ON APPROXIMATION PROPERTIES FOR NON-LINEAR INTEGRAL OPERATORS

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ABSTRACT. We investigate the problem of pointwise convergence of the family of non-linear integral operators

\[ L_\lambda(f, x) = \int_a^b f^m(t)K_{\lambda,m}(x, t)dt, \]

where \( \lambda \) is a real parameters, \( K_{\lambda,m}(x, t) \) is non-negative kernel and \( f \) is the function in \( L_1(a, b) \). We consider two cases where \((a, b)\) is a finite interval and when is the whole real axis.

1. Introduction

In [6] the concept of singularity was extended to cover the case of nonlinear integral operators,

\[ T_wf(s) = \int_G K_w(t - s, f(t))dt, \quad s \in G \]

the assumption of linearity of the operators being replaced by an assumption of a Lipschitz condition for \( K_w \) with respect to the second variable. Recently, Swiderski and Wachnicki [7] investigated the pointwise convergence of the operators \( T_wf \) in \( L_p(-\pi, \pi) \) and \( L_p(\mathbb{R}) \) at the points, where \( x_0 \) is a point of continuity and a Lebesque point of \( f \).

In [2], Karsli studied both the pointwise convergence and the rate of pointwise convergence of above operators on a \( \mu - \text{generalized} \) Lebesque point for \( f \in L_1(a, b) \) as \( (x, \lambda) \to (x_0, \lambda_0) \). In [1], it is studied the rate of convergence at a point \( x \), which has a discontinuity of the first kinds as \( \lambda \to \lambda_0 \). In [4] they obtained convergence results and rate of approximation for functions belonging to BV-spaces by mean of nonlinear convolution integral operators.

The aim of the article is to obtain pointwise convergence results for a family of non-linear operators of the form

\[ L_\lambda(f, x) = \sum_{m=1}^N \int_a^b f^m(t)K_{\lambda,m}(x, t)dt \]

where \( K_{\lambda,m}(x, t) \) is a family of kernels depending on \( \lambda \). We study convergence of the family (1) at every Lebesque point of the function \( f \) in the spaces of \( L_1(a, b) \) and \( L_1(-\infty, \infty) \).

Now we give the following definition.

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**Definition 1.** (Class A): We take a family \((K_{\lambda})_{\lambda \in \Lambda}\) of functions \(K_{\lambda,m}(x,t) : R \times R \to R\). We will say that the function \(K_{\lambda}(x,t)\) belongs the class A, if the following conditions are satisfied.

a) \(K_{\lambda,m}(x,t)\) is a non-negative function defined for all \(t \in (a, b)\) and \(\lambda \in \Lambda\).

b) As function of \(t\), \(K_{\lambda,m}(x,t)\) is non-decreasing on \([a, x]\) and non-increasing on \([x, b]\) for any fixed \(x\).

c) For any fixed \(x\), \(\lim_{\lambda \to \infty} b \int_{a}^{b} K_{\lambda,m}(x,t) dt = C_m\).

d) For every \(m, 1 \leq m \leq N\) and every \(y \neq x\), \(\lim_{\lambda \to \infty} K_{\lambda,m}(x,y) = 0\).

2. Main Result

We are going to prove the family of non-linear integral operators (1) with the positive kernel convergence to the functions \(f \in L_1(a,b)\)

**Theorem 1.** Suppose that \(f \in L_1(a,b)\) and \(f\) is bounded on \((a,b)\). If non-negative the kernel \(K_{\lambda,m}\) belongs to Class A, then, for the operator \(L_\lambda(f,x)\) which is defined in (1)

\[ \lim_{\lambda \to \infty} L_\lambda(f,x_0) = \sum_{m=1}^{N} C_m f^m(x_0) \]

holds at every \(x_0\)-lebesgue point of \(f\) function

**Proof.** For integral (1) we can write

\[ L_\lambda(f,x_0) - \sum_{m=1}^{N} C_m f^m(x_0) = \sum_{m=1}^{N} \int_{a}^{b} f^m(t) - f^m(x_0) K_{\lambda,m}(x_0,t) dt \]

\[ + \sum_{m=1}^{N} f^m(x_0) \left[ \int_{a}^{b} K_{\lambda,m}(x_0,t) dt - C_m \right] \]

and in view of a)

\[ \left| L_\lambda(f,x_0) - \sum_{m=1}^{N} C_m f^m(x_0) \right| \leq \sum_{m=1}^{N} \int_{a}^{b} \left| f^m(t) - f^m(x_0) \right| K_{\lambda,m}(x_0,t) dt \]

\[ + \sum_{m=1}^{N} \left| f^m(x_0) \right| \left[ \int_{a}^{b} K_{\lambda,m}(x_0,t) dt - C_m \right] \]

\[ = I_1(x_0, \lambda) + I_2(x_0, \lambda). \]

It is sufficient to show that terms on right hand side of the last inequality tend to zero as \(\lambda \to \infty\). By property c), it is clear that \(I_2(x_0, \lambda)\) tends to zero as \(\lambda \to \infty\).

Now we consider \(I_1(x_0, \lambda)\). For any fixed \(\delta > 0\), we can write \(I_1(x_0, \lambda)\) as follow.
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\[ I_1(x_0, \lambda) = \sum_{m=1}^{N} \left[ \int_{a}^{x_0-\delta} + \int_{x_0}^{x_0+\delta} + \int_{x_0}^{b} \right] |f^m(t) - f^m(x_0)| K_{\lambda,m}(x_0, t) dt \]

(2)

\[ = I_{11}(x_0, \lambda, m) + I_{12}(x_0, \lambda, m) + I_{13}(x_0, \lambda, m) + I_{14}(x_0, \lambda, m) \]

Firstly we shall calculate \( I_{11}(x_0, \lambda, m) \), that’s

\[ I_{11}(x_0, \lambda, m) = \sum_{m=1}^{N} \int_{a}^{x_0-\delta} |f^m(t) - f^m(x_0)| K_{\lambda,m}(x_0, t) dt. \]

By the condition b), we have

\[ I_{11}(x_0, \lambda, m) \leq \sum_{m=1}^{N} K_{\lambda,m}(x_0, x_0 - \delta) \left\{ \int_{a}^{x_0-\delta} |f^m(t)| dt + \int_{a}^{x_0-\delta} |f^m(x_0)| dt \right\} \]

and

\[ \leq \sum_{m=1}^{N} K_{\lambda,m}(x_0, x_0 - \delta) \left\{ \|f^m\|_{L^1(a,b)} + |f^m(x_0)| (b - a) \right\} \]

In the same way, we can estimate \( I_{14}(x_0, \lambda, m) \). From property b)

\[ I_{14}(x_0, \lambda, m) \leq \sum_{m=1}^{N} K_{\lambda,m}(x_0, x_0 + \delta) \left\{ \int_{x_0}^{b} |f^m(t)| dt + \int_{x_0}^{b} |f^m(x_0)| dt \right\} \]

\[ \leq \sum_{m=1}^{N} K_{\lambda,m}(x_0, x_0 + \delta) \left\{ \|f^m\|_{L^1(a,b)} + |f^m(x_0)| (b - a) \right\}. \]

(4)

On the other hand, Since \( x_0 \) is a lebesque point of \( f \), for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ \int_{x_0}^{x_0+h} |f(t) - f(x_0)| dt < \varepsilon h \]

(5)

and

\[ \int_{x_0-h}^{x_0} |f(t) - f(x_0)| dt < \varepsilon h \]

(6)

for all \( 0 < h \leq \delta \). Now let’s define a new function as follows,

\[ F(t) = \int_{x_0}^{t} |f(u) - f(x_0)| du. \]

Then from (5), for \( t - x_0 \leq \delta \) we have

\[ F(t) \leq \varepsilon (t - x_0). \]
Also, since $f$ is bounded, there exists $M > 0$ such that
\[|f^m(t) - f^m(x_0)| \leq |f(t) - f(x_0)| M\]
is satisfied. Therefore, we can estimate $I_{13}(x_0, \lambda, m)$ as follows.
\[
I_{13}(x_0, \lambda, m) \leq M \sum_{m=1}^{N} \int_{x_0}^{x_0 + \delta} |f(t) - f(x_0)| \, K_{\lambda,m}(x_0,t) \, dt
\]
\[
\leq M \sum_{m=1}^{N} \int_{x_0}^{x_0 + \delta} K_{\lambda,m}(x_0,t) \, dF(t).
\]
We apply integration by part, then we obtain the following result.
\[
|I_{13}(x_0, \lambda, m)| \leq M \sum_{m=1}^{N} \left\{ F(x_0 + \delta, x_0) K_{\lambda,m}(x_0 + \delta, x_0) + \int_{x_0}^{x_0 + \delta} F(t) d(-K_{\lambda,m}(x_0,t)) \right\}.
\]
Since $K_{\lambda,m}$ is decreasing on $[x_0, b]$, it is clear that $-K_{\lambda,m}$ is increasing. Hence its differential is positive. Therefore, we can write
\[
|I_{13}(x_0, \lambda, m)| \leq M \sum_{m=1}^{N} \left\{ \varepsilon \delta K_{\lambda,m}(x_0 + \delta, x_0) + \varepsilon \int_{x_0}^{x_0 + \delta} F(t) (-K_{\lambda,m}(x_0,t)) \right\}.
\]
Integration by parts again we have the following inequality
\[
|I_{13}(x_0, \lambda, m)| \leq \varepsilon M \sum_{m=1}^{N} \int_{x_0}^{x_0 + \delta} K_{\lambda,m}(x_0,t) \, dt
\]
\[
\leq \varepsilon M \sum_{m=1}^{N} \int_{x_0}^{b} K_{\lambda,m}(x_0,t) \, dt. \tag{7}
\]
Now, we can use similar method for evaluation $I_{12}(x_0, \lambda, m)$. Let
\[
G(t) = \int_{t}^{x} |f(y) - f(x)| \, dy.
\]
Then, the statement
\[
dG(t) = -|f(t) - f(x_0)| \, dt.
\]
is satisfied. For $x_0 - t \leq \delta$, by using (6), it can be written as follows
\[
G(t) \leq \varepsilon |x_0 - t|
\]
Hence, we get
\[
I_{12}(x_0, \lambda, m) \leq M \sum_{m=1}^{N} \int_{x_0 - \delta}^{x_0} |f(t) - f(x_0)| K_{\lambda,m}(x_0,t) \, dt.
\]
Then, we shall write
\[
|I_{12}(x_0, \lambda, m)| \leq M \sum_{m=1}^{N} \left[ - \int_{x_0 - \delta}^{x_0} K_{\lambda,m}(x_0,t) \, dG(t) \right].
\]
By integration of parts, we have
\[
|I_{12}(x, \lambda, m)| \leq M \sum_{m=1}^{N} \left\{ G(x - \delta K_{\lambda,m}(x - \delta, x) + \int_{x-\delta}^{x} G(t) d_t(K_{\lambda,m}(x, t)) \right\}.
\]

From (6), we obtain
\[
|I_{12}(x_0, \lambda, m)| \leq M \sum_{m=1}^{N} \left\{ \varepsilon \delta K_{\lambda,m}(x_0, x_0 - \delta) + \varepsilon \int_{x_0 - \delta}^{x_0} (x_0 - t) d_t(K_{\lambda,m}(x_0, t)) \right\}.
\]

By using integration of parts again, we find
\[
|I_{12}(x_0, \lambda, m)| \leq \varepsilon M \sum_{m=1}^{N} \int_a^b K_{\lambda,m}(x_0, t) dt.
\]

Combined (7) and (8), we get
\[
|I_{12}(x_0, \lambda, m)| + |I_{13}(x_0, \lambda, m)| \leq 2\varepsilon M \sum_{m=1}^{N} \int_a^b K_{\lambda,m}(x_0, t) dt.
\]

Finally, from (3), (4) and (9), the terms on right hand side of these inequalities tend to 0 as \( \lambda \to \infty \). That’s
\[
\lim_{\lambda \to \infty} L_{\lambda}(f, x_0) = \sum_{m=1}^{N} C_m f^m(x_0).
\]

Thus, the proof is completed. \( \square \)

In this theorem, especially it may be \( a = -\infty \) and \( b = \infty \). In this case, we can give the following theorem.

**Theorem 2.** Let \( f \in L_1(-\infty, \infty) \) and \( f \) is bounded. If non-negative the kernel \( K_{\lambda,m} \) belongs to Class A and satisfies also the following properties,

\[
\lim_{\lambda \to \infty} \int_{-\infty}^{x-\delta} K_{\lambda,m}(t, x) dt = 0
\]

and

\[
\lim_{\lambda \to \infty} \int_{x+\delta}^{\infty} K_{\lambda,m}(t, x) dt = 0,
\]

then the statement
\[
\lim_{\lambda \to \infty} L_{\lambda}(f, x) = f(x).
\]

is satisfied at almost every \( x \in \mathbb{R} \).
According to the conditions d), (10) and (11), we find that

\[ A \]

integration is written the form

\[ A \]

It is sufficient to show that

\[ A \]

\[ \lambda \rightarrow 0 \] as

\[ A \]

\[ A \]

\[ A \]

\[ \lambda \rightarrow \infty. \]

We can write

\[ \lambda \rightarrow \infty. \]

Proof. We can write

\[ |L_\lambda(f, x) - \sum_{m=1}^{N} C_m f^m(x)| \leq \sum_{m=1}^{N} \int_{-\infty}^{\infty} |f^m(t) - f^m(x)| K_{\lambda,m}(x, t) dt \]

\[ + \sum_{m=1}^{N} |f^m(x)| \int_{-\infty}^{\infty} K_{\lambda,m}(x, t) dt - C_m | \]

\[ = A_1(x, \lambda) + A_2(x, \lambda). \]

It is clear that \( A_2(x, \lambda) \rightarrow 0 \) as \( \lambda \rightarrow \infty. \)

For a fixed \( \delta > 0 \), we divide the integral \( A_1(x, \lambda) \) of the form

\[ A_1(x, \lambda) = \sum_{m=1}^{N} \left[ \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{\infty} \right] |f^m(t) - f^m(x)| K_{\lambda,m}(x, t) dt \]

\[ = A_{11}(x, \lambda, m) + A_{12}(x, \lambda, m) + A_{13}(x, \lambda, m) + A_{14}(x, \lambda, m). \]

\( A_{12}(x, \lambda, m) \) and \( A_{13}(x, \lambda, m) \) integrals are calculated as above the proof. For proof, it is sufficient to show that \( A_{11}(x, \lambda, m) \) and \( A_{14}(x, \lambda, m) \) tend to zero as \( \lambda \rightarrow \infty. \)

Firstly, we consider \( A_{11}(x, \lambda, m) \). Since \( f \) is bounded and by the property b), this integration is written the form

\[ A_{11}(x, \lambda, m) \leq M \sum_{m=1}^{N} \int_{-\infty}^{x-\delta} |f(t) - f(x)| K_{\lambda,m}(x, t) dt \]

\[ \leq M \sum_{m=1}^{N} K_{\lambda,m}(x, x-\delta) \left\{ \int_{-\infty}^{x-\delta} |f(t)| \right\} + M |f(x)| \sum_{m=1}^{N} \int_{-\infty}^{x-\delta} K_{\lambda,m}(x, t) dt \]

\[ \leq ||f||_{L_1(-\infty, \infty)} M \sum_{m=1}^{N} K_{\lambda,m}(x, x-\delta) + M |f(x)| \sum_{m=1}^{N} \int_{-\infty}^{x-\delta} K_{\lambda,m}(x, t) dt. \]

In addition to, we obtain the inequality

\[ A_{14}(x, \lambda, m) \leq M \sum_{m=1}^{N} \int_{x+\delta}^{\infty} |f(t) - f(x)| K_{\lambda,m}(x, t) dt \]

\[ \leq ||f||_{L_1(-\infty, \infty)} M \sum_{m=1}^{N} m K_{\lambda,m}(x, x+\delta) + M |f(x)| \sum_{m=1}^{N} \int_{x+\delta}^{\infty} K_{\lambda,m}(x, t) dt. \]

According to the conditions d), (10) and (11), we find that \( A_{11}(x, \lambda, m) + A_{14}(x, \lambda, m) \rightarrow 0 \) as \( \lambda \rightarrow \infty. \) This completes the proof.

\[ \square \]

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