Pressure Responses of a Vertically Hydraulic Fractured Well in a Reservoir with Fractal Structure

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Abstract

We obtain an analytical solution for the pressure-transient behavior of a vertically hydraulic fractured well in a heterogeneous reservoir. The heterogeneity of the reservoir is modeled by using the concept of fractal geometry. Such reservoirs are called fractal reservoirs. According to the theory of fractional calculus, a temporal fractional derivative is applied to incorporate the memory properties of the fractal reservoir. The effect of different parameters on the computed wellbore pressure is fully investigated by various synthetic examples.

Keywords: vertically hydraulic fractured well; fractal geometry; fractal reservoir; fractional derivatives.

1 Introduction

Hydraulic fracturing plays an important role in improving the productivity of damaged wells and wells producing. The vertical plane fracture is created by injecting fluid into the formation and then filling with propping agents, such as propants, to prevent closure. In practical terms, two types of fractured well are considered: infinite (high) or finite (low) conductivity vertical fracture. In case of infinite conductivity fracture, it is assumed that the fluid flows along the fracture without any pressure drop. Finite conductivity fracture occurs when the pressure drop along the fracture plane is not negligible.

The classical diffusion equation has been used to explain the pressure responses of a well in a reservoir, which is assumed to be homogeneous at all scales. However, recent studies show that the homogeneity assumption is not valid in most cases \cite{1,7}. Due to this fact, fractal geometry has been used as an effective tool to describe the heterogeneities of these reservoirs, which are called fractal reservoirs \cite{8,11}. Since the diffusion process of fractal reservoirs is history dependent, and the anomalous diffusion properties of fractal reservoirs cannot be fully described by the fractal model, the concept of fractional derivative has been used to incorporate the memory of the fluid flow \cite{12,15}. Our main aim here is to analyze the pressure behavior of a well with an infinite...
conductivity vertical fracture in a fractal reservoir. An infinite radial system is considered in order to analyze the effects of different parameters on the well response.

The paper is organized as follows. After this brief introduction, the mathematical model is formulated in Section 2. A summary of the nomenclature used appears in Appendix A. The analytical solution to the model is provided in Section 3. Section 4 discusses how the well responses to different reservoir parameters. The main conclusions of our study are given in Section 5.

2 Model description

A schematic diagram of a vertically hydraulic fractured well is shown in Figure 1. Figure 2 shows the geometry of flow lines near the fractured well. Before discussing the mathematical model of transport process, we define three variables:

- the dimensionless pressure
  \[ p_D = \frac{2\pi k_w h (p_i - p(r, t))}{q \mu}; \]
- the dimensionless time
  \[ t_D = \frac{k_w r_{wd} \theta t}{\phi_w \mu c (2L_f)^2}; \]

Figure 1: Geometry of a vertically hydraulic fractured well.

Figure 2: Linear and pseudo-radial flow regimes near an infinite conductivity fracture.
the dimensionless radius

\[ r_D = \frac{r}{2L_f} \].

See Appendix A for the description of all the quantities involved. The best known and most useful model to describe the pressure behavior of fractal reservoirs was firstly proposed by Metzler et al. [1]. Similarly to Camacho-Velázquez et al. [4], from here on we call generalized diffusion equation to the fractal-fractional diffusion (FFD) equation

\[
\frac{1}{r_D^\theta} \frac{\partial^2 p_D}{\partial r_D^2} + \frac{\beta}{r_D^{\theta+1}} \frac{\partial p_D}{\partial r_D} = \frac{\partial^\gamma p_D}{\partial t_D^\gamma},
\]  

(1)

where \( \beta = d_{mf} - \theta - 1 \). The parameter \( d_{mf} \) denotes the mass fractal dimension, while \( \theta \) represents the conductivity index. Mass fractal dimension is responsible for the reservoir structure, and the conductivity index explains the diffusion process in the reservoir. The Caputo fractional order derivative is used to introduce \( \frac{\partial^\gamma p_D}{\partial t_D^\gamma} \):

\[
\frac{\partial^\gamma p_D}{\partial t_D^\gamma} = \frac{1}{\Gamma(m - \gamma)} \int_0^{t_D} (t_D - \tau)^{m - \gamma - 1} p_D^{(m)}(\tau) d\tau,
\]  

(2)

where \( \gamma \in \mathbb{R}^+ \), \( \lceil \gamma \rceil = m \in \mathbb{Z}^+ \), and \( \Gamma \) represents the Gamma function, that is,

\[
\Gamma(\nu) = \int_0^\infty e^{-\tau} \tau^{\nu-1} d\tau.
\]  

(3)

The order \( \gamma \) of the fractional derivative is related to the conductivity index by \( \gamma = 2/(2 + \theta) \).

3 Analytical solution

It is assumed that the pressure distribution of the reservoir is uniform and constant at initial time:

\[
p_D(r_D, 0) = 0.
\]  

(4)

To obtain the line source solution of equation (1), we take \( r_w \to 0^+ \). The inner boundary condition without wellbore storage and skin effects can then be written as

\[
r_D^\beta \frac{\partial p_D(r_D, t_D)}{\partial r_D} \bigg|_{r_D \to 0^+} = -1.
\]  

(5)

The outer boundary condition for the infinite reservoir is given by

\[
\lim_{r_D \to \infty} p_D(r_D, t_D) = 0.
\]  

(6)

Taking the Laplace transform to both sides of (4), and then using (4), we obtain that

\[
\frac{1}{r_D^\theta} \frac{\partial^2 \tilde{p}_D}{\partial r_D^2} + \frac{\beta}{r_D^{\theta+1}} \frac{\partial \tilde{p}_D}{\partial r_D} = s^\gamma \tilde{p}_D,
\]  

(7)

where \( s \) is the Laplace transform variable. The dependent variable \( \tilde{p}_D \) denotes the Laplace transform of \( p_D \), and is a function of \( r_D \) and \( s \). In the Laplace space, the inner boundary condition takes the form

\[
r_D^\beta \frac{\partial \tilde{p}_D(r_D, s)}{\partial r_D} \bigg|_{r_D \to 0^+} = -\frac{1}{s}
\]  

(8)

while the outer boundary condition is given by

\[
\lim_{r_D \to \infty} \tilde{p}_D(r_D, s) = 0.
\]  

(9)
Using the substitutions $\bar{p}_D = r_D^{(1-\beta)/2}\bar{W}$ and $x = (2s^{\gamma/2}r_D(2+\theta)/2)/(2 + \theta)$, and after a slight manipulation, we conclude that (7) is equivalent to
\[
x^2 \frac{\partial^2 \bar{W}}{\partial x^2} + x \frac{\partial \bar{W}}{\partial x} - (x^2 + \nu^2)\bar{W} = 0,
\]
where $\nu = \frac{1 - \beta}{2 + \theta}$. Equation (10) is Bessel’s equation, which has the general solution
\[
\bar{W}(x,s) = AI_{\nu}(x) + BK_{\nu}(x).
\]
Thus, the dimensionless pressure function in Laplace space can be written as
\[
\bar{p}_D(r_D,s) = r_D^{(1-\beta)/2} \left[ AI_{\nu} \left( \frac{2s^{\gamma/2}}{2 + \theta} r_D^{(2+\theta)/2} \right) + BK_{\nu} \left( \frac{2s^{\gamma/2}}{2 + \theta} r_D^{(2+\theta)/2} \right) \right].
\]
Application of the outer boundary condition (9) to (12) yields $A = 0$. Therefore, (12) reduces to
\[
\bar{p}_D(r_D,s) = Br_D^{(1-\beta)/2}K_{\nu} \left( \frac{2s^{\gamma/2}}{2 + \theta} r_D^{(2+\theta)/2} \right).
\]
Based on (13), the inner boundary condition (8) can be written as
\[
-B \lim_{r_D \to 0^+} s^{\gamma/2} r_D^{(1+\beta+\theta)/1} K_{\nu-1} \left( \frac{2s^{\gamma/2}}{2 + \theta} r_D^{(2+\theta)/2} \right) = -\frac{1}{s}.
\]
Equation (14) can be simplified by using the formula
\[
K_{\nu}(x) \approx \frac{\Gamma(\nu)}{2} \left( \frac{2}{x} \right)^{\nu}, \quad \nu > 0,
\]
valid for small arguments $0 < x \ll \sqrt{1 + \nu}$. Indeed, having in mind that $K_{\nu}(x) = K_{-\nu}(x)$, and by making use of (15), we reduce (14) to the following expression:
\[
B \lim_{r_D \to 0^+} s^{\gamma/2} \frac{\Gamma(1-\nu)}{2^{\nu}} \left( \frac{2 + \theta}{2s^{\gamma/2}} \right)^{1-\nu} = \frac{1}{s}.
\]
Consequently,
\[
B = \frac{2}{\Gamma(1-\nu)(2 + \theta)^{1-\nu} s^{1+\nu\gamma/2}}.
\]
Equation (13) can be written as
\[
\bar{p}_D(x_D,y_D,s) = B \left( \sqrt{(x_D - \alpha_1)^2 + (y_D - \alpha_2)^2} \right)^{(1-\beta)/2} \times K_{\nu} \left( \frac{2s^{\gamma/2}}{2 + \theta} \right) \left( \sqrt{(x_D - \alpha_1)^2 + (y_D - \alpha_2)^2} \right)^{(2+\theta)/2}.
\]
We obtain the pressure drop function, in a fractal reservoir with a hydraulic fracture across the well, by integrating (13) with respect to $\alpha_1$ from $-1/2$ to $1/2$:
\[
\bar{p}_{Df}(x_D,y_D,s) = B \int_{-1/2}^{1/2} \left[ \sqrt{(x_D - \alpha_1)^2 + (y_D - \alpha_2)^2} \right]^{(1-\beta)/2} \times K_{\nu} \left( \frac{2s^{\gamma/2}}{2 + \theta} \right) \left( \sqrt{(x_D - \alpha_1)^2 + (y_D - \alpha_2)^2} \right)^{(2+\theta)/2} d\alpha_1.
\]
Since the computation of the pressure along the fracture is favorable, it can be assumed that \( y_D = \alpha_2 \). Thus, the pressure drop function (19) reduces to

\[
\bar{\rho}_D(x_D, 0, s) = B \int_{-1/2}^{1/2} \left( \sqrt{(x_D - \alpha_1)^2} \right)^{(1-\beta)/2} K_{\nu} \left( \frac{2s^{\gamma/2}}{2 + \theta} \sqrt{(x_D - \alpha_1)^2} \right)^{(2+\theta)/2} d\alpha_1, \tag{20}
\]

where \( |x_D| \leq 1/2 \). By changing variables, and after further manipulations, (20) can be written as

\[
\bar{\rho}_D(x_D, 0, s) = \frac{1}{s^{\gamma/2}} \left( \frac{2 + \theta}{2s^{\gamma/2}} \right)^{a-1} \left( \int_0^{(2s^{\gamma/2}/(2+\theta))^{(1/2-x_D)^{(2+\theta)/2}}} z^{a-1} K_{\nu}(z) dz \right.
\]

\[
\left. + \int_0^{(2s^{\gamma/2}/(2+\theta))^{(1/2+x_D)^{(2+\theta)/2}}} z^{a-1} K_{\nu}(z) dz \right) \tag{21}
\]

with \( a = (4 - d_{mf} + \theta)/(2 + \theta) \). The integrals in (21) can be computed by the following formula:

\[
\int z^{a-1} K_{\nu}(z) dz = -\frac{2^{-\nu-1} \pi z^{-\nu} \csc(\pi \nu)}{(\nu-a)\Gamma(1-\nu)} 1_F_2 \left( \frac{a - \nu}{2}; 1 - \nu, \frac{a - \nu}{2} + 1; \frac{z^2}{4} \right) \nonumber
\]

\[
- \frac{2^{-\nu-1} \pi z^{a+\nu} \csc(\pi \nu)}{(a+\nu)\Gamma(1+\nu)} 1_F_2 \left( \frac{a + \nu}{2}; 1 + \nu, \frac{a + \nu}{2} + 1; \frac{z^2}{4} \right), \tag{22}
\]

where \( \nu \notin \mathbb{Z} \) and \( 1_F_2 \) represents the generalized hypergeometric function. For the Euclidean model, that is, the particular case \( \nu = 0 \) and \( a = 1 \), the presented integrals can be evaluated by

\[
\int K_0(z) dz = \frac{\pi z}{2} \left( K_0(z)L_{-1}(z) + K_1(z)L_0(z) \right) \tag{23}
\]

or

\[
\int K_0(z) dz = \frac{\pi z}{2} \left( K_0(z)L_1(z) + K_1(z)L_0(z) \right) + zK_0(z), \tag{24}
\]

where \( L_{-1}(z) \), \( L_0(z) \) and \( L_1(z) \) are modified Struve functions given by

\[
L_{\nu}(z) = \frac{z^{\nu+1}}{2\nu \sqrt{\pi} \Gamma \left( \nu + \frac{3}{2} \right)} 1_F_2 \left( 1; \frac{3}{2}, \nu + \frac{3}{2}; \frac{z^2}{4} \right), \quad -\nu - \frac{3}{2} \notin \mathbb{N}. \tag{25}
\]

The inverse transform operation is easily carried out by using the Gaver–Stehfest algorithm [16].

### 4 Model responses

To show the pressure-transient behavior of a fractured well, we use (21) to compute the wellbore pressure response at \( x_D = 0 \) (uniform flux solution), without wellbore storage and skin effects. The well intercepts the center of a single vertical fracture plane. According to the uniform flux fracture assumption, the flow rate per unit of fracture surface is constant along the fracture length. Figure 3 shows the different behavior of a well response in a homogenous and fractal reservoir. Generally, the fracture structure works as a sink, which enforces the fluid to go towards the fracture with hyperbolical flow geometry. The geometry of fracture enforces the flow lines to be perpendicular to the fracture plane and the pressure-transient response defines a linear flow in the reservoir. After the linear flow regime, when the effect of fluid flow reaches the two ends of the fracture, the geometry of flow can be identified, for a small time, by a hyperbolical flow regime. Ultimately, the characteristic radial flow regime is observed from the well response. The linear flow regime can be identified by two straight-lines of pressure and its logarithmic derivative at early times (see Figure 3). As can be seen from Figure 3, these two parallel straight-lines have a half slope during the linear flow regime in a conventional reservoir. The specialized analysis (Cartesian coordinates) on Figure 4 shows that with a plot of the pressure change versus the square root
of the elapsed time, the linear flow can be identified with a straight-line intercepting the origin. The results of Figure 4 show that the slope of the straight-line is decreasing with the increasing of the parameter $\nu$. In other words, in a more complex reservoir, described by a larger value of the conductivity index $\theta$ and a smaller value of the mass fractal dimension $d_{mf}$, the linear flow is identified with a smaller slope of the straight-line in the specialized plot. Figure 3 indicates that the radial flow regime (at late times) is identified by the straight-line shape of the logarithmic derivative in log-log scale and that the reservoirs with more complex structures (larger values of $\nu$) have the straight-lines with larger slopes. Moreover, Figure 4 shows that in a fractal reservoir, at late times during the radial flow regime, the pressure response and its logarithmic derivative are parallel straight-lines. This fact indicates that the pressure and its logarithmic derivative can be expressed by two power-law functions with the same power.

Figure 3: Responses for a fractured well without wellbore storage and skin effects. Log-log analysis of $p_D$ and $dp_D/d\ln(t_D)$ versus $t_D$ with $d_{mf} = \{2, 1.95, 1.9, 1.85, 1.8\}$, $\theta = \{0, 0.1, 0.3, 0.5, 0.7\}$ and $\gamma = \{1, 0.9524, 0.8696, 0.8000, 0.7407\}$.

The reservoir rock porosity is a statistical property of the system, and can be expressed as $\phi(r) = \phi_v (r/L_r)^{d_{mf}-2}$, where $\phi_v$ is a reference porosity value at the reference length $L_r$. Particularly, the reference length is the wellbore radius, and the porosity of a fractal reservoir can be written as $\phi(r) = \phi_w (r/r_w)^{d_{mf}-2}$. The reference porosity value can be assumed to be equal to the porosities obtained from log and core data. However, based on log and core porosities, the reasonable oil-in-place estimates indicate that the porosity variations are not big, and can be considered equal to the constant reference porosity. It can be therefore immediately concluded that the mass fractal dimension $d_{mf}$ is equal to the Euclidean dimension, that is, $d_{mf} = 2$. On the other hand, because $k(r) = k_v (r/L_r)^{d_{mf}-\theta-2}$ and, particularly, $k(r) = k_w (r/r_w)^{d_{mf}-\theta-2}$, the deficiencies of average permeability calculations indicate that the dynamical properties of the system have an important role in the flowing fluid through the reservoir and in all stages of production. Since $d_{mf} = 2$, the permeability variations can be expressed by $k(r) = k_w (r/r_w)^{-\theta}$. So, it requires investigating the effect of the conductivity index $\theta$ variations on the well response. As can be seen from Figure 5 by increasing the $\theta$ values, the pressure responses and their logarithmic derivatives increase. For a constant-rate production, more pressure drops in reservoirs with a more complicated diffusion process (larger $\theta$) indicates that the diffusion is slower in these types of reservoirs. Therefore, application of the conventional model solutions to such reservoirs may lead the analyst...
Figure 4: Specialized analysis of $p_D$ versus $\sqrt{t_D}$ with $d_{mf} = \{2, 1.95, 1.9, 1.85, 1.8\}$, $\theta = \{0, 0.1, 0.3, 0.5, 0.7\}$ and $\gamma = \{1, 0.9524, 0.8696, 0.8000, 0.7407\}$.

to a wrong interpretation.

Figure 5: Responses for a fractured well without wellbore storage and skin effects. Log-log analysis of $p_D$ and $dp_D/d\ln(t_D)$ versus $t_D$ with $d_{mf} = 2$, $\theta = \{0, 0.1, 0.3, 0.5, 0.7\}$ and $\gamma = \{1, 0.9524, 0.8696, 0.8000, 0.7407\}$.
5 Concluding remarks

We used fractional calculus to model an infinite conductivity vertically fractured well in a fractal reservoir with more realistic behavior than the conventional (homogeneous) reservoirs. An analytical solution was obtained, allowing us to analyze and interpret the well test data taken from these types of wells and reservoirs. On the basis of our results, the simulated examples and the discussion presented, the following conclusions can be expressed:

- the Euclidean interpretation model cannot be used to analyze the pressure-transient behavior of fractal reservoirs;
- the obtained solution can be used as a practical tool to analyze the effects of dynamical properties of the system;
- according to the shortcomings of the average permeability estimates, the analytical solution can be used to determine the conductivity index parameter $\theta$ and then the permeability variations.

A Nomenclature

| Symbol | Meaning                              | Units    |
|--------|--------------------------------------|----------|
| $c$    | compressibility                      | vol/vol/atm |
| $d_{mf}$ | mass fractal dimension            |          |
| $1F_2$ | generalized hypergeometric function|          |
| $h$    | reservoir thickness                 | cm       |
| $I_\nu$ | modified Bessel function of the first kind of order $\nu$ |          |
| $k$    | permeability                         | darcy    |
| $k_w$  | permeability at the wellbore        | darcy    |
| $K_\nu$ | modified Bessel function of the second kind of order $\nu$ |          |
| $L_f$  | fracture half-length                | cm       |
| $L_\nu$ | modified Struve function of order $\nu$ |          |
| $p$    | pressure                             | atm      |
| $p_D$  | dimensionless pressure              | atm      |
| $p_i$  | initial pressure                    | atm      |
| $q$    | production rate                     | cc/sec   |
| $r$    | radial distance                     | cm       |
| $r_D$  | dimensionless radius                |          |
| $r_w$  | wellbore radius                     | cm       |
| $r_{wD}$ | dimensionless wellbore radius      |          |
| $t$    | time                                 | sec      |
| $t_D$  | dimensionless time                  |          |
| $s$    | Laplace image space variable        |          |
| $\gamma$ | fractional derivative order          |          |
| $\Gamma$ | Gamma function                      |          |
| $\theta$ | conductivity index                |          |
| $\mu$  | viscosity                            | cP       |
| $\phi$ | porosity (or void fraction)        |          |
| $\phi_w$ | porosity at the wellbore            |          |
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