Continuous Emission of A Radiation Quantum

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Abstract. It is in accordance with such experiments as single photon self-interference that a photon, conveying one radiation energy quantum "\(\hbar \times \text{frequency}\)", is spatially extensive and stretches an electromagnetic wave train. A wave train, hence an energy quantum, can only be emitted (or absorbed) by its source (or absorber) gradually. In both two processes the wave and "particle" attributes of the radiation field are simultaneously prominent, where an overall satisfactory theory has been lacking; for the latter process no known theoretical description currently exists. This paper presents a first principles treatment, in a unified framework of the classical and quantum mechanics, of the latter process, the emission (similarly absorption) of a single radiation quantum based on the dynamics of the radiation-emitting source, a charged oscillator, which is itself extensive across the potential well in which it oscillates. During the emission of one single radiation quantum, the extensive charged oscillator undergoes a continuous radiation damping and is non-stationary. This process is in this work treated using a quasi stationary approach, whereby the classical equation of motion, which directly facilitates the correspondence principle for a particle oscillator, and the quantum wave equation are established for each sufficiently brief time interval. As an inevitable consequence of the division of the total time for emitting one single quantum, a fractional Planck constant \(\hbar\) is introduced. The solutions to the two simultaneous equations yield for the charged oscillator a continuously exponentially decaying Hamiltonian that is at the same time quantised with respect to the fractional-\(\hbar\) at any instant of time; and the radiation wave field emitted over time stretches a wave train of finite length. The total system of the source and radiation field maintains at any time (integer \(n\) times) one whole energy quantum, \((n\times)\ \hbar \times \text{frequency}\), in complete accordance with the standard notion of quantum mechanics and experiment.
1. Introduction
According to the classical electrodynamic theory (J C Maxwell, 1873), the electromagnetic radiation fields are manifestly waves. The theory satisfactorily accounts for such processes as diffraction, interference and superposition. According to quantum theory laid foundation to by M Planck in 1900, electromagnetic radiation fields consist at the scale \( h \) of energy quanta, and these are manifestly “particles” (M Planck, 1900; Einstein, 1905). The theory is successful especially for such processes where there presents an energy transfer between the radiation field emitter and an external absorber, in general by one energy quantum (or photon), \( h\omega \) or by integral \( n \) multiples of \( n h\omega \) at a time. A photon, as well as a quantum matter particle, is in the current interpretation of the quantum theory regarded as a statistical point particle. While the classical electrodynamics and the quantum electrodynamics (Dirac, 1927) have proven extremely successful where the wave and the particle attributes are separately prominent, the two essentially parallel theories present apparent clashes where the wave and “particle” attributes are simultaneously prominent[1, 2]. A classical example of this is the single photon self-interference in a double slit as demonstrated in experiments [2]. Here, the clash is a logical one: a photon \( (h\omega) \), being regarded as a statistical point particle, logically can not pass two slits at the same time. A realistic theoretical representation of the (self) interference would require the radiation quantum to be depicted as a coherent (radial) wave train.

A separate example is the intermediate process of emission (or absorption) of a radiation quantum \( h\omega \), or photon; this is closely relevant to the single photon self-interference and will be the central concern of this paper. Based on the classical electrodynamics as well as experiment, the radiation field is gradually and continuously emitted (or absorbed) by the source (or target) charge over a time duration \( \sim 10^{-8} \) s [1], and therefore stretches a coherent wave train of a definite phase, a necessary pre-condition for producing single photon self interference. A requisite quantisation of the field however is not facilitated in this theory. According to the quantum theory and experiment, the radiation energy transfer from its emitting charge to an external absorber is by one energy quantum (or less frequently by integral multiples of a quantum) at a time. The radiation energy quantum, when pictured as a statistical point particle and hence a non-divisible unit, on the other hand, can only be emitted (or absorbed) instantaneously (the observational finite elapsing time of a radiation decay instead is attributed entirely to the statistical transitions of many sources, such as atoms, see e.g. H G Kuhn [1]). Similarly here, there presents a pronounced clash between the two theories. The clash, as we have seen from the two examples, arises primarily from the current interpretation of the quantum theory rested on an over simplifying statistical point particle picture, on top of an otherwise rigorous mathematical framework of the quantum mechanics.

The wave and particle pictures may be readily reconciled with one another if we regard the radiation wave field (assuming an angular frequency \( \omega \)) as being at any one time distributed across a finite distance, whence a wave train of a finite (non-zero) length and yet having one fixed quantum of energy, \( h\omega \), as a result of wave amplitude quantisation. This description, in the case of matter wave, is entirely consistent with the corresponding mathematical quantum wave function solution \( (\Psi(x, t)) \) to, for example, the Schrödinger equation. Namely, \( \Psi(x, t) \) is extensive at any one time \( t \) and the integration of energy density, being proportional to \( |\Psi(x, t)|^2 \), across (the space interval enclosing) the wave train leads to one (or integral \( n \) times one) energy quantum. For the matter wave, an intrinsically extensive, internally electrodynamic (IED) scheme has been recently proposed by the author (see e.g. a recent review in [3a]), in terms of which the primary outstanding difficulties, including ones associated with wave-particle duality, attributable mainly to the statistical point particle picture in the current interpretation of the quantum theory, may be overcome. The IED description for matter wave can be in principle translated to one for an (existing) photon, although indirectly and for stationary state only in its current form, by imposing together the solutions to the Maxwell’s equations for radiation field
and to the Schrödinger equation for the radiation-emitting charge. This is mainly because when
the continuous emission of one energy quantum \(\hbar\omega\) is in question, there requires a treatment
which can facilitate both (1) a non stationary process and (2) a differentiable \(\hbar\omega\). The two
features are as yet not incorporated in the exiting IED representation. Currently, to the author’s
knowledge a direct theory for the intermediate emission process of single radiation quantum does
not exist. The existing time-dependent perturbation quantum theory deals with transitions
between sharply defined stationary levels without regarding the intermediate processes from
which the stationary levels are finally reached.

In this paper, we give a direct, first principles and relativistic treatment of the intermediate
process of the emission of an electromagnetic radiation quantum in an unified framework of the
classical and quantum mechanics. The treatment is instrumented by a quasi-stationary quasi-
harmonic approach to the radiation-emitting source, a charged oscillator, and the introduction
of a fractional Planck constant \(\hbar\) which justification is to be provided through its capability of
achieving a consistency both within the overall theory representation and between the theory and
experiment. We elucidate in Sec. 2 in a unified framework the classical and quantum equations
of motion and solutions, in Sec. 3 the unified classical and quantum solutions underlining the
continuous emission of a radiation quantum, and in Sec. 4 the transition probability. In the case of
a charged matter particle oscillator, the relativistic radiation, in the sense of including both the
thermal and rest-mass energy radiations, is facilitated in terms of the IED particle model in
Sec. 5.

2. Equations of motion

2.1 Quasi harmonic motion We consider an extensive charged object such as a quantum particle
or liquid-like entity (see Sec. 5), of mass \(M\) and charge \(q\), located about an equilibrium position
\(R_0(X_{10}, X_{20}, X_{30})\) in the three dimensional \((R^3)\) vacuum. The object was endowed with a
mechanical energy in the past time by an external driving force which has ceased action before
time \(t = 0\). From time \(t = 0\) the object is in spontaneous motion under the action of an elastic
restoring force \(F = -\nabla V = -\beta \mathbf{\mathcal{U}} \) along \(X_j\)-direction \((j = 1, 2, 3)\), where \(V(\mathbf{\mathcal{U}}) = \frac{1}{2} \beta \mathbf{\mathcal{U}}^2\) and
\(\mathcal{U} = (X_{j_1}(t) - X_{j_0})\hat{X}_j\) is the displacement of its mass centre \(X_{j_1}\) at time \(t\) from \(X_{j_0}\). In addition,
the object is acted on by a radiation damping force \(F_r = -\alpha M \frac{d\mathbf{\mathcal{U}}}{dt}\) apparently attributable to
a viscous (elastic) vacuum medium; \(\alpha (> 0)\) is a damping factor or decay rate.

The Hamiltonian of the system acted on by the force \(F_r\) will be time dependent. Supposing
that \(\alpha\) is small so that its Hamiltonian density is in any brief time interval \(\Delta t\) constant,
the extensive object will behave in \(\Delta t\) effectively as if a rigid object (whence facilitating the
correspondence principle in the case of a quantum particle). We may thus write down the
Newtonian equation of motion for the rigid-like extensive object as a whole, \(M \frac{d^2\mathbf{\mathcal{U}}}{dt^2} - (F_r + F) = 0\). Or,

\[
\frac{d^2\mathbf{\mathcal{U}}}{dt^2} + \alpha \frac{d\mathbf{\mathcal{U}}}{dt} + \omega^2 \mathbf{\mathcal{U}} = 0, \quad \omega^2 = \frac{\beta}{M}. \tag{1}
\]

Equation (1) has the general complex damped harmonic oscillation solution,

\[
\mathbf{\mathcal{U}}(t) = A e^{-\left(\frac{\alpha}{2} + i\omega t\right)t}, \tag{2}
\]

where \(A\) is the oscillation amplitude at \(t = 0\); \(\omega^2 = \sqrt{\omega_0^2 - \alpha^2/4}\); \(\mathbf{\mathcal{U}}(t) = \text{Re} [\mathbf{\mathcal{U}}^c] = A e^{-\frac{\alpha t}{2}} \cos(\omega t)\) gives the physical displacement. The initial phase is not relevant here and
is in (2) set to zero. Associated with the solution \(\mathbf{\mathcal{U}}\), the oscillator has at any time \(t\) a
kinetic energy \(\delta_{\text{kin}}(t) = \frac{1}{2} M (\frac{d\mathbf{\mathcal{U}}(t)}{dt})^2 = \frac{1}{2} M \omega^2 A^2 e^{-\alpha t} (\frac{\omega^2}{\omega})^2 \sin^2(\omega t) + \frac{\omega^2}{\omega} \cos(\omega t)^2\), elastic potential
energy \( V(\psi) = \frac{1}{2} \beta \psi'^2(t) = \frac{1}{2} M \omega^2 A^2 e^{-\alpha t} \cos \omega t \) and total mechanical energy or Hamiltonian, assuming \( \alpha << \omega \) and thus \( \omega = \omega' \),

\[
\mathcal{E}(t) = \mathcal{E}_{\text{kin}}(t) + V(\psi) = E e^{-\alpha t} \rho_0(t) \triangleq \frac{1}{2} M \omega^2 |\psi'|^2, \quad E = \frac{1}{2} M \omega^2 A^2, \\
\rho_0(t) = \left[ 1 + \frac{\alpha^2}{4 \omega^2} \right] \cos^2 \omega t + \left( 1 - \frac{\alpha^2}{4 \omega^2} \right) \sin^2 \omega t + \alpha \omega' \sin \omega t \cos \omega t \right] \triangleq |e^{-i \omega t}|^2 = 1.
\]

Here, \( |\psi'|^2 \triangleq A^2 e^{-\alpha t} |e^{-i \omega t}|^2 = A^2 e^{-\alpha t}, |e^{-i \omega t}|^2 = \cos^2 \omega t + \sin^2 \omega t = 1 \). The dynamical variables \( \omega, M, \beta, \alpha, A, \delta, \) etc. in general all contain a relativistic effect, which evaluation will be illustrated for the IED oscillator in Sec. 5.3. Identical energy solutions may be obtained from solving Maxwell’s equations for the radiation field emitted by charged object (Appendix A).

Under the condition \( \alpha << \omega \) which is well fulfilled in the radiation experiments of interest here, about any time \( t \) there will in general exist a brief time interval \( \Delta t \) satisfying \( \frac{\Delta t}{\omega} << \Delta t << \frac{2 \pi}{\omega} \), so that during \( \Delta t \) the displacement (2) effectively has a constant amplitude \( A e^{-\frac{\omega}{2}(t+t+\Delta t)} \) and hence is quasi harmonic. Accordingly, as further combined with Eq. (3a), the extensive oscillator has in \( \Delta t \) effectively a constant Hamiltonian, and hence is quasi stationary, agreeing with our pre-condition for establishing Eq. (1).

2.2 Quasi stationary flow motion Alternately, we may directly describe the extensive oscillator of Sec. 2.1 by a linear probability density \( \rho(x_j,t) = |\psi(x_j,t)|^2 \) along the \( X_j \) direction, where \( \psi \) is a complex function for the same reason as \( \psi^\ast \) is complex and will serve as a natural independent variable of the Hamiltonian similarly as \( \psi^\ast \) in (3a). The coordinate \( x_j \) is related to \( \psi^\ast \) as \( x_j = X_j - X_{j0} = \Delta \mathcal{U} + \psi^\ast \), where \( \Delta \mathcal{U} = X_j - X_{j0} \). With \( \psi^\ast \) given by (2), and necessarily \( \Delta \mathcal{U} = \Delta A e^{-\alpha t} \) for the oscillator being assumed rigid-body like, we have

\[
x_j = A' e^{-\frac{\omega}{2} t} \cos(\omega t), \quad A' = (\Delta A + A).
\]

The \( \psi^\ast \) motion of the oscillator is associated with a \( \rho \)-flow motion in the given \( x_j \)-direction, with a flow velocity \( v \) and flow rate

\[
j = \rho v = -D \left[ \psi^\ast (\nabla \psi) - (\nabla \psi^\ast) \psi \right], \quad D = \frac{\eta}{b M}
\]

Here, \( D \) is an imaginary diffusion constant (a general derivation is given in [3b]); \( b = 2 \) for an oscillator whose \( \psi^\ast \) motion is described by the Newtonian equation (1) as is with \( \psi_q, \psi_d \) of Sec. 5.1, and \( b = 1 \) by the Maxwell’s equations as is with the \( \psi_{r,q} \), Sec. 5.2. \( \eta(t) \) is a new real variable to be determined.

Equation (1) holds, in each brief time interval \( \Delta t \) here, only if \( \rho \) satisfies in \( \Delta t \) the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla j - \mathcal{Q} \rho = 0,
\]

where \( \mathcal{Q} = \frac{V(x_j)}{m} - \frac{\dot{V}(x_j)}{m} \). Substituting the expressions for \( \rho, j, D \) and \( V \), Eq. (6) is decomposed into two second order differential equations for \( \psi, \psi^\ast \), which for \( \psi \) and \( b = 2 \) is given as

\[
i \eta \frac{\partial \psi}{\partial t} = H \psi, \quad H = -\frac{\eta^2}{2M} \nabla^2 + V(x_j), \quad V(x_j) = \frac{1}{2} M \omega^2 x_j^2.
\]

the equation for \( \psi_{r,q} \) for the case of \( b = 1 \) will be given by (37b), Sec. 5.2. It is easily seen that the Hamiltonian \( H \) in (7) is separable into a term \( (H_0) \) associated with a potential \( V_0(x_j) \)
dependent only on $\bar{x}_j = A\cos(\omega t)$ and independent on radiation, and a term $(H_i)$ with a potential $V_i$ describing the source–radiation interaction, i.e.,

$$H = H_0 + H_i, \quad H_0 = -\frac{\eta^2 \nabla^2}{2M} + V_0(\bar{x}_j), \quad V_0(\bar{x}_j) = \frac{1}{2} M \omega^2 \bar{x}_j^2,$$

$$H_i = V_i = V(x_j) - V_0(\bar{x}_j) = -V_0(\bar{x}_j)(1 - e^{-\alpha t}).$$

(8)

If up to the initial time $t = 0$ the oscillator has emitted no (net) radiation and we set for $t = 0$ (as for $t < 0$),

$$2\pi \eta(0) = \eta(0) = h,$$

(9)

Eq. (7) identifies then with the usual Schrödinger equation, describing the quasi stationary harmonic oscillator during a brief time interval $\Delta t$ about $t = 0$ here. And the Eq. (37), Sec. 5, later describes the total wave equation[3c] for the IED particle system. Suppose that upon external perturbation the oscillator begins at $t = 0$ a (net) emission of radiation and is to emit one energy quantum $\hbar \omega$. This is necessarily a gradual process based on experimental indications discussed in Sec. 1. Of the $\hbar \omega$, $\omega$ is an intensive quantity and, when as the natural frequency of a specified mass and potential system mainly of interest in this paper, is in general a fixed value —if disregarding possible small broadening due to particle velocity variation, as known in theory and experiment. So during the intermediate process of emitting one radiation quantum $\hbar \omega$, $\eta(t)$ is inevitably the time-dependent counterpart of $\hbar$.

For the extensive oscillator is quasi stationary and hence its $H$ is effectively constant in $\Delta t$, $\psi(x_j, t)$ must be factorisable as $\psi(x_j, t) = \phi(x_j) \theta(t)$, $\theta = e^{-i\frac{\eta^2}{2}t}$ and $\psi = \phi e^{-i\frac{\eta^2}{2}t}$. Placing $\psi$ in (7), we obtain $-i^2 \eta^2 \eta \phi \theta = H \phi \theta$, or $\mathcal{E} \phi = H \phi$. The last equation may be rewritten as, with the substitutions of $\xi_j \equiv \sqrt{\frac{A_2}{\eta}} x_j$ and accordingly $\phi(\xi_j) = e^{-\xi_j^2/2} \phi(\xi_j)$,

$$\nabla^2 \phi - 2 \xi_j \nabla \phi + 2n \phi = 0,$$

$$\mathcal{E}(t) \rightarrow \mathcal{E}_n(t) = (n + \frac{1}{2}) \eta(t) \omega$$

(10)

Except that $\eta(t)$ is in place of $\hbar$, Eq. (10a) formally is identical to the time-independent Schrödinger equation for harmonic oscillator and can thus be solved accordingly with respect to $\eta(t)$.

3. Continuous emission of a radiation quantum

$\mathcal{E}$ of (3a), and $\mathcal{E}_n$ of (10b) after subtracting the $n = 0$ term, $E_0$ (which does not radiate and hence not present in the Newtonian Eq. 3), are the same energy, hence $\mathcal{E}(t) = \mathcal{E}_n(t) - E_0$. This is rewritten as, with (3a) for $\mathcal{E}$ and (10b) for $\mathcal{E}_n$,

$$\mathcal{E}_n(t) - E_0 = E_n(r_0(t)) e^{-\alpha t} = E_n a(t) = n \eta(t) \omega \quad (a)$$

$$E_n = \mathcal{E}_n(0) = \frac{1}{2} M \omega^2 A_n^2 + E_0 = n \hbar \omega + E_0 \quad (b)$$

$$a(t) = e^{-\alpha t} \quad (c)$$

$$A_n = \sqrt{n} A_1, \quad A_1 = \left(\frac{2b}{M \omega}\right)^{1/2} \quad (d)$$

$$\eta(t) = \frac{M \omega A_n^2}{2n} = e^{-\alpha t} \quad (e)$$

(11)

where $n = 1, 2, \ldots$. In (11b), or (3), $M$ and $\omega$ are constants characteristic of the oscillator and potential system; $A$ only is subject to change upon excitation and thus to quantisation, hence
\( \mathcal{A} \rightarrow \mathcal{A}_n \). Eqs. (11d) follow from the last two equations of (11b), and Eq. (11e) from the last two equations of (11a), where \( \frac{\mathcal{M} \omega \mathcal{A}^2}{2n} = \hbar \) from (11d). For the (quasi harmonic) oscillator undergoing electromagnetic radiation here, \( E_0 = 0 \) based on comparison with the empirical Planck energy equation. (One might also interpret this as the consequence that the ground level energy \( E_0 \) is finite but is never emitted as electromagnetic radiation.)

With (11d) in (4) and (2), we obtain \( x_j \rightarrow x_{jn} = \frac{\Delta \mathcal{A}_n + \mathcal{A}_n}{\mathcal{A}_n} \mathcal{Y}_n \) and

\[
\mathcal{U} \rightarrow \mathcal{U}_n = \mathcal{A}_n e^{-\frac{1}{2}t} \cos(\omega t) = (2\pi B_n \eta)^{1/2} \cos(\omega t),
\]

where \( B_n = \frac{n}{\pi \mathcal{M} \omega} \). With (11a) for \( \mathcal{E}_n(t) \), \( x_j \rightarrow x_{jn} \), \( \xi_j \rightarrow \xi_{jn} = \sqrt{\mathcal{M} \omega \eta} x_{jn} \), and the standard Hermit polynomial solution for \((\varphi \rightarrow \varphi) \) or the normalised \( \mathcal{H}_n \), we obtain the total eigen function solution for (10a)

\[
\psi_n(\xi_{jn}, t) = \phi_n(\xi_{jn}) \theta_n(t) = C_n \mathcal{H}_n(\xi_{jn}) e^{\frac{i}{2} \xi_{jn}^2} e^{-in\omega t}
\]

Accordingly, \( \rho(\xi_{jn}, t) \equiv |\psi_n(\xi_{jn}, t)|^2 = |\phi_n(\xi_{jn})|^2 \) which is independent of time, implying that the extensive oscillator moves as a rigid-object in the way that its Hamiltonian density \( \mathcal{E}_n |\psi_n(\xi_{jn}, t)|^2 \) during a brief time interval \( \Delta t \) is everywhere constant in time, as presumed in Sec. 2.1. Supposing that \( \psi_n(x_{jn}, t) \) is normalised in \([-\frac{L}{2}, \frac{L}{2}] \), where \( L \sim \mathcal{A} \), there is then at any time \( t \)

\[
\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathcal{E}_n(t) |\psi_n(x_{jn}, t)|^2 dx_{jn} = \mathcal{E}_n(t) \int_{-\frac{L}{2}}^{\frac{L}{2}} |\phi_n(x_{jn})|^2 dx_{jn} = \mathcal{E}_n(t)
\]

The charged oscillator of a time dependent \( \mathcal{E}_n(t) \) begins according to (11a) at time \( t = 0 \) a (net) radiation-emission, assuming \( \alpha > 0 \). At a later time \( t \) (assuming less than an equilibrium time \( t_o \) which may in question for e.g. an oscillator enclosed between reflection walls), it will have emitted a total amount of radiation energy given as, with \( E_{r,n} \equiv E_n \),

\[
\mathcal{E}_{r,n}(t) = E_n - \mathcal{E}_n(t) = (1 - e^{-\alpha t}) E_n = a_r(t) E_{r,n} = n \eta_r(t) \omega
\]

\[
a_r(t) = \frac{\mathcal{E}_{r,n}(t)}{E_{r,n}} = 1 - a(t) \quad (b)
\]

\[
\eta_r(t) = h(1 - e^{-\alpha t}) = h - \eta(t) \quad (c)
\]

It follows from Eqs. (11) that during the intermediate process of emission of a radiation quantum, the \( \mathcal{E}_n(t) \), being \( \propto \mathcal{Y}_n^2(t) \) (dotted line in Fig 1a), of the quasi-harmonic oscillator is at any instant of time quantised with respect to an exponentially decaying fractional-\( \hbar \), \( \eta(t) \) (solid line in Fig 1a). In this specific way, the oscillator emits electromagnetic radiation gradually and continuously with time. At the same time, as follows from Eqs. (15), the emitted radiation field \( \psi_{r,n} \) has a Hamiltonian \( \mathcal{E}_{r,n}(t) \) that is quantised with respect to an exponentially increasing fractional-\( \hbar \), \( \eta_r(t) \) (dashed line in Fig 1b). The wave field \( \psi_{r,n} \) emitted over time, being propagated in the vacuum medium at the speed of light \( c \) outward from the source, therefore stretches an extensive wave train, of a length \( L'_r(t) = ct \) at time \( t \). The sum of the two fractional-\( \hbar \)'s of the source and radiation two-component system is at any time equal to

\[
\eta(t) + \eta_r(t) = h e^{-\alpha t} + h(1 - e^{-\alpha t}) = h
\]

(16)

(16)

(16)

(16)

(16)

(16)

(16)

(16)
modulated accordingly in amplitudes. Used for the plot: i.e. a constant consisting of μ oscillator

Figure 1. Continuous emission of a radiation quantum in a (n =) 2 → 1 and 1 → 0 transition of a charged oscillator. The intermediate processes are characterised by an exponentially decreasing and increasing fractional Planck constants of the oscillator and its radiation field as functions of time t. ω(t) and η(t) given by Eqs. (11e) and (15c), shown by the solid and long-dashed curves in graphs (a) and (b). The sinusoidal oscillation displacement squares ϵ_2(t)/B_n and ϵ_1(t)/B_n given after Eq. (12), short-dashed curves in (a) and (b), are modulated accordingly in amplitudes. Used for the plot: ω = 2π/10, α_y = 0.03, t_1 = 2.3T, t_1 ≥ 200 – 2.3T. (c) is a schematic illustration of Fig 1a (dotted lines), and of Fig 1b (curly solid lines) as the permanently emitted electromagnetic wave trains of energy quanta hωq’s, or photons, being probed by detectors D.

i.e. a constant consisting of n quanta.

The energy difference (ΔE_n,n−1) between adjacent stationary levels n and n′ = n − 1 (given by Eq. 18b below) is always one whole energy quantum hω, in complete accordance with the standard notion of quantum mechanics and with overall experiments. This result may be also stated as that, the difference action ΔE_n,n−1 x 2π/ω between two stationary levels is always equal to one full Planck constant h. This is shown in [3d] to be the combined consequence of the least action principle and second law of thermodynamics, i.e. the maximum entropy condition.

If the given charged oscillator (μ) is situated between (fixed) reflection walls, and no other absorbers or perturbing fields present, the radiation wave field will be reflected back to μ, be re-absorbed, and then re-emitted by it, iteratively. After an equilibrium time t = t_o, the re-emission and re-absorption of radiation will reach equilibrium, the δ_r,n(t_o), a(t_o), η(t_o) etc. will be independent of time. We thus have a oscillator (as source) and radiation field two-component system that is as a whole in stationary state and carry n multiples of the one energy quantum hω, nhω, a situation as described by Eq. (17). The radiation field, being not charged nor undergoing energy transfer to external absorber (detector), is evidently not observable to an external observer (detector). The above in particular is the scheme by which IED particle maintains as a distinct, stationary quantum system (Sec. 5).

4. Transition time and probability

Of the two-component system above, the radiated fields will fail to be re-absorbed by their emission-oscillator μ if (i) no reflection walls present within distance of reach, (ii) another charged oscillator (as absorber) μ′ nearby begins to absorb the radiation emitted by μ, and/or (iii) the oscillator μ is externally strongly disturbed away from stationary state. The radiation field will then become manifestly permanently emitted by μ.

Consider that the oscillator μ and radiation total system is at initial time t = t_i, after having previously undergone an equilibrium time t_o, in the stationary level n. Subjected to any of the circumstances (i)–(iii), the system begins from time t = t_i to undergo a permanent emission of a radiation quantum, transforming at final time t = t_f = t_i + t_n,n′ to the stationary level n’. Because in between t_o and t_i all the time dependent functions a, a_o etc remain the same, we may generally set t_i = t_o; t_i = t_o = 0 gives an oscillator with no radiation at initial time. So at
$t_f = t_o + t_{n,n'}$, the energy of the two-component system $\mu$ will have reduced by a total amount given based on Eqs. (1a,b), (15a) and (17) in two alternative ways as

$$
\Delta E_{tot,n,n'} = \Delta E_{tot,n}(t_i) - \Delta E_{tot,n}(t_f) = \left[ (e^{-\alpha t_o} + a_r(t_o)) - (e^{-\alpha t_f} + a_r(t_f)) \right] E_n
$$

$$
= e^{-\alpha t_o}(1 - e^{-\alpha t_{n,n'}}) n\hbar \omega = \Delta E_{n,n'} \quad (a); \quad \Delta E_{n,n'} = E_n - E_{n'} = (n - n') \hbar \omega \quad (b)
$$

(18)

In going from the second to the third of Eqs. (18a) we assumed that during the transition any new radiation emitted by the source contributes to the permanently emitted radiation (i.e. $\Delta E_{n,n'}$) only and not to $a_r E_n$, so $a_r(t_o) = a_r(t_f) = 1 - e^{-\alpha t_r}$. Further from the identity relation $\Delta E_{n,n'} = \Delta E_{n,n'}$ we obtain the transition time or lifetime evaluated from a finite $t_i (= t_0)$ and from $t_i = t_0$ respectively

$$
t_{n,n'} = t_f - t_o = \frac{1}{\alpha} \ln \frac{n e^{-\alpha t_o}}{n' (1 - \frac{n}{n'}(1-e^{-\alpha t_o}))},
$$

$$
t_{n,n'} = t_f - t_o = \frac{1}{\alpha} \ln \frac{n}{n'}
$$

(19)

$t_{n,n'}$, with $n' = n - 1$ say, informs the literary time span for the exchange of one photon between two systems $\mu$ and $\mu''$ rather than a statistical average, for in accordance to the fundamental quantum principle that one energy quantum or photon can either be exchanged between two systems as a whole, or not exchanged at all. For an oscillator endowed with $n$ energy quanta $(E_n = n\hbar \omega)$ at the beginning, the total time $t_r$ required for transiting to final $n' = 0$ level is

$$
t_r = \sum_{n'=1}^{n} t_{n',n'-1} = \frac{1}{f \alpha}, \quad \frac{1}{f} = \sum_{n'=1}^{n} \ln \frac{n'}{n' - 1}
$$

(20)

These have the mean values $\langle t_r \rangle = \int_0^{\infty} f e^{-\alpha t} dt = \frac{1}{\alpha} \quad (\text{the mean life time}) \quad \text{and} \quad \langle \frac{1}{f} \rangle = 1$. Since for the systems considered $\alpha \ll \omega$, or $\frac{2 \pi}{\alpha} \sim t_{n,n-1} >> \frac{2\pi}{\omega} = \tau$, during a transition time $t_{n,n-1}$ the oscillator in general continuously oscillates a large $\frac{t_{n,n-1}}{\tau} >> 1$ number of oscillation cycles.

Suppose there presenting a large $N$ number of identical oscillators that at initial time all lie at energy level $n$ and will decay to final level $n'$ statistically via $n \to n'$ transitions, elapsing a transition time $t_{n,n'}$ given by (19) each, and emitting a total $N_{n,n'}$ energy quanta or photons. The total apparent elapsing time is $T = \sum_{i=1}^{N} t_{n,n'} = N t_{n,n'}$. So $N_{n,n'} = \frac{T}{t_{n,n'}} = N$. The probability per unit time that any oscillator makes a $n \to n'$ transition and emits one photon is thus given as

$$
\alpha_{n,n'} = \frac{dN_{n,n'}}{N_{n,n'} dt_{n,n'}} = \frac{1}{\alpha} \frac{T}{t_{n,n'}} = \frac{1}{\alpha} \frac{N}{\ln \frac{n'}{n}}
$$

(21)

Specifically if $n = 1$ and $n' = n - 1 = 0$, Eqs. (19b) or similarly (19a), and (21) give

$$
t_{1,0} = \infty, \quad \alpha_{1,0} = \frac{1}{t_{1,0}} = 0;
$$

(22)

and for $n = 2$, $n' = 1$, $t_{2,1} = \ln 2 / \alpha$, $\alpha_{2,1} = \alpha / \ln 2$, etc. For a fixed $\alpha$, $t_{n,n-1}$ reduces, and $\alpha_{n,n-1}$ increases with increasing $n$.

The transition probability may be more generally expressed in the usual terms of a source–radiation interaction Hamiltonian, our $H_I$ given by Eq. (8d) earlier. The total and the unperturbed Hamiltonians $H(= H_0 + H_I)$ and $H_0$ are given by Eqs. (8a) and (b). The
total (or ensemble \([3e]\)) wave function at any time \(t\) is of the general form \(\psi_{en}(x_j, t) = \sum_m b_m(t)\psi_m(x_j)e^{-i\frac{E_m}{\hbar}t}\), where \(b_m(t)\) is the amplitude of state \(m\); \(\psi_{en}\) clearly is also a solution to Eq. (7a). Substituting (8a) for \(H\) and the equation above for \(\psi_{en}\) in the corresponding equation of (7a) we obtain

\[
\sum_m (H_0 + H_1)b_m(t)\psi_m(x_j)e^{-i\frac{E_m}{\hbar}t} = i\hbar \sum_m \left[ b_m(t)\psi_m(x_j)e^{-i\frac{E_m}{\hbar}t} - \frac{i\delta_m}{\eta}b_m(t)\psi_m(x_j)e^{-i\frac{E_m}{\hbar}t}\right]
\]

Subtracting \(\sum_m H_0\psi_m(x_j, t) = \sum_m \delta_m\psi_m(x_j, t)\), multiplying \(\psi_k(x_j)\) from the left and integrating over all \(x_j\), we obtain, for the eigen functions \(\psi_m(x_j)\)'s being orthogonal,

\[
\sum_m b_m(t) \int \psi_k^*(x_j)H_1\psi_m(x_j)dx_j e^{-i\frac{E_m}{\hbar}t} = i\hbar \sum_m b_m(t) \int \psi_k^*(x_j)\psi_m(x_j)dx_j e^{-i\frac{E_m}{\hbar}t}.
\]

Or

\[
\int b_n(t)\psi_k^*(x_j)H_1\psi_n(x_j)dx_j e^{-i\frac{E_m}{\hbar}t} = i\hbar (t)\psi_k(t)e^{-i\frac{E_k}{\hbar}t},
\]

assuming the oscillator to be in a definite energy state \(n\) at initial time \(t_i\), and thus \(b_n = 0\) for all \(m \neq n\). Based on the solutions in Sec. 3, throughout the intermediate process of quasi stationary transition from level \(n\) to \(k\) here across a time duration \((t_i, t_f)\), there present one component wave function which maintains precisely the same as at initial time,

\[
\psi_n(x_j, t) = \psi_n(x_j)e^{-i\frac{E_n}{\hbar}t} = \psi_n(x_j)e^{-i\frac{E_n}{\hbar}t};
\]

and the other (in a fashion as discussed after Eq. 18) as at the final time \(t_f\), \(\psi_k(x_j, t) = \psi_k(x_j)e^{-i\frac{E_k}{\hbar}t} = \psi_k(x_j)e^{-i\frac{E_k}{\hbar}t}\). And the amplitudes of the two quasi stationary component states reduces and increases respectively with time as \(a_n(t)\) and \(a_r,n - a_k(t) = 1 - a_n(t)\). So with \(b_n(t) = a_n(t) = e^{-\alpha t}\) (accordingly \(b_k(t) = a_k(t) = 1 - b_n(t)\); i.e. we are here facilitated with the explicit time dependent functions \(b_n(t), b_k(t)\) instead of the usual perturbation approach), and \(\eta = \hbar e^{-\alpha t}\) of (11c), denoting \(\omega_{kn} = \frac{\delta_k(t) - \delta_n(t)}{\eta(t)} = \frac{E_k - E_n}{\eta},\) (24) is rewritten as

\[
\dot{b}_k = \frac{1}{\hbar} H_{t_{kn}} e^{i\omega_{kn}t},
\]

\[
H_{t_{kn}} = \int \psi_k^*(x_j)H_1\psi_n(x_j)dx_j = -V_{0_0}(1 - e^{-\alpha t}) \int \left(\frac{x_j}{A_n e^{-\frac{\alpha t}{2}}}\right)^2 \psi_k^*(x_j)\psi_n(x_j)dx_j = H_{i:o} \chi_{kn},
\]

\[
\chi_{kn} = \int \left(\frac{x_j}{A_n e^{-\frac{\alpha t}{2}}}\right)^2 \psi_k^*(x_j)\psi_n(x_j)dx_j,
\]

where \(V_{0_0} = \frac{1}{2}\omega^2\hbar^2A_l^2, H_{i:o} = V_{0_0}(1 - e^{-\alpha t}).\) Notice that \(H_{i:o}\), the amplitude of source–radiation interaction Hamiltonian, or similarly \(H_j\), is gradually switched on from zero at initial time \(t_i\) to maximum \(V_{0_0}\) at final time \(t_f\); and this is not necessarily a small or perturbation quantity. For an oscillator with an initial Hamiltonian \(E_1 = 2V_{0_0}\) and a corresponding a final time radiation-source interaction Hamiltonian \(H_{i:o} = V_{0_0}(1 - 0),\) for example, \(H_{i:o}\) is equal just to the (maximum) potential or binding energy \(V_{0_0}\) of the (quasi) Harmonic oscillator.

The instantaneous transition probability per unit time in a brief time interval say \((t, t + \tau)\), with \(\tau << t_{n,k}\), in which \(H_{i:o} = V_{0_0}(1 - e^{-\alpha t})\) is essentially a constant, is given as

\[
P_{n,k} = \frac{1}{\tau} \frac{|b_k|^2}{b_k} = \int_0^{\tau} \frac{H_{t_{kn}}(e^{i\omega_{kn}t} - 1)}{\hbar\omega_{kn}} dt,
\]

(27)
Suppose that $\omega_{kn}$ may have a finite continuous dispersion. The probability per unit time integrated over the entire possible dispersion range $(-\infty, \infty)$ of the transition energy $E_k = E_k - E_n = h\omega_{kn}$, as evaluated at $\frac{1}{2} \tau$ at which $b_k = \frac{H_{tkn}}{\hbar \omega_{kn}} e^{i\omega_{kn}(\frac{\tau}{2})} 2i \sin(\omega_{kn}\frac{\tau}{2})$, is

$$
\int_{-\infty}^{+\infty} \mathcal{P}_{n,k} dE_{kn} = \frac{1}{\tau} \int_{-\infty}^{+\infty} |b_k|^2 dE_{kn} = \frac{4}{\tau \hbar^2} \int_{-\infty}^{+\infty} |H_{tkn}|^2 \frac{2 \sin^2 \frac{\omega_{kn} \tau}{2}}{\omega_{kn}^2} d(h\omega_{kn})
$$

where $y = \frac{1}{2} \omega_{kn} \tau; \frac{\sin y^2}{y^2} \propto \delta(y)$, and $\int_{-\infty}^{+\infty} \frac{\sin y^2}{y^2} dy = \pi$. (27) after dividing $E_{kn}$ out is by definition equal to the $\alpha_{n,n'}$ earlier for $n' = k$. Except with $H_t(t)$ being here an explicit function of time and describing the instantaneous source-radiation interaction at any time $t$ during the emission of a radiation quantum, the conclusions (25)–(27) are formally as given based on the standard perturbation approach (see e.g. [4]). The source-radiation interaction Hamiltonian at the final completion of emission of one quantum is simply found at time $t_f$.

5. IED particle oscillator

5.1 The kinetic and the total charge oscillations  Consider as two inter-related applications that the (extensive quasi-harmonic) charged oscillator of Secs. 2.1–2.2 firstly represents an usual charged oscillatory quantum particle of a charge $q$ and mass $m$. The particle has an oscillation displacement $\mathcal{U}_d = \mathcal{U}_d(x) = X_0$ at its mass centre along the $X$ direction under the actions of an elastic force $F_d = -\nabla V_d = -\beta_d \mathcal{U}_d$ due to an applied potential $V_d = \frac{1}{2} \beta_d \mathcal{U}_d^2$ (Fig. 2a), and a radiation damping force $F_{rd} = -\alpha_d m \frac{d\mathcal{U}_d}{dt}$. Its kinetic motion is associated with a wave function $\psi_d(x,t)$, where $x = X - X_0$. The equations of motion are given directly by substitutions of $\mathcal{U}_d$, $\alpha_d$, $\omega_d$, $\beta_d$, $m$, $\psi_d$, $H_d$, $\eta_d$ and $x$ for $\mathcal{U}$, $\alpha$, $\omega$, $\beta$, $\psi$, $H$, $\eta$ and $x_j$ in Eqs. (1) and (7),

(1.a) \[ \frac{d^2\mathcal{U}_d}{dt^2} + \alpha_d \frac{d\mathcal{U}_d}{dt} + \omega_d^2 \mathcal{U}_d = 0; \quad \omega_d^2 = \frac{\beta_d}{m} \]

(7.a) \[ i\eta_d \frac{\partial \psi_d}{\partial t} = H_d \psi_d, \quad H_d = -\frac{\eta_d^2}{2m} \nabla_d^2 + \frac{1}{2} m \omega_d^2 x^2 \]

The solutions to (28)–(29) are given directly by substitutions of the corresponding variables $n_d$, $\omega_d$, $m$, $A_n$, $a_d$, $\eta_d$ and $\alpha_d$ for $n$, $\omega$, $M$, $A_n$, $a$, $\eta$ and $\alpha$ in Eqs. (2)–(3), (9) and (10)–(20), Sec. 3. Of specific interest here, the Hamiltonian is given upon the substitutions above in Eqs. (11), divided for $t \leq 0$ and $t \geq 0$ according to if any of the circumstances (i)–(iii) of Sec. 4 is onsets, as

$$
\mathcal{E}_n(t) \rightarrow \mathcal{E}_n(t) = \begin{cases} E_{n_d} = \frac{1}{2} m \omega_d^2 A_{n_d}^2 = n_d \hbar \omega_d & (t < 0) \\ \mathcal{E}_n(t) = a_d(t) E_{n_d} = n_d \eta_d(t) \omega_d & (t \geq 0) \end{cases}
$$

where $a_d(t) = e^{-\alpha_d t}$, $\eta_d(t) = \hbar e^{-\alpha_d t}$. By (30a), up to time $t = 0$ the particle is in accelerated ($\mathcal{U}_{n_d}$) motion but is in stationary state, with a quantised Hamiltonian $E_{n_d}$. As such, the accelerated (charged) particle does not radiate. The accelerated $\mathcal{U}_{n_d}$ motion is instead to manifestly augment the frequency of the total wave from $\Omega_q$ by an average factor $\gamma$ (given by Eq. 50b below) to $\omega_q = \gamma \Omega_q$, and accordingly modulate the plane electromagnetic wave $\Xi_{r,n_q}$ into $\psi_{r,n_q} = \Xi_{r,n_q} \psi_{n_d}$ (Sec. 5.2). Under any of the circumstances (i)–(iii) of Sec. 4 only, the charged particle will emit thermal radiation according to (30b).

We shall be interested also in the rest-mass radiation of the particle. As a viable scheme for including the rest-mass radiation here, and for representing the extensive quantum wave of a
matter particle in general, we shall represent the particle [assuming for simplicity (effectively) single charged and non-composite, like the electron] in terms of the IED particle model along with a vacuuonic vacuum proposed in [3] based on overall experiments. An IED particle of charge $q$ is composed of a minute oscillatory charge $\psi$ (as source) and the total electromagnetic radiation field $E'(r, t), B'(r, t)$ emitted by the charge. So, besides the oscillation of the IED particle itself described by Eqs. (28)–(29) in the $X$ direction here, there is simultaneously an internal oscillation of its generating charge $q$ (the $\psi_q$ along $Z$ direction below), and accordingly the dynamical process of its radiation field (the $\psi_{r,q}$ in Sec. 5.2). The IED particle oscillation consists in the oscillation of the charge–radiation field system as a whole (Sec. 5.3).

The vacuum is by construction (on experimental basis[3a,b]) filled of electrically neutral but polarisable vacuuons that are densely and disorderly packed with a mean separation distance $b_v \sim 1 \cdot 10^{-18}$ m. The vacuuons, polarised by the given charge $q$ situated in an interstice $i$ of the vacuuons centred about $R_{i0}$, produce at the charge $q$ a vacuum potential

$$V_{vq}(\mathcal{U}_q) = V_{vq}(0) + \sum_n \frac{1}{n!} \nabla^n V_{vq}(R_{i0}) \mathcal{U}_q^n = V_{vq0} + V_q, \quad V_q = \frac{1}{2} \beta_q \mathcal{U}_q^2,$$  \hspace{1cm} (31)$$

where $V_{vq0} = V_{vq}(0)$ and $\beta_q = \nabla^2 V_{vq}$; $\mathcal{U}_q(t) = R_{ci}(t) - R_{i0}$ is the displacement of the charge’s mass center $R_{ci}$ from $R_{i0}$ along a direction perpendicular to the maximum intensity of its radiation wave (the $\psi_{r,q}$ later, or the $\psi_d$ above along the $X$-direction); let this be the $Z$ axis (Fig. 2.b); so $\mathcal{U}_q(t) = Z_{ci}(t) - Z_{i0}$. The approximation in (31a) is given for $\mathcal{U}_q$ being relatively small.

![Figure 2: A quasi harmonic IED particle oscillator of oscillation $\mathcal{U}_{q} (\text{Figure 2a})$, which contains a simultaneous internal oscillation $\mathcal{U}_{d} (\text{Figure 2b})$, transits from initial thermal energy level $n_d = 1$, with an eigen function $\psi_{d}$ (solid curves in Figure 2a) plotted after Eq. (13), to a final level $n_d = 0$ ($\psi_{d}$ not shown). At the end one energy quantum $\hbar \omega_d$ is emitted.](image)

The charge $q$ is a spinning liquid-like entity (or vortex) of a linear dimension $\sim b_v \sim 1 \times 10^{-18}$ m; a point $Z$ on it has a displacement $z = Z - Z_{i0}$, and probability density $\psi_q(z, t)$. The charge has a zero rest mass. It however has a total mechanical energy (or Hamiltonian) $E_q$ endowed in the past time, upon the action of an external driving force which has ceased action before time $t = 0$. The charge of the present time is thus an inertial system moving about in the vacuum, spontaneously propelled by its own inertial force $F_{inr}$. Defining $F_{inr} \equiv M_q \frac{d^2\mathcal{U}_q}{dt^2}$ in direct analogy to the Newtonian inertia for the usual matter-particles, we obtain $M_q$ as a proportionality constant, or a manifestly dynamical mass upon mapping on to a non-viscous vacuum. Subjected to the vacuum potential $V_{vq}$ of Eq. (31) about the fixed site $Z_{i0}$ here, the motion of the charge is resisted, by an elastic resistive force $F_{vq} = \frac{\partial V_{vq}}{\partial \mathcal{U}_q} = -\beta_q \mathcal{U}_q$, and in addition a radiation damping force $F_{r,q} = -M_q \alpha_q \frac{d\mathcal{U}_q}{dt}$ in the viscous elastic vacuum. The corresponding equations of motion are given directly by substitutions of $\mathcal{U}_q, \alpha_q, \omega_q, \beta_q, M_q, \psi_q,$
The total wave motion

The charge oscillator of motion, Eqs. (28)–(29) and (32)–(33). The solutions for the two oscillators are accordingly.

\[ \text{motion in a three-dimensional vacuum, generates along any radial direction two oppositely travelling, and } \psi \text{ to } \omega \text{ to } \eta \text{ and } n \text{ and } \alpha \text{ to } \eta \text{ and } \alpha \text{ in Eqs. (11) as} \]

\[ H_q, q, \text{ and } z \text{ for } H, \alpha, \omega, \beta, M, \psi, H, q \text{ and } x_j \text{ in Eqs. (1) and (7)} \]

\[ \frac{d^2 q}{dt^2} + \alpha \frac{dq}{dt} + \omega_q^2 q = 0, \quad \omega_q^2 = \frac{\beta_q}{M_q}; \]

\[ i \hbar \frac{\partial q}{\partial t} = H q, \quad H = -\frac{\hbar q^2}{2M_q} \nabla_z^2 + \frac{1}{2} M_q \omega_q^2 z^2 \]

The Hamiltonian solution to Eqs. (32)–(33) is given similarly by direct substitutions of \( q, \psi \text{ and } H \) in Eqs. (7) as

\[ \delta_n \rightarrow \delta_n = \frac{1}{2} M_q \omega_q^2 A^2 \eta \phi(t) = n \hbar q(t) \omega_q, \quad a_q(t) = e^{-\alpha_q t}, \quad \eta_q(t) = \hbar e^{-\alpha_q t} \]

In sum, for the IED model, there are two simultaneous orthogonal (modes of) oscillations \( \psi = (\psi_0, \psi_1) \), and two corresponding probability density \( |q(t)|^2 \)–flow motions in the \((X_j, X_j') = (X, Z)\) directions, executed by two apparent oscillators, the IED particle oscillating itself and the IED-particles’s source charge \( q \), of the masses \( M = (m, M_q) \) and a common charge \( q \). The \( d \) and \( q \) oscillators each are extensive in their respective potential wells \( V = (V_d, V_q) \) of a common quadrature form and are described by the common forms of equations of motion, Eqs. (28)–(29) and (32)–(33). The solutions for the two oscillators are accordingly formally commonly given by Eqs. (2)–(3), (9), and (10)–(20).

### 5.2 The total wave motion

The charge oscillator \( q \), oscillating according to (32)–(33) along the \( Z \) direction in a three-dimensional vacuum, generates along any radial \( r \)-direction two oppositely travelling, and \( v_d \)-motion resultant Doppler-differentiated (Sec. 5.3) electromagnetic wave fields \( E^1(r, t), B^1(r, t) \) travelling with velocities \( c \) and \(-c \) parallel (\( j = 1 \)) and antiparallel (\( j = 2 \)) to \( \mathbf{v}_d(t) \). Of direct relevance to the resultant IED particle is the superposed total wave field \( E(r, t) = E^1(r, t) + E^1(-r, t) \); accordingly \( B = -\frac{E}{c} \). The \( E, B \) fields in the case of \( V_d = 0 \) identify with the solutions (A.3)–(A.4) to the Maxwell’s equations, Appendix A. In so far as the particle’s coherent wave motion, and the associated total radiation power which is a constant independent of \( r \) (Eq. A.10, Appendix A), are mainly in question here, the radial radiation waves \( E(r, t) \) may be in effect furthermore (i) represented by the maximum-intensity wave field along the particle’s motion \( X \)-direction and (ii) solved in regions where \( \rho_q = j_q = 0 \) only. For the general case of a finite \( V_d \), wave fields are of the general complex forms

\[ E^c_x(x, t) = a_{q}^{1/2} \mathbf{E}_q \psi_{r,q} \frac{x, t}{c} \zeta, \quad B^c_y(x, t) = -\frac{\mathbf{E}_q \psi_{r,q} \frac{x, t}{c}}{c} \gamma \]

given similarly as in (A.8), where \( \psi_{r,q}(x, t) = \psi_{r,q} \frac{x, t}{c} + \psi_{r,q} \frac{x, t}{c} \) is a complex dimensionless transverse wave and is to be solved in the presence of \( V_d \).

The radiation field has a zero rest mass (similarly as its generating charge in the IED model) but a finite dynamical mass derived based on the following consideration. Suppose that at time \( t \) the ratio of the radiation Hamiltonian (\( \delta_{r,q}(t) \)) to the total Hamiltonian of the charge-radiation system (\( \delta_{\text{tot},q}(t) \)) is \( a_{r,q}(t) \). Regardless of the applied \( V_d \), the total radiated electromagnetic wave train (of the wave function \( \psi_{r,q} \) and total length \( L_{r,q} \)), propagating at the constant velocity \( c \) in the vacuum, has according to Newtonian mechanics an (intrinsic) total linear momentum \( P_{r,q}(t) = \int \frac{\mathbf{E}_q \psi_{r,q} \frac{x, t}{c} \zeta}{c} dx \) = \( mc \) multiplied by \( a_{r,q}^{1/2}(t) \), and kinetic energy \( a_{r,q}(t) |E_{r,q,kin}(t)|^2 = a_{r,q}(t) \frac{\Omega_{r,q}}{2m} = a_{r,q}(t) \frac{1}{2} mc^2 \)

(in the electromagnetic energy expression in the brackets the factor \( \frac{1}{2} \) in front of \( \epsilon_0 \) is because only the \( E \) field does work and not the \( B \)). \( m \) manifestly represents the relativistic dynamical...
mass of the wave train in a non viscous vacuum (as is assumed in Newtonian mechanics). In the vacuuum vacuum representation we adopt in this section, \( m \) represents a coefficient proportional to the resistive force \( F_{\nu r} \) of the viscous elastic vacuum bulk against the wave train motion, \( \langle P_{r,q} \rangle = \int F_{\nu r} dt = \int_0^c \eta m c^2 dt \). Accordingly, the wave train has in addition an elastic vacuum potential energy \( a_{r,q}(t) (V_{r,q}(x)) = a_{r,q}(t) E_{r,q,kin}(t) = a_{r,q}(t) \frac{1}{2} mc^2 \), and therefore a total (intrinsic, i.e. excluding \( a_{r,q}(t) V_{d0} \)) Hamiltonian

\[
H = a_{r,q} E_{r,q}(t) - a_{r,q}(V_{d0}) = a_{r,q} \langle E_{r,q,kin}(t) \rangle + a_{r,q} \langle V_{r,q,0}(x) \rangle = a_{r,q}(t) 2 \times \frac{1}{2} mc^2 = a_{r,q}(t) mc^2
\]

Or \( \hbar \omega_q - \langle V_{d0} \rangle = mc^2 \) (36)
given after substituting the Eq. (40) below for \( E_{r,q}(t) \) and dividing \( a_{r,q}(t) \) out. In the formal sense of the Eq. later, \( m \) represents the dynamic mass of the IED particle.

The Maxwell’s equations for the fields (35) lead in regions where \( \rho_q = j_q = 0 \) to the wave equation

\[
\frac{\partial^2 \psi_{r,q}}{\partial x^2} = (c^2 + \frac{V_d}{m}) \nabla^2 \psi_{r,q}.
\]

This further reduces to, by combining with the identity relation \( m = \frac{\hbar \omega_q - V_d}{c^2} = \gamma M \) described by Eqs. (36b) (or 41) and (50) and with the procedure described in [3c] except with \( \eta_{r,q} \) replacing \( \hbar \) in the final result,

\[
i \eta_{r,q} \frac{\partial \psi_{r,q}}{\partial t} = H_p \psi_{r,q}, \quad H_p = \frac{-\eta_{r,q}^2}{m} \nabla^2 + V_d
\]

(37)

It is easily seen that Eq. (37) is associated with a continuity equation,

\[
\frac{\partial \rho_{r,q}}{\partial t} + \nabla (-D_p \nabla \rho_{r,q}) = \left( \frac{V_d}{\eta_{r,q}} - \frac{V_d}{\eta_{r,q}} \right) \rho_{r,q} = 0,
\]

given by substitutions of \( \rho_{r,q} = \psi_{r,q}(x,t) \), \( b = 1, m, D_p = \frac{\eta_{r,q}^2}{m}, V_d \) and \( x_j, b, M, D, V \) and \( x \) in (6).

Using the procedure of [3c] except with \( \hbar \) being here replaced by the \( \eta_{r,q} \), with \( \psi_{r,q} = \Xi_{r,q}, \psi_d \). Eq. (37) may be decomposed into two separate wave equations for \( \psi_{r,q} \) and \( \Xi_{r,q} \). The wave equation for \( \psi_d(x,t) \), with a Hamiltonian \( H_d = H_p - H_p^0 = -\frac{\eta_{r,q}^2}{m} \frac{\partial^2}{\partial x^2} + V_d \) is just the Eq. (29). \( \Xi_{r,q} \) is the total radiation field emitted by the charge \( q \) when oscillating about a fixed site, i.e. \( v_d(t) = 0 \), and is described by the wave equation

\[
i \eta_{r,q} \frac{\partial \Xi_{r,q}}{\partial t} = H_p^0 \Xi_{r,q}, \quad H_p^0 = -\frac{\eta_{r,q}^2}{M} \frac{\partial^2}{\partial x^2},
\]

(38)

(38) has the solutions

\[
\Xi_{r,q}(x,t) = e^{i \left( \frac{\partial^0_{r,q,n} x}{\eta_{r,q}} - \frac{\partial^0_{r,n,q} t}{\eta_{r,q}} \right)} \Xi_{r,n,q}(t), \quad \partial^0_{r,n,q} = \frac{\partial^0_{r,n,q}}{c},
\]

(39)

The total Hamiltonian of the radiation field is given after (15) as

\[
E_{r,n} \rightarrow E_{r,n,q}(t) = a_{r,q}(t) n_q \hbar \omega_q = n_q \eta_{q,r}(t) \omega_q, \quad a_{r,q}(t) = 1 - \alpha q(t) = 1 - e^{-\alpha_q t}, \quad \eta_{r,q}(t) = \hbar - \eta_q(t) = \hbar(1 - e^{-\alpha_q t})
\]

(40)

5.3 The charge and radiation-wave total system

The minute liquid-like charge \( q \) and the resulting radiation wave \( \psi_{r,q} \) are maintained as one system, the IED particle, by the repeated radiation re-absorption and re-emission scheme commented after Eq. (16). The total Hamiltonian \( H_{tot,n,q}(t) \) of the IED particle thus is at any time \( t \) carried a fraction \( a_{q}(t) \) by
the charge oscillator $q$, and $a_{r,q}(t)$ by the total radiation field $\psi_{r,q}$. With the $\mathcal{E}_{n_q}$ of the charge and $\mathcal{E}_{r,n_q}$ of its radiation field given by (34) and (40), we obtain, for $n_q = 1$,

$$
\mathcal{E}_{tot,1_q}(t) = \mathcal{E}_{1_q}(t) + \mathcal{E}_{r,1_q}(t) = a_q(t)E_{1_q} + a_{r,q}(t)E_{r,1_q}
$$

$$
\equiv E_{1_q} \equiv E_{r,1_q} = \frac{1}{2}\mathcal{M}_q\omega^2_{1_q}A_1^2_q = h\omega_q = mc^2 + V_d0.
$$

(41)

The one energy quantum $\hbar\omega_q$ of the $q$–$\psi_{r,q}$ system will, applying the general quantum mechanical principle once again, either not be absorbed by an external source at all, or be absorbed as a whole upon an energy exchange.

As elucidated in [3b], an oscillatory charge $q$ (of a zero rest mass) at the stationary level $n_q = 1$ in $V_q$ gives rise to a stationary electron ($e$) if $q = -e$, and a proton ($p$) if $q = +e$. A transition of the $-e$ or $+e$ oscillator from initial level $n_q = 1$ to a final $n'_q = 0$ (in $V_q$) in the vacuum corresponds to a spontaneous decay of the particle $e$ or $p$, over an infinite transition time $t_{1,0} = \infty$ according to Eqs. (19)–(22), in direct accordance with the empirical fact that the proton and electron have under normal conditions infinite lifetimes. (Nevertheless, the $e$ or $p$ may transit to the ground state $n_q = 0$ if the quadratic $V_q$ condition is strongly distorted, as e.g. would be the case when an anti-particle presents nearby, leading to a pair annihilation.) In a vacuum composed of densely packed vacuuons, the $V_q$ of each site extends only about half way to its neighbouring site and is whereof superseded by $V_q$ of the neighbouring site. So there is no stationary state for $n_q > 1$. The charge of an existing electron, proton or an composite particle of these, carries already one quantum of the relativistic mass energy, $h\omega_q$, of the particle and can not absorb another $h\omega_q$. So, the charge of another such existing particle can only permanently emit its $E_{tot,1_q}$, in a quasi stationary process, through pair annihilation; and the emitted gamma photon can only be absorbed by another "bare" charge (a vacuuleon composing a vacuuon) out of the vacuum.

If any of the circumstances (i)–(iii) of Sec. 4 sets in (at time $t = 0$) so that the IED particle is perturbed away from the stationary level $n_d$, the IED particle oscillator will now (tending to stabilise in a lower and stationary level) manifestly emit thermal radiation ($\psi_{r,n_d}$) according to Maxwellian electrodynamics, or equivalently the quasi harmonic solution Eq. (30b). The emitted thermal radiation energy is accordingly given as $\mathcal{E}_{r,n_d}(t) = E_{n_d} - \mathcal{E}_{n_d}(t) = E_{n_d}(1 - e^{-a_d t}) = \eta_{r,d}\omega_d$, where $\eta_{r,d}(t) = h - \eta_d(t)$ given after (15) and (30). The maximum intensity, of the thermal radiation is in a direction perpendicular to $V_d$, hence lying along a line in the $Y - Z$ plane, passing $(X_0, Z_0)$ in Fig 2.

The sum of the two terms above, $\mathcal{E}_{tot,n_d}(t) = \mathcal{E}_{n_d}(t) + \mathcal{E}_{r,n_d}(t) = E_{n_d}e^{-a_d t} + E_{n_d}(1 - e^{-a_d t})$, represents two briefly co-existing components during the transition only, rather than a distinct system given in (41), for the thermal frequency $\omega_d$ is not unique but is one out of a continuous spectrum and the energy quantum $\hbar\omega_d$ can be readily absorbed/emitted by another particle oscillator in the surrounding. At the end of the transition time $t_{n_d,n_d-1}$, the energy exchanged is given based on (18) as $\Delta\mathcal{E}_{n_d,n_d-1} = E_{n_d}(1 - e^{-a_d t_{n_d,n_d-1}}) = \Delta E_{n_d,n_d-1} = \hbar\omega_d$, i.e. one whole energy quantum. For the thermal electromagnetic radiation here, the result $t_{r,1,0} = \infty$ of Eqs. (19)–(22) implies that the IED particle oscillator will always maintain at least a "zero point" (ground-state) thermal energy, $E_{0_d}$.

We finally evaluate the relativistic effect due to the $v_{n_d}$ (or $V_{n_d}$) motion of the particle in $n_d$th thermal level combining with the thermal and total energy solutions already obtained. The stationary harmonic oscillation (given for $a_d = 0$) of the (IED) particle is an accelerated motion. The instantaneous velocity $v_{n_d}(t)$, accordingly the instantaneous relativistic mass $m(t)$, kinetic energy and linear momentum

$$
\mathcal{E}_{kin,n_d}(t) = \frac{p_{n_d}^2(t)}{2m(t)} = \frac{1}{2}m(t)\omega_d^2 A_{n_d}^2 \sin^2(\omega_d t), \quad p_{n_d}(t) = m(t)v_{n_d}(t),
$$

(42)
and potential energy $V_{nd}(t)$ (as a short-hand denotation of $V_{nd}(x(t))$ here) as measured in the laboratory frame (S), each vary with time. In a brief time interval about $t$, $V_{nd}(t)$ is effectively constant. Thus the relationship between the instantaneous total (internal) Hamiltonian $\mathcal{H}_{tot.1q} - V_{nd}(t)$ (with $\mathcal{H}_{tot.1q} = \hbar \omega_q$ as given in Eq. (41)) and the $p_{nd}(t)$ may be written according to the solution for a IED particle of constant potential (see e.g. [3c]), or equivalently the Einstein mass energy relation, as

$$[\hbar \omega_q(t) - V_{nd}(t)]^2 = m^2c^4 + p_{nd}^2(t)c^2,$$

where

$$\omega_q(t) = \gamma(t)\Omega_q, \quad m(t) = \gamma(t)M, \quad V_{nd}(t) = \gamma(t)V_{nd}^0, \quad p_{nd}(t) = \gamma(t)p_{nd}^0 = \gamma(t)M v_{nd}(t),$$

$$\gamma(t) = \frac{1}{\sqrt{1 - \frac{v_{nd}^2(t)}{c^2}}}, \quad \gamma^2(t) = 1 + \frac{\langle v_{nd}^2(t) \rangle}{c^2};$$

$\Omega_q$, $M$, $V_{nd}^0(t)$ and $p_{nd}^0(t)$ are the corresponding rest-mass values measured in $S$.

The time average of (43) is $(\langle [\hbar \omega_q(t) - V_{nd}(t)]^2 \rangle = (m^2(t))c^4 = M^2c^4 + \langle p_{nd}^2(t) \rangle c^2$. The first two expressions develop as $\langle [\hbar \omega_q(t) - V_{nd}(t)]^2 \rangle = \langle \gamma^2(t) \rangle [\hbar \Omega_q - V_{nd}^0]^2$ and $\langle m^2(t) \rangle = \langle \gamma^2(t) \rangle M^2$, where

$$\langle \gamma^2(t) \rangle = \frac{1}{1 - \frac{v_{nd}^2(t)}{c^2}} = 1 + \frac{\langle v_{nd}^2(t) \rangle}{c^2} + \frac{\langle v_{nd}^4(t) \rangle}{c^4} + \ldots$$

The $\langle v_{nd}^2(t) \rangle$, $\langle v_{nd}^4(t) \rangle$, etc. may be readily individually evaluated as

$$\langle v_{nd}^2(t) \rangle = \frac{1}{2m(t)} \omega_q^2 \int_0^{\tau_d} \langle \sin^2 \omega_q dt \rangle = \frac{1}{2} \omega_q^2 \alpha_{nd}^2,$$

and $\langle v_{nd}^4(t) \rangle = \frac{1}{2} \omega_q^4 \alpha_{nd}^4$ etc., where $\int_0^{\tau_d} \sin^2 \omega_q dt = \frac{\tau_d}{2}$, $\int_0^{\tau_d} \sin^4 \omega_q dt = \frac{3}{8} \tau_d$, etc. Multiplying (46) by $\langle v_{nd}^2(t) \rangle$ and in turn adding 1 on each side, we obtain $\langle \gamma^2(t) \rangle \langle v_{nd}^2(t) \rangle + 1 = 1 + \frac{\langle v_{nd}^2(t) \rangle}{c^2} + \frac{\langle v_{nd}^4(t) \rangle}{c^4} + \ldots$, or

$$\langle \gamma^2(t) \rangle = 1 + \frac{\langle v_{nd}^2(t) \rangle}{c^2} = 1 + \frac{\langle v_{nd}^2(t) \rangle}{c^2}.$$

The last of Eqs. (48) follows from the equality of the left side of Eqs. (48) with that of the time average of Eq. (45b), $\langle \gamma^2(t) \rangle = 1 + \langle \gamma^2(t) \rangle \langle \omega_q^2(t) \rangle$. With Eqs. (48), $\langle p_{nd}^2(t) \rangle = \langle \gamma^2(t) \rangle M^2 \langle v_{nd}^2(t) \rangle$ is written as

$$\langle p_{nd}^2(t) \rangle = \langle \gamma^2(t) \rangle M \langle v_{nd}^2(t) \rangle.$$

Combining with (49), the root mean of Eqs. (43) is developed as $\hbar \omega_q - V_{nd} = mc^2 = \sqrt{M^2c^4 + \langle \gamma^2(t) \rangle M^2 \langle v_{nd}^2(t) \rangle}$, where

$$m = \sqrt{\langle m^2(t) \rangle} = \gamma M, \quad \gamma = \sqrt{\langle \gamma^2(t) \rangle}, \quad v_{nd} = \sqrt{\langle v_{nd}^2(t) \rangle}.$$

Accordingly the time averages of $\omega_q(t)$, $p_{nd}(t)$, etc. are given by $\omega_q = \sqrt{\langle \omega_q^2(t) \rangle} = \gamma \Omega_q$, $p_{nd} = \sqrt{\langle p_{nd}^2(t) \rangle} = mv_{nd}$, etc. Hence $E_{1q} = \hbar \omega_q$, where $E_{1q}^0 = \hbar \Omega_q$, $E_{nd} = n\hbar \omega_d = \frac{1}{2}mv_{nd}^2 + V_{nd} = \gamma E_{nd}^0$, where $E_{nd}^0 = n\hbar \Omega_d = \frac{1}{2}Mv_{nd}^2 + V_{nd}^0$. 


In sum, the dynamical variables $m, \omega, \rho, q$ etc of the particle in harmonic, hence accelerated motion in the in $X$ direction here relative to the laboratory frame $S$ are each on average augmented by the factor $\gamma$ as measured in $S$. Accordingly, as measured in $S$, the space and time variables $x, \nu_{nd}, A_{nd}$ and $t$ of the particle in the particle’s motion $X$ direction are each contracted by the average factor $\gamma$ from their rest values (indicated by the superscript 0 as elsewhere if not specified otherwise) $x^0$, $\nu_{nd}^0$, $A_{nd}^0$ and $t^0$ as (Lorentz-Einstein transformation)

$$\frac{x}{x^0} = \frac{\nu_{nd}}{\nu_{nd}^0} = \frac{A_{nd}}{A_{nd}^0} = \frac{t}{t^0} = \frac{1}{\gamma}. $$

The space variables $z, \nu_q$, etc. in the transverse direction are particularly here the projections of the longitudinally contracted (by $\gamma$) elastic deformation of the vacuum. These are therefore proportionally contracted by the same factor $\gamma$ from their rest values: $\frac{z}{z^0} = \frac{\nu_q}{\nu_q^0} = \frac{A_0}{A_0^0} = \frac{1}{\gamma}$. Finally, $M_q = \gamma M_q^0, (\beta_d, \beta_q) = \gamma^3 (\beta_d^0, \beta_q^0), (\alpha_d, \alpha_q) = \gamma (\alpha_d^0, \alpha_q^0)$ given as derivative relations based on Eqs. (28b), (32b), (41) and (30) combined with the foregoing transformation relations.

The author thanks Professor C Burdik for kindly inviting the author for presenting a contribution at the XXIst Int Conf on Integrable Systems and Quantum Symmetries in Prague, June, 2013, during which stay the author had very much enjoined extensive scientific discussions with several of the participants. Several Swedish professors have kindly communicated with the author regarding academic funding possibilities for the research. The author’s this research is privately financed by P-I Johansson (emeritus scientist, Uppsala Univ.).

**Appendix A. Radiation damping based on solution to Maxwell equations**

**A.1 Electromagnetic Hamiltonian quantisation and damping** For a self-contained illustration of the subject we derive in this appendix the electromagnetic equations for $\psi_r$ (the radiation Hamiltonian), $\alpha$, $F_r$, and $H_1$ based on solutions to the Maxwell’s equations. The results are formally mostly well known; although, the following solutions will be obtained in terms of the position $V_A$ satisfied condition with the superposed radiation electric field electromagnetic waves are Doppler-differentiated as in Sec. 5.2, we shall be concerned mainly the case where the charged oscillator has also an (instantaneous) linear motion, and thus its radiated case.

Consider the quasi-stationary charged oscillator as specified in Sec. 2.1, oscillating about $n = 0$ at time $t = 0$ along a specified $X_j$ direction, which we set as the $Z$ axis here. In the case where the charged oscillator has also an (instantaneous) linear motion, and thus its radiated electromagnetic waves are Doppler-differentiated as in Sec. 5.2. Similarly as the $E_1, B^1$, the $E, B$ fields are governed by the Maxwell’s equations $\nabla \cdot E = \frac{\partial \rho}{\partial t}, \nabla \times B = -\frac{\partial E}{\partial t}, \nabla \cdot B = 0, \nabla \times E = -\frac{\partial B}{\partial t}$, where $c^2 = \frac{1}{\mu_0} = \frac{1}{\epsilon_0}$; $\rho_q$ and $j_q$ are the charge and current densities of the radiation-emitting charged oscillator. In regions where $\rho_q = j_q \equiv 0$, choosing here the radiation gauge $f = -\int \Phi dt, \Phi$ being the Coulomb potential of $q, \rho_q$ such that $\nabla \cdot A = 0$ and (the vector potential) $A$ will be a transverse field while $E, B$ maintain unaltered, furthermore setting for simplicity $V = 0$, the Maxwell’s equations lead to the wave equations

$$\nabla^2 Y - \frac{1}{c^2} \frac{\partial^2 Y}{\partial t^2} = 0, \quad Y = A, E, B. \tag{A.1}$$

Eqs. (A.1) have the solutions given in the spherical polar coordinates as, under the easily satisfied condition $A/r < < 1$ for the systems of interest here,

$$A_0 = \frac{q A \omega}{4 \pi \epsilon_0 \mu_0 c^2},$$

$$A(r, t) = -\frac{a^{1/2} A_0 r_0 \sin \theta \sin (k \cdot r - \omega t)}{r}, \quad E(r, t) = -\nabla (\Phi + \frac{\partial f}{\partial t}) - \frac{\partial (A - \nabla f)}{\partial t} = -\frac{\partial A}{\partial t} = \frac{a^{1/2} E_0 r_0 \sin \theta \cos \Phi (\hat{r} - t)}{r} \frac{\hat{r}}{A}, \quad \hat{r} = \frac{E_0 r_0 \sin \theta}{A} \frac{\hat{r}}{A}, \quad \hat{r} = \frac{E_0 r_0 \sin \theta}{A} \frac{\hat{r}}{A}. \tag{A.2}$$

The space variables $z, \nu_q$, etc. in the transverse direction are particularly here the projections of the longitudinally contracted (by $\gamma$) elastic deformation of the vacuum. These are therefore proportionally contracted by the same factor $\gamma$ from their rest values: $\frac{z}{z^0} = \frac{\nu_q}{\nu_q^0} = \frac{A_0}{A_0^0} = \frac{1}{\gamma}$. Finally, $M_q = \gamma M_q^0, (\beta_d, \beta_q) = \gamma^3 (\beta_d^0, \beta_q^0), (\alpha_d, \alpha_q) = \gamma (\alpha_d^0, \alpha_q^0)$ given as derivative relations based on Eqs. (28b), (32b), (41) and (30) combined with the foregoing transformation relations.

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Consider the quasi-stationary charged oscillator as specified in Sec. 2.1, oscillating about position $r = 0$ at time $t = 0$ along a specified $X_j$ direction, which we set as the $Z$ axis here. In the case where the charged oscillator has also an (instantaneous) linear motion, and thus its radiated electromagnetic waves are Doppler-differentiated as in Sec. 5.2. Similarly as the $E_1, B^1$, the $E, B$ fields are governed by the Maxwell’s equations $\nabla \cdot E = \frac{\partial \rho}{\partial t}, \nabla \times B = -\frac{\partial E}{\partial t}, \nabla \cdot B = 0, \nabla \times E = -\frac{\partial B}{\partial t}$, where $c^2 = \frac{1}{\mu_0} = \frac{1}{\epsilon_0}$; $\rho_q$ and $j_q$ are the charge and current densities of the radiation-emitting charged oscillator. In regions where $\rho_q = j_q \equiv 0$, choosing here the radiation gauge $f = -\int \Phi dt, \Phi$ being the Coulomb potential of $q, \rho_q$ such that $\nabla \cdot A = 0$ and (the vector potential) $A$ will be a transverse field while $E, B$ maintain unaltered, furthermore setting for simplicity $V = 0$, the Maxwell’s equations lead to the wave equations

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$$A_0 = \frac{q A \omega}{4 \pi \epsilon_0 \mu_0 c^2},$$

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\[ E_0 = A_0 \omega = \frac{qA_0 \omega^2}{4\pi \varepsilon_0 c^2}, \quad E_{r \theta} = \frac{E_0 r_0 \sin \theta}{r}, \]

\[ \Psi_r(r, t) = a^{1/2} A \cos(k \cdot r - \omega t) \hat{\theta}; \quad (A.3) \]

\[ B = \nabla \times (A - \nabla f) = -\frac{E}{c} \hat{\phi}; \quad (A.4) \]

Here, \( k = k \hat{r}, k = \omega/c. \) \( r_0 \) represents an (effective) radius of the oscillator. \( a^{1/2}(t)E_0 \) is the radiation-damped amplitude of a spherical wave front at radius \( r_0 \) and angle \( \theta = \frac{\pi}{2} \) at time \( t \), and \( E_0 \) is the amplitude without damping. \( a \) is to be (separately) determined. For either of the two oscillators of Sec. 5.2 or any other compatible system, \( \omega = \gamma \Omega, A = A^0/\gamma, M = \gamma M^0 \) and \( r = (x^0/\gamma^2 + y^0/\gamma^2 + z^0/\gamma^2)^{1/2} \) are the relativistic values of \( \Omega, A^0, M^0 \) and \( r^0 \) as in Sec. 5.2. The transverse coordinates \( y, z \), representing the projections of longitudinally contracted vacuum for the same (IED) particle here, are contracted similarly as discussed in Sec. 5.3. According to the relations, \( E_0 = \gamma^2 E_0^0 \) and \( E = \gamma^2 E^0 \) in the above, and \( E = \gamma E^0, t_r = t_r^0/\gamma \) and \( L_r = L_r^0/\gamma \) later, are the relativistic values of \( E_0^0, E^0, t_r^0, \) and \( L_r^0 \).

\( \Psi_r(r, t) \) of Eq. (A.3) represents the (quasi harmonic) oscillation displacement of a constituent of the vacuum, a coupled polarised vacuum according to [3], along \( Z \) direction at position \( r \) at time \( t \), generated in response to the perturbation by \( \Psi_r(0, t - \frac{r}{c}) \) of the charged oscillator at \( r = 0 \) and \( t = \frac{r}{c} \). Accordingly, as follows directly from the last relation of Eqs. (A.3a), \( E(r, t) \) is an electromagnetic representation of this vacuum deformation \( \Psi_r(r, t) \). The radiated time-dependent electromagnetic Hamiltonian \( \mathcal{H}_r(r, \theta, t) \) therefore necessarily consists of a kinetic oscillation term \( \mathcal{H}_r, \theta, t \) and in addition, an elastic vacuum potential term \( V_r(r, \theta, t) \). The two terms and hence their sum may be written down by observing both the usual form of electromagnetic energy equation and the underlying dynamics, given for per volume within a volume element \( dV \) about position \( r \) at time \( t \), as

\[ \frac{d\mathcal{H}_{r, \theta}}{dV} = \frac{1}{2} \left[ c_0 E^2(r, \theta, t, -\hat{\phi}) + \frac{1}{\mu_0} B^2(r, \theta, t, -\hat{\phi}) \right] = \frac{1}{2} \left[ \frac{E^2(r, \theta, t)}{\omega^2} + \frac{1}{\mu_0} \frac{B^2(r, \theta, t)}{\omega^2} \right] \]

\[ = c_0 E_{r \theta}^2 \sin^2[k \cdot r - \omega t] = M_{r r}^e \omega^2 E_{r \theta}^2 \sin^2[k \cdot r - \omega t], \quad M_{r r}^e = \frac{c_0}{\omega^2} \]

\[ \frac{dV_{r r}}{dV} = \frac{1}{2} \left[ c_0 E^2(r, \theta, t) + \frac{1}{\mu_0} B^2(r, \theta, t) \right] = c_0 E^2(r, \theta, t) = c_0 E_{r \theta}^2 \cos^2[k \cdot r - \omega t] \]

\[ \frac{d\mathcal{H}_r}{dV} = \frac{d(\mathcal{H}_{r, \theta} + V_r)}{dV} = c_0 E_{r \theta}^2 a_1 \sin^2[k \cdot r - \omega t] + \cos^2[k \cdot r - \omega t] \]

\[ = c_0 E_{r \theta}^2 \sin^2[\theta \cdot \mathcal{E}_r^2(r, t) \frac{r^2}{2}], \quad (A.5a) \]

\[ \mathcal{E}_r^2(r, t) = a^{1/2} E_0 e^{-i(k \cdot r - \omega t)} \hat{\theta} = a^{1/2} E_0 \hat{\psi}_r(r, t) \hat{\theta} = \frac{E_0}{\mathcal{A}} \mathcal{U}_r^e(r, t) \hat{\theta}, \]

\[ \psi_r(r, t) = e^{-i(k \cdot r - \omega t)}, \quad \mathcal{U}_r^e(r, t) = a^{1/2} \mathcal{A} \psi_r(r, t) \quad (A.8) \]

In the above, the first of Eqs. (A.5a) preserves the usual form of electromagnetic energy equation except with the cosine function being shifted by a phase \(-\frac{\pi}{2}\). The second of Eqs. (A.5a) is alternatively and equivalently expressed as the time rate of \( \mathcal{E} \); that is, \( \mathcal{H}_{r, \theta} \) is now proportional to \( \mathcal{E}^2 \) and a corresponding "electromagnetic inertial mass" \( \mathcal{M}_{r r}^e \), by making analogy to the kinetic energy of an oscillator. It is rather the elastic potential term (A.6), which does not present in the usual empty vacuum representation, that formally directly corresponds to the usual (phenomenological) electromagnetic energy equation.

The "complex Poynting vector", defined as the vector intensity of the electromagnetic radiation Hamiltonian \( d\mathcal{H}_r(r, \theta, t) \) passing per unit time per unit area through a differential...
cross-section area \( d\sigma = \frac{dV}{dr} = r^2 \sin \theta d\theta d\phi \) along \( r(\theta, \phi) \) direction, is given as

\[
\Gamma(r, \theta, \phi) = \frac{d\sigma}{dt d\sigma} = \frac{d\sigma}{dV} \frac{\epsilon_0 r_0^2 \sin^2 \theta |E_x(r, t)|^2 c}{r^2} \frac{d}{dt} \\tag{A.9}
\]

The negative time rate of the Hamiltonian of the oscillator, \(-\frac{d\mathcal{E}(t)}{dt}\), is equal to the total radiation power passing a sphere of radius \( r \), i.e.,

\[
-\frac{d\mathcal{E}(t)}{dt} = P(t) \equiv \frac{d\mathcal{E}_r(t)}{dt} = \int_0^\infty \int_0^{2\pi} \mathcal{I}(r, \theta, \phi) r^2 \sin \theta d\theta d\phi = s_0 \epsilon_0 |E_x(r, t)|^2 c \\tag{a}
\]

\[
= \frac{q^2 \omega^4 A_0^2}{6\pi\epsilon_0 c^3} \frac{q^2 |\mathbb{E}_r^c|^2}{6\pi\epsilon_0 c^3} \\tag{b}
\]

\[
= \frac{q^2 \omega^4 A_0^2 a \frac{1}{2} M}{6\pi\epsilon_0 c^3} \frac{q^2 \omega^2 \mathcal{E}^c}{3\pi\epsilon_0 c^3 M} = \alpha \mathcal{E}, \\tag{c}
\]

\[
\alpha = \frac{q^2 \omega^2}{3\pi\epsilon_0 c^3 M} = \frac{q^2 \omega^3 A_0^2}{6\pi\epsilon_0 c^3} = \frac{4q\epsilon_0 E_0}{3\lambda M c} \\tag{A.10}
\]

where \( s_0 = \frac{8\pi}{r_0^2} \) represents the apparent area passed by the spherical wave \( \mathbb{E} \) at \( r = r_0 \). Eqs. (A.10c) are given after multiplying and then dividing the right sides of (A.10.b) by \( \frac{1}{2} M \), and substituting into it by \( \mathcal{E} = \frac{1}{2} M \omega^2 A_0^2 a(t) \) given by (3a). The second expression of Eqs. (A.11) is given by substituting Eq. (11b) for \( M \); and the third by substituting (A.3a) for \( E_0 \). Eqs. (A.10b) and (A.11) give the well-known Larmor formula in complex form here and formula for damping factor. Integrating (A.10c,a) over time \((0, t)\) gives \( \mathcal{E}(t)\big|_0^t = -\alpha t\big|_0^t, -\mathcal{E}(t)\big|_0^t = \mathcal{E}_r(t)\big|_0^t \). Combining with the quantisation results of (11a)–(b) for the same energies \( \mathcal{E}, E \) as here, we obtain

\[
\mathcal{E}(t) \rightarrow \mathcal{E}(t) = E_n e^{-\alpha t} = E_n a(t), \quad E_n = \mathcal{E}(0) = \frac{1}{2} M \omega^2 A_0^2, \quad a(t) = e^{-\alpha t} \\tag{A.12}
\]

\[
\mathcal{E}_r(t) \rightarrow \mathcal{E}_r(t) = E_{r,n}(1 - e^{-\alpha t}) = E_{r,n} a_r(t) \quad E_{r,n} = \mathcal{E}_{r,n}(\infty) = L_r s_0 \epsilon_0 E_0^2, \quad a_r(t) = 1 - e^{-\alpha t},
\]

where \( L_r = \frac{1}{\alpha} c = t_r c = \frac{3\pi\epsilon_0 c^3}{q^2 \omega^2}, \quad t_r = \frac{1}{\alpha}, \quad E_{0n} = \frac{q^2 A_1 \omega^2}{4\pi^2 \epsilon_0 c^2 \omega}. \\tag{A.13}
\]

The equations for \( a, a_r \) above are the same as Eqs. (11c), (15b). \( t_r \) of (A.13) is identical to the mean transition time \( \langle t \rangle = \int_0^\infty \frac{e^{-\alpha t} dt}{\int_0^\infty e^{-\alpha t} dt} = \frac{1}{\alpha} \). So \( L_r \) represents an average length of the radiated electromagnetic wave train.

Specifically for a charge oscillator \( q = +e \) or \(-e\) which oscillates in the vacuum magnetic potential field \( V_q \) and generates an electron or proton based on Eqs. 31–33, Sec. 5, accordingly with the substitutions of \( |\pm e|, \mathbb{M}_q = \frac{2\hbar}{A_1^2 \omega_q}, \omega_q = \frac{mc^2}{\hbar} \) for \( q, \mathbb{M}, \omega \), Eqs. (A.11), (A.13b), (a) and (c) are written as

\[
\alpha_q = \frac{e^2 \omega_q^2}{3\pi\epsilon_0 c^3 \mathbb{M}_q} = \frac{e^2 (\frac{mc^2}{\hbar})^2}{3\pi\epsilon_0 c^3 \left(\frac{2\hbar}{A_1^2 \omega_q}\right)} = \frac{e^2 m^3 c^2 A_1^2}{6\pi\epsilon_0 \hbar^4}, \quad t_{r,q} = \frac{1}{\alpha_q}, \quad L_{r,q} = ct_{r,q} = \frac{6\pi\epsilon_0 \hbar^4}{e^2 m^3 c^2 A_1^2}, \\tag{A.14}
\]

\[
E_{01_q} = \frac{e A_{1q} \omega_q^2}{4\pi\epsilon_0 r_0 c^2} = \frac{e A_{1q} m^2 c^2}{4\pi\epsilon_0 r_0 \hbar^2}
\]

Calculated values of \( \alpha_q, t_{r,q}, L_{r,q} \) and \( E_{01_q} \) based on Eqs. (A.14) for the charge oscillators \(-e\) and \(+e\), which generate an IED electron and proton, are given in Table A1, where the \( \mathbb{M}_q \) and
The oscillator has a potential energy \( V \) amounting to a negative source-radiation interaction potential, \( F(\omega) \), where \( \omega \) lies in the region \( [\pi/2, \pi] \). Values from \([3b]\), with the free parameter \( f \) set to 1.

For obtaining the last of Eqs. (A.15) we used (A.11) and (A.14b), where we have set \( r_0 = A_{1q}; \) clearly \( A_{1q} \sim \frac{\bar{q}}{\hbar}, \) \( b_q \sim 1 \times 10^{-18} \) m being the inter-vacuum distance estimated based on experiment \([3b]\).

\( \omega_q(= \frac{m \omega^2}{\hbar}) \) values (second and third columns in the table) are as input data. In comparison, for an electron oscillator described by Eqs. (28)–(29), Sec. 5, with an oscillation frequency \( \omega_d = 2\pi \cdot 10^{14} \) s, corresponding evaluations give \( t_{r,e} = 2.02 \times 10^{-7} \) s, \( L_{r,e} = 60.6 \); and for a proton oscillator with the same oscillation frequency, \( t_{r,p} = 3.71 \times 10^{-7} \) s, \( L_{r,p} = 1.11 \times 10^5 \) m. Common with all of the examples, the condition \( \alpha << \omega \) is well satisfied.

### A.2 Radiation–charge electromagnetic interaction force and work

In the mechanical representation of Sec. 2.1 the radiation damping force \( F_r \) acts on a charged oscillator through the deformation of a viscous elastic vacuum medium, \( \mathcal{U}_r, \) and \( \mathcal{X} \) whose electromagnetic counterpart is \( \mathcal{E} \) (Eq. A.3). We shall below derive the electromagnetic counterpart of the \( F_r, \) \( F_{em} \). Consider that a charged oscillator \( \mu \) of charge \( q \) is emitting radiation of radiation electric field \( \mathcal{E}(r, \theta, t) \) at position \( r \) at time \( t \); our specific attention is the \( r \) lying in the region \([-r_0, r_0]\) occupied by the oscillator here. \( \mu \) must inevitably in turn be acted by \( \mathcal{E} \) a Coulomb force given for per unit length along the X axis and per unit cross-sectional area as, with (A.2) for \( \mathcal{E} \),

\[
F_e(x, t) = -q\mathcal{E}(r, \frac{\pi}{2}, t) = -\alpha^{1/2}q^2\mathcal{A}\omega^2\cos[kx - wt]\frac{\zeta}{4\pi\epsilon_0rc^2}
\]

\[
= -M\left(\frac{q^2\omega^2}{3\pi\epsilon_0c^3M}\right)\frac{3\alpha^{1/2}2\mathcal{A}\cos[\omega(\frac{\pi}{2} - t)]\bar{\zeta}}{4r} = -3\alpha\mathcal{M}\mathcal{W}(x, \frac{\pi}{2}, t) \tag{A.15}
\]

For obtaining the last of Eqs. (A.15) we used (A.11) and \( \mathcal{U}(t) \) for \( \mathcal{W}(x, \frac{\pi}{2}, t) = a^{1/2}A\cos[\omega(\frac{\pi}{2} - t)] \).

Substituting into the last of Eqs. (A.15) the identity relation \( \mathcal{W}(t) = \frac{d\mathcal{U}(t)}{\omega dt} \) (where \( t' = t - \frac{\pi}{2\omega} \)), and \( r = r_0 \) and \( \omega = 2\pi c/\lambda \), we obtain

\[
\frac{s_0}{r_0\lambda}F_e = -\alpha\mathcal{M}\frac{d\mathcal{W}}{dt} \tag{A.16}
\]

where \( s_0 = \frac{8\pi r_0^2}{\lambda} \) as before. The right side of (A.16) is just the expression for \( F_r(= -\alpha\mathcal{M}\frac{d\mathcal{U}}{dt}) \) of Sec. 2.1, hence

\[
F_{em}(= F_r) = \frac{s_0F_e}{r_0\lambda} = \frac{s_0q\mathcal{E}}{r_0\lambda} = -\frac{s_0q\omega\mathcal{A}}{r_0\lambda} \tag{A.17}
\]

We shall next derive the source-radiation interaction potential \( (V_f) \) from the work done by \( F_r \) or \( F_{em} \). \( F_{em} \) does to the oscillator \( \mu \) a dissipative work \( W_f \); and \( F_e \), hence \( F_{em} \) or \( F_r \), is a conservative force (i.e., it depends on the position of the charge in the \( \mathcal{E} \) field only). So \( W_f \) amounts to a negative source-radiation interaction potential, \( V_f \). Suppose that at time \( t = 0 \), the oscillator has a potential energy \( V(t = 0) = V_0 = \frac{1}{2}M\omega^2\mathcal{W}^2 \) (\( V(t) \) is a short hand notion of \( V(\mathcal{W}(t)) \) used in this and the next sub-section), with \( \mathcal{W} = \frac{\mathcal{U}}{a^{1/2}} \), which is undamped. The
corresponding work done during one oscillation cycle $2\pi \tau / \tau = 2\pi \int_0^t \mathrm{d}t$, so is given as, with $d\mathcal{W} / dt = -\omega \mathcal{W} (x, t - \pi / 2\omega)$,

$$\Delta V_i(0) = -4 \times \frac{2\pi (\tau / 4)}{\tau} \int_{\mathcal{W}(0)=0}^{\mathcal{W}(0)=A_n} F_r(t')d\mathcal{W}(t') = 2\pi \int_{\mathcal{W}(0)=0}^{\mathcal{W}(0)=A_n} M\alpha \omega \mathcal{W}(t')d\mathcal{W}(t')$$

$$\Delta V_i(0) = -\left(\frac{1}{2}M\omega^2\mathcal{W}^2(0)\right)2\pi \frac{\alpha \tau}{2\pi} = -V_0 \alpha \tau \hat{\pi} - V_0 (1 - e^{-\alpha \tau})$$  \hspace{1cm} (A.18)

The last equality holds for $\alpha \tau << 1$, a condition characteristic with the quasi harmonic oscillation here. The total potential at time $t = \tau$ thus is, $V(\tau) = V(0) + \Delta V_i(\tau) = V_0 - V_0 (1 - e^{-\alpha \tau}) = V_0 e^{-\alpha \tau}$. Similarly, for $\tau \leq t' \leq 2\tau$, $\Delta V_i(2\tau) = -V(\tau)(1 - e^{-\alpha \tau})$, $V(2\tau) = V(\tau) + \Delta V_i(\tau) = V(\tau)e^{-\alpha \tau} = (V_0 e^{-\alpha \tau})e^{-\alpha \tau} = V_0 e^{-2\alpha \tau}$; and so forth. Finally, for $t - \tau \leq t' \leq t$, $\Delta V_i(\tau) = -V(t - \tau)(1 - e^{-\alpha \tau})$,

$$V(t) = V(t - \tau) + \Delta V_i(t) = (V_0 e^{-\alpha (t - \tau)})e^{-\alpha \tau} = V_0 e^{-\alpha t} = V_0 + V_i(t)$$

$$H_i = V_i(t) = -V_0 (1 - e^{-\alpha \tau}) = -\frac{1}{2}M\omega^2x_j^2(1 - e^{-\alpha \tau}) = -\hbar \omega \cos^2(\omega t)(1 - e^{-\alpha t})$$  \hspace{1cm} (A.19)

Eqs. (A.19) and $H_0 = H - H_i$ are the same as Eqs. (7.b), (8c), and (8a-b), Sec. 2.2.

A.3 Radiation-charge interaction Hamiltonian expressed in $A$ Suppose that due to the action of the $V_j(t)$ above, a photon is emitted by its source during a particular transition time $t_r = 2\pi / \omega$, and we want to express $H_i = V_i$ using $A$. For $\alpha t_r = \alpha \tau << 1$, Eq. (A.19b) reduces to

$$H_i = V_i(\tau) = -\hbar \omega \cos^2(\omega \tau)(1 - (1 - \alpha \tau \frac{1}{2}\alpha^{2\tau^2} - \ldots)) \hat{\pi} - 2\pi \alpha$$

or \hspace{1cm} $H_i = -2\pi \hbar \left(\frac{4qroA \cdot (vk)}{3Amc}\right) - \frac{\hbar q}{\omega} \left(\frac{8\pi r_0}{3AQ}\right) A$  \hspace{1cm} (A.20)

The first of Eqs. (A.20) is given after substituting (A.11) for $\alpha$, $E_0 = A_0 \omega \approx A_\omega$ from (A.3a) and $\omega = vk$; and the second $k = i\nabla x_j$. If the radiation is emitted by the charge oscillator $q$ along the $Z$ direction as specified in Sec. 5, substituting the corresponding variables $z$, $2Mq$, $A_q$, $\omega_q = kc$ and $c$ for $x_j$, $M$, $A$, $\omega$ and $v$, (A.20) is written as

$$H_i = \frac{i\hbar q}{2A_q} A \cdot \nabla_z = \frac{i\hbar q}{2A_q} A \cdot \nabla_z = \frac{i\hbar q}{2A_q} \left(\frac{8\pi r_0}{3A_q}\right) A$$  \hspace{1cm} (A.21)

If alternatively the radiation is emitted by the charged particle oscillator $d$ along the $X$ direction, substituting accordingly $x$, $m$, $A_d$, $\omega_d = \frac{1}{2}v_{n_d}k_d$, $\frac{1}{2}v_{n_d}$, and $k_d$ for $x_j$, $M$, $A$, $\omega$, $v$, and $k$, (A.20) is written as

$$H_i = \frac{i\hbar q}{m} \left(\frac{8\pi r_0}{3A_q}\right) \frac{\nabla_x}{m} = \frac{i\hbar q}{m} \left(\frac{8\pi r_0}{3A_q}\right) \frac{\nabla_x}{m},$$

$$A'' = P_r \times A, \hspace{1cm} P_r = \frac{8\pi r_0}{6(A_d/\nu_{n_d)}} \hat{y}$$  \hspace{1cm} (A.22)

The electromagnetic radiation fields, the vector field $A$ in the above, emitted (or absorbed) by either the oscillator $q$ or $d$ is oriented along the transverse $Z$ direction and propagated in the $X$ direction. So when expressed into the dot product form in the second to third of Eqs. (A.22), an apparent $A''$ is involved given after the projection of $A$ by $P_r$.  

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The same result (A.22) for a charged particle oscillator \( d \) is in the usual derivation given by considering the effect of \( \mathbf{A} \) and \( \Phi \) on the particle’s linear momentum. Assume that \( \mathbf{A} \) is along the transverse \( Z \) direction as in the above and is projected on to the \( X \) direction into \( \mathbf{A}'' \) according to Eq. (A.22). Under the action of \( \mathbf{A}'' \), the linear momentum of the charged oscillator is reduced from \( p_d \) by an amount \( \Delta p_d = -q\mathbf{A}'' \) to \( p'_d = p_d + \Delta p_d = p_d - q\mathbf{A}'' \). The Hamiltonian operator \( H_d \), subtracted by \( V_{d0} \), is thus given as, with the substitutions \( p_d \cdot (q\mathbf{A}'') = \mathbf{A}'' \cdot p_d - \hbar \nabla \cdot \mathbf{A}'' \) and \( p_d = \frac{\hbar}{i} \nabla \),

\[
H_d - V_{d0} = \frac{1}{2m}(p_d - q\mathbf{A}'')^2 + q\Phi = \frac{1}{2m}[p_d \cdot p_d - 2q\mathbf{A}'' \cdot p_d + i\hbar \nabla \cdot \mathbf{A}'' + q^2\mathbf{A}''^2] + q\Phi
\]

The last of (A.23) is given by considering the situation that \( \nabla \cdot \mathbf{A}'' = 0 \) for \( \mathbf{A}'' \) is essentially of uniform amplitude along the radiation path; \( \frac{q^2\mathbf{A}''^2}{2m} \) is a higher order term and thus omitted; and the static \( \Phi \) field produced by the charge \( q \) does not act on \( q \) itself, thus \( q\Phi = 0 \). The remaining last term in the last of Eqs. (A.23) gives just the \( H_d \).

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