Horizon fractalization in black strings ungravity

Í. D. D. Carvalho\textsuperscript{1,a}, J. Furtado\textsuperscript{2,b}, R. R. Landim\textsuperscript{1,c}, G. Alencar\textsuperscript{1,d}

\textsuperscript{1} Universidade Federal do Ceará, Fortaleza, Ceará, Brazil
\textsuperscript{2} Universidade Federal do Cariri, Juazeiro do Norte, Ceará, Brazil

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Abstract In this paper, we study the scalar (tensor) and vector unparticle corrections for cosmic and black strings. Initially, we consider a static cosmic string ansatz from which we obtain the solution in terms of first- and second-kind Bessel functions. We also obtain the solution for a black string in the unparticle scenario. We identify two regimes, namely, a gravity-dominated regime and an ungravity-dominated regime. In the gravity-dominated regime, the black string solution recovers the usual solution for black strings. The Hawking temperature is also studied in both regimes. As in the static and rotating black hole, we find a fractalization of the event horizon. This points to the fact that fractalization is a natural consequence of unparticles. Finally, we study the thermodynamics of the black string in the ungravity scenario by computing the entropy, heat capacity, and free energy. For both cases, we find that, depending on the region of the parameter $d_U$, phase transitions are possible.

1 Introduction

The unusual properties of nontrivial scale invariance of matter in the infrared (IR) regime was first studied by Banks and Zaks [1]. Georgi suggested [2,3] a coupling between the standard model (SM) fields and this scale-invariant sector of a high-energy physics described by the Banks–Zaks (BZ) fields. The interactions between the standard model fields and the BZ fields are mediated by the exchange of highly massive particles $M_U$, obeying the scale-suppressed non-renormalizable Lagrangian density

$$\mathcal{L} = \frac{1}{M_U^k} \mathcal{O}_{\text{SM}} \mathcal{O}_{\text{BZ}},$$

where $\mathcal{O}$ stands for the field operators of dimension $d_{\text{SM}}$ and $d_{\text{BZ}}$, while $k = d_{\text{SM}} + d_{\text{BZ}} - 4$ in order to guarantee a dimensionless action. It was argued by Georgi [3] that below some scale energy limit ($\Lambda_U$), the interactions between the standard model and the hidden sector become stronger. As the scale invariance is manifested, massive particles are not allowed, and this sector was dubbed “unparticles,” since the notion of mass in this sector is meaningless.

Since the proposal of the unparticle physics, several papers have been published in particle physics [4–6], condensed matter analogue models [7,8], and extra dimensions [9], among others. A very interesting work by Gaete and Spallicci [10] studies the unparticle dynamics in a standard quantum field theory scenario, and the authors construct effective actions for scalar, gravitational, and vector gauge unparticles. Also, several works have been carried out in cosmological and astrophysical scenarios, including studies with black holes [11–14] and white dwarfs [15,16]. One of these works on black holes draws attention to the fact that the exterior event horizon presents a surface with fractal dimension equal to $d_U$ [11]. Recently, some of the present authors generalized the work of [11] to rotating black holes and found that the fractalization is also present [14]. It is also notable that a recent paper on the Casimir effect with unparticles shows a fractalization in the parallel plates dimension [17]. The dimensional reduction seems to be a natural behavior in high energy, and understanding it is fundamental at this regime scale [18–20]. Recently, Frassino and Panella discovered that the quantum behavior of such a nonlocal field theory in $d$ dimensions can be described in terms of local action in $d + 1$ dimensions, which can be quantized using the canonical operator formalism, though giving up local commutativity. In the context of ungravity, they found that what emerges naturally from the standard quantization of the bulk theory via the canonical operator formalism is that the two-point Feynman
propagator for the brane field can be expressed as an integral over the mass of the corresponding Feynman propagator of an ordinary massive scalar field with a specific spectral density. Their method thus has a connection with the scale invariance theory proposed by Georgi (unparticle model), showing that the two approaches are characterized by the same particle content and are, therefore, equivalent [21].

Objects such as cosmic strings may have been formed due to phase transitions in the early universe [22]. The spacetime geometry related to an infinitely long and straight cosmic string is characterized by a planar angle deficit on the two-surface orthogonal to the string. In addition, the spacetime is flat locally, except on the top of the string. Cosmic strings can also be described in the realm of a classical field theory when the energy-momentum tensor related to the vortex configuration of the Higgs–Maxwell system [23] couples to the Einstein equations.

On the other hand, black strings are higher-dimensional solutions of the Einstein equations in D-dimensional spacetime [24]. These vacuum objects can be understood as symmetrized black holes under translations, with the event horizon topologically equivalent to $S_2 \times R$ (or $S_2 \times S_1$ in the case of black rings), and spacetime is asymptotically $M_{D-1} \times S_1$ for a zero cosmological constant or $\text{AdS}_{D-1} \times S_1$ for a negative one [25,26]. A black string is a particular case of a black $p$-brane solution (with $p = 1$), where $p$ is the number of extra spatial dimensions to ordinary space. Hence, a $p$-brane gives rise to a $(p + 1)$-dimensional world-volume in spacetime. In string theory, a black string is described by a D1-brane surrounded by a horizon. It is widely known that since an open string propagates through spacetime, its endpoints must lie on a D2-brane on which it satisfies the Dirichlet boundary condition, and the dynamics on the D-brane world-volume is described by a gauge field theory [27,28]. In some models, our universe is understood as a higher-dimensional world-brane: thus, the gravitational collapse of galactic matter would produce black holes lying on the brane. Black string solutions are nevertheless unstable to long-wavelength perturbations (at least in asymptotically flat spaces) since the localized black hole is entropically preferred to a long segment of string. The string’s horizon therefore has a tendency to form a line of black holes. However, in AdS, the space acts as a confining box which prevents fluctuations of long wavelengths from developing. For instance, in a Randall–Sundrum domain wall in a five-dimensional AdS spacetime, the black string is stable at least far from the AdS horizon. Near the horizon, the string is likely to pinch off to become a stable shorter cigar-like singularity [29].

In this paper, we study the unparticle corrections for cosmic and black strings in $1+3$ dimensions. The paper is organized as follows. In the next section, we present a brief review of the unparticle theory. In Sect. 3, we study the unparticle corrections for cosmic strings, and in Sect. 4, we obtain the black string solution in the unparticle scenario. In Sect. 5, we study the thermodynamic properties (we consider the generalization to different dimensions just to identify the effective dimensions according to the thermodynamic properties) of black strings in the unparticle physics domain, and in Sect. 6, we draw our conclusions.

## 2 Unparticle theory

The ungravity action is [11]

$$S_u = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \kappa^2 \left( \frac{\kappa^2}{(2d_U-1) \sin (\pi d_U)} \right)^{2-d_U} \left( \frac{D^2}{\Lambda^2} \right)^{1-d_U} R,$$

where $\kappa^2 = 8\pi G$, $\kappa^2$ represents the ungravitational Newton constant, $A_{d U} = 16\pi G \Gamma(d_U+1/2) \left[ (2\pi)^{2d_U} \Gamma(2d_U) \Gamma((d_U-1)^{-1} \right]$ [11], and $D^2$ is the generally covariant d’Alembertian operator. The Einstein field equations (EFE) are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T^\mu_{\nu} + \kappa^2 \frac{A_{d U}}{\sin (\pi d_U)} \left( \frac{1}{D^2} \right)^{1-d_U} T^\mu_{\nu},$$

(3)

The second term of the right side of the EFE is the unparticle correction of the energy-momentum tensor. In Ref. [12], an exact solution of the EFE was found for vector particles, and this solution is Reissner–Nordström-like. In this paper, we will consider an axially symmetric ansatz for the metric, $(g_{\mu\nu}) = \text{diag}(g_{RR}(r), g_{r\phi}(r), g_{\phi\phi}(r), g_{zz}(r))$, since we are interested in studying solutions of cosmic strings and black strings with unparticle corrections. Moreover, we adopt the metric’s signature $(-1,1,1,1)$, so we consider the energy-momentum tensor as $(T^\mu_{\nu}) = \text{diag}(-\rho, p, p, p)$, which can be represented by

$$T^\mu_{\nu} = (\rho + p_{\phi}) u^\mu u_\nu + (p - p_{\phi}) w^\mu w_\nu + (p_{\phi} - p) Z^\mu Z_\nu + p_{\phi} \delta^\mu_{\nu},$$

(4)

where $(u_\mu) = (\sqrt{-g_{rr}}, 0, 0, 0)$ is the four-velocity field, and $(w_\mu) = (0, \sqrt{-g_{rt}}, 0, 0)$ and $(Z_\mu) = (0, 0, 0, \sqrt{-g_{zz}})$ are unit spacelike vectors.

Let us consider a static string which has an energy density $\rho(r) = \mu(2\pi r)^{-1} \delta(r) = \mu \delta(x) \delta(y)$, where $\mu$ is the linear density of mass. We can compute the unparticle contribution in Eq. (3) by calculating the term $T^\mu_{\nu}$. We achieve...
this by switching to isotropic (free-falling), Cartesian-like coordinates for computational convenience,

\[
\left(-\frac{1}{D^2}\right)^{1-d_U} T_0^0 = -\mu \left(-\frac{1}{V^2} \right)^{1-d_U} \delta (\vec{X})
\]

\[
= -\mu \left(-\frac{1}{V^2} \right)^{1-d_U} \frac{1}{(2\pi)^2} \int d^2 k e^{i \vec{k} \cdot \vec{X}}
\]

\[
= -\mu \frac{1}{(2\pi)^2} \int d^2 k \left(\frac{1}{k^2}\right)^{1-d_U} e^{i \vec{k} \cdot \vec{X}},
\]

where \( \vec{X} \) is a vector in a direction perpendicular to the string. Hence, by applying the Schwinger representation for \((1/k^2)^{1-d_U}\), we obtain

\[
\left(\frac{1}{k^2}\right)^{1-d_U} \frac{1}{\Gamma(1-d_U)} \int_0^\infty ds s^{d_U-2} e^{-sk^2},
\]

yielding for Eq. (5)

\[
\left(-\frac{1}{D^2}\right)^{1-d_U} T_0^0 = -\mu \frac{1}{(2\pi)^2} \int d^2 k \left[ \frac{1}{\Gamma(1-d_U)} \int_0^\infty ds s^{d_U-2} e^{-sk^2} \right] e^{i \vec{k} \cdot \vec{X}}.
\]

Finally, when we compute this integration, we find

\[
\left(-\frac{1}{D^2}\right)^{1-d_U} T_0^0 = -\mu \frac{2^{d_U-2}}{\pi} \frac{\Gamma(d_U)}{\Gamma(1-d_U)} \left(\frac{1}{X^2}\right)^{d_U} ,
\]

which allows us to write a proper expression for the energy density of a static string with an unparticle as

\[
\rho(r) = \frac{\mu}{2\pi} \frac{\delta(r)}{r} + \mu \frac{\kappa^2}{k^2} \frac{A_{d_U}}{\sin \pi d_U} \times \frac{\Lambda_{d_U}^{2-2d_U}}{(2d_U - 1)} \frac{2^{2d_U-2}}{\pi} \frac{\Gamma(d_U)}{\Gamma(1-d_U)} \left(\frac{1}{r}\right)^{2d_U}.
\]

When we replace \( A_{d_U} = 16\pi^{5/2} \Gamma(d_U + 1/2) [(2\pi)^2 d_U] \) \( \Gamma(2d_U) \Gamma(1-d_U)^{-1} \) in Eq. (9), we find

\[
\rho(r) = \frac{\mu}{2\pi} \left[ \frac{\delta(r)}{r} + \frac{R^{2d_U-2}}{r^{2d_U}} \right],
\]

where

\[
R = \left[ \frac{\kappa^2}{k^2} \frac{8\Lambda_{d_U}^{2-2d_U}}{\sin \pi d_U} (2d_U - 1) \pi^{2d_U - 5/2} \frac{\Gamma(d_U)}{\Gamma(1-d_U) \Gamma(2d_U) \Gamma(d_U + 1/2)} \left(\frac{1}{r}\right)^{2d_U} .
\]

Also, it is important to highlight that the \( R \) parameter sets a proper unparticle length scale so that, similarly to the authors in [11,14], we are able to distinguish between two regimes. The gravity-dominated (GD) regime is defined by \( R \ll r \) and the ungravity-dominated (UGD) regime by \( R \gg r \). Note that in the GD regime, the second term of (10) vanishes, and we recover the usual cosmic string energy density.

In the next section, we will present the solution of a static cosmic string with unparticle corrections.

### 3 Cosmic string with unparticle corrections

In this section, we will study the spacetime solution which is static and has cylindrical symmetry. Consider the ansatz of the line element as follows

\[
ds^2 = -dt^2 + dr^2 + b(r)^2 d\phi^2 + dz^2.
\]

The components of the Ricci tensor are

\[
R_{\mu\nu} = R_{\mu\nu}^{0} = \delta_{[\mu} \Gamma_{\nu]\lambda}^{\alpha} + \Gamma_{[\alpha}^{\sigma} \Gamma_{\nu]\mu]^{\beta},
\]

where \( V_{[\mu\nu]} = V_{\mu\nu} - V_{\nu\mu} \). The Einstein Tensor is

\[
G_{\mu\nu} = \frac{1}{b(r)} \frac{d^2 b(r)}{dr^2} \left[ \frac{\delta(r)}{r} + R^{2d_U-2} - 2d_U \right] \left(\frac{1}{r}\right) = 0 .
\]

This equation can be rewritten as

\[
\frac{d^2 b(r)}{dr^2} + \frac{\mu k^2}{2\pi} \left[ \frac{\delta(r)}{r} + R^{2d_U-2} - 2d_U \right] b(r) = 0 .
\]

Now, we need to properly solve Eq. (15). Let us perform the following variable transformation \( b(r) = \sqrt{r} f(r) \), such that Eq. (15) becomes

\[
\left\{ \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left[ \frac{\mu k^2}{2\pi} \frac{\delta(r)}{r} + \frac{\mu k^2}{2\pi} R^{2d_U-2} - \frac{1}{4r^2} \right] f \right\} \sqrt{r} = 0 .
\]

First, we need to solve this equation for \( r > 0 \). After this, we impose the condition that when \( R/r \to 0 \), we must find the cosmic string spacetime. When we consider \( r > 0 \), Eq. (16) simplifies to

\[
\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left[ \frac{\mu k^2}{2\pi} R^{2d_U-2} - \frac{1}{4r^2} \right] f = 0 .
\]

Now, we can perform a transformation of variable \( x = \sqrt{\frac{\mu k^2}{2\pi} R^{-2d_U-1}} r^{1-d_U} \), allowing Eq. (17) to be written as

\[
\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} + \left[ 1 - \frac{\left( \frac{1}{2d_U-2} \right)}{x^2} \right] f = 0 .
\]
This is the Bessel differential equation. The solution of \ref{eq:18} depends on \((\frac{1}{2dU-2})\). When we define \(\xi \equiv 1/(2dU-2)\) as a positive real number, \ref{eq:18} becomes

\[
d\frac{2}{dx^2}f + \frac{1}{2x} \frac{df}{dx} + \left[1 - \frac{\xi^2}{x^2}\right]f = 0, \tag{19}
\]

whose solution is

\[
f(x) = C_1 J_\xi(x) + C_2 N_\xi(x). \tag{20}
\]

Here, \(J_\xi(x)\) and \(N_\xi(x)\) are the first- and second-kind Bessel functions of order \(\xi\), respectively. Let us present some considerations regarding the boundary conditions. Initially, we want the condition that when \(R/r \to 0\), \(b(r) = (1 - 8G\mu)^{1/2}r\), that is, the spacetime becomes the cosmic string spacetime. Since \(x\) is proportional to \((R/r)^{2dU-1}\) and \(b(r) = \sqrt{r} f(x(r))\), we can study \ref{eq:20} when \(x \ll 1\). In regime \(x \ll 1\) and \(\xi > 0\), \ref{eq:20} can be written as

\[
f(x) = C_1 \frac{1}{\Gamma(\xi + 1)} \left(\frac{x}{2}\right)^{-\xi} C_2 \frac{\Gamma(\xi)}{\pi} \left(\frac{x}{2}\right)^{-\xi}. \tag{21}
\]

Replacing \(x = \sqrt{\frac{\mu x^2}{2\pi (dU - 1)^d} f^{dU-1}}\) in \ref{eq:21}, when we impose the limit \(R/r \ll 1\), we will find \(f(r) = (1 - 8G\mu)^{1/2}R\). Hence, \(C_1 = 0\) and

\[
C_2 = -(1 - 8G\mu)^{1/2} \frac{\pi}{\Gamma(\xi)} \left(\frac{\sqrt{2\pi}}{2\pi}\right)^\xi R^{1/2}.
\]

Consequently, the component \(g_{\phi\phi}\) of the metric is

\[
b^2(r) = (1 - 8G\mu)^2 \left[\frac{\pi^2}{\Gamma^2(\xi)} \sqrt{2\pi} (4G\mu)^\xi \frac{R}{r}\right] N_\xi^2 \times \left(4\xi \sqrt{G\mu} \left(\frac{R}{r}\right)^{\frac{1}{\pi}}\right). \tag{22}
\]

To understand the structure of this spacetime, that is, how the unparticles modify the cosmic string spacetime, we must analyze the curvature scalars. The Ricci scalar for the ansatz \(ds^2 = -dr^2 + d\rho^2 + b^2(r) d\phi^2 + dz^2\) is \(R = -\frac{2dU}{b^2}\). Then the Ricci scalar for our solution is given by

\[
R = \frac{\mu k^2}{\pi} \left[\frac{\delta(r)}{r} + \frac{1}{r^2} \left(\frac{R}{r}\right)^{2dU-2}\right], \tag{23}
\]

where we used \ref{eq:16}.

4 Black string with unparticle corrections

Let us consider the following line element for the black string

\[
ds^2 = -f(r) dr^2 + \frac{1}{f(r)} dr^2 + r^2 d\phi^2 + \alpha^2 r^2 dz^2, \tag{24}
\]

where \(t \in (-\infty, \infty), \) the radial coordinate \(r \in [0, \infty), \) the angular coordinate \(\phi \in [0, 2\pi), \) and the axial coordinate \(z \in (-\infty, \infty). \) The \(\alpha\) parameter is considered as \(\alpha^2 = -\Lambda/3.\)

For the black string in the ungravity scenario, the Einstein–Hilbert effective action is composed of \ref{eq:2} plus the cosmological constant contribution, i.e.,

\[
S_u = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{1 + \frac{A_{\mu}}{(2dU - 1) \sin(\pi dU)} \kappa^2_g \left(\frac{R^2}{\Lambda^2}\right)^{1-dU} \left(1 - \frac{R^2}{\Lambda^2}\right)^{-1} \right\} R - 2\Lambda. \tag{25}
\]

The Einstein equation \ref{eq:3} remains valid; however, the left-hand side will be modified due to the different ansatz for the metric in \ref{eq:24}. The EFE for this ansatz are

\[
G_i^r - 3\alpha^2 = \frac{1}{r^2} \frac{df(r)}{dr} + \frac{f(r)}{r^2} - 3\alpha^2, \tag{26}
\]

\[
G_r^\phi - 3\alpha^2 = \frac{1}{r^2} \frac{df(r)}{dr} + \frac{f(r)}{r^2} - 3\alpha^2, \tag{27}
\]

\[
G_\phi^\phi - 3\alpha^2 = \frac{1}{2} \frac{d^2f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr} - 3\alpha^2, \tag{28}
\]

\[
G_z^z - 3\alpha^2 = \frac{1}{2} \frac{d^2f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr} - 3\alpha^2. \tag{29}
\]

We can see that the energy-momentum tensor for the ansatz of \ref{eq:24} is \(T_{\mu}^\nu = -\rho(r) \text{diag}(1, 1, 0, 0) + p_i(r) \text{diag}(0, 0, 1, 1)\), where \(p_i = p_\phi = p_z\). This way we can find \(f(r)\) by solving \(G_i^r - 3\alpha^2 = -k^2 \rho(r)\), where we find

\[
f(r) = \alpha^2 r^2 - 4\frac{\mu}{ar} \pm \frac{\kappa^2 \mu}{2\pi (2dU - 3)} \left(\frac{R}{r}\right)^{2dU-2}, \tag{30}
\]

with \(R\) given by \ref{eq:11}. Note that the unparticle black string solution is divergent for \(dU = 3/2\). The scalar unparticle case is associated with the positive sign in \ref{eq:30}, while the vector case is negative, see (Fig. 1).

The above solution for the black string in the ungravity scenario recovers the usual black string solution \cite{30} in the GD regime, i.e.,

\[
f(r) = \left(\alpha^2 r^2 - 4\frac{\mu}{ar}\right). \tag{31}
\]

Also, it is important to highlight here that the unparticle black string may exhibit naked singularity, therefore violating the cosmic censorship, for the scalar case when \(3/2 < dU < 2\).
5 Thermodynamic properties of the black string with unparticle corrections

To simplify the calculations, we can rewrite Eq. (30) as
\[ f(r) = h(r) + \mu s(r), \]
where
\[ h(r) = \alpha^2 r^2, \]
\[ s(r) = \frac{4}{ar} \left[ -1 \pm \frac{\alpha r k^2}{8\pi (2dU - 3)} \left( \frac{R}{r} \right)^{2dU-2} \right]. \]

Our black string solution in the ungravity scenario has
horizon curves defined by \( f(r_h) = 0 \), so \( \mu = -\frac{h(r_h)}{s(r_h)} \), which gives us
\[ \mu = -\frac{\alpha^3 r_h^3}{4} \left[ -1 \pm \frac{\alpha r_k^2}{8\pi (2dU - 3)} \left( \frac{R}{r_h} \right)^{2dU-2} \right]^{-1}. \]

In the GD regime, the mass parameter \( \mu \) reduces to \( \mu = \frac{\alpha^3 r_h^3}{4} \), which is the usual mass parameter for the black string in general relativity. In [14] the authors found that for an
unparticle black hole, the mass parameter becomes negative for the vector unparticle case. For the unparticle black string, as we can see in Fig. 2, both vector and scalar cases present the possibility of a negative mass parameter. For the scalar unparticle case, we have a negative mass parameter for values of \( 3/2 < dU < 2 \). For the vector unparticle case, the mass parameter becomes negative for \( 1 < dU < 3/2 \).

The authors in [12] do not take into consideration the regions where the mass parameter is negative. However, it is known that a negative mass parameter in a black hole solution suggests a nontrivial topology [31]. Therefore, we can expect that these negative mass regions are also related to nontrivial topologies of the black string.

As we have the solution for the static black string in the ungravity scenario given by (30), we are able to study the thermodynamics of the black string by computing the Hawking temperature by means of \( T_H = \frac{1}{4\pi f'(r_h)} \), where \( r_h \) is the radius of the horizon. Note that in terms of \( h(r) \) and \( s(r) \), we find 
where we used $\mu = -\frac{h(r_h)}{s(r_h)}$. When we substitute the expressions for $h(r_h)$ and $s(r_h)$, we find

$$T_H = \frac{3\alpha^2 r_H}{4\pi} \begin{bmatrix} -1 & \frac{\alpha r_H^2}{12\pi(2d_U - 3)} \left( \frac{r_H}{r_H} \right)^{2d_U - 2} \\ 1 & \frac{\alpha r_H^2}{8\pi(2d_U - 3)} \left( \frac{r_H}{r_H} \right)^{2d_U - 2} \end{bmatrix},$$

(35)

The usual black string Hawking temperature, i.e., $T_H = \frac{3\alpha^2 r_H}{4\pi}$, is recovered in the GD regime and in the UGD regime, we find

$$T_H = \frac{\alpha^2 2d_U}{4\pi} r_H,$$

(37)

for scalar and vectorial case. The Hawking temperature for the unparticle black string presents similar behavior for both scalar and vector unparticle cases. The behavior of the Hawking temperature as a function of $r$ for the scalar unparticle case for some values of $d_U$ is presented in Fig. 3. We can see from Fig. 3 that when $1 < d_U < 3/2$, the temperature increases slowly with the radius in comparison with the usual black string temperature. When $3/2 < d_U < 2$, the unparticle correction promotes an increase in the temperature in comparison with the black string usual case.

Note that the temperature is approximately proportional to $D - 1$, where $D$ is the dimension of the spacetime. So, when we confront Eq. (37) with this fact, we can see the fractalization of the event horizon with

$$d_H = 2d_U - 1,$$

(38)

where $d_H = D - 2$. To reinforce the fractalization of the event horizon, we can compute the density of entropy by length $S$ although $dS = d\mu / T_H$, which in terms of $h(r)$ and $s(r)$,

$$dS = -\frac{4\pi}{s(r_h)} \frac{d\mu}{dH},$$

(39)

where we used Eq. (35) and again $\mu = -\frac{h(r_h)}{s(r_h)}$. So we find

$$dS = \frac{\pi \alpha r_h}{1 + \frac{\alpha R^2}{8\pi(3 - 2d_U)} \left( \frac{r_H}{r_h} \right)^{2d_U - 3}}.$$

(40)

Now, we want to analytically find the entropy by length of the black string with the unparticle. It can be done by integration of Eq. (40), but it is necessary to analyze the regions due to the possible singularity of this equation. Up to now we have discussed the scalar and vector cases simultaneously, but now, for the sake of simplicity we will consider them separately.

5.1 Scalar case

Let us begin by considering the case when $d_U < 3/2$. In this case we can rewrite Eq. (40) as

$$dS = \frac{\pi \alpha r_h}{1 + \frac{\alpha R^2}{8\pi(3 - 2d_U)} \left( \frac{r_H}{r_h} \right)^{2d_U - 3}},$$

(41)

So, for the case $d_U < 3/2$ there is no singularity. By using the identity

$$\int \frac{x \, dx}{1 + A x^\gamma} = \frac{x^2}{2\gamma} {}_2F_1 \left( 1, \frac{2}{\gamma}; 1 + \frac{2}{\gamma}; \mp A x^\gamma \right) + \text{const.},$$

(42)

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function and $\gamma > 0$, the integration of Eq. (41) can be written as

$$S = \frac{2\pi \alpha r_H^2}{4} {}_2F_1 \left( 1, \frac{2}{3 - d_U}; 1 + \frac{2}{3 - d_U}; \frac{R}{r_H} \right)^{2d_U - 3}.$$

(43)

Note that in the GD regime ($R \to 0$), the argument of hypergeometric function is proportional to $R^{2d_U - 2}$, which approaches 0 when $R \to 0$. That is, the entropy in GD recovers $S = \frac{2\pi \alpha r_H^2}{4}$, as shall be seen.

The heat capacity and the free energy of the system are defined by
The heat capacity can be obtained directly from Eqs. (34) and (36), which yields

\[ C_V = \frac{d\mu}{dT_H} \] (44)

\[ F = \mu - T_H S. \] (45)

The behavior of the Hawking temperature, entropy, heat capacity, and free energy for \( d_U = 1.4 \) (a) and for \( d_U = 1.6 \) (b) is depicted in Fig. 4.

The heat capacity can be obtained directly from Eqs. (34) and (36), which yields

\[ C_V = \frac{8\pi^2(-3 + 2d_U)\alpha(r_h R)^2}{96(3 - 2d_U)^2\pi^2 R^4 + 8\pi(-9 + 27d_U - 20d_U^2 + 4d_U^3)\kappa^2 r_h^3 R^2 \alpha (\frac{R}{r_h})^{2d_U-3} + d_U \alpha^2 \kappa^4 r_h^6 (\frac{R}{r_h})^{4d_U}}. \] (46)

The free energy can also be obtained from (34), (36), and (43), yielding

\[ F = -\frac{\alpha^3 r_h^3}{4\alpha \kappa^2 r_h^3 - 2d_U R^{2d_U} + 32\pi(3 - 2d_U) R^2} \left[ 8\pi(2d_U - 3)^2 R^2 + \alpha d_U \kappa^2 r_h^3 \left( \frac{R}{r_h} \right)^{2d_U} + 12\pi(3 - 2d_U) R^2 \right] \]

\[ \times \text{hypergeometric}_{2}^{1} \left( \frac{2}{3 - 2d_U}; 1 + \frac{2}{3 - 2d_U}; \frac{\alpha \kappa^2 r_h^{3 - 2d_U} R^{2d_U - 2}}{24\pi - 16\pi d_U} \right). \] (47)

The behavior of the Hawking temperature, entropy, heat capacity, and free energy for \( d_U < 3/2 \) is depicted in Fig. 4.

Now, we need to investigate the case when \( d_U > 3/2 \). In this case, there is a singularity in Eq. (40) when

\[ r_0 = \left[ \frac{\alpha R \kappa^2}{8\pi(2d_U - 3)} \right]^{\frac{1}{2d_U - 3}} R. \] (48)

It is easy to see that we have two disjoint regions, and when \( r_h > r_0 \), the differential of entropy by length, Eq. (40), approaches infinity (minus infinity). First, let us compute the entropy for \( r_h > r_0 \). We can rewrite Eq. (40) as

\[ dS = \left[ 1 - \left( \frac{r_0}{r_h} \right)^{2d_U - 3} \right]. \] (49)

We can change the variable \( z = \left( r_h/r_0 \right)^{2d_U - 3} \), and Eq. (49) becomes

\[ dS = -\frac{\alpha \pi r_0^2}{(2d_U - 3)} z^{-\frac{(2d_U - 1)}{2d_U - 3}} \left[ 1 - z \right]. \] (50)

We can use the identity

\[ \int x^{\pm c} \frac{dx}{1 - x} = \frac{x^{1+\pm c}}{1 \pm c} \text{hypergeometric}_{2}^{1} \left( \frac{2}{3 - 2d_U}; 1 + \frac{2}{3 - 2d_U}; x \right) + \text{const.}, \] (51)

where \( c > 0 \). Therefore, when we integrate Eq. (49), we find

\[ S = \frac{2\pi \alpha r_0^2}{4} \text{hypergeometric}_{2}^{1} \left( \frac{2}{3 - 2d_U}; 1 + \frac{2}{3 - 2d_U}; \left( \frac{r_0}{r_h} \right)^{2d_U - 3} \right) \] (52)

Note that Eq. (52) recovers \( S = \frac{2\pi \alpha r_0^2}{4} \) in the GD regime.

The ultimate case is when \( r_h < r_0 \). We can rewrite Eq. (40) as

\[ dS = -\alpha \pi \left( \frac{r_h}{r_0} \right)^{2d_U - 3} \left[ 1 - \left( \frac{r_h}{r_0} \right)^{2d_U - 3} \right]. \] (53)
When we change the variable \( z = (r_h/r_0)^{2d_U-3} \), Eq. (53) becomes

\[
\text{d}S = -\frac{\alpha \pi r_0^2}{(2d_U - 3)} \frac{z^{2d_U-3}}{1 - z} \text{d}z.
\]  

(54)

So, when we integrate Eq. (54) using Eq. (51), we find

\[
S = -\frac{\pi \alpha r_0^2}{(2d_U - 1)} \left( \frac{r_h}{r_0} \right)^{2d_U-1}
\times \mathcal{F}_1 \left( 1, \frac{2d_U - 1}{2d_U - 3}; \frac{4(2d_U - 1)}{2d_U - 3}; \left( \frac{r_h}{r_0} \right)^{2d_U-3} \right).
\]  

(55)

Note that in the UGD regime, the dependence of \( r_h \) in Eq. (55) becomes \( r_h^{d_U} = r_h^{2d_U-1} \). Then \( d_h = 2d_U - 1 \), which reinforces the fractalization and is in agreement with Eq. (37).

The heat capacity can be obtained straightforwardly from Eqs. (34) and (36), yielding

\[
C_v = \frac{8\pi^2 \alpha (2d_U - 3) R^2 r_h^{2d_U+2} \left( 12\pi (2d_U - 3) R^2 r_h^{2d_u} - \alpha \kappa^2 r_h \right) R^{2d_U}}{\alpha^2 \kappa^2 R^{2d_U} + \alpha (2d_U - 3)(2d_U - 3)(2d_U - 1) \kappa^2 r_h^2 R^2 (r_h R)^{2d_U} + 96\pi^2 (3 - 2d_U)^2 R^4 r_h^{4d_U}}.
\]  

(56)

The free energy can also be obtained from (34), (36), and (43), from which it follows that

\[
F = \frac{\alpha^3 r_h \left( \frac{r_h}{r_0} \right)^{2d_U-3} \left( 1 + \frac{2}{2d_U-3} \right) \left( \alpha \kappa^2 r_h \right)^{2d_U} - 12\pi (2d_U - 3) R^2 r_h^{2d_U} + 4\pi (3 - 2d_U)^2 R^2 r_h^{2d_U+2} \right)}{2(2d_U - 3) \left( 8\pi (2d_U - 3) R^2 r_h^{2d_U} - \alpha \kappa^2 r_h \right) R^{2d_U}} .
\]  

(57)

where \( \mathcal{F}_1(a, b, z) \) is the incomplete Euler beta function.

As is widely known, the thermodynamic stability of black holes (black strings for our case) is directly related to the sign of the heat capacity. A positive heat capacity indicates that the system is thermodynamically stable, while its negativity implies thermodynamic instability. As we can see in Fig. 4, the value of \( d_U \) controls the black string stability, where the black string is stable in the ungravity regime for \( d_U < 3/2 \) and unstable for \( d_U > 3/2 \).

### 5.2 Vector case

Let us consider the case when \( d_U > 3/2 \), which for the vector case avoids the singularity, so that Eq. (40) can be written as

\[
\text{d}S = \frac{\pi \alpha r_h \text{d}r_h}{1 + \frac{\alpha \kappa^2 R}{2\pi(2d_U-3)} \left( \frac{R}{r_h} \right)^{2d_U-3}} .
\]  

(58)

We can clearly see that, by virtue of the similarity between the expressions (58) and (41), the same analysis performed for the scalar case when \( d_U < 3/2 \) can be done for the vector case when \( d_U > 3/2 \). Also, the results for the scalar case when \( d_U > 3/2 \) are identical to those of the vector case when \( d_U < 3/2 \).

Therefore, the thermodynamic quantities depicted in Fig. 4a for the scalar case when \( d_U < 3/2 \) also describe the thermodynamic quantities for the vector case when \( d_U > 3/2 \). Likewise, the thermodynamic quantities depicted in Fig. 4b for the scalar case when \( d_U > 3/2 \) also describe the thermodynamic quantities for the vector case when \( d_U < 3/2 \).

### 6 Conclusion

In this paper we study the unparticle corrections for cosmic and black strings. Initially we discussed some general features about the unparticle physics, from which we obtained a general expression for the energy density for a string. We were able to identify a proper length scale \( R \) associated with the unparticle scenario. This length scale defines two regimes, namely, a gravity dominated-regime and an ungravity-dominated regime. We find that the unparticle corrects the a string source by

\[
\rho(r) = \frac{\mu}{2\pi} \left[ \frac{\delta(r)}{r} + \frac{R^{2d_U-2}}{r^{2d_U}} \right] .
\]  

(59)

With the above source, we first considered a static cosmic string ansatz from which we obtained the solution in terms of first- and second-kind Bessel functions. The structure of the cosmic string spacetime in the unparticle scenario was studied by means of the Ricci scalar.

However, in order to study fractalization, we must consider the black hole analogous to cylindrical symmetry. In order to do this, we first obtained the solution for the black string in the unparticle scenario directly from the Einstein field equations. The solutions were studied for both scalar and vector unparticle cases, and in both cases the solution is divergent when \( d_U = 3/2 \). In both cases the unparticle black string solution recovers the usual black string solution in the gravity-dominated regime, as expected. Also, it is important to highlight that in the black string, all the cases considered may exhibit a naked singularity, depending on the range of...
$d_U$. For the unparticle black hole, this was a feature present only for the vector case [14].

The thermodynamic properties of the black string with the unparticle corrections were also addressed. We verified that in either the scalar unparticle case or the vector case, there are regions with negative mass parameters. For the scalar case, the mass becomes negative when $3/2 < d_U < 2$, while for the vector case the mass becomes negative when $1 < d_U < 3/2$. As pointed out in [31], black holes (black strings, for our case) with negative mass can exist and present nontrivial topology. By investigating the Hawking temperature, we could see that both scalar and vector cases present very similar behavior. When $1 < d_U < 3/2$, the temperature increases slowly with the radius in comparison with the usual black string temperature. When $3/2 < d_U < 2$, the unparticle correction promotes an increase in the temperature in comparison with the black string usual case.

With the above solution, we analyze whether, similarly to the unparticle black hole cases, a fractalization can be present. We found that for the black string in the UGD regime, we have an effective dimension given by $d_H = 2d_U - 1$. We should point out that for the static and rotating black hole, the fractal dimension is given by $d_H = 2d_U$. The fractalization can also be obtained from the entropy, which was analytically obtained for the scalar and vector cases. For both cases it is given by the hypergeometric confluent functions. In the UGD regime we again find a fractal dimension given by $d_H = 2d_U - 1$. Entropy and other thermodynamic quantities also give important information about the stability of the system. We perform our analysis separately for $d_U > 3/2$ and $d_U < 3/2$ in order to avoid singularities. We compute the specific heat and free energy for the scalar case, and plots are depicted in Fig. 4 for a particular choice of parameters. We could see that the value of the $d_U$ controls the black string stability in the ungravity scenario, where the black string is stable for $d_U < 3/2$ and unstable for $d_U > 3/2$.

Finally, we point out that the source (59) will give modified light deflection and could be found in the present era of high-precision astrophysical measurements. It would also be interesting to study whether the rotating case gives the same fractal dimension as in the black hole case. In fact, the study of the fractal dimension in unparticles can be extended to many other objects. A more ambitious project would be to show that horizon fractalization is always present in unparticle gravity.

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