DRIFT AND MATRIX COEFFICIENTS FOR DISCRETE GROUP EXTENSIONS OF COUNTABLE MARKOV SHIFTS

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Abstract. There has been much interest in generalizing Kesten’s criterion for amenability in terms of a random walk to other contexts, such as determining amenability of a deck covering group by the bottom of the spectrum of the Laplacian or entropy of the geodesic flow. One outcome of this work is to generalise the results to so-called discrete group extensions of countable Markov shifts that satisfy a strong positive recurrence hypothesis. The other outcome is to further develop the language of unitary representation theory in this problem, and to bring some of the machinery developed by Coulon–Dougall–Schapira–Tapie [Twisted Patterson-Sullivan measures and applications to amenability and coverings, arXiv:1809.10881, 2018] to the countable Markov shift setting. In particular we recast the problem of determining a drop in Gurevič pressure in terms of eventual almost sure decay for matrix coefficients, and explain that a so-called twisted measure “finds points with the worst decay.” We are also able locate the results of Dougall–Sharp [Anosov flows, growth rates on covers and group extensions of subshifts, Inventiones Mathematicae, 223, 445–483, 2021] within this framework.

1. Introduction

It has been known for some time that one can connect certain structural properties of a group to properties exhibited by a random walk on the group. (A path in a random walk is the sequence $s_1 \cdots s_n \in G$, for $n \in \mathbb{N}$, with each $s_i \in G$, $i \in \mathbb{N}$, independently sampled according to a probability $p$ on $G$. Here, we only consider $G$ to be countable and finitely generated.) Kesten’s criterion for the radius of convergence of a (symmetric) random walk [14] states that if the probability $p$ is symmetric and has support generating $G$ then the radius of convergence of the (symmetric) random walk is equal to one if and only if $G$ is amenable. Motivated by hyperbolic dynamics (where independence is not given) it is natural to relax the independence of the products $s_1 \cdots s_n$ and instead suppose they are given by a $G$-extension (also called a group extension, see subsection 1.1) of a strongly positively recurrent countable Markov shift (CMS). (A random walk can be understood as a $G$-extension of a subshift of finite type.) It is also natural to remove the symmetry requirement entirely, as we later explain for the example of Anosov flows. The radius of convergence of the random walk is naturally realised as the exponential of a Gurevič pressure of the $G$-extension. The notion of pressure is important to geometric examples — it is directly (in cases admitting a symbolic coding) or indirectly (by considering a transliteration in Patterson–Sullivan theory) related to critical exponents of groups acting on sufficiently negatively curved spaces. There has been much interest in extending the Kesten criterion to dynamical or geometric settings. For example, the early work of Brooks [3] connects amenability of a covering deck group with the bottom of the spectrum of the Laplacian. In more recent times the advances began in earnest with Stadlbauer’s criterion for the Gurevič pressure of a (weakly symmetric) group extension [18] which characterises amenability of $G$ in terms of the Gurevič pressure of a group extension by $G$ under the following restrictions: the dynamics for the group extension must be transitive, the CMS must satisfy a
finiteness condition called “the big images and preimages property”, and the Hölder potential and group extension must satisfy a weak symmetry hypothesis.

Amenability of a countable group can be understood in a combinatorial manner from the equivalent notion due to Følner [12]. Yet it can be more enlightening to phrase in terms of unitary representations. Indeed, to determine whether there can be a gap in Gurevič pressure over a family of extensions, one is led to consider a spectral gap condition for the left regular representation in $\ell^2(G/H)$, for $H \leq G$. These were observations of Dougall [7] and Coulou–Dal’bo–Sambusetti [4] who independently gave a gap criterion for extensions of subshifts of finite type with similar “visibility hypotheses”. (The visibility hypothesis replaces transitivity, which cannot be taken to be uniform in such problems.) We again emphasise that the weak symmetry hypothesis is required for one direction of the criterion.

Let us now comment on “symmetry”. A flow is by definition invertible, and in this way there is a mapping of an increasing-time orbit to a decreasing-time orbit. The geodesic flow has extra symmetry given in the existence of a natural smooth conjugacy of the positive time flow (defined by vector field $X$) with the negative time flow (defined by vector field $-X$), preserving both an invariant volume and the measure of maximal entropy. (This can be encoded to a weak symmetry hypothesis for a symbolic coding.) Dougall and Sharp [9] study the entropy of an Anosov flow in a covering manifold — in the context of Anosov flows one can already have a drop in entropy for the homology cover [16]. A particular case of their technique characterises the radius of convergence of a finitely supported random walk on an amenable group $G$ as being equal to the radius of convergence for the abelianisation of $G$, with the restriction that the semigroup $G_p^+$ generated by the support of the probability $p$ is in fact the whole of $G$. (In particular there is no symmetry requirement.) This was proved in the framework of transitive group extensions of subshifts of finite type (and in that context $G_p^+ = G$ implies transitivity of the group extension). The characterisation of the radius of convergence for an amenable group had not been seen before the work of [9]. We expand on the technique of [9] to give Theorem 3.16, and compare this with a ratio limit theorem (see subsection 2.2).

Finally, we elaborate on the work of Coulon, Dougall, Schapira and Tapie [5], which sought to characterise critical exponents $h_{\Gamma'}$ for $\Gamma' \leq \Gamma$ in terms of the Patterson–Sullivan machinery for the action of $\Gamma$ on a proper Gromov hyperbolic space $X$. They introduced a so-called twisted Patterson–Sullivan measure and characterised equality $h_{\Gamma'} = h_{\Gamma}$ in terms of the twisted measure coinciding with the usual Patterson–Sullivan measure for $\Gamma$. This geometric setting has local compactness that a CMS does not, and so one of our aims is to extend the result to this setting.

We reinterpret arguments and results of [5] in terms of the existence of a thermodynamic $G$-density $\phi[t] : G \to \mathbb{R}$, and to show that the parameters at which this thermodynamic $G$ density $\phi[t]$ exists controls the eventual decay of $\langle \rho(s_1 \cdots s_n)f, v \rangle$ along typical paths $s_1 \cdots s_n$ in a group extension $T_s$ of a CMS. And that diagonal coefficients associated to $\phi[t]$ are naturally related to integrals of functions with respect a “(twisted) measure” $\nu_\phi^c$ on $\Sigma^+$. We hope that this exposition will increase the accessibility of these techniques and grow interest in this novel relation between hyperbolic dynamics and unitary representations.

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1.1. Countable Markov shifts and discrete group extensions. More detail for the definitions is given in Section 12. Let $\sigma : \Sigma^+ \to \Sigma^+$ be a (one-sided) countable Markov shift with alphabet $W_1$. (In particular $\Sigma^+ \subset W_1^\mathbb{N}$ but need not be a full-shift.) As usual, we call a word $w \in (W_1)^k$ admissible if it appears as a subword of some $x \in \Sigma^+$. The two-sided shift is denoted $\sigma : \Sigma \to \Sigma$. Cylinder sets given by $[w_1 \cdots w_k] = \{ x = (x_0, x_1, \ldots) \in \Sigma^+ : x_{i-1} = w_i, i = 1, \ldots, k \}$ form a basis for the topology on $\Sigma^+$. We always assume that $\sigma$ is mixing. Let $R : \Sigma^+ \to \mathbb{R}_+$ with $\log R$ locally Hölder continuous (we use multiplicative notation, writing $R_n(x) = R(x) \cdots R(\sigma^{n-1}x)$). We will also view $R$ as a function on $\Sigma$. The Gurevič pressure $P(\log R, \sigma)$ is

\begin{equation}
P(\log R, \sigma) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in [B] : \sigma^n x = x} R_n(x).
\end{equation}

The function $\hat{R} = \exp(-P(\log R, \sigma))R$ has $P(\log \hat{R}, \sigma) = 0$, and $\hat{R}$ satisfies the same recurrence properties as $R$. Therefore we always assume that $P(\log R, \sigma) = 0$.

The countable Markov shift and $R : \Sigma^+ \to \mathbb{R}$ is said to be positive recurrent (following Sarig [20]) if there is a constant $M_B$ with

$$
\sum_{x \in \Sigma^+ : \sigma^nx = x, x_0 = B} R_n(x) \in [M_B^{-1}, M_B]
$$

for all $n \in \mathbb{N}$. We also ask that $R$ is strongly positively recurrent (see subsection 12.2 which discusses this condition as it was given by Sarig in [21]), which may be equivalently stated as having that for any $B \in W_1$

$$
\exp P(\log R, \sigma) = 1 > \gamma\text{(SPR)} := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in [B] : \sigma^n x = x, x \neq B \text{ if } i \notin \mathbb{N}} R_n(x).
$$

(For a subshift of finite type, any $R$ with $\log R$ being Hölder continuous is strongly positively recurrent — note that we assumed that $\sigma$ is mixing.)

The shift-invariant probability equilibrium state is denoted $\mu = \mu_R$. (We also identify it as a shift-invariant measure for the two-sided shift space $\Sigma$.) We should think of the data of $\sigma, \Sigma^+ \subset W_1, R, \mu$ as being fixed and to then vary the skew products that follow.

Let $G$ a discrete countable group. A map $s : W_1 \to G$ defines a $s$-skew product with phase space $\Sigma \times G$ and dynamical system

$$
T_s(x, g) = (\sigma x, s(x_0)^{-1}g).
$$

The skew product is isomorphic to a countable Markov shift. We also call $T_s$ a discrete group extension, or $G$-extension. We lift $R$ to $\Sigma^+ \times G$ by defining $R(x, g) = R(x)$ (and use the same notation for the function on domain $\Sigma^+$ and on domain $\Sigma^+ \times G$). For $w \in W_1$, $w = w_1 \cdots w_n$, define $\bar{g}(w) = s(w_1) \cdots s(w_n)$. Then

$$
T_s^n(x, g) = (\sigma x, \bar{g}(x_0 \cdots x_{n-1})^{-1}g).
$$

The Gurevič pressure $P(\log R, T_s)$ is

$$
\limsup_{n \to \infty} \frac{1}{n} \log \sum_{(x, e) \in [B] \times \{ e \} : T_s^n(x, e) = (x, e)} R_n(x),
$$

where $e$ is the identity element of $G$. When $T_s$ is transitive the definition is independent of $B$. We always have $P(\log R, T_s) \leq P(\log R, \sigma) = 1$.

The basic example is given by a random walk $p : S \to [0,1]$ on a group $G$ in which we assume $S = S^{-1}$ generates $G$. We let $\Sigma^+ = S^\mathbb{N}$, $R((s_0, s_1, \ldots)) = p(s_0)$ and $s(w) = w$, once we identify the alphabet with the group elements. Then $T_s$
describes the $p$-random walk. The logarithm of the Gurević pressure coincides with the
radius of convergence of the random walk $\limsup_{n \to \infty} p^{*n}(e)^{1/n}$.

A different example to have in mind is $F_{a,b}$, the free group with generators
$\{a, a^{-1}, b, b^{-1}\}$, and $\Sigma = \{x \in \{a, a^{-1}, b, b^{-1}\}^\mathbb{N} : x_{i+1} \neq x_i \forall i \in \mathbb{N}\}$, identifying
the alphabet with the group elements. The skew product $\Sigma \times \xi(P)$ describes the
radius of convergence of the random walk $\limsup_{n \to \infty} p^{*n}(\xi)$. Following [7], [4] we say that $\xi$ satisfies the
visibility hypothesis if there is a finite set $K$ of $G$ so that every $g \in G$ may
be expressed $g = k_1 \xi(w_1) \cdots \xi(w_n)k_2$ for $k_1, k_2 \in K$ and some admissible word
$w_1 \cdots w_n$. We expect the visibility hypothesis to be satisfied in settings where there is
a coding of a geometric problem; the visibility hypothesis is used in Coulon–
Dal’bo–Sambusetti [4] to characterize a uniform gap in critical exponents of certain
isometric actions of word hyperbolic groups, and a more restrictive version is used
in Dougall [7] to characterize a uniform gap in Gurević entropy for certain geodesic
flows. One of the insights of [4] and [7] is that one should understand a family of
skew products in $G/H$ in terms of unitary representation of $G$.

1.2. Unitary representations and matrix coefficients. A comprehensive text
on unitary representations is [1]. Let $G$ be a discrete group. For a (real) Hilbert
space $\mathcal{H}$ with inner product $(\cdot, \cdot)_\mathcal{H}$, we denote by $U(\mathcal{H})$ the unitary operators
from $\mathcal{H}$ to itself. (Recall that an operator $U$ is unitary if $(Uv, Uw) = (v, w)_{\mathcal{H}}$
for all $v, w \in \mathcal{H}$.) A unitary representation of $G$ (in $\mathcal{H}$) is a homomorphism
$\rho : G \to U(\mathcal{H})$. The prototypical example is given by the real Hilbert space
$\ell^2(G) = \ell^2(G, \mathbb{R}) = \{f : G \to \mathbb{R} : \sum_{g \in G} |f(g)|^2 < \infty\}$ and left regular representa-
tion $\lambda$ defined by $[\lambda(g)]f(x) = f(g^{-1}x)$. (One could consider the complex Hilbert space $\ell^2(G, \mathbb{C})$, but all the representations we consider preserve the real cone and
so in this case there is nothing of interest added by complexifying.) For a subgroup
$H \leq G$ we write $G/H$ for the cosets of $H$ in $G$. We refer to to the unitary representa-
tion $\lambda_H$ of $G$ in $\ell^2(G/H)$ defined by $[\lambda_H(g)]f(xH) = f(g^{-1}xH)$ as the quotient representa-
tion. (We are equally tempted to call $\lambda_H$ a permutation representation
since one of the key features is that $\lambda_H$ permutes an orthonormal basis. See section
13.) The scope of this paper is restricted to unitary representations $\rho$ in $\mathcal{H}$ that are
either a quotient representation or a countable direct sum of quotient representa-
tions. It should be noted that in this paper we rely heavily on the $G$-invariant cone of non-negative functions $\ell^2_+(G/H)$ in $\ell^2(G/H)$.

When $G$ is infinite, there does not exist a vector $v \in \ell^2(G)$ invariant by all of
$G$. A group is said to be amenable if the left regular representation $\lambda$ has almost
invariant vectors: meaning that for every $\varepsilon > 0$ and every finite set $S$ there is a
unit vector $v \in \ell^2(G)$ with $\|\rho(s)v - v\|_{\ell^2(G)} \leq \varepsilon$ for all $s \in S$. One also says that
a unitary representation $\rho$ in $\mathcal{H}$ weakly contains the trivial representation, written $1 \preceq \rho$, if for every $\varepsilon > 0$ and every finite set $S$ there is a unit vector $v \in \mathcal{H}$ with
$\|\rho(s)v - v\|_{\mathcal{H}} \leq \varepsilon$ for all $s \in S$. If $G/H_n$, $n = 1, 2, \ldots$, is a sequence of non-amenable
quotients (and so $1 \preceq \lambda_{H_n}$) one may ask whether $1 \preceq \sum_{n=1}^\infty \lambda_{H_n}$.

We use the notation of convolution throughout the paper: for two functions
$f, \phi : G \to \mathbb{R}$ we write $\phi * f$ for the function $\phi * f(x) = \int \phi(h)f(h^{-1}g)dm$, where $m$ is the
counting measure in $G$ (or Haar measure, recalling that $G$ is a countable discrete
group). For a representation $\pi$ of $G$ write $\phi * \pi f$ for the vector $\int \phi(h)\pi(h)f dm$.
Frequently one sees $\pi(\phi)f$ or $\pi_\phi f$; we prefer $\phi * \pi f$ in this setting because the
unitary representations are all discrete function spaces $\ell^2(G/H)$. We also use the
notation $\phi^*$ for the function $\phi^*(g) = \phi(g^{-1})$; the operator $\pi(\phi) = \sum_{h \in G} \phi(h)\pi(h)$
has adjoint $\pi(\phi^*)$ (recall that the Hilbert spaces are real).
Fixing the representation $\rho$ of $G$ in $\mathcal{H}$, a vector $f \in \mathcal{H}$ gives rise to a *(diagonal)* matrix coefficient $\psi_f : G \to \mathbb{R}$ defined by $\psi_f(g) = \langle \rho(g) f, f \rangle_{\mathcal{H}}$. Matrix coefficients play an important role in the unitary representation theory of groups (where the theory is at all tractable) such as for compact groups, Abelian groups and semi-simple Lie groups. Here our groups are always countable and equipped with the discrete topology. When $G$ contains an infinite cyclic subgroup one can find $f \in \ell^2(G)$ so that $\psi_f$ is not integrable ($\psi_f \notin \ell^1(G)$) or square integrable ($\psi_f \notin \ell^2(G)$).

In a different direction, if $\phi \in \ell^1(G)$ then $\int \psi_f(g) \phi(g) dm(g) = \langle \phi * \rho, f \rangle_{\ell^2(G)}$ is always well-defined and finite, i.e. $\psi_f$ is integrable with respect to the density $\phi m$.

2. Thermodynamic $G$-densities

Throughout the data of the countable Markov shift and log-Hölder function $R$ are fixed, as is $s : W_1 \to G$. Recall that we assume $P(\log R, \sigma) = 0$. The ideology is that properties of a unitary representation of $G$ relate to statistics of group extensions, and vice-versa. To this end, we need a mathematical object by which to reveal this relationship. We introduce a “thermodynamic $G$-density”, a family of non-negative functions $\phi[t] : G \to G$ associated to a convergence parameter $t \in \mathbb{R}$, and show a relationship to certain statistics of a unitary representation.

**Remark 2.1.** The emphasis is to begin with the function $\phi[t]$ and then for each unitary representation $\rho$ in $\mathcal{H}$ and $f \in \mathcal{H}$ consider the vectors $\phi[t] * \rho f$, rather than go directly to the operator $\rho_+(\phi[t])$. The operator $\rho_+(\phi[t])$ appears (in a different guise) in two other contexts. In the work of [5] the *twisted Poincaré series* could be transcribed to $\rho_+(\phi[t])$. When the underlying skew product describes a random walk we have that $\sum_{a,b \in W_1} \rho_+(\phi^{a,b}[t])$ coincides with the Neumann series of the random walk operator (for $t$ large enough). One might consider the Neumann series for the transfer operator for $T_s$ but this has disadvantage that one has to worry about the whole space $\Sigma^+ \times G$ — this is the topic of future work.

Recall that we write $W_n^{A,a}$ for admissible words $w = w_1 \cdots w_n$ of length $n$ with first letter $w_1 = A$ and last letter $w_n = a$.

**Definition 2.2** (Thermodynamic $G$-density). Let $A, a \in W_1$ and $x \in \sigma[a]$. Let $c : \mathbb{N} \to [0, \infty)$ with subexponential growth. Define the partial sums $\phi_{c \leq N}^{A,a}[t] : G \to G$ by

$$\phi_{c \leq N}^{A,a}[t](g) = \sum_{n=1}^{N} t^{-n} c_n \sum_{w \in W_n^{A,a}} R_n(w x).$$

For any $t > 0$ such that $\sup_N \phi_{c \leq N}^{A,a}[t](g) < \infty$ for each $g \in G$, we call $\phi_{c}^{A,a}[t] : G \to G$

$$\phi_c^{A,a}[t](g) = \sum_{n=1}^{\infty} t^{-n} \sum_{w \in W_n^{A,a}} R_n(w x).$$

a thermodynamic $G$-density.

In the case that $c$ is identically 1 we abbreviate to $\phi_{c \leq N}^{A,a}[t](g)$ and $\phi_c^{A,a}[t]$ respectively. If $\Sigma^+$ is not compact we insist that $a A$ is admissible, and $x \in [A]$. We continue the exposition for a fixed $A, a, x$, and use the shorter notation $\phi[t] = \phi^{A,a}[t]$ (and $\phi_c[t] = \phi_c^{A,a}[t]$). When $T_s$ is transitive (and $\log R$ is Hölder) the convergence parameters that follow are independent of these choices. (Or if $\rho$ corresponds to the quotient representation $\ell^2(G/H)$ then we ask for $T_s$ to be transitive.) In the case that $\mathcal{H} = \ell^2(G/H)$ we write $\delta_h$ to denote the indicator function at a coset $h \in G/H$, and $\delta_e$ denotes the indicator function at the coset corresponding to $H$. 

Let $\rho$ and $\mathcal{H}$ be a countable sum of quotient representations of $G$. (We encompass the group extension $\mathfrak{s}_H : W_1 \to G/H$ using the quotient representation for $H \leq G$). We make use of the cone of non-negative functions $\ell^2_+(G/H) = \{ f \in \ell^2(G/H) : \forall g \in G f(g) \geq 0 \}$ and denote $\mathcal{H}_+$ the corresponding cone in $\mathcal{H}$. Recall that a vector $f \in \mathcal{H}$ gives rise to a matrix coefficient $h \mapsto \langle \rho(h)f, f \rangle_\mathcal{H}$. We define convergence parameters according to the integrability of matrix coefficients in the following way.

**Definition 2.3** (Hierarchy of convergence parameters). For $f \in \mathcal{H}_+$ denote

$$\gamma(f) = \inf \left\{ t > 0 : \sup_{N \in \mathbb{N}} \int (\rho(h)f, f)_\mathcal{H} d\phi_{\leq N}[t](h) < \infty \right\},$$

$$\gamma(\phi^* \ast \rho f) = \inf \left\{ t > 0 : \sup_{N \in \mathbb{N}} \int (\rho(h)f, f)_\mathcal{H} d\phi_{\leq N}[t] * \phi_{\leq N}[t](h) < \infty \right\},$$

$$= \inf \left\{ t > 0 : \sup_{N \in \mathbb{N}} \langle \phi_{\leq N}[t] *_{\rho} f, \phi_{\leq N}[t] *_{\rho} f \rangle_\mathcal{H} < \infty \right\},$$

$$\gamma(\phi \ast_{\rho} f) = \inf \left\{ t > 0 : \sup_{N \in \mathbb{N}} \int (\rho(h)f, f)_\mathcal{H} d\phi_{\leq N}[t] * \phi_{\leq N}[t](h) < \infty \right\},$$

$$= \inf \left\{ t > 0 : \sup_{N \in \mathbb{N}} \langle \phi_{\leq N}[t] *_{\rho} f, \phi_{\leq N}[t] *_{\rho} f \rangle_\mathcal{H} < \infty \right\}.$$ 

We say that the **pressure of** $f$ is $-\log \gamma(f)$. We say that $\gamma(\phi *_{\rho} f)$ is the **decay exponent** for $f$.

We justify the terminology by Proposition 3.1 and Lemma 3.5. Let us comment on the hierarchy of convergence — the verification of the statements follows later. We simplify here to $\mathcal{H} = \ell^2(G)$, and so $f \in \ell^2_+(G)$.

1. The hierarchy is increasing: $\gamma(f) \leq \gamma(\phi^* \ast \rho f) \leq \gamma(\phi \ast_{\rho} f)$.
2. For $t > \gamma(\text{SPR})$, $\phi_{\leq N}[t]$ belongs to $\ell^1(G)$.
3. For $t > \gamma(\delta_e)$, $\phi_{\leq N}[t] = \lim_{N \to \infty} \phi_{\leq N}[t]$ is well-defined as an element of $G^R$.
4. For $t > \gamma(\phi *_{\rho} f)$, $\phi_{\leq N}[t] *_{\rho} f \in \ell^1(G)$.

Most textbooks featuring convolutions of functions explain that for $f \in \ell^2(G)$, we have that the convolution $v * f$ satisfies $v * f \in \ell^2(G)$ if $v \in \ell^1(G)$. The condition that $v \in \ell^1(G)$ is not optimal — later Theorem 3.13 will tell us that if $G$ is non-amenable there is $t < 1$ so that $\phi(t) = \phi[t] * \delta_e \in \ell^1(G)$ despite $\phi(t) \notin \ell^1(G)$. We use the positivity of the functions throughout this paper. We check the next lemma in Section 13.

**Lemma 2.4.** Let $f \in \mathcal{H}_+$. If $\sup_{N \in \mathbb{N}} \| \phi_{\leq N}[t] *_{\rho} f \| < \infty$ then $\phi_{\leq N}[t] *_{\rho} f$ strongly converges to $\phi_{\leq N}[t] *_{\rho} f$ as $N \to \infty$, and $\phi_{\leq N}[t] *_{\rho} f \in \mathcal{H}_+$.

**Remark 2.5.**

1. The vector norm $\| \phi(t) \|_{\ell^2(G)}$ should not be confused with the operator norm $\| \rho_\ast(\phi(t)) \|_{\text{op}} := \sup_{f \neq 0} \frac{\| \rho_\ast(\phi(t))f \|_{\ell^2(G)}}{\| f \|_{\ell^2(G)}}$.
2. We only make use of the unitarity of the representation $\rho$ for results involving the largest parameter $\gamma(\phi *_{\rho} f)$. A hidden consequence of unitarity is the strong convergence of $\phi_{\leq N}[t]$; in particular $\langle \phi_{\leq N}[t], \phi_{\leq N}[t] \rangle \to \langle \phi[t], \phi[t] \rangle$ as $N \to \infty$. For $t > \gamma(\phi^* \ast \rho f)$ we know only that $\langle \phi_{\leq N}[t], \phi_{\leq N}[t] \rangle$ has a limit, and our manipulations only utilise the homomorphism property of $\rho$ and that $\rho$ preserves the non-negative cone.

Secondly, we explain the effect of the choice of $A, a$. For two vectors $v, w \in \ell^2_+(G)$ we say that $v \leq w$ if $v(g) \leq w(g)$ for every $g \in G$. 


Lemma 2.6. For each $B$ there are a constants $\text{Const.(}B)$, $K_B$ and a group elements $h_B,g_B$ with

$$\tag{2.1} \text{Const.(}B)^{-1} \delta_{h_B} \ast \phi^{A,a}_{c \leq N-K_B}[t] \leq \phi^{B,a}_{c \leq N}[t] \leq \text{Const.(}B) \delta_{g_B} \ast \phi^{A,a}_{c \leq N+K_B}[t].$$

If $T_s$ is transitive then we may assume $g_B = e = h_B$.

In Section 13 we also check other basic facts, such as the connectedness of the parameters $t$ for which the defining integrals converge, and that transitivity implies $0 < \gamma(f) = \gamma(\phi^* \ast_p f)$.

2.1. Examples. For the Abelian group $\mathbb{Z}$ and vector $f = \delta_0$ we have

$$\int \langle \rho(k)\delta_0, \delta_0 \rangle_{\ell^2(\mathbb{Z})} d\phi \leq N \leq N [t](k) = \int \sum_{n=1}^{N} t^{-n} \sum_{w \in \mathcal{W}_{A,a}^{n}} R_n(wx) \exp(2\pi i g(w)\alpha) d\alpha$$

$$=: \int \zeta \leq N(t, \alpha) d\alpha$$

The implied series $\zeta(t, \alpha)$ converges for all $t \geq |\exp(P(\log R + 2\pi i a, \sigma))|$. At $t = \exp(P(\log R, \sigma)) = 1$ the series $\zeta(1, 0) = \infty$, but this does not guarantee that the integral over all $\alpha$ is infinite. For a transitive random walk on $\mathbb{Z}$ that does not have zero mean we have $\gamma(\delta_0) < 1$ — see [26].

Let $S$ be a generating set for a group $G$, and $P : S \rightarrow [0, 1]$. Then for the $G$-extension corresponding to the random walk we have

$$\phi[t](g) = \sum_{n \in \mathbb{N}} t^{-n} \mu^n(g).$$

In the case that $G = F_{a,b}$, $S = \{a, b, a^{-1}, b^{-1}\}$ and $P = 1/4$, one can compute asymptotics for $\mu^{2n}(g)$ (see [26]) and in particular we deduce that $\gamma(\delta_e) = \sqrt{3}/2$.

This example is elaborated on in section 4.

2.2. Related notions: ratio limit theorems and local limit theorems. It is worth mentioning related concepts in the context of random walks since we find our constructions to be reminiscent. We quote a ratio limit theorem from the book of Woess [26]. Set $\gamma = \lim_{n \rightarrow \infty}(p^n(e))^{1/n}$ and note that this coincides with our definition of $\gamma(\delta_e)$.

Proposition 2.7 (Ratio Limit Theorem, Theorem 5.6 (and Corollary 5.8) of [26]). Suppose that $G$ is Abelian (or nilpotent) and $P$ is a finitely supported measure whose support generates $G$ as a semigroup. Then there is a unique $\gamma$-harmonic function $\Psi : G \rightarrow \mathbb{R}$ with

$$\lim_{n \rightarrow \infty} \frac{p^n(x)}{p^n(e)} = \frac{1}{\Psi(x)}$$

for all $x \in G$.

Let us elaborate on the case where $G$ is nilpotent: the assumption that the support of $p$ generates $G$ as a semigroup forces that the harmonic function $\Psi$ for $G$ is also harmonic for the abelianisation of $G$. Recent work of Benoist [2] finds a new harmonic function for the Heisenberg group for a probability whose support does not generate $G$ as a semigroup (and the ratio limit theorem stated in [2] is of a different nature than what is stated in Proposition 2.7). In section 4 we discuss further the setting of a random walk when $p$ is finitely supported, (aperiodic,) and with the assumption that the semigroup generated by the support is the whole of $G$ (this last assumption implies transitivity of the group extension). When $G$ is Abelian we show in Proposition 4.2 how to directly recover the function $\Psi$ (but not the ratio limit theorem) from Lemma 3.9. Theorem 3.16 also recovers the function $\Psi$ and, notably, is valid even for amenable groups.
A stronger result to mention is a local limit theorem (to keep this discussion light we do not mention the local limit theorem for Abelian groups). For the free group $F_r$ on $r$ generators one can find $C\Psi$ with

$$\limsup_{n \to \infty} \frac{p^n(x)}{\gamma^n n^{-3/2}} = C\Psi(x) > 0$$

for all $x \in F_r$ (Theorem 6.8 of [26] and attributed to Lalley, and also in this isotropic case to Picardello [23]). We give a calculation to explain how one recovers $\Psi$ (but not the local limit theorem) in the case of the simple random walk in Section 4.

3. Results

3.1. Preliminaries. We motivate the study of the thermodynamic $G$-density by its connection to growth statistics for discrete group extensions.

Let us begin with Gurevič pressure and notion of decay of matrix coefficients along $\mu$ typical paths.

**Proposition 3.1.** Assume that $T_\delta$ is transitive. If $\exp P(\log R, T_\delta) > \gamma(\text{SPR})$ then

$$\gamma(\delta_c) = \exp P(\log R, T_\delta),$$

where $\delta_c$ is the indicator function at the identity for the left regular representation $\lambda$ in $\ell^2(G)$.

The proof is found in Section 12.

A random walk on a group is said to be transient (see [26]) if the probability $p : G \to [0, 1]$ has $\sum_{n=1}^\infty p^n(e) < \infty$. We reimagine this as a statement about decay of pairs of vectors in $\ell^2(G)$. Ultimately, the idea is that if the left regular representation $\lambda$ in $\ell^2(G)$ does not weakly contain the trivial representation ($G$ is non-amenable) then pairs of vectors are forced to have a so-called statistical-dynamical eventual decay.

**Definition 3.2.** Assume $\Sigma$ is compact. We say that vectors $f, v \in H_+$ have a (statistical-dynamical) eventual decay if there is $\gamma < 1$ so that for $\mu$ almost every $x \in \Sigma$ there is $N$ with

$$n \geq N \implies \langle \rho(g(x_n \cdots x_1))f, v\rangle_H \leq \gamma^n.$$

We say that $\rho, H$ has uniform (statistical-dynamical) eventual decay exponent if we can take $\gamma < \gamma_0$ for some $\gamma_0 < 1$ independent of $f, v \in H_+$.

**Remark 3.3.** One may also consider decay for the products $\rho(g(x_1 \cdots x_n))f$ (and indeed we do in Lemma 3.11 as the measure there is only defined on the one-sided shift). Lemma 3.5 is also valid for products $\rho(g(x_1 \cdots x_n))f$.

**Remark 3.4.** Note that the Definition 3.2 is scalar independent. The significance of the decay statement is given by the arbitrariness of $v$. We illustrate this with the example of a symmetric random walk on a group.

Consider $f = \delta_c$ and $v = \delta_c$. The property that $f = \delta_c$ and $v = \delta_c$ have almost sure eventual $\gamma$-decay is equivalent to asking that the $\mu$ measure of points $(x,e)$ that $T_\delta$-return infinitely often to $\Sigma^+ \times G$ is zero. The (assumed symmetric) random walk on $\mathbb{Z}^d$ is transient for $d > 2$ [26], and so $f = \delta_0$, $v = \delta_0$ have eventual decay for any $\gamma < 1$.

On the other hand, in $\mathbb{Z}^d$ and $f = \delta_0$ we can find a function $v$ for which $f,v$ do not have statistical-dynamical decay (with respect to a symmetric random walk).

The function $v_t(k) = |k|^{-t}$ belongs to $\ell^2(\mathbb{Z})$ if $t > \frac{1}{2}$, and similarly $v_t(\bar{k}) = ||\bar{k}||_1^{-t}$ belongs to $\ell^2(\mathbb{Z}^d)$ if $d - 2t < 2$. Suppose that $s$ takes values in the unit $|| \cdot ||_1$-ball. Let $\gamma < 1$. Choose $M = M_s$ so that $v_t(\bar{h}) > \gamma ||\bar{h}||_1$ for $\bar{h}$ in the complement to the $|| \cdot ||_1$-ball of radius $M$. Then $\langle \lambda(g(x_n \cdots x_1))f, v_t\rangle_{\ell^1(\mathbb{Z}^d)} > \gamma^n$ for any $n > M$. 

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In an arbitrary amenable group, with some extra work we can show that for every \( \gamma \) there are vectors \( f = v \), and \( v = v_\gamma \) which have cannot have decay rate faster than \( \gamma \). In this way, for \( G \) amenable, \( \lambda, \ell^2(G) \) does not have uniform (statistical-dynamical) eventual decay. (The argument presented for \( \mathbb{Z} \) relied on polynomial growth and gave the stronger conclusion of exhibiting vectors not having eventual decay.) See Lemma 4.1.

**Lemma 3.5.** Assume \( \Sigma \) is compact. Let \( f \in \mathcal{H}_+ \). If \( \gamma(\phi *_f f) < 1 \) then for every \( \gamma \in (\gamma(\phi *_f f), 1) \) and every \( v \in \mathcal{H}_+ \) we have that for \( \mu \) almost every \( x \in \Sigma \) there is \( N \) with

\[
 n \geq N \implies \langle \rho(g(x_\gamma \cdots x_{\gamma-1})) f, v \rangle_\mathcal{H} \leq \gamma^n
\]

**Remark 3.6.** In the case that \( \Sigma \) is not compact one should expect a restriction on \( x \) returning to a small open set.

**Proof of Lemma 3.5.** Let \( f, v \in \mathcal{H}_+ \) be arbitrary. We denote the sets

\[
 E_n = \{ x \in \Sigma : \langle g(x_\gamma \cdots g(x_{\gamma-1}) f, v \rangle_\mathcal{H} \geq \gamma^n \},
\]

\[
 \limsup E_n = \{ x \in \Sigma : \langle g(x_\gamma \cdots g(x_{\gamma-1}) f, v \rangle_\mathcal{H} \geq \gamma^n, n \text{ i.o.} \}.
\]

Our goal is to use the Borel-Cantelli Lemma to deduce that \( \limsup E_n \) has zero \( \mu \) measure. Negative-coordinate cylinder sets are denoted

\[
 [u_\gamma \cdots u_{\gamma-1}] = \{ x \in \Sigma : x_{\gamma-i} = u_{\gamma-i}, i = 1, \cdots n \}.
\]

The Gibbs property 12.13 states that there is a constant \( C > 0 \) with

\[
 \mu([w_\gamma]) \leq CR_n(wy)
\]

for any \( w \) of length \( n \) and \( y \in \sigma^n[w_\gamma] \). We compute

\[
 \mu(E_n) = \sum_{A, a \in W_1} \sum_{w \in W_n^{A,a}} \mu([w_\gamma \cdots w_{\gamma-1}])
\]

\[
 \leq \sum_{A, a \in W_1} \sum_{w \in W_n^{A,a}} \mu([w_\gamma \cdots w_{\gamma-1}]) \gamma^{-n} \langle \rho(g(w_1) \cdots g(w_n)) f, v \rangle_\mathcal{H}
\]

\[
 \leq C \sum_{A, a \in W_1} \sum_{w \in W_n^{A,a}} R_n(w_1 \cdots w_n) \gamma^{-n} \langle \rho(g(w_1) \cdots g(w_n)) f, v \rangle_\mathcal{H}
\]

\[
 \leq C \sum_{A, a \in W_1} \langle \phi_n^{A,a}[\gamma] *_f f, v \rangle_\mathcal{H},
\]

with \( \phi_n^{A,a}[t](g) := \sum_{w \in W_n^{A,a}, g(w) = g} t^{-n} R_n(w_1 \cdots w_n) \). Then

\[
 \sum_{n=1}^{\infty} \mu(E_n) \leq \sum_{A, a \in W_1} \langle \phi_n^{A,a}[\gamma] *_f f, v \rangle_\mathcal{H} \leq \sum_{A, a \in W_1} \| \phi_n^{A,a}[\gamma] *_f f \|_\mathcal{H} \| v \|_\mathcal{H}.
\]

By Lemma 2.4 we know that \( \| \phi_n^{A,a}[\gamma] *_f f \|_\mathcal{H} \) is finite for \( \gamma > \gamma(\phi *_f f) \), thus giving the convergence case of the Borel-Cantelli Lemma as desired.

**3.2. (Limits of) matrix coefficients of the thermodynamic \( G \)-density and twisted measures.** We refer the reader to Subsection 7.1 and Definition 7.1 for the definition of a measure twisted by a generalised multiplicative cocycle. We are not able to work with \( \phi[t] \) but rather must increase it with a slowly increasing function.

**Lemma 3.7.** Let \( f \in \mathcal{H}_+ \). There is a slowly increasing function \( f : \mathbb{N} \to \infty \) and sequence \( t_k \to \gamma(\phi *_f f) \) as \( k \to \infty \) with the following. The function \( \Upsilon_c : G \to \mathbb{R} \)

\[
 \Upsilon_c(g) = \lim_{k \to \infty} \frac{\langle \rho(g) \phi_c[t_k] *_f f, \phi_c[t_k] *_f f \rangle_\mathcal{H}}{\langle \phi_c[t_k] *_f f, \phi_c[t_k] *_f f \rangle_\mathcal{H}}.
\]
is well-defined, and there is a twisted measure \( \nu^{\phi * f}_c \) on \( \Sigma^+ \) (that is finite on cylinders) with
\[
(\gamma(\phi * f))^{n} \nu^{\phi * f}_c(R_n^{-1}1_{[w]}) = \Upsilon_c(\varphi(w))
\]
for all \( w \in \mathcal{W}^A_n \).

**Remark 3.8.** The measure in 3.7 can be thought of as being analogous to the twisted measure in [5]. However, there are some caveats. The setting of [5] is geometric SPR, and they define the measure according to a sequence \( f \) maximising the twisted operator, whereas in this exposition \( f \) is turned into a group action invariant current by a generalised product with itself. In the absence of symmetry \( \nu^{\phi * f}_c \) is not easily related to a shift invariant measure. (For an example with symmetry see also equation 4.2.)

Recall that \( \log R \) is assumed to be (locally) Lipschitz with respect to the \( \theta \)-metric (see equation 12.1).

**Lemma 3.9.** Assume \( \Sigma^+ \) is compact, and assume \( T_\theta \) is transitive. Let \( f \in \mathcal{H}_+ \). There is a slowly increasing function \( c : \mathbb{N} \to \infty \) and sequence \( t_k \to \gamma \) as \( k \to \infty \) with the following. The function \( \Upsilon_c : G \to \mathbb{R}^+ \)
\[
\Upsilon_c(g) = \limsup_{k \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g) \phi_{c \leq N} [t_k] * \rho f, \phi_{c \leq N} [t_k] * f \rangle_H}{\langle \phi_{c \leq N} [t_k] * \rho f, \phi_{c \leq N} [t_k] * f \rangle_H}
\]
is well-defined, and there is a shift invariant finite measure \( \mu^{\phi * f}_c \) on \( \Sigma^+ \)
\[
C^{-\theta^{w \wedge \gamma}} \Upsilon_c(\varphi(w)) \leq (\gamma(f))^{n} \mu^{\phi * f}_c(R_n^{-1}1_{[w]}) \leq C^{\theta^{w \wedge \gamma}} \Upsilon_c(\varphi(w))
\]
for some constant \( C > 1 \).

**Remark 3.10.** (1) The \( c \) appearing in Lemmas 3.7 and 3.9 can be different. (And indeed the \( \gamma \) can be different.)

(2) The functions \( \Upsilon_c, \Upsilon_c \) implicitly depend on the letters \( A, a \) for which \( \phi = \phi^{A,a} \).

(3) The function \( \Upsilon_c \) is defined by a limit (along \( t_k \)) of matrix coefficients. We can check that \( \Upsilon_c \) is positive definite \( \langle \sum_{i,j \in I} a_i \alpha_j \Upsilon_c(g_j^{-1}g_i) \rangle \geq 0 \), from which \( \Upsilon_c \) is a matrix coefficient for a unitary representation (see [23] or [10]).

Section 14 gives an explicit example.

We give an interpretation that the \( \nu^{\phi * f}_c \) measure finds points with the slowest decay exponent — for a reversed ordering of the products in Lemma 3.5. We make this more precise in the case of a symmetric random walk on the free group where we are able to express section \( \Upsilon_c(g) \) in terms of an exponential — see Section 4. In general we can only say that a typical point has decay bounded by \( \Upsilon_c(g) \). And that \( \Upsilon_c(g) \) is typically decays no faster than \( \gamma \).

**Lemma 3.11.** Assume \( \Sigma^+ \) is compact and that \( T_\theta \) is transitive. Let \( f \in \mathcal{H}_+ \). If \( \gamma(\phi * f) < 1 \) then for every \( \epsilon > 0 \) and every \( v \in \mathcal{H}_+ \) we have that for \( \nu^{\phi * f}_c \) almost every \( x \in \Sigma^+ \) there is \( N \) with
\[
n \geq N \implies \langle \rho(\varphi(x_1 \cdots x_n))f, v \rangle_H \leq (1 - \epsilon)^{-n} \Upsilon_c(\varphi(x_1 \cdots x_n))
\]
In addition for \( \nu^{\phi * f}_c \) almost every \( x \in \Sigma^+ \) there is \( N \) with
\[
n \geq N \implies \Upsilon_c(\varphi(x_1 \cdots x_n)) \geq n^{-2} (\gamma(\phi * f))^n.
\]

**Remark 3.12.** (1) We present a calculation for a random walk on the free group (see Section 4). In this case \( \gamma^n = 2^n 3^{-n/2} \) whereas it can be shown that \( \Upsilon_c \) has infinitely many \( g \) with \( \Upsilon_c(g) \geq 3^{-|g|/2} \). In fact a certain linear combination relating to \( \Upsilon_c(g) \) (taking into account different choices of \( A, a \)) is equal to \( (1 + |g|/2)3^{-|g|/2} \). Then 3.3 implies that a \( \nu^{\phi}_c \) typical point \( x \) has
that $g(x_1 \cdots x_n)$ is not reduced in a quantitative way, which we compare with the known drift for $\mu$ random walk.

(2) If $\gamma(\phi * \rho \, f) = \gamma(f)$ then, upon identifying $\mu^\phi_{\Sigma^+ * f}$ with a shift invariant measure on $\Sigma$ we can show that for every $\gamma(f) < \gamma$ and every $v \in H_+$ we have that for $\mu^\phi_{\Sigma^+ * f}$ almost every $x \in \Sigma$ there is $N$ with

\[ n \geq N \implies \langle \rho(g(x_{-n} \cdots x_{-1}))f, v \rangle_H \leq (1 - \epsilon)^{-n} \chi_{c,v}(g(x_{-n} \cdots x_{-1})). \]

and

\[ n \geq N \implies \langle \rho(g(x_{-n} \cdots x_{-1}))f, v \rangle_H \leq (1 - \epsilon)^{-n} \chi_{c,v}(g(x_1 \cdots x_n)). \]

In addition for $\nu_{\Sigma^+ * f}$ almost every $x \in \Sigma^+$ there is $N$ with

\[ n \geq N \implies \chi_{c}(g(x_{-n} \cdots x_{-1})) \geq n^{-2}(\gamma(\phi * \rho \, f))^n. \]

\textbf{Proof of Lemma 3.11.} Let $f, v \in H_+$ be arbitrary. For brevity write $\gamma = \gamma(\phi * \rho \, f)$. Set

\[ E_n = \{ x \in \Sigma : \langle \rho(g(x_{-n} \cdots x_{-1}))f, v \rangle_H \geq \chi_{c}(g(x_{-n} \cdots x_{-1}))(1 - \epsilon)^{-n} \}, \]

\[ \text{lim sup } E_n = \{ x : \langle \rho(g(x_{-n} \cdots x_{-1}))f, v \rangle_H \geq \chi_{c}(g(x_{-n} \cdots x_{-1}))(1 - \epsilon)^{-n}, \text{ n i.o.} \}. \]

Lemma 3.7 is stated only for cylinders $wA$, however Lemma 7.11 upgrades to

\[ \nu_{\Sigma^+ * f}([u]) \leq CR_n(wy)\chi_{c}(g(u))\gamma^{-n} \]

for any $u$ of length $n$ and $y \in \sigma^n[u]$. We compute

\[ \nu_{\Sigma^+ * f}(E_n) = \sum_{A, a \in W_1} \sum_{w \in W_n^{A,a}} \nu_{\Sigma^+ * f}([w]) \]

\[ \sum_{w \in W_n^{A,a}} \gamma^{-n}(1 - \epsilon)^n R_n(wx)\gamma^{-n} \]

Since $\gamma(1 - \epsilon)^{-1} > \gamma(\phi * \rho \, f)$ the convergence case of the Borel-Cantelli Lemma follows.

For the second part set

\[ E_n = \{ x \in \Sigma^+ : \chi_{c}(g(x_{-n} \cdots x_{-1})) \leq \gamma^n n^{-2} \}. \]

We have

\[ \nu_{\Sigma^+ * f}(E_n) = \sum_{A, a \in W_1} \sum_{w \in W_n^{A,a}} \nu_{\Sigma^+ * f}([w]) \]

\[ \leq \sum_{A, a \in W_1} \sum_{w \in W_n^{A,a}} CR_n(wy)\chi_{c}(g(u))\gamma^{-n} \]

\[ \leq C \sum_{A, a \in W_1} \sum_{w \in W_n^{A,a}} R_n(wx)\frac{1}{n^2}. \]

The series $\sum_{n=1}^{\infty} n^{-2}$ is summable and so we conclude the convergence case of the Borel-Cantelli Lemma.
3.3. Theorems. Lemma 3.7 is the main ingredient to our first two theorems.

Theorem 3.13. Assume that $T_s$ is transitive. We have the following:

- If $G$ is non-amenable then $P(\log R, T_s) < 0$.
- If $\rho$ does not weakly contain the trivial representation then $\rho, H$ has uniform eventual decay exponent $\gamma_0 = \sup_{0 \neq f \in \mathcal{H}^+} \gamma(\phi * f) < 1$.

Theorem 3.14. Assume that $s$ satisfies the visibility hypothesis. Let $F$ be a family of normal subgroups of $G$. We have

$$1 \not\in \bigoplus_{H \in F} \ell^2(G/H) \implies \sup \{ P(\log R, T_{s_H}) : H \in F \} < 0.$$  

Remark 3.15. (1) In section 4 we give an example to see that the decay statement of Theorem 3.13 is non-trivial.

- Theorem 3.14 only uses normality for the definition of $T_{s_H}$ as a group extension. Transitivity is not required as the proof relies on the study of $\gamma(\phi * f)$ for $f$ the countable sum of $\ell^2(G/H)$.

Assume now that $\Sigma$ is compact. For amenable groups the work of [9] shows that the Gurevič pressure $P(\log R, T_s)$ is equal $P(\log R + \psi)$ for a unique real one-dimensional representation $\pi : G \to \mathbb{R}$ and $\psi = \pi \circ s$. (In other words: let $\bar{\psi}_{ab}$ be the composition of $s$ with $G \to G/[G : G] \cong \mathbb{Z}^d \oplus F \to \mathbb{Z}^d$, with $F$ the finite torsion group. Then there is a unique $\xi \in \mathbb{R}$ that determines $\psi(x) = (\xi, \bar{\psi}_{ab}(x))_{\mathbb{R}^d}.$)

Theorem 3.16. Assume $\Sigma^+$ is compact and assume $T_s$ is transitive. If $G$ is amenable then the measure $\mu_{s, s_s}^\ast$ from Lemma 3.9 is the equilibrium state for $R \exp \psi$. In particular

$$\Upsilon_{c, \ast}(g) = \exp \psi(g).$$

Remark 3.17. In section 4 we discuss further the setting of a random walk when $p$ is finitely supported, (aperiodic,) and with the assumption that the semigroup generated by the support is the whole of $G$ (this last assumption implies transitivity of the group extension). When $G$ is Abelian we show in Proposition 4.2 how to recover the function $\Psi$ of the ratio limit theorem as stated in Proposition 2.7 (but not recover the ratio limit theorem) from Lemma 3.9 by

$$\Upsilon_{c, \ast}(g) = \frac{1}{\Psi(g)}.$$  

Theorem 3.16, notably, is valid even for amenable groups.

4. (Symmetric) Random walks

We vary $G$, a (symmetric) finite generating set $S$, and the probability $p : S \to [0, 1]$, but fix that the structure of the group extension to describes the random walk. Namely, $\Sigma = S^\mathbb{Z}$ is the full shift on a (symmetric) finite generating set $S$ of $G$, $R = p : S \to [0, 1]$, and $s$ identifies the formal letter $a \in W_1$ with the group element it represents. We assume that the semigroup generated by $S$ is equal to $G$. In this way the group extension is always transitive.

The structure of a random walk is useful as

$$\sum_{B, b \in W_1} \langle \phi^{B, b}[t], \delta_c \rangle = \sum_{n=1}^\infty p^\ast c(e) t^{-n}. \tag{4.1}$$

When $S$ is symmetric $\phi$ and $\phi^\ast$ coincide. Yet $\nu_c^\phi$ need not coincide with $\mu_c^\phi$ as

$$\nu_c^\phi([b]) = \lim_{q \to \infty} \frac{\langle \phi_c^{b, a}[t_q], \phi_c^{A, a}[t_q] \rangle}{\langle \phi_c^{A, a}[t_q], \phi_c^{A, a}[t_q] \rangle}. $$
whereas
\[
\mu_c^\phi([b]) = \lim_{q \to \infty} \frac{\langle \phi_c^{b,a}[t_q], \sum_{D \in W_1} \phi_c^{D,a}[t_q] \rangle}{\langle \phi_c^{A,a}[t_q], \phi_c^{A,a}[t_q] \rangle}.
\]

Let \(\nu_c^{A,a;D,a}\) be the measure with
\[
\nu_c^{A,a;D,a}([b]) = \lim_{q \to \infty} \frac{\langle \phi_c^{b,a}[t_q], \phi_c^{D,a}[t_q] \rangle}{\langle \phi_c^{A,a}[t_q], \phi_c^{A,a}[t_q] \rangle}.
\]

Let \(\mu_c^d\) be the measure with
\[
\mu_c^d([b]) = \lim_{q \to \infty} \frac{\langle \phi_c^{d,\gamma}[t_q], \sum_{B \in W_1} \phi_c^{B,a}[t_q] \rangle}{\langle \phi_c^{A,a}[t_q], \phi_c^{A,a}[t_q] \rangle}.
\]

Then
(4.2)\[
\sum_{D \in W_1} \nu_c^{A,a;D,a} = \mu_c^a,
\]

with mass
\[
C_{a,a}^{A} = \sum_{D \in W_1} \nu_c^{A,a;D,a}(1) = \lim_{q \to \infty} \frac{\langle \sum_{B \in W_1} \phi_c^{b,a}[t_q], \sum_{D \in W_1} \phi_c^{D,a}[t_q] \rangle}{\langle \phi_c^{A,a}[t_q], \phi_c^{A,a}[t_q] \rangle}.
\]

Let us also point out that \(T_c^{B,b;D,d}(g) = \lim_{q \to \infty} \frac{\langle \rho(g)\phi_c^{b,b}[t_q], \phi_c^{D,d}[t_q] \rangle}{\langle \phi_c^{A,a}[t_q], \phi_c^{A,a}[t_q] \rangle}\) has
(4.3)\[
T_c^{B,b;D,d}(\varphi(w)) = \gamma^{|w|}p^{-|w|}\nu_c^{b,d;D,d}([wB]).
\]

(See also equation 4.5.)

The condition \(\mathbb{1} \preceq \rho\) can be characterized by operator norms: namely \(\mathbb{1} \preceq \rho\) implies that for any \(F : G \to \mathbb{R}\) we have \(\|\rho_*(F)\|_\text{op} = \|\mathbb{1}_{*}(F)\|_\text{op} = \|F\|_{\ell^1(G)}\). In particular \(\|\rho_*(\phi[f])\|_\text{op} \to \infty\) as \(t \to 1\), but this is weaker than the existence of \(f\) with \(\gamma(\phi * f) = 1\).

We first check the non-triviality of the decay statement by giving an amenable example that does not have decay.

**Lemma 4.1.** Assume that \(S\) is symmetric. Suppose that \(G\) is amenable and \(\rho = \lambda\) the left regular representation. Then for every \(\gamma < 1\) there is \(v = v_\gamma\) so that \(\langle \rho(\varphi(w)v), v \rangle \geq \gamma^{|w|}\) for \(w\) with \(|w|\) sufficiently large.

**Proof.** As \(G\) is amenable and \(p\) symmetric we deduce from equation 4.1 that \(\gamma(\phi * \delta_c) = 1\). Then (using for instance equation 4.2) \(\mu_c^\phi\) coincides with \(\mu = \mu_p\) the equilibrium state for \(p\). Using Lemma 3.7 we have
\[
\mu_c^\phi([aWA]) = p^{|W|} \lim_{q \to \infty} \frac{\langle \rho(\varphi(W))\phi_c[t_q], \phi_c[t_q] \rangle}{\langle \phi_c[t_q], \phi_c[t_q] \rangle}
\]

On the other hand, since \(\mu([aWA]) = p^{|W|+2}\) we must have
\[
p^2 = \lim_{q \to \infty} \frac{\langle \rho(\varphi(W))\phi_c[t_q], \phi_c[t_q] \rangle}{\langle \phi_c[t_q], \phi_c[t_q] \rangle}.
\]

Let \(\eta < 1\) which will be determined in terms of \(\gamma < 1\). Define
\[
v_\eta = \sum_{q=1}^{\infty} \frac{\phi_c[t_q]}{\|\phi_c[t_q]\|^2}
\]
For any \( g = \mathcal{g}(W) \) we have
\[
\langle \rho(g)v_n, v_\eta \rangle_{\ell^2(G)} = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \eta^n \eta^m \frac{\langle \rho(g)\phi_c[t_n], \phi_c[t_m] \rangle_{\ell^2(G)}}{\|\phi_c[t_n]\|_{\ell^2(G)} \|\phi_c[t_m]\|_{\ell^2(G)}} \\
\geq \sum_{n \in \mathbb{N}} (\eta^2)^n \frac{\langle \rho(g)\phi_c[t_n], \phi_c[t_n] \rangle_{\ell^2(G)}}{\|\phi_c[t_n]\|_{\ell^2(G)}} \\
\geq \sum_{n \in \mathbb{N}; n \geq |u|} (\eta^2)^n p^2
\]
We assume \( \eta \) is sufficiently close to 1 so that \( \sum_{n \in \mathbb{N}; n \geq k} (\eta^2)^n p^2 > \gamma^k \) for all \( k \) larger than some \( K \).

We now explain directly from the definitions how to obtain the harmonic function from the Ratio Limit Theorem.

**Proposition 4.2.** If \( G \) is Abelian then
\[
\Upsilon_{c,s}(g) = \frac{1}{\Psi(g)}
\]
for \( \Psi \) in the Ratio Limit Theorem 2.7.

**Proof.** When \( G \) is Abelian we can rearrange terms
\[
\Upsilon_{c,s}(g) = \limsup_{k \to \infty} \lim_{N \to \infty} \frac{\langle \phi_{c \in N}[t_k], \phi_{c \in N}[t_k] \rangle_{F}}{\phi_{c \in N}[t_k] \phi_{c \in N}[t_k] f, \rho(g)^{-1} f} \mathcal{H}
\]
In particular given that we assume \( \mathcal{H} = \ell^2(G) \), \( f = \delta_c \) and that the group extension is for a random walk, we have
\[
\Upsilon_{c,s}(g) = \limsup_{k \to \infty} \lim_{N \to \infty} \frac{\langle \phi_{c \in N}[t_k], \delta_g \rangle_{\mathcal{H}}}{\langle \phi_{c \in N}[t_k], \delta_g \rangle_{\mathcal{H}}}.
\]
(In checking this we use Equation 6.1 and divergence of the denominator.)

Use the notation \( F_1 \preceq C, F_2 \) to mean \( C^{-1} F_1 \leq F_2 \leq C F_1 \). The ratio limit theorem for random walks (Proposition 2.7) says that there are \( \epsilon_n(g) \to 0 \) as \( n \to \infty \) with \( \Psi(g)p^n(g) = (1 + \epsilon_n(g))p^n(e) \). Substituting this into equation 4.4 gives
\[
\Upsilon_{c,s}(g) \preceq (1 + \epsilon'_s(g)) \frac{1}{\Psi(g)} \limsup_{k \to \infty} \frac{\sum_{n > M} (c \ast e)n t_k^{-n} p^n(e)}{\sum_{n > M} (c \ast e)n t_k^{-n} p^n(e)},
\]
with \( \epsilon'_M(g) = \sup_{n > M} |\epsilon_n(g)| \), and again using divergence of the denominator and boundedness of \( \sum_{n \leq M} (c \ast e)n t_k^{-n} p^n(e) \) for \( M \) fixed. Since \( M \) is arbitrary and \( \epsilon'_M(g) \to 0 \) the conclusion follows.

We now specialise to the free group \( F_{a,b} \) with free basis \( S = \{a, a^{-1}, b, b^{-1}\} \).

Consider the uniform random walk \( p(s) = 4^{-1} \) for each \( s \in S \). One can determine the radius of convergence in this case. (To do the calculation, it is helpful to represent returns to the identity in terms of Catalan numbers.) See also [23].

**Proposition 4.3** (Theorem 3 in [15]; Corollary 3.6. in [26]). For \( S^n, s, p \) given by the simple random walk on the free group \( G = F_{a,b} \) we have
\[
\gamma(\delta_c) = \sqrt{3} \over 2.
\]

There is a special class of functions \( \Psi : F_{a,b} \to \mathbb{R} \) called the spherical functions — the reference we use for this discussion is the exposition of Figà-Talamanca and Picardello [11]. By definition a spherical function \( \Psi : F_{a,b} \to \mathbb{R} \) is constant on every sphere (is radial) and is multiplicative on the convolutional algebra of radial functions (and one tends to normalise so that \( \Psi(e) = 1 \)). Alternatively, the spherical functions are precisely eigenfunctions \( p * \Psi = \gamma(\Psi) \Psi \) with \( p \) the uniform
probability on the free basis, and can be characterized in terms the Poisson-Kernel (we explain a particular case in section 14). Our construction of \( \phi_c[t] = \phi^{c,a}[t] \) and \( \Upsilon_c = \Upsilon^{c,A,a} \) is not optimized to be radial, but we claim that there is a natural linear combination that is spherical for the free group. As we have specialised to the symmetric setting we have \( \Upsilon_c = \Upsilon_{c,*} \).

**Lemma 4.4.** For any \( c \) the function \( \Upsilon = \sum B,b,D,d \in W_1 \Upsilon^{B,b,D,d} \) is spherical, and moreover

\[
\Upsilon(g) = C \left( 1 + \frac{|g|}{2} \right) 3^{-|g|/2}
\]

for some constant \( C > 0 \).

**Proof.** It is easy to see that \( \Upsilon \) is constant on spheres from

\[
\# \{ w_n \in S^n, u_m \in S^m : gw_n = u_m \} = \# \{ w_n \in S^n, u_m \in S^m : \hat{g}w_n = \hat{u}_m \},
\]

for each \( g, \hat{g} \) with \( |g| = |\hat{g}| \).

If a function \( \hat{\Psi} \), constant spheres, satisfies the equation \( p \ast \hat{\Psi} = r \hat{\Psi} \) then \( \hat{\Psi}(g) \) is completely determined by an induction from the identity term \( \hat{\Psi}(e) \). We already know from \([11]\) that the function \( \Psi(g) = \left( 1 + \frac{|g|}{2} \right) 3^{-|g|/2} \) has \( p \ast \Psi = 2^{-1}3^{1/2} \Psi \). It is enough to check that \( \Upsilon \) has \( p \ast \Upsilon = 2^{-1}3^{1/2} \Upsilon \).

Equation 4.3 implies that

\[
(4.5) \quad \Upsilon(g(w)) = \sum_{B,a',D,a'' \in W_1} \gamma^{[w]} p^{[w]} \nu^{a';D,a''} ([wB]).
\]

Consequently for any \( s \in S \) and letting \( g = g(w) \) we have

\[
\Upsilon(sg) = \gamma p^{-1} \sum_{B,a',D,a'' \in W_1} \gamma^{[w]} p^{-[w]} \nu^{a';D,a''} ([swB]).
\]

But also

\[
\sum_{D \in W_1} \nu^{a';D,a''} ([swB]) = \nu^{s^{-1};a''} ([wB]).
\]

In this way

\[
\sum_{s \in S} p(s) \Upsilon(sg) = \gamma \sum_{B,a',a'' \in W_1} \sum_{s \in S} \gamma^{[w]} p^{-[w]} \nu^{a';s^{-1},a''} ([wB]) = \gamma \Upsilon(g).
\]

It follows that \( \Upsilon \) is a constant multiple of \( \Psi \). \( \square \)

**Remark 4.5.** (1) We see in the proof of Lemma 4.4 that the coincidence between \( \Upsilon \) and a constant multiple of \( \Psi \) is entirely due to the eigenvalue. There is not an immediate comparison with the local limit theorem since it concerns \( \langle \rho(g) \phi[t] \ast \delta_c \rangle \) and not \( \langle \rho(g) \phi[t] \ast \delta_c, \phi[t] \ast \delta_c \rangle \).

(2) For the unacquainted, we include in section 14 that the curious-looking function \( \Psi(g) = \left( 1 + \frac{|g|}{2} \right) 3^{-|g|/2} \) is a matrix coefficient for the boundary representation.

Lemma 4.4 has consequences for the lower and upper bound on rate of decay.

**Corollary 4.6.** For each \( \epsilon > 0 \), a \( \mu^\epsilon \) typical point has

\[
\langle \rho(g(x_n \cdots x_{-1})) f, v \rangle \leq \frac{\|g(x_n \cdots x_{-1})\| (1 + \epsilon)^{n \sqrt{3}}}{\sqrt{3}^{u(x_n \cdots x_{-1})}} 2^n
\]

for \( n \) sufficiently large (depending on \( x \)).

For each \( \epsilon > 0 \), a \( \mu \) typical point has

\[
\langle \rho(g(x_n \cdots x_{-1})) f, v \rangle \leq \frac{(1 + \epsilon)^n \sqrt{3}^n}{2^n}
\]

for \( n \) sufficiently large (depending on \( x \)).
It is well-known that the drift \( l(p) \) for the simple random walk \( p \) on the free group on 2 generators is equal to \( 1/2 \) (using that \( l(p) = \int b(g, \xi) dp(g)d\nu(\xi) \) with \( b \) the Busemann additive cocycle — see Proposition 2.2. of [13]). That says that for a \( \mu \) typical \( x \) we have
\[
\lim_{N \to \infty} \frac{1}{n} |g(x_{-n} \cdots x_{-1})| = \frac{1}{2}.
\]
This gives in particular a lower bound on the amount of cancellation in \( g(x_{-n} \cdots x_{-1}) \). For a \( \mu^x \) typical point we have quantitatively more cancellation along a subsequence.

**Lemma 4.7.** A \( \mu^x \) typical point \( x \) has
\[
\limsup_{n \to \infty} \frac{1}{n} |g(x_{-n} \cdots x_{-1})| \leq \frac{2 \log 2 - \log 3}{\log 3} < \frac{1}{2}.
\]

**Proof.** We know from Lemma 3.11 together with Remark 3.12, and the symmetry of the probability, that a \( \mu^x \) typical \( x \) has
\[
\Upsilon_c(\bar{\Upsilon}(x_{-n} \cdots x_{-1})) \geq \sqrt{3}^{-n} 2^{-n} n^{-2}
\]
for all \( n > N_x \).

Since \( \bar{\Upsilon} \) is a linear combination of \( \Upsilon^{B,k;D,d}_c \) it follows in particular for \( \Upsilon^{A,a;A,a}_c \) that there is a constant \( C > 0 \) for which a \( \mu^x \) typical \( x \) has
\[
\bar{\Upsilon}(\bar{\Upsilon}(x_{-n} \cdots x_{-1})) \geq C \Upsilon^{A,a;A,a}_c(\bar{\Upsilon}(x_{-n} \cdots x_{-1})) \geq \sqrt{3}^{-n} 2^{-n} n^{-2}
\]
for \( n \) larger than some \( N_x \).

Now using the identity for \( \bar{\Upsilon} \) and writing \( g = \bar{\Upsilon}(x_{-n} \cdots x_{-1}) \), this gives for every \( \epsilon > 0 \) and \( N \) sufficiently large
\[
(\sqrt{3})^{-(n+|g|)} \geq (1 + \epsilon)^n 2^{-n}
\]
Equivalently
\[
\frac{|g|}{2} \log 3 + \frac{n}{2} \log 3 \leq n \log 2 + n \log(1 + \epsilon).
\]

Let \( \epsilon_k \) be a sequence with \( \epsilon_k \to 0 \) as \( k \to \infty \). Then we have a full measure set \( E \) for which
\[
\limsup_{n \to \infty} \frac{1}{n} |g(x_{-n} \cdots x_{-1})| \leq \frac{2 \log 2 - \log 3}{\log 3} + \frac{\log(1 + \epsilon_k)}{\log 3}.
\]
for every \( k \). The conclusion follows. (One also checks by computation that \( \frac{2 \log 2 - \log 3}{\log 3} < \frac{1}{2} \).)

5. One sided twisted measures: preliminary constructions

In sections 2 and 3 the notation is descriptive of the objects. We consider a fixed \( f \in \mathcal{H}_x \) and a fixed slowly increasing function \( \epsilon \) (that has to satisfy certain properties), and so their dependency is suppressed in the notation that follows. We de-clutter notation by writing \( \ast \) for \( \ast_p \), and \( Q[t], Q_{\leq N}[t] \in \mathcal{H} \) and \( \gamma(Q) \) in place of either \( \phi_x[t] \ast f, \phi_{c \leq N}[t] \ast f \) and \( \gamma(\phi \ast f) \) or \( \phi_x[t] \ast f, \phi_{c \leq N}[t] \ast f \) and \( \gamma(\phi^* \ast f) \) respectively. In this section we allow \( \Sigma^+ \) to be non-compact, however it should be noted that a convergence of the twisted measure for \( Q[t] = (\phi_{c \leq N}[t])^* \ast f \) will ultimately only be verified under the assumption that \( \Sigma^+ \) is compact.
5.1. Weighted Dirac mass construction.

Definition 5.1. (Approximating measures) Assume that the slowly increasing function $c : \mathbb{N} \to \mathbb{R}$ has
\[
\lim_{N \to \infty} \langle \phi_{c \leq N}[t] * f, Q_{\leq N}[t] \rangle \to \infty \text{ as } t \to \gamma(Q)
\]
and
\[
\lim_{N \to \infty} \langle f, Q[t] \rangle \to 0 \text{ as } t \to \gamma(Q).
\]

The normalising factor is $F(Q)(t) = \lim_{N \to \infty} \langle \phi_{c \leq N}[t] * f, Q_{\leq N}[t] \rangle$. Formally define, for $t > \gamma(Q)$ and $N \in \mathbb{N}$,
\[
\nu_{c,N}^Q[t] = \sum_{B \in \mathcal{W}} \sum_{n=1}^{N} t^{-n} c_n \sum_{v \in \mathcal{W}^n_{B,a}} R_n(v,x) \frac{\langle \rho(g(v)) f, Q_{\leq N}[t] \rangle}{F(Q)[t]} D(vx),
\]
in which $D(z)$ is the Dirac mass at $z \in \Sigma^+$, and $x \in \Sigma^+$ is chosen with $x \in [A]$ and $\sigma x$ not containing $A$ (this choice relates to a remainder term in Proposition 5.6 and relates to the choice of first returns in the proof of Theorem 6.6).

For technical reasons we will also make reference to the following family of measures, defined with respect to a group element $g \in G$ and slowly increasing function $d : \mathbb{N} \to \mathbb{R}_+$. Formally define, for $t > \gamma(Q)$ and $N \in \mathbb{N}$,
\[
(5.1) \nu_{d,N}^{Q^{-1}}[t] = \sum_{B \in \mathcal{W}} \sum_{n=1}^{N} t^{-n} d_n \sum_{v \in \mathcal{W}^n_{B,a}} R_n(v,x) \frac{\langle \rho(g(v)) f, Q_{\leq N}[t] \rangle}{F(Q)[t]} D(vx)
\]
for the same $x \in \Sigma^+$ as in Definition 5.1. (Recall that $Q$ is defined in terms of the slowly increasing function $c$.)

Remark 5.2. For $Q_{\leq N}[t] = \phi_{c \leq N}[t] * f$ let us remark that the well definedness of $\nu_{c,N}^Q[t]$ follows in the same way as checking for $\nu_N[t]$ given in equation 5.3. The proof is subsumed in Theorem 6.6.

One could check whether $\nu_{c,N}^Q[t]$ is well-defined for $Q_{\leq N}[t] = \phi_{c \leq N}[t] * f$ but ultimately we have no method to verify whether a limit in $N$ is well-defined, unless $\Sigma^+$ is compact.

Lemma 5.3. Let $g \in G$. For each $B$ we have that the linear functional $\nu_{c,N}^{Q,g}[t]$ has
\[
\sup_{t > \gamma(Q)} \sup_{N \in \mathbb{N}} \nu_{c,N}^{Q,g}[t](|B|) < \infty.
\]

If $T$ is transitive then
\[
\liminf_{t \to \gamma(Q)} \sup_{N \in \mathbb{N}} \nu_{c,N}^{Q,g}[t](|B|) = \liminf_{t \to \gamma(Q)} \nu_{c,N}^{Q,g}[t](|B|) > 0.
\]

In general we have
\[
\sup_{N \in \mathbb{N}} \nu_{c,N}^{Q,g}[t](|A|) = \lim_{N \to \infty} \nu_{c,N}^{Q,g}[t](|A|) = 1
\]
for all $t > \gamma(Q)$.

If the family $\nu_{c,N}^{Q,g}[t]$ is tight in $N \in \mathbb{N}$, for a fixed $t$, then $\nu_{c,N}^{Q,g}[t]$ converge to a limit measure, for a fixed $t$.

Proof. It is immediate that
\[
\nu_{c,N}^{Q,g}[t](|B|) = \frac{\langle \phi_{c \leq N}[t] * f, Q_{\leq N}[t] \rangle}{F(Q)[t]}.
\]
If $Q = \phi_c * f$ then we use Lemma 2.4 and the Cauchy-Schwarz inequality to get the upper bound
\[
\nu_{c,N}^Q[t][|B|] \leq \frac{\|\phi_c B[a][t] * f\|\|Q_c[t]\|}{F(Q)[t]} \leq \text{Const.}(B),
\]
using Lemma 2.6 in the last inequality. If $Q_c = \phi_c^* * f$ then we use Lemma 2.6 to get the upper bound
\[
\nu_{c,N}^Q[t][|B|] \leq \text{Const.}(B) \frac{\langle \phi_c A,a [t] * f, Q \leq N [t] \rangle}{F(Q)[t]} \leq \text{Const.}(B) F(Q)[t].
\]
The lower bound is seen using
\[
\nu_{c,N}^Q[t][|B|] = \frac{\langle \phi_c B,a [t] * f, Q \leq N [t] \rangle}{F(Q)[t]}
\]
\[
\geq \text{Const.}(B)^{-1} \frac{\langle \phi_c A,a [t] * f, Q \leq N [t] \rangle}{F(Q)[t]}
\]
\[
\geq \text{Const.}(B)^{-1} \frac{\langle \phi_c A,a [t] * f, Q \leq N [t] \rangle}{F(Q)[t]} - \text{Const.}(B)^{-1} \frac{\langle \phi_c A,a [t] * f, Q \leq N [t] \rangle}{F(Q)[t]}
\]
We check that the linear functionals $\nu_{c,N}^q Q[t]$ converge as $N \to \infty$. (We check that they are Cauchy.) In the first case $Q = \phi_c * f$. For arbitrary $N, M$ we have
\[
\nu_{c,N}^q Q[t][|B|] - \nu_{c,M}^q Q[t][|B|]
\]
\[
= \frac{\langle \phi_c B,a [t] * f - \phi_c B,a [t] * f, \rho(g)^{-1} \phi_c[t] * f \rangle}{\|\phi[t]_c * f\|^2}
\]
\[
\leq \frac{\|\phi_c B,a [t] * f - \phi_c B,a [t] * f\|}{\|\phi[t]_c * f\|}.
\]
Strong convergence (Lemma 2.4) gives that $\|\phi_c B,a [t] * f - \phi_c B,a [t] * f\| \leq \epsilon$ for $N, M$ sufficiently large.
In the second case $Q = \phi_c^* * f$. For each $g$ we use Lemma 2.6 to show that
\[
\nu_{c,N}^q Q[t][|B|] - \nu_{c,M}^q Q[t][|B|]
\]
\[
\leq \text{Const.}(B)^{-1} \frac{\langle \phi_c B,a [t] * f, Q \leq N [t] \rangle}{F(Q)[t]}
\]
The result follows upon verifying the claim that for a fixed $t > \gamma(Q)$ we have
\[
\frac{\langle \phi_c N - r \leq n \leq N [t] * f, \phi_c^* N [t] * f \rangle}{F(Q)[t]} \to 0 \text{ as } N \to \infty.
\]
Were false, there would be a sequence $(N_j)_{j \in \mathbb{N}}$ with
\[
\frac{\langle \phi_c N - r \leq n \leq N [t] * f, \phi_c^* N [t] * f \rangle}{F(Q)[t]} > C,
\]
for some constant $C > 0$. In particular for $J > j$,
\[
\frac{\langle \phi_c N - r \leq n \leq N [t] * f, \phi_c^* N [t] * f \rangle}{F(Q)[t]} > C.
\]
Now assuming $N_j - N_{j-1} > r$ we would conclude
\[
\frac{\langle \phi_c n \leq N_J [t] * f, \phi_c^* N_J [t] * f \rangle}{F(Q)[t]} \geq \sum_{j=1}^{J} \frac{\langle \phi_c N - r \leq n \leq N_J [t] * f, \phi_c^* N_J [t] * f \rangle}{F(Q)[t]} > JC.
\]
This cannot hold for arbitrarily many $J$ whilst $t$ is fixed.
Now if the family of $\nu^{\sigma^{-1}_c Q}_N[t]$ is tight in $N$ we deduce that the limit linear functional is a measure. (See corollary 6.7.)

We give the definition of a twisted measure conditional on the existence of limits of $\nu^{\sigma^{-1}_c Q}_N[t]$.

**Definition 5.4.** Let $(t_q)_{q \in \mathbb{N}}$ be a sequence with $\lim_{q \to \infty} t_q = \gamma(Q)$ and for which $\nu^{\sigma^{-1}_c Q}_{t_q} \ (t_q)$ (exists and) converges as $q \to \infty$ for every $q$. We call $\nu_\phi$ the limit of $\nu^{\sigma^{-1}_c Q}_{t_q}$ as $q \to \infty$ a $Q^\ast \ast \phi$ one-sided (twisted) measure. We also use the notation $\nu^{\phi \sigma^0}_{t_q}$ for the limit of $\nu^{\phi \sigma^0}_{t_q}$ as $q \to \infty$.

The outcome of subsection 6.2 is to show that limit points exist (under certain hypotheses). The outcome of section 7 is to check that these measures agree with the notion of twisted in Definition 7.1. Before proceeding any further we show how to obtain the eigenmeasure $\nu$ in this way. Define, for $t > 1$,

$$
\nu_N[t] = \sum_{B \in \mathcal{W}_1} \sum_{n=1}^{N} t^{-n} \sum_{u \in \mathcal{W}_n^{B,a}} R_n(u\zeta) \frac{1}{\xi[t]} D(u\zeta),
$$

recalling that $\xi[t] = C^A\alpha[t]$. In due course we will check that these measure are well-defined, and show the existence of an accumulation point.

**Lemma 5.5.** Suppose that $\nu[t]$ is a weak* limit of $\nu_N[t]$ as $N \to \infty$ and $\hat{\nu}$ is a weak* limit $\nu_N[t_k]$ as $k \to \infty$. We have

$$
L^* \hat{\nu} = \hat{\nu}.
$$

**Proof.** Let $w = w_1 \cdots w_r$ be arbitrary. First observe that

$$
L(1[w])(z) = \sum_{u \in \mathcal{W}_1:t(u,x_0) = 1} R(u\zeta) 1[w_1 w_2 \cdots w_r](u\zeta) = R(w_1 z) 1[w_1 \cdots w_r](z)
$$

and so $L(1[w])(z)$ is bounded and continuous. We have, for any presumed limit along $t_k \to 1$ as $k \to \infty$,

$$
\hat{\nu}(L(1[w])) = \lim_{k \to \infty} \lim_{N \to \infty} \sum_{B \in \mathcal{W}_1} \sum_{n=1}^{N} t_k^{-n} \sum_{u \in \mathcal{W}_n^{B,a}} R_n(u\zeta) \frac{1}{\xi[t]} R(w_1 u\zeta) 1[w_2 \cdots w_n](u\zeta)
$$

$$
= \lim_{k \to \infty} \sum_{b \in \mathcal{W}_1} \sum_{n=1}^{\infty} t_k^{-n} \sum_{u \in \mathcal{W}_n^{b,a}} R_n(w_1 \cdots w_n u\zeta) \frac{1}{\xi[t]}
$$

$$
= \lim_{k \to \infty} \lim_{N \to \infty} \sum_{b \in \mathcal{W}_1} \sum_{n=2}^{N+1} t_k^{-(n-1)} \sum_{u \in \mathcal{W}_n^{b,a}} R_{n+1}(u_1 \cdots w_n u\zeta) \frac{1}{\xi[t]}
$$

$$
= \hat{\nu}(1[w])
$$

using the fact that $\xi[t]$ diverges to obtain the last equality. \hfill \square

### 2. Basic estimates for the approximating measures.

**Proposition 5.6.** (The main equality). Let $wB \in \mathcal{W}$ with $|w| = r$. For any $k \geq r$ we have

$$
\nu^{\sigma^{-1}_c Q}_N[t](t^r R_r^{-1} 1_{[wB]}) = \sum_{n=1}^{N-r} t^{-n} c_n \sum_{u \in \mathcal{W}_n^{b,a}} \frac{\langle \rho(\phi(w)) \rho(\phi(v)) f, \rho(\phi)^{-1} Q_{c \leq N}[t] \rangle}{\langle \phi_{c \leq N}[t] * f, Q_{c \leq N}[t] \rangle} R_n(u\zeta)
$$

$$
+ \sum_{n=1}^{r} t^{-n} c_n \sum_{b \in \mathcal{W}_1} \sum_{u \in \mathcal{W}_n^{b,a}} R_n(u\zeta) \frac{\langle \rho(\phi(v)) f, Q_{c \leq N}[t] \rangle}{\langle \phi_{c \leq N}[t] * f, Q_{c \leq N}[t] \rangle} t^r 1_{[wB]}(v\zeta)
$$
Recall that Proposition 5.6 gives an expression for the term $\nu_n$ that the sequence of approximating measures is tight we will need to check some

$$
\nu_{c,N}^{-1}Q[t](t^r R_r^{-1}I_{[wA]}) = \sum_{n=r+1}^N t^{-n}c_n \sum_{v_1,\ldots,v_r \in \mathcal{W}_n^{d,a}} R_n(v_1 \cdots v_r x) \frac{\langle \rho(g(v_1 \cdots v_r x))f, Q_{c \leq N}[t] \rangle}{\langle \phi_{c \leq N}[t] * f, Q_{c \leq N}[t] \rangle} \frac{t^r \|_{[wA]}(v_1 \cdots v_r x)}{R_r(v_1 \cdots v_r x)}
$$

and

$$
\sum_{n=r+1}^N t^{-n}c_n \sum_{v_1,\ldots,v_r \in \mathcal{W}_n^{d,a}} R_n(v_1 \cdots v_r x) \frac{\langle \rho(g(v_1 \cdots v_r x))f, Q_{c \leq N}[t] \rangle}{\langle \phi_{c \leq N}[t] * f, Q_{c \leq N}[t] \rangle} \frac{t^r \|_{[wA]}(v_1 \cdots v_r x)}{R_r(v_1 \cdots v_r x)}
$$

Proof. Write $wB = duB$ where $d \in W$. We have

$$
\nu_{c,N}^{-1}Q[t](t^r R_r^{-1}I_{[wA]})
$$

In Definition 12.5 we gave a local Gibbs definition which is satisfied by an $R$ conformal measure. Eventually Lemma 8.1 will give a RHS local Gibbs inequality for limits of the twisted measure when $Q = \phi_c * f$. In order to even check that the sequence of approximating measures is tight we will need to check some approximation of the RHS local Gibbs inequality.

Lemma 5.7 ($\gamma$-heavy RHS local Gibbs when $Q = \phi_c * f$). Suppose that $Q = \phi_c * f$. For each $B$ there is a constant $C(B)$ so that for any $BwA \in W$ we have

$$
\nu_{c,N}^{-1}Q[t](|wA|) \leq R_r(wA) t^{-r} C(B) \sup_{n \in \mathbb{N}} \left( \frac{c_{n+r}}{c_n} \right),
$$

where $r = |w|$ and $g \in G$ is arbitrary.

Proof. Write $\gamma = \gamma(Q)$. Strong convergence tells us that $F(Q)[t] = \|\phi_c[t] * f\|^2$. Recall that Proposition 5.6 gives an expression for the term $\nu_{c,N}^{-1}Q[t](\gamma^r R_r^{-1}I_{[wA]}).$

As we chose $x$ to not contain $A$ this forces that the second series is 0 except at $n = r$ from which

$$
\sum_{n=1}^r t^{-n}c_n \sum_{b \in \mathcal{W}_1 \in \mathcal{W}_n^{d,a}} R_n(vx) \frac{\langle \rho(g(v)x)f, \phi_{c \leq N}[t] * f \rangle}{\| \phi_c[t] * f \|^2} \frac{t^r \|_{[wA]}(vx)}{R_r(vx)}
$$

We give an upper bound for the first series in Proposition 5.6 by

$$
\left( \sup_{n \geq r} \frac{c_{n+r}}{c_n} \right) \sum_{n=1}^N t^{-n}c_n \sum_{v_1,\ldots,v_r \in \mathcal{W}_n^{d,a}} \frac{\langle \rho(g(v_1 \cdots v_r x))f, \phi_{c \leq N}[t] * f \rangle}{\phi_c[t] * f} R_n(vx)
$$

$$
= \left( \sup_{n \geq k} \frac{c_{n+r}}{c_n} \right) \frac{\langle \rho(g(v_1 \cdots v_r x))f, \phi_{c \leq N}[t] * f \rangle}{\phi_c[t] * f} R_n(vx)
$$

Using local Hölder continuity and recalling that we fixed $x \in [A]$ gives

$$
\nu_{c,N}^{-1}Q[t](\gamma^r R_r^{-1}I_{[wA]}) \geq \text{const.}(wA) \gamma^r R_r^{-1}(wA) \nu_{c,N}^{-1}Q[t](|wA|).
$$

Recalling const.$(wA)$ does not depend on $w$, the Lemma follows. \qed
6. One-sided (twisted) measure: existence

In this section we collect the machinery to show existence of the one-sided (twisted) measures. In particular we must check existence of the slowly increasing functions, and will verify the existence of accumulation points by checking that the family of approximating measures are tight.

6.1. The slowly increasing function. The assumption that $R$ is recurrent (and consequent divergence of the return series 12.11) is useful in the construction of the $R$-conformal measure (Lemma 5.5). We can always force a series to diverge at its radius of convergence by increasing the summands with a slowly increasing function. This observation is used widely in the literature on conformal measures (for instance [22], [24], [6]). The result is stated as Proposition 15.1.

Lemma 6.1. There is a slowly increasing $c$ with

\[
\int (\rho(h)f,f)d\phi_c^*[t] \cdot \phi_c[t] < \infty \text{ for } t > \gamma(\phi \ast \rho f)
\]

and

\[
\int (\rho(h)f,f)d\phi_c^*[t] \cdot \phi_c[t] \to \infty \text{ as } t \to \gamma(\phi \ast \rho f).
\]

Proof. In particular the Lemma asks us to check that, for any slowly increasing $c$ we have

\[
\int (\rho(h)f,f)d\phi_c^*[t] \cdot \phi_c[t] < \infty \iff \int (\rho(h)f,f)d\phi_c^*[t] \cdot \phi_c[t] < \infty.
\]

This is seen immediately from the following. Since $c$ is slowly increasing, for each $\delta < 1$ there is $C_\delta$ with

\[
\langle \phi_{c \leq N}[t] * f, \phi_{c \leq N}[t] * f \rangle \leq \sum_{m \leq N} \sum_{n \leq N} C_\delta \langle \phi[\delta^{-1}t] * f, \phi[\delta^{-1}t] * f \rangle
\]

\[
\leq C_\delta \langle \phi[\delta^{-1}t] * f, \phi[\delta^{-1}t] * f \rangle < \infty.
\]

Now we check that $\gamma := \gamma(\phi \ast \rho f)$ is the abscissa of convergence of the real series

\[
\pi[t] = \sum_{N=1}^{\infty} t^{-N} \sum_{m+n=N} \sum_{V \in \mathcal{W}_{m,n}^{A_s}} \sum_{v \in \mathcal{W}_{m}^{A_s}} R_m(Vx) R_n(vx) \langle \rho(g(V))^{-1} \rho(g(v)) f, f \rangle.
\]

Write

\[
\pi_{\leq N}[t] = \sum_{K \leq N} t^{-K} \sum_{m+n=K} \sum_{V \in \mathcal{W}_{m,n}^{A_s}} \sum_{v \in \mathcal{W}_{m}^{A_s}} R_m(Vx) R_n(vx) \langle \rho(g(V))^{-1} \rho(g(v)) f, f \rangle
\]

Set $\phi_n[t](g) = \sum_{w \in \mathcal{W}_{m,n}^{A_s}} g(w) R_n(wx)$. It is easy to see that

\[
\pi_{\leq N}[t] \leq \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} \langle \phi_n[t] * f, \phi_m[t] * f \rangle = \langle \phi_{\leq N}[t] * f, \phi_{\leq N}[t] * f \rangle
\]

On the other hand

\[
\langle \phi_{\leq N}[t] * f, \phi_{\leq N}[t] * f \rangle = \sum_{k \leq 2N} \sum_{m+n=k} \langle \phi_n[t] * f, \phi_m[t] * f \rangle
\]

\[
\leq \sum_{k \leq 2N} \sum_{m+n=k} \langle \phi_n[t] * f, \phi_m[t] * f \rangle = \pi_{\leq 2N}[t]
\]

It follows that the abscissa of convergence of $\pi[t]$ is $\gamma$.

Using Proposition 15.1, choose $d_\gamma$ slowly increasing so that

\[
\pi^d[t] = \sum_{N=1}^{\infty} t^{-N} d_\gamma \sum_{m+n=N} \sum_{V \in \mathcal{W}_{m,n}^{A_s}} \sum_{v \in \mathcal{W}_{m}^{A_s}} R_m(Vx) R_n(vx) \langle \rho(g(V))^{-1} \rho(g(v)) f, f \rangle
\]
diverges at $\gamma$. Now write $c_n = d_{2n}$, and observe that $c$ inherits the slowly increasing property from $d$. Then for $m + n = N$ we have $c_m c_n = d_{2m} d_{2n} \geq d_N$, giving

$$
\pi^d_{\leq N}[t] = \sum_{K \leq N} t^{-K} d_K \sum_{m + n = K} \sum_{V \in \mathcal{W}_m^{A_a}} \sum_{v \in \mathcal{W}_n^{A_a}} R_m(Vx)R_n(vx) (\rho(\mathfrak{g}(V))^{-1} \rho(\mathfrak{g}(v)))f, f
\leq \sum_{K \leq N} t^{-K} \sum_{m + n = K} c_m c_n \sum_{V \in \mathcal{W}_m^{A_a}} \sum_{v \in \mathcal{W}_n^{A_a}} R_m(Vx)R_n(vx) (\rho(\mathfrak{g}(V))^{-1} \rho(\mathfrak{g}(v)))f, f
\leq \langle \phi_c[t] * f, \phi_c[t] * f \rangle.
$$

The conclusion follows. \hfill \square

The divergence statement for $\phi^*$ involves more attention. We begin by verifying that a slowly increasing function does not increase the convergence parameter.

**Lemma 6.2.** Let $c : \mathbb{N} \to \mathbb{R}_+$ be subexponentially increasing. Then

$$
\sup_{N \in \mathbb{N}} \langle \phi_{c \leq N}[t] * f, f \rangle < \infty \iff \sup_{N \in \mathbb{N}} \langle \phi_{\leq N}[t] * f, f \rangle < \infty.
$$

If $c$ is slowly increasing then

$$
\sup_{N \in \mathbb{N}} \langle \phi_{c \leq N}[t] * f, \phi_{c \leq N}[t] * f \rangle < \infty \iff \sup_{N \in \mathbb{N}} \langle \phi_{\leq N}[t] * f, \phi_{c \leq N}[t] * f \rangle < \infty.
$$

If $T_a$ is transitive then $\gamma(\phi^* * f) = \gamma(f)$.

**Proof.** Fix $f$. To begin with we only ask that $c : \mathbb{N} \to \mathbb{R}_+$ has subexponential growth $\limsup_{n \to \infty} \frac{1}{n} \log c_n = 0$. (If $c_n$ is slowly increasing then $\log c_n - \log c_k \leq \log \frac{c_n}{c_{n-1}} + \cdots + \log \frac{c_k+1}{c_k} \leq (n-k) \log \gamma$ for $\gamma$ arbitrarily close to 1 and $k = k_\gamma$.)

We have

$$
\int (\rho(g)f, f) d\phi_{\leq N}[t](g) = \langle \phi_{c \leq N}[t] * \rho f, f \rangle =: \eta_{c \leq N}[t],
$$

and

$$
\int (\rho(g)f, f) d\phi_{c \leq N}[t](g) = \langle \phi_{c \leq N}[t] * \rho f, f \rangle =: \eta_{c \leq N}[t].
$$

Note $\eta_c[t] = \lim_{N \to \infty} \eta_{c \leq N}[t]$, $\eta^f[t] = \lim_{N \to \infty} \eta_{\leq N}[t]$ are real power series in $t^{-1}$. For each $\delta < 1$ there is $C_\delta$ with $\eta_c[t] \leq C_\delta \eta_c[\delta^{-1}t]$. The first statement follows.

Now set

$$
\langle \phi_{\leq N}[t] * \phi_{\leq N}[t] * \rho f, f \rangle =: \alpha_{\leq N}[t],
$$

and

$$
\langle \phi_{c \leq N}[t] * \phi_{c \leq N}[t] * \rho f, f \rangle =: \alpha_{c \leq N}[t].
$$

To be clear $\alpha_c[t] = \lim_{N \to \infty} \alpha_{c \leq N}[t]$ does not have form of a real power series in $t^{-1}$, it is

$$
\alpha_c[t] = \lim_{N \to \infty} \sum_{K=1}^{\infty} t^{-K} \sum_{m+n=K, m \leq N, n \leq N} \sum_{V \in \mathcal{W}_m^{A_a}} \sum_{v \in \mathcal{W}_n^{A_a}} R_m(uv)R_n(vx) (\rho(\mathfrak{g}(u))\rho(\mathfrak{g}(v)))f, f.
$$

However setting

$$
\pi_{c \leq N}[t] = \sum_{K=1}^{N} t^{-K} \sum_{m+n=K} c_m c_n \sum_{V \in \mathcal{W}_m^{A_a}} \sum_{v \in \mathcal{W}_n^{A_a}} R_m(uv)R_n(vx) (\rho(\mathfrak{g}(u))^{-1} \rho(\mathfrak{g}(v)))f, f,
$$

we have that $\pi_c = \lim_{N \to \infty} \pi_{c \leq N}[t]$ is a real power series in $t^{-1}$, and we check that

$$
\pi_{c \leq N}[t] \leq \alpha_{c \leq N}[t] \leq \pi_{c \leq 2N}[t].
$$

Assuming that $Aa$ are admissible we have

$$
\pi_{c \leq 2N}[t] \leq \eta_{c+2N}[t].
$$

(6.1)
By assumption \(c_n\) is slowly increasing, from which we deduce that \(\log(c \ast c)_n \leq \log n + \log c_n\) and so \(c \ast c\) grows subexponentially. Using the first of the lemma we deduce that for \(t > \gamma(\phi \ast f)\) we have \(\alpha_c[t]\) is finite.

We conclude by mentioning that

\[
\eta[t] \leq C\alpha_c[t]
\]

follows when \(T_a\) is transitive. This tells us that \(\gamma(\phi^* \ast f) = \gamma(f)\).

We are now ready to prove the divergence statements. Let us mention separately the case \(f = \delta_e\) which follows easily.

**Proposition 6.3.** Assume that \(T_a\) is transitive. There is \(c\) with

\[
\lim_{N \to \infty} \langle \phi_{c \leq N}[t] \ast \delta_e, \phi_{c \leq N}[t] \ast \delta_e \rangle \to \infty \text{ as } t \to \gamma(\delta_e)
\]

and

\[
\lim_{N \to \infty} \frac{\langle \phi_{c \leq N}[t] \ast \delta_e, \phi_{c \leq N}[t] \ast \delta_e \rangle}{\langle \phi_{c \leq N}[t] \ast \delta_e \rangle} \to 0 \text{ as } t \to \gamma(\delta_e).
\]

**Proof.** Note that the transitivity hypothesis ensures that \(\gamma(\phi^* \ast \delta_e) = \gamma(\delta_e)\).

Choose \(c\) to be a slowly increasing function with \(\langle \phi_{c}[t] \ast \delta_e, \delta_e \rangle \to \infty \text{ as } t \to \gamma\) (recall that \(\eta[t]\) in the proof of Lemma 6.2 is a real power series in \(t^{-1}\)). Then immediately we have \(\lim_{N \to \infty} \langle \phi_{c \leq N}[t] \ast \phi_{c \leq N}[t] \ast \delta_e, \delta_e \rangle \to \infty \text{ as } t \to \gamma\). We have that

\[
\langle \phi_{c \leq N}[t] \ast \delta_e, \phi_{c \leq N}[t] \ast \delta_e \rangle \geq \langle \phi_{c \leq N}[t] \ast \delta_e, \delta_e \rangle \langle \phi_{c \leq N}[t] \ast \delta_e \rangle.
\]

It follows that

\[
\lim_{N \to \infty} \frac{\langle \phi_{c \leq N}[t] \ast \delta_e, \phi_{c \leq N}[t] \ast \delta_e \rangle}{\langle \phi_{c \leq N}[t] \ast \delta_e \rangle} = \lim_{N \to \infty} \frac{\langle \phi_{c}[t] \ast \delta_e, \delta_e \rangle}{\langle \phi_{c}[t] \ast \delta_e \rangle^2} \to 0 \text{ as } t \to \gamma(\phi^* \ast \delta_e).
\]

\(\square\)

In general we use strong positive recurrence to check that the convolution \(\phi_{c \leq N}[t] \ast \phi_{c \leq N}[t]\) “is bigger” than \(\phi_{c \leq N}[t]\). In order to do this we need to use the second outcome of Proposition 15.1 that says that we can choose \(c\) with \(c_{n+k} \leq c_n c_k\) for all \(n, k \in \mathbb{N}\). This helps us estimate the convolution of \(c\) with itself.

**Lemma 6.4.** Assume that \(T_a\) is transitive. There is \(c\) with

\[
\lim_{N \to \infty} \langle \phi_{c \leq N}[t] \ast f, \phi_{c \leq N}[t] \ast f \rangle \to \infty \text{ as } t \to \gamma(f)
\]

and

\[
\lim_{N \to \infty} \frac{\langle \phi_{c \leq N}[t] \ast f, \phi_{c \leq N}[t] \ast f \rangle}{\langle \phi_{c \leq N}[t] \ast f \rangle} \to 0 \text{ as } t \to \gamma(f).
\]

**Proof.** The number of ways a word \(w \in \mathcal{W}_m^{A \ast a}\) can be written as \(w = uw\), for words in \(u \in \mathcal{W}_n^{A \ast a}\), \(v \in \mathcal{W}_k^{A \ast a}\) with \(n + k = m\), depends on the number of times an orbit in \(w\) returns to \(Aa\). We have

\[
\phi_{c \leq N}[t] \ast \phi_{c \leq N}[t] \geq C^{-1} \sum_{m=1}^{N} t^{-m} \sum_{w \in \mathcal{W}_m^{A \ast a}} R_m(wx) \delta_{2}(w) \sum_{j \leq N : w_{m-j} = Aa} c_{j} c_{m-j}
\]

\[
\geq C^{-1} \sum_{m=1}^{N} t^{-m} c_m \sum_{w \in \mathcal{W}_m^{A \ast a}} R_m(wx) \delta_{2}(w) \# \{j \leq N : w_{m-j} = Aa\}.
\]

Fix \(M\). Consider \(\mathcal{W}_m^{A \ast a}(< M)\) those words with \(\# \{j \leq N : w_{m-j} = Aa\} < M\). By strong positive recurrence we have

\[
\sum_{m=1}^{N} c_m t^{-m} \sum_{w \in \mathcal{W}_m^{A \ast a}(< M)} R_m(wx) = A_M(t)
\]
converges for \( t \in (\gamma(SPR) + \epsilon, \infty) \) and in particular \( \sup_{t \in [\gamma(f), \gamma(f) + 1]} A_M(t) < \infty \).

We also note that
\[
\sum_{m=1}^{N} t^{-m} c_m \sum_{w \in W_{A,s}(<M)} R_m(\omega) \langle \delta_{\omega(w)} * f, f \rangle \leq A_M(t).
\]

It follows that
\[
\limsup_{N \to \infty} \frac{\langle \phi_{c \leq N}[t] \ast \phi_{c \leq N}[t] \ast f, f \rangle}{\langle \phi_{c}[t] \ast f, f \rangle} \geq C^{-1} M \langle \phi_{c \leq N}[t] \ast f, f \rangle - C^{-1} A_M(t),
\]

i.e.
\[
\limsup_{N \to \infty} \frac{\langle \phi_{c \leq N}[t] \ast \phi_{c \leq N}[t] \ast f, f \rangle}{\langle \phi_{c}[t] \ast f, f \rangle} \geq C^{-1} M - C^{-1} A_M(t).
\]

Using the divergence of \( \langle \phi_{c}[t] \ast f, f \rangle \) gives the conclusion.

\[\square\]

**Remark 6.5.** If \( d, d : \mathbb{N} \to \mathbb{R} \) have that \( \lim_{n \to \infty} d_n/c_n = 1 = \lim_{n \to \infty} d_n/c_n \) then for the \( c \) in Lemma 6.4 we also have
\[
\lim_{N \to \infty} \frac{\langle \phi_{c}[t] \ast f, f \rangle}{\langle \phi_{c \leq N}[t] \ast f, f \rangle} \to 0 \text{ as } t \to \gamma(f).
\]

### 6.2. Tightness results

If \( \Sigma^+ \) is compact then any collection of measures with bounded mass is tight, and by Lemma 5.3 we know this for \( Q = \phi^* \ast f \) and \( \nu^g \phi^* \).

We now let \( \Sigma^+ \) be a countable Markov shift and assume that \( R \) is strongly positively recurrent. We can only handle the case \( Q = \phi \ast f \).

**Theorem 6.6.** Assume that \( Q[t] = \phi[t] \ast f \). Assume that
\[
t \mapsto \sum_{r=1}^{\infty} \sup_{n \in \mathbb{N}} \left( \frac{c_{n+r}}{c_r} \right) \sum_{w \in W_{A,s}(*)} R_r(\omega)
\]
converges at \( t = \gamma(Q) \). For every \( B \) and for every \( \epsilon > 0 \) there is a compact set \( K \) for which
\[
\nu_{c,N}^{g^{-1}Q}[t](\Sigma^+ \cap [B]) \leq \epsilon
\]
and
\[
\nu_N[t']((\Sigma^+ \cap [B]) \leq \epsilon
\]
for every \( N \in \mathbb{N} \), \( g \in G \) and \( t > \gamma(Q), t' > 1 \).

**Corollary 6.7.** Assume that \( \gamma(Q) > \gamma(SPR) \) and that \( c \) is subexponentially increasing. For each \( t > \gamma(Q) \) there are measures \( \nu_{c,N}^{g^{-1}Q}[t] \) on \( \Sigma^+ \), finite on cylinders, with \( \nu_{c,N}^{g^{-1}Q}[t] \to \nu_c^{g^{-1}Q}[t] \) as \( N \to \infty \) in the weak* topology. There is a sequence \( t_k \to \gamma(Q) \) and measures \( \nu_{c,N}^{g^{-1}Q} \) on \( \Sigma^+ \), finite on cylinders, with \( \nu_{c,N}^{g^{-1}Q}[t_k] \to \nu_c^{g^{-1}Q} \) as \( k \to \infty \) in the weak* topology. In addition \( \nu_c^{Q}([A]) = 1 \).

**Proof.** The hypotheses to the corollary imply that for \( \beta < 1 \)
\[
\sum_{r=1}^{\infty} \sup_{n \in \mathbb{N}} \left( \frac{c_{n+r}}{c_r} \right) \sum_{w \in W_{A,s}(*)} R_r(\omega) \leq C \sum_{r=1}^{\infty} (\beta t)^{-r} \sum_{w \in W_{A,s}(*)} R_r(\omega)
\]
which converges for \( \beta \gamma(Q) > \gamma(SPR) \).

One does have to be careful about extracting accumulation points when \( \Sigma^+ \) is non-compact — it is the tightness result of Theorem 6.6 that implies the existence of accumulation points. Let us sketch of the details of this well-known mechanism. Suppose \( m_k \) are tight in \( k \in \mathbb{N} \). For a fixed \( B \) and compact set \( \mathcal{K}_n \) in the tightness criterion, we have that the measures \( m_k \) restricted to \( \mathcal{K}_n \cap [B] \) have an accumulation point that is a positive measure. So, for some subsequence \( m_{k_{n}(q)} \) converge as \( q \to \infty \) to a positive measure. Using nesting of \( \mathcal{K}_n \) we may assume that \( k_n(q) \) is a subsequence of \( k_{n-1}(q) \). Then setting \( q_n = k_n(n) \) we have that \( m_{q_n} \) converges for any \( \mathcal{K}_n \), and in particular the limit measure is well-defined on the union, which is
We use a similar diagonal sequence have convergence along a sequence which works for all $B$. We use a similar diagonal sequence to deduce convergence along a sequence $t_k \to \gamma(Q)$ which works for all $g$. To see that the limit is finite on cylinders, we use the first part of Lemma 5.3 which tells us that the measure of a cylinder is bounded from above in $t$.

We proceed in a similar fashion as Sarig [20]. In our case we rely on Lemma 5.7 to estimate the measure of cylinders.

Without loss of generality we may assume that $\Sigma^+ \subset \mathbb{N}^\mathbb{N}$; that is $x \in \Sigma$ is bounded from above in the local Hölder constant.

We write $\hat{c} : \mathbb{N} \to \mathbb{R}$

$$\hat{c}_r = \sup_{n \in \mathbb{N}} \left( \frac{c_{n+r}}{c_{r}} \right)$$

We write

$$\eta^A[t] := \sum_{n=1}^{\infty} t^{-n} \hat{c}_n \sum_{w \in A_n} R_n(Aw_1 \cdots w_{n-1}x),$$

and

$$\xi^B[t] := \sum_{n=1}^{\infty} t^{-n} \hat{c}_n \sum_{w \in B_n} R_n(Bw_1 \cdots w_{n-1}x),$$

recalling that $x \in [A]$ (see Definition 5.1).

The set $\mathcal{K}((M_p))$ is compact. And

$$\Sigma^+ - \mathcal{K}((M_p)) = \{ x \in \Sigma : \exists q \in \mathbb{N} x_q > M_q \}.$$

We will always assume that $M_k \leq M_k+1$.

For brevity we write $m_N[t] = \nu_{c,N}^{g^{-1}Q}[t]$. The only tool we use is Lemma 5.7 which applies uniformly in $g$.

Claim 6.8. Assuming $M_p$ are large enough, if $z \in \text{supp}(m_N[t])$ and $z \in (\Sigma - \mathcal{K}((M_p))) \cap [B]$, it must be that either

1. $z \in [u]$, with $u \in \mathcal{B}(\geq M_1)$, or
2. $z \in [uvw]$ with $u \in \mathcal{B}$, $v \in \mathcal{A}^j$, and $w \in \mathcal{A}(\geq M_j)$, for some $j \in \mathbb{N}$.

Recall the constant $C(B)$ appearing in Lemma 5.7, which we may assume exceeds the local Hölder constant.

Claim 6.9. We have

1. For $u \in \mathcal{B}(\geq M_1)$, $|u| = r + 1$,

$$m_N[t](|u|) \leq C(B)t^{-r} \hat{c}_r R_r(ux)$$

j. For $u \in \mathcal{B}(\geq M_1)$, $|u| = r + 1$, $v \in \mathcal{A}^j$, $|v| = p$, $w \in \mathcal{A}_k(\geq M_j)$, for some $j \in \mathbb{N}$;

$$m_N[t](|uvw|) \leq C(B)^{j} t^{-k} \hat{c}_k R_k(Awx)t^{-r} \hat{c}_r R_r(ux)t^{-p} \hat{c}_p R_p(Avx)$$

We use that $C(B)$ exceeds the local Hölder constant and that $\hat{c}_{m+k} \leq \hat{c}_k \hat{c}_m$. 
Claim 6.10. We have

1: 

$$m_N[t](\{[u] : u \in B_k(\geq M_1), k \in \mathbb{N}\}) \leq C(B)^2 \sum_{k \in \mathbb{N}} t^{k-1} \hat{c}_{k-1} \sum_{w \in B_k(\geq M_1)} R_{k-1}(wx)$$

2: 

$$m_N[t](\{[uvw] : v \in A^j, u \in A_k(\geq M_j), j, k \in \mathbb{N}\}) \leq C(B)^2 \xi^B(t) (C(B)\eta^A[t])^{2j+1} \sum_{k \in \mathbb{N}} t^k \hat{c}_k \sum_{w \in A_k(\geq M_k)} R_k(Awx)$$

Claim 6.11. For every \( \lambda > 0 \) there is choice of \( K_\lambda \) (uniform in \( t > \gamma(Q) \)) so that

$$\sum_{k \in \mathbb{N}} t^k \hat{c}_k \sum_{w \in A_k(\geq K_\lambda)} R_k(wx) \leq \lambda$$

and

$$\sum_{k \in \mathbb{N}} t^k \hat{c}_k \sum_{w \in B_{k+1}(\geq K_\lambda)} R_k(wx) \leq \lambda.$$

Proof of Theorem 6.6. Let \( \epsilon > 0 \). For \( q \in \mathbb{N} \), set

$$\lambda_q = \frac{\epsilon}{2q} (C(B))^{-1} (C(B)\eta^A[t])^{-2q-1}$$

and let \( K_{\lambda_q} \) be given as in the claim. Set \( M_q = K_{\lambda_q} \). Then

$$m_N[t](\Sigma - K((M_p))) \leq \sum_{q \in \mathbb{N}} C(B) (C(B)\eta^A[t])^{2q+1} \sum_{k \in \mathbb{N}} t^k \hat{c}_k \sum_{w \in A_k(\geq M_k)} R_k(wx) \leq \epsilon \sum_{q \in \mathbb{N}} \frac{1}{2^q} \leq \epsilon \quad \square$$

7. Twist by cocycle, and (limit of) matrix coefficients

Section 6 verifies cases where the twisted measures (as in Definition 5.4) exist. In this section we continue to elaborate on their properties. First we give a digression into the terminology of twisted measures. Let us recall that the construction of our twisted measures originates (albeit in a different form) in [5], where the twisted measure is operator valued. In [5] one may understand a “twisting” phenomena taking place in the operator space, whereas here we discuss a real-valued counterpart.

7.1. Local branches, multiplicative cocycles, and twisted measures. The map \( \sigma : \Sigma^+ \to \Sigma^+ \) is not invertible but on any cylinder \([w] \) with \(|w| = n \) the local branch \( \sigma^{(w)} : [w] \to \sigma^n[w] \), \( \sigma^{(w)}(z) = \sigma^n(z) \) has a local (left) inverse \( \tau^{(w)} : \sigma^n[w] \to [w] \). The measures \((\tau^{(w)})_* m, (\sigma^{(w)})_* m\) have the defining property

$$\int F(\tau^{(w)})_* m = \int F o \tau^{(w)}(z) \mathbb{1}_{\sigma^n[w]} dm(z), \quad \int F(\tau^{(w)})_* m = \int F o \sigma^{(w)}(z) \mathbb{1}_{[w]} dm(z).$$

Choosing \( F = \mathbb{1}_{[w]} \) we have

$$(\tau^{(w)})_* m([w]) = m(\sigma^n[w]), \quad (\sigma^{(w)})_* m([w]) = m([ww]),$$

if \( ww \) is admissible. The measure \((\tau^{(w)})_* m\) that is supported in \([w]\) and so it makes sense to ask whether it is absolutely continuous to \( m \) restricted to \([w]\); in this way the Radon-Nikodym derivative \( \frac{d(\tau^{(w)})_* m}{dm}(z) \) is only defined in \([w]\). Whereas the
measure \((\sigma^w)_* m\) that is supported in \(\sigma^n[w]\) and the Radon-Nikodym derivative 
\[ \frac{d(\sigma^w)_* m}{dm}(z) = \left( \frac{d(\tau^w)_* m}{dm}(\tau(w)z) \right)^{-1} \]

In general we have no reason to be able to extend \(\tau^w\) and \(\sigma^w\) to a group action (compare with section 14). We are, however, able to generalise the cocycle Radon-Nikodym derivate aspect of the group action using the structure of a group extension \(s : \mathcal{W}_1 \to G\). We introduce some terminology.

**Definition 7.1.** Let \(m\) be a probability measure on \(\Sigma^+\). If there exist \(\gamma > 0\) and \(h : G \times \Sigma^+ \to \mathbb{R}\) satisfying

\[ \frac{d(\tau^w)_* m}{dm}(z) = \gamma^n R_n(z)^{-1} h(\sigma(w), z), \tag{7.1} \]

for every \(w \in \mathcal{W}_n\) and \(z \in [w]\) then we say that \(m\) is twisted by \(h\). We call such an \(h\) a generalised multiplicative cocycle.

In [5] the word “twisted” can be thought of as referring to a unitary twist. Here we use the term “twist” to mean twisted by \(h\). We will check that the measure \(m = \nu_Q^\circ\) is twisted in the sense of Definition 7.1, for \(Q = \phi * f, \phi^\tau * f\).

Let us conclude the digression with the following. We have the identity

\[ h(e, z) = h(g^{-1}, z) h(\sigma(w), \tau(w)z) \]

whenever \(g = \sigma(w)\) and \(z \in \sigma^n[w]\). This is of interest because, on the one hand for \(g = \sigma(w), |w| = n\),

\[ \int_{\sigma^n[w]} \frac{h(g^{-1}, z)}{h(e, z)} dm(z) \int_{\sigma^n[w]} \frac{1}{h(g, \tau(w)z)} dm(z) = \int_{\sigma^n[w]} \frac{1}{h(\sigma(w), \tau(w)z)} \frac{1}{h(g(w), \tau(w)z)} dm(z) \]

\[ = \int_{\sigma^n[w]} \frac{1}{h(\sigma(w), z)} d(\tau(w)_* m)(z) \]

\[ = \int_{\sigma^n[w]} \frac{1}{h(\sigma(w), z)} \frac{d(\tau(w)_* m)}{dm}(z) dm(z) \]

\[ = \int_{\sigma^n[w]} \gamma^n R_n^{-1}(z) dm(z). \]

And on the other hand when \(m = \nu_Q^\circ\) we evaluate the final term in terms of (limits of) matrix coefficients at \(g\).

### 7.2. Technical lemmas.

**Lemma 7.2.** Assume \(\gamma(Q) > \gamma(SPR)\). Then \(\nu_{d,N}^{-1}Q[t]\) converge to a measure, finite on cylinders, as \(N \to \infty\). In this way

\[ \nu_{d,N}^{-1}Q[t] = \sum_{b \in \mathcal{W}^+} \sum_{n=1}^{\infty} t^{-n} d_n \sum_{v \in \mathcal{W}_n^+} R_n(vx) \frac{(\rho(\sigma(v))) f, \rho(g)^{-1}Q[t]}{F(Q)[t]} D(vx) \]

is well defined, and for each \(B\) the \(\nu_{d,N}^Q[t]\) measure of \([B]\) is bounded uniformly in \(t > \gamma(Q)\).

**Proof.** Let \(C = \max(\sup \left\{ \frac{d_k}{c_k} : c_k \neq 0 \right\}, \sup \left\{ \frac{c_k}{d_k} : d_k \neq 0 \right\})\). We can transfer any cylinder bounds for \(\nu_{c,N}^Q[t]\) to \(\nu_{d,N}^Q[t]\) since \(C^{-1} \nu_{d,N}^Q[t](E) \leq \nu_{c,N}^Q[t](E) \leq C \nu_{d,N}^Q[t](E)\) for every open set. \(\square\)
"The main equality" of Proposition 5.6 can be interpreted as saying that
\[
\int F dt^k R_k^{-1} \nu_{c,N}^Q [t] = \nu_{d,N}^Q [t] (F) + \rem_{d,N;[1,r]}^g (t) (t^k R_k^{-1} F) + \rem_{d,N;[N-r,N]}^g (t) (F)
\]
with \(d_n = 1_{[r,\infty)} (n) c_n + r\) and \(\rem_{d_N}^g (t)\) the measure
\[
\rem_{d,N}^g (t) = \sum_{n \in I} d_n t^{-n} \sum_{v \in W^0_n} (\rho (v) f, \rho (g)^{-1} Q_{\leq N} [t]) / F(Q) [t].
\]

**Proposition 7.3.** For any \(k\)
\[
\nu_{d,N}^g (t) = \nu_{d,N}^g (t) + \rem_{d,N;[1,k]}^g (t)
\]
with \(d_n = 1_{[k,\infty)} (n) c_n\)

We first check that the two remainder terms go zero.

**Lemma 7.4.** For each \(g, r, K\) and \(d\), we have
\[
\rem_{d,N;[1,r]}^g (t) \to 0 \text{ as } t \to \gamma.
\]

**Proof.** We have
\[
\rem_{d,N;[1,r]}^g (t) [B] = \frac{\langle \phi_{d,N;[1,r]}^g * f, \rho (g)^{-1} Q_{\leq N} [t] \rangle}{F(Q) [t]}
\]
First note that \(d/c\) is bounded in the range \([1, r]\) so
\[
\rem_{d,N;[1,r]}^g (t) [B] \leq C (\phi_{d,N;[1,r]}^g * f, \rho (g)^{-1} Q_{\leq N} [t]) / F(Q) [t],
\]
for some \(C > 0\). Second, use Lemma 2.4 to show that
\[
\rem_{d,N;[1,r]}^g (t) [B] \leq \text{Const.} (B) C (\rho (h_B) \phi_{e,N;[1,r]}^g * f, \rho (g)^{-1} Q_{\leq N} [t]) / F(Q) [t],
\]
for some \(h_B\).

If \(T_a\) is transitive we may assume \(h_B = g^{-1}\) and then use that
\[
\frac{\langle \phi_{e,N;[1,r]}^g * f, Q [t] \rangle}{F(Q) [t]} \to 0 \text{ as } t \to \gamma,
\]
using Lemma 6.4 and that \(r, K_B\) are fixed.

If \(Q = \phi_e * f\) then we use the Cauchy-Schwarz inequality and then see that
\[
\frac{\| \phi_{e,N;[1,r]}^g * f \| \| Q_{\leq N} [t] \|}{F(Q) [t]} \to 0 \text{ as } t \to \gamma,
\]
using Lemma 6.1 and that \(r, K_B\) are fixed. \(\square\)

**Lemma 7.5.** For each \(g, r\) and \(d\) with \(d/c\) bounded, we have
\[
\rem_{d,N;[N-r,N]}^g (t) \to 0 \text{ as } N \to \infty.
\]

**Proof.** We have
\[
\rem_{d,N;[N-r,N]}^g (t) [B] = \frac{\langle \phi_{d,N;[N-r,N]}^g * f, \rho (g)^{-1} Q_{\leq N} [t] \rangle}{F(Q) [t]}
\]
The easier case is \(Q = \phi_e * f\). We have the upper bound
\[
\rem_{d,N;[N-r,N]}^g (t) [B] \leq C (B) \frac{\rho (h_B) \phi_{e,N;r-N;[N-r,N]}^g * f, \rho (g)^{-1} Q_{\leq N} [t])}{F(Q) [t]}
\]

\[
\rem_{d,N;[N-r,N]}^g (t) [B] \leq \left( \sup_{n \in [N-r,N]} d_n \right) \text{Const.} (B) \frac{\rho (h_B) \phi_{e,N;r-N;[N-r,N]}^g * f, \rho (g)^{-1} Q_{\leq N} [t])}{F(Q) [t]}
\]
Using Cauchy-Schwarz gives
\[
\text{rem}_{d,N}[\mathcal{N}][N-r,N][t](|[B]|) \leq \left( \sup_{n \in [N-r,N]} \frac{d_n}{c_n} \right) \text{Const.}(B) \frac{\|\phi^{A,a}_{c,N+r \leq N+K_b} \ast f\|_{\mathcal{Q} \leq N}[t]}{F(Q)[t]}.
\]

We know that \(\|\phi^{A,a}_{c,N+r \leq N+K_b} \ast f\| \to 0\) as \(N \to \infty\) by strong convergence of \(\phi^{c \leq N} \ast f\) to \(\phi \ast f\). It follows that \(\text{rem}_{d,N}[\mathcal{N}][N-r,N][t](|[B]|) \to 0\) as \(N \to \infty\).

In the other case \(Q = \phi^{c} \ast f\) and we assume that \(T_a\) is transitive. We obtain
\[
\text{rem}_{d,N}[\mathcal{N}][N-r,N][t](|[B]|) \leq \left( \sup_{n \in [N-r,N]} \frac{d_n}{c_n} \right) \text{Const.}(B) \frac{\langle \phi^{A,a}_{c,N+r \leq N+K_b}[t] \ast f, \phi^{c \leq N}[t] \ast f \rangle}{F(Q)[t]}.
\]

We use 5.1 to conclude. \(\Box\)

Now we check how the limit of \(\nu^{q^{-1}Q}_d\) depends (or doesn’t depend) on \(d\).

**Lemma 7.6.** If \(\frac{d}{d_k} \to 1\) as \(n \to \infty\) then \((\nu^{q^{-1}Q}_d[t_k])\) converge as \(k \to \infty\), the limit measure has \(\nu^{q^{-1}Q}_0 = \nu^{q^{-1}Q}_d\).

**Proof.** Let the cylinder \([w]\) be arbitrary. Let \(\epsilon > 0\) be arbitrary. Using the hypothesis on \(d\), choose \(K\) sufficiently large with \(c_n/d_n \leq 1 + \epsilon\) for \(n \geq K\). Using Proposition 7.3 we have
\[
\nu^{q^{-1}Q}_d[wB] = \nu^{q^{-1}Q}_N[wB] + \text{rem}_{d,N}[wB][t].
\]

Now,
\[
\nu^{q^{-1}Q}_d[wB] \leq (1 + \epsilon)\nu^{q^{-1}Q}_N[wB]
\]
We use Lemma 7.4 to conclude that the remainder term vanishes as \(q \to \infty\). Since \(\epsilon\) was arbitrary the conclusion follows. \(\Box\)

**Remark 7.7.** The propositions and lemmas 7.3 7.4 7.5 7.6 are all valid when we replace \(\rho\) with the anti-homomorphism \(\rho^*(g) = \rho(g)^{-1}\). This observation will save us from almost identical arguments in section 10.

**Lemma 7.8.** If \(T_a\) is transitive then \(\nu^{q^{-1}Q}_c\) is absolutely continuous with respect to \(\nu^{q^{-1}Q}_c\).

**Proof.** It is enough to check that for each \(w\) we have
\[
\nu^{q^{-1}Q}_c(R^{-1}_B[wB]) \leq C\nu^{q^{-1}Q}_c(R^{-1}_B[wB]).
\]

There is a constant \(K_g\), Const.(\(g\)) with
\[
\nu^{q^{-1}Q}_c[wB][t] - \text{rem}^{q^{-1}Q}_{c,N}[wB][t] = \frac{\langle \rho(\hat{g}(w))\phi^{B,a}_{c \leq N-|w|}[t] \ast f, \rho(g)(Q)_{c \leq N}[t] \ast f \rangle}{F(Q)[t]}
\]
\[
\leq \text{Const.}(g) \frac{\langle \rho(\hat{g}(w))\phi^{B,a}_{c \leq N-|w|}[t] \ast f, Q_{N+K_g}[t] \ast f \rangle}{F(Q)[t]}
\]
\[
\leq \text{Const.}(g) \frac{\langle \rho(\hat{g}(w))\phi^{B,a}_{c \leq N+K_a-|w|}[t] \ast f, Q_{N+K_g}[t] \ast f \rangle}{F(Q)[t]}
\]
\[
\leq \text{Const.}(g) \left( \nu^{q^{-1}Q}_{c,N+K_g}[w] - \text{rem}^{q^{-1}Q}_{c,N+K_g}[w][t] \right).
\]

Using the results on remainders (Lemma 7.4) the conclusion follows. \(\Box\)
7.3. The twist and the (limits of) matrix coefficients.

Lemma 7.9. If $T_h$ is transitive then $\nu^Q_c$ is twisted in the sense of Definition 7.1.

Proof. For brevity write $m = \nu^Q_c$, $m^g = \nu^{g^{-1}Q}_c$ and $\gamma = \gamma(Q)$. The measure $m_w$ defined by

$$\int F dm_w = \int F \gamma^{-n} R_n d(\tau(w)) * m$$

has $m_w = m^g$ for a certain $d$ and $g = g(w)$. In addition $m^g$ is absolutely continuous with respect to $m$ by Lemma 7.8. Since $m_w(F) = m_w(F)$ for every $w, v$ with $g(w) = g(v) = g$ it follows that $\frac{dm}{dm^g}(z) = \frac{dm}{dm^g}(z) = h'(g, z)$, i.e. a function indexed by the group element. Now using the chain rule for Radon-Nikodym derivatives we have

$$h'(g, z) = \frac{dm_w}{dm}(z) = \frac{d\gamma^{-n} R_n(\tau(w)) * m}{d(\tau(w)) * m}(z) = \gamma^{-n} R_n \frac{d(\tau(w)) * m}{dm}(z).$$

It is now easier to write the conclusions for the (limits of) matrix coefficients.

Lemma 7.10. For any $wA$ we have

$$\int_{[wA]} \gamma(Q)^n R^{n-1} c d\nu_c^Q = \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g(w)) \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}.$$ 

If $Q = \phi_c * f$ then

$$\int_{[wA]} \gamma(Q)^n R^{n-1} c d\nu_c^Q = \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g(w)) \phi_{c,[t_q]} * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c,[t_q]} * f, Q_{\leq N}[t_q] \rangle}.$$ 

Lemma 7.11. Suppose $T_h$ is transitive. We have

$$\int_{[wB]} \gamma(Q)^n R^{n-1} c d\nu_c^Q \leq C_B \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g(w)) \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}.$$ 

If $Q = \phi_c * f$ then

$$\int_{[wB]} \gamma(Q)^n R^{n-1} c d\nu_c^Q \leq C_B \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g(w)) \phi_{c,[t_q]} * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c,[t_q]} * f, Q_{\leq N}[t_q] \rangle}.$$ 

Proof of Lemmas 7.10 and 7.11. By the main equality and Lemmas 7.4 and 7.5,

$$\nu^Q_c(\gamma(Q)^n R^{-1} \chi_{[wB]}) = \nu^{g^{-1}Q}_c([B]),$$

for $g = g(w)$. We are always using that fact that $R^{-1} \chi_{[wB]}$ is bounded and continuous so that we can deduce its integral with respect to the weak limit $\nu^Q_c$. We know that

$$\nu^{g^{-1}Q}_c([B]) = \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g) \phi^{B,g}_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}.$$ 

and since the numerator and denominator have bounded limits we may say

$$\nu^{g^{-1}Q}_c([B]) = \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g) \phi^{B,g}_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}.$$ 

In the case that $Q = \phi_c * f$ we use strong convergence to say, for $B = A$,

$$\lim_{N \to \infty} \lim_{M \to \infty} \frac{\langle \rho(g) \phi_{c, M}[t_q] * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c, M}[t_q] * f, Q_{\leq N}[t_q] \rangle} = \frac{\langle \rho(g) \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}.$$ 

If $T_h$ is transitive then using Lemma 2.6

$$\nu^{g^{-1}Q}_c([B]) \leq C_B \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g) \phi^{A,g}_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}{\langle \phi_{c, N}[t_q] * f, Q_{\leq N}[t_q] \rangle}.$$ 

completing the proof. \qed
7.4. Further estimates. We check that the (limits of) matrix coefficients are non-trivial.

Corollary 7.12. Assume $\Sigma^+$ compact. Suppose $T_\nu$ is transitive. For each $g$ we have
\[
\lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g)\phi_{c \leq N[t_q]} \ast f, Q_{\leq N}[t_q] \ast f \rangle}{\langle \phi_{c \leq N}[t_q] \ast f, Q_{\leq N}[t_q] \ast f \rangle} > 0.
\]
If $\gamma(Q) < 1$ then for every $\epsilon > 0$ there exist $h$ with
\[
\lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g)\phi_{c \leq N[t_q]} \ast f, Q_{\leq N}[t_q] \ast f \rangle}{\langle \phi_{c \leq N}[t_q] \ast f, Q_{\leq N}[t_q] \ast f \rangle} < \epsilon.
\]
Proof. Choose $w$ with $g(w) = g$. Then
\[
\lim_{q \to \infty} \sup_{wA} \frac{\langle \rho(g)\phi_{c \leq N[t_q]} \ast f, \phi_{c}[t_q] \ast f \rangle}{\| \phi_{c}[t_q] \ast f \|^2} \geq C\nu^Q_c(1) \geq C\nu^Q_c(1_{wA})
\]
for some constant $C$ and any $v$. If we choose $v$ with $g(wA) = e$ then we get positive lower bound for $\nu^Q_c(v_{wA})$.

For the second part we have that there is some constant $C$ so that for each $n$
\[
\gamma(Q) = \sum_{b \in W_1} \sum_{w \in W_n^{b,A}} \gamma(Q)^{-n} R_n(wE) \lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(g)(w)\phi_{c \leq N[t_q]} \ast f, Q_{\leq N}[t_q] \ast f \rangle}{\langle \phi_{c \leq N}[t_q] \ast f, Q_{\leq N}[t_q] \ast f \rangle}.
\]
Since the $\sum_{b \in W_1} \sum_{w \in W_n^{b,A}} \gamma(Q)^{-n} R_n(wE)$ diverges in $n$, we have that for $h$ with
\[
\lim_{q \to \infty} \lim_{N \to \infty} \frac{\langle \rho(h)\phi_{c \leq N}[t_q] \ast f, Q_{\leq N}[t_q] \ast f \rangle}{\langle \phi_{c \leq N}[t_q] \ast f, Q_{\leq N}[t_q] \ast f \rangle}
\]
for any $w \in W_n^{b,A}$, for any $wA$ and $|w| = r$ we have
\[
\nu^Q_c([wA]) \leq C R_x(wE) \gamma^{-r}
\]

8. The local RHS Gibbs property and absolute continuity at the maximal parameter

These manipulations are only valid for the case $\nu^Q_c$ measure. We set $\gamma = \gamma(\phi_c \ast f)$. Using that $Y_c(g) \leq 1$ we have the following lemma.

Lemma 8.1 (RHS local Gibbs). There is a constant $C$ so that for any word $w$ with admissible $wA$ and $|w| = r$ we have
\[
\nu^Q_c([wA]) \leq C R_x(wE) \gamma^{-r}
\]

Theorem 8.2. If $\gamma = 1$ then $\nu^Q_c$ is absolutely continuous with respect to $\nu$.

Proof. Fix $B$. Let $\epsilon > 0$. We must find $\eta > 0$ for which $\nu^Q_c(E \cap [B]) < \epsilon$, for any open set $E$ with $\nu(E \cap [B]) < \eta$. For brevity we assume always that $E \subset [B]$.

Recall in Theorem 6.6 the construction of a compact set $K_\epsilon = K((M_p)_p)$ with $\nu^Q_c(E) < \epsilon$ and $\nu(E) < \epsilon$ for $E \subset \Sigma - K_\epsilon$. On the other hand, for $E \subset K_\epsilon$ we have the following.

Claim 8.3. There is a finite set $S \subset W_1$ and a sequence $w(k) \in W^{s,a}$, $s \in S$ with
\[
\nu^Q_c(E) = \sum_{k=1}^{\infty} \nu^Q_c([w(k)])
\]
\[
\nu(E) = \sum_{k=1}^{\infty} \nu([w(k)])
\]

Proof of claim. Inductively define $u(B, n)$ to be the shortest word with $[Bu(B, n)A] \subset E - \cup_{s \leq n}[Bu(B, k)A]$. Either the sequence terminates or $|u(B, n)| \to \infty$ by compactness of $K_\epsilon$. Then, up to measure zero sets, $E = \cup_{u}[Bu(B, n)A]$.
Lemma 9.2. Assume that 
\[ \int \sum w A \] 
each \[ \mu \] 
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In conclusion, using the Dominated Convergence Theorem, 
Proof. Let \( H \in L^1(\mu) \), bounded, and \( B \in \mathcal{W}_1 \). As \( \mu \) is ergodic we have 
pointwise convergence for \( \mu \) almost every \( z \). In particular 
\[ \mathbb{1}_B(z) \frac{1}{N} \sum_{n=0}^{N-1} H \circ \sigma^n(z) \rightarrow \mu(H) \mathbb{1}_B(z) \text{ as } N \rightarrow \infty; \]
pointwise convergence for \( \mu \) almost every \( z \). And we have the domination 
\[ \mathbb{1}_B(z) \frac{1}{N} \sum_{n=0}^{N-1} H \circ \sigma^n(z) \leq \|H\|_\infty \mathbb{1}_B(z), \]
As \( \nu_{\nu,\phi,f} \) is absolutely continuous with respect to \( \mu \) we have that \( \mathbb{1}_B H \circ \sigma^n \in L^1(\nu_{\nu,\phi,f}) \) and pointwise convergence of Birkhoff sums \( \nu_{\nu,\phi,f} \)-almost surely. We check integrability of the domination 
\[ \int \mathbb{1}_B(z) \frac{1}{N} \sum_{n=0}^{N-1} H \circ \sigma^n(z) \nu_{\nu,\phi,f}(z) \leq \|H\|_\infty \nu_{\nu,\phi,f}(\{B\}) < \infty. \]
In conclusion, using the Dominated Convergence Theorem, 
\[ \int \mathbb{1}_B(z) \frac{1}{N} \sum_{n=0}^{N-1} H \circ \sigma^n(z) \nu_{\nu,\phi,f}(z) \mu(H) \int \mathbb{1}_B(z) \nu_{\nu,\phi,f} \text{ as } N \rightarrow \infty. \]

Let us give a different description of \( \sigma_\nu^* \nu_{\nu,\phi,f}. \) Suppose \( awA \) is admissible. We have 
\[ \mathbb{1}_{[aw:A]} \circ \sigma = \sum_{B \in \mathcal{W}_1} \sum_{v \in \mathbb{W}_{B,a}} \mathbb{1}_{[aw:A]} \cdot \]
. By Lemma 8.1 we have 
\[ \int \mathbb{1}_{[aw:A]}(\sigma^* z) \nu_{\nu,\phi,f}(z) \]
\[ \leq \limsup_{k \rightarrow \infty} \sum_{B \in \mathcal{W}_1} \sum_{v \in \mathbb{W}_{B,a}} CR_{\nu,\nu} R_{[aw:A]}(awx) (\rho(\bar{\phi}(w)) \phi[t_k] * f, \rho(\bar{\phi}(v))^{-1} \phi[t_k] * f) \]
\[ \frac{1}{\| \phi[t_k] * f \|^2}. \]
Lemma 9.2. Assume that \( \gamma(\phi * f) = 1 \). Let \( Q \) be a finite subset of \( \mathbb{W}^{awA} \), and for each \( awA \in Q \) let \( q(w) \in (0, \infty) \). There is a constant \( C > 0 \) with 
\[ \int \sum_{awA \in Q} q(w) R_{[aw:A]} \mathbb{1}_{[aw:A]} \circ \sigma^* (z) \mathbb{1}_{[B]}(z) \nu_{\nu,\phi,f}(z) \leq C \sum_{awA \in Q} q(w) \mathbb{1}_{[aw:A]} \rho(\bar{\phi}(w)) \]
Proof.

$$\int \sum_{aw \in Q} \frac{q(w)}{R_{[aw]}(aw; x)} 1_{[w]}(z) \sigma^r(z) 1_{[B]}(z) d\nu_c^{\phi * f}(z)$$

$$= \lim_{k \to \infty} \max\left[ \sum_{aw \in Q} \frac{q(w)}{R_{[aw]}(aw; x)} 1_{[w]}(z) \sigma^r(z) 1_{[B]}(z) d\nu_c^{\phi * f}(z) \right]$$

$$\leq \lim_{k \to \infty} \max\left[ \sum_{aw \in Q} \frac{q(w)}{R_{[aw]}(aw; x)} 1_{[w]}(z) \sigma^r(z) 1_{[B]}(z) d\nu_c^{\phi * f}(z) \right]$$

Now we use the fact that $t_k > \gamma(\phi * f) = 1$ to conclude

$$\sup_{v \in \mathcal{H}^+} \frac{\gamma(\phi * f)}{\gamma(\phi * v)} < \infty.$$

□

Proposition 9.3. There is $v \in \mathcal{H}^+$ with

$$\sup_{v \in \mathcal{H}^+} \gamma(\phi * f) = \gamma(\phi * v).$$

Proof. Let $f_k$ be unit vectors approaching the supremum. Set $v = \sum_{k=1}^\infty 2^{-k} f_k$. By positivity $\int \langle \rho(g), v \rangle H_d(\phi * f)(t) \geq 2^{-2k} \int \langle \rho(g), f_k \rangle H_d(\phi * f)(t)$. In particular, for any $t > \gamma(\phi * f)$, we have $\int \langle \rho(g), v \rangle H_d(\phi * f)(t) \geq 2^{-2k} \int \langle \rho(g), f_k \rangle H_d(\phi * f)(t)$. In conclusion $\gamma(\phi * f) = \gamma(\phi * f)$ for every $k$. □

Proof of Theorem 3.13. By Proposition 9.3 it suffices to prove that the supremum of $\gamma(\phi * v)$ is strictly less than 1. We proceed by contradiction, assume that $f$ is some vector with $\gamma(\phi * f) = 1$. Using transitivity we may choose $Q$ and $q(w)$ with $\sum_{aw \in Q} q(w) = M_p$, where $M_p$ is a symmetric random walk operator on $G$. In particular there is some $\beta < 1$ with $\|M_p\| < \beta^k$.

Set $H = \sum_{aw \in Q} R_{[aw]}(aw; x) 1_{[aw; x]}$. Note that $H$ is bounded because $Q$ is finite. By Lemma 9.1,

$$\int \frac{1}{N} \sum_{n=0}^{N-1} H \circ \sigma^n(z) 1_{[B]}(z) d\nu_c^{\phi * f}(z) \to \mu(H) \nu_c^{\phi * f}(\{B\})$$

as $N \to \infty$. We compute that

$$\mu(H) = \sum_{aw \in Q} \frac{q(w)}{R_{[aw]}(aw; x)} 1_{[aw; x]} h d\nu$$

$$\geq \sum_{aw \in Q} \frac{q(w)}{R_{[aw]}(aw; x)} (h(z) - \beta^{w \wedge x}) \nu([w])$$

$$\geq (h(z) - \beta^{w \wedge x}) C^{-1}.$$

This is a contradiction to the upper bound $C \beta^k$. □

Proof of Theorems 3.14. Aiming for a contradiction, we assume that $\sup H \gamma(\phi * 1_{[\epsilon; w]}) = 1$. Let $\rho$ be the direct sum representation. Then there is $f$ in the direct
Definition 10.1. We denote the concatenation for a finite set $\mathcal{R}$ denoted $Y.y$, $\mu$ is therefore well defined. By definition, the mass of sided shift space $\Sigma$, and we will need to assume that the phase space is compact.

Lemma 10.2. There is weak convergence of limiting measure.

For fixed $x$, an arbitrary $z$, $(\mu \circ \sigma^n(z))_t = \mu_t(\sigma\circ f_t(z))$. For any subset $\Lambda$, $\mu$ accumulates points. We check convergence along $\mu$ weak limit coincides with $\mu_t(\sigma\circ f_t(z))$. The normalizing factor is $m_{\lambda}((\cdot)^{\lambda})_{t} = 1$ for all $\lambda$. Using the visibility hypothesis we may choose $\mu_t(\sigma\circ f_t(z))$. In order to construct the shift invariant measure we prefer to work in the two sided shift space $\Sigma$, and we will say that a sequence $Y.y$ tends to 1 as $N \to \infty$. Again, we have arrived at a contradiction. 

10. A SHIFT INVARIANT $\phi^*$ (TWISTED) MEASURE

In order to construct the shift invariant measure we prefer to work in the two sided shift space $\Sigma$, and we will need to assume that the phase space is compact. Therefore, assume throughout that $\Sigma$ is a subshift of finite type. We also always assume that $T_a$ is transitive.

We set up some notation. There is a basis of open sets given by cylinders $[u_1, \ldots, u_m, v_1, \ldots, v_n] = \{ z \in \Sigma : z_i = v_{i+1}, 0 \leq i \leq n-1, z_i = u_{m+1-i}, 1 \leq i \leq m \}$ For any subset $\Lambda \subset \Sigma$ we will say that a sequence $y : \Lambda \to \mathcal{W}_1$ is admissible if $\tau(y_i, y_{i+1}) = 1$ for all $i \in \Lambda$. For $Y : \mathcal{W}_0 \to \mathcal{W}_1$ and $u = u_1, \ldots u_m$ a finite word we denote the concatenation $Y.u : \mathcal{W}_0 \to \mathcal{W}_1$, $(Y.u)_i = u_{m+1-i}$ if $1 \leq i \leq m$ and $(Y.u)_i = Y_{m+1-i}$ otherwise. Similarly, for $y : \mathcal{W}_0 \to \mathcal{W}_1$ and $v = v_1, \ldots v_n$ a finite word we denote the concatenation $vy : \mathcal{W}_0 \to \mathcal{W}_1$, $(vy)_i = v_{i+1}$ if $0 \leq i \leq n-1$ and $v_{i+1} = y_{i-n}$ otherwise. The concatenation of $Y : \mathcal{W}_0 \to \mathcal{W}_1$ and $y : \mathcal{W}_0 \to \mathcal{W}_1$ is denoted $Y.y : \mathcal{W}_0 \to \mathcal{W}_1$, $(Y.y)_i = y_i$ if $i < 0$ and $(Y.y)_i = Y_i$ if $i \geq 0$. We assume that $R : \Sigma \to \mathbb{R}$ depends only on future coordinates, meaning for any $z = Y.y$, $z' = Y'.y$, we have $R(z) = R(z')$. We denote the common value as $R(y)$. The Dirac mass at an arbitrary $z \in \Sigma$ is denoted $D(z)$.

The normalizing factor is $F(\phi^*)[t] = \lim_{N \to \infty} \langle \phi_{\leq N}[t] \ast \phi_{\leq N}[t] \ast f, f \rangle$.

Definition 10.1. For $t > \gamma(\phi, f)$ and $N \in \mathbb{N}$ denote,

$$\mu_{c,N}^{\phi^* f}[t] = \sum_{B,b \in \mathcal{W}_1, \tau(B,b) = 1} \sum_{n=1}^{N} t^{-n} c_n \sum_{m=1}^{N} t^{-m} c_m \sum_{\nu \in \mathcal{W}_0^{\neq B}} R_{\nu}(u) \sum_{w \in \mathcal{W}_0^{\neq A}} R_{\nu}(u \nu) \frac{\rho(\nu \nu)}{F(\phi^*)[t]} D(Xu, uw)$$

for fixed $x : \mathcal{W}_0 \to \mathcal{W}_1$, $X : \mathcal{W}_0 \to \mathcal{W}_1$ chosen with $x_i = A$ uniquely at $i = 0$, and $X_i = a$ uniquely at $i = -1$.

The measure $\mu_{c,N}^{\phi^* f}[t]$ is a positive finite linear combination of Dirac masses and is therefore well defined. By definition, the mass of $\mu_{c,N}^{\phi^* f}[t]$ is bounded for each $t$ and the mass of $[a,A]$ tends to 1 as $N \to \infty$. By weak* compactness there are accumulation points. We check convergence along $N$ and a rearrangement for the limiting measure.

Lemma 10.2. There is weak convergence of $\mu_{c,N}^{\phi^* f}[t]$ as $N \to \infty$. In addition the weak limit coincides with $\mu_{c}^{\phi^* f}[t]$ defined by

$$\mu_{c}^{\phi^* f}[t] = \sum_{N=1}^{\infty} t^{-N} \sum_{w \in \mathcal{W}_0^{\neq A}} R_{N}(uw) \frac{\rho(\nu \nu)}{F(\phi^*)[t]} D(Xuw, \nu)$$
We aim to show that $\hat{\mu}_{c,N}^f[t]$ weak* converge as $N \to \infty$. Indeed for $t$ fixed, any cylinder $E$ has that $\mu_{c,N}^f[t](E)$ is a monotonically increasing bounded sequence in $N$.

The limit of the measure of any cylinder $E$ is

$$
\frac{1}{F(\phi^*)[t]} \lim_{M \to \infty} \sum_{N=1}^{M} t^{-N} \sum_{m,n \leq N} \sum_{B,b \in W_1 \cap B(b)} \sum_{v \in W_{n}^{B,b}} R_{m+n}(uvx) \langle \rho(\varphi(uv))f,f \rangle \frac{c_n c_m \sigma_n^k D(x,uvx)(E)}{F(\phi^*)[t]}
$$

It is clear there is an lower (resp. upper) bound by $\hat{\mu}_{M/2}(E)$ (resp. $\hat{\mu}_{M}(E)$) given by

$$
\hat{\mu}_{M}(E) = \frac{1}{F(\phi^*)[t]} \lim_{M \to \infty} \sum_{N=1}^{M} t^{-N} \sum_{w \in W_{N}^{A,a}} R_{N}(wx) \langle \rho(\varphi(w))f,f \rangle \sum_{k=1}^{N-1} c_k c_{N-k} \sigma_n^k D(x,uvx)(E).
$$

Hence for any cylinder $E$,

$$
\mu_{c,M}^f[t](E) \geq \mu_{M/2}^f[t](E) \geq \mu_{c,M/2}^f[t](E).
$$

Using convergence along $M$ and $M/2$ the conclusion follows. 

**Definition 10.3.** Let $c$ be slowly increasing with

$$
\lim_{N \to \infty} \frac{\langle \phi_{c,N}[t] * \phi_{c,N}[t] * f,f \rangle}{\langle \phi_{c}[t] * f,f \rangle} \to \infty \quad \text{and} \quad \lim_{N \to \infty} \frac{\langle \phi_{\infty}[t] * \phi_{\infty}[t] * f,f \rangle}{\langle \phi_{\infty}[t] * f,f \rangle} \to \infty \quad \text{as} \quad t \to \gamma(f).
$$

Any accumulation point of $\mu_{c}^{\phi^*}f[t_k]$ as $t_k \to \gamma(f)$ is denoted $\mu_{c}^{\phi^*}$ and called a $\phi^*$ (twisted) measure.

**10.1. Shift invariance.**

**Theorem 10.4.** Any $\phi^*$ (twisted) measure $\mu_{c}^{\phi^*}$ is $\sigma$-invariant.

The proof of shift invariance will rely on a technical lemma regarding the slowly increasing function. For $z \in \Sigma$ set

$$
\mathbb{P}(z,c \ast c,N) = \sum_{k=1}^{N-1} c_k c_{N-k} \sigma_n^k D(z).
$$

Then

$$
\mu_{c}^{\phi^*}f[t] = \sum_{N=1}^{\infty} t^{-N} \sum_{w \in W_{N}^{A,a}} R_{N}(wx) \langle \rho(\varphi(w))f,f \rangle \frac{\mathbb{P}(z,c \ast c,N) - \sigma_{c} \mathbb{P}(z,c \ast c,N)}{F(\phi^*)[t]}.
$$

**Claim 10.5.** There is $\gamma_k \to 0$ with

$$
|\mathbb{P}(z,c \ast c,N)(E) - \sigma_{c} \mathbb{P}(z,c \ast c,N)(E)| \leq \gamma_k \mathbb{P}(z,c \ast c,N)(E) + 6k(c_N c_k + c_N c_k)
$$

**Proof.** Write $\tau_{-1}c_m = \tau_{m-1}c_n = c_{n+1} - c_n$.

$$
\sigma_{c} \mathbb{P}(z,c \ast c,N) = \sum_{k=1}^{N} c_k c_{N-(k-1)} \sigma_n^k D(z)
$$

$$
= \sum_{k=1}^{N-1} \tau_{-1}c_k \tau_{+1}c_{N-k} \sigma_n^k D(z) \leq \tau_{-1}c_1 \tau_{+1}c_{N-1} \sigma_n^k D(z) + \tau_{-1}c_1 \tau_{+1}c_{0} \sigma_n^k D(z)
$$

We aim to show that $\sum_{k=1}^{N-1} \tau_{-1}c_k \tau_{+1}c_{N-k} \sigma_n^k D(z) = \mathbb{P}(z,c \ast c \ast \tau_{+1}c,N)$ is close to $\mathbb{P}(z,c \ast c,N)$.

Since $c$ is increasing we have $\tau_{-1}c_k \leq c_k$ and $c_k \leq \tau_{+1}c_k$, giving

$$
(c \ast c - \tau_{-1}c \ast \tau_{+1}c)_N \leq \tau_{-1}c \ast \tau_{+1}c - c \ast c)_N \leq [c \ast (\tau_{+1}c - c)]_N.
$$
Choose $K$ sufficiently large so that $c_{n+1} - c_n - 1 \leq \epsilon c_{n+1}$ and $c_{n+1} - c_n - 1 \leq \epsilon c_n$ for $n > K$. Then
\[
[(c - \tau_1c) * e]_N \leq \epsilon c * c_N - K(c_k - \tau_1c_k)c_{N-k} \leq \epsilon c * c_N + 2Kc_Kc_N
\]
and
\[
[c * (\tau_1c - e)]_N \leq \epsilon c * c_N - \sum_{k=N-K}^N c_k(\tau_1c_{N-k} - c_{N-k}) \leq \epsilon c * c_N + 2(1+ \epsilon)Kc_Kc_Nc_k
\]

Putting these together gives
\[
\|P(z, c * c, N)(E) - \sigma_*P(z, c * c, N)(E)\| \leq \|P(z, c * c, N)(E) - P(z, c * \tau_1c, N)(E)\| + c_1c_N \leq \epsilon P(z, c * c, N)(E) + 2(1+ \epsilon)Kc_Kc_N + c_1c_N
\]

Proof of Theorem 10.4. Let $E$ be an open set. We will show that for every $\epsilon > 0$ we have $|\mu^{\phi * f}[t] - \sigma_*\mu^{\phi * f}[t](E)| \leq \epsilon$ for $t$ sufficiently close to $\gamma(f)$. Assuming the claim, choose $k$ with $2\gamma_k \max_{t \geq \gamma(f)} \|\mu^{\phi * f}[t](E)\| \leq \epsilon$. We have
\[
|\mu^{\phi * f}[t] - \sigma_*\mu^{\phi * f}[t](E)| \leq \gamma_k\mu^{\phi * f}[t](E) + \sum_{N=1}^{\infty} t^{-N} \sum_{w \in \mathbb{W}_a} R_N(w, x) \frac{\langle \mu(w) \rangle f, f}{F(\phi^*)[E]} 6k c_N c_k
\]
\[
= \gamma_k\mu^{\phi * f}[t](E) + 6k \frac{\langle \phi_e[t] \rangle f, f}{F(\phi^*)[E]}
\]
By Lemma 6.4, when $k$ is fixed, for $t$ sufficiently close to $\gamma(f)$ we have $6k \frac{\langle \phi_e[t] \rangle f, f}{F(\phi^*)[E]} \leq \frac{\epsilon}{2}$, whence
\[
|\mu^{\phi * f}[t] - \sigma_*\mu^{\phi * f}[t](E)| \leq \epsilon.
\]

10.2. The (limit of) matrix coefficients. In the goal of checking conformal properties it is easier to work with a measure that is one-sided. Recall that $R$ is Lipschitz in the $\theta$-metric (see equation 12.1). It follows that there is a constant $C > 1$ so that for any $u$ and any word $b = b_1 \cdots b_p$ with $ub$ admissible we have
\[
C^{-\theta^p} \leq \frac{R_{[u]}(uy)}{R_{[u]}(uz)} \leq C^{\theta^p}
\]
for all $y, z \in [b_1 \cdots b_p]$.

For brevity write $m = \pi_*\mu^{\phi * f}$ and $\text{Lip}(p) = C^{\theta^p}$.

Lemma 10.6. The measures $\sigma^{(a)}_e m$ and $\nu^{\phi * f}$ are equivalent. For any $w$ with $|w| = n$ and $awA$ admissible we have,
\[
\text{Lip}(p)^{-1} \leq \frac{\int [awA] R_{[a]}^{-1} \sigma^{\phi * f}}{\int [awA] R_{[a]}^{-1} \sigma dm} \leq \text{Lip}(p)
\]
for $p = w \land x$.

Proof. Let $awA$ be given with $|w| = k$. Write $p = w \land x$. Both statements are verified upon checking the second. Suppose $w = w'B$. From the definition of $m$ we
have
\[ m(R_k^{-1}1_{[aw:A]}) = \mu_{c}^{\phi * f}(R_k^{-1} \circ \sigma^k 1_{[aw:A]}) \]
\[ = \lim_{N \to \infty} \sum_{B, b \in \mathcal{W}_1} \sum_{n=1}^{N} t^{-n} c_n \sum_{m=1}^{N} t^{-m} c_m \]
\[ \sum_{v \in \mathcal{W}_1^a} R_n(vx) \sum_{u_1 \ldots u_m \in \mathcal{W}_1^a} R_m(uv) \frac{(\rho(\bar{g}(v)) \bar{f}, f)}{F(\phi^*)[t]} 1_{[aw:A]}(Xu.vx) \]

Let \( g = \bar{g}(w) \). Using Hölder continuity we have an upper bound (lower bound) by a \( \text{Lip}(p)^{-1} (\text{Lip}(p)) \) multiple of the \( \text{lim sup} \) (\( \text{lim inf} \)) of
\[ \sum_{m=k}^{N} t^{-m} c_m \sum_{u \in \mathcal{W}_1^a} R_{m-k}(uax) \frac{(\rho(g) \phi_{c,N} f, \phi_{c,N} f)}{F(\phi^*)[t]} \]
\[ + \text{rem}_{c,N;1,k}(R_k^{-1}) \]

where \( \text{rem}_{c,N;1,k} \) is defined for the anti-homomorphism \( \rho^* \); and the first term is equal to \( \nu_{c,N;1,k}^{\phi^*}([A]) \) + \( \text{rem}_{c,N;1,k}^{\phi^*}([A]) \) defined for the anti-homomorphism \( \rho^* \) (see by Remark 7.7). It follows that
\[ \text{Lip}(p)^{-1} \lim_{q \to \infty} \lim_{N \to \infty} \frac{(\rho(g) \phi_{c,N} f, \phi_{c,N} f)}{F(\phi^*)[t]} \]
\[ \leq m(R_k^{-1}1_{[aw:A]}) \]
\[ \leq \text{Lip}(p) \lim_{q \to \infty} \lim_{N \to \infty} \frac{(\rho(g) \phi_{c,N} f, \phi_{c,N} f)}{F(\phi^*)[t]} \]

Now by Lemma 7.10 and since \( g = \bar{g}(w) \),
\[ \lim_{q \to \infty} \lim_{N \to \infty} \frac{(\rho(\bar{g}(w)) \phi_{c,N} f, \phi_{c,N} f)}{F(\phi^*)[t]} = \nu_{c}^{\phi^* f}(R_k^{-1}1_{[aw:A]}). \]

**Remark 10.7.** If \( R \) is depends on one letter and \( \Sigma \) is the full shift then in fact \( \sigma_{(a)}^* m = \nu_{c}^{\phi^* f} \).

**Corollary 10.8.**
\[ \text{Lip}(p)^{-2} \lim_{q \to \infty} \lim_{N \to \infty} \frac{(\rho(\bar{g}(w)) \phi_{c,N} [t_q] f, \phi_{c,N} [t_q] f)}{(\phi_{c,N} [t_q] f, \phi_{c,N} [t_q] f)} \]
\[ \leq m(\text{Lip}(p)^{-1}1_{[aw:A]}) \]
\[ \leq \text{Lip}(p)^{-1} \lim_{q \to \infty} \lim_{N \to \infty} \frac{(\rho(\bar{g}(w)) \phi_{c,N} [t_q] f, \phi_{c,N} [t_q] f)}{(\phi_{c,N} [t_q] f, \phi_{c,N} [t_q] f)} \]

11. **Amenability implies \( \phi \cdot \phi \) (twisted) measure is Gibbs**

11.1. **The periodic point variant.** In the goal of finding a shift invariant measure it would be more obvious to define a measure supported on periodic points. In order to use equidistribution arguments from the thermodynamic formalism we will find ourselves preferring combinations of periodic points. Define
\[ \bar{\mu}_{c,N}[t] = \sum_{n=1}^{N} t^{-n} \sum_{n=m+k} c_m c_k \sum_{z \in \Sigma^m z = z} R_n(z) \frac{(\rho(\bar{g}(z_0 \ldots z_n)) f, f)}{F(\phi^*)[t]} D(z). \]
Denote $d_n = \sum_{m+k=n} c_m c_k$ and

$$\tilde{\phi}_{d \leq N}[t](g) = \sum_{n=1}^{N} t^{-n} \sum_{m+k=n} c_m c_k \sum_{z \in \Sigma^m z = \#(z_1 \cdots z_n) = g} R_n(z)g(z_1 \cdots z_n).$$

It is clear that any accumulation point of $\tilde{\mu}^{\phi_{s_{d}} f}[t]$ has mass $\lim_{N \to \infty} \tilde{\phi}_{d \leq N}[t] f, f$. We check absolute continuity and boundedness of the mass of the measure. We use the notation

$$F \asymp C \iff C^{-1} F \leq H \leq CF,$$

for a constant $C > 1$.

**Lemma 11.1.** There is a constant $C$ so that for any cylinder $[WB, bw]$ we have

$$\mu^{\phi_{s} f}_c ([WB, bw]) \asymp_C \tilde{\mu}^{\phi_{s} f}_c ([WB, bw]).$$

**Proof.** Write $\mu^{\phi_{s} f}_c = \mu^{A, a, x}$ (with $A, a, x$ defining $\phi$). To begin with we denote $E = [WB, bw]$. For each $D \in \mathcal{W}_1$ choose a one-sided $x_D \in \{D\} \cap \Sigma^+$. Using Hölder continuity we have,

$$\tilde{\mu}^{\phi_{s} f}_c [t](E) = \sum_{n=1}^{N} t^{-n} \sum_{m+k=n} c_m c_k \sum_{z : \sigma^m z = z} R_n(z) \frac{\langle \rho(g(z_1 \cdots z_n), f, f) \rangle_{F(\phi^*)[t]}}{F(\phi^*)[t]} \mathbb{1}_E(\sigma^m z)$$

$$\sum_{D^i, D^i : \tau(D^i, D) = 1} \sum_{u \in \mathcal{W}_{d}^n, v \in \mathcal{W}_{k}^n} R_{m+k}(uvx_D) \langle \rho(g(uv)), f, f \rangle \mathbb{1}_E((uv)_\infty u, v(uv)_\infty)$$

where $(uv)_\infty$ (resp. $(vu)_\infty$) is the right-infinite (resp. left-infinite) concatenation of $uv$ (resp. $vu$). Now recall that $E = [WB, bw]$, and suppose that $|WB| = P$, $|bw| = p$. Then

$$\sum_{D^i, D^i : \tau(D^i, D) = 1} \mu^{D^i, D^i, x_D}[t](E) - \frac{\langle b_{A, B} \rho(t) * f, \phi_{c \leq N}^t \rho(t) * f \rangle}{F(\phi^*)[t]} - \frac{\langle b_{A, B} \rho(t) * f, \phi_{c \leq p}^t \rho(t) * f \rangle}{F(\phi^*)[t]}$$

$$\leq \sum_{n=1}^{2N} t^{-n} \sum_{m+k=n} c_m c_k \sum_{D^i, D^i : \tau(D^i, D) = 1} \sum_{u \in \mathcal{W}_{d}^n} \sum_{v \in \mathcal{W}_{k}^n} R_{m+k}(uvx_D) \langle \rho(VBwbv), f, f \rangle \mathbb{1}_E((uv)_\infty u, v(uv)_\infty)$$

The limit in $N$ and $2N$ agrees. Taking a limit in $t$ makes the remainder term vanish. Then it is straightforward to check that $\mu^{D^i, D^i, x_D}$ is equivalent to $\mu^{A, a, x}$. \hfill \Box

For the remainder of this section we have only results for the case $f = s$. Recall that $\phi_{c} * \delta_{c} = \phi_{c}$ and in particular $\langle \phi_{c}[t] * f, \phi_{c\leq N}[t] * f \rangle = \langle \phi_{c}[t] * f, \phi_{c\leq p}[t] * f \rangle$.

For amenable groups the work of [9] shows that the Gurevič pressure $P(log R, T_S)$ is equal $P(log R + \psi)$ for a unique real one-dimensional representation $\pi : G \to \mathbb{R}$ and $\psi = \pi \circ s$. (In other words: let $\overline{\psi}_{ab}$ be the composition of $s$ with $G \to G/[G : G] \cong \mathbb{Z}^d \oplus F \to \mathbb{Z}^d$, with $F$ the finite torsion group. Then there is a unique $\xi \in \mathbb{R}$ that determines $\psi(x) = (\xi, \overline{\psi}_{ab}(x))_{\mathbb{Z}^d}$.) We use similar ideas to [16] describing drift for abelian extensions; and the same ideas behind the equidistribution result of [9]
Our goal is to prove Theorem 3.16: if \( G \) is amenable then any \( \phi^* \) (twisted) measure \( \mu_{\phi^*} \) is equal to the equilibrium state \( \mu_{\text{exp} \psi} \); moreover for each \( g \)

\[
\lim_{k \to \infty} \lim_{N \to \infty} \langle \rho(g) \phi_{c \leq N}[t_k], \phi^*_{c \leq N}[t_k] \rangle = \psi(g).
\]

We use the following large deviation estimate

**Lemma 11.2.** Let \( K \) be a weak* compact set not containing \( \mu_{\text{exp} \psi} \). Then

\[
\frac{1}{n} \log \sum_{\{ (u) = x \in \text{Per}(n) : \mathcal{g}(u) = e \}} R_n(x) - \beta < P(\log R + \psi)
\]

*Proof.* We allow ourselves to use \( \nu \) to denote an arbitrary shift invariant measure (previously it was reserved for the \( R \) conformal measure). Set

\[
\rho = \inf_{\nu} \sup_{F} \left( \int F d\nu - P(\log R + \psi + \psi) \right)
\]

We claim that \( \rho + P(\log R + \psi) > 0 \). First, for any \( F \)

\[
\left( \int F d\nu - P(F + \log R + \psi) + P(\log R + \psi) \right)
\]

\[
= \left( P(F + \log R + \psi) - \int F + \psi + \log Rd\nu \right) + \left( P(\log R + \psi) - \int \psi + \log Rd\nu \right)
\]

and so

\[
\sup_{F} \left( \int F d\nu - P(F + \log R + \psi) + P(\log R + \psi) \right)
\]

\[
= - \inf_{F} \left( P(F + \log R + \psi) - \int F + \psi + \log Rd\nu \right) + \left( P(\log R + \psi) - \int \psi + \log Rd\nu \right)
\]

\[
= -h(\nu) + \left( P(\log R + \psi) - \int \psi + \log Rd\nu \right) > 0
\]

Since \( \inf_{F} \left( P(F + \log R + \psi) - \int F + \psi + \log Rd\nu \right) = \inf_{G} \left( P(G) - \int G d\nu \right) = h(\nu) \) and is strictly positive by uniqueness of the equilibrium state. The lower bound is uniform in \( \nu \) in \( K \) by lower semi-continuity.

Now, by definition of \( \rho \), for every \( \nu \in K \) we have

\[
\sup_{F} \int F d\nu - P(F + \log R + \psi) > \rho
\]

and so we may choose \( \gamma > 0 \) and \( F \) with

\[
\int F d\nu - P(F + \log R + \psi) > \rho - \gamma
\]

We deduce that

\[
K \subset \left\{ \nu : \int F d\nu - P(F + \log R + \psi) > \rho - \gamma \right\},
\]

and since \( K \) is weak* compact there are \( F_1, \ldots, F_k \) with

\[
K \subset \bigcup_{i=1}^{k} \left\{ \nu : \int F_i d\nu - P(F_i + \log R + \psi) > \rho - \gamma \right\}.
\]

We need the following two observations

\[
\tau_{x,n} = \sum_{k=0}^{n-1} D_{\sigma^{k} x} \in K \implies \exp(F^n(x) - nP(F_i + \log R + \psi) - n\rho + n\gamma) \geq 1,
\]

\[
(w) = x \in \text{Per}(n), \mathcal{g}(w) = e \implies \psi^n(x) = 0.
\]
Putting this together gives
\[
\sum_{\{w=x \in \text{Per}(n) \mid \underline{g}(w)=e, \frac{1}{n} \sum_{s \in K} D_{s,x} \in K\}} R_n(x) \\
\leq \sum_{\{w=x \in \text{Per}(n) \mid \underline{g}(w)=e, \frac{1}{n} \sum_{s \in K} D_{s,x} \in K\}} R_n(x) \exp(-nP(F_i + \log \rho + \psi) - n(\rho - \gamma) + F^n_i) \\
\leq \exp(-nP(F_i + \log \rho + \psi) - n(\rho - \gamma)) \sum_{\{w=x \in \text{Per}(n) \mid \underline{g}(w)=e, \frac{1}{n} \sum_{s \in K} D_{s,x} \in K\}} R_n(x) \exp(F^n_i + \psi^n)
\]
So
\[\frac{1}{n} \log \sum_{\{w=x \in \text{Per}(n) \mid \underline{g}(w)=e, \frac{1}{n} \sum_{s \in K} D_{s,x} \in K\}} R_n(x) \leq -nP(F_i + \log \rho + \psi) - \rho + \gamma + P(\log R + \psi + F_i) \leq -\rho + \gamma \leq -\epsilon + P(\log R + \psi) + \gamma.
\]

Proof of Theorem 3.16. Let \( H : \Sigma \rightarrow \mathbb{R} \) be continuous and non-negative. Let \( K = \{ m : | \int Hdm - \int H\mu_{R\exp \psi} | \geq \epsilon \} \), a compact set that clearly does not contain \( \mu_{R\exp \psi} \). Writing \( \tau_{x,n} = \frac{1}{n} \sum_{k=0}^{n-1} D_{\sigma^kx} \),

\[
t^n \sum_{x \in \text{Per}(n), \psi_n(x)=0, \tau_{x,n} \in K} R_n(x) \int Hd\tau_{x,n} \leq \|H\|_{\infty} t^n \sum_{x \in \text{Per}(n), \psi_n(x)=0, \tau_{x,n} \in K} R_n(x),
\]

\[
\sum_{n=1}^{\infty} d_n t^n \sum_{x \in \text{Per}(n), \psi_n(x)=0, \tau_{x,n} \in K} R_n(x) \int Hd\tau_{x,n} \leq \|H\|_{\infty} \sum_{n=1}^{\infty} d_n t^n \sum_{x \in \text{Per}(n), \psi_n(x)=0, \tau_{x,n} \in K} R_n(x),
\]

By Lemma 11.2 the series on the right converges at \( t = P(\log R + \psi) \), denote the value as \( C \).

Therefore,
\[
\bar{\mu}_c^{\phi^\ast}[t](H) = \frac{1}{F(\phi^\ast)[t]} \sum_{n=1}^{\infty} d_n t^n \sum_{(w) = x \in \text{Per}(n)} \langle \rho(\underline{g}(w)) \delta_c, \delta_c \rangle R_n(x) \int Hd\tau_{x,n} \\
\leq \frac{\|H\|_{\infty}}{F(\phi^\ast)[t]} \sum_{n=1}^{\infty} d_n t^n \sum_{(w) = x \in \text{Per}(n), \underline{g}(w)=e, \tau_{x,n} \in K} R_n(x) \\
+ \frac{1}{F(\phi^\ast)[t]} \sum_{n=1}^{\infty} d_n t^n \sum_{(w) = x \in \text{Per}(n), \underline{g}(w)=e, \tau_{x,n} \notin K} R_n(x) \left( \int Hd\mu_{R\exp \psi} + \epsilon \right) \\
\leq \frac{\|H\|_{\infty} C}{F(\phi^\ast)[t]} + \left( \int Hd\mu_{R\exp \psi} + \epsilon \right) \lim_{N \rightarrow \infty} \frac{\bar{\phi}_{\ell \leq N}[t](\epsilon)}{F(\phi^\ast)[t]} \mu^{\phi^\ast}(H) = \int Hd\mu_{R\exp \psi}.
\]

A lower bound follows similarly. We conclude
\[
\lim_{k \rightarrow \infty} \frac{\bar{\phi}_{\ell \leq N}[t_k](\epsilon)}{\phi_{\ell \leq N}[t_k](\epsilon)} \left( \frac{\bar{\mu}^{\phi^\ast}(H)}{\mu^{\phi}(H)} \right) = \int Hd\mu_{R\exp \psi}.
\]

Up to scaling, the measures coincide. (Finiteness of the scaling is given by Lemma 11.1.) Now it follows that \( \bar{\mu}^\phi, \bar{\mu}^\phi_e \) and \( \mu_{R\exp \psi} \) are equivalent measures, and by ergodicity are proportional.

We are ready to harvest the (limit of) matrix coefficients property. For every \( p \) we can choose \( u, v \) with \( |u| = |v| = n \geq p, u \land v \geq p, u \land x \geq p \) and \( \underline{g}(u) = g, \underline{g}(v) = v \)},
\(\varphi(v) = e\). Now we have, on the one hand using 12.18, 
\[
\frac{\mu_{R^{\exp \psi}}(R^{-1}_n1_{[w]})}{\mu_{R^{\exp \psi}}(R^{-1}_n1_{[v]})} \in \exp(\psi(g))[\alpha_p^{-1}, \alpha_p].
\]

And on the other hand using Corollary 10.8
\[
\frac{\mu^*_{\delta_n^s}(R^{-1}_n1_{[w]})}{\mu^*_{\delta_n^s}(R^{-1}_n1_{[v]})} \in \lim_{N \to \infty} (\phi)_{\phi \leq N}[k], \phi^*_c \leq N [t_k])^{-1} [(1 + \epsilon(t_k)) \text{Lip}(p)^{-2}, (1 - \epsilon(t_k)) \text{Lip}(p)^2],
\]
where \(\epsilon(t_k) \to 0\). Since \(p\) was arbitrary this completes the proof. \(\square\)

12. Countable Markov shifts and strong positive recurrence

In this section we expand on the basic machinery for countable Markov shifts. Our use of the notion of strong positive recurrence is non-standard, and moreover our choice of “first-return series” is non-standard. It will be important to make clear the allowed estimations regarding local Hölder continuity. Equilibrium states in a CMS need not satisfy the Gibbs property but it is still possible to estimate ratios of certain cylinders, as we make clear.

12.1. Basic definitions. Let \(x \in \mathbb{N}^\mathbb{Z}\); that is, \(x\) is a bi-infinite sequence in the countable set \(\mathbb{N}\). We will often write \(\mathcal{W}_1\) in place of \(\mathbb{N}\), and describe \(\mathcal{W}_1\) as the alphabet of the CMS. As is common usage, we use \(x_i\) to denote the \(i\)th element in the sequence. In the theory of Markov shifts, it is usual to write \(x = \ldots x_{-1}, x_0 x_1 \ldots\); that is, to separate the negative coordinates from the non-negative coordinates by a period. In order to define a CMS it is useful to make reference to a transition matrix \(\tau: \mathbb{Z} \times \mathcal{W}_1 \to \{0, 1\}\). The (two-sided) countable Markov shift (with transition matrix \(\tau\)) is
\[
\Sigma = \{x \in \mathcal{W}^\mathbb{Z}: \tau(x_i, x_{i+1}) = 1\}.
\]

We use \(\sigma\) to denote the left shift \(\sigma: \Sigma \to \Sigma\), \((\sigma x)_i = x_{i+1}\). We always assume that the dynamics are transitive for \(\sigma: \Sigma \to \Sigma\). The (two-sided) countable Markov shift (with transition matrix \(\tau\)) is
\[
\Sigma^+ = \{x \in \mathcal{W}_1^N: \tau(x_i, x_{i+1}) = 1 \forall i \in \mathbb{N}\}.
\]

We can project from \(\Sigma\) to \(\Sigma^+\) by “forgetful” map. Many of the constructions that follow naturally pass from \(\Sigma\) to \(\Sigma^+\).

We equip \(\Sigma\) with the product topology, which can be metrized: write \(x \wedge y = \min(k+1: x_i = y_i, -k \leq i \leq k)\), and \(d(x, y) = 2^{-x \wedge y}\). We use some convenient notation for a basis of open balls,
\[
[\mathcal{W}_1 \cdots \mathcal{W}_m, w_1 \cdots w_n] = \{x \in \Sigma: x_j = w_j, j = 0, \ldots, n, x_{-m-1+i} = W_i, i = 1, \ldots, m\}
\]
where \(w_j, w_{j+1}, W_1, W_{i+1} \in \mathcal{W}_1, \tau(w_j, w_{j+1}) = 1, \tau(W_i, W_{i+1}) = 1, \tau(W_m, w_0) = 1, \)
\(i = 0, \ldots, n-1, j = 1, \ldots, m-1\). We say that \(w = w_1 \cdots w_n, W = W_1 \cdots W_n, Ww\)
are admissible, and \(|w| = n, |W| = m\). For \(B, b \in \mathcal{W}_1\) we write
\[
\mathcal{W}^B_{n, b} = \{Bu_1 \cdots u_{n-2} b: u_j, u_{j+1} \in \mathcal{W}_1, \tau(u_j, u_{j+1}) = 1, j = 1, \ldots, n-3\}
\]
and
\[
\mathcal{W}^B_{n, b} = \{Bu_1 \cdots u_n b: u_j, u_{j+1} \in \mathcal{W}_1, \tau(u_j, u_{j+1}) = 1, j = 1, \ldots, n, n \in \mathbb{N}\}
\]

Let \(R^+: \Sigma^+ \to \mathbb{R}_+\) be a positive function. We say that \(\log R^+\) is locally Hölder continuous if there is \(\theta < 1\) for which
\[
\text{Var}_n(\log R^+) = \sup \{|\log R^+(x) - \log R^+(y)|: x \wedge y = n\}
\]
has
\[
(12.1) \quad \text{Var}_n(\log R^+) \leq C \theta^n
\]
for \( n > 1 \). It should be noted that \( \log R^+ \) can be unbounded on any cylinder when \( \Sigma^+ \) is non-compact. We lift \( R^+ \) to \( \Sigma \) by defining \( R(\cdots x_{-1}, x_1 \cdots) = R^+(x_1 \cdots) \). We will simply write \( R \) for both functions.

We write \( \text{const.}(Bb) \) for the constant with

\[
(\text{const.}(Bb))^{-1} \leq \frac{R_{k+1}(uBby)}{R_{k+1}(uBbz)} \leq \text{const.}(Bb)
\]

for any admissible \( uBb, |u| = k, |b| = |B| = 1 \), and \( y,z \in \sigma[b] \). Local Hölder continuity implies that \( \text{const.}(Bb) \) does not depend on \( B,b \). Nevertheless we use this notation to remind us to condition on two letters, as local Hölder demands.

12.2. **Strong positive recurrence by discriminants.** Following [20] we say that \( R \) is positive recurrent if there is a constant \( M_a \) with

\[
\sum_{x: \sigma^n x = x, x_0 = B} R_n(x) \in [M_B^{-1} \exp nP(\log R), M_B \exp nP(\log R)]
\]

for all \( n \in \mathbb{N} \).

The definition of strong positive recurrence is in introduced by Sarig [21] in terms of discriminants. Let us borrow some of the notation for this discussion. We switch to additive notation (e.g. \( R = \exp r \) for \( r \) locally Hölder continuous). Write \( \varphi_a = \mathbb{1}_{[a]}(x) \inf \{ n \geq 1 : \sigma^n x \in [a] \} \),

\[
Z_n(r) = \sum_{\sigma^n x = x} (\exp r)_n(x) \mathbb{1}_{[a]}(x) ; Z_n^*(r) = \sum_{\sigma^n x = x} (\exp r)_n(x) \mathbb{1}_{[a = n]}(x)
\]

\[
P(r) := P(r, \sigma) = \limsup_{n \to \infty} \frac{1}{n} \log Z_n(r) ; -p^* := \limsup_{n \to \infty} \frac{1}{n} \log Z_n^*(r)
\]

We write \( P(\tau) \) for the pressure in the system induced on returning to \( a \). (So \( \tau = \exp(\sum_{k=0}^{\infty} r \circ \sigma^n \circ \pi) \), and \( \pi \) maps the induced phase space to \( [a] \subset \Sigma^+ \).)

The Discriminant Theorem (Theorem 2 of [21]) states that \( P(\tau + p) = 0 \) has a unique solution \( p = -P(r) \) if \( r \) is not transient (“not transient” is the same as “recurrent”). The first return series controls the range of \( p \) for which \( P(\tau + p) < \infty \), indeed \( \sup \{ p : P(\tau + p) < \infty \} = p^* \). The \( a \)-discriminant of \( r \) is \( \Delta_a[r] = \sup \{ P(\tau + p) : p < p^* \} \). Non-transience implies that \( \Delta_a[r] \geq 0 \), and indeed \( P(\tau - P(r)) = 0 \). Following Sarig [21], one says that \( r \) is strongly positively recurrent if \( \Delta_a[r] > 0 \).

(At \( \Delta_a[r] = 0 \) the behaviour can be either positive recurrence or null recurrence.)

The main thing for us to note is Proposition 3 that \( P(\tau + p) \) is strictly increasing in \( (-\infty, p^*) \). Now if \( P(r) \neq -p^* \) then there is some \( -P(r) < t < p^* \) with \( 0 < P(\tau + t) < \infty \) from which \( \Delta_a[r] > 0 \). Conversely, if \( P(r) = -p^* \) then \( \Delta_a[r] = 0 \). In conclusion, strong positive recurrence is equivalent to

\[
P(r) \neq -p^*,
\]

and this is the formulation we will use.

We will use the notation

\[
-p^* = \gamma(\text{SPR}).
\]

**Return series.**

**Definition 12.1.** We will say that \( \eta \) is a power series in \( t^{-1} \) if \( \eta \) is a function of a real variable \( t \in [0, \infty) \) defined as

\[
\eta[t] = \lim_{N \to \infty} \sum_{n=0}^{N} t^{-n} a_n
\]

with \( a_n \in [0, \infty) \).
Observe that \( \eta[t] \) coincides with the Dirichlet series \( \sum_{n=1}^{\infty} \exp(-sn)a_n \) evaluated at \( s = \log t \) and so we borrow terminology such as abscissa of convergence but applied to the variable \( t = \exp(s) \). As \( \eta(t - \epsilon) \geq \eta[t] \) for any \( 0 < t - \epsilon < t \) we have that

\[
\{ t > 0 : \eta[t] < \infty \} \subset [\sigma, \infty),
\]

\[
(0, \sigma) \subset \{ t > 0 : \eta[t] = \infty \},
\]

for some \( \sigma \in [0, \infty] \) which we call the abscissa of convergence (of the power series in \( t^{-1} \)).

Let \( R : \Sigma^+ \to \mathbb{R}_+ \) with \( \Sigma^+ \) some CMS and \( \log R \) locally Hölder continuous. Recall we always assume that \( R \) has \( P(\log R, \sigma) = 0 \). To begin with we do not even ask that \( R \) is recurrent — in this way the discussion applies to both \( \sigma : \Sigma \to \Sigma \) and to the skew product \( T_\sigma : \Sigma \times G \to \Sigma \times G \), upon realising an isomorphism with a CMS. For a letter \( B \in W_1 \), the \((B\text{-conditioned)} \)return series is

\[
(12.7) \quad t \mapsto \sum_{n=1}^{\infty} t^{-n} \sum_{x: \sigma^n x = x, x_0 = B} R_n(x),
\]

where \( R_n(z) = R(z)R(\sigma z) \cdots R(\sigma^{n-1}z) \). Using local Hölder continuity 12.7 has the same abscissa of convergence as the \((B\text{-conditioned)} \)return series

\[
(12.8) \quad t \mapsto \sum_{n=1}^{\infty} t^{-n} \sum_{b: \tau(b,B) = 1} \sum_{w \in W_n^{B,b}} R_n(wBy),
\]

with fixed initial condition \( y \in \sigma[B] \). And indeed if \( \sigma \) is transitive then the abscissa of convergence is equal to \( \exp P(\log R, \sigma) = 1 \).

The periodic \( B \) first return series,

\[
(12.9) \quad t \mapsto \sum_{n=1}^{\infty} t^{-n} \sum_{x: \sigma^n x = x, x_0 = B, x_i \neq B, 0 < i < n} R_n(x),
\]

is within a constant multiple of

\[
(12.10) \quad \sum_{n=1}^{\infty} t^{-n} \sum_{b: \tau(b,B) = 1} \sum_{w \in W_n^{B,B}} R_n(wBy).
\]

We say that \( R \) has a growth gap if 12.9, or equivalently 12.10, converges for \( t = 1-\epsilon \), for some \( \epsilon > 0 \). The existence of a growth gap will allow us to consider two-letter conditioned returns.

Fix \( A, a \in W_1 \) with \( aA \) admissible. Define the \((A, a \text{ conditioned)} \)return series as

\[
(12.11) \quad \zeta^{A,a}[t] = \sum_{n=1}^{\infty} t^{-n} \sum_{w \in W_n^{A,a}} R_n(wAx),
\]

with initial condition \( Ax \in \Sigma^+ \).

**Lemma 12.2.** If \( R \) has a growth gap then

\[
\{ t > 0 : \zeta^{A,a}[t] < \infty \} = 1.
\]

If \( R \) is recurrent then \( \zeta^{A,a}[1] = \infty \).
Proof. Notice that
\[
\sum_{n=1}^\infty t^{-n} \sum_{w \in \mathcal{W}_n^{A,a}} R_n(wA)x \sum_{m=1}^\infty t^{-m} \sum_{b \tau(b,A)=1} \sum_{u \in \mathcal{W}_m^{A,b}} R_m(uA)x
\geq \text{const.}(aA) \sum_{n=1}^\infty t^{-n} \sum_{w \in \mathcal{W}_n^{A,a}} R_{n+m}(wuA)x
= \text{const.}(aA) \sum_{n=1}^\infty t^{-m} \sum_{b \tau(b,A)=1} \sum_{u \in \mathcal{W}_m^{A,b}} R_{n+m}(vuA)x
\geq \text{const.}(aA) \sum_{k=1}^\infty t^{-k} \sum_{b \tau(b,A)=1} \sum_{v \in \mathcal{W}_k^{A,b}} R_k(vuA)x.
\]

This says that conditioning on one or two letters contains the same information in the case that \( R \) is has a growth gap. We update the notion of first returns. We write \( \mathcal{A}_n = \{Aw : w \in \mathcal{W}_{n-2}, w/w_{i+1} \neq Aa, i = 1, \ldots, n - 3 \} \). Define
\[
\eta^{A,a}[t] := \sum_{n=1}^\infty t^{-n} \sum_{w \in \mathcal{A}_n} e^{R_n(wx)}.
\]
Then \( \gamma(\text{SPR}) \geq \inf \{ t > 0 : \eta^{A,a}[t] < \infty \} \).

We summarize what we have learnt.

Lemma 12.3. Let \( R : \Sigma^+ \to \mathbb{R}_+ \) with \( \Sigma^+ \) a mixing CMS and \( \log R \) locally Hölder continuous. Assume \( R \) is strongly positively recurrent. Then \( \gamma(\text{SPR}) < \exp P(\log R) \) and
\[
(\gamma(\text{SPR}), \infty) \subset \{ t > 0 : \eta^{A,a}[t] < \infty \},
(\gamma(\text{SPR}), \infty) \subset \{ t > 0 : \zeta^{A,a}[t] < \infty \},
(1, \infty) = \{ t > 0 : \zeta^{A,a}[t] < \infty \}.
\]

We are now able to prove Proposition 3.1.

Proof of Proposition 3.1. \( \exp P(\log R, T_a) > \gamma(\text{SPR}) \) implies that \( R : \Sigma^+ \times G \to \mathbb{R}_+ \) (viewed as a CMS) has a growth gap. In particular by Lemma 12.2, for any fixed \( h \)
\[
\inf \{ t > 0 : \zeta^{A,a}(x,\{a,h\})[t] < \infty \} = \exp P(\log R, T_a).
\]
But also \( \phi[t]/(e) \) is bounded from above by the \( (A,e) \)-conditioned periodic return series, and so \( \gamma(\phi^* + \delta_e) \leq \exp P(\log R, T_a) \).

12.4. Equilibrium states. Recall that we assume \( P(\log R, \sigma) = 0 \). Positive recurrence hypothesis guarantees that the transfer operator \( L \) has an eigenfunction \( Lh = h \), with \( h \) a positive locally Hölder continuous and eigenmeasure \( L^\ast \nu = \nu \); see [20]. The equilibrium state \( d\mu = hd\nu \) is finite, \( \sigma \) invariant and ergodic.

When \( \Sigma \) is compact \( \mu \) has the Gibbs property (see [25]): there is a constant \( C > 0 \) with
\[
C^{-1} R_n(\mu) \leq \mu([w]) \leq CR_n(\mu)
\]
for any \( w \) of length \( n \) and \( \gamma \in \sigma^n[w] \). In general we cannot expect to have the Gibbs property in the CMS setting but we are able to make use of the conformal property for \( \nu \).
Definition 12.4. A measure $\hat{\nu}$ is said to be $R$-conformal if there is $\lambda$ with
\begin{equation}
(12.14) \quad \int R_n^{-1} 1_{[wb]} d\hat{\nu} = \lambda^{-n} \hat{\nu}([b])
\end{equation}
for any $w \in W_n$ and $b \in W_1$ with $wb$ admissible.

In particular $\nu$ is $R$-conformal with $\lambda = 1$. We also need a local version of a Gibbs inequality

Definition 12.5. A measure $\hat{\nu}$ has a RHS local Gibbs inequality if there is a constant $C_b$ with
\begin{equation}
(12.15) \quad \hat{\nu}([wb]) \leq C_b R_n(wbx)
\end{equation}
for any admissible $wb$ with $|w| = n$ and for any $x \in \sigma[b]$.

A measure $\hat{\nu}$ has a local Gibbs inequality if there is a constant $C_b$ with
\begin{equation}
(12.16) \quad C_b^{-1} R_n(wbx) \leq \hat{\nu}([wb]) \leq C_b R_n(wbx)
\end{equation}
for any admissible $wb$ with $|w| = n$ and for any $x \in \sigma[b]$.

We check that $\nu$ has a local Gibbs inequality: inside the conformal property 12.14 we substitute $R_n(wbx) \leq \text{const.}(w_n b) R_n(wby)$ using 12.2. This gives
\begin{equation}
(12.17) \quad \text{const.}(w_n b)^{-1} R_n(wbx) \leq \frac{\nu([wb])}{\nu([b])} \leq \text{const.}(w_n b) R_n(wbx)
\end{equation}
for any $x \in \sigma[b]$.

We make a similar estimate for the equilibrium state $\mu$. First recall the standard manipulations to check shift invariance:
\[ \int f \circ \sigma^n d\mu = \int f \circ \sigma^n h d\nu = \int f \circ \sigma^n (L^*)^n \nu = \int f L^n h d\nu = \int f h d\nu. \]

Now we check the integral of $R_n^{-1} 1_{[w]}$,
\[ \int R_n^{-1} 1_{[w]} d\mu = \int R_n^{-1} 1_{[w]} h d\nu = \int L^n(R_n^{-1} 1_{[w]} h) d\nu = \int h(w \cdot) d\nu \leq h(w \xi) + |h|_{\beta^n}, \]
where $|h|_{\beta}$ is the local Hölder constant for $h$.

Suppose $R \exp \psi$ is another locally Hölder strongly positively recurrent function and that $\psi$ depends only on one letter (such is the case for Abelian extensions). Write $\nu'$ the eigenmeasure of $R \exp \psi$, $\lambda = \exp P(\log R + \psi, \sigma)$, $h'$ the eigenfunction. We have
\[ \int R_n^{-1} 1_{[w]} h' d\nu' = \lambda^{-n} \int L^n(R_n^{-1} 1_{[w]} h') d\nu' = \lambda^{-n} \int \exp \psi^n(w) h'(w \cdot) d\nu' \leq \exp \psi^n(w) \lambda^{-n} (h'(w \xi) + |h'|_{\beta^n}) \]
and a lower bound given by $\exp \psi^n(w) \lambda^{-n} (h'(w \xi) - |h'|_{\beta^n})$. Then in particular for $u, v$ with $u \wedge v = p$, some fixed $\xi \in \sigma^p([u])$, and
\[ \alpha_p = \frac{(h'(u \xi) + |h'|_{\beta^p})}{(h'(u \xi) - |h'|_{\beta^p})}, \]
we have

\[ \alpha_p^{-1} \exp \psi^n(u) \exp -\psi^n(v) \leq \frac{\int R_n^{-1} \mathbb{1}_[\alpha \rho] h' d\nu'}{\int R_n^{-1} \mathbb{1}_[\alpha \rho] h' d\nu'} \leq \alpha_p \exp \psi^n(u) \exp -\psi^n(v) \]

(12.18)

13. CONVERGENCE IN \( \ell^2 \)

In this section we present the basic properties relating to convergence of the thermodynamic densities. The main feature is that \( \rho \) permutes an orthonormal basis of the Hilbert space. We use the notation \( \delta_h \) to denote the indicator function on the coset \( h \in G/H \).

We begin by showing that for \( t > \gamma(\text{SPR}) \), \( \phi_{\leq N}[t] \) belongs to \( \ell^1(G) \). Recall that for any \( f : G \to \mathbb{R} \) we define \( ||f||_{\ell^1(G)} = \sum_{g \in G} |v(g)| \). (The definition merely uses a countable series of non-negative terms.) Substituting \( v = \phi_{\leq N}[t] \) in the definition gives

\[ ||\phi_{\leq N}[t]||_{\ell^1(G)} = \zeta^{A_a}[t], \]

which we know to be finite by Lemma 12.3.

By definition, if \( t > \gamma(\phi, \delta_e) \) then \( \sup_{N \in \mathbb{N}} \phi_{\leq N}[t](e) < \infty \). Using transitivity, this implies that for each \( g \) we have \( \sup_{N \in \mathbb{N}} \phi_{\leq N}[t](g) < \infty \). A bounded series of non-negative terms in \( \mathbb{R} \) converges and hence \( \phi[t](g) \) is well-defined (in whichever way we arrange the series). It is clear that \( \phi[t](g) \leq \zeta^{A_a}[t] \) and so \( \gamma(\phi, \delta_e) \leq 1 \).

We have the identity

\[ ||\phi[t]||_{\ell^1(G)} = \sum_{g \in G} \phi[t](g) = \zeta^{A_a}[t]. \]

So \( ||\phi[t]||_{\ell^1(G)} < \infty \) for \( t > 1 \), but \( ||\phi[t]||_{\ell^1(G)} = \zeta^{A_a}[t] = \infty \) for \( t \leq 1 \).

Proof of Lemma 2.4. Recall that in a Hilbert space we say that \( X_N \) weakly converges to \( X_\infty \) if \( \langle X_N - X_\infty, v \rangle_H \to 0 \) as \( N \to \infty \) for every vector \( v \). And strongly converge if \( \|X_N - X_\infty\|_H \to 0 \) as \( N \to \infty \). The reader may already be familiar with this fact: if \( \|X_N\|_H \) is uniformly bounded and \( X_N \) weakly converge to \( X_\infty \) on an orthonormal basis then in fact \( X_N \) strongly converge to \( X_\infty \). We provide the details of this argument. Denote the orthonormal basis vectors permuted by \( \rho \) as \( e_i \) for \( i \in \mathbb{N} \).

The argument is applied to \( X_N = \phi_{\leq N}[t] \ast f \) and \( X_\infty \) formally defined by \( \langle X_\infty, e_i \rangle_H = \lim_{N \to \infty} \langle \phi[t] \ast f, e_i \rangle_H \). Let us check that \( X_\infty \) is the weak limit with respect to the orthonormal basis \( e_i \). Let \( v \in H_+ \) be arbitrary (it is sufficient to check against vectors in \( H_+ \)). We will show that \( \langle X_N, v \rangle_H \) are a Cauchy sequence. Write \( v = \sum_{q=1}^{\infty} a_q e_q \). Choose \( Q \) with

\[ \| \sum_{n=Q}^{\infty} a_n e_n \|_H \leq \epsilon/2. \]

Then we have

\[ \langle X_N - X_M, v \rangle_H \leq \sum_{q \leq Q} a_q \langle X_N - X_M, e_q \rangle_H + \langle X_N - X_M, \sum_{q > Q} a_q e_q \rangle_H. \]

Using the Cauchy-Schwarz inequality we have

\[ \langle X_N - X_M, \sum_{q > Q} a_q e_q \rangle_H \leq \|X_N - X_M\|_H \| \sum_{q > Q} a_q e_q \|_H \leq \epsilon \sup_{n \in \mathbb{N}} \|X_n\|_H \]

Now for any weak limit \( X_\infty^w \) (i.e. \( \langle X_N - X_\infty^w, v \rangle_H \to 0 \)) we also have

\[ \epsilon \geq \langle X_N - X_\infty^w, v \rangle_H \geq \langle X_N - X_\infty^w, a_i e_i \rangle_H, \]

so that \( \langle X_\infty^w, e_i \rangle_H = \langle X_\infty, e_i \rangle_H \) for each \( i \). In conclusion, \( X_N \) weakly converges to \( X_\infty \).
Recall that the norm is lower semicontinuous:
\[ \|X_\infty\|_H = \sup_v \langle X_\infty, v \rangle_H = \sup_N \lim_{N \to \infty} \langle X_N, v \rangle_H \leq \liminf_{n \to \infty} \|X_n\|_H, \]
using Cauchy-Schwarz in the last line. It follows that \( \|X_\infty\|_H < \infty \). On the other hand we have by the monotonicity
\[ \|X_\infty\|_H \geq \limsup_{n \to \infty} \|X_n\|_H. \]

Denote \( P_{i>Q} \) the projection on to the span of \( \{e_i \colon i > Q\} \). We have \( \|P_{i>Q}(X_\infty - X_N)\|_H \leq \|P_{i>Q}X_\infty\|_H + \|P_{i>Q}X_N\|_H \leq 2\|P_{i>Q}X_\infty\|_H, \) by monotonicity. Using that \( \|X_\infty\|_H < \infty \), we can choose \( Q \) large enough to make \( \|P_{i>Q}X_\infty\|_H < \epsilon/2 \), for some \( \epsilon \). (Indeed \( \|X_\infty\|_H < \infty \) implies that \( \sum_{i \in \mathbb{N}} |\langle X_\infty, e_i \rangle|^2 \) is a convergent series.) For a fixed \( Q \) we can choose \( N \) large enough with
\[ \|P_{i\leq Q}(X_\infty - X_N)\|_H^2 = \sum_{q \leq Q} \|X_\infty - X_N, e_q\|_H^2 \leq \epsilon \]
In conclusion \( \|X_N - X_\infty\|_H \to 0 \) as \( N \to \infty \). That is, we have strong convergence of \( X_N \) to \( X_\infty \).

\[ \square \]

**Proof of Lemma 2.6.** We need to upper bound \( \phi^{B,a} \) in terms of \( \phi^{A,a} \). Choose \( u_B \) with \( Au_B B \) admissible, in this way any \( v \in W_n^{B,a} \) has \( Au_B v \in W_n^{A,a} \). Moreover,
\[ R_{[Au_B]+n}(Au_B vx) = R_{[Au_B]}(Au_B vx)R_n(vx) \geq \text{const.} \cdot (u_B B)^{-1} R_{[Au_B]}(Au_B Bz)R_n(vx) \]
for some fixed \( z \in \sigma[B] \). Let \( C \) be the constant with \( c_n \leq Cc_{n-|u_B|} \). We deduce that
\[ \sum_{n=1}^N t^{-n} c_n \sum_{v \in W_n^{B,a}} R_n(vx)(\rho(g(v)) f, \delta_h) \]
\[ \leq \frac{\text{const.} \cdot (u_B B)}{R_{[u_B]+1}(Au_B Bz)} \sum_{n=1}^N t^{-n} c_n \sum_{v \in W_n^{B,a}} \langle \phi_{c_i \leq N+|u_B|}[t] * f, \rho(g(Au_B)) \delta_h \rangle \]
\[ \leq C \frac{\text{const.} \cdot (u_B B)}{R_{[u_B]+1}(Au_B Bz)} t^{[u_B]+1} \langle \phi_{c_i \leq N+|u_B|}[t] * f, \rho(g(Au_B)) \delta_h \rangle \]
The result follows.

\[ \square \]

**Lemma 13.1.** Transitivity implies \( \gamma(f) > 0 \).

**Proof.** We only need two distinct periodic orbits, represented by \( u, v \) with \( g(u) = g(v) = e \), to create non-trivial Gurevič pressure.

\[ \square \]

**14. The boundary measure for the free group**

We explain the action of \( F_{a,b} \) on its visual boundary in terms of a subshift of finite type, the related unitary representation and spherical function.

Identify the visual boundary of \( F_{a,b} \) with \( \Sigma^t \), then an element \( x \) is an infinite reduced word, and for any \( g \in F_{a,b} \) we have that \( g^{-1}x \) is an infinite word but may not be reduced. The \( F_{a,b} \) action is defined as \( g^{-1} \cdot x = y \) where \( y \) is the infinite reduced form of \( g^{-1}x \). The \( F_{a,b} \) action does not preserve the Markov measure \( \nu \) associated to \( R = 1/3 \) but it does preserve the measure class. This gives rise to a (unitarizable) representation of \( F_{a,b} \) in \( L^2(\Sigma^t, \nu) \). Set \( c(g, x) = \frac{d_{F_{a,b}}}{dx}(x); \) this can easily be computed as \( c(g, x) = 3^{-q(g,x)} \) where \( q(g, x) = |g| \) if \( g^{-1}x \) is already reduced, and
\[ q(g, x) = |g| - 2(g^{-1} \wedge x) \] in general. To see this it is enough to check cylinders: for \( m > n, u = u_1 \cdots u_m \) and \( g_1 \cdots g_n \), we have \( g^{-1} u = g_n^{-1} \cdots g_{k+1}^{-1} u_{k+1} \cdots u_m \) so that
\[
\int \mathbb{1}_u(y) dg_u \nu(y) = \int \mathbb{1}_u(y) d\nu(g^{-1} \cdot y)
\]
\[
= \int \mathbb{1}_u(g \cdot x) d\nu(x) = \int \mathbb{1}_{g^{-1} u}(x) d\nu(x)
\]
\[
= \frac{1}{4} 3^{-|u| + k - n + k}
\]
The conclusion follows upon observing that \( g^{-1} \wedge u = k \) and \( |g| = n \). Denote \( \pi_{1/2} \) for the unitary representation
\[ \pi_{1/2}(g) F(x) = c(g, x)^{1/2} F(g^{-1} \cdot x). \]
Let us compute the matrix coefficient \( g \mapsto \langle \pi_{1/2}(g) \mathbb{1}, \mathbb{1} \rangle_{L^2(\Sigma^+, \nu)} \), where \( \mathbb{1} \) denotes the constant unit function. We have
\[ \langle \pi_{1/2}(g) \mathbb{1}, \mathbb{1} \rangle_{L^2(\Sigma^+, \nu)} = \int \Sigma^+ c(g, x)^{1/2} d\nu(x) \]
Write \( E_k \) for the set with \( q(g, x) = |g| - 2k \); or equivalently \( x \) with \( g^{-1} \wedge x = k \). The measure of \( E_k \) takes values:
\[ \nu(E_k) = \begin{cases} 
\frac{3}{4}, & \text{for } k = 0; \\
\frac{1}{3} 3^{-k + 1} \frac{2}{3}, & \text{for } 0 < k < |g| \\
\frac{1}{3} 3^{-|g| + 1}, & \text{for } k = |g|.
\end{cases} \]
Therefore
\[
\langle \pi_{1/2}(g) \mathbb{1}, \mathbb{1} \rangle_{L^2(\Sigma^+, \nu)} = \frac{3}{4} 3^{-|g|} + \frac{6}{4} 3^{-|g| + 1} \sqrt{3}^{3|g|} + \sum_{k=1}^{r} \frac{1}{2} 3^{-|g| + 2k} \frac{1}{3} 3^{-k}
\]
\[
= \frac{3}{2} 3^{-|g|} + \frac{1}{2} \sqrt{3}^{-|g|} \sum_{k=1}^{r} \frac{1}{3} 3^{2k} 3^{-k}
\]
\[
= \frac{3}{2} 3^{-|g|} + \frac{1}{2} \sqrt{3}^{-|g|} ((|g| - 1)
\]
\[
= \sqrt{3}^{-|g|} \left( \frac{3}{2} + \frac{1}{2} (|g| - 1) \right)
\]
Then \( g \mapsto \langle \pi_{1/2}(g) \mathbb{1}, \mathbb{1} \rangle_{L^2(\Sigma^+, \nu)} \) is identically \( g \mapsto (1 + \frac{|g|}{2}) \sqrt{3}^{-|g|} \).

Spherical functions for the free group are explored in more detail by Figà-Talamanca and Picardello [11].

15. The slowly increasing function

A formula for the slowly increasing function is given in [6]. We present the details needed to verify it works.

**Proposition 15.1.** For any real series \( \eta(t) = \sum_{n=1}^{\infty} t^{-n} B_n \) there is a slowly increasing function \( c : \mathbb{N} \to \infty \) so that \( \eta_c(t) = \sum_{n=1}^{\infty} t^{-n} c_n B_n \) has \( \eta_c(\gamma) = \infty \) at \( \gamma = \lim \sup B_n^{1/n} \) and satisfies
\[ c_{n+k} \leq c_n c_k. \]

**Proof.** By hypothesis,
\[ \lim_{n \to \infty} B_n^{1/n} = \gamma, \sum_{n=1}^{\infty} \gamma^{-n} B_n < \infty. \]
First shift so that
\[ D_n = \gamma^{-n} B_n \]
and now the problem is

\[ \limsup_{n \to \infty} D_n^{1/n} = 1, \quad \sum_{n=1}^{\infty} D_n < \infty. \]

Certainly \( D_n \) are decreasing (in order to have convergence in Eq 15.1) so \( D_n^{-1} \) are increasing. A naive idea would be to set \( c_n = D_n^{-1} \) so that every summand is 1! (Far too optimistic!) A slowly increasing function necessarily has \( \limsup c_n^{1/n} = 1 \), but this is only sufficient, and so equation 15.1 is not enough. Failing this we might think to choose \( c_n = D_{q(n)}^{-1} \) with \( q(n) \leq n \). In order to have

\[ \sum_{n=1}^{\infty} D_n D_{q(n)}^{-1} = \infty \]

it is sufficient that the equation \( q(n) = n \) is satisfied for infinitely many \( n \) (no matter how sparse the subset may be!). For instance if \( q(n) \) were constant in the range \([N, N + M] \) then certainly

\[ D_{q(n+k)}^{-1} \leq D_{q(n)}^{-1} \leq D_{q(n)}^{-1}(1 + \epsilon(k)). \]

provided \( n \in [N, N + M] \), \( n + k \in [N, N + M] \). (But this is too optimistic!)

The solution is to take a sparse set (a collection \( N_r \) having \( N_r, N_{r+1}^{-1} \to 0 \)) and “linearly interpolate” in such a way to make the extension of \( c_{N_r} = D_{N_r}^{-1} \) slowly increasing. For \( n \in [N_r, N_{r+1}] \), set \( \alpha_r(n) \in [0, 1] \),

\[ \alpha_r(n) = \frac{N_{r+1} - n}{N_{r+1} - N_r}. \]

(No \( \alpha_r(N_r) = 1, \alpha_r(N_{r+1}) = 0 \).) Set

\[ c_n = (D_{N_r}^{-1})^{n_{N_r(n)}} (D_{N_{r+1}}^{-1})^{n(1 - \alpha_r(n))}. \]

For brevity write \( d(r) = N_r^{-1} \log D_{N_r}^{-1} \), whence

\[ \frac{1}{n} \log c_n = \alpha_r(n)d(r) + (1 - \alpha_r(n))d(r + 1). \]

We may assume \( r \) are chosen with \( D_{N_r}^{-1} > 1 \) (so \( d(r) > 0 \)) and with \( N_r^{-1} \log D_{N_r}^{-1} \) monotonically decreasing to 0 (so \( 0 < d(r + 1) < d(r) \)). It is immediate that \( \frac{1}{n} \log c_n \to 0 \).

We check the slowly increasing condition. Let \( k \) be arbitrary. We must show that

\[ \lim_{n \to \infty} \log c_n - \log c_{n-k} = 0. \]

Equivalently, that

\[ \lim_{n \to \infty} \log c_n - \frac{n}{n-k} \log c_{n-k} = 0. \]

If \( n, n-k \in [N_r, N_{r+1}] \) then

\[ \frac{1}{n} \log c_n - \frac{1}{n-k} \log c_{n-k} = \frac{[\alpha_r(n) - \alpha_r(n-k)]d(r) - (\alpha_r(n) - \alpha_r(n-k))d(r+1)}{n-k}. \]

When \( n \) is large enough (so \( r, N_r \) are large enough) we have

\[ \frac{1}{n} \log c_n - \frac{1}{n-k} \log c_{n-k} = \frac{[\alpha_r(n) - \alpha_r(n-k)]e^{-\delta} - [\alpha_r(n) - \alpha_r(n-k)]e - \delta}{n-k} = -\delta[\alpha_r(n) + \alpha_r(n-k)]. \]

Note that

\[ \alpha_r(n) - \alpha_r(n-k) = \frac{k}{N_{r+1} - N_r} = \frac{k}{1 - \frac{N_r}{N_{r+1}}} = \frac{k}{N_{r+1}} (1 - \epsilon_r). \]

So

\[ |\log c_n - \frac{n}{n-k} \log c_{n-k}| \leq N_{r+1} \delta[\alpha_r(n) + \alpha_r(n-k)] \leq k\delta(1 - \epsilon_r). \]
The reason for interpolation is to cover the disjoint ranges. It is sufficient to check for $k = 1$. Now if $n = N_r$ and $n - 1 < N_r$ we have
\[
\frac{c_n^{1/n}}{c_{n-1}^{1/(n-1)}} = \left( D_{N_r-1}^{-1} \right)^{1/n} \left( D_{N_r}^{-1} \right)^{-1/n}.
\]
So
\[
\frac{1}{n} \log c_n - \frac{1}{n-1} \log c_{n-1} = \frac{1}{N_r} d(r) + \frac{1}{N_r} d(r - 1).
\]
Since $d(r)$ tends to 0 the conclusion follows.

Now that we have convinced ourselves of the divergence and slowly varying property it still remains to check that products have $c_n c_k \geq c_{n+k}$ ($\log c_{n+k} \leq \log c_n + \log c_k$). We can write
\[
\log c_{n+k} = \left( \sum_{q=1}^{n} \alpha_r(n) d(r) + (1 - \alpha_r(n)) d(r - 1) \right) + \left( \sum_{q=1}^{k} \alpha_r(n) d(r) + (1 - \alpha_r(n)) d(r - 1) \right).
\]
Since $d(r) > 0$ are decreasing in $r$, any linear combination has
\[
td(r - 2) + (1-t)d(r - 1) > sd(r - 1) + (1-s)d(r),
\]
giving the conclusion when $n + k \in [N_r, N_{r+1}]$ and $n, k < N_r$. We also have monotonicity. If $n, n + k \in [N_r, N_{r+1}]$ then $\alpha_r(n) \leq \alpha_r(n + k)$ whence $\alpha_r(n) d(r) + (1 - \alpha_r(n)) d(r - 1) < \alpha_r(n + k) d(r) + (1 - \alpha_r(n + k)) d(r - 1)$. \hfill \Box

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