ON SOME EQUIVALENT NORMS IN SOBOLEV SPACES ON BOUNDED DOMAINS AND ON THE BOUNDARIES

BIENVENIDO BARRAZA MARTÍNEZ, JONATHAN GONZÁLEZ OSPINO, AND JAIRO HERNÁNDEZ MONZÓN

ABSTRACT. We consider the equivalence of some norms in Sobolev spaces on bounded domains of \( \mathbb{R}^d \) and also in Sobolev spaces on the boundaries of those domains.

1. Introduction

In this note we will consider a bounded domain (i.e., a bounded open and connected subset) \( \Omega \) of \( \mathbb{R}^d \), \( d \in \mathbb{N} \), with enough regular boundary \( \partial \Omega \) (this regularity will be made precise later). We will present a relatively general result about the equivalence of norms in the scalar Sobolev space \( W^{k,p}(\Omega) \) with \( k \in \mathbb{N} \) and \( 1 \leq p < \infty \). The main result follows strongly the proof of Theorem 7.1 in [4]. For a domain \( \Omega \subset \mathbb{R}^d \) (bounded or unbounded) the usual Sobolev space \( W^{m,p}(\Omega) \) for \( m \in \mathbb{N} \) and \( 1 \leq p \leq \infty \), is the subspace of \( L^p(\Omega) \) consisting of all complex functions \( u \in L^p(\Omega) \) such that its distributional (weak) derivatives \( \partial^\alpha u \), with \( \alpha \in \mathbb{N}^d_0 \) and \( |\alpha| \leq m \), belong to \( L^p(\Omega) \). A standard norm in \( W^{m,p}(\Omega) \) for \( 1 \leq p < \infty \) is given by

\[
\|u\|_{m,p} := \|u\|_{m,p,\Omega} := \left( \sum_{|\alpha| \leq m} \int_{\Omega} |\partial^\alpha u(x)|^p \, dx \right)^{1/p} \quad (u \in W^{m,p}(\Omega)).
\]

The Sobolev space \( W^{m,\infty}(\Omega) \) is usually endowed with the norm

\[
\|u\|_{m,\infty} := \|u\|_{m,\infty,\Omega} := \max_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty(\Omega)} \quad (u \in W^{m,\infty}(\Omega)).
\]

Now, for \( m \in \mathbb{R}, \ m > 0, \ m \notin \mathbb{Z} \), and \( 1 \leq p < \infty \), the Sobolev space \( W^{m,p}(\Omega) \) (also called Sobolev-Slobodetskii spaces) is the subspace of \( W^{[m],p}(\Omega) \), where \([m]\) denotes the integer part of \( m \), of functions \( u \) such that for \( \alpha \in \mathbb{N}^d_0 \) with \( |\alpha| = [m] \) the following holds

\[
\int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{d+p([m]-[m])}} \, dx \, dy < \infty.
\]

In this case the usual norm in \( W^{m,p}(\Omega) \) is given by

\[
\|u\|_{m,p} := \|u\|_{m,p,\Omega} := \left( \|u\|_{m,p,\Omega}^p + \sum_{|\alpha| = [m]} \int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{d+p([m]-[m])}} \, dx \, dy \right)^{1/p}.
\]
for \( u \in W^{m,p}(\Omega) \).
The Sobolev space \( W_0^{m,p}(\Omega) \), for \( m \geq 0 \) and \( p \geq 1 \), is defined as the closure of \( C^\infty_c(\Omega) \) in \( W^{m,p}(\Omega) \).
Finally, for \( m < 0 \) and \( p \geq 1 \), \( W^{m,p}(\Omega) \) is defined as the topological dual of \( W^{-m,q}_0(\Omega) \), where \( q := \frac{p}{p-1} \).

2. About the regularity of the boundaries of domains

In this section we will make precise the concept of continuous and smooth boundary for a bounded domain in \( \mathbb{R}^d \). Let \( \Omega \) a bounded domain in \( \mathbb{R}^d \) with boundary \( \partial \Omega \).

**Definition 2.1.** The boundary \( \partial \Omega \) is called continuous if there exist real numbers \( a > 0 \), \( b > 0 \), a system of local coordinates \( \{(x_1^r, \ldots, x_{d-1}^r, x_d^r) =: (x_1^r, x_d^r) : r = 1, \ldots, m\} \) and continuous functions \( a_r : \overline{\Delta_r} \to \mathbb{R}, r = 1, \ldots, m \), where \( \Delta_r \) are the open cubes in \( \mathbb{R}^{d-1} \) defined by \( \Delta_r := \{(x_1^r, \ldots, x_{d-1}^r) : |x_j^r| < a, j = 1, \ldots, d-1\} \), such that for each point on the boundary \( \partial \Omega \) there is an open neighborhood \( V \), such that for some \( r \in \{1, \ldots, m\} \) the following holds (see Fig. 1)

\[
V \cap \partial \Omega = \{(x_1^r, a_r(x_1^r)) : x_1^r \in \Delta_r\},
\]
\[
V \cap \Omega = \{(x_1^r, x_d^r) : x_1^r \in \Delta_r, a_r(x_1^r) < x_d^r < a_r(x_1^r) + b\},
\]
\[
V \cap (\mathbb{R}^d \setminus \overline{\Omega}) = \{(x_1^r, x_d^r) : x_1^r \in \Delta_r, a_r(x_1^r) - b < x_d^r < a_r(x_1^r)\}.
\]

**Remark 2.2.** Note that \( \partial \Omega \) is continuous if locally it is the graph of a continuous function defined in a subset of \( \mathbb{R}^{d-1} \).
Definition 2.3. If the functions \( a_r \) in the Definition 2.1 are Lipschitz continuous it is said that \( \Omega \) has a Lipschitz boundary or that the boundary \( \partial \Omega \) is lipschitzian. Usually it is said also that the domain \( \Omega \) belongs to the class \( \mathcal{N}^{0,1} \).

Remark 2.4. If \( \Omega \in \mathcal{N}^{0,1} \), then a normal vector exists almost everywhere on \( \partial \Omega \) (cf. [4], Lemma 4.2 on pag. 83). In fact, since the functions \( a_r \) in Definition 2.1 are Lipschitz continuous, they are differentiable almost everywhere in their domains. Therefore, for each part \( \partial \Omega_r \) of the \( \partial \Omega \), represented locally as the graph of the functions \( a_r \) for some \( r \in \{1, \ldots, m\} \), the gradient \( \nabla a_r \) exists almost everywhere in \( \Delta_r \) and since \( \partial \Omega_r \) is a level set of the function \( \Delta_r \ni x_r' \mapsto a_r(x_r') - x^d_r \), the vector \( (\nabla a_r(x_r'), -1) = (\frac{\partial}{\partial x^1_r} a_r(x_r'), \ldots, \frac{\partial}{\partial x^{d-1}_r} a_r(x_r'), -1) \) defined almost everywhere in \( \partial \Omega_r \), is normal to \( \partial \Omega_r \), pointing to the exterior of \( \Omega \). Then the outer unit normal vector to \( \partial \Omega_r \) is given by

\[
\nu := \nu(x_r') := (1 + |\nabla a_r(x_r')|^2)^{-1/2} (\nabla a_r(x_r'), -1) = \\
1 + \left( \frac{\partial}{\partial x^1_r} a_r(x_r') \right)^2 + \cdots + \left( \frac{\partial}{\partial x^{d-1}_r} a_r(x_r') \right)^2 \\
\cdot \left( \frac{\partial}{\partial x^1_r} a_r(x_r'), \ldots, \frac{\partial}{\partial x^{d-1}_r} a_r(x_r'), -1 \right)
\]

almost everywhere on \( \partial \Omega_r \).

Definition 2.5 (Domains of class \( \mathcal{N}^{k,\mu} \)). Let \( k \in \mathbb{N}_0 \) and \( 0 \leq \mu \leq 1 \). It is said that the domain \( \Omega \) in Definition 2.1 belong to the class \( \mathcal{N}^{k,\mu} \) if the functions \( a_r \), \( r = 1, \ldots, m \), given in that definition are of class \( C^{k,\mu}(\Delta_r) \), i.e., if \( a_r \) together with its derivatives of order \( \leq k \) are Hölder continuous with exponent \( \mu \) in \( \Delta_r \), which means that for each \( \alpha \in \mathbb{N}_0^{d-1} \) there is a constant \( c \) such that for all \( x_r', y_r' \in \Delta_r \) the estimate

\[
|\partial^\alpha a_r(x_r') - \partial^\alpha a_r(y_r')| \leq c|x_r' - y_r'|^\mu
\]

holds.

Remark 2.6. Note that the case \( k = 0 \) and \( \mu = 1 \) is the Lipschitz case mentioned previously above. In case that \( \mu = 0 \) we also say that \( \Omega \) is a domain of class \( C^k \). The notion of continuous boundary given in the first definition of this section corresponds to the case \( k = 0 \) and \( \mu = 0 \).

3. Lebesgue and Sobolev spaces on the boundary

Let \( \Omega \) a bounded domain in \( \mathbb{R}^d \) with continuous boundary \( \partial \Omega \). The notations in this section refer to those given in Definition 2.1\(^1\).

\(^1\)The spaces defined in this section are independent on the local system of coordinates choosen in Definition 2.1. The corresponding norms related to each local system of coordinates are all equivalents.
Definition 3.1. Let $1 \leq p \leq \infty$. It is said that a complex function $f$ defined almost everywhere on $\partial \Omega$ (which means that $x'_r \mapsto f(x'_r, a_r(x'_r))$ is defined almost everywhere in $\Delta_r$, $r = 1, \ldots, m$) belongs to the space $L^p(\partial \Omega)$ if the function $x'_r \mapsto f_r(x'_r) := f(x'_r, a_r(x'_r))$ belongs to $L^p(\Delta_r)$ for each $r \in \{1, \ldots, m\}$. The space $L^p(\partial \Omega)$ is a Banach space with the norm given by

$$
\|f\|_{p, \partial \Omega} := \|f\|_{L^p(\partial \Omega)} := \left(\sum_{r=1}^m \|f_r\|_{L^p(\Delta_r)}^p\right)^{1/p}, \quad \text{if } 1 \leq p < \infty
$$

and

$$
\max_{r=1,\ldots,m} \|f_r\|_{L^\infty(\Delta_r)}, \quad \text{if } p = \infty.
$$

If $\Omega \in \mathcal{N}^{0,1}$, the space $L^p(\partial \Omega)$, with $1 \leq p < \infty$, can be endowed with another useful norm, equivalent to the norm in (3.1), which is given in terms of a boundary integral. Next we define the boundary integral for a function in $L^1(\partial \Omega)$.

Definition 3.2. Let $\Omega \in \mathcal{N}^{0,1}$. With the notations of Definition 2.1, let

$$V_r := \{(x'_r, x^d_r) \in \mathbb{R}^d : x'_r \in \Delta_r, a_r(x'_r) - b < x^d_r < a_r(x'_r) + b\}, \quad r = 1, \ldots, m.$$

Now, let $\{\varphi_r\}_{r=1}^m$ a partition of the unity on $\partial \Omega$ subordinate to the cover $\{V_r\}_{r=1}^m$, i.e., for each $r = 1, \ldots, m$, $\varphi_r \in C^\infty_c(V_r)$, $0 \leq \varphi_r \leq 1$, and it holds $\sum_{r=1}^m \varphi_r(x) = 1$ for all $x \in \partial \Omega$.

For a function $f \in L^1(\partial \Omega)$ we have $f = \sum_{r=1}^m \varphi_r f$ and we define

$$
\int_{\partial \Omega} f \, d\sigma := \sum_{r=1}^m \int_{\Delta_r} f(x'_r, a_r(x'_r)) \varphi_r(x'_r, a_r(x'_r)) \sqrt{1 + |\nabla a_r(x'_r)|^2} \, dx'_r.
$$

Proposition 3.3. Let $\Omega \in \mathcal{N}^{0,1}$ and $1 \leq p < \infty$. The functional

$$
(3.2) \quad f \mapsto \left(\int_{\partial \Omega} |f|^p \, d\sigma\right)^{1/p}
$$

is a norm in $L^p(\partial \Omega)$, equivalent to the norm given in (3.1).

Proof. See [4], Lemma 1.2., pag. 116. \qed

The following is a standard definition for the Sobolev spaces on the boundary $\partial \Omega$.

Definition 3.4 (Sobolev spaces on the boundary). Let $k \geq 0$, $1 \leq p \leq \infty$ and $\Omega \in \mathcal{N}^{[k]-1,1}$, where $[k]$ is the smallest integer greater than or equal to $k$. The Sobolev space $W^{k,p}(\partial \Omega)$ is the subspace of $L^p(\partial \Omega)$ consisting of all functions $f \in L^p(\partial \Omega)$ such that $f_r \in W^{k,p}(\Delta_r)$ for $r = 1, \ldots, m$. The space $W^{k,p}(\partial \Omega)$ is endowed with the norm

$$
(3.3) \quad \|f\|_{k,p, \partial \Omega} := \|f\|_{W^{k,p}(\partial \Omega)} := \left\{ \begin{array}{ll}
\left(\sum_{r=1}^m \|f_r\|_{k,p, \Delta_r}^p\right)^{1/p}, & \text{if } 1 \leq p < \infty \\
\max_{r=1,\ldots,m} \|f_r\|_{k,\infty, \Delta_r}, & \text{if } p = \infty.
\end{array} \right.
$$
With this norm \( W^{k,p}(\partial \Omega) \) is a Banach space.

## 4. Imbeddings and Traces

**Theorem 4.1** (Sobolev imbedding theorem). Let \( \Omega \in \mathcal{H}^{0,1} \), \( p \geq 1 \) and \( kp > d \). Then \( W^{k,p}(\Omega) \hookrightarrow C^{0,\mu}(\Omega) \), where

\[
\mu = \begin{cases} 
  k - \frac{d}{p}, & \text{if } k - \frac{d}{p} < 1, \\
  < 1, & \text{if } k - \frac{d}{p} = 1, \\
  = 1, & \text{if } k - \frac{d}{p} > 1.
\end{cases}
\]

**Proof.** See Theorem 3.8 in [4], pag. 66.

**Theorem 4.2** (A first trace theorem). Let \( \Omega \in \mathcal{H}^{0,1} \). For \( 1 \leq p < d \), or for \( q \geq 1 \) if \( p = d \), there exists a continuous linear mapping \( Z : W^{1,p}(\Omega) \to L^q(\partial \Omega) \) such that \( Zu = u|_{\partial \Omega} \) if \( u \in C^\infty(\Omega) \).

**Proof.** Since \( L^{q_2}(\partial \Omega) \hookrightarrow L^{q_1}(\partial \Omega) \) for \( 1 \leq q_1 \leq q_2 \), the result follows from Theorem 4.2 on page 79 and Theorem 4.6 on page 81 of [4]. Cf. also with Theorem 5.36 in [1].

From Theorems 4.1 and 4.2 above, we have

**Theorem 4.3** (A second trace theorem). Let \( \Omega \in \mathcal{H}^{0,1} \) and \( p \geq 1 \). There is a bounded linear mapping \( \gamma_0 : W^{1,p}(\Omega) \to L^p(\partial \Omega) \) such that \( \gamma_0 u = u|_{\partial \Omega} \) if \( u \in C^\infty(\Omega) \).

Usually the mapping \( \gamma_0 \) is called trace map of order zero. The map \( \gamma_0 : W^{1,p}(\Omega) \to L^p(\partial \Omega) \) is not surjective, but it holds that \( \gamma_0(W^{1,p}(\Omega)) \) is dense in \( L^p(\partial \Omega) \), whenever \( p \geq 1 \) and \( \Omega \in \mathcal{H}^{0,1} \) (see Theorem 4.9 on page 82 in [4]).

A further useful result about traces is the following.

**Theorem 4.4** (A third trace theorem). Let \( k \in \mathbb{N} \), \( p > 1 \), \( \Omega \in \mathcal{H}^{k-1,1} \) and \( u \in W^{k,p}(\Omega) \). If \( l \in \mathbb{N}_0 \) is such that \( l \leq k - 1 \), then \( \gamma_0 \partial^\alpha \nu u \in W^{k-l-\frac{1}{p},p}(\partial \Omega) \) for all \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| = l \), and

\[
\|\partial^\alpha u\|_{k-l-\frac{1}{p},p,\partial \Omega} \leq \text{const.} \|u\|_{k,p,\Omega},
\]

where

\[
\partial^\alpha u = \sum_{|\alpha| = l} \frac{l!}{\alpha!} \nu^\alpha \gamma_0 \partial^\alpha u,
\]

where \( \nu \) is the outer normal on \( \partial \Omega \).

**Proof.** See [4], Th. 5.5, pag. 95.

From now on we will omit sometimes the notation \( \gamma_0 u \) and write simply \( u \).
5. Main results about equivalence of norms in Sobolev spaces on bounded domains

The following is the main result of this notes.

**Theorem 5.1.** Let \(d, k, l \in \mathbb{N}, 1 \leq p < \infty, \Omega \subset \mathbb{R}^d\) a bounded domain with Lipschitz boundary \(\partial \Omega\), \(\{f_i\}_{i=1}^l\) a set of seminorms in \(W^{k,p}(\Omega)\) such that

i) For each \(i = 1, \ldots, l\), there exists \(C_i \geq 0\) with \(f_i(v) \leq C_i\|v\|_{k,p}\) for all \(v \in W^{k,p}(\Omega)\).

ii) For \(v \in P_{k-1} := \big\{ \sum_{|\alpha| \leq k-1} C_\alpha x^\alpha : C_\alpha \in \mathbb{R}, x \in \mathbb{R}^d \big\}\) it holds that
\[
\sum_{i=1}^l f_i^p(v) = 0 \implies v = 0.
\]

Then,
\[
(5.1) \quad u \mapsto \|u\|_{k,p}' := \left( \sum_{i=1}^l f_i^p(u) + |u|^p_{k,p} \right)^{1/p},
\]
with
\[
(5.2) \quad |u|_{k,p} : = \left( \sum_{|\alpha| = k} \int_{\Omega} |\partial^\alpha u(x)|^p \, dx \right)^{1/p},
\]
is a norm in \(W^{k,p}(\Omega)\), equivalent to the standard one given in (1.1).

**Proof.** It is clear, due to i), that there exists \(b > 0\) such that
\[
\|u\|_{k,p}' \leq b\|u\|_{k,p} \quad (u \in W^{k,p}(\Omega)).
\]
Now suppose that there does not exist a constant \(a > 0\) such that
\[
a\|u\|_{k,p} \leq \|u\|_{k,p}'
\]
for all \(u \in W^{k,p}(\Omega)\). Then, for each \(n \in \mathbb{N}\), there exists \(u_n \in W^{k,p}(\Omega)\) with \(\|u_n\|_{k,p} = 1\) and such that
\[
(5.3) \quad \frac{1}{n} > \left( \sum_{i=1}^l f_i^p(u_n) + |u_n|_{k,p}^p \right)^{1/p}.
\]
Therefore, for each multiindex \(\alpha \in \mathbb{N}^d_0\) with \(|\alpha| = k\), we have
\[
(5.4) \quad \partial^\alpha u_n \to 0 \quad \text{in } L^p(\Omega), \quad \text{when } n \to \infty.
\]
Theorem 6.3 in [4] ensures that the identity mapping \(Id : W^{1,p}(\Omega) \to L^p(\Omega)\) is compact, which implies that the identity mapping \(Id : W^{k,p}(\Omega) \to W^{k-1,p}(\Omega)\) is also compact. Since \((u_n)_{n \in \mathbb{N}}\) is a bounded sequence in \((W^{k,p}(\Omega), \|\cdot\|_{k,p})\), there exists a subsequence \((u_{n_m})_{m \in \mathbb{N}}\) of \((u_n)_{n \in \mathbb{N}}\), which converges in the space \((W^{k-1,p}(\Omega), \|\cdot\|_{k-1,p})\). Let \(u := \lim_{m \to \infty} u_{n_m}\), where the limit is taken in \(W^{k-1,p}(\Omega)\).

We assert that \(u \in W^{k,p}(\Omega), \partial^\alpha u = 0\) for all multiindex \(\alpha\) with \(|\alpha| = k\) and that
Remark 5.2. It is clear that the functional $\| \cdot \|_{k,p}$ given in Theorem 5.1 is a seminorm in $W^{k,p}(\Omega)$. Theorem 3.2 in [3] implies that it is in fact a norm in $W^{k,p}(\Omega)$, because if $\|u\|_{k,p} = 0$ for $u \in W^{k,p}(\Omega)$, then $\partial^{\alpha}u = 0$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$ and therefore $u \in P_{k-1}$. We would have also $\sum_{i=1}^{l} f_i^p(u) = 0$ and then $u = 0$ by assumption ii).
Corollary 5.3. Let $d, k \in \mathbb{N}$, $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^d$ a bounded domain with sufficiently regular boundary $\partial \Omega$ (at least $\Omega \in \mathcal{A}^{[k-\frac{d}{2}]-1}$), $\nu$ the outer normal on $\partial \Omega$, $\Gamma \subset \partial \Omega$ with $\sigma(\Gamma) \neq 0$, where $\sigma$ is the $(d-1)$-dimensional Lebesgue surface measure. Furthermore, suppose that $\Gamma$ is not contained in a hyperplane of $\mathbb{R}^d$. Then, the functional

$$u \mapsto \left( \sum_{i=0}^{k-1} \int_{\Gamma} |\partial^i u|^p d\sigma + \sum_{|\alpha|=k} \int_{\Omega} |\partial^\alpha u|^p dx \right)^{1/p}$$

is a norm in $W^{k,p}(\Omega)$, equivalent to the standard one $\| \cdot \|_{k,p}$.

Proof. For $i = 0, \ldots, k - 1$, let $f_i$ be defined by

$$f_i(u) := \left( \int_{\Gamma} |\partial^i u|^p d\sigma \right)^{1/p} \quad (u \in W^{k,p}(\Omega)).$$

It is easy to see that the functional $f_i$, $i = 0, \ldots, k - 1$, is a seminorm in $W^{k,p}(\Omega)$. Furthermore, due to equation (4.1) in Trace theorem 4.4, we have that there exists $C_i \geq 0$ such that

$$f_i(v) \leq \left( \int_{\partial \Omega} |\partial^i v|^p d\sigma \right)^{1/p} \leq \text{const.} \|\partial^i v\|_{k-i-\frac{1}{p}, p, \partial \Omega} \leq C_i \|v\|_{k,p, \Omega}$$

for all $v \in W^{k,p}(\Omega)$.

On the other side, let $v \in P_{k-1}$, $v(x) = \sum_{|\beta| \leq k-1} c_\beta x^\beta$ such that $\sum_{i=0}^{k-1} f_i^p(v) = 0$. This implies $\partial^i v = 0$ on $\Gamma$ for each $i = 0, \ldots, k - 1$. We recall that $\partial^\alpha x^\beta = \alpha!(\beta/\alpha)! x^{\beta-\alpha}$ with $(\beta/\alpha)! := \frac{\beta!}{\alpha!(\beta-\alpha)!}$ if $\alpha \leq \beta$, $(\beta/\alpha)! = 0$ otherwise (see [5], pag. 18). Now, if $|\alpha| = |\beta|$ and $\alpha \neq \beta$, then there exists $j \in \{1, \ldots, d\}$ such that $\alpha_j > \beta_j$, otherwise $\alpha_i \leq \beta_i$ for all $i \in \{1, \ldots, d\}$ and since $\alpha \neq \beta$, we would have $\alpha_i < \beta_i$ for some $l \in \{1, \ldots, d\}$ and then $|\alpha| < |\beta|$ which contradicts $|\alpha| = |\beta|$. Therefore, if $|\alpha| = |\beta|$ we have $\partial^\alpha x^\beta = 0$ if $\alpha \neq \beta$ and $\partial^\alpha x^\beta = \alpha!$ if $\alpha = \beta$. We recall also from (4.2) that

$$\partial^i v = \sum_{|\alpha|=i} \frac{i!}{\alpha!} \partial^\alpha v \nu^\alpha.$$

Then

$$0 = \partial^{k-1} v = \sum_{|\alpha|=k-1} \frac{(k-1)!}{\alpha!} \partial^\alpha v \nu^\alpha = \sum_{|\alpha|=k-1} \frac{(k-1)!}{\alpha!} \nu^\alpha \sum_{|\beta| \leq k-1} c_\beta \partial^\alpha x^\beta$$

$$= \sum_{|\alpha|=k-1} \frac{(k-1)!}{\alpha!} \nu^\alpha \sum_{|\beta| \leq k-1} c_\beta \partial^\alpha x^\beta = \sum_{|\alpha|=k-1} \frac{(k-1)!}{\alpha!} \nu^\alpha c_\alpha \alpha!$$

$$= \sum_{|\alpha|=k-1} (k-1)! c_\alpha \nu^\alpha.$$
Since $\Gamma$ is not contained in a hyperplane, the powers $\nu^\alpha$ are linear independent and then $c_\alpha = 0$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k - 1$. Therefore $v(x) = \sum_{|\beta| \leq k-2} c_\beta x^\beta$. Similarly $\partial^k v = 0$ implies $c_\beta = 0$ for all $\beta \in \mathbb{N}_0^d$ with $|\beta| = k - 2$. In this form we obtain that $c_\beta = 0$ for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq k - 1$, i.e., $v = 0$.

In consequence, we have proved that the functionals $f_i, i = 0, \ldots, k - 1$, satisfy the assumptions of Theorem 5.1 and we conclude that

$$u \mapsto \left( \sum_{i=0}^{k-1} f_i^p(u) + |u|_{k,p}^p \right)^{1/p} = \left( \sum_{i=0}^{k-1} \int_{\Gamma} |\partial^i v|^p d\sigma + \sum_{|\alpha| = k} \int_{\Omega} |\partial^\alpha u|^p dx \right)^{1/p}$$

is a norm in $W^{k,p}(\Omega)$, equivalent to the norm $\| \cdot \|_{k,p}$.

**Theorem 5.4** (Generalized Poincaré inequality). Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected with Lipschitz boundary $\partial \Omega$ (i.e. $\Omega \in \mathcal{W}^{0,1}$). Moreover, let $1 < p < \infty$ and let $M \subset W^{1,p}(\Omega)$ be nonempty, closed and convex. Then the following assertions are equivalent for every $u_0 \in M$:

1. There exists a constant $C_0 < \infty$ such that for all $\xi \in \mathbb{R}$,

$$u_0 + \xi \in M \implies |\xi| \leq C_0.$$

2. There exists a constant $C < \infty$ with

$$\|u\|_{L^p(\Omega)} \leq C (\|\nabla u\|_{L^p(\Omega)} + 1) \quad (u \in M).$$

If $M$ in addition, is a cone with apex 0, i.e. if

$$u \in M, r \geq 0 \implies ru \in M,$$

then the inequality in the assertion (2) can be replaced with

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad (u \in M).$$

**Proof.** See [2, 8.16 on pag. 242.] □

**Corollary 5.5.** Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected with Lipschitz boundary $\partial \Omega$. Moreover let $1 < p < \infty$, $\Gamma \subset \partial \Omega$ with $\sigma(\Gamma) \neq 0$, where $\sigma$ is the $(d - 1)$-dimensional Lebesgue surface measure, and

$$W^{1,p}_\Gamma(\Omega) := \{ u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Gamma \}.$$ 

Then there exists a constant $C < \infty$ such that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad (u \in W^{1,p}_\Gamma(\Omega)).$$

**Proof.** Let $M$ be defined by

$$M := \left\{ u \in W^{1,p}(\Omega) : \int_{\Gamma} u \, d\sigma = 0 \right\}.$$
Then $M \subset W^{1,p}(\Omega)$ is nonempty because $0 \in M$, closed because $u \mapsto \int_{\Gamma} u \, d\sigma : W^{1,p}(\Omega) \to \mathbb{C}$ is continuous, and convex because of the linearity of this functional. Now let $u_0 \in M$ and take $C_0 := 0$. For all $\xi \in \mathbb{R}$ we have

$$u_0 + \xi \in M \implies 0 = \int_{\Gamma} (u_0 + \xi) \, d\sigma = \int_{\Gamma} u_0 \, d\sigma + \xi \sigma(\Gamma) = \xi \sigma(\Gamma).$$

Then $\xi = 0$ which implies $|\xi| \leq C_0$. Since $M$ is a cone with apex $0$, we have in virtue of Theorem 5.4 that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad (u \in M).$$

Let now $u \in W^{1,p}(\Omega)$ and define $\tilde{u} := u - \frac{1}{\sigma(\Gamma)} \int_{\Gamma} u \, d\sigma$. Then $\tilde{u} \in M$ and we have

$$\|\tilde{u}\|_{L^p(\Omega)} \leq C \|\nabla \tilde{u}\|_{L^p(\Omega)}.$$

Then

$$\|u\|_{L^p(\Omega)} - \frac{\mu(\Omega)^{1/p}}{\sigma(\Gamma)} \int_{\Gamma} u \, d\sigma \leq \left\| u - \frac{1}{\sigma(\Gamma)} \int_{\Gamma} u \, d\sigma \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

where $\mu(\Omega)$ is the $d$-dimensional Lebesgue measure of $\Omega$. Therefore

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} + \frac{\mu(\Omega)^{1/p}}{\sigma(\Gamma)} \left| \int_{\Gamma} u \, d\sigma \right|$$

for all $u \in W^{1,p}(\Omega)$. In particular if $u \in W^{1,p}_{\Gamma}(\Omega)$ we have

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

because in this case $\int_{\Gamma} u \, d\sigma = 0$. \hfill \Box

**Remark 5.6.** In virtue of Corollary 5.5 we have that the functional $u \mapsto \|\nabla u\|_{L^p(\Omega)}$ is a norm in $W^{1,p}_{\Gamma}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{k,p,\Omega}$ (compare with Corollary 5.3).

6. **About equivalence of norms in Sobolev spaces on the boundary**

**Theorem 6.1.** Let $1 \leq p < \infty$ and $\Omega$ a bounded domain in $\mathbb{R}^2$ with sufficiently regular boundary $\partial \Omega$ (at least $\Omega \in \mathcal{C}^{2,1}$). Then, the functional

$$u \mapsto \left( \|u\|_{1,p,\partial \Omega} + \|\partial^2 u\|_{p,\partial \Omega} \right)^{1/p},$$

where $\tau$ is the unit tangential vector on $\partial \Omega$, is a norm in $W^{2,p}(\partial \Omega)$, equivalent to the standard norm $\|\cdot\|_{2,p,\partial \Omega}$ given in (3.3).
Proof. Let $u \in W^{2,p}(\partial \Omega)$. With the notations of Sections 2 and 3 we have

$$
\|u\|_{2,p,\partial \Omega}^p = \sum_{r=1}^m \|u_r\|_{2,p,\Delta_r}^p
= \sum_{r=1}^m \sum_{j=0}^2 \left\| \left( \frac{d}{dx_1^r} \right)^j u_r \right\|_{L^p(\Delta_r)}^p
= \sum_{r=1}^m \sum_{j=0}^1 \left\| \left( \frac{d}{dx_1^r} \right)^j u_r \right\|_{L^p(\Delta_r)}^p + \sum_{r=1}^m \left\| \left( \frac{d}{dx_1^r} \right)^2 u_r \right\|_{L^p(\Delta_r)}^p
= \sum_{r=1}^m \|u_r\|_{1,p,\Delta_r}^p + \sum_{r=1}^m \left\| \left( \frac{d}{dx_1^r} \right)^2 u_r \right\|_{L^p(\Delta_r)}^p,
$$

where $[x_r^1 \mapsto u_r(x_r^1) := u(x_r^1, a_r(x_r^1))] \in W^{2,p}(\Delta_r) = W^{2,p}([-a, a])$, with $a_r \in C^{1,1}(\overline{\Delta_r}) = C^{1,1}([-a, a])$, $r = 1, \ldots, m$. Note that $\Delta_r = (-a, a)$ for all $r = 1, \ldots, m$.

Now, fix $r \in \{1, \ldots, m\}$. Taking into account that $\overline{\Delta_r} \ni x_r^1 \mapsto (x_r^1, a_r(x_r^1))$ is a parametrization of a part of $\partial \Omega$, we have that $(1, a'_r(x_r^1))$ is a tangent vector to $\partial \Omega$ on that part. Set $\theta_r(x_r^1) := |(1, a'_r(x_r^1))| = \sqrt{1 + (a'_r(x_r^1))^2}$. Furthermore, the weak (or distributional) derivative $\left( \frac{d}{dx_1^r} \right)^2 u_r$ is almost everywhere equal to the corresponding usual classical derivative in $\Delta_r$ (see Theorem 2.2. in [4], pag. 55). Then, it holds almost everywhere in $\Delta_r$ that

$$
\left( \frac{d}{dx_1^r} \right)^2 u_r(x_r^1)
= \frac{d}{dx_1^r} \left[ \nabla u(x_r^1, a_r(x_r^1)) \cdot (1, a'_r(x_r^1)) \right]
= \frac{d}{dx_1^r} \left[ \theta_r(x_r^1) \partial_r u(x_r^1, a_r(x_r^1)) \right]
= \theta'_r(x_r^1) \partial_r u(x_r^1, a_r(x_r^1)) + \theta_r(x_r^1) \nabla(\partial_r u)(x_r^1, a_r(x_r^1)) \cdot (1, a'_r(x_r^1))
= \theta'_r(x_r^1) \partial_r u(x_r^1, a_r(x_r^1)) + \theta_r(x_r^1)^2 \partial^2_r u(x_r^1, a_r(x_r^1))
= \theta'_r(x_r^1) \frac{d}{dx_1^r} u(x_r^1, a_r(x_r^1)) + \theta_r(x_r^1)^2 \partial^2_r u(x_r^1, a_r(x_r^1)).
$$
Since \( a_r \in C^{2,1}(\Delta_r) \), there are constants \( c^1_r, c^2_r \) and \( c^3_r \) such that
\[
\left| \frac{\theta'_r(x^1_r)}{\theta_r(x^1_r)} \right| \leq c^1_r,
\]
(6.4)
\[0 < c^2_r \leq \theta_r(x^1_r)^2 \leq c^3_r.\]
Then, due to (6.3) we have with
\[[\partial^2_r u]_r(x^1_r) := \partial^2_r u(x^1_r, a_r(x^1_r))\]
that
\[
\left\| \left( \frac{d}{dx^1_r} \right)^2 u_r \right\|_{L^p(\Delta_r)} \leq \left\| \frac{\theta'_r}{\theta_r} \frac{d}{dx^1_r} u_r \right\|_{L^p(\Delta_r)} + \left\| \theta'_r [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}
\leq c^1_r \left\| \frac{d}{dx^1_r} u_r \right\|_{L^p(\Delta_r)} + c^3_r \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}.
\]
Therefore
\[
\left\| \left( \frac{d}{dx^1_r} \right)^2 u_r \right\|_{L^p(\Delta_r)}^p \leq 2(c^1_r)^p \left\| \frac{d}{dx^1_r} u_r \right\|_{L^p(\Delta_r)}^p + 2(c^3_r)^p \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}^p.
\]
From (6.2), with \( c_1 := \max_{r=1, \ldots, m} 2(c^1_r)^p \) and \( c_3 := \max_{r=1, \ldots, m} 2(c^3_r)^p \), we obtain
\[
\left\| u \right\|_{L^p(\Delta_r)}^p \leq \left\| u \right\|_{1,p,\partial \Omega}^p + m \sum_{r=1}^m \left\| \frac{d}{dx^1_r} u_r \right\|_{L^p(\Delta_r)}^p + m \sum_{r=1}^m \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}^p
\leq \left\| u \right\|_{1,p,\partial \Omega}^p + m \sum_{r=1}^m \left\| u_r \right\|_{1,p,\Delta_r}^p + m \sum_{r=1}^m \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}^p
= (1 + c_1) \left\| u \right\|_{1,p,\partial \Omega} + m \sum_{r=1}^m \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}^p
\leq \max \{1 + c_1, c_3\} \left\| u \right\|_{1,p,\partial \Omega} + \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}^p.
\]
With \( \tilde{c}_1 := (\max \{1 + c_1, c_3\})^{1/p} \), it holds
\[
\left\| u \right\|_{2,p,\partial \Omega} \leq \tilde{c}_1 \left( \left\| u \right\|_{1,p,\partial \Omega} + \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)} \right)^{1/p}.
\]
On the other side, again from (6.3), we have
\[
c^2_r \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)} \leq \left\| \theta'_r [\partial^2_r u]_r \right\|_{L^p(\Delta_r)} \leq \left\| \left( \frac{d}{dx^1_r} \right)^2 u_r \right\|_{L^p(\Delta_r)} + c^4_r \left\| \frac{d}{dx^1_r} u_r \right\|_{L^p(\Delta_r)},
\]
which implies
\[
(c^2_r)^p \left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}^p \leq 2p \left\| \left( \frac{d}{dx^1_r} \right)^2 u_r \right\|_{L^p(\Delta_r)}^p + 2p(c^1_r)^p \left\| \frac{d}{dx^1_r} u_r \right\|_{L^p(\Delta_r)}^p,
\]
i.e.,
\[
\left\| [\partial^2_r u]_r \right\|_{L^p(\Delta_r)}^p
\]
\[
\leq \max \left\{ \frac{2^p}{(c_r^2)^p}, \frac{2^p(c_1^1)^p}{(c_r^2)^p} \right\} \left( \left\| \left( \frac{d}{d x_r^1} \right)^2 u_r \right\|_{L^p(\Delta_r)}^p + \left\| \frac{d}{d x_r^1} u_r \right\|_{L^p(\Delta_r)}^p \right) \\
\leq \max \left\{ \frac{2^p}{(c_r^2)^p}, \frac{2^p(c_1^1)^p}{(c_r^2)^p} \right\} \left\| u_r \right\|_{2,p,\Delta_r}^p.
\]
Therefore, with \( c_2 := \max_{r=1,\ldots,m} \left\{ \frac{2^p}{(c_r^2)^p}, \frac{2^p(c_1^1)^p}{(c_r^2)^p} \right\} \), we have
\[
\left\| u \right\|_{1,p,\partial \Omega}^p + \left\| \partial^2 u \right\|_{p,\partial \Omega}^p = \left\| u \right\|_{1,p,\partial \Omega}^p + \sum_{r=1}^m \left\| \partial^2 u_r \right\|_{L^p(\Delta_r)}^p \\
\leq \left\| u \right\|_{1,p,\partial \Omega}^p + c_2 \sum_{r=1}^m \left\| u_r \right\|_{2,p,\Delta_r}^p = \left\| u \right\|_{1,p,\partial \Omega}^p + c_2 \left\| u \right\|_{2,p,\partial \Omega}^p \\
\leq (1 + c_2) \left\| u \right\|_{2,p,\partial \Omega}^p.
\]
With \( \widetilde{c}_2 := (1 + c_2)^{1/p} \) it holds
\[
(6.6) \quad \left( \left\| u \right\|_{1,p,\partial \Omega}^p + \left\| \partial^2 u \right\|_{p,\partial \Omega}^p \right)^{1/p} \leq \widetilde{c}_2 \left\| u \right\|_{2,p,\partial \Omega}.
\]
From (6.5) and (6.6) follows the result. \( \square \)

7. Some useful interpolation type estimates for traces on Sobolev spaces

In this section we use as norm in \( L^p(\partial \Omega) \), for a domain \( \Omega \subset \mathbb{R}^d \), the norm given in (3.2).

**Proposition 7.1.** Let \( \mathbb{R}^d_+ := \{ x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > 0 \} \) and \( u \in C^1_c(\mathbb{R}^d_+) \) with \( 1 \leq p < \infty \). The following estimate holds
\[
(7.1) \quad \left\| u \right\|_{p,\partial \mathbb{R}^d_+} \leq p^{1/p} \left\| u \right\|_{p,\mathbb{R}^d_+}^{(p-1)/p} \left\| u \right\|_{1,p,\mathbb{R}^d_+}^{1/(p-1)}.
\]

**Proof.** Let \( \delta > 0 \) be such that \( \text{supp } u \subset B(0; \delta) \cap \overline{\mathbb{R}^d_+} \) and \( \nu = (\nu_1, \ldots, \nu_d) \) the outer normal on \( \partial[B(0; \delta) \cap \mathbb{R}^d_+] \) (see Fig. 2). Then, by virtue of Gauß theorem of divergence and Hölder inequality it holds
\[
\int_{\partial \mathbb{R}^d_+} |u|^p \, d \sigma = \int_{\mathbb{R}^{d-1}} |u|^p \, dx' = - \int_{\mathbb{R}^{d-1}} |u|^p (-1) \, dx' = - \int_{\partial[B(0; \delta) \cap \mathbb{R}^d_+]} |u|^p \nu_d \, d \sigma \\
= - \int_{B(0; \delta) \cap \mathbb{R}^d_+} \partial_{x_d} (|u|^p) \, dx = - \int_{\mathbb{R}^d_+} \partial_{x_d} (u \overline{u})^{p/2} \, dx \\
= - \frac{p}{2} \int_{\mathbb{R}^d_+} (u \overline{u})^{\frac{p-2}{2}} \left[ (\partial_{x_d} u) \overline{u} + u \partial_{x_d} \overline{u} \right] \, dx
\]
Fig. 2. Domain $\mathbb{R}_+^d := \{ x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > 0 \}$ and supp $u$.

$$
= -p \int_{\mathbb{R}_+^d} |u|^{p-2} \text{Re}[(\partial_{x_d} u)\overline{u}] \, dx = -p \text{Re} \int_{\mathbb{R}_+^d} |u|^{p-2}(\partial_{x_d} u)\overline{u} \, dx \\
\leq p \int_{\mathbb{R}_+^d} |u|^{p-2} |\partial_{x_d} u||u| \, dx = p \int_{\mathbb{R}_+^d} |u|^{p-1}|\partial_{x_d} u| \, dx \\
\leq p \left( \int_{\mathbb{R}_+^d} |u|^p \, dx \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}_+^d} |\partial_{x_d} u|^p \, dx \right)^{\frac{1}{p}} \\
\leq p \|u\|_{p,\mathbb{R}_+^d}^{p-1} \|u\|_{1,p,\mathbb{R}_+^d},
$$

which implies (7.1). \qed

**Proposition 7.2.** Let $\Omega$ a bounded domain in $\mathbb{R}^d$, $\Omega \Subset \mathbb{R}^{1,1}$, $1 \leq p < \infty$ and $u \in C^1(\Omega)$. Then, it holds

$$
\|u\|_{p,\partial \Omega} \leq c_p \|u\|_{p,\Omega} \|u\|_{1,p,\Omega},
$$

with $c_p$ a constant, which depends on $p$, but not on $u$.

**Proof.** With the notations of Section 2, let

$$
V_r := \{(x', x^d) \in \mathbb{R}^d : x' \in \Delta_r, a_r(x'_r) - b < x^d_r < a_r(x'_r) + b\}, \quad r = 1, \ldots, m.
$$

Furthermore let $V_0 \subset \Omega$, $V_0$ abierto, such that $\overline{\Omega} \subset \bigcup_{r=0}^m V_r$. Choose a $C^\infty$ partition of the unity $\{\varphi_r\}_{r=0}^m$ on $\overline{\Omega}$ subordinate to the cover $\{V_r\}_{r=0}^m$, i.e., $\varphi_r \in C^\infty_c(V_r)$, $0 \leq \varphi_r \leq 1$ and supp $\varphi_r \subset V_r$ for $r = 0, 1, \ldots, m$. Moreover $\sum_{r=0}^m \varphi_r(x) = 1$ for all $x \in \overline{\Omega}$. Then, $u = \sum_{r=0}^m u\varphi_r$ on $\overline{\Omega}$ with $u\varphi_r \in C^1_c(V_r^+)$. For
$r = 0, 1, \ldots, m$, where $V^+_r := V_r \cap \Omega$.

Now, for $r = 1, \ldots, m$, we will consider the transformation of coordinates $V_r \ni (x'_r, x''_r) \longleftrightarrow (y'_r, y''_r)$ given by

$$y_r = \Phi_r(x_r) := \begin{cases} y^i_r := x^i_r, & i = 1, \ldots, d-1, \\ y^d_r := x^d_r - a_r(x'_r), & \end{cases}$$

with inverse transformation

$$x_r = \Phi_r^{-1}(y_r) := \begin{cases} x^i_r = y^i_r, & i = 1, \ldots, d-1, \\ x^d_r = y^d_r + a_r(y'_r). & \end{cases}$$

Let

$$w^r(y_r) := u(y'_r, y''_r + a_r(y'_r)) \varphi_r(y'_r, y''_r + a_r(y'_r)), \quad r = 1, \ldots, m,$$

for $y_r \in \Phi_r(V^+_r)$. We have $w^r \in C^1_c(\Phi_r(V^+_r))$ and therefore, extending by zero outside of $\Phi_r(V^+_r)$, $w^r \in C^1_c(\mathbb{R}^d_+)$. Then, it follows (with several constants $c_r, c^1, c^2_p$, etc., which can depend on $p$, but not on $u$)

$$\|u\|_{p, \partial \Omega}^p = \left\| \sum_{r=0}^m u \varphi_r \right\|_{p, \partial \Omega}^p \leq m \sum_{r=1}^m \|u \varphi_r\|_{p, \partial \Omega}^p = m \sum_{r=1}^m \int_{\partial \Omega} |u \varphi_r|^p d\sigma$$

$$= m \sum_{r=1}^m \int_{\Delta_r} |u(x'_r, a_r(x'_r)) \varphi_r(x'_r, a_r(x'_r))|^p \sqrt{1 + |\nabla a_r(x'_r)|^2} dx'_r$$

$$\leq m \sum_{r=1}^m c_r \int_{\mathbb{R}^{d-1}} |u(y'_r, a_r(y'_r)) \varphi_r(y'_r, a_r(y'_r))|^p dy'_r$$

$$\leq c^1 \sum_{r=1}^m \int_{\mathbb{R}^{d-1}} |w^r(y'_r, 0)|^p dy'_r$$

$$\leq c^1 p \sum_{r=1}^m \|w^r\|_{p, \mathbb{R}^d_+}^{p-1} \|w^r\|_{1, p, \mathbb{R}^d_+} \quad \text{(Prop. 7.1)}$$

$$= c^2_p \sum_{r=1}^m \|w^r\|_{p, \Phi_r(V^+_r)}^{p-1} \|w^r\|_{1, p, \Phi_r(V^+_r)}$$

$$\leq c^3_p \sum_{r=1}^m \|u \varphi_r\|_{p, V^*_r}^{p-1} \|u \varphi_r\|_{1, p, V^*_r}$$

(Th. 4.1 in [6], p. 80).

Now,

$$\|u \varphi_r\|_{p, V^*_r} \leq \|u\|_{p, \Omega}$$

and

$$\|u \varphi_r\|_{1, p, V^*_r} = \|u \varphi_r\|_{p, V^*_r} + \|\nabla (u \varphi_r)\|_{p, V^*_r} \leq c^4_r \|u\|_{1, p, \Omega}.$$
Therefore,
\[ \|u\|_{p,\partial \Omega}^p \leq c_p^3 \sum_{r=1}^{m} (c_r^4)^{1/p} \|u\|_{p,\Omega}^{p-1} \|u\|_{1,p,\Omega}. \]

From this follows (7.2) with \( c_p : = \left( c_p^3 \sum_{r=1}^{m} (c_r^4)^{1/p} \right)^{1/p}. \)

**Proposition 7.3.** Let \( \Omega \) a bounded domain in \( \mathbb{R}^d \), \( \Omega \in \mathcal{C}^{1,1} \), \( 1 \leq p < \infty \) and \( u \in W^{1,p}(\Omega) \). Then, it holds

\[ \|u\|_{p,\partial \Omega} \leq c_p \|u\|_{p,\Omega}^{p-1} \|u\|_{1,p,\Omega}^\frac{1}{p} \]
with \( c_p \) a constant, which depends on \( p \), but not on \( u \).

**Proof.** Let \((u_n)_{n \in \mathbb{N}}\) a sequence of functions of \( C^\infty(\overline{\Omega}) \) such that \( u_n \to u \) in \( W^{1,p}(\Omega) \) whenever \( n \to \infty \). Due to trace theorem 4.3 we have that there exists a constant \( c \), such that
\[ \|u_n - u\|_{p,\partial \Omega} \leq c \|u_n - u\|_{1,p,\Omega} \quad (n \in \mathbb{N}). \]

Then, \( \|u_n - u\|_{p,\partial \Omega} \to 0 \) when \( n \to \infty \).

Now, from Proposition 7.2 it follows that
\[ \|u_n\|_{p,\partial \Omega} \leq c_p \|u_n\|_{p,\Omega}^{p-1} \|u_n\|_{1,p,\Omega}^{\frac{1}{p}} \quad (n \in \mathbb{N}). \]

Making \( n \to \infty \) we obtain (7.3). \( \square \)

From Proposition 7.3 follow also the following estimates.

**Proposition 7.4.** Let \( \Omega \) a bounded domain in \( \mathbb{R}^d \), \( \Omega \in \mathcal{C}^{2,1} \), \( 1 \leq p < \infty \) and \( u \in W^{2,p}(\Omega) \). The following estimate holds:

\[ \|\partial \nu u\|_{p,\partial \Omega} \leq \tilde{c}_p \|u\|_{p,\Omega}^{p-1} \|u\|_{2,p,\Omega}^\frac{1}{p} \]
with \( \tilde{c}_p \) being a positive constant independent of \( u \), where \( \nu \) is the outer normal on \( \partial \Omega \).

**Proof.** For \( u \in C^2(\overline{\Omega}) \), due to Proposition 7.3, the following estimates hold:
\[ \|\partial \nu u\|_{p,\partial \Omega} = \left\| \sum_{j=1}^{d} \nu_j \partial_j u \right\|_{p,\partial \Omega} \leq \sum_{j=1}^{d} \|\nu_j \partial_j u\|_{p,\partial \Omega} \leq \max_{\partial \Omega} |\nu| \sum_{j=1}^{d} \|\partial_j u\|_{p,\partial \Omega} \]
\[ \leq c_p \max_{\partial \Omega} |\nu| \sum_{j=1}^{d} \|\partial_j u\|_{p,\Omega}^{p-1} \|\partial_j u\|_{1,p,\Omega}^{\frac{1}{p}} \leq \tilde{c}_p \|u\|_{1,p,\Omega} \|u\|_{2,p,\Omega}^{\frac{1}{p}}. \]
where \( \tilde{c}_p : = d c_p \max_{\partial \Omega} |\nu| \), with \( c_p \) the constant of Proposition 7.3. Then, (7.4) is also true for \( u \in W^{2,p}(\Omega) \) due to the density of \( C^2(\overline{\Omega}) \) in \( W^{2,p}(\Omega) \). \( \square \)
Proposition 7.5. Let $\Omega$ a bounded domain in $\mathbb{R}^d$, $\Omega \in \mathcal{N}^{3,1}$, $1 \leq p < \infty$ and $u \in W^{3,p}(\Omega)$. The following estimate holds:

\begin{equation}
\|\Delta u\|_{p,\partial\Omega} \leq \hat{c}_p \|u\|_{2,p,\Omega}^{\frac{p-1}{p}} \|u\|_{3,p,\Omega}^{\frac{1}{p}}
\end{equation}

with $\hat{c}_p$ being a positive constant independent of $u$.

Proof. Let $u \in C^3(\overline{\Omega})$. Then, similarly as the proof of Proposition 7.4 and using again Proposition 7.3 (or Proposition 7.2), we have

\[
\|\Delta u\|_{p,\partial\Omega} = \left\| \sum_{j=1}^{d} \partial^2_j u \right\|_{p,\partial\Omega} \leq \sum_{j=1}^{d} \left\| \partial^2_j u \right\|_{p,\partial\Omega} \\
\leq c_p \sum_{j=1}^{d} \left\| \partial^2_j u \right\|_{p,\partial\Omega}^{\frac{p-1}{p}} \|\partial^2_j u\|_{1,p,\Omega}^{\frac{1}{p}} \leq d c_p \|u\|_{2,p,\Omega}^{\frac{p-1}{p}} \|u\|_{3,p,\Omega}^{\frac{1}{p}}.
\]

Due to the density of $C^3(\overline{\Omega})$ in $W^{3,p}(\Omega)$ we obtain that the estimate (7.5) holds also for $u \in W^{3,p}(\Omega)$, with $\hat{c}_p := d c_p$, where $c_p$ is the constant of Proposition 7.3.

Proposition 7.6. Let $\Omega$ a bounded domain in $\mathbb{R}^d$, $\Omega \in \mathcal{N}^{4,1}$, $1 \leq p < \infty$ and $u \in W^{4,p}(\Omega)$. The following estimate holds:

\begin{equation}
\|\partial_\nu \Delta u\|_{p,\partial\Omega} \leq \tilde{c}_p \|u\|_{3,p,\Omega}^{\frac{p-1}{p}} \|u\|_{4,p,\Omega}^{\frac{1}{p}}
\end{equation}

with $\tilde{c}_p$ being a positive constant independent of $u$.

Proof. Similarly as the proof of Proposition 7.4 we obtain that for $u \in C^4(\overline{\Omega})$ the estimate (7.6) holds. Then, due to the density of $C^4(\overline{\Omega})$ in $W^{4,p}(\Omega)$, the estimate (7.6) holds also for $u \in W^{4,p}(\Omega)$, with the constant $\tilde{c}_p$ being the same of Proposition 7.4.

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B. Barraza Martínez, Universidad del Norte, Departamento de Matemáticas y Estadística, Barranquilla, Colombia
Email address: bbarraza@uninorte.edu.co

J. González Ospino, Universidad del Norte, Departamento de Matemáticas y Estadística, Barranquilla, Colombia
Email address: gjonathan@uninorte.edu.co

J. Hernández Monzón, Universidad del Norte, Departamento de Matemáticas y Estadística, Barranquilla, Colombia
Email address: jahernan@uninorte.edu.co