Finite groups of symplectic automorphisms
of hyperkähler manifolds of type $K3^{[2]}$

Gerald Höhn
Department of Mathematics, Kansas State University
Geoffrey Mason
Department of Mathematics, University of California at Santa Cruz

September 2014

Abstract
We determine the possible finite groups $G$ of symplectic automorphisms of a
hyperkähler manifold which is deformation equivalent to the second Hilbert
scheme of a K3 surface. We prove that $G$ is isomorphic to a subgroup of either
the Mathieu group $M_{23}$ having at least four orbits in its natural permutation
representation on 24 elements, or one of two groups $3^{1+4}:2.2^2$ and $3^4:A_6$ associated to $S$-lattices in the Leech lattice. We describe in detail those $G$ which
are maximal with respect to these properties, and (in most cases) we determine
all deformation equivalence classes of such group actions. We also compare our
results with the predictions of Mathieu Moonshine.

Contents
1 Introduction 2
2 Background on $K3^{[2]}$ 5
  2.1 Integral lattices ........................................... 5
  2.2 Automorphisms of $K3^{[2]}$ ................................. 6
3 S-lattices and subgroups of $Co_0$ 7
4 Geometric conditions 9
  4.1 Admissible conjugacy classes ................................. 11
  4.2 Symplectic actions of 2-groups ............................. 21

*Supported by the NSF
1 Introduction

A hyperkähler manifold is a compact $4n$-dimensional Riemannian manifold with holonomy group contained in $\text{Sp}(n)$. Such a manifold is of type $K3^[[2]]$ if it is deformation equivalent to the second Hilbert scheme of a K3 surface. An example of a K3 surface is the Fermat quartic $Y \subset \mathbb{CP}^3$ given by the equation $x_0^4+x_1^4+x_2^4+x_3^4 = 0$. An isometry of a hyperkähler manifold is called a symplectic automorphism. See [Hu1] for a review of basic properties of hyperkähler manifolds.

In the present paper we determine and study those finite groups $G$ which can occur as groups of symplectic automorphisms of hyperkähler manifolds of type $K3^[[2]]$. Recent work of Mongardi [Mo1] shows that $G$ is isomorphic to a subgroup of the Conway group $\text{Co}_0$, the group of isometries of the Leech lattice $\Lambda$. Moreover, the fixed-point sublattice $\Lambda^G$ must have rank at least 4. Mongardi also gave (loc. cit.) restrictions on the possible automorphisms of prime order. In a well-known paper [Mu], Mukai showed that a finite group of symplectic automorphisms of a K3 surface...
is isomorphic to a subgroup of the Mathieu group $M_{23}$ having at least five orbits on its defining action on 24 elements. Our main result is the following theorem, which may be regarded as a higher-dimensional analog of Mukai’s result.

**Theorem A.** Let $G$ be a finite group of symplectic automorphisms of a hyperkähler manifold of type $K3^{[2]}$. Then $G$ is isomorphic to one of the following:

(a) a subgroup of $M_{23}$ with at least four orbits in its natural action on 24 elements,
(b) a subgroup of one of two subgroups $3^{1+4}:2.2^2$ and $3^4:A_6$ of $Co_0$ associated to $S$-lattices inside the Leech lattice.

There are 13 isomorphism classes of subgroups of type (a) that are maximal in the poset of all such groups. We will describe them and the two maximal groups of type (b) in detail.

By explicit construction, Mukai showed (loc. cit.) that every subgroup of $M_{23}$ that satisfies the conditions of his theorem indeed occurs as a group of symplectic automorphisms of a $K3$ surface. These groups also act on the corresponding Hilbert schemes, thereby providing examples of groups $G$ as in part (a) of Theorem A and examples explicitly realizing several more of the maximal groups are known. We will establish the full analog of Mukai’s result, namely:

**Theorem B.** Each group $G$ in Theorem A can be realized as group of symplectic automorphisms of some hyperkähler manifold of type $K3^{[2]}$.

Hashimoto has classified [Ha] all the deformation equivalence classes of finite symplectic group actions on $K3$ surfaces. He found that for each group permitted by Mukai’s theorem, there is a unique such class except for five cases where there are two such classes. We obtain a similar result for $K3^{[2]}$.

**Theorem C.** There are at least $243$ deformation classes of finite symplectic group actions on hyperkähler manifold of type $K3^{[2]}$.

We can deduce Theorem C from the following purely lattice theoretic result:

**Theorem D.** Let $L$ be the unique even, integral lattice of signature $(3, 20)$ and discriminant group of order 2. There are at least 243 conjugacy classes of subgroups $G$ of the isometry group $O(L)$ of $L$ such that the orthogonal complement $L_G$ of the fixed-point lattice $L^G$ in $L$ satisfies the following three properties:

(i) $L_G$ is negative-definite;
(ii) $L_G$ contains no vectors of norm $-2$;
(iii) $L_G$ contains no vectors $v$ of norm $-10$ such that $v/2$ is contained in the dual lattice $L^*$. 


The different classes can be read off from Tables 12, 13 and 9.

The methods used in our paper are based on ideas developed by Nikulin, Mukai, Kondō and Hashimoto for K3 surfaces \[ \mathbb{N} \mathbb{M} \mathbb{K} \mathbb{H} \mathbb{L} \mathbb{A} \]. We also use fundamental results on the geometry of hyperkähler manifolds, obtained by many authors in recent decades, including work on the global Torelli theorem due to Huybrechts, Markman and Verbitsky. Recent results of Mongardi are crucial in allowing us to achieve a complete classification.

Much of our interest in the subject matter of the present paper originates from issues surrounding moonshine. Hirzebruch suggested \[ \{HBJ\} \] that the Witten genus of a hypothetical 24-dimensional monster manifold could be related to monstrous moonshine. Furthermore, the equivariant denominator identity of the monster Lie algebra can be interpreted as the equivariant second quantized Witten genus of a monster manifold as noted by the first author \[ \{BH2\} \]. Mathieu Moonshine \[ \{EOT\} \] connects the Mathieu group \( M_{24} \) with the complex elliptic genus of a K3 surface. It seems natural to investigate geometric questions dealing with the equivariant second quantized complex elliptic genus of a K3 surface. See also \[ \{Ch\} \] for a physical interpretation. Moreover, Mathieu Moonshine is closely related to a multiplicative version of Moonshine for \( M_{24} \) investigated by the second author \[ \{Ma1\} \]. Important input also came from recent work of Gaberdiel, Hohenegger and Volpato \[ \{GHV\} \], where the lattice approach of Kondō for K3 surfaces was partially generalized to sigma models on K3 surfaces.

The paper is organized as follows. In Section 2 we cover required background about integral lattices and hyperkähler manifolds of type \( K3^{[2]} \). In Section 3 we discuss the conjugacy classes of subgroups \( G \subseteq \mathbb{C}o_0 \) such that \( \text{rk} \,(\Lambda^G) \geq 4 \). Building on Mongardi’s work, we show in Section 4 that there are exactly 15 conjugacy classes of elements in \( \mathbb{C}o_0 \) that can occur as symplectic automorphisms of \( K3^{[2]} \). This is achieved by applying the equivariant Atiyah-Singer index formula to Hirzebruch’s \( \chi_y \)-genus. In Section 5 a group-theoretic analysis based on this conjugacy restriction then shows that the only groups satisfying the condition and \( \text{rk} \,(\Lambda^G) \geq 4 \) are those described in parts (a) and (b) of Theorem \( \{A\} \) or certain groups of order 12, 24, 48, 16, 32 or 64. Apart from a certain (inevitable) amount of computer calculation, the methods here are an extension of those used in \[ \{Ma2\} \] to study the corresponding problem for K3 surfaces. In Section 6 we show that there are exactly 198 conjugacy classes of such groups in \( \mathbb{C}o_0 \). In Section 7 we determine — apart from a few cases — the conjugacy classes of groups in \( O(L) \) that arise from these 198 conjugacy classes, while in section 8 we determine which of these conjugacy classes arise from symplectic group actions on some \( K3^{[2]} \). In the final section, we compare the equivariant complex elliptic genus of a \( K3^{[2]} \) with the predictions of Mathieu Moonshine applied to the second quantized elliptic genus. In the appendix, we itemize the conjugacy classes of subgroups \( G \subseteq \mathbb{C}o_0 \) found in Section 6 together with additional information about \( G \) and the corresponding lattices \( L^G \).

Acknowledgments. The first author thanks D. Huybrechts for discussions which partially motivated this work, and G. Mongardi for answering questions about his work. We are
grateful to T. Creutzig, M. Gaberdiel, V. V. Nikulin and R. Volpato for useful discussions. The first author enjoyed the hospitality of the Hausdorff Research Institute of Mathematics in Bonn during the early stages of this work, and we are both indebted to the Simons Center for inviting us to the workshop on Mock Modular Forms, Moonshine, and String Theory during the Fall 2013, where part of this work was done. The Simons Foundation also provided us with a license for the computer algebra system Magma, and G. Nebe and D. Lorch helped us by providing certain Magma procedures.

2 Background on \( K3^{[2]} \)

2.1 Integral lattices

We introduce some notation related to integral lattices and record some results that we will need.

Let \( A = (A, q) \) be a finite quadratic space, i.e., a finite abelian group \( A \) together with a quadratic form \( q : A \rightarrow \mathbb{Q}/2\mathbb{Z} \). We denote by \( O(A) \) its orthogonal group, i.e., the subgroup of \( \text{Aut}(A) \) that leaves \( q \) invariant.

Let \( L \) be an even integral lattice, with dual lattice \( L^* \). The discriminant group \( L^*/L \) is equipped with the discriminant form \( q_L : L^*/L \rightarrow \mathbb{Q}/2\mathbb{Z} \), \( x + L \mapsto \langle x, x \rangle \mod 2\mathbb{Z} \). This turns \( L^*/L \) into a finite quadratic space, called the discriminant space of \( L \) and denoted by \( A_L := (L^*/L, q_L) \). We let \( O(L) := \text{Aut}(L) \) be the automorphism group (i.e. group of isometries) of \( L \).

Automorphisms in \( O(L) \) induce orthogonal transformations of the discriminant space \( A_L \). This leads to the short exact sequence

\[
1 \rightarrow O_0(L) \rightarrow O(L) \rightarrow O(L) \rightarrow 1,
\]

where \( O(L) \) is the subgroup of \( O(A_L) \) induced by \( O(L) \) and \( O_0(L) \) consists of the automorphism of \( L \) which act trivially on \( A_L \).

Suppose that \( L \) is an even integral lattice and \( G \subseteq O(L) \) a group of automorphisms. We call \( (L, G) \) a group lattice. Suppose that \( L' \) is another even integral lattice and \( \iota : L \rightarrow L' \) an isomorphism of lattices, i.e. a bijective isometry. Set

\[
\iota[G] := \{ \iota \circ g \circ \iota^{-1} \mid g \in G \}.
\]

Then \( (L', \iota[G]) \) is a group lattice and \( (L, G), (L', \iota[G]) \) are isomorphic group lattices. Upon identifying \( L \) and \( L' \), this just means that \( G \) and \( G' \) are conjugate subgroups of \( O(L) \). To be more precise, \( (L, G) \) and \( (L', \iota[G]) \) are isomorphic objects in a category whose objects are group lattices and whose morphisms \( \iota : (L, G) \rightarrow (L', G') \) are isometric embeddings \( \iota : L \rightarrow L' \) with the property that

\[
\iota[G] \subseteq \{ g' \in G' \mid g' \text{ leaves } \iota(L) \text{ invariant} \}.
\]
The invariant and coinvariant lattices of a group lattice \((L, G)\) are defined as follows:

\[
L^G = \{ x \in L \mid gx = x \text{ for all } g \in G \}, \\
L_G = \{ x \in L \mid (x, y) = 0 \text{ for all } y \in L^G \}
\]

respectively. The invariant and the coinvariant lattice are primitive sublattices of \(L\), i.e., \(L/L^G\) and \(L/L_G\) are free abelian groups. The restriction of the \(G\)-action to \(L_G\) turns it into a group lattice \((L_G, G)\).

As a matter of notation, by \(L(n)\) we will mean a lattice \(L\) with norms scaled by an integer \(n\). We also note that the genus of an even integral lattice \(L\) is determined by the quadratic space \(A_L\) together with the signature of \(L\), cf. [Ni].

### 2.2 Automorphisms of \(K3^2\)

In this subsection, we fix the following notation: \(X\) is a hyperkähler manifold of type \(K3^2\), \(\text{Aut} X\) is the group of symplectic automorphisms of \(X\), and \(G \subseteq \text{Aut} X\) is a finite subgroup.

The second integral cohomology \(L := H^2(X, \mathbb{Z})\) admits a non-degenerate symmetric integral bilinear form \((\cdot, \cdot)\), called the Beauville-Bogomolov form, with respect to which \(L\) is isomorphic to the lattice \(E_8(-1)^2 \oplus U^3 \oplus \langle -2 \rangle\) of signature \((3, 20)\). Here, \(E_8(-1)\) denotes the unique even, unimodular, negative-definite lattice of rank 8, \(U\) the hyperbolic plane and \(\langle -2 \rangle \cong A_1(-1)\) the 1-dimensional lattice generated by a vector of norm \(-2\). The discriminant space \((A_L, q_L) \cong (A_{A_1}, -q_{A_1})\) has order order 2. The group \(O(L)\) is trivial, i.e., \(O(L)\) acts trivially on \(A_L\).

There is an injective map ([Mo1, HS, Be])

\[
\nu : \text{Aut}(X) \rightarrow O(L)
\]

by which we may, and shall, identify \(G\) with its image in \(O(L)\).

**Theorem 2.1** (Mongardi [Mo2], Lemma 3.5). The coinvariant lattice \(L_G\) has the following properties:

(i) \(L_G\) is a negative definite lattice.

(ii) \(L_G\) contains no vectors of norm \(-2\).

It is also shown that \(L_G\) is contained in the Picard lattice of \(X\).

Recall that the Leech lattice \(\Lambda\) is the unique positive-definite, even, unimodular lattice of rank 24 without roots. Its automorphism group is the Conway group \(\text{Co}_0\).

**Theorem 2.2.** There is an embedding of group lattices \((L_G(-1), G) \rightarrow (\Lambda, \text{Co}_0)\) such that \((L_G(-1), G) \cong (\Lambda_G, G)\).
Proof: The discriminant form \( q_L \) of \( L \) is the negative of the discriminant form \( q_{A_1} \) of the root lattice \( A_1 = \langle 2 \rangle \). This permits us to extend the lattice \( L \oplus A_1 \) by the coset \((x, y) \in A_L \oplus A_{A_1}, (x \neq 0, y \neq 0)\) to the unique even unimodular lattice \( M \) of signature \((4, 20)\), thus providing an primitive embedding of \( L \) into \( M \). Since \( O(L) \) is trivial, the \( G \)-action extends to \( M \), thereby fixing the sublattice \( L_G \oplus A_1 \subseteq M \). Because \( L_G \) is negative-definite, \( L_G \oplus A_1 \subseteq M \) has signature \((4, 20 - \text{rk} L_G)\). In particular, we can find a 4-dimensional positive-definite subspace \( \Pi \subset M \otimes \mathbb{R} \) such that \( L_G = \Pi^\perp \cap L \). It was shown in [GHV] (see also [Hu2], Prop. 2.2) that because \( L_G \) is negative-definite, \( G \) can be embedded into the Conway group (a result first shown in [Mo1]). Indeed, the proof actually shows that this corresponds to an embedding \((L_G(-1), G) \rightarrow (\Lambda, \text{Co}_0)\) of group lattices with \((L_G(-1), G) \cong (\Lambda_G, G)\).

Note that because \( \text{rk} L_G \leq 20 \) then \( \text{rk} \Lambda_G \geq 4 \). Theorem 2.2 allows us to identify \( G \) with a subgroup of \( \text{Co}_0 \).

We will see in Section 6 that for a given group lattice \((L_G, G)\), the resulting embedding of \( G \rightarrow \text{Co}_0 \) is unique up to conjugation in \( \text{Co}_0 \).

3 S-lattices and subgroups of \( \text{Co}_0 \)

We have seen in the previous section that a finite group \( G \) of symplectic automorphisms of a hyperkähler manifold \( X \) of type \( K3^{[2]} \) defines a subgroup \( G \subseteq \text{Co}_0 \) with the property that \( \text{rk}(\Lambda_G) \geq 4 \). In this section, we will establish some general results about such \( G \).

Recall [Co] that the \( 2^{24} \) cosets comprising \( \Lambda/2\Lambda \) have representatives \( v \) which may be chosen to be short vectors, i.e., \((v, v) \leq 8\). More precisely, if \((v, v) \leq 6\) then \{\(v, -v\}\) are the only short representatives of \( v + 2\Lambda\); if \((v, v) = 8\) then the short vectors in \( v + 2\Lambda\) comprise a coordinate frame \{\(\pm w_1, \ldots, \pm w_{24}\)\} consisting of pairwise orthogonal vectors \(w_j\) of norm 8. In particular, if \(u \in \Lambda\) then \(u = v + 2w\) for some \(v, w \in \Lambda\) and \(v\) a short vector, and if \((v, v) \leq 6\) then \(v\) is unique up to sign.

It is well-known that \( \text{Co}_0 \) acts transitively on coordinate frames, the (setwise) stabilizer of one such being the monomial group \(2^{12}:M_{24}\).

A sublattice \( S \subseteq \Lambda \) is an \( S \)-lattice if, for every \( u \in S \), the corresponding short vector \( v \) satisfies \((v, v) \leq 6\) and furthermore \(w \in S\). This concept, which was introduced by Curtis [Cu], will be very useful.

The following Theorem depends in an essential way on a result of Allcock [Al]. See also [GHV].

Theorem 3.1. Suppose that \( G \subseteq \text{Co}_0 \) is a subgroup with \( \text{rk} \Lambda^G \geq 4 \). One of the following holds.

(a) \( G \) leaves a coordinate frame invariant,
(b) \( \Lambda^G \) is an \( S \)-lattice of rank 4,
(c) \( \Lambda^G \) is contained in a \( G \)-invariant \( S \)-lattice of rank larger than 4.
Proof: Let $L := \Lambda^G$. We may assume that (a) does not hold. Suppose that $u \in L$ with $u = v + 2w$ and $(v,v) \leq 8$. Then $v + 2\Lambda = u + 2\Lambda$ is $G$-invariant, and since $G$ leaves no coordinate frame invariant then we have $(v,v) \leq 6$. Thus $G$ acts on $\{\pm v\}$. We claim that $v \in L \cup L^\perp$. For if $v \notin L$ there is a $g \in G$ such that $g(v) = -v$. Then for $x \in L$ we obtain $(x,v) = (g(x),g(v)) = (x,-v)$, showing that $v \in L^\perp$.

We use results of Allcock [Al], especially (a special case of) Lemma 4.8 (loc. cit.) which we state as follows: suppose that $L \subseteq \Lambda$ is a primitive sublattice of rank at least 4 with the property that if $u \in L$ with $u = v + 2w$ and $(v,v) \leq 8$, then $(v,v) \leq 6$ and $v \in L \cup L^\perp$. Then $L$ is contained in an $S$-lattice. The previous paragraph establishes that $L = \Lambda^G$ satisfies these properties, so $L$ is contained in an $S$-lattice. Because the family of $S$-lattices containing $L$ is closed under intersection and $G$-conjugation, there is a $G$-invariant $S$-lattice that contains $L$. We denote it by $S$.

If $\text{rk } S = 4$ then $L \subseteq S$ has finite index because of our assumption that $\text{rk } L \geq 4$. Then $L = S$ because $L$ is primitive, and we are in case (b) of the Theorem. Otherwise $\text{rk } S \geq 5$ and (c) holds. This completes the proof of the Theorem.

We now draw some more detailed conclusions concerning the subgroups $G \subseteq \text{Co}_0$ using Theorem 3.1. This depends on Curtis’s classification of $S$-lattices [Cu]. See also [GV] for a related discussion.

**Theorem 3.2.** There are exactly five conjugacy classes of subgroups $G \subseteq \text{Co}_0$ such that $\text{rk } \Lambda^G \geq 4$, $G$ is the full (pointwise) stabilizer of $\Lambda^G$ in $\text{Co}_0$, and $G$ fixes no coordinate frame. They are described as follows.

(i) $\Lambda^G$ is an $S$-lattice of rank 6 and $G \cong 3^{1+4}.2$,
(ii) $\text{rk } \Lambda^G = 5$ and $G \cong 3^{1+4}.2.2$,
(iii) $\Lambda^G$ is an $S$-lattice of rank 4 and $G \cong 3^4.A_6$,
(iv) $\Lambda^G$ is an $S$-lattice of rank 4 and $G \cong 5^{1+2}.4$,
(v) $\text{rk } \Lambda^G = 4$ and $G \cong 3^{1+2}.2^2$.

Proof: Set $L := \Lambda^G$. By Theorem 3.1 there is a $G$-invariant $S$-lattice $S$ with $L \subseteq S$. Let $N$ be the (pointwise) stabilizer of $S$. Because $S$ is $G$-invariant then $G$ normalizes $N$, so since $N$ fixes $L$ pointwise then $N \leq G$. Set $\widetilde{G} := G/N \subseteq \text{Aut}_{\text{Co}_0}(S)$.

The possibilities for $S$ are as follows ([Cu], [CCNPW]):

| $\text{rk } S$ | $N$ | $\text{Aut}_{\text{Co}_0}(S)$ |
|---|---|---|
| 4 | $3^4.A_6$ | $2 \times (S_3 \times S_3).2$ |
| 4 | $5^{1+2}.4$ | $2 \times S_5$ |
| 6 | $3^{1+4}.2$ | $2 \times U_4(2).2$ |

If $S/L$ is finite then $L = S$ because $L$ is primitive, so we have $G = N$ and cases (i), (iii) or (iv) apply. Thus from now on we will assume that $S/L$ is not finite. In
particular, we have $\text{rk} \ S \geq 5$, whence $S$ is the $S$-lattice of rank 6 by Curtis [Cu], moreover $\text{rk} \ L = 4$ or 5. We also have $3^{1+4} \cong O^2(N) \leq G$ and $|\tilde{G}| > 1$.

From the table, the group of isometries of $S$ induced within $\text{Co}_0$ is the group $\{\pm 1\} \times W(E_6)$ (which is actually the full group of isometries). Indeed, a generator of the direct factor $\pm 1$ acts on $S$ as $-1$, and $W(E_6) \cong U_4(2).2$ is the Weyl group of type $E_6$. The lattice $\frac{1}{\sqrt{3}}S$ is isometric to the weight lattice of type $E_6$, the roots corresponding to short vectors of norm 6 in $S$.

If $\text{rk} \ L = 5$ then $\tilde{G}$ fixes a hyperplane pointwise, so that $\tilde{G} \cong \mathbb{Z}_2$ is generated by a reflection in a hyperplane of $S$ orthogonal to a norm 6 vector, and any two such hyperplanes are conjugate in the Weyl group. This is case (ii).

The case $\text{rk} \ L = 4$ requires more care. If $L$ is an $S$-lattice then from the table, it must be that $L$ is of the first kind, i.e. with stabilizer $3^4.A_6$. Although there is a containment of $S$-lattices of this kind ([Cu] or Tables 12 and 13 below), in our set-up we have $3^{1+4} \leq G$, whereas $3^4.A_6$ has no such normal subgroup. Thus $L$ is not an $S$-lattice, and in the proof of Theorem 3.1 we showed that in this situation we can find a nonzero short vector $v \in L^\perp$. Then in the orthogonal $GF(2)$-space $\Lambda := \Lambda/2\Lambda$, $v$ maps onto a nonzero element $\bar{v} \in \text{rad}(\bar{T})$, so that $\bar{T}$ is a 4-dimensional degenerate subspace of the nondegenerate 6-dimensional orthogonal space $\mathfrak{S}$. ($\mathfrak{T}$, $\mathfrak{S}$ are the images of $L$, $S$ respectively in $\mathfrak{T}$.) The pointwise stabilizer of such a degenerate subspace in the full isometry group $O_6^-(2)$ of $\mathfrak{S}$ is a 2-group, and as a result it follows that $\tilde{G}$ is also a 2-group. Since no element of order 4 in $\text{Aut}_{\text{Co}_0}(S)$ fixes a rank 4 sublattice pointwise, then $\tilde{G} \cong \mathbb{Z}_2^k$ for some $k \geq 1$.

Suppose that $\tilde{G} \not\subseteq W(E_6)$. There is a unique conjugacy class of involutions $t$ (of type $-2A$) in $\{\pm 1\} \times W(E_6) \setminus W(E_6)$ fixing a sublattice in $S$ of rank $\geq 4$ ([CCNPW]) and the rank is exactly 4. Now $t$ commutes with a subgroup $R \subseteq \text{Aut}_{\text{Co}_0}(S)$ of order 9 such that $\bar{T} = [\mathfrak{S}, R]$. Because $R$ has odd order, this forces $\bar{T}$ to be nondegenerate, contradiction. So this case does not occur.

The remaining possibility is $\tilde{G} \subseteq W(E_6)$. The only involutions in the Weyl group fixing a sublattice of $S$ of rank $\geq 4$ pointwise are those of type $2B$ and $2C$ ([CCNPW]), the fixed-point ranks being 4 and 5 respectively. Moreover, the product of a pair of distinct commuting Weyl reflections (type $2C$) is of type $2B$. It follows that $L$ is the sublattice of $S$ fixed pointwise by an involution of type $2B$ (so that all such sublattices are conjugate in $\text{Aut}_{\text{Co}_0}(S)$), or equivalently by a pair of commuting Weyl reflections (so that $|\tilde{G}| \geq 4$). Moreover, any subgroup of $\text{Aut}_{\text{Co}_0}(S)$ strictly containing $\tilde{G}$ has fixed sublattice of rank no greater than 3, whence $\tilde{G} \cong \mathbb{Z}_2^2$. This is case (v), and the proof of the theorem is complete.

4 Geometric conditions

In this section, we use geometric arguments to obtain strong restrictions on the group-theoretic properties of finite groups of symplectic automorphism of a hyperkähler manifold of type $K3^{[2]}$. 

9
In the first subsection, we determine which conjugacy classes of $\text{Co}_0$ can arise as symplectic automorphisms. We refer to these conjugacy classes, and the elements in them, as the **admissible conjugacy classes** and **admissible elements** respectively. The remaining conjugacy classes and elements are called **inadmissible**. The main result (Theorem 4.9) asserts that there are just 15 admissible conjugacy classes. We also determine the structure of the fixed-point set of the admissible elements and the action on the normal bundle.

In the second subsection, we show that a 2-group of symplectic automorphisms has order at most $2^7$.

### 4.1 Admissible conjugacy classes

As explained in the previous section, a finite group $G$ of symplectic automorphisms of a hyperkähler manifold of type $K3^2$ can be identified with a subgroup of $\text{Co}_0$. After making this identification, the primitive embedding of the coinvariant lattice $L_G$ into the Leech lattice $\Lambda$ is such that $\text{rk}(\Lambda^g) \geq 4$. Obviously then, we have $\text{rk}(\Lambda^g) \geq 4$ for every $g \in G$.

We start with Table 1, which lists the 42 conjugacy classes $[g]$ of $\text{Co}_0$ that satisfy the condition $\text{rk}(\Lambda^g) \geq 4$, together with some supplementary data. It transpires that such a $g$ is uniquely specified by the triple (order of $(g)$, Trace$(g)$, Trace$(g^2)$), and this is the entry in the first column of the table. In what follows, we often identify a conjugacy class using this triple. The second column is the **Frame shape** of $g$, the third column gives $\text{rk}(\Lambda^g)$, the fourth column the **torsion-invariants** of $A_{\Lambda^g} = (\Lambda^g)^*/\Lambda^g$, and the fifth column (‘powers’) the nontrivial prime powers of $g$. Column six records whether $g$ belongs (up to conjugacy) to the monomial subgroup $2^{12} : M_{24} \subseteq \text{Co}_0$ (* indicates that it does). Finally, in the seventh column (‘excluded’) the symbol $3.x$ refers to the Lemma or Theorem 3.x below by which the inadmissible elements are excluded.

Columns 1, 2, 3, 5 and 6 in Table 1 can be read-off from the Atlas [CCNPW]. The structure of the fixed-point lattice has been investigated in [KT1, KT2, La, HL]. The table was also verified using Magma [Mag] together with a realization of $\text{Co}_0$ as a matrix group.

The following observation is clear:

**Remark 4.1.** If a conjugacy class $[g]$ is inadmissible, then so is $[h]$ whenever $g$ is a power of $h$.

The lattice-theoretic set-up leads to a condition on the discriminant group.

**Lemma 4.2.** Let $A_{\Lambda^g} = (\Lambda^g)^*/\Lambda^g$ be the discriminant group of $\Lambda^g$ with quadratic form $q_{A^g} : A_{\Lambda^g} \to \mathbb{Q}/\mathbb{Z}$. Suppose that $\text{rk}(A_{\Lambda^g}) = \text{rk}(\Lambda^g)$. Then the following hold:

1. $g$ has **Frame shape** $1^{m_1}2^{m_2}\ldots$ if its characteristic polynomial (considered as a linear transformation of $\Lambda \otimes \mathbb{R}$) is $(t - 1)^{m_1} (t^2 - 1)^{m_2}\ldots$. Each $m_i \in \mathbb{Z}$. 

---

10
| class $[g]$ | Frame shape $r k \Lambda^g$ | $A_{A^g}$ powers | $2^{12}:M_{24}$ excluded |
|-------------|----------------------------|------------------|--------------------------|
| (1, 24, 24) | $1^{24}$ 24 $1$ | * | – |
| (2, 8, 24)  | $2^8 2^1$ 16 $2^8$ | * | – |
| (2, 0, 24)  | $2^{12}$ 12 $2^{12}$ | * | – |
| (2, −8, 24) | $2^{16}/1^8$ 8 $2^8$ | * | 111 |
| (3, 6, 6)   | $1^6 3^6$ 12 $3^6$ | * | – |
| (3, 0, 0)   | $3^8$ 8 $3^8$ | * | 13 |
| (3, −3, −3) | $3^3/1^3$ 6 $3^5$ | No | – |
| (4, 8, −8)  | $1^8 4^8/2^8$ 8 $2^8$ (2, −8, 24) | * | 111 113 |
| (4, 4, 8)   | $1^4 2^4 4^4$ 10 $2^4 4^4$ (2, 8, 24) | * | – |
| (4, 0, 8)   | $2^{14}$ 8 $2^4 1^4$ (2, 8, 24) | * | 111 |
| (4, 0, −8)  | $4^8/2^4$ 4 $2^2 4^2$ (2, −8, 24) | * | 111 |
| (4, 0, 0)   | $4^6$ 6 $4^6$ (2, 0, 24) | * | 111 113 |
| (4, −4, 8)  | $2^6 4^4/1^4$ 6 $2^4 4^4$ (2, 8, 24) | * | 111 113 |
| (5, 4, 4)   | $1^4 5^4$ 8 $5^4$ | * | – |
| (5, −1, −1) | $5^5/1$ 4 $5^3$ | No | 113 |
| (6, 5, −3)  | $1^5 3.6^4/2^4$ 6 $3^5$ (2, 8, 24), (3, −3, −3) No | – |
| (6, 4, 6)   | $1^2 2^6 3^3/3^4$ 6 $2^6 1^1$ (2, −8, 24), (3, 6, 6) | * | 111 113 |
| (6, 2, 6)   | $1^2 2^3 3^6 2^2$ 8 $2^4 6^2$ (2, 8, 24), (3, 6, 6) | * | – |
| (6, 0, 6)   | $2^6 3$ 6 $2^6 3$ (2, 0, 24), (3, 6, 6) | * | 111 |
| (6, 0, 0)   | $6^2$ 6 $6^2$ (2, 0, 24), (3, 0, 0) | * | 111 113 113 |
| (6, −2, 6)  | $2^6 4^4/1^2 3^2$ 4 $2^2 6^2$ (2, −8, 24), (3, 6, 6) | * | 111 113 113 |
| (6, −1, −3) | $3^6 1^1/1.2$ 4 $3^2 6^2$ (2, 8, 24), (3, −3, −3) No | 113 |
| (6, −4, 6)  | $2^3 4^6/1^4$ 6 $2^1 6^5$ (2, 8, 24), (3, 6, 6) | * | 113 |
| (7, 3, 3)   | $1^7 3$ 6 $7^3$ | * | – |
| (8, 4, 0)   | $1^4 8^4/2^4 4^2$ 4 $2^2 4^2$ (4, 0, −8) | * | 111 |
| (8, 0, 8)   | $2^4 4^4$ 4 $4^4$ (4, 8, −8) | * | 111 113 |
| (8, 2, 4)   | $1^2 2.4.8^2$ 6 $2^4 1^4 8^2$ (4, 4, 8) | * | – |
| (8, 0, 0)   | $8^4/2^4$ 4 $4^8 2^4$ (4, 0, 8) | * | 111 113 |
| (8, −2, 4)  | $2^1 4^4 8^2/1^2$ 4 $2^1 4^4 8^2$ (4, 4, 8) | * | 111 |
| (9, 3, 3)   | $1^9 3^3/2^3$ 4 $3^9 1^3$ (3, −3, −3) No | – |
| (10, 3, −1) | $1^3 5.10^2/2^2$ 4 $5^3$ (2, 8, 24), (5, −1, −1) No | 111 |
| (10, 2, 4)  | $1^2 2.10^3/5^2$ 4 $2^3 10^1$ (2, −8, 24), (5, 4, 4) | * | 111 113 |
| (10, 0, 4)  | $2^4 10^2$ 4 $2^2 10^2$ (2, 0, 24), (5, 4, 4) | * | 111 |
| (10, −2, 4) | $2^5 10^2/1^2$ 4 $2^1 10^3$ (2, 8, 24), (5, 4, 4) | * | 113 |
| (11, 2, 2)  | $1^2 11^2$ 4 $1^2 1^2$ | * | – |
| (12, 2, 2)  | $1^2 4.6^2 12^3/3^2$ 4 $2^2 4^1 12^1$ (4, −4, 8), (6, 2, 6) | * | 111 |
| (12, 2, −2) | $1^2 3^2 4^2 12^2/2^2 6^2$ 4 $2^6 2^2$ (4, 8, −8), (6, −2, 6) | * | 111 113 |
| (12, 1, 5)  | $1^2 2.3.12^3/4^2$ 4 $3^1 6^2$ (4, 4, 8), (6, 5, −3) No | – |
| (12, 0, 2)  | $2.4.6.12$ 4 $2^2 12^2$ (4, 0, 8), (6, 2, 6) | * | 111 |
| (12, −2, 2) | $2^2 3^2 4.12^2/1^2$ 4 $2^4 6^1 12^2$ (4, 4, 8), (6, 2, 6) | * | 111 113 |
| (14, 1, 3)  | $2^1 7.14$ 4 $14^2$ (2, 8, 24), (7, 3, 3) | * | – |
| (15, 1, 1)  | $1^3 5.15$ 4 $15^4 1^1$ (3, 6, 6), (5, 4, 4) | * | – |
(a) The discriminant group of $L^g$ has index 2 in $A_{A^g}$.

(b) $q_{A^g}$ has one of the values $\frac{1}{2}, \frac{3}{2}$ in its image.

We defer the proof until Section 7.

**Lemma 4.3.** Conjugacy classes of type $(2, -8, 24)$, $(3, 0, 0)$, $(4, 0, 0)$, $(4, 8, -8)$, $(6, -1, -3)$, $(6, 0, 0)$, $(6, -4, 6)$, $(6, -2, 6)$, $(6, 4, 6)$, $(8, 0, 8)$, $(8, 0, 0)$, $(10, -2, 4)$, $(10, 2, 4)$ and $(12, 2, -2)$ are inadmissible.

**Proof:** Inspection of the structure of $A_{A^g}$ in Table 1 together with Lemma 4.2 (a) eliminates types $(3, 0, 0)$, $(4, 0, 0)$, $(6, -1, -3)$, $(6, 0, 0)$, $(8, 0, 8)$ and $(8, 0, 0)$. The additional types $(2, -8, 24)$, $(4, 8, -8)$, $(6, -4, 6)$, $(6, -2, 6)$, $(6, 4, 6)$, $(10, -2, 4)$, $(10, 2, 4)$, and $(12, 2, -2)$ are excluded by Lemma 4.2 (b) since a computer calculation shows that $q_{A^g}$ has neither the value $\frac{1}{2}$ nor $\frac{3}{2}$ in its image. \hfill \Box

Suppose that $g$ is a symplectic automorphism of a hyperkähler manifold of type $K3[2]$. We say that $g$ is of $K3$-type if there is a symplectic automorphism $h$ of a $K3$-type such that $g$ is conjugate in $Co_0$ to the element defined by $h$. We also say that the elements and conjugacy classes in $Co_0$ corresponding to $g$ are themselves of $K3$-type. For symplectic automorphisms of $K3$ surfaces we have $\text{rk} \ (\Lambda^g) \geq 5$. [Mu], while the analog of part a) Lemma 4.2 is $\text{rk} \ (A_g) \leq \text{rk} \ (\Lambda^g) - 2$. Then examination of Table 1 establishes the next Remark:

**Remark 4.4.** There are 8 conjugacy classes in $Co_0$ of $K3$-type, namely $(1, 24, 24)$, $(2, 8, 24)$, $(3, 6, 6)$, $(4, 4, 8)$, $(5, 4, 4)$, $(6, 2, 6)$, $(7, 3, 3)$ and $(8, 2, 4)$. \hfill \Box

Nikulin [Ni] first proved that the order of a (finite order) symplectic automorphism of $K3$ is at most 8. See also [Ma2], [Mu].

To exclude further cases beyond Lemma 4.3 we apply the equivariant Atiyah-Singer theorem to the Hirzebruch $\chi_y$-genus of the hyperkähler manifold $X$ of type $K3[2]$. Let $g \in \text{Aut}(X)$ have finite order $n$ and let

$$
\chi_y(g; X) := \sum_{p, q=0}^{4} (-1)^q \text{Tr}(g|H^{p,q}(X)) y^p
$$

be the equivariant $\chi_y$-genus.

**Lemma 4.5.** Let $t = \text{Tr}(g|H^{1,-1}(X))$ and $s = \text{Tr}(g^2|H^{1,1}(X))$. Then

$$
\chi_y(g; X) = 3 - 2ty + \frac{6 + t^2 + s}{2} y^2 - 2ty^3 + 3y^4.
$$

**Proof:** Inspection of the Hodge diamond of $X$ shows that the only nontrivial contributions one has to know are those coming from $H^{1,1}(X)$ and $H^{2,2}(X)$. The remainder then follow from the symmetries of the Hodge diamond, which holds equivariantly.
But $H^{2,2}(X) \cong \mathbb{C} \oplus S^2H^{1,1}(X)$. Together with the formula for the character of a symmetric square, this gives the result. For further details, see Camere [Ca].

We note that $\text{Tr}(g|\Lambda) = \text{Tr}(g|H^{1,1}(X)) + 3$. Moreover from Table 1 we see that $t$ and $s$ are rational integers.

There is a basic result regarding the structure of the fixed-point set $X^g$.

**Theorem 4.6** (cf. Camere [Ca], Proposition 3). *The fixed-point set of a finite symplectic automorphism of a hyperkähler manifold is the disjoint union of finitely many components, which are themselves hyperkähler manifolds. The centralizer of such an automorphism acts by symplectic automorphisms on the fixed-point set.*

Since the only connected 4-dimensional hyperkähler manifolds are K3-surfaces and complex 2-tori, it follows that the fixed-point set of a non-trivial finite symplectic automorphism on an 8-dimensional hyperkähler manifold consists of isolated points, complex 2-tori and K3-surfaces.

The occurrence of 2-tori can sometimes be excluded by the following geometric result of Mongardi.

**Theorem 4.7** (Mongardi [Mo1], Proposition 5.1.4). *Let $g$ be a symplectic automorphism of finite order of a hyperkähler manifold $X$ of type $K3[2]$, and suppose that $X^g$ contains a torus. Then $\text{rk}(L^g) \leq 6$.***

**Corollary 4.8.** Suppose that $g$ lies in one of the Conway classes $(2,8,24), (2,0,24), (3,6,6), (3,0,0), (4,8,-8), (4,4,8), (4,0,8), (5,4,4)$ or $(6,2,6)$. Then the components of $X^g$ are isolated fixed-points or K3 surfaces.

**Proof:** For all of these choices of $g$, $\text{rk} L^g = \text{rk} \Lambda^g - 1 \geq 7$ (cf. column 3 of Table 1), and the Theorem applies.

To compute $\chi_y(g; X)$ using the equivariant Atiyah-Singer index theorem, we have to know the possible eigenvalues for the action of $g$ on the normal bundle in $X$ of a component $F$ of $X^g$. Let $\zeta = e^{2\pi i/n}$ (where $g$ has order $n$).

Since the structure group of the tangent bundle of $X$ can be reduced to $Sp(2) \subset SU(4) \subset U(4)$ there are the following possibilities:

$F$ is an isolated fixed-point $p$. The possible eigenvalues for $g$ acting on $T_p X$ are $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j}), 0 < i \leq j \leq n/2$.

$F$ is a K3 surface or 2-torus. The possible eigenvalues for $g$ acting on the normal bundle $N$ of $F$ in $X$ are $(\zeta^i, \zeta^{-i}), 0 < i \leq n/2$.

By the equivariant fixed-point theorem (cf. [HBJ]) one has the following formula:

$$
\chi_y(g; X) = \sum_{F \subset X^g} \prod_{k=1}^{\dim F} \frac{x_k(1 + ye^{-x_k})}{1 - e^{-x_k}} \prod_{k=1}^{4-\dim F_C} \frac{1 + y\lambda_k e^{-x_k'}}{1 - \lambda_k e^{-x_k'}} [F],
$$

(1)
the sum running over the components $F$ of $X^g$. The $x_k$ and $x'_k$ are the formal roots of the total Chern classes of the tangent and normal bundle of $F$ respectively, and the $\lambda_k$ are the eigenvalues of $g$ acting on the normal bundle. We will evaluate the right-hand-side for each type of fixed-point component and $g$-action on the normal bundle.

For an isolated fixed-point $p$ of type $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j})$, the contribution is

$$f_{i,j} := \frac{1 + y\zeta^i}{1 - \zeta^i} \cdot \frac{1 + y\zeta^{-i}}{1 - \zeta^{-i}} \cdot \frac{1 + y\zeta^j}{1 - \zeta^j} \cdot \frac{1 + y\zeta^{-j}}{1 - \zeta^{-j}}.$$

For a complex 2-dimensional surface $F$, the total Chern class of $F$ is $c(TF) = (1 + x_1)(1 + x_2) = 1 + c_2(TF)$ since $c_1(TF) = 0$. A short calculation expanding $e^{-x_k}$ up to order 2 shows that the contribution from the tangent bundle in (1) is the factor

$$h_0 := \frac{x_1(1 + ye^{-x_1})}{1 - e^{-x_1}} \cdot \frac{x_2(1 + ye^{-x_2})}{1 - e^{-x_2}} = (1 + y)^2 + (2 - 20y + 2y^2) \cdot \frac{c_2(F)}{12}.$$

For the normal bundle $N$ with $g$ acting with eigenvalues $(\zeta^i, \zeta^{-i})$, we have to express

$$h_i := \frac{1 + y\zeta^i e^{-x_1}}{1 - \zeta^i e^{-x_1}} \cdot \frac{1 + y\zeta^{-i} e^{-x_2}}{1 - \zeta^{-i} e^{-x_2}}$$

in the total Chern class $c(N) = (1 + x'_1)(1 + x'_2)$ of the normal bundle. One obtains

$$h_i = -\frac{\zeta^i + y + \zeta^2i y + \zeta^i y^2}{(\zeta^i - 1)^2} - \frac{\zeta^i(\zeta^i + 1)(y + 1)^2}{(\zeta^i - 1)^3} \cdot x'_1 - \frac{\zeta^2i(\zeta^2i + 4\zeta^i + 1)(y + 1)^2}{2(\zeta^i - 1)^4} \cdot (x'_1)^2,$$

where we have also used that $x'_1 + x'_2 = c_1(N) = c_1(TX)|_F - c_1(TF) = 0$. Note that if $g$ is an involution, the linear term for $x'_1$ vanishes. Otherwise, $N$ splits canonically into two eigenspace bundles and $x'_1$ is well-defined in this case.

Assume that there are $a_{i,j}$ isolated fixed-points of type $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j})$ and $b_i$ fixed-point components which are K3 surfaces of type $(\zeta^i, \zeta^{-i})$. The right hand side of (1) equals

$$\sum_{i,j} a_{i,j} \cdot f_{i,j} + \sum_i \sum_{F \subset \Phi_i} h_0 h_i[F] + \sum_i \sum_{F \subset \Psi_i} h_0 h_i[F],$$

where $\Phi_i$ and $\Psi_i$ denote the set of fixed-point components $F \subset X^g$ which are K3-surfaces resp. 2-tori of type $(\zeta^i, \zeta^{-i})$. Using $- (x'_1)^2 = c_2(N) = c_2(TX)|_F - c_2(TF)$ and $c_2(TF)[F] = 24$ for $F$ a K3 surface resp. $c_2(TF)[F] = 0$ for $F$ a 2-tori, we see that for fixed $i$, the sum $\sum_{F \subset \Phi_i \cup \Psi_i} h_0 h_i[F]$ depends only on $b_i$ and the sum $C_i := \sum_{F \subset \Phi_i \cup \Psi_i} c_2(TX|_F)[F]$. Thus (1) becomes

$$\chi_g(g; X) = \sum_{0 < i \leq n/2} a_{i,j} \cdot f_{i,j} + \sum_{0 < i \leq n/2} b_i \cdot b_i + \sum_{0 < i \leq n/2} C_i \cdot \gamma_i, \quad (2)$$
with explicit polynomials $\beta_i$ and $\gamma_i$ in $y$ and $\zeta$. These give $3^\varphi(n)$ linear equations (possibly trivial and linearly dependent) for the integers $a_{i,j}$, $b_i$ and $C_i$ since there are $3$ independent rational coefficients in the palindromic polynomial $\chi_g(g; X)$, and the right-hand-side is a polynomial in $y$ with coefficients in the cyclotomic field $Q(\zeta)$ of degree $\varphi(n)$ over $Q$. In addition, the $a_{i,j}$ and $b_i$ are non-negative.

If $h$ is a power of $g$ then the fixed-point configurations of $g$ and $h$ and the actions on the normal bundles are related. As before, let $n$ be the order of $g$ and let $h = g^k$ for $k | n$, $k < n$. Consider an isolated fixed-point $p$ for which $g$ acts with eigenvalues $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j})$ in the tangent space. Then $p$ is also a fixed-point of $h$ and $h$ acts with eigenvalues $(\zeta^{ik}, \zeta^{-ik}, \zeta^{jk}, \zeta^{-jk})$ in the tangent space. If both, $\zeta^{ik}$ and $\zeta^{jk}$, are different from 1 then $p$ is also an isolated fixed-point of $h$. If one of them is equal to 1, then $p$ belongs to a 4-dimensional fixed-point set, i.e. a K3 surface or a 2-torus. The case that $\zeta^{ik} = \zeta^{jk} = 1$ is impossible, since otherwise $h = 1$. If $p$ is a fixed-point of $g$ belonging to a higher-dimensional fixed-point component $F$ of $g$, then $h$ acts with the $(\zeta^{ik}, \zeta^{-ik})$ in the normal bundle and $\zeta^{ik}$ must be different from 1, i.e., $\zeta$ is necessarily a primitive $n$-th root of unity. Note that $h$ can have additional fixed-point components besides the one described above.

It will turn out that for all classes of Table 1 there is at most one possible fixed-point configuration. Thus by analyzing the fixed-point structure and the action of $g$ on the normal bundle, we can apply the information previously obtained to all non-trivial powers of $g$. This gives several restrictions on the possible fixed-point components and the eigenvalues.

For a given conjugacy class, our approach now is to solve the resulting system of linear equations and inequalities. This can be done in a straightforward way with the help of a computer, although the number of equations and variables will become quite large. From these calculations, together with some additional geometric results, we obtain the following main theorem:

**Theorem 4.9.** A symplectic automorphism $g$ of finite order of a hyperkähler manifold of type $K3^{[2]}$ belongs to one of the 15 Co0 conjugacy classes $(1, 24, 24), (2, 8, 24), (3, 6, 6), (3, -3, -3), (4, 4, 8), (5, 4, 4), (6, 2, 6), (6, 5, -3), (7, 3, 3), (8, 2, 4), (9, 3, 3), (11, 2, 2), (12, 1, 5), (14, 1, 3), (15, 1, 1)$. If $g$ is of type $(2, 8, 24)$, the fixed-point set contains a unique K3-surface; if $g$ is of type $(3, -3, -3)$, the fixed-point set contains a 2-torus; for all other $g \neq 1$, the fixed-point set consists of isolated fixed-points. The complete description of the fixed-point sets and the action on the normal bundles is given in Table 2.

In the following, we discuss the proof in more detail.

For involutions, the fixed-point formula was first used by Camere. She obtained the following result, which we verified with our program.

**Theorem 4.10** (Camere [Ca]). Let $g$ be a symplectic involution of a hyperkähler manifold $X$ of type $K3^{[2]}$. Then $g$ is of type $(2, 0, 24)$, $(2, 6, 24)$ or $(2, 8, 24)$ and the corresponding fixed-point sets are as follows:
Table 2: Admissible classes and corresponding fixed point configurations

| class of g   | # of components of a certain type | prime powers |
|-------------|-----------------------------------|--------------|
| (1, 24, 24)* | X                                 |              |
| (2, 8, 24)*  | $28 \times (-1, -1)$, $K3$       |              |
| (3, 6, 6)*   | $27 \times (\zeta_3, \zeta_3)$   |              |
| (3, −3, −3)†| $T^2$                             |              |
| (4, 4, 8)*   | $8 \times [(i, i), (i, -1)]$     | (2, 16, 8)*  |
| (5, 4, 4)*   | $(\zeta_5, \zeta_5), (\zeta_5^2, \zeta_5^2), 12 \times (\zeta_5, \zeta_5^2)$ | (2, 8, 24)*, (3, 6, 6)* |
| (6, 2, 6)*   | $(\zeta_6, \zeta_6), 6 \times (\zeta_6, \zeta_6^2)$ | (2, 8, 24)*, (3, 6, 6)* |
| (6, 5, −3)† | $10 \times (\zeta_6, \zeta_6^3), 6 \times (\zeta_6^2, \zeta_6^3)$ | (2, 8, 24)*, (3, −3, −3)† |
| (7, 3, 3)*   | $3 \times [(\zeta_7, \zeta_7^2) (\zeta_7^2, \zeta_7^2) (\zeta_7^2, \zeta_7^2)]$ | (2, 8, 24)*, (3, −3, −3)† |
| (8, 2, 4)*   | $2 \times [(i, \zeta_8), (i, \zeta_8^3), (\zeta_8, \zeta_8^3)]$ | (4, 4, 8)* |
| (9, 3, 3)†  | $3 \times [(\zeta_9, \zeta_9^3), (\zeta_9^2, \zeta_9^3), (\zeta_9^3, \zeta_9^3)]$ | (3, −3, −3)† |
| (11, 2, 2)   | $(\zeta_{11}, \zeta_{11}^2), (\zeta_{11}, \zeta_{11}^4), (\zeta_{11}^3, \zeta_{11}^4), (\zeta_{11}^3, \zeta_{11}^4), (\zeta_{11}^3, \zeta_{11}^4)$ |              |
| (12, 1, 5)† | $(\zeta_{12}, \zeta_{12}^3), (\zeta_{12}^2, \zeta_{12}^3), 2 \times (\zeta_{12}^2, \zeta_{12}^3)$ | (6, 5, −3)†, (4, 4, 8)* |
| (14, 1, 3)   | $(\zeta_{14}, \zeta_{14}^3), (\zeta_{14}^2, \zeta_{14}^3), (\zeta_{14}^2, \zeta_{14}^3)$ | (2, 8, 24)*, (7, 3, 3)* |
| (15, 1, 1)   | $(\zeta_{15}, \zeta_{15}^3), (\zeta_{15}^2, \zeta_{15}^3)$ | (3, 6, 6)*, (5, 4, 4)* |

Notation: * means element is of K3-type and contained in $M_{24}$; † means element is not in $2^{12}:M_{24}$.

(2, 0, 24): 12 isolated fixed-points and at least one complex torus,
(2, 6, 24): 36 isolated fixed-points and at least one complex torus,
(2, 8, 24): 28 isolated fixed-points, one K3 surface and an undetermined number of complex tori.

Combining this with the other information, we obtain

Lemma 4.11 (Mongardi [Mo1], Theorem 6.2.3). The symplectic automorphisms of order 2 have type (2, 8, 24).

Proof: The case (2, −8, 24) of Table I cannot occur by Camere’s theorem, (2, 6, 24) cannot occur since it is absent from Table I and (2, 0, 24) is excluded by Corollary 4.8. The only remaining possibility from Table I is (2, 8, 24).

Another application of Corollary 4.8 shows if $g$ is an involution, then the components of $X^g$ are either isolated fixed-points or K3 surfaces. Therefore we have:

Lemma 4.12. If $g$ is a symplectic automorphism of $X$ of even order, then the components of $X^g$ are either isolated fixed-points or K3 surfaces.
Next we consider symplectic automorphisms of order 4.

**Proposition 4.13.** Let $g$ be an order 4 symplectic automorphism of a hyperkähler manifold $X$ of type $K3[2]$. Then $g$ is of type $(4,4,8)$, and $X^g$ consists of 16 isolated fixed-points. There are 8 fixed-points with eigenvalues $(i, -i, i, -i)$ and 8 fixed-points with eigenvalues $(i, -i, -1, -1)$.

**Proof:** Since $g^2$ has order 2, we know from Lemma 4.11 that $g^2$ has type $(2,8,24)$, whence $g$ is of type $(4,4,8)$, $(4,0,8)$ or $(4,-4,8)$ (cf. Table 1). By Lemma 4.12, the only fixed-point components are isolated fixed-points or K3 surfaces. Since $g^2$ is not the identity we have $a_{2,2} = b_2 = 0$. Thus we must solve eqn. (2) for the four variables $a_{1,1}$, $a_{1,2}$, $b_1$ and $C_1$. For the right-hand-side of (2) we obtain

$$\frac{a_{1,1}}{4}(1+y^2)^2 + \frac{a_{1,2}}{8}(-1+y)^2(1+y^2) - b_1(11(1+y^4) + 58(y+y^3) + 70y^2) + \frac{C_1}{8}(1+y)^4.$$  

The left-hand-side is given by Lemma 4.5 where the value for $t$ and $s$ can be read off from the type $(n,t,s)$ of $g$.

For $g$ of type $(4,4,8)$ one obtains three linear equations

\[
\begin{align*}
\frac{1}{4} a_{1,1} + \frac{1}{8} a_{1,2} - 11 b_1 + \frac{1}{2} C_1 &= 3, \\
-\frac{1}{4} a_{1,2} - 58 b_1 + 2 C_1 &= -2, \\
\frac{1}{2} a_{1,1} + \frac{1}{4} a_{1,2} - 70 b_1 + 3 C_1 &= 6
\end{align*}
\]

with the solutions $a_{1,1} = \frac{3}{2}(C_1 + 12)$, $a_{1,2} = \frac{1}{3}(24 - 5C_1)$, and $b_1 = \frac{1}{21} C_1$. Using the inequalities $a_{1,1} \geq 0$, $a_{1,2} \geq 0$, $b_1 \geq 0$ for integral $C_1$ shows that $C_1 \in \{0,1,2,3,4\}$. Only for $C_1 = 0$ do we obtain integer solutions $a_{1,1} = a_{1,2} = 8$ and $b_1 = 0$. In particular, $g$ must have isolated fixed-points.

The same approach for $g$ of type $(4,0,8)$ or $(4,-4,8)$ gives no solutions. (Moreover, this also holds for the cases of type $(4,8,-8)$, $(4,0,-8)$, and $(4,0,0)$, though they are already excluded.) This completes the proof of the proposition. □

We remark that the 8 fixed-points of an order four element $g$ with eigenvalues $(i, -i, -1, -1)$ necessarily lie on the $K3$ fixed-point component of $g^2$, and $g$ acts on this K3 surface with 8 isolated fixed-points. This is in agreement with results of Nikulin and Mukai for symplectic automorphisms of K3 surfaces.

A similar approach will handle the elements of order 8:

**Proposition 4.14.** Let $g$ be a symplectic automorphism of a hyperkähler manifold $X$ of type $K3[2]$ of order 8. Then $g$ is of $K3$-type $(8,2,4)$, acting with 6 isolated fixed-points and eigenvalues as in Table 4.

**Proof:** By Proposition 4.13, $g^2$ has type $(4,4,8)$, whence $g$ is of type $(8,2,4)$ or $(8,-2,4)$ by Table 1. Moreover, since $g^2$ has to act with isolated fixed-points, the same is true for $g$, and $a_{1,4} = a_{2,4} = a_{3,4} = a_{4,4} = 0$. Since $g^4 \neq 1$ then $a_{2,2} = 0$. Solving eqn. (1) for the remaining variables $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,3}$, and $a_{3,3}$ gives for
$g$ of type $(8, 2, 4)$ a unique solution $a_{1,1} = 0, a_{1,2} = 2, a_{1,3} = 2, a_{2,3} = 2, a_{3,3} = 0,$ and for $g$ of type $(8, 2, -4)$ a unique solution $a_{1,1} = 1, a_{1,2} = 2, a_{1,3} = -4, a_{2,3} = 2, a_{3,3} = 1.$ The latter case is impossible since $a_{1,3}$ is negative, and the Proposition is proved.

Note that for an element $g$ of order 8, $g^4$ is an involution which has a K3 surface as a fixed-point component on which $g$ acts with 4 fixed-points with eigenvalues $(i, -i)$, as required.

At this point, application of Remark 4.1, Lemmas 4.3 and 4.11, and Propositions 4.13 and 4.14 leaves only types $(5, -1, -1), (10, 3, -1)$ and $(12, -2, 2)$ from Table 1 to be eliminated. To deal with the prime 5 we use:

**Proposition 4.15** (Mongardi [Mo1], Thm. 6.2.9). Let $g$ be a symplectic automorphism of a hyperkähler manifold of type $K3^{[2]}$ of order 5. Then $g$ is of type $(5, 4, 4)$ acting with 14 isolated fixed-points and eigenvalues as in Table 2.

The eigenvalues for $g$ have been implicitly determined in the proof of [Mo1], Thm. 6.2.9. We confirmed the calculation with our computer program.

Mongardi’s result means that elements of type $(5, -1, -1)$ are inadmissible, and those of type $(10, 3, -1)$ are also inadmissible because they have squares of type $(5, -1, -1)$ (cf. Table 1).

**Proposition 4.16.** Let $g$ be a symplectic automorphism of a hyperkähler manifold of type $K3^{[2]}$ of order 12. Then $g$ is of type $(12, 1, 5)$ acting with 4 isolated fixed-points and eigenvalues as in Table 3.

**Proof:** The only conjugacy classes for $g$ having $g^3$ of type $(4, 4, 8)$ and $g^2$ of type $(6, 2, 6)$ or $(6, 5, -3)$ are those of type $(12, -2, 2)$ and $(12, 1, 5)$, respectively. Since $g^3$ acts by isolated fixed-points the same holds for $g$. Since $g^2$ acts by isolated fixed-points we have $a_{i,6} = 0 (1 \leq i \leq 6).$ Since $g^6 \neq 1,$ $a_{2,2}, a_{2,4}, a_{4,4}$ also vanish, and since $g^4 \neq 1$ then $a_{3,3} = 0.$ In addition, if $g$ is of type $(12, -2, 2)$ then $g^4$ has only isolated fixed-points and $a_{1,3} = a_{2,3} = a_{3,4} = a_{3,5} = 0.$

For $g$ of type $(12, -2, 2),$ [2] has no solution such that the remaining seven variables are non-negative. For $g$ of type $(12, 1, 5),$ the only solution such that the remaining eleven variables are non-negative and integral is $a_{1,3} = 1, a_{3,5} = 1, a_{2,3} = 2$ and 0 for the other eight variables. The proposition follows.

For an order 12 element $g$, $g^6$ is an involution with a K3 surface as one of the fixed-point components. $g$ acts on this with 2 fixed-points, as required. $g^4$ has order 3 element and a 2-torus as one fixed-point component on which $g$ acts with 4 fixed-points.

This completes the proof that the admissible classes of symplectic automorphisms are those listed in Theorem 4.9.

Next we discuss the fixed-point configuration for the classes in Table 2 not yet covered. The cases when $g$ has order 3, 7 or 11 are covered by Mongardi [Mo1] (the
fixed-point sets are also described) and we merely state the result in these cases. By
our method based on (2), we can determine the eigenvalues for the action of $g$ on
the normal bundle in all cases. Mongardi’s result for $g$ of order 3 is as follows:

**Proposition 4.17** (Mongardi [Mo1], Theorem 6.2.4, Proposition 6.2.8). The ad-
missible elements $g$ of order 3 are the K3-type $(3, 6, 6)$ acting with 27 isolated fixed-
points, and type $(3, -3, -3)$ acting with a 2-torus as fixed-point set and eigenvalues
as in Table 2. □

Note that type $(3, 0, 0)$ is inadmissible by Lemma 4.3. Mongardi’s proof for the
type $(3, -3, -3)$ (and also if $g$ has order 7 or 11) uses Theorem 1.2 of [BN]. The
eigenvalues are all uniquely determined.

**Proposition 4.18.** The admissible $g$ of order 6 are the K3-type $(6, 2, 6)$ and type
$(6, 5, -3)$, each acting with isolated fixed-points and eigenvalues as in Table 2.

**Proof:** We have already shown that the only possibilities are types $(6, 2, 6)$ and
$(6, 5, -3)$.

In the first case, $g^2$ is of type $(3, 6, 6)$ acting with isolated fixed points. Hence $g$
must act by isolated fixed points, too. In the second case, $g^2$ is of type $(3, -3, -3)$
and has a 2-torus as fixed-point set. Furthermore, $g^3$ is of type $(2, 8, 24)$ and contains
only isolated fixed points or a K3 surface in its fixed-point set. Thus $g$ can only have
isolated fixed points.

We also see that the numbers $a_{2, 2} = a_{3, 3} = 0$ for $g$ (otherwise $g^2$ or $g^3$ would be
the identity).

For type $g$ of type $(6, 2, 6)$, (2) has the solution $a_{1, 1} = 1 + a_{1, 3}/8$, $a_{1, 2} = 6 - 9 a_{1, 3}/8$,
$a_{2, 3} = 0$, implying that $a_{1, 3} = 0$, $a_{1, 1} = 1$, $a_{1, 2} = 6$. For type $g$ of type $(6, 2, 6)$, we
get the solution $a_{1, 1} = -5/4 + a_{1, 3}/8$, $a_{1, 2} = 45/4 - 9 a_{1, 3}/8$ and $a_{2, 3} = 6$, which
implies $a_{1, 3} = 10$, $a_{1, 1} = 0$ and $a_{1, 2} = 0$. □

In both cases, $g^3$ has a K3 surface as a fixed-point component on which $g$ acts
with 6 isolated fixed-points, as required. For $g$ of type $(6, 5, -3)$, $g^2$ has a 2-torus as
one fixed-point component on which $g$ acts with 16 isolated fixed-points. □

**Proposition 4.19.** An admissible elements of order 9 has type $(9, 3, 3)$, acting with
9 isolated fixed-points and eigenvalues as in Table 2.

**Proof:** From Table 1 there is only one possible type of element $g$ of order 9.

Since $g^3$ is of type $(3, -3, -3)$ with a 2-torus as fixed-point set, we have to
consider the nine variables $a_{1, 3}$, $a_{2, 3}$, $a_{3, 4}$, $b_1$, $b_2$, $b_4$, $C_1$, $C_2$ and $C_4$ which might be
nonzero. Equation (2) provides us with the unique solution $a_{1, 3} = a_{2, 3} = a_{3, 4} = 3$,
$b_1 = b_2 = b_4 = 0$ and $C_1 = C_2 = C_4 = 0$ for non-negative $a_{i, j}$ and $b_i$. □

**Proposition 4.20.** An admissible element of order 15 has type $(15, 1, 1)$, acting
with 2 isolated fixed-points and eigenvalues as in Table 2.
Proof: The statement about the type is clear. By considering \( g^7 \) of type \((3, 6, 6)\) and \( g^3 \) of type \((5, 4, 4)\) it is also clear that \( g \) acts with isolated fixed-points and 
\[ a_{1,1}, a_{2,3}, a_{3,4}, a_{3,5}, a_{3,6}, a_{3,7}, a_{1,6}, a_{2,6}, a_{4,6}, a_{5,6}, a_{6,6}, a_{6,7} \text{ and } a_{1,5}, a_{2,5}, a_{3,5}, a_{4,5}, a_{5,5}, a_{5,6}, a_{5,7} \] 
all vanish. Then has the general solution \( a_{1,1} = (-a_{4,7} - a_{7,7})/5, a_{1,2} = a_{4,7}, a_{1,4} = (5 - 4a_{1,7} + 26a_{7,7})/5, a_{1,7} = (a_{4,7} - 24a_{7,7})/5, a_{2,2} = a_{7,7}, a_{2,4} = (a_{4,7} - 24a_{7,7})/5, a_{2,7} = (5 - 6a_{1,7} + 14a_{7,7})/5, a_{4,4} = (-a_{4,7} - a_{7,7})/5. \]
The only non-negative integral solution occurs when \( a_{4,7} = a_{7,7} = 0 \), leading to the eigenvalues as in Table 2.

The two fixed-points of \( g \) are the two distinguished fixed-points of \( g^3 \) with eigenvalues \((\zeta_5, \zeta_5^4, \zeta_5^7)\) and \((\zeta_5^2, \zeta_5^3, \zeta_5^4, \zeta_5^7)\).

Proposition 4.21 (Mongardi \cite{Mongardi}, Proposition 6.2.15). An admissible element of order 7 has type \((7, 3, 3)\), acting with 9 isolated fixed-points with eigenvalues as in Table 2.

Proposition 4.22. An admissible element \( g \) of order 14 has type \((14, 1, 3)\), acting with 3 isolated fixed-points and eigenvalues as in Table 2.

Proof: The statement about the type is again clear. By considering \( g^7 \) of type \((2, 8, 24)\) and \( g^2 \) of type \((7, 3, 3)\) it follows that \((2)\) contains the variables \( a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}, a_{1,6}, a_{2,3}, a_{2,5}, a_{3,3}, a_{3,4}, a_{3,5}, a_{3,6}, a_{4,5}, a_{5,5}, a_{5,6} \). The solution of the corresponding system of linear equations depends on the variables \( a_{1,5}, a_{1,6}, a_{3,4}, a_{3,5}, a_{3,6}, a_{4,5}, a_{5,5}, a_{5,6} \). The only non-negative integral solution is obtained for \( a_{1,4} = a_{2,3} = a_{5,6} = 1 \), with the remaining variables vanishing. This verifies the entry for \( g \) in Table 2.

If \( g \) has order 14 then \( g^7 \) is an involution with a K3 surface as a fixed-point component on which \( g \) acts with 3 fixed-points, as required.

Proposition 4.23 (Mongardi \cite{Mongardi}, Proposition 6.2.16). An admissible element of order 11 has type \((11, 2, 2)\), acting with 5 isolated fixed-points and eigenvalues as in Table 2.

This completes the proof of Theorem 4.9.

In Section 8 we will show that there are hyperkähler manifold \( X \) of type \( K3^{[2]} \) realizing symplectic automorphisms of each of the fifteen admissible types described in Theorem 4.9 and Table 2.

4.2 Symplectic actions of 2-groups

In this subsection we will show:

Theorem 4.24. Suppose that \( G \) is a finite 2-group of symplectic automorphisms of a hyperkähler manifold \( X \) of type \( K3^{[2]} \). Then \( |G| \leq 2^7 \).
For K3 surfaces, it follows from \([\text{Ma2}]\) that a finite 2-group of symplectic automorphisms is isomorphic to a subgroup of the 2-Sylow subgroup of \(M_{23}\) and thus has order at most \(2^7\). A short proof is given in \([\text{Ma2}]\): starting from the fact that a symplectic involution on a K3 surface has exactly 8 isolated fixed-points, it is shown that the centralizer of an involution in a group of symplectic automorphisms group must be a subgroup of \(\hat{A}_8\), the unique non-split extension of the alternating group \(A_8\) by \(Z_2\). The result then follows courtesy of the fact that the 2-Sylow subgroups of \(\hat{A}_8\) and \(M_{23}\) are isomorphic.

Assume for the rest of the subsection that \(G\) satisfies the assumptions of Theorem 4.24. We know that \(G\) is (isomorphic to) a subgroup of \(Co_0\). Let \(V := \Lambda \otimes Q\) be the linear space obtained from \(\Lambda\) by extension of scalars. The group \(G\) acts on \(V\). Let \(\chi\) be the character of \(G\) furnished by \(V\). After Theorem 4.9 we know that the following holds:

(a) Each \(g \in G\) has order \(n = 1, 2, 4\) or \(8\) and \(\chi(g) = 24, 8, 4\) or \(2\), respectively.

(b) \(\dim V^G \geq 4\).

The following formula is well-known:

\[
\dim V^G = |G|^{-1} \sum_{g \in G} \chi(g).
\]

We refer to this result as the invariant subspace formula (i.s.f.). We turn to the proof of Theorem 4.24.

Step 1. \(|G| \leq 2^8\).

Let \(t \in Z(G)\) be a central involution, and consider the fixed-point set \(X^t\). By Theorem 4.9 we know that \(X^t\) has a unique component that is a K3 surface, call it \(Y\). We claim that \(G/\langle t \rangle\) acts faithfully on \(Y\). Once this is established, we will know that \(G/\langle t \rangle\) is a finite 2-group of symplectic automorphisms of \(Y\), whence, as previously explained, \(|G/\langle t \rangle| \leq 2^7\) and Step 1 is completed.

By way of contradiction, assume that \(G/\langle t \rangle\) does not act faithfully on \(Y\). Then there is \(t \in A \subseteq G\) such that \(|A| = 4\) and \(A\) acts trivially on \(Y\). \(A\) cannot be cyclic, because elements of order 4 have discrete fixed-points by Theorem 4.9. Therefore, \(A = \langle s, t \rangle \cong Z_2^2\). Since both \(s\) and \(t\) leave \(Y\) fixed pointwise, they each act on the tangent space at a point \(p \in Y\), and this consists of the tangent space of \(Y\) at \(p\) and the restriction of the normal bundle. Both \(s\) and \(t\) act trivially on the first piece, and as \(-1\) on the second. Therefore \(st\) acts trivially on both, and hence is the identity. As \(s\) and \(t\) are involutions then \(s = t\), which is the required contradiction.

Step 2. \(|G| \leq 2^7\).

Assume, by way of contradiction, that this is false. By Step 1 we may, and shall, assume that \(|G| = 2^8\). Let \(t \in Z(G)\) be a central involution, and let \(Y\) be the K3 component of \(X^t\) as in Step 1. We proved before that \(H := G/\langle t \rangle\) is a group of
symplectic automorphisms of $Y$. Since $H$ has order $2^7$ it is isomorphic to a Sylow 2-subgroup of $M_{23}$ \cite{Mu, Ma}. We show that this leads to a contradiction.

$H$ contains exactly 5 abelian subgroups of order 16, namely 3 isomorphic to $\mathbb{Z}_4^2$ and 2 isomorphic to $\mathbb{Z}_2^4$. These maximal abelian subgroups generate a subgroup $J \subseteq H$ of order $2^6$ and $Z(J) \cong \mathbb{Z}_2^4$ is the intersection of any two of them. ($J$ is the Thompson subgroup of $H$. See \cite{Ma} for further discussion of this subgroup and its relevance in the present context.)

First let $t \in B \subseteq G$ be such that $B/\langle t \rangle \cong \mathbb{Z}_4^2$. If $B$ is abelian then $B \cong \mathbb{Z}_8 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4^2$, and in either case an easy application of the i.s.f. yields a contradiction. So $B$ is nonabelian, whence $B' = \langle t \rangle$. If $a, b \in B$ we have $[a, b] = a^{-1}b^{-1}ab \in \langle t \rangle$. Hence $[a^2, b] = a^{-2}b^{-1}a^{-2}b = a^{-1}[a, b]b^{-1}ab = a^{-1}[a, b]b^{-1}a = 1$, the latter equality because $[a, b] = [b, a^{-1}] = 1$ or $t$. This shows that all squares $a^2$ in $B$ commute with all elements in $B$, i.e., they are contained in $Z(B)$. Now it is clear that $\langle a^2 \mid a \in B \rangle$ contains $t$ and projects (modulo $\langle t \rangle$) to the subgroup of $B/\langle t \rangle$ generated by the involutions of this quotient. But from the previous paragraph, this is precisely the group $Z(J)$. Let us define $Z_0 \subseteq G$ to be the inverse image of $Z(J)$ in $G$. Thus what we have established in this paragraph is that $Z_0 \subseteq Z(B)$.

This analysis applies to each of the three groups $B$ such that $B/\langle t \rangle \cong \mathbb{Z}_4^2$. Therefore, $Z_0$ is contained in the center of the subgroup of $G$ that they generate. Call this group $J_0$. It is the inverse image in $G$ of the Thompson subgroup $J \subseteq H$. Thus we have $Z_0 \subseteq Z(J_0)$.

Finally, let $t \in D \subseteq G$ be such that $D/\langle t \rangle \cong \mathbb{Z}_2^4$. (There are 2 such subgroups in $G$.) We have $D/\langle t \rangle \subseteq J$, therefore $Z_0 \subseteq D \subseteq J_0$. By the i.s.f., $D$ cannot be abelian. Since $Z_0 \subseteq Z(D)$, $|Z_0| = 8$ and $D/\langle t \rangle \cong \mathbb{Z}_2^4$ it is easily seen that $D \cong \mathbb{Z}_2^2 \times D_8$, $\mathbb{Z}_2^4 \times Q_8$, or $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times D_8)$, and the i.s.f. yields a contradiction ($\dim V^G$ turns out to be a half-integer in all three cases). This completes the proof of Step 2, and with it that of Theorem \ref{main_theorem}.

Something more can be said about the possible isomorphism types of $G$. Let $t$ be any involution in $G$ and $C_G(t)$ its centralizer. Since $X^t$ contains a $K3$-surface as component, $C_G(t)/\langle t \rangle$ acts faithfully on the $K3$-surface, and is therefore isomorphic to a 2-subgroup of $M_{23}$.

**Theorem 4.25.** Let $G$ be a 2-group such that for every involution $t$, the quotient $C_G(t)/\langle t \rangle$ is isomorphic to a subgroup of $M_{23}$. Then $|G| \leq 2^7$, and $G$ belongs to one of 70 possible isomorphism types.

We verified this by checking the condition for all 2-groups of order at most 256 \cite{OB} using Magma.

## 5 Isomorphism classes of admissible subgroups in $\text{Co}_0$

In this section we describe the isomorphism classes of subgroups $G \subseteq \text{Co}_0$ with the following two properties:
1. $G$ consists of admissible elements.
2. $\text{rk } \Lambda^G \geq 4$.

We call these subgroups admissible. We note that an admissible element generates an admissible cyclic subgroup.

One of the main results of this section is a sufficient condition for $G$ to be isomorphic to a subgroup of $M_{23}$: namely, that every element of $G$ lies in one of the 11 admissible conjugacy classes that meet $2^{12}:M_{24}$ and $|G|$ is not a 2-group. If $G$ is a 2-group then it lies in $2^{12}:M_{24}$ because the monomial group contains a Sylow 2-subgroup of $\text{Co}_0$. The third type occurs precisely when $G$ contains elements in one of the four admissible conjugacy classes $(3, -3, -3)$, $(6, 5, -3)$, $(9, 3, 3)$ or $(12, 1, 5)$ which do not meet $2^{12}:M_{24}$.

The maximal subgroups in the first two cases will be explicitly described. In the third case, $G$ has to be one of the four groups described in Theorem 3.2 (i), (ii), (iii) or (v).

5.1 Admissible subgroups related to $M_{23}$

In this subsection we assume that $G \subseteq \text{Co}_0$ has the following properties

1. $G$ is admissible;
2. $G$ contains no elements of type $(3, -3, -3)$;
3. $G$ is not a 2-group.

As main result we have:

**Theorem 5.1.** $G$ is isomorphic to a subgroup of one of the following 13 groups:

(a) $L_2(11)$,
(b) $\mathbb{Z}_2 \times L_2(7)$,
(c) $\mathbb{Z}_2^3:L_2(7)$,
(d) $A_7$,
(e) $L_3(4)$,
(f) $(\mathbb{Z}_3 \times A_5):\mathbb{Z}_2$,
(g) $\mathbb{Z}_2^4:A_6$,
(h) $\mathbb{Z}_2^4:S_5$,
(i) $M_{10}$,
(j) $S_6$,
(k) $\mathbb{Z}_3^3:QD_{16}$,
(l) $\mathbb{Z}_2^4:(S_3 \times S_5)$,
(m) $Q(\mathbb{Z}_3^3, \mathbb{Z}_2)$, $|Q| = 2^6$. 

23
In each case, $G$ is isomorphic to a subgroup of $M_{23}$.

Before beginning the proof we make some remarks. In the following proof, it will be convenient to occasionally assume in addition that a Sylow 2-subgroup of $G$ has order at most $2^7$. We have already proved this if $G$ acts on a hyperkähler manifold of type $K3^{[2]}$ (Theorem 4.24), but as we have seen, this is an $a priori$ stronger requirement than merely assuming that $G$ is admissible. During the course of the proof, whenever the condition that a Sylow 2-subgroup has order no greater than $2^7$ is invoked, we will indicate how to complete the argument without using this property.

From now on, then, we assume that $G$ satisfies (5.1) and has a Sylow 2-subgroup of order at most $2^7$. Of the 15 admissible types enumerated in Theorem 4.9 (cf. Table 1) those of types $(3, -3, -3)$, $(6, 5 - 3)$, $(9, 3, 3)$ and $(12, 1, 5)$ all have some power of type $(3, -3, -3)$. Since this type is excluded by assumption, all four of these classes are excluded. There remain 11 admissible conjugacy classes, these being precisely the admissible classes that meet the monomial group $2^{12}:M_{24}$. Indeed, each of these classes meets $M_{23}$ and the class is characterized by the order $n$ of the elements that it contains ($n = 1, \ldots, 8, 11, 12, 15$). The upshot is that for $g \in G$, $\chi(g)$ is equal to the trace of the corresponding element in $M_{23}$ in the usual permutation representation of degree 24. Here, $\chi$ is the character of the representation of $G$ on $V = \Lambda \otimes Q$ as in (4.2). This observation, coupled with the invariant-subspace-formula (i.s.f.) (3), which of course holds for any finite group, is used frequently in what follows.

We turn to the proof of Theorem 5.1, which will be divided into several cases. Before beginning the first case, we establish a preliminary estimate for $|G|$.

**Lemma 5.2.** $|G|$ divides $2^7.3^2.5.7.11$.

**Proof:** We only need to deal with the odd primes $p$. Note that $G$ contains no elements of order $p^2$. Therefore a Sylow $p$-subgroup $P \subseteq G$ has exponent at most $p$. If $|P| = p^k$, the i.s.f. gives $\dim V^P = (24 + (p^k - 1)r)/p^k = r + (24 - r)/p^k$, where $r$ is the trace of an element of order $p$. It follows that $(24 - r)/p^k$ is an integer, and because $r > 0$ (cf. Table 1) then $p^k \leq 23$, and Step 1 follows immediately.

**Case 1.** $11||G|$. We will show in this easiest case that $G$ is isomorphic to a subgroup of $L_2(11)$.

Let $P \subseteq G$ be a Sylow 11-subgroup. Thus $P \cong Z_{11}$, and since there are no elements of order $11k$ ($k \geq 2$) then $C_G(P) = P$. The i.s.f. shows that there are no dihedral subgroups of order 22, whence $|N_G(P)| = 11$ or 55.

In the first case, $G$ has a normal 11-complement by Burnside’s normal $p$-complement theorem, call it $Q$. Thus $G = QP$ with $\gcd(|Q|, 11) = 1$. Since $C_G(P) = P$ then there must be a prime $p = 2, 3, 5$ or 7 and a subgroup $E \subseteq Q$ with $E \cong Z_p^k$ for some $k \geq 0$ such that $P$ normalizes and acts faithfully on $E$. Moreover, if $Q \neq 1$ then we can choose $E \neq 1$. Setting $H = EP$, the i.s.f. applied to $H$ shows
that the only possibility is \( E = 1 \), so that \( H = 1 \) and \( G = P \) is a Sylow 11-subgroup of \( L_2(11) \).

The second possibility is \(|N_G(P)| = 55\). The previous argument shows that \( P \) cannot normalize any nontrivial subgroups of \( G \) of order coprime to 11. So if \( G \) is solvable then \( G = N(P) \) has order 55 and is the normalizer of a Sylow 11-subgroup of \( L_2(11) \). Finally, assume that \( G \) is nonsolvable. A minimal normal subgroup \( N \leq G \) cannot have order coprime to 11, so it is simple and contains \( P \). By the Frattini argument, \( G = NN_G(P) \). If \( N_G(P) \) is not contained in \( N \) then \( P \) is self-normalizing in \( N \), so \( N \) cannot be simple by the same Burnside theorem, contradiction. Thus \( N_G(P) \subseteq N \) and \( G = N \) is simple. Because \(|G| \) divides \( 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \), any one of several classification theorems will tell us that \( G \cong L_2(11) \). This completes Case 1.

**Case 2.** \( 7|\lvert G\rvert \). We will establish

\[ G \text{ is isomorphic to a subgroup of } \mathbb{Z}_2 \times L_2(7), \mathbb{Z}_2^3 : L_2(7), A_7 \text{ or } L_3(4). \]

Let \( P \cong \mathbb{Z}_7 \) be a Sylow 7-subgroup.

**Lemma 5.3.** \( G \) has no subgroup \( H \) of any of the following types: dihedral of order 14, order 28, order \( 2^4 \cdot 7 \).

**Proof:** The i.s.f. eliminates the dihedral group. Suppose that \(|H| = 28\). Since there are no elements of order 28 and no dihedral group of order 14, then \( H \) is either \( \mathbb{Z}_2 \times \mathbb{Z}_{14} \) or \( \mathbb{Z}_7 : \mathbb{Z}_4 \). The first possibility is eliminated by the i.s.f. The second possibility does not hold either, because calculation (e.g. by Magma) shows that \( \chi \) cannot be the character of a rational representation of such a group (the Schur index is greater than 1). Suppose that \(|H| = 2^4 \cdot 7\). Since there are no dihedral groups of order 14 or abelian groups of order 28 then a 2-Sylow-subgroup \( Q \subseteq H \) is normal (Burnside’s theorem again), moreover \(|C_Q(P)| = 2\). The only possibility is \( Q \cong \mathbb{Z}_2^4 \), and one verifies by applying the i.s.f. to \( QP \) that this is impossible. □

Now suppose that \( Q \neq 1 \) is a subgroup of order coprime to 7 and normalized by \( P \). Then \( P \) leaves invariant a Sylow \( r \)-subgroup of \( Q \) for each prime divisor \( r \) of \(|Q|\). Let \( R \) be such a Sylow \( r \)-subgroup. If \( r \geq 3 \) then \(|Q| \leq r^2 \), so \( P \) acts trivially on \( R \), thereby producing elements of order \( 7r \), contradiction. So \( Q = R \) is a 2-group. We have \(|C_Q(P)| \leq 2\) by Lemma 5.3 (there are no subgroups of order 28). Suppose that \( Q \) is extra-special. Then \( P \) acts faithfully on \( Q \), and we have \(|Q| = 2^{1+6k}\). The (unique) faithful irreducible representation of \( Q \) has dimension \( 2^{3k} \), so we must have \( k = 1 \), and the i.s.f. applied to \( QP \) yields a contradiction. Therefore \( Q \) is not extra-special. So if \(|Q| \geq 4 \), there is a noncyclic characteristic elementary abelian 2-group of \( Q \), call it \( E \). Because \(|C_E(P)| \leq 2\) then \( P \) acts faithfully on \( E \), so \(|E| \geq 8\). If \( P \) centralizes \( Z(Q) \) then \( EZ(Q)/P \) contains a group of order \( 2^4 \cdot 7 \), contradiction. So \( P \) does not centralize \( Z(Q) \), whence we may, and shall, take \( E \subseteq Z(Q) \). The same argument then shows that \( C_Q(P) = 1 \). If \(|Q| \geq 16 \) we must then have \(|Q| = 2^{3k}, k \geq 2 \), and the i.s.f. again yields a contradiction. So in fact \( Q = E \cong \mathbb{Z}_2^3 \). Thus we have so far shown:

Suppose there is \( 1 \neq Q \subseteq G \) normalized by \( P \) with \( \gcd(|Q|, 7) = 1 \).
Then \( Q \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_2^3 \), and in the second case \( C_Q(P) = 1 \).

Now assume that \( \mathbb{Z}_2^3 \cong Q \leq G \). If \( G \) is solvable we easily find (there being no dihedral subgroup of order 14 by Lemma 5.3) that \( G \cong \mathbb{Z}_2^3: \mathbb{Z}_7 \) or \( \mathbb{Z}_2^3:(\mathbb{Z}_7: \mathbb{Z}_3) \). Suppose that \( G \) is nonsolvable. Since \( C_G(Q) \) has order coprime to 7 we must have \( C_G(Q) = Q \). Since \( \text{Aut}(Q) \cong L_3(2) \cong L_2(7) \) is a minimal simple group, the only possibility is \( G/N \cong L_2(7) \), so that \( G \) occurs in a short exact sequence \( 1 \to \mathbb{Z}_2^3 \to G \to L_2(7) \to 1 \).

It is well-known (and easy to see) that this extension splits and is unique up to isomorphism. Note that the possible solvable groups with \( \mathbb{Z}_2^3 \leq G \) are all contained in this nonsolvable example.

Assume that \( |Q| = 2 \). If \( G \) is solvable, it follows from what has already been established that \( G \) is isomorphic to a subgroup of \( \mathbb{Z}_2 \times (\mathbb{Z}_7: \mathbb{Z}_3) \). Finally, suppose \( G \) is nonsolvable. Then \( G/Q \) is simple of order divisible by 7, so \( G/Q \cong L_2(7), A_7, A_8, L_3(4) \). Since there are no elements of order 10, only the first possibility survives. Therefore, \( G \cong \mathbb{Z}_2 \times L_2(7) \) or \( SL_2(7) \). In the second case the i.s.f. fails, leaving only the first possibility. Note once again that all solvable possibilities are contained in the nonsolvable case.

The last possibility is that there is no nontrivial normal subgroup of \( G \) of order coprime to 7. Then a minimal normal subgroup \( N \leq G \) is simple and contains \( P \). The only possibilities for \( N \) (taking into account that \( |G| \) divides \( 2^6 \cdot 3^2 \cdot 5 \cdot 7 \) with no elements of order 9) are

\[
N \cong L_2(7), \ A_7, \ A_8, \ L_3(4).
\]

We claim that \( A_8 \) cannot occur. Indeed, \( A_8 \leq M_{23} \) acts with only 3 orbits on the 24 points. Therefore, applying the i.s.f. to our abstract group \( G \cong A_8 \) will certainly yield \( \dim V^G = 3 \), contradiction. We assert that \( G = N \) in the other three cases, so assume this is false. If \( N \cong L_3(4) \), we have \( V = V^G \oplus U \) where \( U \) is \( G \)-invariant and irreducible of dimension 20. \( (L_3(4) \) has no irreducible representation of dimension less than 20 \([\text{CCNPW}]\). Since \( |G| \) is not divisible by 27, we must have \( |G/N| = 4 \), and every nontrivial coset of \( N \) in \( G \) contains an involution \( t \) such that \( \text{Tr}_U(t) \neq 4 \) (loc. cit). Because \( 8 = \chi(t) = 4 + \text{Tr}_U(t) \), this is a contradiction. On the other hand, if \( N \cong L_2(7) \) or \( A_7 \) then \( G \cong \text{PGL}_2(7) \) or \( S_7 \) respectively, and since both of these groups contain a dihedral group of order 14 they cannot occur either. This completes the proof that \( G = N \cong L_2(7), A_7 \) or \( L_3(4) \). Because \( L_2(7) \leq A_7 \) there is no need to include \( L_2(7) \) on the list of possible groups, and the analysis of Case 2 is done.

Before taking up Case 3 we interpolate a Lemma.

**Lemma 5.4.** Suppose that \( |G| = 2^k \cdot 5 \). Then either \( G \cong \mathbb{Z}_5: \mathbb{Z}_4 \) or \( O_2(G) \cong \mathbb{Z}_2^4 \).

**Proof:** Let \( P \) be a Sylow 5-subgroup of \( G \). \( G \) is necessarily solvable, and assuming that the first stated possibility does not hold we have \( Q := O_2(G) \neq 1 \). Since there are no elements of order 10 then \( Z(Q) \) contains a \( P \)-invariant subgroup \( E \cong \mathbb{Z}_2^3 \). We will obtain a contradiction if \( Q \neq E \). Indeed, in the contrary case we have \( |Q| \geq 2^8 \) because \( C_Q(P) = 1 \), and this is impossible because a Sylow 2-subgroup has order \( \leq 2^7 \). (To avoid this assumption, choose a \( P \)-invariant subgroup of order \( 2^8 \) in \( Q \),

26
call it \( D \), where \( E \subseteq D \) and \( D \) has exponent 4. Apply the i.s.f. to \( D \) to see that there must be elements of order 10 in \( PE \), contradiction.)

\( \Box \)

**Case 3.** \( \exists \mathbb{Z}_2^4 \cong N \leq G \), and \( 5| |G| \). We will establish

\[ G \text{ is isomorphic to a subgroup of } \mathbb{Z}_2^4 : S_5 \text{ or } \mathbb{Z}_2^4 : A_6. \]

First, we can check using the i.s.f. that \( \dim V^N = 9 \) and \( \dim V^E = 10 \) for every hyperplane \( E \subseteq N \). This means implies that if \( V = W \oplus V^N \) is a \( G \)-invariant decomposition, then \( \dim W = 15 \) and (considered as \( N \)-module) \( W \) is the sum of the 15 distinct nontrivial irreducible characters of \( N \). Therefore, we can choose a 1-dimensional subspace of \( V^G \), call it \( V_0 \) so that \( U := W \oplus V_0 \) is both \( G \)-invariant and a free \( N \)-module. There is a unique set of 1-dimensional spaces \( V_i := CV_i \) \((0 \leq i \leq 15)\) spanning \( W \oplus V_0 \) and afford the irreducible characters of \( N \), and \( G \) permutes the \( V_i \) among themselves. Furthermore, we readily find that \( G = N; H \) is a split extension and that \( G \) acts as a transitive permutation group on \( U \) with point-stabilizer \( H \). Applying Lemma \[ \text{with } G \text{ replaced by } O_2(G)P \text{ (}P \text{ a Sylow 5-subgroup of } G\text{)}, \] we conclude that \( N = O_2(G) \), i.e., \( O_2(H) = 1 \). Furthermore, the i.s.f. shows that \( G \) cannot contain any abelian groups of order \( 2^5 \) and rank at least 4. In particular, \( N \) is self-centralizing in \( G \) whence \( H \) is isomorphic to a subgroup of \( \text{Aut}(N) \cong L_4(2) \cong A_8 \). From the analysis of Case 2 it also follows that \( \gcd(| H |, 7) = 1 \), and by the i.s.f. there is no subgroup isomorphic to \( \mathbb{Z}_2^4 \cdot \mathbb{Z}_{15} \).

From these reductions, it follows that if \( G \) is solvable then \( G \) is isomorphic to a subgroup of \( \mathbb{Z}_2^4 : (\mathbb{Z}_5 : \mathbb{Z}_4) \), which is itself a subgroup of \( \mathbb{Z}_2^4 : S_5 \). If \( G \) is nonsolvable then \( H \) has a normal subgroup \( K \cong A_5 \) or \( A_6 \), and in the latter case we have \( H \cong A_6 \) or \( S_6 \). We can eliminate the latter possibility much as in the \( L_3(4) \) case handled earlier. Indeed, if \( H \cong S_6 \) then \( V \) decomposes into a transitive permutation representation of \( K \) of dimension 15 (corresponding to its action on the involutions of \( N \)), plus a 4-dimensional fixed-point subspace, plus a 5-dimensional irreducible in \( V^N \). Let \( t \in H \) be an involution acting on this 5-dimensional space with trace \(-1\) (such a \( t \) always exists). \( t \) has trace 7 or 3 on the 15-dimensional space, so that \( \chi(t) = 10 \) or \( 6 \), contradiction. Suppose that \( K \cong A_5 \). Because there are no subgroups of the form \( \mathbb{Z}_2^4 : \mathbb{Z}_{15} \) then \( G \) is isomorphic to a subgroup \( \mathbb{Z}_2^4 : S_5 \). This completes Case 3.

**Case 4.** \( 5| |G| \), \( \gcd(|G|, 77) = 1 \). We show in this case that either Case 3 holds, or

\[ G \text{ is isomorphic to a subgroup of } (\mathbb{Z}_3 \times A_5) : \mathbb{Z}_2, S_6 \text{ or } M_{10}. \]

If \( O_2(G) \neq 1 \) then by Lemma \[\text{we are in Case 3 and there is nothing to prove. Hence, we may assume that } O_2(G) = 1. \] If there is a nontrivial normal subgroup of order prime to 5 it must then be a 3-group, call it \( R \leq G \). Then \( P \cong \mathbb{Z}_5 \) acts trivially on \( R \) since \( |R| \leq 9 \). There is no abelian subgroup of order 45 by the i.s.f. so \( R = O_3(G) \cong \mathbb{Z}_5 \). Let \( M/R \) be a minimal normal subgroup of \( C_G(R)/R \). If \( (\gcd(|M|, 5) = 1 \) then \( M/R \) is a 2-group and it centralizes \( R \) and therefore descends to a yield a nontrivial normal 2-subgroup of \( G \), contradiction. Thus \( P \subseteq M \). If \( M \)
is solvable then $M = P \times R \cong \mathbb{Z}_{15}$ and $G \subseteq \mathbb{Z}_{15}:\mathbb{Z}_4$. If $M$ is nonsolvable then $M/R \cong A_5$, $M \cong \mathbb{Z}_3 \times A_5$, and $G$ is isomorphic to a subgroup of $(\mathbb{Z}_3 \times A_5):\mathbb{Z}_2$. This contains the solvable group described earlier in the paragraph.

Finally, assume that a minimal nontrivial normal subgroup $N$ contains $P$. If it is equal to $P$ then $G$ is isomorphic to a subgroup of $\mathbb{Z}_5:\mathbb{Z}_4$. Otherwise, $N$ is simple, so that $N \cong A_5$ or $A_6$ and $G = A_5, S_5$ or $A_6 \subseteq G \subseteq \text{Aut}(A_6)$. In the latter case, since there are no elements of order 10 then we cannot have $G = \text{PGL}_2(9)$ or $\text{Aut}(A_6)$. Therefore, $G \cong A_6, M_{10}$ or $S_6$. (The latter two groups correspond to the two cosets of $\text{Aut}(A_6)/A_6$ distinct from that corresponding to $\text{PGL}_2(9)$). This completes the analysis of Case 4.

We have now completed the proof of Theorem 5.1 in case $|G|$ is divisible by one of 7, 9 or 11. This leaves us with

**Case 5.** $|G| = 2^f 3^2$. We will show that either $G$ is contained in one of the groups occurring in Cases 1–4, or else it is a subgroup of one of

$$\mathbb{Z}_3^2:QD_{16}, \mathbb{Z}_2^4(S_3 \times S_3) \text{ or } Q:(\mathbb{Z}_3^2:\mathbb{Z}_2) \text{ with } |Q| = 2^6. $$

First note that $G$ is solvable of 3-length 1. Because there are no elements of order 9 then a Sylow 3-subgroup $R \subseteq G$ is isomorphic to $\mathbb{Z}_3^2$ and $G = RT$ where $T$ is a Sylow 2-subgroup of $G$. Suppose first that $O_2(G) = 1$. Then $T$ acts faithfully on $R$ and is therefore isomorphic to a subgroup of $\text{Aut}(R) = \text{GL}_2(3)$. This latter group has Sylow 2-subgroup $QD_{16}$ (quasidihedral), so $G$ is isomorphic to a subgroup of $\mathbb{Z}_3^2:QD_{16}$.

Next assume that $O_2(G) \neq 1$, $S := O_3(G) \neq 1$. Since there is no abelian subgroup of order 18 by the i.s.f. we have $|Q| = 3$. Similarly, because there is no subgroup $\mathbb{Z}_3 \times A$ with $|A| \geq 16$ by the i.s.f., we have $|Q| = 4$. Then $G$ is isomorphic to a subgroup of $(\mathbb{Z}_3 \times A_4):\mathbb{Z}_2$, and this group is contained in $(\mathbb{Z}_3 \times A_5):\mathbb{Z}_2$.

Now suppose that $O_3(G) = 1$. Because $R$ is self-centralizing and there is no $\mathbb{Z}_3 \times A$, $|A| = 16$ as before, then $Q := O_2(G)$ has order $4^k$ where $2 \leq k \leq 4$ is the number of subgroups $U \subseteq R$ of order 3 satisfying $C_Q(U) \neq 1$ (in which case $C_Q(U) \cong \mathbb{Z}_3^2$). Since there is no subgroup isomorphic to $\mathbb{Z}_3^2$, it follows easily that either $k = 2$ and $Q \cong \mathbb{Z}_3^2$, or else $k \geq 3$ and $Z(Q) \cong \mathbb{Z}_3^2$. Moreover, $G = QH$ where $H := N_G(R)$.

**Lemma 5.5.** If $\mathbb{Z}_3 \cong U \subseteq R$ then either $C_H(U) = R$ or $C_Q(U) = 1$.

**Proof:** Assume false. Set $Q_0 := C_Q(U) \cong \mathbb{Z}_3^2$ and $R = U \times U_0$. Then $U_0$ acts on $Q_0$, and since $C_Q(R) = R$ then $C_{Q_0}(U_0) = 1$. Now $8|C_G(U)$. Since $C_Q(R) = 1$ we can choose an involution $t \in C_H(U)$. Since $R \subseteq H$ then $t$ normalizes $R$ and we may, and shall, choose $t$ so that it normalizes $U_0$. Since $t$ commutes with $U$ but not $R$ then $t$ must act as the inverting automorphism of $U_0$. Now $U_0(t) \cong S_3$ acts on $Q_0$. Since $t$ commutes with $Q_0$ then so does $U_0 = [U_0, t]$, that is $Q_0 = C_{Q_0}(U_0)$. This contradicts the earlier statement that $C_{Q_0}(U_0) = 1$, and the proof of the Lemma is complete. □
We now consider the various possibilities for the integer $k$ defined, as before, by the equality $|Q| = 4^k$.

**Case 5(a).** $k = 2$. Here, $Q \cong \mathbb{Z}_2^4$ and $H$ is isomorphic to a subgroup of $\mathbb{Z}_2^3:QD_{16}$. However, because $k = 2$ then a Sylow 2-subgroup $T_0$ of $H$ cannot act transitively (by conjugation) on the subgroups of $R$ of order 3. As a consequence, $T_0$ is neither $QD_{16}$ itself, nor is it $\mathbb{Z}_8$ or $Q_8$. It is thus a subgroup of $D_8$. If it is $D_8$ then we can find a $\mathbb{Z}_3 \cong U \subseteq R$ which satisfies $C_Q(U) \neq 1$, $C_H(U) \neq R$, against Lemma 5.5. Therefore, $T_0 \cong \mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{Z}_2^2$. Furthermore, the conditions of Lemma 5.5 have to be satisfied. If $T_0 \cong \mathbb{Z}_4$ then $G = \mathbb{Z}_2^2:(\mathbb{Z}_3^2 : \mathbb{Z}_4)$ is a subgroup of $\mathbb{Z}_2^4:A_6$, which occurs in Case 5(b). If $T_0 \cong \mathbb{Z}_2^2$ then $G = \mathbb{Z}_2^2:(S_3 \times S_3)$, while if $T_0 \cong \mathbb{Z}_2$, then $G$ is isomorphic to a subgroup of $G = \mathbb{Z}_2^3:(S_3 \times S_3)$. This completes Case 5(a).

**Case 5(b).** $k = 3$. Here, $Q$ is the product of three subgroups $Q_i := C_Q(U_i) \cong \mathbb{Z}_2^2$ ($i = 0, 1, 2$) for three distinct subgroups $U_i \subseteq R$ or order 3. We may, and shall, take $Q_0 = Z(Q)$. Suppose that $QR$ is a proper subgroup of $G$. Then the 2-Sylow subgroup $T_0$ of $H$ is nontrivial. $T_0$ normalizes $U_0 = C_R(Z(Q))$, and no nonidentity element can act trivially by Lemma 5.5. Thus $T_0 = \langle t \rangle$ has order 2. Now $t$ must also normalize the unique subgroup $U_3 \subseteq R$ of order 3 satisfying $C_Q(U_3) = 1$. We claim that $t$ also inverts $U_3$. Otherwise, $t$ centralizes $U_3$, so $U_3$ acts on $C_Q(t)$. Then $C_Q(t)$ has order 16 and contains $Z(Q)$, from which it follows that $C_Q(t)$ is abelian. Then $\langle t \rangle \times C_Q(t)$ is abelian of order 32, and the fixed-point formula shows that this is not possible. This completes the proof that $t$ inverts $U_3$. (It follows from these facts that the Sylow 2-subgroup $T = Q\langle t \rangle$ of $G$ is isomorphic to a Sylow 2-subgroup of $M_{23}$.) In any case, $G \cong Q:\langle \mathbb{Z}_2^2 \times \mathbb{Z}_2 \rangle$, and Case 5(b) is finished.

**Case 5(c).** $k = 4$. This case cannot hold because a Sylow 2-subgroup has order $\leq 2^7$. To avoid this assumption, proceed as follows. $Q$ is the product of the four group $Q_i := C_Q(U_i) \cong \mathbb{Z}_2^2$, where the $U_i$ are the four subgroups of $R$ of order 3. We easily see as before that $Q$ has exponent 4. So if it has a elements of order 4 then $\dim V^Q = (1/2^8)(24 + 4a + (2^8 - 1 - a)8) = (16 - 4a + 2^{11})/2^8$ and $a \equiv 4 \pmod{64}$. As before, we also have $9\alpha$, and it follows that $a = 324 + 576n$ for some integer $n \geq 0$. Since $324 > 2^8$ this is impossible, and the proof is finished.

This completes the discussion of the case when $3^2||G|$.

**Case 6.** $|G| = 2^f \cdot 3$. We choose a computational approach for the case when $3||G|$.

After Lemma 5.2 we have $|G|\leq 384$. We use the library of small groups $[\texttt{BEO}]$ in Magma to find those $G$ satisfying $3||G||384$ and possessing a 20-dimensional character $\chi - 4 \cdot 1$, $\chi(g)$ being determined by the order of $g$. With the exception of a single group $X_{24}$ (library entry (24,7)), all others are contained in one of the 13 groups of Theorem 5.1. See also Table 3.

$X_{24}$ has eight 1-dimensional and four 2-dimensional irreducible characters. In the representation with character $\chi - 4 \cdot 1$, all irreducible characters appear with multi-
Table 3: Isomorphism classes of $2^f \cdot 3$-groups

| order   | 3 | 6 | 12 | 24 | 48 | 96 | 192 | 384 | total |
|---------|---|---|----|----|----|----|-----|-----|-------|
| no. of groups | 1 | 2 | 5  | 15 | 52 | 231| 1543| 20169| 22018 |
| and correct representation | 1 | 2 | 4  | 5  | 8  | 6  | 8   | 4   | 38    |
| contained in $M_{23}$ | 1 | 2 | 4  | 4  | 8  | 6  | 8   | 4   | 37    |
| others contained in $Co_0$ | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0     |

Multiplicity 1 apart from one 2-dimensional character which has multiplicity 3. However, two of the 2-dimensional irreducible representations have Schur index 2, implying that the representation affording $\chi - 4 \cdot 1$ cannot be rational. Consequently, $G$ cannot be contained in $Co_0$. This completes the proof of Theorem 5.1

That $G$ is indeed isomorphic to a subgroup of $M_{23}$ is implicit in the preceding proofs (and can be verified by a Magma calculation). Indeed, there is an alternate characterization of this set of groups which we state by reformulating the previous result in terms of the usual permutation action of $M_{23}$ on a set $\Omega$ of 24 elements.

**Theorem 5.6.** Let $V = \mathbb{Q}^{24}$, and consider the set $\mathcal{T}$ of isomorphism classes of finite subgroups $G \subseteq \text{GL}(V)$ maximal with respect to the properties

1. $|G|$ is divisible by at least one of 3, 5, 7, 11,
2. $G$ consists of elements $g$ of order 1–8, 11, 14, 15 and $\text{Tr}_V(g) = \text{number of elements of } \Omega \text{ fixed by an element in } M_{23} \text{ of the same order}$,
3. $\dim V^G \geq 4$.

Then $\mathcal{T}$ is precisely the set of isomorphism classes of subgroups of $M_{23}$ maximal with respect to having at least four orbits on $\Omega$.

**Proof:** By Theorem 5.1 any $G$ satisfying properties 1, 2 and 3 is isomorphic to a subgroup of $M_{23}$ with at least four orbits on $\Omega$. On the other hand, a subgroup $G \subseteq M_{23}$ with at least four orbits satisfies the condition 2, and condition 3 is satisfied if, and only if, $G$ has at least four orbits on $\Omega$. It is easily checked (e.g. with Magma) that the thirteen groups given in Theorem 5.1 are all the isomorphism types of subgroups $G$ of $M_{23}$ which are maximal with respect to the property of having at least four orbits on $\Omega$. Thus any such group also satisfies condition 1.

**Remark 5.7.** The $M_{23}$-conjugacy classes of the groups $G$ in $\mathcal{T}$ are not unique. For the groups $A_7$, $\mathbb{Z}_3^2 : L_2(7)$ and $\mathbb{Z}_4^3 : S_5$ there are two conjugacy classes. For $\mathbb{Z}_3^4 : S_5$, the orbit structure on $\Omega$ is different for the two conjugacy classes, whence they are also not conjugate in $M_{24}$.
Table 4: Isomorphism classes of 2-groups

| order | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | total |
|-------|---|---|---|---|----|----|----|-----|-----|-------|
| no. of 2-groups | 1 | 1 | 2 | 5 | 14 | 51 | 267 | 2328 | 56092 | 58761 |
| of exponent ≤ 8 | 1 | 1 | 2 | 5 | 13 | 45 | 234 | 2093 | 53529 | 55923 |
| and correct representation | 1 | 1 | 2 | 5 | 12 | 12 | 6 | 1 | 0 | 40 |
| contained in $M_{23}$ | 1 | 1 | 2 | 5 | 9 | 7 | 5 | 1 | 0 | 31 |
| others contained in $\text{Co}_0$ | 0 | 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 4 |

5.2 Admissible 2-groups

Suppose $G$ only has elements of orders 1, 2, 4 or 8. Then $G$ is necessarily a 2-group, and we already know from Theorem 4.24 that if $G$ comes from an effective symplectic group action on a hyperkähler manifold of type $K3^{[\mathbb{C}]}$ then $|G| \leq 2^7$.

An immediate consequence of Theorem 5.10 below is that $|G| \leq 2^7$ holds under the weaker assumption that $G$ is admissible. Thus the results in subsection 5.1 hold for all admissible groups and for this reason we eschewed any prior discussion of the 2-structure of $G$. We will describe a computational approach, which seem unavoidable if one wants a complete proof of the Theorem that is not too long.

**Proposition 5.8.** Let $G$ be a 2-group of exponent at most 8 having a complex representation $\rho$ such that $\text{Tr}\rho(g) = \chi(g') - 4$, where $g' \in M_{23}$ has the same order as $g$. Then $G$ is isomorphic either to a subgroup of $M_{23}$, or to one of 9 additional groups of order 16, 32 or 64 described in Table 5.

**Proof:** Note that $\dim \rho = \chi(1) - 4 = 20$. We first use the library of small groups in Magma to find the 2-groups $G$ of order at most 256 and exponent at most 8 [OB]. Then we check to see if $G$ has a 20-dimensional representation of the correct type.

The result is given in Table 4. As expected, there are no such groups of order 256, and a unique group of order 128, namely a Sylow 2-subgroup of $M_{23}$. We also list in Table 4 the number of 2-groups contained in $M_{23}$.

To explain Table 5, recall that a normal subgroup $A \trianglelefteq G$ which is maximal with respect to being abelian, is necessarily self-centralizing. Thus $G/A$ is isomorphic to a subgroup of $\text{Aut}(A)$. In the present situation, the existence of the complex representation means that $A$ is necessarily isomorphic to a subgroup of one of $\mathbb{Z}_2^4$, $\mathbb{Z}_4 \times \mathbb{Z}_2^2$, $\mathbb{Z}_4^2$ or $\mathbb{Z}_8 \times \mathbb{Z}_2$. Table 5 lists a choice of $A$ for each of the nine isomorphism

---

*Magma Commands:* `D:=SmallGroupDatabase(); value:=[20, 4, 99, 0, 99, 99, 99, -2]; [ G : G in [SmallGroup(o, n) : n in [1..NumberOfSmallGroups(o)]] | (x1) : x in Classes(G) subset {1, 2, 4, 8} and IsCharacter(CharacterRing(G) ! value[x1]) : x in Classes(G))];`
Table 5: The nine non-excluded 2-groups

| No. | order | Group library | Symbol   | A       | G/A  |
|-----|-------|---------------|----------|---------|------|
| 1   | 16    | (16, 4)       | $\Gamma_2c_2$ | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 2   | 16    | (16, 5)       | $\mathbb{Z}_8 \times \mathbb{Z}_2$ | $\mathbb{Z}_8 \times \mathbb{Z}_2$ | 1     |
| 3   | 16    | (16, 10)      | $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ | $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ | 1     |
| 4   | 32    | (32, 8)       | $\Gamma_7a_3$ | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $\mathbb{Z}_4$ |
| 5   | 32    | (32, 30)      | $\Gamma_4c_1$ | $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ | $\mathbb{Z}_2$ |
| 6   | 32    | (32, 32)      | $\Gamma_4c_3$ | $\mathbb{Z}_4^2$ | $\mathbb{Z}_2$ |
| 7   | 32    | (32, 35)      | $\Gamma_4a_3$ | $\mathbb{Z}_4^2$ | $\mathbb{Z}_2$ |
| 8   | 32    | (32, 50)      | $\Gamma_5a_2$ | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $\mathbb{Z}_4^2$ |
| 9   | 64    | (64, 36)      | $\Gamma_{23}a_3$ | $\mathbb{Z}_4^2$ | $\mathbb{Z}_4$ |

classes of $G$ not contained in $M_{23}$. Some choices of $A$ (elementary abelian and cyclic) are not represented in Table 5, meaning that the corresponding $G$ is contained in $M_{23}$. The group library number refers to [BEO], the symbol for the non-abelian groups is as in [HS].

Lemma 5.9. The number of conjugacy classes of abelian subgroups $A \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4^2$, and $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ in $2^{12}:M_{24}$ containing only admissible elements is 8, 0, 6 and 7, respectively.

Proof: We construct all abelian subgroups of rank at most 3 by the following procedure. We first select representatives $a$ for each of the 7 conjugacy classes in $2^{12}:M_{24}$ of the correct Conway types. For each of those elements, we select representatives $b$ for the conjugacy classes of its centralizer in $2^{12}:M_{24}$ and check if the subgroups $H = \langle a, b \rangle$ generated by $a$ and $b$ contains only elements of admissible Conway types. In total, there are 26 such conjugacy classes of subgroups $H$. Then we select representatives $c$ for the conjugacy classes of the centralizer of the subgroup $H$ and determine the conjugacy classes of subgroups $K = \langle H, c \rangle$ in $2^{12}:M_{24}$. There are 41 such conjugacy classes of subgroups $K$ containing only admissible elements. The resulting number of conjugacy classes for each choice of $A$ is as stated in the Lemma.

Theorem 5.10. Suppose $G$ is an admissible 2-group. Then either $G$ is isomorphic to a subgroup of $M_{23}$, or is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2^2$, or is contained in a group of order 32 or 64 described below (Table 5, nos. 8 and 9).

Proof: We have to check which of the nine groups $G$ of Proposition 5.8 (cf. Table 5) are isomorphic to subgroups of $\text{Co}_0$ and contain only admissible elements. Note that because the monomial subgroup $2^{12}:M_{24}$ contains a Sylow 2-subgroup of $\text{Co}_0$, the relevant calculations can be carried out in the monomial group.
For each conjugacy class of abelian subgroups $A \subseteq 2^{12}:M_{24}$ as in Lemma 5.9, we compute the normalizer $N(A)$ in $2^{12}:M_{24}$ and check if it contains a group $G$ as in Table 3 containing only admissible elements. This is done by selecting a representative $d$ of each conjugacy class of elements in $N(A)$ such that $d^4$ is in $A$ and determining the isomorphism type of $\langle A, d \rangle$. In the case of group no. 8 in Table 5, we also select in addition all elements $e$ of $N(A)$ such that $e^2$ and the commutator $[e, d]$ is in $A$ and determine the isomorphism type of $\langle A, d, e \rangle$. It transpires that only groups no. 3, 4, 8, and 9 occur. Moreover, group no. 4 is a subgroup of group no. 9.

6 Conjugacy classes of admissible subgroups in $\text{Co}_0$

In the previous section, we determined by largely theoretical arguments the abstract isomorphism types of admissible subgroups of $\text{Co}_0$.

For the classification of group lattices $(\Lambda_G, G)$, we will enumerate the conjugacy classes of admissible subgroups $G \subseteq \text{Co}_0$. The following main result of this section depends heavily on computer calculations:

**Theorem 6.1.** There are 198 conjugacy classes of admissible subgroups $G \subseteq \text{Co}_0$. In particular, there are exactly 22 classes which are maximal (with respect to containment), which are described as follows:

(a) thirteen subgroups of $M_{23}$ (Table 4);
(b) two groups $3^4.A_6$ and $3^{1+4}:2^2$ related to $S$-lattices;
(c) two groups of order 48 and five $2$-groups (Table 7).

Detailed information about all of these conjugacy classes can be found in Table 12 in the appendix. The corresponding group lattices $(\Lambda_G, G)$, which we will discuss in the next section, are pairwise nonisomorphic (cf. Table 13 in the appendix).

**Remark:** There are just 82 admissible conjugacy classes of $G$ containing only $K3$-classes of elements. The corresponding group lattices $(L_G, G)$ were first determined by Hashimoto [Ha], who used the 23 Niemeier lattices with roots rather than the Leech lattice that we use here. These 82 classes are in 1-1 correspondence with the combinatorial structure of symplectic group actions on a K3 surface, as determined by Xiao [Xi].

In the following, we describe our method of computing admissible conjugacy classes of subgroups $G \subseteq \text{Co}_0$ using Magma. We first realized $\text{Co}_0$ as a group of integral $24 \times 24$ matrices starting from an explicit description of the Leech lattice. This realization was used to determine the conjugacy classes of $\text{Co}_0$ and to compute the dimension of $\Lambda^G$ for a subgroup $G \subseteq \text{Co}_0$. In addition, a realization as permutation group on the 196560 minimal vectors of $\Lambda$ together with an explicit isomorphism with the matrix group realization was constructed. This realization was used to
check if two subgroups of Co₀ are conjugate. Computations with both realizations are relatively time consuming and had to be minimized.

By Theorem 3.2, an admissible subgroup \(G\) is either conjugate to a subgroup of the monomial group \(2^{12}:M_{24}\) or a subgroup of one of three groups related to \(S\)-lattices. We identified \(2^{12}:M_{24}\) inside the permutation representation of Co₀ by computing the stabilizer of 1104 norm 4 vectors \((\pm u \pm v)/2\), where \(\pm u\) and \(\pm v\) run through a coordinate frame of \(\Lambda\). For the \(S\)-lattices, we determined the pointwise stabilizers by using the explicit realizations given by Curtis [Cu]. For the \(S\)-lattice of rank 6, we computed in addition the normalizer of the stabilizer in Co₀. For calculations inside \(2^{12}:M_{24}\), we used a realization as a permutation group on the 48 elements. We also determined an explicit isomorphism between this permutation realization of \(2^{12}:M_{24}\) and the above described frame stabilizer inside Co₀. For calculations inside the three \(S\)-lattice groups, we constructed a permutation representation of much smaller degree together with an explicit isomorphism with the corresponding subgroups in Co₀. This allowed us to determine their complete subgroup lattice. For each conjugacy class of subgroups we constructed the corresponding subgroup of Co₀ and selected the admissible one. Finally, we determined the Co₀-conjugacy classes.

For the subgroups \(G \subseteq 2^{12}:M_{24}\) we distinguished two cases: 2-groups and non-2-groups. For the non-2-groups, we were able to construct the corresponding part of the subgroup lattice of \(2^{12}:M_{24}\) completely. Starting from \(2^{12}:M_{24}\), we constructed inductively via the order all maximal subgroups of fixed order of the already found conjugacy classes. Then we checked for a given order all the found maximal subgroups for conjugacy in \(2^{12}:M_{24}\). Since the number of required conjugacy checks is growing almost quadratically with the number of such subgroups, we first determined for each subgroup the size of each of its conjugacy classes, using this as a numerical indicator for non-conjugacy and thereby reducing the number of conjugacy checks in \(2^{12}:M_{24}\). Overall, there are 279,343 classes of such subgroups in \(2^{12}:M_{24}\). Just 280 of these classes contain only admissible elements and of these, 241 classes are admissible. They belong to 94 Co₀-conjugacy classes.

It follows from Theorem 5.6 that the admissible subgroups of \(2^{12}:M_{24}\) of order divisible by 3, 5, 7, or 11 are isomorphic to subgroups of the set \(\mathcal{T}\) of isomorphism classes of subgroups of \(M_{23}\) maximal with respect to having at least four orbits on \(\Omega\). In most cases, there exist several \(2^{12}:M_{24}\) conjugacy classes for a group in \(\mathcal{T}\). With respect to Co₀-conjugacy however, our calculations show:

**Theorem 6.2.** For each of the groups in \(\mathcal{T}\), there exists a unique Co₀-conjugacy class inside \(2^{12}:M_{24}\). □

However, for groups of order \(2^f.3\) there are further possibilities:

**Theorem 6.3.** Suppose that \(G \subseteq 2^{12}:M_{24}\) is admissible and not a 2-group. Then \(G\) is conjugate in Co₀ to a subgroup of either a group in the set \(\mathcal{T}\) of subgroups of \(M_{23}\) (cf. Theorem 5.6), or one of two conjugacy classes of groups of order 48. □
Although the two groups of order 48 project isomorphically to subgroups in $M_{23} \subseteq M_{24}$, they are not conjugate.

We collect basic information about the groups in $\mathcal{T}$ in Table 6 and the two groups of order 48 in Table 7. The entries of the first four columns is self-explanatory, the column “torsion” gives the structure of the discriminant group $(\Lambda^G)^*/\Lambda^G$. The last three columns give information on all $2^{12}:M_{24}$-conjugacy classes of $G$. To explain this, let $E = Z^2_2$ and $M = M_{24}$, so that the monomial group is $E:M$. Then in Table 6 and 7, $A = G \cap E$ and $P = G/A \subseteq M$. The seventh column describes the orbits of $P$ on $\Omega$, and a star in the last column indicates that $G \cong A.P$ is not a subgroup $A:P$ of $E:M$.

For the 2-subgroups of $2^{12}:M_{24}$, we were unable to compute the corresponding part of the subgroup lattice because the number conjugacy classes became too large. Instead we determined only the admissible one. First we selected a 2-Sylow subgroup $P \subseteq 2^{12}:M_{24}$ and hence $Co_0$. All $P$-conjugacy classes of admissible 2-groups $G \subseteq P$ were determined, starting with the trivial group and successively adding further elements. Finally, we tested for $2^{12}:M_{24}$-conjugacy and $Co_0$-conjugacy. The numbers of such conjugacy classes of groups for a given order are listed in Table 8.

In particular, we obtained the following result:

**Theorem 6.4.** Let $G \subseteq Co_0$ be an admissible 2-group. Then $G$ is conjugate to a subgroup of either a group in the set $\mathcal{T}$ of subgroups of $M_{23}$ as described in Theorem 5.6, or of one of 5 conjugacy classes of 2-groups listed in Table 7. □

**Remarks:** Theorem 6.4 implies Theorem 5.10. Since the computations for Theorem 6.4 take much longer and both computations are independent, we have presented both of them.

Although the 2-group with entry #2 in the library of groups of order 16 is one of the 5 maximal groups in Theorem 6.4, it is also isomorphic to a subgroup of $M_{23}$.

As a final step, we took all the admissible conjugacy classes of $Co_0$ obtained from the $S$-lattices and from $2^{12}:M_{24}$, rechecked for $Co_0$-conjugacy, and determined the corresponding subgroup lattice structure. The result is Theorem 6.1 and the information given in Table 12 in the appendix.

### 7 Subgroups of $O(L)$

In this section, we investigate which conjugacy classes of subgroups $G$ and lattices $\Lambda_G$ found in the previous section can indeed be realized via symmetries of the lattice $L = H^2(X, Z) \cong E_8(-1)^2 \oplus U^3 \oplus \langle -2 \rangle$.

For most of the realizable groups $G$, we will also determine the exact number of corresponding isomorphism classes of embeddings $(L_G, G) \rightarrow (L, O(L))$. 35
Table 6: Maximal admissible groups contained in $M_{23}$

| No. | $G$             | $|G|$ | $rkA^G$ | Torsion          | $2^{12}:M_{24}$-classes | orbits       |
|-----|-----------------|------|---------|------------------|-------------------------|-------------|
| 1   | $L_2(11)$       | 660  | 4       | $[11,11]$        | $[660,1]$               | $[1,1,11,11]$|
| 2   | $L_3(4)$        | 20160| 4       | $[2,42]$         | $[20160,1]$             | $[1,1,1,21]$|
| 3   | $A_7$           | 2520 | 4       | $[105]$          | $[2520,1]$              | $[1,1,7,15]$|
| 4   | $Z_2^3:L_2(7)$  | 1344 | 4       | $[4,28]$         | $[1344,1]$              | $[1,7,8,8]$  |
|     |                 |      |         |                  |                         |             |
|     |                 |      |         |                  |                         |             |
| 5   | $Z_2 \times L_2(7)$ | 336 | 4       | $[14,14]$        | $[336,1]$               | $[1,2,7,14]$|
|     |                 |      |         |                  |                         |             |
| 6   | $Z_2^4:A_6$     | 5760 | 4       | $[4,24]$         | $[5760,1]$              | $[1,1,6,16]$|
|     |                 |      |         |                  |                         |             |
| 7   | $Z_2^4:S_5$     | 1920 | 4       | $[4,40]$         | $[1920,1]$              | $[1,1,2,20]$|
|     |                 |      |         |                  |                         |             |
|     |                 |      |         |                  |                         |             |
| 8   | $S_6$           | 720  | 4       | $[6,30]$         | $[720,1]$               | $[2,6,6,10]$|
|     |                 |      |         |                  |                         |             |
| 9   | $M_{10}$        | 720  | 4       | $[2,60]$         | $[720,1]$               | $[1,2,6,15]$|
| 10  | $(Z_3 \times A_5):Z_2$ | 360 | 4       | $[15,15]$        | $[360,1]$               | $[1,3,5,15]$|
| 11  | $Q(Z_3^2:Z_2)$  | 1152 | 4       | $[8,24]$         | $[1152,1]$              | $[1,3,4,16]$|
|     |                 |      |         |                  |                         |             |
|     |                 |      |         |                  |                         |             |
| 12  | $Z_2^4:(S_3 \times S_3)$ | 576 | 4       | $[12,24]$        | $[576,1]$               | $[2,3,3,16]$|
|     |                 |      |         |                  |                         |             |
|     |                 |      |         |                  |                         |             |
| 13  | $Z_2^3:QD_{16}$ | 144  | 4       | $[6,36]$         | $[144,1]$               | $[1,2,9,12]$|
Table 7: Maximal admissible groups contained in $2^{12}:M_{24}$ but not in $M_{23}$

| No. | $G$       | $|G|$ | rk$A^G$ | Torsion | $2^{12}:M_{24}$-classes | orbits |
|-----|-----------|------|---------|---------|-------------------------|--------|
| 1   | $< 48,49 >$ | 48   | 5       | $[2,2,2,6,12]$ | $[48,1]$ | $[2,2,6,6,8]$ | *      |
|     |           |      |         |         | $[24,2]$ | $[1,1,3,3,8]$ | *      |
|     |           |      |         |         | $[24,2]$ | $[1,1,2,6,6]$ | *      |
|     |           |      |         |         | $[6,8]$  | $[1,1,2,3,3,6,6]$ | *      |
| 2   | $< 48,32 >$ | 48   | 4       | $[2,2,4,12]$  | $[48,1]$ | $[2,6,8]$  | *      |
|     |           |      |         |         | $[24,2]$ | $[1,1,6,8]$  | *      |
|     |           |      |         |         | $[24,2]$ | $[1,1,2,6,6]$ | *      |
|     |           |      |         |         | $[24,2]$ | $[1,1,2,6,6]$ | *      |
| 3   | $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ | 16   | 7       | $[2,2,2,4,4,4]$ | $[16,1]$ | $[2,2,2,2,4,4,8]$ | *      |
|     |           |      |         |         | $[8,2]$  | $[1,1,1,2,2,8]$ | *      |
|     |           |      |         |         | $[8,2]$  | $[1,1,2,2,4,8]$ | *      |
|     |           |      |         |         | $[8,2]$  | $[1,1,2,2,2,2,2,8]$ | *      |
|     |           |      |         |         | $[8,2]$  | $[1,1,2,2,2,2,2,8]$ | *      |
|     |           |      |         |         | $[4,4]$  | $[1,1,1,2,2,2,2,4,4]$ | *      |
|     |           |      |         |         | $[2,8]$  | $[1,1,1,1,1,1,1,2,2,2,2,2,2]$ | *      |
| 4   | $\mathbb{Z}_4^2$ | 16   | 6       | $[2,2,4,4,4,4]$ | $[16,1]$ | $[4,4,4,4,4,4]$ | *      |
|     |           |      |         |         | $[8,2]$  | $[1,1,2,2,4,4]$ | *      |
|     |           |      |         |         | $[4,4]$  | $[1,1,1,1,2,2,2,2,2,4,4]$ | *      |
| 5   | $\Gamma_3 a_2$ | 32   | 6       | $[2,2,4,4,4,4]$ | $[8,4]$  | $[1,1,2,2,2,2,2,2,2,8]$ | *      |
| 6   | $< 32,8 >$ | 32   | 4       | $[2,2,8,8]$  | $[8,4]$  | $[1,1,2,2,2,4,4,8]$ | *      |
|     |           |      |         |         | $[8,4]$  | $[1,1,2,2,2,4,4,8]$ | *      |
| 7   | $< 64,36 >$ | 64   | 4       | $[2,2,2,16]$ | $[16,4]$ | $[1,1,1,4,8,8]$ | *      |

Table 8: Conjugacy classes of admissible 2-groups

| order | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | total |
|-------|---|---|---|---|----|----|----|-----|-----|-------|
| # $P$-classes | 1 | 27 | 317 | 1312 | 2190 | 803 | 239 | 30 | 0 | 4919 |
| # $2^{12}:M_{24}$-classes | 1 | 2 | 7 | 27 | 63 | 34 | 20 | 3 | 0 | 157 |
| # Co0-classes | 1 | 1 | 2 | 6 | 15 | 10 | 6 | 1 | 0 | 42 |
For each of the 198 Co₀-classes of admissible groups G, we first determine the isomorphism classes of coinvariant lattices \( L_G \cong \Lambda_G(-1) \). This is easily done with Magma. There are 69 such lattices, listed in Table 13 in the appendix.

The lattices \( K \) in Table 13 are naturally divided into three main types depending on the relation between \( \text{rk} K \) and \( \text{rk} A_K \). Setting \( \alpha(K) := 24 - \text{rk} K - \text{rk} A_K \), we find the following possibilities:

(a) 54 lattices with \( \alpha(K) \geq 2 \): these are the 13 lattices corresponding to the maximal subgroups of \( M_{23} \) with at least four orbits and the 41 lattices coming from symplectic group actions on K3 surfaces as classified in [Ha].

(b) 4 lattices with \( \alpha(K) = 1 \): two of them are corresponding to the two maximal S-lattice groups, the other two are corresponding to certain subgroups.

(c) 11 lattices with \( \alpha(K) = 0 \): they correspond to 2-groups or groups of order 2\(^f\).3 in \( 2^{12}:M_{24} \) not conjugate to a subgroup of \( M_{23} \).

Apart from the non-canonically defined 2-adic genus symbol, Table 10.2 of [Ha] seems to be in complete agreement with our Tables 12 and 13.

For certain lattices \( K \), there is more than one group such that \( K \cong L_G \). For such \( K \) we list all classes of groups \( G \) with \( K \cong L_G \) in Table 14. For the K3 cases, this again is in accord with Table 10.4 of [Ha]. Let \( G(K) \) be the classes of groups \( G \) from Table 12 which have the same \( K \cong L_G \).

**Theorem 7.1.** The set \( G(K) \) contains a unique maximal class \( G_{\text{max}}(K) \). The other classes \( H \in G(K) \) correspond to subgroups of \( G_{\text{max}}(K) \). More precisely, the classes in \( G(K) \) can be identified with the \( O(K) \)-conjugacy classes of subgroups \( H \) of \( O_0(K) \) which have a trivial fixed-point lattice \( K^H \) and are contained in a conjugacy class \( G_{\text{max}}(K) \). Furthermore, for the lattices of type (a) and (b) one has \( G_{\text{max}}(K) = O_0(K) \). □

To study embeddings \( (L_G, G) \to (L, O(L)) \), we have to find a lattice \( L_G \) of rank \( 23 - \text{rk} L_G \) and an extension of \( L_G \oplus L_G \) to \( L \). Such an extension is described by a glue code \( C \subseteq A_{L_G} \oplus A_{L_G} \) which has to be an isotropic subspace with respect to the quadratic form \( q_{L_G} \otimes q_{L_G} \) for \( A_{L_G} \oplus A_{L_G} \). The resulting extension \( K_C \supseteq L_G \oplus L_G \) is isomorphic to \( L \) exactly when the discriminant form on \( A_{K_C} \) is isomorphic to the discriminant form on \( A_L \), since there is only one lattice in the genus of \( L \). Two lattices \( K_C \) and \( K_{C'} \) determine \( O(L) \)-conjugate sublattices \( L_G \) inside \( L \) if, and only if, \( C \) and \( C' \) are in the same orbit for the action of \( O(L_G) \times O(L_G) \) on \( A_{L_G} \oplus A_{L_G} \) induced by the natural action of \( O(L_G) \times O(L_G) \). For a fixed lattice \( K_C \), the \( O(L) \)-conjugacy classes of \( G \subseteq O_0(L_G) \) are given by the \( F \)-conjugacy classes where \( F \subseteq O(L_G) \) is the image of the projection on the first factor of the stabilizer of \( C \) under the action of \( O(L_G) \times O(L_G) \).
Since $L^G$ and $L_G$ are both primitive sublattices of $L$, the code $C$ must have the form $C = \{(x, y) \mid x \in A_{L_G}, y \in A_{L_G}\}$ and satisfy $(x, 0) \in C \Rightarrow x = 0$ and $(0, y) \in C \Rightarrow y = 0$. Since $L$ has a discriminant group $A_L$ of order 2, this implies that $|C|^2 = 2|A_{L_G}||A_{L_G}|$, and either

1. $C = \{(x, y) \mid x \in A_{L_G}\}$ where $\gamma : A_{L_G} \to A_{L_G}$ is a group monomorphism with $q_{L_G} \circ \gamma = -q_{L_G}$, or

2. $C = \{(\gamma'(y), y) \mid y \in A_{L_G}\}$ where $\gamma' : A_{L_G} \to A_{L_G}$ is a group monomorphism with $q_{L_G} \circ \gamma' = -q_{L_G}$.

Assume we are in case (1). Then $|A_{L_G}/\gamma(A_{L_G})| = 2$. Since $\gamma(A_{L_G})$ is a non-degenerate subspace of $A_{L_G}$ with respect to $q_{L_G}$, there is an orthogonal decomposition $A_{L_G} = \gamma(A_{L_G}) \oplus \gamma(A_{L_G})$ and $\gamma(A_{L_G})$ is generated by an element $v_\gamma \in A_{L_G}$ of order 2 that satisfies $q|_{v_\gamma} = q_L$. Thus in order to describe the $(L^G) \times (L^G)$ orbits of codes $C$, we first determine the $(L^G)$ orbits of splittings $A_{L_G} = \gamma(A_{L_G}) \oplus (v_\gamma)$ and then the $(L^G) \times S$-orbits of maps $\gamma : A_{L_G} \to \gamma(A_{L_G})$ as in (1), where $S$ is the stabilizer of $v_\gamma$ in $(L^G)$ acting on $\gamma(A_{L_G})$. Fixing some $\gamma$ allows us to identify $S$ with a subgroup of $O(A_{L_G})$ and the $(L^G) \times S$-orbits with the double cosets of the pair $(\bar{O}(L_G), S)$ in $O(A_{L_G})$.

For case (2), $\gamma'(A_{L_G})$ is a subgroup of index 2 in $A_{L_G}$ and there is an orthogonal decomposition $A_{L_G} = \gamma'(A_{L_G}) \oplus \gamma'(A_{L_G})$ where $\gamma'(A_{L_G})$ is generated by an element $w_{\gamma'} \in A_{L_G}$ of order 2 with $q|_{w_{\gamma'}} = q$. So to describe the $(L^G) \times (L^G)$ orbits of codes $C$, we first determine the $(L_G)$ orbits of splittings $A_{L_G} = \gamma'(A_{L_G}) \oplus (w_{\gamma'})$ and then the $S \times (L^G)$-orbits of maps $\gamma' : A_{L_G} \to \gamma(A_{L_G})$ as in (2), where now $S$ is the stabilizer of $w_{\gamma'}$ in $(L^G)$ acting on $\gamma'(A_{L_G})$. Fixing a $\gamma'$ again permits us to identify $(L^G)$ with a subgroup of $O(\gamma'(A_{L_G}))$ and the $S \times (L^G)$-orbits with the double cosets of the pair $(S, (L^G))$ in $O(A_{L_G})$.

For the above discussion see also [Ni], especially Proposition 1.5.1.

We are now ready for the proof of Lemma 4.2. We will apply the preceding discussion with $G = \langle g \rangle$.

**Proof:** We have

$$
\alpha(L^g) := 24 - \text{rk } L_g - \text{rk } A_{L_g} = 24 - \text{rk } A_g - \text{rk } A_{A_g} = \text{rk } A_g^\perp - \text{rk } A_{A_g} = 0.
$$

This means that $L_g$ is a lattice of type (c), and the glueing of $L_g$ with $L^g$ is described by case (2), i.e., by an injective map $\gamma' : A_{L^g} \to A_{L_g} \cong A_{A_g}$. This proves part (a) of Lemma 4.2.

For part (b), note that $A_{L^g} = \gamma'(A_{L^g}) \oplus (w_{\gamma'})$. Since $q_{L_g}(w_{\gamma'}) = \frac{3}{2}$, it follows that $q_{L^g}$, and therefore also $q_{A_g}$, has $\frac{3}{2}$ in its image. \(\square\)

Returning to a general group $G$, we proceed according to the above discussion for a fixed pair $(L_G, G)$ by the following five steps:

1. We fix one of the constructions (1) or (2).
2. We determine all lattices in the genus for $L^G$ which is uniquely determined by the genus of $L_G$.

3. We determine the $O(L^G)$-orbits of splittings $A_{L^G} = \gamma(A_{L_G}) \oplus \langle v_\gamma \rangle$ resp. the $O(L_G)$-orbits of splittings $A_{L_G} = \gamma'(A_{L_G}) \oplus \langle w_{\gamma'} \rangle$.

4. We determine the double cosets for $O(L_G) - S$ in $O(A_{L_G})$ (construction (1)) resp. for $S - O(L^G)$ in $O(A_{L_G})$ (construction (2)).

5. We determine the $O(L)$-conjugacy classes of $G$ for a fixed double coset.

**Construction (1).** We first note that we only have to consider lattices of type (a) or (b). Indeed, for lattices of type (c) we have $rk L^G = 23 - rk L_G < 24 - rk L_G = rk A_{L_G}$, whence we cannot embed $A_{L_G}$ into $A_{L_G}$ using $\gamma$.

First we consider lattices $L_G$ of rank 20. From Table 13 we see that there are 13 lattices of type (a) corresponding to the maximal $G \subseteq M_{23}$ with at least four orbits in the usual action on 24 letters, and the two lattices of type (b) corresponding to the two maximal $S$-lattice groups. In this case, $L^G$ has to be positive-definite of rank 3 and the quadratic form $q_{L^G}$ is equivalent to $q_{L_G} \oplus q_L$. Recall (proof of Lemma 4.2) that $q_L(x) = \frac{3}{2}$ for the non-zero element $x$ in $A_L$. This uniquely determines the genus of $L^G$, and the corresponding lattices $L^G$ can be read off from the Brandt-Intrau tables of positive definite ternary forms [III]. The result is listed in Table 9.

In the case when $G \cong M_{10}$ and $L^G$ is the lattice with Gramian matrix Diag$(2,4,30)$, there are two $O(L^G)$-orbits of splittings $A_{L_G} = \gamma(A_{L_G}) \oplus \langle v_\gamma \rangle$, whereas in all other cases there is a unique $O(L^G)$-orbit of splittings.

We assert that there is a unique $O(L_G) - S$ double coset in $O(A_{L_G})$. If $G \not\cong S_6$ then $O(L_G)$ coincides with $O(A_{L_G})$, and the assertion follows. A computation shows that the same result holds if $G \cong S_6$.

Finally, let $F \subseteq O(L_G)$ be the projection onto the first factor of the stabilizer in $O(L_G) \times S$ of the identity element under the double coset action. Let $F \subset O(L_G)$ be the inverse image of $F$ under the natural projection. We consider the conjugation action of $F$ on the set of subgroups $H \subseteq O_0(L_G)$ with $L_G^H = \{0\}$. For lattices of type (a), the orbits agree with the $O(L_G)$ classes, whereas for the two lattices of type (b), there are more orbits. We list the number of conjugacy classes $H$ for both groups in the last two columns of Table 9.

Now let $L_G$ be of rank smaller than 20. In this case, $L^G$ has rank at least 4 and must be indefinite. We claim that the lattice $L^G$ exists and is unique. Indeed, the 41 lattices $L_G$ of type (a) are sublattices of the K3-lattice $N \cong E_8(-1)^2 \oplus U^3$. If $N^G$ is an orthogonal complement of $L^G$ in $N$ then we let $L^G = N^G \oplus A_1(-1)$. If the uniqueness criterion of Theorem 1.7 of [Ha] (which follows from Eichler's theory of Spinor genera, cf. [M]) applies to $N^G$ then it also applies to $L^G$. Therefore, the explicit verification in Section 6 of [Ha] establishes uniqueness for all 30 lattices $L^G$ of type (a) and rank $> 4$. For the 11 lattices $L^G$ of rank 4, we apply Theorem 1.7
of \[\mathbb{H}_3\] directly. For the two lattices \(L_G\) of type (b) and rank 19 and 18, one gets for \(L_G\) the two lattices with Gramian matrix
\[
\begin{pmatrix}
0 & 3 & 0 & 0 \\
3 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 \\
0 & 0 & 0 & 6
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 3 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 6 & 0
\end{pmatrix},
\]
respectively. Again, the uniqueness follows from Theorem 1.7 of \[\mathbb{H}_3\].

**Conjecture 7.2.** For \(L_G\) a lattice of type (a) or (b) and \(\text{rk} \, L_G \geq 4\), we have \(O(L_G) = O(A_{L_G})\).

By applying the criterion from Theorem 1.14.2 in \[\mathbb{N}_3\], one checks that the conjecture holds for all \(L_G\) of type (a) and (b) in Table 13, except for the ten cases with number
\(n = 18, 26, 31, 39, 41, 47, 48, 54, 55, 61\).

Unfortunately, most lattice function of Magma are presently implemented for definite lattices only. So we cannot verify this conjecture by a computer calculation without extra programming work. One could do this more theoretically as in \[\mathbb{H}_3\], Section 7, by using the strong approximation theorem, but we do not investigate this here.

If Conjecture 7.2 holds, it is clear that there is a single \(O(L_G)\)-orbit of orthogonal splittings \(A_{L_G} = \gamma'(A_{L_G}) \oplus \langle w_\gamma' \rangle\). Furthermore, we would have \(S = O(A_{L_G})\) and \(F = O(L_G)\). It then follows from Theorem 7.1 that for each \((L_G, G)\), there is a unique \(O(L)\)-conjugacy class of subgroups \(G\).

**Remark:** Since Conjecture 7.2 holds in particular for the three K3-lattices \(L_G\) of rank 19 (numbers 44, 46 and 52 in Table 13) for which there are two lattices in the genus of \(N_G\), it follows that the corresponding symplectic group actions on K3 are not deformation equivalent. However, the induced symplectic group actions on \(K3^{[2]}\) are so.

Our calculations have shown:

**Theorem 7.3.** Let \(G \subseteq \text{Co}_0\) be an admissible subgroup. Then \((L_G, G)\) can be realized as the coinvariant lattice for a subgroup \(G \subseteq O(L)\) by construction (1) if, and only if, \(\alpha(L_G) \geq 1\). If \(\text{rk} \, L_G = 20\), the number of corresponding conjugacy classes of \(G \subseteq O(L)\) can be read off from Table 18. For \(\text{rk} \, L_G < 20\), there is a unique such class, provided that either \(L_G\) is not one of the cases no. 18, 26, 31, 39, 41, 47, 48, 54, 55, 61 in Table 18 or Conjecture 7.2 holds.

**Construction (2).** We first study the \(O(L_G)\)-orbits of splittings \(A_{L_G} = \gamma'(A_{L_G}) \oplus \langle w_\gamma' \rangle\). Thus we searched for elements \(w_\gamma' \in A_{L_G}\) of order 2 with \(q_{L_G}(w_\gamma') = 3/2\).
Table 9: Conjugacy classes in $O(L)$: rank $L^G = 3$, construction (1)

| No. | $G$                  | $|A_L^G|$ | $|O(L^G)|$ | $|\bar{O}(L^G)|$ | $|O(A_L^G)|$ | $L^G$                                           | $|O(L^G)|$ | $|\bar{O}(L^G)|$ | $|O(A_L^G)|$ | $\#H$ | $\#orbs$ |
|-----|----------------------|----------|-----------|------------------|------------|------------------------------------------------|----------|------------------|------------|------|---------|
| 1   | $L_2(11)$            | 121      | 15840     | 24               | 24         | $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix}$ | 8        | 4               | 24         | 3   | 3       |
| 2   | $L_3(4)$             | 84       | 483840    | 24               | 24         | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 10 & 4 \\ 0 & 4 & 10 \end{pmatrix}$ | 8        | 4               | 24         | 1   | 1       |
| 3   | $A_7$                | 105      | 20160     | 8                | 8          | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 10 & 70 \\ 0 & 2 & 0 \end{pmatrix}$ | 8        | 4               | 8          | 1   | 1       |
| 4   | $Z_3^2: L_2(7)$      | 112      | 21504     | 16               | 16         | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 2 & 10 \end{pmatrix}$ | 4        | 4               | 16         | 3   | 3       |
| 5   | $Z_2 \times L_2(7)$ | 196      | 10752     | 32               | 32         | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ | 16       | 8               | 32         | 3   | 3       |
| 6   | $Z_2^3 : A_6$        | 96       | 92160     | 16               | 16         | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 24 \end{pmatrix}$ | 8        | 4               | 16         | 7   | 7       |
| 7   | $Z_2^3 : S_5$        | 160      | 30720     | 16               | 16         | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ | 8        | 8               | 16         | 2   | 2       |
| 8   | $S_6$                | 180      | 23040     | 32               | 96         | $\begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 24       | 24              | 96         | 1   | 1       |
| 9   | $M_{10}$             | 120      | 5760      | 8                | 8          | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 8        | 4               | 16         | 1   | 1       |
| 10  | $(Z_3 \times A_5) : Z_2$ | 225   | 17280     | 48               | 48         | $\begin{pmatrix} 4 & 1 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix}$ | 8        | 8               | 48         | 5   | 5       |
| 11  | $Q(Z_3^2 : Z_4)$    | 192      | 36864     | 32               | 32         | $\begin{pmatrix} 2 & 6 & 0 \\ 0 & 2 & 14 \end{pmatrix}$ | 8        | 8               | 64         | 10  | 10      |
| 12  | $Z_3^2 : (S_3 \times S_3)$ | 288    | 36864     | 64               | 64         | $\begin{pmatrix} 0 & 6 & 0 \\ 0 & 0 & 24 \end{pmatrix}$ | 8        | 8               | 64         | 2   | 2       |
| 13  | $Z_2^3 : QD_{16}$   | 216      | 3456      | 24               | 24         | $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 10 & 0 \\ 0 & 0 & 12 \end{pmatrix}$ | 8        | 8               | 48         | 2   | 2       |
| 14  | $3^{1+4} : 2.2^2$   | 108      | 186624    | 96               | 96         | $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ | 48       | 48              | 288        | 7   | 8       |
| 15  | $3^4 : A_6$         | 81       | 4199040   | 144              | 144        | $\begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ | 24       | 24              | 144        | 40  | 71      |
Table 10: Conjugacy classes in $O(L)$: rank $L^G = 3$, construction (2)

| No. | $G$            | $|A_L|$ | $|O(L_G)|$ | $|O(L_G^*)|$ | $|O(A_L)|$ | $L^G$ | $|O(L^G)|$ | $|O(L_G^*)|$ | $|O(A_L)|$ |
|-----|----------------|-------|----------|------------|----------|-------|----------|------------|----------|
| 1   | $M_{10}$       | 120   | 5760     | 8          | 8        |       | 8        | 4          | 8        |
|     |                |       |          |            |          | $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 8 \end{pmatrix}$ | $\begin{pmatrix} 8 & 4 & 0 \\ 4 & 2 & 2 \\ 2 & 4 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 8 \\ 2 & 1 & 6 \end{pmatrix}$ | $\begin{pmatrix} 8 & 4 & 8 \\ 8 & 8 & 8 \end{pmatrix}$ |
| 2   | $Z_2^3:QD_{16}$| 216   | 3456     | 24         | 24       |       | 24       | 8          | 8        |
|     |                |       |          |            |          | $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ | $\begin{pmatrix} 8 & 0 & 0 \\ 2 & 1 & 10 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix}$ | $\begin{pmatrix} 8 & 8 & 24 \\ 8 & 8 & 24 \end{pmatrix}$ |
| 3   | $\langle 32, 36 \rangle$ | 256   | 196608   | 256        | 256      |       | 16       | 8          | 32       |
|     |                |       |          |            |          | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ | $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 16 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 10 \end{pmatrix}$ | $\begin{pmatrix} 16 & 8 & 32 \\ 16 & 8 & 32 \end{pmatrix}$ |
| 4   | $\langle 64, 36 \rangle$ | 128   | 36864    | 96         | 96       |       | 96       | 16         | 4        |
|     |                |       |          |            |          | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ | $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 16 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 10 \end{pmatrix}$ | $\begin{pmatrix} 16 & 8 & 32 \\ 16 & 8 & 32 \end{pmatrix}$ |

(mod 2). Such elements exist only for the eleven lattices $L_G$ with the numbers

$$n = 9, 14, 18, 20, 27, 28, 33, 38, 40, 45, 50$$

in Table 13 and in each case there is a unique $O(L_G)$-orbit.

The discriminant form of $L^G$ is equal to $-q_{L^G}|w^*_\gamma$, so in each case the genus of $L^G$ is uniquely determined. For the four cases $n = 58, 45, 38$ and 27 with $rk L_G = 20$, Table 10 lists $L^G$ together with additional information concerning $O(L_G)$ and $O(L_G^*)$. In the other seven cases $L^G$ is indefinite, and one easily checks that $L^G$ exists and is unique in these cases.

For reasons which will become apparent in the next section, we will not further analyze the exact number of $O(L)$-conjugacy classes of realizations of the group lattice $(L_G, G)$ inside $L$. We have established:

**Theorem 7.4.** Let $G \subseteq Co_0$ be an admissible subgroup. Then $(L_G, G)$ can be realized as the coinvariant lattice for a subgroup $G \subseteq O(L)$ by construction (2) if, and only if, $L_G$ is one of cases no. 9, 14, 18, 20, 27, 28, 33, 38, 40, 45, 50 in Table 13.

We note that there are admissible subgroup $G \subseteq Co_0$ for which $(L_G, G)$ cannot be realized in this way.

**Remark:** We also have applied our above method to prove the uniqueness of the conjugacy class of $G$ in the isometry group $O(N)$ of the K3-lattice $N$ for the 11 K3-lattices $(L_G, G)$ of rank 19 and the corresponding 14 rank 3 lattices $N^G$. This provides a somewhat more systematic approach for these cases compared to Section 8.2 of [Ha].

### 8 Symplectic actions on $K3^{[2]}$

In this section, we will show that the groups $G \subseteq O(L)$ that can be realized by isometries of a hyperkähler manifold of type $K3^{[2]}$ are exactly those for which $L_G$ is
a lattice of type (a) or (b).

Regarding birational maps of a hyperkähler manifold of type $K3[2]$, there is the following characterization by Mongardi ([Mo2], Thm. 3.6):

**Theorem 8.1.** Let $G \subseteq O(L)$ be a finite subgroup, and suppose that the coinvariant lattice $L_G$ is negative-definite and contains no roots of norm $-2$. Then $G$ is induced by a group of birational transformations of some hyperkähler manifold of type $K3[2]$.

Thus we have:

**Corollary 8.2.** The conjugacy classes of groups $G \subseteq O(L)$ found in Section 7 arise as finite group of birational transformations of some hyperkähler manifold of type $K3[2]$.

We note that there are further subgroups of $G \subseteq O(L)$ which arise from birational transformations. The groups $G \subseteq O(L)$ in Section 7 have been constructed from admissible subgroups $G \subseteq Co_0$ only.

Based on work on the global Torelli theorem for hyperkähler manifolds, Mongardi gave the following criterion regarding symplectic automorphisms of type $K3[n]$ ([Mo5], Thm. 4.1) which we state here for the case $K3[2]$:

**Theorem 8.3.** Let $G \subseteq O(L)$ be a finite group. Then $G$ is induced by a group of symplectic automorphisms for some manifold $X$ of type $K3[2]$ if, and only if, the following holds:

- $L_G$ is negative-definite;
- $L_G$ contains no numerical wall divisor.

The numerical wall divisors for $K3[n]$ have been discussed in [BHT] and [Mo5] based on work in [BM]. For manifolds of type $K3[2]$ one has ([Mo5], Prop. 2.12):

**Theorem 8.4.** Let $X$ be a manifold of type $K3[2]$. Then the numerical wall divisors are the following vectors in the Picard sublattice of $L$:

- vectors $v$ of norm $v^2 = -2$;
- vectors $v$ of norm $v^2 = -10$ and $v/2 \in L^*$.

Since $L_G$ is a sublattice of the Picard lattice, we have to check for which of the groups $G \subseteq O(L)$ from Section 7 the lattice $L_G$ contains no vectors $v$ of norm $-10$ such that $v/2 \in L^*$.

**Proposition 8.5.** If $L$ is obtained from $L_G \oplus L_G'$ by gluing construction (1) in Section 7 then $L_G$ contains no numerical wall divisor. If $L$ is obtained by gluing construction (2) then $L_G$ contains a numerical wall divisor if, and only if, it contains a vector $v$ of norm $-10$ such that $v/2 = w' \gamma$ in $A_L$.

44
Theorem 7.4 we searched randomly for vectors $v$ unique in each case there is a numerical wall divisor. With this in mind, for each of the eleven lattices can assume that $G$ action of a group $G$ of $K$ hyperkähler manifolds of type $G$.

Proof: By Theorem 8.4, we have to check if $(L_G, G)$ contains no numerical wall divisor.

In case (1), $L$ contains vectors of the form $(a, b) \in L_G \oplus L_G$ and the cosets $(x, \gamma(x)), x \in A_{L_G}$. Thus a vector $(v/2, 0) \in L_G^* \oplus (L_G)^*$ is contained in $L^*$ if and only if $(v/2, L_G^*) \subset \mathbb{Z}$, i.e. $v/2 \in L_G$. But this is impossible since the norm of $v/2$ is $-5/2$ if $v$ has norm $-10$.

In case (2), $L_G = \gamma'(A_{L_G}) \oplus \langle w_{\gamma'} \rangle, L/(L_G \oplus L_G) = \{ (\gamma'(y), y) \mid y \in A_{L_G} \}$ so that $(w_{\gamma'}, 0)$ generates $L = L^*/L$. Thus a vector $(v/2, 0) \in L_G^* \oplus (L_G)^*$ is contained in $L^*$ if and only if $(v/2, \gamma'(A_{L_G})) = 0$, i.e. $v/2 = w_{\gamma'}$ in $A_{L_G}$.

Theorem 8.6. Let $G \subseteq C_0$ be an admissible subgroup. Then $G$ is induced by a group of symplectic automorphisms for some hyperkähler manifold $X$ such that $(\Lambda_G(-1), G) \cong (L_G, G)$ if, and only if, $\alpha(L_G) \geq 1$.

Proof: If there exists a realization of $(L_G, G)$ as coinvariant lattice by a group $G \subseteq O(L)$ using construction (1), then by Proposition 8.3 $L_G$ contains no numerical wall divisor.

We will show that any realization of $(L_G, G)$ using construction (2) contains a numerical wall divisor. With this in mind, for each of the eleven lattices $L_G$ as in Theorem 7.4 we searched randomly for vectors $v$ in the dual lattice $L_G^*$ of norm $-5/2$ such that $2v \in L_G$. We always found such a $v$. As discussed in the last section, in each case there is a unique $O(L_G)$-orbit of norm 3/2 vectors $w_{\gamma'}$ in $A_{L_G}$, i.e. we can assume that $v/2 = w_{\gamma'}$. Thus by Proposition 8.3 $L_G$ contains a numerical wall divisor.

It follows therefore from Theorem 8.3 that $G$ is induced by a group of symplectic automorphisms if, and only if, there is a realization of $(L_G, G)$ using construction (1). By Theorem 7.4 these are exactly the admissible groups for which $\alpha(L_G) \geq 1$.

Combining Theorem 8.6 with Theorem 6.11 we have:

Theorem 8.7. The finite groups $G$ arising as symplectic automorphisms of a hyperkähler manifold of type $K3^{[2]}$ are:

(a) subgroups of $M_{23}$ with at least four orbits in the natural action on 24 elements,
(b) subgroups of $3^{1+4}:2.2^2$ and $3^4:A_6$ associated to the corresponding $S$-lattices.

At this juncture, we have established Theorem A and Theorem B.

Let $X_i$ be hyperkähler manifolds of type $K3^{[2]}$ with finite groups of symplectic automorphism $G_i \subseteq \text{Aut} X_i$ ($i = 1, 2$). We say that $(X_1, G_1)$ is deformation equivalent to $(X_2, G_2)$ if there exists a flat family $\chi \rightarrow B$ over a connected base $B$ with hyperkähler manifolds of type $K3^{[2]}$ as fibers, together with a fiberwise symplectic action of a group $G$ such that $(X_1, G_1)$ and $(X_2, G_2)$ are isomorphic to the action of $G$ at certain fibers.

The following result follows from [Mo4]; cf. the proof of Corollary 5.2 in [Mo6].
Theorem 8.8. Two hyperkähler manifolds $X_i$ of type $K3^{[2]}$ with symplectic automorphism groups $G_i$ ($i = 1, 2$), are equivariantly deformation equivalent if, and only if, the associated group lattices $(L_i, G_i)$ are isomorphic.

Together with Theorem 8.6 and the enumeration results from the previous section, this proves Theorem C. Indeed, there are $198 - 13 = 185$ group lattices $(L, G)$ that can be realized. For the 88 groups $G$ with $\text{rk} L = 20$, there are 146 conjugacy classes in $O(L)$, as summing up the last column in Table 9 shows, giving at least $185 - 88 + 146 = 243$ conjugacy classes in all. If Conjecture 7.2 holds, there are also exactly 243 deformation classes.

We finally note that Theorem D is equivalent to Theorem 8.6 by Theorem 8.3, Theorem 8.4 and Theorem 8.8.

Remarks: The admissible conjugacy classes can also be determined with the help of Theorem 8.3 and Theorem 8.4 instead of the method we used in Section 4.1. However, we will need the exact fixed-point configuration (described in Table 2) in the final section. Also, Theorem 8.3 became only available after main parts of the paper had been written. Theorem 8.6 was conjectured in [Mo6].

Explicit examples of group actions. It is known that certain maximal groups can be realized as symplectic actions on a $K3^{[2]}$ through induced actions on Fano schemes of lines of certain cubic fourfolds $S \subseteq \mathbb{CP}^5$, cf. [Fu], [Mo1] Ch. 4 and the references therein. We collect these examples in Table 11. In addition, we consider two apparently new examples, with $G \cong 3^{1+4} : 2.2^2$ and $M_{10}$. These are discussed in the next few paragraphs.

Concerning $G \cong 3^{1+4} : 2.2^2$, first note that the cubic polynomial $f$ in Table 11 is invariant under an obvious action of $H = (3^2.S_3)^2.S_2 \subseteq \text{GL}(6, \mathbb{C})$ given by permutations and multiplication of the coordinates by cube roots of unity. In addition, $f$ is invariant under the unimodular matrix

$$\alpha := \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & 1 \\ 1 & 1 & 1 \\ \omega^2 & \omega & 1 \\ \omega & \omega^2 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \quad (\omega = e^{2\pi i/3}).$$

An element in $\text{GL}(6, \mathbb{C})$ leaving $f$ invariant acts symplectically on the Fano scheme of $S$ if, and only if, it is in $\text{SL}(6, \mathbb{C})$ (cf. [Fu]). Calculations show that the projections of $H \cap \text{SL}(6, \mathbb{C})$ and $\alpha$ into $\text{PSL}(6, \mathbb{C})$ generate a group isomorphic to $G$. Finally, it can be verified that the resulting cubic fourfold $S$ is smooth by solving the equations

$$f = \partial f/\partial x_0 = \cdots = \partial f/\partial x_5 = 0,$$

confirming that 0 is an isolated singularity of $f$.

For the case $G \cong M_{10}$, we start with the 6-dimensional representation of the
Table 11: Fano schemes of cubic fourfolds

| Group  | Equation for $S$ |
|--------|------------------|
| $L_2(11)$ | $x_0^3 + x_1^2 x_5 + x_2^2 x_4 + x_3^2 x_2 + x_4^2 x_1 + x_5^2 x_3$ |
| $A_7$ | $x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - (x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^3$ |
| $(\mathbb{Z}_3 \times A_5): Z_2$ | $x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_0 + x_4^3 + x_5^3$ |
| $3^4: A_6$ | $x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3$ |
| $3^{1+4}: 2.2^2$ | $x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + 3(i - 2e^{\pi i/6} - 1)(x_0 x_1 x_2 + x_3 x_4 x_5)$ |
| $M_{10}$ | $x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + \lambda \cdot \bar{\sigma}_3(x_0, \ldots, x_5)$. |

The group $3.A_6 \subseteq \text{SL}(6, \mathbb{C})$ is given by the generators

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & \omega & 1 \\
\omega^2 & 1 & \omega \\
\omega & \omega^2 & 1
\end{pmatrix}
$$

from the Atlas of finite group representations [W]. In addition, we choose the unimodular matrix

$$
\beta := \frac{1}{\sqrt{6}} \begin{pmatrix}
1 & \omega & \omega^2 & \omega & 1 & \omega \\
\omega & 1 & \omega & \omega & \omega & \omega \\
\omega^2 & \omega & 1 & \omega & \omega & \omega^2 \\
1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\
\omega & \omega^2 & \omega^2 & \omega & 1 & \omega^2 \\
\omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega & 1
\end{pmatrix}
$$

which normalizes $3.A_6$. We found the following cubic polynomial $f$ invariant under the action of $3.A_6$ and $\beta$:

$$
f = (x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3) + 1/5(-3\zeta^7 - 3\zeta^5 + 3\zeta^4 - 3\zeta^3 + 6\zeta - 3) \times
\left[
x_1 x_2 x_3 + x_1 x_2 x_4 + (\zeta^4 - 1)x_1 x_2 x_5 + x_1 x_2 x_6 + (\zeta^4 - 1)x_1 x_3 x_4 + x_1 x_3 x_5 \\
x_1 x_3 x_6 + (\zeta^4 - 1)x_1 x_4 x_5 - \zeta^4 x_1 x_4 x_6 - \zeta^4 x_1 x_5 x_6 + (\zeta^4 - 1)x_2 x_3 x_4 \\
+ (\zeta^4 - 1)x_2 x_3 x_5 - \zeta^4 x_2 x_3 x_6 + x_2 x_4 x_5 + x_2 x_4 x_6 - \zeta^4 x_2 x_5 x_6 + x_3 x_4 x_5 \\
- \zeta^4 x_3 x_4 x_6 + x_3 x_5 x_6 + x_4 x_5 x_6
\right]
$$

with $\zeta = e^{2\pi i/24}$. The projection of $3.A_6(\beta)$ into $\text{PSL}(6, \mathbb{C})$ is isomorphic to $M_{10}$. Again, we verified that $S$ is smooth.

It would be of interest to find explicit realizations for the nine remaining maximal groups of Theorem A not treated here.
9 Connections with Mathieu Moonshine

Suppose that \( g \in M_{24} \) belongs to one of the 11 admissible classes. In this section we compare the equivariant complex elliptic genus \( \chi_g(q;g,LX) \) of a hyperkähler manifold \( X \) of type \( K3 \) with the prediction of the Mathieu Moonshine applied to the second quantized complex elliptic genus of a \( K3 \) surface. See Theorem 9.3 below for a precise statement. This connection was one of our main motivations for studying symplectic symmetries of \( K3 \).

9.1 Mathieu Moonshine

Recall [Hi] that a complex genus in the sense of Hirzebruch is a graded ring homomorphism from the complex bordism ring into some other graded ring \( R \). For a \( d \)-dimensional complex manifold \( X \) and a holomorphic vector bundle \( E \) on \( X \), the \( \chi_y \)-genus twisted by \( E \) is

\[
\chi_y(X,E) := \sum_{p=0}^{d} \chi(X, \Lambda^p T^* \otimes E) y^p.
\]

where \( \chi(X, E) = \sum_{q=0}^{d} (-1)^q \dim H^q(X, \mathcal{O}(E)). \)

The complex elliptic genus can be formally defined as the \( S^1 \)-equivariant \( \chi_y \)-genus of the loop space of a manifold:

\[
\chi_y(q,LX) := (-y)^{-d/2} \chi(y, \bigotimes_{n=1}^{\infty} \Lambda_y q^n T^* \otimes \bigotimes_{n=1}^{\infty} \Lambda_{y-1} q^n T \otimes \bigotimes_{n=1}^{\infty} S_q(T^* \oplus T))
\]

taking values in \( \mathbb{Q}[y^{1/2}, y^{-1/2}][[q]] \). If the first Chern class vanishes, this is the Fourier expansion of a Jacobi form of weight 0 and index equal to one half of the complex dimension of \( X \) [H11]. For automorphisms \( g \) of \( X \) one has the corresponding equivariant elliptic genus \( \chi_g(q;g,LX) \) where we let \( \chi(g;X,E) = \sum_{q=0}^{d} (-1)^q \Tr(g|H^q(X, \mathcal{O}(E))). \)

For a \( K3 \) surface \( Y \), the elliptic genus \( \chi_y(q,LY) \) is the Jacobi form

\[
2 \phi_{0,1}(z; \tau) = 8 \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2
\]

of weight 0 and index 1.

Eguchi, Ooguri and Tachikawa observed [EOT] that the decomposition of \( 2 \phi_{0,1}(z; \tau) \) into the (expected) characters of the \( N = 4 \) super algebra at central charge \( c = 6 \) has multiplicities which are sums of dimensions of irreducible representations of \( M_{24} \).

The observation of Eguchi, Ooguri and Tachikawa suggests the existence of a graded \( M_{24} \)-module \( K = \bigoplus_{n=0}^{\infty} K_n q^{n-1/8} \) whose graded character is given by the
decomposition of the elliptic genus into characters of the $\mathcal{N} = 4$ super algebra. Subsequently, analogues of McKay-Thompson series in monstrous moonshine were proposed in several works [CH, CD, EH1, GHV, GHV2]. The corresponding McKay-Thompson series for $g \in M_{24}$ are of the form

$$\Sigma_g(q) = q^{-1/8} \sum_{n=0}^{\infty} \text{Tr}(g|K_n) q^n = \frac{e(g)}{24} \Sigma(q) - \frac{f_g(q)}{\eta(q)^3}. \quad (5)$$

Here, $\Sigma = \Sigma_e$ is the graded dimension of $K$ (an explicit mock modular form), $e(g)$ is the character of the 24-dimensional permutation representation of $M_{24}$, $f_g$ is a certain explicit modular form of weight 2 on a congruence subgroup $\Gamma_0(N_g)$, and $\eta$ is the Dedekind eta function. Gannon has shown [Ga] that these McKay-Thompson series indeed determine a graded $M_{24}$-module.

In [CH], Creutzig and the first author have shown that for symplectic automorphisms of a K3 surface, the McKay-Thompson series of Mathieu Moonshine determines the equivariant elliptic genus:

**Theorem 9.1.** Let $g$ be a finite symplectic automorphism of a K3 surface $Y$. Then the equivariant elliptic genus and the character determined by the McKay-Thompson series of Mathieu moonshine agree, i.e. one has

$$\chi_{-y}(g; q, L_Y) = \frac{e(g)}{12} \phi_{0,1} + f_g \phi_{-2,1},$$

where $g$ is considered on the right-hand-side as an element in $M_{24}$.

Here, $\phi_{-2,1}$ is the Jacobi form

$$\phi_{-2,1} = y^{-1}(1 - y)^2 \prod_{n=1}^{\infty} \frac{(1 - q^n y)^2 (1 - q^n y^{-1})^2}{(1 - q^n)^4} \quad (6)$$

of weight $-2$ and index 1.

### 9.2 The second quantized elliptic genus and its relation to Hilbert schemes of K3

The elliptic genus of an orbifold $X/G$ for a finite group $G$ acting on a complex manifold $X$ is defined by

$$\chi_{y}(q, \mathcal{L}(X/G)) := \frac{1}{|G|} \sum_{g,h \in G \, |gh| = 1} \chi_{y}(g; q, \mathcal{L}_h X),$$

where $\mathcal{L}_h(X)$ is the $h$-twisted loop space and $\chi_{y}(g; q, \mathcal{L}_h X)$ is determined by formally applying the equivariant Atiyah-Singer index theorem.

For a space $X$, let $\exp(pX) := \sum_{n=0}^{\infty} X^n/S_n \cdot p^n$ be the generating series of its symmetric powers. It follows from calculations by Verlinde, Verlinde, Dijkgraaf and
Moore [VVDM] that the second quantized elliptic genus $\chi_{-y}(q, L \exp(pX))$ is, up to an automorphic correction factor, the Borcherds lift of $\chi_{-y}(q, LX)$. Explicitly one has

$$\chi_{-y}(q, L \exp(pX)) = \prod_{n>0, m \geq 0, \ell} \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} c(4nm - \ell^2) (p^n q^m y^\ell)^k \right),$$

where $\chi_{-y}(q, LX) = \sum_{n, \ell \in \mathbb{Z}} c(4n - \ell^2) q^n y^\ell$.

For K3 surfaces, there is the following connection between the orbifold elliptic genus of symmetric powers and the Hilbert schemes conjectured by [VVDM].

**Theorem 9.2** (Borisov and Libgober [BL]). Let $Y$ be a K3 surface. Then

$$\chi_y(q, L \exp(pY)) = \sum_{n=0}^{\infty} \chi_y(q, LY^{[n]}) p^n.$$

\[\square\]

If $g$ acts on $X$ then there is an induced action of $g$ on $X^n/S_n$ since the diagonal action of $g$ on $X^n$ commutes with the $S_n$-action. There is then the following equivariant generalization:

$$\chi_{-y}(g; q, L \exp(pX)) = \prod_{n>0, m \geq 0, \ell} \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} c_g(4nm - \ell^2) (p^n q^m y^\ell)^k \right),$$

where $\chi_{-y}(g; q, LX) = \sum_{n, \ell \in \mathbb{Z}} c_g(4n - \ell^2) q^n y^\ell$.

Alternatively, we may take this directly as the definition of the equivariant second quantized elliptic genus.

For a K3 surface $Y$, we can use the McKay-Thompson series of Mathieu Moonshine to define $\chi_{-y}(g; q, LY)$ for all $g \in M_{24}$ by the formula in Theorem 9.1. Then the previous formula allows us to also define the equivariant second quantized elliptic genus for $g \in M_{24}$. We prove:

**Theorem 9.3.** Let $g \in M_{24}$ be a finite symplectic automorphism of order 1, 3, 4, 5, 7, 8 or 11 acting on a hyperkähler manifold $X$ of type $K3^{[2]}$. Then the equivariant elliptic genus $\chi_{-y}(g; q, LX)$ and the coefficient of $p^2$ in the equivariant second quantized elliptic genus determined by the McKay-Thompson series of Mathieu moonshine agree.

There are 11 classes $g \in M_{24}$ acting on a $K3^{[2]}$. If $g = 1$, the theorem follows from Theorem 9.2. For $g$ acting by symplectic automorphisms on a K3 surface, there is probably an equivariant generalization, however this currently seems to be unknown. This would not, in any case, apply to the three classes of order 11, 14 and 15 which do not correspond to symplectic automorphisms of K3. We have verified the result also for the four cases 2, 6, 14 or 15 for the first coefficients (up to the order 4 in $q$).
To prove Theorem 9.3, we start by showing that both the equivariant elliptic genus and the $p^2$ coefficient of the equivariant second quantized elliptic genus are weak Jacobi forms.

**Proposition 9.4.** Let $N = \text{ord}(g)$ and assume $N > 2$. Then $\chi_{-y}(g; q, \mathcal{L}X)$ is a weak Jacobi form on $\Gamma_0(N)$ of weight 0 and index 2.

**Proof:** The proof is similar to that of Lemma 4.2 in [CH]. Because $g$ has order $\geq 3$, it follows from Table 2 that $X^g$ consists only of isolated fixed-points $\{p_i\}$. Set

$$\varphi(u; \tau) := \vartheta_1(u; \tau) \eta(\tau)^{-3} = -i(y^{1/2} - y^{-1/2}) \prod_{n=1}^{\infty} (1 - yq^n)(1 - y^{-1}q^n)(1 - q^n)^{-2}.$$ 

The fixed-point formula gives (cf. also equation (1) in the case of $\chi_{-y}(g; X)$):

$$\chi_{-y}(g; q, \mathcal{L}X) = \sum_{p_i} \frac{\varphi(u + \frac{m_i}{N}; \tau) \varphi(u - \frac{m_i}{N}; \tau)}{\varphi(\frac{m_i}{N}; \tau) \varphi(-\frac{m_i}{N}; \tau)} \cdot \frac{\varphi(u + \frac{m_i}{N}; \tau) \varphi(u - \frac{m_i}{N}; \tau)}{\varphi(\frac{m_i}{N}; \tau) \varphi(-\frac{m_i}{N}; \tau)},$$

where the pair $\{n_i, m_i\}$ for a given $p_i$ can be read off from Table 2. Namely, $\{\zeta^{n_i}, \zeta^{-n_i}, \zeta^{m_i}, \zeta^{-m_i}\}$ with $\zeta$ a primitive $N$-th root of unity are the eigenvalues of $g$ acting at the tangent space of $p_i$.

In order to check the Jacobi transformation property, we consider the action of $(\mathbb{Z}/N\mathbb{Z})^*$ (see the proof of Lemma 4.2 in [CH]). The fixed-point of type $\{\zeta^{n_i}, \zeta^{m_i}\}$ is mapped by $d \in (\mathbb{Z}/N\mathbb{Z})^*$ to $\{(\zeta^{n_i})^d, (\zeta^{m_i})^d\}$, and it is clear from Table 2 that this induces a permutation action of $(\mathbb{Z}/N\mathbb{Z})^*$ on $\{p_i\}$. Therefore, the above expression for $\chi_{-y}(g; q, \mathcal{L}X)$ is left invariant. \hfill \Box

**Proposition 9.5.** Let $N = \text{ord}(g)$ and assume $N \neq 6, 14, 15$. Then the coefficient of $p^2$ in the equivariant second quantized elliptic genus $\chi_{-y}(g; q, \mathcal{L}\exp(pX))$ is a weak Jacobi form for $\Gamma_0(N)$ of weight 0 and index 2.

**Proof:** According to M. Raum [Ra] (Theorem 1.2),

$$\Phi_g := \frac{pqy}{(n,m,\ell)>0} \exp \left( - \sum_{k=1}^{\infty} \frac{1}{k} c_{g,k} (4nm - \ell^2) (p^n q^m y^\ell)^k \right)$$

is a Siegel modular form of degree 2 of a certain weight $k_g$ on the subgroup $\Gamma_0^2(N) \subseteq Sp(4, \mathbb{Z})$. Here, the product runs over triples of integers $(n, m, \ell)$ and $(n, m, \ell) > 0$ means that $n > 0$, or $n = 0$ and $m > 0$, or $n = m = 0$ and $\ell < 0$. This implies that the coefficient $\psi_{g,n}$ of $p^n$ in the Fourier-Jacobi expansion

$$\Phi_g^{-1} = \sum_{n=0}^{\infty} \psi_{g,n} p^n$$

is a Jacobi form of weight $-k_g$ and index $m$ for $\Gamma_0(N)$. 

51
To compare $\Phi_g^{-1}$ with the second quantized elliptic genus, we write

$$\Phi_g^{-1} = \chi_y(g; q, \mathcal{L} \exp(pY)) \cdot \alpha_g^{-1}$$

so that

$$\alpha_g = \prod_{(m, \ell) \geq 0} \prod_{d \mid N} \exp \left( - \sum_{k=1}^{\infty} \frac{1}{k} c_p (-\ell^2) (q^m y^\ell)^k \right) \prod_{(m, \ell) \geq 0} \prod_{d \mid N} \left( 1 - (q^m y^\ell)^d \right)^{d^{-1} \sum_{e \mid d} \mu(d/e) c_{p^e} (-\ell^2)}.$$

The only contributions come from terms $c_p (-\ell^2)$ for $\ell \in \{0, \pm 1\}$, which are determined by the Hodge structure of the K3 surface. One has $c_p(-1) = 2$ and $c_p(0) = e(h) - 4$ where $e(h)$ is the number of fixed-points of the element $h \in M_{24}$ acting on 24 elements.

We claim that

$$\alpha_g = p \eta_g \phi_{-2,1}$$

(cf. [Ch], Section 4) where $\eta_g$ is the usual twisted eta-product for $g$. (If $g$ has cycle shape $a_1^{b_1} a_2^{b_2} \ldots$, $\eta_g(q) := \eta(q^{a_1})^{b_1} \eta(q^{a_2})^{b_2} \ldots$) Indeed, the contribution for $m > 0$, $\ell = \pm 1$ and the constant term $-4$ of $c_p(0)$ gives the product formula for $\phi_{-2,1}$ as in equation (6) up to the leading $y$-factors. The contribution for $m > 0$, $\ell = 0$ is the twisted eta-product $\eta_g$. Note that $d^{-1} \sum_{e \mid d} \mu(d/e) c_{p^e}$ counts the number of $d$-cycles of $g$. The contribution for $m = 0$ and $\ell = -1$ gives the missing $y$-factors, and the factor $p$ comes from the factor $qy^p$ in front of $\Phi_g$.

Since the weight of $\eta_g$ is $k_g + 2$ [Mas1] and $\phi_{-2,1}$ has weight $-2$ and index 1, it follows that the coefficient $\psi_{g,n-1} \cdot \alpha_g$ of $p^n$ in $\chi_y(g; q, \mathcal{L} \exp(pY))$ is a Jacobi form of weight 0 and index $n$ for $\Gamma_0(N)$. □

**Proof of Theorem 9.3.** Let $J_{k,m}(\Gamma_0(N))$ be the space of weak holomorphic Jacobi forms for $\Gamma_0(N)$ of weight $k$ and index $m$. It follows from [AC] Prop. 6.1 that

$$J_{0,2}(\Gamma_0(N)) \cong M_0(\Gamma_0(N)) \times M_2(\Gamma_0(N)) \times M_4(\Gamma_0(N)),$$

where $M_l(\Gamma_0(N))$ is the space of holomorphic modular forms of weight $l$ for $\Gamma_0(N)$. The isomorphism is defined by

$$(f_0, f_2, f_4) \mapsto f_0 \phi_{0,1}^2 + f_2 \phi_{0,1} \phi_{-2,1} + f_4 \phi_{-2,1}^2$$

for $(f_0, f_2, f_4) \in M_0(\Gamma_0(N)) \times M_2(\Gamma_0(N)) \times M_4(\Gamma_0(N))$.

By using explicit bases for the spaces of modular forms of weights 0, 2 and 4 on $\Gamma_0(N)$, the result follows from Proposition 9.4 and Proposition 9.5 by checking the equality for sufficiently many coefficients. This was carried through using Magma. □

Our calculation shows that Theorem 9.3 will also hold for the remaining four classes $g$ of order 2, 6, 14 and 15, once the Jacobi form property has been verified. See [PV] for work in this direction.
## A Tables of admissible groups and their lattices

The appendix contains three tables, listing all admissible subgroups of \( C_0 \) and information about the coinvariant lattices \( L_G \).

Table 12 lists the 198 classes of admissible subgroups \( G \) of \( C_0 \) sorted by size. The entry of the first two columns is clear. The third column lists the abstract isomorphism type of the group either by name or the number of the small group library \[\text{BEO}\]. The fourth column has an entry “K3” if \((L_G; G)\) is one of the 82 group lattices of Table 10.2 of \[\text{Ha}\], arising from a symplectic action on a K3 surfaces; an “M” if the group is inside \( M_{23} \) and realized; an “S” if it is only realized by an \( S \)-lattice group; and “−” if the group is not realized. The fifth column records the dimension of \( L_G \), the sixth column gives the number of the unique largest group with the same lattice \( L_G \), and the last column lists the numbers of all admissible groups containing \( G \) as a maximal subgroup.

Table 13 lists the 69 isomorphism types of lattices \( L_G \) for admissible groups \( G \). The second column gives the number (from Table 12) of the unique largest group with the same lattice \( L_G \), while the next three columns are self explanatory. Column \( \det \) gives the determinant of \( L_G \), and column \( A_{L_G} \) the structure of the discriminant group. In the Genus column we provide the genus symbol of \( L_G \) as defined in \[\text{CS}\] (we omit the signature since is directly determined by the rank). The last column provides information about the realization: if \( L_G \) is one of the 82 lattices of Table 10.3 of \[\text{Ha}\], the entry is “K3 \#n” where \( n \) is the K3 group number; if \( L_G \) is one of the 13 lattices belonging to the groups in Theorem 5.1, the entry is “max \#n” where \( n \) is the \( n \)-th group of that theorem; if \( L_G \) belongs to (maximal) \( S \)-groups, the entry is “(max) S-lattice”; if \( G \) can be realized in \( O(L) \) the entry is “\( O(L) \)”;

Finally, in Table 14 we list all lattices \( L_G \) for which there are several admissible groups with the same lattice \( L_G \). The first column gives the number of the lattice as in Table 13. The second column gives the number of groups with the same lattice \( L_G \). The last column lists all such groups by their number as in Table 12.

### Table 12: Conjugacy classes of admissible groups

| No. | order | \( G \) | Type | dim | fix | minimal overgroups |
|-----|-------|---------|------|-----|-----|-------------------|
| 1   | 1     | #1 K3   | 0    | 1   | 2, 3, 4, 7, 11, 23 |
| 2   | 2     | #1 K3   | 8    | 2   | 5, 6, 8, 9, 10, 22, 31 |
| 3   | 3     | #1 K3   | 12   | 3   | 8, 9, 18, 19, 21, 25, 32, 54 |
| 4   | 3     | #1 S    | 18   | 170 | 10, 18, 20, 21 |
| 5   | 4     | #1 K3   | 14   | 5   | 12, 13, 14, 15, 17, 26, 28, 30, 53, 79 |
| 6   | 4     | #2 K3   | 12   | 6   | 13, 14, 16, 24, 25, 27, 29 |
| 7   | 5     | #1 K3   | 16   | 22  | 22, 32, 99, 114, 169 |
| 8   | 6     | #2 K3   | 16   | 24  | 24, 27, 29, 30, 48, 51, 52, 55, 57, 58, 67, 83 |
| 9   | 6     | #1 K3   | 14   | 9   | 24, 48, 49, 50, 51, 56, 102 |
| 10  | 6     | #2 S    | 18   | 170 | 26, 28, 49, 52 |
Table 12: Conjugacy classes of admissible groups

| No. | order | G  | Type | dim | fix | minimal overgroups |
|-----|-------|----|------|-----|-----|-------------------|
| 11  | 7     | #1 | K3   | 18  | 54  | { 31, 54, 100 }   |
| 12  | 8     | #1 | K3   | 18  | 40  | { 33, 40, 44, 45, 109 } |
| 13  | 8     | #3 | K3   | 15  | 13  | { 40, 46, 47, 56, 60, 111 } |
| 14  | 8     | #2 | K3   | 16  | 47  | { 33, 34, 36, 38, 39, 41, 42, 43, 44, 46, 47 } |
| 15  | 8     | #4 | K3   | 17  | 68  | { 42, 43, 45, 46, 57 } |
| 16  | 8     | #5 | K3   | 14  | 16  | { 35, 36, 37, 41, 47, 58, 100 } |
| 17  | 8     | #4 | K3   | 17  | 17  | { 39, 40, 45, 55, 59, 110 } |
| 18  | 9     | #2 | S    | 18  | 170 | { 49, 61, 62, 63, 64, 65, 66 } |
| 19  | 9     | #2 | K3   | 16  | 50  | { 48, 50, 62, 66, 78 } |
| 20  | 9     | #1 | S    | 20  | 198 | { 63, 64, 65 } |
| 21  | 9     | #2 | S    | 20  | 198 | { 51, 52, 62 } |
| 22  | 10    | #1 | K3   | 16  | 22  | { 53, 67, 102, 136, 181 } |
| 23  | 11    | #1 | M    | 20  | 177 | { 99 } |
| 24  | 12    | #4 | K3   | 16  | 24  | { 60, 80, 81, 86, 88, 132, 177 } |
| 25  | 12    | #3 | K3   | 16  | 25  | { 56, 58, 78, 85, 87, 102, 160 } |
| 26  | 12    | #1 | S    | 19  | 185 | { 59, 82, 130 } |
| 27  | 12    | #5 | —    | 18  | 27  | { 90, 93 } |
| 28  | 12    | #2 | S    | 20  | 191 | { 59, 127, 131 } |
| 29  | 12    | #5 | K3   | 18  | 112 | { 60, 78, 80, 89, 92 } |
| 30  | 12    | #1 | K3   | 18  | 112 | { 60, 82, 84, 91, 94, 101, 129 } |
| 31  | 14    | #2 | M    | 20  | 162 | { 83 } |
| 32  | 15    | #1 | M    | 20  | 164 | { 67 } |
| 33  | 16    | #6 | K3   | 19  | 168 | { 69, 70, 71, 75 } |
| 34  | 16    | #2 | —    | 18  | 34  | { } |
| 35  | 16    | #14 | —   | 15  | 35  | { 93 } |
| 36  | 16    | #10 | —   | 17  | 36  | { } |
| 37  | 16    | #14 | K3   | 15  | 37  | { 77, 85, 92, 114 } |
| 38  | 16    | #2 | K3   | 18  | 150 | { 69, 72, 87 } |
| 39  | 16    | #12 | K3   | 18  | 150 | { 71, 72, 73, 89 } |
| 40  | 16    | #8 | K3   | 18  | 40  | { 71, 86, 135, 178 } |
| 41  | 16    | #3 | K3   | 17  | 77  | { 72, 76, 77, 84 } |
| 42  | 16    | #12 | —   | 18  | 42  | { 90 } |
| 43  | 16    | #12 | —   | 18  | 74  | { 70, 74 } |
| 44  | 16    | #6 | —    | 19  | 44  | { 73 } |
| 45  | 16    | #9 | K3   | 19  | 168 | { 71, 91, 94 } |
| 46  | 16    | #13 | K3   | 17  | 68  | { 68, 69, 71, 74 } |
| 47  | 16    | #11 | K3   | 16  | 47  | { 68, 72, 75, 76, 77, 88 } |
| 48  | 18    | #3 | K3   | 18  | 81  | { 81, 95, 98, 142, 156 } |
| 49  | 18    | #3 | S    | 18  | 170 | { 95, 96, 97 } |
| 50  | 18    | #4 | K3   | 16  | 50  | { 79, 81, 96, 98, 112 } |
| 51  | 18    | #3 | S    | 20  | 198 | { 80, 95, 98 } |
Table 12: Conjugacy classes of admissible groups

| No. | order | $G$ | Type | dim | fix | minimal overgroups |
|-----|-------|-----|------|-----|-----|-------------------|
| 52  | 18    | #5  | S    | 20  | 198 | { 80, 82, 95 }    |
| 53  | 20    | #3  | K3   | 18  | 53  | { 101, 132, 158, 178 } |
| 54  | 21    | #1  | K3   | 18  | 54  | { 83, 140, 141 }  |
| 55  | 24    | #3  | K3   | 19  | 86  | { 86, 89, 94 }    |
| 56  | 24    | #12 | K3   | 17  | 56  | { 88, 112, 120, 124, 132, 140, 163, 176 } |
| 57  | 24    | #3  | K3   | 19  | 143 | { 90, 91, 122 }   |
| 58  | 24    | #13 | K3   | 18  | 88  | { 84, 88, 92, 93, 122, 125, 141 } |
| 59  | 24    | #4  | S    | 20  | 191 | { 151 }           |
| 60  | 24    | #8  | K3   | 18  | 112 | { 112, 113, 121, 123, 152 } |
| 61  | 27    | #3  | S    | 18  | 170 | { 97, 116, 117, 119 } |
| 62  | 27    | #5  | S    | 20  | 198 | { 95, 98, 117, 118 } |
| 63  | 27    | #4  | S    | 20  | 198 | { 115, 116 }      |
| 64  | 27    | #4  | S    | 20  | 198 | { 115, 117 }      |
| 65  | 27    | #2  | S    | 20  | 198 | { 115, 116 }      |
| 66  | 27    | #5  | S    | 18  | 170 | { 96, 115, 118, 119 } |
| 67  | 30    | #2  | M    | 20  | 164 | { 101, 142 }      |
| 68  | 32    | #49 | K3   | 17  | 68  | { 106, 108, 122 } |
| 69  | 32    | #11 | K3   | 19  | 168 | { 106, 124 }      |
| 70  | 32    | #8  | –    | 20  | 70  | {}               |
| 71  | 32    | #44 | K3   | 19  | 168 | { 106, 123 }      |
| 72  | 32    | #31 | K3   | 18  | 150 | { 103, 105, 106, 107 } |
| 73  | 32    | #8  | –    | 20  | 107 | { 107 }           |
| 74  | 32    | #50 | –    | 18  | 74  | {}               |
| 75  | 32    | #7  | K3   | 19  | 168 | { 104, 106 }      |
| 76  | 32    | #6  | K3   | 18  | 108 | { 104, 105, 108 } |
| 77  | 32    | #27 | K3   | 17  | 77  | { 103, 104, 108, 120, 121, 125, 136 } |
| 78  | 36    | #11 | K3   | 18  | 112 | { 112, 134, 142, 184 } |
| 79  | 36    | #9  | K3   | 18  | 79  | { 109, 110, 111, 129, 130, 131, 163, 172 } |
| 80  | 36    | #12 | S    | 20  | 198 | { 113, 126 }      |
| 81  | 36    | #10 | K3   | 18  | 81  | { 111, 126, 128, 164, 173 } |
| 82  | 36    | #7  | S    | 20  | 198 | { 113, 161 }      |
| 83  | 42    | #2  | M    | 20  | 162 | { 162 }           |
| 84  | 48    | #30 | K3   | 19  | 157 | { 121, 147, 149 } |
| 85  | 48    | #50 | K3   | 17  | 85  | { 120, 125, 134, 150 } |
| 86  | 48    | #29 | K3   | 19  | 86  | { 123 }           |
| 87  | 48    | #3  | K3   | 18  | 150 | { 124, 150 }      |
| 88  | 48    | #48 | K3   | 18  | 88  | { 121, 143, 148, 162, 179 } |
| 89  | 48    | #32 | M    | 20  | 187 | { 123, 144 }      |
| 90  | 48    | #32 | –    | 20  | 90  | {}               |
| 91  | 48    | #28 | M    | 20  | 196 | { 149 }           |
| 92  | 48    | #49 | K3   | 19  | 157 | { 121, 134, 144, 145, 146 } |
| No. | order | $G$ | Type | dim | fix | minimal overgroups |
|-----|-------|-----|------|-----|-----|-------------------|
| 93  | 48    | #49 | –    | 19  | 93  | {}                |
| 94  | 48    | #28 | M    | 20  | 187 | { 123 }           |
| 95  | 54    | #12 | S    | 20  | 198 | { 126, 137, 139 } |
| 96  | 54    | #13 | S    | 18  | 170 | { 130, 131, 138, 139 } |
| 97  | 54    | #8  | S    | 18  | 170 | { 127, 137, 138 } |
| 98  | 54    | #13 | S    | 20  | 198 | { 126, 128, 129, 139 } |
| 99  | 55    | #1  | M    | 20  | 177 | { 177 }           |
| 100 | 56    | #11 | M    | 20  | 188 | { 141 }           |
| 101 | 60    | #7  | M    | 20  | 164 | { 164 }           |
| 102 | 60    | #5  | K3   | 18  | 102 | { 132, 142, 163, 177, 182, 183, 195 } |
| 103 | 64    | #242| K3   | 18  | 150 | { 133, 144, 146, 150 } |
| 104 | 64    | #32 | K3   | 19  | 168 | { 133, 147, 158, 172 } |
| 105 | 64    | #35 | K3   | 19  | 168 | { 133 }           |
| 106 | 64    | #136| K3   | 19  | 168 | { 133, 149 }      |
| 107 | 64    | #36 | –    | 20  | 107 | {}                |
| 108 | 64    | #138| K3   | 18  | 108 | { 133, 143, 145, 148 } |
| 109 | 72    | #39 | M    | 20  | 135 | { 135 }           |
| 110 | 72    | #41 | K3   | 19  | 110 | { 135, 151, 178, 197 } |
| 111 | 72    | #40 | K3   | 19  | 111 | { 135, 152, 179 } |
| 112 | 72    | #43 | K3   | 18  | 112 | { 157, 164, 192, 193 } |
| 113 | 72    | #22 | S    | 20  | 198 | { 175 }           |
| 114 | 80    | #49 | K3   | 19  | 183 | { 136 }           |
| 115 | 81    | #13 | S    | 20  | 198 | { 153, 155 }      |
| 116 | 81    | #8  | S    | 20  | 198 | { 153 }           |
| 117 | 81    | #7  | S    | 20  | 198 | { 137, 155, 160 } |
| 118 | 81    | #15 | S    | 20  | 198 | { 139, 155, 169 } |
| 119 | 81    | #12 | S    | 18  | 170 | { 138, 153, 154, 155 } |
| 120 | 96    | #227| K3   | 18  | 120 | { 148, 156, 157, 168, 183 } |
| 121 | 96    | #195| K3   | 19  | 157 | { 157, 165, 166, 167, 182 } |
| 122 | 96    | #204| K3   | 19  | 143 | { 143, 145, 149 } |
| 123 | 96    | #190| M    | 20  | 187 | { 167 }           |
| 124 | 96    | #64 | K3   | 19  | 168 | { 168 }           |
| 125 | 96    | #70 | K3   | 19  | 148 | { 146, 147, 148, 156 } |
| 126 | 108   | #38 | S    | 20  | 198 | { 152, 159 }      |
| 127 | 108   | #15 | S    | 20  | 191 | { 186 }           |
| 128 | 108   | #40 | S    | 20  | 198 | { 152, 159, 160 } |
| 129 | 108   | #37 | S    | 20  | 198 | { 152, 161 }      |
| 130 | 108   | #37 | S    | 19  | 185 | { 151, 161, 185 } |
| 131 | 108   | #36 | S    | 20  | 191 | { 151, 186 }      |
| 132 | 120   | #34 | K3   | 19  | 132 | { 164, 179, 190, 193 } |
Table 12: Conjugacy classes of admissible groups

| No. | order | $G$ | Type | dim | fix | minimal overgroups |
|-----|-------|-----|------|-----|-----|-------------------|
| 133 | 128   | #931 | K3   | 19  | 168 | { 165, 166, 167, 168 } |
| 134 | 144   | #184 | K3   | 19  | 157 | { 156, 157, 174 }   |
| 135 | 144   | $\mathbb{Z}_3^2:Q_{16}$ | M | 20  | 135 | {}                |
| 136 | 160   | #234 | K3   | 19  | 183 | { 158, 182, 183 }   |
| 137 | 162   | #10  | S    | 20  | 198 | { 171, 176 }        |
| 138 | 162   | #46  | S    | 18  | 170 | { 170, 171 }        |
| 139 | 162   | #52  | S    | 20  | 198 | { 159, 161, 171, 181 } |
| 140 | 168   | #42  | K3   | 19  | 140 | { 162, 188, 193, 197 } |
| 141 | 168   | #43  | M    | 20  | 188 | { 188 }            |
| 142 | 180   | #19  | M    | 20  | 164 | { 164 }            |
| 143 | 192   | #1493 | K3  | 19  | 143 | { 165, 188 }       |
| 144 | 192   | #1024 | M  | 20  | 187 | { 167, 174 }       |
| 145 | 192   | #201 | M    | 20  | 196 | { 165, 182 }       |
| 146 | 192   | #1009 | M  | 20  | 187 | { 166, 174 }       |
| 147 | 192   | #184 | M    | 20  | 187 | { 166 }            |
| 148 | 192   | #955 | K3   | 19  | 148 | { 166, 173, 188, 190 } |
| 149 | 192   | #1492 | M  | 20  | 196 | { 165 }            |
| 150 | 192   | #1023 | K3 | 18  | 150 | { 168, 174, 183 }   |
| 151 | 216   | #161 | S    | 20  | 191 | { 191 }            |
| 152 | 216   | #158 | S    | 20  | 198 | { 175, 176 }       |
| 153 | 243   | #57  | S    | 20  | 198 | { 180 }            |
| 154 | 243   | #65  | S    | 18  | 170 | { 170, 180 }       |
| 155 | 243   | #51  | S    | 20  | 198 | { 171, 180, 184 }  |
| 156 | 288   | #1025 | M  | 20  | 173 | { 173 }            |
| 157 | 288   | #1026 | K3 | 19  | 157 | { 172, 173, 187 }  |
| 158 | 320   | #1635 | M  | 20  | 190 | { 190 }            |
| 159 | 324   | #167 | S    | 20  | 198 | { 175, 184 }       |
| 160 | 324   | #160 | S    | 20  | 198 | { 176, 184 }       |
| 161 | 324   | #163 | S    | 20  | 198 | { 175, 194 }       |
| 162 | 336   | $Z_2 \times L_2(7)$ | M | 20  | 162 | {}                |
| 163 | 360   | #118 | K3   | 19  | 163 | { 178, 179, 193, 196, 197, 198 } |
| 164 | 360   | $(Z_3 \rtimes A_5):Z_2$ | M | 20  | 164 | {}                |
| 165 | 384   | #5603 | M  | 20  | 196 | { 196 }            |
| 166 | 384   | #5678 | M  | 20  | 187 | { 187 }            |
| 167 | 384   | #18133 | M | 20  | 187 | { 187 }           |
| 168 | 384   | #18135 | K3 | 19  | 168 | { 187, 190, 196 }  |
| 169 | 405   | #15  | S    | 20  | 198 | { 181 }            |
| 170 | 486   | #249 | S    | 18  | 170 | { 185, 186, 189 }  |
| 171 | 486   | #166 | S    | 20  | 198 | { 189, 192, 195 }  |
| 172 | 576   | #8652 | M  | 20  | 196 | { 196 }            |
Table 12: Conjugacy classes of admissible groups

| No. | order | $G$         | Type | dim | fix | minimal overgroups |
|-----|-------|-------------|------|-----|-----|-------------------|
| 173 | 576   | $\mathbb{Z}_2^3:(S_3 \times S_3)$ | M    | 20  | 173 | {}               |
| 174 | 576   | #5129       | M    | 20  | 187 | { 187 }          |
| 175 | 648   | #722        | S    | 20  | 198 | { 192 }          |
| 176 | 648   | #704        | S    | 20  | 198 | { 192 }          |
| 177 | 660   | $L_2(11)$   | M    | 20  | 177 | {}               |
| 178 | 720   | $M_{10}$    | M    | 20  | 178 | {}               |
| 179 | 720   | $S_6$       | M    | 20  | 179 | {}               |
| 180 | 729   | #321        | S    | 20  | 198 | { 189 }          |
| 181 | 810   | #101        | S    | 20  | 198 | { 195 }          |
| 182 | 960   | #11358      | M    | 20  | 196 | { 196 }          |
| 183 | 960   | #11357      | K3   | 19  | 183 | { 190, 196, 197 }|
| 184 | 972   | #877        | S    | 20  | 198 | { 192, 195 }    |
| 185 | 972   | #776        | S    | 19  | 185 | { 191, 194 }    |
| 186 | 972   | #777        | S    | 20  | 191 | { 191 }         |
| 187 | 1152  | $2^6(\mathbb{Z}_2^3: \mathbb{Z}_2^3)$ | M    | 20  | 187 | {}               |
| 188 | 1344  | $\mathbb{Z}_2^3:L_2(7)$ | M    | 20  | 188 | {}               |
| 189 | 1458  | #1229       | S    | 20  | 198 | { 194 }         |
| 190 | 1920  | $\mathbb{Z}_2^4:S_5$ | M    | 20  | 190 | {}               |
| 191 | 1944  | $3^{1+4}:2.2^2$ | S    | 20  | 191 | {}               |
| 192 | 1944  | #3877       | S    | 20  | 198 | { 198 }         |
| 193 | 2520  | $A_7$       | M    | 20  | 193 | {}               |
| 194 | 2916  | $3^4:(3^2:\mathbb{Z}_4)$ | S    | 20  | 198 | { 198 }         |
| 195 | 4860  | $3^4:A_5$   | S    | 20  | 198 | { 198 }         |
| 196 | 5760  | $\mathbb{Z}_2^4:A_6$ | M    | 20  | 196 | {}               |
| 197 | 20160 | $L_3(4)$    | M    | 20  | 197 | {}               |
| 198 | 29160 | $3^4:A_6$   | S    | 20  | 198 | {}               |
Table 13: Isometry types of coinvariant lattices $L_G$

| No. | G-No. | $|G|$ | Symbol | Rank | Det | $A_L$ | Genus | Type |
|-----|-------|------|--------|------|-----|-------|-------|------|
| 1   | 1     | 1    | 1      | 0    | 1   | 1     | K3 # 0 |
| 2   | 2     | 2    | $\mathbb{Z}_2$ | 8 | 256 | $2^8$ | $2^{1+8}$ | K3 # 1 |
| 3   | 3     | 3    | $\mathbb{Z}_3$ | 12 | 729 | $3^6$ | $3^{1+6}$ | K3 # 2 |
| 4   | 4     | 4    | $\mathbb{Z}_4$ | 14 | 1024 | $2^{2+4}$ | $2^{2+2+4}$ | K3 # 4 |
| 5   | 6     | 4    | $\mathbb{Z}_2^2$ | 12 | 1024 | $2^{6+4}$ | $2^{2+6+4}$ | K3 # 3 |
| 6   | 9     | 6    | $\mathbb{Z}_6$ | 14 | 972 | $3^{2+6}$ | $2^{1+2+3+5}$ | K3 # 6 |
| 7   | 13    | 8    | $D_8$ | 15 | -1024 | $4^5$ | $4^{1+5}$ | K3 # 15 |
| 8   | 16    | 8    | $\mathbb{Z}_2^3$ | 14 | 1024 | $2^{6+4}$ | $2^{2+6+4}$ | K3 # 14 |
| 9   | 17    | 8    | $Q_8$ | 17 | -512 | $2^{3+2}$ | $2^{1+3+8}$ | K3 # 12 |
| 10  | 22    | 10   | $D_{10}$ | 16 | 625 | $5^4$ | $5^{1+4}$ | K3 # 16 |
| 11  | 24    | 12   | $D_{12}$ | 16 | 1296 | $6^4$ | $2^{2+4}+4$ | K3 # 18 |
| 12  | 25    | 12   | $A_4$ | 16 | 576 | $2^{4+2}$ | $2^{2+4}+2$ | K3 # 17 |
| 13  | 27    | 12   | $\mathbb{Z}_2 \times \mathbb{Z}_6$ | 18 | 1728 | $2^{6+3}$ | $2^{6+3}$ | K3 # 19 |
| 14  | 34    | 16   | $\mathbb{Z}_4^2$ | 18 | 1024 | $2^{2+4}$ | $2^{1+4}+4$ | O(L) |
| 15  | 35    | 16   | $\mathbb{Z}_2^4$ | 15 | -1024 | $2^{8+1}$ | $2^{1+8+4}$ | K3 # 21 |
| 16  | 36    | 16   | $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ | 17 | -1024 | $2^{4+3}$ | $2^{1+4}+4$ | K3 # 26 |
| 17  | 37    | 16   | $\mathbb{Z}_3^2$ | 15 | -512 | $2^{6+1}$ | $2^{6+1}$ | K3 # 33 |
| 18  | 40    | 16   | $\Gamma a_2$ | 18 | 512 | $2^{4+2}$ | $2^{1+4}+1+2$ | K3 # 22 |
| 19  | 42    | 16   | $\Gamma a_2$ | 18 | 1024 | $2^{2+4}$ | $2^{2+4}$ | K3 # 30 |
| 20  | 44    | 16   | $\Gamma d$ | 19 | -512 | $2^{3+2}$ | $2^{3+3}+2$ | O(L) |
| 21  | 47    | 16   | $\Gamma a_1$ | 16 | 1024 | $2^{2+4}$ | $2^{2+4}$ | K3 # 32 |
| 22  | 50    | 18   | $A_{3,3}$ | 16 | 729 | $3^{2+3}$ | $3^{1+3}$ | K3 # 33 |
| 23  | 53    | 20   | $\text{Hol}(\mathbb{Z}_4)$ | 18 | 500 | $5^{1+2}$ | $2^{5+1}+3$ | K3 # 34 |
| 24  | 54    | 21   | $\mathbb{Z}_7: \mathbb{Z}_3$ | 18 | 343 | $7^3$ | $7^{1+3}$ | K3 # 35 |
| 25  | 56    | 24   | $S_4$ | 17 | -576 | $4^{1+2}$ | $4^{1+2}+2$ | K3 # 36 |
| 26  | 68    | 32   | $\Gamma a_1$ | 17 | -1024 | $4^5$ | $4^{1+5}$ | K3 # 40 |
| 27  | 70    | 32   | $\Gamma a_3$ | 20 | 256 | $2^{8+2}$ | $2^{2+8+2}+2$ | O(L) |
| 28  | 74    | 32   | $\Gamma a_2$ | 18 | 1024 | $2^{2+4}$ | $2^{2+4}$ | O(L) |
| 29  | 77    | 32   | $\Gamma a_1$ | 17 | -512 | $2^{2+4}$ | $2^{2+4}$ | K3 # 39 |
| 30  | 79    | 36   | $3^{2} \mathbb{Z}_4$ | 18 | 324 | $3^{1+6}+1$ | $2^{1+6}+2+3+9$ | K3 # 46 |
| 31  | 81    | 36   | $S_{3,3}$ | 18 | 972 | $3^{2+6}+1$ | $2^{2+3}+3+9$ | K3 # 48 |
| 32  | 85    | 48   | $2^4 \mathbb{Z}_3$ | 17 | -384 | $2^{4+2}$ | $2^{4+2}$ | K3 # 49 |
| 33  | 86    | 48   | $T_{4,8}$ | 19 | -384 | $2^{8+2}+4$ | $2^{8+2+4}$ | K3 # 50 |
| 34  | 88    | 48   | $\mathbb{Z}_2 \times S_4$ | 18 | 576 | $2^{12+2}$ | $2^{12+2}$ | K3 # 51 |
| 35  | 90    | 48   | $\mathbb{Z}_2 \times SL_2(3)$ | 20 | 192 | $2^{4+2}12$ | $2^{12}12$ | K3 # 52 |
| 36  | 93    | 48   | $\mathbb{Z}_2^2 \times A_4$ | 19 | -576 | $2^{6+1}+12$ | $2^{6+1}+12$ | K3 # 53 |
| 37  | 102   | 60   | $A_5$ | 18 | 300 | $10^{3+30}$ | $2^{2+3}+15+2$ | K3 # 55 |
| 38  | 107   | 64   | $\Gamma a_3$ | 20 | 128 | $2^{3+16}$ | $2^{3+16}$ | O(L) |
| 39  | 108   | 64   | $\Gamma a_4$ | 18 | 512 | $4^{8+1}$ | $4^{8+1}$ | K3 # 56 |
| 40  | 110   | 72   | $M_9$ | 19 | -216 | $2^{16+18}+1$ | $2^{16+18}$ | K3 # 63 |
Table 13: Isometry types of coinvariant lattices $L_G$

| No. | G-No. | $|G|$ | Symbol | Rank | Det | $A_{L_G}$ | Genus | Type |
|-----|-------|------|--------|------|-----|----------|-------|------|
| 41  | 111   | 72   | $N_{72}$ | 19   | -324| $3^2 36^1$ | $4_1^{+1}3^{+2}9^{-1}$ | K3 # 62 |
| 42  | 112   | 72   | $A_{13}$  | 18   | 432 | $3^1 12^2$ | $4_2^{+2}3^{-3}$ | K3 # 61 |
| 43  | 120   | 96   | $2^4 D_6$  | 18   | 384 | $2^2 4^1 24^1$ | $2_2^{+2}4_7^{+1}8_1^{-1}3^{-1}$ | K3 # 65 |
| 44  | 132   | 120  | $S_5$    | 19   | -300| $5^1 60^1$ | $4_5^{+1}3^{+15}5^{-2}$ | K3 # 70 |
| 45  | 135   | 144  | $Z_3^2:QD_{16}$ | 20   | 216 | $6^1 36^1$ | $2_1^{+1}4_1^{+1}3^{+1}9^{-1}$ | max # 13 |
| 46  | 140   | 168  | $L_2(7)$ | 19   | -196| $7^1 28^1$ | $4_1^{+1}7^{+2}$ | K3 # 74 |
| 47  | 143   | 192  | $T_{192}$ | 19   | -192| $2^2 12^1$ | $4_7^{+2}3^{+1}$ | K3 # 77 |
| 48  | 148   | 192  | $H_{192}$ | 19   | -384| $4^2 24^1$ | $4_2^{+2}8_1^{+1}3^{-1}$ | K3 # 76 |
| 49  | 150   | 192  | $4^2 A_4$ | 18   | 256 | $2^2 8^2$ | $2_2^{+2}8_6^{-2}$ | K3 # 75 |
| 50  | 157   | 288  | $A_{14}$ | 19   | -288| $2^1 6^{12}4^1$ | $2_1^{+2}8_1^{+1}3^{+2}$ | K3 # 78 |
| 51  | 162   | 336  | $Z_2 \times L_2(7)$ | 20   | 196 | $14^2$ | $2_1^{+2}7^{+2}$ | max # 5 |
| 52  | 163   | 360  | $A_6$    | 19   | -180| $3^1 60^1$ | $4_5^{+1}3^{+25}5^{+2}$ | K3 # 79 |
| 53  | 164   | 360  | $(Z_3 \times A_5):Z_2$ | 20   | 225 | $15^2$ | $3^{+1}2^{+5}5^{-2}$ | max # 10 |
| 54  | 168   | 384  | $F_{384}$ | 19   | -256| $4^1 8^2$ | $4_1^{+1}8^{+2}$ | K3 # 80 |
| 55  | 170   | 486  | $3^{1+4}2^2$ | 18   | 243 | $3^5$ | $3^{+5}$ | S-lattice |
| 56  | 173   | 576  | $Z_2^3 \times (S_4 \times S_3)$ | 20   | 288 | $12^1 24^1$ | $4_2^{+2}8_1^{+1}3^{+2}$ | max # 12 |
| 57  | 177   | 660  | $L_2(11)$ | 20   | 121 | $11^2$ | $11^{+2}$ | max # 1 |
| 58  | 178   | 720  | $M_{10}$ | 20   | 120 | $2^1 60^1$ | $2_5^{+1}4_1^{+1}3^{+1}15^{+1}$ | max # 9 |
| 59  | 179   | 720  | $S_6$    | 20   | 180 | $6^1 30^1$ | $2_2^{+2}3^{+25}5^{+1}$ | max # 8 |
| 60  | 183   | 960  | $M_{30}$ | 19   | -160| $2^2 40^1$ | $2_2^{+2}8_1^{+1}15^{-1}$ | K3 # 81 |
| 61  | 185   | 972  | $3^{1+4}2^2$ | 19   | -162| $3^6^1$ | $2^{+1}3^{+4}$ | S-lattice |
| 62  | 187   | 1152 | $Q(Z_3^2:Z_2)$ | 20   | 192 | $8^1 24^1$ | $8_5^{+2}3^{-1}$ | max # 11 |
| 63  | 188   | 1344 | $Z_2^3 \times L_2(7)$ | 20   | 112 | $4^2 28^1$ | $4_2^{+2}7^{+1}$ | max # 4 |
| 64  | 190   | 1920 | $Z_3^2 S_5$ | 20   | 160 | $4^1 40^1$ | $4_1^{+1}8_1^{+1}5^{-1}$ | max # 7 |
| 65  | 191   | 1944 | $3^{1+4}2^2$ | 20   | 108 | $3^6^2$ | $2_2^{+2}3^{+3}$ | max S-lattice |
| 66  | 193   | 2520 | $A_7$    | 20   | 105 | $10^1 5^3 1$ | $3^{+1}5^{+1}7^{+1}$ | max # 3 |
| 67  | 196   | 5760 | $Z_2^4 A_6$ | 20   | 96  | $4^1 24^1$ | $4_5^{+1}8_1^{+1}3^{+1}$ | max # 6 |
| 68  | 197   | 20160| $L_3(4)$ | 20   | 84  | $2^1 42^1$ | $2_2^{+2}3^{+1}7^{-1}$ | max # 2 |
| 69  | 198   | 29160| $3^6 A_6$ | 20   | 81  | $3^9^1$ | $3^{+2}9^{+1}$ | max S-lattice |
Table 14: The 34 coinvariant lattices $L_G$ with several groups

| No. | #G | G-No. |
|-----|----|-------|
| 69  | 40 | {198, 195, 194, 192, 189, 184, 181, 180, 176, 175, 171, 169, 161, 160, 159, 155, 153, 152, 139, 137, 129, 128, 126, 118, 117, 116, 115, 113, 98, 95, 82, 80, 65, 64, 63, 62, 52, 51, 21, 20} |
| 67  | 7  | {196, 182, 172, 165, 149, 145, 91} |
| 65  | 7  | {191, 186, 151, 131, 127, 59, 28} |
| 64  | 2  | {190, 158} |
| 63  | 3  | {188, 141, 100} |
| 62  | 10 | {187, 174, 167, 166, 147, 146, 144, 123, 94, 89} |
| 61  | 3  | {185, 130, 26} |
| 60  | 3  | {183, 136, 114} |
| 57  | 3  | {177, 99, 23} |
| 56  | 2  | {173, 156} |
| 55  | 12 | {170, 154, 138, 119, 97, 96, 66, 61, 49, 18, 10, 4} |
| 54  | 11 | {168, 133, 124, 106, 105, 104, 75, 71, 69, 45, 33} |
| 53  | 5  | {164, 142, 101, 67, 32} |
| 51  | 3  | {162, 83, 31} |
| 50  | 5  | {157, 134, 121, 92, 84} |
| 49  | 6  | {150, 103, 87, 72, 39, 38} |
| 48  | 2  | {148, 125} |
| 47  | 3  | {143, 122, 57} |
| 45  | 2  | {135, 109} |
| 42  | 5  | {112, 78, 60, 30, 29} |
| 39  | 2  | {108, 76} |
| 38  | 2  | {107, 73} |
| 34  | 2  | {88, 58} |
| 33  | 2  | {86, 55} |
| 31  | 2  | {81, 48} |
| 29  | 2  | {77, 41} |
| 28  | 2  | {74, 43} |
| 26  | 3  | {68, 46, 15} |
| 24  | 2  | {54, 11} |
| 22  | 2  | {50, 19} |
| 21  | 2  | {47, 14} |
| 18  | 2  | {40, 12} |
| 11  | 2  | {24, 8} |
| 10  | 2  | {22, 7} |
References

[AI] D. Allcock, *Orbits in the Leech Lattice*, Experimental Mathematics **14** No. 4 (2005), 491–509.

[AI] H. Aoki and T. Ibukiyama, *Simple graded rings of Siegel modular forms, differential operators and Borcherds products*, Int. J. Math. **16** (2005), 249–279.

[BHT] A. Bayer, B. Hassett, Y. Tschinkel, *Mori cones of holomorphic symplectic varieties of K3 type*, [arXiv:1307.2291](http://arxiv.org/abs/1307.2291).

[BM] A. Bayer, E. Macrì, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, [arXiv:1301.6968](http://arxiv.org/abs/1301.6968).

[Be] A. Beauville, *Some remarks on Kähler manifolds with $c_1 = 0**, Progr. Math., 39, Birkhäuser Boston, Boston, MA, 1983, 1–26.

[BEO] H. U. Besche, B. Eick, E. A. O’Brien, *A millennium project: constructing small groups*, Internat. J. Algebra Comput. **12** (2002), 623–644.

[BNS] S. Boissière, M. Nieper-Wißkirchen and A. Sarti, *Smith theory and irreducible holomorphic symplectic manifolds*, J. Topol. **6** (2013), 361–390, [arXiv:1204.4118](http://arxiv.org/abs/1204.4118).

[BL] L. Borisov, A. Libgober, *McKay correspondence for elliptic genera*, Ann. of Math. (2) **161** (2005), 1521–1569.

[BI] H. Brandt and O. Intrau, *Tabellen reduzierter positiver ternärer quadratischer Formen*, Abh. Sächs. Akad. Wiss. Math.-Nat. Kl. **45** (1958) No. 4, 261 pages.

[OB] E. A. O’Brien, *The groups of order 256*, J. Algebra **142**, 1991.

[Ca] C. Camere, *Symplectic involutions of holomorphic symplectic four-folds*, Bull. London Math. Soc. **44** (2012), 687–702, [arXiv:1010.2607](http://arxiv.org/abs/1010.2607).

[Ch] M. C. N. Cheng, *K3 Surfaces, $N = 4$ Dyons, and the Mathieu Group $M_{24}$*, Comm. Num. Theor. Phys. **4** (2010) 623–657.

[Co] J. H. Conway, Three lectures on exceptional groups. Finite simple groups (Proc. Instructional Conf., Oxford, 1969), pp. 215–247.

[CCNPW] J. H. Conway et al, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[CS] J. H. Conway and N. Sloane, *Sphere Packings, Lattices and Groups*, Third edition. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Grundlehren der Mathematischen Wissenschaften, 290. Springer-Verlag, New York, 1999.

[CD] M. C. N. Cheng and J. F. R. Duncan, *On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine*, preprint (2011), [arXiv:1110.3859](http://arxiv.org/abs/1110.3859).
[CH] T. Creutzig and G. Höhn, *Mathieu Moonshine and the Geometry of K3 Surfaces*, arXiv:1309.2671. To appear in CNTP 8 (2014).

[Cu] R. Curtis, *On Subgroups of .0. I. Lattice Stabilizers*, J. of Alg. 27 (1973), 549–573.

[EH1] T. Eguchi and K. Hikami, *Note on twisted elliptic genus of K3 surface*, Phys. Lett. B 694 (2011), 446–455, arXiv:1008.4924.

[EH2] T. Eguchi and K. Hikami, *Twisted elliptic genus for K3 and Borcherds Product*, arXiv:1112.5928.

[EOT] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group M_24*, Exper. Math. 20 (2011) 91–96.

[Fu] L. Fu, *Classification of polarized symplectic automorphisms of Fano varieties of cubic fourfolds*, arXiv:1303.2241.

[Ga] T. Gannon, *Much ado about Mathieu*, arXiv:1211.5531 [math.RT].

[GHV] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu twining characters for K3*, J. High Energy Phys. (2010), no. 9, 058, 20 pp, arXiv:1006.0221.

[GHV2] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu Moonshine in the elliptic genus of K3*, JHEP 1010 (2010) 062, 24 pp.

[GV] M. R. Gaberdiel and R. Volpato, *Mathieu moonshine and orbifold K3s*, arXiv:1206.5143.

[HS] M. Hall and J.K. Senior, *The groups of order 2^n (n ≤ 6)*, McMillan, New York, 1964.

[HL] K. Harada and M.-L. Lang, *On some sublattices of the Leech lattice*, Hokkaido Math. J. 19 (1990), 435–446.

[Ha] K. Hashimoto, *Finite symplectic actions on the K3 lattice*, Nagoya Math. J. 206 (2012), 99–153.

[HS] B. Hassett and Y. Tschinkel, *Hodge theory and Lagrangian planes on generalized Kummer fourfolds*, Mosc. Math. J. 13 (2013), 33–56.

[Hi] F. Hirzebruch, *Topological methods in algebraic geometry*. Third enlarged edition. New appendix and translation from the second German edition by R. L. E. Schwarzenberger, with an additional section by A. Borel. Die Grundlehren der Mathematischen Wissenschaften, Band 131 Springer-Verlag New York, Inc., New York 1966 x+232 pp.

[HBJ] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and modular forms*, Aspects of Mathematics, E20, With appendices by Nils-Peter Skoruppa and by Paul Baum, Friedr. Vieweg & Sohn, Braunschweig, 1992.
[Hö1] G. Höhn, Komplexe elliptische Geschlechter und $S^1$-äquivariante Kobordismustheorie, Diploma thesis, Bonn 1991, arXiv:math/0405232.

[Hö2] G. Höhn, Elliptic genera of symmetric products and generalized Kac-Moody Lie algebras, Workshop on ‘Verallgemeinerte Kac-Moody-Algebren’ organized by R. Borcherds and P. Slodowy, Tagungsbericht 1998/29, Oberwolfach Digital Archive.

[Hu1] D. Huybrechts, Compact hyper-Kähler manifolds: basic results, Invent. Math. 135 (1999), 63–113.

[Hu2] D. Huybrechts, On derived categories of $K3$ surfaces, symplectic automorphisms and the Conway group, arXiv:1309.6528.

[K] S. Kondō with an appendix by S. Mukai, Niemeier lattices, Mathieu groups and finite groups of symplectic automorphisms of $K3$ surfaces, Duke Math. Journal 92 (1998), 593–603.

[KT1] T. Kondo and T. Tasaka, The theta functions of sublattices of the Leech lattice., Nagoya Math. J. 101 (1986), 151–179.

[KT2] T. Kondo and T. Tasaka, The theta functions of sublattices of the Leech lattice. II, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), 545–572.

[La] M.-L. Lang, On a question raised by Conway-Norton, J. Math. Soc. Japan 41 (1989), 263–284.

[Mag] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), 235–265.

[Ma1] G. Mason, $M_{24}$ and certain automorphic forms, in J. McKay, ed., Finite Groups — Coming of Age, vol. 45 Contemp. Math., pp. 223-244, Amer. Math. Soc., Providence, 1985.

[Ma2] G. Mason, Symplectic Automorphisms of $K3$-surfaces (after S. Mukai and V. V. Nikulin), CWI Newslett. No. 13 (1986), 3–19.

[Mat] S. Wolfram, Mathematica: A System for Doing Mathematics by Computer, Second Edition. Redwood City, Addison-Wesley, 1991.

[Mo1] G. Mongardi, Automorphisms of Hyperkähler manifolds, Ph.D. thesis, Roma ???, arXiv:1303.4670.

[Mo2] G. Mongardi, Symplectic involutions on deformations of $K3^{[2]}$, Centr. Eur. J. Math. 10 (2012) 1472–1485.

[Mo3] G. Mongardi, On symplectic automorphisms of hyperkähler fourfolds of $K3^{[2]}$ type, Michigan Math. J. 62 (2013) 537–550.

[Mo4] G. Mongardi, On natural deformations of symplectic automorphisms of manifolds of $K3^{[n]}$ type, C. R. Math. Acad. Sci. Paris 351 (2013), 561–564, arXiv:1304.6630.
[Mo5] G. Mongardi, *A note on the Kähler and Mori cones of hyperkähler manifolds*, arXiv:1307.0393.

[Mo6] G. Mongardi, *Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$ type*, arXiv:1405.3232.

[Mu] S. Mukai, *Finite group of automorphisms of $K3$ surfaces and the Mathieu group*, Inv. Math. 94 (1988) 183–221.

[Ni] V. V. Nikulin, Integral symmetric bilinear forms and some of their geometric applications. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 111–177.

[PV] D. Persson and R. Volpato, *Second Quantized Mathieu Moonshine*, arXiv:1312.0622.

[Ra] M. Raum, *$M_{24}$-twisted product expansions are Siegel Modular forms*, Commun. Number Theory Phys. 7 (2013), 469–495, arXiv:1208.3453.

[VVDM] R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Comm. Math. Phys. 185 (1997) 197–209.

[W] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott, *ATLAS of Finite Group Representations, Version 3* available at http://brauer.maths.qmul.ac.uk/Atlas/v3/.

[Xi] G. Xiao, *Galois covers between $K3$ surfaces*, Annales de l’institut Fourier 46 (1996) 73–78.