Order reduction for problems with traveling wave solutions to reaction–diffusion systems

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Abstract. The paper is devoted to the order reduction for traveling wave problems for reaction-diffusion systems. The use of the method of invariant manifolds of singularly perturbed systems allows us to replace the study of the traveling wave of the original PDE system with the analysis of their profiles in the ODE system with a lower order.

1. Introduction
The paper deals with the traveling wave problems for reaction–diffusion systems. Such problems naturally arise in chemistry, biology, physics, ecology [1–15]. If the propagation velocity of the traveling waves is constant, their study is reduced to the analysis of their profiles in the corresponding systems of ordinary differential equations.

We consider processes characterized by small diffusion, which leads to the appearance of singular perturbations in the corresponding ODE systems. An important point of the study is the possibility of order reduction of such systems preserving the essential properties of the qualitative behavior of its solutions. For this goal, a geometric approach based on the apparatus of slow invariant manifolds is effectively used [16–27]. The essence of this approach consists of the decomposition of the original system into the fast subsystem and the independent slow subsystem. The slow subsystem is a projection of the original system onto its slow invariant manifold. Moreover, the slow subsystem, of a lesser order, reflect the behavior of the original models to a high order of accuracy when the slow invariant manifold is attractive. A mathematical justification of this method one can find in [27] and references therein.

We demonstrate this approach via the Belousov–Zhabotinsky–type reaction.

2. Mathematical Background
Consider a reaction–diffusion system in one spatial dimension

$$\frac{\partial \mathbf{u}}{\partial t} = \varepsilon D \frac{\partial^2 \mathbf{u}}{\partial s^2} + \mathbf{F}(\mathbf{u}),$$

(1)

where $\mathbf{u} = \mathbf{u}(s, t) \in \mathbb{R}^n$, $s \in \mathbb{R}$ is the spatial coordinate, $t > 0$ is time, $D$ is the real diagonal matrix of diffusion coefficients, and $\mathbf{F}(\mathbf{u})$ is the vector of reaction terms. The parameter $0 < \varepsilon \ll 1$ represents the relative sizes of the diffusion.
We are looking for traveling wave solutions to (1), which propagate at constant speed $c > 0$ preserving their spatial profile.

After transforming to a co-moving coordinate $\xi = s - ct$, equation (1) takes the following form

$$-c \frac{d\tilde{u}}{d\xi} = \varepsilon D \frac{d^2\tilde{u}}{d\xi^2} + F(\tilde{u}),$$

(2)

where $\tilde{u} = \tilde{u}(\xi) = u(s,t)$. Introducing $v = d\tilde{u}/d\xi$, from (2) we get

$$\frac{d\tilde{u}}{d\xi} = v,$$

$$\varepsilon D \frac{dv}{d\xi} = -cv - F(\tilde{u}).$$

(3)

Periodic solutions, as well as heteroclinic and homoclinic trajectories of the system (3), correspond to traveling waves of the original system (1).

The application of the geometric theory of invariant manifolds allows us to replace the study of the full system (3) by the analysis of a system of lower dimension. The order reduction occurs due to the decomposition of (3) in the vicinity of the invariant surface into the fast subsystem and the independent slow subsystem [27]. This independent slow subsystem is the projection of the original system onto its slow invariant manifold. If the slow invariant manifold is stable (or attractive), then the study of the system (3) can be replaced by the analysis of the slow subsystem.

Recall that the degenerate system for (3) is

$$\frac{d\tilde{u}}{d\xi} = v,$$

$$0 = -cv - F(\tilde{u}).$$

(4)

Note that (4) is equivalent to the equation

$$-c \frac{d\tilde{u}}{d\xi} = F(\tilde{u}),$$

(5)

which follows from (2) as $\varepsilon = 0$.

The second equation in (4) determines the slow surface of (3). The slow surface can be considered as a zero-order approximation ($\varepsilon = 0$) of the slow invariant manifold of the system [23, 24, 27]: the slow invariant manifold of the system can be found as an asymptotic expansion

$$v = v(\tilde{u}, \varepsilon) = v_0(\tilde{u}) + \varepsilon v_1(\tilde{u}) + O(\varepsilon^2),$$

where

$$-cv_0 = F(\tilde{u}).$$

On this slow manifold, the flow of the system (3) is described by

$$\frac{d\tilde{u}}{d\xi} = v(\tilde{u}, \varepsilon).$$

(6)

If the slow invariant manifold $v = v(\tilde{u}, \varepsilon)$ is attractive, then the analysis of the system (3) can be replaced by the analysis of (6) with a high degree of accuracy. Thus, problems with traveling wave solutions to (1) can be reduced to the study of periodic solutions, heteroclinic and homoclinic trajectories of the system (6).
3. Order Reduction for the Reaction-diffusion Model

Let us consider the reaction-diffusion system with two reacting components, i.e., when \( \mathbf{u} = (x, y)^T \), which is describing by the following dimensionless equations:

\[
\begin{align*}
\frac{\partial x}{\partial t} &= \varepsilon \frac{\partial^2 x}{\partial s^2} + f(x, y), \\
\frac{\partial y}{\partial t} &= \varepsilon \frac{\partial^2 y}{k \partial s^2} + g(x, y).
\end{align*}
\]

where

\[
f(x, y) = \frac{\alpha (\nu_0 + x^\gamma)}{1 + x^\gamma} - x(1 + y),
\]

\[
g(x, y) = x(\beta + y) - \delta y,
\]

\(x\) and \(y\) are the dimensionless concentrations of the reacting components; \(\alpha, \beta, \gamma, \delta,\) and \(\nu_0\) are the dimensionless positive parameters, moreover, \(\beta > 1\) and \(\gamma > 1\) \[28\].

Following the algorithm described above, when searching for traveling wave solutions to (7), we look for solutions,

\[
\begin{align*}
x(s, t) &= \varphi(s - ct) = \varphi(\xi), \\
y(s, t) &= \psi(s - ct) = \psi(\xi).
\end{align*}
\]

From (7) and (8) we get

\[
\begin{align*}
-c \frac{d\varphi}{d\xi} &= \varepsilon \frac{d^2\varphi}{d\xi^2} + \frac{\alpha (\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi), \\
-c \frac{d\psi}{d\xi} &= \varepsilon \frac{d^2\psi}{k d\xi^2} + \varphi(\beta + \psi) - k\delta\psi,
\end{align*}
\]

or, in the form of (3),

\[
\begin{align*}
\frac{d\varphi}{d\xi} &= p, \\
\frac{d\psi}{d\xi} &= q, \\
\varepsilon \frac{dp}{d\xi} &= -cp - \frac{\alpha (\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi), \\
\varepsilon \frac{dq}{d\xi} &= -ckq - k\varphi(\beta + \psi) + k\delta\psi.
\end{align*}
\]

The corresponding degenerate system is

\[
\begin{align*}
\frac{d\varphi}{d\xi} &= p, \\
\frac{d\psi}{d\xi} &= q, \\
0 &= -cp - \frac{\alpha (\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi) := h_1, \\
0 &= -ckq - k\varphi(\beta + \psi) + k\delta\psi := h_2.
\end{align*}
\]
The last two equations give the unique solution

\[ p = P_0(\varphi, \psi) = -\frac{\alpha(\nu_0 + \varphi^\gamma)}{c(1 + \varphi^\gamma)} + \frac{1}{c} \varphi(1 + \psi), \]

\[ q = Q_0(\varphi, \psi) = -\frac{1}{c} \varphi(\beta + \psi) + \frac{\delta}{c} \psi, \]

which determines the slow surface of (9). The slow surface is attractive since [27]

\[ \text{tr} B(\varphi, \psi) < 0, \quad \det B(\varphi, \psi) > 0, \]

where

\[ B = \begin{pmatrix} \frac{\partial h_1}{\partial \psi} & \frac{\partial h_1}{\partial \varphi} \\ \frac{\partial h_2}{\partial \psi} & \frac{\partial h_2}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} -c & 0 \\ 0 & -ck \end{pmatrix}. \]

Due to the geometric theory of singular perturbations in an \( \varepsilon \)-neighborhood of the slow surface, there exists an attractive slow invariant manifold, which can be represented in the form

\[ p = P_0(\varphi, \psi, \varepsilon) = P_0(\varphi, \psi) + \varepsilon P_1(\varphi, \psi) + O(\varepsilon^2), \]

\[ q = Q(\varphi, \psi, \varepsilon) = Q_0(\varphi, \psi) + \varepsilon Q_1(\varphi, \psi) + O(\varepsilon^2). \]

To calculate the first–order approximation to the slow invariant manifold we substitute (12) into the invariance equations

\[ \varepsilon \left( \frac{\partial P}{\partial \varphi} P + \frac{\partial P}{\partial \psi} Q \right) = -cP - \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi), \]

\[ \varepsilon \left( \frac{\partial Q}{\partial \varphi} P + \frac{\partial Q}{\partial \psi} Q \right) = -ckQ - k\varphi(\beta + \psi) + k\delta \psi, \]

which follows from (9). Hence, we obtain

\[ \varepsilon \left( \frac{\partial P_0}{\partial \varphi} P_0 + \frac{\partial Q_0}{\partial \psi} Q_0 \right) + O(\varepsilon^2) = -c \left( P_0 + \varepsilon P_1 + O(\varepsilon^2) \right) - \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi), \]

\[ \varepsilon \left( \frac{\partial Q_0}{\partial \varphi} P_0 + \frac{\partial Q_0}{\partial \psi} Q_0 \right) + O(\varepsilon^2) = -ck \left( Q_0 + \varepsilon Q_1 + O(\varepsilon^2) \right) - k\varphi(\beta + \psi) + k\delta \psi. \]

Equating the coefficients of the first power of \( \varepsilon \) and taking into account (11), from these equations we get

\[ P_1(\varphi, \psi) = -\frac{1}{c^2} \left[ \frac{\alpha \varphi^\gamma - 1(1 - \nu_0)}{(1 + \varphi^\gamma)^2} - 1 - \psi \right] \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) - \varphi^2(\beta + \psi) + \delta \varphi \psi, \]

\[ Q_1(\varphi, \psi) = -\frac{1}{c^2 k} \left[ (\beta + \psi) \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) + (\varphi - \delta) (\varphi(\beta + \psi) - \delta \varphi) \right]. \]

Thus, the first–order approximation to the slow motion of (9) is described by

\[ \frac{d\varphi}{d\xi} = -\frac{\alpha(\nu_0 + \varphi^\gamma)}{c(1 + \varphi^\gamma)} + \frac{1}{c} \varphi(1 + \psi) \]

\[ - \frac{\varepsilon}{c^2} \left[ \frac{\alpha \varphi^\gamma - 1(1 - \nu_0)}{(1 + \varphi^\gamma)^2} - 1 - \psi \right] \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) - \varphi^2(\beta + \psi) + \delta \varphi \psi \]

\[ + O(\varepsilon^2), \]

\[ \frac{d\psi}{d\xi} = -\frac{1}{c} \varphi(\beta + \psi) + \frac{\delta}{c} \psi \]

\[ - \frac{\varepsilon}{c^2 k} \left[ (\beta + \psi) \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) + (\varphi - \delta) (\varphi(\beta + \psi) - \delta \varphi) \right] + O(\varepsilon^2). \]
System (13) has no singular perturbations and its order is lower compared to (9) which essentially simplifies the analysis.

4. Traveling Waves
As was noted, the wave properties of the original system (1) are determined by the properties of the system (3) under $\varepsilon \to 0$, i.e., by the properties of (4). For $\varepsilon = 0$, the singular points of (3) coincide with the equilibria of (5) and are determined by $\mathbf{F}(\tilde{\mathbf{u}}) = 0$.

Let us consider (9) for $\alpha = 12$, $\beta = 1.5$, $\gamma = 3$, $\delta = 1.7$, and $\nu_0 = 0.01$. In this case, there are three equilibria of the corresponding degenerate system (10): the unstable node $P_1$, the saddle $P_2$, and the unstable focus $P_3$ (Figure 1). Note, that these equilibria are the projections of the singular points $\tilde{P}_1$, $\tilde{P}_2$, and $\tilde{P}_3$ of (9) onto $(\varphi, \psi)$–plane.

Adding the corresponding asymptotic boundary conditions to (10), we can obtain a heteroclinic trajectory connecting the equilibria $P_1$ and $P_2$ (Figure 2). Moreover, the system (9) has a solution tending to the unstable equilibrium point $\tilde{P}_1$ as $\xi \to -\infty$ and to $\tilde{P}_2$ as $\xi \to +\infty$. This solution determines the profile of the traveling waves of the system (7) propagating at constant speed $c > 0$.

Figures 3 and 4 demonstrate the heteroclinic trajectories of the systems (10) (see the blue line) and (13) (see the red line), as well as the $(\varphi, \psi)$–projection of the corresponding trajectory of the system (9) (see the black line). All these trajectories are very close to each other, which means that the reduced systems preserve the essential properties of the qualitative behavior of the original system.

5. Conclusions
We have discussed how the method of invariant manifolds of singularly perturbed systems helps to reduce traveling wave problems for a reaction-diffusion system. We have shown that the traveling wave problems for the original PDE system can be reduced to the study of the projection of this system onto its slow invariant manifold.
Figure 3. The heteroclinic trajectories of (10) (blue line) and (13) (red line), and \((\varphi, \psi)\)– projection of the corresponding heteroclinic trajectory of (9) (black line).

Figure 4. Plots of \(\varphi = \varphi(\xi)\) (black solid line) and \(\psi = \psi(\xi)\) (blue dash line) for \(\tilde{P}_1 \rightarrow \tilde{P}_2\) heteroclinic trajectory of (9).

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