Anomalous Quantum Hall Effect on Sphere

Ahmed Jellal

The Abdus Salam International Centre for Theoretical Physics,
Strada Costiera 11, 34014 Trieste, Italy

and

Theoretical Physics Group, Faculty of Sciences, Chouaib Doukkali University,
Ibn Maächou Road, P.O. Box 20, 24000 El Jadida, Morocco

Abstract

We study the anomalous quantum Hall effect exhibited by the relativistic particles living on two-sphere $S^2$ and submitted to a magnetic monopole. We start by establishing a direct connection between the Dirac and Landau operators through the Pauli–Schrödinger Hamiltonian $H^\text{SP}_S$. This will be helpful in the sense that the Dirac eigenvalues and eigenfunctions will be easily derived. In analyzing $H^\text{SP}_S$ spectrum, we show that there is a composite fermion nature supported by the presence of two effective magnetic fields. For the lowest Landau level, we argue that the basic physics of graphene is similar to that of two-dimensional electron gas, which is in agreement with the planar limit. For the higher Landau levels, we propose a $SU(N)$ wavefunction for different filling factors that captures all symmetries. Focusing on the graphene case, i.e. $N = 4$, we give different configurations those allowed to recover some known results.
1 Introduction

Since its discovery, the quantum Hall effect (QHE) [1] is always bringing new surprises. In fact, recently with the new technological progress, one can observe the effect in graphene [2, 3], which is a two-dimensional (2D) projection of graphite. It is governed by a Hall conductivity of the form

$$\sigma_{H}^{g} = 4 \left( l + \frac{1}{2} \right) \frac{e^2}{h}, \quad l = 0, \pm 1, \pm 2 \ldots$$

which was predicted theoretically [4, 5]. It is different from that characterizing the integer quantum Hall effect (IQHE) exhibited by two-dimensional electron gas (2DEG) in the presence of an external magnetic field [6]:

$$\sigma_{H}^{SC} = \frac{l e^2}{h}$$

where $l \in \mathbb{N}$. The anomalous Hall conductivity $\sigma_{H}^{g}$ can be seen as a consequence of different effects. Indeed, the prefactor 4 reflects the two-fold spin and two-fold valley degeneracy in the graphene band structure. However, the term $\frac{1}{2}$ comes from the Berry phase due to the pseudospin (or valley) precession when a massless (chiral) Dirac particle exercises cyclotron motion. Moreover, $\sigma_{H}^{g}$ provides a direct evidence for the relativistic nature of the charge carriers in graphene [7]. This offers unexpected bridge between condensed matter physics and quantum electrodynamics, more detail can be found in [8].

The above new challenge offered a laboratory for different investigations. Theoretically, many works have been reported on the subject and extended to the fractional quantum Hall effect (FQHE) [9]. In fact, a set of integer and FQHE states that break the $SU(4)$ spin/valley symmetry has been constructed [10]. Also it is shown that the lowest Landau level (LLL) FQHE in graphene, in the large Zeeman energy limit, is equivalent to the LLL in 2DEG [11]. Using the composite fermion (CF) theory [12], different developments have been appeared dealing with a possible FQHE in graphene. More precisely, a direct and immediate mapping between IQHE resulting from CF of graphene and FQHE has been established [13]. New FQHE states that have no analogue in 2DEG have been predicted [14] and also compressible states at $\nu = \pm \frac{1}{3}$ and $\nu = \pm \frac{2}{3}$ have been discussed [15]. Furthermore, Using the Halperin theory [16] for spin in the conventional QHE, a $SU(N)$ wavefunction has been constructed and related discussions have been reported by Goerbig and Regnault [17]. Other important investigations related to the subject can be found in [18, 19] as well as [20]. For early works, one may see the papers cited in [21].

Motivated by different analysis of the anomalous fractional quantum Hall effect (AFQHE) on plane, we discuss its realization on two-sphere $S^2$. As far as we know, such consideration can be argued by the symmetry conservation. Recalling that the Laughlin theory [22] is translationally invariant but not rotationally. To solve such problem, Haldane [23] analyzed the Hall system on $S^2$ and constructed a theory that possess all symmetries. This leads to conclude that the study of AFQHE on the compact surface will be interesting. Certainly this will bring new ideas and show the difference as well as similarities with respect to the standard case.

To do our task, we consider a mathematical formalism governed by the Pauli–Schrödinger and Dirac Hamiltonian’s. The spectral properties of the first one have been considered at many occasions where the asymptotic behavior of the magnetic field at infinity has been discussed [24]. Also the eigenprojector kernels have been analytically studied [25]. The second one has been analyzed in different contexts, e.g. [26, 27]. These investigations show that both Hamiltonian’s are interesting from mathematical point view. Their extensions to physics have been done for different motivations,
The anomalous QHE on plane. In the present paper, we give another way to show the physical realization of both Hamiltonian’s. This concern a survey of the anomalous QHE on $S^2$.

It is well-known that the Landau problem is the cornerstone of the conventional QHE. However, for the anomalous QHE, one could start with the Dirac Hamiltonian because particles are relativistic. Both of systems are connected, which is due to the fact that the anomalous Hall conductivity can be understood from the Landau level structures of Dirac particles. Because of this connection, we start by considering the Pauli–Schrödinger Hamiltonian describing particles on $S^2$ in the presence of a magnetic monopole. Its matrix elements are obtained to be dependent on two different effective magnetic fields. This will offer another way to make contact with the composite fermions. From this analysis, we derive the Dirac eigenvalues as well as the corresponding eigenfunctions.

We give some discussions about the Pauli–Schrödinger spectrum. Indeed, in determining such spectrum we show that its matrix elements are expressed in terms of the Landau Hamiltonian. Each element describes a subsystem subjected to an effective magnetic field. The sum of both parts gives exactly the external magnetic field. We argue that these fields are similar to those felt by the composite fermions in 2DEG [12]. This interpretation allows us to define an effective filling factor and establish different links with other theories.

We start our analysis of the anomalous QHE by considering LLL. Indeed, to discuss similarities and differences between FQHE on 2DEG and graphene, we evaluate some physical quantities. In doing these, we show that AFQHE in both systems is similar when the particle are restricted to live on LLL. To argue this, we evaluate the incompressibility through the correlation function and determine the density of particles. As interesting results, we conclude that the basic physics in the graphene LLL is the same as in 2DEG LLL for the Hall system on $S^2$. This coincides with the planar limit analysis [11].

As far as the higher Landau levels are concerned, Goerbig and Regnault [17] proposed a wavefunction to describe $SU(4)$ FQHE in graphene sheet. This is seen as a direct extension of the Halperin wavefunction [16]. We note that the proposed state is related to our ground state configuration realized in terms of the matrix model and non-commutative Chern–Simons theories. To consider the higher Landau levels on $S^2$, we build a $SU(N)$ wavefunction generalizing those of Haldane [23] and corresponding to different filling factors

$$\nu = q_i K_{ij}^{-1} q_j^{-1}$$

where $K_{ij}$ is $N \times N$ matrix and $q_i$ is a vector. To make contact with graphene, we analyze the $N = 4$ case and show its relations to others theories. For this, we make different appropriate choices of matrices to reproduce specific filling factors.

The paper is organized as follows. In section 2, we discuss the anomalous IQHE on the plane, which can be done by introducing the corresponding Hamiltonian formalisms. In section 3, we introduce the covariant derivatives acting on p-forms of $S^2$ to write algebraically and analytically the Pauli–Schrödinger Hamiltonian $H^{PS}_S$. Using an unitary transformation, we show that $H^{PS}_S$ can be mapped in terms of that of Landau. This mapping will be used to derive the Dirac spectrum. In section 4, we show that the Pauli–Schrödinger system is exhibiting a composite fermion behavior where the effective magnetic fields will be defined. Subsequently, we discuss AFQHE on $S^2$ in terms of the density of particles and two-point function. We generalize the Haldane wavefunction and reproduce different filling factors in section 5. We conclude and give some perspectives in the final section.
2 Anomalous QHE on plane

Before developing our main ideas, it is relevant to discuss the anomalous IQHE on the plane and emphasize its differences with respect to the conventional QHE. Since this new challenge is due to a manifestation of the relativistic particles, it is natural to introduce the Dirac formalism in 2D. This can be done by considering the Pauli–Schrödinger Hamiltonian and resorting its eigenvalues and eigenfunctions in terms of those of Landau.

2.1 Pauli–Schrödinger Hamiltonian

Let us consider one-relativistic particle living on the plane \((x, y)\) in presence of a perpendicular magnetic field \(B\). The Pauli–Schrödinger Hamiltonian for such system can be written as

\[
H_{PS}^p = \frac{1}{2m} \left[ \vec{\sigma} \cdot \left( \vec{p} - \frac{e}{c} \vec{A} \right) \right]^2
\]

where the Pauli matrices \(\vec{\sigma}\) satisfy the usual relations

\[
\{\sigma_i, \sigma_j\} = 2\delta_{ij}, \quad [\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k.
\]

It will be clear that the square root of \(H_{PS}^p\) gives exactly the Dirac Hamiltonian in 2D. This connection will be used in the next in order to achieve our purpose.

For simplicity, it is convenient to express \(H_{PS}^p\) in terms of the Landau Hamiltonian \(H_L^p\). Indeed, by choosing the symmetric gauge

\[
\vec{A} = \frac{B}{2} (-y, x)
\]

we show that \(H_{PS}^p\) can be mapped as

\[
H_{PS}^p = \left( \begin{array}{ccc} H_L^p & 0 \\ 0 & H_L^p \end{array} \right) - B \left( \begin{array}{ccc} 1 & 0 \\ 0 & -1 \end{array} \right)
\]

where \(H_L^p\), in complex notation \(z = x + iy\), takes the form

\[
H_L^p = -\left\{ \frac{\partial^2}{\partial z \partial \bar{z}} + B \left( \frac{\partial}{\partial z} - \frac{\bar{z}}{\partial \bar{z}} \right) - B^2 \right\}.
\]

Hereafter, we set the fundamental constants \((e, c, \hbar, m)\) to one. Clearly, the spectrum and eigenfunctions of \(H_{PS}^p\) can be derived from that of \(H_L^p\). This will play an important role when we consider the present system on \(S^2\).

One can also use the algebraic approach to write another version of \(H_{PS}^p\). This can be done by introducing the second order differential operators

\[
D_p = \partial + i\frac{B}{2} \text{ext}(\theta_p), \quad D_p^* = \partial^* - i\frac{B}{2} \text{ext}(\theta_p)^*
\]

in terms of the gauge potential

\[
\theta_p = i(\bar{z}dz - zd\bar{z}).
\]

Recalling that, the wavefunctions of \(H_{PS}^p\) are type differential one-forms of \(\mathbb{C}\) and each one has two component spinors. They are

\[
\Psi = \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right).
\]
The term \( \text{ext}(\theta_p) \) acts on the states as follows

\[
\text{ext}(\theta_p) \Psi = \theta_p \land \Psi.
\]

Therefore, we can write \( H_p^{\text{PS}} \) in terms of the covariant derivatives as

\[
H_p^{\text{PS}} = D^*_p D_p + D_p D^*_p.
\]

With this we now have two possibilities to get the eigenvalues and eigenfunctions of \( H_p^{\text{PS}} \). This can be done either algebraically or analytically.

### 2.2 Dirac Hamiltonian

The above tools can be applied to analyze the anomalous QHE in graphene. Indeed in such systems, the two Fermi points, each with a two-fold band degeneracy, can be described by a low-energy continuum approximation with a four-component envelope wavefunction whose components are labeled by a Fermi-point pseudospin \( = \pm 1 \) and a sublattice forming an honeycomb. Specifically, the Hamiltonian for one-pseudospin component can be obtained from (4) under some considerations. This is \( ^{[30, 31]} \)

\[
H_p^D = v_F \left( \begin{array}{cc} 0 & \Pi_x - i \Pi_y \\ \Pi_x + i \Pi_y & 0 \end{array} \right)
\]

where \( v_F \approx \frac{e c}{100} \) is the Fermi velocity and the many-body effects are neglected. The conjugate momentum, in the symmetric gauge, are given by

\[
\Pi_x = p_x - \frac{B}{2} y, \quad \Pi_y = p_y + \frac{B}{2} x.
\]

Using the \( H_p^D \) form, we can establish a link to \( H_p^{\text{PS}} \). To proceed, let us return to (6) and write explicitly the raising and lowering operators as

\[
D^*_p = -2 \partial_z + \frac{B}{2} \bar{z}, \quad D_p = 2 \partial_z + \frac{B}{2} z
\]

which satisfy the commutation relation

\[
[D_p, D^*_p] = 2B.
\]

These can be used to write \( H_p^D \) as

\[
H_p^D = i v_F \left( \begin{array}{cc} 0 & D_p \\ D^*_p & 0 \end{array} \right).
\]

Its spectrum can be determined in a simple way if we introduce its square. This is

\[
\left( H_p^D \right)^2 = v_F^2 \left( \begin{array}{cc} D_p D^*_p & 0 \\ 0 & D^*_p D_p \end{array} \right).
\]

Hereafter we set \( v_F \) to one. \( \left( H_p^D \right)^2 \) is related to the Pauli–Schrödinger Hamiltonian \( ^{[4]} \) up to some multiplicative constants. It is clear that, \( \left( H_p^D \right)^2 \) is written in terms of the diagonal form of the Landau Hamiltonian \( ^{[5]} \). Therefore, its spectrum can easily be obtained.
Before deriving the Dirac eigenvalues and eigenfunctions, we make a general statement. Indeed, assuming that the eigenvalue equation
\[ \hat{O}^2 \psi = \lambda^2 \psi \] (17)
is satisfied for a given operator \( \hat{O} \) and a vector \( \psi \). Thus, one can simply check that the vector
\[ \phi^\pm = \pm \lambda \psi + \hat{O} \psi \] (18)
is an eigenvector of \( \hat{O} \) and verify
\[ \hat{O} \phi^\pm = \pm \lambda \phi^\pm. \] (19)
Therefore, we underline that for \( \phi^\pm \neq 0 \), the eigenvalue of \( \hat{O} \) is given by \( \pm \lambda \). This can be applied to the Dirac Hamiltonian and its square. For this, we start by solving the equation
\[ \left( \hat{H}_D^2 \right)^2 \Psi = E \Psi. \] (20)
Since that \( \left( \hat{H}_D^2 \right)^2 \) has to do with the Landau Hamiltonian \( \hat{L} \), the wavefunctions \( \Psi \) should be written in an appropriate form. They are
\[ \Psi_{m,n} = \begin{pmatrix} \psi_{m-1,n} \\ \psi_{m,n} \end{pmatrix} \] (21)
where the eigenfunctions \( \psi_{m,n} \)
\[ \psi_{m,n}(z) = \frac{(-1)^m \sqrt{B^m m!}}{\sqrt{2^{n+1} \pi (m+n)!}} z^{\frac{n}{2}} L_n^m \left( \frac{z \cdot \bar{z}}{2} \right) e^{-\frac{B}{4} z \cdot \bar{z}}, \quad m, n = 0, 1, 2 \cdots \] (22)
are describing particles living on \( \mathbb{R}^2 \) and subjected to the magnetic field \( B \). The corresponding Landau levels are given by
\[ (E_D^2)_m = B (2m + 1). \] (23)
Using the general argument stated before, we can show that the normalized eigenfunctions of \( \hat{H}_D^2 \) take the form
\[ \Psi_{m \neq 0,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\text{sgn}(m) i \psi_{|m|-1,n} \\ \psi_{m,n} \end{pmatrix} \] (24)
with the convention \( \text{sgn}(0) = 0 \). Note that, the zero-mode wavefunction is
\[ \Psi^{(0,n)} = \begin{pmatrix} 0 \\ \psi_{0,n} \end{pmatrix}. \] (25)
Their energy levels read as
\[ (E_D)_m = \text{sgn}(m) \sqrt{2B|m|}. \] (26)
Compared to the usual Landau eigenvalues \( [23] \), we firstly note that there is a missing factor, i.e. \( \frac{1}{2} \). This is due to the chirality, which in our case is related to the magnetic field and is contributing by an amount of \( \pm \frac{1}{2}. \) Secondly unlike the 2DEG LLL, its graphene analogue is characterized by a zero-energy and a state given in \( [25]. \)
2.3 Anomalous FQHE on plane

The Dirac formalism can be employed to show that the Hall conductivity on graphene is different from that on 2DEG. In fact, for non-interacting electrons, when both the spin and the pseudospin degeneracies are present, the Hall plateaus are

$$\sigma_H = 4 \left( l + \frac{1}{2} \right) \frac{e^2}{2\pi}, \quad l = 0, \pm1, \pm2, \cdots .$$  \hspace{1cm} (27)

This result has been theoretically predicted \[4, 5\] and experimentally seen \[2, 3\]. It can be also obtained by using the thermodynamical analysis of the Hall particles on graphene \[32, 33\]. On the other hand, (27) can be interpreted as follows. The number 4 is accounting the degeneracy of spin and valley where each one is contributing by 2. $\frac{1}{2}$ is coming from the Berry phase contribution that is equal $\pi$. This makes difference with respect to the conventional QHE.

An important peculiarity of the Landau levels for massless Dirac fermions is the existence of zero-energy states, i.e. $m = 0$ in (26). Clearly, this is unlike the usual 2DEG with parabolic bands where the first Landau level is shifted by $B$. The existence of $m = 0$ in (26) leads to an anomalous QHE instead of the conventional IQHE. This anomaly is provided by the famous Atiyah–Singer index theorem \[34\]. A connection between this theorem for the Dirac operator on a compact coset space and higher dimensional QHE can be found in \[35\].

Returning now to discuss FQHE in graphene. In doing so, Let us consider $M$-particles in LLL, which of course means that all $m_i = 0$ with $i = 1, \cdots, M$ and each $m_i$ corresponds to the spectrum (24–26). The total wavefunction of zero-energy Landau level (25) can be written in terms of the Slater determinant. This is

$$\psi(z, \bar{z}) = \epsilon^{i_1 \cdots i_M} z_1^{n_{i_1}} \cdots z_M^{n_{i_M}} \exp \left( -\frac{B}{4} \sum_i M |z_i|^2 \right) .$$  \hspace{1cm} (28)

where $\epsilon^{i_1 \cdots i_M}$ is the fully antisymmetric tensor and $n_i$ are integers. It is relevant to write this wavefunction as Vandermonde determinant. We have

$$\psi(z, \bar{z}) = \text{const} \prod_{i<j}^M (z_i - z_j) \exp \left( -\frac{B}{4} \sum_i M |z_i|^2 \right) .$$  \hspace{1cm} (29)

This can be interpreted by remembering the Laughlin wavefunction

$$\psi_{\text{Laugh}}(z, \bar{z}) = \prod_{i<j}^M (z_i - z_j)^{2l+1} \exp \left( -\frac{B}{4} \sum_i M |z_i|^2 \right) .$$  \hspace{1cm} (30)

It is well-known that it has many interesting features and good ansatz to describe the fractional QHE at the filling factor $\nu = \frac{1}{2l+1}$, with $l$ is integer value. It is clear that (29) is nothing but the first Laughlin state that corresponds to $\nu = 1$. Actually, (29) is describing the first quantized Hall plateau of the integer QHE. Note that (30) can also be written as

$$|l\rangle = \left\{ \epsilon^{i_1 \cdots i_N} z_1^{n_{i_1}} \cdots z_N^{n_{i_N}} \right\}^{2l+1} |0\rangle .$$  \hspace{1cm} (31)

Consequently, the wavefunction for the particles in the graphene LLL are identical to those of the 2DEG LLL. We conclude that the basic physics in both systems is the same. More discussion about this issue can be found in \[11\].
For the higher Landau levels, different attends are proposed in dealing with the anomalous QHE in graphene. Among them, we cite the $SU(4)$ wavefunction realized by Boerbig and Regnault [17] as a candidate to describe the effect. In doing this, they extended the Halperin wavefunction [16] to take account of the spin and valley degree of freedom. This allowed them to discuss different issues and predict some filling factors. This wavefunction is related to that we have proposed [28], in general way, by using the matrix model and non-commutative Chern–Simons theories. In section 4, we give different discussions about the matter.

3 Sphere analysis

After describing the anomalous QHE on plane $\mathbb{R}^2$, we now consider the case of two-sphere $\mathbb{S}^2$. In fact, we are attending to construct the same effect and therefore generalize all obtained results by considering graphene as a spherical manifold. In doing this task, we should first establish some mathematical tools. This will allow us to write the Pauli–Schrödinger Hamiltonian in terms of that of Landau. Using the same technical as for $\mathbb{R}^2$, we derive the Dirac eigenvalues as well as its eigenfunctions. These materials will be the subject of the present section.

3.1 Pauli–Schrödinger Hamiltonian on sphere

To generalize the Pauli–Schrödinger Hamiltonian from $\mathbb{R}^2$ to $\mathbb{S}^2$ of radius unit, we first start by defining some tools on this compact surface. Indeed, the $\mathbb{S}^2$ Khäehlerian metric and the corresponding volume measure are

$$ds^2_s = \frac{4}{(1 + z \cdot \bar{z})^2} dz \otimes d\bar{z}, \quad d\mu_s(z) = \frac{4d\mu_p(z)}{(1 + z \cdot \bar{z})^2}. \quad (32)$$

They go to the standard distance and the Lebesgue measure $4d\mu_p(z) = dzd\bar{z}$ on the planar limit. With (32), the inner product on $\mathbb{S}^2$ of two functions $\Psi_1$ and $\Psi_2$ in the Hilbert space reads as

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{\mathbb{S}^2} d\mu_s(z) \bar{\Psi}_1 \times \Psi_2. \quad (33)$$

Because of its relation to magnetic field, one has to define the vector potential on $\mathbb{S}^2$. This can be done by introducing the differential one-form $\theta_s$, such as

$$\theta_s(z) = \frac{i(\bar{z}dz - zd\bar{z})}{1 + z \cdot \bar{z}}. \quad (34)$$

Combining this with the Dirac quantization

$$\int_{\mathbb{S}^2} F = \int_{\mathbb{S}^2} dA = 2\pi k \quad (35)$$

we end up with the required vector potential

$$A = \frac{k}{2} \frac{i(\bar{z}dz - zd\bar{z})}{1 + z \cdot \bar{z}} \equiv \frac{k}{2} \theta_s(z). \quad (36)$$

Its complex components are given by

$$A_z = \frac{k}{2} \frac{\bar{z}}{1 + z \cdot \bar{z}}, \quad A_{\bar{z}} = -i\frac{k}{2} \frac{z}{1 + z \cdot \bar{z}}. \quad (37)$$

This is showing an interesting relation between the integer $k$ and magnetic field $B$, which will be used in discussing QHE. It is

$$k = 2B. \quad (38)$$

This also express the Landau level degeneracies of $N$-particles. Moreover, it can be used to define the filling factor, such as

$$\nu = \frac{N}{k}. \quad (39)$$

There are two ways to write down the Pauli–Schrödinger Hamiltonian $H_{sPS}$ on $S^2$. Indeed algebraically, it can be done by introducing the suitable covariant derivatives in terms of $\theta_s$. In analogy to $R^2$, we realize them as

$$D_s = \partial + i\frac{k}{2} (\text{ext}\theta_s), \quad D_s^* = \partial^* - i\frac{k}{2} (\text{ext}\theta_s)^*. \quad (40)$$

which act on the $p$-forms of $S^2$. Therefore, the required Hamiltonian can be mapped as

$$H_{sPS} = D_s^* D_s + D_s D_s^*. \quad (41)$$

This mapping has two advantages. Indeed, firstly it can be used to make a group theory analysis of the present work in similar way to those have been developed in [36, 37]. Secondly, it allows us to apply the spectral theory approach [38] in order to deal with different issues. This requires to introduce a $H_{sPS}$ form in terms of the local coordinates $(z, \bar{z})$ instead of operators (40). To obtain such form, it is convenient to consider the corresponding smooth differential one-forms as two component spinors

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad (42)$$

which define a Hilbert space of the form

$$\mathcal{H}_s = L^2 (S^2, d\mu_p) dz \oplus L^2 (S^2, d\mu_p) d\bar{z}. \quad (43)$$

By acting (41) on (42), we find an elliptic operator

$$H_{sPS} = \begin{pmatrix} H_s^{B_1,B} & 0 \\ 0 & H_s^{B_2,B} \end{pmatrix} - \frac{k}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (44)$$

The operator $H_{s}^{\alpha,\beta}$ is related to the Landau Hamiltonian on $S^2$ and they coincide under some conditions as we will see next. It is given by

$$H_{s}^{\alpha,\beta} = (1 + z \cdot \bar{z}) \left\{ - (1 + z \cdot \bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - \alpha z \frac{\partial}{\partial z} + \beta \bar{z} \frac{\partial}{\partial \bar{z}} \right\} + \alpha \beta z \cdot \bar{z}. \quad (45)$$

where the fields $B_i$ reads as

$$B_1 = B - 2, \quad B_2 = B + 2. \quad (46)$$

We end this part by citing three remarks. Firstly, the mapping (41) is helpful in sense that we can derive the basic features of $H_{sPS}$ from those of the Landau Hamiltonian on $S^2$. Secondly, the $H_{sPS}$ form is consistent in describing particles on a spherical graphene. In fact, recalling that graphene is resulting from an unification of two-subsystems. They form an honeycomb and each part is governed by a component of $H_{sPS}$. Globally, the system can be seen as composite fermions, we will return to discuss these matters.
3.2 Spectrum of the Hamiltonian $H^L_s$

Staring from (44), it is clear that we first need to determine the Landau spectrum on $S^2$. To do this, we introduce the corresponding Landau Hamiltonian. It can simply be obtained by requiring $\alpha = \beta \equiv \frac{k}{2}$ in (45), such as

$$H^L_s = (1 + z \cdot \bar{z}) \left\{ - (1 + z \cdot \bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - \frac{k}{2} \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) \right\} + \frac{k^2}{4} z \cdot \bar{z}. \quad (47)$$

Firstly, (47) has been used by Haldane [23] to build a theory for the filling factor $\nu = \frac{1}{2l+1}$ on $S^2$ that generalizes the Laughlin one [22] on $R^2$. Subsequently, it has been generalized to the higher dimensional spaces by Karabali and Nair [36] as well as the Bergman ball [37, 38].

Solving the eigenvalue problem

$$H^L_s \Psi = E \Psi \quad (48)$$

one can show that the eigenvalues are given by

$$E^s_m = \frac{k}{2} (2m + 1) + m(m + 1), \quad 0 \leq m \leq k + 1. \quad (49)$$

To derive the corresponding eigenfunctions, we express $D_s$ and $D^*_s$ in terms of the local coordinates $(z, \bar{z})$. Thus from (37) and (40), one can write

$$D_s = (1 + z \cdot \bar{z}) \frac{\partial}{\partial z} - \frac{k}{2} z, \quad D^*_s = -(1 + z \cdot \bar{z}) \frac{\partial}{\partial \bar{z}} - \frac{k}{2} \bar{z} \quad (50)$$

with the condition

$$[D_s, D^*_s] = k. \quad (51)$$

As usual, we start by determining the fundamental state. This can be obtained by integrating the lowest Landau condition

$$D_s \Psi = 0. \quad (52)$$

After calculation, we find

$$\Psi(z, \bar{z}) = (1 + z \cdot \bar{z})^{-\frac{k}{2}} h(z) \quad (53)$$

where $h(z)$ is holomorphic function of $z$ and must be a polynomial of order $\leq k$. Therefore, other eigenfunctions can simply be obtained by successive application of the raising operators on (53). Doing this process, we obtain

$$\Psi_m(z, \bar{z}) = (D^*_s)^{k-m}(D^*_s)^{k-m+1} \cdots (D^*_s)^{k-1} \left( (1 + z \cdot \bar{z})^{-\frac{k}{2}} h(z) \right). \quad (54)$$

These define a Hilbert space of dimension $k + 2m + 1$. Note that, (54) can explicitly be written in terms of the coordinates $(z, \bar{z})$.

Since the above results generalize those of the Landau problem on the flat surface, it is relevant to check the asymptotic behavior. This can be achieved by requiring the planar limit. In this case, we recover the usual energy levels on plane as well as the corresponding eigenfunctions seen before.
3.3 Spectrum of the Hamiltonian $H_s^{PS}$

The above results will be employed to derive the spectrum of $H_s^{PS}$. For this, we can split the mother eigenvalue equation

$$H_s^{PS} \Psi = E \Psi \quad (55)$$

into two daughter’s where the functions $\Phi$ (42) form a Hilbert space $H_s$, $\Phi_1$ and $\Phi_2$ are in the space $L^2(S^2, d\mu_p)$. To clarify our statement, we can introduce an appropriate unitary transformation. It is generated by a $2 \times 2$ matrix

$$U = \begin{pmatrix} 1 + z \cdot \bar{z} & 0 \\ 0 & 1 + z \cdot \bar{z} \end{pmatrix}. \quad (56)$$

Now we can easily establish an explicit relation between $H_s^{PS}$ and the Landau Hamiltonian $H_s^L$. Therefore, we show that $H_s^{PS}$ can be written as

$$H_s^{PS} = U^{-1} \begin{pmatrix} H_s^{L,B_1} & 0 \\ 0 & H_s^{L,B_2} \end{pmatrix} - \begin{pmatrix} 0 & -k_2 - 1 \\ k_2 - 1 & 0 \end{pmatrix} U. \quad (57)$$

Clearly, (56) tells us that any function $\tilde{\Phi}_i \in L^2(S^2, d\mu_s)$ can be written in terms of another one belongs to the space $L^2(S^2, d\mu_s)$. This is

$$\tilde{\tilde{\Phi}}_i = (1 + z \cdot \bar{z}) \Phi_i. \quad (58)$$

Starting from (56), one can see that $H_s^{\alpha,\beta}$ transforms as

$$H_s^{\alpha,\beta} \Psi = (1 + z \cdot \bar{z})^{-1} H_s^L \frac{\alpha + \beta}{2} [(1 + z \cdot \bar{z}) \Psi] \quad (59)$$

where $H_s^L \frac{\alpha + \beta}{2}$ is nothing but the Landau Hamiltonian on $S^2$ with a magnetic field of the form $\frac{\alpha + \beta}{2}$, which is equal to $B$.

With the above tools, we are able to determine the spectrum of $H_s^{PS}$. Indeed, instead of (55) we have two eigenvalue equations given by

$$H_s^{L,B_1} \tilde{\Psi}_1 = (E + 4B_1) \tilde{\Psi}_1, \quad H_s^{L,B_2} \tilde{\Psi}_2 = (E - 4B_2) \tilde{\Psi}_2 \quad (60)$$

where $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are in $L^2(S^2, d\mu_s)$. It is clear that the whole eigenvalue problem becomes that of Landau Hamiltonian and then the required spectrum can easily be deduced. Indeed, after some calculation, we show that the eigenvalues of $H_s^{L,B_1}$ are

$$E^+_m = Bm + m(m - 1), \quad 0 \leq m \leq k_1 + 1. \quad (61)$$

In contrast, for the second magnetic field, we find

$$E^-_{m'} = B(m' + 1) + (m' + 1)(m' + 2), \quad 0 \leq m' \leq k_2 + 1. \quad (62)$$

To obtain the $H_s^{PS}$ spinors, we start by evaluating theirs components. This can be done by considering the lowering and raising operators in terms of $k_i$ fields. They are

$$D_{k_i} = (1 + z \cdot \bar{z}) \frac{\partial}{\partial \bar{z}} - \frac{k_i}{2} z, \quad (D_{k_i}^*)^* = -(1 + z \cdot \bar{z}) \frac{\partial}{\partial z} - \frac{k_i}{2} \bar{z}. \quad (63)$$
Thus, the ground-state is solution of the equation

\[ D_{ki} \chi = 0. \]  

(64)

This yields to the function

\[ \chi(z, \bar{z}) = (1 + z \cdot \bar{z})^{-\frac{1}{2} + \frac{m}{2} + \frac{1}{2}} \{ 1 + z \cdot \bar{z} - \frac{k_i}{2} - m - 1 \} \]  

(65)

where \( h(z) \) is an holomorphic function of degree \( \leq k_i \) on \( S^2 \). As far as (60) is concerned, the eigenfunctions can be obtained by a successive application of the corresponding raising operators. Indeed, for the first spectrum, we have

\[ \Psi_m(z, \bar{z}) = (1 + z \cdot \bar{z})^{-1} \left\{ (D_{S}^{k_1})^{*}_{\frac{m}{2} + 1} (D_{S}^{k_1})^{*}_{\frac{m}{2} + 2} \cdots (D_{S}^{k_1})^{*}_{\frac{m}{2} + m - 2} [1 + z \cdot \bar{z}]^{-\frac{1}{2} - m + 1} h(z) \right\}. \]  

(66)

Similarly, we obtain for the second Hamiltonian

\[ \Psi_{m'}(z, \bar{z}) = (1 + z \cdot \bar{z})^{-1} \left\{ (D_{S}^{k_2})^{*}_{\frac{m' + 1}{2} - \frac{1}{2}} (D_{S}^{k_2})^{*}_{\frac{m' + 2}{2}} \cdots (D_{S}^{k_2})^{*}_{\frac{m' + m'}{2} + m'} [1 + z \cdot \bar{z}]^{-\frac{1}{2} - m' - 1} h(z) \right\}. \]  

(67)

Finally, the Hilbert space of the Pauli–Schrödinger Hamiltonian reads as

\[ H_{PS} = \left\{ \Psi_{m, m'} = \begin{pmatrix} \Psi_m \\ \Psi_{m'} \end{pmatrix} \in L^2(S^2, d\mu_{\rho}) \oplus L^2(S^2, d\mu_{\rho}), \quad H_{PS}^{\pm} \Psi_{m, m'} = E_{m, m'}^{\pm} \Psi_{m, m'} \right\}. \]  

(68)

Note that, with these results one can also discuss the conventional QHE and see the effect of adding the spin as a degree of freedom. This can be done, for instance, by evaluating the density of particles and two-point function, but this discussion is out of the scope of the present paper.

### 3.4 Dirac spectrum on two-sphere

Motivated by the fact that the Dirac Hamiltonian \( H_{S}^{D} \) is the basic tool that can be used to discuss the anomalous QHE, we have to build this later on \( S^2 \). This can be realized by applying the same technical as it has been done for the \( \mathbb{R}^2 \) case. More precisely, we suggest to write the \( H_{S}^{D} \) in terms of differential operators those are related to the Pauli–Schrödinger formalism on \( S^2 \). This suggestion allows us to make an easily derivation of the \( H_{S}^{D} \) eigenvalues and the corresponding eigenfunctions.

The Dirac operator on \( S^2 \) can be written in different forms \cite{39}. For our purpose, it is convenient to introduce the Hamiltonian for one-pseudospin component as

\[ H_{S}^{D} = i \begin{pmatrix} 0 & D_{S} \\ D_{S}^{*} & 0 \end{pmatrix} \]  

(69)

where lowering \( D_{S} \) and the raising \( D_{S}^{*} \) operators are those given in (63), which verify the commutation relations similar to (51). Clearly, \( H_{S}^{D} \) coincides exactly with its partner in the planar limit. Now it is easy to see that the square of \( H_{S}^{D} \)

\[ \left( H_{S}^{D} \right)^2 = \begin{pmatrix} D_{S}D_{S}^{*} & 0 \\ 0 & D_{S}^{*}D_{S} \end{pmatrix} \]  

(70)

is related the Pauli–Schrödinger Hamiltonian on \( S^2 \). More precisely, \( H_{S}^{D} \) is expressed in terms of the Landau Hamiltonian \( H_{S}^{L} \) on \( S^2 \). This tells us that the spectrum of the (69) can be deduced from that of \( H_{S}^{L} \).
Consequently, using the general statement shown before one can find the eigenvalues

$$E_m^D = \pm \sqrt{|m(k + m)|}$$

(71)

with the condition $m \geq 1$ when $k = 0$ The corresponding eigenfunctions are

$$\Psi_m = \begin{pmatrix} -\text{sgn}(m)i\psi_{|m|-1} \\ \psi_{|m|} \end{pmatrix}$$

(72)

where the wavefunctions $\psi_{|m|}$ are those given in (54). In addition, there is a zero-energy mode whose eigenfunction is given by

$$\Psi_0 = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}.$$  

(73)

This corresponds to the graphene LLL. In fact, these results will applied to discuss the anomalous FQHE. More precisely, we analysis the incompressibility in LLL as well as the particle density. Furthermore, we show that how the obtained eigenfunction can be used to construct a $SU(N)$ wavefunction and its links to other theories. All these materials will be the subject of two last sections.

4 Anomalous QHE on sphere

We discuss the anomalous QHE on two-sphere $S^2$. For this, we focus on two aspects of QHE, which concern the composite fermion picture and the incompressibility. For the first one, we show that the Pauli–Schrödinger system has a composite fermion nature. For the second one, we end up with a result that has been showed by Karabali and Nair [36] in analyzing the conventional QHE on $S^2$.

4.1 Composite fermion picture

To give an explanation of the filling factors beyond the Laughlin states, Jain has introduced the composite fermion formalism [12]. In fact, they are new kind of particles appeared in condensed matter physics to provide an explanation of the behavior of particles moving in the plane when a strong magnetic field $B$ is present. Particles possessing $2p$, with $p \in N^*$, flux quanta (vortices) can be thought of being composite fermions. One of the most important features of them is they feel effectively a magnetic field of the form

$$B^* = B \pm 2p\Phi_0\rho$$

(74)

where in our convention the unit flux is $\Phi_0 = 2\pi$. Recalling the relation between the filling factor $\nu$ and $B$, one can define a similar quantity for the field $B^*$. This can be written as

$$\nu^* = 2\pi \frac{\rho}{B^*}.$$ 

(75)

It is clear that from (74), the factors $\nu$ and $\nu^*$ can be linked to each other through

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}.$$ 

(76)

This relation has been used to deal with different issues related to QHE in 2DEG. More discussions about the mapping (76) and its applications can be found in [12].
As an immediate consequence of (76), we can map the anomalous FQHE in terms of IQHE in graphene. Indeed, it is easy to see

\[ \nu^G = 2 \frac{2n + 1}{4p(2n + 1) \pm 1}. \]  

(77)

This result has been obtained in different contexts, for instance see [13]. It is obvious that for \( n = 0 \), we obtain Laughlin states and the anomalous IQHE can be recovered by fixing \( p = 0 \). Moreover, (77) tells us that the Jain’s series is quite different from the 2DEG. Therefore, it is interesting to focus on FQHE in graphene.

The above results can be linked to the present work. Otherwise, the composite fermion picture exists in our analysis. Indeed, if one look at the obtained fields in rearranging the Pauli–Schrödinger Hamiltonian, one can notice that both fields \( B^\pm \) behave as effective fields in similar way to the composite fermion field \( B^* \) given in (74). To clarify this statement, firstly recall that after applying an external field \( B \), we finally got \( B^\pm \). Secondly, the Dirac quantization for these field can be deduced from that of \( B \). Thus, we have

\[ k_1 = k - 4, \quad k_2 = k + 4. \]  

(78)

This is should be valid for any integer value \( l \) because their summation gives the whole external magnetic field

\[ k_1 + k_2 = k. \]  

(79)

Thirdly, let us write the fields as

\[ k^\pm = k \pm l \]  

(80)

where we have set \(- \equiv 1\) and \(+ \equiv 2\). Now it becomes clear that there is a shift with respect to \( B \), which is analogue to (74). Therefore, (80) shows that the present system behaves as a collection of composite fermions where \( l \) is attributed to vortices. Moreover, we can define the filling factor \( \nu \) in terms of the quantity \( \nu^\pm \), such as

\[ \nu = \frac{\nu^\pm}{2p\nu^\pm \pm 1}. \]  

(81)

Adopting the definition (75), we can express \( \nu^\pm \) as

\[ \nu^\pm = 2\pi \frac{\rho}{B^\pm} \equiv 2\pi \rho l_B^2 \pm \]  

(82)

where \( 2\pi l_B^2 \) is the Hall droplet area of the composite fermions submitted to \( B^\pm \). At this level, one can give some interpretations to (81) by switching on \( \nu^\pm \) to different values. Indeed, for \( \nu^\pm = 1 \) we recover Laughlin states [22] and for \( \nu^\pm \) integer we end up with Jain sequences [12].

4.2 Graphene LLL

Particles in LLL are confined in a potential that is grand enough to neglect the kinetic energy. This produces a gap such that particles are not allowed to jump to the next level. LLL is rich and contains many interesting features that are relevant in discussing QHE. With this, it is interesting to consider the present system on LLL. Before talking about the graphene LLL for two-sphere, let us make some discussions about LLL of the Landau and Pauli–Schrödinger Hamiltonian’s.
We keep particles in LLL and investigate their basic features. We start by giving the spectrum for one particle in LLL, which can be obtained just by fixing \( m = 0 \) in the previous analysis. This gives the ground state in (53) that has the energy

\[
E_0 = B.
\]  

(83)

This value coincides with that corresponds to the Landau problem on the plane. More discussions about the issue can be found in Karabali and Nair work [36].

One particle spectrum in LLL can be generalized to that for \( M \)-particles. It is obvious that the total energy is given by

\[
E_M = MB.
\]  

(84)

The corresponding wavefunction can be constructed as the Slater determinant. This is

\[
\Psi_M(z) = \epsilon^{i_1 \cdots i_M} \Psi_{i_1}(z_{i_1})\Psi_{i_2}(z_{i_2}) \cdots \Psi_{i_M}(z_{i_M})
\]  

(85)

where each \( \Psi_{i_j}(z_{i_j}) \) has the form given in (53). This is similar to the wavefunction (28) on the plane and corresponds to the filling factor \( \nu = 1 \). Other Laughlin states can be obtained as we have given in (31).

Now let us turn to the Pauli–Schrödinger Hamiltonian \( H_{PS} \). First, we emphasis an important behavior of its spectrum. Indeed, unlike the Landau Hamiltonian, \( H_{PS} \) has an isolated eigenvalue and in this case, the Hilbert space becomes

\[
H_{PS}^D = \left\{ \begin{pmatrix} \Phi_0 \\ 0 \end{pmatrix} , \quad \Phi_0 = (1 + z \cdot \bar{z})^{-\frac{k}{2}} \right\}.
\]  

(86)

It is describing the first component LLL of \( H_{PS} \). In contrast, the second LLL is governed by the energy

\[
E_{\nu}^- = B + 2 \equiv B_2
\]  

(87)

which is not equal to zero. However this is nothing but that corresponding to plane results for a composite fermion subjected to the magnetic field \( B_2 \). Therefore, all discussions reported on QHE in \( S^2 \) can be applied to the second component of \( H_{PS} \).

At this point, we return to analysis the anomalous QHE on \( S^2 \). As we noticed before, the graphene LLL is described by a zero-energy mode that corresponds to the wavefunction (53). Since this is also the same state for the Landau Hamiltonian on \( S^2 \) then we can construct the same Laughlin states (85) for the filling factor \( \frac{1}{1+T} \). Therefore, we have the same results as those obtained in the conventional QHE on \( S^2 \). In fact, we can show that the density of particles is constant and the the probability of finding two particles at the same position is zero. These basically the results obtained by Karabali and Nair [36] in analyzing the conventional QHE on the projective complexes spaces \( \mathbb{C}P^d \), with \( d \) is an integer.

We now underline that there is no difference between the graphene LLL and the 2DEG LLL for a compact surface. Indeed, the definition (75) tells us that the density of particles is an important ingredient. To get QHE, this parameter should be kept constant by varying the magnetic field. For its relevance, we evaluate the density, which is

\[
\rho = \frac{M}{4\pi r^2} = \frac{BM}{4\pi k}
\]  

(88)
for a two-sphere of volume $4\pi r^2$. In the thermodynamic limit, i.e. $(M,k \to \infty)$, it goes to the finite quantity

$$\rho \sim \frac{B}{2\pi}. \quad (89)$$

This is exactly the density of particles on flat geometry and therefore corresponds to the fully occupied state $\nu = 1$.

In QHE, the quantized plateaus come from the realization of an incompressible liquid. This property is important since it is related to the energy. It means that by applying an infinitesimal pressure to an incompressible system the volume remains unchanged \[40\]. This condition can be checked for our system by considering two-point function and integrating over all particles except two. This is

$$I(z_{i1}, z_{i2}) = \int_{S^2} d\mu_s(z_3, z_4, \ldots, z_M) [\Psi_M(z)]^* \Psi_M(z). \quad (90)$$

It is easy to see that $I(z_{i1}, z_{i2})$ reads as

$$I(z_{i1}, z_{i2}) \sim |\Psi_{0,i1}|^2 |\Psi_{0,i2}|^2 - |(\Psi_{0,i1})^* \Psi_{0,i2}|^2. \quad (91)$$

This can also be evaluated in the planar limit as

$$I(z_{i1}, z_{i2}) \sim 1 - \exp \left[-k|\vec{x}_1 - \vec{x}_2|^2\right] \quad (92)$$

which tells us that the probability of finding two particles at the same position is zero, as it should be. This is analogue to the result obtained by Karabali and Nair \[36\] for the conventional FQHE.

The above result showed that there is no difference between the QHE at LLL in both systems: graphene and 2DEG. This suggests to analyze the higher Landau levels and show if there is any difference between them. To reply this inquiry, one may go straightforwardly to generalize the Haldane wavefunction for the conventional QHE to that for graphene.

### 5 SU($N$) Wavefunction

Using different theoretical arguments, some authors suggested a possible FQHE in graphene. As we have seen before, this will be natural if we look at the anomalous IQHE as a product of collective behavior of the composite fermions instead of particles. Moreover, different wavefunctions have been proposed to describe AFQH for different filling factors. Among them, we cite that constructed by Goerbig and Regnault \[17\] to solve some problems brought by other theories like for instance that developed in \[11\]. However, this wavefunction is sharing many common features with our early proposal \[28\]. On the other hand, the Goerbig and Regnault states are a direct extended version of those of the Laughlin \[22\] as well as Halperin \[16\]. These states are also translationally invariant but they are not rotationally. To overcome this problem, one may apply the Haldane \[23\] picture to deal with the anomalous FQHE at the filling factor \[29\]

$$\nu = q_i K_{ij}^{-1} q_j \quad (93)$$

where $K_{ij}$ is an $N \times N$ matrix and $q_i$ is a vector. In fact, this will include different fractions suggested recently for FQHE in graphene and allows us to make contact with different proposals.
5.1 Generic case

To construct a general wavefunction, we use the obtained result so far. In fact, the starting point is to note that from the Haldane wavefunction one can realize that of Halperin on $S^2$. In doing so, we consider two sectors labeled by $(m)$ and $(n)$. This is the case for instance in graphene where there are two subsystems forming a honeycomb. Let us define $\psi^{(m,n)}$ as a tensor product

$$\psi^{(m,n)} = \psi^{(m)} \otimes \psi^{(n)}$$  \hspace{1cm} (94)

where each $\psi^{(m)}$ is given by (53). Assuming that the condition between matrix elements $K_{ij} = K_{ji}$ is fulfilled, a natural way to construct the required wavefunction is

$$|\Psi\rangle = N \prod_{m=1}^{N} \left[ \epsilon^{i_1 \ldots i_M} \psi^{(m)}_{i_1} \psi^{(m)}_{i_2} \ldots \psi^{(m)}_{i_M} \right] K_{mm} - K_{mn} \prod_{n=1}^{N} \left[ \epsilon^{j_1 \ldots j_M} \psi^{(n)}_{i_1} \psi^{(n)}_{i_2} \ldots \psi^{(n)}_{i_M} \right] K_{nn} - K_{mn} N \prod_{m<n} \left[ \epsilon^{k_1 \ldots k_{M+m+n}} \psi^{(m,n)}_{k_1} \psi^{(m,n)}_{k_2} \ldots \psi^{(m,n)}_{k_{M+m+n}} \right] K_{mn} |0\rangle.$$  \hspace{1cm} (95)

We can show that $|\Psi\rangle$ verifies the constraint

$$L_{tot}|\Phi\rangle = 0$$  \hspace{1cm} (96)

where the total angular momenta is given by

$$L_{tot} = \sum_{i} L_{i}.$$  \hspace{1cm} (97)

The components of each $L_{i}$ can be written in terms of the global coordinates $u_{i}$ and $v_{i}$ as

$$L_{i}^+ = u_{i} \frac{\partial}{\partial u_{i}}, \quad L_{i}^- = v_{i} \frac{\partial}{\partial u_{i}}, \quad L_{i}^z = \frac{1}{2} \left( u_{i} \frac{\partial}{\partial v_{i}} - v_{i} \frac{\partial}{\partial u_{i}} \right)$$  \hspace{1cm} (98)

forming a closed Lie algebra of the $SU(2)$ group. On the other hand, novel about this vacuum configuration is that one can interpret the term

$$N \prod_{m<n} \left[ \epsilon^{k_1 \ldots k_{M+m+n}} \psi^{(m,n)}_{k_1} \psi^{(m,n)}_{k_2} \ldots \psi^{(m,n)}_{k_{M+m+n}} \right] K_{mn}$$  \hspace{1cm} (99)

as an inter-layer correlation. In conclusion, our configuration could be a good ansatz for the ground states of FQHE in graphene. This will be clarified soon.

To write the above wavefunction in terms of the global coordinates, one may recall the Haldane realization [23] of the Laughlin states [22] on $S^2$. This is

$$\Psi_{H}^{l} = \prod_{i<j}^{M} (u_{i}v_{j} - u_{j}v_{i})^{2l+1}$$  \hspace{1cm} (100)

where $u$ and $v$ are given by

$$u = \cos \left( \frac{\theta}{2} \right) \exp \left( \frac{i}{2} \varphi \right), \quad v = \sin \left( \frac{\theta}{2} \right) \exp \left( \frac{i}{2} \varphi \right).$$  \hspace{1cm} (101)
These are related to the real coordinates \((x^1, x^2, x^3)\) on \(S^2\) through
\[
x^i = \rho u^1 \sigma^i u
\]  
(102)

where \(\sigma^i\) are the Pauli matrices. Using the standard stereographic mapping, we express the global coordinates in terms of \((z, \bar{z})\) as
\[
u_i = \frac{1}{1 + z_i \cdot \bar{z}_i}, \quad v_i = \frac{\bar{z}_i}{1 + z_i \cdot \bar{z}_i}.
\]  
(103)

At this level, we have all ingredients to do our job. Indeed, we start by defining a new complex variable
\[
\zeta_i = \begin{cases} 
z^{(m)}_i & \text{for } i = 1, \ldots, M \\
z^{(n)}_{i-M} & \text{for } i = M + 1, \ldots, 2M
\end{cases}
\]  
(104)

assuming that the particle numbers are equal, i.e. \(M_1 = M_2 = M\), and recalling the antisymmetric Vandermonde determinant for the fully occupied state
\[
\prod_{i<j} (z_i - z_j) = \text{det} \begin{pmatrix} z_1^{M-j} & \cdots & z_{M-1}^M \end{pmatrix} = \epsilon^{i_1 \cdots i_M} z_{i_1}^0 \cdots z_{i_M}^{M-1}.
\]  
(105)

Therefore in the planar limit, (95) can be projected on the complex plane as
\[
\Psi_{\text{plane}} = \prod_{m=1}^{N} \left[ \epsilon^{i_1 \cdots i_M} \left( z^{(m)}_{i_1} \right)^0 \cdots \left( z^{(m)}_M \right)^{M-1} \right] K_{11} - K_{12} \prod_{n=1}^{N} \left[ \epsilon^{j_1 \cdots j_M} \left( z^{(n)}_{j_1} \right)^0 \cdots \left( z^{(n)}_M \right)^{M-1} \right] K_{22} - K_{12} \prod_{m<n} \left[ \epsilon^{k_1 \cdots k_{2M}} \left( z^{(m)}_{k_1} \right)^0 \cdots \left( z^{(m)}_{k_{2M}} \right)^{2M-1} \right] K_{12} \Psi_0.
\]  
(106)

It can be written in the standard form as
\[
\Psi_{\text{plane}} = \prod_{m=1}^{N} \prod_{i<j} \left( z^{(m)}_i - z^{(m)}_j \right) K_{1m} \prod_{n=1}^{N} \prod_{i<j} \left( z^{(n)}_i - z^{(n)}_j \right) K_{1n} \prod_{m<n} \prod_{i,j} \left( z^{(m)}_i - z^{(n)}_j \right) K_{mn} \Psi_0.
\]  
(107)

This exactly coincides with that constructed by Goerbig and Regnault [17] and what we have proposed [28] in terms of the matrix model and non-commutative Chern–Simson theories. Using the mapping (103), it is easy to show that (95) takes the form
\[
\Psi_{(K_{mm}, K_{nn}, K_{mn})} = \prod_{m=1}^{N} \prod_{i<j} \left( u^{(m)}_i v^{(m)}_j - u^{(m)}_j v^{(m)}_i \right) K_{mm} \prod_{n=1}^{N} \prod_{i<j} \left( u^{(n)}_i v^{(n)}_j - u^{(n)}_j v^{(n)}_i \right) K_{nn}
\]  
(108)

\[
\prod_{m<n} \prod_{i,j} \left( u^{(m)}_i v^{(m)}_j - u^{(n)}_j v^{(n)}_i \right) K_{mn} \Psi_0.
\]  
(109)

It clear that, the first and second part of (108) are nothing but two copies of Haldane wavefunctions for two systems without interaction between each other. However, the third term is showing the inter-layer correlation
\[
\prod_{m<n} \prod_{i,j} \left( u^{(m)}_i v^{(n)}_j - u^{(n)}_j v^{(m)}_i \right) K_{mn}.
\]  

The wavefunction (108) is a good candidate for describing the anomalous FQHE in graphene. Next focusing on the \(N = 4\) case, we will illustrate (108) by giving different applications. For this, we give some configurations those lead to recover some filling factors.
5.2 \( N = 4 \) case

As stated before, the factor 4 has different independent origins and together contribute in the anomalous QHE. In fact, it can be regarded as a manifestation of two spin states as well as two Dirac points.

Illustration 1: Considering the two layers and treating them as additional degrees of freedom, the \( \nu = \frac{1}{2} \) state was predicted by Yoshioka, MacDonald and Girvin [41]. They made a straightforward generalization of the Laughlin states to those with the filling factor

\[ \nu = \frac{2}{k + l} \]  

(110)

where \( k \) and \( l \) are integers. This can be obtained from our generalization on \( S^2 \) by taking the configuration

\[ K = \begin{pmatrix} k & l \\ l & k \end{pmatrix}, \quad q = \begin{pmatrix} 1 & -1 \end{pmatrix} \]  

(111)

which is leading to the wavefunction

\[
\Psi_{(k,l)} = \prod_{m=1}^{4} \prod_{i<j}^{M} \left( u_{i}^{(m)} v_{j}^{(m)} - u_{j}^{(m)} v_{i}^{(m)} \right)^{k} \prod_{n=1}^{4} \prod_{i<j}^{M} \left( u_{i}^{(n)} v_{j}^{(n)} - u_{j}^{(n)} v_{i}^{(n)} \right)^{k} \]  

(112)

\[
\prod_{m<n}^{4} \prod_{i,j}^{M} \left( u_{i}^{(m)} v_{j}^{(m)} - u_{j}^{(m)} v_{i}^{(m)} \right)^{l} \Psi_{0}. \]  

(113)

It is clear that this is corresponding to different filling factors. Indeed, choosing \( k = 3 \) and \( l = 1 \), we recover the FQHE \( \nu = \frac{1}{2} \) state corresponding to

\[
\Psi_{(3,1)} = \prod_{m=1}^{4} \prod_{i<j}^{M} \left( u_{i}^{(m)} v_{j}^{(m)} - u_{j}^{(m)} v_{i}^{(m)} \right)^{3} \prod_{n=1}^{4} \prod_{i<j}^{M} \left( u_{i}^{(n)} v_{j}^{(n)} - u_{j}^{(n)} v_{i}^{(n)} \right)^{3} \]  

(114)

Moreover, \( \nu = \frac{2}{5} \) can be recovered by setting for instance \( k = 2 \) and \( l = 1 \). Other filling factors can be derived in similar way.

Illustration 2: By adopting the Halperin picture [16] for the conventional FQHE, another interesting result can be obtained. Indeed, in the context of single-layered unpolarized QH systems, the labels \( m \) and \( n \) can be considered as an analogue of spin. Following this idea, our graphene system can be seen as mixing layers of particles with spin up and spin down. As a consequence, for \( k = 3 \) and \( l = 2 \), we get the unpolarized Halperin wavefunction with the filling factor \( \frac{2}{5} \)

\[
\Psi_{(3,2)} = \prod_{m=1}^{4} \prod_{i<j}^{M} \left( u_{i}^{(m)} v_{j}^{(m)} - u_{j}^{(m)} v_{i}^{(m)} \right)^{3} \prod_{n=1}^{4} \prod_{i<j}^{M} \left( u_{i}^{(n)} v_{j}^{(n)} - u_{j}^{(n)} v_{i}^{(n)} \right)^{3} \]  

(115)

\[
\prod_{m<n}^{4} \prod_{i,j}^{M} \left( u_{i}^{(m)} v_{j}^{(m)} - u_{j}^{(m)} v_{i}^{(m)} \right)^{2} \Psi_{0}. \]  

(116)

This can be seen as the wavefunction of a system of \( M \)-particles with spin parallel \( m = \uparrow \) and \( M \)-particles with spin anti-parallel \( n = \downarrow \) to the external magnetic field.
Illustration 3: Finally, we give a configuration that allows us to derive other interesting states like $\frac{8}{19}$. Indeed, by requiring

$$K = \begin{pmatrix} k & l \\ l & k \end{pmatrix}, \quad q = \begin{pmatrix} 2 & -2 \end{pmatrix}$$

(115)

we find another sequence of the filling factors

$$\nu = \frac{8}{m+n}.$$  

(116)

This is including the state $\frac{8}{19}$ that may not be described by some theories. It can be recovered by fixing for instance $k = 10$ and $l = 9$.

6 Conclusions

We have analytically analyzed the Dirac Hamiltonian with a magnetic field for particle living on two-sphere $S^2$. To do this, we have introduced the spectral properties of the Pauli–Schrödinger Hamiltonian $H_{PS}^S$ and explored its relationship to the Landau problem. This has been done by making use of an appropriate transformation, which is generated by a $2 \times 2$ matrix. The matrix elements of $H_{PS}^S$ have been seen as two Hamiltonian’s describing two systems submitted to different portion of the external magnetic field. Subsequently, after mapping the Dirac operator in terms of Landau Hamiltonian, it was easy to derive its eigenvalues as well as the corresponding eigenfunctions.

We have shown that the Pauli–Schrödinger system is exhibiting a composite fermion behavior. This has been done by establishing a contact between our subfields $B^\pm$ and those felt by composite fermions in the Jain picture [12]. Moreover, using the stand definition, we have written a filling factor in terms of $B^\pm$ and linked to different sequences. In fact, we have reproduced the Laughlin states for $\nu^\pm = 1$ and those of Jain for $\nu^\pm$ integer.

Before discussing the anomalous fractional quantum Hall effect (AFQHE) on $S^2$, we have made some notes on its partner on plane. For this we have introduced some discussions related the anomalous integer Hall conductivity and its extended version to AFQHE on $\mathbb{R}^2$. Subsequently, we have moved to $S^2$ in order to show what are the similarities and differences between AFQHE in two-dimensional electron gas (2DEG) and graphene. In the beginning we have focused on the lowest Landau level (LLL), i.e. $m = 0$ in Dirac spectrum, where two physical quantities have been evaluated. These concerned the incompressibility as well as the density of particles. These allowed us to conclude that the basics physics in the graphene LLL is the same as in the 2DEG LLL.

To analyze the high Landau levels, we have extended the Haldane wavefunction on $S^2$ to $SU(N)$ states those showing many interesting properties. This has been done by using our early work [28] related to the matrix model and non-commutative Chern–Simons theories. Our states allowed us to generalize their partners on the planar limit [17] and describe different filling factors. Among them, we cite $\frac{8}{19}$ that may not, in some instance, be described by the $SU(4)$ composite fermion wavefunction [11].

The present investigation of the anomalous QHE on $S^2$ can not remain at this level. In fact, many questions can be extracted and solved in different ways. Firstly, one may think to make a group theory analysis of the present results in similar way to that have been reported by Karabali and Nair [30] as
well as our works [37]. Secondly, it might of interest to consider the anomalous effect on the three-sphere $S^3$ by applying an approach used by Nair and Randjbar-Daemi [42] to underline what makes differences with respect to the conventional QHE. Still also other questions where some of them are under preparation.

Acknowledgment

This work was completed during a visit of AJ to High Energy Section of the Abdus Salam International Centre for Theoretical Physics (AS-ICTP). He would like to thank Professor S. Randjbar-Daemi for his kind invitation. The author is indebted to the referee for his constructive comment.

References

[1] For instance see R.E. Prange and S.M. Girvin (editors), “The Quantum Hall Effect”, (Springer, New York 1990).

[2] K.S. Novoselov, A.K. Greim, S.V. Morosov, D. Jiang, M.I. Katsnelson, V.I. Grigorieva, L. Levy, S.V. Dubonos and A.A. Firsov, *Nature* **438** (2005) 197.

[3] Y. Zhang, Y.W. Tan, H.L. Störmer and P. Kim, *Nature* **438** (2005) 201.

[4] Y. Zheng and T. Ando, *Phys. Rev.* **B65** (2002) 245420.

[5] V.P. Gusynin and S.G. Sharapov, *Phys. Rev. Lett.* **95** (2005) 146801.

[6] K. von Klitzing, G. Dorda and M. Pepper, *Phys. Rev. Lett.* **45** (1980) 494.

[7] K. Yang, *Solid State Communications* **143** (2007) 27, cond-mat/0703757

[8] M.I. Katsnelson and K.S. Novoselov, *Solid State Communications* **143** (2007) 3, cond-mat/0703374.

[9] D.C. Tsui, H.L. Störmer and A.C. Gossard, *Phys. Rev. Lett.* **48** (1982) 1559.

[10] K. Yang, S. Das sarma and A.H. MacDonald, *Phys. Rev.* **B74** (2006) 075423.

[11] C. Töke, P.E. Lammert, V.H. Crespi and J.K. Jain *Phys. Rev.* **74** (2006) 235417.

[12] J.V. Jain, *Phys. Rev. Lett.* **63** (1989) 199; *Phys. Rev.* **B41** (1990) 7653; *Adv. Phys.* **41** (1992) 105; O. Heinonen (editor), ”Composite Fermions: A Unified View of Quantum Hall Regime”, (World Scientific, 1998).

[13] N.M.R. Peres, F. Guinea and A.H. Castro Neto, *Phys. Rev.* **B73** (2006) 125411.

[14] C. Töke and J.K. Jain, *SU(4) Composite Fermions in Graphene: New Fractional Quantum Hall States*, cond-mat/0701026.

[15] D.V. Khveshchenko, *Phys. Rev.* **B75** (2007) 153405.
[16] B.I. Halperin, Helv. Phys. Acta. 56 (1983) 75.

[17] M.O. Goerbig and N. Regnault, Phys. Rev. B75 (2007) 241405, cond-mat/0701661.

[18] M.O. Goerbig, R. Moessner and B. Douçot, Phys. Rev. B74 (2006) 161407.

[19] V. Apalkov and T. Chakraborty, Phys. Rev. Lett. 97 (2006) 126801.

[20] C.P. Burgess and B.P. Dolan, Phys. Rev. B76 (2007) 113406, cond-mat/0612269.

[21] J. Gonzalez, F. Guinea and A.H. Vozmediano, Phys. Rev. Lett. 69 (1992) 172; Nucl. Phys. B406 (1993) 771.

[22] R.B. Laughlin, Phys. Rev. Lett. 50 (1989) 1559.

[23] F.D. Haldane, Phys. Rev. Lett. 51 (1983) 605.

[24] I. Shigekawa, J. Func. Anal. 101 (1991) 255.

[25] A. Ghanmi, J. Phys. 38: Math. Gen. (2005) 1917.

[26] R. Comporesi and A. Higuchi, J. Geom. Phys. 20 (1996) 1.

[27] C. Bär, J. Math. Soc. Japan 48 (1996) 69.

[28] A. Jellal and M. Schreiber, J. Phys. 37: Math. Gen. (2004) 3147.

[29] X.G. Wen and A. Zee, Phys. Rev. B46 (1992) 2290.

[30] N.H. Shon and T. Ando, J. Phys. Soc. Jpn. 67 (1998) 2421.

[31] G.W. Semenoff, Phys. Rev. Lett. 53 (1984) 2449.

[32] C.G. Beneventano, P. Giacconi, E.M. Santangelo and R. Soldati, The Quantum Hall Effect in Graphene Samples and the Relativistic Dirac Effective Action, hep-th/0701095.

[33] A. Jellal and Y. Khedif, An Approach for Anomalous Quantum Hall Effect, in progress.

[34] M. Kaku, "Introduction to Superstrings", (Springer, New York, 1988).

[35] B.P. Dolan, JHEP 0305 (2003) 018, hep-th/0304037.

[36] D. Karabali and V.P. Nair, Nucl. Phys. B641 (2002) 533, hep-th/0203264; J. Phys. A39: Math. Gen. (2006) 12735, hep-th/0606161.

[37] M. Daoud and A. Jellal, Nucl. Phys. B764 (2007) 109, hep-th/0605289; Int. J. Geo. Meth. Mod. Phys. 4 (2007) 0407, hep-th/0605290.

[38] A. Jellal, Nucl. Phys. B (2005) 554, hep-th/0505095.

[39] A. Pinzul and A. Stern, Phys. Lett. B512 (2001) 217, hep-th/0103206.
[40] Z.F. Ezawa, "Quantum Hall Effects: Field Theoretical Approach and Related Topics" (World Scientific, Singapore 2000).

[41] D. Yoshioka, A.H. MacDonald and S.M. Girvin, *Phys. Rev. B* **39** (1989) 1932.

[42] V.P. Nair and S. Randjbar-Daemi, *Nucl. Phys. B* **679** (2004) 447, [hep-th/0309212](http://arxiv.org/abs/hep-th/0309212).