GLOBAL COEFFICIENT RING IN THE NILPOTENCE CONJECTURE

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ABSTRACT. In this note we show that the nilpotence conjecture for toric varieties is true over any regular coefficient ring containing $\mathbb{Q}$.

In [G] we showed that for any additive submonoid $M$ of a rational vector space with the trivial group of units and a field $k$ with $\text{char } k = 0$ the multiplicative monoid $\mathbb{N}$ acts nilpotently on the quotient $K_i(k[M])/K_i(k)$ of the $i$th $K$-groups, $i \geq 0$. In other words, for any sequence of natural numbers $c_1, c_2, \ldots \geq 2$ and any element $x \in K_i(k[M])$ we have $(c_1 \cdots c_j)_*(x) \in K_i(k)$ for all $j \gg 0$ (potentially depending in $x$). Here $c_*$ refers to the group endomorphism of $K_i(k[M])$ induced by the monoid endomorphism $M \to M, m \mapsto m^c$, writing the monoid operation multiplicatively.

The motivation of this result is that it includes the known results on (stable) triviality of vector bundles on affine toric varieties and higher $K$-homotopy invariance of affine spaces. Here we show how the mentioned nilpotence extends to all regular coefficient rings containing $\mathbb{Q}$, thus providing the last missing argument in the long project spread over many papers. See the introduction of [G] for more details.

Using Bloch-Stienstra’s actions of the big Witt vectors on the $NK_i$-groups [St] (that has already played a crucial role in [G], but in a different context), Lindel’s technique of étale neighborhoods [L], van der Kallen’s étale localization [K], and Popescu’s desingularization [Sw], we show

**Theorem 1.** Let $M$ be an additive submonoid of a $\mathbb{Q}$-vector space with trivial group of units. Then for any regular ring $R$ with $\mathbb{Q} \subset R$ the multiplicative monoid $\mathbb{N}$ acts nilpotently on $K_i(R[M])/K_i(R)$, $i \geq 0$.

**Conventions.** All our monoids and rings are assumed to be commutative. $X$ is a variable. The monoid operation is written multiplicatively, denoting by $e$ the neutral element. $\mathbb{Z}_+$ is the additive monoid of nonnegative integers. For a sequence of natural numbers $c = c_1, c_2, \ldots \geq 2$ and an additive submonoid $N$ of a rational space $V$ we put

$$N^c = \lim_{\to} \left( N \xrightarrow{-c_1} N \xrightarrow{-c_2} \cdots \right) = \bigcup_{j=1}^{\infty} N^{\frac{1}{c_1 \cdots c_j}} \subset V.$$
Lemma 2. Let $M$ be a finitely generated submonoid of a rational vector space with the trivial group of units. Then $M$ embeds into a free commutative monoid $\mathbb{Z}_r^+$. For the stronger version of Lemma 2 with $r = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes M)$ see, for instance, [BG, Proposition 2.15(e)]. (In [BG] the monoids as in Lemma 2 are called the affine positive monoids.)

Lemma 3. Let $F$ be a functor from rings to abelian groups, $H$ be a monoid with the trivial group of units, and $\Lambda = \bigoplus_H \Lambda_h$ be an $H$-graded ring (i. e. $\Lambda_h \Lambda_{h'} \subset \Lambda_{hh'}$). Then we have the implication

$$F(\Lambda) = F(\Lambda[H]) \implies F(\Lambda_e) = F(\Lambda).$$

The special case of Lemma 3 when $H = \mathbb{Z}_+^+$ is known as the Swan-Weibel homotopy trick and the proof of the general cases makes no real difference, see [G, Proposition 8.2].

Lemma 4. Theorem 1 is true for any coefficient ring of the form $S^{-1}k[\mathbb{Z}_+^+]$ where $k$ is a field of characteristic 0 and $S \subset k[\mathbb{Z}_+^+]$ is a multiplicative subset.

In the special case when $k$ is a number field Lemma 4 is proved in Step 2 in [G, §8], but word-by-word the same argument goes through for a field $k$ provided the nilpotence conjecture is true for the monoid rings with coefficients in $k$.

Notice. The reason we state the result in [G] only for number fields is that the preceding result in [G] is the validity of the nilpotence conjecture for such coefficient fields. Actually, the proof of Theorem 1 is an étale version of the idea of interpreting the globalization problem for coefficient rings in terms of the $K$-homotopy invariance, used for Zariski topology in [G, §8].

Finally, in order to explain one formula we now summarize very briefly the Bloch-Stienstra action of the ring of big Witt vectors $W(\Lambda)$ on

$$NK_i(\Lambda) = \operatorname{Coker}(K_i(\Lambda) \to K_i(\Lambda[X])).$$

For the details the reader is referred to [St].

The additive group of $W(R)$ can be thought of as the multiplicative group of formal power series $1 + X \Lambda[[X]]$. It has the decreasing filtration by the ideals $I_p(R) = (1 + X^{p+1} \Lambda[[X]])$, $p = 1, 2, \ldots$, and every element $\alpha(X) \in W(\Lambda)$ admits a convergent series expansion in the corresponding additive topology $\alpha(X) = \Pi_n(1 - \lambda_m X^m)$, $\lambda_m \in \Lambda$. To define a continuous $W(\Lambda)$-module structure on $NK_i(\Lambda)$ it is enough to define the appropriate action of the Witt vectors of type $1 - \lambda X^m$, satisfying the condition that every element of $NK_i(\Lambda)$ is annihilated by some ideal $I_p(W(\Lambda))$. Finally, such an action of $1 - \lambda X^m$ on $NK_i(\Lambda)$ is provided by the composite map in
the upper row of the following commutative diagram with exact vertical columns:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
NK_i(\Lambda[X]) & NK_i(\Lambda[X]) & NK_i(\Lambda[X]) & NK_i(\Lambda[X]) & NK_i(\Lambda[X]) & NK_i(\Lambda[X]) \\
K_i(\Lambda[X]) & K_i(\Lambda[X]) & K_i(\Lambda[X]) & K_i(\Lambda[X]) & K_i(\Lambda[X]) & K_i(\Lambda[X]) \\
K_i(\Lambda) & K_i(\Lambda) & K_i(\Lambda) & K_i(\Lambda) & K_i(\Lambda) & K_i(\Lambda) \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

where:

1. \(m_*\) corresponds to scalar extension through the \(\Lambda\)-algebra endomorphism \(\Lambda[X] \to \Lambda[X], X \mapsto X^m\),
2. \(m^*\) corresponds to scalar restriction through the same endomorphism \(\Lambda[X] \to \Lambda[X]\),
3. \(\lambda_*\) corresponds to scalar extension through the \(\Lambda\)-algebra endomorphism \(\Lambda[X] \to \Lambda[X], X \mapsto \lambda X\).
4. \(m \cdot -\) is multiplication by \(m\).

A straightforward check of the commutativity of the appropriate diagrams, based on the description above, shows that for a ring homomorphism \(f : \Lambda_1 \to \Lambda_2\) we have

\[
f_*(\alpha z) = f_*(\alpha)f_*(z), \quad \alpha \in W(\Lambda_1), \quad z \in NK_i(\Lambda_1),
\]

where the same \(f_*\) is used for the both induced homomorphisms

\[
W(\Lambda_1) \to W(\Lambda_2) \quad \text{and} \quad NK_i(\Lambda_1) \to NK_i(\Lambda_2).
\]

**Proof of Theorem 1.** Since \(K\)-groups commute with filtered colimits there is no loss of generality in assuming that \(M\) is finitely generated. Then by Lemma 2 \(R[M]\) admits a \(\mathbb{Z}_+\)-grading

\[
R[M] = \bigoplus_{\mathbb{Z}_+} R_j, \quad R_e = R.
\]

In particular, by the Quillen local-global patching for higher \(K\)-groups [V], we can without loss of generality assume that \(R\) is local.

**Notice.** Actually, the local-global patching proved in [V] is for the special case of polynomial extensions. However, the more general version for graded rings is a straightforward consequence via the Swan-Weibel homotopy trick, discussed above.

By Popescu’s desingularization [Sw] and the same filtered colimit argument we can further assume that \(R\) is a regular localization of an affine \(k\)-algebra for a field
with char $k = 0$. In this situation Lindel has shown [L, Proposition 2] that there is a subring $A \subset R$ of the form $k[Z^d_+], \mu \in \max(k[Z^d_+])$, $d = \dim R$, such that

(2) $R$ is étale over $A$.

Notice. Lindel's result is valid in arbitrary characteristic under the conditions that the residue field of $R$ is a simple separable extension of $k$, which is automatic in our situation because char $k = 0$.

Using again that $K$-groups commute with filtered colimits, the validity of Theorem 1 for $R$ is easily seen to be equivalent to the equality

(3) $K_i(R) = K_i(R[M^c])$

for every sequence of natural numbers $c = c_1, c_2, \ldots \geq 2$.

Next we show that (3) follows from the condition

(4) $NK_i(R[M^c]) = 0$.

In fact, by the filtered colimit argument we have

$K_i(R[M^c]) = K_i(R[M^c])[X] \Rightarrow K_i(R[M^c]) = K_i(R[M^c])[Z^c_+] (= \lim_{\to} K_i(R[M^c])[Z^c_+]).$

On the other hand, by Lemma 2 the ring $R[M^c]$ has a $Z^c_+$-grading:

$$R[M^c] = \bigoplus_{Z^c_+} S_j, \quad S_e = R.$$  

So by Lemma 3 we have (3).

To complete the proof it is enough to show (4) assuming (2).

By the base change property the ring extension $A[M] \subset R[M] = A[M] \otimes_A R$ is étale. Then by van der Kallen’s result [K, Theorem 3.2] we have the isomorphism of $W(R[M])$-modules

(5) $NK_i(R[M]) = W(R[M]) \otimes_{W(A[M])} NK_i(A[M]).$

Notice. Van der Kallen proves his formula (the étale localization) for a modified tensor product that takes care of the filtrations on $W(A[M])$ and $W(R[M])$ by the ideals $I_p(\cdot)$. But the presence of the characteristic 0 subfield $k$ yields (via the ghost map) the infinite product presentations $W(A[M]) = \prod_{N} A[M]$ and $W(R[M]) = \prod_{N} R[M]$ and, in particular, makes checkable the appropriate compatibility of the two filtrations: $I_p(R[M]) = I_p(A[M]) W(R[M])$; see the discussion before [K, Theorem 3.2].

Pick an element $z \in NK_i(R[M])$. By (5) it admits a representation of the form

$$z = \sum_q \alpha_q \bar{y}_q, \quad \alpha_q \in W(R[M]), \quad y_q \in NK_i(A[M]),$$

where the bar refers to the image in $NK_i(R[M])$. 
By Lemma 4 we know that Theorem 1 is true for $A$. Therefore, $(c_1 \cdots c_j)_*(y_q) = 0$ for all $q$ provided $j \gg 0$. In particular, (1) implies
\[
(c_1 \cdots c_j)_*(z) = \sum_q (c_1 \cdots c_j)_*(a_q)(c_1 \cdots c_j)_*(y_q) = 0, \quad j \gg 0.
\]
Since $z$ was an arbitrary element the filtered colimit argument shows (4). □

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