ISOGENY CLASSES OF ABELIAN VARIETIES OVER FUNCTION FIELDS

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ABSTRACT. We study finiteness problems for isogeny classes of abelian varieties over an algebraic function field $K$ in one variable over the field of complex numbers. In particular, we construct explicitly a non-isotrivial absolutely simple abelian fourfold $X$ over a certain $K$ such that the isogeny class of $X \times X$ contains infinitely many mutually non-isomorphic principally polarized abelian varieties. (Such examples do not exist when the ground field is finitely generated over its prime subfield.)

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1. Introduction

Let $K$ be a field, $\bar{K}$ its separable closure, $\text{Gal}(K) = \text{Gal}(\bar{K}/K)$ the (absolute) Galois group of $K$. Let $X$ be an abelian variety over $K$, $\text{End}_K(X)$ its ring of $K$-endomorphisms and $\text{End}_K^0(X) := \text{End}_K(X) \otimes \mathbb{Q}$.

Let us consider the set $\text{Isog}(X, K)$ of $K$-isomorphism classes of abelian varieties $Y$ over $K$ such that there exists a $K$-isogeny $Y \to X$. If $\ell$ is a prime different from $\text{char}(K)$ then let us consider the set $\text{Isog}(X, K, \ell)$ of $K$-isomorphism classes of abelian varieties $Y$ over $K$ such that there exists an $\ell$-isogeny $Y \to X$ that is defined over $K$. (Recall that $\ell$-isogeny is an isogeny, whose degree is a power of $\ell$.) If $m$ is a positive integer then we write $\text{Isog}_m(X, K, \ell)$ for the subset of $\text{Isog}(X, K, \ell)$ that consists of all (isomorphism) classes of $Y$ with a $K$-polarization of degree $m$.

When $K$ is finitely generated over its prime subfield, Tate [32] conjectured the finiteness of $\text{Isog}_m(X, K, \ell)$. This conjecture played a crucial role in the proof of Tate's conjecture on homomorphisms and the semisimplicity of the Galois representations in the Tate modules of abelian varieties [32, 28, 24, 33, 34, 35, 10, 11, 20, 39].

If $K$ is finitely generated over the field $\mathbb{Q}$ of rational numbers then it was proven by Faltings [11] that $\text{Isog}(X, K)$ is finite. (See also [10, 38].) When $K$ is finite, the finiteness of $\text{Isog}(X, K)$ is proven in [36]. (See [24, 33, 35, 36, 20] for a discussion of the case when $K$ is infinite but finitely generated over a finite field.)

The aim of this note is to discuss the situation when $K$ is an algebraic function field in one variable over $\mathbb{C}$. (In other words, $K$ is finitely generated and has transcendence degree 1 over $\mathbb{C}$.) In order to state our results, notice that one may view $K$ as the field of rational functions on a suitable irreducible smooth algebraic complex curve $S$ such that $X$ becomes the generic fiber of an abelian scheme $f : X \to S$. Let $s \in S$ be a complex point of $S$, let $X_s$ be the fiber of $f$ at $s$ and $H_1(X_s, \mathbb{Q})$ its first rational homology group. (Recall that $X_s$ is a complex abelian variety, whose dimension coincides with $\text{dim}(X_s)$.) There is the natural
of the fundamental group $\pi_1(S, s)$ of $S$. Deligne [6] proved that this representation is completely reducible and therefore its centralizer $D_f := \text{End}_{\pi_1(S, s)}(H_1(X_0, \mathbb{Q}))$ is a (finite-dimensional) semisimple $\mathbb{Q}$-algebra. On the other hand, every $u \in \text{End}_K(X)$ extends to an endomorphism of $X$ and therefore induces a certain endomorphism $u_s$ of $X_0$. This gives rise to the embeddings

$$\text{End}_K(X) \hookrightarrow \text{End}(X_0) \hookrightarrow \text{End}_\mathbb{Q}(H_1(X_0, \mathbb{Q})),$$

whose composition extends by $\mathbb{Q}$-linearity to the embedding

$$\text{End}^0_K(X) \hookrightarrow \text{End}_\mathbb{Q}(H_1(X_0, \mathbb{Q})),$$

whose image lies in $D_f$. Further, we identify $\text{End}^0_K(X)$ with its image in $D_f$. Our main result (Theorem 4.1) may be restated as follows.

The set $\text{Isog}(X, K)$ is infinite if and only if $D_f \neq \text{End}^0_K(X)$.

If $D_f \neq \text{End}^0_K(X)$ then the set $\text{Isog}(X, K, \ell)$ is infinite for all but finitely many primes $\ell$ (see Theorem 4.3). In addition, if $X$ is principally polarized over $K$ then the set $\text{Isog}_1(X^2, K, \ell)$ is infinite for all but finitely many primes $\ell$ congruent to 1 modulo 4 (see Corollary 4.6).

In order to describe other results of this paper (Sect. 10), let us further assume that every homomorphism over $\overline{\mathbb{C}(S)}$ between $X$ and “constant” abelian varieties (defined over $\mathbb{C}$) is zero. If $\dim(X) \leq 3$ then Deligne [6] proved that $D_f = \text{End}^0_K(X)$: this implies that in this case $\text{Isog}(X, K)$ is finite. On the other hand, Faltings [9] constructed a four-dimensional $X$ with $D_f \neq \text{End}^0_K(X)$; in his example(s) $X$ is the generic fiber of an universal family of abelian fourfolds with level $n \geq 3$ structure over a Shimura curve.

We prove that if $\dim(X) = 4$, all endomorphisms of $X$ are defined over $K$ and $\text{Isog}(X, K)$ is finite then $\text{End}^0_K(X)$ is a CM-field of degree 4 (see Theorem 10.1). Almost conversely, if $X$ is a fourfold, $\text{End}^0_K(X)$ is a CM-field of degree 4 and all endomorphisms of $X$ are defined over $K$ then there exists a finite algebraic extension $L/K$ such that $\text{Isog}(X \times_K L, L)$ is infinite (see Theorem 10.3). A rather explicit example of the latter case (with infinite $\text{Isog}(X \times_K L, L)$) is provided by the field $K = \mathbb{C}(\lambda)$ of rational functions in independent variable $\lambda$, the jacobian $X$ of the genus 4 curve $y^5 = x(x - 1)(x - \lambda)$ and the overfield $L = \mathbb{C}(\sqrt[5]{\lambda}, \sqrt[5]{\lambda - 1})$ (see Example 10.6).

The paper is organized as follows. Section 2 contains basic notation and useful facts about abelian varieties. In Section 3 we discuss abelian schemes over curves and corresponding monodromy representations. In Section 4 we state the main results. In Sections 5 and 6 we discuss non-isotrivial abelian schemes and isogenies of abelian schemes respectively. The next two sections contains the proofs of main results. Section 9 contains auxiliary results about quaternions. In Section 10 we deal with isogeny classes of four-dimensional abelian varieties.

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2. Abelian Varieties

Let $X$ be an abelian variety of positive dimension over a field $K$ of characteristic zero. If $n$ is a positive integer then we write $X_n$ for the kernel of multiplication by $n$ in $X(K_s)$. It is well-known [23] that $X_n$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2\dim(X)$; it is also a Galois submodule in $X(K)$. We write $K(X_n)$ for the field of definition of all points of order $n$; clearly, $K(X_n)/K$ is a finite Galois extension, whose Galois group is canonically identified with the image of $\text{Gal}(K)$ in $\text{Aut}(X_n)$. We write $\text{Id}_X$ for the identity automorphism of $X$. We write $\text{End}(X)$ for the ring of all $K$-endomorphisms of $X$ and $\text{End}^0(X)$ for the corresponding $\mathbb{Q}$-algebra $\text{End}(X) \otimes \mathbb{Q}$. We have

$$\mathbb{Z} : \text{Id}_X \subset \text{End}(X) \subset \text{End}^0(X).$$

Remark 2.1. Let $X'$ be an abelian variety over $K$. If $X$ and $X'$ are $K$-isogenous then every $K$-isogeny $X \to X'$ gives rise to a bijection between $\text{Isog}(X,K)$ and $\text{Isog}(X',K)$. In particular, the sets $\text{Isog}(X,K)$ and $\text{Isog}(X',K)$ are either both finite or both infinite.

Let $\ell$ be a prime.

Lemma 2.2. Suppose that $\ell$ is congruent to 1 modulo 4. If $X$ admits a principal polarization over $K$ and $\text{Isog}(X,K,\ell)$ is infinite then $\text{Isog}_1(X^2,K,\ell)$ is also infinite.

Proof. Clearly, every abelian variety $Y$ over $K$ that admits an $\ell$-isogeny $Y \to X$ over $\mathbb{C}(S)$ also admits a $K$-polarization, whose degree is a power of $\ell$. In addition, there exists an $\ell$-isogeny $X \to Y$ over $K$; its dual $Y^t \to X^t \cong X$ is an $\ell$-isogeny that is also defined over $K$. Since $\ell$ is congruent to 1 modulo 4 then $\sqrt{-1} \in \mathbb{Z}_\ell$ and it follows from Remark 5.3.1 on pp. 314–315 of [38] that $Y \times Y^t$ is principally polarized over $K$. Clearly, there is an $\ell$-isogeny $Y \times Y^t \to X \times X = X^2$ that is defined over $K$. On the other hand, for any given abelian variety $Z$ over $K$ there are, up to an isomorphism, only finitely many abelian varieties over $K$ that are isomorphic over $K$ to an abelian subvariety of $Z$ [16]. It follows that $\text{Isog}_1(X^2,K,\ell)$ is infinite.

Remark 2.3. Let $L/K$ be a finite Galois extension with Galois group $\text{Gal}(L/K)$. Let $Y$ be an abelian variety over $K$ and $\text{Aut}_L(Y)$ be its group of $L$-automorphisms. It is well-known that $\text{Aut}_L(Y)$ is an arithmetic group; it follows from a theorem of Borel-Serre [1] that the corresponding first noncommutative Galois cohomology set $H^1(\text{Gal}(L/K), \text{Aut}_L(Y))$ is finite, i.e., the set of $(K$-isomorphism classes of) $L/K$-forms of $Y$ is finite. This implies that if $\text{Isog}(X \times_K L, L)$ (resp. $\text{Isog}(X \times_K L, L, \ell)$) is finite then $\text{Isog}(X,K)$ (resp. $\text{Isog}(X,K,\ell)$) is also finite [39].

Abusing notation, we sometime write $\text{Isog}(X,L,\ell)$, $\text{Isog}(X,L,\ell)$ and $\text{Isog}_m(X,L,\ell)$ instead of $\text{Isog}(X \times_K L, L, \ell)$, $\text{Isog}(X \times_K L, L, \ell)$ and $\text{Isog}_m(X \times_K L, L, \ell)$ respectively.
3. Abelian schemes and monodromy representations

Let $S$ be an irreducible smooth (but not necessarily projective) algebraic curve over $\mathbb{C}$. We write $\mathcal{C}(S)$ for the field of rational functions on $S$ and $\overline{\mathcal{C}(S)}$ for its algebraic closure. Let $f : \mathcal{X} \to S$ be a polarized abelian scheme of positive relative dimension $d$ over $S$. Let $\eta$ be the generic point of $S$ and $\mathcal{X}$ be the generic fiber; it is a $d$-dimensional abelian variety over $k(\eta) = \mathcal{C}(S)$.

An abelian variety $Z$ over $\mathcal{C}(S)$ is called constant if there exists an abelian variety $W$ over $\mathcal{C}$ such that $Z$ is isomorphic to $W \times_{\mathcal{C}} \mathcal{C}(S)$ over $\mathcal{C}(S)$.

Recall that $\mathcal{X}$ is called isotrivial if there exists a finite étale cover $S' \to S$ (with non-empty $S'$) such that the pullback $\mathcal{X}_{S'} = \mathcal{X} \times_S S'$ is a constant abelian scheme over $S'$. We say that $\mathcal{X}$ is weakly isotrivial if (under the same assumptions on $S' \to S$) $\mathcal{X}_{S'}$ contains a non-zero constant abelian subscheme.

**Theorem 3.1.**

(i) $\mathcal{X}$ is isotrivial if and only if there exists an abelian variety $W$ over $\mathcal{C}$ such that the $\mathcal{C}(S)$-abelian varieties $X$ and $W \times_{\mathcal{C}} \mathcal{C}(S)$ are isomorphic over $\overline{\mathcal{C}(S)}$.

(ii) $\mathcal{X}$ is weakly isotrivial if and only if there exist an abelian variety $W$ of positive dimension over $\mathcal{C}$ and an abelian subvariety $Z \subset X \times_{\mathcal{C}(S)} \mathcal{C}(S)$ such that the $\mathcal{C}(S)$-abelian varieties $W \times_{\mathcal{C}} \mathcal{C}(S)$ and $Z$ are isomorphic.

We prove Theorem 3.1 in Section 5.

We write $R_1 f_* \mathbb{Z}$ for the corresponding local system of the first integral homology groups $H_1(\mathcal{X}_s, \mathbb{Z})$ of the fibres $\mathcal{X}_s = f^{-1}(s)$ on $S$. If $g : Y \to S$ is another polarized abelian scheme over $S$ with generic fiber $f$ then the natural map

$$\text{Hom}_S(Y, \mathcal{X}) \to \text{Hom}_{\mathcal{C}(S)}(Y, X)$$

is bijective. We write $R_1 f_* \mathbb{Q}$ for the corresponding local system $R_1 f_* \mathbb{Z} \otimes \mathbb{Q}$ of the first rational homology groups $H_1(\mathcal{X}_s, \mathbb{Q}) = H_1(\mathcal{X}_s, \mathbb{Z}) \otimes \mathbb{Q}$ of the fibres $\mathcal{X}_s = f^{-1}(s)$ on $S$ and $\pi_1(S, s) \to \text{Aut}(H_1(\mathcal{X}_s, \mathbb{Z})) \subset \text{Aut}(H_1(\mathcal{X}_s, \mathbb{Q}))$ for the corresponding monodromy representation.

**Remark 3.2.** An abelian scheme $f : \mathcal{X} \to S$ is isotrivial if and only if the image of the monodromy representation $\pi_1(S, s) \to \text{Aut}(H_1(\mathcal{X}_s, \mathbb{Z}))$ is finite [14] (see also [6, Sect. 4.1.3.3]).

### 3.3. Rigidity and specialization

It follows from the rigidity lemma [22, Ch. 6, Sect. 1, Cor. 6.2] that the natural homomorphism

$$\text{Hom}_S(Y, \mathcal{X}) \to \text{Hom}(Y_s, \mathcal{X}_s), \quad u \mapsto u_s$$

is injective. In particular, the natural ring homomorphism $\text{End}_S(\mathcal{X}) \to \text{End}(\mathcal{X}_s)$ is also injective. It is well-known that the natural homomorphisms

$$\text{Hom}(Y_s, \mathcal{X}_s) \to \text{Hom}(H_1(Y_s, \mathbb{Z}), H_1(\mathcal{X}_s, \mathbb{Z})), \quad \text{End}(\mathcal{X}_s) \to \text{End}(H_1(\mathcal{X}_s, \mathbb{Z}))$$

are embeddings. Taking the compositions, we get the embeddings

$$\text{Hom}_S(Y, \mathcal{X}) \to \text{Hom}(H_1(Y_s, \mathbb{Z}), H_1(\mathcal{X}_s, \mathbb{Z})), \quad \text{End}_S(\mathcal{X}) \to \text{End}(H_1(\mathcal{X}_s, \mathbb{Z}));$$

the images of $\text{Hom}_S(Y, \mathcal{X})$ and of $\text{End}_S(\mathcal{X})$ lie in $\text{Hom}_{\pi_1(S, s)}(H_1(Y_s, \mathbb{Z}), H_1(\mathcal{X}_s, \mathbb{Z}))$ and $\text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Z}))$ respectively. Further we will identify $\text{End}_S(\mathcal{X})$ with its image in $\text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Z}))$ and $\text{Hom}_S(Y, \mathcal{X})$ with its image in $\text{Hom}_{\pi_1(S, s)}(H_1(Y_s, \mathbb{Z}), H_1(\mathcal{X}_s, \mathbb{Z}))$ respectively.
We have
\[ \text{Hom}^0_\mathcal{S}(\mathcal{Y}, \mathcal{X}) := \text{Hom}_\mathcal{S}(\mathcal{Y}, \mathcal{X}) \otimes \mathbb{Q} \subset \text{End}_\mathcal{S}(\mathcal{X}) \otimes \mathbb{Q} \subset \text{End}_\mathcal{S}(\mathcal{X}) \]
\[ \text{End}^0_\mathcal{S}(\mathcal{X}) := \text{End}_\mathcal{S}(\mathcal{X}) \otimes \mathbb{Q} \subset \text{End}_\mathcal{S}(\mathcal{X}) \]

Notice that a theorem of Grothendieck [12, p. 60] (see also [6, Sect. 4.1.3.2]) implies that
\[ \text{Hom}_\mathcal{S}(\mathcal{Y}, \mathcal{X}) = \text{Hom}(\mathcal{Y}, \mathcal{X}) \cap \text{Hom}_{\mathcal{S}}(H_1(\mathcal{Y}, \mathbb{Q}), H_1(\mathcal{X}, \mathbb{Q})). \]
\[ \text{End}_\mathcal{S}(\mathcal{X}) = \text{End}(\mathcal{X}) \cap \text{End}_{\mathcal{S}}(H_1(\mathcal{X}, \mathbb{Q})). \]

3.4. Let \( \Gamma_s \) be the image of \( \pi_1(S, s) \) in \( \text{Aut}(H_1(\mathcal{X}_s, \mathbb{Q})) \), let \( G_s \) be the Zariski closure of \( \Gamma_s \) in \( \text{GL}_Q(H_1(\mathcal{X}_s, \mathbb{Q})) \) and \( G_s^0 \) its identity component. By a theorem of Deligne [6, Cor. 4.2.9], \( G_s^0 \) is a semisimple algebraic \( \mathbb{Q} \)-group. Recall [2, Ch. 1, Sect. 1.2] that \( G_s^0 \) has finite index in \( G_s \). It follows that the intersection \( \Gamma^0_s := \Gamma_s \cap G_s^0 \) is a normal subgroup of finite index in \( \Gamma_s \) and this index divides \([G_s : G_s^0]\).

The groups \( \Gamma_s \) and \( G_s \) do not depend on \( s \) in the following sense. Recall that \( S \) is arcwise connected with respect to the complex topology. If \( t \) is another point of \( S \) then every path \( \gamma \) in \( S \) from \( s \) to \( t \) defines an isomorphism \( \gamma_* : H_1(\mathcal{X}_s, \mathbb{Z}) \cong H_1(\mathcal{X}_t, \mathbb{Z}) \) such that \( \Gamma_t = \gamma_* \Gamma_s \gamma_*^{-1} \) and therefore
\[ G_t = \gamma_* G_s \gamma_*^{-1}, \quad G_t^0 = \gamma_* G_s^0 \gamma_*^{-1}. \]

Let \( \text{Hdg}(\mathcal{X}_s) \subset \text{GL}_Q(H_1(\mathcal{X}_s, \mathbb{Q})) \) be the Hodge group of \( \mathcal{X}_s \) [23] (see also [37, 18, 19]). Recall that \( \text{Hdg}(\mathcal{X}_s) \) is a connected reductive algebraic \( \mathbb{Q} \)-group and its centralizer \( \text{End}_{\text{Hdg}(\mathcal{X}_s)}(H_1(\mathcal{X}_s, \mathbb{Q})) \) in \( \text{End}_Q(H_1(\mathcal{X}_s, \mathbb{Q})) \) coincides with \( \text{End}^0(\mathcal{X}_s) \).

The following assertion is a special case of a theorem of Deligne [37, Th. 7.3] (see also [7, Prop. 7.5], [17]).

**Theorem 3.5.** For all \( s \in S \) outside a countable set, \( G^0_s \) is a normal algebraic subgroup in \( \text{Hdg}(\mathcal{X}_s) \).

**Definition 3.6.** We say that a point \( s \in S \) is in general position (with respect to \( \mathcal{X} \to S \)) if \( G_s^0 \) is a normal algebraic subgroup in \( \text{Hdg}(\mathcal{X}_s) \). It follows from Theorem 3.5 that every \( s \) outside a countable set is in general position. In particular, a point in general position always does exist.

**Corollary 3.7.** Suppose that \( s \) is in general position and \( G_s \) is connected. Then
\[ \text{End}^0(\mathcal{X}_s) = \text{End}^0_\mathcal{S}(\mathcal{X}) = \text{End}^0_{C(S)}(X). \]

If, in addition, \( \mathcal{X} \) is not isotrivial and \( \text{Hdg}(\mathcal{X}_s) \) is a \( \mathbb{Q} \)-simple algebraic group then
\[ \text{End}^0_{C(S)}(X) = \text{End}^0_\mathcal{S}(\mathcal{X}) = \text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Q})). \]
Proof. We have \( G_s = G_s^0 \subset \text{Hdg}(X_s) \). Clearly,
\[
\text{End}_{\pi_1(S,s)}(H_1(X_s, Q)) = \text{End}_{G_s}(H_1(X_s, Q)) \supset \text{End}_{\text{Hdg}(X_s)}(H_1(X_s, Q)) = \text{End}^0(X_s).
\]

It follows that \( \text{End}^0_s(X') = \text{End}^0_s(X_s) \cap \text{End}_{\pi_1(S,s)}(H_1(X_s, Q)) = \text{End}^0_s(X_s) \). Assume now that \( X' \) is not isotrivial and \( \text{Hdg}(X'_s) \) is \( \mathbb{Q} \)-simple. By Remark 3.2, \( \Gamma_s \) is infinite and therefore \( G_s \) has positive dimension. Since \( s \) is in general position, \( G_s \) is normal in \( \text{Hdg}(X'_s) \). Now the simplicity of \( \text{Hdg}(X'_s) \) implies that \( \text{Hdg}(X'_s) = G_s \) and therefore
\[
\text{End}^0(X'_s) = \text{End}_{G_s}(H_1(X_s, Q)) = \text{End}_{\pi_1(S,s)}(H_1(X_s, Q)) = \text{End}^0_s(X) = \text{End}_{\mathcal{C}(S)}(X).
\]

\[
\Box
\]

3.8. Base change. Let \( S' \to S \) be a finite étale cover and \( X' = X \times_S S' \) the corresponding abelian \( S \)-scheme. If a point \( s' \in S' \) lies above \( s \in S \) then (in obvious notations)
\[
X'_{s'} = X_s, \; H_1(X_s, \mathbb{Z}) = H_1(X'_{s'}, \mathbb{Z}), \; H_1(X_s, \mathbb{Q}) = H_1(X'_{s'}, \mathbb{Q}), \; \text{Hdg}(X'_{s'}) = \text{Hdg}(X_s).
\]

The fundamental group \( \pi_1(S', s) \) of \( S' \) is a subgroup of finite index in \( \pi_1(S, s) \) and therefore \( \Gamma_{s'} \) is a subgroup of finite index in \( \Gamma_s \). It follows that \( G_{s'} \) is a subgroup of finite index in \( G_s \); in particular, \( G_{s'}^0 = G_s^0 \). This implies that \( s \) is in general position with respect to \( X' \) if and only if \( s' \) is in general position with respect to \( X' \). It also follows that if \( G_s \) is connected then \( G_{s'} = G_s \).

On the other hand, let \( \Gamma^0 \subset \Gamma_s \) be a subgroup of finite index (e.g., \( \Gamma^0 = \Gamma_s^0 \)). Clearly, its Zariski closure lies between \( G_s^0 \) and \( G_s^0 \); in particular, if \( \Gamma^0 \subset \Gamma_s^0 \) then this closure coincides with \( G_s^0 \). Let \( \pi^0 \subset \pi_1(S, s) \) be the preimage of \( \Gamma^0 \); it is a subgroup of finite index in \( \pi_1(S, s) \). Let \( S^0 \to S \) be a finite étale map of irreducible smooth algebraic curves attached to \( \pi^0 \) with \( \pi_1(S^0) = \pi^0 \). Clearly, the degree of \( S^0 \to S \) coincides with the index \( [\Gamma_s : \Gamma^0] \).

Notice that if \( \Gamma^0 \) is normal in \( \Gamma_s \), then \( S^0 \to S \) is Galois with Galois group \( \Gamma_s/\Gamma^0 \).

Fix a point \( s_0 \in S^0 \) that lies above \( s \). Then the image of the monodromy representation \( \pi_1(S^0, s_0) = \pi^0 \to \text{Aut}(H_1(X_s, Q)) \) attached to the abelian \( S^0 \)-scheme \( X \times_S S^0 \) coincides with \( \Gamma^0 \). In other words, \( \Gamma_{s_0} = \Gamma^0 \) and therefore \( G_{s_0} = G_s^0 \) is connected if \( \Gamma^0 \subset \Gamma_s \).

Example 3.9. Let \( n \geq 3 \) be an integer and \( \Gamma^n \subset \Gamma_s \) the kernel of the reduction map modulo \( n \)
\[
\Gamma_s \subset \text{Aut}(H_1(X_s, \mathbb{Z})) \twoheadrightarrow \text{Aut}(H_1(X_s, \mathbb{Z}/n\mathbb{Z})).
\]

Here \( H_1(X_s, \mathbb{Z}/n\mathbb{Z}) = H_1(X_s, \mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \) is the first integral homology group of \( X_s \) with coefficients in \( \mathbb{Z}/n\mathbb{Z} \). Clearly, \( \Gamma^n \) is a (normal) subgroup of finite index in \( \Gamma_s \). On the other hand, since \( \Gamma^n \subset 1 + n\text{End}(H_1(X_s, \mathbb{Z})) \), Zariski closure of \( \Gamma^n \) is connected \([31, \text{Prop. 2.6}]\). This implies that this closure coincides with \( G_s^0 \).

Remark 3.10. (i) It is known \([30]\) that all endomorphisms of \( X \) are defined over \( L := \mathcal{C}(S)(X_3) \). Clearly, \( L/\mathcal{C}(S) \) is a finite Galois extension and all points of order 3 on \( X \) are \( L \)-rational. It follows from Néron-Ogg-Shafarevich criterion \([29]\) that \( L/\mathcal{C}(S) \) is unramified at all points of \( S \). This implies that if \( S' \) is the normalization of \( S \) in \( K \) then \( S' \to S \) is finite étale. Clearly, \( \mathcal{C}(S') = L \).
(ii) Suppose that \( G_s \) is connected. Then \( \text{End}^0(X) = \text{End}^0_{C(S)}(X) \), i.e., all endomorphisms of \( X \) are defined over \( C(S) \). Indeed, first,

\[
\text{End}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Q})) = \text{End}_{\pi_1}(H_1(X_s, \mathbb{Q})) = \text{End}_{G_s}(H_1(X_s, \mathbb{Q}))
\]

and therefore \( \text{End}^0_{C(S)}(X) = \text{End}^0(X) \cap \text{End}_{G_s}(H_1(X_s, \mathbb{Q})) \). Second, pick a point \( s' \in S' \) that lies above \( s \). We have \( X_{s'} = X_s \), \( G_s = G_{s'} \), and

\[
\text{End}^0(X) = \text{End}_{C(S')}^0(X) = \text{End}^0(X_s') \cap \text{End}_{\pi_1(S',s')}(H_1(X_{s'}, \mathbb{Q})) = \text{End}^0(X_s) \cap \text{End}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Q})) = \text{End}_{C(S)}(X).
\]

3.11. Isogenies. It is well-known that \( u \in \text{Hom}_{C(S)}(Y,X) \) is an isogeny (resp. an \( \ell \)-isogeny) if and only if there exist \( v \in \text{Hom}_{C(S)}(X,Y) \) and a positive integer \( n \) such that the compositions \( uv \) and \( vu \) are multiplications by \( n \) (resp. by \( \ell^n \)) in \( X \) and \( Y \) respectively.

We say that \( u \in \text{Hom}_S(Y,X) \) is an isogeny (resp. an \( \ell \)-isogeny) of abelian schemes if the induced homomorphism of generic fibers \( u_0 : Y \to X \) is an isogeny of abelian varieties.

Clearly, \( u \) is an isogeny (resp. an \( \ell \)-isogeny) of abelian schemes if and only if there exist \( v \in \text{Hom}_S(X,Y) \) and a positive integer \( n \) such that the compositions \( uv \) and \( vu \) are multiplications by \( n \) (resp. by \( \ell^n \)) in \( X \) and \( Y \) respectively (see [4, Sect. 7.3]).

If \( u \in \text{Hom}_S(Y,X) \) is an isogeny (resp. an \( \ell \)-isogeny) of abelian schemes then it is clear that the induced homomorphism \( u_s : Y_s \to X_s \) is an isogeny (resp. an \( \ell \)-isogeny) of the corresponding complex abelian varieties [4, Sect. 7.3].

Conversely, suppose that \( u \in \text{Hom}_S(Y,X) \) and assume that \( u_s \in \text{Hom}(Y_s, X_s) \) is an isogeny (resp. an \( \ell \)-isogeny). Then there exist an isogeny \( w_s : X_s \to Y_s \) (resp. an \( \ell \)-isogeny) and a positive integer \( n \) such that the compositions \( w_su_s \) and \( u_sw_s \) are multiplications by \( n \) (resp. by \( \ell^n \)) in \( X_s \) and \( Y_s \) respectively. Since \( u_s \in \text{Hom}_{\pi_1(S,s)}(H_1(Y_s, \mathbb{Z}), H_1(X_s, \mathbb{Z})) \) and therefore

\[
u_s^{-1} \in \text{Hom}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Z}) \otimes \mathbb{Q}, H_1(Y_s, \mathbb{Z}) \otimes \mathbb{Q}),
\]

it follows that \( w_s \in \text{Hom}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Z}), H_1(Y_s, \mathbb{Z})) \). This implies that

\[
\text{Hom}(X_s, Y_s) \cap \text{Hom}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Z}), H_1(Y_s, \mathbb{Z}))
\]

and therefore there exists \( v \in \text{Hom}_S(X,Y) \) with \( v_s = w_s \). It follows that \( (uv)_s \) and \( (vu)_s \) are multiplications by \( n \) (resp. by \( \ell^n \)) in \( X_s \) and \( Y_s \) respectively. By the rigidity lemma (Sect. 3.3), \( uv \) and \( vu \) are multiplications by \( n \) (resp. by \( \ell^n \)) in \( X \) and \( Y \) respectively. This implies that \( u \) and \( v \) are isogenies (resp. \( \ell \)-isogenies) of abelian schemes.

3.12. Semisimplicity. Recall [6, Sect. 4.2] that the monodromy representation

\[
\pi_1(S,s) \to \text{Aut}(H_1(X_s, \mathbb{Z}) \subset \text{Aut}(H_1(X_s, \mathbb{Z}) \otimes \mathbb{Q})) = \text{Aut}_\mathbb{Q}(H_1(X_s, \mathbb{Q}))
\]

is completely reducible and therefore its centralizer

\[
D = D_\ell := \text{End}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Q})) \subset \text{End}_\mathbb{Q}(H_1(X_s, \mathbb{Q}))
\]

is a finite-dimensional semisimple \( \mathbb{Q} \)-algebra. In addition, the center \( E \) of \( D \) lies in \( \text{End}_S(X) \otimes \mathbb{Q} \) [6, Sect. 4.4.7]. It follows that \( E \) lies in the center \( C \) of \( \text{End}^0_{C(S)}(X) \).
Notice that $\text{End}_S(\mathcal{X}) \otimes \mathbb{Q} = \text{End}_{\mathbb{C}(S)}(X) \otimes \mathbb{Q}$. So, we have the inclusion of finite-dimensional semisimple $\mathbb{Q}$-algebras

$$\text{End}_{\mathbb{C}(S)}(X) \otimes \mathbb{Q} =: \text{End}^0_S(\mathcal{X}) \subset D \subset \text{End}_\mathbb{Q}(H_1(\mathcal{X}_s, \mathbb{Q}));$$

in addition, $E$ lies in $C$. It follows easily that if $D$ is commutative then $\text{End}^0_S(\mathcal{X}) = D$.

**Remark 3.13.** Assume that $X$ is simple. Then:

(i) $\text{End}^0_{\mathbb{C}(S)}(X)$ is a division algebra. Since the center $E$ of $D_f$ lies in $\text{End}^0_{\mathbb{C}(S)}(X)$ (Sect. 3.12), it has no zero divisors. This implies that $E$ is a field and therefore $D$ is a central simple $E$-algebra. It follows that the $H_1(\mathcal{X}_s, \mathbb{Q})$ is an isotypic $\pi_1(S, s)$-module, i.e., is either simple or isomorphic to a direct sum of several copies of a simple module.

(ii) Notice that $\text{End}^0_S(\mathcal{X}) = D$ if and only if $\text{End}_S(\mathcal{X}) = \text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Z}))$. It is also known [6, Cor. 4.4.13] that $\text{End}^0_S(\mathcal{X}) = D$ when $d \leq 3$ and $\mathcal{X}$ is not weakly isotrivial.

**Remark 3.14.** Faltings [9, Sect. 5] has constructed a principally polarized abelian scheme $f : \mathcal{X} \rightarrow S$ with $d = 4$ that is not weakly isotrivial, $\Gamma_s = \Gamma^n$ for a certain integer $n \geq 3$ (in notations of Example 3.9) and $\text{End}^0_S(\mathcal{X}) \neq D$. It follows from arguments in Example 3.9 that in Faltings’ example $G_s$ is connected.

## 4. Main results

Our main result is the following statement.

**Theorem 4.1.** Let $f : \mathcal{X} \rightarrow S$ be a polarized abelian scheme of positive relative dimension $d$. Then the following conditions are equivalent:

(i) $D_f \neq \text{End}^0_S(\mathcal{X})$.

(ii) The set $\text{Isog}(X, \mathbb{C}(S))$ is infinite.

Theorem 4.1 is an immediate corollary of the following two statements.

**Theorem 4.2.** Let $f : \mathcal{X} \rightarrow S$ be a polarized abelian scheme of positive relative dimension $d$. If $D_f = \text{End}^0_S(\mathcal{X})$ then $\text{Isog}(X, \mathbb{C}(S))$ is finite.

**Theorem 4.3.** Let $f : \mathcal{X} \rightarrow S$ be a polarized abelian scheme of positive relative dimension $d$. Suppose that $D = D_f \neq \text{End}^0_S(\mathcal{X})$ (in particular, $D$ is noncommutative). If $\ell$ is a prime such that $D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is isomorphic to a direct sum of matrix algebras over fields then there exist a positive integer $d_0 < 2d$ and a sequence $\{Z(m)\}_{m=1}^\infty$ of $\mathbb{C}(S)$-abelian varieties $Z(m)$ that enjoy the following properties:

(i) For each positive integer $m$ there exists an $\mathbb{C}(S)$-isogeny $Z(m) \rightarrow X$ of degree $\ell^{(2d-d_0)m}$.

(ii) Let $Y$ be an abelian variety over $\mathbb{C}(S)$ and let $M_Y$ be the set of positive integers $m$ such that $Z(m)$ is isomorphic to $Y$ over $\mathbb{C}(S)$. Then $M_Y$ is either empty or finite. In other words, if $M$ is an arbitrary infinite set of positive integers then there exists an infinite subset $M_0 \subset M$ such that for $m \in M_0$ all $Z(m)$ are mutually non-isomorphic over $\mathbb{C}(S)$.

**Remark 4.4.** Clearly, $D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is isomorphic to a direct sum of matrix algebras over fields for all but finitely many primes $\ell$.

We prove Theorems 4.3 and 4.2 in Sections 8 and 7 respectively.
Corollary 4.5. Suppose that a point \( s \in S \) is in general position and \( \text{Hdg}(\mathcal{X}_s) \) is \( \mathbb{Q} \)-simple. If \( \mathcal{X} \) is not weakly isotrivial then for all finite algebraic extension \( L/C(S) \) the set \( \text{Isog}(X, L) \) is finite.

Proof. Replacing \( S \) by its normalization in \( L \) and using (3.8), we may assume without loss of generality that \( L = K \). Choosing an integer \( n \geq 3 \), replacing \( S \) by \( S_n \) and applying (3.8) and Remark 2.3, we may assume without loss of generality that \( G_s \) is connected. It follows from Corollary 3.7 that

\[
\text{End}_{C(S)}(X) = \text{End}_{\mathbb{Q}}^0(\mathcal{X}) = \text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Q})).
\]

Now the result follows from Theorem 4.1.

Corollary 4.6. Suppose that \( D_f \neq \text{End}_{\mathbb{Q}}^0(\mathcal{X}) \). If \( X \) admits a principal polarization over \( C(S) \) then \( \text{Isog}_1(X^2, C(S), \ell) \) is infinite for all but finitely many primes \( \ell \) congruent to 1 modulo 4.

Proof. Let us pick a prime \( \ell \) such that \( D_f \otimes \mathbb{Q}_\ell \) is a direct sum of matrix algebras over fields. It follows from Theorem 4.3 that \( \text{Isog}(X, C(S), \ell) \) is infinite. Applying Lemma 2.2, we conclude that \( \text{Isog}_1(X^2, C(S), \ell) \) is infinite if \( \ell \) is congruent to 1 modulo 4. Now the result follows from Remark 4.4.

Till the end of this Section we assume that \( K \) is a field of algebraic functions in one variable over \( C \), i.e., \( K \) is finitely generated and of degree of transcendency 1 over \( C \).

Theorem 4.7. Let \( X \) be an abelian variety over \( K \) of positive dimension \( d \). Suppose that the \( \overline{K}/C \)-trace of \( X \) is zero. If \( d \leq 3 \) then \( \text{Isog}(X, K) \) is finite.

Proof of Theorem 4.7. There exists a smooth irreducible algebraic \( C \)-curve \( S \) such that \( K = C(S) \) and there exists a polarized abelian scheme \( \mathcal{X} \to S \) of relative dimension \( d \leq 3 \), whose generic fiber coincides with \( X \) [4, Sect.1.4, p.20]. Pick a point \( s \in S(C) \). Recall [15, Ch. VIII, Sect. 3] that the vanishing of the \( C(S)/C \)-trace means that \( X \) does not contain over \( C(S) \) a non-zero abelian subvariety that is isomorphic over \( C(S) \) to a (constant) abelian variety of the form \( W \times_{C C(S)} \mathcal{V} \) where \( W \) is a complex abelian variety. By Theorem 3.1, this means \( X \) is not weakly isotrivial. It follows from results of Deligne (Remark 3.13(ii)) that \( \text{End}_S(\mathcal{X}) = \text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Z})) \) and therefore \( D_f := \text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Q})) = \text{End}_{\mathbb{Q}}^0(\mathcal{X}) \). Now the assertion follows from Theorem 4.3.

Corollary 4.8. Let \( X \) be an abelian variety over \( K \) that is isogenous over \( \overline{K} \) to a product of abelian varieties of dimension \( \leq 3 \). If \( \overline{K}/C \)-trace of \( X \) is zero then \( \text{Isog}(X, K) \) is finite.

Proof. Replacing if necessary, \( K \) by its finite Galois extension and using Remarks 2.3 and 2.1, we may assume without loss of generality that \( X \) is isomorphic over \( K \) to a product \( Y_1 \times \cdots \times Y_r \) of abelian varieties \( Y_i \)'s of dimension \( \leq 3 \) over \( K \).

There exists a smooth irreducible algebraic \( C \)-curve \( S \) such that \( K = C(S) \) and there exist polarized abelian schemes \( \mathcal{Y}_i \to S \), whose generic fiber coincides with \( Y_i \) [4, Sect.1.4, p.20]. If \( f : \mathcal{X} \to S \) is the fiber product of \( \mathcal{Y}_i \) then the generic fiber of the abelian \( S \)-scheme \( \mathcal{X} \) coincides with \( X \).

Pick a point \( s \in S(C) \). Since all \( \dim(Y_i) \leq 3 \), it follows from results of Deligne [6, Cor. 4.4.13] applied to all pairs \( (\mathcal{Y}_i, \mathcal{Y}_j) \) that \( \text{End}_S(\mathcal{X}) = \text{End}_{\pi_1(S, s)}(H_1(\mathcal{X}_s, \mathbb{Z})) \).
and therefore \( D_f := \text{End}_{\mathbb{Q},(H_1(X, \mathbb{Q}))} = \text{End}_{\mathbb{Q}}(X) \). Now the assertion follows from Theorem 4.3. □

**Corollary 4.9.** Let \( X \) be a four-dimensional abelian variety over \( K \). Suppose that the \( K/\mathbb{Q} \)-trace of \( X \) is zero. If \( X \) is not absolutely simple then \( \text{Isog}(X, K) \) is finite.

**Proof.** Replacing if necessary, \( K \) by its finite Galois extention and using Remarks 2.3 and 2.1, we may assume without loss of generality that \( X \) is isomorphic over \( K \) to a product \( Y \times T \) of abelian varieties \( Y \) and \( T \) of positive dimension over \( K \). Since \( 4 = \text{dim}(X) = \text{dim}(Y) + \text{dim}(T) \), we conclude that both \( \text{dim}(Y) \) and \( \text{dim}(T) \) do not exceed 3. One has only to apply Corollary 4.8. □

**Example 4.10.** Let \( f : \mathcal{X} \to S \) be the Faltings’ example (Remark 3.14) and let \( X \) be its generic fiber, which is a principally polarized four-dimensional abelian variety over \( \mathbb{C}(S) \). Since \( \mathcal{X} \) is not weakly isotrivial, it follows from Theorem 3.1 that the \( \mathbb{C}(S)/\mathbb{Q} \)-trace of \( X \) is zero. Since \( D_f \neq \text{End}_{\mathbb{Q}}(X) \), it follows that \( \text{Isog}(X, \mathbb{C}(S)) \) is infinite and \( \text{Isog}_{\mathbb{C}}(X^2, \mathbb{C}(S), \ell) \) is also infinite for all but finitely many primes \( \ell \) congruent to 1 modulo 4. Recall that \( G_x \) is connected (3.14) and therefore all endomorphisms of \( X \) are defined over \( \mathbb{C}(S) \), thanks to Remark 3.10. It follows from Corollary 4.9 that \( X \) is absolutely simple.

We discuss isogeny classes of absolutely simple abelian fourfolds in Section 10.

5. **Non-isotrivial Abelian schemes**

**Proof of Theorem 3.1.** In one direction the assertion is almost obvious. Indeed, if \( \mathcal{X}_S \) is a constant abelian scheme (resp. contains a non-zero constant abelian subscheme) \( W \times_{\mathbb{C}} S' \) where \( W \) is an abelian variety over \( \mathbb{C} \) then \( X \times_{\mathbb{C}(S)} \mathbb{C}(S') \) is isomorphic to \( W \times_{\mathbb{C}} \mathbb{C}(S') = (W \times_{\mathbb{C}} \mathbb{C}(S)) \times_{\mathbb{C}(S)} \mathbb{C}(S') \) (resp. contains an abelian subvariety that is defined over \( \mathbb{C}(S') \) and is isomorphic to \( W \times_{\mathbb{C}} \mathbb{C}(S') \) over \( \mathbb{C}(S') \)).

(By the way, we did not use an assumption that \( S' \to S \) is étale.)

In the opposite direction, let \( W \) be an abelian variety over \( \mathbb{C} \), let \( W_{\bar{\eta}} \) be the constant abelian variety \( W \times_{\mathbb{C}} \mathbb{C}(S) \) and \( \bar{u} : W_{\eta} \times_{\mathbb{C}(S)} \mathbb{C}(S) \hookrightarrow X \times_{\mathbb{C}(S)} \mathbb{C}(S) \) an embedding of abelian varieties over \( \mathbb{C}(S) \). Let \( L = \mathbb{C}(S)(X_3) \) be the field of definition of all points of order 3 on \( X \) and \( S' \) is the normalization of \( S \) in \( L \).

By Remark 3.10(i), \( S' \to S \) is a finite etale map of smooth irreducible curves, \( \mathbb{C}(S') = L \) and all points of order 3 on \( X \) are \( L \)-rational. Clearly, all torsion points of \( W_{\eta} \) are \( \mathbb{C}(S) \)-rational; in particular, all points of order 3 on \( W_{\eta} \) are defined over \( L \). It follows from results of [30] that all \( \mathbb{C}(S) \)-homomorphisms between \( W_{\eta} \) and \( X \) are defined over \( \mathbb{C}(S') \); in particular, \( \bar{u} \) is defined over \( \mathbb{C}(S') \), i.e., there exists an embedding \( u : W_{\eta} \times_{\mathbb{C}(S)} \mathbb{C}(S') \hookrightarrow X \times_{\mathbb{C}(S)} \mathbb{C}(S') \) of abelian varieties over \( \mathbb{C}(S') \) such that \( \bar{u} \) is obtained from \( u \) by extension of scalars from \( \mathbb{C}(S') \) to \( \mathbb{C}(S) \). Notice that the \( \mathbb{C}(S') \)-abelian varieties \( X \times_{\mathbb{C}(S)} \mathbb{C}(S') \) and \( W \times_{\mathbb{C}} \mathbb{C}(S') \) are generic fibers of abelian \( S' \)-schemes \( \mathcal{X}_{S'} \) and \( \mathcal{X} \times_{\mathbb{C}} S' \) respectively. It follows that \( u \) extends to a certain homomorphism of abelian \( S' \)-schemes \( W \times_{\mathbb{C}} S' \to \mathcal{X}_{S'} \), which we denote by \( u_{S'} \). If \( u \) is an isomorphism of generic fibers (the isotrivial case) then \( u_{S'} \) is an isomorphism of the corresponding abelian schemes; in particular, \( \mathcal{X}_{S'} \) is a constant abelian scheme. If \( u \) is not an isomorphism, it is still a closed embedding; in particular, the image \( Y \) of \( u : W \times_{\mathbb{C}} \mathbb{C}(S') \to X \times_{\mathbb{C}(S)} \mathbb{C}(S') \) is an abelian subvariety in \( X \times_{\mathbb{C}(S)} \mathbb{C}(S') \) and the abelian varieties \( Y \) and \( W \times_{\mathbb{C}} \mathbb{C}(S') \) are isomorphic over \( \mathbb{C}(S') \). Let \( \mathcal{Y} \) be the schematic closure of the image \( Y \) in \( \mathcal{X}_{S'} \).
It follows from Corollary 6 on p. 175 of [4] that $Y$ is the Néron model of $Y$ over $S'$. Since $W \times_C C(S') \cong Y$ over $C(S')$ and the Néron model of $W \times_C C(S')$ over $S'$ is $W \times S'$, we conclude that $X_{S'}$ contains an abelian subscheme isomorphic to the constant abelian scheme $W \times_C S'$.

\[ \square \]

6. ISOGENY CLASSES OF ABELIAN SCHEMES

Let $X \to S$ be a polarized abelian scheme of positive relative dimension $d$. Let us consider the category $\text{Is}(\mathcal{X})$, whose objects are pairs $(\mathcal{Y}, \alpha)$ that consist of an abelian scheme $g : \mathcal{Y} \to S$ and an isogeny $\alpha : Y \to X$ of abelian schemes and the set of morphisms $\text{Mor}(\mathcal{Y}_1, \mathcal{Y}_2, (\mathcal{Y}_1, \alpha_1), (\mathcal{Y}_2, \alpha_2)) := \text{Hom}_S(\mathcal{Y}_1, \mathcal{Y}_2)$ for any pair of objects $(\mathcal{Y}_1, \alpha_1)$ and $(\mathcal{Y}_2, \alpha_2)$. Let us consider the category $\text{Is}_s(\mathcal{X})$, whose objects are pairs $(\Lambda, i)$ that consist of an $\pi_1(S, s)$-module $\Lambda$, whose additive group is isomorphic to $\mathbb{Z}^{2d}$ and an embedding $i : \Lambda \to H_1(X_s, \mathbf{Z})$ of $\pi_1(S, s)$-modules and the set of morphisms $\text{Mor}_{\text{Is}_s(\mathcal{X})}(\Lambda_1, \Lambda_2), (\Lambda_1, i_1), (\Lambda_2, i_2))$ is the set

$$ \{ a \in \text{End}_s(\mathcal{X}_s) | \exists u \in \text{End}_s(\mathcal{X}_s) \text{ such that } i_2a = ui_1 \}. $$

**Theorem 6.1.** The functor

$$ \Psi_s : \text{Is}(\mathcal{X}) \to \text{Is}_s(\mathcal{X}), (\mathcal{Y}, \alpha) \mapsto (H_1(\mathcal{Y}_s, \mathbf{Z}), \alpha_s : H_1(\mathcal{Y}_s, \mathbf{Z}) \to H_1(X_s, \mathbf{Z})), $$

$$ (\gamma : \mathcal{Y}_1 \to \mathcal{Y}_2) \mapsto \gamma_s \in \text{Hom}(\mathcal{Y}_1, \mathcal{Y}_2) \cap \text{Hom}_{\pi_1(S, s)}(H_1(\mathcal{Y}_1, \mathbf{Z}), H_1(\mathcal{Y}_2, \mathbf{Z})) $$

is an equivalence of categories.

**Proof.** First, we need to check that $\gamma_s \in \text{Mor}_{\text{Is}_s(\mathcal{X})}(H_1(\mathcal{Y}_1, \mathbf{Z}), H_1(\mathcal{Y}_2, \mathbf{Z}))$, i.e. there exists $u \in \text{End}_s(\mathcal{X}_s)$ such that $\alpha_2 \gamma_s = u \alpha_1 s$. (Recall that $\alpha_1 : \mathcal{Y}_1 \to X, \alpha_2 : \mathcal{Y}_2 \to X$ are isogenies of abelian schemes and $\gamma_s \in \text{Hom}(\mathcal{Y}_1, \mathcal{Y}_2)$.) Clearly, both $\alpha_1 s : \mathcal{Y}_1 \to X_s, \alpha_2 s : \mathcal{Y}_2 \to X_s$ are isogenies of abelian varieties. Then $u := \alpha_2 s \gamma_s \alpha_1 s^{-1} \in \text{End}_s(\mathcal{X}_s)$ satisfies $\alpha_2 \gamma_s = u \alpha_1 s$. Second, the injectiveness and surjectiveness of $\text{Hom}(\mathcal{Y}_1, \mathcal{Y}_2) \to \text{Hom}(\mathcal{Y}_1, \mathcal{Y}_2) \cap \text{Hom}_{\pi_1(S, s)}(H_1(\mathcal{Y}_1, \mathbf{Z}), H_1(\mathcal{Y}_2, \mathbf{Z}))$ follow from the rigidity lemma and Grothendieck’s theorem (Sect. 3.3) respectively.

Let $(\Lambda, i_s)$ be an object of $\text{Is}_s(\mathcal{X})$, i.e., a $\pi_1(S, s)$-module $\Lambda$, whose additive group is isomorphic to $\mathbb{Z}^{2d}$ and an embedding $i_s : \Lambda \to H_1(X_s, \mathbf{Z})$ of $\pi_1(S, s)$-modules. In order to check the essential surjectiveness of $\Psi_s$, we need to construct an abelian scheme $g : \mathcal{Y} \to S$, an isogeny $\alpha : \mathcal{Y} \to X$ and an isomorphism $\phi_s : \Lambda \cong H_1(\mathcal{Y}_s, \mathbf{Z})$ of $\pi_1(S, s)$-modules such that $\phi_s = \alpha s i_s$. In order to do that, recall that the $\pi_1(S, s)$-module $\Lambda$ defines a certain local system $\mathcal{U}$ of free $\mathbf{Z}$-modules of rank $2d$ on $S$, whose fiber at $s$ coincides with $\Lambda$. In addition, $i_s$ defines an embedding of local systems $i : \mathcal{U} \to R_1 f_* \mathbf{Z}$, whose fiber at $s$ coincides with our “original” $i_s$. Rank arguments imply that the corresponding embedding

$$ i : \mathcal{U} \otimes \mathbf{Q} \to R_1 f_* \mathbf{Z} \otimes \mathbf{Q} = R_1 f_* \mathbf{Q} $$

is, in fact, an isomorphism. This allows us to provide $\mathcal{U}$ with the structure (induced by $R_1 f_* \mathbf{Z}$) of the holomorphic family of polarized Hodge structures of type $(-1, 0) + (0, -1)$ [6, Sect. 4.4]. The equivalence of the category of polarized abelian schemes and the category of holomorphic families of polarized Hodge structures of type $(-1, 0) + (0, -1)$ over $S$ [6, Sect. 4.4.2 and 4.4.3] (based on results of [3]) implies that there exist an abelian scheme $g : \mathcal{Y} \to S$, a homomorphism $\alpha \in \text{Hom}_S(\mathcal{Y}, \mathcal{X})$ and an isomorphism of local systems $\phi_S : \mathcal{U} \cong R_1 g_* \mathbf{Z}$ such that $\alpha \psi_S = i$. Taking the fiber of the latter equality at $s$, we get the desired $\alpha_s(\psi_S)_s = i_s$. Clearly, $\alpha_s \in \text{Hom}(\mathcal{Y}_s, \mathcal{X}_s)$ induces an isomorphism $H_1(\mathcal{Y}_s, \mathbf{Q}) \cong H_1(\mathcal{X}_s, \mathbf{Q})$ and therefore
is an isogeny. Applying results of Section 3.11, we conclude that \( \alpha \) is an isogeny of abelian schemes. \( \Box \)

**Remark 6.2.** The degree of \( \alpha \) coincides with the index \([H_1(\mathcal{X}_s, \mathbb{Z}) : i_s(\Lambda)]\).

### 7. Finite isogeny classes

Let \( L \) (resp. \( L_Q \)) be the image of the group algebra \( \mathbb{Z}[\pi_1(S, s)] \) (resp. of \( \mathbb{Q}[\pi_1(S, s)] \)) in \( \text{End}(H_1(\mathcal{X}_s, \mathbb{Z})) \) (resp. in \( \text{End}_Q(H_1(\mathcal{X}_s, \mathbb{Q})) \)) induced by the monodromy representation. Clearly, \( L \) is an order in the \( \mathbb{Q} \)-algebra \( L_Q \).

It follows from Jackobson’s density theorem and the semisimplicity of the monodromy representation (over \( \mathbb{Q} \)) that \( L_Q \) is a semisimple \( \mathbb{Q} \)-algebra that coincides with the centralizer \( \text{End}_{D_f}(H_1(\mathcal{X}_s, \mathbb{Q})) \) of \( D_f \) and

\[
\text{End}_{L_Q}(H_1(\mathcal{X}_s, \mathbb{Q})) = D_f.
\]

Clearly, each \( \pi_1(S, s) \)-stable \( \mathbb{Z} \)-lattice in \( H_1(\mathcal{X}_s, \mathbb{Q}) \) is an \( L \)-module, whose additive group is isomorphic to \( \mathbb{Z}^{2d} \).

#### 7.1. Finite isogeny classes

It follows from the Jordan-Zassenhaus theorem [27, Theorem 26.4] that there are, up to an isomorphism, only finitely many \( L \)-modules, whose additive group is isomorphic to \( \mathbb{Z}^{2d} \).

**Proof of Theorem 4.2.** Since \( D_f = \text{End}_\mathbb{Q}^0(\mathcal{X}) \), we have

\[
D_f = \text{End}_\mathbb{Q}^0(\mathcal{X}) \subset \text{End}^0(\mathcal{X}_s) \subset \text{End}_Q(H_1(\mathcal{X}_s, \mathbb{Q})).
\]

Thanks to Theorem 6.1, it suffices to check that the set of isomorphism classes of objects in \( \text{Is}_s(\mathcal{X}) \) is finite. Thanks to the Jordan-Zassenhaus theorem (Sect. 7.1), it becomes an immediate corollary of the following statement.

**Lemma 7.2.** Suppose that \((\Lambda_1, i_1)\) and \((\Lambda_2, i_2)\) are objects in \( \text{Is}_s(\mathcal{X}) \).

Then \( i_1(\Lambda_1) \) and \( i_2(\Lambda_2) \) are \( L \)-submodules in \( H_1(\mathcal{X}_s, \mathbb{Z}) \), whose additive group is isomorphic to \( \mathbb{Z}^{2d} \). If the \( L \)-modules \( i_1(\Lambda_1) \) and \( i_2(\Lambda_2) \) are isomorphic then \((\Lambda_1, i_1)\) and \((\Lambda_2, i_2)\) are isomorphic.

**Proof of Lemma 7.2.** Recall that \( \Lambda_1 \) and \( \Lambda_2 \) are \( \pi_1(S, s) \)-modules, whose additive groups are isomorphic to \( \mathbb{Z}^{2d} \) and \( i_1 : \Lambda_1 \hookrightarrow H_1(\mathcal{X}_s, \mathbb{Z}) \), \( i_2 : \Lambda_2 \hookrightarrow H_1(\mathcal{X}_s, \mathbb{Z}) \) are embeddings of \( \pi_1(S, s) \)-modules. Clearly, \( i_1 : \Lambda_1 \cong i_1(\Lambda_1) \), \( i_2 : \Lambda_2 \cong i_2(\Lambda_2) \) are isomorphisms of \( \pi_1(S, s) \)-modules. Both \( i_1(\Lambda_1) \) and \( i_2(\Lambda_2) \) are \( L \)-submodules in \( H_1(\mathcal{X}_s, \mathbb{Z}) \).

Assume that there exists an isomorphism of \( L \)-modules: \( \alpha : i_1(\Lambda_1) \cong i_2(\Lambda_2) \). Clearly, \( \alpha \) is an isomorphism of \( \pi_1(S, s) \)-modules. Extending \( \alpha \) by \( \mathbb{Q} \)-linearity, we obtain an isomorphism

\[
u : i_1(\Lambda_1) \otimes \mathbb{Q} \cong i_2(\Lambda_2) \otimes \mathbb{Q}
\]

of \( L \otimes \mathbb{Q} \)-modules. Recall that \( L \otimes \mathbb{Q} = L_Q \), \( i_1(\Lambda_1) \otimes \mathbb{Q} = i_2(\Lambda_2) \otimes \mathbb{Q} = H_1(\mathcal{X}_s, \mathbb{Q}) \). We have

\[
u \in \text{End}_{L_Q}(H_1(\mathcal{X}_s, \mathbb{Q})) \cap \text{Aut}_\mathbb{Q}(H_1(\mathcal{X}_s, \mathbb{Q})) = D_f \cap \text{Aut}_\mathbb{Q}(H_1(\mathcal{X}_s, \mathbb{Q})).
\]

Since \( D_f \subset \text{End}^0(\mathcal{X}_s) \), we obtain that \( u \in \text{End}^0(\mathcal{X}_s) \). Since \( u \in \text{End}^0(\mathcal{X}_s) \subset \text{End}_Q(H_1(\mathcal{X}_s, \mathbb{Q})) \) is an automorphism of the finite-dimensional \( \mathbb{Q} \)-vector space \( H_1(\mathcal{X}_s, \mathbb{Q}) \), we have \( u^{-1} \in \text{End}^0(\mathcal{X}_s)^* \), i.e., \( u \in \text{End}^0(\mathcal{X}_s)^* \). Now if we put

\[
\alpha := i_2^{-1} \alpha i_1 : \Lambda_1 \to i_1(\Lambda_1) \to i_2(\Lambda_2) \to \Lambda_2
\]
then \( a \in \text{Hom}_{D_1(S_s)}(A_1, A_2) \) is an isomorphism of \( \pi_1(S, s) \)-modules and \( i_2a = u_{i_1}, \ i_1a^{-1} = u^{-1}i_2 \). This implies that \( a \) is an isomorphism of \( (A_1, i_1) \) and \( (A_2, i_2) \).

\[ \square \]

8. INFINITE ISOGENY CLASSES

We deduce Theorem 4.3 from the two following auxiliary statements.

**Lemma 8.1.** Let \( V \) be a finite-dimensional vector space over a field \( k \), let \( \text{Id}_V \) be the identity automorphism of \( V \), let \( A \subset \text{End}_k(V) \) a \( k \)-subalgebra that contains \( k \cdot \text{Id}_V \) and is isomorphic to a direct sum of matrix algebras over fields. Let \( B \subset A \) be a semisimple \( k \)-subalgebra that contains the center of \( A \) but does not coincide with \( A \).

Then there exists a proper subspace \( W \subset V \) that enjoys the following properties:

(i) There does exist \( a \in A \) such that \( a(V) = W \).

(ii) There does not exist \( b \in B \) such that \( b(V) = W \).

In order to state the next lemma, let us choose a prime \( \ell \) and consider the natural \( \mathbb{Q}_\ell \)-linear representation

\[
\pi_1(S, s) \to \text{Aut}(H_1(X_s, \mathbb{Q})) \subset \text{Aut}_{\mathbb{Q}_\ell}(H_1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \subset \text{Aut}_{\mathbb{Q}_\ell}(H_1(X_s, \mathbb{Q}_\ell));
\]

here \( H_1(X_s, \mathbb{Q}_\ell) = H_1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \). Clearly, this representation remains semisimple and the centralizer of \( \pi_1(S, s) \) in \( \text{End}_{\mathbb{Q}_\ell}(H_1(X_s, \mathbb{Q}_\ell)) \) coincides with \( D_\ell \otimes \mathbb{Q}_\ell \).

Let us put \( D = D_\ell \). We have

\[
\text{End}_S^0(X) \otimes \mathbb{Q}_\ell \subset D \otimes \mathbb{Q}_\ell \subset \text{End}_{\mathbb{Q}_\ell}(H_1(X_s, \mathbb{Q}_\ell)).
\]

Clearly, \( E \otimes \mathbb{Q}_\ell \) is the center of \( D \otimes \mathbb{Q}_\ell \) and lies in \( C \otimes \mathbb{Q}_\ell \), which is the center of \( \text{End}_S^0(X) \otimes \mathbb{Q}_\ell \). It is also clear that both \( \mathbb{Q}_\ell \)-algebras \( D \otimes \mathbb{Q}_\ell \) and \( \text{End}_S^0(X) \otimes \mathbb{Q}_\ell \) are semisimple finite-dimensional.

**Lemma 8.2.** Let \( \ell \) be a prime, \( W \) a proper \( \pi_1(S, s) \)-stable \( \mathbb{Q}_\ell \)-vector space of \( H_1(X_s, \mathbb{Q}_\ell) \) and \( d_0 = \text{dim}_{\mathbb{Q}_\ell}(W) \). Then one of the following two conditions holds:

(1) There exists \( u \in D \otimes \mathbb{Q}_\ell \) with \( u(H_1(X_s, \mathbb{Q}_\ell)) = W \).

(2) There exists a sequence \( \{Z(m)\}_{m=1}^{\infty} \) of \( \mathbb{C}(S) \)-abelian varieties \( Z(m) \) that enjoy the following properties:

(i) For each positive integer \( m \) there exists a \( \mathbb{C}(S) \)-isogeny \( Z(m) \to X \) of degree \( \ell^{(2d-d_0)m} \).

(ii) Let \( Y \) be an abelian variety over \( \mathbb{C}(S) \) and let \( M_Y \) be the set of positive integers \( m \) such that \( Z(m) \) is isomorphic to \( Y \) over \( \mathbb{C}(S) \). Then \( M_Y \) is either empty or finite.

**Remark 8.3.** The statement (and the proof) of Lemma 8.2 is inspired by [32, Prop. 1, pp. 136–137].

**Proof of Theorem 4.3 (modulo Lemmas 8.1 and 8.2).** Let us apply Lemma 8.1 to

\[ k = \mathbb{Q}_\ell, V = H_1(X_s, \mathbb{Q}_\ell), A = D \otimes \mathbb{Q}_\ell, B = \text{End}_S^0(X) \otimes \mathbb{Q}_\ell. \]

We conclude that there exist a proper \( \mathbb{Q}_\ell \)-vector subspace \( W \) in \( H_1(X_s, \mathbb{Q}_\ell) \), an element \( v \in D \otimes \mathbb{Q}_\ell \) with \( v(H_1(X_s, \mathbb{Q}_\ell)) = W \) but there does not exist an element \( u \in \text{End}_S^0(X) \otimes \mathbb{Q}_\ell \) with \( u(H_1(X_s, \mathbb{Q}_\ell)) = W \). Since \( v \) lies in \( D \otimes \mathbb{Q}_\ell \), it commutes
with $\pi_1(S,s)$ and therefore $W$ is $\pi_1(S,s)$-stable. Now the result follows from Lemma 8.2.

**Proof of Lemma 8.2.** Let us consider the $\pi_1(S,s)$-stable $\mathbb{Z}_l$-lattice $H_1(\mathcal{X}_s, \mathbb{Z}_l) = H_1(\mathcal{X}_s, \mathbb{Z}) \otimes \mathbb{Z}_l$ in the $\mathbb{Q}_l$-vector space $H_1(\mathcal{X}_s, \mathbb{Q}_l)$. Notice that for each positive integer $m$ we have a canonical isomorphism of free $\mathbb{Z}/\ell^m\mathbb{Z}$-modules

$$H_1(\mathcal{X}_s, \mathbb{Z}_l)/\ell^m H_1(\mathcal{X}_s, \mathbb{Z}_l) = H_1(\mathcal{X}_s, \mathbb{Z})/\ell^m H_1(\mathcal{X}_s, \mathbb{Z}) = H_1(\mathcal{X}_s, \mathbb{Z}/\ell^m\mathbb{Z})$$

(induced by the canonical isomorphism $\mathbb{Z}/\ell^m\mathbb{Z} = \mathbb{Z}_l/\ell^m\mathbb{Z}_l$) that commutes with the actions of $\pi_1(S,s)$.

The intersection $T := W \cap H_1(\mathcal{X}_s, \mathbb{Z}_l)$ is a $\pi_1(S,s)$-stable free pure $\mathbb{Z}_l$-submodule of rank $d_0$ in $H_1(\mathcal{X}_s, \mathbb{Z}_l)$. Clearly, $T \subset W$ and the natural map $T \otimes \mathbb{Z}_l \mathbb{Q}_l \to W$ is an isomorphism of $\mathbb{Q}_l$-vector spaces. The image $T_m$ of $T$ in $H_1(\mathcal{X}_s, \mathbb{Z}/\ell^m\mathbb{Z})$ is a free $\mathbb{Z}/\ell^m\mathbb{Z}$-submodule of rank $d_0$. The preimage of $T_m$ in $H_1(\mathcal{X}_s, \mathbb{Z}_l)$ coincides with $T + \ell^m H_1(\mathcal{X}_s, \mathbb{Z}_l)$.

We write $\Lambda_m$ for the preimage of $T_m$ in $H_1(\mathcal{X}_s, \mathbb{Z})$. Clearly, $T + \ell^m H_1(\mathcal{X}_s, \mathbb{Z}_l)$ contains $\Lambda_m$ and the natural map $\Lambda_m \otimes \mathbb{Z}_l \to T + \ell^m H_1(\mathcal{X}_s, \mathbb{Z}_l)$ is an isomorphism of free $\mathbb{Z}_l$-modules. Notice that $\Lambda_m$ is a $\pi_1(S,s)$-stable subgroup of index $\ell^m(2d-d_0)$ in $H_1(\mathcal{X}_s, \mathbb{Z})$ and contains $\ell^m H_1(\mathcal{X}_s, \mathbb{Z}_l)$. It follows from Theorem 6.1 that there exists an abelian scheme $h_m : Z(m) \to S$ and an isogeny $\gamma(m) : Z(m) \to \mathcal{X}$ of abelian schemes of degree $\ell^m(2d-d_0)$ such that $\gamma(m)_* (h_1(Z(m), s)) = \Lambda_m$. (Here $Z(m)$ is the fiber of $Z(m)$ over $s$.) Since $\Lambda_m$ contains $\ell^m H_1(\mathcal{X}_s, \mathbb{Z})$, there exists an isogeny $\gamma(m)^*_s : X_s \to Z(m)_s$ of degree $\ell^{md_0}$ such that the compositions

$$\gamma(m)_* \gamma(m)^*_s : X_s \to Z(m)_s \to X_s, \quad \gamma(m)^*_s \gamma(m)_s : Z(m)_s \to X_s \to Z(m)_s$$

coincide with multiplication(s) by $\ell^m$. Clearly,

$$\gamma(m)_* (h_1(Z(m)_s, \mathbb{Z}_l)) = \gamma(m)_* (h_1(Z(m)_s, \mathbb{Z}) \otimes \mathbb{Z}_l) = \Lambda_m \otimes \mathbb{Z}_l = T + \ell^m H_1(\mathcal{X}_s, \mathbb{Z}_l).$$

We also have $H_1(Z(m)_s, \mathbb{Z}) \supset \gamma(m)^*_s (h_1(X_s, \mathbb{Z})) \supset \ell^m H_1(Z(m)_s, \mathbb{Z})$ and therefore

$$H_1(Z(m)_s, \mathbb{Z}_l) \supset \gamma(m)^*_s (h_1(X_s, \mathbb{Z}_l)) \supset \ell^m H_1(Z(m)_s, \mathbb{Z}_l) = \ell^m H_1(Z(m)_s, \mathbb{Z}_l).$$

Since $\gamma(m)_s \in \text{Hom}_{\pi_1(S,s)}(H_1(Z(m)_s, \mathbb{Z}_l), H_1(X_s, \mathbb{Z}_l))$, we have

$$\gamma(m)_s^{-1} \in \text{Hom}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Q}), H_1(Z(m)_s, \mathbb{Q})).$$

It follows that $\gamma(m)^*_s = \ell^m \gamma(m)^{-1}$ lies in

$$\text{Hom}_{\pi_1(S,s)}(H_1(X_s, \mathbb{Q}), H_1(Z(m)_s, \mathbb{Q})) \cap \text{Hom}(H_1(X_s, \mathbb{Z}), H_1(Z(m)_s, \mathbb{Z})).$$

This implies that $\gamma(m)^*_s$ coincides with the “fiber over” $s$ of a certain isogeny of abelian schemes $\gamma(m)^*_s : \mathcal{X} \to Z(m)$ of degree $\ell^{md_0}$. The rigidity lemma implies that the compositions

$$\gamma(m) \gamma(m)^*_s : \mathcal{X} \to Z(m) \to \mathcal{X}, \quad \gamma(m)^*_s \gamma(m) : Z(m) \to \mathcal{X} \to Z(m)$$

coincide with multiplication(s) by $\ell^m$. The generic fiber $Z(m)$ of $Z(m)$ is a $\mathcal{C}(S)$-abelian variety and $\gamma(m)$ and $\gamma(m)^*$ induce $\mathcal{C}(S)$-isogenies $\gamma(m)_s : Z(m) \to X$, $\gamma(m)^*_s : X \to Z(m)$. Their degrees are $\ell^m(2d-d_0)$ and $\ell^{md_0}$ respectively. Their composition(s)

$$\gamma(m)_s \gamma(m)^*_s : X \to Z(m) \to X, \quad \gamma(m)^*_s \gamma(m)_s : Z(m) \to X \to Z(m)$$


Let us consider the composition with respect to the \( \ell \)
therefore the \( v_i \) follows that for all \( \ell \)
and therefore \( v_i \) extends to an isomorphism of abelian schemes \( \mathbb{Z}(n) \rightarrow \mathbb{Z}(i) \), which we denote by \( v_i \). Since \( v_i \) is an isomorphism, its specialization \( v_{i,s} \) at \( s \) satisfies
\[
v_{i,s}(H_1(\mathbb{Z}(n), \mathbb{Z}_\ell)) = H_1(\mathbb{Z}(i), \mathbb{Z}_\ell).
\]
Let us consider the composition
\[
u_i := \gamma(i) v_i \gamma(n) : \mathcal{X} \rightarrow \mathbb{Z}(n) \rightarrow \mathbb{Z}(i) \rightarrow \mathcal{X}.
\]
We have
\[
u_i \in \text{End}_S(\mathcal{X}) \subset \text{End}_S(\mathcal{X}) \otimes \mathbb{Z}_\ell \subset \text{End}_S(\mathcal{X}) \otimes \mathbb{Q}_\ell = \text{End}_S(\mathcal{X}) \otimes \mathbb{Q}_\ell.
\]
Since
\[
H_1(\mathbb{Z}(n), \mathbb{Z}_\ell) \supset \gamma(n)^s(H_1(\mathcal{X}, \mathbb{Z}_\ell)) \supset \ell^n H_1(\mathbb{Z}(n), \mathbb{Z}_\ell),
\]
we conclude that
\[
H_1(\mathbb{Z}(i), \mathbb{Z}_\ell) \supset v_i \gamma(n)^s(H_1(\mathbb{Z}(i), \mathbb{Z}_\ell)) \supset \ell^n H_1(\mathbb{Z}(i), \mathbb{Z}_\ell)
\]
and therefore
\[
\ell^i H_1(\mathcal{X}, \mathbb{Z}_\ell) + T = \gamma(i)(H_1(\mathbb{Z}(i), \mathbb{Z}_\ell)) \supset v_i \gamma(n)(H_1(\mathcal{X}, \mathbb{Z}_\ell)) \supset \ell^n H_1(\mathbb{Z}(i), \mathbb{Z}_\ell).
\]
It follows that for all \( i \in I \),
\[
\ell^i H_1(\mathcal{X}, \mathbb{Z}_\ell) + T \supset \gamma(i) v_i \gamma(n)(H_1(\mathcal{X}, \mathbb{Z}_\ell)) = u_i H_1(\mathcal{X}, \mathbb{Z}_\ell) \supset \ell^{n+1} H_1(\mathcal{X}, \mathbb{Z}_\ell) + \ell^n T.
\]
Recall that \( \text{End}_S(\mathcal{X}) = \text{End}_{\mathbb{C}(S)}(\mathcal{X}) \) is a free commutative group of finite rank and therefore the \( \mathbb{Z}_\ell \)-lattice \( \text{End}_S(\mathcal{X}) \otimes \mathbb{Z}_\ell \) in \( \text{End}_S(\mathcal{X}) \otimes \mathbb{Q}_\ell \) is a compact metric space with respect to the \( \ell \)-adic topology. So, we can extract from \( \{u_i\}_{i \in I} \) a subsequence \( \{u_j\}_{j \in J} \) that converges to a limit
\[
u \in \text{End}_S(\mathcal{X}) \otimes \mathbb{Z}_\ell \subset \text{End}_{\mathbb{Z}_\ell}(H_1(\mathcal{X}, \mathbb{Z}_\ell)).
\]
We may assume that there is a sequence of nonnegative integers \( \{m_j\}_{j \in J} \) that tends to infinity and such that for all \( j \), \( u - u_j \in \ell^{m_j} \cdot \text{End}_S(\mathcal{X}) \otimes \mathbb{Z}_\ell \). In particular, \( (u - u_j)(H_1(\mathcal{X}, \mathbb{Z}_\ell)) \subset \ell^{m_j} \cdot H_1(\mathcal{X}, \mathbb{Z}_\ell) \). It follows that
\[
u(H_1(\mathcal{X}, \mathbb{Z}_\ell)) = \{\lim u_j(c) \mid c \in H_1(\mathcal{X}, \mathbb{Z}_\ell)\}.
\]
This implies easily that \( u(H_1(\mathcal{X}, \mathbb{Z}_\ell)) \subset T \). On the other hand, if \( t \in \ell^n T \) then for each \( j \in J \) there exists \( c_j \in H_1(\mathcal{X}, \mathbb{Z}_\ell) \) with \( u_j(c_j) = t \). Since \( H_1(\mathcal{X}, \mathbb{Z}_\ell) \) is a compact metric space with respect to the \( \ell \)-adic topology, we can extract from \( \{c_j\}_{j \in J} \) a subsequence \( \{c_k\}_{k \in K} \) that converges to a limit \( c \in H_1(\mathcal{X}, \mathbb{Z}_\ell) \). Then
\[
u(c) = \lim u_k(c) = \lim u_k(c_k) = t.
\]
It follows that \( \ell^n T \subset u(H_1(\mathcal{X}, \mathbb{Z}_\ell)) \subset T \). This implies that \( u(H_1(\mathcal{X}, \mathbb{Q}_\ell)) = u(H_1(\mathcal{X}, \mathbb{Z}_\ell)) \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell = T \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell = W \).

\( \square \)
Proof of Lemma 8.1. **Step 1. Reduction to the case of simple A.** Suppose that the semisimple \( k \)-algebra \( A \) splits into a direct sum \( A = A_1 \oplus A_2 \) of non-zero semisimple \( k \)-algebras \( A_1 \) and \( A_2 \). Let \( e_1 \) and \( e_2 \) be the identity elements of \( A_1 \) and \( A_2 \) respectively. Clearly, \( \text{Id}_V = e_1 + e_2, e_1 e_2 = e_2 e_1 = 0, e_1^2 = e_1, e_2^2 = e_2 \). It is also clear that both \( e_1 \) and \( e_2 \) lie in the center of \( A \) and therefore in the center of \( B \). Let us put

\[
V_1 = e_1 V, V_2 = e_2 V, B_1 = e_1 B = B e_1 \subset A_1, B_2 = e_2 B = B e_2 \subset A_2;
\]

we have \( V = V_1 \oplus V_2, B = B_1 \oplus B_2 \). In addition, \( A_1 \) acts trivially on \( V_2 \) and \( A_2 \) acts trivially on \( V_1 \). So, we may view \( A_1 \) as a subalgebra of \( \text{End}_k(V_1) \) and \( A_2 \) as a subalgebra of \( \text{End}_k(V_2) \) respectively. Obviously, the center of \( A_i \) lies in the center of \( B_i \) for both \( i = 1, 2 \). Clearly, either \( B_1 \neq A_1 \) or \( B_2 \neq A_2 \). It is also clear that the validity of the assertion of Lemma 8.1 for \((V_1, A_1, B_1)\) with \( A_1 \neq B_1 \) implies its validity for \((V, A, B)\). It follows that it suffices to prove Lemma 8.1 under an additional assumption that \( A \) is simple. So, further we assume that \( A \) is a simple \( k \)-algebra and therefore is isomorphic to a matrix algebra \( M_r(E) \) of size \( r \) over a field \( E \). Since \( E \) is the center of \( M_r(E) \), it is a finite algebraic extension of \( k \). The field \( E \) lies in the center of \( D \), i.e., \( D \) is an \( E \)-algebra. It follows easily that a semisimple \( k \)-algebra \( D \) is a also a semisimple \( E \)-algebra.

**Step 2. Reduction to the case of simple \( V = E^r \).** Since all \( M_r(E) \)-modules of finite \( k \)-dimension are isomorphic to direct sums of finite number of copies of the standard module \( E^r \), we may assume that \( V = E^r, A = M_r(E) = \text{End}_E(V) \). Clearly, if \( W \) is a \( k \)-vector subspace in \( V \) then one can find \( a \in \text{End}(V) \) with \( u(V) = W \) if and only if \( W \) is a \( E \)-vector subspace in \( V \). So, in order to prove Lemma 8.1, it suffices to prove the following statement.

**Proposition 8.4.** Let \( V \) be a vector space of finite dimension \( r \) over a field \( E \). Let \( B \subset \text{End}_E(V) \) be a semisimple \( E \)-subalgebra that contains \( \text{Id}_V \). If \( B \neq \text{End}_E(V) \) then there exists a proper \( E \)-vector subspace \( W \) of \( V \) that enjoys the following property: there does not exist \( b \in B \) with \( b(V) = V \).

**Proof of Proposition 8.4.** Suppose that \( B \) is not simple, i.e. \( B \) splits into a direct sum \( B = B_1 \oplus B_2 \) of non-zero summands \( B_1 \) and \( B_2 \). Let \( e_1 \) and \( e_2 \) be the identity elements of \( A_1 \) and \( A_2 \) respectively. Clearly \( V \) splits into a direct sum \( V = e_1 V \oplus e_2 V \) of non-zero \( B \)-stable subspaces \( e_1 V \) and \( e_2 V \). It is also clear that for each \( b \in B \) the image

\[
bV = bV_1 \oplus bV_2, \ bV_1 \subset V_1, bV_2 \subset V_2.
\]

In particular, if \( W = \text{Id}_V \) is any choice of \( b \in B \). This proves Proposition 8.4in the case of non-simple \( B \). So further we assume that \( B \) is a simple \( E \)-algebra. Let \( F \) be the center of \( B \). Clearly, \( F \) is a field that contains \( E \cdot \text{Id}_V \). If \( F \neq E \cdot \text{Id}_V \) then every \( b(V) \) is an \( F \)-vector space; in particular, its \( k \)-dimension is divisible by the degree \( [F : E \cdot \text{Id}_V] \), which is greater than 1. So, none of \( E \)-subspaces \( W \) of \( E \)-dimension 1 is of the form \( b(V) \). So, without loss of generality, we may assume that \( F = E \cdot \text{Id}_V \), i.e., \( B \) is a central simple \( F \)-algebra. Then there exist a division algebra \( H \) of finite dimension over its center \( E \) and a positive integer \( m \) such that \( B \) is isomorphic to the matrix algebra \( M_m(H) \). It follows from the classification of modules over central simple algebras that the left \( B = M_m(H) \)-module \( V \) is isomorphic to a direct sum of finitely many copies of the standard module \( H^m \).
Clearly, $V$ carries a natural structure of right $H$-module in such a way that every $b(V)$ is a right $H$-submodule. In particular, $\dim_E(bV)$ is divisible by $\dim_E(H)$. This implies that if $H \neq E$ then none of $E$-subspaces $W$ of $E$-dimension 1 is of the form $bV$. So, without loss of generality, we may assume that $H = E$ and $B \cong M_n(E)$.

Since $B \subset \text{End}_E(V)$ but $B \neq \text{End}_E(V)$, we conclude that $m < r$. Now dimension arguments imply that the $B = M_n(E)$-module $V = E^r$ is isomorphic to the direct sum of $r/m$ copies of the standard module $E^m$; in particular, $m \mid r$. This implies that $\dim_E(bV)$ is divisible by $r/m$. Since $m < r$, none of $E$-subspaces $W$ of $E$-dimension 1 is of the form $bV$.

\[ \square \]

9. QUATERNIONS AND SU(2)

This stuff is (or should be) well-known. However, I was unable to find a proper reference. (However, see [26, Lecture 1, Example 12].)

Let $E$ be a field of characteristic zero, $F/E$ its quadratic extension. We write $\sigma : z \mapsto \bar{z}$ for the only nontrivial $E$-linear automorphism of $F$, which is an involution. Fix a non-zero element $i \in E$ such that $\sigma(i) = -i$. Clearly,

\[ F = E + E \cdot i, \quad 0 \neq -a := i^2 \in E. \]

Let $V$ be a two-dimensional $E$-vector space. We write $\text{Aut}_F^1(V)$ for the group of $F$-linear automorphisms $u$ of $V$ with $\det_F(u) = 1$. Here $\det_F : \text{Aut}_F(V) \to F^*$ is the determinant map. If $\omega : V \times V \to F$ is a map then we write $\text{Aut}_F(V, \omega)$ for the group of all $F$-linear automorphisms $u$ of $V$ such that $\omega(ux, uy) = \omega(x, y) \forall x, y \in V$ and put

\[ \text{Aut}_F^1(V, \omega) := \text{Aut}_F(V, \omega) \cap \text{Aut}_F^1(V). \]

Let $\psi : V \times V \to F$ be a non-degenerate $F$-sesquilinear Hermitian form. Let us reduce $H$ to a diagonal form, i.e., pick $e_1 \in V$ with $\psi(e_1, e_1) \neq 0$ and let $V_2$ be the orthogonal complement in $V$ to $e_1$ with respect to $\psi$. The non-degeneracy of $\psi$ implies that $V_2$ is one-dimensional and the restriction of $\psi$ to $V_2$ is also non-degenerate, i.e., $\psi$ does not vanish on non-zero elements of $V_2$. It follows that if $e_2$ is a non-zero element of $V_2$ then $\{e_1, e_2\}$ is an orthogonal basis of $V$ then

\[ \psi(z_1e_1 + z_2e_2, w_1e_1 + w_2e_2) = b_1z_1w_1 + b_2z_2w_2 \quad \forall \ z_1, z_2, w_1, w_2 \in F \]

where $0 \neq b_1 := \psi(e_1, e_1) \in E$, $0 \neq b_2 := \psi(e_2, e_2) \in E$. Let us put $\text{discr}(\psi) = b_1b_2$. It is known [5, Ch. 9, §2] that the (multiplicative) class of $\text{discr}(\psi)$ in $E^*/N_{F/E}(F^*)$ does not depend on the choice of basis. Here $N_{F/E} : F \to E$ is the norm map. Let us put $b = b_2/b_1 \in F^*$ and consider the Hermitian form

\[ \psi' := \frac{1}{b_1^2} \psi : V \times V \to F, \]

\[ \psi'(z_1e_1 + z_2e_2, w_1e_1 + w_2e_2) = z_1w_1 + z_2w_2 \quad \forall \ z_1, z_2, w_1, w_2 \in F. \]

Clearly, the classes of $b$ and $\text{discr}(\psi)$ in $E^*/N_{F/E}(F^*)$ do coincide and

\[ \text{Aut}_F^1(V, \psi') = \text{Aut}_F^1(V, \psi). \]

Let us consider the cyclic algebra [25, Sect. 15.1]

\[ D = (F, \sigma, -b) \cong (F, \sigma, -\text{discr}(\psi)). \]
Recall that $D$ is the four-dimensional central simple $E$-algebra that contains $F$ and coincides as (left) $F$-vector space with $F \oplus F \cdot j$ where $j$ is an element of $A'$ such that $j^2 = -b$, $jzj^{-1} = \sigma(z) = \bar{z}$ $\forall z \in F$. If we put $k := ij$ then

\[ A = F \oplus F \cdot j = E \cdot 1 \oplus E \cdot i \oplus E \cdot j \oplus E \cdot k \]

with

\[ i^2 = -a, j^2 = -b, k^2 = -ab, k = ij = -ji, ik = -aq, kj = -bi = -jk. \]

Clearly, $D = \left( \frac{-a-b}{E} \right)$ (see [25, Sect. 15.4]). Let us consider the standard $F$-linear involution $D \to D, q \mapsto q$ that sends $i, j, k$ to $-i, -j, -k$ respectively. Clearly, $\bar{\bar{q}} = q\bar{q} = x^2 + ay^2 + bs^2 + abt^2 \subset E \forall q = x \cdot 1 + y \cdot i + s \cdot j + t \cdot k; x, y, s, t \in E$.

Let us consider the “quaternionic” Hermitian $D$-sesquilinear form

\[ \phi_D : D \times D \to D, (q_1, q_2) \mapsto q_1\bar{q}_2. \]

Taking the compositions of $\phi_D : D \times D \to D$ with the projection maps

\[ D = F \oplus F \cdot j \to F; D = F \oplus F \cdot j \to F, \]

we get $E$-bilinear forms

\[ H_D : D \times D \to F, A_D : D \times D \to F \]

defined by

\[ \phi_D(q_1, q_2) = H_D(q_1, q_2) \cdot 1 + A_D(q_1, q_2) \cdot j. \]

Clearly, both $H_D$ and $A_D$ are $F$-linear with respect to first argument. Since

\[ \phi_D(q_1, q_2) = \phi_D(q_2, q_1), \quad \phi_D(q, q) = q\bar{q} \in E \subset F, \]

we have

\[ H_D(q_1, q_2) = H_D(q_2, q_1), \quad A_D(q, q) = 0 \forall q_1, q_2, q \in D. \]

This means that $H_D$ is an Hermitian $F$-sesquilinear form on $D$ and $A_D$ is an alternating $F$-bilinear form on $D$.

Clearly, $A_D$ is not identically zero and therefore is non-degenerate, since the $F$-dimension of $D$ is 2. This implies that the symplectic group Aut$_F(D, A_D)$ coincides with Aut$_F^1(V)$ and therefore

\[ \text{Aut}_F(D, \phi_D) = \text{Aut}_F(D, H_D) \cap \text{Aut}_F(D, A_D) = \text{Aut}_F(D, H_D) \cap \text{Aut}_F^1(V) = \text{Aut}_F^1(D, H_D). \]

Clearly, $F \cdot 1$ and $F \cdot j$ are mutually orthogonal with respect to $H_D$. On the other hand

\[ H_D(z_1, 1, z_2, 1) = z_1\bar{z}_1, \quad H_D(z_1, j, z_2, j) = z_1\bar{z}_2 = z_1\bar{z}_2, \quad H_D(z_1, j, z_2, j) = z_1\bar{z}_2 + b \bar{z}_1 z_2 \quad \forall z_1, z_2 \in F. \]

This implies that the isomorphism of $F$-vector spaces

\[ \kappa : V \cong D, zv_1 + wv_2 \mapsto z \cdot 1 + w \cdot j \]

is an isomorphism of Hermitian $F$-vector spaces $(V, \psi')$ and $(D, H_D)$, i.e.,

\[ H_D(\kappa(v_1), \kappa(v_2)) = \psi'(v_1, v_2) \quad \forall v_1, v_2 \in V. \]

For every $q \in D$ we denote by $R(q)$ the $F$-linear operator $D \to D, d \mapsto d \cdot q$. Clearly,

\[ R(1) = \text{Id}_D, \quad R(q_1 q_2) = R(q_2)R(q_1), \quad R(xq_1 + yq_2) = xR(q_1) + yR(q_2) \]
for all $q_1, q_2 \in D; \ x, y \in E$. I claim that
\[
\text{Aut}_F(D, \phi_D) = \{ R(q) \mid q\bar{q} = 1 \}.
\]

Indeed, if $q\bar{q} = 1$ then
\[
\phi_D(R(q)q_1, R(q)q_2) = \phi_D(q_1q_2) = q_1q_2 = q_1q_2\bar{q}_2 = q_1\bar{q}_2 = \phi_D(q_1, q_2),
\]
i.e., $R(q)$ preserves $\phi_D$. On the other hand, if a $E$-linear automorphism $u$ of $D$

preserves $\phi_D$ then $u(1)u(1) = \phi_D(u(1), u(1)) = \phi_D(1, 1) = \bar{1} = 1 \cdot 1 = 1$ and $u' = R(u(1))^{-1}u$ also preserves $\phi_D$ and satisfies $u'(1) = 1$. This implies that for all $q \in D$,
\[
q = q \cdot 1 = q = q \cdot \bar{1} = \phi_D(q, 1) = \phi_D(u'(q), u'(1)) = u'(q) \cdot \bar{1} = u'(q),
\]
i.e., $u'$ is the identity map $\text{Id}_D$ and therefore $u = R(u(1))$. Since $\text{Aut}_F(D, \phi_D) = \text{Aut}_F^1(D, H_D)$, we conclude that $\text{Aut}_F^1(D, H_D) = \{ R(q) \mid q\bar{q} = 1 \}$.

Viewing $D$ as the left $D$-module, we get an embedding $D \subset \text{End}_E(D)$. Clearly, (the algebra) $D$ (of left multiplications) commutes with all right multiplications $R(q)$ and therefore with $\text{Aut}_F^1(D, H_D)$ in $\text{End}_E(D)$.

**Lemma 9.1.** The centralizer $D$ of $\text{Aut}_F^1(D, H_D)$ in $\text{End}_E(D)$ coincides with $D$.

**Proof.** Clearly, $D$ contains $D$. Let us pick non-zero integers $n$ and $m$ such that $n^2 + a \neq 0$, $m^2 + b \neq 0$ and put
\[
q_1 = \frac{(n^2 - a) \cdot 1 + 2n \cdot \bar{1}}{n^2 + a}, \quad q_2 = \frac{(m^2 - b) \cdot 1 + 2m \cdot \bar{j}}{m^2 + b}.
\]

One may easily check that $q_1\bar{q}_1 = 1 = q_2\bar{q}_2$. Clearly,
\[
R(q_1) = \frac{n^2 - a}{n^2 + a} \text{Id}_V + \frac{2n}{n^2 + a} R(1), \quad R(q_2) = \frac{m^2 - b}{m^2 + b} \text{Id}_V + \frac{2m}{m^2 + b} R(j)
\]
and therefore $D$ commutes with $R(q_1)$ and $R(q_2)$. Since $n$ and $m$ do not vanish, we conclude that $D$ commutes with right multiplications $R(1)$ and $R(j)$. Clearly, $R(1)R(1) = R(k)$. This implies that $D$ commutes with $R(k)$ and therefore commutes with all $R(q)$ ($q \in D$). Now if $u(q) = u(1) \in D$. For all $q \in D$ we have $u(q) = u(R(q)(1)) = R(q)(u(1)) = R(q)(z) = zq$, i.e., $u = z \in D \subset \text{End}_E(D)$. \hfill \Box

Recall that the Hermitian $F$-vector spaces $(V, \phi')$ and $(D, H_D)$ are isomorphic. Taking into account that $\text{Aut}_F^1(V, \psi) = \text{Aut}_F^1(V, \psi')$ and applying Lemma 9.1, we obtain the following statement.

**Theorem 9.2.** Let us view $V$ as a four-dimensional $E$-vector space. Then the centralizer $D$ of $\text{Aut}_F^1(V, \psi)$ in $\text{End}_E(V)$ is a central simple four-dimensional $E$-algebra isomorphic to the four-dimensional central simple $E$-algebra $D = \left( \frac{-a-b}{E} \right)$.

## 10. Abelian fourfolds

**Theorem 10.1.** Let $S$ be a smooth irreducible algebraic curve over $C$ and $f : X \to S$ a polarized abelian scheme of relative dimension 4 with generic fiber $X$.

Suppose that $X$ is not weakly isotrivial and $\text{Isog}(X, C(S))$ is infinite. Then:

(i) $X$ is absolutely simple.

(ii) The center $E$ of $D_f$ is a real quadratic field.
(iii) $D_f$ is a quaternion division $E$-algebra that is unramified at one infinite place of $E$ and ramified at the other infinite place.

(iv) $\text{End}_{\mathbb{C}(S)}^0(X)$ is either $E$ or a CM-field of degree $4$.

(v) $\text{End}^0(X)$ is a CM-field of degree $4$.

(vi) Let $s \in S$ and assume that $G_s$ is connected. Then $\text{End}_{\mathbb{C}(S)}^0(X) = \text{End}^0(X)$ is a CM-field of degree $4$.

Proof. The absolute simplicity of $X$ follows from Corollary 4.9, which proves (i). This implies that $\text{End}_{\mathbb{C}(S)}^0(X)$ has no zero divisors. Since $E$ is isomorphic to a subalgebra of $\text{End}_{\mathbb{C}(S)}^0(X)$, we conclude that $E$ is a number field and $[E : \mathbb{Q}]$ divides $2\dim(X) = 8$. Since $\text{Isog}(X, \mathbb{C}(S))$ is infinite, it follows from Theorem 4.1 that $D_f \neq \text{End}^0_S(X)$. By the last sentence of 3.12, $D_f \neq E$, i.e., $D_f$ is a non-commutative central simple $E$-algebra. Since $\dim_E(D_f)$ divides

$$\dim_E(H_1(X_s, \mathbb{Q})) = \frac{\dim_{\mathbb{Q}}(H_1(X_s, \mathbb{Q}))}{[E : \mathbb{Q}]} = \frac{8}{[E : \mathbb{Q}]}$$

we conclude that $8/[E : \mathbb{Q}]$ is not square-free. It follows that $E$ is either $\mathbb{Q}$ or a quadratic field. On the other hand, Deligne [6, Prop. 4.4.11] proved that if $E$ is either $\mathbb{Q}$ or an imaginary quadratic field then $D_f = \text{End}^0_S(X)$. It follows that $E$ is a real quadratic field (which proves (ii)) and $\dim_E(D_f) = 4$. It follows from [6, Prop. 4.4.11] combined with the inequality $D_f \neq \text{End}^0_S(X)$ that $D_f$ is unramified at one infinite place of $E$ and ramified at the another one. This rules out the possibility that $D_f$ is a matrix algebra over $E$. It follows that $D_f$ is a quaternion division $E$-algebra, which proves (iii).

We have

$$E \subset \text{End}^0_S(X) \subset D_f, \quad \text{End}^0_S(X) \neq D_f.$$  

Since $\dim_E(D_f) = 4$, we conclude that either $\text{End}^0_S(X) = E$ or $\text{End}^0_S(X)$ is a field of degree $4$ that contains $E$. Since there is an embedding $E \hookrightarrow \mathbb{R}$ such that $D_f \otimes_E \mathbb{R}$ is the standard quaternion $\mathbb{R}$-algebra, either $E = \text{End}^0_S(X)$ or $\text{End}^0_S(X)$ is a degree four field that is not totally real. Since $\text{End}^0_S(X) = \text{End}_{\mathbb{C}(S)}^0(X)$, we conclude that either $E = \text{End}^0_S(X)$ or $\text{End}^0_S(X) = \text{End}_{\mathbb{C}(S)}^0(X)$ or $\text{End}^0_S(X) = \text{End}_{\mathbb{C}(S)}^0(X)$ is a CM-field of degree $4$, which proves (iv).

Let $s \in S$ be a point in general position and assume that $G_s$ is connected. Suppose that $\text{End}_{\mathbb{C}(S)}^0(X) = E$. We need to arrive to a contradiction. It follows from the first assertion of Corollary 3.7 that $\text{End}^0(X_s) = \text{End}^0_S(X) = \text{End}_{\mathbb{C}(S)}^0(X)$.

This implies that $\text{End}^0(X_s) = E$ is a real quadratic field. It follows from [18, 4.2, p. 566] that $\text{Hdg}(X_s)$ is $\mathbb{Q}$-simple. It follows from the first assertion of Corollary 3.7 that $\text{End}_{\mathbb{C}(S)}^0(X) = \text{End}^0_S(X) = \text{End}_{\mathbb{C}(S)}^0(X)$ (see Subsect. 3.8, Example 3.9 and Remark 3.10). By Remark 2.3, $\text{Isog}(X \times_{\mathbb{C}} L, L)$ is
as above, we view \( SU(V, \psi) \) where \( SU(V, \psi) \) is the complex conjugation. This implies that \( \phi = End_U(V, \psi) \) such that \( \phi = End_U(V, \psi) \) is an 8-dimensional \( \mathbb{Q} \)-vector space and \( \Gamma_s \subset \text{Aut}_F(V) \). We can do better, using the polarization on \( X \) that induces a polarization on \( X \), whose Riemann form gives rise to a non-degenerate alternating \( \pi_1(S, s) \)-invariant \( \mathbb{Q} \)-bilinear form \( \phi : V \times V \to \mathbb{Q} \). Since the complex conjugation \( e \mapsto \bar{e}' \) on \( F \) is the only positive involution on the CM-field \( F = End^0(S, s) \), all Rosati involutions on \( End^0(S, s) = F \) coincide with the complex conjugation. This implies that \( \phi(ex, y) = \phi(x, \bar{y}) \forall x, y \in V, e \in F \) where \( e \mapsto \bar{e}' \) is the complex conjugation on \( F \). Pick a non-zero element \( \alpha \in E \) with \( \alpha' = -\alpha \). Then there exists a unique (non-degenerate) \( F \)-Hermitian form

\[
\psi : V \times V \to F
\]
such that \( \phi(x, y) = \text{tr}_{F/\mathbb{Q}}(\alpha \psi(x, y)) \forall x, y \in V \) [8, Sect. 9]. Here \( \text{tr}_{F/\mathbb{Q}} : F \to \mathbb{Q} \) is the trace map. Since \( \phi \) is \( \pi_1(S, s) \)-invariant, \( \psi \) is also \( \pi_1(S, s) \)-invariant and therefore is \( \Gamma_s \)-invariant. Let us consider the unitary group

\[
U(V, \psi) \subset \text{GL}(V)
\]
of the \( F \)-vector space \( V \) relative to \( \psi \). A priori \( U(V, \psi) \) is an algebraic group over \( E \), but we regard it as an algebraic \( \mathbb{Q} \)-group, i.e., take its Weil restriction over \( \mathbb{Q} \). In particular, \( U(V, \psi)(\mathbb{Q}) = \text{Aut}_F(V, \psi) \). Clearly, \( \Gamma_s \subset \text{Aut}_F(V, \psi) = U(V, \psi)(\mathbb{Q}) \) and therefore \( \Gamma_s \subset U(V, \psi) \). The semisimplicity of \( \text{G}_s^0 \) implies that \( \text{G}_s^0 \subset U(V, \psi) \) where \( U(V, \psi) \) is the special unitary group of the \( F \)-vector space \( V \) relative to \( \psi \). As above, we view \( SU(V, \psi) \) as an algebraic \( \mathbb{Q} \)-(sub)group (of \( U(V, \psi) \)). Clearly,

\[
SU(V, \psi)(\mathbb{Q}) = \text{Aut}_F^1(V, \psi) := \{ u \in \text{Aut}_F(V, \psi) \mid \det_F(u) = 1 \}.
\]
Example 10.6. Recall that the 5th cyclotomic field is a smooth genus zero affine curve of degree 4. Notice that µ and λ get a finite étale Galois cover whose order divides S. Remark 10.5. Applying Lemma 10.4 to the abelian scheme readily. In order to prove the assertion (i), recall (Sect. 3.4) that Γ corresponds monodromy representation coincides with Γ with X and that if Gs ⊂ SU(V, ψ) then Df ≠ Endf(X). Applying Theorem 4.4.1, we obtain the following statement.

Lemma 10.4. If Gs ⊂ SU(V, ψ) then Isog(X, C(S)) is infinite.

End of the proof of Theorem 10.3. Since Gs ⊂ SU(V, ψ), the assertion (ii) follows readily. In order to prove the assertion (i), recall (Sect. 3.4) that Γs := Γs ∩ Gs is a normal subgroup of finite index in Γs. Since Gs ⊂ SU(V, ψ), it follows that detF(Γs) ⊂ µF. This implies that if we put Γ0 := {g ∈ Γs | detF(g) = 1} then Γ0 is a normal subgroup in Γs and the quotient Γs/Γ0 is a finite cyclic subgroup, whose order divides rF. Using the construction of Section 3.8 applied to Γ0, we get a finite étale Galois cover S0 → S with Galois group Γs/Γ0 and abelian S0-scheme X0 = X × S0 such that if s0 ∈ S0 lies above s then the image Σs0 of the corresponding monodromy representation coincides with Γ0. (Here we identify Xs with Xs0 and H1(Xs0, Q) with H1(Xs, Q) = V.)Clearly, Γs0 = Γ0 ⊂ SU(V, ψ).

Applying Lemma 10.4 to the abelian S0-scheme X0 = X × S S0, we conclude that Isog(X, C(S0)) is infinite. In order to finish the proof of (i), notice that the field extension C(S0)/C(S) is normal and its Galois group coincides with Γs/Γ0.

Remark 10.5. We keep the notation and assumptions of Theorem 10.3.

1. Hdg(Xs) = U(V, ψ) [18, Sect. 7.5]. It follows from Deligne’s Theorem 3.5 that Gs0 = SU(V, ψ).
2. It follows from Theorem 9.2 that the centralizer EndSU(V, ψ)(V) of SU(V, ψ) in EndQ(V) is a four-dimensional central simple E-algebra Df containing F.
3. Suppose that Gs = Gs0. It follows that Df contains Df and therefore does not coincide with F = End3(V). However, the center E of Df lies in F. Since F ⊂ Df ⊂ Df and the center of Df is E, we conclude that E ⊂ F. This means that either E = Q or E = E, because E is a quadratic field. In both cases Df is a central simple E-algebra. Applying Proposition 4.4.11 of [6], we conclude that E ≠ Q, i.e., E = E. Since dimQ(Df) must divide 8 and dimQ(Df) = 8, we conclude that Df = Df. Applying again the same Proposition, we conclude that Df is ramified at one infinite place of E and unramified at another one. In particular, Df is not isomorphic to the matrix algebra M2(E), i.e., Df is a quaternion (division) E-algebra.

Example 10.6. Recall that the 5th cyclotomic field F := Q(µ5) is a CM-field of degree 4. Notice that µF is a cyclic group of order 10, i.e., rF = 10. Let us consider a smooth genus zero affine curve

\[ S = \mathbb{A}^1 \setminus \{0, 1\} = \mathbb{P}^1 \setminus \{0, 1, \infty\} \]

with coordinate λ and a family \( \mathcal{E} \rightarrow S \) of genus four smooth projective curves over S defined by the equation

\[ y^5 = x(x - 1)(x - \lambda) \]
and the corresponding family of jacobians \( f : \mathcal{J} \to S \). Clearly, \( \mathcal{J} \) is a (principally) polarized abelian \( S \)-scheme of relative dimension four, \( \mathbb{C}(S) = \mathbb{C}(\lambda) \) is the field of rational functions in one variable over \( \mathbb{C} \) and the generic fiber \( J \) of \( f \) is the jacobian of the \( \mathbb{C}(\lambda) \)-curve \( y^5 = x(x - 1)(x - \lambda) \) of genus four. It follows from \([13, \text{Lemma } 2.2 \text{ and Prop. } 2.7]\) that \( \mathcal{J} \) is not isotrivial and
\[
\text{End}^0(J) = \text{End}^0_{\mathbb{C}(\lambda)}(J) = F.
\]
This implies that \( J \) is absolutely simple and therefore \( \mathcal{J} \) is not weakly isotrivial.

By Theorem 10.3(i), there exists a cyclic extension \( L/\mathbb{C}(\lambda) \) that has degree \( r \) dividing 10, is unramified outside \( \{0, 1, \infty\} \) and such that \( \text{Isog}(\mathcal{J}, L) \) is infinite. Using Kummer theory, we obtain easily that there are nonnegative integers \( a \) and \( b \) such that
\[
0 \leq a < r \leq 10, \quad 0 \leq b < r \leq 10
\]
and \( L = \mathbb{C}(\lambda)(\sqrt[10]{\lambda^a} (\lambda - 1)^b) \). Clearly, \( \mathbb{C}(\lambda) \subset L \subset \mathbb{C}(\sqrt[10]{\lambda}, \sqrt[10]{\lambda - 1}) \). It follows from Remark 2.3 that \( \text{Isog}(\mathcal{J}, \mathbb{C}(\sqrt[10]{\lambda}, \sqrt[10]{\lambda - 1})) \) is infinite. Now Corollary 4.6 implies that \( \text{Isog}(J^2, \mathbb{C}(\sqrt[10]{\lambda}, \sqrt[10]{\lambda - 1}, \ell)) \) is infinite for all but finitely many primes \( \ell \) congruent to 1 modulo 4.

Notice that \( \mathbb{C}(\sqrt[10]{\lambda}, \sqrt[10]{\lambda - 1}) \) is the field of the rational functions on the affine Fermat curve \( S' : u^{10} - v^{10} = 1, \; u \neq 0, v \neq 0 \) with \( u = \sqrt[10]{\lambda}, v = \sqrt[10]{\lambda - 1} \) and \( S' \to S, \; \lambda = u^{10} \) is the corresponding finite étale cover.

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