Ground-States of Two Directed Polymers

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Abstract

Joint ground states of two directed polymers in a random medium are investigated. Using exact min-cost flow optimization the true two-line ground-state is compared with the single line ground state plus its first excited state. It is found that these two-line configurations are (for almost all disorder configurations) distinct implying that the true two-line ground-state is non-separable, even with 'worst-possible' initial conditions. The effective interaction energy between the two lines scales with the system size with the scaling exponents 0.39 and 0.21 in 2D and 3D, respectively.

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I. INTRODUCTION

The physics of disordered systems has attracted a lot of attention due to the discovery that the free energy of extended objects - lines, surfaces and so on - has singular corrections because of the domination of zero-temperature or ground-state effects [1]. The paradigm of such systems is a directed polymer in a random medium (DPRM). In this particular example the object minimizes its energy which is determined by two competing forces: the elastic energy cost of wandering on one hand and the energy gain using energetically favorable pins in the environment on the other hand. The result is super-diffusive behavior, and constrained energy fluctuations. The phase space of the DPRM problem is very rich depending on the nature of the correlations in the disorder and the dimensionality. In low enough dimensions the physics is (at arbitrary temperatures) governed by the so-called zero-temperature fixed point if the noise has weak enough correlations including the uncorrelated case. The case with one transverse dimension becomes exactly solvable in terms of the roughness and energy fluctuation exponents, due to a mapping to the Kardar-Parisi-Zhang equation [2]. The values are $\zeta_2 = 2/3$ and $\theta_2 = 1/3$, which fulfill the exponent relation $2\zeta_d - 1 = \theta_d$. In the 3(=2+1)-dimensional case the roughness exponent is approximately 0.62.

In this paper we study the problem of two (not necessarily directed) polymers in a joint random medium (TPRM) with mutual interactions [3–8] and focus on the repulsive strong coupling limit, i.e. hard core interaction. The work is related both to the question of the physics of flux-lines in high-$T_c$-semiconductors in the low field limit, and to the field-theoretical issues due to the importance of the DP interaction energy. The physics of the problem is in general very similar to that of the one-line case [3–8] but shows interesting twists if one tries to understand the problem in the light of individual, independent objects. In particular, we are going to consider by numerical, exact min-cost flow optimization computations the difference in energy between the TPRM problem, the single-line ground-state and the 'first excited state'. The last one is given by adhering to a hierarchical picture,
in which the first polymer is first optimized given a disorder configuration, and then the
next one is added by applying a hard-core repulsion to the bonds already taken up. The
procedure gives us two energies to compare with the true TPRM ground state energy \( E_2 \):
the single line ground-state energy doubled, \( 2E_1 \), and the sum of the ground-state energy \( E_1 \)
and the energy of the first excited state \( E'_1 \) in the single line problem. The two energy dif-
ferences, \( E_2 - 2E_1 \) and \( E_2 - E_1 - E'_1 \) define an interaction energy of the two polymers. In an
earlier paper Tang [5] studied the TPRM in hierarchical lattices and in two dimensions with
binary disorder. His main conclusion was, for the physically more relevant real-space case,
that the probability for an interaction energy exactly equal to zero (with binary disorder)
decayed much faster than expected, the exponent being -2/3 instead of the -1/3 expected
based on single-DP geometric arguments. We study both the interaction energies discussed
above. We also comment on the topology of the TPRM ground-state. One of the main
conclusions of our paper is that the TPRM ground-state is non-separable at least in the
particular geometry we use. This means that the optimization of the TPRM ground-state
can not be done in two quasi-independent steps.

The structure of the paper is as follows. In section two we formulate the problem and
outline the relevant scaling exponents to be studied later. Section three discusses the nu-
merical method. In section four we give the numerical data concerning the scaling behavior.
Finally section five finishes the paper with conclusions.

II. TWO DIRECTED POLYMERS IN A RANDOM MEDIUM

The continuum Hamiltonian for the TPRM problem is written in all generality as

\[
H = \int_0^L \Gamma_1(\nabla h_1(x))^2 + \Gamma_2(\nabla h_2(x))^2 + V_r(x, h_1) + V_r(x, h_2) + V_{\text{int}} dx.
\]  (1)

The Hamiltonian describes the physics of two elastic lines (subscripts 1 and 2) in the pres-
ence of the random potential \( V_r \) which is sample-to-sample the same for both lines. The
longitudinal coordinate is labeled with \( x \) while the transverse coordinate (which can be a
vector) is \( h_1 \) or \( h_2 \). In the following we shall consider only two 'identical' lines, that is the line stiffnesses \( \Gamma_i \) are taken to be finite and equal. The random potential \( V_r \) describes point disorder and therefore the correlator \( \langle V_r(x, h)V_r(x', h') \rangle \propto \delta(x - x')\delta(h - h') \).

The interaction potential \( V_r \) gives rise to a variety of phenomena. First, for ground-state problems the case of an attractive potential is obviously trivial: the two lines will localize to the same ground-state. In this paper we are going to deal with a hard-core interaction between the lines 1 and 2. This implies a delta-function-like \( V_r \sim V_0\delta(x_1 - x_2)\delta(h_1 - h_2) \) with \( V_0 \rightarrow \infty \) so that overlap between the lines is strictly excluded. Would one allow for e.g. a finite \( V_0 \) then the one-line ground-state would act as a pinning defect and the physics would slowly cross-over from the hard-core case to that of two independent lines as \( V_0 \) is decreased.

The simplest scaling picture for the TPRM in the presence of a hard-core interaction \( V_r \) consists of two independent directed polymers one being in the one-line global minimum and the second being in the first local minimum or the first excited state. This picture implies that the TPRM ground-state would be separable, that is it could be constructed by a successive optimization procedure. This turns out to be false, but the construction gives a definition for the effective interaction energy

\[
V_{\text{int,eff}} = E_1 + E'_1 - E_2 \sim L^{\theta_V}
\]

where \( E_1 \) refers to the single-DP ground-state energy in a particular sample, \( E'_1 \) to the first excited state, \( E_2 \) is the true TPRM ground-state energy and \( L \) is the system size to be defined below in section IV. \( \theta_V \) defines a scaling exponent for this particular form of the interaction energy. Recall that one has \( E_1 \sim AL + \bar{A}L^{\theta_1} + \ldots \) and that the same is expected of \( E'_1 \) as well where \( A, \bar{A} \) are disorder and dimension-dependent non-universal pre-factors. The argument is, however, essentially based on the claim that in the DPRM problem there is only one energy scale, that governed by the DPRM energy fluctuation exponent \( \theta \) and is therefore only qualitative. For \( E_2 \) it is to be expected that the scaling is of the same form \( E_2 \sim BL + \bar{B}L^{\theta_2} \) where the exponent \( \theta_2 \) measures the energy fluctuations of the TPRM.
ground-state. The ensemble-averaged $V_{int,eff}$ allows one to note that since the energy and its fluctuations have as an upper bound the separable trial ground-state $\theta_V$ should be limited from above by $\theta_1$.

Likewise, the interaction energy can be described by the energy of the TPRM ground-state minus twice the single line energy, i.e.

$$\delta E_2 = E_2 - 2E_1 \sim L^{\theta_E}.$$  \hfill (3)

Here $\theta_E$ defines another scaling exponent characterising the TRPM ground-state. One has naturally $\delta E_2 + V_{int,eff} = E'_1 - E_1 > 0$ and in particular if the single-line problem has two geometrically independent, energetically degenerate solutions then the sum is zero. Since $\delta E_2$ is positive semi-definite sample-to-sample, a lower limit for $\theta_2$ is $\theta_1$ and therefore by this dual construction one would expect that $\theta_2 = \theta_1$. In this work we do not consider the roughness properties of the two-line system but note that for it one would likewise expect that $\zeta_2 = \zeta_1$. Figure (2) shows examples from two and three dimensions of situations in which the true TPRM ground-state is non-separable, i.e. it can not be constructed out of the states with energies $E_1$ and $E'_1$ and has thus a non-zero $V_{int,eff}$.

**III. NUMERICAL METHOD**

Here we define the lattice version of the continuum model of two random polymers with hard core interactions in a random environment introduced in the preceeding section. We formulate it in such a way that the connection to a minimum cost flow problem becomes obvious [11,12], for which powerfull algorithms from combinatorial optimization exist that find exact ground states in polynomial time [13].

Consider the energy function

$$H(x) = \sum_{(ij)} e_{ij} \cdot x_{ij},$$  \hfill (4)

where $\sum_{(ij)}$ is a sum over all bonds $(ij)$ joining site $i$ and $j$ of a $d$-dimensional lattice, e.g. a rectangular ($L^{d-1} \times H$) lattice, with periodic boundary conditions (b.c.) in $d - 1$
space direction and free b.c. in one direction. The bond energies $e_{ij} \geq 0$ are quenched random variables that indicate how much energy it costs to put a segment of a polymer on a specific bond $(ij)$. The variables describing the two polymers are $x_{ij} \in \{0, 1\}$ (for hard core interactions), $x_{ij} = 1$ if there is a polymer passing bond $(ij)$ and zero otherwise. For the configuration to form lines on each site of the lattice all incoming flow should balance the outgoing flow, i.e. the flow is divergence free

$$\nabla \cdot \mathbf{x} = 0,$$  \hspace{1cm} (5)

where $\nabla \cdot$ denotes the lattice divergence. Obviously the flux-line has to enter, and to leave, the system somewhere. We attach all sites of one free boundary to an extra site (via energetically neutral arcs, $e = 0$), which we call the source $s$, and the other side to another extra site, the target, $t$ as indicated in fig. 1a. Now one can push one line through the system by inferring that $s$ has a source strength of $+1$ and that $t$ has a sink strength of $-1$, i.e.

$$(\nabla \cdot \mathbf{x})_s = +N \quad \text{and} \quad (\nabla \cdot \mathbf{x})_t = -N,$$  \hspace{1cm} (6)

with $N = 1$. Thus, the 1-line problem consists in minimizing the energy (4) by finding a flow $\mathbf{x}$ in the network (the lattice plus the two extra sites $s$ and $t$) fulfilling the constraints (4) and (5). Naively one would expect that the 2-line problem consists simply in adding a second line to the 1-line configuration, avoiding the bonds already occupied due to the hard core interaction we consider here. A glance at Fig. 1a convinces us that this is not correct and actually the main issue of the present paper is to provide evidence that the correct solution of the TPRM problem is significantly different from what one gets when assuming the separability of the ground state.

The first key ingredient to treat the two-line problem (and the $N$-line problem in general) is that one does not work with the original network but with the residual network corresponding to the actual flux-line configuration, which contains also the information about possibilities to send flow backwards (now with energy $-e_{ij}$ since one wins energy by reducing $x_{ij}$), i.e. to modify the actual flow. Suppose that we put one flux-line along
a shortest path \( P(s,t) \) from \( s \) to \( t \), which means that we set \( x_{ij} = 1 \) for all arcs on the path \( P(s,t) \). Then the residual network is obtained by reversing all arcs and inverting all energies along this path, indicating that here we cannot put any further flow in the forward direction (since we assume hard-core interaction, i.e. \( x_{ij} \leq 1 \)), but can send flow backwards by reducing \( x_{ij} \) on the forward arcs by one unit. This procedure is sketched in Figure 1.

The second key ingredient is the introduction of a so-called potential \( \varphi \) that fulfills the relation

\[
\varphi(j) \leq \varphi(i) + e_{ij}
\]

for all arcs \((ij)\) in the residual network, indicating how much energy \( \varphi(j) \) it would \textit{at least} take to send one unit of flow from \( s \) to site \( j \), IF it would cost an energy \( \varphi(i) \) to send it to site \( i \). With the help of these potentials one defines the reduced costs

\[
c_{ij}^\varphi = e_{ij} + \varphi(i) - \varphi(j) \geq 0.
\]

The last inequality, which follows from the properties of the potential \( \varphi \) actually ensures that there is no loop \( \mathcal{L} \) in the current residual network (corresponding to a flow \( x \)) with negative total energy, since \( \sum_{(ij) \in \mathcal{L}} e_{ij} = \sum_{(ij) \in \mathcal{L}} c_{ij}^\varphi \), implying that the flow \( x \) is optimal [12].

The idea of the \textit{successive shortest path algorithm} is to start with an empty network, i.e. \( x^0 = 0 \), which is certainly an optimal flow for \( N = 0 \), and set \( \varphi = 0 \), \( c_{ij}^0 = e_{ij} \). One now successively adds FL to the system using the following iteration: Suppose we have an optimal \( N - 1 \)-line configuration corresponding to the flow \( x^{N-1} \). The current potential is \( \varphi^{N-1} \), the reduced costs are \( c_{ij}^{N-1} = e_{ij} + \varphi^{N-1}(i) - \varphi^{N-1}(j) \) and we consider the residual network \( G_c^{N-1} \) corresponding to the flow \( x^{N-1} \) with the reduced costs \( c_{ij}^{N-1} \geq 0 \). The iteration leading to an optimal \( N \)-line configuration \( x_{ij}^N \) is

1. Determine shortest distances \( d(i) \) from \( s \) to all other nodes \( i \) with respect to the reduced costs \( c_{ij}^{N-1} \) in the residual network \( G_c^{N-1} \).

2. For all nodes \( i \) update the potential: \( \varphi^N(i) = \varphi^{N-1}(i) + d(i) - d(t) \).
3. Let \( P(s, t) \) denote a shortest path from node \( s \) to \( t \). To obtain \( x_{ij}^N \) increase (decrease) by one unit the flow variables \( x_{ij}^{N-1} \) on all forward (backward) arcs \( (ij) \) along \( P(s, t) \).

(see Fig. [1]). Note that due to the the fact that the numbers \( d(i) \) are shortest distances one has again \( c_{ij}^N \geq 0 \), i.e. the flow \( x^N \) is indeed optimal. To estimate the complexity of this algorithm it is important to note that it is not necessary to determine shortest paths from \( s \) to all other nodes in the network; a shortest path from \( s \) to \( t \) is sufficient if one updates the potentials in a slightly different way [12]. Thus, the complexity of each iteration is the same as that of Dijkstra’s algorithm for finding shortest paths in a network, which is \( \mathcal{O}(M^2) \) for a naive implementation (\( M \) is the number of nodes in the network). We find, however, for the cases we consider (\( d \)-dimensional lattices) it roughly scales linearly in \( M = L^d \). Thus, for \( N \) flux-lines the complexity of this algorithm is \( \mathcal{O}(NL^d) \).

In Figure 2 we show the true ground state configuration for a specific disorder configuration in 2d and in 3d and compare it with the one-line ground state plus the first excited state (the latter defined as the ground state in the network that is left when the bonds occupied by the one-line ground state are excluded). This is a typical example in which the two two-line configuration in 2d and in 3d are distinct.

**IV. RESULTS**

For the actual computations reported in the following we set the height of the system \( H \) equal to its lateral size \( L \), i.e. \( H = L \), yielding a square geometry in 2d and a cubic one in 3d) and considered system sizes from \( L = 16 \) to \( L = 256 \) in 2d and from \( L = 8 \) to \( L = 64 \) in 3d. For each system size the results are averaged over \( N = 12000 \) (2D) and \( N = 8000 \) (3D) disorder configurations, and quantities like \( \mathcal{O} = E_1, E_2, \delta E_2, V_{int, eff} \) denote disorder averages from now on.

We expect the various exponents that we estimate to be independent of the actual disorder we put in (as long it is uncorrelated and does not have algebraic tails), nevertheless we took two different probability distributions for the bond energies: 1) a uniform distribution...
for which \(P(e_{ij}) = 1\) for \(e_{ij} \in [0, 1]\) and 0 otherwise; 2) a *binary* distribution in which \(e_{ij}\) is 1 with probability \(p\) and 0 with probability \(1 - p\).

### A. Two dimensions

Figure 3 shows the scaling of the two-line system energy and energy fluctuations for both a uniform distribution for the \(e_{ij}\)'s and a binary one with \(qp = 0.8\). As expected, the scaling of the total energy \(E_2\) is linear and the fluctuations \(\delta E_2\) scale with an exponent \(\theta_2\) with \(\theta_2 \simeq \theta_1\), the one-line energy fluctuation exponent. This adheres to the picture that the energetics of the DP problem are in general dictated by the one-line exponent.

In Figure 4 we show the probability that \(\delta E_2 = 0\) as a function of system size. This measures the true degeneracy of the two-line system as the joint ground-state can be obtained from two independent minima with the same energy. \(P(\delta E_2 = 0) \sim L^{-a_1}\) with \(a_1 = 0.63 \pm 0.03\) which is compatible with to \(a_1 = 2/3\) adhering thus to Tang’s result which indicated \(a_1 = 1 - \theta\). One can compare this with the scaling of \(P(V_{int,eff} = 0)\), which scales with an exponent \(a_2 = 0.15 \pm 0.02\) for both distributions \((P \sim L^{-a_2})\). Similarly to Tang’s conjecture, we are left with a picture which explains the frequency of separable ground-states (with \(\delta E_2 = 0\)) by a picture in which the two lines belong to two neighboring trees in the energy landscape. This means that one considers an inverted structure in which the two lines end up next to each other but belonging to two different trees (starting from \(x = L\)) with the same energy. Meanwhile the interaction energy in general shows increasing entanglement with a probability for a separable GS that decays with a novel exponent \(a_2 = 0.15\).

Figures 5 and 6 discuss further the scaling of the mean interaction energies \(\delta E_2\) and \(V_{int,eff}\) for the both distributions. We find the exponents \(\theta_E \sim 0.39 \pm 0.03\) and \(\theta_V \sim 0.39 \pm 0.03\), respectively. For both these quantities we seem to obtain that the effective scaling exponents are slightly higher than the one- or two-line energy fluctuation exponents as such. However, as shown in figure 7 we can collapse the energy probability distributions for \(\delta E_2\) and \(V_{int,eff}\) by using a two-exponent collapse. Note that this is different from the
simple collapse using $\theta_E$ and $\theta_V$, however the two exponents combined make it so that the averages scale with $\theta_E$ and $\theta_V$.

**B. Three dimensions**

Figure 8 shows the scaling of three-dimensional case again for both a uniform distribution for the $e_{ij}$’s and a binary one with $p = 0.8$ for the case of the two-line system energy and energy fluctuations. As expected, the scaling of the total energy $E_2$ is linear and the fluctuations $\delta E_2$ scale with an exponent $\theta_2$ with $\theta_2 \approx \theta_1 \approx 0.24$, the one-line energy fluctuation exponent in three dimensions.

In Figure 8 we show the probability that $\delta E_2 = 0$ as a function of system size. In 3D, for binary disorder, $P(\delta E_2 = 0) \sim L^{-a_1}$ with $a_1 = 0.25 \sim \theta_1$ in contrast with the geometric picture valid in 2D. The scaling of $P(V_{int, eff} = 0)$ can not be described with a unique exponent and we find $a_2 = 0.11 \pm 0.01$ for binary, and $a_2 = 0.05 \pm 0.01$ for continuous disorder ($P \sim L^{-a_2}$). Again the interaction energy in general shows increasing entanglement with a probability for a separable GS that decays with novel exponents $a_1, a_2$.

Figures 10 and 11 discuss further the scaling of the mean interaction energies $\delta E_2$ and $V_{int, eff}$ for the continuous distributions. The 3D exponents become $\theta_E = 0.26 \pm 0.02$ and $\theta_V = 0.21 \pm 0.02$. As shown in figure 7 we can collapse the energy probability distributions for $\delta E_2$ and $V_{int, eff}$ by using as in 2D a two-exponent collapse. For binary disorder the collapse of the data makes sense in both cases, for continuous we restrict ourselves to $\delta E_2$.

**V. CONCLUSIONS**

In this paper we have investigated the joint ground-state of two directed polymers in a random medium, the TPRM problem. The main questions addressed here are whether the scaling of the TPRM can be described with the one-line exponents and an associated picture of behavior and if not so when. Unsurprisingly it turns out that $V_{int, eff}$ as defined
here seems to result in an *independent* exponent that can not be explained by the one-line scaling arguments. This is natural since it measures the difference of the true TPRM ground-state to the 'Ansatz’ of two separable states and is thus the first non-analytic and non-trivial correction characterizing the unique nature of the TPRM problem. On the other hand some of the features of the TPRM energetics, like the degeneracy of $\delta E_2$ are clearly related to the single-line picture in two dimensions. In three dimensions this is no longer true. We lack a geometrical explanation for the scaling of the degeneracy exponent $a_2$ in this higher-dimensional case.

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FIG. 1. Sketch of the successive shortest path algorithm for the solution of the minimum cost flow problem described in the text.  
(a) Network for $N = 0$, the numbers are the reduced costs $e_{ij}$, red for downward and right arcs, blue for upward and left arcs. The red line is a shortest path from $s$ to $t$.  
(b) $G_c^0$ with the updated node potentials.  
(c) $G_c^0$ with the updated reduced costs. The green line is a shortest path.  
(d) OPtimal flow configuration for $N = 2$. 

FIGURES
FIG. 2. a): Two-polymer ground-state in 2D, b) the same system but with the first (1-line GS) frozen first. c) the TPRM GS in 3D, d) as b) but in 3D. In both the 2D and the 3D comparisons the disorder landscape is the same.

FIG. 3. Energy $E_2$ (circles) and energy fluctuations $\Delta E_2$ (squares) of the TPRM problem in two dimensions in a log-log plot. We show data for binary disorder ($e_{ij} \in \{0, 1\}$) (filled symbols) and the uniform distribution of $e_{ij}$'s (open symbols). One expects $E_2 \propto L$ and $\Delta E_2 \propto L^\theta$, correspondingly the straight lines have slopes 1 (top) and 1/3 (bottom).
FIG. 4. \( P(\delta E_2 = 0) \) (squares) and \( P(V_{\text{int,eff}}) = 0 \) (circles) vs. \( L \) in 2d in a log-log plot. Data for both binary (filled symbols) and uniform (open symbols) distribution of the bond energies \( e_{ij} \). The data follow the relations \( P(\delta E_2 = 0) \propto L^{-a_1} \) and \( P(V_{\text{int,eff}}) = 0 \propto L^{-a_2} \) with \( a_1 \) and \( a_2 \) given by the slopes of the straight lines: \( a_1 = 0.63 \ (\approx 1 - \theta) \) and \( a_2 = 0.15 \).

FIG. 5. \( \delta E_2 \) in 2d for binary (filled symbols) and uniform (open symbols) distribution of the bond energies \( e_{ij} \) in a log-log plot. It is \( \delta E_2 \propto L^{\theta_E} \) with \( \theta_E = 0.39 \pm 0.02 \).
FIG. 6. $V_{\text{int,eff}}$ in 2d for binary (filled symbols) and uniform (open symbols) distribution of the bond energies $e_{ij}$ in a log-log plot. It is $V_{\text{int,eff}} \propto L^{\theta_V}$ with $\theta_V 0.39 \pm 0.02$.

FIG. 7. Scaling plots of the probability distributions of $\delta E_2$ and $V_{\text{int,eff}}$ in 2d for binary (filled symbols) and uniform (open symbols) distribution of the bond energies $e_{ij}$ in a log-log plot.
FIG. 8. Energy $E_2$ (circles) and energy fluctuations $\Delta E_2$ (squares) of the TPRM problem in three dimensions in a log-log plot. We show data for binary disorder ($e_{ij} \in \{0, 1\}$) (filled symbols) and the uniform distribution of $e_{ij}$’s (open symbols). One expects $E_2 \propto L$ and $\Delta E_2 \propto L^\theta$, correspondingly the straight lines have slopes 1 (top) and 0.24 (bottom).

FIG. 9. $P(\delta E_2 = 0)$ (squares) and $P(V_{int,eff} = 0)$ (circles) vs. $L$ in 3d in a log-log plot. Data for both binary (filled symbols) and uniform (open symbols) distribution of the bond energies $e_{ij}$. The data follow the relations $P(\delta E_2 = 0) \propto L^{-a_1}$ and $P(V_{int,eff} = 0) \propto L^{-a_2}$ with $a_1$ and $a_2$ given by the slopes of the straight lines: $a_1 = 0.25$ and $a_2 = 0.11, 0.05$ for binary and continuous disorder, respectively.
FIG. 10. $\delta E_2$ in 3d for binary (filled symbols) and uniform (open symbols) distribution of the bond energies $e_{ij}$ in a log-log plot. It is $\delta E_2 \propto L^{\theta_E}$ with $\theta_E = 0.26 \pm 0.02$.

FIG. 11. $V_{\text{int,eff}}$ in 3d for binary (filled symbols) and uniform (open symbols) distribution of the bond energies $e_{ij}$ in a log-log plot. It is $V_{\text{int,eff}} \propto L^{\theta_V}$ with $\theta_V = 0.21 \pm 0.02$. 
FIG. 12. Scaling plots of the probability distributions of $\delta E_2$ and $V_{int,eff}$ in 3d for the uniform distribution of the bond energies $e_{ij}$ in a log-log plot.