$\mathcal{N} = 4$ Superconformal Characters and Partition Functions

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Character formulae for positive energy unitary representations of the $\mathcal{N} = 4$ superconformal group are obtained through use of reduced Verma modules and Weyl group symmetry. Expansions of these are given which determine the particular representations present and results such as dimensions of superconformal multiplets. By restriction of variables various ‘blind’ characters are also obtained. Limits, corresponding to reduction to particular subgroups, in the characters isolate contributions from particular subsets of multiplets and in many cases simplify the results considerably. As a special case, the index counting short and semi-short multiplets which do not form long multiplets found recently is shown to be related to particular cases of reduced characters. Partition functions of $\mathcal{N} = 4$ super Yang Mills are investigated. Through analysis of these, exact formulae are obtained for counting $\frac{1}{2}$ and some $\frac{1}{4}$-BPS operators in the free case. Similarly, partial results for the counting of semi-short operators are given. It is also shown in particular examples how certain short operators which one might combine to form long multiplets due to group theoretic considerations may be protected dynamically.

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1. Introduction

The dynamics of $\mathcal{N} = 4$ super-Yang Mills theories is highly constrained by the large supersymmetry group present in this case. Invariance of the classical action under $\mathcal{N} = 4$ superconformal transformations persists as a symmetry of these theories at the quantum level both perturbatively and non-perturbatively thanks to their remarkable ultra-violet properties.

Moreover, with a $U(N)$ or $SU(N)$ gauge group, in the large $N$ limit the theory should admit an effective string description that has found a concrete realisation in the holographic AdS/CFT correspondence conjectured by Maldacena some time ago [1]. More recently there has been much interest in the potential integrability of $\mathcal{N} = 4$ super-Yang Mills theories at the planar level, which is related to the existence of an infinite number of non-local classically conserved charges for type IIB superstrings on $AdS_5 \times S^5$. This has found applications in the study of states with large $R$-charge or spin in terms of various spin chains that are expected to be the holographic counterpart of various string solitons (for recent reviews of spin chains and integrability see [2,3]).

In terms of determining the full spectrum of operators in general unfortunately at weak ’t Hooft coupling, the regime amenable to a (perturbative) superconformal field theory analysis, the $AdS_5 \times S^5$ background where the string propagates is highly curved and the spectrum of its excitations can only be extrapolated from large radius that corresponds to large ’t Hooft coupling, where an effective supergravity description is available. Nevertheless, invoking the higher spin symmetry enhancement present for zero coupling [4,5] allows the precise matching of the spectrum of single particle states of the type IIB superstring on $AdS_5 \times S^5$ with the spectrum of single-trace gauge invariant operators in $\mathcal{N} = 4$ super-Yang Mills. Turning on interactions with a non zero coupling, all but the $\frac{1}{2}$-BPS multiplets combine into long multiplets of the $\mathcal{N} = 4$ superconformal group $PSU(2,2|4)$ and acquire anomalous dimensions. The bulk counterpart is a Higgs mechanism, termed La Grande Bouffe, whereby higher spin fields absorb lower spin Goldstone fields and become massive - see [6,7,8,9], and also the review [10].

A similar analysis for the spectrum of multi-particle states of supergravity or more generally string theory on $AdS_5 \times S^5$, that appear in various guises as multi-graviton states, giant gravitons, wrapped D3-branes, and AdS black holes, and are expected to be dual to multi-trace gauge invariant operators, is still in its infancy.

The purpose of the present investigation is to develop a framework which allows the decomposition of the spectrum of $\mathcal{N} = 4$ super-Yang Mills theories into positive energy unitary representations of the $\mathcal{N} = 4$ superconformal group. The large superconformal symmetry allows for a rich variety of BPS and short/semi-short multiplets. An index pro-
posed recently should count the ‘unpaired’ short multiplets that cannot combine into long ones. The index is ‘topological’ in the sense that it is invariant under continuous deformations that preserve $\mathcal{N} = 4$ superconformal symmetry such as turning on or switching off interactions. It can be thus reliably computed at weak coupling. The information stored in the index is however limited since, as we will show, there exist certain short operators which would be allowed to combine into long multiplets due to purely representation considerations, but which, according to the analysis of [11,12], are in fact prevented from doing so dynamically. Indeed Morales has shown that these operators are protected in some cases at one loop [13]. A mismatch of the macroscopic entropy of some class of $\frac{1}{8}$-BPS and $\frac{1}{16}$-BPS AdS black holes with the counting of corresponding operators in the gauge theory as revealed by the index [14] is also in contradiction with the naive expectation that this counts all such operators and that all potential long multiplets gain anomalous dimensions for non zero coupling.

The spectrum of operators, either as single trace or multi-trace operators, in $\mathcal{N} = 4$ superconformal theories may be determined in terms of various partition functions involving operators constructed from the fundamental fields. For particular classes of short or semi-short supermultiplets these may be restricted to particular subsectors. For single trace operators these are equivalent to various spin chains on which the dilatation operator $D$, whose eigenvalues determine the scaling dimensions, takes an integrable form. For the interacting large $N$ superconformal theory there should then perhaps be the possibility of determining the partition function for non zero coupling, but such issues are not addressed here. In any event in order to disentangle the particular supermultiplets present it is necessary to consider expansions in terms of the characters for the different irreducible representations. Only genuine unconstrained long multiplets may develop anomalous dimensions with non zero coupling.

In this paper we attempt to construct explicit compact, as far as possible, formulae for the characters for the various short and semi-short multiplets of $\mathcal{N} = 4$ superconformal symmetry. The results are obtained by application of a similar technique to that outlined in [15] for obtaining the Weyl character formula for compact Lie groups by using Verma modules. Verma modules are vector spaces constructed by the action of lowering operators in the Lie algebra on highest weight states. The representation space is then formed by acting on the Verma modules with the elements of the Weyl symmetry group. This procedure is extended here to the $\mathcal{N} = 4$ superconformal group. The analysis depends only on the algebraic properties of the $\mathcal{N} = 4$ superconformal group. Various appropriate shortening conditions are applied on the highest weight state, at particular values of the conformal scale dimension $\Delta$, and these lead to the different short and semi-short super-
multiplets for which characters are obtained.\footnote{Mathematical results \cite{16} for characters for atypical representations, when shortening conditions apply, of superalgebras are not straightforward. Our procedure is not valid in all cases, but it has been carefully checked in applications here.} A more general, although in some respects similar, discussion is found in \cite{17}.

The plan of the paper is as follows. In section 2 we describe standard mathematical results for the characters of Lie algebras as obtained from Verma modules in a fashion which allows generalisation to the superconformal case subsequently. Although this introduces some convenient notation this is not essential for later applications. In section 3 we then describe how character formulae for positive energy unitary representations of the $\mathcal{N} = 4$ superconformal group may be obtained, taking account of various possible shortening conditions for supermultiplets. It is shown how to use these results to obtain the decomposition of long multiplets in terms of semi-short multiplets in a simple fashion. In section 4 the character formulae are applied in particular cases for unitary supermultiplets of interest and, for special choices of variables (‘fugacities’) we obtain the ‘blind’ characters that encode less information, eventually only keep track of the scaling dimensions of the components of the supermultiplets. In section 5 we show that limits exist which reduce the characters to those for various subgroups of the superconformal group $PSU(2,2|4)$ and isolate contributions from subsets of multiplets of the full spectrum allowed by superconformal symmetry. In particular, we identify a limit that exposes the characters contributing to the index in \cite{14}. In section 6 we describe the basic supermultiplet formed by the fundamental fields and how this may be reduced to different subsectors corresponding to particular subgroups of $PSU(2,2|4)$ which involve subsets of the elementary fields. In section 7 we investigate partition functions of $\mathcal{N} = 4$ super-Yang Mills in different sectors and obtain exact formulae for counting $\frac{1}{2}$ and some $\frac{1}{4}$-BPS operators in the free case and partial results for the counting of semi-short operators. These are applied both for general multi-trace operators and also for those dual to supergravity fields which are relevant in the strong coupling limit. Results for the index are also obtained in this context. Finally in section 8 we discuss further when semi-short operators are protected in the interacting theory, making connections with specific examples. The index is applied in particular cases and it is argued that all semi-short operators dual to supergravity fields should be protected although in some cases they may form long multiplets.

Some details are referred to four appendices. Appendix A describes the Lie algebra for $PSU(2,2|4)$ and lists its relevant subgroups related to different shortening conditions. Appendix B contains formulae for the expansion of infinite products in terms of Schur polynomials that are useful in the analysis of partition functions. Appendix C contains various tables of the semi-short operators that are required by the expansion of partition
functions for the first few levels, as discussed in the main text. In addition appendix D analyses the product of two characters for the fundamental representation and obtains a decomposition into irreducible multiplets.

2. Verma Modules and Characters

We here outline, without any proofs, how characters are obtained from Verma modules for Lie algebras in a form which is appropriate for our later discussion of superconformal groups.

For $\mathrm{Sl}(2)$, with generators $J_\pm, J_3$ having standard commutation relations, the Verma module $\mathcal{V}_j$ is spanned by states $(J_\pm)^N|j\rangle^\text{h.w.}$ for $N = 0, 1, 2, \ldots$, where the highest weight state satisfies $J_+|j\rangle^\text{h.w.} = 0$, $J_3|j\rangle^\text{h.w.} = j|j\rangle^\text{h.w.}$. The corresponding character for $\mathcal{V}_j$ may then be expressed, for general $j$, as a formal trace involving a sum over all eigenvalues for $J_3$ in $\mathcal{V}_j$

$$C_j(x) = \tilde{\text{tr}}_{\mathcal{V}_j}(x^{2J_3}) = \sum_{N=0}^{\infty} x^{2j-2N} = \frac{x^{2j+2}}{x^2 - 1}.$$ (2.1)

For $2j \in \mathbb{N}$, $\mathcal{V}_{j-1} \subset \mathcal{V}_j$ and, with the $SU(2)$ hermiticity requirements for $J_\pm, J_3$, $\mathcal{V}_{j-1}$ contains null states. A finite dimensional space is obtained by considering $\mathcal{V}_j = \mathcal{V}_j/\mathcal{V}_{j-1}$, removing also null states, and the associated representation of $\mathrm{Sl}(2)$ corresponds to the standard spin-$j$ unitary irreducible representation for $SU(2)$. The corresponding character for this representation on $\mathcal{V}_j$ is given by a well defined trace

$$\chi_j(x) = \text{tr}_{\mathcal{V}_j}(x^{2J_3}) = C_j(x) - C_{j-1}(x) = \frac{x^{2j+1} - x^{-2j-1}}{x - x^{-1}}.$$ (2.2)

The Weyl group $W$ in this case is $S_2 \simeq \mathbb{Z}_2$ with elements $\{e, \sigma\}$ where $\sigma^2 = e$. Defining $\sigma f(x) = f(x^{-1})$ for any function $f$ then the character for the irreducible spin $j$ representation may also be written as

$$\chi_j(x) = \mathcal{W}^{S_2}C_j(x) = C_j(x) + C_j(x^{-1}), \quad \mathcal{W}^{S_2} = e + \sigma.$$ (2.3)

Of course it is easy to see that $\dim(\mathcal{V}_j) = \chi_j(1) = 2j + 1$, the standard dimension of the spin $j$ representation.

A general simple Lie algebra is decomposed into an abelian Cartan subalgebra with generators $H = (H_1, H_2, \ldots, H_r)$ and remaining generators $\{E_\alpha\}$ defined by roots $\alpha \in \Phi$ where $[H_i, E_\alpha] = \alpha_i E_\alpha$. We require $[E_\alpha, E_{-\alpha}] = \alpha^\vee \cdot H$ with $\alpha^\vee = 2\alpha/\alpha^2$ the coroots. Dividing the root space into positive and negative roots, $\Phi = \Phi_+ \cup \Phi_-$, a highest weight
state $|\Lambda\rangle^\text{h.w.}$ satisfies $E_\alpha |\Lambda\rangle^\text{h.w.} = 0$, $\alpha \in \Phi_+$ and $H|\Lambda\rangle^\text{h.w.} = \Lambda|\Lambda\rangle^\text{h.w.}$. The associated Verma module $\mathcal{V}_\Lambda$ is then defined in terms of the basis of states

$$\prod_{\alpha \in \Phi_+} (E_{-\alpha})^{N_\alpha} |\Lambda\rangle^\text{h.w.}, \quad N_\alpha = 0, 1, 2, \ldots,$$

for a particular choice of ordering of $E_{-\alpha}$. For simple roots $\alpha_i$ and corresponding fundamental weights $\varpi_i$, where $\alpha_i \lor \cdot \varpi_j = \delta_{ij}$, we can expand $\Lambda = \sum_i \lambda_i \varpi_i$. For $\tilde{H}_i = \alpha_i \lor \cdot H$, which with $E_{\pm \alpha_i}$ forms a standard $\text{Sl}(2)$ algebra, we define the character for the Verma module (2.4) by

$$C_\Lambda(\varpi) = \tilde{\text{tr}}_{\mathcal{V}_\Lambda} (\prod_i x_i^{\tilde{H}_i}) = \sum_{\{N_\alpha\}} \prod_{\alpha \in \Phi_+} x_i^{\lambda_i - N_\alpha \sum_j n_{\alpha j} K_{ji}}$$

$$= \prod_{i=1}^r x_i^{\lambda_i} \prod_{\alpha \in \Phi_+} \left(1 - \prod_j x_j^{\sum_k n_{\alpha k} K_{kj}}\right)^{-1},$$

where $\alpha = \sum_i n_{\alpha_i} \alpha_i$ for $\alpha \in \Phi_+$ and $K_{ji} = \alpha_j \lor \cdot \alpha_i \lor$ are the elements of the Cartan matrix.

In a similar fashion to the $\text{Sl}(2)$ case for $\lambda_i \in \mathbb{N}$ the Verma module is reducible since there are states in the Verma module $\mathcal{V}_\Lambda$ also satisfying the highest weight conditions. Such states arise when $\Lambda') = \Lambda - \sum_{\tilde{\alpha} \in \Phi_+} N_{\tilde{\alpha}} \tilde{\alpha}$ for some $N_{\tilde{\alpha}}$ and where $\Lambda')$ denotes the action of an element $\sigma$ of the Weyl group $\mathcal{W}$ on $\Lambda$. $\mathcal{W}$ is generated by reflections corresponding to the simple roots, $\sigma_\varpi = e - \sum_{\alpha_\varpi} \alpha_\varpi \cdot \varpi$, $\sigma_i^2 = e$, and is determined by the relations, for $i \neq j$, $(\sigma_i \sigma_j)^{2+K_{ij}} = e$ (excluding the group $G_2$). For any $\sigma$ we define $\Lambda') = \sigma(\Lambda + \rho) - \rho$, with $\rho = \frac{1}{2} \sum_{\tilde{\alpha} \in \Phi_+} \tilde{\alpha} = \sum_{i=1}^r \varpi_i$. Hence, as for $\text{Sl}(2)$, $\mathcal{V}_{\Lambda')} \subset \mathcal{V}_{\Lambda}$. A vector space $\mathcal{V}_{\Lambda}$ for a finite dimensional representation may be formed from $\mathcal{V}_{\Lambda}$ by an extension of the procedure for $\text{Sl}(2)$. For any $\sigma \in \mathcal{W}$ the length $\ell(\sigma)$ is defined as the minimal number of elementary reflections $\sigma_i$ in a product necessary to generate $\sigma$, there is further a unique element $\sigma_{\text{max}}$ such that $\ell(\sigma_{\text{max}}) = \ell_{\text{max}}$. Defining $V_{\Lambda}(\rho) = (\oplus_{\sigma : \ell(\sigma) = \rho} V_{\Lambda})/V_{\Lambda}(\rho+1)$, with $V_{\Lambda}(\ell_{\text{max}}) = V_{\Lambda_{\text{max}}}$, then $\mathcal{V}_{\Lambda} = V_{\Lambda}(0)$. The associated character for this representation is

$$\chi_{\Lambda}(\varpi) = \text{tr}_{\mathcal{V}_{\Lambda}} (\prod_i x_i^{\tilde{H}_i}) = \sum_{\sigma \in \mathcal{W}} (-1)^{\ell(\sigma)} C_{\Lambda'}(\varpi),$$

which is the standard Weyl character formula for the representation with Dynkin labels $\Lambda = [\lambda_1, \ldots, \lambda_r]$.

For later application we obtain more explicit expressions for the characters of $\text{Gl}(n)$ and then $\text{Sl}(n)$. In an orthonormal basis the generators are $R^i_j$, with $1 \leq i, j \leq n$, and satisfy the Lie algebra $[R^i_j, R^k_l] = \delta^k_j R^i_l - \delta^i_l R^k_j$. The Cartan subalgebra generators for $\text{Gl}(n)$ are then $H = (R^1_1, \ldots, R^n_n)$ and $[H, R^i_j] = (e_i - e_j)R^i_j$ with $\{e_1, \ldots, e_n\}$.
a set of \( n \)-dimensional orthonormal unit vectors given by \( (e_i)_j = \delta_{ij} \). In this case \( e_i - e_j \in \Phi_+ \) for \( 1 \leq i < j \leq n \) and a corresponding set of simple roots is given by \( e_i - e_{i+1} \) for \( i = 1, \ldots, n - 1 \). Representations are obtained from highest weight states \( |\ell_1, \ell_2, \ldots, \ell_n\rangle_{h.w.} \) satisfying \( R^{i}_{j}|\ell_1, \ldots, \ell_n\rangle_{h.w.} = 0 \), \( i < j \), and \( H|\ell_1, \ldots, \ell_n\rangle_{h.w.} = \ell|\ell_1, \ldots, \ell_n\rangle_{h.w.} \). for \( \ell = (\ell_1, \ldots, \ell_n) \). The corresponding Verma module \( \mathcal{V}_\ell \) is spanned by

\[
\prod_{i>j} (R^{i}_{j})^{N_{ij}} |\ell_1, \ell_2, \ldots, \ell_n\rangle_{h.w.}, N_{ij} = 0, 1, 2, \ldots \). For \( u = (u_1, \ldots, u_n) \) the Verma module character may be defined by a sum over all eigenvalues of \( H \)

\[
C_\ell(u) = \prod_{\{N_{ij}\} = 1}^{n} \left( u_i^{\ell_i - \sum_{j>i} N_{ij} + \sum_{j<i} N_{ij}} \right) = \prod_{i=1}^{n} u_i^{\ell_i} \prod_{j<k}^{\infty} \sum_{N_{jk}=0} \left( \frac{u_k}{u_j} \right)^{N_{jk}} \Delta(u)^{-1},
\]

which converges for \( u_{i+1} < u_i \) and where \( \Delta(u) \) is the Vandermonde determinant,

\[
\Delta(u) = \det [u_i^{-1}] = \prod_{1 \leq i < j \leq n} (u_i - u_j) .
\]

Under a rescaling \( C_\ell(\lambda u) = \lambda \sum \ell_i C_\ell(u) \).

For \( Sl(n) \) the generators satisfy \( \sum_{i} R^{i}_{i} = 0 \) and a basis for the Cartan subalgebra is given by \( H_i = R^{i}_{i} - R^{i+1}_{i+1} \), \( i = 1, \ldots, n - 1 \). Since the Verma module character \( (2.7) \) satisfies \( C_\ell(u) \rightarrow (u_1 \ldots u_n)^c C_\ell(u) \) for \( \ell \rightarrow \ell + c \sum \ell_i \), for any constant \( c \), we may impose for the \( Sl(n) \) case \( \prod_{i=1}^{n} u_i = 1 \), so that \( \ell \) is then arbitrary up to \( \ell \rightarrow \ell + c \sum \ell_i \). This condition on \( u \) may be realised by letting \( u_1 = x_1 \), \( u_n = 1/x_n \), and \( u_i = x_i/x_{i-1} \) for \( i = 2, \ldots, n - 1 \) and then \( C_\ell(u) = C_{\lambda(x)} \) where \( \lambda_i = \ell_i - \ell_{i+1}, i = 1, \ldots, n - 1 \).

For the simple roots considered above \( e_i - e_{i+1} \) it is easy to see that \( \sigma_i e_i = e_{i+1}, \sigma_i e_{i+1} = e_i \) and \( e_i e_j = e_j \) otherwise, so that the Weyl group \( W \simeq S_n \) the usual permutation group. Taking \( \rho = (n-1, n-2, \ldots, 0) \) and \( \ell' = \sigma(\ell + \rho) - \rho \) it is straightforward to show that \( C_{\ell'}(u) = \text{sign}(\sigma) C_{\ell}(su) \) where \( \sigma(u_1, \ldots, u_n) = (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \) is a simple permutation and \( \text{sign}(\sigma) = \pm 1 \) for even, odd permutations, so that \( \Delta(\sigma u) = \text{sign}(\sigma) \Delta(u) \). Hence applying \( (2.6) \) in this case the character for the finite dimensional representation obtained when we take \( \ell_i - \ell_{i+1} \in \mathbb{N} \) is

\[
\chi_\ell(u) = \sum_{\sigma \in S_n} \text{sign}(\sigma) C_{\ell'}(u) = \sum_{\sigma \in S_n} C_{\ell}(\sigma u) = \det [u_i^{\ell_i - j + n - j}] \Delta(u)^{-1},
\]

which are Schur polynomials, expressing compactly the characters for \( Gl(n) \) and also \( Sl(n) \) with \( \prod_{i} u_i = 1 \). The result \( (2.9) \) may also be expressed, if \( \sigma f(u) = f(\sigma u) \), as

\[
\chi_\ell(u) = \mathfrak{m}^{S_n} C_\ell(u), \quad \mathfrak{m}^{S_n} = \sum_{\sigma \in S_n} \sigma .
\]
A particular example which is useful later is

\[ \chi(p,0,\ldots,0)(u) = \sum_{\alpha_i > 0} \prod_{i=1}^n u_i^{\alpha_i}. \]  

(2.11)

For calculating products involving \( Sl(2) \) and \( Gl(n) \) or \( Sl(n) \) characters we may use

\[ (x^n + x^{-n})\chi_j(x) = \chi_{j+\frac{1}{2}n}(x) + \chi_{j-\frac{1}{2}n}(x), \quad \chi_j(x) = -\chi_{-j-1}(x), \]  

(2.12)

and results such as

\[ \sum_{\sigma \in S_n/\mathcal{S}} u_{\sigma(1)}^{r_1} \ldots u_{\sigma(n)}^{r_n} \chi_{\ell}(u) = \sum_{\sigma \in S_n/\mathcal{S}} \chi_{\ell+r_1e_{\sigma(1)}+\ldots+r_ne_{\sigma(n)}}(u), \quad r_i = 0, 1, 2, \ldots, \]  

(2.13)

with \( \mathcal{S} \subset S_n \) defined by \( u_{\sigma(1)}^{r_1} \ldots u_{\sigma(n)}^{r_n} = u_1^{r_1} \ldots u_n^{r_n} \). With the relations

\[ \chi_{\ell}(u) = (-1)^{\text{sign}(\sigma)} \chi_{\ell^\sigma}(u), \quad \sigma \in S_n, \]  

(2.14)

and also for \( \chi_j \) under \( j \rightarrow -j - 1 \) in (2.12), we may ensure all resulting characters in the expansion have \( j \geq 0, \ell_i \geq \ell_{i+1} \), requiring also \( \chi_{-\frac{1}{2}} = 0 \) and \( \chi_\ell = 0 \) if \( \ell = \ell^\sigma \) for \( \text{sign}(\sigma) = -1 \). For \( Gl(4) \), which is relevant later, the Weyl reflections are generated by

\[ \ell^\sigma_1 = (\ell_2-1, \ell_1+1, \ell_3, \ell_4), \quad \ell^\sigma_2 = (\ell_1, \ell_3-1, \ell_2+1, \ell_4), \quad \ell^\sigma_3 = (\ell_1, \ell_2, \ell_4-1, \ell_3+1). \]  

(2.15)

For positive energy unitary representations of non compact groups, such as the conformal group \( SO(d,2) \), in order to obtain the corresponding characters it is necessary to restrict the action of elements of the Weyl group to those corresponding to a maximal semi-simple compact subgroup, for the conformal group this is \( SO(d) \). The Verma module in this case is obtained from a highest weight state for \( O(d) \otimes O(2) \) which is an eigenvector of the dilatation operator \( D \) with eigenvalue \( \Delta \) and is annihilated by the generators for special conformal transformations \( K_\mu \). A basis for the Verma module is then constructed by the action of arbitrary products of the momentum operators \( \prod_\mu (P_\mu)^{N_\mu} \), as well lowering operators for the Lie algebra of \( SO(d) \). A detailed discussion was given in [18]. There are various possible shortening conditions when appropriate subsets of certain combinations of the generators \( P_\mu \) annihilate the highest weight state for particular \( \Delta \) which lead to reduced Verma modules. The characters for the corresponding unitary representations are then obtained by acting with the reduced Weyl symmetry group on the reduced Verma module characters.
3. Characters for unitary irreducible representations of the $N = 4$ superconformal group

The conformal group $SU(2, 2)$ algebra consists of translation generators $P_{\alpha \dot{\alpha}}$, special conformal generators $\tilde{K}^{\alpha \dot{\alpha}}$, dilatations $D$ and $SU(2)\times SU(2)$ spin generators $J_3, J_\pm, \tilde{J}_3, \tilde{J}_\pm$, for $\alpha, \dot{\alpha} = 1, 2$ spinor indices. The superconformal extension $PSU(2, 2|4)$ has in addition supercharges $Q^i_\alpha, \tilde{Q}_{i\dot{\alpha}}$, along with their superconformal extensions $S_i^\alpha, \tilde{S}_{i\dot{\alpha}}$, for $SU(4)$ indices $i = 1, \ldots, 4$, and also $SU(4)_R$ $R$-symmetry generators $R^i_j$. The details of the conformal algebra are given in appendix A.\[3\]

A generic highest weight primary state for this superalgebra $|\Delta; k, p, q; j, \bar{j}\rangle^{h.w.}$ has conformal dimension $\Delta$ belongs to the spin $SU(2) \otimes SU(2)$ representation $(j, \bar{j})$ and the $SU(4)$ representation with Dynkin labels $[k, p, q]$ and satisfies

\[
(K_{\alpha \dot{\alpha}}, S^i_\alpha, \tilde{S}^{i\dot{\alpha}}, J_+, \tilde{J}_+, R^i_{i+1})|\Delta; k, p, q; j, \bar{j}\rangle^{h.w.} = 0,
\]

\[
(D; H_1, H_2, H_3, J_3, \tilde{J}_3)|\Delta; k, p, q; j, \bar{j}\rangle^{h.w.} = (\Delta; k, p, q; j, \bar{j})|\Delta; k, p, q; j, \bar{j}\rangle^{h.w.},
\]

with $H_i$ the Cartan generators for $SU(4)$. The corresponding Verma module $V_{(\Delta; k, p, q; j, \bar{j})}$ is then spanned by the states

\[
\prod_{i,j,k,l=1,\ldots,4, k>j} \prod_{\alpha, \dot{\alpha}, \beta, \dot{\beta}=1,2} (P_{\alpha \dot{\alpha}})^{N_{\alpha \dot{\alpha}}}(Q^i_\beta)^{n_{i\beta}}(\tilde{Q}_{j\dot{\beta}})^{\tilde{n}_{j\dot{\beta}}}(J_-)^N(\tilde{J}_-)^{\bar{N}}(R^k_l)^{N_{kl}}|\Delta; k, p, q; j, \bar{j}\rangle^{h.w.},
\]

for $N_{\alpha \dot{\alpha}}, N, \bar{N}, N_{kl} = 0, 1, 2, \ldots$ and $n_{i\beta}, \tilde{n}_{j\dot{\beta}} = 0, 1$. The character for the Verma module $V_{(\Delta; k, p, q; j, \bar{j})}$ is expressed in terms of variables $s, u_1, u_2, u_3, u_4, x, \bar{x}$, with $\prod_i u_i = 1$, so that in the series expansion of the character then $s^{2\Delta}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3}u_4^{\ell_4}x^{2j}\bar{x}^{2\bar{j}}$, for $k = \ell_1 - \ell_2, p = \ell_2 - \ell_3, q = \ell_3 - \ell_4$, corresponds to the highest weight state. In general in translating between orthonormal basis labels $(\ell_1, \ldots, \ell_4)$ and Dynkin labels $[k, p, q]$ for $SU(4)$ representations then we may take

\[
\ell_1 = k + p + q, \quad \ell_2 = p + q, \quad \ell_3 = q, \quad \ell_4 = 0,
\]

without loss of generality.

The action of the generators on the highest weight state, as in (3.2), introduces further factors according to $P_{\alpha \dot{\alpha}} \rightarrow s^2 x^{\pm 1}\bar{x}^{\pm 1}$, along with

\[
Q^1_\alpha \rightarrow s u_1 x^{\pm 1}, \quad Q^2_\alpha \rightarrow s u_3 x^{\pm 1}, \quad Q^3_\alpha \rightarrow s u_3 x^{\pm 1}, \quad Q^4_\alpha \rightarrow s u_4 x^{\pm 1},
\]

As compared to [19] $D \rightarrow iD$ so that $D$ here, although anti-hermitian, has real eigenvalues.
\[ \bar{Q}_{4\alpha} \rightarrow su_{4}^{-1}x^{\pm 1}, \quad \bar{Q}_{3\alpha} \rightarrow su_{3}^{-1}x^{\pm 1}, \quad \bar{Q}_{2\alpha} \rightarrow su_{2}^{-1}x^{\pm 1}, \quad \bar{Q}_{1\alpha} \rightarrow su_{1}^{-1}x^{\pm 1}. \] (3.5)

In the above \( \alpha = 1, 2 \) correspond to \( x, x^{-1} \) and \( \dot{\alpha} = 1, 2 \) to \( \bar{x}^{-1}, \bar{x} \) respectively.

By using (2.1), (2.7) we may easily determine the Verma module character, which is given by a formal trace,

\[ C(\Delta; k, p, q; j, \bar{j}) (s; u; x, \bar{x}) = \sum_{n_{\epsilon}, \eta} (u \bar{x}^{\epsilon} x^{\eta})^{n_{\epsilon}} \sum_{i, j} (s \bar{u} x^{\epsilon} u^{\eta})^{n_{\epsilon}} \bar{s} u x^{\epsilon}, \] (3.6)

where \( u = (u_1, u_2, u_3, u_4) \) with \( u_1 u_2 u_3 u_4 = 1 \). For a general long multiplet, with all possibilities for \( n_{\epsilon}, n_{\eta}, \bar{n}_{\eta} \) as in (3.6) included, we have assuming (3.3)

\[ C(\Delta; k, p, q; j, \bar{j}) (s; u; x, \bar{x}) = s^{2\Delta} C(\Delta; k, p, q; j, \bar{j}) (s; u; x, \bar{x}) P(s, x, \bar{x}) Q(su, x) \bar{Q}(s^{-1}u, \bar{x}), \] (3.7)

where

\[ P(s, x, \bar{x}) = \prod_{\epsilon, \eta = \pm 1} (1 - 2x^{\epsilon} x^{\eta})^{-1}, \quad Q(u, x) = \prod_{i = 1}^{4} (1 + u_{i} x^{\epsilon}) \quad \bar{Q}(u, \bar{x}) = \prod_{i = 1}^{4} (1 + u_{i}^{-1} x^{\eta}). \] (3.8)

According to our prescription the character of the irreducible long representation is then given by a trace over a representation space \( V(\Delta; k, p, q; j, \bar{j}) \)

\[ \chi(\Delta; k, p, q; j, \bar{j}) (s, u, x, \bar{x}) = \prod_{\epsilon, \eta = \pm 1} (1 - 2x^{\epsilon} x^{\eta})^{-1}. \] (3.9)

since \( P(s, x, \bar{x}), Q(su, x), \bar{Q}(s^{-1}u, \bar{x}) \) are invariant under the action of the Weyl symmetriser. Here \( \mathfrak{w}^{S_{2}} \) imposes symmetry under \( x \rightarrow x^{-1} \) and \( \mathfrak{w}^{S_{2}} \) under \( \bar{x} \rightarrow \bar{x}^{-1} \). Factoring off \( P(s, x, \bar{x}) \) in (3.9) and setting \( s, u_{i}, x, \bar{x} = 1 \) then this gives the usual dimension formula for the conformal primary states in a long multiplet as \( 2^{16}(2j + 1)(2\bar{j} + 1)d_{[k, p, q]} \) where \( d_{[k, p, q]} \) is the dimension of the \( SU(4) \) irreducible representation labelled by \([k, p, q]\),

\[ d_{[k, p, q]} = \chi(\epsilon_{1}, \ldots, \epsilon_{4}) (1, 1, 1, 1) = \frac{1}{12} (k+1)(p+1)(q+1)(k+p+2)(p+q+2)(k+p+q+3). \] (3.10)

More generally for the conjugate fundamental representation of \( GL(4) \) we should take \( u_{i}^{-1} \rightarrow \prod_{j \neq i} u_{j} \).
In (3.3) the factors $Q, \bar{Q}, P$ can be decomposed into $SU(4)_R \otimes SU(2)_J \otimes SU(2)_J$ characters according to

$$Q(su, x) = \sum_{r=0}^{4} s^r \chi_{(1^r 0^{4-r})}(u) \chi_j(x), \quad \bar{Q}(s^{-1} u, \bar{x}) = \sum_{r=0}^{4} s^r \chi_{(1^r 0^{4-r})}(u) \chi_{\bar{j}}(\bar{x}),$$

$$P(s, x, \bar{x}) = \sum_{r=0}^{\infty} \frac{s^{2r}}{1 - s^4} \chi_{\frac{r}{2}}(x) \chi_{\frac{r}{2}}(\bar{x}), \quad j_0 = j_4 = 0, j_1 = j_3 = \frac{1}{2}, j_2 = 1.$$

(3.11)

There are various possible shortening conditions [20, 19] which fall into essentially three classes for unitary representations of $PSU(2, 2|4)$ which we label by $t, \bar{t}$, the fraction of the $Q, \bar{Q}$ supercharges which are eliminated from the Verma module in the generic case. For the semi-short conditions from [19] we have

$$\left(Q^2 - \frac{1}{2j+1} J_- Q^1 \right) |\Delta; k, p, q; j, \bar{j}\rangle^{h.w.} = 0, \quad \Delta = 2 + 2j + \frac{1}{2}(3k + 2p + q), \quad t = \frac{1}{8},$$

$$\left(\bar{Q}_{42} + \frac{1}{2\bar{j}} \bar{J}_{-}\bar{Q}_{44} \right) |\Delta; k, p, q; j, \bar{j}\rangle^{h.w.} = 0, \quad \Delta = 2 + 2\bar{j} + \frac{1}{2}(k + 2p + 3q), \quad \bar{t} = \frac{1}{8}.$$  

(3.12)

The conditions (3.12) may be extended to $Q^i_2$ for $i = 1, 2$ if $k = 0$, $i = 1, 2, 3$ if $k = p = 0$ and conversely to $\bar{Q}_{i1}$ for $i = 3, 4$ if $q = 0$ and $i = 2, 3, 4$ if $p = q = 0$. For short multiplets there are two cases given by

$$Q^1_{\alpha} |\Delta; k, p, q; 0, \bar{j}\rangle^{h.w.} = 0, \quad \Delta = \frac{1}{2}(3k + 2p + q), \quad t = \frac{1}{4},$$

$$\bar{Q}_{4\bar{\alpha}} |\Delta; k, p, q; 0, j\rangle^{h.w.} = 0, \quad \Delta = \frac{1}{2}(k + 2p + 3q), \quad \bar{t} = \frac{1}{4}.$$

(3.13)

and

$$Q^i_{\alpha} |\Delta; 0, p, q; 0, j\rangle^{h.w.} = 0, \quad i = 1, 2, \Delta = \frac{1}{2}(2p + q), \quad t = \frac{1}{2},$$

$$\bar{Q}_{j\bar{\alpha}} |\Delta; k, p, 0; j, 0\rangle^{h.w.} = 0, \quad j = 3, 4, \Delta = \frac{1}{2}(k + 2p), \quad \bar{t} = \frac{1}{2}.$$  

(3.14)

The conditions listed in (3.12) become lower bounds for $\Delta$ for unitary long representations where $\Delta$ is not determined in terms of other parameters. There are also conditions for which both $t, \bar{t}$ are non zero, for $t = \bar{t} = \frac{1}{8}$ then $j - \bar{j} = \frac{1}{2}(q - k)$, for $t = \bar{t} = \frac{1}{4}$ and $t = \bar{t} = \frac{1}{2}$ then $j = \bar{j} = 0$ with $k = q$ and $k = q = 0$ respectively. We may also impose conditions which involve $Q^i_{\alpha}$ and $\bar{Q}_{j\bar{\alpha}}$ for some $\alpha, \bar{\alpha}$ and $i = j$ but since $\{Q^i_{\alpha}; \bar{Q}_{j\bar{\alpha}}\} = 2\delta^i_j P_{\alpha\bar{\alpha}}$ there are then constraints on the highest weight state involving various components of the momentum operator as well. The unitary cases of interest in this paper, which cover all gauge invariant operators, are given by $(t, \bar{t}) = (0, 0), (\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, 0), (\frac{1}{8}, \frac{1}{8}), (\frac{1}{8}, \frac{1}{8})$ along with conjugate representations $(0, \frac{1}{4}), (0, \frac{1}{8}), (\frac{1}{8}, \frac{1}{4})$. In terms of the notation
in [19] for various supermultiplets we have for the significant cases:

\begin{align*}
\text{long:} & \quad t = \bar{t} = 0, \quad A_{[k,p,q](j,j)}^\Delta, \\
\text{short:} & \quad t = \bar{t} = \frac{1}{4}, \quad B_{[q,p,q](0,0)}^{\frac{1}{2}, \frac{3}{2}}, \quad t = \bar{t} = \frac{1}{2}, \quad B_{[0,p,0](0,0)}^{\frac{1}{2}, \frac{3}{2}}, \quad (3.15) \\
\text{semi-short:} & \quad t = \bar{t} = \frac{1}{8}, \quad C_{[k,p,q](j,j)}^{\frac{1}{4}, \frac{3}{4}}, \quad C_{[0,p,0](j,j)}^{\frac{3}{4}, \frac{5}{4}}, \quad C_{[0,0,0](j,j)}^{1,1},
\end{align*}

where \( C^{\frac{1}{4}, \frac{3}{4}} \) and \( C^{1,1} \) are semi-short multiplets with \( Q^i_2 \sim 0, Q_k \sim 0 \), as in (3.12), with \( i = 1, 2, k = 3, 4 \) and \( i, k = 1, 2, 3, 4 \) respectively. Other conditions than (3.12), (3.13) and (3.14) are possible but they are not consistent with unitary representations.\(^4\) They are related to the above by elements of the Weyl group, corresponding to the full superconformal group, which do not belong to the Weyl group for the compact subgroup used in our construction.

The prescription for obtaining the characters for unitary irreducible representations of the \( \mathcal{N} = 4 \) superconformal group corresponding to the various short and semi-short supermultiplets is an extension of that used for the conformal group in [18]. For a highest weight state satisfying the conditions in (3.12), (3.13) or (3.14) a reduced Verma module \( \mathcal{V}^{t,\bar{t}} \subset \mathcal{V}_{(\Delta,k,p,q;j,j)} \) may be constructed, similarly to (3.2), by the action of \( \{J_-, \bar{J}_-, R^j_i\} \), \( 1 \leq j < i \leq 4 \), along with an appropriate subset of \( \{\tilde{P}_\alpha, Q^i_\beta, \tilde{Q}_{i\bar{\beta}}\} \), \( \alpha, \beta, \bar{\alpha}, \bar{\beta} = 1, 2 \). As a consequence of (3.12) the contribution of the supercharges \( Q^1_2 \) or \( Q_{41} \) are expressible in terms of other contributions present in the Verma module so that these operators may be removed from the basis in (3.2). For (3.13) or (3.14) the supercharges \( Q^i_\alpha \) and/or \( \tilde{Q}_{j\bar{\alpha}} \) are removed either for \( i = 1 \) or \( i = 1, 2 \) and/or \( j = 4 \) or \( j = 3, 4 \). The character for the reduced Verma module \( \mathcal{V}^{t,\bar{t}} \) is then directly obtained and the character of the corresponding unitary irreducible representation is then given by the action of the Weyl symmetriser \( \mathcal{W}^2_{\mathfrak{S}_2} \mathcal{W}^2_{\mathfrak{S}_2} \mathcal{W}^3_{\mathfrak{S}_2} \mathcal{W}^3_{\mathfrak{S}_2} \) associated with the maximal compact subgroup \( SU(4)_R \otimes SU(2)_j \otimes SU(2)_{\bar{j}} \) on the character for the Verma module for \( \mathcal{V}^{t,\bar{t}} \). As in the usual Weyl character formula discussed in section 2 this corresponds to a trace over the space \( \mathcal{V}^{t,\bar{t}}_{(\Delta,k,p,q;j,j)} \) on which unitary representations are defined.

Previously [19] those \( SU(4)_R \otimes SU(2)_j \otimes SU(2)_{\bar{j}} \) representations present in the full short or semi-short supermultiplet were determined by using the Racah-Speiser algorithm for considering the tensor products of the representations carried by the various supercharges and that carried by the highest weight state where shortening conditions were applied to remove particular sets of supercharges in accord with (3.12), (3.13) and (3.14).

\(^4\) For those cases where only one condition was imposed there are also \((\frac{1}{8}, 0) \leftrightarrow C^{\frac{1}{4}, 0}, (0, \frac{1}{3}) \leftrightarrow C^{0, \frac{3}{4}}, (0, \frac{1}{3}) \leftrightarrow C^{0, \frac{3}{4}} \) (which are sometimes referred to as \( \frac{1}{12} \) BPS), and also \((\frac{1}{4}, 0) \leftrightarrow B^{\frac{1}{2}, 0}, (0, \frac{1}{2}) \leftrightarrow B^{0, \frac{1}{2}} \). Multiplets which were both short and semi-short are denoted by \((\frac{1}{4}, \frac{1}{8}) \leftrightarrow D^{\frac{1}{4}, \frac{3}{4}}, (\frac{1}{8}, \frac{1}{4}) \leftrightarrow D^{\frac{3}{4}, \frac{1}{4}} \).

\(^5\) Non unitary representation were considered in [21][22].
The present approach gives equivalent results in terms of characters, which of course is as expected since the Racah-Speiser algorithm may be proved using Weyl symmetrisers acting on Verma module characters. Without attempting a more rigorous proof along the lines of [18], we endeavour to show in the next section and later that the procedure yields characters which agree with the analysis in [19] and also with other results in the literature.

The expressions for characters found by the prescription described may also be derived using similar techniques on analytic superspace [23] on which all positive energy irreducible representations are carried by unconstrained superfields [24] thus providing some simplification. Consequently the problem is to some extent reduced to that of finding \( Gl(2|2) \) characters as extended Schur polynomials, which were written out in an expanded form in [23].

For various short and semi-short multiplets the character for the associated reduced Verma module \( V^{t,i} \) is thus obtained by restricting some \( n_{\bar{e}j}, n_{ie}, \bar{n}_{i\bar{e}} \) in (3.10) to be zero depending on which of \( P_{\alpha\alpha}, Q^t_\alpha, \bar{Q}_{i\bar{a}} \) are to be omitted from the operators generating \( V(\Delta, k, p, q; j, \bar{j}) \). For example from (3.12) either \( n_{12} = 0 \) or \( \bar{n}_{41} = 0 \). This prescription then requires that all such unitary multiplets of \( PSU(2, 2|4) \) have character expressible in the form

\[
\chi^{t,i}_{(\Delta, k, p, q; j, \bar{j})}(s; u; x, \bar{x}) = \text{tr}_{V^{t,i}_{(\Delta, k, p, q; j, \bar{j})}} \left( s^{2D} u_1 H_1 + H_2 + H_3 u_2 H_2 + H_3 u_3 H_3 x^2 J_3 \bar{x}^2 J_3 \right) = s^{2\Delta} \bar{m}^2 \bar{S}_4 \bar{m} \bar{S}_2 \left( C_{(\ell_1, \ell_2, \ell_3, \ell_4)}(u) C_{j}(x) C_{j}(\bar{x}) \frac{Q(s, x) \bar{Q}(s^{-1} u, \bar{x})}{Q_{t}(s, x) \bar{Q}_{t}(s^{-1} u, \bar{x})} \right),
\]

for appropriate \( \Delta, k, p, q, j, \bar{j} \) compatible with (3.12), (3.13) and (3.14) where

\[
Q_t(u, x) = \begin{cases} 
1, & \text{if } t = 0, \\
(1 + u_1 x^{-1}), & \text{if } t = \frac{1}{4}, \\
(1 + u_1 x)(1 + u_1 x^{-1}), & \text{if } t = \frac{1}{2}, \\
(1 + u_1 x)(1 + u_1 x^{-1})(1 + u_2 x)(1 + u_2 x^{-1}), & \text{if } t = \frac{3}{4},
\end{cases}
\]

and

\[
\bar{Q}_{\bar{t}}(u, \bar{x}) = \begin{cases} 
1, & \text{if } \bar{t} = 0, \\
(1 + u_4 \bar{x}^{-1}), & \text{if } \bar{t} = \frac{1}{8}, \\
(1 + u_4 \bar{x})(1 + u_4 \bar{x}^{-1}), & \text{if } \bar{t} = \frac{1}{4}, \\
(1 + u_4 \bar{x})(1 + u_4 \bar{x}^{-1})(1 + u_3^{-1} \bar{x})(1 + u_3^{-1} \bar{x}^{-1}), & \text{if } \bar{t} = \frac{3}{8},
\end{cases}
\]

Trivially of course \( \chi^{0,0}_{(\Delta, k, p, q; j, \bar{j})}(s, u, x, \bar{x}) = \chi^{(\Delta, k, p, q; j, \bar{j})}(s, u, x, \bar{x}) \) as in (3.9).

In the following it is sometimes convenient to allow for more general \( \Delta, \ell_1, \ell_2, \ell_3, \ell_4, j, \bar{j} \) in most character formulae so as to reveal simplifications but we stress that for characters
corresponding to positive energy unitary representations then we must restrict the values appropriately.

Using the character formulae allows an easy derivation of various identities for semi-short and short supermultiplets found earlier \[19\]. Since \( C_{(\ell_1, \ell_2, \ell_3, \ell_4)}(u) = u_1 C_{(\ell_1, \ell_2, \ell_3, \ell_4)}(u) \) and \( C_{j-\frac{1}{2}}(x) = x^{-1} C_j(x) \) it is trivial to see from (3.16) and (3.17) that

\[
\chi_{(\Delta, k, p, q; j, \bar{j})}^0, \bar{\ell} (s; u; x, \bar{x}) = \chi_{(\Delta, k, p, q; j, \bar{j})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x}) + \chi_{(\Delta + \frac{1}{2}; k, p, q; j, \bar{j} - \frac{1}{2})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x}),
\]

(3.19)
since the Weyl symmetriser is linear. Similarly

\[
\chi_{(\Delta, k, p, q; j, \bar{j})}^{t, 0} (s; u; x, \bar{x}) = \chi_{(\Delta, k, p, q; j, \bar{j})}^{t, \frac{1}{2}} (s; u; x, \bar{x}) + \chi_{(\Delta + \frac{1}{2}; k, p, q; j, \bar{j} - \frac{1}{2})}^{t, \frac{1}{2}} (s; u; x, \bar{x}).
\]

(3.20)
For a long multiplet at the unitarity threshold \( \Delta = 2 + k + p + q + j + \bar{j} \) and also \( k - q = 2(\bar{j} - j) \) these results may be combined to give a decomposition of the character in terms of four \( \chi_{\frac{1}{2} \bar{\ell}, \frac{1}{2} \ell} \) characters,

\[
\chi_{(\Delta, k, p, q; j, \bar{j})} (s; u; x, \bar{x}) = \chi_{(\Delta, k, p, q; j, \bar{j})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x})
+ \chi_{(\Delta + \frac{1}{2}; k, p, q; j, \bar{j} - \frac{1}{2})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x}) + \chi_{(\Delta + \frac{1}{2}; k, p, q; j, \bar{j} + \frac{1}{2})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x})
+ \chi_{(\Delta + \frac{1}{2}; k, p, q; j, \bar{j} - \frac{1}{2})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x}).
\]

(3.21)

From (3.16) and (3.17) we may also write

\[
\chi_{(\Delta, k, p, q; -\frac{1}{2}, \bar{j})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x}) = s^{2\Delta} P(s, x, \bar{x})
\times \mathbb{W}^2_{q \mathbb{W}^2_{S_2} \mathbb{W}^S_{2}} \left\{ C_{(\ell_1, \ell_2, \ell_3, \ell_4)}(u)(1 + su_1 x) C_{\frac{1}{2}}(x) C_{\frac{1}{2}}(x) \mathcal{Q}(s u, x) \mathcal{Q}(s^{-1} u, \bar{x}) \mathcal{Q}(s u, x) \mathcal{Q}(s^{-1} u, \bar{x}) \right\},
\]

(3.22)
since \( P(s, x, \bar{x}) \) is invariant under \( \mathbb{W}^2_{q \mathbb{W}^2_{S_2} \mathbb{W}^S_{2}} \). The expressions involving \( \mathcal{Q}(s u, x) \) and \( \mathcal{Q}(s u, x) \) are also invariant under \( \mathbb{W}^S_{2} \mathbb{W}^S_{2} \mathcal{C}_{\frac{1}{2}}(x) = 0 \). Further the term \( su_1 x \) may be absorbed by \( \ell_1 \rightarrow \ell_1 + 1 \) and \( x C_{\frac{1}{2}}(x) = C_0(x) \) giving then

\[
\chi_{(\Delta, k, p, q; -\frac{1}{2}, \bar{j})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x}) = \chi_{(\Delta + \frac{1}{2}; k, p, q; 0, \bar{j})}^{\frac{1}{2}, \bar{\ell}} (s; u; x, \bar{x}).
\]

(3.23)
Following a similar argument

\[
\chi_{(\Delta, k, p, q; j, -\frac{1}{2})}^{t, \frac{1}{2}} (s; u; x, \bar{x}) = \chi_{(\Delta + \frac{1}{2}; k, p, q; 0, \bar{j})}^{t, \frac{1}{2}} (s; u; x, \bar{x}),
\]

(3.24)
and hence

\[
\chi_{(p + 2q + 1; p, q; -\frac{1}{2}, -\frac{1}{2})}^{\frac{1}{2}, \frac{1}{2}} (s; u; x, \bar{x}) = \chi_{(p + 2q + 2; p, q + 1; 0, 0)}^{\frac{1}{2}, \frac{1}{2}} (s; u; x, \bar{x}).
\]

(3.25)
Hence we may identify
\[ C_{\frac{1}{4}; \frac{3}{4}}^{\frac{1}{4}; \frac{3}{4}}(\frac{1}{4}; \frac{3}{4}) \simeq B_{\frac{1}{4}; \frac{3}{4}}^{\frac{1}{4}; \frac{3}{4}}(0,0) \cdot (3.26) \]

The results (3.23) and (3.24) may be used in (3.19) and (3.20) when \( j = 0 \) and \( \bar{j} = 0 \) respectively. In general they show how \( t, \bar{t} = \frac{1}{4} \) characters are a special case of those for \( t, \bar{t} = \frac{1}{8} \).

As described in [19] the result (3.21) is a consequence of the representation for a long multiplet, at unitarity threshold, containing four superconformal primary operators from which \( (\frac{1}{8}, \frac{1}{8}) \) semi-short multiplets can be constructed. This may be described by the diagrams

\[
\begin{align*}
\mathcal{O}_{\frac{3}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{1}{8}}(k, p, q) & \quad \mathcal{O}_{\frac{1}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{3}{8}}(0,0) \\
\mathcal{O}_{\frac{3}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{3}{8}}(k+1, p, q) & \quad \mathcal{O}_{\frac{1}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{3}{8}}(0,0) \\
\mathcal{O}_{\frac{3}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{3}{8}}(k+1, p, q+1) & \quad \mathcal{O}_{\frac{1}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{3}{8}}(0,0) \\
\mathcal{O}_{\frac{3}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{3}{8}}(k+1, p, q+1) & \quad \mathcal{O}_{\frac{1}{8}; \frac{3}{8}}^{\frac{1}{8}; \frac{3}{8}}(0,0)
\end{align*}
\]

where \( \sqcup \) in the first case represents the action of the \( Q_{12} \) supercharge and \( \sqcap \) that of the \( \bar{Q}_{11} \) supercharge, in the second they represent \( \epsilon^{\alpha \beta} Q_{\alpha} Q_{\beta} \) and \( \epsilon^{\dot{\alpha} \dot{\beta}} \bar{Q}_{\dot{\alpha}} \bar{Q}_{\dot{\beta}} \), and we have ignored other descendant operators. The \( j, \bar{j} = 0 \) case may be obtained by interpreting operators with \( j, \bar{j} = -\frac{1}{2} \) as \( t, \bar{t} = \frac{1}{4} \)-BPS operators, in accord with the results (3.23, 3.24) and (3.25).

For later discussion it is convenient to adapt the treatment of the semi-short conditions shown in (3.12), following [37], so that the conditions are just \( \delta = \bar{\delta} = 0 \) where
\[
\delta = \Delta - 2j - \frac{3}{2}(3k + 2p + q) , \quad \bar{\delta} = \Delta - 2\bar{j} - \frac{3}{2}(k + 2p + 3q) . \tag{3.28}
\]

For any state satisfying (3.1) we define a corresponding state
\[
|\Delta + \frac{1}{2}; k + 1, p, q; j + \frac{1}{2}, \bar{j}\rangle_{\text{h.w.}} = Q_{1}^{1} |\Delta; k, p, q; j, \bar{j}\rangle_{\text{h.w.}} , \tag{3.29}
\]
which is easily seen to satisfy the same conditions as in (3.1) \( (S_{1}^{a} |\Delta; k, p, q; j, \bar{j}\rangle_{\text{h.w.}} = 0 \) for \( i = 2, 3, 4 \) follows from \( R_{1}^{1} |\Delta; k, p, q; j, \bar{j}\rangle_{\text{h.w.}} = 0 \) for \( i = 2, 3, 4 \), similarly for \( S_{1}^{2} \), except that \( S_{1}^{1} \) is omitted and instead
\[
Q_{1}^{1} |\Delta; k, p, q; j, \bar{j}\rangle_{\text{h.w.}} = 0 . \tag{3.30}
\]

From (3.29) we may easily find an inverse relation,
\[
S_{1}^{1} |\Delta; k, p, q; j, \bar{j}\rangle_{\text{h.w.}} = 2(4j + \delta) |\Delta - \frac{1}{2}; k - 1, p, q; j - \frac{1}{2}, \bar{j}\rangle_{\text{h.w.}} . \tag{3.31}
\]
For this state we may then impose
\[ Q_{12}^1|\Delta; k, p, q; j, \bar{j}\rangle^{\text{h.w.}} = 0 \Rightarrow \Delta = 2j + \frac{1}{2}(3k + 2p + q), \] (3.32)
which, by considering \( S_1^1 Q_{12}^1 Q_{11}^1 |\Delta; k, p, q; j, \bar{j}\rangle^{\text{h.w.}} = 0 \) and using \( \{S_1^1, Q_{12}^1\} = 4J_-, \{S_1^1, Q_{11}^1\} = 4J_3 + 2D - 3H_1 - 2H_2 - H_3 \), is equivalent to the \( t = \frac{1}{8} \) condition in (3.12). Hence for analysing \( t = \frac{1}{8} \) semi-short multiplets it is sufficient to consider states satisfying \( \delta = 0 \). If \( j = \bar{j} = 0 \) then (3.31) shows that \( |\Delta; k, p, q; 0, 0\rangle^{\text{h.w.}} \) becomes a highest weight state. For \( \bar{t} = \frac{1}{8} \) a similar analysis is possible while in the \( t = \bar{t} = \frac{1}{8} \) case we may consider
\[ |\Delta + 1; k + 1, p, q + 1; j + \frac{1}{2}, \bar{j} + \frac{1}{2}\rangle^{\text{h.w.}} = \bar{Q}_{42} Q_{11}^1 |\Delta; k, p, q; j, \bar{j}\rangle^{\text{h.w.}}, \] (3.33)
and we omit \( S_1^1, \bar{S}^{42} \) from the conditions in (3.1) and they are replaced by \( Q_{11}^1, \bar{Q}_{42} \). Imposing \( Q_{12}^1 |\Delta; k, p, q; j, \bar{j}\rangle^{\text{h.w.}} = \bar{Q}_{41} |\Delta; k, p, q; j, \bar{j}\rangle^{\text{h.w.}} = 0 \), which requires \( \delta = \bar{\delta} = 0 \) or \( \Delta = j + \bar{j} + k + p + q, j - \bar{j} = \frac{1}{2}(q - k) \), is then equivalent to the combined \( t = \bar{t} = \frac{1}{8} \) conditions in (3.12). For \( j = \bar{j} = 0, k = q \) then \( |\Delta; q, p, q; 0, 0\rangle^{\text{h.w.}} \) becomes a \( t = \bar{t} = \frac{1}{4} \) highest weight state. If we denote \( \bar{O}'_{\frac{r}{s}} \) as the operator corresponding to these alternative highest weight conditions, with \( \Delta = j + \bar{j} + k + p + q, j - \bar{j} = \frac{1}{2}(q - k) \), then instead of (3.27) we have an equivalent diagram,
\[
\begin{array}{c}
\bar{O}'_{\frac{r}{s}} \downarrow \quad \bar{O}'_{\frac{r}{s}} \\
\uparrow \quad \uparrow \downarrow \quad \uparrow \\
\bar{O}'_{\frac{r}{s}} \downarrow \quad \bar{O}'_{\frac{r}{s}} \\
\uparrow \quad \uparrow \\
\bar{O}'_{\frac{r}{s}} \\
\end{array}
\] (3.34)
where \( \bar{O}'_{[k, p, q](0, \bar{j})} = \bar{O}'_{[k, p, q](0, j)} \), similarly \( \bar{O}'_{\frac{r}{s}} = \bar{O}'_{\frac{r}{s}} \) for \( j = 0 \), and \( \bar{O}'_{[q, p, q](0, 0)} = \bar{O}'_{[q, p, q](0, 0)} \).

4. Characters in Special Cases

In this section we consider in more detail characters for the various short and semi-short multiplets discussed in the previous section and attempt, where possible, to simplify them and to write them in a more explicit fashion, allowing particular cases to be considered subsequently. This then allows us to determine ‘blind’ characters (such as where the variables \( u_i, x, \bar{x} = 1 \) so that there is then a dependence just on \( s \)). With these results we are
able to greatly simplify dimension formula for multiplets given in \( \text{(3.15)} \), \( C^{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}[0, 0, 0](j, j) \approx C^{\frac{1}{4}, \frac{1}{4}}[0, p, 0](j, j) \) and \( C^{\frac{1}{4}, \frac{1}{4}}[0, 0, 0](j, j) \approx C^{1, 1}[0, 0, 0](j, j) \).

**Basic Supermultiplets**

Our results may be illustrated first by considering the multiplet for the fundamental fields in \( \mathcal{N} = 4 \) superconformal theories which corresponds to \( B^{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}[0, 1, 0](0, 0) \). The character is

\[
\begin{align*}
\chi^{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}_{(1;0,1,0,0,0)}(s; u, x, \bar{x}) & = D_0(s, x, \bar{x}) \chi_{(1,1,0,0)}(u) \\
+ D_{\frac{1}{2}}(s, x, \bar{x}) \chi_{(1,1,1,0)}(u) + \overline{D}_{\frac{1}{2}}(s, x, \bar{x}) \chi_{(1,0,0,0)}(u) \\
+ D_1(s, x, \bar{x}) + \overline{D}_1(s, x, \bar{x}) ,
\end{align*}
\]

where

\[
\begin{align*}
D_j(s, x, \bar{x}) & = s^{2j+2}(\chi_j(x) - s^2 \chi_{j-\frac{1}{2}}(x) \chi_{\frac{1}{2}}(x) + s^4 \chi_{j-1}(x)) P(s, x, \bar{x}) , \\
\overline{D}_j(s, x, \bar{x}) & = s^{2j+2}(\chi_j(\bar{x}) - s^2 \chi_{j-\frac{1}{2}}(\bar{x}) \chi_{\frac{1}{2}}(\bar{x}) + s^4 \chi_{j-1}(\bar{x})) P(s, x, \bar{x}) ,
\end{align*}
\]

are the characters corresponding to the spin-\( j \) chiral, respectively, spin-\( \bar{j} \) anti-chiral free field representations of the conformal group in 4 dimensions \( \mathbb{R}^4 \). \( D_0 \) corresponds to a free scalar field of conformal dimension 1, \( D_{\frac{1}{2}} \) to a free spin-\( \frac{1}{2} \) chiral fermion of dimension \( \frac{3}{2} \), \( \overline{D}_{\frac{1}{2}} \) to a free spin-\( \frac{1}{2} \) anti-chiral fermion of dimension \( \frac{3}{2} \) and \( D_1, \overline{D}_1 \) to free chiral and anti-chiral vector fields of conformal dimension 2 so that the combination corresponds to a free Maxwell field. We may also easily determine

\[
\begin{align*}
\chi_{(1,0,0,0)}(u) & = u_1 + u_2 + u_3 + u_4 , \\
\chi_{(1,1,0,0)}(u) & = u_1 - 1 + u_2 - 1 + u_3 - 1 + u_4 - 1 , \\
\chi_{(1,1,1,0)}(u) & = u_1 u_2 + u_1 u_3 + u_1 u_4 + u_2 u_3 + u_2 u_4 + u_3 u_4 ,
\end{align*}
\]

assuming \( \prod_{i=1}^4 u_i = 1 \), corresponding to the 4, \( \overline{4} \) and 6 representations of \( SU(4) \) respectively.

The simplest gauge invariant \( \frac{1}{2} \) BPS multiplet is \( B^{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}[0, 2, 0](0, 0) \), when the chiral primary operators belong to a 20 dimensional representation of \( SU(4)_R \). This supermultiplet contains the energy momentum tensor as well as the \( SU(4)_R \) current. To expand the character we use the characters for a long representation of \( SO(4, 2) \)

\[
A_{\Delta, j, \bar{j}}(s, x, \bar{x}) = s^{2\Delta} \chi_j(x) \chi_{\bar{j}}(\bar{x}) P(s, x, \bar{x}) , \tag{4.4}
\]

with unitarity requiring \( \Delta \geq j + \bar{j} + 2 \), and also that for conserved currents

\[
D_{j, \bar{j}}(s, x, \bar{x}) = s^{2j+2j+4} (\chi_j(x) \chi_{\bar{j}}(\bar{x}) - s^2 \chi_{j-\frac{1}{2}}(x) \chi_{\bar{j}-\frac{1}{2}}(\bar{x})) P(s, x, \bar{x}) . \tag{4.5}
\]
Then we may obtain for the superconformal character
\[
\chi_{(2;0,2,0,0,0)}^{1/2,1/2}(s, u, x, \bar{x}) = A_{2,0,0}(s, x, \bar{x}) \chi_{(2,2,0,0,0)}(u)
\]
\[
+ A_{2,1,0}^{1/2}(s, x, \bar{x}) \chi_{(2,1,1,0)}(u) + A_{2,0,1}^{1/2}(s, x, \bar{x}) \chi_{(2,1,0,0)}(u)
\]
\[
+ A_{3,0,0}(s, x, \bar{x}) \left( \chi_{(2,0,0,0)}(u) + \chi_{(2,2,0,0)}(u) \right)
\]
\[
+ \left( A_{3,1,0}(s, x, \bar{x}) + A_{3,0,1}(s, x, \bar{x}) \right) \chi_{(1,1,0,0)}(u)
\]
\[
+ D_{1/2,1/2}(s, x, \bar{x}) \chi_{(2,1,1,0)}(u)
\]
\[
+ \left( A_{2,1,0}^{1/2}(s, x, \bar{x}) + D_{1/2,1/2}(s, x, \bar{x}) \right) \chi_{(1,1,0,0)}(u)
\]
\[
+ \left( A_{2,0,1}^{1/2}(s, x, \bar{x}) + D_{1/2,1/2}(s, x, \bar{x}) \right) \chi_{(1,0,0,0)}(u).
\]

Characters for short multiplets

More generally we consider initially characters \( t = \bar{t} = \frac{1}{4}, \frac{1}{2} \) requiring \( k = q, j = \bar{j} = 0 \), \( \Delta = p + 2q \), with \( q = 0 \) when \( t = \bar{t} = \frac{1}{2} \). In this case in (3.16) the action of \( \mathfrak{m}^{S_2} \mathfrak{m}^{S_2} \) becomes simple, since all factors are invariant under \( x \leftrightarrow x^{-1}, \bar{x} \leftrightarrow \bar{x}^{-1} \). Hence the general result reduces to
\[
\chi_{(p+2q;p,q;0,0)}^{t,t}(s; u; x, \bar{x}) = s^{2p+4q} P(s, x, \bar{x})
\]
\[
\times \mathfrak{m}^{S_4} \left( C_{(p+2q,p+q,q,0)}(u) \mathcal{P}_n (s, u, x) \bar{\mathcal{P}}_{\bar{n}}(s^{-1}u, \bar{x}) \right),
\]
where
\[
\mathcal{P}_n (u, x) = \prod_{i=N}^{4} \prod_{\varepsilon = \pm 1} (1 + u_i x^\varepsilon), \quad \bar{\mathcal{P}}_{\bar{n}}(u, \bar{x}) = \prod_{j=1}^{\bar{N}} \prod_{\eta = \pm 1} (1 + u_j^{-1} \bar{x}^\eta),
\]
are invariant under \( \mathfrak{m}^{S_2}, \mathfrak{m}^{S_2} \) and
\[
N_\frac{1}{2} = 2, \quad N_\frac{3}{2} = 3, \quad \bar{N}_\frac{1}{2} = 3, \quad \bar{N}_\frac{3}{2} = 2.
\]

For \( t = \frac{1}{2} \) the character may be calculated by expanding the factors \( \mathcal{P}_3(s, u, x) \) and \( \bar{\mathcal{P}}_2(s^{-1}u, \bar{x}) \) in (4.7) giving
\[
\chi_{(p;0,0;0,0)}^{1/2,1/2}(s; u; x, \bar{x})
\]
\[
= s^{2p} P(s, x, \bar{x}) \sum_{a,b,c,d=0}^{2} s^{a+b+c+d} \chi_{j_a}(\bar{x}) \chi_{j_b}(\bar{x}) \chi_{j_c}(x) \chi_{j_d}(x) \chi_{(p-a,p-b,c,d)}(u),
\]
with the notation here
\[
j_0 = j_2 = 0, \quad j_1 = \frac{1}{2}.
\]
For \( x = 1 \), \( \chi_{j_0}(1) = \binom{2}{0}^2 \). With the aid of (2.14) and (2.15) this may be simplified further
\[
\chi_{(p;0,p;0,0,0)}(s; u; x, \bar{x}) = s^{2p} P(s, x, \bar{x}) \sum_{0 \leq a \leq b \leq 2} \sum_{0 \leq c \leq d \leq 2} s^{a+b+c+d} \chi_{j_{ba}}(\bar{x}) \chi_{j_{cd}}(x) \chi_{(p-a,p-b,c,d)}(u),
\]
\[
j_{0} = j_{22} = j_2 = 1, \quad j_{10} = j_{21} = \frac{1}{2}, \quad j_{11} = 1.
\]
Noting that \( \mathcal{P}_3(u, 1) \mathcal{P}_2(u, 1) = (u_1 u_2)^{-2} \prod_{i=1}^{4} (1 + u_i)^2 \) we may also straightforwardly obtain
\[
\chi_{(p;0,p;0,0,0)}(s; u; 1, 1)/P(s, 1, 1)|_{s \rightarrow 1} = \prod_{i=1}^{4} (1 + u_i)^2 \chi_{(p-2,p-2,0,0)}(u),
\]
which immediately gives \( 2^8 d_{[0,p-2,0]} \) for the number of conformal primary operators as in [19].

For the special cases \( p = 1, 2 \) applying (2.14) leads to further reductions and gives the results (4.1) and (4.6). For \( p = 0 \) from (4.12) we get
\[
\chi_{(p;0,0,0,0,0)}(s; u; x, \bar{x}) = P(s, x, \bar{x}) (1 - (s^2 + s^6) \chi_{\frac{1}{2}}(x) \chi_{\frac{1}{2}}(\bar{x}) + s^4 (\chi_{1}(x) + \chi_{1}(\bar{x})))
\]
\[
= 1,
\]
in accord with \( B_{[0,0,0](0,0)}^{\frac{1}{2} \frac{1}{2}} = I \), the trivial representation.

For \( t = \frac{1}{4} \) in (4.17) we may expand \( \mathcal{P}_2(su, x) \mathcal{P}_3(s^{-1} u, \bar{x}) \) in a similar fashion as above to obtain, with the definitions (4.11),
\[
\chi_{(p+2q;0,p;0,0)}(s; u; x, \bar{x}) = s^{2p+4q} P(s, x, \bar{x})
\]
\[
\times \sum_{a,b,c,d,e,f=0}^{2} s^{a+b+c+d+e+f} \chi_{j_{ba}}(\bar{x}) \chi_{j_{cd}}(x) \chi_{j_{ef}}(x) \chi_{(p-a,p-b,c,d)}(u).
\]
Except for \( q = 1 \) there are no additional simplifications as in (4.12). Furthermore using \( \mathcal{P}_2(u, 1) \mathcal{P}_3(u, 1) = (u_1 u_2 u_3)^{-4} (1 + u_2)^2 (1 + u_3)^2 \prod_{i=1}^{4} (1 + u_i)^2 \) we have for \( s \rightarrow 1 \)
\[
\chi_{(p+2q;0,p;0,0)}(s; u; 1, 1)/P(s, 1, 1)|_{s \rightarrow 1}
\]
\[
= \prod_{i=1}^{4} (1 + u_i)^2 \sum_{a,b=0}^{2} \binom{2}{a} \binom{2}{b} \chi_{(p+2q-2,p+2q+a-2)}(u).
\]

\[6\] For \( p = 1 \), the singleton representation, this is zero but the counting of physical degrees of freedom for fields satisfying equations of motion is different in this case.
For \( u_i \to 1 \) this gives \( 2^o_\sum_{a,b} \binom{2}{a} \binom{2}{b} d_{i \{q-a,p+a-b,q+b-2\}} \) for the number of conformal primary operators, which may be shown to agree with the corresponding formula for the \( \frac{1}{4} \)-BPS multiplet given in [19].

When \( q = 0, \chi^{\frac{1}{4}, \frac{1}{4}} \) may be reduced to \( \chi^{\frac{1}{4}, \frac{1}{2}} \) characters. Using in (4.7), from (4.8), \( P_2(u, x)P_3(u, \bar{x}) = \prod_{\varepsilon=\pm 1}(1+u\varepsilon x) \prod_{\eta=\pm 1}(1+u_3^{-1} \bar{x} \eta) P_3(u, x) P_2(u, \bar{x}) \) the action of \( \mathfrak{m}^S_4 \) may be simplified for a subgroup \( S_2 \times S_2 \subset S_4 \), generated by the permutations (12), (34), to

\[
\mathfrak{m}^{S_2 \times S_2}(\prod_{\varepsilon=\pm 1}(1+su_2\varepsilon x)\prod_{\eta=\pm 1}(1+su_3^{-1} \bar{x} \eta) C_{(p,p,0,0)}(u)) = \mathfrak{m}^{S_2 \times S_2}(\sum_{r=0}^{2}(\frac{2}{r})(-1)^r s^{2r} C_{(p+r,p+r,0,0)}(u)),
\]

(4.17)

since other factors in (4.7) are invariant under this \( S_2 \times S_2 \). Hence we obtain

\[
\chi^{\frac{1}{4}, \frac{1}{4}}(p; 0, p, 0, 0, 0) (s; u; x, \bar{x}) = \sum_{r=0}^{2}(\frac{2}{r})(-1)^r \chi^{\frac{1}{4}, \frac{1}{2}}(p+r, p+r, 0, 0, 0) (s; u; x, \bar{x}),
\]

(4.18)

with a corresponding decomposition for \( B^{\frac{1}{4}, \frac{1}{4}}_{[0, p, 0]}(0, 0) \).

For the case \( t = \frac{1}{4}, \bar{t} = 0, j = 0 \) (4.7) is modified to

\[
\chi^{\frac{1}{4}, 0}(\frac{1}{4}(3k+2p+q); k, p, q, 0, j) (s; u; x, \bar{x}) = s^{3k+2+p+q} P(s, x, \bar{x}) \bar{Q}(s^{-1}u, \bar{x}) \chi_j(\bar{x})
\]

\[
\times \mathfrak{m}^{S_2}(C(k+p+q, p+q, q, 0)P_2(s, u, x)).
\]

(4.19)

As before we may obtain an expansion

\[
\chi^{\frac{1}{4}, 0}(\frac{1}{4}(3k+2p+q); k, p, q, 0, j) (s; u; x, \bar{x})
\]

\[
= s^{3k+2+p+q} P(s, x, \bar{x}) \bar{Q}(s^{-1}u, \bar{x}) \chi_j(\bar{x}) \sum_{a,b,c=0}^{2} s^{a+b+c} \chi_{j_a}(x) \chi_{j_b}(x) \chi_{j_c}(x)
\]

\[
\times \chi(k+p+q, p+q+a, q+b, c)(u).
\]

(4.20)

The dimension of conformal primary states for the \( (\frac{1}{4}, 0) \) multiplet is obtained by factoring \( P(s, 1, 1) \) out of the last expression and then setting \( s = 1, u_i = 1 \), using (3.10). The result agrees with a comparable result in [19].

The forms exhibited in (4.20), (4.15) and (4.12), along with (3.10), allow for easy determination of the ‘blind’ partition functions when \( u_i, x, \bar{x} = 1 \). For the \( t = \frac{1}{4} \) case we find from (4.20) that

\[
\chi^{\frac{1}{4}, 0}(\frac{1}{4}(3k+2p+q); k, p, q, 0, j) (s; 1, 1, 1, 1; 1, 1) = \frac{1}{12} s^{3k+2+p+q} \frac{(1 + s)^7}{(1 - s)^4} (2j+1)(p+1)(q+1)(p+q+2)
\]

\[
\times \left( (k+1)(k+p+2)(k+p+q+3)
\right.
\]

\[
+ (5(k+1)(k+p+2)(k+p+q+3) - 2(k+2)(k+p+3)(k+p+q+4)) s
\]

\[
+ (5(k-1)(k+p)(k+p+q+1) - 2(k-2)(k+p-1)(k+p+q)) s^2
\]

\[
+ (k-1)(k+p)(k+p+q+1) s^4 \right).
\]

(4.21)
from (4.12) we may find directly for the \( \frac{1}{4}\)-BPS multiplet that

\[
\chi_{(p+2; q, p, q, 0, 0)}(s; 1, 1, 1, 1; 1, 1) = \frac{1}{12} s^{2p+4q} \left(\frac{1+s}{1-s}\right)^3 (p+1) \\
\times \left((q+1)(p+q+2) + 2(q(p+q+1) - 2s + (q-1)(p+q)s^2)\right) \\
\times \left((q+1)(p+q+2)(p+2q+3) + \frac{1}{5}(2(q+2)(3p+3q+1)(3p+6q+5) + (q-3)(3p+3q+10)(3p+6q+5) - 140)s \\
+ \frac{1}{5}(2(q-2)(3p+3q+5)(3p+6q+1) + (q+3)(3p+3q-4)(3p+6q+1) + 140)s^2 \\
+ (q-1)(p+q)(p+2q-1)s^3\right),
\]

(4.22)

and, finally, we have for the \( \frac{1}{2}\)-BPS multiplet from (4.12) that

\[
\chi_{(p; 0, p, 0, 0)}(s; 1, 1, 1, 1; 1, 1) = \frac{1}{12} s^{2p} \left(\frac{1+s}{1-s}\right)^4 (p+2 + (p-2)s) \\
\times \left((p+1)(p+2)(p+3) + 3(p-2)(p+1)(p+3)s \\
+ 3(p-3)(p-1)(p+2)s^2 + (p-3)(p-2)(p-1)s^3\right).
\]

(4.23)

For \( p = 1 \), \( \chi_{(\frac{1}{2}; 0, 1, 0, 0)}(s; 1, 1, 1, 1; 1, 1) = 2s^2(3 - s)/(1 - s)^3 \) which is the standard result for the fundamental multiplet.

**Characters for semi-short multiplets**

Corresponding to the \( (\frac{1}{4}, 0) \) case for which \( \Delta = 2 + 2j + \frac{1}{2}(3k + 2p + q) \) then in (3.16) we may obtain an expansion in terms of \( (\frac{1}{4}, 0) \) characters by reducing the dependence on \( u \) to expressions of the form (4.13)

\[
\chi_{(\Delta, k, p, q, 0; j)}(s; u; x, \bar{x}) \\
= s^{2\Delta} P(s, x, \bar{x}) Q(s^{-1} u, \bar{x}) \\
\times \Omega_s \Omega_{s_{2}} \Omega_{s_{3}} ((1 + su_1 x) C_{(k+p+q,p+q,0)}(u) C_j(x) C_{\bar{j}}(\bar{x}) P_2(su, x)) \\
= \chi_{(\Delta, k, p, q, 0; 0)}(s; u; x, \bar{x}) \chi_j(x) + \chi_{(\Delta + \frac{1}{2}, k+1, p, q, 0; j)}(s; u; x, \bar{x}) \chi_{j+1}(x),
\]

(4.24)

so that previous results (4.20) for \( \chi_{(\Delta, k, p, q, 0; j)} \) may be used although with different \( \Delta \). Setting \( j = -\frac{1}{2} \) gives (3.23) for \( \bar{t} = 0 \).

For the \( (\frac{1}{8}, \frac{1}{8}) \) semi-short multiplet with \( \Delta = 2 + j + k + p + q \) and \( k - q = 2(j - j) \) then a similar expansion may be obtained from (3.16) in terms of \( (\frac{1}{4}, \frac{1}{4}) \) characters for more general \( \Delta, k, p, q, \).
\[ \chi_{(\Delta, k, p, q; j, j)}(s; u; x, \bar{x}) = \chi_{(\Delta, k, p, q; 0, 0)}(s; u; x, \bar{x})\chi_j(x)\chi_j(\bar{x}) + \chi_{(\Delta + \frac{1}{2}, k + 1, p, q; 0, 0)}(s; u; x, \bar{x})\chi_{j + \frac{1}{2}}(x)\chi_j(\bar{x}) + \chi_{(\Delta + \frac{1}{2}, k + 1, p, q + 1; 0, 0)}(s; u; x, \bar{x})\chi_j(x)\chi_{j + \frac{1}{2}}(\bar{x}) + \chi_{(\Delta + 1, k + 1, p, q + 1; 0, 0)}(s; u; x, \bar{x})\chi_{j + \frac{1}{2}}(x)\chi_{j + \frac{1}{2}}(\bar{x}). \] (4.25)

For \( j = \bar{j} = -\frac{1}{2} \) in (4.25) we simply obtain (3.25).

Other cases considered in [19] are for the \( C^{1,\frac{1}{2}} \), \( C^{1,1} \) semi-short multiplets, which we now show reduce to the \( (\frac{1}{8}, \frac{1}{2}) \) case above.

For the \( C^{1,\frac{1}{2}} \) semi-short multiplet with \( \Delta = 2 + 2j + p, \bar{j} = \bar{j} \) and \( k, q = 0 \) then \( Q^1_2, Q^2_2, \bar{Q}_3, \bar{Q}_4 \) are missing from the full Verma module (3.2) so that the character formula becomes

\[ \chi_{(\Delta, 0, p, 0; j, j)}(s; u; x, \bar{x}) = s^{2\Delta}P(s, x, \bar{x}) W^{S_2 \times S_2} \left( C_j(x)C_j(\bar{x})\omega^{S_2}(X_p(s; u; x, \bar{x})) \right), \]

\[ X_p(s; u; x, \bar{x}) = C_{(p, p, 0, 0)}(u) \prod_{i=1}^2 (1 + su_i x_i) \prod_{k=3}^4 (1 + su^{-1}_k x^{-1}) P_3(s, x)\bar{P}_2(s^{-1}u, \bar{x}). \] (4.26)

The corresponding expression for \( \chi_{(\Delta, 0, p, 0; j, j)}(s; u; x, \bar{x}) \) obtained from (3.16) is identical with (3.24) except that it involves

\[ W^{S_2}(1 + su_2 x^{-1})(1 + su_3^{-1}x^{-1})X_p(s; u; x, \bar{x}). \] (4.27)

Nevertheless this leads to the same result as (1.26) as a consequence of

\[ W^{S_2 \times S_2}(1 + su_2 x^{-1})(1 + su_3^{-1}x^{-1})C_{(p, p, 0, 0)}(u) = W^{S_2 \times S_2}(C_{(p, p, 0, 0)}(u)) = \frac{(u_1 u_2)^{p+2}}{(u_1 - u_3)(u_1 - u_4)(u_2 - u_3)(u_2 - u_4)}. \] (4.28)

for \( S_2 \times S_2 \) defined in the same fashion as in (1.17). This then implies

\[ \chi_{(2+2j+p, 0, p, 0; j, j)}(s; u; x, \bar{x}) = C^{1,\frac{1}{2}}_{[0, p, 0; (j, j)]}(s; u; x, \bar{x}), \] (4.29)

demonstrating \( C^{1,\frac{1}{2}}_{[0, p, 0; (j, j)]} \simeq C^{1,\frac{1}{2}}_{[0, p, 0; (j, j)]} \), as was also noted in [13].

For the \( C^{1,1} \) multiplet, for which \( \Delta = 2 + 2j \) and \( k, p, q = 0 \), then \( Q^i_2, \bar{Q}_k \) for all \( i, k \) are removed from the operators generating the Verma module and hence we must also
drop the contribution of $P_{21}$ arising from their anticommutator in (3.2). Thus (3.16) must be modified in this case and takes the form

$$\chi^{c,1}_{(2+2j;0,0,0;0,j,j)}(s;u;x,\bar{x}) = s^{4+4j} P(s, x, \bar{x}) \mathfrak{M}_2 \mathfrak{M}_2 S_2 \left( \prod_{i=1}^{4} (1 + su_i x) (1 + su_i^{-1} \bar{x}) (1 - s^2 x^{-1} \bar{x}^{-1}) C_j(x) C_j(\bar{x}) \right).$$

(4.30)

Expanding in powers of $x, \bar{x}$ and using the definitions (4.3) and (4.5) we easily obtain the succinct form

$$\chi^{c,1}_{(2+2j;0,0,0;0,j,j)}(s;u;x,\bar{x}) = \sum_{0 \leq m,n \leq 4} D_{j+\frac{1}{2} m,j+\frac{1}{2} n}(s, x, \bar{x}) \chi(1^{m} 0^{4-m})(u) \chi(1^{n} 0^{4-n})(u),$$

(4.31)

which involves solely contributions corresponding to conserved currents. As a particular limit from (4.30) we have

$$\chi^{c,1}_{(2+2j;0,0,0;0,j,j)}(1;1,1,1;1;x,\bar{x}) = P(1, x, \bar{x})(1 + x) (1 + x^{-1}) (1 + x^{-1}) (1 + \bar{x}^{-1}) \times (\chi_{j+1}(x) \chi_{j+1}(\bar{x}) - \chi_{j+\frac{1}{2}}(x) \chi_{j+\frac{1}{2}}(\bar{x})).$$

(4.32)

so that, factoring off $P(1, x, \bar{x})$ and setting $x = \bar{x} = 1$, the dimension of the corresponding multiplet is $2^8 (4j + 5)$ which agrees with [19]. We also have

$$\chi^{c,1}_{(2+2j;0,0,0;0,j,j)}(s;1,1,1,1;1,1) = s^{4+4j} \frac{(1+s)^3}{(1-s)^4} (2j+1 + (2j+4)s)(2j+1 + 5s - (2j+4)s^2).$$

(4.33)

To verify $C^{1,1}_{[0,0,0](j,j)} \approx C^{1,1}_{[0,0,0](j,j)}$ in [19], we set $p = 0$ in (4.26) and use

$$\mathfrak{M}_S^4 (C_{(0,0,0,0)}(u) \prod_{i=3}^{4}(1 + su_i x^{-1}) \prod_{k=1}^{2}(1 + su_k^{-1} \bar{x}^{-1})) = 1 - s^2 x^{-1} \bar{x}^{-1},$$

(4.34)

to ensure that it is then identical with (4.30) and so,

$$\chi^{c,1}_{(2+2j;0,0,0;0,j,j)}(s;u;x,\bar{x}) = \chi^{c,1}_{(2+2j;0,0,0;0,j,j)}(s;u;x,\bar{x}).$$

(4.35)

Other characters which are readily obtainable are those for multiplets with mixtures of shortening and semi-shortening conditions, denoted $\mathcal{D}^{s,i}, \mathcal{D}^{s,\bar{i}}$ in [19]. For the $(\frac{1}{1}, \frac{1}{8})$ semi-short multiplet for which $\Delta = \frac{1}{2}(3k + 2p + q)$ and $j = 0, \bar{j} = \frac{1}{2}(k - q) - 1$ then the general formula gives,

$$\chi^{c,1,\frac{1}{2},\frac{1}{2}}_{(\Delta;k,p,q;0,j)}(s;u;x,\bar{x}) = s^{2\Delta} P(s, x, \bar{x}) \mathfrak{M}_2 \mathfrak{M}_2 S_2 \left( (1 + su_4^{-1} \bar{x}) C_{(k+p+q,p+q,0)}(u) C_j(x) \mathcal{P}_2(su, x) \mathcal{P}_3(s^{-1} u, \bar{x}) \right).$$

(4.36)
This can be expanded as
\[
\chi^{\frac{1}{4},\frac{1}{2}}_{k,\Delta,p,q;0,j}(s;u;x,\vec{x}) = \chi^{\frac{1}{4},\frac{1}{2}}_{k,\Delta,p,q;0,0}(s;u;x,\vec{x})\chi_j(\vec{x}) + \chi^{\frac{1}{4},\frac{1}{2}}_{k,\Delta+\frac{1}{2},p,q+1;0,0}(s;u;x,\vec{x})\chi_{j+\frac{1}{2}}(\vec{x}),
\]
where (4.15) may be used for \(\chi^{\frac{1}{4},\frac{1}{2}}\) with an extension to more general \(\Delta, k, p, q\). For \(j = -\frac{1}{2}\) (4.37) reproduces (3.24).

5. Reduction of Characters in Different Limits

Here we consider various limits for the \(PSU(2,2|4)\) characters, as obtained in sections 3 and 4, which restrict them to different subgroups. These restrictions ensure that particular short and semi-short multiplet characters survive to give non-zero contribution and also that in most cases the expressions for the characters greatly simplify. Furthermore the expression for the general long multiplet character in (3.9) has factors \(1 + su_1x^{-1}\) and \(1 + su_4^{-1}\vec{x}^{-1}\) which are not present in the characters for the different short multiplets that are given by (3.16) with (3.17) when \(t, \bar{t}\) are non zero. These factors, which survive in particular limits, may then be set to zero to eliminate contributions for multiplets which have \(t\) and/or \(\bar{t}\) zero, and hence remove all long multiplets. The result corresponds to the index in \([14]\).

We consider the different cases in turn making use of the analysis in appendix A of the relevant subgroups of \(G_{t,\bar{t}} \subset PSU(2,2|4)\), which correspond to the various shortening conditions in section 3 and are the symmetry groups for differing reduced sectors of the theory. The limits are obtained by re-expressing the basic trace \(\text{tr}(s^{2D}u_1H_1+H_2+H_3u_2H_2+H_3u_3H_3x^{2J_3}\bar{x}^{2\bar{J}_3})\) in terms of different linear combinations of operators \(\hat{D}, \ldots\) with corresponding variables \(\hat{s}, \ldots\) and then requiring appropriate variables to vanish when the character reduces to one for the reduced group \(G_{t,\bar{t}}\). For a variable \(h\) which contributes to the trace in the form \(h^{2\mathcal{H}}\), where \(\mathcal{H}\) has a positive semi-definite spectrum, the limit \(h \to 0\) ensures that the trace is reduced to a subspace on which \(\mathcal{H}\) has zero eigenvalue. Furthermore for a long multiplet, which is reducible to semi-short multiplets, the limit for suitable cases gives zero, realising the index defined in \([14]\).

In each case considered we first list \((t, \bar{t})\) and the residual group \(G_{t,\bar{t}}\) which is relevant for the particular limit. The limits are applied to \(\chi^{t,\bar{t}}\) in this case and also for \(\chi^{\frac{1}{4},\frac{1}{2}}\) which is then restricted to \(p = 1\), corresponding to the fundamental or singleton representation \(\mathcal{F} = B^{\frac{1}{2},\frac{1}{2}}_{[0,1,0],[0,0]}\).

(i) \(t = \bar{t} = \frac{1}{2}\), \(G_{\frac{1}{2},\frac{1}{2}} = U(1)_D\)
In this case we write the trace in the form

$$\text{tr} \left( h^{2H} \hat{s}^{2D} \hat{u}_1 H_1 \hat{u}_3 H_3 x^{2J_3} \bar{x}^{2\bar{J}_3} \right), \quad \hat{H} = D - \frac{1}{2} (H_1 + 2H_2 + H_3),$$

and then consider the limit $h \to 0$. Applying this to the character for the $\frac{1}{2}$-BPS multiplet, from (1.7) with $t = \bar{t} = \frac{1}{2}$, we obtain

$$\lim_{h \to 0} \chi_{(p,0,0,0,0)}^{\frac{1}{2}}(h \hat{s}; u_h; x, \bar{x}) = \chi_p^{U(1)}(\hat{s}) = \hat{s}^{2p},$$

$$u_h = (h^{-1} \hat{u}_1, h^{-1} \hat{u}_2, h \hat{u}_3, h \hat{u}_4), \quad \hat{u}_1 \hat{u}_2 = \hat{u}_3 \hat{u}_4 = 1.$$  \hspace{1cm} (5.2)

Trivially we have, when $p = 1$,

$$\chi_p^{U(1)}(\hat{s}) = \hat{s}^2.$$ \hspace{1cm} (5.3)

(ii) $t = \bar{t} = \frac{1}{4}$, $G_{\frac{1}{4}, \frac{1}{4}} = SU(2) \otimes U(1)_{H_+}$

The trace is now rewritten as

$$\text{tr} \left( h^{2H_0} \hat{h}^{2\hat{H}_0} \hat{u}_2 H_2 u^{H_+} x^{2J_3} \bar{x}^{2\bar{J}_3} \right), \quad H_+ = H_1 + H_2 + H_3,$$

$$\mathcal{H}_0 = D - \frac{1}{2} (3H_1 + 2H_2 + H_3), \quad \mathcal{H}_0 = D - \frac{1}{2} (H_1 + 2H_2 + 3H_3),$$ \hspace{1cm} (5.4)

and we now consider the limit $h, \hat{h} \to 0$. For the $\frac{1}{4}$-BPS multiplet character

$$\lim_{h, \hat{h} \to 0} \chi_{(p+2q,p,q,0,0)}^{\frac{1}{4}}(h \hat{h} \hat{s}; u_{h, \hat{h}}; x, \bar{x}) = \chi_{(p,q)}^{U(2)}(u; \hat{u}_2, \hat{u}_3) = u^{p+2q} \chi_{(p,0)}(\hat{u}_2, \hat{u}_3),$$

$$u_{h, \hat{h}} = (h^{-3} \hat{h}^{-1} u, h h^{-1} \hat{u}_2, h \hat{h}^{-1} \hat{u}_3, h \hat{h}^3 u^{-1}), \quad \hat{u}_1 \hat{u}_2 = 1,$$ \hspace{1cm} (5.5)

which is a $U(2)$ character. Applying this limit to $\chi_{(p,0,0,0,0)}^{\frac{1}{4}}$ gives the same result with $q = 0$. In particular for $p = 1$

$$\chi_{\underline{1}, \underline{2}}^{U(2)}(u, \hat{u}_2, \hat{u}_3) = u(\hat{u}_2 + \hat{u}_3).$$ \hspace{1cm} (5.6)

(iii) $t = \frac{1}{4}, \bar{t} = 0$, $G_{\frac{1}{4}, 0} = SU(2|3)$

The trace is re-expressed as

$$\text{tr} \left( h^{2H_0} \hat{s}^{2\hat{D}} \hat{u}_2 H_2 + H_3 \hat{u}_3 H_3 x^{2J_3} \bar{x}^{2\bar{J}_3} \right), \quad \hat{D} = \frac{3}{2} D - \frac{1}{4} (3H_1 + 2H_2 + H_3),$$ \hspace{1cm} (5.7)

with $\mathcal{H}_0$ as in (5.3). The limit $h \to 0$ reduces this trace to a $SU(2|3)$ character. Applying this limit to $\chi_{\frac{1}{4}, 0}$ as in (4.19) gives

$$\lim_{h \to 0} \chi_{\frac{1}{4}, 0}^{\frac{1}{4}}(h \hat{s}; u_h; x, \bar{x}) = \chi_{(3k+2p+q), p, q, j}^{SU(2|3)}(\hat{s}; \hat{u}, \bar{x}),$$

$$u_h = ((h \hat{s})^{-3}, h \hat{s} \hat{u}), \quad \hat{u} = (\hat{u}_2, \hat{u}_3, \hat{u}_4), \quad \hat{u}_2 \hat{u}_3 \hat{u}_4 = 1,$$ \hspace{1cm} (5.8)
where
\[
\chi^\text{SU(2|3)}_{(\kappa,p,q,\bar{q})}(\hat{s}; \hat{u}; \bar{x}) = \hat{s}^{2\kappa} \chi_{(p+q,q,0)}(\hat{u}) \chi_{\bar{f}}(\bar{x}) \prod_{i=2}^{3} \prod_{\eta=\pm 1} \left(1 + \hat{s} \hat{u}_i^{-1} \bar{x}^\eta \right). \tag{5.9}
\]

The same limit for \(\chi^\text{SU(2|3)}_{(p;0,p,0;0,0)}\) gives
\[
\lim_{\bar{h} \to 0} \chi^\text{SU(2|3)}_{(p;0,0,0;0,0)}(\bar{h} \bar{s}^{\frac{3}{2}}; u_h; x, \bar{x}) = \chi^\text{SU(2|3)}_{\text{short,p}}(\hat{s}; \hat{u}; \bar{x}), \tag{5.10}
\]
where
\[
\chi^\text{SU(2|3)}_{\text{short,p}}(\hat{s}; \hat{u}; \bar{x}) = \hat{s}^{2p} \mathcal{M}_1(C_{(p,0,0)}(\hat{u}) \prod_{\eta=\pm 1} (1 + \hat{s} \hat{u}_2^{-1} \bar{x}^\eta))
= \hat{s}^{2p} \left( \chi_{(p,0,0)}(\hat{u}) + \hat{s} \chi_{(p-1,0,0)}(\hat{u}) \chi^{\frac{1}{2}}(\bar{x}) + \hat{s}^2 \chi_{(p-2,0,0)}(\hat{u}) \right), \tag{5.11}
\]
and the \(SU(3)\) characters are all given by (2.11). Here the associated \(SU(2|3)\) representation satisfies a shortening condition. In the particular case of \(p = 1\)
\[
\chi^\text{SU(2|3)}_\mathfrak{c}(\hat{s}; \hat{u}; \bar{x}) = \hat{s}^2 (\hat{u}_2 + \hat{u}_3 + \hat{u}_4 + \hat{s} (\bar{x} + \bar{x}^{-1})). \tag{5.12}
\]

(iv) \(t = \frac{1}{4}, \bar{t} = \frac{1}{8}, \mathcal{G}_{\frac{1}{4}, \frac{1}{8}} = SU(1|2) \otimes U(1)_H\)

The trace is now rewritten as
\[
\text{tr} \left( h^2 \mathcal{H}_0 \bar{h}^2 \bar{\mathcal{H}} \bar{s}^{2\hat{D}} \bar{u}_2 H_2 u H^+ x^{2\hat{J}_3} \right), \quad \hat{D} = D + 2\bar{J}_3 - \frac{1}{2} (H_1 - H_3),
\]
\[
\bar{\mathcal{H}} = D - 2\bar{J}_3 - \frac{1}{2} (H_1 + 2H_2 + 3H_3), \tag{5.13}
\]
with \(\mathcal{H}_0, H_+\) defined in (5.4). With the result in (4.36), and requiring the necessary conditions on \(\Delta\) from (3.12) and (3.13), we have
\[
\lim_{h, \bar{h} \to 0} \chi^\frac{\mathcal{H}}{4} (\frac{\mathcal{H}}{4}(3k+2p+q); k, p, q; 0, \bar{q}) \left( h \bar{h} \hat{s}; u_h, \bar{h}; x, \bar{h}^{-2} \hat{s}^2 \right) = u^{k+p+q+1} \chi^\text{SU(1|2)}_{(2k+p, p)}(\hat{s}; \hat{u}_2, \hat{u}_3),
\]
\[
u_{h, \bar{h}} = (\bar{h}^{-3} \bar{h}^{-1} \hat{s}^{-1} u, \bar{h} \bar{h}^{-1} \hat{s} \hat{u}_2, \bar{h} \bar{h}^{-1} \hat{s} \hat{u}_3, \bar{h} \bar{h}^{-1} \hat{s} \hat{u}_3), \quad \bar{u}_2 \bar{u}_3 = 1, \tag{5.14}
\]
with
\[
\chi^\text{SU(1|2)}_{(\kappa, p)}(\hat{s}; \hat{u}_2, \hat{u}_3) = \hat{s}^{2\kappa} \chi_{(p,0)}(\hat{u}_2, \hat{u}_3) \prod_{i=2,3} (1 + \hat{s}^2 \hat{u}_i^{-1}). \tag{5.15}
\]
For \(\chi^\frac{\mathcal{H}}{4,0}\) as \(\bar{h} \to 0\), with \(\bar{\delta}\) as in (3.28),
\[
\chi^\frac{\mathcal{H}}{4,0} (\frac{\mathcal{H}}{4}(3k+2p+q); k, p, q; 0, \bar{q}) \left( h \bar{h} \hat{s}; u_h, \bar{h}; x, \bar{h}^{-2} \hat{s}^2 \right) \sim \bar{h}^{2\bar{\delta}} (1 + u) u^{k+p+q+1} \chi^\text{SU(1|2)}_{(2k+p, p)}(\hat{s}; \hat{u}_2, \hat{u}_3). \tag{5.16}
\]
Even for \(\bar{\delta} = 0\) we may set \(u = -1\) to remove all \(\chi^\frac{\mathcal{H}}{4,0}\) contributions.
Applying this limit for $\chi_{(p;0,0,0,0)}^{1/2}$ gives

$$
\lim_{h,\bar{h} \to 0} \chi_{(p;0,0,0,0)}^{1/2}(\hbar \hat{s}; u_{h,\bar{h}}; x, \bar{h}^{-2} \hat{s}^2) = u^p \chi_{\text{short}, p}^{SU(1/2)}(\hat{s}; \hat{u}_2, \hat{u}_3),
$$

(5.17)

For $p = 1$

$$
\chi_{\mathcal{F}}^{SU(1/2)}(\hat{s}; \hat{u}_2, \hat{u}_3) = \hat{s}^2(\hat{u}_2 + \hat{u}_3 + \hat{s}^2).
$$

(5.18)

$$(\nu) \ t = \tilde{t} = \frac{1}{8}, \ G_{\mathfrak{h}, \mathfrak{h}} = PSU(1, 1|2) \ltimes U(1)_{H_-} \otimes U(1)_{H_+}
$$

The trace here has the form

$$
\text{tr}(h^{2\mathcal{H}} \bar{h}^{2\bar{\mathcal{H}}} \hat{s}^2 \hat{d}_2^2 u^H v^{H_-}), \quad H_+ = H_1 + H_2 + H_3, \quad H_- = H_1 - H_3,
$$

$$
\mathcal{H} = D - 2J_3 - \frac{1}{2}(3H_1 + 2H_2 + H_3), \quad \bar{\mathcal{H}} = D - 2\bar{J}_3 - \frac{1}{2}(H_1 + 2H_2 + 3H_3),
$$

(5.19)

and we consider the double limit $h, \bar{h} \to 0$. Assuming both conditions on $\Delta$ in (3.12) are satisfied then the result for $\chi^{1/2}$ is

$$
\lim_{h,\bar{h} \to 0} \chi_{(\Delta;k,p,q;j,\bar{j})}^{1/2}(\hbar \hat{s}; u_{h,\bar{h}}; h^{-2} \hat{s}, \bar{h}^{-2} \hat{s}) = u^{k+p+q+2} \chi_{(k+p+q+2j+2, p, k-q)}^{PSU(1,1|2) \ltimes U(1)}(\hat{s}; \hat{u}_2, \hat{u}_3; v),
$$

$$
u_{h,\bar{h}} = (h^{-3} \bar{h}^{-1} uv, h^{-1} v^{-1} \hat{u}_2, h^{-1} v^{-1} \hat{u}_3, h\bar{h} u^{-1} v), \quad \hat{u}_2 \hat{u}_3 = 1,
$$

(5.20)

where

$$
\chi^{PSU(1,1|2) \ltimes U(1)}_{(\kappa, p, m)}(\hat{s}; \hat{u}_2, \hat{u}_3; v) = v^m \hat{s}^{2\kappa} \frac{\hat{s}}{2\pi} \chi_{(p,0)}^{SU(1/2)}(\hat{u}_2, \hat{u}_3) \prod_{i=2}^3 \prod_{\epsilon=\pm 1} (1 + \hat{s}^2 v^\epsilon \hat{u}_i).
$$

(5.21)

For a general long multiplet as $h, \bar{h} \to 0$

$$
\chi_{(\Delta;k,p,q;j,\bar{j})}(\hbar \hat{s}; u_{h,\bar{h}}; h^{-2} \hat{s}, \bar{h}^{-2} \hat{s})
$$

$$
\sim h^{\Delta} \bar{h}^{\bar{\Delta}} (1 + uv)(1 + uv^{-1}) u^{k+p+q+2} \chi_{(\Delta+j+j+2, p, k-q)}^{PSU(1,1|2) \ltimes U(1)}(\hat{s}; \hat{u}_2, \hat{u}_3; v),
$$

(5.22)

with $\delta, \bar{\delta}$ defined in (3.28). At the unitarity threshold, $\delta = \bar{\delta} = 0$, the long multiplet character in this case vanishes if $uv = -1$ or $uv^{-1} = -1$. If both $uv = uv^{-1} = -1$ then $\chi^{1/2,0}$ and $\chi^{0,1/2}$ characters vanish in this limit for $\delta = 0$ and $\bar{\delta} = 0$ respectively.

The $\frac{1}{2}$-BPS multiplet character in this limit satisfies

$$
\lim_{h,\bar{h} \to 0} \chi_{(p;0,0,0,0)}^{1/2}(\hbar \hat{s}; u_{h,\bar{h}}; h^{-2} \hat{s}, \bar{h}^{-2} \hat{s}) = u^p \chi_{\text{short}, p}^{SU(1,1|2) \ltimes U(1)}(\hat{s}; \hat{u}_2, \hat{u}_3; v),
$$

(5.23)

\footnote{This result is equivalent to a combination of formulae in \cite{ref}.}
where
\[ \chi_{\text{short},p}^{PSU(1,1|2) \times U(1)}(\hat{s}; \hat{u}_2, \hat{u}_3; v) = \frac{\hat{s}^{2p}}{1 - \hat{s}^4} \left( \chi(p,0)(\hat{u}_2, \hat{u}_3) + \hat{s}^2 \chi(\frac{1}{4})(\hat{u}_2, \hat{u}_3) + \hat{s}^4 \chi(p-2,0)(\hat{u}_2, \hat{u}_3) \right). \] (5.24)

For \( p = 1 \)
\[ \chi_F^{PSU(1,1|2) \times U(1)}(\hat{s}; \hat{u}_2, \hat{u}_3; v) = \frac{\hat{s}^2}{1 - \hat{s}^4} (\hat{u}_2 + \hat{u}_3 + \hat{s}^2(v + v^{-1})). \] (5.25)

(vi) \( t = \frac{1}{\hat{s}}, \bar{t} = 0, \mathcal{G}^{1,6}_{\frac{1}{4},0} = PSU(1,2|3) \times U(1)_R \)
The trace here has the form
\[ \text{tr}(h^{2\mathcal{H}} \hat{s}^{2\bar{D}} \hat{u}_2 H_2 + H_3 \hat{u}_3 H_3 x^{2\bar{J}_3} u^R), \quad R = 3H_1 + 2H_2 + H_3, \bar{D} = D + J_3, \] (5.26)
with \( \mathcal{H} \) as defined in (5.19). Applying the limit \( h \to 0 \) to \( \chi_{\hat{s},0}^{\hat{s}} \) gives
\[ \lim_{h \to 0} \chi_{h}^{(2+2j+\frac{1}{2}(3k+2p+q);k,p,q;j,j)}(h \hat{s}; u_h; h^{-2}\hat{s}, \bar{x}) \]
\[ = \chi_{PSU(1,2|3) \times U(1)}(\frac{1}{2}(3k+2p+q)+3j,3k+2p+q,3p,q;\bar{x})(\hat{s}, u; \hat{u}, \bar{x}), \]
\[ u_h = (h^{-3}u^3, hu^{-1}\bar{u}), \quad \hat{u} = (\hat{u}_2, \hat{u}_3, \hat{u}_4), \quad \hat{u}_2 \hat{u}_3 \hat{u}_4 = 1, \] (5.27)
where
\[ \chi_{PSU(1,2|3) \times U(1)}^{PSU(1,2|3) \times U(1)}(\hat{s}, u; \bar{x}) = \frac{\hat{s}^{2\kappa} u^r}{(1 - \hat{s}^3 \bar{x})(1 - \hat{s}^3 \bar{x}^{-1})} \chi_{PSU(1,2|3) \times U(1)}(\hat{s}, u; \bar{x}) \]
\[ \times \prod_{i=2}^{4} \left( (1 + \hat{s}^2 u^{-1} \hat{u}_i) \prod_{\eta=\pm 1} \left( 1 + u \hat{u}_i^{-1} \bar{x}^{\eta} \right) \right), \] (5.28)
where \( \chi_{PSU(1,2|3) \times U(1)}(\hat{s}, u; \bar{x}) \) is a SU(3) character. For a long multiplet for \( h \to 0 \)
\[ \chi_{(\Delta; k,p,q;j,j)}(h \hat{s}; u_h; h^{-2}\hat{s}, \bar{x}) \sim h^{2\delta}(1 + u^3) \chi_{PSU(1,2|3) \times U(1)}(h \hat{s}; u; \bar{x}), \] (5.29)
so that \( u = -1 \) removes such contributions even at threshold \( \delta = 0 \).

For the \( \frac{1}{2} \)-BPS character we find in this limit
\[ \lim_{h \to 0} \chi_{(p;0,0,0;0)}^{PSU(1,2|3) \times U(1)}(h \hat{s}; u_h; h^{-2}\hat{s}, \bar{x}) = \chi_{\text{short},p}^{PSU(1,2|3) \times U(1)}(\hat{s}, u; \hat{u}, \bar{x}), \] (5.30)
for

$$\chi_{\text{short}, p}^{PSU(1,2|3) \times U(1)}(\hat{s}, u; \hat{u}; \hat{x}) = \frac{(\hat{s} u)^{2p}}{(1 - \hat{s}^3 \hat{x})(1 - \hat{s}^3 \hat{x}^{-1})} \mathfrak{m}^p \mathfrak{S}_3 \left( \sum_{j=3,4} C_{(p,0,0)}(\hat{u}) \prod_{\eta=\pm 1} \left( 1 + \hat{s}^2 u^{-1} \hat{u}_j \right) \prod_{\eta=\pm 1} \left( 1 + \hat{s} u u_2^{-1} \hat{x}^\eta \right) \right)$$

$$= \frac{(\hat{s} u)^{2p}}{(1 - \hat{s}^3 \hat{x})(1 - \hat{s}^3 \hat{x}^{-1})} \sum_{a=0}^2 (\hat{s} u)^a \chi_{j_a}(\hat{x}) \times \left( \chi_{(p-a,0,0)}(\hat{u}) + \hat{s}^2 u^{-1} \chi_{(p-a,1,0)}(\hat{u}) + \hat{s}^4 u^{-2} \chi_{(p-a-1,0,0)}(\hat{u}) \right),$$

using the notation in [11]. To calculate the $SU(3)$ characters in (5.31) we may use (2.11) and

$$\chi_{(p,1,0)}(\hat{u}) = \sum_{i=2}^4 \hat{u}_i^{-1} \chi_{(p-1,0,0)}(\hat{u}) - \chi_{(p-2,0,0)}(\hat{u}).$$

When $p = 1$, using $\chi(-1,0,0) = \chi(-2,0,0)$, $\chi(-1,1,0) = -1$, we obtain for the reduced character for the fundamental representation,

$$\chi^{PSU(1,2|3) \times U(1)}_{\mathcal{F}}(\hat{s}, u; \hat{u}; \hat{x}) = \frac{(\hat{s} u)^2 (\sum_{i=2}^4 \hat{u}_i + \hat{s}^2 u^{-1} \sum_{i=2}^4 \hat{u}_i^{-1} + \hat{s} u(\hat{x} + \hat{x}^{-1}) + \hat{s}^4 u^{-2}(1 - u^3))}{(1 - \hat{s}^3 \hat{x})(1 - \hat{s}^3 \hat{x}^{-1})}.$$

For $u = -1$, corresponding to the index, and also taking $\hat{u}_2 = v$, $\hat{u}_3 = w/v$, $\hat{u}_4 = 1/w$, this is identical with the result in [14] obtained by direct calculation of all contributions to the trace.

6. Free Fields

The fundamental fields for $\mathcal{N} = 4$ supersymmetric Yang Mills are six scalars $X_r$, four chiral fermions (gauginos) $\lambda_{i\alpha}$, $\bar{\lambda}^i_{\dot{\alpha}}$ and gauge field strengths $F_{\alpha\beta} = F_{\beta\alpha}$, $\bar{F}_{\dot{\alpha}\dot{\beta}} = \bar{F}_{\dot{\beta}\dot{\alpha}}$, all belonging to the adjoint representation of the relevant gauge group and to the $[0,1,0], [1,0,0], [0,0,1]$ and trivial $SU(4)_R$ representations respectively. By using $SU(4)$ gamma matrices it is convenient to let $X_r \rightarrow \varphi_{ij} = -\varphi_{ji}$. The character for this representation is given by (1.1) and the non zero action of the supercharges on these fields, at the

\footnote{Comparing with (5.3) of [14] we should take $\hat{s}^3 = x^2$, $\hat{x} = e^x$, $u^3 = x e^{\frac{1}{4}(\mu_1 + \mu_2 + \mu_3)}$, $\hat{u}_2 = e^{\frac{1}{4}(2\mu_1 - \mu_2 - \mu_3)}$, $\hat{u}_3 = e^{\frac{1}{4}(2\mu_2 - \mu_1 - \mu_3)}$, $\hat{u}_4 = e^{\frac{1}{4}(2\mu_3 - \mu_1 - \mu_2)}$.}
origin $x = 0$, is given by

$$[Q_i^1, \varphi_{kl}] = 2\delta_i^k \lambda_l \alpha, \quad [\bar{Q}_{i\dot{a}}, \varphi_{kl}] = \varepsilon_{ijk\dot{a}} \bar{\lambda}_{\dot{a}}, \quad \{Q_i^1, \lambda_j\beta\} = \delta_j^i F_{\alpha\beta},$$

$$\{Q_i^1, \lambda_j\alpha\} = 2i \partial_{\alpha\beta} \varphi_{ij}, \quad \{Q_i^1, \bar{\lambda}_{\dot{a}}^i\} = \varepsilon_{ijk\dot{a}} i \partial_{\alpha\dot{a}} \varphi_{kl}, \quad \{Q_j^\dot{a}, \bar{\lambda}_{\dot{a}}^i\} = \delta_i^j F_{\dot{a}\alpha\dot{b}},$$

$$\{S_i^\beta, \lambda_j\alpha\} = 4\delta_i^\beta \varphi_{ij}, \quad \{\bar{S}_{i\dot{a}}, \bar{\lambda}_{\dot{a}}^i\} = -2\delta_{i\dot{a}}^\beta \varepsilon_{ij\dot{a}} \varphi_{kl},$$

$$[Q_{i\dot{a}}, F_{\alpha\beta}] = 2i \partial_{(\alpha\beta)} \lambda_{i\dot{a}}, \quad [Q_i^1, \bar{F}_{\alpha\beta}] = 2i \partial_{\alpha\beta} (\bar{\lambda}^i_{\dot{a}}),$$

$$[S_i^\alpha, F_{\beta\gamma}] = 8\delta^\alpha_{(\beta} \lambda_{\gamma)} \bar{F}_{\beta\gamma} = -8\delta_{(\beta}^\alpha \lambda_{\gamma)}.$$

As is well known the algebra closes subject to the equations of motion $\partial \dot{\alpha} \lambda_{i\alpha} = 0$, $\partial \dot{\alpha} F_{\alpha\beta} = 0$, and their conjugates, which also imply $\partial^2 \varphi_{kl} = 0$.

The basic gauge singlet operators are formed by products of multiple traces over adjoint indices of products of the fundamental fields with derivatives. These form the various possible supermultiplets each of which arises from a unique highest weight operator together with descendants. For the various short multiplets it is possible to restrict to just those elementary fields annihilated by the relevant supercharges as described in section 3. Below we list in turn the crucial cases for different possible $t, \bar{t}$ the associated supercharges which have a trivial action and then the remaining subgroup $G_{t, \bar{t}} \subset PSU(2, 2|4)$. Following this the corresponding kernel in terms of the fundamental fields, which is easily determined from (6.1) and forms a representation of $G_{t, \bar{t}}$, is displayed.

$$t = \frac{1}{8}, Q_1, S_1^2, PSU(1, 2|3) \times U(1)_R;$$

$$\partial^{n_1}_1 \partial^{n_2}_{12} (Z, Y, X, \lambda_{i1}, \bar{\lambda}_{1\dot{a}}, F_{11}), \quad i = 2, 3, 4, \dot{\alpha} = 1, 2, n_1, n_2 = 0, 1, 2, \ldots, \quad (6.2)$$

where

$$Z = \varphi_{34}, \quad Y = \varphi_{42}, \quad X = \varphi_{23}, \quad \partial_{11} \bar{\lambda}_{12} = \partial_{12} \bar{\lambda}_{11}.$$  

(6.3)

Derivatives are present in (6.2) since $[Q_1, P_{i\dot{a}}] = [S_1^2, P_{i\dot{a}}] = 0$. The weights associated with each field are listed in Table 1, along with their elementary contribution to the character which is identified with a particular letter, up to identifications, for later application in partition functions. The Cartan subalgebra for $PSU(1, 1|2)$ is $(\hat{D}; H_2, H_3; \bar{J}_3)$, with $\hat{D} = D + J_3$ and the $U(1)$ generator $R = 3H_1 + 2H_2 + H_3$ and the associated supercharges are $Q^1, S^1_J, Q_{j\dot{a}}, S_{\dot{a}i}, i, j = 2, 3, 4$. $Z, Y, X$ form a 3 representation and $\lambda_{i1}, i = 2, 3, 4$ a 3 representation of $SU(3)$, their $SU(3)$ weights $(H_2, H_3)$ can be read off from Table 1.

$$t = \bar{t} = \frac{1}{8}, Q_1, S_1^2, \bar{Q}_{41}, \bar{S}^{11}, PSU(1, 1|2) \times U(1)_{H_\cdot} \otimes U(1)_{H_+};$$

$$\partial^{n_2}_{12} (Z, Y, \lambda_{41}, \bar{\lambda}_{12}), \quad n = 0, 1, 2, \ldots. \quad (6.4)$$

The Cartan generators for $PSU(1, 1|2)$ are here $\hat{D} = D + J_3 + \bar{J}_3, H_2$ and $H_\cdot = H_1 - H_3, H_+ = H_1 + H_2 + H_3$, and the algebra contains supercharges $Q^1, S^1_J, \bar{Q}_{j2}, \bar{S}^{12}, i, j = 2, 3$. 

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\[ t = \frac{1}{4}, Q_1^\alpha, S_1^\beta, SU(2|3); \]

\[ Z, \ Y, \ X, \ \bar{\lambda}_1^1. \]

(6.5)

The Cartan generators for \( SU(2|3) \) are \( J_3, H_2, H_3 \) and \( \hat{D} = \frac{3}{2}D - \frac{1}{3}(3H_1 + 2H_2 + H_3) \) with supercharges \( \bar{Q}_{j\bar{\beta}}, \bar{S}_{i\dot{\alpha}}, i, j = 2, 3, 4. \)

\[ t = \frac{1}{4}, \bar{t} = \frac{1}{8}, Q_1^\alpha, S_1^\beta, \bar{Q}_{4\dot{\alpha}}, \bar{S}_{4\dot{\alpha}}, SU(1|2) \otimes U(1)_{H_+}; \]

\[ Z, \ Y, \ \bar{\lambda}_1^2. \]

(6.6)

The Cartan generator for \( SU(1|2) \) are \( H_2, \hat{D} = D + 2\bar{J}_3 - \frac{1}{2}H_-, \) and \( H_\pm \) are as previously and the supercharges are \( Q_{j2}, S_{i2}, i, j = 2, 3. \)

\[ t = \bar{t} = \frac{1}{4}, Q_1^\alpha, S_1^\beta, \bar{Q}_{4\dot{\alpha}}, \bar{S}_{4\dot{\alpha}}, SU(2) \otimes U(1)_{H_+}; \]

\[ Z, \ Y. \]

(6.7)

The Cartan generator for \( SU(2), \) for which \((Z,Y)\) form doublet, is \( H_2 \) and \( H_+ \) is as previously.

\[ t = \bar{t} = \frac{1}{2}, Q_1^\alpha, S_j^\beta, \bar{Q}_{k\dot{\alpha}}, S_{l\dot{\alpha}}, i, j = 1, 2, k, l = 3, 4, U(1)_D; \]

\[ Z. \]

(6.8)

Table 1

| Field | \((\Delta; H_1, H_2, H_3; J_3, \bar{J}_3)\) | \(s^{2\Delta}u_1^{H_1+H_3}u_2^{H_2+H_3}u_3^{x2J_3\bar{x}2J_3}\) | Letter |
|-------|---------------------------------|---------------------------------|--------|
| \(Z\)  | \((1; 0, 1, 0; 0, 0)\)            | \(s^2u_1u_2\)                  | \(z\)  |
| \(Y\)  | \((1; 1, -1, 1; 0, 0)\)           | \(s^2u_1u_3\)                  | \(y\)  |
| \(X\)  | \((1; 1, 0, -1; 0, 0)\)           | \(s^2u_1u_4\)                  | \(x\)  |
| \(\lambda_{41}\) | \((\frac{3}{2}; 0, 0, 1; \frac{1}{2}, 0)\) | \(s^3u_4^{-1}x\)               | \(b\)  |
| \(\lambda_{31}\) | \((\frac{3}{2}; 0, 1, -1; \frac{1}{2}, 0)\) | \(s^3u_3^{-1}x\)               | \(c\)  |
| \(\lambda_{21}\) | \((\frac{3}{2}; 1, -1, 0; \frac{1}{2}, 0)\) | \(s^3u_2^{-1}x\)               | \(d\)  |
| \(\bar{\lambda}_{12}\) | \((\frac{3}{2}; 1, 0, 0; 0, \frac{1}{2})\) | \(s^3u_1\bar{x}\)             | \(a\)  |
| \(\bar{\lambda}_{11}\) | \((\frac{3}{2}; 1, 0, 0; 0, -\frac{1}{2})\) | \(s^3u_1\bar{x}^{-1}\)         | \(\bar{a}\) |
| \(F_{11}\) | \((2; 0, 0, 0; 1, 0)\)            | \(s^4x^2\)                     | \(a\bar{a}b^2y^{-2}z^{-2}\) |
| \(\partial_{12}\) | \((1; 0, 0, 0; \frac{1}{2}, \frac{1}{2})\) | \(s^2x\bar{x}\)               | \(ab^{-1}y^{-1}z^{-1}\) |
| \(\partial_{11}\) | \((1; 0, 0, 0; \frac{1}{2}, -\frac{1}{2})\) | \(s^2x\bar{x}^{-1}\)          | \(\bar{a}by^{-1}z^{-1}\) |

Relations: \(xyz = a\bar{a}, xb = yc = zd. \)

Each case listed above defines a sector of operators formed by multiple traces over gauge indices of products of the fields in each set, with appropriate derivatives in the \((\frac{1}{8}, 0)\)
and \((\frac{1}{5}, \frac{1}{8})\) cases. With an \(SU(N)\) gauge group then in each trace with \(n\) field operators we require \(n \geq 2\) and also \(n\) is restricted by removal trace identities present for finite \(N\), thus \(\text{tr}(Z^n)\) for \(n > N\) is expressible in terms of products of traces with \(\sum_i n_i = n, \quad n_i \leq N\). Restricting to multi-trace operators of the fields listed above in each case provides a basis for constructing the various potential short and semi-short supermultiplets. In the \((\frac{1}{2}, \frac{1}{2})\) sector a basis for \(k\)-trace, \(k = 1, 2, \ldots\), operators is given by

\[
\text{tr}(Z^{n_1}) \cdots \text{tr}(Z^{n_k}), \quad \sum_i n_i = n.
\]

These are all \(\frac{1}{2}\)-BPS operators belonging to the \([0, n, 0]\) representation. In the \(t = \bar{t} = \frac{1}{4}\) sector a basis of \(k\)-trace operators is

\[
\text{tr}(\prod_j Z^{n_{1j}} Y^{m_{1j}}) \cdots \text{tr}(\prod_j Z^{n_{kj}} Y^{m_{kj}}), \quad \sum_j (n_{ij} + m_{ij}) = n_i, \quad \sum_{ij} m_{ij} = m,
\]

where there is now a choice of ordering within each trace. Those related by application of the \(SU(2)\) lowering operator to the operators given \((6.9)\) are part of \(\frac{1}{4}\)-BPS multiplets, the rest are potential superconformal primary operators for \(\frac{1}{4}\)-BPS multiplets belonging to the \([m, n - 2m, m]\) representation. They may become long multiplets with anomalous dimensions if joined with other semi-short multiplets as in \((3.27)\). The \(m = 1\) case is always protected (for a single trace this is part of the \(\frac{1}{2}\)-BPS multiplet).

Other examples are similarly constructed for the \(t = \frac{1}{4}, \quad t = \bar{t} = \frac{1}{8}\) and \(t = \frac{1}{8}\) sectors.

In the various limits described in section 5 it is important to recognise that only those letters \(z, y, \ldots\) listed in table 1 survive which correspond to operators which are present in each sector. We here rephrase some of the previous results in terms of these letters.

\(t = \bar{t} = \frac{1}{2}\). Here only \(z = s^2 u_1 u_2 = \hat{s}^2\) survives and we have from \((5.2)\)

\[
\hat{\chi}_p^{\frac{3}{5}, \frac{4}{5}}(z) = \chi_p^{U(1)}(\hat{s}) = z^p, \quad \hat{\chi}_1^{\frac{3}{5}, \frac{4}{5}}(z) = z.
\]

\(t = \bar{t} = \frac{1}{4}\). Here \((z, y) = u(\hat{u}_2, \hat{u}_3)\) survive and we have from \((5.3)\)

\[
\hat{\chi}_p^{\frac{1}{5}, \frac{4}{5}}(z, y) = \chi_{(p, 0)}^{U(2)}(\hat{s}; \hat{u}_2, \hat{u}_3) = \chi_{(p, 0)}(z, y), \quad \hat{\chi}_1^{\frac{1}{5}, \frac{4}{5}}(z, y) = z + y.
\]

\(t = \frac{1}{4}, \quad \bar{t} = 0\). Here \((z, y, x) = \hat{s}^2 \hat{u}, (a, \bar{a}) = \hat{s}^3(\bar{x}, \bar{x}^{-1}), \quad zyx = a\bar{a}\). From \((5.11)\)

\[
\hat{\chi}_p^{\frac{1}{5}, 0}(z, y, x; a, \bar{a}) = \chi_{\text{short,} p}^{SU(2)}(\hat{s}; \hat{u}; \bar{x})
\]

\[
= \chi_{(p, 0, 0)}(z, y, x) + (a + \bar{a})\chi_{(p-1, 0, 0)}(z, y, x) + \chi_{(p-1, 1, 1)}(z, y, x),
\]

\[
\hat{\chi}_1^{\frac{1}{5}, 0}(z, y, x; a, \bar{a}) = z + y + x + a + \bar{a}.
\]
\( t = \frac{1}{4}, \bar{t} = \frac{1}{8} \). Here \((z, y) = u\hat{s}^2(\hat{u}_2, \hat{u}_3), a = u\hat{s}^4 \). From (5.14)

\[
\hat{\chi}_p^{\frac{1}{3}, \frac{1}{3}}(z, y; a) = u^p \hat{\chi}_{\text{short, } p}^{SU(1|2)}(\hat{s}, \hat{u}_2, \hat{u}_3) = \chi_{(p,0)}(z, y) + a \chi_{(p-1,0)}(z, y),
\]

\[
\hat{\chi}_1^{\frac{1}{3}, \frac{1}{3}}(z, y; a) = z + y + a.
\] (6.14)

For \( u = -1 \), appropriate for the index, \( a = -zy \).

\( t = \bar{t} = \frac{1}{8} \). Here \((z, y) = u\hat{s}^2(\hat{u}_2, \hat{u}_3), a = u\hat{s}^4, b = uv^{-1}\hat{s}^4 \). From (5.24)

\[
\hat{\chi}_p^{\frac{1}{3}, \frac{1}{3}}(z, y; a, b) = u^p \hat{\chi}_{\text{short, } p}^{PSU(1,1|2) \times U(1)}(\hat{s}, \hat{u}_2, \hat{u}_3; v)
\]

\[
= \frac{1}{1 - \sigma} \left( \chi_{(p,0)}(z, y) + (a + b) \chi_{(p-1,0)}(z, y) + ab \chi_{(p-2,0)}(z, y) \right),
\] (6.15)

\[
\hat{\chi}_1^{\frac{1}{3}, \frac{1}{3}}(z, y; a, b) = \frac{1}{1 - \sigma} (z + y + a + b), \quad \sigma = \frac{ab}{zy}.
\]

For \( uv^{\pm 1} = -1 \), appropriate for the index, \( a = b = -zy \).

\( t = \frac{1}{8}, \bar{t} = 0 \). Here \((z, y, x) = u^2\hat{s}^2\hat{u}, (a, \bar{a}) = u^3\hat{s}^3(\bar{x}, \bar{x}^{-1}), (d, c, b) = u\hat{s}^4(\hat{u}_2^{-1}, \hat{u}_3^{-1}, \hat{u}_4^{-1}) \). From (5.31) with \( \lambda = u^{-3} = b/yz \)

\[
\hat{\chi}_p^{\frac{1}{3}, 0}(z, y, x; a, \bar{a}, b) = \hat{\chi}_{\text{short, } p}^{PSU(1,2|3) \times U(1)}(\hat{s}, u; \hat{u}, \bar{x})
\]

\[
= \frac{1}{(1 - \lambda a)(1 - \lambda \bar{a})} 
\left( \chi_{(p,0,0)}(z, y, x) + \lambda \chi_{(p,1,0)}(z, y, x) + \lambda^2 \chi_{(p,1,1)}(z, y, x) 
\right.

\[
+ (a + \bar{a}) \left( \chi_{(p-1,0,0)}(z, y, x) + \lambda \chi_{(p-1,1,0)}(z, y, x) + \lambda^2 \chi_{(p-1,1,1)}(z, y, x) \right) 
\]

\[
+ \chi_{(p-1,1,1)}(z, y, x) + \lambda \chi_{(p-1,2,1)}(z, y, x) + \lambda^2 \chi_{(p-1,2,2)}(z, y, x) \right),
\]

\[
\hat{\chi}_1^{\frac{1}{3}, 0}(z, y, x; a, \bar{a}, b) = \frac{1}{(1 - \lambda a)(1 - \lambda \bar{a})} 
\left( z + y + x + a + \bar{a} 
\right.

\[
+ \lambda (yz + zx + xy) + xyz(\lambda^2 - \lambda) \right).
\] (6.16)

For \( u = -1 \), appropriate for the index, \( \lambda = -1 \).

7. Partition functions

Here we analyse partition functions at large \( N \) for free \( \mathcal{N} = 4 \) super Yang-Mills making use of results from the discussion of characters in previous sections. We endeavour to decompose them into linear sums of characters in order to identify the operator content of the theory.
In the general case these partition functions\textsuperscript{9} involve the single particle partition function which is given by,

$$Z(s; u; x, \bar{x}) = \chi_{(1; 0, 1, 0; 0, 0)}^s(s; u; -x, -\bar{x})$$  \hspace{1cm} (7.1)$$

and is expanded in detail in (4.4). The partition function for single trace operators, for $SU(N)$ theories as $N \to \infty$, is then given by

$$Z_{s.t.}(s; u; x, \bar{x}) = \sum_{n=2}^{\infty} \frac{1}{n} \sum_{d|n} \phi(d) Z(s^n; u^n; x^n, \bar{x}^n)^{n/d}$$

$$= -\sum_{d=1}^{\infty} \frac{\phi(d)}{d} \log \left(1 - Z(s^d; u^d; x^d, \bar{x}^d)\right) - Z(s; u; x, \bar{x}),$$  \hspace{1cm} (7.2)

where $\phi(n)$ is the Euler totient function, being the number of integers relatively prime to and smaller than $n$ with $\phi(1) = 1$. This formula reflects the modification of counting as a consequence of cyclic symmetry of the trace. For $U(N)$ theories the lower limit of the first sum in (7.2) becomes 1 so that the last term in the second line is missing.

The multi-trace partition function is expressed in terms of this by

$$Z_{m.t.}(s; u; x, \bar{x}) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} Z_{s.t.}(s^n; u^n; x^n, \bar{x}^n)\right)$$

$$= \exp \left(-\sum_{n=1}^{\infty} \frac{1}{n} Z(s^n; u^n; x^n, \bar{x}^n)\right) \prod_{m=1}^{\infty} \frac{1}{1 - Z(s^m; u^m; x^m, \bar{x}^m)},$$  \hspace{1cm} (7.3)

which is achieved using $\sum_{d|n} \phi(d) = n$. The modification for $U(N)$ theories is simply that the pre-factor in the last line of (7.3) is missing.

We will also be interested in the partition function over operators whose fundamental fields are completely symmetrised within the trace. The single trace partition function over such completely symmetrised operators is given by

$$Z_{s.t., sym.}(s; u; x, \bar{x}) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} Z(s^n; u^n; x^n, \bar{x}^n)\right) - Z(s; u; x, \bar{x}) - 1,$$  \hspace{1cm} (7.4)

and the multi-trace partition function over such operators is

$$Z_{m.t., sym.}(s; u; x, \bar{x}) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} Z_{s.t., sym.}(s^n; u^n; x^n, \bar{x}^n)\right).$$  \hspace{1cm} (7.5)

\textsuperscript{9} Note that what we refer to as a ‘partition function’ is defined by a supertrace, as appropriate for determining an index. This is reflected by the sign change $(x, \bar{x}) \to -(x, \bar{x})$ in the characters which introduces a sign $(-1)^F$ for fermion number $F$. 

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For $U(N)$ theories the $Z(s; u; x, \bar{x})$ term in (7.4) is not subtracted.

We also consider the single trace partition function over $\frac{1}{2}$-BPS operators, given by,

$$Z_{s.t., \frac{1}{2}-\text{BPS}}(s; u; x, \bar{x}) = \sum_{p=2}^{\infty} \chi_{(p,0,0;0,0)}^{\frac{1}{2}, \frac{1}{2}}(s; u; -x, -\bar{x}) , \quad (7.6)$$

along with the multi-trace partition function over operators formed from these (i.e. multi-trace operators constructed by multiplying together single-trace $\frac{1}{2}$-BPS operators and descendants). In the large $N$, large coupling limit, these multi-trace operators correspond to supergravity multi-particle states via the AdS/CFT correspondence, and hence we label the partition function in this case by ‘sugra’. The resulting partition function is then given by

$$Z_{\text{sugra}}(s; u; x, \bar{x}) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} Z_{s.t., \frac{1}{2}-\text{BPS}}(s^n; u^n; x^n, \bar{x}^n) \right) . \quad (7.7)$$

The operator content in the different sectors labelled by $t, \bar{t}$ is determined by using the reduced characters discussed in section 5. The relevant partition functions $Z^{t, \bar{t}}$ can be obtained from the general partition function by taking the appropriate limits described there. In each case we seek expansions of the form

$$Z^{t, \bar{t}} = \sum_{\mathcal{M}} N_{\mathcal{M}} \hat{\chi}^{t, \bar{t}}_{\mathcal{M}} , \quad (7.8)$$

in terms of the appropriate reduced characters $\hat{\chi}^{t, \bar{t}}_{\mathcal{M}}$ corresponding to different supermultiplets $\mathcal{M}$, the integers $N_{\mathcal{M}}$ then determine the numbers of such multiplets in each sector of the theory. The supermultiplets which are accessible in the expansion (7.8) depend on $t, \bar{t}$, it is of course crucial that the resulting $N_{\mathcal{M}}$ are consistent between different sectors. This follows quite simply since both $Z^{t, \bar{t}}$ and $\hat{\chi}^{t, \bar{t}}_{\mathcal{M}}$ are related by setting various letters to zero.

$t = \bar{t} = \frac{1}{2}$ operators

The operators here are constructed from the single field $Z$. The single particle partition function is determined in (3.11),

$$Z^{\frac{1}{2}, \frac{1}{2}}(z) = \hat{\chi}^{\frac{1}{2}, \frac{1}{2}}_{1}(z) = z . \quad (7.9)$$

In this sector it is easy to see from (7.2), since $\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln(1 - x^k) = -x/(1 - x)$, (7.4) and (7.3)

$$Z^{\frac{1}{2}, \frac{1}{2}}_{s.t.}(z) = Z^{\frac{1}{2}, \frac{1}{2}}_{s.t., \text{sym.}}(z) = Z^{\frac{1}{2}, \frac{1}{2}}_{s.t., \frac{1}{2}-\text{BPS}}(z) = \frac{z^2}{1 - z} = \sum_{n=2}^{\infty} z^n . \quad (7.10)$$
Trivially for single trace operators there is just one symmetric $\frac{1}{2}$-BPS operator represented by $\text{tr}(Z^n)$ for $n = 2, 3, \ldots$

The multi-trace partition function is then simply given from (7.3) by

$$Z_{\text{m.t.}}^{\frac{1}{2}, \frac{1}{2}}(z) = \prod_{m=2}^{\infty} \frac{1}{1 - z^m},$$

and it follows immediately from (7.5) and (7.7), using (7.10), that

$$Z_{\text{m.t.}}^{\frac{1}{2}, \frac{1}{2}}(z) = Z_{\text{m.t., sym.}}^{\frac{1}{2}, \frac{1}{2}}(z) = Z_{\text{sugra}}^{\frac{1}{2}, \frac{1}{2}}(z).$$

The character expansion is here

$$Z_{\text{m.t.}}^{\frac{1}{2}, \frac{1}{2}}(z) = \sum_{n=0}^{\infty} N_{\frac{1}{2}, \frac{1}{2} - \text{BPS}}^{\text{m.t., n}} z^n. \quad (7.13)$$

Using

$$\prod_{k=1}^{\infty} \frac{1}{1 - z^k} = \sum_{n=0}^{\infty} p(n) z^n, \quad (7.14)$$

where $p(n)$ is the number of unrestricted partitions of the positive integer $n$ into a sum of strictly positive integers where the order of summands is irrelevant, with $p(0) = 1$, $p(-1) = 0$, we have

$$N_{\frac{1}{2}, \frac{1}{2} - \text{BPS}}^{\text{m.t., n}} = p(n) - p(n - 1), \quad n = 2, 3, \ldots, \quad (7.15)$$

which counts the number of $\frac{1}{2}$-BPS operators of conformal dimension $n$, $N_{0, \frac{1}{2} - \text{BPS}}^{\frac{1}{2}} = 1$ corresponds to the identity operator. The corresponding operators for the first few cases are summarised in the Table 2.

| $(\frac{1}{2}, \frac{1}{2})$ primary operators | Operators |
|---|---|
| $\Delta$ | $R_{\{0,0\}, [0, p, 0]}^{(0,0)}$, $R_{\{0,0\}, [0, 2, 0]}^{(0,0)}$, $R_{\{0,0\}, [0, 3, 0]}^{(0,0)}$, $2 R_{\{0,0\}, [0, 4, 0]}^{(0,0)}$, $2 R_{\{0,0\}, [0, 5, 0]}^{(0,0)}$, $4 R_{\{0,0\}, [0, 6, 0]}^{(0,0)}$, $4 R_{\{0,0\}, [0, 7, 0]}^{(0,0)}$, $7 R_{\{0,0\}, [0, 8, 0]}^{(0,0)}$ |
$t = \bar{t} = \frac{1}{4}$ operators

The relevant fields are the scalars $Z, Y$ with associated letters $z, y$. Here the single particle partition function from (6.12) is

$$Z^{\frac{1}{4},\frac{1}{4}}(z, y) = \hat{\chi}^{\frac{1}{4},\frac{1}{4}}(z, y) = z + y. \quad (7.16)$$

We also have from (7.4) and (7.6),

$$Z^{\frac{1}{4},\frac{1}{4}}_{s.t., sym.}(z, y) = Z^{\frac{1}{4},\frac{1}{4}}_{s.t., \frac{1}{2}-BPS}(z, y) = \sum_{n=2}^{\infty} \chi(n, 0)(z, y) = \frac{1}{(1 - z)(1 - y)} - z - y - 1. \quad (7.17)$$

This shows that all completely symmetric single trace operators are in fact the $\frac{1}{2}$-BPS ones which are related by $SU(2)$ lowering operators to those in the $(\frac{1}{2}, \frac{1}{2})$ sector. For general single trace operators from (7.2) in this case

$$Z^{\frac{1}{4},\frac{1}{4}}_{s.t.}(z, y) = -\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln(1 - z^k - y^k) - z - y. \quad (7.18)$$

For multi-trace operators the partition function becomes

$$Z^{\frac{1}{4},\frac{1}{4}}_{m.t.}(z, y) = (1 - z)(1 - y) \prod_{k=1}^{\infty} \frac{1}{1 - z^k - y^k}. \quad (7.19)$$

Using from (7.17) $Z^{\frac{1}{4},\frac{1}{4}}_{s.t., sym.}(z, y) = \sum_{k,l=0, k+l \geq 2}^{\infty} z^k y^l$ we also get

$$Z^{\frac{1}{4},\frac{1}{4}}_{m.t., sym.}(z, y) = Z^{\frac{1}{4},\frac{1}{4}}_{sugra}(z, y) = \prod_{k,l=0, k+l \geq 2}^{\infty} \frac{1}{1 - z^k y^l}. \quad (7.20)$$

To determine the decomposition into contributions corresponding to $\frac{1}{4}$-BPS multiplets $B_{[q,p,q]}^{\frac{1}{4},\frac{1}{4}}(0,0)$ multiplets we use from (5.3) the character expressed in terms of $z, y$,

$$\hat{\chi}_{(p+2q,p,q;0,0)}^{\frac{1}{4},\frac{1}{4}}(z, y) = \chi_{(p+q,q)}(z, y) = (zy)^q \frac{z^{p+1} - y^{p+1}}{z - y}, \quad (7.21)$$

which is just a $U(2)$ character. The required expansion is then

$$Z^{\frac{1}{4},\frac{1}{4}}_{m.t.}(z, y) = \sum_{n,m=0}^{\infty} N^{\frac{1}{4},\frac{1}{4}}_{m.t., nm} \chi_{(n,m)}(z, y), \quad (7.22)$$

$$Z^{\frac{1}{4},\frac{1}{4}}_{m.t., sym.}(z, y) = \sum_{n,m=0}^{\infty} N^{\frac{1}{4},\frac{1}{4}}_{m.t., sym., nm} \chi_{(n+m,m)}(z, y),$$
where the coefficients determine the number of primary operators belonging to the representation \( \mathcal{R}_{[m,n,m]}^{(0,0)} \). Expressions for the coefficients in some cases are obtained in appendix B. These give the following results

\[
\begin{align*}
N_{\text{m.t.},n0}^{\frac{1}{2}\frac{1}{2}} &= N_{\text{m.t.},\text{sym},n0}^{\frac{1}{2}\frac{1}{2}} = p(n) - p(n - 1), \quad n = 2, 3, \ldots, \\
N_{\text{m.t.},n1}^{\frac{1}{2}\frac{1}{2}} &= N_{\text{m.t.},\text{sym},n1}^{\frac{1}{2}\frac{1}{2}} = p(n) + p(n + 1) - p(n + 2), \quad n = 3, 4, \ldots,
\end{align*}
\]

(7.23)

Clearly from (7.13) \( N_{\text{m.t.},n0}^{\frac{1}{2}\frac{1}{2}} = N_{\text{m.t.},n}^{\frac{1}{2}\text{BPS}} \) which reflects that for \( m = 0 \) these are just the \( \frac{1}{2} \)-BPS operators in this sector. The operators corresponding to \( m = 1 \) are genuine \( \frac{1}{4} \)-BPS operators, for \( m \geq 2 \) these may combine with other operators to form long multiplets as discussed further later.

For single trace operators we clearly have from (7.17) \( N_{\text{s.t.},\text{sym},n0}^{\frac{1}{2}\frac{1}{2}} = 1, n = 2, 3, \ldots \) as expected since the relevant operators are all part of \( \frac{1}{2} \)-BPS multiplets. For general single trace operators from (7.18)

\[
\begin{align*}
N_{\text{s.t.},n0}^{\frac{1}{2}\frac{1}{2}} &= 1, \quad N_{\text{s.t.},n1}^{\frac{1}{2}\frac{1}{2}} = 0, \quad N_{\text{s.t.},n2}^{\frac{1}{2}\frac{1}{2}} = 1 + \left[ \frac{1}{2} n \right], \quad n = 0, 1, \ldots,
\end{align*}
\]

(7.24)

The last case corresponds to operators \( \text{tr}(YZ^rYZ^{n+2-r}), r = 0, 1, \ldots, 1 + \left[ \frac{1}{2} n \right], \) with the symmetric sum part of the \( \frac{1}{2} \)-BPS multiplet formed from \( \text{tr}(Z^n) \).

Corresponding to the results for the expansions (7.22), apart from the \( \frac{1}{2} \)-BPS operators, we list the required \( \frac{1}{4} \)-BPS operators for the first few cases in Table 3.

| \( \Delta \) | Symmetric operators | Remaining operators |
|---|---|---|
| 4 | \( \mathcal{R}_{[2,0,2]}^{(0,0)} \) | (1)\( \mathcal{R}_{[2,1,2]}^{(0,0)} \) |
| 5 | \( \mathcal{R}_{[2,1,2]}^{(0,0)} \) \( \mathcal{R}_{[1,3,1]}^{(0,0)} \) \( \mathcal{R}_{[2,2,2]}^{(0,0)} \) | (1)\( \mathcal{R}_{[2,1,2]}^{(0,0)} \) |
| 6 | \( \mathcal{R}_{[1,4,1]}^{(0,0)} \) \( 3 \mathcal{R}_{[2,2,2]}^{(0,0)} \) \( 3 \mathcal{R}_{[1,5,1]}^{(0,0)} \) \( 4 \mathcal{R}_{[2,3,2]}^{(0,0)} \) | 3(2)\( \mathcal{R}_{[2,2,2]}^{(0,0)} \) (1)\( \mathcal{R}_{[3,0,3]}^{(0,0)} \) |
| 7 | \( 2 \mathcal{R}_{[3,1,3]}^{(0,0)} \) \( 3 \mathcal{R}_{[3,1,3]}^{(0,0)} \) \( 4 \mathcal{R}_{[2,3,2]}^{(0,0)} \) | 4(2)\( \mathcal{R}_{[2,3,2]}^{(0,0)} \) 3(2)\( \mathcal{R}_{[3,1,3]}^{(0,0)} \) |
| 8 | \( 4 \mathcal{R}_{[3,1,3]}^{(0,0)} \) \( 8 \mathcal{R}_{[2,4,2]}^{(0,0)} \) \( 3 \mathcal{R}_{[3,2,3]}^{(0,0)} \) \( 4 \mathcal{R}_{[4,0,4]}^{(0,0)} \) | 8(3)\( \mathcal{R}_{[2,4,2]}^{(0,0)} \) 7(3)\( \mathcal{R}_{[3,2,3]}^{(0,0)} \) |

\( \frac{1}{4} \)-BPS primary operators with conformal dimensions \( \Delta \) belonging to representations \( \mathcal{R}_{[q,p,q]}^{(0,0)} \) as obtained from expansion of partition function. When present numbers of single trace operators are listed in parenthesis.
The results here are in accord with the explicit construction of $\frac{1}{4}$-BPS operators in various cases in [27].

\[ t = \bar{t} = \frac{1}{8} \] and \[ t = \frac{1}{4}, \bar{t} = \frac{1}{8} \] semi-short operators

The fields in these sectors are listed in (6.4) and (6.3). In terms of the variables \(z, y, a, b\) then the single particle partition function from (6.15) in the \(z, y, a, b\) is sufficient to count \((\bar{a}, \bar{b})\) semi-short multiplets. For \(\bar{a} = \bar{b} = 0\) the operators \(\tilde{O}_{[:k:p:q]}^{[:j]}\) corresponding to the operators which are simply those multi-trace operators obtained by products of single trace \(1, 4\)-operators.

Applying (7.26) the partition function for single trace operators is

\[ Z_{\text{s.t.}}(z, y, a, b) = -\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln \left(1 - \frac{z^k + y^k - a^k - b^k}{1 - \sigma^k}\right) - \frac{z + y - a - b}{1 - \sigma}. \]  
(7.26)

Corresponding to (7.6) the single trace partition function for \(\frac{1}{2}\)-BPS operators is

\[ Z_{\text{s.t.,} \frac{1}{2}-\text{BPS}}(z, y, a, b) = \sum_{p=2}^{\infty} \chi_{\frac{1}{2}, \frac{1}{2}}(z, y; -a, -b) \]

\[ = \frac{1}{1 - \sigma} \left(\frac{(1 - a)(1 - b)}{(1 - z)(1 - y)} - 1 - z - y + a + b\right), \]

using (6.13). With (7.25) the multi-trace partition function for this sector is given by

\[ Z_{\text{m.t.}}(z, y, a, b) = \prod_{n=0}^{\infty} \frac{(1 - z \sigma^n)(1 - y \sigma^n)}{(1 - a \sigma^n)(1 - b \sigma^n)} \]

\[ \times \prod_{m=1}^{\infty} \frac{1 - \sigma^m}{1 - \sigma^m - z^m - y^m + a^m + b^m}. \]

The partition function corresponding to the supergravity sector of the AdS dual theory corresponds to operators which are simply those multi-trace operators obtained by products of single trace \(\frac{1}{2}\)-BPS operators. This partition function is obtained from (7.7) by using (7.27),

\[ Z_{\text{sugra}}(z, y, a, b) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} Z_{\text{s.t.,} \frac{1}{2}-\text{BPS}}(z^n, y^n, a^n, b^n)\right) \]

\[ = \prod_{n=0}^{\infty} \frac{\prod_{k,l=0,k+l\geq 1}(1 - a z^k y^l \sigma^n)(1 - b z^k y^l \sigma^n)}{\prod_{k,l=0,k+l\geq 2}(1 - z^k y^l \sigma^n) \prod_{k,l=0}(1 - ab z^k y^l \sigma^n)}. \]

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In order to decompose these partition functions in terms of contributions for different semi-short multiplets by a character expansion it is first necessary to subtract the contribution of $\frac{1}{2}$-BPS multiplets. Using (7.13) this is given by

$$
Z_{\text{m.t.}}^{\frac{1}{4}, \frac{1}{8}}(z, y, a, b) = \sum_{n=0}^{\infty} \left( p(n) - p(n-1) \right) \chi_n^{\frac{1}{4}, \frac{1}{8}}(z, y; -a, -b) \\
= \frac{(z - a) (y - b)}{(1 - \sigma) z - y} \prod_{m=2}^{\infty} \frac{1}{1 - z^m} + z \leftrightarrow y.
$$

From (7.26) and (7.27)

$$
Z_{\text{sugra}}^{\frac{1}{4}, \frac{1}{8}}(z, y, z, b) = - \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln (1 - y^k) - y = Z_{\text{sugra,} \frac{1}{2}-\text{BPS}}^{\frac{1}{4}, \frac{1}{8}}(z, y, z, b) = \frac{y^2}{1 - y},
$$

with similar results for $z = b$ and $y = a, b$. In consequence we have

$$
Z_{\text{m.t.}}^{\frac{1}{4}, \frac{1}{8}}(z, y, z, b) = Z_{\text{sugra}}^{\frac{1}{4}, \frac{1}{8}}(z, y, z, b) = Z_{\text{m.t.}}^{\frac{1}{4}, \frac{1}{8}}(z, y, z, b) = \prod_{m=2}^{\infty} \frac{1}{1 - y^m},
$$

and analogously for $z = b$ and $y = a, b$.

Initially we set $b = 0$, so that no derivatives are present, and consider expansions determining the contributions of $(\frac{1}{4}, \frac{1}{8})$ multiplets. The relevant characters are obtained from (5.14) and (5.15). After subtracting the $\frac{1}{2}$-BPS contribution, given by (7.30) for $b = 0$, then (7.32) shows that the result vanishes for $z, y = a$ and we can write

$$
Z_{\text{m.t.}}^{\frac{1}{4}, \frac{1}{8}}(z, y, a, 0) - Z_{\text{m.t.}}^{\frac{1}{4}, \frac{1}{8}}(z, y, a, 0) \\
= \frac{(1 - z)(1 - y)}{1 - a} \prod_{m=1}^{\infty} \frac{1}{1 - z^m - y^m + a^m} - \frac{1}{z - y} \left( (z - a) \prod_{k=2}^{\infty} \frac{1}{1 - z^k} - (y - a) \prod_{k=2}^{\infty} \frac{1}{1 - y^k} \right) \\
= (z - a)(y - a) \sum_{n,m=0}^{\infty} \sum_{2j=-1}^{\infty} N_{\text{m.t.},nm,\bar{j}}^{\frac{1}{4}, \frac{1}{8}}(-a)^{2j+1} \chi_{(n+m,m)}(z, y),
$$

$$
Z_{\text{sugra}}^{\frac{1}{4}, \frac{1}{8}}(z, y, a, 0) - Z_{\text{m.t.}}^{\frac{1}{4}, \frac{1}{8}}(z, y, a, 0) \\
= \frac{\prod_{k,l=0,k+l \geq 1}^{\infty} (1 - a z^k y^l)}{\prod_{k,l=0,k+l \geq 2}^{\infty} (1 - z^k y^l)} - \frac{1}{z - y} \left( (z - a) \prod_{k=2}^{\infty} \frac{1}{1 - z^k} - (y - a) \prod_{k=2}^{\infty} \frac{1}{1 - y^k} \right) \\
= (z - a)(y - a) \sum_{n,m=0}^{\infty} \sum_{2j=-1}^{\infty} N_{\text{sugra,}nm,\bar{j}}^{\frac{1}{4}, \frac{1}{8}}(-a)^{2j+1} \chi_{(n+m,m)}(z, y).
$$

Here $N_{\text{m.t.},pq,\bar{j}}^{\frac{1}{4}, \frac{1}{8}}$ and $N_{\text{sugra},pq,\bar{j}}^{\frac{1}{4}, \frac{1}{8}}$ are then the number of $t = \frac{1}{4}$, $\bar{t} = \frac{1}{8}$ multiplets which contribute to the relevant partition functions and whose primary operators belong to the
representation labelled by $R_{[q+2\bar{q}+q,p,q]}^{(0,\bar{j})}$ and has a scale dimension $\Delta = 3 + 3\bar{j} + p + 2q$. It is easy to see, taking $a = 0$ when only the $\bar{j} = -\frac{1}{2}$ term survives in the sum, that by comparing with the expansion (5.22) we have

$$N_{\text{m.t.},nm,-\frac{1}{2} \frac{1}{2}} = N_{\text{m.t.},nm,0}^{\frac{3}{4}}$$

$$N_{\text{sugra},nm,-\frac{1}{2} \frac{1}{2}} = N_{\text{sugra},nm,0}^{\frac{3}{4}}$$

(7.34)

in accord with (3.24). For this case, as shown later, $N_{\text{sugra},nm,\bar{j}} = N_{\text{m.t.},\text{symm.,nm},\bar{j}}$ involving operators formed from symmetrised traces. Analytic formulae for the expansion coefficients are harder to obtain but for the lowest dimension operators results are given in Table 4.

### Table 4

| $\Delta - \bar{j}$ | Symmetric operators | Remaining operators | $(\frac{1}{4}, 0)$ primary | $(\frac{1}{4}, 0)$ descendant |
|---------------------|---------------------|---------------------|-----------------------------|-------------------------------|
| 3                   |                     |                     | $(1)R_{[2,0,0]}^{(0,0)}$ | $R_{[3,1,0]}^{(0,0)}$        |
| 4                   |                     |                     | $(1)R_{[2,1,0]}^{(0,0)}$ | $R_{[3,2,0]}^{(0,0)}$        |
| 5                   | $R_{[3,2,1]}^{(0,0)}$ |                     | $(1)R_{[2,0,0]}^{(0,0)}$, $R_{[3,0,0]}^{(0,0)}$, $R_{[3,1,0]}^{(0,0)}$, $R_{[4,0,0]}^{(0,0)}$ | $R_{[3,1,0]}^{(0,0)}$, $(1)R_{[4,0,1]}^{(0,0)}$ |
| 6                   |                     | $R_{[3,2,1]}^{(0,0)}$ | $(1)R_{[2,0,0]}^{(0,0)}$, $R_{[3,0,0]}^{(0,0)}$, $R_{[3,1,0]}^{(0,0)}$, $R_{[4,0,0]}^{(0,0)}$ | $R_{[3,1,0]}^{(0,0)}$, $(1)R_{[4,0,1]}^{(0,0)}$ |
| 7                   | $R_{[3,2,1]}^{(0,0)}$ | $R_{[3,2,1]}^{(0,0)}$ | $(1)R_{[2,0,0]}^{(0,0)}$, $R_{[3,0,0]}^{(0,0)}$, $R_{[3,1,0]}^{(0,0)}$, $R_{[4,0,0]}^{(0,0)}$ | $R_{[3,1,0]}^{(0,0)}$, $(1)R_{[4,0,1]}^{(0,0)}$ |
| 8                   | $2R_{[3,1,1]}^{(0,0)}$, $R_{[4,1,2]}^{(0,0)}$ | $R_{[3,2,1]}^{(0,0)}$ | $(1)R_{[2,0,0]}^{(0,0)}$, $R_{[3,0,0]}^{(0,0)}$, $R_{[3,1,0]}^{(0,0)}$, $R_{[4,0,0]}^{(0,0)}$ | $R_{[3,1,0]}^{(0,0)}$, $(1)R_{[4,0,1]}^{(0,0)}$ |

$(\frac{1}{4}, 0)$ primary operators belonging to representations $R_{[q+2\bar{q}+2,p,q]}^{(0,\bar{j})}$ as obtained from expansion of partition function. When present numbers of single trace operators are listed in parenthesis.

In the general case we consider expansions of the form

$$Z_{\text{m.t.}}^{\frac{1}{4} \frac{1}{2}}(z, y, a, b) - Z_{\text{m.t.}}^{\frac{1}{4} \frac{1}{2}}(z, y, a, b) = \sum_{nm,j,\bar{j}} N_{\text{m.t.},nm,j}^{\frac{1}{4} \frac{1}{2}} \hat{\chi}_{nm,j}(z, y, -a, -b),$$

$$Z_{\text{sugra}}^{\frac{1}{4} \frac{1}{2}}(z, y, a, b) - Z_{\text{m.t.}}^{\frac{1}{4} \frac{1}{2}}(z, y, a, b) = \sum_{nm,j,\bar{j}} N_{\text{sugra},nm,j}^{\frac{1}{4} \frac{1}{2}} \hat{\chi}_{nm,j}(z, y, -a, -b),$$

(7.35)

where, using the results (5.20) and (5.21) for the characters in the relevant limit,

$$\hat{\chi}_{nm,j}(z, y, a, b) = \frac{(z + a)(y + a)(z + b)(y + b)}{1 - \sigma} \alpha^{2j + 1} \beta^{2j + 1} \chi_{(n+m,m)}(z, y).$$

(7.36)
The left hand sides in (7.33) has a factor \((z-a)(y-a)(z-b)(y-b)\) as a consequence of (7.32) and the symmetry under \(z \leftrightarrow y, a \leftrightarrow b\). In the summation \(n = 0, 1, 2, \ldots, j, j = -\frac{1}{2}, 0, \frac{1}{2}, \ldots\) while \(m\) may take negative values, \(m \geq -2j - 2, -2j - 2\). The expansion coefficients (7.33) \(N_{m.t.,nm,jj}^{\frac{1}{2},\frac{1}{2}}\) and \(N_{sugra,nm,jj}^{\frac{1}{2},\frac{1}{2}}\) then give the numbers of semi-short primary operators in the representation \(R_{[m+2j+2,n,m+2j+2]}^{(j,j)}\), along with their descendants, determined by the partition function. It is easy to see that

\[
N_{m.t.,nm,jj}^{\frac{1}{2},\frac{1}{2}} = N_{sugra,nm,jj}^{\frac{1}{2},\frac{1}{2}}, \quad N_{sugra,nm,jj}^{\frac{1}{2},\frac{1}{2}} = N_{sugra,nm,jj}^{\frac{1}{2},\frac{1}{2}},
\]

(7.37)

and, setting \(b = 0\),

\[
N_{m.t.,nm,\frac{1}{2},-\frac{1}{2}j}^{\frac{1}{2},\frac{1}{2}} = N_{m.t.,nm+1,\frac{1}{2}j}^{\frac{1}{2},\frac{1}{2}}, \quad N_{sugra,nm,\frac{1}{2},-\frac{1}{2}j}^{\frac{1}{2},\frac{1}{2}} = N_{sugra,nm+1,\frac{1}{2}j}^{\frac{1}{2},\frac{1}{2}},
\]

(7.38)

As emphasised in section 3 semi-short multiplets may combine to form long multiplets for which there are no shortening conditions, as shown diagrammatically in (3.27) or (3.34). It is crucial, in order to determine when anomalous dimensions are allowed in the interacting theory, to identify when semi-short multiplets may combine in this fashion. All semi-short multiplets which are present in the expansion of \(Z_{m.t.}^{\frac{1}{2},\frac{1}{2}}\) after subtracting those coming from \(Z_{sugra}^{\frac{1}{2},\frac{1}{2}}\) do so combine. To demonstrate this we note that (as first observed in §)

\[
Z_{s.t.}^{\frac{1}{2},\frac{1}{2}}(z, y, zy, b) = Z_{s.t.}^{\frac{1}{2},\frac{1}{2}}(z, y, zy, b),
\]

(7.39)

with a corresponding formula in the multi-trace case following directly from this,

\[
Z_{m.t.}^{\frac{1}{2},\frac{1}{2}}(z, y, zy, b) = Z_{m.t.,sugra}^{\frac{1}{2},\frac{1}{2}}(z, y, zy, b).
\]

(7.40)

Identical results to (7.39) and (7.40) also follow for \(b = zy\). Following from (7.40) we may then write

\[
Z_{m.t.}^{\frac{1}{2},\frac{1}{2}}(z, y, a, 0) - Z_{sugra}^{\frac{1}{2},\frac{1}{2}}(z, y, a, 0) = \left(1 - \frac{zy}{a}\right)(z-a)(y-a) \sum_{n,m,j} \hat{N}_{nm,j}^{\frac{1}{2},\frac{1}{2}}(-a)^{2j+1} \chi_{(n+m,m)}(z, y),
\]

(7.41)

and also

\[
Z_{m.t.}^{\frac{1}{2},\frac{1}{2}}(z, y, a, b) - Z_{sugra}^{\frac{1}{2},\frac{1}{2}}(z, y, a, b) = \left(1 - \frac{zy}{a}\right) \left(1 - \frac{zy}{b}\right) \sum_{n,m,j} \hat{N}_{long,nm,jj}^{0,0} \chi_{nm,jj}(z, y, -a, -b).
\]

(7.42)

Using the result (5.16) the right hand side in (7.41) involves only an expansion in terms of \(t = \frac{1}{4}, \bar{t} = 0\) characters at the unitarity threshold where \(\bar{\delta}, \delta\), as defined in (3.28),
are both zero. As a consequence all \((\frac{1}{4}, 1)\) operators in the right-most column of Table 3, can combine with certain \((\frac{1}{4}, \frac{1}{8})\) semi-short operators in the third column of Table 4 to form \((\frac{1}{4}, 0)\) multiplets. Then the remaining \((\frac{1}{4}, \frac{1}{8})\) operators are in pairs, in the third and fourth columns of Table 4, as necessary to construct further \((\frac{1}{4}, 0)\) \(\frac{1}{8}\)-BPS multiplets. These combinations of multiplets are a reflection of the character identities, following from (3.21) and (3.24),

\[
\chi_{(\Delta;2+2j+q,p,q;0,j)}^{\frac{1}{4},0} = \chi_{(\Delta;2+2j+q,p,q;0,j)}^{\frac{1}{4},\frac{1}{2}} + \chi_{(\Delta+\frac{1}{4},\frac{1}{2}+2j+q,p,q+1;0,j-\frac{1}{2})}^{\frac{1}{4},0} \tag{7.43}
\]

where \(\Delta = 3 + 3j + p + 2q\). In terms of the coefficients in (7.33) we have

\[
N_{m.t.,nm,j}^{\frac{1}{4},\frac{1}{8}} - N_{sugra,nm,j}^{\frac{1}{4},\frac{1}{8}} = \tilde{N}_{nm,j}^{\frac{1}{4},0} + \tilde{N}_{n m-1,j+\frac{1}{2}}^{\frac{1}{4},0} \tag{7.44}
\]

In a similar fashion (7.22) shows that in (7.42) only characters for long multiplets with \(\delta = \bar{\delta} = 0\) are present in the expansion. This indicates that all primary operators which are not supergravity dual states can kinematically join together to form long operators, as in (3.27). Assuming, as discussed in the next section, such operators actually do combine dynamically and that they are the only ones that do so in this sector we may then read off the number of long multiplets in the interacting theory with anomalous dimensions for each representation. Hence from (7.42)

\[
N_{m.t.,nm,j}^{\frac{1}{4},\frac{1}{8}} - N_{sugra,nm,j}^{\frac{1}{4},\frac{1}{8}} = \tilde{N}_{long,nm,j}^{0,0} + \tilde{N}_{long,n m-1,j+\frac{1}{2}}^{0,0} + \tilde{N}_{long,n m-2,j+\frac{1}{2} j+\frac{1}{2}}^{0,0} \tag{7.45}
\]

For \(b = 0\) setting \(a = zy\) removes all contributions which may combine to form \((\frac{1}{4}, 0)\) multiplets in (7.33). Thus

\[
Z_{sugra}(z, y, zy, 0) - Z_{m.t.}^{\frac{1}{4},\frac{1}{8}}(z, y, zy, 0)
= \frac{1}{1-zy} \prod_{k=2}^{\infty} \frac{1}{(1-z^k)(1-y^k)} - \frac{z(1-y)}{z-y} \prod_{k=2}^{\infty} \frac{1}{1-z^k} + \frac{y(1-z)}{z-y} \prod_{k=2}^{\infty} \frac{1}{1-y^k}
= (1-z)(1-y)(zy)^2 \sum_{n,m} \chi_{(n+m,m)}(z, y), \tag{7.46}
\]

where, assuming (7.34),

\[
I_{nm}^{\frac{1}{4},\frac{1}{8}} = \sum_{r=-1}^{m} (-1)^{r+1} N_{sugra,n m-r,\frac{1}{2} r}^{\frac{1}{4},\frac{1}{8}}, \hspace{1cm} n \geq 0, m \geq -1. \tag{7.47}
\]
Here $I_{\frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}}$ is an index [14] whose magnitude determines a lower bound for the number of possible $t = \frac{1}{4}$, $\ell = \frac{1}{8}$ multiplets. For this index we find

$$I_{\frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}} = p(n+m+2)p(m+2) - \sum_{r=1}^{m+2} (p(n+m+4-r) - p(n+m+2-r))p(m+2-r). \quad (7.48)$$

Particular results are easily found, for the lowest $n, m$,

| $I_{\frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}}$ |
|---|---|---|---|---|---|---|
| $n$ | $m$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
| 0 | 0 | 1 | 0 | 4 | -2 | 14 |
| 1 | 0 | 1 | 2 | 4 | 6 | 17 |
| 2 | 0 | 3 | 2 | 12 | 7 | 27 |
| 3 | 1 | 4 | 7 | 16 | 20 | 70 |
| 4 | 1 | 8 | 9 | 29 | 29 | 92 |
| 5 | 3 | 11 | 17 | 41 | 55 | 176 |

The result here are in accord with the numbers of symmetric operators listed in Tables 3 and 4.

To analyse those $(\frac{1}{8}, \frac{1}{8})$ multiplets which cannot form long or $(\frac{1}{8}, 0)$ or $(0, \frac{1}{8})$ multiplets we may set $a = b = zy$ when we obtain

$$Z_{\text{sugra}}(z, y, zy, zy) - Z_{\text{m.t.}}^{\frac{1}{4}, \frac{1}{8}}(z, y, zy, zy)$$

$$= \frac{1}{1 - zy} \prod_{k=2}^{\infty} \frac{1}{1 - z^k(1 - y^k)(1 - z^k y^k)}$$

$$- \frac{1}{(1 - zy)(z - y)} \left( z(1 - y)^2 \prod_{k=2}^{\infty} \frac{1}{1 - z^k} - y(1 - z)^2 \prod_{k=2}^{\infty} \frac{1}{1 - y^k} \right) \quad (7.49)$$

$$= \frac{(1 - z)^2(1 - y)^2}{1 - zy} (zy)^2 \sum_{n,m} I_{\frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}} \chi(n+m,m)(z, y) ,$$

where

$$I_{\frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}} = \sum_{r,s=-1}^{m} (-1)^{r+s} N_{\text{sugra}, n-m-r-s=2, \frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}}, \quad n \geq 0, m \geq -1. \quad (7.50)$$

When $r, s = -1$ it is necessary to use (7.38) and (7.34). For the special case $m = -1$, from (7.47) and (7.50),

$$I_{\frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}} = I_{\frac{1}{4}, \frac{1}{8}}^{\frac{1}{4}, \frac{1}{8}} = N^{\frac{1}{4}, \frac{1}{8}}_{\text{m.t., sym}, n_{11}}, \quad (7.51)$$

which is the number of $\frac{1}{4}$-BPS multiplets belonging to the representation $R^{(0,0)}_{[1, n_{11}]}$, given explicitly in (7.23). As remarked in [19] such multiplets are always protected in that they cannot be combined to form long multiplets. In other cases expansion of (7.49) gives
A third class of operators which are interesting to examine are the operators formed from products of symmetrised traces. For the cases discussed previously these were identical to the operators present in the supergravity limit. For \((\frac{1}{8}, \frac{1}{8})\) operators, which generate semi-short representations, there are symmetric operators which are not present in supergravity limit (the simplest example being the Konishi operator). For completely symmetric single trace operators the partition function in this sector is given by (7.4)

\[
Z_{1/8, 1/8} = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} Z_{1/8, 1/8}^{1/2}(z^m, y^m, a^m, b^m) \right) - Z_{1/8, 1/8}^{1/2}(z, y, a, b) - 1
\]

\[
= \prod_{n=0}^{\infty} \frac{(1-a \sigma^n)(1-b \sigma^n)}{(1-z \sigma^n)(1-y \sigma^n)} - \frac{1}{1-\sigma}(z + y - \sigma - a - b + 1).
\]

The multi-trace operators then formed from symmetric single trace operators may then be counted from (7.5) using the partition function,

\[
Z_{1/8, 1/8}^{1/2}(z, y, a, b) = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} Z_{1/8, 1/8}^{1/2}(z^m, y^m, a^m, b^m) \right),
\]

Since, from (7.52), \(Z_{1/8, 1/8}^{1/2}(z, y, a, 0) = Z_{s.t., sym}^{1/2}(z, y, a, 0)\), as given in (7.27), then also \(Z_{m.t., sym}^{1/8}(z, y, a, 0) = Z_{s.t.}^{1/8}(z, y, a, 0)\), so that the supergravity and symmetric operator partition functions are again identical in the \((\frac{1}{8}, \frac{1}{8})\) sector. Furthermore as \(Z_{1/8, 1/8}^{1/2}(z, y, z, b) = Z_{s.t.}^{1/8}(z, y, a, 0)\), as in (7.31), we may replace \(Z_{m.t.}^{1/8}(z, y, a, b) \rightarrow Z_{m.t., sym}^{1/8}(z, y, a, b)\) in (7.35) with \(N_{1/8, 1/8} \rightarrow N_{1/8, 1/8}^{1/8}\), to give the spectrum of symmetric primary operators, excluding those belonging to \(\frac{1}{2}\)-BPS multiplets, in the \((\frac{1}{8}, \frac{1}{8})\) sector of \(\mathcal{N} = 4\) superconformal Yang Mills theory.

The representation content for the various highest operators that arise can be conve-
nently summarised by the following spaces

\[ S_{\text{sugra}} = \bigoplus_{nm, jj} N_{\text{sugra}, nm, jj}^\frac{1}{2}, \frac{1}{2} R_{[m+2j+2, n, m+2j+2]}^{(j, j)} ; \]
\[ S_{\text{sym.}} = \bigoplus_{nm, jj} N_{\text{sym.}, nm, jj}^\frac{1}{2}, \frac{1}{2} R_{[m+2j+2, n, m+2j+2]}^{(j, j)} ; \]
\[ S_{\text{long}} = \bigoplus_{nm, jj} N_{\text{long}, nm, jj}^0, 0 R_{[m+2j+2, n, m+2j+2]}^{(j, j)} ; \]
\[ S_{\text{free}} = \bigoplus_{nm, jj} N_{\text{sym.}, nm, jj}^\frac{1}{2}, \frac{1}{2} R_{[m+2j+2, n, m+2j+2]}^{(j, j)} ; \]

(7.54)

where \( S_{\text{sugra}} \) is then formed from the set of representations, with multiplicities, for highest weight states of supergravity dual multiplets (i.e. those formed from tensor products of \( \frac{1}{2} \)-BPS states), \( S_{\text{sym.}} \) correspondingly from the set of highest weight representations of supermultiplets formed from symmetric traces other than those in \( S_{\text{sugra}} \), \( S_{\text{long}} \) from the set of highest weight representations for long supermultiplets, and \( S_{\text{free}} \) from the set of all highest weight \( (\frac{1}{2}, \frac{1}{2}) \) representations for all supermultiplets in the free theory. We claim that these spaces of operators are all nested as follows.\textsuperscript{10}

\[ S_{\text{sugra}} \subset S_{\text{sym.}} ; \quad S_{\text{sym.}}/S_{\text{sugra}} \subset S_{\text{long}} \subset S_{\text{free}} ; \]

(7.55)

requiring \( 0 \leq N_{\text{sym.}, nm, jj}^\frac{1}{2}, \frac{1}{2} - N_{\text{sugra}, nm, jj}^\frac{1}{2}, \frac{1}{2} \leq \tilde{N}_{\text{long}, nm, jj}^0, 0 \). The assumption here that all symmetric operators, other than those dual to the supergravity sector, are primary operators for long multiplets is discussed further in the next section.

The results, obtained using the above formulae for character expansions, in the simplest cases are presented in appendix C. Table 6 lists those operators dual to the supergravity sector obtained from \( N_{\text{sugra}, nm, jj}^\frac{1}{2}, \frac{1}{2} \) in \((7.35)\), Tables 7 and 8 list those obtained from \((7.42)\) which form primary operators for long multiplets, divided into those which are formed from symmetric traces other than those in the supergravity sector and also those which are not, while Table 9 lists all remaining \( (\frac{1}{2}, \frac{1}{2}) \) operators which are descendants for those in Tables 7 and 8. Tables 10 and 11 give the same information as in Tables 7 and 8 but for single trace operators.

\( t = \frac{1}{4}, \bar{t} = 0 \) operators

In this sector the operators are constructed from the fields in \((6.5)\), with associated letters \( z, y, x, a, \bar{a} \). The single particle partition function is given, from \((6.13)\), by

\[ Z_{1/4, 0}^{1/2, 0}(z, y, x, a, \bar{a}) = \lambda_{1/2, 0}^{1/4, 0}(z, y, x; -a, -\bar{a}) = z + y + x - a - \bar{a} . \]

(7.56)

The multi-trace partition function then takes the form

\[ Z_{\text{m.t.}}^{1/4, 0}(z, y, x, a, \bar{a}) = \frac{(1 - z)(1 - y)(1 - x)}{(1 - a)(1 - \bar{a})} \prod_{m=1}^\infty \frac{1}{1 - z^m - y^m - x^m + a^m + \bar{a}^m} . \]

(7.57)

\textsuperscript{10} Previously discussed cases are much simpler. There we have \( S_{\text{sugra}} = S_{\text{sym.}} \subset S_{\text{free}} \).
We also find from (7.4) and (7.6),
\[ Z^{\frac{1}{4},0}_{\text{s.t., sym.}}(z, y, x, a, \bar{a}) = Z^{\frac{1}{4},0}_{\text{s.t., BPS}}(z, y, x, a, \bar{a}) = \sum_{p=2}^{\infty} \hat{\chi}^{\frac{1}{4},0}_p(z, y, x; -a, -\bar{a}) \]
\[ = \frac{(1 - a)(1 - \bar{a})}{(1 - z)(1 - y)(1 - x)} - z - y - x + a + \bar{a} - 1 \]
\[ = \sum_{k, l, m \geq 0, r, s = 0, 1} z^k y^l x^m (-a)^r (-\bar{a})^s, \]
using (6.13) for the $\frac{1}{2}$-BPS characters. In this sector symmetric single trace operators are identical with $\frac{1}{2}$-BPS operators. It follows that for the multi-trace partition functions
\[ Z^{\frac{1}{4},0}_{\text{m.t., sym.}}(z, y, x, a, \bar{a}) = Z^{\frac{1}{4},0}_{\text{sugra}}(z, y, x, a, \bar{a}), \]
with explicitly
\[ Z^{\frac{1}{4},0}_{\text{m.t., sym.}}(z, y, x, a, \bar{a}) = \prod_{k, l, m \geq 0, k + l + m \geq 1} (1 - a z^k y^l x^m)(1 - \bar{a} z^k y^l x^m) \]
\[ \times (1 - z^k y^l x^m) \prod_{k, l, m \geq 0, k + l + m \geq 2} (1 - a \bar{a} z^k y^l x^m). \]

To disentangle the contributions of various supermultiplets we require expansions of the form
\[ Z^{\frac{1}{4},0}_{\text{m.t.}}(z, y, x, a, \bar{a}) = \sum_{p=2}^{\infty} N^{\frac{1}{4}}_{\text{m.t., p}} \hat{\chi}^{\frac{1}{4},0}_p(z, y, x; -a, -\bar{a}) \]
\[ + \sum_{p, q = 0}^{\infty} \sum_{2j = -1}^{\infty} N^{\frac{1}{4}}_{\text{m.t., pq}, j} \chi^{\frac{1}{4},0}_{\text{semi, pq}, j}(z, y, x; -a, -\bar{a}) \]
\[ + \sum_{k, p, q = 0}^{\infty} \sum_{2j = 0}^{\infty} N^{\frac{1}{4}}_{\text{m.t., } [k, p, q], j} \hat{\chi}^{\frac{1}{4},0}_{[k, p, q], j}(z, y, x; -a, -\bar{a}), \]

involving $\frac{1}{2}$-BPS characters given in (6.13) as well as
\[ \hat{\chi}^{\frac{1}{4},0}_{[k, p, q], j}(z, y, x; a, \bar{a}) = (xyz)^{\frac{1}{4}(k-q-2j-4)} \chi_{(p+q, q, 0)}(z, y, x) \chi_{(2j, 0)}(a, \bar{a}) \]
\[ \times (z + a)(y + a)(x + a)(z + \bar{a})(y + \bar{a})(x + \bar{a}), \]
which is obtained from (5.8) and (5.9) involving $U(3)$ and $U(2)$ characters $\chi_{(p+q, q, 0)}$ and $\chi_{(2j, 0)}$, and also the reduced characters calculated from (4.36),
\[ \chi^{\frac{1}{4},0}_{\text{semi, pq}, j}(z, y, x; a, \bar{a}) = \mathcal{M}_{xy}^{S_1} \mathcal{M}_{aa}^{S_2} ((yz)^{-1} C_{(p+q, q, 0)}(z, y, x) C_{(2j, 0)}(a, \bar{a}) \]
\[ \times (z + a)(y + a)(x + a)(z + \bar{a})(y + \bar{a})(x + \bar{a}), \]

In (7.62) the Weyl symmetrisers over $x, y, z$ and $a, \bar{a}$ act so as to generate an expression for this semi-short character in terms of a sum over standard $U(3)$ and $U(2)$ characters. In the
limit \( x, \bar{a} \to 0 \), with \( \bar{a}/x = yz/a \), the expansion reduces to that in (7.33) and (7.41). With a similar expansion to (7.60) for \( Z_{m t s y m}^{\frac{1}{4},0}(z, y, x, a, \bar{a}) \), in addition to the \((\frac{1}{4}, \frac{1}{4})\) and \((\frac{1}{4}, \frac{1}{8})\) operators tabulated previously, the first few necessary primary \((\frac{1}{4}, 0)\) \( \frac{1}{8}\)-BPS operators are listed in Table 5.

| \( \Delta - \bar{j} \) | Symmetric operators | Remaining operators |
|------------------------|------------------|-------------------|
| 6                      | \( \mathcal{R}^{(0,0)}_{[4,0,0]} \) | 4(2)\( \mathcal{R}^{(0,0)}_{[4,0,0]} \) |
| 7                      | \( \mathcal{R}^{(0,0)}_{[4,1,0]} \) | 7(3)\( \mathcal{R}^{(0,0)}_{[4,1,0]} \), 4(2)\( \mathcal{R}^{(0,1)}_{[5,0,0]} \) |
| 8                      | 3\( \mathcal{R}^{(0,0)}_{[4,2,0]} \) | 24(8)\( \mathcal{R}^{(0,0)}_{[4,2,0]} \), 12(5)\( \mathcal{R}^{(0,0)}_{[5,0,1]} \) |
|                        |                  | 22(8)\( \mathcal{R}^{(0,0)}_{[5,1,0]} \), 8(3)\( \mathcal{R}^{(0,1)}_{[6,0,0]} \) |

\((\frac{1}{4}, 0)\) primary operators belonging to representations \( \mathcal{R}^{(0,j)}_{[k,p,q]} \), with \( k - q > 2 + 2j \), as obtained from expansion of partition function. When present numbers of single trace operators are listed in parenthesis.

\( t = \frac{1}{8}, \bar{t} = 0 \) semi-short operators

The fields for this sector are listed in (6.2). All the various \((t, \bar{t})\) considered above are subsumed as special cases by setting various letters to zero. We have not attempted in this paper to extend at the same level of detail the previous discussion to this sector, it is considered in [14]. Of course the results already obtained are a necessary corollary of the expansions of the partition functions for this sector in terms of appropriate characters although the algebraic complexity increases significantly.

We list below a few salient formulae in the notation of this paper for future reference.

The basic single particle partition function is given by (6.16), with \( xyz = a\bar{a} \) and \( \lambda = b/yz \),

\[
Z_{\frac{1}{4},0}(z, y, x, a, \bar{a}, \lambda) = \hat{\chi}_{1,0}^{\frac{1}{4},0}(z, y, x; -a, -\bar{a}, -b) \\
= \frac{1}{(1 - \lambda a)(1 - \lambda \bar{a})} (z + y + x - a - \bar{a} - \lambda(yz + zx + xy) + xyz(\lambda^2 + \lambda)).
\]  

(7.63)

The corresponding single trace partition function for \( \frac{1}{2}\)-BPS operators is

\[
Z_{s.t., \frac{1}{2}-BPS}^{\frac{1}{4},0}(z, y, x, a, \bar{a}, \lambda) = \sum_{p=2}^{\infty} \hat{\chi}_{p}^{\frac{1}{4},0}(z, y, x; -a, -\bar{a}, -b).
\]  

(7.64)
To calculate this starting from (6.16) we may use
\[
\sum_{p=1}^{\infty} \hat{\chi}_p \hat{\delta}^0(z, y, x; -a, -\bar{a}, -b)
\]
\[
= \frac{1}{(1 - \lambda a)(1 - \lambda \bar{a})} \wp_{xyz}^s \left( \frac{(1 - z^{-1}a)(1 - z^{-1}\bar{a})(1 - \lambda x)(1 - \lambda y)}{(z - y)(z - x)(y - x) \frac{z^3y}{1 - z}} \right)
\]
\[
= \frac{1}{(1 - \lambda a)(1 - \lambda \bar{a})(1 - x)(1 - y)(1 - z)} \times \left( (x + y + z)(1 - \lambda xyz) - (\lambda + 1)(xy + yz + zx - 2xyz) + \lambda^2xyz(1 + xyz) \right.
\]
\[
\left. - (a + \bar{a})(1 - \lambda(xy + yz + zx - xyz) + \lambda^2xyz) \right)
\]
\[
(7.65)
\]
With this we may obtain
\[
Z_{s.t., BPS}^{\frac{1}{3}, 0}(z, y, x, a, \bar{a}, 1) = \frac{x}{1 - x} + \frac{y}{1 - y} + \frac{z}{1 - z} - \frac{a}{1 - a} - \frac{\bar{a}}{1 - \bar{a}} + \frac{(1 - x)(1 - y)(1 - z)}{(1 - a)(1 - \bar{a})} - 1, \tag{7.66}
\]
agreeing with comparable results in [14].

8. Further Remarks

The results of the analysis of partition functions in the previous section and also in [14] may be summarised in part as follows

- In the \((\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, 0)\) sectors, which are composed of what are generally referred to as \(\frac{1}{2}\), \(\frac{1}{4}\), \(\frac{1}{8}\)-BPS operators, as well as the \((\frac{1}{4}, \frac{1}{8})\) sector, the set of operators obtained from symmetrised traces is identical with the supergravity dual operators formed by products of corresponding \(\frac{1}{2}\)-BPS single trace operators.

- In the \((\frac{1}{8}, \frac{1}{8})\) sector supergravity dual operators are a subset of the symmetric operators.

- All operators in the \((\frac{1}{8}, \frac{1}{8})\) and \((\frac{1}{4}, \frac{1}{8}), (\frac{1}{4}, \frac{1}{8})\) sectors, other than those dual to supergravity fields, are potentially part of long multiplets and may gain an anomalous dimension in the interacting theory. Highest weight symmetric operators in the \((\frac{1}{8}, \frac{1}{8})\) sector, other than those which are supergravity dual operators, are primary operators for potential long multiplets.

In general in the analysis of \(\mathcal{N} = 4\) SYM a central issue is whether particular operators are protected in the interacting theory or gain an anomalous scale dimension for non zero
coupling. Our discussion in the previous section is essentially kinematic, determining when there are appropriate combinations to form long multiplets which may then have an anomalous dimension for $\lambda \neq 0$. It is then a dynamical question as to whether this occurs in each case. The results for the different sectors formed from the basic elementary fields of $\mathcal{N} = 4$ SYM, described in section 6 depends on the superconformal transformations given for free field in (6.1). In an interacting theory these are modified and in the $(\frac{1}{8}, \frac{1}{8})$ sector we expect

$$\{Q_{12}^i, \lambda_{41}\} \sim [Z, Y], \quad \{Q_{41}, \bar{\lambda}_{12}\} \sim [Z, Y]. \quad (8.1)$$

For $t = \frac{1}{4}$, $\bar{t} = 0$ we have also $\{Q_{12}^i, \lambda_{31}\} \sim [X, Z]$, $\{Q_{12}^i, \lambda_{21}\} \sim [Y, X]$. In consequence the supercharges $Q_{12}^i, \bar{Q}_{41}$ no longer annihilate the $(\frac{1}{8}, \frac{1}{8})$ primary operators, as in the free theory for a semi-short multiplet, but generate the operators necessary to complete a long multiplet. The descendants which are formed, assuming the supercharges act as in (8.1) and also when acting on covariant derivatives of fields inside a trace, involve commutators and so exclude operators formed by symmetrised traces in accord with the kinematic observations made above.

All operators other than those dual to supergravity fields are therefore expected to combine into long multiplets not only kinematically but dynamically implying that they then have non zero anomalous dimensions. Furthermore a natural conjecture is that those ‘supergravity’ operators which may also combine to form long operators kinematically nevertheless remain protected dynamically. These conclusions follow from an analytic superspace analysis [12] for the classical interacting theory and a recent paper [28] gives further evidence for this claim in the $\frac{1}{8}$-BPS sector.

For $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS, operators constructed from chiral fields as in (6.8), (6.7) and (6.6), the statements concerning protected operators follow from the notion of the chiral ring [29] composed of operators formed by multiple traces of the basic chiral fields modulo relations requiring that commutators such as $[Z, Y] = [Z, \bar{\lambda}_{1 \dot{\alpha}}] = 0$ or anti-commutators $\{\bar{\lambda}_{12}, \bar{\lambda}_{12}\} = 0$. The counting of such operators is identical to that for symmetrised traces of $Z, Y, X, \bar{\lambda}_{1 \dot{\alpha}}$ and these operators form a closed set under multiplication with scale dimensions additive.

To justify these claims we review some evidence from perturbative calculations and also from application of the AdS/CFT correspondence for the large $N$ strong coupling limit.

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11 In (6.1) for the interacting theory the algebra is modified by

$$\{Q^i_{\alpha}, \lambda_{j\beta}\} = \delta^i_j F_{\alpha\beta} + \frac{1}{2} g \varepsilon_{\alpha\beta\gamma} \varepsilon^{klm} \varphi_{jk} \varphi_{lm}, \quad \{Q_{j\dot{\alpha}}, \lambda^i_{\dot{\beta}}\} = \delta^i_j F_{\dot{\alpha}\dot{\beta}} + \frac{1}{2} g \varepsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \varepsilon^{klm} \varphi_{jk} \varphi_{lm},$$

and otherwise derivatives are replaced by covariant derivatives.
Historically the first protected operator to be identified in this context \([30]\) belonged to the representation \(R^{(0,0)}_{[0,2,0]}\) with canonical conformal dimension 4, denoted by \(D_{20'}\) in \([31]\). In free theory this gives a highest weight state for a semi-short multiplet but, as in (3.27), it may combine potentially with others to form a long multiplet. If we identify the primary \(\frac{1}{2}\)-BPS operators \(O_{rs} = \text{tr}(X_r X_s) - \frac{1}{6} \delta_{rs} \text{tr}(X_t X_t)\) then this operator is a double trace operator of the form \(O_{rt} O_{st} - \frac{1}{6} \delta_{rs} O_{tu} O_{tu}\). The anomalous dimension was shown to vanish at one-loop \([31]\). In addition a procedure for counting states to determine whether long multiplets may be formed was also described in \([3\]) (whimsically called the Eratosthenes’ super sieve by analogy to finding prime numbers, this procedure of successively removing superconformal descendants by hand is equivalent to an expansion in terms of supermultiplet characters). The authors of \([14]\) devoted some attention to this example and were able to show there are no available operators of higher conformal dimensions that \(D_{20'}\) can be combined with to become long and therefore allow it to acquire an anomalous dimension.

These results for \(D_{20'}\), and other related operators, may also be arrived at by applying the index \(I_{nm}^{\frac{1}{4}, \frac{1}{2}}\) for the simple values of \(m, n\). For \(m = -1\) the index, from (7.51), just counts the number of \(\frac{1}{4}\)-BPS operators transforming in \(R^{(0,0)}_{[1,n,1]}\) representation which, as stressed in \([19]\), cannot combine into long multiplets and must remain short. For \(m = 0\) from (7.50), considering only operators dual to supergravity fields for which the index may be non zero,

\[
I_{n0}^{\frac{1}{4}, \frac{1}{2}} = N_{\text{m.t.sym.}, n2}^{\frac{1}{4}, \frac{1}{2}} (R^{(0,0)}_{[2,n,2]}) - 2 N_{\text{sugra}, n0,0}^{\frac{1}{4}, \frac{1}{2}} (R^{(0,0)}_{[2,n,0]}) + N_{\text{sugra}, n0,0}^{\frac{1}{4}, \frac{1}{2}} (R^{(0,0)}_{[0,n,0]}), \tag{8.2}
\]

where we list in parenthesis the representations of the primary operators of each short/semi-short multiplet and we have used the symmetry under conjugation \(N_{\frac{1}{2}, \frac{5}{2}} = N_{\frac{5}{2}, \frac{1}{2}}\). The non zero numbers for the first few cases are as follows,

\[
\begin{align*}
I_{00}^{\frac{1}{4}, \frac{1}{2}} &= N_{\text{m.t.sym.}, 02}^{\frac{1}{4}, \frac{1}{2}} = 1, & I_{10}^{\frac{1}{4}, \frac{1}{2}} &= N_{\text{m.t.sym.}, 12}^{\frac{1}{4}, \frac{1}{2}} = 1,
I_{20}^{\frac{1}{4}, \frac{1}{2}} &= 4, & N_{\text{m.t.sym.}, 22}^{\frac{1}{4}, \frac{1}{2}} &= 3, & N_{\text{sugra}, 20,00}^{\frac{1}{4}, \frac{1}{2}} &= 1,
I_{30}^{\frac{1}{4}, \frac{1}{2}} &= 5, & N_{\text{m.t.sym.}, 32}^{\frac{1}{4}, \frac{1}{2}} &= 4, & N_{\text{sugra}, 20,00}^{\frac{1}{4}, \frac{1}{2}} &= 1,
I_{40}^{\frac{1}{4}, \frac{1}{2}} &= 11, & N_{\text{m.t.sym.}, 42}^{\frac{1}{4}, \frac{1}{2}} &= 8, & N_{\text{sugra}, 40,00}^{\frac{1}{4}, \frac{1}{2}} &= 3,
I_{50}^{\frac{1}{4}, \frac{1}{2}} &= 15, & N_{\text{m.t.sym.}, 22}^{\frac{1}{4}, \frac{1}{2}} &= 11, & N_{\text{sugra}, 50,00}^{\frac{1}{4}, \frac{1}{2}} &= 4,
I_{60}^{\frac{1}{4}, \frac{1}{2}} &= 25, & N_{\text{m.t.sym.}, 62}^{\frac{1}{4}, \frac{1}{2}} &= 19, & N_{\text{sugra}, 60,00}^{\frac{1}{4}, \frac{1}{2}} &= 1, & N_{\text{sugra}, 60,00}^{\frac{1}{4}, \frac{1}{2}} &= 8.
\end{align*}
\tag{8.3}
\]

For \(n = 2, 3, 4, 5\) there are no negative contributions to the index so that the \(\frac{1}{4}\)-BPS operators listed must all be protected as well as the associated \((\frac{1}{8}, \frac{1}{8})\) primary operators for semi-short multiplets, \(n = 2\) corresponds to the operator \(D_{20'}\) mentioned above. The
results for $n = 2, 3, 4$ are in agreement with calculations which take into account non-planar operator mixing of multi-trace operators formed by the fields $Z,Y$ by applying a long-range version of the $SU(2)$ spin chain $[32]$. In each case as expected there are the required number of operators with protected scale dimensions. Taking $n = 6$ would however be a less trivial case to consider in future investigation since then it is possible to form a long multiplet. For $m = 1$ the index chain is longer,

$$
I_{n1}^{\frac{1}{3}, \frac{1}{8}} = N_{m.t.\text{sym.},n3}^{\frac{1}{3}, \frac{1}{8}} (\mathcal{R}_{[3,n,3]}^{0,0}) - 2N_{\text{sugra},n1,0}^{\frac{1}{3}, \frac{1}{8}} (\mathcal{R}_{[3,n,1]}^{0,0}) + 2N_{\text{sugra},n0,\frac{1}{4}}^{\frac{1}{3}, \frac{1}{8}} (\mathcal{R}_{[3,n,0]}^{0,0})
$$

$$
+ N_{\text{sugra},n-1,00}^{\frac{1}{3}, \frac{1}{8}} (\mathcal{R}_{[1,n,1]}^{0,0}) - 2N_{\text{sugra},n-2,\frac{1}{4},0}^{\frac{1}{3}, \frac{1}{8}} (\mathcal{R}_{[0,n,1]}^{0,0}) + N_{\text{sugra},n-3,\frac{1}{4},\frac{1}{4}}^{\frac{1}{3}, \frac{1}{8}} (\mathcal{R}_{[0,n,0]}^{0,0})
$$

(8.4)

For $n = 0$ all contributions are zero whereas for $n = 1,2$

$$
I_{11}^{\frac{1}{3}, \frac{1}{8}} = 3, \quad N_{m.t.\text{sym.},13}^{\frac{1}{3}, \frac{1}{8}} = 2, \quad N_{\text{sugra},1-1,00}^{\frac{1}{3}, \frac{1}{8}} = 1,
$$

$$
I_{21}^{\frac{1}{3}, \frac{1}{8}} = 3, \quad N_{m.t.\text{sym.},23}^{\frac{1}{3}, \frac{1}{8}} = 3, \quad N_{\text{sugra},21,0}^{\frac{1}{3}, \frac{1}{8}} = 1, \quad N_{\text{sugra},2-1,00}^{\frac{1}{3}, \frac{1}{8}} = 2.
$$

(8.5)

The second case allows for the possibility of forming a long multiplet.

However the corresponding $\frac{1}{4}$-BPS operators, of scaling dimension 8 and belonging to the $\mathcal{R}^{(0,0)}_{[3,2,3]}$ representation, have been analysed by Morales [13] following the methods of [32]. It transpires that three symmetric operators corresponding to $\mathcal{O}_{[3,2,3]}^{\frac{1}{4}, \frac{1}{8}}$ have vanishing anomalous dimension at one-loop, including for non-planar interactions as stressed above, although only two would be necessary to satisfy the index (along with one $(\frac{1}{8}, \frac{1}{8})$ operator belonging to $\mathcal{R}_{[1,2,1]}^{(0,0)}$). The analysis is non trivial. It is necessary to include in the basis for the spin chain all multi-trace operators formed by fields $Z^5 Y^3$ and also from $Z^8$, which yields 7 protected $\frac{1}{4}$-BPS operators, $Z^7 Y$ giving additionally only protected $\frac{1}{4}$-BPS operators, 4 in all, and $Z^6 Y^2$ producing 8 more protected (for a total of 19 protected including the previous ones) as well as 8 unprotected operators. The operators formed from $Z^5 Y^3$ then give 10 operators beyond those which are $SU(2)$ partners of the ones already discussed. In order to resolve the mixing, it is necessary to diagonalise a $37 \times 37$ matrix or, equivalently, find the zeroes of the degree 37 characteristic polynomial! Luckily there are 22 zero eigenvalues which corresponds to the 19 protected operators counted above plus 3 more corresponding to all three $\mathcal{O}_{[3,2,3]}^{\frac{1}{4}, \frac{1}{8}}$ operators formed from symmetric traces.

Although tentative this seems to be an indication that the dynamics of $\mathcal{N} = 4$ super-conformal Yang Mills theory is more strictly constrained than required just by $PSU(2,2|4)$ representation theory. Of course one loop results for one operator are not sufficient to demonstrate such conclusions. In [33] a specific triple trace scalar operator of canonical conformal dimension 6 (called $T_6 = \mathcal{O}_{rs} \mathcal{O}_{st} \mathcal{O}_{tr}$) was shown to have vanishing anomalous dimension at one loop despite its being part of a long unprotected multiplet. In both these
one loop examples, attempting a two-loop calculation may help sort out such issues but this is a formidable task. Our expectation is that the vanishing of the anomalous dimension of $T_{6}$ is a one-loop accident, while all three $O_{[3,2,3]}^{\frac{1}{3}}(0,0)$ operators remain \textit{dynamically} protected.

\textit{Beyond the chiral ring}

If we look beyond the chiral ring at semi-short $(\frac{1}{8}, \frac{1}{8})$ operators, or $(\frac{1}{8}, 0)$ operators the situation is more complicated. However, in \cite{11,12} all operators which remain semi-short in the classical interacting theory were classified with the aid of analytic superspace. This classification, which is expected to be valid in the quantum theory at non zero coupling assuming no quantum anomalies in the action of supercharges, is straightforward: an operator is short/semi-short in the classical interacting theory if and only if it is short/semi-short in the free theory and is constructed from $\frac{1}{2}$-BPS operators.

With this classification of operators for weakly interacting $\mathcal{N} = 4$ SYM, inherited from the classical superconformal theory, it is trivial that the partition function, restricted to short/semi-short operators, is the same at weak and strong coupling, since they are calculated in exactly the same way. At weak coupling the partition function is defined as in (7.7). At strong coupling, on the other hand, according to the AdS/CFT correspondence, the partition function is that of a gas of free gravitons together with superpartners on AdS/CFT. The single particle partition function of these states is represented by (7.6), which is equal to the single particle partition function of $\frac{1}{2}$-BPS operators in the field theory, and the partition function of a free gas of such states is defined precisely as in (7.7).

Applying these arguments independently shows that $D_{20}'$ is non-renormalised: it is semi-short in the free theory and is constructed from two $\frac{1}{2}$-BPS operators (up to mixing). More generally for $O_{r_{1}...r_{n}}^{(n)} = \text{tr}(X_{r_{1}}...X_{r_{n}}) - \text{traces}$, the single trace $\frac{1}{2}$-BPS primary operators formed from the basic six component scalars $X_{r}$ for representation $[0, n, 0]$, then all semi-short supergravity dual primary operators belonging to the representations $R_{[0,p,0]}^{(0,0)}$, given by symmetric traceless tensors of rank $p$ $O_{r_{1}...r_{p}}$, are formed from products of two or more $O^{(n)}$, $\sum n = p + 2$, with one pair of indices contracted and all remaining $p$ free indices symmetrised and traces subtracted. For representations $R_{[1,p-2,1]}^{(0,0)}$, described by tensor fields $O_{r,r_{1}...r_{p-1}} = O_{r,(r_{1}...r_{p-1})}$ where $O_{(r,r_{1}...r_{p-1})} = 0$ and contractions of any pair of indices are zero, a similar construction of all corresponding semi-short operators is possible. Starting from a product $\prod n O^{(n)}$, $\sum n = p + 2$, one pair of indices is again contracted and another pair is antisymmetrised before $p - 1$ free indices are symmetrised.

\textsuperscript{12} Mixing in the quantum theory will mean the precise definition of the protected operators may be different from this, but the counting should remain the same.
and traces subtracted. The indices which are contracted or antisymmetrised must come from different $O^{(n)}$. This procedure gives the two semi-short operators $O_{[1,2],[0,0]}^{\frac{1}{3},\frac{1}{3}}$ formed from $O^{(2)}$ and $O^{(2)}O^{(4)}$, which are therefore protected.

There is in fact an alternative way to show that some semi-short operators are non-renormalised at non zero coupling. This involves considering three point functions for two $\frac{1}{2}$-BPS operators $O_{[0,p,0]}^{\frac{1}{2},\frac{1}{2}} \equiv O^{(p)}$ and an operator $O$ of the form $\langle O^{(p)} O^{(q)} O \rangle$. This was analysed in [34] and shown to be non zero only for

\[ O = O_{[0, p + q - 2r, 0]}^{\frac{1}{2},\frac{1}{2}}, \quad r = 0, \ldots, q, \quad O_{[s, p + q - 2r, s]}^{\frac{1}{2},\frac{1}{2}}, \quad r = 0, \ldots, q, s = 1, \ldots q - r, \]
\[ O_{[s, p + q - 2r - 2s, r]}^{\frac{1}{2},\frac{1}{2}}, \quad r = 1, \ldots, q, s = 0, \ldots q - r, \]
\[ O_{[s, p + q - 2r - 2s, s]}^{0,0}, \quad r = 2, \ldots, q, s = 0, \ldots q - r, \]

(8.6)

assuming $p \geq q$ and for the long operator $O^{0,0}$ we must have $\Delta \geq 2 + p + q - 2r + 2j$. For the semi-short operators $O_{[s, p + q - 2r - 2s, s]}^{\frac{1}{2},\frac{1}{2}}$, $r = 1$ is special so that if $O$ has twist $\Delta - 2j = p + q$, the associated three point function is non zero only if $O$ is semi-short and has vanishing anomalous dimension. This was used in [34] to prove the non-renormalisation of the operator $D_{20}^\lambda$ and further demonstrates that any short/semi-short operator of twist $p + q$ in the operator product of $O^{(p)}O^{(q)}$ is protected. Equivalent results may also be obtained using superconformal Ward identities for the operator product expansion applied to the four point function $\langle O^{(p)}O^{(q)}O^{(p)}O^{(q)} \rangle$ [22, 35]. Hence performing a conformal partial wave analysis on this four point function in the free theory implies that all semi-short operators of twist $p + q$ which are present in the operator product expansion for $O^{(p)}O^{(q)}$ must be protected in the interacting theory. This analysis demonstrates that semi-short operators belonging to the representations $R^{(j,j)}_{[q,p,q]}$ must be present for any $j, p, q$ although it is harder to disentangle the number of such operators in each case in this fashion.

Construction of semi-short operators

As has been discussed in section 6 the construction of $(\frac{1}{8}, \frac{1}{8})$ semi-short operators can be reduced to operators formed from the fields in [6.4]. In terms of the letters in Table 1 these correspond to words $a^s b^{l+1} z^u y^v$, with $u \geq v$, where the associated operator belongs to the representation $R_{[v+s, u-v, v+l]}^{(s, t, \frac{1}{2})}$. To illustrate some examples we adopt the notation

\[ Z^i = (Z, Y), \quad \lambda = \lambda_{41}, \quad \bar{\lambda} = \bar{\lambda}_{2}^1, \quad \vartheta = \vartheta_{12}, \quad S_i = (S_2^1, S_3^1), \quad \bar{S}^i = (\bar{S}_{22}^2, \bar{S}_{32}^3), \]

(8.7)

so that from (6.1)

\[ [S_i, \partial^n Z^j] = 2in \delta_{ij} \partial^{n-1} \bar{\lambda}, \quad [\bar{S}^i, \partial^n Z^j] = -2in \varepsilon^{ij} \partial^{n-1} \lambda, \]
\[ \{S_i, \partial^n \lambda\} = -4(n+1) \varepsilon_{ij} \partial^n Z^j, \quad \{\bar{S}^i, \partial^n \bar{\lambda}\} = 4(n+1) \partial^n Z^i, \]

(8.8)
as well as \( \{ S_i, \partial^n \bar{\lambda} \} = \{ \bar{S}^i, \partial^n \lambda \} = 0 \). For each word \( a^{s+1}b^{t+1}z^u y^v \) there are various possible operators formed by one or more traces which are annihilated by \( S_i, \bar{S}^j \) which may also be required to be \( SU(2) \) highest weight states. If only \( Z, Y \), without any derivatives, are present these are just \( \frac{1}{4} \) or \( \frac{1}{2} \)-BPS operators. To discuss the simplest operators in this sector we consider

\[
\sum_{s=0}^{n+1} \left( \alpha_s^{(n)} \partial^s \bar{\lambda} \otimes_{\pm} \partial^{n-s} \lambda - \beta_s^{(n)} i \epsilon_{ij} \partial^s Z^i \otimes_{\pm} \partial^{n+1-s} Z^j \right), \quad \alpha_{n+1} = 0, \quad \beta_{n+1-s}^{(n)} = \mp \beta_s^{(n)},
\]

(8.9)

where \( \otimes_{\pm} \) represents the symmetric/antisymmetric tensor product for \( SU(N) \) matrices. Imposing the conditions that this commutes with \( S_i, \bar{S}^i \) gives the relations \( (n-s+1)\alpha_s^{(n)} = -(s+1)\beta_{s+1}^{(n)} \), \( (s+1)\alpha_s^{(n)} = (n-s+1)\beta_{s+1}^{(n)} \) and hence we may take

\[
\alpha_s^{(n)} = \left( \frac{n+1}{s} \right) \left( \frac{n+1}{s+1} \right) (-1)^s, \quad \beta_s^{(n)} = \left( \frac{n+1}{s} \right)^2 (-1)^s,
\]

(8.10)

with \( n \) even, odd according to the two signs in (8.9).

Hence for the word \( ab \) there are semi-short singlet operators of twist 2, corresponding to the representations \( R_{[0,0,0]}^{(\frac{1}{2} n, \frac{1}{2} n)} \) in Table 7, represented in this sector by

\[
\sum_{s=0}^{n+1} \left( \alpha_s^{(n)} \text{tr} \left( \partial^s \bar{\lambda} \partial^{n-s} \lambda \right) - \beta_s^{(n)} i \epsilon_{ij} \text{tr} \left( \partial^s Z^i \partial^{n+1-s} Z^j \right) \right), \quad n = 0, 2, \ldots \tag{8.11}
\]

For \( n = 0 \) this corresponds to the well known Konishi scalar. The result (8.11) is equivalent to the results for twist 2 operators in [9]. Similarly for the semi-short operators in Table 7 for the representations \( R_{[0,1,0]}^{(\frac{1}{2} n, \frac{1}{2} n)} \) we have

\[
\sum_{s=0}^{n+1} \left( \alpha_s^{(n)} \text{tr} \left( Z [\partial^s \bar{\lambda}, \partial^{n-s} \lambda]_\mp \right) - \beta_s^{(n)} i \epsilon_{ij} \text{tr} \left( Z [\partial^s Z^i, \partial^{n+1-s} Z^j]_\pm \right) \right), \tag{8.12}
\]

with \( [X, Y]_\mp = XY \mp YX \) and the two cases in (8.12) require \( n \) even/odd. Other multiple trace operators may be similarly found. Thus there are four \( \Delta = 4 \) semi-short operators for \( R_{[0,2,0]}^{(0,0)} \) represented by,

\[
\text{tr} (Z \bar{\lambda}) \text{tr} (Z \lambda) - 2 i \epsilon_{ij} \text{tr} (Z Z^i) \text{tr} (Z \partial Z^j), \quad \text{tr} (Z Z) \text{tr} (\bar{\lambda} \lambda - 2 i \epsilon_{ij} Z^i \partial Z^j), \quad \text{tr} (Z Z \left[ \bar{\lambda}, \lambda \right] - 2 i \epsilon_{ij} \left[ Z^i, \partial Z^j \right]) \tag{8.13}
\]

in accord with Tables 6,7,8. The remaining \( \Delta = 4 \) semi-short operator necessary according to Table 8 for the representation \( R_{[1,0,1]}^{(0,0)} \) corresponds to the single trace operator \( \epsilon_{kl} \text{tr} (Z^k Z^l \left[ \bar{\lambda}, \lambda \right] - 2 i \epsilon_{ij} \left[ Z^i, \partial Z^j \right]) \).
A privileged set of multi-trace operators are those constructed from $\frac{1}{2}$-BPS single trace operators. The simplest example in the \((\frac{1}{8}, \frac{1}{8})\) sector are those double trace operators composed of \([0, 2, 0]\) $\frac{1}{2}$-BPS operators formed from the descendants of $\text{tr}(ZZ)$. There are two cases to consider, an $SU(2)$ singlet

$$
O^{(n)}_0 = \sum_{s=0}^{n+2} \left( \frac{1}{2} \alpha_{0,s}^{(n)} \partial^s \text{tr}(\lambda \lambda - i \varepsilon_{ij} \partial Z^i Z^j) \partial^{n-s} \text{tr}(\lambda \lambda - i \varepsilon_{kl} \partial Z^k Z^l)
+ \beta_{0,s}^{(n)} i \varepsilon_{ij} \partial^s \text{tr}(\lambda Z^i) \partial^{n+1-s} \text{tr}(\lambda Z^j)
+ \frac{1}{4} \gamma_{0,s}^{(n)} \varepsilon_{ik} \varepsilon_{jl} \partial^s \text{tr}(Z^i Z^j) \partial^{n+2-s} \text{tr}(Z^k Z^l) \right),
$$

(8.14)

where $\alpha_{0,s}^{(n)} = \alpha_{0,n-s}^{(n)}$, $\gamma_{0,s}^{(n)} = \gamma_{0,n+2-s}^{(n)}$ and the highest weight state for a $SU(2)$ triplet

$$
O^{(n)}_1 = \sum_{s=0}^{n+1} \left( \frac{1}{2} \alpha_{1,s}^{(n)} \partial^s \text{tr}(ZZ) \partial^{n-s} \text{tr}(\lambda \lambda - i \varepsilon_{ij} \partial Z^i Z^j) - \beta_{1,s}^{(n)} \partial^s \text{tr}(\lambda Z) \partial^{n-s} \text{tr}(\lambda Z)
+ \frac{1}{4} \gamma_{1,s}^{(n)} \varepsilon_{ij} \partial^s \text{tr}(Z^j Z) \partial^{n+1-s} \text{tr}(Z^j Z) \right),
$$

(8.15)

with $\gamma_{1,s}^{(n)} = -\gamma_{1,n+1-s}^{(n)}$. The operators in (8.14) and (8.15) correspond to the representations $R_{\frac{1}{2}(n+1), \frac{1}{2}(n+1)}$ and $R_{\frac{1}{2}n, \frac{1}{2}n}$ in Table 6. By using

$$
[S_i, \partial^n \text{tr}(\lambda \lambda - i \varepsilon_{kl} \partial Z^k Z^l)] = 2(n + 3) \varepsilon_{ij} \partial^n \text{tr}(\lambda Z^j),
$$
$$
[S^i, \partial^n \text{tr}(\lambda \lambda - i \varepsilon_{kl} \partial Z^k Z^l)] = 2(n + 3) \partial^n \text{tr}(\lambda Z^i),
$$

\begin{align*}
\{S_i, \partial^n \text{tr}(\lambda Z^j)\} &= 2n \delta^i_j \partial^{n-1} \text{tr}(\lambda \lambda - i \varepsilon_{kl} \partial Z^k Z^l) - 2(n + 2) \varepsilon_{ik} \partial^n \text{tr}(Z^k Z^j), \\
\{S^i, \partial^n \text{tr}(\lambda Z^j)\} &= 2n \varepsilon_{ij} \partial^{n-1} \text{tr}(\lambda \lambda - i \varepsilon_{kl} \partial Z^k Z^l) + 2(n + 2) \partial^n \text{tr}(Z^i Z^j), \\
[S_i, \partial^n \text{tr}(Z^j Z^k)] &= 2in \partial^{n-1}(\delta^i_j \text{tr}(\lambda Z^k) + \delta^i_k \text{tr}(\lambda Z^j)), \\
[S^i, \partial^n \text{tr}(Z^j Z^k)] &= -2in \partial^{n-1}(\varepsilon^{ij} \text{tr}(\lambda Z^k) + \varepsilon^{jk} \text{tr}(\lambda Z^i)),
\end{align*}

(8.16)

we may determine the conditions for $O^{(n)}_0$ and $O^{(n)}_1$ to be superconformal primary operators. These give

$$
\alpha_{0,s}^{(n)} = \binom{n+3}{s} \binom{n+3}{s+3} (-1)^s, \quad \beta_{0,s}^{(n)} = \binom{n+3}{s} \binom{n+3}{s+2} (-1)^s,
$$
$$
\gamma_{0,s}^{(n)} = \binom{n+3}{s} \binom{n+3}{s+1} (-1)^s,
$$
$$
\alpha_{1,s}^{(n)} = \binom{n+3}{s} \binom{n+1}{s+1} (-1)^s, \quad \gamma_{s,s}^{(n)} = \binom{n+3}{s} \binom{n+1}{s} (-1)^s,
$$
$$
\beta_{1,s}^{(n)} = \frac{n+3}{n+2} \binom{n+2}{s} \binom{n+2}{s+2} (-1)^s,
$$

(8.17)

55
where to satisfy the symmetry conditions we must take \( n \) to be even. It is easy to verify that \( \mathcal{O}_1^{(0)} \) is a linear combination of the double trace operators in (8.13).

The operator \( \mathcal{O}_1^{(0)} \) represents the protected operator \( D_{20} \). All the operators \( \mathcal{O}_0^{(n)}, \mathcal{O}_1^{(n)} \) are still annihilated by \( Q_{12}, \bar{Q}_{41} \) in the interacting theory since they are formed from products of \( \frac{1}{2} \)-BPS operators and their descendants and the action of the supercharges on these short multiplets cannot be changed (although the detailed form of the the \( \frac{1}{2} \)-BPS operators in terms of elementary fields may modified) and hence are protected. In (8.14) and (8.13) the operators corresponding to the \([0, 2, 0]\) short multiplet in this sector are unaffected by interactions except for the descendant \( \text{tr}(\bar{\lambda}\lambda - i\varepsilon_{ij} D Z^i Z^j) \) where it is necessary to take \( \partial \to D \), a \( SU(N) \) covariant derivative (derivatives outside the trace are not changed). Extending (8.1) to

\[
\{Q_{12}, \lambda\} = g \varepsilon_{ij} [Z^i, Z^j], \quad [Q_{12}, D Z^i] = g i [\bar{\lambda}, Z^i],
\]

\[
\{\bar{Q}_{41}, \bar{\lambda}\} = -g \varepsilon_{ij} [Z^i, Z^j], \quad [\bar{Q}_{41}, D \bar{Z}_i] = g i [\lambda, \bar{Z}_i],
\]  

(8.18)

it is easy to see that \( \text{tr}(\bar{\lambda}\lambda - i\varepsilon_{ij} D Z^i Z^j) \) commute with \( Q_{12}, \bar{Q}_{41} \) for non zero \( g \). This is in contrast to operators such as those in (8.11) and (8.12) which are therefore part of long multiplets.

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The generators of the \( \mathcal{N} = 4 \) superconformal group \( PSU(2, 2|4) \) consist of the usual Lorentz transformations \( M_{\alpha\beta} \), translations \( P_a \), special conformal translations \( K_a \), dilatations \( D \), and \( SU(4)_R \) \( R \)-symmetry generators \( R^i_j \) together with supercharges \( Q^i_\alpha, \bar{Q}^i_{\dot{\alpha}} \) and their superconformal partners \( S^i_\alpha, \bar{S}^i_{\dot{\alpha}} \), for \( i, j = 1, 2, 3, 4 \). In a spinorial basis \( P_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}} P_a, \bar{K}^{\dot{\alpha}\alpha} = (\bar{\sigma}^a)_{\dot{\alpha}\alpha} K_a, M_{\alpha\beta} = -\frac{1}{4} i (\sigma^a\bar{\sigma}^b)_{\alpha\beta} M_{ab}, \bar{M}^{\dot{\alpha}\beta} = -\frac{1}{4} i (\bar{\sigma}^a\sigma^b)_{\dot{\alpha}\beta} M_{ab} \) we may write

\[
\mathcal{M}_A^B = \left( M^\alpha_\beta + \frac{1}{2} \delta^\alpha_\beta D, \quad \bar{M}^{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \bar{D}_{\dot{\alpha}\dot{\beta}} \right), \quad Q^i_A = (Q^i_\alpha, \bar{Q}^i_{\dot{\alpha}}), \quad Q^i_B = (S^i_\beta, \bar{Q}^i_{\dot{\beta}}),
\]

and their various commutators, anticommutators are then determined by

\[
\begin{align*}
[M^B_C, M^D_P] &= \delta^B_C M^D_P - \delta^D_P M^C_B, & [R^i_j, R^k_l] &= \delta^k_l R^i_j - \delta^i_j R^k_l, \\
[M^B_C, Q^i_A] &= \delta^B_C Q^i_A - \frac{1}{2} \delta^B_C Q^i_C, & [M^B_C, \bar{Q}^i_A] &= -\delta^B_C \bar{Q}^i_A + \frac{1}{2} \delta^B_C \bar{Q}^i_C, \\
[R^i_j, Q^k_A] &= \delta^k_j Q^i_A - \frac{1}{2} \delta^k_j Q^k_A, & [R^i_j, \bar{Q}^k_A] &= -\delta^k_j \bar{Q}^i_A + \frac{1}{2} \delta^k_j \bar{Q}^k_A, \\
\{Q^i_A, \bar{Q}^k_B\} &= 4(\delta^i_j M^B_A - \delta^A_B R^i_j), & \{Q^i_A, Q^j_B\} &= 0, & \{\bar{Q}^i_A, \bar{Q}^j_B\} &= 0,
\end{align*}
\]

for \( \delta_A^B = \left( \begin{smallmatrix} \delta^\alpha_\beta & 0 \\ 0 & \delta^{\dot{\alpha}}_{\dot{\beta}} \end{smallmatrix} \right) \). In terms of the usual angular momentum generators we have

\[
[M^\alpha_\beta] = \left( \begin{array}{cc} J_3 & J_+ \\ J_- & -J_3 \end{array} \right), \quad [\bar{M}^{\dot{\alpha}}_{\dot{\beta}}] = \left( \begin{array}{cc} \bar{J}_3 & \bar{J}_+ \\ \bar{J}_- & -\bar{J}_3 \end{array} \right),
\]

and for \( SU(4) \) there is the decomposition

\[
[R^i_j] = \left( \begin{array}{ccc} \frac{1}{4}(3H_1+2H_2+H_3) & E^+_1 & [E^+_1, E^+_2] \\ -[E^-_1, E^-_2] & E^-_2 & [E^-_2, E^-_3] \\ [E^-_1, [E^-_2, E^-_3]] & -[E^-_2, E^-_3] & E^-_3 \end{array} \right),
\]

with \( E^\pm_i \) the raising/lowering operators for the simple roots.

The standard hermiticity requirements are

\[
(M^\alpha_\beta)^\dagger = (\tau M \tau)^\alpha_\beta, \quad (R^i_j)^\dagger = R^i_j, \quad (Q^i_A)^\dagger = (\bar{Q}^i_{\dot{A}})^\dagger, \quad \tau = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).
\]

Thus \( D^\dagger = -D \) and \( (M^\alpha_\beta)^\dagger = \bar{M}^{\dot{\alpha}}_{\dot{\beta}} \), interchanging \( SU(2)_J \) and \( SU(2)_{\dot{J}} \). However for states formed by the action of local field operators on the vacuum the two point function defines a scalar product leading to modified hermiticity conditions \([13]\). For an operator \( O \) we define an alternative conjugation operation by

\[
O^+ = U^{-1} O^\dagger U, \quad U = U^\dagger = \exp \left( \frac{i}{2} (P_0 - K_0) \right).
\]
With this definition dotted and undotted indices are still interchanged but

\[ D^+ = D, \quad P_{\alpha\alpha}^+ = -\langle\sigma_0\rangle_{\alpha\beta} \tilde{K}^{\beta\beta}(\sigma_0)_{\beta\dot{\alpha}}, \quad (M^\alpha_{\dot{\alpha}})^+ = (\bar{\sigma}_0)^{\dot{\beta}\gamma} M_{\gamma\delta}(\sigma_0)_{\delta\dot{\alpha}}, \quad (\tilde{M}^\beta_{\dot{\alpha}})^+ = (\sigma_0)_{\dot{\gamma}\dot{\delta}} \tilde{M}^{\dot{\delta}\dot{\beta}}(\sigma_0), \quad Q^i_{\alpha} = S_i^\beta(\sigma_0)_{\beta\dot{\alpha}}, \quad \tilde{Q}_{i\alpha}^+ = -\langle\sigma_0\rangle_{\alpha\beta} \tilde{S}_j^{\beta}, \]

where \( \sigma_0\bar{\sigma}_0 = 1 \) and we may take \( \sigma_0 = \bar{\sigma}_0 = 1 \).

Corresponding to the various shortening conditions labelled by \( t, \bar{t} \), described in section 3, there are corresponding reductions to subgroups which we list in turn.

\[ PSU(2,2|4) \supset (SU(1|1) \otimes PSU(1,2|3)) \ltimes U(1)_R, \quad t = \frac{1}{8}. \]

The generators of \( SU(1|1) \) are \( Q_{12}^1, S_{12}^1, H \) with

\[ \{Q_{12}^1, S_{12}^1\} = 2H, \quad [H, Q_{12}^1] = [H, S_{12}^1] = 0, \quad H = D - 2J_3 - 2R_{1+}. \]

The conditions \((A.7)\) require \( H \) to have a positive spectrum. For \( PSU(1,2|3) \) we may write the generators in a similar fashion to \((A.1)\)

\[ \tilde{\mathcal{M}}_A^B = \left(\begin{array}{cc} \frac{1}{2} \tilde{D} & \frac{1}{2} P_{1\dot{\beta}} \\ \frac{1}{2} \bar{K}^{\dot{\beta}1} M_{\dot{\alpha}\dot{\beta}} - \frac{1}{8} \delta_{\dot{\alpha}\dot{\beta}} \tilde{D} \end{array}\right), \quad \tilde{R}_j^i = R_j^i + \frac{1}{3} \delta_j^i R_1^1, \quad Q_A^i = \left(\begin{array}{c} \bar{Q}_i^1 \\ S_{i\dot{\alpha}}^1 \end{array}\right), \quad \bar{Q}_j^B = (S_j^1 \quad \bar{Q}_{j\dot{\beta}}), \quad i, j = 2, 3, 4, \quad \tilde{D} = D + J_3, \]

and the algebra is as in \((A.2)\), with the obvious modification \( \frac{1}{4} \rightarrow \frac{1}{3} \), except for

\[ \{Q_A^i, \bar{Q}_j^B\} = 4(\delta^i_j \tilde{\mathcal{M}}_A^B - \delta_A^B \tilde{R}_j^i) - \frac{2}{3} \delta^i_j \delta_A^B H, \]

where \( H \) is here a central extension. The \( U(1) \) with generator \( R = 4R_1^1 \) and plays the role of an external automorphism with the action

\[ [R, Q_{12}^1] = 3Q_{12}^1, \quad [R, S_{12}^1] = -3S_{12}^1, \quad [R, Q_A^i] = Q_A^i, \quad [R, \bar{Q}_A^i] = -\bar{Q}_A^i. \]

\[ PSU(2,2|4) \supset (SU(1|1) \otimes SU(1|1) \otimes PSU(1,1|2)) \ltimes (U(1)_H \otimes U(1)_H), \quad t = \bar{t} = \frac{1}{8}. \]

In this case first \( SU(1|1) \) is as above in \((A.8)\) but the second \( SU(1|1) \) has generators \( \bar{Q}_{41}, \bar{S}_{41}, \bar{H} \) with

\[ \{\bar{Q}_{41}, \bar{S}_{41}\} = -2\bar{H}, \quad [\bar{H}, \bar{Q}_{41}] = [\bar{H}, \bar{S}_{41}] = 0, \quad \bar{H} = D - 2\bar{J}_3 + 2R_{4+}, \]

with \( \bar{H} \) also positive. The generators of \( PSU(1,1|2) \) can be expressed as

\[ \tilde{\mathcal{M}}_A^B = \left(\begin{array}{cc} \frac{1}{2} \tilde{D} & \frac{1}{2} P_{12} \\ \frac{1}{2} \bar{K}_{21} & -\frac{1}{2} \tilde{D} \end{array}\right), \quad \tilde{R}_j^i = R_j^i + \frac{1}{2} \delta_j^i (R_{1+}^1 + R_{4+}^1), \quad [\bar{R}_j^i] = \left(\begin{array}{cc} \frac{1}{2} H_2 & E_{2+} \\ E_{2-} & -\frac{1}{2} H_2 \end{array}\right), \quad Q_A^i = \left(\begin{array}{c} \bar{Q}_j^1 \\ S_{i\dot{\alpha}}^1 \end{array}\right), \quad \bar{Q}_j^B = (S_j^1 \quad \bar{Q}_{j\dot{\beta}}), \quad i, j = 2, 3, \quad \tilde{D} = D + J_3 + \bar{J}_3. \]
The corresponding algebra is evident except for
\[ \{ Q^i_A, Q^j_B \} = 4(\delta^i_j M^A_B - \delta^i_A \tilde{R}^j_B) - \delta^i_j \delta^A_B (H - \bar{H}). \] (A.14)

The two $U(1)$ automorphisms are generated by $R^1$ and $R^4$ or equivalently to
\[ H_+ = R^1 - R^4 = H_1 + H_2 + H_3, \quad H_- = 2(R^1 + R^4) = H_1 - H_3. \] (A.15)

The non zero commutators are then
\[ [H_\pm, Q^i_2] = Q^i_2, \quad [H_\pm, S^i_1] = -S^i_1, \quad [H_\pm, \bar{Q}_{4i}] = \pm \bar{Q}_{4i}, \quad [H_\pm, \bar{S}^{4i}] = \mp \bar{S}^{4i}, \]
\[ [H_+, Q^i_A] = [H_+, \bar{Q}^i_A] = 0, \quad [H_-, Q^i_A] = -Q^i_A, \quad [H_-, \bar{Q}^i_A] = \bar{Q}^i_A. \] (A.16)

$PSU(2, 2|4) \supset SU(2|1) \otimes SU(2|3), \ t = \frac{1}{4}$.

Here the generators of $SU(2|1)$ are $M^\alpha_{\beta}, H_0, Q^1_{\alpha}, S^1_{\beta}$ with
\[ \{ Q^1_{\alpha}, S^1_{\beta} \} = 4M^\alpha_{\beta} + 2\delta^\alpha_{\beta} H_0, \quad [H_0, Q^1_{\alpha}] = -Q^1_{\alpha}, \quad [H_0, S^1_{\beta}] = S^1_{\beta}. \] (A.17)

The generators for $SU(2|3)$ are then $\tilde{M}^\alpha_{\beta}, \tilde{D}, \tilde{R}^i_j, \tilde{S}^{i\alpha}, \tilde{Q}^i_{\beta}$ for $i, j = 2, 3, 4$ and
\[ \{ \tilde{S}^{i\alpha}, \tilde{Q}^i_{\beta} \} = 4(\delta^i_j \tilde{M}^\alpha_{\beta} - \delta^\alpha_{\beta} \tilde{R}^i_j) - \frac{4}{3} \delta^\alpha_{\beta} \delta^i_j \tilde{D}, \quad [\tilde{D}, \tilde{S}^{i\alpha}] = -\frac{1}{2} \tilde{S}^{i\alpha}, \quad [\tilde{D}, \tilde{Q}^i_{\beta}] = \frac{1}{2} \tilde{Q}^i_{\beta}, \quad \tilde{D} = \frac{3}{2} D - R_1^1. \] (A.18)

$PSU(2, 2|4) \supset (SU(2|1) \otimes SU(1|1) \otimes SU(1|2)) \rtimes U(1)_{H_+}, \ t = \frac{1}{4}, \ t = \bar{t} = \frac{1}{8}$.

The first $SU(2|1)$ is as in (A.17) and $SU(1|1)$ as in (A.12). $SU(1|2)$ has generators
\[ \tilde{R}^i_j, \tilde{D}, \tilde{E}^{i\alpha}, \tilde{Q}^{i\alpha}, \tilde{H}_i, i, j = 2, 3, \text{ where } H_+ \text{ is as in (A.15), } \tilde{R}^i_j \text{ as in (A.13)} \]
\[ \{ \tilde{E}^{i\alpha}, \tilde{Q}^{i\alpha} \} = -4 \tilde{R}^i_j - 2\delta^i_j \tilde{D}, \quad \tilde{D} = D + 2 \tilde{H}_3 - \frac{1}{2} H_-, \quad \tilde{E}^{i\alpha} = -\tilde{S}^{i\alpha}, \quad \tilde{[\tilde{D}, \tilde{Q}_{j2}] = \tilde{Q}_{j2}, \ i, j = 2, 3.} \] (A.19)

The action of $H_+$ can be determined from (A.16).

$PSU(2, 2|4) \supset (SU(2|1) \otimes SU(2|1) \otimes SU(2)) \rtimes U(1)_{H_+}, \ t = \bar{t} = \frac{1}{4}$.

The first $SU(2|1)$ is as previously in (A.17) whereas the second has generators
\[ \tilde{M}^\alpha_{\beta}, H_0, S^{4\alpha}, Q_{4\beta} \text{ with} \]
\[ \{ \tilde{S}^{4\alpha}, \tilde{Q}_{4\beta} \} = 4\tilde{M}^\alpha_{\beta} - 2\delta^\alpha_{\beta} \tilde{H}_0, \quad \tilde{H}_0 = D + 2 R^4, \]
\[ [\tilde{H}_0, \tilde{Q}_{4\beta}] = -\tilde{Q}_{4\beta}, \quad [\tilde{H}_0, \tilde{S}^{4\alpha}] = \tilde{S}^{4\alpha}, \] (A.20)
and the $SU(2)$ corresponds to $\tilde{R}^i_j$ which is defined in (A.13) and corresponds to the generators $H_2, E_2^\pm$. The additional $U(1)$ automorphism generated by $H_+$ in (A.13) has the action

$$[H_+, Q^1_\alpha] = Q^1_\alpha, \ [H_+, S_1^\alpha] = -S_1^\alpha, \ [H_+, \bar{Q}_4\hat{\alpha}] = \bar{Q}_4\hat{\alpha}, \ [H_+, \bar{S}^4\hat{\alpha}] = -\bar{S}^4\hat{\alpha}. \ (A.21)$$

$PSU(2,2|4) \supset (PSU(2,2) \otimes PSU(2,2) \otimes U(1)_{\tilde{H}}) \ltimes U(1)_D$, $t = \frac{1}{2}$ and $t = \bar{t} = \frac{1}{2}$.

The generators of the $PSU(2,2)$ factors are $M^{\alpha}_\beta, \tilde{R}^i_j, Q^i_\alpha, S_j^\alpha$ for $i, j = 1, 2$ and $\tilde{M}^{\alpha}_\beta, \bar{R}^k_l, \bar{S}^k\hat{\alpha}, \bar{Q}_{l\hat{\beta}}$ for $k, l = 3, 4$, where

$$\{Q^i_\alpha, S_j^\beta\} = 4(\delta^i_j M^{\alpha}_\beta - \delta^{\alpha}_\beta \tilde{R}^i_j) + 2\delta^i_j \delta^{\alpha}_\beta \tilde{H}, \ \tilde{R}^i_j = R^i_j + \frac{1}{2} \delta^i_j (R_3^3 + R_4^4),$$

$$\{\bar{S}^k\hat{\alpha}, \bar{Q}_{l\hat{\beta}}\} = 4(\delta^k_l \tilde{M}^{\alpha}_\beta - \delta^{\alpha}_\beta \bar{R}^k_l) - 2\delta^k_l \delta^{\alpha}_\beta \bar{H}, \ \bar{R}^k_l = R^k_l + \frac{1}{2} \delta^k_l (R_1^1 + R_2^2), \ (A.22)$$

with $\bar{H}$ a central charge

$$\bar{H} = D - \frac{1}{2}(H_1 + 2H_2 + H_3), \ \{\bar{H}, Q^i_\alpha\} = [\bar{H}, S_j^\beta] = [\bar{H}, \bar{S}^k\hat{\alpha}] = [\bar{H}, \bar{Q}_{l\hat{\beta}}] = 0. \ (A.23)$$

For this case $D$ generates an external automorphism with

$$[D, Q^i_\alpha] = \frac{1}{2} Q^i_\alpha, \ [D, \bar{Q}_{l\hat{\beta}}] = \frac{1}{2} \bar{Q}_{l\hat{\beta}}, \ [D, S_j^\beta] = -\frac{1}{2} S_j^\beta, \ [D, \bar{S}^k\hat{\alpha}] = -\frac{1}{2} \bar{S}^k\hat{\alpha}. \ (A.24)$$

### Appendix B. Expansions in Schur Polynomials

Here for application in the decomposition of partition functions we discuss how to expand a general symmetric function $f(z, y) = f(y, z)$ in terms of 2 variable Schur polynomials $\chi_{(n+m,m)}(z, y)$, defined as in (B.21),

$$f(z, y) = \sum_{n,m \geq 0} N_{n,m} \chi_{(n+m,m)}(z, y), \ (B.1)$$

where we require $f(z, y)$ to have a power series expansion in $z, y$. It is important to note the symmetry relations

$$\chi_{(n+m,m)}(z, y) = -\chi_{(m-1,n+m+1)}(z, y) \Rightarrow N_{n,m} = -N_{-n-2,n+m+1}, \ N_{-1,m} = 0. \ (B.2)$$

Writing

$$f(z, y) = \sum_{r=0}^\infty f_r(z) y^r, \ \chi_{(n+m,m)}(z, y) = (zy)^m \sum_{r=0}^n y^r z^{n-r}, \ (B.3)$$

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we easily get
\[ \sum_{n=0}^{\infty} \sum_{m \geq 0, s-n} N_{n,m} n^{s-m} = f_s(z). \]  
(B.4)

Using (B.2) this can be rearranged as
\[ \sum_{n=-s-1}^{\infty} N_{n,s} n^{s} = \hat{f}_s(z), \]  
(B.5)

for
\[ \hat{f}_0(z) = f_0(z), \quad \hat{f}_s(z) = f_s(z) - \frac{1}{z} f_{s-1}(z), \quad s = 1, 2, \ldots. \]  
(B.6)

For application in section 7 we consider first, as in (7.19),
\[ f(z, y) = (1 - z)(1 - y) \prod_{k=1}^{\infty} \frac{1}{1 - z^k - y^k}. \]  
(B.7)

The definitions (B.3) and (B.5) give
\[ f_0(z) = (1 - z) \prod_{k=1}^{\infty} \frac{1}{1 - z^k}, \quad \hat{f}_1(z) = \left( z + 1 - \frac{1}{z} \right) \prod_{k=1}^{\infty} \frac{1}{1 - z^k}, \]  
\[ \hat{f}_2(z) = \frac{2z^2}{1 - z^2} \prod_{k=1}^{\infty} \frac{1}{1 - z^k}, \]  
\[ \hat{f}_3(z) = \left( \frac{z}{(1 - z)^2} - \frac{z}{1 - z^2} + \frac{1 - z}{1 - z^3} - \frac{1}{z} \right) \prod_{k=1}^{\infty} \frac{1}{1 - z^k}. \]  
(B.8)

Similarly with, as in (7.20),
\[ g(z, y) = \prod_{k+l \geq 2}^{\infty} \frac{1}{1 - z^k y^l}. \]  
(B.9)

we get
\[ g_0(z) = (1 - z) \prod_{k=1}^{\infty} \frac{1}{1 - z^k}, \quad \hat{g}_1(z) = \left( z + 1 - \frac{1}{z} \right) \prod_{k=1}^{\infty} \frac{1}{1 - z^k}, \]  
\[ \hat{g}_2(z) = \frac{z^2}{1 - z^2} \prod_{k=1}^{\infty} \frac{1}{1 - z^k}, \]  
\[ \hat{g}_3(z) = \left( \frac{1}{(1 - z)(1 - z^6)} - \frac{z}{1 - z^6} - \frac{1}{z} \right) \prod_{k=1}^{\infty} \frac{1}{1 - z^k}. \]  
(B.10)

Expansion of these gives the results in (7.23).
If we consider the corresponding expression for single trace operators as in (7.26)

\[ h(z, y) = -\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln(1 - z^k - y^k) - z - y , \]  

(B.11)

we find

\[ h_0(z) = \frac{z^2}{1 - z} , \quad \hat{h}_1(z) = 0 , \quad \hat{h}_2(z) = \frac{z^2}{(1 - z)(1 - z^2)} \] 

\[ \hat{h}_3(z) = \frac{2z^3}{(1 - z)(1 - z^2)(1 - z^3)} - \frac{z}{1 - z^2} - \frac{1}{z} . \]  

(B.12)

Expansion gives (7.24).

For the purposes of counting \( \frac{1}{4} \)-BPS operators for large \( R \)-symmetry charges \( p, q \) a different approach may be more relevant. An alternative expression for \( N_{n,m} \) in (B.1) is obtained by employing the orthogonality relation,

\[ \frac{1}{8\pi^2} \oint \oint \chi_{(n+m,m)}(z, y) \chi_{(p+q,q)}(z^{-1}, y^{-1})(z^{-1} - y^{-1})^2 \, dz \, dy = \delta_{np} \delta_{mq} - \delta_{n-p-2} \delta_{m+p+1} , \]  

(B.13)

with contours encircling the origin. This relation is consistent with (B.2) and follows from an orthogonality relation given in [36] for Jack polynomials for which Schur polynomials are a special case. Hence

\[ N_{nm} = \frac{1}{8\pi^2} \oint \oint \chi_{(n+m,m)}(z, y) f(z^{-1}, y^{-1})(z^{-1} - y^{-1})^2 \, dz \, dy . \]  

(B.14)
Appendix C. Tables

Here we list primary \((1/3, 1/3)\) semi-short operators obtained from partition functions in terms of their \(SU(4)_R \otimes SU(2)_J \otimes SU(2)_J\) representations \(R^{(j,j)}\) and scale dimension \(\Delta\).

Table 6

| \(\Delta\) | Operators in \(S_{\text{sugra}}\) |
|---|---|
| 4 | \(R^{(0,0)}\) |
| 5 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 6 | \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 13/2 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 7 | \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 15/2 | \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 8 | \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |

Table 7

| \(\Delta\) | Operators in \(S_{\text{sym./sugra}}\) |
|---|---|
| 2 | \(R^{(0,0)}\) |
| 3 | \(R^{(0,0)}\) |
| 4 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 9/2 | \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 5 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 11/2 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 6 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 13/2 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 7 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 15/2 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
| 8 | \(R^{(1,1)}\), \(R^{(1,1)}\), \(R^{(0,0)}\), \(R^{(0,0)}\), \(R^{(0,0)}\) |
Table 8

| Δ   | Semi-short \((\frac{1}{2}, \frac{1}{2})\) primary operators |
|-----|------------------------------------------------------|
| 4   | \(R_{[0,1,0]}, R_{[1,0,0]}, R_{[0,0,0]}\) |
| 5   | \(R_{[1,0,1]}, R_{[1,1,0]}, R_{[0,0,2]}\) |
| 11/2| \(3R_{[1,1,1]}, R_{[2,0,2]}, R_{[0,2,0]}\) |
| 6   | \(2R_{[1,1,0]}, R_{[2,1,0]}, R_{[0,1,2]}, R_{[0,2,1]}\) |
| 13/2| \(3R_{[1,0,1]}, 8R_{[1,2,1]}, 5R_{[2,0,1]}, 7R_{[1,1,2]}\) |
| 7   | \(3R_{[0,1,0]}, 2R_{[2,1,1]}, 7R_{[1,0,2]}\) |
| 15/2| \(5R_{[2,0,0]}, 2R_{[2,2,0]}, R_{[2,0,1]}, 5R_{[2,1,0]}\) |
| 8   | \(12R_{[2,0,0]}, 44R_{[2,1,1]}, 12R_{[2,0,1]}, 2R_{[0,0,1]}, 8R_{[2,2,1]}\) |
|     | \(2R_{[2,2,0]}, 5R_{[0,0,2]}, 8R_{[1,1,1]}, 77R_{[1,1,0]}\) |
|     | \(13R_{[0,0,0]}, 18R_{[2,0,0]}, 17R_{[0,0,0]}, 28R_{[2,2,0]}\) |

Table 9

| Δ   | Semi-short \((\frac{1}{2}, \frac{1}{2})\) primary operators |
|-----|------------------------------------------------------|
| 9/2 | \(R_{[1,1,0]}, R_{[2,0,0]}, R_{[0,1,0]}\) |
| 5   | \(R_{[1,0,1]}, 4R_{[0,0,0]}\) |
| 11/2| \(R_{[1,1,0]}, R_{[1,1,0]}, 7R_{[1,0,2]}, 7R_{[2,0,1]}\) |
| 6   | \(2R_{[1,1,0]}, 4R_{[1,0,1]}, 2R_{[2,0,0]}, 3R_{[0,1,2]}\) |
| 13/2| \(2R_{[0,1,1]}, 2R_{[1,0,1]}, 2R_{[1,2,0]}\) |
| 7   | \(R_{[0,1,1]}, 7R_{[2,0,1]}, 16R_{[1,0,3]}, 7R_{[2,1,0]}\) |
| 15/2| \(12R_{[1,0,3]}, 26R_{[2,0,2]}, 12R_{[0,0,0]}, 9R_{[0,0,0]}\) |
| 8   | \(5R_{[0,1,0]}, 36R_{[2,2,0]}, 22R_{[1,1,1]}\) |
|     | \(2R_{[0,0,2]}, 17R_{[2,0,1]}, 17R_{[2,0,1]}\) |
|     | \(2R_{[2,0,0]}, 3R_{[0,1,1]}, 3R_{[1,0,1]}, 6R_{[0,1,0]}\) |
|     | \(6R_{[1,0,0]}, 6R_{[1,0,0]}, 6R_{[1,0,0]}\) |

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### Table 10

| $\Delta$ | (0, 0) primary single trace operators in $S_{\text{sym.}}/S_{\text{sugra}}$ |
|----------|-------------------------------------------------------------------|
| 2        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 3        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 4        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 5        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| $\frac{11}{2}$ | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 6        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| $\frac{13}{2}$ | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 7        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| $\frac{15}{2}$ | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |

### Table 11

| $\Delta$ | (0, 0) primary single-trace operators in $S_{\text{int.}}/(S_{\text{sym.}}/S_{\text{sugra}})$ |
|----------|--------------------------------------------------------------------------------------------------|
| 4        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 5        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| $\frac{11}{2}$ | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 6        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| $\frac{13}{2}$ | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| 7        | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |
| $\frac{15}{2}$ | $R_{[0,0]}^{(0,0)}$, $R_{[0,1]}^{(1,1)}$, $R_{[1,0]}^{(0,1)}$, $R_{[1,1]}^{(1,1)}$ |

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Appendix D. Use of Characters for Product of Fundamental Representation

For many cases characters are very convenient for determining the decomposition of products of representations into irreducible components. We show here how this can be achieved for the product of two fundamental representations $F \otimes F$, $F \equiv B_{[0,1,0]}^{±}$. The character in this case was obtained in (4.1), it can be written more compactly in the form

$$\chi_F(s; u; x, \bar{x}) = \chi_{(1;0,1,0;0,0)}(s; u; x, \bar{x})$$

$$= \sum_{n=0}^{4} D_{1-\frac{1}{2}n} (s, x, \bar{x}) \chi_{(1^{4-n}0^n)}(u), \text{ if } D_{-j} = \overline{D}_j, j = 0, \frac{1}{2}, 1. \tag{D.1}$$

The relations $D_{\frac{1}{2}} = D_{-\frac{1}{2}}$, $\overline{D}_0 = D_0$ follow directly from (4.2) using (2.12), otherwise for $D_{-1}$ this is a just a notational convention. Nevertheless we may then write the product of two $D_j$ for $j = 0, \pm \frac{1}{2}, \pm 1$ in the convenient form [18]

$$D_j D_{j'} = \sum_{q \geq 0} D_{\frac{1}{2}q+j+q-j} + \delta_{j,1} \delta_{j',1} D_1 + \delta_{j,-1} \delta_{j',-1} \overline{D}_1, \tag{D.2}$$

for $j >, j' = j, j'$ if $j > j'$, otherwise $j >, j' = j', j$, and also using (1.5) with the special cases

$$D_{j,0} = A_{j+2,0,0}, \quad D_{0,j} = A_{j+2,0,j}, \quad D_{j,-\frac{1}{2}} = A_{j+\frac{1}{2}, j-\frac{1}{2}, 0}, \quad D_{-\frac{1}{2}, j} = A_{j+\frac{1}{2}, 0, j-\frac{1}{2}},$$

$$D_1 + D_{1,-1} = \overline{D}_1 + D_{-1,1} = A_{2,0,0}, \quad D_{0,-\frac{1}{2}} = D_{-\frac{1}{2}, 0} = 0, \quad \text{ (D.3)}$$

where $A_{\Delta, j, j}$ is defined in (4.4). Applying (D.2) we easily obtain

$$\chi_F^2 = \sum_{q \geq 0} \left( \sum_{0 \leq n < m \leq 4} 2 D_{\frac{1}{2}(q-n)+1, \frac{1}{2}(q+m)-1} \chi_{(1^{4-n}0^n)} \chi_{(1^{4-m}0^m)} + \sum_{0 \leq n \leq 4} D_{\frac{1}{2}(q-n)+1, \frac{1}{2}(q+m)-1} \chi_{(1^{4-n}0^n)} \chi_{(1^{4-m}0^m)} \right) + D_1 + \overline{D}_1. \tag{D.4}$$

Using (4.31) we have

$$\chi_F^2 = \sum_{q \geq 0} \chi_{(q,0,0,0;\frac{1}{2}q-1,\frac{1}{2}q)}$$

$$= D_{1,1} - D_{-\frac{1}{2}, -\frac{1}{2}} - D_{0,0} - D_{\frac{1}{2}, \frac{1}{2}} + D_1 + \overline{D}_1$$

$$+ D_{0,0} \chi_{(1^{2}0^2)} + D_{\frac{1}{2}, 0} \chi_{(1^{3}0)} + \chi_{(1^{3}0^2)} \chi_{(1^{3}0)} + \chi_{(1^{3}0^2)} \chi_{(1^{3}0)} + \chi_{(1^{3}0^2)} \chi_{(1^{3}0)} + \chi_{(1^{3}0^2)} \chi_{(1^{3}0)}$$

$$+ \left( D_{\frac{1}{2}, 1} + D_{0,-1} - D_{-\frac{1}{2}, \frac{1}{2}} - D_{-\frac{1}{2}, -\frac{1}{2}} \right) \chi_{(1^{2}0^2)}$$

$$+ \left( D_{1,0} + D_{0,0} - D_{\frac{1}{2}, -\frac{1}{2}} - D_{-\frac{1}{2}, \frac{1}{2}} \right) \chi_{(1^{3}0^2)}.$$

$$= \chi_{(2,0,2,0,0;0,0)}.$$
where this may be shown to be equal to (4.1) by using (D.3) and the \( SU(4) \) decomposition into irreducible representations given by

\[
\begin{align*}
\chi(1,1,0,0) \chi(1,1,0,0) &= \chi(2,2,0,0) + \chi(2,1,1,0) + 1, \\
\chi(1,0,0,0) \chi(1,1,1,0) &= \chi(2,1,1,0) + 1, \\
\chi(1,1,0,0) \chi(1,0,0,0) &= \chi(2,1,0,0) + \chi(1,1,1,0), \\
\chi(1,1,0,0) \chi(1,1,1,0) &= \chi(2,2,1,0) + \chi(1,1,1,0), \\
\chi(1,0,0,0) \chi(1,0,0,0) &= \chi(2,0,0,0) + \chi(1,1,0,0), \\
\chi(1,1,1,0) \chi(1,1,1,0) &= \chi(2,2,2,0) + \chi(1,1,0,0). \\
\end{align*}
\]  

(D.6)

The result (D.5) is in accord with the tensor product \([37]\)

\[
\mathcal{F} \otimes \mathcal{F} = B^{\frac{1}{2}, \frac{1}{2}}_{[0,0,0](0,0)} \oplus B^{\frac{1}{2}, \frac{1}{2}}_{[1,0,1](0,0)} \oplus \bigoplus_{q \geq 1} \mathcal{V}_q, \quad \mathcal{V}_q = C^{1,1}_{(q,0,0)(\frac{1}{2}(q-1), \frac{1}{2}(q-1))}.
\]  

(D.7)

where we note that \( \mathcal{V}_0 \simeq B^{\frac{1}{2}, \frac{1}{2}}_{[1,0,1](0,0)} \), as in (3.20).

An interesting extension in this context is to consider the perturbative expansion of partition functions where \( D = D_0 + \lambda D_2 + O(\lambda^2) \), in the notation of [38]. The first correction to \( O(\lambda) \) is given by

\[
\langle D_2 \rangle = \text{tr}_{\mathcal{F} \otimes \mathcal{F}} \left( D_2 x^{2D_0} u_1 H_1 + H_2 + H_3 u_2 H_2 + H_3 u_3 H_3 x^2 J_3 \bar{x}^2 J_3 \right),
\]  

(D.8)

restricting to \( \mathcal{F} \otimes \mathcal{F} \). With the decomposition in (D.7) and noting that \( D_2 \mathcal{V}_q = h(q) \mathcal{V}_q \) where \( h(q) = \sum_{r=1}^{q} 1/r \) are harmonic numbers, we may obtain that from (4.30) and using \( \sum_{q \geq 1} h(q) x^q = -\ln(1 - x)/(1 - x) \),

\[
\langle D_2 \rangle = \sum_{q=1}^{\infty} h(q) \chi^{1,1}_{(q,0,0; \frac{1}{2} q-1, \frac{1}{2} q-1)}(s, u, x, \bar{x})
\]

\[
= - \mathcal{M}^{S_2 \bar{S}_2} \mathcal{M}^{S_2} \prod_{i=1}^{4} (1 + su_i x)(1 + su_i^{-1} \bar{x})
\]

\[
\times \frac{\ln(1 - s^2 x \bar{x})}{(1 - x^2)(1 - \bar{x}^2)(1 - s^2 x \bar{x})^2(1 - s^2 x^{-1} \bar{x})(1 - s^2 x \bar{x}^{-1})},
\]

\[\text{where } \mathcal{M}^{S_2 \bar{S}_2} \mathcal{M}^{S_2} \text{ imposes symmetry under } x \leftrightarrow x^{-1}, \bar{x} \leftrightarrow \bar{x}^{-1} \text{. This can be evaluated for } x, \bar{x} = 1 \text{ in the form}
\]

\[
\langle D_2 \rangle |_{x, \bar{x} = 1} = - \frac{1}{(1 - s^2)^6} \sum_{n, m=0}^{4} s^{n+m} \chi_{(1^n 0^4-n)}(u) \chi_{(1^m 0^6-m)}(u)
\]

\[
\times \left( (1 - n + (n - 3)s^2)(1 - m + (m - 3)s^2) \ln(1 - s^2) \right.
\]

\[
+ \left. (1 - n - m + (n + m - 6)s^2) s^2 \right).
\]  

(D.10)
Similar results to (D.10) were given in [39,40] where \( \langle D_2 \rangle \) was of prime importance in the calculation of one-loop corrections to the Hagedorn temperature, although the calculations there were restricted to particular sectors.
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