Nice Banach Modules and Invariant Subspaces

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Abstract

Let $A$ be a semisimple unital commutative Banach algebra. We say that a Banach $A$-module $M$ is nice if every proper closed submodule of $M$ is contained in a closed submodule of $M$ of codimension 1. We provide examples of nice and non-nice modules.

MSC: 46J10, 47A15
Keywords: Banach algebras, Banach modules, Invariant subspaces

1 Introduction

In this article, all vector spaces are assumed to be over the field $\mathbb{C}$ of complex numbers. As usual, $\mathbb{R}$ is the field of real numbers, $\mathbb{N}$ is the set of all positive integers, $\mathbb{Z}$ is the set of integers and $\mathbb{Z}_+$ is the set of non-negative integers. For a Banach space $X$, $L(X)$ stands for the algebra of bounded linear operators on $X$, while $X^*$ is the space of continuous linear functionals on $X$. For $T \in L(X)$, its dual is denoted $T^*: T^*f(x) = f(Tx)$ for every $f \in X^*$ and every $x \in X$.

Throughout this article $A$ stands for a unital commutative semisimple Banach algebra. It is well-known and is a straightforward application of the Gelfand theory [2, 1] that for an ideal $J$ in $A$, $J = A \iff J$ is dense in $A \iff \kappa|_J \neq 0$ for every $\kappa \in \Omega(A)$, where $\Omega(A)$ is the spectrum of $A$, that is, $\Omega(A)$ is the set of all (automatically continuous) non-zero algebra homomorphisms from $A$ to $\mathbb{C}$ (endowed with the $*$-weak topology). Equivalently, every proper ideal in $A$ is contained in a closed ideal of codimension 1.

Let $\Omega^+(A)$ be the set of all algebra homomorphisms from $A$ to $\mathbb{C}$. That is, $\Omega^+(A) = \Omega(A)$ together with the identically zero map from $A$ to $\mathbb{C}$. The main purpose of this paper is to draw attention to possible extensions of the above fact to Banach $A$-modules. Clearly, each $\kappa \in \Omega^+(A)$ gives rise to the 1-dimensional $A$-module $\mathbb{C}_\kappa$ being $\mathbb{C}$ with the $A$-module structure given by the multiplication $a\lambda = \kappa(a)\lambda$ for every $a \in A$ and $\lambda \in \mathbb{C}$. It is also rather obvious that we have just listed all the 1-dimensional $A$-modules up to an isomorphism.

Definition 1.1. Let $M$ be a Banach $A$-module. A character on $M$ is a non-zero $\varphi \in M^*$ such that there exists $\kappa \in \Omega^+(A)$ making $\varphi$ into an $A$-module morphism from $M$ to $\mathbb{C}_\kappa$.

Obviously, the kernel of a character on a Banach $A$-module $M$ is a closed $A$-submodule of $M$.

Definition 1.2. Let $M$ be a Banach $A$-module. We say that $M$ is nice if for every proper closed submodule of $M$ is contained in a closed submodule of codimension 1. Equivalently, $M$ is nice if and only if for every proper closed submodule $N$ of $M$, there is a character $\varphi$ on $M$ such that $\varphi$ vanishes on $N$.

The general question we would like to raise is:

Question 1.3. Characterize nice Banach $A$-modules.

The remark we started with ensures that $A$ is nice as an $A$-module. In this paper we just present examples of nice and non-nice modules. Before even formulating the results, I would like to put forth my personal motivation for even looking at this question. Assume for a minute that $A$ is a subalgebra of $L(X)$ for some Banach space $X$. We allow the norm topology of $A$ to be stronger (not necessarily strictly) than
the topology defined by the norm inherited from \( L(X) \). The multiplication \((A, x) \mapsto Ax\) defines a Banach \( \mathfrak{A}\)-module structure on \( X \). What are the characters on \( X \)? Why, one easily sees that they are exactly the common eigenvectors of \( A^* \) for \( A \in \mathfrak{A} \). What are the \( \mathfrak{A}\)-submodules of \( X \)? They are exactly the invariant subspaces for the action of \( \mathfrak{A} \) on \( X \). Thus the \( \mathfrak{A}\)-module \( X \) is nice exactly when every non-trivial closed \( \mathfrak{A}\)-invariant subspace of \( X \) is contained in a closed \( \mathfrak{A}\)-invariant hyperplane. Thus \( X \) being a nice \( \mathfrak{A}\)-module translates into a strong and important property of the lattice of \( \mathfrak{A}\)-invariant subspaces. Note that under relatively mild extra assumptions on \( \mathfrak{A} \), the nicety of \( X \) results in every closed \( \mathfrak{A}\)-invariant subspace being the intersection of a collection of characters on \( X \) thus providing a complete description of the lattice of \( \mathfrak{A} \). A byproduct of this observation is the following easy example of a non-nice module.

**Example 1.4.** Let \( \Omega \) be a non-empty compact subset of \( \mathbb{C} \) with no isolated points and \( \mu \) be a finite \( \sigma \)-additive purely non-atomic Borel measure on \( \mathbb{C} \), whose support is exactly \( \Omega \). The pointwise multiplication equips \( L^2(\Omega, \mu) \) with the structure of a Banach \( C(\Omega) \)-module. This module is non-nice.

**Proof.** The \( C(\Omega) \)-module \( L^2(\Omega, \mu) \) does have plenty of closed submodules. For instance, every Borel subset \( A \) of \( \Omega \) satisfying \( \mu(A) \neq 0 \) and \( \mu(\Omega \setminus A) \neq 0 \) generates a closed non-trivial submodule \( M_A = \{ f \in L^2(\Omega, \mu) : f \text{ vanishes outside } A \} \). On the other hand, we can always pick \( f \in C(\Omega) \) satisfying \( \mu(f^{-1}(\lambda)) = 0 \) for every \( \lambda \in \mathbb{C} \). In this case the dual of the multiplication by \( f \) operator on \( L^2(\Omega, \mu) \) has empty point spectrum. Due to the above remark, our module possesses no characters at all (while possessing non-trivial closed submodules) and therefore can not possibly be nice.

In the positive direction we have the following two rather easy statements.

**Proposition 1.5.** The finitely generated free \( \mathfrak{A}\)-module \( \mathfrak{A}^n \) is nice.

**Proposition 1.6.** Let \( \Omega \) be a Hausdorff compact topological space and \( X \) be a Banach space. Then the \( C(\Omega) \)-module \( C(\Omega, X) \) is nice, where \( C(\Omega, X) \) carries the natural norm \( \| f \| = \sup \{ \| f(\omega) \|_X : \omega \in \Omega \} \) and the module structure is given by the pointwise multiplication.

Note that Example 1.4 is rather cheatish since the non-nicety comes from the lack of characters. A really interesting situation is when a non-nice module possesses a separating set of characters. The following result says that this is quite possible. Recall that the Sobolev space \( W^{1,2}[0,1] \) consists of the functions \( f : [0, 1] \to \mathbb{C} \) absolutely continuous on any bounded subinterval of \( I \) and such that \( f' \in L^2[0,1] \). The space \( W^{1,2}[0,1] \) with the inner product

\[
\langle f, g \rangle_{1,2} = \int_0^1 (f(t)\overline{g(t)} + f'(t)\overline{g'(t)}) \, dt
\]

is a separable Hilbert space. We denote \( \| f \|_{1,2} = \sqrt{\langle f, f \rangle_{1,2}} \). Apart from being a Hilbert space, \( W^{1,2}[0,1] \) is also a Banach algebra with respect to the pointwise multiplication (if one strives for the submultiplicativity of the norm together with the identity \( \| 1 \| = 1 \), he or she has to pass to an equivalent norm).

We say that a function \( f \) defined on \([0, 1]\) and taking values in a Banach space \( X \) is absolutely continuous if there exists an (automatically unique up to a Lebesgue-null set) Borel measurable function \( g : [0, 1] \to X \) such that

\[
\int_0^1 \| g(t) \| \, dt < +\infty \quad \text{and} \quad \int_0^x g(t) \, dt = f(x) \quad \text{for each } x \in [0, 1],
\]

where the second integral is considered in the Bochner sense. We denote the function \( g \) as \( f' \). If \( H \) is a Hilbert space. The symbol \( W^{1,2}([0,1], H) \) stands for the space of absolutely continuous functions \( f : [0, 1] \to H \) such that

\[
\int_0^1 \| f'(t) \|^2 \, dt < +\infty.
\]

The space \( W^{1,2}([0,1], H) \) with the inner product

\[
\langle f, g \rangle = \int_0^1 (\langle f(t), g(t) \rangle_H + \langle f'(t), g'(t) \rangle_H) \, dt
\]
is a Hilbert space and is separable if $H$ is separable. In any case if $\{e_\alpha\}_{\alpha \in A}$ is an orthonormal basis of $H$, then the space $W^{1,2}(0,1,H)$ is naturally identified with the Hilbert direct sum of $|A|$ copies of $W^{1,2}(0,1)$: $f \mapsto \{f_\alpha\}_{\alpha \in A}$, where $f_\alpha(t) = (f(t), e_\alpha)_H$. It is also clear that $W^{1,2}(0,1,H)$ is naturally isomorphic to the Hilbert space tensor product of $W^{1,2}(0,1]$ and $H$. Clearly, $W^{1,2}(0,1,H)$ is a Banach $W^{1,2}(0,1]$-module. This module possesses a lot of characters. Indeed, if $t \in [0,1]$ and $x \in H$, then the functional $f \mapsto \langle f(t), x \rangle_H$ is a character on $W^{1,2}(0,1,H)$. Moreover, these characters do separate points of $W^{1,2}(0,1,H)$.

**Theorem 1.7.** Let $H$ be a Hilbert space. Then the $W^{1,2}(0,1]$-module $W^{1,2}(0,1,H)$ is nice if and only if $H$ is finite dimensional.

## 2 Proof of Proposition 1.6

It is easy to see that a character on $C(\Omega, X)$ is exactly a functional of the form

$$\kappa_{\omega, \varphi}(f) = \varphi(f(\omega)), \quad \text{where } \omega \in \Omega \text{ and } \varphi \in X^* \setminus \{0\}. \quad (2.1)$$

The following lemma describes all closed submodules of $C(\Omega, X)$.

**Lemma 2.1.** Let $M$ be a $C(\Omega)$-submodule of $C(\Omega, X)$ and for each $\omega \in \Omega$ let $M_\omega = \{f(\omega) : f \in M\}$. Then the closure $\overline{M}$ of $M$ in $C(\Omega, X)$ satisfies

$$\overline{M} = \overline{M}, \quad \text{where } \overline{M} = \{f \in C(\Omega, X) : f(\omega) \in \overline{M}_\omega \text{ for each } \omega \in \Omega\}, \quad (2.2)$$

with $\overline{M}_\omega$ being the closure in $X$ of $M_\omega$.

**Proof.** Since $M \subseteq \overline{M}$ and $\overline{M}$ is closed, we have $\overline{M} \subseteq \overline{M}$. Let $f \in \overline{M}$ and $\varepsilon > 0$. The desired equality will be verified if we show that there is $g \in M$ such that $\|f - g\| < \varepsilon$. Indeed, in this case $\overline{M} \subseteq \overline{M}$ and therefore $\overline{M} = \overline{M}$.

Take $\omega \in \Omega$. Since $M_\omega$ is dense in $\overline{M}_\omega$, there is $g_\omega \in M$ such that $\|f(\omega) - g_\omega(\omega)\|_X < \varepsilon$. Then $V_\omega = \{s \in \Omega : \|f(s) - g_\omega(s)\|_X < \varepsilon\}$ is an open subset of $\Omega$ containing $\omega$. Thus $\{V_\omega\}_{\omega \in \Omega}$ is an open covering of $\Omega$. Since for every open covering of a Hausdorff compact topological space, there is a finite partition of unity consisting of continuous functions and subordinate to the covering, there are $\omega_1, \ldots, \omega_n \in \Omega$ and $\rho_1, \ldots, \rho_n \in C(\Omega)$ such that

$$0 \leq \rho_j(s) \leq 1 \quad \text{for every } 1 \leq j \leq n \text{ and } s \in \Omega;$$

$$\rho_j(s) = 0 \quad \text{whenever } 1 \leq j \leq n \text{ and } s \in \Omega \setminus V_{\omega_j};$$

$$\rho_1(s) + \ldots + \rho_n(s) = 1 \quad \text{for each } s \in \Omega. \quad (2.3)$$

Now we set $g = \rho_1 g_{\omega_1} + \ldots + \rho_n g_{\omega_n}$. Since $M$ is a $C(\Omega)$-module and $g_\omega \in M$, we have $g \in M$. Using (2.3) together with the inequality $\|f(s) - g_\omega(s)\|_X < \varepsilon$ for $s \in V_{\omega_j}$, we easily see that $\|f(s) - g(s)\|_X < \varepsilon$ for each $s \in \Omega$. Hence $g \in M$ and $\|f - g\| < \varepsilon$, which completes the proof.

**We are ready to prove Proposition 1.6.** Let $M$ be a closed submodule of $C(\Omega, X)$ such that none of the characters on $C(\Omega, X)$ vanishes on $M$. According to (2.1), the latter means that every $M_\omega = \{f(\omega) : f \in M\}$ is dense in $X$ and therefore $\overline{M}_\omega = X$ for each $\omega \in \Omega$. Since $M$ is closed, Lemma 2.1 says that $M = C(\Omega, X)$. The proof is complete.

## 3 Proof of Propositions 1.5

We start with the following easy observation. Let $\kappa \in \Omega(\mathbb{A})$. Then the $\mathbb{A}$-module morphisms $\psi : \mathbb{A}^n \to \mathbb{C}_\kappa$ are all given by

$$\varphi_c(a_1, \ldots, a_n) = \sum_{j=1}^n c_j \kappa(a_j), \quad \text{where } c \in \mathbb{C}^n.$$
**Proposition 3.1.** Let \( n \in \mathbb{N} \) and \( M \) be an \( \mathbb{A} \)-submodule of the free \( \mathbb{A} \)-module \( \mathbb{A}^n \). Assume also that none of the characters on \( \mathbb{A}^n \) vanishes on \( M \). Then \( M = \mathbb{A}^n \).

**Proof.** We use induction with respect to \( n \). The case \( n = 1 \) is trivial (see the remark at the very start of the article). Assume now that \( n > 2 \) and that the conclusion of Proposition 1 holds for every smaller \( n \). We interpret \( \mathbb{A}^n \) as \( \mathbb{A}^n = \mathbb{A} \times \mathbb{A}^{n-1} \). The induction hypothesis easily implies that that the projection of \( M \) onto \( \mathbb{A}^{n-1} \) is onto. Let \( J \subseteq \mathbb{A} \) be defined by \( M \cap (\mathbb{A} \times \{0\}) = J \times \{0\} \). Then \( J \) is an ideal in \( \mathbb{A} \). If \( J = \mathbb{A} \), we can factor out the first component in the product \( \mathbb{A} \times \mathbb{A}^{n-1} = \mathbb{A}^n \) and then use the induction hypothesis to conclude that \( M = \mathbb{A}^n \). Thus it remains to consider the case \( J \neq \mathbb{A} \). Then there is \( \in \Omega(\mathbb{A}) \) such that \( J \subseteq \ker \in \mathbb{A} \). Using the definition of \( J \), and the facts that \( M \) is an \( \mathbb{A} \)-module, \( M \) projects onto the entire \( \mathbb{A}^{n-1} \) and \( \in \mathbb{A} \) vanishes on \( J \), we can define \( \psi : \mathbb{A}^{n-1} \rightarrow \mathbb{C} \) by the rule \( \psi(b) = \in(a) \) if \( (a, b) \in M \subseteq \mathbb{A} \times \mathbb{A}^{n-1} \). It is easy to see that \( \psi \) is a well-defined continuous linear functional and that \( \psi : \mathbb{A}^{n-1} \rightarrow \mathbb{C}_\in \) is an \( \mathbb{A} \)-module morphism. According to the above display there are \( \mathbb{A}^n \) such that \( \psi(a_1, \ldots, a_{n-1}) = \sum_{j=1}^{n-1} c_j \in(a_j) \) for every \( a_1, \ldots, a_{n-1} \in \mathbb{A} \). By definition of \( \psi \), we now see that \( \psi \) vanishes on \( M \), where \( \psi \) is defined by the formula \( \psi(a_1, \ldots, a_n) = \sum_{j=1}^n c_j \in(a_j) \) with \( c_n = -1 \).

By the above display, \( \psi : \mathbb{A}^n \rightarrow \mathbb{C}_\in \) is an \( \mathbb{A} \)-module morphism. Since \( c_n \neq 0 \), \( \psi \neq 0 \) and therefore \( \psi \) is a character on \( \mathbb{A}^n \). We have produced a character on \( \mathbb{A}^n \) vanishing on \( M \), which contradicts the assumptions. Thus the case \( J \neq \mathbb{A} \) does not occur, which completes the proof. \( \square \)

4 Proof of Theorem 1.7

In this section, for a function \( f \) on an interval \( I \) of the real line \( \|f\|_2 \) will always denote the \( L^2 \)-norm of \( f \) (with respect to the Lebesgue measure), while \( \|f\|_\infty \) always stands for the \( L^\infty \)-norm of \( f \).

**Lemma 4.1.** Let \( -\infty < \alpha < \beta < +\infty \), \( a, b \in \mathbb{C} \) and \( \varepsilon > 0 \). Then there exists \( f \in C^1[\alpha, \beta] \) such that \( f(\alpha) = f(\beta) = 0 \), \( f'(\alpha) = a \), \( f'(\beta) = b \) and \( \|f\|_\infty < \varepsilon \).

**Proof.** Let \( \varphi \in C^1[0, \infty) \) be a monotonically non-increasing function such that \( \varphi(0) = 1 \), \( \varphi'(0) = 0 \) and \( \varphi(x) = 0 \) for \( x \geq 1 \). For any \( \delta \in (0, \frac{\beta-\alpha}{2}) \) let

\[
f_\delta(x) = \begin{cases} 
0 & \text{if } x \in (\alpha + \delta, \beta - \delta), \\
(a(x - \alpha)) \varphi((x - \alpha)/\delta) & \text{if } x \in [\alpha, \alpha + \delta), \\
b(x - \beta) \varphi((\beta - x)/\delta) & \text{if } x \in (\beta - \delta, \beta].
\end{cases}
\]

Obviously, \( f_\delta \in C^1[\alpha, \beta] \), \( f_\delta'(\alpha) = f_\delta(\beta) = 0 \), \( f_\delta'(\alpha) = a \), \( f_\delta'(\beta) = b \) and \( \|f\|_\infty \leq \delta \max\{|a|, |b|\} \). Hence the function \( f = f_\delta \) for \( \delta < \varepsilon / \max\{|a|, |b|\} \) satisfies all desired conditions. \( \square \)

**Lemma 4.2.** Let \( K \subset [0, 1] \) be a nowhere dense compact set, \( a \in C(K) \), \( f \in C[0, 1] \) and \( \varepsilon > 0 \). Then there exists \( g \in C^1[0, 1] \) such that \( g'|_K = a \) and \( \|g - f\|_\infty < \varepsilon \).

**Proof.** Since \( C^1[0, 1] \) is dense in the Banach space \( C[0, 1] \), we can, without loss of generality, assume that \( f \in C^1[0, 1] \). Since any continuous function on \( K \) admits a continuous extension to \( [0, 1] \) (one can apply, for instance, the Tietze theorem [4]), there exists \( h \in C[0, 1] \) such that \( h(x) = a(x) - f'(x) \) for any \( x \in K \). Let \( \delta > 0 \). Since \( K \) is nowhere dense, there exist

\[0 = \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n = 1\]

such that \( \beta_j - \alpha_j < \varepsilon \) for any \( j = 1, \ldots, n \) and \( K \subset \bigcup_{j=1}^n I_j \), where \( I_j = [\alpha_j, \beta_j] \). Let

\[a_j = \int_{\alpha_j}^{\beta_j} h(t) \, dt \text{ for } 1 \leq j \leq n - 1.\]
By Lemma 4.1, for \(1 \leq j \leq n - 1\), there is \(\varphi_j \in C^1[\beta_j, \alpha_{j+1}]\) such that \(\varphi_j(\beta_j) = \varphi_j(\alpha_{j+1}) = 0\), \(\varphi_j'(\beta_j) = h(\beta_j) + \frac{a_j}{\alpha_{j+1}-\beta_j}\), \(\varphi_j'(\alpha_{j+1}) = h(\alpha_{j+1}) + \frac{a_j}{\alpha_{j+1}-\beta_j}\) and \(\|\varphi_j\|_\infty < \delta\). Consider the function
\[
\psi(x) = \begin{cases} 
\int_0^x h(t) \, dt & \text{if } x \in [\alpha_j, \beta_j], \ 1 \leq j \leq n, \\
\varphi_j(x) + \frac{a_j(x-\alpha_{j+1})}{\beta_j-\alpha_{j+1}} & \text{if } x \in (\beta_j, \alpha_{j+1}), \ 1 \leq j \leq n - 1.
\end{cases}
\]
The values of \(\varphi_j'\) at \(\beta_j\) and \(\alpha_{j+1}\) were chosen in such a way that \(\psi \in C^1[0, 1]\). Moreover, \(\psi'|_{I_j} = h\) for \(1 \leq j \leq n\). Hence, \(\psi + f\)'\(|_K = a\). Let us estimate \(\|\psi\|_\infty\). If \(1 \leq j \leq n - 1\) and \(x \in [\beta_j, \alpha_{j+1}]\), then \(|\psi(x)| \leq \delta + |a_j| \leq \delta + |\beta_j - \alpha_j| h\|_\infty \leq \delta(1 + h\|_\infty)\). If \(1 \leq j \leq n\) and \(x \in [\alpha_j, \beta_j]\), then \(|\psi(x)| \leq |\beta_j - \alpha_j| h\|_\infty \leq \delta h\|_\infty\). Hence \(\|\psi\|_\infty \leq \delta(1 + h\|_\infty)\). Choose \(\delta < \varepsilon/(1 + h\|_\infty)\) and denote \(g = \psi + f\). Then \(g'|_K = a\) and \(\|g - f\|_\infty = \|\psi\|_\infty < \varepsilon\).

**Lemma 4.3.** Let \(K \subset [0, 1]\) be a nowhere dense compact set and \(\varepsilon > 0\). Then there exists \(f \in C(K)\) such that
\[
\int_K f(t) \, dt = 0 \quad \text{and} \quad \|\chi + g\|_2 \leq \varepsilon,
\]
where \(g(x) = \int_{K \cap [x, 1]} f(t) \, dt\) and \(\chi\) is the indicator function of \(K\) (\(\chi(x) = 1\) if \(x \in K\) and \(\chi(x) = 0\) if \(x \in [0, 1] \setminus K\)).

**Proof.** If the Lebesgue measure \(\mu(K)\) of \(K\) is zero, the statement is trivially true since the function \(f \equiv 0\) satisfies the desired conditions for any \(\varepsilon > 0\). Thus, we can assume that \(\mu(K) > 0\). Let \(n \in \mathbb{N}\). Since \(K\) is nowhere dense and has positive Lebesgue measure, we can choose \(n \in \mathbb{N}\) and \(\alpha_k, \beta_k, a_k, b_k, u_k, v_k \in [0, 1] \setminus K\) for \(1 \leq k \leq n\) in such a way that
\[
\begin{align*}
\alpha_k < \beta_k < a_k < b_k < u_k < v_k & \quad \text{for } 1 \leq k \leq n \quad \text{and} \quad v_k - 1 < \alpha_k \quad \text{for } 2 \leq k \leq n, \\
0 < \mu(K \cap [\alpha_k, \beta_k]) < \frac{\varepsilon^2}{16n} & \quad \text{and} \quad 0 < \mu(K \cap [u_k, v_k]) < \frac{\varepsilon^2}{16n} \quad \text{for } 1 \leq k \leq n, \\
\mu\left(\bigcup_{k=1}^n [\alpha_k, v_k] \setminus K\right) < \frac{\varepsilon^2}{8}. & \quad \text{(4.1)}
\end{align*}
\]
Consider the function \(f : K \to \mathbb{R}\) defined by the formula
\[
f(x) = \begin{cases} 
\frac{1}{\mu(K \cap [\alpha_k, \beta_k])} & \text{if } x \in K \cap [\alpha_k, \beta_k], \ 1 \leq k \leq n; \\
\frac{1}{\mu(K \cap [u_k, v_k])} & \text{if } x \in K \cap [u_k, v_k], \ 1 \leq k \leq n; \\
0 & \text{otherwise.}
\end{cases}
\]
Obviously \(f \in C(K)\) and
\[
\int_K f(t) \, dt = \sum_{k=1}^n \int_{K \cap [\alpha_k, \beta_k]} f(t) \, dt - \int_{K \cap [u_k, v_k]} f(t) \, dt = \sum_{k=1}^n (1 - 1) = 0.
\]
Let \(g : [0, 1] \to \mathbb{R}\) be defined by
\[
g(x) = \int_{K \cap [x, 1]} f(t) \, dt.
\]
From the definition of \(f\) it follows that \(|g(x)| \leq 1\) for any \(x \in [0, 1]\), \(g(x) = -1\) if \(x \in \bigcup_{k=1}^n [\beta_k, u_k]\) and \(\chi(x) = g(x) = 0\) if \(x \in [0, 1] \setminus \bigcup_{k=1}^n [\alpha_k, v_k]\). Hence the set \(\Omega = \{x \in [0, 1] : g(x) + \chi(x) \neq 0\}\) is contained in the union
\[
\Omega_1 = \left(\bigcup_{k=1}^n [\alpha_k, v_k] \setminus K\right) \cup \left(\bigcup_{k=1}^n (\alpha_k, \beta_k) \cap K\right) \cup \left(\bigcup_{k=1}^n (u_k, v_k) \cap K\right).
\]
Therefore
\[
\|g + \chi\|_2^2 = \int_0^1 (g(x) + \chi(x))^2 \leq 4\mu(\Omega) \leq 4\mu(\Omega_1).
\]
Using (4.1) and (4.2), we see that \(\mu(\Omega_1) \leq \varepsilon^2/4\). Hence \(\|g + \chi\|_2 \leq \varepsilon\). \(\square\)
Lemma 4.4. Let \( \{e_n\}_{n \in \mathbb{Z}_+} \) be an orthonormal basis in a separable Hilbert space \( H \) and scalar sequences \( \{\gamma_n\}_{n \in \mathbb{N}} \) and \( \{\delta_n\}_{n \in \mathbb{N}} \) be such that
\[
\sum_{n=1}^{\infty} (|\gamma_n|^2 + |\delta_n|^2) < \infty. \tag{4.3}
\]

Let also \( f_0 = e_0 + \sum_{n=1}^{\infty} \gamma_n e_n \) and \( f_n = e_n - \delta_n e_0 \) for \( n \in \mathbb{N} \). Then the linear span of \( \{f_n : n \in \mathbb{Z}_+\} \) is dense in \( H \) if and only if
\[
\sum_{n=1}^{\infty} \gamma_n \delta_n \neq -1. \tag{4.4}
\]

Proof. Condition (4.3) implies that the linear operator \( T : H \to H \) such that \( T e_0 = \sum_{n=1}^{\infty} \gamma_n e_n \) and \( T e_n = -\delta_n e_0 \) for \( n \in \mathbb{N} \) is bounded. Since the range of \( T \) is at most two-dimensional, \( T \) is compact. By the Fredholm theorem [3], the operator \( S = I + T \) has dense range if and only if \( S \) is injective. Since \( S e_n = f_n \) for \( n \in \mathbb{Z}_+ \), the linear span of \( \{f_n\}_{n \in \mathbb{Z}_+} \) is dense in \( H \) if and only if the operator \( S \) injective.

The equation \( S x = 0 \), \( x \in H \) can be rewritten as
\[
\langle x, e_0 \rangle \left( 1 + \sum_{n=1}^{\infty} \gamma_n \delta_n \right) = 0 \quad \text{and} \quad \langle x, e_n \rangle = \gamma_n \langle x, e_0 \rangle \quad \text{for any } n \in \mathbb{N}.
\]

If \( \sum_{n=1}^{\infty} \gamma_n \delta_n \neq -1 \), the first equation implies \( \langle x, e_0 \rangle = 0 \) and the rest yield \( \langle x, e_n \rangle = 0 \) for each \( n \in \mathbb{N} \).

Thus in this case \( x = 0 \). That is, \( S \) is injective and therefore the linear span of \( \{f_n : n \in \mathbb{Z}_+\} \) is dense in \( H \). If \( \sum_{n=1}^{\infty} \gamma_n \delta_n = -1 \), the system of the equations in the above display has the non-zero solution \( x = x_0 + \sum_{n=1}^{\infty} \gamma_n e_n \in H \). Hence \( S \) is not injective and therefore the linear span of \( \{f_n : n \in \mathbb{Z}_+\} \) is non-dense. \( \square \)

We are ready to prove Theorem 1.7. First, if \( n \in \mathbb{N} \) and \( H \) is \( n \)-dimensional, then \( W^{1,2}([0,1],H) \) is isomorphic to the free \( W^{1,2}([0,1]) \)-module with \( n \) generators and the nicety of \( W^{1,2}([0,1],H) \) follows from Proposition 1.5. It is easy to see that a direct (module) summand of a nice module is nice. Thus the proof of Theorem 1.7 will be complete if we verify that \( W^{1,2}([0,1],\ell_2) \) is non-nice. In order to do this, we have to construct a proper closed \( W^{1,2}([0,1]) \)-submodule \( M \) of \( W^{1,2}([0,1],\ell_2) \) such that none of the characters on \( W^{1,2}([0,1],\ell_2) \) vanishes on \( M \). Now we shall do just that.

Pick a nowhere dense compact set \( K \subset [0,1] \) of positive Lebesgue measure and let \( \chi \) be the indicator function of \( K \). By Lemma 4.3 there exists \( A_n \in C(K) \) such that for any \( n \in \mathbb{N} \),
\[
\int_K A_n(x) \, dx = 0, \tag{4.5}
\]
\[
\|B_n + \chi\|_2 < 2^{-n}, \quad \text{where} \quad B_n(x) = \int_{K \cap [x,1]} A(t) \, dt. \tag{4.6}
\]

We also set \( A_0 = 0, B_0 = 0 \) and \( S_0 = 1 \). By Lemma 4.2 there exist \( S_n \in C^1[0,1] \) such that
\[
S'_n|_K = A_n \quad \text{and} \quad \|S_n - 1\|_\infty < 2^{-n} \quad \text{for each } n \in \mathbb{N}. \tag{4.7}
\]

Denote \( \rho_n = n^2(S_n - S_{n-1}) \) for \( n \in \mathbb{N} \). Then \( \rho_n \in C^1[0,1] \) and according to (4.7),
\[
\|\rho_n\|_\infty \leq n^2(\|S_n - 1\|_\infty + \|S_{n-1} - 1\|_\infty) \leq n^2(2^{1-n} + 2^{-n}) = 3n^2 2^{-n} \quad \text{for each } n \in \mathbb{N}. \tag{4.8}
\]
Let also \( \{e_n\}_{n \in \mathbb{Z}_+} \) be the standard orthonormal basis in \( \ell_2 \). Consider the functions \( f^{[n]} \in W^1_2([0, 1], \ell_2) \) defined by the formulas

\[
f^{[0]}(x) = e_0 + \sum_{n=1}^{\infty} n^{-2} e_n \quad \text{and} \quad f^{[n]}(x) = e_n - \rho_n(x)e_0 \quad \text{for} \quad n \in \mathbb{N}.
\]

Let now \( M \) be the closed \( W^{1,2}[0, 1] \)-submodule of \( W^{1,2}([0, 1], \ell_2) \) generated by the set \( \{f^{[n]} : n \in \mathbb{Z}_+\} \). Equivalently, \( M \) is the closed linear span in \( W^{1,2}([0, 1], \ell_2) \) of the set \( \{\varphi f^{[n]} : n \in \mathbb{Z}_+, \ \varphi \in W^{1,2}[0, 1]\} \).

It is easy to see that every character on \( W^{1,2}([0, 1], \ell_2) \) has the shape

\[
\varphi_{t,y}(f) = \langle f(t), y \rangle_H, \quad \text{where} \ t \in [0, 1] \quad \text{and} \ y \in \ell_2 \setminus \{0\}.
\]

Thus in order for every character on \( W^{1,2}([0, 1], \ell_2) \) not to vanish on \( M \) it is necessary and sufficient for \( M_t = \{f(t) : f \in M\} \) to be dense in \( \ell_2 \) for every \( t \in [0, 1] \). Let \( t \in [0, 1] \). By definition of \( \rho_n \) and (4.7), we have

\[
\sum_{n=1}^{\infty} n^{-2}\rho_n(t) = \lim_{m \to \infty} \sum_{n=1}^{m} (S_n(t) - S_{n-1}(t)) = \lim_{m \to \infty} (S_m(t) - S_0(t)) = 0 \neq -1. \quad (4.9)
\]

By Lemma 4.4 with \( \gamma_n = n^{-2} \) and \( \delta_n = \rho_n(t) \), the linear span of \( \{f^{[n]}(t)\}_{n \in \mathbb{Z}_+} \) is dense in \( \ell_2 \). Since \( f^{[n]} \in M \), \( M_t \) is dense in \( \ell_2 \). Thus none of the characters on \( W^{1,2}([0, 1], \ell_2) \) vanishes on \( M \). It remains to verify that \( M \neq W^{1,2}([0, 1], \ell_2) \). Consider \( g_n \in W^1_2[0, 1]^* \) for \( n \in \mathbb{Z}_+ \), defined by the formula

\[
g_n(\varphi) = \int_K (\rho_n \varphi)'(x) \, dx, \quad \text{where} \ \rho_0 \text{ is assumed to be identically 1.}
\]

We start with estimating the norms of the functionals \( g_n \). Clearly,

\[
g_n(\varphi) = \int_K \rho_n(x)\varphi'(x) \, dx + \int_K \rho_n'(x)\varphi(x) \, dx \quad \text{for any} \ \varphi \in W^{1,2}[0, 1]. \quad (4.10)
\]

Since \( \rho_n'(x) = n^2(S_n'(x) - S_{n-1}'(x)) = n^2(A_n(x) - A_{n-1}(x)) \) for \( x \in K \), we have

\[
\int_K \rho_n(x)\varphi(x) \, dx = n^2 \int_K (A_n(x) - A_{n-1}(x))\varphi(x) \, dx = n^2 \int_0^1 (B_{n-1}(x) - B_n(x))\varphi(x) \, dx.
\]

By (4.5) and (4.6),

\[
B_n(0) = B_n(1) = 0 \quad \text{for} \quad n \in \mathbb{Z}_+.
\]

Integrating by parts and using the above display, we obtain

\[
\int_K \rho_n(x)\varphi(x) \, dx = n^2 \int_0^1 (B_{n-1}(x) - B_n(x))\varphi(x) \, dx = n^2 \int_0^1 (B_n(x) - B_{n-1}(x))\varphi'(x) \, dx.
\]

This formula together with (4.10) yields

\[
|g_n(\varphi)| \leq \|\varphi\|_2 (\|\rho_n\|_2 + n^2\|B_n - B_{n-1}\|_2) \quad \text{for} \quad n \in \mathbb{N}.
\]

Since \( \|\rho_n\|_2 \leq 3n^22^{-n} \) and \( \|B_n - B_{n-1}\|_2 \leq \|B_n + \chi\|_2 + \|B_{n-1} + \chi\|_2 \leq 21^{-n} + 2^{-n} = 3 \cdot 2^{-n} \), we have

\[
|g_n(\varphi)| \leq 6n^22^{-n}\|\varphi\|_{1,2}. \quad \text{Hence} \quad \|g_n\| \leq 6n^22^{-n} \quad \text{for each} \ n \in \mathbb{N}. \quad \text{Therefore} \ \sum_{n=0}^{\infty} \|g_n\|^2 < \infty. \quad \text{Thus the formula}
\]

\[
g(h) = \sum_{n=0}^{\infty} g_n(h_n)
\]

defines a continuous linear functional on \( W^{1,2}([0, 1], \ell_2) \), where, as usual, \( h_n(t) = \langle h(t), e_n \rangle \). Since \( g_0 \neq 0 \), we have \( g \neq 0 \). In order to show that \( M \neq W^{1,2}([0, 1], \ell_2) \), it suffices to verify that \( g(h) = 0 \) for any \( h \in M \).
For this it is enough to check that \( g(\varphi f^{[n]}) = 0 \) for every \( \varphi \in W^{1,2}[0,1] \) and \( n \in \mathbb{Z}_+ \). First, let \( n \in \mathbb{N} \). Then by definition of \( g_n \), we immediately have

\[
g(\varphi f^{[n]}) = g_n(\varphi) - g_0(\rho_n \varphi) = 0.
\]

It remains to prove that \( g(\varphi f^{[0]}) = 0 \). Using the uniform convergence of the series \( \sum_{n=1}^{\infty} n^{-2} \rho_n \) provided by the estimate (4.8), we have

\[
\sum_{n=1}^{\infty} n^{-2} \rho_n(x) = 0.
\]

On the other hand, using (4.7) and the equality \( S_0 = 1 \), we have

\[
\sum_{n=1}^{x} n^{-2} \rho_n(x) = \sum_{n=1}^{x} (S'_n(x) - S'_{n-1}(x)) = S'_m(x) = A_m(x) \quad \text{for each} \quad x \in K.
\]

Hence

\[
g(\varphi f^{[0]}) = \int_K \varphi'(x) \, dx + \lim_{m \to \infty} \int_K \varphi(x) A_m(x) \, dx. \tag{4.11}
\]

Integrating by parts, we obtain

\[
\int_K \varphi(x) A_m(x) \, dx = -\int_0^1 \varphi(x) B'_m(x) \, dx = \int_0^1 \varphi'(x) B_m(x) \, dx = \int_0^1 \varphi'(x) (B_m(x) + \chi(x)) \, dx - \int_K \varphi'(x) \, dx.
\]

According to (4.11) and the above display,

\[
g(\varphi f^{[0]}) = \lim_{m \to \infty} \int_0^1 \varphi'(x) (B_m(x) + \chi(x)) \, dx. \tag{4.12}
\]

By (4.6) and (4.12), \( g(\varphi f^{[0]}) = 0 \) for each \( \varphi \in W^{1,2}[0,1] \). Thus \( g(h) = 0 \) for every \( h \in M \) and therefore \( M \neq W^{1,2}([0,1], \ell_2) \). The proof of Theorem 1.7 is complete.

5 Remarks

One can easily generalize Theorem 1.7 by taking most any algebra of smooth functions instead of \( W^{1,2}[0,1] \). For example, following the same route of argument with few appropriate amendments one can show that if \( X \) is an infinite dimensional Banach space and \( k \in \mathbb{N} \), then \( C^k([0,1], X) \) as a \( C^k([0,1], \ell_2) \)-module is non-nice. We opted for \( W^{1,2}([0,1], H) \) to make a point that even the friendly Hilbert space environment does not save the day.

Theorem 1.7 says that there are weird proper closed submodules of \( W^{1,2}([0,1], \ell_2) \) which are not contained in any closed submodule of codimension 1. The following question remains wide open.

Question 5.1. Characterize closed submodules of \( W^{1,2}([0,1], \ell_2) \).

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