Performance Measure of Fuzzy Queue using Pentagonal Fuzzy Numbers

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Abstract. In this paper, we propose a performance measure using \( \alpha \)-cut method in terms of crisp value for fuzzy queues \( F M/F M/1 \). The arrival and service rate are fuzzy numbers which represented by pentagonal membership functions. Fuzzy queues are much more reasonable than the commonly used crisp queues in many real life situations. The \( \alpha \)-cut for linear pentagonal fuzzy numbers are applied. To illustrate the approach, a numerical example is solved for pentagonal fuzzy number.

1. Introduction
Queueing models have great extent applications in service organizations as well as manufacturing firms. Various customers get service by different types of serves according to specific queue discipline within the context of queuing theory, the inter arrival time and service times are required to follow certain distributions. Fuzzy Logic was initiated by Zadeh [10]. Fuzzy queuing model was first introduced by Lie and Lee [5], further developed this model by many authors namely Buckely [2], Negi and Lee [6], Chen [4] and Srinivasan [7], Shanmugasundaram and Venkatesh [8].

Also fuzzy queuing models have been described by such researchers like Chen [3, 4] analyzed bulk arrival queuing model with fuzzy parameters and varying batch sizes using Zadehs extension principle and recently he developed \( F M/F M/1 \) queuing models. Apurba Panda and Madhumangal Pal [1] introduced the pentagonal fuzzy number. Thamotharan [9] studied multi server queueing model in triangular and trapezoidal fuzzy numbers using \( \alpha \) cuts in 2016. In practical, the queuing model, the arrival rate and service rates are vague datas. Vagueness can be determined precisely by using fuzzy membership functions. It is a suitable model used to represent a service oriented problem, where customers arrive randomly to receive some service, the service time being also a random variable. Hence the classical queuing model has more applications if it is expanded using fuzzy models. Here the parameters, fuzzy arrival rate and fuzzy service rate are best described by linguistic terms very high, high, low, very low and moderate.

2. Fuzzy Set Theory
Fuzzy systems are global approximations which accepts partial membership functions. Fuzzy set theory generalizes the classical set theory. The primary benefit of fuzzy systems theory is to approximate system behavior where analytic functions or numerical relations do not exist. In 1937 by Max Black, with his studies in vagueness which is an exercise for logical analysis, then
with the introduction of fuzzy sets by Zadeh [10]. The idea proposed by Lotfi Zadeh suggested membership function which is having more appropriateness when there is an uncertain situation. The notion of a fuzzy set provides a relative grade of membership which is a convenient point of departure for the construction of a theoretical framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability.

2.1. Fuzzy set
Let \( X \) be a nonempty set. A fuzzy set \( A \) in \( X \) is characterized by its membership function \( \mu_A : X \to [0,1] \) and \( \mu_A(x) \) is interpreted as the degree of membership of element \( x \) in fuzzy set \( A \) for each \( x \in X \). It is clear that \( A \) is completely determined by the set of tuples \( A = \{(x, \mu_A(x)) \mid x \in X\} \). Frequently we will write \( A(x) \) instead of \( \mu_A(x) \).

2.2. \( \alpha \)-cut
The \( \alpha \)-cut for a fuzzy set \( A \) are shown by \( A_\alpha \) for \( \alpha \in [0,1] \) are defined to be \( A_\alpha = \{x \mid \mu_A(x) \geq \alpha, x \in X\} \), where \( X \) is the universal set. Upper and lower bounds for any \( \alpha \)-cut \( A_\alpha \) are shown by \( \sup A_\alpha \) and \( \inf A_\alpha \) respectively. \( \sup A_\alpha \) is denoted by \( A_\alpha^R \) and \( \inf A_\alpha \) is denoted by \( A_\alpha^L \).

2.3. Pentagonal fuzzy set with membership function
A pentagonal fuzzy number \( \tilde{A} = (a_1, a_2, a_3, a_4, a_5) \) should satisfy the following conditions

- \( \mu_{\tilde{A}}(x) \) is a continuous function in the interval \([0,1]\).
- \( \mu_{\tilde{A}}(x) \) is strictly increasing and continuous function on \([a_1, a_2]\) and \([a_2, a_3]\).
- \( \mu_{\tilde{A}}(x) \) is strictly decreasing and continuous function on \([a_3, a_4]\) and \([a_4, a_5]\).

2.4. Linear pentagonal fuzzy number with asymmetry
A linear pentagonal fuzzy number with asymmetry is written as
\[
\tilde{A}_{\text{LAS}} = (a_1, a_2, a_3, a_4, a_5; r, s)
\]
whose membership function is written as
\[
\mu_{\tilde{A}_{\text{LAS}}} = \begin{cases} 
  \frac{x - a_1}{a_2 - a_1}, & \text{if } a_1 \leq x \leq a_2 \\
  1 + (1-r) \frac{x - a_2}{a_3 - a_2}, & \text{if } a_2 \leq x \leq a_3 \\
  1, & \text{if } x = a_3 \\
  1 - (1-s) \frac{a_4 - x}{a_4 - a_3}, & \text{if } a_3 \leq x \leq a_4 \\
  s \frac{a_5 - x}{a_5 - a_4}, & \text{if } a_4 \leq x \leq a_5 \\
  0, & \text{if } x > a_5 
\end{cases}
\]

Note:

(i) If \( r = s \) the asymmetry pentagonal fuzzy number becomes symmetry pentagonal fuzzy number.

(ii) For asymmetry pentagonal fuzzy number may be \( r < s \) or \( r > s \).
2.5. \( \alpha \)-cut or parametric form of linear pentagonal fuzzy number

\( \alpha \)-cut or parametric form of linear pentagonal fuzzy number is represented by the formulae

\[
A_\alpha = \{x \in X \mid \mu_{\tilde{A}_{LAS}}(x) \geq \alpha\}
\]

\[
A^{1L}(\alpha) = a_1 + \frac{\alpha}{r}(a_2 - a_1), \quad \text{for } \alpha \in [0, r]
\]

\[
A^{2L}(\alpha) = a_2 + \frac{1 - \alpha}{1 - r}(a_3 - a_2), \quad \text{for } \alpha \in [r, 1]
\]

\[
A^{2R}(\alpha) = a_4 - \frac{1 - \alpha}{1 - s}(a_4 - a_3), \quad \text{for } \alpha \in [s, 1]
\]

\[
A^{1R}(\alpha) = a_5 - \frac{\alpha}{s}(a_5 - a_4), \quad \text{for } \alpha \in [0, s]
\]

where \( A^{1L}(\alpha), A^{2L}(\alpha) \) are increasing functions with respect to \( \alpha \) and \( A^{1R}(\alpha), A^{2R}(\alpha) \) are decreasing functions with respect to \( \alpha \).

3. Solution Procedure

Let us consider a single server \( FM/FM/1 \) queuing system first come first served discipline. The inter arrival time \( A \) and service times \( S \) are described by the following fuzzy sets:

\[
A = \{(a, \mu_\tilde{A}(a)) \mid a \in X\}
\]

\[
S = \{(s, \mu_\tilde{S}(a)) \mid s \in Y\}
\]

Here \( X \) is the set of the inter arrival time and \( Y \) is the set of the service time. \( \mu_\tilde{A}(a) \) is the membership function of the inter arrival time and \( \mu_\tilde{S}(a) \) is membership function of the service time.

The \( \alpha \)-cuts of inter arrival time, service time are represented as

\[
\tilde{A}(\alpha) = \{a \in X \mid \mu_\tilde{A}(a) \geq \alpha\}
\]

\[
\tilde{S}(\alpha) = \{s \in Y \mid \mu_\tilde{S}(a) \geq \alpha\}
\]

Using these \( \alpha \)-cuts we have to define the membership function as in equation (1).

This queue adapts a first-come first served discipline and consider an infinite source population where both the arrival time and the service times follows Poisson and exponential distributions with parameters \( \lambda \) and \( \mu \) respectively, which are fuzzy variables rather than crisp values.

4. Interval analysis for arithmetic definitions

Let \([a, b]\) and \([c, d]\) be two interval numbers. The basic arithmetic operations are defined as follows.

1. \([a, b] + [c, d] = [a + c, b + d]\),
2. \([a, b] - [c, d] = [a - d, b - c]\),
3. \([a, b] \ast [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]\),
4. \([a, b] \div [c, d] = [a, b] \ast \left[\frac{1}{d} \frac{1}{c}\right]\) provided that 0 does not belong to \([c, d]\),
5. \(\alpha[a, b] = [\alpha a, \alpha b]\) for \(\alpha > 0\) and \([\alpha b, \alpha a]\) for \(\alpha < 0\).
5. Numerical Example
Consider $FM/FM/1$ queue where both arrival rate and service rate are Pentagon fuzzy numbers represented by $\tilde{\lambda} = [3, 4, 5, 6, 7]$ and $\tilde{\mu} = [11, 12, 13, 14, 15]$ per hour respectively.

To derive the membership function of $\tilde{L} = \frac{x}{y - x}$, $\tilde{W} = \frac{\tilde{L}}{x}$ and $\tilde{P}_b = \frac{x}{y}$ where

$$x = A_\alpha = \begin{cases} 
3 + 1.4286\alpha, & \text{for } \alpha \in [0, 0.7] \\
3.333\alpha + 1.6667, & \text{for } \alpha \in [0.7, 1] \\
7.5 - 0.25\alpha, & \text{for } \alpha \in [0.6, 1] \\
7 - 1.6667\alpha, & \text{for } \alpha \in [0, 0.6]
\end{cases}$$

$$y = S_\alpha = \begin{cases} 
11 + 1.4286\alpha, & \text{for } \alpha \in [0, 0.7] \\
3.333\alpha + 9.6666, & \text{for } \alpha \in [0.7, 1] \\
15.5 - 2.5\alpha, & \text{for } \alpha \in [0.6, 1] \\
15 - 1.6667\alpha, & \text{for } \alpha \in [0, 0.6]
\end{cases}$$

The following performance are obtained:

- The mean number of customers in the system $\tilde{L} = \frac{\tilde{\lambda}}{\tilde{\mu} - \tilde{\lambda}}$
- The mean waiting time in the system $\tilde{W} = \frac{\tilde{L}}{\tilde{\lambda}}$
- The probability that the server is busy $\tilde{P}_b = \frac{\tilde{\lambda}}{\tilde{\mu}}$
- The idle probability $\tilde{P}_0 = 1 - \tilde{P}_b$

![Figure 1. The $\alpha$ of $\tilde{L}$](image-url)
6. Conclusion
In this paper pentagonal fuzzy number and $\alpha$-cut operations using addition, subtraction and multiplication has been discussed. The expected number of customers and waiting time in the system for fuzzy queue has been discussed with numerical example. The pentagonal fuzzy number plays a significant role in solving the problems and also easy to apply in the real life problems.

References
[1] Apurba Panda, Madhumangal Pal 2015 A study on pentagonal fuzzy number and its corresponding matrices Pacific Sciences Review B: Humanities and Social Sciences 1 131-139
[2] Buckely J J 1990 Elementary Queueing Theory based on Possibility Theory Fuzzy Sets and Systems 37 43-52.
[3] Chen S P 2005 Parametric Nonlinear Programming Approach to Fuzzy Queues with Bulk Service European Journal of Operations Research 163 434-444
[4] Chen S P 2006 A Mathematics Programming Approach to the Machine Interference Problem with Fuzzy Parameters Applied Mathematics and Computation 174 374-387
[5] Lie R J and Lee E S 1989 Analysis of Fuzzy Queues Computers and Mathematics with Applications 17(7) 1143-1147.
[6] Negi D S and Lee E S, 1992, Analysis and Simulation of Fuzzy Queue Fuzzy sets and Systems 46 321-330
[7] Srinivasan R 2014 Fuzzy Queueing Model using DSW algorithm International Journal of Advanced research in Mathematics and Application 1(1) 57-62
[8] Shanmugasundaram S and Venkatesh BB 2015 Fuzzy Multi Server Queueing Model through DSW algorithm International Journal of Latest Trends in Engineering and Technology 5(3) 452-457
[9] Thamotharan S 2016 A Study on Multi Server Fuzzy Queueing Model in Triangular and Trapezoidal Fuzzy Numbers using α-Cuts International Journal of Science and Research 5(1) 226-230
[10] Zadeh L A 1965 Fuzzy Sets Information and Control 8 338-353