A graceful entrance to braneworld inflation

James E. Lidsey and David J. Mulryne

Astrology Unit, School of Mathematical Sciences,
Queen Mary, University of London, London E1 4NS, U.K.

Positively-curved, oscillatory universes have recently been shown to have important consequences for the pre-inflationary dynamics of the early universe. In particular, they may allow a self-interacting scalar field to climb up its potential during a very large number of these cycles. The cycles are naturally broken when the potential reaches a critical value and the universe begins to inflate, thereby providing a ‘graceful entrance’ to early universe inflation. We study the dynamics of this behaviour within the context of braneworld scenarios which exhibit a bounce from a collapsing phase to an expanding one. The dynamics can be understood by studying a general class of braneworld models that are sourced by a scalar field with a constant potential. Within this context, we determine the conditions a given model must satisfy for a graceful entrance to be possible in principle. We consider the bouncing braneworld model proposed by Shtanov and Sahni and show that it exhibits the features needed to realise a graceful entrance to inflation for a wide region of parameter space.

I. INTRODUCTION

The inflationary scenario is presently the favoured model for large-scale structure formation in the universe. (For reviews, see). Inflation arises whenever the universe undergoes a phase of accelerated expansion. Within the context of relativistic cosmology, a necessary and sufficient condition for acceleration is a violation of the strong energy condition, i.e., \( \rho + 3p < 0 \). This is equivalent to the condition \( \gamma < 2/3 \), where \( \gamma \equiv (\rho + p)/\rho \) defines the equation of state parameter. In the simplest versions of the scenario, such a condition is realised by a scalar ‘inflaton’ field that is rolling sufficiently slowly down its self-interaction potential, \( \dot{\phi}^2 < V(\phi) \).

Given the recent successes of inflation when confronted with observations of the Cosmic Microwave Background (CMB) radiation and high redshift surveys, a key question to address is how the initial conditions for successful inflation were established in the very early universe. This question was recently examined within the context of Loop Quantum Cosmology (LQC), which is the application of Loop Quantum Gravity to symmetric states. (For reviews, see). In this scenario, semi-classical modifications to the standard Friedmann equations cause a minimally coupled scalar field to behave as an effective phantom fluid below a critical value of the scale factor, \( a_* \), with an effective equation of state, \( \gamma_{\text{eff}} < 0 \). This causes a collapsing, isotropic, spatially closed Friedmann-Robertson-Walker (FRW) universe to undergo a non-singular bounce into an expansionary phase. On the other hand, if the field satisfies the strong energy condition above \( a_* \), the universe will eventually recollapse. The overall result, therefore, is that the universe oscillates between cycles of contraction and expansion.

If the field’s kinetic energy dominates its potential energy during these cycles, it will effectively behave as a massless field with its value either increasing or decreasing monotonically with time. In principle, therefore, the field may gradually roll up its potential (assuming implicitly that it is evolving along a region of the potential that is increasing in magnitude). As a result, the potential energy will gradually increase during each successive cycle, with the net result that it becomes progressively harder to satisfy the strong energy condition on scales above \( a_* \). Eventually, therefore, a cycle will be reached when the strong energy condition becomes violated during the expansionary phase of the cycle (\( a > a_* \)), and this will initiate a phase of inflation. This implies that slow-roll inflation may be possible even if the field is initially located in a region of the potential that would not generate accelerated expansion, such as a minimum. This effect has been termed a ‘graceful entrance’ to inflation.

A key feature of the dynamics described above is that the universe is able to oscillate because the effective equation of state satisfies \( \gamma_{\text{eff}} < 2/3 \) for \( a < a_* \) and \( \gamma_{\text{eff}} > 2/3 \) for \( a > a_* \). Since these are rather weak conditions, it is important to investigate whether similar effects are possible in other cosmological scenarios, such as the braneworld paradigm. This is the purpose of the present paper. The braneworld scenario is motivated by string/M-theory and has attracted considerable attention in recent years. (See for reviews). Our observable four-dimensional universe is interpreted as a co-dimension one brane propagating in a five- (or higher-) dimensional ‘bulk’ space. A number of bouncing braneworld models have been developed to date. In the model proposed by Shtanov and Sahni (S-S), for example, the extra bulk dimension is timelike and this results in modifications to the effective four-dimensional Friedmann equation that induce a non-singular bounce.

The paper is organized as follows. In Section II, we outline how a phase space analysis for a field with a constant potential can yield valuable insight into the cosmic dynamics that leads to a graceful entrance for inflation. We then proceed in Section III to determine the
necessary and sufficient conditions for graceful entrance when the Friedmann equation has an arbitrary dependence on the energy density of the universe. We consider the S-S braneworld scenario as an explicit example in Sections IV–VI and determine the regions of parameter space where a graceful entrance is possible. We conclude with a discussion in Section VII.

II. THE PHASE SPACE DESCRIPTION OF GRACEFUL ENTRANCE TO INFLATION

The oscillatory dynamics of isotropic universes sourced by a scalar field with a constant potential proves very useful when determining the necessary conditions for a graceful entrance to inflation for a given cosmological model. In general, the dynamics is determined by three evolution equations for the scale factor, the Hubble parameter and the velocity of the scalar field, respectively. In addition, there is the Friedmann equation, which represents a constraint. This implies that the dynamics can be expressed as a two-dimensional system of equations and presented on a two-dimensional phase space.

In the LQC scenarios considered previously, it proved convenient to parametrize the phase space in terms of the scale factor and Hubble parameter $[18]$. The qualitative nature of the phase space that can result in a graceful entrance to inflation is illustrated in Fig. 1. For a positive potential taking values in the range $0 < V < V_{\text{crit}}$, where $V_{\text{crit}}$ is some critical value that is determined by the parameters of the model, the phase space contains a centre and a saddle equilibrium point. Two types of behavior are therefore possible. Trajectories that are sufficiently close to the centre encircle it and represent trajectories that undergo eternal oscillations. On the other hand, trajectories which pass above the saddle point represent initially collapsing universes which evolve through a bounce into an eternal (de Sitter) inflationary era. This is illustrated qualitatively in the top panel of Fig. 1.

If the value of the potential is increased, the effect on the phase space is such that the centre and saddle point are positioned closer to one another. At the critical value $V = V_{\text{crit}}$, the points merge and disappear. The phase space for $V > V_{\text{crit}}$ therefore contains no equilibrium points and all trajectories represent initially contracting universes which bounce and subsequently inflate, as shown in the lower panel of Fig. 1.

This simplified phase space structure enables us to develop a graceful entrance mechanism for more realistic cosmologies, where the potential is field-dependent. Let us assume that the scalar field is initially located in a region of the potential such that $V(\phi) < V_{\text{crit}}$ and that the universe is undergoing oscillations about the centre equilibrium point. The dynamics in this regime will be dominated by the field’s kinetic energy and we will further suppose that the field’s velocity is such that it moves up the potential. If the change in the magnitude of the potential is negligible over a given cycle, this cycle can still be represented as a trajectory in the phase space associated with an instantaneous value of the potential, as illustrated in the top panel of Fig. 1. Over a large number of cycles, however, this ‘instantaneous phase space’ will become significantly modified. Indeed, if the potential energy increases monotonically as the field evolves, the region of phase space in which oscillations take place, as indicated by the region enclosed by the dotted line, and trajectories which evolve into an inflationary phase. The centre equilibrium point about which oscillations occur is marked with a circle, the separatrix is represented by the dotted line and the saddle equilibrium point occurs at the self-intersection of the separatrix. In the lower panel ($V > V_{\text{crit}}$), there exist no centre or saddle equilibrium points and all trajectories evolve into an inflationary phase.

FIG. 1: Illustrating the general features of the phase space discussed in the text that are relevant to a graceful entrance to inflation. The variables are $\{H, a\}$ and all trajectories evolve in an anti-clockwise direction. The top diagram corresponds to $0 < V < V_{\text{crit}}$ and the bottom panel to $V > V_{\text{crit}}$. In the former case, the important features for a graceful entrance are an area of phase space in which oscillations take place, as indicated by the region enclosed by the dotted line, and trajectories which evolve into an inflationary phase. The centre equilibrium point about which oscillations occur is marked with a circle, the separatrix is represented by the dotted line and the saddle equilibrium point occurs at the self-intersection of the separatrix. In the lower panel ($V > V_{\text{crit}}$), there exist no centre or saddle equilibrium points and all trajectories evolve into an inflationary phase.
At this point the trajectory in the phase space rapidly evolves into the inflationary regime corresponding to the lower panel of Fig. 1. Hence, the value $V_{\text{crit}}$ sets the energy scale at the onset of inflation. Once inflation begins, the field’s kinetic energy will rapidly tend to zero as the field slows down, reaches a point of maximum displacement and rolls back down the potential. Slow-roll inflation will then arise if the potential satisfies the usual slow-roll constraints.

Although the dynamics may be further complicated by the introduction of additional matter sources [19], the above description outlines how a field can evolve up its self-interaction potential while the universe undergoes oscillations. This forms the basis for a graceful entrance mechanism that naturally generates the conditions for slow-roll inflation. A phase plane analysis is important since it highlights the relevance of the centre and saddle equilibrium points. In particular, a sufficient but not necessary condition for cyclic behaviour is the existence of a centre equilibrium point, while a saddle point is required to separate those regions of phase space where cyclic and non-cyclic behaviour takes place. Furthermore, the disappearance of the saddle point at a critical value of the potential is key to the graceful entrance mechanism.

In the following Section, therefore, we will consider the conditions for the existence of saddle and centre equilibrium points in cosmologies described by a set of generalized Friedmann equations.

III. GENERAL CONDITIONS FOR CENTRE AND SADDLE EQUILIBRIUM POINTS

A. Relativistic Cosmology

We wish to study classes of cosmological models that are motivated by the braneworld paradigm and to determine whether such models display the general characteristics required for a graceful entrance to inflation. However, we will first consider the relativistic cosmology based on classical Einstein gravity, since this system provides a suitable framework for considering more general models. Specifically, we will consider a positively-curved FRW cosmology sourced by an effective perfect fluid with models. Specifically, we will consider a positively-curved FRW cosmology sourced by an effective perfect fluid with an arbitrary equation of state, $p_{\text{eff}} = \frac{\gamma_{\text{eff}} - 1}{\gamma_{\text{eff}}} \rho_{\text{eff}}$, where it is assumed implicitly that the equation of state parameter is a known function of the scale factor, $\gamma_{\text{eff}} = \gamma_{\text{eff}}(a)$.

The Friedmann and fluid equations for this model are given by

$$H^2 = \frac{8\pi G \rho_{\text{eff}}}{3} - \frac{1}{a^2}, \tag{1}$$

$$\dot{\rho}_{\text{eff}} = -3H\gamma_{\text{eff}}\rho_{\text{eff}}, \tag{2}$$

where

$$\dot{a} = Ha, \tag{3}$$

and Eqs. (1) and (2) fully determine the cosmic dynamics. Differentiating Eq. (1) with respect to cosmic time implies that

$$\dot{H} = -\frac{3\gamma_{\text{eff}}}{2}H^2 + \left(1 - \frac{3\gamma_{\text{eff}}}{2}\right) \frac{1}{a^2}, \tag{4}$$

and Eqs. (3) and (4) then describe a closed dynamical system, where the Friedmann equation (1) represents a constraint. The equilibrium points of this system arise whenever $\dot{a} = \ddot{a} = 0$, which implies that

$$\gamma_{\text{eff}}(a_{\text{eq}}) = \frac{2}{3}, \quad H(a_{\text{eq}}) = 0. \tag{5}$$

The stability of the system (3)-(4) can be determined by linearising about the equilibrium points and evaluating the corresponding eigenvalues. It is straightforward to show that the eigenvalues are given by

$$\lambda^2 = -\frac{3}{2} \left[ \frac{1}{a^2} \frac{d\gamma_{\text{eff}}}{\ln a} \right]_{a_{\text{eq}}}. \tag{6}$$

In general, therefore, a necessary and sufficient condition for the equilibrium point to be a centre is

$$\lambda^2 < 0, \quad H = 0, \quad \gamma_{\text{eff}} = \frac{2}{3}, \quad \frac{d\gamma_{\text{eff}}}{\ln a} > 0, \tag{7}$$

whereas the corresponding condition to be a saddle is

$$\lambda^2 > 0, \quad H = 0, \quad \gamma_{\text{eff}} = \frac{2}{3}, \quad \frac{d\gamma_{\text{eff}}}{\ln a} < 0. \tag{8}$$

It is interesting to note that under the assumptions we have made, centre and saddle points are the only types of equilibrium points permitted.

B. Braneworld Scenarios

The four-dimensional cosmological dynamics for a wide class of positively-curved FRW braneworld scenarios can be modeled in terms of a generalized Friedmann equation of the form

$$H^2 = \frac{8\pi G \rho_{\text{eff}}}{3} L^2(\rho) + f(a) - \frac{1}{a^2}, \tag{9}$$

where $\rho$ is the total energy density of the matter confined to the brane. The function $L(\rho)$ is assumed to be positive-definite and parametrizes the departure of the model from the standard relativistic behaviour, $H^2 \propto \rho$. The function $f(a)$ is a function of the scale factor and, in a braneworld context, usually parametrizes the effects of a bulk black hole on the four-dimensional dynamics [32, 33, 34]. If the matter fields are confined to the brane, they satisfy the standard conservation equation

$$\dot{\rho} = -3H\gamma \rho. \tag{10}$$
implies immediately that a collapsing brane world-volume is able to undergo energy density in the Friedmann equation (13) implies that the effective equation of state parameter is given by
\[ \gamma_{\text{eff}} = \frac{\rho(1 - \rho/\sigma)a^4 + m/2\pi\ell_{\text{Pl}}^2}{\rho(1 - \rho/2\sigma)a^4 + 3m/8\pi\ell_{\text{Pl}}^2}. \]

Since we are interested in whether a graceful entrance to inflation can occur in this model, we will determine the equilibrium points that arise when the matter confined to the brane corresponds to a scalar field that is rolling along a constant potential, \( V \). This is equivalent to a matter source comprised of a massless scalar field and a cosmological constant. The equation of state for the system is given by
\[ \gamma = 2 \left( 1 - \frac{1}{\rho} \right). \]

In general, the equilibrium points for this system will arise whenever Eqs. (7) or (8) are satisfied. It follows from Eqs. (13) and (16) that such points occur when
\[ \frac{8\pi\ell_{\text{Pl}}^2}{3} \left( \rho - \frac{\rho^2}{2\sigma} \right) + \frac{m}{a^4} = \frac{1}{a^2}, \]
and
\[ \gamma \left( \rho - \frac{\rho^2}{\sigma} \right) - \frac{2}{3} \left( \rho - \frac{\rho^2}{2\sigma} \right) = -\frac{m}{4\pi\ell_{\text{Pl}}^2} \frac{1}{a^2}. \]

For finite values of the scale factor, Eq. (17) may be simplified after substitution of Eq. (15):
\[ \frac{1}{a^2} = 4\pi\ell_{\text{Pl}}^2 \rho \left[ \frac{4}{3} - \frac{2}{3} \gamma + \gamma \left( \frac{\rho}{\sigma} - 1 \right) \right], \]
and Eq. (19) may then be employed to express Eq. (17) in the form of a quartic equation in the energy density:
\[ 8\pi\ell_{\text{Pl}}^2 m \left[ \frac{2\rho_{eq}^2}{3\sigma} - \left( \frac{1}{3} + \frac{V}{\sigma} \right) \rho_{eq} + V \right]^2 - \frac{5\rho_{eq}^2}{6\sigma} + \left( \frac{2}{3} + \frac{V}{\sigma} \right) \rho_{eq} - V = 0. \]

The solutions to Eq. (20) yield the values of the energy density at the equilibrium points and the corresponding value of the scale factor can then be deduced directly from Eq. (19). For physical solutions, one must ensure that the energy density and scale factor are positive and real at each equilibrium point.

The dark radiation on the brane can significantly influence the dynamics of the system. In view of this, we consider separately the cases where this radiation is present or absent in the following Sections.

### IV. SHTANOV-SAHNI BRANEWORLD

The Shtanov-Sahni (S-S) braneworld scenario embeds a co-dimension one brane with a negative tension \( \sigma \) in a five-dimensional Einstein space sourced by a positive cosmological constant, where the fifth dimension is timelike. The effective Friedmann equation on the brane is given by
\[ H^2 = \frac{8\pi\ell_{\text{Pl}}^2}{3} \left[ \rho - \frac{\rho^2}{2\sigma} \right] + \frac{m}{a^4} - \frac{1}{a^2}, \]
where \( \sigma \equiv -\tilde{\sigma} \). If the bulk space is not conformally flat, the projection of the five-dimensional Weyl tensor induces an effective ‘dark radiation’ term on the brane, parametrized by the constant \( m \). A conformally flat bulk corresponds to \( m = 0 \). Under quite general conditions, a negative quadratic dependence on the energy density in the Friedmann equation implies that a collapsing brane world-volume is able to undergo a non-singular bounce.

Comparison of the Friedmann equations \( \rho_{\text{eff}} = \rho L^2 + \frac{3}{8\pi\ell_{\text{Pl}}^2} f(a) \) and \( H^2 = \frac{8\pi\ell_{\text{Pl}}^2}{3} \left[ \rho - \frac{\rho^2}{2\sigma} \right] + \frac{m}{a^4} - \frac{1}{a^2} \) implies immediately that
\[ L^2 = \left( 1 - \frac{\rho}{2\sigma} \right), \quad f(a) = \frac{m}{a^4}, \]
and substituting Eq. (14) into Eq. (12) implies that the effective equation of state parameter is given by
\[ \gamma_{\text{eff}} = \frac{\rho(1 - \rho/\sigma)a^4 + m/2\pi\ell_{\text{Pl}}^2}{\rho(1 - \rho/2\sigma)a^4 + 3m/8\pi\ell_{\text{Pl}}^2}. \]
and this equation can be solved in terms of the field’s kinetic energy:

\[ \dot{\phi}_{eq}^2 = \frac{4V - 4\sigma + 2 \sqrt{9V^2 - 18V\sigma + 4\sigma^2}}{5} \]  

(22)

It follows that there can be at most two static (physical) solutions to the Friedmann equations (19) and (20), depending on the value of the potential. If \( V < 0 \), there is only one solution to Eq. (21) with \( \dot{\phi}^2 > 0 \), implying there is only one equilibrium point in the phase space. For \( V = 0 \), there is one solution with \( \dot{\phi}^2 > 0 \) and a second, but physically uninteresting, point where the field’s kinetic energy vanishes and the scale factor diverges. For a positive potential, on the other hand, there are two real roots to Eq. (21) if the condition

\[ 9V^2 - 18V\sigma + 4\sigma^2 > 0 \]  

(23)

is satisfied. The values of the potential which bound the region of parameter space in which there are no real solutions are therefore given by

\[ V = \left( 1 \pm \frac{\sqrt{3}}{3} \right) \sigma . \]  

(24)

We find that for \( 0 < V < V_{crit} \), where

\[ V_{crit} = \left( 1 - \frac{\sqrt{3}}{3} \right) \sigma , \]  

(25)

there are two real solutions to Eq. (21). Moreover the roots are physical, since they are positive and lead to real values of the scale factor. The roots merge and disappear at \( V_{crit} \). Although other solutions to Eq. (21) arise when the potential is greater than the positive branch of Eq. (21), these do not correspond to physical equilibrium points, either because the solution requires \( \dot{\phi}^2 < 0 \) or the value of the scale factor is imaginary.

Hence, there are two static equilibrium points for \( 0 < V < V_{crit} \). The nature of these points can be determined from condition (9) after substituting Eqs. (15), (17) and (21) and we find that the eigenvalues are given by

\[ \lambda^2 = 4\pi\ell_p^2\dot{\phi}^2 \left[ 4 - \frac{1}{\sigma} \left( 5\dot{\phi}^2 + 4V \right) \right] . \]  

(26)

This implies that a given point will correspond to a centre if \( \dot{\phi}^2 > \frac{4}{\sigma}(\sigma - V) \), otherwise it will represent a saddle. Consequently, for physical roots to Eq. (21), the field’s kinetic energy has a value

\[ \dot{\phi}_{eq}^2 = \frac{4V + 4\sigma + 2\sqrt{9V^2 - 18V\sigma + 4\sigma^2}}{5} , \]  

(27)

at the centre equilibrium point, whereas it takes the value

\[ \dot{\phi}_{eq}^2 = \frac{4V + 4\sigma - 2\sqrt{9V^2 - 18V\sigma + 4\sigma^2}}{5} , \]  

(28)

VI. EFFECTS OF DARK RADIATION

In the case where dark radiation plays a dynamical role in the Friedmann equation (19), the full quartic expression given by Eq. (21) must be solved to determine the nature of the equilibrium points. One must also ensure that the scale factor and kinetic energy of the field take positive and real values at these points. The con-
FIG. 2: The top panel represents the phase space in the variables \( \{H, \phi, a\} \), when \( m = 0, \sigma = 0.05 \) and \( 0 < V < V_{\text{crit}} \), where \( V_{\text{crit}} \) is defined in Eq. (25). Numerical values are given in Planck units. The surface defined by the Friedmann equation (13), with \( \rho = \dot{\phi}^2/2 + V \), is also shown and all phase space trajectories lie on this surface. The compactified coordinates \( x \equiv \arctan(H), y \equiv \arctan(\ln \dot{\phi}) \) and \( z \equiv \arctan(\ln a) \) have been employed in order to show the entire phase space. The lower panel represents the two-dimensional projection of this space onto the plane spanned by the variables \( \{H, \dot{\phi}\} \). The centre equilibrium point is denoted by the solid circle and the separatrix is represented by the dotted line. The saddle point occurs at the point where the separatrix self-intersects. The axes have been compactified using the coordinate change \( x = \arctan(H) \) and \( y = \arctan(\ln \dot{\phi}) \).

Consequently, the surface defined by Eq. (25) consists of two disconnected pieces if \( m < m_{\text{crit}} \), whereas it is a single surface for \( m > m_{\text{crit}} \). It is possible that one of the two surfaces may correspond to a region where \( \dot{\phi}^2 < 0 \), in which case it would be unphysical. However, the qualitative dynamics of the universe will clearly be radically different above and below the critical value \( m_{\text{crit}} \) and we therefore proceed to consider each regime in turn.

FIG. 3: As for Fig. 2 but now for a potential \( V > V_{\text{crit}} \). There are no finite equilibrium points in the phase space in this region of parameter space.

\[
- \left( \frac{\sqrt{m}}{a^2} - \frac{1}{2\sqrt{m}} \right)^2 = \frac{4\pi \ell_{\text{Pl}}^2 \sigma}{3} - \frac{1}{4m} .
\]  

Provided \( V < \sigma \), the hyperboloid’s topology depends only the relative values of the dark radiation parameter, \( m \), and the brane tension, \( \sigma \). Indeed, there is a critical value of \( m \) at which the topology changes and this is given by

\[
m_{\text{crit}} = \frac{3}{16\pi \ell_{\text{Pl}}^2 \sigma} .
\]
is the same for all and dashed lines, respectively. The qualitative behaviour of the potential, \( V \), their stability was determined from Eq. (6). (20) correspond to physical values of the scale factor, and (19) was also employed to verify that the solutions to Eq. 7 illustrates the position and nature of the physically relevant equilibrium points as a function of the field’s kinetic and potential energies for specific values of the dark radiation parameter and the brane tension. Eq. 19 was also employed to verify that the solutions to Eq. 20 correspond to physical values of the scale factor, and their stability was determined from Eq. 9. The results are shown in Fig. 4 for \( m = 1 \) and \( \sigma = 0.05 \), where the locations of the saddle and centre equilibrium points for given field energies are represented by the solid and dashed lines, respectively. The qualitative behaviour is the same for all \( m < m_{\text{crit}} \). There is a critical value of the potential, \( V_{\text{crit1}} \), such that there are three real roots to the quartic equation (20) when \( 0 < V < V_{\text{crit1}} \). These correspond to three distinct equilibrium points, two of which are centres whereas the third is a saddle point. At \( V = V_{\text{crit1}} \), however, the saddle and one of the centre points merge and, for \( V > V_{\text{crit1}} \), these points correspond to unphysical roots. Consequently, there is only one equilibrium point for \( V > V_{\text{crit1}} \) and this is a centre. The qualitative evolution of the universe in the regimes \( V < V_{\text{crit2}} \) and \( V > V_{\text{crit2}} \) is shown in Figs. 5 and 6, respectively, where the trajectories have been calculated by numerically integrating the field equations for the specific choices \( m = 1 \) and \( \sigma = 0.05 \). The topology of the Friedmann surface in the upper panels of these figures, together with Eq. 20, implies that the entire phase space can be represented as a single two-dimensional plot by employing cylindrical polar coordinates. These are shown in the lower panels of the figures. The dynamics in the upper sectors of the Friedmann surface in Figs. 5 and 6 is qualitatively similar to the case where no dark radiation is present \((m = 0)\). The lower sector of the Friedmann surface contains the new centre equilibrium point that arises when \( m \neq 0 \). This point is real for all values of the potential. Hence, as shown in Figs. 5 and 6, the universe remains trapped in an indefinite cycle of expansion and contraction if it is initially located in the lower half of the Friedmann surface. This behaviour can be understood since a sufficiently large (positive) cosmological constant introduces a large negative effective cosmological constant in the Friedmann equation 13 as a consequence of the quadratic term in the energy density. This will prevent the universe from entering a phase of accelerated inflationary expansion. Indeed, although is is not relevant for the graceful entrance mechanism, it can be shown that if \( V \) exceeds yet another critical value, the upper region of the Friedmann surface is no longer physical since the field’s kinetic energy becomes negative. In effect, therefore, the size of the universe is bounded from above.

By comparing the qualitative dynamics of the universe when no dark radiation is present (Figs. 2 and 3) to that illustrated in the upper sectors of the Friedmann surfaces in Figs. 5 and 6, we may infer that a graceful entrance to inflation, as outlined in Section II, may in principle occur for suitable choices of initial conditions.

A. \( m < m_{\text{crit}} \)

Since the analytic expressions for the solutions to the quartic equation (20) are not particularly illuminating, we have employed them graphically to illustrate how the nature of the equilibrium points depends on the field’s kinetic and potential energies for specific values of the dark radiation parameter and the brane tension. Eq. (19) was also employed to verify that the solutions to Eq. (20) correspond to physical values of the scale factor, and their stability was determined from Eq. (9). The results are shown in Fig. 4 for \( m = 1 \) and \( \sigma = 0.05 \), where the locations of the saddle and centre equilibrium points for given field energies are represented by the solid and dashed lines, respectively. The qualitative behaviour is the same for all \( m < m_{\text{crit}} \). There is a critical value of the potential, \( V_{\text{crit1}} \), such that there are three real roots to the quartic equation (20) when \( 0 < V < V_{\text{crit1}} \). These correspond to three distinct equilibrium points, two of which are centres whereas the third is a saddle point. At \( V = V_{\text{crit1}} \), however, the saddle and one of the centre points merge and, for \( V > V_{\text{crit1}} \), these points correspond to unphysical roots. Consequently, there is only one equilibrium point for \( V > V_{\text{crit1}} \) and this is a centre.

The qualitative evolution of the universe in the regimes \( V < V_{\text{crit2}} \) and \( V > V_{\text{crit2}} \) is shown in Figs. 5 and 6, respectively, where the trajectories have been calculated by numerically integrating the field equations for the specific choices \( m = 1 \) and \( \sigma = 0.05 \). The topology of the Friedmann surface in the upper panels of these figures, together with Eq. (20), implies that the entire phase space can be represented as a single two-dimensional plot by employing cylindrical polar coordinates. These are shown in the lower panels of the figures.

B. \( m > m_{\text{crit}} \)

We may adopt a similar approach for \( m > m_{\text{crit}} \). Fig. 4 illustrates the position and nature of the physically relevant equilibrium points as a function of the field’s kinetic and potential energies for the specific choices \( m = 1.4 \) and \( \sigma = 0.05 \). The qualitative nature of this plot remains unaltered for all \( m > m_{\text{crit}} \). It follows from Fig. 4 that there is only a single saddle equilibrium point when the magnitude of the potential is below a critical value \( V_{\text{crit3}} \). There is then a finite range of values of the potential, \( V_{\text{crit3}} \leq V \leq V_{\text{crit4}} \), for which there are no physical equilibrium points. For \( V > V_{\text{crit4}} \), on the other hand, there exists both a centre and a saddle equilibrium point, and at still higher values of the potential, the saddle point disappears once more. However, the equilibrium points which occur above \( V_{\text{crit4}} \) are not relevant to the graceful entrance scenario.

Figs. 8 and 9 illustrate the Friedmann surface and and the corresponding phase trajectories for \( V < V_{\text{crit3}} \) and \( V_{\text{crit3}} < V < V_{\text{crit4}} \), respectively. In Fig. 8, the saddle point represents a divide between cyclic and inflationary behaviour, although it should be noted that the cyclic dynamics does not occur around a centre equilibrium point in this case. For \( V > V_{\text{crit3}} \), the saddle point disappears and all trajectories eventually evolve into an
FIG. 5: The top panel represents the phase space in the variables \(\{H, \phi, a\}\), when \(m = 1.0, \sigma = 0.05\) and \(0 < V < V_{\text{crit}}^2\). The surface defined by the Friedmann equation (13), with \(\rho = \frac{\dot{\phi}^2}{2} + V\), is also plotted. The phase space trajectories lie on this surface. The axes have been compactified using the rescalings \(x = \arctan(H), y = \arctan(\ln \dot{\phi})\) and \(z = \arctan(\ln a)\). The lower panel illustrates a corresponding two-dimensional phase space. It follows from Eq. (29) that there exists an axis of symmetry parallel to the \(a\)-axis through the point \(H = 0, \dot{\phi}^2 = 2\sigma - 2V\). Cylindrical polar coordinates can then be defined by using this axis. Specifically, we define \(a = a, X = R\cos \theta\) and \(Y = R\sin \theta\), where \(X = \frac{\sqrt{8\sqrt{2}\sigma} \dot{\phi}^2}{\sqrt{3}} \left(\frac{\dot{\phi}^2}{2\sqrt{2}\sigma} - \left(\frac{\sqrt{2}}{\sqrt{2}\sigma} - \frac{V}{\sqrt{2}\sigma}\right)\right)\), \(Y = H\), and \(R = \left(\frac{\sqrt{\frac{m}{\sigma}}}{2\sqrt{m}} - \frac{1}{2\sqrt{m}}\right)^2 + \frac{4\sqrt{2}\sigma}{3} - \frac{1}{\sqrt{m}}\). The two-dimensional plot is then presented in the \(\{\theta, a\}\) plane, where we have once more compactified the \(a\)-axis using \(y = \arctan(\ln a)\). This diagram is essentially a projection of the top panel about the axis defined above, with the points at \(\theta = \pi\) identified with those at \(\theta = -\pi\). The shaded areas mark the unphysical regions in this coordinate system. The centre equilibrium points are identified by a solid circle and the separatrix is represented by a dotted line. The saddle point occurs at the point where the separatrix intersects with itself.

FIG. 6: As in Fig. 5, but now for a potential \(V > V_{\text{crit}}^2\). There are no saddle points and only one centre equilibrium point in the phase plane.

FIG. 7: As for Fig. 4 but now for the case where \(m > m_{\text{crit}}\). Numerical values chosen for the parameters are \(m = 1.4\) and \(\sigma = 0.05\) in Planck units.
in the previous subsection, it can be shown that for a sufficiently large value of the potential, the Friedmann surface becomes modified in such a way that the surface is bounded to always lie below a critical value of the scale factor. This implies that the size of the universe is bounded from above and such behaviour follows once more due to the presence of an effective negative cosmological constant in the Friedmann equation.

The question of whether a graceful entrance to inflation is possible if \( m > m_{\text{crit}} \) is more difficult to answer. The key features of a cyclic region and a saddle point in the phase space, which disappear at a critical value of the potential, are indeed present. However, there is a further complication. For \( m < m_{\text{crit}} \), the cyclic dynamics is always dominated by the field’s kinetic energy, but this is no longer the case when \( m > m_{\text{crit}} \).

Further insight may be gained from Figs. 8 and 9. The shaded gray areas (red in the online versions) represent the regions of phase space where the field’s kinetic energy is negative, with the boundaries corresponding to the limit \( \dot{\phi}^2 = 0 \). For a realistic (i.e. sufficiently flat but field-dependent) potential, this implies that on a trajectory which passes sufficiently close to these boundaries in the ‘instantaneous phase space’, the field’s kinetic energy will be so small that the gradient of the potential will become significant. This will result in the kinetic energy of the field falling to zero and the field turning around on the potential before it has climbed sufficiently far up the potential to drive a successful phase of slow-roll inflation. In the previous cases, such behaviour was only possible in the region of parameter space relevant to inflation, or in a region disconnected from it, such as the lower section of the Friedmann surface. In the present case, however, the two regions of the previously disjointed Friedmann surface are connected. In effect, therefore, the turn around in the field may occur too soon for inflation to occur if \( m > m_{\text{crit}} \).

More specifically, let us assume as before that we have a realistic inflationary potential, with a magnitude initially in the range \( 0 < V < V_{\text{crit}3} \), and that the universe
begins in the oscillatory region of the phase space. As the universe oscillates, the field climbs its potential. In terms of Fig. 8, the instantaneous phase space is therefore altered, with the saddle point moving to successively smaller values of the scale factor, and the shaded (physically forbidden) regions growing in size. Ultimately, a point will be attained when the universe’s trajectory either moves outside the region of the oscillations or it passes too close to the boundary of the lower, shaded region. If the former behaviour arises, the graceful entrance dynamics applies as in the previous scenarios discussed above. On the other hand, in the latter case, the field may turn around on the potential without inflation occurring, since the scale factor will still be small at this stage and consequently the curvature term in the Friedmann equation will be significant. Such behaviour is qualitatively similar to the effect discussed in the LQC scenario when perfect fluid matter sources are introduced into the system. The issue of whether a graceful entrance to inflation may occur is therefore sensitive to the initial conditions.

VII. DISCUSSION

In this paper, we have examined the criteria that a cosmological model must satisfy in order for it to undergo a ‘graceful entrance’ into a phase of inflation, whereby a scalar field is able to move up its interaction potential whilst the universe undergoes a large number of oscillatory cycles. Our approach has been to consider an idealized model, where the matter is comprised of a scalar field evolving along a constant potential. This system provides a good approximation to more realistic scenarios where the potential is field-dependent if the change in the potential is sufficiently small over a large number of cycles. It therefore provides us with a methodology for identifying whether a particular cosmological scenario will meet the necessary requirements for graceful entrance.

The mechanism we have outlined is very generic. The important ingredients are that the phase space should exhibit both a centre and saddle equilibrium point when the magnitude of the potential $V$ falls below a critical value $V_{\text{crit}}$. These points gradually move towards each other as $V$ increases and eventually merge when $V = V_{\text{crit}}$. Above this scale, the points should disappear from the phase space. From the physical viewpoint, this behaviour arises because the potential now dominates the field’s kinetic energy, $V > \dot{\phi}^2$, and consequently drives a phase of inflationary expansion. It follows that the critical value $V_{\text{crit}}$ sets the energy scale for the onset of inflation.

We have developed a framework for investigating a general class of braneworld models in this context that are characterized by a Friedmann equation with an arbitrary dependence on the energy density. Such a class of models can be represented as a standard, relativistic Friedmann universe by reinterpreting the effective equation of state of the matter on the brane. As a concrete example, we focused on the Shtanov-Sahni braneworld reference, where the extra bulk dimension is timelike. We found that the question of whether the braneworld oscillates or expands indefinitely is dependent on the field’s kinetic and potential energies, as well as the dark radiation and the brane tension, $\sigma$. If the dark radiation is negligible, the phase space has a similar structure to that outlined above and a graceful entrance to inflation can therefore be realised, where the energy scale at the onset of inflation is given by $V_{\text{crit}} \approx \sigma$. The dark radiation can have a significant effect on the dynamics, however, and the question of whether a graceful entrance is possible when this term is present is sensitive to the initial conditions.

In conclusion, therefore, we have presented a nonsingular, oscillating braneworld that can in principle exhibit a graceful entrance to inflation. We anticipate that such behaviour will apply for a wide range of collapsing braneworld scenarios that are able to undergo nonsingular bounces (see, e.g., [27]) and, moreover, will not be strongly dependent on the precise form of the inflaton potential. Similar dynamics was recently shown to arise in models inspired by loop quantum cosmology and it is interesting that such behaviour is also possible in braneworld models motivated by string/M-theory.

Finally, our analysis also provides the basis for realising the recently proposed emergent universe scenario in a braneworld context. In this scenario, the initial state of the universe is postulated to be either an Einstein static universe or one that is oscillating about an asymptotically flat section, where $V < V_{\text{crit}}$ as $\phi \to -\infty$, but subsequently increases above $V_{\text{crit}}$ once the value of the field has increased beyond a certain value. If the field is initially located in the plateau region of the potential and moving in the appropriate direction, it will eventually reach the section of the potential which rises above $V_{\text{crit}}$, thus initiating inflation. Within the framework of classical Einstein gravity, the emergent universe scenario suffers from severe fine-tuning, since the static universe corresponds to a saddle point in the phase space. However, a mechanism which allows for a graceful entrance to inflation provides a natural realization of this scenario with less fine tuning and a greater degree of freedom in the choice of inflationary potential.

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