinants

Let $\phi$ be a complex $N \times N$ matrix. Here we give an elegant trick [20] to compute the traces

$$\phi_n \equiv \text{tr} \phi^n$$

for $n > N$, given $\phi_p$ for $p \leq N$.

Consider the determinant $P_N(t) \equiv Det(1 - t\phi)$, where $t$ is a complex variable. It is a polynomial of degree $N$ in $t$ and its term of degree $N$ is $(-1)^N Det(\phi)$, and we have

$$\log(P_N(t)) = \text{tr} \log(1 - t\phi).$$

Both sides of this identity can be expanded in $t$ in some finite neighbourhood of zero. The method consists in identifying the coefficients of the two series. The $N$ first orders determine the coefficients of $P_N(t)$ from the $\phi_n$’s , $n \leq N$. Then, the higher orders directly express any $\phi_n$, $n > N$ as a function of the $\phi_p$’s, $p \leq N$. Note that instead of computing $Det(\phi)$ from the order $N$ coefficient of $P_N$, one
can alternatively compute $\phi_N$ given $\text{Det}(\phi)$, which is convenient for SU(N) group matrices.

If applied to $\phi = A$, an element of the SU(3) algebra, (in which case $A_1 = 0$), this technique gives $A_n$, $n > 3$ in terms of $A_2$ and $A_3$ taken as independent variables. The first non trivial identities are

$$
A_4 = \frac{1}{2}(A_2)^2, \\
A_5 = \frac{5}{6}A_2A_3, \\
A_6 = \frac{1}{4}(A_2)^3 + \frac{1}{3}(A_3)^2, \\
A_7 = \frac{7}{12}A_3(A_2)^2, \\
A_8 = \frac{1}{8}(A_2)^4 + \frac{4}{9}(A_3)^2A_2,
$$

and the determinant is

$$
\text{Det}A = \frac{2A_3}{3!}.
$$

In what follows, we will have to manipulate monomials of the elementary scalar fields $A^\alpha(x)$, defined for SU(N) through

$$
A(x) = \sum_{\alpha=1}^{N^2-1} A^\alpha(x)\lambda^\alpha
$$

where the traceless basis $\lambda^\alpha$ is subject to the normalization

$$
\text{tr} \lambda^\alpha \lambda^\beta = \frac{1}{2}\delta_{\alpha\beta}.
$$

On this basis the anti-commutators read

$$
\{\lambda^\alpha, \lambda^\beta\} = c_{\alpha\beta} 1_N + \sum_{\gamma=1}^{N^2-1} d_{\alpha\beta\gamma} \lambda^\gamma,
$$

with real and totally symmetric tensors $c$ and $d$. With these normalizations and notations, we have

$$
\text{tr} A^2 = \sum_{\alpha\beta=1}^{N^2-1} \text{tr} [\lambda^\alpha \lambda^\beta] A^\alpha A^\beta = \frac{1}{2} \sum_{\alpha\beta=1}^{N^2-1} \delta_{\alpha\beta} A^\alpha A^\beta, \\
\text{tr} A^3 = \sum_{\alpha\beta\gamma=1}^{N^2-1} \text{tr} [\lambda^\alpha \lambda^\beta \lambda^\gamma] A^\alpha A^\beta A^\gamma = \frac{1}{4} \sum_{\alpha\beta\gamma=1}^{N^2-1} d_{\alpha\beta\gamma} A^\alpha A^\beta A^\gamma.
$$
The projection property
\[
\sum_{\alpha=1}^{N^2-1} \lambda_{ab}^\alpha \lambda_{cd}^\alpha = \frac{1}{2} \left( \delta_{ac} \delta_{bd} - \frac{1}{N} \delta_{ab} \delta_{cd} \right)
\]  
(46)

can be used to derive that for any pair \(X, Y\) of complex \(N \times N\) matrices the following identities hold:
\[
\sum_{\alpha} \text{tr} \lambda^\alpha X \text{tr} \lambda^\alpha Y = \frac{1}{2} \left( \text{tr} XY - \frac{1}{N} \text{tr} X \text{tr} Y \right),
\]
(47)
\[
\sum_{\alpha} \text{tr} X \lambda^\alpha Y \lambda^\alpha = \frac{1}{2} \left( \text{tr} X \text{tr} Y - \frac{1}{N} \text{tr} XY \right).
\]
(48)

In what follows we specialize to \(N = 3\).

B Correlators of Composite Operators

By gauge invariance, \(A_2(x)\) and \(A_3(x)\) can be chosen as the two effective degrees of freedom. By \(R_\tau\)-symmetry, the even and odd sectors under \(A \to -A\) decouple. Here we derive consequences of the assumption that their dynamics is determined at leading order by their given vacuum expectation values \(A_2\) and 0 respectively and their connected two-body correlations \(A_{2,2}(x)\) and \(A_{3,3}(x)\).

We use a Wick like treatment to express the higher order connected correlation functions and averages through the quantities mentioned above. If \(n = 2p\), any \(A^\alpha(x)\) is assigned to belong to a pair \(A_2(x)\), then considered as a free field denoted \(S(x)\). So each monomial is replaced by a sum over all such pairings, and each of the \(p\) pairs is subject to the substitution \(W_2\),
\[
W_2 : \quad A^\alpha A^\beta \to \delta_{\alpha\beta} \frac{1}{4} S(x),
\]
(49)
leading to a monomial of degree \(p\) in \(S(x)\). If \(n = 2p + 1\), one first performs the \(p - 1\) possible substitutions \(W_2\) (the result after \(p\) substitutions would transform as an octet under \(SU(3)\) and thus vanishes), which yield a monomial necessarily proportional to \(S^{p-1}(x) \text{tr} A^3(x)\). There we apply the substitution \(W_3\),
\[
W_3 : \quad \text{tr} A^3(x) \to P(x),
\]
(50)
where \(P(x)\) is also considered as a free field.
Once the local operators have been expressed in terms of $S(x)$ and $P(x)$, any average is obtained by using

\begin{align}
\langle S(x) \rangle &= A_2, \\
\langle S(x) S(0) \rangle &= A_{2,2}(x) + A_2^2, \\
\langle P(x) S(0) \rangle &= 0, \\
\langle P(x) \rangle &= 0, \\
\langle P(x) P(0) \rangle &= A_{3,3}(x). 
\end{align}

These rules generalize the way how in Section 4 we computed averages involving $A_4(x)$. Let us now detail calculations involving $A_5(x)$.

From Eqs. (51-55), we have

\begin{equation}
6 \frac{A_5(x)}{5} = \sum_{\alpha \beta} A^\alpha A^\beta \text{ tr } \lambda^\alpha \lambda^\beta \sum_{\gamma,\delta,\epsilon} A^\gamma A^\delta A^\epsilon \text{ tr } \lambda^\gamma \lambda^\delta \lambda^\epsilon.
\end{equation}

We apply rule $W_2$ to the right hand side. The contraction of $\alpha$ with $\beta$ produces $S(x) P(x)$ once. Using Eqs. (61, 62, 63, 64), one finds that the 6 $W_2$—contractions of either $\alpha$ or $\beta$ with either one of the three other indices contribute each the same amount

\begin{equation}
\frac{1}{4} S(x) \times \frac{1}{2} \text{ tr } A^3(x),
\end{equation}

that is from rule $W_3$

\begin{equation}
\frac{1}{8} S(x) P(x).
\end{equation}

We thus arrive to the substitution

\begin{equation}
A_5(x) \rightarrow \frac{5}{6} \left( 1 + \frac{6}{8} \right) S(x) P(x) = \frac{35}{24} S(x) P(x),
\end{equation}

which we perform in the two point correlations $A_{3,5}(x) \equiv \langle A_3(x) A_5(0) \rangle$ and $A_{5,5}(x) \equiv \langle A_5(x) A_5(0) \rangle$ to get

\begin{align}
A_{3,5}(x) &= \frac{35}{24} \langle P(x) P(0) S(0) \rangle, \\
A_{5,5}(x) &= \left( \frac{35}{24} \right)^2 \langle P(x) P(0) S(x) S(0) \rangle.
\end{align}

According to Eqs. (61-63), these averages are given by

\begin{align}
A_{3,5}(x) &= \frac{35}{24} A_2 A_{3,3}(x), \\
A_{5,5}(x) &= \left( \frac{35}{24} \right)^2 A_{3,3}(x) \left( A_2^2 + A_{2,2}(x) \right).
\end{align}
As a last application, we derive the value of \( A_{3,3}(0) \equiv \langle A^3_3(x) \rangle \). By definition

\[
A^2_3(x) = \sum_{\alpha \beta \gamma} A^\alpha A^{\beta} A^{\gamma} \text{tr} \lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} \sum_{\delta \epsilon \zeta} A^\delta A^\epsilon A^\zeta \text{tr} \lambda^{\delta} \lambda^\epsilon \lambda^\zeta,
\]

where all the fields are taken at the same point \( x \). Applying all possible \( W_2 \) substitutions and using Eqs. (47, 48) leads to the substitution

\[
A^2_3(x) \rightarrow \frac{5S^3(x)}{64},
\]

and averaging via Eq. (51) provides the final result (29).

References

[1] P. Ginsparg, *Nucl. Phys.* B170 (1980) 388.

[2] T. Appelquist and R. Pisarski, *Phys. Rev.* D23 (1981) 2305.

[3] S. Nadkarni, *Phys. Rev.* D27 (1983) 917; *Phys. Rev.* D38 (1988) 3287; *Phys. Rev. Lett.* 60 (1988) 491.

[4] N. P. Landsman, *Nucl. Phys.* B322 (1989) 498.

[5] T. Reisz, *Z. f. Phys.* C53 (1992) 169.

[6] P. Lacock, D. E. Miller and T. Reisz, *Nucl. Phys.* B369 (1992) 501.

[7] L. Kärkkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, *Phys. Lett.* B282 (1992) 121.

[8] L. Kärkkäinen, P. Lacock, B. Petersson and T. Reisz, *Nucl. Phys.* B395 (1993) 733.

[9] K. Kajantie, M. Laine, K. Rummukainen, M. Shaposhnikov, *Nucl. Phys.* B503 (1997) 357.

[10] O. Philipsen, “Static correlation lengths in QCD at high temperature and finite density”, [hep-lat/0011019](http://arxiv.org/abs/hep-lat/0011019).
[11] P. Bialas, A. Morel, B. Petersson, K. Petrov and T. Reisz, “High Temperature 3D QCD: Dimensional Reduction at Work”, Nucl. Phys. B581 (2000) 477.

[12] P. Arnold and L.G. Yaffe, Phys. Rev. D52 (1995) 7208.

[13] L. Kärkkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, Nucl. Phys. B418 (1994) 3.

[14] T. Reisz, “Dimensionally Reduced SU(2) Yang-Mills Theory is Confined”, in Quantum Field Theoretical Aspects of High Energy Physics, 230-235, B. Geyer and E.M. Ilgenfritz Eds., Frankenthal 1993.

[15] K. Kajantie, M. Laine, J. Peisa, A. Rajantie, K. Rummukainen and M. Shaposhnikov, Phys. Rev. Lett. 79 (1997) 3130.

[16] K. Kajantie, M. Laine, A. Rajantie, K. Rummukainen, and M. Tsypin, JHEP 9811 (1998) 11.

[17] F. Karsch, M. Oevers and P. Petreczky, Phys. Lett. B442 (1998) 291.

[18] S. Datta and S. Gupta, Nucl. Phys. B534 (1998) 392 Phys. Lett. B471 (2000) 382.

[19] A. Hart, O. Philipsen, J.D. Stack and M. Teper, Phys. Lett. B396 (1997) 217.

[20] A. Hart and O. Philipsen, Nucl. Phys. B572 (2000) 243.

[21] G. Parisi, Statistical Field Theory (Addison-Wesley, New York, 1988)

[22] M. Alford and M. Gleiser Phys. Rev. D48 (1993) 2838; J. Borrill and M. Gleiser, Nucl. Phys. B483 (1997) 416.

[23] C. Legeland, PhD Thesis, “Aspects of (2+1) Dimensional Lattice Gauge Theory” (University of Bielefeld, Germany, September 1998).

[24] E. D’Hoker, Nucl. Phys B201 (1982) 401.

[25] W. Buchmuller and O. Philipsen Phys. Lett. B397 (1997) 112.

[26] We thank M. Bauer for providing us with this simple trick.

[27] R. Pisarski, “Quark-Gluon Plasma as a condensate of SU(3) Wilson Lines”, hep-ph/0006205. K. Kajantie, M. Laine, J. Peisa, A. Rajantie, K. Rummukainen, M. Shaposhnikov Phys. Rev. Lett. 79 (1997) 3130.
Figure 1: The on-axis correlations $A_{n,m}(r)$ at $T/T_c = 1.97$ ($\beta_3 = 29$), versus the distance in units of $1/T$. The even cases $[n, m] = [2, 2], [2, 4]$ and $[4, 4]$ all have the same shape, and the odd cases $[3, 3], [3, 5]$ and $[5, 5]$, again similar with each other in shape, are steeper.
Figure 2: Residue factorization: data for the quantity $X_2$, Eq. (19) at $T/T_c = 1.97$ ($\beta_3 = 29$) versus the distance in units of $1/T$. It approaches one at large distances. The quantity $\tilde{X}_2$ corresponds to our interpretation (Section 4, Eq. (35)) of the deviation from one of $X_2$ at shorter distances.
Figure 3: Residue factorization: data for the quantity $X_3$, Eq. (13) $T/T_c = 1.97$ ($\beta_3 = 29$) versus the distance in units of $1/T$. It approaches one at large distances. The quantity $\tilde{X}_3$ corresponds to our interpretation (Section 4, Eq. (36)) of the deviation from one of $X_3$ at shorter distances.
Figure 4: Same as in Fig. 2 at $T/T_c = 5.7$ ($\beta_3 = 84$).
Figure 5: Same as in Fig. 3 at $T/T_c = 5.7$ ($\beta_3 = 84$).
Figure 6: The residues $R_i$ defined by Eq. (30) stay close to one in the whole temperature range.
Figure 7: Plot of $4A_{24}/5A_{2}A_{22}$ at $T/T_c = 1.97$ ($\beta_3 = 29$). This quantity is one if Eq. (31) exactly holds.