Four algorithms for propositional forgetting

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Four algorithms for propositional forgetting

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Abstract

Four algorithms for propositional forgetting are compared. The first performs all possible resolutions and deletes the clauses containing a variable to forget. The second forgets a variable at time by resolving and then deleting all clauses that resolve on that variable. The third outputs the result of all possible linear resolutions on the variables to forget. The fourth generates a clause from the points of contradiction of a backtracking search. The latter emerges as the winner, with the second and first performing well in some cases. The linear resolution algorithm is inadequate, at least in the current implementation.

1 Introduction

Logical forgetting is reducing the amount of information by eliminating the parts that are related to some kind of entities such as variables, objects or conditions [Del17]. Propositional, modal, temporal, action, defeasible, description and first-order logics are fields of application [Boo54, Moi07, Del17, vDHLM09, FAS+20, EF07, RHPT14, AEW12, KWW09, EIS+06, LR94, ZZ11]. Others are answer set programming [WZZZ14, GKL16], argumentation theory [BDR20], belief revision [NCL07] and circumscription [WWWZ15]. In the propositional case, it is also called variable elimination [SP04] and is the core of one of the earliest mechanisms for automated reasoning [DP60]. In modal logics, it is often associated with its dual, uniform interpolation [KWW09].

The applications of logical forgetting include relevance [LLM03], updating [NCL07], automated reasoning [DP60, DR94, SP04], privacy preserving [GKLW17], limited reasoning agents [FHMV95, RHPT14], knowledge focusing [EKI19] and consistency restoring [LM10].

Automated reasoning highlights the two sides of logical forgetting. The algorithm by Davis and Putnam [DP60] establishes the satisifiability of a propositional formula by repeatedly eliminating variables while preserving the satisfiability of the formula. Bucket elimination [DR94, RKL+21] improves this mechanism. The NiVER system [SP04] removes variables only if it does not take too much time or produce too large a formula, which would slow down the overall process. Running time is central.

Yet, satisfiability of a single formula is not always the final goal of automated reasoning. The same logical knowledge base may be queried many times as to whether it entails or is consistent with other information. As done by materialized views in databases [MRSR01], if such queries are often about some specific topics, it may be convenient to isolate the

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relevant part of the knowledge base. The running time of forgetting is counterbalanced by
the speed-up it produces among many queries.

More generally, the aim of forgetting dictates its efficiency requirements. Forgetting may
be required for legal reasons [GKLW17], where applications are usually not time constrained. 
Forgetting may be done to tell whether some variables are related or influence some others; 
the result is yes/no; even the size of the result of forgetting is not that important.

An algorithm performing a certain number of steps consumes at most a unit of memory 
in each. Time caps memory. Not the other way around. Memory can be reused. As an 
example, a trivially memory-efficient algorithm for propositional satisfiability loops over all 
possible evaluations of the variables, checking the satisfaction of the formula against each. 
The only required memory is that needed for storing the interpretation, the formula and the 
value of each subformula according to the interpretation. Yet, the running time is linear in 
the number of possible interpretations, which is exponential in the number of variables.

In practice, a computer may be run some more if time allows, but cannot be indefinitely 
increased in memory. Hitting the memory limit prevents any output to be generated. Ironi-
cally, this may even happen when the output is very small. As an example, the satisfiability 
algorithm by Davis and Putnam [DP60] may exponentially blow up the input formula by 
iteratively eliminating variables, but the output is always a single bit: either the formula is 
satisfiable or it is not. The output is always small when forgetting most of the variables, 
as it is when focusing information on a specific topic or querying is required only on few 
variables.

Depending on the goal of forgetting, its result may not even need to be stored explicitly. 
For example, it may be discarded once a variable is found to be influenced by certain others. 
It is also not needed if its final purpose is to establish whether it is equivalent to a given 
other formula [Lib20].

Memory requirements hit all three algorithms currently defined for forgetting in proposi-
tional logics: double substitution, prime implicate selection and variable elimination by 
resolution.

Double substitution was first introduced by Boole [Boo54] as variable marginalization: 
a variable is first assumed true, then assumed false; forgetting is the disjunction of the two 
cases. Prime implicate selection [LLM03, Proposition 19] generates all prime implicates and 
selects only the ones that do not contain any variable to forget. Variable elimination [DW13, 
Wan15, Del17] resolves and discards all clauses that contain the variables to forget.

All three methods may take exponential space. This is particularly bad as the theory 
predicts that it is never necessary [Lib20]: forgetting is at the second level of the poly-
nomial hierarchy [Sto76], which implies its membership to PSPACE. It can always be done in 
polynomial space.

A naive but polynomial-space mechanism proves the claim: iterating over all possible 
values of the variables to remember. Each one is tested for inconsistency with the input 
formula. If so, the clause made by the negation of the literals it satisfies is generated. Each 
satisfiability check takes linear space. Iterating over the values of the variables is linear in 
space as well. Overall, forgetting this way only takes a linear amount of space. It is still 
not a good way to forget since it always takes exponential running time, even when other 
algorithms would be fast, such as when forgetting few variables. A similar issue occurs in 
Answer Set Programming: the semantical forms of forgetting require looping over all possible 
answer sets the result may have, if implemented literally; for this reason, syntactical methods
are looked for [BGKL19].

Whether forgetting can always be done in polynomial space is known: yes. Both in theory and in practice. The open problem is to forget in polynomial space as quickly as possible, not taking exponential time in all cases.

Four algorithms are compared:

**Resolution closure** All clauses are resolved; the subsumed ones are removed. The ones comprising variables to remember are output. It is correct because only entailed clauses are produced this way. It is complete because the prime implicates of forgetting are the minimally entailed clauses not containing a variable to forget [LLM03]. This mechanism is a baseline for forgetting: it does not require anything else than generating minimal clauses by resolution and selecting some by a simple rule. It is useful to show how better the others fare.

**Variable elimination** All clauses containing a variable to forget are iteratively resolved. The results are clauses not containing that variable. These and the other clauses of the formula make the result of forgetting. Variable marginalization (replacing a variable with true and false and disjoining) gives the same result when applied to a set of clauses, if the result is converted back to a set of clauses.

Variable elimination by resolution was introduced by Davis and Putnam [DP60] to establish the satisfiability of a formula: when all variables are removed, it produces no clauses if the formula is satisfiable and an empty clause otherwise. It was later employed on some of the variables to speed up the following call to a satisfiability checker [SP04]. When applied on a given set of variables, it forgets them from the formula [DW13, Wan15, Del17].

Research on propositional satisfiability concluded that this mechanism takes exponential space on some formulae when forgetting all variables regardless of the order of removal of variables [Gal77]. This is exactly one of the cases where superpolynomial space should be avoided: the result is small, but the memory necessary for producing it is large.

**Linear resolution** Variable elimination is the same as ordered resolution. Proof complexity proves that it is surpassed on proof size by linear resolution [Goe92, BEGJ00, BP07]. This makes linear resolution a reasonable candidate for forgetting.

**Backtracking** Partial models of a formula are found by backtracking over the possible values of the variables. In reverse, backtracking finds partial models falsifying the formula. These are dual to entailed clauses. The ones including variables to forget make the result of forgetting. They are obtained by delaying the assignment of the variables to forget. The resulting algorithm is guaranteed to only take polynomial working space. Unit propagation [DG84] speeds it up.

These four algorithms are detailed in a following section each. Section 7 describes their implementation. A number of experiments have been performed to assess their computational properties: time and memory used and size of result. These experiments and their results are in Section 8. Section 9 presents some considerations on how an arbitrary proof method could be adapted to forgetting. Section 10 presents some conclusive remarks.
The backtracking algorithm proves superior or comparable to the others on most considered measures. It is the quicker and takes comparable or less memory than the best of the others, depending on how memory is measured. Its main drawback is that it rebuilds its output from the semantics of the formula, disregarding its syntax; this output may not be the most intuitive expression of forgetting.

2 Preliminaries

The logic considered in this article is propositional Boolean logic over a finite alphabet. It is based on a finite set of variables. A literal is a variable or its negation. A clause is a set of literals, and is interpreted as their disjunction: either one of them is true; it is written using the symbol $\lor$ between literals. A formula is a set of clauses, and is interpreted as their conjunction: all of them are true. An empty clause is a clause containing no literals. It is denoted by $\bot$ and is always false by definition.

For every clause $C$, the notation $\neg C$ is used for the set of literals that are exactly the opposite of each literal in $C$. For example, $\neg(a \lor \neg b)$ is $\{\neg a, b\}$. In the other way around, if $I$ is a partial interpretation, then $\neg I$ is the clause that is falsified exactly by $I$. For example, $\neg\{\neg a, b\}$ is $a \lor \neg b$.

Forgetting a variable is removing that variable from the formula while maintaining the semantic of the formula on all other variables. The typical definition is that the result entails another formula on the other variables if and only if the original formula does. This is the same with mutual consistency instead of entailment.

Resolution [Rob65] is a rule that derives a clause $C \lor D$ from two clauses $C \lor x$ and $D \lor \neg x$. The resulting clause is called the resolvent of the two and is denoted by $C \ast D$. It logically follows from them. This makes resolution a correct inference rule. Obtaining a clause $C$ from a set of clauses $F$ by resolving clauses an arbitrary number of times is denoted $F \vdash C$. Since resolution is correct, a consequence is $F \models C$. Resolution is also refutationally complete: if $F \models \bot$ then $F \vdash \bot$.

Resolution is entailment-complete [SCL69]: if $F \models C$ then $F \vdash C'$ where $C' \subseteq C$. This property is lost if resolution is restricted in certain ways that are still refutationally complete, such as resolving variables in a certain order [AM05].

3 Resolution closure

An equivalent definition of forgetting is the set of prime implicates of the formula that comprise the variables to remember [LLM03]. Resolution generates all prime implicates of the formula, among others [SCL69]. The ones comprising variables to remember express forgetting.

In order to reduce the number of generated clauses, the non-minimal ones are removed multiple times. This is correct since a clause that contains another is entailed by it. At each step, all pairs of clauses that resolve are resolved. Their resolvents are added to the formula, which is then simplified by the removal of non-minimal clauses.

Algorithm 1 (Forget by Closure)

\begin{verbatim}
forget_close(Formula F, Variables V)
\end{verbatim}
1. \( R = \emptyset \)

2. \( N = F \)

3. while \( N \neq R \)
   
   (a) \( R = N \)
   
   (b) for each \( C \lor x, D \lor \neg x \in R \) for some variable \( x \)
       
       i. \( N = N \cup \{ C \lor x \land D \lor \neg x \} \)
   
   (c) \( N = \{ C \in N \mid \neg \exists C' \in N, C' \subset C \} \)

4. return \( \{ C \in R \mid \exists x \in V. x \in C \lor \neg x \in C \} \)

At each step, the current formula is copied into a new set \( R \). All pairs of clauses of \( R \) that resolve are resolved: \( C \lor x \land D \lor \neg x \) denotes the resolvent of \( C \lor x \) and \( D \lor \neg x \), which is added to \( N \). This set is then minimized. If nothing changes, the loop ends. The return value is the set of clauses of \( R \) not containing any variable to forget.

Minimizing at every step takes time but limits the inflation of the formula. Not doing it would unnecessarily enlarge \( N \) by redundant clauses, which may resolve into even more clauses in the following steps.

The algorithm is very simple. Apart from resolution and minimization, it only takes few lines of code. A couple of additional data structures would improve it.

The first is an index of the clauses by literals. For each literal, a set stores all clauses that contain it. This speeds up finding the clauses that resolve. It negates the need of checking every pair of clauses of the formula. If a clause is in the set of \( x \) and another in the set of \( \neg x \), they are guaranteed to resolve. The only required check is whether their resolvent is tautology.

The second is an index of the clauses by size. It simplifies minimizing. A clause no longer needs to be checked against all others, only against the smaller ones. Starting from the smallest increases the chances of detecting redundant clauses earlier.

### 4 Variable elimination

A way to forget a variable from a formula is to remove all clauses containing that variable after resolving them in all possible ways. This is called variable elimination or resolving that variable out. It was introduced by Davis and Putnam [DP60] to establish the satisfiability of a formula. Eliminating a variable from a formula preserves all consequences of the formula that do not contain the variable. This includes contradiction: a formula entails contradiction if and only if it does so after eliminating all variables.

The method was applied to forgetting in the Horn case by Delgrande and Wasserman [DW13] and in the general case by Wang [Wan15] and Delgrande [Del17]. Only a change is required: not all variables are removed, only the ones to forget.

**Algorithm 2 (Forget by Elimination)**

\[
\text{forget_eliminate(Formula } F, \text{ Variables } V)\]
1. for each $x \in V$:

   (a) $P = \{C \in F \mid x \in C\}$
   
   (b) $N = \{C \in F \mid \neg x \in C\}$
   
   (c) $F = F \setminus (P \cup N) \cup \{C \ast D \mid C \in P, D \in N\}$

2. return $F$

Proof complexity classes variable elimination as an exponential-size method: for some formulae, the size of a minimal resolution proof is exponential [Gal77]. This is unconditional: it does not depend on any complexity theory assumption like the non-equality of the complexity classes P and NP. A proof being exponential means that the clauses become exponentially many when eliminating all variables.

The algorithm has a long history, starting more than half a century ago [DP60, RD00, OWB21]. Even its version that removes only a subset of the variables [SP04] predates its application to propositional forgetting. The original algorithm by Davis and Putnam includes two optimizations (unit propagation and pure literals) that are irrelevant to proof complexity and are neglected by the adaptation to forgetting.

The program is implemented using the same resolution subroutine of the others. It could be sped up by indexing the clauses by the literals they contain: for each variable, two sets store the clauses that respectively include the variable and its negation. These sets would make finding the clauses to resolve trivial: each clause of the first set is resolved with each of the second. The non-tautological results are kept, the original clauses discarded.

5 Linear resolution

Variable elimination is a form of resolution, called directional resolution [DR94]. Adapting it to forgetting is straightforward, by resolving on variables to forget only. On the downside, it is not the best resolution strategy, both from the theoretical and practical point of view [Gal77, Goe92, BEGJ00, DR94]. Better resolution policies exists. A commonly used one is linear resolution [Lov70, Luc70, ZS71, Ino92, BJ16]. Adapting it to forgetting is equally simple, but requires a separate proof of correctness.

Linear resolution hinges around a center clause. This is initially a clause of the input formula. At each step, the policy is to always resolve the center clause with a side clause, which is either a clause of the input formula or a previous center clause. The result is the new center clause. This mechanism is refutationally complete: if a formula is unsatisfiable, the empty clause follows from a linear resolution.

Forgetting requires all clauses of the original formula to be linearly resolved in all possible ways. Running time decreases by reducing the number of lines to explore. Forgetting allows for two reductions:

- only resolving on variables to forget;
- stopping when the center clause does not contain any variable to forget.
Like all restrictions of resolution, linear resolution only generates clauses entailed by the original formula. The variant for forgetting only generates clauses comprising variables to remember. The result is a formula on the variables to remember that is entailed by the original formula. This is not enough. A further requirement is that every formula on the variables to remember that is entailed by the original formula is also entailed by the result of forgetting. Proving this condition only on the clauses comprising all variables to remember suffices [Lib20].

A clause that contains all variables is a full clause [AK18]. A clause that contains all variables to remember is a full remembrance clause. Linear resolution with the two additional restrictions for forgetting is proved to generate a subset of every full remembrance clause that is entailed by the formula.

A full remembrance clause \( C \) is entailed by \( F \) if and only if \( F \cup \neg C \models \bot \), where \( \neg C \) is the set comprising the negation of the literals of \( C \):

\[
\neg C = \{ \neg l \mid l \in C \}
\]

Clauses entailed by others can be removed from a formula without altering its semantics. This is the case for clauses that contain others. Therefore, \( F \cup \neg C \models \bot \) still holds after removing from \( F \) all clauses that contain a literal in \( \neg C \), the opposite of a literal of \( C \).

\[
G = \{ D \in F \mid \not\exists l \in C . \neg l \in D \}
\]

The entailment \( G \cup \neg C \models \bot \) holds since \( G \) is equivalent to \( F \). Therefore, the empty clause follows from \( G \cup \neg C \) by linear resolution. This is denoted \( G \cup \neg C \vdash \bot \). The claim is that such a linear resolution can be altered to derive a subset of \( C \) from \( G \) while obeying the two above restriction.

\[
? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad \bot
\]

The derivation starts from a clause of \( G \cup \neg C \) and iteratively resolves it with other clauses until it is empty. These unknown clauses and their resolvents are denoted as question marks in the figure. If none of them contain \( l \) or \( \neg l \), then \( G \vdash \bot \), which means that \( \bot \) expresses forgetting.

Otherwise, some clauses in the derivation contain \( l \) or \( \neg l \). These literals are removed by some resolution steps since the final clause \( \bot \) does not contain them. By construction, \( l \) is negative only in the clause \( \neg l \). This clause only resolves with a clause containing \( l \), and the result does not contain these literals. The conclusion is that the only clause containing \( \neg l \) in the derivation is the clause \( \neg l \) itself. No other clause contains \( \neg l \) together with other literals.

At some point, resolution removes \( l \) since the final clause is empty. Since \( \neg l \) is the only clause containing \( \neg l \) that occurs in the derivation, this step resolves it with a clause \( D \lor l \).

The resolution of \( \neg l \) with a clause \( D \lor l \) may occur several times in the derivation. The first occurrence is considered. The center clause may be \( \neg l \) or \( D \lor \neg l \). The first case can be reduced to the second: since \( \neg l \) is also a clause of \( G \cup \neg C \), everything before it can be cut out, and this clause swapped with the other.
This change turns $\neg l$ from a center clause to a side clause.

The general situation is that $\neg l$ is a side clause and $D \lor l$ is a center clause. Resolving them gives $D$. The following clauses are denoted $E$, $F$, and so on.

All clauses that resolve with $D$ also resolve with $D \lor l$. Removing the resolution of $D \lor l$ and $\neg l$ turns $D$ into $D \lor l$, but all following resolutions are still valid. The only difference is that $l$ remains in the following clauses.

A following step may still resolve $l$ with $\neg l$, but the same procedure applies again. This resolution is removed and $l$ added to the following clauses. The final clause is added $l$.

Since $\neg l$ is not in this derivation, it can be removed from the formula $G \cup \neg C$. The result is a derivation from $G \cup \neg C \setminus \{\neg l\}$ to $\bot \lor l$.

The same mechanism can be iterated over all literals of $\neg C$. If $\neg l \in \neg C$ is not involved in the derivation, it can be removed from the formula. Otherwise, removing all resolutions with $\neg l$ adds $l$ to the derivation result. The result is a derivation from $G$ to a subset of $C$.

Since all resolutions on a clause of $C$ have been removed and $C$ contains all variables to remember, this derivation only resolves on the variables to forget. It meets the first condition: only resolve on variables to forget.

The second condition is that resolution stops on clauses comprising variables to remember. The contrary is that a center clause $D$ that is not the final one of the derivation contains only variables to remember. Since $D$ is not the final clause, it resolves with a side clause $E$.  

8
Since $D$ only contains variables to remember, the resolution with $E$ is on one of them. But all resolutions on variables to remember have been removed from the derivation. This is a contradiction.

This proves that if $C$ is an entailed clause comprising all variables to remember, a subset $C'$ of its is at the end of a derivation that only resolves on variables to forget and stops at the first clause comprising variables to remember.

The center clause of a linear resolution may be resolved with an arbitrary clause of the formula or any previous clause in the line. Each choice is a new derivation to explore. Efficiency increases by reducing their number while still maintaining refutation completeness. This is achieved by either restricting the clauses in the line or the resolving literals. The first possibility is for example exploited by $s$-linear resolution and merge resolution, the second by A-ordering resolution or the various kinds of $SL$-resolution.

An A-ordering is just an arbitrary linear order of the variables. A center clause is always resolved on its maximal variable.

This restriction extends to forgetting if the variables to forget are larger than the variables to remember. If a center clause contains at least a variable to forget then resolution is on one of them. Otherwise, resolution stops because the center clause contains only variables to remember. This ensures the first condition for forgetting: only resolve on variables to forget. The conclusion is that every A-ordering that compares the variables to forget over the variables to remember can be used for forgetting, since a subset of every full remembrance clause is derived by resolution with the ordering.

The complete algorithm for forgetting by A-ordering linear resolution follows.

**Algorithm 3 (Forget by A-Ordering Linear Resolution)**

$\text{forgetlinear}(\text{Clause } C, \text{ Formula } F, \text{ Variables } V)$$

1. $X = \text{Var}(C) \cap V$
2. if $X = \emptyset$
   
   (a) return \{C\}
3. $x = \text{max}(X)$
4. let $l \in C$ be such that $\text{Var}(l) = x$
5. $R = \emptyset$
6. for each $D \in F$ such that $-l \in D$:
   
   (a) $E = C \ast D$
   (b) $R = R \cup \text{forgetlinear}(E, S, V)$
7. return $R$

$\text{forget_linear}(\text{Formula } F, \text{ Variables } V)$

1. $R = \emptyset$
2. for each $C \in F$:
   
   (a) $R = R \cup \text{forget_linear}(C, F, V)$
3. return $R$

6 Backtracking

The satisfiability of a propositional formula can be established by backtracking: each variable is first set to true and then to false; for each value, a recursive call establishes whether the formula is consistent with that value.

The parameters of the recursive calls are the formula and the current values of some variables. The base case is when the formula evaluates to true or false.

Every pair of recursive subcalls adds assignments that are incompatible ($x = \text{true}$ and $x = \text{false}$ conflict with each other) and cover all possible cases ($\text{true}$ and $\text{false}$ are the only values a Boolean variable $x$ may take). Because of this, the partial assignments in the leaves are incompatible with each other and cover all cases.

A partial assignment $I$ is denoted as the set of literals it sets to true, like $\{x, \neg y, z\}$. It contradicts a formula $F$ if and only if $F \cup l \models \bot$. This condition is the same as $F \models \neg \land I$. A further reformulation is $F \models \lor \{\neg l \in I\}$. The consequent of this implication is denoted $\neg I$, the clause comprising the negation of all literals satisfied by $I$. The conclusion is $F \models \neg I$. As an example, $I = \{x, \neg y, z\}$ contradicting $F$ is the same as $F \models \neg x \lor y \lor \neg z$.

The collection of the clauses that are the negation of the partial interpretations that falsify the formula during a backtracking search is equivalent to the original formula [Cas96, Sch96]. Equivalence is required by forgetting, but is not enough. The absence of variables to forget is also necessary. The clauses produced this way may contain them instead.

The simplest solution is to first assign the variables to remember and then the variables to forget. Clauses are output not in the base case of recursion, but in the top points of contradiction in the recursive tree, and only if the partial assignment does not include any variable to forget.

The following figure is an example of a backtracking trace, where each node is a recursive call and edges link calling calls to called calls. The variables to remember are $x$ and $y$, the ones to forget $z$ and $w$. Each internal node of the tree is inscribed the variable assigned there and labeled with the literals that are true according to the passed partial assignment. Mark $X$ is for formula false and $=$ for formula true according to the partial interpretation.
The negation of every partial assignment falsifying the formula is a clause entailed by the formula. An example is the node labeled \( \{x, \neg y, \neg z\} \) is marked X. The negation of this partial assignment is the clause \( \neg x \lor y \lor z \). This clause is entailed by the formula, but cannot be in the result of forgetting because it contains \( z \), a variable to forget. The formula being true in its sibling \( \{x, \neg y, z\} \) makes it true in their parent \( \{x, \neg y\} \). The formula is false in \( \{x, y\} \) and true in its sibling \( \{x, \neg y\} \). This is a top point of contradiction. The negation of its partial interpretation \( \{x, y\} \) is \( \neg x \lor \neg y \). This clause is part of the result of forgetting.

This example illustrates the main points of the algorithm:

- first assign the variables to remember, then the variables to forget;
- a clause is generated when the formula is false in a node and true in its sibling; in the example: false in \( \{x, y\} \), true in \( \{x, \neg y\} \);
- it is generated only if the partial interpretation assigns no variable to forget; in the example: it is not generated in \( \{x, \neg y, \neg z\} \), although it falsifies the formula and its sibling \( \{x, \neg y, z\} \) does not;
- the clause comprises the literals the partial assignment falsifies: \( \{x, y\} \) produces \( \neg x \lor \neg y \).

The second and last point alone generate a formula made of implicates [Cas96, Sch96]. Forgetting requires the addition of the first and third point: assign variables in a certain order and do not generate some clauses.

The main contributor to the efficiency of modern backtracking-based algorithms is unit propagation. If all literals of a clause are falsified but one, that one has to be true for the formula to be satisfied. The evaluation of the variable that makes the literal true is added to the others.

This improvement does not work as it is for forgetting. The algorithm may still be setting the values to the variables to remember when unit propagation generates a value for a variable to forget.

An alternative view of unit propagation explains why this is a problem. Setting a variable to a certain value corresponds to evaluating that variable in the next step; the wrong value
falsifies the formula immediately. An example is forgetting $y$ from a formula that contains $\neg x \lor y$ and the first variable that is assigned is $x$. In the subcall where $x$ is true, all literals of $\neg x \lor y$ are false but $y$. Consequently, $y$ is the next variable that is assigned.

The branch $\{x, \neg y\}$ closes immediately because it falsifies $\neg x \lor y$. No further recursive calls are made. Search is only necessary in the other branch $\{x, y\}$. If the formula is true in some descendant, it is true in $\{x, y\}$ and false in $\{x, \neg y\}$. This is a top point of unsatisfaction. The clause $\neg x \lor y$ comprising the negation of the literals in $\{x, \neg y\}$ is entailed by the formula. Yet, it is not generated because it contains $y$, a variable to forget.

This is correct: an entailed clause is not generated if it contains a variable to forget. The problem in the example is that no other entailed clause $\neg x \lor z$ is generated even if $z$ is a variable to remember.

The clause $\neg x \lor z$ is not generated in the subtree of $\{\neg x\}$ because all assignments there contain $\neg x$: the clauses they produce contain $x$ instead of $\neg x$. It is not generated in the subtree of $\{x\}$ either: since $\{x, \neg y\}$ is marked X, all remaining calls are descendants of $\{x, y\}$. A partial assignment that falsifies the formula may contain $z$, but it also contains $y$, a variable to forget.
The clause that would be generated in \( \{x, y, \ldots, z\} \) is \( \neg x \lor \neg y \lor \cdots \lor \neg z \), but is not generated because it contains \( y \), a variable to forget. However, this variable is not necessary: the interpretation \( \{x, \ldots, z\} \) also falsifies the formula since the missing literal \( y \) is entailed by \( x \) anyway. In terms of clauses, \( \neg x \lor \neg y \lor \cdots \lor \neg z \) and \( \neg x \lor y \) resolve into \( \neg x \lor \cdots \lor \neg z \). This clause is entailed and does contain the variable to forget \( y \). It is a clause to generate.

The same outcome is obtained graphically by delaying the evaluation of \( y \) to the end.

The highest node where the formula is false and is true in the sibling is \( x, \ldots, z \). The generated clause is \( \neg x \lor \cdots \lor \neg z \).
Delaying branching on $y$ solves the problem in theory, but is inconvenient in practice. Immediately evaluating $y$ to true as normally done by unit propagation might generate other variables by unit propagation, with a possibly exponential reduction of the recursion tree.

This is why the assignment of true to $y$ is added straight away. When the formula is false in a node and true in the other, the assignment to $y$ is discarded. The generated clause does not contain this variable.

The treatment is different when $y$ is a variable to remember. Unit propagation is the same as branching on $y$ right after $x$. An algorithm can be implemented exactly this way: by branching first on the variables to remember that occur in unit clauses. One subcall returns false. If the other returns true, a clause is generated as usual. This way, generating a clause does not need a specialized rule when contradiction comes from unit propagation.

Operatively, when a variable $y$ is in a unit clause:

- if $y$ is a variable to forget, its value is added to the partial model;
- otherwise, $y$ is chosen as the next variable to assign;
- the generated clauses only include the literals on the variables to remember.

The complete algorithm follows. Its starting point is the empty partial model, which assigns no variable.

The forget_backtracking($I, F, V$) subroutine returns a formula expressing forgetting $V$ from $F \cup I$. This formula is always $\{⊥\}$ if contradictory and $\emptyset$ if tautologic.

The propagate($I, F, V$) subroutine performs unit propagation on the variables in $V$ only. It returns the updated model and the simplified formula; the latter is always $\{⊥\}$ if contradictory and $\emptyset$ if satisfied by $I$.

**Algorithm 4 (Forget by Backtracking)**

```plaintext
forget_backtracking(PartialModel $I$, Formula $F$, Variables $V$)

1. $I, F = \text{propagate}(I, F, V)$
2. if $F = \emptyset$ or $F = \{⊥\}$
   (a) return $F$
3. if $\exists x \in \text{Var}(F) \setminus V$ such that $\text{Var}(C) = \{x\}$ for some $C \in F$
   (a) $B = x$
4. else if $\text{Var}(F) \setminus \text{Var}(I) \setminus V \neq \emptyset$
   (a) $B = \text{a variable in } \text{Var}(F) \setminus \text{Var}(I) \setminus V$
5. else
   (a) $B = \text{a variable in } \text{Var}(F) \setminus \text{Var}(I)$
6. $P = \text{forget_backtrack}(I \cup \{B\}, F, V)$
```

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7. \( N = \text{forget\_backtrack}(I \cup \{\neg B\}, F, V) \)

8. if \( P = \{\bot\} \) and \( N = \{\bot\} \)
   
   (a) return \( \{\bot\} \)

9. if \( B \in V \)
   
   (a) return \( \emptyset \)

10. \( C = \bot \)

11. if \( T = \{\bot\} \)
    
    (a) \( C = \lor(\{\neg l \mid l \in I, \var(l) \notin V\} \cup \{\neg B\}) \)

12. if \( F = \{\bot\} \)
    
    (a) \( C = \lor(\{\neg l \mid l \in I, \var(l) \notin V\} \cup \{B\}) \)

13. return \((P \cup N \cup \{C\}) \setminus \{\bot\}\)

\textit{forget\_backtracking}(\textit{Formula} F, \textit{Variables} V)

1. return \textit{forget\_backtracking}(\emptyset, F, V)

7 Implementation

The four algorithms are implemented in Python [VRD11]. They are currently available in the repository https://github.com/paololiberatore/four.

Variables are single characters or html-like entities like &name;. A sequence abc stands for the clause \( a \lor b \lor c \). The forms ab->cd and ef=gh are allowed.

The programs output the result of forgetting, self-reported time and memory and tracing information. Every output line but the result of forgetting starts with the character #. The second character in the line is a space for tracing information. Time and memory are reported as \#T=number and \#M=number. The numbers are not the exact time and memory consumption. They do not include constant factors and are about the best way to perform all operations rather than the way they are implemented. For example, if a certain part of the algorithm could be performed in linear time, the printed time may be the size of the formula. It neglects implementation details.

The programs print multiple resource usage information lines \#T=number and \#M=number. The first is the ideal duration of a certain operation; the total time is their sum. The second is the ideal memory usage at a certain time; the overall memory requirement is their maximum.
8 Comparison

The four algorithms are compared on random formulae of three clauses each. Each literal in each clause is randomly chosen from a fixed alphabet. A time limit of 10 seconds is set for each formula. Time and memory as reported by the programs is stored, as well as actual time in seconds and maximal memory usage in kilobytes according to the operating system. Ten random formulae are generated for every combination of total variables, variables to forget and clauses.

Real time and memory usage tell which programs are better than which, but neglects unimplemented optimizations. Self-reported time and memory keep them into account. Being asymptotic, these measures are better suited for determining how the resources used by a single algorithm change among the tests rather than comparing different algorithms test by test. Real time and memory usage are for comparison between different programs; self reporting time and memory usage is for comparing each algorithm with itself on different inputs.

The experiments have been performed on all combinations of three to ten total variables, zero to all variables to forget, and clauses ranging from one to five times the total number of variables. All following diagrams are summaries of the results of these tests.

The real time spent by the programs depends on the number of total variables as shown in Figure 1. Each point is the result of a test, positioned according to its total number of variables and running time. As expected, all programs take more on larger formulae. The close and elimination algorithms show an exponential-like increase, which is not so marked for the backtracking algorithm. The close algorithm hits the timeout of ten seconds for some formulae of nine variables, while the elimination and backtracking algorithms do not. The behavior of the linear resolution algorithm is confounded by the timeouts reached already for a very small number of variables. On absolute values, the backtracking algorithm is the clear winner.

Figure 2 depicts the real memory the programs use, depending on the total number of variables. Timeouts are shown as 14000, a value larger than any other in these experiments. The close algorithm is exponential as long as it does not time out. The elimination algorithm takes less memory, but the increase is still exponential. The backtracking algorithm is almost unaffected by the increase in the number of variables. The linear algorithm is hard to evaluate due to the timeouts; when it does not time out, it is even better than backtracking.

Figure 3 shows the size the output produced by the algorithms in relation to the number of clauses of the input. They all follow a bell curve, but the shape and the position of their peak depend on the algorithm. Since the output is a simplification of the input, few input clauses are expected to produce few output clauses. The following decrease is not so obvious. The limit case shows an explanation: the formula that contains all possible clauses is inconsistent, and all inconsistent formulae are equivalent to very short ones, like $\{x, \neg x\}$. When the formula is not inconsistent but contains many clauses, an algorithm may be able to combine many of them to form shorter clauses, such as when $a \lor b \lor c$ and $a \lor b \lor \neg c$ resolve into $a \lor b$. The ability to do so depends on the algorithm, which explains the differences in the bell curves. The size of the output is not plotted if the algorithm timeouts since the output would be incomplete anyway.

Figure 4 shows the number of timeouts of the linear algorithm depending on the number of variables and the fraction of variables to forget. The first graph shows the number of timeouts
Figure 1: Real time, depending on the total number of variables
Figure 2: Actual memory used, depending on the total number of variables
Figure 3: Size of output, depending on the size of input
Figure 4: Timeouts of the linear algorithm
and total tests for each number of variables. The second graph is the percentage of timeouts over the total tests. The increase looks almost linear, meaning that the percentage of failed tests increases linearly with the number of variables. The graph showing the number of timeouts in relation to the fraction of variables to forget groups them in blocks: for example, the height of the bar at 0.5 is the number of timeouts when the fraction of variables to forget is between 0.45 included and 0.55 excluded. The graph exhibits an irregular increase. The irregularity may be done to an uneven distribution of the fractions on the abscissa. More significant is the increase: it suggests that the linear algorithm looses efficiency as the fraction of variables to forget increases.

The above are direct comparisons between the algorithms. They are based on the actual time and memory usage and size of the output. They are quantitative measure on a common scale. The individual numbers can be directly compared: an algorithm took 0.390 seconds, another took 5.660 seconds.

What follows is a comparison of the ideal efficiency of the algorithms. It is based on the self-reported time and memory usage, which disregards multiplicative constants and keeps into account unimplemented optimizations. They do not allow for a number-to-number comparison between the algorithms. Yet, they show how the time and efficiency of the algorithms relates to the input parameters (total number of variables, number of variables to forget, number of clauses).

All plots are for eight variables, the maximal number when at least three of the four algorithms did not time out. The timeouts of the linear algorithm are omitted since self-reported time and memory are incomplete measures anyway in such cases.

Figure 5 shows how the number of input clauses affects the running time. The number of total variables and variables to forget are fixed to respectively eight and four. Different numbers lead to similar plots. A common trait is that all algorithms spend more time on larger formulae, as expected: a large formula takes more time to be processed. Yet, how time depends on size varies a lot between algorithms. The curve of the close algorithm steadily increases. The curve of the elimination algorithm keeps increasing; it also exhibits a concentration of points at the bottom of the diagram, with a few outliers. The linear resolution algorithm suffers from many timeouts as the number of clauses increases; yet, its behavior appears qualitatively similar to the elimination algorithm. The backtracking algorithm is the least affected by the increase in the number of clauses; the distribution looks like a logarithmic one, with an initial raise that slows down quickly.

Figure 6 shows the self-reported running time depending on the number of variables to forget when the total number of variables and the number of clauses are respectively fixed at eight and eighteen. The close algorithm is not influenceed by the number of variables to forget; this is expected, since the slowest part of the algorithm is the generation of the resolution closure, which does not even look at the variables to forget. The elimination algorithm takes more time at the intermediate proportion of the variables to forget, but the extent is not so marked as expected, and many difficult outliers emerge in the left-to-middle of the diagram. A similar distribution is seen for the linear resolution algorithm, with outliers in the right of the diagram. The backtracking algorithm is unique: many variables to forget are slightly easier than few, the contrary to the others; the dependency is small and is more evident for many clauses; this is why the diagram is shown for thirty clauses instead of eighteen.

The dependency of memory on the number of clauses is shown in Figure 7. The curve
Figure 5: Self-reported time, depending on the number of input clauses
Figure 6: Self-reported time, depending on the number of variables to forget
Figure 7: Self-reported memory usage, depending on the number of clauses
of the close algorithm is similar to that of time, with a steady increase. The elimination algorithm exhibits a slightly superlinear increase in memory with the number of clauses; points are distributed uniformly, and are more scattered when the clauses are many; none is a real outlier. The linear resolution algorithm shows a linear increase with a very uniform distribution of points. The diagram for the backtracking algorithm looks weird at a first sight: its superior bound is a flat line, its inferior bound is bell-shaped; the way the algorithm works explains the first: the upper limit is due to the recursive subcalls being limited by the number of variables; the lower limit is due to the hardest cases showing up for a middle number of clauses, a phenomenon known as phase transition [XHL12]. Overall, the diagrams for memory are similar to the corresponding ones for time except for the backtracking algorithm.

The memory used by the programs depending on the number of variables to forget is shown in Figure 8. The close algorithm does not depend on the number of variables to forget, with points distributed uniformly. The elimination algorithm is similar, with a slight increase in the middle. The linear resolution algorithm is similar. The backtracking algorithm is bounded in memory by the number of variables, which explains the flat upper limit; some easy points are seen for many variables to forget, but overall the difference is small.

This concludes the interpretation of the individual tests. Some algorithm is faster than others, some algorithm uses less time, some algorithm generates the shortest input. Yes, but which algorithm is the best?

Which algorithm is the best? Looking at actual time and memory, the winner is the backtracking algorithm. It requires the least actual time and self-reported memory, while being on the same scale as the close and elimination algorithms on the size of the output.

The close algorithm surprisingly produces the shortest outputs. Yet, the other algorithms may still have a role if this measure is significant, as for example the elimination algorithm is good for small inputs and the backtracking algorithm is comparable for large inputs.

When the clarity of the resulting formula is important, the close and eliminate algorithm win because they use the syntactic form of the input clauses. The backtracking algorithm does not; it is semantical: it checks the effect of the input clauses on the consistency of partial interpretations. For example, when forgetting \( b \) from \( \{a \rightarrow b, b \rightarrow c, c \rightarrow d, d \rightarrow a\} \), it may produce \( \{a \rightarrow d, d \rightarrow c, c \rightarrow a\} \), instead of \( \{a \rightarrow c, c \rightarrow d, d \rightarrow a\} \), which is more similar to the input formula and is the result of the eliminate algorithm.

9 Any given algorithm

Variable elimination and linear resolution are two forms of resolution, but others exist, including many restrictions to linear resolution. This raises the question: is forgetting possible in all of them? As an example, can s-linear resolution be used for forgetting?

Variable elimination and linear resolution both forget variables by only resolving on them, stopping when none is left. The same restrictions work for s-linear resolution.

The set of clauses generated this way expresses forgetting. This is proved by showing that a subset of every full remembrance clause that is entailed by the formula is generated. The full remembrance clauses are the clauses that contain all variables to forget.

Being entailed by the formula \( F \), the negation of such a clause \( C \) is inconsistent with it: \( F \cup \neg C \models \bot \). Since s-linear resolution is refutationally complete, \( G \cup \neg C \vdash \bot \) follows: the
Figure 8: Self-reported memory usage, depending on the number of variables to forget
empty clause is generated from \( G \cup \neg C \) by s-linear resolution. This derivation is turned into \( G \vdash C \) where \( C' \) is a subset of \( C \) by the transformation used for linear resolution, but this transformation does not always maintain s-linearity. A center clause \( E \) is only allowed to be resolved with a previously generated clause \( D \) if the result \( E' \) is a subset of \( E \).

The transformation used for linear resolution removes resolutions of center clauses with the negation \( \neg l \) of a literal of \( C \). The effect is that \( l \) is added to some of the following clauses. If neither \( D \) nor \( E \) change, the condition \( E' \subset E \) does not change since both \( E' \) and \( E \) stay the same. The same happens if both are added \( l \), or if \( E \) is added \( l \) and \( D \) is not. The only case where a violation occur is when \( D \) is added \( l \) and \( E \) is not.

The condition of s-linearity is violated since \( E' \lor l \) contains \( l \) while \( E \) does not. Yet, since \( l \) is in \( D \lor l \) but is not in \( E \), it is resolved with \( \neg l \) in between.

The transformation also removes this resolution and adds \( l \) to all following clauses. The result is that \( E \) is added \( l \).

The condition \( E' \lor l \subset E \lor l \) holds as a consequence of \( E' \subset E \). While removing a single resolution with \( \neg l \) may violate s-linearity, removing all others restores it.

Not only directional and linear resolution, but also A-order linear resolution and s-linear resolution forget variables. Yet, each of these methods requires a separate proof. A common point is that correctness is easy: resolution only produces entailed clauses, and all entailed clauses comprising variables to remember are correct when forgetting. Correctness is guaranteed by resolution. Completeness is not.

A single proof of completeness for all forms of resolution would be ideal. The second-best choice is reusing or adapting existing proofs. For example, many resolution restrictions are
entailment-complete: they generate a subset of every clause entailed by the formula [SCL69, Ino92, dV99]. Restricting to the clauses comprising only the variables to remember turns them into forgetting algorithms.

A misleading solution emerges: forget by whichever form of resolution that is entailment-complete. This solution is misleading because entailment completeness is a. not required, and b. not desirable.

Directional resolution is not entailment complete. Neither is A-ordering linear resolution [AM05]. Yet, they both forget variables.

The proof that linear resolution for forgetting works shows what is really required. The starting point is an entailed full remembrance clause $C$, a clause comprising exactly the variables to remember. The conclusion is that linear resolution produces a subset of $C$. This is not entailment-completeness. It is entailment-completeness on full remembrance clauses. Only clauses comprising all variables to remember need to be generated, not all of them.

An example shows the difference: forgetting $b$ from $\{a \lor b, \neg b \lor c, \neg c \lor d\}$. Both variable elimination and A-ordering linear resolution only resolve the first two clauses $a \lor b$ and $\neg b \lor c$ since their resolving variable $b$ is the only one to forget. The result is $a \lor c$. Neither $a \lor d$ nor any subset of its is generated, in spite of being entailed by the formula.

Completeness is not even required on prime implicates (the clauses that are entailed while none of theirs subsets is). The previous example provides a counterexample, since $a \lor d$ is such a prime implicate.

Entailment completeness is not required. Neither directional not A-ordering linear resolution meet it. Resolution by closure does. A trivial and inefficient method is entailment-complete.

Resolution by closure resolves clauses in all possible ways and outputs the ones comprising variables to remember. When forgetting $b$ from $\{a \lor b, \neg b \lor c, \neg c \lor d\}$, it produces $a \lor c$, but also $a \lor d$. The latter clause is redundant. Forgetting does not require it, as it is entailed by $a \lor c$ and $\neg c \lor d$. Entailment-completeness is not only unnecessary, it is not even desirable. A shorter formula expressing forgetting is better than a larger one.

The proofs of correctness of linear resolution and its restrictions all start from a full remembrance clause. When forgetting no variables, these are the full clauses, the clauses that contain all variables. The generation of a subset of all full clauses is necessary. The question is whether it is also sufficient: is a restriction of resolution complete for forgetting if it is entailment-complete for full clauses?

The existing proofs do not help. Their starting point is refutational completeness: if $F \cup \neg C \models \bot$ then $F \cup \neg C \vdash \bot$. If a formula is contradictory, the empty clause is generated by resolution. The proofs then proceed by turning $F \cup \neg C \vdash \bot$ into $F \vdash C'$ with $C' \subseteq C$. This change is not guaranteed to maintain all possible restrictions of resolution. It does for the specific case of linear resolution, for A-ordering linear resolution and s-linear resolution, but is not guaranteed in general. For some restrictions, removing the literals of $\neg C$ from the derivation may violate the restriction.

All four implemented algorithms are resolution, even if some do not look so (backtracking). The question is whether every resolution restriction can be adapted for forgetting. No general answer seems possible, as every restriction requires a separate proof of completeness. Some general observations are still possible: entailment completeness is not required nor desirable.
The question is even more general: can every proof system be adapted to forgetting?

A proof system is complete for refutation if it proves $F \vdash \bot$ whenever $F$ is unsatisfiable. If $F \models C$, then $F \cup \neg C \models \bot$. By refutational completeness, $F \cup \neg C \vdash \bot$ follows. This is not enough, as only $F$ is given. Such a proof $F \cup \neg C \vdash \bot$ needs to be converted into $F \vdash C$.

This is not necessary for all clauses $C$ that are entailed by $F$. The full remembrance clauses are enough. If $C$ comprises all variables to remember and is entailed by $F$, the refutation $F \cup \neg C \vdash \bot$ has to be converted into $F \vdash C$. How to do this conversion, and whether it is even possible, depends on the specific proof system.

10 Conclusions

Four algorithms for propositional forgetting are compared. The first three are based on resolution: unrestricted, ordered and linear. The fourth is based on backtracking. They are implemented in Python and run on random formulae. The time and memory usage and the size of the generated output are compared.

The backtracking algorithm wins on most measures. This is unsurprising given the efficiency of modern backtracking-based satisfiability algorithms [LSS16]. Still, forgetting is not the same as satisfiability. It requires not just to prove the existence of a model but to derive a formula that expresses forgetting.

Variable elimination is a good competitor. Linear resolution seems unusable as far as the experiments show. This is unexpected, given the amount of work done on this form of resolution. Yet, only the basic version of linear resolution was implemented, with inverse alphabetic ordering of the variables. Many variants exist. Some of them may prove efficient for forgetting. At the same time, the other algorithms were also implemented in their basic version: variable elimination with alphabetic removing order, backtracking with basic unoptimized unit propagation and choice of the variable to evaluate, resolution closure with no additional data structures. In spite of this, these algorithms did not perform as poorly as linear resolution.

Variable elimination and resolution closure appear to work well on some measures and in some specific cases. For example, resolution closure produces the shortest formulae unless the input formula is not very small, in which case it is surpassed by variable elimination. The difference is however not large. It may be due to the way the algorithms are implemented.

Further optimizing them is a possible direction for future work. Others include the extension to other logics, especially Answer Set Programming, which is relatively similar to propositional logics and forgetting was thoroughly investigated [WZZZ14, GKL16]. Another direction is the adaptation of other algorithms for satisfiability to forgetting.

Resolution and backtracking can be used for forgetting, but they both were designed for propositional satisfiability. Do all algorithms for satisfiability extend to forgetting?

Since forgetting preserves all consequences of the original formula on the variables to remember, it is expressed by a formula comprising only clauses that are entailed by the original formula. If an algorithm produces all consequences of the original formula, it can be adapted to forgetting by selecting only the clauses that do not contain variables to forget.

Being able to generate a subset of every clause entailed by a formula is called entailment completeness [SCL69, Ino92, dV99, AM05]. Unrestricted resolution possesses this property, for example. The algorithm based on resolution closure works this way.
A first possible requirement for an algorithm to be adapted to forgetting is its entailment completeness: if a clause is entailed by the formula, a subset of it is found by the algorithm.

This condition is sufficient for expressing forgetting, but is not necessary. It is too stringent. It does not hold for A-ordering linear resolution, which still expresses forgetting. While A-ordering linear resolution does not generate all minimal clauses entailed by the formula, it generates enough of them. Enough for forgetting. Enough to entail all consequences of the original formula on the variables to remember.

To express forgetting, an algorithm needs not to generate all consequences of the formula. As an example, an algorithm may not generate \( a \lor d \) when forgetting \( b \) from \{\( a \lor b, \neg b \lor c, \neg c \lor d \)\}, even though \( a \lor d \) is a prime implicate. It may still express forgetting as \{\( a \lor c, \neg c \lor d \)\}. Neither variable elimination nor linear resolution output \( a \lor d \), for example. What matters is not that \( a \lor d \) is in the output. What matters is that \( a \lor d \) is entailed by the output. Generating all consequences of the original formula or even just all prime implicates is not required. It may even be a drawback, when it results in oversized outputs.

Another metric for evaluating different proof mechanisms is their proof size. For example, unrestricted resolution proofs are always comparable in size to the backtracking proofs for the same formula, but the converse does not hold for all formulas [BP01]. The minimal length of the proofs is a lower bound on the time used by an algorithm. It is not on memory usage. The backtracking algorithm is a counterexample: a proof is a tree of recursive calls; when a branch is over, the memory it requires can be recycled. Even when measuring time, the size of a proof is only a lower bound. To find a proof of length \( n \), no less than \( n \) steps are necessary. Necessary, not sufficient. Finding the proof may be hard, requiring more than \( n \) steps [Bus12].

Proof size is still an indicator of the memory efficiency of the algorithms when memory cannot be reused. This is the case for directional resolution (variable elimination): viewing a proof as a tree of clauses generated by resolution, a whole level has to be maintained at every time, and the width of a level is related to the overall size of the proof. Other resolution strategies are able to generate smaller proofs. Yet, the experiments on linear resolution excludes it as a practical alternative in spite of its theoretical properties.

A further direction of study is approximate forgetting: instead of maintaining exactly the consequences on the variables to remember, just keeping most of them may be enough. Or the opposite: maintain all of them at the cost of introducing new consequences. Incomplete satisfiability methods such as local search [SKC94, EP19] may help.

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