Bottom Quark Fragmentation
in Top Quark Decay

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Abstract

We study the fragmentation of the $b$ quark in top decay in NLO QCD, within the framework of perturbative fragmentation, which allows one to resum large logarithms $\sim \log(m_t^2/m_b^2)$. We show the $b$-energy distribution, which we compare with the exact $\mathcal{O}(\alpha_S)$ result for a massive $b$ quark. We use data from $e^+e^-$ machines in order to describe the $b$-quark hadronization and make predictions for the energy spectrum of $b$-flavoured hadrons in top decay. We also investigate the effect of NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function on parton- and hadron-level energy distributions.
1 Introduction

For the sake of performing accurate studies of the top-quark properties and a precise measurement of its mass at the present Run II of the Tevatron accelerator and, in future, at the LHC [1] and at the Linear Collider [2], a reliable description of the bottom-quark fragmentation in top decay $t \rightarrow bW$ will be essential.

As shown in [3], the $b$-fragmentation is indeed one of the sources of uncertainty in the measurement of the top mass at the Tevatron, as it contributes to the so-called Monte Carlo systematics. At the LHC, recent studies [4] have suggested that final states with leptons and $J/\psi$, with the $J/\psi$ coming from the decay of a $b$-flavoured hadron and the isolated lepton from the $W$ decay, will be a promising channel to reconstruct the top mass. In [4], the expected experimental error, a result of statistics and systematics, has been estimated to be $\Delta m_t \approx 1 \text{ GeV}$ and the $b$-fragmentation is the largest source of uncertainty, accounting for about 0.6 GeV.

Available tools to describe the $b$-quark hadronization are Monte Carlo event generators, such as HERWIG [5], PYTHIA [6] or ISAJET [7], implementing respectively cluster [8], string [9] and independent-fragmentation [10] models. Monte Carlo programs describe the initial- and final-state multiparton radiation in hadron collisions according to the soft and/or collinear approximation (see, for example, [11]). HERWIG and PYTHIA parton showers have been provided with matrix-element corrections for a few processes, such as top decay [12], in order to allow hard and large-angle parton radiation. The analysis of [4] was in fact performed using the PYTHIA event generator. In [13], the HERWIG event generator was used to perform studies on the top mass reconstruction at the LHC, relying on the $b$-quark fragmentation. The $m_{B\ell}$ invariant mass distribution, where $B$ is a $b$-flavoured hadron and $\ell$ a lepton from the $W$ decay, was exploited in order to fit the top mass.

In this paper we analyse the $b$-quark fragmentation in top decay in the framework of perturbative fragmentation at next-to-leading order (NLO) in QCD. The factorization theorem [14] dictates that, up to power corrections $\sim \mathcal{O}((1/Q)^p)$, with $p \geq 1$ and $Q$ being a characteristic energy scale of the process, a hadron-level cross section can be written as the convolution of a short-distance, perturbative cross section and long-distance, non-perturbative terms, corresponding to initial-state parton distribution functions and/or final-state fragmentation functions. For heavy-quark production, the quark mass $m$ acts as a regulator for the collinear singularity and allows one to perform perturbative calculations. However, fixed-order event shapes or differential distributions typically contain terms like $\alpha_s \log(Q^2/m^2)$, where $Q$ is, for example, the centre-of-mass energy or the heavy-quark transverse momentum. Such terms spoil the convergence of the perturbative expansion and make fixed-order calculations unreliable once $Q$ is much larger than $m$. The method of perturbative fragmentation functions, originally proposed in [15], allows one to resum these large logarithms.

According to the method in [15], heavy quarks are first produced at large transverse momentum $m \ll p_T$, as if they were massless, and afterwards they slow down and...
fragment into a massive object. The perturbative fragmentation function \( D(\mu_F, m) \) expresses the transition of a massless parton into a massive quark at the factorization scale \( \mu_F \).

The value of \( D(\mu_F, m) \) at any scale \( \mu_F \) can be obtained by solving the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations \([16,17]\), once its initial value at a scale \( \mu_0 \) is assigned. The universality of the initial condition and, in general, of the perturbative fragmentation function, already suggested in \([13]\) in the framework of \( e^+e^- \) annihilation, has been recently proved in a completely process-independent way \([18]\). As discussed e.g. in \([19]\) for heavy-quark production at hadron colliders, the perturbative fragmentation formalism yields a weaker dependence of phenomenological observables on the renormalization/factorization scales and on the chosen set of parton distribution functions. Furthermore, the analysis of \([18]\) fully resums the leading (LL) and next-to-leading logarithms (NLL) which are associated with the emission of soft gluons and appear in the initial condition of the perturbative fragmentation function (process independent) and in the parton-level differential massless cross section (process dependent) of \( e^+e^- \) annihilation.

Finally, in order to describe the non-perturbative fragmentation of a parton into a hadron, several phenomenological models have been proposed \([20,21]\), besides the ones which are implemented in Monte Carlo event generators. Non-perturbative fragmentation functions contain parameters which need to be fitted to the experimental data. Since the hadronization mechanism is universal and independent of the perturbative process which produces the heavy quark, one can exploit the existing data on \( e^+e^- \rightarrow b\bar{b} \) events to fit such models and describe the \( b \)-quark non-perturbative fragmentation in other processes, such as top decay.

Perturbative fragmentation functions and non-perturbative hadronization models have been extensively applied to study the physics of \( c \)– and/or \( b \)-flavoured hadrons produced in \( e^+e^- \) annihilation \([22-25]\), hadron collisions \([19,26]\), Deep Inelastic Scattering \([24]\) and \( \gamma p \) collisions \([27]\).

In this work, we apply this method to the \( b \)-fragmentation in top decay. In Section 2 we review the method of perturbative fragmentation functions and apply it to predict the \( b \)-quark energy spectrum in top decay. In Section 3 we analyse the non-perturbative fragmentation of the \( b \) quark and show energy distributions of \( b \)-hadrons in top decay, making use of fits to LEP and SLD data to parametrize the hadronization models. In Section 4 we summarize the main results of our analysis and make comments on possible developments of our study. In Appendices A and B we show details of our calculation and, comparing results for massless and massive \( b \) quarks, we check that our computation is consistent with the initial condition of heavy-quark perturbative fragmentation functions.
2 Perturbative fragmentation and parton-level results

We wish to study $b$-quark production in top decay within the framework of perturbative fragmentation functions. We consider the decay of an on-shell top quark at next-to-leading order in $\alpha_S$

$$t(q) \to b(p_b)W(p_W)(g(p_g))$$

(1)

and define the $b$-quark scaled energy fraction $x_E$ as:

$$x_E = \frac{2p_b \cdot q}{m_t^2}.$$ (2)

Neglecting the $b$ mass, we have $0 \leq x_E \leq 1 - w$, $w$ being $w = m_W^2/m_t^2$. Throughout this paper, we shall make use of the normalized $b$ energy fraction:

$$x_b = \frac{x_E}{1 - w}, \quad 0 \leq x_b \leq 1.$$ (3)

Following [15], the differential width for the production of a massive $b$ quark in top decay can be expressed via the following convolution:

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dx_b}(x_b, m_t, m_W, m_b) = \frac{1}{\Gamma_0} \sum_i \int_{x_b}^1 dz \frac{d\hat{\Gamma}_i}{dz}(z, m_t, m_W, \mu, \mu_F) D_i \left( \frac{x_b}{z}, \mu_F, m_b \right),$$ (4)

where $\Gamma_0$ is the width of the Born process $t \to bW$, $d\hat{\Gamma}_i/dz$ is the differential width for the production of a massless parton $i$ in top decay with energy fraction $z$, $D_i(x_b/z, \mu_F, m_b)$ is the perturbative fragmentation function for a parton $i$ to fragment into a massive $b$ quark. In Eq. (4) $\mu$ and $\mu_F$ are the renormalization and factorization scales respectively, the former associated with the renormalization of the strong coupling constant. In principle, one can use two different values for the factorization and renormalization scales; however, a choice often made consists of setting $\mu = \mu_F$ and we shall adopt this convention for most of the results which we shall show.

The approach of [15] and the factorization on the right-hand side of Eq. (4) are rigorously valid if one can neglect terms behaving like $(m/Q)^p$, where $p \geq 1$, $m$ is the mass of the fragmenting heavy quark and $Q$ is the hard scale of the process. This is indeed our case since the scale of top decay is set by its mass and $m_b/m_t \approx O(10^{-2})$.

The definitions of $d\hat{\Gamma}_i/dz$ and $D_i(x_b/z, \mu_F, m_b)$ are not unique, but they depend on the scheme which is used to subtract the collinear singularities which appear in the massless differential width $d\hat{\Gamma}_i/dz$. In [15], the $\overline{\text{MS}}$ factorization scheme is chosen and we shall stick to it hereinafter.

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1Our assumption $B(t \to bW) = 1$ is consistent with recent measurements of the CDF Collaboration of the ratio $R = B(t \to Wb)/B(t \to Wq)$, where $q$ is a $d$, $s$ or $b$ quark, and the subsequent extraction of the Cabibbo–Kobayashi–Maskawa matrix element $V_{tb}$ [28].
Since we have been assuming \( B(t \rightarrow bW) = 1 \) and the probability to produce a \( b \) quark via the splitting of a secondary gluon is negligible, we shall safely limit ourselves to considering the perturbative fragmentation of a massless \( b \) into a massive \( b \) and, on the right-hand side of Eq. (4), we shall have only the \( i = b \) contribution. We shall then need to evaluate the differential width \( d\hat{\Gamma}_b/dz \) for the production of a massless \( b \) quark in top decay, in dimensional regularization, and subtract the collinear singularity according to the \( \overline{\text{MS}} \) prescription. We obtain the following \( \overline{\text{MS}} \) coefficient function:

\[
\frac{1}{\Gamma_0} \frac{d\hat{\Gamma}_b}{dz} \bigg|_{\overline{\text{MS}}} = \delta(1 - z) + \frac{\alpha_S(\mu)}{2\pi} \hat{A}_1(z),
\]

with

\[
\hat{A}_1(z) = C_F \left\{ \left[ \frac{1 + z^2}{(1 - z)^+} + \frac{3}{2} \delta(1 - z) \right] \log \frac{m_t^2}{\mu_F^2} + 2 \frac{1 + w}{1 + 2w} - 2 \log(1 - w) \right. \\
+ \left. \frac{1 + z^2}{(1 - z)^+} \left[ 4 \log \frac{(1 - w)z}{1 - 2w} - \frac{1}{1 + 2w} \right] - \frac{4z}{(1 - z)^+} \left[ 1 - \frac{w(1 - w)(1 - z)^2}{(1 + 2w)(1 - (1 - w)z)} \right] \right. \\
+ \left. 2(1 + z^2) \left[ \left( \frac{1}{1 - z} \log(1 - z) \right)_+ - \frac{1}{1 - z} \log z \right] \right. \\
+ \left. \delta(1 - z) \left[ 4 \text{Li}_2(1 - w) + 2 \log(1 - w) \log w - \frac{2\pi^2}{3} + \frac{1 + 8w}{1 + 2w} \log(1 - w) \right] - \frac{2w}{1 - w} \log w + \left. \frac{3w}{1 + 2w} - 9 \right\},
\]

where

\[
\text{Li}_2(x) = -\int_0^x \frac{dt}{t} \log(1 - t)
\]

is the Spence function. In Appendix A we shall give more details on the derivation of Eq. (6) and present results for the differential width \( d\Gamma/dx_b \) once the \( b \)-quark mass is fully taken into account.

In order to be consistent at \( \text{NLO} \), in Eq. (3) \( \alpha_S(\mu) \) is to be the strong coupling constant at \( \text{NLO} \) as well:

\[
\alpha_S(\mu) = \frac{1}{b_0 \log(\mu^2/\Lambda^2)} \left\{ 1 - b_1 \log \frac{\log(\mu^2/\Lambda^2)}{b_0 \log(\mu^2/\Lambda^2)} \right\},
\]

with \( b_0 \) and \( b_1 \) given by

\[
b_0 = \frac{33 - 2n_f}{12\pi}, \quad b_1 = \frac{153 - 19n_f}{24\pi^2},
\]

\( \Lambda \) being the typical QCD scale and \( n_f = 5 \) the number of flavours.

We note that the coefficient function (6) contains a term where the strong coupling constant multiplies the logarithm \( \log(m_t^2/\mu_F^2) \). For our calculation to be reliable, we shall
have to require such a logarithm not to be too large, which implies that the factorization scale \( \mu_F \) will have to be chosen of the order of \( m_t \).

In [15], considering heavy-quark production in \( e^+e^- \) annihilation and comparing the massive and massless differential cross sections in the \( \overline{\text{MS}} \) factorization scheme, the authors have obtained the NLO initial conditions at a scale \( \mu_{0F} \) for heavy-quark perturbative fragmentation functions. For the \( b \to b \) transition, it has been found:

\[
D_b(x_b, \mu_{0F}, m_b) = \delta(1-x_b) + \frac{\alpha_S(\mu_0)C_F}{2\pi} \left[ \frac{1 + x_b^2}{1 - x_b} \left( \log \frac{\mu_{0F}^2}{m_b^2} - 2 \log(1 - x_b) - 1 \right) \right]_+ ,
\]

with \( C_F = 4/3 \). In order to avoid large logarithms in Eq. (10), the scale \( \mu_{0F} \) is to be taken of the order of the \( b \) mass. The universality of the initial condition (10) has been lately proved in [18] and one can therefore exploit it to predict the \( b \) fragmentation in top decay as well.

For the sake of completeness, in Appendix A we shall show the result for the differential width \( d\Gamma/dx_b \) once we keep the \( b \) mass only in contributions \( \sim \log(m_t^2/m_b^2) \) and neglect terms proportional to powers of the ratio \( m_b/m_t \). Comparing the result with the massless rate (5), we shall be able to reproduce the initial condition of the \( b \)-quark fragmentation function, which will be a consistency check of our calculation and, at the same time, a confirmation of the validity of Eq. (10) in our context as well.

Assigned the initial condition (10), the value of the perturbative fragmentation function at any other scale \( \mu_F \) can be obtained by solving the DGLAP evolution equations [16,17]:

\[
\frac{d}{d\log \mu_F^2} D_i(x_b, \mu_F, m_b) = \sum_j \int_{x_b}^1 \frac{dz}{z} P_{ij} \left( \frac{x_b}{z}, \alpha_S(\mu_F) \right) D_j(z, \mu_F, m_b),
\]

where

\[
P_{ij}(x_b, \alpha_S(\mu_F)) = \frac{\alpha_S(\mu_F)}{2\pi} P_{ij}^{(0)}(x_b) + \left( \frac{\alpha_S(\mu_F)}{2\pi} \right)^2 P_{ij}^{(1)}(x_b) + \mathcal{O}(\alpha_S^3).
\]

\( P_{ij}^{(0)}(x_b) \) are the Altarelli–Parisi splitting functions [16], and the higher-order terms \( P_{ij}^{(1)}(x_b) \) can be found in [29,30].

As shown in [15], solving the DGLAP equations for the evolution from a scale \( \mu_{0F} \) to \( \mu_F \) allows one to resum potentially-large logarithms \( \sim \alpha(\mu_F) \log(\mu_F^2/\mu_{0F}^2) \). Assuming \( \mu_{0F} \simeq m_b \) and \( \mu_F \simeq m_t \), and considering the splitting functions [12] at \( \mathcal{O}(\alpha_S) \), one resums the leading logarithms \( \sim \alpha_S^2(m_t) \log^2(m_t^2/m_b^2) \). Accounting for \( \mathcal{O}(\alpha_S^2) \) terms in Eq. (13) leads to the inclusion of next-to-leading logarithms \( \sim \alpha_S^{n+1}(m_t) \log^n(m_t^2/m_b^2) \) as well. In this paper, we shall always assume that the \( b \)-quark perturbative fragmentation function evolves with NLL accuracy.

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2For the fragmentation of a gluon, light quark or \( \overline{b} \) quark into a \( b \) quark, see [15].
The DGLAP equations get highly simplified in Mellin space; the solution reads [13]:

\[
D_{i,N}(\mu_F, m_b) = D_{i,N}(\mu_0F, m_b) \exp \left\{ P_N^{(0)} t \right\} + \frac{1}{4\pi^2 b_0} \left[ \alpha_S(\mu_0F) - \alpha_S(\mu_F) \right] \left[ P_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)} \right],
\]

with \( D_{i,N}(\mu_F, m_b) \) being the Mellin transform of the \( x \)-space perturbative fragmentation function

\[
D_{i,N}(\mu_F, m_b) = \int_0^1 dx \ x_b^{N-1} D_i(x_b, \mu_F, m_b)
\]

and the variable \( t \) defined as

\[
t = \frac{1}{2\pi b_0} \log \frac{\alpha_S(\mu_0F)}{\alpha_S(\mu_F)}.
\]

Throughout our analysis, \( i = b \) in Eqs. (11-14) and \( D_{b,N}(\mu_0F, m_b) \) is the \( N \)-space transform of Eq. (10). Expressions for \( D_{b,N}(\mu_0F, m_b) \) and for the Mellin transforms of the NLO splitting functions in Eq. \( P_N^{(0)} \) and \( P_N^{(1)} \) can be found in [13]. We shall have to compute the \( N \)-space transform of the \( \overline{\text{MS}} \) coefficient function (3) and multiply it by Eq. (13) in order to get the Mellin transform of Eq. (4):

\[
\Gamma_N(m_t, m_W, m_b) = \hat{\Gamma}_N(m_t, m_W, \mu, \mu_F) D_{b,N}(\mu_F, m_b),
\]

with

\[
\Gamma_N(m_t, m_W, m_b) = \frac{1}{\Gamma_0} \int_0^1 dx_b \ x_b^{N-1} \frac{d\Gamma}{dx_b}(x_b, m_t, m_W, m_b).
\]

The \( b \)-quark energy distribution in \( x \)-space will finally be obtained by inverting the \( N \)-space result (16) numerically.

Furthermore, in Eq. (10) the coefficient multiplying the strong coupling constant contains terms behaving \( \sim 1/(1-x_b)_+ \) or \( \sim \log(1-x_b)/(1-x_b)_+ \) once \( x_b \to 1 \), which corresponds to behaviours \( \sim \log N \) or \( \sim \log^2 N \) in moment space, for large \( N \). The limit \( x_b \to 1 \ (N \to \infty) \) corresponds to soft-gluon radiation in top decay. Soft LL \( \sim \alpha_3^s(\mu_0) \log^{n+1} N \) and NLL \( \sim \alpha_3^s(\mu_0) \log^{n} N \) contributions in the initial condition of the perturbative fragmentation function have been resummed in [18]. Due to the process independence of the heavy-quark perturbative fragmentation function, we can exploit the result of [18], which we do not report here for the sake of brevity, to resum LL and NLL terms in Eq. (13) in top decay as well.

We wish to present results for the normalized \( b \)-quark energy distribution in top decay using the technique just described. We normalize our plots to the total NLO width \( \Gamma \), obtained neglecting powers \( \sim (m_b/m_t)^p \), whose expression can be found in [31]. In

\[\text{In Appendix B we shall present results for the } N \text{-space counterpart of Eq. (3).}\]
fact, the factorization on the right-hand side of Eq. (4), the DGLAP evolution and the resummation of soft logarithms in the initial condition of the perturbative fragmentation function with NLL accuracy do not affect the total NLO normalization. It can be observed that, as long as one neglects interference between top production and decay, one has:

\[
\frac{1}{\sigma} \frac{d\sigma}{dx_b} = \frac{1}{\Gamma} \frac{d\Gamma}{dx_b},
\]

where \( (1/\sigma)d\sigma/dx_b \) is the normalized differential cross section for the production of a \( b \) quark with energy fraction \( x_b \) via top quarks, independently of the production mechanism. Our results will then be applicable to \( pp \) (Tevatron), \( pp \) (LHC) or \( e^+e^- \) (Linear Collider) collisions. For our numerical study, we shall assume \( m_t = 175 \) GeV, \( m_W = 80 \) GeV, \( m_b = 5 \) GeV and \( \Lambda = 200 \) MeV.

We show our results for the \( x_b \) spectrum in Fig. 1. We plot the \( x_b \) distribution according to the perturbative fragmentation approach, with and without NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function. For the sake of comparison, we also show the exact \( O(\alpha_s) \) result for a massive \( b \) quark, whose analytical expression will be given in Appendix A. We set \( \mu = \mu_F = m_t \) and \( \mu_0 = \mu_0 F = m_b \).

We note that the use of perturbative fragmentation functions has a stronger impact on the \( x_b \) distribution. The fixed-order result lies well below the perturbative fragmentation results for about \( 0.1 \leq x_b \leq 0.9 \) and diverges once \( x_b \to 1 \), due to a behaviour \( \sim 1/(1-x_b)_+ \). Moreover, the full inclusion of powers of \( m_b/m_t \) has a negligible effect on the \( x_b \) spectrum; the dot-dashed and dotted lines in Fig. 1 are in fact almost indistinguishable. As for the perturbative fragmentation results, the distribution with no soft-gluon resummation shows a very sharp peak, though finite, once \( x_b \) approaches unity. This behaviour is smoothed out after we resum the soft NLL logarithms appearing in the initial condition of the perturbative fragmentation function, as the \( b \)-energy spectrum gets softer and shows the so-called ‘Sudakov peak’. Both perturbative fragmentation distributions become negative for \( x_b \to 0 \) and \( x_b \to 1 \), which is a known result, already found for heavy-quark production in \( e^+e^- \) annihilation [18,25]. For \( x_b \to 0 \), the coefficient function (5) contains large logarithms \( \sim \alpha_s(\mu) \log x_b \) which have not been resummed yet. Likewise, in the soft limit \( x_b \to 1 \), Eq. (6) contains contributions \( \sim \alpha_s(\mu)/(1-x_b)_+ \) and \( \sim \alpha_s(\mu)[\log(1-x_b)/(1-x_b)]_+ \). Since \( \alpha_s(m_b) \simeq 2\alpha_s(m_t) \), for \( \mu = m_t \) and \( \mu_0 = m_b \) such terms are smaller than the similar ones which appear in the initial condition of the perturbative fragmentation function (10), but nonetheless they would need to be resummed. As stated in [18], once \( x_b \) gets closer to unity, non-perturbative contributions also become important and should be taken into account.

The region of reliability of the perturbative calculation at large \( x_b \) may be related to the Landau pole in the expression for the strong coupling constant (8), and estimated

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\footnote{As far as soft-gluon resummation is concerned, matching the resummed initial condition to the exact \( O(\alpha_S) \) result (10) (see Eq. (76) in Ref. [18]), guarantees that the total NLO width is left unchanged.}
Figure 1: $b$-quark energy distribution in top decay according to the perturbative fragmentation approach, with (solid line) and without (dashes) NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function, and according to the exact NLO calculation, with (dot-dashes) and without (dots) inclusion of powers of $m_b/m_t$. In the inset figure, we show the same curves on a logarithmic scale.

to be $x_b \lesssim 1 - \Lambda/m_b$.

Figs. 2 and 3 show the dependence of the perturbative fragmentation $x_b$ distributions on the factorization scales $\mu_{0F}$ and $\mu_F$, with (Fig. 2) or without (Fig. 3) soft-gluon resummation. We observe in Fig. 2 (a) that the dependence on the initial scale $\mu_{0F}$ is small when we resum soft logarithms in the initial condition of the $b$-quark perturbative fragmentation function. Little impact is only visible in the neighbourhood of the Sudakov peak and, in particular, the distributions obtained for $\mu_{0F} = m_b$ and $\mu_{0F} = 2m_b$ are indistinguishable from each other. On the contrary, Fig. 3 (a) shows that the $x_b$ spectrum has a remarkable dependence on $\mu_{0F}$ once soft logarithms are not resummed. Comparing Fig. 2 (b) and Fig. 3 (b), we note a quite similar dependence on the choice of $\mu_F$. In fact, $\mu_{0F}$ rather than $\mu_F$ is the scale which enters the expression of the initial condition $D_b(x_b, \mu_{0F}, m_b)$. For $\mu_F$ approaching $\mu_{0F}$, the distribution with no soft resummation gets closer to the fixed-order, unevolved one shown in Fig. 1. We expect that once one includes soft resummation in the $\overline{\text{MS}}$ coefficient function (4), which contains the factorization scale $\mu_F$, the dependence of the $b$-energy spectrum on $\mu_F$ will become weaker as well, as found in [18] for the purpose of $e^+e^-$ annihilation. As a whole, we can state that resumming soft logarithms yields a reduction of the theoretical uncertainty, as the dependence on factorization scales is indeed an estimation of effects of higher-order contributions which we have been neglecting.
Figure 2: (a): $x_b$ spectrum for $\mu_F = m_t$ and $\mu_{0F} = m_b/2$ (dots), $\mu_{0F} = m_b$ (solid) and $\mu_{0F} = 2m_b$ (dashes); (b): $\mu_{0F} = m_b$ and $\mu_F = m_t/2$ (dots), $\mu_F = m_t$ (solid) and $\mu_F = 2m_t$ (dashes). The renormalization scales are kept at $\mu = m_t$ and $\mu_0 = m_b$. All curves include NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function.

Figure 3: As in Fig. 2 but with no soft resummation. Though not visible, all distributions show a finite, sharp peak once $x_b$ is close to 1.
3 Non-perturbative fragmentation and hadron-level results

In this Section we shall present results for the energy distribution of $b$-flavoured hadrons in top decay. We consider the transition $b \to B$, where $B$ is either a meson or a baryon containing a $b$ quark and define the normalized $b$-hadron energy fraction $x_B$ similarly to the parton-level one in Eq. (3).

The energy distribution of a hadron $B$ can be expressed as the convolution of the parton-level spectrum with the non-perturbative fragmentation function $D_{np}^B(x_B)$:

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dx_B}(x_B, m_t, m_W, m_b) = \frac{1}{\Gamma_0} \int_0^1 \frac{dz}{z} \frac{d\Gamma}{dz}(z, m_t, m_W, m_b) D_{np}^B \left( \frac{x_B}{z} \right).$$

In Eq. (19), $(1/\Gamma_0)d\Gamma/dz$ is the parton-level differential width (1) for $x_B = z$ and $D_{np}^B(x_B/z)$ is the non-perturbative fragmentation function describing the hadronization $b \to B$, which is process independent. We can therefore exploit data from $e^+e^- \to b\bar{b}$ processes to predict the $b$-quark hadronization in top decay.

Several models have been proposed to describe the non-perturbative transition from a quark- to a hadron-state. One of the most-commonly used \[22,24,26,27\] consists of a simple power functional form:

$$D_{np}(x; \alpha, \beta) = \frac{1}{B(\beta + 1, \alpha + 1)} (1 - x)\alpha x^\beta,$$

\[B(x, y)\] being the Euler Beta function.

The model of Kartvelishvili et al. \[20\] is still a power law, but with just one free parameter $\delta$:

$$D_{np}(x; \delta) = \frac{1}{(1 + \delta)(2 + \delta)} (1 - x)x^\delta.$$

We expect that if we use the non-perturbative fragmentation function in (20), which has two fittable parameters, we shall be able to get better agreement to the data than when using (21). However, in our analysis we shall try to tune the model (21) as well, in order to investigate how good it is at reproducing the data and how it compares to the other models considered.

Finally, the Peterson model \[21\] describes the transition of a heavy quark into a heavy hadron according to the following non-perturbative fragmentation function:

$$D_{np}(x; \epsilon) = \frac{A}{x[1 - 1/x - \epsilon/(1 - x)].}$$

For an explicit expression of the normalization factor $A$ and of the Mellin transform of Eq. (22), see \[32\]. The $N$-space transforms of Eqs. (20) and (21) are straightforward. The parameters $\alpha$ and $\beta$ in Eq. (20), $\delta$ in Eq. (21) and $\epsilon$ in Eq. (22) will have to be obtained from fits to experimental data.
To predict the $b$-quark hadronization properties, we shall use LEP data from the ALEPH Collaboration [33] and SLD data [34] for $e^+e^-$ collisions at the $Z$ pole, i.e. $\sqrt{s} = 91.2$ GeV. Both data sets refer to weakly-decaying $b$-hadrons [1], however, while the ALEPH $x_B$-data just accounts for $B$ mesons, the SLD set also considers $b$-flavoured baryons, mainly the $\Lambda_b$. In [33], it is stated that the mean values $\langle x_B \rangle$ of ALEPH and SLD are consistent with each other, within the range of systematic and statistical errors. However, no complete statistical analysis aiming at checking the consistency of the $x_B$ distributions has been done yet; therefore some difference is to be expected when comparing fits of non-perturbative models to data actually referring to different $b$-hadron samples, as we do have.

In order for the results of our fits to be applicable to the $b$-hadronization in top decay, we shall have to describe the perturbative process $e^+e^- \to \bar{b}b(g)$ in the same framework as we did for $t \to bW(g)$ when we do the fits. In fact, the factorization on the right-hand side of Eq. (19) and the splitting between perturbative and non-perturbative part is somewhat arbitrary and the parametrization of the non-perturbative model indeed depends on the approach which is used to describe the perturbative, parton-level process and on the values which are chosen for quantities like $\Lambda, m_b$ and for the renormalization and factorization scales. We shall therefore use $\overline{\text{MS}}$ coefficient functions [23] for $e^+e^-$ annihilation into massless quarks and convolute them with the perturbative $b$-quark fragmentation function evolved to NLL accuracy according to the DGLAP equations, possibly including soft-gluon NLL resummation in the initial condition of the perturbative fragmentation function. We shall then obtain a parton-level differential cross section $1/\sigma_0(d\sigma/dx_b)$, equivalent to Eq. (4), which we shall have to convolute with the non-perturbative fragmentation function of the hadronization model considered.

The experimental analyses [33,34] use the PYTHIA Monte Carlo event generator [6] to simulate $e^+e^- \to \bar{b}b$ processes and the subsequent parton showering; as a result, their framework is pretty different from the one which we have been using for top decay. We cannot therefore simply use the parametrizations of the hadronization models as they are reported in [33,34].

An extensive phenomenological study of fragmentation in $e^+e^-$ processes, with more details on fits to LEP and SLD data is currently under way [35]. For the purposes of our paper, we just tune the hadronization models to the data sets for a particular choice of the quantities which enter the perturbative calculation.

We point out that, although we convolute our perturbative result with a non-perturbative, smooth function, problems are still to be expected once $x_B \to 0, 1$. The region $x_B \to 0$ will not be reliably described since the perturbative calculation itself includes unresummed $\sim \log x_b$ terms. For $x_B \to 1$ one should in principle correctly account for all the missing, non-perturbative power corrections. Resumming a class of

---

5By weakly-decaying $b$-hadrons we mean hadrons containing a $b$ quark which decay according to a weak transition. For example, in the decay chain $B^* \to B\gamma \to (D^{(*)}\ell\nu)\gamma$, the $B$ meson rather than the $B^*$ is the weakly-decaying $b$-hadron whose energy fraction is experimentally measurable.
Table 1: Results of fits to $e^+e^- \rightarrow b\bar{b}$ data, using NLO coefficient functions, NLL DGLAP evolution and NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function. We set $\Lambda = 200$ MeV, $\mu_0F = \mu_0 = m_b = 5$ GeV and $\mu_F = \mu = \sqrt{s} = 91.2$ GeV. $\alpha$ and $\beta$ are the parameters in the power law (20), $\delta$ refers to (21), $\epsilon$ to (22).

|        | ALEPH       | SLD         |
|--------|-------------|-------------|
| $\alpha$ | 0.31 ± 0.15 | 1.88 ± 0.42 |
| $\beta$  | 13.21 ± 1.62| 27.04 ± 4.02|
| $\chi^2(\alpha, \beta)/\text{dof}$ | 2.62/14     | 11.12/16    |
| $\delta$ | 20.39 ± 0.77| 18.80 ± 0.60|
| $\chi^2(\delta)/\text{dof}$ | 17.27/15    | 17.46/17    |
| $\epsilon$ | (1.12 ± 0.16) × 10^{-3} | (1.17 ± 0.10) × 10^{-3} |
| $\chi^2(\epsilon)/\text{dof}$ | 22.96/15    | 130.80/17   |

Table 1: Results of fits to $e^+e^- \rightarrow b\bar{b}$ data, using NLO coefficient functions, NLL DGLAP evolution and NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function. We set $\Lambda = 200$ MeV, $\mu_0F = \mu_0 = m_b = 5$ GeV and $\mu_F = \mu = \sqrt{s} = 91.2$ GeV. $\alpha$ and $\beta$ are the parameters in the power law (20), $\delta$ refers to (21), $\epsilon$ to (22).

In order to perform trustworthy fits to the $e^+e^-$ data and acceptable predictions for the $b$-hadron spectrum in top decay, we shall limit ourselves to analysing data not too close to the critical points $x_B = 0, 1$. In particular, we shall consider ALEPH data within the range $0.18 < x_B < 0.94$ and SLD data for $0.18 < x_B < 0.90$, which implies that our predictions for top decay will have to be considered in the same $x_B$ ranges. When doing the fits, we account for both statistical and systematic errors on the data.

In Tables 1 and 2 we report the parameters which correspond to our best fits to the data, along with the $\chi^2$ per degree of freedom. We also investigate the impact of NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function. Standard deviations for best-fit parameters are included as well.

From Table 1, we learn that the use of the power law (20) which has two tunable parameters leads to excellent fits to both ALEPH and SLD data, but the values of $\alpha$ and $\beta$ which minimize the $\chi^2$ are affected by fairly large errors. The model of Kartvelishvili et al. is good at fitting in with the ALEPH and SLD data as well. The Peterson model is marginally consistent with the ALEPH data and unable to reproduce the SLD data. Although we are comparing data samples with different $b$-hadron contents, we observe that the best-fit values of $\epsilon$ and $\delta$ obtained for ALEPH and SLD are compatible within one and two standard deviations respectively. A bigger difference between ALEPH and SLD is found once we try to fit the power law (20).

Comparing Tables 1 and 2 we observe that the implementation of soft-gluon resummation in the perturbative fragmentation function results in statistically-different values of the best-fit parameters. With no soft resummation, using power laws with one
|       | ALEPH             | SLD               |
|-------|-------------------|-------------------|
| $\alpha$ | $0.66 \pm 0.13$ | $2.05 \pm 0.28$ |
| $\beta$  | $12.39 \pm 1.04$ | $22.10 \pm 2.13$ |
| $\chi^2(\alpha, \beta)/\text{dof}$ | $7.12/14$ | $40.23/16$ |
| $\delta$ | $14.97 \pm 0.44$ | $14.57 \pm 0.37$ |
| $\chi^2(\delta)/\text{dof}$ | $13.30/15$ | $58.63/17$ |
| $\epsilon$ | $(2.87 \pm 0.21) \times 10^{-3}$ | $(2.33 \pm 0.19) \times 10^{-3}$ |
| $\chi^2(\epsilon)/\text{dof}$ | $52.76/15$ | $275.69/17$ |

Table 2: As in Table 1, but without resuming soft logarithms.

or two fittable parameters still yields very-good fits to the ALEPH data, while none of the models considered is able to describe consistently the SLD data and the Peterson non-perturbative fragmentation fails in reproducing the ALEPH data as well.

We wish now to predict the $b$-hadron spectra in top decay, using models and parameters which give reliable fits to the $e^+e^-$ data. In order to account for the uncertainties on the parameters in the non-perturbative fragmentation functions, as shown in Tables 1 and 2, for each hadronization model we shall plot two curves which delimit a set of predictions at one-standard-deviation confidence level.

In Figs. 4 and 5 we show the $x_B$ distribution using all three considered models fitted to the ALEPH data (Fig. 4) and the power forms (20) and (21) fitted to SLD (Fig. 5), with all curves including NLL soft resummation in the initial condition of the perturbative fragmentation function. We note that different hadronization models yield statistically-different predictions for $b$-hadron spectra in top decay, within the accuracy of one standard deviation. However, one can show that the $x_B$ distributions according to the models (20) and (21) fitted to the SLD data are consistent within two standard deviations. This result can be related to the use of similar functional forms and to the large errors on $\alpha$ and $\beta$. The prediction obtained fitting the Peterson non-perturbative fragmentation function to the ALEPH data looks pretty different from the others, especially at small and middle values of $x_B$, where the predicted errors are pretty small for all the considered models. Moreover, the Peterson distributions are peaked at slightly-larger values of $x_B$.

In Fig. 6 we compare the ALEPH and SLD predictions according to the power law of Eq. (20) since, as shown in Table 1, this is the only hadronization model where the fitted parameters are statistically different. In fact, the overall shapes of the ALEPH- and SLD-based distributions lead to different predictions within one-standard-deviation accuracy, especially for $x_B \approx 0.7$, with the SLD ones being peaked at smaller $x_B$ values.

6We point out that the correlations between the errors on $\alpha$ and $\beta$ of the non-perturbative fragmentation function (20) are correctly taken into account throughout our analysis and in the plots which we show.
Figure 4: $x_B$ spectrum in top decay, with the hadronization modelled according to a power law (solid lines), the Kartvelishvili et al. (dashes) and the Peterson (dots) model, with the relevant parameters fitted to the ALEPH data. The plotted curves are the edges of bands at one-standard-deviation confidence level. NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function is included. We set $\mu_F = \mu = m_t$ and $\mu_0F = \mu_0 = m_b$.

This result can be checked to be true even at higher confidence level and, as anticipated, can be associated with the different $b$-hadron samples which the two experiments reconstructed. For the sake of comparison, we also show the ALEPH-based prediction without resumming soft logarithms, as Table 2 reports a small $\chi^2$ even in this case. We find that, although the perturbative predictions look rather different according to whether such contributions are resummed or not (see Fig. 4), the hadron-level results agree for $x_B \gtrsim 0.5$. In fact, the convolution with the non-perturbative fragmentation function smears the sharp peak shown by the unresummed result and the parameters $\alpha$ and $\beta$ are accordingly set by the fit to the $e^+e^-$ data in such a way that the two $x_B$-predictions do not differ too much from each other. Statistically-significant differences are nonetheless found for $x_B \lesssim 0.5$, where the predicted bands get narrower.

It is finally interesting to gauge the overall impact of the non-perturbative fragmentation by comparing our parton-level result with one of our hadron-level predictions, in particular the one obtained using the power law with two parameters fitted to the ALEPH data, as it corresponds to our smallest $\chi^2$. From Fig. 4 we learn that the hadronization effects are remarkable and the $x_B$ distribution is considerably softened with respect to the $x_b$ one. This means that, even when one uses a refined perturbative approach, with DGLAP evolution and soft resummation in the initial condition of the perturbative fragmentation function up to NLL accuracy, the role played by the non-perturbative input and hence by the $e^+e^-$ experimental data will still be crucial to
Figure 5: As in Fig. 4, but fitting the hadronization-model parameters to the SLD data.

Figure 6: As in Fig. 4, using the model of Eq. (20) and fitting its parameters to ALEPH (solid lines) and SLD (dashes), including NLL soft resummation in the initial condition of the perturbative fragmentation function. We also show results from the fit to ALEPH, but with no soft resummation (dots).
perform any prediction on $b$-fragmentation in top decay.

4 Conclusions

We discussed the $b$-quark fragmentation in top decay in NLO QCD using the method of perturbative fragmentation, which resums large logarithms $\sim \log(m_t^2/m_b^2)$ which multiply the strong coupling constant in the fixed-order massive calculation.

We computed the NLO differential width of top decay with respect to the $b$-quark energy fraction $x_b$ for a massless $b$, fully including $b$-mass effects and for a massive $b$ quark, but neglecting contributions proportional to powers of the ratio $m_b/m_t$. We determined the $\overline{\text{MS}}$-subtracted coefficient function and checked that our result is consistent with the known expression of the initial condition for heavy-quark perturbative fragmentation functions. We convoluted our $\overline{\text{MS}}$ coefficient function with the process-independent, perturbative fragmentation function for a massive $b$ quark, evolved up to NLL accuracy using the DGLAP equations. We showed results for the $b$ energy-fraction distribution in top decay, which we compared to the fixed-order results for a massive $b$ quark. We found that the use of the perturbative fragmentation approach has a remarkable effect on the parton-level distribution which is smoothed out with respect to the $\mathcal{O}(\alpha_S)$ one, which gets arbitrarily large once $x_b$ approaches unity. We also investigated the impact of the resummation of process-independent next-to-leading logarithms, which appear in the initial condition of the perturbative fragmentation function $D_b(x_b, \mu_{0F}, m_b)$ and are
associated with soft-gluon radiation. We found that it softens the $x_b$ distribution and makes the dependence on the scale $\mu_0^F$ weaker.

We then studied the energy distribution of $b$-hadrons in top decay and fitted some hadronization models to $e^+e^-$ data. We used ALEPH data on $b$-flavoured mesons and SLD data on $b$-flavoured baryons and mesons. In order to perform such fits we described the perturbative process $e^+e^- \rightarrow b\bar{b}(g)$ as we did for $b$-quark production in top decay. Throughout our analysis, we discarded data points where the hadron-level energy fraction is close to $x_B \simeq 0, 1$, since our approach is unreliable in the neighbourhood of these points.

We found that, within our perturbative framework, models which describe the non-perturbative fragmentation according to power laws with one and, in particular, with two fittable parameters lead to good descriptions of ALEPH and SLD data. The Peterson model is marginally consistent with the ALEPH results and unable to describe the SLD data. We also found that, although the ALEPH and SLD data refer to different $b$-hadron samples, the best-fit parameters obtained using the Kartvelishvili or the Peterson model are statistically-consistent within the error ranges. The implementation of NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function yields pretty different fits to the $e^+e^-$ data. An unresummed perturbative calculation still leads to good fits to the ALEPH data once we use power laws to model the hadronization, while it is unable to reliably describe the SLD data.

We then showed distributions of the energy fraction $x_B$ of $b$-flavoured hadrons in top decay, using only models which give reliable descriptions of the $e^+e^-$ data. We found that the models fitted to the ALEPH data yield statistically-distinguishable predictions for the $b$-meson spectrum in top decay. In particular, the Peterson-model distributions lie quite far from the others and are peaked at slightly-larger $x_B$ values. If we model the hadronization using power laws with one or two parameters, fitted to the SLD data sample, the predictions for the $x_B$ spectrum in top decay are compatible within two standard deviations, a result which is due to the large uncertainties on the best-fit values of $\alpha$ and $\beta$. We also compared results obtained using the power law with two parameters, but fitted to the ALEPH or SLD data, and obtained predictions which are statistically different.

We investigated the impact of NLL soft-gluon resummation in the initial condition of the perturbative fragmentation function on hadron-level distributions using ALEPH-based fits. We found a significant impact on the $x_B$ spectrum only at relatively-small $x_B$, while predictions with or without soft resummation are indistinguishable at large $x_B$ values. This result stresses the importance of the non-perturbative input and consequently of the use of the $e^+e^-$ results to perform reliable predictions in top decay. This can be learned from direct comparison between parton- and hadron-level distributions shown throughout the paper.

It will be now very interesting to use the present approach to perform predictions of other observables relying on the $b$-fragmentation in top decay, such as the invariant-
mass distributions used in [4,13] to fit the top mass value. It is clearly mandatory to compare the results obtained in the framework of perturbative fragmentation functions with the ones of Monte Carlo event generators, taking particular care about the induced uncertainty on the top mass measurement. However, for such a comparison to be reliable, Monte Carlo programs will have to be tuned to fit with the experimental data. This is in progress.

Finally, we plan to extend the method developed in [18] in the framework of $e^+e^-$ processes to resum with NLL accuracy also the process-dependent soft logarithms $\sim \alpha_s^n(\mu) \log^{n+1} N$ and $\sim \alpha_s^n(\mu) \log^n N$ appearing in the Mellin transform of the MS coefficient function (5). This further step will allow one to include in the perturbative calculation all soft logarithms, besides the process-independent ones which are present in the initial condition of the perturbative fragmentation function and have been correctly accounted for throughout this paper. It will be interesting to analyse the impact of soft-gluon resummation in the MS coefficient function on the energy spectrum of $b$ quarks and $b$-flavoured hadrons. This is in progress as well.

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A Details of the calculation

We wish to give some details on the calculation of the top-decay differential width at $\mathcal{O}(\alpha_S)$ for massless and massive $b$ quarks. In particular, the comparison of the two results will allow us to obtain the initial condition (10) of the perturbative fragmentation function. We adopt the on-shell mass-renormalization scheme and use dimensional regularization to regulate the ultraviolet and soft singularities and, in the massless case, the collinear divergence as well. We define the parameter $\epsilon$, which is related to the number $d$ of dimensions via $d = 4 - 2\epsilon$.

In the massless case, the differential width will contain a pole $\sim 1/\epsilon$, due to the collinear singularity, which disappears only in the total NLO width. This requires that, in order to get the correct finite term in the normalized $(1/\Gamma_0) d\Gamma_b/dx_b$, with $x_b$ defined in Eq. (3), the Born width $\Gamma_0$ will have to be evaluated in dimensional regularization at
\( O(\epsilon) \). We find:

\[
\Gamma_0 = \frac{\alpha m_t |V_{tb}|^2 (1 - w)^2 (1 + 2w)}{16 \sin^2 \theta_W} \left\{ 1 + \epsilon \left[ -\gamma_E + \log 4\pi - 2 \log(1 - w) + 2 \frac{1+w}{1+2w} \right] \right\},
\]

(23)

where \( \alpha \) is the electromagnetic coupling constant, \( \gamma_E = 0.577216 \ldots \) is the Euler constant, \( \theta_W \) is the Weinberg angle and \( w \) is the ratio \( w = m_W^2/m_t^2 \), already introduced in Section 2. We thus obtain:

\[
\frac{1}{\Gamma_0} \frac{d\hat{\Gamma}_b}{dx_b} = \delta(1 - x_b) + \frac{\alpha S(\mu)}{2\pi} \left\{ C_F \left[ \frac{1 + x_b^2}{(1 - x_b)_+} + \frac{3}{2} \delta(1 - x_b) \right] \right\} \times \left\{ -\frac{1}{\epsilon} + \gamma_E + \log 4\pi \right\} + \hat{A}_1(x_b),
\]

(24)

with \( \hat{A}_1(x_b) \) defined in Eq. (1). We note that \( \hat{A}_1(x_b) \) depends on the scale \( \mu_F \), remnant of the regularization procedure, which disappears only in the total width. We also checked that the integral of Eq. (24) agrees with the result of [31], where the \( O(\alpha_S) \) corrections to the top width have been evaluated in the approximation of a massless \( b \) quark. In order to get the \( \overline{\text{MS}} \) coefficient function (1), by definiteness, we shall have to subtract from Eq. (24) the \( O(\alpha_S) \) term multiplying the characteristic \( \overline{\text{MS}} \) constant \((-1/\epsilon + \gamma_E - \log 4\pi)\).

We wish to derive the NLO differential width with the full inclusion of the \( b \) mass \( m_b \). For this purpose, it is convenient to define the following quantities:

\[
b = \frac{m_b^2}{m_t^2},
\]

(25)

\[
s = \frac{1}{2}(1 + b - w),
\]

(26)

\[
\beta = \frac{\sqrt{b}}{s},
\]

(27)

\[
Q = s \sqrt{1 - \beta^2},
\]

(28)

\[
G_0 = \frac{1}{2} \left[ 1 + b - 2w + \frac{(1-b)^2}{w} \right],
\]

(29)

\[
\Phi(x_b) = s \left[ \sqrt{x_b^2 - \beta^2} - \text{arcth} \left( \frac{\sqrt{x_b^2 - \beta^2}}{x_b} \right) \right],
\]

(30)

where \( x_b \) is the normalized \( b \)-quark energy fraction

\[
x_b = \frac{x_E}{x_{E,\text{max}}} = \frac{x_E}{2s}, \quad \beta \leq x_b \leq 1,
\]

(31)
with \( x_E \) defined in Eq. (2). We find:

\[
1 \frac{d\Gamma}{\Gamma_0 dx_b} = \delta(1-x_b) + \frac{C_F \alpha_s(\mu)}{\pi Q} \left\{ 2s \left[ \text{Li}_2 \left( \frac{2Q}{1-s+Q} \right) - \text{Li}_2 \left( \frac{2Q}{s-b+Q} \right) \right] - \log(s+Q) \left( \log \left( \frac{1-s+Q}{\sqrt{w}} \right) + \log \left( \frac{s-b+Q}{2s(1-\beta)} \right) \right) \right. \\
+ \left. \frac{1}{2} \log(b) \log \left( \frac{s-b+Q}{2s(1-\beta)} \right) + \left( \frac{Q^2}{G_0} + s-b \right) \log \left( \frac{s+Q}{\sqrt{b}} \right) \right. \\
+ \left. (1-b) \log \left( \frac{1-s+Q}{\sqrt{w}} \right) + Q \left( \frac{6(s-b)(s-b)}{wG_0} - 1 \right) \log(b) \right\} - \frac{3}{2} \left[ \frac{1}{G_0} + 1 + \frac{1+b}{2w} \right] (1-x_b) - 1 \\
- 2 \Phi(x_b) \left[ \frac{1}{(1-x_b)_+} + \frac{s}{G_0} \left( \frac{1+b}{2w} \right) (1-x_b) - 1 \right] \\
- \left. 2 \left[ \frac{2s(1-\beta)}{\sqrt{w}} \right] \delta(1-x_b) \right\} \\
- \left[ \frac{2s^2}{G_0} \left( \frac{1-x_b}{1-2sx_b+b} \right) + \frac{s}{G_0} \left( \frac{1+b}{2w} \right) (1-x_b) - 1 \right] \right\}.
\]

In Eqs. (24) and (32) we note the presence of the so-called ‘plus prescription’ \( 1/(1-x_b)_+ \). Such a term arises after one integrates over the phase space for real-gluon radiation. For a massive \( b \) quark, where \( x_{b,\text{min}} \neq 0 \), one makes use of the following expansion:

\[
\frac{(x_b-x_{b,\text{min}})}{(1-x_b)^{1+\epsilon}} = \frac{-1}{\epsilon} \delta(1-x_b) + \frac{1}{(1-x_b)_+} + \mathcal{O}(\epsilon),
\]

with the plus prescription defined as:

\[
\int_{x_{b,\text{min}}}^{1} (g(x_b))_+ h(x_b) dx_b = \int_{x_{b,\text{min}}}^{1} g(x_b) [h(x_b) - h(1)] dx_b.
\]

We observe that in Eq. (30) the quark mass regulates the collinear divergence, therefore, unlike the result with a massless \( b \) quark (24), Eq. (30) does not contain any dependence on the dimensional-regularization quantities \( \epsilon \) or \( \mu_F \).

For the sake of checking the initial condition of the \( b \)-quark perturbative fragmentation function (10), we need to rewrite Eq. (30) neglecting powers of \( m_b/m_t \).

We find:

\[
1 \frac{d\Gamma}{\Gamma_0 dx_b} = \delta(1-x_b) + \frac{\alpha_s(\mu)}{2\pi} A_1(x_b),
\]

with

\[
A_1(x_b) = C_F \left\{ \frac{1+x_b^2}{(1-x_b)_+} + \frac{3}{2} \delta(1-x_b) \right\} \log \frac{m_t^2}{m_b^2}
\]
+ 2 \frac{1 + x_b^2}{(1 - x_b)_+} \log[(1 - w)x_b] - \frac{4x_b}{(1 - x_b)_+} + \frac{4w(1 - w)}{1 + 2w} \frac{x_b(1 - x_b)}{1 - (1 - w)x_b}
+ \delta(1 - x_b) \left[ 4 \text{Li}_2(1 - w) + 2 \log w \log(1 - w) - \frac{2\pi^2}{3} - \frac{2(1 - w)}{1 + 2w} \log(1 - w)
- \frac{2w}{1 - w} \log w - 4 \right]. \tag{36}

We note that in Eq. (36) the strong coupling constant multiplies the large logarithm 
\log(m_t^2/m_b^2), which spoils the reliability of the fixed-order calculation and makes the 
approach of perturbative fragmentation mandatory. Such large logarithms are absent 
in the total NLO width, which can be checked to agree with the result in the literature, 
one once accounts for the mass of the b quark \[36\].

If \( \mu_F \) is of the order of \( m_b \), one can express the perturbative fragmentation function 
\( D(x_b, \mu_F, m_b) \) as a power expansion in \( \alpha_S \):

\[
D(x_b, \mu_F, m_b) = d^{(0)}(x_b) + \frac{\alpha_S(\mu)}{2\pi} d^{(1)}(x_b, \mu_F, m_b) + \mathcal{O}(\alpha_S^2) \tag{37}
\]

Inserting Eqs. (5), (35) and (37) in Eq. (4), and solving for \( d^{(0)} \) and \( d^{(1)} \), one finds:

\[
d^{(0)}(x_b) = \delta(1 - x_b), \tag{38}
\]

\[
d^{(1)}(x_b, \mu_F, m_b) = A_1(x_b) - \hat{A}_1(x_b). \tag{39}
\]

Comparing then \( A_1(x_b) \) and \( \hat{A}_1(x_b) \), it is straightforward getting Eq. (10). It should be 
noted that, although \( A_1(x_b) \) and \( \hat{A}_1(x_b) \) separately depend on \( m_W \) via the ratio \( w \), their 
difference and hence the initial condition for the perturbative fragmentation function do 
not, which is essential for it to be process independent.

## B Coefficient function in Mellin space

For the sake of completeness, we wish to present the result for the Mellin transform of 
the MS coefficient function (5):

\[
\hat{\Gamma}_N(m_t, m_W, \mu, \mu_F) = \frac{1}{\Gamma_0} \int_0^1 dz \, z^{N-1} \frac{d \hat{\Gamma}_b}{dz}^{\text{MS}} (z, m_t, m_W, \mu, \mu_F). \tag{40}
\]

Given \( \Gamma(x) \) the Euler Gamma function, we define the poligamma functions:

\[
\psi_k(x) = \frac{d^k \log \Gamma(x)}{dx^k} \tag{41}
\]

and the combinations

\[
S_1(N) = \psi_0(N + 1) - \psi_0(1), \tag{42}
\]

\[
S_2(N) = -\psi_1(N + 1) + \psi_1(1). \tag{43}
\]
The basic, non-trivial $N$-space transforms of the terms which appear in Eq. (3) are given by:

\[
\int_0^1 dz \frac{z^{N-1}}{(1-z)^+} = -S_1(N-1), \quad (44)
\]

\[
\int_0^1 dz \frac{\log z}{(1-z)^+} z^{N-1} = -\psi_1(N), \quad (45)
\]

\[
\int_0^1 dz \left[ \frac{1}{1-z} \log(1-z) \right]_+ z^{N-1} = \frac{1}{2} \left[ S_1^2(N-1) + S_2(N-1) \right], \quad (46)
\]

\[
\int_0^1 dz \frac{z(1-z)}{1-(1-w)z} z^{N-1} = \frac{2F_1(1,N+1,N+2,1-w)}{N+1} - \frac{2F_1(1,N+2,N+3,1-w)}{N+2}. \quad (47)
\]

We shall then get:

\[
\hat{\Gamma}_N(m_t,m_W,\mu,\mu_F) = 1 + \frac{\alpha_s(\mu)C_F}{2\pi} \left\{ \log \frac{m_t^2}{\mu_F^2} \left[ \frac{1}{N(N+1)} - 2S_1(N) + \frac{3}{2} \right] \right. \\
+ \left[ 1 + 2 \log(1-w) \right] \frac{1}{N(N+1)} - 2\psi_1(N) - 2\psi_1(N+2) \\
+ \frac{4w(1-w)}{1+2w} \left[ \frac{2F_1(1,N+1,N+2,1-w)}{N+1} \\
- \frac{2F_1(1,N+2,N+3,1-w)}{N+2} \right] + S_1^2(N-1) + S_1^2(N+1) \\
+ \left[ S_2(N-1) + S_2(N+1) + 2[1 - 2 \log(1-w)]S_1(N) \right. \\
+ 2 \log w \log(1-w) - 2 \frac{1-w}{1+2w} \log(1-w) - \frac{2w}{1-w} \log w \\
+ 4\text{Li}_2(1-w) - 6 - \frac{2\pi^2}{3} \left. \right\}. \quad (48)
\]

One can show that, for $N \to \infty$:

\[
S_1(N) \sim \psi_0(N) \sim \log N. \quad (49)
\]

Similar large-$N$ behaviour is also shown by the Mellin transform of the initial condition of the perturbative fragmentation function (10), whose most-singular term at large $x_b$ has a $N$-space counterpart analogous to Eq. (49).

**References**

1. M. Beneke, I. Efthymiopoulos, M.L. Mangano, J. Womersley et al., in Proceedings of 1999 CERN Workshop on Standard Model Physics (and more) at the LHC, CERN 2000-004, G. Altarelli and M.L. Mangano eds., p. 419, [hep-ph/0003033](http://arxiv.org/abs/hep-ph/0003033).
2. J.A. Aguilar–Saavedra et al., ECFA/DESY LC Physics Working group, hep-ph/0106315; T. Abe et al., American Linear Collider Working Group, Part 3, Studies of Exotic and Standard Model Physics, hep-ex/0106057; K. Abe et al., ACFA Linear Collider Working group, hep-ph/0109166.

3. DØ Collaboration, B. Abbott et al., Phys. Rev. D58 (1998) 052001; CDF Collaboration, T. Affolder et al., Phys. Rev. D63 (2001) 032003.

4. A. Kharchilava, Phys. Lett. B476 (2000) 73.

5. G. Corcella, I.G. Knowles, G. Marchesini, S. Moretti, K. Odagiri, P. Richardson, M.H. Seymour and B.R. Webber, JHEP 0101 (2001) 010.

6. T. Sjöstrand, Comput. Phys. Commun. 82 (1994) 74; T. Sjöstrand, P. Edén, C. Friberg, L. Lönnblad, G. Miu, S. Mrenna and E. Norrbin, Comput. Phys. Commun. 135 (2001) 238.

7. H. Baer, F.E. Paige, S.D. Protopopescu and X. Tata, hep-ph/0001086.

8. B.R. Webber, Nucl. Phys. B238 (1984) 492.

9. B. Andersson, G. Gustafson, G. Ingelman and T. Sjöstrand, Phys. Rep. 97 (1983) 31.

10. R.D. Field and R.P. Feynman, Nucl. Phys. B136 (1978) 1.

11. G. Marchesini and B.R. Webber, Nucl. Phys. B310 (1988) 461.

12. G. Corcella and M.H. Seymour, Phys. Lett. B442 (1998) 417; E. Norrbin and T. Sjöstrand, Nucl. Phys. B603 (2001) 297.

13. G. Corcella, J. Phys. G26 (2000) 634; G. Corcella, M.L. Mangano and M.H. Seymour, JHEP 0007 (2000) 004.

14. J.C. Collins and D.E. Soper, Ann. Rev. Nucl. Part. Sci. 37 (1987) 383.

15. B. Mele and P. Nason, Nucl. Phys. B361 (1991) 626.

16. G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977) 298.

17. L.N. Lipatov, Sov. J. Nucl. Phys. 20 (1975) 95; V.N. Gribov and L.N. Lipatov, Sov. J. Nucl. Phys. 15 (1972) 438; Yu.L. Dokshitzer, Sov. Phys. 46 (1977) 641.

18. M. Cacciari and S. Catani, hep-ph/0107138.

19. M. Cacciari and M. Greco, Nucl. Phys. B421 (1994) 530.

20. V.G. Kartvelishvili, A.K. Likehoded and V.A. Petrov, Phys. Lett. B78 (1978) 615.
21. C. Peterson, D. Schlatter, I. Schmitt and P.M. Zerwas, Phys. Rev. D27 (1983) 105.

22. G. Colangelo and P. Nason, Phys. Lett. B285 (1992) 167.

23. P. Nason and B.R. Webber, Nucl. Phys. B421 (1994) 473; ibid. B480 (1996) 755 (erratum).

24. M. Cacciari and M. Greco, Phys. Rev. D55 (1997) 7134.

25. P. Nason and C. Oleari, Nucl. Phys. B565 (2000) 245.

26. M. Cacciari, M. Greco, S. Rolli and A. Tanzini, Phys. Rev. D55 (1997) 2736.

27. M. Cacciari and M. Greco, Z. Phys. C69 (1996) 459.

28. CDF Collaboration, T. Affolder et al., Phys. Rev. Lett. 86 (2001) 3233.

29. G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B175 (1980) 27; W. Furmanski and R. Petronzio, Phys. Lett. B97 (1980) 437.

30. E.G. Floratos, D.A. Ross and C.T. Sachrajda, Nucl. Phys. B129 (1977) 66; ibid. B139 (1978) 545 (erratum); ibid. B152 (1979) 493; A. Gonzales-Arroyo, C. Lopez and F.J. Yndurain, Nucl. Phys. B153 (1979) 161; E.G. Floratos, R. Lacaze and C. Kounnas, Nucl. Phys. B192 (1981) 417.

31. C.S. Li, R.J. Oakes and T.C. Yuan, Phys. Rev. D43 (1991) 855.

32. B.A. Kniehl, G. Kramer and M. Spira, Z. Phys. C76 (1997) 689.

33. ALEPH Collaboration, A. Heister et al., Phys. Lett. B512 (2001) 30.

34. SLD Collaboration, K. Abe et al., Phys. Rev Lett. 84 (2000) 4300; D. Dong, private communication.

35. M. Cacciari, private communication.

36. A. Czarnecki, Phys. Lett. B252 (1990) 467.