Leibenzon-Ishlinsky Criterion for the Axisymmetric Problem of Compression of the Pillar

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Abstract. In the framework of the Leibenzon-Ishlinsky approach, the problem of the loss of stability of a pillar of a cylindrical mine working is solved. The pillar material was assumed with an initial anisotropy corresponding to the layered structure. A criterion for loss of stability is being constructed, a solution to the system of differential equations of the problem in the form of combinations of cylindrical and trigonometric functions is determined. From the fact that the determinant of a system of homogeneous algebraic equations is equal to zero, the critical load value is found at which, along with the main continuation of the deformation of the pillar, something else is possible with a changed surface geometry. The influence of the initial anisotropy, the parameters of the pillar (height, radius) on the values of the ultimate load is investigated.

1. Introduction
In [1–7], the problems of loss of stability in structures under the plane deformation (an infinitely long strip, the contour of a cylindrical working, etc.) were considered. The concept of bifurcation was proposed when at some load value \( P = P^* \) there is a possible existence of both - the continuation of the main deformation process and the process that is infinitely close to the main one but with curved boundaries of the studied structure.

Below, the Leibenzon-Ishlinsky approach is proposed to be extended in the case of structural deformation not under conditions of plane deformation, but under conditions of axisymmetric deformation. Moreover, the material is assumed to be not initially isotropic, but initially anisotropic, having a layered structure. In this situation, it is necessary to find such a value of the compressive load \( p = p^* \), acting on the pillar of the working at which bifurcation of the process of its deformation will occur will occur

2. Mathematical model
Let there be a pillar in the shape of a cylinder of radius \( R \) and height \( H \) (Fig. 1). Let the pillar material be deformed in elasticity according to the Hooke law in the form:
\[
\begin{align*}
\sigma_r &= A\varepsilon_r + B\varepsilon_\phi + C\varepsilon_z, \\
\sigma_\phi &= B\varepsilon_r + A\varepsilon_\phi + C\varepsilon_z, \\
\sigma_z &= C\varepsilon_r + A\varepsilon_\phi + D\varepsilon_z, \\
\tau_{rz} &= E\varepsilon_{rz}. 
\end{align*}
\] (1)

Here \( A, B, C, D, E \) - rigidities, where in general case \( A \neq D, B \neq C, E \neq A-B \). From these conditions it follows that (1) describes the behavior of a layered medium where the layers are parallel to the plane \( z = 0 \) because under the same load acting along the axes \( r, \phi, z \), the deformation along the \( z \) axis will be different than along the axes \( r, \phi \) (here \( r, \phi, z \) - the axis of the cylindrical coordinate system).

It is further assumed that the pillar is under the action of compressive load, i.e. tensors of stress and strain until the moment of stability loss have the form

\[
T_\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_z = \begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_z & 0 \\ 0 & 0 & \varepsilon_\phi \end{pmatrix},
\] (2)

where \( \varepsilon_r = \varepsilon_\phi \neq \varepsilon_z \). Stresses satisfy the equilibrium equations:

\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\phi}{r} &= 0, \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{\partial \sigma_z}{\partial z} &= 0,
\end{align*}
\] (3)

strains satisfy Cauchy relations:

\[
\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\phi = \frac{u}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right),
\] (4)

where \( u, v \) - displacements. At the moment of stability loss, to (2) we add the following state

\[\text{Figure 1.} \text{ Mine working pillar under pressure load (Figure 1a). When the load reaches the critical value, the geometry of the surface of the pillar changes (Fig. 1b).}\]
where $\Delta \epsilon_r = \frac{\partial \Delta u}{\partial r}, \ldots$; $\Delta u, \Delta w$ - increments of displacements. Thus, at the moment of stability loss, the stress state is the sum of the states $T_{r}, T_{r\sigma}$, the deformed state is $T_{r} + T_{\Delta \epsilon}$. Following [1-3], on the curved surface of a massive structure, we obtain the following boundary conditions:

$$
\left. \begin{array}{c}
\Delta \sigma_r \big|_r = 0, \\
\Delta \sigma_r \big|_r = \sigma_z \frac{\partial \Delta w}{\partial r},
\end{array} \right\}
$$

(6)

where $\Gamma$ - body surface until stability loss.

The task is to find the stress value $\sigma_z$, at which the quantities $\Delta \sigma_{ij}$ satisfy the equilibrium equations (3), are associated with strain increments $\Delta \epsilon_{ij}$ by the relations of Hooke's law (1), strains $\Delta \epsilon_{kl}$ are expressed through the increment of displacements $\Delta u, \Delta w$ using (4).

Let us give the problem solution. We substitute (4) in (1), (1) in (3). As a result, to determine $\Delta u$ and $\Delta w$, we find the following system of equations:

$$
\begin{aligned}
&\frac{\partial^2 \Delta u}{\partial r^2} + \frac{1}{2} \frac{\partial \Delta u}{\partial r} - \frac{\Delta u}{r^2} + \frac{E}{2A} \frac{\partial^2 \Delta u}{\partial z^2} + \frac{2C + E}{2A} \frac{\partial^2 \Delta w}{\partial r \partial z} = 0, \\
&\frac{\partial^2 \Delta u}{\partial r \partial z} + \frac{1}{2} \frac{\partial \Delta u}{\partial z} + \frac{E}{E + 2C} \left( \frac{\partial^2 \Delta w}{\partial r^2} + \frac{1}{2} \frac{\partial \Delta w}{\partial r} \right) + \frac{2D}{E + 2C} \frac{\partial^2 \Delta w}{\partial z^2} = 0.
\end{aligned}
$$

(7)

We find solution (7) in the form

$$
\Delta u = M Z_1 (\lambda r) \text{ch} \; p z, \quad \Delta w = N Z_0 (\lambda r) \text{sh} \; p z,
$$

(8)

where $Z_0, Z_1$ are cylindrical functions of zero and first order respectively to [8] and $M, N, \lambda, p$ are arbitrary constants.

Substitution of (8) into (7) leads to the following homogeneous system of linear algebraic equations for determining the constants $M$ and $N$:

$$
\begin{aligned}
&-\frac{\lambda^2}{2A} \left( \frac{E}{2} \right)^2 \cdot M - \frac{2C + E}{2A} \lambda p \cdot N = 0, \\
&\lambda p \cdot M + \left( - \frac{E \lambda^2}{E + 2C} + \frac{2D}{E + 2C} \right) \cdot p^2 \cdot N = 0.
\end{aligned}
$$

(9)

For the existence of a nonzero solution (9), it is necessary that the determinant of the system (9) vanish. As a result, we obtain the characteristic equation

$$
\left( \frac{\lambda}{p} \right)^4 - 2 \left( \frac{\lambda}{p} \right)^2 \frac{AD - C^2 - EC}{AE} + \frac{D}{A} = 0.
$$

(10)

for determining $\frac{\lambda}{p}$. We note that for an initially isotropic $A = D, B = C, E = A - B$, therefore equation (10) turns into the simplest one with multiple roots equal to 1:
\[
\left( \frac{\lambda}{p} \right)^4 - 2 \left( \frac{\lambda}{p} \right)^2 + 1 = 0.
\]

Solving (10) we get
\[
\begin{align*}
\left( \frac{\lambda}{p} \right)^2_{1,2} &= \frac{AD - C^2 - EC}{AE} + \sqrt{\frac{AD - C^2 - EC}{AE} - \frac{D}{A}}, \\
\left( \frac{\lambda}{p} \right)^2_{3,4} &= \frac{AD - C^2 - EC}{AE} - \sqrt{\frac{AD - C^2 - EC}{AE} - \frac{D}{A}}.
\end{align*}
\]

From (11) it follows that \( \lambda_1 = -\lambda_2, \quad \lambda_3 = -\lambda_4 \). Therefore, the general solution (7), based on the evenness and oddness of cylindrical functions, \( Z_0, Z_1 \), can be written in the form:
\[
\begin{align*}
\Delta u &= \left[ (M_1 - M_2) Z_1(\lambda r) + (M_3 - M_4) Z_1(\lambda r) \right] \text{ch} \, pz, \\
\Delta w &= \left[ (N_1 + N_2) Z_0(\lambda r) + (N_3 + N_4) Z_0(\lambda r) \right] \text{sh} \, pz,
\end{align*}
\]
where constants \( M_i \) are connected with constants \( N_i \) (\( i = 1, \ldots, 4 \)) by relations
\[
M_i = \left[ \frac{E}{E + 2C} \left( \frac{\lambda}{p} \right)_{i} - \frac{2D}{E + 2C} \left( \frac{p}{\lambda} \right)_{i} \right] N_i,
\]
following from (9).

A few words about evenness and oddness of functions \( \Delta u, \Delta w \). From physical considerations, it is clear that function \( \Delta u \) must be an odd one of the coordinate \( r \), i.e.
\[
\Delta u_{|r=R} = -\Delta u_{|r=-R},
\]
function \( \Delta w \) - an even one of the coordinate \( r \):
\[
\Delta w_{|r=R} = \Delta w_{|r=-R}.
\]

Taking into account the oddness of the function \( Z_1 \) and the evenness of the function \( Z_0 \) [9] on the basis of (12), (14), (13), (15) we obtain that \( M_2 = -M_1, N_1 = N_2 \) by virtue of \( \lambda_1 = -\lambda_2 \), and \( M_3 = -M_4, N_3 = N_4 \) by virtue of \( \lambda_3 = -\lambda_4 \). So
\[
\begin{align*}
\Delta u &= \left[ 2M_1 Z_1(\lambda r) + 2M_3 Z_1(\lambda r) \right] \text{ch} \, pz, \\
\Delta w &= 2 \left[ N_1 Z_0(\lambda r) + N_3 Z_0(\lambda r) \right] \text{sh} \, pz.
\end{align*}
\]

It remains to satisfy the boundary conditions (6) at \( r = R \), where \( R \) is the radius of the pillar. Given (1), we have
\[
\begin{align*}
\Delta \sigma_r = A \frac{\partial \Delta u}{\partial r} + B \frac{\Delta u}{r} + C \frac{\partial \Delta w}{\partial z} \bigg|_{r=R} = 0, \\
\Delta \tau_{ez} = E \left( \frac{1}{2} \left( \frac{\partial \Delta w}{\partial r} + \frac{\partial \Delta u}{\partial z} \right) \right) \bigg|_{r=R} = -P_e \frac{\partial \Delta u}{\partial z},
\end{align*}
\]
where \( \sigma_z = -P_e \) at the moment of the pillar stability loss.

Substituting (16) into (17), we find the equations for finding constants \( N_1, N_3 \):
System (18) under conditions (13) is a homogeneous system of linear equations for determining the constants $\lambda_i$. Its determinant must turn to zero for the existence of a non-zero solution. Expressing the determinant, we have

$$\Delta = \begin{vmatrix} (A_1\lambda_1 + Cp)RZ_0(\lambda_1R) + (A_2\lambda_2 + Cp)RZ_0(\lambda_2R) + (B - A)L_1Z_1(\lambda_1R) + (B - A)L_2Z_1(\lambda_2R) & \vdots & \vdots \end{vmatrix} = 0, \quad (19)$$

where values $L_1, L_2$ follow from (13) at: $M_1 = L_1N_1$, $M_3 = L_3N_3$, (19) – the equation to determine $P_\star$.

This expression includes the quantity $P_\star$ associated with hyperbolic functions using (8). These functions are unbounded with increasing coordinate $z$ at a real value of $P_\star$. To eliminate the unboundedness of the function $\Delta w$, we consider the case when $p = i\frac{\pi}{H}$.

Then $\text{sh} z = \text{sh} i\frac{\pi}{H} z = i\sin \frac{\pi z}{H}$. Considering (20) on the basis of (8) we get $\Delta w\bigg|_{z = 0} = 0$, $\Delta w\bigg|_{z = \frac{\pi}{2H}} = iNZ_0(\lambda R) = -\Delta w\bigg|_{z = -\frac{\pi}{2H}}$, as it should be for the pillar with the origin taken at the center of the pillar.

Thus, when calculating the determinant $\Delta$ according to (19), we obtain in it the value of $P_\star$ equal to (20).

Solving (19) relatively to $P_\star$, we find the expression

$$P_\star = \frac{E}{2} \left[ (L_1p - \lambda_1)Z_1(\lambda_1R)\cdot \Sigma_2 - (L_3p - \lambda_3)Z_1(\lambda_3R)\cdot \Sigma_1 \right]/\left[ (L_3pZ_1(\lambda_3R)\cdot \Sigma_1 - L_1pZ_1(\lambda_1R)\cdot \Sigma_2) \right], \quad (21)$$

where

$$\Sigma_1 = (A_1\lambda_1 + Cp)RZ_0(\lambda_1R) + (B - A)L_1Z_1(\lambda_1R), \quad \Sigma_2 = (A_2\lambda_2 + Cp)RZ_0(\lambda_2R) + (B - A)L_2Z_1(\lambda_2R),$$

$$L_1 = \frac{E}{E + 2C}\left(\frac{\lambda}{p}\right)_1 - \frac{2D}{E + 2C}\left(\frac{\lambda}{p}\right)_1, \quad L_3 = \frac{E}{E + 2C}\left(\frac{\lambda}{p}\right)_3 - \frac{2D}{E + 2C}\left(\frac{\lambda}{p}\right)_3.$$
3. Mathematical modelling results

Dependencies of critical stress $P_c$ on $R/H$ and rigidity parameters are shown on Fig.1

![Figure 1](image.png)

Figure 2. Dependencies of critical stress $P_c$ on relation $R/H$

We get dashed line at $A = 1.33, B = 0.33, C = 0.33, D = 0.83$; solid line $A = 1.75, B = 0.75, C = 0.75, D = 1.25$; dotted line at $A = 1.75, B = 0.75, C = 0.75, D = 0.76$; dash-dot line at $A = 1.75, B = 0.075, C = 0.75, D = 0.76$. Analyzing results we may say that if $R/H < 1$ then the pillar is deformed by the type of «collar», if $R/H > 1$ - by the type of «barrel».

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4. Conclusions

1. The problem of the loss of stability of a cylindrical pillar with the initial anisotropy of the material has been solved.
2. The influence of the anisotropy and geometrical dimensions of the pillar on the critical load value is investigated.

5. References

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