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A Discontinuous-Galerkin Finite-Element Method for Simulation of Packed Bed Chromatographic Processes

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Abstract: Packed bed chromatography is an important unit operation for purification of product molecules in biopharmaceutical processes. Packed bed chromatographic processes are modeled as advection-diffusion-reaction partial differential equations. The advection term strongly dominates the diffusion term. Therefore, specialized numerical methods must be used for efficient simulation of packed bed chromatographic processes. In this paper, we use a discontinuous-Galerkin method on finite-elements for spatial discretization and low storage explicit Runge-Kutta (LSERK) methods for numerical solution of the resulting system of differential equations. We study the numerical solution of deterministic and stochastic models of packed bed chromatographic processes. The stochastic model and its corresponding numerical solution constitute the first step toward systematic monitoring, fault detection, and optimal predictive control of chromatographic processes. It is also an essential ingredient in uncertainty quantification for efficient and robust design and operation of chromatographic processes.

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1. INTRODUCTION

Even though chromatography is a costly separation process, it is extensively used in downstream biopharmaceutical manufacturing (Hong et al., 2018). Consequently, mathematical models for simulation, parameter estimation, design, control, and optimization of chromatographic processes are important. The modeling (Arkell et al., 2018), simulation (Klatt et al., 2000; von Lieres and Andersson, 2010; He et al., 2018; Leweke and von Lieres, 2018; Borg et al., 2013), control (Klatt et al., 2002), and optimization (Püttmann et al., 2013; Holmqvist and Magnusson, 2016; Sellberg, 2018) of chromatographic processes have been extensively studied. Observers for chromatographic processes have also been studied (Alamir and Corrion, 2003; Corrion and Alamir, 2006; Küpper and Engell, 2006; Küpper et al., 2009; Lemoine-Nava and Engell, 2014). Also many processes such as adsorption possess structural similarities to chromatography, such that the numerical techniques for observers in such processes can be transferred to chromatographic processes (Won and Lee, 2012). In this paper, we use high-order methods for efficient numerical simulation of the advection-diffusion partial differential equation that models chromatographic processes (Hesthaven and Warburton, 2008; Lord et al., 2014). We apply this method to handle stochastic inlet boundary conditions as a first step toward extended and unscented Kalman filters for chromatographic processes. Systematic use of such filters allows for monitoring, prediction and fault detection of chromatographic processes. The specialized methods, that we use for simulation in this paper, are also needed in the extended and unscented Kalman filters for continuous-discrete systems.

1.1 Chromatographic Processes

Chromatography is widely used for separation of peptides and proteins in biopharmaceutical processes (Guchon et al., 2006). Analytically, chromatography is used for separating mixtures into components in order to analyze, identify, purify, quantify, and optimize processes (Holmqvist and Magnusson, 2016). In production scale, it is used to isolate and purify the compound of interest. Fig. 1 illustrates a chromatographic column. In liquid column chromatography a cylindrical column is packed with a stationary porous material. A continuous flow of liquid with more components are sent to the inlet of the column in a so-called mobile phase. As illustrated in Fig. 1, this creates a flow that moves through the column from the left to the right. The injected liquid contains the mixture...
of components. The physical and chemical properties of
each component lead to different affinities of the molecules
to the mobile and the stationary phase. This difference
is used to separate the molecules by having them move
along the column at different speeds. Typically, a known
solute with a strong affinity to the stationary phase is
injected to manipulate pH, polarity or another property
that influences the separation of the molecules of interest.

1.2 Motivation

The simulation of multicomponent chromatographic pro-
cesses under non-linear conditions and reaction kinetics
require fast and accurate numerical methods (Javeed et al.,
2011; He et al., 2018; Meyer et al., 2018). These pro-
cesses are typically described by parabolic, non-linear,
convection-dominated, coupled partial differential equa-
tions (Javeed et al., 2011). Due to the underlying physics,
these partial differential equations are advection domi-
nated such that discontinuities in the inputs produces
shocks on either side of the elution bands (Ma and Guio-
chon, 1992). These shocks challenge and cause difficulties
for many numerical methods based on finite-difference
or finite volume discretization. To mitigate this, we use
a high-order nodal discontinuous-Galerkin finite-element
(DG-FE) scheme as an accurate and cost efficient simula-
or a first-step towards observers based on extended and
unscented Kalman filter technology.

1.3 Paper Organization

This paper is organized as follows. Section 2 presents the
equilibrium dispersive model for chromatography that we
study in this paper. Section 3 describes the numerical
discretization of the model by the DG-FE method. Section
4 presents the numerical results and conclusions are made
in Section 5.

2. THE CHROMATOGRAPHIC MODEL

The literature contains a variety of models that describes
the behaviour of liquid chromatographic processes inside
a separation column. Well known models include the
equilibrium dispersive model, the lumped kinetic model,
and the general rate model (Guichon et al., 2006). These
models are non-linear, convection dominated partial differ-
tential equations describing the interaction between chro-
matographic components inside a separation column, each
under varying assumptions. In this work, we consider
the equilibrium dispersive model (ED). The ED model is
derived on six fundamental assumptions (Qamar et al.,
2013): 1) The chromatographic process occurring in the
separation column is isothermal. 2) The column is radially
homogeneous. 3) The stationary phase of the column is
packed with a porous material of equally sized spherical
particles. 4) The mobile phase is incompressible. 5) The
mobile and stationary phase of each component do not
interact with each other. 6) There is axial dispersion in
the process, which causes band broadening of the concen-
tration profiles.

2.1 The Equilibrium Dispersive Model of Chromatography

The ED model is a partial differential equation of the form
(1a) and involves the concentrations of $N_c$ components in
the mobile, $c_i$, and stationary phase, $q_i$. The transition
from mobile to stationary phase is modelled by an equi-
librium adsorption isotherm. In the literature, different
boundary conditions exist to model almost any configu-
rations of the column (Guichon et al., 2006). A Danckwerts
boundary condition is used to model the inlet concentra-
tion profile and a zero Neumann condition is given at the
outlet (Danckwerts, 1953). Given proper initial conditions,
the multi-component ED model is given as

$$\frac{\partial c_i}{\partial t} + \nabla \cdot (D_i \nabla c_i) - v_i \frac{\partial c_i}{\partial z} = 0,$$

$$c_i = c_{i,0},$$

$$\partial c_i / \partial z = 0,$$

$$q_i = q_{i,n}.\]
The competitive Langmuir adsorption isotherm describes a rectangular pulse of injection time $t_{\text{inj}}$ given as
\[
c_{i,\text{inj}} = \begin{cases} c_{i,f}, & 0 \leq t_{\text{inj}} \leq t, \\ 0, & t_{\text{inj}} < t, \end{cases}
\]
with a feed concentration for the $i$th component $c_{i,f}$.

### 2.2 The Equilibrium Isotherm

The equilibrium isotherm describes the distribution of a component between the mobile and the stationary phase in the porous media. This implies that in general $q_i$ is a function $q_i = f(c_1, c_2, \ldots, c_i, \ldots, c_{N_c})$ depending on all the components in the chromatographic column. Therefore $f$ is typically a non-linear function that couples the equations in (1).

The competitive Langmuir adsorption isotherm describes a non-linear interaction between components as (Guichon et al., 2006)
\[
q_i = \frac{a_i c_i}{1 + \sum_{s=1}^{N_c} b_s c_s}.
\]
The linear adsorption isotherm is given as
\[
q_i = a_i c_i.
\]
For low concentrations the competition between components become almost negligible and (3) typically simplifies to the linear isotherm (4).

### 2.3 First-Order Equations

To apply the discretization techniques of the DG-FE method, equation (1a) is rewritten as two first order equations (Bassi and Rebay, 1997). Omitting the index $i$ and defining the functions
\[
f(c, q(c)) := v c - \sqrt{D} q, \\
w(c) := c + F q(c),
\]
equation (1a) takes the form
\[
\frac{\partial w}{\partial t}(c, q(c)) = -\frac{\partial}{\partial z}(f(c, q(c)),
\]
\[
g(c) = \sqrt{D} \frac{\partial c}{\partial z}.
\]
Solving the equations in (7) is then equivalent to solving (1a).

### 3. DISCRETIZATION

The model equations (7) are solved using a method-of-lines approach. This procedure allows for a separate discretization of the spatial and temporal variables. The spatial discretization uses a DG-FEM method (Meyer et al., 2018). This discretization recovers a continuous-time-discrete-space system. This semi-discrete system (27) is discretized in time using a low storage, five stage, fourth order explicit Runge-Kutta (LSERK) method (Hesthaven and Warburton, 2008).

#### 3.1 Spatial Discretization

Consider the spatial domain $\Omega$ and partition it into $N_e$ non-overlapping elements $z \in [z_i^k, z_i^k] = \Omega_k$, where $z_i^k$ and $z_i^{k+1}$ refer to the left and right boundaries on $\Omega_k$. The local solution is approximated on each element, $\Omega_k$, by a polynomial of order $N = N_p - 1$:
\[
\begin{bmatrix} w^k_h(z,t) \\ g^k_h(z,t) \end{bmatrix} = \sum_{n=1}^{N_p} \begin{bmatrix} \hat{w}_n^k(t) \\ \hat{g}_n^k(t) \end{bmatrix} \varphi_n(z) = \sum_{i=1}^{N_p} \begin{bmatrix} w_i^k(z_i^k, t) \\ g_i^k(z_i^k, t) \end{bmatrix} l_i^k(z).
\]

Here the local solution $(w_h^k, g_h^k)$ may be represented in a modal, $\{\varphi_n\}_{n=1}^{N_p}$, or nodal, $\{l_i^k\}_{i=1}^{N_p}$, basis. In this regard, recall that the Lagrange polynomials, $\{l_i^k\}_{i=1}^{N_p}$, satisfy the interpolation property, $l_i^k(z_j) = \delta_{ij}$ (the Kronecker delta function). Given (8) on each of the $N_{e}$ elements, the DG-FEM method approximates the global solution, $(w, g)$, by the $N_{e}$th order piecewise polynomials
\[
\begin{bmatrix} w(z, t) \\ g(z, t) \end{bmatrix} \simeq \sum_{k=1}^{N_p} \begin{bmatrix} w_i^k(z_i^k, t) \\ g_i^k(z_i^k, t) \end{bmatrix} \varphi_k(z).
\]

To determine the approximate solution, $(w_h, g_h)$, the DG-FEM method uses a Galerkin approach. To this end, consider the element-wise, local residuals
\[
z \in \Omega^k : R(w)^k = \frac{\partial w_i^k}{\partial t} + \frac{\partial w_i^k}{\partial z},
\]
\[
R(g)^k = g_i^k - \sqrt{D} \frac{\partial c_i^k}{\partial z},
\]
and define the approximation space of piecewise smooth polynomials of order $N = N_p - 1$
\[
V_k^N := \{v : v_k \in \mathbb{P}_N(\Omega_k), \forall \Omega_k \in \Omega\},
\]

The local, discrete Galerkin approximation to (7) then becomes
\[
\int_{\Omega_k} R(w)^k \varphi_n(z) \, dz = 0, \quad (12a)
\]
\[
\int_{\Omega_k} R(g)^k \varphi_n(z) \, dz = 0, \quad (12b)
\]
where $\{\varphi_n\}_{n=1}^{N_p-1}$ is an appropriate basis for $V_k^N$. To recover a DG-FEM formulation, the system (12a) is recast to the form (Meyer et al., 2018):
\[
\int_{\Omega_k} \frac{\partial w_i^k}{\partial t} l_j^k \, dz + \int_{\Omega_k} \frac{\partial f_i^k}{\partial z} l_j^k \, dz = \int_{\Omega_k} \hat{n} \cdot (f_i^k - f^*) l_j^k \, dz,
\]
\[
\frac{1}{\sqrt{D}} \int_{\Omega_k} g_i^k l_j^k \, dz + \int_{\partial \Omega_k} \hat{n} \cdot (c_i^k - c^*) l_j^k \, dz = \int_{\Omega_k} \frac{\partial g_i^k}{\partial z} l_j^k \, dz.
\]
Here $c^*(c^-, c^+)$ and $f^*(c^-, c^+, q^-, q^+)$ denote appropriate numerical fluxes, where (-) and (+) refer to interior and exterior interface states of each element, $\Omega_k$. Further, $\partial \Omega_k$ denotes the edges of the $k$th element and $\hat{n} = (-1, 1)^T$ is the outward pointing unit normal.
3.2 Construction of orthonormal basis functions - \{\varphi_n\}_{n=1}^{N_p-1}

Consider the affine mapping \( z : [-1, 1] \to \Omega_k \) given by
\[
z(\xi) = z^k + \frac{1 + \xi}{2} \Delta z,
\]
(14)
where \( \Delta z = \Delta z^k = z^k_{i+1} - z^k_i \) is the equidistant length of the elements, \( \Omega_k \). The orthonormal basis functions, \( \varphi_n \), are then chosen as the normal Legendre polynomials
\[
\varphi_n(\xi) = \tilde{P}_{n-1}(\xi) = \frac{1}{\sqrt{2n+1}} P_{n-1}(\xi),
\]
(15)
where \( P_{n-1} \) are the usual Legendre polynomials. By the map (15), the local, approximate solution is now represented by basis functions defined on the reference element, \( I \). Further, by using the Legendre-Gauss-Lobatto quadrature points on \( I \), the Legendre Vandermonde matrix, \( V \in \mathbb{R}^{N_p \times N_p} \), is well conditioned (Hesthaven and Warburton, 2008). This implies that the the local, element-wise Galerkin operators are given by
\[
M^k = \frac{\Delta z^k}{2} \int_{-1}^{1} t_i^k(\xi) t_j^k(\xi) \, d\xi = \frac{\Delta z^k}{2} M, \quad (16a)
\]
\[
S^k = \int_{-1}^{1} t_i^k(\xi) \frac{d}{d\xi} t_j^k(\xi) \, d\xi = M D \xi, \quad (16b)
\]
where \( M = (V V^T)^{-1} \). The differentiation matrix is computed by \( D \xi = V \xi V^{-1} \), where \( V_{\xi,(i,j)} = \frac{d t_j}{d \xi} \xi_i^k \) is the gradient of the Vandermonde matrix. Note that the affine mapping allows for exact evaluation of the integrals (16) without the use of quadrature rules (Hesthaven and Warburton, 2008). Now, by substitution of the nodal expansion (8) and the operators (16) into (13), the semi-discrete form becomes
\[
M^k \frac{\partial w_h^k}{\partial t} + S^k f_h^k = \mathcal{E} \left[ \hat{n} \cdot (f_h^k - f^*) \right] z^k_i, \quad (17a)
\]
\[
M^k g_h^k = \sqrt{D} S^k c_h^+ - \sqrt{D} \mathcal{E} \left[ \hat{n} \cdot (c_h^k - c^*) \right] z^k_i. \quad (17b)
\]
Here the local operator \( \mathcal{E} \) extracts the surface term and \( \mathcal{E} = (e_1, e_N) \), where \( e_i \) is a \( N_p \) size vector with value 1 at position \( i \). The final semi-discrete form is recovered from (17)
\[
\frac{\partial w_h^k}{\partial t} = -\frac{2}{\Delta z} D \xi c_h^k + \frac{2}{\Delta z} \mathcal{L} \left[ \hat{n} \cdot (f_h^k - f^*) \right] z^k_i, \quad (18a)
\]
\[
g_h^k = \sqrt{D} \frac{2}{\Delta z} D \xi c_h^k - \sqrt{D} \mathcal{E} \left[ \hat{n} \cdot (c_h^k - c^*) \right] z^k_i, \quad (18b)
\]
where \( (w_h^k, g_h^k, f_h^k) \in \mathbb{R}^{N_p} \) is the nodal coefficients on each element, \( \Omega_k \), and \( \mathcal{L} := M^{k-1} \mathcal{E} \) is the lifting matrix.

3.3 Numerical Flux Functions

To capture the physical phenomena of the model (1), the following specifies appropriate numerical fluxes, \( (c^*, g^*, f^*) \). The fluxes may be approximated by a variety of options (Javeed et al., 2011). The following choices are based on considerations presented in Meyer et al. (2018). Given the diffusive nature of band broadening, \( c^* \) is approximated on all element boundaries by a central flux
\[
c^* = \frac{c^- + c^+}{2}. \quad (19)
\]
The field differences on each element interface are then
\[
\hat{n} \cdot (c_h^k - c^*) = \hat{n} \cdot (c^- - c^+). \quad (20)
\]
Given the definition of \( f \) in (5), it is reasonable to consider \( f^* \) as the sum of a convective and diffusive term
\[
f^* = f_{\text{conv}}(c^*, c^+) + f_{\text{diff}}(g^-, g^+). \quad (21)
\]
The convective term is approximated by an upwind flux and the diffusive term by a central flux on all interfaces, except at the left boundary
\[
f^* = \frac{v}{2} (c^- + c^+ + \hat{n} \cdot (c^- + c^+)) + \sqrt{D} \frac{g^- + g^+}{2}. \quad (22)
\]
The field differences on each element interface, excluding the left boundary, is given by
\[
\hat{n} \cdot (f_h^k - f^*) = \hat{n} \cdot \left( \frac{v(c^- + c^+) - \sqrt{D}(g^- - g^+)}{2} - \frac{v(c^- - c^+)}{2} \right). \quad (23)
\]
On the remaining left most interface, the flux is specified as a central flux to ease the implementation of the Robin boundary condition, thereby giving rise to the field difference
\[
\hat{n} \cdot (f_h^k - f^*) = \hat{n} \cdot (f^- - f^+). \quad (24)
\]

3.4 Imposing the Boundary Conditions

The DG-FEM method allows for a practical implementation of the boundary conditions, which is done weakly by defining the exterior ghost states \( (c^+, g^+) \) at the left and right most boundary of \( \Omega \). The Robin boundary condition (1b) at the inlet is implemented by requiring that
\[
c_h^+ = c_h^+ \quad \text{at the left boundary,}\quad f_h^+ = f_h^- + 2v c_h\text{-in}. \quad (25)
\]
Similarly, the Neumann condition (1c) at the outlet is enforced by the requirement that
\[
c_h^+ = c_h^- \quad \text{at the right boundary,} \quad g_h^+ = -g_h^- \quad (26)
\]

3.5 Forward Time Integration

To advance the semi-discrete form (18) in time, the DG-FEM approach uses the chain-rule on \( \frac{\partial}{\partial t} \) in (1a) to obtain
\[
\frac{\partial w_h^k}{\partial t} = \left( I + F \frac{\partial \sigma_k}{\partial \xi} \right)^{-1} \frac{\partial w_h^k}{\partial \xi}. \quad (27)
\]
Now, \( \sigma_k \) and \( w_k \) correspond to row vectors in \( \mathbb{R}^{N_p N_p} \) that contain the stacked nodal values \( (c^k_h, w^k_h) \) for each component \( i \) on each element. \( \frac{\partial \sigma_k}{\partial \xi} \) is the Jacobian in \( \mathbb{R}^{N_p N_p \times N_p N_p} \) of the associated equilibrium isotherm evaluated in the corresponding nodal values of \( c^k \) as diagonals in the Jacobian. \( I \) is the identity matrix. The right hand side is evaluated at every element \( \Omega_k \) for every component \( i \) to produce a set of ODEs that are solved using an LSERK method.

4. NUMERICAL CASE STUDY

To demonstrate the capabilities of the DG-FEM method, the following considers two case studies. To verify the DG-FEM scheme, the first study performs a convergence test for the linear isotherm (4) that compares the error between the known solution given by van Genuchten (1981) and the
computed DG-FEM solution. To represent the inherent uncertainty of the chromatographic process, the second study performs a stochastic simulation of the non-linear Langmuir isotherm with a piecewise normally distributed inlet concentration noise and a noisy initial concentration.

4.1 Case I: One component, linear isotherm

This case study demonstrates numerical convergence of the DG-FEM scheme by verifying the h-p-adaptivity of the error, which is defined by the difference between the numerical and analytical solution, i.e., $e_a = c - c_a$. Table 1 lists the simulation parameters. The injection is a rectangular pulse of $t_f = 1$ of time $t_{inj} = 1$. The final simulation time is $t_f = 4$.

The error is measured in the $L^2(\Omega)$ norm as

$$||e_a||_{\Omega,h} = \sum_{k=1}^{N_e} ||e_a||_{\Omega_k} = \sum_{k=1}^{N_e} \sqrt{\sum_{i=1}^{N_{p,k}} (e_a^i)^2} \frac{\Delta x_i^2}{2} M_{e_a}^i,$$

which is similar to the standard 2-norm. Figure 2 compares the results of h- and p-adaptivity. These are two complementary refinement techniques. In h-adaptivity, the polynomial order, $N$, is constant and the mesh is adjusted. In p-adaptivity, $N$ is kept constant while the polynomial order is increased on each element. In particular, p-adaptivity proves superior to h-adaptivity in terms of DOFs and accuracy.

4.2 Case II: Two components, stochastic, non-linear isotherm

This case study investigates the effect of introducing process noise on the inlet concentration of the column. The inlet concentration $c_{1,f}$ perturbed

$$c_{1,in}^{\Delta w} = \begin{cases} c_{1,f} + \Delta w^c, & 0 \leq t_{inj} \leq t, \\ \Delta w^c, & t_{inj} < t, \end{cases}$$

and the piecewise noise is sampled from $\Delta w \sim \mathcal{N}(0,1)$, where $\Delta w^c = \Delta w > 0$. Table 2 lists the simulation parameters. The injection profiles are noisy, rectangular pulses of time $t_{inj} = 12$ with final simulation time $t_f = 50$. Figure 3 compares the noisy, inlet concentrations of component 1 and 2 with the respective, predicted states at the outlet. Compared to the deterministic case, the stochastic model results show a significant variation in the predicted states at the column outlet. Table 3 displays the RMSE values that are calculated based on the deterministic and stochastic band profiles for times $t_j$. As Table 3 shows, the stochastic band profile for $c_1$ shows large variations until $t = 23$, before it collapses and dissipates complementary to the deterministic prediction. In particular, a large gap between both simulations for $c_2$ occurs at $t_{23}$ to $t_{23}$. As a result, the area $A_{c2,\text{out}}$ and $A_{c2,\text{out}}$ is $7.75 \times 10^4$ and $3.73 \times 10^4$ deviates significantly.
5. CONCLUSION

This paper presents a high-order DG-FE method for efficient and accurate simulation of chromatographic processes that can be described by the ED model. This is motivated by the observation that high-order methods should be considered when simulating convection-dominated nonlinear partial differential equations such as (1). As the main contribution, this paper considers the effects of uncertainties in the inlet concentrations. The stochastic simulation demonstrates significant deviations between the deterministic and stochastic states and outlet concentrations. The stochastic chromatographic simulation model is a first step towards observers based on extended and unscented Kalman filter technology.

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