ON TURÁN NUMBERS FOR DISCONNECTED HYPERGRAPHS

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(Received September 22, 2022; revised March 26, 2023; accepted March 29, 2023)

Abstract. We introduce the following simpler variant of the Turán problem: Given integers $n > k > r \geq 2$ and $m \geq 1$, what is the smallest integer $t$ for which there exists an $r$-uniform hypergraph with $n$ vertices, $t$ edges and $m$ connected components such that any $k$-subset of the vertex set contains at least one edge? We prove some general estimates for this quantity and for its limit, normalized by $\binom{n}{r}$, as $n \to \infty$. Moreover, we give a complete solution of the problem for the particular case when $k=5$, $r=3$ and $m \geq 2$.

1. Introduction

Given $r \geq 2$, an $r$-uniform hypergraph, or $r$-graph for short, is a pair $H = (V, E)$, where $V = V(H)$ is a finite set of nodes or vertices, and $E = E(H)$ is a set of $r$-subsets of $V$, called edges. In particular, a graph is a 2-graph. Given an $r$-graph $F$, the Turán number $\text{ex}(n, F)$ is the largest integer $t$ such that there exists an $r$-graph on $n$ vertices and $t$ edges that does not contain $F$ as a sub-hypergraph.

The Turán problem consists of determining or estimating $\text{ex}(n, K_k^{(r)})$, where $K_k^{(r)}$ is a complete $r$-graph on $k$ vertices, i.e., the hypergraph consisting of all possible $r$-subsets of $V$. Moreover, one is also interested in estimating the limit

$$
\pi_{r,k} := \lim_{n \to \infty} \frac{\text{ex}(n, K_k^{(r)})}{\binom{n}{r}},
$$

*Corresponding author.
†Raffaella Mulas was supported by the Max Planck Society’s Minerva Grant.
Key words and phrases: Turán number, Turán problem, hypergraph.
Mathematics Subject Classification: 05D05.

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where \(^n_r\) is the number of edges of \(K_n^{(r)}\). Note that the edge density of an \(n\)-vertex \(r\)-graph equals the average edge density of its \((n - 1)\)-vertex induced subgraphs. Hence \(ex(n, K_k^{(r)})/\binom{n}{r}\) is non-increasing, which implies that \(\pi_{r,k}\) is well-defined. This simple averaging argument was first introduced by Katona, Nemetz and Simonovits [14]. The Turán problem was introduced in 1941 by Paul Turán [28], who showed that \(\pi_{2,k} = 1/(k - 1)\). More specifically, let \(T_{n,k}\) be the graph on \(n\) vertices such that the vertex set can be written as \(V = V_1 \sqcup \cdots \sqcup V_k\), each \(V_i\) has size either \(\lceil n/k \rceil\) or \(\lfloor n/k \rfloor\), and two vertices form an edge if and only if they belong to different \(V_i\)’s. Turán showed that

\[
\text{ex}(n, K_k^{(2)}) = |E(T_{n,k} - 1)|
\]

and \(T_{n,k-1}\) is the only \(n\)-vertex, \(K_k^{(2)}\)-free graph that achieves this number of edges. Eighty years later, there are still very few results on the Turán’s problem for \(k > r \geq 3\). Turán conjectured that \(\pi_{3,4} = 5/9\) and \(\pi_{3,5} = 3/4\). There are exponentially many constructions achieving the conjectured densities, see [2,9,11,18]. The best upper bound so far for \(\pi_{3,4}\) is 0.561666 by Razborov [24] using flag algebra method. Some classical surveys are [6,12, 15,25], and other related results are presented, for instance, in [1,7,10,13,16, 17,19–23,26,27]. Paul Erdős, who was a close collaborator of Turán, rarely offered prizes for problems that were posed by others [3], offered 500 dollars for determining \(\pi_{r,k}\) for even one single pair \(k > r > 2\), and he offered 1000 dollars for solving the whole set of problems [8]. After Erdős’ death, his close collaborators Fan Chung and Ron Graham declared that they were willing to offer these prizes, as a way to honor him [3]. At the time of writing, Fan Chung is currently in charge of these rewards.

The Turán problem can also be reformulated in a dual way, as follows. Given an \(r\)-graph \(H = (V, E)\), its complement is the hypergraph \(H^c := (V, E^c)\), where

\[
E^c := \{ f \subseteq V : |f| = r \text{ and } f \not\in E \}.
\]

Clearly,

\[
|E^c| = \binom{n}{r} - |E|,
\]

therefore, given \(t \in \mathbb{N}\),

\[
|E| \leq t \iff |E^c| \geq \binom{n}{r} - t.
\]

Also, \(H\) is \(K_k^{(r)}\)-free if and only if \(H^c\) satisfies the condition

\[(1) \quad \forall v_1, \ldots, v_k \in V \implies \exists f \subseteq \{v_1, \ldots, v_k\} : f \in E^c.\]
Hence, $H$ is an optimal solution for the Turán problem, i.e., it maximizes the number of edges among all $K_k^{(r)}$-free $r$-graphs on $n$ nodes, if and only if $H^c$ minimizes the number of edges among all $r$-graphs on $n$ nodes that satisfy (1). Motivated by this dual (and equivalent) formulation, we say that the dual Turán problem consists of determining or estimating

$$T(n, K_k^{(r)}) := \binom{n}{r} - ex(n, K_k^{(r)}),$$

which is the smallest integer $t$ such that there exists an $r$-graph on $n$ vertices and $t$ edges satisfying (1). Further, we let

$$t_{r,k} := \lim_{n \to \infty} \frac{T(n, K_k^{(r)})}{\binom{n}{r}}.$$

Clearly, we have $t_{r,k} = 1 - \pi_{r,k}$. In this paper, we introduce and study a variant of the Turán number concerning the number of connected components. Namely, we let $T(n, K_k^{(r)}; m)$ be the smallest integer $t$ such that there exists an $r$-graph with $n$ vertices, $t$ edges and $m$ connected components that satisfies (1). We also consider the limit

$$t_{r,k}(m) := \lim_{n \to \infty} \frac{T(n, K_k^{(r)}; m)}{\binom{n}{r}}.$$

The existence of this limit can be proved by an argument similar to that of $\pi_{r,k}$. One of our main results is the following theorem, that we shall prove in Section 4.

**Theorem 1.1.** Let $k > r \geq 2$ be integers. If $n \geq k + \binom{k-2}{r-1}$ and

$$T(n, K_k^{(r)}) = T(n, K_k^{(r)}; m),$$

then

$$m \leq \left\lfloor \frac{k - 1}{r - 1} \right\rfloor.$$

In particular, for $k \leq 2r - 2$, Theorem 1.1 implies that the optimal solution of $T(n, K_k^{(r)})$ must be connected when $n$ is large enough.

Note that in the case of graphs, the optimal solutions of $T(n, K_k^{(2)})$ always have $k - 1$ connected components. That is, $T(n, K_k^{(2)}; k - 1) < T(n, K_k^{(2)}; m)$ for any $m < k - 1$. It is natural to ask whether this phenomenon extends to $r$-graphs, i.e., if the optimal solutions of $T(n, K_k^{(r)})$ always have
[(k - 1)/(r - 1)] connected components. We show that the answer to this
question is “no” for r-graphs when k - 1 is a multiple of r - 1:

**Theorem 1.2.** For integers r ≥ 3, k ≥ 1 and n ≥ (r - 1)k + 1 + (r - 1)k - 1
such that k | n, there exists an integer m < k such that

\[ T(n, K^{(r)}_{(r-1)k+1}; k) \geq T(n, K^{(r)}_{(r-1)k+1}; m). \]

Moreover, we determine \( t_{3,2m+1}(m) \) and \( t_{3,2m+2}(m) \) in terms of \( t_{3,4} \):

**Theorem 1.3.** For integers m ≥ 1,

\[ t_{3,2m+1}(m) = \frac{1}{m^2}, \quad \text{and} \quad t_{3,2m+2}(m) = (m - 1 + t_{3,4}^{-\frac{1}{2}})^{-2}. \]

We point out that it is unclear whether a similar result also holds for
r ≥ 4. For example, we do not know if \( t_{4,7}(2) = 1/8 \). This is because the current best lower bound for \( t_{4,6} \), to the best of our knowledge, is 1/10, which is smaller than 1/8.

**Structure of the paper.** In Section 2 we give further definitions and we
prove some general results that will be needed throughout the paper. In
Section 3 we give a complete solution of the problem for the case when k = 5,
r = 3 and m ≥ 2. In Section 4 we prove Theorem 1.1, and in Section 5 we
prove Theorem 1.2 and Theorem 1.3. Finally, in Section 6 we propose some
open questions.

**2. Basic definitions and general results**

In the introduction we gave several definitions regarding hypergraphs
and the Turán problem. In this section, we give further definitions and we
prove some general results that hold for any k > r ≥ 3.

**Definition 2.1.** Given a hypergraph \( H = (V, E) \) and a vertex \( v \in V \),
we let \( H \setminus \{v\} := (V \setminus \{v\}, E \setminus \{v\}) \), where \( E \setminus \{v\} := \{e \in E : v \notin e\} \).

**Definition 2.2.** Given an r-graph \( H = (V, E) \), an r-subset of V is a
non-edge of H if it does not belong to E, or equivalently, if it belongs to \( E^c \).

Given an hypergraph H, we let \( C(H) \) denote the number of its connected
components.

**Definition 2.3.** Let \( H = (V, E) \) be an hypergraph. An independent set
of H is a subset of V which does not contain any edge of H. The independence number of H, denoted \( \alpha(H) \), is the maximum size of an independent set.

The independence sequence of H, denoted \( S(H) \), is the multiset of size
\( C(H) \) whose members are the independence numbers of each component.
Given a multiset $S$ whose elements are positive integers, we let $|S|$ denote the number of entries in $S$, and we let $\|S\|$ denote the sum of all entries of $S$. That is, if $S = \{s_1, s_2, \ldots, s_t\}$, then $|S| := t$ and $\|S\| := s_1 + s_2 + \cdots + s_t$. Moreover, given two multisets $S$ and $S'$, we let $S \cup S'$ denote the union of $S$ and $S'$. For instance, $\{3, 1\} \cup \{2, 2, 1\} = \{3, 2, 2, 1, 1\}$. Given a positive integer $m$, we let $m \cdot S$ denote the multiset union of $m$ copies of $S$. For example, $3 \cdot \{2, 1\} = \{2, 2, 2, 1, 1, 1\}$.

We also let $\tilde{T}(n, r; S)$ be the smallest number of edges in an $n$-vertex $r$-graph $H$ such that $S(H) = S$, and we let

$$\tilde{t}_r(S) := \lim_{n \to \infty} \frac{\tilde{T}(n, r; S)}{(\binom{n}{r})}.$$ 

The existence of this limit can be proved by a simple averaging argument similar to that for $\pi_{r,k}$. When $S = \{s\}$, we write $\tilde{T}(n, r; s) = \tilde{T}(n, r; \{s\})$ and $\tilde{t}_r(s) = \tilde{t}_r(\{s\})$ for short. One can then check that

$$t_{r,k}(m) = \min_{S: |S| = m, \|S\| = k-1} \tilde{t}_r(S).$$

The following lemmas will be needed throughout the paper.

**Lemma 2.4.** Let $S^{(1)}$, $S^{(2)}$ and $S$ be multisets. If

$$\tilde{t}_r(S^{(1)}) \geq \tilde{t}_r(S^{(2)}),$$

then

$$\tilde{t}_r(S^{(1)} \cup S) \geq \tilde{t}_r(S^{(2)} \cup S).$$

**Proof.** It suffices to show that $\tilde{T}(n, r; S^{(1)} \cup S) \geq \tilde{T}(n, r; S^{(2)} \cup S) + o(n^r)$. For any optimal solution $H$ of $\tilde{T}(n, r; S^{(1)} \cup S)$, let $H_1$ be the induced subgraph of $H$ such that $S(H_1) = S^{(1)}$. Since

$$\tilde{T}(n, r; S^{(1)}) \geq \tilde{T}(n, r; S^{(2)}) + o(n^r),$$

there is an $r$-graph $H_2$ on the same vertex set as $H_1$ such that $S(H_2) = S^{(2)}$ and $e(H_1) \geq e(H_2) + o(n^r)$. Let $H'$ be the $r$-graph obtained from $H$ by replacing $H_1$ with $H_2$. Then, we have $S(H') = S^{(2)} \cup S$ and $e(H') \leq e(H) + o(n^r)$. Therefore, $\tilde{T}(n, r; S^{(1)} \cup S) \geq \tilde{T}(n, r; S^{(2)} \cup S) + o(n^r)$. \square

**Lemma 2.5.** For integers $b \geq a > r > 1$,

$$\binom{a}{r} + \binom{b}{r} < \binom{a-1}{r} + \binom{b+1}{r}.$$
Proof. We use the fact that
\[
\binom{a}{r} = \binom{a-1}{r} + \binom{a-1}{r-1}
\]
and, similarly,
\[
\binom{b+1}{r} = \binom{b}{r} + \binom{b}{r-1}.
\]
This implies that
\[
\binom{a}{r} + \binom{b}{r} < \binom{a-1}{r} + \binom{b+1}{r} \iff \binom{a-1}{r-1} < \binom{b}{r-1},
\]
which is true since \(b > a - 1\). □

Lemma 2.6. For \(i \in \{1, \ldots, m\}\), let \(t_i > 0\), and \(p_i \in (0,1]\) be real numbers such that \(\sum_{i=1}^{m} p_i = 1\). Then, for any integer \(r \geq 2\),
\[
(2) \quad \sum_{i=1}^{m} t_i \cdot p_i^r \geq \left( \sum_{i=1}^{m} t_i^{\frac{1}{r-1}} \right)^{-r+1}.
\]
Equality holds if and only if \(p_i = t_i^{\frac{1}{r-1}} / \sum_{j=1}^{m} t_j^{\frac{1}{r-1}}\).

Proof. Let \(c_i := t_i^{\frac{1}{r-1}}\), and let \(C := \sum_{i=1}^{m} c_i\). Then,
\[
(*) := \sum_{i=1}^{m} t_i \cdot p_i^r = \sum_{i=1}^{m} c_i \cdot \left( \frac{p_i}{c_i} \right)^r = C \sum_{i=1}^{m} \frac{c_i}{C} \cdot \left( \frac{p_i}{c_i} \right)^r.
\]
Hence, by Jensen’s inequality, we have
\[
(*) \geq C \left( \sum_{i=1}^{m} \frac{c_i}{C} \cdot \frac{p_i}{c_i} \right)^r = C^{-r+1},
\]
and equality holds if and only if
\[
\frac{p_1}{c_1} = \frac{p_2}{c_2} = \cdots = \frac{p_m}{c_m}.
\]
Together with \(\sum_{i=1}^{m} p_i = 1\), this implies that \(p_i = c_i/C\) for all \(i \in \{1, \ldots, m\}\). □

The next lemma establishes an equation for \(\hat{t}_r(S)\).
**Lemma 2.7.** Let \( r \geq 3 \) be an integer, and let \( S = \{s_1, s_2, \ldots, s_m\} \) be a multiset such that \( s_i \geq r - 1 \) for all \( i \in \{1, \ldots, m\} \). Then,

\[
\tilde{t}_r(S) = \left( \sum_{i=1}^{m} \tilde{t}_r(s_i)^{-\frac{1}{r-1}} \right)^{-r+1}.
\]

**Proof.** Let \( G_n \) be an \( r \)-graph on \( n \) vertices whose independence sequence is \( S \) such that \( e(G_n) = \tilde{\mathcal{T}}(n, r; S) \), and let \( n_i \) be the number of vertices in the component of \( G_n \) corresponding to \( s_i \), for \( 1 \leq i \leq m \). Then, by definition,

\[
e(G_n) = \frac{\sum_{i=1}^{m} \tilde{\mathcal{T}}(n_i, r; s_i)}{\binom{n}{r}} = \sum_{i=1}^{m} \frac{n_i}{\binom{n}{r}} \tilde{\mathcal{T}}(n_i, r; s_i).
\]

It is not hard to check that

\[
\liminf_{n \to \infty} \left( \frac{n_i}{\binom{n}{r}} \tilde{\mathcal{T}}(n_i, r; s_i) \right) = \liminf_{n \to \infty} \left( \frac{n_i}{n} \right)^r \tilde{t}_r(s_i);
\]

when \( \liminf(n_i/n) > 0 \), this is clearly true; and when \( \liminf(n_i/n) = 0 \), both sides are zero. Hence, by Lemma 2.6,

\[
\tilde{t}_r(S) = \liminf_{n \to \infty} \frac{e(G_n)}{\binom{n}{r}} \geq \liminf_{n \to \infty} \sum_{i=1}^{m} \left( \frac{n_i}{n} \right) \tilde{t}_r(s_i) \geq \left( \sum_{i=1}^{m} \tilde{t}_r(s_i)^{-\frac{1}{r-1}} \right)^{-r+1}.
\]

On the other hand, we can construct a sequence of \( r \)-graphs whose density converges to \( \left( \sum_{i=1}^{m} \tilde{t}_r(s_i)^{-\frac{1}{r-1}} \right)^{-r+1} \) by taking the union of the optimal solutions of \( \tilde{\mathcal{T}}([np_i], r; s_i) \), for \( 1 \leq i \leq m \), where

\[
p_i = \frac{\tilde{t}_r(s_i)^{-\frac{1}{r-1}}}{\sum_{j=1}^{m} \tilde{t}_r(s_j)^{-\frac{1}{r-1}}}.
\]

We can therefore conclude that \( \tilde{t}_r(S) = \left( \sum_{i=1}^{m} \tilde{t}_r(s_i)^{-\frac{1}{r-1}} \right)^{-r+1} \). \( \square \)

**3. Solution for \( r = 3, k = 5 \) and \( m \geq 2 \)**

In this section we solve the problem for the case when \( r = 3, k = 5 \), and \( m \geq 2 \). This will serve as a motivation for our next results.

**Lemma 3.1.** Let \( H = (V, E) \) be a \( K_5^{(3)} \)-free 3-graph on \( n \geq 6 \) nodes. Then, \( H^c \) has at most three connected components. Moreover,

- If \( H^c \) has three connected components, then \( H^c \) is given by two isolated vertices together with a complete 3-graph on \( n - 2 \) nodes.

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• If $H^c$ has two connected components and it has no isolated vertices, then each connected component of $H^c$ is a complete 3-graph.

• If $H^c$ has two connected components and one of them is an isolated vertex, then the other one is $K_4^{(3)}$-free in $H$.

**Proof.** We first assume, for the sake of a contradiction, that $H^c$ has at least four connected components $V_1, \ldots, V_4$. Let $v_1 \in V_1, \ldots, v_4 \in V_4$, and let $v_5 \in V$ belong to any of the connected components. Since $H$ is $K_5^{(3)}$-free, $H^c$ satisfies (1) and therefore there exists $f \subseteq \{v_1, \ldots, v_5\}$ such that $f \in E^c$. Since $|f| = 3$ and since $V_1, \ldots, V_4$ are not joined by any edge, this gives a contradiction. Hence, $H^c$ has at most three connected components.

If $H^c$ has three connected components, then clearly one component has at least one edge and therefore at least 3 vertices.

Assume, for the sake of a contradiction, that $H^c$ has three connected components $V_1, V_2, V_3$ and at least two of them, say $V_1$ and $V_2$, have cardinality larger than 1. Then, we can pick $v_1, v_2 \in V_1, v_3, v_4 \in V_2$ and $v_5 \in V_3$. By (1), there exists $f' \subseteq \{v_1, \ldots, v_5\}$ such that $f' \in E^c$. Since $f'$ has cardinality 3, this gives a contradiction. Therefore, if $H^c$ has three connected component, then two of them are given by isolated vertices. Now, assume that $H^c$ has three connected components $V_1, V_2, V_3$, where $V_2$ and $V_3$ are given by isolated vertices. Given $v_1, v_2, v_3 \in V_1, v_4 \in V_2, v_5 \in V_3$, (1) implies that $\{v_1, v_2, v_3\} \in E^c$. Since this holds for all $v_1, v_2, v_3 \in V_1, V_1$ is a complete 3-graph on $n-2$ nodes.

Now, assume that $H^c$ has two connected components $V_1, V_2$, and it has no isolated vertices. Since edges have cardinality 3, there cannot be connected components of cardinality 2, therefore $|V_i| \geq 3$ for $i = 1, 2$. Let $v_1, v_2, v_3 \in V_1$, and $v_4, v_5 \in V_2$. By (1), there exists $f'' \subseteq \{v_1, \ldots, v_5\}$ such that $f'' \in E^c$. Since there are no edges between $V_1$ and $V_2$, this implies that $f = \{v_1, v_2, v_3\}$. This is true for all $v_1, v_2, v_3 \in V_1$, and the same reasoning can be applied to $V_2$. Hence, both $V_1$ and $V_2$ are complete 3-graphs.

Finally, assume that $H^c$ has two connected components and one of them is an isolated vertex $v_1$. Let $v_2, \ldots, v_5 \in V \setminus \{v_1\}$. If $v_2, \ldots, v_5$ form a complete 3-graph on 4 vertices in $H$, then $v_1, \ldots, v_5$ form a complete 3-graph on 5 vertices in $H$, which leads to a contradiction. Hence, $V \setminus \{v_1\}$ is $K_4^{(3)}$–free in $H$. □

Lemma 3.1 allows us to prove the following

**Theorem 3.2.** Let $H = (V, E)$ be a $K_5^{(3)}$-free 3-graph on $n \geq 6$ nodes such that $H^c$ has more than one connected component.

• If $n = 2m$, then

$$|E^c| \geq 2 \cdot \binom{m}{3},$$

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or equivalently

\[ |E| \leq m^2(m - 1), \]

with equality if and only if \( H^c \) is the disjoint union of two copies of \( K_m^{(3)} \).

- If \( n = 2m + 1 \), then

\[ |E^c| \geq \binom{m}{3} + \binom{m + 1}{3}, \]

or equivalently

\[ |E| \leq m^3 + \frac{m^2}{2} - \frac{m}{2}, \]

with equality if and only if \( H^c \) is the disjoint union of \( K_m^{(3)} \) and \( K_{m+1}^{(3)} \).

**Proof.** We consider three cases.

1. **Case 1**: \( H^c \) has more than two connected components. By Lemma 3.1, \( H^c \) is given by two isolated vertices together with a complete 3-graph on \( n - 2 \) nodes. Therefore,

\[ |E^c| = \binom{n - 2}{3}. \]

It is easy to check that, for \( n = 2m \),

\[ |E^c| = \binom{n - 2}{3} > 2 \cdot \binom{m}{3} \]

and, for \( n = 2m + 1 \),

\[ |E^c| = \binom{n - 2}{3} > \binom{m}{3} + \binom{m + 1}{3}. \]

2. **Case 2**: \( H^c \) has two connected components and no isolated vertices. By Lemma 3.1, each connected component of \( H^c \) is a complete 3-graph. Hence, if they have cardinality \( c_1 \) and \( c_2 \), respectively, then

\[ |E^c| = \binom{c_1}{3} + \binom{c_2}{3}. \]

By Lemma 2.5, for \( n = 2m \) this is minimized precisely by \( c_1 = c_2 = m \), while for \( n = 2m + 1 \) this is minimized by \( c_1 = m \) and \( c_2 = m + 1 \).
Case 3: $H^c$ has two connected components and one of them is an isolated vertex $\hat{v}$. By Lemma 3.1, $H \setminus \{\hat{v}\}$ is $K^{(3)}_4$-free. Therefore, using the bound $\text{ex}(n-1, K^{(3)}_4) \leq (0.63) \cdot \left(\frac{n-1}{3}\right)$ in [4,6],
\[
|E| = \text{deg} \hat{v} + |E(H \setminus \{\hat{v}\})| \\
\leq \left(\frac{n-1}{2}\right) + \text{ex}(n-1, K^{(3)}_4) < \left(\frac{n-1}{2}\right) + (0.63) \cdot \left(\frac{n-1}{3}\right).
\]
If $n = 2m$, then one can check that
\[
|E| < \left(\frac{2m-1}{2}\right) + (0.63) \cdot \left(\frac{2m-1}{3}\right) < m^2(m-1).
\]
Similarly, if $n = 2m + 1$, then
\[
|E| < \left(\frac{2m}{2}\right) + (0.63) \cdot \left(\frac{2m}{3}\right) < m^3 + \frac{m^2}{2} - \frac{m}{2}.
\]
This proves the claim. □

Note that the last result implies that $3/4 = t_{3.5}(2) \leq t_{3.5}(m)$ for $m \geq 2$. Moreover, Turán conjectured that the disjoint unions of two complete $r$-graphs are optimal for $r = 3$ and $k = 5$ (in the general case when $m \geq 1$). However, as discussed in [25], counterexamples are known for any odd $n \geq 9$. For $n$ even, no counterexample was found yet, and the conjecture has proved to be true up to $n = 12$.

4. Number of connected components

In this section we prove Theorem 1.1. As a preliminary result, we first prove a theorem showing that the optimal solutions of $T(n, K^{(r)}_k)$ have no isolated vertices, if $n$ is big enough.

**Lemma 4.1.** For integers $k > r \geq 2$ and $n \geq k + \left(\frac{k-2}{r-1}\right)$, let $H = (V, E)$ be a $K^{(r)}_k$-free $r$-graph with $n$ vertices and $\text{ex}(n, K^{(r)}_k)$ edges. Then, $H^c$ has no isolated vertices.

**Proof.** Suppose, for the sake of a contradiction, that $H^c$ contains an isolated vertex $u$, and let $H' := H \setminus \{u\}$. Then, $H'$ is a $K^{(r)}_{k-1}$-free $r$-graph on $n-1$ vertices. Moreover, since $H$ is an optimal solution for the Turán problem, if any edge is added to $H$, then $H$ is not $K^{(r)}_k$-free anymore. Therefore, if any edge is added to $H'$, then $H'$ is not $K^{(r)}_{k-1}$-free anymore. Hence, given $f = \{v_1, \ldots, v_r\} \in E(H'^c)$, there exist $v_{r+1}, \ldots, v_{k-1} \in V \setminus \{u\}$ such
that there are no other edges among \( v_1, \ldots, v_{k-1} \) in \( H^c \) other than \( f \). Therefore, in particular, there are no edges in \( H^c \) among \( v_2, \ldots, v_{k-1} \). Let now \( w_1, \ldots, w_l \in V \setminus \{u, v_2, \ldots, v_{k-1}\} \), for some \( l \geq 1 \). Then, for each \( i \in \{1, \ldots, l\} \), there exists \( e_i \in E(H^c) \) such that \( e_i \subset \{w_i, v_2, \ldots, v_{k-1}\} \) and \( w_i \in e_i \).

If \( l > \binom{k-2}{r-1} \), which is possible since \( n \geq k + \binom{k-2}{r-1} \), there exist \( i,j \in \{1, \ldots, l\} \) such that \( i \neq j \) and \( e_i \setminus \{w_i\} = e_j \setminus \{w_j\} =: U \).

Let now \( H'' \) be the \( r \)-graph obtained from \( H \) by deleting \( U \cup \{u\} \) and adding \( U \cup \{w_i\} \) and \( U \cup \{w_j\} \) as edges. Then, \( H'' \) is \( K_k^{(r)} \)-free. In fact,

- Any \( k \)-set not containing \( U \) is not \( K_k^{(r)} \) since \( H \) is \( K_k^{(r)} \)-free;
- Any \( k \)-set containing \( U \cup \{u\} \) contains the non-edge \( U \cup \{u\} \);
- For any \( k \)-set \( W \) containing \( U \) but not \( \{u\} \), fix a vertex \( v \in U \). Note that \( W \setminus \{v\} \) is a \( (k-1) \)-set. This implies that there exists a non-edge of \( H' \) contained in \( W \setminus \{v\} \), which is neither \( U \cup \{w_i\} \) nor \( U \cup \{w_j\} \), and which is also a non-edge of \( H'' \).

Therefore, \( H'' \) is an \( K_k^{(r)} \)-free \( r \)-graph with \( n \) nodes and more edges than \( H \), which is a contradiction. \( \square \)

**Proof of Theorem 1.1.** Let \( H \) be an optimal solution of \( T(n, K_k^{(r)}) \).

By Lemma 4.1, \( H \) has no isolated vertices. Hence, the independence number of each connected component of \( H \) is at least \( r - 1 \). Suppose, for the sake of a contradiction, that \( m > \lfloor (k - 1)/(r - 1) \rfloor \). Then \( \alpha(H) \geq (r - 1)m > k - 1 \), which contradicts to \( \alpha(H) \leq k - 1 \). \( \square \)

### 5. Turán numbers for disconnected 3-graphs

This section is dedicated to the proofs of Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.2.** The proof is divided into two cases.

**Case 1:** The optimal solution of \( T(n, K_{(r-1)k+1}^{(r)}; k) \) has an isolated vertex. In this case, by Lemma 4.1, we have

\[
T(n, K_{(r-1)k+1}^{(r)}; k) > T(n, K_{(r-1)k+1}^{(r)}; k - 1).
\]

**Case 2:** The optimal solution of \( T(n, K_{(r-1)k+1}^{(r)}; k) \) does not contain isolated vertices. In this case, the \( k \) components are all complete \( r \)-graphs. Hence, by Lemma 2.5, these components all have size \( n/k \), implying that

\[
T(n, K_{(r-1)k+1}^{(r)}; k) = k \left( \frac{n}{r} \right) \cdot \frac{n}{k}.
\]

But we also know that \( T(n, K_{(r-1)k+1}^{(r)}; 1) \leq k \left( \frac{n}{r} \right), \) by the construction of Sidorenko [25, Construction 4 in Section 4.2]. \( \square \)
For the proof of Theorem 1.3, we need the following key lemma.

**Lemma 5.1.** For any integer \( l \geq 2 \),
\[
\tilde{t}_3(\{l + 1\} \cup (l - 1) \cdot \{1\}) > \tilde{t}_3(l \cdot \{2\}),
\]
\[
\tilde{t}_3(\{l + 2\} \cup (l - 1) \cdot \{1\}) > \tilde{t}_3(\{3\} \cup (l - 1) \cdot \{2\}).
\]

**Proof.** Due to de Caen [5], we have
\[
t_{r,l+1} \geq \frac{1}{(r-1)}.\]

On the other hand, the construction of Sidorenko [25, Construction 4 in Section 4.2] gives
\[
\tilde{t}_r(l) \leq \frac{(r - 1)^2}{l^2}.
\]
Hence for any integer \( l \geq 2 \),
\[
\frac{1}{(l^2)} \leq t_{3,l+1} \leq \tilde{t}_3(l) \leq \frac{4}{l^2}.
\]

Therefore, by definition and inequality (3)
\[
\tilde{t}_3(\{l + 1\} \cup (l - 1) \cdot \{1\}) = \tilde{t}_3(l + 1) \geq \frac{1}{\binom{l+1}{2}},
\]
\[
\tilde{t}_3(\{l + 2\} \cup (l - 1) \cdot \{1\}) = \tilde{t}_3(l + 2) \geq \frac{1}{\binom{l+2}{2}}.
\]

On the other hand, by Lemma 2.7 and inequality (3)
\[
\tilde{t}_3(l \cdot \{2\}) = l^{-2} < \frac{1}{\binom{l+1}{2}},
\]
\[
\tilde{t}_3(\{3\} \cup (l - 1) \cdot \{2\}) = (l - 1 + (\tilde{t}_3(3))^{-\frac{1}{2}})^{-2} \leq \left( \frac{2l+1}{2} \right)^{-2} < \frac{1}{\binom{l+2}{2}}. \quad \square
\]

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** For any multiset \( S \) such that \(|S| = m\) and \( \|S\| = 2m \), if there exists an element of \( S \) which is larger than 2, then there exists an integer \( l > 2 \) such that \( \{l\} \cup (l - 2) \cdot \{1\} \) is a subset of \( S \). Let \( S' \) be a multiset obtained from \( S \) by replacing \( \{l\} \cup (l - 2) \cdot \{1\} \) by \( (l - 1) \cdot \{2\} \). Then, by Lemma 2.4 and Lemma 5.1, \( \tilde{t}_3(S') < \tilde{t}_3(S) \). Also, note that the number of
2’s in $S$ is smaller than that in $S'$. Hence, by repeating the above argument finitely many times and Lemma 2.7, we obtain $\tilde{t}_3(S) \geq \tilde{t}_3(m \cdot \{2\}) = 1/m^2$. Therefore,

$$t_{3,2m+1}(m) = \min_{S:|S|=m, |S|=2m} \tilde{t}_3(S) = \tilde{t}_3(m \cdot \{2\}) = \frac{1}{m^2}.$$ 

Similarly, we have

$$t_{3,2m+2}(m) = \min_{S:|S|=m, |S|=2m+1} \tilde{t}_3(S)$$

$$= \tilde{t}_3(\{3\} \cup (m-1) \cdot \{2\}) = (m-1 + \tilde{t}_3(3)^{-\frac{1}{2}})^{-2}.$$ 

By Theorem 1.1, the optimal solution of $T(n,K_4^{(3)})$ must be connected when $n$ is large enough. This implies $t_{3,4} = \tilde{t}_3(3)$. Therefore,

$$t_{3,2m+2}(m) = (m - 1 + t_{3,4}^{-\frac{1}{2}})^{-2}. \quad \Box$$

6. Open questions

We conclude by formulating some open questions.

**Question 1.** Can we improve the bound $n \geq k + \binom{k-2}{r-1}$ in Theorem 1.1?

**Question 2.** For $m < k$, can we explicitly express $t_{3,2k+1}(m)$ or $t_{3,2k+2}(m)$ in terms of the $t_{3,l}$’s?

**Question 3.** Can we prove results similar to Theorem 1.3 for $r \geq 4$?

**Acknowledgement.** The authors are grateful to the anonymous referees for the comments and suggestions that have greatly improved the first version of this paper.

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