In this paper, we consider a single-index mixed model with longitudinal data. A new set of estimating equations is proposed to estimate the single-index coefficient. The link function is estimated by using the local linear smoothing. Asymptotic normality is established for the proposed estimators. Also, the estimator of the link function achieves optimal convergence rates; and the estimators of variance components have root-$n$ consistency. These results facilitate the construction of confidence regions/intervals and hypothesis testing for the parameters of interest. Some simulations and an application to real data are included.
1 Introduction

Consider the single-index mixed model

\[ Y_{ij} = g(X_{ij}^T \beta_0) + \alpha_i + \epsilon_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \]  

(1.1)

where \( \alpha_i \) and \( \epsilon_{ij} \) are independent mean zero random variables with variances \( \sigma^2_\alpha > 0 \) and \( \sigma^2_\epsilon > 0 \), respectively, \( g(\cdot) \) is an unknown link function, and \( \beta_0 \) is a \( p \times 1 \) vector of unknown parameters. For the sake of identifiability, it is often assumed that \( \| \beta_0 \| = 1 \) and the first nonzero component of \( \beta_0 \) is positive, where \( \| \cdot \| \) denotes the Euclidean metric. Let \( Y_i = (Y_{i1}, \ldots, Y_{im})^T \), \( X_i = (X_{i1}, \ldots, X_{im})^T \), \( \epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{im})^T \) and \( G(X_i \beta_0) = (g(X_{i1}^T \beta_0), \ldots, g(X_{im}^T \beta_0))^T \). The model implies that the \( Y_i \) are independent with \( E(Y_i | X_i) = G(X_i \beta_0) \) and \( \text{cov}(Y_i | X_i) = V = \sigma^2_\alpha 1_m 1_m^T + \sigma^2_\epsilon I_m \), where \( 1_m \) is an \( m \times 1 \) vector of ones and \( I_m \) is the \( m \times m \) identity matrix.

We address the general problem of estimating the parameter \( \beta_0 \), the function \( g(\cdot) \), and the variance components \( \sigma^2_\alpha \) and \( \sigma^2_\epsilon \) simultaneously when \( m \) is fixed. We will show in Section 3 that the variance components \( \sigma^2_\alpha \) and \( \sigma^2_\epsilon \) can be estimated at the parametric rate \( O_P(n^{-1/2}) \) which allows us to treat them as known when we derive the theoretical results for \( \beta_0 \) and \( g(\cdot) \) in Sections 2 and 3.

The single-index model is an important tool in multivariate nonparametric regression, which can avoid the so-called “curse of dimensionality” by searching a univariate index of the multivariate covariate \( X \) to capture important features of high-dimensional data. The single-index model has been applied in a variety of fields, such as discrete choice analysis in econometrics and dose-response models in biometrics (Härdle et al. 1993). In the cross-sectional data, many authors have studied the statistical inference problem of the single-index model, and reported many results, for example, Li (1991), Ichimura (1993), Zhu and Ng (1995), Xia and Li (1999), Naik and Tsai (2000), Hristache, Juditsky and Spokoiny (2001), Xia et al. (2002), Stute and Zhu (2005), Xia (2006), and Xue and Zhu (2006). Meanwhile, the estimation problem of the partially linear single-index model has been widely addressed as well by Carroll et al. (1997), Yu and Ruppert (2002), Xia and Härdle (2006), Zhu and Xue (2006), Wang et al. (2010) and others. These reported methods have been proven to be useful and effective for the independent data. On the other hand, to our knowledge, the
method to treat correlated data, which are commonly seen in econometrics and biometrics, is lacking in literature. In this paper, such models will be developed and reported.

Longitudinal data are perhaps the most well-known type of correlated data. There are already extensive literature on the generalized linear, nonparametric and semiparametric mixed models for longitudinal data, see, for example, Zeger and Diggle (1994), Jiang (1998), Zhang, et al. (1998), Jiang (1999), Ruckstuhl, Welsh and Carroll (2000), Jiang and Zhang (2001), Jiang, Jia and Chen (2001), Ke and Wang (2001), Cai, Cheng and Wei (2002), Wu and Zhang (2002), Liang, Wu and Carroll (2003), Zhang and Lin (2003), Gu and Ma (2005), Hall and Maiti (2006), Jiang (2006) and Field, Pang and Welsh (2008), among others. However, literature on the applications of single-index models for longitudinal/panel data is limited. Honorá and Kyriazidou (2000) and Carro (2007) proposed some estimating methods for dynamic panel data discrete choice models. Bai et al. (2009) studied the single-index models for longitudinal data, where they proposed a procedure to estimate the single-index component and the link function based on the combination of the penalized splines and quadratic inference functions. Liang and Zeger (1986) proposed an extension of the generalized linear models to the analysis of longitudinal data. They introduced the generalized estimating equations (GEE) that gave consistent estimates of the regression parameters and their variance under mild assumptions on the time dependence. The GEE were derived without specifying the joint distribution of a subject’s observations yet they reduced to the score equations for multivariate Gaussian outcomes. In this paper, we apply the idea of GEE to the single-index mixed models with longitudinal data. To estimate the single-index coefficient $\beta_0$, we propose a new set of estimating equations which take the constraint $\|\beta_0\| = 1$ into account. The estimator based on these estimating equations outperform previous ones, as summarized below. First, our estimation procedure does not specify a form for both the distribution of random effect and the joint distribution of the repeated measurements. Second, we introduce estimating equations that give the root-$n$ consistent estimate of $\beta_0$ under week assumptions on the joint distribution. Third, we construct the root-$n$ consistent estimates of the variance components $\sigma^2_\varepsilon$ and $\sigma^2_\alpha$. It allows us to consider the construction of confidence regions and hypothesis testing for $\beta_0$. Lastly,
we also obtain the asymptotic normality and the uniform convergence rate of the estimator of \( g(\cdot) \). Our algorithm is numerically fast and stable.

The rest of the paper is organized as following. In Section 2, we elaborate on the methodology. Section 3 presents the asymptotic properties for all proposed estimators. Section 4 reports the results of simulation studies and one real example. The proofs of the main theorems are relegated to the Appendix.

## 2 Estimation method

### 2.1 Estimations of the parametric and nonparametric components

If \( g \) were known, we could estimate \( \beta_0 \) by minimizing

\[
R_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} \{Y_i - G(X_i \beta)\}^T W(X_i \beta) V^{-1} \{Y_i - G(X_i \beta)\}
\]

for \( \beta \) with \( \|\beta\| = 1 \), where \( W(X_i \beta) = \text{diag}\{w(X_{i1}\beta), \ldots, w(X_{im}\beta)\} \), and \( w(\cdot) \) is a bounded weight function with a bounded support \( U_w \), which is introduced to control the boundary effect. For simplicity and convenience, we assume that \( dw(u)/du = 0 \). Especially, we can take \( w(\cdot) = I_{[-a,a]}(\cdot) \), for some constant \( a > 0 \). This is a restricted least squares problem. We now use the constraint \( \|\beta_0\| = 1 \) to transfer the restricted least squares to the unrestricted least squares, which makes it possible to search for the solution of the estimating equations over a restricted region in the Euclidean space \( R^{p-1} \). For this, we need to calculate the derivative of \( g(X_i^T \beta) \) at point \( \beta_0 \). Note that \( \|\beta_0\| = 1 \) means that the true value \( \beta_0 \) is the boundary point of the unit sphere. The function \( g(X_i^T \beta) \) does not have the derivative at point \( \beta_0 \). For this, we suggest the popularly used delete-one-component method (Wang et al., 2010). The detail is as follows. Without loss of generality, we may assume that the true parameter \( \beta_0 \) has a positive component (otherwise, consider \( -\beta_0 \)), say \( \beta_{0r} > 0 \) for \( \beta_0 = (\beta_{01}, \ldots, \beta_{0p})^T \) and \( 1 \leq r \leq p \). For \( \beta = (\beta_1, \ldots, \beta_p)^T \), let \( \beta^{(r)} = (\beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_p)^T \) be a \( p-1 \) dimensional parameter vector after removing the \( r \)th component \( \beta_r \) in \( \beta \). Then the true parameter \( \beta_0^{(r)} \) must satisfy the constraint \( \|\beta_0^{(r)}\| < 1 \), and \( \beta \) is infinitely differentiable in a
neighborhood of $\beta_0^{(r)}$. The Jacobian matrix of $\beta$ with respect to $\beta^{(r)}$ is defined as

$$J_{\beta^{(r)}} = \frac{\partial \beta}{\partial \beta^{(r)}} = (\gamma_1, \ldots, \gamma_p)^T,$$

(2.1)

where $\gamma_s (1 \leq s \leq p, s \neq r)$ is a $p-1$ dimensional unit vector with $s$th component 1, and $\gamma_r = -(1-\|\beta^{(r)}\|^2)^{-1/2}\beta^{(r)}$. Let $X_{ij} = (X_{ij1}, \ldots, X_{ijp})^T$ and $X_{ij}^{(r)} = (X_{ij1}, \ldots, X_{ij(r-1)}, X_{ij(r+1)}, \ldots, X_{ijp})^T$. Then we have $X_{ij}^T\beta = X_{ij}^{(r)}\beta^{(r)} + (1-\|\beta^{(r)}\|^2)^{1/2}X_{ijr}$, which is a function of $\beta^{(r)}$. When $g$ is known, we can obtain an estimator of $\beta_0^{(r)}$ by solving

$$Q_n(G, \beta^{(r)}) \equiv \frac{1}{n} \sum_{i=1}^{n} J_{\beta^{(r)}}^T X_i^T G^r_{\Delta}(X_i\beta) W(X_i\beta) V^{-1} \{Y_i - G(X_i\beta)\} = 0$$

(2.2)

for $\beta^{(r)}$, where $G^r_{\Delta}(X_i\beta) = \text{diag}\{g'(X_{i1}\beta), \ldots, g'(X_{im}\beta)\}$. An iteratively reweighted least squares algorithm is widely used for solving this system of equations. Given a current estimate $\tilde{\beta}^{(r)}$ with $\|\tilde{\beta}^{(r)}\| = 1$, compute

$$\hat{\beta}^{(r)} = \tilde{\beta}^{(r)} + B_n^{-1}(G, \tilde{\beta}^{(r)}) Q_n(G, \tilde{\beta}^{(r)})$$

and $\tilde{\beta} = \hat{\beta}^{(r)}/\|\hat{\beta}^{(r)}\|$, where

$$B_n(G, \beta^{(r)}) \equiv \frac{1}{n} \sum_{i=1}^{n} J_{\beta^{(r)}}^T X_i^T G^2_{\Delta}(X_i\beta) W(X_i\beta) V^{-1} X_i J_{\beta^{(r)}}.$$

This iteratively reweighted least squares algorithm solves (2.2) and is identical to the Fisher’s method of scoring version of the Newton-Raphson algorithm for solving these estimating equations. Using $\|\beta_0\| = 1$ and $\|\beta\| = 1$, we can prove

$$\beta - \beta_0 = J_{\beta_0^{(r)}}(\beta^{(r)} - \beta^{(r)}) + O_P(n^{-1}).$$

Thus, we can obtain an iterative formula for estimating $\beta$ when $g$ is known, that is

$$\hat{\beta}^* = \tilde{\beta} + J_{\tilde{\beta}_0}^{-1}(G, \tilde{\beta}) Q_n(G, \tilde{\beta})$$

(2.3)

and $\tilde{\beta}^* = \hat{\beta}^*/\|\hat{\beta}^*\|$, where the initial value of $\beta_0$, say $\|\tilde{\beta}_0\| = 1$, can be obtain by fitting the linear model. Then, set $\tilde{\beta} = \tilde{\beta}_0$ and iterate until convergence.

Since we assume that the link function $g$ is unknown, it must be estimated. Given an initial estimate $\tilde{\beta}_0$ of $\beta_0$, we can easily compute a nonparametric estimates $\hat{g}$ and $\hat{g}'$ of $g$ and
We employ the local linear smoother (Fan and Gijbels, 1996) to obtain estimators of the link function $g$ and its derivative $g'$. Specifically, for a kernel function $K(\cdot)$ on the real set $\mathbb{R}$ and a bandwidth sequence $h = h_n$ tending to 0, define $K_h(\cdot) = h^{-1}K(\cdot/h)$. For a fixed $\beta$, the local linear smoother aims at minimizing the weighted sum of squares

$$
\sum_{i=1}^{n} \sum_{j=1}^{m} \{Y_{ij} - d_0 - d_1(X_{ij}^T \beta_0 - u)\}^2 K_h(X_{ij}^T \beta_0 - u)
$$

with respect to the parameters $d_\nu$, $\nu = 0, 1$. Let $\hat{d}_0$ and $\hat{d}_1$ be the solutions to the weighted least squares problem (2.4). The local linear estimators for $g(u)$ and $g'(u)$ are defined as $\hat{g}(u; \beta_0) = \hat{d}_0$ and $\hat{g}'(u; \beta_0) = \hat{d}_1$ at the fixed point $\beta_0$. It follows from the theory of least squares that

$$
(\hat{g}(u; \beta_0), h\hat{g}'(u; \beta_0))^T = S_n^{-1}(u; \beta_0)\xi_n(u; \beta_0),
$$

where

$$
S_n(u; \beta_0) = \begin{pmatrix}
S_{n,0}(u; \beta_0) & S_{n,1}(u; \beta_0) \\
S_{n,1}(u; \beta_0) & S_{n,2}(u; \beta_0)
\end{pmatrix}
$$

and

$$
\xi_n(u; \beta_0) = (\xi_{n,0}(u; \beta_0), \xi_{n,1}(u; \beta_0))^T
$$

with

$$
S_{n,l}(u; \beta_0) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{X_{ij}^T \beta_0 - u}{h}\right)^l K_h(X_{ij}^T \beta_0 - u)
$$

and

$$
\xi_{n,l}(u; \beta_0) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} Y_{ij} \left(\frac{X_{ij}^T \beta_0 - u}{h}\right)^l K_h(X_{ij}^T \beta_0 - u)
$$

for $l = 0, 1, 2$.

The estimator $\hat{g}$ is called pooled estimator in existing literatures, for example Lin and Carroll (2000), Ruckstuhl, Welsh and Carroll (2000), and Xue (2010). As pointed out in these literatures, the simple pooled estimator which ignores the dependence structure performs very well asymptotically.

When $g$ is unknown, we can also obtain an estimator of $\beta^{(r)}_0$ by solving the estimating equations $Q_n(\hat{G}, \beta^{(r)}) = 0$, where $\hat{G}(X_i \beta_0) = (\hat{g}(X_{i1}^T \beta_0), \ldots, \hat{g}(X_{im}^T \beta_0))^T$ and $\hat{G}'_A(X_i \beta) = \ldots$
\[
\text{diag}\{\hat{g}'(X_{i1} \beta), \ldots, \hat{g}'(X_{im} \beta)\} \text{ for } i = 1, \ldots, n. \]

We propose the use of an alternating algorithm; first estimating \( \beta_0 \), and then the link function \( g \), repeating these until certain criterion is met. Given \( \hat{g} \) and \( \hat{g}' \), we use the scoring algorithm (2.3) to estimate \( \beta_0 \), that is

\[
\hat{\beta} = \bar{\beta} + J_{\beta(\cdot)} B_n^{-1}(\hat{G}, \bar{\beta}) Q_n(\hat{G}, \bar{\beta})
\]

and \( \hat{\hat{\beta}} = \hat{\beta}/\|\hat{\beta}\| \); given the estimate of \( \beta_0 \), we used the pooled estimate (2.5) to get a new estimate of the link function \( g \).

With \( \hat{\beta} \), the final estimator of \( g \) can be defined by \( \hat{g}^*(u) = \hat{g}(u; \hat{\beta}) \). The asymptotic result for the estimate of link function \( g \) follows from Theorems 1 and 2, and the result for the estimate of parameter \( \beta_0 \) is established in Theorem 3.

**Remark 1** We consider a homoscedastic model of (1.1). While the estimation procedure can be extended to heteroscedastic errors. In addition, the single-index assumption in (1.1) can be readily extended to multiple indices through Sliced Inverse Regression (SIR) or its variants, but the estimation of the multivariate link function \( g \) would encounter the curse of high dimensionality. In many applications, since no more than three indices will be needed, the approach in this paper can indeed be extended in practice to multiple indices.

### 2.2 Estimations of the variance components

The estimation of the nonparametric component and the asymptotic variances of all the estimators depends on the variance components, thus we need to exhibit consistent estimators of the variance components.

A useful approach to estimate the variance components is to pretend that the residuals are of mean zero and have the covariance matrix same as if \( g(\cdot) \) were known. If we assume that the random effects \( \alpha_i \) and the errors \( \varepsilon_{ij} \) are Gaussianly distributed, then the observation \( Y_i \) have independent \( N(G(X_i \beta_0), V) \) distributions. Replacing \( g(\cdot) \) and \( \beta_0 \) with their estimators \( \hat{g}(\cdot) \) and \( \hat{\beta} \), respectively, the Gaussian “likelihood” for \( \sigma^2_\varepsilon \) and \( \sigma^2_\alpha \) can be written as

\[
-n(m - 1) \log(\sigma^2_\varepsilon) - n \log(\sigma^2_\varepsilon + m\sigma^2_\alpha) - \frac{m}{\sigma^2_\varepsilon + m\sigma^2_\alpha} \sum_{i=1}^{n}(\bar{Y}_i - \bar{\hat{g}}_i)^2
\]

\[
-\frac{1}{\sigma^2_\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\{ Y_{ij} - \hat{g}(X_{ij}^T \hat{\beta}) - (\bar{Y}_i - \bar{\hat{g}}_i) \right\}^2,
\]
where $\bar{Y}_i = m^{-1} \sum_{j=1}^{m} Y_{ij}$ and $\bar{g}_i = m^{-1} \sum_{j=1}^{m} \hat{g}(X_{ij}^T \hat{\beta})$. This "likelihood" is maximized at

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{n(m-1)} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\{ Y_{ij} - \hat{g}(X_{ij}^T \hat{\beta}) - (\bar{Y}_i - \bar{g}_i) \right\}^2,$$

$$\hat{\sigma}_\alpha^2 = \frac{1}{n} \sum_{i=1}^{n} (\bar{Y}_i - \bar{g}_i)^2 - \hat{\sigma}_\varepsilon^2 / m,$$

when $\hat{\sigma}_\alpha^2 > 0$, and at $\hat{\sigma}_\alpha^2 = 0$ and

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\{ Y_{ij} - \hat{g}(X_{ij}^T \hat{\beta}) \right\}^2,$$

otherwise. It can be shown that the resulting estimators have the same convergence rate as if $g(\cdot)$ and $\beta_0$ actually were known. The result will be given in next section.

Alternatively, we can abandon the "likelihood" and employ a method of moments device to get the estimators (2.9)–(2.11). We can also adjust for the loss of degrees of freedom due to estimating $g(\cdot)$, and obtain the estimators of $\sigma^2_\varepsilon$ and $\sigma^2_\alpha$. The details can be found in Ruckstuhl, et al. (2000).

### 3 Main Results

We now study the asymptotic behavior of the estimators for the nonparametric component $g$ as well as the parametric components $\beta_0$, $\sigma^2_\alpha$ and $\sigma^2_\varepsilon$. We first list the following regularity conditions:

(C1) The joint density of $(X_{i1}^T, \ldots, X_{im}^T)^T$ exists, the marginal density $f_j(u)$ of $X_{ij}^T \beta$ and the joint density $f_{j1,j2}(u, s)$ of $(X_{ij}^T \beta, X_{ij}^T \beta)$, for any $j_1 \neq j_2$, are continuously differentiable for $u_0 \in U_w$ and $(u_0, s_0) \in U_w \times U_w$, respectively, and there exists a $j$ such that $f_j(u)$ is bounded away from 0, uniformly for $u \in U_w$ and $\beta$ near $\beta_0$, where $U_w$ is the support of $w(u)$.

(C2) The function $g(u)$ has two bounded and continuous derivatives, and $g_{2r}(u)$ satisfies a Lipschitz condition of order 1 on $U_w$, where $g_{2r}(u)$ is the $r$th component of $g_2(u)$, and $g_2(u) = \mathbb{E}(X_{ij} | X_{ij}^T \beta_0 = u)$, $1 \leq r \leq p$.

(C3) The kernel $K(\cdot)$ is a bounded and symmetric probability density function with bounded support, and satisfies the Lipschitz condition of order 1 and $\int u^2 K(u) du \neq 0$. 

8
There exists an $r = \max\{4, s\}$ such that $E(|X_{ij}|^r) < \infty$, $E(|\alpha_i|^r) < \infty$ and $E(|\varepsilon_{ij}|^r) < \infty$, and for some $\epsilon < 2 - s^{-1}$ such that $n^{2\epsilon - 1}h \rightarrow \infty$, $i = 1, \ldots, j = 1, \ldots, m$.

(C5) $nh^3 / \log(1/h) \rightarrow \infty$ and $nh^4 \rightarrow 0$ as $n \rightarrow \infty$.

(C6) $B = E\left[ J_{\beta_0^{(r)}}^T X_1^T G^2(X_1 \beta_0) W(X_1 \beta_0) V^{-1} X_1 J_{\beta_0^{(r)}} \right]$ is a positive definite matrix.

**Remark 2** Condition (C1) ensures that the denominators of $\hat{g}(u; \beta)$ and $\hat{g}'(u; \beta)$ are, with high probability, bounded away from 0 for $t \in U_w$ and $\beta$ near $\beta_0$. (C2) is the standard smoothness condition. (C3) is the usual assumption for second-order kernels. (C4) is a necessary condition for the asymptotic normality of an estimator. (C5) is the usual condition for bandwidth. (C6) ensures that the limiting variances for the estimator $\hat{\beta}$ exist.

Let $B_n = \{ \beta \in B : \| \beta - \beta_0 \| \leq c_1 n^{-1/2} \}$ for some positive constant $c_1$. The definition is motivated by the fact that, since we anticipate that $\hat{\beta}$ is root-$n$ consistent, we should look for a solution of the equations $Q_n(\hat{g}, \beta^{(r)}) = 0$ which involves $\beta^{(r)}$ distant from $\beta_0^{(r)}$ by order $n^{-1/2}$. Similar restriction was also made by Härdle, Hall and Ichimura (1993) and Xia and Li (1999). Denote $\mu_l = \int u^l K(u) du$ and $\nu_l = \int K^l(u) du$, $l = 1, 2$.

The following theorems state the asymptotic behavior of the estimators proposed in Section 2. We first give the uniform convergence rates for the estimators $\hat{g}$ and $\hat{g}'$ respectively.

**Theorem 1** Suppose that conditions (C1)–(C4) hold. Then

$$\sup_{u \in U_w, \beta \in B_n} |\hat{g}(u; \beta) - g(u)| = O_P \left( (nh/\log n)^{-1/2} + h^2 \right)$$

and

$$\sup_{u \in U_w, \beta \in B_n} |\hat{g}'(u; \beta) - g'(u)| = O_P \left( (nh^3/\log n)^{-1/2} + h \right).$$

The following Theorem 2 shows the asymptotic normality of estimator $\hat{g}$. 

9
Theorem 2. Suppose that conditions (C1)–(C4) hold. If \( nh^5 = O(1) \), then for any \( u \in U \) and \( \tilde{\beta} \) such that \( \| \tilde{\beta} - \beta_0 \| = O_p(n^{-1/2}) \), we have

\[
\sqrt{nh}\{ \hat{g}(u; \tilde{\beta}) - g(u) - b(u) \} \xrightarrow{D} N(0, \sigma^2(u)).
\]

where \( b(u) = (1/2)h^2 \mu g''(u) \), and \( \sigma^2(u) = (\sigma^2_\alpha + \sigma^2_\epsilon)\nu_0 / \sum_{j=1}^m f_j(u) \).

If further assume that \( nh^5 \to 0 \), then

\[
\sqrt{nh}\{ \hat{g}(u; \tilde{\beta}) - g(u) \} \xrightarrow{D} N(0, \sigma^2(u)).
\]

In Theorems 1 and 2, when we start with \( \sqrt{n} \)-consistent estimator for \( \beta_0 \), \( \hat{g} \) has uniform convergence rate and asymptotic normality. Numerous examples of \( \sqrt{n} \)-consistent estimators already exist in the literature. For instance, Hall (1989) showed that one can obtain a \( \sqrt{n} \)-consistent estimator for \( \beta_0 \) using projection pursuit regression. Under the linearity condition that is slightly weaker than elliptical symmetry of \( X \), Li (1991), Hsing and Carroll (1992) and Zhu and Ng (1995) proved that SIR, proposed by Li (1991), leads to a \( \sqrt{n} \)-consistent estimator of \( \beta_0 \). Xia et al. (2002) proposed the minimum average variance estimation (MAVE) and Xia (2006) proposed a refined version of MAVE, and both methods can provide \( \sqrt{n} \)-consistent estimators for the single-index \( \beta_0 \).

Theorem 3. Suppose that conditions (C1)–(C6) hold. If the \( r \)th component of \( \beta_0 \) is positive, then

\[
\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{D} N \left( 0, J_{\beta_0} B^{-1} A B^{-1} J_{\beta_0}^T \right),
\]

where \( A = E \left[ J_{\beta_0}^T \{ X_1 - G_1(X_1 \beta_0) \} G^2(X_1 \beta_0) W^2(X_1 \beta_0) V^{-1} \{ X_1 - G_1(X_1 \beta_0) \} J_{\beta_0} \right] \) with \( G_1(X_1 \beta_0) = (g_1(X_{11} \beta_0), \ldots, g_1(X_{1m} \beta_0))^T \) and \( g_1(u) = E(X_{1j} | X_{1j}^T \beta_0 = u) \), and \( B \) is defined in condition (C6).

From Theorems 2 and 3, we can obtain the following corollary 1.
COROLLARY 1 Suppose that conditions (C1)–(C6) hold. Then, for any \( u \in \mathcal{U}_w \),

\[
\sqrt{nh}\{\hat{g}^*(u) - g(u)\} \xrightarrow{D} N(0, \sigma^2(u)).
\]

where \( \sigma^2(u) \) is defined in Theorem 2.

From Theorem 3, we obtain an asymptotic result regarding the angle between \( \hat{\beta} \) and \( \beta_0 \), which can be used to study issues of sufficient dimension reduction.

COROLLARY 2 Suppose that the conditions of Theorem 3 hold. Then

\[
|\hat{\beta}^T\beta_0| - 1 = O_P(n^{-1/2}),
\]

where \( |\hat{\beta}^T\beta_0| \) is the absolute inner product. Their inner product represents the cosine of the angle between the two directions.

The following theorem provides the convergence rates of the estimators of \( \sigma^2_\alpha \) and \( \sigma^2_\varepsilon \), respectively.

THEOREM 4 Suppose that conditions (C1)–(C6) hold. Then

\[
\hat{\sigma}^2_\varepsilon - \sigma^2_\varepsilon = O_P\left(n^{-1/2}\right),
\]

\[
\hat{\sigma}^2_\alpha - \sigma^2_\alpha = O_P\left(n^{-1/2}\right).
\]

To construct confidence regions for \( \beta_0 \), a plug-in estimator of the limiting variance of \( \hat{\beta} \) is needed. We define the following estimators \( \hat{B} \) and \( \hat{A} \) of \( B \) and \( A \), respectively, by

\[
\hat{B} = B_n(\hat{g}, \hat{\beta}) \quad \text{and} \quad \hat{A} = \frac{1}{n} \sum_{i=1}^{n} J_{\hat{\beta}(\cdot)}^T \{X_i - \hat{G}_1(X_i; \hat{\beta})\}^T \hat{G}^2_\Delta(X_i; \hat{\beta}) \hat{W}^2(X_i; \hat{\beta}) \hat{V}^{-1} \{X_i - \hat{G}_1(X_i; \hat{\beta})\} J_{\hat{\beta}(\cdot)},
\]
where \( \hat{V} = \hat{\sigma}^2_{\alpha} I_m + \hat{\sigma}^2_{\tilde{\varepsilon}} I \), \( \hat{G}_\Delta(X_i; \hat{\beta}) = \text{diag}\{\hat{g}'(X_{i1}; \hat{\beta}), \ldots, \hat{g}'(X_{im}; \hat{\beta})\} \), \( \hat{G}_1(X_i; \hat{\beta}) = (\hat{g}_1(X_{i1}; \hat{\beta}), \ldots, \hat{g}_1(X_{im}; \hat{\beta}))^T \) with \( \hat{g}_1(u; \hat{\beta}) = \sum_{i=1}^n \sum_{j=1}^m \hat{W}_{nij}(u; \hat{\beta})X_{ij} \), which is the estimator of \( g_1(u) = E(X_{ij}|X_{ij}; \beta_0) = u, \)

\[
W_{nij}(u; \hat{\beta}) = \frac{n^{-1} K_h(X_{ij}; \hat{\beta} - u)(S_{n,2}(u; \hat{\beta}) - (X_{ij}; \hat{\beta} - u)S_{n,1}(u; \hat{\beta}))}{S_{n,0}(u; \hat{\beta})S_{n,2}(u; \hat{\beta}) - S_{n,1}(u; \hat{\beta})},
\]

and \( S_{n,2}(u; \hat{\beta}) \) is defined in (2.6). It is easy to prove that \( J_{\hat{\beta}(\cdot)} \xrightarrow{P} J_{\beta_0}(\cdot), \hat{B} \xrightarrow{P} B \) and \( \hat{A} \xrightarrow{P} A \). Then for any \( p \times l \) matrix \( H \) of full rank with \( l < p \), Theorem 2 implies that

\[
\left(n^{-1}H^T J_{\hat{\beta}(\cdot)} \hat{B}^{-1} \hat{A} \hat{B}^{-1} J_{\hat{\beta}(\cdot)}^T H\right)^{-1/2} H^T (\hat{\beta} - \beta_0) \xrightarrow{D} N(0, I_l).
\]

We use Theorem 10.2d in Arnold (1981) to obtain the following limiting distribution.

**Theorem 5** Suppose that the conditions of Theorem 2 hold. Then

\[
(\hat{\beta} - \beta_0)^T H \left(n^{-1}H^T J_{\hat{\beta}(\cdot)} \hat{B}^{-1} \hat{A} \hat{B}^{-1} J_{\hat{\beta}(\cdot)}^T H\right)^{-1} H^T (\hat{\beta} - \beta_0) \xrightarrow{D} \chi^2_l.
\]

Theorem 5 can be used to construct the large sample confidence region or interval for the parameter \( \beta_0 \).

Applying Corollary 1, we can construct pointwise confidence interval for \( g(u_0) \) at a fixed point \( u_0 \in U_u \). However, we need to use the plug-in estimators for the asymptotic bias and covariance. Obviously, the asymptotic bias and covariance of \( \hat{g}(u_0) \) are dependent on \( \sigma^2_{\tilde{\varepsilon}}, \sigma^2_{\alpha} \) and \( f_j(u_0) \). \( \sigma^2_{\tilde{\varepsilon}} \) and \( \sigma^2_{\alpha} \) have been estimated in (2.9) and (2.10). The estimator of \( f_j(u_0) \), \( j = 1, \ldots, m \), is defined by

\[
\hat{f}_j(u_0) = \frac{1}{nh} \sum_{i=1}^n K((X_{ij} - u_0)/h).
\]

Thus, we can derive \( \hat{\sigma}^2(u_0) \) by replacing \( f_j(u_0) \), \( \sigma^2_{\tilde{\varepsilon}} \) and \( \sigma^2_{\alpha} \) by their consistent estimators \( \hat{f}_j(u_0) \), \( \hat{\sigma}^2_{\tilde{\varepsilon}} \) and \( \hat{\sigma}^2_{\alpha} \) respectively. Therefore, \( \hat{\sigma}^2(u_0) \) is a consistent estimator of \( \sigma^2(u_0) \). By Corollary 1, we have

\[
\sqrt{nh}\{\hat{g}(u_0; \hat{\beta}) - g(u_0)\}/\hat{\sigma}(u_0) \xrightarrow{D} N(0, 1)
\]

Using above result, we can obtain an approximate \( 1 - \alpha \) confidence interval for \( g(u_0) \).
4 Concluding remarks

In this paper we have investigated the inference of single-index mixed models with longitudinal data. We use local linear regression smoothing to estimate the link function, and use the generalized estimating equations to estimate the parametric components. We also construct the estimators of the variance components. The proposed method avoids the need for multivariate distribution by only assuming a functional form on the marginal distribution for each time. The covariance structure across time is treated as a nuisance. A key feature of our approach is that we transform a restricted least squares problem to an unrestricted least squares problem by solving the estimating equations to estimate the parametric components. The asymptotic variance of our estimator for parametric components is the same as that obtained by Wang et al. (2010) in pure single-index models.

In longitudinal studies, sometimes the covariance structure is very complex; that is, the covariance matrix of outcome variable may be of a general form, allowing $V$ to have $\frac{1}{2}m(m-1)$ parameters. Our method can be extended to study this type of problem. In particular, the estimators obtained using our method will be efficient only if the observations on a subject are independent. The estimating equations described in this paper can be considered as an extension of the quasi-likelihood to the case where the second moment cannot be fully specified in terms of the expectation but rather additional correlation parameters must be estimated. It is the independence across subjects that allows us to consistently estimate these nuisance parameters where this could not be done otherwise.

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