Groupoid symmetry, constraints and quantization of General Relativity

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Abstract

The purpose of the current paper is twofold: to provide a conceptual link between the quantization framework based on Lie integration of algebroids proposed by N.P. Landsman [7] and the dynamical formulation of the Einstein’s equation, and to clarify how from the relevant groupoid the Poisson bracket between constraints of the Hamiltonian formulation emerges as an algebroid bracket. To do so, we adapt the groupoid proposed by C. Blohmann, M. Fernandes and A. Weinstein in the paper "Groupoid symmetry and constraints in General Relativity" [1] by using a different, in our view simpler and more natural, diffeological structure. As a result, we get a different algebroid associated to it, such that the algebroid bundle can be understood as a configuration space of the dynamical formulation of General Relativity, the bracket structure between its sections being of the desired form. We point out the conceptual advantages of this perspective, as well as some interesting issues that require further investigation. We also discuss some of the difficulties that need to be overcome to apply our approach to quantize the Einstein’s theory in this sense. This is an extract from the Master’s thesis [4] written under the supervision of Prof. Klaas Landsman.

Keywords General Relativity · ADM formalism · groupoid symmetry · constraint bracket · algebroid of hypersurface deformations · diffeology in physics · quantum gravity

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1 Introduction

Consider a finite-dimensional physical system on a configuration space given by the tangent bundle $TQ$. The spatial, ‘external’ symmetry of this system can be described by a pair groupoid $Q \times Q \Rightarrow Q$, where a pair $(a, b) \in Q \times Q$ can be understood as possible initial and final ‘position’ of the system and taking the whole pair groupoid means "it is possible for the system to evolve from one ‘position’ to any other". The tangent bundle $TQ$ can be seen as the algebroid $A(Q \times Q) \cong TQ$ of this pair groupoid, with the dynamics of the system given through a lagrangian function $L : A(Q \times Q) \to \mathbb{R}$. The quantum counterpart of this system can in turn be understood as given by the non-commutative $C^*$-algebra generated by the pair groupoid, since we have [7]:

$$C^*(Q \times Q) \cong \mathcal{K}(L^2(Q)),$$

where $\mathcal{K}(L^2(Q))$ is the algebra of compact operators of square-integrable functions on $Q$. Quantization of a simple system with a configuration space $TQ$, seen as replacing the Poisson algebra of classical observables by the $C^*$-algebra of quantum observables, can be then thought of as replacing the commutative algebra $C^*(A^*(G), \mathbb{R})$, where $G$ is the groupoid describing the external symmetry of the system, by the non-commutative $C^*$-algebra $C^*(G)$ generated by this groupoid. Instead of $C^*(A^*(G), \mathbb{R})$, we can also consider the algebra of functions vanishing at infinity on the dual of the algebroid as the algebra of classical observables. It turns out [7], that it is isomorphic as a $C^*$-algebra to the one generated by the algebroid:

$$C_0(A^*(G)) \cong C^*(A(G)),$$

when we consider $A(G)$ as a Lie groupoid with fiber-wise addition as partial multiplication. Quantization can be performed by taking the classical limit in turn replacing the global structure of a groupoid $G$ by its linearized, infinitesimal algebroid version.

We will argue that the pair groupoid description of external symmetries at the classical level can be extended to grasp the symmetry of the initial value formulation of the Einstein’s equation. Following [1], given a manifold $\Sigma$ (space), we define a groupoid $\Theta(\Sigma) \Rightarrow \Sigma$ (which is almost a pair groupoid) over the space of so-called $\Sigma$-universes $\Sigma$ consisting of space-like embeddings of $\Sigma$ as a hypersurface in a lorentzian manifold, quotient by the isometries respecting the embedding. Since we want to discover the algebroid $A(\Theta(\Sigma))$ associated to this geometric structure, we need to put some kind of smooth structure on $\Theta(\Sigma)$ and $\Sigma$ - to this end we make use of the diffeological framework, although in a different, in our view simpler and more straightforward way, than it was originally done in [1]. As a result, we get a different algebroid $A(\Theta(\Sigma))$ that provides us with the following:

1. the structure of the algebroid $A(\Theta(\Sigma))$ provides a natural context for considering the dynamical formulation of the Einstein’s equation – the lagrangian that comes from projecting the Einstein-Hilbert action is naturally interpreted as a functional on the algebroid bundle, and

2. the Lie bracket of sections of $A(\Theta(\Sigma))$ produces the Poisson bracket structure between the ADM constraints.

In the light of the 1976 paper "Geometrodynamics regained" by A. Hojman, K. Kuchař and C. Teitelboim [3], where the authors claim that in some sense the ADM constraints can be recovered if we assume their bracket structure, this means that the Einstein’s theory is in a way dynamically empty – the dynamics is already present in the kinematical setting of the groupoid. Making their points mathematically and conceptually sound could be one of the direction of development of our approach, the other one being an attempt to generalize the construction of the groupoid $C^*$-algebras to the diffeological setting, which today is an open problem.
2 The groupoid

In order to consider Einstein’s equation as an evolution problem, we break the 4-dimensional invariance of the theory by choosing an initial hypersurface. The resulting “3 + 1” diffeomorphism invariance forces us to consider classes of embedded hypersurfaces as a solution to the initial value problem. Indeed, any two Cauchy developments of the same initial data are isometric in the neighbourhood of the initial value hypersurface and to get a concrete space-time we need to choose the lapse function and shift vector field that fixes the way in which the neighbouring time slices are to be ’glued’ together into a 4-dimensional, lorentzian manifold. In this sense, considering space as a Riemannian manifold, the space-time is only given up to this local isometries and hence it is natural to consider the position space of our system to consist of classes of hypersurfaces. In order to grasp the global groupoid structure we will require isometries to be defined on the whole space-time.

Following [1], given a (fixed) 3-dimensional manifold \( \Sigma \), we define the space of \( \Sigma \)-universes:

**Definition 1.** The space of \( \Sigma \)-universes \( \mathcal{U}(\Sigma) \) consists of classes of embeddings \( i: \Sigma \hookrightarrow (M, g) \), where:

- \((M, g)\) is a 4-dimensional, connected, lorentzian manifold,
- \( i: \Sigma \hookrightarrow M \) is a proper embedding of \( \Sigma \) into \( M \) as a space-like hypersurface,

and two embeddings are equivalent iff there is an orientation-preserving isometry making a commutative diagram. We will write \( i \sim i' \) and \([i] = [i'] \in \mathcal{U}(\Sigma)\).

**Remark.** Note that to each class a unique Riemannian metric tensor \( \gamma \in \text{Riem}(\Sigma) \) is associated. Indeed, for \( i \sim i' \) we have:

\[
\gamma := i^*g = (\psi \circ i')^*g = (i')^*(\psi^*g) = (i')^*g' = \gamma',
\]

where we used that \( \psi \) is an isometry. However, the 2nd fundamental form \( k := -\frac{1}{2}L_\xi \gamma \) is not necessary preserved by such isometries [3].

It seems natural to consider the pair groupoid \( \mathcal{U}(\Sigma) \times \mathcal{U}(\Sigma) \Rightarrow \mathcal{U}(\Sigma) \) as the one describing the external symmetry structure of the system. But if we want to keep the interpretation that this groupoid should correspond to possible pairs of initial and final ’positions’ of the system, there should be, at least in principle, a way to evolve one hypersurface from a given pair to the other. Since we assume space-times to be connected, taking those pairs that have the same target manifold seems to be the right generalization. We also need to take care of the equivalence classes, so that our groupoid will respect ”3 + 1” diffeomorphism invariance and in the end we get a groupoid over \( \mathcal{U}(\Sigma) \). We will then consider the classes of pairs of such embeddings, and again following [1] we define:

**Definition 2.** A groupoid of \( \Sigma \)-evolutions \( \Theta(\Sigma) \Rightarrow \mathcal{U}(\Sigma) \) is given by:

- A space \( \Theta(\Sigma) \) consists of classes of pairs of embeddings \((i_1, i_0) : \Sigma \hookrightarrow M\) are embeddings as before and two pairs are equivalent iff there is an orientation-preserving isometry making the inner and outer triangles in the diagram:

```
\[
\begin{array}{c}
\Sigma
\end{array}
\begin{array}{c}
M
\end{array}
\begin{array}{c}
M'
\end{array}
\begin{array}{c}
\Sigma
\end{array}
\begin{array}{c}
M
\end{array}
\begin{array}{c}
M'
\end{array}
\]
\]
```

commute. We will write \([i_1, i_0] \sim [i'_1, i'_0] \) and \([i_1, i_0] = [i'_1, i'_0] \in \Theta(\Sigma)\).

We write the pairs in \((1, 0)\)-order for better compatibility with the groupoid multiplication.
Similarly to the pair groupoid setting, we define:

- the source and target projections \( s, t : Σ(Σ) \to Σ(Σ) \) by \([i] \mapsto t \) and \([i, i] \mapsto s \),
- the inclusion \( ε : Σ(Σ) \to Σ(Σ) \) by \( ε([i]) := [i, i] \),
- the inversion map \( I : Σ(Σ) \to Σ(Σ) \) by \( I([i, i]) := [i_0, i_1] \),
- partial multiplication in \( Σ(Σ) \) by \([i, i_0] \ast [j_1 = i_0, j_0] := [i_1, j_0] \in Σ(Σ) \).

**Remark.** Since the classes \([i, i] \) and \([i] \) are exactly the same thing, the map \( ε \) is injective, and since \([i, i] \) is always in \( Σ(Σ) \), the maps \( s \) and \( t \) are surjective.

Notice that a \( Σ \)-evolution \([i_1, i_0] \) correspond to a diffeomorphism \( i_1 \circ i_0^{-1} : i_0(Σ) \to i_1(Σ) \), given up to isometry and understood as a possible evolution of \( Σ \). Moreover, \( Σ \)-evolutions can be multiplied iff there is a space-time in which both of them can be represented and the middle embeddings coincide. The partial multiplication in \( Σ(Σ) \) then describes the composition of evolutions: \([i_1, i_0] \ast [i_0, j_0] = [i_1, j_0] \) corresponds to \((i_1 \circ i_0^{-1}) \circ (i_0 \circ j_0^{-1}) = i_1 \circ j_0^{-1} \).

Hence, the pair groupoid symmetry of systems with dynamics given on the tangent bundle is naturally generalized to the context of the initial value approach to General Relativity by the groupoid \( Σ(Σ) \to Σ(Σ) \) of \( Σ \)-evolutions over the space \( Σ(Σ) \) of \( Σ \)-universes just described.

### 3 The diffeology

In order to define the algebroid associated to the groupoid \( Σ(Σ) \to Σ(Σ) \), we need to equip the spaces \( Σ(Σ) \) and \( Σ(Σ) \) with structures that allow us to define tangent spaces and mappings between them. Given that we are dealing with quotients of infinite-dimensional spaces of mappings with varying codomains, it is difficult to imagine the possibility of using standard manifold structures. Instead, we will equip those spaces with diffeological structures, which provide a simple but powerful framework that will serve our purpose perfectly. Let us first briefly recall some basic definition.

**Definition 3.** A **parametrization** of a set \( X \) is a map \( φ : U \to X \), where \( U \) is an open subset of euclidean space \( \mathbb{R}^n \).

**Definition 4.** A **diffeological space**, denoted \((X, D_X)\) is a pair of sets, where \( X \) is the space we are concerned with and \( D_X \) is a set of parametrizations of \( X \), called plots, subject to the following conditions:

i. constant maps are plots:

\[ φ : U \to X, \quad φ(u) = x \in X \quad \forall u \in U \quad ⇒ \quad φ \in D_X, \]

ii. \( D_X \) is closed under composition with smooth maps between open subsets of euclidean spaces:

\[ φ : U \to X, \quad f : U' \to U \in C^∞(U', U) \quad ⇒ \quad φ \circ f : U' \to X \in D_X, \]

iii. compatible plots defined on an open cover of an open subset of euclidean space can be uniquely glued together to give another plot:

\[ φ : U \to X, \quad U = \bigcup_{i ∈ I} U_i \quad φ_i := φ|_{U_i} \in D_X \quad ⇔ \quad φ : U \to X \in D_X. \]

Among other perspectives, the notion of a diffeological space can be regarded as generalization of that of a manifold:

\[ \text{For the detailed treatment of these notions, we refer to our thesis [4] and the standard textbook [5].} \]
Remark. Manifolds are diffeological spaces, when plots are defined to be those parametrizations that are smooth (in the manifold sense).

Definition 5. We call the diffeology described above the manifold diffeology.

Definition 6. We call a function between diffeological spaces a smooth map if its composition with any plot on the source space gives a plot on the target space:

\[ f : X \to Y \in C^\infty(X,Y) \iff f \circ \phi : U_\phi \to Y \in D_Y \forall \phi : U_\phi \to X \in D_X. \]

Remark. Smooth maps between manifolds are precisely smooth in the above sense, when manifolds are considered as diffeological spaces [3].

Remark. Plots are exactly those parametrizations that are smooth when the open subsets of Euclidean spaces are equipped with manifold diffeologies.

Definition 7. The subspace diffeology is a diffeology \( D_{A\subseteq X} \) given on a subset \( A \subseteq X \) of a diffeological space \( (X,D_X) \) by taking plots to be those parametrizations that composed with the inclusion \( i : A \hookrightarrow X \) gives plots on \( X \):

\[ \phi : U \to A \in D_{A\subseteq X} \iff i \circ \phi : U \to X \in D_X. \]

Remark. Restriction of a smooth map to the subset is again smooth for the subspace diffeology [1].

The diffeological structure is very flexible in the sense that it can be naturally extended to products, quotients and function spaces:

Definition 8. The product diffeology \( D_{X \times Y} \) is given on a cartesian product \( X \times Y \) of diffeological spaces \( (X,D_X) \) and \( (Y,D_Y) \) by taking plots to be those parametrizations which, when composed with projections, are plots on the factors:

\[ \phi : U \to X \times Y \in D_{X \times Y} \iff \pi_X \circ \phi \in D_X \& \pi_Y \circ \phi \in D_Y, \]

where \( \pi_X \) and \( \pi_Y \) are canonical projections.

Definition 9. The quotient diffeology is a diffeology \( D_{\pi(X)} \) given on an image \( \pi(X) \) of a surjective function \( \pi \) defined on a diffeological space \( (X,D_X) \) by taking plots to be those parametrizations that are given by composition of \( \pi \) with a plot on \( X \):

\[ \phi : U \to \pi(X) \in D_{\pi(X)} \iff \phi = \pi \circ \psi, \ \psi \in D_X. \]

Example 10. Given an equivalence relation on a diffeological space, the canonical projection \( \pi : X \to X/\sim \) provides a diffeology on the quotient space.

Definition 11. The functional diffeology is a diffeology \( D_{C^\infty(X,Y)} \) given on a set of smooth functions \( C^\infty(X,Y) \) between two diffeological spaces by taking plots to be those parametrizations for which the evaluation map is smooth:

\[ \phi : U \to C^\infty(X,Y) \in D_{C^\infty(X,Y)} \iff \text{ev}_\psi : U \times X \to Y \in C^\infty(U \times X,Y), \]

where \( \text{ev}_\psi : U \times X \ni (u,x) \mapsto \phi(u)(x) \in Y, \) and we put standard (manifold) diffeology on \( U \) and product diffeology on \( U \times X. \)

Given a diffeological space, we can think of all kinds of tangent structures, usually defined similarly to the manifold setting. For example, tangent vectors can be defined as equivalence classes of curves:

Definition 12. The curve on a diffeological space \( (X,D_X) \) is a smooth map from an open interval in \( \mathbb{R} \) containing zero to the set \( X \):

\[ c : I \to X, \ 0 \in I \in \partial(\mathbb{R}). \]

If in addition we have \( c(0) = x \in X \), we call \( c \) a curve through \( x. \)
Definition 13. A tangent space $T_xX$ at point $x \in X$ in a diffeological space $(X, D_X)$ is an equivalence class of curves through $x$, where we consider two curves equivalent if they can be represented by a single curve up to a reparametrization of their domains that respects the speed at zero:

$$c : I \to X \in C^\infty(I, X)_x, \quad c' : I' \to X \in C^\infty(I', X)_x.$$  

We say that $c$ and $c'$ are equivalent at $x \in X$, denoted $c \sim_x c'$, if and only if:

$$\exists s : I \to J \in C^\infty(I, J), \quad s' : I' \to J \in C^\infty(I', J), \quad k : J \to X \in C^\infty(J, X)_x$$

such that the diagram:

$$\begin{array}{ccc}
I & \xrightarrow{s} & J \\
\downarrow{c} & & \downarrow{k} \\
X & \xrightarrow{\alpha} & \text{commutes and the curves } s \text{ and } s' \text{ define the same tangent vector to the interval } J \text{ at zero: } s'(0) = s(0).
\end{array}$$

Remark. The above definition agrees with the standard one when the diffeological space $(X, D_X)$ is a manifold [1].

We can now make $G(\Sigma) \to U(\Sigma)$ a diffeological groupoid. The space of $\Sigma$-universes is a quotient of a subspace of the space of mappings from $\Sigma$ with varying codomains. It is then natural to consider the quotient diffeology. For the space of mappings itself we can think of a functional diffeology if we consider manifold diffeologies on $\Sigma$ and the target space-times and restrict ourselves to those parametrizations, for which locally the latter does not change. We define:

$$\text{Definition 15. Given a smooth map } f \in C^\infty(X, Y) \text{ between diffeological spaces, we define the tangent map } T f : TX \to TY \text{ point-wisely by:}$$

$$T f : [c] \mapsto [f \circ c] \in T_{f(x)}Y.$$  

This is, roughly speaking, all the diffeology we need: we have seen that this framework easily deals with functional spaces, does not fear the quotients, and can support tangent structures.

4 The diffeological groupoid

The concept of a Lie group can be generalized to groupoids by requiring that the base space and the space of arrows are manifolds, the source and target maps are submersions, and rest of the structure maps are smooth [3]. Diffeological groupoids are defined similarly, although we do not expect from the projections $s$ and $t$ anything more than smoothness:

$$\text{Definition 16. A diffeological groupoid is a groupoid such that:}$$

i. the arrow space $G$ and a base $Q$ are diffeological spaces,

ii. the projections $s$ and $t$, partial multiplication, the inclusion map $\varepsilon$ and the inverse map $I$ are smooth.

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ii. the projections $s$ and $t$, partial multiplication, the inclusion map $\varepsilon$ and the inverse map $I$ are smooth.
Definition 17. A parametrization $\phi : U \to \U(\Sigma)$ is a plot iff:

i. (quotient diffeology) $\phi : u \mapsto [i_u],$

ii. (fix the target) $\forall u \in U \ \exists u \in V_u \in \mathcal{O}(U) : \ M_{i_u} \equiv M_{v_u} \ \forall v \in V_u.$

iii. (functional diffeology) $\forall u \in U \ the \ map \ ev_\phi : V_u \times \Sigma \ni (v, x) \mapsto i_v(x) \in M_{i_u} \ is \ smooth.$

To equip the space of arrows $\Theta(\Sigma)$ with a diffeological structure, we will go along similar lines: before taking the quotient, we consider the diffeology on the space of pairs of embeddings by firstly assuming that the target space-time is locally fixed and secondly that the local evaluation maps are smooth:

Definition 18. A parametrization $\phi : U \to \Theta(\Sigma)$ is a plot iff:

i. (quotient diffeology) $\phi : u \mapsto [(i_1)_u, (i_0)_u],$

ii. (fix the target) $\forall u \in U \ \exists u \in V_u \in \mathcal{O}(U) : \ M_{(i_1)_u} = M_{(i_0)_u} \equiv \ M_{(i_0)_u} = M_{(i_1)_u} \ \forall v \in V_u.$

iii. (functional diffeology) $\forall u \in U \ the \ map \ ev_\phi : V_u \times \Sigma \ni (v, x) \mapsto (i_1)_u(x), (i_0)_u(x)) \in M_{(i_1)_u} \times M_{(i_0)_u} \ is \ smooth.$

When written down, smoothness of the structure maps of $\Theta(\Sigma) \to \U(\Sigma)$ becomes trivial:

Lemma 1. With the diffeologies just defined, $\Theta(\Sigma) \to \U(\Sigma)$ becomes a diffeological groupoid.

Proof. Notice first that since we are using quotient diffeologies, all the plots come from compositions with the projections and hence we can focus on smooth parametrizations of the space of embeddings and the space of pairs of embeddings ($\U(\Sigma)$ and $\Theta(\Sigma)$ before taking quotients). Then we see that:

- the projections $s, t : \Theta(\Sigma) \to \U(\Sigma)$ are smooth, since for any parametrization $\phi$ of the space of pairs of embeddings we have $ev_{t \circ \phi} = \pi_0 \circ ev_\phi$ and $ev_{t \circ \phi} = \pi_1 \circ ev_\phi$, which are smooth if only $ev_\phi$ is,

- the inclusion is smooth since for any parametrization $\phi$ of the space of embeddings we have $ev_{s \circ \phi} = ev_\phi \times ev_\phi$, which is smooth iff $ev_\phi$ is,

- the inversion map is smooth, since for any parametrization $\phi$ of the space of pairs of embeddings we have $ev_{inv \circ \phi} = inv \circ ev_\phi$, where $inv$ denotes exchanging the product factors, which is smooth iff $ev_\phi$ is,

- since the set of composable arrows in $\Theta(\Sigma)$ is naturally equipped with the product-subspace diffeology, any parametrization $\phi$ of this space is a plot iff the relevant 3-factor evaluation maps

  $ev_\phi : V_u \times \Sigma \ni (v, x) \mapsto ((i_1)_u(x), (i_0)_u(x)), (j_0)_u(x)) \in M_{(i_1)_u} \times M_{(i_0)_u} \times M_{(j_0)_u}$

  are smooth, and the map $ev_{inv \circ \phi}$ is smooth iff $\pi_0 \circ ev_\phi$ and $\pi_2 \circ ev_\phi$ are.

5 The algebroid

The construction of a Lie algebroid from a given Lie groupoid can, to some extent, be generalized to the diffeological setting [4]. For our purpose however, we will not need the full power of the general theory. Hence we will briefly recall the smooth case and pinpoint the subtleties that we have to take
we have a foliation $\mathcal{F}$ where:

Given a Lie groupoid $G \Rightarrow Q$, we can think of the manifold $Q$ as being embedded into $G$ via the inclusion $\varepsilon : Q \hookrightarrow G$. Thus, we can consider the normal sub-bundle $N^0 \epsilon(Q)$ of the tangent space along $\varepsilon(Q)$. This can be understood as a bundle over $Q$ itself – we call it the algebroid associated to the groupoid $G \Rightarrow Q$ and denote it by $A(G) \rightarrow Q$. Sections of the algebroid bundle are equipped with the bracket structure that comes from the commutator on the space of vector fields over $G$ and a fiber-wisely defined map $a : A(G) \rightarrow TQ \subset TG_{\varepsilon(Q)}$ that preserves it, called an anchor. For further details, we refer to our thesis [4] or a standard textbook [8].

In diffeological setting, the situation is similar in many ways, but a bit more subtle, as we will shortly see. To investigate the algebroid associated to our groupoid we need to reformulate a little the procedure above.

Firstly, since we do not have a notion of a normal bundle let us notice that in the manifold setting the bundle $N^0 \epsilon(Q)$ is isomorphic to the kernel of the tangent map of one of the projections, say the source map: $A(G) = N^0 \epsilon(Q) \cong \ker(T s)|_{\varepsilon(Q)}$, which is meaningful also in the diffeological setting.

Secondly, the resulting bundle is not necessarily interpretable as a bundle over the base space – this happens when the diffeological structure on the base is constructed as a quotient diffeology and hence will be of some importance in our case.

Let us now get back to the groupoid in question. Since we claim that the algebroid bundle $A(\mathfrak{G}(\Sigma))$ provides a proper generalization of the tangent space $TQ$ for the dynamical approach to General Relativity, let us here briefly recall the Lagrangian formulation of the Einstein’s equation. When we assume that the space-time $M$ is globally hyperbolic, i.e. it admits a Cauchy surface $i(\Sigma) = S \subset M$, we have a foliation $\mathbb{R} \times \Sigma \cong M$, and hence an isometry $F : \mathbb{R} \times \Sigma \rightarrow M$ such that for any $t \in \mathbb{R}$ $F_t : \Sigma \ni x \mapsto F(t, x) \in M$ is a proper embedding. The Einstein-Hilbert action

$$S = \int_M \sqrt{-g} \, R(g),$$

can be then rewritten in the projected form [5]:

$$S = \int_M \sqrt{-g} \, R(g) = \int_{\mathbb{R}} dt \int_{\Sigma} \phi \sqrt{\gamma} \{\mathcal{L} R(\gamma) + tr(k^2) - (tr(k)^2)\},$$

where:

- $\gamma(t)$ is the Riemannian metric tensor inherited on $\Sigma_t$ from $M$,
- the shift vector field and lapse function $(X(t), \phi(t))$ come from the normal-tangent decomposition of the deformation vector field given by the evolution of $\Sigma$ in $M$ with respect to $t \in \mathbb{R}$:

$$TF(\partial_t) = X(t) + \phi(t) \hat{n}(t),$$

where $\hat{n}(t)$ is a vector normal to $\Sigma_t$.
- the 2nd fundamental form $k$ is given as a function of the lapse, shift, $\gamma$ and $\gamma$ through $[3]$:

$$k(\phi, X, \gamma, \dot{\gamma}) := -\frac{1}{2} \mathcal{L}_{\hat{n}(t)} \gamma + \frac{1}{2\phi(t)} \{\mathcal{L}_{X(t)} \gamma - \gamma(\tau)\},$$

\[\text{Diffeological tangent bundles are cones but not necessary vector spaces} \, [1].\]

\[\text{The choice of source or target projection here differs among the authors. We prefer to use s so that the algebroid describes infinitesimal arrows with fixed origins. The anchor is then simply given by the tangent map to the other projection } a := T r.\]
• traces are taken with respect to the metric: \( tr = t r \).  

The lagrangian of General Relativity is then a real functional of the form \([9]\):

\[
L : T (Riem(\Sigma) \times (\mathcal{D}(\Sigma) \oplus \mathcal{F}(\Sigma))) \ni (\gamma, X, X, \phi, \phi) \mapsto \int_{\Sigma} \phi \sqrt{\gamma} \left[ R(\gamma) + tr(k^2) - (tr(k))^2 \right] \in \mathbb{R}.
\] (1)

If we want, as in the motivating example, to understand the algebroid bundle as a domain of the lagrangian, we need:

\[
A(\mathfrak{g}(\Sigma)) \cong T (Riem(\Sigma) \times (\mathcal{D}(\Sigma) \oplus \mathcal{F}(\Sigma))),
\]

which is our first claim.

5.1 The tangent bundle \( T \Omega(\Sigma) \)

As a warm up, let us take a look at the tangent bundle \( T \Omega(\Sigma) \). The tangent space at \([i_0] \in \Omega(\Sigma)\) consists of classes of smooth curves \( c \in D_{\Omega(\Sigma)} \) of the form:

\[
c : I \ni \tau \mapsto [i_\tau] \in \Omega(\Sigma).
\]

Such a curve is fully determined by its evaluation map:

\[
ev_c : I \times \Sigma \ni (\tau, x) \mapsto i_\tau(x) \in M_{i_0},
\]

and describes a deformation of the hypersurface \( i_0(\Sigma) \subset M_{i_0} \) up to isometries of \( M_0 \) preserving it, i.e. two curves:

\[
c : I \ni \tau \mapsto [i_\tau], \quad ev_c : I \times \Sigma \to M_{i_0} \quad \& \quad c' : I \ni \tau \mapsto [i'_{\tau}], \quad ev_{c'} : I \times \Sigma \to M_{i'_0}
\]

are indistinguishable iff there is an isometry \( \psi : M_{i_0} \to M_{i'_0} \) such that:

\[
ev_{c'} = \psi \circ ev_c.
\]

The curve \( c \) on \( \Omega(\Sigma) \) then does not see the neighbourhood of the embedded hypersurface, as it is subject to arbitrary isometric changes. Notice here, that because of this huge group of isometries we quotient out in \( \Omega(\Sigma) \), neither the normal vector field \( \dot{n}(\tau) \), nor the change of the inherited Riemannian tensor with respect to the curve parameter: \( \gamma(\tau) := i_\tau g_{i_0} \) is preserved. The tangent vector \( v = [c] \in T_{[i_0]} \Omega(\Sigma) \) is given by a class of curves that arise as a reparametrization of \( c \) that preserves the speed of change with respect to the parameter \( \tau \) (see Definition [13]). It is then identified with the vector field along \( i_0(\Sigma) \), defined by:

\[
v = \dot{X}(i_0(x)) := \frac{\partial}{\partial \tau} ev_c(\tau, x) \bigg|_{\tau=0},
\]

which can be decomposed into a normal and tangential part \( 6\) to give:

\[
v = X + \phi \dot{\hat{n}} \in \mathcal{D}(\Sigma) \oplus \mathcal{F}(\Sigma).
\]

Because of the quotient diffeology, the natural 'starting point' of the vector \([c]\) is the embedded hypersurface \( i_0(\Sigma) \), and not the class \([i_0] \in \Omega(\Sigma)\). Notice, that embedding \( \Sigma \) into \( M_0 \) and 'quotienting out' the details of the particular space-time boils down to equipping \( \Sigma \) with a metric tensor \( \gamma = i_0^* g_{i_0} \) and since this metric tensor is the same for any representative in \([i_0] \in \Omega(\Sigma)\), it is natural to consider \( T \Omega(\Sigma) \) as a bundle over \( Riem(\Sigma) \). We then get:

\[\text{6}\text{Notice here, that the vector space operations on } T M_{i_0} \mid_{[i_0]} \text{ commute with the decomposition.}\]
Lemma 2. The (diffeological) tangent space $T\mathcal{U}(\Sigma)$ to the space $\mathcal{U}(\Sigma)$ can be seen as a vector bundle of the form:

$$T\mathcal{U}(\Sigma) \cong \text{Riem}(\Sigma) \times \left( \mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma) \right).$$

The sections of the tangent bundle of $\mathcal{U}(\Sigma)$, i.e. vector fields $\sigma \in \mathcal{X}(\mathcal{U}(\Sigma))$, are given through their flows, i.e. at each point $[t] \in \mathcal{U}(\Sigma)$ we have a curve $\alpha[t]$ such that $\sigma_{[t]} = [\alpha[t]]$, so in particular we always have $\alpha[t](\tau = 0) = [t]$. Notice here, that given another vector at $[i_0]$, defined by a curve $c'$:

$$v' = \tilde{X}'(i_0(x)) := \frac{\partial}{\partial \tau} \left. ev_{\alpha}(\tau, x) \right|_{\tau = 0},$$

we have an isometry $\psi: M_{i_0} \rightarrow M_{i_0}'$ that allows us to consider them both in one and the same space-time. The commutator of sections $[\sigma, \sigma']$ is then calculated point-wisely by the commutator of those vector fields, evaluated on $i_0(\Sigma)$:

$$[\sigma, \sigma']_{[t]} = [\alpha[t], \alpha'[t]] = \left[ \frac{\partial}{\partial \tau} ev_{\alpha_{[t]}}, \frac{\partial}{\partial \tau} ev_{\alpha'[t]} \right] \bigg|_{\tau = 0} \in T_{[t]}\mathcal{U}(\Sigma).$$

5.2 The tangent bundle $T\mathcal{G}(\Sigma)$

Since the bracket of the sections of our algebroid $A(\mathcal{G}(\Sigma))$ comes from the commutator of sections of the tangent bundle $T\mathcal{G}(\Sigma)$, we need to take a look at that one as well. The tangent space at $[i_1, i_0] \in \mathcal{G}(\Sigma)$ consists of classes of smooth curves $c \in D_{\mathcal{G}(\Sigma)}$ of the form:

$$c: I \ni \tau \mapsto [(i_1, \tau), (i_0, \tau)] \in \mathcal{G}(\Sigma).$$

Again, such a curve is fully determined by its evaluation map:

$$ev_{c} : I \times \Sigma \ni (\tau, x) \mapsto (\tau_1, \tau_0) \in M_{\tau_1} \times M_{\tau_0},$$

and describes a deformation of a pair of hypersurfaces $(i_1)_{\Sigma_0}, (i_0)_{\Sigma_0} \subset M_{\tau}$ up to isometries of $M_{\tau}$ preserving them.

Similarly to the vectors in $T\mathcal{U}(\Sigma)$ just described, the vector $[c] = (v, w)$ can be seen as a pair of vector fields:

$$(v, w) := \left( \frac{\partial}{\partial \tau} ev_{c_1}(\tau, x) \bigg|_{\tau = 0}, \frac{\partial}{\partial \tau} ev_{c_0}(\tau, x) \bigg|_{\tau = 0} \right),$$

where $ev_{c_1} := \pi_1 \circ ev_c$ and $ev_{c_0} := \pi_0 \circ ev_c$. These can again be decomposed into a normal and tangential part, and we get:

Lemma 3. The (diffeological) tangent space $T\mathcal{G}(\Sigma)$ to the space $\mathcal{G}(\Sigma)$ can be seen as a vector bundle of the form:

$$T\mathcal{G}(\Sigma) \cong T\mathcal{U}(\Sigma) \times T\mathcal{U}(\Sigma) \cong \text{Riem}(\Sigma) \times \left( \mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma) \right) \times \text{Riem}(\Sigma) \times \left( \mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma) \right).$$

The vector fields $\sigma \in \mathcal{X}(\mathcal{G}(\Sigma))$, are again given through their flows, i.e. at each point $[i_1, i_0] \in \mathcal{G}(\Sigma)$ we have a curve $\alpha[i_1, i_0]$ such that $\sigma_{[i_1, i_0]} = [\alpha[i_1, i_0]]$. The commutator of sections $[\sigma, \sigma']$ is then given by a pair of commutators:

$$[\sigma, \sigma']_{[i_1, i_0]} = \left[ \alpha[i_1, i_0], \alpha'[i_1, i_0] \right] = \left[ \frac{\partial}{\partial \tau} ev_{\alpha_{[i_1, i_0]}}(\tau, x), \frac{\partial}{\partial \tau} ev_{\alpha'[i_1, i_0]}(\tau, x) \right] \bigg|_{\tau = 0} = \left[ \frac{\partial}{\partial \tau} ev_{\alpha_{[i_1, i_0]}}(\tau, x), \frac{\partial}{\partial \tau} ev_{\alpha'[i_1, i_0]}(\tau, x) \right] \bigg|_{\tau = 0}.$$
5.3 The algebroid bundle \( A(\mathcal{G}(\Sigma)) \)

Let us first take a look at our motivating example – the system with the external symmetry structure of a pair groupoid \( Q \times Q \rightarrow Q \) – the kernel of \( T\sigma s \) along \( e(Q) = \{(x, y) \in Q \times Q \mid x = y\} \) consists of those vectors that are given by the curves that are constant on the second slot:

\[
c : I \ni \tau \mapsto (p(\tau), p(0)) \in Q \times Q.
\]

A tangent vector generated by a curve as above is just a tangent vector at \( p(0) \in Q \), considering all the curves of this form gives the whole tangent bundle and hence \( A(Q \times Q) \cong TQ \). The lagrangian that gives the dynamics of the system is then a functional on the algebroid bundle.

Since our \( \mathcal{G}(\Sigma) \rightarrow \mathcal{U}(\Sigma) \) is not a pair groupoid, things get a little more complicated – in particular, \( A(\mathcal{G}(\Sigma)) \not\cong T\mathcal{U}(\Sigma) \). Let us discover what happens in this case.

The kernel of \( T\sigma s \) consists of those vectors that are given by the curves of the form:

\[
c : I \rightarrow \mathcal{G}(\Sigma), \quad \lbrack c \rbrack \in \text{ker}(T\sigma s)|_{\mathcal{U}(\Sigma)} \quad \Rightarrow \quad c : I \ni \tau \mapsto \lbrack i_{\tau}, i_{0} \rbrack \in \mathcal{G}(\Sigma).
\]

For each such curve we have the associated evaluation map:

\[
ev_{c} : I \times \Sigma \ni (\tau, x) \mapsto (i_{\tau}(x), i_{0}(x)) \in M_{i_{0}} \times M_{i_{0}},
\]

which is smooth due to the diffeology on \( \mathcal{G}(\Sigma) \). Since the right factor does not depend on \( \tau \), there is no more data in the curve \( c \) than there is in the evaluation map:

\[
ev_{c_{1}} := \pi_{1} \circ ev_{c} : I \times \Sigma \ni (\tau, x) \mapsto i_{\tau}(x) \in M_{i_{0}}.
\]

However, since it is a curve on \( \mathcal{G}(\Sigma) \), we need to consider points along the curve as elements of \( \mathcal{G}(\Sigma) \), and hence, for each \( \tau \in I, c(\tau) \in \mathcal{G}(\Sigma) \) is given by \( i_{\tau}, i_{0} : \Sigma \ni \gamma \mapsto i_{0}(\gamma) \) only up to the isometries of \( M_{i_{0}} \) fixing both embeddings. It then describes a small deformation of the hypersurface \( i_{0}(\Sigma) \) in \( M_{i_{0}} \) given up to those isometries, which is precisely the way we want to see them to keep the 3 + 1 diffeomorphism invariance. Unlike in the case of curves on \( \mathcal{U}(\Sigma) \), the curves of this form do see the neighbourhoods of the embedded hypersurfaces they are centered on – that is because the isometry group that we quotient out in \( \mathcal{G}(\Sigma) \) is considerably smaller. Also, unlike for the general curves on \( \mathcal{G}(\Sigma) \), we have now fixed one of the embedding, the other being it's smooth deformation. They can get arbitrarily close, which makes the structure of the tangent vectors in \( A(\mathcal{G}(\Sigma)) \) more interesting.

Let us explain.

Notice first, that a curve \( c : I \rightarrow \mathcal{G}(\Sigma) \) can be understood as providing a unique foliation of a particular neighbourhood of \( i_{0}(\Sigma) \) in the following way. Consider the natural embedding:

\[
\iota : \Sigma \hookrightarrow I \times \Sigma =: M_{\Sigma},
\]

where the metric \( g_{\Sigma} \) on \( M_{\Sigma} \) is given by a pullback of the one on \( M_{i_{0}} \) through the evaluation map \( g_{\Sigma} := ev_{c_{1}} \circ g_{i_{0}} \). Locally, we then have an isometry:

\[
\psi := ev_{c_{1}} : I \times \Sigma \rightarrow M_{i_{0}},
\]

such that \( \psi(0, x) = i_{0}(x) \) and \( \psi(\tau, x) = i_{\tau}(x) \) for all \( \tau \in I \). This provides a unique choice of a representative in \( [i_{\tau}, i_{0}] \) for each \( \tau \in I \). Since this construction is uniquely given by the curve \( c \), it can be identified with this representation on \( (I \times \Sigma, g_{\Sigma}) \), seen as a way of "smearing out" \( (\Sigma, \gamma = i_{0}(g_{i_{0}})) \). The curve on \( \mathcal{G}(\Sigma) \) then describes a deformation of a hypersurface in a fixed neighbourhood, unlike the curves on \( \mathcal{U}(\Sigma) \). It can be understood as a curve \( \tau \mapsto [i_{\tau}] \in \mathcal{U}(\Sigma) \) represented on a neighbourhood straightened with respect to the \( \tau \) slicing. The vector \( v = \lbrack c \rbrack \) can be then thought of as an infinitesimal
neighbourhood of \( \mathcal{I}(\Sigma) \) in \((M_L, g_L)\), linearized in the sense of the \( \tau \) parameter. Let us then take a closer look at it.

The neighbourhood \((M_L, g_L)\) is uniquely determined by the curve on the space of Riemannian metrics on \( \Sigma \) and the deformation vector field of this foliation:

\[
I \ni \tau \mapsto \gamma(\tau) = \mathcal{L}_\gamma g_L = (\psi^{-1}\circ \iota_t)^* g_0 = \hat{c}_\tau g_0 \in \text{Riem}(\Sigma),
\]

\[
\frac{\partial}{\partial \tau} \in \mathcal{X}(\gamma)(\tau) + \phi(\tau)\hat{\eta}(\tau).
\]

We recognize \( X(\tau) \in \mathcal{F}(\Sigma) \) as the shift vector field and \( \phi(\tau) \in \mathcal{F}(\Sigma) \) the lapse function. Finally, we can identify the curve \( \tilde{c} : I \rightarrow \mathcal{G}(\Sigma) \) with the following one:

\[
\tilde{c} : I \ni \tau \mapsto (\gamma(\tau), X(\tau), \phi(\tau)) \in \text{Riem}(\Sigma) \times (\mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma)),
\]

and hence the tangent vector \( \nu = [c] \) consists of the following data:

\[
A(\mathcal{G}(\Sigma))|_{\nu} \ni \nu = (\gamma, \dot{\gamma}, X, \dot{X}, \phi, \dot{\phi}) \in T\text{Riem}(\Sigma) \times (\mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma)),
\]

where:

\[
\gamma := \frac{\partial}{\partial \tau} \gamma(\tau)|_{\tau=0} \in S^2(\Sigma), \quad X := \frac{\partial}{\partial \tau} X(\tau)|_{\tau=0} \in \mathcal{X}(\Sigma), \quad \phi := \frac{\partial}{\partial \tau} \phi(\tau)|_{\tau=0} \in \mathcal{F}(\Sigma).
\]

We then arrive at:

**Lemma 4.** The algebroid \( A(\mathcal{G}(\Sigma)) \) of hypersurface deformations associated to the diffeological groupoid \( \mathcal{G}(\Sigma) \Rightarrow \mathcal{U}(\Sigma) \) of \( \Sigma \)-evolutions over the space of \( \Sigma \)-universes \( \mathcal{U}(\Sigma) \) can be seen as a vector bundle of the form:

\[
A(\mathcal{G}(\Sigma)) \cong T\text{Riem}(\Sigma) \times (\mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma)) \cong \text{Riem}(\Sigma) \times S^2(\Sigma) \times (\mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma))^2.
\]

Interestingly, we see that \( A(\mathcal{G}(\Sigma)) \cong T\text{Riem}(\Sigma) \times (\mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma)) \cong TT\mathcal{U}(\Sigma) \), i.e. under this interpretation the algebroid \( A(\mathcal{G}(\Sigma)) \) can be understood as a double tangent bundle of \( \mathcal{U}(\Sigma) \). Hence there is a reason for \( A(\mathcal{G}(\Sigma)) \) to also be seen as a bundle over the space \( \text{Riem}(\Sigma) \):

\[
A(\mathcal{G}(\Sigma)) \cong \text{Riem}(\Sigma) \times S^2(\Sigma) \times (\mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma))^2
\]

\[
\begin{array}{c}
\downarrow \pi_0
\\
\text{Riem}(\Sigma)
\end{array}
\]

We have then shown that replacing the tangent bundle \( T\text{Riem}(\Sigma) \cong \text{Riem}(\Sigma) \times S^2(\Sigma) \) by the algebroid \( A(\mathcal{G}(\Sigma)) \) of hypersurface deformations provides a natural context for considering the lagrangian formulation of the Einstein’s equation: in this setting, the lagrangian becomes a functional on the algebroid bundle, just as in our motivating example the lagrangian function was given on \( TQ \cong A(Q \times Q) \). There is no need to introduce the lapse and shift as additional variables. In the initial value approach, we naturally consider \( \Sigma \) being equipped with a metric \( \gamma \) as a result of embedding it as a Cauchy surface into a, not yet existing but to be determined by the field equations, lorentzian manifold. Our approach can be then interpreted as a result of taking properly into account the fact that in order to specify the position in this context, we need to specify not only the inherited Riemannian metric tensor \( \gamma \) on \( \Sigma \), but also the slice \( \Sigma \) is to be glued together with the neighbouring one, which is specified by the lapse and shift. The freedom of choice of those, caused by “3 + 1” diffeomorphism invariance, have now been naturally absorbed into the structure of our algebroid. We have then proved our first claim – the algebroid \( A(\mathcal{G}(\Sigma)) \), generating deformations of the embedded hypersurfaces while respecting the indefiniteness of the metric of the ambient manifold, can be seen as a domain of the lagrangian of General Relativity.
5.4 The Lie bracket of $\Gamma(A(\Theta(\Sigma)))$

Performing the Legendre transform on the lagrangian $[1]$ gives the so-called ADM hamiltonian. The lapse function and shift vector field are then treated as additional variables, and since their time derivatives do not appear in the lagrangian, they give rise to the constraints $[1]$:

\[
C_{\text{mo}}(\gamma, \pi) = -2 \text{div}(\pi) = 0, \\
C_{\text{en}}(\gamma, \pi) = -3 R(\gamma) + tr(\pi^2) - \frac{1}{2}(tr(\pi))^2 = 0,
\]
where the divergences and traces are taken with respect to the metric:

\[
\text{div}_C = \text{div}, \quad tr_C = tr, \quad \text{and} \quad \pi \text{ is the variable conjugate to } \dot{\gamma}.
\]

These constraints can be made into functions on the phase space $T^*\text{Riem}(\Sigma)$ by pairing with vector field $X$ and a function $\phi$ on $\Sigma$ $[1]$:

\[
C_{(X, \phi)} : (\gamma, \pi) \mapsto \int_{\Sigma} \sqrt{\gamma} \{ \gamma(X, C_{\text{mo}}(\gamma, \pi)) + \phi \cdot C_{\text{en}}(\gamma, \pi) \}. \tag{2}
\]

Defying $C_{(X, \phi)} = C_{(X, 0)} + C_{(0, \phi)} = : C_X + C_\phi$, the Poisson bracket structure of the constraints can be shown to be $[1]$:

\[
\{C_X, C_Y\} = C_{[X,Y]}, \tag{3}
\]
\[
\{C_X, C_\phi\} = C_{X\phi}, \tag{4}
\]
\[
\{C_\phi, C_\psi\} = C_{\phi\grad\psi - \grad\phi}. \tag{5}
\]

Interestingly, the structure above is being recovered as a bracket of constant sections of our algebroid, the precise link between those structures remaining to be understood$[\S]$. Let us then derive the bracket of $A(\Theta(\Sigma))$, using the notation from the previous paragraph.

Our derivation is based on a simple observation. Namely, notice that the metric tensor on $M_\Sigma[1]$ has the following gaussian form with respect to the $\tau$ parameter:

\[
g_{\tilde{\gamma}}(\tau) = -d\tau \otimes d\tau + \tilde{\gamma}(\tau),
\]
where $\tilde{\gamma}(\tau)$ is the zero extension of $\gamma(\tau)$ along $\{\tau\} \times \Sigma \in M_\Sigma$. The Lie derivative of $g_{\tilde{\gamma}}$ with respect to the deformation vector field $X + \phi \tilde{n}$ takes here an especially simple form:

\[
\mathcal{L}_{X + \phi \tilde{n}} g_{\tilde{\gamma}} = \mathcal{L}_{\partial_\tau} g_{\tilde{\gamma}} = \mathcal{L}_{\partial_\tau}(-d\tau \otimes d\tau + \tilde{\gamma}) = \mathcal{L}_{\partial_\tau} \tilde{\gamma} = \mathcal{L}_X \gamma + \phi \mathcal{L}_{\tilde{n}} \tilde{\gamma},
\]
where $\mathcal{L}_X \gamma$ is understood as a zero extension (we omit here the tilde symbol) and the last equality holds since $M_\Sigma$ is 'straightened' with respect to $\tau$, i.e. it is a family of slices parametrized by $\tau$ and glued together by $X$ and $\phi$. As we will see, this equation gives a constraint on the form of the deformation vector fields associated to curves that generate vectors in $A(\Theta(\Sigma))$. This will allow us to calculate their commutator that comes from the tangent structure of the ambient space-times, even though, as we have seen, the data of those vectors are given solely on $\Sigma$. We will call the vector fields that satisfy the above equation $g$-gaussian$[\S]$.

**Definition 19.** A vector field $v$ defined in the neighbourhood $U$ of a space-like hypersurface $i(\Sigma)$, where $i : \Sigma \rightarrow M$ is a proper embedding and $(M, g)$ a connected, lorentzian manifold is called $g$-gaussian if it satisfies:

\[
\mathcal{L}_v g = \mathcal{L}_X \gamma + \phi \tilde{\gamma}, \tag{6}
\]
where $X + \phi \tilde{n}$ is the tangent-normal decomposition of $v$ along $i(\Sigma)$ and $\tilde{\gamma} = \mathcal{L}_{\tilde{n}} \tilde{\gamma}$.

---

$^7$See page 11.

$^8$Our definition is different that the one given in $[1]$ but, as we will shortly see, the two are equivalent.
Let us now investigate the properties of \( g \)-gaussian vector fields. In gaussian normal coordinates of a small enough neighbourhood of \( i(\Sigma) \) we have:

\[
g = -dt \otimes dt + \tilde{\gamma},
\]

where \( \tilde{\gamma} \) is again the zero extension of \( \gamma \). Further, now \( \partial_t = \hat{n} \) and when an arbitrary vector field \( v \in \mathcal{X}(U) \) is decomposed for \( v =: X + \phi \partial_t \), we get:

\[
\mathcal{L}_X(-dt \otimes dt) = 0,
\]

\[
\mathcal{L}_{\phi \partial_t}(-dt \otimes dt) = -2d\phi \otimes dt,
\]

\[
\mathcal{L}_X \tilde{\gamma} = \mathcal{L}_X \gamma + 2i_{\frac{d\phi}{dt}} \tilde{\gamma} \otimes dt,
\]

\[
\mathcal{L}_{\phi \partial_t} \tilde{\gamma} = \phi \mathcal{L}_{\partial_t} \tilde{\gamma} = \phi \tilde{\gamma}.
\]

Hence, calculating the Lie derivative of \( g \) with respect to \( v \) gives \([1]\):

\[
\mathcal{L}_v g = \mathcal{L}_{X+\phi \partial_t}(-dt \otimes dt + \tilde{\gamma})
\]

\[
= \mathcal{L}_X(-dt \otimes dt) + \mathcal{L}_\gamma \tilde{\gamma} + \mathcal{L}_{\phi \partial_t}(-dt \otimes dt) + \mathcal{L}_{\phi \partial_t} \tilde{\gamma}
\]

\[
= \mathcal{L}_X \gamma + 2i_{\frac{d\phi}{dt}} \tilde{\gamma} \otimes dt - 2d\phi \otimes dt + \phi \tilde{\gamma}.
\]

We then see that \( \gamma \) is \( g \)-gaussian if and only if in gaussian normal coordinates we have

\[
i_\frac{d\phi}{dt} \tilde{\gamma} \otimes dt = d\phi \otimes dt,
\]

which reads \([1]\):

\[
\frac{\partial \phi}{\partial t} = 0 \quad \& \quad \frac{\partial X}{\partial t} = \text{grad}_\gamma \phi,
\]

or, in coordinate-free form \([1]\),

\[
\hat{n}(\phi) = 0 \quad \& \quad [\hat{n}, X] = \text{grad}_\gamma \phi.
\]

Hence, any vector field \( X + \phi \hat{n} \) along \( i(\Sigma) \) can be uniquely extended to a \( g \)-gaussian vector field on a gaussian neighbourhood \( U \supset i(\Sigma) \) by solving the above equations. Such an extension depends on the metric \( \gamma = i^* g \), which is the reason the bracket of our algebroid is point-dependent.

Let us now return to the algebroid \( \mathcal{A}(\mathcal{G}(\Sigma)) \). We have seen, that the deformation vector fields \( \frac{d}{dt} ev_c = X(\tau) + \phi(\tau) \hat{n} \) associated to curves \( c \) such that or \( v = [c] \in \mathcal{A}(\mathcal{G}(\Sigma))_{[\phi]} \) are \( g \)-gaussian, and hence they need to satisfy the above relations. The bracket of sections of the algebroid bundle \( \mathcal{A}(\mathcal{G}(\Sigma)) \) comes from the commutator of vector fields over \( \mathcal{G}(\Sigma) \), which we have already investigated – it is given point-wisely by the commutator of the vector fields generated by the evaluation maps. The bracket in \( \Gamma(\mathcal{G}(\Sigma)) \) is then given, again point-wisely, by the commutator of the extensions of the data \( (\Sigma, \gamma, X, \phi) \) to the \( g \)-gaussian vector field on the neighbourhood of the embedded hypersurface.

Let us then take another vector, say \( u = [c'] \in \mathcal{A}(\mathcal{G}(\Sigma))_{[\psi]} \), such that:

\[
v = (\gamma = \gamma', \gamma, X, \phi, \psi) \in \mathcal{A}(\mathcal{G}(\Sigma))_{\gamma},
\]

\[
u = (\gamma' = \gamma', \gamma', Y, \psi, \psi) \in \mathcal{A}(\mathcal{G}(\Sigma))_{\gamma'},
\]

The commutator of the two then reads \([3, 5]\):

\[
[v, u]_{\gamma} = [X + \phi \hat{n}, Y + \psi \hat{n}]_{\gamma}
\]

\[
= [X, Y] + [X, \psi \hat{n}]_{\gamma} + [\phi \hat{n}, Y]_{\gamma} + [\phi \hat{n}, \psi \hat{n}]
\]

\[
= [X, Y] + X(\psi) \hat{n} - \psi[X, \gamma] + \phi[\hat{n}, Y]_{\gamma} - Y(\phi) \hat{n}
\]

\[
= [X, Y] + \phi \text{grad}_\gamma \psi - \psi \text{grad}_\gamma \phi + (X(\psi) - Y(\phi)) \hat{n},
\]

which corresponds to \([3, 5]\). We then conclude the following:
Lemma 5. The bracket structure between sections of the algebroid $A(\mathfrak{g}(\Sigma))$ of hypersurface deformations equals the Poisson bracket structure of the ADM constraints, when the former are point-wisely decomposed into the normal and tangential part, and the latter naturally paired with functions and vector fields on a hypersurface as in \([2]\).

6 Discussion

We have shown how from the relatively simple, very natural and conceptually well motivated geometric structure of the groupoid $\mathfrak{g}(\Sigma) = \Omega(\Sigma)$, the correct setting for stating the initial value problem for the Einstein’s equation emerges. More precisely, the algebroid is of the form:

$$A(\mathfrak{g}(\Sigma)) \cong \mathcal{T}(\mathfrak{g}(\Sigma) \times (\mathfrak{D}(\Sigma) \oplus \mathfrak{F}(\Sigma))) \cong \mathcal{T}(\mathfrak{g}(\Sigma) \times (\mathfrak{D}(\Sigma) \oplus \mathfrak{F}(\Sigma))^2 \bigg|_{\mathcal{R}(\Sigma)}$$

and the lagrangian associated to the Einstein-Hilbert action is a functional on $A(\mathfrak{g}(\Sigma))$:

$$L : A(\mathfrak{g}(\Sigma)) \ni (\gamma, \varphi, X, X, \phi, \phi) \mapsto \int_\Sigma \phi \sqrt{\text{det} R} (\gamma) + tr(k^2) - (tr(k))^2 \in \mathbb{R}.$$ 

Moreover, the algebroid $A(\mathfrak{g}(\Sigma))$ is naturally understood to generate deformations of the embedded hypersurfaces respecting the indefiniteness of the metric of the ambient manifold.

We have also shown that the Lie bracket structure of the algebroid bundle is point-dependent and equals the structure given by the Poisson bracket of the ADM constraints. More precisely, for $(\gamma, X, \phi)$ and $(\gamma, Y, \psi)$ being (parts of the) data of vectors $v$ and $u$ in $A(\mathfrak{g}(\Sigma))$, the bracket $[u, v]_\gamma$ is given by the commutator of $g$-gaussian extensions of $(\gamma, X, \phi)$ and $(\gamma, Y, \psi)$ evaluated on the embedded hypersurface, and we have:

$$[u, v]_\gamma = \lbrack X + \phi \hat{n}, Y + \psi \hat{n} \rbrack_\gamma = \lbrack X, Y \rbrack + \phi \text{grad}_Y \psi - \psi \text{grad}_Y \phi + (X(\psi) - Y(\phi)) \hat{n},$$

which reflects the bracket of constraint functions given by pairing $C_{mo}$ and $C_{en}$ with a vector field in $\mathfrak{D}(\Sigma)$ and a function in $\mathfrak{F}(\Sigma)$, according to \([2]\). The bracket structure is then recovered by the commutator of the deformation vector fields when we consider them as respecting the ”$3+1”$ diffeomorphism invariance and hence arising as $g$-gaussian extensions of the lapse and shift. We are now working towards understanding the precise link between those structures, which might be included in the final version of this paper. Let us mention here some intuitions that we have in this context.

Notice first, that equation \(6\) can be understood as an infinitesimal version of the ”path independence principle” (PIP) introduced in \([5]\) applied to the metric tensor. Indeed, the PIP states that the change of any quantity defined over a space-time measured on two different hypersurfaces does not depend on its evolution path. In particular, stretching and normal evolution should commute:

$$\mathcal{L}_v = \mathcal{L}_X + \phi \mathcal{L}_{\dot{n}}, \quad (8)$$

where $v$ is the deformation vector field, and $\phi$ and $X$ re lapse&shift as before. This can be treated as a basic assumption about the evolution in the diffeomorphism-invariant context. The equation above, which we call ”infinitesimal path independence principle” (IPIP), applied to the metric tensor yields a restriction on the deformation vector fields to $g$-gaussian ones and the bracket structure follows, as we have seen. If the constraints can be recovered form their Poisson bracket structure, as claimed in \([5]\), and the origin of the latter can be explained, as above, the theory of General Relativity could be derived from the IPIP through \([6]\).
The precise connection between the commutator of \( g \)-gaussian evolution vector fields and the Poisson bracket structure of the constraints can be approached by generalizing the Hamilton’s equation

\[
\frac{df(t)}{dt} = \{f, H\},
\]

where \( f \) is a function on a phase space and \( H \) the Hamiltonian, to the context of hypersurface evolution. Inspired by [5], we think of the dynamics in this case as being given by the 4-dimensional space-time deformations and generated by both the scalar energy constraint, \( C_{en} \), which generates the evolution in the normal direction, and the vector momentum constraint, \( C_{mo} \), which generates the ‘tangent’ evolution. We then generalize the above equation by

\[
\mathcal{L}_v f = \{f, C_{[X, \phi]}\},
\]

where \( C_{[X, \phi]} \) is a function that comes from pairing the momentum and energy constraints with a lapse function and a shift vector field, according to (2), and \( f \) is a function on space-time. When the (IPIP) is assumed, we get

\[
\mathcal{L}_v f = \mathcal{L}_X f + \phi \mathcal{L}_{\hat{n}} f = X(f) + \phi \hat{n}(f) = \{f, C_{[X, \phi]}\},
\]

which in some sense collapses to (9) when we take \((X, \phi) = (0, 1)\), since then we have \( C_{[0,1]} = C_{en} \) and \( \hat{n} = \partial_t \). However, there are issues that we are still confused about concerning the precise link between the dynamics of an object in space with respect to time, the dynamics of the space-time itself and how these should be coupled to each other. The Poisson bracket structure of the constraints, however, does not depend on the dynamics of fields defined over space-time [5], which makes those issues even more exciting and intriguing.

Let us just mention here some questions that we find interesting to investigate in this context:

- Does \( \mathcal{R}(\Sigma) \Rightarrow \mathcal{U}(\Sigma) \) collapse to \( \Sigma \times \Sigma \Rightarrow \Sigma \) when we pass to Galilean context in some sense?
- What happens when \( \Sigma \) becomes small?
- Can (7) be linked to the vacuum Einstein’s equation through the ideas of [5]?

Moreover, and maybe most importantly, our approach to the dynamical formulation of the Einstein’s equation provides a link to the quantization framework proposed by N.P. Landsman [7] and mentioned in the introduction. It is based on the observation that the \( \mathcal{C}^* \)-algebra generated by a pair groupoid \( TQ \Rightarrow Q \) is isomorphic to the algebra of compact operators of square-integrable functions on \( Q \):

\[
\mathcal{C}^*(TQ) \cong \mathcal{K}(L^2(Q)).
\]

Hence, quantization of a simple system with a configuration space \( TQ \) can be understood as replacing the commutative Poisson algebra \( C^*(T^*Q, \mathbb{R}) \) with the above algebra of operators. Since for a pair groupoid \( TQ \cong A(Q \times Q) \), we can think of classical observable as the smooth real functions on the dual of the algebroid associated to the groupoid that describes the external symmetry structure of the system, while the quantum observables are elements of the \( \mathcal{C}^* \)-algebra generated by this groupoid. It is then interesting to ask what kind of algebra can be generated by our groupoid in a similar way that \( \mathcal{C}^* \)-algebras are generated from Lie groupoids. In the smooth case, this construction involves invariant measures and convolution products of functions on the groupoid, which are structures still missing in the general diffeological setting. Nevertheless, it is interesting to ask if this construction can in some way be generalized and give rise to a non-commutative algebra, understood as the algebra of quantum observables. If this can be done, we would eventually arrive at a quantum description of gravity.

\[\text{10See [5], pg 108, eq. 4.1.}\]
Let us also mention here the paper [2], where the authors of [1] are on the search of the generalization of the concept of a momentum map from the context of Lie algebras to Lie algebroids, in order to eventually apply it to the case of General Relativity. However, the generalization concerns with the smooth case only, the paper dealing with GR being in preparation.

Last but not least, we would like to reflect a bit at this point on "the problem of time" in the context of quantizing a diffeomorphism-invariant theory. In quantization frameworks for theories on a fixed background, like the one described above, time plays a special role being a dimension distinguished from the space coordinates. Indeed, the origin of the tangent bundle $TQ$ can be understood as a consequence of adapting the Galilean concept of time – tangent vectors are geometric representation of the direction of movement in time. Alternatively, in our algebroid we consider deformations of hypersurfaces in space-time (taking care of the diffeomorphism invariance by the relevant quotients), being however also parametrized by a single real number. As a result, the tangent vectors have a different interpretation – they give 4-dimensional deformation vector fields, or pairs $(X, \phi) \in \mathcal{X}(\Sigma) \oplus \mathcal{F}(\Sigma)$ and a pair $(\gamma, k) \in \text{Riem}(\Sigma) \oplus S^2(\Sigma)$. The shift $X$ represents the stretching of space and the lapse $\phi$ is the speed of passing time, the previous case being recovered for $(X, \phi; \gamma, k) = (0, 1, 1, 0)$. It seems that our algebroid approach is a way to grasp the idea of evolution in a situation when the notion of time has not been specified, or is understood much broader than the Galilean sense, a perspective fit to deal with the diffeomorphism invariance of the theory.

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