LONG-TIME BEHAVIOR FOR A CLASS OF WEIGHTED EQUATIONS WITH DEGENERACY

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Abstract. In this paper we study the existence and some properties of the global attractors for a class of weighted equations when the weighted Sobolev space $H_{1,a}^0(\Omega)$ (see Definition 1.1) cannot be bounded embedded into $L^2(\Omega)$. We claim that the dimension of the global attractor is infinite by estimating its lower bound of $\mathbb{Z}_2$-index. Moreover, we prove that there is an infinite sequence of stationary points in the global attractor which goes to 0 and the corresponding critical value sequence of the Lyapunov functional also goes to 0.

1. Introduction. Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial \Omega$, we consider the long time behavior of a class of degenerate parabolic equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \text{div}(a(x)\nabla u) - \lambda u + |u|^{p-2}u &= 0, & \text{in } \Omega \times \mathbb{R}^+, \\
u &= 0, & \text{on } \partial \Omega \times \mathbb{R}^+, \\
u(x,0) &= u_0 \in L^2(\Omega), & \text{in } \Omega,
\end{aligned}
$$

(1)

where $p \geq 2$ and $a(x)$ satisfied the following assumption

(A1): $a(x) \in L^\infty(\Omega)$ and $a(x) = 0$ for $x \in \Sigma$, $a(x) > 0$ for $x \in \overline{\Omega}\setminus\Sigma$, where $\Sigma$ is a closed subset of $\Omega$ with $\text{meas}(\Sigma) = 0$.

For our problem, it is natural to look for solutions in weighted Sobolev spaces $H_{1,a}^0(\Omega)$ which is defined as follows.

Definition 1.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $a(x)$ satisfy (A1). $H_{1,a}^0(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$
\|u\|_{H_{1,a}^0} = \left( \int_\Omega a(x)|\nabla u|^2 \right)^{\frac{1}{2}}.
$$

(2)

Furthermore, $H_{1,a}^0(\Omega)$ is a Hilbert space with respect to the scalar product

$$
\langle u, v \rangle = \int_\Omega a(x)\nabla u \cdot \nabla v dx.
$$
In addition, since our problem is mainly dependent on the properties of the norm of weighted Sobolev space $H^{1, a}_0(\Omega)$, the following assumption is always needed.

$(A_2)$ The embedding $H^{1, a}_0(\Omega) \hookrightarrow L^r(\Omega)$ is compact for some $r \in [1 + \infty)$.

Usually, this assumption $(A_2)$ can be easily satisfied. For example, $(A_2)$ holds if $a(x)$ satisfies

$$\int_{\Omega} \frac{1}{a(x)^\alpha} \, dx < \infty,$$

which will be proved in Appendix A.

Under the condition of $(A_2)$, if $r \geq 2$, the problem is completely similar to the usual reaction-diffusion equations which is $a(x) \equiv 1$. However, if $1 \leq r < 2$ and $H^{1, a}_0(\Omega)$ cannot even be bounded embedded into $L^2(\Omega)$ the problem is very different from the usual case.

**Remark 1.** A model for the weight $a$ is $|x|^\beta$ for $2 < \beta < N$ as $\alpha = n$ in (3). And it is obvious that there are a lot of functions $u$ which belong to $H^{1, a}_0(\Omega)$ but not to $L^2(\Omega)$, for example, $u(x) = \frac{\alpha(x)}{|x|^\alpha} \omega$, $x \in \Omega$ and $\alpha(x) \in C^1_0(\Omega)$.

For degenerate parabolic PDEs when $H^{1, a}_0(\Omega)$ cannot be bounded embedded into $L^2(\Omega)$, not so much is known about the long-time behavior although there are a lot of work for it when $H^{1, a}_0(\Omega)$ can be compactly embedded into $L^2(\Omega)$ (see [1]-[14]). Here, we are particularly interested in the case that $1 \leq r < 2$ in $(A_2)$ and $H^{1, a}_0(\Omega)$ can not even be bounded embedded into $L^2(\Omega)$ to consider the existence and properties of the global attractor for Eq. (1). The problem is far from being just technical and the degeneration in this case can lead to essentially new types of attractors which are not observable in ‘regular’ equation in bounded domains. As shown, the global attractors of the degenerate equations are infinite-dimensional.

Since Efendiev & Zelik have given the first example of a physically relevant dissipative system with an infinite-dimensional global attractor in [6], more and more attention is being paid to some dissipative system with an infinite-dimensional global attractor, see [6]-[9]. In order to study such attractors one usually uses the concept of Kolmogorov’s $\varepsilon$-entropy, see [6]-[10]. For some symmetric dynamical system with Lyapunov function in a Banach space, some authors in [15, 19, 21, 22, 20] have shown that $Z_2$-index is a useful tool to estimate the lower bounded of the dimension of the attractor. In this paper, we consider the degenerate case of (1) using the symmetry of its Lyapunov function, as shown, which provides another example that the dimension of global attractor is infinity. First, we prove the existence of the global solution and the global attractor of (1). And then we use $Z_2$ index theory to show that the dimension of the global attractor of (1) is infinite. Finally, we prove that there is an infinite sequence of equilibrium points in the global attractor, in particularly which converges to 0.

This paper is organized as follows. In section 2, we prove the existence and uniqueness of solution, and in section 3 we further prove the existence of a global attractor. In section 4 we estimate the lower bound of $Z_2$-index of a subset of the global attractor and show that its dimension is infinite. Finally we show in section 5 that there exists an infinite sequence of stationary points which goes to 0 and that the critical value sequence of the Lyapunov functional also goes to 0.

2. **Existence of the global solutions.** We now study the existence of the weak solutions of Eq. (1) defined by the integral equality in a weighted Sobolev space. For
convenience, throughout we denote \( \Omega_T = \Omega \times [0, T] \), \( V = L^2(0, T; H^1_0(\Omega)) \cap L^p(\Omega_T) \) and \( V^* \) the dual space of \( V \), respectively. In addition, let \( \| \cdot \|_p \) be the norm of \( L^p(\Omega) \) \((p \geq 1)\), \(|u|\) be the modular (or absolute value) of \( u \), \( C \) be the arbitrary positive constants, which may be different from line to line and even in the same line.

**Definition 2.1.** A function \( u(x, t) \) is called a weak solution of (1) on \([0, T]\) iff

\[
\begin{align*}
& u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \cap L^p(0, T; L^p(\Omega)) \\
& u|_{t=0} = u_0 \text{ almost everywhere in } \Omega \text{ such that} \\
& \int_0^T \int_\Omega \left( \frac{\partial u}{\partial t} + a \nabla u \nabla \xi - \lambda u \xi + |u|^{p-2} u \xi \right) \, dx \, dt = 0
\end{align*}
\]

holds for all test functions \( \xi \in V \).

As usual, in order to study the degenerate case, we prove the existence and uniqueness of the global solution to problem (1) by use of approximating problem (1) by non-degenerate ones.

**Theorem 2.2.** Suppose \( \Omega \subset \mathbb{R}^n \) is an open bounded domain with smooth boundary. Then for any \( u_0 \in H^1_0(\Omega) \cap L^p(\Omega) \) and \( T > 0 \) there exists a unique solution \( u \) of (1) which satisfies

\[
\begin{align*}
& u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \cap L^p(\Omega_T) \\
& \text{The mapping } u_0 \mapsto u(t) \text{ is continuous in } L^2(\Omega).
\end{align*}
\]

**Proof.** For any \( 0 < \varepsilon < 1 \), we choose \( u_{\varepsilon, 0} \in C_c^\infty(\Omega) \) such that \( \|u_{\varepsilon, 0}\|_{L^\infty(\Omega)} \) are uniformly bounded with respect to \( \varepsilon \), and

\[
u_{\varepsilon, 0} \to u_0 \text{ in } L^2(\Omega).
\]

Consider the problem

\[
\begin{align*}
\frac{\partial u_\varepsilon}{\partial t} - \text{div}(a_\varepsilon(x) \nabla u_\varepsilon) - \lambda u_\varepsilon + |u_\varepsilon|^{p-2} u_\varepsilon &= 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\
u_\varepsilon &= 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
|u_\varepsilon(x, 0)| &= u_{\varepsilon, 0} \quad \text{in } \Omega,
\end{align*}
\]

where \( a_\varepsilon(x) = a(x) + \varepsilon \).

According to the classical theory on parabolic equations (see for example [4, 5, 17, 18]), we know the problem admits a unique weak solution \( u_\varepsilon \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \cap L^p(0, T; L^p(\Omega)) \). Here \( u_\varepsilon \) is called a weak solution of the problem (4), if for any \( \varphi \in C_0^\infty(\Omega) \), we have

\[
\int_0^T \int_\Omega \left( \frac{\partial u_\varepsilon}{\partial t} \varphi + a_\varepsilon \nabla u_\varepsilon \nabla \varphi - \lambda u_\varepsilon \varphi + |u_\varepsilon|^{p-2} u_\varepsilon \varphi \right) \, dx \, dt = 0
\]

and \( u_{\varepsilon}|_{t=0} = u_{\varepsilon, 0} \) almost everywhere in \( \Omega \).

We do some estimates on \( u_\varepsilon \) in the following.

Multiplying (4) by \( u_\varepsilon \) and integrating over \( \Omega \), we get

\[
\frac{1}{2} \frac{d}{dt} \|u_\varepsilon\|^2_2 + \int_\Omega a_\varepsilon |\nabla u_\varepsilon|^2 \, dx - \lambda \int_\Omega |u_\varepsilon|^2 \, dx + \int_\Omega |u_\varepsilon|^p \, dx = 0.
\]

We can use Young’s inequality to write

\[
\frac{1}{2} \frac{d}{dt} \|u_\varepsilon\|^2_2 + \int_\Omega a_\varepsilon |\nabla u_\varepsilon|^2 \, dx + \frac{1}{2} \int_\Omega |u_\varepsilon|^p \, dx \leq \left( \frac{1}{2\lambda} \right)^{\frac{2}{p-2}} |\Omega|.
\]
where $|\Omega| = \int_{\Omega} dx$, The Gronwall lemma implies

\[ u_\varepsilon \] is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$ with respect to $\varepsilon$

for any $T \in \mathbb{R}$. So a subsequence still denoted as $u_\varepsilon$ and $u$ in $L^\infty(0, T; L^2(\Omega))$ can be found such that

\[ u_\varepsilon \rightarrow u \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \] weak-star.

Integrating (7) both sides between 0 and $T$, we may get

\[ \int_0^T a|\nabla u_\varepsilon|^2 dx \leq \int_0^T a|\nabla u_\varepsilon|^2 dx + \varepsilon \int_0^T |\nabla u_\varepsilon|^2 dx dt \]

\[ \leq \int_0^T a|\nabla u_\varepsilon|^2 dx dt \leq C(|\Omega|, \lambda) \] (8)

and

\[ \int_0^T \int_{\Omega} |u_\varepsilon|^p dx dt \leq C \]

with $C$ independent of $\varepsilon$.

We now extract a weakly convergent subsequence, denoted also by $u_\varepsilon$ for convenience, with

\[ u_\varepsilon \rightarrow u \quad \text{in} \quad L^2(0, T; H^1_{0,a}(\Omega)) \] weakly,

\[ u_\varepsilon \rightarrow u \quad \text{in} \quad L^p(0, T; L^p(\Omega)) \] weakly.

Therefore, to obtain the existence, it suffices for us to show

\[ \int_0^T \int_{\Omega} (\frac{\partial u}{\partial t} + a(x) \nabla u \nabla \varphi - \lambda u \varphi + |u|^{p-2} u \varphi) dx dt = 0 \quad \text{for} \quad \varphi \in C_0^\infty(\Omega). \] (9)

From (8) we can infer that

\[ \varepsilon \int_0^T \int_{\Omega} |\nabla u_\varepsilon||\nabla \varphi| dx dt \leq \frac{\varepsilon^{4/3}}{2} \int_0^T \int_{\Omega} |\nabla u_\varepsilon|^2 dx dt + \frac{\varepsilon^{2/3}}{2} \int_0^T \int_{\Omega} |\nabla \varphi|^2 dx dt \rightarrow 0 \]

as $\varepsilon \rightarrow 0$. Multiplying (4) by $\varphi$ and letting $\varepsilon \rightarrow 0^+$ it is immediate that (9) holds and $u$ is a weak solution of Problem (1).

To prove uniqueness and continuous dependence, let $u_0$ and $v_0$ be in $H^1_{0,a}(\Omega) \cap L^p(\Omega)$ and consider $w(t) = u(t) - v(t)$. Then

\[ \frac{\partial w}{\partial t} - \text{div}(a(x) \nabla w) = \lambda w - |u|^{p-2} u + |v|^{p-2} v, \quad w(0) = u_0 - v_0, \]

and multiplying by $w$ and integrating over $\Omega$ gives

\[ \frac{1}{2} \frac{d}{dt} ||w||_2^2 + ||w||_{H^1_{0,a}}^2 = \lambda ||w||_2^2 - (|u|^{p-2} u - |v|^{p-2} v, w). \] (10)

Note that there exists a constant $C > 0$ such that

\[ (|u|^{p-2} u - |v|^{p-2} v, w) = \int_{\Omega} \int_v^u (p-1)|s|^{p-2} dsdwdx \]

\[ = \int_{\Omega} (p-1)|v + \theta w|^{p-2}|w|^2 dx \geq -C||w||_2. \]

(10) shows that

\[ \frac{1}{2} \frac{d}{dt} ||w||_2^2 \leq C||w||_2^2. \]

Integrating this gives the uniqueness and continuity in $L^2(\Omega)$. \hfill \Box
Now we can use these solutions to define a semigroup \( \{S(t)\}_{t \geq 0} \) in \( L^2(\Omega) \), which is a closed bounded absorbing set in \( L^2(\Omega) \), by setting
\[
S(t)u_0 = u(t),
\]
which is continuous on \( u_0 \) in \( L^2(\Omega) \).

3. Existence of the global attractor. Our goal in this section is to show that the semigroup associated with the degenerate equation (1) possesses the global attractor in \( L^2(\Omega) \). We start with the existence of bounded absorbing sets.

**Theorem 3.1.** The semigroup \( \{S(t)\} \) associated to the system (1) possesses a bounded absorbing set in \( L^2(\Omega) \), \( L^p(\Omega) \) and \( H^1_0(\Omega) \), respectively, i.e., for any bounded subset \( B \) in \( H^1_0(\Omega) \cap L^p(\Omega) \), there exists a constant \( T(||u_0||_{H^1_0(\Omega) \cap L^p}) \), such that
\[
||u(t)||^2_2 \leq \rho_0
\]
and
\[
||u(t)||^p_p + \int_{\Omega} a(x)|\nabla u(t)|^2 \leq \rho_1 \quad \text{for all} \quad t \geq T, \quad u_0 \in B,
\]
where both \( \rho_0 \) and \( \rho_1 \) are positive constants independent of \( B \), \( u(t) = S(t)u_0 \).

**Proof.** Multiplying (1) by \( u \), integrating over \( \Omega \), we get
\[
\frac{d}{dt}||u||^2_2 + \int_{\Omega} a(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + \int_{\Omega} |u|^p dx = 0. \tag{11}
\]
Note that \( p \geq 2 \), from (11) we have
\[
\frac{d}{dt}||u||^2_2 + C \int_{\Omega} |u|^2 dx \leq C(\lambda).
\]
Using the Gronwall lemma, there exists a constant \( T_0(||u_0||_{H^1_0(\Omega) \cap L^p}) \), such that
\[
||u(t)||^2_2 \leq \rho_0 \quad \text{for} \quad t \geq T_0.
\]
Taking \( t \geq T_0 \) and integrating (11) on \([t, t + 1]\) gives
\[
||u(t + 1)||^2_2 + \int_t^{t+1} \left( \int_{\Omega} a(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + \int_{\Omega} |u|^p dx \right) dt = ||u(t)||^2_2.
\]
With an application of the existence of the bounded absorbing sets in \( L^2(\Omega) \), it follows that for any \( t \geq T_0 \),
\[
\int_t^{t+1} \left( \int_{\Omega} a(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + \frac{2}{p} \int_{\Omega} |u|^p dx \right) dt \leq C(\lambda, \rho_0). \tag{12}
\]
On the other hand, multiplying (1) by \( u_t \) and integrating over \( \Omega \), we obtain
\[
||u_t||^2_2 + \frac{d}{dt} \left( \int_{\Omega} a(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + \frac{2}{p} \int_{\Omega} |u|^p dx \right) = 0. \tag{13}
\]
By the uniform Gronwall lemma, (13) implies that for \( t \geq T_0 + 1 \)
\[
\int_{\Omega} a(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + \frac{2}{p} \int_{\Omega} |u|^p dx \leq C(\lambda, \rho_0).
\]
Then we have
\[
\int_{\Omega} a(x)|\nabla u|^2 dx + \frac{2}{p} \int_{\Omega} |u|^p dx \leq C(\lambda, \rho_0).
\]
Now, taking \( \rho_1 = C(\lambda, \rho_0) \) and \( T = T_0 + 1 \), we complete the proof of Theorem 3.1. □
By the uniform compactness method introduced in [4, 5, 17, 18] and $H_0^1(\Omega)$ is compactly embedding in $L^r(\Omega)$, we know that the absorbing set $B$ in $H_0^1(\Omega)$ is compact in $L^r(\Omega)$. Thanks to the $L^p(\Omega)$ interpolation inequality $\|u\|_{L^p} \leq \|u\|_{L^p}^{(r-2)/(r-p)} + \|u\|_{L^2}^{2(r-p)/2(r-2)}$, we can have

**Theorem 3.2.** For each $\lambda > 0$, the semigroup $\{S(t)\}_{t \geq 0}$ generated by the weak solution of Eq. (1) possesses a global attractor $A$ in $L^2(\Omega)$, i.e., $A$ is compact, invariant in $L^2(\Omega)$ and attracts the bounded sets of $L^2(\Omega)$ in the topology of $L^2(\Omega)$.

4. **$Z_2$-index of the global attractor.** Our aim of this section is to show that the dimension of the global attractor obtained in Section 4 is infinite. We start with the properties of $Z_2$-index and two critical lemmas.

**Lemma 4.1.** Let $X = L^2(\Omega) \cap H_0^{1,\alpha}(\Omega)$ and $M = \{ u \in X \mid \|u\|_{L^2(\Omega)} = 1 \}$. Suppose that $H_0^{1,\alpha}(\Omega)$ cannot be bounded embedded into $L^2(\Omega)$, i.e.,

$$\inf_{u \in M} \int_{\Omega} a(x) |\nabla u|^2 \, dx = 0.$$  \hspace{1cm} (14)

Then

$$\inf_{u \in M \cap X_n^\perp} \int_{\Omega} a(x) |\nabla u|^2 \, dx = 0$$

holds for any finite dimensional subspace $X_n$ of $X$, where $X_n^\perp = \{ u \in X \mid \int_{\Omega} uv \, dx = 0, \forall v \in X_n \}$ and $X$ is endowed with the norm $\|u\|_X = \left( (\|u\|_{H_0^{1,\alpha}(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2)^{1/2} \right)$.

**Proof.** Assume by contradiction there exist some $n_0$ and an $n_0$ dimensional subspace $X_{n_0}$ of $X$ such that

$$\inf_{u \in M \cap X_{n_0}^\perp} \int_{\Omega} a(x) |\nabla u|^2 \, dx = \alpha > 0,$$

which implies that $X_{n_0}^\perp$ with the norm of $H_0^{1,\alpha}(\Omega)$ is bounded embedded into $L^2(\Omega)$, that is,

$$\|u\|_{L^2(\Omega)} \leq \frac{1}{\alpha} \|u\|_{H_0^{1,\alpha}(\Omega)}, \quad \forall u \in X_{n_0}^\perp.$$  \hspace{1cm} (15)

So $X_{n_0}^\perp$ is a closed subspace of $L^2(\Omega) \cap H_0^{1,\alpha}(\Omega)$ with respect to the norm of $H_0^{1,\alpha}(\Omega)$ and $X_{n_0}^\perp$ is a finite codimensional subspace of $X$.

Let

$$Y = \left\{ u \in X \mid \int_{\Omega} a(x) \nabla u \nabla w \, dx = 0, \forall w \in X_{n_0}^\perp \right\},$$

then

$$X = X_{n_0}^\perp \oplus Y.$$  \hspace{1cm} (14)

We write for any $u \in X$

$$u = u_1 + u_2, \quad u_1 \in X_{n_0}^\perp, \quad u_2 \in Y.$$

From the definition of $Y$ and (15), it follows that for any $u \in M$

$$\int_{\Omega} a(x) |\nabla u|^2 \, dx$$

$$= \int_{\Omega} a(x) |\nabla u_1|^2 \, dx + \int_{\Omega} a(x) |\nabla u_2|^2 \, dx \geq \alpha \|u_1\|_{L^2(\Omega)}^2 + C \|u_2\|_{L^2(\Omega)}^2$$

$$\geq \min\{\alpha, C\} \left( \|u_1\|_{L^2(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 \right) \geq \frac{\min\{\alpha, C\}}{2} \left( \|u_1\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)} \right)^2.$$
\[ \geq \frac{\min\{\alpha, C\}}{2} \|u\|_{L^2(\Omega)}^2 = \frac{\min\{\alpha, C\}}{2} > 0. \]

This contradicts the assumption (14). \(\square\)

**Lemma 4.2.** Let \(X\) and \(M\) be defined as in Lemma 4.1. Suppose that \(H^1_0(\Omega)\) cannot be bounded embedded into \(L^2(\Omega)\), then for any \(\varepsilon > 0\) and any \(n\) there exists some \(n\)-dimensional subspace \(X_n\) of \(X\), such that

\[ \sup_{u \in X_n \cap M} \int_{\Omega} a(x)|\nabla u|^2 \, dx < \varepsilon. \]

**Proof.** For any fixed \(n\) and any \(\varepsilon > 0\), in view of the assumption that \(H^1_0(\Omega)\) cannot be bounded embedded into \(L^2(\Omega)\), \(u_1 \in M\) can be chosen such that

\[ \int_{\Omega} a(x)|\nabla u_1|^2 \, dx < \frac{\varepsilon}{n}. \]

Let \(X_1 = \text{span}\{u_1\}\), from Lemma 4.1, \(u_2 \in M \cap X_1^\perp\) can be chosen such that

\[ \int_{\Omega} a(x)|\nabla u_2|^2 \, dx < \frac{\varepsilon}{n}. \]

Now, let \(X_2 = \text{span}\{u_1, u_2\}\), Lemma 4.1 implies that \(u_3 \in M \cap X_2^\perp\) can be chosen such that

\[ \int_{\Omega} a(x)|\nabla u_3|^2 \, dx < \frac{\varepsilon}{n}. \]

Continuing inductively in this way, let \(X_{i-1} = \text{span}\{u_1, \ldots, u_{i-1}\}\) \((1 < i \leq n+1)\), and \(u_i\) can be chosen in \(M \cap X_{i-1}^\perp\) such that

\[ \int_{\Omega} a(x)|\nabla u_i|^2 \, dx < \frac{\varepsilon}{n}. \]

Then for any \(u \in X_n \cap M\), which can be written to be \(u = \sum_{i=1}^n \alpha_i u_i\) and

\[ \|u\|_{L^2(\Omega)}^2 = \int_{\Omega} |u|^2 \, dx = \sum_{i=1}^n \alpha_i^2 = 1. \]

Hence, for each \(i\), \(\alpha_i^2 \leq 1.\)

So

\[ \int_{\Omega} a(x)|\nabla u|^2 \, dx \]
\[ = \int_{\Omega} a(x) \left( \sum_{i=1}^n \alpha_i \nabla u_i, \sum_{i=1}^n \alpha_i \nabla u_i \right) \, dx \]
\[ = \int_{\Omega} a(x) \sum_{i,j=1}^n \alpha_i \alpha_j \nabla u_i \nabla u_j \, dx \]
\[ \leq \frac{1}{2} \sum_{i,j=1}^n (\alpha_i^2 a(x)|\nabla u_i|^2 + \alpha_j^2 a(x)|\nabla u_j|^2) \, dx < \varepsilon. \]

The proof is complete. \(\square\)

Now we estimate the attractor’s lower bound of \(Z_2\)-index, and then show that the dimension of the attractor of (1) is infinite.
Theorem 4.3. Suppose that $H_0^{1,d}(\Omega)$ cannot be bounded embedded into $L^2(\Omega)$ and $A$ is the global attractor of (1), then for any $n \in \mathbb{Z}$ there exists an $\alpha_n > 0$, which is small enough, such that

$$\gamma \left( A \cap E^{-1}((-\infty, -\alpha_n]) \right) \geq n,$$

where

$$E(u) = \frac{1}{2} \int_{\Omega} a(x)|\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx + \frac{1}{p} \int_{\Omega} |u|^p dx$$

and

$$E^{-1}((-\infty, -\alpha_n]) = \{ u \in X \mid E(u) \leq -\alpha_n \}.$$

Proof. For any fixed $n \in \mathbb{Z}$, from Lemma 4.2 for $\varepsilon = \frac{1}{2}$ there exists the $n$-dimensional subspace $X_n$ of $X$ such that

$$\sup_{u \in X_n \cap M} \int_{\Omega} a(x)|\nabla u|^2 dx < \varepsilon = \frac{\lambda}{2}$$

where $M$ is from Lemma 4.1, and which implies that

$$\int_{\Omega} a(x)|\nabla u|^2 dx \leq \varepsilon \int_{\Omega} |u|^2 dx = \frac{\lambda}{2} \int_{\Omega} |u|^2 dx$$

for any $u \in X_n$. So for any $u \in X_n$, we have

$$E(u) = \frac{1}{2} \int_{\Omega} a(x)|\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx + \frac{1}{p} \int_{\Omega} |u|^p dx \leq -\frac{\lambda}{4} \int_{\Omega} |u|^2 dx + \frac{1}{p} \int_{\Omega} |u|^p dx.$$

Since each equipped norm in a finite dimensional subspace is equivalent, the norms $(\int_{\Omega} u^2 dx)^{\frac{1}{2}}$ and $(\int_{\Omega} u^p dx)^{\frac{1}{p}}$ are equivalent refined in $X_n$, that is, there exist $C_1, C_2 > 0$ such that

$$C_1 \|u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} \leq C_2 \|u\|_{L^p(\Omega)} \quad \text{for any } u \in X_n.$$

Hence,

$$E(u) \leq -\frac{\lambda}{4} \|u\|_{L^2(\Omega)}^2 + \frac{1}{pC_1^p} \|u\|_{L^2(\Omega)}^p = \left( -\frac{\lambda}{4} + \frac{1}{pC_1^p} \|u\|_{L^2(\Omega)}^{p-2} \right) \|u\|_{L^2(\Omega)}^2.$$

Thus there exist an $\alpha_n > 0$ and a $\delta > 0$ small enough such that

$$E(u) \leq -\alpha_n$$

when $u \in X_n$ and $\|u\|_{L^2(\Omega)} = \delta$.

Now we show that

$$\gamma \left( A \cap E^{-1}((-\infty, -\alpha_n]) \right) \geq n.$$

Set $B = \{ u \in X_n \mid \|u\|_{L^2(\Omega)} = \delta \}$, $B(t) = S(t)B$, $t \geq 0$. Then from the properties of $Z_2$-index we know

$$\gamma(\omega(B)) \geq \gamma(B) = n \quad (16)$$

where $\omega(B)$ is the $\omega$-limit set of $B$.

In virtue of the decreasing of $E(S(t)u)$ in $t$, we know

$$E|_{\omega(B)} \leq E|_{S(t)B} \leq -\alpha_n.$$

It follows that

$$\omega(B) \subset A \cap E^{-1}((-\infty, -\alpha_n]).$$

The monotonicity of $Z_2$-index follows from (16) and (17) that

$$\gamma \left( A \cap E^{-1}((-\infty, -\alpha_n]) \right) \geq n.$$
Hence the proof is complete. □

Note that any compact set \( E \) with fractal dimension \( \dim_F E = n \) can be mapped in to \( \mathbb{R}^{2n+1} \) by a linear (odd) Hölder continuous one-to-one projector, see [11]. So we have the following result.

**Corollary 1.** Under the above assumptions, the fractal dimension of the global attractor \( \dim \mathcal{A} = +\infty \).

5. **Multiple equilibrium points in the global attractor.** In this section, we show the existence of multiple equilibria in the global attractor for our problem (1). We will need the following deformation lemma in [22].

**Lemma 5.1** ([22]). Let \( \{ S(t) \}_{t \geq 0} \) be a continuous semigroup on Banach space \( X \). Assume that \( S(\cdot) \) possesses a \( C^0 \) even Lyapunov function \( E \) on \( X \) and a global attractor \( \mathcal{A} \). Set

\[
K = \{ u \in \mathcal{A} : S(t)u = u, \ \forall t \geq 0 \}
\]

and

\[
K_c = \{ u \in K : E(u) = c \}.
\]

Then for any \( \delta \) neighborhood \( \mathcal{N}(K_c, \delta) \) of \( K_c \), there exist \( \varepsilon > 0 \) and \( t_\varepsilon > 0 \) such that

\[
S(t_\varepsilon) [(E_{c+\varepsilon} \cap \mathcal{A}) \setminus \mathcal{N}(K_c, \delta)] \subset E_{c-\varepsilon},
\]

where \( E_\alpha = \{ u \in X \mid E(u) \leq \alpha \} \).

**Theorem 5.2.** Under the assumptions in Lemma 5.1, there are infinite equilibria in \( \mathcal{A} \cap E^{-1}((-\infty, 0)) \).

**Proof.** Set

\[
c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} E(u), \quad k = 1, 2, \ldots.
\]

where \( \Gamma_k = \{ A \subset \mathcal{A} : A \text{ is closed, } \mathcal{A} = -A \text{ and } \gamma(A) \geq k \} \). Then

(i) For each \( c_k \), there exists some equilibrium points \( u_k^* \in X \) of (1) such that \( E(u_k^*) = c_k \);

(ii) \( -\infty < c_k \leq c_{k+1} < 0, \quad k = 1, 2, \ldots \)

(iii) if there exists some \( k_0 \) such that \( c_{k_0} = c_{k_0+1} = \ldots = c_{k_0+p} \), then

\[
\gamma(K_{c_{k_0}}) \geq p + 1,
\]

where \( K_{c_{k_0}} = \{ u \in \mathcal{A} \mid E(u) = c_{k_0}, S(t)u = u, \forall t \geq 0 \} \).

To prove (i), we suppose by contradiction that there exists a \( k_0 \) such that for any \( u \in \mathcal{A} \) and \( S(t)u = u \) but \( E(u) \neq c_{k_0} \), that is, \( K_{c_{k_0}} = \emptyset \). Then Lemma 5.1 implies that there exist some \( \varepsilon > 0 \) and \( t_\varepsilon > 0 \) such that

\[
S(t_\varepsilon)(\mathcal{A} \cap E_{c_{k_0}+\varepsilon}) \subset E_{c_{k_0} - \varepsilon}.
\]

In addition, by the definition of \( c_{k_0} \), there exists some \( A \in \Gamma_{k_0} \) such that

\[
c_{k_0} \leq \sup_{u \in A} E(u) < c_{k_0} + \varepsilon.
\]

Hence,

\[
E(S(t_\varepsilon)A) \leq c_{k_0} - \varepsilon.
\]

On the other hand, it follows from \( \gamma(S(t_\varepsilon)A) \geq \gamma(A) \geq k_0 \) that \( S(t_\varepsilon)A \in \Gamma_{k_0} \). Also by the definition of \( c_{k_0} \), we know that

\[
\sup_{u \in S(t_\varepsilon)A} E(u) \geq c_{k_0}.
\]
which contradicts (19).

(ii) is immediate since \( \gamma(\{0\}) = \infty \) and \( \Gamma_{1+1} \subset \Gamma_1 \).

To prove the third assertion, we also suppose by contradiction that \( \gamma(K_{c_{k_0}}) \leq p \). Then there exist a \( \delta \) neighborhood \( N(K_{c_{k_0}}, \delta) \) of \( K_{c_{k_0}} \) such that

\[
\gamma(N(K_{c_{k_0}}, \delta)) = \gamma(K_{c_{k_0}}) \leq p
\]

since \( K_{c_{k_0}} \) is compact.

By the definition of \( c_{k_0+p} \), there exists an \( A \subset \Gamma_{c_{k_0}+p} \) such that

\[
c_{k_0} \leq \sup_{u \in A} E(u) < c_{k_0} + \varepsilon.
\]

On the one hand, the sub-additivity of \( Z_2 \)-index implies that

\[
\gamma(A \setminus N(K_{c_{k_0}}, \delta)) \geq \gamma(A) - \gamma(N(K_{c_{k_0}}, \delta)) \geq k_0 + p - p = k_0.
\]

Therefore,

\[
\gamma[S(t) (A \setminus N(K_{c_{k_0}}, \delta))] \geq k_0, \quad \forall \ t > 0.
\]

On the other hand, Lemma 5.1 implies that there exist some \( \varepsilon > 0 \) and \( t_\varepsilon > 0 \) such that

\[
S(t_\varepsilon) [(A \cap E_{c_{k_0}+\varepsilon}) \setminus N(K_{c_{k_0}}, \delta)] \subset E_{c_{k_0}-\varepsilon}.
\]

So (21) and (22) imply

\[
S(t_\varepsilon) (A \setminus N(K_{c_{k_0}}, \delta)) \subset E_{c_{k_0}-\varepsilon}.
\]

which contradict the definition of \( c_{k_0} \).

So

\[
\gamma(K_{c_{k_0}}) \geq p + 1 \geq 2
\]

as \( p \geq 1 \). It follows that the semigroup \( \{S(t)\}_{t \geq 0} \) possesses infinite pairs of different fixed points in \( K_{c_{k_0}} \), then there are infinite equilibria in \( A \cap E^{-1}((-\infty, 0)) \).

\[\square\]

**Lemma 5.3.** Assume that \( H^{1,a}_0(\Omega) \) is compactly embedded into \( L^r(\Omega) \), \( 1 < r < 2 \).

Let \( \Gamma_k = \{ A \subset H^{1,a}_0(\Omega) \mid \|u\|_{L^r(\Omega)} = 1, \forall u \in A, A \text{ is closed}, A = -A \text{ and } \gamma(A) \geq k \} \) and

\[
\beta_k = \inf_{A \in \Gamma_k} \sup_{u \in A} \int_{\Omega} a(x) |\nabla u|^2 dx,
\]

\( k = 1, 2, \ldots \).

Then \( \beta_k \to +\infty \) as \( k \to \infty \). Here \( \gamma(A) \) denotes the \( z_2 \)-index of \( A \).

**Proof.** Set

\[
M_0 = \left\{ u \in H^{1,a}_0(\Omega) \mid \|u\|_{L^r(\Omega)} = 1 \right\}
\]

and

\[
a_0 = \inf_{u \in M_0} \int_{\Omega} a(x)|\nabla u|^2 dx.
\]

Then there exists \( \varphi_0 \in M_0 \) such that

\[
\alpha_0 \leq \int_{\Omega} a(x)|\nabla \varphi_0|^2 dx < \alpha_0 + 1.
\]

Now we can choose \( f_0 \in (L^r(\Omega))^* \) such that \( \|f_0\| = 1 \) and \( f_0(\varphi_0) = 1 \). Let \( X_1 = \ker f_0 \), then for any \( \varphi \in X_1 \) we have

\[
\|\varphi_0 - \varphi\|_{L^r(\Omega)} \geq f_0(\varphi_0 - \varphi) = f_0(\varphi_0) = 1.
\]
Set $M_1 = M_0 \cap X_1$ and $\alpha_1 = \inf_{u \in M_1} \int_{\Omega} a(x)|\nabla u|^2 dx$, then there exists $\varphi_1 \in M_1$ such that
\[
\alpha_1 \leq \int_{\Omega} a(x)|\nabla \varphi_1|^2 dx < \alpha_1 + 1
\]
and
\[
\|\varphi_0 - \varphi_1\|_{L^r(\Omega)} \geq 1.
\]
Now we can choose $f_1 \in X_1^*$ such that $\|f_1\| = 1$ and $f_1(\varphi_1) = 1$. Then for any $\varphi \in \ker f_1 = X_2$, we have
\[
\|\varphi_1 - \varphi\|_{L^r(\Omega)} \geq f_1(\varphi_1 - \varphi) = f_1(\varphi_1) = 1.
\]
Next, set $M_2 = M_1 \cap X_2$ and $\alpha_2 = \inf_{u \in M_2} \int_{\Omega} a(x)|\nabla u|^2 dx$. Then there exists $\varphi_2 \in M_2$ such that
\[
\alpha_2 \leq \int_{\Omega} a(x)|\nabla \varphi_2|^2 dx < \alpha_2 + 1
\]
and
\[
\|\varphi_2 - \varphi_i\|_{L^r(\Omega)} \geq 1 \quad i = 0, 1.
\]
Continuing inductively in this way, set $M_n = M_{n-1} \cap X_n$, we can construct a sequence
\[
\alpha_n = \inf_{u \in M_n} \int_{\Omega} a(x)|\nabla \varphi_n|^2 dx, \quad n = 1, 2, \ldots
\]
and
\[
\|\varphi_n - \varphi_m\|_{L^r(\Omega)} \geq 1 \quad \text{if } n \neq m.
\]
We claim that $\alpha_n \to \infty$ as $n \to \infty$.

In fact, assume by contradiction that $0 \leq \alpha_n \leq M$, $n = 1, 2, \ldots$, then
\[
\int_{\Omega} a(x)|\nabla \varphi_n|^2 dx \leq \alpha_n + 1 \leq M + 1.
\]
Hence, $\{\varphi_n\}_{n=1}^\infty$ has a convergent subsequence in $L^r(\Omega)$ since $H_0^{1,a}$ is compactly embedded into $L^r(\Omega)$, $1 < r < 2$, which contradicts to the choice of the $\{\varphi_n\}_{n=1}^\infty$.

So $\alpha_n \to \infty$ as $n \to \infty$.

Next we will show that $\beta_k \to +\infty$ as $k \to \infty$.

Set $N_k = \bigcap_{i=1}^k \ker f_i$, then $M = N_k \oplus \text{span}\{\varphi_0, \ldots, \varphi_{k-1}\}$. From the property of $Z_2$-index of a set, we know that $A \cap N_k \neq \emptyset$ if $A \subset M_0$ and $\gamma(A) \geq k + 1$. Furthermore, $A \cap M_k \neq \emptyset$. Then
\[
\sup_{u \in A} \int_{\Omega} a(x)|\nabla u|^2 dx \geq \inf_{u \in M_k} \int_{\Omega} a(x)|\nabla u|^2 dx = \alpha_k.
\]
And so
\[
\inf_{A \in \Gamma_k} \sup_{u \in A} \int_{\Omega} a(x)|\nabla u|^2 dx \geq \alpha_k \to \infty \quad \text{as } n \to \infty,
\]
as claimed. \hfill \Box

**Theorem 5.4.** $c_k \to 0$ as $k \to \infty$ where $c_k$ is given by (18). Furthermore, $u_k^* \to 0$ in $L^p(\Omega)$ as $k \to \infty$.

**Proof.** From the proof of Theorem 5.2, for each $c_j$, there exists some equilibrium points $u_j^* \in X$ of (1) such that
\[
0 > c_j = E(u_j^*) = \frac{1}{2} \int_{\Omega} a(x)|\nabla u_j^*|^2 dx - \frac{\lambda}{2} |u_j^*|^2 + \frac{1}{p} |u_j^*|^p dx.
\] (25)
On the other hand, by the definition of $c_j$ there exists a $j$-dimensional subspace $X_j$ of $X$ such that
\[
c_j \geq \inf_{u \in X_j} \left\{ \frac{1}{2} \int_{\Omega} a(x) |\nabla u|^2 - \frac{\lambda}{2} |u|^2 + \frac{1}{p} |u|^p dx \right\}.
\] (26)

Moreover, we can choose $v_j \in X_j^\perp$ such that
\[
\inf_{u \in X_j} \left\{ \frac{1}{2} \int_{\Omega} a(x) |\nabla u|^2 - \frac{\lambda}{2} |u|^2 + \frac{1}{p} |u|^p dx \right\} = \frac{1}{2} \int_{\Omega} a(x) |\nabla v_j|^2 - \frac{\lambda}{2} |v_j|^2 + \frac{1}{p} |v_j|^p dx.
\] (27)

In addition, the definition of $\beta_j$ implies that
\[
\int_{\Omega} a(x) |\nabla v_j|^2 dx \geq \beta_j \left( \int_{\Omega} |v_j|^r dx \right)^{\frac{2}{r}} \text{ for } v_j \in X_j^\perp.
\] (28)

(25)-(28) show that
\[
0 > c_j \geq \frac{\beta_j}{2} \left( \int_{\Omega} |v_j|^r dx \right)^{\frac{2}{r}} - \frac{\lambda}{2} \int_{\Omega} |v_j|^2 dx + \frac{1}{p} \int_{\Omega} |v_j|^p dx
\]
\[
\geq \int_{\Omega} |v_j|^r dx \left( \beta_j \left( \int_{\Omega} |v_j|^r dx \right)^{\frac{2}{r}} - \frac{\lambda}{2} (2p)^{r/p} \right) + \frac{1}{2p} \int_{\Omega} |v_j|^p dx,
\] (29)

where we used
\[
\int_{\Omega} |v_j|^2 dx \leq \frac{1}{2p} \int_{\Omega} |v_j|^p dx + (2p)^{r/p} \int_{\Omega} |v_j|^r dx.
\] (30)

However, by the argument of Lemma 5.3, $\beta_j \to \infty$, it is definite that
\[
\int_{\Omega} |v_j|^r \to 0 \text{ as } j \to \infty.
\] (31)

Now, using (30) again, we have
\[
0 > \frac{1}{2} \int_{\Omega} a(x) |\nabla v_j|^2 - \frac{\lambda}{2} |v_j|^2 + \frac{1}{p} |v_j|^p dx
\]
\[
\geq \frac{1}{2} \int_{\Omega} a(x) |\nabla v_j|^2 + \frac{1}{2p} \int_{\Omega} |v_j|^p dx - \frac{\lambda}{2} (2p)^{r/p} \int_{\Omega} |v_j|^r ds.
\]

Thanks to (31), we can infer that
\[
\frac{1}{2} \int_{\Omega} a(x) |\nabla v_j|^2 - \frac{\lambda}{2} |v_j|^2 + \frac{1}{p} |v_j|^p dx \to 0 \text{ as } j \to \infty.
\]

Note that (25)-(27) imply that
\[
0 > c_j \geq \frac{1}{2} \int_{\Omega} a(x) |\nabla u_j|^2 - \frac{\lambda}{2} |u_j|^2 + \frac{1}{p} |u_j|^p dx,
\]
it follows that $c_k \to 0$ as $k \to \infty$, as claimed.

Next we prove $u_k^* \to 0$ in $L^p(\Omega)$ as $k \to \infty$.

Observe that $u_j^*$ is an equilibrium point, we know
\[
\int_{\Omega} a(x) |\nabla u_j^*|^2 - \lambda |c_j|^2 + |u_j^*|^p dx = 0.
\] (32)

So (25) and (32) derive that
\[
(1 - \frac{1}{p}) \int_{\Omega} |u_j^*|^p dx = 2c_j.
\] (33)
It’s immediate that
\[ \int_{\Omega} |u_{j}^{*}|^{p} \, dx \to 0 \quad \text{as} \quad j \to \infty, \quad (34) \]
since \( c_{j} \to 0 \) as \( j \to \infty \). The proof is complete. \( \square \)

Appendix A. In this section, we briefly discuss some properties of weighted Sobolev spaces \( H^{1,a}_{0}(\Omega) \) mentioned in Section 1 when \( a(x) \) satisfied the assumptions \((A1)\) and \((3)\).

In general there is no inclusion relationship between \( H^{1,a}_{0}(\Omega) \) and the standard Sobolev space \( H^{1}_{0}(\Omega) \), but we have the following embedding results for the space \( H^{1,a}_{0}(\Omega) \).

**Theorem 5.5.** Suppose that \( \Omega \) is bounded domain in \( \mathbb{R}^{n} \) and \( a(x) \) satisfies \((A1)\) and \((3)\). Then the following embeddings hold:

(i) \( H^{1,a}_{0}(\Omega) \) is continuously embedding into \( W^{1,\frac{2n}{n+\alpha}}_{0}(\Omega) \);

(ii) \( H^{1,a}_{0}(\Omega) \) is continuously embedding into \( L^{2^{*}_{\alpha}}(\Omega) \);

(iii) \( H^{1,a}_{0}(\Omega) \) is compactly embedding into \( L^{r}(\Omega) \) as \( 1 \leq r < 2^{*}_{\alpha} = \frac{2n}{n-2+\alpha} \).

**Proof.** The Hölder inequality yields
\[
\int_{\Omega} |\nabla u|^{\frac{2n}{n+\alpha}} \, dx = \int_{\Omega} \frac{1}{[a(x)]^{\frac{n}{n+\alpha}}} [a(x)]^{\frac{n}{n+\alpha}} |\nabla u|^{\frac{2n}{n+\alpha}} \, dx \\
\leq \left( \int_{\Omega} \frac{1}{[a(x)]^{\frac{n}{n+\alpha}}} \, dx \right)^{\frac{n}{n+\alpha}} \left( \int_{\Omega} [a(x)] |\nabla u|^{2} \, dx \right)^{\frac{n+\alpha}{n}} .
\]
Noticing the assumption \((3)\), we complete (i).

The other results (ii) and (iii) of Theorem 5.5 are followed by the standard embedding theorems. \( \square \)

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**REFERENCES**

[1] C. T. Anh and P. Q. Hung, Global attractors for a class of degenerate parabolic equations, *Acta Mathematica Vietnamica*, 34 (2009), 213–231.

[2] C. T. Anh, N. M. Chuong and T. D. Ke, Global attractors for the m-semiflow generated by a quasilinear degenerate parabolic equations, *J. Math. Anal. Appl.*, 363 (2010), 444–453.

[3] C. T. Anh and T. D. Ke, Long-time behavior for quasilinear parabolic equations involving weighted \( p \)-Laplacian operators, *Nonlinear Anal.*, 71 (2009), 4415–4422.

[4] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, Studies in Mathematics and its Applications, 25. North-Holland Publishing Co., Amsterdam, 1992.

[5] J. W. Cholewa and T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, London Mathematical Society Lecture Note Series, 278. Cambridge University Press, Cambridge, 2000.

[6] M. Efendiev and S. Zelik, Finite- and infinite-dimensional attractors for porous media equations, *Proc. London Math. Soc. (3)*, 96 (2008), 51–57.

[7] M. A. Efendiev and M. Ôtani, Infinite-dimensional attractors for parabolic equations with \( p \)-Laplacian in heterogeneous medium, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28 (2011), 565–582.

[8] M. Efendiev, A. Miranville and S. Zelik, Infinite-dimensional exponential attractors for nonlinear reaction-diffusion systems in unbounded domains and their approximation, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460 (2004), 1107–1129.
[9] M. Efendiev, Infinite-dimensional exponential attractors for fourth-order nonlinear parabolic equations in unbounded domains, *Math. Meth. Appl. Sci.*, 34 (2011), 939–949.

[10] J. K. Hale, L. T. Magalhães and W. M. Oliva, *An Introduction to Infinite Dimensional Dynamical Systems-Geometric Theory*, Applied Mathematical Sciences, 47. Springer-Verlag, New York, 1984.

[11] B. R. Hunt and V. Y. Kaloshin, Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces, *Nonlinearity*, 12 (1999), 1263–1275.

[12] N. I. Karachalios and N. B. Zographopoulos, Convergence towards attractors for a degenerate Ginzburg-Landau equation, *Z. Angew. Math. Phys.*, 56 (2005), 11–30.

[13] N. I. Karachalios and N. B. Zographopoulos, On the dynamics of a degenerate parabolic equation global bifurcation of stationary states and convergence, *Calc. Var. Partial Differential Equations*, 25 (2006), 361–393.

[14] N. I. Karachalios and N. B. Zographopoulos, Global attractors and convergence to equilibrium for degenerate Ginzburg-Landau and parabolic equations, *Nonlinear Anal.*, 63 (2005), e1749–e1768.

[15] F. Li, B. You and C. K. Zhong, Multiple equilibrium points in global attractors for some \( p \)-Laplacian equations, *Applicable Analysis*, 97 (2018), 1591–1599.

[16] A. Miranville and S. Zelik, Finite-dimensionality of attractors for degenerate equations of elliptic-parabolic type, *Nonlinearity*, 20 (2007), 1773–1797.

[17] J. C. Robinson, *Infinite-Dimensional Dynamical Systems. An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[18] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Second edition, Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997.

[19] B. You, F. Li and C. K. Zhong, The existence of multiple equilibrium points in a global attractor for some \( p \)-Laplacian equation, *J. Math. Anal. Appl.*, 418 (2014), 626–637.

[20] J. Zhang, C. K. Zhong and B. You, The existence of multiple equilibrium points in global attractors for some symmetric dynamical systems II, *Nonlinear Anal. Real World Appl.*, 36 (2017), 44–55.

[21] C. K. Zhong and W. S. Niu, On the \( Z_2 \) index of the global attractor for a class of \( p \)-Laplacian equations, *Nonlinear Anal.*, 73 (2010), 3698–3704.

[22] C. K. Zhong, B. You and R. Yang, The existence of multiple equilibrium points in global attractor for some symmetric dynamical systems, *Nonlinear Anal. Real World Appl.*, 19 (2014), 31–44.

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