An exactly soluble model with tunable p-wave paired fermion ground states

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Abstract – Motivated by the work of Kitaev, we construct an exactly soluble spin-\(\frac{1}{2}\) model on a honeycomb lattice whose ground states are identical to \(\Delta_1 p_x + \Delta_1 p_y + i(\Delta_2 p_x + \Delta_2 p_y)\)-wave paired fermions on a square lattice, with tunable pairing order parameters. We derive a universal phase diagram for this general \(p\)-wave theory which contains a gapped A-phase and a topologically non-trivial B-phase. We show that the gapless condition in the B-phase is governed by a generalized inversion (G-inversion) symmetry under \(p_x \leftrightarrow \Delta_1 p_y\). The G-inversion symmetric gapless B-phase near the phase boundaries is described by \((1+1)\)-dimensional gapless Majorana fermions in the asymptotic long-wavelength limit, i.e. the \(c = 1/2\) conformal field theory. The gapped B-phase has G-inversion symmetry breaking and is the weak pairing phase described by the Moore-Read Pfaffian. We show that in the gapped B-phase, vortex pair excitations are separated from the ground state by a finite energy gap.

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Introduction. – The low-energy excitations of a topologically non-trivial phase have remarkable properties. A well-known example is the quasihole excitation of the Laughlin state in the fractional quantum Hall effect (FQHE), which carries fractional charge and anyon statistics. The most intriguing possibility of a topological phase of matter is the non-Abelian FQHE proposed by Moore and Read [1] for even-denominator filling factors, \(e.g.\ \nu = \frac{5}{2}\) [2]. The quasiparticle excitations, vortices in the Moore-Read Pfaffian wave function, have non-Abelian statistics [1] which plays a fundamental role in topological quantum computation [3,4]. The key ingredient in the Moore-Read Pfaffian state is that the topologically nontrivial part of the wave function is asymptotically the same as the pair wave function in a \(p_x + ip_y\)-wave fermion paired state [5] in the weak pairing phase [6]. The existence of the exotic non-Abelian statistics is thus likely a generic property of the more tangible time-reversal symmetry (T-symmetry) breaking \(p\)-wave pairing states.

Recently, Kitaev constructed a spin-\(\frac{1}{2}\) model with link-dependent Ising couplings on the honeycomb lattice [4]. Kitaev showed that the model is equivalent to a bilinear Majorana fermion model and is thus exactly soluble. A topological non-trivial gapless phase (the B-phase) was discovered. (A Jordan-Wigner transformation to a model with two-Majorana fermions for this model has been proposed in [7].) In the presence of a T-symmetry breaking term, the B-phase becomes gapped and exhibits vortex excitations obeying non-Abelian statistics. The model also has a topologically trivial, gapped A-phase. The two phases are separated by a topological phase transition via a gapless critical state. These properties strongly resemble the weak and strong pairing phases and the critical state in the \(p_x + ip_y\)-wave paired states of spinless fermions [6]. The interconnections among the Kitaev model, the \(p\)-wave paired fermions, and the Moore-Read Pfaffian and its excitations have not been well understood previously. In particular, it is important to understand the universal properties among these systems, analogous to finding the universality class in statistical mechanics models. In this paper, we show that the Kitaev model is a special case of a broader class of two-dimensional spin-\(\frac{1}{2}\) models whose ground states are equivalent to general paired fermion states in the \(p\)-wave channel. Indeed, the vortex-free Kitaev Hamiltonian maps to an exact BCS fermion pairing model with \(i(p_x + p_y)\)-wave attractions on a square lattice [8]. Our generalized model includes both the \(p_x + ip_y\)-wave paired states and the original...
Kitaev model as special limits. It is an exactly soluble model with minimal three and four-spin interactions. We show that the vortex-free ground states of this model are described by $\Delta_1 x p_x + \Delta_1 y p_y + i(\Delta_2 x p_x + \Delta_2 y p_y)$-wave paired fermion states with tunable pairing order parameters $\Delta_{ab}$ on a square lattice. We find that the structure of the phase diagram is determined by the geometry of the underlying Fermi surface. It contains both topologically trivial (A) and nontrivial (B) phases. The A-phase is always gapped and corresponds to the strong pairing phase. The B-phase can be either gapped or gapless even if T-symmetry is broken. We find that gapless excitations in the B-phase is protected by a generalized inversion (G-inversion) symmetry under $p_x \leftrightarrow \Delta_1 x p_y$ and the emergence of a gapped B-phase is thus tied to G-inversion symmetry breaking. For instance, the $p_x + i p_y$-wave paired state is gapped while $p_y + i p_x$-wave paired state is gapless although they both break the T-symmetry. The critical states of the A-B phase transition remains gapless whether or not T- and G-inversion symmetries are broken, indicative of its topological nature. Indeed, if all $\Delta_{ab}$ are tuned to zero, the topological A-B phase transition is from a band insulator to a free Fermi gas. The Fermi surface shrinks to a point zero at criticality.

We show that the gapped B-phase is a weak pairing state while the G-inversion symmetric ground states are extended. The gapless phase was not well-understood before. We show that the effective theory near the phase boundary corresponds to $(1+1)$-dimensional massless Majorana fermions in the long-wavelength limit, i.e., a $c=1/2$ conformal field theory or the 2-dimensional Ising model. The vortex excitations are important in the family of Kitaev models since the vortex excitations may obey anyon statistics [4]. The vortex excitation energies have been numerically estimated in the A-phase [9] and the B-phase [10]. We study the vortex excitations in the gapped B-phase in the continuum limit and show that the vortex-pair excitations cost a finite energy. This is consistent with the results of numerical calculations [10] and suggests that vortex excitations may have well-defined statistics.

The generalization of the Kitaev model. – We extend the Kitaev model on the honeycomb lattice by introducing minimal three- and four-spin terms in the Hamiltonian,

$$H = -J_x \sum_{x \text{-links}} \sigma_x^x \sigma_j^x - J_y \sum_{y \text{-links}} \sigma_y^y \sigma_j^y - J_z \sum_{z \text{-links}} \sigma_z^z \sigma_j^z - \kappa_x \sum_b \sigma_b^x \sigma_{b+e_x+e_y}^x - \kappa_x \sum_w \sigma_w^x \sigma_{w+e_x+e_y}^x$$

$$- \kappa_y \sum_b \sigma_b^y \sigma_{b+e_x+e_y}^y - \kappa_y \sum_w \sigma_w^y \sigma_{w+e_x+e_y}^y - \lambda_x \sum_b \sigma_b^z \sigma_{b+e_x+e_y}^z - \lambda_x \sum_w \sigma_w^z \sigma_{w+e_x+e_y}^z$$

$$- \lambda_y \sum_b \sigma_b^z \sigma_{b+e_x+e_y}^z - \lambda_y \sum_w \sigma_w^z \sigma_{w+e_x+e_y}^z,$$

(1)

where $\sigma^{x,y,z}$ are Pauli matrices, $x$, $y$, $z$-links are shown in fig. 1 (upper panel), " $w$" and " $b$" label the white and black sites of lattice, and $e_x$, $e_y$, $e_z$ are the positive unit vectors, which are defined as, e.g., $e_{12} = e_z$, $e_{23} = e_x$, $e_{61} = e_y$. $J_x,y,z$, $\kappa_{x,y}$, and $\lambda_{x,y}$ are tunable real parameters. The original Kitaev model has $\kappa_0 = \lambda_0 = 0$, $\alpha = x,y$. Adding a T-symmetry breaking external magnetic field corresponds to $\kappa_x = \kappa_y = \kappa \neq 0$ and a $\kappa_z$-term [4,7]. It is important to note that the generalized Hamiltonian maintains the $Z_2$ gauge symmetry acted by a group element, e.g.,

$$W_P = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^z$$

with $[H, W_P] = 0$. In fact, one can construct $Z_2$ gauge invariant spin models with higher multi-spin terms, e.g., $\sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^z$, $\sigma_5^x \sigma_6^y \sigma_7^z \sigma_8^z$, and so on. One can also add the “$z$”-partner of the $\kappa_{x,y}$ and $\lambda_{x,y}$ terms so that the model becomes more symmetric. However, adding these terms or not will not affect our result in this paper as we will explain later.

We now write down the Majorana fermion representation of this spin model. Let $b_{x,y,z}$ and $c$ be the four kinds of Majorana fermions with $b_{x,y,z}^2 = 1$ and $c^2 = 1$. The spin operator is given by

$$\hat{a}^a = \frac{i}{2} \left( b_a c - \frac{1}{2} c \epsilon_{abc} b_b b_c \right).$$

Restricting to the physical Hilbert space, one needs to require [4]

$$D = b_y b_z c = 1$$

and thus $\hat{a}^a = i b_a c$. The Hamiltonian now reads

$$H = i \sum_a \sum_{a \text{-links}} J_a b_a^a c_j^a - i \sum_b K_{b,b+e_x+e_y} b_b c_{b+e_x+e_y}$$

$$- i \sum_w K_{w+e_x+e_y} c_{w+e_x+e_y} c_{w+e_x+e_y} c_{w+e_x+e_y}$$

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\[-i \sum_b K_{b}^{x}\xi_{s,b}e^{i\varepsilon_{s,b}}e^{2\varepsilon_{s,b}+\varepsilon_{s}} + i \sum_w K_{w}^{x}\xi_{w,s}e^{i\varepsilon_{w,s}}e^{2\varepsilon_{w,s}+\varepsilon_{s}} + y\text{-partners},\]

where \(K_{b}^{x}\) is related to the hopping between the \(b\)-links. The pairing of the link fermions reflects the \(\lambda\)'s positive.

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\[H_{0} = i\tilde{\lambda}_{x} \sum_{s}(\xi_{s,b}e^{-\varepsilon_{s,b}} - \varepsilon_{s,b}e^{-\varepsilon_{s,b}}),\]

\[+i\tilde{\lambda}_{y} \sum_{s}(\xi_{s,b}e^{-\varepsilon_{s,b}} + \varepsilon_{s,b}e^{-\varepsilon_{s,b}}),\]

\[+i\kappa_{s} \sum_{s}(\xi_{s,b}e^{-\varepsilon_{s,b}}e^{2\varepsilon_{s,b}+\varepsilon_{s}} + \varepsilon_{s,b}e^{-\varepsilon_{s,b}}e^{2\varepsilon_{s,b}+\varepsilon_{s}}) + y\text{-partners},\]

where \(\tilde{\lambda}_{x} = \frac{\tilde{\lambda}_{x}}{2}\) and \(\tilde{\lambda}_{y} = \frac{\tilde{\lambda}_{y}}{2}\).

\[d_{s} = \frac{(\xi_{s,b} + ic_{s,w})}{2}, \quad d_{s}^{\dagger} = \frac{(\xi_{s,b} - ic_{s,w})}{2},\]

\[H_{0} \text{ becomes}\]

\[H_{0} = J_{x} \sum_{s}(d_{s}^{\dagger}d_{s} - 1/2) + J_{y}(d_{s}^{\dagger}d_{s} + \varepsilon_{s} - d_{s}d_{s}^{\dagger} + \varepsilon_{s}),\]

\[+i\tilde{\lambda}_{x} \sum_{s}(d_{s}^{\dagger}d_{s} - \varepsilon_{s} - d_{s}d_{s}^{\dagger} + \varepsilon_{s}),\]

\[+i\kappa_{s} \sum_{s}(d_{s}^{\dagger}d_{s} + \varepsilon_{s}) + (d_{s}^{\dagger}d_{s} + \varepsilon_{s}) + y\text{-partners}.\]

This is a quadratic model of spinless fermions \(d_{s}\) on the square lattice (fig. 2, left panel) with general \(p\)-wave pairing. If we include the \(z\) partner of the three and four spin terms in eq. (1), we have additional corresponding terms in eq. (4) which are the next-nearest neighbor terms in the square lattice. These terms will not qualitatively affect our result. Returning to Majorana fermions, a link fermion is a superposition of two Majorana fermions in a link. Therefore, the paring of the link fermions reflects the “pairing” of Majorana fermions.

After a Fourier transformation, eq. (4) becomes

\[H_{0} = \sum_{p} \xi_{p}d_{p}^{\dagger}d_{p} + \frac{\Delta_{1,p}}{2}(d_{p}^{\dagger}d_{-p} + d_{p}d_{-p}) + \frac{\Delta_{2,p}}{2}(d_{p}^{\dagger}d_{-p} - d_{p}d_{-p}),\]

where the dispersion and the pairing functions are

\[\xi_{p} = J_{z} - \tilde{\lambda}_{x}\cos px - \tilde{\lambda}_{y}\cos py,\]

\[\Delta_{a,p} = \Delta_{a}\sin px + \Delta_{a}\sin py, \quad a = 1, 2,\]

with \(\Delta_{1,1}(\xi) = \kappa_{0}(\xi)\) and \(\Delta_{2,2}(\xi) = \lambda_{0}(\xi)\). We have thus shown that the ground state of the extended Kitaev model in eq. (1) are equivalent to general \(p\)-wave paired fermion states. The quasiparticle excitations are given by the BdG equations

\[E_{p}^{u,v} = \xi_{p}u_{p} - \Delta_{p}v_{p}, \quad E_{p}^{v,u} = -\xi_{p}v_{p} - \Delta_{p}u_{p},\]

where \(E_{p} = \sqrt{\xi_{p}^{2} + (\Delta_{1,p})^{2} + (\Delta_{2,p})^{2}}\) is the dispersion, \(\Delta_{p} = \Delta_{1,p} + i\Delta_{2,p}\), and \((u_{p}, v_{p})\) are the coherence factors with \(|u_{p}|^{2} = \frac{1}{2}(1 + \frac{\xi_{p}}{E_{p}})|v_{p}|^{2} = \frac{1}{2}(1 - \frac{\xi_{p}}{E_{p}})\) and \(v_{p} = -(E_{p} - \xi_{p})/\Delta_{p}^{*}\).

Phase diagram in terms of the \(p\)-wave states. – We now turn to the properties of this \(p\)-wave paired state of link fermions. It is instructive to consider the free fermion dispersion. The condition \(\xi_{p} = 0\) defines a topological transition between a bound insulator and a metal with a Fermi surface in the absence of pairing. The solution is given by \(|\cos p_{x}^{*}| = |\cos p_{y}^{*}| = 1\), \(p_{x}^{*} = (0, 0), (\pm \pi, 0), (\pm \pi, 0), (\pm \pi, \pm \pi)\), where \(J_{z} + J_{x} + J_{y} = 0\). Without loss of the generality, one considers only \(J_{z} > 0\) and \(J_{x} > 0\). Then, \(\xi_{p}^{*} = 0\) corresponds to the inner triangle of the (1,1,1)-cross-section in the \((J_{z}, J_{y}, J_{x})\)-space (see fig. 2 (right panel)). Notice that the \(p\)-wave pairing gap functions \(\Delta_{a,p}\) vanish at \(p^{*}\) and therefore, this triangle is the gapless critical boundary separating the A and B phases. Outside the triangle, \(\xi_{p}^{*} > 0\). Thus the A-phase is gapped. In the limit \(p \rightarrow p^{*}\), the pair correlation \(g_{p} \equiv v_{p}/u_{p}\) is analytic near \(p^{*}\), implying tightly bound pairs in positional space and hence the A-phase as the strong-pairing phase (see below for detailed discussions [6]).
global structure of the phase diagram is invariant in the
generalized \((\hat{J}_x, \hat{J}_y, \hat{J}_z)\)-space because our minimal three-
and four-spin extension does not change the topology of the
underlying Fermi surface.

The nature of the B-phase is much more intriguing.
Inside the triangle, \(\xi_p, \Delta_{1,p} \) and \(\Delta_{2,p} \) can be zero individually.
The gapless condition \((E_p = 0)\) requires all three to be zero at a common \(p^*\). This can only be achieved if i) one of the \(\Delta_{i,p} = 0 \) or ii) \(\Delta_{1,p} \propto \Delta_{2,p} \). If either i) or ii) is true, \(\xi_p \) and \(\Delta_{p} \) can vanish simultaneously, i.e. \(E_p = 0\) at \(p^*\), and the paired state is gapless. Otherwise, the B-phase is gapped. Note that contrary to conventional
symmetrybreaking, with \(\eta \) in the BCS wave function \(\xi^\dagger \eta_x \eta^\dagger \eta_y \) with \(\eta = \frac{\Delta}{\Delta_x} \) with \(a = 1 \) or 2 and for nonzero \(\Delta \). We refer to this as a generalized inversion \((G\text{-inversion})\) symmetry since it reduces to the usual mirror reflection when \(\eta = 1\). This (projective) symmetry protects the gapless nature of fermionic excitations and may be associated with the underlying quantum order [12]. kitaev’s original model has \(\Delta_{1,i} = 0\), and is thus G-inversion invariant and gapless. The magnetic field perturbation [4] breaks this G-inversion symmetry and the fermionic excitation becomes gapped. A special case with G-inversion symmetry breaking is \(\Delta_{p} \propto \sin p_x + \sin p_y\), i.e. the \(p_x + ip_y\)-wave paired state discussed by Read and Green in the continuum limit [6].

The Fermi-fermion ground-state wave function in the general \(p\)-wave paired state can be written down as a Pfaffian as for a 2D Fermi-fermion system by putting \(\psi(r_1, \ldots, r_N) = 1/(2\sqrt{N/2})\times \prod_{i=1}^{\infty} g(p_{\pm (\delta_{x,i} - \delta_{y,i})})\), where \(g(r)\) is the “pair correlation”, i.e. the Fourier transform of \(g = v_F/v_p\) in the BCS wave function \(\Omega = \prod_p \nu_p |1/2 \times \exp\left(\frac{i}{2} \sum_{p} gdp_{p}^{\dagger} dp_{p}\right)|0\). The wave function exhibits very different behaviors in the long-wavelength limit in different parameters [13]. In the A-phase, \(\xi_p > 0\) as \(p \rightarrow 0\), thus \(\xi_p \propto \Delta_p\). The analyticity of \(g\) leads to \(g(0) \propto e^{-\pi \epsilon^2}\) as in the strong pairing phase of a pure \(p_x + ip_y\) state [6]. In the gapped B-phase with G-inversion symmetry breaking, \(\xi_p < 0\) as \(p \rightarrow 0\). Defining \(p_i^1 = \Delta_{i,p} p_i\) with \(a = 1,2\) and \(i = x,y\), it follows that \(g_p \propto \frac{i}{\sin \frac{\pi}{p_i^1} + \sin \frac{\pi}{p_i^2}}\), leading to \(g(r) = \frac{1}{r^{1+2p_i}}\) with \(x' = 1/2\xi_p x\) and thus a weak-pairing phase. Identifying \(z' = x' + ix'\), we see that the ground state of the gapped B-phase corresponds exactly to the Moore-Read Pfaffian. It is easy to show that, \(u\) and \(v^*\) obey the same BDG equation in this general p-wave paired state, such that the anti-particle of the quasiparticle \(\psi = (u, v)\) is itself, i.e., it is Majorana fermion obeying Dirac equations in 2+1 dimensions [6].

We now discuss the nature of the gapless B-phase in the general phase with G-inversion symmetry. In this case, \(E_p = 0\) at \(p = \pm p^*\) which are the solutions of \(\xi_p = 0\) and, say, \(\Delta_p = \Delta_{1,p} = 0\). At \(p^*\), the fermion dispersions are generally given by 2D Dirac cones. However, by a continuous variation of the parameters, one can realize a dimensional reduction near the phase boundary where the effective theory is in fact a \((1+1)\)-dimensional conformal field theory in the long-wavelength limit. Let us consider parameters that are close to the critical line with \([\sin p^2_x] < \cos p^2_x\) where \(g_0 = \cos \xi_p \Delta_{1,p} \cos p^2_x + q_0 \Delta_{1,p} \cos p^2_x = \cos \xi_p\) with \(q = p - p^*\). Doing the Fourier transform, we find

\[
\delta(E) = \int dq_x dq_y e^{iq_x x + iq_y y} \text{sgn}(q_x^0) = \frac{\delta(y)}{x^v}.
\]
for arbitrary $p$, $(u_p, v_p)$ maps the $p$-sphere to the upper hemisphere and the winding number is zero. That is, the topological number $\nu = 0$ in the strong pairing phase.

In the weak pairing phase, $u_p \rightarrow 0$ and $v_p \rightarrow 1$ as $p \rightarrow 0$. This means that the winding number is nonzero (at least wrapping once). For our case, the winding number can be directly calculated and is given by

$$\nu = \frac{1}{4\pi} \int dp_x dp_y \psi^* \cdot (\partial_{p_x} n_p \times \partial_{p_y} n_p) \psi = 1.$$  \hspace{1cm} (9)

Defining $P(p) = \frac{1}{2}(1 + n_p \cdot \sigma)$, which is the Fourier component of the projection operator to the negative spectral space of the Hamiltonian, this winding number can be identified as the spectral Chern number defined by Kitaev [4]

$$\nu = \frac{1}{2\pi i} \int \text{Tr}[P_-(\partial_{p_x} n_p \cdot \partial_{p_y} n_p - \partial_{p_y} n_p \cdot \partial_{p_x} n_p)] dp_x dp_y,$$  \hspace{1cm} (10)

where $P_+ = I - P$ is a projective operator. This spectral Chern number vanishes in the strong pairing A-phase but takes an integer value in the weak pairing B-phase. Thus, the phase transition from A to B is a topological phase transition.

**Vortex excitations.** We have discussed the universal behaviors of the ground state. We now turn to discuss the $Z_2$ vortex excitation in the spin model which corresponds to setting $W_p = -1$ for a given plaquette. The Hamiltonian in the Majorana fermion representation is bilinear and the energies of the vortices can be estimated both in the A-phase [9] and the B-phase [10]. However, it remained difficult to obtain analytical solutions of the wave functions with two well-separated vortices. We have shown that the ground-state sector is equivalent to the $p_x + ip_y$ pairing theory for fermions on the square lattice. Therefore, the Pfaffian state is the ground-state wave function in the continuum limit in the weak pairing phase. Our strategy is to evaluate the energy of the trial wave function containing vortices above the Pfaffian state in the continuum limit. For two well-separated half-vortices located at $w_1$ and $w_2$ shown in fig. 1 (lower panel), the Moore-Read trial wave function has been studied well [1,6] and is given by

$$\Psi(z_1, \ldots z_N ; w_1, w_2) \propto \text{Pf}(g'(z_1, z_j; w_1, w_2)),$$

$$g'(z_1, z_2; w_1, w_2) \propto \frac{(z_1 - w_1)(z_2 - w_2) + (w_1 + w_2)}{z_1 - z_2}.$$  \hspace{1cm} (11)

The second quantized state corresponding to this wave function reads

$$|w_1, w_2\rangle \propto \exp \left\{ \frac{1}{2} \sum_{r_1, r_2} g'(z_1, z_2; w_1, w_2) d_{r_1}^{+} d_{r_2} \right\}.$$  \hspace{1cm} (12)

Performing a Fourier transformation, we have

$$|w_1, w_2\rangle \propto \exp \left\{ \frac{1}{2} \sum_{K} g'_K(K) d_K^{+} d_{-K} \right\},$$

where $K = k_2 - k_1$ and $K = k_1 + k_2$ are the relative and the total momenta of the pairs and $g'_K(K)$ is the Fourier transform of $\delta H$. One can show that,

$$g'_K(K) \sim \left( \frac{1}{k} - \frac{A}{w_1 w_2 k^2} \right) \delta(K) + \frac{1}{k} \left( \frac{B}{w_1 w_2 k^2} - \frac{(w_1 + w_2) C}{w_1 w_2 k^2} \right) = g'_K(K) + \frac{1}{k} \tilde{g}(K),$$  \hspace{1cm} (13)

where $A$, $B$ and $C$ are positive constants and $\tilde{g}(K)$ is independent of $K$. Thus,

$$|w_1, w_2\rangle \propto \exp \left\{ \frac{1}{2} \sum_{k} g'_K(K) d_k^{+} d_{-K} \right\}.$$

Such a vortex pair is shown in fig. 1 where the red $z$-links have $u_{w_1} = -1$ and all others have $u_{w_2} = 1$. The corresponding Hamiltonian can be written as $H = H_0 + \delta H$, where $H_0$ is the vortex-free Hamiltonian and $\delta H$ is the vortex part. The latter is expressed as a sum of the pairing and chemical potential terms according to eq. (5) over the red $z$-links extending in the $\xi$-direction (i.e., the direction with $x = y$) between the vortices. It is straightforward to show that $\delta H$ has the following expectation value in the vortex state:

$$\langle w_1, w_2 | \delta H | w_1, w_2 \rangle \propto \sum_{p_\xi} i e^{i w_1 p_\xi} - e^{i w_2 p_\xi} \langle p_\xi, \bar{p}_\xi \rangle = 0, $$  \hspace{1cm} (14)

where $f(p_\xi, \bar{p}_\xi)$ is an analytical function of $p_\xi + \bar{p}_\xi$. This means that there is no continuum spectrum above the vortex pairs and then the vortex pairs are also separated from other higher-energy excitations. On the other hand, one can check that since $|H_0 + \sum_{K,k} g'_K(K \neq 0) d_k^{+} d_{-K} = 0$, the $K \neq 0$ sector does not play a nontrivial role in calculating the energy $E_v$ of such a vortex pair. The latter is given by

$$E_v = \langle w_1, w_2 | H | w_1, w_2 \rangle = \langle w_1, w_2 | H_0 | w_1, w_2 \rangle$$

$$= \sum_{k} E_k |u_k\delta g_k|^2 \langle w_1 w_2 | d_k^{+} d_k | w_1 w_2 \rangle$$

$$= \sum_{k} E_k |u_k\delta g_k|^2 / (1 + |g_k^{(0)}|^2),$$  \hspace{1cm} (15)

where $g_k^{(0)} = g_k' + \frac{i}{k} \left( \frac{B}{w_1 w_2 k^2} - \frac{(w_1 + w_2) C}{w_1 w_2 k^2} \right) |k\rightarrow 0$ and $\delta g_k = g_k - g_k^{(0)}$. Physically, the factor $\langle w_1 w_2 | d_k^{+} d_k | w_1 w_2 \rangle = 1 - |g_k^{(0)}|^2 / (1 + |g_k^{(0)}|^2)$ in eq. (15) is the quasihole distribution when the two vortices are located at $w_1$ and $w_2$. Thus, $E_v$ indeed corresponds to the energy cost to excite the vortex pair. We have evaluated the vortex pair energy $E_v$ in different limits. First, if $w_1$ and $w_2$ were sent to infinity before $K \rightarrow 0$, then $g_k^{(0)} \rightarrow g_k$ and we
recover the ground state. Second, if $K \rightarrow 0$, while $w_1$ and $w_2$ remain finite, then $E_k |\psi_{g,k}\rangle^2/(1 + |g_k^0|^2) \sim E_k |\psi_{k}\rangle^2$, which tends to $|k|^2$ in both the small and large $k$ limits. The excitation energy is thus high but finite due to the short distance cut-off. (Recall that $g_k < 0$ in the B-phase.) It is independent of $w_1$ and $w_2$ and as a result the vortices are deconfined. The third case is when the vortices are far from the origin such that $|Kw_1|$ and $|Kw_2|$ are finite. In the short-distance, large-$k$ limit, $E_k |\psi_{g,k}\rangle^2/(1 + |g_k^0|^2) \sim |k|^{-4}$. On the other hand, in the long-wavelength limit with $|k| \rightarrow 0$, $|\psi_{k}\rangle \sim |k|$, $|g_k^0| \rightarrow |k|^{-3}$ and $E_k \rightarrow$ constant such that $E_k |\psi_{g,k}\rangle^2/(1 + |g_k^0|^2) \sim |k|^2 \rightarrow 0$. As a result, the vortex pair energy $E_v$ is free of infrared divergences and is only weakly dependent on $w_1 - w_2$. Therefore, the vortices are also deconfined. Finally, since $E_k$ increases with the pairing parameters $\kappa$ and $\lambda$ in eq. (4), $E_v$ is expected to increase with increasing paring gap parameters. These results are consistent with the finite size numerical calculations of the vortex pair energy in the lattice model [10]. Our analytical results suggest that the vortex pair described by eq. (11), while costing a high energy in the bulk, corresponds to low energy excitations near the edge of the system.

We note an important difference between this $p$-wave theory and a conventional $p$-wave superfluid: Instead of spontaneously breaking the $U(1)$ symmetry in an usual $p$-wave superfluid, only the discrete $Z_2$ symmetry is broken and the $U(1)$ symmetry is absent in the present model. The vortices studied here are thus $Z_2$ vortices instead of $U(1)$ vortices. As a consequence, in the gapped B-phase, the vortices are in the deconfinement phase [14] instead of being logarithmically confined in the $p$-wave superfluid. This fact can be easily seen because $\Delta_{xx(y)}$ are real and there is no $U(1)$ phase factor whose gradient gives rise to a vector field of the vortex. Our analysis of the vortex pair energy also shows this difference.

The finiteness of $E_v$ and the vanishing of $\langle \delta H \rangle = 0$ imply that the vortex excitations are separated either from the ground state or other excitations. This is consistent with analysis of Read and Green on the $U(1)$ vortex excitations in the $p$-wave paired state [6,15]. To determine how close the Moore-Read vortex state is to the exact vortex excitations in this model requires a numerical calculation of the overlapping between the exact eigenstates and the Moore-Read vortex wave functions. A more important question is the realization of the Read-Moore four-vortex state which has a two-fold degeneracy with the vortices obeying non-Abelian statistics [1]. We leave these studies to future works.

Conclusions. – We have constructed an exactly soluble spin model with two-, three- and four-spin couplings on a honeycomb lattice. The ground state sector of this model on the honeycomb lattice is mapped to a $p$-wave paired state of the link fermions on a square lattice with general pairing parameters. Based on the general $p$-wave paired states, we analyzed the phase diagram of the system and the properties of topologically different phases. We found that our phase diagram is universal and includes both the Kitaev model and the Pfaffian state in its universality class.

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