General existence proof for rest frame systems in asymptotically flat space-time

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Abstract

We report a new result on the nice section construction used in the definition of rest frame systems in general relativity. This construction is needed in the study of non trivial gravitational radiating systems. We prove existence, regularity and non-self-crossing property of solutions of the nice section equation for general asymptotically flat space-times. This proves a conjecture enunciated in a previous work.

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1 Introduction

The rest frame equation, or “nice section” equation[11], for asymptotically flat space-times, is an equation which relates quantities defined at null infinity (scri or I). The solutions of this equation are a family of privileged sections of scri, which represent the analog of having a frame of reference[12][13].

The physical motivations for the nice section construction have been explained already in the literature[11][12]. We just mention here that at present the description of astrophysical gravitational radiation in terms of the sources, is carried out through the so called quadrupole radiation formula[8].

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The construction of the nice sections provides a necessary tool, that we ex-
pect to be useful for the discussion of gravitational radiation and its relation
with the notion of multipole moments, including the cases of systems which
suffer a non-trivial back reaction due to gravitational radiation, or when the
lapse of time considered is relatively long, when compared to the dynamical
characteristic time. Those cases can not be dealt with the quadrupole radi-
ation formula, which has been deduced for an instant of time, and for weak
fields.

In what follows we will concentrate in the mathematical aspects of the
nice section construction, and we will provide a general existence proof of
them; which demonstrates that the conjecture that we have enu-
nicated in ref. [12] is true.

Let us start by recalling the basic structure of scri. Roughly speak-
ing scri is the three dimensional manifold that corresponds to the end points
when one takes the limits to infinity of the future directed null geode-
sic. The topology of a complete scri is $\mathbb{R} \times S^2$, and the natural coordinates are the
so called Bondi coordinates; which are labeled by $(u, \zeta, \bar{\zeta})$, where $(\zeta, \bar{\zeta})$ are
complex stereographic coordinates of the sphere $S^2$, and $u$ takes values in $\mathbb{R}$. Given an arbitrary real function $\gamma(\zeta, \bar{\zeta})$ of the sphere, we say that $u = \gamma(\zeta, \bar{\zeta})$
defines a section of $\mathcal{L}$.

The symmetry group of scri (i.e. the group of coordinate transformations
which preserves the asymptotic geometric structure of the space-time) is the
Bondi-Metzner-Sach (BMS) group. It is defined by the following relations:

\begin{align}
\tilde{u} &= K(u - \gamma(\zeta, \bar{\zeta})), \\
\tilde{\zeta} &= \frac{a\zeta + b}{c\zeta + d},
\end{align}

with

$$ad - bc = 1, \quad K = \frac{1 + \zeta\bar{\zeta}}{|a\zeta + b|^2 + |c\zeta + d|^2}$$

where $a, b, c, d$ are complex constant, and $\gamma$ is a real function on the sphere.

We express the supermomentum at a section $u = \text{constant}$ in the following
way:

$$P_{lm} = -\frac{1}{\sqrt{4\pi}} \int Y_{lm}(\zeta, \bar{\zeta})\Psi(u, \zeta, \bar{\zeta})dS^2,$$

where $dS^2$ is the surface element of the unit sphere; $Y_{lm}$ are the spherical
harmonics;

$$\Psi = \Psi_2 + \sigma \dot{\sigma} + \partial^2 \bar{\sigma};$$
with $\Psi_2$ and $\sigma$ being the leading order asymptotic behavior of the second Weyl tensor component and the Bondi shear respectively; where we are using the GHP notation\cite{5} for the edth operator of the unit sphere, and an over dot means $\partial/\partial u$.

Similarly the total Bondi energy-momentum vector at a given section of null infinity can be expressed by

$$
P^a = -\frac{1}{4\pi} \int l^a(\zeta, \bar{\zeta})\Psi(u, \zeta, \bar{\zeta})dS^2, \quad (4)
$$

where

$$(l^a) = \left(1, \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{i(1 + \zeta\bar{\zeta})}, \frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}\right). \quad (5)$$

In fact the Bondi momentum is a linear combination of the first four component ($l = 0, 1$) of the supermomentum.

The rest mass $M$ of the section is given by

$$M = \sqrt{P^aP_a}, \quad (6)$$

where the indices are moved with the Lorenzian flat metric $\eta_{ab}$ at scri\cite{10}.

The BMS group (1) and the supermomentum (2) are the basic ingredients for the definition of the nice section equation.

A rest frame system in special relativity is a coordinate system in which the total four momentum has only temporal component. Given such a system we can perform a translation and we still have a rest frame system.

This idea can be translated to the structure of scri; where the supermomentum takes the role of the momentum in special relativity. With this idea in mind, the nice sections are defined in such a way that the supermomentum has only temporal component. This means that the integrand $\Psi$ at a nice section is a constant, i.e., it does not depend on the angular coordinates $(\zeta, \bar{\zeta})$.

The function $\Psi$ transforms under a BMS transformation in the following way\cite{11}

$$\tilde{\Psi} = \frac{1}{K^3}(\Psi - \bar{\sigma}^2 \bar{\bar{\sigma}}^2 \gamma), \quad (7)$$

where $\tilde{\Psi} = \Psi_2 + \bar{\sigma}^2 \gamma + \bar{\bar{\sigma}}^2 \bar{\gamma}$.

This means that given an initial arbitrary section, a nice section can be determined by the appropriate BMS transformation such that $\tilde{\Psi}$ is constant. Therefore we can express the nice section equation by

$$\bar{\sigma}^2 \bar{\bar{\sigma}}^2 \gamma = \Psi(u = \gamma, \zeta, \bar{\zeta}) + K^3(\gamma, \zeta, \bar{\zeta})M(\gamma), \quad (8)$$
where the conformal factor $K$ can be related to the Bondi momentum by

$$K = \frac{M}{P^a l_a},$$

(9)

and $P^a$ is evaluated at the section $u = \gamma$; which is calculated through the integral $\int u$. 

Since the operator $\bar{\partial}^2 \bar{\partial}^2$ occurs very frequently in this work we use the notation $D = \bar{\partial}^2 \bar{\partial}^2$, and we note also that

$$D \equiv \bar{\partial}^2 \bar{\partial}^2 = \frac{1}{4} \Delta^2 - \frac{1}{2} \Delta$$

(10)

where $\Delta$ is the Laplacian of the unit sphere.

We say that a supertranslation $\gamma$ (or equivalently a section of $\mathcal{I}$) satisfies the “nice section” equation if the function $\tilde{\Psi}$ evaluated in $\gamma$ is equal to the total mass $M$; this is exactly the content of equation (8).

Note that the nice section equation involves also a Lorenz rotation, as it can be seen by the appearance of the scalar $K$ in the equation.

Equation (8) is our fundamental equation, we want to investigate the existence and properties of solutions of this equation.

We say that a real function $x = x(\zeta, \bar{\zeta})$ on the sphere is a translation if $Dx = 0$. Note that this implies that $x$ has an expansion in spherical harmonics with $l = 0, 1$. An arbitrary regular function $\gamma$ can be decomposed in

$$\gamma = x + y,$$

where $y$ has an expansion with $l \geq 2$. Given an arbitrary $x$, equation (8) is an equation for $y$.

The main result of this paper is the following Theorem:

**Theorem 1.1** If $\Psi$ is a smooth function on $\text{scri}$, the total energy $P^0$ is bounded by the constant $E_0$, the total mass $M$ is bounded from below by $M \geq M_0$, $M_0 > 0$; and the gravitational energy density flux $|\dot{\sigma}|^2 \leq \lambda$; where the constant $\lambda$ satisfies

$$\lambda < \frac{\sqrt{27}}{4} (1 + (2C_K)^4)^{-1},$$

and $C_K$ is given by

$$C_K = \frac{E_0}{M_0} + \sqrt{\frac{E_0^2}{M_0^2} - 1};$$
then

(i) For every translation \(x\) there exists a solution \(y\) of equation (8), and \(y\) is a smooth function on the sphere.

(ii) The solutions \(\gamma = x + y(x)\) are continuous in the 4-parameter translation \(x\), and if \(x_1\) and \(x_2\) are two translations such that the difference, \(x_2 - x_1\), corresponds to a future directed time like vector, then

\[
\gamma(x_1, \zeta, \bar{\zeta}) < \gamma(x_2, \zeta, \bar{\zeta}).
\]

Since the mass \(M\) is a decreasing function of \(u\), the constant \(M_0\) is the final rest mass as \(u \to \infty\). Note that while \(M_0\) is Lorentz invariant \(P^0\) is not; in other words the inequalities depend on the particular Bondi system being used.

It is important to recall that all the hypothesis of the theorem are in terms of physical quantities; namely the total mass \(M\), the energy \(P^0\) and the gravitational energy density flux \(|\hat{\sigma}|^2\). Theorem 1.1 essentially says that the solutions of equation (8) exist and have the expected physical properties, when the gravitational radiation of the space-time is not too high.

As it was pointed out in [12], there exist another choices for the supermomentum \(\Psi\) in the literature. We want to analyze here these other possibilities. One of them is the Geroch supermomentum

\[
\Psi_G = \Psi_2 + \sigma \hat{\sigma} + \frac{1}{2}(\bar{\sigma}^2 \sigma - \bar{\sigma} \bar{\sigma}^2 \sigma);
\]

and another one is the Geroch-Winicour supermomentum

\[
\Psi_{GW} = \Psi_2 + \sigma \hat{\sigma} - \bar{\sigma} \bar{\sigma}^2 \sigma. \tag{11}
\]

The function \(\Psi_G\) is invariant under supertranslation, then it will not give an equation for \(\gamma\). Note that \(\Psi_G\) vanishes for Minkowski space-time, but equation (8) is not trivial in this space-time; its solutions are the shear-free sections (\(\sigma = 0\)). As it is well known, the sections with \(\sigma = 0\) in Minkowski space-time represent a natural way to isolate the Poincare group from the BMS group of scri, and this is precisely the aim of the ‘nice section’ equation in the general case. These considerations rule out \(\Psi_G\).

The Geroch-Winicour supermomentum \(\Psi_{GW}\) could in principle be used to construct an alternative equation. But the supermomentum \(\Psi\) satisfies the following remarkable equation

\[
\dot{\Psi} = |\hat{\sigma}|^2, \tag{12}
\]

\footnote{We take this opportunity to correct a mistake in the articles \[13\] and \[11\], the supermomentum \(\Psi\) is not the same as the Geroch-Winicour supermomentum, as it is clear from the expression \[11\].}
which relates in a simple way the time derivative of $\Psi$ with the gravitational radiation flux. This equation allows us to write the hypothesis of the theorems in terms only of the total mass $M$, the energy $P^0$ and the gravitational energy density flux $|\dot{\sigma}|^2$. It might be also possible to prove similar theorems for an equation based on $\Psi_{GW}$, but they would certainly look much more complicated.

In section 2 we prove the existence part (i) of the theorem. In section 3 we prove the part (ii). Finally, for the sake of completeness, we remind in the Appendix the Schauder fixed point theorem, which is the main tool for the existence proof of section 2.

## 2 Existence Proof

The equation (8) is an elliptic, nonlinear, equation on the compact manifold $S^2$. The main tool for proving existence of solutions of this type of equations is the Schauder fixed point Theorem 3.2 (see for example page 380 of ref. [9] for an elementary treatment of a similar equation with the same technique).

Let us define the function $f$ as the right-hand side of equation (8)

$$f(x + y, \zeta, \bar{\zeta}) = \Psi(x + y, \zeta, \bar{\zeta}) + K^3 M(x + y, \zeta, \bar{\zeta}).$$

(13)

The first step is to prove that the function $f$ satisfies the appropriate inequality under the hypothesis of Theorem 1.1; this will be proved in the following Lemma.

**Lemma 2.1** Assume that $\Psi$ is a smooth function on scri, the gravitational radiation flux satisfies $|\dot{\sigma}|^2 \leq \lambda$, the total mass $M \geq M_0 > 0$ and the total energy $P^0 \leq E_0$. Then

$$|f| \leq \lambda|y| + C,$$

(14)

where the constant $C$ does not depend on $y$.

**Proof:** Take $u_0$ an arbitrary, fixed, real number. From equation (12) we have

$$\Psi(\gamma, \zeta, \bar{\zeta}) = \int_{u_0}^{\gamma} |\dot{\sigma}|^2 (u', \zeta, \bar{\zeta}) du' + \Psi(u_0, \zeta, \bar{\zeta}).$$

We use the hypothesis $|\dot{\sigma}|^2 \leq \lambda$ to obtain

$$|\Psi| \leq \lambda|y| + C_1,$$

(15)
where

\[ C_1 = \sup_{\mathcal{S}} \{ \lambda |x - u_0| + |\Psi(u_0, \zeta, \bar{\zeta})| \}. \]

Note that the constant \( C_1 \) depends on \( x \) and \( u_0 \) but not on \( y \).

Looking at the expression (13) we see that for proving the inequality (14) we need only to find bounds for \( K \) and \( M \) in terms of the constant \( M_0 \) and \( E_0 \). The last one follows immediately since \( M \leq P^0 \) and by hypothesis we have \( P^0 \leq E_0 \), then

\[ M \leq E_0. \] (16)

Define the velocity vector \( V^a \) by \( V^a = P^a / M \). Note that \( V^a V_a = 1 \). We use the definition of \( K \), equation (9), to obtain

\[ K = \frac{1}{V^a l_a}. \] (17)

The function \( V^a l_a \) satisfies the following inequality

\[ |V^a l_a| \geq |V^0| - |V^i l_i| \geq |V^0| - \sqrt{V^i V_i}. \]

We have used that \( l^i l_i = 1 \), with \( i = 1, 2, 3 \). Then we use that \( (V^0)^2 - V^i V_i = 1 \) to obtain

\[ |V^a l_a| \geq |V^0| - \sqrt{(V^0)^2} - 1. \] (18)

The right-hand side of this inequality is a monotonically decreasing function of \( |V^0| \). By hypothesis we have the bound

\[ V^0 = \frac{P^0}{M} \leq \frac{E_0}{M_0}. \] (19)

We use the bound (19) in equation (18) to obtain

\[ |V^a l_a| \geq \frac{C_0}{M_0} - \sqrt{\frac{C_0^2}{M_0^2}} - 1; \]

and then by equation (17) we get the desired bound for \( K \):

\[ |K| \leq C_K, \] (20)

where \( C_K \) is given by

\[ C_K = \left( \frac{E_0}{M_0} - \sqrt{\frac{E_0^2}{M_0^2}} - 1 \right)^{-1} = \frac{E_0}{M_0} + \sqrt{\frac{E_0^2}{M_0^2}} - 1. \]
Using the bounds (15), (16) and (20) we obtain the final inequality for the function $f$:

$$|f| \leq \lambda |y| + C,$$

where

$$C = C_1 + E_0 C^3_K.$$

With this Lemma we are in position to prove the following existence Theorem:

**Theorem 2.1** Assume that $\Psi$ is a smooth function on $scri$, the total energy $P^0 \leq E_0$, the total mass $M \geq M_0 > 0$ and the gravitational energy density flux $|\dot{\sigma}|^2 \leq \lambda$ with

$$\lambda < \sqrt{\frac{27}{4}}.$$

Then for every translation $x$ there exists a solution $y$ of equation (8), and $y$ is a smooth function on the sphere.

**Proof:** Consider the mapping $A$ on the space of continuous functions on the sphere $A : C^0(S^2) \rightarrow C^0(S^2)$ defined as follows

$$A(y) = D^{-1}f(x + y, \zeta, \bar{\zeta}),$$

where $D^{-1}$ is the inverse of $D$. First note that $A$ is well defined for every translation $x$, since for an arbitrary $x + y$ the function $f$ is orthogonal to the kernel of $D$, and then it is in the domain of $D^{-1}$. This is not obvious and it is essential, otherwise equation (8) would be inconsistent. For a proof see [12].

A solution of equation (8) is a fixed point $y = A(y)$ of the mapping $A$. To prove that such fixed point exists we will use the Schauder fixed point Theorem 3.2. In the following we will prove that the mapping $A$ satisfies the hypothesis of this Theorem.

First we prove that $A$ is continuous in the $C^0$ norm. By definition

$$DA(y) = f(x + y, \zeta, \bar{\zeta}),$$

we take the $L^2$ norm in both sides of this equation

$$||DA(y)||_{L^2} = ||f(x + y, \zeta, \bar{\zeta})||_{L^2} \leq \sqrt{4\pi}||f(x + y, \zeta, \bar{\zeta})||_{C^0}.$$
The last inequality follows because \( f \) is continuous. We use the inequality
\[
||Dy||_{L^2} \geq \sqrt{4\pi \sqrt{27/4}} ||y||_{C^0},
\]
which is nothing but a particular case of the elliptic regularity inequalities for the elliptic operator \( D \), to obtain
\[
||A(y)||_{C^0} \leq \sqrt{4/27} ||f(x + y, \zeta, \tilde{\zeta})||_{C^0}.
\]
(22)

Since \( D^{-1} \) is a linear operator we obtain also the inequality
\[
||A(y_1) - A(y_2)||_{C^0} = ||D^{-1}(f(y_1) - f(y_2))||_{C^0} \leq \sqrt{4/27} ||(f(y_1) - f(y_2))||_{C^0}.
\]
(23)

Since by hypothesis \( f \) is continuous as a function of \( y \) then by (23) \( A \) is also continuous.

Let \( B \) be the closed ball in \( C^0(S^2) \) of radius \( C_2 \); \( B \) is clearly a closed, convex subset of the Banach space \( C^0(S^2) \). We want to prove that \( A \) maps \( B \) into itself when the constant \( C_2 \) is appropriately chosen. Take \( y \in B \), using the inequality (22) and Lemma 2.1 we have
\[
||A(y)||_{C^0} \leq \sqrt{4/27}(\lambda |y| + C) \leq \sqrt{4/27}(\lambda C_2 + C).
\]
(24)

Take \( C_2 = \alpha C \), where \( \alpha \) satisfies
\[
\lambda + \frac{1}{\alpha} \leq \sqrt{27/4};
\]
this \( \alpha \) exists because we have assumed that \( \lambda < \sqrt{27/4} \). Then from the inequality (24) we obtain
\[
||A(y)||_{C^0} \leq C_2.
\]
Therefore we have proved that \( A \) maps the ball \( B \) into itself.

It remains to be proved that the set \( A(B) \) is precompact. The set \( A(B) \) is a subset of \( H^4(S^2) \). This is a consequence of the elliptic character of the 4th order operator \( D \) and the standard elliptic regularity theorems (see for example [1] [14]).

Since \( H^4(S^2) \) is compactly imbedded in \( C^0(S^2) \) (see for example [7]) it follows that \( A(B) \) is precompact in \( C^0(S^2) \).
The hypothesis of Theorem 3.2 are satisfied, and then there exists a solution \( y = y(\zeta, \bar{\zeta}) \) in \( C^0(S^2) \).

Since \( y \) is in \( C^0(S^2) \) and \( f \) is smooth, then \( f(x + y, \zeta, \bar{\zeta}) \) is in \( L^2(S^2) \). Therefore by the elliptic regularity theorems \( y \in H^4(S^2) \); and then, by the Sobolev imbedding Theorem, it is in \( C^2(S^2) \); but then \( f(x + y, \zeta, \bar{\zeta}) \) is in \( H^2(S^2) \). We use elliptic regularity and induction to conclude that \( y \) is in \( C^\infty(S^2) \). ■

Note that for the existence proof we only need \( \Psi \) to be in \( C^0 \); if we only require this, then by the elliptic regularity theory one would only obtain a \( C^2 \) solution \( y \).

3 Uniqueness and non self-crossing properties of the solutions

Here we want to analyze the dependence of the solution \( y(x, \zeta, \bar{\zeta}) \) in terms of the translation \( x \). In the previous section we have shown that given an arbitrary \( x \) there exists a solution \( y(x, \zeta, \bar{\zeta}) \), which is smooth in the variables \((\zeta, \bar{\zeta})\). Now we want to prove that under more restrictive conditions this solution is unique. Moreover, we also want to prove that \( y(x, \zeta, \bar{\zeta}) \) is continuous in \( x \). This means that the function \( y(x, \zeta, \bar{\zeta}) \) defines a family of solutions parameterized by \( x \). We also prove that this family has the following non self-crossing property: if \( x_1 \) and \( x_2 \) are two translations such that the difference \( x_2 - x_1 \) corresponds to a future directed timelike vector, then

\[
\gamma(x_1, \zeta, \bar{\zeta}) < \gamma(x_2, \zeta, \bar{\zeta}).
\]

This implies that the solutions of equation (8) generated by a time translation do not cross among them.

We begin with the following auxiliary Lemma.

**Lemma 3.1** Assume that the hypothesis of Lemma 2.7 holds. Then

\[
\|f(\gamma_2) - f(\gamma_1)\|_{L^2} \leq \sqrt{4\pi} C_{f'}\|\gamma_2 - \gamma_1\|_{C^0},
\]

where

\[
C_{f'} = \lambda \left( 1 + (2C_K)^4 \right).
\] (25)

**Proof:** Consider \( f \) as a mapping \( f : C^0(S^2) \to L^2(S^2) \), since \( \Psi \) is smooth then \( f \) is Fréchet differentiable, and by the mean value theorem (see for example [2]) we obtain the following bound

\[
\|f(\gamma_2) - f(\gamma_1)\|_{L^2} \leq \sup_{t \in (0,1)} \|f'(\gamma_1 + t(\gamma_2 - \gamma_1))\|_{L(C^0, L^2)}\|\gamma_2 - \gamma_1\|_{C^0},
\]
where $\mathcal{L}(C^0, L^2)$ denotes the operator norm, and $f'$ is the Fréchet derivative of $f$. We have to calculate the bound for $f'$. The Fréchet derivative $f'$ is given explicitly by:

$$f'(\gamma)\delta\gamma = \dot{\Psi}(\gamma)\delta\gamma + \left(\frac{4P^a}{M} - 3Kl^a\right)K^3\delta P_a,$$

where

$$\delta P_a = -\frac{1}{4\pi} \int l_a \dot{\Psi}(\gamma)\delta\gamma dS.$$

And the operator norm of $f'$ is

$$\|f'(\gamma)\|_{\mathcal{L}(C^0, L^2)} = \sup\{\|f'(\gamma)\|_{L^2}; \|\delta\gamma\|_{C^0} = 1\}.$$

We use the bounds $E_0, M_0, \lambda$ and the bound (20) for $K$ to obtain

$$\|f'(\gamma)\|_{\mathcal{L}(C^0, L^2)} \leq \sqrt{4\pi}C_f',$nong where

$$C_f' = \lambda \left(1 + (2C_K)^4\right),$$

and we have used the inequality

$$\frac{1}{C_K} \leq K \leq C_K.$$

$\blacksquare$

**Theorem 3.1** Assume that the hypotheses of Theorem 2.1 hold, and in addition that

$$\lambda < \frac{\sqrt{27}}{4}(1 + (2C_K)^4)^{-1}. \quad (26)$$

Then the solutions $\gamma = x + y(x)$ are continuous in the $4$-parameter translation $x$. Moreover, if $x_1$ and $x_2$ are two translations such the difference $x_2 - x_1$ corresponds to a future directed time like vector, equivalently $x_2 - x_1 > 0$; then

$$\gamma(x_1, \zeta, \bar{\zeta}) < \gamma(x_2, \zeta, \bar{\zeta}),$$

for all $(\zeta, \bar{\zeta})$ in the sphere.
Proof: Let \( x_1 + y(x_1, \zeta, \bar{\zeta}) \) and \( x_2 + y(x_2, \zeta, \bar{\zeta}) \) be two solutions of the equation (8); taking the difference we obtain
\[
D(y(x_2) - y(x_1)) = f(x_2 + y(x_2), \zeta, \bar{\zeta}) - f(x_1 + y(x_1), \zeta, \bar{\zeta}).
\]
Let us take the \( L^2 \) norm on both sides of this equation
\[
||D(y(x_2) - y(x_1))||_{L^2} = ||f(x_2 + y(x_2), \zeta, \bar{\zeta}) - f(x_1 + y(x_1), \zeta, \bar{\zeta})||_{L^2},
\]
then using Lemma 3.1 and the inequality (21) we obtain
\[
\sqrt{27/4}||y(x_2) - y(x_1)||_{C^0} \leq C_f ||x_2 + y(x_2) - (x_1 + y(x_1))||_{C^0}.
\]
Since by hypothesis we have inequality (26), it is deduced that \( C_f < \sqrt{27/4} \) and we obtain
\[
||y(x_2) - y(x_1)||_{C^0} \leq \frac{C_f}{\sqrt{27/4} - C_f} ||x_2 - x_1||_{C^0};
\]
in this way we have proved that the function \( y(x) \) is continuous in \( x \). Note that
\[
\frac{C_f}{\sqrt{27/4} - C_f} < 1,
\]
then for \( x_1 < x_2 \) we obtain
\[
||y(x_2) - y(x_1)||_{C^0} < ||x_2 - x_1||_{C^0}. \tag{27}
\]
The fact that \( x_2 - x_1 \) corresponds to a future directed timelike vector, is equivalent to the statement that we can choose a coordinates system (by means of a Lorentz rotation) in which \( x_2 - x_1 \) is a positive constant. Then \( ||x_2 - x_1||_{C^0} = x_2 - x_1 \) in this coordinate system and from the inequality (27) we conclude that
\[
\gamma(x_2, \zeta, \bar{\zeta}) > \gamma(x_1, \zeta, \bar{\zeta}).
\]
It is important to remark that after one has proved the last inequality, it is possible to come back to the original Bondi system by the inverse of the original transformation which does not change the inequality. That is, this inequality is also true in the original Bondi system. ■

Appendix: Schauder Fixed Point Theorem

We use the following version of the Schauder Theorem (see ref. \[7\] page 280.)

**Theorem 3.2** Let \( B \) be a closed convex set in a Banach space \( V \) and let \( A \) be a continuous mapping of \( B \) into itself such that the image \( A(B) \) is precompact (i.e. the closure of \( A(B) \) is compact). Then \( A \) has a fixed point.
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