THE COHOMOLOGY RING AWAY FROM 2 OF CONFIGURATION SPACES ON REAL PROJECTIVE SPACES

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Abstract. Let \( R \) be a commutative ring with unit where 2 is invertible. We compute the \( R \)-cohomology ring of the configuration space \( \text{Conf}(\mathbb{R}P^m, k) \) of \( k \) ordered points in the \( m \)-dimensional real projective space \( \mathbb{R}P^m \). The method is based on the observation that the configuration space of \( k \) ordered orbits in the \( m \)-dimensional sphere (with respect to the antipodal action) is a \( 2^k \)-fold covering of \( \text{Conf}(\mathbb{R}P^m, k) \). Our results imply that, for odd \( m \), the Leray spectral sequence for the inclusion \( \text{Conf}(\mathbb{R}P^m, k) \subset (\mathbb{R}P^m)^k \) collapses after its first non-trivial differential, just as in the case of a complex algebraic variety. The method also allows us to handle the \( R \)-cohomology ring of the configuration space of \( k \) ordered points in the punctured real projective space \( \mathbb{R}P^m - * \). Finally we compute the Lusternik-Schnirelmann category and topological complexity of some of the auxiliary orbit configuration spaces.

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1. Main results

The goal of this work is to give a description of the cohomology ring away from 2 of configuration spaces of pairwise distinct ordered points in (either regular or punctured) real projective spaces. Our main results are stated next where \( k \) and \( n \) stand for integers greater than 1.

Theorem 4.7 Suppose \( R \) is a commutative ring with unit where 2 is invertible. For \( n \geq 2 \) odd, there is an \( R \)-algebra isomorphism

\[
H^*(\text{Conf}(\mathbb{R}P^n, k); R) \cong \Lambda(t_n) \otimes R[C^+]/\mathcal{K},
\]

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where \( i_n \) has degree \( n \) and is the image of the generator in \( \mathbb{R}P^n \) under the projection on the first coordinate \( \text{Conf}(\mathbb{R}P^n, k) \overset{\pi_1}{\to} \mathbb{R}P^n \), the generators in \( C^+ \) have degree \( n - 1 \) and are detailed at the beginning of Section 4, and the relations \( K \) are specified in Theorem 4.5.

**Theorem 4.13.** Let \( R \) be a commutative ring with unit where 2 is invertible. For \( n \) even, there is an \( R \)-algebra isomorphism

\[
H^*(\text{Conf}(\mathbb{R}P^n, k); R) \cong \Lambda(\omega) \otimes R[C]/J,
\]

where \( J \) is the ideal generated by the relations in Lemma 4.10, \( \omega \) is a generator of degree \( 2n - 1 \) specified in Theorem 2.3, and the set of generators \( E \) have degree \( 2n - 2 \) and are defined just above Lemma 4.10.

**Remark 1.1.** Theorem 4.7 and the known description of the cohomology ring of configuration spaces on spheres ([6, 8]) imply that there is a ring isomorphism

\[
H^*(\text{Conf}(S^n, k); R) \cong H^*(\text{Conf}(\mathbb{R}P^n, k); R)
\]

provided \( n \) is odd. Compare with Remark 4.6. But there is no such an isomorphism if \( n \) is even, in view of Theorem 4.13.

It is interesting to look at the above results in terms of the Leray spectral sequence for the inclusion \( \text{Conf}(\mathbb{R}P^n, k) \hookrightarrow (\mathbb{R}P^n)_k \). To fix ideas, we take cohomology with rational coefficients in the following considerations. Recall from [16] (see also [3, 13]) that, for an oriented closed manifold \( M \) of dimension \( m \), the \( E_2 \)-term and the first non-trivial differential \( \delta_m \) in the Leray spectral sequence for the inclusion \( \text{Conf}(M^m, k) \hookrightarrow (M^m)_k \) depend only on the rational cohomology ring and orientation class of \( M \). Motivated by what happens when \( M \) is a complex algebraic variety, Totaro asks in [16] about a potential general collapsing of this spectral sequence after its \( E_m \) stage. Felix and Thomas prove in [10] such a collapsing when \( M \) is rationally formal, but they also prove that the collapsing fails when \( M \) is a simply connected orientable manifold carrying suitable non-trivial Massey products. Part of the motivation behind Theorems 4.7 and 4.13 came from a desire of finding non-simply connected instances for which the collapsibility phenomenon fails. However Theorem 4.13 now shows that real projective spaces are not such examples: since \( S^n \) is formal, \( E_{m+1} = E_\infty \) in the Leray spectral sequence for \( \text{Conf}(S^n, k) \hookrightarrow (S^n)_k \). Then Remark 1.1 implies the corresponding collapsibility behavior for orientable real projective spaces.

Even though our methods do not make direct use of the Fadell-Neuwirth fibration

\[
\text{Conf}(\mathbb{R}P^m - \ast, k) \to \text{Conf}(\mathbb{R}P^m, k + 1) \to \mathbb{R}P^m,
\]

our approach to Theorems 4.7 and 4.13 allows us to get information on the cohomology ring of configuration spaces on punctured real projective spaces.

**Theorem 5.1.** Let \( R \) be a commutative ring with unit where 2 is invertible. For \( n \geq 2 \) odd,

there is an \( R \)-algebra isomorphism

\[
H^*(\text{Conf}(\mathbb{R}P^n - \ast, k); R) \cong R[C^+]/K.
\]

**Theorem 5.4.** Let \( R \) be a commutative ring with unit where 2 is invertible. For \( n \geq 2 \) even,

there is an \( R \)-algebra isomorphism

\[
H^*(\text{Conf}(\mathbb{R}P^n - \ast, k); R) \cong R[E']/J',
\]

where the generators \( E' \) and the relations \( J' \) are detailed in Section 5.

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1. We thank Professor Totaro for kindly pointing out that this is dealt with in [10].
Since the cohomology groups described here are $R$-free of rank independent of the actual ring $R$, we deduce:

**Corollary 1.2.** There is no odd torsion in the integral cohomology of $\text{Conf} (\mathbb{R}P^n, k)$ and $\text{Conf} (\mathbb{R}P^n - \star, k)$.

Longini and Salvatore show in [15] that there are 3-dimensional twisted lens spaces having the same homotopy type, but whose $k$-points configuration spaces fail to be homotopy equivalent for all $k \geq 2$. As a consequence of the main results of this article, we have a new family of examples for which the homotopy invariance of configuration spaces fails, namely, $F(\mathbb{R}P^n, k)$ and $F(\mathbb{R}P^{n+1} - \star, k)$ are not homotopy equivalent when $k \geq 3$, even though $\mathbb{R}P^n \simeq \mathbb{R}P^{n+1} - \star$. In fact, these configuration spaces do not have isomorphic cohomology groups. For instance, for $n$ odd and $k \geq 3$, Theorem 4.7 implies $H^{n-1}(\text{Conf}(\mathbb{R}P^n, k)) \neq 0$ while, by Theorem 5.4, $H^{n-1}(\text{Conf}(\mathbb{R}P^{n+1} - \star, k)) = 0$. In fact, our results imply that $\text{Conf}(\mathbb{R}P^n, k)$ and $\text{Conf}(\mathbb{R}P^{n+1} - \star, k)$ are not stably homotopy equivalent. Such an observation illustrates the importance of the additional hypotheses in [1, Theorem A] where Aouina and Klein prove the stable homotopy invariance of configuration spaces assuming that the manifolds to which one takes configurations are not only homotopy equivalent but are closed (PL) manifolds of a fixed dimension. Likewise, our results (and a standard argument using the Serre spectral sequence) show that $\text{Conf}(\mathbb{R}P^n, k)$ and $\text{Conf}(\mathbb{R}P^{n+1} - \star, k)$ cannot have homotopy equivalent loop spaces, thus illustrating the importance of the (implicit) additional hypotheses in [14, Theorem 0.1].

The results described in this introductory section are proved by means of a careful cohomological analysis of certain covering projections of the above configuration spaces. In each case, the covering spaces are given by suitable orbit configuration spaces (defined in the next section). Our viewpoint corrects and extends the method used in [17]. In addition, our results fix a couple of errors in the descriptions given in [9] for some of these cohomology rings (see Remarks 2.4 and 3.6).

2. Preliminaries

In this section, $R$ will denote a commutative ring with unit where 2 is not necessarily invertible, and all cohomology rings will be considered with coefficients in $R$ unless otherwise stated. Also, throughout this section $n$ will denote an integer greater than 1.

Recall there is a description, due to Cohen ([4]), of the cohomology of $\text{Conf}(\mathbb{R}^n, k)$ as the $R$-algebra generated by elements $A'_{i,j}$, for $1 \leq j < i \leq k$, subject to the relations

$$A'_{r,j}A'_{r,i} = A'_{i,j}(A'_{r,i} - A'_{r,j}).$$

These generators satisfy $A'_{i,j} = p'_{i,j}(*_{n-1})$, where the maps $p'_{i,j} : \text{Conf}(\mathbb{R}^n, k) \rightarrow S^{n-1}$ are given by

$$p'_{i,j}(x_1, \ldots, x_k) = \frac{x_i - x_j}{\|x_i - x_j\|}$$

and $*_{n-1}$ denotes the cohomology fundamental class of $S^{n-1}$.

Given a topological space $X$ and a group $G$ acting on it, the configuration space of $k$ orbits on $X$ is

$$\text{Conf}_G(X, k) = \{(x_1, \ldots, x_k) \in X^k | Gx_i \neq Gx_j \text{ if } i \neq j\}.$$
Note that if $G$ is the trivial group, we recover the definition of usual ordered configuration spaces. We also have the following analogue of the Fadell-Neuwirth fibrations for ordinary configuration spaces ([17]): If $X$ is a manifold with $G$ acting properly discontinuously on it, and the orbit space $X/G$ is again a manifold then, for $l \leq k$, the projection

$$\text{Conf}_G(X, k) \to \text{Conf}_G(X, l)$$

onto the first $l$ coordinates is a locally trivial bundle with fiber $\text{Conf}_G(X - Q^G_l, k - l)$, where $Q^G_l$ denotes the union of $l$ disjoint orbits.

Consider the antipodal action of the group $\mathbb{Z}_2$ on the sphere $S^n$. The cohomology algebra of $\text{Conf}_{\mathbb{Z}_2}(S^n, k)$ was determined in [17] for $n \geq 2$, albeit with some minor corrections required. We start by addressing the needed corrections, and extending the argument to $n \geq 2$.

Xicotencatl’s method is to look at the Serre spectral sequence associated to the fibration

$$\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1) \approx \text{Conf}_{\mathbb{Z}_2}(S^n - Q^\mathbb{Z}_2_1, k - 1) \to \text{Conf}_{\mathbb{Z}_2}(S^n, k) \to S^n,$$

where the arrow on the right is the projection onto the first coordinate, and the homeomorphism on the left is induced by the stereographic projection $S^n - Q^\mathbb{Z}_2_1 \cong \mathbb{R}^n - \{0\}$. Note that the homeomorphism of the fiber is such that the action of $\mathbb{Z}_2$ on $\mathbb{R}^n - \{0\}$ is not the antipodal action; in this case, the action is given by

$$\tau(x) = -\frac{x}{\|x\|^2}.$$

In turn, Xicoténcatl computes the cohomology of the fiber in (2) using, in an inductive way, the Serre spectral sequence associated to the fibration

$$(\mathbb{R}^n - \{0\}) - Q^\mathbb{Z}_2_{k-2} \to \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1) \to \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 2).$$

The system of coefficients in (3) is trivial and the spectral sequence collapses (see Remark 10 in [9] for a discussion of these facts for $n = 2$ —the most interesting case). There results an $R$-module isomorphism

$$H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1)) \cong M_1 \otimes M_2 \otimes \cdots \otimes M_{k-1}$$

where the tensor product corresponds to the cohomology ring structure. Here $M_i$ is an $R$-free module generated by a zero dimensional class 1 and by $(n - 1)$-dimensional spherical classes $\{A_{i,0}\} \cup \{A_{i,j}, A_{i,-j}\}_{1 \leq j < i}$. We next describe these generators and recall their multiplicative relations.

For $0 \leq |j| < i < k$, define maps $p_{i,j} : \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1) \to S^{n-1}$ given by:

$$p_{i,0}(x_1, \ldots, x_{k-1}) = \frac{x_i}{\|x_i\|},$$

$$p_{i,j}(x_1, \ldots, x_{k-1}) = \frac{x_i - x_j}{\|x_i - x_j\|},$$

$$p_{i,-j}(x_1, \ldots, x_{k-1}) = \frac{x_i - \tau x_j}{\|x_i - \tau x_j\|},$$

where the last two formulas hold for $j > 0$. Define $A_{i,0} = p_{i,0}(t_{n-1})$, $A_{i,j} = p_{i,j}(t_{n-1})$, and $A_{i,-j} = p_{i,-j}(t_{n-1})$. Put

$$A = \{A_{i,j} \mid 1 \leq j < i < k\} \cup \{A_{i,-j} \mid 1 \leq j < i < k\} \cup \{A_{i,0} \mid 1 \leq i < k\}.$$
Then, \( H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1)) \) is the graded commutative \( R \)-algebra generated by the set \( \mathcal{A} \) subject to the relations

(a) For \( 0 \leq j < i < k \),
\[
A^2_{i,j} = A^2_{i,-j} = 0.
\]

(b) For \( 1 \leq i < r < k \)
\[
A_{r,0}A_{r,i} = A_{i,0}(A_{r,i} - A_{r,0}),
\]
\[
A_{r,0}A_{r,-i} = (-1)^nA_{i,0}(A_{r,-i} - A_{r,0}),
\]
\[
A_{r,i}A_{r,-i} = (-1)^nA_{i,0}(A_{r,-i} - A_{r,i}).
\]

(c) For \( 1 \leq j < i < r < k \)
\[
A_{r,j}A_{r,i} = A_{i,j}(A_{r,i} - A_{r,j}),
\]
\[
A_{r,j}A_{r,-i} = (-1)^n(A_{j,0} + A_{i,0} - A_{i,-j})(A_{r,-i} - A_{r,j}),
\]
\[
A_{r,i}A_{r,-j} = (-1)^nA_{i,j}(A_{r,-j} - A_{r,i}),
\]
\[
A_{r,-j}A_{r,-i} = (-1)^n(A_{i,0} - A_{i,j} + (-1)^nA_{j,0})(A_{r,-i} - A_{r,-j}).
\]

It should be noted that the relations found in [17] contain a small typographical error, namely, the product \( A_{r,j}A_{r,-i} \) is not equal to \( (-1)^n(A_{j,0} + A_{i,0} - A_{i,-j})(A_{r,-i} - A_{r,j}) \), as it is claimed there, but rather to the expression in (c) above. Since [17] offers little detail on the actual derivation of the multiplicative relations above, for the sake of completeness, we next give the full argument giving the correct relation for \( A_{r,j}A_{r,-i} \). Our method differs slightly from the one sketched in [17].

In order to obtain the relation \( A_{r,j}A_{r,-i} = (-1)^n(A_{j,0} + A_{i,0} - A_{i,-j})(A_{r,-i} - A_{r,j}) \), we start by considering the map \( \alpha : \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3) \rightarrow \text{Conf}(\mathbb{R}^n, 3) \) given by \( \alpha(x, y, z) = (x, \tau y, z) \). We clearly have
\[
\alpha^*(A'_{3,2}) = A_{3,-2}
\]
\[
\alpha^*(A'_{3,1}) = A_{3,1},
\]
and we next compute \( \alpha^*(A'_{2,1}) \). For ease of notation, let \( N : \mathbb{R}^n - \{0\} \rightarrow S^{n-1} \) be the normalization map. Define maps \( f_{i,j} : S^{n-1} \rightarrow \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3) \) by
\[
f_{1,0}(x) = (x, 2e, 3e), \quad f_{2,0}(x) = (e, \frac{x}{2}, 3e), \quad f_{2,1}(x) = (e, e + \frac{x}{2}, 3e),
\]
\[
f_{2,-1}(x) = (e, -e + \frac{x}{2}, 3e), \quad f_{3,0}(x) = (e, \frac{3}{2}e, \frac{x}{2}), \quad f_{3,1}(x) = (e, \frac{3}{2}e, e + \frac{x}{2}),
\]
\[
f_{3,2}(x) = (e, \frac{3}{2}e, \frac{3}{2}e + \frac{x}{4}), \quad f_{3,-1}(x) = (e, \frac{3}{2}e, -e + \frac{x}{4}), \quad f_{3,-2}(x) = (e, \frac{3}{2}e, -\frac{3}{2}e + \frac{x}{4}),
\]
where \( e = (1, 0 \ldots, 0) \in \mathbb{R}^n \). It can be easily verified that these maps are such that
\[
p_{r,s}f_{i,j} \simeq \begin{cases} 
\text{identity,} & \text{if } r = i \text{ and } s = j; \\
\text{constant,} & \text{otherwise};
\end{cases}
\]
so the degrees of the compositions \( p'_{2,1} \alpha f_{i,j} : S^{n-1} \to S^{n-1} \) are the coefficients of \( \alpha^*(A'_{2,1}) \) in terms of the basis \( \mathcal{A} \). We have

\begin{align*}
(9) \quad p'_{2,1} \alpha f_{1,0}(x) &= N\left(\frac{-e}{2} - x\right) = -N\left(\frac{e}{2} + x\right), \\
(10) \quad p'_{2,1} \alpha f_{2,0}(x) &= N(-2x - e) = -N(2x + e), \\
(11) \quad p'_{2,1} \alpha f_{2,1}(x) &= N(\tau(e + \frac{x}{2}) - e), \\
(12) \quad p'_{2,1} \alpha f_{2,-1}(x) &= N(\tau(-e + \frac{x}{2}) - e) = -N(-e + \frac{x}{2} + \| -e + \frac{x}{2} \|^2 e), \\
(13) \quad p'_{2,1} \alpha f_{3,0}(x) &= p'_{2,1} \alpha f_{3,1}(x) = p'_{2,1} \alpha f_{3,2}(x) = p'_{2,1} \alpha f_{3,-1}(x) = p'_{2,1} \alpha f_{3,-2}(x).
\end{align*}

The maps in (9) and (10) are obviously homotopic to the antipodal map. The map in (11) is homotopic to the constant map since \( p'_{2,1} \alpha f_{2,1} \) is not a surjective map; indeed, \( e \) is not in the image because \( e \) is not enclosed by the image of \( \tau(e + \frac{x}{2}) \). The maps in (13) are all obviously constant maps. Identifying the degree of the map in (12) requires some work: Let \( F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be the map given by

\[ F(t, x) = F(t, (t_1, t_2, \ldots, t_n)) = (tt_1, t_2, \ldots, t_n). \]

Note that \( F(1, x) = x \) and \( F(-1, x) \) is \( x \) reflected across the hyperplane \( t_1 = 0 \). As maps \( S^{n-1} \to S^{n-1} \), we have

\[ N\left( -e + \frac{x}{2} + \| -e + \frac{x}{2} \|^2 e \right) = N\left( F\left( -1, \frac{x}{2}\right) + \left( -1 + \| -e + \frac{x}{2} \|^2 e \right) \right) \simeq N\left( F\left( -1, \frac{x}{2}\right) + \left( -1 + \| -e + \frac{x}{2} \|^2 e \right) \right). \]

The homotopy is given by \( N(F(t, \frac{x}{2}) + (-1 + \| -e + \frac{x}{2} \| e)) \) for \( t \in [-1, 1] \), and it is well defined: suppose there exist \( t \in [-1, 1] \) and \( x = (t_1, \ldots, t_n) \in S^{n-1} \) such that \( F(t, \frac{x}{2}) + (-1 + \| -e + \frac{x}{2} \|^2 e) = 0 \). Then \( F(t, \frac{x}{2}) = (1 - \| -e + \frac{x}{2} \|^2 e) = 0 \) and we have \( \frac{t_1}{2} = 1 - \| -e + \frac{x}{2} \|^2 = 1 - \| e - \frac{x}{2} \|^2 = \frac{3}{4} \), so \( t = \frac{3}{2} > 1 \).

Suppose \( t_1 = 1 \). Then \( \frac{3}{2} = 1 - \| -e + \frac{x}{2} \|^2 = 1 - \| -\frac{x}{2} \|^2 = \frac{3}{4} \), so \( t = \frac{3}{2} > 1 \).

Suppose \( t_1 = -1 \). Then \( \frac{5}{2} = 1 - \| -e - \frac{x}{2} \|^2 = 1 - \| -\frac{3x}{2} \|^2 = \frac{5}{4} \), so \( t = \frac{5}{2} > 1 \).

Both assumptions lead to a contradiction, so the homotopy is well defined. Now we will prove that, as maps \( S^{n-1} \to S^{n-1} \),

\[ N\left( F\left( -1, \frac{x}{2}\right) + \left( -1 + \| -e + \frac{x}{2} \|^2 e \right) \right) \simeq N\left( F\left( -1, \frac{x}{2}\right) \right). \]

Consider the homotopy \( N(F(-1, \frac{x}{2}) + t(-1 + \| -e + \frac{x}{2} \|^2 e)) \) with \( t \in [0, 1] \). This homotopy is well defined: suppose there exist \( t \in [0, 1] \) and \( x = (t_1, \ldots, t_n) \in S^{n-1} \) such that \( F(-1, \frac{x}{2}) + t(-1 + \| -e + \frac{x}{2} \|^2 e) = 0 \). Then \( F(-1, \frac{x}{2}) = t(1 - \| -e + \frac{x}{2} \|^2 e) = 0 \) and so we have \( -\frac{x}{2} = t(1 - \| -e + \frac{x}{2} \|^2) \) and \( t_i = 0 \) for \( i > 1 \). The latter condition, in turn, implies \( t_1 = \pm 1 \).

Suppose \( t_1 = 1 \). Then \( \frac{3}{2} = t(1 - \| -e + \frac{x}{2} \|^2) = t(1 - \| -\frac{x}{2} \|^2) = \frac{3}{4} \), so \( t = \frac{3}{2} < 0 \).
Suppose \( t_1 = -1 \). Then \( \frac{1}{2} = t(1 - \|e - \frac{t}{e}\|)^2 = t(1 - \|\frac{3t}{2}\|)^2 = -t\frac{5}{4} \), so \( t = -\frac{3}{5} < 0 \).

Both assumptions lead to a contradiction, so the homotopy is well defined. Therefore
\[
p'_{2,1} \alpha f_{2,-1}(x) \simeq -N \left( F \left( -1, \frac{x}{2} \right) \right) = -N \left( F(-1, x) \right)
\]
which is clearly a map of degree \((-1)^{n+1}\).

Having understood the maps \( p'_{2,1} \alpha f_{i,j}(x) \), we can read off the expression for \( \alpha^*(A'_{2,1}) \), namely, \( \alpha^*(A'_{2,1}) = (-1)^n (A_{1,0} + A_{2,0} - A_{2,-1}) \). Then, by applying \( \alpha^* \) to the relation
\[
A'_{3,1} A'_{3,2} = A'_{2,1} (A'_{3,2} - A'_{3,1}),
\]
we get
\[
A_{3,1} A_{3,-2} = (-1)^n (A_{1,0} + A_{2,0} - A_{2,-1}) (A_{3,-2} - A_{3,1}),
\]
which is the second relation asserted in \((c)\) above in the case \((r, i, j) = (3, 2, 1)\). The general case follows by applying the maps
\[
\pi_{r, i, j} : F_{\mathbb{Z}_2}(\mathbb{R}^n - 0, k - 1) \rightarrow F_{\mathbb{Z}_2}(\mathbb{R}^n - 0, 3)
\]
given by \( \pi_{r, i, j}(x_1, \ldots, x_{k-1}) = (x_j, x_i, x_r) \), and which evidently satisfy
\[
\pi_{r, i, j}^*(A_{1,0}) = A_{j, 0}, \quad \pi_{r, i, j}^*(A_{2,0}) = A_{i, 0}, \quad \pi_{r, i, j}^*(A_{2,1}) = A_{i, j},
\]
\[
\pi_{r, i, j}^*(A_{2,-1}) = A_{i, -j}, \quad \pi_{r, i, j}^*(A_{3,0}) = A_{r, 0}, \quad \pi_{r, i, j}^*(A_{3,1}) = A_{r, j},
\]
\[
\pi_{r, i, j}^*(A_{3,2}) = A_{r, i}, \quad \pi_{r, i, j}^*(A_{3,-1}) = A_{r, -j}, \quad \pi_{r, i, j}^*(A_{3,-2}) = A_{r, -i}.
\]
The other product relations in \([a] \quad [b] \quad [c] \) can be obtained in a similar way by suitably changing the map \( \alpha \). These relations imply that the cohomology of the fiber in \((2)\) is additively generated by products of the form \( A_{i_1, j_1} \cdots A_{i_r, j_r} \) where \( i_l < i_l' \) if \( l < l' \). Furthermore, such products are in fact an additive basis in view of \((1)\). In summary, we have the following theorem.

**Theorem 2.1 ([17])**. For \( n \geq 2 \), there is an \( R \)-algebra isomorphism
\[
H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1)) \cong R[A]/I,
\]
where \( I \) denotes the ideal generated by the relations \([a] \quad [b] \quad [c] \) above, and \( A \) is defined in \((\emptyset)\).

Now we determine the cohomology ring of the total space of \((2)\), following Section 4 of [17]. Since \( n \geq 2 \), \( S^n \) is simply connected and we have trivial coefficients on the Serre spectral sequence. Also, \( S^n \) has torsion free cohomology, therefore
\[
E_{2}^{p,q} \cong H^p(S^n; H^q(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))) \cong H^p(S^n) \otimes H^q(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1)).
\]
For \( n \) odd, a nowhere vanishing vector field on \( S^n \) easily yields a section for \((2)\), consequently the spectral sequence collapses, and we have the following theorem:

**Theorem 2.2 ([9] Proposition 14, [17 Proposition 5.2(a)])**. For \( n > 2 \) odd, there is an \( R \)-algebra isomorphism \( H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong H^*(S^n) \otimes H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1)) \).

Note that the multiplicative structure in \( E_\infty = E_2 \), which is just the tensor product of the multiplicative structures for the base and fiber, already gives the multiplicative structure of \( H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k)) \), by dimensional considerations —recall \( n \) is an odd integer greater than 1.
For $n$ even, Xicoténcatl shows that the differential $d_n^{0,n-1}: E_n^{0,n-1} \to E_n^{n,0}$ is determined by $d_n(A_{i,j}) = 2\tau_n$ for all $A_{i,j} \in \mathcal{A}$ (see also [9] Proposition 13]). In particular, if the characteristic of $R$ is 2, the conclusion of (and argument for) Theorem 2.2 holds also for any even $n$ (the case $n = 2$ requires an additional argument based on Brown representability, see the proof of Theorem 2.3 below). We close the section with a description of the $R$-cohomology algebra of $\text{Conf}_{\mathbb{Z}_2}(S^n, k)$ for $n$ even under the additional hypothesis—in force throughout the rest of this section—that the characteristic of $R$ is either zero (e.g. $R = \mathbb{Z}$ or $R = \mathbb{Q}$) or an odd integer (e.g. $R = \mathbb{Z}_t$, odd $t$), so that the map $2: R \to R$ given by multiplication by 2 is injective.

It will be convenient to make a change of basis by defining $B_{i,j} = A_{i,j} - A_{1,0}$, for $|j| < i < k$, and

$$\mathcal{B} = \{ B_{i,j} \mid |j| < i < k \text{ and } 1 < i \}.$$  

A straightforward computation shows that a product of two given elements in $\mathcal{B}$ satisfies the exact same relation holding for the product of the corresponding two elements in $\mathcal{A}$ (keeping in mind that, by definition, $B_{1,0} = 0$). Let us denote by $J$ the resulting set of relations among the $B_{i,j}$'s. It is also clear that a new basis for $H^\ast(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))$ is obtained from the basis described just before Theorem 2.1 by replacing each factor $A_{i,j}$ with $i > 1$ by the corresponding $B_{i,j}$. In these conditions, the hypothesis on the characteristic of $R$, and the fact that the differential sends every $A_{i,j}$ to $2\tau_n$ imply that $\mathcal{B}$ is a basis for the kernel of $d_n^{0,n-1}$. More generally, let $\mathbb{K} = \ker d_n^{0,n-1}$ denote the (free) $R$-module generated by $\mathcal{B}$, and let $\mathbb{K}^j$ denote the $R$-module generated by products of $j$ factors in $\mathbb{K}$, where $\mathbb{K}^0$ and $\mathbb{K}^1$ are set to be $R$ and 0 respectively. Then a basis for $\mathbb{K}^j$ is given by the degree $(n-1)$ elements in the above modified basis for $H^\ast(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))$ which do not contain the factor $A_{1,0}$ (e.g. $\mathbb{K}^{k-1} = 0$). It is then clear that the only non-trivial terms in the $(n+1)$-stage of the spectral sequence are given by

$$E_0^{0,j(n-1)} = \mathbb{K}^j;$$
$$E_n^{0,j(n-1)} = \tau_n A_{1,0} \mathbb{K}^{j-1} \oplus (\tau_n \mathbb{K}^j)_2;$$

where $(-)_2$ denotes the mod 2 reduction of the given module (that is, tensoring with $\mathbb{Z}_2$). There are no extension problems in the spectral sequence since its $p = 0$ column is $R$-free. Further, just as with Theorem 2.2 if $n > 2$, the multiplicative structure of the cohomology of the total space follows by dimensional considerations from that for the $E_\infty$-term of the spectral sequence. In fact:

**Theorem 2.3.** Assume that the characteristic of $R$ is either zero or an odd integer. For even $n \geq 2$ there is an isomorphism of graded $R$-algebras

$$H^\ast(\text{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong R[\mathcal{B}]/J \otimes \Lambda(\lambda, \omega)/(2\lambda, \lambda \omega)$$

where $\lambda$ and $\omega$ are represented in the spectral sequence by $\tau_n$ and $\tau_n A_{1,0}$, respectively.

**Proof.** It only remains to argue the assertion about the multiplicative structure when $n = 2$. (The issue is mentioned without explanation by Feichtner and Ziegler on the first half of page 100 in [9].) The point is that, for any even $n$, $2B_{1,j}^2 = 0$ by anticommutativity. But for $n = 2$ we need to rule out the possibility that, as an element in $H^\ast(\text{Conf}_{\mathbb{Z}_2}(S^n, k))$, the square of a 1-dimensional class $B_{i,j}$ agrees with the 2-dimensional 2-torsion class $\lambda$. This follows from Brown representability when the coefficients are $\mathbb{Z}$. For other coefficients $R$ the
assertion holds since the definition of the classes \( B_{i,j} \) is natural with respect to the canonical ring morphism \( \mathbb{Z} \rightarrow R \).

Note \( R[\mathcal{B}]/J = \bigoplus_{0 \leq j \leq k-2} \mathbb{K}^j \), a basis of which has already been described. In the \( E_\infty \) term of the spectral sequence, this \( R \)-subalgebra corresponds to the left hand side tower supported by 1. Besides, two additional “copies” of this tower show up: one copy (tensored with \( \mathbb{Z}_2 \)) is supported by \( \lambda \); another copy (shifted one level up) is supported by \( \omega \):

\[ \begin{array}{c}
(k-1)(n-1) \\
(k-2)(n-1) \\
\vdots \\
2(n-1) \\
n-1
\end{array} \]

\[ \begin{array}{c}
\bullet \\
\bullet \\
\vdots \\
\bullet \\
\omega
\end{array} \]

\[ \begin{array}{c}
1 \\
\lambda
\end{array} \]

**Figure 1.** The \( E_\infty \) term.

**Remark 2.4.** The additive version of Theorem 2.3 is obtained in [17, Theorem 5.2, items (b) and (c)] assuming implicitly \( n > 2 \). On the other hand, for \( n > 2 \), the multiplicative relations among generators in Theorems 2.1 and 2.3 correct those found in [9]. In fact, the multiplicative relations described by Feichtner and Ziegler in [9, Proposition 11] for their generators in the cohomology of \( \text{Conf}_{\mathbb{Z}_2}(S^n, k-1) \) lead to inconsistencies. We illustrate the problem using Feichtner-Ziegler’s notation, which the reader is assumed to be familiar with. (In particular, the notation for the fiber in (2) will momentarily change to \( \mathcal{F}_{\phi}(\mathbb{R}^k \setminus \{0\}, n) \). Take \( 1 \leq i < j \leq n \), and let \( k \) be odd (so that the generators \( c_i, c^+_{i,j}, c^-_{i,j} \) are even dimensional and, therefore, commute without introducing signs). Then Lemma 7 and Proposition 11 in [9] imply

\[
c_{i,j}^+ c_{i,j}^+ + c_i (c_{i,j}^+ + c_{i,j}^-) = 0 = A_i(0) = A_i\left(c_{i,j}^+ c_{i,j}^- + c_i (c_{i,j}^+ + c_{i,j}^-)\right) = c_{i,j}^+ c_{i,j}^- - c_i (c_{i,j}^+ + c_{i,j}^-).
\]

This yields \( c_i c_{i,j}^+ + c_i c_{i,j}^- = 0 \), if we work with integral coefficients. However the latter relation contradicts Proposition 8(2) in [9].

3. \((\mathbb{Z}_2)^k\)-Action

In this section, \( R \) will denote a commutative ring with unit where 2 is (still) not necessarily invertible. In [17], the action of the group \((\mathbb{Z}_2)^k\) on \( H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k)) \) induced via antipodal maps on each coordinate was determined for \( k \leq 3 \), with most details omitted; here we generalize Xicoténcatl’s result for all \( k \), providing full details in typical cases, and correcting the description for \( k = 3 \).
Let us denote by $\epsilon_i : \text{Conf}_{\mathbb{Z}_2}(S^n, k) \to \text{Conf}_{\mathbb{Z}_2}(S^n, k)$ the antipodal map on the $i$-th coordinate and, by abuse of notation, its induced map in cohomology. We will work with the Serre spectral sequence of (2), and determine the action of $(H^n(\mathbb{Z}_2))^k = \langle \epsilon_1, \epsilon_2, \ldots, \epsilon_k \rangle$ on the cohomology of the total space by understanding the action of each $\epsilon_i$ on the cohomology of the base and the fiber. We first state the main results of this section (dealing with the action on the fiber), and then we recall (from [17]) the details on how $(H^n(\mathbb{Z}_2))^k$ can be thought of as acting on the fiber of (2).

**Theorem 3.1.** For $n \geq 2$, the action of $(\mathbb{Z}_2)^k$ on $H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))$ is given by

$$
\epsilon_i A_{i,j} = \begin{cases} 
(-1)^{n-1} A_{j,0} - A_{i,0} + A_{i,j} & \text{if } l = 1, j > 0; \\
-A_{i,j,0} - A_{i,0} + A_{i,j} & \text{if } l = 1, j < 0; \\
-A_{i,0} & \text{if } l = 1, j = 0, i \geq 1; \\
A_{i,-j} & \text{if } l > 1, |j| = l - 1; \\
(-1)^n A_{i,0} & \text{if } l > 1, i = l - 1, j = 0; \\
(-1)^n A_{j,0} + (-1)^n A_{i,0} + (-1)^{n-1} A_{i,-j} & \text{if } l > 2, i = l - 1, j > 0; \\
A_{i,j,0} + (-1)^n A_{i,0} + (-1)^{n-1} A_{i,|j|} & \text{if } l > 2, i = l - 1, j < 0; \\
A_{i,j} & \text{otherwise.}
\end{cases}
$$

**Theorem 3.2.** For $n \geq 2$ even, the action of $(\mathbb{Z}_2)^k$ on the permanent cycles

$$
\mathbb{K}^* \subseteq H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))
$$

is given by

$$
\epsilon_i B_{i,j} = \begin{cases} 
-B_{i,j,0} - B_{i,0} + B_{i,j} & \text{if } l = 1, |j| > 0; \\
-B_{i,0} & \text{if } l = 1, j = 0, i > 1; \\
B_{i,-j} & \text{if } l > 1, |j| = l - 1; \\
B_{i,j,0} + B_{i,0} - B_{i,-j} & \text{if } l > 2, i = l - 1, |j| > 0; \\
B_{i,j} & \text{otherwise.}
\end{cases}
$$

Note that $B_{1,0} = 0$ in (16), and that the formulas in (16) are the same ones as those in (15) for $n$ even and replacing each $A$ with $B$.

Unlike the maps $\epsilon_l$ for $l > 1$, $\epsilon_1$ does not preserve the fiber in (2). Indeed, $\epsilon_1$ covers the antipodal map. This issue is dealt with in [17] by using the rotation

$$
R = \begin{pmatrix} I_{n-1} & 0 \\
0 & -I_2 \end{pmatrix} \in SO(n + 1)
$$

that interchanges the north and south poles $N = (0, \ldots, 0, 1), S = (0, \ldots, 0, -1) \in S^n$. In detail, the restriction of $R$ to $S^n$ is $\mathbb{Z}_2$-equivariant and it is $\mathbb{Z}_2$-equivariantly isotopic to the identity, therefore it induces a map $R \times k : \text{Conf}_{\mathbb{Z}_2}(S^n, k) \to \text{Conf}_{\mathbb{Z}_2}(S^n, k)$ homotopic to the identity such that the following diagram commutes:

$$
\begin{array}{ccc}
\text{Conf}_{\mathbb{Z}_2}(S^n, k) & \overset{R \times k \circ \epsilon_1}{\longrightarrow} & \text{Conf}_{\mathbb{Z}_2}(S^n, k) \\
\pi_1 \downarrow & & \downarrow \pi_1 \\
S^n & \overset{-R}{\longrightarrow} & S^n.
\end{array}
$$
Since \(-R\) fixes the north pole, \(R^{xk} \circ \epsilon_1\) — which is homotopic to \(\epsilon_1\) — restricts to a map on the corresponding fiber. This allows us to understand the effect of \(R^{xk} \circ \epsilon_1\) (and, consequently, of \(\epsilon_1\)) on the spectral sequence. With this in mind note that, after removing the poles and taking into account the stereographic projection, the map \(R\) induces a map \(\tilde{R} : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R}^n - \{0\}\), given by

\[
\tilde{R}(x) = \frac{x}{\|x\|^2},
\]

where \(\tilde{x} = (t_1, \ldots, t_{n-1}, -t_n)\) for \(x = (t_1, \ldots, t_n)\). Thus, the action of \(R^{xk} \circ \epsilon_1\) restricted to the fiber is given by \(\tilde{R}^{(k-1)}\), so the action of \(\epsilon_1\) on the cohomology of the fiber is the same as the action of the map \(R^n - \{0\} \rightarrow \mathbb{R}^n - \{0\}\), which from now on we will also denote by \(\epsilon_1\). The remaining actions restricted to the fiber are given by \(\epsilon_1(x_1, \ldots, x_{k-1}) = (x_1, \ldots, \tau x_{l-1}, \ldots, x_{k-1})\) for \(1 < l \leq k\). The maps \(\epsilon_1, \epsilon_2, \ldots, \epsilon_k : \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \rightarrow \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)\) are related as follows:

**Lemma 3.3.** For \(n\) odd, \(\epsilon_1 \simeq \epsilon_2 \cdots \epsilon_k\). For \(n\) even, \(\epsilon_1 \simeq h^{x(k-1)}\epsilon_2 \cdots \epsilon_k\), with \(h : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R}^n - \{0\}\) given by \(h(x) = \tilde{x}\).

**Proof.** Let \(g, f : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R}^n - \{0\}\) be the maps \(f(x) = \frac{x}{\|x\|^2}\) and \(g(x) = -x\). We have that any \(T \in O(n)\) is \(\mathbb{Z}_2\)-equivariant (with \(\mathbb{Z}_2 = \langle \tau \rangle\)):

\[
T(\tau x) = T\left(\frac{-x}{\|x\|^2}\right) = \frac{-T(x)}{\|x\|^2} = \frac{-T(x)}{\|T(x)\|^2} = \tau T(x).
\]

This, coupled with the injectivity of \(T\), implies that \(T^{x(k-1)}\) sends orbit configurations to orbit configurations. Therefore \(h^{x(k-1)}, g^{x(k-1)}\) are maps of orbit configurations spaces. We also have that \(f\) is injective, and it is also \(\mathbb{Z}_2\)-equivariant:

\[
f(\tau x) = f\left(\frac{-x}{\|x\|^2}\right) = \frac{-x}{\|x\|^2} - \frac{x}{\|x\|^2} = \tau f(x),
\]

therefore \(f^{x(k-1)}\) is a map between orbit configurations spaces. Note that \(\tau = gf\) and \(R = hf\). Therefore we have \(\epsilon_1 = \tilde{R}^{x(k-1)} = (hf)^{x(k-1)} = h^{x(k-1)} f^{x(k-1)}\). For \(n\) odd, it is known that there is a homotopy through \(O(n)\) between \(g\) and \(h\), so we have \(g^{x(k-1)} \simeq h^{x(k-1)}\) as maps of orbit configurations spaces, therefore

\[
\epsilon_1 = h^{x(k-1)} f^{x(k-1)} \simeq g^{x(k-1)} f^{x(k-1)} = \tau^{x(k-1)} = \epsilon_2 \cdots \epsilon_k
\]
as maps of orbit configuration spaces. For \(n\) even, there is a homotopy through \(O(n)\) between \(g\) and \(h\) and the identity. Therefore

\[
\epsilon_1 = h^{x(k-1)} f^{x(k-1)} \simeq h^{x(k-1)} g^{x(k-1)} f^{x(k-1)} = h^{x(k-1)} \tau^{x(k-1)} = h^{x(k-1)} \epsilon_2 \cdots \epsilon_k
\]
as maps of orbit configurations spaces. \(\square\)

Of course, Theorem 3.1 can be used to give a description of the effect in cohomology of the map \(h^{x(k-1)} : \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \rightarrow \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)\) that arises in Lemma 3.3 for \(n\) even. We omit the details as we will not have occasion of using such information.
Yet, in the next section we will need to describe the behavior of the map \( h^{x(k-1)} \) on the permanent cycles \( \mathbb{K}^* \) of the previous section.

Note that \( \epsilon_1 \) acts as multiplication by \((-1)^{n+1}\) on the generator of the cohomology of the base space of \((\mathcal{B})\), and that \( \epsilon_l \) acts trivially on said generator for \( l > 1 \). Thus we have the following description of the action of \((\mathbb{Z}_2)^k\) on the total space of \((\mathcal{B})\).

**Corollary 3.4.** For \( n > 1 \) odd, the action of \((\mathbb{Z}_2)^k\) on

\[
H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong H^*(S^n) \otimes H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1)) = \Lambda(\iota_n) \otimes \mathcal{A}/I
\]

is the tensor product of the corresponding actions on each factor of the tensor product.

**Corollary 3.5.** Assume that the characteristic of \( R \) is either zero or an odd integer. For \( n \geq 2 \) even, the action of \((\mathbb{Z}_2)^k\) on

\[
H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong R[\mathcal{B}]/J \otimes \Lambda(\lambda, \omega)/(2\lambda, \lambda\omega)
\]

satisfies

\[
\epsilon_l(\lambda) = \begin{cases} 
-\lambda, & \text{if } l = 1; \\
\lambda, & \text{if } l > 1,
\end{cases}
\]

\[
\epsilon_l(\omega) = \omega, \quad \forall \ l \geq 1,
\]

and restricts to the action of \((\mathbb{Z}_2)^k\) on \( \mathcal{B} \) stated in Theorem 3.2.

Theorem 3.2 is a straightforward consequence of the definitions and Theorem 3.1. In turn, it suffices to prove the latter result in the special case \( k = 3 \). Indeed, on the one hand, Theorem 3.1 is elementary for \( k = 2 \). On the other, for \( k \geq 3 \) and \( 0 < j < i < k \), the map \( \pi_{i,j} : \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1) \to \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2) \) given by \( \pi_{i,j}(x_1, \ldots, x_{k-1}) = (x_j, x_i) \) sends \( A_{1,0}, A_{2,0}, A_{2,1}, \) and \( A_{2,-1} \) respectively to \( A_{j,0}, A_{i,0}, A_{i,j}, \) and \( A_{i,-j} \), whereas, for \( 1 \leq l \leq k \), \( \pi_{i,j} \) fits in the commutative diagram

\[
\begin{array}{ccc}
\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1) & \xrightarrow{\epsilon_l} & \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1) \\
\downarrow{\pi_{i,j}} & & \downarrow{\pi_{i,j}} \\
\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2) & \xrightarrow{\epsilon} & \text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2),
\end{array}
\]

where

\[
\bar{\epsilon}(x, y) = \begin{cases} 
\epsilon_3(x, y), & \text{if } i = l - 1; \\
\epsilon_2(x, y), & \text{if } j = l - 1; \\
\epsilon_1(x, y), & \text{if } l = 1; \\
(x, y), & \text{otherwise}.
\end{cases}
\]
Recall the maps $p$ we will denote by together with the corresponding maps compositions of the following set of equalities—which corrects the action reported in Table 2 of [17]:

The rest of this section is devoted to proving Theorem 3.1 in the case $k = 3$, i.e. to the proof of the following set of equalities—which corrects the action reported in Table 2 of [17]:

$$\begin{align*}
\epsilon_1 A_{1,0} &= -A_{1,0}, \\
\epsilon_1 A_{2,0} &= -A_{2,0}, \\
\epsilon_1 A_{2,1} &= (-1)^{n-1} A_{1,0} - A_{2,0} + A_{2,1}, \\
\epsilon_2 A_{2,-1} &= -A_{1,0} - A_{2,0} + A_{2,-1}, \\
\epsilon_2 A_{1,0} &= (-1)^{n} A_{1,0}, \\
\epsilon_2 A_{2,0} &= A_{2,0}, \\
\epsilon_2 A_{2,1} &= A_{2,-1}, \\
\epsilon_3 A_{2,-1} &= A_{2,1}, \\
\epsilon_3 A_{1,0} &= A_{1,0}, \\
\epsilon_3 A_{2,0} &= (-1)^{n} A_{2,0}, \\
\epsilon_3 A_{2,1} &= (-1)^{n} A_{1,0} + (-1)^{n} A_{2,0} + (-1)^{n-1} A_{2,-1}, \\
\epsilon_3 A_{2,-1} &= A_{1,0} + (-1)^{n} A_{2,0} + (-1)^{n-1} A_{2,1}.
\end{align*}$$

Recall the maps $p_{i,j}$ and $f_{r,s}$ introduced in [5] and [7]. By abuse of notation, for $|j| < i \leq 2$, we will denote by $f_{i,j}$ the composition $\pi_{2,1} f_{i,j} : S^{n-1} \to \text{Conf}_{Z_2}(\mathbb{R}^n - \{0\}, 2)$. These maps, together with the corresponding maps $p_{r,s}$, detect the generators for $\text{Conf}_{Z_2}(\mathbb{R}^n - \{0\}, 2)$ in the sense of [8]. To prove the above set of relations, we will compute the degree of the compositions

$$S^{n-1} \xrightarrow{f_{i,j}} \text{Conf}_{Z_2}(\mathbb{R}^n - \{0\}, 2) \xrightarrow{\epsilon_{i,j}} \text{Conf}_{Z_2}(\mathbb{R}^n - \{0\}, 2) \xrightarrow{p_{r,s}} S^{n-1}$$

for $0 < l \leq 3$, $|j| < i \leq 2$, and $|s| < r \leq 2$. We start by computing the action of $\epsilon_1$.

(1) $\epsilon_1 A_{1,0}$: We have

$$p_{1,0} \epsilon_1 f_{1,0}(x) = \bar{x}, \quad p_{1,0} \epsilon_1 f_{2,0}(x) = e, \quad p_{1,0} \epsilon_1 f_{2,1}(x) = e, \quad p_{1,0} \epsilon_1 f_{2,-1}(x) = e.$$

The first map is a reflection and the rest are constant maps, therefore

$$\deg(p_{1,0} \epsilon_1 f_{1,0}) = -1, \quad \deg(p_{1,0} \epsilon_1 f_{2,0}) = 0, \quad \deg(p_{1,0} \epsilon_1 f_{2,1}) = 0, \quad \deg(p_{1,0} \epsilon_1 f_{2,-1}) = 0.$$

Thus, $\epsilon_1 A_{1,0} = -A_{1,0}$.

(2) $\epsilon_1 A_{2,0}$: Clearly, $p_{2,0} \epsilon_1 f_{1,0}(x) = N(\bar{y})$, which implies $\deg(p_{2,0} \epsilon_1 f_{1,0}) = 0$. Note that $N(\bar{y}) = N(y)$ for all $y \in \mathbb{R}^n - \{0\}$, therefore

$$\begin{align*}
p_{2,0} \epsilon_1 f_{2,0}(x) &= N(\bar{x}) = \bar{x}, \\
p_{2,0} \epsilon_1 f_{2,1}(x) &= N(\bar{y} + \frac{x}{2}) = N(e + \frac{\bar{x}}{2}), \\
p_{2,0} \epsilon_1 f_{2,-1}(x) &= N(\bar{y} - e + \frac{x}{2}) = N(-e + \frac{\bar{x}}{2});
\end{align*}$$

The second and third maps are not surjective, therefore we have

$$\deg(p_{2,0} \epsilon_1 f_{2,0}) = -1, \quad \deg(p_{2,0} \epsilon_1 f_{2,1}) = 0, \quad \deg(p_{2,0} \epsilon_1 f_{2,-1}) = 0.$$

Thus $\epsilon_1 A_{2,0} = -A_{2,0}$.
(3) $\epsilon_1 A_{2,1}$: We have
\[
p_{2,1}\epsilon_1 f_{1,0}(x) = N\left(\frac{e}{2} - \bar{x}\right) = -N(\bar{x} - \frac{e}{2}) \simeq -\bar{x},
\]
therefore
\[
\deg(p_{2,1}\epsilon_1 f_{1,0}) = (-1)^{n-1}.
\]
We also have
\[
p_{2,1}\epsilon_1 f_{2,0}(x) = N(2\bar{x} - e) = N(\bar{x} - \frac{e}{2}),
\]
so
\[
\deg(p_{2,1}\epsilon_1 f_{2,0}) = -1.
\]
Recall the map $F$ defined earlier, given by $(t, (t_1, \ldots, t_n)) \mapsto (tt_1, t_2, \ldots, t_n)$. We have
\[
p_{2,1}\epsilon_1 f_{2,1}(x) = N(\tilde{R}(e + \frac{x}{2}) - e) = N(e + \frac{F(t, x)}{2} - ||e + \frac{e}{2}||^2 e) \simeq N(e + \frac{F(t, x)}{2}) - ||e + \frac{e}{2}||^2 e) \simeq N(\frac{F(t, x)}{2}).
\]
The first homotopy is given by $N(e + \frac{F(t, x)}{2} - ||e + \frac{e}{2}||^2 e)$ with $t \in [-1, 1]$. As before, we have to check that this homotopy is well defined: suppose there exist $t \in [-1, 1]$ and $x = (t_1, \ldots, t_n) \in S^{n-1}$ such that $F(t, \frac{x}{2}) + (1 - ||e + \frac{e}{2}||^2)e = 0$. Then $F(t, \frac{x}{2}) = (-1 + ||e + \frac{e}{2}||^2)e$ and so we have $\frac{H_s}{2} = -1 + ||e + \frac{e}{2}||^2$ and $t_i = 0$ for $i > 1$. This, in turn, implies $t_1 = \pm 1$.

Suppose $t_1 = 1$. Then $\frac{t}{2} = -1 + ||e + \frac{e}{2}||^2 = -1 + \frac{3e}{2}$, so $t = 1 > 1$.

Suppose $t_1 = -1$. Then $\frac{t}{2} = -1 + ||e + \frac{e}{2}||^2 = -1 + \frac{3e}{2}$, so $t = -1 > 1$.

Both assumptions lead to a contradiction, so the homotopy is well defined. The second homotopy is $N(\frac{F(-1, x)}{2} + t(1 - ||e + \frac{e}{2}||^2)e)$, with $t \in [0, 1]$. Let us verify that this homotopy is well defined: suppose there exist $t \in [0, 1]$ and $x = (t_1, \ldots, t_n) \in S^{n-1}$ such that $\frac{F(-1, x)}{2} + t(1 - ||e + \frac{e}{2}||^2)e = 0$. Then $\frac{F(-1, x)}{2} = t(-1 + ||e + \frac{e}{2}||^2)e$ and so we have $-\frac{t}{2} = t(-1 + ||e + \frac{e}{2}||^2)$ and $t_i = 0$ for $i > 1$. This, in turn, implies $t_1 = \pm 1$.

Suppose $t_1 = 1$. Then $-\frac{t}{2} = t(-1 + ||e + \frac{e}{2}||^2) = t(-1 + \frac{3e}{2}) = t\frac{5}{4}$, so $t = -\frac{5}{4} < 0$.

Suppose $t_1 = -1$. Then $\frac{t}{2} = t(-1 + ||e + \frac{e}{2}||^2) = t(-1 + \frac{3e}{2}) = -t\frac{5}{4}$, so $t = -\frac{5}{4} < 0$.

Both assumptions lead to a contradiction, consequently the homotopy is well defined. Therefore $p_{2,1}\epsilon_1 f_{2,1}$ is homotopic to a composition of two reflections. Thus
\[
\deg(p_{2,1}\epsilon_1 f_{2,1}) = 1.
\]
Finally, since $e$ is not enclosed by the image of $\tilde{R}(e + \frac{x}{2})$, we have that the map $p_{2,1}\epsilon_1 f_{2,-1}(x) = N(\tilde{R}(e + \frac{x}{2}) - e)$ is not surjective. Therefore
\[
\deg(p_{2,1}\epsilon_1 f_{2,-1}) = 0,
\]
and we conclude that $\epsilon_1 A_{2,1} = (-1)^{n-1}A_{1,0} - A_{2,0} + A_{2,1}$.

(4) $\epsilon_1 A_{2,-1}$: We clearly have
\[
p_{2,-1}\epsilon_1 f_{1,0}(x) = N\left(\frac{x}{2} - \tau(\bar{x})\right) = N(\bar{x} + \frac{e}{2}) \simeq \bar{x}
\]
and
\[
p_{2,-1}\epsilon_1 f_{2,0}(x) = N(2\bar{x} + e) = N(\bar{x} + \frac{e}{2}) \simeq \bar{x}.
\]
Therefore
\[ \deg(p_{-1}^{} \epsilon_1 f_{2,0}) = \deg(p_{-1}^{} \epsilon_1 f_{1,0}) = -1. \]

On the other hand, since \(-e\) is not enclosed by the image of \(\tilde{R}(e + \frac{\epsilon}{2})\), \(p_{-1}^{} \epsilon_1 f_{2,1}(x) = N(\tilde{R}(e + \frac{\epsilon}{2}) + e)\) is not surjective. Therefore
\[ \deg(p_{-1}^{} \epsilon_1 f_{2,1}) = 0. \]

Finally,
\[ p_{-1}^{} \epsilon_1 f_{2,-1}(x) = N(\tilde{R}(-e + \frac{\epsilon}{2}) + e) \simeq N(-e + \frac{F(-,x)}{2} + \| -e + \frac{\epsilon}{2} \|^2 e) \simeq x, \]
where the first homotopy is given by \(N(-e + \frac{F(t,x)}{2}) + \| -e + \frac{\epsilon}{2} \|^2 e)\) with \(t \in [-1, 1]\), and the second one by \(N(\frac{F(-,x)}{2} + t(-1 + \| -e + \frac{\epsilon}{2} \|^2 e))\), with \(t \in [0, 1]\). We can show that these homotopies are well defined in a similar fashion to the previous case. Therefore,
\[ \deg(p_{-1}^{} \epsilon_1 f_{2,-1}) = 1, \]
And we conclude that \(\epsilon_1 A_{2,-1} = -A_{1,0} - A_{2,0} + A_{2,-1}\).

From now on we will just record the results of the computations, without writing out the details, for these computations are entirely analogous to the computation of the action of \(\epsilon_1\).

Next we consider \(\epsilon_2\).

1. \(\epsilon_2 A_{1,0}\): We have
\[ p_{1,0}^{} \epsilon_2 f_{1,0}(x) = -x, \quad p_{1,0}^{} \epsilon_2 f_{2,0}(x) = -e, \quad p_{1,0}^{} \epsilon_2 f_{2,1}(x) = -e, \quad p_{1,0}^{} \epsilon_2 f_{2,-1}(x) = -e, \]
therefore
\[ \deg(p_{1,0}^{} \epsilon_2 f_{1,0}) = (-1)^n, \quad \deg(p_{1,0}^{} \epsilon_2 f_{2,0}) = 0, \quad \deg(p_{1,0}^{} \epsilon_2 f_{2,1}) = 0, \quad \deg(p_{1,0}^{} \epsilon_2 f_{2,-1}) = 0. \]

Thus, \(\epsilon_2 A_{1,0} = (-1)^n A_{1,0}\).

2. \(\epsilon_2 A_{2,0}\): We have
\[
\begin{align*}
P_{2,0}^{} & \epsilon_2 f_{1,0}(x) = e, \\
P_{2,0}^{} & \epsilon_2 f_{2,0}(x) = N(\frac{\epsilon}{2}) = x, \\
P_{2,0}^{} & \epsilon_2 f_{2,1}(x) = N(e + \frac{\epsilon}{2}) \simeq 0, \\
P_{2,0}^{} & \epsilon_2 f_{2,-1}(x) = N(-e + \frac{\epsilon}{2}) \simeq 0;
\end{align*}
\]
therefore
\[ \deg(p_{2,0}^{} \epsilon_2 f_{1,0}) = 0, \quad \deg(p_{2,0}^{} \epsilon_2 f_{2,0}) = 1, \quad \deg(p_{2,0}^{} \epsilon_2 f_{2,1}) = 0, \quad \deg(p_{2,0}^{} \epsilon_2 f_{2,-1}) = 0. \]

Thus, \(\epsilon_2 A_{2,0} = A_{2,0}\).

3. \(\epsilon_2 A_{2,1}\): We have
\[
\begin{align*}
P_{2,1}^{} & \epsilon_2 f_{1,0}(x) = N(2e + x) \simeq 0, \\
P_{2,1}^{} & \epsilon_2 f_{2,0}(x) = N(\frac{\epsilon}{2} + e) \simeq 0, \\
P_{2,1}^{} & \epsilon_2 f_{2,1}(x) = N(e + \frac{\epsilon}{2} + e) \simeq 0, \\
P_{2,1}^{} & \epsilon_2 f_{2,-1}(x) = N(-e + \frac{\epsilon}{2} + e) = N(\frac{\epsilon}{2}) = x;
\end{align*}
\]
therefore
\[ \deg(p_{2,1}^{} \epsilon_2 f_{1,0}) = 0, \quad \deg(p_{2,1}^{} \epsilon_2 f_{2,0}) = 0, \quad \deg(p_{2,1}^{} \epsilon_2 f_{2,1}) = 0, \quad \deg(p_{2,1}^{} \epsilon_2 f_{2,-1}) = 1. \]
Thus $\epsilon_2 A_{2,1} = A_{2,-1}$.

(4) $\epsilon_2 A_{2,-1}$: Note that $\epsilon_2^2 = \text{identity}$. Application of the previous case yields $\epsilon_2 A_{2,-1} = A_{2,1}$.

Next, $\epsilon_3$.

(1) $\epsilon_3 A_{1,0}$: We have
$$p_{1,0} \epsilon_3 f_{1,0}(x) = x, \quad p_{1,0} \epsilon_3 f_{2,0}(x) = e, \quad p_{1,0} \epsilon_3 f_{2,1}(x) = e, \quad p_{1,0} \epsilon_3 f_{2,-1}(x) = e;$$
therefore
$$\deg(p_{1,0} \epsilon_3 f_{1,0}) = 1, \quad \deg(p_{1,0} \epsilon_3 f_{2,0}) = 0, \quad \deg(p_{1,0} \epsilon_3 f_{2,1}) = 0, \quad \deg(p_{1,0} \epsilon_3 f_{2,-1}) = 0.$$
Thus, $\epsilon_3 A_{1,0} = A_{1,0}$.

(2) $\epsilon_3 A_{2,0}$: We have
$$\begin{align*}
p_{2,0} \epsilon_3 f_{1,0}(x) &= N(-\frac{e}{2}), \\
p_{2,0} \epsilon_3 f_{2,0}(x) &= N(\tau(\frac{e}{2})) = -x, \\
p_{2,0} \epsilon_3 f_{2,1}(x) &= N(\tau(e + \frac{e}{2})) \cong 0, \\
p_{2,0} \epsilon_3 f_{2,-1}(x) &= N(\tau(-e + \frac{e}{2})) \cong 0;
\end{align*}$$
therefore
$$\deg(p_{2,0} \epsilon_3 f_{1,0}) = 0, \quad \deg(p_{2,0} \epsilon_3 f_{2,0}) = (-1)^n, \quad \deg(p_{2,0} \epsilon_3 f_{2,1}) = 0, \quad \deg(p_{2,0} \epsilon_3 f_{2,-1}) = 0.$$
Thus, $\epsilon_3 A_{2,0} = (-1)^n A_{2,0}$.

(3) $\epsilon_3 A_{2,1}$: We have
$$\begin{align*}
p_{2,1} \epsilon_3 f_{1,0}(x) &= N(-\frac{e}{2} - x) = -N(\frac{e}{2} + x) \cong -x, \\
p_{2,1} \epsilon_3 f_{2,0}(x) &= N(-2x - e) = -N(2x + e) \cong -x, \\
p_{2,1} \epsilon_3 f_{2,1}(x) &= N(\tau(e + \frac{e}{2}) - e) \cong 0,
\end{align*}$$
therefore
$$\deg(p_{2,1} \epsilon_3 f_{1,0}) = (-1)^n, \quad \deg(p_{2,1} \epsilon_3 f_{2,0}) = (-1)^n, \quad \deg(p_{2,1} \epsilon_3 f_{2,1}) = 0.$$
Lastly,
$$\begin{align*}
p_{2,1} \epsilon_3 f_{2,-1}(x) &= N(\tau(-e + \frac{e}{2}) - e) \cong N(e + \frac{F(-1,-x)}{2} - \|e + x\|^2 e) \\
&\cong N\left(\frac{F(-1,-x)}{2}\right) = -F(-1,x),
\end{align*}$$
where the first homotopy is given by $N(e + \frac{F(t,-x)}{2} - \|e + x\|^2 e)$ with $t \in [-1,1]$, and
the second one is given by $N\left(\frac{F(-1,-x)}{2} + t(1 - \|e + x\|^2 e)\right)$, with $t \in [0,1]$. Therefore
$$\deg(p_{2,1} \epsilon_3 f_{2,-1}) = (-1)^{n-1}.$$
And we conclude $\epsilon_3 A_{2,1} = (-1)^n A_{1,0} + (-1)^n A_{2,0} + (-1)^{n-1} A_{2,-1}$.

(4) $\epsilon_3 A_{2,-1}$: Note that $\epsilon_3^2 = \text{identity}$. By our previous computations,
$$A_{2,1} = \epsilon_3(\epsilon_3 A_{2,1}) = \epsilon_3((-1)^n A_{1,0} + (-1)^n A_{2,0} + (-1)^{n-1} A_{2,-1}) = (-1)^n A_{1,0} + A_{2,0} + (-1)^{n-1} \epsilon_3 A_{2,-1}.$$
Therefore $\epsilon_3 A_{2,-1} = A_{1,0} + (-1)^n A_{2,0} + (-1)^{n-1} A_{2,1}$.
Remark 3.6. Theorems 3.1 and 3.2 correct results in [9]. The situation is closely related to our discussion, in Remark 2.4, of the existence of inconsistencies with the determination in [9] of a presentation for the cohomology ring of the fiber and base spaces in (2). As described next, the problem can be traced back to the description in [9] Lemma 7 of the action of the various $\epsilon_i$ on cohomology rings. To simplify the explanation, once again we adopt momentarily Feichtner-Ziegler’s notation in [9]—which the reader is assumed to be familiar with. The proof of Lemma 7(iv) in [9] is based on the asserted equality $(A_2 \circ A_1)^* (c_{1,2}^+) = (-1)^k c_{1,2}^+$ whose proof, in turn, is reduced to showing that the obvious map

\[(17) \quad A_2 \circ A_1 : \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) \rightarrow \mathcal{M}(\{U_1, U_2, U_{1,2}^+\})\]

satisfies

\[(18) \quad (A_2 \circ A_1)^* (\tilde{c}_{1,2}) = (-1)^k \tilde{c}_{1,2}.\]

(Note that (17) is not to be understood as a composition of maps from $\mathcal{M}(\{U_1, U_2, U_{1,2}^+\})$ to itself.) Feichtner-Ziegler’s argument for (18) then proceeds by considering the central sphere $S$ (of radius $\sqrt{2}$) in $\{U_{1,2}^+ \setminus \{0\}$ which retracts from $\mathcal{M}(\{U_1, U_2, U_{1,2}^+\})$ (with retraction $p$). It is observed that

\[(19) \quad \text{(17) restricts on } S \text{ as the antipodal map}\]

and, from this, (18) is concluded. But such a conclusion is clearly flawed: The assertion in (19) is right, and gives the (strict) commutativity of the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\text{antipodal}} & \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) \\
\downarrow & & \downarrow (A_2 \circ A_1) \\
S & \xrightarrow{\text{antipodal}} & \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}).
\end{array}
\]

But (18) cannot be drawn from this, since the map induced in cohomology by the inclusion $S \hookrightarrow \mathcal{M}(\{U_1, U_2, U_{1,2}^+\})$ has a nontrivial kernel. Indeed, $H^{k-1}(\mathcal{M}(\{U_1, U_2, U_{1,2}^+\}))$ is free of rank 3, while $H^{k-1}(S)$ is free of rank 1. Instead, what would certainly give (18) is the existence of a commutative diagram (at least up to homotopy)

\[
\begin{array}{ccc}
\mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) & \xrightarrow{p} & S \\
\downarrow (A_2 \circ A_1) & & \downarrow \text{antipodal} \\
\mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) & \xrightarrow{p} & S.
\end{array}
\]

But (18) is false according to Theorem 3.7, so that such a diagram is impossible.

4. $(\mathbb{Z}_2)^k$-INVARIANTS

In this section $R$ will denote a commutative ring with unit where 2 is invertible, and $n$ will be an integer greater than or equal to 2. First we will compute the $(\mathbb{Z}_2)^k$-invariants in $H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))$ starting with the case $n$ odd, assumption that will be in force until Theorem 4.7.
For $0 < i < k$ we let $C_{i,0}^+$ stand for $A_{i,0}$, and for $0 < j < i < k$ we define
\[
C_{i,j}^+ = A_{i,j} + A_{i,-j} - A_{i,0},
\]
\[
C_{i,j}^- = -A_{i,j} + A_{i,-j} - A_{j,0}.
\]

For ease of notation, for a positive $j$ we will also use the notation $C_{i,j}$ and $C_{i,-j}$ to stand respectively for $C_{i,j}^+$ and $C_{i,j}^-$. Put $C^+ = \{C_{i,j}^+ | 1 \leq j < i \leq k\}$, $C^- = \{C_{i,j}^- | 1 \leq j < i < k\}$, $C_0 = \{C_{i,0} | 1 \leq i < k\}$ and $C = C^+ \cup C^- \cup C_0$. Clearly, $C$ is a basis for $H^{n-1}(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ with inverse change of basis map given by
\[
A_{i,j} = \frac{C_{i,j}^+ - C_{i,j}^- + C_{i,0} - C_{j,0}}{2},
\]
\[
A_{i,-j} = \frac{C_{i,j}^+ + C_{i,j}^- + C_{i,0} + C_{j,0}}{2},
\]
\[
A_{i,0} = C_{i,0},
\]
for $0 < j < i < k$. These formulas make it clear that $H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ is additively generated by the products
\[
C_{i_{1,j_1}} \cdots C_{i_{r,j_r}}
\]
with $|j_l| < i_l < k$ for $l = 1, \ldots, r$. Our first goal is to show that, in fact, an additive basis is formed by such products that satisfy in addition
\[
(20) \quad i_l < i_{l'} \text{ if } l < l'.
\]

Example 4.1. For $n \geq 2$ odd, the multiplicative relations among the $A_{i,j}$’s yield
\[
C_{3,2}C_{3,0} = -A_{2,0}A_{3,0} + A_{3,-2}A_{3,0} - A_{3,0}A_{3,2}
\]
\[
= -A_{2,0}A_{3,-2} + A_{2,0}A_{3,0} - A_{2,0}A_{3,2}
\]
\[
= -C_{3,2}C_{2,0}.
\]

Example 4.2. For odd $n$, the multiplicative relations among the $A_{i,j}$’s yield
\[
C_{4,3}C_{4,2} = -A_{2,0}A_{4,-3} + A_{4,-3}A_{4,-2} + A_{2,0}A_{4,0} - A_{4,-2}A_{4,0} - A_{4,-3}A_{4,2} + A_{4,0}A_{4,2}
\]
\[
- A_{2,0}A_{4,3} + A_{4,-2}A_{4,3} - A_{4,2}A_{4,3}
\]
\[
= A_{2,0}A_{4,-3} - A_{3,-2}A_{4,-3} + A_{3,2}A_{4,-3} - A_{4,-2}A_{4,-2} + A_{3,0}A_{4,-2} - A_{3,2}A_{4,-2}
\]
\[
- A_{2,0}A_{4,0} + A_{3,-2}A_{4,2} - A_{3,0}A_{4,2} + A_{3,2}A_{4,2} - A_{2,0}A_{4,3} + A_{3,-2}A_{4,3} - A_{3,2}A_{4,3}
\]
\[
= -C_{4,3}C_{4,2} - C_{4,2}C_{4,3}^+ - C_{3,2}C_{3,2} - C_{3,2}C_{3,0} - C_{2,0}C_{4,0}.
\]

Therefore, by the previous example,
\[
C_{4,3}C_{4,2} = -C_{3,2}C_{4,3} - C_{4,2}C_{3,2} - C_{2,0}C_{4,0}.
\]

Note that the set of products in (20) satisfying (21) is in bijective correspondence with the basis described just before Theorem 2.1. Using Nakayama’s lemma we see that the former set will be in fact an additive basis of $H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}), k-1)$ as long as it additively generates. In turn, the latter condition follows directly from the fact that the products $A_{i_{1,j_1}} \cdots A_{i_{r,j_r}}$ satisfying the condition (21) form an additive basis, from the explicit form of the relations expressing the $A_{i,j}$’s in terms of the $C_{i,j}$’s, and from the relations in item 2 of Lemma 4.3 below—which generalizes the calculation illustrated in Example 4.1. The proof of the lemma is straightforward and left to the reader.
Lemma 4.3. For $n \geq 2$ odd, the elements of $C$ satisfy the following multiplicative relations:

1. For $0 < j < i < r < k$,
   \[
   C^{+}_{r,i}C^{+}_{r,j} = -C^{+}_{i,j}C^{+}_{r,i} + C^{+}_{i,j}C^{+}_{r,i},
   \]
   \[
   C^{-}_{r,i}C^{-}_{r,j} = -C^{-}_{i,j}C^{-}_{r,i} - C^{-}_{i,j}C^{-}_{r,i} - C^{-}_{i,j}C^{-}_{r,i} - C^{-}_{i,j}C^{-}_{r,i}.
   \]

2. For $0 < i < r < k$,
   \[
   C^{+}_{r,i}C_{r,0} = -C_{i,0}C^{-}_{r,i},
   \]
   \[
   C^{-}_{r,i}C_{r,0} = -C_{i,0}C^{-}_{r,i}.
   \]

3. For $0 \leq j < i < k$,
   \[
   (C^{+}_{i,j})^2 = 0,
   \]
   \[
   (C^{-}_{i,j})^2 = 0,
   \]
   \[
   C^{+}_{i,j}C^{-}_{i,j} = -C_{j,0}C_{i,0}.
   \]

The advantage of using $C$ over $A$ to compute invariants becomes apparent when describing the action of $(\mathbb{Z}_2)^k$ as a straightforward verification yields

\[
\epsilon_l C^{\pm}_{ij} = C^{\pm}_{ij} \text{ for all } 0 < j < i \text{ and all } l,
\]

\[
\epsilon_l C^{-}_{ij} = \begin{cases} -C^{-}_{ij}, & \text{if } i = l - 1 \text{ or } j = l - 1; \\
C^{-}_{ij}, & \text{otherwise}, \end{cases}
\]

\[
\epsilon_l C_{i,0} = \begin{cases} -C_{i,0}, & \text{if } i = l - 1 \text{ or } l = 1; \\
C_{i,0}, & \text{otherwise}. \end{cases}
\]

Theorem 4.4. Suppose $R$ is a commutative ring with unit where 2 is invertible. For $n \geq 2$ odd, the $(\mathbb{Z}_2)^k$-invariants in $H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))$ are multiplicatively generated by the set $C^+$. In fact, and additive basis of the invariants is formed by the products (20) satisfying (21) and $j_l > 0$ for $l = 1, \ldots, r$.

Proof. Let $x \in H^{m(n-1)}(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k - 1))$ be an invariant. We will show that each of the basis elements appearing with a nontrivial coefficient in the expression of $x$ as a linear combination of the basis of products (20) satisfying (21) has no factors belonging to $C^-$ or $C_0$. Write

\[
x = \sum a_{I}C_{i_1,j_1} \cdots C_{i_m,j_m},
\]

where each coefficient $a_I$ is non-zero and the summation runs over some multi-indices $I = ((i_1, j_1), \ldots, (i_m, j_m))$ such that $|j_l| < i_l$ and $i_l < j_l$ if $l < l'$. Note that given our description of the $(\mathbb{Z}_2)^k$-action on $C$, each monomial $C_{i_1,j_1} \cdots C_{i_m,j_m}$ is sent to a multiple of itself under the action of any element in $(\mathbb{Z}_2)^k$. Since 2 is invertible, this means that each term $C_{i_1,j_1} \cdots C_{i_m,j_m}$ appearing in (22) is invariant. Fix $I$, and consider the corresponding invariant monomial $z = C_{i_1,j_1} \cdots C_{i_m,j_m}$. Suppose that the set of integers $i$ such that we have a factor of the form $C_{i,j}^-$ in $z$ is non-empty, and let $i_0$ be the greatest element of this set. By applying $\epsilon_{i_0+1}$
to $z$ we get that $-z = \epsilon_{i_0 + 1}z = z$, which is a contradiction, so $z$ has no factors belonging to $C^-$. An entirely analogous argument shows that there are no factors belonging to $C_0$ in $z$ either. \hfill \Box

**Theorem 4.5.** Suppose $R$ is a commutative ring with unit where $2$ is invertible. For $n \geq 2$ odd, there is an $R$-algebra isomorphism

$$H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))^{(\mathbb{Z}_2)^k} \cong R[C^+]/K,$$

where $K$ is the ideal generated by the elements $C_{i,j}^+$, $C_{r,i}^+ C_{r,j}^+ - C_{i,j}^+(C_{r,i}^+ - C_{r,j}^+)$ for $0 < j < i < r < k$.

**Proof.** Lemma 4.3 gives an obvious ring map (with domain in $R[C^+]/K$). This is an isomorphism since it sets a bijective correspondence between the basis described in Theorem 4.4 and the usual basis in the domain. \hfill \Box

**Remark 4.6.** Note that the second relation in the preceding Theorem is identical to the known relation (1). In particular, the cohomology ring described in Theorem 4.5 is isomorphic to the cohomology ring of the standard configuration space of $k-1$ ordered points in $\mathbb{R}^n$.

Since the canonical projection $S^n \to \mathbb{P}^n$ induces a $(\mathbb{Z}_2)^k$ covering space $\text{Conf}_{\mathbb{Z}_2}(S^n, k) \to \text{Conf}(\mathbb{P}^n, k)$, Theorem 2.2, the fact that the $(\mathbb{Z}_2)^k$-action on $H^*(S^n)$ is trivial for odd $n$, and the preceding theorem imply the following result:

**Theorem 4.7.** Suppose $R$ is a commutative ring with unit where $2$ is invertible. For $n \geq 2$ odd, there is an $R$-algebra isomorphism

$$H^*(\text{Conf}(\mathbb{P}^n, k)) = H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k))^{(\mathbb{Z}_2)^k} \cong \Lambda(\epsilon_n) \otimes R[C^+]/K.$$

Next, we describe $(\mathbb{Z}_2)^k$-invariant permanent cycles in $K^* \subseteq H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ for $n$ even, assumption that will be in force throughout the rest of the section. For $0 < j < i < k$ define

$$D_{i,j}^+ = B_{i,j} + B_{i,-j} - B_{i,0} - B_{j,0},$$

$$D_{i,j}^- = B_{i,j} - B_{i,-j},$$

$$D_{i,0} = B_{i,0},$$

and, for $|j| < i$ with $i \geq 2$, let

$$D_{i,j} = \begin{cases} 
D_{i,j}^+ & \text{if } j > 0; \\
D_{i,|j|}^- & \text{if } j < 0; \\
D_{i,0} & \text{if } j = 0.
\end{cases}$$

(Recall $B_{1,0} = 0$.) Put $D = \{D_{i,j}^+ | 0 < j < i < k\} \cup \{D_{i,j}^- | 0 < j < i < k\} \cup \{D_{i,0} | 1 < i < k\}$. With this notation, the action described in (16) takes the form

$$\epsilon_1 D_{i,j}^+ = \begin{cases} 
-D_{i,j}^+, & \text{if } i = l - 1; \\
D_{i,j}^+, & \text{otherwise},
\end{cases}$$

$$\epsilon_1 D_{i,j}^- = \begin{cases} 
-D_{i,j}^-, & \text{if } j = l - 1; \\
D_{i,j}^-, & \text{otherwise},
\end{cases}$$

20
and
\[ \epsilon_l D_{i,0} = \begin{cases} -D_{i,0}, & \text{if } l = 1; \\ D_{i,0}, & \text{otherwise}. \end{cases} \]

Clearly, \( \mathcal{D} \) forms a basis for \( \mathbb{K} \) with inverse change of basis given by
\[
B_{i,j} = \frac{D_{i,j}^+ + D_{i,j}^- + D_{i,0} + D_{j,0}}{2},
\]
\[
B_{i,-j} = \frac{D_{i,j}^+ - D_{i,j}^- + D_{i,0} + D_{j,0}}{2},
\]
\[
B_{i,0} = D_{i,0},
\]
for \( 0 < j < i < k \). We leave to the reader the verification of the following multiplicative relations among the elements of \( \mathcal{D} \):

**Lemma 4.8.** Let \( R \) be a commutative ring with unit where 2 is invertible. Suppose \( n \geq 2 \) even. The elements of \( \mathcal{D} \) satisfy the following multiplicative relations:

1. For \( 0 < j < i < r < k \),
   \[
   D_{r,i}^+ D_{r,j}^+ = D_{i,j}^- D_{r,j}^- - D_{i,j}^- D_{r,i}^- - D_{j,0} D_{i,0} + D_{j,0} D_{r,0} - D_{i,0} D_{r,0},
   \]
   \[
   D_{r,i}^- D_{r,j}^- = D_{i,j}^+ (D_{r,j}^+ - D_{r,i}^+),
   \]
   \[
   D_{r,i}^- D_{r,j}^- = D_{i,j}^+ (D_{r,j}^- - D_{r,i}^-),
   \]
   \[
   D_{r,i}^- D_{r,j}^- = -D_{i,j}^- D_{r,i}^- + D_{i,j}^+ D_{r,j}^-.
   \]

2. For \( 0 < i < r < k \),
   \[
   D_{r,i}^+ D_{r,0} = -D_{i,0} D_{r,i}^+,
   \]
   \[
   D_{r,i}^- D_{r,0} = -D_{i,0} D_{r,i}^-.
   \]

3. For \( 0 < j < i < k \),
   \[
   (D_{i,j}^+)^2 = 0,
   \]
   \[
   (D_{i,j}^-)^2 = 0,
   \]
   \[
   (D_{i,0})^2 = 0,
   \]
   \[
   D_{i,j}^+ D_{i,j}^- = 0.
   \]

By repeated applications of Lemma 4.8 we see that \( \mathbb{K}^* \) is additively generated by products of the form
\[
D_{i_1,j_1} \cdots D_{i_r,j_r},
\]
where
\[
i_l < i_{l'} \text{ if } l < l'.
\]

The set of these products is in bijective correspondence with the basis consisting of products \( B_{i_1,j_1} \cdots B_{i_r,j_r} \) satisfying condition (24), and so by Nakayama’s lemma the set of products of the form (23) satisfying (24) is an additive basis of the permanent cycles in \( H^*(\text{Conf}_{22}(\mathbb{R}^n - \{0\}, k - 1)) \).
Remark 4.9. The previous discussion and our description of \( \epsilon_l(D_{i,j}^\pm) \) easily yield that the map \( h^{x(k-1)} \) in Lemma 3.3 acts on the permanent cycles in \( \mathbb{K}^m \) as multiplication by \( (-1)^m \).

Next we define elements which are clearly \( (\mathbb{Z}_2)^k \)-invariants; in fact we will show in Theorem 4.12 below that they are multiplicative generators for all \( (\mathbb{Z}_2)^k \)-invariants. For \( 0 < j < i < r < k \), put

\[
I_{r,i,j}^+ = D_{i,j}^+ D_{r,i}^-,
\]
\[
I_{r,i,j}^- = D_{i,j}^- D_{r,j}^-,
\]

and for \( 1 < j < i < k \) put

\[
I_{i,j,0} = D_{j,0} D_{i,0}.
\]

For \( j > 0 \), we will sometimes write \( I_{r,i,j} \) and \( I_{r,i,-j} \) instead of \( I_{r,i,j}^+ \) and \( I_{r,i,j}^- \) respectively.

Accordingly, we will sometimes write \( I_{r,i,j,0}^+ \) or even \( I_{r,i,j,0}^- \) as a substitute for \( I_{i,j,0} \). Let \( \mathcal{E}^+ = \{ I_{r,i,j}^+ \mid 0 < j < i < r < k \} \), \( \mathcal{E}^- = \{ I_{r,i,j}^- \mid 0 < j < i < r < k \} \), \( \mathcal{E}_0 = \{ I_{i,j,0} \mid 1 < j < i < k \} \) and \( \mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- \cup \mathcal{E}_0 \). We leave to the reader the verification of the following result:

Lemma 4.10. Let \( R \) be a commutative ring with unit where \( 2 \) is invertible. Suppose \( n \geq 2 \) even. The elements of \( \mathcal{E} \) satisfy the relations listed below. Relations [a] through [c] express a product \( I_{r,i,j}^\pm J_{s,a,b}^\pm \) with

\[
0 \leq j < i < r < k, \quad 0 \leq b < a < s < k, \quad 2 \leq i, \quad 2 \leq a, \quad \text{and} \quad r \leq s
\]
as a linear combination of such products satisfying in addition

\[
r < s \quad \text{and} \quad a \not\in \{r, i\}.
\]

Those relations are listed according to the several possibilities for the indices \( r, i, s, a, \) and \( b \) when they satisfy (22) but not (26).

(a) \( r = s \)

(a.a) \( j \neq 0 \neq b \)

(a.a.a) \( \mid\{i, j, a, b\} = 4 \) (can assume \( a < i \))

(a.a.a.a) \( b < a < j \)

\[
I_{r,i,j}^+ I_{r,a,b}^+ = I_{j,a,b}^+ (I_{r,i,a}^- - I_{r,i,a}^+ + I_{r,i,j}^+) + (I_{i,b,0} - I_{i,j,a} - I_{i,j,0} - I_{j,b,0}) I_{r,a,b}^+,
\]
\[
I_{r,i,j}^- I_{r,a,b}^- = I_{j,a,b}^- I_{r,i,j}^- + I_{r,a,b}^-,
\]
\[
I_{r,i,j}^+ I_{r,a,b}^- = I_{j,a,b}^+ (I_{r,i,j}^+ - I_{r,i,b}) + (I_{i,j,b} - I_{i,j,b}^+ - I_{j,b,0} + I_{b,0} - I_{i,j,0}) I_{r,a,b}^-,
\]
\[
I_{r,i,j}^- I_{r,a,b}^+ = I_{j,a,b}^- I_{r,i,a}^- - I_{r,a,b}^+ I_{r,i,j}^-,
\]

(a.a.a.a.b) \( b < j < a \)

\[
I_{r,i,j}^+ I_{r,a,b}^+ = (I_{a,j,b} - I_{a,j,b}^+ + I_{a,b,0} - I_{a,j,0} - I_{j,b,0}) (I_{r,i,a}^- - I_{r,i,a}^+ - I_{r,i,j}^+),
\]
\[
+ I_{i,j,b}^+ (I_{r,a,b}^- - I_{r,a,b}^+) + (I_{i,b,0} - I_{i,j,0} - I_{j,b,0}) I_{r,a,b}^+,
\]
\[
I_{r,i,j}^- I_{r,a,b}^- = -I_{a,j,b}^+ (I_{r,i,a}^- - I_{r,i,a}^+ - I_{r,i,j}^+) I_{r,a,b}^+ - I_{r,a,b}^- I_{r,i,j}^+.
\]

(b) \( i = j \)
(a.a.a.c) \( j < b < a \)

\[
I_{r,i,j}^+ I_{r,a,b}^- = (I_{a,b,j}^+ - I_{a,b,j}^- - I_{a,j,0}^- + I_{a,b,0}^- + I_{b,j,0}^-)(I_{r,i,a}^+ - I_{r,i,a}^- - I_{r,i,j}^-) + I_{i,b,j}^-(I_{r,a,j}^+ - I_{r,a,j}^-) + (I_{i,b,0}^- - I_{i,j,0}^- + I_{b,j,0}^-)I_{r,a,b}^-
\]

\[
I_{r,i,j}^- I_{r,a,b}^- = -I_{i,b,j}^- I_{r,a,b}^- - I_{a,b,j}^+ I_{r,i,j}^- + I_{i,b,j}^+(I_{r,a,b}^+ - I_{r,a,b}^-) + (I_{i,j,0}^- - I_{i,b,0}^- + I_{b,j,0}^-)I_{r,a,b}^-
\]

\[
I_{r,i,j}^- I_{r,a,b}^+ = I_{i,b,j}^+(I_{r,a,j}^+ - I_{r,a,j}^-) + (I_{a,b,j}^- - I_{b,j,0}^- + I_{a,j,0}^- - I_{a,b,0}^-)I_{r,i,j}^-.
\]

(\text{a.a.b}) \(|\{i, j, a, b\}| = 3\) (can assume \(a \leq i\))

\[I_{r,i,j}^\pm I_{r,i,b}^\pm = 0.\]

(\text{a.a.b.a}) \(a = i\) (can assume \(b < j\))

\[I_{r,i,j}^\pm I_{r,i,b}^\pm = 0.\]

(\text{a.a.b.b}) \(a = j\)

\[I_{r,i,j}^\pm I_{r,j,b}^\pm = 0.\]

(\text{a.a.b.c}) \(b = i\) is impossible.

(\text{a.a.b.d}) \(b = j\)

\[I_{r,i,j}^\pm I_{r,a,j}^\pm = 0.\]

(\text{a.a.c}) \(|\{i, j, a, b\}| = 2\)

\[I_{r,i,j}^\pm I_{r,a,j}^\pm = 0.\]

(\text{a.b}) \(j = 0 \neq b\) (the case \(j \neq 0 = b\) is symmetric)

(\text{a.b.a}) \(|\{i, a, b\}| = 3\)

(\text{a.b.a.a}) \(i < b < a\)

\[
I_{r,i,0} I_{r,a,b}^+ = I_{b,i,0} I_{r,a,b}^+,
\]

\[
I_{r,i,0} I_{r,a,b}^- = I_{b,i,0} I_{r,a,b}^-.
\]

(\text{a.b.a.b}) \(b < i < a\) or \(b < a < i\)

\[
I_{r,i,0} I_{r,a,b}^+ = -I_{b,i,0} I_{r,a,b}^+,
\]

\[
I_{r,i,0} I_{r,a,b}^- = -I_{b,i,0} I_{r,a,b}^-.
\]

(\text{a.b.b}) \(|\{i, a, b\}| = 2\)

(\text{a.b.b.a}) \(a = i\)

\[I_{r,i,0} I_{r,i,b}^\pm = 0.\]

(\text{a.b.b.b}) \(b = i\)

\[I_{r,i,0} I_{r,a,i}^\pm = 0.\]

(\text{a.c}) \(j = 0 = b\) (can assume \(a \leq i\))

\[I_{r,i,0} I_{r,a,j} = 0.\]

(b) \(a = r < s\)

(\text{b.a}) \(j \neq 0 \neq b\)

(\text{b.a.a}) \(|\{i, j, b\}| = 3\)

\[\text{23}\]
(b.a.a.a) $j < i < b$
\[
\begin{align*}
I_{r,i,j}^+ I_{s,r,b}^+ &= I_{b,i,j}^+ (I_{s,r,b}^+ - I_{s,r,i}^+),
I_{r,i,j}^- I_{s,r,b}^- &= I_{b,i,j}^- I_{s,r,b}^- + I_{r,i,j}^+ I_{s,r,b}^+,
I_{r,i,j}^+ I_{s,r,b}^- &= I_{b,i,j}^- I_{s,r,b}^- + I_{r,i,j}^+ I_{s,b,i}^-,
I_{r,i,j}^- I_{s,r,b}^+ &= I_{b,i,j}^+ (I_{s,r,b}^- - I_{s,r,j}^-).
\end{align*}
\]

(b.a.a.b) $j < b < i$
\[
\begin{align*}
I_{r,i,j}^+ I_{s,r,b}^+ &= (I_{i,b,j}^- - I_{i,b,j}^+ - I_{b,j,0} + I_{i,j,0} - I_{i,b,0}) (I_{s,r,i}^+ - I_{s,r,b}^+),
I_{r,i,j}^- I_{s,r,b}^- &= I_{r,i,j}^+ I_{s,b,j}^+ - I_{r,i,j} I_{s,r,b}^+,
I_{r,i,j}^+ I_{s,r,b}^- &= (I_{r,i,j}^- - I_{r,i,b}^+ I_{s,b,j}^+ + (-I_{i,b,j}^- + I_{i,b,j}^+ + I_{b,j,0} - I_{i,j,0} + I_{i,b,0}) I_{s,r,b}^-,
I_{r,i,j}^- I_{s,r,b}^+ &= I_{i,b,j}^- (I_{s,r,j}^+ - I_{s,r,b}^+).
\end{align*}
\]

(b.a.a.c) $b < j < i$
\[
\begin{align*}
I_{r,i,j}^+ I_{s,r,b}^+ &= (I_{i,j,b}^- - I_{i,j,b}^+ - I_{j,b,0} + I_{i,j,0} - I_{i,b,0}) (I_{s,r,b}^+ - I_{s,r,i}^+),
I_{r,i,j}^- I_{s,r,b}^- &= I_{r,i,j}^+ I_{s,j,b}^+ - I_{r,i,j} I_{s,r,b}^+,
I_{r,i,j}^+ I_{s,r,b}^- &= (I_{r,i,j}^- - I_{r,i,b}^+ I_{s,j,b}^+ + (-I_{i,j,b}^- + I_{i,j,b}^+ - I_{j,b,0} + I_{i,j,0} - I_{i,b,0}) I_{s,r,b}^-,
I_{r,i,j}^- I_{s,r,b}^+ &= I_{i,j,b}^- (I_{s,r,j}^+ - I_{s,r,b}^+).
\end{align*}
\]

(b.a.b) $|\{i, j, b\}| = 2$
\[
(b.a.b.a) b = i
\]
\[
I_{r,i,j}^+ I_{s,r,i}^+ = 0.
\]

(b.a.b.b) $b = j$
\[
I_{r,i,j}^+ I_{s,r,j}^+ = 0.
\]

(b.b) $j = 0 \neq b$
\[
(b.b.a) |\{i, b\}| = 2
\]
\[
(b.b.a.a) i < b
\]
\[
I_{r,i,j}^+ I_{s,r,b}^+ = I_{b,i,j}^+ I_{s,r,b}^+,
I_{r,i,j}^- I_{s,r,b}^- = I_{b,i,j}^- I_{s,r,b}^-.
\]

(b.b.a.b) $b < i$
\[
I_{r,i,j}^+ I_{s,r,b}^+ = -I_{b,i,j}^+ I_{s,r,b}^+,
I_{r,i,j}^- I_{s,r,b}^- = -I_{b,i,j}^- I_{s,r,b}^-.
\]

(b.b.b) $|\{i, b\}| = 1$
\[
I_{r,i,j}^+ I_{s,r,i}^+ = 0.
\]

(b.c) $j \neq 0 = b$
\[
I_{r,i,j}^+ I_{s,r,0}^+ = I_{r,i,j}^+ I_{s,j,0},
I_{r,i,j}^- I_{s,r,0}^- = I_{r,i,j}^- I_{s,j,0}.
\]
(b.d) \( j = 0 = b \)
\[ I_{r,0} I_{s,0} = 0. \]

(c) \( a = i < r < s \)
(c.a) \( j \neq 0 \neq b \)
(c.a.a) \( j < b \)
\[ I^+_{r,i,j} I^+_{s,i,b} = (I^-_{i,b,j} - I_{b,j,0} + I_{i,j,0} - I_{i,b,0} - I_{i,b,j}) I^-_{s,r,i}, \]
\[ I^-_{r,i,j} I^-_{s,i,b} = I^-_{r,b,j} I^-_{s,i,b} + I^+_{r,i,j} I^+_{s,b,j}, \]
\[ I^+_{r,i,j} I^-_{s,i,b} = (I^-_{r,i,j} - I^+_{r,i,b}) I^+_{s,b,j}, \]
\[ I^-_{r,i,j} I^+_{s,i,b} = I^-_{r,b,j} (I^+_{s,i,b} - I^+_{s,i,j}). \]

(c.a.a.b) \( b < j \)
\[ I^+_{r,i,j} I^+_{s,i,b} = (I^-_{i,j,b} + I^+_{i,b,j} + I_{j,b,0} - I_{i,b,0} + I_{i,j,0}) I^-_{s,r,i}, \]
\[ I^-_{r,i,j} I^-_{s,i,b} = I^-_{r,b,j} I^-_{s,i,b} + I^+_{r,i,j} I^+_{s,b,j}, \]
\[ I^+_{r,i,j} I^-_{s,i,b} = (I^+_{r,i,j} - I^{-1}_{r,i,b}) I^+_{s,b,j}, \]
\[ I^-_{r,i,j} I^+_{s,i,b} = I^+_{r,b,j} (I^+_{s,i,b} - I^+_{s,i,j}). \]

(c.b) \( j = 0 \neq b \)
\[ I_{r,0} I_{s,i,b} = I_{r,b,0} I^+_{s,i,b}, \]
\[ I_{r,0} I_{s,i,b}^- = I_{r,b,0} I^-_{s,i,b}. \]

(c.c) \( j \neq 0 \neq b \)
\[ I^+_{r,i,j} I_{s,i,0} = I^+_{r,i,j} I_{s,j,0}, \]
\[ I^+_{r,i,j} I_{s,i,0}^- = I^+_{r,i,j} I_{s,j,0}. \]

(c.d) \( j = 0 \neq b \)
\[ I_{r,i,0} I_{s,i,0} = 0. \]

(d) For \( 0 \leq j < t < s < i < r < k \),
\[ I^+_{i,s,j} I^-_{r,t,j} = I^-_{s,t,j} I^+_{r,i,j}, \]
\[ I^-_{i,t,j} I^-_{r,s,j} = -I^-_{s,t,j} I^-_{r,i,j}. \]

(e) For \( 0 < j < i < t < s < r < k \),
\[ I^-_{s,t} I^+_{r,i,j} = I^+_{t,i,j} I^-_{r,s,i}, \]
\[ I^+_{s,i} I^-_{r,t,i} = -I^-_{t,i,j} I^-_{r,s,i}. \]

Remark 4.11. Relations (d) and (e) in the previous lemma are not a consequence of the multiplicative relations among the elements in \( D \), but rather a consequence of the fact that, in some cases, there are different alternatives for associating four \( D \)'s to form a product of two \( I \)'s.
The relations in Lemma 4.10 imply that every product
\[(27)\quad I_{r_1,i_1,j_1} \cdots I_{r_m,i_m,j_m}
\] with \(|j_l| < i_l < r_l < k\) and \(1 < i_l \) for \(l = 1, \ldots, m\)
can be written as a linear combination of products of the form (27) satisfying in addition
\[(28)\quad r_i < r_{i'} \text{ if } l < l',
\[(29)\quad \text{the sets } \{i_l, r_l\}, \text{ with } 1 \leq l \leq m, \text{ are pairwise disjoint},
\[(30)\quad \text{if } j_a = j_b \leq 0, \text{ say with } r_a < r_b, \text{ then in fact } r_a < i_b,
\[(31)\quad \text{if } j_a > 0 \text{ and } i_a = -j_b, \text{ then in fact } r_a < i_b.

**Theorem 4.12.** Suppose \( R \) is a commutative ring with unit where 2 is invertible. For \( n \geq 2 \) even, the \((\mathbb{Z}_2)^k\)-invariants in \( \mathbb{K}^* \subseteq H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)) \) are multiplicatively generated by the set \( \mathcal{E} \). In fact, an additive basis for the invariants is given by all products of the form (27) satisfying (28)–(31).

**Proof.** Suppose \( m \) odd and let \( x \in \mathbb{K}^m \) be an invariant. By Remark 4.9, we have that \( x = \epsilon_1 x = -\epsilon_2 \cdots \epsilon_k x = -x \), so \( x = 0 \).

Suppose now \( m \) is even and, as above, let \( x \in \mathbb{K}^* \) be an invariant. Write
\[ x = \sum a_I D_{i_1,j_1} \cdots D_{i_m,j_m}, \]
where each \( a_I \) is non-zero and the summation runs over all multi-indices
\[ I = ((i_1, j_1), \ldots, (i_m, j_m)) \]
such that \(|j_l| < i_l \) for \( l = 1, \ldots, m \) and \( i_l < i_{l'} \) if \( l < l' \). Recall that an \( \epsilon_l \) sends each monomial 
\( D_{i_1,j_1} \cdots D_{i_m,j_m} \) to a multiple of itself, therefore, since 2 is invertible, each \( D_{i_1,j_1} \cdots D_{i_m,j_m} \)
is invariant. Fix \( I \), and consider the corresponding monomial \( z = D_{i_1,j_1} \cdots D_{i_m,j_m} \). Note that the action of \( \epsilon_l \) on \( z \) implies that an even number of factors in \( z \) are of the form \( D_{l,0} \).

Further, such factors can be matched in pairs to yield a product of the form
\[(32)\quad I_{i_1,j_l,0} I_{i_2,j_2,0} \cdots \text{ where } j_1 < i_1 < j_2 < i_2 < \cdots.\]
Likewise, for each \( l \) between 2 and \( k \), we have two possibilities:

1. There is no factor \( D_{l,-1,*} \) in \( z \) (e.g. if \( l = 2 \)). In this case, there is an even number of factors of the form \( D_{l,-1,-1} \), because otherwise we would have \( z = \epsilon_l z = -z \).
2. There is (exactly) one factor \( D_{l,-1,*} \) in \( z \). In this case, there is an odd number of factors of the form \( D_{l,-1,1} \).

The first case allows us to associate products of the form \( D_{i,j}^+ D_{r,j}^- \), and the second allows us to associate a product of the form \( D_{i,j}^+ D_{r,j}^- \) and products of the form \( D_{i,j}^- D_{r,j}^+ \). Further, just as with (32), the new matchings can be done so to yield, together with (32), a unique expression of \( D_{i_1,j_1} \cdots D_{i_m,j_m} \) as a product of the form (27) satisfying in addition (28)–(31).

The above analysis shows that the \((\mathbb{Z}_2)^k\)-invariants in \( \mathbb{K}^* \) are generated by the products of the form (27) satisfying in addition (28)–(31). In fact, this is a basis, since such generators are a subset of the additive basis of \( \mathbb{K}^* \) given by the products (23) satisfying (24). \( \square \)

We arrive at the complete description of the invariants for the case \( n \) even.
**Theorem 4.13.** Let $R$ be a commutative ring with unit where 2 is invertible. For $n$ even, there is an $R$-algebra isomorphism

$$H^*(\text{Conf}(\mathbb{RP}^n, k)) = H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k))^{(\mathbb{Z}_2)^k} \cong \Lambda(\omega) \otimes R[\mathcal{E}]/J,$$

where $J$ is the ideal generated by the relations in Lemma 4.10.

**Proof.** Since we are assuming that 2 is a unit in $R$, the isomorphism of Theorem 2.3 reduces to

$$H^*(\text{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong \Lambda(\omega) \otimes R[\mathcal{E}]/J.$$

Recall from Corollary 3.5 that $\omega$ is fixed by the action of $(\mathbb{Z}_2)^k$. The result follows from Theorem 4.12 which implies that the subring of $\mathbb{Z}_2$-invariants in the tensor factor $R[\mathcal{E}]/J$ has the presentation $R[\mathcal{E}]/J$. \qed

5. Punctured real projective spaces

In this section we maintain the assumption that $n$ is an integer greater than or equal to 2, and that $R$ will denote a commutative ring with unit where 2 is invertible. Note that the canonical projection $S^n \to \mathbb{RP}^n$ induces a $(\mathbb{Z}_2)^k$-covering space

$$\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \approx \text{Conf}_{\mathbb{Z}_2}(S^n - Q_1^{\mathbb{Z}_2}, k) \to \text{Conf}(\mathbb{RP}^n - *, k)$$

which, given that 2 is invertible in $R$, induces an isomorphism

$$H^*(\text{Conf}(\mathbb{RP}^n - *, k)) \cong H^*(\text{Conf}_{\mathbb{Z}_2}(S^n - Q_1^{\mathbb{Z}_2}, k))^{(\mathbb{Z}_2)^k} \cong H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))^{(\mathbb{Z}_2)^k}.$$

In order to reuse the notation of the previous section when computing invariants, we will consider that the group action in this section is that of the subgroup $(\mathbb{Z}_2)^k = \langle \epsilon_2, \ldots, \epsilon_{k+1} \rangle < (\mathbb{Z}_2)^{k+1}$ on $\mathbb{R}^n - \{0\} \approx S^n - Q_1^{\mathbb{Z}_2}$. We can do this because in this case each $\epsilon_i$ preserves fibers, so there is no need for the correcting rotation $R$ used at the beginning of Section 3. In practice this means that, when computing invariants, we just have to ignore the action of $\epsilon_1$. Thus Lemma 3.3 and Theorem 4.5 yield:

**Theorem 5.1.** Let $R$ be a commutative ring with unit where 2 is invertible. For $n \geq 2$ odd, there is an $R$-algebra isomorphism

$$H^*(\text{Conf}(\mathbb{RP}^n - *, k)) \cong H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))^{(\mathbb{Z}_2)^k} \cong R[C^+]/\mathcal{K}.$$

Note that the role of the parameter $k$ in Theorem 4.5 changes here to $k + 1$. For instance, the generators $C^+_{i,j}$ of $C^+$ are now defined for $0 < j < i \leq k$.

For $n$ even, we have only computed invariants in the permanent cycles $\mathbb{K}^*$, but we now have to account for all the invariants in the cohomology of $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$. Start by noticing that the considerations following Theorem 2.2 and Lemma 4.8 show that $H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ is multiplicatively generated by $A_{1,0}$ and the elements $D_{i,j}$ with $|j| < i \leq k$ subject only to the relations in Lemma 4.8 together with $A_{1,0}^2 = 0$. Further, an additive basis is given by all products of the form (23) and products of the form

$$A_{1,0}D_{i_1,j_1} \cdots D_{i_r,j_r}$$

satisfying (24).
It is natural to expect now more invariants than those found in the previous section. In fact, all the elements in the set

\[ \mathcal{E}' = \{ I_{r,i,j}^+ \mid 0 < j < i < r \leq k \} \cup \{ I_{r,i,j}^- \mid 0 < j < i < r \leq k \} \cup \{ D_{i,0} \mid 1 < i \leq k \} \cup \{ A_{1,0} \} \]

are clearly \((\mathbb{Z}_2)^k\)-invariant. Before showing these generate all other invariants, we describe their multiplicative relations. First of all, while all relations in Lemma 4.10 are clearly inherited (albeit with upper bound \(k+1\) instead of \(k\) for indices \(r, i, j\)), the relations involving terms of the form \(I_{i,j,0}\) are evidently not in the most primitive form. Instead, we have the following easy-to-check relations:

**Lemma 5.2.** Let \(R\) be a commutative ring with unit where 2 is invertible. Suppose \(n \geq 2\) even. For \(0 < j < i < r \leq k\) we have:

\[
I_{r,i,j}^+ D_{i,0} = I_{r,i,j}^+ D_{j,0},
\]

\[
I_{r,i,j}^+ D_{r,0} = I_{r,i,j}^+ D_{j,0},
\]

\[
I_{r,i,j}^- D_{i,0} = I_{r,i,j}^- D_{j,0},
\]

\[
I_{r,i,j}^- D_{r,0} = I_{r,i,j}^- D_{j,0}.
\]

Therefore, any product of the form

\[
D_{s_1,0} \cdots D_{s_{m'},0} I_{r_1,i_1,j_1} \cdots I_{r_m,i_m,j_m}
\]

with \(0 < |j_l| < i_l < r_l \leq k\) for \(l = 1, \ldots, m\) and \(1 < s_l \leq k\) for \(l = 1, \ldots, m'\) or

\[
A_{1,0} D_{s_1,0} \cdots D_{s_{m'},0} I_{r_1,i_1,j_1} \cdots I_{r_m,i_m,j_m}
\]

with \(0 < |j_l| < i_l < r_l \leq k\) for \(l = 1, \ldots, m\) and \(1 < s_l \leq k\) for \(l = 1, \ldots, m'\)

can be written as a linear combination of products of the form (34) or (35) satisfying

\[
s_l < s_{l'} \text{ if } l < l',
\]

and

\[
s_l \notin \{r_1, \ldots, r_m\} \cup \{i_1, \ldots, i_m\} \text{ for } l = 1, \ldots, m',
\]
as well as conditions (28)–(31).

**Theorem 5.3.** Let \(R\) be a commutative ring with unit where 2 is invertible. For \(n \geq 2\) even, the \((\mathbb{Z}_2)^k\)-invariants in \(H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))\) are multiplicatively generated by the set \(\mathcal{E}'\). Moreover, an additive basis is given by products of the form (34) together with products of the form (35) all of which satisfy (36), (37), and (28)–(31).

**Proof.** The proof is almost the same as the proof of Theorem 4.12 except for two differences:

1. We ignore the action of \(\epsilon_1\). This, however, only removes the condition of having an even number of terms \(D_{i,0}\), and now we can just associate all terms \(D_{i,0}\).
2. We add \(A_{1,0}\) as a potential factor to all monomials. This does not affect our previous proof, because \(A_{1,0}\) is already an invariant.

We thus get:
Theorem 5.4. Let $R$ be a commutative ring with unit where 2 is invertible. For $n \geq 2$ even, there is an $R$-algebra isomorphism
\[ H^*(\text{Conf}(\mathbb{R}P^n - *, k)) \cong H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))^{(\mathbb{Z}_2)k} \cong R[\mathcal{E}']/\mathcal{J}', \]
where $\mathcal{J}'$ is the ideal generated by the relations in Lemma 4.10 not involving a term $I_{i,j,0}$ together with the relations of Lemma 5.2 and the relation $A^2_{1,0} = 0$.

6. Lusternik-Schnirelmann category and topological complexity

The results in the previous sections are now used to study the category and all the higher topological complexities of the auxiliary orbit configuration space $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ for $n > 2$ (hypothesis that will be in force throughout this section, unless explicitly noted otherwise). In what follows all references to cohomology use integer coefficients—omitting the coefficients from the notation.

The homotopy exact sequence associated to the fibrations in [3] inductively yield that $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ is $(n-2)$-connected. Further, from [1], the cohomology of this space is torsion-free, and vanishes above dimension $k(n-1)$. Therefore $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ has the homotopy type of a cell complex $X$ which is $(n-2)$-connected and $k(n-1)$-dimensional (see [12, Section 4.C]). Then the upper bounds in [5, Theorem 1.50] for the Lusternik-Schnirelmann category, and in [2, Theorem 3.9] for the higher topological complexities immediately yield
\[ \text{cat}(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \leq k \text{ and } TC_s(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \leq sk. \]

The former inequality is in fact sharp, as the product $A_{1,0} \cdots A_{k,0} \in H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ is non-zero. We thus get:

Corollary 6.1. For $n > 2$, $\text{cat}(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) = k$.

Alternatively, we could use the observation that the rule $A'_{i,j} \mapsto A_{i-1,j-1}$, $1 \leq j < i \leq k+1$, determines a ring monomorphism
\[ H^*(\text{Conf}(\mathbb{R}^n, k+1)) \hookrightarrow H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)). \]

Further, since these rings are torsion-free, we also get a ring monomorphism
\[ H^*(\text{Conf}(\mathbb{R}^n, k+1))^{(\mathbb{Z}_2)k} \hookrightarrow H^*(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))^{(\mathbb{Z}_2)k}. \]

Consequently, the $s$-th zero-divisors cup-length of $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ is bounded from below by the $s$-th zero-divisors cup-length of $\text{Conf}(\mathbb{R}^n, k+1)$.

Corollary 6.2. Let $n > 2$. Then $TC_s(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) = sk$ if $n$ is odd, whereas, if $n$ is even, $TC_s(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \in \{sk - 1, sk\}$.

Note that item (d) of Theorem 2.2 in [7] implies that the indetermination by one unit in the case with an even $n$ in Corollary 6.2 is resolved in terms of the $s$-th zero-divisors cup-length of $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$:

Corollary 6.3. Let $n > 2$ (any parity). Then $TC_s(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ agrees with the $s$-th zero-divisors cup-length of $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$.

\footnotetext{2}{We use the reduced versions of these homotopy invariants, so that the category and all the higher topological complexities of a contractible space are 0.}
Since $H^\ast(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)))$ and $H^\ast(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^{n+2} - \{0\}, k)))$ differ only by degree scaling, Corollary 6.3 implies that, for fixed $s$ and $k$, $\text{TC}_s(\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ depends only on the parity of $n$. In particular, the indeterminacy by one in Corollary 6.2 could be settled by considering the situation for a single value of $n$. In our setting, $n = 4$ would be the most reasonable instance to explore. However, for the analogous situation in [7] and [11], $n = 2$ is the right choice, in view of the well known splitting

\begin{equation}
\text{Conf}(\mathbb{R}^2, k) \simeq X \times S^1
\end{equation}

with $X$ a CW complex of dimension $k - 2$. Indeed, standard cohomology considerations give $\text{TC}_s(\text{Conf}(\mathbb{R}^{2m}, k)) \in \{s(k - 1) - 1, s(k - 1)\}$, and then (39) implies that the actual answer is given by the lower value.

We close the paper with an interesting challenge: Note that, just as above, the indeterminacy by one in Corollary 6.2 would be resolved with the smallest value (and the restriction $n > 2$ in this section would be waived) by answering affirmatively the following analogue of (39):

Is it true that $\text{Conf}_{\mathbb{Z}_2}(\mathbb{R}^2 - \{0\}, k)$ has the homotopy type of a product $S^1 \times X$ for some CW complex $X$ of dimension $k - 1$?

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