In this paper, we introduce and analyze the approximation properties of bivariate generalization for the family of Kantorovich type exponential sampling series. We derive the basic convergence result and Voronovskaya type theorem for the proposed sampling series. Using logarithmic modulus of smoothness, we establish the quantitative estimate of order of convergence for the Kantorovich type exponential sampling series. Furthermore, we study the convergence results for the generalized Boolean sum (GBS) operator associated with bivariate Kantorovich exponential sampling series. At the end, we provide a few examples of kernels to which the presented theory can be applied along with the graphical representation and error estimates.

KEYWORDS
GBS operators, Kantorovich type exponential sampling series, Mellin B-continuous, Mellin transform, mixed modulus of smoothness

MSC CLASSIFICATION
41A35, 30D10, 94A20, 41A25

1 INTRODUCTION

The exponential sampling formula was first studied by a group of physicists Bertero and Pike [1] and Gori [2]. They established a series representation for the class of Mellin band-limited functions using the samples exponentially spaced on the positive real line. For \( f: \mathbb{R}_+ \to \mathbb{C} \) and \( c \in \mathbb{R} \), the exponential sampling formula is given by

\[
(E_{c,T} f)(x) := \sum_{k=-\infty}^{\infty} \text{lin}_c(e^{-kx})(e^k) f(e^k),
\]

where \( \text{lin}_c(x) = \frac{x^c - x^{-c}}{2\pi i \log x} = x^{-c} \text{sinc}(\log x) \) with continuous extension \( \text{lin}_c(1) = 1 \). The above exponential sampling formula is known as Mellin version of the WKS sampling formula (see Butzer and Stens [3]). Butzer and Jansche [4] started the study of exponential series by a more mathematical point of view exploiting the Mellin analysis (also see previous studies [5–7]). An independent study of Mellin theory was studied by Mamedov [8]. We also refer to Bardaro et al. [9, 10] for advancements in the Mellin theory. Bardaro et al. generalized the notion of exponential sampling formula and studied convergence properties of the generalized exponential sampling series and their linear combination in Bardaro et al. [11] and Balsamo and Mantellini [12], respectively. The approximation properties for these series were
studied in Mellin–Lebesgue spaces in Bardaro et al. [13] while the approximation properties for the bivariate generalized exponential sampling operators have been investigated in Bardaro et al. [14]. In order to reduce the time jitter error, the Kantorovich version of the generalized exponential sampling series was introduced and studied in Kumar and Shivam [15]. Let \( x \in \mathbb{R}_+, \ w \geq 0 \) and \( k \in \mathbb{Z} \), the Kantorovich type exponential sampling series is defined as (see Kumar and Shivam [15])

\[
(I^k_w f)(x) := \sum_{k=-\infty}^{+\infty} \chi(e^{-k/w}) \left( w \int_{k}^{k+1} f(e^u) du \right),
\]

where \( f : \mathbb{R}_+ \to \mathbb{R} \) is locally integrable such that the above series is absolutely convergent for every \( x \in \mathbb{R}_+ \). This provides an useful tool to approximate not necessarily continuous but Lebesgue integrable functions by using its sample values at the nodes \( (e^k/w)_{n=0} \), \( k \in \mathbb{Z} \). To improve the order of approximation for the above family, the linear combination of these operators was considered in Bajpeyi and Kumar [16]. Further, the generalized form of Kantorovich exponential sampling series was studied in Aral et al. [17]. The approximation of discontinuous signals by the Kantorovich exponential sampling series was discussed in Kumar et al. [18], and its inverse approximation results are given in Bajpeyi et al. [19].

The Kantorovich type modification of sampling operators has been a topic of deep interest due to its wide applications in approximation theory. The Kantorovich type modification for the classical sampling series was first introduced and studied in the setting of Orlicz spaces in Bardaro et al. [20]. The study of associated GBS operators. Thispaper deals with the convergence properties of bivariate Kantorovich exponential sampling series along with the study of associated GBS operators. This is organized as follows. We introduce some basic definitions, conditions, and preliminary results in Section 2. The basic convergence theorem, Voronovskaya type theorem, and its quantitative estimates in terms of modulus of continuity for the proposed family of operators have been derived in Section 3. In Section 4, we define the GBS-bivariate Kantorovich exponential sampling series and establish rate of approximation using the notion of Mellin–Bögel continuous and Mellin–Bögel differentiable functions. We also provide a few examples based on the presented theory along with graphical representation and error estimates in Section 5.

## 2 PRELIMINARIES

Let \( \mathbb{R}_+ \) denote the set of all positive real numbers and \( \mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+ \). Let \( C^0(\mathbb{R}_+^2) \) denote the space of all continuous functions with up to \( r \)th-order partial derivatives are continuous and bounded on \( \mathbb{R}_+^2 \) equipped with the supremum norm \( \| f \|_\infty := \sup_{(x,y) \in \mathbb{R}_+^2} |f(x,y)| \). For convenience, we set \( C(\mathbb{R}_+^2) := C^0(\mathbb{R}_+^2) \). For any \( \delta > 0 \) and \( (x,y) \in \mathbb{R}_+^2 \), \( B_\delta(x,y) \) denotes the open ball of radius \( \delta \) and centered at \( (x,y) \) and defined by

\[
B_\delta(x,y) := \{(u,v) \in \mathbb{R}_+^2 : \sqrt{(x-u)^2 + (y-v)^2} < \delta \}.
\]

A function \( f : \mathbb{R}_+^2 \to \mathbb{C} \) is called log-uniformly continuous on \( \mathbb{R}_+^2 \) if for any given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |f(x,y) - f(u,v)| < \epsilon \) whenever \( \sqrt{(\log x - \log u)^2 + (\log y - \log v)^2} < \delta \), for any \( (x,y), (u,v) \in \mathbb{R}_+^2 \). We denote \( \mathscr{C}(\mathbb{R}_+^2) \) containing all log-uniformly continuous and bounded functions defined on \( \mathbb{R}_+^2 \). Moreover, \( L^\infty(\mathbb{R}_+^2) \) denotes the space of all bounded functions on \( \mathbb{R}_+^2 \). The notion of Mellin partial derivatives (see Bardaro et al. [14]) of any function \( f : \mathbb{R}_+^2 \to \mathbb{C} \) with respect to variables \( x, y \) is defined as

\[
\theta_x f := \frac{\partial f}{\partial x} \quad \text{and} \quad \theta_y f := \frac{\partial f}{\partial y},
\]

where \( \theta_x f := \frac{\partial f}{\partial x} \) and \( \theta_y f := \frac{\partial f}{\partial y} \) are the partial derivatives of \( f \) with respect to \( x \) and \( y \) respectively.
For $h = (h_1, h_2) \in \mathbb{N}_0^2$, the Mellin partial derivatives of order $r$, where $r = |h| = h_1 + h_2$ is given by

$$\theta^r_{x_1, y_2} f := \theta^{h_1}_x (\theta^{h_2}_y f).$$

In particular, for $r = 2$, we have $\theta^2_x f := \theta_x (\theta_x f)$ and $\theta^2_y f := \theta_y (\theta_y f)$. Let $f : \mathbb{R}_+^2 \to \mathbb{C}$ be such that $f \in C^{(r)}(\mathbb{R}_+^2)$, $r \in \mathbb{N}$. For $(x, y), (s, t) \in \mathbb{R}_+^2$, the bivariate Taylor formula in the Mellin setting (see Bardaro et al. [14]) is given by

$$f(xs, yt) = f(x, y) + (\theta_x \log s + \theta_y \log t) f(x, y) + \cdots + \frac{(\theta_x \log s + \theta_y \log t)^r}{(r - 1)!} f(x, y) + R_r(s, t),$$

where $R_r(s, t) := h(s, t)(\log^2 s + \log^2 t)^\frac{r}{2}$ and $\lim_{(s,t)\to(1,1)} h(s, t) = 0$.

Assume that $\chi \in C(\mathbb{R}_+^2)$ is fixed. Then, for any $\eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p = (p_1, p_2) \in \mathbb{N}_0^2$ with $|p| = p_1 + p_2 = \eta$, we define the algebraic moments of order $\eta$ as

$$m_{(p_1, p_2)}(\chi, u, v) := \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \chi(e^{-k}u, e^{-j}v)(k - \log u)^{p_1}(j - \log v)^{p_2},$$

and the absolute moments by

$$M_{(p_1, p_2)}(\chi) := \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-k}u, e^{-j}v)||k - \log u|^{p_1}|j - \log v|^{p_2}.$$

Also, we define $M_{\eta}(\chi) := \max_{|p| = \eta} M_{(p_1, p_2)}(\chi)$.

**Remark 2.1.** We can easily see that for $\xi, \eta \in \mathbb{N}_0$ with $\xi < \eta$, $M_{\xi}(\chi) < \infty$ implies that $M_{\eta}(\chi) < \infty$. Indeed, for $p_1 + p_2 = \xi$, we have

$$\sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-k}u, e^{-j}v)||k - \log u|^{p_1}|j - \log v|^{p_2}$$

$$\leq \sum_{(k,j)\in B_1(\log u, \log v)} |\chi(e^{-k}u, e^{-j}v)||k - \log u|^{p_1}|j - \log v|^{p_2}$$

$$+ \sum_{(k,j)\not\in B_1(\log u, \log v)} |\chi(e^{-k}u, e^{-j}v)||k - \log u|^{p_1}|j - \log v|^{p_2}$$

$$:= I_1 + I_2.$$

We can see that $I_1 \leq 4||\chi||_\infty$. For $I_2$, we have $|j - \log v| > \frac{1}{2}$, or $|k - \log u| > \frac{1}{2}$. Now, we can write

$$I_2 \leq \sum_{|u-\log x| > \frac{1}{2}} \sum_{|v-\log y| > \frac{1}{2}} |\chi(e^{-k}u, e^{-j}v)| \frac{|k - \log u|^{\eta-p_1}}{|u-\log x|^{\eta-2}} \frac{|j - \log v|^{p_2}}{|v-\log y|^{\eta-2}}$$

$$+ \sum_{|u-\log x| > \frac{1}{2}} \sum_{|v-\log y| > \frac{1}{2}} |\chi(e^{-k}u, e^{-j}v)| \frac{|k - \log u|^{p_1}}{|u-\log x|^{\eta-2}} \frac{|j - \log v|^{\eta-p_1}}{|v-\log y|^{\eta-2}}$$

$$\leq 2^{\eta-\xi+1} M_{\eta}(\chi) < \infty.$$
(K2) \( M_2(\chi) < \infty \), and for every \( \gamma > 0 \), we have

\[
\lim_{w \to +\infty} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-k}x^w, e^{-j}y^w)| |k - w \log x|^{p_1} |j - w \log y|^{p_2} = 0,
\]

uniformly for \((x, y) \in \mathbb{R}_+^2\), where \( p_1 + p_2 = 2 \).

Let \( \psi \) denote the class of kernels satisfying assumptions (K1) and (K2). Then, for \( \chi \in \psi \) and \( w > 0 \), the bivariate Kantorovich exponential sampling series is defined by

\[
(I^\chi_w f)(x, y) := \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w) \left( w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} f(e^u, e^v) \, du \, dv \right),
\]

(2.1)

where \( f : \mathbb{R}_+^2 \to \mathbb{R} \) is a locally integrable function such that the above series are convergent \( \forall (x, y) \in \mathbb{R}_+^2 \). In view of condition (K2), the series (2.1) are well defined for the class of bounded functions on \( \mathbb{R}_+^2 \).

3 | CONVERGENCE RESULTS

In this section, we prove the pointwise and uniform convergence result for the sampling series \((I^\chi_w f)_{w>0}\). Further, a Voronovskaya type asymptotic formula for the sampling series \((I^\chi_0 f)_{w>0}\) is derived by exploiting the bivariate Mellin–Taylor’s formula.

**Theorem 3.1.** Let \( \chi \in \psi \) and \( f \in L^\infty(\mathbb{R}_+^2) \). Then, the series \((I^\chi_w f)\) converges to \( f \) at \((x, y) \in \mathbb{R}_+^2\), where \( f \) is log-continuous. Further, for \( f \in \mathcal{C}(\mathbb{R}_+^2) \), we have

\[
\lim_{w \to +\infty} ||I^\chi_w f - f||_\infty = 0.
\]

**Proof.** In view of condition (K1), we write

\[
|I^\chi_w f(x, y) - f(x, y)| \leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-k}x^w, e^{-j}y^w)| w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} |f(e^u, e^v) - f(x, y)| \, du \, dv
\]

\[
= \left( \sum_{\frac{k}{w}}^{\frac{k+1}{w}} \sum_{\frac{j}{w}}^{\frac{j+1}{w}} \chi(e^{-k}x^w, e^{-j}y^w) \right) w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} |f(e^u, e^v) - f(x, y)| \, du \, dv
\]

\[
:= I_1 + I_2.
\]

Since \( f \) is log continuous, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |f(e^u, e^v) - f(x, y)| < \epsilon \), whenever \( \sqrt{(u - \log x)^2 + (v - \log y)^2} < \delta \). Let \( w' \) be fixed such that \( \frac{1}{w} < \frac{\delta}{2} \) for every \( w > w' \). Now, for \( u \in \left[ \frac{k}{w}, \frac{k+1}{w} \right] \) and \( v \in \left[ \frac{j}{w}, \frac{j+1}{w} \right] \) and \( w > w' \), we have

\[
|u - \log x| \leq |u - \frac{k}{w}| + \left| \frac{k}{w} - \log x \right| < \delta,
\]

and

\[
|v - \log y| \leq |v - \frac{j}{w}| + \left| \frac{j}{w} - \log y \right| < \delta.
\]
whenever $|k/w - \log x| < \frac{\delta}{2}$ and $|j/w - \log y| < \frac{\delta}{2}$. This gives $I_1 < eM_0(\chi)$ for sufficiently large $w$. Subsequently, $I_2$ can be estimated as

$$I_2 \leq 2\|f\|_{\infty} \sum \sum_{(\frac{k}{w}, \frac{j}{y}) \in B_s(log x, log y)} \left| \chi(e^{-k}x^w, e^{-j}y^w) \right|.$$ 

If $(\frac{k}{w}, \frac{j}{w}) \notin B_s(log x, log y)$, we have $\sqrt{(k - w \log x)^2 + (j - w \log y)^2} > \frac{w\delta}{2}$. This gives

$$I_2 \leq \frac{8\|f\|_{\infty}}{w^2\delta^2} \sum \sum_{(\frac{k}{w}, \frac{j}{y}) \in B_s(log x, log y)} \left| \chi(e^{-k}x^w, e^{-j}y^w) \right| \left| (k - w \log x)^2 + (j - w \log y)^2 \right|.$$ 

Using (K2), we obtain $I_2 \to 0$ as $w \to \infty$. Combining the estimates $I_1 - I_2$, we get the desired result. This proves the first part of the theorem, and the second part can be proved analogously by using the fact that choice of $\delta$ is arbitrary. □

**Theorem 3.2.** Let $f \in C^1(\mathbb{R}^2_+)$ locally at $(x, y)$ and $\chi \in \Psi$ be the kernel such that $m_{1,0}(\chi, u, v) = m_{0,1}(\chi, u, v) = 0$. Then, we have

$$\lim_{w \to \infty} w \left[ (I_0^w f)(x, y) - f(x, y) \right] = \frac{1}{2} \left[ \theta_x f(x, y) + \theta_y f(x, y) \right].$$

**Proof.** Since $f \in C^1(\mathbb{R}^2_+)$, using the bivariate Mellin–Taylor’s formula (see Bardaro et al. [14]), we can write

$$f(e^u, e^v) = f(x, y) + \theta_x f(x, y)(u - \log x) + \theta_y f(x, y)(v - \log y) + h \left( \frac{e^u}{x}, \frac{e^v}{y} \right) \left( (u - \log x) + (v - \log y) \right),$$

where $h$ is a bounded function such that $\lim_{(k, t) \to (1, 1)} h(s, t) = 0$. Applying $I_0^w$ on both sides, we get

$$(I_0^w f)(x, y) - f(x, y) = \sum_{k = -\infty}^{\infty} \sum_{j = -\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w) w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} \left[ \theta_x f(x, y)(u - \log x) 
+ \theta_y f(x, y)(v - \log y) + h \left( \frac{e^u}{x}, \frac{e^v}{y} \right) \left( (u - \log x) + (v - \log y) \right) \right] du dv 
= I_1 + I_2 + I_3.$$ 

It is easy to see that

$$I_1 = \sum_{k = -\infty}^{\infty} \sum_{j = -\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w) w^2 \theta_x f(x, y) \frac{\left( k + 1 \right)}{2w} - \left( \frac{k}{w} \right) \left( \frac{w}{w} - \log x \right)^2 
= \frac{\theta_x f(x, y)}{2w} \sum_{k = -\infty}^{\infty} \sum_{j = -\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w) \left[ 1 + 2(k - w \log x) \right].$$

Since $m_{1,0}(\chi, u, v) = 0$, we get $I_1 = \frac{\theta_x f(x, y)}{2w}$. Similarly, we obtain $I_2$ as

$$I_2 = \frac{w\theta_y f(x, y)}{2w} \sum_{k = -\infty}^{\infty} \sum_{j = -\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w) \left[ \left( \frac{j + 1}{w} \right) - \left( \frac{j}{w} \right) \right] \left( j - \log y \right)^2 
= \frac{\theta_y f(x, y)}{2w} \sum_{k = -\infty}^{\infty} \sum_{j = -\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w) \left( 1 + 2(j - w \log y) \right) = \frac{\theta_y f(x, y)}{2w}. $$

Theorem 3.2.
In order to estimate $I_3$, let $\epsilon > 0$ be fixed then there exists $\delta > 0$ such that $|h(s, t)| < \epsilon$ whenever $|s - 1| < \delta$ and $|t - 1| < \delta$. Moreover, let $w'$ be fixed in such a way that $\frac{1}{w'} < \frac{\delta}{2}$ for every $w > w'$. $I_3$ estimated as

$$|I_3| \leq \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \left| \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} |\chi(e^{-k/x}e^{-j/y})| \right| w^2 \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \left[ h \left( \frac{\epsilon}{x}, \frac{\epsilon}{y} \right) \right] (u - \log x) du dv + \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \sum \left| \chi(e^{-k/x}e^{-j/y}) \right| w^2 \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \left[ h \left( \frac{\epsilon}{x}, \frac{\epsilon}{y} \right) \right] (v - \log y) dv du : = I_3' + I_3''. $$

Now, we write $I_3' = J_1 + J_2$, where

$$J_1 = \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \left| \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \chi(e^{-k/x}e^{-j/y}) \right| w^2 \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \left[ h \left( \frac{\epsilon}{x}, \frac{\epsilon}{y} \right) \right] (u - \log x) du dv ,$$

and

$$J_2 = \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \left| \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \chi(e^{-k/x}e^{-j/y}) \right| w^2 \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \int_{\frac{1}{w}}^{\frac{1}{w}+\frac{1}{w}} \left[ h \left( \frac{\epsilon}{x}, \frac{\epsilon}{y} \right) \right] (v - \log y) dv du .$$

We estimate $J_1$ as follows.

$$J_1 \leq \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \left| \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \chi(e^{-k/x}e^{-j/y}) \right| w^2 \frac{\epsilon}{2w} \left( \left( \frac{k + 1}{w} - \log x \right)^2 + \left( \frac{k}{w} - \log x \right)^2 \right) = \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \left| \chi(e^{-k/x}e^{-j/y}) \right| \frac{\epsilon}{2w} (1 + 2|k - w \log x|) \leq \frac{\epsilon}{2w}(M_0(\chi) + 2M_1(\chi)).$$

Similarly, we obtain $J_2$ as

$$J_2 \leq \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \left| \sum_{(\frac{1}{w}, \frac{1}{w}) \in B_3(\log x, \log y)} \chi(e^{-k/x}e^{-j/y}) \right| \frac{\epsilon}{2w} (1 + 2|j - w \log y|) \leq \frac{\epsilon}{2w}(M_0(\chi) + 2M_1(\chi)).$$
We now find the estimate of $I_3''$. Since $h$ is a bounded function, we assume $K > 0$ such that $|h(t, s)| \leq K$. For $(\frac{k}{w}, \frac{j}{w}) \notin B_2(\log x, \log y)$, we have $|(k - w \log x)^2 + (j - w \log y)^2| > \frac{w^4}{4}$. This gives

$$I_3'' \leq \sum_{(\frac{k}{w}, \frac{j}{w}) \neq B_2(\log x, \log y)} \left| \chi(e^{-k}x^w, e^{-j}y^w) \right| \frac{K}{w} \left( 1 + (k - w \log x) + (j - w \log y) \right)$$

$$\leq \frac{4K(1 + w\delta)}{w^4\delta^2} \sum_{(\frac{k}{w}, \frac{j}{w}) \neq B_2(\log x, \log y)} \left| \chi(e^{-k}x^w, e^{-j}y^w) \right| \left| (k - w \log x)^2 + (j - w \log y)^2 \right| .$$

Now, using $(K_2)$, we easily obtain $wI_3'' \to 0$ as $w \to \infty$. Therefore, we have $\lim_{w \to \infty} wI_3 = 0$; hence, the required result follows.

**Remark 3.1.** The condition that $f$ is bounded on $\mathbb{R}^2_+$ in Theorem 3.2 can be relaxed by assuming that there are positive constants $a, b$ such that $|f(x, y)| \leq a + b|\log^2 x + \log^2 y|$, $\forall x, y \in \mathbb{R}^2_+$.

First, we show that the series $(I_{w,f}^x)$ is well defined for such $f$. Indeed,

$$|I_{w,f}^x(x)| \leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-k}x^w, e^{-j}y^w)| w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} |f(e^u, e^v)| du dv$$

$$\leq \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-k}x^w, e^{-j}y^w)| w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} (a + b|u^2 + v^2|) du dv$$

$$\leq M_0(\chi) \left( a + b \left( |\log^2 x| + |\log^2 y| \right) \right) + \frac{b}{w} M_0(\chi) \left( |\log x| + |\log y| \right)$$

$$+ \frac{2b}{w} \left( M_{1,0}(\chi) + M_{0,1}(\chi) \right) + \frac{2b}{3w^2} M_{0,0}(\chi) + \frac{b}{w^2} \left( M_{1,0}(\chi) + M_{0,1}(\chi) + M_{2,0}(\chi) + M_{0,2}(\chi) \right) .$$

This shows that the series $(I_{w,f}^x)_{w>0}$ is absolutely convergent in $\mathbb{R}^2_+$. For any fixed $(x, y) \in \mathbb{R}^2_+$, we define

$$P_1(u, v) := f(x, y) + (\theta_x f)(x, y)(u - \log x) + (\theta_y f)(x, y)(v - \log y) .$$

From the bivariate Taylor’s formula in terms of Mellin derivatives up to first-order term, we can write as $h \left( \frac{e^x}{x}, \frac{e^y}{y} \right) = \frac{f(e^u, e^v) - P_1(u, v)}{(u - \log x) + (v - \log y)}$, where $h$ is a function such that $\lim_{(v - \log y) \to 0} (u - \log x) = 0$. This implies that $h$ is bounded in $\delta$-neighborhood of $(\log x, \log y)$, that is, $h$ is bounded for $(u, v) \in B_\delta(\log x, \log y)$. Now, for $(u, v) \notin B_\delta(\log x, \log y)$, we have

$$|h(e^u x^{-1}, e^v y^{-1})| \leq \frac{|f(e^u, e^v)|}{|u - \log x| + |v - \log y|} + \frac{|P_1(u, v)|}{|u - \log x| + |v - \log y|}$$

$$\leq \frac{a + b|u^2 + v^2|}{|u - \log x| + |v - \log y|} + \frac{|P_1(u, v)|}{|u - \log x| + |v - \log y|} .$$

This shows that $h$ is also bounded for $(u, v) \notin B_\delta(\log x, \log y)$. This concludes that $h$ is bounded on $\mathbb{R}^2_+$. Now, we can proceed in the similar manner as in the proof of Theorem 3.2 to get the same asymptotic formula.

For $f \in \mathcal{E}(\mathbb{R}^2_+)$, the logarithmic modulus of continuity is defined by (see Bardaro et al. [14])

$$\omega(f, \delta_1, \delta_2) := \sup \{|f(x, y) - f(u, v)| : |\log x - \log u| \leq \delta_1, |\log y - \log v| \leq \delta_2, \delta_1, \delta_2 \in \mathbb{R}^+ \}. $$

The above definition was motivated by the definition of logarithmic modulus of continuity in one variable introduced in Bardaro et al. [12]. Also, one can see that the bivariate logarithmic modulus of continuity satisfies the properties of its univariate form (see Bardaro and Mantellini [10]). In order to derive the required estimate, we utilize the following properties:
a. For every $\delta_1 > 0$, $\delta_2 > 0$, $\omega(f, \delta_1, \delta_2) \to 0$ as $\delta_1 \to 0$ and $\delta_2 \to 0$.

b. For any $(x, y), (u, v) \in \mathbb{R}^2_+$, we have

$$|f(x, y) - f(u, v)| \leq \omega(f, \delta_1, \delta_2) \left(1 + \frac{|\log x - \log u|}{\delta_1}\right) \left(1 + \frac{|\log y - \log v|}{\delta_2}\right).$$

Now, we obtain a quantitative estimate of the convergence of series (2.1) for $f \in \mathscr{C}(\mathbb{R}^2_+)$.

**Theorem 3.3.** Let $f \in \mathscr{C}(\mathbb{R}^2_+)$. Then, for any $(x, y) \in \mathbb{R}^2_+$, we have

$$|(I_w^t f)(x, y) - f(x, y)| \leq \omega(f, \delta_1, \delta_2) \left[M_0(\chi) \left(1 + \frac{1}{2\delta_1 w} + \frac{1}{2\delta_2 w} + \frac{1}{4\delta_1 \delta_2 w^2}\right)\right] + \omega(f, \delta_1, \delta_2) \left[M_{1,0}(\chi) \frac{1}{\delta_1 w} \left(1 + \frac{1}{2w\delta_2}\right)\right]$$

$$+ \omega(f, \delta_1, \delta_2) \left[M_{0,1}(\chi) \frac{1}{\delta_1 w} \left(1 + \frac{1}{2w\delta_2}\right) + M_{1,1}(\chi) \frac{1}{\delta_1 \delta_2 w^2}\right],$$

for any $w > 0$ and $\delta_1 > 0$, $\delta_2 > 0$.

**Proof.** Using property (b), we can write

$$|(I_w^t f)(x, y) - f(x, y)| \leq \omega(f, \delta_1, \delta_2) \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-kx}w, e^{-jy}w)| w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} \left(1 + \frac{|u - \log x|}{\delta_1}\right) \left(1 + \frac{|v - \log y|}{\delta_2}\right) du dv$$

$$= \omega(f, \delta_1, \delta_2) \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-kx}w, e^{-jy}w)| w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} \left[1 + \frac{|u - \log x|}{\delta_1} + \frac{|v - \log y|}{\delta_2} + \frac{|u - \log x|}{\delta_1} \frac{|v - \log y|}{\delta_2}\right] du dv$$

$$=: J_1 + J_2 + J_3 + J_4.$$

It is easy to see that $J_1 = M_0(\chi)\omega(f, \delta_1, \delta_2)$. Now, we estimate $J_2$.

$$J_2 = \omega(f, \delta_1, \delta_2) \frac{1}{\delta_1} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-kx}w, e^{-jy}w)| w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} |u - \log x| \ du \ dv$$

$$= \frac{\omega(f, \delta_1, \delta_2)}{2\delta_1 w} \left[M_0(\chi) + 2M_{0,1}(\chi)\right].$$

Similarly, we obtain $J_3 = \frac{\omega(f, \delta_1, \delta_2)}{2\delta_2 w} \left[M_0(\chi) + 2M_{0,1}(\chi)\right]$. Finally, we evaluate $J_4$.

$$J_4 = \omega(f, \delta_1, \delta_2) \frac{1}{\delta_1 \delta_2} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\chi(e^{-kx}w, e^{-jy}w)| w^2 \int_{\frac{k}{w}}^{\frac{k+1}{w}} \int_{\frac{j}{w}}^{\frac{j+1}{w}} |u - \log x| |v - \log y| \ du \ dv$$

$$= \frac{\omega(f, \delta_1, \delta_2)}{4\delta_1 \delta_2 w^2} \left[M_0(\chi) + 2M_{1,0}(\chi) + 2M_{2,0}(\chi) + 4M_{1,1}(\chi)\right].$$

On combining the estimates $J_1 - J_4$, we get the desired estimate. \qed

**Remark 3.2.** For fixed $w > 0$, if we put $\delta_1 = \delta_2 = \frac{1}{w}$, then the estimate in Theorem 3.3 is as follows:

$$|(I_w^t f)(x, y) - f(x, y)| \leq \frac{\kappa}{4} \omega \left(f, \frac{1}{w}, \frac{1}{w}\right), \quad \kappa := \left(9M_0(\chi) + 6M_{1,0}(\chi) + 6M_{0,1}(\chi) + 4M_{1,1}(\chi)\right).$$
4 | GB S -BIVARIATE KANTOROVICH SAMPLING SERIES

Bögel pioneered the notion of $B$-continuous and $B$-differentiable functions and furnished significant results [37]. Badea et al. [38, 39] established the well-known Korovkin theorem for $B$-continuous functions. Dobrescu and Matei [40] showed that $B$-continuous function can be approximated uniformly by $B$-continuous functions associated to Bernstein polynomials on bounded domain. Since then, various authors have contributed significantly in this direction; see, for example, previous studies [41–47].

For $X, Y \subseteq \mathbb{R}_+$, we call a function $f : X \times Y \to \mathbb{R}$ as Mellin $B$-continuous (Mellin--Bögel continuous) at $(u, v) \in X \times Y$ if

$$\lim_{(x,y) \to (u,v)} \Delta_{(x,y)} f [e^u, e^v; x, y] = 0,$$

where $\Delta_{(x,y)} f [e^u, e^v; x, y] = f(x, y) - f(x, e^v) - f(e^u, y) + f(e^u, e^v)$. The function $f : X \times Y \to \mathbb{R}$ is Mellin $B$-bounded if there exists $\lambda > 0$ such that $|\Delta_{(x,y)} f [e^u, e^v; x, y]| \leq \lambda$ for every $(x, y), (u, v) \in X \times Y$. Throughout this paper, we denote $\mathcal{B}_b (\mathbb{R}_+^2)$ and $\mathcal{B}_b (\mathbb{R}_+^2)$ as the space of all Mellin $B$-bounded and Mellin $B$-continuous functions on $\mathbb{R}_+^2$ with the usual sup-norm $\| \cdot \|_{\infty}$, respectively.

Now, for any $f \in \mathcal{B}_b (\mathbb{R}_+^2)$, the GBS of bivariate Kantorovich exponential sampling series is defined as

$$I^f_{w} (f; x, y) := I^f_{w} (f(x, v) + f(u, y) - f(u, v); x, y),$$

for all $(x, y), (u, v) \in \mathbb{R}_+^2$.

We estimate the order of convergence for the family of operators $I^f_{w} f$ using Bögel frame of modulus of continuity for $f \in \mathcal{B}_b (\mathbb{R}_+^2)$. For any $\delta_1, \delta_2 > 0$, the mixed modulus of smoothness in the Mellin sense is defined as

$$\omega_B (f; \delta_1, \delta_2) := \sup \{| \Delta_{(x,y)} f [s, t; x, y]| : | \log s - \log x| < \delta_1, | \log t - \log y| < \delta_2 \},$$

for every $(x, y), (u, v) \in \mathbb{R}_+^2$. This definition is inspired by the definition of classical mixed modulus of smoothness (see previous studies [38, 39, 45]). The basic properties of above modulus of smoothness can be deduced easily by following the similar arguments given in Badea et al. [38, 39].

**Theorem 4.1.** Let $f \in \mathcal{B}_b (\mathbb{R}_+^2)$. Then, the following estimate holds:

$$|I^f_{w} (f; x, y) - f(x, y)| \leq \left( 1 + \frac{A_1}{\delta_1} + \frac{A_2}{\delta_2} + \frac{A_3}{\delta_1 \delta_2} \right) \omega_B (f; \delta_1, \delta_2),$$

where $A_1 = \frac{1}{2w} (M_{0,0} + 2M_{1,0}), A_2 = \frac{1}{2w} (M_{0,0} + 2M_{0,1}), A_3 = \frac{1}{4w} (M_{0,0} + 2M_{1,0} + 2M_{0,1} + 4M_{1,1})$.

**Proof.** Using the property $\omega_B (f; a \delta_1, b \delta_2) \leq (1 + a)(1 + b)\omega_B (f; \delta_1, \delta_2)$ for $a, b > 0$, we write

$$|\Delta_{(x,y)} f [e^u, e^v; x, y]| \leq \omega_B (f; |u - \log x|, |v - \log y|) \leq \left( 1 + \frac{|u - \log x|}{\delta_1} \right) \left( 1 + \frac{|v - \log y|}{\delta_2} \right) \omega_B (f; \delta_1, \delta_2)$$

for any $(x, y), (u, v) \in \mathbb{R}_+^2$, and $\delta_1, \delta_2 > 0$. Now, applying the series $(I^f_{w})$ on $\Delta_{(x,y)} f [e^u, e^v; x, y]$, we obtain

$$I^f_{w} (f; x, y) = f(x, y) - I^f_{w} (\Delta_{(x,y)} f [e^u, e^v; x, y]).$$

From (4.2), we have

$$|I^f_{w} (f; x, y) - f(x, y)| \leq \left[ 1 + \frac{I^f_{w} (|u - \log x|; x, y)}{\delta_1} + \frac{I^f_{w} (|v - \log y|; x, y)}{\delta_2} \right] \omega_B (f; \delta_1, \delta_2).$$
Using the definition (2.1), we obtain
\[
I_w^g(|u - \log x|; x, y) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w)w^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u - \log x| du \, dv
\]
\[
= \frac{1}{2w} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w)[1 + 2(w \log x - k)]
\]
\[
= \frac{1}{2w} (M_{0,0} + 2M_{1,0}).
\]

Similarly, we get \( I_w^g(|v - \log y|; x, y) = \frac{1}{2w} (M_{0,0} + 2M_{0,1}). \) Finally, we have
\[
I_w^g(|u - \log x| |v - \log y|; x, y) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w)w^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u - \log x| |v - \log y| du \, dv
\]
\[
= \frac{1}{2w} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \chi(e^{-k}x^w, e^{-j}y^w)[1 + 2(w \log y - k)](1 + 2(w \log x - k))
\]
\[
= \frac{1}{4w^2} (M_{0,0} + 2M_{1,0} + 2M_{0,1} + 4M_{1,1}).
\]

On substituting these estimates, we get the desired result.

Now, we define a Mellin B-differential (Mellin–Bögel differential) function. A function \( f : X \times Y (\subseteq \mathbb{R}_+^2) \to \mathbb{R} \) is said to be Mellin B-differential at \((u, v)\) if the following limit exists finitely:
\[
\lim_{(x,y) \to (e^u, e^v)} \frac{\Delta_{(x,y)} f[e^u, e^v; x, y]}{(u - \log x)(v - \log y)}.
\]

The Mellin B-differential of \( f \) at any point \((u, v)\) is represented as \( \theta_B(f; u, v) \). Moreover, we denote \( \mathcal{B}_b(\mathbb{R}_+^2) \) as the space of all Mellin B-differentiable function on \( \mathbb{R}_+^2 \).

**Theorem 4.2.** Let \( f \in \mathcal{B}_b(\mathbb{R}_+^2) \) and \( \theta_B f \in \mathcal{B}(\mathbb{R}_+^2) \). Then, for each \((x, y) \in \mathbb{R}_+^2\), we have
\[
|I_w^g(f; x, y) - f(x, y)| \leq E_1 (3\|\theta_B f\|_\infty + \omega_B(f; \delta_1, \delta_2)) + \left( \frac{E_2}{\delta_1} + \frac{E_3}{\delta_2} + \frac{E_4}{\delta_1 \delta_2} \right) \omega_B(\theta_B f; \delta_1, \delta_2),
\]
where
\[
E_1 = \frac{1}{4w^2} (M_{0,0} + 2M_{1,0} + 2M_{0,1} + 4M_{1,1}), \quad E_2 = \frac{1}{4w^2} (M_{0,0} + 3M_{2,0} + 3M_{1,0} + 2M_{0,1} + 6M_{2,1} + 6M_{1,1}), \quad E_3 = \frac{1}{4w^2} (M_{0,0} + 3M_{2,0} + 2M_{0,1} + 4M_{2,1} + 6M_{1,0} + 4M_{1,1} + 9M_{2,1} + 9M_{1,1}).
\]

**Proof.** Since \( f \in \mathcal{B}_b(\mathbb{R}_+^2) \), we have \( \Delta_{(x,y)} f[e^u, e^v; x, y] = (u - \log x)(v - \log y)\theta_B f(p, q) \), where \( p \in (\log x, u) \) and \( q \in (\log y, v) \). Using the definition of \( \Delta_{(x,y)} f[e^u, e^v; x, y] \), we write
\[
\theta_B f(p, q) = \Delta_{(x,y)} \theta_B f(p, q) + \theta_B f(p, y) + \theta_B f(x, q) - \theta_B f(x, y).
\]

Using the above equality and the fact that \( \theta_B f \in \mathcal{B}(\mathbb{R}_+^2) \), we have
\[
|I_w^g(\Delta_{(x,y)} f[e^u, e^v; x, y]; x, y)|
= |I_w^g((u - \log x)(v - \log y)\theta_B f(p, q); x, y)|
\leq I_w^g(|u - \log x||v - \log y|\Delta_{x,y} f(p, q); x, y)
+ I_w^g(|u - \log x||v - \log y|(|\theta_B f(p, y)| + |\theta_B f(x, q)| + |\theta_B f(x, y)|); x, y)
\leq I_w^g(|u - \log x||v - \log y|\omega_B(\theta_B f; |p - \log x|, |q - \log y|))
+ 3\|\theta_B f\|_\infty I_w^g(|u - \log x||v - \log y|).
Using the monotonicity of mixed modulus of smoothness $\omega_B$, we write

$$|	ilde{I}_w^\delta(f; x, y) - f(x, y)| \leq 3\|\theta_B f\|_\infty I_{w,0}^\delta (|u - \log x| |v - \log y|; x, y)$$

$$+ \left[ I_{w,0}^\delta (|u - \log x|^2 |v - \log y|; x, y) + I_{w,0}^\delta (|u - \log x| |v - \log y|^2; x, y) + I_{w,0}^\delta (|u - \log x|^2 |v - \log y|^2; x, y) \right] \frac{1}{\delta_2} \omega_B (\theta_B f; \delta_1, \delta_2).$$

From the definition (2.1), we obtain

$$I_{w,0}^\delta (|u - \log x| |v - \log y|; x, y) = \frac{1}{4w^2} \left[ M_{0,0} + 2M_{1,0} + 2M_{0,1} + 4M_{1,1} \right].$$

$$I_{w,0}^\delta (|u - \log x|^2 |v - \log y|; x, y) = \frac{1}{6w^2} \left[ M_{0,0} + 3M_{2,0} + 3M_{1,0} + 2M_{0,1} + 6M_{2,1} + 6M_{1,1} \right].$$

$$I_{w,0}^\delta (|u - \log x| |v - \log y|^2; x, y) = \frac{1}{6w^2} \left[ M_{0,0} + 3M_{2,0} + 3M_{1,0} + 2M_{0,1} + 6M_{2,1} + 6M_{1,1} \right].$$

$$I_{w,0}^\delta (|u - \log x|^2 |v - \log y|^2; x, y) = \frac{1}{9w^2} \left[ M_{0,0} + 3M_{2,0} + 3M_{1,0} + 2M_{0,1} + 9M_{2,2} + 9M_{1,2} + 9M_{1,1} \right].$$

Using these estimates, we obtain the required result.

Now, we study the degree of approximation for the series by $B$-continuous functions belonging to a Lipschitz class (see Acar and Kajla [41]). The Lipschitz class in the Mellin frame is defined as

$$\text{Lip}_K = \{ f \in \mathcal{C}_b(\mathbb{R}_+^2); |\Delta_{(x,y)} f [s, t; x, y]| \leq K |\log s - \log x| |\log t - \log y|, L \in \mathbb{R}_+ \},$$

for $(u, v), (x, y) \in \mathbb{R}_+^2$.

**Theorem 4.3.** Let $f \in \text{Lip}_K$. Then, the following holds:

$$|\tilde{I}_w^\delta(f; x, y) - f(x, y)| \leq \frac{K}{4w^2} (M_{0,0} + 2M_{1,0} + 2M_{0,1} + 4M_{1,1}).$$

where $K$ is the Lipschitz constant.

**Proof.** From the definition (2.1), we can write

$$I_{w,0}^\delta (f; x, y) = I_{w,0}^\delta (f(x, u) + f(u, y) - f(u, v))$$

$$= I_{w,0}^\delta (f(x, y) - \Delta_{(x,y)} f [e^u, e^v; x, y]; x, y)$$

$$= f(x,y)I_{w,0}^\delta (1; x, y) - I_{w,0}^\delta (\Delta_{(x,y)} f [e^u, e^v; x, y]; x, y).$$

Since $f \in \text{Lip}_K$ and using the estimate of $I_{w,0}^\delta (|u - \log x| |v - \log y|; x, y)$, we have

$$|\tilde{I}_w^\delta(f; x, y) - f(x, y)| \leq |I_{w,0}^\delta (\Delta_{(x,y)} f [e^u, e^v; x, y]; x, y)|$$

$$\leq K I_{w,0}^\delta (|u - \log x| |v - \log y|; x, y)$$

$$\leq \frac{K}{4w^2} (M_{0,0} + 2M_{1,0} + 2M_{0,1} + 4M_{1,1}).$$

Thus, the proof is completed. □
5 | EXAMPLES OF THE KERNELS

In this section, we provide a few examples of the kernel function in the setting of Mellin theory satisfying assumptions (K1) and (K2). We start with the well-known Mellin B-spline kernel. For \( x \in \mathbb{R}^+ \), the \( n \)-th order Mellin B-spline function is defined as (see Bardaro et al. [12])

\[
B_n(x) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \frac{n}{2} + \log x - j \right)^{n+1}.
\]

\( B_n(x) \) is compactly supported for every \( n \in \mathbb{N} \). The Mellin transformation of \( B_n \) is given by

\[
\hat{M}[B_n](c + is) = \left( \frac{\sin \left( \frac{\pi}{2} \right)}{\left( \frac{s}{2} \right)^n} \right), \quad c = 0, \ s \neq 0.
\]

Indeed, \( \hat{B}_n(x) \) satisfies assumptions (K1) and (K2) for every \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^+_1 \); see Bardaro et al. [12] and Kumar and Shivam [15]. We can produce the multivariate kernel using univariate kernel functions as follows:

\[
\hat{\chi}^n(x_1, x_2, \ldots, x_n) := \prod_{i=1}^{n} \hat{\chi}(x_i),
\]

where \( x_i \in \mathbb{R}^+_1 \) and \( \chi \) is any univariate kernel. For \( n = 2 \), we construct the bivariate kernel in variables \( x \) and \( y \) by

\[
\hat{\chi}^2(x, y) := \chi(x) \chi(y), \quad (x, y) \in \mathbb{R}^+_1.
\]

\[
\text{FIGURE 1} \quad \text{This figure shows the approximation of } f(x, y) \text{ (blue) by the series } (I_w^\chi f)_{w>0} \text{ for } w = 10 \text{ and } 40 \text{ (yellow and pink, respectively).}
\]

[Colour figure can be viewed at wileyonlinelibrary.com]

\[
\text{TABLE 1} \quad \text{Error estimation (up to four decimal points) in the approximation of } f(x, y) \text{ by } I_w^\chi f(x, y) \text{ for } w = 10, 40.
\]

| \( x \) | \( y \) | \( |f(x, y) - I_{w=10}^\chi f(x, y)|\) | \( |f(x, y) - I_{w=40}^\chi f(x, y)|\) |
|---|---|---|---|
| 0.2 | 0.1 | 0.0033 | 0.0008 |
| 0.6 | 0.5 | 0.0119 | 0.0028 |
| 1.1 | 0.9 | 0.0366 | 0.0092 |
| 1.9 | 1.8 | 0.0185 | 0.0052 |
FIGURE 2  This figure shows the convergence of $g(x, y)$ (green) by the series $(I_w^2 g)_{w>0}$ for $w = 8$ and $35$ (blue and pink, respectively). [Colour figure can be viewed at wileyonlinelibrary.com]

| $x$  | $y$  | $|g(x, y) - I_8^w g(x, y)|$ | $|g(x, y) - I_{35}^w g(x, y)|$ |
|-----|-----|--------------------------|--------------------------|
| 1.3 | 1.6 | 0.6079                   | 0.1254                   |
| 1.9 | 1.7 | 0.3958                   | 0.1036                   |
| 2.8 | 2.4 | 0.7695                   | 0.1057                   |
| 3.6 | 3.9 | 2.4188                   | 0.5949                   |

TABLE 2  Error estimation (up to four decimal points) in the convergence of $g(x, y)$ by $I_w^2 g(x, y)$ for $w = 5, 35$.

Exploiting the above construction, we define the bivariate $B$-spline kernel by

$$
\hat{B}_2^2(x, y) = \hat{B}_2(x)\hat{B}_2(y) = \begin{cases} 
(1 + \log x)(1 + \log y), & \text{if } e^{-1} < x, y < 1, \\
(1 - \log x)(1 + \log y), & \text{if } 1 < x < e, e^{-1} < y < 1, \\
(1 + \log x)(1 - \log y), & \text{if } e^{-1} < x < 1, 1 < y < e, \\
(1 - \log x)(1 - \log y), & \text{if } 1 < x, y < e, \\
0, & \text{if otherwise.}
\end{cases}
$$

We show the approximation of $f(x, y) = \sin(x^2 - y^2)$, $(x, y) \in [0, 2] \times [0, 2]$ by $(I_w^2 f)_{w>0}$ using $\hat{B}_2^2(x, y)$ as the kernel function (see Figure 1 and Table 1).

Next, we show the convergence of the function $g(x, y) = y^2 + \cos(\pi x)$, $(x, y) \in [1, 4] \times [1, 4]$ by the series $(I_w^2 g)_{w>0}$ (see Figure 2 and Table 2).

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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