Implied value-at-risk and model-free simulation

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Abstract
We propose a novel model-free approach for extracting the risk-neutral quantile function of an asset using options written on this asset. We develop two applications. First, we show how for a given stochastic asset model our approach makes it possible to simulate the underlying terminal asset value under the risk-neutral probability measure directly from option prices. Specifically, our approach outperforms existing approaches for simulating asset values for stochastic volatility models such as the Heston, the SVI, and the SABR models. Second, we estimate the option implied value-at-risk (VaR) and the option implied tail value-at-risk (TVaR) of a financial asset in a direct manner. We also provide an empirical illustration in which we use S&P 500 Index options to construct an implied VaR Index and we compare it with the VIX Index.

Keywords Implied value-at-risk · Model-free simulation under risk-neutral probability measure · Exact simulation · Stochastic volatility models · SABR model · Heston model

1 Introduction

The knowledge of the prices of European call and put options on an underlying asset $S$ at a given maturity $T$ across a continuum of strikes completely determines the probability distribution under the risk-neutral probability measure (also known in the literature as risk-neutral distribution) of the underlying asset at maturity, $S_T$. This result appeared already in the...
The approach has then been applied and improved by several authors such as Jackwerth and Rubinstein (1996), Aït-Sahalia and Lo (2000), and Bondarenko (2003). All these authors rely on the relationship linking the first derivative (resp. second derivative) of the option price with respect to the strike price $K$ to the cumulative distribution (resp. density) function of the underlying asset. Using this approach, it is possible to get the probability distribution function under the risk-neutral probability measure of the underlying asset. By inversion (which can be done numerically), one then obtains an estimate for the risk-neutral quantiles, and therefore of the implied value-at-risk (VaR) and the implied tail value-at-risk (TVaR), also known as expected shortfall (ES) or as conditional value-at-risk (CVaR). In the literature, there are various studies that use the above mentioned approach to estimate the implied VaR and the implied TVaR. We list a few of these: (a) Ahlawat (2012) extracts the probability distribution of the underlying asset price from option prices and then uses Monte Carlo simulations; (b) Barone-Adesi (2016) uses the calculation of the first derivative of the European put option with respect to its strike price (see Barone-Adesi et al., 2019a, 2019b for the empirical analysis); and (c) Mitra (2015), after showing the link with the Breeden and Litzenberger (1978) relation, derives a closed-form relationship between option prices, VaR, and TVaR.

In this paper, we propose an alternative approach, which makes it possible to bypass numerical inversion. The essential idea is to go directly from the knowledge of the option prices to the computation of the implied VaR at various levels (i.e., quantiles of the risk-neutral distribution of the underlying asset price) and of the implied TVaR. To do so, we use the expression of a quantile as the minimizer to an optimization problem over a weighted sum of prices of call and put options. Our approach makes it then possible to simulate the underlying asset price at some given maturity date under the risk-neutral probability measure in a model-free way given the prices of a large number of options written on this underlying asset. We then get an "exact" simulation method that can be used for simulating the terminal asset value for any model in which closed-form or semi-closed-form expressions for the option prices are known but in which the simulation of the underlying distribution is notoriously difficult. For instance, simulating the underlying asset value in a stochastic volatility model can be difficult or possibly biased due to the path-dependent nature of the underlying process, such as in the case of the Heston and the SABR models.

As a matter of fact, there are several approaches for simulation in these models. Some methods include discretization schemes, such as the implicit Milstein method (see Kahl & Jäckel, 2006) or Euler method, as explained in Broadie and Kaya (2006, Section 2, p. 218), all of which can be used for Heston (-like) models. An approximation (based on Fourier methods and copulas) for the cumulative distribution function of the time-integrated variance is used in the Monte Carlo method proposed by Leitao et al. (2017) for the SABR model. Continuous-time Markov chains are used to approximate a (general) latent stochastic variance process by Cui et al. (2021). The approaches based on approximations and discretization suffer from bias; a survey of bias associated with these approaches can be found in Lord et al. (2010) for the Heston model and in Cai et al. (2017) and Leitao et al. (2017) for the SABR model. To reduce the bias Van der Stoep et al. (2014) use a model independent numerical scheme for the local volatility component, which is able to avoid the mismatch between the stochastic volatility model and the European options prices. Various authors, including Broadie and Kaya (2006, Section 3, p. 219) which use Fourier inversion approaches and conditioning and Cai et al. (2017), provide exact and semi-exact simulation schemes to address the issue of bias. Our paper proposes an alternative and novel approach to exact simulation under a model for which option prices are available. Specifically, our approach makes it possible to
simulate in a fairly straightforward manner the path-dependent asset price $S_T$ without the need to rely on the simulation of a discretized path, and thus avoiding any issue of bias.

In Sect. 2, we express the quantile under the risk-neutral probability measure as the solution to an optimization problem using options prices as inputs. We illustrate this expression within a Black–Scholes model for which a closed-form expression for the quantiles is known. In Sect. 3, we describe the model-free simulation method that allows to simulate the underlying asset prices directly from the observed market prices without the need to differentiate with respect to the strike and without inversion of the probability distribution function. Furthermore, we show how to simulate the terminal value of an asset under Black–Scholes, SABR and Heston models. In Sect. 4, we show how to estimate the implied VaR and the implied TVaR directly from observed option prices. In addition, we also present a brief empirical case study; we use pricing data of options written on the S&P 500 Index to construct an implied VaR Index and we compare it with the VIX Index. Section 5 concludes.

### 2 Implied quantile from options prices

We assume an arbitrage-free market so that a risk-neutral probability measure $\mathbb{Q}$ exists. Furthermore, $S_T$ represents the underlying asset price at maturity $T$ and $r$ indicates a constant, continuously compounded risk-free interest rate. As introduced in Newey and Powell (1987) and Bellini et al. (2014) the quantile of $S_T$ at confidence level $\alpha \in (0, 1)$, denoted by $q_{S_T}(\alpha)$, can be seen as the minimizer of a piecewise-linear loss function. Specifically,

$$ q_{S_T}(\alpha) := \arg\min_{K \in \mathbb{R}} \left\{ \alpha e^{rT} c(K) + (1 - \alpha) e^{rT} p(K) \right\}, $$

where $c(K)$ and $p(K)$ denote the prices of call and put options with respective strike price $K$, i.e.,

$$ c(K) = e^{-rT} \mathbb{E}^\mathbb{Q} \left[ \max (S_T - K, 0) \right] $$

and

$$ p(K) = e^{-rT} \mathbb{E}^\mathbb{Q} \left[ \max (K - S_T, 0) \right]. $$

**Proposition 1** (Computation of $q_{S_T}(\alpha)$ from option prices) The quantile at confidence level $\alpha$, $q_{S_T}(\alpha)$, can be computed as either the minimizer of a weighted sum of call and put prices as stated in Eq. (1), or by taking the First Order Condition (hereafter F.O.C.) from Eq. (1) to express the minimization problem in terms of “Greeks”:

$$ \text{solve } K \in \mathbb{R} \text{ such that } \frac{1 - \alpha}{\alpha} = -\frac{c'(K)}{p'(K)}, $$

where $c'(K) = \frac{d c(K)}{d K} < 0$ and $p'(K) = \frac{d p(K)}{d K} > 0$.

**Proof** The minimization approach follows from (1). Then, (2) is the F.O.C. for this minimization problem. It writes as

$$ \text{solve } K \in \mathbb{R} \text{ such that } \alpha c'(K) + (1 - \alpha) p'(K) = 0. $$

We then factorize with $\alpha$ and we note that the denominator in (2) is never equal to 0 as $c'(K) < 0$ and $p'(K) > 0$, as a respective no-arbitrage property of call and put prices.

Note that the first derivatives of call and put option prices with respect to the strike price are known as “Dual Delta” in the financial industry (see Wystup, 2017 for more details).
Remark 1  The quantile of $S_T$ at confidence level $\alpha$ can thus be obtained in two different ways: either as a solution to the optimization problem (1) or as the solution to Eq. (2). Note that, in case of options traded in financial markets as input, Eq. (1) is the only feasible (practical) solution since it is not possible to derive the First Order Condition. Instead, if option prices are computed with closed-form or semi-closed-form expressions obtained from stochastic models, both (1) and (2) can be implemented. Of course, in the latter case, the two methods provide the same results but they may differ in terms of computational effort and ease of implementation. For the sake of clarity, our exact simulation scheme (ObSim) discussed in Sect. 3 is based on Eq. (1) since, in our opinion, it is more intuitive and immediate to comprehend and implement. In Sect. 3.4, however, we discuss a faster implementation of ObSim where we rely on Eq. (2) instead of Eq. (1) and we point out some advantages and disadvantages.

2.1 Implied quantile in the Black–Scholes model

We use the Black–Scholes model to illustrate Proposition 1, although in this model there is a closed-form expression for the quantile and thus no need for our approach as such. Recall that in the Black–Scholes model the prices of European-style call and put options are given respectively by

$$
c_{BS}(K) = S_0 N(d_1) - K e^{-rT} N(d_2)
$$

$$
p_{BS}(K) = K e^{-rT} N(-d_2) - S_0 N(-d_1),
$$

in which $d_1 = \frac{\ln(S_0/K)}{\sigma \sqrt{T}} + \frac{r + \frac{1}{2} \sigma^2}{\sigma \sqrt{T}} T$, $d_2 = d_1 - \sigma \sqrt{T}$, $S_0$ represents the underlying asset price at time zero, $\sigma$ the volatility and $N(\cdot)$ the cumulative distribution function of a standard normal distribution. From Proposition 1, the quantile of $S_T$ at level $\alpha$, $q_{ST}(\alpha)$, solves

$$
q_{ST}(\alpha) = \arg\min_{K \in \mathbb{R}} \left\{ \alpha \ e^{rT} c_{BS}(K) + (1 - \alpha) \ e^{rT} p_{BS}(K) \right\},
$$

or as the solution of the F.O.C. (2) where the dual deltas are known explicitly,

$$
c'_{BS}(K) = -e^{-rT} N(d_2) \quad \text{and} \quad p'_{BS}(K) = e^{-rT} N(-d_2).
$$

Using the fact that $1 - N(d_2) = N(-d_2)$, the F.O.C. (2) writes as

$$
\alpha = N(-d_2).
$$

Replacing $d_2$ by its expression and solving for $K$, we find that

$$
K = S_0 \exp \left( N^{-1}(\alpha) \sigma \sqrt{T} + \left( r - \frac{1}{2} \sigma^2 \right) T \right),
$$

where $N^{-1}(\alpha)$ indicates the $\alpha$-quantile of the standard normal distribution. This is exactly the expression of the quantile, $q_{ST}(\alpha)$, of the underlying asset price at time $T$ with confidence level $\alpha$. Indeed, expression (5) follows also immediately from the well-known expression of the stock price at time $T$ in the Black–Scholes model, $S_T = S_0 \exp \left\{ (r - \frac{1}{2} \sigma^2) T + \sigma W_T \right\}$, where $W_T$ is a Wiener process at time $T$. 

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2.2 Implied quantile in the SABR model

One of the assumptions of the Black–Scholes model is that the volatility is constant. Unfortunately, in the real world, this assumption is also a limitation since markets show different volatilities for different levels of strike prices. One solution to overcome such a problem is the use of stochastic volatility models. Among them, one important model is the Stochastic Alpha Beta Rho (SABR) model, introduced by Hagan et al. (2002), which aims to replicate the volatility smile and the skew observed in the financial market. Hagan et al. (2002) provide in their work a closed-form expression of an approximation of call and put prices using perturbation techniques. The prices of European call ($c_{sabr}$) and put ($p_{sabr}$) options in the SABR model can be approximated by the Black (1976) formula in which the appropriate implied volatility is used:

$$c_{sabr} = e^{-rT} [F_0 N(d_1) - KN(d_2)]$$
$$p_{sabr} = e^{-rT} [KN(-d_2) - F_0 N(-d_1)],$$

where $F_0$ is the current forward price, $d_1 = \frac{\ln(F_0/K) + \frac{1}{2} \sigma_{sabr}^2 T}{\sigma_{sabr} \sqrt{T}}$, $d_2 = d_1 - \sigma_{sabr} \sqrt{T}$, and where we use the expression of the implied volatility, $\sigma_{sabr}$, as it is done for instance by Korn and Tang (2013) and in industry practice by Rouah (2007). For strikes $K \neq F_0$, its expression is given by

$$\sigma_{sabr} = \frac{v_0}{(F_0 K)^{(1-\beta)/2}} \left\{ 1 + \left[ \frac{(1-\beta)^2}{24} \frac{v_0^2}{(F_0 K)^{1-\beta}} + \frac{3\rho \lambda v_0}{4 (F_0 K)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} \lambda^2 \right] T \right\}^{-1/2} \chi(z),$$

in which $z = \frac{1}{v_0} (F_0 K)^{(1-\beta)/2} \ln \left( \frac{F_0}{K} \right)$, $\chi(z) = \ln \left( \frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho} \right)$, $\beta$ is the exponent, $v_0$ is the current volatility, $\lambda$ denotes the volatility of volatility, and $\rho$ the correlation coefficient of Wiener processes (see Hagan et al., 2002 for further details). When the strike $K = F_0$, the formula of the implied volatility becomes

$$\sigma_{sabr} = \frac{v_0}{F_0^{1-\beta}} \left\{ 1 + \left[ \frac{(1-\beta)^2}{24} \frac{v_0^2}{F_0^{2-2\beta}} + \frac{3\rho \lambda v_0}{4 F_0^{1-\beta}} + \frac{2-3\rho^2}{24} \lambda^2 \right] T \right\}^{-1/2} \chi(z),$$

As known in the literature the Black’s formula can be used to price European options on the underlying asset assuming that the relationship $F_0 = S_0 e^{rT}$ holds and $r$ indicates a constant, continuously compounded risk-free interest rate. As a consequence, from Proposition 1, the quantile of the underlying asset price at time $T$ with confidence level $\alpha$ can be described in the following way:

$$q_{ST}(\alpha) = \arg\min_{K \in \mathbb{R}} \left\{ \alpha e^{rT} c_{sabr}(K) + (1-\alpha) e^{rT} p_{sabr}(K) \right\}.$$

Although a closed-form solution using the SABR model is not as straightforward to find as in the case of Black–Scholes [see Eq. (5)], it is possible to determine $q_{ST}(\alpha)$ as the root of an equation without implementing the optimization problem. Indeed, applying the F.O.C. stated in Proposition 1, we obtain

$$\alpha = N(-d_2) + F_0 n(d_1) \sqrt{T} \frac{\partial \sigma_{sabr}}{\partial K}$$

(7)
where \( n(\cdot) \) denotes the density of the standard normal distribution, and where we note the similarity with the case of the Black–Scholes model in (4) for which the derivative of the volatility with respect to the strike is equal to 0 as it is constant volatility. See “Appendix A.1” for an expression of the derivative of \( \sigma_{sabr} \) with respect to the strike price, i.e., \( \frac{\partial \sigma_{sabr}}{\partial K} \). Solving Eq. (7) for \( K \) then yields an expression for \( q_{ST}(\alpha) \).

**Remark 2** (SVI model) A model that is closely related to the SABR model is the Stochastic Volatility Inspired (SVI) model (see Gatheral, 2004; Gatheral & Jacquier, 2014 for further details). In a similar way as in the case of the SABR model, the SVI parameterization of the implied volatility has a closed-form expression, and thus allows for closed-form expressions of calls and puts using the Black–Scholes formula. Thus, from Proposition 1, the quantile of \( S_T \) at the confidence level \( \alpha \) can also be computed for the SVI model in a similar way as what we present here for the SABR model.

### 2.3 Implied quantile in the Heston model

Another widely-known and used stochastic volatility model is the Heston (1993) model in which the instantaneous variance follows the Cox–Ingersoll–Ross (CIR) process. Let \( \kappa \) represent the rate of mean reversion, \( \theta \) be the long-run variance, \( \nu_0 \) the initial variance, \( \lambda \) the volatility of volatility, and \( \rho \) the correlation coefficient between the dynamics of the stock price and its volatility. Under the Heston model it is possible to price European options by means of the characteristic function, which is known in a closed-form formula. Using the same representation presented in Gilli et al. (2019)\(^1\) (see also references therein for a complete overview), the European call price under the Heston model thus writes as

\[
c_H(K) = S_0 \Pi_1 - e^{-rT} K \Pi_2,
\]

and the price of a European put option is calculated by means of put-call parity

\[
p_H(K) = c_H(K) - S_0 + e^{-rT} K.
\]

Here, \( S_0 \) denotes the current underlying asset price, while the \( \Pi_j, j = 1, 2 \), are defined as

\[
\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i\omega \ln(K)} \phi_H(\omega - i)}{i\omega \phi_H(-i)} \right) d\omega
\]

\[
\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i\omega \ln(K)} \phi_H(\omega)}{i\omega} \right) d\omega,
\]

in which \( i \) is the imaginary unit, \( \text{Re}(\cdot) \) gives the real part of a complex number, and \( \phi_H(\omega) \) is the Heston characteristic function of the logarithm of the underlying asset price defined by

\[
\phi_H(\omega) = e^{A(\omega) + B(\omega) + C(\omega)},
\]

\(^1\) We use the same representation due to the use of NMOF R package (see Schumann, 2021 for details). In this way we are able to show in Sect. 3 that it is possible to take advantage of libraries/packages developed by other users.
where

\[ A(\omega) = i\omega (\ln S_0 + rT) \]

\[ B(\omega) = \frac{\theta \kappa}{\lambda^2} \left( (\kappa - \rho \lambda i\omega - d) T - 2 \ln \left( \frac{1 - ge^{-dT}}{1 - g} \right) \right) \]

\[ C(\omega) = \frac{v_0 \lambda}{\lambda^2} (\kappa - \rho \lambda i\omega - d) \left( 1 - e^{-dT} \right) \frac{1}{1 - ge^{-dT}} \]

\[ d = \sqrt{(\rho \lambda i\omega - \kappa)^2 + \lambda^2 (i\omega + \omega^2)}, \quad \text{and} \quad g = \frac{\kappa - \rho \lambda i\omega - d}{\kappa - \rho \lambda i\omega + d}. \]

In particular \( \phi_H(-i) = S_0 e^{rT} \). As explained in the work of Gilli et al. (2019), \( \Pi_1 \) and \( \Pi_2 \) are calculated by numerical integration for the Heston characteristic function in order to compute the European call price.

From Proposition 1 we find under the Heston model the quantile \( q_{ST}(\alpha) \) either as

\[ q_{ST}(\alpha) = \arg\min_{K \in \mathbb{R}} \left\{ \alpha e^{rT} c_H(K) + (1 - \alpha) e^{rT} p_H(K) \right\}, \quad (8) \]

or by solving for \( K \) the following equation, which corresponds to the F.O.C. of Eq. (8),

\[ \alpha = \frac{1}{2} + \frac{S_0 e^{rT}}{\pi} \int_0^\infty \Re \left( -\frac{\phi_H(\omega - i)}{K^{1+i\omega} \phi_H(-i)} \right) \, d\omega - \frac{K}{\pi} \int_0^\infty \Re \left( \frac{\phi_H(\omega)}{K^{1+i\omega}} \right) \, d\omega - \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-i\omega \ln(K) \phi_H(\omega)}}{i\omega} \right) \, d\omega. \quad (9) \]

3 Simulation

We use Eq. (1) as the basis of our model-free simulation, and propose the following approach to simulate the underlying asset price at a given maturity date directly using the available option prices as input.

As first step we generate random numbers \( u_i, i = 1, \ldots, m \) from a standard uniformly distributed random variable \( U \), i.e., \( U \sim U(0, 1) \). Then, we compute the \( q_{ST}(u_i) \) by applying Eq. (1) (the role of \( \alpha \) is played by the \( u_i \)). The \( q_{ST}(u_i) \) are then the simulated realizations of the \( ST \) under the risk-neutral probability measure. In Algorithm 1, we provide the simulation scheme of the underlying asset price \( ST \) for a general stochastic model.

The approach, which we name Options-based Simulation (hereafter also ObSim), permits to simulate the underlying asset prices from traded options or from options prices calculated by a closed-form or semi-closed formula, and has two considerable advantages. The first advantage is the opportunity to simulate the underlying asset prices under the risk-neutral probability measure in a model-free way, and the second advantage is the possibility of having an “exact” simulation method for stochastic models. As explained in Broadie and Kaya (2006) a standard approach for simulating asset values in the case of a (path-dependent) continuous-time stochastic process is to approximate the path of the stochastic process by a discrete-time stochastic process, causing a bias in the simulation results (due to the discretization). Instead “exact” simulation methods, a relevant subject of study among the applied probability community (see Beskos & Roberts, 2005 for further details), are schemes that are free from the need of discretization procedures. Consequently, for instance, this means that the simulation of a (path-dependent) asset price \( ST \) does not need to rely on the simulation of a discretized path and thus avoids the bias issues discussed for instance in Andersen (2008) and Lord et al.
(2010) for the Heston model and in Cai et al. (2017) and Leitao et al. (2017) for the SABR model.

The numerical experiments are performed in R software with an Intel Core i7-8565U CPU @ 1.80GHz and a 16GB RAM memory.

**Algorithm 1:** (ObSim) Model-free Simulation.

- **Model selection:** Choice of the model to compute call and put prices;
- Initialization of relevant parameters;
- Create a vector $V_K$ of a wide range of discretized values for $K$;

**Function Implied Quantile ($\alpha$):**

```plaintext
for index starting from 1 to the length of $V_K$ do
    Solve the right-hand side of Eq. (1);
end
return Value of $K$ where minimum is reached;
```

**Simulation:**

```plaintext
Generation of $m$ random numbers $u_i$ from Uniform Distribution $\mathcal{U}(0, 1)$;
for index from 1 to $m$ do
    return Implied quantile ($u_i$);
end
```

### 3.1 Exact simulation in the Black–Scholes model

To verify the validity of our approach, we first provide an example in the Black–Scholes model. We simulate $S_T$ using the ObSim method associated to option prices given by the Black–Scholes formula, with a sample of random numbers $m = 100$ replicated 10,000 times and the following parameters: $r = 0$, $T = 0.5$, $\sigma = 0.2$, $S_0 = 100$. To practically solve the minimization problem (1), we use an interval of strike prices. Clearly, to ensure the minimum is obtained the interval size has to be chosen wide enough. We did not investigate this point in detail but rather experimented with testing various interval sizes. In this regard, we underline that there is a trade-off between the interval size and the execution speed: performing a simulation with a wide interval of strike prices and a large sample of random numbers can be time-consuming (e.g., the Heston model. See Sect. 3.4 for further details).

We plot the histogram of the simulated values against the theoretical density (red line in Fig. 1).

To further assess the quality of our approach we evaluate also the European call prices as $c(K) = e^{-rT} E^Q [\max (S_T - K, 0)]$ in which $S_T$ is computed by the ObSim method and we compare the outcomes against the prices obtained by the closed-form formula. Indeed if the European call prices obtained by our approach are close to the prices of closed-form expression, then the distribution of the underlying asset price $S_T$ is correct, validating, as consequence, our simulation method.² Table 1 displays the pricing results for European call options obtained via Options-based Simulation approach with a sample of random numbers $m = 10^6$, and by means of Black–Scholes formula (BS Formula). As expected, the European

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² Knowledge of all call prices is equivalent to the knowledge of the distribution of the underlying asset (see Breeden & Litzenberger, 1978).

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Fig. 1 Histogram of simulated values of $S_T$ in the Black–Scholes model versus the theoretical density (red line) with parameters: $S_0 = 100$, $T = 0.5$, $r = 0$, $\sigma = 0.2$, $m = 100$, and 10,000 replications

Table 1 Prices of European call options in the Black–Scholes model

| Strike, $K$ | BS formula Price | Options-based simulation (ObSim) Price | 95% C.I. |
|------------|------------------|----------------------------------------|---------|
| 94         | 8.9822           | 8.9765                                 | [8.9552, 8.9978] |
| 96         | 7.7526           | 7.7460                                 | [7.7259, 7.7661] |
| 98         | 6.6376           | 6.6302                                 | [6.6114, 6.6491] |
| 100        | 5.6372           | 5.6289                                 | [5.6113, 5.6465] |
| 102        | 4.7492           | 4.7404                                 | [4.7241, 4.7567] |
| 104        | 3.9693           | 3.9604                                 | [3.9455, 3.9754] |
| 106        | 3.2913           | 3.2828                                 | [3.2691, 3.2965] |

Input parameters: $S_0 = 100$, $T = 0.5$, $r = 0$, $\sigma = 0.2$, and $m = 10^6$

call prices for different levels of strike prices, $K \in [94, 106]$, and its 95% confidence interval are close to the Black–Scholes prices.

3.2 Exact simulation in the SABR model

We now simulate the underlying asset price $S_T$ via the ObSim approach associated to the option pricing formula under the SABR model (see Sect. 2.2) with a sample of random numbers $m = 10^6$ and the same parameters for the “Equity derivatives market” case presented in Cai et al. (2017, Section 4.1, p. 940) and Cui et al. (2021, Section 5.2, p. 1058), that is: $v_0 = 0.3$, $\beta = 0.6$, $\rho = 0$, $\lambda = 0.2$, $F_0 = 100$, $r = 0$, and $T = 1$.

Table 2 compares the pricing results for European call options for several strike prices, $K \in [94, 106]$, between the formula provided in Hagan et al. (2002), i.e., Hagan formula, and the ObSim approach. The results obtained from our simulation method are close to the results of Hagan formula.

Remark 3 The formulas used for the call and put prices in the SABR model (see Sect. 2.2) are approximations and it turns out that the SABR model built on these prices is not always arbitrage-free. As a consequence, given the prices of calls and puts, there may not exist a
Table 2. Prices of European call options in the SABR model. Input parameters: \( v_0 = 0.3, \beta = 0.6, \rho = 0, \lambda = 0.2, F_0 = 100, r = 0, T = 1, \) and \( m = 10^6 \).

| Strike, \( K \) | Hagan formula Price | Options-based simulation (ObSim) Price | 95% C.I.                   |
|------------------|----------------------|----------------------------------|--------------------------|
| 94               | 6.23083              | 6.22408                          | [6.21546, 6.23270]       |
| 96               | 4.52561              | 4.51997                          | [4.51208, 4.52787]       |
| 98               | 3.05943              | 3.05543                          | [3.04857, 3.06229]       |
| 100              | 1.90301              | 1.90055                          | [1.89494, 1.90616]       |
| 102              | 1.08033              | 1.07911                          | [1.07482, 1.08339]       |
| 104              | 0.55771              | 0.55713                          | [0.55406, 0.56020]       |
| 106              | 0.26208              | 0.26146                          | [0.25939, 0.26353]       |

distribution for \( S_T \) that is consistent with all of them. In this case, the quantile function does not exist and the results exposed in Proposition 1 cannot be applied (e.g., the F.O.C. may not have a unique root). The arbitrage in the SABR model comes from very low level of strike prices (see Hagan et al., 2014 for further details). For example, in the simulation performed in the example in Table 2, call prices are convex on the interval of strike prices \( 10^{-3} \leq K \leq 10^3 \). Hence, in this example, it turns out that a distribution function can be found for almost the entire domain, which allows us to construct a histogram of simulated values for \( S_T \) based on our approach.

An alternative solution is to switch from the Black option pricing formula using the SABR model to the so-called Shifted SABR model (see “Appendix A.2” and Hagan et al., 2014 for more details), which uses the Bachelier model to price options. Using the latter combination the convexity of call and put prices is preserved for low strike prices leading thus to arbitrage-free prices and consequently a quantile function can always be determined.

We provide a further example in which we simulate the underlying asset prices at maturity, i.e., \( S_T \), using the ObSim approach associated to the option pricing formulas mentioned above with a sample of random numbers \( m = 100 \) replicated 10,000 times, and using the same parameters for the “normal SABR model” case presented in Leitao et al. (2017, Section 5, p. 475), except for \( F_0 = 100 \), which are: \( v_0 = 0.1, \beta = 0, \rho = -0.8, \lambda = 0.4, r = 0, \) and \( T = 0.5 \). In Fig. 2, we display the densities of simulated \( S_T \) obtained with the ObSim method associated to the Black option pricing formula under the SABR model (solid red line) and the Bachelier option pricing formula (hereafter Bachelier formula) under the Shifted SABR model (dotted black line). Figure 2 shows that both densities of simulated \( S_T \) are identical, highlighting that both approaches work in a range of strike prices not close to zero.

Table 3 displays the prices of European call options for different strike prices, \( K \in [99.7, 100.1] \), obtained with the ObSim approach associated to the Bachelier option pricing formula under the Shifted SABR model with a sample of \( m = 10^6 \) random numbers and by means of the Hagan formula and the Bachelier formula.

### 3.3 Exact simulation in the Heston model

We simulate the underlying asset prices at maturity \( S_T \) using the ObSim method associated to the option pricing formula under the Heston model described in Sect. 2.3 with a sample of random numbers \( m = 100 \) replicated 10,000 times. We opt for the same parameters
Fig. 2 Densities of simulated values of $S_T$ obtained with the ObSim approach associated to the SABR model (solid red line) and to the Shifted SABR model (dotted black line), respectively, with parameters $v_0 = 0.1$, $\beta = 0$, $\rho = -0.8$, $\lambda = 0.4$, $F_0 = 100$, $r = 0$, $T = 0.5$, $m = 100$, and 10,000 replications.

Table 3 Prices of European call options in the SABR model and the Shifted SABR model

| Strike, K | Hagan formula Price | Bachelier formula Price | ObSim with Bachelier formula Price | 95% C.I.       |
|-----------|---------------------|------------------------|----------------------------------|----------------|
| 99.7      | 0.30006             | 0.30006                | 0.29993                          | [0.29979, 0.30007]          |
| 99.8      | 0.20052             | 0.20052                | 0.20040                          | [0.20026, 0.20054]          |
| 99.9      | 0.10442             | 0.10442                | 0.10432                          | [0.10419, 0.10444]          |
| 100.0     | 0.02822             | 0.02822                | 0.02818                          | [0.02811, 0.02825]          |
| 100.1     | 0.00117             | 0.00117                | 0.00116                          | [0.00115, 0.00117]          |

Input parameters: $v_0 = 0.1$, $\beta = 0$, $\rho = -0.8$, $\lambda = 0.4$, $F_0 = 100$, $r = 0$, $T = 0.5$, and $m = 10^6$.

used in Albrecher et al. (2007, Section 4, p. 11) where the Feller condition ($2\kappa\theta > \lambda^2$) is satisfied: $S_0 = 100$, $v_0 = 0.04$, $\kappa = 1.5$, $\theta = 0.04$, $\lambda = 0.3$, $\rho = -0.9$, $T = 1$, and $r = 0.025$. The prices of European call and put options with the characteristic function under the Heston model (hereafter Heston formula) are calculated thanks to the NMOF R package (see Schumann, 2021).

Figure 3 shows the histogram of simulated values for $S_T$ and Table 4 exhibits the pricing results for European call options for different strikes, $K \in [94, 106]$, obtained by means of the Heston formula and through Options-based Simulation approach with the same inputs used for the simulation except for $m = 10^6$. The European call prices achieved through the ObSim approach are close to the prices of the Heston formula.

3.4 Fast simulation scheme

The simulation scheme proposed in Algorithm 1 (ObSim), which uses Eq. (1), other than being the only possible solution in case of traded options on financial markets, is intuitive to understand and immediate to implement. In case of call and put prices assessed by means of closed-form formulas, from a computational point of view, simulating the asset values using Eq. (1) requires in principle little computational effort, as efficient libraries/packages...
Fig. 3 Histogram of simulated values of $S_T$ in the Heston model with parameters: $S_0 = 100$, $v_0 = 0.04$, $\kappa = 1.5$, $\theta = 0.04$, $\lambda = 0.3$, $\rho = -0.9$, $T = 1$, $r = 0.025$, $m = 100$ and 10,000 replications

Table 4 Prices of European call options in the Heston model

| Strike, $K$ | Heston formula Price | Options-based simulation (ObSim) Price | 95% C.I. |
|------------|----------------------|--------------------------------------|---------|
| 94         | 12.69837             | 12.68338                             | [12.65946, 12.70729] |
| 96         | 11.36463             | 11.35106                             | [11.32838, 11.37375] |
| 98         | 10.09554             | 10.08320                             | [10.06178, 10.10461] |
| 100        | 8.89487              | 8.88403                              | [8.86394, 8.90413] |
| 102        | 7.76626              | 7.75695                              | [7.73821, 7.77570] |
| 104        | 6.71309              | 6.70543                              | [6.68807, 6.72280] |
| 106        | 5.73834              | 5.73221                              | [5.71624, 5.74817] |

Input parameters: $S_0 = 100$, $v_0 = 0.04$, $\kappa = 1.5$, $\theta = 0.04$, $\lambda = 0.3$, $\rho = -0.9$, $T = 1$, $r = 0.025$, and $m = 10^6$

which compute the option prices and perform the optimization are broadly available. Despite this advantage, the proposed simulation scheme could still suffer from slow execution speed. Indeed, performing the simulation in the Heston model using Eq. (1) requires an important computational effort because for all strikes call option prices need to be computed which unfortunately is a time-consuming process under the Heston model (see Table 5).

Therefore, we also provide an alternative for ObSim, which we label as Fast ObSim. This approach can be summarized as follows. To compute the quantiles of the underlying terminal asset value at any confidence level $\alpha$, as a first step (S1 in Table 5), we compute once the right-hand side of (2) for a very large number of possible strike prices $K$ and tabulate all the values. Then, as second step (S2 in Table 5), one needs to generate random numbers $u_i$, $i = 1, \ldots, m$ from a standard uniform distribution and to look up for each $u_i$ (playing the role of $\alpha$ in (2)) the corresponding $K$-value in the table (simple linear interpolation is used here as well) to obtain the simulated realizations of the $S_T$ under the risk-neutral probability measure. Note that the tabulation of all values obtained by means of Eq. (2) is a crucial step to obtain a very fast simulation method.

To show the difference between ObSim and Fast ObSim in terms of execution speed, Table 5 exhibits the results in terms of computation time for the Fast ObSim and the ObSim
Table 5: Execution speed in seconds for the Fast ObSim and the ObSim approaches in Black–Scholes, SABR and Heston models

| Unit: seconds | Fast ObSim [Eq. (2)] | ObSim [Eq. (1)] |
|--------------|----------------------|----------------|
|              | S1       | S2       | Total (as S1 + S2) |
| BS model     | 0.002    | 6.058   | 6.060             | 52.65  |
| SABR model   | 2.280    | 6.131   | 8.411             | 130.51 |
| Heston model | 35.534   | 6.157   | 41.691            | 282.13 |

For the Fast ObSim method, we provide the time needed to run the first step (S1), the second step (S2), and the total (S1 + S2). The step size for S1 is set equal to 0.01 with a number of discrete points equal to 17,501 and a sample of $m = 10^5$ random numbers was used.

approaches in Black–Scholes, SABR, and Heston models. The parameters for the Black–Scholes and Heston models are the same specified respectively in Sects. 3.1 and 3.3, while the parameters for SABR model corresponds to those mentioned in the first part of Sect. 3.2.

The results illustrate that the Fast ObSim method is a faster simulation scheme than the ObSim method. In addition, it is important to highlight that increasing the execution speed clearly depends on the chosen level of discretization, which leads to a trade-off between the execution speed, the accuracy of the simulation as well as the degree of bias.

4 Implied risk measures

From the literature, we know that option prices contain useful information about the distribution of future returns (e.g., Breeden & Litzenberger, 1978), also called forward-looking information. Financial practitioners and investors are very interested in having such information as can be seen from the success of VIX and SKEW indices in the market (see Barone-Adesi, 2016).

VaR and TVaR are broadly used in risk management (e.g., for setting capital requirements) and computed under the real-world probability measure. It is of interest to study them under the risk-neutral probability measure and to potentially use them as risk indicators. VIX is seen as a forward looking volatility, while the implied VaR, i.e., the VaR under the risk-neutral probability measure, may carry some forward-looking information about the real-world VaR. In this section we derive implied VaR and implied TVaR for a risky asset by using the option prices that are available for this asset.

We use the convention that we compute the risk of an asset based on its loss distribution. Specifically, let $V_0$ be the initial amount of wealth invested in a long position of a financial risky asset. The value of wealth invested at time $T$ is $V_T = V_0 \left( \frac{S_T}{S_0} - 1 \right)$ where $S_T$ and $S_0$ are respectively the price at time $T$ and the initial price of the risky asset at the time of investment. We then define the loss as $L = -V_T / V_0$. We obtain that $L = 1 - (S_T / S_0)$ and note that it has a continuous distribution function (as we assumed that $S_T$ is continuously distributed).

**Remark 4** While we focus on the estimation of VaR and TVaR of $L$ based on option information, it is also possible to estimate general distortion risk measures. Indeed, under some technical assumption, a distortion risk measure can be expressed as a weighted sum of quan-

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3 It is possible to obtain a faster simulation (0.008 seconds for 100,000 simulations) with respect to the Fast ObSim and the ObSim methods using the closed-form expression for the quantile [see Eq. (5)].
tiles. Thus if an estimate of the VaRs can be determined, then it is possible to estimate any distortion risk measure. For a formal definition of a distortion risk measure and further details, we refer to e.g. Dhaene et al. (2012) and references therein.

**Definition 4.1** (Value-at-risk (VaR)) Using the same notation as in McNeil et al. (2015), VaR at the confidence level \( \alpha \in (0,1) \) for the random loss \( L \) is defined as:

\[
\text{VaR}_\alpha(L) = \inf \{ \ell \in \mathbb{R} : \mathbb{Q}(L \leq \ell) \geq \alpha \},
\]

where \( \mathbb{Q}(L \leq \ell) = F_L(\ell) \) and \( F_L \) is the cumulative distribution function of \( L \) under the probability \( \mathbb{Q} \).

In order to compute the implied VaR, we relate the loss \( L \) to the underlying asset price \( S_T \), as options are written on \( S_T \).

**Proposition 2** (Implied VaR) Provided Proposition 1 and Definition 4.1, the implied VaR at the confidence level \( \alpha \in (0,1) \) for \( L \) is given by the following relationship:

\[
\text{VaR}_\alpha(L) := q_{L}(\alpha) = 1 - \frac{q_{S_T}(1 - \alpha)}{S_0}
\]

where \( q_{S_T}(1 - \alpha) \) is computed from Proposition 1.

**Proof** Starting from the definition \( \mathbb{Q}(L \leq q_{L}(\alpha)) = \alpha \), we can substitute \( L \) with \( 1 - (S_T/S_0) \) and rearrange the expression to obtain (11).

The VaR has mainly two important drawbacks: the impossibility to provide information about the severity of losses and the lack of subadditivity. To address such shortcomings, financial practitioners and academics often use TVaR, which is formally defined as follows.

**Definition 4.2** (Tail value-at-risk (TVaR)) The TVaR at the confidence level \( \alpha \in (0,1) \) for the random loss \( L \) is defined as

\[
\text{TVaR}_\alpha(L) = \mathbb{E}[L|L \geq \text{VaR}_\alpha(L)].
\]

Similar to Proposition 2 for the implied VaR, we show how to compute the implied TVaR using pricing information of options written on \( S_T \).

**Proposition 3** (Implied TVaR) The implied TVaR at the confidence level \( \alpha \in (0,1) \) for \( L \) can be computed as:

\[
\text{TVaR}_\alpha(L) = 1 - \frac{1}{S_0} \left( q_{S_T}(1 - \alpha) - \frac{1}{1 - \alpha} e^{rT} p \left( q_{S_T}(1 - \alpha) \right) \right)
\]

or equivalently using the relationship (11),

\[
\text{TVaR}_\alpha(L) = \text{VaR}_\alpha(L) + \frac{1}{1 - \alpha} e^{rT} p \left( q_{S_T}(1 - \alpha) \right) \frac{1}{S_0},
\]

where \( p \left( q_{S_T}(1 - \alpha) \right) \) is the European put price in which now the quantile at confidence level \( 1 - \alpha \) of \( S_T \), i.e., \( q_{S_T}(1 - \alpha) \), reflects the strike price.

Note that Eqs. (13) and (14) are already present in some way in the literature, here we specifically relate prices of options that are written on \( S_T \) to the VaR and TVaR of the random loss \( L = 1 - (S_T/S_0) \).
Proof Starting from Definition 4.2 of TVaR for a loss distribution using option prices we have that:

\[ \text{TVaR}_\alpha(L) = \mathbb{E}[L | L \geq \text{VaR}_\alpha(L)], \]

and substituting \( L \) with \( 1 - (S_T / S_0) \) and \( \text{VaR}_\alpha(L) \) with Eq. (11) we have

\[ \text{TVaR}_\alpha(L) = 1 - \frac{1}{S_0} \mathbb{E}
\left[ S_T | S_T \leq q_{S_T} (1 - \alpha) \right]. \]

Then:

\[ \text{TVaR}_\alpha(L) = 1 - \frac{1}{S_0} \left( \frac{\mathbb{E}
\left[ S_T 1\{S_T \leq q_{S_T} (1 - \alpha)\} \right]}{Q(\alpha)} \right) \]

\[ = 1 - \frac{1}{S_0} \left( \frac{\int_{-\infty}^{q_{S_T} (1 - \alpha)} x f(x)dx}{1 - \alpha} \right) \]

\[ = 1 - \frac{1}{S_0} \left( \frac{\int_{-\infty}^{q_{S_T} (1 - \alpha)} (q_{S_T} (1 - \alpha) - x) f(x)dx}{1 - \alpha} + q_{S_T} (1 - \alpha) \right), \tag{15} \]

where \( f \) is the density of \( S_T \) under \( Q \).

Note that \( \int_{-\infty}^{q_{S_T} (1 - \alpha)} (q_{S_T} (1 - \alpha) - x) f(x)dx \) can be seen as the compounded price of a put option written on \( S_T \) with strike price \( q_{S_T} (1 - \alpha) \). Recall indeed that the price of a European put option with strike \( K \) is

\[ p(K) = e^{-rT} \mathbb{E}^Q \left[ (K - S_T)_+ \right] = e^{-rT} \int_{-\infty}^{K} (K - S_T) f(S_T)dS_T, \]

and consequently, \( e^{rT} p(K) = \int_{-\infty}^{K} (K - S_T) f(S_T)dS_T \). Then from Eq. (15) we obtain (14) in Proposition 3. \( \square \)

4.1 Empirical implied VaR using S&P 500 Index options

In this subsection, we present a small empirical case study in which we use options on the S&P 500 Index (hereafter also SPX) to construct an implied VaR Index and we briefly compare it with the well-known CBOE Volatility Index (also known as VIX Index), i.e., an index of implied volatility based on 30-day SPX options [for further references regarding the comparison between option-implied quantile and option-implied expectile curves with the VIX Index, see Bellini et al. (2018, 2021) as well as some of the references therein].

The call and put options on the S&P 500 Index are selected in order to have a 30-day expiration and consequently to design a 30-day implied VaR. More precisely, in a similar way as done by the Chicago Board Options Exchange (CBOE) to construct the VIX Index, we only consider SPX options having a maturity date that is more than 23 days but less than 37 days away and that expire on Friday. In such a way we include options on S&P 500 Index with standard expiration dates, i.e., 3rd Friday of the month, and weekly expiration dates.

The data set, provided by Chicago Board Options Exchange, is composed of 53,460 options and daily closing levels of VIX Index and covers the period 29/11/2019–15/04/2020. The implied VaR is computed for each quote date using Proposition 2 and Fig. 4 exhibits the behaviour of VIX Index (red line) and of implied VaR at two different confidence levels \( a = 0.90 \) (green line) and \( a = 0.99 \) (blue line).
Fig. 4 Time series of VIX Index (red line) and implied VaR Index at confidence levels $\alpha = 0.90$ (green line) and $\alpha = 0.99$ (blue line) for the period 29/11/2019–15/04/2020. The left axis refers to level of the VIX Index (at closing) and the right axis indicates the level of the implied VaR we compute. The shadow area displays the period of the first major lockdown during the COVID-19 pandemic. (Color figure online)

From Fig. 4 it is possible to observe a strong initial increase of VIX Index, implied VaR at 90% and implied VaR at 99% at the end of February and with a more acute phase at the beginning of the shadow area, which corresponds to the start of the first lockdown (dated 09/03/2020). It appears that all curves move in parallel meaning that the implied VaR Index we construct provides an alternative to the VIX Index for measuring market fear. A more fine-grained study is however needed to assess the advantages and disadvantages of either approaches.

5 Conclusion

In this paper, we propose a new model-free approach for extracting the risk-neutral quantile function of an asset directly from option prices written on this asset. We develop two applications: the simulation of the underlying terminal asset value for a given stochastic asset model and the direct estimation of implied VaR and implied TVaR using options prices.

As for the latter application, the novel approach makes it possible to directly compute the implied VaR and implied TVaR directly from the knowledge of option prices, without relying on the estimation of the probability distribution function under the risk-neutral probability measure as a first step and then to invert it in a second step.

For the former application, the approach allows to have an “exact” simulation for those stochastic asset models in which a closed-form or semi-closed formula for the option prices are known. As a consequence, our approach permits to perform, for instance, the simulation of the underlying asset process at maturity in stochastic volatility models such as the Heston and the SABR models for which the simulation is notoriously difficult and possibly biased. The results obtained in Sect. 3 corroborate the performance of our approach for the simulation.
We expect that our approach will be useful in practice, as it makes it possible to simulate the underlying terminal asset value under the risk-neutral probability measure in a model-free way directly from observed prices of a large number of options traded in financial markets. Moreover, an empirical illustration shows that our results make it possible to construct an implied V aR Index as an alternative to the well-known VIX Index. A detailed comparison between both indices and assessing the extent by which implied V aR (derived under the risk neutral probability measure) yields information about the real-world V aR is left for further study.

Appendix A

A.1 Derivative of the implied volatility with respect to the strike price in the SABR model

The derivative of the implied volatility with respect to the strike price using the SABR model is given by the following equation

\[
\frac{\partial \sigma_{sabr}}{\partial K} = \frac{1}{2\pi} \left( \beta - 1 \right) v_0 F_0 \lambda T \left[ v_0 \left( \beta - 1 \right)^2 \left( F_0 K \right)^{\beta - 2} + 3 \beta \rho \lambda \left( F_0 K \right)^{\beta - 3/2} \right]
\]

\[
- \frac{\eta \chi (z)}{\psi \left( \frac{1}{640K} \right) - \frac{\left( \beta - 1 \right)^2}{24K} + \frac{1}{K \ln^2 \left( \frac{F_0}{K} \right)} \chi (z) + \eta \delta (z) \right] \left( \eta \chi (z) \right)^2,
\]

where

\[
z = \frac{\lambda}{v_0} \left( F_0 K \right)^{(1 - \beta)/2} \ln \left( \frac{F_0}{K} \right),
\]

\[
\delta (z) = \frac{-F_0 \lambda \left( \beta - 1 \right) \ln \left( \frac{F_0}{K} \right) + 2 \xi (z)}{2v_0^2 \sqrt{1 - 2 \rho z + z^2} \left( \sqrt{1 - 2 \rho z + z^2} + z - \rho \right)},
\]

\[
\xi (z) = \left[ v_0 \left( \sqrt{1 - 2 \rho z + z^2} - \rho \right) \left( F_0 K \right)^{(-\beta - 1)/2} + \lambda \left( F_0 K \right)^{-\beta} \ln \left( \frac{F_0}{K} \right) \right],
\]

\[
\chi (z) = \ln \left( \frac{\sqrt{1 - 2 \rho z + z^2} + z - \rho}{1 - \rho} \right),
\]

\[
\psi = \frac{1}{24} \lambda \left( v_0^2 \left( \beta - 1 \right)^2 T \left( F_0 K \right)^{\beta - 1} \left( F_0 K \right)^{\beta - 1} \right)
\]

\[
+ \frac{1}{24} \lambda \left( 6v_0 \beta \rho T \lambda \left( F_0 K \right)^{\beta - 1/2} - 3 \rho^2 T \lambda^2 + 2 \left( T \lambda^2 + 12 \right) \right),
\]

\[
\eta = \frac{1}{\ln \left( \frac{F_0}{K} \right)} + \frac{\left( 1 - \beta \right)^2}{24} \ln \left( \frac{F_0}{K} \right) + \frac{\left( 1 - \beta \right)^4}{1920} \ln^3 \left( \frac{F_0}{K} \right).
\]

Note that Eq. (A1) is defined when \( K \neq F_0 \). When \( K = F_0 \), we adopt the following solution in order to differentiate at this point:

\[
\frac{\partial}{\partial K} (\sigma_{sabr} (K)) = \lim_{\varepsilon \to 0} \frac{\sigma_{sabr} (K + \varepsilon) - \sigma_{sabr} (K - \varepsilon)}{2\varepsilon}.
\]
A.2 The Bachelier option pricing formula under the shifted SABR model

The prices of European call and put options using the Bachelier model are

$$c_{BA C}(K) = e^{-rT} \left[ (F_0 - K) N \left( \frac{F_0 - K}{\sigma_{BA C} \sqrt{T}} \right) + n \left( \frac{K - F_0}{\sigma_{BA C} \sqrt{T}} \right) \sigma_{BA C} \sqrt{T} \right],$$

$$p_{BA C}(K) = e^{-rT} \left[ (K - F_0) N \left( \frac{K - F_0}{\sigma_{BA C} \sqrt{T}} \right) + n \left( \frac{F_0 - K}{\sigma_{BA C} \sqrt{T}} \right) \sigma_{BA C} \sqrt{T} \right],$$

where $N(\cdot)$ and $n(\cdot)$ denote the cumulative standard normal distribution and the density of the standard normal distribution, respectively. As in Hagan et al. (2014) the implied Bachelier volatility (also called implied normal volatility) $\sigma_{BA C}$ under the Shifted SABR model when $\beta = 0$, in which we neglect higher terms in the Taylor expansion, is

$$\sigma_{BA C} = v_0 \left( \frac{\zeta}{\chi(\zeta)} \right) \left\{ 1 + \frac{1}{24} \left( 2 - 3 \rho^2 \right) \lambda^2 \right\} T,$$

where

$$\zeta = \frac{\lambda}{v_0} (F_0 - K), \quad \text{and} \quad \chi(\zeta) = \ln \left( \frac{\sqrt{1 - 2\rho \zeta + \zeta^2} + \zeta - \rho}{1 - \rho} \right).$$

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