INVERSE ELASTIC SCATTERING FOR A RANDOM POTENTIAL

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Abstract. This paper is concerned with an inverse scattering problem for the time-harmonic elastic wave equation with a rough potential. Interpreted as a distribution, the potential is assumed to be a microlocally isotropic generalized Gaussian random field with the covariance operator being described by a classical pseudo-differential operator. The goal is to determine the principal symbol of the covariance operator from the scattered wave measured in a bounded domain which has a positive distance from the domain of the potential. For such a rough potential, the well-posedness of the direct scattering problem in the distribution sense is established by studying an equivalent Lippmann–Schwinger integral equation. For the inverse scattering problem, it is shown with probability one that the principal symbol of the covariance operator can be uniquely determined by the amplitude of the scattered waves averaged over the frequency band from a single realization of the random potential. The analysis employs the Born approximation in high frequency, asymptotics of the Green tensor for the elastic wave equation, and microlocal analysis for the Fourier integral operators.

1. Introduction and statement of the main result

The scattering problems for elastic waves have significant applications in diverse scientific areas such as geophysical exploration and nondestructive testing [18, 31]. In medical diagnostics, elastography is an emerging imaging modality that seeks to determine the mechanical properties of elastic media from their response to exciting forces [28]. By mapping the elastic properties and stiffness of soft tissues, it can give diagnostic information about the presence or status of disease [12]. Driven by these applications, the underlying inverse problems, which are to determine the medium properties based on the elastic wave equation, have been extensively studied and many mathematical results are available, especially for the uniqueness [9, 13, 14, 30]. We refer to [2] for a comprehensive account of mathematical methods in elasticity imaging.

Stochastic modeling has been widely adopted to handle problems involving uncertainties and randomness. In the research area of wave propagation, the wave fields may not be deterministic but rather are described by random fields due to the uncertainties for the media and/or the environments. Therefore, it is more appropriate to consider the stochastic wave equations to describe the wave motion. In addition to the ill-posedness and nonlinearity, stochastic inverse problems have substantially more difficulties than their deterministic counterparts. The random parameters to be determined in stochastic inverse problems can not be characterized by a particular realization, but instead, by its statistics, such as expectation and covariance. Hence, the relationship between these statistics and the wave fields needs to be established. In general, the statistics of the data for the wave fields are required, which significantly increases the computational cost since a large number of realizations is needed. It is an important and challenging problem to determine the statistics of the random parameters through fewer realizations of the wave fields.

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The paper is concerned with an inverse scattering problem for the time-harmonic elastic wave equation with a random potential in two and three dimensions. Specifically, we consider the stochastic elastic wave equation
\[
\mu \Delta u + (\lambda + \mu) \nabla \cdot u + \omega^2 u - V(\cdot, \omega) u = -\delta_\omega \alpha \quad \text{in } \mathbb{R}^d,
\]  
where \( d = 2 \) or \( 3 \), \( u \in \mathbb{C}^d \) is the displacement, \( \omega > 0 \) is the angular frequency, \( \lambda \) and \( \mu \) are the Lamé constants satisfying \( \mu > 0 \) and \( \lambda + 2\mu > 0 \) such that the second-order partial differential operator \( \Delta^* := \mu \Delta + (\lambda + \mu) \nabla \nabla \cdot \) is strongly elliptic (cf. \([27, 10.4]\)), \( \delta_\omega \) is the Dirac delta function centered at the source point \( y \in \mathbb{R}^d \), and \( \alpha \) is a unit polarization vector in \( \mathbb{R}^d \). The randomness of (1.1) comes from the potential \( V \), which represents a linear load inside a known homogeneous and isotropic elastic solid and is considered to be a generalized Gaussian random field. Throughout, the potential \( V \) is required to satisfy the following assumption.

**Assumption 1.** Assume that the centered random potential \( V \) has the form \( V(x, \omega) = \rho(x) \omega^{-\theta} \), where \( \rho \) is a microlocally isotropic Gaussian random field of order \( -m \) in \( \mathbb{R}^d \), \( \theta > \frac{m-1}{2} \), and \( D \) is a bounded domain with a locally Lipschitz boundary. It follows from Definition 2.1 that \( \rho \) has the principal symbol \( \phi(x)|\xi|^{-m} \) with \( \phi \in C^\infty_0(D) \) and \( \phi \geq 0 \). Hence, the potential \( V \) is also a microlocally isotropic Gaussian random field of order \( -m \) with the principal symbol \( \omega^{-2\theta} \phi(x)|\xi|^{-m} \).

We point out that the random potential \( V \) depends on the frequency \( \omega \) in Assumption 1. This particular form is convenient to prove the convergence of the Born series in the high frequency regime. The frequency-dependent model is inspired by \([16]\), where an inverse acoustic scattering problem in half-space with an impedance boundary condition was considered; the impedance function, depending on the frequency, was assumed to be a microlocally anisotropic random field.

The displacement of the total field \( u \in \mathbb{C}^d \) in (1.1) can be decomposed into
\[
u(x, y) = u^i(x, y) + u^s(x, y),
\]
where \( u^s \) represents the scattered field and \( u^i \) is the incident field given by
\[
u^i(x, y) = G_d(x, y, \omega) \alpha, \quad x \neq y.
\]
Here, \( G_d(x, y, \omega) \in \mathbb{C}^{d \times d} \) denotes the Green tensor for the Navier equation. Explicitly,
\[
G_d(x, y, \omega) = \frac{1}{\mu} \Phi_d(x, y, \kappa_s) I + \frac{1}{\omega} \nabla_x \nabla_x^T \left[ \Phi_d(x, y, \kappa_s) - \Phi_d(x, y, \kappa_p) \right],
\]
where \( I \) is the \( d \times d \) identity matrix, \( \kappa_p \) and \( \kappa_s \) are known as the compressional and shear wavenumbers, respectively, and are defined by \( \kappa_p = c_p \omega \) and \( \kappa_s = c_s \omega \) with \( c_p = (\lambda + 2\mu)^{-\frac{1}{2}} \) and \( c_s = \mu^{-\frac{1}{2}} \), and \( \Phi_d(x, y, \kappa) \) is the fundamental solution for the \( d \)-dimensional Helmholtz equation and is given by
\[
\Phi_2(x, y, \kappa) = \frac{\mathrm{i} I_0^{(1)}(\kappa|x - y|)}{4 \pi |x - y|} \quad \text{and} \quad \Phi_3(x, y, \kappa) = \frac{\mathrm{e}^{\mathrm{i} \theta (x - y)}}{4 \pi |x - y|}.
\]
Here \( I_0^{(1)} \) is the Hankel function of the first kind with order zero. For a scalar function \( \varphi \) defined in \( \mathbb{R}^d \), let
\[
\nabla_x \nabla_x^T \varphi = \begin{bmatrix}
\partial^2_{x_1 x_1} \varphi & \cdots & \partial^2_{x_1 x_d} \varphi \\
\vdots & \ddots & \vdots \\
\partial^2_{x_d x_1} \varphi & \cdots & \partial^2_{x_d x_d} \varphi
\end{bmatrix}.
\]

Since the elastic wave equation (1.1) is imposed in the whole space \( \mathbb{R}^d \), an appropriate radiation condition is needed to complete the problem formulation. By the Helmholtz decomposition (cf. \([8, \]

Appendix B), the scattered field $u^s$ can be decomposed into two parts: the compressional wave component $u_p^s$ and the shear wave component $u_s^s$, which read

$$u^s = u_p^s + u_s^s \quad \text{in } \mathbb{R}^d \setminus \overline{D}.$$  

The Kupradze–Sommerfeld radiation condition requires that $u_p^s$ and $u_s^s$ satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} (\partial_r u_p^s - i\kappa_p u_p^s) = 0, \quad \lim_{r \to \infty} r^{\frac{d-1}{2}} (\partial_r u_s^s - i\kappa_s u_s^s) = 0, \quad r = |x|. \quad (1.3)$$

There are few results for the inverse scattering problems in random settings due to the extra challenge of randomness and uncertainties. For inverse random source scattering problems, when the source is driven by an additive white noise, effective mathematical models and efficient computational methods have been proposed for the stochastic acoustic and elastic wave equations [4–7, 23, 24]. To determine the unknown parameters in the above models, in general, the data of the expectation and variance for the measured wave field is needed, and hence a fairly large number of realizations of the random source is required. If the source is described as a generalized Gaussian random field, it was proved in [20, 21] for the stochastic acoustic and elastic wave equations. It was shown that the principal symbol of the covariance operator can be uniquely determined by the amplitude of the wave field in [20, 21] for the stochastic acoustic and elastic wave equations. To obtain the backscattered far-field data associated with the plane wave and the scattered wave field in inverse scattering problems, when the source is a random potential, a related work can be found in [16], where an inverse scattering problem in a half-space with an impedance boundary condition was studied where the impedance function is modeled as a generalized Gaussian random field.

In this work, we study both the direct and inverse scattering problems for the stochastic elastic wave equation (1.1) along with the radiation condition (1.3). Given the random potential $V$, which may be so rough that it can only be interpreted as a distribution, the direct scattering problem is to determine the displacement $u$ which satisfies (1.1) and (1.3) in an appropriate sense. Using Green’s theorem and the Kupradze–Sommerfeld radiation condition, we deduce an equivalent Lippmann–Schwinger integral equation. Based on the Fredholm alternative theorem and the unique continuation principle, the Lippmann–Schwinger equation is shown to have a unique solution in the Sobolev space with a positive smoothness index. The inverse scattering problem is to determine the function $\phi(x)$, which represents the micro-correlation strength of the potential $V$, from the scattered field measured in a bounded domain $U$ which has a positive distance from $D$, i.e., $U \subset \mathbb{R}^d \setminus \overline{D}$.

It is clear to note from the elastic wave equation (1.1) that the displacement $u$ depends on the observation point $x$, the location of the source point $y$, the angular frequency $\omega$, and the unit polarization vector $a$. To express explicitly the dependence of $u$ on these quantities, we write $u(x, y, \omega, a)$, $u^i(x, y, \omega, a)$, $u^s(x, y, \omega, a)$, and $u_j(x, y, \omega, a)$ in the Born series (cf. (4.1) and (4.2) for the definition of $u_j$) for $u(x, y)$, $u^i(x, y)$, $u^s(x, y)$, and $u_j(x, y)$, respectively. Moreover, when the observation point $x$ coincides the source point $y$, for simplicity, we write $u^s(x, \omega, a)$ and $u_j(x, \omega, a)$ for $u^s(x, x, \omega, a)$ and $u_j(x, x, \omega, a)$, respectively.

The following theorem concerns the uniqueness of the inverse scattering problem and is the main result of this paper.

**Theorem 1.1.** Let $p \in (\frac{d}{2}, 2)$, $V$ satisfy Assumption 1 with $m \in (\frac{2d}{p} + d - 4, d]$ and $U \subset \mathbb{R}^d \setminus \overline{D}$ be a bounded domain having a locally Lipschitz boundary and a positive distance from $D$. Then for all
\( x \in U \), it holds almost surely that
\[
\lim_{Q \to \infty} \frac{1}{Q - 1} \int_1^Q \omega^{m+2d-2d+6} \sum_{j=1}^d |u^s(x, \omega, a_j)|^2 \, d\omega = C_d \int_{\mathbb{R}^d} \frac{1}{|x - \zeta|^{2d-2}} \phi(\zeta) \, d\zeta, \tag{1.4}
\]
where
\[
C_d = \begin{cases} 
\left( \frac{s^d - m + c_p^d}{(2^{m+6} \pi^2)}, & d = 2, \\
\left( \frac{e_s^d - m + c_p^d}{(2^{m+8} \pi^4)}, & d = 3,
\end{cases}
\]
and \( a_1, \ldots, a_d \) are orthonormal vectors. Moreover, the function \( \phi \) can be uniquely determined from the integral equation (1.4) for all \( x \in U \).

Since the scattered field \( u^s \) depends on the realization of the random potential \( V \), the scattering data given on the left hand side of (1.4) is random for any finite \( Q \). However, (1.4) indicates that the scattering data is statistically stable when \( Q \) approaches infinity, i.e., it is independent of the realization of the potential. The main result demonstrates that the function \( \phi \) can be uniquely determined by the amplitude of \( d \) scattered fields averaged over the frequency band, which are generated by a single realization of the random potential. Here, the \( d \) scattered fields are excited by the incident waves \( G_d a_1, \ldots, G_d a_d \), where \( a_1, \ldots, a_d \) are orthonormal vectors in \( \mathbb{R}^d \). The proof of the main result is quite technical. The analysis employs the Born approximation in high frequency, asymptotics of the Green tensor for the elastic wave equation, and microlocal analysis for the Fourier integral operators.

For readability, we briefly sketch the steps of the proof for the main result. As mentioned above, the scattering problem (1.1) and (1.3) can be equivalently formulated as a Lippmann–Schwinger integral equation which admits a unique solution. A careful analysis shows that the Born series of the Lippmann–Schwinger integral equation \( \sum_{j=0}^{\infty} u_j \) (cf. (4.1) and (4.2) for the definition of \( u_j \)) converges to the unique solution to the direct scattering problem when the angular frequency \( \omega \) is sufficiently large. Hence, the scattered field \( u^s \) can be written as
\[
u^s = u_1 + b, \quad b = \sum_{j=2}^{\infty} u_j,
\]
For the first item \( u_1 \), by employing the asymptotic expansions of the Green tensor and microlocal analysis for the Fourier integral operators via a multiple coordinate transformation, we show in Theorems 5.1 and 6.1 that
\[
\lim_{Q \to \infty} \frac{1}{Q - 1} \int_1^Q \omega^{m+2d-2d+6} \sum_{j=1}^d |u_1(x, \omega, a_j)|^2 \, d\omega = C_d \int_{\mathbb{R}^d} \frac{1}{|x - \zeta|^{2d-2}} \phi(\zeta) \, d\zeta. \tag{1.5}
\]
For the second item \( b \), by means of estimating the order with respect to the angular frequency \( \omega \), we deduce in Sections 5.2 and 6.2 that
\[
\lim_{Q \to \infty} \frac{1}{Q - 1} \int_1^Q \omega^{m+2d-2d+6} |b(x, \omega, a)|^2 \, d\omega = 0. \tag{1.6}
\]
Finally, the main result follows from (1.5)–(1.6) and the Cauchy–Schwartz inequality.

The paper is organized as follows. In Section 2, we briefly introduce the microlocally isotropic generalized Gaussian random fields and present some of their properties. Section 3 concerns the well-posedness of the direct scattering problem. We show that the direct problem is equivalent to a Lippmann–Schwinger integral equation which is uniquely solvable for a distributional potential. In Section 4, the Born series is studied for the Lippmann–Schwinger integral equation in the high frequency regime. Sections 5 and 6 are devoted to the inverse scattering problems in two and three dimensions, respectively. The paper is concluded with some general remarks and directions for future work in Section 7.
2. Generalized Gaussian random fields

In this section, we give a brief introduction to the microlocally isotropic generalized Gaussian random fields. Let $C_0^\infty(\mathbb{R}^d)$ be the set of smooth functions with compact support, and $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ be the space of test functions, which is $C_0^\infty(\mathbb{R}^d)$ equipped with a locally convex topology. The dual space $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^d)$ of $\mathcal{D}$ is called the space of distributions on $\mathbb{R}^d$ and is equipped with a weak-star topology (cf. [1]). Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space.

A function $\rho$ is said to be a generalized random field if, for each $\bar{\omega} \in \Omega$, the realization $\rho(\bar{\omega})$ belongs to $\mathcal{D}'(\mathbb{R}^d)$ and the mapping

$$\bar{\omega} \in \Omega \mapsto \langle \rho(\bar{\omega}), \psi \rangle \in \mathbb{R} \quad (2.1)$$

is a random variable for all $\psi \in \mathcal{D}$, where $\langle \cdot, \cdot \rangle$ denotes the dual product between $\mathcal{D}'$ and $\mathcal{D}$. The distributional derivative of $\rho \in \mathcal{D}'$ is defined by

$$\langle \partial_{x_j} \rho, \psi \rangle = -\langle \rho, \partial_{x_j} \psi \rangle \quad \forall \psi \in \mathcal{D}, \quad j = 1, \ldots, d.$$

A generalized random field is said to be Gaussian if (2.1) defines a Gaussian random variable for all $\psi \in \mathcal{D}$.

For a generalized random field $\rho \in \mathcal{D}'$, we can define its expectation $\mathbb{E}\rho \in \mathcal{D}'$ and covariance operator $Q_\rho : \mathcal{D} \to \mathcal{D}'$ as follows:

$$\langle \mathbb{E}\rho, \psi \rangle := \mathbb{E}(\rho, \psi) \quad \forall \psi \in \mathcal{D},$$

$$\langle Q_\rho \psi_1, \psi_2 \rangle := \text{Cov}(\langle \rho, \psi_1 \rangle, \langle \rho, \psi_2 \rangle) = \mathbb{E}[(\langle \rho, \psi_1 \rangle - \mathbb{E}(\rho, \psi_1))(\langle \rho, \psi_2 \rangle - \mathbb{E}(\rho, \psi_2))] \quad \forall \psi_1, \psi_2 \in \mathcal{D}.$$

It follows from the continuity of $Q_\rho$ and the Schwartz kernel theorem that there exists a unique kernel function $K_\rho(x, y)$ satisfying

$$\langle Q_\rho \psi_1, \psi_2 \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_\rho(x, y) \psi_1(x) \psi_2(y) dx dy \quad \forall \psi_1, \psi_2 \in \mathcal{D}.$$

The following definition can be found in [19] on the microlocally isotropic generalized Gaussian random fields.

**Definition 2.1.** A generalized Gaussian random field $\rho$ on $\mathbb{R}^d$ is called microlocally isotropic of order $-m$ in $D$ with $m \geq 0$ if the realizations of $\rho$ are almost surely supported in $D$ and its covariance operator $Q_\rho$ is a classical pseudo-differential operator having an isotropic principal symbol $\phi(x)|\xi|^{-m}$ with $\phi \in C_0^\infty(\mathbb{R}^d)$, $\text{supp} \phi \subset D$ and $\phi \geq 0$.

Without loss of generality, we choose the bounded domain $D$ which not only contains the support of $\rho$ almost surely but also has a locally Lipschitz boundary.

To have a better understanding of microlocally isotropic Gaussian random fields, we give an example by introducing the centered fractional Gaussian fields (cf. [25, 26]) defined by

$$f_m(x) := (-\Delta)^{-\frac{m}{2}} \hat{W}(x), \quad x \in \mathbb{R}^d, \quad (2.2)$$

where $(-\Delta)^{-\frac{m}{2}}$ is the fractional Laplacian and $\hat{W} \in \mathcal{D}'$ denotes the white noise. It is shown in [25] that the kernel $K_{f_m}(x, y)$ of $f_m$ is isotropic since its value depends only on the distance between $x$ and $y$, and $f_m$ is a microlocally isotropic Gaussian random field of order $-m$ and satisfies Definition 2.1 with $\phi \equiv 1$. In particular, if $m \in (d, d + 2)$, the fractional Gaussian field $f_m$ defined above is a translation of a classical fractional Brownian motion. More precisely,

$$\tilde{f}_m(x) := \langle f_m, \delta_x - \delta_0 \rangle, \quad x \in \mathbb{R}^d$$

has the same distribution as the classical fractional Brownian motion with Hurst parameter $H = \frac{m-d}{2} \in (0, 1)$ up to a multiplicative constant.

The regularity and kernel functions can be obtained for the microlocally isotropic Gaussian random fields by using the relationship between them and the fractional Gaussian fields defined in (2.2). It is clear to note that the fractional Gaussian field $f_m$ defined by (2.2) has the same regularity as the
Lemma 2.3. Let $\rho$ be a microlocally isotropic Gaussian random field of order $-m$ in $D$ with $m \in [0, d + 2)$.

(i) If $m \in (d, d + 2)$, then $\rho \in C^{0, \alpha}(D)$ almost surely for all $\alpha \in (0, \frac{m-d}{2})$.

(ii) If $m \in [0, d]$, then $\rho \in W^{\frac{m-d}{2} - \epsilon, p}(D)$ almost surely for any $\epsilon > 0$ and $p \in (1, \infty)$.

Moreover, the kernels for both $\rho$ in Definition 2.1 and $f_m$ defined in (2.2) are isotropic and have the same order. The following result gives the explicit expressions of the kernels for $f_m$. The proof can be found in [26].

Lemma 2.2. Let $f_m$ be a fractional Gaussian field defined by (2.2). Denote $H = \frac{m-d}{2}$. The kernel function $K_{f_m}$ of $f_m$ has the following form:

(i) If $m \in (0, \infty)$ and $H$ is not a nonnegative integer, then

$$K_{f_m}(x, y) = C_1(m, d)|x - y|^{2H},$$

where $C_1(m, d) = 2^{-m}\pi^{-\frac{d}{2}}\Gamma\left(\frac{d-m}{2}\right)/\Gamma\left(\frac{m}{2}\right)$ with $\Gamma(\cdot)$ being the Gamma function.

(ii) If $m \in (0, \infty)$ and $H$ is a nonnegative integer, then

$$K_{f_m}(x, y) = C_2(m, d)|x - y|^{2H}\ln|x - y|,$$

where $C_2(m, d) = (-1)^{H+1}2^{m+1}\pi^{-\frac{d}{2}}(H!\Gamma\left(\frac{m}{2}\right))$.

(iii) If $m = 0$, then

$$K_{f_m}(x, y) = \delta(x - y),$$

where $\delta(\cdot)$ is the Dirac delta function centered at 0.

We conclude this section by giving the kernel function of a microlocally isotropic Gaussian random field in Definition 2.1, which has the form

$$K_{\rho}(x, y) = \phi(x)K_{f_m}(x, y) + h(x, y),$$

where $\phi K_{f_m}$ is the leading term with strength $\phi$ and $h$ is a smooth residual (cf. [19]).

3. The Direct Scattering Problem

According to Lemma 2.2, if $m \in (d, d + 2)$, the random potential $V$ is a Hölder continuous function almost surely and has enough regularity such that the scattering problem (1.1) and (1.3) is well-posed in the traditional sense (cf. [8]). However, if $m \in [0, d]$, then the random potential $V(\cdot, \omega) \in W^{\frac{m-d}{2} - \epsilon, p}(D)$ is a distribution, and the elastic wave equation (1.1) should be considered in the distribution sense instead. In this section, we study the well-posedness of the scattering problem (1.1) and (1.3) under Assumption 1 with $m \in [0, d]$ by considering the equivalent Lippmann–Schwinger integral equation.

In the sequel, we denote by $X := X^d = \{g = (g_1, \cdots, g_d)^\top : g_j \in X, \ \forall \ j = 1, \cdots, d\}$ the Cartesian product vector space of $X$, and use the notation $W^{r,p} := (W^{r,p}(\mathbb{R}^d))^d$ and $H^r := W^{r,2}$ for simplicity. The notation $a \lesssim b$ or $a \gtrsim b$ stands for $a \leq Cb$ or $a \geq Cb$, where $C$ is a positive constant whose value is not required but should be clear from the context.
3.1. The Lippmann–Schwinger integral equation. Based on the Green tensor $G_d$ given in (1.2) and given a source point $y \in \mathbb{R}^d$, the Lippmann–Schwinger integral equation takes the form

$$u(x, y) + \int_D G_d(x, z, \omega)V(z, \omega)u(z, y)dz = G_d(x, y, \omega)a, \quad x \in \mathbb{R}^d, \quad x \neq y. \quad (3.1)$$

For a fixed $y \in \mathbb{R}^d$, define two scattering operators $H_\omega$ and $K_\omega$ by

$$(H_\omega u)(x) := [H_\omega u(\cdot, y)](x) = \int_{\mathbb{R}^d} G_d(x, z, \omega)u(z, y)dz$$

and

$$(K_\omega u)(x) := [K_\omega u(\cdot, y)](x) = \int_{\mathbb{R}^d} G_d(x, z, \omega)V(z, \omega)u(z, y)dz,$$

which have the following properties.

**Lemma 3.1.** Let $p \in (\frac{d}{2}, 2)$ and $V$ satisfy Assumption 1 with $m \in (\frac{2d}{p} + d - 4, d]$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded set and $\mathcal{V} \subset \mathbb{R}^{d^2}$ be a bounded open set with a locally Lipschitz boundary.

(i) The operator $H_\omega : H_0^{-\beta}(\mathcal{O}) \to H_0^\beta(\mathcal{V})$ is bounded for any $\beta \in (0, 2 - \frac{d}{2})$.

(ii) The operator $H_\omega : W_0^{-\gamma,p}(\mathcal{O}) \to W_0^{\gamma,q}(\mathcal{V})$ is compact for any $\gamma \in (0, 2 - \frac{d}{p})$ and $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

(iii) The operator $K_\omega : W_0^{\gamma,q}(\mathcal{V}) \to W_0^{\gamma,q}(\mathcal{V})$ is compact for any $\gamma \in (\frac{d-m}{2}, 2 - \frac{d}{p})$ and $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** The result in (i) follows directly from Lemma 4.2 in [20]. For (ii), note that for $p \in (\frac{d}{2}, 2)$ and $\gamma \in (0, 2 - \frac{d}{p})$, there exists $\beta = 2 - \frac{d}{2}$ such that $\gamma < \beta$, $\frac{1}{2} - \frac{\gamma - \beta}{d} < \frac{1}{q}$ and the embeddings

$$W_0^{-\gamma,p}(\mathcal{O}) \hookrightarrow H_0^{-\beta}(\mathcal{O})$$

and

$$H_0^\beta(\mathcal{V}) \hookrightarrow W_0^{\gamma,q}(\mathcal{V})$$

are compact according to the Kondrachov compact embedding theorem (cf. [1]). Combining the result in (i), we show that $H_\omega$ is compact from $W_0^{-\gamma,p}(\mathcal{O})$ to $W_0^{\gamma,q}(\mathcal{V})$.

Next we show (iii). For the operator $K_\omega$, it follows from Lemma 2.2 that $V(\cdot, \omega) \in W^{m-\frac{d}{2} - \epsilon, p'}$ for any $\epsilon > 0$ and $p' > 1$. Then for any $p \in (\frac{d}{2}, 2)$ and $\gamma \in (\frac{d-m}{2}, 2 - \frac{d}{p})$, by the Kondrachov compact embedding theorem, there exists some $p' > 1$ satisfying $\frac{1}{p'} > \frac{m-\frac{d}{2} - \epsilon + \gamma}{d}$ with $p' = \frac{p}{2-p}$ such that

$$W_0^{m-\frac{d}{2} - \epsilon, p'}(D) \hookrightarrow W_0^{-\gamma,p}(D)$$

and $V(\cdot, \omega) \in W_0^{-\gamma,p}(D)$. Hence, for any $v \in W_0^{\gamma,q}(\mathcal{V})$, we obtain from [19, Lemma 2] that $V(\cdot, \omega)v \in W^{-\gamma,p}$ with

$$\|V(\cdot, \omega)v\|_{W^{-\gamma,p}} \lesssim \|V(\cdot, \omega)\|_{W^{-\gamma,p}}\|v\|_{W^{\gamma,q}},$$

where

$$\|v\|_{W^{\gamma,q}} := \left( \sum_{j=1}^{d} \|v_j\|_{W^{\gamma,q}}^2 \right)^{\frac{1}{2}}, \quad v = (v_1, \cdots, v_d)^T.$$

As a result, we have $K_\omega v = H_\omega(V(\cdot, \omega)v) \in W_0^{\gamma,q}(\mathcal{V})$, which, together with (ii), implies that $K_\omega$ is compact from $W_0^{\gamma,q}(\mathcal{V})$ to $W_0^{\gamma,q}(\mathcal{V})$. \hfill \Box

The following result gives the well-posedness of the Lippmann–Schwinger integral equation (3.1).
Theorem 3.2. Let \( p \in (\frac{d}{2}, 2) \) and \( V \) satisfy Assumption 1 with \( m \in (\frac{2d}{p} + d - 4, d] \). Then the Lippmann–Schwinger integral equation (3.1) admits a unique solution \( u \in W^{\gamma,q}_{\text{loc}} \) almost surely with \( \gamma \in (\frac{d - m}{2}, 2 - \frac{d}{2}) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. Let \( \mathcal{V} \subset \mathbb{R}^d \) be any bounded open set with a locally Lipschitz boundary. By the definition of the operator \( K_{\omega} \), the Lippmann–Schwinger integral equation (3.1) can be written as

\[
[(I + K_{\omega})u(\cdot, y)](x) = G_d(x, y, \omega)a, \quad x \in \mathbb{R}^d,
\]

where \( y \in \mathbb{R}^d \) is fixed and \( I \) is the identity operator. It follows from Lemma 3.1 that the operator \( I + K_{\omega} : W^{\gamma,q}(\mathcal{V}) \to W^{\gamma,q}(\mathcal{V}) \) is Fredholm. Moreover, it is shown in [20, Lemma 4.1] that \( G_d(\cdot, y, \omega) \in (W^{1,p'}(\mathcal{V}))^{d \times d} \) with \( p' \in (1, 3 - \frac{d}{2}) \). Choosing \( p' = 3 - \frac{d}{2} - \epsilon \) with \( \epsilon > 0 \) being sufficiently small, we obtain from the Kondrachov compact embedding theorem that the embedding \( W^{1,p'}(\mathcal{V}) \hookrightarrow W^{\gamma,q}(\mathcal{V}) \) is compact, which indicates that the right-hand side of (3.2) satisfies \( (I + K_{\omega})u = 0 \).

By the Fredholm alternative theorem, the Lippmann–Schwinger integral equation (3.2) has a unique solution \( u \in W^{\gamma,q}(\mathcal{V}) \) if

\[
(I + K_{\omega})u = 0 \quad (3.3)
\]

has only the trivial solution \( u \equiv 0 \), which has been proved in [21, Theorem 3.3]. \( \square \)

3.2. Well-posedness. Now we show the existence and uniqueness of the solution of (1.1) in the distribution sense by utilizing the Lippmann–Schwinger integral equation.

Theorem 3.3. Let \( p \in (\frac{d}{2}, 2) \) and \( V \) satisfy Assumption 1 with \( m \in (\frac{2d}{p} + d - 4, d] \). The elastic wave equation (1.1) together with the radiation condition (1.3) is well-defined in the distribution sense, and admits a unique solution \( u \in W^{\gamma,q}_{\text{loc}} \) almost surely with \( \gamma \in (\frac{d - m}{2}, 2 - \frac{d}{2}) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. First we show the existence of the solution of (1.1). It suffices to show that the solution of the Lippmann–Schwinger integral equation (3.1) is also a solution of (1.1) in the distribution sense.

Suppose that \( u^* \in W^{\gamma,q}_{\text{loc}} \) is the solution of (3.1) and satisfies

\[
u^*(x, y) + \int_{\mathbb{R}^d} G_d(x, z, \omega)V(z, \omega)u^*(z, y)dz = G_d(x, y, \omega)a, \quad x \in \mathbb{R}^d.
\]

Since the Green tensor \( G_d \) is the fundamental solution for the operator \( \Delta^* + \omega^2 I \), we have

\[
(\Delta^* + \omega^2 I)G_d(\cdot, y, \omega) = -\delta_y I,
\]

where the Dirac delta function \( \delta(\cdot) \) is a distribution, i.e., \( \delta \in \mathcal{D}' \). Hence, we get for any \( \psi \in \mathcal{D} \) that

\[
\langle (\Delta^* + \omega^2 I)G_d(\cdot, y, \omega), \psi \rangle = -\langle \delta_y I, \psi \rangle = -\psi(y).
\]

For any \( \psi \in \mathcal{D} \), a simple calculation yields

\[
\langle \mu \Delta u^* + (\lambda + \mu) \nabla \nabla \cdot u^* + \omega^2 u^* - V(\cdot, \omega)u^*, \psi \rangle
\]

\[
- \int_{\mathbb{R}^d} (\Delta^* + \omega^2 I) G_d(\cdot, z, \omega)V(z, \omega)u^*(z, y)dz, \psi
\]

\[
+ \langle (\Delta^* + \omega^2 I) G_d(\cdot, y, \omega)a, \psi \rangle - \langle V(\cdot, \omega)u^*, \psi \rangle
\]

\[
= - \int_{\mathbb{R}^d} (V(z, \omega)u^*(z, y)) \nabla \langle (\Delta^* + \omega^2 I) G_d(\cdot, z, \omega), \psi \rangle dz
\]

\[
+ a^\top \langle (\Delta^* + \omega^2 I) G_d(\cdot, y, \omega), \psi \rangle - \langle V(\cdot, \omega)u^*, \psi \rangle
\]

\[
= - \int_{\mathbb{R}^d} (V(z, \omega)u^*(z, y)) \psi(z)dz - a^\top \psi(y) - \langle V(\cdot, \omega)u^*, \psi \rangle
\]

\[
= - \langle \delta_y a, \psi \rangle.
\]
which implies that \( \mathbf{u}^* \in W^{\gamma,q}_{\text{loc}} \) is also a solution of (1.1) and shows the existence of the solution of (1.1) according to Theorem 3.2.

The uniqueness of the solution of (1.1) is obtained by using the same procedure as that of the Lippmann–Schwinger equation. It requires to show that if \( a = 0 \), then any solution \( \mathbf{u} \) of the homogeneous equation (1.1) in the distribution sense is also a solution of (3.3) with \( a = 0 \), i.e., \( \mathbf{u} \equiv 0 \). In fact, let \( \mathbf{u} \) be any solution of (1.1) with \( a = 0 \). Then \( \mathbf{u} \) satisfies

\[
\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = V(\cdot, \omega) \mathbf{u}
\]

in the distribution sense, where \( V(\cdot, \omega) \in W^{m-d,-\gamma,p'}_0(D) \leftrightarrow W^{-\gamma,\tilde{p}}_0(D) \) for some \( p' > 1 \) and \( \tilde{p} = \frac{p}{2-p} \), \( \mathbf{u} \in W^{\gamma,q}_{\text{loc}} \) and \( V(\cdot, \omega) \mathbf{u} \in W^{-\gamma,p} \) according to the proof of Lemma 3.1. Let \( B_r \) be an open ball with radius \( r \) large enough such that \( D \subset B_r \). It follows from the proof of Theorem 3.2 that \( 1_{B_r} G_d(\cdot, y, \omega) \in (W^{\gamma,q})^{d \times d} \), where \( 1_{B_r} \) denotes the characteristic function. Hence, we get

\[
\int_{B_r} G_d(x, z, \omega) \left[ \Delta^* \mathbf{u}(z) + \omega^2 \mathbf{u}(z) \right] dz = \int_{B_r} G_d(x, z, \omega) V(z, \omega) \mathbf{u}(z) dz. \tag{3.4}
\]

Denote by \( T \) the operator that maps \( \mathbf{u} \) to the left-hand side of (3.4). For \( \psi \in \mathcal{D} \), by the similar arguments as those in the proof of [20, Lemma 4.3], we obtain

\[
(T\psi)(x) = -\psi(x) + \int_{\partial B_r} [G_d(x, z, \omega) P \psi(z) - PG_d(x, z, \omega) \psi(z)] ds(z),
\]

where \( P \) is the generalized stress vector on \( \partial B_r \), defined by \( P\psi := \mu \frac{\partial \psi}{\partial \nu} + (\lambda + \mu)(\nabla \cdot \psi) \nu \) with \( \nu \) being the unit outward normal vector on the boundary \( \partial B_r \). Since \( \mathbf{u} \) can be approximated by smooth functions, we have

\[
-\mathbf{u}(x) + \int_{\partial B_r} \left[ G_d(x, z, \omega) P \mathbf{u}(z) - PG_d(x, z, \omega) \mathbf{u}(z) \right] ds(z) = \int_{B_r} G_d(x, z, \omega) V(z, \omega) \mathbf{u}(z) dz.
\]

Letting \( r \to \infty \) and using the radiation condition, we get

\[
\mathbf{u}(x) = -\int_{\mathbb{R}^d} G_d(x, z) V(z, \omega) \mathbf{u}(z) dz,
\]

which indicates that \( \mathbf{u} \) is also a solution of the Lippmann–Schwinger equation (3.1) with \( a = 0 \), and hence \( \mathbf{u} \equiv 0 \) according to Theorem 3.2. \( \square \)

4. The Born series

The results in the previous section indicate that the original scattering problem (1.1) and (1.3), which is interpreted in the distribution sense, is equivalent to the Lippmann–Schwinger integral equation (3.1). In the sequel, we may just focus on the Lippmann–Schwinger integral equation (3.1).

To get an explicit expression of the solution, we consider the Born sequence of the Lippmann–Schwinger integral equation

\[
\mathbf{u}_j(x, y) = [-K_\omega \mathbf{u}_{j-1}(\cdot, y)](x), \quad j \in \mathbb{N}, \tag{4.1}
\]

where the leading term is

\[
\mathbf{u}_0(x, y) = G_d(x, y, \omega) a. \tag{4.2}
\]

In this section, the goal is to prove that the Born series \( \sum_{j=0}^{\infty} \mathbf{u}_j \) converges to the solution \( \mathbf{u} \) for sufficiently large \( \omega \).
4.1. **Estimates of the scattering operators.** Before showing the convergence of the Born series, we first introduce a weighted space which is equipped with a weighted $L^p$-norm (cf. [22]). For any $\delta \in \mathbb{R}$, let

$$L^p_\delta(\mathbb{R}^d) := \{ f \in L^1_{\text{loc}}(\mathbb{R}^d) : \| f \|_{L^p_\delta} < \infty \},$$

which is denoted by $L^p_\delta$ for short and is equipped with the norm

$$\| f \|_{L^p_\delta} := \| (1 + | \cdot |^2)^{\frac{\delta}{2}} f \|_{L^p} = \left( \int_{\mathbb{R}^d} (1 + |x|^2)^{\frac{\delta p}{2}} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Define the space

$$H^{s,p}_\delta(\mathbb{R}^d) := \{ f \in \mathcal{S} : (I - \Delta)^{\frac{\delta}{2}} f \in L^p_\delta \},$$

which is denoted by $H^{s,p}_\delta$ for short if there is no ambiguity and is equipped with the norm

$$\| f \|_{H^{s,p}_\delta} = \| (I - \Delta)^{\frac{\delta}{2}} f \|_{L^p_\delta}.$$

Here $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$ which is the space of all rapidly decreasing functions. For simplicity, let $H^s_\delta := H^{2s}_\delta$. For any $s \in \mathbb{R}$ and $\delta \in [0, 1]$, it is easy to verify that

$$\| f \|_{H^s_\delta} = \| (I - \Delta)^{\frac{\delta}{2}} f \|_{L^2_\delta} = \| (1 + | \cdot |^2)^{\frac{\delta}{2}} (I - \Delta)^{\frac{\delta}{2}} f \|_{L^2}. \quad (4.3)$$

where we have used \cite[Theorem 13.5]{11} to obtain the inequality.

Based on these weighted norms, the operators $H_\omega$ and $K_\omega$ can be estimated as follows.

**Lemma 4.1.** For any $s \in (0, \frac{1}{2})$ and $\epsilon > 0$, the following estimates hold:

$$\| H_\omega \|_{\mathcal{L}(H^{-s}_\delta, H^s)} \lesssim \omega^{-1+2s}, \quad (4.4)$$

$$\| H_\omega \|_{\mathcal{L}(H^{-s}_\delta, L^\infty)} \lesssim \omega^{s+\epsilon+\frac{d}{2}-1}. \quad (4.5)$$

**Proof.** The Green tensor $G_d(x, y)$ satisfies

$$\mu \Delta G_d(x, y) + (\lambda + \mu) \nabla \nabla \cdot G_d(x, y) + \omega^2 G_d(x, y) = -\delta(x-y)I \quad \text{in } \mathbb{R}^d. \quad (4.6)$$

Taking the Fourier transform on both sides of (4.6) with respect to $x - y$ leads to

$$-\mu |\xi|^2 \hat{G}_d(\xi) - (\lambda + \mu) \xi \cdot \xi^T \hat{G}_d(\xi) + \omega^2 \hat{G}_d(\xi) = -I,$$

where $\xi = (\xi_1, \ldots, \xi_d)^T$. A simple calculation gives

$$\hat{G}_d(\xi) = \frac{c_n^2 c_p^2}{(|\xi|^2 - c_n^2 \omega^2)(|\xi|^2 - c_p^2 \omega^2)} A_d(\xi), \quad (4.7)$$

where the matrix

$$A_2 = \begin{bmatrix} \mu |\xi|^2 - \omega^2 + (\lambda + \mu) \xi_1^2 & -\lambda + \mu \xi_1 \xi_2 \\ -(\lambda + \mu) \xi_1 \xi_2 & \mu |\xi|^2 - \omega^2 + (\lambda + \mu) \xi_1^2 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} \mu |\xi|^2 - \omega^2 + (\lambda + \mu) (\xi_2^2 + \xi_3^2) & -(\lambda + \mu) \xi_1 \xi_2 & -\lambda + \mu \xi_1 \xi_3 \\ -(\lambda + \mu) \xi_1 \xi_2 & \mu |\xi|^2 - \omega^2 + (\lambda + \mu) (\xi_2^2 + \xi_3^2) & -(\lambda + \mu) \xi_2 \xi_3 \\ -(\lambda + \mu) \xi_1 \xi_3 & -(\lambda + \mu) \xi_2 \xi_3 & \mu |\xi|^2 - \omega^2 + (\lambda + \mu) (\xi_1^2 + \xi_2^2) \end{bmatrix}$$

Clearly, the matrix function $\hat{G}_d$ has the same form and hence the same singularity as $\hat{G}_2$. Therefore, we only give the proof of (4.4) for the case $d = 2$. The proof can be obtained similarly by following the same procedure for the case $d = 3$. 

Let \( f = (f_1, f_2)^T \in C_0^\infty \) and \( g = (g_1, g_2)^T \in C_0^\infty \). We have from the Parseval identity that

\[
\langle H_\omega f, g \rangle = \int_{\mathbb{R}^2} \tilde{H}_\omega f(\xi) \tilde{g}(\xi) d\xi = \int_{\mathbb{R}^2} \tilde{G}_2(\xi) \hat{f}(\xi) \hat{g}(\xi) d\xi
\]

\[
= \int_{\mathbb{R}^2} \left( [\tilde{G}_{2,11}(\xi) \hat{f}_1(\xi) + \tilde{G}_{2,12}(\xi) \hat{f}_2(\xi)] \hat{g}_1(\xi) + [\tilde{G}_{2,21}(\xi) \hat{f}_1(\xi) + \tilde{G}_{2,22}(\xi) \hat{f}_2(\xi)] \hat{g}_2(\xi) \right) d\xi,
\]

(4.8)

where \( \tilde{G}_{2,ij} \) denotes the \((i, j)\)-entry of \( \tilde{G}_2 \). Noting that each term in (4.8) has the same singularity at the points \( |\xi| = c_s \omega \) and \( |\xi| = c_p \omega \), we only need to estimate the terms

\[
\int_{\mathbb{R}^2} \tilde{G}_{2,11}(\xi) \hat{f}_1(\xi) \hat{g}_1(\xi) d\xi = c_s^2 c_p^2 \int_{\mathbb{R}^2} \frac{\mu |\xi|^2 - \omega^2 + (\lambda + \mu) \xi_2^2}{(|\xi|^2 - c_s^2 \omega^2)(|\xi|^2 - c_p^2 \omega^2)} \hat{f}_1(\xi) \hat{g}_1(\xi) d\xi,
\]

(4.9)

\[
\int_{\mathbb{R}^2} \tilde{G}_{2,12}(\xi) \hat{f}_2(\xi) \hat{g}_1(\xi) d\xi = c_s^2 c_p^2 \int_{\mathbb{R}^2} \frac{- (\lambda + \mu) \xi_1 \xi_2}{(|\xi|^2 - c_s^2 \omega^2)(|\xi|^2 - c_p^2 \omega^2)} \hat{f}_2(\xi) \hat{g}_1(\xi) d\xi,
\]

(4.10)

and the other two terms can be estimated similarly.

Define the Bessel potential operator \( J^s \) by

\[
J^s h(x) = \mathcal{F}^{-1}[(1 + |\xi|^2)^s \hat{h}(\xi)](x) \quad \forall s \in \mathbb{R}, \ h \in S,
\]

where \( \mathcal{F}^{-1} \) is the inverse Fourier transform. To deal with the singularity, we split the whole space \( \mathbb{R}^2 \) into three parts:

\[
\Omega_1 := \{ \xi \in \mathbb{R}^2 : |\xi| - c_s \omega < \varepsilon_1 \omega \},
\]

\[
\Omega_2 := \{ \xi \in \mathbb{R}^2 : |\xi| - c_s \omega > \varepsilon_1 \omega \ \text{and} \ |\xi| - c_p \omega < \varepsilon_2 \omega \},
\]

\[
\Omega_3 := \{ \xi \in \mathbb{R}^2 : |\xi| - c_s \omega > \varepsilon_1 \omega \ \text{and} \ |\xi| - c_p \omega > \varepsilon_2 \omega \},
\]

where \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) are two constants.

First we estimate (4.9). Let

\[
I_j := c_s^2 c_p^2 \int_{\Omega_j} \frac{\mu |\xi|^2 - \omega^2 + (\lambda + \mu) \xi_2^2}{(|\xi|^2 - c_s^2 \omega^2)(|\xi|^2 - c_p^2 \omega^2)} \hat{f}_1(\xi) \hat{g}_1(\xi) d\xi
\]

\[
= c_s^2 c_p^2 \int_{\Omega_j} \frac{\mu |\xi|^2 - \omega^2 + (\lambda + \mu) \xi_2^2}{(|\xi|^2 - c_s^2 \omega^2)(|\xi|^2 - c_p^2 \omega^2)} (1 + |\xi|^2)^s \mathcal{F}^{-s} f_1(\xi) \mathcal{F}^{-s} g_1(\xi) d\xi, \quad j = 1, 2, 3.
\]

For the term \( I_3 \), using the definition of \( \Omega_3 \) and noting

\[
|\lambda + \mu) \xi_2^2| = |(\lambda + 2\mu) \xi_2^2 - \mu \xi_2^2| \leq (\lambda + 2\mu) |\xi|^2 - \omega^2 + \mu |\xi|^2 - \omega^2 + 2\omega^2
\]

and

\[
\frac{\mu |\xi|^2 - \omega^2 + (\lambda + \mu) \xi_2^2}{(|\xi|^2 - c_s^2 \omega^2)(|\xi|^2 - c_p^2 \omega^2)} \leq \frac{1}{|\xi|^2 - c_s^2 \omega^2} + \frac{1}{|\xi|^2 - c_p^2 \omega^2} + \frac{\omega^2}{|\xi|^2 - c_s^2 \omega^2},
\]

we get

\[
|I_3| \lesssim \int_{\Omega_3} \left[ \frac{(1 + |\xi|^2)^s}{|\xi|^2 - c_s^2 \omega^2} + \frac{(1 + |\xi|^2)^s}{|\xi|^2 - c_p^2 \omega^2} + \frac{\omega^2 (1 + |\xi|^2)^s}{|\xi|^2 - c_s^2 \omega^2}|(1 + |\xi|^2)^s| \right] |\mathcal{F}^{-s} f_1(\xi)||\mathcal{F}^{-s} g_1(\xi)| d\xi
\]

\[
\lesssim \omega^{-2+2s} \int_{\Omega_3} |\mathcal{F}^{-s} f_1(\xi)||\mathcal{F}^{-s} g_1(\xi)| d\xi
\]

\[
\lesssim \omega^{-2+2s} \| f_1 \|_{H^{-s}} \| g_1 \|_{H^{-s}},
\]

where in the second step we have used the following estimates: if \( |\xi| < (c_p - \varepsilon_2) \omega \), then

\[
\frac{(1 + |\xi|^2)^s}{|\xi|^2 - c_s^2 \omega^2} \leq \frac{(1 + |\xi|^2)^s}{\varepsilon_2 \omega (|\xi| + c_p \omega)} \leq \frac{(1 + (c_p - \varepsilon_2) \omega)^s}{\varepsilon_2 c_p \omega^2} \lesssim \omega^{-2+2s};
\]
if $|\xi| > (c_p + \varepsilon_2)\omega$ in $\Omega_3$, then

$$\frac{(1 + |\xi|^2)^s}{|\xi|^2 - c_p^2\omega^2} \leq \frac{(1 + |\xi|^2)^s}{\varepsilon_2\omega(|\xi| + c_p\omega)} \leq \frac{(2|\xi|^2)^s}{\varepsilon_2\omega|\xi|^2} \leq \frac{1}{\omega|\xi|^{1-2s}} \leq \omega^{-2+2s}.$$ 

For the term $I_1$, we have

$$I_1 = c_p^2 \int_{\Omega_1} \frac{1}{|\xi|^2 - c_p^2\omega^2} (1 + |\xi|^2)^s \widehat{f}_1(\xi) \widehat{g}_1(\xi) \, d\xi$$

$$+ c_s^2 c_p^3 \int_{\Omega_1} (\frac{\lambda + \mu}{c_s^2\omega^2}) (1 + |\xi|^2)^s \widehat{f}_1(\xi) \widehat{g}_1(\xi) \, d\xi$$

$$= I_{11} + I_{12}.$$ 

For $\xi \in \Omega_1$, we can choose $\varepsilon_1$ small enough such that $|\xi| - c_p\omega \geq c\omega$ for some $c > 0$, and follow similarly the estimate of $I_2$ to get

$$|I_{11}| \lesssim \omega^{-2+2s} \|f_1\|_{H^{-s}} \|g_1\|_{H^{-s}}.$$ 

To estimate $I_{12}$, we make the following change of variables:

$$\xi^* = \xi + \frac{2\varepsilon_1\omega - |\xi|}{|\xi|} \tilde{\xi} = 2c_s\omega \tilde{\xi} - \xi,$$

where $\tilde{\xi} := \xi/|\xi|$. It can be easily verified that the change of variables maps the domain $\Omega_{11} := \{\xi \in \mathbb{R}^2 : c_s\omega - \varepsilon_1\omega < |\xi| < c_s\omega\}$ to the domain $\Omega_{12} := \{\xi \in \mathbb{R}^2 : c_s\omega < |\xi| < c_s\omega + \varepsilon_1\omega\}$, and the Jacobian for the change of variables is

$$J(\xi) = \frac{2c_s\omega}{|\xi|} - 1.$$ 

Using the fact $\Omega_1 = \Omega_{11} \cup \Omega_{12} \cup \{\xi \in \mathbb{R}^2 : |\xi| = c_s\omega\}$ with $\{\xi \in \mathbb{R}^2 : |\xi| = c_s\omega\}$ being a set of zero measure, we obtain

$$I_{12} = c_s^2 c_p^2 \int_{\Omega_{11} \cup \Omega_{12}} \frac{(\lambda + \mu)\xi_2^2}{(|\xi|^2 - c_s^2\omega^2)(|\xi|^2 - c_p^2\omega^2)} (1 + |\xi|^2)^s \widehat{f}_1(\xi) \widehat{g}_1(\xi) \, d\xi$$

$$+ c_s^2 c_p^3 \int_{\Omega_{12}} (\frac{\lambda + \mu}{c_s^2\omega^2}) (1 + |\xi|^2)^s \widehat{f}_1(\xi) \widehat{g}_1(\xi) \, d\xi$$

$$= c_s^2 c_p^2 \int_{\Omega_{12}} m_1(\xi, \omega) (1 + |\xi|^2)^s \widehat{f}_1(\xi) \widehat{g}_1(\xi) \, d\xi$$

$$+ c_s^2 c_p^3 \int_{\Omega_{12}} m_2(\xi, \omega) [1 + |\xi|^2]^s \widehat{f}_1(\xi) \widehat{g}_1(\xi) - (1 + |\xi|^2)^s \widehat{f}_1(\xi) \widehat{g}_1(\xi)] J(\xi) \, d\xi$$

$$=: I_{13} + I_{14},$$

where

$$m_1(\xi, \omega) = \frac{(\lambda + \mu)\xi_2^2}{(|\xi|^2 - c_s^2\omega^2)(|\xi|^2 - c_p^2\omega^2)} + \frac{(\lambda + \mu)\xi_2^2}{(|\xi|^2 - c_s^2\omega^2)(|\xi|^2 - c_s^2\omega^2)} J(\xi),$$

$$m_2(\xi, \omega) = \frac{(\lambda + \mu)\xi_2^2}{(|\xi|^2 - c_s^2\omega^2)(|\xi|^2 - c_s^2\omega^2)}.$$

For $\xi \in \Omega_{12}$, it is not difficult to show that $\xi^* \in \Omega_{11}$ with $|\xi^*|^2 = 4c_s^2\omega^2 + |\xi|^2 - 4c_s\omega|\xi|$. Then there exists a constant $C > 0$ such that

$$\frac{(\lambda + \mu)\xi_2^2}{|\xi|^2 - c_p^2\omega^2} \leq C.$$
Moreover, we can choose \( g \) which leads to

\[
|I_{13}| \lesssim \omega^{-2} \|f_1\|_{H^{-s}} \|g_1\|_{H^{-s}}.
\]

The item \( I_{14} \) can be decomposed as

\[
I_{14} = \frac{c^2_p}{4} \int_{\Omega_{12}} m_2(\xi, \omega)(1 + |\xi|^2)^s - (1 + |\xi|^2)^s J(\xi) d\xi
\]

which shows that

\[
|I_{15}| \lesssim \omega^{-2+2s} \|f_1\|_{H^{-s}} \|g_1\|_{H^{-s}}.
\]

To estimate \( I_{16} \) and \( I_{17} \), we employ the following characterization of \( W^{1,p}(\mathbb{R}^d) \) introduced in [15].

**Lemma 4.2.** For \( 1 < p \leq \infty \), the function \( u \in W^{1,p}(\mathbb{R}^d) \) if and only if there exist \( g \in L^p(\mathbb{R}^d) \) and \( C > 0 \) such that

\[
|u(x) - u(y)| \leq C|x - y|(g(x) + g(y)).
\]

Moreover, we can choose \( g = M(|\nabla u|) \), where \( M \) is defined by

\[
M(f)(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy
\]

and is called the Hardy–Littlewood maximal function of \( f \).
For \( f_1 \in C_0^\infty \), we have \( \tilde{J}^{-s}f_1 \in \mathcal{S} \subset H^1 \). An application of Lemma 4.2 gives
\[ \left| \tilde{J}^{-s}f_1(\xi^*) - \tilde{J}^{-s}f_1(\xi) \right| \lesssim \| \mathcal{F}((\nabla \tilde{J}^{-s}f_1)(\xi^*) + M((\nabla \tilde{J}^{-s}f_1)(\xi)) \| \|. \tag{4.11} \]
By [29, Theorem 2.1], we get
\[ \| M((\nabla \tilde{J}^{-s}f_1))\|_{L^2} \lesssim \| \nabla \tilde{J}^{-s}f_1\|_{L^2} \lesssim (I - \Delta)^{\frac{1}{2}} \tilde{J}^{-s}f_1\|_{L^2} \]
which may further lead to the estimate
\[ \| (I - \Delta)^{\frac{1}{2}} (1 + |\cdot|^2)^{-\frac{3}{4}} \hat{f}_1(\cdot)\|_{L^2} = \| f_1\|_{H_1^{-s}}, \tag{4.12} \]
where (4.3) is used in the last step. Combining (4.11) and (4.12) gives
\[ |I_{16}| \lesssim \omega^{-1+2s} \int_{\Omega_{12}} \| M((\nabla \tilde{J}^{-s}f_1)(\xi^*) + M((\nabla \tilde{J}^{-s}f_1)(\xi)) \| \tilde{J}^{-s}g_1(\xi) d\xi \]
and noting
\[ \| (\lambda + \mu)\xi_1\xi_2 \| \leq \frac{(\lambda + 2\mu)|\xi|^2 - \omega^2}{2} + \frac{\mu|\xi|^2 - \omega^2}{2} + \omega^2, \]
we have
\[ |II_3| \lesssim \omega^{-2+2s} \| f_2\|_{H^{-s}} \| g_1\|_{H^{-s}}. \]
As for the estimate of
\[ II_1 = -c_3^2 \int_{\Omega_{12}} \frac{\xi_2}{(|\xi|^2 - c_3^2\omega^2)(|\xi|^2 - c_3^2\omega^2)} \| \tilde{J}^{-s}f_2(\xi) \tilde{J}^{-s}g_1(\xi) d\xi, \]
it is similar to that of I_{12} and admits
\[ |II_1| \lesssim \omega^{-1+2s} \| f_2\|_{H^{-s}} \| g_1\|_{H^{-s}}, \]
which may further lead to the estimate
\[ |II_2| \lesssim \omega^{-1+2s} \| f_2\|_{H^{-s}} \| g_1\|_{H^{-s}}. \]
Combining the above estimates yields
\[ \left| \int_{\mathbb{R}^2} \mathcal{G}_{2,11}(\xi) \hat{f}_1(\xi) \hat{g}_1(\xi) d\xi \right| \lesssim \omega^{-1+2s} \| f_2\|_{H^{-s}} \| g_1\|_{H^{-s}}. \]
It follows from the estimates of (4.9)--(4.10) that (4.8) has the following estimate:

$$|\langle H_\omega f, g \rangle| \lesssim \omega^{-1+2s}\|f\|_{H^{-s}_1}\|g\|_{H^{-s}_1} \quad \forall f, g \in C_0^\infty.$$  

This result can be extended for any $f, g \in H^{-s}_1$ since $C_0^\infty$ is dense in $H^{-s}_1$. The density argument can be found in [22, Theorem 2.2]. It then completes the proof of (4.4).

To prove (4.5), let $f = (f_1, \ldots, f_d) \in C_0^\infty$. We have

$$\left( H_\omega f \right)(x) = \int_{\mathbb{R}^d} G_d(x, y)f(y)dy$$

$$= \int_{\mathbb{R}^d} (1 + |\xi|^2)\hat{f}(x, \xi)\hat{f}(\xi)dx$$

$$= \int_{\mathbb{R}^d} (1 + |\xi|^2)\hat{f}(x, \xi)\hat{G}_d(x, \xi)\hat{f}(\xi)dx$$

where $\hat{G}_d(x, \xi)$, different from $\hat{G}_d(\xi)$, denotes the Fourier transform of $G_d(x, y)$ obtained by taking the Fourier transform on both sides of (4.6) with respect to $y$ and satisfies

$$-\mu|\xi|^2\hat{G}_d(x, \xi) - (\lambda + \mu)\xi \cdot \xi \hat{G}_d(x, \xi) + \omega^2\hat{G}_d(x, \xi) = -e^{-ix\cdot\xi}I.$$  

Comparing the above equation with (4.7), we get $\hat{G}_{d,ij}(x, \xi) = e^{-i\xi\cdot\theta}\hat{G}_{d,ij}(\xi)$. It follows from the same arguments as those for the item (4.9) that

$$\left| \int_{\mathbb{R}^d} (1 + |\xi|^2)\hat{f}(x, \xi)\hat{G}_d(x, \xi)\hat{f}(\xi)dx \right|$$

$$= \int_{\mathbb{R}^d} G_d(\xi)(1 + |\xi|^2)^{\frac{s+\frac{4}{d}}{2}}\hat{G}_d(\xi)e^{-ix\cdot\xi}(1 + |\xi|^2)^{\frac{4-s}{2}}d\xi$$

$$\lesssim \frac{1}{\omega^{1-s+\frac{d}{2}}}\|f_1\|_{H^{-s}_1}$$

$$\lesssim \|f_1\|_{H^{-s}_1},$$

where we have used the fact that the function

$$g(\xi) := e^{-ix\cdot\xi}(1 + |\xi|^2)^{\frac{4-s}{2}}$$

satisfies $g \in H^1$ for any $x \in \mathbb{R}^d$. The estimates can be similarly obtained for the other three items in (4.13). Therefore we have

$$\|H_\omega f\|_{L^\infty} \lesssim \omega^{s+\epsilon+\frac{d}{2}-1}\|f\|_{H^{-s}_1},$$

which completes the proof of (4.5).

Based on the estimates for the operator $H_\omega$, the following results present the estimates for the operator $K_\omega$.

**Lemma 4.3.** Let $s \in (0, \frac{1}{2})$, $p \in (0, \frac{2d}{d+s})$, and $V$ satisfy Assumption 1 with $m \in (\frac{4d}{p} - d - 2s, d]$. Then the following estimates hold almost surely:

$$\|K_\omega\|_{L(H^{-s}_1)} \lesssim \omega^{1+2s-\theta},$$

$$\|K_\omega\|_{L(H^{-s}_1, L^\infty)} \lesssim \omega^{s+\epsilon+\frac{d}{2}-1-\theta}.$$
Proof. For any \( u \in H^s_\omega \), it holds \( K_\omega u = H_\omega(V(\cdot, \omega)u) \) with \( H_\omega \) being a bounded operator from \( H^{s-\delta}_1 \) to \( H^s_\omega \) according to Lemma 4.1.

We first claim that \( V(\cdot, \omega)u \in H^{s-\delta}_1 \) for any \( u \in H^s_\omega \). Note that \( V(\cdot, \omega) \in W_0^{-\gamma, \delta}(D) \) with \( \gamma \in \left( \frac{d-m}{2}, s + d(1 - \frac{2}{p}) \right) \) and \( \tilde{p} = \frac{p}{2-p} \) based on the same procedure as the proof of Lemma 3.1. For any \( u, v \in S \), define \( \langle V(\cdot, \omega)u, v \rangle := \langle V(\cdot, \omega), \tilde{u} \cdot v \rangle \) and a cutoff function \( \tilde{\vartheta} \in C_0^\infty \) whose support \( \tilde{D} \) has a locally Lipschitz boundary and \( \vartheta(x) = 1 \) if \( x \in D \subset \tilde{D} \). It is easy to verify that

\[
|\langle V(\cdot, \omega)u, v \rangle| = |\langle (I - \Delta)^{-\gamma} V(\cdot, \omega), (I - \Delta)^{\gamma} (\vartheta \tilde{u}) \cdot (\vartheta v) \rangle|
\]

It follows from the fractional Leibniz principle with \( \tilde{p}' \) satisfying \( \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1 \) and \( \tilde{q} \) satisfying \( \frac{1}{\tilde{q}} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \) that

\[
\| (I - \Delta)^{\gamma} (\vartheta \tilde{u}) \cdot (\vartheta v) \|_{L^{\tilde{q}'}(\tilde{D})} \leq \| \vartheta \tilde{u} \|_{L^{\tilde{p}'}(\tilde{D})} \| \vartheta v \|_{W^{\gamma, \delta}(\tilde{D})} + \| \vartheta \tilde{u} \|_{L^{\tilde{p}'}(\tilde{D})} \| \vartheta v \|_{W^{\gamma, \delta}(\tilde{D})},
\]

where \( \tilde{q} = \frac{2p}{3p-1} \). Then the condition \( \gamma \in \left( \frac{d-m}{2}, s + d(1 - \frac{2}{p}) \right) \) leads to \( \frac{1}{\tilde{q}} > \frac{1}{2} - \frac{m}{d} \), which implies \( H^s(\tilde{D}) \hookrightarrow W^{\gamma, \delta}(\tilde{D}) \). Hence

\[
|\langle V(\cdot, \omega)u, v \rangle| \leq \| V(\cdot, \omega) \|_{W^{-\gamma, \delta}(\tilde{D})} \| \vartheta \tilde{u} \|_{W^{\gamma, \delta}(\tilde{D})} \| \vartheta v \|_{W^{\gamma, \delta}(\tilde{D})} \leq \omega^{-\theta} \| \vartheta \tilde{u} \|_{W^{\gamma, \delta}(\tilde{D})} \| \vartheta v \|_{W^{\gamma, \delta}(\tilde{D})}.
\]

Using the facts that \( \| \vartheta \tilde{u} \|_{H^s(\tilde{D})} \leq \| u \|_{H^{s-\delta}_1} \leq \| u \|_{H^s_\omega} \) and that \( S \) is dense in \( H^s_\omega \) proved in [22, Theorem 2.2], we get almost surely that

\[
|\langle V(\cdot, \omega)u, v \rangle| \preceq \omega^{-\theta} \| u \|_{H^{s-\delta}_1} \| v \|_{H^{s-\delta}_1} \quad \forall \, u, v \in H^s_\omega,
\]

which completes the claim. Then the following two estimates hold almost surely:

\[
\| K_\omega u \|_{H^s_\omega} \leq \| H_\omega \|_{L(H^{s-\delta}_1, H^s_\omega)} \| V(\cdot, \omega)u \|_{H^{s-\delta}_1} \leq \omega^{-1+2s-\theta} \| u \|_{H^{s-\delta}_1}
\]

and

\[
\| K_\omega u \|_{L^\infty} \leq \| H_\omega \|_{L(H^{s-\delta}_1, L^\infty)} \| V(\cdot, \omega)u \|_{H^{s-\delta}_1} \leq \omega^{s+\epsilon + \frac{d}{2} - 1 - \theta} \| u \|_{H^{s-\delta}_1},
\]

which complete the proof. \( \square \)

4.2. Convergence of the Born series. Let assumptions in Lemma 4.3 hold and \( U \subset \mathbb{R}^d \backslash \overline{D} \) be a bounded measurement domain which has a locally Lipschitz boundary and a positive distance from \( D \). This section is to show the convergence of the Born series defined in (4.1).

It follows from (4.1) that

\[
[(I + K_\omega) \sum_{j=0}^N u_j(\cdot, y)](x) = u_0(x, y) + (-1)^N [K_\omega^{N+1} u_0(\cdot, y)](x). \tag{4.14}
\]

Note that

\[
[K_\omega u_0(\cdot, y)](x) = \int_D G_d(x, z, \omega)V(z, \omega)u_0(z, y)dz \quad \forall \, x, y \in U,
\]

where \( u_0(z, y) = G_d(z, y, \omega)a \) and \( G_d(z, y, \omega) \) is smooth for any \( z \in D \) and \( y \in U \). We begin with the estimate for \( u_0 \).

Lemma 4.4. Let \( U \subset \mathbb{R}^d \backslash \overline{D} \) be a boundary domain having a locally Lipschitz boundary and a positive distance to \( D \). For any fixed \( y \in U \) and \( s \in (0, 1) \), the following estimate holds:

\[
\| u_0(\cdot, y) \|_{H^{s-\delta}_1(D)} \preceq \omega^{\frac{d}{2} - \frac{3}{2} + s}.
\]
Proof. For any \( y \in U \), it is easy to check that
\[
\|u_0(\cdot, y)\|_{L^2(D)} = \|G_d(\cdot, y, \omega)a\|_{L^2(D)} \lesssim \omega^{\frac{4}{2} - \frac{3}{2}},
\]
\[
\|u_0(\cdot, y)\|_{H^1(D)} = \|G_d(\cdot, y, \omega)a\|_{H^1(D)} \lesssim \omega^{\frac{2}{2} - \frac{3}{2}}.
\]
Utilizing the interpolation inequality \([17]\), we get
\[
\|u_0(\cdot, y)\|_{H^{s-1}(D)} = \|(1 + |\cdot|^2)^{-\frac{s}{2}}(I - \Delta)^{\frac{s}{2}}u_0(\cdot, y)\|_{L^2(D)} \lesssim \|u_0(\cdot, y)\|_{H^s(D)}
\]
\[
\lesssim \|u_0(\cdot, y)\|_{L^2(D)}^{1-s}\|u_0(\cdot, y)\|_{H^1(D)}^s \lesssim \omega^{\frac{4}{2} - \frac{3}{2} + s},
\]
which completes the proof. \( \square \)

By Lemma 4.4, we have
\[
\|K_{\omega}^{N+1}u_0\|_{H^{s-1}(U)} \lesssim \omega^{(-1+2s-\theta)(N+1)}\|u_0(\cdot, y)\|_{H^{s-1}(D)} \lesssim \omega^{(-1+2s-\theta)(N+1)+\frac{4}{2} - \frac{3}{2} + s} \to 0
\]
as \( N \to \infty \). Combining the above estimate with (4.14) leads to
\[
(I + K_{\omega}) \sum_{j=0}^{\infty} u_j = u_0 \quad \text{in} \quad H^{s-1}(U).
\]

Note also that \( G_d(\cdot, y, \omega) \in (L^2_{\text{loc}} \cap W^{1,p'}_{\text{loc}})^{d \times d} \) for any \( p' \in (1, 3 - \frac{d}{2}) \). Choosing \( p' = 3 - \frac{d}{2} - \epsilon \) for sufficient small \( \epsilon > 0 \), we may follow the same proof as that of Theorem 3.2 and get \( W^{\gamma,q'}(U) \hookrightarrow H^s(U) \), which implies that \( u_0(\cdot, y) \in H^s(U) \hookrightarrow W^{\gamma,q}(U) \) and \( (I + K_{\omega})^{-1}u_0 = u \) in \( W^{\gamma,q}(U) \). Hence, the Born series converges to the unique solution \( u \) of (1.1) in \( W^{\gamma,q}(U) \) and
\[
u = \sum_{j=0}^{\infty} u_j. \tag{4.15}
\]

Moreover,
\[
\|u - \sum_{j=0}^{N} u_j\|_{L^\infty(U)} \lesssim \sum_{j=N+1}^{\infty} \|K\omega^j u_0\|_{L^\infty(U)}
\]
\[
\leq \sum_{j=N+1}^{\infty} \|K\omega\|_{L(H^s, L^\infty(U))} \|K\omega\|_{L(H^s, L^\infty(U))} \|u_0(\cdot, y)\|_{H^{s-1}(D)}
\]
\[
\lesssim \omega^{s+\frac{4}{2} - 1 - \theta - (-1+2s-\theta)N + \frac{3}{2} - \frac{3}{2} + s} \to 0 \quad \text{as} \quad N \to \infty, \tag{4.16}
\]
which implies that the equation (4.15) also holds in \( L^\infty(U) \).

5. THE INVERSE PROBLEM IN TWO DIMENSIONS

In this section, we study the inverse problem in two dimensions and reconstruct the strength \( \phi \) of the random potential \( V \).

We consider the case \( y = x \) and recall that the notations \( u^s(x, \omega, a) \) and \( u_j(x, \omega, a) \) stand for \( u^s(x, x, \omega, a) \) and \( u_j(x, x, \omega, a) \), respectively. Then we rewrite (4.15) in terms of the scattered field
\[
u^s(x, \omega, a) = u_1(x, \omega, a) + b(x, \omega, a),
\]
where
\[
b(x, \omega, a) = \sum_{j=2}^{\infty} u_j(x, \omega, a).
\]
5.1. The analysis of \( u_1 \). This subsection is devoted to the analysis of the leading term \( u_1 \). Explicitly, it is given by

\[
 u_1(x, \omega, a) = -\int_D V(z, \omega) G_2(x, z, \omega)^2 adz, \quad x \in U.
\] (5.1)

**Theorem 5.1.** Let assumptions in Theorem 3.3 hold and \( U \subset \mathbb{R}^2 \setminus \overline{D} \) be a boundary domain having a locally Lipschitz boundary and a positive distance to \( D \). Then for all \( x \in U \), it holds

\[
 \lim_{Q \to \infty} \frac{1}{Q} \int_Q \int_{|\omega|^{m+2+2\theta}} \sum_{j=1}^2 |u_1(x, \omega, a_j)|^2 d\omega = C_2 \int_{\mathbb{R}^2} \frac{1}{|x-\xi|^2} d\zeta \quad \text{a.s.,}
\]

where \( a_1 \) and \( a_2 \) are two orthonormal vectors in \( \mathbb{R}^2 \), and \( C_2 \) is the constant given in Theorem 1.1.

Before giving the proof of Theorem 5.1, we first introduce the truncation of the Green tensor \( G_2 \) and some a priori estimates. Let \( H_n^{(1)} \) be the Hankel function of the first kind with order \( n \), which has the following asymptotic expansion (cf. [3]):

\[
 H_n^{(1)}(c) = \sum_{j=0}^N b_j^{(n)} c^{-(j+\frac{1}{2})} e^{i(c-\frac{n\pi}{2})} + O(|c|^{-N-\frac{3}{2}}), \quad c \in \mathbb{C}, \ |c| \to \infty,
\] (5.2)

where \( b_0^{(n)} = \frac{1+i}{\sqrt{\pi}} \) and

\[
b_j^{(n)} = \frac{(1+i)^{j+1}}{\sqrt{\pi}n^j j!} \prod_{l=1}^j \left( 4n^2 - (2l-1)^2 \right), \quad j \geq 1.
\]

For the sufficiently large argument \( c = \kappa |z| \), define the truncated Hankel function

\[
 H_{n,N}^{(1)}(c) := \sum_{j=0}^N b_j^{(n)} c^{-(j+\frac{1}{2})} e^{i(c-\frac{n\pi}{2})},
\]

It follows from (5.2) that

\[
|H_n^{(1)}(\kappa |z|) - H_{n,N}^{(1)}(\kappa |z|)| \lesssim \kappa^{-N-\frac{3}{2}} |z|^{-N-\frac{3}{2}},
\]

\[
|\nabla_z [H_n^{(1)}(\kappa |z|) - H_{n,N}^{(1)}(\kappa |z|)]| \lesssim \kappa^{-N-\frac{3}{2}} |z|^{-N-\frac{3}{2}}.
\] (5.3) (5.4)

By (1.2), a straightforward calculation shows that the Green tensor \( G_2 \) can be rewritten as

\[
 G_2(x, y, \omega) = \left\{ \frac{i}{4\mu} H_0^{(1)}(\kappa_s |x-y|) - \frac{i}{4\omega^2} \frac{1}{|x-y|} \left[ \kappa_s H_1^{(1)}(\kappa_s |x-y|) - \kappa_p H_1^{(1)}(\kappa_p |x-y|) \right] \right\} \mathbf{I} + \frac{i}{4\omega^2} \frac{1}{|x-y|^2} \left[ \kappa_s^2 H_2^{(1)}(\kappa_s |x-y|) - \kappa_p^2 H_2^{(1)}(\kappa_p |x-y|) \right] (x-y)(x-y)^\top,
\]

where \( x-y = (x_1-y_1, x_2-y_2)^\top \). Denote by \( G_2^{(N)} \) the truncation of the Green tensor \( G_2 \). Explicitly,

\[
 G_2^{(N)}(x, y, \omega) := \left\{ \frac{i}{4\mu} H_{0,N}^{(1)}(\kappa_s |x-y|) - \frac{i}{4\omega^2} \frac{1}{|x-y|} \left[ \kappa_s H_{1,N}^{(1)}(\kappa_s |x-y|) - \kappa_p H_{1,N}^{(1)}(\kappa_p |x-y|) \right] \right\} \mathbf{I} + \frac{i}{4\omega^2} \frac{1}{|x-y|^2} \left[ \kappa_s^2 H_{2,N}^{(1)}(\kappa_s |x-y|) - \kappa_p^2 H_{2,N}^{(1)}(\kappa_p |x-y|) \right] (x-y)(x-y)^\top.
\]
Let $G_{2,ij}$ and $G_{2,ij}^{(2)}$ be the $(i,j)$-entry of $G_2$ and $G_2^{(2)}$, respectively. Using (5.2)–(5.4), we have

\[ |G_{2,ij}(x,y,\omega)| \lesssim \omega^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}, \quad |\nabla_x G_{2,ij}(x,y,\omega)| \lesssim \omega^{\frac{1}{2}}|x-y|^{-\frac{1}{2}}, \]
\[ |G_{2,ij}^{(2)}(x,y,\omega)| \lesssim \omega^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}, \quad |\nabla_x G_{2,ij}^{(2)}(x,y,\omega)| \lesssim \omega^{\frac{1}{2}}|x-y|^{-\frac{1}{2}}, \]
\[ |G_{2,ij}(x,y,\omega) - G_{2,ij}^{(2)}(x,y,\omega)| \lesssim \omega^{-\frac{1}{2}}|x-y|^{-\frac{3}{2}}, \quad |\nabla_x (G_{2,ij}(x,y,\omega) - G_{2,ij}^{(2)}(x,y,\omega))| \lesssim \omega^{-\frac{1}{2}}|x-y|^{-\frac{3}{2}}. \]  

(5.5)

Replacing $G_2$ by $G_2^{(2)}$ in (5.1), we define

\[ u^{(2)}_1(x,\omega, a) = - \int_D V(z,\omega)G_2^{(2)}(x,z,\omega)^2dz, \quad x \in U. \]  

(5.6)

For the difference $u_1 - u^{(2)}_1$, we have the following estimate.

**Lemma 5.2.** Under the assumptions in Theorem 5.1, it holds for $x \in U$ that

\[ |u_1(x,\omega, a) - u^{(2)}_1(x,\omega, a)| \leq C\omega^{-3-\theta} \text{ a.s.,} \]

where the constant $C$ depends on the distance between $U$ and $D$.

**Proof.** Using (5.1) and (5.6), for $x, y \in U$ and $z \in D$, we obtain

\[ |u_1(x,\omega, a) - u^{(2)}_1(x,\omega, a)| \leq |\int_D V(z,\omega)(G_2(x,z,\omega) - G^{(2)}_2(x,z,\omega))G_2(x,z,\omega)dz| \]
\[ + \left| \int_D V(z,\omega)G_2^{(2)}(x,z,\omega)(G_2(x,z,\omega) - G^{(2)}_2(x,z,\omega))dz \right| 
\[ =: J_1 + J_2. \]

For $J_1$, we have from (5.5) that

\[ J_1 \leq \|V(\cdot,\omega)\|_{H^0_0(D)}\|G_2(x,\cdot,\omega) - G^{(2)}_2(x,\cdot,\omega)\|_{H^1(D)} \]
\[ \leq \omega^{-\theta}\|\rho\|_{H^0_0(D)}\|\nabla(G_2(x,\cdot,\omega) - G^{(2)}_2(x,\cdot,\omega))\|_{L^2(D)^{2\times 2}}\|G_2(x,\cdot,\omega)a\|_{L^\infty(D)} \]
\[ + \|G_2(x,\cdot,\omega) - G^{(2)}_2(x,\cdot,\omega)\|_{L^2(D)^{2\times 2}}\left( \|G_2(x,\cdot,\omega)a\|_{L^\infty(D)} + \|\nabla G_2(x,\cdot,\omega)a\|_{L^\infty(D)} \right) \]
\[ \lesssim \omega^{-\theta}\left[ \omega^{-\frac{3}{2}}\omega^{-\frac{1}{2}} + \omega^{-\frac{1}{2}}(\omega^{-\frac{1}{2}} + \omega^{\frac{1}{2}}) \right] \left( \int_D |x-z|^{-7}dz \right)^{\frac{1}{2}} \sup_{z \in D} |x-z|^{-\frac{1}{2}} \]
\[ \lesssim \omega^{-3-\theta}, \]

where we use the fact that there is a positive distance between $U$ and $D$. Similarly, we can prove that $J_2 \lesssim \omega^{-3-\theta}$. Thus, the proof is completed. \(\square\)

Let $u^{(2)}_1 = (u^{(2)}_{1,1}, u^{(2)}_{1,2})^\top$, where

\[ u^{(2)}_{1,k} = - \sum_{i,j=1}^2 \int_D V(z,\omega)G^{(2)}_{2,ki}(x,z,\omega)G^{(2)}_{2,ij}(x,z,\omega)a_jdz, \quad k = 1, 2 \]

and $a_j$ is the component of the vector $a$. It follows from a straightforward calculation that

\[ \mathbb{E}(\bar{u}^{(2)}_1(x,\omega_1, a) \cdot \bar{u}^{(2)}_1(x,\omega_2, a)) = \omega^{-\theta} \omega^{-\theta} \sum_{k,i,j=1}^2 a_ja_j \times \]
\[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G^{(2)}_{2,ki}(x,z,\omega_1)G^{(2)}_{2,ij}(x,z,\omega_1)G^{(2)}_{2,ki}(x,z',\omega_2)G^{(2)}_{2,ij}(x,z',\omega_2) \mathbb{E}(\rho(z)\rho(z'))dzdz', \]  

(5.7)
where the entries $G_{2,k}^{(2)}$ in $G_2^{(2)}$ can be expressed by

$$G_{2,k}^{(2)}(x,z,\omega) = \frac{1}{4} \sum_{j=0}^{2} \left[ \frac{b_j^{(0)} c_j^{-j+\frac{1}{2}} \delta_{kl}}{\omega^{j+\frac{1}{2}} |x-z|^{j+\frac{1}{2}}} + \frac{i b_j^{(1)} c_s^{-j+\frac{1}{2}} \delta_{kl}}{\omega^{j+\frac{1}{2}} |x-z|^{j+\frac{1}{2}}} - \frac{b_j^{(2)} c_s^{-j+\frac{1}{2}} (x_k-z_k) (x_l-z_l)}{\omega^{j+\frac{1}{2}} |x-z|^{j+\frac{1}{2}}} \right] e^{i\rho |x-z|}$$

and $\delta_{kl}$ is the Kronecker delta function which equals to 1 when $k = l$ and vanishes when $k \neq l$.

Substituting the expression of $G_{2,k}^{(2)}$ into (5.7), we see that $\mathbb{E}(u_1^{(2)}(x,\omega_1,a) \cdot u_1^{(2)}(x,\omega_2,a))$ is a linear combination of the following type of integral

$$I_2(x,\omega_1,\omega_2) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{|c_1|\omega_1 |x-z| - c_2 \omega_2 |x-z'|} F_2(z,z',x) \mathbb{E}(\rho(z)\rho(z')) dz dz',$nabla$

where $c_1, c_2 \in \{c_s, c_{s+}, c_p, 2c_p\}$ and

$$F_2(z,z',x) := \frac{(x_1-z_1)^{d_1} (x_2-z_2)^{d_2} (x_1-z_1')^{d_1} (x_2-z_2')^{d_2}}{|x-z|^{d_1} |x-z'|^{d_2}}.$$

The estimate of $I_2$ is technical and is given in Appendix for both the two- and three-dimensional cases.

**Corollary 5.3.** For $\omega_1, \omega_2 \geq 1$, the following estimates hold uniformly for $x \in U$:

$$|\mathbb{E}(u_1^{(2)}(x,\omega_1,a) \cdot u_1^{(2)}(x,\omega_2,a))| \leq C_M (\omega_1 \omega_2)^{-1-\theta} (\omega_1 + \omega_2)^{-m} (1 + |\omega_1 - \omega_2|)^{-M}, \quad (5.8)$$

$$|\mathbb{E}(u_1^{(2)}(x,\omega_1,a) \cdot u_1^{(2)}(x,\omega_2,a))| \leq C_M (\omega_1 \omega_2)^{-1-\theta} (\omega_1 + \omega_2)^{-M} (1 + |\omega_1 - \omega_2|)^{-m}, \quad (5.9)$$

where $M \in \mathbb{N}$ is arbitrary and $C_M$ is a constant depending on $M$.

**Proof.** Since $\mathbb{E}(u_1^{(2)}(x,\omega_1,a) \cdot u_1^{(2)}(x,\omega_2,a))$ is a linear combination of $I_2$, where the coefficient of the highest order is $(\omega_1 \omega_2)^{-1-\theta}$, it follows from Lemma A.1 that the estimate (5.8) holds. A simple calculation shows that $\mathbb{E}(u_1^{(2)}(x,\omega_1,a) \cdot u_1^{(2)}(x,\omega_2,a))$ is a linear combination of the following type of integral

$$\tilde{I}_2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{|c_1|\omega_1 |x-z| + c_2 \omega_2 |x-z'|} F_2(z,z',x) \mathbb{E}(\rho(z)\rho(z')) dz dz',$nabla$

with the coefficient $(\omega_1 \omega_2)^{-1-\theta}$. Clearly, $\tilde{I}_2$ is analogous to $I_2$ except that $\omega_2$ in $I_2$ is replaced by $-\omega_2$ in $\tilde{I}_2$. Following the same proof as that for the estimate of $I_2$, we may show that

$$|\tilde{I}_2(x,\omega_1,\omega_2)| \leq C_M (\omega_1 + \omega_2)^{-M} (1 + |\omega_1 - \omega_2|)^{-m},$$

which implies that (5.9) holds and completes the proof. \hfill \Box

Now we are in the position to show the proof of Theorem 5.1.

**Proof.** Rewriting $u_1 = u_1^{(2)} + (u_1 - u_1^{(2)})$, we only need to show that

$$\lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \sum_{j=1}^{2} |u_1^{(2)}(x,\omega,a_j)|^2 d\omega = C_2 \int_{\mathbb{R}^2} \frac{1}{|x-\zeta|^2} \phi(\zeta) d\zeta, \quad (5.10)$$

$$\lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} |u_1(x,\omega,a) - u_1^{(2)}(x,\omega,a)|^2 d\omega = 0, \quad (5.11)$$

$$\lim_{Q \to \infty} \frac{2}{Q-1} \int_1^Q \omega^{m+2+2\theta} \text{Re} [u_1^{(2)}(x,\omega,a)(u_1(x,\omega,a) - u_1^{(2)}(x,\omega,a))] d\omega = 0. \quad (5.12)$$
For (5.10), it follows from a straightforward calculation that
\[
\mathbb{E}|\mathbf{u}_1^{(2)}(x, \omega, \mathbf{a})|^2 = \left| \frac{1}{4\sqrt{\pi}} \right|^4 \omega^{-2-2\theta} \sum_{j=1}^2 a_j a_j \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{E}[\rho(z)\rho(z')] \sum_{k,i=1}^2 \mathbb{E} 
\]
\[
\frac{2}{c_s^2} \left( \frac{\delta_{kj}}{|x-z|^2} - \frac{\delta_{ji}}{|x-z|^2} \right) e^{i\kappa a_i |x-z|} + \frac{2}{c_p^2} \left( \frac{\delta_{kj}}{|x-z|^2} - \frac{\delta_{ji}}{|x-z|^2} \right) e^{i\kappa a_i |x-z|} 
\]
\[
+ O(\omega^{-(m+3+2\theta)}),
\]
which, together with Lemma A.1, gives
\[
\mathbb{E}|\mathbf{u}_1^{(2)}(x, \omega, \mathbf{a})|^2 = T_2(x, \omega, \mathbf{a}) \omega^{-(m+2+2\theta)} + O(\omega^{-(m+3+2\theta)}). \tag{5.13}
\]

Here
\[
T_2(x, \omega, \mathbf{a}) = 2^{-6-m} \pi^{-2} c_s^{-6-m} \int_{\mathbb{R}^2} \frac{1}{|x-\xi|^2} - \sum_{i,j=1}^2 \frac{(x_i - \xi_i)(x_j - \xi_j)}{|x-\xi|^4} a_ia_j \phi(\xi) d\xi 
\]
\[
+ 2^{-6-m} \pi^{-2} c_p^{-6-m} \int_{\mathbb{R}^2} \sum_{i,j=1}^2 \frac{(x_i - \xi_i)(x_j - \xi_j)}{|x-\xi|^4} a_ia_j \phi(\xi) d\xi 
\]
with \(a_i\) being the \(i\)th component of the vector \(\mathbf{a}\). Let \(\mathbf{a}_1 = (a_{11}, a_{12})^T\) and \(\mathbf{a}_2 = (a_{21}, a_{22})^T\) be two orthonormal vectors, i.e., there exists some angle \(\alpha\) such that \(\mathbf{a}_1 = (\cos \alpha, \sin \alpha)^T\) and \(\mathbf{a}_2 = (-\sin \alpha, \cos \alpha)^T\). It then holds \(a_{11}^2 + a_{21}^2 = 1, a_{12}^2 + a_{22}^2 = 1\) and \(a_{11}a_{12} + a_{21}a_{22} = 0\), which lead to

\[
\sum_{j=1}^2 T_2(x, \omega, a_j) = C_2 \int_{\mathbb{R}^2} \frac{1}{|x-\xi|^2} \phi(\xi) d\xi.
\]

It follows from (5.13) and the above equation that we have
\[
\sum_{j=1}^2 \mathbb{E}|\mathbf{u}_1^{(2)}(x, \omega, a_j)|^2 = C_2 \int_{\mathbb{R}^2} \frac{1}{|x-\xi|^2} \phi(\xi) d\xi \omega^{-(m+2+2\theta)} + O(\omega^{-(m+3+2\theta)}),
\]
which gives
\[
\lim_{Q \to \infty} \frac{1}{Q-1} \int_{Q}^{Q} \omega^{m+2+2\theta} \sum_{j=1}^2 \mathbb{E}|\mathbf{u}_1^{(2)}(x, \omega, a_j)|^2 d\omega = C_2 \int_{\mathbb{R}^2} \frac{1}{|x-\xi|^2} \phi(\xi) d\xi.
\]

To prove (5.10), based on the above equation, it suffices to prove that
\[
\lim_{Q \to \infty} \frac{1}{Q-1} \int_{1}^{Q} Y(x, \omega, \mathbf{a}) d\omega = 0, \tag{5.14}
\]
where \(Y(x, \omega, \mathbf{a})\) is defined by
\[
Y(x, \omega, \mathbf{a}) = \omega^{m+2+2\theta} \left[ |\mathbf{u}_1^{(2)}(x, \omega, \mathbf{a})|^2 - \mathbb{E}|\mathbf{u}_1^{(2)}(x, \omega, \mathbf{a})|^2 \right]
\]
\[
\omega^{m+2+2\theta} \left\{ (\text{Re}u_1^{(2)}(x, \omega, a))^2 - \mathbb{E}[\text{Re}u_1^{(2)}(x, \omega, a)]^2 \right\} + \omega^{m+2+2\theta} \left\{ (\text{Im}u_1^{(2)}(x, \omega, a))^2 - \mathbb{E}[\text{Im}u_1^{(2)}(x, \omega, a)]^2 \right\}.
\]

Note that
\[
\mathbb{E}(Y(x, \omega_1, a)Y(x, \omega_2, a)) = F_1 + F_2 + F_3 + F_4,
\]
where
\[
F_1 = \omega_1^{m+2+2\theta} \omega_2^{m+2+2\theta} \mathbb{E} \left\{ (\text{Re}u_1^{(2)}(x, \omega_1, a))^2 - \mathbb{E}[\text{Re}u_1^{(2)}(x, \omega_1, a)]^2 \right\} \times \left[ (\text{Re}u_1^{(2)}(x, \omega_2, a))^2 - \mathbb{E}[\text{Re}u_1^{(2)}(x, \omega_2, a)]^2 \right],
\]
\[
F_2 = \omega_1^{m+2+2\theta} \omega_2^{m+2+2\theta} \mathbb{E} \left\{ (\text{Re}u_1^{(2)}(x, \omega_1, a))^2 - \mathbb{E}[\text{Re}u_1^{(2)}(x, \omega_1, a)]^2 \right\} \times \left[ (\text{Im}u_1^{(2)}(x, \omega_2, a))^2 - \mathbb{E}[\text{Im}u_1^{(2)}(x, \omega_2, a)]^2 \right],
\]
\[
F_3 = \omega_1^{m+2+2\theta} \omega_2^{m+2+2\theta} \mathbb{E} \left\{ (\text{Im}u_1^{(2)}(x, \omega_1, a))^2 - \mathbb{E}[\text{Im}u_1^{(2)}(x, \omega_1, a)]^2 \right\} \times \left[ (\text{Re}u_1^{(2)}(x, \omega_2, a))^2 - \mathbb{E}[\text{Re}u_1^{(2)}(x, \omega_2, a)]^2 \right],
\]
\[
F_4 = \omega_1^{m+2+2\theta} \omega_2^{m+2+2\theta} \mathbb{E} \left\{ (\text{Im}u_1^{(2)}(x, \omega_1, a))^2 - \mathbb{E}[\text{Im}u_1^{(2)}(x, \omega_1, a)]^2 \right\} \times \left[ (\text{Im}u_1^{(2)}(x, \omega_2, a))^2 - \mathbb{E}[\text{Im}u_1^{(2)}(x, \omega_2, a)]^2 \right].
\]

The expression of \(u_1^{(2)}\) given in (5.6) implies that both \(\text{Re}u_1^{(2)}\) and \(\text{Im}u_1^{(2)}\) are centered Gaussian random fields. Applying Lemma 2.3 in [20] and Corollary 5.3, we obtain
\[
F_1 = 2\omega_1^{m+2+2\theta} \omega_2^{m+2+2\theta} \left[ \mathbb{E}(\text{Re}u_1^{(2)}(x, \omega_1, a)\text{Re}u_1^{(2)}(x, \omega_2, a)) \right]^2
\]
\[
= \frac{1}{2} \omega_1^{m+2+2\theta} \omega_2^{m+2+2\theta} \left\{ \mathbb{E} \left[ \text{Re}(u_1^{(2)}(x, \omega_1, a)u_1^{(2)}(x, \omega_2, a)) + \text{Re}(u_1^{(2)}(x, \omega_1, a)u_1^{(2)}(x, \omega_2, a)) \right] \right\}^2
\]
\[
\leq \mathbb{E} \left[ \left( (\omega_1 \omega_2)^{1+\theta}(\omega_1 + \omega_2)^M (1 + |\omega_1 - \omega_2|)^m \right)^2 \right],
\]
\[
\leq \mathbb{E} \left[ \left( |\omega_1 - \omega_2|^m + (1 + |\omega_1 - \omega_2|)^{-M} \right)^2 \right].
\]

By the similar arguments, we can obtain the same estimate for \(F_2, F_3, \) and \(F_4.\) Thus, an application of Lemma 2.4 in [20] implies
\[
\lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q Y(x, \omega, a) d\omega = 0,
\]
which shows that (5.14) holds and therefore (5.10) holds.

For (5.11), by Lemma 5.2, we obtain from the fact \(m \leq d = 2\) that
\[
\frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} |u_1(x, \omega, a) - u_1^{(2)}(x, \omega, a)|^2 d\omega
\]
Using the H"older inequality gives

\[ Q \equiv \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m-4}d\omega = \frac{1}{m-3} \frac{Q^{m-3} - 1}{Q-1} \to 0 \quad \text{as } Q \to \infty. \]

Combining (5.10)–(5.11) and the Hölder inequality, we may easily verify (5.12) and complete the proof. \hfill \Box

### 5.2. The analysis of \( b \)

Recalling \( b = \sum_{j=2}^\infty u_j \), we have from (4.16) and Lemmas 4.3 and 4.4 that

\[ \|b\|_{L^\infty(U)} = \|u - u_0 - u_1\|_{L^\infty(U)} \lesssim \omega^{s+\epsilon-\theta-(1/2+\epsilon) - 1/2} = \omega^{1/4+4s+\epsilon-2\theta}. \]

Then for any sufficiently small \( \epsilon > 0 \), we may choose \( s \in (0, \frac{1}{2}) \) such that \( m - 2\theta + 8s < 1 \) and

\[ \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \|b(x, \omega, a_j)\|^2 d\omega \lesssim \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta + (\frac{1}{4} + 4s + \epsilon - 2\theta)} d\omega \lesssim \frac{Q^{m-2\theta + 8s + 2\epsilon} - 1}{Q-1} \to 0 \quad \text{as } Q \to \infty. \]

We mention that such an \( s \) exists since \( 1 + 2\theta - m > 0 \) for \( \theta > \frac{m-1}{2} \).

### 5.3. The proof of Theorem 1.1

Based on the analysis of \( u_1 \) and \( b \), we are now able to prove the main result: the strength \( \phi \) in the principal symbol of the covariance operator \( Q_V \) can be uniquely determined by the amplitude of two scattered fields averaged over the frequency band with probability one. Here, the two scattered fields are associated with the incident waves given by \( G_2(x, y) a_1 \) and \( G_2(x, y) a_2 \) for any two orthonormal vectors \( a_1 \) and \( a_2 \).

Recall that the scattered field \( u^s \) can be written as

\[ u^s(x, \omega, a_j) = u_1(x, \omega, a_j) + b(x, \omega, a_j), \quad j = 1, 2, \]

where \( u_1 \) and \( b \) satisfy for \( x \in U \) that

\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \sum_{j=1}^2 |u_1(x, \omega, a_j)|^2 d\omega = C_2 \int_{\mathbb{R}^2} \frac{1}{|x - \zeta|^2} \phi(\zeta) d\zeta, \]

\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} |b(x, \omega, a_j)|^2 d\omega = 0. \]

Using the Hölder inequality gives

\[ \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \text{Re} \left[ u_1(x, \omega, a_j) b(x, \omega, a_j) \right] d\omega \lesssim \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} |u_1(x, \omega, a_j)||b(x, \omega, a_j)| d\omega \lesssim \left[ \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} |u_1(x, \omega, a_j)|^2 \right]^\frac{1}{2} \left[ \frac{1}{Q-1} \int_1^Q \omega^{m+2} |b(x, \omega, a_j)|^2 \right]^\frac{1}{2} \to 0 \]

as \( Q \to \infty \). Hence, we obtain

\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \sum_{j=1}^d |u^s(x, \omega, a_j)|^2 d\omega = \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \sum_{j=1}^n |u_1(x, \omega, a_j) + b(x, \omega, a_j)|^2 d\omega \]

\[ = \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \sum_{j=1}^2 \left\{ |u_1(x, \omega, a_j)|^2 + |b(x, \omega, a_j)|^2 + 2\text{Re}[u_1(x, \omega, a_j) b(x, \omega, a_j)] \right\} d\omega \]

\[ \lesssim \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} \sum_{j=1}^d |u^s(x, \omega, a_j)|^2 d\omega. \]

Therefore, we conclude that

\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2+2\theta} |u^s(x, \omega, a_j)|^2 d\omega \to 0 \quad \text{as } Q \to \infty. \]
\[ C_2 \int_{\mathbb{R}^2} \frac{1}{|x - \zeta|^2} \phi(\zeta) d\zeta. \]

It follows from [20, Lemma 3.8] that the function \( \phi \) can be uniquely determined from the integral equation (1.4) for all \( x \in U \). The proof is completed.

6. The Inverse Problem in Three Dimensions

In this section, we consider the inverse problem in three dimensions and present some parallel results to the two-dimensional case. Let the scattered field

\[ u^s(x, \omega, a) = u_1(x, \omega, a) + b(x, \omega, a), \]

where

\[ b(x, \omega, a) = \sum_{j=2}^{\infty} u_j(x, \omega, a), \quad x \in U. \]

6.1. The Analysis for \( u_1 \). Consider the leading term of the scattered field

\[ u_1(x, \omega, a) = -\int_D V(z, \omega) G_3(x, z, \omega)^2 a z, \quad z \in U, \]

which satisfies the following estimate.

**Theorem 6.1.** Let assumptions in Theorem 3.3 hold and \( U \subset \mathbb{R}^3 \setminus \overline{D} \) be a boundary domain having a locally Lipschitz boundary and a positive distance to \( D \). Then for all \( x \in U \), it holds

\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2\beta} \sum_{j=1}^{3} |u_1(x, \omega, a_j)|^2 d\omega = C_3 \int_{\mathbb{R}^3} \frac{1}{|x - \zeta|^4} \phi(\zeta) d\zeta, \]

where \( a_1, a_2, a_3 \) are three orthonormal vectors in \( \mathbb{R}^3 \) and \( C_3 \) is the constant given in Theorem 1.1.

**Proof.** Since

\[ \omega^{m+2\beta} |u_1(x, \omega, a)|^2 = \omega^{m+2\beta} |\mathbb{E}[u_1(x, \omega, a)]|^2 + Y(x, \omega, a), \]

where

\[ Y(x, \omega, a) = \omega^{m+2\beta} \left( |\mathbb{E}[u_1(x, \omega, a)]|^2 - 1 |u_1(x, \omega, a)|^2 \right), \]

it suffices to show that

\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2\beta} \sum_{j=1}^{3} |\mathbb{E}[u_1(x, \omega, a_j)]|^2 d\omega = C_3 \int_{\mathbb{R}^3} \frac{1}{|x - \zeta|^4} \phi(\zeta) d\zeta, \] (6.1)

\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q Y(x, \omega, a) d\omega = 0. \] (6.2)

Since the proof of (6.2) is exactly the same as the two-dimensional case, we omit it here. To show (6.1), we first rewrite the Green tensor \( G_3 \) into the following form:

\[ G_3(x, y, \omega) = \frac{1}{\mu} \Phi_3(x, y, \kappa_s) I + \frac{1}{\omega^2} \nabla_x \nabla_x^T (\Phi_3(x, y, \kappa_s) - \Phi_3(x, y, \kappa_p)) \]

\[ = \frac{1}{4\pi} \left[ \frac{c_s^2}{|x - y|} I - \frac{c_p^2(x - y)(x - y)^T}{|x - y|^3} + \frac{i c_s}{\omega |x - y|^2} I - \frac{3 i c_p (x - y)(x - y)^T}{\omega |x - y|^4} \right] \]

\[ - \frac{1}{\omega^2} I + \frac{3 |x - y|(x - y)^T}{\omega |x - y|^5} e^{i c_p |x - y|} \]

\[ + \frac{1}{4\pi} \left[ \frac{c_s^2}{|x - y|^3} I - \frac{i c_p}{\omega |x - y|^2} I + \frac{3 i c_p (x - y)(x - y)^T}{\omega |x - y|^4} \right] \]
where
\[ \frac{1}{\omega^2|x-y|^2} I - \frac{3(x-y)(x-y)^T}{\omega^2|x-y|^5} \] \exp[ic\omega|x-y|]. \tag{6.3} \]
Substituting (6.3) into \( u_1 \), we get that \( \mathbb{E}|u_1(x,\omega,a)|^2 \) is a linear combination of the integral \( I_3(x,\omega,\omega) \) which is estimated in Appendix. It follows from Lemma A.1 that
\[ \mathbb{E}|u_1(x,\omega,a)|^2 = T_3(x,\omega,a)\omega^{-(m+2\theta)} + O(\omega^{-(m+2\theta+1)}), \tag{6.4} \]
where
\[ T_3(x,\omega,a) = 2^{-m-8} \pi^{-4} s_8^{-m} \int_{\mathbb{R}^3} \left[ \frac{1}{|x-\zeta|^4} - \sum_{i,j=1}^3 \frac{(x_i-\zeta_i)(x_j-\zeta_j) a_i a_j}{|x-\zeta|^6} \right] \phi(\zeta) d\zeta \]
\[ + 2^{-m-8} \pi^{-4} s_8^{-m} \int_{\mathbb{R}^3} \left[ \sum_{i,j=1}^3 \frac{(x_i-\zeta_i)(x_j-\zeta_j) a_i a_j}{|x-\zeta|^6} \right] \phi(\zeta) d\zeta, \]
where \( a_i \) is the \( i \)th component of the vector \( a \). Let \( a_1 = (a_{11}, a_{12}, a_{13})^T, a_2 = (a_{21}, a_{22}, a_{23})^T \) and \( a_3 = (a_{31}, a_{32}, a_{33})^T \) be three orthonormal vectors. It is easy to verify that
\[ a_{1i}^2 + a_{2i}^2 + a_{3i}^2 = 1, \quad i = 1, 2, 3 \]
and
\[ a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, \]
which lead to
\[ \sum_{j=1}^3 T_3(x,\omega,a_j) = C_3 \int_{\mathbb{R}^3} \frac{1}{|x-\zeta|^4} \phi(\zeta) d\zeta. \]
Thus we have from (6.4) that
\[ \sum_{j=1}^3 \mathbb{E}|u_1(x,\omega,a_j)|^2 = C_3 \int_{\mathbb{R}^3} \frac{1}{|x-\zeta|^4} \phi(\zeta) d\zeta \omega^{-(m+2\theta)} + O(\omega^{-(m+2\theta+1)}). \tag{6.5} \]
Using (6.5), we obtain
\[ \lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2\theta} \sum_{j=1}^3 \mathbb{E}|u_1(x,\omega,a_j)|^2 d\omega = C_3 \int_{\mathbb{R}^3} \frac{1}{|x-\zeta|^4} \phi(\zeta) d\zeta, \]
which shows that (6.1) holds. It follows from Lemma 3.8 in [20] again that the knowledge of integral \( \int_{\mathbb{R}^3} \frac{1}{|x-\zeta|^4} \phi(\zeta) d\zeta \) for all \( x \in U \) can uniquely determine the strength function \( \phi \), which completes the proof. \( \square \)

6.2. The analysis of \( b \). Recalling \( b = \sum_{j=2}^\infty u_j \), we have from (4.16) and Lemma 4.3 that
\[ \|b\|_{L^\infty(U)} = \|u - u_0 - u_1\|_{L^\infty(U)} \lesssim \omega^{s+\epsilon+\frac{1}{2}-\theta+(-1+2s-\theta)+s} = \omega^{-\frac{1}{2}+4s+\epsilon-2\theta}. \]
For any sufficiently small \( \epsilon > 0 \), we may choose \( s \in (0, \frac{1}{2}) \) such that \( m - 2\theta + 8s < 1 \) and
\[ \frac{1}{Q-1} \int_1^Q \omega^{m+2\theta} |b(x,\omega,a_j)|^2 d\omega \lesssim \frac{1}{Q-1} \int_1^Q \omega^{m+2\theta+(-\frac{1}{2}+4s+\epsilon-2\theta)^2} d\omega \lesssim \frac{Q^{-m-2\theta+8s+2\epsilon}}{Q-1} \to 0 \quad \text{as} \quad Q \to \infty. \]
In fact, such an \( s \) exists since \( 1 + 2\theta - m > 0 \) for \( \theta > \frac{m-1}{4} \).
We may repeat essentially the proof in Section 5 to show Theorem 1.1 for the three-dimensional case. The details are omitted for brevity.
We have studied the scattering problems for the time-harmonic elastic wave equation with a potential in two and three dimensions. The potential is assumed to be a microlocally isotropic generalized Gaussian random field whose covariance is a classical pseudo-differential operator. For such a distribution potential, we deduce the equivalence between the direct scattering problem and the Lippmann–Schwinger integral equation which is shown to have a unique solution. Employing the Born approximation in the high frequency regime and microlocal analysis for the Fourier integral operators, we establish the connection between the principal symbol of the covariance operator for the random potential and the amplitude of the scattered field generated by a single realization of the random potential. Based on the identity, we obtain the uniqueness for the recovery of the micro-correlation strength of the random potential.

In this work, the random potential is assumed to be given as

\[ V(x, \omega) = \rho(x) \omega^{-\theta}, \quad \theta \geq \frac{m-1}{2}. \]

Recalling \( m \in \left( \frac{2d}{p} + d - 4, d \right] \) and \( p \in \left( \frac{4}{3}, 2 \right) \) in Theorem 1.1, we observe that

\[
\begin{align*}
    m &\in \left\{ \left( \frac{4}{p} - 2, 2 \right], \quad d = 2, \\
    &\in \left( \frac{4}{p} - 1, 3 \right], \quad d = 3 \\
\end{align*}
\]

which imply that the largest possible interval of \( m \) is \((0, 2] \) in two dimensions and \((2, 3] \) in three dimensions. Hence it is possible that \( \theta = 0 \) for an appropriately chosen \( m \) in two dimensions; the potential function \( V \) can be independent of the frequency. However, \( \theta \) is always positive in three dimensions; the potential function \( V \) is a negative power function of the frequency. It is unclear whether the frequency-dependence assumption of \( V \) can be removed in three dimensions. To remove the assumption, it is necessary to carry on a more sophisticated analysis on the higher order terms in the Born series. For example, it is desirable to get the explicit dependence of \( u_2 \) on \( \omega \) for sufficiently large \( \omega \).

This paper discusses the point source excitation and uses the near-field data for the inverse problem. Along this line, we intend to study the scattering problems by using the plane wave illumination and the far-field pattern as the data. Another interesting problem is to investigate the case where both the source and potential are random. We will report the progress on these problems elsewhere in the future.

**Appendix A**

We consider the following integral which is used in Sections 5 and 6:

\[
I_d(x, \omega_1, \omega_2) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(c_1 \omega_1 |x-z| - c_2 \omega_2 |x-z'|)} F_d(z, z', x) \Phi(\rho(z) \rho(z')) dz dz',
\]

where \( c_1, c_2 \in \{ 2c_c, c_c + c_p, 2c_p \} \) and

\[
F_d(z, z', x) := \frac{(x - z_1)^{d_1} \cdots (x - z_d)^{d_d}(x_1 - z'_1)^{d_1} \cdots (x_1 - z'_d)^{d_d}}{|x - z|^{d_1} |x - z'|^{d_2}}.
\]

**Lemma A.1.** For \( \omega_1, \omega_2 \geq 1 \), the following estimate holds uniformly for \( x \in U \):

\[
|I_d(x, \omega_1, \omega_2)| \leq C_M(\omega_1 + \omega_2)^{-m}(1 + |\omega_1 - \omega_2|)^{-M}, \quad (A.1)
\]

where \( M \in \mathbb{N} \) is an arbitrary integer and the positive constant \( C_M \) depends on \( M \). Moreover, if \( \omega_1 = \omega_2 = \omega \), then the following identity holds:

\[
I_d(x, \omega, \omega) = R_d(x, \omega) \omega^{-m} + O(\omega^{-(m+1)}), \quad (A.2)
\]

where

\[
R_d(x, \omega) = \frac{2^m}{(c_1 + c_2)^m} \int_{\mathbb{R}^d} e^{i(c_1 - c_2) \omega |x-\zeta|} \frac{(x_1 - \zeta_1)^{d_1+1} \cdots (x_d - \zeta_d)^{d_d+1}}{|x - \zeta|^{d_1+d_2}} \Phi(\zeta) d\zeta.
\]
Proof. For $x \in U$, it holds

$$I_d(x, \omega_1, \omega_2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(c_1\omega_1|x-z| - c_2\omega_2|x-z'|)} F_d(z, z', x)\mathbb{E}(\rho(z)\rho(z'))dzdz',$$

where $B_1(z, z', x) = K_\rho(z, z')\vartheta(x)$ with $K_\rho$ being the kernel function of $\rho$ and $\vartheta(x) \in C_0^\infty$ such that $\vartheta|_U \equiv 1$ and supp$(\vartheta) \subset \mathbb{R}^d \setminus \overline{D}$. Since $\rho$ is an isotropic Gaussian random field of order $-m$, we can represent $K_\rho$ in terms of its symbol by

$$K_\rho(z, z') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(z, z')\cdot \xi} \sigma(z, \xi) d\xi,$$

where $\sigma \in S_{-m}^{-m}(\mathbb{R}^d \times \mathbb{R}^d)$ is the symbol of the covariance operator $Q_\rho$ with $S_{-m}^{-m}(\mathbb{R}^d \times \mathbb{R}^d)$ being the space of symbols of order $-m$ which is defined by

$$S_{-m}^{-m}(\mathbb{R}^d \times \mathbb{R}^d) := \left\{ (a(x, \xi), \mathbb{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d) : |\partial_\xi^\alpha \partial_x^\beta| \leq C_{\alpha, \beta}(1 + |\xi|)^{-m-|\alpha|} \right\}.$$

Here $\alpha, \beta$ are multiple indices with $|\alpha| := \sum_{j=1}^d \alpha_j$ for $\alpha = (\alpha_1, ..., \alpha_d)^\top$. Therefore

$$B_1(z, z', x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(z, z')\cdot \xi} \sigma_1(z, x, \xi) d\xi,$$

where $\sigma_1(z, x, \xi) = \sigma(z, \xi)\vartheta(x) \in S_{-m}^{-m}$ has the principal symbol $\sigma_1^p(z, x, \xi) = \phi(z)|\xi|^{-m}\vartheta(x)$. Moreover, $B_1$ is a conormal distribution in $\mathbb{R}^{3d}$ of the Hörmander type and is compactly supported in $D^d := D \times D \times \text{supp}(\vartheta)$, which has conormal singularity on the surface $S := \{(z, z', x) \in \mathbb{R}^{3d} : z - z' = 0\}$, and is invariant under the change of coordinates (cf. [17]).

To estimate the integral $I_d$, we apply the coordinate transformations $\tau, \eta$ and $\gamma$ which are introduced in [25].

Define the invertible transformation $\tau : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ by $\tau(z, z', x) = (g_h, h, x)$, where $g = (g_1, \cdots, g_d)$ and $h = (h_1, \cdots, h_d)$ with

$$g_1 = \frac{1}{2} |(x - z) - (x - z')|, \quad g_2 = \frac{1}{2} |x - z| \arcsin \left( \frac{z_1 - x_1}{|x - z|} \right) - |x - z'| \arcsin \left( \frac{z'_1 - x_1}{|x - z'|} \right),$$

$$h_1 = \frac{1}{2} |(x - z) + (x - z')|, \quad h_2 = \frac{1}{2} |x - z| \arcsin \left( \frac{z_1 - x_1}{|x - z|} \right) + |x - z'| \arcsin \left( \frac{z'_1 - x_1}{|x - z'|} \right)$$

if $d = 2$, and

$$g_1 = \frac{1}{2} |(x - z) - (x - z')|, \quad h_1 = \frac{1}{2} |(x - z) + (x - z')|,$$

$$g_2 = \frac{1}{2} |x - z| \arccos \left( \frac{z_3 - x_3}{|x - z|} \right) - |x - z'| \arccos \left( \frac{z'_3 - x_3}{|x - z'|} \right),$$

$$h_2 = \frac{1}{2} |x - z| \arccos \left( \frac{z_3 - x_3}{|x - z|} \right) + |x - z'| \arccos \left( \frac{z'_3 - x_3}{|x - z'|} \right),$$

$$g_3 = \frac{1}{2} |x - z| \arctan \left( \frac{z_2 - x_2}{z_1 - x_1} \right) - |x - z'| \arctan \left( \frac{z'_2 - x_2}{z'_1 - x_1} \right),$$

$$h_3 = \frac{1}{2} |x - z| \arctan \left( \frac{z_2 - x_2}{z_1 - x_1} \right) + |x - z'| \arctan \left( \frac{z'_2 - x_2}{z'_1 - x_1} \right)$$

if $d = 3$. We then get

$$I_d(x, \omega_1, \omega_2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(c_1\omega_1+c_2\omega_2)|x-z| + (c_1\omega_1-c_2\omega_2)|x-z'|}(c_1\omega_1+c_2\omega_2)|x-z| + (c_1\omega_1-c_2\omega_2)|x-z'|) F_d(z, z', x)B_1(z, z', x)dzdz'.$$
and the residual \( r \) here which completes the proof of (A.1).

The way to get the detailed expression of \( B_2 \) is exactly the same as the procedure used in [25], which is based on the transformations \( \eta \) defined by \( \eta(z, z', x) = (v, v, x) \) with \( v = z - z' \) and \( w = z + z' \), and \( \gamma := \eta \circ \tau^{-1} : (0, h, x) \mapsto (v, w, x) \). We decompose the coordinate transform \( \gamma \) into two parts \( \gamma = (\gamma_1, \gamma_2) \) where \( \gamma_1(0, h, x) = v \) is the \( \mathbb{R}^d \)-valued function and \( \gamma_2(0, h, x) = (w, x) \) is the \( \mathbb{R}^{2d} \)-valued function. The Jacobian \( \gamma' \) corresponding to the decomposition of the variables is given by

\[
\gamma' = \begin{bmatrix}
\gamma'_{11} & \gamma'_{12} \\
\gamma'_{21} & \gamma'_{22}
\end{bmatrix}
= \begin{bmatrix}
\partial_g \gamma_1 & \partial_{(h,x)} \gamma_1 \\
\partial_g \gamma_2 & \partial_{(h,x)} \gamma_2
\end{bmatrix}.
\]

Finally, we get

\[
B_2(g, h, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\imath \xi \cdot \sigma_2(h, x, \xi)} d\xi,
\]

where the principal symbol of \( \sigma_2 \) has the form

\[
\sigma_2^p(h, x, \xi) = \phi \left( \frac{\omega(0, h, x)}{2} \right) \delta(x) \left| (\gamma'_{11}(0, h, x))^{-1} \xi \right|^m |\det(\gamma'_{11}(0, h, x))|^{-1} L^r(0, h, x).
\] (A.3)

Here \( \alpha := \frac{h_2}{h_1}, \beta := \frac{h_3}{h_1}, \)

\[
w(0, h, x) = \begin{cases}
2h_1(\sin \alpha, \cos \alpha) + 2x, & d = 2, \\
2h_1(\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha) + 2x, & d = 3,
\end{cases}
\]

and the residual \( r_2 := \sigma_2 - \sigma_2^p \in S_{1,0}^{m-1} \). Note that \( B_2(g, h, x) = [F^{-1} \sigma_2(h, x, \cdot)](g) \). Hence,

\[
I_d(x, \omega_1, \omega_2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\imath ((c_1 \omega_1 + c_2 \omega_2) g + e_1 + (c_1 \omega_1 - c_2 \omega_2) h - e_1)} [F^{-1} \sigma_2(h, x, \cdot)](g) dgdh
\]

\[
= \int_{\mathbb{R}^d} e^{\imath (c_1 \omega_1 - c_2 \omega_2) h_1} \sigma_2(h, x, -(c_1 \omega_1 + c_2 \omega_2) e_1) dh
\]

\[
= \frac{1}{(i(c_1 \omega_1 - c_2 \omega_2))^{M}} \int_{\mathbb{R}^d} e^{\imath (c_1 \omega_1 - c_2 \omega_2) h_1} \partial_{h_1}^M \sigma_2(h, x, -(c_1 \omega_1 + c_2 \omega_2) e_1) dh,
\]

where \( |\partial_{h_1}^M \sigma_2(h, x, -(c_1 \omega_1 + c_2 \omega_2) e_1)| \leq C_M |c_1 \omega_1 + c_2 \omega_2|^{-m} \). Consequently,

\[
|I_d(x, \omega_1, \omega_2)| \leq C_M (1 + |\omega_1 - \omega_2|)^{-M} (\omega_1 + \omega_2)^{-m},
\]

which completes the proof of (A.1).

For (A.2), letting \( \omega_1 = \omega_2 = \omega \), we have from (A.3) that

\[
I_d(x, \omega, \omega) = \int_{\mathbb{R}^d} e^{\imath (c_1 - c_2) \omega h_1} \sigma_2(h, x, -(c_1 + c_2) \omega e_1) dh
\]

\[
= \int_{\mathbb{R}^d} e^{\imath (c_1 - c_2) \omega h_1} \sigma_2^p(h, x, -(c_1 + c_2) \omega e_1) dh + O(\omega^{-(m+1)})
\]
= \int_{\mathbb{R}^d} e^{i(c_1-c_2)\omega h_1 \phi \left( \frac{w(0, h, x)}{2} \right)} \theta(x) \left| (\gamma'_{11}(0, h, x))^{-1} (c_1 + c_2)\omega e_1 \right|^{-m} |\det(\gamma'_{11}(0, h, x))|^{-1} L^\tau(0, h, x) dh + O(\omega^{-(m+1)}).

If \( d = 2 \), by the expression of \( \gamma'_{11}(0, h, x) \), we have for any \( x \in U \) that

\[
I_2(x, \omega, \omega) = \frac{2^{m-2}}{(c_1 + c_2)^m \omega^m} \int_{\mathbb{R}^2} e^{i(c_1-c_2)\omega h_1 \phi \left( \frac{w(0, h, x)}{2} \right)} L^\tau(0, h, x) dh + O(\omega^{-(m+1)}),
\]

where

\[
L^\tau(0, h, x) = F_2(\tau^{-1}(0, h, x)) |\det((\tau^{-1})'(0, h, x))|.
\]

Here

\[
F_2(\tau^{-1}(0, h, x)) = \frac{(-h_1 \sin \alpha)^{d_{11}+d_{21}} (-h_1 \cos \alpha)^{d_{12}+d_{22}}}{h_1^{d_1+d_2}}
\]

and \(|\det((\tau^{-1})'(0, h, x))| = 4\) are proved in [25, Proposition 4.1]. To simplify the expression of \( I_2 \), we consider another coordinate transformation \( \vartheta : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
\vartheta(h) = \zeta := h_1(\sin \alpha, \cos \alpha) + x,
\]

which has the Jacobian

\[
\det(\vartheta') = \begin{vmatrix} \sin \alpha - \alpha \cos \alpha & \cos \alpha \\ \alpha \sin \alpha + \cos \alpha & -\sin \alpha \end{vmatrix} = -1.
\]

Then for any \( x \in U \), we have

\[
I_2(x, \omega, \omega) = \frac{2^m}{(c_1 + c_2)^m \omega^m} \int_{\mathbb{R}^2} e^{i(c_1-c_2)\omega h_1 \phi \left( \frac{w(0, h, x)}{2} \right)} \frac{(-h_1 \sin \alpha)^{d_{11}+d_{21}} (-h_1 \cos \alpha)^{d_{12}+d_{22}}}{h_1^{d_1+d_2}} dh + O(\omega^{-(m+1)})
\]

If \( d = 3 \), then for any \( x \in U \),

\[
I_3(x, \omega, \omega) = \frac{2^{m-3}}{(c_1 + c_2)^m \omega^m} \int_{\mathbb{R}^3} e^{i(c_1-c_2)\omega h_1 \phi \left( \frac{w(0, h, x)}{2} \right)} \frac{L^\tau(0, h, x)}{|\sin \alpha|} dh + O(\omega^{-(m+1)}),
\]

where

\[
L^\tau(0, h, x) = F_3(\tau^{-1}(0, h, x)) |\det((\tau^{-1})'(0, h, x))|.
\]

Here

\[
F_3(\tau^{-1}(0, h, x)) = \frac{(-h_1 \sin \alpha \cos \beta)^{d_{11}+d_{21}} (-h_1 \sin \alpha \sin \beta)^{d_{12}+d_{22}} (-h_1 \cos \alpha)^{d_{13}+d_{23}}}{h_1^{d_1+d_2}}
\]

and \(|\det((\tau^{-1})'(0, h, x))| = 8 \sin^2 \alpha\) are proved in [25, Theorem 4.5]. By defining the transformation \( \vartheta : \mathbb{R}^3 \to \mathbb{R}^3 \) with

\[
\vartheta(h) = \zeta := h_1(\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha) + x,
\]

whose Jacobian satisfies

\[
|\det(\vartheta^{-1})'| = \left| \det \left( \frac{\gamma'_{11}(0, h, x)}{2} \right) \right|^{-1} = \frac{1}{|\sin \alpha|},
\]

we finally get for any \( x \in U \) that

\[
I_3(x, \omega, \omega) = \frac{2^{m-3}}{(c_1 + c_2)^m \omega^m} \int_{\mathbb{R}^3} e^{i(c_1-c_2)\omega h_1 \phi \left( \frac{w(0, h, x)}{2} \right)} \frac{8 \sin^2 \alpha}{|\sin \alpha|} dh + O(\omega^{-(m+1)}).
\]
\[
\frac{(-h_1 \sin \alpha \cos \beta)^{d_1} + d_2}{h_1^{d_1 + d_2}} \frac{(-h_1 \sin \alpha \sin \beta)^{d_1 + d_2}}{-h_1 \cos \alpha} \frac{d_2}{d_1 + d_2} \frac{dh}{O(\omega^{-(m+1)})}
\]
\[
\left(\frac{2^n}{(c_1 + c_2)^{m+n}} \int_{\mathbb{R}^3} e^{i(c_1 - c_2)\omega|x - \zeta|} \phi(\zeta) \right) \frac{(x_1 - \zeta_1)^{d_1 + d_2}}{|x - \zeta|^{d_1 + d_2}} \frac{(x_2 - \zeta_2)^{d_1 + d_2}}{(x_3 - \zeta_3)^{d_1 + d_2}} \frac{d\zeta}{O(\omega^{-(m+1)})},
\]
which completes the proof. \qed

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