Positive operator valued measures covariant with respect to an irreducible representation

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Abstract

Given an irreducible representation of a group $G$, we show that all the covariant positive operator valued measures based on $G/Z$, where $Z$ is a central subgroup, are described by trace class, trace one positive operators.

1 Introduction

It is well known [2, 6] that, given a square-integrable representation $\pi$ of a unimodular group $G$ and a trace class, trace one positive operator $T$, the family of operators

$$Q(X) = \int_X \pi(g)T\pi(g^{-1})d\mu_G(g),$$

defines a positive operator valued measure (POVM) on $G$ covariant with respect to $\pi$ ($\mu_G$ is a Haar measure on $G$). In this paper, we prove that all the covariant POVMs are of the above form for some $T$. More precisely, we show this result for non-unimodular groups and for POVMs based on the quotient space $G/Z$, where $Z$ is a central subgroup.
Let $G$ be a locally compact second countable topological group and $Z$ be a central closed subgroup. We denote by $G/Z$ the quotient group and by $\dot{g} \in G/Z$ the equivalence class of $g \in G$. If $a \in G$ and $\dot{g} \in G/Z$, we let $a \left[ \dot{g} \right] = \dot{a} \dot{g}$ be the natural action of $a$ on the point $\dot{g}$.

Let $\mathcal{B}(G/Z)$ be the Borel $\sigma$-algebra of $G/Z$. We fix a left Haar measure $\mu_{G/Z}$ on $G/Z$. Moreover, we denote by $\Delta$ the modular function of $G$ and of $G/Z$.

By representation we mean a strongly continuous unitary representation of $G$ acting on a complex and separable Hilbert space, with scalar product $\langle \cdot , \cdot \rangle$ linear in the first argument.

Let $(\pi, \mathcal{H})$ be a representation of $G$. A positive operator valued measure $Q$ defined on $G/Z$ and such that

1. $Q(G/Z) = I$;
2. for all $X \in \mathcal{B}(G/Z)$,
   \[ \pi(g)Q(X)\pi(g^{-1}) = Q(g[X]) \quad \forall g \in G \]

is called $\pi$-covariant POVM on $G/Z$.

Given a representation $(\sigma, \mathcal{K})$ of $Z$, we denote by $(\lambda^\sigma, P^\sigma, \mathcal{H}^\sigma)$ the imprimitivity system unitarily induced by $\sigma$. We recall that $\mathcal{H}^\sigma$ is the Hilbert space of ($\mu_G$-equivalence classes of) functions $f : G \rightarrow \mathcal{K}$ such that

1. $f$ is weakly measurable;
2. for all $z \in Z$,
   \[ f(\sigma(z^{-1})g) = f(g) \quad \forall g \in G; \]
3. \[
   \int_{G/Z} \|f(g)\|^2_K \, d\mu_{G/Z}(\dot{g}) < +\infty
\]

with scalar product
\[
   \langle f_1, f_2 \rangle_{\mathcal{H}^\sigma} = \int_{G/Z} \langle f_1(g), f_2(g) \rangle_K \, d\mu_{G/Z}(\dot{g}).
\]

The representation $\lambda^\sigma$ acts on $\mathcal{H}^\sigma$ as
\[
   (\lambda^\sigma(a)f)(g) := f(a^{-1}g) \quad g \in G
\]
for all $a \in G$. The projection valued measure $P^\sigma$ is given by
\[
   (P^\sigma(X)f)(g) := \chi_X(\dot{g})f(g) \quad g \in G.
\]
for all $X \in \mathcal{B}(G/Z)$, where $\chi_X$ is the characteristic function of the set $X$.

We recall some basic properties of square integrable representations modulo a central subgroup. We refer to Ref. [1] for $G$ unimodular and $Z$ arbitrary and to Ref. [4] for $G$ non-unimodular and $Z = \{e\}$. Combining these proofs, one obtains the following result.

**Proposition 1** Let $(\pi, \mathcal{H})$ be an irreducible representation of $G$ and $\gamma$ be the character of $Z$ such that

$$\pi(z) = \gamma(z) I_{\mathcal{H}} \quad \forall z \in Z.$$  

The following facts are equivalent:

1. there exists a vector $u \in \mathcal{H}$ such that

$$0 < \int_{G/Z} |\langle u, \pi(g) u \rangle_{\mathcal{H}}|^2 \, d\mu_{G/Z}(\dot{g}) < +\infty; \quad (1)$$

2. $(\pi, \mathcal{H})$ is a subrepresentation of $(\lambda^\gamma, \mathcal{H}^\gamma)$.

If any of the above conditions is satisfied, there exists a selfadjoint injective positive operator $C$ with dense range such that

$$\pi(g) C = \Delta(g)^{-\frac{1}{2}} C \pi(g) \quad \forall g \in G,$$

and an isometry $\Sigma : \mathcal{H} \otimes \mathcal{H}^* \rightarrow \mathcal{H}^\gamma$ such that

1. for all $u \in \mathcal{H}$ and $v \in \text{dom} C$

$$\Sigma(u \otimes v^*)(g) = \langle u, \pi(g) Cv \rangle_{\mathcal{H}} \quad g \in G,$$

2. for all $g \in G$

$$\Sigma(\pi(g) \otimes I_{\mathcal{H}^*}) = \lambda(g) \Sigma,$$

3. the range of $\Sigma$ is the isotypic space of $\pi$ in $\mathcal{H}^\gamma$.

If Eq. (1) is satisfied, $(\pi, \mathcal{H})$ is called square-integrable modulo $Z$. The square root of $C$ is called formal degree of $\pi$ (see Ref. [4]). In particular, when $G$ is unimodular, $C$ is a multiple of the identity.
2 Characterization of $Q$

We fix an irreducible representation $(\pi, \mathcal{H})$ of $G$ and let $\gamma$ be the character such that $\pi|_Z = \gamma I_{\mathcal{H}}$. The following theorem characterizes all the POVM on $G/Z$ covariant with respect to $\pi$ in terms of positive trace one operators on $\mathcal{H}$.

**Theorem 2** The irreducible representation $\pi$ admits a covariant POVM based on $G/Z$ if and only if $\pi$ is square-integrable modulo $Z$.

In this case, let $C$ be the square root of the formal degree of $\pi$. There exists a one-to-one correspondence between covariant POVMs $Q$ on $G/Z$ and positive trace one operators $T$ on $\mathcal{H}$ given by

$$\langle Q_T(X)v, u \rangle_{\mathcal{H}} = \int_X \langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} d\mu_{G/Z}(g)$$

for all $u, v \in \text{dom} C$ and $X \in \mathcal{B}(G/Z)$.

**Proof.** Let $Q$ be a $\pi$-covariant POVM. According to the generalized imprimitivity theorem there exists a representation $(\sigma, \mathcal{K})$ of $Z$ and an isometry $W : \mathcal{H} \to \mathcal{H}^\sigma$ intertwining $\pi$ with $\lambda^\sigma$ such that

$$Q(X) = W^* P^\sigma(X) W$$

for all $X \in \mathcal{B}(G/Z)$.

Define the following closed invariant subspace of $\mathcal{K}$

$$\mathcal{K}_\gamma = \{ v \in \mathcal{K} \mid \sigma(z)v = \gamma(z)v \}.$$

Let $\sigma_1$ and $\sigma_2$ be the restrictions of $\sigma$ to $\mathcal{K}_\gamma$ and $\mathcal{K}_\gamma^\perp$ respectively. The induced imprimitivity system $(\lambda^\sigma, P^\sigma, \mathcal{H}^\sigma)$ decomposes into the orthogonal sum

$$\mathcal{H}^\sigma = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_2}.$$

If $f \in \mathcal{H}^\sigma$ and $z \in Z$, then

$$(\lambda^\sigma(z)f)(g) = f(z^{-1}g) = f(gz^{-1}) = \sigma(z)f(g) \quad g \in G.$$

On the other hand, if $u \in \mathcal{H}$ and $z \in Z$, we have

$$(\lambda^\sigma(z)Wu)(g) = (W\pi(z)u)(g) = \gamma(z)(Wu)(g) \quad g \in G.$$

It follows that $(Wu)(g) \in \mathcal{K}_\gamma$ for $\mu_G$-almost every $g \in G$, that is, $Wu \in \mathcal{H}^{\sigma_1}$. So it is not restrictive to assume that

$$\sigma = \gamma I_{\mathcal{K}}$$
for some Hilbert space $\mathcal{K}$. Clearly, we have

$$\mathcal{H}^\sigma = \mathcal{H}^\gamma \otimes \mathcal{K}, \quad \lambda^\sigma = \lambda^\gamma \otimes I_{\mathcal{K}}.$$  

In particular, $\pi$ is a subrepresentation of $\lambda^\gamma$, hence it is square-integrable modulo $Z$.

Due to Prop. 1, the operator $W' = (\Sigma^* \otimes I_{\mathcal{K}})W$ is an isometry from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{K}$ such that

$$W'\pi(g) = \pi(g) \otimes I_{\mathcal{H}^* \otimes \mathcal{K}} \quad g \in G.$$  

Since $\pi$ is irreducible, there is a unit vector $B \in \mathcal{H}^* \otimes \mathcal{K}$ such that

$$W' u = u \otimes B \quad \forall u \in \mathcal{H}.$$  

Let $(e_i)_{i \geq 1}$ be an orthonormal basis of $\mathcal{H}$ such that $e_i \in \text{dom} C$, then

$$B = \sum_i e_i^* \otimes k_i,$$

where $k_i \in \mathcal{K}$ and $\sum_i \|k_i\|^2_{\mathcal{K}} = 1$.

If $u \in \text{dom} C$, one has that

$$(Wu)(g) = [(\Sigma \otimes I_{\mathcal{K}})(u \otimes B)](g)$$

$$= \sum_i \Sigma(u \otimes e_i^*)(g) \otimes k_i$$

$$= \sum_i \langle u, \pi(g)Ce_i \rangle_{\mathcal{H}} \otimes k_i$$

$$= \sum_i \langle Ce_i \rangle_{\mathcal{H}} \otimes k_i$$

$$= \sum_i (e_i^* \otimes k_i)(C\pi (g^{-1}) u),$$

where the series converges in $\mathcal{H}^\sigma$. On the other hand, for all $g \in G$ the series $\sum_i (e_i^* \otimes k_i)(C\pi (g^{-1}) u)$ converges to $BC\pi (g^{-1}) u$, where we identify $\mathcal{H}^* \otimes \mathcal{K}$ with the space of Hilbert-Schmidt operators. By unicity of the limit

$$(Wu)(g) = BC\pi (g^{-1}) u \quad g \in G.$$  

If $u, v \in \text{dom} C$, the corresponding covariant POVM is given by

$$\langle Q(X)v, u \rangle_{\mathcal{H}} = \langle P^\sigma(X)Wv, Wu \rangle_{\mathcal{H}^\sigma}$$

$$= \int_{G/Z} \chi_X(\hat{g}) \langle BC\pi (g^{-1}) v, BC\pi (g^{-1}) u \rangle_{\mathcal{H}} d\mu_{G/Z}(\hat{g})$$

$$= \int_X \langle TC\pi (g^{-1}) v, C\pi (g^{-1}) u \rangle_{\mathcal{H}} d\mu_{G/Z}(\hat{g}),$$

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where
\[ T := B^*B \]
is a positive trace class trace one operator on \( \mathcal{H} \).

Conversely, assume that \( \pi \) is square integrable and let \( T \) be a positive trace class trace one operator on \( \mathcal{H} \). Then
\[ B := \sqrt{T} \]
is a (positive) operator belonging to \( \mathcal{H}^* \otimes \mathcal{H} \) such that \( B^*B = T \) and \( \|B\|_{\mathcal{H}^* \otimes \mathcal{H}} = 1 \). The operator \( W \) defined by
\[ Wv := (\Sigma \otimes I_{\mathcal{H}})(v \otimes B) \quad \forall v \in \mathcal{H} \]
is an isometry intertwining \( (\pi, \mathcal{H}) \) with the representation \( (\lambda^\sigma, \mathcal{H}^\sigma) \), where \( \sigma = \gamma I_{\mathcal{H}} \).

Define \( Q_T \) by
\[ Q_T(X) = W^*P^\sigma(X)W \quad X \in \mathcal{B}(G/Z). \]

With the same computation as above, one has that
\[ \langle Q_T(X)u, v \rangle_{\mathcal{H}} = \int_X \langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} d\mu_{G/Z}(\dot{g}) \]
for all \( u, v \in \text{dom} \mathcal{C} \).

Finally, we show that the correspondence \( T \mapsto Q_T \) is injective. Let \( T_1 \) and \( T_2 \) be positive trace one operators on \( \mathcal{H} \), with \( Q_{T_1} = Q_{T_2} \). Set \( T = T_1 - T_2 \).
Since \( \pi \) is strongly continuous, for all \( u, v \in \text{dom} \mathcal{C} \) the map
\[ G/Z \ni \dot{g} \mapsto \langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} = \Delta(\dot{g})^{-1} \langle T\pi(g^{-1})Cv, \pi(g^{-1})Cu \rangle_{\mathcal{H}} \in \mathbb{C} \]
is continuous. Since
\[ \int_X \langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} d\mu_{G/Z}(\dot{g}) = \langle [Q_{T_1}(X) - Q_{T_2}(X)]v, u \rangle_{\mathcal{H}} = 0 \]
for all \( X \in \mathcal{B}(G/Z) \), we have
\[ \langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} = 0 \quad \forall \dot{g} \in G/Z. \]

In particular,
\[ \langle TCv, Cu \rangle_{\mathcal{H}} = 0, \]
so that, since \( \mathcal{C} \) has dense range, \( T = 0 \). \( \blacksquare \)
Remark 3 Scutaru shows in Ref. [6] that there exists a one-to-one correspondence between positive trace one operators on $\mathcal{H}$ and covariant POVMs $Q$ based on $G/Z$ with the property

$$\text{tr } Q(K) < +\infty$$

for all compact sets $K \subset G/Z$. Theorem 2 shows that every covariant POVM $Q$ based on $G/Z$ shares property (3).

Remark 4 If $G$ is unimodular, then $K = \lambda I$, with $\lambda > 0$, and one can normalize $\mu_{G/Z}$ so that $\lambda = 1$. Hence,

$$Q_T(X) = \int_X \pi(g) T \pi(g^{-1}) \, d\mu_{G/Z}(\dot{g}) \quad \forall X \in \mathcal{B}(G/Z),$$

the integral being understood in the weak sense.

Remark 5 If $T = \eta^* \otimes \eta$, with $\eta \in \text{dom } C$ and $\|\eta\|_{\mathcal{H}} = 1$, we observe that

$$\langle Q_T(X) v, u \rangle_{\mathcal{H}} = \int_X \langle C \pi(g^{-1}) v, \eta \rangle_{\mathcal{H}} \langle \eta, C \pi(g^{-1}) u \rangle_{\mathcal{H}} \, d\mu_{G/Z}(\dot{g})$$

$$= \int_X \langle v, \pi(g) C\eta \rangle_{\mathcal{H}} \langle \pi(g) C\eta, u \rangle_{\mathcal{H}} \, d\mu_{G/Z}(\dot{g})$$

$$= \int_X (W_{C\eta}v)(g) (W_{C\eta}u)(g) \, d\mu_{G/Z}(\dot{g})$$

for all $v, u \in \text{dom } C$, where $W_{C\eta} : \mathcal{H} \rightarrow \mathcal{H}^\gamma$ is the wavelet operator associated to the vector $C\eta$. In particular,

$$Q_T(X) = W_{C\eta}^* P^\gamma(X) W_{C\eta}.$$

3 Two examples

3.1 The Heisenberg group

The Heisenberg group $H$ is $\mathbb{R}^3$ with composition law

$$(p, q, t) (p', q', t') = \left( p + p', q + q', t + t' + \frac{pq' - qp'}{2} \right).$$

The centre of $H$ is

$$Z = \{(0, 0, t) \mid t \in \mathbb{R}\},$$
and the quotient group $G/Z$ is isomorphic to the Abelian group $\mathbb{R}^2$, with projection

$$q(p, q, t) = (p, q).$$

The Heisenberg group is unimodular with Haar measure

$$d\mu_{G/Z}(p, q) = \frac{1}{2\pi} dpdq.$$

Given an infinite dimensional Hilbert space $\mathcal{H}$ and an orthonormal basis $(e_n)_{n \geq 1}$, let $a, a^*$ be the corresponding ladder operators. Define

$$Q = \frac{1}{\sqrt{2}}(a + a^*)$$
$$P = \frac{1}{\sqrt{2i}}(a - a^*)$$

It is known [2, 5] that the representation

$$\pi(p, q, t) = e^{i(t+pQ+qP)}$$

is square-integrable modulo $Z$ and $C = 1$.

It follows from Theorem 2 that any $\pi$-covariant POVM $Q$ based on $\mathbb{R}^2$ is of the form

$$Q(X) = \frac{1}{2\pi} \int_X e^{i(pQ+qP)}Te^{-i(pQ+qP)}dpdq, \quad X \in \mathcal{B}(\mathbb{R}^2)$$

for some positive trace one operator on $\mathcal{H}$. Up to our knowledge, the complete classification of the POVMs on $\mathbb{R}^2$ covariant with respect to the Heisenberg group has been an open problem till now.

### 3.2 The $ax + b$ group

The $ax + b$ group is the semidirect product $G = \mathbb{R} \times \mathbb{R}_+$, where we regard $\mathbb{R}$ as additive group and $\mathbb{R}_+$ as multiplicative group. The composition law is

$$(b, a)(b', a') = (b + ab', a a').$$

The group $G$ is nonunimodular with left Haar measure

$$d\mu_G(b, a) = a^{-2} dbda$$

and modular function

$$\Delta(b, a) = \frac{1}{a}.$$
Let $\mathcal{H} = L^2((0, +\infty), dx)$ and $(\pi^+, \mathcal{H})$ be the representation of $G$ given by
\[
[\pi^+ (b, a) f](x) = a^\frac{1}{2} e^{i2\pi bx} f(ax) \quad x \in (0, +\infty).
\]
It is known [5] that $\pi$ is square-integrable, and the square root of its formal degree is
\[
(Cf)(x) = \Delta (0, x)^{\frac{1}{2}} f(x) = x^{-\frac{1}{2}} f(x) \quad x \in (0, +\infty)
\]
acting on its natural domain.

By means of Theorem 2 every POVM based on $G$ and covariant with respect to $\pi^+$ is described by a positive trace one operator $T$ according to Eq. 2. Explicitely, let $(e_i)_{i \geq 1}$ be an orthonormal basis of eigenvectors of $T$ and $\lambda_i \geq 0$ be the corresponding eigenvalues. If $u \in L^2((0, +\infty), dx)$ is such that $x^{-\frac{1}{2}} u \in L^2((0, +\infty), dx)$, the $\pi^+$-covariant POVM corresponding to $T$ is given by
\[
\langle Q_T (X) u, u \rangle_{\mathcal{H}} = \int_X \langle TC\pi^+ (g^{-1}) u, C\pi^+ (g^{-1}) u \rangle_{\mathcal{H}} d\mu_G (g)
\]
\[
= \int_X \sum_i \lambda_i \left| \langle C\pi^+ (g^{-1}) u, e_i \rangle_{\mathcal{H}} \right|^2 d\mu_G (g)
\]
\[
= \sum_i \lambda_i \int_X \left| \int_{\mathbb{R}_+} x^{-\frac{1}{4}} a^{-\frac{1}{4}} e^{-\frac{2\pi ibx}{a}} u \left( \frac{x}{a} \right) e_i(x) dx \right|^2 a^{-2} db da.
\]

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