Comment on Anomaly Matching in \( N = 1 \) Supersymmetric QCD

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An attempt is made at constructing a systematic approach to the anomaly matching problem of non-Abelian electric-magnetic duality in \( N = 1 \) supersymmetric QCD. The strategy we employ is somewhat analogous to anomaly analyses of grand unified models where the anomaly cancellation becomes more transparent if one embeds \( SU(5) \) multiplets into a multiplet of (anomaly-free) \( SO(10) \). A complication arises in the treatment of \( U_{AF}(1) \) matching, where \( U_{AF}(1) \) is an anomaly-free \( R \) symmetry. It is noted that a relatively systematic analysis of the anomaly matching is possible if one considers the formal breaking sequence of color gauge symmetry, \( SU(N_f)_c \rightarrow SU(N_c)_c \times SU(\tilde{N}_c)_c \) with \( N_f = N_c + \tilde{N}_c \), where \( N_f \) represents the number of massless quarks.

§1. Introduction

The investigation of non-Abelian electric-magnetic duality of \( N = 1 \) supersymmetric QCD was initiated by Seiberg.\(^1,2\) (For reviews, see Refs. 3) and 4). The basic ingredients of this analysis are holomorphicity, decoupling, and the 't Hooft anomaly matching condition.\(^5\)

In this note we comment on some aspects of anomaly matching. We concentrate on anomaly matching, without considering other important physical quantities such as holomorphicity, superpotential and decoupling. This is done mainly because anomaly matching provides a very definite mathematical framework: Finding a solution to anomaly matching provides a good starting point of analysis, and one may then exercise one’s imagination with regard to the possible physical meaning of the obtained solution. Our motivation for this analysis is to find a more systematic approach to the anomaly matching problem. The strategy we employ is somewhat analogous to the anomaly analyses applied to the conventional grand unification models. For example, the anomaly cancellation in the \( SU(5) \) scheme is rather miraculous, but if one embeds all the multiplets appearing in the \( SU(5) \) model into the (anomaly-free) \( SO(10) \) model, the anomaly cancellation becomes more systematic and transparent.\(^6\)

However, the present approach as it stands is more involved. One of the main reasons lies in the subtle properties of the anomaly free \( U_{AF}(1) \) symmetry related to \( R \)-symmetry. From a supersymmetry viewpoint, the \( U_{AF}(1) \) charge may be regarded as condensed in the vacuum in the sense that the constant SUSY transformation parameter carries a non-trivial \( U_{AF}(1) \) charge by definition. Stated differently, the \( U_{AF}(1) \) charge does not commute with the basic generators of \( N = 1 \) supersymmetry, though the \( U_{AF}(1) \) charge commutes with the Hamiltonian. Also, if one decouples one of the massless quarks by giving it a mass, the entire \( U_{AF}(1) \) charge assignment of the remaining quarks is reshuffled. In this respect, \( U_{AF}(1) \) is quite different from...
other global symmetries such as the flavor symmetry $SU(N_f)_L \times SU(N_f)_R$.

It turns out to be relatively easy to achieve an intuitive understanding of all anomaly matchings except for that involving the triangle $(U_R^{AF}(1))^3$. We thus propose to examine the solutions obtained by tentatively dropping the requirement of $(U_R^{AF}(1))^3$ anomaly matching and requiring only the anomaly matching linear in $U_R^{AF}(1)$, which is equivalent to imposing the existence of conserved $U_R^{AF}(1)$ current (with spurious “leptons”) without gauging it. The physical relevance of gauging $U_R^{AF}(1)$, which does not commute with the generators of $N = 1$ supersymmetry, will be commented on later.

§2. Minimal model with $SU(N)_c$ gauge theory

Following the analyses in Refs. 1) and 2), we consider the fermion contents of $N = 1$ supersymmetric QCD with color $SU(N_c)$ for $N_c < N_f$ by denoting the quantum numbers related to $SU(N_c) \times SU(N_f)_L \times SU(N_f)_R \times U(1) \times U_R^{AF}(1)$ in the order

$$\Psi_Q : (N_c) \quad \begin{pmatrix} N_f, 1, 1, -\frac{N_c}{N_f} \end{pmatrix},$$
$$\Psi_Q : (\bar{N}_c) \quad \begin{pmatrix} 1, N_f, 1, -\frac{N_c}{N_f} \end{pmatrix},$$
$$\lambda_{N_c} : (N_c^2 - 1) \quad (1, 1, 0, 1),$$

where we write only the (left-handed) fermion components of quark scalar multiplets, $Q$ and $\bar{Q}$, and the gaugino $\lambda_{N_c}$ of the gauge vector multiplet. Here $U(1)$ represents the baryon number. We classify $\Psi_{\bar{Q}}$ as $\bar{N}_f$ of $SU(N_f)_R$. The above multiplet is dual to the (magnetic) $N = 1$ supersymmetric theory with $SU(\bar{N}_c)_c$ gauge symmetry given by (with the same notation)

$$\Psi_{\bar{q}} : (\bar{N}_c) \quad \begin{pmatrix} \bar{N}_f, 1, \frac{\bar{N}_c}{\bar{N}_f}, -\bar{N}_f \end{pmatrix},$$
$$\Psi_{\bar{q}} : (\bar{N}_c) \quad \begin{pmatrix} 1, N_f, \frac{\bar{N}_c}{\bar{N}_f}, -\bar{N}_f \end{pmatrix},$$
$$\lambda_{\bar{N}_c} : (\bar{N}_c^2 - 1) \quad (1, 1, 0, 1),$$
$$\Psi_T : (1) \quad \begin{pmatrix} N_f, \bar{N}_f, 0, \frac{\bar{N}_c - N_c}{\bar{N}_f} \end{pmatrix},$$

with $N_c + \bar{N}_c = N_f$. Here $\Psi_T$ represents the fermion component of the meson scalar multiplet $T$ formed of $Q \bar{Q}$, and $q$ and $\bar{q}$ stand for quarks in the dual theory. We assign $\Psi_q$ and $\Psi_{\bar{q}}$ to $(\bar{N}_c)$ and $(\bar{N}_c)$ of color $SU(\bar{N}_c)$, respectively, for later convenience by departing from the convention used in the original reference 2), but this does not change the physics.
2.1. \(SU(N_c) \times SU(N_f - N_c) \rightarrow SU(N_f)\)

Instead of comparing the anomalies associated with the global symmetries \(SU(N_f)_L \times SU(N_f)_R \times U_B(1) \times U^A_F(1)\) in the above mutually dual multiplets directly, we here propose to compare the anomalies of the set of fields (with \(N_f = N_c + \tilde{N}_c\))

\[
\Psi_Q : (N_c) \left( N_f, 1, 1, -\frac{N_c}{N_f} \right) \rightarrow (N_f) \left( N_f, 1, 0, \frac{\tilde{N}_c - N_c}{N_f} \right),
\]

\[
\bar{\Psi}_Q : (\tilde{N}_c) \left( N_f, 1, -\frac{\tilde{N}_c}{N_f}, \frac{N_c}{N_f} \right) \rightarrow (\tilde{N}_f) \left( 1, \tilde{N}_f, 0, \frac{\tilde{N}_c - N_c}{N_f} \right),
\]

\[
\Psi_T : (1) \left( N_f, \tilde{N}_f, 0, \frac{\tilde{N}_c - N_c}{N_f} \right) \rightarrow (1) \left( N_f, \tilde{N}_f, 0, \frac{\tilde{N}_c - N_c}{N_f} \right).
\]

with those of

\[
\lambda_{N_c} : (N_c^2 - 1)(1, 1, 0, 1) \rightarrow (N_f, \tilde{N}_f) \left( 1, \tilde{N}_f, 0, \frac{\tilde{N}_c - N_c}{N_f} \right),
\]

\[
\bar{\lambda}_{\tilde{N}_c} : (\tilde{N}_c^2 - 1)(1, 1, 0, -1) \rightarrow (N_f, \tilde{N}_f) \left( 1, \tilde{N}_f, 0, \frac{\tilde{N}_c - N_c}{N_f} \right).
\]

Namely, we move the set of fields \(\Psi_q, \bar{\Psi}_q\) and \(\lambda_{N_c}\) from one side of the duality relation to the other; at the same time, we replace all the moved fields by their “anti-fields”, which are defined by reversing all the quantum numbers, including \(U^AF(1)\) charge. Note that this “anti-field” differs from the physical charge conjugated field, since chirality is not reversed in this definition. (We here utilize the freedom represented by the overall constant factor of the \(R\)-charge assignment in a given theory.) The fact that this re-arrangement does not change the anomaly matching condition can be understood in the path integral approach.\(^5\) for example. We then multiply both sides of the duality relation by the path integral

\[
\int d\bar{\mu} \exp \left[ i \int (L(\Psi_q, \bar{\Psi}_q, \lambda_{N_c}) \right] d^4x,
\]

where \(L\) is the Lagrangian for the “anti-fields” defined above. Since the path integral \(\int d\bar{\mu} \exp \left[ i \int (L + L^A) d^4x \right] \) is anomaly-free for all the global symmetries, we maintain the equivalence of the anomaly matching condition by this procedure. From an anomaly matching viewpoint, it is simpler to move all the Lagrangians to one side of the duality relation. In this case, the anomaly matching becomes equivalent to anomaly cancellation without introducing spurious “leptons”. We thus have a better analogy to the anomaly cancellation in grand unification schemes. In fact, the global symmetry \(SU(N_f)_L \times SU(N_f)_R \times U_B(1)\) behaves analogously. However, the symmetry \(U^AF(1)\) behaves differently in many respects; for example, there exists a drastic reshuffling of the quantum number assignment. The reason we compare (3) and (4) becomes clear below.

For the relations in Eq. (3), we first classify the fields according to their representation of \(SU(N_f)_L \times SU(N_f)_R\), which turns out to be the sole well-defined
classification symmetry in the present model. We then combine the fields pair-wise (for example, $\Psi_Q + \bar{\Psi}_q$) into multiplets of the color gauge symmetry of $SU(N_f)$.

Note that $N_f = N_c + \tilde{N}_c$. From a superfield viewpoint, this operation symbolically represents

$$\int \mathcal{L}(\Psi_Q, \bar{\Psi}_q, \lambda_{N_c})(x, \theta_1, \bar{\theta}_1) d^4x d^4\theta_1 + \int \mathcal{L}(\bar{\Psi}_q, \Psi_Q, \lambda_{\tilde{N}_c})(x, \theta_2, \bar{\theta}_2) d^4x d^4\theta_2$$

$$\rightarrow \int \mathcal{L}(\Psi_Q + \bar{\Psi}_q, \Psi_Q, \bar{\Psi}_q, \lambda_{N_c + \tilde{N}_c})(x, \theta, \bar{\theta}) d^4x d^4\theta.$$ 

The Grassmann parameters $\theta_1$ and $\theta_2$ are transformed oppositely under $R$-symmetry, as is indicated in the left-hand side of (3), by using the freedom of the overall constant factor for the $R$-charge assignment within a given theory. This operation (6) causes a complete reshuffling of the $U_R^{AF}(1)$ charge assignment in the combined theory in (3).

In (3) the number of the quark freedom is kept fixed, but the gauge degrees of freedom and the gaugino freedom are changed. Within this pair-wise combination, we can match anomalies associated with $U_B(1)(SU(N_f))_L^2$, $U_B(1)(SU(N_f))_R^2$, $(SU(N_f))_L^3$, $(SU(N_f))_R^3$, $U_R^{AF}(1)(SU(N_f))_L^2$, $U_R^{AF}(1)(SU(N_f))_R^2$, $(U_B(1))^2 \times U_R^{AF}(1)$ and $U_R^{AF}(1)$-gravitational anomaly (i.e., the freedom counting) except for $U_R^{AF}(1)^3$. For example, for the combination $\Psi_Q + \bar{\Psi}_q$ we have the coefficients of anomalies:

$$U_B(1)(SU(N_f))^2 : N_c \times 1 + \tilde{N}_c \times \left(-\frac{N_c}{N_f}\right) = 0 \rightarrow 0,$$

$$(SU(N_f))_L^3 : N_c + \tilde{N}_c = N_f \rightarrow N_f,$$

$$U_R^{AF}(1)(SU(N_f))^2 : N_c \left(-\frac{N_c}{N_f}\right) + \tilde{N}_c \left(\frac{\tilde{N}_c}{N_f}\right) \rightarrow N_f \left(\frac{\tilde{N}_c - N_c}{N_f}\right),$$

$$U_R^{AF}(1)(U_B(1))^2 : N_c \left(-\frac{N_c}{N_f}\right) \times 1 + \tilde{N}_c \frac{N_c}{N_f} - \frac{N_c}{N_c} = 0 \rightarrow 0,$$

$$U_R^{AF}(1)\text{-gravitational} : N_c \left(-\frac{N_c}{N_f}\right) + \tilde{N}_c \left(\frac{\tilde{N}_c}{N_f}\right) \rightarrow N_f \left(\frac{\tilde{N}_c - N_c}{N_f}\right).$$ 

Similarly one can confirm the anomaly matching for other combinations in Eq. (3).

The anomaly of $U_R^{AF}(1)$ related to the instanton of color gauge fields becomes

$$\Psi_Q + \bar{\Psi}_q : \frac{N_f}{2} \left(-\frac{N_c}{N_f}\right) \mathcal{P}(SU(N_c)) + \frac{N_f}{2} \left(\frac{\tilde{N}_c}{N_f}\right) \mathcal{P}(SU(\tilde{N}_c))$$

$$\rightarrow \frac{N_f}{2} \left(\frac{\tilde{N}_c - N_c}{N_f}\right) \mathcal{P}(SU(N_f)), \quad (8)$$

where $\mathcal{P}(SU(N_c))$, for example, stands for the Pontryagin index for $SU(N_c)$ color gauge fields. A similar relation for $\Psi_Q + \bar{\Psi}_q$ together with that for

$$\lambda_{N_c} + \tilde{\lambda}_{\tilde{N}_c} : N_c \mathcal{P}(SU(N_c)) - \tilde{N}_c \mathcal{P}(SU(\tilde{N}_c)) \rightarrow N_f \left(\frac{N_c - \tilde{N}_c}{N_f}\right) \mathcal{P}(SU(N_f)), \quad (9)$$
shows that the assignment of $U^A_R(1)$ charge is consistent; that is, the sum of twice of Eqs. (8) and (9) vanishes on both sides of correspondence. The color singlet component in the right-hand side of $\lambda_{N_c} + \bar{\lambda}_{\bar{N}_c}$ in Eq. (3) does not contribute in this calculation. It is confirmed that the $U^A_R(1)$ charge assignment has a unique solution in (3) if one assumes the appearance of a minimum set of fields as in (3), in other words, if we allow no degenerate fields with respect to $SU(N)_c \times SU(N_f)_L \times SU(N_f)_R \times U_B(1)$ which can be distinguished only by different $U^A_R(1)$ charges. The anomaly $(U^A_R(1))^3$ is not matched within the pair-wise combination, although overall it is matched in Eq. (3), and thus it is non-trivial from the present point of view.

If one looks at the right-hand side of the correspondence in Eqs. (3) and (4) and compares the multiplets appearing there, all anomaly matchings including $(U^A_R(1))^3$ are manifest if one remembers $N_f = N_c + \bar{N}_c$. The anomaly-free condition of $U^A_R(1)$ related to instantons is also manifestly satisfied. [With regard to calculational rules of anomalies, see (7)–(9).] Note that the assignment of $U^A_R(1)$ charge to all the fields is arbitrary up to a common overall constant. [The absolute normalization of $R$ charge becomes relevant when one analyzes the superconformal theory and relations such as $d \geq \frac{3}{2} |R|$, since one has to assign a proper (length) dimension to the Grassmann parameter in such analysis.] Through this embedding into a larger gauge group, one can understand the anomaly matching condition in the original duality relation between (1) and (2) in a more systematic way.

### 2.2. $SU(N_f)_c \rightarrow SU(N_c)_c \times SU(N_f - N_c)_c$

We now look at the above correspondence in Eqs. (3) and (4) in reversed order, namely from the right-hand side to the left-hand side. In this case, the extra color singlet component of the gluino field of $SU(N_f)_c$ in (3) is identified as an “anti-field” of the baryon in the duality relation for the case $N_c = N_f, N = 1$ supersymmetric QCD. More precisely, in the notation of $SU(N)_c \times SU(N_f)_L \times SU(N_f)_R \times U_B(1) \times U^A_R(1)$

$$
\Psi_Q : (N_f) \left( N_f, 1, 0, \frac{\bar{N}_c - N_c}{N_f} \right), \\
\Psi_{\bar{Q}} : (\bar{N}_f) \left( 1, N_f, 0, \frac{\bar{N}_c - N_c}{N_f} \right), \\
\lambda_{N_f} : (N^2_f - 1) \left( 1, 1, 0, -\frac{\bar{N}_c - N_c}{N_f} \right)
$$

is dual to

$$
\Psi_T : (1) \left( N_f, N_f, 0, \frac{\bar{N}_c - N_c}{N_f} \right), \\
B : (1) \left( 1, 1, 0, \frac{\bar{N}_c - N_c}{N_f} \right)
$$
with $N_f = N_c + \tilde{N}_c$. We regard $\lambda_{N_f}$ and the “anti-field” of baryon $B$ to combine into $(N_f, \tilde{N}_f)(1, 1, 0, -\frac{\tilde{N}_c - N_c}{N_f})$ in the right-hand side of (3), and thus the anomaly matching in (10) and (11) is manifest. Note that the assignment of $U_{AF}^R(1)$ charge is arbitrary up to a common overall constant factor.

For $N_c = N_f, N = 1$ supersymmetric QCD, it has been argued \(^1\) that one has non-perturbative constraint with a QCD mass scale $\Lambda$

$$\det T - B\bar{B} = \Lambda^{2N_f}, \quad (12)$$

where $T$ and $B$ are meson and baryon scalar multiplets, respectively, and the above duality relation corresponds to the case

$$\langle T \rangle = 0, \quad \langle B \rangle = -\langle \bar{B} \rangle = \Lambda^{N_f}. \quad (13)$$

Therefore, the baryon number is condensed in the vacuum; this explains the peculiar assignment of vanishing baryon number to all the fields in (10) and (11). When one formally breaks the color symmetry as $SU(N_f)_c \to SU(N_f - \tilde{N}_c)_c \times SU(\tilde{N}_c)_c$, the baryon $B$ (and $\bar{B}$) dissociates. (An assumption to this effect is made in Ref. 2.) At the same time, one can assign definite baryon numbers to the fields as in the correspondence (3) and (4). Namely, each field can pick up (basically arbitrary) baryon number from the vacuum in a way to be consistent with the anomaly matching (or anomaly-free condition if one includes spurious “leptons”). Note that the baryon numbers are arbitrary up to a common overall constant factor.

The peculiar behavior of $U_{AF}^R(1)$ may also be understood in a manner similar to the spontaneously broken baryon number. The $U_{AF}^R(1)$ charge is condensed in the vacuum in the sense that the constant Grassmann parameter carries the charge, and each particle can pick up an arbitrary value from the vacuum in the above formal symmetry breaking in such a manner as to be consistent with the anomaly condition.

From an anomaly matching point of view, we thus have two physically realizable models in one equivalence class. We may picture a formal color symmetry breaking

$$SU(N_f)_c \to SU(N_f - \tilde{N}_c)_c \times SU(\tilde{N}_c)_c \quad (14)$$

and that the duality relation

$$\mathcal{L}(Q, \bar{Q}, \lambda_{N_f}) \sim \mathcal{L}(T, B) \quad (15)$$

is transformed into

$$\mathcal{L}(\Psi_Q, \bar{\Psi}_Q, \lambda_{N_c}) + \bar{\mathcal{L}}(\Psi_q, \bar{\Psi}_q, \lambda_{\tilde{N}_c}) \sim \mathcal{L}(T), \quad (16)$$

which in turn suggests the (electric-magnetic) duality relation

$$\mathcal{L}(\Psi_Q, \bar{\Psi}_Q, \lambda_{N_c}) \sim \mathcal{L}(\Psi_q, \bar{\Psi}_q, \lambda_{\tilde{N}_c}) + \mathcal{L}(T) \quad (17)$$

or

$$\mathcal{L}(\Psi_Q, \bar{\Psi}_Q, \lambda_{N_c}) + \bar{\mathcal{L}}(T) \sim \mathcal{L}(\Psi_q, \bar{\Psi}_q, \lambda_{\tilde{N}_c}). \quad (18)$$
In this picture, it is more natural to assign the color quantum number $(\tilde{N}_c)$ of $SU(\tilde{N}_c)c$ to $\Psi_q$, as in our assignment in (2).

At this point we have no physical meaning assigned to the formal color symmetry breaking sequence in (14), except for providing mnemonics for a systematic anomaly matching. Nevertheless, the peculiar behavior of the baryon number and $U_{AF}(1)$ symmetry in (3) and (10) is quite suggestive, and it might be found to acquire some significance in future analyses of duality. The $N_c = N_f$ case is the simplest from the viewpoint of anomaly matching, and it is likely that it will play a pivotal role in such analysis.

Anomaly matching is an equivalence relation up to a set of fields which are anomaly-free by themselves. It would therefore be sensible to classify the solutions of anomaly matching by restricting the possible set of allowed fields. We tentatively classify a solution as a minimal set if we have no fields that are degenerate with respect to the quantum numbers of $SU(N_c) \times SU(N_f)_L \times SU(N_f)_R \times U_B(1)$ and which are distinguished only by different $U_{AF}(1)$ charges. Otherwise, we classify a solution as a non-minimal set. As for anomaly matching, we first impose only anomaly matching linear in $U_{AF}(1)$ and examine the $U_{AF}(1)^3$ anomaly later. The example we have discussed to this point, i.e., $N = 1$ supersymmetric QCD, is classified as a minimal set. In fact we found that the minimal set in (3) automatically satisfies the $U_{AF}(1)^3$ anomaly matching also. On the other hand, the example considered by Kutasov and Schwimmer 8) belongs to a non-minimal set in this classification.

§3. Non-minimal model with $SU(N)_c$ gauge theory

To be specific, the model in Ref. 8) contains the following fermion contents: the initial theory is a modification of $N = 1$ supersymmetric QCD with color gauge symmetry $SU(N)_c$ obtained by adding an extra chiral field $X$ in the adjoint representation of $SU(N)_c$. In the notation of $SU(N)_c \times SU(N_f)_L \times SU(N_f)_R \times U_B(1) \times U_{AF}(1)$, we have the fermion components

\[ \Psi_Q : (N_c) \left( N_f, 1, 1, -\frac{2}{k+1} \frac{N_c}{N_f} \right), \]
\[ \Psi_{\bar{Q}} : (\bar{N}_c) \left( 1, \bar{N}_f, -1, -\frac{2}{k+1} \frac{N_c}{N_f} \right), \]
\[ \Psi_X : (N_c^2 - 1) \left( 1, 1, 0, \frac{1-k}{k+1} \right), \]
\[ \lambda_{N_c} : (N_c^2 - 1) \left( 1, 1, 0, 1 \right), \]  \hspace{1cm} (19)

where $k$ represents a positive integer. The above multiplet is dual to the (magnetic) $N = 1$ supersymmetric theory with $SU(\tilde{N}_c)c$ gauge symmetry given by (with the same notation)

\[ \Psi_q : (\tilde{N}_c) \left( \tilde{N}_f, 1, \frac{N_c}{N_f}, -\frac{2}{k+1} \frac{\tilde{N}_c}{N_f} \right), \]
\[ \psi_q : (\tilde{N_c}) \left( 1, N_f, -\frac{N_c}{N_f}, -\frac{2}{k+1} \tilde{N_c} \right), \]
\[ \psi_Y : (\tilde{N_c}^2 - 1) \left( 1, 1, 0, \frac{1-k}{k+1} \right), \]
\[ \lambda_{\tilde{N}_c} : (\tilde{N}_c^2 - 1) (1, 1, 0, 1), \]
\[ \psi_{T_j} : (1) \left( N_f, \tilde{N}_f, 0, 1 - \frac{4}{k+1} \frac{N_c}{N_f} + \frac{2}{k+1} (j-1) \right), \quad j = 1 \cdots k \quad (20) \]

with \( N_c + \tilde{N}_c = kN_f \). Here \( \psi_{T_j} \) stands for the fermion component of the \( j \)th meson scalar multiplet formed of \( Q(X) \)\(^{-1}Q \). \( \psi_Y \) is a counterpart of \( \psi_X \). In this scheme, we have \( k \) meson scalar multiplets \( \psi_{T_j} \) that can be distinguished only by their \( U^A_R(1) \) charges, and thus classified as a non-minimal set.

This non-minimal property becomes more apparent if one considers the counterparts of Eqs. (3) and (4) in the present example,

\[ \psi_Q + \bar{\psi}_q : (N_c + \tilde{N}_c) \left( N_f, 1, 0, \frac{2}{k+1} \frac{N_c}{N_f} - \frac{\tilde{N}_c}{N_f} \right), \]
\[ \psi_Q + \bar{\psi}_q : (N_c + N_c) \left( 1, \tilde{N}_f, 0, \frac{2}{k+1} \frac{N_c}{N_f} - \frac{\tilde{N}_c}{N_f} \right), \]
\[ \psi_X + \bar{\psi}_Y + \lambda_{N_c} + \bar{\lambda}_{\tilde{N}_c} : (N_c + \tilde{N}_c, N_c + \tilde{N}_c) \left( 1, 1, 0, -\frac{2}{k+1} \frac{\tilde{N}_c}{kN_f} \right) \quad (21) \]

where all the anomalies except \( U^A_R(1)^3 \) are matched within each combination of fields. This is confirmed by calculations similar to those involving (7)–(9), and

\[ \psi_{T_j} : (1) \left( N_f, \tilde{N}_f, 0, 1 - \frac{4}{k+1} \frac{N_c}{N_f} + \frac{2}{k+1} (j-1) \right), \quad j = 1 \cdots k. \quad (22) \]

If one recalls that \( N_c + \tilde{N}_c = kN_f \) and that the average of the \( U^A_R(1) \) charge for \( \psi_{T_j} \) is \( \frac{2}{k+1} \frac{N_c}{N_f} \), which is relevant for anomaly matching linear in \( U^A_R(1) \), all the anomaly matching between (21) and (22), except for \( U^A_R(1)^3 \), is nearly self-evident if we recall the calculations in (7)–(9).

It turns out that the \( U^A_R(1)^3 \) anomaly matching is not satisfied in (21) by this minimal set of fields. There are many ways to remedy this problem. One may, for example, add

\[ \psi_{Z_i} : (1)(1, 1, 0, \alpha_i), \quad i = 1, 2, 3, \quad (23) \]

with \( \alpha_1 + \alpha_2 + \alpha_3 = 0 \), to \( \psi_X + \bar{\psi}_Y + \lambda_{N_c} + \bar{\lambda}_{\tilde{N}_c} \) in (21), or we may replace the last line in (21) by

\[ (N_c + \tilde{N}_c, N_c + \tilde{N}_c)(1, 1, 0, \beta_i), \quad i = 1, 2, 3, \quad (24) \]

with \( \beta_1 + \beta_2 + \beta_3 = -\frac{2}{k+1} \frac{N_c}{kN_f} \). In either case, if one suitably chooses 3 real parameters \( \alpha_i \) or \( \beta_i \), one can adjust the \( U^A_R(1)^3 \) anomaly freely by keeping the
anomalies linear in $U^\text{AF}_R(1)$ fixed. For example, the $U^\text{AF}_R(1)^3$ anomaly for (23) is proportional to $\alpha_1^3+\alpha_2^3+\alpha_3^3$, which is controlled by the signature of $\alpha_3$ if one chooses $\alpha_1 \simeq \alpha_2$. A characteristic of these modifications is that one needs to introduce a rather large number of color-singlet and flavor-singlet fields, which have no clear physical meaning in the context of supersymmetric QCD.

From an anomaly matching point of view, the correspondence in (21) together with (24) suggests that the $N_f$ flavor $SU(kN_f)_c$ supersymmetric QCD, with 2 color-adjoint flavor-singlet matter fields added, could be dual to $\Psi_{\ell_f}$ and 3 color-singlet flavor-singlet fields, for example. The baryon number assignment in (21) suggests the condensation of quark scalar multiplets, somewhat analogous to (13). Apparently, physical considerations other than those of anomaly matching are needed here to see if there are two physically realizable dual models, based on $SU(kN_f)_c$ as well as on $SU(N_c)_c$ and $SU(\tilde{N}_c)_c$, in this equivalence class of anomaly matching.

§4. Discussion

When one considers the color gauge group $SO(N)_c$, the above formal color symmetry breaking in (14) is replaced by

$$SO(N_f+4)_c \to SO(N_c)_c \times SO(\tilde{N}_c)_c,$$

(25)

where $N_f$ stands for the number of massless quarks and $N_c = N_f + 4 - \tilde{N}_c$. Note that $SO(N_f+4)_c$ theory is known to be a counterpart of $SU(N_f)_c$ theory. In this case, we consider the correspondence by denoting $SO(N)_c$ and $SU(N)_f \times U^\text{AF}_R(1)$ quantum numbers of the fermionic components as

$$Q : (N_f+4) \left( N_f, \frac{N_f+2\tilde{N}_c-N_c}{N_f+4} \right) \to \begin{cases} (N_c) \left( N_f, \frac{2N_c}{N_f} \right) \\ (\tilde{N}_c) \left( N_f, -\frac{2\tilde{N}_c}{N_f} \right) \end{cases}$$

and its dual

$$\lambda : \left( \frac{(N_f+4)(N_f+3)}{2} \right) \left( 1, -\frac{\tilde{N}_c-N_c}{N_f+4} \right) \to \begin{cases} \left( \frac{N_c(N_c-1)}{2} \right) (1, 1) \\ \left( \frac{\tilde{N}_c(\tilde{N}_c-1)}{2} \right) (1, -1) \end{cases}$$

(26)

and

$$M^{ij} : (1) \left( \frac{N_f(N_f+1)}{2}, \frac{\tilde{N}_c-N_c}{N_f} \right) \to (1) \left( \frac{N_f(N_f+1)}{2}, -\frac{\tilde{N}_c-N_c}{N_f} \right),$$

(27)

where $N_f + 4 = N_c + \tilde{N}_c$, and $M^{ij}$ stands for a meson field formed of $Q_i Q^j$. $[SU(N)_f \times U^\text{AF}_R(1)$ is basically $\gamma_5$-symmetry]. Again, note that the $U^\text{AF}_R(1)$ assignment is arbitrary up to a common overall factor. We have a minimal set of fields for the anomaly matching of $SU(N)^2 \times U^\text{AF}_R(1)$, $SU(N)^3$ and $U^\text{AF}_R(1)$-gravitational, without imposing $U^\text{AF}_R(1)^3$ matching, when one looks at the correspondences in (26) along the arrows: The solution thus obtained automatically satisfies the $U^\text{AF}_R(1)^3$ anomaly matching also on both sides of (26) and (27), as is confirmed by explicit calculations. We again have two physically realizable dual models, based...
on $SO(N_f + 4)_c$ as well as on $SO(N_c)_c$ and $SO(\tilde{N_c})_c$, in one equivalence class of anomaly matching.

As for the gauging of $R$-symmetry, it is known that it is consistent only within the framework of supergravity,\(^9\) though the gauging of $R$-symmetry is not inevitable in supergravity. It is interesting that the $U^A_F(1)^3$ anomaly matching, which physically suggests the gauging of $R$-symmetry, imposes a non-trivial constraint on the dynamics of supersymmetric QCD.

In conclusion, we have commented on a specific aspect of the non-Abelian electric-magnetic duality in $N = 1$ supersymmetric QCD. It has been shown that anomaly matching is not mysterious, but, rather, it exhibits a certain regularity. It is hoped that the present note stimulates further thinking about the non-Abelian electric-magnetic duality.

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