ON THE BOUNDED COHOMOLOGY OF LIE GROUPS

I. CHATTERJI, G. MISLIN, CH. PITTET, AND L. SALOFF-COSTE

1. INTRODUCTION

1.1. On Borel cohomology with $\mathbb{Z}$-coefficients. Let $G$ be a connected Lie group. Recall that the radical $\sqrt{G}$ of $G$ is its largest connected solvable normal subgroup. Our main result is:

**Theorem 1.1.** Let $G$ be a connected Lie group. The following conditions are equivalent.

1. The radical $\sqrt{G}$ of $G$ is linear.
2. Each Borel cohomology class of $G$ with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.
3. Each Borel cohomology class of $G$ of degree two with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.

The equivalent conditions of Theorem 1.1 admit geometric (Theorem 1.2, Condition 6) and topological (Theorem 1.2, Condition 2) counterparts.

Theorem 1.1 leads to a generalization (Corollary 1.6) of Gromov’s boundedness theorem [23, Section 1.3, p. 23] on characteristic classes of flat bundles.

Before stating our results in more details in Subsection 1.4, we recall some background works.

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1.2. Flat bundles and bounded cohomology. Milnor has shown in [36] that an \( SL(2, \mathbb{R}) \)-bundle \( P \) over the closed orientable surface \( \Sigma \) of genus \( g > 0 \) has Euler number satisfying

\[
|\langle c(P), [\Sigma_g] \rangle| \leq g - 1
\]

if and only if it admits a flat connection. Let \( G \) be a connected Lie group and let \( c \in H^2(BG, \mathbb{Z}) \) be a class of degree 2 in the cohomology of its classifying space \( BG \). Using the fact that the classifying map of a flat bundle factorizes through the classifying space \( BG \) endowed with the discrete topology, Dupont has shown in [19, Proposition 2.2] that for a flat principal \( G \)-bundle \( P \) over \( \Sigma \) the absolute value of the characteristic number \( \langle c(P), [\Sigma_g] \rangle \) is bounded above by the integer part of \((4g - 2)\|c^\delta\|_\infty\):

\[
|\langle c(P), [\Sigma_g] \rangle| \leq \lfloor (4g - 2)\|c^\delta\|_\infty \rfloor.
\]

In the case \( G = SL(2, \mathbb{R}) \) and \( c \) is the Euler class, \( \|c^\delta\|_\infty = 1/4 \). Hence Inequality 2 and the knowledge of \( \|c^\delta\|_\infty \) imply together Inequality 1. (The upper bound \( \|c^\delta\|_\infty \leq 1/4 \) follows from the bound of the area of hyperbolic triangles, see [19, Formula 4.3] and [10, Section 3.1.3], or can be directly deduced from [36, Lemma 3]. The lower bound \( \|c^\delta\|_\infty \geq 1/4 \), follows from Inequality 2 and the existence, for each \( g > 1 \), of a flat \( SL(2, \mathbb{R}) \)-bundle over \( \Sigma_g \) with Euler number \( 1 - g \), see [36, Theorem 2].)

The above results raise the following general problem, first studied in degree two by Dupont in [19], then in any degree by Gromov in [23]. For which Lie groups does the image of the natural map

\[
H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}),
\]

induced by \( G^\delta \to G \), consist of bounded classes? For \( G \) a connected Lie group, the natural map \( H^*(BG, \mathbb{Z}) \to H^*(BG^\delta, \mathbb{Z}) \) is always injective, (see Milnor [37, Cor. 1]), contrary to the case of \( \mathbb{R} \) coefficients, where it is never injective, unless \( G \) is contractible (it factors through continuous cohomology \( H_c^*(G, \mathbb{R}) \), which is a finite-dimensional \( \mathbb{R} \)-algebra by van Est’s Theorem). In the case \( G_a(\mathbb{C}) \) is the complex Lie group associated to an algebraic group \( G_a \) defined over \( \mathbb{C} \), Grothendieck has shown in [25, Théorème 7.1] that for all fields \( k \) of characteristic zero, the map

\[
H^d(BG_a(\mathbb{C}), k) \to H^d(BG_a(\mathbb{C})^\delta, k),
\]

is zero for \( d > 0 \): the characteristic classes of flat bundles over finite complexes with \( G_a(\mathbb{C}) \) as structure group are torsion classes (see Proposition 3.14). The quotient of the three dimensional complex Heisenberg group by the subgroup generated by a non-trivial central element shows
that the statement is not true for a general (i.e. not necessary algebraic) connected complex Lie group [25, Remarques 7.5, b]). In [21], Goldman considers the same example $H/Z$ (over $\mathbb{R}$) to exhibit non-bounded characteristic classes (in this example, $H^2(B(H/Z), \mathbb{R}) \cong \mathbb{R}$ injects into $H^2(B(H/Z)^\delta, \mathbb{R})$). Dupont has shown in [19, Theorem 4.1] that characteristic classes of degree 2 of semi-simple Lie groups with finite center are bounded. He has also given explicit upper bounds in the case of some simple groups (see also [16] and [17]). In the case $G$ is the real Lie group associated to a linear algebraic group defined over $\mathbb{R}$, Gromov has shown in [23, Section 1.3, p. 23], that all characteristic classes are bounded. The result has been improved by Bucher-Karlsson in [11, Theorem 4]: each characteristic class of $G$ can be represented by a cocycle whose set of values on all singular simplices of $BG^\delta$ is finite.

1.3. Borel cohomology of topological groups. In [41, Theorem 10], Moore has generalized the Eckmann-Eilenberg-Mac Lane theorem to locally compact groups. Namely, he has shown that if $G$ is a locally compact separable group and if $A$ is a polish abelian $G$-module, then the second cohomology group of Borel cochains on $G$ with values in $A$ is isomorphic to the group of equivalence classes of topological group extensions of $G$ by the $G$-module $A$ (see also Theorem 3.17):

$$H^2_B(G, A) \cong \text{Ext}_{top}(G, A).$$

In the case $G$ is a Lie group, a result of Wigner (see [47, Theorem 4] and [1, Section 7]) implies that the cohomology of Borel cocycles on $G$ with values in $\mathbb{Z}$ (with the trivial $G$-action) is naturally isomorphic to the singular cohomology of the classifying space of $G$ with integer coefficients:

$$H^*_B(G, \mathbb{Z}) \cong H^*_c(BG, \mathbb{Z}).$$

1.4. Main results. Building on the works recalled above, we obtain the following theorems.

**Theorem 1.2.** Let $G$ be a connected Lie group. The following conditions are equivalent.

1. The radical $\sqrt{G}$ of $G$ is linear.
2. The closure of the commutator subgroup of $\sqrt{G}$ is simply-connected.
3. Each Borel cohomology class of $G$ with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.
4. Each Borel cohomology class of $G$ of degree two with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.
(5) The class in $H^2_B(G, \pi_1(G))$ defined by the universal cover of $G$ can be represented by a Borel bounded cocycle.

(6) The natural inclusion $\pi_1(G) \to \tilde{G}$ of the fundamental group of $G$ into the universal cover of $G$ is undistorted.

Remarks 1.3.

(1) The equivalence between Conditions 1 and 2 in Theorem 1.2 is not new: a theorem of Goto [22, Theorem 5] states that a connected solvable Lie group $S$ is linear if and only if $\pi_1\left([S, S]\right) = 0$.

(2) The proof we shall give, shows that one can relax the boundedness hypothesis in Conditions 3, 4, and 5, of Theorem 1.2: assuming that the representative cocycle has sub-linear growth, leads in each case to an equivalent condition. Similarly, assuming that the distortion of $\pi_1(G) \to \tilde{G}$ is sub-linear, is equivalent to Condition 6 of Theorem 1.2 (see Definition 2.14).

(3) Theorem 1.1 assumes integer coefficients. Let us emphasis that this is in contrast with the case of real coefficients by recalling the following well-known points about the forgetful map in Borel bounded cohomology with $\mathbb{R}$-coefficients:

(a) $H^*_B(G, \mathbb{R}) \to H^*_B(G, \mathbb{R})$ is not onto for $G$ a connected solvable Lie group for which the right hand side does not vanish for all positive degrees (the left hand side is 0).

(b) $H^*_B(G, \mathbb{R}) \to H^*_B(G, \mathbb{R})$ is not onto in degree 3 for $G$ the universal cover of $SL(2, \mathbb{R})$ (see example [39, 9.3.11 (ii), page 127]), thus, in general it is not onto for semi-simple Lie groups either.

(c) $H^*_B(G, \mathbb{R}) \to H^*_B(G, \mathbb{R})$ is conjectured to be onto for semi-simple Lie groups with finite center (see [12, Conjecture 18.1, page 56]). The conjecture has been proved for Hermitian Lie groups with finite center (see [28]).

One ingredient in the proof of Theorem 1.2 is the following basic result on the cohomology algebra $H^*(BG, \mathbb{R})$ of a connected Lie group.

**Theorem 1.4.** Let $G$ be a connected Lie group and let $\sqrt{G}$ be its radical. Then $H^*(BG, \mathbb{R})$ is generated as an algebra by $H^2(BG, \mathbb{R})$ together with the image $\text{Im}(H^*(B(G/\sqrt{G}), \mathbb{R}) \to H^*(BG, \mathbb{R}))$.

There exist virtually connected (i.e., with finitely many connected components) Lie groups with non-linear radical and with all classes in $H^2_B(G, \mathbb{Z})$ bounded (see Example 3.8). Hence Condition 4 of Theorem 1.2 does not imply Condition 1 of Theorem 1.2 for virtually connected Lie groups, but the following holds true.
Theorem 1.5. Let $G$ be a virtually connected Lie group. If the radical of $G$ is linear then each Borel cohomology class of $G$ with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.

As a corollary, we obtain the following generalization of Gromov’s [23, Section 1.3, p. 23] and Bucher-Karlsson’s [11, Theorem 4] theorems.

Corollary 1.6. Let $G$ be a virtually connected Lie group with linear radical. Each class in the image of the natural map $H^* (BG, \mathbb{R}) \to H^* (BG^\delta, \mathbb{R})$ can be represented by a cocycle whose set of values on all singular simplices of $BG^\delta$ is finite.

Let us prove that Corollary 1.6 follows from Theorem 1.5.

Proof. The group $H^*_B (G, \mathbb{Z})$ is naturally isomorphic to the singular cohomology $H^* (BG, \mathbb{Z})$ of the classifying space $BG$ of $G$ (cf. [47] and [1, Section 7]). The following natural diagram commutes,

$$
\begin{array}{ccc}
H^*_B (G, \mathbb{Z}) & \longrightarrow & H^* (BG, \mathbb{Z}) \cong H^* (BG, \mathbb{R}) \\
\downarrow & & \downarrow \\
H^*_b (G^\delta, \mathbb{Z}) & \longrightarrow & H^* (G^\delta, \mathbb{R}) \longrightarrow H^* (BG^\delta, \mathbb{R}),
\end{array}
$$

and the natural isomorphism $H^* (G^\delta, \mathbb{R}) \to H^* (BG^\delta, \mathbb{R})$ preserves the pseudo-norms (cf. [23, Section 3.3, p. 49]). Hence, Theorem 1.5 implies that each characteristic class $x^\delta$ in the image of the composition $H^* (BG, \mathbb{Z}) \to H^* (BG, \mathbb{R}) \to H^* (BG^\delta, \mathbb{R})$ is bounded. Lemma 3.2 shows that $x^\delta$ can be represented by a cocycle with finite range. As $H^* (BG, \mathbb{R}) \cong H^* (BG, \mathbb{Z}) \otimes \mathbb{R}$, we conclude that each characteristic class in the image of $H^* (BG, \mathbb{R}) \to H^* (BG^\delta, \mathbb{R})$ can be represented by a cocycle with finite range. \qed

In Corollary 1.6, the hypothesis of linearity on the radical is needed; we will give for every $i > 0$ an example of a $2i$-dimensional connected Lie group $G_i$ with non-linear radical such that for all $d$ with $1 \leq d \leq i$, $H^{2d} (BG^\delta_i, \mathbb{R})$ contains elements in the image of $H^{2d} (BG_i, \mathbb{R}) \to H^{2d} (BG^\delta_i, \mathbb{R})$ which cannot be represented by bounded cocycles (Example 3.16).

Generalizing to all connected Lie groups a construction of Goldman in [21], we shall prove:

Theorem 1.7. Let $G$ be a connected Lie group. If the commutator subgroup of the radical of $G$ is not simply-connected relative to its analytic topology, then there exists a class of degree 2 in the image of the
natural map \( H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}) \) which can’t be represented by a bounded cocycle.

We do not know if the condition

\[
\pi_1\left([\sqrt{G}, \sqrt{G}]\right) \neq 0,
\]

is the only obstruction to the boundedness of all characteristic classes:

**Question 1.** Let \( G \) be a connected Lie group. Is it true that each class in the image of the natural map \( H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}) \) can be represented by a cocycle whose set of values on all singular simplices of \( BG^\delta \) is finite, if and only if the commutator subgroup of the radical of \( G \) is simply-connected relative to its analytic topology?

**Remark 1.8.** (1) The condition

\[
\pi_1\left([\sqrt{G}, \sqrt{G}]\right) \neq 0,
\]

is equivalent to the existence in \( \pi_1(G) \) of an infinite cyclic subgroup which is distorted in \( \tilde{G} \). Starting with a non-trivial element in the fundamental group of the commutator subgroup of the radical, the construction in the proof of Proposition 2.26 produces an infinite cyclic distorted subgroup in \( \pi_1(G) \subset \tilde{G} \). The converse implication also holds \([18]\).

(2) Let \( N \) be a normal analytic subgroup of a connected Lie group \( G \). The inclusion \( N \subset \overline{N} \) induces a monomorphism

\[
\pi_1(N) \to \pi_1(\overline{N}),
\]

(see Lemma 2.6). In the case \( N = [\sqrt{G}, \sqrt{G}] \), if we assume that the monomorphism is an isomorphism, Corollary 1.6 together with Theorem 1.7 bring a positive answer to Question 1. See Remark 2.12 for an example of a connected solvable Lie group \( G \) with \([G,G]\) simply-connected but \([\overline{G},\overline{G}]\) not.

Question 1 has a positive answer if \( G \) is solvable:

**Theorem 1.9.** (Compare with Goldman \([21]\).) Let \( G \) be a connected solvable Lie group. The following are equivalent:

(1) The commutator group \([G,G]\) is simply-connected.

(2) Flat principal \( G \)-bundles over finite complexes are virtually trivial.

(3) The natural map \( H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}) \) is 0 in all positive degrees.

(4) The image of the natural map \( H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}) \) consist of bounded cohomology classes.
1.5. **Structure of the paper.** The paper is devoted to proving the theorems stated in Subsection 1.4. To this end, most of the results recalled and proved in Sections 2 and 3, are needed. Each subsection of these two sections begins with a summary of the main results it contains. The reader who wishes to understand quickly the main ideas of the paper may read Section 4 first, which separately presents the proof of each implication of each theorem of Subsection 1.4, and refer herself/himself to Sections 2 and 3 when needed.

Theorem 1.2 has roots in [14, Proposition 5.5 and Lemma 6.3], where some of the authors of the present paper needed to work with Borel cocycles associated to distorted and undistorted central extensions.

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2. **On the geometry and topology of Lie groups**

2.1. **Preliminaries and facts on Lie groups.** In this subsection we discuss general facts about the topology of Lie groups (Lemmata 2.2, 2.3 and 2.5) and their algebraic structure (Proposition 2.8, Lemmata 2.7, 2.9, 2.13). We also recall linearity criteria for Lie groups (Theorem 2.1 and Proposition 2.10).

To fix notation, we recall some basic facts on Lie groups; a good reference is [43]. Let $G$ be a connected (real) Lie group. Then $G$ admits a Levi decomposition $G = \sqrt{G} \cdot L(G)$ with $\sqrt{G}$ the radical of $G$ (the maximal connected solvable normal subgroup) and $L(G)$ a Levi subgroup (a maximal connected semi-simple subgroup; it is unique up to conjugation). The intersection $\sqrt{G} \cap L(G)$ is a totally disconnected subgroup of $G$, which is discrete in $L(G)$ but in general not discrete in $\sqrt{G}$. In case $G$ is simply-connected, $\sqrt{G} \cap L(G) = \{e\}$. Hence $G$ is the semi-direct product $G = \sqrt{G} \rtimes L(G)$. A group $G$ is called linear, if it admits a faithful representation $G \to GL(n, \mathbb{R})$. In case $G$ is a connected Lie group, there is a closed normal subgroup $\Lambda(G)$, the linearizer of $G$, such that $G/\Lambda(G)$ is linear and such that any Lie homomorphism $G \to H$ with $H$ linear factors through $G/\Lambda(G)$ (for the structure of the linearizer see [31]). If $G$ is connected and semi-simple, $\Lambda(G)$ is a central discrete subgroup and any quotient of $G/\Lambda(G)$ is linear too [43, Ch. 5, §3, Thm. 8]. In case $G$ is connected and solvable,
Λ(\(G\)) is a central torus; a quotient of a linear solvable Lie group need not be linear. Concerning the linearity of Lie groups, the following theorem is basic.

**Theorem 2.1.** (Malcev [34].) A connected Lie group \(G\) is linear if and only if its radical \(\sqrt{G}\) and Levi subgroup \(L(G)\) are.

In the sequel we will also deal with not necessarily connected Lie groups \(G\); we will write \(G^0\) for the connected component of \(G\). By the radical \(\sqrt{G}\) of \(G\) we still mean a maximal connected normal solvable subgroup of \(G\), thus \(\sqrt{G} = \sqrt{G^0}\). A group \(G\) is called *virtually connected*, if \(G/G^0\) is a finite group. In case \(G_a\) is a connected linear algebraic group defined over \(\mathbb{R}\), its Lie group of \(\mathbb{R}\)-points \(G_a(\mathbb{R})\) is virtually connected; it might fail to be connected as a Lie group. On the other hand every connected linear reductive Lie group \(G\) is, as a Lie group, isomorphic to \(G_a(\mathbb{R})^0\) for some connected linear real algebraic group \(G_a\) (see [33]). The following simple observation is used later.

**Lemma 2.2.** Let \(G\) be a contractible Lie group and \(H < G\) a connected subgroup. Then \(H\) is contractible too.

**Proof.** Recall that any connected Lie group is homotopy equivalent to a maximal compact subgroup. Let \(K\) be a maximal compact subgroup of \(H\). Then \(K\) is contained in a maximal compact subgroup of \(G\), which is \(\{e\}\). Thus \(K = \{e\}\) and therefore \(H\) is contractible. \(\square\)

Since a maximal compact subgroup of a connected solvable Lie group \(S\) is a torus, \(S\) is contractible if and only if \(S\) is simply-connected. It follows that every connected subgroup of a simply-connected solvable Lie group is contractible.

**Lemma 2.3.** Let \(N\) be a connected closed normal subgroup of a connected Lie group \(G\). Then the inclusion \(N \hookrightarrow G\) induces a short exact sequence \(0 \to \pi_1(N) \to \pi_1(G) \to \pi_1(G/N) \to 0\).

**Proof.** The long exact sequence in homotopy reads

\[
\cdots \to \pi_2(G/N) \to \pi_1(N) \to \pi_1(G) \to \pi_1(G/N) \to 0.
\]

Because \(G/N\) is a Lie group, \(\pi_2(G/N) = 0\) (see [9]), and the result follows. \(\square\)

**Remark 2.4.** Since \(\mathbb{R}\) is divisible, \(\text{Hom}(\cdot, \mathbb{R})\) is an exact contravariant functor on abelian groups. Since the fundamental group of a Lie group is abelian, Lemma 2.3 implies that the inclusion \(N \hookrightarrow G\) induces a surjection \(\text{Hom}(\pi_1(G), \mathbb{R}) \twoheadrightarrow \text{Hom}(\pi_1(N), \mathbb{R})\).
Lemma 2.5. Let $N$ be a closed connected normal subgroup of a connected Lie group $G$. There are two exact sequences with commutative squares

$$
\begin{array}{ccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \longrightarrow & \Gamma \cap \tilde{N} & \longrightarrow & \tilde{N} & \longrightarrow & N & \longrightarrow & 1,
\end{array}
$$

were $\tilde{G}$ and $\tilde{N}$ are the respective universal covers of $G$ and $N$, whereas $\Gamma$ and $\Gamma \cap \tilde{N}$ are central discrete subgroups, isomorphic to the respective fundamental groups of $G$ and $N$, and the vertical arrows are injections of closed normal subgroups.

Proof. Let $p : \tilde{G} \to G$ be the universal cover of $G$. The connected component $C$ of the identity in $p^{-1}(N)$ is a closed connected normal subgroup of $\tilde{G}$ and the restriction of $p$ to $C$ is a cover of $N$. According to Lemma 2.3 the group $C$ is simply-connected. Hence $C = \tilde{N}$, the universal cover of $N$. \qed

Lemma 2.6. Let $p : \tilde{G} \to G$ be the universal cover of a connected Lie group $G$. Let $\tilde{N}$ be an analytic normal subgroup of $\tilde{G}$. Let $N$ denote the projection $p(\tilde{N})$ of $\tilde{N}$ into $G$ with the quotient topology defined by the restriction of $p$ to $\tilde{N}$. The inclusion $N \subset \overline{N}$, induces a monomorphism $\pi_1(N) \to \pi_1(\overline{N})$.

Proof. The subgroup $\tilde{N}$ is closed (see [38, Chapter 2 (5.14)]). The restriction of $p$ to $\tilde{N}$ is a cover of $N$. Hence, according to Lemma 2.3, the group $\tilde{N}$ is the universal cover of $N$. Let $X$ be any path-connected subspace of $G$ containing $N$ (for example $X = \overline{N}$). Let us show that the inclusion $N \subset X$, induces a monomorphism $\pi_1(N) \to \pi_1(X)$. Let $c : S^1 \to N$ be a loop such that $c(0) = e$ and assume that there is a continuous map $h : D^2 \to X \subset G$ from the disk to $X$ which extends $c$. Let $\tilde{h} : D^2 \to \tilde{G}$ be the unique lift of $h$ such that $\tilde{h}(0) = \tilde{e}$. The restriction $\tilde{c}$ of $\tilde{h}$ to $S^1$ is the unique lift in $\tilde{N}$ of $c$ with $\tilde{c}(0) = \tilde{e}$. \qed

Lemma 2.7. Let $G$ be a connected nilpotent Lie group. Let $\phi : \mathbb{R} \to G$ be a one-parameter subgroup. If $\phi(t_0)$ is central for some $t_0 \neq 0$, then $\phi(t)$ is central for all $t$.

Proof. As the universal cover of a Lie group $G$ is an extension of $G$ with discrete central kernel, an easy argument shows that it is enough to prove the lemma when the nilpotent group $G$ is simply-connected.
In this case, the exponential map
\[ \exp : g \to G \]
is a diffeomorphism. In exponential coordinates the Lie group multiplication is given by the Campbell-Hausdorff formula
\[ xy = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \ldots \]
for all \( x, y \in g \). As the Lie algebra \( g \) is nilpotent, the above expression has finitely many terms. Hence if \( g \in G \) is given, the equation
\[ g \phi(t) g^{-1} = e \]
for all \( t \in \mathbb{R} \). □

**Proposition 2.8.** A connected nilpotent Lie group \( N \) has a central torus as its unique maximal compact subgroup.

**Proof.** Since a connected compact subgroup of \( N \) is a torus, and since the union of all 1-dimensional tori in a given torus \( T \) is a dense subset of \( T \), it suffices to prove that every 1-dimensional torus \( S^1 < N \) is central. Choose a surjective homomorphism \( \phi : \mathbb{R} \to S^1 < N \) and \( t_0 \in \mathbb{R} \setminus \{0\} \) with \( \phi(t_0) = e \). Then the results follows by applying Lemma 2.7. □

Recall that the nilradical of a Lie group is the largest connected, normal, nilpotent subgroup.

**Lemma 2.9.** Let \( K \) be a compact subgroup of a connected solvable Lie group \( G \). Let \( T_N \) be the maximal compact subgroup of the nilradical \( N \) of \( G \). If \( [G, G] \cap K \neq \{e\} \), then there exists a non-trivial one-parameter subgroup in \( T_N \cap [G, G] \).

**Proof.** Let \( k \in [G, G] \cap K \) be a non-trivial element. Let \( \mathfrak{D} \) be the Lie algebra of the (not necessarily closed) analytic subgroup \( [G, G] \). Since \( G \) is solvable, \( [G, G] \) is nilpotent and the exponential map
\[ \exp : \mathfrak{D} \to [G, G] \]
is surjective (see [30]). Let \( X \in \mathfrak{D} \) such that \( \exp(X) = k \) and let \( \phi : \mathbb{R} \to [G, G] \) be the corresponding one-parameter subgroup \( \phi(t) = \exp(tX) \). Since \( [G, G] \) is a normal nilpotent connected subgroup of \( G \), it is contained in the nilradical \( N \) of \( G \). Let \( T_N \) be the maximal compact subgroup of \( N \). We hence have that \( [G, G] \cap K \subseteq T_N \). Since \( k = \exp(X) \in T_N \), it implies that \( \phi(Z) \subseteq T_N \). The 1-parameter subgroup \( \phi(\mathbb{R}) \), being at bounded distance from \( \phi(Z) \) (for any left-invariant Riemannian metric on \( G \)), is bounded and hence the closure of \( \phi(\mathbb{R}) \) in \( N \) is a compact subgroup. We conclude that \( \phi(\mathbb{R}) \subseteq T_N \cap [G, G] \). □
Proposition 2.10. Let \( G \) be a connected solvable Lie group. The following are equivalent:

1. The group \( G \) is linear.
2. The group \( \pi_1([G,G]) \) is trivial.
3. For every maximal compact subgroup \( K < G \), \( [G,G] \cap K = \{e\} \).
4. For every maximal compact subgroup \( K < G \), \( [G,G] \cap K = \{e\} \).
5. There is a maximal compact subgroup \( K < G \) and a closed 1-connected normal subgroup \( H \leq G \) such that \( G = HK \).
6. For every maximal compact subgroup \( K < G \) there is a 1-connected closed normal subgroup \( H < G \) such that \( G = HK \) and \( H \cap K = \{e\} \).

Remark 2.11. In case the group \( G \) is nilpotent, the equivalent conditions of Proposition 2.10 are also equivalent to the following one: the group is a direct product \( G = T \times N \) where \( T \) is the unique maximal compact subgroup of \( G \) and where \( N \) is nilpotent contractible. This follows from Lemma 2.8.

Proof of Proposition 2.10. Conditions (5) and (6) are equivalent since any two maximal compact subgroups are conjugate and the equivalence of (5) and (1) was proved by Malcev in [34]. The equivalence of (1) and (2) was proved by Gotô, see [22, Thm.5]. That (5) implies (3) and therefore (4) follows from \( [G,G] \subset H \) (to check this inclusion remember that \( K \) is abelian), which implies that \( [G,G] \) is contractible. Since \( [G,G] \) is homotopy equivalent to its maximal torus, (3) implies (2) and it remains to show that (4) implies (3). If (3) does not hold, the maximal compact subgroup \( T \) of \( [G,G] \) is a non-trivial torus with \( T \cap [G,G] = \{e\} \) by assumption. Thus there is a continuous embedding

\[
T \times [G,G] \rightarrow [G,G],
\]

which implies that \( \{e\} \times [G,G] \) is dense in \( T \times [G,G] \), which is a contradiction. \( \square \)

Remark 2.12. The equivalent conditions of Proposition 2.10 are not equivalent to the group \( \pi_1([G,G]) \) being trivial. Indeed, the following is an example of a connected solvable Lie group \( G \) with \([G,G]\) simply-connected but \([G,G]\) not. Let \( H \) be the 3-dimensional Heisenberg group and consider \( H \times S^1 \). Its center is \( \mathbb{R} \times S^1 \). Take the discrete central subgroup \( \mathbb{Z} \) generated by \((1,t)\) with \( 1 \) generating \( \mathbb{Z} \) in \( \mathbb{R} \), and \( t \) of infinite order in \( S^1 \); this central subgroup of \( H \times S^1 \) is discrete. Let us define \( G := (H \times S^1)/\mathbb{Z} \). It is a nilpotent connected Lie group with \([G,G]\) homeomorphic to \( \mathbb{R} \), embedded in the maximal torus \( S^1 \times S^1 \) of \( G \) in a
dense way. It follows that $\pi_1([G,G])$ is trivial but $\pi_1([\overline{G},G]) = \mathbb{Z} \times \mathbb{Z}$.

Thus $G$ is not linear (its linearizer is $\overline{G,G}$, a central 2-torus). We would like to warn the reader that in several places in the literature one finds an incorrect statement saying that a connected solvable Lie group $G$ is linear if and only if $[G,G]$ is simply-connected (for instance, see [46, Ch. 2, Thm. 7.1]); the correct statement is that $G$ is linear if and only if the closure $[G,G] < G$ is simply-connected.

**Lemma 2.13.** Let $G$ be a connected Lie group and let $\sqrt{G}$ be its radical. If $Q = G/\sqrt{G}$ is simply-connected, then the short exact sequence

$$1 \to \sqrt{G} \to G \to Q \to 1$$

splits, and therefore we have a semi-direct product, $G = \sqrt{G} \rtimes Q$.

**Proof.** This is a classical result in the case $G$ itself is simply-connected (see [30]). The proof in our case is reduced to the simply-connected case by considering the following two exact sequences with commutative squares,

$$
\begin{array}{ccccccc}
1 & \to & \sqrt{G} & \to & G' & \to & Q & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \sqrt{G} & \to & \hat{G} & \to & \hat{G}/\sqrt{G} & \to & 1,
\end{array}
$$

where $\hat{G}$ is the universal cover of $G$. The map $\phi$ is an isomorphism because it is a connected covering of a simply-connected space. Thus, if $\sigma$ is a splitting for $\hat{G} \to \hat{G}/\sqrt{G}$, then $p \circ \sigma \circ \phi^{-1}$ is a splitting for $G \to Q$. □

### 2.2. Topological $A$-extensions

In this subsection we define topological $A$-extensions. Moore’s theorem (see Theorem 3.17) shows that they are closely related to 2-dimensional Borel cohomology.

A central extension of topological groups $0 \to A \to E \to G \to 1$ will be referred to as a topological $A$-extension. In this setting, we always assume that the groups are locally compact and second countable and that the monomorphism $A \to E$ has closed image. We write $\text{Ext}_{\text{top}}(G,A)$ for the group of isomorphism classes of topological $A$-extensions of $G$, where two extensions $0 \to A \to E_1 \to G \to 1$ and $0 \to A \to E_2 \to G \to 1$ are called isomorphic if there is an isomorphism of topological groups $\varphi : E_1 \to E_2$ making the following diagram

$$
\begin{array}{ccccccc}
0 & \to & A & \to & E_1 & \to & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A & \to & E_2 & \to & G & \to & 1
\end{array}
$$
commute:

\[
\begin{array}{c}
E_1 \\
\downarrow \phi \\
0 \longrightarrow A \longrightarrow E_2 \longrightarrow G \longrightarrow 1.
\end{array}
\]

In case \( G \) is a connected Lie group and \( A \) is discrete, a topological \( A \)-extension is just a covering space \( E \to G \) with covering transformation group \( A \). Such an \( E \) is determined by a homomorphism \( \phi : \pi_1(G) \to A \), with

\[
E = A \times \tilde{G} = (A \times \tilde{G})/\pi_1(G)
\]

with \( \tilde{G} \) the universal cover of \( G \), where \( \pi_1(G) \) acts on \( \tilde{G} \) via deck transformations and on \( A \) via \( \phi \). It follows that there is a natural isomorphism

\[
\text{Ext}_{\text{top}}(G, A) \cong \text{Hom}(\pi_1(G), A).
\]

2.3. Distorted and undistorted central extensions. In this subsection, we first recall some basic facts about the distortion of subgroups. The main results are then the following. We prove that in a topological \( A \)-extension of a Lie group, the distortion of \( A \) is a lower bound for the growth of any Borel cocycle defining the extension (Proposition 2.20). Then we establish an algebraic criterion to decide when certain central subgroups of a simply-connected solvable Lie group are distorted (Proposition 2.23). Finally, we show that if the radical of a connected Lie group \( G \) is not linear, then the fundamental group of \( G \) is distorted in the universal cover of \( G \) (Proposition 2.26).

**Definition 2.14.** Let \( A \) and \( E \) be two locally compact, compactly generated groups. We denote by \( L_S \), resp. \( L_U \), the word length associated to a symmetric relatively compact generating set \( S \) of \( A \), resp. \( U \) of \( E \). Assume that \( A \) is a subgroup of \( E \). We say that \( A \) is undistorted in \( E \) if the identity map is a quasi-isometry between \( (A, L_S) \) and \( (A, L_U|_A) \). Otherwise we say that the subgroup \( A \) is distorted in \( E \).

**Remarks 2.15.**

1. Under the hypothesis of the above definition, Gromov defines in [24, Chapter 3], the distortion function as

\[
\text{DISTO}(r) := \text{diam}_A(A \cap B_E(r)) / r, \quad \forall r > 0.
\]

One checks that \( A \) is undistorted exactly when the function \( \text{DISTO} \) is bounded.

2. We say that the distortion of \( A \) is at least linear if there exist \( a > 0 \) and \( R > 0 \) such that for all \( r \geq R \), we have

\[
ar \leq \text{DISTO}(r).
\]
We say that the distortion is sub-linear if it is not at least linear.

(3) Having a bounded, or unbounded, or sub-linear, etc., distortion function, is a well defined property of the couple \( A < E \), i.e. does not depend on the choice of the relatively compact symmetric generating sets.

**Definition 2.16.** Let \( A \) and \( E \) be locally compact, compactly generated groups. A topological \( A \)-extension \( E \) is called undistorted, resp. distorted, if \( A \) is undistorted, resp. distorted, in \( E \).

**Lemma 2.17.** Let \( p : G \rightarrow Q \) be a continuous homomorphism between locally compact, compactly generated groups. Let \( H < G \) be a compactly generated subgroup of \( G \) and assume that \( H \cap \ker(p) = \{e\} \). If \( H \) is distorted in \( G \), then so is \( p(H) \) in \( Q \).

**Proof.** Let \( S \) be a compact symmetric generating set of \( H \). Then \( p(S) \) is a compact symmetric generating set of \( p(H) \). Let \( h \in H \). As \( p \) is a homomorphism, \( L_{p(S)}(p(h)) \leq L_S(h) \). As \( H \cap \ker(p) = \{e\} \), we also have \( L_{p(S)}(p(h)) \geq L_S(h) \). The proof now follows because a homomorphism is \( C \)-Lipschitz with respect to word metrics (for \( C \geq 1 \) a constant depending on \( p \) and on generating sets) and because the projection of a path between the identity in \( G \) and \( h \in H \) is still a path between the identity in \( Q \) and \( p(h) \in p(H) \). \( \square \)

**Lemma 2.18.** Let \( 0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1 \), be an exact sequence of second countable locally compact groups, with \( A \) discrete in \( E \). Let \( \sigma \) be a Borel section of the projection \( p : E \rightarrow G \). Let \( \mu_E \) be a Haar measure on \( E \). There exists a Borel subset \( X \) of \( E \) with the following properties:

- (1) \( X \) is relatively compact,
- (2) \( \mu_E(X) > 0 \),
- (3) \( X = X^{-1} \),
- (4) \( \sigma(p(X)) \) is relatively compact.

**Proof.** Let \( U \) be an open symmetric relatively compact subset of \( E \), small enough such that the covering projection \( p \) restricted to \( U \), is a homeomorphism onto its image \( V = p(U) \). Let \( \mu_E \), resp. \( \mu_G \), be the Haar measure on \( E \), resp. on \( G \), such that \( \mu_E(U) = 1 \), resp. such that \( \mu_G(V) = 1 \). The uniqueness of the Haar measure, together with the fact that the homomorphism \( p \) is a local homeomorphism, imply that for any Borel subset \( A \) of \( U \), we have \( \mu_E(A) = \mu_G(p(A)) \) (see [7, Intégration, chapitre 7, paragraphe 2, numéro 7, proposition 10, p. 60]). Let \( E = \bigcup_{n \in \mathbb{N}} K_n \), be an exhaustion of \( E \) by compact subsets, such that \( K_n \subseteq K_{n+1} \). Let \( B_n = \{ v \in V : \sigma(v) \in K_n \} = V \cap \sigma^{-1}(K_n) \).
It is a Borel subset of $G$ because $\sigma$ is a Borel map. Notice that
\[
\lim_{n \to \infty} \mu_G(V \setminus B_n) = 0.
\]
We conclude from this, and from the fact that the modular function of $G$ is bounded on $V$, that there exists $N \in \mathbb{N}$, such that $\mu_G(V \setminus B_N) < 1/2$, and such that $\mu_G((V \setminus B_N)^{-1}) < 1/2$. Let
\[
Y = V \setminus ((V \setminus B_N) \cup (V \setminus B_N)^{-1}).
\]
The set $Y$ has the following properties:

1. $Y$ is relatively compact,
2. $\mu_G(Y) > 0$,
3. $Y = Y^{-1}$,
4. $\sigma(Y) \subseteq K_N$.

Let $X = p^{-1}(Y) \cap V$. The set $X$ has the required properties. \qed

**Remark 2.19.** A projection of separable locally compact groups always admits a locally bounded Borel section [32, Lemma 2]. The point in the above lemma is that the given Borel section $\sigma$ is not assumed to be locally bounded.

The following proposition has first been proved by Gersten [20], in the setting of finitely generated groups.

**Proposition 2.20.** Let $0 \to A \to E \to G \to 1$, be an exact sequence of Lie groups, with $E$ connected, $A$ finitely generated discrete (hence central) in $E$. Let $\sigma$ be a Borel section of the projection $p : E \to G$, with $\sigma(e) = e$. Let $c : G \times G \to A$, $c(g, g') = \sigma(g)\sigma(g')\sigma(gg')^{-1}$ be the corresponding cocycle. If $c$ is bounded, then $A$ is undistorted in $E$.

**Proof.** Let $x \in E$. There exist a unique element $a = a(x) \in A$ and a unique element $g = g(x) \in G$ such that $x = a\sigma(g)$. Let $X \subseteq E$ be as in Lemma 2.18. The set $X(\sigma(p(X)))^{-1}$, is relatively compact in $E$. This implies, together with the hypothesis that $A$ is discrete in $E$, that there exists a finite symmetric generating set $S$ of $A$ with the property that if $x \in X$, then $a(x) \in S$.

As $\mu_E(X) > 0$, the set $XX^{-1} = XX$ contains a neighborhood of the identity of $E$ (see [27, Chapter XII, Section 61, Exercise 3, p. 268]). As $E$ is connected, the set $X$ generates $E$.

Let $a \in A$ of length relative to $X$ equal to $n$. Hence, there exist $x_1, \ldots, x_n \in X$ such that
\[
a = x_1 \cdots x_n = a_1\sigma(g_1) \cdots a_n\sigma(g_n) = a_1 \cdots a_n c(g_1, g_2) \cdots c(g_1 \cdots g_{n-1}, g_n)\sigma(g_1 \cdots g_n).
\]
Applying $p$ to the above equality, we get
\[ e = p(a_1 \cdots a_n c(g_1, g_2) \cdots c(g_1 \cdots g_{n-1}, g_n) \sigma(g_1 \cdots g_n)) = g_1 \cdots g_n. \]
Since $\sigma(e) = e$, we see that the length of $a$ relative to $S$ is bounded by $n + (n - 1) \sup \{L_s(c(g,g'))|L_{p(X)}(g) < n, L_{p(X)}(g') \leq 1\}$. □

**Remark 2.21.** The proof of Proposition 2.20 shows in fact that the distortion of $A$ in $E$ is bounded by the growth of $c$.

**Proposition 2.22.** (Gromov [24, 3B2].) Let $N$ be a simply-connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $x \in [\mathfrak{n}, \mathfrak{n}] \setminus \{0\}$. Then the one-parameter subgroup $t \mapsto \exp(tx)$, is at least linearly distorted in $N$.

The statement in [24, 3B2] which implies Proposition 2.22 is not proved. For a proof of Proposition 2.22, we refer the reader to [45, Prop. 4.1] or [44]. A more conceptual proof, in the spirit of [24], follows from the existence of a homothety, relative to a Carnot-Caratheodory metric on $N$, in the case $N$ is graded. The general case is reduced to the graded case using [8, Theorem 1.3].

**Proposition 2.23.** Let $0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{a} \to 0$ be an exact sequence of Lie algebras over $\mathbb{R}$ with $\mathfrak{n}$ nilpotent and $\mathfrak{a}$ abelian. Let $\mathfrak{z} \subseteq \mathfrak{n}$ be a subalgebra which is central in $\mathfrak{g}$. Let $G$ be the simply-connected Lie group whose Lie algebra is $\mathfrak{g}$ and let $Z$ be the analytic subgroup of $G$ corresponding to $\mathfrak{z}$. Then either $\mathfrak{z}$ is a direct factor in $\mathfrak{g}$, or there exists a one-parameter subgroup in $Z$ which is distorted in $G$.

**Proof.** Let $T \in \mathfrak{g}$ be a regular element, and let $\mathfrak{n}(T, \mathfrak{g})$ be the associated Cartan subalgebra [15, Chapitre VI, Paragraphe 4, 2, Proposition 9, p. 387]. The subalgebra $\mathfrak{n}(T, \mathfrak{g})$ is the big kernel of $ad(T)$; that is, there is an integer $N \in \mathbb{N}$, big enough such that $\mathfrak{n}(T, \mathfrak{g}) = \ker(ad^N(T)) = \ker(ad^{N+1}(T))$. Let $i(T, \mathfrak{g})$ be the small image of $ad(T)$; that is, there is an integer $M \in \mathbb{N}$, big enough such that $i(T, \mathfrak{g}) = ad^M(T)(\mathfrak{g}) = ad^{M+1}(T)(\mathfrak{g})$. Hence $\mathfrak{g}$ decomposes as a direct sum of $ad(T)$-invariant subspaces,
\[ \mathfrak{g} = \mathfrak{n}(T, \mathfrak{g}) \oplus i(T, \mathfrak{g}). \]

The derivation $ad(T)$ has a semi-simple part $ad_s(T)$, that gives $\mathfrak{n}(T, \mathfrak{g}) = \ker(ad_s(T))$, and $i(T, \mathfrak{g}) = ad_s(T)(\mathfrak{g})$, see [15, Chapitre VI, Paragraphe 4, 2, Proposition 8, p. 385]. It follows that
\[ [\mathfrak{n}(T, \mathfrak{g}), i(T, \mathfrak{g})] \subseteq i(T, \mathfrak{g}), \]
because if $X \in \mathfrak{n}(T, \mathfrak{g})$ and $Y \in i(T, \mathfrak{g})$, we can choose $\tilde{Y} \in \mathfrak{g}$ such that $ad(T)(\tilde{Y}) = Y$, hence
\[ [X, Y] = [X, ad_s(T)(\tilde{Y})] + [ad_s(T)(X), \tilde{Y}] = ad_s(T)([X, \tilde{Y}]). \]
We note that the Cartan subgroup $C < G$ associated to $n(T, g)$ is, by Lemma 2.2, a simply-connected nilpotent group, because $G$ is a simply-connected solvable Lie group and therefore contractible. If the intersection $\mathfrak{z} \cap [\mathfrak{n}(T, g), n(T, g)]$ is non-trivial, then according to Proposition 2.22, there is a one-parameter subgroup in $Z$ which is distorted in the Cartan subgroup $C < G$. This obviously implies that $Z$ is distorted in $G$ as well, and the proof is finished in this case.

So we can assume that $\mathfrak{z} \cap [\mathfrak{n}(T, g), n(T, g)] = \{0\}$. We notice that as the subalgebra $\mathfrak{z}$ is central in $g$, it is contained in $n(T, g)$. Hence we can choose a complement $V$ for $\mathfrak{z} \oplus [\mathfrak{n}(T, g), n(T, g)]$ in $n(T, g)$. Obviously $\mathfrak{m} = [n(T, g), n(T, g)] \oplus V$ is a subalgebra of $n(T, g)$ and hence $n(T, g) = \mathfrak{z} \times \mathfrak{m}$. Let $u = [n, n] \cap n(T, g)$.

We can assume that $\mathfrak{z} \cap u = \{0\}$. Because otherwise, according to Proposition 2.22, there is a one-parameter subgroup in $Z$ which is distorted in the analytic subgroup of $G$ associated to $u$, hence in $G$ as well. Let $q : \mathfrak{z} \times \mathfrak{m} \to \mathfrak{m}$ be the projection onto the second factor. Let $V'$ be a complement for $u \cap \mathfrak{m}$ in $u$, that is $(u \cap \mathfrak{m}) \oplus V' = u$. If $q(V') \cap ([\mathfrak{m}, \mathfrak{m}] + u \cap \mathfrak{m}) \neq \{0\}$, we claim that there is a one-parameter subgroup in $Z$ which is distorted in $G$. To see why, let $v \in V'$ such that $q(v) = x + y \neq 0$, with $x \in [\mathfrak{m}, \mathfrak{m}]$ and $y \in u \cap \mathfrak{m}$. The element $z = v - (x + y)$ belongs to $\mathfrak{z}$. Notice that $z \neq 0$, because otherwise we would have $v = x + y \in V' \cap (u \cap \mathfrak{m}) = \{0\}$. Similarly $x \neq 0$, because otherwise $z = v - y \in u \cap \mathfrak{z} = \{0\}$. Since $x \in [\mathfrak{m}, \mathfrak{m}]$, the one-parameter subgroup $t \mapsto \exp(tx)$ is distorted in $G$, according to Proposition 2.22. Since $v - y \in u \subseteq [n, n]$, the one-parameter subgroup $t \mapsto \exp(t(v - y)x)$ is distorted in $G$, according to Proposition 2.22. The sub-algebra of $g$ spanned by $x$ and $z$ is isomorphic to $\mathbb{R}^2$. Hence, as $z = (-x) + (v - y)$, the one-parameter subgroup $t \mapsto \exp(tz)$ is also distorted in $G$. (The geometric picture is the following. To reach the element $\exp(n^2z)$ where $n$ is large, we start from the identity in $G$ and we first reach with a path in $\exp([\mathfrak{m}, \mathfrak{m}])$, of length linear in $n$, the point $\exp(n^2(-x))$. Then we follow a path between $\exp(n^2(-x))$ and $\exp(n^2z)$, obtained as the left-translated in $G$ by $\exp(n^2(-x))$ of a path in $\exp([n, n])$ between the identity and $\exp(n^2(v - y))$ and of length linear in $n$.)

Hence we assume that $q(V') \cap ([\mathfrak{m}, \mathfrak{m}] + u \cap \mathfrak{m}) = \{0\}$, and we will show that $\mathfrak{z}$ is a direct factor in $g$. We consider a complement $W$ for $q(V') \oplus ([\mathfrak{m}, \mathfrak{m}] + u \cap \mathfrak{m})$ in $\mathfrak{m}$. Hence

$$\mathfrak{m} = ([\mathfrak{m}, \mathfrak{m}] + u \cap \mathfrak{m}) \oplus q(V') \oplus W.$$
We define the sum of subspaces, 
\[ \tilde{m} = ([m, m] + u \cap m) + V' + W. \]
We claim that \( 3 \times \tilde{m} \cong n(T, g) \). The inclusion \( 3 + \tilde{m} \subseteq n(T, g) \) is obvious. The opposite inclusion is true because if \( x \in n(T, g) \), then \( x = z + y \) with \( z \in 3 \) and \( y \in m \), and we can write \( y = y_1 + y_2 + y_3 \) with \( y_1 \in [m, m] + u \cap m \), \( y_2 \in q(V') \), and \( y_3 \in W \). Let \( v \in V' \) such that \( q(v) = y_2 \). Define \( z' = v - y_2 \in 3 \). Hence \( x = z + y = (z - z') + y_1 + v + y_3 \), with \( z - z' \in 3 \), \( y_1 \in [m, m] + u \cap m \), \( v \in V' \), and \( y_3 \in W \). The sum is direct because,
\[
\dim(\tilde{m}) \leq \dim([m, m] + u \cap m) + \dim(V') + \dim(W) \\
= \dim([m, m] + u \cap m) + \dim(q(V')) + \dim(W) = \dim(m).
\]
As the subspace \( \tilde{m} \) of \( n(T, g) \) is a sub-algebra (because it contains \([m, m] = [n(T, g), n(T, g)]\) ), and as \( 3 \) is central, we obtain a direct product as claimed. What we gained in replacing \( m \) with \( \tilde{m} \), is that the latest contains \( u \) because \( u = (u \cap m) \oplus V' \subseteq \tilde{m} \). This will be crucial in finishing the proof.

We have:
\[
g = n(T, g) \oplus i(T, g) = 3 \oplus \tilde{m} \oplus i(T, g).
\]
The proof will be finish if we show that \( \tilde{m} \oplus i(T, g) \) is a sub-algebra of \( g \). Let \( x, x' \in \tilde{m} \) and \( y, y' \in i(T, g) \). We have,
\[
[x + y, x' + y'] = [x, x'] + [x, y'] + [y, x'] + [y, y'].
\]
Hence we have to show that \([y, y'] \in \tilde{m} \oplus i(T, g) \). As \( i(T, g) \subseteq \ker(p) = n \), we have \([i(T, g), i(T, g)] \subseteq [n, n] \). As \( n \) is an ideal of \( g \), it is preserved by \( ad(T) \). Hence, as \( ad(T) \) is a derivation it also preserves \([n, n] \). Let \( n(T, [n, n]) \), resp. \( i(T, [n, n]) \), be the big kernel, resp. the small image, of the restriction of \( ad(T) \) to \([n, n] \). We have
\[
[n, n] = n(T, [n, n]) \oplus i(T, [n, n]).
\]
Now we can conclude because \( n(T, [n, n]) \subseteq n(T, g) \cap [n, n] = u \subseteq \tilde{m} \) and \( i(T, [n, n]) \subseteq i(T, g) \).

**Remark 2.24.** In fact, the proof of Proposition 2.23 shows that if the distortion function of the central one-parameter subgroup under consideration is not bounded, then it grows at least linearly.

**Lemma 2.25.** Let \( V \) be a simply-connected abelian Lie group with a left-invariant Riemannian metric and let \( \Gamma \) be a lattice of \( V \). Let \( t \mapsto \exp(tX) \) be a one-parameter subgroup of \( V \) which projects to a dense subgroup of \( V/\Gamma \). Let \( \Gamma = \mathbb{Z} \oplus A \) be a splitting of \( \Gamma \) and let \( z \) be
a generator of the \( \mathbb{Z} \)-factor. Then there are constants \( C > 1, 0 < \alpha < 1 < \beta \), such that for each \( n \in \mathbb{Z} \), there exists \( t \in \mathbb{R} \), satisfying \( \alpha n < |t| < \beta n \) such that in the cylinder \( V/A \) equipped with the Riemannian metric locally isometric to \( V \), we have \( d(nz, \exp(tX)) \leq C \) (where \( nz \), and \( \exp(tX) \) are viewed in the cylinder \( V/A \)).

**Proof.** It is enough to prove the lemma in the case \( V = \mathbb{R}^d \) with the usual coordinates and metric, \( \Gamma = \mathbb{Z}^d \) generated by the canonical basis vectors, \( t \mapsto \vec{t}X \) with dense image in \( \mathbb{R}^d/\mathbb{Z}^d \), \( \mathbb{Z}^d = \mathbb{Z} \oplus \mathbb{Z}^{d-1} \), and \( z \) the first vector of the canonical basis. As \( t \mapsto \vec{t}X \) has dense image in \( \mathbb{R}^d/\mathbb{Z}^d \), the vector \( X \) is not orthogonal to \( z \). We may assume that \( X \) and \( z \) are in the same connected component of the complement of the hyper-plane orthogonal to \( z \). Let \( 0 \leq \theta < \pi/2 \) be the angle between \( X \) and \( z \). We may assume \( n \in \mathbb{N} \). We choose \( t = z^n/\cos \theta \). In the cylinder \( \mathbb{R}^d \setminus \{0\} \times \mathbb{Z}^d \), we have \( d(nz, tX) \leq \sqrt{2}/2 \). \( \square \)

**Proposition 2.26.** Let \( G \) be a connected Lie group and let \( K \) be a compact subgroup of the radical \( \sqrt{G} \) of \( G \). If \( [\sqrt{G}, \sqrt{G}] \cap K \neq \{e\} \), then the fundamental group of \( G \) is distorted in the universal cover of \( G \). Also there exists a distorted topological \( \mathbb{Z} \)-extension \( E \) of \( G \).

**Proof.** For this proof let us set \( R = \sqrt{G} \). According to Lemma 2.9, there is a non-trivial one-parameter subgroup \( \phi : \mathbb{R} \to [R, R] \cap T_N \), where \( T_N \) is the maximal compact subgroup of the nilradical \( N \) of \( G \), which is also the nilradical of \( R \). The closure of \( \phi(\mathbb{R}) \) in \( T_N \) is a torus \( T \subseteq T_N \). Notice that \( T_N \) is central in \( G \) because \( T_N \) is a normal subgroup of \( G \) with discrete automorphism group. According to Lemma 2.5, the sequence of inclusions of closed connected normal subgroups \( T \subseteq R \subseteq G \) induces the commutative diagram,

\[
\begin{array}{cccccc}
1 & \to & \Gamma & \to & \tilde{G} & \to & G & \to & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \to & \Gamma \cap \tilde{R} & \to & \tilde{R} & \to & R & \to & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \to & \Gamma \cap \tilde{T} & \to & \tilde{T} & \to & T & \to & 1,
\end{array}
\]

where \( \tilde{G}, \tilde{R}, \) and \( \tilde{T} \), are resp. the universal covers of \( G, R, \) and \( T \), where \( \Gamma, \Gamma \cap \tilde{R}, \) and \( \Gamma \cap \tilde{T} \), are central discrete subgroups, resp. isomorphic to the fundamental groups of \( G, R, \) and \( T \), and where the vertical arrows are injections of closed normal subgroups. Let \( \tilde{\phi}(t) = \exp(tX) \) be the one-parameter subgroup of \( \tilde{T} \) which covers \( \phi \). Let \( \mathfrak{r} \) denote the Lie algebra of \( \tilde{R} \). The vector \( X \neq 0 \) belongs to \([\mathfrak{r}, \mathfrak{r}] \) and is central in the
Lie algebra of $\tilde{G}$, hence we can apply Proposition 2.23, with $g = \mathfrak{r}$, $n = [\mathfrak{r}, \mathfrak{r}]$, $a = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$, and $z = \mathbb{R}X$ to deduce that the one-parameter subgroup $\tilde{\phi}$ is distorted in $\tilde{R}$. As the image of $\tilde{\phi}$ in $\tilde{T} \cong \mathbb{R}^{\dim(T)}$ is a line and as $\Gamma \cap \tilde{T} \cong \mathbb{Z}^{\dim(T)}$ is cocompact in $\tilde{T}$, we deduce that $\Gamma \cap \tilde{T}$ is distorted in $\tilde{R}$, hence in $\tilde{G}$ as well. As $\Gamma \cap \tilde{T}$ is undistorted in $\Gamma$, this shows that $\Gamma$ is distorted in $\tilde{G}$.

To prove the existence of a distorted topological $\mathbb{Z}$-extension of $G$, let us choose a non-zero element $z_0$ in the free abelian group $\Gamma \cap \tilde{T}$. It is possible to choose a direct summand $\mathbb{Z}$ in the finitely generated abelian group $\Gamma$, such that a generator $z$ of the direct summand $\mathbb{Z}$ shares a non-zero power with $z_0$: there exist $m, n \in \mathbb{N}$, such that $z_0^m = z^n$. This implies that the infinite cyclic groups $(z_0)$ and $(z)$ lie at bounded distance from each other in $\tilde{G}$ (with respect to any left-invariant Riemannian metric on $\tilde{G}$). Let $B$ be a complement of $(z) = \mathbb{Z}$ in $\Gamma$ and let $E = \tilde{G}/B$. Let $p : \tilde{G} \to E$ denote the canonical projection. The connected Lie group $E$ is a topological $\mathbb{Z}$-extension of $G$, with kernel generated by the image $p(z) \in E$. The proof will be finished if we prove that the kernel $(p(z)) \cong \mathbb{Z}$ is distorted in $E$. As $\tilde{T}$ is the universal cover of the closure $T$ of $\phi$, and as $\tilde{\phi}$ is distorted in $\tilde{G}$, we deduce from Lemma 2.17 that the embedding of $p(\tilde{\phi})$ in $E$ is distorted in $E$, and from Lemma 2.25 (applied with $(p(z_0)) = \mathbb{Z}$) that $(p(z)) = \mathbb{Z}$ itself is distorted in $E$.

**Remark 2.27.** (1) The above proof shows, under the hypothesis of Proposition 2.26, that the distortion function of the fundamental group in the universal cover of $G$ grows at least linearly. Similarly, the kernel of the distorted topological $\mathbb{Z}$-extension $E$ of $G$ is at least linearly distorted.

(2) In the example of Remark 2.12, the fundamental group $\pi_1(G)$ is distorted in the universal cover $\tilde{G}$ of $G$, but each infinite cyclic subgroup of $\pi_1(G)$ is undistorted in $\tilde{G}$. This last condition implies that the image of the one-parameter subgroup $\phi$ (in the proof of Proposition 2.26) is not closed.

### 3. On Borel cohomology

#### 3.1. The relationship between the various cohomology groups.

In this subsection we recall the relationship we need between various cohomology groups. Recall that a map $f : X \to Y$ of topological spaces, is Borel if it is measurable with respect to the $\sigma$-algebras generated by the open subsets of $X$ resp. $Y$. For $G$ any topological group, we write
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For its cohomology based on Borel cocycles (resp. cocycles which are Borel bounded, continuous, continuous bounded, or just bounded). We refer the reader to [40–42], [26], and [39] for the definition and functorial properties of these cohomology theories. For the coefficient group A we take a metric abelian group; if A is finitely generated abelian, we always assume that the metric corresponds to the word metric coming from a finite symmetric generating set. All groups will usually be supposed to be separable and locally compact, with topology given by a complete metric (occasionally we will also consider non-separable groups like \( \mathbb{R}^\delta \)).

Our main object of study is the forgetful map \( H^*_B(G, \mathbb{Z}) \to H^*_B(G, \mathbb{Z}) \) for the case of a virtually connected Lie group. Note that the target group \( H^*_B(G, \mathbb{Z}) \) is naturally isomorphic to the singular cohomology \( H^*(BG, \mathbb{Z}) \) of the classifying space \( BG \) of \( G \) (cf. [47] and [1, Section 7]). There are also canonical isomorphisms \( H^*_c(G, \mathbb{R}) \cong H^*_B(G, \mathbb{R}) \) (see [13, Section 2.3, (2i), p. 15] which refers to [3, Section 4]). We write \( G^\delta \) for \( G \) considered as a discrete group. We have a canonical isomorphism \( H^*(G^\delta, \mathbb{R}) \cong H^*(BG^\delta, \mathbb{R}) \). The relationship between the various cohomology groups can be expressed by the following natural commutative diagram:

\[
\begin{array}{ccc}
H^*_B(G, \mathbb{Z}) & \to & H^*_B(G, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^*_c(G, \mathbb{R}) & \to & H^*_c(G, \mathbb{R}) \\
\downarrow & & \downarrow \\
H^*_b(G^\delta, \mathbb{Z}) & \to & H^*_b(G^\delta, \mathbb{R}) & \to & H^*(G^\delta, \mathbb{R}).
\end{array}
\]

The map \( \phi \) from the upper right corner is defined as the composition:

\[
(3) \quad \phi : H^*(BG, \mathbb{R}) \cong H^*(BG, \mathbb{Z}) \otimes \mathbb{R} \cong H^*_B(G, \mathbb{Z}) \otimes \mathbb{R} \to H^*_B(G, \mathbb{R}).
\]

3.2. **On d-dimensional Borel cohomology.** In this subsection, we recall and prove several general properties in bounded cohomology (Lemmata 3.1, 3.2, 3.5, 3.9, Corollaries 3.4, 3.6, Proposition 3.7). In particular, we prove that if \( \Gamma \) is a cocompact lattice in a virtually connected Lie group \( G \), then a class in the continuous cohomology of \( G \), whose restriction to \( \Gamma \) is bounded, admits a continuous bounded representative cocycle (Proposition 3.10). We then use this fact to deduce from Gromov’s theorem [23, Section 1.3, p. 23], that the forgetful map \( H^*_B(G, \mathbb{Z}) \to H^*_B(G, \mathbb{Z}) \), is surjective for \( G \) a semi-simple linear Lie
group (Proposition 3.11). We then recall and prove the geometric interpretation of the vanishing of real characteristic classes in positive degrees: if \( G \) is a virtually connected Lie group then the natural map \( H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}) \) is zero in positive degree, if and only if all integral primary characteristic classes of flat principal \( G \)-bundles over finite complexes are torsion classes (Proposition 3.14). Finally we give examples of unbounded characteristic classes of any (even) degree (Example 3.16).

**Lemma 3.1.** Let \( G \) be a topological group and \( x \in H^d_B(G, \mathbb{Z}) \) be such that for some \( n > 0 \), \( nx \) is bounded. Then \( x \) is bounded as well.

**Proof.** The short exact coefficients sequence, \( 0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0 \), yields long exact cohomology sequences, for Borel bounded cohomology as well as for Borel cohomology:

\[
H^{d-1}_B(G, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{n} H^d_B(G, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{n} H^d_B(G, \mathbb{Z}) \xrightarrow{n} H^d_B(G, \mathbb{Z}/n\mathbb{Z}).
\]

Using that the vertical maps are isomorphisms for \( \mathbb{Z}/n\mathbb{Z} \)-coefficients, the result follows from a simple diagram chase. \( \square \)

**Lemma 3.2.** Let \( G \) be a topological group. Let \( x^\delta \) be a characteristic class of \( G \) in the image of the composition \( H^d(BG, \mathbb{Z}) \to H^d(BG, \mathbb{R}) \to H^d(BG^\delta, \mathbb{R}) \). If \( x^\delta \) is bounded then it admits a representative cocycle whose set of values on the singular simplices of \( BG^\delta \) is a finite subset of \( \mathbb{Z} \).

**Proof.** The exact sequence of coefficients \( 0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0 \) admits a bounded section. Let us focus on the following two horizontal exact sequences with commutative squares, it induces.

\[
H^d(BG^\delta, \mathbb{Z}) \to H^d(BG^\delta, \mathbb{R}) \to H^d(BG^\delta, S^1).
\]

By hypothesis, \( x^\delta \) has a representative cocycle with integer values, hence it goes to zero in \( H^d(BG^\delta, S^1) \). By hypothesis, it is the image of an element \( w \in H^d_b(BG^\delta, \mathbb{R}) \). A simple diagram chase shows that \( w \) is in the image of \( H^d_b(BG^\delta, \mathbb{Z}) \to H^d_b(BG^\delta, \mathbb{R}) \). \( \square \)

**Lemma 3.3.** Let \( p : H \to G \) be a covering map of connected Lie groups. Then the induced map \( H^*(BG, \mathbb{R}) \to H^*(BH, \mathbb{R}) \) is surjective.
Proof. As \( p \) is a \( \pi_1(G) \)-Galois covering and as \( \pi_1(G) \) is finitely generated abelian, we can factor \( p \) in a sequence of connected covering spaces \( H = X_0 \to X_1 \to \cdots X_n = G \) with \( X_i \to X_{i+1} \) a cyclic covering. If \( X_i \to X_{i+1} \) is a finite covering, the induced map \( H^*(BX_{i+1}, \mathbb{R}) \to H^*(BX_i, \mathbb{R}) \) is an isomorphism, because the fiber of \( BX_i \to BX_{i+1} \) is \( \mathbb{R} \)-acyclic. In case \( X_i \to X_{i+1} \) is an infinite cyclic covering, there is a circle fibration \( S^1 \to BX_i \to BX_{i+1} \) with associated Gysin sequence with \( \mathbb{R} \)-coefficients

\[
H^d(BX_{i+1}) \to H^{d+2}(BX_{i+1}) \to H^{d+2}(BX_i) \xrightarrow{\theta(d)} H^{d+1}(BX_{i+1})
\]

in which \( \theta(d) \) is the zero map, because the real cohomology of a connected Lie group is concentrated in even dimensions. It follows that \( H^*(BX_{i+1}, \mathbb{R}) \to H^*(BX_i, \mathbb{R}) \) is surjective and therefore the composite map \( H^*(BG, \mathbb{R}) \to H^*(BH, \mathbb{R}) \) is surjective too. \( \square \)

This now yields the following useful corollary.

**Corollary 3.4.** Let \( p : H \to G \) be a covering map of connected Lie groups. If all Borel cohomology classes in degree \( d \) with \( \mathbb{Z} \) coefficients for \( G \) are bounded, then they are all bounded for \( H \) as well.

**Proof.** Using the natural isomorphism

\[
H^d_B(L, \mathbb{Z}) \cong H^*(BL, \mathbb{Z})
\]

for \( L \) a connected Lie group, we have a natural commutative diagram

\[
\begin{array}{ccc}
H^d_B(G, \mathbb{Z}) & \xrightarrow{p^*_B} & H^d_B(H, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^d_B(G, \mathbb{Z}) = H^d(BG, \mathbb{Z}) & \xrightarrow{p^*_B} & H^d_B(H, \mathbb{Z}) = H^d(BH, \mathbb{Z}).
\end{array}
\]

In view of Lemma 3.3 we know that \( p^*_B \) maps onto a subgroup of finite index. Therefore, if all Borel cohomology classes in degree \( d \) for \( G \) are bounded, then for every \( x \in H^d_B(H, \mathbb{Z}) \) there is an \( m > 0 \) so that \( mx \) is bounded. But this implies by Lemma 3.1 that already \( x \) is bounded. \( \square \)

**Lemma 3.5.** Let \( G \) be an arbitrary topological group. Then \( x \in H^*_B(G, \mathbb{Z}) \) is bounded, if its image \( y \in H^*_B(G, \mathbb{R}) \) in real Borel cohomology is bounded (i.e., lies in the image of \( H^*_B(G, \mathbb{R}) \to H^*_B(G, \mathbb{R}) \)).

**Proof.** The short exact sequence \( 0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0 \) of topological groups admits a Borel bounded section \( S^1 \to \mathbb{R} \) and gives therefore rise
to long exact cohomology sequences and a commutative diagram

\[ \cdots H^{d-1}_{BB}(G, S^1) \longrightarrow H^d_{BB}(G, \mathbb{Z}) \longrightarrow H^d_{BB}(G, \mathbb{R}) \longrightarrow H^d_{BB}(G, S^1) \cdots \]

A simple diagram chase completes the proof.

**Corollary 3.6.** If \( G \) is a compact Lie group, then all elements of \( H^*_B(G, \mathbb{Z}) \) are bounded.

**Proof.** Let \( x \in H^*_B(G, \mathbb{Z}) \). Then its image \( y \in H^*_B(G, \mathbb{R}) \) is obviously bounded, because by van Est’s theorem, the continuous cohomology of a compact Lie group vanishes in positive dimensions, and \( H^*_B(G, \mathbb{R}) = H^*_c(G, \mathbb{R}) \) by Wigner [47, Theorem 3, p. 91].

**Proposition 3.7.** Let \( G \) be a virtually connected Lie group and let \( G^0 \) be the connected component of the identity. If \( x \in H^*_B(G, \mathbb{Z}) \) restricts to a bounded class in \( H^*_B(G^0, \mathbb{Z}) \) then \( x \) itself is bounded. If \( y \in H^*(BG^\delta, \mathbb{R}) \) restricts to a class in \( H^*(B(G^0)^\delta, \mathbb{R}) \) which has a representing cocycle which takes only finitely many values, then \( y \) has such a representative too.

**Proof.** Assume \( x \in H^*_B(G, \mathbb{Z}) \) restricts to a bounded class in \( H^*_B(G^0, \mathbb{Z}) \). It suffices by Lemma 3.5 to show that the image \( u = \gamma(x) \in H^*_B(G, \mathbb{R}) \) of \( x \) is bounded. We have a natural commutative diagram with horizontal arrows \( \beta \) and \( \delta \) induced by restriction:

\[
\begin{array}{ccc}
H^d_{BB}(G, \mathbb{Z}) & \overset{\alpha}{\longrightarrow} & H^d_{BB}(G, \mathbb{R}) \\
\downarrow & & \Downarrow \phi \\
x \in H^d_B(G, \mathbb{Z}) & \overset{\gamma}{\longrightarrow} & H^d_B(G, \mathbb{R}) \\
\end{array}
\]

That the horizontal arrows \( \beta \) and \( \delta \) are isomorphisms follows from the Lyndon-Hochschild-Serre spectral sequences for the short exact sequences \( G^0 \to G \to G/G^0 \), using that the Borel bounded cohomology and the Borel cohomology with \( \mathbb{R} \) coefficients vanishes for the finite group \( G/G^0 \) in positive dimensions (for the Lyndon-Hochschild Serre spectral sequence in continuous, resp. continuous bounded, cohomology see [3] resp. [39, Chapter IV, Section 12]; as we have mentioned at the beginning of Section 3, there are natural isomorphisms \( H^*_B(G, \mathbb{R}) \cong H^*_c(G, \mathbb{R}) \), resp. \( H^*_B(G, \mathbb{R}) \cong H^*_c(G, \mathbb{R}) \)). From our assumption it follows that \( \delta(u) = v \) is bounded, say \( v = \psi(w) \) for some \( w \in H^d_{BB}(G^0, \mathbb{R}) \). By averaging with respect to the \( G/G^0 \)-action we...
can form \( \overline{w} = \frac{1}{|G_0^0|} \sum g w \) where the sum is taken over a set of coset representatives of \( G^0 \) in \( G \). Then \( \overline{w} \in H^d_{B\partial}(G^0, \mathbb{R})^{G/G^0} \) and \( \psi(\overline{w}) = v \). Thus \( \beta^{-1}(\overline{w}) \) is a bounded representative for \( u \) and we are done with the first case.

The case of \( H^*(BG^\delta, \mathbb{R}) \) can be dealt with in a similar way, using the fact that the restriction map induces an isomorphism \( H^*(BG^\delta, \mathbb{R}) \to H^*(B(G^\delta)^0, \mathbb{R})^{G/G^0} \). \( \square \)

**Example 3.8.** The following is an example of a virtually connected Lie group \( G \) with radical \( \sqrt{G} \) non-linear (actually \( \sqrt{G} = G^0 \)), but all cohomology classes in \( H^2_B(G, \mathbb{Z}) \) bounded and with an unbounded class in \( H^2_B(G^0, \mathbb{Z}) \); thus the converse of Proposition 3.7 does not hold, and in our Theorem 1.2 the implication (2) \( \Rightarrow \) (1) would not hold if we assumed \( G \) only to be virtually connected. Take \( H \) to be the three dimensional real Heisenberg group. It admits an involution \( T : H \to H \) which induces multiplication by -1 on the center of \( H \). In matrix notation,

\[
T : \begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1 \\
\end{pmatrix} \to \begin{pmatrix}
1 & -a & -c \\
0 & 1 & b \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

Then \( T \) preserves the infinite cyclic central subgroup \( \mathbb{Z} \) generated by any chosen non-zero central element. Thus \( T \) induces an involution on \( H/\mathbb{Z} \), which induces multiplication by -1 on \( H^2(B(H/\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z} \). It follows that the semi-direct product \( G := (H/\mathbb{Z}) \rtimes_T \mathbb{Z}/2\mathbb{Z} \) has \( H^2_B(G, \mathbb{Z}) \) finite, hence bounded according to Lemma 3.1 (it is a finite group isomorphic to \( H^2(BO(2), \mathbb{Z}) \cong H^2(P^2(\mathbb{R}), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \), because \( H^2_B(G, \mathbb{Z}) \cong H^2(BG, \mathbb{Z}) \) and \( G \) has \( O(2) \) as a maximal compact subgroup). But it follows from Theorem 1.2 that 0 \( \neq x \in H^2_B(G^0, \mathbb{Z}) \cong H^2(BG^0, \mathbb{Z}) \cong \mathbb{Z} \) is not bounded, because \( \sqrt{G^0} = G^0 \) and the closed subgroup \( [\sqrt{G^0}, \sqrt{G^0}] = \mathbb{R}/\mathbb{Z} \) is not simply-connected.

**Lemma 3.9.** Let \( A \) be a finitely generated abelian group. If \( f : G \to Q \) is a homomorphism of Lie groups and a homotopy equivalence of the underlying topological spaces, then the induced map \( f^* : H^*_B(Q, A) \to H^*_B(G, A) \) is an isomorphism. In particular, if all classes of \( H^*_B(Q, A) \) are bounded, then the same is true for \( H^*_B(G, A) \).

**Proof.** The map \( f \) induces a homotopy equivalence of classifying spaces \( Bf : BG \to BQ \). Indeed, it induces an isomorphism at the level of homotopy groups and the spaces in question have the homotopy type of CW-complexes. On the other hand, \( H^*_B(G, A) \) is naturally isomorphic to \( H^*(BG, A) \) (cf. [47]). This proves that \( f^* \) is an isomorphism. If \( c \)
is a bounded representative of $[c] \in H^*_B(Q, A)$, then $f^*c$ is a bounded representative of $f^*[c]$. This concludes the proof. \qed

For functorial properties of (bounded) continuous cohomology, we refer the reader to [26, Chapter III] and [39]. The following result also appears in Bucher-Karlsson [10, p. 60].

**Proposition 3.10.** Let $G$ be a locally compact group with a cocompact lattice $\Gamma$. Let $\text{res} : H^*_c(G, \mathbb{R}) \to H^*(\Gamma, \mathbb{R})$ be the restriction map and let $x \in H^*_c(G, \mathbb{R})$. If $\text{res}(x)$ is bounded then $x$ admits a continuous, bounded cocycle representative.

**Proof.** There is a commutative diagram:

$$
\begin{array}{ccc}
H^*_c(G, \mathbb{R}) & \xrightarrow{F} & H^*_c(G, \mathbb{R}) \\
\downarrow \text{tr} & & \downarrow \text{tr} \\
H^*_b(\Gamma, \mathbb{R}) & \xrightarrow{F} & H^*(\Gamma, \mathbb{R}),
\end{array}
$$

where $F$ denotes forgetful maps from bounded cohomologies and $\text{tr}$ denotes the transfer maps (cf. [39, p. 107]). The equality

$$
\text{tr} \circ \text{res} = \text{id},
$$

holds on $H^*_c(G, \mathbb{R})$ (as well as on $H^*_b(G, \mathbb{R})$). By hypothesis, there exists $y \in H^*_b(\Gamma, \mathbb{R})$ such that $F(y) = \text{res}(x)$. A continuous, bounded cocycle representative of the transfer of $y$ exists by definition and

$$
F(\text{tr}(y)) = \text{tr}(F(y)) = \text{tr}(\text{res}(x)) = x.
$$

Recall the map $\phi : H^*(BG, \mathbb{R}) \to H^*_c(G, \mathbb{R})$ from Subsection 3.1, Equation 3.

**Proposition 3.11.** (Compare with Bucher-Karlsson [10, p. 60].) Let $G$ be a Lie group of the form $G_a(\mathbb{R})^0$ for some semi-simple linear algebraic $\mathbb{R}$-group $G_a$.

1. Every element in the image of $\phi : H^*(BG, \mathbb{R}) \to H^*_c(G, \mathbb{R})$ admits a continuous, bounded cocycle representative.

2. The forgetful map $H^*_B(G, \mathbb{Z}) \to H^*_B(G, \mathbb{Z})$ is surjective.

**Proof.** Let $x \in H^*(BG, \mathbb{R})$. Its image $\phi(x)$ maps to the element $x^\delta \in H^*(BG^\delta, \mathbb{R}) \cong H^*(G^\delta, \mathbb{R})$. The proof of Gromov’s result [23, Section 1.3, p. 23] by Bucher-Karlsson [11] works in a semi-algebraic setting, hence applies to $G = G_a(\mathbb{R})^0$. Therefore, the class $x^\delta$ is bounded.
According to [4], there exists a cocompact lattice \( \Gamma \) in \( G \). As the arrows of the following commutative diagram,

\[
\begin{array}{ccc}
H_c^*(G, \mathbb{R}) & \xrightarrow{\text{res}} & H_b^*(G^\delta, \mathbb{R}) \\
\downarrow & & \downarrow \\
H_b^*(G^\delta, \mathbb{R}) & \xrightarrow{\text{res}} & H^*(\Gamma, \mathbb{R}),
\end{array}
\]

preserve boundedness, the class \( \text{res}(\phi(x)) \in H^*(\Gamma, \mathbb{R}) \) is also bounded. Proposition 3.10 shows that the class \( \phi(x) \) has a continuous, bounded representative. This proves the first statement of the proposition.

To prove the second statement, let \( x \in H_b^*(G, \mathbb{Z}) \). According to Lemma 3.5, it is enough to show that the image \( y \in H_b^*(G, \mathbb{R}) \) of \( x \) is bounded. In view of the commutative diagram of Subsection 3.1, and the first statement of the proposition, the class \( y \) admits a continuous, bounded cocycle representative. □

**Lemma 3.12.** Let \( W \) be a CW-complex. If a class \( x \in H^d(W, \mathbb{R}) \) in the real singular cohomology of \( W \) is non-zero, then there exists a finite subcomplex \( F \subset W \) such that the restriction of \( x \) to \( F \) is a non-zero element of \( H^d(F, \mathbb{R}) \).

**Remark 3.13.** The corresponding statement with \( \mathbb{R} \) replaced by \( \mathbb{Z} \) is not true [35, p. 1212].

**Proof.** We prove the lemma. We have:

\[
H^d(W, \mathbb{R}) \cong \text{Hom}_\mathbb{R}(H_d(W, \mathbb{R}), \mathbb{R}) \cong \text{Hom}_\mathbb{R}(\lim \rightarrow H_d(F_\alpha, \mathbb{R}), \mathbb{R}) \\
\cong \lim \rightarrow \text{Hom}_\mathbb{R}(H_d(F_\alpha, \mathbb{R}), \mathbb{R}) \cong \lim \rightarrow H^d(F_\alpha, \mathbb{R}),
\]

where the inductive, resp. projective, limit is taken over the inductive system of finite subcomplexes \( F_\alpha \) of \( W \), resp. over the projective system \( \text{Hom}_\mathbb{R}(H_d(F_\alpha, \mathbb{R})) \).

□

**Proposition 3.14.** Let \( G \) be a virtually connected Lie group.

1. If the natural map \( H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}) \) is zero in positive degree, then all integral characteristic classes of flat principal \( G \)-bundles over finite complexes are torsion classes.

2. If \( c(EG) \in H^*(BG, \mathbb{Z}) \) is a characteristic class with the property that for any flat principal \( G \)-bundle \( P \) over any finite complex \( X \), the class \( c(P) \in H^*(X, \mathbb{Z}) \) is a torsion class, then \( c(EG) \) is in the kernel of the composition

\[
H^*(BG, \mathbb{Z}) \to H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}).
\]
Proof. Let $P$ be a flat principal $G$-bundle over a finite complex $X$. We have a commutative diagram:

\[
P \longrightarrow EG^\delta \longrightarrow EG \\
\downarrow \downarrow \downarrow \downarrow \\
X \longrightarrow BG^\delta \longrightarrow BG,
\]

where vertical arrows are projections of principal $G$-bundles. It induces the commutative diagram:

\[
H^*(BG, \mathbb{Z}) \longrightarrow H^*(BG^\delta, \mathbb{Z}) \longrightarrow H^*(X, \mathbb{Z}) \\
\downarrow \downarrow \downarrow \\
H^*(BG, \mathbb{R}) \longrightarrow H^*(BG^\delta, \mathbb{R}) \longrightarrow H^*(X, \mathbb{R}),
\]
in which characteristic classes are preserved. As $X$ is a finite complex, the kernel of the vertical arrow on the right is the torsion subgroup.

To prove the first claim of the proposition, let $c(P) \in H^*(X, \mathbb{Z})$ be a characteristic class. By definition it is the image of $c(EG) \in H^*(BG, \mathbb{Z})$. The commutativity of the diagram and the hypothesis that the map $H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R})$ is zero imply together that $c(P)$ is a torsion class. This proves the first part of the proposition.

To prove the second claim of the proposition, we consider $c(EG)^\delta \in H^*(BG^\delta, \mathbb{Z})$ and assume that its image in $H^*(BG^\delta, \mathbb{R})$ is not zero. According to Lemma 3.12, there exists a finite sub-complex $X \subset BG^\delta$, such that our characteristic class $c(P) \in H^*(X, \mathbb{Z})$ defined by the flat principal $G$-bundle induced by the composition $X \subset BG^\delta \to BG$ is not a torsion class. This concludes the proof of the proposition. \)

We recall the following fact about transfer in cohomology between covering spaces.

Lemma 3.15. Let $Y \to X$ be a $d$-fold covering space of CW-complexes. There is a transfer map:

\[tr : H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z}),\]

such that the composition

\[H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z}),\]

is multiplication by $d$. Thus, if $x \in H^*(X, \mathbb{Z})$ maps to $0$ in $H^*(Y, \mathbb{Z})$, then $d \cdot x = 0$.

Proof. Choose a CW-structure on $X$ and lift it to $Y$ so that each $n$-cell in $X$ has precisely $d$ counter images in $Y$. Define $tr$ on the cellular
cochain level by mapping a cellular $n$-cochain $f$ on $Y$ to the cellular $n$-cochain $g$ on $X$ defined by

$$g(\sigma) = \sum f(\sigma_i),$$

where $\sigma_i$ runs over the $d$ counter images of $\sigma$. The rest is obvious. □

The following family of examples of Lie groups with non-linear radical shows that one can have unbounded characteristic classes in any positive (even) degree. (The basic idea in the construction appears in [21] and also in [25, Remarques 7.5, b].)

Example 3.16. Let $G_i$ be the $i$-fold cartesian product of Heisenberg quotients $H/\mathbb{Z}$, where $H$ is the 3-dimensional Heisenberg group and $\mathbb{Z} < H$ a central subgroup. Note that the maximal compact subgroup of $H/\mathbb{Z}$ is $S^1$ so that $H^*(BG_i, \mathbb{R})$ is a polynomial algebra generated by 2-dimensional classes $x_j$, $1 \leq j \leq i$, with the property that the map induced by the $j$-th injection $H/\mathbb{Z} \to G_i$ maps $x_j$ to a non-zero element in $H^2(B(H/\mathbb{Z}), \mathbb{R})$. According to Goldman [21] there is a flat $H/\mathbb{Z}$-bundle over the 2-torus $T^2$ given by a homomorphism $\phi: \mathbb{Z} \times \mathbb{Z} \to H/\mathbb{Z}$ such that $\phi^*: H^2(B(H/\mathbb{Z}), \mathbb{R}) \to H^2(T^2, \mathbb{R})$ is non-zero. Taking $i$-fold products, we get a flat $G_i$-bundle of the $2i$-torus $T^{2i}$, with associated map $H^*(BG_i, \mathbb{R}) \to H^*(T^{2i}, \mathbb{R})$ mapping the product $y_d := x_1 \cdots x_d$ non-trivially into $H^{2d}(T^{2i}, \mathbb{R})$, where $1 \leq d \leq i$. It follows that the images $y_d \in H^{2d}(BG_i, \mathbb{R})$ of $y_d$ are all non-trivial. But they cannot have bounded cocycle representatives, because the bounded cohomology of the nilpotent group $G_i$ vanishes in positive dimensions.

3.3. On 2-dimensional Borel cohomology. In this subsection we first recall Moore’s theorem (Theorem 3.17) which generalizes the Eckmann-Eilenberg-Mac Lane theorem to locally compact groups. In Lemmata 3.18, 3.20, 3.21, we establish some properties of 2-cocycles associated to central extensions. In Proposition 3.23 we show that the 2-dimensional Borel cohomology with $\mathbb{Z}$-coefficients of a Lie group $G$ is bounded if and only if the class defined by the universal cover of $G$ is bounded. Lemma 3.24 is a 2-dimensional Künneth formula in Borel bounded cohomology with $\mathbb{Z}$-coefficients. Proposition 3.25 shows that for a connected normal closed subgroup $N$ of a connected Lie group $G$, each real class of degree 2 on the classifying space $BN$ is the restriction of a class on $BG$. The subsection ends with a proof based on Moore’s theorem, of the surjectivity of the forgetful map $H^2_B(G, \mathbb{Z}) \to H^2_B(G, \mathbb{Z})$, for $G$ a real linear algebraic group.

A topological $A$-extension $0 \to A \to E \to G \to 1$ always admits a Borel section $\sigma$ (cf. [32]). We denote $c_\sigma: G \times G \to A$ the Borel
2-cocycle

\[ c_\sigma(g, g') = \sigma(g)\sigma(g')\sigma(gg')^{-1}. \]

The following statement is a special case of [41, Theorem 10]:

**Theorem 3.17.** (Moore [41, Theorem 10].) Let \( G \) and \( A \) be locally compact separable groups with \( A \) abelian. The map

\[
\text{Ext}_{\text{top}}(G, A) \to H^2_B(G, A)
\]

\[
\{0 \to A \to E \to G \to 1\} \mapsto [c_\sigma],
\]

is an isomorphism.

**Lemma 3.18.** Let \( G \) be a topological group and let \( A \) be a topological abelian group. Let \( c : G \times G \to A \) be an inhomogeneous Borel 2-cocycle (that is a Borel map such that, for all \( x, y, z \in G \),

\[
dc(x, y, z) = c(y, z) - c(xy, z) + c(x, yz) - c(x, y) = 0).\]

Assume that \( c \) is cohomologous to \( c_\tau \), where \( \tau \) is a Borel section of a topological \( A \)-extension \( E \) of \( G \) (if \( G \) and \( A \) are locally compact and separable, such an extension with such a section always exist according to Theorem 3.17). Then there is a Borel section \( \sigma \) of \( E \), such that \( c = c_\sigma \).

**Proof.** By hypothesis, there exists a Borel map \( b : G \to A \), such that \( c = c_\tau + db \). Recall that the coboundary operator on a (inhomogeneous) 1-cochain \( b \) (and with \( A \) a the trivial \( G \)-module) is \( db(g, g') = b(g') - b(gg') + b(g) \). The Borel cocycle \( c_\sigma \), associated to the Borel section \( \sigma : G \to E \), defined as \( \sigma(g) = \tau(g)b(g) \), is equal to \( c \).

**Remark 3.19.** Topological \( \mathbb{R} \)-extensions are topologically split and can therefore be described by a continuous cocycle \( G \times G \to \mathbb{R} \).

**Lemma 3.20.** Let \( p : G \to Q \) and \( \pi : \tilde{Q} \to Q \) be surjective continuous homomorphisms between Hausdorff topological groups. (We view \( \pi : \tilde{Q} \to Q \) as a principal ker \( \pi \)-bundle.)

1. The total space of the pull-back \( p^*(\tilde{Q}) \to G \) of \( \tilde{Q} \to Q \) is a closed subgroup of \( G \times \tilde{Q} \) and there is a short exact sequence of topological groups

\[
1 \to \ker(p) \to p^*(\tilde{Q}) \to \tilde{Q} \to 1.
\]

2. Assume that both \( \ker \pi \) in \( \tilde{Q} \) and \( \ker(p) \) in \( G \) are central. If the extension

\[
0 \to \ker(\pi) \to \tilde{Q} \to Q \to 1
\]
can be defined by a Borel bounded cocycle, then the same is true for the extension
\[ 0 \to \ker(\pi) \to p^*(\tilde{Q}) \to G \to 1. \]

Proof. To prove (1), recall that by definition,
\[ p^*(\tilde{Q}) = \{(g, \tilde{q}) \in G \times \tilde{Q} \text{ such that } p(g) = \pi(\tilde{q})\}. \]
We embed \( \ker(p) \) in \( p^*(\tilde{Q}) \) by composing the inclusions \( \ker(p) \subset G \subset G \times \{e\} \). The map \( \tilde{p} : p^*(\tilde{Q}) \to \tilde{Q} \), defined as \( (g, \tilde{q}) \mapsto \tilde{q} \), gives the wanted exact sequence. To prove (2), we apply Lemma 3.18 in order to obtain a Borel section \( \sigma : Q \to \tilde{Q} \) of \( \pi \) such that the associated cocycle \( c_\sigma \) is bounded. Then \( \sigma^* : G \to p^*(\tilde{Q}) \) defined as \( g \mapsto (g, \sigma p(g)) \), is a Borel section of the projection \( p^*(\tilde{Q}) \to G \) and its associated cocycle satisfies
\[ c_{\sigma^*}(g, g') = (e, c_\sigma(p(g), p(g'))). \]

\[ \square \]

Lemma 3.21. Let \( p : X \to Y \) and \( q : Y \to Z \) be two surjective homomorphisms of groups with central kernels. Let \( \sigma \) be a section of \( p \), and \( \tau \) one for \( q \). Assume that the associated cocycles \( c_\sigma \) and \( c_\tau \) have finite range. If the image under \( \sigma \) of the kernel of \( q \) is central in \( X \), then the cocycle \( c_{\sigma \tau} \) associated to the section \( \sigma \tau \) of \( qp \) has finite range.

Proof. Since \( \sigma (\ker(q)) \) is central, the kernel of \( qp \) is central so that \( qp : X \to Z \) is a central \( \ker(qp) \) extension. One verifies that
\[ c_{\sigma \tau}(x, y) = c_\sigma(\tau(x), \tau(y)) \cdot \sigma(c_\tau(x, y)) \cdot [c_\tau(c_\sigma(x, y), \tau(xy))]^{-1}. \]
Thus \( c_{\sigma \tau} \) is a product of three functions each of which takes only finitely many values. It follows that \( c_{\sigma \tau} \) has finite range. \( \square \)

Remark 3.22. Consider the following commutative diagram of groups
\[ G \xrightarrow{p} E \xleftarrow{r} Q. \]
with \( q \) and \( r \) surjective. If \( \sigma \) is a section of \( q \) then \( p\sigma \) is a section of \( r \) and
\[ c_{p\sigma}(x, y) = p\sigma(x)p\sigma(y)(p\sigma(xy))^{-1} = p(\sigma(x)\sigma(y)(\sigma(xy))^{-1}) = (p \circ c_\sigma)(x, y). \]

Proposition 3.23. Let \( G \) be connected Lie group. The following conditions are equivalent.
(1) The class in $H^2_B(G, \pi_1(G))$, corresponding (in the sense of Theorem 3.17) to the universal cover of $G$, is bounded.

(2) All classes in $H^2_B(G, \mathbb{Z})$ are bounded.

(3) For every finitely generated abelian group $A$, all classes in $H^2_B(G, A)$ are bounded.

Proof. We first prove that (1) implies (3). Let $x \in H^2_B(G, A)$ and let $E$ be a central extension of $G$ with central discrete kernel $A$ corresponding to $x$. Let $E^0$ be the connected component of $E$. The restriction of the projection $E \to G$ to $E^0$ is a connected cover of $G$. Hence, we obtain a commutative diagram of covering groups,

\[
\begin{array}{ccc}
\tilde{G} & \longrightarrow & E \\
\downarrow & & \downarrow \\
G, & & \\
\end{array}
\]

where $\tilde{G}$ is the universal cover of $G$. Applying the hypothesis together with Remark 3.22, we deduce the existence of a Borel section $\tau : G \to E^0 \subseteq E$ of the projection $E \to G$ such that the corresponding cocycle $c_\tau : G \times G \to A$ is bounded. As $[c_\tau] = x$, we conclude that $x$ is bounded.

The fact that (3) implies (2) is trivial and obviously (3) implies (1), since $\pi_1(G)$ is a finitely generated abelian group. It remains to see that (2) implies (3). Let $A = \mathbb{Z}^n \times F$ be a finitely generated abelian group, $F$ a finite group, and let $c : G \times G \to A$ be a 2-cocycle; it has components $c_i : G \times G \to \mathbb{Z}$, for $1 \leq i \leq n$ and a component $c_F$, taking values in $F$. The latter is obviously bounded. By assumption, the cocycles $c_i$ are cohomologous to bounded cocycles $d_i$. Thus, the cocycle $G \times G \to A$ with components $d_i$ and $c_F$ is bounded and cohomologous to $c$, finishing the proof. 

Lemma 3.24. If $G$ and $H$ are two virtually connected Lie groups such that the maps $H^2_{Bb}(G, \mathbb{Z}) \to H^2_B(G, \mathbb{Z})$ and $H^2_{Bb}(H, \mathbb{Z}) \to H^2_B(H, \mathbb{Z})$ are onto, then so is the map $H^2_{Bb}(G \times H, \mathbb{Z}) \to H^2_B(G \times H, \mathbb{Z})$.

Proof. Because $\pi_1(BG) = \pi_0(G)$ is finite, $H^1(BG, \mathbb{Z}) = 0$, and similarly $H^1(BH, \mathbb{Z}) = 0$. By the Künneth formula in singular cohomology, we conclude that the natural map $BG \sqcup BH \to BG \times BH = B(G \times H)$ induces an isomorphism $\psi$ at the level of degree 2 cohomology with $\mathbb{Z}$ coefficients. On the other hand, the inclusion $i_G : G \to G \times H$ induces a surjection $i_G^* : H^2_B(G, \mathbb{Z}) \to H^2_B(G \times H, \mathbb{Z})$. The following diagram commutes because for each component of the vertical
maps we have a natural commutative diagram:

\[
\begin{array}{ccc}
H^2_{Bb}(G \times H, \mathbb{Z}) & \longrightarrow & H^2_{Bb}(G \times H, \mathbb{Z}) = H^2(B(G \times H), \mathbb{Z}) \\
epi & & \text{iso} \\
\downarrow & & \psi \\
H^2_{Bb}(G, \mathbb{Z}) \oplus H^2_{Bb}(H, \mathbb{Z}) & \longrightarrow & H^2(BG, \mathbb{Z}) \oplus H^2(BH, \mathbb{Z}).
\end{array}
\]

This shows that the map \(H^2_{Bb}(G \times H, \mathbb{Z}) \rightarrow H^2_{Bb}(G, \mathbb{Z}) \oplus H^2_{Bb}(H, \mathbb{Z})\) is onto as well. \(\square\)

**Proposition 3.25.** Let \(G\) be a connected Lie group and let \(N\) be a closed connected normal subgroup of \(G\). The natural map

\[H^2(BG, \mathbb{R}) \rightarrow H^2(BN, \mathbb{R}),\]

induced by the inclusion \(N \subset G\), is onto.

**Proof.** There is a commutative diagram:

\[
\begin{array}{ccc}
H^2(BG, \mathbb{Z}) & \longrightarrow & H^2(BN, \mathbb{Z}) \\
\cong & & \cong \\
\downarrow & & \downarrow \\
\text{Hom}(\pi_1(G), \mathbb{Z}) & \longrightarrow & \text{Hom}(\pi_1(N), \mathbb{Z}).
\end{array}
\]

Tensoring with \(\mathbb{R}\), we see that the proof will be finished if we show that

\[\text{Hom}(\pi_1(G), \mathbb{R}) \rightarrow \text{Hom}(\pi_1(N), \mathbb{R})\]

is onto. But this is the case according to Remark 2.4. \(\square\)

**Theorem 3.26.** Let \(G_a\) be a linear algebraic group defined over \(\mathbb{R}\). Let \(K_a\) and \(S_a\) be algebraic subgroups of \(G_a\) defined over \(\mathbb{R}\) with the following properties.

1. The Lie group \(K_a(\mathbb{R})\) is compact.
2. The Lie group \(S_a(\mathbb{R})\) is simply-connected.
3. \(G_a(\mathbb{R}) = S_a(\mathbb{R})K_a(\mathbb{R})\).
4. \(S_a(\mathbb{R}) \cap K_a(\mathbb{R}) = \{e\}\).

Let \(A\) be a finitely generated abelian group. Let \(G\) be a virtually connected Lie group. Assume that the connected Lie groups \(G^0\) and \(G_a(\mathbb{R})^0\) are isomorphic. Then the natural map,

\[H^2_{Bb}(G, A) \rightarrow H^2_{Bb}(G, A),\]

is surjective.

**Proof.** Thanks to Lemma 3.7, we see that we can assume that \(G\) is connected. In this case \(G = G^0 = G_a(\mathbb{R})^0\). According to Proposition 3.23, it is enough to prove that the universal cover \(\tilde{G} \rightarrow G\) of \(G\) admits a Borel section whose associated cocycle is bounded. Let
Let \( K = K_a(\mathbb{R})^0 \) and \( S = S_a(\mathbb{R}) = S_a(\mathbb{R})^0 \). Let \( \tilde{K} = p^{-1}(K) \). Since \( A := \pi_1(G) \subseteq \tilde{K} \), the regular \( A \)-cover
\[
0 \to A \to \tilde{G} \to G \to 1,
\]
restricts to
\[
0 \to A \to \tilde{K} \to K \to 1.
\]
Let \( \tilde{S} \) be the connected component of the identity of \( p^{-1}(S) \). The subgroup \( \tilde{S} \subset \tilde{G} \) is closed. Hence it is a Lie subgroup. Since \( S \) is simply-connected, the restriction of the covering map \( p \) to \( \tilde{S} \) is an isomorphism between \( \tilde{S} \) and \( S \). In what follows, if \( s \in S \), we will denote by \( \tilde{s} \) the unique element of \( \tilde{S} \) such that \( p(\tilde{s}) = s \). From this isomorphism and from the hypothesis \( S \cap K = \{ e \} \), we deduce that \( \tilde{S} \cap \tilde{K} = \{ e \} \). Since \( G = SK \), we have \( \tilde{G} = \tilde{S}\tilde{K} \). There is a commutative diagram
\[
\begin{array}{ccc}
\tilde{S}\tilde{K} & \longrightarrow & SK \\
\downarrow & & \downarrow \\
\tilde{K} & \longrightarrow & K
\end{array}
\]
where the horizontal arrows are \( A \)-regular covers and the vertical ones are projections of trivial left \( S \)-principal bundles. In other terms, \( \tilde{G} \) and \( G \) are trivial left \( S \)-principal bundles over \( \tilde{K} \) and \( K \) resp., and \( \tilde{G} = p^*(G) \) is induced by \( G \), where \( p : \tilde{K} \to K \) denotes the restriction of \( p : \tilde{G} \to G \) to \( \tilde{K} \).

Since \( K \) is the connected component of the identity of the real points of an affine algebraic group, we can identify \( K \) with a connected component of an algebraic subvariety of a Euclidean space \( \mathbb{R}^N \). Since \( p : \tilde{K} \to K \) is a covering map, for each \( x \in K \subseteq \mathbb{R}^N \), there is a radius \( \epsilon_x > 0 \) such that the cover is trivial over \( B(x, \epsilon_x) \cap K \), and hence we can choose a continuous section \( \sigma_x : B(x, \epsilon_x) \cap K \to \tilde{K} \) of \( p \). Since \( K \) is compact, we can choose a finite trivializing cover
\[
K = \bigcup_{i=1}^{m} K_i,
\]
where \( K_i = B(x_i, \epsilon_i) \cap K \). Let
\[
A_i = K_i \setminus \bigcup_{j<i} K_j.
\]
Each set \( A_i \) is semi-algebraic, and the sets \( A_1, \ldots, A_m \) form a partition of \( K \). As each \( A_i \) is contained in \( K_i \), we obtain by restriction a continuous section \( \sigma_i : A_i \to \tilde{K} \) of \( p \). We extend each section \( \sigma_i \) by zero.
outside of $A_i$ and we define a section $\sigma : K \to \tilde{K}$ by $\sigma(x) = \sum_{i=1}^m \sigma_i(x)$. We define

$$\tau : G \to \tilde{G}, \ g = sk \mapsto \tilde{s}\sigma(k).$$

The map $\tau$ is a Borel section of $p$ and is $S$-equivariant: for $s \in S$ and $x \in G$, $\tau(sx) = \tilde{s}\tau(x)$. Let $c : G \times G \to A$ be the corresponding 2-cocycle. That is, $c$ is defined by the equation

$$\tau(g)\tau(h) = c(g,h)\tau(gh) \ \text{for all} \ g, h \in G.$$ 

Note that $c$ is a Borel map. It remains to show that $c$ takes only finitely many values in $A$.

As $G = SK$, taking inverses, we also have that $G = KS$. Hence each element $g$ of $G$ can uniquely be written as $g = ks$ with $k \in K$ and $s \in S$. Let

$$\mu : K \times S \to G, \ (k, s) \mapsto ks,$$

denotes the multiplication and let $\pi(\mathbb{R}) : G \to G/K$ be the canonical projection. The quotient space $G/K$ is naturally homeomorphic to $(G_a/K_a)(\mathbb{R})$, the connected topological space of $\mathbb{R}$-points of the quotient variety $G_a/K_a$, and $\pi : G_a \to G_a/K_a$ is a morphism of $\mathbb{R}$-varieties \cite[Ch. 2, Thm. 6.8]{6}. Consider the following morphisms

$$\theta : S_a \subseteq G_a \to G_a/K_a,$$

between $\mathbb{R}$-varieties \cite[Ch. 2, Prop. 6.12]{6}. Since $G = SK$ and $S \cap K = 1$ the above $\mathbb{R}$-morphism induces a homeomorphism $S \to G/K$. Hence the map

$$q_S : G \to S, \ sk \mapsto s,$$

(the composition of $G \to G/K$ with the inverse of $S \to G/K$) is induced by a morphism of $\mathbb{R}$-varieties. Let $s = q_S \circ \mu$. Similarly, the homeomorphism

$$K \to S \setminus G,$$

is obtained by restricting a morphism of $\mathbb{R}$-varieties to topological connected components. We define

$$q_K : G \to K, \ sk \mapsto k,$$

and $k = q_K \circ \mu$. Let $k \in K$ and $s \in S$, there is a unique $s' \in S$ and a unique $k' \in K$ such that $ks = s'k'$. By definition of $s$ and $k$, $s' = s(k, s)$ and $k' = k(k, s)$, hence

$$ks = s(k, s)k(k, s).$$

Let us define

$$f_1 : \tilde{K} \times \tilde{K} \times \tilde{K} \times \tilde{S} \times \tilde{S} \to \tilde{G}$$

$$(u, v, w, p, q) \mapsto upvw^{-1}q^{-1}.$$
This is a continuous function. Let us define
\[ f_2 : K \times K \times K \times S \times S \rightarrow \tilde{K} \times \tilde{K} \times \tilde{K} \times \tilde{S} \times \tilde{S} \]
\[ (x, y, z, s, t) \mapsto (\sigma(x), \sigma(y), \sigma(z), \tilde{s}, \tilde{t}) \].

This is not a continuous function because \( \sigma \) is not necessarily continuous, but the restriction of \( f_2 \) to each semi-algebraic set \( A_i \times A_j \times A_l \times S \times S, 1 \leq i, j, l \leq m \), is continuous. Let us define
\[ f_3 : K \times K \times S \rightarrow K \times K \times K \times S \times S \]
\[ (x, y, s) \mapsto (x, y, k(x, s)y, s, s(x, s)) \].

This is an \( \mathbb{R} \)-morphism, hence it is continuous and the inverse images
\[ f_3^{-1}(A_i \times A_j \times A_l \times S \times S), 1 \leq i, j, l \leq m, \]
are again semi-algebraic sets. Semi-algebraic sets have only a finite number of connected components [2], say \( C_1, \ldots, C_n \) in our case. The sets \( C_1, \ldots, C_n \) form a partition of \( K \times K \times S \) (since the \( f_3^{-1}(A_i \times A_j \times A_l \times S \times S) \) do, and the latter are partitioned by their connected components) and the restriction of \( f = f_1 \circ f_2 \circ f_3 \) to each \( C_i \) is continuous.

It is easy to check that \( f(x, y, s) = c(x, sy) \) for all \( (x, y, s) \in K \times K \times S \) (we defined the functions \( f_i \) in order to obtain this equality). Since \( \tau \) is \( S \)-equivariant and \( A \) is central, \( c(sx, ty) = c(x, ty) \) for all \( s, t \in S \) and \( x, y \in K \). We conclude that the image of \( c \) is equal to the image of \( f \). As \( A \) is discrete, it follows that the cardinality of the image of \( f \) is bounded by \( n \) because \( K \times K \times S \) is partitioned by the connected sets \( C_1, \ldots, C_n \). \( \square \)

**Corollary 3.27.** Let \( G \) be a virtually connected Lie group with \( G^0 \) semi-simple (not necessarily linear). Then \( G \) has all its Borel cohomology in degree 2 with \( \mathbb{Z} \)-coefficients bounded.

**Proof.** We first reduce to the case of the connected component \( G^0 \) by Proposition 3.7. Dividing by the center, we reduce then to the case we reduce to the case of \( G^0/Z(G^0) \) by using Corollary 3.4. Now \( G^0/Z(G^0) := H \) is a connected linear semi-simple Lie group. According to [33] or [48, Proposition 3.1.6], \( H \) is as a Lie group isomorphic to \( G_n(\mathbb{R})^0 \) for some connected real linear algebraic group \( G_n \). It is well known that such a group admits a decomposition satisfying the assumptions of Theorem 3.26 and we are done. \( \square \)

**Remark 3.28.** In case \( G \) is a Lie group of Hermitian type, it was proved in [13, Prop. 7.7] that the comparison map \( H^2_{Bb}(G, \mathbb{Z}) \rightarrow H^2_{B}(G, \mathbb{Z}) \) is actually an isomorphism.
4. Proofs of the main results

4.1. Proof of Theorem 1.4. We first need a lemma.

**Lemma 4.1.** Let $G$ be a connected Lie group and $\sqrt{G}$ its radical. Then the restriction map $H^*(BG, \mathbb{R}) \to H^*(B\sqrt{G}, \mathbb{R})$ is surjective.

**Proof.** According to Proposition 3.25, the map $H^2(BG, \mathbb{R}) \to H^2(B\sqrt{G}, \mathbb{R})$ is surjective. Because $\sqrt{G}$ is homotopy equivalent to a torus, the cohomology $H^*(B\sqrt{G}, \mathbb{R})$ is a polynomial algebra on 2-dimensional classes. Therefore, since the restriction map $H^*(BG, \mathbb{R}) \to H^*(B\sqrt{G}, \mathbb{R})$ is a map of algebras, it is surjective. □

**Proof.** We prove Theorem 1.4. Let $I^*$ denote the image of the map $H^*(B(G/\sqrt{G}), \mathbb{R}) \to H^*(BG, \mathbb{R})$. According to Lemma 4.1, the restriction map $H^*(BG, \mathbb{R}) \to H^*(B\sqrt{G}, \mathbb{R})$ is surjective and therefore the fiber in the fibration $B\sqrt{G} \to BG \to B(G/\sqrt{G})$ is totally nonhomologous to zero relative to $\mathbb{R}$. We conclude from Borel’s Theorem [5, Thm. 14.2]) that $H^*(BG, \mathbb{R})/\langle I^+ \rangle$ maps isomorphically onto $H^*(B\sqrt{G}, \mathbb{R})$, where $\langle I^+ \rangle$ stands for the ideal generated by the classes $I^+ < I^*$ of positive degree. Using the well-known fact that for a connected Lie group $L$, $H^*(BL, \mathbb{R})$ is a polynomial algebra on even dimensional generators (see [5, Thm. 18.1]) we now prove our claim on the generation of $H^*(BG, \mathbb{R})$ by checking it for generators in even degrees. Thus, assume that $x \in H^{2d}(BG, \mathbb{R})$ is a generator with $d > 1$. Then $x$ maps to an element $y \in H^{2d}(B\sqrt{G}, \mathbb{R})$, which is a product of $d$-dimensional classes $y_i$. By choosing counter images $x_i$ in $H^2(BG, \mathbb{R})$ for the $y_i$’s, their product will be an element $z \in H^{2d}(BG, \mathbb{R})$ with $x - z \in \langle I^+ \rangle$, say $x - z = \sum a_j b_j$ with $a_j \in I^+$ and $b_j \in H^{<2d}(BG, \mathbb{R})$. Thus $x = z + \sum a_j b_j$ lies in the subring generated by $H^2(BG, \mathbb{R})$ together with $\langle I^+ \rangle$, proving the theorem. □

4.2. Proof of Theorem 1.2.

4.2.1. If the class defined by the universal cover is bounded then the fundamental group is undistorted.

**Proof.** We prove that (5) implies (6). Let $A \cong \pi_1(G)$ denotes the fundamental group of $G$. We have a topological $A$-extension:

$$0 \to A \to \tilde{G} \to G \to 1,$$

where $\tilde{G}$ denotes the universal cover of $G$. Let $\tau$ be a Borel section of the projection $\tilde{G} \to G$. By hypothesis the class $[c_\tau] \in H^2_B(G, A)$ is
bounded. According to Lemma 3.18, there exists a Borel section $\sigma$ of $\tilde{G} \to G$ such that $c_\sigma$ is bounded. We may modify $\sigma$ at the identity so that $\sigma(e) = e$. It is easy to check that the new cocycle $c_\sigma$ associated to the modified section $\sigma$ is still bounded. Proposition 2.20 implies that $A$ is undistorted in $\tilde{G}$.

4.2.2. If the fundamental group is undistorted then the radical is linear.

**Proof.** We prove that (6) implies (1). Let $\sqrt{G}$ be the radical of $G$. According to Proposition 2.10, the non-linearity of $\sqrt{G}$ implies the existence of a maximal compact subgroup $K$ of $\sqrt{G}$ such that $[\sqrt{G}, \sqrt{G}] \cap K \neq \{e\}$. Hence, Proposition 2.26 applies, and we deduce that the fundamental group is distorted. \[\square\]

4.2.3. If the 2-dimensional cohomology is bounded then the radical is linear.

**Proof.** We show that (4) implies (1). Let us assume that $\sqrt{G}$ is not linear. We will show that $H^2_B(G, \mathbb{Z})$ contains a class with no Borel bounded representative. According to Proposition 2.10, the non-linearity of $\sqrt{G}$ implies the existence of a maximal compact subgroup $K$ of $\sqrt{G}$ such that $[\sqrt{G}, \sqrt{G}] \cap K \neq \{e\}$. Hence, Proposition 2.26 applies, and we deduce the existence of a distorted topological $\mathbb{Z}$-extension

$$0 \to \mathbb{Z} \to E \to G \to 1.$$ 

Consider a Borel cocycle $c$, such that $[c] \in H^2_B(G, \mathbb{Z})$ is defined by the extension $E$ of $G$. We claim that $c$ is not bounded. Assume this is not the case. Lemma 3.18 shows that $c = c_\sigma$, where $\sigma$ is some Borel section of $E \to G$ (and we may modify $\sigma$ at the identity so that $\sigma(e) = e$, keeping the modified cocycle $c_\sigma$ bounded). Proposition 2.20 implies that the extension $0 \to \mathbb{Z} \to E \to G \to 1$ is undistorted, contradicting the assumption. This proves that $c$ is unbounded. \[\square\]

4.2.4. If the radical is linear then the 2-dimensional cohomology is bounded. We first prove the following lemma.

**Lemma 4.2.** Let $G$ be a connected Lie group. Assume that the radical $\sqrt{G}$ of $G$ is linear. Then there is a closed contractible subgroup $V < \sqrt{G}$ which is normal in $G$ such that $\sqrt{G}/V$ is a torus $T$ and the covering space $\xi$ induced from the universal cover $\tilde{L} \to G/\sqrt{G}$ via the canonical projection $G/V \to G/\sqrt{G}$ is of the form $\xi : T \times \tilde{L} \to G/V$.

**Proof.** Let $H$ be the covering space of $G$ induced from the universal cover $\tilde{G}$ of $G/\sqrt{G}$. Then, according to (1) of Lemma 3.20 and Lemma 2.13, $H = \sqrt{G} \rtimes \tilde{G}$, with $\tilde{L}$ semi-simple. Let $W$ be the linearizer of $\tilde{L}$;
it is discrete and central in $\tilde{L}$. Also $W$ is central in $H$ because $W$ lies in the kernel of the adjoint representation of $H$. So we can form the quotient

$$Q := \sqrt{G} \rtimes (\tilde{L}/W),$$

which is a linear Lie group by Theorem 2.1. According to Hochschild [30, XVIII.4, Thm. 4.3], the group $Q$ contains a contractible normal closed solvable subgroup $X$ with $Q/X$ linear reductive. Since $X < \sqrt{G}$ we can think of $X$ as a subgroup of $H = \sqrt{G} \rtimes \tilde{L}$, and this subgroup is normal, since $W$ is central. Under the covering projection $H \to G$, this subgroup $X$ maps isomorphically onto a closed normal subgroup $V < G$.

As $\tilde{L}$ is simply-connected, and as $T$ is the radical of $E$, the sequence splits (again by Lemma 2.13). Since $T$ is central, we deduce that $E$ is a direct product $E = T \times \tilde{L}$.

\[ \square \]

\textbf{Proof.} We prove that (1) implies (4). By Lemma 4.2, $G$ contains a normal contractible subgroup $V$, hence $G \to G/V$ is a homotopy equivalence, and thus $H^*_B(G/V, \mathbb{Z}) \to H^*_B(G, \mathbb{Z})$ an isomorphism by Lemma 3.9. It suffices therefore to prove (4) for $G/V$. In the notation of Lemma 4.2, we have covering spaces $\mathbb{R}^n \times \tilde{L} \to T \times \tilde{L} \to G/V$. Note that $\mathbb{R}^n \times \tilde{L}$ is the universal cover of $G/V$. By Proposition 3.23, it suffices to show that the cocycle defining this universal cover can be chosen bounded. For this, we appeal to Lemma 3.21, with $X = \mathbb{R}^n \times \tilde{L}$, $Y = T \times \tilde{L}$ and $Z = G/V$. The map $p : X \to Y$ is the product of $\mathbb{R}^n \to T$ with $\text{Id} : \tilde{L} \to \tilde{L}$ and we conclude from Lemma 3.24 and Corollary 3.6 that the Borel cocycle associated with $p$ can be chosen to be bounded. For $q : Y \to Z$ we observe that $q$ is obtained by pulling back $\tilde{L} \to G/\sqrt{G}$ over $G/V$. According to (2) of Lemma 3.20, it suffices to prove that the universal cover $\tilde{L} \to G/\sqrt{G}$ has a Borel section $\sigma$ such that the associated cocycle $c_{\sigma}$ is bounded. But this case has already been dealt with in Corollary 3.27 and we are done. \[ \square \]

\textbf{4.2.5.} \textit{If the 2-dimensional cohomology is bounded then all the cohomology is bounded.}
Proof. We prove that (4) implies (3). Since for every \(d\), \(H^d(BG, \mathbb{Z})\) is a finitely generated abelian group, we see that the \(\mathbb{Z}\)-subalgebra \(A^* \subseteq H^*(BG, \mathbb{R})\) generated by \(H^2(BG, \mathbb{Z})\) together with the image \(J^*\) of the inflation map \(H^*(B(G/\sqrt{G}), \mathbb{Z}) \to H^*(BG, \mathbb{Z})\) satisfies, in view of Theorem 1.4, \(A^* \otimes \mathbb{R} = H^*(BG, \mathbb{R})\). Thus \(A^d \subseteq H^d(BG, \mathbb{Z})\) has finite index for every \(d\). To show that \(H^d(B_b(G, \mathbb{Z}) \to H^d(B(G, \mathbb{Z})\) is onto it suffices therefore by Lemma 3.1 to show that the elements of \(A^d\) all lie in the image of \(H^d(B_b(G, \mathbb{Z})\to H^d(B(G, \mathbb{Z})\). From the definition of \(A^d\) it suffices therefore to show that:

(a) The map \(H^2(B_b(G, \mathbb{Z}) \to H^2(B(G, \mathbb{Z})\) is onto.

(b) The map \(H^*_{B_b}(G/\sqrt{G}, \mathbb{Z}) \to H^*_{B}(G/\sqrt{G}, \mathbb{Z})\) is onto.

Now (a) holds, because of our assumption (2). To prove (b), we consider the Lie group \(L\) obtained by dividing \(G/\sqrt{G}\) by its center. By Corollary 3.4 it suffices to show that \(H^*_{B_b}(L, \mathbb{Z}) \to H^*_{B}(L, \mathbb{Z})\) is onto. But this follows from the second statement in Proposition 3.11, by observing that \(L\) is as a Lie group isomorphic to the topological connected component of the identity of a Lie group \(G_a(\mathbb{R})\) for some real linear algebraic group \(G_a\) (cf. \([33]\) or \([48]\)).

4.2.6. Remaining implications. That (3) implies (4) is a tautology. That (4) implies (5) follows from Proposition 3.23. That (1) is equivalent to (2) follows from Proposition 2.10.

4.3. Proof of Theorem 1.5.

Proof. In view of Proposition 3.7 we may assume that \(G\) is connected. Hence Theorem 1.5 follows from Theorem 1.2.

4.4. Proof of Theorem 1.7.

4.4.1. Goldman’s construction. Recall that if \(G\) is a connected Lie group and if \(P\) is a \(G\)-principal bundle over a \(CW\)-complex \(B\), there is a characteristic class:

\[ o_2(P) \in H^2(B, \pi_1(G)), \]

defined by obstruction theory. (More precisely, let \(e^{(2)}\) be a 2-cell of the 2-skeleton \(B^{(2)}\) of \(B\). There is, up to homotopy, a unique way of getting \(P|_{B^{(2)}}\) by gluing \(P|_{B^{(2)} \setminus \text{interior}(e^{(2)})}\) with \(e^{(2)} \times G\). The gluing is determined by a map \(\partial e^{(2)} \to G\). The homotopy classes \(c(e^{(2)}) \in \pi_1(G)\) of these maps define a cellular cocycle whose class is \(o_2(P)\).)

Proposition 4.3. (Compare with Goldman [21].) Let \(G\) be a connected solvable Lie group. If \(\pi_1([G,G]) \neq 0\), then there exists a class of degree 2 in the image of the natural map \(H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R})\) which can’t be represented by a bounded cocycle.
Proof. Let $p : \tilde{G} \to G$ be the universal cover of $G$. The commutator subgroup $[\tilde{G}, \tilde{G}]$ is closed connected and normal in $\tilde{G}$. Lemma 2.3 implies that $[\tilde{G}, \tilde{G}]$ is simply-connected. Obviously $p([\tilde{G}, \tilde{G}]) = [G, G]$, hence the restriction of $p$ to $[\tilde{G}, \tilde{G}]$ is the universal cover of the subgroup $[G, G]$ of $G$. (Here the topology on the group $[G, G]$ is the quotient topology induced by the restriction of $p$ to $[\tilde{G}, \tilde{G}]$.) Hence we obtain two exact sequences with commutative squares:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \pi_1(G) & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \pi_1([G, G]) & \longrightarrow & [\tilde{G}, \tilde{G}] & \longrightarrow & [G, G] & \longrightarrow & 1.
\end{array}
$$

Each vertical arrow is an inclusion (but $[G, G] \subset G$ is not necessarily closed). A connected solvable Lie group has the homotopy type of its maximal compact subgroup which is a torus. Hence, by hypothesis, there exists an element $z \in \pi_1([G, G]) \subset \text{Ker}(p)$ of infinite order. Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be elements of $\tilde{G}$ such that

$$
z = \prod_{k=1}^g [a_k, b_k].$$

Let $\Sigma_g$ denote the orientable closed surface of genus $g$ and let

$$
\Gamma = \pi_1(\Sigma_g) = (A_1, \ldots, A_g, B_1, \ldots, B_g : \prod_{k=1}^g [A_k, B_k]),
$$

be its fundamental group with the usual presentation. Let $h : \Gamma \to G$ be the homomorphism defined by the conditions $h(A_k) = p(a_k), h(B_k) = p(b_k)$. Let $P_h = \Gamma \backslash (\Sigma_g \times G)$ be the flat principal $G$-bundle over $\Sigma_g$ with $h$ as holonomy. In other words, the bundle $P_h$ is the pullback of the universal principal $G$-bundle $EG$ by the map $Bh : B\Gamma \to BG$. By construction:

$$
o_2(P_h)([\Sigma_g]) = z.
$$

As $z$ has infinite order in the finitely generated abelian group $\pi_1(G)$, we can choose a homomorphism $q : \pi_1(G) \to \mathbb{R}$ between coefficients such that $q(z) = 1$. Let $q_* : H^2(BG, \pi_1(G)) \to H^2(BG, \mathbb{R})$ be the induced map. We have $q_*(o_2(P_h)([\Sigma_g])) = 1$. The map $Bh$ factors through $BG^g$:

$$
\Sigma_g \simeq B\Gamma \to BG^g \to BG.
$$
It induces the commutative diagram:

\[
P_h \longrightarrow E \longrightarrow \Sigma_g \overset{\delta}{\longrightarrow} B G
\]

where vertical arrows are projections of principal $G$-bundles. It also induces the commutative diagram:

\[
\begin{array}{c}
H^2(BG, \pi_1(G)) \longrightarrow H^2(BG^\delta, \pi_1(G)) \longrightarrow H^2(\Sigma_g, \pi_1(G)) \\
\downarrow \downarrow \downarrow \\
H^2(BG, \mathbb{R}) \longrightarrow H^2(BG^\delta, \mathbb{R}) \longrightarrow H^2(\Sigma_g, \mathbb{R}),
\end{array}
\]

in which characteristic classes are preserved and where vertical arrows are induced by the coefficients homomorphism $q$. The image $x^\delta \in H^2(BG^\delta, \mathbb{R})$ of $x := q_0(o_2(EG)) \in H^2(BG, \mathbb{R})$ is not equal to zero because it is sent to $q_0(P_h)$ and we have proved that this class is non-zero by evaluating it on $[\Sigma_g]$. The class $x^\delta \in H^2(BG^\delta, \mathbb{R}) \cong H^2(G^\delta, \mathbb{R})$ can’t be bounded because a solvable Lie group is amenable also with respect to its discrete topology [29, Volume 1, Chapter IV, Paragraph 17, (17.14)]. Hence its real bounded cohomology vanishes in positive dimensions. \hfill $\square$

**Proof.** We prove Theorem 1.7. We apply Proposition 4.3 to the group $\sqrt{G}$. We conclude that there exists a class $x \in H^2(B\sqrt{G}, \mathbb{R})$ with $x^\delta \in H^2(B(\sqrt{G})^\delta, \mathbb{R})$ not bounded. According to Proposition 3.25, the map $H^2(BG, \mathbb{R}) \rightarrow H^2(B\sqrt{G}, \mathbb{R})$ is onto. Let $y \in H^2(BG, \mathbb{R})$ be a counter image of $x$. The commutativity of the diagram:

\[
\begin{array}{c}
H^2(BG, \mathbb{R}) \longrightarrow H^2(B\sqrt{G}, \mathbb{R}) \\
\downarrow \downarrow \downarrow \\
H^2(BG^\delta, \mathbb{R}) \longrightarrow H^2(B(\sqrt{G})^\delta, \mathbb{R}),
\end{array}
\]

implies that $y^\delta$ is unbounded. \hfill $\square$

4.5. **Proof of Theorem 1.9.**

4.5.1. *Condition 1 implies Condition 2.*

**Proof.** That (1) implies (2) is one of the implications of Goldman’s result in [21], so we don’t prove it but we show with an example that the finiteness assumption on the complexes in (2), which is not explicitly stated as an hypothesis in [21] (but used in the proof), is needed.
Let $G = SO(2, \mathbb{R})$. The bundle $EG^\delta \to BG^\delta$ induced by the map $BC^\delta \to BG$ is a flat principal $G$-bundle. Although $[G, G] = 1$, this bundle is not virtually trivial because the image $e^\delta \in H^2(BG^\delta, \mathbb{Z})$ of the Euler class $e \in H^2(BG, \mathbb{Z})$ has infinite order [37, Cor. 1]. □

4.5.2. Condition 2 implies Condition 3.

Proof. Let $c(EG) \in H^*(BG, \mathbb{Z})$ be a characteristic class. Let $P$ be a flat principal $G$-bundle over a finite complex $X$. By hypothesis there exists a finite cover $X_d \to X$, say of degree $d$, such that the pullback $P_d$ of $P$ to $X_d$ is trivial. Hence, according to Lemma 3.15, we have $d \cdot c(P) = 0$. We apply (2) of Proposition 3.14 and conclude that $c(EG)$ is sent to zero in $H^*(BG^\delta, \mathbb{R})$. As $H^*(BG, \mathbb{Z}) \otimes \mathbb{R} = H^*(BG, \mathbb{R})$ this shows that the map $H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R})$ is zero. □

4.5.3. Remaining implications. That (3) implies (4) is a tautology. That (4) implies (1) follows from Theorem 1.7.

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