ON MINIMAL DECAY AT INFINITY OF HARDY-WEIGHTS

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Abstract. We study the behaviour of Hardy-weights for a class of variational quasilinear elliptic operators of $p$-Laplacian type. In particular, we obtain necessary sharp decay conditions at infinity on the Hardy-weights in terms of their integrability with respect to certain integral weights. Some of the results are extended also to nonsymmetric linear elliptic operators. Applications to various examples are discussed as well.

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1. Introduction

The problem of finding a function $W \geq 0$ such that a given nonnegative operator $L$ in a domain $\Omega \subset \mathbb{R}^N$ satisfies, in certain sense, the inequality

$$L \geq W$$

(1.1)

has been intensively studied in the past decades. For a detailed analysis of these so-called Hardy-type inequalities we refer to the monographs [1, 8, 11] and references therein. The function $W$ is usually called a Hardy-weight for the operator $L$.

The aim of the present paper is to study the ‘decay’ at infinity of Hardy-weights for a class of variational quasilinear operators, the so-called $(p,A)$–Laplacians with a potential term, in external domains, see Section 2 for a detailed definition. For practical purposes, and also from the theoretical point of view, it is very natural to address the question of ‘how large’ the weight function $W$ might be. One would of course like to make $W$ as large as possible in order to optimize inequality (1.1) (see [5, 6]). However, there are certain constrains, depending on $L$ and $\Omega$, which have to be respected.

In this note we study one of such constrains, namely the behaviour of $W$ at infinity. Roughly speaking we show that the Hardy-weights cannot decay too slowly at infinity. More precisely, we establish necessary decay conditions on $W$ in terms of $L^1$ integrability of $W$ (at infinity) with respect to integral weights which are related to positive solutions of the equation $Lu = 0$. This is done in Section 3.1 for critical operators (see Theorem 3.1), and in Section 3.2 for subcritical operators (see Theorem 3.2). In Section 3.3 we show that the results can be extended also to a certain class of linear nonsymmetric operators (see, Theorem 3.5).

In Section 2 we introduce the necessary notation and recall some results previously obtained in the literature on some of which we rely in the proofs of our main theorems. The latter are formulated and proved in Section 3. In the closing Section 4 we illustrate the sharpness of our decay conditions by several examples.
2. Preliminaries

2.1. Notation. Let $\Omega$ be a domain in $\mathbb{R}^N$, $1 < p < \infty$, $N \geq 2$. Throughout the paper we use the following notation.

- We denote by $\infty$ the ideal point which is added to $\Omega$ to obtain the one-point compactification of $\Omega$.
- We write $\Omega_1 \Subset \Omega_2$ if the set $\Omega_2$ is open in $\Omega$, the set $\overline{\Omega_1}$ is compact and $\overline{\Omega_1} \subset \Omega_2$.
- Let $g_1, g_2$ be two positive functions defined in a domain $D$. We say that $g_1$ is equivalent to $g_2$ in $D$ (and use the notation $g_1 \asymp g_2$ in $D$) if there exists a positive constant $C$ such that
  \[ C^{-1} g_2(x) \leq g_1(x) \leq C g_2(x) \quad \text{for all } x \in D. \]
- The open ball of radius $r > 0$ centered at $y$ is denoted by $B_r(y) := \{ x \in \mathbb{R}^N : |x - y| < r \}$.
- We denote by $\text{diam}(\omega)$ the diameter of the set $\omega \subset \mathbb{R}^N$.

2.2. The quasilinear case. Let $\Omega$ be a domain in $\mathbb{R}^N$, $1 < p < \infty$, $N \geq 2$. In this note we consider the quasilinear operator $L(A, V)$ acting on $W^{1,p}_\text{loc}(\Omega)$ of the form
  \[ L(A, V)u := -\nabla \cdot ((|\nabla u|^{p-2} A \nabla u) + V|\nabla u|^{p-2} u), \quad (2.1) \]
and the following associated functional defined on $C^\infty_c(\Omega)$
  \[ Q_{A,V}[\varphi] := \int_{\Omega} (|\nabla \varphi|^p_A + V|\varphi|^p) \, dx, \quad (2.2) \]
where $A : \Omega \to \mathbb{R}^N \times \mathbb{R}^N$ is a symmetric matrix, $V : \Omega \to \mathbb{R}$ is a potential function, and
  \[ |\xi|_{A(x)} := \langle \xi, A(x) \xi \rangle^{1/2} \quad x \in \Omega, \xi \in \mathbb{R}^N. \]
We assume that $A$ is locally bounded and locally uniformly positive definite, i.e., for any $K \Subset \Omega$ there exists $\Lambda_K > 0$ such that
  \[ \Lambda_K^{-1} |\xi|^2 \leq |\xi|_{A(x)}^2 \leq \Lambda_K |\xi|^2 \quad \forall x \in K, \forall \xi \in \mathbb{R}^N, \quad (2.3) \]
and $V$ belongs to a certain local Morrey space $M^q_{\text{loc}}(p; \Omega)$ which is defined below (see [14]).

**Definition 2.1.** Let $q \in [1, \infty]$ and let $\omega \Subset \mathbb{R}^N$. Given a measurable function $f : \omega \to \mathbb{R}$, we set
  \[ \|f\|_{M^q(\omega)} := \sup_{f \in \omega} \sup_{r < \text{diam}(\omega)} \left\{ r^{-N(q-1)/q} \int_{B_r(y) \cap \omega} |f| \, dx \right\}. \quad (2.4) \]
We write $f \in M^q_{\text{loc}}(\Omega)$ if for any $\omega \Subset \Omega$ it holds $\|f\|_{M^q(\omega)} < \infty$. We then define
  \[ M^q_{\text{loc}}(p; \Omega) := \begin{cases} 
    M^q_{\text{loc}}(\Omega) & \text{with } q > \frac{N}{p} \text{ if } p < N, \\
    L^1_{\text{loc}}(\Omega) & \text{if } p > N, 
  \end{cases} \quad (2.5) \]
and for $p = N$ we write $f \in M^q_{\text{loc}}(N; \Omega)$, if for some $q > N$ and any $\omega \Subset \Omega$ it holds
  \[ \|f\|_{M^q(N; \omega)} := \sup_{f \in \omega} \sup_{r < \text{diam}(\omega)} \left\{ \log \frac{\text{diam}(\omega)}{r} \left( \frac{\text{diam}(\omega)}{r} \right)^{q-1} \int_{B_r(y) \cap \omega} |f| \, dx \right\} < \infty. \quad (2.6) \]
We are now ready to introduce our regularity hypotheses on the coefficients of the operator 
$L(A,V)$. Throughout the paper we assume that

the matrix $A$ satisfies (2.3), and the potential $V \in M_{\text{loc}}^q(p;\Omega)$.  (H0)

In the case $1 < p < 2$, we make the following stronger hypothesis:

$A \in C^{0,\gamma}_{\text{loc}}(\Omega;\mathbb{R}^{N^2})$ satisfies (2.3), and $V \in M_{\text{loc}}^q(\Omega)$, where $q > N$.  (H1)

**Definition 2.2.** The functional $Q_{A,V}$ is said to be

1) nonnegative in $\Omega$ (in short, $Q_{A,V} \geq 0$ in $\Omega$) if

$$Q_{A,V}[\varphi] := \int_{\Omega} (|\nabla \varphi|^p + V|\varphi|^p) \, dx \geq 0 \quad \text{for all } \varphi \in C^\infty_c(\Omega).$$  (2.7)

2) subcritical in $\Omega$ if there exists a nonzero nonnegative weight function $W \in M_{\text{loc}}^q(p;\Omega)$, called a Hardy-weight, such that

$$Q_{A,V}[\varphi] \geq \int_{\Omega} W|\varphi|^p \, dx \quad \text{for all } \varphi \in C^\infty_c(\Omega).$$  (2.8)

3) critical in $\Omega$ if $Q_{A,V} \geq 0$ in $\Omega$, but $Q_{A,V}$ does not admit any Hardy-weight.

4) supercritical in $\Omega$ if $Q_{A,V} \not\geq 0$ in $\Omega$ (that is, there exists $\varphi \in C^\infty_c(\Omega)$ such that $Q_{A,V}[\varphi] < 0$).

**Definition 2.3.** We say that the operator $L(A,V)$ is nonnegative in $\Omega$ if the equation

$L(A,V)w = 0$ in $\Omega$ admits a positive weak supersolution $\tilde{u} \in W^{1,p}_{\text{loc}}(\Omega)$.

First, we recall the following Allegretto-Piepenbrink-type theorem.

**Theorem 2.4** (Allegretto-Piepenbrink-type theorem [14, Theorem 4.3]). Suppose $A$ and $V$ satisfy hypothesis (H0). Then the following assertions are equivalent:

- The functional $Q_{A,V}$ is nonnegative on $C^\infty_c(\Omega)$.
- The equation $L(A,V)w = 0$ in $\Omega$ admits a positive solution $v \in W^{1,p}_{\text{loc}}(\Omega)$.
- The equation $L(A,V)w = 0$ in $\Omega$ admits a positive supersolution $\tilde{v} \in W^{1,p}_{\text{loc}}(\Omega)$.

**Definition 2.5.** A sequence $\{\varphi_k\} \subset C^\infty_c(\Omega)$ is called a null-sequence with respect to the nonnegative functional $Q_{A,V}$ in $\Omega$ if

a) $\varphi_k \geq 0$ for all $k \in \mathbb{N}$,

b) there exists a fixed open set $K \subset \Omega$ such that $\|\varphi_k\|_{L^p(K)} = 1$ for all $k \in \mathbb{N}$,

c) $\lim_{k \to \infty} Q_{A,V}[\varphi_k] = 0$.

We call a positive function $\phi \in W^{1,p}_{\text{loc}}(\Omega)$ a ground state of $Q_{A,V}$ in $\Omega$ if $\phi$ is an $L^p_{\text{loc}}(\Omega)$ limit of a null-sequence.

We have [13, Theorem 4.15]:

**Theorem 2.6.** Suppose that $Q_{A,V}$ is nonnegative on $C^\infty_c(\Omega)$ with $A$ and $V$ satisfying hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$. Then $Q_{A,V}$ is critical in $\Omega$ if and only if $Q_{A,V}$ admits a null-sequence. Moreover, in this case the equation $L(A,V)u = 0$ admits (up to multiplicative constant) a unique positive supersolution $\phi$. Furthermore, $\phi$ is a ground state.
If \( Q_{A,V} \) is subcritical in \( \Omega \), then the set of all Hardy-weights is convex. Moreover, \( W \) is an extreme point of this set if and only if \( Q_{A,V-W} \) is critical. This indicates that critical Hardy-weights are rare, and in general difficult to be determined concretely. The papers \[5, 6\] are devoted to the search of a class of optimal Hardy-weights, that is, Hardy-weights that are ‘as large as possible’ in the following sense.

**Definition 2.7.** Suppose that \( Q_{A,V} \geq 0 \) in \( \Omega \). Assume that a nonzero nonnegative function \( W \) satisfies the following Hardy-type inequality

\[
Q_{A,V}[\varphi] \geq \lambda \int_{\Omega} W|\varphi|^p \, dx \quad \forall \varphi \in C_c^\infty(\Omega),
\]

with some \( \lambda > 0 \). We say that \( W \) is an optimal Hardy-weight for the operator \( Q_{A,V} \) in \( \Omega \) if the following conditions hold true.

- (Criticality) The functional \( Q_{A,V-W} \) is critical in \( \Omega \). In particular, \( Q_{A,V-W} \) admits a ground state \( \phi \) in \( \Omega \).
- (Null-criticality) The functional \( Q_{A,V-W} \) is null-critical in \( \Omega \) with respect to \( W \), that is, \( \phi \notin L^p(\Omega, W \, dx) \).
- (Optimality at infinity) \( \lambda = 1 \) is also the best constant for inequality (2.9) restricted to functions \( \varphi \) that are compactly supported in any fixed neighborhood of infinity in \( \Omega \).

For the \( p \)-Laplacian in ‘exterior’ domains we have

**Theorem 2.8** ([6, Theorem 6.1]). Let \( \Omega \) be a \( C^{1,\alpha} \) domain (not necessarily bounded), where \( 0 < \alpha \leq 1 \). Let \( U \Subset \Omega \) be an open \( C^{1,\alpha} \) subdomain of \( \Omega \), and consider \( \Omega := \Omega \setminus U \). Denote by \( \bar{\infty} \) the infinity in \( \Omega \), and assume that \(-\Delta_p\) admits a nonconstant positive \( p \)-harmonic function \( G \) in \( \tilde{\Omega} := \Omega \setminus U \) satisfying the following conditions

\[
\lim_{x \to \partial U} G(x) = \gamma_1, \lim_{x \to \bar{\infty}} G(x) = \gamma_2,
\]

where \( \gamma_1 \neq \gamma_2 \), and \( 0 \leq \gamma_1, \gamma_2 \leq \infty \). Denote

\[
m := \min\{\gamma_1, \gamma_2\}, \quad M := \max\{\gamma_1, \gamma_2\}.
\]

Define positive functions \( v_1 \) and \( v \), and a nonnegative weight \( W \) on \( \tilde{\Omega} \) as follows:

(a) If \( M < \infty \), assume further that either \( m = 0 \) or \( p \geq 2 \), and let

\[
v_1 := (G - m)(M - G), \quad v := v_1^{(p-1)/p} = [(G - m)(M - G)]^{(p-1)/p},
\]

and

\[
W := \left( \frac{p-1}{p} \right)^p \left| \frac{\nabla G}{v_1} \right|^p \left| m + M - 2G \right|^{p-2}[2(p-2)v_1 + (M-m)^2].
\]

(b) If \( M = \infty \), define

\[
v_1 := (G - m), \quad v := v_1^{(p-1)/p} = (G - m)^{(p-1)/p},
\]

and

\[
W := \left( \frac{p-1}{p} \right)^p \left| \frac{\nabla G}{v_1} \right|^p.
\]
Then the following Hardy-type inequality holds true
\[ \int_{\tilde{\Omega}} |\nabla \varphi|^p \, dx \geq \int_{\tilde{\Omega}} W |\varphi|^p \, dx \quad \forall \varphi \in C^\infty_c(\tilde{\Omega}), \] (2.13)
and \( W \) is an optimal Hardy-weight for \(-\Delta_p\) in \( \tilde{\Omega} \).

Moreover, up to a multiplicative constant, \( v \) is the unique positive supersolution of the equation \( L(1, -W)w = 0 \) in \( \tilde{\Omega} \).

2.3. The linear case. In the same token, we consider also linear (not necessarily symmetric) second-order elliptic operators \( P \) with real coefficients in divergence form:
\[ Pu := -\nabla \cdot \left[ A(x) \nabla u + u \tilde{b}(x) \right] + b(x) \cdot \nabla u + c(x)u \quad x \in \Omega. \] (2.14)
We assume that \( A \) satisfied hypothesis (H0), and \( b \) and \( \tilde{b} \) are measurable vector fields in \( \Omega \) of class \( L^q_{\text{loc}}(\Omega, \mathbb{R}^N) \) and \( c \) is a measurable function in \( \Omega \) of class \( L^{q/2}_{\text{loc}}(\Omega, \mathbb{R}) \) for some \( q > N \).

By a solution \( v \) of the equation \( Pu = 0 \) in \( \Omega \), we mean that \( v \in W^{1,2}_{\text{loc}}(\Omega) \) and satisfies the equation in the weak sense. Subsolutions and supersolutions are defined similarly.

We denote by \( P^* \) the formal adjoint operator of \( P \) on its natural space \( L^2(\Omega, dx) \). If \( b = \tilde{b} \), then the operator \( P \) is symmetric in the space \( L^2(\Omega, dx) \), and we call this setting the linear symmetric case. We note that if \( P \) is symmetric and \( b \) is smooth enough, then \( P \) is in fact a Schrödinger-type operator of the form \( Pu = -\nabla \cdot \left( A(x) \nabla u \right) + \tilde{c}u \), where \( \tilde{c} = c - \nabla \cdot b \).

Remark 2.9. Our results hold true also when \( P \) is of the form
\[ P = -\sum_{i,j=1}^N a_{ij}(x) \partial_i \partial_j + \sum_{i=0}^N b_i(x) \partial_i + c(x). \] (2.15)
In this case we should assume that the coefficients \( a_{ij}, b_i \) and \( c \) are Hölder continuous and that the quadratic form
\[ \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j, \quad \xi \in \mathbb{R}^N \]
is positive definite for all \( x \in \Omega \). In this framework we consider classical solutions and supersolutions.

Definition 2.10. We say that the operator \( P \) is
\( \begin{enumerate} \item \text{nonnegative in} \ \Omega \ \text{(and we write} \ P \geq 0 \ \text{in} \ \Omega) \ \text{if the equation} \ Pu = 0 \ \text{in} \ \Omega \ \text{admits a positive (super)solution.} \) \item \text{subcritical in} \ \Omega \ \text{if there exists a function} \ W \geq 0 \ \text{in} \ \Omega \ \text{such that} \ P - W \geq 0 \ \text{in} \ \Omega. \ \text{Such a weight} \ W \ \text{is called a Hardy-weight for the operator} \ P \ \text{in} \ \Omega. \ \text{If} \ P \geq 0 \ \text{in} \ \Omega, \ \text{but} \ P \ \text{does not admit any Hardy-weight, then} \ P \ \text{is said to be critical in} \ \Omega. \end{enumerate} \)

For more details concerning criticality theory, see for example the review article [13] and references therein. In particular, we need the following result.

Lemma 2.11. The following claims hold true;
\( \begin{enumerate} \item \text{The operator} \ P \ \text{is critical in} \ \Omega \ \text{if and only if} \ P^* \ \text{is critical in} \ \Omega. \ \item \text{The operator} \ P \ \text{is critical in} \ \Omega \ \text{if and only if the equation} \ Pu = 0 \ \text{in} \ \Omega \ \text{admits (up to a multiplicative constant) a unique positive supersolution called the Agmon ground state (or in short ground state).} \end{enumerate} \)
The operator $P$ is subcritical in $\Omega$ if and only if $P$ admits a positive minimal Green function $G_\Omega^P(x, y)$ in $\Omega$.

**Definition 2.12.** Let $K \Subset \Omega$. We say that a positive solution $u$ of the equation $Pw = 0$ in $\Omega \setminus K$ is a positive solution of the operator $P$ of minimal growth in a neighborhood of infinity in $\Omega$, if for any compact set $K \Subset K_1 \Subset \Omega$ with a smooth boundary and any positive supersolution $v$ of the equation $Pw = 0$ in $\Omega \setminus K_1$, $v \in C((\Omega \setminus K_1) \cup \partial K_1)$, the inequality $u \leq v$ on $\partial K_1$ implies that $u \leq v$ in $\Omega \setminus K_1$.

**Remark 2.13.** Note that
(1) If $P$ is subcritical in $\Omega$, then for any fixed $y \in \Omega$, the positive minimal Green function $G_\Omega^P(\cdot, y)$ is a positive solution of the equation $Pu = 0$ in $\Omega \setminus \{y\}$ of minimal growth in a neighborhood of infinity in $\Omega$.
(2) On the other hand, in the critical case, the ground state of $P$ in $\Omega$ is a positive solution of the equation $Pu = 0$ in $\Omega$ of minimal growth in a neighborhood of infinity in $\Omega$.

**Definition 2.14.** A Hardy-weight $W$ for a subcritical operator $P$ in $\Omega$ is said to be **optimal** if the following three properties hold:

- (Criticality) $P - W$ is critical in $\Omega$. Denote by $\phi$ and $\phi^*$ the ground states of $P$ and $P^*$, respectively.
- (Null-criticality) $W \notin L^1(\Omega, \phi \phi^* dx)$.
- (Optimality at infinity) For any $\lambda > 1$ the operator $P - \lambda W \not\geq 0$ in any neighborhood of infinity in $\Omega$.

The following theorem is a version of [5, Theorem 4.12] (cf. the discussion therein); we omit its proof since it can be obtained by a slight modification of the proof in [5].

**Theorem 2.15.** Let $P$ be a subcritical operator in $\Omega$, and let $0 \leq \varphi \in C_c^\infty(\Omega)$. Consider the Green potential

$$G_\varphi(x) := \int_\Omega G_\Omega^P(x, y) \varphi(y) \, dy,$$

where $G_\Omega^P$ is the minimal positive Green function. Let $u$ be a positive solution of the equation $Pu = 0$ in $\Omega$ satisfying

$$\lim_{x \to \infty} \frac{G_\varphi(x)}{u(x)} = 0,$$

where $\infty$ is the ideal point in the one-point compactification of $\Omega$. Consider the positive supersolution

$$v := \sqrt{G_\varphi u}$$

of the operator $P$ in $\Omega$. Then the associated Hardy-weight

$$W := \frac{Pv}{v},$$

is an optimal Hardy-weight with respect to $P$ in $\Omega$. Moreover,

$$W = \frac{1}{4} \left| \nabla \log \left( \frac{G}{u} \right) \right|^2_A \quad \text{in } \Omega \setminus \text{supp } \varphi.$$
3. Main results

3.1. The quasilinear critical case. We suppose that $L(A,V)$ is nonnegative in $\Omega$, in other words (by Theorem 2.4),

$$Q_{A,V}[\varphi] := \int_{\Omega} (|\nabla \varphi|_A^p + V|\varphi|^p) \, dx \geq 0 \quad \forall \varphi \in C_c^\infty(\Omega). \quad (3.1)$$

Let $K \Subset \Omega$ be a compact set of positive measure with smooth boundary. Then by [14], Proposition 4.18], the operator $L(A,V)$ is subcritical in $K^c := \Omega \setminus K$. Hence, there exists a Hardy-weight $W \geq 0$, which depends on $\Omega, K, A$ and $V$, such that a Hardy-type inequality

$$Q_{A,V}[\varphi] = \int_{K^c} (|\nabla \varphi|_A^p + V|\varphi|^p) \, dx \geq \int_{K^c} W|\varphi|^p \, dx \quad \forall \varphi \in C_c^{\infty}(K^c) \quad (3.2)$$

holds true.

The following theorem answers the question how large (in a neighborhood of infinity in $\Omega$) the Hardy-weight $W$ in (3.2) might be if $L(A,V)$ is critical in $\Omega$.

**Theorem 3.1.** Suppose that $L(A,V)$ is critical in $\Omega$ with $A$ and $V$ satisfying either hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$, and let $\phi \in W^{1,p}_{\text{loc}}(\Omega)$ be the corresponding ground state satisfying the normalization condition $\|\phi\|_{L^p(K)} = 1$. Let $K \Subset \Omega$ be a compact set of positive measure with smooth boundary.

Then for any $W \geq 0$ satisfying (3.2) and any compact set $K$ such that $K \Subset K \Subset \Omega$ we have $W \in L^1(K^c, \phi^p \, dx)$.

**Proof.** Let $K \Subset K \Subset \Omega$. In view of our ellipticity assumption (2.3) it is possible to choose $f \in C^1(\Omega)$ satisfying

$$f = \begin{cases} 
0 & \text{in } K, \\
1 & \text{in } K^c, 
\end{cases} \quad 0 \leq f \leq 1,$$

and such that $|\nabla f(x)|_A \leq C_0$ holds for all $x \in \Omega$. Since $L(A,V)$ is critical in $\Omega$, there exists a null-sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subset C_c^{\infty}(\Omega)$ such that $\varphi_n \geq 0$, $\|\varphi_n\|_{L^p(K)} = 1$ for all $n \in \mathbb{N}$, and

$$\int_{\Omega} (|\nabla \varphi_n|_A^p + V|\varphi_n|^p) \, dx \to 0 \quad \text{as } n \to \infty. \quad (3.3)$$

Moreover, by density and inequality (3.2) we have

$$\int_{K^c} (|\nabla (f \varphi_n)|_A^p + V|f \varphi_n|^p) \, dx \geq \int_{K^c} W|f \varphi_n|^p \, dx \geq \int_{K^c} W|\varphi_n|^p \, dx \quad (3.4)$$

Since $(a+b)^p \leq 2^{p-1}a^p + 2^{p-1}b^p$ holds for all $a,b > 0$ and $p \geq 1$, we obtain the upper bound

$$\int_{K^c} (|\nabla (f \varphi_n)|_A^p + V|f \varphi_n|^p) \, dx = \int_{K^c} (|\nabla \varphi_n|_A^p + V|\varphi_n|^p) \, dx + \int_{K \setminus K^c} (|\nabla (f \varphi_n)|_A^p + V|f \varphi_n|^p) \, dx$$

$$= \int_{\Omega} (|\nabla \varphi_n|^p_A + V|\varphi_n|^p) \, dx - \int_{K} (|\nabla \varphi_n|^p_A + V|\varphi_n|^p) \, dx + \int_{K \setminus K^c} (|\nabla (f \varphi_n)|_A^p + V|f \varphi_n|^p) \, dx$$

$$\leq \int_{\Omega} (|\nabla \varphi_n|^p_A + V|\varphi_n|^p) \, dx + 2^{p-1} \int_{K} |\nabla \varphi_n|^p_A \, dx + 2 \int_{K} |V| |\varphi_n|^p \, dx + 2^{p-1}C_0^p \int_{K} |\varphi_n|^p \, dx.$$

Hence, there exists $C_p > 0$ such that

$$\int_{K^c} W|\varphi_n|^p \, dx \leq \int_{\Omega} (|\nabla \varphi_n|^p_A + V|\varphi_n|^p) \, dx + C_p \int_{K} (|\nabla \varphi_n|^p_A + V||\varphi_n|^p + |\varphi_n|^p) \, dx. \quad (3.5)$$
On the other hand, by [14] Theorem 4.12 it then follows that the sequence \( \{ \varphi_n \} \) converges in \( L^p_{\text{loc}} \) and almost everywhere in \( \Omega \) to \( \phi \) and that
\[
\phi, |\nabla \phi| \in L^\infty_{\text{loc}}(\Omega) \quad \text{if} \quad 1 < p < 2.
\]
Moreover, if we write
\[
\varphi_n = \psi_n \phi,
\]
then the sequence \( \psi_n \) is bounded in \( W^{1,p}_{\text{loc}}(\Omega) \) and \( \nabla \psi_n \rightarrow 0 \) in \( L^p_{\text{loc}}(\Omega) \), see [14] Proposition 4.11]. In view of the Rellich-Kondrachov theorem, Hölder inequality and (3.6) it thus follows that \( \varphi_n \) is bounded in \( W^{1,p}_{\text{loc}}(\Omega) \). Therefore,
\[
\limsup_{n \to \infty} \int_K \left( |\nabla \varphi_n|^p_A + |\varphi_n|^p \right) \, dx < \infty.
\]
This in turn implies, again by using the Rellich-Kondrachov theorem, that \( \varphi_n \) is bounded in \( L^r_{\text{loc}}(\Omega) \) for all \( r \in [p, \infty] \) if \( p > N \), all \( r \in [p, \infty] \) if \( p = N \) and all \( r \in [p, p^*] \) if \( p < N \), where \( p^* = \frac{pN}{N-p} \). By assumption, we have \( p q' < p^* \). Then by the Hölder inequality
\[
\limsup_{n \to \infty} \int_K \left( |V| |\varphi_n|^p \right) \, dx \leq \limsup_{n \to \infty} \left( \int_K |V|^q \, dx \right)^{\frac{1}{q}} \left( \int_K |\varphi_n|^{pq'} \, dx \right)^{\frac{1}{p'}} < \infty.
\]
To complete the proof we note that in view of the pointwise a.e. convergence of \( \varphi_n \) to \( \phi \) and the Fatou lemma it holds
\[
\liminf_{n \to \infty} \int_{K^c} W|\varphi_n|^p \, dx \geq \int_{K^c} W \liminf_{n \to \infty} |\varphi_n|^p \, dx = \int_{K^c} W|\phi|^p \, dx.
\]
This in combination with (3.3), (3.5), (3.7) and (3.8) gives
\[
\int_{K^c} W|\phi|^p \, dx < \infty,
\]
and the claim follows.

\[\square\]

3.2. The quasilinear subcritical case. We have

**Theorem 3.2.** Suppose that \( L(A,V) \) of the form (2.1) is subcritical in \( \Omega \) with \( A \) and \( V \) satisfying either hypothesis (H0) if \( p \geq 2 \), or (H1) if \( 1 < p < 2 \). Let \( K \Subset \Omega \) be a compact set of positive measure with smooth boundary, and let \( \phi \in W^{1,p}_{\text{loc}}(\Omega \setminus K) \) be a positive solution of the equation \( L(A,V)[u] = 0 \) in \( \Omega \setminus K \) of minimal growth in a neighborhood of infinity in \( \Omega \).

Then for any Hardy-weight \( W \geq 0 \) for \( L(A,V) \) in \( \Omega \), and any compact set \( K \) such that \( K \Subset K \Subset \Omega \), we have \( W \in L^1(K^{\infty}, \phi^p \, dx) \).

**Proof.** Suppose that \( L(A,V) \) is subcritical in \( \Omega \), and let \( W \) be a Hardy-weight for \( L(A,V) \) in \( \Omega \). So,
\[
Q_{A,V}[\varphi] \geq \int_{\Omega} W|\varphi|^p \, dx \quad \text{for all} \quad \varphi \in C^\infty_c(\Omega).
\]

Let \( K \Subset \Omega \) be a compact set of positive measure with smooth boundary, and let \( \phi \in W^{1,p}_{\text{loc}}(\Omega \setminus K) \) be a positive solution of the equation \( L(A,V)[u] = 0 \) in \( \Omega \setminus K \) of minimal growth in a neighborhood of infinity in \( \Omega \).

Let \( V_c \in C^\infty_c(K) \) be a nonnegative potential such that \( L(A,V - V_c) \) is critical in \( \Omega \) with a ground state \( \psi \) [14] Proposition 4.19]. Since both \( \psi \) and \( \phi \) are positive solutions of the
equation $L(A, V)[u] = 0$ in $\Omega \setminus K$ of minimal growth in a neighborhood of infinity in $\Omega$ [14, Theorem 5.9], it follows that $\psi \preceq \phi$ in $K^c$. On the other hand, in light of (3.9), we have

$$
Q_{A,V - V_c}[\varphi] = Q_{A,V}[\varphi] = \int_{K^c} \left( |\nabla \varphi|^p + V|\varphi|^p \right) \, dx \geq \int_{K^c} W|\varphi|^p \, dx \quad \forall \varphi \in C^\infty_c(K^c) \tag{3.10}
$$

Consequently, Theorem 3.1 implies that $W \in L^1(K^c, \psi^p \, dx) = L^1(K^c, \phi^p \, dx)$. □

**Remark 3.3.** The conditions on the decay of $W$ at infinity given by Theorem 3.2 could be compared with the behaviour at infinity of the optimal Hardy-weight given by Theorem 2.8. For example, let $V = 0$ and $A = 1$ so that $L = -\Delta_p$ and assume that $L$ is subcritical in $\Omega$. Let $W$ be an optimal Hardy-weight for $L$, and let $v$ be the ground state of the critical operator $L(1, -W)$. Then by the null-criticality with respect to $W$

$$
\int_{\Omega \setminus K} W(x) \, v^p(x) \, dx = \infty \tag{3.11}
$$

holds for all $K \Subset \Omega$. This is of course not in contradiction with Theorem 3.2 because $v$ is larger than the function $\phi$ considered in Theorem 3.2. For example, if $v$ is given by the supersolution construction as in Theorem 2.8 (or [5, Theorem 1.5]), then $v = \phi^{(p-1)/p}$ and by (3.11) we have $W \not\in L^1(K^c, \phi^{p-1} \, dx)$, while by Theorem 3.2 $W \in L^1(K^c, \phi^p \, dx)$.

As a straightforward consequence of Theorem 3.1 we have:

**Corollary 3.4.** Suppose that $L(A, V)$ of the form (2.1) is subcritical in $\Omega$ with $A$ and $V$ satisfying either hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$. Let $W \geq 0$ be a Hardy-weight for $L(A, V)$ such that (3.9) holds true. If $W$ is null-critical, then $W$ is also optimal at infinity in the sense of Definition 2.7.

**Proof.** Let $\phi \in W^{1, p}_{\text{loc}}(\Omega)$ be the ground state of the critical operator $L(A, V - W)$ in $\Omega$. Assume that $W$ is not optimal at infinity. Then there exists a neighborhood of infinity in $\Omega$, which we denote by $\Omega_\infty$, and a constant $\lambda > 1$ such that

$$
Q_{A,V}[\varphi] \geq \lambda \int_{\Omega} W|\varphi|^p \, dx \quad \forall \varphi \in C^\infty_c(\Omega_\infty). \tag{3.12}
$$

Let $K \Subset \Omega$ be a compact set such that $K^c \subset \Omega_\infty$. Then

$$
Q_{A,V-W}[\varphi] \geq (\lambda - 1) \int_{\Omega} W|\varphi|^p \, dx \quad \forall \varphi \in C^\infty_c(K^c).
$$

Hence by Theorem 3.1 we have $W \in L^1(K^c, \phi^p \, dx)$ for any compact set $K \Subset \Omega$ with $K \Subset K$. Since $W \in L^1(K, \phi^p \, dx)$ by density and inequality (3.10), it follows that $W \in L^1(\Omega, \phi^p \, dx)$ which is in contradiction with the null-criticality of $W$. □

### 3.3. The linear critical case

We have

**Theorem 3.5.** Let $P$ be an elliptic operator of the form (2.14) (or (2.15)), and assume that $P$ is critical in $\Omega$. Let $\phi$ and $\phi^*$ be the ground states, in $\Omega$, of $P$ and $P^*$, respectively. Let $K \Subset \Omega$ be a compact set of positive measure with smooth boundary.

Let $W \geq 0$ be a Hardy-weight for $P$ in $K^c$. Then for any $K$ such that $K \Subset K \Subset \Omega$ we have $W \in L^1(K^c, \phi \phi^* \, dx)$. 

Proof. The operator $P$ is subcritical in $K^c$. Let $G_P^{K^c}(x,y)$ be the minimal positive Green function of $P$ in $K^c$, and let $W \geq 0$ be a Hardy-weight for $P$ in $K^c$. Then by [12, Lemma 3.1]
\[
\int_{K^c} G_P^{K^c}(x,z) W(z) G_P^{K^c}(z,y) \, dz < \infty \quad \forall x, y \in K^c.
\] (3.13)
On the other hand, by Remark 2.13, for any compact $K$ such that $K \subset \Omega$, we have for fixed $x, y \in K^c \setminus K^c$
\[
G_P^{K^c}(x, \cdot) \asymp \phi^*, \quad G_P^{K^c}(\cdot, y) \asymp \phi \quad \text{in } K^c.
\]
Hence, it follows from (3.13) that for fixed $x, y \in K^c \setminus K^c$ it holds
\[
\int_{K^c} W(z) \phi(z) \phi^*(z) \, dz \leq C \int_{K^c} G_P^{K^c}(x,z) W(z) G_P^{K^c}(z,y) \, dz < \infty. \quad \Box
\]

3.4. The linear subcritical case. We have

Theorem 3.6. Consider a linear operator $P$ of the form (2.14) (or (2.15)), satisfying the corresponding local regularity assumption mentioned in Subsection 2.3. Assume that $P$ is subcritical in $\Omega$. Let $K \subset \Omega$ be a compact set of positive measure with smooth boundary, and let $\phi, \phi^* \in W^{1,2}_{\text{loc}}(\Omega \setminus K)$ be positive solutions of the equation $Pu = 0$ and respectively, $P^*u^* = 0$ in $\Omega \setminus K$ of minimal growth in a neighborhood of infinity in $\Omega$.

Then for any Hardy-weight $W \geq 0$ for $P$ in $\Omega$, and any compact set $K$ such that $K \subset \Omega$, we have $W \in L^1(K^c, \phi \phi^* \, dx)$.

Proof. The proof is similar to the proof of Theorem 3.2 and therefore it is omitted. $\Box$

Similar to the quasilinear case, we have the following consequence of Theorem 3.5

Corollary 3.7. Let $P$ be a subcritical elliptic operator of the form (2.14) (or (2.15)), and let $W \geq 0$ be a Hardy-weight for $P$ in $\Omega$. If $P - W$ is null-critical with respect to $W$ in $\Omega$, then $W$ is also optimal at infinity in the sense of Definition 2.14.

Proof. Let $\phi$ and $\phi^*$ be the ground states, in $\Omega$, of $P - W$ and $P^* - W$, respectively. Let $K \subset \Omega$ be a compact set of positive measure with smooth boundary, and assume to the contrary that for some $\lambda > 1$ the operator $P - \lambda W$ is nonnegative in $\Omega \setminus K$. In other words, $(\lambda - 1)W$ is a Hardy-weight for $P - W$ in $\Omega \setminus K$. Then it follows from Theorem 3.5 that for any $K$ such that $K \subset \Omega$, we have $W \in L^1(K^c, \phi \phi^* \, dx)$, but this contradicts the null-criticality of $P - W$ with respect to $W$. $\Box$

Remark 3.8. The integrability conditions in the present section are only necessary conditions for $w$ to be a Hardy-weight, as the following elementary example demonstrates.

Consider the operator $P := -\Delta$ on $\mathbb{R}^N$, where $N \geq 2$ and the weight $W := 1$. Then a positive harmonic function $u := -\Delta$ on $\mathbb{R}^N$, where $N \geq 2$ and the weight $W := 1$. Then a positive harmonic function $u := -\Delta$ on $\mathbb{R}^N$ satisfies $u(x) \asymp |x|^{2-N}$, which is in $L^2(B_R(0))$ for $R$ sufficiently large if and only if $N \geq 5$. On the other hand, in any dimension, $W = 1$ is not a Hardy-weight for the Laplacian in $B_R(0)$.

For sufficient conditions, as well as necessary and sufficient conditions for $W$ to be a Hardy-weight in the linear case, see [12] and references therein.
4. Examples

4.1. Example 1. Consider the case $\Omega = \mathbb{R}^N$ and $L(A,V) = -\Delta_p$, i.e. the $p$-Laplacian. So, $A = 1$ is the identity matrix, $V = 0$ and assume that $p \geq N \geq 2$. It is well-known that $-\Delta_p$ is critical in $\mathbb{R}^N$ if and only if $p \geq N$, and therefore, $\phi = 1$ is its ground state.

We first consider the example $p = N$ and $K := B_R(0)$. Theorem 2.8 with $G(x) := \log(|x|/R)$ directly implies the following Leray-type inequality (cf. [5, Example 13.1]).

**Lemma 4.1.** Let $K = B_R(0) \subset \mathbb{R}^N$ and let $p = N > 1$. Then for any $\varphi \in C_c^\infty(K^c)$ it holds

$$\int_{K^c} |\nabla \varphi|^N \, dx \geq \left( \frac{N - 1}{N} \right)^N \int_{K^c} \frac{|\varphi|^N}{|x|^N (\log(|x|/R))^N} \, dx.$$  \hspace{1cm} (4.1)

Moreover,

$$W(x) := \left( \frac{N - 1}{N} \right)^N \frac{1}{|x|^N (\log(|x|/R))^N}$$

is an optimal Hardy-weight for $-\Delta_p$ in $K^c$ with $v(x) := \log \frac{N-1}{N}(|x|/R)$ being the ground state for the operator $L(1, -W)$ in $K^c$.

**Remark 4.2.** Inequality (4.1) is well-known (see e.g. [11, Thm. 1.14]), while the optimality of $W$ follows from Theorem 2.8. Note also that in the linear case $p = 2$ inequality (4.1) can be generalized, with suitable modifications, to Laplace operators with Robin boundary conditions on $\partial K$, see [7].

Here the optimal Hardy-weight $W$ is not in $L^1(K^c, dx)$, but Theorem 3.1 implies that $W \in L^1(K^c, dx)$ for any $K$ such that $K \subset K \subset \mathbb{R}^N$. Indeed, for any $\rho > R$ such that $B_{\rho}(0) \subset K$ we have

$$\int_{K^c} W \, dx \leq \int_{B_{\rho}(0)^c} \frac{1}{|x|^N (\log(|x|/R))^N} \, dx \asymp \int_{\rho/R}^\infty \frac{dr}{r \log r} < \infty.$$  \hspace{1cm} (4.2)

Theorem 3.1 now implies that the logarithmic factor in (4.1) cannot be removed. This is in contrast with the well-known (optimal) Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla \varphi|^p \, dx \geq \frac{N - p}{p} \left( \int_{\mathbb{R}^N} \frac{|\varphi|^p}{|x|^p} \, dx \right)^{\frac{p}{p - N}} \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}),$$  \hspace{1cm} (4.3)

which holds for $N \neq p$ (see for example, [6, Example 4.7]). Note also that by the optimality of $W$ it follows that the weight function $W$ satisfies $\int_{K^c} |v|^N W \, dx = \infty$, where $v(x) = \log \frac{N-1}{N}(|x|/R)$, c.f. (3.11). Indeed, we have

$$\int_{K^c} |v|^N W \, dx = \left( \frac{N - 1}{N} \right)^N \int_{K^c} \frac{1}{|x|^N (\log(|x|/R))^N} \, dx \asymp \int_{\delta}^\infty \frac{dr}{r \log r} = \infty,$$  \hspace{1cm} (4.4)

for some $\delta > 1$. 

**Remark 3.9.** Since criticality theory and in particular, the results concerning Hardy-type inequalities are valid in the setting of second-order elliptic operators on noncompact Riemannian manifolds [5,6,13], it follows that the results of the paper hold true also when $\Omega$ is a noncompact Riemannian manifold of dimension $N \geq 2$. 
In the case $p > N$, we apply Theorem 2.8 with $G(x) := |x|^{(p-N)/(p-1)}$. This gives

$$W(x) := \left(\frac{p-N}{p}\right)^p \frac{1}{|x|^p \left(1 - (R/|x|)^{(p-N)/(p-1)}\right)}.$$ 

In agreement with Theorem 3.1, we have

$$\int_{K^c} W \, dx < \infty,$$

for any $K$ such that $B_R(0) \subset K$.

4.2. Example 2. Let $\Omega = \mathbb{R}^N \setminus \{0\}$, $2 \leq p = (N+1)/2$, $A = \mathbb{I}$, and

$$V(x) = -C_{p,N} \frac{1}{|x|^p}, \quad \text{where} \quad C_{p,N} := \frac{|N-p|^p}{p} = C_{p,1} \quad \text{is the Hardy constant.}$$

Here we have

$$L(A,V)u = Lu = -\Delta_p u + V|u|^{p-2}u = -\Delta_p u - \frac{|N-p|^p}{p} \left|u^{p-2}u \right| \frac{|x|^p}{|x|^p},$$

and the optimality of the integral weight in [1,2] implies in particular, that the operator $L$ is critical in $\Omega$. A straightforward calculation shows that $\phi(x) = |x|^{(p-N)/p}$ is the ground state for $L$ in $\Omega$. The null-criticality, implies that

$$\int_{K^c} W \frac{|\phi|^p}{|x|^p} \, dx = C_{p,N} \int_{K^c} \frac{|x|^{p-N}}{|x|^p} \, dx \times \int_{\delta} \frac{dr}{r} = \infty.$$ 

On the other hand, if we set $K = \overline{B_R(0)}$, then

$$\int_{K^c} \left|\nabla \varphi\right|^p \, dx \geq C_{p,1} \int_{K^c} \frac{|\varphi|^p}{(|x|-R)^p} \, dx \quad \forall \varphi \in C^\infty_c(K^c),$$

where the constant $C_{p,1}$ is optimal, see e.g., [10, Section 4] and [9]. Since $C_{p,N} = C_{p,1}$, it follows that

$$\int_{K^c} \left(\left|\nabla \varphi\right|^p - C_{p,1} \frac{|\varphi|^p}{|x|^p}\right) \, dx \geq \int_{K^c} W(x) \varphi^p \, dx \quad \forall \varphi \in C^\infty_c(K^c),$$

with the Hardy-weight

$$W(x) := C_{p,1} \left(\frac{1}{(|x|-R)^p} - \frac{1}{|x|^p}\right) > 0 \quad \text{in} \ K^c.$$ 

It is now easy to verify that

$$\int_{K^c} W \phi^p \, dx < \infty,$$

for any $K \subset K \subset \mathbb{R}^N$. 

4.3. **Example 3.** Let $L$, $Ω$, and $K$ be as in Example 2 with $p = 2$ and $N \geq 3$. Let $\phi(x) = |x|^{(2-N)/2}$ and $ψ(|x|) := ϕ(|x|) \log(|x|/R)$. Using the supersolution construction with $ϕ$ and $ψ$ in $K^c$, we obtain the following optimal Hardy inequality in $K^c$

$$\int_{K^c} \left( |\nabla ϕ|^2 - \left( \frac{N-2}{2} \right)^2 |ϕ|^2 \right) dx \geq \int_{K^c} W |ϕ|^2 dx \quad ∀ϕ ∈ C^∞_c(K^c),$$

with ground state $ϕ_K := (ϕψ)^{1/2}$, and

$$W(x) := \frac{1}{4|x|^2 \left( \log(|x|/R) \right)^2} > 0.$$  

Recall that $ϕ$ is a ground state of the critical operator $L$ in $Ω$. It is now easy to verify that as claimed in Theorem 3.1 for any $K \Subset Ω$ and any $ρ > R$ such that $B_ρ(0) ⊂ K$ we have

$$\int_{K^c} W |ϕ|^2 dx < \int_{δ}^{∞} \frac{dr}{r \log r} < ∞,$$

where $δ = ρ/R > 1$. On the other hand, the optimality of $W$ in $K^c$ implies that there exists $δ > 1$ such that

$$\int_{K^c} W |ϕ_K|^2 dx = \int_{K^c} W |ϕ|^2 \log(|x|/R) dx < \int_{δ}^{∞} \frac{dr}{r \log r} = ∞,$$

demonstrating the sharpness of Theorem 3.1.

4.4. **Example 4.** Let $Ω = \mathbb{R}^N$, $A = I$ the identity matrix, $V = 0$ and $p < N$. In this case we have that $L(A,V) = −Δ_p$ is subcritical in $\mathbb{R}^N$. The fundamental solution $|x|^{(p-N)/(p-1)}$ with a singular point at the origin is a positive $p$-harmonic function of minimal growth at infinity in $\mathbb{R}^N$. Assume further that $N ≥ 3$. Then the following Hardy inequality holds true with a subcritical Hardy-weight

$$\int_{\mathbb{R}^N} |\nabla ϕ|^p dx \geq \left( \frac{N-p}{p} \right)^{p} \int_{\mathbb{R}^N} \frac{|ϕ|^p}{1 + |ϕ|^p} dx \quad ∀ϕ ∈ C^∞_c(\mathbb{R}^N).$$

By Theorem 3.2 we have

$$\int_{|x| > R} \left( \frac{|x|^{(p-N)/(p-1)}}{1 + |x|^p} \right)^p dx \asymp \int_{R}^{∞} r^{(1-N)/(p-1)} dr < ∞.$$  

The optimality at infinity of the Hardy-weight $C_{p,N} |x|^{-p}$ in $\mathbb{R}^N \setminus \{0\}$ with a ground state $|x|^{(p-N)/p}$ implies, on the other hand, that

$$\int_{|x| > R} \left( \frac{|x|^{(p-N)/p}}{|x|^p} \right)^p dx \asymp \int_{R}^{∞} r^{-1} dr = ∞.$$  

4.5. **Example 5.** (cf. [2] [3] [4]) Let $Ω$ be a $C^1,α$-bounded domain in $\mathbb{R}^N$ and let $A = I$ be the identity matrix, and $p = 2$. Fix $x_0 ∈ Ω$, and denote $G(x) := G_Ω^{Δ}(x,x_0)$. Let $K ⊂ Ω$ be a compact set of positive measure with smooth boundary and suppose that $K$ is large enough such that $G(x) < 1$ in $K^c$. For $t ∈ (0,1)$ define

$$X_0(t) = 1, \quad X_1(t) = (1 − \log(t))^{-1},$$

and

$$X_{i+1}(t) = X_1(X_i(t)), \quad ∀i ≥ 1.$$
For $i \geq 0$, and $x \in K^c$, let
\[
W_i(x) := \frac{\|\nabla G(x)\|^2}{4G^2(x)} \sum_{k=0}^{i} \prod_{j=0}^{k} X_j^2(G(x)), \quad u_i(x) := \left(\frac{G(x)}{\prod_{j=0}^{i} X_j(G(x))^\alpha}\right)^{1/2},
\]
and let $W_i = 0$ in $K$. Then $W_i$ is a Hardy-weight for $-\Delta$ in $\Omega$ (obtained by the supersolution construction), and $u_i$ is a positive solution of the equation
\[
(-\Delta - W_i)u = 0
\]
of minimal growth in a neighborhood of infinity in $\Omega$. Let
\[
R_i(x) = W_{i+1}(x) - W_i(x) = \frac{\|\nabla G(x)\|^2}{4G^2(x)} \prod_{j=0}^{i+1} X_j^2(G(x)).
\]
Then $R_i$ is a Hardy-weight for $-\Delta - W_i$ in $K^c$ and a straightforward calculation shows that
\[
\int_{K^c} u_i^2(x) R_i(x) \, dx = \int_{K^c} \frac{\|\nabla G(x)\|^2}{4G(x)} \prod_{j=0}^{i} X_j(G(x)) X_{i+1}^2(G(x)) \, dx
\]
\[
\leq \int_{0}^{1} \prod_{j=0}^{i} X_j(t) X_{i+1}(t) \frac{dt}{t} \leq \int_{1}^{\infty} \frac{1}{(1 + \log t)^2} \frac{dt}{t} < \infty,
\]
in agreement with Theorem 3.2.

4.6. Example 6. (cf. [9]) Let $\Omega$ be a $C^{1,\alpha}$-bounded domain in $\mathbb{R}^N$, and fix $1 < p < \infty$. Let
\[
\delta_\Omega : \Omega \to \mathbb{R}_+ \text{ be the distance function to } \partial \Omega.
\]
Then there exists $\Omega^\prime \subset \Omega$, a neighborhood of infinity in $\Omega$ such that the following Hardy inequality holds
\[
\int_{\Omega^\prime} |\nabla \varphi|^p \, dx \geq C_{p,1} \int_{\Omega^\prime} \frac{|\varphi|^p}{\delta_\Omega^p} \, dx \quad \forall \varphi \in C_c^\infty(\Omega^\prime).
\]
A positive solution $u$ for $L(1, -C_{p,1}\delta_\Omega^{-p})$ of minimal growth at infinity in $\Omega$ behaves like $\delta_\Omega^{(p-1)/p}$ (see [9]). Hence, for $W := C_{p,1}\delta_\Omega^{-p}$ we obtain $\int_{\Omega^\prime} W^{(\delta_\Omega^{(p-1)/p})} \, dx = \infty$. In particular, by Theorem 3.2 we have
\[
\lambda_\infty(-\Delta_p, \delta_\Omega^{-p}, \Omega) := \sup \{ \lambda \in \mathbb{R} | \exists K \in \Omega \text{ s.t. } -\Delta_p - \lambda \delta_\Omega^{-p} \geq 0 \text{ in } \Omega \setminus K \} = C_{p,1}.
\]
On the other hand, let $\lambda < C_{p,1}$, then $W_\lambda := (C_{p,1} - \lambda)(\delta_\Omega)^{-p}$ is a Hardy-weight for the subcritical operator $L_\lambda := L(1, -\lambda \delta_\Omega^{-p})$ in $\Omega^\prime$. Any positive solution $v$ for $L_\lambda$ of minimal growth at infinity in $\Omega$ behaves like $\delta_\Omega^{\alpha_{\lambda}(\lambda)}$, where $\alpha(\lambda) > (p - 1)/p$ is a solution of the transcendent equation $\lambda = (p - 1)\alpha^{p-1}(1 - \alpha)$, see [9]. Note that $\alpha(\lambda) \to (p - 1)/p$ as $\lambda \to C_{p,1}$. Hence, we have $\int_{\Omega^\prime} W_\lambda^{(\delta_\Omega^{\alpha_{\lambda}(\lambda)})} \, dx < \infty$, demonstrating again the sharpness of Theorem 3.2.

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