Mathematical model of the vibratory mill body movement with biharmonic oscillations

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Abstract. Vibratory mills are widely used in various industries such as construction, mining, chemical, energy ones in order to obtain powders with the size of a particle less than 5 microns. Vibratory mills with harmonic oscillations of a cylindrical body are most widely used in world practice. The disadvantage of such mills is the presence of stagnant zones in the central part of the load which reach 40% of the total mass of milling bodies. This significantly reduces the efficiency of the grinding process. In recent years, vibratory mills with two drives have been used. The body of these mills makes biharmonic vibrations; as a result, the volume of stagnant zones decreases to 15–20%. We have proposed a new design of a vibratory mill equipped with two imbalanced vibration drives, the axes of which are shifted in different directions from the horizontal axis of the mill, passing through the center of mass of the oscillating system. This article discusses the method that allows one to calculate the trajectory of the mill body in polar coordinates. This method is contemplated for the first time. The resulting equation of the motion trajectory of the vibromill body takes into account: the geometrical parameters of the cylindrical body; body weight and load; weight of imbalances; eccentricities of imbalances; imbalance angular velocity; the forces created by imbalances; imbalance offset angles; spring stiffness and damping coefficient.

1. Introduction
At the moment, vibratory mills with a cylindrical body filled with 80–90% of milling bodies, which is installed on elastic supports, are most often used in various industries. The mill drive consists of an imbalance vibrator mounted on the horizontal axis of the mill body, or on its periphery. In this case, the mill body performs harmonic oscillations. One of the significant drawbacks of such construction of vibratory mills is that in the central part in the cross-section of the load, stagnant zones are formed, reaching up to 40% of the total mass of grinding bodies. In addition, there is a segregation of milling bodies and the material to be ground. Large milling bodies and small particles of material are moved to the upper part of the load, and small milling bodies and large particles of material are moved to the lower part of the load. All this significantly reduces the efficiency of the grinding process in a vibromill [1, 2].

In recent years, vibratory mills with biharmonic oscillations, equipped with two imbalanced vibratory drives, have been used [3–5]. A significant difference of such designs of vibromills is that the vibromill body moves along trajectories in the form of a triangle, square, or circular, depending on the mass of the imbalance, frequency and phase of their movement. In such mills, the volume of stagnant zones...
decreases, but the grinding bodies move only in the cross section of the cylindrical drum of the mill [5, 6].

We have developed a vibromill design which design scheme is shown in figure 1. In the proposed design of the vibratory mill, the imbalances are shifted at some distance from the center of mass of the oscillating mass (mill body and load). Such a constructive solution provides an additional occurrence of the rotational moment which value depends on the magnitude of the external imbalance offset from the axis passing through the center of mass of the oscillating system [7].

![Design scheme for description of the oscillations of the vibratory mill body.](image)

**Figure 1.** Design scheme for description of the oscillations of the vibratory mill body.

2. Results
At present, there are no mathematical models to calculate the parameters of forced vibrations of the vibratory mill body [5–10].

For descriptions of vibrations of a cylindrical body of a vibratory mill, created by two imbalances relative to the axis of symmetry of the cylindrical body, we introduce a polar coordinate system $\rho, \varphi$ in the plane perpendicular to the axis of the cylinder.

Due to axial symmetry, the deviation of the cross-sectional plane from the equilibrium position will depend only on the coordinate $\rho$.

The kinetic energy of the oscillating body is:

$$T = \frac{M\dot{\rho}^2}{2}, \quad (1)$$

where $M$ is the total mass of the mill body together with the load (grinding bodies and the material to be ground).

The potential energy of an oscillating body is:

$$U = \frac{K_0 \rho^2}{2}, \quad (2)$$

where $K_0$ is the stiffness coefficient of the elastic elements.

Based on (1) and (2) we compose the Lagrange function:

$$L = T - U = \frac{M\ddot{\rho}^2}{2} - \frac{K_0 \rho^2}{2} \quad (3)$$

From now on, the dot above the letter means taking the derivative with respect to time “$t$”.

The Lagrange function (3) must satisfy the following equation:
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = f_1 + f_2 \cdot \cos(\varphi_1 - \varphi_2) + f_3
\] (4)

where \( f_1 \) and \( f_2 \) are the forces that affect the mill body in the radial direction, created by vibroexciters, separated at a distance of “a” and “b”, respectively, from the horizontal plane (\( a > b \)) (figure 1), \( f_3 \) is the force of resistance to movement of the body; \( \varphi_1 \) and \( \varphi_2 \) are the angles formed by vibration exciters with a horizontal plane, their values are respectively equal to:

\[
\varphi_1 = \arcsin \frac{a}{R_0},
\]

(5)

\[
\varphi_2 = \arcsin \frac{b}{R_0},
\]

(6)

where \( R_0 \) is the internal radius of the drum of the vibromill’s cylindrical body. Considering that:

\[
f_1 = m_1 r_1 \omega_1^2 \sin \omega_1 t,
\]

(7)

\[
f_2 = m_2 r_2 \omega_2^2 \sin \omega_2 t,
\]

(8)

\[
f_3 = -\alpha \rho,
\]

(9)

where \( m_1, m_2 \) are the total unbalanced masses of the rotating parts of the first and second vibration exciters; \( r_1, r_2 \) – the eccentricities of unbalanced rotating masses \( m_1, m_2 \), respectively; \( \omega_1, \omega_2 \) – the angular velocity (frequency) of rotation of the first and second vibration exciters; \( \alpha \) is the damping coefficient.

Substituting (3) into (4), taking into account (7) - (9), leads to the following result:

\[
M \ddot{\rho} + \alpha \dot{\rho} + K_0 \rho = m_1 r_1 \omega_1^2 \sin \omega_1 t + m_2 r_2 \omega_2^2 \sin \omega_2 t \cos(\varphi_1 - \varphi_2).
\]

(10)

In order to simplify the equation (10), we introduce the following notation:

\[
\lambda = \frac{\alpha}{M'}
\]

(11)

\[
\omega_0^2 = \frac{K_0}{M'}
\]

(12)

\[
f_{01} = \frac{m_1 r_1 \omega_1^2}{M}
\]

(13)

\[
f_{02} = \frac{m_2 r_2 \omega_2^2 \cos(\varphi_1 - \varphi_2)}{M}
\]

(14)

Taking into account (11) - (14), equation (10) will take the following form:

\[
\ddot{\rho} + \lambda \dot{\rho} + \omega_0^2 \rho = f_{01} \sin \omega_1 t + f_{02} \sin \omega_2 t.
\]

(15)

Based on the principle of superposition, we look for the solution of equation (15) in the form:

\[
\rho(t) = y_1(t) + y_2(t).
\]

(16)

where the desired functions \( y_1(t) \) and \( y_2(t) \) are the solution of the following equations:

\[
\ddot{y}_1 + \lambda \dot{y}_1 + \omega_0^2 y_1 = f_{01} \sin \omega_1 t,
\]

(17)

\[
\ddot{y}_2 + \lambda \dot{y}_2 + \omega_0^2 y_2 = f_{02} \sin \omega_2 t.
\]

(18)

Since equations (17), (18) are of the same type, that is why we are looking for a solution of the following equation:

\[
\ddot{y}_n + \lambda \dot{y}_n + \omega_0^2 y_n = f_{0n} \sin \omega_n t, \quad n = 1, 2.
\]

(19)

From a mathematical point of view, equation (19) is a linear heterogeneous second-kind equation, the solution has the form:

\[
y_n = y_0 + y_r,
\]

(20)
where $y_0$ is a general solution of a homogeneous equation:

$$
\ddot{y}_0 + \lambda \dot{y}_0 + \omega_0^2 y_0 = 0, \quad (21)
$$

$y_r$ – a particular solution of the heterogeneous equation (19) which is sought in the following form:

$$
y_r = A_n \cos \omega_0 t + B_n \sin \omega_0 t. \quad (22)
$$

Based on (22), we calculate:

$$
\ddot{y}_r = -A_n \omega_n \sin \omega_n t + B_n \omega_n \cos \omega_n t, \quad (23)
$$

$$
\ddot{y}_r = -A_n \omega_n^2 \cos \omega_n t - B_n \omega_n^2 \sin \omega_n t. \quad (24)
$$

Substituting (22) - (24) into equation (19) leads to the following equation:

$$
-A_n \omega_n^2 \cos \omega_n t - B_n \omega_n^2 \sin \omega_n t - \lambda A_n \omega_n \sin \omega_n t +
+ \lambda B_n \omega_n \cos \omega_n t + \omega_0^2 A_n \cos \omega_n t + \omega_0^2 B_n \sin \omega_n t
= f_0 \sin \omega_n t.
$$

Based on the expression (25), we can obtain the following system of algebraic equations for the indeterminates $A_n$ and $B_n$:

$$
\begin{cases}
A_n = (\omega_0^2 - \omega_n^2) + \omega_n \lambda B_n = 0, \\
-\lambda \omega_n A_n + (\omega_0^2 - \omega_n^2) B_n = f_0 \sin \omega_n t.
\end{cases} \quad (26)
$$

To find the solution of the system (26), it is necessary to calculate three second-order determinants:

$$
\Delta = \begin{vmatrix}
\omega_0^2 - \omega_n^2 & \lambda \omega_n \\
-\lambda \omega_n & \omega_0^2 - \omega_n^2
\end{vmatrix} = (\omega_0^2 - \omega_n^2)^2 + \lambda^2 \omega_n^2, \quad (27)
$$

$$
\Delta_1 = \begin{vmatrix}
0 & \lambda \omega_n \\
\omega_0^2 - \omega_n^2 & 0
\end{vmatrix} = -\lambda \omega_n f_0, \quad (28)
$$

$$
\Delta_2 = \begin{vmatrix}
\omega_0^2 - \omega_n^2 & 0 \\
-\lambda \omega_n & f_0
\end{vmatrix} = f_0 (\omega_0^2 - \omega_n^2). \quad (29)
$$

Based on the formulas (27) - (29), we find:

$$
A_n = \frac{\Delta_1}{\Delta} = -\frac{\lambda \omega_n f_0}{(\omega_0^2 - \omega_n^2)^2 + \lambda^2 \omega_n^2}, \quad (30)
$$

$$
B_n = \frac{\Delta_2}{\Delta} = -\frac{f_0 (\omega_0^2 - \omega_n^2)}{(\omega_0^2 - \omega_n^2)^2 + \lambda^2 \omega_n^2}. \quad (31)
$$

Using the same mathematical transformations, we transform (22):

$$
y_r = \sqrt{A_n^2 + B_n^2} \left( \frac{A_n}{\sqrt{A_n^2 + B_n^2}} \cos \omega_0 t + \frac{B_n \sin \omega_n t}{\sqrt{A_n^2 + B_n^2}} \right) =
$$

$$
= c_n \left( \sin \psi_n \cos \omega_0 t + \cos \psi_n \sin \omega_n t \right) = c_n \sin (\omega_0 t + \psi_n), \quad (32)
$$

where: $c_n = \sqrt{A_n^2 + B_n^2}$, $\psi_n = \arctg \frac{A_n}{B_n}. \quad (33)$

To find the general solution of the homogeneous equation (21), it is necessary to make the characteristic equation:

$$
x^2 + \lambda x + \omega_0^2 = 0. \quad (35)
$$

The characteristic equation (35) has two different solutions:
\[ x_{1,2} = \frac{-\lambda}{2} \pm \frac{\sqrt{\lambda^2 - \omega_0^2}}{2}. \]  

(36)

Due to the fact that the stiffness coefficient \( K_0 \) is greater than the damping coefficient \( \lambda \), therefore, (36) can be represented as:

\[ x_{1,2} = \frac{-\lambda}{2} \pm \frac{\sqrt{4\omega_0^2 - \lambda^2}}{2}. \]  

(37)

According to (37), the characteristic equation (35) has two complex conjugate solutions (37); on that basis the general solution of the differential equation (21) has the form:

\[ y_0 = c \left( c_{01} \cos t + c_{02} \sin t \right), \]  

(38)

where \( c = \frac{\sqrt{4\omega_0^2 - \lambda^2}}{2} \).

According to (38) with \( \frac{\lambda t}{2} \gg 1 \) or \( t \gg \frac{2}{\lambda} \).

(39)

Obviously, condition (39) is satisfied at steady state of motion of the cylindrical body of the vibromill. Therefore, the functional dependence (38) with arbitrary values of \( c_{01} \) and \( c_{02} \) will tend to its limit value equal to zero \((y_0 \rightarrow 0)\).

Consequently, for periods of time satisfying (39), the body of the vibratory mill will perform forced oscillations defined by the relation:

\[ \rho(t) = c_1 \sin(\omega_1 t + \psi_1) + c_2 \sin(\omega_2 t + \psi_2), \]  

(40)

where:

\[ c_1 = \frac{\lambda^2 \omega_1^2 f_{01}^2}{\sqrt{\left(\omega_0^2 - \omega_1^2\right)^2 + \lambda^2 \omega_1^2}} + \frac{f_{01}^2 (\omega_0^2 - \omega_1^2)^2}{\lambda^2 \omega_1^2}, \]  

\[ \psi_1 = \arctg \left( -\frac{\lambda \omega_1}{\omega_0^2 - \omega_1^2} \right), \]  

(41)

\[ c_2 = \frac{f_{02}}{\sqrt{\left(\omega_0^2 - \omega_2^2\right)^2 + \lambda^2 \omega_2^2}} \]  

\[ \psi_2 = \arctg \left( -\frac{\lambda \omega_2}{\omega_0^2 - \omega_2^2} \right), \]  

(42)

(43)

(44)

3. Conclusion

Thus, the obtained expressions (40) - (44) describe the forced oscillations of the body of the vibratory mill as a result of two vibration exciters affection installed on both sides of the cylindrical body (figure 1). The given method of calculating of the trajectory of the vibratory mill cylindrical body allows one to calculate the trajectory of the mill body in polar coordinates. This method takes into account all the main parameters of the vibratory mill: design, technological, energy and kinematic parameters.

The geometrical parameters of the mill body are taken into account as well as body weight and load; imbalance masses, their eccentricities and angular velocities; stiffness of elastic elements and damping coefficient.
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