Quantum Metal–Superconductor Transition: A Local Field Theory Approach

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The zero temperature, or quantum, metal-superconductor phase transition is studied in disordered systems in dimension greater than two. An effective local field theory is developed that keeps all soft modes or fluctuations explicitly. A simple renormalization group analysis is used to exactly determine the quantum critical behavior at this transition.

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I. INTRODUCTION

Recently there has been much interest in quantum phase transitions. Occurring at $T = 0$, these transitions provide new insight into the possible physical phases of systems at low temperature. The first quantum phase transition studied in detail was the ferromagnetic transition in itinerant electron system at zero temperature. Hertz argued in 1976 that the transition was mean-field-like in the physically interesting dimension $d = 3$.

This simple mean-field description was later shown to be incorrect. The reason for this breakdown is the existence of soft or massless modes other than the order parameter fluctuations. These modes were being neglected in Hertz’s theory. In disordered systems these modes are diffusive, and they couple to the order parameter fluctuations and modify the critical behavior. Technically, if these additional soft modes are integrated out, they lead to a long-ranged interaction and a nonlocal field theory. It was argued that once this effect is taken into account, all other fluctuation effects are suppressed by the long-range nature of the interactions and that the critical behavior is governed by a fixed point that is Gaussian, but does not yield mean-field exponents.

Similar argument was used to describe the normal metal-to-superconductor quantum phase transition at $T = 0$. In this case the usual finite temperature superconducting phase transition is driven to zero temperature by nonmagnetic disorder where the additional soft modes come from particle-hole excitations. Again, it was argued that the critical behavior found at this quantum phase transition could be exactly determined using the same technique as in Refs. [1,2].

The theory developed in Refs. [1,2] however, suffered from one major drawback: Since the additional soft modes were integrated out in order to obtain a description entirely in terms of the order parameter fluctuations, the effective field theory that was derived was nonlocal and not suitable for perturbatively calculating effects that depend on all of the soft modes in the system. The analysis in Refs. [1,2] was therefore restricted to power counting arguments at tree level to show that all non-Gaussian terms are irrelevant in a RG sense. However, relying entirely on tree-level power counting can be dangerous. In particular, logarithmic corrections can be easily missed. Later on logarithmic corrections were indeed found in the description of the quantum ferromagnetic transition.

It is the motivation of this paper to keep all the relevant soft modes and to construct an effective local field theory for the metal-superconductor transition so that the exact behavior at this quantum phase transition can be determined. Unlike the quantum ferromagnetic transition discussed above, we will show that the previous results for the metal-superconductor transition, though from nonlocal field theory treatment, are still valid. The inherent reason for that is explained in detail.

This paper is organized as follows. In Sec. II we use methods developed in Refs. [3,4] to derive an effective local theory for disordered electron systems that explicitly separates massive modes from soft ones, and keeps all of the latter. In Sec. III we give a renormalization group analysis of this model. In Sec. IV we discuss our results.

II. EFFECTIVE LOCAL FIELD THEORY

A local field theory will be given in this section to describe the normal metal-to-superconductor quantum phase transition at $T = 0$. All relevant soft modes will be contained in this field theory. We start from a general model of interacting electrons with quenched disorder and an attractive Cooperon interaction amplitude. We then introduce the superconducting order parameter and separate massive and soft modes. After integrat-
ing out the massive modes, we obtain a effective local field theory that describe the coupling between the superconducting fluctuations and the soft or massless diffusive modes.

A. Composite field theory

The general partition function of the interacting, disordered electrons can be given in the form of Grassmann fields \( \tilde{\psi} \) and \( \psi \):

\[
Z = \int D[\tilde{\psi}, \psi] e^{S[\tilde{\psi}, \psi]} .
\]

with the action \( S \) being

\[
S = -\int_{0}^{\beta} d\tau \int d\mathbf{x} \sum_{\sigma} \bar{\psi}_{\sigma}(\mathbf{x}, \tau) \frac{\partial}{\partial \tau} \psi_{\sigma}(\mathbf{x}, \tau) - \int_{0}^{\beta} d\tau H(\tau) .
\]

We denote the spatial position by \( \mathbf{x} \), and the imaginary time by \( \tau \). \( H(\tau) \) is the Hamiltonian in imaginary time representation, \( \beta = 1/T \) is the inverse temperature, \( \sigma = 1, 2 \) denotes spin labels. As what we have done in previous papers, we integrate out the Grassmann fields and rewrite the theory in terms of complex-number fields \( Q \) and \( \tilde{Q} \). With the help of the following isomorphism,

\[
B_{12} = \frac{i}{2}
\begin{pmatrix}
-\psi_{1\uparrow}^{\dagger}\psi_{2\uparrow} - \psi_{1\downarrow}^{\dagger}\psi_{2\downarrow} - \psi_{1\uparrow}^{\dagger}\psi_{2\downarrow} - \psi_{1\downarrow}^{\dagger}\psi_{2\uparrow} \\
-\psi_{1\uparrow}^{\dagger}\psi_{2\downarrow} - \psi_{1\downarrow}^{\dagger}\psi_{2\uparrow} + \psi_{1\uparrow}^{\dagger}\psi_{2\uparrow} + \psi_{1\downarrow}^{\dagger}\psi_{2\downarrow} \\
\psi_{3\uparrow}^{\dagger}\psi_{4\uparrow} + \psi_{3\downarrow}^{\dagger}\psi_{4\downarrow} + \psi_{3\uparrow}^{\dagger}\psi_{4\uparrow} + \psi_{3\downarrow}^{\dagger}\psi_{4\downarrow} \\
-\psi_{3\uparrow}^{\dagger}\psi_{4\uparrow} - \psi_{3\uparrow}^{\dagger}\psi_{4\downarrow} - \psi_{3\downarrow}^{\dagger}\psi_{4\uparrow} - \psi_{3\downarrow}^{\dagger}\psi_{4\downarrow}
\end{pmatrix}
\end{pmatrix}
\approx Q_{12} ,
\]

we exactly rewrite the partition function as

\[
Z = \int D[\tilde{\psi}, \psi] e^{S[\tilde{\psi}, \psi]} \int D[Q] e^{S(Q - B)} = \int D[\tilde{\psi}, \psi] e^{S[\tilde{\psi}, \psi]} \int D[Q] D[\tilde{Q}] e^{\Gamma(\tilde{Q} - B)} = \int D[Q] D[\tilde{Q}] e^{A[Q, \tilde{Q}]} .
\]

Here \( \tilde{\pi} \) is an auxiliary bosonic matrix field that plays the role of a Lagrange multiplier. The reason to do so is that the rewritten action is particularly suited for the separation of massive and soft modes. We then decouple the particle-particle spin-singlet interaction by means of a Hubbard-Stratonovich transformation. Denoting the Hubbard-Stratonovich field by \( \Psi \), the partition function becomes

\[
Z = \int D[Q, \tilde{Q}, \Psi] e^{\tilde{A}[Q, \tilde{Q}, \Psi]} ,
\]

where the action

\[
\tilde{A}[Q, \tilde{Q}, \Psi] = A_{\text{dis}}[Q] + A_{\text{int}}^{(s)}[Q] + A_{\text{int}}^{(t)}[Q]
\]

\[
\quad + \frac{1}{2} \text{Tr} \ln \left( G_{0}^{-1} - i\tilde{A} \right) + \text{Tr} \left( \tilde{A}Q \right)
\]

\[
\quad - \int d\mathbf{x} \sum_{\alpha, n} \sum_{r=1, 2} \bar{\Psi}_{n}(\mathbf{x}) \Psi_{n}^{\alpha}(\mathbf{x})
\]

\[
\quad + i\sqrt{2T} |\Gamma^{(c)}| \int d\mathbf{x} \sum_{\alpha} \sum_{n=1, 2} e^{\Psi_{n}^{\alpha}(\mathbf{x})}
\]

\[
\quad \times \sum_{m} \text{tr} \left[ (\tau_{r} \otimes \sigma_{0}) Q_{m, m+n}^{\alpha}(\mathbf{x}) \right] .
\]

(2.4b)

with \( \text{Tr} \) denoting a trace over all degrees of freedom, including the continuous real space position, and \( \text{tr} \) a trace over all discrete degrees of freedom that are not summed over explicitly. \( \Gamma^{(c)} < 0 \) is the attractive Cooperon interaction amplitude. The first three terms in Eq. (2.4b) have the following forms,

\[
A_{\text{dis}}[Q] = \frac{1}{\pi N F T_{e}} \int d\mathbf{x} \text{tr} \left( Q(\mathbf{x}) \right)^{2} ,
\]

\[
A_{\text{int}}^{(s)} = \frac{T T^{(s)}}{2} \int d\mathbf{x} \sum_{r=0, 3} \sum_{n_{1}, n_{2}, m} \sum_{\alpha} \text{tr} \left[ (\tau_{r} \otimes \sigma_{0}) Q_{n_{1}, n_{1}+m}^{\alpha}(\mathbf{x}) \right] \text{tr} \left[ (\tau_{r} \otimes \sigma_{0}) Q_{n_{2}+m, n_{2}}^{\alpha}(\mathbf{x}) \right] ,
\]

\[
A_{\text{int}}^{(t)} = \frac{T T^{(t)}}{2} \int d\mathbf{x} \sum_{r=0, 3} \sum_{n_{1}, n_{2}, m} \sum_{\alpha} \sum_{i=1}^{3} \text{tr} \left[ (\tau_{r} \otimes \sigma_{i}) Q_{n_{1}, n_{1}+m}^{\alpha}(\mathbf{x}) \right] \text{tr} \left[ (\tau_{r} \otimes \sigma_{i}) Q_{n_{2}+m, n_{2}}^{\alpha}(\mathbf{x}) \right] ,
\]

Finally,

\[
G_{0}^{-1} = -\partial_{\mathbf{r}} + \nabla^{2} / 2m + \mu ,
\]

is the inverse Green operator.

Physically, the Hubbard-Stratonovich field \( \Psi \) can be related to the superconducting, or Cooper pair, order parameter, \( \Psi \sim \psi_{\uparrow} \psi_{\downarrow} \).

B. Soft modes

Now we are ready to separate the massive and soft modes. Our calculations follow the same procedure in previous papers. Here we just simply quote the results.
The effective action has the form

\[
A[q, \Psi, \Delta P, \Delta \Lambda] = A_{NL\sigma M}[q] + \delta A[\Delta P, \Delta \Lambda, q] \\
- \int dx \sum_{\alpha} \sum_{n} \sum_{r=1,2} \rho \Psi_{n}^{\alpha}(x) \rho \Psi_{n}^{\alpha}(x) \\
+ \frac{i}{\sqrt{\pi T}} |K^{(c)}| \int dx \sum_{\alpha} \sum_{n} \sum_{r=1,2} \rho \Psi_{n}^{\alpha}(x) \\
\times \sum_{m} \text{tr} (\tau \odot s_{0}) \left[ \hat{Q}_{\alpha}^{m,-m+n}(x) \right] \\
+ \frac{4}{\pi N_{F}} (S\Delta P S^{-1})^{\alpha\alpha}_{m,-m+n}(x) .
\] (2.5)

Here \( K^{(c)} = \pi N_{F}^{2} \Gamma^{(c)}/8 \). The matrix \( S \) can be treated as \( S = 1 \) when we neglect the coupling between massless modes \( q \) and these massive fluctuations \( \Delta P \) and \( \Delta \Lambda \), which is irrelevant for the purpose of the current paper.

\( A_{NL\sigma M} \) is the known action of the nonlinear sigma model:

\[
A_{NL\sigma M} = A_{(c)}^{(s)}[\pi N_{F}\hat{Q}/4] + A_{(c)}^{(t)}[\pi N_{F}\hat{Q}/4] \\
+ \frac{-1}{2G} \int dx \left( \nabla^{2}Q(x) \right)^{2} \\
+ 2\bar{H} \int dx \left( \left( \Omega \hat{Q}(x) \right) \right) .
\] (2.6a)

with \( A_{(c)}^{(s)} \) from Eq. (2.4d), \( A_{(c)}^{(t)} \) from Eq. (2.4e), and \( \Omega \) a frequency matrix with elements

\[
\Omega_{12} = (\tau_{0} \odot s_{0}) \delta_{12} \omega_{n} .
\] (2.6b)

Here \( Q \) is subject to the following constraints,

\[
\hat{Q}^{2}(x) = 1 \otimes \tau_{0} , \quad \hat{Q}^{\dagger} = \hat{Q} , \quad \text{tr} \hat{Q}(x) = 0 .
\] (2.6c)

and then can be write in a block matrix form as

\[
\hat{Q} = \begin{pmatrix} \sqrt{1 - q q} & -\sqrt{1 - q q} \\ q & -q \end{pmatrix} .
\] (2.6d)

where the matrix \( q \) has elements \( q_{mn} \), whose frequency labels are restricted to \( n \geq 0, m < 0 \). Symmetry analysis with Ward identities ensures that the matrix \( q \) are massless, which are diffusive in disordered systems. The coupling constants \( G \) and \( H \) are proportional to the inverse conductivity, \( G \propto 1/\sigma \), and the specific heat coefficient, \( H \propto \gamma \equiv \lim_{T \rightarrow 0} C_{V}/T \), respectively.

\( \delta A \) contains the corrections to the nonlinear sigma model that were given in Ref. [10]. We list explicitly the terms that are bilinear in the massive fluctuations \( \Delta P \) and \( \Delta \Lambda \), but do not contain couplings between the massive modes and \( q \).

\[
\delta A^{(2)} = A_{\text{dis}}[\Delta P] + \int dx dr \left( \Delta \Lambda(x) \Delta P(x) \right) \\
+ i \frac{1}{\sqrt{\pi T}} K^{(c)} \int dx \sum_{\alpha} \sum_{n} \sum_{r=1,2} \rho \Psi_{n}^{\alpha}(x) \\
\times \sum_{m} \text{tr} (\tau \odot s_{0}) \left[ \hat{Q}_{\alpha}^{m,-m+n}(x) \right] .
\] (2.7)

with \( A_{\text{dis}}^{(s)} \) from Eq. (2.4f).

Let us first integrate out \( \Delta P \) and \( \Delta \Lambda \) in a Gaussian approximation. A additional quadratic contribution in terms of the order parameter field \( \Psi \) will be obtained from Eq. (2.7) and the last term in Eq. (2.6). Combining it with the \( \Psi^{2} \) term in Eq. (2.5) yields a term

\[
A_{G}[\Psi] = - \sum_{n} \sum_{r=1,2} \sum_{m} \rho \Psi_{n}^{\alpha}(k) \\
\times \left[ 1 + 2\Gamma_{(c)}(k, \Omega_{n}) \right] \rho \Psi_{n}^{\alpha}(-k) ,
\] (2.8a)

where

\[
\Gamma_{(c)}(k, \Omega_{n}) = T \sum_{n_{1}n_{2}} \delta_{n_{1}+n_{2},n} D_{n_{1}n_{2}}(k) ,
\] (2.8b)

is given in terms of

\[
D_{nm}(k) = \varphi_{nm}(k) \left[ 1 - \frac{1}{2\pi N_{F} \tau_{el}} \varphi_{nm}(k) \right]^{-1} .
\] (2.8c)

with

\[
\varphi_{nm}(k) = \frac{1}{V} \sum_{p} G_{sp}(p, \omega_{n}) G_{sp}(p + k, \omega_{m}) .
\] (2.8d)

Here \( G_{sp} \) is the saddle-point Green function from the inverse of Eq. (2.4d). The Theta-function in Eq. (2.8b), which restricts the frequency sum to frequencies that both have the same sign. For small frequencies and wavenumbers, the calculation shows that

\[
A_{G}[\Psi] = - \sum_{k} \sum_{\alpha} \sum_{n} \sum_{r=1,2} \rho \Psi_{n}^{\alpha}(k) u_{2, \rho \Psi_{n}^{\alpha}(-k) ,
\] (2.9a)

with

\[
u_{2} = 1 + O(k^{2}, \Omega_{n}) .
\] (2.9b)

Below we will see the wavenumber and frequency corrections indicated in Eq. (2.8d) are irrelevant for the critical behavior.

Now we can write an effective local action including only soft modes and the superconducting order-parameter fluctuations. The action has the form of

\[
\tilde{A}_{\text{eff}} = A_{G}[\Psi] + A_{NL\sigma M}[q] + A_{c}[\Psi, q] .
\] (2.10a)

Here the nonlinear sigma model part of the action, \( A_{NL\sigma M} \), has been given in Eqs. (2.4), and \( A_{c} \) represents the coupling between \( \Psi \) and \( q \),

\[
A_{c}[\Psi, q] = \frac{i}{\sqrt{\pi T}} K^{(c)} \int dx \sum_{\alpha} \sum_{n} \sum_{r=1,2} \rho \Psi_{n}^{\alpha}(x) \\
\times \sum_{m} \text{tr} \left[ (\tau \odot s_{0}) \hat{Q}_{\alpha}^{m,-m+n}(x) \right] .
\] (2.10b)
For the simplicity, we rewrite the coupling action as

\[ \mathcal{A}_c[b, q] = i \sqrt{\pi T |K^{(c)}|} \int d\mathbf{x} \text{ tr} \left( b(\mathbf{x}) \dot{Q}(\mathbf{x}) \right) . \]  

(2.10c)

Here we define a field

\[ b_{12}(\mathbf{x}) = \sum_{r=1,2} (\tau_r \otimes s_0) r \delta_{12}(\mathbf{x}) , \]  

(2.11a)

with components

\[ r \delta_{12}(\mathbf{k}) = \delta_{\alpha_1 \alpha_2} \sum_n \delta_{n,1+n_2} \Psi_n^\dagger \Psi_n^\alpha(k) . \]  

(2.11b)

Higher-order corrections can be shown be irrelevant the same way as in Ref. 9 Using Eq. (2.6d) in Eq. (2.10c), and integrating out the massive \( P \)-fluctuations, obviously leads to a series of terms coupling \( \Psi \) and \( q, \Psi \) and \( q^2 \), etc. We thus obtain \( \mathcal{A}_c[\Psi, q] \) in form of a series

\[ \mathcal{A}_c[\Psi, q] = \mathcal{A}_{\Psi-q} + \mathcal{A}_{\Psi-q^2} + \ldots \]  

(2.12a)

The first term in this series is obtained by just replacing \( Q \) by \( q \) in Eq. (2.10c),

\[ A_{\Psi-q} = i c_1 T^{1/2} \int d\mathbf{x} \text{ tr} \left( b(\mathbf{x}) q(\mathbf{x}) \right) \]  

(2.12b)

with \( c_1 = \sqrt{\pi |K^{(c)}|} \). The next term in this expansion yields

\[ A_{\Psi-q^2} = i c_2 \sqrt{T} \int d\mathbf{x} \text{ tr} \left( b(\mathbf{x}) q(\mathbf{x}) q^\dagger(\mathbf{x}) \right) \]  

(2.12c)

with \( c_2 = c_1/16 \). Terms of higher order in \( q \) in this expansion will turn out to be irrelevant for determining the critical behavior at the quantum phase transition.

**III. RENORMALIZATION GROUP ANALYSIS**

**A. Gaussian Propagators**

For the purpose of renormalization group analysis, we first need to determine the behaviors of Gaussian Propagators. The Gaussian or second-order action can be written as follows,

\[ \mathcal{A}^{(2)}[\Psi, q] = - \sum_{\mathbf{k}} \sum_{\alpha} \sum_{n=1,2} \text{ tr} \Psi_n^\alpha(\mathbf{k}) u_2(\mathbf{k}) \Psi_n^{-\alpha}(\mathbf{k}) \]

\[ - \frac{4}{G} \sum_{\mathbf{k}} \sum_{i=1,2,3,4} \sum_{r=1,2} \Psi_n^\alpha(\mathbf{k}) i \delta_{i12}(\mathbf{k}) \Gamma_{1234}^{(2)}(\mathbf{k}) \delta_{r12}(\mathbf{k}) \]

\[ -8i \sqrt{\pi T |K^{(c)}|} \sum_{\mathbf{k}} \Psi_n^\alpha(\mathbf{k}) \delta_{i12}(\mathbf{k}) b_{12}(\mathbf{k}) \]  

(3.1a)

where the bare two-point \( q \) vertex has the form

\[ \Gamma_{1234}^{(2)}(\mathbf{k}) = - \delta_{i13} \delta_{24} (k^2 + G \Omega_{n_1-n_2}) + \delta_{i1+2,3+4} \times \delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} 4\pi T G \delta k_c , \]  

(3.1b)

\[ \frac{4}{G} \sum_{\mathbf{k}} \frac{1}{\Omega_{n_1+n_2}^2} (2) \]  

(3.1c)

\[ \times \delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} 4\pi T G K_s , \]  

(3.1d)

with \( K_s = -\pi N^2 \Gamma^{(s)} / 8 \) and \( K_t = -\pi N^2 \Gamma^{(t)} / 8 \). Note that there is an additional repulsive interaction, \( \delta k_c \), in Eq. (3.1b), which comes from the one-loop renormalization of the action. We choose to take this effect into account at Gaussian order. Alternately, it would arise as a higher order disorder effect. For a complete discussion of this term we refer elsewhere. Here we note that it is this term that drives the superconducting transition temperature to zero, and leads to a quantum metal-superconductor phase transition.

If the fermion \( q \) fields are integrated out, an effective action including only the superconducting order parameter can be obtained. In the long wavelength and low frequency limit,

\[ A^{(2)}[\Psi] = - \sum_{\mathbf{k}} \sum_{\alpha} \sum_{n=1,2} \text{ tr} \Psi_n^\alpha(\mathbf{k}) \]

\[ \left( \left( u_2(\mathbf{k}) + \frac{I K^{(c)}}{H \ln \frac{\Omega_0}{|\Omega_n|^2 + k^2 / G H}} \right) \frac{1 + \delta k_c}{I \ln \frac{\Omega_0}{|\Omega_n|^2 + k^2 / G H}} \right) \Psi_n^{-\alpha}(\mathbf{k}) \]

\[ \approx - \sum_{\mathbf{k}} \sum_{\alpha} \sum_{n=1,2} \text{ tr} \Psi_n^\alpha(\mathbf{k}) \]

\[ \left( t + \frac{|K^{(c)}|}{\delta k_c} \right) \frac{1}{\ln \frac{\Omega_0}{|\Omega_n|^2 + k^2 / G H}} \Psi_n^{-\alpha}(\mathbf{k}) , \]  

(3.2)

Here \( \Omega_0 \) is a frequency cutoff on the order of the Debye frequency, and \( t \equiv u_2 - \frac{|K^{(c)}|}{\delta k_c} \) denotes the distance from the mean field or Gaussian critical point. \( A^{(2)}[\Psi] \) is the Gaussian order parameter field theory that was considered in Ref. 7.

For the coupled field theory it is straightforward to compute the two-point correlation functions. For the superconducting order parameter correlations we obtain

\[ \langle \Psi_n^\alpha(\mathbf{k}) \Psi_m^\beta(\mathbf{p}) \rangle = \delta_{\mathbf{k}-\mathbf{p}} \delta_{n,m} \delta_{\alpha \beta} \frac{1}{2} M_n(\mathbf{k}) , \]  

(3.3a)

with

\[ M_n(\mathbf{k}) = \frac{1}{t + \frac{|K^{(c)}|}{\delta k_c} \frac{1}{\ln \frac{\Omega_0}{|\Omega_n|^2 + k^2 / G H}}} . \]  

(3.3b)
Similarly, we find the fermionic propagators
\[ \langle \eta(q_{12}(k))\eta(q_{34}(p)) \rangle = \delta_{k,-p} \delta_{ij} \frac{G}{8} i^{(2)}_{12,34}(k), \tag{3.4a} \]
where
\[ i^{(2)}_{12,34}(k) = \delta_{i3} \delta_{24} D_{n1-n2}(k) - \delta_{1-2,3-4} \delta_{\alpha_1 \alpha_2} \delta_{\alpha_1 \alpha_3} \times 2\pi TGK^{(i)} D_{n1-n2}(k) D_{n1-n2}(k), \tag{3.4b} \]
and
\[ 1.2^{(2)}_{12,34}(k) = -\delta_{i3} \delta_{24} D_{n1-n2}(k) + \delta_{1+2,3+4} \delta_{\alpha_1 \alpha_2} \delta_{\alpha_1 \alpha_3} \times 4\pi TGK^{(i)} D_{n1-n2}(k) D_{n3-n4}(k) \times \frac{1}{1 + 4\pi TGK^{(i)} \ln \frac{\lambda}{|a| + k^2/8Gh^2}}. \tag{3.4c} \]
Here \( D^{(i)} \) is the spin-singlet propagator, which in the limit of long wavelengths and small frequencies reads
\[ D_n^{(i)}(k) = \frac{1}{k^2 + G(H + K^{(i)})\Omega_n}. \tag{3.4d} \]

B. Gaussian level renormalization group

The standard momentum-shell renormalization group (RG) technique will be employed. We use \( b \) as the RG length rescaling factor, and we rescale the wave number and two fields straightforwardly via
\[ k \rightarrow k'/b, \tag{3.5a} \]
\[ \Psi_{n}(k) \rightarrow b^{(2-\eta_k)/2}\Psi_{n}^i(k'), \tag{3.5b} \]
\[ q_{nm}(k) \rightarrow b^{(2-\eta_k)/2}q_{nm}^i(k'). \tag{3.5c} \]
The rescaling of imaginary time, frequency, or temperature is less straightforward. In general, there are two different time scales in the problem, namely, one that is associated with the critical order-parameter fluctuations, and one that is associated with the soft fermionic fluctuations. Therefore, we allow for two different dynamical exponents, \( \zeta_{\Psi} \) and \( \zeta_q \). The temperature may then get rescaled via
\[ T \rightarrow b^{-\zeta_{\Psi}} T', \tag{3.5d} \]
or via
\[ T \rightarrow b^{-\zeta_q} T', \tag{3.5e} \]
How these various exponents should be chosen is discussed below.

In the tree, or zero-loop, approximation the RG equations for the parameters in our field theory are determined as
\[ t' = b^{2-\eta_k} t, \tag{3.6a} \]
\[ \frac{1}{G^H T_{\Psi}} = b^{-2}, \tag{3.6b} \]
\[ \frac{1}{G} = \frac{b^{-\eta_k}}{\bar{G}}, \tag{3.6c} \]
\[ H^T_{q'} = b^{2-\eta_k} H T_q, \tag{3.6d} \]
\[ \eta_q = 0, \tag{3.7a} \]
\[ \zeta_{\Psi} = 2 \tag{3.7b} \]
\[ \zeta_q = 2 \tag{3.7c} \]
Here we find that two dynamical exponents, \( \zeta_{\Psi} \) and \( \zeta_q \), have the same value, which is different from the ferromagnetic systems. We further choose
\[ \eta_q = 2, \tag{3.7d} \]
which is implied within the logarithmic structure of Eq. With these choices, we find that
\[ c_2' = b^{4+\delta}/c_2. \tag{3.7e} \]
As in the ferromagnetic case, there is a critical fixed point where \( c_2 \) is marginal, and the fermions are diffusive, with exponents given by Eqs. However, in contrast to the magnetic case, the coupling constant \( c_2 \) of the term \( A_{\Psi - q^2} \) is RG irrelevant for all \( d \geq 2 \), and so are all higher terms in the expansion in powers of \( q \). We therefore conclude that the Gaussian critical behavior is exact. No additional logarithmic corrections exist here. The most important technical difference that leads to the irrelevance of \( c_2 \) for this quantum phase transition, while for the quantum ferromagnetic transition it was marginal, is that the time scales for the order-parameter fluctuations and the fermions, respectively, are the same. This renders inoperative the mechanism that led the possibility of \( c_2 \) being marginal as in the ferromagnetic case. Physically, the very long range interaction between the order-parameter fluctuations stabilizes the Gaussian critical behavior. This is in agreement with the fact that long-ranged order parameter correlations in classical systems stabilize mean-field critical behavior.
are
\[ \eta_\Psi = 2 - \frac{\ln \ln b^2}{\ln b}, \]  
\[ \nu = \frac{\ln b}{\ln \ln b^2}, \]  
\[ \gamma = 1. \]  
(3.8a)
(3.8b)
(3.8c)
When \( b \to \infty \) we have \( \eta_\Psi = 2 \) and \( \nu = \infty \).

IV. CONCLUSION

We have investigated the quantum metal-superconductor phase transition in the present paper on the basis of an effective local field theory. With a simple renormalization group analysis, we have determined the critical behavior at the quantum metal-superconductor phase transition. In contrast to the disordered ferromagnetic case studied earlier, we showed that the previous results obtained with a nonlocal field theory were correct. The reason is that the two dynamical exponents, \( z_\Psi \) and \( z_q \), are exactly the same for the disordered metal-superconductor quantum phase transition, which physically comes from the strong additional soft modes, or particle-hole excitations, at zero temperature.

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