Rigid Invariance in Gauge Theories

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Abstract

Rigid gauge invariance comprises the symmetry content for physical quantities in a local gauge theory. Its derivation from BRS invariance is thus crucial for determining the physical consequences of the symmetry.

* To appear in the “Proceedings of the XXIst Conference on Group Theor. Methods in Physics, Goslar, Juli 1996”.
† supported by Deutsche Forschungsgemeinschaft
1 Introduction

Local gauge invariance and its translation into BRS invariance define gauge models in the sense that they permit the proof of unitarity. They govern the behaviour of the unphysical modes which – in perturbation theory – have to be introduced in order to maintain Lorentz invariance and locality. What they do not tell in the general case is which symmetry relations survive quantization and renormalization. The existence of conserved currents and charges has to be inferred from the rigid gauge invariance or (in the BRS case) from local Ward identities associated with the rigid symmetry. In this note we shall sketch some of the more relevant cases.

2 Quantum electrodynamics

The classical action

\[ \Gamma = \int -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2\alpha} (\partial A)^2 + \bar{\psi}(i\partial^\mu \gamma_\mu - m + eA^\mu \gamma_\mu)\psi \]  

satisfies the local Ward identity (WI)

\[ w\Gamma \equiv (-\partial \frac{\partial}{\partial A} - ie\bar{\psi} \frac{\delta}{\delta \psi} + ie\psi \frac{\delta}{\delta \psi})\Gamma = -\frac{1}{\alpha} \Box \partial A \]  

and the rigid Ward identity

\[ \mathcal{W} \Gamma \equiv \int \hat{w}\Gamma = 0 \]  

\[ \hat{w} \equiv -ie\bar{\psi} \frac{\delta}{\delta \psi} + ie\psi \frac{\delta}{\delta \psi} \]  

Rewriting the local WI for the general Green functions one can prove that

\[ \Box \partial A^{Op} = 0, \]  

i.e. \( \partial A^{Op} \) is a free field operator and can thus be decomposed into positive and negative frequency part, a decomposition which in turn allows to single out physical states by the condition

\[ (\partial A)^{(-)} | phys > = 0. \]
This guarantees the absence of negative norm states and leads to the Hilbert space of physical states via the formation of equivalence classes (members differ only by zero norm states) and closure.

For the perturbative existence of QED it is now crucial that for every regularization or renormalization scheme the local WI (2) can be proven if $e \to e' = e + o(\hbar), m \to m' = m + o(\hbar)$ and suitable counter terms are added to $\Gamma$. Because then (5) again follows, now to all orders of perturbation theory, and (6) together with the construction of the state space is possible also.

The rigid WI (3) expresses the symmetry content of the theory which just means conservation of the electric charge. It also expresses the conservation of the respective current

$$\hat{\omega} \Gamma = \partial^\mu j_\mu$$

whose 0-th component gives rise to the charge once it is integrated over 3-space. Obviously the WI (3) follows from (2) by integration and (7) differs from (2) by $-\frac{1}{\alpha} \square \partial A + \partial^2 \frac{\phi}{m^2}$. Hence the local WI (2) encodes also all information about the symmetry of the theory.

This property is a peculiarity of QED which does not hold in models where BRS invariance plays a role. These are the models with non-abelian gauge group and (or) spontaneous symmetry breaking.

## 3 The abelian Higgs model

The classical action

$$\Gamma_{\text{inv}} = \int \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^*(D^\mu \phi) + \frac{1}{2} m_H^2 \phi^* \phi - \frac{1}{2} m_2^2 e^2 (\phi^* \phi)^2 \right)$$

is invariant under the $\text{U}(1)$-transformation

$$\delta_\omega \phi = i e \omega \phi, \quad \delta_\omega A_\mu = 0.$$  

It describes a vector field $A_\mu$ of mass $m$ in interaction with the Higgs field $\phi_1$ of mass $m_H$ and the would-be Goldstone field $\phi_2$. But due to the spontaneously broken character of the symmetry the unphysical modes of $A_\mu$ interact and
there is no local gauge WI which would characterize the model and guarantee unitarity. It rather has to be replaced by BRS invariance expressed via a Slavnov–Taylor identity: After introducing the Faddeev–Popov ghosts, a Lagrange multiplier field $B$ and suitably coupled external fields we arrive at

\begin{align}
    sA_\mu &= \partial_\mu c \quad s\bar{c} = B \\
    sc &= 0 \quad sB = 0 \\
    s\phi &= iec\phi
\end{align}

for the elementary fields in the tree approximation and at

\[ s(\Gamma) \equiv \int \partial_\mu c \frac{\delta \Gamma}{\delta A_\mu} + B \frac{\delta \Gamma}{\delta \bar{c}} + \frac{\delta \Gamma}{\delta Y_i} \frac{\delta \Gamma}{\delta \phi_i} = 0. \]  

(12)

For calculational purposes one prefers a ’t Hooft type gauge fixing

\[ \frac{\delta \Gamma}{\delta B} = \xi B + \partial A + \xi_A m\phi_2 \] 

(13)

which however breaks the naive rigid invariance as defined by (10). The construction of higher orders thus requires some more machinery (i.e. further external fields) and in particular leads to a deformation of the classical rigid invariance in a well-specified sense. I.e. if one imposes physical normalization conditions for the vector and Higgs mass and their wave function renormalizations, then all normalizations of the wave function of $\phi_2$ at finite momentum lead to a rigid WI of the type

\[ \mathcal{W}\Gamma \equiv \int (-\phi_2 \frac{\delta}{\delta \phi_1} + (1 + u)(\phi_1 - \hat{\xi}_A \frac{m}{e} \frac{\delta}{\delta \phi_2})\Gamma = 0 \] 

(14)

(for all external fields = 0). $u = o(\hbar)$ parametrizes the deformation. It is crucial for the derivation of (14) that the rigid WI operator $\mathcal{W}$ is symmetric with respect to the ST identity (12). The coupling has been fixed by some 3-point-function. It is clear that to (14) is associated a local WI. Although it does not define the model and does not yield unitarity it is nevertheless useful. One can show [1] that it has the form

\[ (e(1 + a)w(x) - \partial \frac{\delta}{\delta A})\Gamma = \Box B \] 

(15)
(Here \( f \) \( w(x) = \mathcal{W} \).) If one refines the argument by varying the gauge parameter \( \xi \) into a Grassmann variable \( \chi \), adds this variation to the ST identity and ensures this enlarged identity to all orders one has control over the gauge parameter dependence. It turns out [2] that the coefficient \( a \) in (15) is independent of the gauge parameter. Hence one can like in QED define the coupling via the requirement “validity of an exact local WI to all orders”

\[
(\mathcal{W}(x) - \partial \frac{\delta}{\delta A}) \Gamma = \Box B.
\]  

(16)

A conserved current is defined via (16) by

\[
\mathcal{W}(x) \Gamma = \partial \mu j_\mu \cdot \Gamma
\]

(17)

The associated charge does, however, not exist since the symmetry is spontaneously broken.

4 QCD

As an example of a Yang-Mills theory with unbroken gauge group we look at QCD i.e. \( SU(3) \) and multiplets of fermions in the fundamental representation. The classical invariant action has the form

\[
\Gamma_{\text{inv}} = \int \left( -\frac{1}{4g^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i D^\mu \gamma_\mu - m) \psi \right)
\]

(18)

\[
F_{\mu\nu} \equiv \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \right) \tau^a
\]

(19)

\[
D_\mu \psi \equiv \left( \partial_\mu - ie \tau^a A_\mu^a \right) \psi
\]

\[
[\tau^a, \tau^b] = if^{abc} \tau^c
\]

Since classically rigid invariance

\[
\mathcal{W}^a \Gamma_{\text{inv}} \equiv i \int (A_\mu^b f^{bac} \frac{\delta}{\delta A_\mu^c} - \bar{\psi} i \tau^a \frac{\delta}{\delta \psi} + \psi i \tau^a \frac{\delta}{\delta \bar{\psi}}) \Gamma = 0
\]

(20)

is not broken one tries to impose (20) to all orders. A theorem due to BRS [3,4] guarantees that this is for (semi-) simple gauge groups indeed possible (in any renormalization scheme). In a second step one establishes then BRS invariance by way of a ST identity:

\[
s(\Gamma) \equiv \int \left( \frac{\delta \Gamma}{\delta \phi} \frac{\delta \Gamma}{\delta A} + B \frac{\delta \Gamma}{\delta c} + \frac{\delta \Gamma}{\delta \sigma} \frac{\delta \Gamma}{\delta c} + \frac{\delta \Gamma}{\delta Y} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta \Gamma}{\delta \psi} \frac{\delta \Gamma}{\delta Y} \right) = 0
\]

(21)
The existence of a local WI and its relation to a conserved current becomes a rather subtle question because the $\phi\Pi$ ghosts do not couple minimally to the vector field. A suitable device for their construction is an external field $\tilde{A}_\mu$, called background field, which varies under BRS in a further Grassmann field $\tilde{c}_\mu$ [5]:

$$s\tilde{A}_\mu = \tilde{c}_\mu$$ (22)

(All of them transform according to the adjoint representation under rigid symmetry.) Taking also gauge parameter variation into account one can derive [6]

$$(gw^a - \partial \frac{\delta}{\delta A^a} - \partial \frac{\delta}{\delta \tilde{A}^a})\Gamma = 0,$$ (23)

$$w^a \equiv \int \sum_f f^{bac} \frac{\delta}{\delta \phi^c} + i\bar{\psi}\tau^a \frac{\delta}{\delta \psi} - i\psi \tau^a \frac{\delta}{\delta \bar{\psi}} \phi : A, \tilde{A}, B, c, \tilde{c}, \tilde{c}_\mu (24)$$

Here, remarkably enough, one can again define the coupling $g$ by the requirement “validity of exact local WI”. (“Exact” meaning that it holds in this form without quantum corrections.)

### 5 The electroweak standard model

The symmetry group of the electroweak standard model (SM) is $SU(2) \times U(1)$, hence not semi-simple. It is furthermore spontaneously broken bringing about the difficulty of identifying an unbroken $U(1)$ subgroup which eventually yields the electric charge. The most systematic way of proceeding is to impose and establish first of all the BRS invariance associated with the gauge group, hence to forget about rigid invariance and choosing a gauge fixing and $\phi\Pi$-sector so general, that one can encompass all possible situations [7]. In order to analyze the possible forms of rigid transformations in higher orders one writes down a set of WI operators $W^a$ ($a = +, -, Z, A$) constrained only by formal charge conservation and by commutation with the ST identity. Then one requires an $SU(2) \times U(1)$ algebra for them and solves for the most general representation matrices in all relevant field sectors (vector, scalar, spinor). Under the simplifying assumption of CP-invarince (no family mixing) we have identified all free parameters appearing in $W^a$ with free one’s appearing in the general solution of the ST identity [8]. Like
in the abelian Higgs model it is then clear how normalization conditions lead to deformed rigid WI and the associated deformed algebra.

Acknowledgments

K.S. thanks the organizers for the invitation to this nice conference.

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