Controllability of a Nonlinear Voter Model

Arastoo Azimi\textsuperscript{a}, Mohsen Aghazadeh Shiran\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Mechanical Engineering Department, Amirkabir University of Technology, Tehran, Iran
\textsuperscript{b} Electrical Engineering Department, Iran University of Science and Technology, Tehran, Iran

### Abstract

Converging and emerging behavior of different phenomena call for a strong tool to describe the behavior of multi-agent systems. Economics and quantitative sociology are two of the most important areas where these behavioral patterns are observed. In this paper, a brief introduction to statistical physics is provided as a tool to understand and investigate the upper-mentioned phenomena. Some of the available interaction models used in quantitative sociology have been presented to get an idea of how they work. Using one of these models, a discrete nonlinear dynamic system is presented. Main focus of this paper is providing necessary tools and solutions for discrete nonlinear dynamical systems.

### 1. Introduction

In today’s world, the fact that there is no specific boundary between different realms of science is obvious for everyone. At the moment, scientists and researchers try not only to master their area of specialty but also to grasp some details from other realms and if necessary, use them in their own favor. Vast variety of interdisciplinary research topics such as Thermo-plasticity, Neuro-computation, Quantitative sociology, Thermoelectricity and etc. prove the fact that all the basic sciences are inseparable in higher levels of research. One of the broadly used concepts (or area of studies), which is of great importance to every experimental and computational branch of science, is Statistical Physics or Statistical Mechanics. Statistical physics \[1\] started from the point where physicists were able to measure and formulate the behavior of gases but could not explain that behavior. They needed a much stronger tool which could start observing and defining the behavior from molecule levels of material and develop to the entire volume. This strong tool was using statistics and its concepts in determining the upper mentioned criteria.

It is worth reminding that statistical physics considers each particle of the matter, and uses simple conservation laws \[2\] for them (by some simple assumptions). Next, the scientist tries to understand and formulate the behavior \[3, 4\] of the whole complex based on these assumptions and formulations. This point of view with which the properties of the whole structure or complex are obtained by considering particles (as individual) is the most powerful aspect of statistical physics which is used in chemistry (crystallization behavior of some materials \[5\]), material sciences (time dependent growth of lattices \[6\]), and even in social sciences (social dynamics).

In social phenomena, the basic constituents are not particles but humans, and every individual interacts with a limited number of peers, usually negligible compared to the total number of people in the system \[7\]. There are transitions from disorder to order \[8\], like the spontaneous formation of a common language and culture or the emergence of consensus about a specific issue. There are examples of scaling and universality. These macroscopic phenomena naturally call for a statistical physics approach to social behavior, i.e., the attempt to understand regularities at large scale as collective effects of the interaction among single individuals, considered as relatively simple entities.

\* Corresponding author: m.aghazadeh.sh@gmail.com

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No one knows precisely the dynamics of a single individual, nor the way he interacts with others. But in this respect, statistical physics brings an important added value. In most situations, qualitative and even some quantitative properties of large-scale phenomena do not depend on the microscopic details of the process. Only higher level features, such as symmetries, dimensionality, or conservation laws are relevant for the global behavior [9]. With this concept of universality in mind, one can approach the modelling of social systems, trying to include only the simplest and most important properties of single individuals and looking for qualitative features exhibited by models in global scales.

In this paper, the usage of statistical physics in social sciences is explained in length. First, the concept of order and disorder will be illustrated in short. Then, the geometry types with which people are connected with each other are explained. This is followed by interaction models used in social sciences. Next, a mathematical model will be designed. In this paper focus is on the models that have received more attention in the physics literature, pointing out analogies as well as differences between them.

2. Order and Disorder

The common theme of social dynamics is the understanding of the transition from an initial disordered state to a configuration that displays order (at least partially) [10]. This is done using a paradigmatic example of order-disorder transitions in physics, the one exhibited by the Ising model [11] for Ferro magnets. Beyond its relevance as a physics model, the Ising Ferro magnet can be seen as a simple model for opinion dynamics, with agents influenced by the state of the majority of their interacting partners.

Considering a collection of \( N \) spins \( s_i \) that can have two values \( \pm 1 \). Each spin is energetically pushed to be aligned with its nearest neighbors. The total energy is:

\[
H = -\frac{1}{2} \sum_{\langle i,j \rangle} s_i s_j
\]

in which the sum runs on the pairs of nearest-neighbors spins. Among the possible types of dynamics, the most common (Metropolis) takes as an elementary move a single spin flip that is accepted with probability \( \exp(-\Delta E / kBT) \), where \( \Delta E \) the change in energy and \( T \) is the Temperature [12]. Ferromagnetic interactions in Eq. (1) drive the system toward one of the two possible ordered states, with all positive or all negative spins.

3. Topology

An important aspect always present in social dynamics is topology, i.e., the structure of the interaction network describing who is interacting with whom, how frequently, and with what intensity [13], [14]. Agents are thus supposed to sit on vertices (nodes) of a network, and the edges (links) define the possible interaction patterns. Type of the connecting model is one of the key factors of defining a society, which is another area of research. If one node is isolated, this node will not be able to affect the society no matter how strong it might be. Choosing wrong topology means wrong modelling of the phenomena and wrong answers.

Furthermore, topology of the system is the place where the society is chosen to be deterministic or stochastic by adding concepts of probability to the topology. Some agents might be totally disconnected from the others because of the low intensity of their connections (low values of probability). So, different societies in different problems might need different models based on the concepts of that problem to gain the highest accuracy possible. For instance, imagine a society whose number of nodes changes as time goes by. For this society, adopting a static network model will be absolutely wrong since it will not be able to present the most important characteristic of the society, the dynamic size.

4. Opinion Dynamics

The dynamics of agreement or disagreement among individuals is complex because the individuals are [15]. Statistical physicists working on opinion dynamics aim at defining the opinion states of a population and the elementary processes that determine transitions between such states, for which some scientists have used language measure theories to model each individual decision making processes [16], [17]. In any mathematical model, opinion has to be a variable, or a set of variables, i.e., a collection of numbers. If opinions can be represented by numbers, the challenge is to find an adequate set of mathematical rules to describe the mechanisms responsible for their evolution and changes.

In the past decade, physicists have started to work actively in opinion dynamics, and many models have been designed. In this paper focus is on the models that have received more attention in the physics literature, pointing out analogies as well as differences between them.
4.1. Voter Model

The voter model has been named in this way for the very natural interpretation of its rules in terms of opinion dynamics; for its extremely simple definition, however, the model has also been thoroughly investigated in fields quite far from social dynamics, such as probability theory and population genetics. Voter dynamics was first considered in 1973 as a model for the competition of species. The definition is extremely simple: each agent is endowed with a binary variable \( s = \pm 1 \). At each time step, an agent \( i \) is selected along with one of its neighbors \( j \) and \( s_i = s_j \), i.e., the agent takes the opinion of the neighbor. This update rule implies that agents imitate their neighbors. Starting from a disordered initial condition, voter dynamics tends to increase the order of the system.

There are other models developed for different problems and purposes such as the Sznajd Model and Social Impact Theory which are not the scope of this paper. So far, every concept related to social dynamics has been covered. In the following section, a nonlinear model for the voter model will be presented.

5. Development of the Non-Linear Voter Model

Competition dynamic between two species (species 0 and 1) is considered. We assume that a site in the model receives information primarily from itself and its four nearest neighbors (Figure 1). It is assumed that each location or site contains a single individual of either species 0 or species 1; No empty cells are allowed. The dynamics is simple: the identity (0 or 1) of the individual at a particular location depends upon the identity of the individual at that location in the previous time step and the identity of its four neighbors (the von Neumann neighborhood in CA).

A mean-field approximation of the system is adopted by considering an infinite lattice on which sites are mixed randomly after each time step. On such a lattice, the proportion of neighborhoods with a specified number of individuals of species 1 is given by the binomial expansion; and since the lattice is infinite, we can write a deterministic system describing \( x_i \), the proportion of sites occupied by species 1 as follows:

\[
x_{t+1} = 5p_1x_t(1 - x_t)^4 + 10p_2x_t^2(1 - x_t)^3 + 10(1 - p_2)x_t^2(1 - x_t)^2 + 5(1 - p_1)x_t^4(1 - x_t) + x_t^5
\]

Which is a discrete dynamic system whose equilibrium points can be found by satisfying the equilibrium condition, \( x_{t+1} = x_t \).

Figure 1. Neighborhood used in the model

Now, the question is whether or not there are approaches with which these sorts of systems can be studied and investigated from Control Engineering point of view. For continuous systems, Lyapunov Functions bring a fairly easy solution to related problems such as stability. In the following section, control of discrete systems will be explained. To do this, wherever needed, mathematical backgrounds have been provided.

6. Nonlinear Discrete Control Systems

In Control Engineering, controllability is one of the very first concepts to consider. In general, controllability is investigating the fact that the system is capable of starting from any initial condition, and get to any final state using a set of controlling inputs. So, in this study, controllability conditions are discussed and presented.

To begin with, consider a general nonlinear discrete 1-D system with time-invariant coefficients as below:

\[
x(i + 1) = f(x(i), u(i))
\]
where \( i \in Z^+ \), and \( Z^+ \) is a set of non-negative integers, \( x(i) \in R^n \) is a state vector at the point \( i \), \( u(i) \in R^m \) is a state control vector at the point \( i \), and \( f: R^n \times R^m \rightarrow R^n \) is a given function [18], [19].

Let \( U \subset R^m \) be a given arbitrary set. The sequence of controls:

\[
\mathcal{U}_p = \{ u(i); 0 \leq i \leq p, u(i) \in U \}
\]

is called an admissible sequence of controls. The set of all such admissible sequences of controls forms the so-called admissible set of controls \( \mathcal{U}_p \). In the sequel, we shall also use the following notation: \( \Omega^p \subset R^m \) is a neighborhood of zero, \( U^p \subset R^m \) is a closed convex cone with vertex at zero and \( U^{co} = U^p \cap \Omega^p \).

The initial condition for the nonlinear vector difference Eq. 3 is given by

\[
x(0) = x_0 \in R^n
\]

where \( x_0 \) is a known vector. Now, the controllability question becomes: For a given initial condition (5) and for an arbitrary admissible sequence of controls, does there exist a unique solution of the nonlinear difference equation (3), which may be computed by successive iterations? To answer this question, first, an associated linear discrete system will be considered with time invariant coefficients.

\[
x(i + 1) = Ax(i) + Bu(i)
\]

Defined for \( i \geq 0 \), where \( A \) and \( B \) are \( n \times n \) and \( n \times m \) dimensional constant matrices, respectively. For the linear discrete system (Eq. 6) we can define the so called transition matrix \( A^i \). Using \( A^i \), we can express the solution \( x(i) \) of Eq. 6 for \( i > 0 \) in the following compact form:

\[
x(i) = A^i x_0 + \sum_{j=0}^{i-1} A^{i-j-1} Bu(j)
\]

For linear discrete system (Eq. 6), controllability matrix is defined as follows:

\[
W_p = [A^{p-1}B, A^{p-2}B, \ldots, AB, B]
\]

**Definition 1.** The system (3) is said to be globally U-controllable in a given interval \([0, p]\) if for the controllability matrix \( W_p \) is of rank \( n \), which is similarly the same controllability condition for continuous time-invariant linear systems. However, this condition does not suffice for the system in hand regarding its limitation to linear systems. For nonlinear systems, using some approximation methods taking from nonlinear function analysis, it is possible to obtain a linearized system and practice the linear approach studied so far.

In short, if the function \( f(x, u) \) is continuously differentiable with respect to all its arguments in some neighborhood of zero in the product space \( R^n \times R^m \), by defining \( A = f'_x(0,0) \) and \( B = f'_u(0,0) \), the nonlinear system can be formulated as a linear system around \((0,0)\). So, all the dynamical characteristics related to the nonlinear system could be investigated around \((0,0)\).

7. Simulations

In this section, the aim is to investigate whether or not opinion evolution in a society can be controlled and guided to a desired state. To do so, a specific society with a variety of opinions inside is chosen. In quantitative sociology, biased people (nodes who never change their opinions) can be considered as control inputs.

Climate change is one of the important subjects in today’s world which needs serious attention because not only does it threaten the future of the planet earth, but also it is hard to prevent it. As a result, knowing about people’s opinion at the present, and how it would change as time goes by can be a big help [20]. Hence, it is chosen as a practical application for our study. It deals with people’s perceptions of climate change and provides a wide range of data regarding people’s mind state about different matters, Figure 2. The study results are as follows:

In which, “Perceived Instrumentality (PI)” is concerned with the fact that whether or not participant think they can be effective in preventing climate change by changing their own behavior regarding energy use. However, the “Concern (C)” factor determines how concerned participants are about the climate change which is sometimes referred to as “global warming”. Furthermore, “Preparedness to Reduce Energy Use (PRE)” considers the extent to which participants are ready to greatly reduce their energy use to help tackle climate change. In addition,
“Uncertainty (U)” measures the extent to which participants are uncertain that climate change is really happening. Finally, “Perceived Local Vulnerability (PLV)” asks the participants about the extent to which they think their local area is likely to be affected by climate change. The data collected for these factors are presented in Table 1.

With all the explanations and data presented, everything is ready for Monte Carlo simulations. We will start by constructing a society in which every member has an opinion state vector such as $O_i = [PI C PRE U PLV]^T$ (with $i$ denoting the $i$th node). In order to initialize the opinion state of each node, the percentages available in Table 1 are used. It is worth noting that each opinion state is going to be presented with a number as 5 being the “Strongly Agree” state and 1 being the “Strongly Disagree” state. For instance, in the case of PI the opinion state vectors of 17% of the society will be given number 5, 46% will be given number 4, 12% will be given number 3, 17% will be given number 2, and finally, 8% will be given number 1 just to make the process of presentation and simulation easier. To finish constructing the society, the same procedure has to be adopted to initialize the other rows of the agents’ opinion state vectors.

For the simulation process, an M-file is written in MATLAB software program which constructs the society, does the interactions based on the Voter model, and recalculates the percentages of opinion states for each involving factor. In this study, first, we will run the code for the uninfluenced society, and then for the influenced society. Each type will be run 1000 times to reduce, or in some case eliminate the effects of randomness of the process. Additionally, because a society as large as 100 agents is studied, in each run, 10000 interactions are considered, which is a safe number of interactions to let the society reach its equilibrium phase.

### Table 1

| Factors                        | Opinion States and Percentage of People in Different Opinion States |
|--------------------------------|--------------------------------------------------------------------|
|                                | Strongly Agree | Tend to Agree | Neither Agree nor Disagree | Tend to Disagree | Strongly Disagree |
| Perceived Instrumentality      | 17             | 46            | 12                         | 17              | 8                |
| Preparedness to Reduce Energy  | 15             | 50            | 16                         | 2               | 1                |
| Uncertainty                    | 6              | 22            | 12                         | 35              | 25               |
| Perceived Local Vulnerability  | 13             | 40            | 16                         | 21              | 10               |
| Concern                        | Very Concerned | Fairly Concerned | Not Very Concerned | Not at All Concerned |
|                                | 28             | 43            | 19                         | 10              |
8. Results

In this part, the results of the computer simulations are presented. It is necessary to know that only results of the PI are presented since there is no space for all of them. Figure 3 represents the initial state of the society (blue bars), the results of the uncontrolled society (green bars), and the results related to the controlled society (red bars). These results easily answer the previously asked question regarding controllability of the whole society’s opinion. In (a) the control input used is 9-1-1-1-1, and in (b) the control input is 1-1-1-1-9.

The difference between red bars and green bar in both graphs not only indicates controllability of the dynamical system, but also shows the proportionality of the final output with the input units.

![Figure 3. Results of the simulation for PI](image)

9. Conclusions

In this paper, concept of order and disorder, quantitative sociology, and related areas such as topology and interaction models are presented followed by a nonlinear voter dynamic model in a lattice which turned out to be a nonlinear discrete system. To tackle the problem of nonlinearity, a specific approach is presented which includes linearization of the system around an equilibrium point. Then, Monte Carlo simulations are adopted to simulate the opinion evolution of a specific society. The results show the controllability of the dynamical system. Results also show that bigger inputs in different states will cause a bigger change in that state.

There are a few studies on the controllability of opinion dynamics in social settings. For example, Karan and Chakraborty [21] introduced a fundamentally new interaction model, derived mathematical equations, and investigated the controllability of opinion propagation in social settings. The researchers studied the controllability of the system both mathematically and with simulations. In another article by the mentioned researchers [22], the controllability of the influenced Sznajd model has been studied in depth. The findings of both studies confirm the findings of the present paper regarding controllability of opinion dynamic in social systems.

It is worth mentioning that the application of opinion dynamics is not limited to studying controllability of the system or the evolution of an opinion in a social setting. The concepts presented in this area could be applied in different practical fields. In [23], [24], researchers use the simple concepts of opinion sharing to investigate the evacuation dynamics of a large crowd from a building in life threatening conditions; Their groundbreaking models and findings can be used in emergency evacuations and save precious lives. These applications further prove the importance of the field of opinion dynamics. Future studies can and should focus on the application part of the said concepts.

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