Feynman problem in the noncommutative case

José F. Cariñena† and Héctor Figueroa‡

† Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain
‡ Departamento de Matemáticas, Universidad de Costa Rica, 2060 San Pedro, Costa Rica

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Abstract

In the context of the Feynman’s derivation of electrodynamics, we show that noncommutativity allows other particle dynamics than the standard formalism of electrodynamics.

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1 Introduction

Feynman procedure [1] to obtain Maxwell’s equations in electrodynamics has been reviewed under different kind of settings, and several nontrivial and interesting generalizations are possible, see for instance [2] [3] [4] [5] [6] [7] [8]. In general the locality property that different coordinates commute is assumed. However, as pointed out by Jackiw [9], Heisenberg suggested in a letter to Peierls [10] that spatial coordinates may not commute, Peierls communicated the same idea to Pauli [11], who told it to Oppenheimer; eventually the idea arrived to Snyder [12] who wrote the first paper on the subject. On the other hand, the existence of a minimal length beyond which no strict localization is possible, the importance of the physics in noncommutative planes, the noncommutative Landau problem, Peierls substitution, and the fact that noncommutative field theory is relevant not only in string theory but also in condensed matters, motivated a new interest on the subject during the last years.

Due to this increasing interest in noncommutative field theories, it is worthwhile to consider the noncommutative version of such procedure, where locality no longer holds, which has a better chance to find new kinds of particle dynamics, which after all, according to Dyson [11], was the original aim of Feynman. Such considerations were actually done in [13], but the argument given there seems to be inadequate or incomplete for two reasons: they only considered the case where the nonlocality is described by a coordinate independent Moyal Bracket, whereas nowadays the
non–constant (i.e. coordinate dependent) noncommutative spaces are gaining a lot of attention in the noncommutative realm, because of the appearance of such type of noncommutativity in various contexts specially in string theory. Among the papers that invoke variable noncommutativity are \[14, 15, 16, 17, 18, 19, 20, 21\]. On the other hand, the treatment in \[13\] is somewhat sloppy and the main conclusions are not correct, as we shall point out later on.

To avoid unnecessary complications due to operator ordering, we shall only discuss the classical analogue of the Feynman procedure in the noncommutative case. Accordingly, the appropriate setting would be in terms of Poisson brackets, which is regarded as the classical limit of the commutator of quantum observables. But we shall explore the different possibilities arising from the dependence of the fundamental brackets on the different sets of variables involved.

Let \( \mathcal{F}(M) \) be the algebra of functions (the algebra of classical observables) on a manifold \( M \) (the classical state space). A Poisson structure on \( M \) is a real skew symmetric bilinear map \( \{\cdot, \cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M) \) satisfying the Jacobi identity:

\[
\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0, \quad \forall F, G, H \in \mathcal{F}(M),
\]

and such that the map \( X_F = \{\cdot, F\} \) is a derivation of the Lie algebra \( \mathcal{F}(M) \), for each \( F \in \mathcal{F}(M) \), in other words, \( X_F \) is a vector field, usually called a Hamiltonian vector field, and \( F \) is said to be the Hamiltonian of \( X_F \). This second property, called the Leibnitz’ rule, is important as there are many examples of Lie algebra structures on \( \mathcal{F}(M) \) that do not satisfy the Leibnitz’ rule.

In particular, if \( \xi^a \) denotes a set of local coordinates on \( M \), then, using the summation index convention,

\[
X_F = X_F(\xi^a) \frac{\partial}{\partial \xi^a} = \{\xi^a, F\} \frac{\partial}{\partial \xi^a},
\]

hence

\[
\{F, G\} = X_G(F) = \{\xi^a, G\} \frac{\partial F}{\partial \xi^a}.
\]

Thus,

\[
\{\xi^a, G\} = -\{G, \xi^a\} = -\{\xi^b, \xi^a\} \frac{\partial G}{\partial \xi^b} = \{\xi^a, \xi^b\} \frac{\partial G}{\partial \xi^a},
\]

and the local coordinate expression of the Poisson Bracket becomes

\[
\{F, G\} = \{\xi^a, \xi^b\} \frac{\partial G}{\partial \xi^b} \frac{\partial F}{\partial \xi^a}.
\]

Therefore to compute the Poisson bracket of any pair of functions is enough to know the fundamental Poisson brackets

\[
\Lambda^{ab} = \{\xi^a, \xi^b\}.
\]

Moreover, the value of \( \{F, G\} \) at a point \( m \in M \) does not depend on \( F \) and \( G \) but on \( dF \) and \( dG \), as explicitly shown in \[1\], hence from the Poisson structure we get a twice contravariant skew symmetric tensor

\[
\Lambda(dF, dG) := \{F, G\}.
\]

Indeed, if \( \bar{\xi} = \phi(\xi) \) is another set of local coordinates on \( M \), then,

\[
\bar{\Lambda}^{ab} = \{\bar{\xi}^a, \bar{\xi}^b\} = \{\phi^a, \phi^b\} = \{\xi^c, \xi^d\} \frac{\partial \phi^a}{\partial \xi^c} \frac{\partial \phi^b}{\partial \xi^d} = \Lambda^{cd} \frac{\partial \phi^a}{\partial \xi^c} \frac{\partial \phi^b}{\partial \xi^d},
\]

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so the components of $\Lambda$ change like the local coordinates of the twice contravariant skew symmetric tensor with coordinate expression

$$\Lambda = \Lambda^{ab} \frac{\partial}{\partial \xi^a} \wedge \frac{\partial}{\partial \xi^b}.$$  

The tensor $\Lambda$ is called a Poisson tensor. We are using the convention that in the local expression of the wedge product only summands whose subindex on the left hand side term is smaller than the subindex on the right hand side term appear.

For any function $H \in \mathcal{F}(M)$ the integral curves of the dynamical vector field $X_H$ are precisely determined by the solutions of the system of differential equations

$$\frac{d\xi^a}{dt} = \{\xi^a, H\},$$

and the dynamical evolution of a function $F$ in $M$ is given by

$$\frac{dF}{dt} = \{F, H\},$$

or in local coordinates

$$\frac{dF}{dt} = \Lambda^{ab} \frac{\partial F}{\partial \xi^a} \frac{\partial H}{\partial \xi^b}.$$  

In terms of $\Lambda$ the Hamiltonian vector field associated to $F$ is given by

$$X_F G = -\Lambda(dF, dG), \quad \forall G \in C^\infty(M).$$

Furthermore, the Jacobi identity is equivalent to the vanishing of the Schouten Bracket of $\Lambda$ with itself $[2]$. 

### 2 The velocity independent case

In this section we study the Feynman argument in the framework of a tangent bundle, in the case where the bracket is nonlocal; in other words, we do not suppose that the variables on the configuration space commute. So we assume that the Poisson manifold $M$ is the tangent bundle $TQ$ of a $n$-dimensional configuration space $Q$, with local coordinates $x^i, \dot{x}^i$, for $i = 1, \ldots, n = \dim Q$. Thus a general Poisson bracket on $TQ$ is locally given by

$$\{F, G\} = \{x^i, x^j\} \frac{\partial G}{\partial x^j} \frac{\partial F}{\partial x^i} + \{x^i, \dot{x}^j\} \frac{\partial G}{\partial \dot{x}^j} \frac{\partial F}{\partial x^i} + \{\dot{x}^i, x^j\} \frac{\partial G}{\partial x^j} \frac{\partial F}{\partial \dot{x}^i} + \{\dot{x}^i, \dot{x}^j\} \frac{\partial G}{\partial \dot{x}^j} \frac{\partial F}{\partial \dot{x}^i}.$$  

Although we shall concentrate on autonomous systems, our arguments can be extended to more general contexts. We first consider a bracket such that

$$\{x^i, x^j\} = g_{ij}(x),$$

where $g_{ij}$ is an arbitrary skewsymmetric matrix of functions, fulfilling the constraints that a Poisson bracket satisfying the Leibniz rule impose. In other words, we examine the possibility of a
bracket without the locality property; a condition needed, for instance, in a classical description of a massless particle \[22\]. We also require

\[ m \{ x^i, \dot{x}^j \} = \delta_{ij}, \] (5)

so this part of the Poisson bracket is the same as in the commutative case, considered by Feynman.

Now, the Jacobi identity

\[ \{ x^i, \{ x^j, \dot{x}^k \} \} + \{ \dot{x}^k, \{ x^j, x^i \} \} + \{ x^j, \{ \dot{x}^k, x^i \} \} = 0 \]

entails, upon using (5), and \( \partial g_{ij} / \partial \dot{x}^k = 0 \),

\[ 0 = \{ \dot{x}^k, g_{ij} \} = \{ \dot{x}^k, x^l \} \frac{\partial g_{ij}}{\partial x^l} + \{ \dot{x}^k, \dot{x}^l \} \frac{\partial g_{ij}}{\partial \dot{x}^l} = -\frac{1}{m} m \frac{\partial g_{ij}}{\partial x^k}. \] (6)

Thus, the matrix \( g_{ij} \) is a constant skewsymmetric \( 3 \times 3 \) matrix, and nonconstant matrices will only be possible if one assume dependence of \( g \) on the dotted variables, but we explore this possibility in the next section.

In other words, we are assuming that the Poisson tensor \( \Lambda \) is given by

\[ \Lambda = g_{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{1}{m} \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial \dot{x}^i} + A_{ij}(x, \dot{x}) \frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial \dot{x}^j}, \] (7)

where the functions \( A_{ij} \) are skewsymmetric functions to be determined.

To continue with Feynman’s argument we further assume Newton’s equations:

\[ m \ddot{x}^j = F^j(x, \dot{x}), \] (8)

whose solutions are the integral curves of the vector field \( \Gamma \) with coordinate expression

\[ \Gamma = \dot{x}^i \frac{\partial}{\partial x^i} + \frac{1}{m} F^j(x, \dot{x}) \frac{\partial}{\partial \dot{x}^j}. \]

In other words, we assume that the equations of motion can be written as

\[ \frac{dx^i}{dt} = \{ x^i, H \} = \dot{x}^i, \] (9)

\[ \frac{d\dot{x}^j}{dt} = \{ \dot{x}^i, H \} = \frac{1}{m} F^i(x, \dot{x}), \] (10)

with \( \{ \cdot, \cdot \} \) a Poisson bracket to be determined. Note however that as we assumed the nonlocality property of the Poisson bivector, such bivector cannot be associated to a symplectic structure defined by a regular Lagrangian, because the locality assumption is equivalent to the vanishing of the symplectic form \( \omega_L \) on a pair of vertical fields, which is a necessary condition for the existence of a regular Lagrangian \[23, 24\].

Now, if we restrict ourselves to the case \( Q = \mathbb{R}^3 \), we can define a field \( B \), that, in analogy with the commutative case, we may call the magnetic field, by means of

\[ A_{ij}(x, \dot{x}) = \{ \dot{x}^i, \dot{x}^j \} = \frac{1}{m^2} \varepsilon_{ijk} B_k(x, \dot{x}). \] (11)

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We require $\Gamma$ to be Hamiltonian, in particular $\Gamma$ is a derivation of the Poisson algebra structure. Applying $\Gamma$ to (5) we obtain

$$0 = m\{\dot{x}^i, \dot{x}^j\} + \{x^i, F^j\},$$

where we use the second order condition: $\Gamma x^i = \dot{x}^i$, therefore

$$\{x^i, F^j\} = -m\{\dot{x}^i, \dot{x}^j\} = m\{\dot{x}^j, \dot{x}^i\} = -\{x^i, F^i\},$$

i.e. $\{x^i, F^j\}$ is skewsymmetric and there exists $B_k(x, \dot{x})$ such that

$$\{x^i, F^j\} = -\frac{1}{m} \varepsilon^{ijk} B_k,$$  \hspace{1cm} (12)

where $\varepsilon^{ijk}$ denotes the fully skewsymmetric Levi–Civita tensor, for which $\varepsilon^{123} = 1$; so, for instance,

$$B_3 = -m\{x^1, F^2\} = m^2\{\dot{x}^1, \dot{x}^2\}. \hspace{1cm} (13)$$

Now, the Jacobi identities with one position and two velocities entail

$$\{x^i, B_j\} = 0,$$

and the local expression (2) gives

$$0 = \{x^i, B_j\} = g_{ik} \frac{\partial B_j}{\partial x^k} + \frac{1}{m} \frac{\partial B_j}{\partial \dot{x}^i}. \hspace{1cm} (14)$$

In the commutative case, i.e. when $g_{ik} \equiv 0$, (14) implies that $B_j$ is independent of the $\dot{x}$’s, but in our setting this is not necessarily true. However, notice that, for instance

$$\{\dot{x}^3, B_3\} = m^2\{\dot{x}^3, \{\dot{x}^1, \dot{x}^2\}\}.\hspace{1cm}$$

Thus, the Jacobi identity with three different velocities gives

$$\{\dot{x}^i, B_i\} = 0. \hspace{1cm} (15)$$

Once again the local expression of the Poisson bracket gives

$$m\{\dot{x}^i, B_j\} = -\frac{\partial B_j}{\partial x^i} + m\{\dot{x}^i, \dot{x}^k\} \frac{\partial B_j}{\partial \dot{x}^k} = -\frac{\partial B_j}{\partial x^i} + \frac{1}{m} \varepsilon_{ijk} B_k \frac{\partial B_j}{\partial \dot{x}^j},$$

and then we can rewrite (15) as

$$\text{div} B = -\frac{1}{m} B \cdot \hat{\nabla} \times B,$$ \hspace{1cm} (16)

upon using the notation $\hat{\nabla} = (\partial/\partial \dot{x}^1, \partial/\partial \dot{x}^2, \partial/\partial \dot{x}^3)$. This is the equation that replaces the Maxwell equation $\text{div} B = 0$ describing the absence of monopoles in the noncommutative case.

In the particular case when the field $B$ is independent of the $\dot{x}$’s, the previous equation (16) reduces indeed to the such Maxwell equation

$$\text{div} B = 0.$$
Now, we mentioned already that \( \mathbf{B} \) may very well depend on the variables \( \dot{x} \), but even if we assume that the field \( \mathbf{B} \) is independent of the \( \dot{x}'s \), from (14) we see that \( \mathbf{B} \) can still depend on the variables \( x \), since the matrix \( g_{ij} \), being a constant skew-symmetric \( 3 \times 3 \) matrix, is singular. Therefore, the conclusion in [13] that the conditions (4), (5) and (8) entail static Maxwell equations is wrong. One of the problems in [13] is that in the noncommutative space that they are using, the conclusion in [13] that the conditions (4), (5) and (8) entail static Maxwell equations is wrong. One of the problems in [13] is that in the noncommutative space that they are using, which is neither explicitly defined nor described, it is not clear at all the meaning of the variables \( \dot{x} \).

On the other hand, in the quest of an equation similar to the second Maxwell equation, we define another field \( \mathbf{E} \), the electric field, by \( E^i = F^i - \varepsilon_{jkl} \dot{x}^k B_l \). This makes sense in the commutative case because, there, \( \mathbf{B} \) is certainly independent of the \( \dot{x}'s \) and, as we shall see in a moment, (12) implies that \( \mathbf{F} \) is at most linear in the \( \dot{x}'s \) variables, but again this is not necessarily what happens in our setting, even if we assume independence of \( \mathbf{B} \) on the \( \dot{x}'s \) variables. Indeed, from (12) and (14) we obtain

\[
\{ x^i, E_j \} = \{ x^i, F^j - \varepsilon_{jkl} \dot{x}^k B_l \} = \{ x^i, F^j \} - \varepsilon_{jkl} \{ x^i, \dot{x}^k \} B_l - \varepsilon_{jkl} \dot{x}^k \{ x^i, B_l \} = \{ x^i, F^j \} - \frac{1}{m} \varepsilon_{jkl} B_l = 0;
\]

therefore, as claimed, in the commutative case the field \( \mathbf{E} \) so defined is independent of the velocities.

Following the commutative case, we apply the vector field \( \Gamma \) to (13) (which boils down to take the derivative with respect to \( t \) of that equation):

\[
\dot{x}^l \frac{\partial B_k}{\partial x^l} + \frac{1}{m} F^l \frac{\partial B_k}{\partial \dot{x}^l} = \frac{m^2}{2} \varepsilon_{ijk} \left( \{ \dot{x}^i, F^j \} + \{ F^i, \dot{x}^j \} \right) = m \varepsilon_{ijk} \{ F^i, \dot{x}^j \} = m \varepsilon_{ijk} \{ E^i, \dot{x}^j \} + \varepsilon_{ilm} \dot{x}^l \{ B_n, \dot{x}^j \}.
\]

Now, the local expressions of the brackets give

\[
m \varepsilon_{ijk} \{ E^i, \dot{x}^j \} = m \varepsilon_{ijk} \left( \{ x^i, \dot{x}^j \} \frac{\partial E^i}{\partial x^l} + \{ \dot{x}^i, \dot{x}^j \} \frac{\partial E^i}{\partial \dot{x}^l} \right) = \varepsilon_{ijk} \left( \frac{\partial E^i}{\partial x^l} + \frac{1}{m} \varepsilon_{ijn} B_n \frac{\partial E^i}{\partial \dot{x}^l} \right) = \varepsilon_{ijk} \left( \frac{\partial E^i}{\partial x^l} + \frac{1}{m} (\delta_{il} \delta_{kn} - \delta_{ln} \delta_{kl}) B_n \frac{\partial E^i}{\partial \dot{x}^l} \right)
\]

Moreover,

\[
m \varepsilon_{ijk} \varepsilon_{ilm} \dot{x}^l \{ B_n, \dot{x}^j \} = m (\delta_{jl} \delta_{kn} - \delta_{jn} \delta_{kl}) \{ \dot{x}^i, \dot{x}^j \} B_n = -m \{ \dot{x}^k, \dot{x}^j \} B_j = -\frac{1}{m} \varepsilon_{kjl} B_l B_j = 0,
\]

\[\]
on account of (11). Also, using (15), we have

\[ m \varepsilon_{ijk} \varepsilon_{ilm} \dot{x}^n \{ B_l, \dot{x}^j \} = m (\delta_{jn} \delta_{kl} - \delta_{jl} \delta_{kn}) \dot{x}^n \{ B_l, \dot{x}^j \} \]

\[ = m (\dot{x}^n \{ B_k, \dot{x}^n \} - \dot{x}^k \{ B_l, \dot{x}^l \}) \]

\[ = m \dot{x}^n \{ B_k, \dot{x}^n \} \]

\[ = m \left( \dot{x}^n \{ x^l, \dot{x}^n \} \frac{\partial B_k}{\partial \dot{x}^l} + \dot{x}^n \{ \dot{x}^l, \dot{x}^n \} \frac{\partial B_k}{\partial \dot{x}^l} \right) \]

\[ = \dot{x}^l \frac{\partial B_k}{\partial x^l} + \frac{1}{m} \dot{x}^n \varepsilon_{lm} B_r \frac{\partial B_k}{\partial \dot{x}^l} \]

\[ = \dot{x}^l \frac{\partial B_k}{\partial x^l} + \frac{1}{m} F_l \frac{\partial B_k}{\partial \dot{x}^l} - \frac{1}{m} E_l \frac{\partial B_k}{\partial \dot{x}^l} . \]

Collecting all together, we see that (17) reduces to

\[ \varepsilon_{ijk} \frac{\partial E^i}{\partial x^j} = \frac{1}{m} \left( E^l \frac{\partial B_k}{\partial \dot{x}^l} + B_l \frac{\partial E^l}{\partial \dot{x}^k} - B_k \frac{\partial E^l}{\partial \dot{x}^l} \right) , \tag{18} \]

or in other form,

\[ (\text{rot } E)_k + \frac{1}{m} \left( (E \cdot \nabla) B_k + B \cdot \frac{\partial E}{\partial \dot{x}^k} - \text{div } B \right) = 0 , \]

which is what replaces the Maxwell equation corresponding to Faraday’s law, in the setting suggested at the beginning of this section.

Finally, we point out that had we assumed that the fields \( B \) and \( E \) do not depend on the \( \dot{x} \)'s (so \( F \) is actually a Lorentz force), then (16) and (18) would exactly be the usual Maxwell equations, so in the limit we have a smooth transition into the commutative case, contrary to what is claimed in (13). However, here the Lorentz force condition would be an extra assumption, not a consequence as in the commutative case considered in (11).

### 3 Velocity dependent Poisson brackets

We now return to the case where the matrix \( g_{ij} = g_{ij}(x, \dot{x}) \) in (4) also depends on the variables \( \dot{x} \). Then \( g_{ij} \) no longer need to be a constant matrix, as now (6) rather imposes on \( g_{ij} \) the condition

\[ 0 = - \frac{1}{m} \frac{\partial g_{ij}}{\partial x^k} + \{ x^k, x^l \} \frac{\partial g_{ij}}{\partial \dot{x}^l} . \]

Moreover, the Jacobi identity on \( x^i, x^j \) and \( x^k \) reduces to

\[ 0 = \{ x^i, g_{jk} \} + \{ x^k, g_{ij} \} + \{ x^j, g_{ki} \} \]

\[ = g_{il} \frac{\partial g_{jk}}{\partial x^l} + g_{ki} \frac{\partial g_{ij}}{\partial x^l} + g_{jl} \frac{\partial g_{ki}}{\partial x^l} + \frac{1}{m} \left( \frac{\partial g_{jk}}{\partial \dot{x}^i} + \frac{\partial g_{ij}}{\partial \dot{x}^k} + \frac{\partial g_{ki}}{\partial \dot{x}^j} \right) , \]

which gives exactly one more constraint on the \( g_{ij} \)'s, since the skew symmetry property of \( g_{ij} \) entails that a permutation of the indexes gives the same equation as for \( i = 1, j = 2 \) and \( k = 3 \).
when the permutation is even, and negative the expression if the permutation is odd. Note, however, that
\[ \Gamma g_{ij} = \Gamma \{ x^i, x^j \} = \{ x^i, x^j \} + \{ x^i, \Gamma x^j \} = \{ x^i, \dot{x}^j \} + \{ \dot{x}^i, x^j \} = 0, \]

implying that the \( g_{ij} \)'s are constants of the motion.

Furthermore, in the previous section we did not use the fact that the \( g \)'s were constant, therefore by the same token we obtain also for \( g_{ij}(x, \dot{x}) \) the generalized Maxwell equations (16) and (18).

On the other hand, even though condition (5) simplified matters quite a bit, it may be useful, in some settings, to modify also this condition. Thus we now address the problem when
\[
\{ x^i, x^j \} = g_{ij}(x, \dot{x}), \tag{19}
\]
and
\[
m\{ \dot{x}^i, \dot{x}^j \} = \delta_{ij} + f_{ij}(x, \dot{x}), \tag{20}
\]
where \( f_{ij} \) is another matrix compatible with the Poisson bracket properties, which now impose several relations among the \( g_{ij} \)'s and the \( f_{ij} \)'s, again the \( g_{ij} \)'s need not be constants.

In principle, there is no need to impose a special condition on \( f_{ij} \), but the parallelism with the computation of the previous section is more transparent if one assumes, as we do, that \( f_{ij} \) is skewsymmetric. In [13] a particular instance of this situation was considered, but they assumed that the variables \( \dot{x} \) are functions of the \( x^i \)'s, a hypothesis without much physical justification, they assume a special form of the \( f_{ij} \)'s which is completely unnecessary, and they place their argument in the constant noncommutative case.

In other words, we are now replacing (7) by the general Poisson tensor
\[
\Lambda = g_{ij}(x, \dot{x}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{1}{m}(\delta_{ij} + f_{ij}(x, \dot{x})) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \dot{x}^j} + A_{ij}(x, \dot{x}) \frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial \dot{x}^j},
\]
and the problem is to determine the functions \( A_{ij} \) and the Hamiltonian \( H \) given the functions \( g_{ij} \) and \( f_{ij} \), and the equations of motion (9) and (10), or Newton’s equation
\[ m\ddot{x}^j = F^j(x, \dot{x}). \]

Once more, when applying the vector field \( \Gamma \) to (20) we obtain
\[ \Gamma f_{ij} = m\{ \dot{x}^i, \dot{x}^j \} + \{ x^i, F^j \}, \]
therefore
\[ \{ \dot{x}^i, \dot{x}^j \} = \frac{1}{m}(\Gamma f_{ij} - \{ x^i, F^j \}), \]
and since \( f_{ij} \) is skewsymmetric,
\[ \{ x^i, F^j \} = -\{ x^j, F^i \}, \]
so a field \( B \) can be defined as in (11) or (12), and exactly the same computations can be performed, leading to some equations a bit more involved, but similar to (16) and (18). We see no point in repeating the calculations.
In this context the equations of motion become

\[
\dot{x}^i = \{x^i, x^j\} \frac{\partial H}{\partial x^j} + \{x^i, \dot{x}^j\} \frac{\partial H}{\partial \dot{x}^j},
\]

\[
F^i = \{\dot{x}^i, x^j\} \frac{\partial H}{\partial x^j} + \{\dot{x}^i, \dot{x}^j\} \frac{\partial H}{\partial \dot{x}^j},
\]

which are more complicated than the classical ones, but, in principle, a Hamiltonian description is still possible in the noncommutative setting.

We conclude that noncommutativity does allow other dynamics than the standard formalism of electrodynamics.

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