Cohomological classification of Ann-functors

Nguyen Tien Quang and Dang Dinh Hanh

April 27, 2009

Abstract

Regular Ann-functor classification problem has been solved with Shukla cohomology. In this paper, we would like to present a solution to the above problem in the general case and in the case of strong Ann-functors with, respectively, Mac Lane cohomology and Hochschild cohomology.

Mathematics Subject Classifications (2000): 18D10, 13D03.

Key words: Ann-category, Ann-functor, classification, Mac Lane ring cohomology, Hochschild cohomology.

1 Introduction

The definition of Ann-categories was presented by N.T. Quang [7] in 1987, which is regarded as a categorization of ring structure. Each Ann-category $\mathcal{A}$ is Ann-equivalent to its reduced Ann-category. This Ann-category is of the type $(R, M, h)$, where $R$ is a ring of congruence classes of objects of $\mathcal{A}$, $M = Aut(0)$ is the $R$-bimodule and $h$ is a 3-cocycle in $Z^3_{Mac}(R, M)$ (due to Mac Lane [6]). Then, there exists a bijection between the set of congruence classes of Ann-categories of the type $(R, M)$ and the cohomology group $H^3_{Mac}(R, M)$ (see [11]).

For the regular Ann-categories (whose the commutativity constraint satisfies $c_{X,X} = id$), then in the above bijection, the group $H^3_{Mac}(R, M)$ is replaced with the Shukla cohomology group $H^3_{Sh}(R, M)$ [14].

In 2006 [4], M. Jibladze and T. Pirashvili presented the definition of categorical rings as a slightly modified version of the definition of Ann-categories and classified them by Mac Lane ring cohomology. However, in [10] authors have showed that, it has not been proved whether the $R$-bimodule structure on $M$ can be deduced from the axiomatics of categorical rings.

The Ann-functor classification problem has been solved for regular Ann-categories with Shukla cohomology [1, 12]. In this paper, we present a solution for this problem in the general case via low-dimensional cohomology groups of Mac Lane ring cohomology. In proper, Hochschild algebra cohomology used to classification of strong Ann-functor.

In this paper, for convenience, sometimes we denote by $XY$ the tensor product of the two objects $X$ and $Y$, instead of $X \otimes Y$. 
2 Preliminaries

2.1 The basic concepts

The definition of Ann-categories was presented in [7, 10, 11]. We always suppose that \( A \) is an Ann-category with a system of constraints:

\[ (a^+, c, (O, g, d), a, (I, l, r), L, R). \]

**Definition 1.** Let \( A \) and \( A' \) be Ann-categories. An Ann-functor from \( A \) to \( A' \) is a triple \((F, \tilde{F}, \tilde{F}')\), where \( (F, \tilde{F}) \) is a symmetric monoidal functor respect to the operation \( \oplus \), \((F, \tilde{F}')\) is an \( A \)-functor (i.e. an associativity functor) respect to the operation \( \circ \), satisfies the two following commutative diagrams:

\[
\begin{align*}
F(X(Y \oplus Z)) & \quad \tilde{F} \quad FX.F(Y \oplus Z) \quad \text{id}_S \tilde{F} \quad FX(FY \oplus FZ) \\
F(XY \oplus XZ) & \quad F \quad F(XY) \oplus F(XZ) \quad \tilde{F} \circ \tilde{F} \quad FX.FY \oplus FX.FZ \\
F((X \oplus Y)Z) & \quad \tilde{F} \quad F(X \oplus Y).FZ \quad \tilde{F} \circ \text{id} \quad (FX \oplus FY).FZ \\
F(XZ \oplus YZ) & \quad \tilde{F} \quad F(XZ) \oplus F(YZ) \quad \tilde{F} \circ \tilde{F} \quad FX.FZ \oplus FY.FZ
\end{align*}
\]

The commutation of the above diagrams are called the compatibility of the functor \( F \) with the distributivity constraints of the two Ann-categories \( A, A' \).

We call \( \varphi: F \rightarrow G \) an Ann-morphism between two Ann-functors \((F, \tilde{F}, \tilde{F}')\) and \((G, \tilde{G}, \tilde{G}')\) if it is an \( \oplus \)-morphism as well as an \( \circ \)-morphism.

An Ann-functor \((F, \tilde{F}, \tilde{F}') : A \rightarrow A'\) is called an Ann-equivalence if there exists an Ann-functor \((F', \tilde{F}', \tilde{F}'') : A' \rightarrow A\) and natural isomorphisms \( \alpha : F \circ F' \cong \text{id}_{A'} \), \( \alpha' : F' \circ F \cong \text{id}_A \).

By Theorem 8 [9], an Ann-functor \((F, \tilde{F}, \tilde{F}') : A \rightarrow A'\) is an Ann-equivalence iff \( F \) is a categorical equivalence.

Note that, similar to a ring homomorphism, an Ann-functor \( F \) is not required \( F(1) \cong 1' \). Moreover, note that: in the Definition 1, it is only required that \((F, \tilde{F})\) is an AC-functor (i.e. an \( \oplus \)-functor which is compatible with the associativity and commutativity constraints). Indeed, since \((A, \oplus), (A', \oplus)\) are Gr-categories, each A-functor is compatible with the unitality constraints.

2.2 The third Mac Lane ring cohomology group \( H^3_{Mac}(R, M) \)

Let \( R \) be a ring and \( M \) be an \( R \)-bimodule. From the definition of Mac Lane ring cohomology [6], we may obtain the description of the elements of cohomology group \( H^3_{Mac}(R, M) \).

The group \( Z^3_{Mac}(R, M) \) of 3-cochains of the ring \( R \), with coefficients in \( R \)-bimodules \( M \), consists of quadruples \((\sigma, \alpha, \lambda, \rho)\), functions:

\[ \alpha, \lambda, \rho : R^3 \rightarrow M \]

and \( \sigma : R^4 \rightarrow M \) satisfy the following relations:

\[ \begin{align*}
\text{M1.} & \quad x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)t = 0 \\
\text{M2.} & \quad \alpha(x, z, t) + \alpha(y, z, t) - \alpha(x + y, z, t) + \rho(xz, yz, t) - \rho(x, y, zt) + \rho(x, y, z)t = 0
\end{align*} \]
\[ -\alpha(x, y, t) - \alpha(x, z, t) + \alpha(x, y + z, t) + x\rho(y, z, t) - \rho(xy, xz, t) \\
-\lambda(x, y, t) + \lambda(x, y, z) + \lambda(x, y, z) = 0 \]

\[ \alpha(x, y, z) + \alpha(x, y, z + t) + x\lambda(y, z, t) - \lambda(xy, yz, yt) = 0 \]

\[ \lambda(x, z, t) + \lambda(y, z, t) - \lambda(x + y, z, t) + \rho(x, y, z) + \rho(x, y, t) - \rho(xy, xz, t) + \sigma(xz, xt, yz, yt) = 0 \]

\[ \lambda(x, a, b) + \lambda(x, c, d) - \lambda(x, a + c, b + d) - \lambda(x, a, c) - \lambda(x, b, d) \]
\[ + \lambda(x, a + b, c + d) - x\sigma(a, b, c, d) + \sigma(xa, xb, xc, xd) = 0 \]

\[ \rho(a, b, x) + \rho(c, d, x) - \rho(a + c, b + d, x) - \rho(a, c, x) - \rho(b, d, x) \]
\[ + \rho(a + b, c + d, x) - \sigma(ax, bx, cx, dx) + \sigma(a, b, c, d)x = 0 \]

\[ \sigma(a, b, c, d) + \sigma(x, y, z, t) - \sigma(a + x, b + y, c + z, d + t) + \sigma(a, b, x, y) + \sigma(c, d, z, t) \]
\[ - \sigma(a + c, b + d, x + z, y + t) + \sigma(a, c, x, z) + \sigma(b, d, y, t) - \sigma(a + b, c + d, x + y, z + t) = 0 \]

\[ \alpha(0, y, z) = \alpha(x, 0, z) = \alpha(x, y, 0) = 0 \]

\[ \sigma(0, 0, z, t) = \sigma(x, y, 0, t) = \sigma(x, 0, z, 0) = \sigma(x, 0, 0, t) = 0. \]

The subgroup \( B^3_{MaL}(R, M) \subset Z^3_{MaL}(R, M) \) of 3-coboundaries consists of the quadruples \((\sigma, \alpha, \lambda, \rho)\) such that there exist the maps \( \mu, \nu : R^2 \to M \) satisfying:

\[ \sigma(x, y, z, t) = -\mu(x, y) - \mu(z, t) = \mu(x + y, z + t) + \mu(x, z) + \mu(y, t) \]

\[ \alpha(x, y, z) = xv(y, z) - \nu(xy, z) + \nu(xy, z) - \nu(xy, z) \]

\[ \lambda(x, y, z) = \nu(x, y) + \nu(x, z) - \nu(x + y, z) + \nu(x + y, z) - \nu(x, xy, z) \]

\[ \rho(x, y, z) = \nu(x, y) + \nu(x + y, z) + \nu(xy, xz, t) - \nu(xy, xz, t). \]

Finally, \( H^3_{MaL}(R, M) = Z^3_{MaL}(R, M)/B^3_{MaL}(R, M) \).

Each Ann-category \( \mathcal{A} \) of the type \((R, M)\) having the structure \( f \) is a family \( f = (\xi, \eta, \alpha, \lambda, \rho) \), where \( \xi, \alpha, \lambda, \rho : R^3 \to M \) and \( \eta : R^2 \to M \) are functions satisfying 17 the equations (see Proposition 5.8 [11]). Now, we define a function \( \sigma : R^4 \to M \), given by:

\[ \sigma(x, y, z, t) = \xi(x + y, z, t) - \xi(x + y, z + t) + \eta(y, z) + \xi(x, y, z) - \xi(x + y, z) \]

This function is respect to the associativity-commutativity constraint \( v \) in the Ann-category \( \mathcal{A} \), where

\[ v = v_{X,Y,Z,T} : (X \oplus Y) \oplus (Z \oplus T) \longrightarrow (X \oplus Z) \oplus (Y \oplus T) \]

is given by commutative diagram following:

\[ \begin{array}{ccc}
(X \oplus Y) \oplus (Z \oplus T) & \xrightarrow{\alpha_{+}} & ((X \oplus Y) \oplus Z) \oplus T \\
\downarrow v & & \downarrow \alpha_{+} \otimes T \\
(X \oplus Z) \oplus (Y \oplus T) & \xrightarrow{\alpha_{+}} & ((X \oplus Z) \oplus Y) \oplus T \\
\end{array} \]

The quadruple \( h = (\sigma, \alpha, \lambda, \rho) \) is a 3-cocycle of the ring \( R \) with coefficients in \( R \)-bimodule \( M \) due to Mac Lane (Theorem 7.2 [11]) and therefore each reduced Ann-category is of the form \((R, M, h)\).

3 An equivalence criterion of an Ann-functor

First, we show a characterized property of Ann-functors, which is related to the associativity-commutativity constraint \( v \).

3

3
**Definition 2.** Let $\mathcal{A}$, $\mathcal{A}'$ be symmetric monoidal $\oplus-$categories. Then, the $\oplus-$functor $(F, \tilde{F}) : \mathcal{A} \to \mathcal{A}'$ is called compatible with the constraints $v, v'$ if the following diagram commutes for all $X, Y, Z, T \in \mathcal{A}$

\[
\begin{array}{c}
F((X \oplus Y) \oplus (Z \oplus T)) \xrightarrow{F} F(X \oplus Y) \oplus F(Z \oplus T) \xrightarrow{F \circ F} (FX \oplus FY) \oplus (FZ \oplus FT) \\
F((X \oplus Z) \oplus (Y \oplus T)) \xrightarrow{F} F(X \oplus Z) \oplus F(Y \oplus T) \xrightarrow{F \circ F} (FX \oplus FZ) \oplus (FY \oplus FT)
\end{array}
\]

(1)

Then **Lemma 3.1.** Let $\oplus-$functor $(F, \tilde{F}) : \mathcal{A} \to \mathcal{A}'$ be compatible with the unitivity constraints. Then $(F, \tilde{F})$ is an $AC-$functor iff it is compatible with the constraints $v, v'$.

**Proof.** The necessary condition was presented by D. B. A. Epstein (Lemma 1.5 [2]).

Now, assume that the diagram (1) commutes. To prove that the pair $(F, \tilde{F})$ is compatible with the commutativity constraints, we consider the following Diagram 1.

In the Diagram 1, the region (I) commutes thanks to the naturality of the morphism $v$, the regions (II) and (IV) commute since $(F, \tilde{F})$ is compatible with the unicity constraints, the regions (III) and (VII) commute thanks to the coherence theorem in a symmetric monoidal category, the regions (VI) and (VIII) commute thanks to the naturality of $\tilde{F}$, the outside region commutes by the diagram (1). Hence, the region (V) commutes. So $(F, \tilde{F})$ is compatible with the commutativity constraints.

Next, we consider the following Diagram 2.

In the Diagram 2, the region (I) commutes thanks to the naturality of the morphism $v$; the first component of the region (II) commutes since $(F, \tilde{F})$ is compatible with unicity constraints, the second one commutes thanks to the composition of morphisms, so the region (II) commutes; the regions (III) and (X) commute thanks to the coherence in a symmetric monoidal category; the first component of the region (IV) commutes thanks to the composition of morphisms, the second one commutes since $(F, \tilde{F})$ is compatible with unicity constraints, so the region (IV) commutes; the region (V) and (VII) commute...
thanks to the composition of morphisms; the regions (VIII) and (IX) commute thanks to
the naturality of $\tilde{F}$; the outside region commutes thanks to the diagram (1). Therefore, the
region (V) commutes, i.e., the pair $(F, \tilde{F})$ is compatible with associativity constraints.

\begin{center}
Diagram 2
\end{center}

\[ F((X \oplus O) \oplus (Y \oplus Z)) \]
\[ F((X \oplus Y) \oplus (O \oplus Z)) \]

\[ F(g \oplus id) \]
\[ F(id \oplus d) \]

\[ F(g') \]
\[ F(id \oplus \tilde{F}) \]

\[ \tilde{F} \]
\[ \tilde{F} \]

\[ (X) \]
\[ (I) \]

\[ a' \]
\[ \gamma^{-1} \]

\[ (VII) \]
\[ (III) \]

\[ \tilde{F} \oplus F(d^{-1}) \]

\[ (X) \]
\[ (I) \]

\[ \tilde{F} \oplus \tilde{F} \]

\[ (F \oplus d') \]
\[ \gamma^{-1} \]

\[ \tilde{F} \oplus \tilde{F} \]

\[ (F \oplus (O \oplus FZ)) \]
\[ (F \oplus (Y \oplus FZ)) \]

\[ (F \oplus (Y \oplus O)) \]
\[ (F \oplus (O \oplus FZ)) \]

\[ (F \oplus (Y \oplus O)) \]
\[ (F \oplus (O \oplus FZ)) \]

\[ (F \oplus (Y \oplus Z)) \]
\[ (F \oplus (O \oplus Z)) \]

\[ (F \oplus (Y \oplus O)) \]
\[ (F \oplus (O \oplus FZ)) \]

\[ (F \oplus (Y \oplus O)) \]
\[ (F \oplus (O \oplus FZ)) \]

\[ F_0 : \Pi_0(A) \rightarrow \Pi_0(A') ; \quad F_1 : \Pi_1(A) \rightarrow \Pi_1(A') \]

\[ clsX \rightarrow clsFX \]
\[ u \mapsto \gamma^{-1}_{F_0}(Fu) \]

satisfying

\[ F_1(su) = F_0(s)F_1(u); \quad F_1(us) = F_1(u)F_0(s) \]

\[ \square \]

**Proposition 3.2.**

In the definition of Ann-functors, the condition that $(F, \tilde{F})$ is an symmetric monoidal
$\oplus$-functor is equivalent to the two following conditions:

1. $(F, \tilde{F})$ is compatible with the unitality constraints respect to the operation $\oplus$,
2. $(F, \tilde{F})$ is compatible with the constraints $v, v'$.

*Proof.* Directly deduced from Lemma 3.1. \[ \square \]

4 Ann-functors and the low-dimensioned cohomology
groups of rings due to Mac Lane

4.1 Ann-functors of the type $(p, q)$

Now, we will show that each Ann-functor $(F, \tilde{F}, \tilde{F}) : A \rightarrow A'$ induces a Ann-functor $\overline{F}$
on their reduced Ann-categories, and this correspondence is 1-1. First, we have

**Theorem 4.1.** (Theorem 4.6 [11]) Let $A$ and $A'$ be Ann-categories. Then, each Ann-
functor $(F, \tilde{F}, \tilde{F}) : A \rightarrow A'$ induces the pair of ring homomorphisms:

\[ F_0 : \Pi_0(A) \rightarrow \Pi_0(A') ; \quad F_1 : \Pi_1(A) \rightarrow \Pi_1(A') \]

\[ clsX \rightarrow clsFX \]
\[ u \mapsto \gamma^{-1}_{F_0}(Fu) \]

satisfying

\[ F_1(su) = F_0(s)F_1(u); \quad F_1(us) = F_1(u)F_0(s) \]
where \( \Pi_1(A) \) is regarded as a ring with the null multiplication. Furthermore, \( F \) is an equivalence if \( F_0, F_1 \) are isomorphisms.

The pair \((F_0, F_1)\) is called the pair of induced homomorphisms of the Ann–functor \((F, \tilde{F}, \bar{F})\). If \( S, S' \) are, respectively, the reduced Ann–categories of \( A, A' \) then the functor \( \mathcal{F} : S \to S' \) given by

\[
\mathcal{F}(s) = F_0(s), \quad \mathcal{F}(s, u) = (F_0 s, F_1 u)
\]

is called the reduced functor of \((F, \tilde{F}, \bar{F})\) on reduced Ann–categories.

**Proposition 4.2.** Let \( \mathcal{F} \) be the induced functor of the Ann–functor \((F, \tilde{F}, \bar{F}) : A \to A'\). Then the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
H \downarrow & & \downarrow G' \\
S & \xrightarrow{\mathcal{F}} & S'
\end{array}
\]

commutes, where \( H, G' \) are canonical Ann-equivalences, and therefore \( \mathcal{F} \) induces an Ann-functor.

**Proof.** This Proposition is naturally extended of Proposition 2 [8]. \( \square \)

**Definition 3.** Let \( S = (R, M, h), S' = (R', M', h') \) be Ann–categories. A functor \( F : S \to S' \) is called a functor of the type \((p, q)\) if

\[
F(x) = p(x), \quad F(x, a) = (p(x), q(a)),
\]

where \( p : R \to R' \) is a ring homomorphism and \( q : M \to M' \) is a group homomorphism satisfying

\[
q(ax) = p(x)q(a), q(ax) = q(a)p(x),
\]

for \( x \in R, a \in M \).

**Proposition 4.3.** Let \( A = (R, M, h), A' = (R', M', h') \) be Ann-categories and \((F, \tilde{F}, \bar{F})\) is an Ann-functor from \( A \) to \( A' \). Then, \((F, \tilde{F}, \bar{F})\) is a functor of the type \((p, q)\).

**Proof.** For \( x, y \in R \), we have

\[
\tilde{F}_{x,y} : F(x) \oplus F(y) \to F(x \oplus y), \quad \bar{F}_{x,y} : F(x) \otimes F(y) \to F(x \otimes y)
\]

are morphisms in the Ann-category \( A' \). Hence, \( F(x) + F(y) = F(x + y) \) and \( F(x).F(y) = F(xy) \), so the map \( p : R \to R' \) given by \( p(x) = F(x) \) is a ring homomorphism.

Assume that \( F(x, a) = (p(x), q_a(a)) \). Since \((F, \tilde{F})\) is a Gr-functor, according to Theorem 5 [8], \( q_a = q \) for all \( x \in R \). Moreover \( q \) is a group homomorphism:

\[
q(a + b) = q(a) + q(b)
\]

for all \( a, b \in M \).

Since \((F, \tilde{F})\) is a \( \otimes \)-functor, the following diagram

\[
\begin{array}{ccc}
F x \otimes F y & \xrightarrow{\tilde{F}} & F(x \otimes y) \\
F((x,a)) \otimes F((y,b)) \downarrow & & \downarrow F((x,a) \otimes (y,b)) \\
F x \otimes F y & \xrightarrow{\tilde{F}} & F(x \otimes y)
\end{array}
\]
commutes, for all morphisms \((x, a), (y, b)\). So, we have:

\[
F((x, a) \otimes (y, b)) = F(x, a) \otimes F(y, b)
\]

\[
\Leftrightarrow q_{xy}(ay + xb) = q_x(a)F(y) + F(x)q_y(b)
\]

Applying \(q_x = q_y = q_{xy} = q\) to the relation (3), we have:

\[
q(ay + xb) = q(a)F(y) + F(x)q(b)
\]

Applying \(x = 1\) to (4), we have:

\[
q(ay) = q(a)F(y) = q(a)p(y)
\]

Applying \(y = 1\) to (4), we have:

\[
q(xb) = F(x)q(b) = p(x)q(b)
\]

If \(R'\)-bimodule \(M'\) is regarded as an \(R\)-bimodule thanks to the actions \(xa' = p(x)a', a'x = a'p(x)\), from the equations (2), (5), (6) we may show that \(q : M \rightarrow M'\) is a homomorphism between \(R\)-bimodules. \(\square\)

4.2 Classification of Ann-functors

The existence problem of Ann-functors between Ann-categories has been solved for the regular Ann-categories (Theorem 5.1 [13], Theorem 4.2 [1]) thanks to Shukla cohomology. In this section, we will solve that problem in the general case.

**Definition 4.** If \(F : (R, M, h) \rightarrow (R', M', h')\) is a functor of the type \((p, q)\), then \(F\) induces 3-cocycles \(h = q, h = q(h), h'^* = p^*h' = h'p\), for example

\[
\sigma^* = (p(x), p(y), p(z), p(t))
\]

\[
\sigma = (x, y, z, t).
\]

The function \(k = p^*h' - q, h\) is called an obstruction of the functor of the type \((p, q)\). Then we have

**Theorem 4.4.** The functor \(F : (R, M, h) \rightarrow (R', M', h')\) of the type \((p, q)\) is an Ann–functor iff the obstruction \(k = 0\) in \(H^3_{Ann}(R, M')\).

**Proof.** Let \((F, \tilde{F}, \bar{F}) : (R, M, h) \rightarrow (R', M', h')\) be an Ann–functor of the type \((p, q)\). Since \(\tilde{F}_{x,y} = (\bullet, \mu(x, y)), \bar{F}_{x,y} = (\bullet, \nu(x, y))\) where \(\mu, \nu : R^2 \rightarrow M'\), we may identify \(F, \tilde{F}, \bar{F}\) with \(\mu, \nu\) and call \(\mu, \nu\) the pair of associated functions with \(F, \tilde{F}, \bar{F}\). According to Lemma 3.1, \((F, \tilde{F}, \bar{F})\) is compatible with the pair of constraints \((\nu, \nu')\), i.e. the diagram (1) commutes, so we have:

7. \(\sigma^*(x, y, z, t) - \sigma(x, y, z, t) = \mu(x, y) + \mu(z, t) - \mu(x + z, y + t) - \mu(x, z)
\]

\[
- \mu(y, t) + \mu(x + y, z + t)
\]

Since \(F\) is compatible with the associativity constraint of multiplication, the distributivity constraints of Ann-categories \(A\) and \(A'\), we have:

8. \(\alpha^*(x, y, z) - \alpha(x, y, z) = z\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z\)

9. \(\lambda^*(x, y, z) - \lambda(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\mu(y, z) - \mu(xy, xz)\)

10. \(\rho^*(x, y, z) - \rho(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \mu(x, y)z - \mu(xz, yz)\)
From the equations (7)-(10), we have:

\[ h'^* - h_* = \delta g \quad (11) \]

where \( g = (-\mu, \nu) \). Hence the obstruction of the functor \( F \) vanishes in the cohomology group \( H^3_{\text{MaL}}(R, M) \).

Conversely, assume that the obstruction of the functor \( F \) vanishes in the cohomology group \( H^3_{\text{MaL}}(R, M') \). Then there exists a 2-cochain \( g = (\mu, \nu) \) such that \( h'^* - h_* = \delta g \). Take \( \bar{F}, \tilde{F} \) be functor morphisms associated with the functions \(-\mu, \nu\), we can verify that \((F, \bar{F}, \tilde{F})\) is an Ann-functor. \( \square \)

**Theorem 4.5.** If there exists an Ann-functor \((F, \bar{F}, \tilde{F}) : A \to A'\), of the type \((p, q)\) then:

1. There exists a bijection between the set of congruence classes of Ann-functors of the type \((p, q)\) and the cohomology group \( H^2_{\text{MaL}}(R, M') \) of the ring \( R \) with coefficients in \( R \)-bimodule \( M' \).

2. There exists a bijection \( \text{Aut}(F) \to Z^1_{\text{MaL}}(R, M') \) between the group of automorphisms of the Ann-functor \( F \) and the group \( Z^1_{\text{MaL}}(R, M') \).

**Proof.** (a) Suppose that there exists \((F, \bar{F}, \tilde{F}) : A \to A'\), which is an Ann-functor of the type \((p, q)\). According to Theorem 4.4, we have

\[ h'^* - h_* = 0. \]

Hence, there exists a 2-cochain \( k \) such that

\[ h'^* - h_* = \delta k. \]

Fix 2-cochain \( k \). Now, we assume that

\[ (G, \bar{G}, \tilde{G}) : (R, M, h) \to (R', M', h') \]

is an Ann-functor of the type \((p, q)\). Then, from the proof of the Theorem 4.4, we have

\[ h'^* - h_* = \delta g. \]

Hence, \( k - g \) is a 2-cocycle. Consider the correspondence:

\[ \Phi : \text{class}(G) \mapsto \text{class}(k - g) \]

from the set of the congruence classes of Ann-functors of the type \((p, q)\) to the group \( H^2_{\text{MaL}}(R, M') \).

First, we prove that the above correspondence is a map. Indeed, suppose that

\[ (G', \bar{G}', \tilde{G}') : (R, M, h) \to (R', M', h') \]

is also an Ann-functor of the type \((p, q)\) and \( u : G \to G' \) is an Ann-morphism. Since \( u \) is an \( \oplus \)-morphism as well as an \( \otimes \)-morphism, we have:

\[ g' = g - \delta(u) \quad (12) \]

So

\[ k - g' = k - g + \delta(u). \]
Thus $k - g = k - g' \in H^2_{MaL}(R, M')$.

Now, we prove that $\Phi$ is an injection. Assume that

$$(G, \tilde{G}, \tilde{G}'), (G', \tilde{G}', \tilde{G}') : (R, M, h) \to (R', M', h')$$

are Ann-functors of the type $(p, q)$ and satisfying

$$k - g = k - g' \in H^2_{MaL}(R, M').$$

Then, there exists an 1-cochain $u$ such that

$$k - g = k - g' - \delta(u)$$

That means

$$g' = g - \delta(u).$$

Hence, the following diagrams:

\[
\begin{array}{ccc}
G(x) \oplus G(y) & \xrightarrow{\bar{G}} & G(x \oplus y) \\
\downarrow u_x \oplus u_y & & \downarrow u_x \oplus u_y \\
G'(x) \oplus G'(y) & \xrightarrow{\bar{G}'} & G'(x \oplus y)
\end{array}
\]

\[
\begin{array}{ccc}
G(x) \otimes G(y) & \xrightarrow{\tilde{G}} & G(x \otimes y) \\
\downarrow u_x \otimes u_y & & \downarrow u_x \otimes u_y \\
G'(x) \otimes G'(y) & \xrightarrow{\tilde{G}'} & G'(x \otimes y)
\end{array}
\]

commute, it means that $u : G \to G'$ is an Ann-morphism. Therefore,

$$\text{class}(G) = \text{class}(G').$$

Finally, we must prove that the correspondence $\Phi$ is a surjection. Indeed, assume that $g$ is an arbitrary 2-cocycle. We have:

$$\delta(k - g) = \delta k - \delta g = \delta k = h'^* - h^*.$$

Then, according to Theorem 4.4, there exists an Ann-functor

$$(G, \tilde{G}, \tilde{G}) : (R, M, h) \to (R', M', h')$$

of the type $(p, q)$, and the pair of isomorphisms $\tilde{G}, \tilde{G}$ associated with the 2-cochain $k - g$.

Clearly, $\Phi(G) = \mathcal{A}$. So $\Phi$ is a surjection.

(b) Assume that $F = (F, \bar{F}, \bar{F}) : (R, M, h) \to (R', M', h')$ is an Ann-functor of the type $(p, q)$ and $u \in \text{Aut}(F)$. Then, from the equation (12) with $g' = g$, we have $\delta(u) = 0$, i.e.,

$$u \in Z^1_{MaL}(R, M').$$

5 Ann-functors and Hochschild cohomology

In this section, we will consider special Ann-functors which are related to the low-dimensioned Hochschild groups.

Following, we will find a condition for the existence of Ann-functors of the form

$$F = (F, id, \bar{F}) : (R, M, h) \to (R', M', h')$$

of the type $(p, 0)$, where $p : R \to R'$ is a ring homomorphism.

Suppose that there exists an Ann-functor

$$F = (F, id, \bar{F} = \nu) : (R, M, h) \to (R', M', h')$$

of the type $(p, 0)$. Then, the equations (7) - (10) turn into:
13. $\sigma'(x, y, z, t) = 0$
14. $\alpha'(x, y, z) = xy + yz + xz$  
15. $\lambda'(x, y, z) = x^2 + y^2 + z^2$  
16. $\rho'(x, y, z) = x^2 + y^2 + z^2$

and the Theorem 4.4 turns into

**Corollry 5.1.** Let $p : R \rightarrow R'$ be a ring homomorphism. There exists an Ann-functor $(F, id, \tilde{F})$ from $(R, M, h)$ to $(R', M', h')$ of the type $(p, 0)$ iff $h'^* = 0 \in H^3_{Hochs}(R, M')$.

Each cocycle of $Z$-algebras due to Hochschild is a multi-linear function. This suggests us the following definition:

**Definition 5.** An Ann-functor

$$(F, id, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$$

of the type $(p, 0)$ is called a **strong** Ann-functor if the function $\nu : R^2 \rightarrow M'$ corresponding to $\tilde{F}$ is bi-additive.

If $\nu$ is a normal bi-additive function then $\nu$ is a 2-cocycle of the $Z$-algebra $R$ with coefficients in $R$-bimodule $M'$ due to Hochschild. Then, in the equations (13)-(16), $\alpha'^*$ is a normal multi-linear function and other functions are equal to 0. So, we can identify $h'^* = \alpha'^* = \delta(\nu)$.

in which $\delta(\nu)$ is an 3-coboundary of the ring $R$ with coefficients in $R$-bimodule $M'$ due to Hochschild. Then, we have the following proposition, as a direct corollary of Theorem 4.4.

**Proposition 5.2.** Let $F : (R, M, h) \rightarrow (R', M', h')$ be a functor of the type $(p, 0)$. There exists a strong Ann-functor $(F, id, \tilde{F})$ iff its cohomology class $h'^* = 0$ in the cohomology group $H^3_{Hochs}(R, M')$.

**Theorem 5.3.** If there exists an strong Ann-functor $(F, id, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$, of the type $(p, 0)$, then:

1. There exists a bijection between the set of congruence classes of strong Ann-functors of the type $(p, 0)$ and the cohomology group $H^2_{Hochs}(R, M')$ of the ring $R$ with coefficients in $R$-bimodule $M'$.

2. There exists a bijection

$$\text{Aut}(F) \rightarrow Z^1_{Hochs}(R, M')$$

between the group of automorphisms of the Ann-functor $F$ and the group $Z^1_{Hochs}(R, M')$.

**Proof.** (a) The restriction $\Phi_H$ of the map $\Phi$, refered in Theorem 4.5, on the set of congruence classes of strong Ann-functors, gives us an injection to the group $H^2_{Hochs}(R, M')$. Moreover, it is easy to see that $\Phi_H$ is also a surjection.
(b) Assume that \( F = (F, id, \bar{F}) : (R, M, h) \rightarrow (R', M', h') \) is an strong \( Ann \)–functor of the type \((p, 0)\) and \( u \in Aut(F) \). Then \( u \) is bi-linear respect to the \( \oplus \). So \( u \in Z^1_{Hochs}(R, M') \). The converse also holds.

\[ \square \]

References

[1] T. P. Dung, Doctoral dissertation, Hanoi, Vietnamese, 1992.

[2] D. B. A. Epstein, Functors between tensored categories, Invent. math. 1, 221-228 (1966).

[3] G. Hochschild, On the cohomology groups of an associative algebra, Ann. of Math. (2) 46, (1945), 58-67.

[4] M. Jibladze and T. Pirashvili, Third Mac Lane cohomology via categorical rings, arxiv. math. KT/0608519 v1, 21 Aug 2006.

[5] S. Mac Lane, Extensions and Obstruction for rings, Illinois J. Mathematics, 2 (1958), 316-345.

[6] S. Mac Lane, Homologie des anneaux et des modules, Colloque de Topologie algébrique, Louvain (1956), 55-88.

[7] N. T. Quang, Doctoral dissertation, Hanoi, Vietnamese, 1988.

[8] N. T. Quang, On Gr-functors between Gr-categories: Obstruction theory for Gr-functors of the type \((\varphi, f)\), arXiv: 0708.1348 v2 [math.CT] 18 Apr 2009

[9] N. T. Quang and P. L. Hong Anh, On monoidal equivalences and Ann-equivalences, arXiv: 0705.0736 v1 [math. CT] 5 May 2007.

[10] N. T. Quang, D. D. Hanh and N. T. Thuy, On the axiomatics of Ann-categories, JP Journal of Algebra, Number Theory and applications, Vol 11, No 1, 2008, 59 - 72.

[11] N. T. Quang, Structure of Ann-categories, arXiv. 0805. 1505 v3 [math. CT] 6 Apr 2009.

[12] N. T. Quang, Ann-categories and the Mac Lane-Shukla cohomology of rings, Abelian groups and modules No 11,12 (Russian), 166 - 183, Tomsk. Gos. Univ., Tomsk, 1994 .

[13] N. T. Quang, Structure of Ann-categories and Mac Lane-Shukla cohomology, East-West, J. of Math. 5 (2003), 51-66.

[14] U. Shukla, Cohomologie des algèbres associatives, Ann. Sci. cole Norm. Sup. (3) 78 (1961), 163 - 209.

Dept. of Mathematics, Hanoi National University of Education, Viet Nam

Email: nguyenquang272002@gmail.com
ddhanhthanh@gmail.com