COMPUTING THE WRITHE OF A KNOT

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ABSTRACT
We study the variation of the Tait number of a closed space curve according to its different projections. The results are used to compute the writhe of a knot, leading to a closed formula in case of polygonal curves.

Keywords: PL knot, Tait number, writhe, lattice knot.

1. Introduction
Since Crick and Watson’s celebrated article in 1953, the local structure of DNA is well understood; its visualization as a double helix suggests the mathematical model of a ribbon in $\mathbb{R}^3$. But what about the global structure of DNA, that is, what kind of closed curve does the core of the ribbon form? It appears that these curves can show a great complexity (supercoiling) and are of central importance in understanding DNA (see [1] and [2]).

In 1961, Călugăreanu [3] made the following discovery: take a ribbon in $\mathbb{R}^3$, let $Lk$ be the linking number of its border components, and $Tw$ its total twist; then the difference $Lk - Tw$ depends only on the core of the ribbon. This real number, later called writhe by Fuller [4], is of great interest for biologists, as it gives a measure of supercoiling in DNA.

Several techniques have been developed to estimate the writhe of a given space curve, particularly by Aldinger, Klapper and Tabor [5]. The aim of the present paper is to give a new way of computing the writhe (see proposition 3). Moreover, this method will lead to a closed formula for the writhe of any polygonal space curve.

2. Definitions
Definition 1. A polygonal knot (or PL knot) is the union of a finite number of
segments in $\mathbb{R}^3$, homeomorphic to $S^1$. A point $x \in K$ is either a vertex or an interior point of $K$.

Let us fix a PL knot $K$. Take a point $\xi$ in $S^2$; let $d_\xi$ be the oriented vector line containing $\xi$ and $p_\xi: K \to \mathbb{R}^2$ the orthogonal projection with kernel $d_\xi$.

**Definition 2.** The map $p_\xi$ is a good projection (or generic projection) if, for all $v$ in $\mathbb{R}^2$, $p_\xi^{-1}(v)$ is empty, one point, or two interior points of $K$.

Clearly, the good projections form an open dense subset $O_K$ of $S^2$; given $\xi \in O_K$, we get a diagram of the PL knot $K$ by pointing out which segment lies on the top of the other at double points, according to the orientation of $d_\xi$.

![Fig. 1.](image.png)

To each double point $v$ of a diagram, it is possible to give a sign $s(v) = \pm 1$ by the following rule: we (temporarily) orientate $K$, and set $s(v) = +1$ if the oriented upper strand has to be turned counterclockwise to coincide with the lower one, $s(v) = -1$ in the other case (see figure 1). The sign obviously does not depend on the orientation of the knot.

**Definition 3.** Let $K$ be a PL knot and $\xi \in O_K$; the Tait number of $K$ relatively to $\xi$ is the integer $T_K(\xi) = \sum_v s(v)$, where the sum runs through all double points of the diagram associated with $\xi$.

Given $K$, let us define a function $T_K: S^2 \to \mathbb{R}$ by

$$T_K(\xi) = \begin{cases} T_K(\xi), & \text{if } \xi \in O_K; \\ 0, & \text{otherwise}. \end{cases}$$

This function is not continuous, but it is integrable (see proposition 1).

**Definition 4.** The writhing number (or writhe) of a PL knot $K$ is the real number $W_r(K) = \frac{1}{4\pi} \int_{S^2} T_K(\xi) \, d\xi$.

Since $S^2 \setminus O_K$ is of measure zero, we can extend the Tait number in any way without changing the value of the writhe; the first proposition will give a very natural way to do so.

The study of $T_K(\xi)$ requires one more object: the indicatrix of a knot, that we define now. Let $K$ be a PL knot; an orientation of $K$ allows us to number its
there exists \( K \) segments of \( D \). This time, only they are illustrated in figure 2.

The (spherical) indicatrix \( I \) of a PL knot is the oriented curve of \( S^2 \) given by \( I = \Gamma \cup -\Gamma \).

It is important to notice that the indicatrix may run several times on a fixed arc of great circle.

We can easily check that if \( \xi \) exists \( p_\xi \) is locally injective if and only if \( \xi \in S^2 \setminus I \).

3. The Results

Proposition 1. If \( \xi \) and \( \xi' \) in \( \mathcal{O}_K \) belong to the same connected component of \( S^2 \setminus I \), then \( T_K(\xi) = T_K(\xi') \).

Remarks.

- It is possible to extend the definition of the Tait number to all locally injective projections in a very natural way by setting \( T_K(\xi) = T_K(A) \), where \( T_K(A) \) is the value of the Tait number on the connected component \( A \) of \( S^2 \setminus I \) containing \( \xi \).
- This proposition directly implies that \( T_K \) is integrable on \( S^2 \).

Proof. Let \( \Omega \) be a connected component of \( S^2 \setminus I \), and \( \xi, \xi' \in \Omega \cap \mathcal{O}_K \). If \( \Omega \cap \mathcal{O}_K \) were connected, a simple continuity argument would do, but this is not the case in general. Indeed, \( \Omega \setminus (\Omega \cap \mathcal{O}_K) \) is the union of a finite number of curves on \( S^2 \) that can separate \( \xi \) and \( \xi' \). We need to describe all these curves, and check that crossing them does not change the value of \( T_K \).

Let us first consider the non-generic projections that do not send coplanar segments onto the same line; we call \( C_{(m,n)} \) the set of points \( \xi \in S^2 \) such that there exists \( v \in \mathbb{R}^2 \) with \( p_\xi^{-1}(v) \) consisting of \( m \) interior points and \( n \) vertices of \( K \). It is easy to check that if \( m \geq 4 \) or \( n \geq 2 \), \( C_{(m,n)} \) is a finite number of points, as well as \( C_{(2,1)} \) and \( C_{(3,1)} \). Since \( C_{(0,0)} \), \( C_{(1,0)} \) and \( C_{(2,0)} \) are the constituents of \( \mathcal{O}_K \), the only potential trouble makers are \( C_{(1,1)} \) and \( C_{(3,0)} \) (see figure 2).

We also have to study projections sending \( k \geq 2 \) coplanar (pairwise non-adjacent) segments of \( K \) onto a line \( d \) of the plane. Let \( D_{(m,n)} \) be the set of \( \xi \in S^2 \) such that there exists \( v \in d \) with \( p_\xi^{-1}(v) \) consisting of \( m \) interior points and \( n \) vertices of \( K \). This time, only \( D_{(0,0)} \), \( D_{(0,1)} \) and \( D_{(1,0)} \) need to be considered.

Thus, there are five types of non-generic projections that can separate \( \xi \) and \( \xi' \); they are illustrated in figure 2.

We now study \( \Delta T = T_K(\eta) - T_K(\eta') \), where \( \eta \) and \( \eta' \) are good projections on each side of a curve in \( S^2 \) formed by one type of non-generic projection. Crossing \( C_{(1,1)} \) (resp. \( C_{(3,0)} \)) corresponds to the second (resp. third) Reidemeister move; the Tait number being unchanged by these transformations, the first two cases are
settled. It is trivial that $\Delta T = 0$ for $D_{(1,0)}$, while $D_{(0,1)}$ can be seen as $k$ second Reidemeister moves.

It remains to show that $\Delta T = 0$ for $D_{(0,0)}$. If $k = 2$, all the possible cases are described in figure 3, each time, $\Delta T = 0$. For $k \geq 3$, the segments being pairwise non-adjacent, we can apply the same argument $(k-1)$ times. This concludes the proof.

What about the behavior of $T_K(\xi)$ when $\xi$ crosses the indicatrix? Let $\alpha$ be an open segment of the indicatrix $I$ on $S^2$, $p_1$ and $p_2$ two distinct points on $\alpha$. Let $n$ be the algebraic number of times that the indicatrix runs from $p_1$ to $p_2$. We will say that $\xi \in S^2 \setminus \alpha$ is to the north of $\alpha$ if $(p_1 \times p_2) \cdot \xi$ is strictly positive, to the south if it is strictly negative.

**Proposition 2.** Let $\Omega_0$ be the component of $S^2 \setminus I$ to the south of $\alpha$, $\Omega_1$ to the north; then: $T_K(\Omega_1) = T_K(\Omega_0) + n$.

**Proof.** To simplify the exposition, we will give the demonstration only when $\alpha$ is covered once by the indicatrix.

Since $K$ is polygonal, $I$ is the union of a finite number of arcs of great circles; $\alpha$ is an open arc of great circle produced by two adjacent segments $A$ and $B$ of $K$ (we will say: the site $AB$). Let $U$ be an open path-connected subset of $S^2$ such that $U \setminus (U \cap \alpha) \subset O_K$, and let us take $\xi_0 \in \Omega_0 \cap U, \xi_1 \in \Omega_1 \cap U$, $c$ a path in $U$ joining $\xi_0$ to $\xi_1$, crossing $\alpha$ at $\xi$ (see figure 4). Since the indicatrix runs only once through
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\(\alpha\) and since \(\text{Im}(c) \setminus \xi \subset \mathcal{O}_K\), the site \(AB\) alone influences the variation \(\Delta T\) of the Tait number along \(c\).

\[\begin{array}{c}
\pi_{AB} \\
A \quad B
\end{array}\]

\[\begin{array}{c}
\xi_0 \\
\xi_1
\end{array}\]

\(\alpha\)

\[U \]

\[\xi_0 \quad \xi_1\]

\(\xi\)

\[\pi_{AB}\]

Fig. 4.

Thus, we have to look at the site projected by \(p_{\xi_0}\) and by \(p_{\xi_1}\), and check that

\[T(\xi_1) = T(\xi_0) + 1.\]

\[\begin{array}{c}
A \\
\pi_{AB} \\
\xi_0 \\
\xi_1
\end{array}\]

\[\xi_0 \quad \xi_1\]

\(\xi_0 \quad \xi_0\)

\(\pi_{AB}\)

Fig. 5.

Let \(N = |\xi - \epsilon; \xi + \epsilon|\) be a neighborhood of \(\xi\) in \(\alpha\) such that the indicatrix travels only once along \(N\). One or several segments adjacent to \(A\) or \(B\) can remain in the plane \(\pi_{AB}\) generated by \(A\) and \(B\), but these segments lie in the shaded area in figure 5. Indeed, any segment outside the shaded area would make the indicatrix cover \(N\) one more time.

Clearly, the transformation illustrated in figure 5 does not change \(T(\xi_0)\) and \(T(\xi_1)\); therefore, we only need to consider the case where the segments adjacent to \(A\) and \(B\) leave the plane \(\pi_{AB}\).

Let us check all the different possibilities. By proposition 1, the four cases illustrated in figure 7 are sufficient. They correspond to the first Reidemeister
move, and we see that \( T(\xi_1) = T(\xi_0) + 1 \).

**Corollary.** Let \( \xi, \xi' \in S^2 \setminus I \); then \( T_K(\xi) = T_K(\xi') + n \), where \( n \) stands for the intersection number of the indicatrix with a path joining \( \xi \) to \( \xi' \).

The previous results were stated for PL knots; nevertheless, they remain true for a wider class of knots.

**Definition 6.** A piecewise \( C^2 \) knot is the image of a closed space curve \( \gamma: [0; 1] \to \mathbb{R}^3 \) twice continuously differentiable everywhere except on a finite number of points, satisfying for all \( t_0 \in [0; 1] \):

(a) \( \lim_{t \to t_0^+} \dot{\gamma}(t) \) and \( \lim_{t \to t_0^-} \dot{\gamma}(t) \) exist and are non-zero;

(b) \( \lim_{t \to t_0^+} \frac{\dot{\gamma}(t)}{||\dot{\gamma}(t)||} \neq -\lim_{t \to t_0^-} \frac{\dot{\gamma}(t)}{||\dot{\gamma}(t)||} \).

The indicatrix of a piecewise \( C^2 \) knot \( K \) is defined in the obvious way, and it is always possible to approach \( K \) with a sequence \( \{K_n\} \) of PL knots such that \( I_{K_n} \to I_K \). Hence, proposition 1 and 2 are true for piecewise \( C^2 \) knots, as well as the corollary.

**Illustration.** The picture shown on figure 8 was obtained by Akos Dobay at the University of Lausanne. It represents \( S^2 \) via cylindrical coordinates with \( 360 \times 900 \) evaluations of the function \( T_K \) associated with a given smooth trefoil knot. For this
particular space curve, these calculations provide a kind of “experimental check” of our results.

4. The Writhe of a Knot

Let $K$ be a piecewise $C^2$ knot in $\mathbb{R}^3$; since $T_K(\xi) = T_K(-\xi)$ for all $\xi$ in $S^2 \setminus I$,

$$W_r(K) = \frac{1}{4\pi} \int_{S^2} T_K(\xi) \, d\xi = \frac{1}{2\pi} \int_{S^2} T_K(\xi) \, d\xi.$$ 

By the corollary, we get the following formula, related to proposition 4 from Aldinger et al.:

**Proposition 3.** Let $K$ be a piecewise $C^2$ knot, $I$ its indicatrix, $A_0, A_1, \ldots, A_r$ the connected components of a hemisphere minus $I$, and $T_K(\xi_0)$ the Tait number of $K$ relatively to some $\xi_0 \in A_0$. Then, the writhe of $K$ is given by

$$W_r(K) = T_K(\xi_0) + \frac{1}{2\pi} \sum_{i=1}^{r} n_i \cdot \text{area}(A_i),$$

where $n_i$ stands for the intersection number of the indicatrix with a path joining $A_0$ to $A_i$. \hfill \Box

Let us now suppose that $K$ is a $PL$ knot. In this case, the $A_i$ are domains of $S^2$ delimited by arcs of great circles, that is, geodesics. By Gauss-Bonnet: $\text{area}(A_i) = 2\pi - \sum j \theta_{ij}$, where the $\theta_{ij}$ are the exterior angles of $A_i$ (see figure 9). Given $K$,
the computation of these angles is very easy. With the notations of the previous proposition, we get:

**Proposition 4.** The writhe of a PL knot $K$ is given by

$$Wr(K) = T_K(\xi_0) + \frac{1}{2\pi} \sum_{i=1}^{r} n_i \cdot (2\pi - \sum_j \theta_{ij}).$$

For lattice knots, the calculation is immediate.

**Definition 6.** A lattice knot is a PL knot on a cubic lattice in $\mathbb{R}^3$.

Let $K$ be a lattice knot; its indicatrix divides a hemisphere into four connected components $A_1, \ldots, A_4$, each of area $\frac{\pi}{2}$. Hence:

$$Wr(K) = \frac{1}{2\pi} \sum_{i=1}^{4} T_K(A_i) \cdot \frac{\pi}{2} = \frac{T_K(A_1) + T_K(A_2) + T_K(A_3) + T_K(A_4)}{4}.$$

Using only the first proposition, we have proved:

**Proposition 5.** If $K$ is a lattice knot, then $4 \cdot Wr(K)$ is an integer.

Furthermore, using the second proposition, it is an easy exercise to implement a program computing the writhe, given a diagram of the lattice knot.

**Final Remark**

The average crossing number of a PL knot $K$ is defined by

$$A(K) = \frac{1}{4\pi} \int \int_{S^2} |T_K(\xi)| \, d\xi.$$

Our method also applies to the computing of this number, leading to a closed formula. Here, all the curves on $S^2$ corresponding to the second Reidemeister move have to be taken into account, as well as the indicatrix $I$. 
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