Multivector Solutions to the Hyper-Holomorphic Massive Dirac Equation

William M. Pezzaglia Jr.

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Department of Physics
Santa Clara University
Santa Clara, CA 95053
U.S.A.
Email: wpezzaglia@scuacc.scu.edu

Abstract
Attention is given to the interface of mathematics and physics, specifically noting that fundamental principles limit the usefulness of otherwise perfectly good mathematical general integral solutions. A new set of multivector solutions to the meta-monogenic (massive) Dirac equation is constructed which form a Hilbert space. A new integral solution is proposed which involves application of a kernel to the right side of the function, instead of to the left as usual. This allows for the introduction of a multivector generalization of the Feynman Path Integral formulation, which shows that particular “geometric groupings” of solutions evolve in the manner to which we ascribe the term “quantum particle”. Further, it is shown that the role of usual \( i \) is subplanted by the unit time basis vector, applied on the right side of the functions.

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1 Introduction

As a physicist, I am like ‘a stranger in a strange land’ where familiar names have different meanings. It become increasingly clear that the practitioners hypercomplex analysis, and those of multivector physics do not occupy the same “Clifford” space. This is I believe a manifestation of the classical chasm which unfortunately often exists between the disciplines of applied mathematics and theoretical physics. My attempts to bridge this gap by conversation at this conference often ran aground. Lacking background (or interest) in physics, many mathematicians are reluctant to be drawn into unfamiliar territory (and of course visa versa). For brevity I will continue from my own point of view, but it should be clear that many of the statements I make are equally true if you interchange the words “mathematician” and “physicist”. Further, any parochial statements are a function of deliberate hyperbolism.
There is perhaps an unconscious assumption that the interface of mathematics and physics is a no-man’s land, which is not to be ventured into under threat of being accused, by one’s colleagues, of defection to the other side. For example, the theoretical physicist that treads too close to the border runs the risk of having his work dismissed as “mere reformulation”, or worse as just mathematics, i.e. not ‘real’ physics. The point is well taken. While learning more math will enable the physicist to communicate better with mathematicians, it will not necessarily lead to new or even better physics. This is because physics is much more than the study of the subset of mathematical equations that is isomorphic to physical phenomena. On the other hand, mathematicians are far more than nit-picking grammarians that clean up the details after physicists have done the real work of creating new language (e.g. Newton’s calculus) as a side effect of their struggles to comprehend the universe.

Rather than dismiss practitioners of the opposition because they lack sophistication in our own language, it behooves us to acknowledge that mathematics and physics are different disciplines with distinct agendas and criteria. An example in point was the ‘open problem book’ of the conference, in which mathematicians posed concise questions for which an answer is yet unknown (or even known to exist, as in Fermat’s last theorem) but presumably can be solved through clever logical deduction. It was difficult to come up with an analogous “physics” question. As recently discussed by Romer[14], such questions can usually only be answered by experimental verification (e.g. does the Top Quark exist), or an application of a physical principle. It was the downfall of Galileo that the ‘new science’ had new physical laws that could not be derived by logical extension or verified by mathematical proof. Given this, it is somewhat of a mystery that physical principles can be so conveniently expressed in mathematical form[1].

The goal of this paper is twofold. Specifically we consider the more general meta-monogenic equation (in non-Euclidean spacetime), which is of more interest (to physicists) than the more limited monogenic equation, yet has not received nearly as much attention in Clifford analysis. Secondly, this paper will be used as a vehicle to address the interface of mathematics and physics, by providing associations between mathematical concepts and physical principles.

2 Algebraic Notation

We seek to use a geometric algebra which will describe the empirically known properties of physical spacetime. Specifically, the mathematical structure chosen should encode physical principles (e.g. Lorentz covariance).

2.1 Classical Galilean Space and The Pauli Algebra

Before the theory of relativity, physicists thought “space” was intrinsically three dimensional. Physical quantities [such as the electric field vector $\mathbf{E} = \mathbf{E}(\mathbf{X}, t)$] would be represented as a function of a position Gibbs vector $\mathbf{X}$ and scalar time $t$,

$$(\mathbf{X}, t) = x\sigma_1 + y\sigma_2 + z\sigma_3 + t\sigma_0,$$  \hspace{1cm} (2.1)

where $\sigma_0 = 1$ is the unit scalar. The $3 \oplus 1$ notation of eq. (2.1) matches our perception, which insists upon seeing time as a scalar, different from 3D space. Equivalently,
we say that the domain \((\hat{X}, t)\) belongs to \(\mathbb{R}^3 \oplus \mathbb{R}\); it is an ordered pair of a three space Gibbs vector and a scalar. This structure (i.e. setting \(\sigma_0 = 1\)) was assumed by many of the contributors at this conference (n.b. plenary presentations).

The Clifford algebra generated by the three mutually anticommuting basis elements \(\{\sigma_1, \sigma_2, \sigma_3\}\) is \(\text{End} \mathbb{R}^3\), commonly called the Pauli algebra, with matrix representation of \(2 \times 2\) complex matrices \(\mathbb{C}(2)\). While this algebra and the \(3 \oplus 1\) notation is sufficient for many physical applications (see for example Baylis[7]), there are points where it will however lead to ambiguities (n.b. time reversal), which can only be resolved with the full \(4\)D concept.

### 2.2 Minkowski Spacetime and Majorana Algebra

The principle of Lorentz invariance demands that \((\hat{X}, t)\) is an element of a four dimensional domain, rather than \(3 \oplus 1\). Algebraically this means that the basis element associated with time \([e.g. \sigma_0 \text{ of eq. (2.1)}]\) must anticommute with the other three mutually anticommuting basis vectors. In order to avoid confusion, we will introduce a new set of symbols: \(\{e_\mu\}\) as the basis vectors of \(4\)D space, with \(e_4\) (instead of \(e_0\)) as the fourth basis element. Hence the coordinate four-vector will be expressed in boldface lower case (no overhead arrow),

\[
x = (x, y, z, t) = x^\mu e_\mu = x e_1 + y e_2 + z e_3 + t e_4,
\]

where the Einstein summation notation is assumed on the repeated index: \(\mu = 1, 2, 3, 4\).

The metric of time is empirically known to be of the opposite sign as that of the \(3\)D positional portion. Mathematically, we would say that the orthogonal space is either \(\mathbb{R}^{3,1}\) or \(\mathbb{R}^{1,3}\), corresponding to metric signatures \((+++\) or \((- - -)\) respectively (physicists refer to these respectively as the east coast or west coast metrics). In the former case, the norm of the position four-vector would be,

\[
||x|| = x_\mu x^\mu = x^2 + y^2 + z^2 - t^2,
\]

where \(x_4 = -x^4 = t\), but \(x_i = x^i\) for \(i = 1, 2, 3\). Physically, this hyperbolic nature is interpreted as a manifestation of the principle of causality. For an “event” at the origin to be the source or cause of another event at \(x\), the separation four-vector between them must have norm \(||x||^2 < 0\) in the \(\mathbb{R}^{3,1}\) space (or \(||x||^2 > 0\) in the \(\mathbb{R}^{1,3}\) space). Equivalently, we say that the vector connecting an event with its cause is timelike, or lies inside the light cone (the hypersurface for which \(||x||^2 = 0\).

The Clifford algebra generated by the four basis vectors in the “east coast” \((+++-)\) metric is known as the Majorana algebra, with matrix representation \(\mathbb{R}(4) = \text{End} \mathbb{R}^{3,1}\). This algebra is different to that generated by the other metric choice (used for example by Hestenes[2]), which has inequivalent matrix representation \(\mathbb{H}(2) = \text{End} \mathbb{R}^{1,3}\), i.e. \(2 \times 2\) quaternionic matrices[18]. Regardless, either algebra has 16 basis elements, but neither contains the global commuting \(i\), which is usually required by physicists for standard relativistic quantum mechanics[21]. Those theories use the “classic” Dirac algebra, which has five anticommuting generators \(\{\gamma^\mu\}\), corresponding to complex matrix representation \(\mathbb{C}(4) = \mathbb{C} \otimes \mathbb{H}(2) = \mathbb{C} \otimes \mathbb{R}(4)\).

The “projection” or spacetime split from \(4\)D \(\rightarrow 3 \oplus 1\) can be encoded algebraically in \(\mathbb{R}(4)\),

\[
-x \cdot e_4 = \hat{X} + t,
\]

3
where the 3D Pauli algebra generators \( \sigma_j \) of eq. (2.1) are related to the Majorana generators: \( \sigma_j = e_4 e_j \) for \( j = 1, 2, 3 \).

### 2.3 Automorphisms and Conservation Laws

Fundamental to physical theories are conservation laws, e.g. conservation of energy. These are intimately related via Noether’s Theorem[8] to the physical principles being invariant under certain transformation (e.g. rotations). The “allowed” symmetries should correspond mathematically to those generated by the automorphism group of the geometric algebra.

Of particular interest are the orthogonal transformations associated with hyperbolic rotations between space and time. These have the physical interpretation of connecting two reference systems, one at rest and the other moving at velocity \( \vec{V} = \frac{d \vec{X}}{dt} \). A particle of mass \( m \) at rest in the “moving” frame, will be perceived as moving at velocity \( \vec{V} \) in the “rest” frame, with four-vector momentum \( \vec{p}, \rho(\vec{p}) = e^4 \Gamma e_4 \rho(\vec{p}) = L m e_4 L^{-1} \),

\[
\mathcal{L} = \exp \left( \frac{r \hat{\beta}}{2} \right) = \frac{-pe_4 + m}{\sqrt{2m(E + m)}} = \frac{\vec{P} + E + m}{\sqrt{2m(E + m)}},
\]

\[
E = p^4 = -e_4 \cdot \vec{p},
\]

\[
\vec{P} = E \vec{V} = e_4 \wedge \vec{p},
\]

\[
\tanh(r) = \|\vec{V}\| = E^{-1} \|\vec{P}\|,
\]

\[
\hat{\beta} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{\vec{P}}{\|\vec{P}\|}.
\]

The term \( \mathcal{L} \) is called the (half-angle) Lorentz Boost operator, where \( r \) is the rapidity associated with the velocity. The unit bivector \( \hat{\beta} \) points in the direction of the 3D velocity \( \vec{V} \), or 3D momentum \( \vec{P} \). The factor of \( e_4 \) in the middle of eq. (2.5b) causes the spacetime split [see eq. (2.4)] of the four-momentum, where the fourth component \( p^4 \) is physically interpreted as the energy \( E \) in eq. (2.5c).

We define the following algebra involutions,

\[
\mathcal{T}(\Gamma) = -e_1 e_2 e_3 \Gamma e_1 e_2 e_3,
\]

\[
\mathcal{P}(\Gamma) = -e_4 \Gamma e_4,
\]

which correspond geometrically to reflections. The first inverts timelike geometry, hence is called Time Reversal in physics. Basis elements which are invariant under this involution (e.g. \( e_1 e_2 \)) are called spacelike, while those which acquire a minus sign (e.g. \( e_1 e_4 \)) timelike. We will have more use for the second, the contrapositive inversion of 3D space called the Parity Transformation[7], under which many (but not all) physical laws are invariant. The composition of the two is the main involution of the algebra,

\[
\mathcal{PT}(\Gamma) = \mathcal{T} \mathcal{P}(\Gamma) = -e_1 e_2 e_3 e_4 \Gamma e_1 e_2 e_3 e_4,
\]

inverting the odd geometry (i.e. vectors and trivectors).
The anti-involutions associated with eq. (2.6b) and (2.6c) are respectively the “dagger” and “bar” (equivalently the Hermitian and Dirac conjugates respectively). They are related,
\[ \Gamma = -e_4 \Gamma^\dagger e_4, \quad (2.7a) \]
\[ \bar{\epsilon}_\mu = -\epsilon_\mu, \quad (2.7b) \]
\[ e_4^\dagger = -e_4, \quad e_j^\dagger = +e_j \quad (j = 1, 2, 3), \quad (2.7c) \]
where the “bar” reverses the order of all elements and inverts the basis vectors.

The multivector bilinear form \( \Psi \Psi \) has the advantage of being Lorentz invariant (as the Dirac bar conjugate operation inverts the bivector generators of Lorentz transformations). On the other hand, the scalar part of this form is not in general positive definite, in contrast to the form \( \Psi^\dagger \Psi \). Both forms will in general have non-scalar portions.

3 Functional Solutions of the Massive Dirac Eqn.

The adjective “massive” should be superfluous, because historically Dirac was describing the electron, a particle with mass. It is therefore a source of confusion for the physicist to encounter the use of the term “Dirac equation” in Clifford analysis sometimes applied to the monogenic equation \( \Box \Psi = 0 \), rather than eq. (3.2a) below. When \( \Psi \) is restricted to be a bivector, the monogenic equation would be called the (sourceless) Maxwell equation, describing the spin-one massless photon (i.e. electromagnetic waves). If \( \Psi \) is projected onto a minimal ideal (i.e. a column spinor), it would describe a spin-half (again massless) neutrino, often called the Majorana equation. These are special cases which have many interesting properties, unfortunately which vanish once mass is included, or a source added (which makes the equation non-homogeneous). In this paper, we will address the more general meta-monogenic or “hyper-holomorphic” situation, in the non-Euclidean spacetime metric case, which so far has not received much attention in Clifford analysis.

3.1 Relativistic Quantum Wave Equations

The massive meta-harmonic equation over the spacetime domain \( \mathbb{R}^{3,1} \) is known as the Klein-Gordon equation
\[ (\Box^2 - m^2) \phi(x) = 0, \quad (3.1) \]
where \( \Box^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2 \) is the d’Alembertian (the 4D generalization of the Laplacian, but in non-Euclidean spacetime). Note that some authors use the symbol \( \Box \) without the square for the d’Alembertian, which makes the use of the same symbol for the factored Dirac operator ambiguous. Physically, eq. (3.1) is interpreted to be the relativistic generalization of Schrödinger’s quantum wave equation. Historically, \( \phi \) was a complex scalar function describing a charged spinless particle. If however the function \( \phi \) is a vector, then eq. (3.1) is called the Proca equation, describing a spin-one particle (the “vector boson”equivalently electromagnetism with mass).
Factoring the Klein-Gordon operator minimally requires four anticommuting algebraic entities $e_\mu$, $$(\Box - m) \Psi(x) = 0, \quad (3.2a)$$ $$\Psi(x) = (\Box + m) \phi(x) = (e^\mu \partial_\mu + m) \phi(x), \quad (3.2b)$$ $$\{e_\mu, e_\nu\} = 2g_{\mu\nu}, \quad (3.2c)$$ where $e_\mu = g_{\mu\nu}e^\nu$, and $\Box = e^\mu \partial_\mu$ is the 4D spacetime gradient (we hesitate to call it the “Dirac operator”, because this term is sometimes used by physicists to refer to the second quantized wavefunction $\Psi$). Following Greider[5], we note that the metric tensor $g_{\mu\nu}$ must have the diagonal values $g_{11} = g_{22} = g_{33} = -g_{44} = +1$ of the east coast signature ($++-)$ if the use of the commuting $i$ is excluded. Note that this means that eq. (3.2a) and most of the following results of this paper are inaccessible if the other metric ($+++-$) with inequivalent algebra $H(2)$ is used, as assumed by Hestenes[2].

3.2 Meta-Monogenic Functions

Real solutions (no $i$) to the meta-harmonic Klein-Gordon eq. (3.1) are of the form: $\phi_\mathbf{p}(x) \sim \cos(p_\mu x^\mu)$. Substituting this eigenfunction into eq. (3.1) gives the characteristic equation, $$m^2 = -p_\mu p^\mu = E^2 - ||\mathbf{P}||^2, \quad (3.4a)$$ known in this case as the Einstein relation between mass $m$, energy $E$ and vector momentum $\mathbf{P}$. Note for a given mass and vector momentum, the energy could be positive or negative, the latter is unphysical. We shall define energy to be positive, $$\pm p^4 = E = +\sqrt{m^2 + ||\mathbf{P}||^2}. \quad (3.4b)$$

Substitution $\phi = \cos(p_\mu x^\mu)$ into eq. (3.2b) yields a multivector solution to the Dirac eq. (3.2a) which can be expressed in the exponential form[3][4], $$\Psi_\mathbf{p}(x) = \exp(-\hat{\mathbf{p}} p_\mu x^\mu) \Lambda, \quad (3.5a)$$ $$\hat{\mathbf{p}} = m^{-1} \mathbf{p} = m^{-1} p^\mu e_\mu, \quad (3.5b)$$ $$\hat{\mathbf{p}}^2 = -1, \quad (3.5c)$$ where $\Lambda$ is an arbitrary geometric factor (the “spin” degrees of freedom). The unit four-velocity $\hat{\mathbf{p}}$ plays the role of the usual $i$ as the generator of quantum phase (when multiplied sinistrally, i.e. on the left). However, each momentum eigenfunction has its own unique generator $\hat{\mathbf{p}}$. Note that eq. (3.5a) is invariant under the replacement of $\mathbf{p} \to -\mathbf{p}$, hence we can restrict eq. (3.4b) to $p^4 = +E$ without any loss of generality, hence: $p_\mu x^\mu = \hat{\mathbf{P}} \cdot \mathbf{X} - Et$.

Consider a solution which is a linear combination of eigenfunctions of eq. (3.5a), restricted to $\Lambda = 1$. This is a reasonable construct to consider because of the superposition principle of quantum theory: the sum of any two physically interpretable solutions will be a reasonably interpretable solution. Although each eigenfunction separately will have a unit quadratic form, $\nabla \Psi = 1$, the new solution will display non-scalar “interference” terms due to the differing generators $\hat{\mathbf{p}}$. These have been interpreted as a possible useful description as a source for mesonic interactions[10].


3.3 Multivectorial Hilbert Space

In order to have a probabilistic interpretation of quantum theory, we need a positive definite norm. A restricted choice of factor \( \Lambda \) in eq. (3.5a) will make the eigenfunctions unitary\[3\],

\[
\Lambda = \Lambda_{\vec{P}} = \sqrt{\frac{m}{(2\pi)^3 E}} \mathcal{L},
\]

(3.6)

where the term \( \mathcal{L} \) is the Lorentz Boost operator of eq. (2.5b). Using the property of the Lorentz operator, \( p_\mathcal{L} = \mathcal{L} m e_4 \) from eq. (2.5a), the unimodular meta-monogenic multivector eigenfunction can be expressed in the alternate form,

\[
\Psi_{\vec{P}}(x) = \Lambda_{\vec{P}} \exp \left( -e_4 p_\mu x^\mu \right),
\]

(3.7a)

\[
= (\Box + m) \Phi_{\vec{P}}(x),
\]

(3.7b)

\[
\Phi_{\vec{P}}(x) = \frac{\exp \left( -e_4 p_\mu x^\mu \right)}{(2\pi)^{3/2} \sqrt{2E(E + m)}}.
\]

(3.7c)

Now we see that each eigenfunction has the same generator of quantum phase, the (positive parity) unit time vector \( e_4 \) when multiplied dextrally (right-side applied). Further the meta-monogenic eigenfunction \( \Psi_{\vec{P}}(x) \) can be written in terms of a multivector solution \( \Phi_{\vec{P}}(x) \) to the Klein-Gordon equation, again complex \( e_4 \). This differs from the work of Hestenes\[2\] and Benn\[6\] which both interpret the negative parity geometric pseudoscalar as playing the role of \( i \). Eigenfunctions in that form will not obey the standard parity relation, which ours do,

\[
\mathcal{P} \left( \Psi_{\vec{P}}(\vec{X}, t) \right) = -e_4 \Psi_{\vec{P}}(\vec{X}, t) e_4 = +\Psi_{-\vec{P}}(-\vec{X}, t).
\]

(3.8)

The unitary meta-mongenic eigenfunctions are orthonormal and complete,

\[
\int d^3 \vec{X} \, \Psi_{\vec{K}}^\dagger(\vec{X}, t) \, \Psi_{\vec{P}}(\vec{X}, t) = \delta_{\vec{K}, \vec{P}},
\]

(3.9a)

\[
\delta(\vec{X} - \vec{Y}) = \int d^3 \vec{P} \, \Psi_{\vec{P}}^\dagger(\vec{X}, t) \, \Psi_{\vec{P}}(\vec{Y}, t).
\]

(3.9b)

Hence these functions are the restricted subset of the solution space that forms a Multivector Hilbert space, where again \( e_4 \) (dextrally applied, i.e. multiplied on the right side) plays the role of the usual \( i \). A general solution in this function subspace (e.g. a “quantum wavepacket”) can be expanded,

\[
\Psi(\vec{X}, t) = \int d^3 \vec{P} \, \Psi_{\vec{P}}(\vec{X}, t) \, C_{\vec{P}},
\]

(3.10a)

\[
C_{\vec{P}} = \int d^3 \vec{X} \, \Psi_{\vec{P}}^\dagger(\vec{X}, t) \, \Psi(\vec{X}, t),
\]

(3.10b)

where the coefficients \( C_{\vec{P}} \) can be thought of as “3D scalars”, complex in \( e_4 \) and must appear on the right side of eq. (3.10a). Note these coefficients turn out to be independent of time \( t \).
The full 16-degree-of-freedom solution to eq. (3.2) is of mixed parity and can be written,

\[ \Psi(\vec{X}, t) = \int d^3\vec{P} \Psi_{\vec{P}}(\vec{X}, t) \mathcal{C}_\vec{P} [f_+(\vec{P}) + \varepsilon f_-(\vec{P})], \]

(3.11a)

\[ \varepsilon = e_1e_2e_3e_4, \]

(3.11b)

where \( f_\pm \) are multivector functions with four degrees of freedom on the basis set consisting of the scalar and the timelike trivectors: \( \{1, \varepsilon e_1, \varepsilon e_2, \varepsilon e_3\} \). These represent the ‘spin’ degrees of freedom, of both parities (indicated by the subscript). This full solution can be interpreted as an algebraic representation of isospin doublet of Dirac bispinors\[9\][10].

4 Integral Meta-Monogenic Solutions

In general, the first-order monogenic equation: \( \Box \Psi = 0 \) [or the generalized Dirac eq. (3.2a)] more directly lends itself to integral solution than the associated second-order harmonic equation [or the generalized Klein-Gordon eq. (3.1)]. For example, time independent (i.e. 3D) “static” electromagnetism can be easily re-expressed in Cauchy integral form, even when non-homogeneous source terms are included. A typical electrostatic example would be to calculate the electric field \( \vec{E} \) at any point inside a region knowing the field on the boundary (equivalently knowing the surface charge density \( d\sigma = \vec{E} \cdot d\vec{A} \)) and the distribution of (scalar) charge \( \rho = \nabla \vec{E} \) inside the region\[16\]. On the other hand, the Bergman kernel would only provide a method of calculating the electric field at a point in the region, if one already knows the complete solution everywhere inside the region. I am not aware of any classical problem which makes use of this circular relationship, however it possibly could be used to generate a perturbation series approximation.

There are additional constraints imposed by fundamental physical laws which further limit the usefulness of integral formulations in describing tangible phenomena. For example, consider the 3D Helmholtz equation: \( (\nabla^2 + \lambda^2)\phi(\vec{X}) = 0 \), of which integral solutions to the factored meta-monogenic form have been recently treated by Shapiro\[15\]. This should provide a good alternative derivation of Kirchhoff diffraction theory, however it will still suffer from the practical limitation of the function only being approximately known on the boundary\[16\]. Alternatively, this equation could be physically interpreted as a factored form of the time-independent quantum Pauli equation (see recent exposition by Adler\[19\]). However, one is then ontologically constrained by the quantum principle which states that only the modulus \( \rho = \phi^\dagger\phi \) of the wavefunction \( \phi = \sqrt{\rho} \exp(i\theta) \) can be measured; the absolute quantum phase \( \theta \) cannot be known. Hence it will be impossible to perform a Cauchy boundary integral except for very special situations. This has influenced the way that physicists approach or “formulate” quantum problems.

4.1 The Propagating Kernel

One common scenario is to assume a physical system’s initial configuration, even though it cannot be directly measured, [e.g. \( \Psi(\vec{X}, t') \) known for all \( \vec{X} \) at \( t' = 0 \)], and then to ask how the system must change with time. The Time Evolution Operator
$U(t; t')$ is defined,

$$\Psi(\vec{X}, t) = U(t; t') \psi(\vec{X}, t), \quad (4.1a)$$

$$U(t, t') = U(t - t') = \exp[-(t - t') (\vec{\nabla} - e_4)], \quad (4.2b)$$

where $\vec{\nabla} = e_4 \wedge = e_4 e^t \partial_t$ is the 3D gradient. It will be assumed that $t > t'$, otherwise one would be talking about ‘retrodiction’, a different problem. An integral representation of eq. (4.1a) is,

$$\Psi(\vec{X}, t) = \int d^3 \vec{X}' F(\vec{X}, t; \vec{X}', t') \psi(\vec{X}', t'), \quad (4.3a)$$

$$F(\vec{X}, t; \vec{X}', t') = U(t, t') \delta(\vec{X} - \vec{X}'), \quad (4.3b)$$

where the Dirac Delta Function $\delta(\vec{X})$ is a distribution of measure one at the origin and zero elsewhere. The kernel $F(x, x')$ of eq. (4.3b) is known in physics as the transformation function\cite{13}, or sometimes the Hyugen Propagator since eq. (4.3a) is a description of Hyugen’s principle: each point on a wave front may be regarded as a new source of waves. The integral is over the 3D hypersurface defined by $t' = \text{constant}$, however it could be generalized to any spacelike surface, i.e. a surface made of points that are not causally connected to each other, and further that $t$ of eq. (4.11a) is greater than the $t'$ of each point on the hypersurface. This amounts to requiring that the surface cannot have any “kinks” in it that would have the surface locally slip inside the “light cone”. The mathematician would equivalently say it must be a Lipschitz surface.

By inspection of eq. (4.3b), the kernel is function only of the difference of the coordinates: $F(\vec{X}, t; \vec{X}', t') = F(x, x') = F(x - x')$. Substituting eq. (3.9a) for the delta function in equation (4.3b) gives a closed integral form,

$$F(\vec{X}, t; \vec{X}', t') = \int d^3 \vec{p} \psi(\vec{p}, t) \psi(\vec{p}, \vec{X}', t'), \quad (4.4a)$$

$$= \int \frac{d^3 \vec{p}}{(2\pi)^3} \exp[-\vec{p} \cdot (x_\mu - x'_\mu)] \vec{p} e_4, \quad (4.4b)$$

$$- (\Box + m) \Delta(x - x') e_4, \quad (4.4c)$$

$$\Delta(x) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \sin(p^\mu x_\mu). \quad (4.4d)$$

From eq. (4.4a) it is clear that: $F^1(x, y) = F(y, x)$. The function $\Delta(x)$ is given by Bjorken\cite{8} to be related to a regular Bessel function inside the light cone. Clearly it is a solution everywhere to the Klein Gordon eq. (3.1), and so it follows from eq. (4.4c) that the propagator kernel is left meta-monogenic,

$$ \Box - m \ F(x, x') = 0, \quad (4.5a)$$

$$ \Box + m \ F(x', x) = 0, \quad (4.5b)$$

where $\boxdot$ operates on $x$, but not on $x'$. What we have derived is a special case of the Cauchy kernel. The usual closed boundry integral reduces to eq. (4.3a) when one assumes that $\Psi(\vec{X}, t) \to 0$ for: $|\vec{X}| \to \infty$ or $t \to \infty$. In quantum theory, one sidesteps the unmeasurable phase by assuming an ‘initial’ solution at $t' \to -\infty$ which is a coherent plane wave. This approximates for example the collimated beam of electrons shooting down the linear accelerator at Stanford.
4.2 Green Function

Another common situation is to know the function over a restricted spacelike surface (e.g., a long thin cylinder, such as an antenna), but different than our previous example, the time dependence of the function is known (e.g., harmonic). This will require the use of a Green function which satisfies: \((\Box - m) G(x) = \delta^4(x)\). When the use of \(i\) has been excluded, the inhomogeneous part will take the form,

\[
G(x) = \int \frac{d^4k}{(2\pi)^4 (m^2 + k^2)} \left[ m \cos(k^\mu x_\mu) + k \sin(k^\mu x_\mu) \right], \quad (4.6a)
\]

\[
= (\Box + m) \Theta(t) \Delta(x), \quad (4.6b)
\]

\[
= \Theta(t) (\Box + m) \Delta(x), \quad (4.6c)
\]

\[
= \Theta(t) F(x) e_4, \quad (4.6d)
\]

\[
= \Theta(t) \int \frac{d^3\vec{p}}{(2\pi)^3} \ p \ exp(-\hat{p} p^\mu x_\mu), \quad (4.6e)
\]

where \(\Theta(t)\) is a step function, which takes on the value of \(+\frac{1}{2}\) for \(t > 0\), and \(-\frac{1}{2}\) for \(t < 0\). Equation (4.6c) follows from eq. (4.6b) if we note \(\delta(t) \Delta(x) = 0\). One can go directly from eq. (4.6a) to eq. (4.6e) by doing the contour integral, however the use of \(i\) has been excluded. The generator of the residue can possibly be taken to be the geometry associated with \(dk^4\), again the (minus) time basis vector \(-e_4\) (see for example Hestenes\[1\]) where there must be attention to its placement. Equivalently we state (without proof) that the unit four-velocity \(\hat{p}\) may be used [inspection of eq. (4.6e) shows this to be consistent].

Inspection of eq. (4.6e) shows that \(\overline{G}(x) = G(-x)\), hence one can show that this Green function also satisfies the adjoint equation: \(G(x)(\Box - m) = \delta^4(x)\), where \(\Box\) is now understood to operate to the left. Hence we may derive the general integral solution,

\[
\Psi(x') = \oint d\Sigma^{\mu} \ G(x', x) \ e_{\mu} \Psi(x), \quad (4.7)
\]

where \(x\) is an interior point. Actually, its abit more complicated than that, one must add a homogeneous term to eq. (4.6a) such that the boundary conditions are met.

4.3 Path Integral Formulation

The Feynman PIF (Path Integral Formulation) of non-relativistic quantum mechanics\[12\] provides an elegant means of bridging the interpretation gap between classical mechanics and quantum formulations. In the non-relativistic case, it asserts that the propagator kernel of eq. (4.3a) can be written,

\[
F(x, x') = \sum \exp \left( \frac{i}{\hbar} S(x, x') \right), \quad (4.8a)
\]

\[
S(x, x') = \int_{x'}^x p^\mu dx_\mu. \quad (4.8b)
\]

The classical action \(S\) is evaluated over the path from \(x'\) to \(x\). The sum in eq. (4.8a) is over all possible classical paths: \(x^\mu = x^\mu(\tau)\), where \(p^\mu = \int dx_\mu\) and \(\tau\) is an affine
parameter called the \textit{proper time}. Classical paths require \( \frac{dx^4}{d\tau} > 0 \), i.e. paths in which particles go backward in time are excluded.

Because of the second order time derivative, the Klein-Gordon eq. (3.1) does not easily lend itself to PIF. The first order Dirac eq. (3.2a) is a better candidate. However, the standard derivation of eq. (4.8a) from the Schrödinger wave equation\[13\] required that each eigenfunction have the same generator of quantum phase, a commuting \( i \) which is now unavailable to us in the real multivector theory. In fact, we have seen from eq. (3.5a) that each eigenfunction (as seen from the left side) has its own unique quantum phase generator \( \hat{p} \), which makes the PIF derivation problematic. In the restricted case of plus parity eigenfunctions however, we saw from eq. (3.7a) that they all have the \textit{same} generator \( e_4 \) when viewed from the right side. We exploit the advantage of multivector wavefunctions (over standard Dirac column spinors) in that we can now consider right-side applied operations. Hence we introduce here a new alternative to eq. (4.3a), the \textit{dextrad propagator},

\[ \Psi(\vec{X}, t) = \int d^3 \vec{X}' \: \Psi(\vec{X}', t') \: H(\vec{X}, t; \vec{X}', t'), \quad (4.9a) \]

\[ H(x, x') = H(x - x') = \int \frac{d^3 \vec{P}}{(2\pi)^3} \exp \left[ -e_4 \: p^\mu (x_\mu - x'_\mu) \right]. \quad (4.9b) \]

It can be easily shown\[3\] that this propagator may be written in the PIF form of eq. (4.8a) with the replacement: \( i \rightarrow -e_4 \). Further, the other half of the solution space, the negative parity solutions (‘antiparticles’) will follow the same relation, except requiring the replacement \( e_4 \rightarrow -e_4 \) in eq. (4.9b) and \( i \rightarrow +e_4 \) in eq. (4.8a).

### 5 Summary

We are not aware of any other work in Clifford analysis that has considered the PIF (Path Integral Formulation) of kernels. This approach suggests that the particular form of eq. (3.7a) is the “special” multigeometric entity which propagates (unchanged) in the manner to which we ascribe the term “quantum particle”. Further, the PIF provides a method to introduce interactions into the quantum equation based upon classical mechanics. Because of eq. (4.9a), we see that the interactions will couple \textit{dextrally} to the wavefunction (multiplied on the right). In particular, the generator of electromagnetic interactions will be \( e_4 \) applied on the right\[10\]. Finally, separate from these physical interpretations, the general mathematical relationship between the “right-side applied” \textit{dextrad kernel} introduced in eq. (4.9a) and the standard “left-side applied” one of eq. (4.3a) [both as an integral solutions to eq. (3.2a)] should be explored. In particular, it is unknown to the author if a general ‘dextrad’ form of eq. (4.7) exists, nor under what conditions it would reduce to eq. (4.9a).

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References

[1] E.P. Wigner, ‘The Unreasonable Effectiveness of Mathematics in Natural Sciences’, in *Symmetries and Reflections- Scientific Essays*, (Ox Bow Press, 1979), pp. 222-237.

[2] D. Hestenes, ‘Spacetime Structure of Weak and Electromagnetic Interactions’, *Found. Phys.*, 12, 153 (1982); ‘Observables, Operators, and Complex Numbers in the Dirac Theory’, *J. Math. Phys.*, 16, 556-72 (1975).

[3] W.M. Pezzaglia, ‘A Clifford Algebra Multivector Reformulation of Field Theory’, (Ph.D. thesis, UC Davis 1993), UMI-83-26101-mc (microfiche from University Microfilms).

[4] M. Ross, ‘Geometric Algebra in Classical and Quantum Physics’, (Ph.D. thesis, UC Davis, 1980), (available on microfiche from University Microfilms), p. 61; K. Greider, private communication (July 28, 1977).

[5] K. Greider, ‘A Unifying Clifford Algebra Formalism for Relativistic Fields’, *Found. Phys. Lett.* 14, 467-506 (1984); ‘Relativistic Quantum Theory with Correct Conservation Laws’, *Phys. Rev. Lett.* 44, 1718-21 (1980); ‘Erratum’, *Phys. Rev. Lett.* 45, 673 (1980).

[6] I.M. Benn, and R.W. Tucker, ‘The Dirac Equation in Exterior Form’, *Commun. Math. Phys.* 98, 53-63 (1985).

[7] J.D. Bjorken, and S.D. Drell, *Relativistic Quantum Mechanics*, (McGraw-Hill, New York, 1964).

[8] J.D. Bjorken, and S.D. Drell, *Relativistic Quantum Fields*, (McGraw-Hill, New York, 1965).

[9] W.M. Pezzaglia, ‘Clifford Algebra Geometric-Multispinor Particles and Multivector-Current Gauge Fields’, *Found. Phys. Lett.* 5, 57-62 (1992).

[10] W.M. Pezzaglia, ‘Dextral and Bilateral Multivector Gauge Field description of Light-Unflavored Mesonic Interactions’, Preprint: CLF-ALG/PEZZ9302 (1993).

[11] D. Hestenes, ‘Multivector Functions’, *J. Math. Anal. Appl.*, 24, 467 (1968).

[12] R. Feynman, and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, New York, 1965).

[13] E.S. Abers and B.W. Lee, ‘Gauge Theories’, *Phys. Reports* C9, 1 (1973).

[14] R.H. Romer, ‘Editorial: Fermat’s last theorem’, *Amer. Jour. Phys.*, 61, 873 (1993).

[15] M.V. Shapiro, ‘The Dirac operator factorizes the Laplacian’, (Preprint 1993, to appear in these proceedings).
[16] J.D. Jackson, Classical Electrodynamics, (John Wiley & Sons, Inc., New York, 2nd edition, 1975).

[17] W.E. Baylis and G. Jones, ‘The Pauli-algebra approach to special relativity’, Phys. Rev. A45, 4293-4302 (1989).

[18] I.R. Porteous, Topological Geometry, (Cambridge Univ. Press, 2nd ed., 1981).

[19] R.J. Adler and R.A. Martin, ‘The electron $g$ factor and factorization of the Pauli equation’, Amer. Jour. Phys. 60, 837-9 (1992); W.E. Baylis, ‘Comment on “The electron $g$ factor and factorization of the Pauli equation” by R.J. Adler and R.A. Martin’, to appear in: Amer. Jour. Phys. (1994).