Modules $M$ such that $\text{Ext}^1_R(M, -)$ Commutes with Direct Limits

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Received: 21 March 2012 / Accepted: 1 October 2012 / Published online: 19 October 2012
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Abstract We will use Watts’s theorem together with Lenzing’s characterization of finitely presented modules via commuting properties of the induced tensor functor in order to study commuting properties of covariant Ext-functors.

Keywords Ext-functor · Direct limit · Hereditary ring

Mathematics Subject Classifications (2010) 16E30 · 16E60 · 18G15

1 Introduction

It is well known that commuting properties of some canonical functors (as Hom or tensor functors induced by a right module) provide important information (about that module) or some important tools in the study of some subcategories for the module category. For instance, H. Lenzing proved in [19, Satz 3] that a right $R$-module $M$ is finitely presented if and only if the functor $\text{Hom}_R(M, -)$ preserves direct limits (i.e. filtered colimits) or the tensor product $- \otimes_R M$ commutes with direct products [19, Satz 2]. These theorems had a great influence in modern algebra: the first result is used to define finitely presented objects in various categories, e.g., [1], while the second is an important ingredient in Chase’s characterization of right coherent rings [11, Theorem 2.1].

Presented by Alain Verschoren.

Research supported by the CNCS-UEFISCDI grant PN-II-RU-TE-2011-3-0065.

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The property that the covariant Hom-functor commutes with direct sums provides us with the notion of small (compact) module [5, p. 54]. This notion is useful in many topics in module theory as generalizations of Morita equivalences [12], almost free modules [24] or the internal structure of the ring [27]. It is well known that every finitely generated module (these are the modules such that the induced covariant Hom-functor commutes with direct unions [29, 24.10]) is small. Moreover, for some important classes of rings (as right noetherian or right perfect by [13, 20]) these two conditions are equivalent, but there are also important types of rings for which there are non-finitely generated small modules (e.g. non-artinian regular simple rings) [14]. It is proved in [26] that every small module is finitely presented if and only if the ring is noetherian. A similar problem can be proposed for the covariant functor Ext$^1_R$. Identify classes of rings R such that a functor Ext$^1_R(M, -)$ commutes with direct unions (or direct limits) if and only if it commutes with direct sums. Corollary 2.5 provides an answer to this problem. Moreover, in Example 2.6 it is shown that these two conditions are not equivalent in general. We mention here that $R$ is right coherent exactly if for every right $R$-module $M$ the functor Ext$^1_R(M, -)$ commutes with direct unions if and only if it commutes with direct limits [7, Corollary 7].

Concerning commuting properties of the derived functors Ext$^*$ and Tor$^*$, it was proved by Brown in [10] that Lenzing’s results can be extended to produce more finiteness conditions. In order to state this results, let us recall from [17, p. 103] that a right $R$ module $M$ is FP$_n$, for a fixed integer $n \geq 0$, if $M$ has a projective resolution which is finitely generated in every dimension $\leq n$.

**Theorem 1.1** [10, Theorem 1], [22, Theorem A] The following are equivalent for a right $R$ module $M$ and a non-negative integer $n$:

1. $M$ is FP$_{n+1}$;
2. Ext$^k_R(M, -)$ commutes with direct limits for all $0 \leq k \leq n$,
3. Tor$^k_R(M, -)$ commutes with direct products for all $0 \leq k \leq n$.

The proof of this theorem is based on Lenzing’s characterizations of finitely presented modules via commuting properties of covariant Hom-functors with direct limits, respectively commuting properties of tensor functors with direct products, and these are applied to obtain independently (1)$\Leftrightarrow$(2) and (1)$\Leftrightarrow$(3). For the case of Ext-functors, this theorem was refined and completed by Strebel in [22]. We can ask if there is a cyclic proof. A connection is suggested in [22, Corollary 1], where it is proved that a right $R$-module $M$ of projective dimension at most $n$ has the property that Ext$^n_R(M, -)$ commutes with direct limits if and only if $M$ has a projective resolution which is finitely generated in dimension $n$.

In the present paper we give such a proof in Theorem 2.4, where modules of projective dimension at most 1 such that the induced covariant Ext$^1$-functor commutes with direct limits are characterized via Lenzing’s characterization of finitely presented modules by commuting properties of the tensor product. This theorem is used to obtain some results from Strebel’s paper [22]. Similar techniques were used by Krause in [18] in order to characterize general coherent functors.

In the end of this introduction let us observe that there are commuting properties which are non-trivial for the Hom-functors but they are trivial for Ext-functors. For
instance, it is well known that the covariant Hom-functors commute with inverse limits. However, as in [17, Example 3.1.8], we can use [6, Theorem 2] to write every module as an inverse limit of injective modules. Therefore

**Proposition 1.2** Let $M$ be a module. The functor $\text{Ext}^1_R(M, -)$ commutes with inverse limits if and only if $M$ is projective.

Dualizing the definition of small modules we obtain the notion of slender module. The structure of these modules can be very complicated (see [15] or [17]). Transferring this approach to the contravariant Ext-functor, we say that the contravariant functor $\text{Ext}_R^n(\cdot, M)$ inverts products if for every countable family $\mathcal{F} = (M_i)_{i < \omega}$ of right $R$-modules the canonical homomorphism $\bigoplus_{i < \omega} \text{Ext}_R^n(M_i, M) \rightarrow \text{Ext}_R^n(\prod_{i < \omega} M_i, M)$ is an isomorphism. However, in the case when $M$ has the injective dimension at most $n$ this property is very restrictive.

**Proposition 1.3** Let $M$ be a right $R$-module of injective dimension at most $n \geq 1$. The functor $\text{Ext}_R^n(\cdot, M)$ inverts products with countably many factors if and only if $M$ is of injective dimension at most $n - 1$.

**Proof** By the dimension shifting formula, it is enough to assume $n = 1$.

In this hypothesis, we start with a module $N$ and with the canonical monomorphism $0 \rightarrow N^{(\omega)} \rightarrow N^{\omega}$. Applying the contravariant Ext-functor and [4, Proposition 2.4] we obtain the natural homomorphism

$$
\text{Ext}_R^1(N, M)^{(\omega)} \cong \text{Ext}_R^1(N^{\omega}, M) \rightarrow \text{Ext}_R^1(N^{(\omega)}, M) \cong \text{Ext}_R^1(N, M)^{\omega},
$$

which coincides with the canonical homomorphism

$$
\text{Ext}_R^1(N, M)^{(\omega)} \rightarrow \text{Ext}_R^1(N, M)^{\omega},
$$

and it is an epimorphism. This is possible only if $\text{Ext}_R^1(N, M) = 0$. \qed

We do not know what happens in Proposition 1.3 in case we do not assume any bound on the injective dimension. It is also an open question when the contravariant Ext$^1$-functor preserves products. In [16] the authors provide an answer for abelian groups, and we refer to [8] for the case of contravariant Hom-functors. Other commuting properties of these functors, for the case of abelian groups, are studied in [3] and [21].

### 2 When Ext Commutes with Direct Limits

Let $R$ be a unital ring and $M$ a right $R$-module. If $\mathcal{S} = (M_i, v_{ij})_{i, j \in I}$ is a direct system of modules and $v_i : M_i \rightarrow \varinjlim M_i$ are the canonical homomorphisms, then for every non-negative integer $n$ there is a canonical homomorphism

$$
\Phi_{\mathcal{S}}^n : \varinjlim \text{Ext}_R^n(M, M_i) \rightarrow \text{Ext}_R^n(M, \varinjlim M_i),
$$

the natural homomorphism induced by the family $\text{Ext}_R^n(M, v_{ij}), i, j \in I$. 

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For the case of direct sums, if we consider $\mathcal{F} = (M_i)_{i \in I}$ a family of modules and we denote by $u_i : M_i \to \bigoplus_{i \in I} M_i$ the canonical homomorphisms then we will obtain a natural homomorphism

$$\Phi^n_{\mathcal{F}} : \bigoplus_{i \in I} \text{Ext}^n_R(M, M_i) \to \text{Ext}^n_R(M, \bigoplus_{i \in I} M_i),$$

induced by the family $\text{Ext}^n_R(M, u_i), i \in I$.

We will use these homomorphisms for the cases $n \in \{0, 1\}$, and we will denote $\Phi^0_{\mathcal{F}} := \Psi^M_{\mathcal{F}}, \Phi^1_{\mathcal{F}} := \Phi^M_{\mathcal{F}}$, respectively $\Phi^0_{\mathcal{F}} := \Psi^M_{\mathcal{F}}, \Phi^1_{\mathcal{F}} := \Phi^M_{\mathcal{F}}$.

We say that $\text{Ext}^n_R(M, -)$ commutes with direct limits (direct sums) if the homomorphisms $\Phi^n_{\mathcal{F}}(\mathcal{M})$ (respectively $\Phi^n_{\mathcal{F}}(\mathcal{M})$) are isomorphisms for all directed systems $\mathcal{F}$ (respectively all families $\mathcal{F}$).

Lenzing’s theorem says us that $M$ is finitely presented if and only if $\Psi^M_{\mathcal{F}}$ are isomorphisms for all $\mathcal{F}$. Moreover, using Theorem 1.1 for $n = 1$, we observe that $M$ is an $FP_2$ module if and only if $\Psi^M_{\mathcal{F}}$ and $\Phi^M_{\mathcal{F}}$ are isomorphisms for all families $\mathcal{F}$. We will start with a slight improvement of this result, replacing the condition “$\Psi^M_{\mathcal{F}}$ is an isomorphism” (i.e. $M$ is finitely presented) by the condition “$M$ is finitely generated”. Therefore in the case of finitely generated modules the hypothesis of [17, Lemma 3.1.6] is sharp.

Following the terminology used in [7], we will call a module $M$ an $fp$-$\Omega^1$-module (fg-$\Omega^1$-module, respectively small-$\Omega^1$-module) if there is a projective resolution

$$(P) : \cdots \to P_2 \to P_1 \overset{\alpha_1}{\to} P_0 \to M \to 0$$

such that the first syzygy $\Omega^1(P) = \text{Im}(\alpha_1)$ is finitely presented (finitely generated, respectively small).

**Lemma 2.1** If $M$ is $fp$-$\Omega^1$ (fg-$\Omega^1$, respectively small-$\Omega^1$) right $R$-modules then there is a projective module $L$ such that $M \oplus L$ is a direct sum of an $FP_2$-module (finitely presented module, respectively finitely generated module) and a projective module.

**Proof** Let $0 \to K \overset{\alpha}{\to} P \overset{\beta}{\to} M \to 0$ be an exact sequence such that $P$ is projective and $K$ is finitely presented (finitely generated, respectively small). If $C$ is a projective module such that $P \oplus C \cong R^I$ is free, then we consider the induced exact sequence $0 \to K \overset{\alpha}{\to} P \oplus C \overset{\beta \oplus 1_C}{\to} M \oplus C \to 0$. Since $K$ is small (recall that every finitely generated module is small) as a right $R$-module there is a finite subset $J$ of $I$ such that $\text{Im}(\alpha) \subseteq R^J$. Then $M \oplus C \cong R^J/\text{Im}(\alpha) \oplus R^{I \setminus J} = H \oplus L$, where $H$ is an $FP_2$-module (finitely presented, respectively finitely generated) and $L$ is projective.

The closure under direct summands of the class of fg-$\Omega^1$-modules can be characterized by the commuting property of the covariant $\text{Ext}^1$-functor with direct unions (i.e. direct system of monomorphisms). We state this result for sake of completeness.

**Theorem 2.2** [7, Theorem 5] The following are equivalent for a right $R$-module $M$:

1. $M$ is a direct summand of an fg-$\Omega^1$-module;
2. The functor $\text{Ext}^1_R(M, -)$ commutes with direct systems of monomorphisms.
For the cases of fp-$\Omega^1$-modules and small-$\Omega^1$-modules we are able to prove a similar result only in the case of finitely generated modules. In fact, it is easy to observe, using [17, Lemma 3.1.6] that the covariant $\text{Ext}^1_R$-functor induced by an fp-$\Omega^1$-modules commutes with direct limits. The converse is true for finitely generated modules [7, Lemma 1]. A similar result is valid for small-$\Omega^1$-modules.

**Theorem 2.3** Let $M$ be a right $R$-module $M$.

1. If $M$ is an fp-$\Omega^1$-module then $\text{Ext}^1_R(M, -)$ commutes with direct limits.
2. If $M$ is a small-$\Omega^1$-module then $\text{Ext}^1_R(M, -)$ commutes with direct sums.

The converses of these statements are true if $M$ is finitely generated.

**Proof** We will prove (2). The proof for (1) follows the same steps. Using Lemma 2.1, we can suppose that $M$ is finitely generated.

If $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ is an exact sequence with $P$ a projective module then for every family $F = (M_i)_{i \in I}$ of right $R$-modules, we have the following useful commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \bigoplus_{i \in I} \text{Hom}_R(M, M_i) & \rightarrow & \bigoplus_{i \in I} \text{Hom}_R(P, M_i) & \rightarrow \\
& & \downarrow \Psi^M \psi & \downarrow \Psi^P \psi & & \\
0 & \rightarrow & \text{Hom}_R(M, \bigoplus_{i \in I} M_i) & \rightarrow & \bigoplus_{i \in I} \text{Hom}_R(P, M_i) & \rightarrow \\
& & \downarrow \Psi^M \phi & & & \\
& \rightarrow & \bigoplus_{i \in I} \text{Hom}_R(K, M_i) & \rightarrow & \bigoplus_{i \in I} \text{Ext}^1_R(M, M_i) & \rightarrow & 0 \\
& & \downarrow \Psi^K \phi & & \downarrow \Phi^M \phi & & \\
& \rightarrow & \text{Hom}_R(K, \bigoplus_{i \in I} M_i) & \rightarrow & \text{Ext}^1_R(M, \bigoplus_{i \in I} M_i) & \rightarrow & 0 \\
\end{array}
\]

whose rows are exact.

Since $M$ and $P$ are finitely generated, the arrows $\Psi^M_{\phi}$ and $\Psi^P_{\phi}$ are isomorphisms by [29, 24.10]. Using Five Lemma we observe that $\Psi^K_{\phi}$ is an isomorphism if and only if $\Phi^M_{\phi}$ is an isomorphism. \qed

We are ready to state the main result of this paper. It states that fp-$\Omega^1$-modules of projective dimension at most 1 can be characterized by commuting properties of the induced covariant $\text{Ext}^1$-functor.

**Theorem 2.4** The following are equivalent for a right $R$-module $M$ of projective dimension 1:

1. $M$ is an fp-$\Omega^1$-module.
2. There is a projective module $L$ such that $M \oplus L$ is a direct sum of an $FP_2$-module ($FP_1$-module) and a projective module.
3. $\text{Ext}^1_R(M, -)$ commutes with direct limits.
4. $\text{Ext}^1_R(M, -)$ commutes with direct sums of copies of $R$.

**Proof** We only need to prove (4) $\Rightarrow$ (1).

Let $M$ be a module of projective dimension at most 1 such that $\text{Ext}^1_R(M, -)$ commutes with direct sums of copies of $R$. Therefore $\text{Ext}^1_R(M, -)$ is a right exact
functor which commutes with direct sums of copies of $R$. By Watts’s theorem [28, Theorem 1] (and its proof) we conclude that the functor $\text{Ext}_R^1(M, -)$ is naturally equivalent to the functor $- \otimes_R \text{Ext}_R^1(M, R)$. It follows that the tensor product functor $- \otimes_R \text{Ext}_R^1(M, R)$ preserves the products, hence that $\text{Ext}_R^1(M, R)$ is a finitely presented left $R$-module.

Let $n$ be a positive integer such that the left $R$-module $\text{Ext}_R^1(M, R)$ is generated by $n$ elements. Using [23, Lemma 6.9] we conclude that there is an exact sequence of right modules $0 \to R^n \to C \to M \to 0$ such that $\text{Ext}_R^1(C, R) = 0$. Starting with this exact sequence we obtain for every cardinal $\kappa$ a commutative diagram

\[
\begin{array}{cccccc}
\text{Hom}(R^n, R^{(\kappa)}) & \longrightarrow & \text{Ext}_R^1(M, R)^{((\kappa))} & \longrightarrow & \text{Ext}_R^1(C, R^{(\kappa)}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Hom}(R^n, R^{(\kappa)}) & \longrightarrow & \text{Ext}_R^1(M, R^{(\kappa)}) & \longrightarrow & \text{Ext}_R^1(C, R^{(\kappa)}) & \longrightarrow & 0
\end{array}
\]

where the vertical arrows are the canonical homomorphisms induced by the universal property of direct sums. Moreover the first vertical is an isomorphism since every finitely generated module is small, while the second is also an isomorphism by our hypothesis. Therefore the third vertical arrow is also an isomorphism. It follows that $\text{Ext}_R^1(C, R^{(\kappa)}) = 0$ for all cardinals $\kappa$. Since $C$ is of projective dimension at most 1, it is not hard to see that $C$ is projective. \qed

**Corollary 2.5** Let $R$ be an right hereditary ring and $M$ a right $R$-module. The following are equivalent:

1. $\text{Ext}_R^1(M, -)$ commutes with direct limits;
2. $\text{Ext}_R^1(M, -)$ commutes with direct sums;
3. $\text{Ext}_R^1(M, -)$ commutes with direct sums of copies of $R$;
4. $M = N \oplus P$, where $N$ is finitely presented and $P$ is projective.

**Proof** Since every right hereditary ring is right coherent, every finitely presented module is an $FP_2$-module. Therefore only (3) $\Rightarrow$ (4) requires a proof.

Suppose that $\text{Ext}_R^1(M, -)$ commutes with direct sums of copies of $R$. Using Theorem 2.4, we observe that there is a projective module $L$ such that $M \oplus L = F \oplus U$ with $F$ finitely presented and $U$ projective. The conclusion follows using the same techniques as in [2]. If $\pi_U : F \oplus U \to U$ is the canonical projection, then $\pi_U(M)$ is projective. Then $F \cap M = \text{Ker}(\pi_U|_M)$ is a direct summand of $M$. Therefore $N = F \cap M$ is a direct summand of $F \oplus U = M \oplus L$. Using this, it is not hard to see that $N = F \cap M$ is a direct summand of $F$. Hence $N$ is finitely presented and $M = N \oplus P$, where $P \cong \pi_U(M)$ is projective. \qed

**Example 2.6** The equivalence (3) $\Leftrightarrow$ (4) from Theorem 2.4 is not valid for general finitely generated modules.
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To see this, it is enough to consider a ring $R$ which has a non-finitely generated small right ideal $I$. For instance, it is proved in [25] that every direct product of infinitely many rings has such an ideal. Since $R/I$ is a finitely generated right $R$-module, we can apply Theorem 2.3 to see that $\text{Ext}^1_R(R/I, -)$ commutes with direct sums, but it does not commute with direct limits.

**Remark 2.7** Corollary 2.5 was proved for abelian groups in [9] without the condition that the isomorphisms $\bigoplus_{i \in I} \text{Ext}^1_R(M, M_i) \cong \text{Ext}^1_R(M, \bigoplus_{i \in I} M_i)$ are the natural ones.

**Corollary 2.8** [22, Corollary 1] Let $M$ be right $R$-module of finite projective dimension $\leq n$. Then $M$ admits a projective resolution which is finitely generated in dimension $n$ if and only if $\text{Ext}^n_R(M, -)$ preserves direct sums of copies of $R$. If one of the equivalent conditions is fulfilled, then the functors $\text{Ext}^n_R(M, -)$ and $- \otimes_R \text{Ext}^n_R(M, R)$ are naturally isomorphic.

**Proof** Let $(P)$ be a projective resolution of $M$. Then the $(n - 1)$-th syzygy $\Omega^{n-1}(P)$ is of projective dimension at most 1. Using the dimension shifting formula we observe that $\text{Ext}^n_R(M, -)$ is naturally isomorphic to $\text{Ext}^1_R(\Omega^{n-1}(P), -)$, and the conclusion follows from Theorem 2.4 and Watts’s theorem. \qed

**Lemma 2.9** Let $L$ be a right $R$-module of projective dimension at most 1 and $K$ a finitely presented submodule of $L$. The module $M = L/K$ has the property that $\text{Ext}^1_R(M, -)$ preserves direct sums of copies of $R$ if and only if $M$ is an $\text{fp-}\Omega^1$-module.

**Proof** Let $I$ be a set. It induces a commutative diagram

$$
\begin{array}{ccccccccc}
\text{Hom}_R(K, R)^{(I)} & \longrightarrow & \text{Ext}^1_R(M, R)^{(I)} & \longrightarrow & \text{Ext}^1_R(L, R)^{(I)} & \longrightarrow \\
\downarrow\psi_K & & \downarrow\phi_M & & \downarrow\phi_L & & \\
\text{Hom}_R(K, R^{(I)}) & \longrightarrow & \text{Ext}^1_R(M, R^{(I)}) & \longrightarrow & \text{Ext}^1_R(L, R^{(I)}) & \longrightarrow \\
& & \downarrow\phi_K & & \downarrow & & \\
& & \text{Ext}^1_R(K, R^{(I)}) & \longrightarrow & \text{Ext}^2_R(M, R^{(I)}) & \longrightarrow & \text{Ext}^2_R(R^{(I)}) & \longrightarrow & 0 \\
\end{array}
$$

where all arrows represent the canonical homomorphisms. Since $K$ is finitely presented, the arrows $\psi_K$ and $\phi_K$ are isomorphisms (for $\phi_K$ we use Theorem 2.3). Moreover, $\phi_M$ is an isomorphism from our hypothesis and it is obvious that the last vertical arrow is also a monomorphism by [22, Lemma 2.2]. Therefore $\text{Ext}^1_R(L, -)$ commutes with direct sums of copies of $R$. By Theorem 2.4, it follows that there is a projective resolution $0 \rightarrow R^n \rightarrow P \rightarrow L \rightarrow 0$. \hfill \$\natural$ Springer
Using all these data we construct a pullback diagram

\[
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
0 & R^n & U & K & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & P & L & 0.
\end{array}
\]

\[
\begin{array}{c}
M \\
\downarrow \\
0 \\
\end{array}
\]

Since \( U \) is finitely presented, the proof is complete. \( \square \)

We also obtain the principal ingredient for the proof of [22, Theorem B].

**Corollary 2.10** [22, Lemma 2.6] Let \( M \) be a right \( R \)-module such that \( \text{Ext}_{R}^{1}(M, -) \) commutes with direct sums of copies of \( R \). Suppose that there is a projective resolution

\[
(P) \quad P_2 \to P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0
\]

such that \( \Omega^2(P) \) is finitely generated. Then there is a projective resolution

\[
(P') \quad P_2 \to P'_1 \to P'_0 \to M \to 0
\]

with \( P'_1 \) finitely generated such that \( \Omega^2(P) = \Omega^2(P') \).

**Proof** We can suppose that there is a set \( I \) such that \( P_1 \cong R^{|I|} \). If \( J \) is a finite subset of \( I \) such that \( \Omega^2(P) \subseteq R^{|J|} \) and \( K = R^{|J|}/\Omega^2(P) \) then we have an exact sequence

\[
0 \to K \oplus R^{|I|} \xrightarrow{\alpha} P_0 \to M \to 0,
\]

where \( \alpha \) is induced by \( \alpha_1 \). If \( M' = P_0/\alpha(R^{|J|}) \) then we have an exact sequence

\[
0 \to K \to M' \to M \to 0.
\]

Observe that \( M' \) is of projective dimension at most 1. By Lemma 2.9, the module \( M \) is an fp-\( \Omega^1 \)-module. Therefore there is an exact sequence
0 → U → Q → M → 0 with Q projective and U finitely presented, and we can construct a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & 0 & 0 & \\
\downarrow & \downarrow & \downarrow & \\
0 & \Omega^2(P) & V & U & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & P_1 & W & Q & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \Omega^1(P) & P_0 & M & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

Since the middle vertical row is splitting, V is a finitely generated projective module. Moreover we can construct an exact sequence 0 → Ω²(P) → V → Q → M → 0, and the proof is complete.

\[\Box\]

Acknowledgements I thank Jan Trlifaj for drawing my attention on Strebel’s work [22]. I also thank to the Referee, whose comments improved this paper.

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