Asymmetric Unification and Disunification

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Abstract

We compare two kinds of unification problems: Asymmetric Unification and Disunification, which are variants of Equational Unification. Asymmetric Unification is a type of Equational Unification where the right-hand sides of the equations are in normal form with respect to the given term rewriting system. In Disunification we solve equations and disequations with respect to an equational theory for the case with free constants. We contrast the time complexities of both and show that the two problems are incomparable: there are theories where one can be solved in Polynomial time while the other is NP-hard. This goes both ways. The time complexity also varies based on the termination ordering used in the term rewriting system.

1 Introduction and Motivation

This is a short introductory survey on two variants of unification, namely asymmetric unification \cite{11} and disunification \cite{2, 8}. We contrast the two in terms of their time complexities for different equational theories, for the case where terms in the input can also have free constant symbols. Asymmetric unification is a new paradigm comparatively, which requires one side of the equation to be irreducible \cite{11}, while disunification \cite{8} deals with solving equations and disequations. Complexity analysis has been performed separately on asymmetric unification \cite{4, 12}, and disunification \cite{2, 6}, but not much work has been done on contrasting the two paradigms. In \cite{11}, it was shown that there are theories which are decidable for symmetric unification but are undecidable for asymmetric unification, so here we investigate this further. Initially, it was thought that the two are reducible to one another \cite{12}, but our results indicate that they are not at least where time complexity is concerned. In our last section we show that the time complexity of asymmetric unification varies depending on the symbol ordering chosen for the theory.

Unification deals with solving symbolic equations. A solution to a unification problem is a unifier, a substitution of certain variables by another expression or term. Often we need to find most general unifiers (mgu).
For example, given two terms \( u = f(a, y) \) and \( v = f(x, b) \), where \( f \) is a binary function symbol, \( a \) and \( b \) are constants, and \( x \) and \( y \) are variables, the substitution \( \sigma = \{ x \mapsto a, y \mapsto b \} \) unifies \( u \) and \( v \).

## 2 Notations and Preliminaries: Term Rewriting Systems, Equational Unification

We assume the reader is accustomed with the terminologies of term rewriting systems (TRS), equational rewriting [1], unification and equational unification [2].

### Term Rewriting Systems: A term rewriting system (TRS) [1] is a set of rewrite rules, where a rewrite rule is an identity \( l \approx r \) such that \( l \) is not a variable and \( \text{Var}(l) \subseteq \text{Var}(r) \). It is often written or denoted as \( l \rightarrow r \). These oriented equations are commonly called rewrite rules. The rewrite relation induced by \( R \) is written as \( \rightarrow_R \).

A term is reducible by a term rewriting system if and only if a subterm of it is an instance of the left-hand side of a rule. In other words, a term \( t \) is reducible modulo \( R \) if and only if there is a rule \( l \rightarrow r \) in \( R \), a subterm \( t' \) at position \( p \) of \( t \), and a substitution \( \sigma \) such that \( \sigma(l) = t' \). The term \( t[\sigma(r)]_p \) is the result of reducing \( t \) by \( l \rightarrow r \) at \( p \). The reduction relation \( \rightarrow_R \) associated with a term rewriting system \( R \) is defined as follows: \( s \rightarrow_R t \) if and only if there exist \( p \in \text{Pos}(s) \) and \( l \rightarrow r \) in \( R \) such that \( t \) is the result of reducing \( s \) by \( l \rightarrow r \) at \( p \), i.e., \( t = s[\sigma(r)]_p \).

A term is in normal form with respect to a term rewriting system if and only if no rule can be applied to it. A term rewriting system is terminating if and only if there are no infinite rewrite chains.

Two terms \( s \) and \( t \) are said to be joinable modulo a term rewriting system \( R \) if and only if there exists a term \( u \) such that \( s \rightarrow^*_R u \) and \( t \rightarrow^*_R u \), denoted as \( s \downarrow t \).

The equational theory \( \mathcal{E}(R) \) associated with a term rewriting system \( R \) is the set of equations obtained from \( R \) by treating every rule as a (bidirectional) equation. Thus the equational congruence \( \approx_{\mathcal{E}(R)} \) is the congruence \( (\rightarrow_R \cup \leftarrow_R)^* \).

A term rewriting system \( R \) is said to be confluent if and only if the following (“diamond”) property holds:

\[
\forall t \forall u \forall v \left( t \rightarrow^*_R u \land t \rightarrow^*_R v \Rightarrow \exists w (u \rightarrow^*_R w \land v \rightarrow^*_R w) \right)
\]

\( R \) is convergent if and only if it is terminating and confluent. In other words, \( R \) is convergent if and only if it is terminating and, besides, every term has a unique normal form.

An equational theory \( \approx_E \) is said to be subterm-collapsing if and only if there exist terms \( s, t \) such that \( s \approx_E t \) and \( t \) is a proper subterm of \( s \). Equivalently, \( \approx_E \) is subterm-collapsing if and only if \( s \approx_E s|_p \) for some term \( s \) and \( p \in \text{Pos}(s) \). If the theory has a convergent term rewriting system \( R \), then it is subterm-collapsing if and only if \( s \rightarrow_R s|_p \) for some term \( s \) and \( p \in \text{Pos}(s) \). An equational
theory is said to be non-subterm-collapsing or simple if and only if it is not subterm-collapsing.

Equational rewriting facilitates incorporation of equational theories such as associativity and commutativity, which basic (“pure”) term rewriting systems cannot handle, since they cannot (often) be turned into terminating rewrite rules. An equational term rewriting system consists of a set of identities \( E \) (which often contains identities such as Commutativity and Associativity) and a set of rewrite rules \( R \). This gives rise to a new rewrite relation \( \rightarrow_{R,E} \), which uses equational matching modulo \( E \) instead of standard matching.

**Example:** Let \( E = \{ (x+y)+z \approx x+(y+z), x+y \approx y+x \} \) and \( R = \{ 0+x \rightarrow x \} \). Then
\[
(a+0) + b \rightarrow_{R,E} a + b
\]
since \( a+0 \) matches with \( 0+x \) modulo \( E \).

A set of equations is said to be in dag-solved form (or d-solved form) if and only if they can be arranged as a list
\[
X_1 = ? t_1, \ldots, X_n = ? t_n
\]
where (a) each left-hand side \( X_i \) is a distinct variable, and (b) \( \forall 1 \leq i \leq j \leq n: X_i \) does not occur in \( t_j \).

**Equational Unification:** Two terms \( s \) and \( t \) are unifiable modulo an equational theory \( E \) iff there exists a substitution \( \theta \) such that \( \theta(s) \approx_E \theta(t) \). The unification problem modulo equational theory \( E \) is the problem of solving a set of equations \( \mathcal{S} = \{ s_1 \approx_E t_1, \ldots, s_n \approx_E t_n \} \), whether there exists \( \sigma \) such that \( \sigma(s_1) \approx_E \sigma(t_1), \ldots, \sigma(s_n) \approx_E \sigma(t_n) \). This is also referred to as semantic unification where equational equivalence \( [3] \) or congruence is considered among the terms being unified, rather than syntactic identity. Some of the standard equational theories used are associativity and commutativity.

A unifier \( \delta \) is more general than another unifier \( \rho \) iff a substitution equivalent to the latter can be obtained from the former by suitably composing it with a third substitution:
\[
\delta \preceq_E \rho \quad \text{iff} \quad \exists \sigma: \delta \circ \sigma =_E \rho
\]

A substitution \( \theta \) is a normalized substitution with respect to a term rewrite system \( R \) if and only for every \( x \), \( \theta(x) \) is in \( R \)-normal form. In other words, terms in the range of \( \theta \) are in normal form. (These are also sometimes referred to as irreducible substitutions.) When \( R \) is convergent, one can assume that all unifiers modulo \( R \) are normalized substitutions.
3 Asymmetric Unification

**Definition 1.** Given a decomposition $(\Sigma, E, R)$ of an equational theory, a substitution $\sigma$ is an asymmetric $R, E$-unifier of a set $Q$ of asymmetric equations $\{s_1 \approx^i t_1, \ldots, s_n \approx^i t_n\}$ iff for each asymmetric equation $s_i \approx^i t_i$, $\sigma$ is an $(E \cup R)$-unifier of the equation $s_i \approx t_i$, and $\sigma(t_i)$ is in $R, E$-normal form. In other words, $\sigma(s_i) \rightarrow^i_{R, E} \sigma(t_i)$.

(Note that symmetric unification can be reduced to asymmetric unification. We could also include symmetric equations in a problem instance.)

**Example:**
Let $R = \{x + a \rightarrow x\}$ be a rewrite system. An asymmetric unifier $\theta$ for $\{u + v \approx v + w\}$ modulo this system is $\theta = \{u \mapsto v, w \mapsto v\}$. However, another unifier $\rho = \{u \mapsto a, v \mapsto a, w \mapsto a\}$ is not an asymmetric unifier. But note that $\theta \preceq_E \rho$, i.e., $\rho$ is an instance of $\theta$, or, alternatively, $\theta$ is more general than $\rho$. This shows that instances of asymmetric unifiers need not be asymmetric unifiers.

4 Disunification

Disunification deals with solving a set of equations and disequations with respect to a given equational theory.

**Definition 2.** For an equational theory $E$, a disunification problem is a set of equations and disequations $\mathcal{L} = \{s_1 \approx_E t_1, \ldots, s_n \approx_E t_n\} \cup \{s_{n+1} \not\approx_E t_{n+1}, \ldots, s_{n+m} \not\approx_E t_{n+m}\}$.

A solution to this problem is a substitution $\sigma$ such that:

$$\sigma(s_i) \approx_E \sigma(t_i) \quad (i = 1, \ldots, n)$$

and

$$\sigma(s_{n+j}) \not\approx_E \sigma(t_{n+j}) \quad (j = 1, \ldots, m).$$

**Example:**
Given $E = \{x + a \approx x\}$, a disunifier $\theta$ for $\{u + v \not\approx_E v + u\}$ is $\theta = \{u \mapsto a, v \mapsto b\}$.

If $a + x \approx x$ is added to the identities $E$, then $\theta = \{u \mapsto a, v \mapsto b\}$ is clearly no longer a disunifier modulo this equational theory.
5 A theory for which asymmetric unification is in \( P \) whereas disunification is NP-complete

Let \( R_1 \) be the following term rewriting system:

\[
\begin{align*}
  h(a) & \rightarrow f(a, c) \\
  h(b) & \rightarrow f(b, c)
\end{align*}
\]

We show that asymmetric unifiability modulo this theory can be solved in polynomial time. The algorithm is outlined in Appendix A (p. 18–21).

However, disunification modulo \( R_1 \) is NP-hard. The proof is by a polynomial-time reduction from the three-satisfiability (3SAT) problem.

Let \( U = \{x_1, x_2, \ldots, x_n\} \) be the set of variables, and \( B = \{C_1, C_2, \ldots, C_m\} \) be the set of clauses. Each clause \( C_k \), where \( 1 \leq k \leq m \), has 3 literals.

We construct an instance of a disunification problem from 3SAT. There are 8 different combinations of \( T \) and \( F \) assignments to the variables in a clause in 3SAT, out of which there is exactly one truth-assignment to the variables in the clause that makes the clause evaluate to false. For the 7 other combinations of \( T \) and \( F \) assignments to the literals, the clause is rendered true. We represent \( T \) by \( a \) and \( F \) by \( b \). Hence for each clause \( C_i \) we create a disequation \( DEQ_i \) of the form

\[
f(x_p, f(x_q, x_r)) \not\approx_{R_1} f(d_1, f(d_2, d_3))
\]

where \( x_p, x_q, x_r \) are variables, \( d_1, d_2, d_3 \in \{a, b\} \), and \( (d_1, d_2, d_3) \) corresponds to the falsifying truth assignment. For example, given a clause \( C_k = x_p \lor x_q \lor x_r \), we create the corresponding disequation

\[
DEQ_k = f(x_p, f(x_q, x_r)) \not\approx_{R_1} f(b, f(a, b)).
\]

We also create the equation \( h(x_j) \approx_{R_1} f(x_j, c) \) for each variable \( x_j \). These make sure that each \( x_j \) is mapped to either \( a \) or \( b \).

Thus for \( B \), the instance of disunification constructed is

\[
S = \left\{ h(x_1) \approx f(x_1, c), h(x_2) \approx f(x_2, c), \ldots, h(x_n) \approx f(x_n, c) \right\} \cup \left\{ DEQ_1, DEQ_2, \ldots, DEQ_m \right\}
\]

**Example:** Given \( U = \{x_1, x_2, x_3\} \) and \( B = \{x_1 \lor \overline{x_2} \lor x_3, \overline{x_1} \lor \overline{x_2} \lor x_3\} \), the constructed instance of disunification is

\[
\left\{ h(x_1) \approx f(x_1, c), h(x_2) \approx f(x_2, c), h(x_3) \approx f(x_3, c), f(x_1, f(x_2, x_3)) \not\approx f(b, f(a, b)), f(x_1, f(x_2, x_3)) \not\approx f(a, f(a, b)) \right\}
\]

Note that membership in \( NP \) is not hard to show since \( R_1 \) is saturated by paramodulation [20].
6 A theory for which disunification is in P whereas asymmetric unification is NP-hard

The theory we consider consists of the following term rewriting system $R_2$:

$$
\begin{align*}
x + x & \rightarrow 0 \\
x + 0 & \rightarrow x \\
x + (y+x) & \rightarrow y
\end{align*}
$$

and the equational theory $AC$:

$$
\begin{align*}
(x+y) + z & \approx x + (y+z) \\
x + y & \approx y + x
\end{align*}
$$

This theory is called ACUN because it consists of associativity, commutativity, unit and nilpotence. This is the theory of the boolean XOR operator. An algorithm for general ACUN unification is provided by Zhiqiang Liu [19] in his Ph.D. dissertation. (See also [11, Section 4].)

Disunification modulo this theory can be solved in polynomial time by what is essentially Gaussian Elimination over $\mathbb{Z}_2$.

Suppose we have $m$ variables $x_1, x_2, \ldots, x_m$, and $n$ constant symbols $c_1, c_2, \ldots, c_n$, and $q$ such equations and disequations to be unified. We can assume an ordering on the variables and constants $x_1 > x_2 > \cdots > x_m > c_1 > c_2 > \cdots > c_n$. We first pick an equation with leading variable $x_1$ and eliminate $x_1$ from all other equations and disequations. We continue this process with the next equation consisting of leading variable $x_2$, followed by an equation containing leading variable $x_3$ and so on, until no more variables can be eliminated. The problem has a solution if and only if (i) there are no equations that contain only constants, such as $c_3 + c_4 \approx c_5$, and (ii) there are no disequations of the form $0 \not\approx 0$. This way we can solve the disunification problem in polynomial time using Gaussian Elimination over $\mathbb{Z}_2$ technique.

Example: Suppose we have two equations $x_1 + x_2 + x_3 + c_1 + c_2 \approx_{R_2AC}^2 0$ and $x_1 + x_3 + c_2 + c_3 \approx_{R_2AC}^2 0$, and a disequation $x_2 \not\approx_{R_2AC}^2 0$.

Eliminating $x_1$ from the second equation, results in the equation $x_2 + c_1 + c_3 \approx_{R_2AC}^2 0$. We can now eliminate $x_2$ from the first equation, resulting in $x_1 + x_3 + c_2 + c_3 \approx_{R_2AC}^2 0$. We can also be eliminated from the disequation $x_2 \not\approx_{R_2AC}^0$, which gives us $c_1 + c_3 \not\approx_{R_2AC}^0$. Thus the procedure terminates with

$$
\begin{align*}
x_1 + x_3 + c_2 + c_3 & \approx_{R_2AC}^0 \\
x_2 + c_1 + c_3 & \approx_{R_2AC}^0 \\
c_1 + c_3 & \not\approx_{R_2AC}^0
\end{align*}
$$
Thus we get
\[
\begin{align*}
  x_2 & \approx_{R_{2,AC}} c_1 + c_3 \\
  x_1 + x_3 & \approx_{R_{2,AC}} c_2 + c_3 
\end{align*}
\]
and the following substitution is clearly a solution:
\[
\{ x_1 \mapsto c_2, x_2 \mapsto c_1 + c_3, x_3 \mapsto c_3 \}
\]

However, asymmetric unification is NP-hard. The proof is by a polynomial-time reduction from the graph 3-colorability problem.

Let \( G = (V,E) \) be a graph where \( V = \{v_1,v_2,v_3,\ldots,v_n\} \) are the vertices, \( E = \{e_1,e_2,e_3,\ldots,e_m\} \) the edges and \( C = \{c_1,c_2,c_3\} \) the color set with \( n \geq 3 \). \( G \) is 3-colorable if none of the adjacent vertices \( \{v_i,v_j\} \in E \) have the same color assigned from \( C \). We construct an instance of asymmetric unification as follows. We create variables for vertices and edges in \( G \): for each vertex \( v_i \) we assign a variable \( y_i \) and for each edge \( e_k \) we assign a variable \( z_k \). Now for every edge \( e_k = \{v_i,v_j\} \) we create an equation \( EQ_k = c_1 + c_2 + c_3 \approx_{AC} y_i + y_j + z_k \). Note that each \( z_k \) appears in only one equation.

Thus for \( E \), the instance of asymmetric unification problem constructed is
\[
S = \{ EQ_1, EQ_2, \ldots, EQ_m \}
\]

If \( G \) is 3-colorable, then there is a color assignment \( \theta : V \to C \) such that \( \theta v_i \neq \theta v_j \) if \( e_k = \{v_i,v_j\} \in E \). This can be converted into an asymmetric unifier \( \alpha \) for \( S \) as follows: We assign the color of \( v_i \), \( \theta(v_i) \) to \( y_i \), \( \theta(v_j) \) to \( y_j \), and the remaining color to \( z_k \). Thus \( \alpha(v_i + v_j + z_k) \approx_{AC} c_1 + c_2 + c_3 \) and therefore \( \alpha \) is an asymmetric unifier of \( S \). Note that the term \( c_1 + c_2 + c_3 \) is clearly in normal form modulo the rewrite relation \( \rightarrow_{R_{2,AC}} \).

Suppose \( S \) has an asymmetric unifier \( \beta \). Note that \( \beta \) cannot map \( y_i \), \( y_j \) or \( z_k \) to 0 or to a term of the form \( u + v \) since \( \beta (y_i + y_j + z_k) \) has to be in normal form or irreducible. Hence for each equation \( EQ_k \), it must be that \( \beta (y_i), \beta (y_j), \beta (z_k) \in \{c_1,c_2,c_3\} \) and \( \beta (y_i) \neq \beta (y_j) \neq \beta (z_k) \). Thus \( \beta \) is a 3-coloring of \( G \).

**Example:** Given \( G = (V,E), V = \{v_1,v_2,v_3,v_4\}, E = \{e_1,e_2,e_3,e_4\}, \) where \( e_1 = \{v_1,v_3\} \), \( e_2 = \{v_1,v_2\} \), \( e_3 = \{v_2,v_3\} \), \( e_4 = \{v_3,v_4\} \) and \( C = \{c_1,c_2,c_3\} \), the constructed instance of asymmetric unification is
\[
\begin{align*}
  EQ_1 &= c_1 + c_2 + c_3 \approx_{AC} y_1 + y_3 + z_1 \\
  EQ_2 &= c_1 + c_2 + c_3 \approx_{AC} y_1 + y_2 + z_2 \\
  EQ_3 &= c_1 + c_2 + c_3 \approx_{AC} y_2 + y_3 + z_3 \\
  EQ_4 &= c_1 + c_2 + c_3 \approx_{AC} y_3 + y_4 + z_4 
\end{align*}
\]
Now suppose the vertices in the graph $G$ are given this color assignment: $\theta = \{v_1 \mapsto c_1, v_2 \mapsto c_2, v_3 \mapsto c_3, v_4 \mapsto c_1\}$. We can create an asymmetric unifier based on this $\theta$ by mapping each $v_i$ to $\theta(v_i)$ and, for each edge $e_j$, mapping $z_j$ to the remaining color from $\{c_1, c_2, c_3\}$ after both its vertices are assigned. For instance, for $e_1 = \{v_1, v_3\}$, since $y_1$ is mapped to $c_1$ and $y_3$ is mapped to $c_2$, we have to map $z_1$ to $c_3$. Similarly for $e_2 = \{v_1, v_2\}$, we map $z_2$ to $c_2$ since $y_1$ is mapped to $c_1$ and $y_2$ is mapped to $c_3$. Thus the asymmetric unifier is

$$\{y_1 \mapsto c_1, y_2 \mapsto c_2, y_3 \mapsto c_2, z_1 \mapsto c_3, z_2 \mapsto c_2, z_3 \mapsto c_1, z_4 \mapsto c_3\}$$

We have not yet looked into whether the problem is in NP, but we expect it to be so.
7 A theory for which ground disunifiability is in P whereas asymmetric unification is NP-hard

This theory is the same as the one mentioned in previous section, ACUN, but with a homomorphism added. It has an AC-convergent term rewriting system, which we call $R_3$:

\[
\begin{align*}
  x + x & \rightarrow 0 \\
  x + 0 & \rightarrow x \\
  x + (y + x) & \rightarrow y \\
  h(x + y) & \rightarrow h(x) + h(y) \\
  h(0) & \rightarrow 0
\end{align*}
\]

7.1 Ground disunification

Ground disunifiability [2] problem refers to checking for ground solutions for a set of disequations and equations. The restriction is that only the set of constants provided in the input, i.e., the equational theory and the equations and disequations, can be used; no new constants can be introduced.

We show that ground disunifiability modulo this theory can be solved in polynomial time, by reducing the problem to that of solving systems of linear equations. This involves finding the Smith Normal Form [13, 17, 16]. This gives us a general solution to all the variables or unknowns.

Suppose we have $m$ equations in our ground disunifiability problem. We can assume without loss of generality that the disequations are of the form $z \neq 0$. For example, if we have disequations of the form $e_1 \neq e_2$, we introduce a new variable $z$ and set $z = e_1 + e_2$ and $z \neq 0$. Let $n$ be the number of variables or unknowns for which we have to find a solution.

For each constant in our ground disunifiability problem, we follow the approach similar to [14], of forming a set of linear equations and solving them to find ground solutions.

We use $h^k x$ to represent the term $h(h(\ldots h(x)\ldots))$ and $H^k = h^{k_1}x + h^{k_2}x + \cdots + h^{k_n}x$ is a polynomial over $\mathbb{Z}_2[h]$.

We have

\[
\begin{align*}
  s_i &= H_{i1} x_1 + H_{i2} x_2 + \ldots + H_{in} x_n, \quad H_{ij} \in \mathbb{Z}_2[h] \\
  t_i &= H'_{i1} c_1 + H'_{i2} c_2 + \ldots + H'_{im} c_l, \quad H'_{ij} \in \mathbb{Z}_2[h]
\end{align*}
\]

where
\{c_1, \ldots, c_l\} \text{ is the set of constants and} \\
\{x_1, \ldots, x_n\} \text{ is the set of variables.}

For each constant \(c_i, 1 \leq i \leq l\), and each variable \(x\), we create a variable \(x^{c_i}\). We then generate, for each constant \(c_i\), a set of linear equations \(S^i\) of the form \(AX = B\) with coefficients from the polynomial ring \(\mathbb{Z}_2[h]\).

The solutions are found by computing the Smith Normal Form of \(A\). We now outline that procedure:\footnote{We follow the notation and procedure similar to Greenwell and Kertzner\cite{13}}

Note that the dimension of matrix \(A\) is \(m \times n\) where \(m\) is the number of equations and \(n\) is the number of unknowns. The dimension of matrix \(B\) is \(m \times 1\). Every matrix \(A\), of rank \(r\), is equivalent to a diagonal matrix \(D\), given by

\[
D = \text{diag}(d_{11}, d_{22}, \ldots, d_{rr}, 0, \ldots, 0)
\]

Each entry \(d_{kk}\) is different from 0 and the entries form a divisibility sequence.

The diagonal matrix \(D\), of size \(m \times n\), is the Smith Normal Form (SNF) of matrix \(A\). There exist invertible matrices \(P\), of size \(m \times m\), and \(Q\), of size \(n \times n\) such that

\[
D = PAQ
\]

and let

\[
D = \text{diag}(d_{11}, d_{22}, \ldots, d_{rr})
\]

be the submatrix consisting of the first \(r\) rows and the first \(r\) columns of \(D\).

Suppose \(AX = B\). We have, from (1),

\[
PAX = PB
\]

Since \(Q\) is invertible we can write

\[
PAQ(Q^{-1}X) = PB
\]

Let \(C = PB\) and

\[
Y = (Q^{-1}X) = \begin{bmatrix} Y \ V \\ Z \end{bmatrix}
\]
with $\mathbf{Y}$ being first $r$ rows of the $n \times 1$ matrix $Y$, and $Z$ the remaining $(n - r)$ rows of $Y$.

\[
C \text{ can be written as } \begin{bmatrix} \mathbf{C} \\ U \end{bmatrix}
\]

with $\mathbf{C}$ the first $r$ rows of $C$, and $U$ a matrix of zeros.

Then $DY = PB = C$ translates into

\[
\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ Z \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ U \end{bmatrix}
\]

We solve for $Y$ in $DY = C$, by first solving $D\mathbf{Y} = \mathbf{C}$:

\[
\begin{bmatrix} d_{11} & \cdots & d_{rr} \\ \vdots & \ddots & \vdots \\ y_1 & \cdots & y_r \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix}
\]

A solution exists if and only if each $d_{ii}$ divides $c_i$. If this is the case let $\widehat{y}_i = c_i/d_{ii}$. Now to find a general solution plug in values of $Y$ in $X = QY$:

\[
\begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \vdots \\ \widehat{y}_r \\ z_{r+1} \\ \vdots \\ z_n \end{bmatrix}
\]

First $r$ columns of $Q$ are referred to as $Q_1$ and remaining $n - r$ columns are referred to as $Q_2$. To find a particular solution, for any $x_j$, we take the dot product of the $j^{th}$ row of $Q_1$ and $(\widehat{y}_1, \ldots, \widehat{y}_r)$.

Similarly, to find a general solution, we take the dot product of the $i^{th}$ row of $Q_1$ with $(\widehat{y}_1, \ldots, \widehat{y}_r)$, plus the dot product of the $i^{th}$ row of $Q_2$, with a vector $(z_{r+1}, \ldots, z_n)$ consisting of distinct variables.

If we have a disequation of the form $x_i \neq 0$, to check for solvability for $x_i$, we first check whether the particular solution is 0. If it is not, then we are done. Otherwise, check whether all the values in $i^{th}$ row of $Q_2$ are identically 0. If it is not, then we have a solution since $z_{r+1}, \ldots, z_n$ can take any arbitrary values. This procedure has to be repeated for all constants.
7.2 Ground Asymmetric Unification

However, asymmetric unification modulo $R_3$ is NP-hard. Decidability can be shown by automata-theoretic methods as for Weak Second Order Theory of One successor (WS1S) \([10,5]\).

In WS1S we consider quantification over finite sets of natural numbers, along with one successor function. All equations or formulas are transformed into finite-state automata which accepts the strings that correspond to a model of the formula \([18,21]\). This automata-based approach is key to showing decidability of WS1S, since the satisfiability of WS1S formulas reduces to the automata intersection-emptiness problem. We follow the same approach here.

For ease of exposition, let us consider the case where there is only one constant $a$. Thus every ground term can be represented as a set of natural numbers. The homomorphism $h$ is treated as a successor function. Just as in WS1S, the input to the automata are column vectors of bits. The length of each column vector is the number of variables in the problem.

$$\Sigma = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

The deterministic finite automata (DFA) are illustrated in Appendix C (p. 24–29). The $+$ operator behaves like the symmetric set difference operator.

Once we have automata constructed for all the formulas, we take the intersection and check if there exists a string accepted by corresponding automata. If the intersection is not empty, then we have a solution or an asymmetric unifier for set of formulas.

This technique can be extended to the case where we have more than one constant. Suppose we have $k$ constants, say $c_1, \ldots, c_k$. We express each variable $X$ in terms of the constants as follows:

$$X = X^{c_1} + \ldots + X^{c_k}$$

effectively grouping subterms that contain each constant under a new variable. Thus if $X = h^2(c_1) + c_1 + h(c_3)$, then $X^{c_1} = h^2(c_1) + c_1$, $X^{c_2} = 0$, and $X^{c_3} = h(c_3)$. If the variables are $X_1, \ldots, X_m$, then we set

$$X_1 = X_1^{c_1} + \ldots + X_1^{c_k}$$
$$X_2 = X_2^{c_1} + \ldots + X_2^{c_k}$$
$$\vdots$$
$$X_m = X_m^{c_1} + \ldots + X_m^{c_k}$$

For example, if $Y$ and $Z$ are set variables and $a, b, c$ are constants, then we can write $Y = Y^a + Y^b + Y^c$ and $Z = Z^a + Z^b + Z^c$ as our terms with constants. For each original variable, say $Z$, we refer to $Z^{c_1}$ etc. as its components for ease of exposition.
If the equation to be solved is \( X = h(Y) \), with \( a, b, c \) as constants, then we create the equations
\[
X^a = h(Y^a), \quad X^b = h(Y^b), \quad X^c = h(Y^c).
\]
However, if the equation is asymmetric, i.e., \( X = \downarrow h(Y) \), then \( Y \) has to be a term of the form \( h^i(d) \) where \( d \) is either \( a, b, \) or \( c \). All components except one have to be 0 and we form the equation \( X^a = \downarrow h(Y^d) \) since \( Y \neq 0 \). The other components for \( X \) and \( Y \) have to be 0.

Similarly, if the equation to be solved is \( X = W + Z \), with \( a, b, c \) as constants, we form the equations
\[
X^a = W^a + Z^a, \quad X^b = W^b + Z^b \quad \text{and} \quad X^c = W^c + Z^c.
\]
If we have an asymmetric equation \( X = \downarrow W + Z \), then clearly one of the components of each original variable has to be non-zero; e.g., in \( W = W^a + W^b + W^c \), all the components cannot be 0 simultaneously. It is ok for \( W^a \) and \( Z^a \) to be 0 simultaneously, provided either one of \( W^b \) or \( W^c \) is non-zero and one of \( Z^b \) or \( Z^c \), is non-zero. For example, \( W = W^b \) and \( Z = Z^c \) is fine, i.e, \( W \) can be equal to its b-component and \( Z \) can be equal to its c-component, respectively, as in the solution \( \{ W \mapsto h^2(b) + h(b), \ Z \mapsto h(c) + c, \ X \mapsto h^2(b) + h(b) + h(c) + c \} \). If \( W^a \) and \( Z^a \) are non-zero, they cannot have anything in common, or otherwise there will be a reduction. In other words, \( X^a, W^a \) and \( Z^a \) must be solutions of the asymmetric equation \( X^a = \downarrow W^a + Z^a \).

Our approach is to design a nondeterministic algorithm. We guess which constant component in each variable has to be 0, i.e., for each variable \( X \) and each constant \( a \), we “flip a coin” as to whether \( X^a \) will be set equal to 0 by the target solution. Now for the case \( X = \downarrow W + Z \), we do the following:

for all constants \( a \) do:
if \( X^a = W^a = Z^a = 0 \) then skip
else if \( W^a = 0 \) then set \( X^a = Z^a \)
if \( Z^a = 0 \) then set \( X^a = W^a \)
if both \( W^a \) and \( Z^a \) are non-zero then set \( X^a = \downarrow W^a + Z^a \)

In the asymmetric case \( X = \downarrow h(Y) \), if more than one of the components of \( Y \) happens to be non-zero, it is clearly an error. (“The guess didn’t work.”). Otherwise, i.e., if exactly one of the components is non-zero, we form the asymmetric equation as described above.

**Nondeterministic Algorithm when we have more than one constant**

1. If there are \( m \) variables and \( k \) constants, then represent each variable in terms of its \( k \) constant components.
2. Guess which constant components have to be 0.
3. Form symmetric and asymmetric equations for each constant.
4. Solve each set of equations by the Deterministic Finite Automata (DFA) construction.

The exact complexity of this problem is open.
8 A theory for which time complexity of Asymmetric Unification varies based on ordering of function symbols

Let $E_4$ be the following equational theory:

\[
\begin{align*}
g(a) & \approx f(a,a,a) \\
g(b) & \approx f(b,b,b)
\end{align*}
\]

Let $R_4$ denote

\[
\begin{align*}
f(a,a,a) & \rightarrow g(a) \\
f(b,b,b) & \rightarrow g(b)
\end{align*}
\]

This is clearly terminating, as can be easily shown by the lexicographic path ordering (lpo) using the symbol ordering $f > g > a > b$. We show that asymmetric unification modulo the rewriting system $R_4$ is NP-complete. The proof is by a polynomial-time reduction from the Not-All-Equal Three-Satisfiability (NAE-3SAT) problem.

Let $U = \{x_1, x_2, \ldots, x_n\}$ be the set of variables, and $C = \{C_1, C_2, \ldots, C_m\}$ be the set of clauses. Each clause $C_k$ has to have at least one true literal and at least one false literal.

We create an instance of asymmetric unification as follows. We represent T by $a$ and F by $b$. For each variable $x_i$ we create the equation

\[
f(x_i, x_i, x_i) \approx_{R_4} g(x_i)
\]

These make sure that each $x_i$ is mapped to either $a$ or $b$. For each clause $C_j = x_p \lor x_q \lor x_r$, we introduce a new variable $z_j$ and create an asymmetric equation $EQ_j$

\[
z_j \approx_{R_4} f(x_p, x_q, x_r)
\]

Thus for any $C$, the instance of asymmetric unification problem constructed is

\[
\mathcal{S} = \left\{ f(x_1, x_1, x_1) \approx g(x_1), \ldots, f(x_n, x_n, x_n) \approx g(x_n) \right\} \cup \left\{ EQ_1, EQ_2, \ldots, EQ_m \right\}
\]

If $\mathcal{S}$ has an asymmetric unifier $\gamma$, then, $x_p, x_q$ and $x_r$ cannot map to all $a$'s or all $b$'s since these will cause a reduction. Hence for $EQ_j$, $\gamma(x_p), \gamma(x_q)$ and $\gamma(x_r)$ should take at least one $a$ and at least one $b$. Thus $\gamma$ is also a solution for NAE-3SAT.

Suppose, all clauses in $C$ have a satisfying assignment. Then $\{x_p, x_q, x_r\}$ cannot all be T or all F, i.e., $\{x_p, x_q, x_r\}$ needs to have at least one true literal and at least one false literal. Thus if $\sigma$ is a satisfying assignment, we can convert $\sigma$ into an asymmetric unifier $\theta$ as follows: $\theta(x_p) := \sigma(x_p)$, the value of $\sigma(x_p)$, $a$ or $b$, is assigned to $\theta(x_p)$. Similarly $\theta(x_q) := \sigma(x_q)$ and $\theta(x_r) := \sigma(x_r)$. 

14
Recall that we also introduce a unique variable $z_j$ for each clause $C_j$ in $C$. Thus if $C_j = \{x_p, x_q, x_r\}$ we can map $z_j$ to $\theta(f(x_p, x_q, x_r))$. Thus $\theta$ is an asymmetric unifier of $S$ and $z_j \approx f(x_p, x_q, x_r)$. Note that $f(x_p, x_q, x_r)$ is clearly in normal form modulo the rewrite relation $R_4$, since $x_p, x_q, x_r$ can’t all be same.

**Example:** Given $U = \{x_1, x_2, x_3, x_4\}$ and $C = \{x_1 \lor x_2 \lor x_3, x_1 \lor x_2 \lor x_4, x_1 \lor x_3 \lor x_4, x_2 \lor x_3 \lor x_4\}$ the constructed instance of asymmetric unification $S$ is

$$
\{ f(x_1, x_1, x_1) \approx g(x_1), f(x_2, x_2, x_2) \approx g(x_2), f(x_3, x_3, x_3) \approx g(x_3), f(x_4, x_4, x_4) \approx g(x_4), \\
    z_1 \approx f(x_1, x_2, x_3), \\
    z_2 \approx f(x_1, x_2, x_4), \\
    z_3 \approx f(x_1, x_3, x_4), \\
    z_4 \approx f(x_2, x_3, x_4) \}
$$

Again, membership in NP can be shown using the fact that $R_4$ is saturated by paramodulation [20].

However, if we orient the rules the other way, i.e., when $g > f > a > b$, we can show that asymmetric unifiability modulo this theory can be solved in polynomial time, i.e., when the term rewriting system is

$$
g(a) \rightarrow f(a, a, a) \\
g(b) \rightarrow f(b, b, b)
$$

Let $R_5$ denote the above term rewriting system. The algorithm is outlined in Appendix B (p. 22-23).
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A Asymmetric Unifiability modulo $R_1$

Recall that the term rewriting system $R_1$ is

\[
\begin{align*}
    h(a) & \rightarrow f(a, c) \\
    h(b) & \rightarrow f(b, c)
\end{align*}
\]

Note that reversing the directions of the rules also produces a convergent system, i.e.,

\[
\begin{align*}
    f(a, c) & \rightarrow h(a) \\
    f(b, c) & \rightarrow h(b)
\end{align*}
\]

is also terminating and confluent. We assume that the input equations are in standard form, i.e., of one of four kinds: $X \approx^2 Y$, $X \approx^2 h(Y)$, $X \approx^2 f(Y, Z)$ and $X \approx^2 d$ where $X, Y, Z$ are variables and $d$ is any constant. Asymmetric equations will have the extra downarrow, e.g., $X \approx^1 h(Z)$.

Our algorithm transforms an asymmetric unification problem to a set of equations in *dag-solved form* along with clausal constraints, where each atom is of the form $\langle$variable$\rangle = \langle$constant$\rangle$). We use the notation $EQ \parallel \Gamma$, where $EQ$ is set of equations in standard form as mentioned above, and $\Gamma$ is a set of clausal constraints. Initially $\Gamma$ is empty.

**Lemma A.1.** (Removing asymmetry) If $s$ is an irreducible term, then $h(s)$ is (also) irreducible iff $s \neq a$ and $s \neq b$.

*Proof.* If $s = a$ or $s = b$, then clearly $h(s)$ is reducible. Conversely, if $s$ is irreducible and $h(s)$ is reducible, then $s$ has to be either $a$ (for the first rule to apply) or $b$ (for the second rule). \[\square\]

Hence we first apply the following inference rule (until finished) that gets rid of asymmetry:

\[
\frac{\langle X \approx^2 h(Y) \rangle \mid \Gamma}{\langle X \approx^2 h(Y) \rangle \mid \Gamma \cup \{\neg(Y = a)\} \cup \{\neg(Y = b)\}}
\]

**Lemma A.2.** (Cancellativity) $h(s) \downarrow_{R_1} h(t)$ iff $s \downarrow_{R_1} t$. Similarly, $f(s_1, s_2) \downarrow_{R_1} f(t_1, t_2)$ iff $s_1 \downarrow_{R_1} t_1$ and $s_2 \downarrow_{R_1} t_2$.

*Proof.* The if part is straightforward. If $s$ and $t$ are joinable, this implies $h(s)$ and $h(t)$ are joinable modulo $R_1$. \[18\]
Only if part: Suppose \( h(s) \) is joinable with \( h(t) \). Without loss of generality assume \( s \) and \( t \) are in normal form. If \( s = t \) then we are done. Otherwise, if \( s \neq t \), since we assumed \( s \) and \( t \) are in normal forms, \( h(s) \) or \( h(t) \) must be reducible. If \( h(s) \) is reducible, then \( s \) has to be either \( a \) or \( b \), which reduces \( h(s) \) to \( f(s, c) \). Then \( h(t) \) must also be reducible and joinable with \( f(s, c) \). Hence \( s \) and \( t \) will be equivalent.

The proof of the second part is straightforward.

**Lemma A.3.** (Root Conflict) \( h(s) \downarrow_{R_1} f(t_1, t_2) \) iff either

\[
\begin{align*}
  s \rightarrow^1 a, t_1 \rightarrow^1 a, t_2 \rightarrow^1 c & \\
  \text{or} \\
  s \rightarrow^1 b, t_1 \rightarrow^1 b, t_2 \rightarrow^1 c.
\end{align*}
\]

**Proof.** The if part is straightforward. If \( s \) and \( t_1 \) reduce to \( a \) (resp., \( b \)) and \( t_2 \) reduces to \( c \), then \( h(a) \) reduces to \( f(a, c) \) (resp., \( f(b, c) \)).

Only if part: Suppose \( h(s) \) is joinable with \( f(t_1, t_2) \) modulo \( R_1 \). We can assume wlog that \( s, t_1, t_2 \) are in normal forms. Then \( h(s) \) must be reducible, i.e., \( s = a \) or \( s = b \). If \( s = a \), then \( t_1 = a \) and \( t_2 = c \); else if \( s = b \), then \( t_1 = b \) and \( t_2 = c \) (from our rules).

Now for \( E \)-unification, we have the inference rules

(a) \[
\frac{\{X \approx^2 V\} \uplus E \parallel \Gamma}{\{X \approx^2 V\} \cup [V/X](E \parallel \Gamma)} \quad \text{if } X \text{ occurs in } E \parallel \Gamma
\]

(b) \[
\frac{E \parallel \{X \approx^2 h(Y), X \approx^2 h(T)\} \parallel \Gamma}{E \parallel \{X \approx^2 h(Y), T \approx^2 Y\} \parallel \Gamma}
\]

(c) \[
\frac{E \parallel \{X \approx^2 f(V, Y), X \approx^2 f(W, T)\} \parallel \Gamma}{E \parallel \{X \approx^2 f(V, Y), W \approx^2 V, T \approx^2 Y\} \parallel \Gamma}
\]

(d) \[
\frac{E \parallel \{X \approx^2 h(Y), X \approx^2 f(U, V)\} \parallel \Gamma}{E \parallel \{U \approx^2 Y, V \approx^2 c, X \approx^2 f(Y, V)\} \parallel \Gamma \cup \{(Y = a) \lor (Y = b)\}}
\]
The above inference rules are applied with rule (a) having the highest priority and rule (d) the lowest.

The following are the failure rules, which, of course, have the highest priority.

\[(F1) \quad \frac{\mathcal{E} \cup \{X \approx^2 d, X \approx^2 f(U, V)\} \parallel \Gamma}{\text{FAIL}} \quad d \in \{a, b, c\}\]

\[(F2) \quad \frac{\mathcal{E} \cup \{X \approx^2 d, X \approx^2 h(V)\} \parallel \Gamma}{\text{FAIL}} \quad d \in \{a, b, c\}\]

\[(F3) \quad \frac{\mathcal{E} \cup \{X \approx^2 c, X \approx^2 d\} \parallel \Gamma}{\text{FAIL}} \quad d \in \{a, b\}\]

\[(F4) \quad \frac{\mathcal{E} \cup \{X \approx^2 b, X \approx^2 a\} \parallel \Gamma}{\text{FAIL}}\]

**Lemma A.4.** \(R_1\) is non-subterm-collapsing, i.e., no term is equivalent to a proper subterm of it.

**Proof.** Since the rules in \(R_1\) are size increasing, no term can be reduced to a proper subterm of it. \(\square\)

Because of the above lemma, we can have an extended occur-check or cycle check \([15]\) as another failure rule.

\[(F5) \quad \frac{\{X_0 \approx^2 s_1[X_1], \ldots, X_n \approx^2 s_n[X_0]\} \cup \mathcal{E} \parallel \Gamma}{\text{FAIL}}\]

where the \(X_i\)'s are variables and \(s_j\)'s are non-variable terms.

Once these inference rules have been exhaustively applied, we are left with a set of equations in *dag-solved form* along with clausal constraints. Thus the set of equations is of the form

\[\left\{ X_1 ≈^2 t_1, \ldots, X_m ≈^2 t_m \right\}\]

where the variables on the left-hand sides are all distinct (i.e., \(X_i \neq X_j\) for \(i \neq j\)). The clausal constraints are either negative unit clauses of the form \(-(Y = a)\) or \(-(Y = b)\) or positive two-literal clauses of the form \((Y = a) \lor (Y = b)\). The solvability of such a system of equations and clauses can be checked in polynomial time.
Steps for polynomial time solvability of equations and clauses:

1. Add to the list of clauses $\Gamma$ more clauses derived from the solved form, to generate $\Gamma'$. For example if we have an equation of the form $X \approx h(Y)$, then $X \neq a$ and $X \neq b$ will be added to $\Gamma'$.

2. Check for satisfiability of $\Gamma'$ by unit resolution with the negative clauses.

Soundness of this algorithm follows from the lemmas A.1 through A.4.

As for termination, we first observe that none of the inference rules introduce a new variable, i.e., the number of variables never increases. With the first inference rule which removes asymmetry, asymmetric equations are eliminated from $E_2$, i.e., the number of asymmetric equations goes down. For the $E$-unification rules, we can see that in each case either the overall size of equations decreases or some function symbols are lost. In rule (a), we replace $X$ by $V$ and are left with an isolated $X$, hence the number of unsolved variables go down. In rules (b) and (d) the number of occurrences of $h$ goes down and in rule (c) the number of occurrences of $f$ goes down.
B  Asymmetric Unifiability modulo \( R_5 \)

Recall that the term rewriting system \( R_5 \) is

\[
\begin{align*}
g(a) & \rightarrow f(a,a,a) \\
g(b) & \rightarrow f(b,b,b)
\end{align*}
\]

We assume that the input equations are in standard form, i.e., of one of four kinds: \( X \approx^? Y \), \( X \approx^? g(Y) \), \( X \approx^? f(U,V,W) \) and \( X \approx^? d \) where \( X,Y,U,V,W \) are variables and \( d \) is any constant. Asymmetric equations will have the extra downarrow, e.g., \( X \approx^? \downarrow g(Y) \).

As in Appendix A, our algorithm transforms an asymmetric unification problem to a set of equations in dag-solved form along with clausal constraints, where each atom is of the form \( (\text{variable} = \text{constant}) \). We use the notation \( EQ \parallel \Gamma \), where \( EQ \) is set of equations in standard form as mentioned above, and \( \Gamma \) is a set of clausal constraints. Initially \( \Gamma \) is empty.

We first apply the following inference rule (until finished) that gets rid of asymmetry:

\[
\frac{\mathcal{E} \cup \{X \approx^? g(Y)\} \parallel \Gamma}{\mathcal{E} \cup \{X \approx^? g(Y)\} \parallel \Gamma \cup \{\neg(Y = a)\} \cup \{\neg(Y = b)\}}
\]

Now for \( E \)-unification, we have the inference rules

(a) \[
\frac{\{X \approx^? V\} \cup \mathcal{E} \parallel \Gamma}{\{X \approx^? V\} \cup [V/X](\mathcal{E}) \parallel [V/X](\Gamma)} \quad \text{if } X \text{ occurs in } \mathcal{E}
\]

(b) \[
\frac{\mathcal{E} \cup \{X \approx^? g(Y), X \approx^? g(T)\} \parallel \Gamma}{\mathcal{E} \cup \{X \approx^? g(Y), T \approx^? Y\} \parallel \Gamma}
\]

(c) \[
\frac{\mathcal{E} \cup \{X \approx^? f(U_1,V_1,W_1), X \approx^? f(U_2,V_2,W_2)\} \parallel \Gamma}{\mathcal{E} \cup \{X \approx^? f(U_1,V_1,W_1), U_1 \approx^? U_2, V_1 \approx^? V_2, W_1 \approx^? W_2\} \parallel \Gamma}
\]

(d) \[
\frac{\mathcal{E} \cup \{X \approx^? g(Y), X \approx^? f(U,V,W)\} \parallel \Gamma}{\mathcal{E} \cup \{U \approx^? Y, V \approx^? Y, W \approx^? Y, X \approx^? f(Y,Y,Y)\} \parallel \Gamma \cup \{(Y = a) \vee (Y = b)\}}
\]
The inference rules are applied in the descending order of priority from (a), the highest, to (d) the lowest. Occurrence of equations of the form $X \approx^? a$ and $X \approx^? f(U, V, W)$ will make the equations unsolvable. Hence we have failure rules as in Appendix A. Since the equational theory is non-subterm-collapsing, we have an extended occur-check or cycle check rule here as well:

$\text{(Cycle-check)} \quad \{X_0 \approx^? s_1[X_1], \ldots, X_n \approx^? s_n[X_0]\} \uplus \mathcal{E} \parallel \Gamma \quad \text{FAIL}$

where the $X_i$'s are variables and $s_j$'s are non-variable terms.

After exhaustively applying these inference rules we are left with a set of equations in dag-solved form along with clausal constraints. Recall that the clausal constraints are either unit clauses of the form $\neg(W = a)$ or $\neg(W = b)$ or positive two-literal clauses of the form $(W = a) \lor (W = b)$. The solvability of such a system of equations and clauses can be checked in polynomial time as described in Appendix A.

Similarly, soundness and termination can be shown as is done in Appendix A.
C Automata Constructions

We illustrate how automata are constructed for each equation in standard form. In order to avoid cluttering up the diagrams the dead state has been included only for the first automaton. The missing transitions lead to the dead state by default for the others. Recall that we are considering the case of one constant $a$. The homomorphism $h$ is treated as successor function.

C.1 $P = Q + R$

\[
\begin{pmatrix}
0 \\ 0 \\
0 \\ 1
\end{pmatrix} , \begin{pmatrix}
0 \\ 1 \\
1 \\ 1
\end{pmatrix} , \begin{pmatrix}
1 \\ 0 \\
0 \\ 0
\end{pmatrix} , \begin{pmatrix}
1 \\ 1 \\
1 \\ 1
\end{pmatrix}
\]

Let $P_i$, $Q_i$ and $R_i$ denote the $i^{th}$ bits of $P$, $Q$ and $R$ respectively. $P_i$ has a value 1, when either $Q_i$ or $R_i$ has a value 1. We need 3-bit alphabet symbols for this equation. For example, if $R_2 = 0$, $Q_2 = 1$, then $P_2 = 1$. The corresponding alphabet symbol is $\begin{pmatrix} P_2 \\ Q_2 \\ R_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Hence, only strings with the alphabet symbols \{ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ \} are accepted by this automaton. Rest of the input symbols like \{ $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ \} go to the dead state $D$ as they violate the XOR property.

Note that the string $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is accepted by automaton. This corresponds to $P = a + h(a)$, $Q = h(a)$ and $R = a$.  

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To preserve asymmetry on the right-hand side of this equation, $Q + R$ should be irreducible. If either $Q$ or $R$ is empty, or if they have any term in common, then a reduction will occur. For example, if $Q = h(a)$ and $R = h(a) + a$, there is a reduction, whereas if $R = h(a)$ and $Q = a$, irreducibility is preserved, since there is no common term and neither one is empty. Since neither $Q$ nor $R$ can be empty, any accepted string should have one occurrence of $\left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right)$ and one occurrence of $\left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right)$.
C.3 \( X = h(Y) \)

We need 2-bit vectors as alphabet symbols since we have two unknowns \( X \) and \( Y \). Note again that \( h \) acts like the successor function. \( q_0 \) is the only accepting state. A state transition occurs with bit vectors \( (1,0) \), \( (0,1) \). If \( Y=1 \) in current state, then \( X=1 \) in the next state, hence a transition occurs from \( q_0 \) to \( q_1 \), and vice versa. The ordering of variables is \( (Y, X) \).

C.4 \( X = \downarrow h(Y) \)

In this equation, \( h(Y) \) should be in normal form. So \( Y \) cannot be either 0 or of the form \( u + v \). Thus \( Y \) has to be a string of the form \( 0^i 10^j \) and \( X \) then has to be \( 0^{i+1} 10^{j-1} \). Therefore the bit vector \( (1,0) \) has to be succeeded by \( (0,1) \).
C.5 An Example

Let \( \{ U = V + Y, W = h(V), Y = h(W) \} \) be an asymmetric unification problem. We need 4-bit vectors and 3 automata since we have 4 unknowns in 3 equations, with bit-vectors represented in this ordering of set variables: \( \begin{bmatrix} V \\ W \\ Y \\ U \end{bmatrix} \).

\[
Y = h(W)
\]

\[
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
0 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
1 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
\]
\[ U = V + Y \]

We include the \( \times \) ("don’t-care") symbol in state transitions to indicate that the values can be either 0 or 1. This is essentially to avoid cluttering the diagrams. Note that here this \( \times \) symbol is a placeholder for the variable \( W \) which does not have any significance in this automaton.
\[ W = h(V) \]

\[
\begin{pmatrix}
0 \\ 0 \\
\times
\end{pmatrix}
\quad \begin{pmatrix}
1 \\ 0 \\
\times
\end{pmatrix}
\quad \begin{pmatrix}
1 \\ \times
\end{pmatrix}
\]

\[ q_0 \quad \rightarrow \quad q_1 \]

**NOTE:** As before, the symbol \( \times \) in the vectors means that the bit value can be either 0 or 1.

The string \[
\begin{pmatrix}
1 \\ 0 \\ 0 \\ 1 \\
1 \\ 0 \\ 1 \\ 0 \\
0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
\]

is accepted by all the three automata. The corresponding asymmetric unifier is

\[
\{ V \mapsto a, W \mapsto h(a), Y \mapsto h^2(a), U \mapsto (h^2(a) + a) \}.
\]