Counting the degrees of freedom of generalized Galileons

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Abstract

We consider Galileon models on curved spacetime, as well as the counterterms introduced to maintain the second-order nature of the field equations of these models when both the metric and the scalar are made dynamical. Working in a gauge invariant framework, we first show how all the third-order time derivatives appearing in the field equations — both metric and scalar — of a Galileon model or one defined by a given counterterm can be eliminated to leave field equations which contain at most second-order time derivatives of the metric and of the scalar. The same is shown to hold for arbitrary linear combinations of such models, as well as their k-essence-like/Horndeski generalizations. This supports the claim that the number of degrees of freedom in these models is only 3, counting 2 for the graviton and 1 for the scalar. We comment on the arguments given previously in support of this claim. We then prove that this number of degrees of freedom is strictly less that 4 in one particular such model by carrying out a full-fledged Hamiltonian analysis. In contrast to previous results, our analyses do not assume any particular gauge choice of restricted applicability.

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I. INTRODUCTION

Scalar-tensor theories are widely used in cosmology and extensions of general relativity, with applications ranging from inflation to the late-time observed acceleration of the Universe, and tests of gravitation. Motivated in part by the ability of some of these theories to give an alternative to dark energy, there has recently been renewed interest in the delineation of theories in which — besides the two degrees of freedom of a massless graviton — there is only one propagating degree of freedom (d.o.f.) stemming from the scalar. Along this line, an important result was achieved by Horndeski who classified all scalar-tensor theories in 4 dimensions having field equations (both for the metric and the scalar) with derivatives of order less than or equal to two \[1\]. Similarly, Ref. \[2\] (see also \[3\] for earlier works) introduced, on flat spacetime and for an arbitrary number of dimensions \(D\), a set of scalar theories with field equations exactly of second order: the Galileons. These theories were later “covariantized”, i.e., put on arbitrary curved spacetime with a dynamical metric, while maintaining the second-order nature of the scalar field equation, as well as yielding metric field equations of the same order \[4, 5\]. This covariantization procedure involves a non-minimal coupling between the curvature and the scalar in the form of very specific counterterms able to remove all higher derivatives from the field equations. Indeed a minimal covariantization of the original Galileon of Ref. \[2\] (i.e., the mere replacement of partial derivatives by covariant derivatives in the action) was shown to lead to third-order derivatives in the field equations.\(^1\) Another relevant work is that of \[6\], which classified all scalar theories having equations of motion of order less than or equal to two on a flat spacetime of arbitrary dimension, and then covariantized these theories. It was shown that the original flat-space time Galileons \[2\], their flat spacetime generalizations \[6\] as well as their covariantization \[4, 6\] (with the meaning above) belong, for a spacetime with 4 dimensions, to the set of Horndeski (as they should according to Horndeski theorem) \[6, 7\]. These theories were also generalized to the case of multiscalars and \(p\)-forms \[8–16\].

Having covariant second-order field equations is \textit{a priori} enough, once diffeomorphism invariance is taken into account, to have just 3 propagating degrees of freedom in vacuum (counting 2 for the metric and 1 for the scalar), and to put the theory on the safe side as far as Ostrogradski’s type of instability is concerned \[17, 18\]. However, to the best of our knowledge, a proper Hamiltonian counting of degrees of freedom in these theories, including the ones contained in the metric, has so far not been carried out (the flat spacetime limit has been analyzed in Refs. \[19, 20\], while some other references start from a gauge-fixed action in which the gauge invariance has not been properly fixed or is explicitly broken \[21–24\]). In fact, the Hamiltonian analysis is complicated by the kinetic mixing (or braiding to use the wording of \[25\]) between the scalar and the metric, and one aim of the present work is to provide a first step towards a proper Hamiltonian treatment of Galileon-like theories.

A second motivation stems from the work of \[21, 22\], building on the earlier works of \[26, 27\], suggesting that despite the presence of higher derivatives in the field equations of minimally covariantized Galileons, the number of propagating degrees of freedom can still be only three due to the presence of some hidden constraint in the theory. (In minimally covariantized Galileons and related models, as stressed in \[4, 5\], the field equations for the

\(^1\) We refer to such models as “minimally covariantized” Galileons, to stress the difference with the “covariantized” Galileons of ref. \[1, 4, 5\] which contain non-minimal coupling to curvature in the form of the counterterms mentioned above.
metric are second order for the metric but contain third-order derivatives of the scalar, and conversely the scalar field equation is second order for the scalar but third-order for the metric.) This appears, of course, to be a perfectly legitimate possibility and it is not hard to build some simple examples with such a property (see e.g. [28]). However, so far, the arguments given in favor of this happening, as proposed in Refs. [21, 22, 27], do not appear to us to be entirely conclusive.

The reasons are the following. First, the Hamiltonian analysis of the corresponding theory has only been carried starting from a gauge fixed action [21–23] where (i) the scalar gradient $\nabla_\mu \phi$ is assumed to give the time flow direction, and (ii) this gauge fixing is not included in the Hamiltonian (i.e., the gauge is explicitly broken to start with). This gauge, usually referred to as the “unitary gauge”, hides all the dynamics of the scalar and it is easy to see that it eliminates all third time derivatives in the field equations (see Refs. [4, 5] and also Sec. III below). In this sense, it is perhaps not surprising that working in this gauge, one finds less degrees of freedom than those expected from an Ostrogradski-type of reasoning. Furthermore, this unitary gauge choice is obviously only possible if the scalar gradient is everywhere time-like (or at least time-like in the vicinity of some would-be Cauchy surface), a situation which only covers rather limited subset of all possibilities. Indeed, it does not allow one to say anything about the Cauchy problem when, on the Cauchy surface, the scalar has a gradient which is not always time-like — a perfectly legitimate choice of initial condition. For instance, a physical situation of major importance of this kind is that of a static and spherically symmetric background, since the hypersurfaces $\phi = \text{const.}$ are not spacelike and cannot be chosen as initial value surfaces. Second, it is well known, e.g. when considering Maxwell theory, that analyzing the d.o.f. content of the theory in a given gauge can be very misleading, in particular when the gauge is explicitly broken to start with. Finally, the covariant reasoning given in [22] (analogous to the one of [27]) appears to us to be incomplete if not incorrect. Indeed, there it is stated that taking an appropriate trace of the metric field equations (which are known to contain third-order time derivatives of the scalar) enables one to extract the third time derivatives of the scalar in terms of second derivatives — and then, inserting this back in the metric field equations, gives a second-order system (and similarly with the scalar field equation). This seems to omit the fact that in this way one can at best eliminate from the field equations all but one third-order time derivatives of the scalar (and similarly for the metric): the reason is that the trace of the metric field equations is itself a field equation which must still be solved, and which still contains a third-order time derivative. Hence, in contrast to the claims in [22, 27], the covariant procedure outlined in those papers appears not to lead to a complete set of equations in which all third-order time derivatives have been eliminated.

Here we will reexamine these issues and argue, in two different ways, that minimally covariantized Galileons indeed propagate less degrees of freedom than expected from the third-order nature of the field equations. Throughout we work in a totally gauge-invariant framework. This paper is organized as follows. In Sec. II we show how the system of $\frac{D(D+1)}{2} + 1$ field equations\(^2\) of the theory considered — namely all minimally covariantized Galileons and independently all the counterterms, as well as any linear combination of them and their Horndeski-like generalizations — can be reduced to an equivalent system with only second-order time derivatives (using, however, a very different procedure from the one given

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\(^2\) This number $\frac{D(D+1)}{2} + 1$ will be quoted several times in the present paper. In $D = 4$ dimensions, it simply reduces to the usual $10 + 1 = 11$ field equations of the metric and the scalar field.
in Refs. [22, 27]). In Sec. III, we present a Hamiltonian analysis of one particular theory in the minimally covariantized Galileon family, namely the simplest non-trivial one (in the sense that its field equations do contain third-order derivatives), to show that the number of constraints is sufficient to yield less than 4 propagating degrees of freedom. A last section gives our conclusions.

II. REMOVAL OF THIRD TIME DERIVATIVES IN THE GENERAL CASE

In a spacetime with metric $g_{\mu\nu}$ in any dimension $D$, Ref. [5] introduced the generalized Galileon Lagrangians $L_{(n+1,p)}$. These involve a total of $2n$ derivatives acting on a product of $n + 1$ scalar fields $\varphi$, and $p$ Riemann tensors $R_{\lambda\mu\nu\rho}$. The Lagrangians with $p = 0$, the $L_{(n+1,0)}$, are the “minimally covariantized” Galileons, whereas the Lagrangians with $p \neq 0$ are called the “counterterms”.

For a given $n$, it was shown that — up to an irrelevant global factor — there exists a unique linear combination

$$\sum_{p=0}^{p_{\text{max}}} C_{(n+1,p)} L_{(n+1,p)}$$

(1)

such that all field equations are of second order. These are the “covariantized” Galileons. Here

$$p_{\text{max}} = \left\lfloor \frac{n-1}{2} \right\rfloor$$

(2)

is the integer part of $\frac{n-1}{2}$, and the constant coefficients $C_{(n+1,p)}$ take a very specific form which may be found in Eq. (37) of [5].

Conversely, any other linear combination (for instance, each of these Lagrangians $L_{(n+1,p)}$ individually when $p_{\text{max}} > 0$) does yield third derivatives in its field equations. More specifically, the scalar field equation contains third time derivatives of the metric tensor, and the Einstein equations contain the third time derivative of the scalar field. To be able to compute the time evolution, it seems thus necessary to specify more initial data on a Cauchy surface, and one expects the existence of more degrees of freedom than just a single scalar field and the two helicities of the graviton.

Although higher-order field equations indeed generically lead to extra degrees of freedom (which are even generically ghost modes [17, 18], implying the instability of the theory), specific examples show that this is not always the case. There may, for instance, exist extra constraints (related or not to some hidden gauge symmetry) which kill some of the modes. Or the few equations involving third (or higher) time derivatives may actually be obtained by differentiating, with respect to time, some independently known second-order field equations. In this case, extra initial data are not necessarily needed on the Cauchy surface. An elegant toy-model of this kind was presented in Sec. 7.1 of [28], and a similar result applies to “mimetic dark matter” [29].

The aim of this section is to show that the field equations of the generalized Galileon Lagrangians $L_{(n+1,p)}$ are of this second kind: all the third time derivatives can be obtained by deriving independently known second-order field equations, and can therefore be removed from all field equations. This supports the main claims of Refs. [21, 23], although our procedure differs from theirs, and does not suffer from the problems mentioned in the Introduction. In particular, we will not fix any gauge in our derivation.
Thus we consider the $D$-dimensional theories defined by the action

$$S = S_{\text{EH}} + S_{\text{Gal}}$$

in curved spacetime, where $S_{\text{EH}} = \int d^Dx \sqrt{-g} R$ is the Einstein-Hilbert action\(^3\), without any factor $c^3/16\pi G$ to simplify our discussion, and where

$$S_{\text{Gal}} = \int d^Dx \sqrt{-g} \left( \sum_{n,p} k_{(n,p)} \mathcal{L}_{(n,p)} \right),$$

with arbitrary constant coefficients $k_{(n,p)}$ — and hence not the specific $\mathcal{C}_{(n,p)}$ discussed above. (We will furthermore consider the Horndeski-like generalization of these theories at the end of the section.) We also define the Galileon energy-momentum tensor as

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{Gal}}}{\delta g^{\mu\nu}},$$

without any factor 2, so that Einstein’s equations (i.e., the field equations for the metric) simply read $G^{\mu\nu} = T^{\mu\nu}$, where $G^{\mu\nu}$ is the Einstein tensor. Finally, we define $\mathcal{E} = \delta S_{\text{Gal}}/\delta \varphi$, so that the scalar field equation reads $\mathcal{E} = 0$.

Note that this scalar field equation is a consequence of Einstein’s equations since, because of the diffeomorphism invariance of action $S$, it follows that

$$\varphi^{\nu} \mathcal{E} = -2 \nabla_{\mu} (G^{\mu\nu} - T^{\mu\nu}),$$

where $\varphi^{\nu} = \nabla^{\nu} \varphi$ denotes the covariant derivative of the scalar field (without writing any semicolon, to simplify; we shall also write $\varphi^{\mu\nu...} = \nabla \nabla \nabla \nabla \nabla \varphi$ in the following). Independently of the diffeomorphism-invariance argument, this can also be checked explicitly for the general Lagrangians $\mathcal{L}_{(n,p)}$ or for particular cases \[^{[4]}\]. Therefore, if we manage to prove that Einstein’s equations can be recast as a set of second-order differential equations (with respect to time), Eq. (6) shows that the third time derivatives of the metric tensor entering the scalar field equation $\mathcal{E} = 0$ should not pose more problems than in the toy-model of Ref. \[^{[28]}\]. It should be noted that at the spacetime points where $\varphi^{\nu}$ happens to vanish, $\mathcal{E}$ can no longer be extracted from (6). However, it is easy to see that in all field equations, third derivatives are always multiplied by a gradient $\varphi^{\nu}$ (and even several of them). Therefore, at the points where $\varphi^{\nu} = 0$, all field equations are at most of second order, and do not pose any problem. In addition to the above argument based on Eq. (6), we will actually prove an even stronger result below: It is also possible to recast the scalar field equation itself as a second-order one (with respect to time), by combining it with the time derivative of another linear combination of Einstein’s equations.

**A. Example: $\mathcal{L}_{(4,0)}$ in $D = 4$ dimensions**

Before attacking the general case of $\mathcal{L}_{(n,p)}$ which is rather technically involved, we begin in this subsection by illustrating how the steps work in $D = 4$ dimensions, focusing on the simplest non-trivial Galileon action, namely $S_{\text{Gal}} = \int d^4x \sqrt{-g} \mathcal{L}_{(4,0)}$.

\[^{3}\] Throughout this paper, we use the sign conventions of Ref. \[^{[30]}\], and notably the mostly-plus signature.
The Lagrangian $\mathcal{L}_{(4,0)}$, which is also the subject of the Hamiltonian analysis of Sec. [III] is given by [5]

$$\mathcal{L}_{(4,0)} = \left(\Box \varphi\right)^2 (\varphi_\mu \varphi^\mu) - 2 \left(\Box \varphi\right) (\varphi_\mu \varphi^{\mu
u} \varphi_\nu) - (\varphi_\mu \varphi^{\mu
u})(\varphi_\nu \varphi^\rho) + 2 (\varphi_\mu \varphi^{\mu
u} \varphi_\nu \varphi^\rho)$$

$$= -\epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\mu_5 \mu_6} \alpha \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6},$$

(7)

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita (fully antisymmetric) tensor in 4 dimensions [see Eq. (25) for its definition in $D$ dimensions]. Its stress-energy tensor is given by

$$T^{\mu\nu}_{(4,0)} = \left(\frac{1}{2} \gamma^{00} \epsilon^{\mu_1 \mu_2 \mu_3 \alpha} \epsilon^{\mu_4 \mu_5 \mu_6} \alpha \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6} - \epsilon^{\mu_1 \mu_3 \mu_5} \epsilon^{\mu_2 \mu_4 \mu_6} \alpha \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6} \alpha \right)$$

$$+ \epsilon^{\mu_1 \mu_3 \mu_5} \epsilon^{\mu_2 \mu_4 \mu_6} \alpha \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6} \sigma.$$

(8)

In the remainder of this subsection we simply denote $T^{\mu\nu}_{(4,0)}$ by $T^{\mu\nu}$.

The first step is to determine explicitly the different order time derivatives which appear in the zero-zero component of the Einstein equation $G^{00} = T^{00}$, recalling that in $G^{00}$ there are at most first time derivatives of the metric. From (8) it follows that

$$T^{00} = \frac{1}{2} \gamma^{00} \epsilon^{\mu_1 \mu_3 \mu_5} \epsilon^{\mu_2 \mu_4 \mu_6} \alpha \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6} - \epsilon^{\mu_1 \mu_3 \mu_5} \epsilon^{\mu_2 \mu_4 \mu_6} \alpha \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6} \alpha$$

$$- 2 \varphi_{\mu_1} \left[\epsilon^{\mu_1 \mu_3 \mu_5 \kappa} \epsilon^{\mu_2 \mu_4} \alpha \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6} \alpha \right] + \epsilon^{\alpha \beta

(9)

from which we immediately see that $T^{00}$ contains no terms in $\ddot{\varphi}$ nor in $\dot{\varphi}_i$ (where latin indices mean spatial components). This latter term could be generated from the term within square brackets, but that would require $\mu_3 = \mu_5 = 0$, in which case the result vanishes by antisymmetry of the Levi-Civita tensor. Thus the highest order time derivative of the scalar field it contains is $\ddot{\varphi}$, whose coefficient can be determined directly from (9) and will be given below. Regarding the metric, there are second-order time derivatives coming from the third-order covariant derivatives of $\varphi$ on the second line, since $\varphi_{\alpha \beta \gamma} \supset - (\partial_\gamma \Gamma^\mu_{\alpha \beta}) \varphi_\mu$. We must therefore take $\mu_3 = 0$ and $\sigma = 0$ to find these, and we obtain

$$T^{00} \bigg|_{\ddot{g}_{ij}} = \frac{\varphi_0}{N^2} \left[\epsilon^{\alpha \beta k} \epsilon^{ij} \varphi_i \varphi_j (\partial_0 \Gamma^\nu_{\alpha \beta}) \varphi_\nu \right],$$

(10)

where $N \equiv 1/\sqrt{-g^{00}}$ is the usual lapse in the ADM decomposition, see Eq. (A1), and $\epsilon^{ijk}$ is the 3-dimensional Levi-Civita tensor related to the 4-dimensional one by

$$\epsilon^{0ijk} = -\frac{\epsilon^{ijk}}{N}.$$

(11)

The subscript $\ddot{g}_{ij}$ on the left hand side of (10) is due to the fact that the Christoffel symbols $\Gamma^\nu_{\alpha \beta}$ only contain first time derivatives of the spatial components of the metric (see Appendix A). More explicitly

$$\left(\partial_0 \Gamma^\nu_{\alpha \beta}\right) \varphi_\nu = -N \varphi^0 (\partial_0 K_{\alpha \beta}) + \text{first-order derivatives},$$

(12)

4 Hence note that $T^{00}$ contains no terms in $\ddot{N}$ or $\ddot{N}_i$ (where $N$ and $N_i$ are the usual lapse and shift in the ADM decomposition). Hence there are no terms in $\ddot{g}_{00}$ nor $\ddot{g}_{0i}$. 6
where $K_{ij}$ is the extrinsic curvature. Thus

$$T^{00}|_{\ddot{g}_{ij}} = -\frac{(\varphi^0)^2}{N} \left( \epsilon^{pqk} \epsilon^{ij} k_{ij} \varphi p \varphi q \dot{K}_{ij} \right).$$

(13)

The second step involves carrying out the same procedure for $\varphi_i T^{0i}$. From (8) with $\mu = 0$ and $\nu = i$ we find

$$\varphi_i T^{0i} = \frac{1}{2} \varphi g^{0i} \left( \epsilon^{\mu_1 \mu_2 \mu_5 \alpha} \epsilon^{\mu_2 \mu_4 \mu_6} \alpha(\varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6}) - \epsilon^{\mu_1 \mu_2 \mu_5 \gamma} \epsilon^{\mu_2 \mu_4 \mu_6} \varphi(i \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6}) \right)
+ \varphi_{ij} \epsilon^{\alpha \beta \gamma} \epsilon^{\mu_1 \mu_2 \mu_3} k(\varphi^0 \varphi_{\mu_1} \varphi_{\mu_2} \varphi_{\mu_3} \varphi_{\mu_4} \varphi_{\mu_5} \varphi_{\mu_6}) \gamma.$$

(14)

Again, it is clear that there are no terms in $\ddot{\varphi}$. Similarly, there are no terms in $\ddot{\varphi}_j$ since one always ends up with a contraction $\epsilon^{ij} \varphi_{ij} = 0$. There are obviously terms in $\ddot{\varphi}$ (see below). Concerning the terms in $\ddot{g}_{ij}$, following the same logic as above, we find that they are given by

$$\varphi_i T^{0i}|_{\ddot{g}_{ij}} = \frac{\varphi m \varphi^p}{N^2} \left[ \epsilon^{ij \gamma} \epsilon^{pq \gamma} k_{ij} \varphi p (\partial_0 \Gamma^\nu_{pq}) \varphi^\nu \right].$$

(15)

It follows from (10) and (15) that $\varphi_i T^{0i}$ and $T^{00}$ contain exactly the same combination of second-order derivatives of the metric.

Furthermore, we recall that the components $G^{0\mu}$ of the Einstein tensor do not involve second time derivatives of the metric. Indeed, the Bianchi identities imply the covariant conservation of the Einstein tensor, $\nabla_\gamma G^{\lambda \mu} = 0$, therefore

$$\partial_0 G^{0\mu} = -\partial_i G^{i\mu} - \mathcal{O}(\Gamma G),$$

(16)

where $\mathcal{O}(\Gamma G)$ means the four terms involving contractions of Christoffel symbols $\Gamma^\lambda_{\mu \nu}$ with the Einstein tensor $G^{\mu \nu}$. Since the right-hand side of Eq. (16) contains at most second time derivatives of the metric tensor, this must be so for the left-hand side $\partial_0 G^{0\mu}$, therefore the components $G^{0\mu}$ contain at most first time derivatives.

Thus we arrive at the first important conclusion that, using Eqs. (10) and (15), the combination of the Einstein equations

$$\varphi^0 \varphi T^{0i} - (\varphi_i \varphi) T^{00} = \varphi^0 \varphi T^{0\mu} - (\varphi_i \varphi^0) T^{00} = \varphi^0 (\varphi_i G^{0\mu}) - (\varphi_i \varphi^0) G^{00}$$

(17)

determines $\ddot{\varphi}$ in terms of first time derivatives. As a result, any $\ddot{\varphi}$ appearing in the equations of motion can be expressed in terms of second time derivatives simply by differentiating (17).

The next step is the determination of the coefficients of these $\ddot{\varphi}$ terms. Starting from (9) and (14) we find

$$\varphi_0 T^{00} = \frac{\varphi_0 \varphi^0}{N^2} \left[ \epsilon^{mnk} \epsilon^{ij} k_{ij} \varphi^m (\partial_0 \Gamma^\nu_{nj}) \varphi^\nu \right] + B \varphi_0 (\varphi^0 \ddot{\varphi}) + \text{lower order derivatives},$$

(18)

$$\varphi_i T^{0\mu} = \frac{\varphi i \varphi^0}{N^2} \left[ \epsilon^{mnk} \epsilon^{ij} k_{ij} \varphi^m (\partial_0 \Gamma^\nu_{nj}) \varphi^\nu \right] - A (\varphi^0 \ddot{\varphi}) + \text{lower order derivatives},$$

(19)

where

$$B = -\frac{1}{N^2} \epsilon^{mnk} \epsilon^{ij} k_{ij} \varphi^m \Gamma^0_{nj}, \quad A = -\frac{1}{N^2} \epsilon^{mnk} \epsilon^{ij} k_{ij} \varphi^m \varphi_{nj}.$$
(Note that $A$ is in fact nothing other than $\mathcal{L}_{(3,0)}$ with only spatial indices.) Hence the second combination of Einstein’s equations

$$B(\phi, T^{0\mu}) + AT^{00} = B(\phi, G^{0\mu}) + AG^{00}$$  \hspace{1cm} (21)

is an equation for the specific combination of $\check{g}_{ij}$ appearing in

$$\varepsilon^{mnk} \varepsilon_{ij} \phi_i \phi_m \phi^0 \check{K}_{nj},$$  \hspace{1cm} (22)

in terms of first time derivatives of the fields.

As a final step we must show that the third-order time derivatives of the metric appearing in the equation of motion $\mathcal{E} = 0$ for the scalar field, are exactly given by the time derivative of the combination appearing in (22). The third-order derivatives of the metric in $\mathcal{E}$ are

$$\mathcal{E} \sim \varepsilon^{\mu_1 \mu_3 \mu_5 \alpha} \varepsilon^{\mu_2 \mu_4 \mu_6 \alpha} \phi_{\mu_1} \phi_{\mu_2} \phi^0 R_{\mu_3 \mu_5 \mu_4 \mu_6; \lambda},$$  \hspace{1cm} (23)

where $R_{\mu_\nu \rho \sigma}$ is the Riemann tensor. To find the $\ddot{g}_{ij}$ terms appearing here, it is sufficient to set $\lambda = 0$, and then the relevant third-order time derivative is simply the derivative of the term in $\ddot{g}$ appearing in

$$\varepsilon^{\mu_1 \mu_3 \mu_5 \alpha} \varepsilon^{\mu_2 \mu_4 \mu_6 \alpha} \phi_{\mu_1} \phi_{\mu_2} \phi^0 R_{\mu_3 \mu_5 \mu_4 \mu_6}. $$  \hspace{1cm} (24)

However, using the results of Appendix A this is nothing other than the combination (22) (up to irrelevant numerical factors). Since this is the contraction appearing in (21), it can thus be expressed in terms of first time derivatives of the field. Thus the third-order time derivatives of the metric appearing in the scalar field equation of motion can be replaced by second-order time derivatives on using the derivative of (21).

Hence we arrive at the conclusion that, despite containing higher-order time derivatives, all 11 equations of motion for this theory can be expressed solely in terms of second-order time derivatives: the derivative of the combination (17) gives $\ddot{\phi}$ in terms of second-order time derivatives, whilst the derivative of the combination (21) gives the required contraction of $\dddot{g}_{ij}$ in terms of second-order time derivatives.

We now generalize these results to arbitrary $D$, $n$ and $p$, which requires us to introduce more powerful notation. We will also consider arbitrary linear combinations of these Lagrangians.

**B. General case: arbitrary $D$, $n$ and $p$**

For the following discussion, the most convenient expression [5] for the Lagrangians $\mathcal{L}_{(n+1,p)}$ uses the Levi-Civita fully antisymmetric tensor

$$\varepsilon^{\mu_1 \mu_2 \ldots \mu_D} = -\frac{1}{\sqrt{-g}} \delta^{[\mu_1}_{1} \delta^{\mu_2}_{2} \ldots \delta^{\mu_D]}_{D},$$  \hspace{1cm} (25)

where the square bracket denotes unnormalized permutations. In any dimension $D \geq n$, we define

$$\mathcal{L}_{(n+1,p)} = -A_{(2n)} \phi_{\mu_1} \phi_{\mu_2} R_{(p)} S_{(q)},$$  \hspace{1cm} (26)

where $A_{(2n)}$ is a compact notation for

$$A_{(2n)}^{\mu_1 \mu_2 \ldots \mu_{2n-1}} \varepsilon^{\mu_1 \mu_3 \mu_5 \ldots \mu_{2n-1} \nu_1 \nu_2 \ldots \nu_{D-n} \varepsilon^{\mu_2 \mu_4 \mu_6 \ldots \mu_{2n} \nu_1 \nu_2 \ldots \nu_{D-n}},$$  \hspace{1cm} (27)
and where

\[ R(p) \equiv (\varphi^\lambda \varphi^\mu)^p \prod_{i=1}^{p} R_{\mu_{4i-1} \mu_{4i+1}}^{\mu_{4i} \mu_{4i+2}} , \]  

(28)

\[ S(q) \equiv \prod_{i=0}^{q-1} \varphi_{\mu_{2n-1-2i} \mu_{2n-2i}} , \]  

(29)

with

\[ q = n - 1 - 2p. \]  

(30)

These definitions assume that \( n, p \) and \( q \) are integers in the ranges

\[ 1 \leq n \leq D, \quad 1 \leq p \leq \left\lfloor \frac{n-1}{2} \right\rfloor , \quad \text{and} \quad 1 \leq q \leq n - 1. \]  

(31)

We also set \( R(0) \equiv 1 \) for \( p = 0 \) and \( S(0) \equiv 1 \) for \( q = 0 \), and use the convention that \( R(p) = 0 \) for \( p < 0 \) and \( S(q) = 0 \) for \( q < 0 \). [The cases \( n = 0 \) and \( n = -1 \), with \( p = 0 \), may also be defined \( \equiv \) as \( L_{(1,0)} = \varphi \) and \( L_{(0,0)} = \text{const.} \), but we do not consider them here since they obviously do not yield higher-order field equations.] The numerical factor \( 1/(D - n)! \) is introduced in Eq. (27) so that \( A_{(2n)} \) keeps the same expression in terms of products of metric tensors in any dimension.

When \( p = 0 \), i.e., without any Riemann tensor involved, these definitions reduce to the Galileons of Ref. [2]. For instance, \( L_{(2,0)} = \varphi^\mu \varphi_\mu \) is the kinetic term of a standard scalar field (though, referring to Eq. (4), one should choose a negative \( k_{(2,0)} \) in order not to have a ghost around an empty and flat background). The cubic Lagrangian \( L_{(3,0)} = \varphi^\mu \varphi^\nu \varphi_\nu - \varphi^\mu \varphi^\nu \varphi_\nu \varphi_\lambda = \frac{3}{2} \varphi^\mu \varphi^\nu \square \varphi + \text{tot. div.} \), is the one obtained in the decoupling limit of the DGP model [31–33]. The quartic Lagrangian was given in Eq. (7). In \( D = 4 \) dimensions, there also exists \( L_{(5,0)} \), written for instance in Eq. (3) of [5]. On the other hand, the Lagrangians (26) with \( p \neq 0 \), involving one or several Riemann tensors, are the “counterterms” introduced in Refs. [4–6, 8, 34] to avoid any third derivative in the field equations. There are only two of them in \( D = 4 \) dimensions, \( L_{(4,1)} \) and \( L_{(5,1)} \), cf. Eqs. (14) and (15) of [5]. Our proof below will be valid for all \( L_{(n+1,p)} \) in any dimension \( D \), as well as linear combinations of them.

1. \( L_{(n+1,p)} \) and their linear combinations

We first focus on a single Lagrangian \( L_{(n+1,p)} \). In order to write its Einstein equations \( G^{\alpha \beta} = T^{\alpha \beta} \) in the simplest way, it will be useful to introduce the following compact notation generalizing (27):

\[ A_{(2n,i)}^{\alpha} \equiv A_{(2n)}^{\mu_1 \mu_2 \ldots \mu_{4i-1} \alpha \mu_{4i+1} \ldots \mu_{2n}} , \]  

(32)

\[ A_{(2n,i,j)}^{\alpha \beta} \equiv A_{(2n)}^{\mu_1 \mu_2 \ldots \mu_{4i-1} \alpha \mu_{4i+1} \ldots \mu_{4j-1} \beta \mu_{4j+1} \ldots \mu_{2n}} , \]  

(33)
where \(2n\) is the rank of the tensor, and \(i\) and \(j\) locate the positions of the indices \(\alpha\) and \(\beta\) which are explicitly indicated. The energy-momentum tensor is then given by

\[
T^\alpha{}^\beta = \left[ \frac{1}{2} g^{\alpha\beta} + p \frac{\varphi^\alpha \varphi^\beta}{\varphi^2} \right] A_{(2n)} - A_{(2n+2,2n+1,2n+2)}^{\alpha\beta} \varphi_{\mu_1} \varphi_{\mu_2} R_{(p)} S_{(q)}
\]

\[
- q A^{(\alpha,4p+4)} \mu_1 \left[ \varphi^\gamma \varphi_{\mu_2} R_{(p)} S_{(q-1)} \right]_{;\mu_4 p+3}
\]

\[
+ \frac{q}{2} A^{\alpha\beta}_{(2n,4p+3,4p+4)} \left[ \varphi^\sigma \varphi_{\mu_1} \varphi_{\mu_2} R_{(p)} S_{(q-1)} \right]_{,\sigma}
\]

\[
+ 2 p A^{(\alpha\beta)}_{(2n,4p+1,4p+2)} \left[ \varphi_{\mu_1} \varphi_{\mu_2} \left( \varphi^2_{\lambda} R_{(p-1)} S_{(q)} \right) \right]_{;\mu_4 p+4 p+1}
\]

\[
- p A^{(\alpha,4p-1)} R_{\mu_4 p+1 \mu_4 p+2 \mu_4 p+3} \varphi_{\mu_1} \varphi_{\mu_2} \left( \varphi^2_{\lambda} R_{(p-1)} S_{(q)} \right),
\]

(34)

where symmetrization over \(\alpha\) and \(\beta\) is assumed [i.e., \(X^{(\alpha\beta)} = (X^{\alpha\beta} + X^{\beta\alpha})/2\) for any tensor], notably on the second and last two lines which are not automatically symmetric. Regarding the first term, note that the factor \(p/\varphi^2\) does not cause any divergence: it vanishes when \(p = 0\), and when \(p \neq 0\) it is multiplied by \((\varphi^2)\) contained in the \(R_{(p)}\) term (28). The tensor \(A^{\alpha\beta}_{(2n+2,2n+1,2n+2)}\) on the first line has \(2n + 2\) free indices, the last two of them being \(\alpha\) and \(\beta\). The corresponding term in \(T^{\alpha\beta}\) originates from the variation with respect to the metric of the contracted indices in Eq. (27). It vanishes if \(D < n + 1\). Note that in the second line, the factor \(\varphi_{\mu_1}\) has been extracted from the square bracket which is covariantly derived with respect to \(\mu_4 p+3\), because \(\varphi_{\mu_1} \mu_4 p+3\) with two odd indices would vanish when contracted with the first antisymmetric \(\varepsilon\) tensor of Eq. (27).

This energy-momentum tensor in Eq. (34) allows us to prove several important lemmas. Let us focus to start with on third derivatives: by suitably permuting and relabeling dummy indices, one may rewrite them as

\[
T_{3\text{rd \, der.}}^{\alpha\beta} = \frac{q(q-1)}{2} A^{\alpha\beta}_{(2n,4p+3,4p+4)} \varphi_{\mu_1} \varphi_{\mu_2} R_{(p)} S_{(q-2)} \varphi^\lambda \varphi_{\mu_4 p+1 \mu_4 p+2 \mu_4 p+3 \mu_4 p+4} \lambda
\]

\[
+ 4 p^2 A^{\alpha\beta}_{(2n,4p+1,4p+2)} \varphi_{\mu_1} \varphi_{\mu_2} R_{(p-1)} S_{(q)} \varphi^\lambda \varphi_{\mu_4 p+4 p+1} \lambda
\]

(35)

This is equivalent to Eq. (34) of Ref. [5]. We can thus arrive at the following conclusions.

First, no third-order derivative of the metric tensor enters \(T^{\alpha\beta}\). Indeed, the Bianchi identities \(R_{\mu[p} \varepsilon_{\rho \sigma]} = 0\) cancel most of the differentiated Riemann tensors in (34), because three of their indices are contracted with the same antisymmetric \(\varepsilon\) tensor of \(A_{(2n)}\). The only non-vanishing ones come from the \(R_{(p)\sigma}\) of the third term, but they exactly cancel with the derivatives of the Riemann tensors generated by permuting the indices of \(S_{(q)\mu_4 p+4 p+1}\) coming from the fourth term.

Second, Eq. (35) also proves that \(T^{00}\) does not contain any \(\widetilde{\varphi}\) nor \(\varphi_i\). Indeed, because \(\alpha = \beta = 0\), one odd index \((4p + 3\) or \(4p + 1\)) and one even index \((4p + 4\) or \(4p + 2\)) must be 0, therefore all other (contracted) indices of the two \(\varepsilon\) tensors in \(A_{(2n)}\) must be spatial. In conclusion, the only third-differentiated scalar fields (35) cannot contain more than one time derivative in \(T^{00}\).

Third, the contraction \(\varphi_{\alpha} T^{\alpha\beta}\) (and in particular \(\varphi_{\alpha} T^{\alpha 0}\), that we will use in the argument below) does not contain any third derivative of the scalar field. Indeed, \(\varphi_{\alpha}\) and \(\varphi_{\mu_1}\) are then contracted with the same antisymmetric \(\varepsilon\) tensor entering \(A_{(2n)}\), in Eq. (35), making it vanish.

We now proceed as in Section (III A) and compute the second time derivatives entering \(\varphi_{\alpha} T^{\alpha 0}\) and \(T^{00}\), using the full expression (34). However, before doing so, and in order to
On the other hand, this combination does contain a term \( \varphi \) which tells us that of the metric as proved in Eq. (35). Indeed, diffeomorphism invariance implies Eq. (6), whereas \( \text{first time derivatives, whereas } \dddot{\varphi} \text{ involves both of which are both 0, meaning that the 2n first must be spatial. Similarly, we define} \]

\[
\mathcal{L}_{(n+1,p)}^{\text{spatial}} \equiv -A^{00}_{(2n+2,2n+2,2n+2,2n+2)} \varphi_{\mu_1} \varphi_{\mu_2} \mathcal{R}(p) \mathcal{S}(q), \tag{36}
\]

which is the analogue of Eq. (26), but where now all contracted indices are spatial (though the covariant derivatives and the Riemann tensors remain D-dimensional). As before, see Eq. (31), \( q = n - 1 - 2p \). Note that \( A^{00}_{(2n+2,2n+2,2n+2,2n+2)} \) has two extra indices relative to \( A_{(2n)} \), namely the last two which are both 0, meaning that the 2n first must be spatial. Similarly, we define

\[
\mathcal{L}_{(n+1,p)}^{\Gamma} \equiv -A^{00}_{(2n+2,2n+2,2n+2,2n+2)} \varphi_{\mu_1} \varphi_{\mu_2} \mathcal{R}(p) \mathcal{S}(q) \Gamma^{00}_{\mu_1 \mu_2 \mu_3 \mu_4}, \tag{37}
\]

which, as before, contains \( n+1 \) scalar fields and \( p \) Riemann tensors, as well as now a Christoffel symbol \( \Gamma_{ij}^{0} \). Notice that all contracted indices again are spatial, because the \((2n+3)\)rd and \((2n+4)\)th indices are imposed to be 0. [Actually, \( \mathcal{L}_{(n+2,p)}^{\text{spatial}} \) contains \(-(n - 2p) \dot{\varphi} \mathcal{L}_{(n+1,p)}^{\Gamma} \).]

On using (34), the second time derivatives entering \( \varphi_0 T^{\alpha 0} \) and \( T^{00} \) can then be written as

\[
\begin{align*}
\varphi_0 T^{\alpha 0}_{2\text{nd t der.}} & = -\varphi_0 (A \dddot{\varphi} + \varphi_0^2 C), \tag{38} \\
T^{00}_{2\text{nd t der.}} & = \varphi_0 (B \dddot{\varphi} - \varphi_0 C), \tag{39}
\end{align*}
\]

where

\[
\begin{align*}
A & \equiv \frac{\tilde{q}(\tilde{q} - 1)}{2} \mathcal{L}_{(n,p)}^{\text{spatial}} + 4p^2 \mathcal{L}_{(n,p-1)}^{\text{spatial}}, \tag{40} \\
B & \equiv \frac{\tilde{q}(\tilde{q} - 1)}{2} \mathcal{L}_{(n-1,p)}^{\Gamma} + 4p^2 \mathcal{L}_{(n-1,p-1)}^{\Gamma}, \tag{41} \\
C & \equiv \frac{\tilde{q} - 1}{4} A^{00}_{(2n+2,2n+2,2n+2,2n+2)} \varphi_{\mu_1} \varphi_{\mu_2} \mathcal{R}(p) \mathcal{S}(q) \tilde{\gamma}_{4\mu_4+5 \mu_4+6} \\
& + 2p^2 A^{00}_{(2n+2,2n+2,2n+2,2n+2)} \varphi_{\mu_1} \varphi_{\mu_2} \mathcal{R}(p) \mathcal{S}(q) \tilde{\gamma}_{4\mu_4+1 \mu_4+2} \tag{42}
\end{align*}
\]

with \( \tilde{q} \equiv n - 1 - 2p \) in these expressions — but the \( q \)'s involved within \( \mathcal{L}_{(n,p)}^{\text{spatial}} \) and \( \mathcal{L}_{(n-1,p)}^{\Gamma} \) depend on their precise indices \((n, p)\) as defined in Eqs. (36) and (37). Note that \( A \) and \( B \) involve at most first time derivatives, whereas \( C \) contains a specific contraction of \( \tilde{\gamma}_{ij} \) with other fields (themselves differentiated at most once with respect to time).

The fact that the same contraction \( C \) of \( \tilde{\gamma}_{ij} \) enters both (38) and (39) will be crucial below. Whilst this looks miraculous, it is actually simply a consequence of the diffeomorphism invariance of the theory, together with the fact that \( T^{\alpha \beta} \) involves at most second derivatives of the metric as proved in Eq. (33). Indeed, diffeomorphism invariance implies Eq. (6), which tells us that \( \varphi \alpha \mathcal{E} = 2 \partial_0 T^{0\alpha} + 2 \partial_\alpha T^{0\alpha} + \mathcal{O} (TT) \). The third time derivatives of the metric entering \( \mathcal{E} \) can thus come only from \( 2 \partial_0 T^{0\alpha} \), proving that all \( T^{0\alpha} \) must contain exactly the same terms in \( \tilde{\gamma}_{ij} \), multiplied by \( \varphi^0 \). Using definition (12), we have explicitly

\[
\begin{align*}
T^{0\alpha}|_{\tilde{\gamma}_{ij}} & = -\varphi^0 \varphi^\alpha C, \text{ while } \mathcal{E}|_{\tilde{\gamma}_{ij}} = -2 \varphi^0 (\partial_0 C)|_{\tilde{\gamma}_{ij}}.
\end{align*}
\]

This dependence of (38) and (39) on the same contraction \( C \) of \( \tilde{\gamma}_{ij} \) allows us to cancel it in the linear combination

\[
\varphi^0 \varphi_0 T^{0\alpha} - \varphi^2 T^{00}. \tag{43}
\]

On the other hand, this combination does contain a term

\[
- \varphi^0 (\varphi^0 A + \varphi^2 B) \dddot{\varphi}. \tag{44}
\]
proportional to the second time derivative of the scalar field. Our lemmas above also prove that all its other terms involve at most first time derivatives of the fields (and up to three spatial derivatives, but they do not pose any difficulty for the Cauchy problem). Now, using the fact that the $G^{00}$ components of the Einstein tensor depend only on first time derivatives of the metric, cf. Eq. (16), we arrive at the following crucial combination of Einstein’s equations:

\[ \varphi^2_\alpha (G^{00} - T^{00}) - \varphi^0_\alpha (G^{\alpha 0} - T^{\alpha 0}) = 0. \]

(45)

This generalizes Eq. (17) for arbitrary $n$ and $p$, and allows us to express $\ddot{\varphi}$ in terms of undifferentiated fields, their spatial derivatives, and their first time derivatives.

We now follow the same logic as in subsection II A. On taking the time derivative of Eq. (45), we have a way to express $\dddot{\varphi}$ in terms of fields which are differentiated at most twice with respect to time. Since $\dddot{\varphi}$ was the only higher-order time derivative entering Einstein’s equations $G^{\alpha \beta} = T^{\alpha \beta}$, we conclude that all of them now become of second order (as far as time is concerned). Moreover, since the linear combination (45) does not depend on the values $n$ and $p$ specifying the model $L^{(n+1,p)}$, we can repeat the same argument for any linear combination $\sum_{n,p} k_{(n,p)} L^{(n,p)}$, and this achieves our proof: Only the fields and their first time derivatives should need to be specified on an initial value surface.

Of course, our argument fails if the coefficient (44) happens to vanish at some spacetime point, since $\dddot{\varphi}$ can no longer be expressed in terms of at most first time derivatives of fields, in such a case. This is notably what happens on a surface where $\varphi_i = 0$, which corresponds precisely to the unitary gauge chosen in Refs. [21–23]. However, no third time derivative enters any field equation when $\varphi_i = 0$, therefore one can keep all of them without any difficulty when this happens. Let us anyway mention that our argument is generic, as required when discussing the well-posedness of the Cauchy problem. The spacetime domains where Eq. (44) may vanish have measure zero, and our proof is thus valid almost everywhere, in the mathematical sense of measure theory.

Let us now show that the scalar field equation $E = 0$ may also be recast as a second-order differential equation with respect to time. We already saw in Eq. (6) that $E$ can be obtained by taking the divergence of Einstein’s equations. However, we now show how the third time derivatives of the metric tensor can be removed from $E$, while still keeping a non-zero term in $\dddot{\varphi}$. For this purpose, consider another linear combination of Einstein’s equations

\[ A(G^{00} - T^{00}) + B \varphi_\alpha (G^{\alpha 0} - T^{\alpha 0}) = 0, \]

(46)

which generalizes Eq. (21) above for arbitrary $n$ and $p$. All second time derivatives of the scalar field cancel in this combination, but it still contains a term

\[ \varphi^0 (\varphi^0 A + \varphi^2 B) C, \]

(47)

where $C$ involves a specific contraction of $\dddot{g}_{ij}$, cf. Eq. (42). Therefore, the linear combination (46) allows us to express this contraction in terms of fields differentiated at most once with respect to time.\footnote{Note that the same coefficient $\varphi^0 (\varphi^0 A + \varphi^2 B)$ enters both Eqs. (44) and (47). At the generic spacetime points where it does not vanish, the two linear combinations of Einstein’s equations (45) and (46) can thus both be used, to extract $\dddot{\varphi}$ and the specific contraction $C$ of $\dddot{g}_{ij}$ in terms of fields differentiated at most once with respect to time.} Or, by taking the time derivative of (46), one may express the contraction
(42), with $\ddot{g}_{ij}$ replaced by $\dddot{g}_{ij}$, in terms of at most second time derivatives. Returning to the scalar field equation $\mathcal{E} = 0$, it turns out that the third time derivatives entering it take precisely the same form, namely:

$$\mathcal{E}_{3rd \ t \ der.} = -\frac{1}{2} \varphi^0 \left[ q(q - 1)A_{(2n,Ap+3,4p+4)}^{00} \varphi_{\mu_1} \varphi_{\mu_2} R_{(p)} S_{(q-2)} \dddot{g}_{\mu_4 p+5 \mu_4 p+6} + 8p^2 A_{(2n,Ap-1,4p)}^{00} \varphi_{\mu_1} \varphi_{\mu_2} R_{(p-1)} S_{(q)} \dddot{g}_{\mu_4 p+1 \mu_4 p+2} \right]. \quad (48)$$

As discussed below Eq. (42), this is a consequence of the diffeomorphism invariance of the theory (and of the absence of third time derivatives of the metric in $T^{\alpha \beta}$). Therefore, all third time derivatives entering $\mathcal{E} = 0$ are exactly canceled by adding to it the time derivative of Eq. (46) multiplied by $2/(\varphi^0 A + \varphi^2 B)$.

This procedure can also be extended to an arbitrary sum of Lagrangians $\sum_{n,p} k_{(n,p)} \mathcal{L}_{(n,p)}$, although this is less obvious than for the Einstein equations, cf. the paragraph below Eq. (45). Indeed, since the coefficients $A$ and $B$ of Eqs. (40) and (41) depend on $n$ and $p$, the linear combination (46) is thus specific to a single case. However, if we denote as $A_{(n,p)}$, $B_{(n,p)}$ and $C_{(n,p)}$ the coefficients $40-12$ corresponding to a given Lagrangian $\mathcal{L}_{(n,p)}$, the above results immediately show that the linear combination

$$\left( \sum_{n,p} k_{(n,p)} A_{(n,p)} \right) (G^{00} - T^{00}) + \left( \sum_{n,p} k_{(n,p)} B_{(n,p)} \right) \varphi_\alpha (G^{\alpha 0} - T^{\alpha 0}) = 0 \quad (49)$$

does not contain any $\ddot{\varphi}$, whereas its terms involving $\dddot{g}_{ij}$ are of the form

$$\varphi^0 \left[ \varphi^0 \left( \sum_{n,p} k_{(n,p)} A_{(n,p)} \right) + \varphi^2 \left( \sum_{n,p} k_{(n,p)} B_{(n,p)} \right) \right] \left( \sum_{n,p} k_{(n,p)} C_{(n,p)} \right). \quad (50)$$

Moreover, all its other terms involve at most first time derivatives. On the other hand, the third time derivatives entering the scalar field equation $\mathcal{E} = 0$ are exactly the same as those of $-2\varphi^0 \sum_{n,p} k_{(n,p)} \delta_0 C_{(n,p)}$, again for the same reasons as explained below Eq. (12). Therefore, it suffices to add to this scalar field equation the time derivative of Eq. (49), multiplied by 2 and divided by the large coefficient within the square brackets of (50), to get a second-order differential equation with respect to time.

Together with our previous proof that the Einstein equations themselves can be recast in terms of at most second time derivatives, we thus arrive at the following powerful result:

All generalized Galileons $\sum_{n,p} k_{(n,p)} \mathcal{L}_{(n,p)}$ in curved spacetime, with arbitrary constant coefficients $k_{(n,p)}$, yield $\frac{D(D+1)}{2} + 1$ field equations which can be combined so that none of them involve more than second time derivatives.

Obviously, this does not prove that all these theories are stable. For instance, it suffices to choose the wrong signs for some of the coefficients $k_{(n,p)}$ to get a ghost scalar degree of freedom, notably in the simplest case $+ \mathcal{L}_{(2,0)}$ with a positive sign. But this shows that the specific combination $\sum_{p=0}^{n_{max}} C_{(n,p)} \mathcal{L}_{(n,p)}$ of Ref. [8] is not safer, nor worse, than any other linear combination. The only difference is that the field equations of these “covariantized” Galileons (11) do not involve any third derivative, even purely spatial, whereas we proved that arbitrary sums $\sum_{n,p} k_{(n,p)} \mathcal{L}_{(n,p)}$ can be cured from their third time derivatives, but they keep other types of third derivatives (spatial, or mixing space and time).
2. Horndeski/k-essence-like generalizations

It was shown in Ref. [6] (see also the review [34]), that the above models can be generalized further. Indeed, the flat-space Lagrangians $\mathcal{L}_{(n+1,0)}$, with $p = 0$, can first be integrated by parts to be rewritten as $-\frac{n+1}{2} \phi_0^2 A^{\mu_3 \mu_4 \ldots \mu_{2n}} S_{(n-1)} + \text{tot. div.}$, and one may then replace the factor $\phi_0^2$ by any function $f(\phi, \phi_0^2)$ without changing the structure of the higher derivatives of the model. Hence the simplest case of $\mathcal{L}_{(2,0)} = \phi_0^2$ generates all k-essence theories $f(\phi, \phi_0^2)$, which also include the tadpole $\mathcal{L}_{(1,0)} = \phi$ and the cosmological constant $\mathcal{L}_{(0,0)} = \text{const.}$ The $p \neq 0$ cases may also be generalized in the same way [6], as

$$- f_{(n+1,p)}(\phi, \phi_0^2) A^{\mu_3 \mu_4 \ldots \mu_{2n}} R_{(p)} S_{(q)}, \quad (51)$$

In $D = 4$ dimensions, one recovers the full class of Horndeski’s theories [1], where the functions $f_{(n+1,p)}$ of $\phi$ and $\phi_0^2$ multiplying the different terms need to have specific relations amongst themselves in order to avoid the appearance of third derivatives in the field equations. Another possible generalization of the $p \neq 0$ cases has been considered in Refs. [21,22], and is obtained by the following set of Lagrangians

$$f_{(n+1,p)}(\phi, \phi_0^2) \mathcal{L}_{(n+1,p)} = - f_{(n+1,p)}(\phi, \phi_0^2) A_{(2n-2)} \phi, \phi S_{(q)}, \quad (52)$$

which are just obtained by multiplying the Lagrangians of Eq. (26) by arbitrary functions $f_{(n+1,p)}$. The novelty claimed by Refs. [21,22] is that even without any relation between such functions, these theories in any case do not generate any second scalar degree of freedom, although their field equations do involve third derivatives, and notably third time derivatives. These references concluded that the most general model in $D = 4$ dimensions depends on six arbitrary functions of $\phi$ and $\phi_0^2$. However, as mentioned in our Introduction, the arguments given in Refs. [21,22] do not appear to be fully convincing.

Our reasoning above can be repeated without much change for these wide classes of theories. Indeed, let us first consider a single Lagrangian of the form (52). It is easy to see that the energy-momentum tensor of such a theory is just obtained by Eq. (34) where one replaces everywhere $\phi_0$ by $f_{(n+1,p)} \phi_0$, and to which one adds the extra term

$$- \frac{\partial f}{\partial X} X^\alpha X^\beta \mathcal{L}_{(n+1,p)}, \quad (53)$$

where $X \equiv \phi_0^2$, and $\mathcal{L}_{(n+1,p)}$ is the generalized Galileon Lagrangian (26) (corresponding to $f = 1$). The three lemmas derived below Eq. (35) follow then immediately: No third derivative of the metric enters $T^{\alpha \beta}$, and neither $T^{00}$ nor $\phi_0 T^{\alpha 0}$ contain any $\ddot{\phi}$ or $\dddot{\phi}$. For the same reasons as explained below Eq. (12), one also finds that all components $T^{\alpha 0}$ of the stress-energy tensor involve the same contraction of $\ddot{g}_{ij}$, namely

$$T^{\alpha 0} |_{\ddot{g}_{ij}} = - \phi_0^2 C + \frac{\partial f}{\partial X} \mathcal{L}_{(n+1,p)} |_{\ddot{g}_{ij}}, \quad (54)$$

Note that the set of theories defined respectively by Lagrangians (51) and (52) can be related to each other by suitable integration by parts and identities between Levi-Civita tensors of the kind discussed in [6].

On the other hand, there do remain some $\ddot{\phi}_i$ in $\phi_0 T^{\alpha 0}$ when considering a Lagrangian of the form (51). However, even in this case, our reasoning also shows that the linear combination (45) does not contain any second time derivative but those of the scalar field.
where $C$ is given in Eq. (42). Therefore, these second time derivatives of the metric still cancel in the same linear combination (45) as above, while it is easy to check that the coefficient of $\ddot{\varphi}$ remains generically non zero (although it is changed with respect to the $f = 1$ case). In conclusion, Eq. (45) still allows us to express $\ddot{\varphi}$ in terms of undifferentiated fields, their spatial derivatives, and their first time derivatives, and the time derivative of this Eq. (45) can thus be used to remove all third time derivatives entering Einstein’s equations. Moreover, since the linear combination (45) does not depend on the precise model (52) under consideration, one can obviously repeat the same argument for any sum of such Lagrangians.

Let us now consider the scalar field equation $\mathcal{E} = 0$ deriving from a Lagrangian (52). It involves third time derivatives of the metric, of the form

$$\mathcal{E}|_{\tilde{g}_{ij}} = -2\varphi^0 \partial_0 \left( f C + \frac{\partial f}{\partial X} \mathcal{L}_{(n+1,p)} \right)|_{\tilde{g}_{ij}}.$$

They are proportional to those entering $\partial_0 T^{00}$, Eq. (54), and this is not a surprise since this is again a consequence of diffeomorphism invariance, as explained below Eq. (42). It thus suffices to construct the unique linear combination of the Einstein equations $(G^{00} - T^{00}) = 0$ and $\varphi_0 (G^{00} - T^{00}) = 0$ such that all $\ddot{\varphi}$ cancel, and the time derivative of this combination allows us to replace all third derivatives entering $\mathcal{E} = 0$ in terms of at most second time derivatives. This conclusion can also be extended to an arbitrary sum of Lagrangians (52), by following the same reasoning as in Eqs. (49)–(50) above. [Let us also recall that for any theory, the scalar field equation $\mathcal{E} = 0$ can always be recovered from the divergence of Einstein’s equations, Eq. (6).]

In conclusion, any sum of Lagrangians $\sum_{n,p} f_{(n,p)}(\varphi, \varphi^2 \lambda) \mathcal{L}_{(n,p)}$, i.e., the k-essence-like extensions of generalized Galileon theories (26), yield field equations which can be recast as a set of $\frac{D(D+1)}{2} + 1$ second-order differential equations (as far as time is concerned). In $D$ dimensions, there are $\lfloor \frac{D+1}{2} \rfloor \lfloor \frac{D}{2} + 1 \rfloor$ classes of models, each depending on an arbitrary function $f_{(n,p)}(\varphi, \varphi^2 \lambda)$. This is consistent with the 6 classes found in Refs. [21–23] for $D = 4$, and would give for instance 30 classes of models in the $D = 10$ dimensions of string theory.

III. HAMILTONIAN ANALYSIS OF THE QUARTIC GALILEON

In this final section, we work in $D = 4$ dimensions and focus solely on the Lagrangian $\mathcal{L}_{(4,0)}$. Our aim is to carry out a Hamiltonian analysis of this particular theory without fixing any gauge, and to show — from a Hamiltonian point of view — that it cannot contain 4 degrees of freedom. As such, the results of this section support the results of the previous section, though the approach is different.

A Hamiltonian analysis of “beyond Horndeski” theories was carried out in [21–23], though in that paper the authors restricted their attention to the unitary gauge, $t = \varphi$. As we shall see below, in the unitary gauge the Hamiltonian analysis is greatly simplified. Indeed, in an arbitrary gauge, the Lagrangian (from which the Hamiltonian is constructed) explicitly contains a term in $\ddot{\varphi}$ multiplied by first time derivatives of the metric: this is the term which generates third time derivatives in the equations of motion. However, this term is also multiplied by spatial derivatives of $\varphi$ which vanish in the unitary gauge. Hence in the unitary gauge the Lagrangian contains no second-order time derivatives. Their presence (in a gauge-invariant calculation, as discussed here) renders a Hamiltonian analysis much more involved as we shall see.
In this section we carry out a Hamiltonian analysis for the non-gauge fixed action

\[ S = \int d^4x \sqrt{-g} \left[ R + \mathcal{L}_{(4,0)} \right], \tag{56} \]

where we recall that \( \mathcal{L}_{(4,0)} \) was given in Eq. (7). However, precisely because of the presence of the \( \ddot{\varphi} \) terms mentioned above, the calculation will quickly become complex and lead to very sizeable expressions. Thus for reasons of clarity, we will not display all expressions in their full gory detail: a courageous reader is referred to Appendix B for more details. In fact, we shall push the calculation only as far as to be able to conclude that the theory necessarily possesses less than 4 Lagrangian degrees of freedom.

A. ADM parametrization and primary constraints

In the ADM parametrization (with lapse \( N \), shift \( N^i \), and spatial metric \( \gamma_{ij} \)), action (56) becomes

\[ S = \int dt d^3x N \sqrt{\gamma} (K_{ij} K^{ij} - K^2 + (3) R) \]
\[ + \int dt d^3x \frac{\sqrt{\gamma}}{N} \varepsilon^{ijk} \varepsilon^{lm} k \left[ - \dot{\varphi} \dot{s}_{ij} s_{lm} - 2 \dot{\varphi} \dot{s}_{ij} s_{00} s_{lm} + 2 \varphi \dot{\varphi} \dot{s}_{00} s_{ij} s_{lm} + 4 \varphi \varphi \dot{s}_{00} s_{ij} s_{lm} \right] \]
\[ + \int dt d^3x \frac{\sqrt{\gamma}}{N} \varepsilon^{ijk} \varepsilon^{lmn} \varepsilon^0_k \left[ 2 \varphi \varphi \dot{s}_{00} s_{ij} s_{lm} - 4 \varphi \varphi \dot{s}_{00} s_{ij} s_{lm} \right] \]
\[ + \int dt d^3x N \sqrt{\gamma} \left( 1 - \frac{N_s N^p}{N^2} \right) \varepsilon^{ijk} \varepsilon^{lmn} s_{ij} s_{lm} s_{kn} \dot{\varphi} \dot{\varphi} \]
\[ \equiv \int d^4x L, \tag{57} \]
\[ \equiv \int d^4x L, \tag{58} \]

where

\[ s_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \varphi, \tag{59} \]

and \( K_{ij} \) is the extrinsic curvature (see Appendix A).

The first line, linear in the lapse, is the usual ADM decomposition of the Einstein Hilbert action in General Relativity (GR). Notice, however, that due to the other terms, the Lagrangian \( L \) is no longer linear either in \( N \) nor \( N^i \). Furthermore, \( L \) generically contains products of second (covariant) derivatives of \( \varphi \) (the \( s_{\mu\nu} \)), which in turn contain Christoffel symbols — which are expressed in terms of time derivatives of the lapse and shifts, see Appendix A Thus the action depends explicitly and non-linearly on \( \dot{N} \) and \( \dot{N}^i \), as opposed to the case of GR. Notice that the term \( \propto \varphi \varphi \dot{s}_{00} s_{ij} \) (mentioned above) in principle generates 3rd order derivatives in the equations of motion, though it vanishes in the unitary gauge.

More generally, the non-linear dependence of action \( S \) on the variables \( s_{\mu\nu} \) makes it very difficult to invert \( \dot{\gamma}_{ij} \) in terms of its conjugate momenta \( \pi^{ij} \equiv \delta L / \delta \dot{\gamma}_{ij} \). To alleviate this problem, we proceed by linearizing the action in second derivatives. [Note that this technique would also work for higher-order Galileon theories, for instance \( L_{(5,0)} \).] That is, we rewrite the action (58) as

\[ \tilde{S} = S + \int d^4x \tilde{\lambda}^{\mu\nu} \left( s_{\mu\nu} - \nabla_\mu \nabla_\nu \varphi \right), \tag{60} \]
where \( s_{\mu\nu} \) and \( \tilde{\lambda}_{\mu\nu} \) are just symmetric tensors considered as dynamical fields. The field \( \tilde{\lambda}_{\mu\nu} \) is a Lagrange multiplier imposing the relation (59). It is straightforward to check that the equations of motion following from the two actions (58) and (60) are equivalent. Notice, however, that the price to pay for this linearization is the introduction of new degrees of freedom: indeed the dynamical fields are now

\[
N, N^i, \gamma_{ij}, \phi, \lambda_{\mu\nu}, s_{\mu\nu},
\]

(61)

where, for computational simplicity we choose to work with

\[
\lambda_{\mu\nu} = N \sqrt{-\gamma} \tilde{\lambda}_{\mu\nu}
\]

(62)

rather than \( \tilde{\lambda}_{\mu\nu} \). Thus there are a total of 31 dynamical fields, together with their 31 conjugate momenta defined by

\[
\pi_N \equiv \frac{\delta L}{\delta \dot{N}}, \quad \pi_i \equiv \frac{\delta L}{\delta \dot{N}^i}, \quad \pi^{ij} \equiv \frac{\delta L}{\delta \dot{\gamma}_{ij}}, \\
\pi_{\phi} \equiv \frac{\delta L}{\delta \dot{\phi}}, \quad \pi^{(\lambda)}_{\mu\nu} \equiv \frac{\delta L}{\delta \dot{\lambda}_{\mu\nu}}, \quad \pi^{(s)}_{\mu\nu} \equiv \frac{\delta L}{\delta \dot{s}_{\mu\nu}}.
\]

(63)

The fields and their conjugate momenta satisfy the standard Poisson-Bracket (PB) relations, for instance \( \{N(x), \pi_N(y)\} = \delta^3(x, y) \) (see Appendix B for the remaining — obvious — relations).

The canonical momenta are determined directly from (57) and (60), and we find

\[
\pi^{(s)}_{\mu\nu} = 0, \quad \pi^{(\lambda)}_{0i} = 0, \quad \pi^{(\lambda)}_{ij} = 0, \quad \pi_i = \lambda^{00} \phi_i, \quad \pi_N = \frac{1}{N} (N^0 \pi^{(\lambda)} - N^i \pi_i),
\]

(64)

where

\[
\pi^{(\lambda)} = \pi^{(\lambda)}_{00} = \dot{\phi}.
\]

(65)

The remaining conjugate momenta \( \pi_{\phi} \) and \( \pi^{ij} \) are given in Appendix B, equations (B4) and (B5) respectively. In particular \( \pi_{\phi} \) depends linearly on both \( \dot{\phi} \) and \( \dot{N} \) [actually on \( \partial_0(\lambda^{00} N) \)]. Hence, given the expression for \( \pi^{(\lambda)} \) in (65), naively one might expect this theory to contain two scalar degrees of freedom, \( \lambda^{00} \) and \( \phi \). The momentum \( \pi^{ij} \) conjugate to \( \dot{\gamma}_{ij} \) depends on \( \dot{\gamma}_{ij} \) (as expected) as well as \( \phi \).

The 23 relations in (64) define 23 primary constraints

\[
\Phi^{(s)}_{\mu\nu} \equiv \pi^{(s)}_{(\mu\nu)} \approx 0, \\
\Phi^{(\lambda)}_{0i} \equiv \pi^{(\lambda)}_{0i} \approx 0, \\
\Phi^{(\lambda)}_{ij} \equiv \pi^{(\lambda)}_{ij} \approx 0, \\
\Phi_i \equiv \pi_i - \lambda^{00} \phi_i \approx 0, \\
\Phi_N \equiv \pi_N - \frac{\lambda^{00}}{N} (\pi^{(\lambda)} - N^i \pi_i) \approx 0.
\]

(66-70)

It is straightforward to see that on shell, all the primary constraints have vanishing Poisson brackets amongst each other. For the following discussion, it will be useful to denote the primary constraints by

\[
\{\Phi^{(0)}_{(s)}, \Phi_i, \Phi_N, \Phi_P\}, \quad P = 1, \ldots, 18.
\]

(71)
That is we separate out $\Phi_{(s)}^{00}$, $\Phi_i$ and $\Phi_N$ from the remaining $\Phi_P$ primary constraints, since, as we shall see below, they must be considered differently from the others. Recall that in GR, $\Phi_i$ and $\Phi_N$ correspond, respectively, to the primary constraints $\pi_i \approx 0$, $\pi_N \approx 0$ which constitute 4 of the 8 first class constraints associated with diffeomorphism invariance. In a similar way, 4 of the primary constraints in (71) (namely, a linear combination of $\Phi_i$ and $\Phi_N$ with the other primary constraints) must also be first-class constraints. There must also be 4 secondary constraints of first class — which, in turn, do not generate tertiary or higher generation constraints — as a consequence of the fact that diffeomorphism invariance is expressed infinitesimally through four independent parameters (the four components of a vector) which are just differentiated once with respect to time [35, 36].

**B. Canonical Hamiltonian and secondary constraints**

The remaining Hamiltonian analysis follows the standard route (see for instance [37]), but is rather involved due to the large number of fields and the intrinsically non-linear nature of the problem. The first step is to calculate the canonical Hamiltonian $H_c$ from which, as a second step we determine the secondary constraints, obtained by imposing the preservation of primary constraints under time evolution. We find

$$H_c = \int d^3x \left\{ N\sqrt{\gamma} \left[ (K_{ij}K^{ij} - K^2) - (3)R \right] + 2(D_iN_j)\pi^{ij} - \lambda^{\mu\nu}s_{\mu\nu} + (\pi_\varphi\pi^\lambda) \right. \right.$$  
$$\left. \left. - (P\pi^\lambda + q)\pi^\lambda - V \right. \right. \right.$$  
$$+ (D_iD_j\varphi)(\lambda^{ij} + 2\lambda^{0i}N^j) - \lambda^{00}\varphi^k [N^q(D_qN_k) + N(\partial_kN)] \right.$$  
$$\left. - \pi_N \left[ N^k(\partial_kN) + \frac{2}{\lambda^{00}}\partial_i(N\lambda^{0i}) \right] \right\},$$

where $P$, $q$ and $V$ are various combinations of functions appearing in the action (57), as defined below. To simplify expressions, we define

$$\mathcal{F}^{ijkm} = \epsilon^{ijk}\epsilon^{lmn}$$

so that

$$P = -\frac{\sqrt{\gamma}}{N}\mathcal{F}^{ijkm}s_{it}s_{jm},$$

$$q = \frac{2\sqrt{\gamma}}{N}\varphi_\ell \left( 2\mathcal{F}^{ijkm}s_{i0}s_{jm} + \epsilon^{ijk}\epsilon^{lmn}N_ks_{lm}s_{jn} \right),$$

$$V = \frac{\sqrt{\gamma}}{2}\varphi_\ell\varphi_\iota \left[ \mathcal{F}^{ijkm}(s_{0m}s_{0i} - s_{00}s_{jm}) - 2\epsilon^{ijk}\epsilon^{lmn}N_ks_{0m}s_{jn} \right]$$  
$$\left. + N\sqrt{\gamma} \left( 1 - \frac{N_pN_p}{N^2} \right) \epsilon^{ijk}\epsilon^{lmn}s_{jm}s_{kn}\varphi_\iota\varphi_\ell, \right.$$  

and the extrinsic curvature $K_{ij}$ is given in terms of the canonical momenta $\pi^{ij}$ by

$$\sqrt{\gamma}K^{ij} = \Lambda^{ij} - \gamma^{ij}\frac{\Lambda}{2},$$

where

$$\Lambda^{ij} = \pi^{ij} - \frac{\pi_N}{2N} \left( N^iN^j + \frac{\lambda^{ij}}{\lambda^{00}} + 2\frac{\lambda^{0(i}N^{j)}}{\lambda^{00}} - \left( \lambda^{0(i}\varphi^{j)} + \lambda^{00}N^{(j}\varphi^{i)} \right).$$

18
Notice that, because of the term linear in $\pi_\varphi$, the canonical Hamiltonian appears at first sight not to be bounded from below, à la Ostrogradski [17]. However, we shall see that $\pi_\varphi$ is in fact not independent of the other fields, because of one of the secondary constraints (to be precise it is the constraint $\mathcal{H}_0$, defined in (85) and given explicitly in equation (B8) in Appendix [13]. As a result the canonical Hamiltonian will actually vanish on shell, as expected for a diffeomorphism-invariant theory, see Eq. (87). Following the standard procedure, the total Hamiltonian is then given by

$$H_T = H_c + \int d^3x \left[ \zeta_{\mu \nu} \Phi_{(s)}^{\mu \nu} + \zeta^{0i} \Phi_{(s)}^{0i} + \zeta^{ij} \Phi_{(s)}^{ij} + \zeta^L \Phi_L + \zeta^N \Phi_N \right],$$

(79)

thus introducing 23 Lagrange multipliers, $\zeta$.

Imposing the conservation of the primary constraints, schematically $\dot{\Phi}(x) = \{\Phi(x), H_T\}$, enables us to determine 23 corresponding secondary constraints (denoted by $\kappa = \dot{\Phi}(x)$). Using Eq. (72) together with the primary constraints in (70), we find

$$\kappa^{0i}_{(s)} = -\frac{2\sqrt{\gamma}}{N} \mathcal{F}^{i\ell j m} \varphi_i \varphi_\ell s_{jm} + \lambda^{00},$$

(80)

$$\kappa^{0i}_{(s)} = 4 \frac{\sqrt{\gamma}}{N} \left( \pi^\lambda \mathcal{F}^{i\ell j k} s_{j k} \varphi_\ell + \mathcal{F}^{i_{\ell j} j} s_{0j} \varphi_k \varphi_\ell - \epsilon^{\ell j k} \epsilon_{\ell m n} N_k s_{jn} \varphi_\ell \varphi_\ell \right) + 2 \lambda^{0i},$$

(81)

$$\kappa^{ij}_{(s)} = \pi^{pq} - 2 \frac{\sqrt{\gamma}}{N} (\pi^\lambda) \mathcal{F}^{i_{pq} j \ell m} s_{jm} + 4 \frac{\sqrt{\gamma}}{N} \pi^\lambda \left\{ \mathcal{F}^{i \ell (pq)} \varphi_\ell s_{00} - \epsilon^{k(p} \epsilon(q) \epsilon_{\ell m n} N_k \varphi_\ell s_{jm} \right\},$$

(82)

$$\kappa^{00}_{(s)} = \pi_0 \mathcal{F}^{i \ell j k} s_{jk} \varphi_\ell + \frac{K_{ij} \pi N}{\lambda^{00}},$$

(83)

$$\kappa^{ij}_{(s)} = 2 s_{00} - 2 (\Pi_i \Pi_j \varphi) N^{ij} - 2 N \partial_i \left[ \frac{\pi N}{\lambda^{00}} \right] + 2 K_{ij} \left[ N^i \pi N + N \varphi_j \right].$$

(84)

The last 4 secondary constraints, denoted by

$$\mathcal{H}_0 = -\{\Phi_N, H_T\}, \quad \mathcal{H}_i = -\{\Phi_i, H_T\},$$

(85)

in analogy with the Hamiltonian and momentum constraints in GR, are given in Appendix [13]. The first, $\mathcal{H}_0$ contains a term in $\pi_\varphi \pi_N$, while the second $\mathcal{H}_i$ contains a term in $D_j \pi_\varphi^i$. In linear combinations with the other (primary and secondary) constraints, they must therefore constitute the 4 remaining secondary first-class generators associated with diffeomorphism invariance.

In analogy with the primary constraints, it will be useful to write the set of secondary constraints as

$$\{\kappa^{(s)}_{0i}, \mathcal{H}_i, \mathcal{H}_0, \kappa_P\}, \quad \tilde{P} = 1, \ldots, 18.$$

(86)

Finally, in terms of the primary and secondary constraints, the canonical Hamiltonian in Eq. (72) can be expressed as

$$H_c = \int d^3x \left\{ N \mathcal{H}_0 + N_i \mathcal{H}_i - (2 \kappa^{00}_{(s)} s_{00} + \kappa^{0i}_{(s)} s_{0i}) - \kappa^{ij}_{(s)} \lambda^{ij} \right.$$

$$+ \Phi_N \left[ K_{ij} \left( N^i N^j + 2 \frac{\lambda^{0i}(N^j)}{\lambda^{00}} + \frac{\lambda^{ij}}{\lambda^{00}} \right) + \frac{2}{\lambda^{00}} \partial_i (N \lambda^{0j}) + N^i \partial_i N \right] \}.$$  

(87)

That is, as expected for a diffeomorphism-invariant theory, the Hamiltonian vanishes on shell.
C. Counting degrees of freedom, first and second-class constraints

The theory has $2 \times 31 = 62$ Hamiltonian degrees of freedom, and so far we have identified 23 primary and 23 secondary constraints. Of these, at least 8 must be first class (due to diffeomorphism invariance). If all the remaining constraints were second class and if there were no tertiary constraints, then at this stage we would conclude that the theory contains

$$62 - (2 \times 8) - (46 - 8) = 8$$

(88)

Hamiltonian degrees of freedom. That is 4 Lagrangian degrees of freedom: 2 for the graviton and 2 scalars.

The next step therefore consists in determining whether or not the remaining primary and secondary constraints are of second class. To do so, as per the standard procedure, one must calculate the $46 \times 46$ matrix of their Poisson brackets. We order the constraints with the primary constraints first followed by the secondary ones. At this stage it is useful to notice that this $46 \times 46$ matrix is of the anti-diagonal form

$$\begin{pmatrix} 0 & A \\ -A & B \end{pmatrix},$$

(89)

where first $23 \times 23$ block vanishes since the primary constraints all commute (see above), and the $23 \times 23$ block $A$ is itself symmetric because of the Jacobi identity:

$$\{\Phi, \kappa_j\} = \{\Phi_i, \{\Phi_j, H_T\}\} = -\{\Phi_j, \{H_T, \Phi_i\}\} = \{H_T, \{\Phi_i, \Phi_j\}\} = +\{\Phi_j, \kappa_i\}.$$  

(90)

We now focus on a subset of the primary and secondary constraints, namely the $\Phi_0$ and $\kappa_0$; the remaining constraints will be discussed later. The Poisson brackets of these constraints define a $36 \times 36$ matrix of the same anti-diagonal form, where the antidiagonal block is the $18 \times 18$ matrix

$$\mathcal{M}_{18} \equiv \{\Phi_0, \kappa_Q\}, \quad (\bar{P}, \bar{Q}) = 1, \ldots, 18.$$  

(91)

The determinant of this $36 \times 36$ matrix is $= -(\det(\mathcal{M}_{18}))^2$, and using the commutators given in appendix [B] one can check that $\det(\mathcal{M}_{18}) \neq 0$. Hence these $18 \times 2 = 36$ constraints are necessarily second class. This also implies that they do not generate any tertiary constraints, because $\{\kappa, H_T\} \approx 0$ actually fix the corresponding Lagrange multipliers $\zeta$ entering the total Hamiltonian in Eq. (79).

However, as soon as we consider the $19 \times 19$ matrix consisting of $M_{18}$ but augmented by the further primary constraint $\Phi_{00}^{(s)}$, and the further secondary constraint $\kappa_{00}^{(s)}$, the determinant vanishes. Hence there exists a linear combination of $\Phi_{00}^{(s)}$ and $\Phi_0$ which commutes with all the $(\Phi_{00}^{(s)}, \Phi_0, \kappa_{00}^{(s)}, \kappa_0)$. The explicit expression for this linear combination, denoted by $\tilde{\Phi}_{00}^{(s)}$, is given in Appendix [B], equation (B16). Similarly we have shown that there exists a linear combination, denoted by $\tilde{\kappa}_{00}^{(s)}$, of $\kappa_{00}^{(s)}$ with the $\kappa_0$’s and the primary constraints which commutes with all the $(\Phi_{00}^{(s)}, \Phi_0, \kappa_{00}^{(s)}, \kappa_0)$.

At this stage, the Hamiltonian analysis of $L_4$ must necessarily proceed in one of the two following ways:
1. Either $\tilde{\Phi}^{00}_0$ and $\tilde{\kappa}^{(s)}_{00}$ also commute with the remaining constraints (namely $\mathcal{H}_0$, $\mathcal{H}_i$, $\Phi_N$ and $\Phi_i$), and hence constitute two more first-class constraints in addition to the $2 \times 4$ implied by diffeomorphism invariance. In this case the theory has 6 Hamiltonian degrees of freedom (2 for the graviton and 1 for the scalar field).

2. Or $\tilde{\Phi}^{00}_0$ and $\tilde{\kappa}^{(s)}_{00}$ do not commute with the remaining constraints (namely $\mathcal{H}_0$, $\mathcal{H}_i$, $\Phi_N$ and $\Phi_i$). In that case there exist a tertiary and possibly quaternary (or higher) constraints. Indeed, the vanishing of the above $19 \times 19$ determinant implies that the Lagrange multiplier $\zeta^{(s)}_{00}$ entering (79) is not fixed by the conservation of the secondary constraint, $\{\tilde{\kappa}^{(s)}_{00}, H_T\} \approx 0$.

Whichever is correct, we arrive at the important conclusion that there are necessarily less than 8 Hamiltonian degrees of freedom.

In order to see whether the first option is the true one, we note that if they are indeed first-class generators, $\tilde{\Phi}^{00}_0$ and $\tilde{\kappa}^{(s)}_{00}$ must generate a new symmetry. However, we have determined the transformations of the fields induced by these generators, and found that the action of the theory in not invariant. Hence, necessarily we arrive at the conclusion that case 2 is correct: there must exist a tertiary (and even perhaps a quaternary) constraint beyond the primary and secondary constraints found above. These are generated by the conservation of $\tilde{\kappa}^{(s)}_{00}$. Unfortunately, though, because of the size of the corresponding Poisson brackets, the computation of these tertiary/quaternary constraints is a cumbersome task we felt unnecessary to attempt.

However, we have carried out an identical calculation in the simpler case of $\mathcal{L}_{(2,0)}$ (using the same technique with our Lagrange multipliers (60), although of course this greatly complicates the calculation in this very simple case). Here we have shown that there are indeed a tertiary constraint generated by the conservation of $\tilde{\kappa}^{(s)}_{00}$, and even a quaternary constraint generated by the conservation of this tertiary one. There are no higher order constraints because the conservation of the quaternary constraints actually imposes the value of the Lagrange multiplier $\zeta^{(s)}_{00}$. It happens that the secondary and tertiary do not commute with each other, and that the primary and the quaternary also do not commute with each other. Therefore, all of these four constraints are actually of second class. As compared to Eq. (88), there are thus 2 extra constraints, which give a final number of 6 Hamiltonian degrees of freedom. That is, 3 Lagrangian degrees of freedom: 2 for the graviton and a single scalar. We expect the same result to hold true also for $\mathcal{L}_{(4,0)}$, although we did not do this explicit calculation because of its complexity, and due to the conclusions of Sec. IV which we felt rendered it unnecessary.

IV. CONCLUSIONS

In this work, we have considered Galileon models in $D$ dimensions, as well as the models defined by the counterterms which have been introduced to maintain the second-order nature of the field equations when both the metric and the scalar are made dynamical. We have first shown that in one given such model, all the third time derivatives which appear in the field equations can be eliminated, leaving a set of $\frac{D(D+1)}{2} + 1$ field equations with at most second time derivatives of the dynamical fields. The same has been shown to hold for an arbitrary linear combination of such models, as well as their k-essence-like generalizations involving
free functions of $\varphi$ and $\varphi^2$. (In $D$ dimensions, these models can depend on $\lfloor \frac{D+1}{2} \rfloor \lfloor \frac{D}{2} + 1 \rfloor$ independent such functions.) This supports the claim made previously [21, 22] that the number of degrees of freedom in these theories is only 3, counting 2 for the graviton and 1 for the scalar. However, it does not provide an absolutely rigorous proof of this claim which would require a detailed Hamiltonian analysis of these models. Such a Hamiltonian analysis has been carried out in the second part of this paper for one of the models under consideration, reaching the conclusion that the number of degrees of freedom is indeed strictly less than 4. It seems, however, very likely, in light of our results, and taking into account the gauge invariance of the theory (which has been fully kept in our analysis, in contrast to previous works) that the final number of degrees of freedom is well only 3. Indeed, using the first part of this paper, one expects that the “reduced” field equations (containing only second derivatives) can still be decomposed into 4 Lagrangian constraints (coming from the invariance of the theory under reparametrization) plus 6 dynamical equations for 6 dynamical metric variable and an other one for the scalar. Note, however, that the extraction of these would be 4 Lagrangian constraints is much trickier than in standard general relativity because it can be checked that, in an ADM language, the field equations contain second time derivatives of the spatial part of the metric, but also of the lapse and the shift. Provided these Lagrangian constraints exist, the gauge invariance would then reduce to two the number of dynamical components in the metric. Rigorously speaking, our Hamiltonian analysis also leaves open the possibility that there is just one tertiary second class constraint (and no quaternary). This would result in an odd (times infinity$^8$) number of second class constraints, which can happen in a field theory (see e.g. [38] and references therein), but seems to us unlikely in a bosonic and Lorentz-invariant theory. This, however, deserves further investigation to be rigorously checked.

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Appendix A: Christoffel symbols in ADM variables

In the ADM notation, the 4D metric and its inverse are written as

$$\begin{align*}
g_{\mu \nu} &= \begin{pmatrix} -N^2 + N_k N^k & N_i \\
N_j & \gamma_{ij} \end{pmatrix}, &
g^{\mu \nu} &= \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\
\frac{N^j}{N^2} & \gamma_{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}. \quad (A1)
\end{align*}$$

$^8$ In classical mechanics, there is always an even number of second-class constraints, but this is no longer true in continuous field theories, where each constraint should actually be understood as an infinity of them, since it is imposed at every point of the Cauchy surface.
where here and in the following all spatial indices $i, j, k, ...$ are raised and lowered with the spatial metric $\gamma_{ij}$. On using (A1) the Christoffel symbols can be calculated. These will contain $\dot{\gamma}_{ij}$, $\dot{N}$ and $\dot{N}^i$. The covariant derivative associated with the metric $\gamma_{ij}$ is denoted by $D$, and $K_{ij}$ is defined by

$$K_{ij} = \frac{1}{2N} (\dot{\gamma}_{ij} - D_i N_j - D_j N_i).$$

(A2)

We find

$$\Gamma^{00}_{00} = \frac{1}{N} \left( \dot{N} + N^i \partial_i N + K_{ij} N^i N^j \right),$$

$$\Gamma^{ij}_{00} = -\frac{\dot{N}}{N} N^j + \dot{N}^j + 2NN^q K_{qk} \left( \gamma^{jk} - \frac{N^k N^j}{2N^2} \right) + N \partial_k N \left( \dot{\gamma}^{jk} - \frac{N^j N^k}{N^2} \right),$$

$$\Gamma^{0i}_{j0} = \frac{1}{N} \left( \partial_i N + K_{jl} N^l \right),$$

$$\Gamma^{il}_{j0} = -\frac{N^l \partial_i N}{N} - \frac{1}{N} N^i N^m K_{jm} + \gamma^{lm} K_{jm} N + \gamma^{lm} D_j N_m,$$

$$\Gamma^{0j}_{il} = \frac{1}{N} K_{jl},$$

$$\Gamma^{nj}_{jl} = -\frac{N^n}{N} K_{jl} + \Gamma^{nj}_{jl}(\gamma),$$

(A3)

where $\Gamma^{nj}_{jl}(\gamma)$ stands for the Christoffel symbols of the spatial metric $\gamma_{ij}$.

**Appendix B: Hamiltonian analysis of $\mathcal{L}_{(4,0)}$**

We give here some of the intermediate expressions required for Sec. III. We will use one abuse of notation in this Appendix (and only here): namely we denote $\gamma_{ij} \phi_j$ by $\phi^i$. The 31 dynamical variables given in (61), namely

$$N, N^i, \gamma_{ij}, \varphi, \lambda_{\mu\nu}, s_{\mu\nu},$$

(B1)

and their conjugate momenta

$$\pi_N, \pi^i, \pi^{ij}, \pi_\varphi, \pi^{(\lambda)}_{\mu\nu}, \pi^{\mu\nu}_{(s)},$$

(B2)

satisfy the Poisson Brackets

$$\{ N(x), \pi_N(y) \} = \delta^3(x, y),$$

$$\{ N^i(x), \pi_j(y) \} = \delta^i_j \delta^3(x, y),$$

$$\{ \gamma_{ij}(x), \pi^{pq}(y) \} = \delta^p_i \delta^q_j \delta^3(x, y),$$

$$\{ \lambda^{00}(x), \pi_{00}^0(y) \} = \delta^3(x, y),$$

$$\{ \lambda^{0i}(x), \pi_{0i}^0(y) \} = \delta^i_j \delta^3(x, y),$$

$$\{ \lambda^{ij}(x), \pi_{pq}^0(y) \} = \delta^i_p \delta^j_q \delta^3(x, y),$$

$$\{ s_{00}(x), \pi^0_{S0}(y) \} = \delta^3(x, y),$$

$$\{ s_{0i}(x), \pi^0_{Si}(y) \} = \delta^i_j \delta^3(x, y),$$

$$\{ s_{ij}(x), \pi^0_{Sj}(y) \} = \delta^i_j \delta^3(x, y),$$

(B3)
with all other commutators vanishing.

Starting from (57), (58) and (60), the conjugate momenta \( \pi_\phi \) and \( \pi_{ij} \) are found to be given by

\[
\pi_\phi = \dot{\lambda}^{00} + \dot{Q} + 2P \pi^\lambda + \frac{\lambda^{00} N}{N} + \frac{K_{ij} Q^{ij}}{N},
\]

\[
\pi_{ij} = \sqrt{\gamma}(K^{ij} - K\gamma^{ij}) + \frac{\varphi_m T^{mij}}{2N} + \frac{Q^{ij} \dot{\varphi}}{2N^2}
\]

where

\[
\dot{Q} = \frac{1}{N}(\partial_i N)\lambda^{00} + \frac{2}{N}\partial_i (N\lambda^{0i}),
\]

\[
Q^{ij} = \lambda^{00} N^i N^j + 2\lambda^{0i} N^j + \lambda^{ij},
\]

\[
T^{mnm} = 2N\lambda^m \left( \gamma^{im} - \frac{N^i N^m}{2N^2} \right) + 2\lambda^0 N \left( \gamma^{im} - \frac{N^i N^m}{2N^2} \right) \lambda^{mn} N^i
\]

On using (69), (70) and (72), one can calculate the secondary constraints

\[
\mathcal{H}_0 = -\{\Phi_N, H_T\},
\]

\[
\mathcal{H}_i = -\{\Phi_i, H_T\}.
\]

The first is given by

\[
\mathcal{H}_0 = \sqrt{\gamma} \left[ (K_{ij} K^{ij} - K^2) - (3) R \right] + K_{ij} B^{ij} + \frac{\pi_\phi \pi_N}{\lambda^{00}} + \frac{\lambda^{00}}{N} s_{00} + \frac{2}{N} \partial_i \left( N\lambda^{0i} \right) + \frac{\lambda^{00}}{N} \partial_i \left( N\lambda^{0i} \right)
\]

\[
+ 2\lambda^{0i} \partial_i \left( \frac{\pi_N}{\lambda^{00}} \right) + \partial_i \left( N\lambda^{00} \phi^i + \pi_N N^i \right) + \frac{\pi_N}{N} N^i (\partial_i N)
\]

\[
+ \frac{\lambda^{00}}{N} \phi^j \dot{N}^j (D_j N_i) - (\partial_i \pi^\lambda) \left( \frac{\lambda^{00} N^i}{N} \right) - \frac{\Phi_N}{N} \left[ K_{ij} \left( N^i N^j + 2\frac{\lambda^{00}(i) N^j}{\lambda^{00}} + \frac{\lambda^{ij}}{\lambda^{00}} \right) + N^k \partial_k N + \frac{1}{\lambda^{00}} \partial_i (N\lambda^{0i}) + \frac{N\pi_\phi}{\lambda^{00}} \right],
\]

where

\[
B_{ij} = \frac{\pi_N}{N} \left( 2N^i N^j + 2\frac{\lambda^{00}(i) N^j}{\lambda^{00}} + \frac{\lambda^{ij}}{\lambda^{00}} \right) + 2\lambda^{00} N^{i} \varphi^i,
\]

and \( P \) and \( q \) were defined in (74) and (75) respectively. The quantities \( V_1 \) and \( V_2 \) are different components (depending on their \( N \)-dependence) of \( V \) defined in Eq. (76)). Namely \( V = V_1 + V_2 \) with

\[
V_1 = \frac{2}{N} \sqrt{\gamma} \phi_\ell \phi_i \left[ \mathcal{F}^{ijm} (s_{0m} s_{0j} - s_{00} s_{jm}) - 2\epsilon^{ijk} \epsilon^{kmn} N_k s_{0m} s_{jm} \right]
\]

\[
- \frac{\sqrt{\gamma}}{N} N_p \epsilon^{ijk} \epsilon^{kmn} s_{jm} s_{kn} \phi_i \phi_\ell,
\]

\[
V_2 = N \sqrt{\gamma} \epsilon^{ijk} \epsilon^{kmn} s_{jm} s_{kn} \phi_i \phi_\ell.
\]
Finally,
\[ H_i(x) = -2D_j \pi^j_i - 2NK_{ij} \left[ \frac{\pi_N}{N} \left( N^j + \frac{\lambda_0^j}{\lambda_{00}} \right) + \lambda_{00} \varphi_j \right] \]
\[ + 2D_j (\lambda_{00} N^j \varphi_i) + 2(D_j D_i \varphi) \lambda_{0j} - \lambda_{00} \varphi_j (D_i N_j) - \pi_N \partial_i N + \lambda_{00} \partial_i \pi^\lambda \]
\[ + \varphi_i \pi_j - \varphi_j (2P \pi^\lambda + q) \]
\[ + \frac{2\sqrt{N}}{N} \epsilon^{lmn} \varphi \epsilon \left[ N_i (\epsilon_{qjk} s_{jm} s_{kn} \varphi_q) - \epsilon^{qj} s_{jn} (\pi^\lambda s_{qm} - 2 \varphi_q s_{0m}) \right]. \] (B12)

We now list the different commutators of primary and secondary constraints, which can be straightforwardly determined from the primary constraints given in (66)-(70) and the secondary constraints given in (80)-(84). In particular, we find that \( \Phi_{(s)}^{00} \) satisfies
\[ \{ \Phi_{(s)}^{00}, \kappa_{ij} \} = \frac{2\sqrt{N}}{N} F^{ijpq} \varphi_p \varphi_q = \{ \Phi_{(s)}^{ij}, \kappa_{(0)}^{00} \}, \] (B13)
with all its remaining commutators (including with the \( \kappa^{(\lambda)} \) vanishing. Then we find
\[ \{ \Phi_{(s)}^{0i}, \kappa_{0q} \} = -\frac{4\sqrt{N}}{N} F^{k\ell_1i} \pi^{k\ell_1} \varphi \epsilon, \]
\[ \{ \Phi_{(s)}^{0i}, \kappa_{pq}^{0q} \} = -\frac{4\sqrt{N}}{N} \varphi \epsilon \left[ F^{\ell(pq)} \pi^\lambda + \epsilon^{fj(p} \pi^{q)} \varphi_j N_j \right], \]
\[ \{ \Phi_{(s)}^{0i}, \kappa_0^{0j} \} = -2\delta^i_j, \]
\[ \{ \Phi_{(s)}^{0i}, \kappa_{pq}^{0j} \} = 0, \]
\[ \{ \Phi_{(s)}^{ij}, \kappa_{pq}^{00} \} = -\delta_{pq} \delta^{ij}, \]
\[ \{ \Phi_{(s)}^{ij}, \kappa_{pq}^{ij} \} = \frac{2\sqrt{N}}{N} \left[ (\pi^\lambda)^2 F^{(pq)(ij)} + 2\pi^\lambda \epsilon^{ik(p} \epsilon^{q)j} \varphi \epsilon N_k - \epsilon^{fj(p} \pi^{q)} \varphi_j \varphi \epsilon (N^2 - N_k N^q) \right], \] (B14)

where this last expression should be understood to be symmetrized over both \( pq \) and \( ij \).

Finally
\[ \{ \Phi_{(s)}^{(\lambda)}, \kappa_0^{0j} \} = \frac{N}{\sqrt{N}} \gamma_{ij} \left[ \left( \frac{\pi_N}{N \lambda_{00}} \right) N_f + \varphi_f \right]^2, \]
\[ \{ \Phi_{(s)}^{(\lambda)}, \kappa_{mn}^{(\lambda)} \} = \frac{\pi_N}{2 \sqrt{N} \lambda_{00}} \left[ \left( \frac{\pi_N}{N \lambda_{00}} \right) N^f + \varphi_f \right] \left[ \gamma_{m} \gamma_{nf} + \gamma_{m} \gamma_{fm} - \gamma_{if} \gamma_{mn} \right], \]
\[ \{ \Phi_{(s)}^{ij}, \kappa_{mn}^{(\lambda)} \} = \frac{\pi_N^2}{4N \sqrt{N} \lambda_{00}^{00}} \left[ \gamma_{m} \gamma_{nj} + \gamma_{mj} \gamma_{ni} - \gamma_{ij} \gamma_{mn} \right]. \] (B15)

From the 19 × 19 matrix discussed in Sec. IIIIC, the following linear combination of primary constraints (denoted by \( \Phi_{(s)}^{00} \), commutes with all the \( \{ \Phi_{(s)}^{ij}, \kappa_{(s)} \} \):
\[ \begin{align*}
\tilde{\Phi}_{(s)}^{00} &= [N^4 - 2N^2 \varphi^j_0 (\pi^\lambda - N^k \varphi_k) + \pi^\lambda (\pi^\lambda - N^k \varphi_k) (\pi^\lambda - 2N^j \varphi_j)] \Phi_{(s)}^{00} \\
&- [N^2 \varphi^2_0 (\pi^\lambda - N^k \varphi_k) + N_j \varphi_j (N^2 \varphi^2_0 (\pi^\lambda - N^k \varphi_k)^2 \varphi_j \Phi_{(s)}^{0i}] \\
&- (\pi^\lambda - N^k \varphi_k)^2 \varphi_i \Phi_{(s)}^{ij} \varphi_j - 2N^2 (\varphi^i \Phi_{(s)}^{ij} \varphi_j - \varphi^2 \Phi^{(\lambda)}_{ij} k). \end{align*} \] (B16)
We have also proved that it commutes with $\Phi_N$ as well as $H_0$. The secondary constraint $\tilde{\kappa}_{00}^{(s)}$ which commutes with all the $\{\Phi_P\}$ as well as $\Phi_N$ is given by the same expression but where, on the right hand side, the primary constraints $\Phi$ are replaced by secondary constraints $\kappa$ (with the same labels). One must also complement this $\tilde{\kappa}_{00}^{(s)}$ with a linear combination of primary constraints so that it commutes with all secondary constraints [see our discussion below Eq. (91)].

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