Hikami boxes and the Sinai billiard.

Daniel L. Miller
Dept. of Physics of Complex Systems,
The Weizmann Institute of Science, Rehovot, 76100 Israel
e-mail fdanill@weizmann.ac.il
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Diagram, known in theory of the Anderson localization as the Hikami box, is computed for the Sinai billiard. This interference effect is mostly important for trajectories tangent to the opening of the billiard. This diagram is universal at low frequencies, because of the particle number conservation law. An independent parameter, which we call phase volume of diffraction, determines the corresponding frequency range. This result suggests that level statistics of a generic chaotic system is not universal.

Any function, which describes the physical properties of a single particle bounded system, is called universal in this work, if it is a function of either frequency or time and has the only one parameter: the mean level spacing \( \Delta \) divided by \( \hbar \).

I. FORMAL DERIVATION.

Following Ref. \( ^1 \) the quantum evolution of the particle inside the two-dimensional billiard is convenient to describe by the scattering matrix \( S(\theta, \theta') \). For given energy \( E = \hbar^2 k^2/(2m) \) the wave numbers of the incident and scattered wave functions are \( \vec{k}_i = (k \cos(\theta), k \sin(\theta)) \) and \( \vec{k}_f = (k \cos(\theta'), k \sin(\theta')) \) correspondingly. The wave reflected \( n \) times from the billiard walls is described by the \( n \)-th power of the matrix \( S \).

In the particular case of the Sinai billiard one needs the product of two matrices \( S = S_2 S_1 \), where \( S_2 \) is the scattering matrix of the square, and \( S_1 \) is the scattering matrix of the circle. A similar decomposition was used for quantization of the Sinai billiard \( ^3 \).

The direct product of two scattering matrices is a density evolution operator. The two ways to pair arguments of two matrices give diffusion and cooperon-like operators:

\[
S_{E+\omega}^n(\theta_1, \theta_2)S_{E}^{\dagger n}(\theta_3, \theta_4) = \sum_{l_1 l_2} e^{i(\theta_1 - \theta_4) l_1 - i(\theta_2 - \theta_3) l_2} \\
\times D_{\omega}^n(\theta_1, l_1; \theta_2, l_2), \quad (1a)
\]

\[
S_{E+\omega}^n(\theta_1, \theta_2)S_{E}^{\dagger n}(\theta_3, \theta_4) = \sum_{l_1 l_2} e^{i(\theta_1 - \theta_3 + \pi) l_1 - i(\theta_2 - \theta_4 - \pi) l_2} \\
\times C_{\omega}^n(\theta_1, l_1; \theta_2, l_2), \quad (1b)
\]

where \( l = [\vec{r} \times \vec{k}]_z \) is the angular momentum measured in units of \( \hbar \), the \( z \)-axis points out of the plane.

Operator \( D \) is a one step classical evolution operator. It is well defined if position of a classical particle on the energy shell of the phase space \( \vec{r}, \vec{k}/\hbar \) is described by \( l, \theta \). This is the case for the Sinai billiard, see Fig. 1, if we agree to compute \( l \) and \( \theta \) for a particle, just before it hits the square. Thus each point in the phase space corresponds to either straight piece of trajectory, for example the point A, or trajectory with one reflection from the circle, for example the point B in Fig. 1.

Each trajectory in the configuration space is a sequence of straight lines (segments), it becomes a sequence of points in the phase space; the reflection law generates a map in the phase space

\[
2 = M(1), \quad 1 \equiv (\theta_1, l_1), \quad 2 \equiv (\theta_2, l_2). \quad (2a)
\]

The density evolution operators Eq. (1) computed within semiclassical and diagonal approximations are \( C \approx C, \) and \( D \approx D, \) where

\[
C_{\omega}^n(1; 2) = D_{\omega}^n(1; 2) = 2\pi e^{i\omega t_{1,2}} \delta(2 - M^n(1)), \quad (2b)
\]

\( M^n \) means \( n \) iterations of the map \( M, \) and \( t_{1,2} \) is the time of flight along the trajectory.

The diagonal approximation is justified when classical actions of different trajectories are not correlated. The action of the trajectory is proportional to its length. The only scale of the length-length correlation function is the system size \( 4l, \) and the scale of the action correlations is, therefore, \( \hbar N_H, \) where \( N_H \) is the number of open channels in the system, it is the effective dimensionality of the scattering matrix \( ^4 \), and it is the analog of the Heisenberg time for maps. This argument provides us the condition

\[
n \ll N_H \text{ or } \omega \gg \Delta/\hbar, \quad (3)
\]

where \( \Delta \) is the mean level spacing.

Under the condition Eq. (3) the density evolution operator must preserve the invariant measure \( \hbar \) and therefore \( \sum_n \text{tr} D_{\omega}^n \) must have single pole at \( \omega = 0 \) or alternatively

\[
\text{tr} D_{\omega=0}^n = 1, \quad n \gg n_*, \quad (4)
\]

This sum rule holds for the Frobenius - Perron operator Eq. (21). In this case it is known also as the Hainay - Ozorio de Almeida sum rule \( ^6 \), and \( n_\star \) characterizes decay of Frobenius - Perron modes.

Dashed lines in Fig. 1 mark the parts of the phase space where the semiclassical approximation fails because of diffraction. The phase volume of diffraction is much
smaller than the phase volume of the system. Therefore \( D \approx D + \delta D \), and the first order correction to the evolution operator is

\[
\delta D^n_{\varphi}(1; 2) = \int d3d4 \sum_{n=1}^{n-1} D^n_{\varphi}(1; 3) \times \left[ F_D(3; 4) - \delta(3; 4) \right] D^n_{\varphi}(3; 2)
\]

(5)

\[
F_D(1; 2) = \delta_{l_1l_2}f_D(l_1, \theta_1 - \theta_2)
\]

(6)

\[
f_D(l, \theta) = \int dl' S_{l+l'/2}S_{l-l'/2}e^{il'\theta},
\]

(7)

where \( \delta(1; 2) = 2\pi\delta_l\delta(\theta_1 - \theta_2) \), \( 3 = (\theta_3, l_3) \), \( 4 = (\theta_4, l_4) \), \( \int d3 = \sum_{l_3} \int \frac{d\theta_3}{2\pi} \), \( S_l = -H^{(1)}(kR)/H^{(2)}(kR) \) is the scattering matrix of a circle. The diffraction coefficient \( f_D(l, \theta) \) for \( kR - \alpha' < l < kR + \alpha'' \), where \( \alpha' \sim \alpha'' \sim (kR)^{1/3} \), may be approximated in a number of ways.\cite{4}\, see Fig. 2. Outside this interval, for \( l < kR - \alpha' \), it must reproduce the map \( f_D(l, \theta) = 2\pi\delta(\theta - \arccos(\frac{l}{kR})) \), and for \( l > kR + \alpha'' \) one has \( f_D(\theta) = 2\pi\delta(\theta) \).

FIG. 1. Configuration and phase space of the Sinai billiard. Four areas in the phase space correspond to the four sets of trajectories hitting four walls of the square. Reflections from walls generate a map. It takes the particle from one area of the phase space and put it to other area.

In order to compute the so-called interference correction one should examine all possible pairing of arguments in the product of \( S \) matrices. The further computation is eventually the same as in Ref.\cite{3} and the correction mixing two diffusion and one cooperon like operators is

\[
\delta D^n(1, 2) = \int d3d4 D(1; 3) D(4; 2) \left[ F_I(3; 4) C(4; 3) + F_I^*(3; 4) C(3; 4) \right]
\]

+ \( D(1; 3) D(3; 4) \left[ F_I(3; 4) C(4; 3) + F_I^*(4; 3) C(3; 4) \right] \)

+ \( D(1; 3) D(4; 2) \left[ F_I(3; 4) C(4; 3) + F_I(3; 4) C(3; 4) \right] \)

(8)

where \( \bar{3} \equiv (\theta_3 + \pi, -l_3) \), \( \bar{4} \equiv (\theta_4 + \pi, -l_4) \). The right hand side of Eq. (8) should be understood as a sum over all possible powers \( n_1 + n_2 + n_3 = n \) of the density evolution operators, i.e. \( \delta D^n \propto D^{n_1}C^{n_2}D^{n_3} \). The diffraction kernels are defined in the vicinity of tangency to the cylindrical mirror:

\[
F_I(l_1, \theta_1; l_2, \theta_2) = \delta_{l_1l_2}f_I(l_1, \theta_1 - \theta_2)
\]

(9)

\[
f_l(\theta) = \int dl' S_{l+l'/2}S_{l-l'/2}e^{il'\theta},
\]

(10)

where \( S_l \) is the same as in Eq. (8), and \( 2Re f_l \) for \( kR = 50 \) is shown in Fig. 2. In order to avoid difficulties with self-tracing trajectories we put \( F_I(1; 2) = F_D(1; 2) = 0 \) for \( l_1 < kR - \alpha' \) and \( l_1 > kR + \alpha'' \).

The choice of the constants \( \alpha', \alpha'' \) is restricted by the particle number conservation law

\[
\int d3d4 \left[ F_I(3; 4) + F_I^*(3; 4) + F_D(3; 4) \right] = 0.
\]

(11)

Indeed, the interference correction Eq. (8) has a form of the interaction of the Frobenius - Perron modes. Therefore, the sum over all diffraction and interference corrections shifts the \( \omega = 0 \) pole of the evolution operator. In order to avoid the contradiction with the particle number conservation law in the form of Eq. (9) the shift must be less or of the order of the mean level spacing \( \Delta \). ( In this case the shift should be neglected because of the condition Eq. (11).) The shift of the pole depends on the integral Eq. (11). After all this integral must be so small, that we put it equal to zero.

The interference correction Eq. (8) contains a small parameter: the probability for a given trajectory to visit the diffraction region twice, with the same value of angular momentum. The probability to find two points of trajectory of length \( n \) in the certain part of the phase space growth like \( n^2 \) and this is expected behavior of all terms in the right hand side of Eqs. (8). Because of Eq. (11) these terms have different signs and the overall result behaves like \( n \). There is a similar effect in theory of disordered metals. The interference correction to the density evolution operator consists from three evolution operators connected by the kernel. This kernel is called the Hikami box; it is proportional to the small factor \( \omega \tau \) because of the well known cancelation.\cite{4}
II. LEVEL STATISTICS

The form-factor of the two-point correlation function of energy levels is expressed in terms of scattering matrix traces:

\[ K(\tau) = \frac{1}{N_H} \text{tr} S_n^2 \approx \frac{n}{N_H} \text{tr} \left[ D_{\omega=0}^n + C_{\omega=0}^n \right], \]

\[ \tau \equiv n/N_H \] \hspace{1cm} (12)

In the first order in \( \tau \) one obtains \( K(\tau) = 2\tau \), because of Eq. (11). Averaging probability to find such a trajectory is

\[ \text{ergodicity of Eq. (4); this universal result is a consequence of the} \]

We will go beyond this approximation by taking into account the interference correction to the both diffusion and cooperon evolution operators. We should introduce the third kind of evolution operators, which would take into account exactly the diffraction corrections Eq. (5). Let us, instead, include these corrections into the definition of the operators \( D \) and \( C \). Then the next order in \( \tau \) correction to the form-factor becomes

\[ \delta K(\tau) = \tau n \int d3 \, d4 \sum_{\omega=0} C_{\omega=0}^n(3; 4) \]

\[ \times \left\{ D_{\omega=0}^{n-\tau'}(4; 3) \text{Re} F_1(3; 4) + D_{\omega=0}^{n-\tau''}(3; 3) F_D(3; 4) \right\} \] \hspace{1cm} (13)

The first term in the braces is the sum over periodic orbits of length \( n \) visiting twice the diffraction region with the same value of angular momentum. The probability to find such a trajectory is

\[ \frac{n(n-1) \alpha}{2} \Omega \frac{1}{N_H} \] \hspace{1cm} (14)

where \( \alpha = \int d1 \, d2 \, F_D(1, 2) \approx (kR)^{1/3} \) is the effective phase volume of the diffraction region, and the \( \Omega \) is the volume of the phase space; it is just twice the perimeter of the billiard \( \Omega = 8ka \).

The second term in the braces in Eq. (13) is the sum over periodic orbits which has a self-tracing piece. The probability to find such a trajectory is

\[ \frac{n(n-1) \alpha}{2} \Omega \frac{1}{N_H} \left(1 - e^{-j\alpha/\Omega}\right) \] \hspace{1cm} (15)

where \( j \) is the length of the self-tracing part of the trajectory. The first term in the braces in Eq. (13) partially cancels the second term, because of Eq. (14). Averaging over \( j \) and assuming \( n \gg 1 \) we obtain

\[ K(\tau) = 2\tau - \tau^2 \left(1 - e^{-N_H\alpha/\Omega}\right) + O(\tau^3) \] \hspace{1cm} (16)

where \( \tau = n/\Omega H \) is the time of mixing being measured in the units of the Heisenberg time. At the moment it is not clear whether we should take in to account the correlation between the classical trajectories on the Ehrenfest time scale.

The linear in \( \tau \) term in the right hand side of Eq. (16) implies level repulsion and it cannot be correct if \( R \) is small and one computes the energy levels perturbatively. The perturbation theory works when \( kR < (9 \cdot 2^6 \pi^2)^{1/8} \), and therefore Eq. (16) is valid under the condition \( kR \gg 1 \). This implies \( \alpha \gg 1 \). Since \( \alpha \approx \hbar^{-1/3} \) and \( \tau_* \approx \hbar \) our result Eq. (16) demonstrates two regimes

\[ K(\tau) = 2\tau - \frac{N_H}{2\Omega} \alpha^3 + O((\alpha \tau)^2) \quad \tau_\ast \ll \tau \ll \alpha^{-1}, \] \hspace{1cm} (18)

\[ K(\tau) = 2\tau - \tau^2 + O(\tau^3) \quad \alpha^{-1} \ll \tau \ll 1. \] \hspace{1cm} (19)

The last expression matches the universal form-factor \( K(\tau) = 2\tau - \tau^2 + \tau^3/2 + O(\tau^4) \). The universality of the form-factor of classically chaotic systems was conjectured on basis of numerical data. Our theory supports this conjecture under the condition of Eq. (13).

The interference correction to the form-factor, Eq. (18), can be obtained on the language of the action correlation. One should pair off the trajectory with two small angle scatterings and the nearby trajectory without these two scatterings. Each small angle reflection changes the phase of the wave by \( \pi/3 \), then the total phase difference between two periodic trajectories is \( 2\pi/3 \). This pair contributes \( e^{2\pi i/3} \) to the probability to return. The time reversal pair contributes \( e^{-2\pi i/3} \) and the sum is just \(-1\), that is the right sign of the interference correction.

The special attention must be paid to the case of the mixed boundary conditions at the opening of the billiard. The phase of the exact scattering matrix is

\[ \phi_l(\kappa) = 2\text{arg} \left[H_l(kR) - \kappa H'_l(kR)\right] + \pi \]

\[ = \phi_l(0) - 2\text{arctan} \left[\frac{\kappa \phi'_l(0)}{2}\right] \] \hspace{1cm} (20)

where \( H_l \) and \( H'_l \) are the Hankel function and its derivative, \( \kappa \) is the parameter, the degree of the mixing. Since \( \phi'_l \approx -\partial \phi/\partial \ell \) one get

\[ \phi_l(\kappa) \approx \phi_{l+\kappa}(0). \] \hspace{1cm} (21)

Therefore the mixed boundary conditions just shift the diffraction edge leaving the phase at the tangency unchanged. For the large values of \( \kappa \) the exact position of the diffraction edge is given by complicated expression, but the physical mechanism of the interference remains essentially the same.
III. GENERAL DISCUSSION

Present work breaks the common believe that the level statistic of the ergodic systems is determined by symmetry. This believe is a result of the analyzes of the effective Lagrangian in theory of disordered solids. At the wave number equal to zero the interaction of the diffusion modes is universal and implies the universality of the level statistics.

However, this universality is not occasional, it is the consequence of the ergodicity and the particle number conservation law. Therefore the interaction of the Liouvillian modes in chaos should be universal too. For example one may introduce the δ-correlated disorder potential as the source of the interference in a chaotic system and obtain the interaction of the Liouvillian modes independent of the potential strength. The same result may be obtained in the model with the smooth disorder potential.

In this work we consider an example of chaotic system, the Sinai billiard, in absence of any disorder potential. In such a system, the interference between classical trajectories takes place because of the hard wall diffraction. In the semiclassical limit the phase volume of diffraction is relatively small. The interaction of the Frobenius-Perron modes (we have introduced the area preserving map instead of the Hamiltonian flow) is proportional to this operator is universal. However, the relevant parameter is the time $\tau_D$, a particle needs to enter the region of diffraction. At low frequencies $\omega \tau_D \ll 1$, or long times $t \gg \tau_D$, the density evolution operator is not any more Frobenius-Perron, because of diffraction. According to our results, the interaction between modes of this operator is universal.

The domain of the universality of the interference effect is, therefore, $\Delta/h \ll \omega \ll \tau_D^{-1}$ and it depends explicitly on the phase volume of diffraction. In the case of the Sinai billiard this domain is large enough and our result supports the universality of the level statistics observed numerically. Generic chaotic system can have so long $\tau_D$ at given energy, that its level statistics will never manifest the universality.

In summary we have found that the hard wall diffraction contributes the $\tau^2$ and possibly the high order terms into the form-factor of the energy levels correlation function of the Sinai billiard.

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