The Performance of the MLE in the Bradley-Terry-Luce Model in $\ell_\infty$-Loss and under General Graph Topologies

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Abstract

The Bradley-Terry-Luce (BTL) model is a popular statistical approach for estimating the global ranking of a collection of items of interest using pairwise comparisons. To ensure accurate ranking, it is essential to obtain precise estimates of the model parameters in the $\ell_\infty$-loss. The difficulty of this task depends crucially on the topology of the pairwise comparison graph over the given items. However, beyond very few well-studied cases, such as the complete and Erdős-Rényi comparison graphs, little is known about the performance of the maximum likelihood estimator (MLE) of the BTL model parameters in the $\ell_\infty$-loss under more general graph topologies. In this paper, we derive novel, general upper bounds on the $\ell_\infty$ estimation error of the BTL MLE that depend explicitly on the algebraic connectivity of the comparison graph, the maximal performance gap across items and the sample complexity. We demonstrate that the derived bounds perform well and in some cases are sharper compared to known results obtained using different loss functions and more restricted assumptions and graph topologies. We further provide minimax lower bounds under $\ell_\infty$-error that nearly match the upper bounds over a class of sufficiently regular graph topologies. Finally, we study the implications of our bounds for efficient tournament design. We illustrate and discuss our findings through various examples and simulations.

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1 Introduction

Simultaneous or ‘global’ ranking of a possibly large set of items is a practical problem that arises naturally in a variety of domains. For example, one may wish to ascertain a ‘best player’ or ‘best team’ in a given sports league. Designing a principled statistical approach to global ranking of items is challenging due to data limitations and complex domain-specific relationships between the underlying items to be ranked.

A popular and practicable solution to estimating global ranking is to utilize pairwise comparison information across the items to be ranked, which is easily accessible across many application domains. The Bradley-Terry-Luce (BTL) model (Bradley and Terry, 1952; Luce, 1959) is a popular statistical model for pairwise comparison data. A similar model was also originally studied in Zermelo (1929). The continued practical and theoretical interest in the BTL model stems from its relatively simple parametric form which provides a good balance between interpretability and tractability for theoretical analysis. The BTL model is domain-agnostic, making it an ideal benchmarking tool across a variety of ranking applications e.g. sports analytics (Fahrmeir and Tutz, 1994; Masarotto and Varin, 2012; Cattelan et al., 2013), and bibliometrics (Stigler, 1994; Varin et al., 2016).

Formally, we can describe the BTL model as follows. Suppose that we have $n$ distinct items, each with a (fixed but unobserved) positive strength or preference score $w_i$, $i \in [n]$, quantifying item $i$’s propensity to beat other items in pairwise comparisons. The BTL model assumes that the comparisons between different pairs are independent and the outcomes of comparisons between any given pair, say item $i$ and item $j$, are independent and identically distributed Bernoulli random variables, with winning probability $p_{ij}$, defined as

$$p_{ij} = P(i \text{ beats } j) = \frac{w_i}{w_i + w_j}, \forall i, j \in [n].$$  (1)
A common reparametrization is to set, for each $i$, $w_i^* = \exp(\theta_i^*)$, where $\theta^* := (\theta_1^*, \ldots, \theta_n^*)^T \in \mathbb{R}^n$. Following convention, we also assume that $\sum_{i \in [n]} \theta_i^* = 0$ for parameter identifiability.

**General pairwise comparison graphs**

Given $n$ items to be compared, the pairwise comparison scheme among them can be expressed through an undirected simple graph $G(V, E)$, where the vertex set $V := [n]$ and the edge set $E := \{(i, j) : i$ and $j$ are compared $\}$ is determined by the comparison scheme. Correspondingly, if we define the directed edge set as $E_d := \{(i, j, k) : (i \text{ beats } j) k \text{ times}\}$, then the induced directed simple graph $G(V, E_d)$ is called a directed comparison graph. It is a classical result (Ford, 1957; Simons and Yao, 1999; Hunter, 2004) that the BTL model is identifiable if and only if $G(V, E)$ is connected, and the maximum likelihood estimator (MLE) of the model parameters exists and is consistent if and only if $G(V, E_d)$ is strongly connected. Henceforth, comparison graph refers to the undirected pairwise comparison graph.

From a statistical theory perspective, much attention in the BTL literature has been paid to two popular estimators, namely the MLE and the spectral method. Typically one is interested in getting sharp bounds for the estimation risk, which could be based on a norm-induced metric $\|\hat{\theta} - \theta^*\|_p$ or a ranking metric, e.g., Kendall’s tau distance (Kendall, 1938). What makes risk analysis of BTL model estimators particularly challenging is a combination of the type of estimation risk loss considered, and the assumptions on the pairwise comparison graph $G(V, E)$ topology.

**Core question of interest**

Among the metrics considered for estimation loss, the $\ell_\infty$-loss is the one that directly connects with ranking metrics, e.g. binary and Hamming top-$k$ (partial) ranking loss (see, e.g. Chen et al., 2019, 2020). Moreover, Chen et al. (2020) show that the MLE attains a sharper minimax rate of the Hamming top-$k$ loss compared to the spectral method.

It is thus natural to study the MLE for the BTL parameters in the $\ell_\infty$-loss, to better understand the risk optimality of the MLE and further justify its use for practical global and partial ranking problems. In this spirit, Yan et al. (2012) focus specifically on proving $\ell_\infty$-error bounds for the BTL MLE for general comparison graphs. However a notable limitation in their setting is that they impose a strictly dense comparison graph assumption, which may be impractical in many real world applications. This leaves a gap in the literature, summarized in the following question:

**Core question:** For the BTL model, how does the MLE perform with respect to the $\ell_\infty$ loss, under much weaker assumptions on the pairwise comparison graph i.e. assuming only that the comparison graph is connected?

Providing a sharp analysis to this question with a detailed comparison to recent theoretical results in the BTL literature motivates our work in this paper.

**Relevant and related literature**

We give a brief overview of the work that addresses the challenge of comparison graph topology in ranking. When the comparison graph is a complete graph, Simons and Yao (1999) give a high-probability upper bound for the $\ell_\infty$ loss, i.e., $\|\hat{\theta} - \theta^*\|_\infty$ and obtain the asymptotic distribution of the MLE. In the setting where the comparison graph follows the Erdős-Rényi graph model, Chen and Suh (2015), Chen et al. (2019), Chen et al. (2020) and Han et al. (2020) derive high-probability upper bounds for the $\ell_\infty$ loss. Moreover, Chen et al. (2019) show that both MLE and spectral method are minimax optimal in terms of the binary top-$k$ ranking loss i.e. whether the items with the highest $k$ out of $n$ preference scores are perfectly identified; Chen et al. (2020) consider a Hamming Loss for top-$k$ items and show that the MLE is minimax optimal compared to the spectral method with differences arising in constant factors.

For a broader class of comparison graphs beyond complete and Erdős-Rényi graph, researchers have studied the explicit dependence of the estimation risk on their graph topology. In particular, Yan et al. (2012) give a high-probability upper bound for the $\ell_\infty$-loss for relatively dense graphs. Hajek et al. (2014); Shah et al. (2016) give a high probability upper bound for the $\ell_2$ or Euclidean loss $\|\hat{\theta} - \theta^*\|_2$, establish upper and lower bounds of $E[\|\hat{\theta} - \theta^*\|_2]$ and show the minimax optimality of the constraint MLE across a wide range of graph topologies. Recently, Agarwal et al. (2018) give sharp upper bounds for a novel spectral method in the $\ell_1$-loss $\|\hat{\pi} - \pi^*\|_1$ for $\pi^* = w^*/\|w^*\|_1$ instead of $\theta^*$. Hendrickx et al. (2019, 2020) propose a novel weighted least square method to estimate $w^*$ and prove a sharp upper bound for their estimator in $E[\sin^2(\hat{w}, w^*)]$ or equivalently in $E[\hat{w}/\|\hat{w}\|_2 - w^*/\|w^*\|_2]^2$, in the sense that this upper bound matches a instance-wise lower bound up to constant factors.

**Contributions**

Our contributions in this paper are fourfold and are summarized as follows:
• **Upper bounds:** We derive a novel upper bound for the $\ell_{\infty}$-error of the regularized MLE in the BTL model allowing for general graph topology. Our upper bounds hold under minimal assumptions on graph topologies, i.e., assuming only that the comparison graph is connected. Given such generality, we show our $\ell_{\infty}$ bound is tighter than existing results under a broad range of graph topologies, and works well in general. A minor corollary of our techniques results in state of the art $\ell_{2}$-loss bound in the case of the Erdős-Rényi graph.

• **Lower bounds:** We derive a lower bound for BTL parameter estimation in $\ell_{\infty}$-loss. We analyze specific graph topologies satisfying certain regularity connectivity conditions under which the BTL MLE is nearly minimax optimal.

• **Implications for tournament design:** We show our upper bounds using the $\ell_{\infty}$-loss satisfy a unique subadditivity property. We demonstrate how our $\ell_{\infty}$ bounds can exploit this property for efficient tournament design.

• **Extension to the unregularized BTL model:** We also extend our upper bounds under $\ell_{\infty}$-loss to the unregularized (‘vanilla’) BTL MLE, which is also frequently used in practice.

Due to the more complicated form of the vanilla BTL MLE upper bounds and space limitations, we present these analogous results and their proofs separately in Appendix A.7. Henceforth, MLE refers to the regularized BTL MLE unless stated otherwise. In addition to our theoretical contributions a core aspect throughout our paper is to emphasize the interpretability of our results, the associated assumptions, and implications for practical ranking tasks.

**Organization of the paper**

The rest of the paper is organized as follows. In Section 2 we present our main results for the upper bound in Theorem 1 and an interpretation of the key components of the bound. In Section 3, we discuss minimax lower bounds using the $\ell_{\infty}$ risk loss in Theorem 3. In Section 4, we show some practical implications of our results in efficient tournament design from a ranking perspective. In Section 5, we conduct extensive numerical simulations to validate the optimality of our bounds compared to related results in the literature.

**Notation**

Throughout this paper we use the following notational conventions. We typically use lowercase for scalars in $\mathbb{R}$ e.g. $(x,y,z,\ldots)$, boldface lowercase for vectors e.g. $(\mathbf{x},\mathbf{y},\mathbf{z},\ldots)$, and boldface uppercase for matrices $(\mathbf{X},\mathbf{Y},\mathbf{Z},\ldots)$. We denote the finite set $\{1,\ldots,n\}:=\{n\}$.

For asymptotics, we denote $x_n \lesssim y_n$ or $x_n = O(y_n)$ and $u_n \gtrsim v_n$ or $u_n = \Omega(v_n)$ if $\forall n$, $x_n \leq c_1 y_n$ and $u_n \geq c_2 v_n$ for some constants $c_1, c_2 > 0$. Similarly, $x_n \asymp y_n$ or $x_n = \Theta(y_n)$ means $\forall n$, $c_1 y_n \leq x_n \leq c_2 y_n$ for some constants $c_1, c_2 > 0$. We denote $\mathbf{e}_i$, as a vector whose entries are all 0 except that the $i$-th entry is 1. $a_n = o(b_n)$ means $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$. We denote $\mathbf{1}_n \in \mathbb{R}^n$ to be a vector of ones.

**2 Upper bounds**

Recall that given $n$ items to be compared, the comparison scheme among them defines the comparison graph $G(V,E)$, where $V = [n]$ and $E = \{(i,j) : i$ and $j$ are compared $\}$. We denote the corresponding adjacency matrix as $A \in \mathbb{R}^{n \times n}$, and its $(i,j)^{th}$ entry is $A_{ij} := 1\{(i,j) \in E\}$. The associated (unnormalized) graph Laplacian is the symmetric, positive-semidefinite matrix $L_{\mathbf{A}} := \mathbf{D} - \mathbf{A}$, where $\mathbf{D} = \text{diag}(n_1,\ldots,n_n)$, with $n_i := \sum_{j=1}^n A_{ij}$ the degree of node $i$. It is well known that the smallest eigenvalue of $L_{\mathbf{A}}$ is 0 with an eigenvector $\mathbf{1}_n$. Let $\lambda_2(L_{\mathbf{A}})$ be the second smallest eigenvalue of $L_{\mathbf{A}}$, known also as the algebraic connectivity of $G$ (Das, 2004), then $G$ is connected if and only if $\lambda_2(L_{\mathbf{A}}) > 0$. Following the standard in the BTL literature we assume a that for each edge $(i,j)$ of the comparison graph, the corresponding items $i$ and $j$ are compared $L$ times, each leading to an independent outcome $y_{ij}^{(l)} \in \{0,1\}$, where $l \in [L]$. If pairs are compared different number of times, we take $L$ to be the smallest number of pairwise comparisons over the edge set, as a worst-case scenario. The corresponding sample averages are denoted with $\bar{y}_{ij} = \frac{1}{L} \sum_{l=1}^{L} y_{ij}^{(l)}$ and are sufficient statistics for the model parameters. The $\ell_2$-regularized MLE of $\theta$ is defined as

$$\hat{\theta}_p = \arg \min_{\theta \in \mathbb{R}^n} \ell_p(\theta;\mathbf{y}), \quad \ell_p(\theta;\mathbf{y}) = \ell(\theta;\mathbf{y}) + \frac{p}{2} ||\theta||_2^2, \quad (2)$$

where $\ell(\theta;\mathbf{y})$ is the negative log-likelihood, given by

$$\ell(\theta;\mathbf{y}) := - \sum_{1\leq i<j\leq n} A_{ij} \{ y_{ij} \log(\psi(\theta_i - \theta_j)) + (1 - y_{ij}) \log[1 - \psi(\theta_i - \theta_j)] \}, \quad (3)$$

and $t \in \mathbb{R} \mapsto \psi(t) = 1/[1 + e^{-t}]$ the sigmoid function.

Under this notational setup, we are ready to state the $\ell_{\infty}$ upper bound of the BTL MLE in Theorem 1.

**Theorem 1.** Assume the BTL model with parameter $\theta^* = (\theta^*_1,\ldots,\theta^*_n)^\top$ such that $\mathbb{1}_n^\top \theta^* = 0$ and a comparison graph $G = G([n],E)$ with adjacency matrix $\mathbf{A}$, algebraic connectivity $\lambda_2(L_{\mathbf{A}})$ and maximum and minimum degrees $n_{\max}$ and $n_{\min}$. Suppose that each pair of items $(i,j) \in E$ are compared $L$ times. Let $K = \max_{i,j} |\theta^*_i - \theta^*_j|$ and $\kappa_E = \max_{(i,j) \in E} |\theta^*_i - \theta^*_j|$ and
set \( \rho \geq c_\rho \kappa^{-2} e^{-2.5 \kappa n^{-1}} n^{-1/2} \). Assume that \( \mathcal{G} \) is connected or \( \lambda_2(\mathcal{L}_A) > 0 \). Then with probability at least \( 1 - O(n^{-4}) \), the regularized MLE \( \hat{\theta}_\rho \) from (2) satisfies

\[
\| \hat{\theta}_\rho - \theta^* \|_\infty \lesssim \frac{e^{2 \kappa E} \max\{ n + r \}}{\lambda_2} \left( \frac{n}{\min\{ \rho \}} + \sqrt{\frac{n}{\max\{ \rho \}}} \right)
\]

\[
+ \frac{e^{2 \kappa E}}{\lambda_2} \sqrt{\frac{n}{\max\{ \rho \}}} \left( \frac{\max\{ \rho \}}{\max\{ \rho \}} \right)
\]

(4)

\[
\| \hat{\theta}_\rho - \theta^* \|_2 \lesssim \frac{e^{2 \kappa E}}{\lambda_2} \left( \frac{n}{\min\{ \rho \}} + \sqrt{\frac{n}{\max\{ \rho \}}} \right)
\]

(5)

where \( \lambda_2 = \lambda_2(\mathcal{L}_A) \), \( r = \kappa_E + \log \kappa \) provided that \( L \leq n^8 e^{5 \kappa E} \max\{ 1, \kappa \} \), and \( L \) is large enough so that the right hand side of Equation (4) is smaller than a sufficiently small constant \( C > 0 \). In particular, if we set \( \rho = \frac{c_\rho}{\kappa} \sqrt{\frac{\max\{ \rho \}}{L}} \) for some \( c_\rho > 0 \), then

\[
\| \hat{\theta}_\rho - \theta^* \|_\infty \lesssim \frac{e^{2 \kappa E} \max\{ n + r \}}{\lambda_2} \left( \frac{n}{\min\{ \rho \}} + \sqrt{\frac{n}{\max\{ \rho \}}} \right)
\]

(6)

As a brief sketch, the proof is based on a gradient descent procedure initialized at \( \theta^{(0)} = \theta^* \) and the idea is to control \( \| \theta^{(T)} - \hat{\theta}_\rho \|_\infty \) using the linear convergence property and \( \| \theta^{(T)} - \theta^* \|_\infty \) using the leave-one-out technique in Chen et al. (2019) and Chen et al. (2020). The extension of the argument to arbitrary topology is non-trivial. The proof is included in Appendix A.2.

**Interpretation of key terms**

The upper bound in (4) contains several distinct terms, which interact with each other in non-trivial ways and express different aspects of the intrinsic difficulty of the estimation task.

- The factor \( \frac{e^{2 \kappa E}}{\lambda_2(\mathcal{L}_A)} \) combines two sources of statistical hardness: the maximal gap in performance \( \kappa_E \) among the ranked items over the edge set \( E \) and the algebraic connectivity \( \lambda_2(\mathcal{L}_A) \) of the comparison graph. It is intuitively clear that the larger the performance gap among the compared items, the more difficult it is to accurately estimate the model parameters. Furthermore, the smaller the algebraic connectivity, the less connected the comparison graph is, due to the presence of bottlenecks\(^1\). This in turn will increase the chance of obtaining a highly erroneous ranking or of gathering data from which a global ranking cannot be elicited at all. The minimal and maximal degrees \( n_{\min} \) and \( n_{\max} \) further quantify the impact of the connectivity of the comparison graph.

- We note that the factor \( \frac{1}{\lambda_2(\mathcal{L}_A)} \) can be equivalently replaced with \( \frac{1}{\lambda_2(\mathcal{L})} \) (see Lemma 8 in Appendix A.2). Here, \( \mathcal{I} := \sqrt{2} \ell_0(\theta^*; y) \) is the Fisher information matrix at \( \theta^* \) and \( \lambda_2(\mathcal{I}) \) its smallest non-zero eigenvalue. The fact that the bound depends on the Fisher information is not too surprising. This is so, since this quantity in exponential families quantifies the curvature of the likelihood and the intrinsic difficulty of estimating \( \theta^* \).

- Our bounds depend on both \( \kappa \) and \( \kappa_E \), which is non-standard in the literature. By definition, \( \kappa_E \leq \kappa \) and in many cases, \( \kappa_E \) can be much smaller than \( \kappa \). We discuss this further in Section 5.

- The term \( r := \kappa_E + \log \kappa \) shows the impact of large \( \kappa \) and \( \kappa_E \). When \( \kappa \lesssim n \) and \( \kappa_E \lesssim \log n \), \( r \) is negligible. We will consider this parameter range throughout the paper unless stated otherwise.

- The term \( \sqrt{\frac{n}{\rho}} \) describes explicitly the impact of a high-dimensional parameter space on the estimation problem in relation to \( L \), the number of samples for each comparison, which can be thought of as a measure of the sample size required for each of the \( n \) parameters. The inverse root dependence on \( L \) is to be expected and, we conjecture, not improvable.

**Remark 1.** In the case of dense graphs, e.g., complete graphs, \( \lambda_2(\mathcal{L}_A) \) is large enough so that even \( L = 1 \) will ensure a consistent estimator as \( n \to \infty \). But for sparse graphs, \( L \) needs to be larger to compensate for weaker connectivity. The assumption that \( L \leq n^8 e^{5 \kappa E} \max\{ 1, \kappa \} \) is a technical condition. There is nothing special in the exponent for \( n \). Any fixed number larger than 8 can be used which will only affect the constants in the bounds. The condition \( L \leq n^8 e^{5 \kappa E} \max\{ 1, \kappa \} \) may seem counter-intuitive, since it places an upper bound on the sample size. But a control over \( L \) is needed because as \( L \) gets larger, the optimal choice of the regularization parameter \( \rho = c_\rho \frac{1}{\kappa} \sqrt{\frac{n_{\max}}{L}} \) gets smaller and, accordingly, the convergence rate of the gradient descent procedure upon which our proof is based degrades. The optimal choice \( \rho = c_\rho \sqrt{\frac{n_{\max}}{L}} \) depends on \( \kappa \), which is unknown before an estimator is produced, however, one can set \( \rho = c_\rho \sqrt{\frac{n_{\max}}{L}} \) and the upper bound will only change by a factor \( \max\{ 1, \kappa \} \) in the first term of Equation (6).

### 2.1 Comparison to other work

To the best of our knowledge, Yan et al. (2012); Hajek et al. (2014); Shah et al. (2016); Negahban et al. (2017);
Agarwal et al. (2018); Hendrickx et al. (2019, 2020) are the only papers that study estimation error for the BTL model on a comparison graph with general topology. Since Negahban et al. (2017); Agarwal et al. (2018); Hendrickx et al. (2019, 2020) estimate the the preference scores \( w^* \) rather than \( \theta^* \), we cannot directly compare our results with theirs because there is no tight two-sided relationship between their metrics of error and ours. Therefore, here we only compare our results to those in Yan et al. (2012); Hajek et al. (2014); Shah et al. (2016). We include detailed comparison between our bound and the other 4 papers in Appendix A.1.

\( \ell_\infty \) loss: Yan et al. (2012) establish an \( \ell_\infty \)-bound of 
\[
\min_{n_{ij}} n_{ij} \sqrt{n \max n/L} \quad \text{where} \quad n_{ij} \text{ is the number of common neighbors of item } i \text{ and item } j \text{ in the comparison graph, under a strong assumption that } n_{ij} \geq cn \text{ for some constant } c \in (0,1). \]
This constraint on graph topology is stronger than ours since it always implies that the comparison graph is dense. In particular, when the comparison graph comes from an Erdős-Rényi model \( ER(n,p) \), the conditions in Yan et al. (2012) implies that \( p \) is bounded away from 0. Thus, \( \min_{n_{ij}} n_{ij} \approx np^2 \), so the bound\(^3\) in Yan et al. (2012) becomes \( \frac{e^{\alpha \epsilon}}{\sqrt{npL}} \), while our bound is \( e^{\epsilon E}/\sqrt{\log n} \sqrt{npL} \). Our bound is tighter for moderate or small \( \kappa_E \), and importantly, allows \( p \) to vanish. Furthermore, in Section 5, we show by some specific examples that \( \min_{n_{ij}} n_{ij} \) could be 0 even for many fairly dense graphs, to illustrate that the Yan et al. (2012) upper bound cannot apply to many realistic settings.

\( \ell_2 \) loss: Hajek et al. (2014); Shah et al. (2016) consider constrained MLE \( \hat{\theta} := \min_{\theta} \| \theta \|_{\infty} \leq B \ell_0(\theta) \) for a known parameter \( B \) such that \( \| \theta \|_{\infty} \leq B \). In our setting, the upper bounds for \( \ell_2 \)-error in Shah et al. (2016) and Hajek et al. (2014) are \( e^{\alpha B}/\sqrt{n \log n/\lambda_2(L_A)} \sqrt{\log n/L} \). Setting aside the fact that their results require stricter conditions than ours, our \( \ell_2 \) bound of order \( e^{\epsilon E}/\sqrt{\lambda_2(L_A)} \sqrt{\max n/L} \) is tighter than theirs for general parameter settings with moderate \( B, \kappa \) and for a broad range of graphs with moderate \( \lambda_2(L_A) \), i.e., not too sparse or irregular.

### 2.2 Special cases of graph topologies

By Theorem 1, for the estimator \( \hat{\theta}_p \) to be consistent, \( L \) needs to be sufficiently large. We can check some common types of comparison graph topologies and see in what order the necessary sample complexity \( N_{\text{comp}} = |E|L \) needs to be, to achieve consistency. The results are summarized in Table 1. Spectral properties of graphs listed here can be found in well-known textbooks (Brouwer and Haemers, 2012). Details behind the calculations and additional results for \( d \)-Caley graphs, and expander graphs are noted in Appendix A.6.

| Graph     | \( \lambda_2(L_A) \) | \( N_{\text{comp}} \) |
|-----------|----------------------|----------------------|
| Complete  | \( n \)               | \( \Omega(n^2) \)     |
| Bipartite | \( (n) \)             | \( \Omega(n^2) \)     |
| Path/Cycle| \( \Theta(n^{-2}) \)  | \( \Omega(e^{4\epsilon E} n^6) \) |
| Star      | 1                     | \( \Omega(e^{4\epsilon E} n^4) \) |
| Barbell   | \( \Theta(n^{-1}) \)  | \( \Omega(e^{2\kappa E} n^3 \log n) \) |

Table 1: Magnitude of \( N_{\text{comp}} \) to ensure the consistency of \( \hat{\theta}_p \) for some special cases of comparison graphs.

**Remark 2.** For the path/cycle graph, star graph, and barbell graph, the necessary sample complexity induced by directly applying our \( \ell_\infty \) bound is larger than the sample complexity induced by the \( \ell_2 \) bound in Shah et al. (2016), however they require more stringent conditions than ours. Additionally, in Section 4, we illustrate that by applying a unique sub-additivity property of \( \ell_\infty \)-loss, we can achieve a much smaller sample complexity in graphs with bottlenecks like the barbell graph. In Sections 2.1 and 5 we show that, except for such extreme graph topologies, our bound is tighter.

**Erdős-Rényi graph:** By a union bound on \( \lambda_2(L_A) \), \( n_{\max} \), and \( n_{\min} \), we can get a straightforward corollary of Theorem 1 in the special and highly studied setting where the comparison graph follows the Erdős-Rényi model \( ER(n,p) \).

**Corollary 2 (Erdős-Rényi graph).** Suppose that the comparison graph comes from an Erdős-Rényi graph \( ER(n,p) \). Assume that \( 1_n \sup \| \theta^* \| \leq n, \kappa_E \leq \log n, L \leq n^3 e^{4\epsilon E} \max\{1, \kappa\}, np > C_1 \log n, \) and \( L \geq C_2 \min\{1, \kappa\} e^{4\epsilon E} n/\log^2 n \) for some sufficiently large constants \( C_1, C_2 > 0 \). Set \( \rho = e^{\kappa}/\sqrt{\log n/\max L} \). Then \( 1_n \hat{\theta}_p = 0 \), and with probability at least \( 1 - O(n^{-4}) \), it holds that

\[
\| \hat{\theta}_p - \theta^* \|_\infty \lesssim e^{2\epsilon E} \sqrt{\frac{1}{np^2 L}} + e^{\epsilon E} \sqrt{\frac{\log n}{npL}}, \tag{7}
\]

\[
\| \hat{\theta}_p - \theta^* \|_2 \lesssim e^{\epsilon E} \frac{1}{\sqrt{Lp}}.
\]

The proof of Corollary 2 can be found in Appendix A.2. For the Erdős-Rényi comparison graph \( ER(n,p) \), the tightest \( \ell_\infty \)-norm error bound \( e^{2\epsilon E} \sqrt{\log n/npL} \) is proved in Chen et al. (2019) and Chen et al. (2020). Han et al. (2020) establish an \( \ell_\infty \)-norm upper bound of \( e^{2\epsilon E} \sqrt{\log n/np} \). Negahban et al. (2017) obtain an \( \ell_2 \)-norm upper bound of \( e^{4\epsilon \log n/np} \) and a lower bound of \( e^{-\kappa 1/p^2} \). Thus the derived \( \ell_2 \)-bound in Corollary 2 in Erdős-Rényi case is minimax optimal.
In this case our derived $\ell_\infty$-bound cannot achieve the rate established in Chen et al. (2019), Chen et al. (2020), though our $\ell_2$-bound exhibits the optimal rate proved in Negahban et al. (2017). The reason why our bound is unable to recover the optimal $\ell_\infty$-rate under a Erdős-Rényi comparison graph is that such a random graph topology exhibits, with high probability, certain concentration properties that are not fully expressed by the algebraic connectivity nor by the maximal/minimal degrees and thus cannot be leveraged in our proofs.

3 Lower bounds

In this section, we derive a minimax lower bound for the $\ell_\infty$ loss. Towards that end, we introduce some new notation. Let $N_{\text{comp}}$ be the total number of comparisons that have been observed, so in our setting, $N_{\text{comp}} = |E|L$ where $|E|$ is number of edges in the comparison graph $\mathcal{G}$. Denote the two items involved in the $i$-th comparison as $(i_1, i_2)$ such that $i_1 < i_2$. Let $\tilde{\mathcal{L}}_A = \frac{1}{N_{\text{comp}}} \sum_{j=1}^{N_{\text{comp}}} (e_i - e_j)(e_i - e_j)^\top$ be the normalized graph Laplacian with pseudo inverse $\tilde{L}_A$ and eigenvalues $0 = \lambda_1(\tilde{\mathcal{L}}_A) \leq \lambda_2(\tilde{\mathcal{L}}_A) \leq \cdots \leq \lambda_n(\tilde{\mathcal{L}}_A)$. With the main notation in place, our minimax lower bound is summarized in the following result.

**Theorem 3.** Assume that the comparison graph $\mathcal{G}$ is connected and the sample size $N_{\text{comp}} \geq \frac{\text{tr}(\tilde{L}_A^2)}{\epsilon e^{-\kappa}}$, any estimator $\hat{\theta}$ based on $N_{\text{comp}}$ comparisons with outcomes from the BTL model satisfies

$$\sup_{\theta^* \in \Theta_{\kappa}} \mathbb{E} \left[ \| \hat{\theta} - \theta^* \|_\infty \right] \geq \frac{e^{-2\kappa}}{n N_{\text{comp}}} \times \max \left\{ n^2, \max_{n' \in \{2, \ldots, n\}} \sum_{i=1}^{n'} \left[ \lambda_i(\tilde{\mathcal{L}}_A) \right]^{-1} \right\}$$

where $\Theta_{\kappa} = \{ \theta \in \mathbb{R}^n : 1_n^\top \theta = 0, \| \theta \|_\infty \leq \kappa \}$.

The proof of Theorem 3 largely leverages the lower bound construction from Theorem 2 in Shah et al. (2016). The main modification in adapting it to our setting is to construct an $\ell_\infty$-packing set. This is done by utilizing the tight topological equivalence of $\ell_\infty$ and $\ell_2$ norms in finite dimensions.

We can compare this lower bound with the upper bound in Theorem 1. In our setting, the comparisons distribute evenly over all pairs, so $N_{\text{comp}} = |E|L$, and $\lambda_i(\tilde{\mathcal{L}}_A) = \frac{1}{|E|} \lambda_i(\mathcal{L}_A)$. Thus, given a comparison graph with $\lambda_2(\tilde{\mathcal{L}}_A) \gtrsim \frac{1}{n}$, the lower bound becomes

$$\sup_{\theta^* \in \Theta_{\kappa}} \mathbb{E} \left[ \| \hat{\theta} - \theta^* \|_\infty \right] \geq e^{-\kappa} \sqrt{\frac{n}{N_{\text{comp}}}}$$

In $ER(n, p)$ case, this lower bound becomes $e^{-\kappa} \sqrt{\frac{1}{npL}}$ which matches the upper bound in Chen et al. (2019). For some “regular” graph topology with $\lambda_2(\mathcal{L}_A) \gtrsim \frac{1}{n}$ like complete graph, expander graph with $\phi = \Omega(n)$ and complete bipartite graph with two partition sets of size $\Omega(n)$, the upper bound becomes

$$\| \hat{\theta}_p - \theta^* \|_\infty \leq e^{2\kappa} \sqrt{\frac{n \log n}{N_{\text{comp}}}}.$$ 

Therefore, when the comparison graph topology is sufficiently regular, our upper bound matches the lower bound up to a log $n$ factor and a factor of $e^{2\kappa}$.

As a final remark, Negahban et al. (2017) show that the minimax lower bound for $\ell_2$-loss and Erdős-Rényi comparison graph $ER(n, p)$ is $e^{-\kappa} \frac{1}{npL}$, which matches our $\ell_2$ upper bound up to a factor of $e^{2\kappa}$.

4 Implications for tournament design

In this section we discuss how our results can be leveraged to construct more efficient tournament design from a ranking perspective in sports leagues.

As discussed in Section 2.2, for some comparison graphs with small $\lambda_2(\mathcal{L}_A)$, the requirement on $L$ and $N_{\text{comp}}$ for consistency is stringent. However, as we show next, we can significantly relax the requirement on the sample complexity $N_{\text{comp}}$ by an adaptive design that sets different $L$ over edges and uses a divide and conquer strategy to estimate $\theta^*$ more efficiently in practice.

**Lemma 4** (Subadditivity of $\ell_\infty$-loss in BTL). Let $I_1, I_2, I_3$ be three subsets of $[n]$ such that $\cup_{j=1}^3 I_j = [n]$ and, for each $j \neq k$, $I_j \nsubseteq I_k$ and for $i = 1, 2, I_1 \cap I_2 \neq \emptyset$. For any vector $\theta \in \mathbb{R}^n$, let $\theta_{(j)} \in \mathbb{R}^{|I_j|}$ be the sub-vector of $\theta$ consisting of the entries indexed by $I_j$, $j = 1, 2, 3$. Let $\theta^*$ be the vector of preference scores in the BTL model over $n$ items and $\theta_{(j)}$ be the MLE of $\theta_{(j)}^*$ for the BTL model involving only items in $I_j$, $j = 1, 2, 3$, assuming that the sub-graphs induced by the $I_j$’s are connected and allowing for a different number of comparisons in each sub-graph. Finally, let $\tilde{\theta} \in \mathbb{R}^n$ be the ensemble MLE based on $\theta_{(1)}, \theta_{(2)}, \theta_{(3)}$. Then

$$d_\infty(\tilde{\theta}, \theta^*) \leq 4 \sum_{i=1}^3 d_\infty(\theta_{(i)}, \theta_{(i)}^*),$$

where $d_\infty(v_1, v_2) := \| v_1 - \text{avg}(v_1) \mathbf{1} - (v_2 - \text{avg}(v_2)) \mathbf{1} \|_\infty$, where $\text{avg}(x) := \frac{1}{|I|} I_n^\top x$ for $x \in \mathbb{R}^n$.

The proof of Lemma 4 is put in Appendix A.5. As we discuss in Section 2.2, for a barbell graph containing two size-$n/2$ complete sub-graphs connected by a single edge, we need $N_{\text{comp}} = \Omega(n^5 \log n)$ for an $o(1)$ error.
bound. From a practical perspective, we note that such a divide and conquer strategy gives flexibility in the number of comparisons in each sub-graph. For example, if we set $L = 1$ for the two complete sub-graphs to get MLEs $\theta_1, \theta_2$, and set $L = n$ for the two items linking the two sub-graphs to get an MLE $\theta_3$, and combine them by shifting $\theta_2$ by the difference of two entries of $\theta_3$, then a total sample complexity $N_{\text{comp}} = \Omega(n^2)$ will ensure $\ell_\infty$-norm error of order $O(e^{2\kappa_\infty} \sqrt{\log n}/\sqrt{n}) = o(1)$, because for a complete graph of size $m$, the $\ell_\infty$-norm error is $O(e^{2\kappa_\infty} \sqrt{\log m}/\sqrt{mL})$.

5 Examples and simulations

In this section, we show with numerical experiments on simulated data that our upper bound is tighter than existing results for a broad range of graph topologies, except for some very sparse and very dense cases. This is in-line with our discussions in Sections 2.1 and 2.2. We explicitly focus our comparisons with the $\ell_\infty$-bound in Yan et al. (2012) here, since their work is closest in spirit to ours. We have also compared our $\ell_\infty$-upper bound with the $\ell_2$-upper bound in Shah et al. (2016) in Appendix A.4 in the appendix. All of our reproducible code is openly accessible.

Dependence on $\kappa / \kappa_E$

In all existing work, the dependence of the upper bound on $\kappa$ or $\kappa_E$ is exponential (i.e., as in $e^\kappa$ or $e^{\kappa_E}$), which could be abnormally large even for moderate values of $\kappa$. Recall that $\kappa$ and $\kappa_E$ are the maximal performance gaps over the whole vertex set $V$, and the edge set $E$, respectively. When the compare scheme is carefully designed, we can make $\kappa_E \ll \kappa$ so that our bound is much tighter than others using $\kappa$. In BTL model, the maximal (pairwise) winning probability is $p_{\text{max}}(\kappa) = 1/(1 + e^{-\kappa})$. To get a sense, $p_{\text{max}}(2.20) = 0.900, p_{\text{max}}(4.59) = 0.990$. A winning probability larger than 0.99 is fairly rare in practice, so it would not be too constraining to set $\kappa = 2.2$ in our simulation. But analytically our result allows $\kappa$ and $\kappa_E$ to diverge with $n$.

In our experiments, we set entries of $\theta^*$ to be $\theta^*_{ij} = \theta^*_i + (i - 1)\delta$ with $\delta = \kappa/(n - 1)$. Under this setting, for some special graphs, e.g., the island graph in Example 5, $\kappa_E$ can be much smaller than $\kappa$, showing an advantage of our upper bound in representing the dependency on the maximal performance gap $\kappa_E$ along the edge set, rather than $\kappa$ the whole vertex set. However, in some cases where the majority of edges has small performance gap and only a few edges have huge edges, the control purely by $\kappa_E$ can again be loose. An interesting future direction is to make upper bounds tighter in such cases by including more structural parameters, like the proportion of small-gap edges. We include some examples in Appendix A.4 for illustration.

Example 5 (Graph with $\min_{i,j} n_{ij} = 0$). In this case, we intend to illustrate that $\min_{i,j} n_{ij}$ could be 0 or quite close to 0 for even fairly dense graphs, making the upper bound in Yan et al. (2012) less effective. Consider a 3-island comparison graph $G$ with $n$ nodes. The induced sub-graphs on node sets $V_1, V_2, V_3$ with $|V_i| = n_i$ are complete graphs, where $V_1 \cap V_2 = \emptyset, V_1 \cup V_2 \cup V_3 = [n], \text{ and } V_i \cap V_j \neq \emptyset \text{ for } i, j = 1, 3$. There is no edge except for those within $V_1, V_2, V_3$. This graph $G$ is connected, and can be fairly dense if we make $n_2$ large, but $\min_{i,j} n_{ij} = 0$ always holds since $V_1 \cap V_3 = \emptyset$ and the two induced sub-graphs are complete. See Figure 1 left panel for a visualization of the adjacency.

Figure 1: Left: Adjacency matrix of a 3-island graph, with yellow indicating 1 and purple indicating 0; $\lambda_2(L_A) = 11.92$. Right: Adjacency matrix of a general island graph, with $n_{\text{island}} = 30, n_{\text{overlap}} = 5, n = 120; \lambda_2(L_A) = 11.19$. Bottom: comparison of real error of MLE and our upper bound using $\kappa$ and $\kappa_E$ while varying $n_{\text{overlap}}$. The upper bound of Yan et al. (2012) is not shown because all graphs satisfy $\min_{i,j} n_{ij} = 0$ and the upper bound is always $\infty$. Note: The error bars are small since the graphs are deterministic and the regularized MLE is very stable.

\footnote{Code repository: https://github.com/MountLee/btl_ell_infinity_general_topo All of the simulation results in this paper were run on a personal laptop with Windows10 OS and Intel Core i7-8550H CPU. The total computation time for a single run is approximately 30 minutes.}
matrix of such a graph.

We can also consider more general island graphs. A general island graph is determined by \( n \), the size of the graph, \( n_{\text{island}} \), the size of island sub-graphs, and \( n_{\text{overlap}} \), the number of overlapped nodes between islands. Each island sub-graph is a complete graph, and there are no edges outside islands. For island graphs, it holds that \( \min_{i,j} n_{ij} = 0 \) and \( \kappa_E \approx n_{\text{island}} / n \). Figure 1 middle panel shows the adjacency matrix of an island graph with \( n_{\text{island}} = 30, n_{\text{overlap}} = 5, n = 120 \). Figure 1 right panel shows the comparison of real error of MLE and our upper bound while varying \( n_{\text{overlap}} \). Every point on the lines is the average of 20 trials. It can be seen that representing the upper bound in \( \kappa_E \) rather than \( \kappa \) can make it much tighter when treating \( \kappa \) and \( \kappa_E \) as divergent parameters instead of constants.

In Example 5, we show a common family of graphs which is fairly dense while \( \min_{i,j} n_{ij} = 0 \), so that the upper bound in Yan et al. (2012) does not hold. Next in Example 6 we consider another family of graphs where their upper bound holds but still looser than our bound.

**Example 6** (Barbell graph with random bridge edges). Consider a generalized Barbell graph \( G \) containing \( n = n_1 + n_2 \) nodes, where the induced sub-graph on nodes \( \{1, \cdots, n_1\} \) and \( \{n_1 + 1, \cdots, n\} \) are complete graphs, and the two sub-graphs are connected by some bridge edges \((i, j)\) for some \( 1 \leq i \leq n_1 \) and \( n_1 + 1 \leq j \leq n \). Denote the set of bridge edges as \( E_i \), then \( |E_i|/(n_1 n_2) \) quantifies the connectivity of \( G \): the larger \( |E_i|/(n_1 n_2) \) is, the denser or more regular \( G \) is.

In Figure 2 we show a comparison of real \( \ell_\infty \)-loss \( \| \hat{\theta} - \theta^* \|_\infty \), and the upper bounds of \( \ell_\infty \)-error in Yan et al. (2012) and our paper. In our experiment, we set \( n_1 = n_2 = 50 \), and randomly link \( |E_i| \) edges between the two complete sub-graphs to make \( |E_i|/(n_1 n_2) \) vary from 0.004 to 0.96. Every point on the lines is the average of 20 trials. Some points near the left end of the orange line are cut because \( \min_{i,j} n_{ij} = 0 \) and the corresponding upper bound is \( \infty \). It can be seen that our upper bound is strictly tighter than Yan et al. (2012) for a wide range \( |E_i|/(n_1 n_2) < 0.8 \), and as expected looser in the extremely dense graph topology.

6 Discussion

In this work we provide a sharp risk analysis of the MLE for the BTL global ranking model, under a more general graph topology, in the \( \ell_\infty \)-loss. This addresses a major gap in the BTL literature, in extending the comparison graph to more general and thus more practical settings. Specifically we derive a novel upper bound for the \( \ell_\infty \) and \( \ell_2 \)-loss of the BTL MLE, showing explicit dependence on the algebraic connectivity of the graph, the sample complexity, and the maximal performance gap between compared items. We also support our analysis by deriving lower bounds for the \( \ell_\infty \)-loss and analyze specific topologies satisfying certain regularity conditions under which the MLE is nearly minimax optimal. Additionally we show that the \( \ell_\infty \)-loss satisfies a unique subadditivity property for the BTL MLE and utilize our derived bounds in the case of efficient tournament design. We note that our upper bound is suboptimal in the cases where the graph topology is extremely sparse or irregular. A good future direction would be to optimize the upper and lower bounds in such comparison graph regimes.

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The Performance of the MLE in the Bradley-Terry-Luce Model in $\ell_\infty$-Loss and under General Graph Topologies: Supplementary Materials

A Appendix

A.1 Comparison of results

This section is a complement to Section 2.1. We will summarize all existing works on the estimation error of Bradley-Terry model in two tables, and then compare our results with the results in Negahban et al. (2017); Agarwal et al. (2018); Hendrickx et al. (2020) in detail.

For simplicity, in Table 2 and Table 3 we use $\kappa$ to replace $B$ for results in Hajek et al. (2014); Shah et al. (2016) as $\kappa \approx B$ when $1^T \theta^* = 0$, and in Table 3 we omit the lower bound as they are usually in fairly complex forms.

| norm   | paper                          | bound                                                                 |
|--------|-------------------------------|----------------------------------------------------------------------|
| $\| \cdot \|_\infty$ | Simons and Yao (1999)          | $p = 1, \lesssim e^\kappa \sqrt{\log n \over nL}$                  |
|        | Yan et al. (2012)              | $\lesssim e^{2\kappa} \sqrt{\log n \over nL}$                      |
|        | Han et al. (2020)              | $\lesssim e^{2\kappa} \sqrt{\log n \over npL}$                     |
|        | Chen et al. (2019), Chen et al. (2020) | $\lesssim e^{2\kappa} \sqrt{\log n \over npL}$                  |
|        | Our work                      | $\lesssim e^{2\kappa} \sqrt{\log n \over npL}$                     |
| $\| \cdot \|_2$   | Hajek et al. (2014)            | $\lesssim e^{8\kappa \log n \over pL}$                             |
|        | Shah et al. (2016)             | $\lesssim e^{8\kappa \log n \over pL}$                             |
|        | Negahban et al. (2017)         | $\lesssim e^{8\kappa \log n \over pL}$                             |
|        | Chen et al. (2019), Chen et al. (2020) | $\lesssim e^{2\kappa} \sqrt{\log n \over nL}$                  |
|        | Our work                      | $\lesssim e^{2\kappa} \sqrt{\log n \over nL}$                     |
| $\sin^2(\cdot, \cdot) (\hat{w})$ | Hendrickx et al. (2020)       | $\lesssim e^{2\kappa} \sqrt{\log n \over nL}$                     |
| $\| \cdot \|_1 (\hat{w})$ | Agarwal et al. (2018)          | $\lesssim e^\kappa \sqrt{\log n \over L}$                        |

Table 2: Comparison of results under ER($n, p$) in literature.

| norm   | paper                          | bound                                                                 |
|--------|-------------------------------|----------------------------------------------------------------------|
| $\| \cdot \|_\infty$ | Yan et al. (2012)              | $\lesssim e^{\nu E \min_{i,j} n_{i,j} \sqrt{n_{\max} \log n \over L}}$ |
|        | Our work                      | $e^{\nu E \min_{i,j} n_{i,j} \sqrt{n_{\max} \log n \over L}} + e^{\nu E \max_{i,j} n_{i,j} \sqrt{n_{\max} \log n \over L}}$ |
| $\| \cdot \|_2$   | Hajek et al. (2014)            | $\lesssim e^{8\kappa \log n \over \lambda_2(Z_A) L}$               |
|        | Shah et al. (2016)             | $\lesssim e^{8\kappa \log n \over \lambda_2(Z_A) L}$               |
|        | Our work                      | $e^{\nu E \max_{i,j} n_{i,j} \sqrt{n_{\max} \log n \over L}}$       |
| $\sin^2(\cdot, \cdot) (\hat{w})$ | Hendrickx et al. (2020)       | $\lesssim e^{2\kappa \nu T(r(Z)) \over L \|w\|_2}$                |
| $\| \cdot \|_1 (\hat{w})$ | Agarwal et al. (2018)          | $\lesssim e^{\nu E \min_{i,j} n_{i,j} \sqrt{n_{\max} \log n \over L}}$ |

Table 3: Comparison of results for a fixed general comparison graph in literature.
In Negahban et al. (2017), they establish an \( \ell_2 \) upper bound for \( \| \hat{\pi} - \pi \|_2 / \| \hat{\pi} \|_2 \) in the order of \( \frac{e^{2.5n}}{\lambda_2(L_{rw})} \sqrt{\frac{n_{\text{max}} \log n}{L}} \), where \( \hat{\pi}(i) := w_i / \sum_j w_j \) with \( w_i = \exp(\theta_i) \), \( \pi \) is the rank centrality estimator of \( \hat{\pi} \), \( \lambda_2 \) refers to the second smallest eigenvalue, \( L_{rw} = D^{-1}A \) (which has the same spectrum as \( D^{-1/2}L_AD^{-1/2} \)), and \( L_A = D - A \). Recall that our \( \ell_2 \) upper bound is for \( \| \hat{\theta} - \theta \|_2 \) and the order is \( e^{2n} / \lambda_2(L_A) \sqrt{n_{\text{max}} / L} \). We can now see that it’s hard to give a general comparison between the two results because 1. for a general graph, there is no precise relationship between \( \lambda_2(L_{rw}) \) and \( \lambda_2(L_A) \); 2. more importantly, for a general model parameter \( \theta \), there is no tight two-sided bound between \( \| \hat{\theta} - \theta \|_2 \) and \( \| \pi - \hat{\pi} \|_2 / \| \hat{\pi} \|_2 \). Although, it would be a very interesting future work to give a tight description of these two relevant pairs of quantities and make a meaningful comparison.

Agarwal et al. (2018) establish an \( \ell_1 \)-norm upper bound for the score parameter \( \hat{\pi}_i := w_i / \sum_j w_j \) with \( w_i = \exp(\theta_i) \). Their bound is of the order \( \frac{n_{\text{avg}}}{\lambda_2(D^{-1}A)n_{\text{min}}} \sqrt{\frac{\log n}{L}} \), where \( n_{\text{avg}} = \sum_{i \in [n]} \hat{\pi}_i n_i \), \( D = \text{diag}(n_1, \ldots, n_n) \), and \( \eta := \log \left( \frac{n_{\text{avg}}}{n_{\text{min}} \pi_{\text{min}}} \right) \) with \( \pi_{\text{min}} = \min_{i \in [n]} \hat{\pi}_i \).

In Hendrickx et al. (2020), they propose a novel weighted least square method to estimate vector \( w \), with \( w_i = \exp(\theta_i) \), and provide delicate theoretical analysis of their method. Their estimator shows a sharp upper bound for \( \mathbb{E} [\sin^2(\hat{w}, w)] \) and equivalently for \( \mathbb{E} \| \hat{w} / \| \hat{w} \|_2 - w / \| w \|_2 \|_2^2 \), in the sense that the upper bound for \( \mathbb{E} [\sin^2(\hat{w}, w)] \) matches an instance-wise lower bound up to constant factors. Such a universal sharp/optimal bound for general graph comparison (although the bound is not in the form of minimax rate) is unique in literature. For convenience of comparison, here we assume \( w_i \)’s are \( O(1) \) (otherwise we can put a factor \( e^{2B} \) in the bound \( \frac{Tr(L_A^1)}{L\| w \|^2} \)). Then their upper bound is of the order \( \frac{Tr(L_A^1)}{L\| w \|^2} \), where \( L_A \) refers to the Moore-Penrose pseudo inverse of the graph Laplacian of the comparison graph. To correct for their different choice of metric, we need to multiply \( \| w \|^2 \) to their bound, and it becomes \( \frac{Tr(L_A^1)}{L} \). On the other hand, the upper bound for expected \( \ell_2 \) loss in Shah et al. (2016) is \( \frac{n}{L\lambda_2(L_A)} \). Since \( \lambda_2(L_A) \) is the smallest positive eigenvalue of \( L_A \), it holds that \( Tr(L_A^1) < n/\lambda_2(L_A) \), and hence the upper bound for expected error in Hendrickx et al. (2020) is tighter than the one in Shah et al. (2016). Although, it should be noted that the loss function in Hendrickx et al. (2020) is not directly comparable to a plain \( \ell_2 \) loss, and we are just doing an approximate comparison.

In our paper, however, we provide a high probability bound for \( \| \hat{\theta} - \theta \|_2 \) in the order of \( \frac{n_{\text{max}} n}{L\lambda_2^2(L_A)} \). It’s usually hard to make a fair comparison between a high probability bound and a bound for expected metrics, but in Hajek et al. (2014); Shah et al. (2016), they also provide a high probability bound in equation (8b) of Theorem 2. As we discussed in Section 2.2 and Section 5, the \( \ell_2 \) bound is not the primary focus of our paper, and although our theoretical analysis is not optimized for \( \ell_2 \) error, our bound is still tighter than the bound in Hajek et al. (2014); Shah et al. (2016) for moderately dense and regular graphs, and is only worse for fairly sparse and irregular graphs.
A.2 Proof in Section 2

Proof of Theorem 1. We will use a gradient descent sequence defined recursively by \( \theta^{(0)} = \theta^\ast \) and, for \( t = 1, 2, \ldots \),

\[
\theta^{(t+1)} = \theta^{(t)} - \eta [\nabla \ell_{\theta}(\theta^{(t)}) + \rho \theta^{(t)}].
\]

Our proof builds heavily on the ideas and techniques developed by Chen et al. (2019) and further extended by Chen et al. (2020), and contains two key steps. In the first step, we control \( \|\theta^{(T)} - \hat{\theta}_\rho\| \) for \( T \) large enough, by leveraging the convergence property of gradient descent for strongly convex functions. In the second step, we control \( \|\theta^{(T)} - \theta^\ast\| \) through a leave-one-out argument. The proof can be sketched as follows:

1. Bound \( \|\theta^{(T)} - \hat{\theta}_\rho\|_\infty \), for large \( T \) using the linear convergence property of gradient descent for strongly-convex and smooth functions.
2. Bound \( \|\theta^{(T)} - \theta^\ast\|_\infty \) for large \( T \) using the leave-one-out argument.
3. Finally, \( \|\hat{\theta}_\rho - \theta^\ast\|_\infty \) is controlled by triangle inequality.

**Step 1.** Bound \( \|\theta^{(T)} - \hat{\theta}_\rho\|_\infty \), for large \( T \).

1. **Linear convergence, orthogonality to \( 1_n \).** We say that a function \( \ell \) is \( \alpha \)-strongly convex if \( \nabla^2 \ell(x) \succeq \alpha I_n \) and \( \beta \)-smooth if \( \|\nabla \ell(x) - \nabla \ell(y)\|_2 \leq \beta \|x - y\|_2 \) for all \( x, y \in \text{dom}(\ell) \). By Lemma 8, we know that \( \ell_{\rho}(\cdot) \) is \( \rho \)-strongly convex and \( (\rho + n_{\max}) \)-smooth. By Theorem 3.10 in Bubeck (2015), we have

\[
\|\theta^{(t)} - \hat{\theta}_\rho\|_2 \leq \left(1 - \frac{\rho}{\rho + n_{\max}}\right)^t \|\theta^{(0)} - \hat{\theta}_\rho\|_2.
\]

Besides, as we start with \( \theta^\ast \) that satisfies \( 1_n^\top \theta^\ast = 0 \), it holds that \( 1_n^\top \theta^{(t)} = 0 \) for all \( t \geq 0 \). To see this, just notice that

\[
\nabla \ell_{\rho}(\theta) = \rho \theta + \sum_{(i,j) \in E} [-\tilde{y}_{ij} + \psi(\theta_i - \theta_j)](e_i - e_j)
\]

and \( 1_n^\top (e_i - e_j) \) for any \( i, j \). Then by \( 1_n^\top \theta^{(t)} = 0, \forall t \) and (10), we have \( 1_n^\top \hat{\theta}_\rho = 0 \).

2. **Control \( \|\theta^\ast - \hat{\theta}_\rho\|_2 \).** By a Taylor expansion, we have that

\[
\ell_{\rho}(\hat{\theta}_\rho; y) = \ell_{\rho}(\theta^\ast; y) + (\hat{\theta}_\rho - \theta^\ast)^\top \nabla \ell_{\rho}(\theta^\ast; y) + \frac{1}{2} (\hat{\theta}_\rho - \theta^\ast)^\top \nabla^2 \ell_{\rho}(\xi; y) (\hat{\theta}_\rho - \theta^\ast),
\]

where \( \xi \) is a convex combination of \( \theta^\ast \) and \( \hat{\theta}_\rho \). By Cauchy-Schwartz inequality,

\[
|(\hat{\theta}_\rho - \theta^\ast)^\top \nabla \ell_{\rho}(\theta^\ast; y)| \leq \|\nabla \ell_{\rho}(\theta^\ast; y)\|_2 \|\theta^\ast - \hat{\theta}_\rho\|_2.
\]

The two inequalities above and the fact that \( \ell_{\rho}(\theta^\ast; y) \geq \ell_{\rho}(\hat{\theta}_\rho; y) \) yield that

\[
\|\theta^\ast - \hat{\theta}_\rho\|_2 \leq \frac{2 \|\nabla \ell_{\rho}(\theta^\ast; y)\|_2}{\rho_{\min}(\nabla^2 \ell_{\rho}(\xi; y))}.
\]

By Lemma 7, \( \|\nabla \ell_{\rho}(\theta^\ast; y)\|_2 \leq \sqrt{\frac{n_{\max}(n+1)}{L}} \). This fact, together with \( \rho_{\min}(\nabla^2 \ell_{\rho}(\xi; y)) \geq \rho \) and \( \rho > \frac{1}{\kappa} \sqrt{\frac{n_{\max}}{L}} \), gives that \( \|\theta^\ast - \hat{\theta}_\rho\|_2 \leq c \kappa \sqrt{n + r} \), for some \( c > 0 \).

3. **Bound \( \|\theta^{(T)} - \hat{\theta}_\rho\|_2 \).** Take \( T = \lceil \kappa^2 e^{4\kappa_E n^6} \rceil \) and remember that \( L \leq \kappa^2 e^{4\kappa_E n^6} \). The previous two steps imply that

\[
\|\theta^{(T)} - \hat{\theta}_\rho\|_2 \leq c(1 - \frac{\rho}{\rho + n_{\max}})^T \kappa \sqrt{n + r} \leq c \exp \left( - \frac{T \rho}{\rho + n_{\max}} \right) \kappa \sqrt{n + r}.
\]
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Let $\tilde{f}_d = \frac{e^{\kappa E}}{\lambda_2(\mathbf{A})} \frac{n_{\text{max}}}{n_{\text{min}}} \sqrt{\frac{n + r}{L}} + \frac{e^{\kappa E}}{\lambda_2(\mathbf{A})} \sqrt{\frac{n_{\text{max}}(\log(n + r))}{L}}$ and consider inequality $e^{-g} \kappa \sqrt{n + r} \leq \tilde{f}_d$. The solution is given by $g \geq \log \kappa + \frac{1}{2} \log(n + r) - \log \tilde{f}_d$ and the inequality holds as long as $g \geq \kappa + 6 \log n + 3 \log \kappa$ since $L \leq \max\{1, \kappa\} e^{3\kappa E n^6}$. Take $g = \kappa + 5 \log n + \log \kappa$, then as long as

$$T \rho \geq 2 n_{\text{max}} n g,$$

it holds that $T \rho > \frac{1}{2} g(\rho + n_{\text{max}})$, then $\|\theta(T) - \hat{\theta}_\rho\|_2 \leq c \exp(-g) \kappa \sqrt{n + r}$ is smaller than $\hat{C}_d \tilde{f}_d$ for some constant $\hat{C}_d$. Since $T = [\kappa^2 e^{3\kappa E} n^6]$ and $\rho \geq \frac{\kappa}{\kappa^2 e^{3\kappa E} n^6}$, we have

$$T \rho \geq c_\rho \kappa e^{\kappa E} n^2 \sqrt{n_{\text{max}}} \geq 2 n_{\text{max}} n g.$$

In conclusion, we have

$$\|\theta(T) - \hat{\theta}_\rho\|_2 \leq \|\theta(T) - \hat{\theta}_\rho\|_2 \leq \hat{C}_d \tilde{f}_d.$$

The arguments above also hold with $\tilde{f}_a = \frac{e^{\kappa E}}{\lambda_2(\mathbf{A})} \sqrt{\frac{n_{\text{max}}(n + r)}{L}}$, i.e., we have $\|\theta(T) - \hat{\theta}_\rho\|_2 \leq \hat{C}_a \tilde{f}_a$ for some constant $\hat{C}_a$.

**Step 2.** Bound $\|\theta(T) - \theta^*\|_\infty$ by a leave-one-out argument.

Denote $\psi(x) = \frac{1}{1 + e^x}$ and $r = \log \kappa + \kappa E$, and define the leave-one-out negative log-likelihood as

$$
\ell_n^{(m)}(\theta) = \sum_{1 \leq i < j \leq n, i,j \neq m} A_{ij} \left[ \frac{1}{\psi(\theta_i - \theta_j)} + \log \frac{1}{\psi(\theta_i - \theta_j)} \right] + \sum_{j \in [n] \setminus \{m\}} A_{mj} \left[ \psi(\theta_m - \theta_j) \log \frac{1}{\psi(\theta_m - \theta_j)} + \psi(\theta_m - \theta_j) \log \frac{1}{\psi(\theta_m - \theta_j)} \right],
$$

so the leave-one-out gradient descent sequence is, for $t = 0, 1, \ldots$,

$$\theta^{(t+1,m)} = \theta^{(t,m)} - \eta \left( \nabla \ell_n^{(m)}(\theta^{(t,m)}) + \rho \theta^{(t,m)} \right).$$

We initialize both sequences by $\theta^{(0)} = \theta^{(0,m)} = \theta^*$ and use step size $\eta = \frac{1}{\rho + n_{\text{max}}}$. By assumption 2, $\lambda_2(\mathbf{A}) > 0$, so we can let $f_a = C_a \frac{e^{\kappa E}}{\lambda_2(\mathbf{A})} \sqrt{\frac{n_{\text{max}}(n + r)}{L}} + \rho \kappa \kappa \sqrt{n}$, $f_b = 10 e^{\kappa E} \frac{\sqrt{n_{\text{max}}}}{n_{\text{min}}}$, $f_c = C_c \frac{e^{\kappa E}}{\lambda_2(\mathbf{A})} \sqrt{\frac{n_{\text{max}}(\log(n + r))}{L}}$, $f_d = f_a + f_c$ with sufficiently large constant $C_c > 0$ and $\hat{C}_c \gg C_c$. By assumption 1, we have $f_c + f_d \leq 0.1$. We will show in Lemma 10, 11, 12, 13 that for all $0 \leq t \leq T = [\kappa^2 e^{3\kappa E} n^6]$

$$\|\theta^{(t)} - \theta^*\|_2 \leq f_a,$$

$$\max_{m \in [n]} \|\theta^{(t,m)} - \theta^*\|_2 \leq f_b,$$

$$\max_{m \in [n]} \|\theta^{(t,m)} - \theta^{(t)}\|_2 \leq f_c,$$

$$\|\theta^{(t)} - \theta^*\|_\infty \leq f_d.$$

When $t = 0$, (12) holds since $\theta^{(0)} = \theta^{(0,m)} = \theta^*$. By Lemma 10, 11, 12, 13, and a union bound, we know that (12) holds for all $0 \leq t \leq T = [\kappa^2 e^{3\kappa E} n^6]$ with probability at least $1 - O(n^{-4})$. Therefore, using the result in step 1, we have

$$\|\hat{\theta}_\rho - \theta^*\|_\infty \leq \|\theta(T) - \theta^*\|_\infty \leq \|\theta(T) - \theta^*\|_2 \leq 2 f_d.$$

As byproduct, we have

$$\|\hat{\theta}_\rho - \theta^*\|_2 \leq \|\hat{\theta}_\rho - \theta(T)\|_2 + \|\theta(T) - \theta^*\|_2 \leq 2 f_a.$$

**Lemma 7.** With probability at least $1 - O(\kappa^{-2} e^{-3\kappa E} n^{-10})$ the gradient of the regularized log-likelihood satisfies

$$\|\nabla \ell_\rho(\theta^*)\|_2^2 \lesssim \frac{\max(n + r)}{L} + \rho \kappa \kappa \sqrt{n}.$$

In particular, for $\rho \geq \frac{1}{\kappa(\theta^*)^2} \sqrt{\frac{n_{\text{max}}}{L}}$, we have $\|\nabla \ell_\rho(\theta^*)\|_2^2 \lesssim \frac{\max(n + r)}{L}$.
\begin{proof}

Triangle inequality gives
\[ \| \nabla \ell_\rho(\theta^*) \|_2 \leq \| \nabla \ell_0(\theta^*) \|_2 + \rho \| \theta^* \|_2. \]

By definition of \( \kappa(\theta^*) \), we have \( \| \theta^* \|_2 \leq \sqrt{n} \kappa(\theta^*) \). For the first term, by Lemma 14 we have
\[ \| \nabla \ell_0(\theta^*) \|_2^2 = \sum_{i=1}^{n} \left[ \sum_{j \in A(i)} \left[ \tilde{y}_{ij} - \psi(\theta_i^* - \theta_j^*) \right] \right]^2 \leq C_1 n_{\text{max}} \left( \frac{n + r}{L} \right). \]
\end{proof}

\textbf{Lemma 8.} Let \( \kappa_E(x) = \max_{(i,j) \in E} |x_i - x_j| \), then \( \forall \theta \in \mathbb{R}^n \),
\[ \lambda_{\text{max}}(\nabla^2 \ell_\rho(\theta; y)) \leq \rho + \frac{1}{2} n_{\text{max}} \]
\[ \lambda_2(\nabla^2 \ell_\rho(\theta; y)) \geq \rho + \frac{1}{4e^{\kappa_E(\theta)}} \lambda_2(\mathcal{L}_A). \]

In particular, we have
\[ \lambda_2(\nabla^2 \ell_\rho(\theta; y)) \geq \rho + \frac{1}{4e^{\kappa_E(\theta^*)}e^{2\|\theta - \theta^*\|_\infty}} \lambda_2(\mathcal{L}_A). \]
\end{proof}

\textbf{Proof.} Use the fact that
\[ \nabla \ell_0(\theta; y) = \sum_{(i,j) \in E} \frac{e^{\theta_i}e^{\theta_j}}{(e^{\theta_i} + e^{\theta_j})^2} (e_i - e_j)(e_i - e_j)^T, \]
and \( \forall (i, j) \in E, \frac{1}{4e^{\kappa_E(\theta)}} \leq \frac{e^{\theta_i}e^{\theta_j}}{(e^{\theta_i} + e^{\theta_j})^2} \leq \frac{1}{4}, \kappa_E(x_1) \leq \kappa_E(x_2) + 2\|x_1 - x_2\|_\infty \). In addition, the largest eigenvalue of graph Laplacian satisfies (Corollary 3.9.2 in \textit{Brouwer and Haemers (2012)})
\[ \lambda_{\text{max}}(\mathcal{L}_A) \leq \max_{(i,j) \in E} (n_i + n_j) \leq 2n_{\text{max}}. \]
\end{proof}

\textbf{Lemma 9.} Provided that (12) holds, then
\[ \max_{m \in [n]} \| \theta^{(t+1,m)} - \theta^* \|_\infty \leq f_c + f_d, \]
\[ \max_{m \in [n]} \| \theta^{(t+1,m)} - \theta^* \|_2 \leq f_c + f_a. \]
\end{proof}

\textbf{Proof.} By triangle inequality, we have
\[ \max_{m \in [n]} \| \theta^{(t,m)} - \theta^* \|_\infty \leq \max_{m \in [n]} \| \theta^{(t,m)} - \theta^{(t)} \|_\infty + \| \theta^{(t)} - \theta^* \|_\infty \leq f_c + f_d \]
\[ \max_{m \in [n]} \| \theta^{(t,m)} - \theta^* \|_2 \leq \max_{m \in [n]} \| \theta^{(t,m)} - \theta^{(t)} \|_2 + \| \theta^{(t)} - \theta^* \|_2 \leq f_c + f_a. \]
\end{proof}

\textbf{Lemma 10.} Suppose (12) holds, and the step size satisfies \( 0 < \eta \leq \frac{1}{\rho + n_{\text{max}}} \). If
\[ f_d \leq 0.1 \quad \text{and} \quad f_a \geq C_a \left( \frac{e^{\kappa_E(\theta^*)}}{\lambda_2(\mathcal{L}_A)} \right) \left[ \frac{n_{\text{max}}(n + r)^3}{L} + \rho \kappa(\theta^*) \sqrt{n} \right], \]
for some large constant \( C_a \), then with probability at least \( 1 - O(\kappa^{-2}e^{-3\kappa E n^{-10}}) \) we have
\[ \| \theta^{(t+1)} - \theta^* \|_2 \leq f_a. \]
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**Proof.** By the form of the gradient descent, we have that
\[
\theta^{(t+1)} - \theta^* = \theta^{(t)} - \eta \nabla \ell_\rho(\theta^{(t)}) - \theta^* = \theta^{(t)} - \eta \nabla \ell_\rho(\theta^{(t)}) - (\theta^* - \eta \nabla \ell_\rho(\theta^*)) = \left[ I_n - \eta \int_0^1 \nabla^2 \ell_\rho(\theta(t)) \, dt \right] (\theta^{(t)} - \theta^*) = \eta \nabla \ell_\rho(\theta^*),
\]
where $\theta(t) = \theta^* + \tau(\theta^{(t)} - \theta^*)$. Letting $H = \int_0^1 \nabla^2 \ell_\rho(\theta(t)) \, dt$, by the triangle inequality,
\[
\|\theta^{(t+1)} - \theta^*\|_2 \leq \|I_n - \eta H(\theta^{(t)} - \theta^*)\|_2 + \|\nabla \ell_\rho(\theta^*)\|_2. \tag{16}
\]
Setting $\kappa_E(x) = \max_{(i,j) \in E} |x_i - x_j|$, then, for sufficiently small $\epsilon$, we have that
\[
\kappa_E(\theta(t)) \leq \kappa_E(\theta^*) + 2\|\theta^{(t)} - \theta^*\|_\infty \leq \kappa_E(\theta^*) + \epsilon. \tag{17}
\]
as long as
\[
2f_d \leq \epsilon. \tag{18}
\]
Then, by Lemma 8 and setting $\epsilon = 0.2$, for any $\tau \in [0, 1],
\[
\rho + \frac{\lambda_2(\mathcal{L}_A)}{10e^{\kappa_E(\theta^*)}} \leq \rho + \frac{\lambda_2(\mathcal{L}_A)}{8e^{\kappa_E(\theta^*)}} \leq \lambda_2(\nabla^2 \ell_\rho(\theta(t))) \leq \lambda_{\max}(\nabla^2 \ell_\rho(\theta(t))) \leq \rho + \frac{1}{2n_{\max}}. \tag{19}
\]
Since $1_n(\theta^{(t)} - \theta^*) = 0$, we obtain that
\[
\|I_n - \eta H(\theta^{(t)} - \theta^*)\|_2 \leq \max\{1 - \eta \lambda_2(\mathcal{L}_A), 1 - \eta \lambda_{\max}(\mathcal{L}_A)\} \|\theta^{(t)} - \theta^*\|_2. \tag{20}
\]
By (19) and the fact that $\eta \leq \frac{1}{\rho + n_{\max}}$, we get
\[
\|I_n - \eta H(\theta^{(t)} - \theta^*)\|_2 \leq (1 - \frac{\eta \lambda_2(\mathcal{L}_A)}{10e^{\kappa_E(\theta^*)}}) \|\theta^{(t)} - \theta^*\|_2. \tag{21}
\]
By Lemma 7 and the induction hypothesis, we have
\[
\|\theta^{(t+1)} - \theta^*\|_2 \leq (1 - \frac{\eta \lambda_2(\mathcal{L}_A)}{10e^{\kappa_E(\theta^*)}}) f_a + C_\eta \left[ \sqrt{\frac{n_{\max}(n + r)}{L}} + \rho \kappa(\theta^*) \sqrt{n} \right] \leq f_a \tag{22}
\]
as long as
\[
f_a \geq C_a \frac{e^{\kappa_E(\theta^*)}}{\lambda_2(\mathcal{L}_A)} \left[ \sqrt{\frac{n_{\max}(n + r)}{L}} + \rho \kappa(\theta^*) \sqrt{n} \right] \tag{23}
\]
for some large constant $C_a$. \hfill \Box

**Lemma 11.** Suppose (12) holds and assume that
\begin{enumerate}
  \item $f_a = C_a\frac{e^{\kappa_E(\theta^*)}}{\lambda_2(\mathcal{L}_A)} \left[ \sqrt{\frac{n_{\max}(n + r)}{L}} + \rho \kappa(\theta^*) \sqrt{n} \right]$, $f_c = C_c\frac{e^{\kappa_E(\theta^*)}}{\lambda_2(\mathcal{L}_A)} \sqrt{\frac{n_{\max}(\log(n + r))}{L}}$ with $C_a \gg C_c$.
  \item $\frac{n_{\min}}{10e^{\kappa_E(\theta^*)}} f_b \geq 3\sqrt{n_{\max}} \frac{1}{4} f_a$.
  \item $f_c + f_d \leq 0.1$.
\end{enumerate}
then as long as the step size satisfies $0 < \eta \leq \frac{1}{\rho + n_{\max}}$, with probability at least $1 - O(\kappa^{-2}e^{-3\kappa_E n^{-10}})$ we have
\[
\max_{m \in [n]} |\theta_n^{(t+1,m)} - \theta_n^*| \leq f_b.$
Proof. Recall that the gradient descent step for leave-one-out estimator $\theta^{(m)}$ is defined as

$$\theta^{(t+1,m)} = \theta^{(t,m)} - \eta \left( \nabla \ell_n^{(m)} \left( \theta^{(t,m)} \right) + \rho \theta^{(t,m)} \right),$$

where

$$\ell_n^{(m)}(\theta) = \sum_{1 \leq i < j \leq n, i \neq m} A_{ij} \left[ \bar{y}_{ij} \log \frac{1}{\psi(\theta_i - \theta_j)} + (1 - \bar{y}_{ij}) \log \frac{1}{1 - \psi(\theta_i - \theta_j)} \right] + \sum_{j \in [n] \setminus \{m\}} A_{mj} \left[ \psi(\theta_m^{*} - \theta^*_j) \log \frac{1}{\psi(\theta_m - \theta_j)} + \psi(\theta^*_m - \theta^*_j) \log \frac{1}{\psi(\theta_m - \theta_j)} \right].$$

Direct calculations give

$$[\nabla \ell_n^{(m)}(\theta)]_m = \sum_{j \in [n] \setminus \{m\}} A_{mj} \left[ \psi(\theta_m^{*} - \theta^*_j)(\psi(\theta_m^{*} - \theta^*_j) - 1) + (1 - \psi(\theta_m^{*} - \theta^*_j))\psi(\theta_m - \theta_j) \right]$$

$$= \sum_{j \in [n] \setminus \{m\}} A_{mj} \left[ -\psi(\theta_m^{*} - \theta^*_j) + \psi(\theta_m - \theta_j) \right].$$

Thus, we have

$$\theta^{(t+1,m)} - \theta^*_m = \left( 1 - \eta \rho - \eta \right) \sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j) \left( \theta^{(t,m)} - \theta^*_m \right) - \rho \eta \theta^*_m + \eta \sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j) (\theta^{(t,m)} - \theta^*_j),$$

where $\xi_j$ is a scalar between $\theta^*_m - \theta^*_j$ and $\theta^{(t,m)} - \theta^{(t,m)}$. Notice that $\psi'(x) = \frac{e^x}{(1 + e^x)^2} \leq \frac{1}{2}$ for any $c \in \mathbb{R}$, thus by Cauchy-Schwartz inequality we have

$$\left| \sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j) (\theta^{(t,m)} - \theta^*_j) \right| \leq \frac{1}{4} \sqrt{n_{\text{max}}} \| \theta^{(t,m)} - \theta^* \|_2. \quad (24)$$

Also, since $\eta \leq \frac{1}{\rho + n_{\text{max}}}$,

$$1 - \eta \rho - \eta \sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j) \geq 1 - \eta \rho - \eta \frac{n_{\text{max}}}{4} \geq 0.$$

Therefore,

$$0 \leq 1 - \eta \rho - \eta \sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j) \leq 1 - \eta \min_{j \in \mathcal{N}(m)} \psi'(\xi_j).$$

Since $\xi_j$ is a scalar between $\theta^*_m - \theta^*_j$ and $\theta^{(t,m)} - \theta^{(t,m)}$, we have

$$\max_{j \in \mathcal{N}(m)} |\xi_j| \leq \max_{j \in \mathcal{N}(m)} |\theta^*_m - \theta^*_j| + \max_{j \in \mathcal{N}(m)} |\theta^*_m - \theta^*_j| - (\theta^{(t,m)} - \theta^{(t,m)})|$$

$$\leq \kappa_\psi(\theta^*) + 2\|\theta^{(t,m)} - \theta^*\|_\infty \leq \kappa_\psi(\theta^*) + \epsilon$$

as long as

$$\|\theta^{(t,m)} - \theta^*\|_\infty \leq f_c + f_d \leq \epsilon/2. \quad (25)$$

Let $\epsilon = 0.2$, then $e^\epsilon \leq 5/4$ and

$$\psi'(\xi_j) = \frac{e^{\xi_j}}{(1 + e^{\xi_j})^2} = \frac{e^{-|\xi_j|}}{(1 + e^{-|\xi_j|})^2} \geq \frac{e^{-|\xi_j|}}{4} \geq \frac{1}{4e^\epsilon} \geq \frac{1}{5e^{\kappa_\psi(\theta^*)}}.$$

By triangle inequality we get

$$|\theta^{(t+1,m)} - \theta^*_m| \leq \left( 1 - \frac{\eta \min_{j \in \mathcal{N}(m)}}{10e^{\kappa_\psi(\theta^*)}} \right) |\theta^{(t,m)} - \theta^*_m| + \rho \eta \|\theta^*\|_\infty + \eta \frac{\sqrt{n_{\text{max}}}}{4} \|\theta^{(t,m)} - \theta^*\|_2$$

$$\leq f_b - \frac{\eta \min_{j \in \mathcal{N}(m)}}{10e^{\kappa_\psi(\theta^*)}} f_b + \eta \rho \epsilon(\theta^*) + \eta \frac{\sqrt{n_{\text{max}}}}{4} (f_a + f_c) \leq f_b \quad (26)$$
as long as 
\[ \frac{n_{\min}}{10e^{\kappa E(\theta^*)}} f_b \geq \rho \kappa (\theta^*) + \frac{\sqrt{n_{\max}}}{4} (f_a + f_c). \]

By assumption, \( f_a = C_a e^{\kappa E(\theta^*)} \left[ \sqrt{n_{\max}(n+r)} + \rho \kappa (\theta^*) \sqrt{n} \right] \), \( f_c = C_c e^{\kappa E(\theta^*)} \sqrt{n_{\max}(\log n + r)} \) with \( C_a \gg \max\{C_c, 1\} \), so
\[ f_a \gg f_c, \quad \text{and} \quad \frac{\sqrt{n_{\max}}}{4} f_a \gg \frac{n_{\max}}{\lambda_2(L_A)} \left[ \sqrt{\frac{n+r}{L}} + \sqrt{\frac{n}{n_{\max}}} \rho \kappa (\theta^*) \right] \geq \rho \kappa (\theta^*). \]

Therefore, a sufficient condition for \( |\theta_m^{(t+1,m)} - \theta_m^*| \leq f_b \) is
\[ \frac{n_{\min}}{10e^{\kappa E(\theta^*)}} f_b \geq 3 \frac{\sqrt{n_{\max}}}{4} f_a, \]
which is satisfied by our assumption.

\[ \square \]

\textbf{Lemma 12.} \textit{Suppose (12) holds with} \( f_c = C_c e^{\kappa E(\theta^*)} \sqrt{n_{\max}(\log n + r)} \) \textit{for some sufficiently large constant} \( C_c \), \( f_d = f_b + f_c \), \textit{then as long as the step size satisfies} \( 0 < \eta \leq \frac{1}{\rho + n_{\max}} \), \textit{with probability at least} \( 1 - O(\kappa^{-2}e^{-3\kappa E n^{-10}}) \) \textit{we have}
\[ \max_{m \in [n]} \|\theta_m^{(t+1,m)} - \theta_m^*\|_2 \leq f_c. \]

\textit{Proof.} By the update rules, we have
\begin{align*}
\theta^{(t+1)} - \theta_m^{(t+1,m)} = &\theta^{(t)} - \eta \nabla \ell_{\rho}(\theta^{(t)}) - \left[ \theta_m^{(t,m)} - \eta \nabla \ell_{\rho}(\theta^{(t,m)}) \right] \\
= &\theta^{(t)} - \eta \nabla \ell_{\rho}(\theta^{(t)}) - \left[ \theta_m^{(t,m)} - \eta \nabla \ell_{\rho}(\theta^{(t,m)}) \right] \\
= &\eta \left[ \nabla \ell_{\rho}(\theta^{(t,m)}) - \nabla \ell_{\rho}(\theta^{(t,m)}) \right] \\
= &v_1 - v_2,
\end{align*}

where
\[ v_1 = \left[ I_n - \eta \int_0^1 \nabla^2 \ell_{\rho}(\theta(\tau)) d\tau \right] (\theta^{(t)} - \theta^{(t,m)}) \quad \text{and} \quad v_2 = \eta \left[ \nabla \ell_{\rho}(\theta^{(t,m)}) - \nabla \ell_{\rho}(\theta^{(t,m)}) \right].\]

Now following the same arguments towards (21), as long as \( \eta \leq \frac{1}{\rho + n_{\max}} \), we can get
\[ \|v_1\|_2 \leq (1 - \frac{\eta \lambda_2(L_A)}{10e^{\kappa E(\theta^*)}})\|\theta^{(t)} - \theta^{(t,m)}\|_2. \]

For \( v_2 \), we know that
\[ \frac{1}{\eta} v_2 = \sum_{i \in [n] \setminus \{m\}} A_{mi} \left[ \psi(\theta_i^{(t,m)} - \theta_m^{(t,m)}) - \bar{y}_{im} \right] - \sum_{i \in [n] \setminus \{m\}} A_{mi} \left[ \psi(\theta_i^{(t,m)} - \theta_m^{(t,m)}) - \psi(\theta_i^* - \theta_m^*) \right] (e_i - e_m). \]

By the form of the derivatives and Lemma 14, we know that with probability at least \( 1 - O(n^{\kappa^{-2}e^{-3\kappa E n^{-10}}}) \),
\begin{align*}
\left\| \frac{1}{\eta} v_2 \right\|_2^2 = &\sum_{i \in [n] \setminus \{m\}} A_{im} \left( \bar{y}_{im} - \psi(\theta_i^* - \theta_m^*) \right)^2 + \sum_{i \in [n] \setminus \{m\}} A_{im} \left( \bar{y}_{im} - \psi(\theta_i^* - \theta_m^*) \right)^2 \\
\leq &\frac{n_{\max}(\log n + r)}{L} + \frac{\log n + n_{\max} + r}{L} \leq \frac{n_{\max}(\log n + r)}{L}.
\end{align*}
Therefore, we have

\[
\|\theta^{(t+1)} - \theta^{(t,m)}\|_2 \leq \|v_1\|_2 + \|v_2\|_2
\]

\[
\leq (1 - \frac{\eta \lambda_2(LA)}{10e^{\kappa E(\theta^*)}})\|\theta^{(t)} - \theta^{(t,m)}\|_2 + C\eta \sqrt{\frac{n_{\max}(\log n + r)}{L}}
\]

\[
\leq (1 - \frac{\eta \lambda_2(LA)}{10e^{\kappa E(\theta^*)}})f_c + C\eta \sqrt{\frac{n_{\max}(\log n + r)}{L}} \leq f_c,
\]

where the last inequality is due to the fact that \(C_c\) is a sufficiently large constant by our assumption and

\[
\frac{\eta \lambda_2(LA)}{30e^{\kappa E(\theta^*)}} f_c \geq C\eta \sqrt{\frac{n_{\max}(\log n + r)}{L}} \iff f_c = C\frac{e^{\kappa E(\theta^*)}}{\lambda_2(LA)} \sqrt{\frac{n_{\max}(\log n + r)}{L}}.
\]

\[
(28)
\]

Lemma 13. Suppose (12) holds and \(f_d \geq f_b + f_c\), then with probability at least \(1 - O(\kappa^{-2}e^{-3\kappa E}n^{-10})\) we have

\[
\|\theta^{(t+1)} - \theta^*\|_\infty \leq f_d.
\]

Proof. By Lemma 11 and Lemma 12 we have

\[
|\theta_m^{(t+1)} - \theta_m^*| \leq |\theta_m^{(t+1)} - \theta_m^{(t+1,m)}| + |\theta_m^{(t+1,m)} - \theta_m^*|
\]

\[
\leq \|\theta^{(t+1)} - \theta^{(t+1,m)}\|_2 + |\theta_m^{(t+1,m)} - \theta_m^*| \leq f_c + f_b \leq f_d,
\]

since \(f_d \geq f_b + f_c\) by our assumption. \(\square\)

Lemma 14. With probability at least \(1 - O(\kappa^{-2}e^{-3\kappa E}n^{-10})\) it holds that

\[
\max_{i \in [n]} \left[ \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right]^2 \leq C \frac{(\log n + r) \cdot n_{\max}}{L},
\]

\[
\sum_{i=1}^{n} \left[ \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right]^2 \leq C \frac{(n + r) \cdot n_{\max}}{L}.
\]

\[
\max_{i \in [n]} \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)]^2 \leq C \frac{\log n + n_{\max} + r}{L},
\]

where \(r = \log \kappa + \kappa E\).

Proof. To prove the first inequality, notice that by Hoeffding’s inequality we have

\[
P\left( \sum_{j < i} A_{ij}[\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right) \geq \sqrt{\frac{8n_{\max}(\log n + r)}{L}} \leq 2\exp\left( - \frac{2L}{n_{\max}} \cdot \frac{8n_{\max}(\log n + r)}{L} \right) = 2\kappa^{-2}e^{-3\kappa E}n^{-12},
\]

where \(r = \log \kappa + \kappa E\). By union bound we know that on an event \(B\) with probability at least \(1 - \kappa^{-2}e^{-3\kappa E}n^{-10}\) we have \(\forall i \in [n], \sum_{j < i} A_{ij}[\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)]^2 \leq \frac{8n_{\max}(\log n + r)}{L} \).

Next we prove the second inequality. Consider the unit ball \(S = \{ v \in \mathbb{R}^n: \sum_{i \in [n]} v_i^2 = 1 \} \) in \(\mathbb{R}^n\). By Lemma 5.2 of Vershynin (2011), we can pick a subset \(\mathcal{U} \subset S\) so that \(\log |\mathcal{U}| \leq cn\) and for any \(v \in S\), there exists a vector \(u \in \mathcal{U}\) such that \(\|u - v\|_2 \leq \frac{1}{2}\). For a given \(v \in S\), pick \(u \in \mathcal{U}\) such that \(\|u - v\|_2 \leq \frac{1}{2}\) and we have

\[
\sum_{i=1}^{n} v_i \left[ \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right] = \sum_{i=1}^{n} u_i \left[ \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right] + \sum_{i=1}^{n} (v_i - u_i) \left[ \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right]
\]

\[
\leq \sum_{i=1}^{n} u_i \left[ \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right] + \frac{1}{2} \left( \sum_{i=1}^{n} \left[ \sum_{j \in \mathcal{N}(i)} [\bar{y}_{ij} - \psi(\theta_i^* - \theta_j^*)] \right]^2 \right).
\]
Taking maximum over $v$ and the left hand side can achieve \( \sqrt{\sum_{i=1}^{n} \left[ \sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)] \right]^2} \), thus we have

\[
\sum_{i=1}^{n} \left[ \sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)] \right] \leq 2 \max_{u \in \mathcal{U}} \sum_{i=1}^{n} u_i \left[ \sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)] \right] = 2 \max_{u \in \mathcal{U}} \sum_{i < j} A_{ij} (u_i - u_j) [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)].
\]

To apply Hoeffding’s inequality and union bound, we should account for \( |\mathcal{U}| \leq e^{cn} \leq e^{cn} \) and for any \( v \in \mathcal{V}_i \), there exists a vector \( u \in \mathcal{U}_i \) such that \( ||u - v||_2 \leq \frac{1}{2} \). For a given \( v \in \mathcal{V}_i \), pick \( u \in \mathcal{U}_i \) such that \( ||u - v||_2 \leq \frac{1}{2} \) and we have

\[
\sum_{j \in N(i)} v_{ij} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)] = \sum_{j \in N(i)} u_{ij} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)] + \sum_{j \in N(i)} (v_{ij} - u_{ij}) [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)] \leq \sum_{j \in N(i)} u_{ij} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)] + \frac{1}{2} \sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)]^2.
\]

Taking maximum over $v$ and the left hand side can achieve \( \sqrt{\sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)]^2} \), thus we have

\[
\sqrt{\sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)]^2} \leq 2 \max_{u \in \mathcal{U}_i} \sum_{j \in N(i)} u_{ij} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)].
\]

Therefore,

\[
\sqrt{\max_{i \in [n]} \sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)]^2} \leq 2 \max_{i \in [n]} \max_{u \in \mathcal{U}_i} \sum_{j \in N(i)} u_{ij} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)].
\]

Now a straightforward application of Hoeffding’s inequality and union bound gives

\[
\max_{i \in [n]} \sum_{j \in N(i)} [\bar{y}_{ij} - \psi(\theta^*_i - \theta^*_j)]^2 \leq C \frac{1}{L} \left[ \log n + n_{\max} + \kappa_E + \log \kappa \right]
\]

with probability at least \( 1 - O(\kappa^{-2}e^{-3\kappa_E n^{-1}}) \). \( \square \)

**Proof of Corollary 2.** For an ER(n, p) graph $G$ with $p \geq c \frac{\log n}{n}$ for some large $c > 0$, it holds with probability at least \( 1 - O(n^{-10}) \) that $G$ is connected, and

\[
\frac{1}{2} np \leq n_{\min} \leq n_{\max} \leq 2np,
\]

and

\[
\lambda_2 (\mathcal{L}_A) = \min_{u \neq 0, \|u\|^2 = 1} u^\top \mathcal{L}_A u \geq \frac{np}{2}.
\]

The proof can be seen in either Chun et al. (2019) or Chen et al. (2020). Thus, by a union bound, we can replace the corresponding quantities in upper bounds in Theorem 1 and get the high probability bounds in Corollary 2. \( \square \)
A.3 Proof in Section 3

Let $F(t) = \frac{1}{1+e^{-t}}$, then the Bradley-Terry model can be written as $p_{ij} = F[(w_i^* - w_j^*)/\sigma]$ with $\sigma = 1$. We have $\max_{i \in [0,2\kappa/\sigma]} F'(t) = F'(0) = 1/4$, so $\zeta$ in Lemma 19 satisfies

$$\zeta = c e^{\frac{c}{2\kappa}}.$$ 

Moreover, we denote $W := \{\theta^* | \|\theta^*\|_\infty \leq B\}$. Since $1^n_\ast \theta^* = 0$, we have $B \geq \kappa := \max_{i,j} |\theta_i^* - \theta_j^*|$ (in general, there is no more information, but for some special cases, e.g., when entries of $\theta^*$ are equal-spaced, we have $\kappa = 2B$). In what follows in this section, we still use quantity $\sigma$ for generality, but keep in mind that in our setting $\sigma = 1$. We still use $B$ to differ it from $\kappa$.

A.3.1 Background and Required Results

Before writing our proof of Theorem 3 we note down specific results from (Shah et al., 2016) and other sources that we will use repeatedly.

**Definition 15** (Pairwise $(\delta, \beta)-$packing set from (Shah et al., 2016)). Let $\theta \in \mathbb{R}^n$ be a parameter to be estimated, as indexed over a class of probability distributions $\mathcal{P} := \{\mathbb{P}_\theta | \theta \in W\}$, and let $\rho : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be a pseudo-metric. Suppose there exist a finite set of $M$ vectors $\{\theta^1, \ldots, \theta^M\}$ such that the following conditions hold:

$$\min_{j,k \in [M]} \rho (\theta^j, \theta^k) \geq \delta \text{ and } \frac{1}{2M} \sum_{j,k \in [M]} D_{KL}(\mathbb{P}_{\theta^j} || \mathbb{P}_{\theta^k}) \leq \beta$$

Then we refer to $\{\theta^1, \ldots, \theta^M\}$ as $(\delta, \beta)-$packing set.

**Lemma 16** (Fano minimax lower bound). Suppose that we can construct a $(\delta, \beta)-$packing set with cardinality $M$, then the minimax risk is lower bounded as:

$$\inf_{\delta} \sup_{\theta^* \in W} \mathbb{E} \left[ \rho \left( \hat{\theta}, \theta^* \right) \right] \geq \frac{\delta}{2} \left( 1 - \frac{\beta + \log 2}{\log M} \right)$$

**Proof.** See (Yu, 1997, Lemma 3) for details. \qed

**Lemma 17** (Equivalence of $\|\cdot\|_\infty$ and $\|\cdot\|_2$ norms). Given any vector $\theta \in \mathbb{R}^n$, with $n \in \mathbb{N}$ fixed, the following inequalities holds:

$$\frac{1}{\sqrt{n}} \|\theta\|_2 \leq \|\theta\|_\infty \leq \|\theta\|_2$$

**Proof.** The result is standard e.g. see (Wendland, 2018, Proposition 2.10) for a more general version and proof. For the sake of completeness we provide a direct proof of the equivalent statement $\|\theta\|_\infty \leq \|\theta\|_2 \leq \sqrt{n} \|\theta\|_\infty$ as follows:

$$\|\theta\|_\infty = \max_{i \in [n]} |\theta_i| = \sqrt{\max_{i \in [n]} |\theta_i|^2} \leq \sqrt{\sum_{i=1}^{n} \theta_i^2} \leq \|\theta\|_2$$

$$\|\theta\|_2 = \sqrt{\sum_{i=1}^{n} \theta_i^2} \leq \sqrt{\sum_{i=1}^{n} \max_{i \in [n]} |\theta_i|^2} = \sqrt{n} \max_{i \in [n]} |\theta_i|^2 = \sqrt{n} \|\theta\|_\infty$$

This proves the lower and upper inequalities respectively, as required. \qed

**Remark 3.** We note that the above inequalities are tight i.e. have optimal constants. In the case of the upper bound consider $\theta = 1_n = (1, \ldots, 1)^T \in \mathbb{R}^n$. So $\|\theta\|_\infty = 1$ and $\|\theta\|_2 = \sqrt{n} = \sqrt{n} \|\theta\|_\infty$ showing the tightness of the upper bound. In the case of the lower bound, consider $\theta = e_1$ i.e. WLOG the first standard basis vector in $\mathbb{R}^n$. We then have that $\|\theta\|_\infty = \|\theta\|_2 = 1$, which shows tightness in the lower bound.
Lemma 18 (Lemma 7 in (Shah et al., 2016)). For any $\alpha \in (0, \frac{1}{4})$, there exists a set of $M(\alpha)$ binary vectors \( \{z^1, \ldots, z^{M(\alpha)}\} \subset \{0, 1\}^n \) such that

\[
\alpha n \leq \left\| z^j - z^k \right\|^2_2 \leq n \quad \text{for all } j \neq k \in [M(\alpha)], \quad \text{and} \\
\langle e_1, z^j \rangle = 0 \quad \text{for all } j \in [M(\alpha)]
\]

Lemma 19 (Lemma 8 in (Shah et al., 2016)). For any pair of quality score vectors $\theta^j$ and $\theta^k$, and for any pair of quality score vectors $\theta^j$ and $\theta^k$, and for

\[
\zeta := \frac{\max_{x \in [0,2B/\sigma]} F'(x)}{F(2B/\sigma)(1-F(2B/\sigma))}
\]

we have

\[
D_{KL}(P_{\theta^j} \mid \mid P_{\theta^k}) \leq \frac{N_{\text{comp}} \zeta}{\sigma^2} (\theta^j - \theta^k)^\top L (\theta^j - \theta^k) =: \frac{N_{\text{comp}} \zeta}{\sigma^2} \left\| \theta^j - \theta^k \right\|^2_L
\]

Lemma 20 (Lemma 14 in (Shah et al., 2016)). The Laplacian matrix $\tilde{\mathcal{L}}_\Lambda$ satisfies the trace constraints:

\[
\text{tr}(\tilde{\mathcal{L}}_\Lambda) = 2
\]

\[
\text{tr}(\tilde{\mathcal{L}}_\Lambda^\top) \geq \frac{n^2}{4}
\]

Proof. See (Shah et al., 2016, Lemma 14) for details.

The challenge - constructing a suitable pairwise packing set that meets Definition 15. The main tool to use here is the Varshamov-Gilbert Lemma.

A.3.2 Sketch of Lower Bound Proof

In brief, we seek a minimax lower bound proof of the same form as (Shah et al., 2016, Theorem 2(a)), except in our case we choose our packing set norm to the $\ell_\infty$ norm, rather than the $\ell_2^2$ in (Shah et al., 2016, Theorem 2(a)). We leverage their construction directly in two main ways. First, we use the slightly modified version of Fano’s Lemma that enables the $(\delta, \beta)$-packing set to be constructed for the $\ell_\infty$ norm, not the $\ell_2^2$ norm, consistent with our high probability upper bound per Lemma 16. Second, we can switch out their use of the $(\delta, \beta)$-packing set in the $\ell_2$ norm to the $(\delta', \beta')$-packing set in the $\ell_\infty$ norm. This is done by using the topological equivalence of norms in finite dimensions per Lemma 17, which is shown to be tight in the dimension per Remark 3. For the sake of clarity, we use much of the same wording as the proof from (Shah et al., 2016, Appendix B), for the convenience of the reader.

A.3.3 Lower Bound Proof - Part I

Our proof follows directly the approach taken from (Shah et al., 2016, Section B.1). The normalized Laplacian $\tilde{\mathcal{L}}_\Lambda$ of the comparison graph is symmetric and positive-semidefinite. We can thus decompose this via diagonalization as $L = U^\top \Lambda U$ where $U \in \mathbb{R}^{n \times n}$ is an orthonormal matrix, and $\Lambda$ is a diagonal matrix of nonnegative eigenvalues $\Lambda_{jj} = \lambda_j(L)$ for each $j \in [n]$. Similar to (Shah et al., 2016, Section B.1) we first prove that the minimax risk is lower bounded by $c \sigma^2 \frac{n_{\text{comp}}}{N_{\text{comp}}}$.

Fix scalars $\alpha \in (0, \frac{1}{4})$ and $\delta > 0$, with values to be specified later. Obtain set of vectors on the Boolean Hypercube \( \{0, 1\}^n \) i.e. \( \{z^1, \ldots, z^{M(\alpha)}\} \) given by Lemma 18, where $M(\alpha)$ is set to be

\[
M(\alpha) := \left[ \exp \left\{ \frac{n}{8} \log 2 + 2\alpha \log 2\alpha + (1 - 2\alpha) \log(1 - 2\alpha) \right\} \right].
\]

Define another set of vectors of the same cardinality \( \{\theta^j \mid j \in [M(\alpha)]\} \) via $\theta^j := \frac{\delta}{\sqrt{n}} U^\top P z^j$, where $P$ is a permutation matrix. The permutation matrix $P$ has the constraint that it keeps the first coordinate constant i.e. $P_{11} = 1$. By construction for each $j \neq k$ we have that
\[
\|\theta^j - \theta^k\|_\infty \geq \frac{1}{\sqrt{n}} \|\theta^j - \theta^k\|_2 \overset{(i)}{=} \frac{1}{\sqrt{n}} \left( \frac{\delta}{\sqrt{n}} \|z^j - z^k\|_2 \right) \overset{(ii)}{=} \delta \sqrt{\frac{\alpha}{n}} \overset{(iii)}{=} \delta \sqrt{\frac{\alpha}{n}} \tag{36}
\]

Here (i) follows from Lemma 17. Additionally (ii) follows since \(\theta^j := \frac{\delta}{\sqrt{n}} U^T P z^j\). In the case of the final inequality (iii), we have \(\frac{\delta}{\sqrt{n}} \|z^j - z^k\|_2 \overset{\text{(32)}}{=} \frac{2n \delta^2}{\pi^2} \overset{\text{(32)}}{=} \frac{\delta^2}{\pi} \) since the set \(\{z^1, \ldots, z^{M(\alpha)}\}\) is binary vectors with a minimum Hamming distance at least \(\alpha n\) using Equation (31). Consider, any distinct \(j, k \in [M(\alpha)]\), then for some subset \(\{i_1, \ldots, i_r\} \subseteq \{2, \ldots, n\}\) with \(\alpha n \leq r \leq n\) it must follow that

\[
\|\theta^j - \theta^k\|^2_{L_\alpha} = \frac{\delta^2}{n} \|U^T P z^j - U^T P z^k\|^2_{L_\alpha} = \frac{\delta^2}{n} \|z^j - z^k\|_{L_\alpha} = \frac{\delta^2}{n} \sum_{m=1}^r \lambda_{i_m}(\tilde{\mathcal{L}}_\alpha)
\]

The last part follows since \(A\) is a diagonal matrix of non-negative eigenvalues with \(\Lambda_{ii} = \lambda_j(L)\). Now for given \(\{a_2, \ldots, a_n\}\) such that \(a_n \leq \sum_{i=2}^n a_i \leq n\) we have that

\[
\frac{1}{\left(\frac{M(\alpha)}{2}\right)} \sum_{j \neq k} \|\theta^j - \theta^k\|^2_{\tilde{\mathcal{L}}_\alpha} = \frac{\delta^2}{n} \sum_{i=2}^n a_i \lambda_i(\tilde{\mathcal{L}}_\alpha)
\]

The permutation matrix \(P\) is chosen such that the last \(n - 1\) coordinates are permuted to have \(a_1 \geq \ldots \geq a_n\) and keep the \(n^{th}\) coordinate fixed. By this particular choice, and using the fact that \(\text{tr}(\tilde{\mathcal{L}}_\alpha) = 2\) we have that:

\[
\frac{1}{\left(\frac{M(\alpha)}{2}\right)} \sum_{j \neq k} \|\theta^j - \theta^k\|^2_{\tilde{\mathcal{L}}_\alpha} = \frac{\delta^2}{n} \frac{n}{n-1} \text{tr}(\tilde{\mathcal{L}}_\alpha) \leq \frac{2\delta^2}{n} \text{tr}(\tilde{\mathcal{L}}_\alpha) = \frac{4\delta^2}{n}
\]

Now by the choice of \(P\) above, we have that for every choice of \(j \in [M(\alpha)]\)

\[
\langle \tilde{\mathcal{L}}_\alpha, \theta^j \rangle = \frac{\delta}{n} e^T z^j = e^T z^j = 0
\]

where the last equality follows from Equation (32). Now the basic condition needs to be verified i.e. did the \(\theta^j\) we chose satisfy the boundedness constraint, to ensure that \(\theta^j \in \mathcal{W}_B\)? Setting \(\delta^2 = \frac{c \sigma^2 n^2}{\zeta N_{\text{comp}}}\), it indeed follows that \(\|\theta^j\|_\infty \leq \frac{\delta}{n} \|z^j\|_2 \overset{(i)}{=} \frac{\delta}{n} \overset{(ii)}{\leq} B\). Here (i) follows since \(z^j \in \{0,1\}^n\). Furthermore (ii) follows from our choice of \(\delta\) and our assumption that \(N_{\text{comp}} \geq \frac{c \sigma^2 \text{tr}(\tilde{\mathcal{L}}_\alpha)}{\zeta B^2}\) with \(c = 0.002\), where Lemma 20 guarantees that \(N_{\text{comp}} \geq \frac{c \sigma^2 n^2}{\zeta B^2}\). We have thus verified that each vector \(\theta_j\) also satisfies the boundedness constraint \(\|\theta^j\|_\infty \leq B\), which is required for membership in \(\mathcal{W}_B\). Finally by Lemma 19 we have that:

\[
D_{\text{KL}}(P_{\theta^j} || P_{\theta^k}) \leq \frac{N_{\text{comp}} \zeta}{\sigma^2} \frac{4\delta^2}{n} = 0.01n \tag{37}
\]

To summarize, we have now constructed a \((\delta', \beta')\)-packing set with respect to the norm \(\rho(\theta^i, \theta^k) := \|\theta^j - \theta^k\|_\infty\)

where \(\delta' = \delta \sqrt{\frac{2}{n}}\) from Equation (36), and \(\beta' = 0.01d\) from Equation (37).

Finally we have by substituting \((\delta', \beta')\) into the pairwise Fano’s lower bound (Lemma 16) that:

\[
\sup_{\theta^* \in \mathcal{W}_B} E \left[ \|\tilde{\theta} - \theta^*\|_\infty \right] \geq \delta \sqrt{\frac{\alpha}{n}} \left( 1 - \frac{0.01n + \log 2}{\log M(\alpha)} \right) = c \sigma \sqrt{\frac{n}{\zeta N_{\text{comp}}}} \left( 1 - \frac{0.01n + \log 2}{\log M(\alpha)} \right)
\]

which yields the claim, after appropriate substitution of \(\delta\) and setting \(\alpha = 0.01\).

For the case of \(n \leq 9\), consider the set of the three \(n\)-length vectors \(z^1 = (0, \ldots, -1), z^2 = (0, \ldots, 0)\) and \(z^3 = (0, \ldots, 0)\). Construct the packing set \((\theta^1, \theta^2, \theta^3)\) from these three vectors \((z^1, z^2, z^3)\) as done above for the case of \(n > 9\). From the calculations made for the general case above, we have for all pairs \(\min_{j \neq k} \|\theta^j - \theta^k\|_\infty \geq \ldots\)
have the same eigenvalues; and inequality (38) follows because the matrices \( \sqrt{\Lambda^T} \) and \( \tilde{\Lambda}^T \) have the same eigenvalues; and inequality (iii) follows from our choice of \( \delta \) and our assumption \( N_{\text{comp}} \geq \frac{\sigma^2 \nu^2(z^n_{\Lambda})}{\delta^2} \) on the sample size with \( c = 0.01 \). We have thus verified that each vector \( \theta^j \) also satisfies the boundedness constraint \( \| \theta^j \|_\infty \leq B \), as required for membership in \( \mathcal{W}_B \). Furthermore, for any pair of distinct vectors in this set, we have:

\[
\| \theta^j - \theta^k \|_{\tilde{\mathcal{L}}_\Lambda}^2 = \frac{\delta^2}{n'} \| z^j - z^k \|_2^2 \leq \delta^2
\]

By Lemma 19 we have that:

\[
D_{KL}(P_{\theta^j} \| P_{\theta^k}) \leq \frac{N_{\text{comp}}}{\sigma^2} \left( \| \theta^j - \theta^k \|_{\tilde{\mathcal{L}}_\Lambda}^2 \right) = 0.01 n'
\]

### A.3.4 Lower Bound Proof - Part II

Given an integer \( n' \in \{2, \ldots, n\} \), and constants \( \alpha \in (0, \frac{1}{2}) \), \( \delta > 0 \), define the integer:

\[
M'(\alpha) := \left\lceil \exp \left( \frac{n'}{2} (\log 2 + 2\alpha \log 2 + (1 - 2\alpha) \log(1 - 2\alpha)) \right) \right\rceil
\]

(38)

Applying Lemma 18 using \( n' \) as the dimension results in a subset \( \{z^1, \ldots, z^{M'(\alpha)}\} \) of the Boolean hypercube \( \{0, 1\}^{n'} \), with specified properties. We then define a finite set of size \( M'(\alpha) \), of \( n \)-length vectors \( \{\tilde{\theta}^1, \ldots, \tilde{\theta}^{M'(\alpha)}\} \) using:

\[
\tilde{\theta}^j = \begin{bmatrix} 0 & (z^j)^\top & \cdots \end{bmatrix}^\top \quad \text{for each } j \in [M(\alpha)]
\]

For each \( j \in [M(\alpha)] \), let us define \( \theta^j := \frac{\delta}{\sqrt{n'}} U^\top \sqrt{\Lambda^T} \tilde{\theta}^j \). For the first standard basis vector \( e_1 \in \mathbb{R}^n \), we then have that \( \langle 1_n, \theta^j \rangle = \frac{\delta}{\sqrt{n'}} \tilde{\Lambda}^T U^\top \sqrt{\Lambda^T} \tilde{\theta}^j = 0 \). Here the main fact used is \( \tilde{\Lambda} A_1 = 0 \). Additionally we have that for any \( j \neq k \), we have that:

\[
\| \theta^j - \theta^k \|_{\tilde{\mathcal{L}}_\Lambda}^2 \geq \frac{1}{n} \| \theta^j - \theta^k \|_2^2 = \frac{\delta^2}{n n'} \left( \theta^j - \tilde{\theta}^k \right)^\top \Lambda^\dagger \left( \theta^j - \tilde{\theta}^k \right) \geq \frac{\delta^2}{n n'} \sum_{i=\{(1-\alpha)n'\}}^{n'} \frac{1}{\lambda_i}
\]

(39)

Now, setting \( \delta^2 = 0.01 \frac{\sigma^2 n'}{N_{\text{comp}}} \) results in:

\[
\| \theta^j \|_\infty \leq \delta \sqrt{\frac{n'}{n'}} \left\| \sqrt{\Lambda^T} \tilde{\theta}^j \right\|_2 \leq \frac{\delta}{\sqrt{n'}} \left\| \sqrt{\Lambda^T} \right\|_2 \left( \sum_{i=\{(1-\alpha)n'\}}^{n'} \frac{1}{\lambda_i} \right) \leq B
\]

(40)

where inequality (i) follows from the fact that \( z^j \) has entries in \( \{0, 1\} \); step (ii) follows because the matrices \( \sqrt{\Lambda^T} \) and \( \tilde{\Lambda}^T \) have the same eigenvalues; and inequality (iii) follows from our choice of \( \delta \) and our assumption \( N_{\text{comp}} \geq \frac{\sigma^2 \nu^2(z^n_{\Lambda})}{\delta^2} \) on the sample size with \( c = 0.01 \). We have thus verified that each vector \( \theta^j \) also satisfies the boundedness constraint \( \| \theta^j \|_{\infty} \leq B \), as required for membership in \( \mathcal{W}_B \). Furthermore, for any pair of distinct vectors in this set, we have:

By Lemma 19 we have that:

\[
D_{KL}(P_{\theta^j} \| P_{\theta^k}) \leq \frac{N_{\text{comp}}}{\sigma^2} \left( \| \theta^j - \theta^k \|_{\tilde{\mathcal{L}}_\Lambda}^2 \right) = 0.01 n'
\]

(41)
Finally we have by substituting \((\delta', \beta')\) into the pairwise Fano’s lower bound (Lemma 16) that:

\[
\sup_{\theta^* \in \mathcal{W}_B} \mathbb{E}\left[\|\hat{\theta} - \theta^*\|_{\infty}\right] \geq \frac{\delta}{\sqrt{n'd}} \sqrt{\frac{1}{2} \sum_{i=\lceil(1-\alpha)n'\rceil}^{n'}} \frac{1}{\lambda_i} \left(1 - \frac{0.01n' + \log 2}{\log M'(\alpha)}\right)
\]

Substituting our choice of \(\delta\) and setting \(\alpha = 0.01\) proves the claim for \(n' > 9\).

For the case of \(n' \leq 9\). Consider the packing set of the three \(n'\)-length vectors \(\theta^1 = \delta U \sqrt{\Lambda^\top}(0, 1, \ldots, 0)\), \(\theta^2 = -\theta^1\) and \(\theta^3 = (0, \ldots, 0)\). Then we have for all pairs \(\min_{j \neq k} \|\theta^j - \theta^k\|_{\infty} \geq \frac{1}{8} \min_{j \neq k} \|\theta^j - \theta^k\|_{2} \geq \frac{\delta}{\lambda_2(L)}\) and \(\max_{j, k} \|\theta^j - \theta^k\|_{2}^2 \leq 4\delta^2\), and as a result \(\max_{j, k} D_{KL}(P_{\theta^j} || P_{\theta^k}) \leq \frac{4N_{\text{comp}} \delta^2}{\sigma^2}\). Choosing \(\delta^2 = \frac{\sigma^2 \log 2}{8N_{\text{comp}}\eta}\) and applying the pairwise Fano’s lower bound (Lemma 16) yields the claim.
A.4 Additional Experiments

In this section, we show some additional results of experiments. A one-sentence summary of this section is, we demonstrate our discussions in Section 2.2 and Section 2.1 by real cases that our upper bound is tighter than those of Yan et al. (2012) and Shah et al. (2016) for general comparison graphs and maximal performance gap $\kappa$, except for some extremely sparse and sense comparison graphs and small $\kappa$.

There are two things we want to note for numerical experiments. First, since all upper bounds are high probability bounds in an asymptotic sense, i.e., they contain some constant factors that are ignorable only for large $n$, so the exact relative tightness of those bounds is hard to show numerically, because in experiments we can only check for certain moderate-magnitude values of $n$.

Another thing that should be noted is the factor $e^{c\kappa}$ or $e^{c\kappa_E}$. Again since we are working with a moderate $n$, this exponential term could be dominating even for a small $\kappa$, and bounds with different leading constants $c$ in the exponents could show huge difference in magnitude. Thus the tightness of a method is significantly affected by this exponential factor. Because of this, our upper bound is definitely tighter than the $\ell_2$ bound in Shah et al. (2016) for general $\kappa$. Moreover, as we show in Example 5, for some special design, $\kappa_E$ can be much smaller than $\kappa$ and makes our bound tighter than those with $e^{c\kappa}$. But for constant $\kappa$ and sufficiently large $n$ this exponential factor is ignorable, and the $\ell_2$ upper bound in Shah et al. (2016), acting as a trivial $\ell_\infty$ upper bound will be tighter than ours under some sparse or irregular comparison graphs, as we discussed in Section 2.2. We will see the same results of comparison in experiments below.

![Figure 3: Comparison under the Barbell graph with random bridge edges. Shah-l2 shows the $\ell_2$ high probability upper bound in Shah et al. (2016). In the left panel Shah-l2 uses the factor $e^{4\kappa}$, while in the right panel it uses factor $e^{2\kappa}$. Some points near the left end of the orange line are cut because $\min_{i,j} n_{ij} = 0$ and the corresponding upper bound is $\infty$. Each point on lines is an average of 20 trials. Upper and lower 0.05 quantiles are presented by colored areas.](image)

![Figure 4: Comparison under the Barbell graph with random bridge edges, with $e^{c\kappa}$ replaced by some constant factor.](image)
Additional results for Example 6. Fig. 3 shows a comparison under the Barbell graph with random bridges of the real $\ell_\infty$ error of the MLE and the 3 upper bounds. Here $n_1 = n_2 = 50$ and $n = 100$. The left panel shows the dominance of $e^{\kappa}$ in Shah’s $\ell_2$ bound. Since in our equal-spaced $\theta^*$ setting, $\kappa = 2B$, so we tighten Shah’s $\ell_2$ bound for a “fair” comparison in the right panel. It doesn’t change the relative order for most bridge density $|E_l|/(n_1n_2)$, but suggests that Shah l2 will be lower than our l-inf for very small bridge density.

To illustrate the relative order of bounds for (very) large $n$ and small $\kappa$, we replace the exponential factors by some common appropriate constant and compare the 3 bounds in Figure 4. It confirms our assertion at the beginning of this section.

Additional results for Example 5. Figure 5 shows a comparison under the island graph of the real $\ell_\infty$ error of the MLE, and the upper bounds of Shah and ours. The bound for Yan et al. (2012) is not shown because $\min_{i,j} n_{ij} = 0$. The results again confirm our assertion.

![Figure 5: Comparison under the island graph. Shah-l2 shows the $\ell_2$ high probability upper bound in Shah et al. (2016). In the left panel Shah-l2 uses the factor $e^{\kappa}$, while in the right panel it uses factor $e^{2\kappa}$. Each point on lines is an average of 20 trials. Upper and lower 0.05 quantiles are presented by colored areas.](image)

A.4.1 Cases where $\kappa_E$ is loose

Consider a path graph of $n$ nodes and edge set $\{(i, i+1) : i \in [n-1]\}$. Assume $\theta^*_i = \theta^*_1 + (i-1)\delta$, then $\kappa_E = \delta$ and $\kappa = (n-1)\delta$. In this case, a factor of $e^{\kappa_E}$ gives tighter control than $e^\kappa$. However, if we add one edge $(1, n)$ into the graph, $\kappa_E$ becomes $(n-1)\delta$ and our upper bound will increase a lot, which is counter-intuitive because the newly-added 1 out of $n$ edges should not affect the estimation accuracy too much. In other words, the bound gets looser after the new edge is added.

The reason is that $\kappa_E$ itself is not enough to provide tight control across all comparison graphs. For instance, to avoid such loose cases, one also needs to take into account the proportion of edges with big performance gaps compared to edges with small performance gaps. Figure 6 shows some numerical results on the impact of big-gap edges. In the left panel, we add edges to the top-right and bottom-left corner of the adjacency matrix so that their performance gaps are big. In the right panel, we switch some pairs of parameters $\{\theta^*_i\}$ so that there are more big-gap edges while keeping the algebraic connectivity. The first switch will switch $\theta^*_1$, $\theta^*_{[n/2]+1}$. The $i$-th switch will switch the pair $\theta^*_i \rightarrow \theta^*_i + \delta$. As an example, taking $n = 8$, $\theta^*_1 = 0$, and $\delta = 1$, 2 switches will make parameters change as

$$(1, 2, 3, 4, 5, 6, 7, 8) \rightarrow (5, 2, 7, 4, 1, 6, 3, 8). \quad (42)$$

In the right panel, as new big-gap edges come in, the algebraic connectivity of the graph gets larger as well, keeping estimation errors in the same level. In the right panel, as we keep the algebraic connectivity constant, the impact of the proportion of big-gap edges is shown more clearly.

Another thing we need to point out is that for cases like path graphs, even if we ignore the factor of $\kappa$, bounding estimation error itself is hard due to the poor connectivity of the graph, as is argued in Shah et al. (2016) (their upper bound for both path and cycle graphs is not optimal). As we have shown in Section 2.2, even for $d$-regular graphs with relatively small $d$, the algebraic connectivity is not big enough and our upper bound cannot match...
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Figure 6: Experiment on the path graph. Left: the $\ell_\infty$ error of the regularized MLE when adding new edges with big performance gaps to the path graph. Right: the $\ell_\infty$ error of the regularized MLE when switch some pairs of the performance parameter $\{\theta^*_i\}$ so that there are more big-gap edges while keeping the algebraic connectivity. The results show that when the proportion of big-gap edges is small, the estimation error would not be affected a lot. Both experiments are under $\kappa \approx 6.9$, equal-gap $\theta^*$, $n = 200$ and $L = 5000$ and based on 40 trials for each hyperparameter, with 0.05 and 0.95 quantiles shown by the shaded area.

The lower bound. But as the comparison graph gets denser and more regular, $\kappa_E$ will get closer to $\kappa$, making the difference between $e^{\kappa_E}$ and $e^\kappa$ not as dramatic, although in finite sample phase the difference can still be big because it is an exponential factor.

The last thing we want is that the estimation itself (without providing a theoretical tight upper bound for the estimation error) under the path graph is hard: one need a huge $L$ to make accurate estimation when $n$ is big, as is shown in Figure 7.

Figure 7: Experiment on the path graph, under $\kappa \approx 6.9$, equal-gap $\theta^*$, $n = 200$ and based on 40 trials for each hyperparameter $L$, with 0.05 and 0.95 quantiles shown by the shaded area.
A.5 Other Supporting Results

**Lemma 21.** Given a $d$-Cayley graph, where $(i, j) \in E$ if and only if $i - j \equiv k (\mod n)$ with $-d \leq k \leq d$, $k \neq 0$. It satisfies $\lambda_2(\mathcal{L}_A) \approx d^3/n^2$.

**Proof.** By definition, a $d$-Cayley graph is a $2d$-regular graph, and it’s well known that (see, e.g. Brouwer and Haemers, 2012) the spectra of its adjacency matrix $A$ is given by

$$\lambda_j(A) = \sum_{k=1}^{d} (\zeta_j^k + \zeta_j^{-k}), \quad \zeta_j := \cos \frac{2\pi j}{n} + \sqrt{-1} \sin \frac{2\pi j}{n},$$

for $j = 1, \ldots, n$. Thus, $\lambda_2(A) = 2d - 2 \sum_{k=1}^{d} \cos(\frac{2\pi k}{n})$. Since $\cos(\frac{2\pi k}{n}) = 1 - 2\sin^2(\frac{\pi k}{n})$ and for $k \leq d < 0.5n$, $\sin(\frac{\pi k}{n}) \in (0, 0.5\pi k, \pi k)$, we have

$$\lambda_2(A) = c^2 \pi^2/n^2 \sum_{k=1}^{d} k^2 = cd^3/n^2,$$

for $c \in (2\pi^2/3, 4\pi^2/3)$. \hfill \(\blacksquare\)

**Lemma 4.** (Subadditivity) Let $I_1, I_2, I_3$ be three subsets of $[n]$ such that $\cup_{j=1}^3 I_j = [n]$ and, for each $j \neq k$, $I_j \not\subseteq I_k$ and for $i = 1, 2, I_i \cap I_3 \neq \emptyset$. For any vector $\theta \in \mathbb{R}^n$, let $\theta_{(j)} \in \mathbb{R}^{|I_j|}$ be the sub-vector of $\theta$ consisting of the entries indexed by $I_j$, $j = 1, 2, 3$. Let $\theta^*_j$ be the vector of preference scores in the BTL model over $n$ items and $\hat{\theta}_{(j)}$ be the MLE of $\theta^*_j$ for the BTL model involving only items in $I_j$, $j = 1, 2, 3$, assuming that the sub-graphs induced by the $I_j$’s are connected and allowing for a different number of comparisons in each sub-graph. Finally, let $\theta \in \mathbb{R}^n$ be the ensemble MLE based on $\theta_{(1)}, \theta_{(2)}, \theta_{(3)}$. Then

$$\frac{1}{4} d_\infty(\hat{\theta}, \theta^*) \leq d_\infty(\hat{\theta}_{(1)}, \theta^*_{(1)}) + d_\infty(\hat{\theta}_{(2)}, \theta^*_{(2)}) + d_\infty(\hat{\theta}_{(3)}, \theta^*_{(3)}),$$

where $d_\infty(v_1, v_2) := \| (v_1 - 2\text{avg}(v_1)) - (v_2 - 2\text{avg}(v_2)) \|_\infty$, where $\text{avg}(x) := \frac{1}{n} 1^\top_n x$.

**Proof of Lemma 4.** First, for each $i = 1, 2, 3$, let’s shift $\hat{\theta}_{(i)}, \theta^*_{(i)}$ by some constant vector $a1_{|I_i|}$ so that

$$\text{avg}(\hat{\theta}_{(i)}) = \frac{1}{|I_i|} 1_{|I_i|}^\top \hat{\theta}_{(i)} = 0, \quad \text{avg}(\theta^*_{(i)}) = 0,$$

and hence

$$d_\infty(\hat{\theta}_{(i)}, \theta^*_{(i)}) = \| \hat{\theta}_{(i)} - \theta^*_{(i)} \|_\infty.$$

Next, for each $i = 1, 2, 3$, let $\tilde{\theta}_{(i)} \in \mathbb{R}^n$ be the augmented version of $\hat{\theta}_{(i)} \in \mathbb{R}^{|I_i|}$, given by

$$\tilde{\theta}_{(i)}(j) = \begin{cases} \hat{\theta}_{(i)}(j), & j \in I_i, \\ 0, & j \notin I_i, \end{cases}$$

where $v(j)$ refers to the $j$-th entry of vector $v$. Similarly, we define

$$\tilde{\theta}^*_{(i)}(j) = \begin{cases} \theta^*_{(i)}(j), & j \in I_i, \\ 0, & j \notin I_i, \end{cases}$$

Now let’s define the ensemble MLE $\hat{\theta}$ and show the subadditivity. The idea is to first fix $\hat{\theta}_{(1)}$, and then shift nonzero entries in $\hat{\theta}_{(2)}$ and $\hat{\theta}_{(3)}$ to comply with the difference in common entries of $I_1, I_3$ and $I_2, I_3$.

Let $S_1 = I_1, S_2 = I_2 \setminus I_1$ (note that we don’t put any constraint on $I_1 \cap I_2$), $S_3 = I_3 \setminus (I_1 \cup I_2)$. We allow $S_3$ to be $\emptyset$, but by the assumption that $I_j \not\subseteq I_k$, we have $S_2 \neq \emptyset$. Since $\cup_{j=1}^3 I_j = [n]$ and $I_i \cap I_3 \neq \emptyset$ for $i = 1, 2$, we
have $\cup_{j=1}^3 S_j = \{u\}$ and $S_i \cap S_k = \emptyset$ for any $i \neq k$. Pick $t_1 \in I_1 \cap I_3$, $t_2 \in I_2 \cap I_3$, and let $\delta_3 = \hat{\theta}_{(1)}(t_1) - \hat{\theta}_{(3)}(t_1)$, $\delta_2 = \hat{\theta}_{(3)}(t_2) - \hat{\theta}_{(2)}(t_2)$. Moreover, define $\hat{\theta}_{(3)}$, $\hat{\theta}_{(2)}$ by

$$
\hat{\theta}_{(3)}(j) = \begin{cases} 
\hat{\theta}_{(3)}(j) + \delta_3, & j \in S_3, \\
0, & j \notin S_3,
\end{cases}
\quad \hat{\theta}_{(2)}(j) = \begin{cases} 
\hat{\theta}_{(2)}(j) + \delta_3 + \delta_2, & j \in S_2, \\
0, & j \notin S_2,
\end{cases}
$$

Let

$$
\hat{\theta} = \hat{\theta}_{(1)} + \hat{\theta}_{(2)} + \hat{\theta}_{(3)},
$$

then it can be seen that

$$
\hat{\theta}(j) = \begin{cases} 
\hat{\theta}_{(1)}(j), & j \in S_1, \\
\hat{\theta}_{(2)}(j) + \delta_3 + \delta_2, & j \in S_2, \\
\hat{\theta}_{(3)}(j) + \delta_3, & j \in S_3,
\end{cases}
$$

To analyze the error, we first notice that for any $v, u \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

$$
d_\infty(u, v) = d_\infty(u, v + a1_n) \leq \|u - (v + a1_n)\|_\infty + \frac{1}{n} \sum_{i=1}^n |u_i - (v_i + a)| \leq 2\|u - (v + a1_n)\|_\infty.
$$

Let $\delta_3^* = \hat{\theta}^*_{(1)}(t_1) - \hat{\theta}^*_{(3)}(t_1)$, $\delta_2^* = \hat{\theta}^*_{(3)}(t_2) - \hat{\theta}^*_{(2)}(t_2)$. Define a new true parameter $\hat{\theta}^*$ by shifting $\theta^*$ via

$$
\hat{\theta}^*(j) = \begin{cases} 
\hat{\theta}^*_{(3)}(j), & j \in S_1, \\
\hat{\theta}^*_{(2)}(j) + \delta_3^* + \delta_2^*, & j \in S_2, \\
\hat{\theta}^*_{(3)}(j) + \delta_3^*, & j \in S_3,
\end{cases}
$$

It can be verified that $\hat{\theta}^* = \theta^* + (\theta^*(t_1) - \hat{\theta}^*(t_1))1_n$. Therefore,

$$
d_\infty(\hat{\theta}, \theta^*) \leq 2\|\theta - \theta^*\|_\infty.
$$

For $j \in S_1$,

$$
|\hat{\theta}(j) - \theta^*(j)| = |\hat{\theta}(j) - \hat{\theta}^*(j)| \leq \|\hat{\theta}(j) - \hat{\theta}^*(j)\|_\infty.
$$

For $j \in S_3$,

$$
|\hat{\theta}(j) - \theta^*(j)| = |\hat{\theta}(j) - \hat{\theta}^*(j)| + |\delta_3 - \delta_3^*| \leq 2\|\hat{\theta}(j) - \hat{\theta}^*(j)\|_\infty + \|\delta_3 - \delta_3^*\|_\infty.
$$

For $j \in S_2$,

$$
|\hat{\theta}(j) - \theta^*(j)| = |\hat{\theta}(j) - \hat{\theta}^*(j)| + |\delta_3 - \delta_3^*| + |\delta_2 - \delta_2^*| \\
\leq \|\hat{\theta}(j) - \hat{\theta}^*(j)\|_\infty + 2\|\delta_3 - \delta_3^*\|_\infty + 2\|\delta_2 - \delta_2^*\|_\infty.
$$

Therefore by definition of $\|\cdot\|_\infty$ and Eq. (44),

$$
\|\hat{\theta} - \theta^*\|_\infty \leq 2\|\hat{\theta}(1) - \theta^*(1)\|_\infty + \|\hat{\theta}(2) - \theta^*(2)\|_\infty + \|\hat{\theta}(3) - \theta^*(3)\|_\infty
$$

$$
= 2(d_\infty(\hat{\theta}(1), \theta^*(1)) + d_\infty(\hat{\theta}(2), \theta^*(2)) + d_\infty(\hat{\theta}(3), \theta^*(3))),
$$

and we have

$$
\frac{1}{4} d_\infty(\hat{\theta}, \theta^*) \leq d_\infty(\hat{\theta}(1), \theta^*(1)) + d_\infty(\hat{\theta}(2), \theta^*(2)) + d_\infty(\hat{\theta}(3), \theta^*(3)).
$$
A.6 Special cases of pairwise graph topologies

By Theorem 1, for the estimator $\hat{\theta}_p$ to be consistent, $L$ needs to be sufficiently large. We can check some common types of comparison graph topologies and see in what order the necessary sample complexity $N_{\text{comp}} = |E|L$ needs to be, to achieve consistency. To simplify results, we assume $e^{2k_E} \lesssim \log n$. Spectral properties of graphs listed here can be found in well-known textbooks (Brouwer and Haemers, 2012). Shah et al. (2016) provide analogous comparisons. Per Section 2.2 we include results for the $d$-Cayley graphs, and expander graphs here. For reader convenience other results from Section 2.2 are also noted below.

**Complete graph:** In this case, $\lambda_2(L_A) = n_{\text{max}} = n_{\text{min}} = n - 1$. Thus, we need $e^{-2k_E} \log n/n = o(L)$. Hence $L = \Omega(1)$ and $N_{\text{comp}} = \Omega(n^2)$.

**Expander graph:** If the comparison graph is a $d$-regular expander graph with edge expansion (or Cheeger number) coefficient $\phi$, then $\lambda_2(L_A) \geq \phi^2/(2d)$ (Alon et al., 2008), $n_{\text{max}} = n_{\text{min}} = d$. Here $\phi$ is defined by $\phi := \min_{\{S,T\}} e(S,T)/|S|$ where $\{S,T\}$ is a partition of the vertex set and $|S| \leq |T|$. We need $d^2e^{2k_E}/\phi^4 \cdot (e^{2k_E}n \vee d \log n) = o(L)$, so $N_{\text{comp}} = \Omega(nd^3e^{2k_E}/\phi^4 \cdot (e^{2k_E}n \vee d \log n))$.

**Complete bipartite graph:** If the comparison graph has two partitioned sets of size $m_1$ and $m_2$ such that $m_1 \leq m_2$, then $\lambda_2(L_A) = m_1$, $n_{\text{max}} = m_2$, $n_{\text{min}} = m_1$. We need $e^{-2k_E}m_2/m_1^2 \cdot [(e^{2k_E}nm_2/m_1^2) \vee \log n] = o(L)$. When $m_1 = \Omega(n)$, we have $N_{\text{comp}} = \Omega(n^2)$.

**d-Cayley graph:** $(i,j) \in E$ if and only if $i-j \equiv k (\mod n)$ with $-d \leq k \leq d$, $k \neq 0$. It is a $d$-regular graph, and $\lambda_2(L_A) \approx d^2/n^2$ (see Appendix A.5 in the supplement), $n_{\text{max}} = n_{\text{min}} = 2d$. Thus, we need $e^{2k_E}n^4/d^6 \cdot (e^{2k_E}n \vee 2d \log n) = o(L)$, so $N_{\text{comp}} = \Omega(e^{2k_E}n^5/d^6 \cdot (e^{2k_E}n \vee 2d \log n))$ for $d = o(n)$ and $N_{\text{comp}} = \Omega(n^2)$ for $d = \Omega(n)$. When $d = 1$, the $d$-Cayley graph is a cycle graph.

**Path or Cycle graph:** When comparisons occur based on a path or cycle comparison graph, then $\lambda_2(L_A) = 2 - 2 \cos(c\pi/n) \approx c^2\pi^2/n^2$, $n_{\text{max}} = 2$, $n_{\text{min}} = c$ with $c = 1$ for path and $c = 2$ for cycle. Thus, we need $e^{4k_E}n^5 = o(L)$, so $N_{\text{comp}} = \Omega(e^{4k_E}n^6)$.

**Star graph:** For a start graph on $n$ node one node has degree $n - 1$ and the remaining have degree 1. In this case, $\lambda_2(L_A) = n_{\text{min}} = 1$, $n_{\text{max}} = n$. We need $e^{4k_E}n^3 = o(L)$, hence $N_{\text{comp}} = e^{4k_E}n^4$.

**Barbell graph:** It contains two size-$n/2$ complete sub-graphs connected by 1 edge, so $\lambda_2(L_A) \approx 1/n$, $n_{\text{max}} = n/2$, $n_{\text{min}} = n/2 - 1$. We need $e^{2k_E}n^3 \log n = o(L)$, and $N_{\text{comp}} = e^{2k_E}n^5 \log n$. 

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A.7 Upper bound for Unregularized/Vanilla MLE

A.7.1 Main theorem

The unregularized or vanilla MLE is defined as

$$
\hat{\theta} := \arg\min_{\theta \in \mathbb{R}^n} \ell(\theta; y),
$$

(46)

where $\ell(\theta; y)$ is the negative log-likelihood, given by

$$
\ell(\theta; y) := -\sum_{1 \leq i < j \leq n} A_{ij} \left\{ y_{ij} \log \psi(\theta_i - \theta_j) + (1 - y_{ij}) \log[1 - \psi(\theta_i - \theta_j)] \right\},
$$

(47)

and $t \in \mathbb{R} \mapsto \psi(t) = 1/[1 + e^{-t}]$ the sigmoid function. To make the expressions of results simpler, we consider the parameter range $\kappa \leq n$ and $\kappa_E \leq \log n$ as discussed in the comments after Theorem 1.

**Theorem 22** (Vanilla MLE). Assume the BT model with parameter $\theta^* = (\theta_1^*, \ldots, \theta_n^*)^\top$ such that $A_n^\top \theta^* = 0$ and a comparison graph $G = G([n], E)$ with adjacency matrix $A$, algebraic connectivity $\lambda_2(A)$ and maximum and minimum degrees $n_{\max}$ and $n_{\min}$, respectively. Suppose that each pair of items $(i, j) \in E$ are compared $L$ times. Let $\kappa = \max_{i,j} |\theta_i^* - \theta_j^*|$ and $\kappa_E = \max_{(i,j) \in E} |\theta_i^* - \theta_j^*|$. Assume that $G$ is connected, or equivalently, $\lambda_2(A) > 0$.

In addition, assume that 1. $\lambda_2(A) L > C e^{2\kappa_E} \max\{n_{\max} \log n, e^{2\kappa_E} n_{\max}^2 / n_{\min}^2\}$ for some large constant $C > 0$, and 2. $\lambda_2(A) \geq 2 e^{2\kappa_E} n_{\max} / n_{\min}$. Then, with probability at least $1 - O(n^{-5})$, the unregularized MLE $\hat{\theta}$ from (46) satisfies

$$
\|\hat{\theta} - \theta^*\|_\infty \lesssim e^{\kappa_E} \sqrt{\frac{n_{\max} \log n}{L^2 n_{\min}^2}} + e^{2\kappa_E} \sqrt{\frac{n_{\max}^3}{L \lambda_2(A)^2 n_{\min}^2}} \left[ 1 + \frac{e^{\kappa_E} \sqrt{\log n}}{n_{\min}} + \frac{e^{2\kappa_E} \sqrt{n_{\max} n}}{\lambda_2(A)n_{\min}} \right],
$$

(48)

and

$$
\|\hat{\theta} - \theta^*\|_2 \lesssim e^{\kappa_E} \sqrt{\frac{n_{\max} n}{L \lambda_2(A)}},
$$

(49)

provided that $L \lesssim n^5$, $\kappa < n$, $\kappa_E \leq \log n$, and the right hand side of Equation (48) is smaller than a sufficiently small constant $C > 0$.

**Remark 4.** The expression of the $\ell_\infty$ bound looks messy, but in the ER($n, p$) case it reduces to the same form as the upper bound of the regularized MLE in Corollary 2. Moreover, for vanilla MLE we require an additional pure topological assumption $\lambda_2(A) \geq 2 e^{2\kappa_E} n_{\max} / n_{\min}$, which is stringent in some sense as it exclude many graphs with small $n_{\min}$. But for such graphs with high degree heterogeneity $n_{\max} / n_{\min}$, it's reasonable that we need some regularity in our objective function. We believe that this condition can be weakened, and the $\ell_\infty$ upper bound can be improved or tightened by improving the proof techniques, which can be a good future direction for researchers.

To prove Theorem 22, we need two lemmas, Lemma 23 anbd 24.

**Lemma 23.** Under the setting of Theorem 22, it holds with probability at least $1 - O(n^{-5})$ that

$$
\|\hat{\theta} - \theta^*\|_\infty \leq 5.
$$

(50)

Following Chen et al. (2020), we decompose the full negative loglikelihood function as

$$
\ell_n(\theta) = \ell_n^{(-m)}(\theta_{-m}) + \ell_n^{(m)}(\theta_{-m}),
$$

(51)

where $\theta_{m} \in \mathbb{R}$ is the $m$-th entry of $\theta$ and $\theta_{-m} \in \mathbb{R}^{n-1}$ is the subvector containing the rest of entries, and the two functions are given by

$$
\ell_n^{(-m)}(\theta_{-m}) = \sum_{1 \leq i < j \leq n, i \neq m} A_{ij} \left\{ y_{ij} \log \frac{1}{\psi(\theta_i - \theta_j)} + (1 - y_{ij}) \log \frac{1}{1 - \psi(\theta_i - \theta_j)} \right\},
$$

$$
\ell_n^{(m)}(\theta_{m} | \theta_{-m}) = \sum_{j \in [n] \setminus \{m\}} A_{mj} \left\{ \bar{y}_{mj} \log \frac{1}{\psi(\theta_{m} - \theta_j)} + (1 - \bar{y}_{mj}) \log \frac{1}{1 - \psi(\theta_{m} - \theta_j)} \right\}.
$$
Let $H^{(-m)} := \nabla^2 \ell_n^{(-m)}(\ell_{-m})$, and
\[
\theta^{(m)} = \arg\min_{\theta \in H} \ell_n^{(-m)}(\theta).
\]

**Lemma 24.** Under the setting of Theorem 22, it holds with probability at least $1 - O(n^{-9})$ that
\[
\max_{m \in [n]} \|\theta^{(m)} - \theta^* - m - a_m 1_{n-1}\|^2 \leq C e^{2\kappa_H n \max_{n}} \frac{\lambda_2(L)^2}{L}.
\]

for some constant $C > 0$, where $a_m = \text{avg} \left(\theta^{(m)} - \theta^* - m\right) := \frac{1}{n-1} 1_{n-1}^T \left(\theta^{(m)} - \theta^* - m\right)$.

**Proof of Theorem 22.** Again we define the leave-one-out negative log-likelihood as
\[
\ell_n^{(m)}(\theta) = \sum_{1 \leq i < j \leq n, i, j \neq m} A_{ij} \left[ \frac{1}{\psi(\theta_i - \theta_j)} + (1 - \bar{y}_{ij}) \log \frac{1}{1 - \psi(\theta_i - \theta_j)} \right] + \sum_{i \in [n] \setminus \{m\}} A_m \left[ \psi(\theta_i^* - \theta_m^*) \log \frac{1}{\psi(\theta_i - \theta_m)} + \psi(\theta_m^* - \theta_i^*) \log \frac{1}{\psi(\theta_m - \theta_i)} \right],
\]

For the $\ell_2$ bound, notice that by Taylor expansion
\[
\ell_n(\hat{\theta}) = \ell_n(\theta^*) + (\hat{\theta} - \theta^*)^T \nabla \ell_n(\theta^*) + \frac{1}{2}(\hat{\theta} - \theta^*)^T H(\xi)(\hat{\theta} - \theta^*),
\]

where $\xi$ is a convex combination of $\hat{\theta}$ and $\theta^*$. By Lemma 23, we have $\|\hat{\theta} - \theta^*\|_2 \leq 5$ and hence $\|\xi - \theta^*\|_\infty \leq 5$. By Lemma 8, we have $\frac{1}{2}(\hat{\theta} - \theta^*)^T H(\xi)(\hat{\theta} - \theta^*) \geq ce^{-\kappa_E} \lambda_2\|\hat{\theta} - \theta^*\|^2_2$ for some constant $c > 0$. By the fact that $\ell_n(\theta^*) \geq \ell_n(\theta)$, Cauchy-Schwartz inequality, and Lemma 7, we have
\[
\|\hat{\theta} - \theta^*\|_2 \leq \frac{e^{\kappa_E}}{\lambda_2} \|\nabla \ell(\theta^*)\|_2 \leq \frac{e^{\kappa_E}}{\lambda_2} \sqrt{\frac{\eta_{\max} n}{L}}.
\]

For the $\ell_\infty$ bound, the proof can be sketched as following steps.

0. By Lemma 23, $\|\hat{\theta} - \theta^*\|_\infty \leq 5$ with probability at least $1 - O(n^{-5})$.

1. By Lemma 24, it holds with probability exceeding $1 - O(n^{-9})$ that,
\[
\max_{m \in [n]} \|\theta^{(m)} - \theta^* - m - a_m 1_{n-1}\|^2 \leq C e^{2\kappa_H n \max_{n}} \frac{\lambda_2(L)^2}{L}.
\]

where $a_m = \text{avg} \left(\theta^{(m)} - \theta^* - m\right) := \frac{1}{n-1} 1_{n-1}^T \left(\theta^{(m)} - \theta^* - m\right)$.

2. Show that on the same event,
\[
\max_{m \in [n]} \|\theta^{(m)} - \hat{\theta} - a_m 1_{n-1}\|^2 \leq C_1 \frac{\max_{m \in [n]} \sum_{i \in [m]} (\bar{y}_{mi} - \psi(\theta^* - \theta_i^*))^2}{\lambda_2(L)^2 e^{-2\kappa_E}} + C_1 \frac{e^{2\kappa_H n \max_{n}}}{\lambda_2(L)^2} \|\hat{\theta} - \theta^*\|_\infty.
\]

Following the same arguments towards Equation (66), we can get
\[
\|\theta^{(m)} - \hat{\theta} - a_m 1_{n-1}\|^2 \leq \frac{e^{2\kappa_E}}{c^3 \lambda_2(L)^2} \|\nabla \ell_n^{(-m)}(\hat{\theta} - m)\|_2^2.
\]

By Equation (51), for each $i \in [n] \setminus \{m\}$, we have
\[
\frac{\partial}{\partial \theta_i} \ell_n^{(-m)}(\theta - m) = \frac{\partial}{\partial \theta_i} \ell_n(\theta) - \frac{\partial}{\partial \theta_i} \ell_n^{(m)}(\theta | \theta - m),
\]
Using the fact that $\nabla \ell_n(\hat{\theta}) = 0$, we get

$$\frac{\partial}{\partial \theta_i} \ell_n^{(-m)}(\theta_m | \theta_m) \bigg|_{\theta = \hat{\theta}} = -\frac{\partial}{\partial \theta_i} \ell_n^{(m)}(\theta_m | \theta_m) \bigg|_{\theta = \hat{\theta}} = -A_{mi} \left[ \hat{y}_{mi} - \psi(\hat{\theta}_m - \hat{\theta}_i) \right].$$

Therefore, we have

$$\|\nabla \ell_n^{(-m)}(\hat{\theta}_m)\|_2^2 = \sum_{i \in [n] \setminus \{m\}} A_{mi} \left[ \hat{y}_{mi} - \psi(\hat{\theta}_m - \hat{\theta}_i) \right]^2 \leq 2 \sum_{i \in [n] \setminus \{m\}} A_{mi} \left( \hat{y}_{mi} - \psi(\theta^*_m - \theta^*_i) \right)^2 + 2 \sum_{i \in [n] \setminus \{m\}} A_{mi} \left( \psi(\theta^*_m - \theta^*_i) - \psi(\hat{\theta}_m - \hat{\theta}_i) \right)^2 \leq 2 \sum_{i \in [n] \setminus \{m\}} A_{mi} \left( \hat{y}_{mi} - \psi(\theta^*_m - \theta^*_i) \right)^2 + 2\|\hat{\theta} - \theta^*\|^2 \sum_{i \in [n] \setminus \{m\}} A_{mi} \leq 2 \sum_{i \in [n] \setminus \{m\}} A_{mi} \left( \hat{y}_{mi} - \psi(\theta^*_m - \theta^*_i) \right)^2 + 4n_{\max}\|\hat{\theta} - \theta^*\|^2 \infty.$$

Now use the fact that $1_n^T \theta^* = 1_n^T \hat{\theta} = 0$, we have

$$\|a_m 1_{n-1} - a_m 1_{n-1}\|_2^2 = (n-1)[\text{avg}(\hat{\theta}_m - \theta^*_m)]^2 = \frac{(\hat{\theta}_m - \theta^*_m)^2}{n-1} \leq \frac{\|\hat{\theta} - \theta^*\|^2_\infty}{n-1}. \tag{54}$$

These results, together with the fact that $\lambda_2(\mathcal{L}_A) \leq 2n_{\max} \leq 2n$, give Equation (54).

3. Show that on the same event,

$$\|\hat{\theta} - \theta^*\|_\infty \cdot C_{4} e^{-\kappa E} n_{\min} \leq \max_{m \in [n]} \sum_{i \in \mathcal{N}(m)} (\hat{y}_{mi} - \psi(\theta^*_m - \theta^*_i)) \bigg| + \sqrt{n_{\max}} \max_{m \in [n]} \|\theta^{(m)}_m - \theta^*_m - a_m 1_{n-1}\|_2 + \sqrt{n_{\max}} \max_{m \in [n]} \|\theta^{(m)}_m - \hat{\theta}_m - a_m 1_{n-1}\|_2. \tag{55}$$

First, define two univariate functions as some proxy of gradient and hessian:

$$g^{(m)}(\theta_m | \theta_m) = \frac{\partial}{\partial \theta_m} \ell_n^{(m)}(\theta_m | \theta_m) = -\sum_{i \in [n] \setminus \{m\}} A_{mi} (\hat{y}_{mi} - \psi(\theta_m - \theta_i))$$

$$h^{(m)}(\theta_m | \theta_m) = \frac{\partial^2}{\partial \theta^2_m} \ell_n^{(m)}(\theta_m | \theta_m) = \sum_{i \in [n] \setminus \{m\}} A_{mi} \psi(\theta_m - \theta_i) \psi(\theta_m - \theta_i).$$

By the definition of $\hat{\theta}$ and the shift invariance of $\ell_n$, we have $\ell_n(\hat{\theta}) \leq \ell_n(\theta)$ for any $\theta \in \mathbb{R}^n$, thus

$$\ell_n(\theta^*_m | \hat{\theta}_m) + \ell_n^{(-m)}(\hat{\theta}_m) \geq \ell_n(\hat{\theta}).$$

This implies

$$\ell_n^{(m)}(\hat{\theta}_m) \geq \ell_n^{(m)}(\hat{\theta}_m | \hat{\theta}_m - \hat{\theta}_m) = \ell_n^{(m)}(\hat{\theta}_m | \hat{\theta}_m) + (\hat{\theta}_m - \theta^*_m) g^{(m)}(\theta_m | \hat{\theta}_m) + \frac{1}{2} (\hat{\theta}_m - \theta^*_m)^2 h^{(m)}(\xi | \hat{\theta}_m),$$

where $\xi$ is a convex combination of $\theta^*_m$ and $\hat{\theta}_m$. By Lemma 23, we have $|\xi - \theta^*_m| \leq |\hat{\theta}_m - \theta^*_m| \leq 5$. Thus for any $i \neq m$ it holds that $|\xi - \hat{\theta}_i| \leq |\xi - \theta^*_m | + |\theta^*_m - \theta^*_i| + |\hat{\theta}_i - \theta^*_i| \leq 10 + \kappa$. By definition of $h^{(m)}$, we have $\frac{1}{2} h^{(m)}(\xi | \hat{\theta}_m) \geq c_2 e^{-\kappa E} n_{\min}$ for some constant $c_2 > 0$. Therefore, we get

$$\frac{(\hat{\theta}_m - \theta^*_m)^2}{(c_2 n_{\min})^2} \leq \frac{\epsilon^{2\kappa E}}{(c_2 n_{\min})^2} g^{(m)}(\theta_m | \hat{\theta}_m).$$

(56)
To bound $|g^{(m)}(\theta^*_m | \hat{\theta}_{-m})|$, we decompose it as

$$ |g^{(m)}(\theta^*_m | \hat{\theta}_{-m})| = \sum_{i \in [n] \setminus \{m\}} A_{mi}(\bar{y}_{mi} - \psi(\theta^*_m - \hat{\theta}_i)) $$

$$ \leq \sum_{i \in [n] \setminus \{m\}} A_{mi}(\bar{y}_{mi} - \psi(\theta^*_m - \theta^*_i)) $$

$$ + \sum_{i \in [n] \setminus \{m\}} A_{mi}(\psi(\theta^*_m - \theta^*_i) - \psi(\theta^*_m - \theta^{(m)}_i + a_m)) $$

$$ + \sum_{i \in [n] \setminus \{m\}} A_{mi}(\psi(\theta^*_m - \theta^{(m)}_i + a_m) - \psi(\theta^*_m - \hat{\theta}_i)). $$

By Cauchy-Schwartz inequality, we can bound (58) and (59) by

$$ \left| \sum_{i \in [n] \setminus \{m\}} A_{mi}(\psi(\theta^*_m - \theta^*_i) - \psi(\theta^*_m - \theta^{(m)}_i + a_m)) \right|^2 \leq n_{\max} ||\theta^{(m)}_m - \theta^*_m - a_m 1_{n-1}||^2_2, $$

$$ \left| \sum_{i \in [n] \setminus \{m\}} A_{mi}(\psi(\theta^*_m - \theta^{(m)}_i + a_m) - \psi(\theta^*_m - \hat{\theta}_i)) \right|^2 \leq n_{\max} ||\theta^{(m)}_m - \theta^*_m - a_m 1_{n-1}||^2_2. $$

Plugging these bounds into Equation (56) and taking maximum over $m \in [n]$ give the desired bound (55).

4. Plug (55) back into (54) and get

$$ \max_{m \in [n]} ||\theta^{(m)}_m - \hat{\theta}_{-m} - a_m 1_{n-1}||^2_2 \leq \frac{e^{2\kappa_E} \max_m \sum_{i \in \mathcal{N}(m)} (\bar{y}_{mi} - \psi^*_m)^2}{\lambda_2(\mathcal{L})^2} + \frac{e^{4\kappa_E} n_{\max} \max_{m \in [n]} \sum_{i \in \mathcal{N}(m)} (\bar{y}_{mi} - \psi^*_m)^2}{\lambda_2(\mathcal{L})^2} $$

$$ + \frac{e^{4\kappa_E} n_{\max} \max_{m \in [n]} ||\theta^{(m)}_m - \theta^*_m - a_m 1_{n-1}||^2_2}{\lambda_2(\mathcal{L})^2} $$

as we assume $\frac{e^{2\kappa_E} \max_m}{\lambda_2(\mathcal{L}) n_{\min}} \leq \frac{1}{2}$, we have

$$ \max_{m \in [n]} ||\theta^{(m)}_m - \hat{\theta}_{-m} - a_m 1_{n-1}||^2_2 \leq \frac{e^{2\kappa_E} (\log n + n_{\max})}{\lambda_2(\mathcal{L})^2 L} + \frac{e^{4\kappa_E} n_{\max} \log n}{\lambda_2(\mathcal{L})^2 n_{\min}^2 L} + \frac{e^{4\kappa_E} n_{\max}^3 \log n}{\lambda_2(\mathcal{L})^2 n_{\min}^2 L}. $$

(60)

5. Plug (60) back into (55) and we can get

$$ ||\hat{\theta} - \theta^*||^2 \leq \frac{e^{2\kappa_E} n_{\max} \log n}{n_{\min}^2 L} + \frac{e^{2\kappa_E} n_{\max}^2 e^{2\kappa_E} n_{\max} n}{\lambda_2(\mathcal{L})^2 L} $$

$$ + \frac{e^{2\kappa_E} n_{\max}^3 \log n}{n_{\min}^2 \lambda_2(\mathcal{L})^2 L} \left[ e^{2\kappa_E} (\log n + n_{\max}) + \frac{e^{4\kappa_E} n_{\max}^3 \log n}{n_{\min}^2} + \frac{e^{6\kappa_E} n_{\max}^3 n_{\max}^3}{\lambda_2(\mathcal{L})^2 n_{\min}^2} \right]. $$

(61)

One term can be reduced and the inequality becomes

$$ ||\hat{\theta} - \theta^*|| \leq e^{\kappa_E} \sqrt{\frac{n_{\max} \log n}{Ln_{\min}^2}} + e^{2\kappa_E} \sqrt{\frac{n_{\max}^3}{L\lambda_2(\mathcal{L})^2 n_{\min}^2}} \left[ 1 + e^{2\kappa_E} \sqrt{\frac{\log n}{n_{\min}^2}} + e^{2\kappa_E} \sqrt{\frac{n_{\max} n}{\lambda_2(\mathcal{L})^2 n_{\min}^2}} \right]. $$

(62)
A.7.2 Proof of Lemmas

Proof of Lemma 23. We will use a gradient descent sequence defined by

$$\theta^{(t+1)} = \theta^{(t)} - \eta [\nabla \ell_n(\theta^{(t)}) + \rho \theta^{(t)}].$$

The leave-one-out negative log-likelihood is defined as

$$\ell^{(m)}(\theta) = \sum_{1 \leq i < j \leq n, i, j \neq m} A_{ij} \left[ \tilde{y}_{ij} \log \frac{1}{\psi(\theta_i - \theta_j)} + (1 - \tilde{y}_{ij}) \log \frac{1}{1 - \psi(\theta_i - \theta_j)} \right]$$

$$+ \sum_{i \in [n] \setminus \{m\}} A_{mi} \left[ \psi(\theta_i^* - \theta_m^*) \log \frac{1}{\psi(\theta_i - \theta_m^*)} + \psi(\theta_m^* - \theta_i^*) \log \frac{1}{\psi(\theta_m^* - \theta_i^*)} \right],$$

so the leave-one-out gradient descent sequence is defined as

$$\theta^{(t+1,m)} = \theta^{(t,m)} - \eta [\nabla \ell^{(m)}(\theta^{(t,m)}) + \rho \theta^{(t,m)}].$$

We initialize both sequences by $\theta^{(0)} = \theta^{(0,m)} = \theta^*$ and set $\rho = \frac{1}{n} \sqrt{\frac{n_{\text{max}}}{L}}$ and step size $\eta = \frac{1}{\lambda_2 n_{\text{max}}}$. We will show that under the assumption $\lambda_2 (\mathcal{L}_A)^2 L > C e^{2\kappa E} \max\{n_{\text{max}} \log n, e^{2\kappa E} n_{\text{max}}^2 / n_{\text{min}}^2 \}$ for some large constant $C > 0$, we have

$$\max_{m \in [n]} \| \theta^{(t,m)} - \theta^{(t)} \|_2 \leq f_1 := C_1 \frac{e^{2\kappa E}}{\lambda_2} \sqrt{\frac{n_{\text{max}} \log n}{L}} \leq 1$$

$$\| \theta^{(t)} - \theta^* \|_2 \leq f_2 := C_2 \frac{e^{2\kappa E}}{\lambda_2} \sqrt{\frac{n_{\text{max}}}{L}} \leq \sqrt{\frac{n}{\log n}}$$

$$\max_{m \in [n]} | \theta^{(t,m)} - \theta^* | \leq f_3 := C_3 \frac{e^{2\kappa E}}{\lambda_2} \frac{n_{\text{max}}}{n_{\text{min}}} \sqrt{\frac{n}{L}} \leq 1.$$  (63)

A useful fact given that (63) holds is that

$$\| \theta^{(t,m)} - \theta^* \|_\infty \leq f_1 + f_2.$$  (64)

We again have the Taylor expansion

$$\theta^{(t+1)} - \theta^{(t+1,m)} = [(1 - \eta \rho) I_n - \eta H(\xi)] (\theta^{(t,m)} - \theta^{(t)}) - \eta [\nabla \ell_n(\theta^{(t,m)}) - \nabla \ell^{(m)}(\theta^{(t,m)})].$$

Now by the fact that $\lambda_{\text{min,}1}(H(\xi)) \geq c_0 e^{-\kappa} \lambda_2$, we have

$$\| [(1 - \eta \rho) I_n - \eta H(\xi)] (\theta^{(t,m)} - \theta^{(t)}) \|_2 \leq (1 - \eta \rho - c_1 \eta \lambda_2) \| \theta^{(t,m)} - \theta^{(t,m)} \|_2$$

for some constant $c_1 > 0$ and the other term can be bounded as

$$\| \nabla \ell_n(\theta^{(t,m)}) - \nabla \ell^{(m)}(\theta^{(t,m)}) \|_2^2$$

$$= \left[ \sum_{j \in [n] \setminus \{m\}} A_{jm} (\tilde{y}_{jm} - \psi(\theta_j^* - \theta_m^*)) \right]^2 + \sum_{j \in [n] \setminus \{m\}} A_{jm} (\tilde{y}_{jm} - \psi(\theta_j^* - \theta_m^*))^2$$

$$\leq C_1 \frac{1}{L} n_{\text{max}} \log n + C_1 \frac{1}{L} (\log n + n_{\text{max}}).$$

Therefore, for

$$\| \theta^{(t+1)} - \theta^{(t+1,m)} \|_2 \leq (1 - c_1 \eta \lambda_2) f_1 + \sqrt{2C_1 \frac{1}{L} n_{\text{max}} \log n} \leq f_1$$

to hold, we need $f_1 > C_1 \frac{e^{2\kappa E}}{\lambda_2} \frac{n_{\text{max}} \log n}{L}$ for some sufficiently large positive constant $C > 0$.

Next, we bound $\| \theta^{(t+1)} - \theta^* \|$. By Tarlor expansion,

$$\theta^{(t+1)} - \theta^* = ((1 - \eta \lambda) I_n - \eta H(\xi)) (\theta^{(t)} - \theta^*) - \eta \lambda \theta^* - \eta \nabla \ell_n(\theta^*).$$
Equation (41) becomes
\[
((1 - \eta\lambda)I_n - \eta H(\xi)) (\theta(t) - \theta^*) \leq (1 - \eta\lambda - c_2\eta\lambda^2) \|\theta(t) - \theta^*\|_2,
\]
and equation (42) becomes
\[
\|\nabla \ell_n(\theta^*)\|_2^2 = \sum_{i=1}^n \left( \sum_{j \in [n] \setminus \{i\}} A_{ij} (\bar{g}_{ij} - \psi(\theta_i^* - \theta_j^*)) \right)^2 \leq C_2 \frac{n n_{\text{max}}}{L},
\]
for some constants $c_2, C_2 > 0$. Therefore,
\[
\|\theta(t+1) - \theta^*\|_2 \leq (1 - c_2\eta\lambda^2)f_2 + \eta \sqrt{C_2 \frac{n n_{\text{max}}}{L}} + \eta \lambda\|\theta^*\|_2.
\]
For $\|\theta(t+1) - \theta^*\| \leq f_2$ to hold, we need $\frac{\lambda\eta}{c_2^2 f_2} C_2 > \frac{n n_{\text{max}}}{L}$ for some sufficiently large constant $C$, which is guaranteed by the definition of $f_2$.

Next, we bound $|\theta_m(t+1,m) - \theta_m^*|$. Note that by the definition of the gradient descent
\[
\theta_m(t+1,m) - \theta_m^* = \left[ 1 - \eta\lambda - \eta \sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j) \right] (\theta_m(t,m) - \theta_m^*) - \lambda\eta\theta_m^* + \eta \sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j)(\theta_j(t,m) - \theta_j^*)
\]
where $\xi_j$ is a scalar between $\theta_m^* - \theta_j^*$ and $\theta_m(t,m) - \theta_j(t,m)$. Since $\|\theta(t,m) - \theta^*\|_{\infty} \leq 3$, we have $|\xi_j - \theta_m^* + \theta_j^*| \leq |\theta_m^* - \theta_j^* - \theta_m(t,m) + \theta_j(t,m)| \leq 6$ and $\|\xi\|_{\infty}$ is bounded. Therefore,
\[
\sum_{j \in [n] \setminus \{m\}} A_{mj} \psi'(\xi_j) \geq c_3 \min_{i \in [n]} n_i
\]
and
\[
\left| \sum_{j \in \mathcal{N}(m)} A_{mj} \psi'(\xi_j)(\theta_j(t,m) - \theta_j^*) \right| \leq \sqrt{\sum_{j \in \mathcal{N}(m)} [A_{mj} \psi'(\xi_j)]^2} \sqrt{\sum_{j \in \mathcal{N}(m)} (\theta_j(t,m) - \theta_j^*)^2}
\leq c_4 \sqrt{n_{\text{max}}} (f_1 + f_2).
\]
for some constant $c_3, c_4 > 0$. Thus we have
\[
|\theta_m(t+1,m) - \theta_m^*| \leq (1 - c_3\eta n_{\text{min}}) + \eta \sqrt{\eta n_{\text{max}}} (f_1 + f_2) + \lambda\eta\theta_m^*.
\]
For $|\theta_m(t+1,m) - \theta_m^*| \leq f_3$ to hold, we need $\frac{\eta n_{\text{min}}}{c_3^2} f_5 > C \sqrt{n_{\text{max}}} f_2$ for some sufficiently large constant $C > 0$, which is ensured by the definition of $f_2, f_3$.

As the last step, we again use the fact that $\ell_\rho(\cdot)$ is $\rho$-strongly convex and $(\rho + n_{\text{max}})$-smooth (see definition in the paragraph before Equation (10)), so by Theorem 3.10 in Bubeck (2015), we have
\[
\|\theta(t) - \hat{\theta}_\rho\|_2 \leq (1 - \frac{\rho}{\rho + n_{\text{max}}})^T \|\theta^* - \hat{\theta}_\rho\|_2.
\]
By a union bound, Equation (63) holds for all $t \leq T$ with probability at least $1 - O(Tn^{-10})$. Triangle inequality implies that
\[
\|\hat{\theta}_\rho - \theta^*\|_{\infty} \leq \|\theta(T) - \hat{\theta}_\rho\|_2 + \|\theta(T) - \theta^*\|_{\infty} \leq (1 - \frac{\rho}{\rho + n_{\text{max}}})^T \sqrt{n} \|\theta_\rho - \theta^*\|_2 + 2
\]
Take $T = n^5$ and remember that $L \lesssim n^5$. If $\rho > n_{\text{max}}$, then $(1 - \frac{\rho}{\rho + n_{\text{max}}})^T \sqrt{n} \leq 2^{-n^5} \sqrt{n} \leq 1/2$. Otherwise, since
\[
(1 - \frac{\rho}{\rho + n_{\text{max}}})^T \sqrt{n} \leq \exp \left( - \frac{T \rho}{\rho + n_{\text{max}}} \right) \sqrt{n},
\]
using the fact that \( \kappa < n \), we have

\[
(1 - \frac{\rho}{n} + n_{\max})^T \sqrt{n} \leq \exp \left(-\frac{T}{\epsilon \kappa} \sqrt{\frac{1}{n_{\max} L}}\right) \sqrt{n} \leq c e^{-n^{3/2}} \leq \frac{1}{2}.
\]

In conclusion, we have \( \|\hat{\theta}_\rho - \theta^*\|_\infty \leq \frac{1}{2} \|\hat{\theta}_\rho - \theta^*\|_\infty + 2 \), thus \( \|\hat{\theta}_\rho - \theta^*\|_\infty \leq 4 \), with probability at least \( 1 - O(n^{-5}) \). \( \square \)

**Proof of Lemma 24.** By definition, \( \theta_{(m)}^* \) is a constrained MLE on a subset of the data, thus by Taylor expansion, for \( \xi \) given by a convex combination of \( \theta_{-m}^* \) and \( \theta_{-m}^{(m)} \), we have

\[
\ell_n^{(-m)}(\theta_{-m}^*) \geq \ell_n^{(-m)}(\theta_{-m}^{(m)})
\]

\[
= \ell_n^{(-m)}(\theta_{-m}^*) + (\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1})^T \nabla \ell_n^{(-m)}(\theta_{-m}^*)
\]

\[
+ \frac{1}{2} (\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1})^T H^{(-m)}(\xi)(\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1}),
\]

where we use the invariant property of \( \ell_n^{(-m)}(\theta_{-m}) \), i.e., \( \ell_n^{(-m)}(\theta_{-m}) = \ell_n^{(-m)}(\theta_{-m} + c1_{n-1}) \). By the fact that \( \|\xi - \theta_{-m}^*\|_\infty \leq \|\theta_{-m}^{(m)} - \theta_{-m}^*\|_\infty \leq 5 \) and Lemma 8, we have

\[
(\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1})^T H^{(-m)}(\xi)(\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1}) \geq c e^{-\kappa E} \lambda_2(\mathcal{L}_{A_{-m}}) \|\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1}\|_2^2.
\]

Applying Cauchy-Schwarz inequality to the expansion and we can get

\[
\|\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1}\|_2 \leq \frac{e^{2\kappa E}}{c^2 \lambda_2(\mathcal{L}_{A_{-m}})^2} \|\nabla \ell_n^{(-m)}(\theta_{-m}^*)\|_2^2.
\]

Where \( A_{-m} \) is the adjacency matrix of the comparison graph with node \( m \) excluded. By the interlacing property of the eigenvalue sequences of Laplacians of graph and its induced subgraph (see, e.g. Brouwer and Haemers, 2012, Proposition 3.2.1), we have \( \lambda_2(\mathcal{L}_{A_{-m}}) \geq \lambda_2(\mathcal{L}_A) \), thus

\[
\|\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1}\|_2 \leq \frac{e^{2\kappa E}}{c^2 \lambda_2(\mathcal{L}_A)^2} \|\nabla \ell_n^{(-m)}(\theta_{-m}^*)\|_2^2.
\]

(66)

Now by Lemma 7, it holds with probability at least \( 1 - O(n^{-10}) \) that

\[
\|\theta_{-m}^{(m)} - \theta_{-m}^* - a_m 1_{n-1}\|_2 \leq C \frac{e^{2\kappa E} R_{\max} N}{\lambda_2(\mathcal{L}_A)^2},
\]

and the conclusion is guaranteed by a union bound. \( \square \)