CLASSES OF FORMS WITT EQUIVALENT TO A SECOND TRACE FORM OVER FIELDS OF CHARACTERISTIC TWO

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Abstract. Let $F$ be a field of characteristic two. We determine all non-hyperbolic quadratic forms over $F$ that are Witt equivalent to a second trace form.

1. Introduction

Let $E/F$ be a finite separable field extension. We define the trace form for this extension by $q(x) = \text{tr}_{E/F}(x^2)$. When the characteristic of $F$ is not equal to 2, the trace form $(E, q)$ is non-degenerate. However, if the characteristic is 2 then $(E, q)$ is degenerate and splits as $\perp V$, with $V$ totally isotropic. It is therefore natural to introduce a modified “second trace form”. To this end one considers for each $a \in E$ its characteristic polynomial

$$p(x, a) = x^n - T_1(a)x^{n-1} + T_2(a)x^{n-2} + \cdots + (-1)^n T_n(a)$$

(whence $T_1(a) = \text{tr}_{E/F}(a)$ and $T_n(a)$ is the norm of $a$). It is clear that $(E, T_2)$ is a quadratic form. When the degree $n$ of the extension is odd this form is necessarily singular. To arrive at a non-degenerate form, two methods have been proposed in the literature. One method, due to Bergé and Martinet [BM], increases the dimension of the space by 1 using the étale $F$-algebra. The other method, due to Revoy [R], reduces the dimension of the space by 1. In this note we will adopt the second method and call such forms 2-trace forms.

We consider the following problem: Which elements $[q] \neq 0$ of the Witt-group $W_q(F)$ are represented by 2-trace forms? Our theorem 3 fully answers this question. Moreover, we will partially answer the same question for $[q] = 0$ (see Prop. 1 and Prop. 2). For fields of characteristic not equal to 2 this problem seems quite more complicated: for partial results concerning generic fields one may consult [CP] and [EHP]; a complete solution for Hilbertian fields is given in [Sch], [KS] and [Wat].

2. The second trace form

As we remarked in the introduction, there are two ways to define a second trace form. In this section we will prove that in fact the corresponding forms are Witt equivalent.

Let $E/F$ be a finite separable field extension. The second trace form $T_{E/F}$ of the extension $E/F$ was defined by Revoy [R] as $(E, T_2)$ if the degree $[E : F]$ is even, and as $(E_0, T_2)$ if the degree is odd, where $T_1$, $T_2$ are given by (1) and $E_0 = \text{Ker} T_1$. 

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It is important to remark that the bilinear form $b_q$ associated to $T_{E/F}$ satisfies the following relations:

\[ b_q(x, y) = T_2(x + y) - T_2(x) - T_2(y) = T_1(xy) - T_1(x)T_1(y), \]  
\[ T_1(x^2) = (T_1(x))^2 \quad \text{and} \quad b_q(x^2, y^2) = b_q(x, y)^2. \]

The assertion is deduced from Theorem 1 above and Theorem 3.5 in [Sa1, p. 150]. In order to illustrate the ideas, we will give the proof in case $[E : F] = 2n + 1$.

Proof. The assertion follows from Theorem 1 above and Theorem 3.5 in [BM, p. 13-14].

In this section we determine all non-hyperbolic quadratic forms over $F$ that are Witt equivalent to some second trace form. Furthermore we give fields where hyperbolic forms are Witt equivalent to a second trace form.

**Theorem 2.** Let $E/F$ be a finite separable field extension, with $[E : F] = 2n + 1$ or $[E : F] = 2n$. Then $T_{E/F} \cong (n-1)\mathbb{H} \perp [1, a]$, for some $a \in F$.

**Proof.** Theorem 2 is deduced from Theorem 1 above and Theorem 3.5 in [BM, p. 13-14].

3. **2-algebraic forms**

A non hyperbolic quadratic form $(V, q)$ over $F$ is 2-algebraic if and only if $(V, q) = r\mathbb{H} \perp (V_a, q_a)$, with $V_a$ an anisotropic plane representing 1.

**Corollary 1.** A non hyperbolic quadratic form $(V, q)$ over $F$ is 2-algebraic if and only if $(V, q) = r\mathbb{H} \perp (V_a, q_a)$, with $V_a$ an anisotropic plane representing 1.

**Example 1.** Let $F = \mathbb{F}_2(a)$ and $E = \mathbb{F}(b)$, where $a^2 + a + 1 = 0$ and $b^3 + b + a = 0$. Then $T_{E/F} = [1, a] \neq [1, 1]$.

In fact, using (2), we see that $(b^2, (1+a)b)$ is a symplectic basis for $E_0 = \text{Ker } T_1$. Since $p(x, (1+a)b) = x^3 + ax + 1 \in \varphi(F)$ and $a \notin \varphi(F)$, we obtain the form $(E_0, T_2) = [1, a] \neq [1, 1]$. 
Corollary 2. If a non hyperbolic quadratic form \((V, q)\) over \(F\) is 2-algebraic then there exists a quadratic extension field \(E\) of \(F\) such that the extension \((V \otimes_F E, q_E)\) is hyperbolic.

Proof: See the proof of Theorem 3 and note that \([1, b] = \mathbb{H}\) over \(E = F(\alpha)\), with \(\alpha^2 + \alpha + b = 0\) (see [Sa1, p. 150]).

Theorem 3. Let \(F = \mathbb{F}_2\) or \(F = \mathbb{F}_2(t)\) with \(t\) transcendental over \(\mathbb{F}_2\). Then hyperbolic quadratics form over \(F\) are 2-algebraic.

Proof. We only need to find an extension \(E\) of \(F\) such that \(T_{E/F}\) is hyperbolic. We first remark that \(p(x) := x^4 + x^3 + 1\) is irreducible over \(\mathbb{F}_2\) and also over \(\mathbb{F}_2(t)\). Let \(\alpha\) be a root of \(p\) and \(E = F(\alpha)\). We decompose the trace form \(T_{E/F}\) with respect to the basis \(\{\alpha, 1 + \alpha\} \cup \{\alpha^2, \alpha + \alpha^2 + \alpha^3\}\). Noting that this basis has the elements conjugate to \(\alpha\), it is easy to recognise that each vector basis is isotropic, and furthermore by (2) we see that it is a symplectic basis. Hence, the space is hyperbolic.

Theorem 4. Let \(F\) be a field. If there exists \(a \in F^*\) and \(n\) odd such that the polynomial \(x^n - a\) is irreducible over \(F[x]\), then hyperbolic quadratics space over \(F\) are 2-algebraic.

Proof. Let \(E = F(\alpha)\), where \(\alpha \in \overline{F}\) and \(\alpha^n = a\). For \(1 \leq k \leq n - 1\), the linear transformation \(f_{\alpha^k} : x \mapsto x\alpha^k\) is given by the matrix \(c_{ij}(k)\), where

\[
c_{ij}(k) = \begin{cases} 1 & \text{if } j = i - k \\ a & \text{if } j = n + i - k \\ 0 & \text{otherwise} \end{cases}
\]

Then for \(1 \leq k \leq n - 1\), \(\alpha^k \in E_0\), because \(c_{ii}(k) = 0\) for each \(i\). Noting that \(T_1(a) = a\) we obtain the decomposition

\[
E_0 = \langle \alpha, \alpha^{n-1} \rangle \perp \langle \alpha^2, \alpha^{n-2} \rangle \perp \cdots \perp \langle \alpha^{n-1}, \alpha \alpha^{n-1} \rangle,
\]

where \((x, y)\) is the space generate by \(x\) and \(y\). Hence, using that \(n \neq 2k\), we deduce that \(T_2(a^k) = 0\) for \(1 \leq k \leq n - 1\), so \((E_0, T_2) = (\mathbb{H}^\perp)\).

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