Local nonglobal minima for solving large-scale extended trust-region subproblems

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Abstract  We study large-scale extended trust-region subproblems (eTRS) i.e., the minimization of a general quadratic function subject to a norm constraint, known as the trust-region subproblem (TRS) but with an additional linear inequality constraint. It is well known that strong duality holds for the TRS and that there are efficient algorithms for solving large-scale TRS problems. It is also known that there can exist at most one local non-global minimizer (LNGM) for TRS. We combine this with known characterizations for strong duality for eTRS and, in particular, connect this with the so-called hard case for TRS. We begin with a recent characterization of the minimum for the TRS via a generalized eigenvalue problem and extend this result to the LNGM. We then use this to derive an efficient algorithm that finds the global minimum for eTRS by solving at most three generalized eigenvalue problems.

Keywords Trust-region subproblem · Linear inequality constraint · Large-scale optimization · Generalized eigenvalue problem

Mathematics Subject Classification 90C26 · 90C30 · 90C46 · 65F15

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1 Introduction

We study large-scale instances of the extended trust-region subproblem, eTRS

\[
\begin{align*}
\min & \quad q(x) := x^T Ax + 2a^T x \\
\text{s.t.} & \quad g(x) := \|x\|^2 - \delta \leq 0, \\
& \quad \ell(x) := b^T x - \beta \leq 0,
\end{align*}
\]

(eTRS)

where \( A \in \mathbb{S}^n \), the space of real \( n \times n \) symmetric matrices, \( a, b \in \mathbb{R}^n \setminus \{0\} \) and \( \beta \in \mathbb{R}, \delta \in \mathbb{R}_{++} \). Here a linear inequality constraint is added onto the standard trust-region subproblem, TRS. The TRS is an important subproblem in trust-region methods for both constrained and unconstrained problems, e.g. [5]. The eTRS problem extends the TRS and is a step toward solving TRS with a general number of inequality constraints. Such problems would be useful for example in the subproblem of finding search directions for sequential quadratic programming (SQP) methods for general nonlinear programming, e.g., [3].

It is known that, surprisingly, strong duality\(^1\) holds for TRS and the global minimizer can be found efficiently and accurately, even though the objective function is not necessarily convex. The early algorithms for moderate sized problems are based on exploiting the positive semidefinite second order optimality conditions using a Cholesky factorization of the Lagrangian, see e.g., [8,16]. These methods were extended to the large-scale case using a parametrized eigenvalue problem, e.g. [9,10,13,17]. A related problem is finding the local non-global minimizer (LNGM) of TRS if it exists, see [15]. See [5] for more extensive details, applications, and background for TRS.

However, strong duality can fail for the eTRS. This is characterized in [2] for the more general two quadratic constraint problem. We show that this is exactly connected to the so-called hard case for TRS. We use this fact to find an efficient approach for finding the global minimizer for eTRS. Recently, a generalized eigenvalue characterization for the TRS optimum is derived in Adachi et al. [1] based on solving a single generalized eigenvalue problem. This algorithm is shown to be extremely efficient for solving the TRS. In this paper we extend this result for the LNGM optimum using the second largest real generalized eigenvalue of a matrix pencil. This provides an efficient procedure for finding the LNGM. From combining the solutions for TRS and LNGM we derive an efficient algorithm for eTRS.

We include a discussion relating strong duality and stability for eTRS. Extensive numerical tests show that our new algorithm is accurate and can solve large-scale problems efficiently.

Related previous work on strong duality and an eigenvalue approach on eTRS appeared in e.g., [11,12,19,20].

\(^1\) The primal and dual values are equal and the dual value is attained.
1.1 Notation and Preliminaries

We let
\[ \lambda_{\min}(A) = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \]
denote the eigenvalues of \( A \) in nondecreasing order, and \( A = Q \Lambda Q^T \) be the orthogonal spectral decomposition of \( A \) with the diagonal matrix of eigenvalues \( \Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n) \). We let \( q_i \) denote the orthonormal columns of the eigenvector matrix \( Q \).

For \( X \in S^n \), we let \( X \succeq 0, \succ 0 \), denote positive semidefiniteness and definiteness, respectively. In addition, we define the vector of ones, \( e \) of appropriate size and \( \text{Diag}(v) \) be the diagonal matrix formed from the vector \( v \).

It is now well known that, surprisingly, the possibly nonconvex TRS problem has the following characterization of optimality with a positive semidefinite Lagrangian Hessian.

**Theorem 1** (Characterization of Global Minimum of TRS, [8, 16]) Define the Lagrangian of TRS, \( L(x, \lambda) := q(x) + \lambda(||x||^2 - \delta) \).

The vector \( x^* \in \mathbb{R}^n \) is a global optimum of TRS if, and only if, there exists \( \lambda^* \in \mathbb{R} \) such that

\[
\begin{align*}
\frac{1}{2} \nabla_x L(x^*, \lambda^*) &= (A + \lambda^* I)x^* + a = 0, \quad \lambda^* \geq 0 \\
\frac{1}{2} \nabla^2_{xx} L(x^*, \lambda^*) &= A + \lambda^* I \succeq 0 \\
&\quad \|x^*\|^2 - \delta \leq 0 \\
&\quad \lambda(||x^*||^2 - \delta) = 0
\end{align*}
\]

\[ \square \]

Now if \( x^* \) is a global minimizer of TRS and \( \nabla^2_{xx} L(x^*, \lambda^*) \) is singular, then

\[ \lambda^* = -\lambda_1 \quad \text{and} \quad 0 \neq a \in \text{Range}(A + \lambda^* I) = (\text{Null}(A + \lambda^* I))^\perp \]

holds and leads to the following definition given in the literature, e.g., [16].

**Definition 1** (Hard Case) The hard case holds for TRS if \( a \) is orthogonal to the eigenspace corresponding to \( \lambda_1 \), \( \text{Null}(A - \lambda_1 I) \).

In addition, the Slater constraint qualification, SCQ, or strict feasibility, can be assumed without loss of generality for feasible instances of eTRS.

**Lemma 1** The eTRS is feasible, respectively strictly feasible, if, and only if

\[ -\sqrt{\delta}\|b\| \leq \beta, \quad \text{respectively} \quad -\sqrt{\delta}\|b\| < \beta. \]

Moreover, if equality holds on the left in (2), then eTRS has the unique feasible (and so optimal) point \( x^* = -\sqrt{\frac{\beta}{\|b\|}} b \).
Proof Consider the convex strictly feasible problem with compact feasible set \( \min_x \{ x^T b : \|x\| \leq \delta \} \). We can differentiate the Lagrangian to get the optimum primal \( x \) and dual \( \lambda 
\)

\[
0 \neq x = -\frac{1}{2\lambda} b, \quad \lambda > 0.
\]

We now have

\[
x^T x = \frac{1}{(2\lambda)^2} \|b\|^2 = \delta \implies 2\lambda = \frac{\|b\|}{\sqrt{\delta}}.
\]

The result now follows by noting that the linear inequality constraint is

\[
x^T b = -\frac{1}{2\lambda} \|b\|^2 \leq \beta
\]

and then substituting for the value found for \( 2\lambda \).

\[\square\]

We note that if the global solution of \( \text{TRS} \) is feasible for \( e\text{TRS} \) then it is clearly optimal. And from the above, we know that it can be found efficiently using a generalized eigenvalue problem. Therefore from this and Lemma 1 we can make the following assumption for the theoretical part of the paper. (We do not make this assumption for the algorithmic part.)

Assumption 1 We assume in this paper that \( e\text{TRS} \) is strictly feasible and that the global solution of \( \text{TRS} \) is infeasible for \( e\text{TRS} \).

1.2 Outline

We continue in Sect. 2 with the details on the \textit{LNGM}. This includes known results from [15] and one of the main results of this paper in Theorem 5, the necessary conditions for a \textit{LNGM} using the second largest real generalized eigenvalue of a matrix pencil. In Sect. 3.1 we discuss necessary and sufficient conditions for strong duality to hold for \( e\text{TRS} \). A discussion on the stability of \( e\text{TRS} \) and resulting stability of our approach is given in Sect. 3.2.

The various optimality conditions for \( e\text{TRS} \) are applied in Sect. 4. Included in this section are outlines of the algorithms for an efficient numerical procedure to find the global optimum of \( e\text{TRS} \). Our numerical results appear in Sect. 5. We provide concluding remarks in Sect. 6.

2 On a local non-global minimizer (LNGM) of TRS

2.1 Background on LNGM

Let \( x^* \) be a global optimal solution of \( e\text{TRS} \). Then the linear constraint is either inactive \( b^T x^* < \beta \) or active \( b^T x^* = \beta \). In the former case, \( x^* \) must be a local (not
global by Assumption 1) minimizer of TRS, i.e., we can have \( x^* \) being a local non-
global minimizer, LNGM, of TRS. We now provide some background on the LNGM.

Lemma 2 If \( A \succeq 0 \), then no LNGM exists.

Proof This is immediate since \( A \succeq 0 \) implies that TRS is a convex problem, i.e., a
problem where local minima are global minima. (It also follows from Theorem 2
below, since \( 0 \leq \lambda^* < -\lambda_1 \).

Therefore, in this section we assume that \( \lambda_1 < 0 \). We continue and present some
known results related to LNGM. Then following the results in [1], we show that the
LNGM can be computed via a generalized eigenvalue problem.

Theorem 2 (Necessary Conditions for LNGM, [15]) Let \( x^* \) be a LNGM. Choose
\( V \in \mathbb{R}^{n \times (n-1)} \) such that \( \left[ \frac{1}{\|x^*\|} x^* \ V \right] \) is orthogonal. Then there exists a unique
\( \lambda^* \in (\max\{0, -\lambda_2\}, -\lambda_1)^2 \) such that
\[
V^T (A + \lambda^* I) V \succeq 0, \quad (A + \lambda^* I) x^* = -a, \quad ||x^*||^2 = \delta. \tag{3}
\]

\( \square \)

Corollary 1 If the so-called hard case holds for TRS, i.e., \( a \) is orthogonal to the
eigenspace corresponding to \( \lambda_1 \), then no LNGM exists.\(^3\)

Proof The proof is given in [15, Lemma 3.2]. We include a separate proof to emphasize
that a stronger result holds as is given in Corollary 2 below.

After an orthogonal transformation if needed, we can assume for simplicity that
\( A = \text{Diag}(\lambda) \) is a diagonal matrix. To obtain a contradiction, we assume that \( a \) is
orthogonal to the smallest eigenvector, \( a^T q_1 = 0 \). From this assumption we have that
the first element \( a_1 = 0 \). From (3) this implies that the first element \( x^*_1 = 0 \) which
yields that the first eigenvector given by the first unit vector \( e_1 \) satisfies \( e_1 = Vu \), for
some \( u \). We have \( u^T V^T (A + \lambda^* I) Vu = \lambda_1 + \lambda^* < 0 \). This contradicts the second
order semidefiniteness condition in (3).

\( \square \)

Corollary 2 If the weak form of the hard case holds for TRS, i.e., \( a \) is orthogonal to some
eigenvector corresponding to \( \lambda_1 \), then no LNGM exists.

Proof The proof of Corollary 1 just needed one eigenvector orthogonal to \( a \).

Now let
\[
\phi(\lambda) := \|(A + \lambda I)^{-1} a\|^2.
\]

\(^2\) We have added the fact that \( \lambda^* > 0 \) whereas only nonnegativity is given in [15, Theorem 3.1]. Strict
complementarity is proved in [14, Prop. 3.5]. In fact, it is easy to see by the second order conditions that
strict complementarity holds as well for the global minimum for TRS in the \( \lambda_1 < 0 \) case.

\(^3\) The hard case arises in algorithms for TRS. The singularity that can arise requires special treatment, see
e.g., [16]. In fact, it can be handled by a shift and deflation step, see [7]. Another approach to handling the
hard case equivalent to the easy case is given in [18].
For
\[
\lambda \in (\max\{0, -\lambda_2\}, -\lambda_1),
\]
Theorem 2 shows that the equation \( \phi(\lambda) = \delta \) is a necessary condition for a LNGM. Furthermore, using the eigenvalue decomposition of \( A \) we have
\[
\phi(\lambda) = \sum_{i=1}^{n} \frac{(q_i^T a)^2}{(\lambda_i + \lambda)^2},
\]
\[
\phi'(\lambda) = -2 \sum_{i=1}^{n} \frac{(q_i^T a)^2}{(\lambda_i + \lambda)^3},
\]
\[
\phi''(\lambda) = 6 \sum_{i=1}^{n} \frac{(q_i^T a)^2}{(\lambda_i + \lambda)^4}. \tag{4}
\]
The Eq. (4) imply that the function \( \phi(\lambda) \) is strictly convex on \( \lambda \in (\max\{0, -\lambda_2\}, -\lambda_1) \) and so it has at most two roots in the interval \( (\max\{0, -\lambda_2\}, -\lambda_1) \). The following theorem states that only the largest root can correspond to a LNGM.

**Theorem 3** ([15, Theorem 3.1])

1. If \( x^* \) is a LNGM, then (3) holds with a unique \( \lambda^* \in (\max\{0, -\lambda_2\}, -\lambda_1) \) and with \( \phi'(\lambda^*) \geq 0 \).
2. There exists at most one LNGM.

\( \square \)

### 2.2 Characterization using a generalized Eigenvalue problem

We now consider the problem of efficiently computing the LNGM. Due to the results in Sect. 2.1 we can make the following two assumptions.

**Assumption 2** 1. The smallest two eigenvalues of \( A \) satisfy
\[
\lambda_1 < \min\{0, \lambda_2\}.
\]
2. The hard case does not hold, i.e., \( a \) is not orthogonal to the eigenspace corresponding to \( \lambda_1 \) which here is \( \text{span}(q_1) \) the span of the eigenvector of \( \lambda_1, a^T q_1 \neq 0 \).

To the best of our knowledge, the only algorithm for computing the LNGM is the one by Martinez [15] which tries to find the largest root of the equation \( \phi(\lambda) = \delta \) for \( \lambda \in (\max\{0, -\lambda_2\}, -\lambda_1) \) via an iterative algorithm. Each step of his approach requires solving an indefinite system of linear equations which can be expensive for large-scale instances. In what follows, we follow on the ideas of [1] and present a new algorithm that shows that the LNGM can be computed efficiently by a generalized eigenvalue problem. Our result is then used to solve large instances of eTRS.
Recently, Adachi et al. [1] designed an efficient algorithm for TRS which solves just one generalized eigenvalue problem. They consider the following $2n \times 2n$ regular symmetric matrix pencil which has $2n$ finite eigenvalues.\(^4\)

$$M(\lambda) = \begin{bmatrix} -I & A + \lambda I \\ A + \lambda I & -\frac{1}{\delta}aa^T \end{bmatrix}.$$ 

We can rephrase Theorem 1 as $x_g^*$ is a global optimal solution of TRS if, and only if, the following system is consistent.

\begin{align}
(A + \lambda_g^* I)x_g^* &= -a, \\
A + \lambda_g^* I &\succeq 0, \quad \lambda_g^* \geq 0 \quad \text{and unique}, \\
||x_g^*||^2 &\leq \delta, \\
\lambda_g^* (||x_g^*||^2 - \delta) &= 0.
\end{align}

Lemma 3 (Generalized Eigenvalue of Pencil, [1]) For every Lagrange multiplier $\lambda_g^* \neq 0$, satisfying the stationarity condition (5a) with equality in the quadratic constraint (5c), we have $\det M(\lambda_g^*) = 0$, i.e., $\lambda_g^*$ is a generalized eigenvalue of the pencil $M(\lambda)$.

\textbf{Proof} The Lemma is proved in [1]. We include a shorter proof.

For simplicity we denote $D = A + \lambda I$ and let $\lambda = \lambda_g^*$ be a Lagrange multiplier satisfying (5a). We can rewrite (with $x = x_g^*$)

\[^{6}\] 

\begin{equation}
\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -I & \frac{1}{\delta}xx^T \\ \frac{1}{\delta}xx^T & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -I & D \\ \frac{1}{\delta}xx^T & \frac{1}{\delta}xx^T \end{bmatrix} = M(\lambda).
\end{equation}

The result follows by observing that the vector $0 \neq \begin{bmatrix} x \\ x \end{bmatrix} \in \text{Null} \left( \begin{bmatrix} -I & \frac{1}{\delta}xx^T \\ \frac{1}{\delta}xx^T & 0 \end{bmatrix} \right)$. \(\square\)

\textbf{Corollary 3} The set of real generalized eigenvalues of $M(\lambda)$ is nonempty. Moreover, if $\det M(\lambda) = 0, \lambda \in \mathbb{R}$, then either $-\lambda$ is an eigenvalue of $A$ or

$$\det \left( \begin{bmatrix} -I & \frac{1}{\delta}xx^T \\ I & -\frac{1}{\delta}xx^T \end{bmatrix} \right) = 0, \quad x = -(A + \lambda I)^{-1}a.$$ 

\textbf{Proof} This follows immediately from Lemma 3 and from (6) in its proof. \(\square\)

The following theorem shows that the global optimal solution of TRS can be obtained via computing an eigenpair of the pencil $M(\lambda)$.

\[^{4}\] The objective function in [1] is $1/2$ our objective function and $I$ in the pencil is a general $B \succ 0$. 

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Theorem 4 (Eigenvalue Characterization of TRS, [1]) Let \((x^*_g, \lambda^*_g)\) be a global optimal solution of TRS with \(||x^*_g||^2 = \delta\). Then the multiplier \(\lambda^*_g\) is equal to the largest real eigenvalue of \(M(\lambda)\). Furthermore, if \(\lambda^*_g > -\lambda_1\), then \(x^*_g\) can be obtained by

\[x^*_g = -\frac{\delta}{a^Ty_2} y_1,\]

where \(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^n\) is an eigenvector for \(M(\lambda^*_g)\) and also we have \(a^Ty_2 \neq 0\).

Theorem 4 establishes that the largest real eigenvalue of \(M(\lambda)\) is the Lagrange multiplier associated with the global optimal solution of TRS. In the following theorem, we prove that if TRS has a LNGM, then the corresponding Lagrange multiplier is the second largest real eigenvalue of \(M(\lambda)\). This is the main result of this section.

Theorem 5 (Eigenvalue Characterization of LNGM) Let \(x^*\) be a LNGM. Then the corresponding Lagrange multiplier \(\lambda^*\) is equal to the second largest real eigenvalue of \(M(\lambda)\). Moreover, \(x^*\) can be computed as \(x^* = -\frac{\delta}{a^Ty_2} y_1\), where \(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\) is an eigenvector for \(M(\lambda^*)\) and we also have \(a^Ty_2 \neq 0\).

Proof From Theorem 2 we have \(\lambda^* \in (\max\{0, -\lambda_2\}, -\lambda_1)\). Moreover, \(||x^*||^2 = \delta\) and it follows from Lemma 3 that \(\lambda^*\) is an eigenvalue of \(M(\lambda)\), i.e. \(\det M(\lambda^*) = 0\).

We know that the hard case does not hold, see Corollary 1. Therefore, by Theorem 4 and the optimality conditions in (5), we get that the largest real eigenvalue of \(M(\lambda)\) is the unique multiplier associated with the global optimal solution of TRS and is the unique root of equation \(\phi(\lambda) - \delta = 0\) in \((-\lambda_1, \infty)\). Moreover, it follows from Theorem 3 that \(\lambda^*\), the multiplier corresponding to the LNGM, is positive and is the largest root of the equation \(\phi(\lambda) - \delta = 0\) in \((-\lambda_2, -\lambda_1)\). Next, note that Lemma 3 implies that \(-\lambda_1\) is not an eigenvalue of \(M(\lambda)\). From the interval considerations for the optimum of TRS and eTRS, this establishes that \(\lambda^*\) is the second largest real eigenvalue of \(M(\lambda)\).

Now let \(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\) be an eigenvector for \(\lambda^*\) for \(M(\lambda)\). We have

\[(A + \lambda^* I) y_2 = y_1, \quad (7)\]
\[(A + \lambda^* I) y_1 = \frac{1}{\delta} a a^T y_2. \quad (8)\]

We first show that \(a^Ty_2 \neq 0\). Suppose by contradiction that \(a^Ty_2 = 0\). Then, since \((A + \lambda^* I)\) is nonsingular, we obtain first that \(y_1 = 0\) from the second equation which then implies \(y_2 = 0\) from the first equation, i.e., we have \(y_1 = y_2 = 0\), a contraction of the fact that \(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\) is an eigenvector. Hence, \(a^Ty_2 \neq 0\). Thus, (8) implies that \(x^* = -\frac{\delta}{a^Ty_2} y_1\) satisfies

\[(A + \lambda^* I) x^* = -a. \quad (9)\]

Moreover, we have
\[ ||x^*||^2 = \frac{\delta^2}{(a^T y_2)^2} y_1^T y_1 = \frac{\delta^2}{(a^T y_2)^2} y_2^T (A + \lambda^* I)(A + \lambda^* I)^{-1} \frac{aa^T}{\delta} y_2 = \delta. \]

\[ \square \]

3 Strong duality and stability for eTRS

3.1 Characterization of strong duality for eTRS

We now provide conditions for strong duality for eTRS. We show that loss of strong duality is directly connected to the hard-case for TRS. In addition, we show that failure of strong duality requires a specific eigenvalue configuration if the global optimum has the linear constraint active.

A necessary and sufficient condition for strong duality of the problem of minimizing a quadratic function over two quadratic inequality constraints, when one of them is strictly convex, is presented in [2]. Since eTRS is a special case of this result, we immediately have the following.

**Theorem 6** (Characterization strong duality eTRS) Strong duality fails for eTRS if, and only if, there exist multipliers \( \lambda, \mu \) such that the following hold:

1. \( \lambda > 0 \) and \( \mu > 0 \);
2. \( A + \lambda I \succeq 0 \), and \( \text{rank}(A + \lambda I) = n - 1 \);
3. The following system of linear equations is consistent.

\[
2(A + \lambda I)x_i = -2a - \mu b, \quad x_i^T x_i = \delta, \quad i = 1, 2, \\
(b^T x_1 - \beta)(b^T x_2 - \beta) < 0.
\] (10)

**Proof** This follows immediately from the characterization in [2, Thm 5.2] for two quadratic constraints, since the affine constraint is a special case of a quadratic constraint. \( \square \)

It is interesting to translate this theorem under our special assumptions and the language of the hard case. In fact, we see that loss of strong duality is directly connected to the hard case in TRS. Note that the hard case is identified by obtaining a feasible solution that satisfies all the optimality conditions except for complementary slackness.

**Corollary 4** Consider the Lagrangian dual of eTRS in the parametric form

\[ d^*_eTRS := \max_{\mu \geq 0} g(\mu), \]

where the dual function, \( g(\mu) \), with \( \lambda \) implicit in \( g \), is a parametric TRS, TRS\( _\mu \), expressed in the dual form

\[ g(\mu) := \max_{\lambda \geq 0} \min_x \left[ L(x, \lambda) + \mu b^T x \right] - \mu \beta. \] (TRS\( _\mu \))
Then strong duality fails for $eTRS$ if, and only if, there exists $\mu > 0$ such that the parametrized $TRS_\mu$ has a hard case solution $x^*_\mu$ that satisfies all the optimality conditions except for complementary slackness, i.e.,

$$\|x^*_\mu\|^2 < \delta, \quad b^T x^*_\mu = \beta.$$  

Proof Since $eTRS$ is a convex problem if $\lambda_1 \geq 0$, without loss of generality we assume that $\lambda_1 < 0$. This implies that the optimal Lagrange multiplier for $TRS_\mu$ satisfies $\lambda > 0$ and moreover there exists an optimal solution on the boundary of the trust region.

The optimality conditions for the parametrized problem at $\mu$ are given as parts of the three conditions in Theorem 6. And the two points $x_i, i = 1, 2$ are on opposite sides of the affine manifold for the linear constraint. We note that necessarily $0 \neq v := x_1 - x_2 \in \text{Null}(A + \lambda I)$. Therefore $v$ is the required eigenvector and this is equivalent to finding the convex combination $x^* = \alpha x_1 + (1 - \alpha)x_2, \alpha \in (0, 1)$ with $b^T x^* = \beta$ and necessarily $\|x^*\| < \delta$.

Therefore, the parametrized $TRS$ has multiple optimal solutions and the hard case holds for the corresponding $TRS_\mu$, i.e., $2a + \mu b \in \text{Range}(A - \lambda_1 I)$, $\lambda^* = -\lambda_1$. ⊓⊔

More details on $\|x^*_\mu\|^2 < \delta$ and the relation with the minimum norm solution $\hat{x} := \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b)$ are discussed in Sect. 4.2.1, where we define the Moore-Penrose generalized inverse, $C^\dagger$. In fact, necessarily $\|x^*_\mu\|^2 = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b) + v$ for $v \in \text{Null}(A - \lambda_1 I)$.

Remark 1 Corollary 4 illustrates the geometry of strong duality in terms of the parametrized $TRS_\mu$. If we start with $\mu = 0$ and let $\mu$ increase, then the corresponding optimal solution of $TRS_\mu$ moves on the boundary of the trust region. If we encounter the boundary of the linear constraint first then strong duality holds. On the other hand if we encounter the hard case at $\mu > 0$ and if we can move using the nullspace $\tilde{x} = x_\mu + v$ so that $\|\tilde{x}\|^2 < \delta, b^T \tilde{x} = \beta$, then strong duality fails.

This means that given a $TRS$ we can characterize all the $b, \beta$ where strong duality would fail using the characterization of the hard case.

Lemma 4 Strong duality fails for $eTRS$ if the LNGM is optimal.

Proof The Lagrangian of $TRS$ is given in (1). The Lagrangian dual of $TRS$ is max$_{\lambda \geq 0} \min_x L(x, \lambda)$. Since the inner problem is a minimization of a quadratic, for it to be finite we get the necessary (hidden) condition that the Hessian of the quadratic $A + \lambda I \succeq 0$. This contradicts the Lagrange multiplier condition for LNGM given in Theorem 3. ⊓⊔

Theorem 7 Suppose that strong duality holds for $eTRS$ and that the optimal solution of $eTRS$ is $x^*$. Then $b^T x^* = \beta$ and $x^*$ is a global optimal solution of $TRS$ after projection onto the linear manifold of the linear constraint.

Proof If $b^T x^* < \beta$, then either $x^*$ is the global minimizer or a LNGM. Since strong duality fails for the LNGM, we conclude that it must be the global minimizer of the $TRS$. But our Assumption 1 means that the global minimizer is infeasible for $eTRS$. 

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If the linear inequality is active, then we have a **TRS** problem after a projection onto the linear manifold and we obtain the global minimizer on this affine manifold.

There are three possibilities for a global optimum for **eTRS**. We know that strong duality fails for **eTRS** if the LNGM is the optimum for **eTRS**; while strong duality holds if the global optimum of **TRS** is also optimal for **eTRS**. The only other possibility is that the global optimum of **eTRS** occurs with the linear constraint active. We now see that it requires a very restrictive eigenvalue configuration for strong duality to fail if the linear constraint is active.

**Theorem 8** Suppose that $x^*$ solves **eTRS** with $b^T x^* = \beta$. Suppose that $\lambda_2 < 0$. Then strong duality holds for **eTRS**.

**Proof** As above, we can construct a full column rank matrix $W$ to represent $\text{Null}(b^T)$. From interlacing of eigenvalues we get that $\lambda_{\text{min}}(W^T A W) < 0$. Therefore, there exists an optimal solution on the boundary of the trust region for the projected problem, i.e., complementary slackness holds. This means that the optimum for **eTRS** is also on the boundary of the trust-region constraint. We can therefore add a multiple of the identity to the Hessians of the original problem and obtain a convex equivalent problem. This shows that strong duality holds. The dual problem is equivalent to perturbing the Hessians to $A - \lambda_1 I$ as long as we subtract the constant $\lambda_1 \beta$.

3.2 Stability for **eTRS**

We now see that the **eTRS** is stable with respect to perturbations in the data.

**Lemma 5** Let $x^*$ be the optimal solution for **eTRS**. Then, under Assumptions 1, and 2, we have that the linear independence constraint qualification, **LICQ**, holds at $x^*$. Moreover, $x^*$ is the unique optimal solution if the second constraint is inactive. Thus unique Lagrange multipliers $\lambda_1^*, \lambda_2^*$ exist for the two constraints, respectively.\(^5\)

**Proof** Suppose that the second constraint is inactive $b^T x^* = \beta$. First we note that the first constraint is active by the $\lambda_1(A) < 0$ assumption and the gradients of the active constraint is $2x^* \neq 0$. Therefore the **LICQ** holds. This immediately implies that $\lambda_1^* > 0$ exists. Moreover, both $x^*$ and the optimal Lagrange multiplier $\lambda^*$ are unique by the $\lambda_1(A) < \lambda_2(A)$ assumption.

If the second constraint is active $b^T x^* = \beta$, then $x^*$ is the optimal solution of the projected problem. If the first constraint is inactive, we are done as $\{b\}$ is a linearly independent set. And it is clear from the geometry that if both constraints are active then the gradients $\{b, 2x^*\}$ are linearly dependent only if strict feasibility fails, a contradiction. Therefore, **LICQ** holds and the multipliers are unique.

**Corollary 5** The **eTRS** is a stable problem (for the objective value) with respect to right-hand side perturbations in the constraints.

\(^5\) The *projected* **eTRS**, $\text{TRS}_{\text{proj}}$ is the **TRS** restricted to the active linear constraint. The optimum does not have to be unique for the $\text{TRS}_{\text{proj}}$, i.e., when the hard case does not hold for **TRS**, it can still hold for the projected problem.
Proof. This follows from standard results in sensitivity analysis, e.g., [6, Sect. 2.3], since the Lagrange multipliers are unique, LICQ is satisfied, and the feasible set is compact.

Remark 2 We note that under the Assumptions 1 and 2, these results on stability along with standard sensitivity results on eigenvalue algorithms imply that our approach is a robust method for solving eTRS.

In addition, strict complementarity can fail for eTRS. If the LNGM is the optimal solution for eTRS, then one can perturb the linear constraint till it becomes active. It is therefore a redundant constraint illustrating that the corresponding Lagrange multiplier can be zero. This would then be a degenerate problem and perturbing the linear constraint further can make the projected trust-region optimal point the optimum for eTRS, i.e., the result is a jump in the optimal solution.

4 Algorithm and subproblems for eTRS

We now describe our proposed method to solve eTRS in Algorithm 1. This finds the global optimal solution for the general problem of eTRS. We include the details about the global minimizer for TRS and the details for the subproblems that need to be solved. We do not assume that the global minimizer of TRS is infeasible in the details of our algorithm, i.e., our algorithm solves the general case.

4.1 Main algorithm

The previous material suggests the following Algorithm 1 for eTRS. Without loss of generality, by Lemma 1, we can assume that strict feasibility holds.

In addition, we see that the cost of the algorithm in the worst case is to find \( \lambda_1, \lambda_2 \) and the eigenvector \( q_1 \) for \( \lambda_1 \); check for strong duality; find the TRS and projected TRS optima or the LNGM and the projected TRS optima.

| Algorithm 1 (Solve the General (strictly feasible) eTRS) |
|--------------------------------------------------------|
| INPUT: \( A \in S^n, a, b \in \mathbb{R}^n, \beta \in \mathbb{R} \) with \(-\sqrt{\delta} < \beta < \beta\). |
| INITIALIZATION: Solve the symmetric eigenvalue problem for \( \lambda_1, \lambda_2 \) and eigenvector \( q_1 \) for \( \lambda_1 \).
| IF: \( \lambda_1 \geq 0 \) or \( \lambda_1 = \lambda_2 \), THEN Strong duality holds; solve TRS for \( x \).
| IF: \( x \) is feasible, THEN it is opt. STOP.
| ELSE: Solve the projected TRS problem for \( x \); it is opt. STOP. |
| ELSE: Check the strong duality condition for eTRS. |
| IF: strong duality holds, THEN solve TRS for \( x \).
| IF: \( x \) is feasible, THEN it is opt. STOP.
| ELSE: Solve the projected TRS problem for \( x \); it is opt. STOP. |
| ELSE: Solve for the projected TRS and the LNGM if it exists; discard LNGM if it is not feasible; choose the \( x \) as the best of the remaining solutions; it is opt. STOP. |
| END: |
| END: |
| OUTPUT: \( x \) is optimizer of eTRS. |

Recall that, if the LNGM exists then we can use Theorem 5 and find it efficiently via the second largest real eigenvalue of the matrix pencil. The other subproblems are now discussed.

\[\square\] Springer
4.2 Subproblems

4.2.1 Verifying strong duality

To specify the value of $\mu$ in Theorem 6, first notice that, for given $\mu$, system (10) is consistent if, and only if, $q_1^T (2a + \mu b) = 0$ where recall $q_1$ is a normalized eigenvector for $\lambda_1$. Next, let us consider the following cases:

1. $q_1^T b = 0$: In this case, we show that strong duality holds for $e_{TRS}$. We show this by contradiction. Suppose that strong duality does not hold for $e_{TRS}$. Then system (10) has two solutions $x_1$ and $x_2$ satisfying $x_i^T x_i = \delta$, $i = 1, 2$, and $(b^T x_1 - \beta)(b^T x_2 - \beta) < 0$ for some $\mu > 0$. Moreover, we know that the solutions $x_1$ and $x_2$ necessarily have the form $x_1 = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b) + z_1$ and $x_2 = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b) + z_2$ where $z_i$, for $i = 1, 2$, is an eigenvector corresponding to $\lambda_1$. By the fact that $b$ is orthogonal to the eigenspace of $\lambda_1$ (if $\lambda_1$ has multiplicity one), we have $b^T x_1 - \beta = b^T x_2 - \beta$, a contradiction to the fact that $(b^T x_1 - \beta)(b^T x_2 - \beta) < 0$, i.e., we have strong duality for $e_{TRS}$.

2. $q_1^T b \neq 0$: In this case, consistency of system (10) and specifically the stationarity equations therein, yields $q_1^T (2a + \mu b) = 0$. This then implies that necessarily $\mu = \frac{-2a q_1^T}{q_1^T b}$. If $\mu = 0$, it follows from Theorem 6 that $e_{TRS}$ enjoys strong duality. If $\mu > 0$, then strong duality does not hold for $e_{TRS}$ if, and only if, system (10) for $\mu = \frac{-2a q_1^T}{q_1^T b}$ has two solutions $x_1$ and $x_2$ satisfying $x_i^T x_i = \delta$, $i = 1, 2$, and $(b^T x_1 - \beta)(b^T x_2 - \beta) < 0$.

To verify whether strong duality holds we suppose that $x_i$, $i = 1, 2$ are as defined in Theorem 6. Clearly, $x_i = x_p + \alpha_i q_1$, $x_p = \frac{1}{2}(A - \lambda_1 I)^\dagger(-2a - \mu b)$ and $\alpha_i$, $i = 1, 2$, are roots of the following quadratic equation.

$$\alpha^2 + 2\alpha q_1^T x_p + x_p^T x_p - \delta = 0.$$ 

The main task in finding $x_i$, $i = 1, 2$, is computing $x_p$. In the sequel, we show that $x_p$ is indeed the solution of a symmetric positive definite linear system. To see this, note that $q_1^T (2a + \mu b) = 0$, we have

$$(A + \gamma q_1 q_1^T - \lambda_1 I)^{-1}(-2a - \mu b) = Q(\Lambda + \gamma e_1 e_1^T - \lambda_1 I)^{-1}Q^T(-2a - \mu b)$$

$$= Q(\Lambda - \lambda_1 I)^\dagger Q^T(-2a - \mu b)$$

$$= (A - \lambda_1 I)^\dagger(-2a - \mu b),$$

where $\gamma$ is a positive constant and $e_1$ is the first unit vector. This implies that $x_p$ can be computed efficiently by applying the conjugate gradient algorithm to the following positive definite system:
\[
2(A + \gamma q_1 q_1^T - \lambda_1 I)x_p = (-2a - \mu b).
\]

However, we note that the perturbation with \( \gamma q_1 q_1^T \) is not required since the right-hand side \((-2a - \mu b) \in \text{Range}(A - \lambda_1 I)\). The MATLAB \texttt{pcg} works fine even though the matrix is singular.

### 4.2.2 Solving the TRS subproblem

The main work of the algorithms lie in solving generalized eigenvalue problems. For the TRS, we use the method of [1] that solves the scaled TRS

\[
\min \frac{1}{2} x^T A x + a^T x \\
\quad x^T B x \leq \delta,
\]

where \( B \) is a positive definite matrix. The algorithm computes one generalized eigenpair and is able to handle the hard case efficiently. Specifically, it is shown that the optimal Lagrange multiplier corresponding to the solution of (11) is the largest real eigenvalue of the \( 2n \times 2n \) matrix pencil \( \tilde{M}(\lambda) = M_0 + \lambda M_1 \), where

\[
\tilde{M}(\lambda) = M_0 + \lambda M_1, \quad M_0 = \begin{bmatrix}
-B & A \\
A & -\frac{aa^T}{\delta}
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
O_{n \times n} & B \\
B & O_{n \times n}
\end{bmatrix}.
\]

As above we have an equivalent result to Lemma 3 that every nonzero KKT multiplier is a generalized eigenvalue of the pencil, \( \det(\tilde{M}(\lambda)) = 0 \).

**Lemma 6** (Generalized Eigenvalue of Pencil, [1, Lemma 3.1]) *For every nonzero KKT multiplier \( \lambda_g^* \neq 0 \) for (11) with equality in the quadratic constraint we have \( \det(\tilde{M}(\lambda_g^*)) = 0 \), i.e., \( \lambda_g^* \) is a generalized eigenvalue of the pencil \( \tilde{M}(\lambda) \).* \( \square \)

**Algorithm 2** (Solve scaled TRS (11), [1].)

1. Solve \( Ax_0 = -a \) by the conjugate gradient algorithm and keep \( x_0 \) if it is feasible, i.e., if \( x_0^T B x_0 \leq \delta \).
2. Compute \( \lambda_g^* \), the largest generalized eigenvalue of the symmetric regular pencil \( M_0 + \lambda M_1 \), and a corresponding eigenvector \( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \), i.e.,

\[
\begin{bmatrix}
-B & A \\
A & -\frac{aa^T}{\delta}
\end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -\lambda_g^* \begin{bmatrix} O_{n \times n} & B \\
B & O_{n \times n}
\end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{12}
\]

3. If \( \|y_1\| \leq \tau \) (default is \( \tau = 10^{-4} \)), then the hard case is detected; run Steps 4 to 6. Else go to Step 7.
4. Compute \( H := (A + \lambda_g^* B + \alpha \sum_{i=1}^d B v_i v_i^T B) \) where \( V = [v_1, ..., v_d] \) is a basis of \( \text{Null}(A + \lambda_g^* B) \) that is \( B \)-orthogonal, i.e., \( V^T B V = I \), \( d = \dim(\text{Null}(A + \lambda_g^* B)) \) and \( \alpha \) is an arbitrary positive scalar.
5. Solve \( Hq = -a \) by the conjugate gradient algorithm.
6. Take an eigenvector \( v \) computed above, and find \( \eta \) such that \( (q + \eta v)^T B(q + \eta v) = \delta \) and return \( x^* = q + \eta v \) as global optimal solution of (11).
7. Set \( x_1 = -\operatorname{sign}(a^T y_2) \sqrt{\frac{\|y_1\|^2}{\|y_1\|^2 + \|y_2\|^2}} \).
8. The global optimal solution of (11) is either \( x_1 \) or \( x_0 \), whichever gives the smallest objective value.
4.2.3 Solving the projected TRS subproblem

We can eliminate the equality constraint $b^T x = \beta$ to solve the projected TRS. For ease of exposition only, we assume that

$$|b_1| \geq |b_2| \geq \ldots \geq |b_r| > 0 = b_{r+1} = \ldots = b_n.$$  

In order to find a basis of $\text{Null}(b^T)$, we define $\bar{b} := (b_2^{-1} \ldots b_r^{-1})^T$ and the matrix

$$W := \begin{bmatrix} -b_1^{-1} e_r^T & 0^T_{n-r} \\ \text{Diag}(b) & 0 \\ 0 & I_{n-r} \end{bmatrix} \in \mathbb{R}^{n \times (n-1)}.$$  

Define a particular solution, $\hat{x}$ satisfying $b^T \hat{x} = \beta, \|\hat{x}\|^2 < \delta$. We choose

$$\hat{x} = \begin{cases} 0, & \text{if } \beta = 0 \\ \frac{\beta}{\|b\|^2} b, & \text{if } \beta \neq 0. \end{cases} \quad (13)$$

Then it is clear that

$$b^T x = \beta \iff x = \hat{x} + W y, \quad \text{for some } y \in \mathbb{R}^{n-1}.$$  

We can now substitute for $x$ into $eTRS$ and eliminate the linear equality constraint. The objective function becomes

$$(\hat{x} + W y)^T A (\hat{x} + W y) + 2 a^T (\hat{x} + W y) = \left[ y^T (W^T A W) y + 2 \left(W^T (a + A \hat{x})\right)^T y \right] + \left[ (A \hat{x} + 2a)^T \hat{x} \right].$$

The constraint becomes

$$y^T (W^T W) y + 2(W^T \hat{x})^T y \leq \delta - \hat{x}^T \hat{x}.$$  

We get the following equivalent problem in the case that the linear constraint is active.

$$\begin{align*}
\min & \ y^T (W^T A W) y + 2(W^T (a + A \hat{x}))^T y \\
\text{s.t.} & \ y^T (W^T W) y + 2(W^T \hat{x})^T y \leq \delta - \hat{x}^T \hat{x} \quad (TRS_{\text{proj}})
\end{align*}$$

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6 In the numerical tests we did not order the elements of $b$ but did divide by the largest component in magnitude.

7 Some scaling issues can arise here. It is preferable to take $\hat{x}$ strictly feasible for the trust-region constraint.

8 We note that the choice $\hat{x} = 0$ simplifies the nonhomogeneous $nTRS$ below.
We let
\[ B := W^T W, \quad \hat{A} := W^T AW, \quad \hat{a} := W^T (a + A \hat{x}), \quad \hat{b} := 2(W^T \hat{x}), \quad \hat{\delta} = \delta - \hat{x}^T \hat{x}. \]

Therefore, we need to solve the nonhomogeneous TRS, \( nTRS \):

\[
\begin{align*}
\min & \quad x^T \hat{A} x + 2\hat{a}^T x \\
\text{s.t.} & \quad x^T B x + \hat{b}^T x \leq \hat{\delta}.
\end{align*}
\]

(\( nTRS \))

By the change of variables
\[ x \leftarrow y + g, \quad \text{with} \quad 2Bg = -\hat{b}, \]
we get
\[
x^T \hat{A} x + 2\hat{a}^T x = (y + g)^T \hat{A}(y + g) + 2\hat{a}^T(y + g) = y^T \hat{A} y + 2(\hat{A}g + \hat{a})^T y + \text{constant}.
\]

and
\[
x^T B x + \hat{b}^T x = (y + g)^T B(y + g) + \hat{b}^T(y + g) = y^T B y + (2Bg + \hat{b})^T y + g^T Bg + b^T g = y^T B y + g^T Bg + b^T g.
\]

We write \( nTRS \) as the scaled homogeneous TRS, \( sTRS \):

\[
\begin{align*}
\min & \quad y^T \hat{A} y + 2(\hat{A}g + \hat{a})^T y \\
\text{s.t.} & \quad y^T B y \leq \hat{\delta} - g^T Bg - \hat{b}^T g.
\end{align*}
\]

(\( sTRS \))

This means we can directly apply the approach in [1] where the scaled TRS is solved using the generalized eigenvalue approach.

Remark 3 When we solve for the optimum in \( sTRS \) using Algorithm 2 we do not form \( B \) explicitly but exploit the rank one update structure of \( W \) and its inverse. This means we can exploit the original sparsity in \( A \) in the objective function and in the,\ now scaled, \( I \) in the original trust-region constraint when performing the matrix-vector multiplications needed for \( eigs \) in MATLAB. Let
\[
\tilde{B} := \text{Diag}(\tilde{b}), \quad \tilde{e} := \begin{pmatrix} e^T_{r-1} \\ 0_{n-r} \end{pmatrix}.
\]

\[ \text{We note again here that if } \beta = 0 \text{ then we can choose } \hat{x} = 0 \text{ and the homogeneous TRS is maintained.} \]
Note that

\[
B = \begin{bmatrix} \bar{B}^2 & 0 \\ 0 & I_{n-r} \end{bmatrix} + b_1^{-2} \bar{e} \bar{e}^T
\]

\[
= \left\{ \begin{bmatrix} \bar{B} & 0 \\ 0 & I_{n-r} \end{bmatrix} + w w^T \right\} \left\{ \begin{bmatrix} \bar{B} & 0 \\ 0 & I_{n-r} \end{bmatrix} + w w^T \right\}
\]

\[
= B^{1/2} B^{1/2}.
\]

We can then find the appropriate rank one update of \([\bar{B} 0 0 I_{n-r}]\) to find the inverse \(B^{-1/2}\). Therefore we can take a diagonal congruence of both sides of (12) and obtain a simple right-hand side of the generalized eigenvalue problem.

5 Numerical results

We now present our numerical results to illustrate the efficiency of the new algorithm. We compare with the exact second order cone and semidefinite programming \(\text{SOCP/SDP}\) reformulation in [4] on some small instances, as this reformulation is not able to handle large instances. This approach contains both second order cone and semidefinite programming constraints.

Hence, for large instances we just report the solution obtained by our new algorithm. All computations were done in MATLAB 8.6.0.267246 (R2015b) on a Dell Optiplex 9020 with 16 GB RAM with Windows 7. To solve the \(\text{SOCP/SDP}\) reformulation, we used SeDuMi 1.3, [21].

5.1 Four classes of test problems

We divide our tests into four classes I,II,III,IV, of test problems. Classes I and II are used to compare our algorithm with the \(\text{SOCP/SDP}\) formulation with no assumptions on the failure of strong duality. These are medium sized problems since the \(\text{SOCP/SDP}\) formulation could not solve the large-scale case. Classes III and IV are used for instances where strong duality holds and fails, respectively. We use different means of guaranteeing that a \(\text{LNGM}\) exists. In each class and dimension we take the average of the results from 10 random instances.

5.1.1 Class I

In this section, we apply our algorithm and the \(\text{SOCP/SDP}\) reformulation to some \(\text{eTRS}\) instances for which the \(\text{LNGM}\) of the corresponding \(\text{TRS}\) is a good candidate for the global optimal solution of \(\text{eTRS}\). To generate the desirable random instances of

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10 The codes are available at URL: \url{www.math.uwaterloo.ca/~hwolkowi//henry/reports/eTRScodesmay2016.d/}.
eTRS, we proceed as follows. First we construct a TRS problem having a local non-global minimizer based on Theorem 5. Then we add the inequality constraint $b^T x \leq \beta$ to enforce that the global minimizer of TRS is infeasible but that the LNGM remains feasible.

Comparison with the SOCP/SDP reformulation is given on some small instances in Table 1. Note that as for our algorithm, the SOCP/SDP approach is successful on instances where strong duality may fail, [4]. We follow [1] and report the relative objective function difference

$$\frac{|q(x^*) - q(x_{best})|}{|q(x_{best})|}$$

accuracy measure, \hspace{1cm} (14)

where $x^*$ is the computed solution by each method and $x_{best}$ is the solution with the smallest objective value between the two algorithms. We report this to five decimals. For each dimension, we have generated 10 eTRS instances. We report the dimension $n$, and the average values of the relative accuracy, the run time in cpu-seconds and we include the time taken for checking the strong duality property of eTRS in Algorithm 1. Moreover, for each dimension, # LNGM denotes the number of test problems among the 10 instances for which our algorithm has detected the LNGM of the corresponding TRS as a global optimal solution of eTRS. It should be noted that the algorithm which gets $x_{best}$ varies from problem to problem and since we are reporting the average of 10 runs, we can have a positive accuracy in both of the accuracy columns of Table 1 (and Tables 2, 5). However, we see that the new algorithm always obtained the better minimum and so the accuracy in that column is zero.

The generated matrix $A$ in the first class of test problems is dense and therefore we do not have $n$ large as the final aim of our method is solving large sparse eTRS instances.

5.1.2 Class II

In this section we test our algorithm on both small and large sparse eTRS instances that are generated differently than in Section 5.1.1. We take advantage of the following lemma from [15] to generate eTRS instances where the LNGM is the global minimum for eTRS.
Table 2 Class II: comparison with SOCP/SDP reformulation; density 0.1

| n   | Accuracy (14) | CPUsec | # LNGM |
|-----|---------------|--------|--------|
| 100 | 0.0           | 4.2588e-09 | 1.697e+00 | 9 |
| 200 | 0.0           | 1.0547e-08 | 1.167e+01 | 6 |
| 300 | 0.0           | 9.3557e-09 | 4.694e+01 | 7 |
| 400 | 0.0           | 3.3775e-09 | 1.287e+02 | 5 |

Lemma 7 (Lemma 3.4 of [15]) Consider the TRS problem. Suppose that $\lambda_1 < 0$, has multiplicity one, and the TRS is an easy case instance. Then there exists $\delta_0 > 0$ such that TRS admits a local non-global minimizer for all $\delta > \delta_0$. \hfill $\square$

The second class of test problems are generated as follows. We generate a random sparse symmetric matrix $A$ via $A = \text{sprandsym}(n, \text{density})$. Next we generate the vector $a$ via $a = \text{randn}(n, 1)$ and make sure that $q_1^T a \neq 0$ where $q_1$ is the eigenvector corresponding to $\lambda_1$, i.e., we get the easy case TRS. Then we set $\delta = 4000$ following Lemma 7. Finally we set $b = 0.9x^*$ and $c = ||b||^2$ to cut off $x^*$, the global optimal solution of the generated TRS instance. We have compared our algorithm with the SOCP/SDP reformulation on the test problems of small size in both runtime and solution accuracy. For each dimension, we have generated 10 eTRS instances and the corresponding numerical results are presented in Table 2, where we report the dimension of the problem $n$, the algorithm run time and the time taken for checking the strong duality property of eTRS, and the accuracy at termination averaged over the 10 random instances. Moreover, for each dimension, # LNGM denotes the number of test problems among 10 instances for which our algorithm has detected the LNGM of the corresponding TRS as a global optimal solution of eTRS. Furthermore, we verified that in all cases, there was a positive duality gap for generated eTRS instances. As in the previous test problems the new algorithm finds higher accuracy solutions in significantly shorter time than the SOCP/SDP reformulation.

Now we turn to solving large sparse eTRS instances. For this class we just report the results of our algorithm since the SOCP/SDP approach could not handle problems of this size. Let $x^*$ be a global optimal solution of eTRS and $\lambda^*$ the corresponding Lagrange multiplier. Depending on the context of the linear constraint being not active or being active, we denote the error in the stationarity condition by $\text{KKT1} := ||(A + \lambda^* I)x^* + a||_\infty$ or the corresponding conditions for the scaled active case, respectively; and the error in complementary slackness by $\text{KKT2} := \lambda^*(||x^*||^2 - \delta)$ or the corresponding condition for the scaled linear active case, respectively. For each dimension, we have generated 10 eTRS instances. In both cases the global optimal solution of eTRS is obtained from solving generalized eigenvalue problems. Numerical results are presented in Table 3.

The following lemma is useful in generating test problems for the next two classes.

Lemma 8 (Generating LNGM) Let $A \in \mathbb{S}^n$ and suppose that $\lambda_1 < \min\{0, \lambda_2\}$. Then there exists a linear term $a$ for which the eigenvector associated with $\lambda_1$ is the LNGM.
Table 3  Class II: large instances; density 0.0001

| n     | Opt. Cond. | CPUsec | # LNGM |
|-------|------------|--------|--------|
|       |            |        |        |
| 10000 | 1.4085e-08 | 1.087  | 4      |
| 20000 | 1.3465e-10 | 2.506  | 6      |
| 40000 | 1.9584e-09 | 10.343 | 2      |
| 60000 | 1.9876e-10 | 13.694 | 4      |
| 80000 | 1.8937e-10 | 26.768 | 5      |
| 100000| 8.5902e-11 | 29.225 | 2      |

Table 4  Class III: density 0.0001

| n     | Opt. Cond. | CPUsec |        |
|-------|------------|--------|--------|
|       |            |        |        |
| 10000 | 5.4076e-14 | 0.313  | 0.118  |
| 20000 | 3.1243e-14 | 0.731  | 0.242  |
| 40000 | 2.0866e-12 | 2.279  | 0.721  |
| 60000 | 8.9301e-14 | 3.827  | 1.448  |
| 80000 | 4.5073e-14 | 5.998  | 2.333  |
| 100000| 9.7731e-14 | 9.727  | 3.820  |

Proof Let \( \mu \in (\max \{0, -\lambda_2\}, -\lambda_1) \). Set \( a = -(A + \mu I_n)q_1 \) where \( q_1 \) is the eigenvector for \( \lambda_1 \) with \( ||q_1||^2 = \delta \). Then for this choice we have the first order stationary conditions. Now let \( \text{Range}(W) = \text{Null}(q_1 q_1^T) \). Then \( W^T (A + \mu I_n)W = \text{diag}(\lambda_2 + \mu, \ldots, \lambda_n + \mu) \). Due to the choice of \( \mu \), the diagonal matrix has all diagonal elements positive. Thus we have the positive definiteness of the reduced Hessian. This implies that \( q_1 \) is the LNGM. \( \square \)

5.1.3 Class III

In this section, we consider a class of large sparse eTRS instances for which strong Lagrangian duality holds while the corresponding TRS has a LNGM which is feasible for eTRS. We generate the TRS using the previous Lemma 8 and set \( b = (A - \lambda_1 I)x \) where \( x = \text{rand}(n, 1) \). This means that \( b^T q_1 = 0 \) implying that we have strong duality property for generated eTRS instances.

Now let \( x^* \) be a global optimal solution of eTRS. Then either \( b^T x^* < \beta \) or \( b^T x^* = \beta \). Since strong duality holds, in the former case, \( x^* \) is the global minimizer of the corresponding TRS. We define KKT1 and KKT2 as the previous section. For each dimension, we have generated 10 eTRS instances and the corresponding numerical results are presented in Table 4. We see that we solve all instances to high accuracy in a short time.
5.1.4 Class IV

For this class also we follow the above Lemma 8 to generate TRS having LNGM. We follow the same procedure as in Section 5.1.3 to obtain $A$, $a$, $\delta$ and LNGM but we set $b = x^* - x_l$ and $\beta = b^T (0.9x_l + 0.1x^*)$ to cut off $x^*$ but leave $x_l$ feasible where $x^*$ and $x_l$ are the global optimal solution and LNGM of the corresponding TRS, respectively. The two Tables 5 and 6 are for small and large instances, respectively. The small instances allow for a comparison with the SOCP/SDP formulation.

6 Conclusion

In this paper we have derived a new necessary condition for the local non-global optimal solution LNGM of the TRS that is based on the second largest real generalized eigenvalue of a matrix pencil. This is then used to derive an efficient algorithm for finding the global minimizer of the extended TRS, the eTRS. We have presented numerical tests to show that our method far outperforms current methods for eTRS. And our method solves large sparse problems which are too large for current methods to be applied. We have included discussions on a characterization of when strong duality holds for eTRS as well as details on the stability of the problem.

It is well known that TRS is important for unconstrained trust region methods, restricted Newton methods, for unconstrained minimization; as well it is important for general minimization algorithms such as sequential quadratic programming (SQP).
methods. For SQP methods it is customary to solve a standard quadratic programming problem for the search direction after using something akin to a quasi-Newton method to guarantee convexity of the objective function. The eTRS we have studied can be viewed as a step toward solving a TRS with multiple linear constraints for the search direction in SQP methods.

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References

1. Adachi, S., Iwata, S., Nakatsukasa, Y., Takeda, A.: Solving the trust region subproblem by a generalized eigenvalue problem. Technical report, Mathematical Engineering, The University of Tokyo, (2015)
2. Ai, W., Zhang, S.: Strong duality for the CDT subproblem: a necessary and sufficient condition. SIAM J. Optim. 19(4), 1735–1756 (2008)
3. Boggs, P.T., Tolle, J.W.: Sequential quadratic programming. Acta Numerica. 4(4), 1–51 (1995)
4. Burer, S., Anstreicher, K.M.: Second-order-cone constraints for extended trust-region subproblems. SIAM J. Optim. 23(1), 432–451 (2013)
5. Conn, A.R., Gould, N.I.M., Toint, PhL: Trust-Region Methods. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2000)
6. Fiacco, A.V.: Introduction to Sensitivity and Stability Analysis in Nonlinear Programming. Mathematics in Science and Engineering, vol. 165. Academic Press, Orlando (1983)
7. Fortin, C., Wolkowicz, H.: The trust region subproblem and semidefinite programming. Optim. Methods Softw. 19(1), 41–67 (2004)
8. Gay, D.M.: Computing optimal locally constrained steps. SIAM J. Sci. Stat. Comput. 2, 186–197 (1981)
9. Gould, N.I.M., Daniel, P., Robinson, P., Sue Thorne, H.: On solving trust-region and other regularised subproblems in optimization. Math. Progr. Comput. 2(1), 21–57 (2010)
10. Hager, W.W., Park, S.: Global convergence of SSM for minimizing a quadratic over a sphere. Math. Comp. 74(251), 1413–1423 (2005)
11. Hsia, Y., Sheu, R.-L.: Trust region subproblem with a fixed number of additional linear inequality constraints has polynomial complexity. Report, Beihang University, Beijing, China (2013)
12. Jeyakumar, V., Li, G.Y.: Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. Math. Progr. 147(1–2, Ser. A), 171–206 (2014)
13. Lampe, J., Rojas, M., Sorensen, D.C., Voss, H.: Accelerating the LSTRS Algorithm. SIAM J. Sci. Comput. 33(1), 175–194 (2011)
14. Lucidi, S., Palagi, L., Roma, M.: On some properties of quadratic programs with a convex quadratic constraint. SIAM J. Optim. 8(1), 105–122 (1998). (Electronic)
15. Martínez, J.M.: Local minimizers of quadratic functions on Euclidean balls and spheres. SIAM J. Optim. 4(1), 159–176 (1994)
16. Mord, J.J., Sorensen, D.C.: Computing a trust region step. SIAM J. Sci. Stat. Comput. 4, 553–572 (1983)
17. Rendl, F., Wolkowicz, H.: A semidefinite framework for trust region subproblems with applications to large scale minimization. Math. Program. 77(2, Ser. B), 273–299 (1997)
18. Rojas, M., Santos, S.A., and Sorensen, D.C.: A new matrix-free algorithm for the large-scale trust-region subproblem. SIAM J. Optim. 11(3):611–646 (2001) (Electronic)
19. Salahi, M., Fallahi, S.: Trust region subproblem with an additional linear inequality constraint. Optim. Lett. 10(4), 821–832 (2016)
20. Salahi, M., Taati, A.: A fast eigenvalue approach for solving the trust region subproblem with an additional linear inequality. Comput. Appl. Math. (2016). doi: 10.1007/s40314-016-0347-3
21. Sturm, J.F.: Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. Optim. Methods Softw. 11/12(1–4), 625–653 (1999). http://sedumi.ie.lehigh.edu