A NUMERICAL METHOD FOR SOLVING NONLINEAR PROBLEMS

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1. Introduction, basic idea and results.

Suppose

\[ A(u) = f, \tag{1.1} \]

\[ A(0) = 0, \tag{1.2} \]

where \( A(u) \) is a nonlinear operator on a Banach space \( X \), which has the G-derivative:

\[ A'(u)w = \lim_{t \to 0} \frac{A(u + tw) - A(u)}{t}, \quad \forall w \in D(A). \tag{1.3} \]

Here \( D(A) \) is the domain of definition of the operator \( A \), which is assumed to be a set, such that if \( u \) and \( w \) belong to it, then for all sufficiently small \( t \in \mathbb{R} \), the element \( u + tw \) belongs to it. The operator \( A'(u) \) is a linear, maybe unbounded, densely defined operator on \( X \). Even when \( A'(u) \) is unbounded, it may be that the following estimate holds:

\[ \|A'(u) - A'(v)\| \to 0 \text{ as } \|u - v\| \to 0. \]

This is usually the case when the unbounded part of \( A'(u) \) is an operator independent of \( u \), see Example 2 in section 2 below.

In this paper we develop an idea from [1] to write (1.1) as an equation:

\[ L(u)u = f, \quad A(u) = L(u)u, \tag{1.4} \]

where \( L(u) \) is a linear operator on \( X \) which depends nonlinearly on \( u(x) \) as on a parameter. We assume that \( L(u) \) has bounded inverse, that is \( L^{-1}(u) \) is a bounded linear operator defined on all of \( X \).

We study an approach described in [1] for solving equation (1.1) written in the form (1.4). This approach differs from the traditional ones (such as Newton’s method and its modifications, and other methods, based on local linearizations of nonlinear operator \( A(u) \) (see [2] for a detailed study of such methods). The difference is in the global nature of equation (1.4) with linear operator \( L(u) \) which depends nonlinearly on \( u \).

In [1] the goal was to construct an analog of Green’s function for nonlinear systems and to suggest numerical schemes, which would be based on the schemes for solving linear equations. The hope is that such schemes may be more efficient and more stable computationally. In [1] an iterative scheme
(1.11)-(1.12) with \( u_0 = f \) was suggested. Analysis of the convergence of this iterative scheme was not
given in [1] but numerical examples in [1] are encouraging. Here we give such an analysis.

In Theorem 1.1 we give sufficient conditions for convergence of an iterative process (1.11)-(1.12) (see
below) for solving equation (1.1). These conditions are formulated in terms of the operator \( L(u) \) defined
in (1.6) below.

In Theorem 1.2 sufficient conditions are given in terms of the G-derivative of \( A(u) \) for the operator
\( L(u) \) to satisfy the conditions of Theorem 1.1.

In section 2 examples are briefly considered.

For linear equation (1.4) one can use various methods for constructing \( L^{-1}(u) \). One writes (1.4) as

\[
  u = L^{-1}(u)f, \tag{1.5}
\]

assuming that \( L(u) \) is boundedly invertible (sufficient conditions for this are given in Theorem 1.2) and
uses an iterative scheme to solve equation (1.5).

Let us first prove that

\[
  L(u) = \int_0^1 A'(tu) \, dt, \tag{1.6}
\]

and that

\[
  A(u) = \int_0^1 A'(tu) \, dt \, u = L(u)u. \tag{1.7}
\]

The following lemma is an immediate consequence of the definition of the G-derivative and its proof is
included for convenience of the reader.

**Lemma 1.1.** Formula (1.7) holds.

**Proof of Lemma 1.1.** Note that

\[
  \int_0^1 \frac{d}{dt} A(tu) \, dt = A(u) - A(0) = A(u), \tag{1.8}
\]

where the assumption (1.2) was used.

On the other hand,

\[
  \frac{d}{dt} A(tu) = \lim_{s \to 0} \frac{A((t+s)u) - A(tu)}{s} = A'(tu)u, \tag{1.9}
\]

by the definition of the G-derivative. It follows from (1.8), (1.9), and the linearity of \( A'(u) \) as an operator
on \( X \), that

\[
  \int_0^1 A'(tu) \, dt \, u = A(u), \tag{1.10}
\]

as claimed.

Lemma 1.1 is proved. \( \square \)

Now the linear operator \( L(u) \) on \( X \) (depending on \( u \) nonlinearly) is well defined by formula (1.6),
and equation (1.4) is well understood as an equation for \( u \) with linear operator \( L(u) \), which depends
nonlinearly on \( u \).

If \( L(u) \) is boundedly invertible, that is, \( L^{-1}(u) \) is a bounded linear operator defined on all of \( X \), then
(1.4) is equivalent to (1.5). Below we use the boundedness of \( L^{-1}(u) \) not for all \( u \in X \) but only in a ball
\( B_R := \|u\| \leq R \), where \( R > 0 \) is some number which will be specified in the proof of Theorem 1.1.

Let us write an iterative process for solving equation (1.5):

\[
  u_{n+1} = L^{-1}(u_n)f, \tag{1.11}
\]

\[
  u_0 = u_0, \tag{1.12}
\]
where \( u_0 \) is an initial approximation which is arbitrary at this moment. In practice it should be chosen taking into account all the a priori information available about the solution. This may improve the rate of convergence of the scheme (1.11)-(1.12).

Let us now analyze the above iterative process.

We have
\[
\|u_{n+1} - u_n\| = \|L^{-1}(u_n) - L^{-1}(u_{n-1})\| \|f\| \leq q\|u_n - u_{n-1}\| \|f\|. \tag{1.13}
\]

Here we have assumed that:
\[
\|L^{-1}(u) - L^{-1}(v)\| \leq q\|u - v\|, \tag{1.14}
\]
for \( u, v \in B_R \), where \( B_R \) is a ball \( \|u\| \leq R, R > 0 \) is a certain number, and \( q > 0 \) is a number independent of \( u, v \in B_R \).

Assume now that
\[
q\|f\| \leq Q < 1. \tag{1.15}
\]

Then
\[
\|u_1 - u_0\| \leq \|L^{-1}(u_0)f - u_0\|, \\
\|u_2 - u_1\| \leq \|L^{-1}(u_1) - L^{-1}(u_0)\| \|f\| \leq q\|f\| \|u_1 - u_0\| \leq Q\|u_1 - u_0\|, \\
\|u_{n+1} - u_n\| \leq \|L^{-1}(u_n) - L^{-1}(u_{n-1})\| \|f\| \leq Q\|u_n - u_{n-1}\| \\
\leq Q^2\|u_{n-1} - u_{n-2}\| \leq \cdots \leq Q^n\|u_1 - u_0\|.
\]

Thus
\[
\|u_{n+1}\| = \left\| u_0 + \sum_{j=1}^{n+1} (u_j - u_{j-1}) \right\| \leq \|u_0\| + \sum_{j=1}^{n+1} (u_j - u_{j-1}) \frac{1 - Q^{n+1}}{1 - Q} \leq \|u_0\| + \frac{\|u_1 - u_0\|}{1 - Q}. \tag{1.16}
\]

It is sufficient to assume that conditions (1.14) -(1.15) hold in a ball \( B_R \) such that \( B_R \supset S \), where
\[
S := \{ u : \|u\| \leq \|u_0\| + (1 - Q)^{-1}\|u_1 - u_0\| \}. \tag{1.17}
\]

If \( B_R \supset S \), then all the members of the sequence \( u_n \) belong to \( S \), and the sequence \( u_n \) converges to the element
\[
u := u_0 + \sum_{j=1}^{\infty} (u_j - u_{j-1}), \quad u \in S. \tag{1.18}
\]

Passing to the limit in (1.11) and using condition (1.14), which implies the continuity (in the operator norm) of \( L^{-1}(u) \) with respect to \( u \), one gets
\[
u = L^{-1}(f), \tag{1.19}
\]
or
\[
A(u) = L(u)u = f. \tag{1.20}
\]

Let us formulate the result.

**Theorem 1.1.** Assume that \( A(u) \) has the G-derivative \( A'(u) \) which is a linear, possibly unbounded, densely defined operator on \( X \), such that for the operator \( L(u) \), defined in equation (1.6), the conditions (1.14) and (1.15) hold for all \( u \) and \( v \) in \( S \), where \( S \) is defined in (1.17). Then equation (1.1) is uniquely solvable in \( S \) and its solution can be obtained by the iterative process (1.11)-(1.12).

**Remark 1.1.** One could choose \( u_0 = f \) in (1.12). However, in practice there may be some information available which allows one to choose a better initial approximation \( u_0(x) \) to the unknown solution \( u \). Note that our conditions (1.14)-(1.15) are the usual conditions for the applicability of the contraction mapping principle to the equation \( u = L^{-1}(f) \).
One could formulate a condition, which implies (1.15), in terms of the bounds on the derivative of $L^{-1}(u)$ with respect to $u$. Namely $L^{-1}(u) = -L^{-1}(u)L(u)L^{-1}(u)$, where the subscript $u$ denotes the $G$-derivative. Therefore (1.14) holds with $q = \sup_u \|L^{-1}(u)L(u)L^{-1}(u)\|$. We do not go into further detail since conditions (1.14), (1.15) are simple and can be verified in some examples (see for instance example 2 in section 2).

Note that an application of the contraction mapping principle directly to the equation (1.1) is not possible in many practical problems in which the operator $A(u)$ is unbounded. Therefore equation (1.5) is much more suitable for an application of the contraction mapping principle in these problems.

Remark 1.2. The inequality, which implies that $S \subset B_R$, is of the form:

$$\|u_0\| + \frac{\|L^{-1}(u_0)f - u_0\|}{1 - q\|f\|} \leq R. \quad (1.21)$$

Solving it for $q$, one gets the following inequality:

$$0 < q < \|f\|^{-1} \left( 1 - \frac{\|L^{-1}(u_0)f - u_0\|}{R - \|u_0\|} \right). \quad (1.22)$$

For a given $f$ ($f \neq 0$ since otherwise $u = 0$ by the assumption (1.2) and the local injectivity of $A(u)$ near $u = 0$, which follows from the assumed bounded invertibility of $L(0)$), one can fix an arbitrary $u_0$ and then choose $R$, $R > \|u_0\|$, sufficiently large, so that

$$\frac{\|L^{-1}(u_0)f - u_0\|}{R - \|u_0\|} < 1, \quad (1.23)$$

and

$$0 < Q < 1, \quad Q := 1 - \frac{\|L^{-1}(u_0)f - u_0\|}{R - \|u_0\|}. \quad (1.24)$$

Then there exists $q > 0$ which satisfies (1.22) and therefore (1.15) holds. In particular, if $u_0 = 0$, and $R > \|L^{-1}(0)f\|$, then equation (1.1) is solvable in the ball $B_R$ provided that (1.14) holds and

$$0 < q < \|f\|^{-1} \left( 1 - \frac{\|L^{-1}(0)f\|}{R} \right). \quad (1.25)$$

Assumptions (1.14)-(1.15) imply the unique solvability of equation (1.1) in the ball $B_R$. This follows from Theorem 1.1, but we give an independent simple proof of this claim.

Indeed, suppose

$$A(u) = L(u)u = f, \quad A(v) = L(v)v = f, \quad u, v \in B_R. \quad (1.26)$$

Then

$$L(u)u = L(v)v, \quad u = (L^{-1}(u) - L^{-1}(v)) f + v,$$

so, using (1.14) and (1.15), one gets

$$\|u - v\| \leq \|L^{-1}(u) - L^{-1}(v)\| \|f\| \leq Q \|u - v\| < \|u - v\|. \quad (1.27)$$

It follows from (1.27) that $u = v$. This proves uniqueness of the solution to (1.1) in the ball $B_R$. Existence of the solution follows from Theorem 1.1.

Now we give sufficient conditions for the existence of the bounded inverse operator $L^{-1}(u)$. We start with
Lemma 1.2. Assume that:

i) \( \| [A'(0)]^{-1} \| \leq p, \)

ii) \( \sup_{0 \leq t \leq 1} \| A'(tu) - A'(0) \| \leq s, \forall u \in B_R := \{ u : \| u \| \leq R \}, \) where \( R > 0 \) is some number,

iii) \( ps < 1. \)

Then

\( \sup_{0 \leq t \leq 1} \| [A'(tu)]^{-1} \| \leq \frac{p}{1 - ps}, \forall u \in B_R. \) \( (1.28) \)

Proof. First note that in (1.28) existence of the bounded linear operator \( [A'(tu)]^{-1} \) is also claimed. One has

\[ A'(tu) = A'(0) + A'(tu) - A'(0) = A'(0) \left[ I + (A'(0))^{-1} (A'(tu) - A'(0)) \right] := A'(0)(I + B). \]

Therefore

\[ [A'(tu)]^{-1} = (I + B)^{-1} [A'(0)]^{-1}, \] \( (1.29) \)

where both inverse operators on the right-hand side of equation (1.29) exist, and by i) and ii), one has:

\( \| B \| \leq ps < 1. \)

Therefore:

\[ \| (I + B)^{-1} \| \leq \frac{1}{1 - ps}. \] \( (1.30) \)

From (1.29), (1.30) and the assumption i), the desired conclusion (1.28) follows. Lemma 1.2 is proved. \( \square \)

Now we want to prove that, under the assumptions of lemma 1.2, the operator \( L^{-1}(u) \) does exist and is bounded.

Since

\[ L(u) = \int_0^1 A'(tu) \, dt, \]

one has

\[ L(u) = A'(0) \left[ I + (A'(0))^{-1} \int_0^1 (A'(tu) - A'(0)) \, dt \right]. \] \( (1.31) \)

Using i) and ii) of Lemma 1.2, and denoting \( [A'(0)]^{-1} := T, \) one gets

\[ L^{-1}(u) = [I + TM]^{-1} T, \] \( (1.32) \)

where

\[ M := \int_0^1 [A'(tu) - A'(u)] \, dt, \quad \| M \| \leq s, \quad \| TM \| \leq ps \leq 1. \] \( (1.33) \)

Therefore

\[ \| (I + TM)^{-1} \| \leq \frac{1}{1 - ps}. \] \( (1.34) \)

We have proved the following:

**Theorem 1.2.** Assume i), ii) and iii) of Lemma 1.2. Then \( L^{-1}(u) \) exists for all \( u \in B_R \) and

\[ \| L^{-1}(u) \| \leq \frac{p}{1 - ps}. \] \( (1.35) \)
2. Examples.

Example 1.
Let \( X = C(D) \), the usual Banach space of continuous in the domain \( D \) functions, \( K(x, y, u) \) and \( K_u(x, y, u) \) be continuous functions on \( D \times D \times \mathbb{R} \), and

\[
A(u) := u + B(u) := u + \int_D K(x, y, u(y)) \, dy, \quad K(x, y, 0) = 0.
\]

We have

\[
A'(u)w = w(x) + \int_D \frac{\partial K}{\partial u}(x, y, u(y)) \, w(y) \, dy,
\]

and

\[
L(u)w = w(x) + \int_D \int_0^1 \frac{\partial K}{\partial u}(x, y, tu(y)) \, dt \, w(y) \, dy = w(x) + \int_D \frac{K(x, y, u(y))}{u(y)} \, w(y) \, dy.
\]

This \( L(u) \) is a continuous linear operator on \( C(D) \) which depends on \( u(y) \) nonlinearly.

Equation

\[
A(u) = f \tag{2.1}
\]

is equivalent to

\[
L(u)u = f, \tag{2.2}
\]

or, if \( L^{-1}(u) \) is a bounded linear operator, to

\[
u = L^{-1}(u)f. \tag{2.3}
\]

An iterative scheme

\[
u_{n+1} = L^{-1}(u_n)f, \quad \text{or} \quad L(u_n)u_{n+1} = f, \quad u_0 = u_0, \tag{2.4}
\]

can be studied and, under suitable assumptions on \( L(u) \), this scheme converges.

Let us briefly outline a proof of the boundedness of \( L^{-1}(u) \) in this example. Assume that \( K(x, y, u) = k(x, y)g(u) \), where \( k(x, y) \) is a continuous selfadjoint kernel, and \( g(u) \) is a continuous function such that \( g(u)/u > 0 \). Assume that \( u(x) > 0 \) is a continuous function in the closure of \( D \). Then \( L(u) \) is a Fredholm-type operator with index zero, and its null-space is trivial, as one can easily check using the above assumptions. By Fredholm’s alternative, \( L^{-1}(u) \) is bounded. If the selfadjoint kernel \( k(x, y) \) has positive eigenvalues, and if \( g'(u) > m > 0 \), then equation (2.1) has no more than one solution. Indeed, if \( A(u) - A(v) = 0 \), then

\[
(u - v, g(u) - g(v)) + (k[g(u) - g(v)], g(u) - g(v)) = 0
\]

where the parentheses denote the inner product in \( L^2(D) \), \( kw := \int_D k(x, y)w(y) \, dy \), and our assumptions imply that both terms in the above equations are nonnegative. Therefore they both vanish, and this implies that \( u = v \) as claimed.

Example 2.
Let

\[
A(u) := -\Delta u + g(x, u) = f(x), \quad \text{in} \ D, \quad u|_S = 0, \quad g(x, 0) = 0, \tag{2.5}
\]

where \( D \subset \mathbb{R}^n \) is a bounded domain with sufficiently smooth boundary \( S \), \( f(x) \) is a given function, \( g(x, u) \) is continuous with respect to \( x \in D \) and \( C^1 \) with respect to \( u \in \mathbb{R} \). As \( X \) we take \( H = L^2(D) \).

One has

\[
A'(u)w = -\Delta w + g'_u(x, u)w,
\]
\[ L(u)w = -\Delta w + \frac{g(x, u(x))}{u(x)} w. \]  
(2.5)

The operator \( L(u) \) here is unbounded and his unbounded part, \(-\Delta\), does not depend on \( u(x) \). Under suitable assumptions the operator \( L(u) \) is boundedly invertible and one can estimate \( L^{-1}(u) \). For example, suppose \( g(x, u) > 0 \) and \( \frac{g(x, u)}{u(x)} = a + q(x) \), where \( a = \text{const} > 0 \), and \( g(x) \) is a nonnegative function which depends on \( u(x) \). In this case

\[ L^{-1}(u) = (-\Delta + a + q(x))^{-1}, \quad \|L^{-1}(u)\| \leq \frac{1}{a}. \]

Here the operator \(-\Delta + a + q(x)\) is defined in \( L^2(D) \) by the Dirichlet boundary condition, or by its domain \( H^2(D) \cap \dot{H}^1(D) \), where \( H^m(D) \) are the Sobolev spaces.

One can get the following estimate:

\[ \|L^{-1}(u) - L^{-1}(v)\| \leq \|L^{-1}(u)\| \|L(u) - L(v)\| \|L^{-1}(v)\| \leq \frac{1}{a^2} \left\| \frac{g(u)}{u} - \frac{g(v)}{v} \right\| \leq \frac{c(g)}{a^2} \|u - v\|. \]
(2.7)

In (2.7) we have assumed that
\[ c(g) = \max_{u \in \mathbb{R}} \left| \left( \frac{g(u)}{u} \right)' \right| < \infty, \]
(2.8)
and have used the following well-known identity:
\[ B_1^{-1} - B_2^{-1} = -B_1^{-1}(B_1 - B_2)B_2^{-1}, \]
(2.9)
with \( B_1 = L(u), B_2 = L(v) \). Thus, the conditions (1.14)-(1.15), sufficient for the convergence of the iterative process (2.4), hold if \( c(g)||f||a^{-2} < 1 \).

**Example 3.**
Let \( D \subset \mathbb{R}^n \) be a bounded domain with a sufficiently smooth boundary \( S \). Consider the problem

\[ \frac{\partial u}{\partial t} - \nabla \cdot [a(u)\nabla u] = f(x, t) \text{ in } D \times [0, T], \]
(2.10)
\[ u|_{t=0} = u_0(x), \]
(2.11)
\[ u|_S = 0. \]
(2.12)

Write this problem as

\[ A(u) := u(x, t) - \int_0^t B(u) \, d\tau = u_0(x) + \int_0^t f(x, t) \, d\tau \]
(2.13)

with \( u = u(x, t) \in C^1 \left( \dot{H}^2(D), [0, T] \right) \), and

\[ B(u) := \nabla \cdot [a(u)\nabla u]. \]
(2.14)

Assume that
\[ a \in C^2(R), \quad 0 < c \leq a \leq m, \quad |a'|, |a''| \leq m, \]
(2.15)
where \( c \) and \( m \) are some positive constants and the primes denote derivatives with respect to \( u \).
One checks that
\[ A'(u)w = w - \int_0^t B'(u)w \, d\tau, \] (2.16)
where
\[ B'(u)w = \nabla \cdot [a'(u)\nabla w] + a'(u)(\nabla^2 u)w + a''(u)(\nabla u)^2 w + a'(u)\nabla u \cdot \nabla w. \] (2.17)

Denote
\[ \gamma(u) := \int_0^u a(v) \, dv. \] (2.18)
Now one checks that
\[ \int_0^1 B'(\alpha u) \, d\alpha w = \nabla \cdot \left[ \frac{a(u) - a(0)}{u} \nabla w \right] + \left( \frac{a(u)}{u} - \frac{\gamma(u)}{u^2} \right) (\Delta u) w \]
\[ + \left( \frac{a(u)}{u} - \frac{\gamma(u)}{u^2} \right) \nabla u \cdot \nabla w + \left( \frac{a'(u)}{u} - \frac{2a(u)}{u^2} + \frac{2\gamma(u)}{u^3} \right) (\nabla u)^2 w. \] (2.19)

Therefore,
\[ L(u)w = \int_0^1 A'(\alpha u) \, d\alpha w = w - \int_0^t \int_0^1 B'(\alpha u) \, d\alpha \, w \, d\tau \]
\[ = w - \int_0^t \left\{ \nabla \cdot \left[ \frac{a(u) - a(0)}{u} \nabla w \right] + \left( \frac{a(u)}{u} - \frac{\gamma(u)}{u^2} \right) (\Delta u) w \right. \]
\[ + \left( \frac{a(u)}{u} - \frac{\gamma(u)}{u^2} \right) \nabla u \cdot \nabla w + \left( \frac{a'(u)}{u} - \frac{2a(u)}{u^2} + \frac{2\gamma(u)}{u^3} \right) (\nabla u)^2 w \}
\[ d\tau \] (2.20)
and this equation defines a linear unbounded operator \( L(u) \), which depends on \( u = u(x,t) \) nonlinearly
as on a parameter.

It seems at first sight that as \( u \to 0 \) one has difficulties in the definition (2.20), since \( u \) is in the
denominator. However, in fact there are no difficulties: one checks using L'Hospital rule, for example,
that the functions
\[ \frac{a(u)}{u} - \frac{\gamma(u)}{u^2} = \frac{ua(u) - \gamma(u)}{u^2} \to \frac{a'(0)}{2} \] (2.21)
and
\[ \frac{a'(u)u^2 - 2ua(u) + 2\gamma(u)}{u^3} \to \frac{a''(0)}{3} \] (2.22)
are bounded as \( u \to 0 \).

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