CAYLEY DIGRAPHS OF MATRIX RINGS OVER FINITE FIELDS

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ABSTRACT. We use the unit-graphs and the special unit-digraphs on matrix rings to show that every $n \times n$ nonzero matrix over $\mathbb{F}_q$ can be written as a sum of two $SL_n$-matrices when $n > 1$. We compute the eigenvalues of these graphs in terms of Kloosterman sums and study their spectral properties; and prove that if $X$ is a subset of $\text{Mat}_2(\mathbb{F}_q)$ with size $|X| > \frac{2n^2}{q}$, then $X$ contains at least two distinct matrices whose difference has determinant $\alpha$ for any $\alpha \in \mathbb{F}_q^*$. Using this result we also prove a sum-product type result: if $A, B, C, D \subseteq \mathbb{F}_q$ satisfy $4 |A||B||C||D| = \Omega(q^{0.75})$ as $q \to \infty$, then $(A - B)(C - D)$ equals all of $\mathbb{F}_q$. In particular, if $A$ is a subset of $\mathbb{F}_q$ with cardinality $|A| > \frac{2}{3}q^2$, then the subset $(A - A)(A - A)$ equals all of $\mathbb{F}_q$. We also recover a classical result: every element in any finite ring of odd order can be written as the sum of two units.

Keywords: Spectral Graph Theory, Matrix Rings, Sum-Product Problem.

AMS 2010 Subject Classification: Primary: 05C50; Secondary: 16U60, 15B33.

1. INTRODUCTION AND STATEMENTS OF RESULTS

Let $R$ be a finite ring with identity, and let $U$ denote the set of units. We define the unit-graph $G$ on $R$ to equal the directed graph (digraph) whose vertex set is $R$, for which there is a directed edge from $a$ to $b$ if and only if $b - a \in U$. This is equivalent to saying that $G$ is the Cayley digraph on $R$ associated to the subset $U$, i.e. $G = \text{Cay}(R, U)$. This digraph can also be viewed as an undirected graph, since the fact that $u \in U \iff -u \in U$ implies that there exists an edge from $a$ to $b$ if and only if there is also an edge from $b$ to $a$.

In this paper we first study these unit-graphs in the special cases where $R$ is a finite simple ring, or equivalently where $R$ is isomorphic to the matrix ring $\text{Mat}_n(\mathbb{F}_q)$ for some finite field $\mathbb{F}_q$. It is easy to see that such graphs are regular, and we show that they are connected as well. If $n = 1$, $R \cong \mathbb{F}_q$, so all such graphs are trivially complete. If $n \geq 2$, it is well known (and also shown here) that any element of $\text{Mat}_n(\mathbb{F}_q)$ can be written as a sum of two invertible matrices, which easily implies that the diameter of the unit-graph on $\text{Mat}_n(\mathbb{F}_q)$ is 2 when $n \geq 2$.

Among other results, we prove in this paper that the adjacency matrix of the unit-graph on $\text{Mat}_n(\mathbb{F}_q)$ has at most $n + 1$ distinct eigenvalues, from which we deduce that the unit-graph on $\text{Mat}_n(\mathbb{F}_q)$ in the case $n = 2$ is strongly regular for any finite field $\mathbb{F}_q$. In addition, we calculate the spectrum and the parameters of these strongly regular graphs for varying $q$, and show that these parameters agree with those of another family of strongly regular graphs, namely Latin square graphs.
Along with studying these unit-digraphs on the rings $R = \text{Mat}_n(\mathbb{F}_q)$, we also define the special unit-digraphs on such rings $R$, by replacing the subsets $U = \text{GL}_n(\mathbb{F}_q)$ by the subsets $U' = \text{SL}_n(\mathbb{F}_q)$, so that these digraphs equal $\text{Cay}(R, U') = \text{Cay}(\text{Mat}_n(\mathbb{F}_q), \text{SL}_n(\mathbb{F}_q))$. We then show that every nonzero element of $\text{Mat}_n(\mathbb{F}_q)$ for $n \geq 2$ can be written as a sum of two $\text{SL}_n$-matrices, and hence these digraphs also are connected with diameter 2, although in this case we show that the corresponding adjacency matrices can have a larger number of distinct eigenvalues (at most $n + q - 1$ of these). Furthermore, we compute the spectrums of these digraphs in terms of Kloosterman sums in the case of $n = 2$ and apply the spectral gap theorem to prove the following:

**Theorem 1.1.** Let $\alpha \in \mathbb{F}_q^*$. If $X, Y \subseteq \text{Mat}_2(\mathbb{F}_q)$ satisfies $\sqrt{|X||Y|} > \frac{2q^3 \sqrt{q}}{q-1}$, then there exists some $M \in X$ and $N \in Y$ such that $M - N$ has determinant $\alpha$. In particular, if $|X| > \frac{2q^3 \sqrt{q}}{q-1}$, then $X$ contains at least two distinct matrices whose difference has determinant $\alpha$. Thus, if $|X| = \Omega(q^{2.5})$ as $q \to \infty$, then it contains at least two distinct matrices whose difference has determinant $\alpha$.

Using this result we also prove that if $A, B, C, D \subseteq \mathbb{F}_q$ satisfy $\sqrt{|A||B||C||D|} = \Omega(q^{0.75})$ as $q \to \infty$, then $(A - B)(C - D)$ equals all of $\mathbb{F}_q$. In particular, if $A$ is a subset of $\mathbb{F}_q$ with cardinality $|A| > \frac{3}{2}q^3$, then the subset $(A - A)(A - A)$ equals all of $\mathbb{F}_q$.

As a separate result, using Artin-Wedderburn Theory together with one of our matrix ring propositions, we show that every element in any finite ring of odd order can be written as the sum of two units.

### 2. Some Linear Algebra

Let $\mathbb{F}_q$ be the finite field of order $q$ and let $\text{Mat}_n(\mathbb{F}_q)$ be the ring of $n \times n$ matrices over $\mathbb{F}_q$. The general linear group $\text{GL}_n(\mathbb{F}_q) = \{ A \in \text{Mat}_n(\mathbb{F}_q) \mid \det(A) \neq 0 \}$ is the group of invertible matrices in $\text{Mat}_n(\mathbb{F}_q)$ under matrix multiplication. We can easily calculate the order of this group by using the fact that a matrix is invertible if and only if its columns (or rows) are linearly independent. If $A \in \text{GL}_n(\mathbb{F}_q)$ and $A = [v_1 \ v_2 \ \cdots \ \ v_n]$ for some column vectors $v_1, v_2, \ldots, v_n \in \mathbb{F}_q^n$, then $v_1$ can be anything but not the zero vector; $v_2$ can be anything but not a scalar multiple of $v_1$; $v_3$ can be anything but not a linear combination of $v_1$ and $v_2$ etc. This means we have $(q^n - 1)$ many possibilities for $v_1$, once we pick $v_1$, $v_2$ has $(q^n - q)$ many possibilities, once we have $v_1$ and $v_2$, $v_3$ has $(q^n - q^2)$ etc. Hence, we have

$$|	ext{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) = q^{n^2} \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right) \cdots \left(1 - \frac{1}{q^n}\right).$$  \hspace{1cm} (1)

We use the same notation with $[7]$ and define

$$\phi(n, q) = \frac{|\text{GL}_n(\mathbb{F}_q)|}{|\text{Mat}_n(\mathbb{F}_q)|} = \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right) \cdots \left(1 - \frac{1}{q^n}\right).$$

\[\text{This result has been known for a while (5), but was previously proven using different and arguably more complicated methods.}\]
Notice that if $\phi(n, q) > \frac{1}{2}$, then every matrix in $\operatorname{Mat}_n(F_q)$ can be written as a sum of two invertible matrices by the pigeonhole principle: For any $A \in \operatorname{Mat}_n(F_q)$, we have $|\operatorname{GL}_n(F_q)| = |A - \operatorname{GL}_n(F_q)|$. Hence if $\phi(n, q) > \frac{1}{2}$, then $\operatorname{GL}_n(F_q) \cap (A - \operatorname{GL}_n(F_q)) \neq \emptyset$, and the result follows.

**Proposition 2.1.** Every element of $\operatorname{Mat}_n(F_q)$ can be written as a sum of two invertible matrices for all $n \geq 1$ and all finite fields $F_q$ as long as $q > 2$.

**Proof.** We will show that $\phi(n, q) > \frac{1}{2}$ under the assumptions and the result will follow from the above discussion. First notice that since each factor of $\phi(n, q)$ is increasing in $q$, $\phi(n, q)$ is increasing in $q$, so the general case will follow from $q = 3$. Since $\phi(n, 3)$ is monotonically decreasing as a function of $n$ and also bounded below by 0, we have $\alpha := \lim_{n \to \infty} \phi(n, 3) = (1 - \frac{1}{3}) (1 - \frac{1}{3^2}) \cdots (1 - \frac{1}{3^r}) \cdots$ exists by the monotone convergence theorem.

$$\begin{align*}
- \log \alpha &= \sum_{k=1}^{\infty} \frac{\left(\frac{1}{3}\right)^k}{k} + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{3^2}\right)^k}{k} + \cdots \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\left(\frac{1}{3}\right)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{1}{3^k}\right)^n = \sum_{k=1}^{\infty} \frac{1}{k(3^k - 1)} \\
&= \frac{1}{2} + \frac{1}{16} + \sum_{k=3}^{\infty} \frac{1}{k(3^k - 1)} \leq \frac{1}{2} + \frac{1}{16} + \frac{10}{9} \sum_{k=3}^{\infty} \frac{1}{3k} \\
&= \frac{1}{2} + \frac{1}{16} + \frac{10}{9} \left(\sum_{k=1}^{\infty} \frac{\left(\frac{1}{3}\right)^k}{k} - \frac{1}{3} - \frac{1}{18}\right) = \frac{9}{16} + \frac{10}{9} \log \left(\frac{3}{2}\right) - \frac{35}{81} \\
\end{align*}$$

Since $\log \left(\frac{3}{2}\right) \approx 0.405$, we have $- \log \alpha < 0.581$ which implies $\alpha > 0.5$ hence the claim follows. $\square$

**Definition 2.1.** Let $F$ be any field. Two $n \times n$ matrices $A, B \in \operatorname{Mat}_n(F)$ are said to be $\operatorname{GL}_n$-equivalent if there exists two invertible matrices $P$ and $Q$ such that $A = PBQ$. $A$ and $B$ are said to be $\operatorname{SL}_n$-equivalent if both $P$ and $Q$ are also in $\operatorname{SL}_n(F)$.

**Theorem 2.2.** Let $F$ be any field and let $A, B \in \operatorname{Mat}_n(F)$.

- $A$ is $\operatorname{GL}_n$-equivalent to $B$ if and only if rank $A = \operatorname{rank} B$.
- $A$ is $\operatorname{SL}_n$-equivalent to $B$ if and only if they have the same rank and determinant.

**Proof.** Since " implies direction is clear for both statements, we will only prove the converses.

Let $A \in \operatorname{Mat}_n(F)$ of rank $r$. First we want to show that by performing a finite number of modified elementary row and column operations on $A$ we can transform it into $D \in \operatorname{Mat}_n(F)$ such that

- $D_{rr} = \begin{cases} 1, & \text{if } r < n \\ \det(A), & \text{if } r = n \end{cases}$
- $D_{ii} = 1$ for $i < r$, and the rest of the entries are zero.
To prove this claim we will use certain modified elementary operations. Adding any constant multiple of a row (column) of $A$ to another row (column) will be called an operation of type 1; multiplying one of the rows (columns) with some nonzero number $\alpha$ and another row (column) with $\frac{1}{\alpha}$ simultaneously will be called an operation of type 2. A result of a type 1 (or type 2) operation on $A$ can also be written as $EA$ or $AE$ (depending on if it is a row or column operation) for some $E \in \text{SL}_n(F)$. Hence, our claim is indeed that $A$ can be transformed into $D$ via the multiplication of $A$ with some $\text{SL}_n$-matrices.

Once the claim is proven, that means there exists some $E_1, \ldots , E_m \in \text{SL}_n(F)$ such that $E_1 \cdots E_j AE_{j+1} \cdots E_m = D$. If $A$ and $B$ have the same rank and determinant, similarly we can add the multiples of $\text{SL}_n$-matrices. Hence the only thing left to show is that $A$ can be transformed into $D$ via the multiplication of $A$ with some matrices in $\text{SL}_n(F)$.

If $A = [0]_{n \times n}$ (or equivalently if rank $A = 0$), then $r = 0$ and $D = A$. Assume $A \neq [0]_{n \times n}$ from now on, so that $r > 0$.

If $n = 1$, then $A = [a]$ for some $a \in F^\ast$. We have $r = 1 = n$ and $\det(A) = a$. Hence we have $D = A$. Assume $n > 1$.

**Step 1**: If $(1,1)$-entry of $A$ i.e. $a_{11}$ is 1, proceed to Step 2.

If $a_{11} = 0$, then since rank $A \neq 0$, there exists some $a_{ij} \neq 0$. Add some multiple of $i^{th}$ row to the $1^{st}$ row so that $(1,j)$-entry will be 1. Then add $j^{th}$ column to the $1^{st}$ column so that $(1,1)$-entry will be 1. Hence this case requires at most two type 1 operations.

If $a_{11} \neq 0,1$, then multiply $A$ with $E_{a_{11}}$ from the left, where $E_{a_{11}}$ is the $n \times n$ identity matrix only with $(1,1)$-entry replaced with $\frac{1}{a_{11}}$ and $(2,2)$-entry replaced with $a_{11}$. This is an example of type 2 operation and the resultant matrix i.e. $E_{a_{11}}A$ has $(1,1)$-entry equal to 1.

**Step 2**: We can add the multiples of $1^{st}$ row to the other rows, and we can add the multiples of $1^{st}$ column to the other columns and eliminate all nonzero entries in the $1^{st}$ row and the $1^{st}$ column with the exception of the 1 in the $(1,1)$-entry. Hence we transformed $A$ into $B$ such that

$$B = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & & & \\
0 & & B' & \\
\vdots & & & & \\
0 & & & & 
\end{bmatrix}_{n \times n}$$
where $B'$ is an $(n-1) \times (n-1)$ matrix. This step requires at most $2n - 2$ many type 1 operations.

**Step 3:** If $\text{rank } B' = 0$ or $B'$ is a $1 \times 1$ matrix, then stop.

Otherwise apply step 1 and 2 on $B'$ this time, and transform $B'$ into

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & & & \\
\vdots & B'' & & \\
0 & & & \end{bmatrix}_{(n-1) \times (n-1)}
\]

and check $B''$ is the zero matrix or $B''$ is a $1 \times 1$ matrix, or not. So, we can continue applying Step 1 and 2 consecutively and find $B''', B'''' \ldots$ etc. until eventually either one of them becomes the zero matrix or a $1 \times 1$ matrix. At the end of this process, if $r < n$ we get $r$ many ones in the diagonal and zeros everywhere else. But if $r = n$, that means $A$ is transformed into some matrix in the form of

\[
D = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & & & \\
0 & & \ddots & & \\
\vdots & & & 1 & \\
0 & & & & \delta
\end{bmatrix}_{n \times n}
\]

Notice that as we transformed $A$ into $D$, we only performed type 1 and type 2 operations on $A$, i.e. we multiplied $A$ with only $\text{SL}_n$-matrices. This operation does not change the determinant. Hence, $\det A = \det D = \delta$ and the claim follows.

Assume $A$ and $B$ have the same rank, but not necessarily the same determinant. Then notice that when we apply the above process, if their determinant is zero, then we get the same $D$ for both of them. But if their determinants are nonzero, then it is easy to show that both $A$ and $B$ are $\text{GL}_n$-equivalent to $n \times n$ identity matrix and this finishes the proof. \qed

In the following remark, we note a small observation which will be very useful later for some of our calculations.

**Remark 2.3.** Let $A$ be $\text{GL}_n$ [resp. $\text{SL}_n$]-equivalent to $B$. $A$ can be written as a sum of two $\text{GL}_n$ [resp. $\text{SL}_n$]-matrices if and only if $B$ can be written as a sum of two $\text{GL}_n$ [resp. $\text{SL}_n$]-matrices.

**Proposition 2.4.** Every element of $\text{Mat}_n(\mathbb{F}_2)$ can be written as a sum of two invertible matrices when $n \geq 2$.

**Proof.** First notice that if we can write a matrix with rank $r$ as a sum of two units (i.e. invertible matrices), then Remark 2.3 combined with the previous theorem implies that every matrix with rank $r$ can be written as a sum of two units for some units. Therefore to prove the result for $n = 2$ and $n = 3$ cases, we write here one arbitrary element from each rank as a sum of two units:
rank 1: \[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} + \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix},
\]

rank 2: \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

rank 3: \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

Besides the only matrix with rank zero is the zero matrix, and it can be written as a sum of the identity matrix and negative of the identity matrix. This calculation completes the result for \( n = 2 \) and \( n = 3 \). We use induction for \( n \geq 4 \). Let \( n \geq 4 \) be fixed. Assume every element of \( \text{Mat}_k(\mathbb{F}_2) \) can be written as a sum of two units in \( \text{Mat}_k(\mathbb{F}_2) \) for \( 1 < k < n \). Let \( A \in \text{Mat}_n(\mathbb{F}_2) \).

**Case 1:** \( A \) is invertible so \( \text{rank } A = n \). Since the \( n \times n \) identity matrix \( I_n \) and \( A \) are \( \text{GL}_n \)-equivalent, we will show that \( I_n \) can be written as a sum of two units and the result will follow for \( A \) by Remark 2.3. We have

\[
I_n = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}_{n \times n} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}_{2 \times 2} \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}_{(n-2) \times (n-2)}.
\]

By the calculation before, we have \( \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = A_1 + A_2 \) for some invertible \( 2 \times 2 \) matrices \( A_1 \) and \( A_2 \).

By induction hypothesis, we have \( \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix} = A_3 + A_4 \) for some invertible \( (n - 2) \times (n - 2) \) matrices \( A_3 \) and \( A_4 \). Hence,

\[
I_n = \begin{bmatrix}
A_1 & 0 \\
0 & A_3
\end{bmatrix} + \begin{bmatrix}
A_2 & 0 \\
0 & A_4
\end{bmatrix}.
\]

Moreover, since \( \det \begin{bmatrix}
A_1 & 0 \\
0 & A_3
\end{bmatrix} = \det(A_1) \det(A_3) \) and similarly for \( \begin{bmatrix}
A_2 & 0 \\
0 & A_4
\end{bmatrix} \), both \( \begin{bmatrix}
A_1 & 0 \\
0 & A_3
\end{bmatrix} \) and \( \begin{bmatrix}
A_2 & 0 \\
0 & A_4
\end{bmatrix} \) are invertible.

**Case 2:** \( A \) is not invertible, then \( \text{rank } A = r < n \). Recall that elementary row and column operations do not change the rank of a matrix, so by performing elementary operations on \( A \), we can transform \( A \) into some \( B \).
such that all of the entries in the \(n\)th row and \(n\)th column of \(B\) are zero. Then
\[
B = \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} A_2 & 0 \\ 0 & -1 \end{bmatrix}
\]
for some \(A_1, A_2 \in \text{GL}_{n-1}(\mathbb{F}_2)\) by the induction hypothesis.

The following corollary recovers a classical result, see \([11], [12]\).

**Corollary 2.5.** Except in the trivial case where \(n = 1\) and \(q = 2\), any element of \(\text{Mat}_n(\mathbb{F}_q)\) can be written as a sum of two invertible matrices.

The corollary follows from Proposition \([2.1]\) and \([2.4]\). Furthermore, we can combine this result with classical finite ring results (Artin-Wedderburn Theory) and recover a classical result (\([5]\)):

**Corollary 2.6.** If \(R\) is a finite ring with identity and if its order is odd, then every element of \(R\) is the sum of two units.

**Proof.** Consider \(R/J\), where \(J\) denotes the Jacobson radical of \(R\). First notice that \(R/J\) is semisimple i.e. \(J (R/J) = 0\). Moreover, since \(R\) is finite, \(R/J\) is finite so \(R/J\) is both left and right Artinian. Artin-Wedderburn theorem implies that \(R/J \cong \text{Mat}_{n_1}(D_1) \times \cdots \times \text{Mat}_{n_r}(D_r)\) for some \(D_1, \ldots, D_r\) division rings (see \([3], [6]\)). Since \(R/J\) is finite, each \(D_i\) has to have finitely many elements. By Wedderburn’s little theorem \(D_i\)’s are finite fields. Therefore, we have \(R/J \cong \text{Mat}_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times \text{Mat}_{n_r}(\mathbb{F}_{q_r})\) for some finite fields \(\mathbb{F}_{q_1}, \ldots, \mathbb{F}_{q_r}\). Since \(R\) has odd order, \(2 \nmid |R/J|\), hence none of the \(\mathbb{F}_{q_i}\)’s is the finite field of order 2. As a result of the previous corollary, every element of \(R/J\) can be written as a sum of two units and this implies every element of \(R\) is the sum of two units, see \([2]\) for more details. \(\square\)

3. Special Graphs on Matrix Rings over Finite Fields

In this section we study the unit-graph and the special unit-digraph on \(\text{Mat}_n(\mathbb{F}_q)\) and prove the rest of the results which we referred to in the introduction earlier.

**Proposition 3.1.** The unit-graph on \(\text{Mat}_n(\mathbb{F}_q)\) is connected and \(|\text{GL}_n(\mathbb{F}_q)|\)-regular.

**Proof.** Let \(A \in \text{Mat}_n(\mathbb{F}_q)\). There is an edge between \(A\) and \(B\) if and only if \(B - A\) is an invertible matrix. That means every time we add an invertible matrix to \(A\), we get a neighbor of \(A\); and they are all distinct as \(A + C_1 = A + C_2\) implies \(C_1 = C_2\). So, the degree of vertex \(A\) is \(|\text{GL}_n(\mathbb{F}_q)|\). To prove the first part of the claim, assume \(\text{Mat}_n(\mathbb{F}_q) \neq \mathbb{F}_2\) and let \(A, B \in \text{Mat}_n(\mathbb{F}_q)\). By Corollary \([2.5]\) we know that \(B - A\) can be written as a sum of two units, i.e. \(B - A = C_1 + C_2\) for some \(C_1, C_2 \in \text{GL}_n(\mathbb{F}_q)\). Then, there is an edge from \(A\) to \(A + C_1\), and there is an edge from \(A + C_1\) to \(B\). This means the graph is connected. \(\square\)
As a side note, notice that this proof implies that the unit-graph on $\text{Mat}_n(F_q)$ has diameter at most two, but it is easy to see that it is actually precisely two when $n \geq 2$, and it has diameter one when $n = 1$.

**Proposition 3.2.** The unit-graph on $\text{Mat}_n(F_q)$ has at most $n + 1$ distinct eigenvalues.

**Proof.** Since the unit-graph on $\text{Mat}_n(F_q)$ is a Cayley digraph, we can find its eigenvectors and eigenvalues using Theorem A.1. Let $\tilde{\chi}$ be a character on $\text{Mat}_n(F_q)$. Then, it can be written as $\tilde{\chi}(s) = \chi(\text{Tr}(As)) = \chi_A(s)$ for some $A \in \text{Mat}_n(F_q)$ where Tr denotes the matrix trace and $\chi$ stands for the canonical character on $F_q$, see [2].

Let $A \in \text{Mat}_n(F_q)$. Then the eigenvalue corresponding to $A$ is explicitly $\lambda_A = \sum_{s \in \text{GL}_n(F_q)} \chi(\text{Tr}(As))$ by Theorem A.1. If $A, B \in \text{Mat}_n(F_q)$ are $\text{GL}_n$-equivalent, i.e. if there exists two invertible matrices $P$ and $Q$ such that $A = PBQ$, then $\lambda_A = \lambda_B$. This a consequence of the following calculation.

$$\lambda_A = \sum_{s \in \text{GL}_n(F_q)} \chi(\text{Tr}(PBQs)) = \sum_{s' \in \text{GL}_n(F_q)} \chi(\text{Tr}(PBs')) = \sum_{s' \in \text{GL}_n(F_q)} \chi(\text{Tr}(Bs'P)) = \sum_{s'' \in \text{GL}_n(F_q)} \chi(\text{Tr}(Bs''))$$

Recall by Theorem 2.2 we know that two matrices are $\text{GL}_n$-equivalent if and only if their rank is the same, this implies we have at most $n + 1$ distinct eigenvalues. □

Let $G$ be a non-empty and not complete regular graph. Recall that $G$ is called a strongly regular graph with parameters $(n, k, a, c)$ if it is $k$-regular, every pair of distinct adjacent vertices has $a$ common neighbors and every pair of distinct nonadjacent vertices has $c$ common neighbors. We note the following well-known fact as a lemma, and its proof can be found on page 220 in [4].

**Lemma 3.3.** A connected regular graph with exactly three distinct eigenvalues is strongly regular.

This lemma combined with Proposition 3.1 and 3.2 yields the following result.

**Corollary 3.4.** The unit-graph on $\text{Mat}_2(F_q)$ is strongly regular for any finite field $F_q$.

Since we found the unit-graph on $\text{Mat}_2(F_q)$ is strongly regular, then one wonders what the parameters are for this graph.

**Theorem 3.5.** The unit-graph on $\text{Mat}_2(F_q)$ is a strongly regular graph with parameters $(q^4 - q^3 - q^2 + q, q^4 - 2q^3 - q^2 + 3q, q^4 - 2q^3 + q)$.

**Proof.** By Proposition 3.1 we know that the unit-graph on $\text{Mat}_2(F_q)$ is $|\text{GL}_2(F_q)|$-regular. By Equation (1) the order of $\text{GL}_2(F_q)$ is $(q^2 - 1)(q^2 - q)$, so the graph is $(q^4 - q^3 - q^2 + q)$-regular.
Now we want to count the number of common neighbors of a pair of nonadjacent vertices. We take \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
and \[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\], since they are nonadjacent. We assume \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) to satisfy \( A \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) and \( A \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).

Then, \( \det A \neq 0 \) and \( A - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a - 1 & b \\ c & d \end{bmatrix} \) should be a unit.

**Case 1:** Exactly two entries of \( A \) are nonzero.

- Let \( A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \) where \( b \) and \( c \) are nonzero. Then, \( A - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & b \\ c & 0 \end{bmatrix} \) should be a unit.
- Let \( A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \) where \( a \) and \( d \) are nonzero. Then, \( A - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a - 1 & 0 \\ 0 & d \end{bmatrix} \) and \( (a-1)d \neq 0 \), which means \( a \neq 1 \). Since \( a \) has \( q-2 \) many different possibilities, in total we have \( (q-1)(q-2) \) many different possibilities for \( A \).

**Case 2:** Exactly three entries of \( A \) are nonzero.

- Let \( A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \) where \( b, c, d \) are all nonzero. Also, \( \begin{bmatrix} -1 & b \\ c & d \end{bmatrix} \) should be a unit. Hence, \( -d - bc \neq 0 \).
- Let \( A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \) where \( a, c, d \) are all nonzero. Also, \( \begin{bmatrix} a - 1 & 0 \\ c & d \end{bmatrix} \) should be a unit. Hence, \( (a-1)d \neq 0 \) i.e. \( a \neq 1 \). Since \( a \) has \( q-2 \) and \( d \) has \( q-1 \) many different possibilities, in total we have \( (q-1)^2(q-2) \) many different possibilities for \( A \).
- Let \( A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \) where \( a, b, d \) are all nonzero. This case is the same with \( A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \) case, so we have in total \( (q-1)^2(q-2) \) many different possibilities for \( A \) in this case.
- Let \( A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \) where \( a, b, c \) are all nonzero. Here, \( \begin{bmatrix} a - 1 & b \\ c & 0 \end{bmatrix} \) should be a unit but since \( b \) and \( c \) are both nonzero, it is automatically satisfied. Hence each one of \( a, b, c \) has \( q-1 \) many different possibilities, in total we have \( (q-1)^3 \) many different possibilities for \( A \).

**Case 3:** None of the entries of \( A \) is zero. We have \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 \) and \( \det \begin{bmatrix} a - 1 & b \\ c & d \end{bmatrix} \neq 0 \).

- Assume \( bcd^{-1} = -1 \). We have \( ad - bc \neq 0 \), i.e. \( a \neq bcd^{-1} \). We also have \( (a-1)d - bc \neq 0 \).
- Assume \( bcd^{-1} \neq -1 \). We have \( q-1 \) many different possibilities for \( b \), \( q-1 \) many for \( c \), \( q-2 \) for \( d \) and \( q-3 \) many for \( a \) (since \( a \neq 0 \), \( a \neq bcd^{-1} \) and \( a \neq bcd^{-1} + 1 \)). We have in total \( (q-1)^2(q-2)(q-3) \) many possibilities for \( A \).
As a result, case 1 gives us $2q^2 - 5q + 3$ many different possibilities for $A$, case 2 gives $4q^3 - 15q^2 + 18q - 7$ and case 3 gives $q^4 - 6q^3 + 13q^2 - 12q + 4$ which sums to $q^4 - 2q^3 + q$.

Now we want to count the number of common neighbors of a pair of adjacent vertices. We pick $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, since they are adjacent. We assume $A - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies $A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then, $\det A \neq 0$ and $A - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a - 1 & b \\ c & d - 1 \end{bmatrix}$ should be a unit.

**Case 1**: Exactly two entries of $A$ are nonzero.

- Let $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ where $b$ and $c$ are nonzero. We have $A - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & b \\ c & -1 \end{bmatrix}$ and $1 - bc \neq 0$ i.e. $b \neq c^{-1}$. Hence $b$ has $q - 1$ and $c$ has $q - 2$ many different possibilities, in total we have $(q - 1)(q - 2)$ many different possibilities for $A$.

- Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ where $a$ and $d$ are nonzero. We have $A - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a - 1 & 0 \\ 0 & d - 1 \end{bmatrix}$ and $(a - 1)(d - 1) \neq 0$ which means both $a \neq 1$ and $d \neq 1$. Since $a$ has $q - 2$ many possibilities and so does $d$, in total we have $(q - 2)^2$ many different possibilities for $A$.

**Case 2**: Exactly three entries of $A$ are nonzero.

- Let $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ where $b, c, d$ are all nonzero. We also have that $\begin{bmatrix} -1 & b \\ c & d - 1 \end{bmatrix}$ is a unit, so $1 - d \neq bc$.

  - □ If $d = 1$, $b$ and $c$ can be anything. So we have $q - 1$ many for $b$ and $q - 1$ many for $c$, in total it is $(q - 1)^2$.

  - □ If $d \neq 1$, then we have $q - 2$ many for $d$, $q - 1$ many for $b$ and $q - 2$ many for $c$, so in total it is $(q - 1)(q - 2)^2$.

- Let $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$. We have that $\begin{bmatrix} a - 1 & 0 \\ c & d - 1 \end{bmatrix}$ is a unit and $(a - 1)(d - 1) \neq 0$ i.e. $a \neq 1$, $d \neq 1$. Since $a$ has $q - 2$, $c$ has $q - 1$ and $d$ has $q - 2$ many different possibilities, in total we have $(q - 2)^2(q - 1)$ many different possibilities for $A$.

Notice that $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ case and $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ case gives us the same count. Similarly, $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ gives the same count. Therefore, in case 2 we have $4q^3 - 18q^2 + 28q - 14$ many possibilities in total.

**Case 3**: None of the entries of $A$ is zero. We have $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$ and $\det \begin{bmatrix} a - 1 & b \\ c & d - 1 \end{bmatrix} \neq 0$.

- $a = 1, d \neq 1$. Since $d$ has $q - 2$, $b$ has $q - 1$ and $c$ has $q - 2$ many different possibilities, in total we have $(q - 2)^2(q - 1)$ many different possibilities for $A$.

- $a \neq 1, d = 1$. This gives the same count with $a = 1, d \neq 1$ case.
\[ a = 1, \, d = 1, \, b \text{ has } q - 1 \text{ and } c \text{ has } q - 2, \text{ we have } (q - 1)(q - 2) \text{ many different possibilities for } A. \]

\[ a \neq 1, \, d \neq 1. \]

\[ a + d = 1. \text{ Notice } a + d = 1 \iff ad + 1 - a - d = ad \iff (a - 1)(d - 1) = ad. \text{ Since } a \text{ has } q - 2, \text{ } d \text{ has } 1, \text{ } b \text{ has } q - 1 \text{ and } c \text{ has } q - 2 \text{ many different possibilities, in total we have } (q - 1)(q - 2)^2 \text{ many different possibilities for } A. \]

\[ a + d \neq 1. \text{ Since } a \text{ has } q - 2, \text{ } d \text{ has } q - 3, \text{ } b \text{ has } q - 1 \text{ and } c \text{ has } q - 3 \text{ many different possibilities, in total we have } (q - 3)^2(q - 2)(q - 1) \text{ many different possibilities for } A. \]

As a result, case 1 gives us \( 2q^2 - 7q + 6 \) many different possibilities for \( A \), case 2 gives \( 4q^3 - 18q^2 + 28q - 14 \) and case 3 gives \( q^4 - 6q^3 + 15q^2 - 18q + 8 \) which sums to \( q^4 - 2q^3 - q^2 + 3q. \)

Note that the last theorem shows us that the parameters of the unit-graph on \( \text{Mat}_2(\mathbb{F}_q) \) agrees with the parameters of another family of strongly regular graphs, Latin square graphs. It is a well-known fact that an orthogonal array \( OA(k, n) \) is equivalent to a set of \( k - 2 \) mutually orthogonal Latin squares. Moreover, the graph defined by \( OA(k, n) \) is strongly regular with parameters

\[ (n^2, (n - 1)k, n - 2 + (k - 1)(k - 2), k(k - 1)), \]

by Theorem 10.4.2 in [4]. Hence, if we let \( n = q^2 \) and \( k = q^2 - q \) then the parameters of \( OA(k, n) \) becomes the same with the parameters of the unit-graph on \( \text{Mat}_2(\mathbb{F}_q) \). In general, a strongly regular graph is not determined by its parameters, and one can wonder that in this case if the unit-graph on \( \text{Mat}_2(\mathbb{F}_q) \) and \( OA(k, n) \) are isomorphic or not. Unfortunately, we do not have a definite answer to that in this paper.

The following lemma is again a well-known fact, we refer the reader to [4] for the proof.

**Lemma 3.6.** Let \( A \) be the adjacency matrix of the \((n, k, a, c)\) strongly regular graph \( G \). Let \( \Delta = (a - c)^2 + 4(k - c) \). Then,

- the largest eigenvalue of \( A \) is \( \lambda_1 = k \) with multiplicity 1;

- the other eigenvalues of \( A \) are \( \lambda_2 = \frac{(a-c)+\sqrt{\Delta}}{2} \) with multiplicity \( m_2 \) and \( \lambda_3 = \frac{(a-c)-\sqrt{\Delta}}{2} \) with multiplicity \( m_3 \) where

\[ m_2 = \frac{1}{2} \left( (n - 1) - \frac{2k + (n - 1)(a - c)}{\sqrt{\Delta}} \right) \text{ and } m_3 = \frac{1}{2} \left( (n - 1) + \frac{2k + (n - 1)(a - c)}{\sqrt{\Delta}} \right) . \]

**Corollary 3.7.** The spectrum of the unit-graph on \( \text{Mat}_2(\mathbb{F}_q) \) is \( \{(q^4 - q^3 - q^2 + q, 1), (q, q^4 - q^3 - q^2 + q), (q - q^2, q^3 + q^2 - q - 1)\} \) where the first entries of the ordered pairs denote the eigenvalues and the second entries are the corresponding multiplicities.

**Proof.** Since we know the parameters of the unit-graph on \( \text{Mat}_2(\mathbb{F}_q) \) from Theorem 3.5, we can plug the parameters in the formulae in the previous lemma, and the result follows. \[ \square \]
**Proposition 3.8.** The formulae
\[ \sum_{s \in \text{GL}_2(\mathbb{F}_q)} \chi(s_{11}) = q - q^2 \quad \text{and} \quad \sum_{s \in \text{GL}_2(\mathbb{F}_q)} \chi(s_{11} + s_{22}) = \sum_{s \in \text{GL}_2(\mathbb{F}_q)} \chi(s_{11})\chi(s_{22}) = q \]
hold for any finite field \( \mathbb{F}_q \) and for any non-trivial character on \( \mathbb{F}_q \).

**Proof.** Consider the unit-graph on \( \text{Mat}_2(\mathbb{F}_q) \). Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). We have
\[ \lambda_A = \sum_{s \in \text{GL}_2(\mathbb{F}_q)} \chi(\text{Tr}(As)) = \sum_{s \in \text{GL}_2(\mathbb{F}_q)} \chi(s_{11}) \quad \text{and} \quad \lambda_B = \sum_{s \in \text{GL}_2(\mathbb{F}_q)} \chi(\text{Tr}(Bs)) = \sum_{s \in \text{GL}_2(\mathbb{F}_q)} \chi(s_{11} + s_{22}). \]
Since matrices of same rank correspond to same eigenvalue, we know the multiplicity of \( \lambda_A \) denoted by \( m_A \) and the multiplicity of \( \lambda_B \) denoted by \( m_B \). But we also have \( q^2 = m_A + m_B + 1 \). This forces \( m_A \) to be the number of rank 1 matrices and \( m_B \) to be \( |\text{GL}_2(\mathbb{F}_q)| \). Hence, \( m_A = q^4 - |\text{GL}_2(\mathbb{F}_q)| - 1 = q^3 + q^2 - q - 1 \) and \( m_B = q^4 - q^3 - q^2 + q \). By Corollary 3.7, we know that \( q - q^2 \) is an eigenvalue with multiplicity \( q^3 + q^2 - q - 1 \) and \( q \) is an eigenvalue with multiplicity \( q^4 - q^3 - q^2 + q \). Hence, \( \lambda_A \) should be \( q - q^2 \) and \( \lambda_B \) should be \( q \). \( \square \)

Since we know every element of \( \text{Mat}_n(\mathbb{F}_q) \) can be written as a sum of two invertible matrices as long as \( \text{Mat}_n(\mathbb{F}_q) \neq \mathbb{F}_2 \), one may wonder similarly if every element of the matrix ring can be written as a sum of two \( \text{SL}_n \)-matrices under some mild hypothesis or not. In the following proposition, we first consider a special case of this question, namely when \( n = 2 \).

**Proposition 3.9.** Every element of \( \text{Mat}_2(\mathbb{F}_q) \) can be written as a sum of two \( \text{SL}_2 \)-matrices.

**Proof.** By Remark 2.3, we know that if we can write a matrix as a sum of two \( \text{SL}_2 \)-matrices, then every \( 2 \times 2 \) matrix with the same rank and determinant can be written as a sum of two \( \text{SL}_2 \)-matrices. We will show that the zero matrix, a matrix with rank one and a matrix with determinant \( \alpha \) for some \( \alpha \in \mathbb{F}_q^* \), in particular \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \), can be written as a sum of two \( \text{SL}_2 \)-matrices and the claim will follow from it.

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

Let \( \alpha \) be any nonzero element of \( \mathbb{F}_q \). Then, \( \alpha^{-1} \) exists and we have:

\[ \begin{bmatrix} 0 & \alpha^{-1} \\ -\alpha & \alpha \end{bmatrix} + \begin{bmatrix} 1 & -\alpha^{-1} \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}. \]
Theorem 3.10. Let \( n \geq 2 \). Every nonzero element of \( \text{Mat}_n(\mathbb{F}_q) \) can be written as a sum of two \( \text{SL}_n \)-matrices. If \( n \) is even or if \( \text{char}(\mathbb{F}_q) = 2 \), then the zero matrix can be written as a sum of two \( \text{SL}_n \)-matrices; otherwise it requires three \( \text{SL}_n \)-matrices.

Proof. First we consider the zero matrix. If \( \det(A) = 1 = \det(-A) \), then the zero matrix can be written as \( A + (-A) \) and we are done. But that happens only when \( \det(A) = (-1)^n \det(A) = \det(-A) \) is satisfied, which is the case either when \( n \) is even or when we work in characteristic 2. Hence, the zero matrix can be written as \( A + (-A) \) when \( n \) is even or \( \text{char}(\mathbb{F}_q) = 2 \). Otherwise, we can write the zero matrix as \( A + (-A) \) for some \( A \in \text{SL}_n(\mathbb{F}_q) \), and once we prove every nonzero matrix can be written as a sum of two \( \text{SL}_n \)-matrices, this implies the zero matrix can be written as a sum of three \( \text{SL}_n \)-matrices.

Now, we focus on the nonzero matrices. We obtained the result for \( n = 2 \) in the previous proposition. We will prove the claim for \( n = 3 \) and then do induction on \( n \). We again note that by Remark 2.3 if we can write a \( n \times n \) matrix as a sum of two \( \text{SL}_n \)-matrices, then every \( n \times n \) matrix with the same rank and determinant can be written as a sum of two \( \text{SL}_n \)-matrices. Let \( \alpha \in \mathbb{F}_q^* \). If the characteristic of \( \mathbb{F}_q \) is 2, then we have

\[
\begin{align*}
\text{rank 1 : } & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{rank 2 : } & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
\text{rank 3 : } & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & \alpha \end{bmatrix}.
\end{align*}
\]

This finishes the proof for characteristic 2 case when \( n = 3 \). If \( \text{char}(\mathbb{F}_q) \neq 2 \), then we have

\[
\begin{align*}
\text{rank 1 : } & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} & \text{rank 2 : } & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
\text{rank 3 : } & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\alpha}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha^{-1} \\ 0 & -\alpha & \frac{\alpha}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha^{-1} \\ 0 & \alpha & 0 \end{bmatrix}.
\end{align*}
\]

This completes the proof for \( n = 3 \) case.
Let \( n \geq 4 \). Assume the claim holds for all \( 2 \leq k < n \). Let \( A \in \text{Mat}_n(F_q) \) of rank \( r \). If \( A \) is the zero matrix, we know how to handle it from the previous discussion. Hence, assume \( A \) is not the zero matrix. By the proof of Theorem 2.2 we know that \( A \) is \( \text{SL}_n \)-equivalent to \( D \in \text{Mat}_n(F_q) \) such that

\[
D_{rr} = \begin{cases} 
1, & \text{if } r < n \\
\det(A), & \text{if } r = n 
\end{cases}
\]

- \( D_{ii} = 1 \) for \( i < r \), and the rest of the entries are zero.

Since \( n \geq 4 \), we have \( n - 2 \geq 2 \). Also, since \( A \) is not the zero matrix, we know \( D_{11} = 1 \). By the induction hypothesis there exist some matrices \( A_i \)'s with determinant 1 such that

\[
D = \begin{bmatrix} A_1 & 0 & 0 \\
0 & A_3 \\
0 & 0 & A_4 \end{bmatrix}
\]

where \( A_1, A_2 \) are both \( (n-2) \times (n-2) \) and \( A_3, A_4 \) are both \( 2 \times 2 \). Hence, the claim follows. \( \square \)

In the following corollary, we state the basic properties of the special unit-digraph on \( \text{Mat}_n(F_q) \).

**Corollary 3.11.** Let \( n \geq 2 \). The special unit-digraph on \( \text{Mat}_n(F_q) \) is connected and its diameter is 2. It can be regarded as a simple (undirected) regular graph if and only if \( n \) is even or \( \text{char}(F_q) = 2 \). The adjacency matrix of the digraph has at most \( n + q - 1 \) distinct eigenvalues.

**Proof.** As we explained in the proof of previous theorem, \( 1 = \det(A) = \det(-A) \), i.e. \( \text{SL}_n(F_q) \) is a symmetric set if and only if \( n \) is even or \( \text{char}(F_q) = 2 \). Hence if \( n \) is even or if \( \text{char}(F_q) = 2 \), then the digraph can be considered as a graph. Connectedness, regularity and the diameter proof is very similar to the proof of Proposition 3.1 and follows from Theorem 3.10. To deduce the number of distinct eigenvalues, let \( A \in \text{Mat}_n(F_q) \). Then recall the eigenvalue corresponding to \( A \) is explicitly \( \lambda_A = \sum_{s \in \text{SL}_n(F_q)} \chi(\text{Tr}(As)) \), and also note that if \( A, B \in \text{Mat}_n(F_q) \) are \( \text{SL}_n \)-equivalent then \( \lambda_A = \lambda_B \) follows from a very similar calculation in the proof of Proposition 3.2. \( \square \)

Earlier in that section we proved that the unit-graph on \( \text{Mat}_2(F_q) \) has at most three distinct eigenvalues and this implied that it is a strongly regular graph. Unfortunately, in general it is not the case for the special unit-graph on \( \text{Mat}_2(F_q) \) except a few cases, but we can still compute its spectrum explicitly using the following theorem:

**Theorem 3.12.** The spectrum of the special unit-graph on \( \text{Mat}_2(F_q) \) consists of \( \lambda = q^3 - q \) with multiplicity 1, \( \mu_0 = -q \) with multiplicity \( q^3 + q^2 - q - 1 \) and \( \mu_{\delta} = q \sum_{\alpha \in F_q^*} \chi(\alpha + \delta \alpha^{-1}) \) with multiplicity \( q^3 - q \) for any \( \delta \in F_q^* \).

**Proof.** The special unit-graph on \( \text{Mat}_2(F_q) \) is a simple regular graph, hence the largest eigenvalue should be the regularity i.e. \( |\text{SL}_2(F_q)| \) with multiplicity 1 by the Perron-Frobenius theorem. Also from a different
perspective, recall that we know
\[ \lambda_A = \sum_{s \in \text{SL}_2(F_q)} \chi(\text{Tr}(As)). \]

If \( A \) is the zero matrix, \( \lambda_A = \sum_{s \in \text{SL}_2(F_q)} \chi(0) = |\text{SL}_2(F_q)| = \frac{|\text{GL}_2(F_q)|}{q - 1} = q^3 - q. \)

Next we want to show that \( \mu_0 \) is the eigenvalue corresponding to rank 1 matrices, and \( \mu_\delta \) is the eigenvalue corresponding to matrices of determinant \( \delta \). To ease our calculations to follow up, first we sort out \( \text{SL}_2 \)-matrices. Let \( s \in \text{SL}_2(F_q) \), then \( s \) should be in one of the following forms.

**Case 1:** Exactly two entries of \( s \) are nonzero. Then, \( s \) should be in the form of \[
\begin{bmatrix}
0 & 0 \\
\alpha & -\alpha^{-1}
\end{bmatrix}
\]
for some \( \alpha \in \mathbb{F}_q^* \).

**Case 2:** Exactly three entries of \( s \) are nonzero. Then, \( s \) should be in the form of \[
\begin{bmatrix}
0 & \beta \\
\alpha & -\alpha^{-1}
\end{bmatrix}
\]
or \[
\begin{bmatrix}
\beta & 0 \\
-\alpha^{-1} & \alpha
\end{bmatrix}
\]
for some \( \alpha, \beta \in \mathbb{F}_q^* \).

**Case 3:** None of the entries of \( s \) is zero. Then, \( s \) should be in the form of \[
\begin{bmatrix}
\alpha & \gamma \\
c & \beta
\end{bmatrix}
\]
where \( c = \gamma^{-1}(\alpha \beta - 1) \) for some \( \alpha, \gamma \in \mathbb{F}_q^* \) and \( \beta \in \mathbb{F}_q \setminus \{\alpha^{-1}\} \).

Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Then we have
\[
\lambda_A = \sum_{s \in \text{SL}_2(F_q)} \chi(\text{Tr}(As)) = \sum_{s \in \text{SL}_2(F_q)} \chi(s_{11}) = \sum_{s \in \text{Case 1}} \chi(s_{11}) + \sum_{s \in \text{Case 2}} \chi(s_{11}) + \sum_{s \in \text{Case 3}} \chi(s_{11}). \tag{2}
\]

We also have
\[
\sum_{s \in \text{Case 1}} \chi(s_{11}) = (q - 1)\chi(0) + \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha)
\]
\[
\sum_{s \in \text{Case 2}} \chi(s_{11}) = (q - 1)^2 \chi(0) + 3 \left[ \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha) \right] (q - 1)
\]
\[
\sum_{s \in \text{Case 3}} \chi(s_{11}) = \left[ \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha) \right] (q - 1)(q - 2).
\]

We can plug the last three equations in Equation \tag{2}, and use the orthogonality relations in character theory (in particular \( \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha) = 0 \)) and we get \( \lambda_A = -q \). The multiplicity is the number of rank 1 matrices, hence it is \( q^3 + q^2 - q - 1 \).
Corollary 3.13. The equation
\[ \lambda_B = \sum_{s \in \text{SL}_2(\mathbb{F}_q)} \chi(\text{Tr}(Bs)) = \sum_{s \in \text{SL}_2(\mathbb{F}_q)} \chi(s_{11} + \delta s_{22}). \] (3)

We also have
\[
\sum_{s \in \text{Case 1}} \chi(s_{11} + \delta s_{22}) = (q - 1)\chi(0) + \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha + \delta^{-1} \alpha^{-1})
\]
\[
\sum_{s \in \text{Case 2}} \chi(s_{11} + \delta s_{22}) = \left[ \sum_{\alpha \in \mathbb{F}_q^*} \chi(\delta \alpha) \right] (q - 1) + (2q - 2) \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha + \delta^{-1} \alpha^{-1}) + (q - 1) \left[ \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha) \right]
\]
\[
\sum_{s \in \text{Case 3}} \chi(s_{11} + \delta s_{22}) = \left[ \sum_{\alpha \in \mathbb{F}_q^*} \left( \sum_{\beta \in \mathbb{F}_q^* \setminus \{\alpha^{-1}\}} \chi(\alpha + \delta \beta) \right) \right] (q - 1) = \left[ \sum_{\alpha \in \mathbb{F}_q^*} (-\chi(\alpha) - \chi(\alpha)\chi(\delta^{-1} \alpha^{-1})) \right] (q - 1).
\]

We can plug the last three equations in Equation (3) and we get \( \lambda_B = q \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha)\chi(\delta^{-1} \alpha^{-1}) \), and the multiplicity follows from the number of matrices of determinant \( \delta \).

\[ \square \]

**Corollary 3.13.** The equation
\[ \sum_{s \in \text{SL}_2(\mathbb{F}_q)} \chi(s_{11} + \delta s_{22}) = \sum_{s \in \text{SL}_2(\mathbb{F}_q)} \chi(s_{11})\chi(\delta s_{22}) = q \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha + \frac{\delta}{\alpha}) \]
holds for any \( \delta \in \mathbb{F}_q \) and any non-trivial character \( \chi \) on \( \mathbb{F}_q \). In particular \( \sum_{s \in \text{SL}_2(\mathbb{F}_q)} \chi(s_{11}) = -q \).

If we calculate the spectrum of the special unit-graph on Mat\(_2(\mathbb{F}_q)\) for some small \( q \) values using Theorem 3.12 we see that this graph is strongly regular for \( q = 2, 3 \) and 4 since we have exactly three distinct eigenvalues in each case. However, it is not strongly regular when \( q = 5 \).

Similar to the definition of special unit-digraph, which is the Cayley digraph on Mat\(_n(\mathbb{F}_q)\) associated to the set of determinant 1 matrices, one can pick an \( \alpha \in \mathbb{F}_q^* \) and define a new Cayley digraph \( G_\alpha \) using the set of matrices with determinant \( \alpha \). That is equivalent to saying that \( G_\alpha \) is a Cayley digraph with the vertex set Mat\(_n(\mathbb{F}_q)\) in which there is a directed edge from \( A \) to \( B \) if and only if \( \det(B - A) = \alpha \). It is easy to show that \( G_\alpha \) is isomorphic to \( G_1 \) for any \( \alpha \in \mathbb{F}_q^* \). Therefore, the spectrum of \( G_\alpha \) is the same with the spectrum of \( G_1 \). Moreover, since each nonzero element gives us a new digraph, we get \( q - 1 \) isomorphic digraphs. This gives us an edge-partition of the unit-graph on Mat\(_n(\mathbb{F}_q)\). Furthermore \( \mu_\delta = q \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha + \delta^{-1} \alpha^{-1}) \) in the previous theorem is exactly \( q \) times the (classical) Kloosterman sum over \( \mathbb{F}_q \). It is a well-known fact that this sum is bounded by the square root law i.e.
\[ \left\| \sum_{\alpha \in \mathbb{F}_q^*} \chi(\alpha + \frac{\delta}{\alpha}) \right\| \leq 2\sqrt{q} \]
for any \( \delta \in \mathbb{F}_q^* \), see e.g. [9]. These ideas yield Theorem 1.1.
Proof. (proof of Theorem 1.1) Let $G_\alpha$ denote the Cayley digraph on $\text{Mat}_2(\mathbb{F}_q)$ associated to the set of determinant $\alpha$ matrices. We apply Theorem A.2 on $G_\alpha$. We have

$$n_s = \frac{n}{|S|} \left( \max_{2 \leq i \leq n} \left\| \sum_{s \in S} \chi_i(s) \right\| \right) = \frac{2q^5 \sqrt{q}}{q^3 - q} < \frac{2q^3 \sqrt{q}}{q - 1}.$$ 

Hence if $\frac{2q^3 \sqrt{q}}{q - 1} < \sqrt{|X||Y|}$, then the result follows from Theorem A.2.

The last result shows that if $|X| = \Omega(q^{2.5})$ as $q \to \infty$, then $X$ contains at least two distinct matrices whose difference has determinant $\alpha$, the next non-example shows that $|X| = \Omega(q^2)$ would not work.

Non-example. Let $X = \{ A \in \text{Mat}_2(\mathbb{F}_q) \mid a_{21} = 0 \text{ and } a_{22} = 0 \}$. Then, notice that $|X| = q^2 < \frac{2q^3 \sqrt{q}}{q - 1}$ and for any $A, B \in X$ we have $\det(A - B) = 0$.

Theorem 1.1 surprisingly yields some results related to sum-product problem:

**Corollary 3.14.** If $A, B, C, D$ are subsets of $\mathbb{F}_q$ satisfying $\sqrt[3]{|A||B||C||D|} > \frac{2q^{5/3}}{\sqrt{q-1}}$, then the subset $(A - B)(C - D)$ equals all of $\mathbb{F}_q$.

Proof. Consider the set of matrices $X = \{ M \in \text{Mat}_2(\mathbb{F}_q) \mid m_{11} \in A, m_{22} \in C \text{ and } m_{21} = 0 \}$ and $Y = \{ M \in \text{Mat}_2(\mathbb{F}_q) \mid m_{11} \in B, m_{22} \in D \text{ and } m_{21} = 0 \}$. We have $|X| = q|A||C|$ and $|Y| = q|B||D|$.

If $\sqrt[3]{|A||B||C||D|} > \frac{2q^{5/3}}{\sqrt{q-1}}$, then $\sqrt{|X||Y|} = q \sqrt[3]{|A||B||C||D|} > \frac{2q^{5/3}}{\sqrt{q-1}}$. Hence, it follows from Theorem 1.1 that for any $\alpha \in \mathbb{F}_q^*$ there exists some $M \in X$ and $N \in Y$ such that $M - N$ has determinant $\alpha$. This implies there exists some $m_{11} \in A$, $n_{11} \in B$, $m_{22} \in C$ and $n_{22} \in D$ such that $(m_{11} - n_{11})(m_{22} - n_{22}) = \alpha$.

**Corollary 3.15.** If $A$ is a subset of $\mathbb{F}_q$ with cardinality $|A| > \frac{3}{2}q^{3/4}$, then the subset $(A - A)(A - A)$ equals all of $\mathbb{F}_q$.

Proof. Consider the set of matrices $E = \{ B \in \text{Mat}_2(\mathbb{F}_q) \mid b_{11}, b_{22} \in A \text{ and } b_{21} = 0 \}$.

First notice that if $q < 9$ the result is easy to show. If $q > 9$, then $|A| > \frac{3}{2}q^{3/4}$ implies $|E| = q|A|^2 > \frac{2q^3 \sqrt{q}}{q - 1}$.

If $|E| = q|A|^2 > \frac{2q^3 \sqrt{q}}{q - 1}$, then it follows from Theorem 1.1 that for any $\alpha \in \mathbb{F}_q^*$ there exists two matrices in $E$ with a difference matrix of determinant $\alpha$. This implies any $\alpha \in \mathbb{F}_q$ can be written as $(b_{11} - c_{11})(b_{22} - c_{22}) = \alpha$ for some matrices $B, C \in E$ and the result follows.
Let $H$ be a finite abelian group and $S$ be a subset of $H$. The Cayley digraph $\text{Cay}(H, S)$ is the digraph whose vertex set is $H$ and there is an edge from $u$ to $v$ (denoted by $u \rightarrow v$) if and only if $v - u \in S$. By definition $\text{Cay}(H, S)$ is a simple digraph with $d^+(u) = d^-(u) = |S|$. Furthermore, if we have a Cayley digraph, then we can find its spectrum easily using characters in representation theory, see [8] for a rigorous treatment on character theory. A function $\chi : H \rightarrow \mathbb{C}$ is a character of $H$ if $\chi$ is a group homomorphism of $H$ into the multiplicative group $\mathbb{C}^*$. If $\chi(h) = 1$ for every $h \in H$, we say $\chi$ is the trivial character. The following theorem is a very important well-known fact, see e.g. [1]:

**Theorem A.1.** Let $\mathcal{A}$ be an adjacency matrix of a Cayley digraph $\text{Cay}(H, S)$. Let $\chi$ be a character on $H$. Then the vector $(\chi(h))_{h \in H}$ is an eigenvector of $\mathcal{A}$, with eigenvalue $\sum_{s \in S} \chi(s)$. In particular, the trivial character corresponds to the trivial eigenvector $1$ with eigenvalue $|S|$.

**Proof.** Let $u_1, u_2, \ldots, u_n$ be an ordering of the vertices of the digraph and let $\mathcal{A}$ correspond to this ordering. Pick any $u_i$. Then we have

$$\sum_{j=1}^{n} A_{ij} \chi(u_j) = \sum_{u_i \rightarrow u_j} \chi(u_j) = \sum_{s \in S} \chi(u_i + s) = \sum_{s \in S} \chi(u_i) \chi(s) = \left[ \sum_{s \in S} \chi(s) \right] \chi(u_i).$$

Notice that we get $|H|$ many eigenvectors as demonstrated in the previous theorem, and they are all distinct since they are orthogonal by character orthogonality, see [8]. This means we know all of the eigenvectors of a Cayley digraph explicitly assuming we know all of the characters of $H$.

Theorem A.2 (viz. spectral gap theorem) below is a very important and widely used tool in graph theory by itself, see [2] for the proof.

**Theorem A.2** (Spectral Gap Theorem For Cayley Digraphs). Let $\text{Cay}(H, S)$ be a Cayley digraph of order $n$. Let $\{\chi_i\}_{i=1,2,\ldots,n}$ be the set of all distinct characters on $H$ such that $\chi_1$ is the trivial one. Define

$$n_* = \frac{n}{|S|} \left( \max_{2 \leq i \leq n} \left\| \sum_{s \in S} \chi_i(s) \right\| \right)$$

and let $X, Y$ be subsets of vertices of $\text{Cay}(H, S)$. If $\sqrt{|X||Y|} > n_*$, then there exists a directed edge between a vertex in $X$ and a vertex in $Y$. In particular if $|X| > n_*$, then there exists at least two distinct vertices of $X$ with a directed edge between them.

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