THE ALEXANDER MODULE OF LINKS AT INFINITY

DAVID CIMASONI

Abstract. Walter Neumann [5] showed that the topology of a “regular”
algebraic curve \( V \subset \mathbb{C}^2 \) is determined up to proper isotopy by some link in
\( S^3 \) called the link at infinity of \( V \). In this note, we compute the Alexander
module over \( \mathbb{C}[t^\pm 1] \) of any such link at infinity.

1. Introduction

The intersection of a reduced algebraic curve \( V \subset \mathbb{C}^2 \) with any suffi-
ciently large sphere \( S^3 \) about the origin in \( \mathbb{C}^2 \) gives a well-defined link called
the link at infinity of \( V \subset \mathbb{C}^2 \). This link at infinity was first introduced by
Walter Neumann and Lee Rudolph [4] and studied further by Neumann [5].
In order to state several of their results, let us recall some terminology. The
fiber \( f^{-1}(c) \) of a polynomial map \( f: \mathbb{C}^2 \to \mathbb{C} \) is called regular if there exists
a neighborhood \( D \) of \( c \) in \( \mathbb{C} \) such that \( f|_{f^{-1}(D)} : D \to D \) is a locally trivial
fibration. An algebraic curve \( V \subset \mathbb{C}^2 \) is regular if it is a regular fiber of
its defining polynomial. One might think that if \( c \) is not a singular value
of \( f \), then \( f^{-1}(c) \) is regular; this is wrong. In fact, the following additional
condition is required: a fiber \( f^{-1}(c) \) is regular at infinity if there exists a
neighborhood \( D \) of \( c \) in \( \mathbb{C} \) and a compact \( K \) in \( \mathbb{C}^2 \) such that \( f \) restricted
to \( f^{-1}(D) \setminus K \) is a locally trivial fibration. It can be proved that \( f^{-1}(c) \) is
regular if and only if it is non-singular and regular at infinity [3].

A first interesting result is that only finitely many fibers of a given \( f \)
are irregular at infinity, and that the regular fibers all define the same link
at infinity up to isotopy: it is called the regular link at infinity of \( f \), and
denoted by \( \mathcal{L}(f, \infty) \). Furthermore, \( \mathcal{L}(f, \infty) \) is a fibered link if and only if all
the fibers of \( f \) are regular at infinity. Finally, Walter Neumann proved the
following striking result: the topology of a regular algebraic curve \( V \subset \mathbb{C}^2 \)
as an embedded smooth manifold is determined by its link at infinity. More
precisely: up to isotopy in \( S^3 \), there exists a unique minimal Seifert
surface \( F \) for \( \mathcal{L}(f, \infty) \), and \( V \) is properly isotopic to the embedded surface
obtained from \( F \) by attaching a collar out to infinity in \( \mathbb{C}^2 \) to the boundary
of \( F \).

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In the present note, we give a closed formula for the Alexander module over \( \mathbb{C}[[t^{\pm 1}]] \) of the regular link at infinity of any polynomial map \( f : \mathbb{C}^2 \to \mathbb{C} \) (Theorem 3.2). The decisive property of \( \mathcal{L}(f, \infty) \) is that it can be seen as the boundary of the fiber \( F \) of a fibered multilink (see [5, Theorem 4]). Furthermore, this multilink can be constructed by iterated cabling and connected sum operations from the unknot, and the Alexander module over \( \mathbb{C}[[t^{\pm 1}]] \) of this type of fibered multilinks is well-known (we recall this result of [2] in Theorem 3.1 below). Therefore, our method will be to consider a fibered multilink with fiber \( F \) and Alexander module \( A \), and to compute the Alexander module of the oriented link \( L = \partial F \) from the module \( A \) (Proposition 2.5). This is achieved by introducing “generalized Seifert forms” for the multilink, and comparing them with the traditional Seifert form for \( L \). The result is then applied to \( \mathcal{L}(f, \infty) \) (Theorem 3.2), and an example concludes the paper.

2. Fibered multilinks

A multilink [2] is an oriented link \( L = L_1 \cup \ldots \cup L_n \) in \( S^3 \) together with an integer \( m_i \) associated with each component \( L_i \), with the convention that a component \( L_i \) with multiplicity \( m_i \) is the same as \( -L_i \) (\( L_i \) with reversed orientation) with multiplicity \( -m_i \). Throughout this paper, we will write \( \mathbf{m} \) for the integers \( (m_1, \ldots, m_n) \), \( d \) for their greatest common divisor, and \( L(\mathbf{m}) \) for the multilink. Of course, a set of multiplicities \( \mathbf{m} \) can be thought of as an element of \( H_1(L) \). If \( X \) denotes the exterior of \( L \), several classical theorems imply that \( H_1(L) \) is isomorphic to \([X, S^1]\), the group of homotopy classes of maps \( X \to S^1 \). As a consequence, assigning a set of multiplicities to an oriented link is a way to specify a preferred infinite cyclic covering \( \tilde{X}(\mathbf{m}) \xrightarrow{p} X \): it is the pullback \( \mathbb{Z} \)-bundle \( \phi^* \exp \), where \( \mathbb{R} \overset{\exp}{\to} S^1 \) is the universal \( \mathbb{Z} \)-bundle and \( X \overset{\phi}{\to} S^1 \) any map in the homotopy class \( \mathbf{m} \):

\[
\begin{array}{ccc}
\tilde{X}(\mathbf{m}) & \longrightarrow & \mathbb{R} \\
p \downarrow & & \exp \\
X & \overset{\phi}{\longrightarrow} & S^1.
\end{array}
\]

Choosing a generator \( t \) of the infinite cyclic group of the covering endows \( H_*(\tilde{X}(\mathbf{m})) \) with a structure of module over \( \mathbb{Z} < t > = \mathbb{Z}[t^{\pm 1}] \), the ring of Laurent polynomials with integer coefficients. Most of these invariants are not interesting: it is easy to prove that \( H_0(\tilde{X}(\mathbf{m})) \simeq \mathbb{Z}[t^{\pm 1}]/(t^d - 1) \), that \( H_2(\tilde{X}(\mathbf{m})) \) is a free module with the same rank as \( H_1(\tilde{X}(\mathbf{m})) \), and of course, that \( H_i(\tilde{X}(\mathbf{m})) = 0 \) for all \( i \geq 3 \). Therefore, the only interesting module is
$H_1(\tilde{X}(m))$: it is called the *Alexander module* of the multilink $L(m)$, and we will denote it by $A(L(m))$. Also, we will write $A(L(m); \mathbb{K})$ for the $\mathbb{K}[t^{\pm 1}]$-module $A(L(m)) \otimes \mathbb{K}[t^{\pm 1}]$, where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{C}$. Given $\mathcal{P}$ an $m \times n$ presentation matrix of $A(L(m))$ (that is, the matrix corresponding to a finite presentation of $A(L(m))$ with $n$ generators and $m$ relations), the greatest common divisor of the $n \times n$ minor determinants of $\mathcal{P}$ is called the *Alexander polynomial* of $L(m)$. This Laurent polynomial, denoted by $\Delta_{L(m)}$, is only defined up to multiplication by units of $\mathbb{Z}[t^{\pm 1}]$. Of course, if a multilink has multiplicities $\pm 1$, it is just an oriented link and these Alexander invariants coincide with the usual Alexander invariants of the corresponding oriented link.

Let us now recall the definition of a very interesting class of multilinks that generalizes the notion of fibered link: a *fibered multilink* is a multilink $L(m)$ such that there exists a locally trivial fibration $X \rightarrow S^1$ in the homotopy class $m \in [X, S^1]$. The oriented surface $F = \varphi^{-1}(1)$ is called the *fiber* of $L(m)$. The diagram

\[
\begin{array}{ccc}
\tilde{X}(m) & \xrightarrow{\Phi} & \mathbb{R} \\
p \downarrow & & \downarrow \exp \\
X & \xrightarrow{\varphi} & S^1 \\
\end{array}
\]

can now be understood as defining the pullback fibration $\Phi = \exp^\ast \varphi$. Since $\mathbb{R}$ is contractible, there exists a homeomorphism $F \times \mathbb{R} \rightarrow \tilde{X}(m)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
F \times \mathbb{R} & \longrightarrow & \tilde{X}(m) \\
\pi \downarrow & & \downarrow \Phi \\
\mathbb{R} & \xrightarrow{\cong} & \mathbb{R} \\
\end{array}
\]

Hence, the generator $\tilde{X}(m) \rightarrow \tilde{X}(m)$ of the infinite cyclic group of the covering $p$ can be seen as the transformation

\[
\begin{array}{ccc}
F \times \mathbb{R} & \longrightarrow & F \times \mathbb{R} \\
(x, z) & \longmapsto & (h(x), z + 1),
\end{array}
\]

where $F \xrightarrow{h} F$ is some homeomorphism, unique up to isotopy, called the *monodromy* of the multilink $L(m)$. We will use the same terminology for the induced automorphism $H_1(F) \xrightarrow{h} H_1(F)$.

**Proposition 2.1.** A presentation matrix of the Alexander module of a fibered multilink is given by $H^T - tI$, where $H$ is any matrix of the monodromy. In
In particular, the Alexander polynomial of a fibered multilink is the characteristic polynomial of the monodromy.

Proof. As seen in the above discussion, there is an isomorphism of \( \mathbb{Z} \)-modules \( H_1(F) \cong H_1(\tilde{X}(m)) \) such that \( t \cdot f(x) = f(h_*(x)) \). Choosing a \( \mathbb{Z} \)-basis \( e_1, \ldots, e_\mu \) of \( H_1(F) \), this gives an exact sequence of \( \mathbb{Z}[t^{\pm 1}] \)-modules

\[
\bigoplus_{i=1}^\mu \mathbb{Z}[t^{\pm 1}] e_i \xrightarrow{h_* - t} \bigoplus_{i=1}^\mu \mathbb{Z}[t^{\pm 1}] e_i \xrightarrow{f_*} H_1(\tilde{X}(m)) \longrightarrow 0,
\]

where \( f_* \) denotes the \( \mathbb{Z}[t^{\pm 1}] \)-linear extension of \( f \). This is a finite presentation of \( H_1(\tilde{X}(m)) \), so \( (H - tI)^T \) is a presentation matrix of this module. \( \square \)

Let \( F \subset S^3 \setminus L \) be the fiber of a fibered multilink \( L(m) \), and let us denote by \( \bar{F} \) the union \( F \cup L \) (see Figure 1 for an illustration of \( F \) and \( \bar{F} \) near a component of the multilink). The Seifert forms associated to \( F \) are the bilinear forms

\[ \alpha_+, \alpha_- : H_1(F) \times H_1(\bar{F}) \to \mathbb{Z} \]

given by \( \alpha_+(x, y) = \ell k(i_+ x, y) \) and \( \alpha_-(x, y) = \ell k(i_- x, y) \), where \( \ell k \) denotes the linking number and \( i_+, i_- : H_1(F) \to H_1(S^3 \setminus \bar{F}) \) the morphisms induced by the push in the positive or negative normal direction off \( F \). We will use the notation \( V_+, V_- \) for matrices of these forms.

As in the usual case of a fibered oriented link, the monodromy can be recovered from the Seifert forms.

**Proposition 2.2.** If a multilink is fibered with fiber \( F \), the matrices \( V_+ \) and \( V_- \) are square and unimodular. Furthermore, a matrix of the monodromy is given by \( H = (V_+ V_-)^T \).
Proof. The bilinear form \( \alpha_+: H_1(F) \times H_1(\mathcal{F}) \to \mathbb{Z} \) can be understood as a homomorphism \( H_1(F) \to \text{Hom}(H_1(\mathcal{F}), \mathbb{Z}) \simeq H^1(\mathcal{F}) \). The composition of this morphism with the Alexander isomorphism \( H^1(\mathcal{F}) \simeq H_1(S^3 \setminus \mathcal{F}) \) is nothing but \( i_+: H_1(F) \to H_1(S^3 \setminus \mathcal{F}) \). The same holds for \( \alpha_- \) and \( i_- \). As a consequence, the Seifert matrix \( V_+ \) (resp. \( V_- \)) with respect to basis \( \mathcal{A} \) of \( H_1(F) \) and \( \mathcal{A}^- \) of \( H_1(\mathcal{F}) \) is equal to the transposed matrix of \( i_+ \) (resp. \( i_- \)) with respect to the basis \( \mathcal{A} \) and \( \mathcal{A}^- \), where \( \mathcal{A}^- \) is the dual basis of \( \mathcal{A} \) via Alexander duality.

Now, the fibration \( S^3 \setminus L \to S^1 \) yields a fibration \( S^3 \setminus \mathcal{F} \to (0,1) \), so \( S^3 \setminus \mathcal{F} \) is homeomorphic to \( F \times (0,1) \). Hence, the maps \( i_+, i_-: F \to S^3 \setminus \mathcal{F} \) are homotopy equivalences, and \( i_+, i_-: H_1(F) \to H_1(S^3 \setminus \mathcal{F}) \) are isomorphisms. Therefore, the matrices \( V_+, V_- \) are unimodular. Finally, the monodromy of a fibered multilink can be defined as the composition \( (i_-)^{-1} \circ (i_+) \). Therefore, a matrix of the monodromy is given by \( H = (V_-)^T V_1^T = (V_+ V_-)^T \).

As an immediate consequence of this proposition, \( H_1(F) \) and \( H_1(\mathcal{F}) \) have the same rank. We need some more information about these modules.

**Lemma 2.3.** Let \( L(\mathbf{m}) \) be a fibered multilink with fiber \( F \) of genus \( g \). For \( i = 1, \ldots, n \), \( F \) has \( d_i = \gcd(m_i, \sum_{j \neq i} m_j k(L_i, L_j)) \) boundary components near \( L_i \). Furthermore, the homology of \( F \) has the form

\[
H_1(F) = G \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{d_i-1} \mathbb{Z} T_i^j \oplus \bigoplus_{j=1}^{d_n - d} \mathbb{Z} T_n^j,
\]

where \( G \) is a free \( \mathbb{Z} \)-module of rank \( 2dg \), and \( T_i^1, \ldots, T_i^{d_i} \) are the boundary components of \( F \) near \( L_i \). Finally,

\[
H_1(\mathcal{F}) = G \oplus \left( \bigoplus_{i=1}^{n} \mathbb{Z} L_i \left/ \sum_{i=1}^{n} \frac{m_i}{d_i} L_i \right. \right) \oplus \mathcal{B},
\]

where \( \mathcal{B} \) is a free \( \mathbb{Z} \)-module of rank \( 1 - n - d + \sum_{i=1}^{n} d_i \).

**Proof.** The fact that \( F \cap N(L_i) \) is a link with \( d_i \) components is very easy to check and well-known (see [2, p. 30]). Since \( F \) consists of \( d \) parallel copies of the fiber of the multilink \( L(\mathbf{m}/d) \), it may be assumed that \( d = 1 \). In this case, \( F \) is a connected oriented surface of genus \( g \) with \( \sum_{i=1}^{n} d_i \) boundary components and the result holds.
We will now compute $H_1(F)$ by induction on $d \geq 1$. Let us assume that $d = 1$. The Mayer-Vietoris exact sequence associated with the decomposition $\overline{F} = F \cup (\overline{F} \cap N(L))$ gives

$$0 \rightarrow H_1(\partial F) \xrightarrow{\varphi_1} H_1(F) \oplus H_1(L) \rightarrow H_1(\overline{F}) \rightarrow H_0(\partial F) \xrightarrow{\varphi_0} H_0(L),$$

where $\varphi_1(T^j_i) = (T^j_i, m_i L_i)$. Using the value of $H_1(F)$, it follows that $(H_1(F) \oplus H_1(L))/\text{Im} \varphi_1 = G \oplus (\bigoplus_{i=1}^n \mathbb{Z} L_i/\sum_i m_i L_i)$. Since the module $\text{Ker} \varphi$ is free of rank $\sum_{i=1}^n (d_i - 1)$, this concludes the case $d = 1$. Let us now consider a fibered multilink $L(m)$ with $\gcd(m_1, \ldots, m_n) = d > 1$. Clearly, $\overline{F} = F' \cup F''$, where $F'$ (resp. $F''$) is the fiber of $L(d)$ (resp. $L(d^{-1}m)$). The associated Mayer-Vietoris sequence together with the case $d = 1$ and the induction hypothesis give the result. □

**Proposition 2.4.** Let $L(m)$ be a fibered multilink. For $i = 1, \ldots, n$, let us note $D_i = \gcd(d_1, \ldots, d_i)$ with $d_i$ as above. Then, the Alexander module of $L(m)$ naturally factors into $A(L(m)) = A_G \oplus A_B$, where

$$A_B = \bigoplus_{i=1}^{n-1} \mathbb{Z}[t^{\pm 1}]/\left(\frac{(t^{D_i} - 1)(t^{d_{i+1}} - 1)}{(t^{D_i} - 1)}\right).$$

**Proof.** As seen above, the fiber $F$ is given by $d$ parallel copies of a connected surface $\overline{F}$ with $\sum_{i=1}^n \frac{d_i}{d}$ boundary components. Let us write $\overline{F} = \overline{G} \cup \overline{B}$, where $\overline{G}$ is a closed surface with a single boundary component, and $\overline{B}$ a planar surface with $1 + \sum_{i=1}^n \frac{d_i}{d}$ boundary components. The Mayer-Vietoris sequence gives $H_1(\overline{F}) = H_1(\overline{G}) \oplus H_1(\overline{B})$. Therefore, $H_1(F) = H_1(\overline{G}) \oplus H_1(B)$, where $G$ (resp. $B$) consists of $d$ parallel copies of $G$ (resp. $B$). Since the monodromy $F \xrightarrow{h} F$ of $L(m)$ is a homeomorphism, the monodromy $H_1(F) \xrightarrow{h_*} H_1(F)$ splits into $h_G \oplus h_B$, where $h_G = (h|G)_*$ and $h_B = (h|B)_*$. Therefore, a matrix $H$ of $h_*$ with respect to some basis $A = A_G \cup A_B$ of $H_1(F) = H_1(\overline{G}) \oplus H_1(B)$ can be written $H = H_G \oplus H_B$. By Proposition 2.1, $A(L(m))$ is presented by

$$H^T - tI = H_G^T \oplus H_B^T - tI = (H_G^T - tI) \oplus (H_B^T - tI).$$

Let us denote by $A_G$ (resp. $A_B$) the $\mathbb{Z}[t^{\pm 1}]$-module presented by $H_G^T - tI$ (resp. $H_B^T - tI$). It remains to compute the module $A_B$.

As seen in Lemma 2.3, a basis of $H_1(B)$ is given by

$$A_B = \langle T^1_1, \ldots, T^1_{d_1}, \ldots, T^n_1, \ldots, T^{d_{n-1}}_n, T^{d_n-1}_n, T^n_1, \ldots, T^{d_n-d}_n \rangle,$$
where $T^1_i, \ldots, T^{d_i}_i$ are the boundary components of $F$ near $L_i$. Clearly, the monodromy cyclically permutes these components, that is: $h_*(T^j_i) = T^{j+1}_i$ for $1 \leq j \leq d_i - 1$ and $h_*(T^{d_i}_i) = T^1_i$. Note that

\[ h_*(T^{d_n-d}_n) = T_n^{d_n-d+1} = \sum_{i=1}^{n-1} T_i^j - \sum_{1 \leq j \leq d_n-d} T_i^j \]

in $H_1(B)$, since $\partial \tilde{F} = \bigcup_{i=1}^{n} \bigcup_{j \equiv 1 \pmod{d}} T^j_i$. Therefore, the matrix of $h_B$ with respect to $A_B$ is given by

\[
H_B = \begin{pmatrix}
P_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & P_{n-1} \\
Q_n & & 
\end{pmatrix},
\]

where $P_i$ is the $d_i \times d_i$-matrix

\[
P_i = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & & 1
\end{pmatrix},
\]

$Q_n$ the $(d_n - d) \times (d_n - d - 1)$-matrix

\[
Q_n = \begin{pmatrix}
0 & \cdots & 0 \\
1 & \cdots & 1 \\
& & 1
\end{pmatrix},
\]

and $v = (v_j)$ a vector such that $v_j = -1$ if $j \equiv 1 \pmod{d}$, $v_j = 0$ else. It is easy to show that $H^T_B - tI$ is equivalent to

\[
\begin{pmatrix}
t^{d_1} - 1 & t^{d_2} - 1 & \cdots & t^{d_{n-1}} - 1 & t^{d_n - 1} \\
\frac{t^{d_1} - 1}{t^{d_1} - 1} & \frac{t^{d_2} - 1}{t^{d_2} - 1} & \cdots & \frac{t^{d_{n-1}} - 1}{t^{d_{n-1}} - 1} & \frac{t^{d_n - 1}}{t^{d_n - 1}}
\end{pmatrix}
\]

as a presentation matrix. It is then an exercise to check that the module $A_B$ presented by $H^T_B - tI$ is equal to $\bigoplus_{i=1}^{n-1} \mathbb{Z}/|t^{\pm 1}| \langle \frac{(t^{d_i}_i-1)(t^{d+i+1}_i-1)}{(t^{d_i}_i+1-1)} \rangle$. \(\square\)

We are finally ready to prove the main result of this paragraph.
**Proposition 2.5.** Let $L(m)$ be a fibered multilink with fiber $F$ and multiplicities $m_i \neq 0$ for all $i$, and let us denote by $L'$ the oriented link given by the boundary $\partial F$ of $F$. Then, the Alexander module of $L'$ over $\mathbb{Q}[t^{\pm 1}]$ is given by

$$A(L'; \mathbb{Q}) = (A_G \otimes \mathbb{Q}[t^{\pm 1}]) \oplus (\mathbb{Q}[t^{\pm 1}]/(t - 1))^{n-1} \oplus (\mathbb{Q}[t^{\pm 1}])^{\sum_{i=1}^{n}(d_i-1)},$$

where $A_G$ is the direct factor of $A(L(m))$ given in Proposition 2.4, and $d_i = \gcd(m_i, \sum_{j \neq i} m_j \ell_k(L_i, L_j))$.

**Proof.** Since $m_i \neq 0$, it follows that $d_i \neq 0$ for all $i$. By Lemma 2.3, one can write

$$H_1(F; \mathbb{Q}) = (G \otimes \mathbb{Q}) \oplus \bigoplus_{i=1}^{n-1} \mathbb{Q}^{d_i} \sum_{j=1}^{d_j} T_{i,j} \oplus \bigoplus_{i=1}^{n-1} \mathbb{Q}^{d_n-d} \sum_{j=1}^{d_j} T_{n,j}$$

$$H_1(F; \mathbb{Q}) = (G \otimes \mathbb{Q}) \oplus \bigoplus_{i=1}^{n-1} \mathbb{Q} m_i L_i \oplus (B \otimes \mathbb{Q}).$$

The matrices $V_+$ and $V_-$ with respect to these basis of $H_1(F; \mathbb{Q})$ and $H_1(F; \mathbb{Q})$ are of the form

$$V_+ = \begin{pmatrix} N & M^T & \ast \\ M & \ell & \ast \\ \ast & \ast & \ast \end{pmatrix}_{2dg \times n-1} \quad \quad V_- = \begin{pmatrix} N^T & M^T & \ast \\ M & \ell^T & \ast \\ \ast & \ast & \ast \end{pmatrix}_{2dg \times n-1}$$

As seen in the proof of Proposition 2.4, a matrix $H$ of the monodromy splits into $H_G \oplus H_B$. Furthermore, the basis of $H_1(F; \mathbb{Q})$ was chosen such that

$$H_B = \begin{pmatrix} I_{n-1} & \ast \\ 0 & \ast \end{pmatrix},$$

where $I_{n-1}$ denotes the identity matrix of dimension $n - 1$. By Proposition 2.23 $V_+ = H^T V_-$, that is,

$$\begin{pmatrix} N & M^T & \ast \\ M & \ell & \ast \\ \ast & \ast & \ast \end{pmatrix} = \begin{pmatrix} H_G^T & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & \ast & \ast \end{pmatrix} \begin{pmatrix} N^T & M^T & \ast \\ M & \ell^T & \ast \\ \ast & \ast & \ast \end{pmatrix} = \begin{pmatrix} H_G^T N^T & H_G^T M^T & \ast \\ H_G^T M & \ell^T & \ast \\ \ast & \ast & \ast \end{pmatrix}.$$
Therefore, we have the equalities

\[ N^T = N H_G , \quad M = M H_G , \quad \ell = \ell^T . \]  

Let us keep them in mind, and turn to the computation of the Alexander module of \( L' \). Since \( F \) has \( d \) connected components, a connected Seifert surface \( F' \) for \( L' \) is obtained from \( F \) via \( d - 1 \) handle attachments. Since \( d_i \neq 0 \) for all \( i \), we can write

\[ H_1(F'; \mathbb{Q}) = (G \otimes \mathbb{Q}) \oplus \bigoplus_{i=1}^{n-1} \mathbb{Q}(d_i T_i^j) \oplus \bigoplus_{j=1}^{d_n-1} \mathbb{Q}(d_n T_n^j) . \]

The Seifert matrix of \( L' \) with respect to this basis has the form

\[
V' = \begin{pmatrix}
N & * & * \\
\ell_1 & \ddots & \ell_1^T \\
\vdots & \ddots & \ddots \\
\ell_n & \ddots & \ell_n^T
\end{pmatrix} = \begin{pmatrix}
2dg & d_1 & d_{n-1} \\
& & \\
& & & 2dg \\
& & & d_1 \\
& & & d_{n-1}
\end{pmatrix},
\]

where \( \ell_i \) denotes \( d_i \) copies of the same line \( \ell_i \) (\( d_n - 1 \) copies if \( i = n \)). A presentation matrix of \( A(L'; \mathbb{Q}) \) is given by \( P' = V' - t(V')^T \). Since \( \sum_{i,j} T_i^j = 0 \) in \( H_1(F') \), it follows that \( \ell_n = - \sum_{i=1}^{n-1} \ell_i \). As a presentation matrix, \( P' \) is therefore equivalent to

\[
\begin{pmatrix}
N - t N^T & * & * \\
\ell_1 (1 - t) & \ddots & \ell_1^T (1 - t) \\
\vdots & \ddots & \ddots \\
\ell_{n-1} (1 - t) & \ddots & \ell_{n-1}^T (1 - t)
\end{pmatrix} = \begin{pmatrix}
N - t N^T & * & * \\
& \ell_1^T (1 - t) & \ldots \ell_{n-1}^T (1 - t) \\
& 0 & \ldots 0
\end{pmatrix},
\]

where the number of zero columns is equal to

\[
\sum_{i=1}^{n-1} (d_i - 1) + (d_n - 1) = \sum_{i=1}^{n} (d_i - 1).
\]

With the notations used above for \( V_+ \) and \( V_- \), this matrix is nothing but

\[
\begin{pmatrix}
N - t N^T & M^T (1 - t) & 0 \ldots 0 \\
M (1 - t) & \ell (1 - t) & 0 \ldots 0
\end{pmatrix}.
\]
Let us note $\tilde{P}' = \tilde{V} - t\tilde{V}^T$, where $\tilde{V} = \begin{pmatrix} N & M^T \\ M & \ell^T \end{pmatrix}$. The computation above shows that $\text{rk} A(L'; \mathbb{Q}) \geq \sum_{i=1}^{n}(d_i - 1)$. The fact that the rank of $A(L'; \mathbb{Q})$ is equal to $\sum_{i=1}^{n}(d_i - 1)$ can be proved by (at least) two distinct methods. By a more subtle analysis of $V_{\pm}$, one can check that $\Delta_{L(m)} = \det \tilde{P}' \cdot \Delta'$ with some factor $\Delta'$; since $L(m)$ is fibered, $\Delta_{L(m)} \neq 0$ so $\det \tilde{P}' \neq 0$ and $\text{rk} A(L'; \mathbb{Q}) = \sum_{i=1}^{n}(d_i - 1)$. A more conceptual proof goes as follows: $L'$ can be thought of as the result of the “splicing” of $L(m)$ with multilinks $L^{(1)}_{(m^{(1)})}, \ldots, L^{(n)}_{(m^{(n)})}$ (see [2, 5]). It can be showed that $\text{rk} A(L^{(i)}_{(m^{(i)})}) = d_i - 1$ for $i = 1, \ldots, n$, and that the rank of the Alexander module is additive under splicing (see [1, Theorem 4.3.1 and Proposition 3.2.4]). Since $L(m)$ is fibered, $\text{rk} A(L(m)) = 0$ and we get the result.

As a consequence, $\tilde{P}'$ is a presentation matrix of the torsion submodule of $A(L'; \mathbb{Q})$. Now, note that

$$(H_G^T \oplus I_{n-1})\tilde{V}^T = \begin{pmatrix} H_G^T & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} N^T & M^T \\ M & \ell^T \end{pmatrix} = \begin{pmatrix} H_G^T N^T & H_G^T M^T \\ M & \ell^T \end{pmatrix}.$$

By the equations (*), this is exactly the matrix $\tilde{V}$. Hence, the torsion submodule of $A(L'; \mathbb{Q})$ is presented by

$$\tilde{P}' = \tilde{V} - t\tilde{V}^T = (H_G^T \oplus I_{n-1})\tilde{V}^T - t\tilde{V}^T = ((H_G^T \oplus I_{n-1}) - tI)\tilde{V}^T.$$

Since $\det \tilde{P}' \neq 0$, $\tilde{V}^T$ is unimodular. Therefore, $\tilde{P}'$ is equivalent as a presentation matrix to $(H_G^T \oplus I_{n-1}) - tI = (H_G^T - tI) \oplus (1-t)I_{n-1}$. This concludes the proof.

3. Application to the Alexander module of links at infinity

In this paragraph, we use Propositions [2.4 and 2.5] to give a closed formula for the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of the regular link at infinity $\mathcal{L} = \mathcal{L}(f, \infty)$ of any polynomial map $f: \mathbb{C}^2 \to \mathbb{C}$. Given such an $f$, there exists a fibered multilink with multiplicities $m_i \neq 0$ and fiber $F$ such that $\mathcal{L} = \partial F$. Furthermore, this multilink is an iterated torus multilink: it can be constructed by iterated cabling and connected sum operations from the unknot. Since the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of iterated torus fibered multilinks is known, the result for $\mathcal{L}$ will follow directly from Propositions [2.4 and 2.5].

To state our result, we must assume that the reader is familiar with splice diagrams (see [2]). Recall that a splice diagram representing a multilink $L(m)$ is a tree $\Gamma$ decorated as follows:
- Some of its leaves (valency one vertices) are drawn as arrow heads and represent components of $L$; they are endowed with the multiplicity $m_i$ of the corresponding component $L_i$ of $L$.

- Each edge has an integer weight at any end where it meets a node (vertex of valency greater than one), and these edge-weights around a fixed node are pairwise coprime.

Associated to each non-arrowhead vertex $v$ of $\Gamma$ is a so-called “virtual component”: this is the additional link component that would be represented by a single arrow at that vertex $v$ with edge-weight 1. Splice diagrams are very convenient to compute linking numbers: given two vertices $v$ and $w$ of $\Gamma$, the linking number of the corresponding components (virtual or “real”) is the product of all the edge-weights adjacent to but not on the shortest path in $\Gamma$ connecting $v$ and $w$.

General splice diagrams as described here encode graph multilinks (that is: multilinks in homology sphere with graph manifold exterior). A multilink in $S^3$ is a graph multilink if and only if it is an iterated torus multilink, so the multilink associated with a polynomial map is encoded by such a splice diagram. Furthermore, Eisenbud and Neumann succeeded in computing the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of any fibered graph multilink $L(m)$ from its splice diagram $\Gamma$. If $L(m)$ has “uniform twists” (this is the case of the multilink associated with a polynomial map), the result goes as follows.

Let us denote by $N$ the set of nodes of $\Gamma$, by $E$ the set of edges connecting two nodes and by $V$ the set of non-arrowhead vertices of $\Gamma$. By cutting an edge $E \in E$ in two, one gets two splice diagrams representing two multilinks; let us denote by $d_E$ the greatest common divisor of the linking numbers of these two multilinks with the virtual component corresponding to the middle of the edge $E$. For every $v \in V$, let $\delta_v$ denote its valency and $m(v)$ the linking number of $L(m)$ with the virtual component corresponding to $v$. Finally, for every node $v \in N$, let $d_v$ be the greatest common divisor of the $d_E$’s of edges $E \in E$ which meet $v$, and of all the $m_i$’s of arrowheads adjacent to $v$.

**Theorem 3.1 (Eisenbud-Neumann [2, Theorem 14.1]).** Let $L(m)$ be a fibered graph multilink with monodromy $h$ and uniform twists, given by a splice diagram $\Gamma$. The Alexander module $A(L(m); \mathbb{C})$ is determined by the following properties:

- The Jordan normal form of $h_*$ consists of $1 \times 1$ and $2 \times 2$ Jordan blocks.
- The characteristic polynomial of $h_*$ is equal to
\[ \Delta(t) = (t^d - 1) \prod_{v \in V} (t^{m(v)|} - 1)^{\delta_v - 2}. \]

- The eigenvalues corresponding to the $2 \times 2$ Jordan blocks are the roots of
\[ \Delta'(t) = (t^d - 1) \prod_{E \in \mathcal{E}} (t^{d_E} - 1) \prod_{v \in \mathcal{N}} (t^{d_v} - 1). \]

Let us now state and prove our final result.

**Theorem 3.2.** Let $f: \mathbb{C}^2 \to \mathbb{C}$ be a polynomial map with regular link at infinity $L = L(f, \infty)$. If $L(m) = L(m_1, \ldots, m_n)$ denotes the multilink associated with $L$, let $d$ be the greatest common divisor of $m_1, \ldots, m_n$, and $d_i = \gcd(m_i, \sum_{j \neq i} m_j \ell k(L_i, L_j))$ for $i = 1, \ldots, n$. Also, let $\Delta(t)$ be the characteristic polynomial of the monodromy of $L(m)$, and $\Delta'(t)$ the polynomial corresponding to the $2 \times 2$ Jordan blocks (as in Theorem 3.1). Then, the Alexander module $A(L; \mathbb{C})$ of $L$ over $\mathbb{C}[t^\pm 1]$ is given by the following properties:

- The rank of $A(L; \mathbb{C})$ is equal to $\sum_{i=1}^n (d_i - 1)$.
- The Jordan normal form of $t$ restricted to the torsion submodule of $A(L; \mathbb{C})$ consists of $1 \times 1$ and $2 \times 2$ Jordan blocks.
- The order ideal of the torsion submodule of $A(L; \mathbb{C})$ is generated by
\[ \widetilde{\Delta}(t) = (t - 1)^{n-1} \frac{(t^d - 1) \Delta(t)}{\prod_{i=1}^n (t^{d_i} - 1)}. \]
- The eigenvalues corresponding to the $2 \times 2$ Jordan blocks are the roots of $\Delta'(t)$.

**Proof.** The regular link at infinity $L$ is given by the boundary $\partial F$ of the fiber of $L(m)$, which has non-zero multiplicities. By Propositions 2.A and 2.B,

\[ A(L; \mathbb{C}) = (A_G \otimes \mathbb{C}[t^\pm 1]) \oplus (\mathbb{C}[t^\pm 1]/(t - 1))^{n-1} \oplus (\mathbb{C}[t^\pm 1])^{\sum_{i=1}^n (d_i - 1)}, \]

where $A(L(m); \mathbb{C}) = (A_G \otimes \mathbb{C}[t^\pm 1]) \oplus \bigoplus_{i=1}^{n-1} \mathbb{C}[t^\pm 1]/ \left( \frac{(t^{d_i} - 1)(t^{d_i+1} - 1)}{t^{d_i+1} - 1} \right)$. Therefore, the rank of $A(L; \mathbb{C})$ is $\sum_{i=1}^n (d_i - 1)$ and the order ideal of its torsion
submodule is generated by
\[(t - 1)^{n - 1} \frac{\Delta(t)}{\prod_{i=1}^{n-1} (t^{D_i-1})(t^{d_i+1-1})} = (t - 1)^{n - 1} \frac{(t^d - 1)\Delta(t)}{\prod_{i=1}^{n}(t^{d_i} - 1)} ,
\]

since \(D_1 = d_1\) and \(D_n = \gcd(d_1, \ldots, d_n) = \gcd(m_1, \ldots, m_n) = d\). Furthermore, \(A_G \otimes \mathbb{C}[t^{\pm 1}]\) contributes to Jordan blocks of dimension at most two (by Theorem 3.1), and \(\bigoplus_{i=1}^{n-1} \mathbb{C}[t^{\pm 1}]\left/ \left( \frac{(t^{D_i-1})(t^{d_i+1-1})}{(t^{D_i+1}-1)} \right) \right/ \) to Jordan blocks of dimension one, since the polynomial \(\frac{(t^{D_i-1})(t^{d_i+1-1})}{(t^{D_i+1}-1)}\) has only simple roots.

We refer to [1, § 5.6] for a different proof of this result. Note that Propositions 2.4 and 2.5 give the Alexander module \(A(L; \mathbb{Q})\) from the module \(A(L(m); \mathbb{Q})\). The problem is that a closed formula for the Alexander module over \(\mathbb{Q}[t^{\pm 1}]\) of a fibered graph multilink remains unknown.

Let us conclude this note with an example.

**Example.** Let \(p, q, r\) be positive integers with \(\gcd(p, r) = 1\) and \(p < (q+1)r\). Consider the polynomial map \(f: \mathbb{C}^2 \to \mathbb{C}\) given by
\[f(x, y) = (x^q y + 1)^r + x^p .\]
As described in [5, p. 451], the associated multilink \(L(m) = L(1, qr)\) is given by the following splice diagram.

Using Theorem 3.1 one easily computes \(\Delta(t) = (t-1)\frac{(t^p-1)}{(t^r-1)}\) and \(\Delta'(t) = 1\). Hence
\[A(L(m); \mathbb{C}) = \mathbb{C}[t^{\pm 1}]\left/ \left( \frac{(t - 1)(t^p - 1)}{(t^r - 1)} \right) \right/ .\]
Furthermore, \(d_1 = \gcd(1, qr^2) = 1\) and \(d_2 = \gcd(qr, r) = r\). Therefore,
\[A(L; \mathbb{C}) = \mathbb{C}[t^{\pm 1}]\left/ \left( \frac{(t - 1)^2(t^r - 1)}{(t^p - 1)(t^r - 1)} \right) \right/ \oplus (\mathbb{C}[t^{\pm 1}])^{r-1} .\]
Note that $L(m)$ is nothing but a torus multilink; on this simple example, it is possible to compute the Alexander modules over $\mathbb{Z}[t^\pm 1]$. Using methods described in [1], one can show that the Alexander module of $L(m)$ is $\mathbb{Z}[t^\pm 1]/\left(\frac{(t^{pr}-1)(t-1)}{(t^{p}-1)}\right)$, and that $A(L) = (\mathbb{Z}[t^\pm 1])^{r-1} \oplus \tilde{A}(L)$, where $\tilde{A}(L)$ is presented by the matrix

$$
\begin{pmatrix}
(t-1)(t^{pr}-1) & q \\
(t^{p}-1)(t^{r}-1) & q(t-1)
\end{pmatrix}.
$$

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