Emergence of scale-free properties in Hebbian networks

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The fundamental ‘plasticity’ of the nervous system (i.e high adaptability at different structural levels) is primarily based on Hebbian learning mechanisms that modify the synaptic connections. The modifications rely on neural activity and assign a special dynamic behavior to the neural networks. Another striking feature of the nervous system is that spike based information transmission, which is supposed to be robust against noise, is noisy in itself: the variance of the spiking of the individual neurons is surprisingly large which may deteriorate the adequate functioning of the Hebbian mechanisms. In this paper we focus on networks in which Hebbian-like adaptation is induced only by external random noise and study spike-timing dependent synaptic plasticity. We show that such ‘HebbNets’ are able to develop a broad range of network structures, including scale-free small-world networks. The development of such network structures may provide an explanation of the role of noise and its interplay with Hebbian plasticity. We also argue that this model can be seen as a unification of the famous Watts-Strogatz and preferential attachment models of small-world nets.

Keywords: small world, Hebbian learning, central nervous system, scale-free network

1. Introduction

In the last few years spike-timing dependent synaptic plasticity (STDP) (see e.g. (Ref. 1) and references therein), which is an extension of the classical Hebbian learning mechanism, has been the subject of intensive research. Recent experiments2,3,4 (for a review, see, e.g. (Ref. 5)) revealed that exact timing and temporal dynamics of the neural activities play a crucial role in forming the neuronal base of plasticity. While it is still an open question, whether the rate of spikes (that is temporal or population averaged spike count) or the exact time pattern of the spikes carries the information, it is broadly accepted in the machine learning literature6,7,8 and is strongly supported in neuronal modelling9 that spike based encoding can be efficient in compression, allows for sparse representation, low energy consumption and that it can be robust against noise. The last property seems to be indispensable knowing the stochastic behavior of the neurons and of the external environment. But if noise should be suppressed, how come that a great part of the signals propagating through several brain regions experienced in different species (ranging from frogs to primates) is con-

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sidered to be internally generated noise. What can be the reason for counteracting the perfect information processing and transmission? One possible role of noise in the nervous system is provided by the recognition that noise can enhance the response of nonlinear systems to weak signals, via a mechanism known as stochastic resonance (see, e.g., (Ref. 12)). However, noisy functioning may have additional roles. For example, it has been shown that synaptic background activity may promote distinguishing very similar inputs. It has been also demonstrated that strict conditions on stability of Hebbian mechanisms can be released by introducing random external noise instead of maintaining competition among neurons over the input sets. In this paper we address the question whether noise may have any impact on structural changes.

In the following, we examine what network structures may emerge in a simplistic neural system by applying pure Hebbian learning. From now on, this neuronal network model will be referred as to HebbNet.

2. Description of HebbNet

We assume that the network is sustained by inputs with no spatio-temporal structure; that is the input is random noise. Our models consist of $N$ number of simplified integrate-and-fire-like ‘neurons’ or nodes. The dynamics of the internal activity is written as

$$\frac{\Delta a_i}{\Delta t} = \sum_j w_{ij} a_j^s + x_i^{(ext)},$$

(1)

for $i = 1, 2, \ldots, N$. ($N$ was 200 in our simulations.) Variable $x_i^{(ext)} \in (0, 1)^N$ denotes the randomly generated input from the environment, $a_i$ is the internal activity of neuron $i$, $w_{ij}$ is the $ij$th element of matrix $W$, i.e., the connection strength from neuron $j$ to neuron $i$. If $\Delta t = 1$ then we have a discrete-time network and each parameter has a time index, or if $\Delta t$ is infinitesimally small then Eq. 1 becomes a set of coupled differential equations. Neuron $j$ outputs a spike (neuron $j$ ‘fires’) when $a_j$ exceeds a certain level, the threshold parameter $\theta$. Spiking means that the output of the neuron $a_j^s$ (superscript $s$ stands for ‘spiking’) is set to 1. After firing, $a_j$ is set to zero at the next time step for the discrete-time network. For the continuous version of Eq. 1, $a_j$ is set to zero after a very small time interval. Amount of excitation received by neuron $i$ from neuron $j$ is $w_{ij} a_j^s$. Equation 1 describes the simplest form of ‘integrate–and–fire’ network models which is still plausible from a neurobiological point of view. Note that if $\Delta t = 1$ and the threshold is set to zero (i.e., if a neuron receives any excitation then it fires and is reset to zero) then Eq. 1 represents ‘binary neurons’ without temporal integration. This can be seen as the simplest model within our framework. Also, if the threshold is kept and if $a_i$ is set to zero before each time step, irrespective of the $i$th neuron fires or not, then the original model of McCulloh and Pitts is recovered.

Beyond the local activity threshold, we also examined the effect of global activity constraint: at each time instant, a given percent of nodes was selected randomly in proportion to the activity $a_i$ for all $i = 1, 2, \ldots, N$. These neurons fired at that time instant. For these two cases, computer simulations showed negligible differences.

Synaptic strengths are modified as follows:

$$\frac{\Delta w_{ij}}{\Delta t} = \sum_{(t_i, t_j)} K(t_j - t_i) a_i^{t_i, s} a_j^{t_j, s},$$

(2)

where $K$ is a kernel function which defines the influence of the temporal activity correlation on synaptic efficacy, $t_i, t_j$ the spiking times of neuron $i$ and $j$, respectively and $a_i^{t_i, s}$ is the firing activity of neuron $i$ at time $t_i$. $\Delta w_{ij}/\Delta t$ may be taken over discrete or over infinitesimally small time intervals. Possible kernels are depicted in Fig. 1. The kernel is a function of the time differences. Because, in our case, the input is noise with no temporal correlation, only the ratio of the positive (strengthening) and the negative (weakening) areas of the kernel function ($r_{A^+/A^-}$) should count. Temporal grouping and reshaping of the kernel would not modify our results as long as the aforementioned ratio is kept constant and the input is pure noise. For this special case, the difference between the two kernel types of Fig. 1 does not have much impact on the temporal evolution of our model network. It should be noted that including inputs with spatiotemporal structure and other known details of synaptic plasticity mechanisms, this kernel shape independence will not hold. Our only constraint on the kernel, namely the constraint that $r_{A^+/A^-} < 1$, is required to constrain weights. This constraint redistributes weight strengths. Empirical data indicate that indeed, there are mechanisms to
redistribute weight strengths; potentiation for weak synapses is favored whereas strong synapses tend to be depressed (see, e.g., (Refs. 14, 16, 17)).

Two temporal kernels as a function of time difference between spiking time of neuron $i$ and $j$ ($t_i - t_j$). Relevant parameter of the shape for noise-sustained systems is the ratio ($r_{A^+/A^-}$) of the areas (sums of positive and negative parts/components) of the kernel, $A^+$ and $A^-$, respectively ($r_{A^+/A^-} = A^+/A^-$).

In the first place, we have been interested in the emerging local and global connectivity structure of $W$. As the network of the connections can be best described by a weighted graph, from now on ‘nodes’ stand for the neurons, while ‘edges’ or ‘directed edges’ denote the connections among them. An insightful way of characterizing graphs has been proposed by Watts and Strogatz. They computed the characteristic path length ($L$), which is the average number of edges on the shortest path in the network. They also computed the clustering coefficient ($C$), which is large if the average local connectivity is large. For more details, see Ref. 18.

In this study, we applied the so called connectivity length measure based on the concept of network efficiency$^{19}$. This measure is more appropriate for weighted networks$^{20}$, equally well applicable for describing global and local properties and offers a unified theoretical background to characterize our system. According to the definition$^{20,21}$, local efficiency between nodes $i$ and $j$ in a weighted network with connectivity matrix $W$ is $\epsilon_{ij} = 1/d_{ij}$, where $d_{ij}$ corresponds to the shortest path length throughout all of the possible paths from neuron $j$ to $i$, where the path length between each connected pair of vertices is the inverse of the weight between them. For graphs with connection strengths of values 0 or 1, $d_{ij}$ corresponds to the shortest distance between nodes $i$ and $j$. The average of these values $E[d_{ij}] = \frac{1}{n(n-1)} \sum_{i \neq j} \epsilon_{ij}$ characterizes the efficiency of the whole network. The local harmonic mean distance for node $i$ is defined as

$$D_h(i) = \frac{n(i)(n(i) - 1)}{\sum_{j,k} \epsilon_{kj}}$$

where $n(i)$ is the number of neurons in subgraph $G^{(i)}$, where subgraph $G^{(i)}$ consists of all nodes $l$ around neuron $i$ with $w_{il} > 0$, $\epsilon_{kj}$ is the inverse of shortest distance between nodes $k$ and $j$ in $G^{(i)}$. $N > n(i)$ arises when weights may become zero. In terms of efficiency, the inverse of this value describes how good the local communication is among the first neighbors of node $i$ with node $i$ removed. That is why this measure can also be regarded as local connectivity length. It is a measure of the fault tolerance of the system. The mean global distance in the network is defined by the following quantity:

$$D_g = \frac{N(N-1)}{\sum_{ij} \epsilon_{ij}}$$

Global distance provides a measure for the size (or the diameter) of the network, which influences the average time of information transfer. That is why, its inverse is used as the (un-normalized) global efficiency. According to the literature$^{20,21}$, local harmonic mean distance measure behaves like $1/C$ (inverse of the clustering coefficient), whereas the global value is a good approximation of $L$ under certain conditions.

Many different networks belong to the same structural family regarded as ‘small-worlds’. Their most characteristic feature is that they are efficient locally and globally, too. While local and global connectedness are useful tools to characterize a network architecture, it is worth investigating the degree distributions of the incoming and outgoing connections as well$^{22}$. They may provide information about the scaling of different properties of the given structure, like the change of the diameter as a function of the number of nodes. One particular subfamily of small-world nets can also be characterized as ‘scale-free’ networks, because their most significant properties scale according to power-law with the connection

**Fig. 1. Kernel functions**

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number distribution. Most scale-free nets are also small-worlds, provided that connection strength is not too sparse and basically no part of the network is isolated.

3. Results

![Log-log plots for different parameters](image)

The four diagrams display typical distributions for parameters (a): $r_{A+}/A^- = 0.1$, $r_{ex} = 0.3$, (b): $r_{A+}/A^- = 0.1$, $r_{ex} = 0.6$, (c): $r_{A+}/A^- = 0.6$, $r_{ex} = 0.3$ and (d): $r_{A+}/A^- = 0.6$, $r_{ex} = 0.75$. Cases (a) and (d) are arbitrary examples from the power law region.

Figure 2 summarizes our findings in different parameter regions. The figure displays the emergence of scale-free nets as a function of the excitation level $r_{ex}$, the average ratio of neurons receiving excitation from the environment, and the ratio of the area of potentiation to the area of depression ($r_{A+}/A^-$) in kernel $K$. The length of the scale-free regions was determined by first plotting the distribution of the sum of the weights of outgoing connections (averaged over 20 runs, each run contains 10000 samples) for every parameter set studied. Results are depicted on loglog plot. Supposing a power-law distribution $\langle P(k^*) \rangle \approx k^* e^{-k^*/\xi}$, where $k^*$ denotes the discretized values of the connection strength), a linear fitting was made to approximate $\gamma$. The width of the scale-free region was estimated by the length of the region with power-law distribution relative to the full length covered on the log scale. Maximum error of the linear fit was set to $10^{-3}$ STD. That is, for 100
discretization points, the width of a region spreading an order of magnitude on the loglog plot is equal to 0.5. Figure 3 shows the corresponding connection matrices. While case (c) resembles a random structure, case (b) seems to be a winners-take-most network, in which only a few neurons dominate over the total amount of the connection strength. However, cases (a) and (d) show strong clustering in a rather sparse structure and therefore correspond to scale-free small world networks characterized by their $\gamma$ values (-1.66 and -1.63, respectively). Figure 3 depicts the corresponding connectivity matrices.

![Connectivity matrices](image)

The four diagrams display connectivity matrices corresponding to the cases in Fig. 2. Cases (a) and (d) are arbitrary examples from the power law region.

With the help of the above introduced connectivity length measures we studied also the emerging network structures as a function of the following parameters: (i) the magnitude of the external excitation and (ii) the strengthening–weakening area ratio ($r_{A+}/A^-$) of kernel $K$. It can be seen that many connection weights have been vanished and it has made possible to talk about ‘subgraphs’ with local connectivity. As an extreme case of the general model, the binary neuron model was also investigated and no important difference were found.

We compared the resulting HebbNet structures with a random net, in which the same weights of the dynamic network have been randomly assigned to different node pairs. Fig. 4 displays the emerging connections of a HebbNet for two different parameter sets. Figure 4 highlights clearly the emerging
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small-world properties, i.e., small local connectivity values (high clustering coefficients) for case (d). Although the global connectivity length was almost the same for all HebbNets and their corresponding random nets, local distances are much smaller in case (d). That is, connectivity structure is sparse but information flow is still fault tolerant and efficient.

![Graph](image)

**Fig. 4. Local connectivity length distances**

Local connectivity length distances in ascending order are shown. For better visualization not all data points are marked and the points are connected with a solid line. Lines with upward triangle markers: STDP learning. Lines with circles: same but randomly redistributed weights. Line with empty (solid) markers: HebbNet of case (c) (case (d)). Global harmonic mean distances for the original and for the randomized networks in case (c) of Fig. 3 (case (d) of Fig. 3) are about the same \( D^h \approx D^r \approx 5.5 \) (\( D^h \approx D^r \approx 10 \)).

The robustness of the network to the external excitation (i.e., the amount of noise input to the network) is illustrated on Fig. 5.

![Graph](image)

**Fig. 5. Average local distance vs. excitation ratio**

A: \( r_{A+}/A^- = 0.1 \), B: \( r_{A+}/A^- = 0.6 \). Diamonds: average local distances for the evolving network. Circles: average local distances for the corresponding random net.

By increasing the excitation level, the average local connectivity length of the random net is drastically increasing, whereas the efficiency of the small-world network shows weak dependencies in the same region. For the network with parameters \( r_{A+}/A^- = 0.1 \) (Fig. 5(A)), there is a sharp cut-off around excitation level 0.55, where local distances suddenly drop, due to the high ratio of excitation. Qualitatively similar behavior can be seen for \( r_{A+}/A^- = 0.6 \) (Fig. 5(B)), but the cut-off is around \( r_{ex} = 0.9 \).

Results demonstrated so far characterize the ‘early’ stages of network development, as the interaction among neurons is weak due to the low connection weight values in all of the above examples. Figure 6 demonstrates that even in case of strong interaction, the found structural properties are present: According to the figure, the power-law behavior is present in a broad range of parameters. For the constant parameter of Fig. 6 (i.e., for \( r_{A+}/A^- = 0.1 \)) we have experienced a convergence of the exponent of the power-law distribution to -1.

![Graph](image)

**Fig. 6. Power-law with significant interaction**

Left: exponent of the power law, right: ratio of the power-law domain (i.e., ratio of the width of power-law distribution region relative to the full length covered on the log scale) as a function of \( r_{ex} \) and excitation threshold \( \theta \). Parameter \( r_{A+}/A^- \) equals to 0.1. Results are averaged over 700 steps. Input from other neurons could exceed the external inputs by a factor of 10. The power-law exponent is about -1 for broad regions of \( \theta \) and \( r_{ex} \). Outside these regions the network may vanish or start oscillating.
4. Discussion and outlook

One of the most exciting findings in recent scientific research is that many complex interactive systems possess a surprising structural and functional property: the emergence of scale-free small-world networks (SFNs) of the building blocks. Such SFNs may be found in distinct fields ranging from metabolic reaction chains to social relation systems\textsuperscript{18,23,24,25,26,20,27}. One may find SFNs in neurobiology as well. For example, the only case of completely mapped neural network of the nematode worm C. elegans\textsuperscript{29} is considered to form a small-world network\textsuperscript{20}. An outstanding example is the Internet, which displays this network structure at the hardware level of servers and also at the level of web pages\textsuperscript{25,26,23}. This fascinating self-organizing system has inspired several studies and models. The original model of the World Wide Web (WWW) by Watts and Strogatz\textsuperscript{18} explored random restructuring of the links among a finite number of ‘nodes’. Barabási and his colleagues introduced the concept of preferential attachment to model the WWW\textsuperscript{24,25}. The idea has been extended to other types of networks\textsuperscript{26} and the focus has been put on the search of general mechanisms underlying the development of these distinct connection systems.

4.1. Relation of HebbNet to other models

Although this paper is intended only to show some experimental (simulation) results on noise induced network structures of simplified neuron models, the results can be related to other, well-known mechanisms, too. In the following we show that under some (strong) constraining assumptions, our model can be transformed to the model of Barabasi et al\textsuperscript{25}, the model of preferential attachment. The following assumptions are made to enable the above-mentioned transition:

(i) Let us suppose that at \( t = 0 \) there are \( N \) nodes, from which only \( n \) nodes \((n << N)\) have at least one connection to other nodes.

(ii) Let the changes in activity and connection strength be discrete by choosing both the weakening and strengthening step of the kernel to be of unit strength.

(iii) Spikings of the cloud of \((N - n)\) isolated nodes can be considered independent and the spiking probability is small. For such isolated nodes, only the external input, the second part of the right hand side of Eq. 1, counts. Furthermore, the coincidence of spiking of two isolated neurons is negligibly small if the temporal kernel is short. At any time instant, when a neuron of the isolated cloud fires the nodes of the connected set may fire or not. If no coincidence occurs then there will be no change in the network. However, such coincidences are much more likely given the connectivity structure between the neurons of the connected set. This is so, because if one neuron fires then there is a chain of firing amongst these neurons. Of th

(iv) In contrast to the cloud, the activity of the connected neurons is strongly dependent on the spiking activity of the ‘neighbors’. If firing starts in the connected cloud of neurons then the first term of the right hand side of Eq. 1 will dominate the resulting firing chain. Input initiates the firing chain, whereas recurrent excitation from other nodes control that chain. In turn, the probability of firing can be taken as (approximately) proportional to the recurrent activity, controlled by the incoming connection distribution.

(v) Having established a connection between two nodes, it is kept steady and may not change by time. This is a strong assumption, which is tacitly assumed by the original model of preferential attachment, too.

This latter constraint does not seem to be realistic in any model. There is no reason that for a growing connection structure should remain steady for old connections. Note, however, that random rewiring of old connections can give rise to scale-free network structure, too. In fact, this rewiring mechanism is the original model of Watts and Strogatz\textsuperscript{18}. As it was noted at the very beginning (see Section ) our model has an intrinsic weight redistributing property originated by the constraint that \( r_{A^+ / A^-} < 1 \). In
turn, the incremental growing of the connected subnet (by connecting new isolated neurons) and the weight redistributing property of HebbNets can be seen as the synthesis of the preferential attachment mechanism with continuous new entries in the model of preferential attachment and the rewiring mechanism of the model of Watts and Strogatz. That is, constraining our model lead to a combination of two models both generating small-world structures. Nonetheless, by means of numerical simulations we have shown that our model can produce such connection structures without the explicit requirement on growing, and without a direct mechanism of weight rewiring.

4.2. Remarks on evolutionary systems

Interestingly, all the listed examples, one way or the other, usually are also regarded as evolutionary systems. In our particular case, the obtained results can also be interpreted in an evolutionary context by reconsidering Edelman’s alternative neuronal group selection theory about the fundamental role of selection during and after development of the nervous system. According to Edelman, a theory to describe a system’s temporal change can be considered as ‘selectionist’, if it includes the following components:

(i) source of diversification leading to variants,
(ii) a means for encounter with an environment not initially categorized,
(iii) a means for differential amplification over some period of time of those variants in a population that have greater adaptive value.

It is no surprise that a system with these features falls into the class of evolutionary systems as far as we look at the system as a whole. In the nervous systems, there are at least two types of temporal changes serving the first requirement: Diversification can occur via the emergence of redundant connectivity during development and via the modification of synaptic efficacy during life-time learning. The main thesis of this paper is to demonstrate how diversification can be realized by noise under STDP rules. The second requirement is fulfilled if the pool of the not yet seen input patterns is not limited.

Now, we can argue that noise in the nervous system has an important role: Noise has no spatialtemporal structure. Thus, obviously it cannot induce ‘learning’ in general sense. However, noise with STDP — according to in our results — gives rise to a search mechanism, which scans at all scales simultaneously. Search in a scale-free manner can be most efficient if no structural formation is known in advance. The searching feature is robust: The noise generated structure is changing rapidly; results depicted in the figures are averaged over several runs. The continuous change induced by noise can be interpreted in the following way. The noise together with the proportionally expressed LTD and LTP mechanisms yields a continuous sparsification and regeneration of the connections. LTD ‘chooses’ sound patterns, whereas LTP helps to ‘forget’ those patterns and maintains a competition amongst patterns. Synchronous patterns or pattern series are quickly learned by HebbNets and approximately stable connectivity patterns may emerge. Noise, in this case, may modify the connectivity strengths and search may be performed ‘around’ an average stable connectivity pattern. Also, the noise may help the system to escape from local minima. Noisy Hebbian learning, in turn, is able to simultaneously learn correlations and make selections among the discovered structures or patterns.

As far as other evolving networks are considered, the profound implication of our result is that local (Hebbian) learning rules may be sufficient to form and maintain an efficient network in terms of information flow. This feature differs from existing models, such as the model on preferential attachment, the global optimization scheme, and also from the original Watts and Strogatz model.

In summary, we have demonstrated that small-world architecture with scale-free domains may emerge in sustained networks under STDP Hebbian learning rule without any other specific constraint on the evolution of the net. According to our results, evolution and plasticity of neural networks may be maintained by noise randomly generated within the central nervous system. We conjecture that the sustained nature of noise and the competition imposed by appropriate $r_{A^+/A^-}$ values are the two relevant components of plasticity and learning. It might be equally important that exponents of HebbNets of neurons with significant interaction are similar in a

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1 Considerations about the third requirement are beyond the scope of the present study.
broad range of parameters providing a system more stable against homeostatic parameter perturbations.

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