Dynamic spin-Hall effect and driven spin helix for linear spin-orbit interactions

Mathias Duckheim¹, Dmitrii L. Maslov², and Daniel Loss³

¹Department of Physics, University of Basel, CH-4056 Basel, Switzerland, and
²Department of Physics, University of Florida, Gainesville, FL 32611-8440, USA

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I. INTRODUCTION

Electron systems with spin-orbit interaction show a variety of spin-electric effects arising from the coupling between (orbital) charge and spin degrees of freedom. The most prominent examples are the spin-Hall effect (SHE) and current-induced spin polarization (CISP), both of which have received substantial interest due to their potential to generate and control spin polarization with electric fields. This type of electrical control is a prerequisite for integrating spin effects into standard lithographic semiconductor structures and, ultimately, utilizing the spin degree of freedom as a carrier of information.

The spin-Hall effect (SHE) manifests itself experimentally as current-induced spin polarization (CISP) at the edges of a Hall-bar (in the absence of a magnetic field). Initial theoretical studies of the SHE in 2D electron systems have focused on linear intrinsic Rashba- and/or Dresselhaus spin-orbit interaction (SOI) and interpreted this boundary spin accumulation in terms of a spin current (defined as a symmetrized product of spin and current densities) flowing transverse to the applied electric field. However, these arguments have been plagued by ambiguities, such as equilibrium spin currents and the absence of spin conservation in systems with intrinsic SOI. Explicit diagrammatic calculations for disordered systems and a more general, non-perturbative argument show that the spin-current is absent in systems with standard linear-in-momentum SOI.

A more straightforward approach is to calculate the quantity directly measured in experiments: the spatially constant spin profiles equivalent to a vanishing spin-Hall effect. We show that an in-plane electric field results in a non-zero ac spin-Hall effect, i.e., it generates a spatially non-uniform out-of-plane polarization even for linear intrinsic spin-orbit interactions. Analyzing different geometries in [001] and [110]-grown quantum wells, we find that although this out-of-plane polarization is typically confined to within a few spin-orbit lengths from the channel edges, it is also possible to generate spatially oscillating spin profiles which extend over the whole channel. The latter is due to the excitation of a driven spin-helix mode in the transverse direction of the channel. We show that while finite frequencies suppress this mode, it can be amplified by a magnetic field tuned to resonance with the frequency of the electric field. In this case, finite size effects at equal strengths of Rashba- and Dresselhaus SOI lead to an enhancement of the magnitude of this helix mode. We comment on the relation between spin currents and boundary conditions.

We derive boundary conditions for the electrically induced spin accumulation in a finite, disordered 2D semiconductor channel. While for DC electric fields these boundary conditions select spatially constant spin profiles equivalent to a vanishing spin-Hall effect, we show that an in-plane ac electric field results in a non-zero ac spin-Hall effect, i.e., it generates a spatially non-uniform out-of-plane polarization even for linear intrinsic spin-orbit interactions. Analyzing different geometries in [001] and [110]-grown quantum wells, we find that although this out-of-plane polarization is typically confined to within a few spin-orbit lengths from the channel edges, it is also possible to generate spatially oscillating spin profiles which extend over the whole channel. The latter is due to the excitation of a driven spin-helix mode in the transverse direction of the channel. We show that while finite frequencies suppress this mode, it can be amplified by a magnetic field tuned to resonance with the frequency of the electric field. In this case, finite size effects at equal strengths of Rashba- and Dresselhaus SOI lead to an enhancement of the magnitude of this helix mode. We comment on the relation between spin currents and boundary conditions.

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tems with SOI were derived microscopically, both in the diffusive\cite{17,19,22,28,29} and ballistic\cite{34} regimes.

It has been shown\cite{22,29} that BCs (for hard-wall spin-conserving boundaries) in disordered\cite{39} systems with linear SOI and for dc electric fields require the spin density to be equal to its value in the bulk, i.e., far away from the boundary, and, thus, lead to a spatially uniform spin profile. This null result is consistent with zero spin currents\cite{13,14,15,16,20}. The experimentally observed dc spin accumulation\cite{35,36} in 2DEGs thus requires an explanation accounting for both extrinsic\cite{22,28} and (cubic\cite{35}) intrinsic effects. That a spin current is finite at finite frequencies and for linear SOIs\cite{29,37,38} however, hints at the presence of boundary spin accumulation in ac solutions. In this article, we focus on the intrinsic mechanism, and show that a dynamic SHE, i.e., boundary spin polarization induced by an ac voltage, is present even in a minimal intrinsic model.

The dynamic SHE arises due to the excitation of spatially non-uniform spin diffusion modes. In the Hall-bar geometry, these modes are excited by a spatially uniform ac electric field and lead to accumulation and spatial oscillations of the spin density close to the boundaries. Analyzing these modes as a function of SOI strengths and in the presence of an external, in-plane magnetic field, we find a spin diffusion mode which is a finite-frequency analog of the persistent spin helix\cite{21,27,32,33}. The relaxation length of this mode -while finite for generic linear SOIs- becomes infinite when the Rashba and Dresselhaus SOI strengths are equal and when the magnetic field is tuned to resonance with the frequency of the electric field. This particularly robust mode, originating from electric-dipole-induced spin resonance (EDSR)\cite{29,37,38,41,42,43,44,45}, gives rise to EDSR (see Sec. VII). Right, (a): a “standard” [001]-grown quantum well with the [110] crystal axis taken along the $x_2$ direction. The bulk polarization $\Omega(\mathbf{eE}) \propto \mathbf{e}_{x_1} \cdot \mathbf{e}_{x_2}$ of a driven spin helix described above.

Dresselhaus SOI -a driven spin helix described above.

The paper is organized as follows. In Sec. II, we introduce our model and formulate the linear response formalism for SHE. In Sec. III we sketch the derivation of the integral equation for the spin density, which is then used to derive the diffusion equation and its boundary conditions. (A more detailed derivation is deferred to appendix A.) In Sec. IV we derive boundary conditions in the presence of ac electric field and comment on the relation between spin currents and these boundary conditions in Sec. V. In Sec. VI we calculate the spatially resolved spin profiles at finite frequencies in various geometries in [001]- and [110]-grown quantum wells. Generation of a driven spin helix under the conditions of EDSR is discussed in Sec. VII.

II. PRELIMINARIES

We consider a disordered 2DEG confined to a quantum well (QW) channel of width $L$ (see Fig. 1) with non-interacting electrons of mass $m$ and charge $e$. The system is described by the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + \Omega(\mathbf{p}) \cdot \mathbf{\sigma} + \mathbf{b}_0 \cdot \mathbf{\sigma} + \mathbf{V}. \quad (1)$$

Here, $\mathbf{p} = (p_1, p_2, 0)$ is the in-plane momentum, $\Omega(\mathbf{p})_i = \Omega_{ij} p_j$ is a linear, vector-valued function of $\mathbf{p}$ describing spin-orbit interaction, $2\mathbf{b}_0 = g\mu_B (B_1, B_2, 0)$ is a magnetic field (equal in magnitude to the Zeeman energy).
applied parallel to the 2DEG, and $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are
the Pauli matrices (and $\sigma^0 = 1$). The disorder potential
$V$ due to static short-ranged impurities randomly dis-
tributed over the channel is characterized by the mean
free path $l = \tau p_F/m$, where $\tau$ is the scattering time and
$p_F$ is the Fermi momentum.

We calculate the impurity-averaged, spatially-
dependent spin density $S^i(r) = \delta(r - \bar{x})$ due to
in-plane ac electric field $E(\omega) = E_0\delta(\omega - \omega_0)$. As it
will be shown below, the overall magnitude of $S$ is
determined by the bulk spin polarization due to CISP
far away from the boundary. We therefore briefly discuss
CISP in different geometries. We define the nominal polariza-

$$S_b \equiv -\nu 2 \Omega (eE(\omega)\tau), \quad (2)$$

(with $\nu = m/2\pi$ being the density of states per spin)
which at zero frequency ($\omega_0 = 0$) coincides with the bulk
polarization. In this case, $S_b$ is simply a paramagnetic
spin response to an effective magnetic field $\Omega (eE_0)\tau$. The
latter is the internal field due to the electrically induced
drift momentum $eE\tau$ and SOI.

Both the magnitude and direction of $S_b$ depend on
the SOI mechanism. We consider two cases (see Fig. 1): the
"standard" [001]- and [110]-grown QW. The Rashba SOI
(with strength $\alpha$) due to an asymmetry in the confine-
ment potential has the same form in both cases and is
assumed to be tunable. The Dresselhaus induced fields
$\Omega_{[001]}$, $\Omega_{[110]}$ are in-plane and out-of-plane in the
[001] and [110]-grown QWs, respectively. For conve-
nience, we define $\xi_\alpha = 2\alpha p_F\tau$, $\xi_\beta = 2\beta p_F\tau$ as the ratios
of the mean free path and spin precession length due the
Rashba and Dresselhaus SOIs, respectively. The vector
couplings of the SOIs are described by

$$\Omega_{[001]} = \begin{pmatrix}
0 & \alpha + \beta & 0 \\
-(\alpha - \beta) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad (3)$$

for case (a) in Fig. 1 and

$$\Omega_{[110]} = \begin{pmatrix}
0 & \alpha & 0 \\
-\alpha & 0 & 0 \\
0 & 0 & \beta
\end{pmatrix} \quad (4)$$

for case (b). In case (a), the bulk polarization $S_b \propto -\mathbf{e} x_1 (\alpha + \beta)$ points along the (negative) $x_1$-axis. When the Rashba-
and Dresselhaus SOIs are of comparable strength, i.e., $\alpha \approx +\beta$ (or $\alpha \approx -\beta$), constructive (de-
structive) interference between the two SOI mechanisms
occurs. In this case, one spin component [along $x_1$ ($x_2$)]
becomes conserved. A similar situation occurs in the
[110]-grown QW, where the out-of-plane spin is conserved
if the Rashba SOI is relatively small. Here the bulk polariza-
tion points out-of-plane and is, thus, easier accessible in
optical measurements.\textsuperscript{2,4,15}

The induced spin density $S^i(r)$ is described by coupled
spin diffusion equations\textsuperscript{15,20,23} which can be derived in

\[ X^{\mu\nu}(r, x) = \frac{1}{2m}\text{tr} \left\{ \sigma^\mu G_{E_F + \omega}^{R}(r, x) \sigma^\nu G_{E_F}^{A}(x, r) \right\}. \]  

Here, $E_F$ is the Fermi energy, tr{$\ldots$} denotes the trace
over spin $s$, $\bar{\nu} = \frac{\hbar}{m} + \Omega_{E}\sigma^k$ is the velocity operator
containing a spin-dependent term, and $G_{E_F}^{R/A}$ is the
impurity-averaged, retarded/advanced Green functions at energy $E$. Note that for $\omega = 0$ the integral equation 5 depends
only on the combination $S - S_b$ so that the spatially uni-
forn spin uniform spin $S = S_b$ is immediate. The uniform spin
profile is equivalent to the absence of the SHE, whose
presence would cause a spatial modulation of $S$ at the
boundary.

\section{DIFFUSION EQUATION}

Far from the sample boundary, the impurity-averaged
Green functions and, hence, the kernel $X^{\mu\nu}(r, x') \approx e^{-|r - x'|/l}$ in Eq. 5 decay on the scale of the mean free path $l$, which is the shortest length scale of the diffusion
problem. The behavior of $S$ on scales larger than $l$
can, thus, be found by expanding: \( S(x) \approx S(r) + (x - r) \partial_r S(r) + \frac{1}{2}(x - r) \delta(x - r) \partial_r \partial_r S(r) \). In this way, one obtains the coupled spin diffusion equation

\[
[-i\omega + \Gamma - D \Delta_r] (S(r) - S_b) = S_b,
\]

where \( D = v_F \Omega / (4\pi) \) is the diffusion constant and \( \Gamma_{\alpha} = \{\text{tr}[(i\Omega)^T]\} \delta_{ij} - \{\text{tr}[i\Omega T]\}2p_F^i \tau \) is the spin relaxation tensor.

We now apply Eq. (7) to the two specific geometries in Fig. 1(a) and (b). Assuming translational invariance along \( e_{x_2} \), we find for the [001]-grown QW with \( E[||\{110\}||e_{x_2}] \)

\[
[-i\omega - D \delta r_{1} + \Gamma_{+} - \Gamma_{-}] (S^1 - S_b) + C_{-} \delta r_{1} S^2 = i\omega S_b, \quad (8)
\]

\[-i\omega - D \delta r_{2} + \Gamma_{+} + \Gamma_{-}] S^3 = 0, \quad (9)
\]

\[-i\omega - D \delta r_{1} + \Gamma_{+} + \Gamma_{-}] S^3 - C_{-} \delta r_{1} S^1 = 0, \quad (10)
\]

where \( \Gamma_{\pm} = 2\pi^2(\alpha \pm \beta)^2, C_{\pm} = 2p_F l(\beta \pm \alpha), \omega_L = 2b_0 \) and \( S_b = -2e\sqrt{E} \delta(\alpha + \beta) \).

In case (b) of a [110]-grown QW with \( b_0, E[||\{110\}||e_{x_2}] \), we find

\[-i\omega - D \delta r_{1} + \Gamma_{+} + \Gamma_{-}] (S^1 - S_b) + C_{-} \delta r_{1} S^2 = i\omega S_b, \quad (11)
\]

\[-i\omega - D \delta r_{2} + \Gamma_{+} + \Gamma_{-}] S^3 = 0, \quad (12)
\]

\[-i\omega - D \delta r_{1} - i\omega + 2\Gamma_{+}] (S^1 - S_b) + C_{-} \delta r_{1} + \omega_{L} - \sqrt{\Gamma_{+} \Gamma_{-}} S^3 = 0, \quad (13)
\]

where \( S_b = -2e\sqrt{E} \delta(\alpha, 0, \beta), \quad \Gamma_{+} = 2\pi^2 \tau_0 \beta^2, \quad \Gamma_{-} = 2\pi^2 \tau_0 \alpha^2, \quad C_{+} = p_F l \beta, \quad C_{-} = p_F l \alpha, \quad \omega_L = 2b_0 \). Note that in this geometry the Dresselhaus SOI adds to \( S_b \), whereas for \( E[||001||] \) the electric field does not couple to the Dresselhaus term.\(^{26}\)

### IV. BOUNDARY CONDITIONS

The diffusion equation (7) has to be supplemented with boundary conditions. These match the bulk solutions of the diffusion equation (7) with the solution of the integral equation Eq. (5) in the region \( 1/p_F \ll x_1 \ll l \) close to the boundary. We follow the approach used in Refs. \(^{14}\) and \(^{29}\). We choose \( x_1 = 0 \) as the boundary and construct the impurity-averaged Green functions \( G^{R/A}(x, x') \), which satisfy the Dyson equation \( \langle x | (\omega - \hat{H}_b - \hat{H}_{\text{SO}} - \Sigma) G^{\ast} | x' \rangle = \delta(x - x') \) with \( \hat{H}_b \) being the Hamiltonian in the absence of SOI and \( \Sigma \) being the self-energy due to impurity scattering.\(^{48,49}\) We, moreover, impose the hard-wall, spin-conserving boundary conditions \( G(x, x') | x_1 = 0 \rightleftharpoons 0 \) for either argument at the boundary.

To 0th order in the SOI, these conditions are satisfied by image constructions \( G^{R/A}_0 = G^{R/A}_{b,0} - G^{R/A}_{b,0} \), where \( G^{R/A}_{b,0} \) is the impurity-averaged Green function in the bulk and \( G^{R/A}_{b,0}(x, x') = G^{R/A}_{b,0}(x, (x' - x_1')) \) is the Green function mirror-reflected at the boundary. Neglecting Friedel oscillations of the self-energy at the boundary, which fall off as \( 1/\sqrt{p_F \tau_0} \), the Green functions \( G^{R/A} \) constructed in this way satisfy the Dyson equation to leading order in \( 1/E \tau_0 \).

To 1st order in \( H_{\text{SO}} \) the Green functions are found as \( G_1 = G_0 \hat{H}_{\text{SO}} G_0 \). By construction, \( G(x, x') = [G_0 + G_1](x, x') \) satisfies the boundary conditions and the Schrödinger equation to linear order in the spin-orbit interaction. Performing a Fourier transform of the Green function \( G(x, x') = \int dp_x G(x, x'|p_x) e^{ip_x(x-x')}/(2\pi) \) along the boundary, we find

\[
G^{R/A}_0 (x_1, x'_1 | p_x) = \frac{\mp i m}{p_E} \left[ e^{\mp ip_E^0 |x_1-x'_1|} - e^{\mp ip_E^0 (x_1+x'_1)} \right],
\]

where \( p_E^0 = \sqrt{2m(E \pm i/2\tau - p_x^2/2m)} \) with \( p_x \) being the momentum along the channel. To first order in \( H_{\text{SO}} \), we find

\[
G^{R/A}_1 (x_1, x'_1 | p_x) = \frac{\mp i m^2 \Omega_{kz} \sigma^k}{p_E^0} (x_1 - x'_1)
\]

\[
\times \left[ e^{\mp ip_E^0 |x_1-x'_1|} - e^{\mp ip_E^0 (x_1+x'_1)} \right] + \ldots,
\]

where the dots stand for additional terms that do not contribute to the integrals below since they are odd in the longitudinal momentum \( p_x \).

We are now in a position to derive boundary conditions using the Green functions from Eqs. (14) and (15). We take the limit \( r \to 0 \) of Eq. (5) and expand \( S^1(x) \approx
$S^i(r) + (x_j - r_j) \frac{\partial}{\partial r_j} S^i(r)$ in the integrand. This yields
\[ 0 = i\omega \tau S_b + (B - 1)(S(0) - S_b) + C_j \frac{\partial}{\partial r_j} S(0), \]
where the coefficients
\[ B^{\mu\nu} = \int_{x'_j > 0} dx'_1 dx'_2 X^{\mu\nu}(x, x') |_{x = 0} \]
\[ C^i_j = \int_{x'_j > 0} dx'_1 dx'_2 X^{i\mu}(x, x')(x'_j - x_j) |_{x = 0} \]
are obtained from the spin-spin correlation function $X^{\mu\nu}$ in Eq. (6) evaluated with the Green’s functions satisfying
\[ \left[ \frac{1}{\Gamma(x)} \right] \]
In this section, we show that a definition of the spin current as the commutator – in contrast to the usual definition
\[ \hat{J}^{\nu}_i \]
where we have neglected terms proportional to $\hat{n}$.
\[ \eta \hat{n} \hat{J}^{\nu}_i(r)|_{r_1 = 0} = 0, \eta = 1, 2, 3. \]
These oscillations. This way, we find the BCs consistent with both the boundary conditions and the diffusion equation [for this see also Ref. 26, Eq. (7), and the boundary conditions, Eq. (19), can be written in terms of the covariant derivative as
\[ -i\omega S^\eta + D\hat{J}^{\nu}_i|_{r_1 = 0} = 0 \]
Thus, spin diffusion with linear SOI has a (formal) analogy to charge diffusion: In charge diffusion, both the diffusion equation $\hat{D} = D\hat{J}$ for the charge density $\rho$ and the BCs $\hat{n} \cdot \hat{J} = 0$ contain the same charge current $\hat{J}$. The current $\hat{J} = \nabla \rho$ is given in terms of the spatial derivative of the density. Analogously, the spin current is given as the SU(2)-covariant derivative of $S^\eta$.

In Ref. 29, in an attempt to identify a spin current directly from the diffusion equation, Eq. (7) was rewritten (for $b_0 = 0$) in the form
\[ -i\omega S^\eta + \left[ \Gamma(S - S_b) \right] S^\eta - D\nabla \cdot \hat{J}^\eta = 0, \]
where the “spin current”
\[ \hat{J}^\eta_i = \frac{\partial}{\partial r_i} \hat{S}^\eta(r) - 4m\Omega_k \epsilon_{knp} \hat{S}^\eta_{ij}(r), \]
however, differs from $\hat{J}^\eta_i$ by a relative factor of 2. This discrepancy is resolved when the definitions Eqs. (20), (21) are used making the introduction of two different spin currents $J$ and $J$ unnecessary.

VI. SOLUTIONS OF THE DIFFUSION EQUATION

First, we obtain a solution of Eq. (7) in an infinitesimal sense. In this case the bulk Green’s functions $G^{R/A}_b(x, x')$ are translation invariant and, thus, $X^{\mu\nu} \equiv \int d^2x' X^{\mu\nu}(r, x')$ becomes independent of $r$. The spatially uniform ansatz
\[ S_\infty = \left[ \Gamma - i\omega \right]^{-1} \Gamma S_b \]
solves both the integral equation [Eq. (15)] and the diffusion equation [Eq. (7)]. The same result for the polarization at finite $\omega$ was found in Ref. 50 using a kinetic equation and in Refs. 37, 38 in the linear response formalism. Remarkably, $S_\infty$ is not simply given by the ac internal field $\Omega(p_d)$ corresponding to ac drift momentum $p_d = eE(\omega)/(1 - i\omega \tau)$, but depends on the spin relaxation rate. Therefore, the deviation of $S_\infty$ from $S_b$ becomes appreciable already at a relatively small frequency $\omega \simeq \Gamma D P$ rather than at a much higher frequency $\omega \simeq \tau^{-1}$, which marks the dispersion of $p_d$. Note also
that there is no dc bulk polarization at $|\alpha| = |\beta|$, if the
limit of $\alpha \to \pm \beta$ is taken before the limit of $\omega \to 0$ [see
Ref. 5] for a discussion of this point.

We now estimate the magnitude of $S_b$. We choose para-

meters similar to the (low-mobility) sample employed in
Ref. 3 except for a higher mobility and a lower sheet
density. With $\alpha = 1.0 \times 10^{-12}$ eVm, sheet density
$n_2 = 1.0 \times 10^{13}$ m$^{-2}$, and transport mean free path $\tau = 5 \times 10^{-13}$ s and choosing $E = 5$ mV/µm, we obtain the
bulk polarization due to Rashba SOI $S_{b,\alpha} \equiv 2\nu eE\tau \alpha = 1.1\mu m^{-2}$, or about 1 spin per $\mu m^2$ ($S_b/n_2 = 0.1\%$). The
magnitude of $S_\infty$ and, as we will see below, the magnitude
of the spatially non-uniform terms in the solution are propor-
tional to $S_b$. Depending on the geometry and on whether the Rashba and Dresselhaus SOIs add con-
 structively or destructively, the overall amplitude of the spin oscillations and edge spin accumulation is modified. In case (a) in Fig. 1 one finds $S_b = S_{b,\alpha}(1 + \beta_{[100]}/\alpha)$ while in case (b) $S_b = S_{b,\alpha}(1,0,\beta_{[110]}/\alpha)$, where $\beta_{[100]}$ and $\beta_{[110]}$ is the Dresselhaus SOI strength in the [001]-
and [110]-grown QW, respectively.

We now focus on the position-dependent spin profile in
a semiconductor channel of finite width. As before, we
assume translational invariance along the channel so that the diffusion equation Eq. (7) becomes an inhomogeneous
ordinary differential equation

$$L(\partial_{r_1})[S(r_1) - S_b] = i\omega\tau S_b$$  \hspace{1cm} (27)

in the transverse coordinate $r_1$, where the differential op-
erator $L(\partial_{r_1})$ is defined by Eqs. (7) and (27). The solution

$$S = S_\infty + e^\theta S_b(r)$$  \hspace{1cm} (28)

consists of the uniform part $S_\infty$, given by Eq. (26) (inhomogeneous solution), and a linear combination of
$k = 1, 2, \ldots, 6$ eigenmodes $S_k = S_k e^{\theta_k r_1}$ satisfying $L(\nabla_r)s(r) = 0$. The wave numbers $\theta_1, \ldots, 6$ (in arbitrary
order) in case (a) are given by

$$\theta_{1,2} = \mp t^{-1}\sqrt{2\tau(\Gamma + i\omega)}$$

$$\theta_{3,4,5,6} = \pm (-1)^{1/2} \frac{1}{2D} \left[ 2D(\Gamma + 2i\omega) - C^2 \right.\
\left. + (-) \mp 2\sqrt{-2D(\Gamma + 2i\omega)C^2 + D^2\Gamma^2} \right]^{1/2}$$  \hspace{1cm} (29)

Some of $\theta_k$ are shown in Fig. 2 as functions of $\alpha/\beta$. The real and imaginary
part of the wave number are responsible for exponentially growing (decaying) and oscillatory
terms in the solution, respectively. The coefficients
$e^{\theta_k}$ are determined by the boundary conditions in the
form $Mc = -(B-\mathbb{1})(S_\infty - S_b), -(B-\mathbb{1})(S_\infty - S_b)$
where $M$ is a $6 \times 6$ matrix obtained by inserting the
general solution into Eq. (19) [see also Eq. (B3) in Appen-
dix B]. The coefficients $e^{\theta_k}$ determine the magnitude of the non-uniform part of $S$, i.e., if all $e^{\theta_k}$ are zero the solution is spatially uniform. Although explicit expressions
for $c$ are too lengthy to be displayed here, the scaling of $c$ with $\omega$ can be found on general grounds. Indeed, all
the entries of the matrix $M^{-1}\text{diag}((B-\mathbb{1}, \ldots, (B-\mathbb{1})$ are
of order 1. The order of magnitude of $c_k$ is thus given by
$|S_\infty - S_b| \approx (\omega/\Gamma)|S_b|$, where the latter holds for
$\omega \lesssim \Gamma$. The non-uniform part of $S$ (proportional to the
c's), thus, scales linearly with $\omega$ for $\omega \ll \Gamma$ and becomes
appreciable at the frequency scale $\omega \approx \Gamma \approx \tau^{-1}$.

A solution for $S$ in a [001]-grown QW (Fig. 1 a) is shown in Fig. 3. The electric field $E$ is along the [110]
axis and the strengths of the Rashba and Dresselhaus SOIs are chosen as $\alpha \approx \beta$, so that the wave numbers (cf.
Fig. 2) are almost imaginary. In this case, oscillations of the out-of-plane spin density $S^z$ extend almost over the
entire channel. Simultaneously with $\alpha$ approaching $-\beta$, however, the internal field $\mathbf{\Omega}(e\mathbf{E}\tau) \propto \alpha + \beta$ and, thus, the
overall amplitude $S_b$ of the spin density becomes small. In other words, suppression of the damping rate $\text{Re} \theta_k \propto |\alpha + \beta|$ close to the special point $\alpha = -\beta$ competes with a suppression of the overall amplitude, so that a purely
oscillatory mode cannot be excited in this geometry.

Figure 4 depicts the polarization profile in a wide [110]-
grown QW as shown in Fig. 1 (b), where the bulk polarization
due to the Dresselhaus SOI is out-of-plane. For a weak Rashba SOI, the wave numbers of the character-
istic modes are almost real, i.e., the modes are strongly
damped. As a result, the polarization close to the bound-
ary is substantially larger than the bulk value given by Eq. (26).
FIG. 5: Polarization in the EDSR geometry $E, B \parallel y$ for case shown in Fig. 1(a) with $S_B = 0.02 \mu m^{-2}$. Upper panel: $\text{Im} S^3(x = 200)$ (black) and $\text{Im} S^4(x = 200)$ (grey curve) are shown as a function of $\omega_L \tau$. Resonance is seen at $\pm \omega_L \tau = \omega_T = 10^{-3}$. Parameters of the [001]-grown QW: $\xi_\beta = -0.08$, $\xi_\alpha = 0.1$, $L = 100 \mu m$. Lower Panel: Density plot of $\text{Re} S^3(x)$, $\text{Im} S^4(x)$ as a function of $x$ and $\omega_L \tau/\xi_\alpha^2$ for the same parameters.

VII. EDSR AND DRIVEN SPIN HELIX

We now focus on electric-dipole-induced spin resonance (EDSR) in the finite Hall bar geometry. We calculate the spin polarization $S$ due to a simultaneous effect of ac electric field and dc magnetic field $b_0$, both along the channel. The directions of the fields are chosen in such a way so that the internal field $\Omega(eE(\omega) \tau)$ and $b_0$ are perpendicular. This geometry is suitable for an observation of electrically driven Rabi oscillations of the spin density between the directions along and opposite to $b_0$.

We focus on case (a) in Fig. 2. The magnetic field $b_0$ leads to an equilibrium polarization (Pauli paramagnetism) $S_{b_0} \propto \hat{x}_2 \omega_L = 2 \hat{b}_0$ in the longitudinal direction of the channel. In addition, the polarization in the bulk of the sample (transverse to $b_0$) is modified. In the geometry of Fig. 2(a) with $b_0 || e_{x_2}$, we find for the bulk polarization

$$S_\infty = \begin{pmatrix} (\omega_L^2 + \Gamma_- (\Gamma_+ + \Gamma_- - i \omega)) \\ 0 \\ -i \omega \omega_L \\
\end{pmatrix}$$

where $\Gamma_\pm = 2 p_\perp^2 r (\alpha \pm \beta)^2$. In the absence of the magnetic field, i.e., for $\omega_L = 0$, Eq. (30) reduces to Eq. (26). Additionally, the characteristic modes change due to $b_0$. The wave numbers $\theta$ are determined by the requirement of vanishing eigenvalues

$$\frac{1}{2} \Gamma_+ + \Gamma_- - D \theta^2 - i \omega \pm \frac{1}{2} \sqrt{\Gamma_+^2 - 4 (\omega_L - \theta C_-)^2} = 0$$

of the differential operator $L(\theta)$ defined by Eqs. (7, 27). We focus on the case of $\alpha \approx -\beta$. Expanding to first order in $\Gamma_+/(\omega_L - \theta C_-) \ll 1$, one finds

$$\theta_{1,2} = \pm l^{-1} \sqrt{2 \tau (\Gamma_+ - i \omega)},$$

$$\theta_{3,4} = \frac{i C_- \pm \sqrt{2 D (\Gamma_+ - 2 i (\omega + \omega_L))}}{2 D},$$

$$\theta_{5,6} = \frac{-i C_- \pm \sqrt{2 D (\Gamma_+ - 2 i (\omega - \omega_L))}}{2 D}.$$ 

At resonance, i.e., for $\omega_L = \omega$, the wave numbers $\theta_{5,6}$ in Eq. (32) become purely imaginary because $\Gamma_+ = 0$ for $\alpha = -\beta$. The modes $s_{5,6}$ are thus completely damped oscillations of the spin density with wave length $\lambda_{ SO} = 1/2 m(\beta - \alpha)$ [cf. Fig 2]. Note that in the considered case of $\alpha = -\beta$ the Hamiltonian commutes with the longitudinal spin $[H, \sigma^z] = 0$, i.e., the $U(1)$-symmetry described in Ref. 32 remains intact; however, the $SU(2)$-symmetry used in Ref. 21 to demonstrate the existence of the persistent spin helix is broken in the presence of $b_0$.

Figure 6 shows a profile of the spin polarization under EDSR conditions. At resonance $(\omega_L = \pm \omega)$, the overall amplitude of the out-of-plane polarization is enhanced. This enhancement becomes particularly strong for $\Gamma_+ \approx 0$ occurring at $\alpha = -\beta$.

Solving the diffusion equation [Eq. (4)] to first order in $\Gamma_+$ for the case $\alpha \approx -\beta$, $\omega \approx \pm \omega_L$, we obtain the following expression for the spin density close to resonance ($\omega \approx \pm \omega_L$)

$$S(r_1) \approx S_\infty + \frac{(S_1^\perp - S_\infty^\perp - i S_\infty^3)}{\sinh (\lambda_{SO} l R)} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} e^{-i r_1/\lambda_{SO}} \cosh (R (L - r_1)/l) \\ -e^{i (L - r_1)/\lambda_{SO}} \cosh (R r_1/l) \end{pmatrix}$$

(33)
where \( R = \sqrt{\tau_+ - 2i(\omega - \omega_L)} \tau \). Equation (33) describes a spin density wave along the transverse direction of the Hall bar with wave length \( \lambda_{SO} \) and an amplitude proportional to \( 1/\sinh(LR/l) \). We discuss this result in more detail below. Inserting Eq. (30) for \( S_{\omega} \) into Eq. (33), setting \( \omega = \omega_L \), and expanding the hyperbolic functions in Eq. (33) for a narrow channel with width \( L \ll \lambda_{SO} = 1/2m(\alpha + \beta) \), one obtains the dominant \( \alpha \)-dependence of \( S \) around the \( \alpha \approx -\beta \) point as

\[
S^3(r_1) \approx K \frac{\alpha + \beta}{(\xi_\alpha + \xi_\beta)^2 + 2\tau \Gamma_{\text{res}}} \times e^{-i r_\alpha / \lambda_{SO}} \left[ e^{i L / \lambda_{SO}} - 1 \right],
\]

where \( K = [-i \omega/\Gamma_\alpha](-2\nu e \tau)(\xi_\beta - \xi_\alpha)/L \) depends only on the combination \( \beta - \alpha \). Here, we introduced a phenomenological linewidth \( \Gamma_{\text{res}} = \Gamma_{\text{res}}^y + 2\Gamma_{\text{res}}^x + O((\omega \tau)^2) \) to model the regularization of the amplitude of \( S^3 \) at \( \alpha + \beta = 0 \), which for \( \Gamma_{\text{res}} = 0 \) would diverge as \( 1/(\alpha + \beta) \). For \( \alpha + \beta = 0 \), the relaxation mechanisms due to linear intrinsic SOIs, which are dominant for generic \( \alpha \neq \pm \beta \), are ineffective, and finite spin relaxation rates \( \Gamma_{\text{res}}^y \) and \( \Gamma_{\text{res}}^x \) of the \( x_1 \) and \( x_2 \) spin components, respectively, are due to an extrinsic or cubic Dresselhaus SOI.

Equation (33) describes a spin density wave \( S^3(r_1) \) at frequency \( \omega \) with a spatial profile of the form \( e^{-i r_\alpha / \lambda_{SO}} \). The real and imaginary parts of \( S^3 \) have stationary nodes separated by the shortest of the two SO lengths, i.e., \( \lambda_{SO} \). In addition, the spin profile is subject to a quantization condition: \( S \) is proportional to a factor \( 1 - e^{i L / \lambda_{SO}} \), which vanishes for \( L = 2\pi N \lambda_{SO} \) (with \( N \) being an integer) and becomes maximal for \( L = (2N + 1)\pi \lambda_{SO} \). The profile described by Eq. (34) arises due to an excitation of the spin helix modes \( s_{3,6} \) under the EDSR conditions. The spatial oscillations of these modes have the same “magic” wave number \( \theta = 1/\lambda_{SO} \) as the static persistent spin helix.\(^{21,22}\) However, whereas the persistent spin helix is time-independent, the spin profiles in Eq. (33) oscillate also in time at each point \( r_1 \) with the frequency \( \omega_0 \) of the applied electric field. The explicit time-dependence, e.g., \( S(r_1, t) \propto \sin \left( r_1 / \lambda_{SO} + \omega_0 t \right) \) for \( L = (2N + 1)\pi \lambda_{SO} \) and for \( \mathbf{E}(t) = E_0 \cos(\omega_0 t) \), is obtained by inverse Fourier transform of Eqs. (33)\(^{21}\). This driven spin helix is a generalization of a static spin helix structure to the time-dependent case.

Spatial quantization due to the Hall-bar boundaries, moreover, leads to further enhancement of the amplitude of the spin helix modes in the EDSR regime. The amplitude \( (\alpha + \beta)/[(\xi_\alpha + \xi_\beta)^2 + 2\tau \Gamma_{\text{res}}] \) is infinite for \( \alpha = -\beta \) in a model with strictly linear SOI, i.e., for \( \Gamma_{\text{res}} = 0 \), but is regularized by the next-to-leading order effects due to cubic Dresselhaus and extrinsic SOIs, giving rise to a finite linewidth \( \Gamma_{\text{res}} \).\(^{22}\) Such an enhancement of the amplitude of the driven spin helix close to the \( \alpha = -\beta \) point in relatively narrow QWs may be observable, e.g., by optical techniques.\(^{22}\)

**VIII. CONCLUSIONS**

In conclusion, we have described several signatures of electrically induced spin polarization and the spin-Hall effect due to linear spin-orbit interactions. We have shown that the spin-Hall effect and edge spin accumulation—while being absent for dc electric fields—becomes finite for time-dependent electric fields. In particular, we have found that boundary effects can extend over the whole sample due to driven spin helix modes for the case of the linear Rashba and Dresselhaus spin-orbit interaction being of equal strengths. The amplitude of these helix modes as a function of the spin-orbit interaction strengths is strongly enhanced due to spatial quantization under the conditions of electric-dipole-induced spin resonance.

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**APPENDIX A: SPIN DIFFUSION EQUATION**

We start from the impurity averaged Kubo formula (for \( E_F \tau \gg 1 \)) for the spin density

\[
S^i(r) = \left[ \begin{array}{c c} R & A \end{array} \right] + \frac{e}{2\pi} \int d^2 x' \delta^{i \nu} \delta(r - x') \left[ \begin{array}{c c} R & A \end{array} \right] \left[ \begin{array}{c} E_j(r) \end{array} \right]
\]

\[
= \frac{e}{2\pi} \int d^2 x' \int d^2 y \gamma^{i \nu} \left( x, x' \right) \gamma^{\nu} \left( x' \right),
\]

where solid lines denote impurity averaged Green’s functions \( G^{R/A}_{E} \) and dashed lines denote correlators of impurity potential.\(^{37,47,52}\) The first term of Eq. (A1) is the “bubble” diagram \( \gamma^{i \nu}(r) = \text{tr} \left\{ \left[ \begin{array}{c} \sigma_\nu \gamma X^{i \mu} \right] \right\} E_j(r) \), i.e., a spin response to the electric field in the absence of vertex corrections. The latter are described by the diffusion \( D^{i \nu} (r, x) \), which is defined by the integral equation

\[
D^{i \nu}(r, x') = \frac{\delta^{i \nu} \delta(r - x')}{2m \tau} + \int d^2 y \left[ \begin{array}{c} \gamma^{i \nu}(r, y) \gamma^{\nu} \left( y, x' \right) \right],
\]

where \( \gamma^{i \nu} \) is given by Eq. (B1). Iterating Eq. (A1) once with the help of Eq. (A2), we find

\[
S^i(r) = \frac{e}{2\pi} \left( 2m \tau \right) \int d^2 x' \left[ D^{i \nu}(r, x') \gamma^{\nu}(x') \right].
\]
Multiplying Eq. (A2) by $\frac{e}{2}\gamma^i(x')$ and integrating over $x'$, we obtain the integral equation for the spin density

$$S^i(r) = \frac{e}{2\pi} \gamma^i(r) + \int d^2x \, X^{ij}(r, x) S^j(x),$$

(A4)

which can be further simplified by partially evaluating the “bubble” term $\gamma^i(r)$ in Eq. (A4). We define the spin-momentum correlation functions $Y_{ij}^{\eta}(r) = \int d^2x \, G^{(ij)}(r, x) X^{kl}(x) S^l_k + 2m\tau e/2\pi Y^{ij}(r) E_j$. We can rewrite Eq. (A4) as

$$S^i(r) - S^i_b = \int d^2x \, X^{ij}(r, x)(S^j(x) - S^j_b)$$

and in the bulk separately. In the bulk, one obviously has $[Y(r) - Y_b] = 0$ and arrives thus at Eq. (5). At the boundary, the Green’s functions $G^{R/A}_0 = G^{R/A}_{b,0} - G^{R/A}_{b,0}$ constructed in Sec. V have to be used to evaluate $\gamma^i(r)$, $Y^{ij}(r)$. Neglecting terms oscillating with a period of $1/\tau$, as described in Sec. IV, one finds $Y^{ij}(r) E_j = Y^{ij}_b E_j$ to linear order in the SOI. Therefore, the last term in Eq. (A7) vanishes. Consequently, Eq. (A7) turns into Eq. (5) and can be used for the derivation of both the bulk diffusion equation and the boundary conditions.

**APPENDIX B: BOUNDARY CONDITIONS**

For the coefficients $B$ and $C$ in Eq. (10) describing a boundary with normal vector $\hat{n}$, we found

$$\delta B^{\mu\nu}(\hat{n}) \equiv [B - \mathbb{1}]^{\mu\nu} = \frac{-2\pi}{\hbar F} \tau \Omega^m(n)\epsilon_{m\nu\mu},$$

(B1)

$$C^\mu_j(\hat{n}) = \frac{-2\pi}{\hbar F} \epsilon^{\mu\nu} \hat{n} \cdot e_j,$$

(B2)

where we neglected terms proportional to $\tau \ll 1$. We define a $6 \times 6$ matrix

$$M = \begin{pmatrix}
(\delta B(\hat{n}) + \theta_1 C(\hat{n})) s_{1,0} e^{\theta_1 r} & \cdots & (\delta B(\hat{n}) + \theta_4 C(\hat{n})) s_{4,0} e^{\theta_4 r} \\
(\delta B(-\hat{n}) + \theta_1 C(-\hat{n})) s_{1,0} e^{-\theta_1 r} & \cdots & (\delta B(-\hat{n}) + \theta_4 C(-\hat{n})) s_{4,0} e^{-\theta_4 r}
\end{pmatrix}$$

(B3)

and a vector $A = (A_0, A_L)$, where $A_{0,L} = \delta B(\pm \hat{n})(S_\infty - S_b)$. Inserting the general solution $S = S_\infty + c_k s_{0,k} e^{\theta_k r}$ into Eq. (10), the BCs can be rewritten as $Mc = -A$.
