Transition from $\nu = 8/5$ to $\nu = 5/3$ in the low Zeeman energy limit

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Abstract

Skyrmions in the FQHE at filling fractions above $\nu = 1/3$ are studied within the anyon model and by exact diagonalization. Relations to the composite fermion theory are pointed out. We find that unpolarized quasiparticles above $\nu = 1/3$ are stable below $B \approx 0.02T$. At low Zeeman energy the polarization in the range $\nu = 8/5 \ldots 5/3$ is found to be a linear function of the filling factor. We also reexamine the energy and wave function of skyrmions at $\nu = 1$ by a new method.

I. INTRODUCTION

The quantum Hall effect continues to surprise. One of the newly found phenomena are skyrmions around filling fraction $\nu = 1$. The question arises naturally if similar phenomena could be observed in the fractional quantum Hall regime.

In this paper we investigate unpolarized quasiparticles above $\nu = 1/3$. They are constructed within the framework of the anyon model. We found the anyon model to be closely related to the composite fermion approach. Our exact diagonalization studies show that unpolarized quasiparticles can be stable only below a critical magnetic field of $0.02T$. In spite of this they can play a prominent role at $\nu$ closer to $2/5$, a state which can be interpreted as their condensate. In particular we find that the polarization should be a linear function of the filling factor in some range below $\nu = 2/5$.

In section II we reexamine skyrmion wave functions at $\nu = 1$ by an exact diagonalization in planar geometry. Translational invariance restricts the number of possible wave functions

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to just two. One is the wave function proposed in [1] and the other candidate is a uniform state. The first candidate turns out as the one with lower energy. Spin-reversed states around $\nu = 1/3$ are studied in section [III]. It is stressed that the unpolarized quasiparticles above $\nu = 1/3$ with one reversed spin are uniform in the thermodynamic limit. We construct a wave function basis for the lowest Landau level of such quasiparticles.

In section [IV] the transition from the spin–singlet $\nu = 2/5$ state to the ferromagnetic state at $\nu = 1/3$ is studied in a hard core model. The polarization appears to be a linear function of the filling fraction. We consider the modifications due to a realistic Coulomb potential. By particle–hole duality this picture also applies to the transition from $\nu = 8/5$ to $5/3$ which should be more accessible experimentally.

II. SKYRMIONS NEAR $\nu = 1$

Let the operator $a^\dagger_l (b^\dagger_l)$ create a spin-up (down) electron in the $l$-th orbital of the lowest Landau level, $z^l \exp(-|z|^2/4)$. The $\nu = 1$ ferromagnetic state is $|1> \equiv \prod_{l=0}^{\infty} b^\dagger_l |0>$. A polarized hole of angular momentum $L_z = h$ can be created by removing one electron from the orbital $h$, $|h> \equiv b_h |1> = \prod_{l \neq h} b^\dagger_l |0>$. In the coordinate representation a state with a hole at $w$ is given by

$$\Psi_{\text{hole}}(w; z_k) = \prod_{k=1}^{\infty} (z_k - w) \prod_{m>n=1}^{\infty} (z_m - z_n) | \downarrow_1 \ldots >.$$  \hspace{1cm} (1)

This is not an eigenstate of the angular momentum $L_z$ relative to the origin. The $|h>$ state of the angular momentum $L_z = h$ is obtained from (1) by the projection

$$\Psi_h(z_k) = \int d^2 w e^{-|w|/4} \Phi^h \Psi_{\text{hole}}(w; z_1, \ldots, z_N) = \lim_{w \to 0} \frac{\partial^h}{\partial_{w^h}} \Psi_{\text{hole}}(w; z_1, \ldots, z_N),$$  \hspace{1cm} (2)

where the equality holds up to a normalization factor. By construction all the states $\Psi_h$ have the same energy provided the two-body interaction is translationally invariant. For Coulomb interaction the gross hole energy is $\varepsilon_\perp = 1.25331 e^2/\kappa l$.

The polarized one hole state in the $\nu = 1$ ferromagnet is completely determined by its angular momentum. For low Zeeman energy a number $R$ of electrons can easily be excited to the up-spin LLL band. Instead of one polarized state we have now a whole subspace of unpolarized states

$$|h_1, \ldots, h_{R+1}; s_1, \ldots, s_{R}> \equiv a^\dagger_{s_1} \ldots a^\dagger_{s_R} b_{h_1} \ldots b_{h_{R+1}} |1>,$$  \hspace{1cm} (3)

where $h_1 < \ldots < h_{R+1}$, $s_1 < \ldots < s_{R}$ and $h_1 + \ldots + h_{R+1} - s_1 - \ldots - s_{R} = L_z = h$. The total charge with respect to the $\nu = 1$ state is still $1e$ and the angular momentum $L_z$ is the same as before. Each of the states (3) has a higher energy than the polarized hole but an appropriate combination of them may have less Coulomb energy.

We now restrict ourselves to the case of one reversed spin, $R = 1$. The general form of the textured hole, restricted just by translational invariance, is

$$(z_1 - w)^K \prod_{k=2}^{\infty} (z_k - z_1)^\alpha (z_k - w)^{2-\alpha} \prod_{m>n=2}^{\infty} (z_m - z_n) | \uparrow_1 \downarrow_2 \ldots >.$$  \hspace{1cm} (4)
Note that all the coordinates appear only in differences of electron coordinates or in differences of electron and skyrmion coordinates. There are two holes in the $\nu = 1$ ferromagnet. $(2 - \alpha)$ holes are localized at $w$ and $\alpha$ holes follow the spin-up electron at $z_1$. The angular momentum of the state $|4\rangle$ relative to $w$ is $L = 1 - K$. States of higher angular momenta with respect to the origin can be generated from $|4\rangle$ by a projection similar to Eq.(2). A family of degenerate states can be constructed in this way for each $K \geq 0$.

Even for definite angular momentum, there still is some freedom left in the choice of $\alpha = 0, 1, 2$. For $K = 0$ there are three states

$$
\prod_{k=2}^{N} (z_k - z_1)^{(2-\alpha)} (z_k - w)^\alpha \prod_{m>n=2}^{N} (z_m - z_n) | \uparrow_1 \downarrow_2 \ldots \downarrow_N > ,
$$

where we have introduced the regulator $N$. The $\alpha = 2$ state is unstable. In the language of second quantization this is just one state $|0, 1; 0 >$ of Coulomb energy $1.12\varepsilon_- > \varepsilon_-$. Thus we are left with the two cases $\alpha = 0$ or $1$.

The $\alpha = 1$ state has been proposed in Ref. on the basis that it is a zero energy eigenstate of the hard-core interaction and that its spin is $S = N/2 - 1$ for large $N$. The $\alpha = 0$ state is also a zero energy eigenstate of the hard-core interaction and, as we prove in Appendix A, it has also spin $S = N/2 - 1$. The $\alpha = 0$ state can not be restricted to $N$ orbitals since the spin-up electron coordinate appears with powers up to $z_1^{(2N-2)}$. This is not a problem for the field theoretical limit $N = \infty$, where the $\alpha = 0$ state is a state with spin reversal. Its charge distribution is uniform in accordance with its topological charge equal to zero.

To verify the form of the skyrmion wave function we performed an exact diagonalization in the space of states $|h_1, h_2; s >$ with $h_1 + h_2 - s = 1$. The dimension of this space is infinite in planar geometry. To make the Hilbert space finite we had to impose a cut-off $h_1, h_2, s \leq M$. The system is thus effectively forced to be ferromagnetic outside the ring of the $M - th$ orbital. The second quantization form of the $\alpha = 1$ state $|4\rangle$ is

$$
\sum_{a=0}^{M-1} \frac{(-1)^a}{\sqrt{a+1}} |0, a+1; a > .
$$

We performed the exact diagonalization for a cut-off in the range $M = 1, \ldots , 30$. For $M = 30$ the overlap of the state $|3\rangle$ with the ground state is 0.915. This derivation is mainly due to the suppression of the tail of $|3\rangle$ by the ferromagnetic cut-off. The states $|0, a+1; a >$ contribute 0.986 to the squared norm of the ground state. The energy is convergent in an algebraic way, as could have been expected for a state like $|3\rangle$, which is localized in an algebraic way itself. The extrapolation with a rational function of $1/M$ to the limit $1/M = 0$ gives an $R = 1$ skyrmion energy of 0.9565(1) $\varepsilon_-$. The number in brackets is the extrapolation error of the last digit. Therefore we expect the $R = 1$ skyrmion to become more stable than the polarized hole at a critical magnetic field of around $70T$.

The imposed cut-off breaks the translational symmetry. In a translationally invariant system the $\alpha = 1$ state $|4\rangle$ is a seed state for a multiplet of degenerate states, which are generated by the operator $L^+ = (\partial_{z_1} + \ldots + \partial_{z_N})$ or equivalently by a projection applied to the $\alpha = 1$ state $|4\rangle$ like in Eq.(2). We checked that this symmetry is restored in the limit of infinite $M$ and that the extrapolation to this limit based on the data for $M \leq 30$ is justified by performing similar diagonalizations in the subspaces of the states $|h_1, h_2; s >$.
with $h_1 + h_2 - s = L$, for $L = 2, \ldots, 10$. The energies extrapolated to $1/M = 0$ are listed in table \[.\] All the states are degenerate up to the extrapolation error. This shows that out extrapolation method is valid in the limit of large $M$ where translational symmetry is restored. This confirms the assumed form \[\] of the wave function. The spin texture is localized in a power law way.

The $\alpha = 0$ state of \[\] proved to be unstable. This state has interesting correlations characteristic for the $(1,1,2)$ Laughlin-Halperin state. One could speculate if a condensate of $\alpha = 0$ holes in the form of the $(1,1,2)$ Halperin-like state could be the ground state at $\nu = 2/3$. Unfortunately this state is known to be unstable in the thermodynamic limit.

III. SKYRMIONS NEAR $\nu = 1/3$

Skyrmions near the filling fraction $\nu = 1$ can be regarded as bound states of spin-up electrons with holes in the $\nu = 1$ spin-down ferromagnet. Similarly skyrmions near the ferromagnetic filling fractions $\nu = 1/3, 1/5$ can be interpreted as bound states of spin-up electrons with an appropriate number of anyons. Whereas the former interpretation is exact, the latter, especially if the anyons are assumed to be pointlike, is an approximation designed to capture the most relevant degrees of freedom.

We studied in detail the simplest case of the quasiparticle skyrmion above $\nu = 1/3$ with one reversed spin $R = 1$. The polarized quasiparticle is a quasielectron of electric charge $-e/3$. Its gross energy is $\varepsilon^{(1/3)}_+ \approx -0.128 \varepsilon^2_M$. The skyrmion is a bound state of a spin-up electron with two quasiholes of electric charge $e/3$. The gross energy of the spin-up electron is zero and the gross energy of the quasihole is $\varepsilon^{(1/3)}_\varepsilon \approx 0.231 \varepsilon^2_M$. The difference between the energies of the skyrmion and the polarized quasiparticle is $\Delta \varepsilon = 2 \varepsilon^{(1/3)}_+ - \varepsilon^{(1/3)}_\varepsilon + \varepsilon_{int} = 0.59 + \varepsilon_{int}$, where $\varepsilon_{int}$ is the interaction energy of the constituents. The point-like anyon model is completely defined if we specify the interaction

$$V(z,w_1,w_2) = \frac{e^2}{9\kappa} \frac{1}{|w_1 - w_2|} - \frac{e^2}{3\kappa} \frac{1}{|z - w_1|} - \frac{e^2}{3\kappa} \frac{1}{|z - w_2|},$$

where $z$ is the coordinate of the spin-up electron and $w_i$ are the quasihole coordinates. If the electron and the anyons are restricted to the lowest Landau level, their Hilbert space can be spanned by the orthonormal basis

$$e_{abc}(z,w_1,w_2) = N_{abc} z^a e^{-|z|^2/4} (\bar{w}_1 - \bar{w}_2)^{1/2} e^{-|w_1|^2/12} e^{-|w_2|^2/12}. \tag{8}$$

The normalization factors $N_{abc}$ and the matrix elements of the Coulomb interaction \[\] in the basis \[\] can be found in Appendix B. The state $e_{000}$ has the lowest energy in the basis \[\]. Its angular momentum is $L_z = 1/3$. To understand how this state mixes with other states in order to form a skyrmion state we performed an exact diagonalization in the subspace $L_z \equiv 1/3 + 2b + c - a = 1/3$ of \[\]. For this subspace to be finite, we had to impose the cut-off $a, 2b, c \leq M$. The maximal cut-off was $M = 17$. We found that 0.9999 of the norm squared of the ground state lies in the subspace spanned by the orthonormal basis

$$e_k^{(0)}(z,w_1,w_2) = N_k^{(0)} z^k e^{-|z|^2/4} (\bar{w}_1 - \bar{w}_2)^{1/2} (\bar{w}_1 + \bar{w}_2)^{b} e^{-|w_1|^2/12} e^{-|w_2|^2/12}. \tag{9}$$
The two anyons are found to be in the state of lowest possible relative angular momentum $1/3$. It seems as if there were just two relevant particles in the problem: the spin-up electron at $z$ and a double quasi-hole at $w_1 + w_2$. The attraction between the electron and the anyons dominates the mutual repulsion between anyons (7). The two anyons try to be as close as possible to the spin-up electron even if this costs some (small) amount of repulsion energy. The spin-up electron coordinate must appear in the ground state wave function as $\prod_{k=2}^{\infty}(z_k - z_1)^2 \prod_{m>n=2}^{\infty}(z_m - z_n)^3 \uparrow_1 \downarrow_2 \ldots >$: for the spin-down electrons the spin-up electron appears as a particle with two attached flux quanta.

Before we proceed, let us try to determine possible skyrmion wave functions on general grounds. Similarly as around $z$ the states (9). In the $k$-th state of the family (13) the spin-up electron sits in the $k$-th

\[
(z_1 - w)^K \prod_{k=2}^{\infty}(z_k - z_1)^{(2-\alpha)}(z_k - w)^{\alpha} \prod_{m>n=2}^{\infty}(z_m - z_n)^3| \uparrow_1 \downarrow_2 \ldots >. \tag{10}
\]

There are two Laughlin quasiholes in addition to the electron with reversed spin. The $\alpha$ quasiholes are localized around $w$ and $(2 - \alpha)$ quasiholes follow the spin-up electron at $z_1$. The angular momentum relative to $w$ is $L_z = \frac{1}{3} - K$ as compared to the $\nu = 1/3$ state. The ground state which we found within the anyon model for $L_z = 1/3$ can be identified with the $K = 0$ and $\alpha = 0$ state

\[
\prod_{k=2}^{\infty}(z_k - z_1)^2 \prod_{m>n=2}^{\infty}(z_m - z_n)^3| \uparrow_1 \downarrow_2 \ldots > = \left[ \prod_{i>j=2}^{\infty}(z_m - z_n)^2 \right] \left\{ \prod_{k=2}^{\infty}(z_k - z_1)^2 \prod_{m>n=2}^{\infty}(z_m - z_n)^3 | \uparrow_1 \downarrow_2 \ldots > \right\}. \tag{11}
\]

The equation shows the way this state is constructed in the composite fermion prescription from the seed state with filled spin-down LLL and one electron in the spin-up LLL. The spin and charge distributions of this state are uniform. We discuss quasiparticle properties of the state (11) in section III.A.

If we restrict ourselves to $K = 0$ it is easy to understand why $\alpha = 0$ is preferred on energetic grounds. The $\alpha = 2$ state is just the single $e_{000}$ state of the anyon model. This state is not an eigenstate of the Coulomb interaction (see Appendix B), so it can not be the ground state. For $\alpha = 1, w = 0$ the wave function (11) can be expanded in powers of $z_1$.

\[
\prod_{k=2}^{\infty}(z_k - z_1)z_k \prod_{m>n=2}^{\infty}(z_m - z_n)^3| \uparrow_1 \downarrow_2 \ldots > = \left[ \prod_{k=2}^{\infty} z_k^2 - z_1 \sum_{l=2,3,4,\ldots}^{\infty} \prod_{k \neq l}^{\infty} z_k^2 + \ldots \right] \prod_{m>n=2}^{\infty}(z_m - z_n)^3 | \uparrow_1 \downarrow_2 \ldots >. \tag{12}
\]

The subsequent terms in this expansion are proportional to the following states in the anyon model

\[
\tilde{e}_k(z, w_1, w_2) = \tilde{N}_k z^k e^{-\frac{|w_1|^2}{2}} (\bar{w}_1 - \bar{w}_2)^3 (\bar{w}_1^k + \bar{w}_2^k) e^{-\frac{|w_1|^2}{2} - \frac{|w_2|^2}{2}}. \tag{13}
\]

A similar expansion in powers of $z_1$ for the $\alpha = 1, w = 0$ wave function (11) gives rise to the states (11). In the $k$-th state of the family (13) the spin-up electron sits in the $k$-th
electronic orbital, one of the anyons is in the 0-th anyonic orbital and the other anyon sits in the \( k \)-th anyonic orbital. As the hole’s magnetic length is \( \sqrt{3} \) that of the electron, the \( k \)-th electronic and anyonic orbitals soon become quite distant with increasing \( k \). The attraction between the electron and each of the anyons becomes weaker with increasing \( k \). On the other hand in the \( k \)-th state of the family (13) there is the spin-up electron in the \( k \)-th orbital and the double hole in the \( k \)-th double hole orbital. The double hole’s magnetic length is just \( \sqrt{3}/2 \) that of the electron, so that the \( k \)-th orbitals are now much closer to each other. The attraction decreases with increasing \( k \) but much slower than for the states (13). The general rule is that for an electron interacting with \( n \) anyons of electric charge \( e/p \), the anyons will tend to group into a \( p \)-particle cluster and \((n-p)\) single anyons. To illustrate this rule let us consider the quasihole skyrmion just below \( \nu = 1/3 \). Its general form which is restricted just by translational invariance is

\[
\prod_{k=2}^{\infty} (z_k - z_1)^{(1-\alpha)} (z_k - w)\prod_{m>n=2}^{\infty} (z_m - z_n)^3 \uparrow_1 \downarrow_2 \ldots > . \tag{14}
\]

There are one spin-up electron and four quasiholes. The rule predicts the \( \alpha = 1 \) state to be the ground state as it is indeed the case. The rule also gives the correct wave functions for skyrmions around \( \nu = 1/5 \).

In the special case of \( p = 1 \), the rule predicts that the \( \alpha = 1 \) state (5) is more stable than the \( \alpha = 0 \) state (5).

A. Lowest Landau level of the unpolarized quasiparticle

The state (11) is just one of the family of degenerate states

\[
\Psi_K(w, z_k) = (z_1 - w)^K \prod_{k=2}^{\infty} (z_k - z_1)^2 \prod_{m>n=2}^{\infty} (z_m - z_n)^3 \uparrow_1 \downarrow_2 \ldots > = \left[ \prod_{i>j=1}^{\infty} (z_i - z_j)^2 \right] \left( \prod_{k=2}^{\infty} (z_k - z_1)^2 \prod_{m>n=2}^{\infty} (z_m - z_n) \right) \uparrow_1 \downarrow_2 \ldots > , \tag{15}\]

with \( K = 0, 1, 2 \ldots \). The states are degenerate because they are related by

\[
\Psi_{K-1}(w, z_k) = -\frac{1}{K} \partial_w \Psi_K(w, z_k) = \frac{1}{K} L^+ \Psi_K(w, z_k) \tag{16}
\]

and the energy should not depend on the choice of \( w \). The anyon model, which is by construction in the field theoretical limit \( N = \infty \), has this particular symmetry too. We checked this by exact diagonalizations in the subspaces spanned by the orthonormal states

\[
e_{k}^{(K)}(z, w_1, w_2) = N_k^{(K)} z^{K+1} e^{\frac{|w_1|^2}{|z|^2}} (\overline{w_1} - \overline{w_2}) (\overline{w_1} + \overline{w_2})^k e^{\frac{|w_1|^2}{|z|^2}} e^{\frac{|w_2|^2}{|z|^2}} \tag{17}\]

in the range of \( K = 0, \ldots, 6 \). The interaction energies extrapolated to \( 1/M = 0 \) are listed in table [1]. They are degenerate up to extrapolation errors. Thus the anyon model correctly reproduces the global symmetry.

In the state \( \Psi_K \) the spin-up electron has been removed from the orbitals 0, \ldots, \( K \) – 1. The states (13) can be interpreted as orbitals in the lowest Landau level of the unpolarized quasielectron. They are obtained in the CF prescription from a filled spin-down LLL and a spin-up electron in the \( K \)-th orbital of the spin-up LLL.
B. Energy of the unpolarized quasiparticle

To get an estimate for the energy gain by depolarization we made a finite size study of electrons in spherical geometry. The one quasiparticle sector at $\nu = 1/3$ is characterized by $m = 3(N - 1) - 1$ flux quanta piercing the sphere. We calculated the lowest eigenstates for Coulomb interaction in the subspace of states with $S_z = N/2 - 1$ for up to 9 electrons. The lowest energy state has always $S = N/2 - 1$, the next state $S = N/2$. The energies are given in table III. The data allow fits with quadratic polynomials in $1/N$ which gives a gap energy for $N \to \infty$ of $0.0009(3)e^2/\varepsilon l$.

Therefore, the unpolarized quasielectron is found to be stable but only at marginally small magnetic fields $B < 0.02T$. It is even less likely to be observed if we take impurities into account. The polarized quasielectron is a localized object, which can form a bound state with an impurity. The unpolarized quasielectron (11) is uniform so its interaction energy with a localized potential is zero to first approximation. The stability of a fluid of quasielectrons can be enhanced when they condense into a Laughlin fluid. In fact, the stable unpolarized $\nu = 2/5$ state can be interpreted as such a condensate.

IV. TRANSITION FROM $\nu = 8/5$ TO $\nu = 5/3$

The ferromagnetic $\nu = 1/3$ ground state is described by the celebrated Laughlin wave function

$$\psi_{1/3}(z_k) = \prod_{m>n=1}^{N} (z_m - z_n)^3 | \downarrow_1 \ldots \downarrow_N> = \left[ \prod_{k>l=1}^{N} (z_k - z_l)^2 \right] \left\{ \prod_{m>n=1}^{N} (z_m - z_n) | \downarrow_1 \ldots \downarrow_N> \right\}.$$  \hspace{1cm} (18)

The second equality shows how this state is constructed from the $\nu = 1$ ferromagnet in the composite fermion theory. In the same framework, the spin-singlet $\nu = 2/5$ is obtained from the spin-singlet $\nu = 2$ state,

$$\psi_{2/5}(z_k) = \prod_{i>j=1}^{N/2} (z_i - z_j)^3 \prod_{k>l=N/2}^{N} (z_k - z_l)^3 \prod_{m=0}^{N/2} \prod_{n=N/2}^{N} (z_m - z_n)^2 | \uparrow_1 \ldots \uparrow_{N/2} \downarrow_{N/2+1} \ldots \downarrow_N> =$$

$$\left[ \prod_{i>j=1}^{N} (z_i - z_j)^2 \right] \left\{ \prod_{k>l=1}^{N/2} (z_k - z_l) \prod_{m>n=N/2}^{N} (z_m - z_n) | \uparrow_1 \ldots \uparrow_{N/2} \downarrow_{N/2+1} \ldots \downarrow_N> \right\},$$  \hspace{1cm} (19)

where we have chosen the spinor component with the spins $1, \ldots, N/2$ up and the spins $N/2+1, \ldots, N$ down. The spin-singlet $\nu = 2/5$ state is the $(3, 3, 2)$ Halperin wave function. The states (18) and (19) are both zero energy eigenstates of the hard core model

$$V(z) = \infty \delta(z) + \lambda \nabla^2 \delta(z)$$  \hspace{1cm} (20)

with pseudopotentials $V_0 = \infty, V_1 > 0$ and $V_k = 0$ for $k \geq 2$.

A fully polarized state at $\nu > 1/3$ can not have zero energy because the electrons are packed too close. More space can be created by reversing a number of spins. An example
of a zero energy eigenstate one flux quantum above \( \nu = 1/3 \) is the unpolarized quasiparticle (11) with one reversed spin

\[
\prod_{k=2}^{N} (z_k - z_1)^2 \prod_{m>n=2}^{N} (z_m - z_n)^3 | \uparrow_1 \downarrow_2 \ldots \downarrow_n > .
\]

(21)

The spin of the state (21) is \( S = \frac{N \uparrow}{2} - 1 \).

Two flux quanta above 1/3 the polarized ground state would contain two quasielectrons. To have zero energy, a number of spins must flip. For just one flipped spin a candidate for a skyrmion is

\[
\prod_{k=2}^{N} (z_k - z_1) \prod_{m>n=2}^{N} (z_m - z_n)^3 | \uparrow_1 \downarrow_2 \ldots \downarrow_n > .
\]

(22)

The energy of this state does not vanish. The zero energy state must contain at least \( R = 2 \) reversed spins, for example

\[
\prod_{i>j=1}^{2} (z_i - z_j)^3 \prod_{k=3}^{N} (z_k - z_l)^2 \prod_{m>n=3}^{N} (z_m - z_n)^3 | \uparrow_1 \uparrow_2 \downarrow_3 \ldots \downarrow_n > .
\]

(23)

The two examples above can be easily generalized. To construct a zero energy eigenstate \( \Phi \) flux quanta above the filling fraction 1/3 one has to flip at least \( \Phi \) spins, \( R \geq \Phi \). An example of zero energy state is

\[
\prod_{i>j=1}^{\Phi} (z_i - z_j)^3 \prod_{k=\Phi+1}^{N} (z_k - z_l)^2 \prod_{m>n=\Phi+1}^{N} (z_m - z_n)^3 | \uparrow_1 \ldots \uparrow_\Phi \downarrow_\Phi+1 \ldots \downarrow_n > .
\]

(24)

The spin of the above state is \( S = \frac{N \uparrow}{2} - \Phi \). The zero energy states have spins in the range of \( S = \frac{N \uparrow}{2} - \Phi, \ldots, 0 \). A state with \( S < \frac{N \uparrow}{2} - \Phi \) can not have zero energy. In particular the spin-singlet state (19) is a unique zero energy state at \( \nu = 2/5 \), where \( \Phi = \frac{N \uparrow}{2} \).

So far the Zeeman energy was set to zero. The ground states for \( S < \frac{N \uparrow}{2} - \Phi \) have positive potential energies proportional to the only relevant quasipotential \( V_1 \), let us call them \( V_1 e(S) \). The total energies of the ground states are \( E(S) = V_1 e(S) - GS \), where \( G \) is the Zeeman energy. The ground state is polarized \( S = \frac{N \uparrow}{2} \) for sufficiently large \( G \). As the Zeeman energy is decreased, the polarization \( P \equiv 2S/N \) decreases from \( P = 1 \) to \( P \leq \frac{(2/5 - \nu)}{(2/5 - 1/3)} \). It can not be any less because \( e(0) = \ldots = e(N \uparrow - \Phi) \) so that for any \( G > 0 \) we have \( E(N \uparrow - \Phi) < \ldots < E(0) \).

We conclude that in the hard core model (20) in the limit of small Zeeman energy \( G = 0^+ \) the polarization depends linearly on the filling fraction

\[
P = \frac{(2/5 - \nu)}{(2/5 - 1/3)}
\]

(25)

in the range \( \nu \in (1/3, 2/5) \). Similar results hold, by particle-hole symmetry, for \( \nu \in (8/5, 5/3) \).

How is this result modified by a more realistic interaction potential and what is its experimental significance? We have seen that the unpolarized quasiparticle (21) is unstable for
realistic magnetic fields. On the other hand, the spin-singlet \( \nu = 8/5 \) state is experimentally observed\(^3\) at fields of around \( B = 5T \). Thus at \( \nu = 8/5 \) the low Zeeman energy limit can be achieved in present experiments. The polarization is bounded from below by (25), since the zero energy band is exactly degenerate in the hard core model. This degeneracy is slightly removed for Coulomb interaction but without decreasing the polarization below (25) at realistic magnetic fields. Thus we expect that the polarization follows the linear pattern (25) in some range above \( \nu = 8/5 \) before it increases faster than linear to \( P = 1 \).

To understand better why the degeneracy of the hard core zero energy band is only slightly removed by the Coulomb interaction, let us observe that all the hard core zero energy states below \( \nu = 2/5 \) can be obtained by inserting quasiholes into the spin-up or spin-down fluids of the \((3,3,2)\) state. The quasiholes interact by Coulomb potential but their electric charge is only \( e/5 \). Let us consider the example of two quasiholes. The spin can be either \( S = 0 \) or \( S = 1 \). The lowest state for a given spin is the one with the highest relative angular momentum of the quasiparticles. For a given number of flux quanta the highest relative angular momenta for \( S = 0 \) and \( S = 1 \) differ only by 1. If these maximal relative angular momenta are large (low quasihole density) the corresponding states are almost degenerate. Degeneracy is removed by the Zeeman energy, which favors the \( S = 1 \) ground state. For more than two quasiholes the ground state for any given spin is the state with the highest possible pairs' angular momenta. By a similar argument as for two holes, the polarization below \( \nu = 2/5 \) (or above \( \nu = 8/5 \)) is given by (25) for low Zeeman energy.

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**APPENDIX A:**

The proof is a generalization of an analogous argument in\(^4\). For any wave-function \( \psi \) its "bosonic" part \( \psi_B \) can be defined by \( \psi = \psi_B \Pi_{m>n=1}^N (z_m - z_n) \). The spin quantum numbers of \( \psi \) and \( \psi_B \) are the same. The bosonic part of the \( \alpha = 0 \) state (4) is

\[
\prod_{k=2}^N (z_k - z_1) \uparrow_1 \downarrow_2 \cdots \downarrow_N \equiv \phi_1(\{z_l\}) \uparrow_1 \downarrow_2 \cdots \downarrow_N > . \tag{A1}
\]

As usual, the exponential factors are omitted and only one spinor component is shown. Other spinor components can be obtained by symmetrization. The expectation value of the operator

\[
S^2 = S_z^2 + \frac{S_+ S_- + S_- S_+}{2} \tag{A2}
\]

in state (A1) can be worked out as

\[
\langle S^2 \rangle = \left( \frac{N}{2} - 1 \right)^2 + \frac{N}{2} + (N - 1) \frac{\langle \phi_1 | \phi_2 \rangle}{||\phi_1|| \ |\phi_2||} . \tag{A3}
\]
The scalar product is understood as the integral \( < f, g > = \int \prod_{k=1}^{N} d^2 z_k \exp(-|z_k|^2/2) f^* g \).

By symmetry \(||\phi_1|| = ||\phi_2||\). The norm can be found to be

\[
||\phi_1||^2 = \pi^N 2^{2N-1} (N-1)! \sum_{s=0}^{N-1} \frac{1}{s!} \tag{A4}
\]

and the unnormalized overlap

\[
< \phi_1 | \phi_2 > = \pi^N 2^{2N-1} N \tag{A5}
\]

so that

\[
< S^2 > \equiv S(S+1) = \left( \frac{N}{2} - 1 \right)^2 + \frac{N}{2} + \frac{N}{(N-2)! \sum_{s=0}^{N-1} \frac{1}{s!}} \tag{A6}
\]

Equation (A6) can be solved with respect to \( S \). The result is \( S = \frac{N}{2} - 1 + O(1/N) \), which was to be demonstrated. The state (A1) has \( S_z = -\frac{N}{2} + 1 \). As such it is in general a combination of eigenstates with \( S = \frac{N}{2} - 1 \) and \( \frac{N}{2} \). In the thermodynamic limit only the \( S = \frac{N}{2} - 1 \) states survives.

**APPENDIX B:**

The normalization factor of the states (8) is

\[
N_{abc}^{-2} = \pi^3 2^{(a-1)} 12^{(2b+c+\frac{1}{2})} \Gamma[a+1] \Gamma[2b + \frac{4}{3}] \Gamma[c+1] \tag{B1}
\]

The matrix elements of the interaction (7) in the basis (8) are

\[
< e_{abc} | V | e_{def} > = \frac{\delta_{ad} \delta_{be} \delta_{cf}}{9 \sqrt{12} \Gamma[2b + \frac{3}{2}]} - \frac{1}{6} N_{abc} N_{def} A_{ad} B_{be} C_{cf} I[a + d, b + e, c + f, |a - d|, |b - e|, |c - f|] \tag{B2}
\]

where the various factors are

\[
A_{ad} = \pi \ i^{(|a-d|)} 2^{(\frac{a+d}{2} - \frac{|a-d|}{2} + 1)} \frac{\Gamma[\frac{a+d}{2} + \frac{|a-d|}{2} + 1]}{\Gamma[|a-d| + 1]},
\]

\[
B_{be} = \pi \ (-i)^{|b-e|} 12^{(\frac{b-e}{2} + 1)} \left( \frac{3}{4} \right)^{|b-e|} \frac{\Gamma[\frac{b+e}{2} + \frac{|b-e|}{2} + 1]}{\Gamma[|b-e| + 1]},
\]

\[
C_{cf} = \pi \ i^{2|c-f|} 12^{(c+f+\frac{4}{3})} \left( \frac{3}{4} \right)^{|c-f|} \frac{\Gamma[a + d + |a - d| + \frac{4}{3}]}{\Gamma[2|c-f| + 1]},
\]

\[
I[A, B, C, a, b, c] = \int_{0}^{\infty} dq \ \frac{q^{(a+2b+c)}}{\exp(2q^2)} \ _1F_1\left[\frac{a - A}{2}, a + 1, \frac{q^2}{2}\right] _1F_1\left[\frac{b - B}{2}, b + 1, \frac{3q^2}{2}\right] _1F_1\left[\frac{c - C}{3}, 2a + 1, \frac{3q^2}{2}\right] \tag{B3}
\]
APPENDIX C:

The normalization factors of the basis (17) are

\[
(N_k^{(K)})^{-2} = \pi^3 2^{(K+k-1)} 12^{(k+\frac{7}{3})} \Gamma\left[\frac{4}{3}\right] \Gamma[k+1] \Gamma[K+k+1] .
\] (C1)

The matrix elements of the interaction (7) in the basis (17) are

\[
\langle \epsilon_k^{(K)} | V | \epsilon_l^{(K)} \rangle = \frac{\Gamma\left[\frac{5}{6}\right]}{9 \sqrt{12} \Gamma\left[\frac{4}{3}\right]} - 2\pi^3 N_k^{(K)} N_l^{(K)} J_K[k+l,|k-l|] \\
\times 2^{(k+l-|k-l|+K+1)} 12^{\left(\frac{k+l+|k-l|}{2}\right)} \left(\frac{4}{3}\right) \frac{\Gamma\left[\frac{4}{3}\right]}{\Gamma^2[|k-l|+1]} \frac{\Gamma\left[\frac{k+l+|k-l|}{2}+1\right]}{\Gamma\left[\frac{k+l+|k-l|}{2}+1+K\right]},
\] (C2)

where

\[
J_K[A, a] = \int_0^{\infty} dq \frac{q^{2a}}{\exp(2q^2)} \frac{1}{\Gamma\left[-\frac{1}{3}, 1, \frac{3q^2}{2}\right]} \frac{1}{\Gamma\left[-\frac{a-A}{2}, a+1, \frac{3q^2}{2}\right]} \frac{1}{\Gamma\left[-\frac{a-A}{2}, a+1, \frac{3q^2}{2}\right]}.
\] (C3)
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TABLES

TABLE I. $R = 1, \nu = 1$ Skyrmion energy for different angular momenta

| L | $E(L) \ [\varepsilon_\nu]$ |
|---|---|
| 1 | 0.9565(1) |
| 2 | 0.9564(2) |
| 3 | 0.9564(1) |
| 4 | 0.9564(1) |
| 5 | 0.9564(1) |
| 6 | 0.9565(1) |
| 7 | 0.9565(1) |
| 8 | 0.9567(3) |
| 9 | 0.9564(2) |
| 10 | 0.9566(3) |

TABLE II. Energy of the unpolarized anyonic quasiparticle at $R = 1, \nu = 1/3$

| K | $E_{\text{int}}(K) \ [\varepsilon_{\nu}^2]$ |
|---|---|
| 0 | -0.395(1) |
| 1 | -0.394(1) |
| 2 | -0.395(2) |
| 3 | -0.392(2) |
| 4 | -0.392(2) |
| 5 | -0.395(1) |
| 6 | -0.393(1) |

TABLE III. Energy per particle of the lowest states of $N$ electrons on a sphere with spin $S = N/2$ and $N/2 - 1$ at filling fraction “1/3 + 1 quasiparticle”. The energy is in units of $e^2/\varepsilon l'$ with $l'$ including a finite size correction $l' = \sqrt{l_0 l}$.

| N | $E \ (S=N/2-1)$ | $E \ (S=N/2)$ |
|---|---|---|
| 5 | -0.405418 | -0.397353 |
| 6 | -0.406033 | -0.399790 |
| 7 | -0.406445 | -0.401230 |
| 8 | -0.406795 | -0.402361 |
| 9 | -0.407090 | -0.403246 |
| $\infty$ | -0.4095(1) | -0.4086(2) |