Superpositions of the Orbital Angular Momentum for Applications in Quantum Experiments

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Abstract

Two different experimental techniques for preparation and analyzing superpositions of the Gaussian and Laguerre-Gaussian modes are presented. This is done exploiting an interferometric method on the one hand and using computer generated holograms on the other hand. It is shown that by shifting the hologram with respect to an incoming Gaussian beam different superpositions of the Gaussian and the Laguerre-Gaussian beam can be produced. An analytical expression between the relative phase and the amplitudes of the modes and the displacement of the hologram is given. The application of such orbital angular momenta superpositions in quantum experiments such as quantum cryptography is discussed.

1 Introduction

In recent years a steadily growing interest in orbital angular momentum states of light can be observed. These light fields which are solutions of the scalar wave equation are mathematically described by
Laguerre-Gaussian modes possessing a helical phase structure. As a result they have one or more phase singularities. The orbital angular momentum carried by these light fields is distinct from the angular momentum associated with polarization, it is quantized in units of $\hbar$ and can be converted into mechanical angular momentum [1]. The possibility of using such light fields for driving micromachines, and their application as optical tweezers and optical traps make them possibly useful [2, 3, 4, 5].

One may be led to believe that the orbital angular momentum of photon pairs created by spontaneous parametric down-conversion is not conserved if using classical detection methods [6]. However from the quantum physics perspective most interesting are those applications which exploit the quantum properties of photons with orbital angular momentum. As already shown [7] spontaneous parametric down-conversion creating pairs of photons conserves orbital angular momentum on the individual photon level. Also, the two down-converted photons in a pair have been demonstrated to be in an entangled state with respect to their orbital angular momentum. As the Laguerre-Gaussian modes can be used to define an infinitely dimensional Hilbert space, orbital angular momentum entangled photons provide access to multi-dimensional entanglement which involves many orthogonal quantum states. Multi-dimensional entangled states are of considerable importance in the field of quantum information and quantum communication enabling, for example, quantum cryptography with higher alphabets.

For these quantum applications of orbital angular momentum it is necessary to be able to analyze orbital angular momentum entangled
states of individual photons and one has to have quantitative measures for the multi-dimensional entanglement. Since analyzing entanglement locally always implies analyzing superpositions it is essential to have experimental techniques for preparing and analyzing superpositions of different orbital angular momentum eigenstates. One quantitative measure for multi-dimensional entanglement is a Bell inequality experiment generalized to more than two dimensions.

In the following we will present techniques, already realized in experiment, for preparing superpositions of Laguerre-Gaussian modes with arbitrary amplitude and phase ratios. This was done employing two different techniques, using computer generated holograms on the one hand and an interferometric method on the other hand. We will also present how these techniques will be applied in a Bell-inequality experiment as mentioned above, which is in progress in our laboratory.

2 Mathematical description of the Laguerre-Gaussian modes

The well-known Gaussian beam is not the only solution of the scalar paraxial wave equation. Also, the Hermite Gaussian (HG) \[8\] and the Laguerre-Gaussian (LG) modes for which the electromagnetic field amplitude is given by

\[
u_{p,l}(r, \theta, z) = \frac{2^{p+l} 1}{\pi (p + |l|)! w(z)} r^{|l|} L_p^{|l|} \left( \frac{2r^2}{w(z)^2} \right) e^{-r^2/w(z)^2} e^{ikrz/2w(z)} e^{-i(2p+|l|+1) \arctan(\frac{z}{\sqrt{R}})} e^{-il\theta} \]

are solutions of this equation. With their two-fold infinite number of the indices both the LG and the HG modes build an orthogonal
basis set for describing any paraxial transversal mode of the free propagation. For our further considerations we will only focus on the LG modes.

An LG mode is characterized by its two indices $p$ and $l$ and by the standard Gaussian beam parameter definitions for the spot size $w(z)$, the radius of wavefront curvature $R(z)$ and the Rayleigh length $z_R$. The $L_p^l(x)$ term in (1) is a generalized Laguerre polynomial. The indices $p$ and $l$ are referred to as the radial and azimuthal mode index respectively, $p + 1$ is the number of radial nodes and $l2\pi$ is the phase variation along a closed path around the beam center (Figure 1). This phase variation which is due to the $e^{-il\theta}$ term in (1) results in a helical structure of the wave front. In consequence there is a phase singularity in the beam center for $l \neq 0$ and in order to satisfy the wave equation the intensity has to vanish there. Therefore such states are also called doughnut modes. Since the Laguerre-Gaussian modes are angular momentum eigenstates they carry an orbital angular momentum of $\hbar l$ per photon. The fact Laguerre Gaussian modes carry an orbital angular momentum was predicted and it was experimentally verified.

As usual in quantum mechanics, external variables describe the quantum state in real space while internal variables refer to additional variables. In that sense it is important to stress that the angular momentum carried by Laguerre Gaussian modes with $l \neq 0$ is an external angular momentum distinct from the internal angular momentum of the photons associated with their spin.
3 Production of the Laguerre-Gaussian modes

There are several experimental methods like cavity induced production, astigmatic mode conversion [10] and the use of computer generated holograms [11] for creating LG modes. In this article we restrict ourselves to describing the use of computer generated holograms.

A hologram is a recording of the interference pattern between the desired field and some reference field. The simplest possible reference field is the plane wave.

\[ R = R_0 e^{i k_x x + i k_z z} \]  

The interference pattern produced by such a beam propagating at an angle \( \xi = \arctan(\frac{k_x}{k_z}) \) and e.g. an \( LG_{01} \) mode propagating along the z-direction can be calculated numerically (Figure 2a). It is a line grating with one dislocation in the center. Now, if this grating is illuminated by the reference beam, which is sufficiently well realized if this hologram is placed at the waist of a Gaussian beam, the \( LG_{01} \) mode is reproduced. Intuitively speaking the phase dislocation exerts a “torque” onto the diffracted beam because of the difference of the local grating vectors in the upper and lower parts of the grating. This “torque” depends on the diffraction order \( n \) and on the number of dislocations \( \Delta m \) of the hologram. Consequently the right and left diffraction orders gain different handedness and the associated orbital angular momentum values differ in their sign. The definition of the sign can be chosen by convention. The modulation of the incoming beam can be either done in the phase using transmission phase
gratings or reflection gratings or in the amplitude using absorption gratings the latter being rather inefficient.

In our experiments we used transmission phase gratings with a diffraction efficiency of 70% after blazing. The binary structure of the hologram in (Figure 2a) is modified by blazing such that it results in a pattern as shown in (Figure 2b). The transmission function of such a hologram in polar coordinates is

$$T(r, \phi) = e^{i \delta \frac{1}{2\pi} \text{mod}(\Delta m \phi - \frac{2\pi}{\Lambda} r \cos \phi, 2\pi)}$$  \hspace{1cm} (3)$$

where $\delta$ is the amplitude of the phase modulation and the second factor in the superscript is the actual pattern of the blazed hologram. $\Lambda$ is the spacing period of the grating and $\text{mod}(a, b) = a - b \text{ Int}(\frac{a}{b})$. As mentioned above the fraction of intensity diffracted into higher orders of the hologram that is for a grating with one dislocation to the higher order LG modes with $l \neq 1$ can be decreased by blazing. But the blazing has no influence on the composition of the output beam’s radial index p-terms. The relative amplitudes of the p-terms depend on the choice of the beam parameters of the input beam to output beam \cite{12}. However for our experimental applications using holograms with one dislocation the relative amplitudes of the $p \neq 0$-terms become negligible.
4 Superpositions of the Gaussian and the Laguerre-Gaussian modes

Central to many, if not all, quantum experiments is the concept of superposition. It is therefore important to be able to both produce and analyze superpositions of the various states of a chosen basis. We therefore discuss now superpositions of the LG modes presented above. There are several experimental methods for producing such superpositions. One simple way is to use a Mach-Zehnder interferometer as sketched in (Figure 3a). After the input mode is split by the first beam splitter the beam of each arm is sent through a hologram causing the desired mode transformations which are in general different transformations in the two arms. For producing superpositions of the Gaussian and the $LG_{01}$ mode it is sufficient to place a hologram with one dislocation only in one of the arms of the interferometer. The two beams are brought together on a second beam splitter where they are superposed. An advantage of this method is that by attenuating each of the arms and using a phase plate superpositions with arbitrary amplitudes and relative phases can be produced without changing the setup. The resulting interference pattern for a superposition of an $LG_{00}$ mode (=Gaussian mode) and an $LG_{01}$ mode is shown in Figure 3b.

However the interferometric preparation of superposition modes has also some disadvantages. The experimental setup becomes too complex and too difficult to control when one has to create and analyze superpositions several times. It would be necessary to keep the relative phase in the interferometer arms stable and one also has to
take the different divergences of the interfering modes into account. A more convenient but less general method for creating superposition modes is to use a displaced hologram[7]. This method is particularly suitable for producing superpositions of an $LG_{0l}$ with the Gaussian mode which may also be seen as an $LG$ mode with $l = 0$. The transmission function of a hologram which is designed to transform a Gaussian mode into an $LG_{01}$ mode is given by (3) with $\Delta m = 1$. In order to transform an incoming Gaussian beam into an $LG_{01}$ beam the hologram should be placed at the waist of the Gaussian beam and the beam should be sent through the center of the hologram where the dislocation is located. The intensity pattern of such an $LG_{01}$ mode possesses a centrally located singularity. By shifting the dislocation out of the beam center step by step one can experimentally observe that the singularity becomes eccentric resulting in the same pattern achieved by the interferometric setup.

Numerical simulations show that there are also higher order $LG_{0l}$ components present in the case when the superposition is produced by a displaced hologram. The traces in the bottom of Figure 5c are corresponding to the amplitudes of the $LG_{0-1}$, $LG_{02}$ and $LG_{0-2}$ modes. This is also required by the unitary of the procedure. Nevertheless the relation (5) between the relative amplitudes and the position of the singularity is still a good approximation because the amplitudes of these higher orders are small. In the case where the beam is sent through a border region of the hologram far away from the dislocation the output beam is again a Gaussian beam because there the hologram acts as an ordinary grating.

An important question is whether in the experiment one actually
observe coherent superpositions rather than incoherent mixtures. The distinction between coherent superposition and incoherent mixture of Gaussian and LG modes is that the latter posses no phase singularity. This is because adding the spatial intensity distributions of these two modes will yield a finite intensity everywhere in the resulting pattern. In contrast, in a coherent superposition the amplitudes are added and therefore the phase singularity must remain and is displaced to an eccentric location (Figure 3b). It will appear at that point where the amplitudes of the two modes are equal with opposite signs. Therefore the radial distance of the singularity from the beam center is a measure of the amplitude ratio of the Gaussian to the LG components whereas the angular position of the singularity is determined by their relative phase.

To obtain quantitative results we calculated the intensity distribution of a normalized superposition mode of an \( LG_{00} \) and an \( LG_{01} \) mode described by

\[
\frac{1}{\sqrt{1+\gamma^2}} \left[ |u_{00}\rangle + \gamma e^{i\varphi} |u_{01}\rangle \right]
\]

(4)

and looked for the position of its phase singularity. Here \( |u_{00}\rangle \) and \( |u_{01}\rangle \) denote the amplitudes of an \( LG_{00} \) and an \( LG_{01} \) respectively as given by \( \text{(1)} \), \( \gamma \) is the relative amplitude of the \( LG_{01} \) mode and \( \varphi \) is the relative phase of the interfering modes. After inserting the corresponding \( LG \)-amplitudes \( \text{(1)} \) into \( \text{(4)} \) we found for the position of the singularity the cylindrical coordinates

\[
r = \frac{w_0}{\gamma \sqrt{2}}, \ \theta = \varphi,
\]

(5)

where \( w_0 \) denotes the waist size of the Gaussian beam.(Figure 3b).
Although this expression only holds for superpositions of $LG_{00}$ and $LG_{0\pm1}$ modes it can easily be generalized to superpositions of higher order LG mode containing more than just one phase singularity.

In order to prove that the eccentric mode is indeed a superposition mode one has to project the superposition onto the orthogonal basis states. Experimentally this was done by sending a Gaussian laser beam (HeNe, 632nm) through a displaced hologram producing a superposition of the Gaussian and the $LG_{01}$ mode. In the next step the output mode was projected onto the Gaussian mode by coupling into a mono-mode optical fiber. Since all other modes have a larger spatial extension, only the Gaussian mode can propagate in the mono-mode fiber and it therefore acts as a filter for the $LG_{0l}$ modes with $l \neq 0$.

Having only a filter transmitting the Gaussian mode the $LG_{01}$ mode had to be identified via an additional step. This was done by introducing a second hologram making a mode transformation on the output beam reducing the azimuthal index $l$ by one before coupling into mono-mode optical fibers (Figure 4). For each position of the displaced hologram the transmitted intensity to the Gaussian and the $LG_{01}$ detector was measured.

The results are shown in Figure 5a and 5b. When the hologram is centered the incoming Gaussian mode is transformed into an $LG_{01}$ mode. As a result the intensity of light coupled into the Gauss-detector is a minimum and the one coupled into the $LG_{01}$-detector a maximum. As the hologram is shifted the intensity at Gauss-detector increases and the intensity at the $LG_{01}$-detector drops. The asymmetry in Figure 5a) is a result of the imperfection of the holograms [13]. However the extinction ratio $e$ for all measurements was always far better than
These results are in agreement with our numerical calculations of the superposition modes (Figure 5c). The action of the hologram on the incoming field is characterized by (8). The transmitted beam directly after the hologram is given by

\[ u_{out}(r, \phi) = T(r, \phi)u_{in}(r, \phi) \]  

(6)

Denoting the relative position of the hologram with respect to the beam center with \((r_0, \phi_0)\) one finds the projection of a transmitted LG0l mode onto the LG0L-mode to be

\[ a^L_{0l}(r_0, \phi_0) = \int_{-\infty}^{+\infty} \int_0^{2\pi} rdrd\phi \left( u_{0L}(r, \phi, 0)e^{-i\Delta m \frac{2\pi}{\Lambda} r \cos \phi} \right)^* T(r-r_0, \phi-\phi_0)u_{l}(r, \phi, 0) \]  

(7)

This expression represents the amplitude of an LG0L-mode when a hologram with one dislocation placed at \((r_0, \phi_0)\) is illuminated by an LG0l mode. The numerical simulation of a setup which corresponds to our experiment is given in Figure 5c.

As mentioned above, the superposition does not only consist of two modes. The output mode will also contain higher order modes (Figure 5c bottom). This is a necessary consequence of the fact that the action of the hologram is unitary. In general, the relative amplitudes of the higher order modes will increase when the hologram has many dislocations or when the single dislocation hologram is illuminated by an LG0l beam with \(|l| > 1\). However they are negligible (Figure 5c bottom) for the case that the hologram is illuminated by an LG01 mode.
5 Conclusion and Outlook

In this work we showed techniques for producing and analyzing superpositions of Laguerre-Gaussian modes. Our experimental results show that it is possible to achieve mode detection of high distinction ratio (1 : 300) with the technique described above. As recently shown the orbital angular momentum of photons is conserved in parametric down-conversion [7] and the down-converted photons are found to be in an entangled state with respect to the orbital angular momentum. Since orbital angular momentum entangled photons can be used as qu-nits they open a practical approach to multi-dimensional entanglement where the entangled states do not only consist of two orthogonal states but of many of them. We expect such states to be of importance for the current efforts in the field of quantum computation and quantum communication. For example, quantum cryptography with higher alphabets could enable one to increase the information flux through the communication channels.

However the ultimate confirmation of the entanglement of the orbital angular momentum will be a Bell inequality experiment generalized to more states [14]. Such an experiment will also give a quantitative measure of the multi-dimensional entanglement produced by parametric down-conversion. Employing the techniques presented here for preparing and analyzing superpositions of orbital angular momentum states such an experiment is presently in progress in our laboratory.

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Figure 1: The wave front (top) and the intensity pattern (bottom) of the simplest Laguerre-Gaussian (LG) or doughnut mode. The azimuthal phase term $e^{-il\theta}$ of the LG modes results in helical wave fronts. The phase variation along a closed path is $2\pi l$. Therefore in order to fulfill the wave equation the intensity has to vanish in the center of the beam.
Figure 2: Computer generated binary a) and blazed b) templates for computer generated holograms with single dislocation. By illuminating these gratings with a Gaussian beam an $LG_{01}$ mode is produced in their first diffraction order. The diffraction efficiency in the desired order can be increased by blazing.
Figure 3: Superpositions of the Gaussian and the $LG_{01}$ mode. Using an interferometer a) with a single dislocation hologram placed in one arm superpositions of the Gaussian and the $LG_{01}$ can be produced. Such superpositions posses an eccentric singularity b) where the radial distance of of the singularity from the beam center is a measure of the amplitude ratio of the Gaussian to the LG components and the angular position of the singularity is determined by their relative phase $\phi$.

Figure 4: Measurement of the Gauss and the $LG_{01}$ components of a superposition mode. The displaced hologram produces a superposition of the Gaussian and the $LG_{01}$ mode. The relative amplitudes are determined by The mode detector which consists of a second hologram and a mono-mode optical fiber makes a scan determining the relative amplitudes of the Gaussian and the $LG_{01}$ mode.
Figure 5: Mode decomposition after a displaced hologram; experimental results 5a), 5b) and simulation 5c). Superpositions of Gaussian and $LG_{01}$ modes were produced by a displaced hologram. For each displacement the intensities at the Gauss-5a) and at the $LG_{01}$-5b) detector were measured. The same results were also achieved in numerical simulation 5c) where a) matches 5a) and b) 5b). The traces at the bottom of 5c) correspond to the amplitudes of the $LG_{01}$, $LG_{02}$ and $LG_{0-2}$ modes which are negligible in this experiment. The unsymmetry in 5a) is due to the imperfection of the hologram.
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