A CHARACTERISATION OF THE BESOV-LIPSCHITZ AND TRIEBEL-LIZORKIN SPACES USING POISSON LIKE KERNELS

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Abstract. We give a complete characterisation of the spaces $\dot{B}^\alpha_{p,q}$ and $\dot{F}^\alpha_{p,q}$ by using a non-smooth kernel satisfying near minimal conditions. The tools used include a Strömberg-Torchinsky type estimate [20] for certain maximal functions and the concept of a distribution of finite growth, inspired by Stein [19]. Moreover, our exposition also makes essential use of a number of refinements of the well-known Calderón reproducing formula. The results are then applied to obtain the characterisation of these spaces via a fractional derivative of the Poisson kernel.

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1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

We aim to characterise the Besov-Lipschitz space $\dot{B}^\alpha_{p,q}$ and the Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}$ using a kernel $\psi$ which satisfies near “minimal” conditions regarding cancellation, smoothness and decay. To facilitate the discussion to follow, we begin by recalling the

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definition of the homogeneous Besov-Lipschitz and Triebel-Lizorkin spaces following Peetre [16, 17] (see also [22]). All functions and distributions are defined on the Euclidean space $\mathbb{R}^n$ unless otherwise stated.

Let $\varphi \in \mathcal{S}$ such that $\text{supp } \hat{\varphi} = \{1/2 \leq |\xi| \leq 2\}$ and for every $\xi \neq 0$

$$\sum_{j \in \mathbb{Z}} \hat{\varphi}(2^{-j}\xi) \hat{\varphi}(2^{-j}\xi) = 1.$$  

(1)

The function $\varphi$ is fixed throughout this article. Given a function $\varphi : \mathbb{R}^n \to \mathbb{C}$ we let $\phi_j(x) = 2^{jn}\phi(2^jx)$ denote the dyadic dilation of $\phi$. Somewhat confusingly, for $t \in \mathbb{R}$ with $t > 0$, we let $\phi_t(x) = t^{-n}\phi(t^{-1}x)$ denote the standard dilation. It should always be clear from the context the type of dilation we are using.

Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. The (homogeneous) Besov-Lipschitz space $\dot{B}_{p,q}^\alpha$ is then defined as the class of all tempered distributions modulo polynomials $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}_{p,q}^\alpha} = \left(\sum_{j \in \mathbb{Z}} \left(2^{j\alpha}\|\varphi_j * f\|_{L^p}\right)^q\right)^\frac{1}{q} < \infty.$$  

(2)

Similarly, for $0 < p, q \leq \infty$, $p \neq \infty$, and $\alpha \in \mathbb{R}$, the (homogeneous) Triebel-Lizorkin space $\dot{F}_{p,q}^\alpha$ is defined as the class of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{F}_{p,q}^\alpha} = \left\|\left(\sum_{j \in \mathbb{Z}} \left(2^{j\alpha}|\varphi_j * f|\right)^q\right)^\frac{1}{q}\right\|_{L^p} < \infty.$$  

(3)

The definition of $\dot{F}_{\infty,q}^\alpha$ is slightly different, the problem is that a naive extension of (3) to the case $p = \infty$ leads to spaces which are not independent of the choice of kernel, and moreover the expected identification $\dot{F}_{\infty,2}^0 = BMO$ fails; see the discussion in Section 5 of [12]. Instead, following the work of [12] we define

$$\|f\|_{\dot{F}_{\infty,q}^\alpha} = \sup_Q \left(\frac{1}{|Q|} \int_Q \sum_{j \geq -\ell(Q)} \left(2^{j\alpha}|\varphi_j * f(x)|\right)^q dx\right)^\frac{1}{q}$$

with the interpretation that when $q = \infty$,

$$\|f\|_{\dot{F}_{\infty,\infty}^\alpha} = \sup_Q \sup_{j \geq -\ell(Q)} \frac{1}{|Q|} \int_Q 2^{j\alpha}|\varphi_j * f(x)| dx,$$

(4)

where the sup is over all dyadic cubes $Q$, and $\ell(Q) = \log_2(\text{side length of } Q)$. The above interpretation for the norm $\| \cdot \|_{\dot{F}_{\infty,\infty}^\alpha}$ is taken from [7] where one can also find the embedding $\dot{F}_{\infty,q}^\alpha \subset \dot{F}_{\infty,\infty}^\alpha = \dot{B}_{\infty,\infty}^\alpha$.

Observe that elements of the quasi Banach spaces $\dot{B}_{p,q}^\alpha$ and $\dot{F}_{p,q}^\alpha$ are equivalence classes of distributions modulo polynomials. However we often make a common (and harmless) abuse of notation by regarding elements of $\dot{B}_{p,q}^\alpha$ and $\dot{F}_{p,q}^\alpha$ as distributions, rather than equivalence classes.
The Besov-Lipschitz and Triebel-Lizorkin scales of spaces arise in many applications in mathematics. In particular, they are of crucial importance in interpolation theory \cite{1,17,22} and contain many well-known function spaces in mathematical analysis. For instance, \( \hat{F}_{p,2}^0 \) is identified with the real-variable Hardy space \( H^p \) of Fefferman-Stein \cite{10}, while \( \hat{F}_{\infty,2}^0 \) is identified with \( BMO \), the space of functions of bounded mean oscillation \cite{12}.

The fundamental and central result in the study of the spaces \( \hat{B}_{p,q}^\alpha \) and \( \hat{F}_{p,q}^\alpha \) is the independence of these function spaces on the choice of kernel function \( \varphi \). Thus if we replace the function \( \varphi \) in (2) and (3) with a different kernel \( \psi \in \mathcal{S} \), then provided \( \psi \) satisfies certain conditions, we have an equivalent norm. This independence was initially established in the pioneering works of Peetre \cite{16,17} for all Besov-Lipschitz spaces and for the Triebel-Lizorkin spaces when \( p < \infty \), and the result applied in particular to band-limited kernels. The method used by Peetre was inspired by the real-variable theory for various maximal functions, developed in the seminal paper \cite{10} by Fefferman and Stein. The result for \( \hat{F}_{\infty,q}^\alpha \) was proved later in \cite{12} (see also \cite{7}). After further partial results by Triebel \cite{21,22}, the essentially optimal conditions, at least in the Schwartz case \( \psi \in \mathcal{S} \), were developed in Bui, Paluszyński and Taibleson in \cite{5}, \cite{6} and \cite{7} where it was shown that we have an equivalent norm for any \( f \in \mathcal{S}' / \mathcal{P} \), provided that the kernel \( \psi \in \mathcal{S} \) satisfies the following:

(I) (Vanishing moments) The kernel \( \psi \) has \([\alpha]\) vanishing moments, thus
\[
\int_{\mathbb{R}^n} x^\kappa \psi(x) dx = 0
\]
for every \( |\kappa| \leq [\alpha] \), with the understanding that no condition is required when \( \alpha < 0 \).

(II) (Tauberian condition) The kernel \( \psi \) satisfies the Tauberian condition; that is, for every \( \xi \in \mathbb{S}^{n-1} \) there exists \( a, b \in \mathbb{R} \) (depending on \( \xi \)) with \( 0 < 2a \leq b \) such that for every \( a \leq t \leq b \)
\[
|\hat{\psi}(t\xi)| > 0.
\]

Here, given \( \alpha \in \mathbb{R} \) we let \([\alpha]\) denote the integer part of \( \alpha \). Note that the conditions (I) and (II) do not require that the kernel \( \psi \) is band-limited. Thus, for example, it is possible to characterise the Besov-Lipschitz and Triebel-Lizorkin spaces with derivatives of the Gaussian kernel \( e^{-|x|^2} \).

In this paper we consider the problem of removing the assumption \( \psi \in \mathcal{S} \). Our key goal is to give conditions on the kernel that are simply stated and easily checked, while still being as close as possible to optimal. Moreover, they should apply in particular to the important case of fractional derivatives of the Poisson kernel \((1 + |x|^2)^{-\frac{n+1}{2}}\).
The first, and somewhat obvious, obstacle in the non-smooth case $\psi \notin S$, is that to define the norms (2) and (3) we need to be able to define the convolution $\psi \ast f$ for arbitrary $f \in S'$. This is clearly not possible unless $\hat{\psi}$ is infinitely differentiable and all its derivatives are slowly increasing. As our main application, the Poisson kernel does not satisfy the last conditions (its Fourier transform is not differentiable at the origin), we need to restrict the class of distributions slightly to a natural class of admissible distributions. To this end, inspired by Stein [19], we introduce the concept of distributions of bounded growth.

Definition 1.1. [Distributions of growth $\ell$] We say $f \in S'$ is a distribution of growth $\ell \geq 0$ if for any $\phi \in S$ we have $\phi \ast f = \mathcal{O}(|x|^\ell)$ (as $|x| \to \infty$).

The importance of this definition is that it allows us to make sense of the convolution $\psi \ast f$ when $\psi \notin S$.

Definition 1.2. Assume $f$ is a distribution of growth $\ell$. Then if $(1 + |\cdot|)^r \hat{\psi} \in L^1$ we define the convolution $\psi \ast f \in S'$ as

$$\psi \ast f(\phi) = \int_{\mathbb{R}^n} \psi(x)(\hat{\phi} \ast f)(x)dx, \quad \phi \in S,$$

where $\hat{\phi}(x) = \phi(-x)$. This definition coincides with the pointwise definition for $\psi \ast f$ when $f = \mathcal{O}(|x|^\ell)$ is locally integrable.

We note that Stein used the concept of a bounded distribution (the case $\ell = 0$ in Definition 1.1) to characterise the Hardy spaces $H^p$ using the Poisson kernel (see [19, 10]).

Before proceeding to state the main theorem we prove, we discuss the key conditions we require on our kernel. To this end, we take parameters $\Lambda \geq 0$ and $m, r \in \mathbb{R}$, and suppose $\hat{\psi} \in L^1(\mathbb{R}^n)$.

(C1) (Cancellation) Let $\hat{\psi} \in C^{n+1+|\Lambda|}(\mathbb{R}^n \setminus \{0\})$ such that for every $|\kappa| \leq n + 1 + |\Lambda|$ we have

$$\partial^\kappa \hat{\psi} = \mathcal{O}(|\xi|^{-|\kappa|}) \quad \text{as } |\xi| \to 0.$$

(C2) (Tauberian condition) The kernel $\psi$ satisfies the Tauberian condition (as in (II) above).

(C3) (Smoothness) Take $\hat{\psi} \in C^{n+1+|\Lambda|}(\mathbb{R}^n \setminus \{0\})$ such that for every $|\kappa| \leq n + 1 + |\Lambda|$ we have

$$\partial^\kappa \hat{\psi} = \mathcal{O}(|\xi|^{-n-m}) \quad \text{as } |\xi| \to \infty.$$

The parameters $\Lambda$ and $m$ roughly correspond to the decay and smoothness we require on a component of our kernel $\psi$. More precisely if $\phi \in S$ has Fourier support away from the origin, then $\hat{\psi} \in C^{n+1+|\Lambda|}(\mathbb{R}^n \setminus \{0\})$ implies that $\psi \ast \phi(x) = \mathcal{O}(|x|^{-n-1-|\Lambda|})$ as $|x| \to \infty$. Thus larger $\Lambda$ requires more decay on the part of $\psi$ with Fourier support away from the origin. Similarly, if (C3) holds, then $\psi \in C^{[m]}(\mathbb{R}^n)$. Thus the larger we take $m$, the smoother the kernel $\psi$ is required to be.
On the other hand, the parameter $r$ and the cancelation condition $(C1)$ are closely related to the vanishing moments condition $(I)$. More precisely, if $\psi \in S$ then it is easy to check that $\psi$ has $[\alpha]$ vanishing moments (i.e. $(I)$ holds), if and only if $(C1)$ holds with $\alpha < r \leq [\alpha] + 1$. Of course if $\psi \not\in S$ then the relationship between $(I)$ and $(C1)$ is somewhat complicated, but roughly speaking $(I)$ requires more spatial decay, while $(C1)$ requires more regularity of $\hat{\psi}$ near the origin. It is worth pointing out that it is possible to prove the characterisations below with $(C1)$ replaced with $(I)$, but this requires more decay on the kernel $\psi$ and leads to less optimal conditions. Instead we have chosen to use conditions on the Fourier transform of $\psi$, as firstly this matches up very well with our intended application to the Poisson kernel, and secondly, in the authors’ opinion the conditions $(C1)$, $(C2)$, and $(C3)$ form an acceptable balance between the sharpness of our result, and the simplicity of its statement.

It is natural to split the characterisation results into two theorems: “Necessary Conditions” and “Sufficient Conditions”. This is due to fact that, as noted in [5], each theorem requires a slightly different set of assumptions. The essential assumption for the former is the cancellation property of the kernel, expressed by the condition $(C1)$, while for the latter the Tauberian condition $(II)$ stated earlier in the introduction is critical. Other conditions, such as the decay at infinity of the kernel in the frequency domain expressed by the smoothness condition $(C3)$, are needed to define the convolution with distributions of finite growth.

In the necessary direction, the statement of our result is expressed using a maximal function introduced in the work of Peetre [16]. Given a kernel $\psi$, if $f \in S'$ is such that each $\psi_j * f$ is a function one defines the Peetre maximal function by

$$
\psi_j^* f(x) = \psi_{j,\lambda} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\psi_j * f(x - y)|}{(1 + 2^j |y|)^{\lambda}}, \quad x \in \mathbb{R}^n,
$$

(5)

where $\lambda > n/p$ in the Besov-Lipschitz case and $\lambda > \max\{n/p, n/q\}$ in the Triebel-Lizorkin case (with $\lambda > n$ for the space $\dot{F}^{\alpha}_{\infty,\infty}$). Unless otherwise stated, the number $\lambda$ satisfies these conditions throughout this work.

In the rest of the paper we write $A \lesssim B$ when there exists a positive constant $C$ such that $A \leq CB$, where $C$ may depend on the parameters such as $n, \alpha, p, q, ...$ but usually not on the variable quantities such as the distribution $f$. When $j, k \in \mathbb{Z}$ we write $j \lesssim k$ to mean that $j \leq k + c$ for some $c \in \mathbb{Z}$ independent of $j$ and $k$.

We can now state our main results. We start with the necessary direction.
Theorem 1.1. Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. Let $\ell \geq 0$ with $\ell > \alpha - \frac{n}{p}$. Assume $(1 + |\cdot|)^{\ell} \psi \in L^1$ and that $\psi$ satisfies the cancellation condition (C1) and smoothness condition (C3) for parameters $\Lambda \geq 0$, $r > \alpha$, and $m > \Lambda - \alpha$.

(i) Let $f \in \dot{B}_{p, q}^\alpha$ and $\Lambda = \frac{\alpha}{p}$. Then there exists a polynomial $\rho$ such that $f - \rho$ is a distribution of growth $\ell$ and we have the inequalities

$$
\left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \left\| \psi_j^* (f - \rho) \right\|_{L^p} \right)^q \right)^{\frac{1}{q}} \lesssim \| f \|_{\dot{B}_{p, q}^\alpha},
$$

and for any $\phi \in S$

$$
\left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \left\| \sup_{t>0} |\phi_t \ast \psi_j^* (f - \rho) \|_{L^p} \right\| \right)^q \right)^{\frac{1}{q}} \lesssim \| f \|_{\dot{B}_{p, q}^\alpha}.
$$

(ii) Similarly if $f \in \dot{F}_{p, q}^\alpha$ and we let $\Lambda = \max\{ \frac{\alpha}{p}, \frac{n}{q} \}$ (with $\Lambda = n$ when $p = q = \infty$), then there exists a polynomial $\rho$ such that $f - \rho$ is a distribution of growth $\ell$ and if $p < \infty$

$$
\left\| \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \psi_j^* (f - \rho) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \lesssim \| f \|_{\dot{F}_{p, q}^\alpha},
$$

and for $p = \infty$

$$
\sup_Q \left( \frac{1}{|Q|} \int_Q \sum_{j > -\ell(Q)} \left( 2^{j\alpha} \psi_j^* (f - \rho)(x) \right)^q dx \right)^{\frac{1}{q}} \lesssim \| f \|_{\dot{F}_{\infty, q}^\alpha}.
$$

Furthermore, if $\phi \in S$, we have for $p < \infty$

$$
\left\| \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \sup_{t>0} |\phi_t \ast \psi_j^* (f - \rho) | \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \lesssim \| f \|_{\dot{F}_{p, q}^\alpha},
$$

and $p = \infty$

$$
\sup_Q \left( \frac{1}{|Q|} \int_Q \sum_{j > -\ell(Q)} \left( 2^{j\alpha} \sup_{t>0} |\phi_t \ast \psi_j^* (f - \rho)(x) | \right)^q dx \right)^{\frac{1}{q}} \lesssim \| f \|_{\dot{F}_{\infty, q}^\alpha}.
$$

Remark 1.1. A few remarks are in order.

(i) When $q = \infty$, the inequality (11) is interpreted similarly to the definition of the $\dot{F}_{\infty, \infty}^\alpha$-norm (1); i.e.,

$$
\sup_Q \sup_{j > -\ell(Q)} \frac{1}{|Q|} \int_Q \left( 2^{j\alpha} \sup_{t>0} |\phi_t \ast \psi_j^* (f - \rho)(x) | \right) dx \lesssim \| f \|_{\dot{F}_{\infty, \infty}^\alpha}.
$$

We adopt this interpretation hereafter in all the theorems and proofs.

(ii) The assumptions on the kernel $\psi$ ensure that each convolution $\psi_j^* (f - \rho)$ is well-defined and moreover is a continuous function (see Theorem 3.1). This is a consequence of two key steps. The first is to show that if $f \in \dot{B}_{p, q}^\alpha$ (or $f \in \dot{F}_{p, q}^\alpha$), then there is a polynomial $\rho$ such that $f - \rho$ is in fact a distribution a growth $\ell > \alpha - \frac{n}{p}$ (see Theorem and Corollary 2.3). Thus we can define the convolution $\psi \ast (f - \rho)$ as a distribution via
Definition 1.1. The second step is to use a version of the Calderón reproducing formula to deduce that the distribution $\psi^*(f - \rho)$ is in fact a continuous function (see Theorem 3.1). It is important to note that both of these steps rely crucially on the fact that we assume $f \in \dot{B}^\alpha_{p,q}$ (or $f \in \dot{F}^\alpha_{p,q}$), and thus have some control over the growth and smoothness of the distribution $f$.

(iii) Given $\phi \in \mathcal{S}$ with $\int \phi \neq 0$ and $0 < p \leq \infty$, the characterisation of the real-variable Hardy space $H^p$ defined by Fefferman and Stein in [10], gives

$$
\left\| \sup_{t > 0} |\phi_t * g| \right\|_{L^p} \approx \|g\|_{H^p}.
$$

(Note that when $1 < p \leq \infty$, $H^p = L^p$ with equivalent norms.) Thus it follows from (7) that

$$
\left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\psi_j^*(f - \rho)\|_{H^p} \right)^q \right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^\alpha_{p,q}}.
$$

(12)

(iv) Since $|\psi_j * g|$ is clearly dominated pointwise by $\psi_j^* g$ we may replace (6) with

$$
\left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\psi_j^*(f - \rho)\|_{L^p} \right)^q \right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^\alpha_{p,q}}.
$$

(13)

Similarly we may replace the Peetre maximal function $\psi_j^*(f - \rho)$ in (8) and (9) with the standard convolution $|\psi_j^* (f - \rho)|$.

We next consider the converse to the above theorem; that is, to find sufficient conditions on the kernel $\psi$ and the distribution $f$ such that the reverse inequalities to those in Theorem 1.1 holds. We emphasise that the results in this sufficient part are the main contribution of this paper. As soon as the assumption $\psi \in \mathcal{S}$ is dropped, one immediately runs into the difficulty of defining the convolution $\psi_j * f$ when $f \in \mathcal{S}'$. The situation is different from the necessary result in Theorem 1.1 where we already knew that $f \in \dot{B}_{p,q}^\alpha$ or $f \in \dot{F}_{p,q}^\alpha$, and therefore the convolution given in Definition 1.2 can be seen to be a continuous bounded function via what is essentially a duality argument (see Theorem 3.1). However, if $f$ is a distribution of growth $\ell \geq 0$, then we have seen it is possible to define $\psi_j * f$ as a distribution under rather mild condition on $\psi$ (see Definition 1.2). Our first result in the sufficient direction makes use of this observation.

**Theorem 1.2.** Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. Assume $f$ is a distribution of growth $\ell \geq 0$. Suppose $(1 + | \cdot |^\ell) \psi \in L^1$ satisfies the Tauberian condition (C2) and there exists $m \in \mathbb{R}$ such that the smoothness condition (C3) holds for $\Lambda \geq 0$.

(i) Let $\Lambda = \max\{\ell, \frac{\alpha}{p}\}$. Then

$$
\|f\|_{\dot{B}^\alpha_{p,q}} \lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\psi_j^* f\|_{H^p} \right)^q \right)^{\frac{1}{q}}.
$$

(14)
(ii) Let \( \Lambda = \max\{\ell, \frac{n}{p}, \frac{n}{q}\} \) and \( \phi \in \mathcal{S} \) with \( \int \phi(x)dx \neq 0 \). If \( p < \infty \) then

\[
\|f\|_{\dot{F}^\alpha_{p,q}} \lesssim \left\| \left( \sum_{j \in \mathbb{Z}} (2^{j \alpha} \sup_{t>0} |\phi_t \ast \psi_j \ast f|)^q \right)^{\frac{1}{q}} \right\|_{L^p} \tag{15}
\]

and in the case \( p = \infty \)

\[
\|f\|_{\dot{F}^\alpha_{\infty,q}} \lesssim \sup_Q \left( \frac{1}{|Q|} \int_Q \sum_{j \geq -\ell(Q)} (2^{j \alpha} \sup_{t>0} |\phi_t \ast \psi_j \ast f(x)|)^q dx \right)^{\frac{1}{q}} \tag{16}
\]

(with \( \Lambda = \max\{n, \ell\} \) when \( p = q = \infty \)).

**Remark 1.2.**

(i) Observe that we are free to choose the smoothness parameter \( m \), thus an alternative way to state the smoothness condition on \( \psi \), is that we simply require \( \partial^\kappa \hat{\psi} \) to be slowly increasing for \( |\kappa| \leq n + 1 + \lfloor \Lambda \rfloor \).

(ii) Theorem 1.2 together with Theorem 1.1 give the following complete characterisation of \( \dot{B}^\alpha_{p,q} \). Let \( \ell > \alpha - \frac{n}{p}, r > \alpha, m > \frac{n}{p} - \alpha \), and \( \Lambda = \max\{\ell, \frac{n}{p}\} \). If the kernel \( (1 + | \cdot |)^\ell \psi \in L^1 \) satisfies the conditions \((C1),(C2),(C3)\), then \( f \in \dot{B}^\alpha_{p,q} \) if and only if \( f \) is a distribution of growth \( \ell \) and \( \left( \sum_{j \in \mathbb{Z}} (2^{j \alpha} \|\psi_j \ast f\|_{H^r})^q \right)^{\frac{1}{q}} < \infty \). A similar comment applies in the Triebel-Lizorkin case.

If we want a version of Theorem 1.2 with \( H^p \) replaced with \( L^p \), we need to assume more on our kernel \( \psi \) to ensure that each \( \psi_j \ast f \) is a measurable function (as opposed to just an element of \( \mathcal{S}' \)). It is worth noting that, under the assumptions of Theorem 1.2 if we assume the right hand side of (14) is finite, then since \( H^p = L^p \) for \( p > 1 \), we may freely replace the \( H^p \) norm with the \( L^p \) norm in (14). Thus at first glance, it may appear that the conditions on \( \psi \) in Theorem 1.2 are sufficient to also deduce an \( L^p \) version of (14).

However this is slightly misleading, as we may only replace \( H^p \) with \( L^p \) after making the a priori assumption that the right hand side of (14) is finite. Without this finiteness assumption, it is not possible to ensure that the distribution \( \psi_j \ast f \) is in fact a function. Thus in general, under the assumptions on \( \psi \) in Theorem 1.2, the norm \( \|\psi_j \ast f\|_{L^p} \) is not defined. Consequently, if our goal is to prove a direct characterisation without any auxiliary assumptions on the distribution \( f \), to ensure that \( \psi_j \ast f \) is a measurable function, we need to make further assumptions on our kernel \( \psi \). One such condition is found in the next theorem.

**Theorem 1.3.** Let \( \alpha \in \mathbb{R} \) and \( 0 < p, q \leq \infty \). Assume \( f \) is a distribution of growth \( \ell \geq 0 \). Suppose \( (1 + | \cdot |)^\ell \psi \in L^1 \) satisfies the Tauberian condition \((C2)\) and that for every \( m \in \mathbb{R} \), the smoothness condition \((C3)\) holds with \( \Lambda \geq \ell \). Then for every \( j \in \mathbb{Z} \) the convolution
$\psi_j * f$ is a continuous function. Moreover, if $\Lambda = \max\{\ell, \frac{n}{p}\}$ then

$$\|f\|_{\dot{B}_{p,q}^\alpha} \lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\psi_j * f\|_{L^p} \right)^q \right)^{\frac{1}{q}}. \quad (17)$$

Similarly, if $\Lambda = \max\{\ell, \frac{n}{p}, \frac{n}{q}\}$ and $p < \infty$ then

$$\|f\|_{\dot{F}_{p,q}^\alpha} \lesssim \left\| \left( \sum_{j \in \mathbb{Z}} (2^{j\alpha} |\psi_j * f|)^q \right)^{\frac{1}{q}} \right\|_{L^p} \quad (18)$$

and in the case $p = \infty$

$$\|f\|_{\dot{F}_{\infty,q}^\alpha} \lesssim \sup_Q \left( \frac{1}{|Q|} \int_Q \sum_{j \geq -\ell(Q)} (2^{j\alpha} |\psi_j * f(x)|)^q dx \right)^{\frac{1}{q}} \quad (19)$$

(with $\Lambda = \max\{n, \ell\}$ when $p = q = \infty$).

**Remark 1.3.**

(i) It is possible to reduce the smoothness assumption slightly, see Theorem 5.1 and the proof of Theorem 1.3 in Section 5. In particular, the smoothness condition $(C3)$ can be replaced with the marginally weaker assumption that $\partial^\kappa \hat{\psi}$ is rapidly decreasing for $|\kappa| \leq \max\{n + 1 + [\ell], [\Lambda]\}$. This is a fairly strong condition on the kernel $\psi$, but a condition of this type seems to be necessary in order for the convolution $\psi * f$ to have a pointwise definition for every $f \in S'$ of growth $\ell$, see Remark 3.2 below. On the other hand, if we instead make further assumptions on the distribution $f$, then it is possible to define the convolution $\psi * f$ without the rapidly decreasing assumption on $\partial^\kappa \hat{\psi}$, see Theorem 5.3 for a result in this direction.

(ii) In the above two theorems, we have restricted the class of distributions to those of growth $\ell$. This condition is natural in light of Theorem 1.1 where it was shown that all elements of $\dot{B}_{p,q}^\alpha$ and $\dot{F}_{p,q}^\alpha$ are, perhaps modulo a polynomial, distributions of growth $\alpha - \frac{n}{p} + \epsilon$ for every $\epsilon > 0$. Thus we do not lose anything by only considering distributions of some finite growth. Observe also that by making $\ell$ smaller, we weaken the condition on $\psi$, but unfortunately require a stronger growth condition on $f$. A good choice for $\ell$, which is suggested by the necessary results, is to take $\ell > \alpha - n/p$.

Theorem 1.1, Theorem 1.2 and Theorem 1.3 provide necessary and sufficient conditions for a distribution to be in the Besov-Lipschitz space or in the Triebel-Lizorkin space. In other words, these theorems provide the characterisations of the function spaces under study.

We now come to our main application of the previous results. Namely we give a characterisation of $\dot{B}_{p,q}^\alpha$ and $\dot{F}_{p,q}^\alpha$ via fractional derivatives of the Poisson kernel. Thus we
consider the case \( \hat{\psi}(\xi) = |\xi|^{\beta} e^{-|\xi|}; \) that is, \( \psi = (-\Delta)^{\beta/2} P, \) and \( P \) is the Poisson kernel on \( \mathbb{R}^n, \)
\[
P(x) = \frac{c_n}{(1 + |x|^2)^{(n+1)/2}}.\]

Note that the Poisson kernel case is one of the main motivations for this work.

**Theorem 1.4.** Let \( \alpha \in \mathbb{R}, \; 0 < p, q \leq \infty. \) Let \( \beta \geq 0, \; \beta > \alpha, \) and define \( \hat{\psi}(\xi) = |\xi|^{\beta} e^{-|\xi|}. \) Let \( \ell \geq 0 \) such that
\[
\alpha - \frac{n}{p} < \ell < \begin{cases} 
\beta + 1 & \frac{\beta}{2} \in \mathbb{N} \\
\beta & \frac{\beta}{2} \notin \mathbb{N}.
\end{cases}
\]
Assume that \( f \) is a distribution of growth \( \ell. \) Then the convolution \( \psi_j \ast f \) is continuous function, and there exists a polynomial \( \rho \) of degree at most \( \lfloor \ell \rfloor \) such that
\[
\left( \sum_{j \in \mathbb{Z}} (2^{j\alpha} \| \psi_j^* (f - \rho) \|_{L^p})^q \right)^{\frac{1}{q}} \lesssim \| f \|_{\dot{B}_{p,q}^\alpha} \lesssim \left( \sum_{j \in \mathbb{Z}} (2^{j\alpha} \| \psi_j \ast f \|_{L^p})^q \right)^{\frac{1}{q}}.
\]

Moreover, in the case \( \ell < \beta, \) we may take \( \rho = 0. \)

**Proof.** It is obvious that \( \psi \) satisfies all the assumptions in Theorem 1.1, Theorem 1.2 and Theorem 1.3 except possibly the integrability condition \( (1 + |.|)^{\nu} \psi \in L^1. \) But this follows readily from Corollary 2.6 in the case \( \frac{\beta}{2} \notin \mathbb{N}, \) and in the case \( \frac{\beta}{2} \in \mathbb{N} \) by observing that \( \psi = (-\Delta)^{\beta/2} P. \) Thus we obtain the required inequalities, up to perhaps a polynomial factor \( \rho \) in the case of the left hand estimates. However as \( f \) and \( f - \rho \) are of growth \( \ell, \) we see that \( \rho \) must be of degree at most \( \lfloor \ell \rfloor. \) The final conclusion follows by noting that if \( \ell < \beta, \) then Corollary 2.5 implies that \( \psi \) has \( \lfloor \ell \rfloor \) vanishing moments. Therefore \( \psi \ast (f - \rho) = \psi \ast f \) and hence we may take \( \rho = 0. \) \( \square \)

**Remark 1.4.**

(i) The previous theorem gives a complete characterisation of \( \dot{B}_{p,q}^\alpha \) in view of the growth properties of elements of distributions belonging to \( \dot{B}_{p,q}^\alpha. \) More precisely, given \( \ell, \beta \) satisfying the conditions in Theorem 1.4, we have shown that \( f \in \dot{B}_{p,q}^\alpha \) if and only if there
exists a polynomial \( \rho \) such that \( f - \rho \) is a distribution of growth \( \ell \) and

\[
\left( \sum_{j \in \mathbb{Z}} (2^{j\alpha} \| \psi_j^* (f - \rho) \|_{L^p})^q \right)^{1/q} < \infty.
\]

A similar comment applies to the Triebel-Lizorkin case.

(ii) More generally, Theorem 1.4 holds when \( \hat{\varphi} = | \cdot |^\beta \hat{\phi} \), where \( \phi \in S \) satisfies the Tauberian condition. The proof is similar to the Poisson kernel case.

(iii) When \( \beta = m \in \mathbb{N} \) in the above theorem, one has \( \psi_t * f = \left( (\partial/\partial t)^m P_t \right) * f = \left( (\partial/\partial t)^m (P_t * f) \right) \) if \( P_t * f \) is defined. This case is historically important as the Poisson kernel was a principal tool used in the classical study of function spaces, in which the mean-value property of the harmonic function \( P_t * f \) is crucial. In fact, the sufficient result for the Poisson kernel case (the right-hand side inequalities in the above theorem) has only been proved in the literature using the mean-value property (see [3], [21]). Moreover, the question of defining the convolution \( \psi_t * f \) was not fully elaborated in these works. Also Theorem 1.4 for non-integer \( \beta \) appears to be new.

While the general outline of our arguments follows the original works [5, 6, 7] and, in the necessary part, also the pioneering paper [16] by Peetre, the non-smooth assumption on the kernel \( \psi \) and its the Fourier transform requires not only substantial technical modifications, but also the introduction of a new concept, the distributions of finite growth. We also benefit from the thesis [8] where some partial results are obtained. Of independent interest is our extension of the Calderón reproducing formula and the Strömberg-Torchinsky estimate in [20] to the non-smooth case. These could be useful in other research in harmonic analysis of function spaces.

The plan for the rest of the paper is as follows. In Section 2 we prove a number of estimates that are used frequently throughout the paper, in particular we state a growth estimate on elements of \( \dot{B}^{\alpha}_{1, \infty} \). Section 3 is devoted to problem of the pointwise definition of the convolution. Section 4 is the main part of the paper, where the necessary tools are developed to prove Theorem 1.2 and Theorem 1.3. Section 5 contains the proofs of our main theorems. In Section 6 we give the proofs of the results in Section 2, namely we prove two versions of the Calderón reproducing formulas on \( S' \) and use these to deduce growth rate for distributions in the Besov-Lipschitz spaces.

We conclude the introduction by a remark. All the main results presented in this paper have continuous versions, in which the sum is replaced by the integral with respect to the dilation variable \( t > 0 \), and the kernel function satisfies the “standard” Tauberian condition (see [5]). We leave the precise formulation as well as modification of the proofs to the interested reader, but note that details in the smooth kernel case can be found in [5, 6, 7]. Moreover, versions of our results should also hold in the weighted case,
where the parameter \( \lambda \) in the Peetre maximal function will depend on the weight function \( w \). Again we refer to the above cited works for a treatment in the smooth kernel case.

An earlier version of this paper was presented at the 2008 Australia-New Zealand Mathematics Convention [4].

2. Preliminary Results

We begin by recalling two versions of the Calderón reproducing formula. These two theorems are classic results that were first used in the study of the homogeneous Besov-Lipschitz spaces by Peetre [17]. (A continuous version of Theorem 2.1 was attributed to A.P. Calderón by the authors of [15].) We collect the proof of the two theorems below in the appendix for easy reference (see Subsection 6.1 below). It is worth noting that our proofs are carried out in the spatial space and are different from [17] (where it was done in the frequency domain). Moreover, our argument gives an explicit definition of the sequence of polynomials \((p_N)\) appearing in the theorems below; see equation (54).

**Theorem 2.1** (Calderón Formula on \( S' \)). Let \( f \in S' \). Then there exists a sequence of polynomials \((p_N)\) such that

\[
f = \lim_{N \to -\infty} \left( p_N + \sum_{j = N + 1}^{\infty} \varphi_j \ast \varphi_j \ast f \right)
\]

with convergence in \( S' \).

We can deduce a more refined version if we make the additional assumption that \( f \in \dot{B}^\alpha_{\infty, \infty} \).

**Theorem 2.2** (Calderón Formula on \( \dot{B}^\alpha_{\infty, \infty} \)). Let \( \alpha \in \mathbb{R} \) and \( f \in \dot{B}^\alpha_{\infty, \infty} \). Then there exist polynomials \( p, p_N \) such that \( \text{deg}(p_N) \leq \lfloor \alpha \rfloor \) and

\[
f - p = \lim_{N \to -\infty} \left( p_N + \sum_{j = N + 1}^{\infty} \varphi_j \ast \varphi_j \ast f \right)
\]

with convergence in \( S' \). Moreover, given any \( \rho \in S \) we have the inequality

\[
\sup_{N < 0 < M} \left| \rho \ast \left( p_N + \sum_{j = N + 1}^{M} \varphi_j \ast \varphi_j \ast f \right)(x) \right| \lesssim 1 + \begin{cases} |x|^\max\{0, \alpha\} & \alpha \not\in \mathbb{N} \\ |x|^\alpha \log(|x|) & \alpha \in \mathbb{N} \end{cases}
\]

In the characterisation results presented in the current paper, we restrict our attention to distributions of finite growth. To see that this restriction is reasonable, we need to show that elements of \( \dot{B}^\alpha_{p,q} \) and \( \dot{F}^\alpha_{p,q} \) have growth of some finite order. This growth is a straightforward application of the bound in Theorem 2.2.

\(^1\)If \( \alpha < 0 \) this statement is vacuous, and we simply have \( p_N = 0 \).
Corollary 2.3 (Growth of distributions in $\dot{B}^\alpha_{p,q}$ and $\dot{F}^\alpha_{p,q}$). Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. If $f \in \dot{B}^\alpha_{\infty,\infty}$ then there exists a polynomial $p$ such that for every $\rho \in \mathcal{S}$ we have

$$|\rho \ast (f - p)(x)| \lesssim 1 + \begin{cases} |x|^\alpha \log |x| & \alpha \in \mathbb{N} \\ |x|^{\max(0,\alpha)} & \alpha \not\in \mathbb{N}. \end{cases}$$

Consequently, if $f \in \dot{B}^\alpha_{p,q}$ or $f \in \dot{F}^\alpha_{p,q}$, then there exists a polynomial $p$ such that $f - p$ is a distribution of growth $\ell$ for every $\ell > \alpha - \frac{2}{p}$ with $\ell \geq 0$.

Proof. The growth bound on $\dot{B}^\alpha_{\infty,\infty}$ follows immediately from Theorem 2.2. To conclude the proof, we simply recall the embedding $\dot{B}^\alpha_{p,q} \subset \dot{B}^{\alpha - \frac{2}{p}}_{\infty,\infty}$. (Note that $\dot{F}^\alpha_{\infty,\infty} \subset \dot{B}^\alpha_{\infty,\infty}$ by an embedding theorem in [7].) \[\square\]

Remark 2.1. When $\alpha > 0$, it is well-known that the characterisation of $\dot{B}^\alpha_{\infty,\infty}$ via differences implies the stronger pointwise growth bound

$$|(f - p)(x)| \lesssim \begin{cases} |x|^\alpha \log |x| & \alpha > 0 \text{ and } \alpha \in \mathbb{N} \\ |x|^{\alpha} & \alpha > 0 \text{ and } \alpha \not\in \mathbb{N}, \end{cases}$$

from which the Corollary follows. On the other hand, in the case $\alpha = 0$, the growth bounds in Corollary 2.3 and Theorem 2.2 appear to be new.

As is common in the study of function spaces via the Calderón formula, we require some control over convolutions of the form $\eta_k \ast \psi_j$ (see for instance the work of Heideman [13]). The precise dilation estimate we need is a refined version of [5, Lemma 2.1] (see also [18, Lemma 1]).

Lemma 2.4. Let $m \in \mathbb{R}, c > 0$ and $N \in \mathbb{N}$. Suppose $\eta \in L^1$ with $\widehat{\eta} \in C^N(\mathbb{R}^n)$ and supp $\widehat{\eta} \subset \{a < |\xi| < b\}$ for some $0 < a < b$. Let $\psi \in \mathcal{S}'$ with $\widehat{\psi} \in C^N(\mathbb{R}^n \setminus \{0\})$.

(i) Assume $\partial^\kappa \widehat{\psi}(\xi) = O(|\xi|^{-m})$ as $|\xi| \to \infty$ for every $|\kappa| \leq N$. Then for any $s \leq ct$ we have

$$|\eta_s \ast \psi_t(x)| \lesssim \left(\frac{t}{s}\right)^{m-n} \frac{t^{-n}}{(1 + t^{-1}|x|)^N}.$$  \hspace{1cm} (20)

(ii) Assume $\partial^\kappa \widehat{\psi}(\xi) = O(|\xi|^{-m})$ as $|\xi| \to 0$ for every $|\kappa| \leq N$. Then for any $t \leq cs$ we have

$$|\eta_s \ast \psi_t(x)| \lesssim \left(\frac{s}{t}\right)^{m} \frac{s^{-n}}{(1 + s^{-1}|x|)^N}.$$  \hspace{1cm} (21)

Proof. Take $c = 1$ for simplicity of notation. The support assumption on $\widehat{\eta}$ implies that the convolution $\eta \ast \psi$ is well-defined (in fact is an $L^\infty$ function). Moreover, for every $|\kappa| \leq N$ and any $x \in \mathbb{R}^n$

$$s^{n-|\kappa|} |x^\kappa \eta_s \ast \psi_t(x)| \lesssim \|x^\kappa \eta \ast \psi_t(x)\|_{L^\infty} \lesssim \|\partial^\kappa [\widehat{\eta}(\xi) \widehat{\psi}_s(\xi)]\|_{L^1} \lesssim \sum_{\gamma \leq \kappa} \left(\frac{t}{s}\right)^{|\gamma|} \int_{a < |\xi| < b} |\partial^\gamma \widehat{\psi}_s(\xi)| d\xi.$$  \hspace{1cm} (22)
In particular, if \( s \leq t \), then assuming \( \partial^\kappa \hat{\psi} = O(|\xi|^{-m}) \) as \( |\xi| \to \infty \), and using the bound \((22)\) we deduce that for every \( |\kappa| \leq N \)
\[
|\langle t^{-1} x \rangle^\kappa \eta_s * \psi_t(x) | \lesssim s^{-n} \left( \frac{s}{t} \right)^{|\kappa|} \sum_{\gamma \leq \kappa} \left( \frac{t}{s} \right)^{|\gamma|} \int_{a \leq |\xi| \leq b} \left| \frac{t}{s} |\xi|^{-m} \right| d\xi \\
\approx t^{-n} \left( \frac{s}{t} \right)^{m-n} \sum_{\gamma \leq \kappa} \left( \frac{s}{t} \right)^{|\kappa| - |\gamma|} \lesssim t^{-n} \left( \frac{s}{t} \right)^{m-n}.
\]

Applying this estimate for \( |\kappa| = 0 \) and \( |\kappa| = N \) we obtain (i).

Similarly, if \( t \leq s \), then assuming \( \partial^\kappa \hat{\psi} = O(|\xi|^{m-|\kappa|}) \) as \( |\xi| \to 0 \) and again applying the bound \((22)\) we have
\[
|\langle t^{-1} x \rangle^\kappa \eta_s * \psi_t(x) | \lesssim s^{-n} \sum_{\gamma \leq \kappa} \left( \frac{t}{s} \right)^{|\kappa|} \left| \int_{a \leq |\xi| \leq b} \left| \frac{t}{s} |\xi|^{m-|\gamma|} \right| d\xi \right| \lesssim s^{-n} \left( \frac{t}{s} \right)^m
\]
which gives (ii).

**Remark 2.2.** It is possible to generalise the previous lemma in the following sense. Suppose \( \|(1 + |x|)^N \eta\|_{L^1} < \infty \) and supp \( \hat{\eta} \subset \{ a < |\xi| < b \} \) for some \( 0 < a < b < \infty \). Then for any \( j, k \in \mathbb{Z} \)
\[
\|(1 + 2^{\min(j,k)} |x|)^N \eta_j * \psi_k\|_{L^p} \leq C_{\psi} 2^{k(n-\frac{p}{\min(p,\infty)})} 2^{-(j-k)m} \tag{23}
\]
where
\[
C_{\psi} \lesssim \begin{cases} 
\sup_{|\gamma| \leq N} \| P_{\geq 1} (x^\gamma \hat{\psi}) \|_{\dot{B}^m_{p,\infty}} & j \geq k \\
\sup_{|\gamma| \leq N} \| P_{\geq 1} (x^\gamma \hat{\psi}) \|_{\dot{B}^m_{p,|\gamma|}} & j \leq k
\end{cases}
\]
and \( P_{\geq 1} \) denotes the restriction to frequencies \( \geq 1 \), i.e. \( \| P_{\geq 1} f \|_{\dot{B}^m_{p,\infty}} = \sup_{j \geq 1} 2^{j\alpha} \| \varphi_j * f \|_{L^p} \). \( P_{\leq 1} \) is defined similarly. Thus we may replace the assumptions in Lemma 2.4 by supposing that \( \psi \) belongs to certain Poised spaces of Besov type (c.f. the work of Peetre [16]).

The inequality \((23)\) follow by an application of Hölder’s inequality, together with the support assumption on \( \eta \) to deduce that
\[
\|(1 + 2^{\min(j,k)} |x|)^N \eta_j * \psi_k\|_{L^p} = 2^{k(n-\frac{p}{\min(p,\infty)})} \|(1 + 2^{\min(j-k,1)} |x|)^N \eta_{j-k} * \psi\|_{L^p} \\
\lesssim 2^{k(n-\frac{p}{\min(p,\infty)})} \sup_{|\gamma| \leq N} \left( 2^{\min((j-k)|\gamma|, -(j-k)|\kappa|)} \right) \left( \| (x^\kappa \eta)_{j-k} * (x^\gamma \psi) \|_{L^p} \right) \\
\lesssim 2^{k(n-\frac{p}{\min(p,\infty)})} 2^{-(j-k)m} \|(1 + |x|)^N \eta\|_{L^1} \sup_{|\gamma| \leq N} \left( \sup_{j' \approx j-k} 2^{j'm+\min(j'+1,1)|\gamma|} \| \varphi_{j'} * (x^{\gamma} \psi) \|_{L^p} \right)
\]
which gives \((23)\) by definition of \( \dot{B}^m_{p,\infty} \).

To illustrate the connection between \((23)\) and Lemma 2.4 note that in the case \( p \geq 2 \), the assumptions on \( \psi \) in Lemma 2.4 implies that the right-hand side of \((23)\) is finite. More precisely, if \( 2 \leq p \leq \infty \) and \( \hat{\psi} \in C^N (\{ |\xi| \geq 1 \}) \) with \( |\partial^\gamma \hat{\psi}(\xi)| \lesssim |\xi|^{-m-n(1-\frac{1}{p})} \) for
\(|\xi| \geq 1\) and \(|\gamma| \leq N\), then an application of the Hausdorff-Young inequality gives for every \(|\gamma| \leq N\)

\[
\|P_{\leq 1}(x^\gamma \psi)\|_{\mathcal{B}_{p,\infty}^{m,\gamma}} \lesssim \sup_{j \geq 1} 2^{jm} \|\partial^j \hat{\psi}\|_{L^p(|\xi| \approx 2^j)} \lesssim \sup_{j \geq 1} 2^{mj} \|\|\xi|^{-m-\frac{\gamma}{p}}\|_{L^p(|\xi| \approx 2^j)} < \infty.
\]

Similarly, if \(\hat{\psi} \in C^N(\mathbb{R}^n \setminus \{0\})\) with \(|\partial^j \hat{\psi}(\xi)| \lesssim |\xi|^{-m-n(1-\frac{1}{p})-|\gamma|}\) for \(0 < |\xi| \lesssim 1\) and \(|\gamma| \leq N\), then

\[
\|P_{\leq 1}(x^\gamma \psi)\|_{\mathcal{B}_{p,\infty}^{m,\gamma}} \lesssim \sup_{j \leq 1} 2^{(m+|\gamma|)j} \|\partial^j \hat{\psi}\|_{L^p(|\xi| \approx 2^j)} < \infty.
\]

Thus in terms of conditions on \(\psi\), (23) implies Lemma 2.4. On the other hand, the disadvantage of (23) is that firstly in certain cases we need more decay on \(\eta\), and secondly the conditions on \(\psi\) are more difficult to verify. As our emphasis is on finding conditions on our kernel which are easy to establish, throughout this article we ignore this slight generalisation and instead make use of Lemma 2.4.

**Remark 2.3.** A typical application of Lemma 2.4 would involve estimating \(\|(1+2^j |x|)^\lambda \eta_j * \psi_k\|_{L^1}\) via an application of Hölder’s inequality to obtain

\[
\|(1+2^j |x|)^\lambda \eta_j * \psi_k\|_{L^1} \lesssim 2^{-jn} \|(1+2^j |x|)^{n+1+|\lambda|} \eta_j * \psi_k\|_{L^\infty}
\]

and then using the \(L^\infty\) bound obtained in Lemma 2.4 which requires \(\hat{\eta} \in C^{n+1+|\lambda|}\). However this argument can be preformed more efficiently by using Plancheral instead of the \(\|u\|_{L^\infty} \leq \|\hat{u}\|_{L^1}\) bound used in the proof of Lemma 2.4. In more detail, we can use

\[
\|(1+2^j |x|)^\lambda \eta_j * \psi_k\|_{L^1} \lesssim \|(1+2^j |x|)^{\frac{\lambda}{p}+|\lambda|+1} \eta_j * \psi_k\|_{L^2} \lesssim \sup_{|\gamma| \leq \frac{n}{2} + |\lambda| + 1} 2^{j\gamma} \|\partial^j_{\xi} (\hat{\eta_j} \hat{\psi}_k)\|_{L^2}
\]

which, after following the proof of Lemma 2.4, only requires \(\hat{\eta} \in C^{\frac{\lambda}{2} + |\lambda| + 1}\). To summerize, it is often possible to replace the assumption \(\hat{\eta} \in C^{n+1+|\lambda|}\) with \(\hat{\eta} \in C^{\frac{n}{2} + |\lambda| + 1}\). A similar comment applies to the differentiability condition on \(\hat{\psi}\).

To apply our characterisation to Poisson like kernels, we need to estimate the spatial decay of \(F^{-1}(|\xi|^{\beta} e^{-|\xi|})\). The required decay is a consequence of the following corollary of Lemma 2.4.

**Corollary 2.5.** Let \(r > \ell \geq 0\) and \(1 \leq p < \infty\). Let \(\psi \in L^p\) and assume supp \(\hat{\psi} \subset \{|\xi| \leq 1\}\). Furthermore, suppose that \(\hat{\psi} \in C^{n+1+|\ell|}(\mathbb{R}^n \setminus \{0\})\) with

\[
\partial^\kappa \hat{\psi}(\xi) = O(|\xi|^{-|\kappa|}) \quad \text{as } |\xi| \to 0
\]

for every \(|\kappa| \leq n + 1 + |\ell|\). Then \((1 + |x|)^r \psi \in L^1\) and moreover \(\psi\) has \(\ell\) vanishing moments.
Proof. We begin by observing that
\[
\sum_{j \leq 1} \| |x|^\ell (\varphi_j * \varphi_j * \psi) \|_{L^1} \lesssim \sum_{j \leq 1} 2^{-j \ell} \| (1 + 2^j |x|) \ell \varphi_j * \psi \|_{L^1} \\
\lesssim \sum_{j \leq 1} 2^{(r-\ell)} < \infty
\]  
(24)
where we used an application of (ii) in Lemma 2.4 (with \( t = 1 \) and \( s = 2^{-j} \)) to deduce that
\[
\| (1 + 2^j |x|) \ell \varphi_j * \psi \|_{L^1} \lesssim 2^{j \ell} 2^{jn} \| (1 + 2^j |x|) \ell (n+1+|\ell|) \|_{L^1} \lesssim 2^{j \ell}.
\]
On the other hand, the assumption \( \psi \in L^p \) together with \( \text{supp} \ \hat{\psi} \subset \{|\xi| \leq 1\} \) implies that we have the pointwise identity\(^2\)
\[
\psi(x) = \sum_{j \leq 1} \varphi_j * \varphi_j * \psi(x)
\]  
(25)
for a.e. \( x \in \mathbb{R}^n \) (in fact as \( \psi \) is smooth, the identity holds for every \( x \in \mathbb{R}^n \)). Consequently, (24) implies that \( |x|^\ell \psi \in L^1 \). Therefore \( \| \psi \|_{L^1} \lesssim \| \psi \|_{L^p} + \| |x|^\ell \psi \|_{L^1} \) and so we deduce that \( (1 + |x|) \ell \psi \in L^1 \). Finally, to check that \( \psi \) has \( \ell \) vanishing moments, we simply note that \( \hat{\psi} \in C^{\ell} (\mathbb{R}^n) \), and hence the decay condition gives \( \partial^\kappa \hat{\psi}(0) = 0 \) for every \( |\kappa| \leq \ell \).
Together with the integrability \((1 + |\cdot|)^\ell \psi \in L^1\), this implies that \( \psi \) has \( \ell \) vanishing moments as claimed.
\[\square\]

Remark 2.4. The previous corollary can be improved somewhat by using Remark 2.3. In particular, we can replace the assumption \( \hat{\psi} \in C^{n+1+|\ell|} (\mathbb{R}^n \setminus \{0\}) \) with the slightly weaker \( \hat{\psi} \in C^{n+1+|\ell|+1} (\mathbb{R}^n \setminus \{0\}) \).

Corollary 2.5 has an immediate application to the Poisson kernel.

Corollary 2.6. Let \( \beta \geq 0 \), and let \( \hat{\psi}(\xi) = |\xi|^\beta e^{-|\ell|} \). Then \((1 + |\cdot|)^\ell \psi \in L^1\) for every \( \ell < \beta \).

Proof. Let \( \chi \in \mathcal{S} \) such that \( \text{supp} \ \chi \subset \{|\xi| \leq 1\} \), and \( \chi = 1 \) in a neighbourhood of the origin. Write \( \hat{\psi} = \hat{\psi} \chi + (1 - \chi) \hat{\psi} = \hat{\theta} + \hat{\mu} \). Then \( \mu \in \mathcal{S} \) and \( \theta \) satisfies the assumptions of Corollary 2.5 with \( r = \beta \). \[\square\]

Finally we make use of the following elementary summation inequalities.

\(^2\)As in the standard proof of the reproducing formula (see (53) in Section 6), there exists \( \phi \in \mathcal{S} \) such that
\[
\psi(x) = \phi_M * \psi(x) + \sum_{M \leq j \leq 1} \varphi_j * \varphi_j * \psi(x)
\]
where we used the support assumption on \( \hat{\psi} \). An application of Hölder’s inequality gives
\[
\| \phi_M * \psi \|_{L^\infty} \leq \| \phi_M \|_{L^{p'}} \| \psi \|_{L^p} \lesssim 2^{M \frac{\beta}{p}}.
\]
and thus, as \( p < \infty \), (25) follows by letting \( M \to -\infty \).
Proposition 2.7. Fix $0 < p, q \leq \infty$ and let $f_k$ be a sequence of measurable functions. If $(a_j)_{j \in \mathbb{Z}} \in \ell^{\min\{p,q,1\}}(\mathbb{Z})$ then we have

$$\left(\sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} |a_{j-k} f_k|^{q}\right)^{1/q}\right)^{1/p} \lesssim \left(\sum_{j \in \mathbb{Z}} |f_j|^{q}\right)^{1/q}.$$ 

Similarly if $(a_j)_{j \in \mathbb{Z}} \in \ell^{\min\{q,1\}}(\mathbb{Z})$ then

$$\left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} |a_{j-k} f_k|^{q}\right)^{1/q}\right)^{1/p} \lesssim \left(\sum_{j \in \mathbb{Z}} |f_j|^{q}\right)^{1/q}.$$ 

Proof. This proposition is a folklore result. The proof is based on Young’s inequality and the inequality

$$\left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} |a_j f_k|^{r}\right)^{1/r}\right)^{1/p} \lesssim \left(\sum_{j \in \mathbb{Z}} |f_j|^{r}\right)^{1/p}$$

which holds whenever $0 < r \leq 1$. We omit the details. \hfill \square

3. Pointwise Definition of the Convolution

The introduction of distributions of finite growth, together with Definition 1.2, makes it possible to define the convolution $\psi \ast f$ as a distribution. However, the characterisation results in Theorems 1.1 and 1.3 require a pointwise definition. In this section we give two sets of sufficient conditions to ensure that $\psi \ast f \in L^1_{\text{loc}}$. The first is via what is essentially a duality argument exploiting the Calderón reproducing formula given in Theorem 2.2. This argument has the advantage that it requires very few assumptions on the kernel $\psi$. On the other hand it is only applicable in the case $f \in \dot{B}_{\alpha,1}^{\alpha,1}$, and thus is not helpful in Theorem 1.3. The second approach is much more general, and works for arbitrary distributions $f \in S'$. However it correspondingly requires much stronger conditions on the kernel $\psi$.

3.1. The case $f \in \dot{B}_{\alpha,\infty}^{\alpha,\infty}$. The key result is the following.

Theorem 3.1. Let $\alpha, \ell \in \mathbb{R}$ with $\ell \geq 0$ and $\ell > \alpha$. Let $f \in \dot{B}_{\alpha,\infty}^{\alpha,\infty}$. Assume $\psi \in \dot{B}_{1,1}^{-\alpha}$ with $(1 + |x|)^{\ell} \psi \in L^1$. Let $p$ be the polynomial given by Theorem 2.2. Then the distribution $\psi \ast (f - p)$ is a bounded continuous function and we have the identity

$$\psi \ast (f - p)(x) = \sum_{j \in \mathbb{Z}} \psi \ast \varphi_j \ast \varphi_j \ast f(x) \quad (26)$$

where the sum converges in $L^\infty$.

Proof. Let $p$ be the polynomial given in Theorem 2.2. The decay assumption on $\psi$ implies that the convolution $\psi \ast (f - p) \in S'$. Define $g$ as

$$g(x) = \sum_{j \in \mathbb{Z}} \psi \ast \varphi_j \ast \varphi_j \ast f(x).$$
The duality estimate \( \sum_j |\psi \ast \varphi_j \ast \varphi_j \ast f(x)| \leq \| f \|_{\dot{B}_{\infty, \infty}^{0}} \| \psi \|_{\dot{B}_{1, 1}^{-\alpha}} \) implies that \( g \) is a bounded continuous function. Thus the theorem would follow by showing that for every \( \rho \in \mathcal{S} \)
\[
\psi \ast (f - p)(\rho) = \int_{\mathbb{R}^n} g(x) \rho(x) dx. \tag{27}
\]

To this end, by definition of the distribution \( \psi \ast (f - p) \), together with the growth bound in Theorem 2.2, the Dominated Convergence Theorem, and the decay condition on \( \psi \), we have for any \( \rho \in \mathcal{S} \)
\[
\psi \ast (f - p)(\rho) = \int_{\mathbb{R}^n} \tilde{\rho} \ast (f - p)(x) \psi(x) dx \\
= \int_{\mathbb{R}^n} \lim_{N \to -\infty} \tilde{\rho} \ast \left(p_N + \sum_{j=N+1}^{\infty} \varphi_j \ast \varphi_j \ast f \right)(x) \psi(x) dx \\
= \lim_{N \to -\infty} \left( \int_{\mathbb{R}^n} \rho(x) \psi \ast p_N(x) dx + \sum_{j=N+1}^{\infty} \int_{\mathbb{R}^n} \psi \ast \varphi_j \ast \varphi_j \ast f(x) \rho(x) dx \right).
\]

We claim that the assumptions on \( \psi \) imply that \( \psi \) has \([\alpha]\) vanishing moments,\(^3\) in other words \( \int x^\gamma \psi = 0 \) for every \( |\gamma| \leq [\alpha] \). Accepting this claim for the moment, we have \( \psi \ast p_N = 0 \) for every \( N \) and hence
\[
\psi \ast (f - p)(\rho) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \psi \ast \varphi_j \ast \varphi_j \ast f(x) \rho(x) dx = \int_{\mathbb{R}^n} g(x) \rho(x) dx
\]
where the last equality follows by the uniform convergence of the sum. Therefore (27) follows as required.

Thus it only remains to show that \( \psi \) has \([\alpha]\) vanishing moments. If \( \alpha < 0 \) there is nothing to prove so we may assume that \( \alpha \geq 0 \). The decay assumption on \( \psi \) implies that \( \hat{\psi} \in C^{[\alpha]}(\mathbb{R}^n) \) and hence using the form of the Taylor series given in [11] we can write
\[
\hat{\psi}(\xi) = \sum_{|\gamma| \leq [\alpha]} \frac{\xi^\gamma}{\gamma!} \partial^n \hat{\psi}(0) + [\alpha] \sum_{|\gamma| = [\alpha]} \frac{\xi^\gamma}{\gamma!} \int_0^1 (1 - t)^{[\alpha] - 1} (\partial^n \hat{\psi}(t\xi) - \partial^n \hat{\psi}(0)) dt.
\]

The continuity of \( \partial^n \hat{\psi} \) at the origin then implies that
\[
\hat{\psi}(\xi) - \sum_{|\gamma| \leq [\alpha]} \frac{\xi^\gamma}{\gamma!} \partial^n \hat{\psi}(0) = o(|\xi|^{[\alpha]}). \tag{28}
\]

On the other hand, given any \( \xi \neq 0 \) we have
\[
|\hat{\psi}(\xi)| \lesssim |\xi|^\alpha \sum_{j \in \log_2(|\xi|)} 2^{-j\alpha} \sup_{2^{-j-1} \leq |\xi| \leq 2^{j+1}} |\hat{\psi}(\xi)| \lesssim |\xi|^\alpha \sum_{j \in \log_2(|\xi|) + 1} 2^{-j\alpha} \| \varphi_j \ast \psi \|_{L^1}.
\]

\(^3\) In fact, the following argument shows that if \((1 + |x|)^{[\alpha]} \psi \in L^1\) and \( \psi \in \dot{B}_{1,q}^{-\alpha} \) for some \( q < \infty \), then \( \psi \) has \([\alpha]\) vanishing moments.
and consequently as $\psi \in \dot{B}_{1,1}^{-\alpha}$, we deduce that $\hat{\psi}(\xi) = o(|\xi|^\alpha)$ as $|\xi| \to 0$. Together with the bound (28) we obtain
\[ \sum_{|\gamma| \leq [\alpha]} \xi^\gamma \partial^\gamma \hat{\psi}(0) = o(|\xi|^{[\alpha]}) \]
which is only possible if $\partial^\gamma \hat{\psi}(0) = 0$ for every $|\gamma| \leq [\alpha]$. Therefore $\psi$ has $[\alpha]$ vanishing moments as claimed. 

□

Remark 3.1. Let $\alpha \in \mathbb{R}$ and suppose $f \in \dot{B}_\infty^{\alpha,\infty}$ and $\psi \in \dot{B}_1^{-\alpha,1}$. It is well known that $\dot{B}_\infty^{\alpha,\infty}$ can be identified with the (topological) dual of $\dot{B}_1^{\alpha,1}$ (see e.g., [2, 17]). Thus $f$ is a continuous linear functional on $\dot{B}_1^{-\alpha,1}$ and furthermore, we have the identity
\[ f(\psi) = \sum_{j \in \mathbb{Z}} \varphi_j * \varphi_j * f(\psi). \]
Thus if we define a convolution $\psi *_d f(x)$ as
\[ \psi *_d f(x) = f(\tau_x \hat{\psi}) \]
we immediately have the pointwise identity
\[ \psi *_d f(x) = \sum_{j \in \mathbb{Z}} \psi * \varphi_j * \varphi_j * f(x) \]
(the convolutions on the righthand side are the standard convolutions between $S$ and $S'$). Since the sum converges uniformly, we see that $\psi *_d f(x)$ is a continuous bounded function. Although this definition of the convolution almost immediately gives the result of Theorem 3.1 it has the drawback that it does not always agree with the standard definition of the convolution. In particular, if for instance $\psi \in L^1$ and $f \in L^\infty$ is a constant, then $\psi *_d f = 0$, however $\psi * f(x) = c \int \psi(y)dy$.

It is natural to ask when the convolution defined directly via duality, $\psi *_d f$, agrees with the definition given in Definition 1.2. The solution is given by the previous theorem.

More explicitly, let $\hat{\mathcal{O}}_0$ be the collection of all $\phi \in S$ such that $0 \not\in \text{supp } \hat{\phi}$. Then as $\hat{\mathcal{O}}_0$ is dense in $\dot{B}_1^{-\alpha}$ (see e.g. [14] [2]), there exists a sequence $\phi^{(k)} \in \hat{\mathcal{O}}_0$ such that $\|\psi - \phi^{(k)}\|_{\dot{B}_{1,1}^{-\alpha}} \to 0$. We then define
\[ f(\psi) = \lim_{k \to \infty} f(\phi^{(k)}). \]
It is easy to check that the limit is independent of the choice of sequence $\phi^{(k)}$, and moreover that the resulting linear functional is continuous (as a map from $\dot{B}_1^{-\alpha}$ to $\mathbb{C}$). In addition, an application of Theorem 2.2 shows that
\[ f(\psi) = \lim_{k \to \infty} f(\phi^{(k)}) = \lim_{k \to \infty} \sum_{j \in \mathbb{Z}} \varphi_j * \varphi_j * f(\phi^{(k)}) = \sum_{j \in \mathbb{Z}} \varphi_j * \varphi_j * f(\psi) \]
where the last line followed from the Dominant Convergence Theorem, the assumption $\psi \in \dot{B}_1^{-\alpha}$, and we used the fact that every $\phi^{(k)} \in \hat{\mathcal{O}}_0$ has infinite vanishing moments (thus annihilates all polynomials).
More precisely, suppose we know in addition that \((1 + |x|)^\ell \psi \in L^1\) for some \(\ell > \alpha\), then for every \(f \in \dot{B}_{\infty,\infty}^\alpha\) there exists a polynomial \(p\) such that we have the pointwise identity
\[
\psi \ast_d f(x) = \psi \ast (f - p)(x).
\]

3.2. The general case \(f \in S'.\) We now drop the assumption \(f \in \dot{B}_{\infty,\infty}^\alpha\), and instead simply assume that \(f\) is a distribution of growth \(\ell\). Our goal is find conditions on \(\psi\) such that the convolution \(\psi \ast f\) defined in Definition 1.2 which belongs to \(S'\), is in fact an element of \(L^1_{\text{loc}}\). One possible solution is to assume \(\psi \in S\), as then \(\psi \ast f \in C^\infty\). However this is far too strong for our purposes, as we would like our characterisation, and thus the pointwise definition, to apply in the case \(\psi \notin S\). The way forward, as in the case of \(f \in \dot{B}_{\infty,\infty}^\alpha\), is to study the convergence of the Calderón reproducing formula. The first step in this direction is the following lemma.

**Lemma 3.2.** Let \(\ell \geq 0\) and assume \(f\) is a distribution of growth \(\ell\). Then there exists \(\beta \geq 0\) depending on \(f\), such that for every \(\phi \in S\) and \(k \in \mathbb{Z}\) we have
\[
|\phi_k \ast f(x)| \lesssim 2^{k|\beta|} (1 + |x|)^\ell.
\]

**Proof.** Define the mapping \(T : S \to L^\infty_\ell\) by \(T(\phi) = \phi \ast f\) where \(L^\infty_\ell\) denotes the weighted \(L^\infty\) space defined by
\[
L^\infty_\ell = \{ \| f(x)(1 + |x|)^{-\ell} \|_{L^\infty} < \infty \}.
\]

Since \(f\) is a distribution of growth \(\ell\), the linear mapping \(T\) is well-defined. We claim that \(T\) is continuous. To see this note that an application of the Closed Graph Theorem (see, for instance, Theorem 1 on page 79 of [24]) reduces the problem to proving that the graph of \(T\)
\[
\{(\phi, T(\phi)) \mid \phi \in S\}
\]
is closed in \(S \times L^\infty_\ell\). Assume \(\phi^{(j)}\) converges to \(\phi\) in \(S\) and \(T(\phi^{(j)})\) converges to some \(g \in L^\infty_\ell\). Then for some \(M > 0\) we have
\[
|T(\phi^{(j)} - \phi)(x)| = |(\phi^{(j)} - \phi) \ast f(x)| \lesssim \sum_{|\alpha|, \gamma \leq M} \| \phi^{(j)}(x - \cdot) - \phi(x - \cdot) \|_{\alpha, \gamma}
\]
\[
\lesssim (1 + |x|)^M \sum_{|\alpha|, \gamma \leq M} \| \phi^{(j)} - \phi \|_{\alpha, \gamma}
\]
and hence \(T(\phi^{(j)})\) converges to \(T(\phi)\) pointwise. Therefore we must have \(T(\phi) = g \in L^\infty_\ell\) and so the graph of \(T\) is closed. Consequently \(T\) is continuous as claimed.

The continuity of \(T\) implies that we can bound \(\| T(\phi) \|_{L^\infty_\ell}\) by a finite number of Schwartz norms of \(\phi\) (see, for instance, Corollary 1 on page 43 of [24]). Thus there exists \(M_1 > 0\) such that
\[
\| T(\phi) \|_{L^\infty_\ell} = \| \phi \ast f \|_{L^\infty_\ell} \lesssim \sum_{|\alpha|, \gamma \leq M_1} \| \phi \|_{\alpha, \gamma}. \tag{29}
\]
To complete the proof, we observe that a simple computation shows that \(\|\phi_k\|_{\alpha, \gamma} \lesssim 2^{k(\ell + |\alpha| - |\gamma|)}\) and hence, using (29), we obtain
\[
\left| \phi_k * f(x) \right| \lesssim 2^{k|\beta|} (1 + |x|)^\ell
\]
for some (possibly large) \(\beta \geq 0\) as required.

We can now prove the following.

**Proposition 3.3.** Let \(\ell \geq 0\). Suppose \((1 + | \cdot |)^\ell \psi \in L^1\) such that \(\hat{\psi} \in C^{n+1+\ell}(\mathbb{R}^n \setminus \{0\})\) with \(\partial^\ell \hat{\psi}\) rapidly decreasing for every \(|\kappa| \leq n + 1 + [\ell]\). Let \(f \in S'\) be a distribution of growth \(\ell\). Then for every \(j \in \mathbb{Z}\) the convolution \(\psi_j * f\) is a well-defined continuous function. Moreover, there exists \(\beta = \beta(f) > 0\) such that for every \(x \in \mathbb{R}^n\)
\[
M_{\ell, \beta}(x, j) = \sup_{k \geq j, y \in \mathbb{R}^n} \frac{|\psi_k \ast f(y)|}{(1 + 2|x - y|)^\ell 2^\beta(j - k)} < \infty. \tag{30}
\]

**Proof.** Fix \(j \in \mathbb{Z}\) and let \(k \geq j\). The assumptions on \(f\) and \(\psi\) imply that \(\psi_k \ast f \in S'\). Thus we can follow the standard proof of the Calderon reproducing formula to deduce the identity
\[
\psi_k \ast f = \phi_k \ast \psi_k \ast f + \sum_{a = k+1}^\infty \varphi_a \ast \varphi_a \ast \psi_k \ast f \tag{31}
\]
where the sum converges in the sense of \(S'\) (see (53) in the proof of Theorem 2.1). To show that \(\psi_k \ast f\) is a continuous function it suffices to prove that the sum converges in \(L^\infty_{\text{loc}}\). An application of Lemma 3.2 shows that there exists \(\beta \geq 0\) such that for every \(\rho \in S\) and \(k \geq j\)
\[
|\rho_k \ast f(x)| \lesssim_j 2^{\beta k}(1 + |x|)^\ell. \tag{32}
\]
Note that by (i) in Lemma 2.1 the assumption that \(\hat{\psi}\) is rapidly decreasing together with the support of \(\hat{\varphi}\) implies that \(\|\varphi_a \ast \psi_k(x)\| \lesssim 2^{(k - a)(\beta + 1)} 2^{an}(1 + 2^\ell |x|)^{-(n + 1 + \ell)}\). Therefore using an application of (32) we deduce the bound
\[
|\varphi_a \ast \varphi_a \ast \psi_k \ast f(x)| \lesssim_j 2^{a \beta} \int_{\mathbb{R}^n} |\varphi_a \ast \psi_k(y)|(1 + |x - y|)^\ell dx \lesssim 2^{(k + 1)\beta} 2^{-a}(1 + |x|)^\ell
\]
and hence the sum in (31) converges uniformly on compact sets. Consequently \(\psi_k \ast f\) is a continuous function. Finally, to deduce the required bound, we note that after another application of (32) we have for every \(k \geq j\)
\[
|\psi_k \ast f(x)| \leq |\psi_k \ast \phi_k \ast f(x)| + \sum_{a > k} |\varphi_a \ast \psi_k \ast \varphi_a \ast f(x)|
\]
\[
\lesssim_j 2^{k\beta}(1 + |x|)^\ell + 2^{k(\beta + 1)} \sum_{a \geq k} 2^{-a}(1 + |x|)^\ell \lesssim 2^{k\beta}(1 + |x|)^\ell
\]
which then gives (30). \(\square\)
Remark 3.2. Lemma 3.2 assures us that for any distribution $f$ of growth $\ell$, there exists a $\beta > 0$ such that $f$ satisfies the conditions of Proposition 3.3. Thus provided we have $\psi \in L^1$ satisfying $(1+|r|)^{\ell}\psi(\cdot) \in L^1$ and $\widehat{\psi} \in C^{n+1+\ell}([R^n \setminus \{0\})$ with, for every $|\kappa| \leq n+1+\ell$ and some $M > \beta$,

$$\partial^\kappa \widehat{\psi}(\xi) = O(|\xi|^{-n-M}) \quad \text{as } |\xi| \to \infty,$$

then the convolution $\psi * f$ is a continuous function. Unfortunately, we have no control over how large $\beta$ is. Thus if we only assume that $f$ is a distribution of (unspecified) finite growth, to ensure $\psi * f$ is a function, we need $\psi$ to satisfy the conditions of Proposition 3.3 for every $\beta$. In particular we need $\widehat{\psi}$ to be rapidly decreasing.

Moreover, some smoothness of $\psi$ is required too. For example, for $\psi * f$ to be a well-defined function for every $f \in S'$ of growth 0, we require $\psi$ to be smooth. To see this take any multi-index $\kappa$ and let $f = \partial^\kappa \delta_0$, where $\delta_0$ is the Dirac Delta function at the origin. Then $f$ is a distribution of growth 0 and by Definition 1.2 for any $\phi \in S$ we must have

$$\psi * f(\phi) = (-1)^{|\kappa|} \int_{R^n} \psi(x) \partial^\kappa \phi(x) dx.$$  

In particular, if $\psi * f \in S'$ is represented by a function $g \in L^1_{loc}$ then for every $\phi \in S$

$$\int_{R^n} \psi(x) \partial^\kappa \phi(x) dx = (-1)^{|\kappa|} \int_{R^n} g(x) \phi(x) dx.$$  

In other words $\psi$ must have $\kappa$ distributional derivatives which are locally integrable. As we can choose $|\kappa|$ to be arbitrarily large, Sobolev embedding then shows that $\psi \in C^{\infty}$.

4. Maximal Inequalities

As in the seminal work of Fefferman and Stein [10], and Peetre [16, 17], the key step in the proof of our characterisation theorems is to obtain certain pointwise maximal inequalities relating $\psi_j * f$ and $\varphi_j * f$. More precisely, assuming for the moment that the convolution $\psi_j * f \in L^1_{loc}$, our goal in this section is to prove an inequality of the form

$$\left(\varphi_j^* f(x) \right)^r \lesssim \sum_{k \geq j} 2^{\delta(j-k)} \int_{R^n} \frac{|\psi_k * f(x-y)|^r}{(1+2^k |y|)^{\lambda r}} 2^{kn} dy$$  

(33)

for some $\delta > 0$, $0 < r < \infty$, and $\lambda$ is as in the definition of the Peetre maximal function (5). The argument used to prove (33) follows a strategy of Strömberg-Torchinsky [20] together with a number of technical refinements. The first of which is the following extension of the Calderón reproducing formula.

**Proposition 4.1.** Let $\ell \geq 0$. Suppose $\psi \in L^1$ satisfies the Tauberian condition with $\widehat{\psi} \in C^{n+1+\ell}([R^n \setminus \{0\})$. There exists $\tilde{\eta}, \tilde{\phi} \in C^{n+1+\ell}([R^n)$ such that for every $g \in L^1_{loc}$ with $g(x) = O(|x|^\ell)$ we have for $k \in Z$ and a.e. $x \in R^n$

$$g(x) = \phi_k * g(x) + \sum_{j=k+1}^{\infty} \eta_j * \psi_j * g(x).$$  

(34)
Moreover supp $\hat{\phi}$ is compact, and supp $\hat{\eta}$ is contained in some annulus about the origin.

**Proof.** We start by observing that there exists an $\eta \in L^1$ satisfying the required conditions, such that for all $\xi \neq 0$

$$\sum_{j \in \mathbb{Z}} \hat{\eta}(2^{-j}\xi) \hat{\psi}(2^{-j}\xi) = 1 \quad (35)$$

The construction of $\eta$ is standard and follows from the following observation: There exist positive numbers $a, b, c$ with $0 < 2a \leq b$ such that for every $\xi \in \mathbb{R}^n$ there exists $j \in \mathbb{Z}$ satisfying $a \leq 2^{-j} |\xi| \leq b$ and

$$|\hat{\psi}(2^{-j}\xi)|^2 \geq c.$$

We refer to [20, Chapter V, Lemma 6] for details of this construction in the smooth case. The modification to the nonsmooth case has been carried out in the thesis [8] (see also [23]).

Define

$$\hat{\phi}(\xi) = \begin{cases} 
\sum_{j \leq 0} \hat{\eta}(2^{-j}\xi) \hat{\psi}(2^{-j}\xi) & \xi \neq 0 \\
1 & \xi = 0.
\end{cases}$$

It is easy to check that $\phi$ satisfies the required conditions and that $\hat{\phi} = 1$ in a neighbourhood of the origin. Moreover we have for any $k, m \in \mathbb{Z}$ with $m > k$

$$\phi_m - \phi_k = \sum_{j=k+1}^{m} \eta_j * \psi_j. \quad (36)$$

Take any $g \in L^1_{loc}$ satisfying $g(x) = O(|x|^{\ell})$. Note that as $\hat{\phi}, \hat{\psi}, \hat{\eta} \in C^{n+1+|\ell|}(\mathbb{R}^n)$ we have $|\phi|, |\psi * \eta| \lesssim (1 + |x|)^{-(n+1+|\ell|)}$ and hence the convolutions $\eta * \psi * g$ and $\phi * g$ are well defined. Moreover since $\phi_m$ forms an approximation to the identity we have $\lim_{m \to \infty} \phi_m * g(x) = g(x)$ for a.e. $x \in \mathbb{R}^n$ (more precisely this holds at every Lebesgue point of $g$). Thus taking the convolution of $g$ with both sides of (36) and letting $m \to \infty$ proves the result. \qed

To prove the maximal function inequality (33), we need to assume the boundedness of a particular auxiliary maximal function, namely, the following variation of the Peetre maximal function

$$M_{\lambda,m}(x, j) = \sup_{y \in \mathbb{R}^n, k \geq j} \frac{|\psi_k * f(y)|}{(1 + 2^j|x - y|)^{\lambda}} 2^{(j-k)m}. \quad (37)$$

Note that if $M_{\lambda,m}(x_0, j)$ is finite for some $x_0 \in \mathbb{R}^n$, then we have $M_{\lambda,m}(x, j) < \infty$ for all $x \in \mathbb{R}^n$. With these definitions at hand we now prove the following theorem which is essentially a non-smooth and discrete version of Theorem 2a in [20, page 61] (see also [6, Lemma 2]).
Theorem 4.2. Let \( 0 < r \leq 1, 0 < \lambda < \infty, \ell \geq 0, \) and \( m, \beta \in \mathbb{R}. \) Assume \((1 + |\cdot|)^{r}\psi(\cdot) \in L^{1}\) satisfies the Tauberian condition. Moreover, suppose that \( \hat{\psi} \in C^{\max\{n+1+|\ell|,|\lambda|+1\}}(\mathbb{R}^{n} \setminus \{0\}) \) and for every \(|\kappa| \leq \max\{|\ell|,|\lambda|\}+1\) we have

\[ \partial^{\kappa} \hat{\psi}(\xi) = O(|\xi|^{-\max\{m,\beta\}}) \quad \text{as } |\xi| \to \infty. \]

Let \( f \) be a distribution of growth \( \ell \) such that for every \( j \in \mathbb{Z} \) the distribution \( \psi_{j} * f \) is a locally integrable function with \( M_{\ell,m}(x,j) < \infty. \)

Then we have the pointwise inequality

\[ (\psi_{*}^{j}f(x))^{r} \lesssim \sum_{k=j}^{\infty} 2^{(j-k)(\beta-\lambda)r} \int_{\mathbb{R}^{n}} \frac{|\psi_{k} * f(x-y)|^{r}}{(1 + 2^{k}|y|)^{\lambda r}} 2^{k\eta} dy \]

(38)

with constant independent of \( f, j, m, \ell \) and \( x. \)

Proof. Fix \( u \geq j. \) The assumption \( M_{\ell,m}(x,j) < \infty \) implies that \( \psi_{u} * f = O(|x|^{\ell}). \) Therefore the Tauberian condition and Proposition 4.1 give

\[ \psi_{u} * f(x) = \phi_{u} * \psi_{u} * f(x) + \sum_{k=u+1}^{\infty} \eta_{k} * \psi_{k} * \psi_{u} * f(x) \]

(39)

with \( \hat{\phi}, \hat{\eta} \in C^{\max\{n+1+|\ell|,|\lambda|+1\}}(\mathbb{R}^{n}) \) and support of \( \hat{\eta} \) is contained in some annulus about the origin. An application of Lemma 2.4 gives

\[ (1 + 2^{u}|x|)^{\lambda}|\eta_{k} * \psi_{u}(x)| \lesssim 2^{-(k-u)\beta}2^{kn} \]

and thus we have the bound

\[ |\eta_{k} * \psi_{k} * \psi_{u} * f(x)| \lesssim \left\| (1 + 2^{u}|\cdot|)^{\lambda} \eta_{k} * \psi_{u} \right\|_{L^{\infty}} \left\| (1 + 2^{u}|x - \cdot|)^{-\lambda} \psi_{k} * f \right\|_{L^{1}} \]

\[ \lesssim 2^{kn}2^{-\beta(k-u)} \left\| (1 + 2^{u}|x - \cdot|)^{-\lambda} \psi_{k} * f \right\|_{L^{1}}. \]

On the other hand, the decay on \( \phi \) shows

\[ |\phi_{u} * \psi_{u} * f(z)| \lesssim \left\| (1 + 2^{u}|\cdot|)^{\lambda} \phi_{u} \right\|_{L^{\infty}} \left\| (1 + 2^{u}|z - \cdot|)^{-\lambda} \psi_{u} * f \right\|_{L^{1}} \lesssim 2^{mn} \left\| (1 + 2^{u}|z - \cdot|)^{-\lambda} \psi_{u} * f \right\|_{L^{1}} \]

and hence via (39) we obtain, for every \( z \in \mathbb{R}^{n} \) and any \( u \geq j, \)

\[ |\psi_{u} * f(z)| \lesssim 2^{(u-j)\beta} \sum_{k=u}^{\infty} 2^{(j-k)\beta} \int_{\mathbb{R}^{n}} \frac{|\psi_{k} * f(y)|}{(1 + 2^{u}|z - y|)^{\lambda}} 2^{kn} dy \]
where the constant depends only on \( \psi, \beta, \) and \( \lambda \) (in particular, it is independent of \( f, j, \ell, \) and \( m \)). Now, since \( k \geq u \geq j \), we have

\[
\frac{|\psi_k * f(y)|}{(1 + 2^u |z - y|)^\lambda} 2^{(j-k)\beta} = \left( \frac{|\psi_k * f(y)|}{(1 + 2^j |x - y|)^\lambda} \right)^r \left( \frac{|\psi_k * f(y)|}{(1 + 2^j |x - y|)^\lambda} \right)^{1-r} \frac{(1 + 2^j |x - y|)^\lambda}{(1 + 2^u |z - y|)^\lambda} \leq \left( \frac{|\psi_k * f(y)|}{(1 + 2^j |x - y|)^\lambda} \right)^r \lambda, \beta \right) \]

and hence using the elementary inequality \((1 + 2^j |y|)^{-1} \leq 2^{-j}(1 + 2^k |y|)^{-1}\) we deduce that

\[
\frac{|\psi_u * f(z)|}{(1 + 2^j |x - z|)^\lambda} 2^{(j-u)\beta} \leq M_{\lambda, \beta}(x, j)^{1-r} \sum_{k=u}^{\infty} 2^{(j-k)\beta r} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1 + 2^j |x - y|)^{\lambda r}} 2^{kn} dy \\
\leq M_{\lambda, \beta}(x, j)^{1-r} \sum_{k=j}^{\infty} 2^{(j-k)(\beta - \lambda) r} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1 + 2^k |x - y|)^{\lambda r}} 2^{kn} dy. \tag{40}
\]

Thus taking the supremum over \( z \in \mathbb{R}^n \) and \( u \geq j \) yields,

\[
M_{\lambda, \beta}(x, j) \leq M_{\lambda, \beta}(x, j)^{1-r} \sum_{k=j}^{\infty} 2^{(j-k)(\beta - \lambda) r} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1 + 2^k |x - y|)^{\lambda r}} 2^{kn} dy. \tag{41}
\]

If we had \( M_{\lambda, \beta}(x, j) < \infty \), then noting that \( \psi_j^* f(x) \leq M_{\lambda, \beta}(x, j) \), we obtain

\[
\left( \psi_j^* f(x) \right)^r \leq \sum_{k=j}^{\infty} 2^{(j-k)(\beta - \lambda) r} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1 + 2^k |x - y|)^{\lambda r}} 2^{kn} dy. \tag{41}
\]

Note that the constant in \((41)\) is independent of \( f, j, m, \ell, \) and \( x \). Therefore it suffices to prove \( M_{\lambda, \beta} < \infty \).

To this end let \( m' = \max\{m, \beta\} \) and \( \lambda' = \max\{\ell, \lambda\} \). Note that by our assumption we have \( M_{\lambda, m'} \leq M_{\ell, m} < \infty \). Moreover, we have \((1 + |z|)^{\lambda'} \eta, (1 + |z|)^{\lambda'} \phi \in L^\infty \) and via Lemma 2.4

\[
2^{(k-u)m'} 2^{kn} (1 + 2^u |x|)^{\lambda'} |\eta_k * \psi_u(x)| < \infty.
\]

Thus repeating the argument used to obtain \((41)\) (with \((\lambda, \beta)\) replaced by \((\lambda', m')\)) we have

\[
|\psi_u * f(y)|^r \leq M_{\lambda', m'}(y, u)^{1-r} \int_{\mathbb{R}^n} \frac{|\psi_k * f(y)|^r}{(1 + 2^u |y - z|)^{\lambda' r}} 2^{un} dz. \tag{42}
\]

Since the right hand side of \((42)\) only gets larger if we decrease \( m' \) and \( \lambda' \), we deduce that \((42)\) in fact holds for \( \lambda' = \lambda \) and \( m' = \beta \) (but with a constant that depends on \( m \), hence...
this argument cannot be used to prove (11) directly). Moreover, as
\[
\frac{2^{(j-u)\beta r}}{(1 + 2^j|x - y|)^{\lambda r}} \times \frac{2^{un}}{(1 + 2^u|y - z|)^{\lambda r}} \times \frac{2^{(u-k)(\beta r - n)}}{(1 + 2^u|x - z|)^{\lambda r}} \leq \frac{2^{jn}}{(1 + 2^j|x - z|)^{\lambda r}}
\]
we have for any \( u \geq j \)
\[
\frac{|\hat{\psi}_u * f(y)|^r}{(1 + 2^j|x - y|)^{\lambda r}} 2^{(j-u)\beta r} \lesssim \sum_{k = u}^{\infty} 2^{(j-k)(\beta r - n)} \int_{\mathbb{R}^n} \frac{|\hat{\psi}_k * f(z)|^r}{(1 + 2^j|x - z|)^{\lambda r}} 2^{jn} dz \lesssim \sum_{k = j}^{\infty} 2^{(j-k)(\beta - \lambda)} \int_{\mathbb{R}^n} \frac{|\hat{\psi}_k * f(z)|^r}{(1 + 2^k|x - z|)^{\lambda r}} 2^{kn} dz.
\]
Therefore, provided the right hand side of (11) is finite, we obtain \( M_{\lambda, \beta} < \infty \) and so (38) follows.

The required maximal inequality (33) is now a corollary of the previous Str"{o}mberg-Torchinsky type estimate, Theorem 4.2 together with another application of the Calderón reproducing formula in Proposition 4.1.

**Corollary 4.3.** Let \( 0 < r, \lambda < \infty, \ell \geq 0, \) and \( m, \beta \in \mathbb{R} \). Assume \((1 + |\cdot|)^{\ell} \psi(\cdot) \in L^1\) satisfies the Tauberian condition. Moreover, suppose that \( \hat{\psi} \in C^{n+1+\max\{[\ell], [\lambda]\}}(\mathbb{R}^n \setminus \{0\}) \) and for every \( |\kappa| \leq \max\{[\ell], [\lambda]\} + 1 \) we have
\[
\partial^{\kappa} \hat{\psi}(\xi) = O(|\xi|^{-\max\{m, \beta\}}) \quad \text{as } |\xi| \to \infty. \tag{43}
\]

Let \( f \) be a distribution of growth \( \ell \) such that for every \( j \in \mathbb{Z} \) the distribution \( \hat{\psi}_j * f \) is a locally integrable function with
\[
M_{\ell, m}(x, j) < \infty.
\]
Then we have the pointwise inequality
\[
\left( \varphi_j^* f(x) \right)^r \lesssim \sum_{k \geq j} 2^{(j-k)(\beta - \lambda)} \int_{\mathbb{R}^n} \frac{|\psi_k * f(x - y)|^r}{(1 + 2^k|y|)^{\lambda r}} 2^{kn} dy
\]
with constant independent of \( f, j, m, \ell \) and \( x \).

**Proof.** Assume \( f \) is a distribution of growth \( \ell \). Then \( \varphi_j * f = O(|x|^{\ell}) \) and so we can apply Proposition 4.1 and obtain
\[
\varphi_j * f(x) = \phi_u * \varphi_j * f(x) + \sum_{k = u + 1}^{\infty} \eta_k * \varphi_j * \psi_k * f(x).
\]
where \( \hat{\eta}, \hat{\varphi}, \hat{\psi} \in C^{n+1+\max\{[\ell], [\lambda]\}}(\mathbb{R}^n) \), supp \( \hat{\varphi} \subset \{ |\xi| < b \} \), and supp \( \hat{\psi} \subset \{ a < |\xi| < b \} \) for some \( a, b > 0 \). Since supp \( \hat{\varphi} \subset \{ 2^{-1} \leq |\xi| \leq 2 \} \), by choosing \( u = j - s \) with \( s \) sufficiently large we have \( \phi_u * \varphi_j = 0 \). Similarly, perhaps choosing \( s \) slightly larger \( \eta_k * \varphi_j = 0 \) for \( k > j + s \). Therefore we have
\[
|\varphi_j * f(x)| \leq \sum_{k = j - s}^{j + s} |\eta_k * \varphi_j * \psi_k * f(x)|. \tag{44}
\]
If \( r \geq 1 \), we simply use an application of Holder’s inequality together with (44) to deduce that

\[
\frac{|\varphi_j * f(x - y)|}{(1 + 2^j |y|)^\lambda} \lesssim \sum_{j \approx k} \int_{\mathbb{R}^n} 2^{-j^\lambda} (1 + 2^j |z - y|)^\lambda |\eta_k * \varphi_j (z - y)| \times \frac{|\psi_k * f(x - z)|}{(1 + 2^j |z|)^\lambda} 2^j \hat{\varphi}_n dz
\]

where we used the decay of \( \varphi \). The require inequality now follows by taking the \( r \)th powers of both sides. On the other hand, if \( 0 < r < 1 \), a similar application of (44) gives

\[
\varphi_j^* f(x) \lesssim \sum_{j \approx k} \| (1 + 2^j |\cdot|)^\lambda \eta_k * \varphi_j \|_{L^1} \psi_k^* f(x) \lesssim \sum_{j \approx k} \psi_k^* f(x).
\]

If we again take \( r \)th powers of both sides, then result follows directly from an application of Theorem 4.2. \[\square\]

Remark 4.1. In the case \( r \geq 1 \), the proof of Corollary 4.3 shows that the decay condition on \( \hat{\psi} \), (43), is not needed. In fact we only need the the smoothness assumption \( \hat{\psi} \in C^{n+1+\max\{|\lambda|,|\ell|\}}(\mathbb{R}^n \setminus 0) \) to ensure that the \( \eta \) given by Proposition 4.1 has sufficient decay.

Remark 4.2. A careful examination of the proof of Theorem 4.2 and Corollary 4.3 shows that we may replace the condition (43) with the slightly weaker condition

\[
\sup_{j \geq 1} \left( 2^j \max\{\beta, m\} \| (1 + |\cdot|^\lambda \varphi_j * \psi \|_{L^\infty} \right) < \infty
\]

(c.f. the “poised spaces of Besov type” introduced by Peetre in [16]). Alternatively, as in Remark 2.2 we may assume that

\[
\sup_{|\gamma| \leq \max\{|\lambda|,|\ell|\}} \| P_{\geq 1} (x^\gamma \psi) \|_{B^{\max\{m, \beta\}}_{\infty, \infty}} < \infty.
\]

5. Proof of Characterisation Theorems

In this section we give the proofs of our main results. We start with the sufficient direction, i.e. Theorems 1.2 and 1.3. The first step is the following preliminary version of Theorem 1.3.

Theorem 5.1. Let \( 0 < p, q \leq \infty, \alpha \in \mathbb{R} \). Assume \( \lambda > \Lambda \geq 0 \) and \( \ell \geq 0 \). Let \( f \) be a distribution of growth \( \ell \) and \( (1 + |\cdot|)^\ell \psi \in L^1 \) satisfying the following:

\begin{enumerate}
  \item[(S1)] the kernel \( \psi \) satisfies the Tauberian condition and we have \( \hat{\psi} \in C^{n+1+\max\{\ell,|\lambda|\}}(\mathbb{R}^n \setminus \{0\}) \);
  \item[(S2)] there exists \( m \geq 0 \) such that for every \( j \in \mathbb{Z} \) the distribution \( \psi_j * f \) is a locally integrable function with \( M_{\ell, m}(x, j) < \infty \);
\end{enumerate}
(S3) there exists $\beta > \Lambda - \alpha$ such that for every $|\gamma| \leq \max\{|\Lambda|, |\ell|\} + 1$

$$\partial^\gamma \hat{\psi} = O(|\xi|^{-\max\{\beta, m\}}) \quad \text{as } |\xi| \to \infty.$$ 

If $\Lambda = \frac{n}{p}$ then

$$\left( \sum_{j \in \mathbb{Z}} (2^{ja} \| \varphi_j * f \|_{L^p})^q \right)^\frac{1}{q} \lesssim \left( \sum_{j \in \mathbb{Z}} (2^{ja} \| \psi_j * f \|_{L^p})^q \right)^\frac{1}{q}$$

with constant independent of $m$ and $f$. Similarly if $\Lambda = \max\{\frac{n}{p}, \frac{n}{q}\}$ and $p < \infty$ then

$$\left\| \left( \sum_{j \in \mathbb{Z}} (2^{ja} \varphi_j * f)^q \right)^\frac{1}{q} \right\|_{L^p} \lesssim \left\| \left( \sum_{j \in \mathbb{Z}} (2^{ja} \psi_j * f)^q \right)^\frac{1}{q} \right\|_{L^p}$$

and in the case $p = \infty$

$$\sup_Q \left( \frac{1}{|Q|} \int_Q \sum_{j > -\ell(Q)} (2^{ja} \varphi_j * f(x))^q dx \right)^\frac{1}{q} \lesssim \sup_Q \left( \frac{1}{|Q|} \int_Q \sum_{j > -\ell(Q)} (2^{ja} \psi_j * f(x))^q dx \right)^\frac{1}{q}$$

where again the implied constant is independent of $m$ and $f$. Note that when $q = \infty$, the previous inequality takes the form

$$\sup \sup_{Q \ni j > -\ell(Q)} \frac{1}{|Q|} \int_Q 2^{ja} \varphi_j * f(x) dx \lesssim \sup \sup_{Q \ni j > -\ell(Q)} \frac{1}{|Q|} \int_Q 2^{ja} \psi_j * f(x) dx,$$

where we require $\Lambda = n$.

**Proof.** The proof follows the arguments used in [5, 6, 7], with Theorem 4.2 replacing [6, Lemma 2]. We only prove the Triebel-Lizorkin case as the Besov-Lipschitz case is similar. As the left-hand side of the inequalities only gets larger if we decrease $\lambda$, we may assume $\max\{\frac{n}{p}, \frac{n}{q}\} = \Lambda < \lambda < \min\{\alpha + \beta, |\Lambda| + 1\}$. Choose $0 < r < \min\{p, q\}$ with $\max\{\frac{n}{p}, \frac{n}{q}\} < \frac{n}{r} < \lambda$. An application of Corollary 4.3 together with a decomposition of $\mathbb{R}^n$ into annuli centred at $x$, gives the pointwise inequality

$$\left( 2^{ja} \varphi_j * f(x) \right)^r \lesssim \sum_{k \geq 1} 2^{-k(\beta + \alpha - \lambda) r} M((2^{(k+j)\alpha} |\psi_{k+j} * f|)^r)(x) \quad (45)$$

where $M(g) = \sup_{R > 0} R^{-n} \int_{|y| < R} |g(x - y)| dy$ denotes the Hardy-Littlewood maximal function, and we used the elementary estimate $\int_{\mathbb{R}^n} \frac{|g(x - y)|}{(1 + 2^{j}|y|)^{2jn}} dy \lesssim M(g)(x)$ which holds...
provided \( N > n \). Therefore, as \( \frac{q}{r}, \frac{p}{r} > 1 \), we deduce that

\[
\left\| \left( \sum_{j \in \mathbb{Z}} (2^j \varphi_j^* f)^q \right)^{\frac{1}{q}} \right\|_{L^p} = \left\| \left( \sum_{j \in \mathbb{Z}} \left[ 2^{j\alpha} \varphi_j^* f \right]^r \right)^{\frac{1}{r}} \right\|_{L^p}^{\frac{1}{r}}
\]

\[
\lesssim \left\| \left( \sum_{j \in \mathbb{Z}} \left[ \sum_{k \geq 1} 2^{-k(\beta + \alpha - \lambda)} M(2^{(k-j)\alpha} \varphi_{k+j}^* f) \right] \right)^{\frac{2}{r}} \right\|_{L^p}^{\frac{1}{r}}
\]

\[
\lesssim \left( \sum_{k \geq 1} 2^{-k(\beta + \alpha - \lambda)} \right) \left\| \left( \sum_{j \in \mathbb{Z}} \left[ M(2^j \varphi_j^* f) \right]^{\frac{2}{r}} \right) \right\|_{L^p}^{\frac{1}{r}}
\]

\[
\lesssim \left\| \left( \sum_{j \in \mathbb{Z}} (2^j |\varphi_j^* f|)^q \right)^{\frac{1}{q}} \right\|_{L^p}
\]

where we used the assumption \( \beta > \lambda - \alpha \) together with vector valued Hardy-Littlewood maximal inequality of Fefferman-Stein [9].

The argument in the case \( p = \infty \) is slightly different and is of a more computational nature. Fix a dyadic cube \( Q \) and let \( x \in Q \). Assume first that \( q < \infty \). An application of Corollary 13 with \( r = q \) gives

\[
\sum_{j \geq -\ell(Q)} (2^j \varphi_j^* f(x))^q \lesssim \sum_{j \geq -\ell(Q)} \sum_{k \geq 1} 2^{-k(\alpha + \beta - \lambda)} \int_{\mathbb{R}^n} \frac{(2^{(j+k)\alpha} |\varphi_{j+k}^* f(x-y)|)^q}{(1 + 2^{(j+k)}|y|)^{\lambda q}} 2^{(j+k)n} dy
\]

\[
\lesssim \sum_{j \geq -\ell(Q)} \int_{\mathbb{R}^n} \frac{(2^{j\alpha} |\varphi_j^* f(x-y)|)^q}{(1 + 2^j|y|)^{\lambda q}} 2^{jn} dy
\]

\[
= \sum_{j \geq -\ell(Q)} \int_{|y| \leq 2^\ell(Q)} \frac{(2^{j\alpha} |\varphi_j^* f(x-y)|)^q}{(1 + 2^j|y|)^{\lambda q}} 2^{jn} dy
\]

\[
+ \sum_{j \geq -\ell(Q)} \sum_{a \geq 1} \int_{|y| \approx 2^{a + \ell(Q)}} \frac{(2^{j\alpha} |\varphi_j^* f(x-y)|)^q}{(1 + 2^j|y|)^{\lambda q}} 2^{jn} dy. \quad (46)
\]

To estimate the first term in (46) we let \( Q^* \) denote a dyadic cube with \( \ell(Q^*) \approx \ell(Q) \) such that \( y + Q \subset Q^* \) for every \( |y| \leq 2^\ell(Q) \). A computation then shows that

\[
\frac{1}{|Q|} \int_Q \sum_{j \geq -\ell(Q)} \int_{|y| \leq 2^\ell(Q)} \frac{(2^{j\alpha} |\varphi_j^* f(x-y)|)^q}{(1 + 2^j|y|)^{\lambda q}} 2^{jn} dy \ dx
\]

\[
\lesssim \sum_{j \geq -\ell(Q)} \int_{|y| \leq 2^\ell(Q)} \frac{2^{jn}}{(1 + 2^j|y|)^{\lambda q}} \frac{1}{|Q^*|} \int_{Q^*} (2^{j\alpha} |\varphi_j^* f(x)|)^q dx \ dy
\]

\[
\lesssim \sup_Q \left( \frac{1}{|Q'|} \int_{Q'} \sum_{j \geq -\ell(Q')} (2^{j\alpha} |\varphi_j^* f(x)|)^q dx \right).
\]
Thus it only remains to control the second term in (46). To this end, observe that for \( j \geq -\ell(Q), a \geq 1, \) and \( |y| \approx 2^{a+\ell(Q)} \) we have

\[
\frac{2^{jn}}{(1 + 2^{\nu} |y|)^{\lambda q}} \lesssim 2^{j(n - \lambda q) - (a + \ell(Q))\lambda q} \lesssim 2^{-(a + \ell(Q))n} \]

where we used the fact that \( \lambda > \frac{n}{q} \). Therefore

\[
\sum_{j \geq -\ell(Q)} \sum_{a \geq 1} \int_{|y| \approx 2^{a+\ell(Q)}} \left( \frac{2^{ja} \psi_j * f(x - y)}{(1 + 2^{\nu} |y|)^{\lambda q}} \right)^{\nu} 2^{jn} dy \]

\[
\lesssim \sum_{a \geq 1} 2^{-(a + \ell(Q))n} \int_{|y| \approx 2^{a+\ell(Q)}} \sum_{j \geq -\ell(Q)} (2^{ja} \psi_j * f(y))^q dy \]

\[
\lesssim \sup_{Q'} \left( \frac{1}{|Q'|} \int_{Q'} \sum_{j \geq -\ell(Q')} (2^{ja} \psi_j * f(x))^q dx \right). \]

These two estimates imply the required inequality when \( q \ll \).

The proof in the case \( p = q = \infty \) is similar, in fact simpler, so we shall be brief. Fix a dyadic cube \( Q \) and let \( x \in Q \) as above. Let \( j \geq -\ell(Q) \). Using Corollary 4.3 with \( r = 1 \) we get

\[
2^{ja} \varphi_j^* f(x) \lesssim \sum_{k \geq j} 2^{-(a + \beta - \lambda)} \int_{\mathbb{R}^n} 2^{ka} \psi_k(x - y) \frac{dy}{(1 + 2^k |y|)^{\lambda q}}. \]

It follows that, by decomposing the \( y \)-integral as before and noting that \( \lambda > n \) in this case, one obtains

\[
\frac{1}{|Q|} \int_Q 2^{ja} \varphi_j^* f(x) dx \lesssim \sup_{Q'} \sup_{k \geq -\ell(Q')} \left( \frac{1}{|Q'|} \int_{Q'} 2^{ka} \psi_k(x) dx \right). \]

The proof of the theorem is thus complete. \( \square \)

The proof of the \( p = \infty \) case in Theorem 5.1 requires the following corollary (c.f. the proof of Lemma 4 and 5 in the work of Rychkov [18]).

**Corollary 5.2.** Let \( 0 < q \ll, \lambda > \frac{n}{q}, \) and \( \lambda > n \) when \( q = \infty \). Let \( k \in \mathbb{Z} \). Then for any dyadic cube \( Q \) we have

\[
\left( \frac{1}{|Q|} \int_Q \sum_{j \geq -\ell(Q)} \left( 2^{ja} \varphi_j^* f(x) \right)^q dx \right)^{\frac{1}{q}} \lesssim (1 + |k|)^{\frac{1}{q}} \left\| f \right\|_{F_{\infty,q}}, \quad q \ll
\]

\[
\sup_{j \geq -\ell(Q) + k} \frac{1}{|Q|} \int_Q 2^{ja} \varphi_j^* f(x) dx \lesssim \left\| f \right\|_{F_{\infty,\infty}}. \]

**Proof.** Assume first that \( q \ll \). An application of Theorem 5.1 with \( \psi = \varphi \) (in which case the assumptions (S1), (S2), and (S3) clearly hold) gives

\[
\left( \sup_{Q'} \frac{1}{|Q'|} \int_{Q'} \sum_{j \geq -\ell(Q')} \left( 2^{ja} \varphi_j^* f(x) \right)^q dx \right)^{\frac{1}{q}} \lesssim \left\| f \right\|_{F_{\infty,q}}. \]
Thus it is enough to show that for \( j < -\ell(Q) \),
\[
\frac{1}{|Q|} \int_{Q} (\varphi_{j}^{*} f(x))^{q} dx \lesssim \frac{1}{|Q|} \int_{Q} (\varphi_{j}^{*} f(x))^{q} dx
\]
where \( Q' \) is a dyadic cube with \( Q \subset Q' \) and \( \ell(Q') = -j \). To this end, note that if \( x, x' \in Q' \), then \(|x - x'| \leq 2^{n-j}\) and hence for any \( y \in \mathbb{R}^{n} \) we have
\[
(1 + 2^{j}|x' - y|)^{\lambda} \lesssim (1 + 2^{j}|x' - x| + 2^{j}|x - y|)^{\lambda} \lesssim (1 + 2^{j}|x - y|)^{\lambda}.
\]
Therefore the definition of \( \varphi_{j}^{*} f \) implies that for any \( x, x' \in Q' \) we have \( \varphi_{j}^{*} f(x) \lesssim \varphi_{j}^{*} f(x') \). Consequently \( \varphi_{j}^{*} f(x) \) is essentially constant on cubes of side lengths \( 2^{-j} \), in particular, we have (47). Thus the result for \( q < \infty \) follows. The modification when \( q = \infty \) is done in a similar manner to the proof of Theorem 5.1.

\[ \square \]

5.1. **Sufficient Conditions.** We now come to the proof of the sufficient direction of our characterisations which are now a straightforward consequence of Theorem 5.1.

**Proof of Theorem 1.3.** We reduce to checking the conditions (S1), (S2), and (S3). The characterisations which are now a straightforward consequence of Theorem 5.1. To check (S1), note that by following the argument leading to (4.1), there exists \( 0 < a < b \) and \( c > 0 \) such that for every \( a \leq t \leq b \) and \( \xi \in \mathbb{S}^{n-1} \)
\[
|\hat{\psi}(t\xi)| \geq c.
\]
Since \( \phi \in \mathcal{S} \) and \( \hat{\phi}(0) = \int \phi \neq 0 \), there exists \( r > 0 \) such that \( |\hat{\phi}(\xi)| > 0 \) for \( |\xi| < r \). Now as \( \hat{\phi}_{s}(\xi) = \hat{\phi}(s\xi) \) we only need to choose \( s < \frac{r}{a} \) to ensure that \( \phi_{s} * \psi \) satisfies the Tauberian condition. Clearly the remaining conditions in (S1) are also satisfied.

To verify (S2), observe that since \( \phi_{s} * f = \mathcal{O}(|x|^{\ell}) \), the convolution \( (\phi_{s} * \psi)_{j} * f \) is well-defined. Furthermore, an application of Lemma 3.2 shows that there exists \( m \) such that
\[
|\phi_{s} * f(x)| \lesssim 2^{k|m}(1 + |x|)^{\ell}
\]
which implies that for any \( x \in \mathbb{R}^{n} \), \( j \in \mathbb{Z} \),
\[
\sup_{y \in \mathbb{R}^{n}, k \geq j} \left| (\phi_{s} * \psi)_{k} * f(y) \right| \lesssim \sup_{y \in \mathbb{R}^{n}, k \geq j} \frac{2^{j-k|m}}{(1 + 2^{j}|x - y|)^{\ell} 2^{(j-k)m}} < \infty.
\]
Thus (S2) holds. Finally, the rapid decay of \( \partial^{\nu} \hat{\phi} \) ensures that (S3) holds provided that \( \partial^{\nu} \hat{\psi} \) is slowly increasing as \(|\xi| \to \infty\).
Therefore, we may apply Theorem 5.1 together with the pointwise bound $|\varphi_j * f(x)| \leq \varphi_j^* f(x)$, to deduce that

$$
\|f\|_{B^{\alpha}_{p,q}} \lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\phi_{sj} * \psi_j * f\|_{L^p} \right)^q \right)^{\frac{1}{q}} 
$$

$$
\lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\sup_{t>0} \phi_t * \psi_j * f\|_{L^p} \right)^q \right)^{\frac{1}{q}} 
$$

$$
\lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\psi_j * f\|_{H^p} \right)^q \right)^{\frac{1}{q}}
$$

where the last line follow from the $H^p$ characterisation of Fefferman-Stein [10]. Similarly, the Triebel-Lizorkin case follows via

$$
\|f\|_{F^{\alpha}_{p,q}} \lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\phi_{sj} * \psi_j * f\|_{L^p} \right)^q \right)^{\frac{1}{q}}
$$

$$
\lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\sup_{t>0} \phi_t * \psi_j * f\|_{L^p} \right)^q \right)^{\frac{1}{q}}
$$

$$
\lesssim \left( \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|\psi_j * f\|_{H^p} \right)^q \right)^{\frac{1}{q}}
$$

An identical computation gives the $p = \infty$ case. Thus the proof of Theorem 1.2 is complete. \qed

**Remark 5.1.** An alternative, more direct proof is possible of the Besov-Lipschitz case in Theorem 1.2. The details are as follows. Assume $f$ is a distribution of growth $\ell$. Choose $\rho \in \mathcal{S}$ with $\hat{\rho}(\xi) = 1$ for $\xi \in \text{supp} \hat{\varphi}$ and let $\frac{n}{p} < \lambda < \left[ \frac{n}{p} \right] + 1$. Then from (44) we deduce that

$$
|\varphi_j * f(x)| \leq \sum_{k=j-s}^{j+s} |\eta_k * \varphi_j * \rho_j * \psi_k * f(x)|
$$

$$
\lesssim \sum_{k=j-s}^{j+s} \sup_{y \in \mathbb{R}^n} \frac{|\rho_j * \psi_k * f(x-y)|}{(1 + 2^j |y|)^\lambda} \int_{\mathbb{R}^n} |\eta_k * \varphi_j(y)|(1 + 2^j |y|)^\lambda dy
$$

$$
\lesssim \sum_{k=j-s}^{j+s} M^{**}_{\lambda}(\psi_k * f)(x)
$$

where

$$
M^{**}_{\lambda}(g)(x) = \sup_{t>0,y \in \mathbb{R}^n} \frac{\rho_t * g(x-y)}{\left(1 + \frac{|y|}{t}\right)^\lambda}
$$

is the maximal function of Fefferman-Stein and we used the fact that $\eta(x) = O(|x|^{-n-1-\left[\frac{n}{p}\right]})$. By the characterisation of $H^p$ by Fefferman-Stein [10] we have

$$
\|M^{**}_{\lambda} g\|_{L^p} \lesssim \|g\|_{H^p}
$$
provided $\lambda > \frac{n}{p}$. Therefore

$$
\|f\|_{B^{\alpha}_{p,q}} = \left(\sum_{j \in \mathbb{Z}} (2^{ja} \|\varphi_j * f\|_{L^p})^q\right)^{\frac{1}{q}} \lesssim \left(\sum_{j \in \mathbb{Z}} (2^{ja} \|M^{\alpha}_{\lambda} (\psi_j * f)\|_{L^p})^q\right)^{\frac{1}{q}}
$$

$$
\lesssim \left(\sum_{k \in \mathbb{Z}} (2^{ka} \|\psi_k * f\|_{H^p})^q\right)^{\frac{1}{q}}.
$$

Hence (14) is proved.

We note that a similar argument gives the corresponding Triebel-Lizorkin version as

$$
\|f\|_{F^{\alpha}_{p,q}} \lesssim \left(\sum_{j \in \mathbb{Z}} (2^{ja} M^{\alpha}_{\lambda} (\psi_j * f))^q\right)^{\frac{1}{q}}
$$

but this does not give (15) as there is no vector valued inequality relating $M^{\alpha}_{\lambda}$ with $\sup_{t>0} |\phi_t * g|$. Thus we cannot directly deduce (15) from (48) and instead need to argue via Theorem 5.1.

Theorem 1.3 required $\partial^\kappa \hat{\psi}$ to be rapidly decreasing to ensure that the convolution $\psi * f$ was a locally integrable function. One way to avoid this fairly strong assumption on the kernel $\psi$, was presented in Theorem 1.2 where we replaced the $L^p$ norm with the Hardy norm $H^p$ which is defined for elements of $S'$. Consequently we only had to make sense of $\psi_j * f$ as an element of $S'$ rather than $L^1_{loc}$. On the other hand, an alternative approach to finding a pointwise definition of the convolution is to instead make further assumptions on $f$. In particular, if we assume that $f$ is a slowly increasing function of order $\ell$, then the convolution $\psi * f$ is well defined as a function without the rapidly decreasing assumption.

This leads to the following version of Theorem 5.1.

**Theorem 5.3.** Let $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, and $\ell \geq 0$. Let $\Lambda \geq 0$ and $\beta > \Lambda - \alpha$. Assume $(1 + |\cdot|)^{-\ell} f \in L^\infty$. Suppose $\psi \in L^1$ satisfies the Tauberian condition with $(1 + |\cdot|)^\ell \psi(\cdot) \in L^1$. Furthermore, assume that $\hat{\psi} \in C^{n+1+\max\{[\ell],[\Lambda]\}}(\mathbb{R}^n \setminus \{0\})$ with

$$
\partial^\kappa \hat{\psi}(\xi) = O(|\xi|^{-\max\{\beta,0\}}) \quad \text{as } |\xi| \to \infty
$$

for $|\kappa| \leq \max\{[\Lambda],[\ell]\} + 1$. If $\Lambda = \frac{n}{p}$ then

$$
\|f\|_{B^{\alpha}_{p,q}} \lesssim \left(\sum_{j \in \mathbb{Z}} (2^{ja} \|\psi_j * f\|_{L^p})^q\right)^{\frac{1}{q}}
$$

Similarly if $\Lambda = \max\{\frac{n}{p}, \frac{n}{q}\}$ and $p < \infty$ then

$$
\|f\|_{F^{\alpha}_{p,q}} \lesssim \left(\sum_{j \in \mathbb{Z}} (2^{ja} |\psi_j * f|)^q\right)^{\frac{1}{q}}
$$

and in the case $p = \infty$

$$
\|f\|_{F^{\alpha}_{\infty,q}} \lesssim \sup_Q \left(\frac{1}{|Q|} \int_Q \sum_{j \gg -\ell(Q)} (2^{ja} |\psi_j * f(x)|)^q dx\right)^{\frac{1}{q}},
$$
with the usual interpretation when \( q = \infty \) (in which case \( \Lambda = n \)).

**Proof.** We begin by observing that for \( k \geq 0 \)
\[
|\psi_k * f(x)| \lesssim (1 + |x|)^\ell
\]
which implies that \( M_{\ell,0}(x,0) < \infty \) and consequently \( M_{\ell,0}(x,j) < \infty \) for every \( j \in \mathbb{Z} \). Therefore result follows from Theorem 5.1. \( \square \)

**Remark 5.2.** Assume \( \alpha > n/p \). Then elements in \( \dot{B}^\alpha_{p,q} \) or in \( \dot{F}^\alpha_{p,q} \) are functions that satisfy the growth condition in Theorem 5.3 with \( \ell = \alpha - n/p \) when \( \alpha - n/p \not\in \mathbb{N} \), and \( \ell > \alpha - n/p \) when \( \alpha - n/p \in \mathbb{N} \) (see Remark 2.1). Hence this theorem readily gives the characterisation of these function spaces without the rapidly decreasing assumption on the Fourier transform of the kernel \( \hat{\psi} \).

5.2. **Necessary Conditions.** We now come to the necessary direction of our characterisation, namely the proof of Theorem 1.1. As in the proof of Theorem 5.1, we follow the maximal function arguments used in the work of Bui-Paluszynski-Taibleson [5, 6, 7]. In addition, in the case \( p = \infty \), we rely also on an argument due to Rychkov [18].

**Proof of Theorem 1.1.** We first show that \( \psi \in \dot{B}^{\alpha}_{1,\infty} \). To this end, the assumptions on \( \psi \) together with Lemma 2.4 imply that
\[
|\phi_j * \psi(x)| \lesssim \begin{cases} 
2^{-jm} & j \geq 0 \\
2^{jr} & j \leq 0
\end{cases} \frac{1}{(1+|x|)^{\alpha + \Lambda + 1}} \quad (49)
\]
It follows that
\[
\|\psi\|_{\dot{B}^{\alpha}_{1,\infty}} \lesssim \sum_{j \geq 0} 2^{-j(\alpha - \frac{n}{p} + m)} + \sum_{j < 0} 2^{j(r - \alpha + \frac{n}{p})} < \infty \quad \text{(as } \alpha - n/p + m > 0, r > \alpha).\]

Let \( f \in \dot{B}^\alpha_{p,q} \) or \( f \in \dot{F}^\alpha_{p,q} \). Then \( f \in \dot{B}^{\alpha - \frac{n}{p}}_{\infty,\infty} \) by a well-known embedding theorem and hence by Theorem 2.2 and Theorem 3.1, after possibly subtracting a polynomial \( \rho \), we see that \( f \) is a distribution of growth \( \ell \), the distribution \( \psi_j * f \) is in fact a bounded continuous function, and moreover for every \( x \in \mathbb{R}^n \) we have the pointwise identity
\[
\psi_k * f(x) = \sum_{j \in \mathbb{Z}^n} \phi_j * \varphi_j * \psi_k * f(x).
\]
Since \( \psi_k * f \) only gets smaller if we increase \( \lambda \) we may assume \( \Lambda < \lambda < \min\{m + \alpha, [\Lambda] + 1\} \) where \( \Lambda = \frac{n}{p} \) in the Besov-Lipschitz case, and \( \Lambda = \max\{\frac{n}{p}, \frac{n}{q}\} \) in the Triebel-Lizorkin case (this is possible as we assume that \( m > \Lambda - \alpha \)). If we now use an application of the above
Calderón formula we obtain
\[
|\psi_k * f(x - z)| \leq \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \frac{|\psi_k * \varphi_j(y)|}{(1 + 2^k|z|)^\lambda} |\varphi_j * f(x - z - y)|\,dy
\leq \sum_{j \in \mathbb{Z}} |\varphi_j f(x)| \int_{\mathbb{R}} \frac{|\psi_k * \varphi_j(y)|}{(1 + 2^k|z|)^\lambda} \,dy.
\]

A change of variables shows that
\[
\int_{\mathbb{R}} \frac{|\psi_k * \varphi_j(y)|}{(1 + 2^j|z + y|)^\lambda} \,dy = \int_{\mathbb{R}} \frac{|\psi_k * \varphi_j(y)|}{(1 + 2^k|z|)^\lambda} \,dy
\]
and hence we have the pointwise estimate
\[
2^{k\alpha} \psi_k f(x) \lesssim \sum_{j \in \mathbb{Z}} a_{j-k} 2^{j\alpha} \varphi_j f(x)
\]
where
\[
a_j = 2^{-j\alpha} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\psi * \varphi_j(y)|}{(1 + |x+y|)^\lambda} \frac{(1 + 2^j|x+y|)}{(1 + |x|)^\lambda} \,dy.
\]

Thus, provided \((a_j) \in \ell^{\min\{p,q,1\}}\) (\((a_j) \in \ell^{\min\{q,1\}}\) in the Triebel-Lizorkin case), the first part of Theorem 1.1 \((6)\) and \((8)\), follows from Proposition 2.7 together with the maximal function characterisation of Peetre \([16]\). In fact, using the obvious inequality
\[
\frac{(1 + 2^j|x+y|)}{(1 + |x|)^\lambda} \lesssim \begin{cases} 2^j(1 + |y|) & j \geq 0 \\ (1 + 2^j|y|) & j \leq 0 \end{cases}
\]
together with the estimates \((49)\), we get
\[
a_j \lesssim \begin{cases} 2^{j(\lambda - m - \alpha)} \int_{\mathbb{R}^n} (1 + |y|)^{\lambda - 1 - [\lambda]} \,dy, & j \geq 0 \\ 2^{j(\alpha - r)} \int_{\mathbb{R}^n} 2^{jn} (1 + |2^jy|)^{-n - 1 - [\lambda]} \,dy, & j \leq 0. \end{cases}
\]
By our assumptions, \(\lambda > [\Lambda] \geq 0, r > \alpha, \) and \(\lambda - m - \alpha < 0\) by our choice of \(\lambda\), we deduce that \((a_j) \in \ell^{\beta}\) for all \(\beta > 0\). Hence \((6)\) and \((8)\) are proved.

It remains to consider the case \(p = \infty\). We provide detail only in the case \(q < \infty\) as the modification when \(q = \infty\) is familiar by now. As in the case \(p < \infty\), an application of \((50)\) together with \((51)\) shows that there exists \(\delta > 0\) such that
\[
2^{k\alpha} \psi_k f(x) \lesssim \sum_{j \in \mathbb{Z}} 2^{-|j|\delta} 2^{(k-j)\alpha} \varphi_{k-j} f(x).
\]
Therefore, Corollary 5.2 gives for any dyadic cube $Q$
\[
\frac{1}{|Q|} \int_Q \left| \sum_{k \geq -\ell(Q)} (2^{k \alpha} \psi_k^*(f(x)))^q \right| dx \lesssim \frac{1}{|Q|} \int_Q \left( \sum_{j \in \mathbb{Z}} 2^{-|j| \delta} 2^{k \alpha} \varphi_{k-j}^* f(x) \right)^q dx
\]
\[
\lesssim \sum_{j \in \mathbb{Z}} 2^{-|j| \delta \min\{1,q\}} \left( \frac{1}{|Q|} \int_Q \left( \sum_{j \in \mathbb{Z}} 2^{k \alpha} \varphi_{k-j}^* f(x) \right)^q dx \right)
\]
\[
\lesssim \left( \sum_{j \in \mathbb{Z}} (1 + |j|)^{2^{-|j| \delta \min\{1,q\}}} \right) \|f\|_{F_{\infty,q}}^q
\]
where, in the second inequality, we also use the $q$-triangle inequality when $q \leq 1$ and Hölder’s inequality when $q > 1$. Thus (5) follows.

We now turn to the proof of (7), (10), and (11). Let $\phi \in S$. Since $\phi_t * f$ satisfies the same properties as $f$ ($\phi_t * f \in \hat{B}_p^q$), we can repeat the proof of (50) to deduce that
\[
2^{k \alpha} |\phi_t * \psi_k * f(x)| \lesssim 2^{k \alpha} \psi_k^*(\phi_t * f)(x) \lesssim \sum_{j \in \mathbb{Z}} a_{k-j} 2^{j \alpha} \varphi_j^*(\phi_t * f)(x)
\]
with constant independent of $t$. If we now follow the arguments leading to (6), (8), and (9), it suffices to show that
\[
\varphi_j^*(\phi_t * f)(x) \lesssim \varphi_j^* f(x).
\]
To this end, let $\mu \in S$ with $\widehat{\mu} = 1$ for $2^{-1} < |\xi| < 2$ and $\sup \widehat{\mu} \subset \{2^{-2} < |\xi| < 4\}$. Then $\varphi_j = \varphi_j * \mu_j$ and so
\[
\frac{|\varphi_j * \phi_t * f(x-y)|}{(1 + 2^j |y|)^\lambda} \lesssim \varphi_j^* f(x) \int_{\mathbb{R}^n} |\mu_j * \phi_t(z)|(1 + 2^j |z|)^\lambda dz.
\]
If $2^j < \frac{1}{t}$ then (52) follows easily by changing the order of integration. On the other hand if $2^j \geq \frac{1}{t}$, then since $\phi \in S$ we use part (i) of Lemma 2.4 to deduce that
\[
|\phi_t * \mu_j(x)| \lesssim \left( \frac{2^{-j} t}{t} \right)^{n+|\lambda|} \frac{t^{-n}}{(1 + |x|)^{n+|\lambda|+1}} \lesssim \frac{2^{jn}}{(1 + 2^j |x|)^{n+|\lambda|+1}}.
\]
Therefore (52) follows and so we obtain (7), (10), and (11).

\[
\square
\]

6. Appendix

6.1. Proof of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Define $\phi \in S$ by letting $\widehat{\phi}(\xi) = \sum_{j \leq 0} \widehat{\phi}(2^{-j} \xi)$ for $\xi \neq 0$ and $\widehat{\phi}(0) = 1$. Then $\widehat{\phi}(\xi) = 1$ for $|\xi| \leq 1$, $\widehat{\phi}(\xi) = 0$ for $|\xi| > 2$ and moreover for any
N < 0 < M we have the identity
\[ \sum_{j=N+1}^{M} \varphi_j \ast \varphi_j(x) = \phi_M(x) - \phi_N(x). \]  
(53)

Let \( f \in \mathcal{S}' \). Then there exists \( a > 0 \) such that for every \( \rho \in \mathcal{S} \)
\[ |f(\rho)| \lesssim \sum_{|\alpha|,|\beta| \leq a} \|\rho\|_{\alpha,\beta} \]
where \( \|\rho\|_{\alpha,\beta} = \sup_{x} |x^\alpha \partial^\beta \rho(x)| \). In particular, for \( N < 0 \) and any \( \kappa \) we have
\[ |\partial^\kappa (\phi_N \ast f)(x)| \lesssim 2^{N|\kappa|} \sum_{|\alpha|,|\beta| \leq a} \sup_{y \in \mathbb{R}^n} |y^\alpha \partial^\beta (\partial^\kappa \phi)_N(x-y)| \]
\[ \lesssim (1 + |x|)^a 2^{N(|\kappa|+n-a)}. \]

For \( N < 0 \), we define the polynomial \( p_N(x) \) as
\[ p_N(x) = \sum_{|\kappa| \leq a-n} \partial^\kappa (\phi_N \ast f)(0)x^\kappa. \]  
(54)
(if \( a < n \) we can just take \( p_N = 0 \) for every \( N \)). By expanding \( \phi_N \ast f \) as a Taylor series about \( x = 0 \) and using the bound on \( \partial^\kappa (\phi_N \ast f) \) obtained above, we have
\[ |\phi_N \ast f(x) - p_N(x)| \lesssim |x|^{a+1-n} \sum_{|\kappa| = a-n+1} \int_0^1 |\partial^\kappa (\phi_N \ast f)(tx)| \, dt \]
\[ \lesssim (1 + |x|)^{2a+n-1} 2^N. \]

Consequently we see that \( \phi_N \ast f - p_N \to 0 \) in \( \mathcal{S}' \) as \( N \to -\infty \). On the other hand, since \( \int \phi = 1 \), we have \( \phi_M \ast f \to f \) in \( \mathcal{S}' \) as \( M \to \infty \). Therefore, the identity (53) gives
\[ \lim_{N \to -\infty} \left( p_N + \sum_{j=N+1}^{\infty} \varphi_j \ast \varphi_j \ast f \right) = \lim_{M \to \infty} \phi_M \ast f - \lim_{N \to -\infty} (\phi_N \ast f - p_N) = f \]
as required. \( \square \)

A similar argument gives Theorem 2.2

Proof of Theorem 2.2. Let \( \phi \) be as in the proof of Theorem 2.1. As previously, the key point is to study the convergence of \( \phi_N \ast f \) as \( N \to -\infty \). Define the polynomials \( p_N(x) \) as
\[ p_N(x) = \sum_{|\gamma| \leq |\alpha|} \frac{x^\gamma}{\gamma!} \partial^\gamma (\phi_N \ast f)(0). \]
The form of the Taylor series remainder given in [11] implies that for any \( N < N' \)
\[
\left| \left( \phi_N * f - p_N \right)(x) - \left( \phi_{N'} * f - p_{N'} \right)(x) \right|
\leq \sum_{j=N+1}^{N'} \left| \varphi_j * \varphi_j * f(x) - \sum_{|\gamma| \leq [\alpha]} \frac{x^\gamma}{\gamma!} \partial^\gamma (\varphi_j * \varphi_j * f)(0) \right|
\lesssim |x|^{[\alpha]} \sum_{j=N+1}^{N'} \sum_{|\gamma| = [\alpha]} \int_0^1 \left| \partial^\gamma (\varphi_j * \varphi_j * f)(tx) - \partial^\gamma (\varphi_j * \varphi_j * f)(0) \right| dt.
\]

If we now observe that
\[
\left| \partial^\gamma (\varphi_j * \varphi_j * f)(tx) - \partial^\gamma (\varphi_j * \varphi_j * f)(0) \right| \lesssim 2^{j|\gamma|} \sup_{0 < t < 1} \| (\partial^\gamma \varphi)(t2^j x + \cdot) - (\partial^\gamma \varphi)(\cdot) \|_{L^1} \| \varphi_j * f \|_{L^\infty}
\lesssim \| f \|_{B^\infty_{2,\infty}} 2^{j|\gamma| - \alpha} \min \{1, |x|2^j\}
\]
we obtain the inequality
\[
\left| \left( \phi_N * f - p_N \right)(x) - \left( \phi_{N'} * f - p_{N'} \right)(x) \right| \lesssim |x|^{[\alpha]} \sum_{j=N+1}^{N'} 2^{j([\alpha] - \alpha)} \min \{1, 2^j |x|\}. \tag{55}
\]

In particular, we have
\[
\left| \left( \phi_N * f - p_N \right)(x) - \left( \phi_{N'} * f - p_{N'} \right)(x) \right| \lesssim |x|^{[\alpha]+1} \sum_{j \leq N'} 2^{j([\alpha]+1 - \alpha)} \lesssim |x|^{[\alpha]+1}2^{N'([\alpha]+1 - \alpha)}.
\]

Consequently \( \phi_N * f - p_N \) forms a Cauchy sequence in \( S' \) as \( N \to -\infty \) and hence converges to some \( g \in S' \). On the other hand it is easy to check that \( \text{supp} \ \widehat{\phi_N * \hat{f}} \subset \{|\xi| \leq 2^{N+1}\} \) and hence we must have \( \text{supp} \ \widehat{\hat{g}} \subset \{0\} \). Therefore \( g = p \) for some polynomial \( p \) and thus, from (53), we deduce that
\[
\lim_{N \to -\infty} \left( \phi_N + \sum_{N+1}^\infty \varphi_j * \varphi_j * f \right) = f - \lim_{N \to -\infty} \left( \phi_N * f - p_N \right) = f - p
\]
as claimed.

It remains to prove the growth bound. To this end, let \( N < 0 \leq M \) and write
\[
p_N + \sum_{j=N+1}^M \varphi_j * \varphi_j * f = \sum_{j=1}^M \varphi_j * \varphi_j * f + p_N + \sum_{j=N+1}^0 \varphi_j * \varphi_j * f.
\]

To control the first term, we note that the support of \( \varphi \) together with an application of Lemma 24 implies that \( \| \rho * \varphi_j \|_{L^1} \lesssim 2^{-j\beta} \) for every \( \beta > 0 \). Thus choosing \( \beta \) sufficiently large we have
\[
\left| \rho * \left( \sum_{j=1}^M \varphi_j * \varphi_j * f \right)(x) \right| \leq \sum_{j \geq 1} \| \rho * \varphi_j \|_{L^1} \| \varphi_j * f \|_{L^\infty} \lesssim \sum_{j \geq 1} 2^{-j(\beta+\alpha)} \lesssim 1.
\]
On the other hand, for the second sum we note that if $\alpha < 0$ then $p_N = 0$ and we can simply write
\[
\sum_{j=N+1}^{0} |\varphi_j * \varphi_j * f(x)| \leq \sum_{j=0}^{0} 2^{-j\alpha} \|f\|_{B\alpha_{\infty,\infty}} \lesssim 1
\]
which gives the required estimate in the case $\alpha < 0$. If $\alpha \geq 0$, we apply (53) followed by (55) to deduce that
\[
|p_N(x) + \sum_{j=N+1}^{0} \varphi_j * \varphi_j * f(x)| \\
\lesssim (1 + |x|)^{\alpha} + |x|^{\alpha+1} \sum_{j=0}^{0} 2^{j(\alpha+1-\alpha)} + |x|^{\alpha} \sum_{j=0}^{0} 2^{j(\alpha-\alpha)} \\
\lesssim 1 + |x|^{\alpha+1} 2^{-((\alpha)+1-\alpha)\log_2(|x|)} + |x|^{\alpha} \begin{cases} \log_2(|x|) & \alpha \in \mathbb{N} \\ 2^{-(\alpha-\alpha)\log_2(|x|)} & \alpha \not\in \mathbb{N} \end{cases}
\]
\[
\lesssim 1 + \begin{cases} |x|^\alpha \log |x| & \alpha \in \mathbb{N} \\ |x|^\alpha & \alpha \not\in \mathbb{N} \end{cases}
\]
As before, taking the convolution with $\rho$ then gives the required estimate. Thus the result follows.

\[\square\]

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