INTERPOLATING SEQUENCES FOR THE BERGMAN SPACE AND THE $\bar{\partial}$-EQUATION IN WEIGHTED $L^p$  

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Abstract. The author showed that a sequence $Z$ in the unit disk is a zero sequence for the Bergman space $A^p$ if and only if the weighted space $L^p(e^{pkZ} \, dA)$ contains a non-zero (equivalently, zero-free) analytic function, where

$$k_Z(z) = \sum_{a \in Z} \frac{(1 - |a|^2)^2 |z|^2}{|1 - az|^2}.$$

Here we show that $Z$ is an interpolating sequence for $A^p$ if and only if it is separated in the hyperbolic metric and the $\bar{\partial}$-equation

$$(1 - |z|^2) \bar{\partial} u = f$$

has a solution $u$ satisfying $\|u\|_{p,Z} \leq C\|f\|_{p,Z}$ for every $f \in L^p(e^{pkZ} \, dA)$, where $\|f\|_{p,Z}$ denotes the $L^p(e^{pkZ})$ norm. This holds for all finite $p \geq 1$, but also for $p < 1$ if $L^p(e^{pkZ} \, dA)$ is suitably modified.

In addition, we show how this relates to a recent criterion for weighted $\bar{\partial}$ estimates by J. Ortega-Cerdà, and to K. Seip's criterion for interpolation.

1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{ z : |z| < 1 \}$ in the complex plane $\mathbb{C}$. For $0 < p < \infty$, let $L^p$ denote the usual Lebesgue space consisting of those measurable functions $f$ on $\mathbb{D}$ such that the norm $\|f\|_p$ defined by $\|f\|_p^p = \int |f|^p \, dA$ is finite. Here $dA$ denotes area measure. Let $A^p$ denote the subspace of $L^p$ consisting of analytic functions.

If $f$ is analytic in $\mathbb{D}$ and not identically zero, let $Z(f)$ denote its zero sequence. This means that $Z(f)$ consists of a listing (in some order) of all the points where $f$ has a zero, each such point being repeated as often as the multiplicity of the zero. We will view $Z(f)$ as a set with multiplicity: an equivalence class of such sequences, where two sequences are equivalent if one is a reordering of the other. We write $Z(f) \subset Z(g)$ if some representative of $Z(f)$ is a subsequence of a representative of $Z(g)$.

A sequence $Z = (a_n, n \in \mathbb{N})$ in $\mathbb{D}$ is an $A^p$ zero sequence if there is a non-zero function $f \in A^p$ such that $Z(f) = Z$, where again we view $Z$ as a set with multiplicity. Necessarily, $Z$ cannot have limit points in $\mathbb{D}$ if it is a zero sequence.

We associate to a sequence $Z$ in $\mathbb{D}$ the normed sequence space $l^p(Z)$ consisting of all sequences $c = (c_a, a \in Z)$ such that the norm $\|c\|_{p,Z}$ defined by $\|c\|_{p,Z}^p = \sum_{a \in Z} |c_a|^p(1 - |a|^2)^2$ is finite (this is a quasi-normed if $p < 1$). The sequence $Z$ is called an interpolating sequence for $A^p$ if the operator $f \mapsto (f(a) : a \in Z)$ takes $A^p$ continuously onto $l^p(Z)$. An interpolating sequence is necessarily a zero sequence for

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A^p, without repetition. A criterion for Z to be interpolating for A^p was provided by K. Seip in [7], and extended by A. Schuster in [9].

If Z is a sequence satisfying \( \sum_{a \in Z} (1 - |a|^2)^2 < \infty \), define the function \( k_Z \) by

\[
 k_Z(z) = \frac{|z|^2}{2} \sum_{a \in Z} \frac{(1 - |a_n|^2)^2}{|1 - a_n z|^2}. 
\]

In [8] it was shown that Z is an A^p zero sequence if and only if there exists a zero-free function \( f \), analytic in the unit disk, such that \( fe^{\bar{z}} \) belongs to \( L^p \). And this is equivalent to there existing a non-zero function with this property. Let \( L^p_Z \) denote the space of all measurable functions \( f \) such that \( fe^{\bar{z}} \) lies in \( L^p \), let \( \|f\|_{p,Z} \) be the \( L^p \) norm of this product, and let \( A^p_Z \) denote the analytic functions in \( L^p_Z \). The main result of [8], then, is that Z is an A^p zero sequence if and only if \( L^p_Z \) contains a non-trivial analytic function, and so \( A^p_Z \) is non-trivial.

The precise result shown in [8] was that if \( \Psi_Z \) denotes the product

\[
 \Psi_Z(z) = \prod \frac{a_n - z}{1 - a_n z} \exp \left( 1 - \frac{a_n - z}{1 - a_n z} \right),
\]

then Z is an A^p zero set if and only if division by \( \Psi_Z \) is an isomorphism from the subspace \( L^p_Z = \{ f \in A^p : Z(f) \subseteq Z \} \) onto \( A^p_Z \). (This is strictly correct only if \( 0 \notin Z \), but a correct version is easily supplied in that case.)

The starting point of our investigations was to try to relate properties of the sequence Z to properties of \( k_Z \) and in particular to properties of \( A^p_Z \) and \( L^p_Z \). Since Z being a zero sequence for \( A^p \) is equivalent to the property that \( L^p_Z \) contains non-trivial analytic functions, one might ask what further properties of \( L^p_Z \) are equivalent to Z being an interpolating sequence. It is natural to conjecture that Z is an interpolating sequence if and only if there are certain \( L^p_Z \) bounds on the solution of the \( \bar{\partial} \)-equation. Part of the reason this is natural is because one direction is rather routine to prove (\( \bar{\partial} \)-estimates imply interpolation).

Let \( w \) denote a weight function on \( \mathbb{D} \), that is, \( w \) is measurable, not necessarily integrable, and \( w(z) > 0 \) for all \( z \in \mathbb{D} \). Let \( L^p(w) \) denote the weighted space of measurable functions with norm \( \|f\|_{p,w} = \int |f|^p w \, dA < \infty \). The \( \bar{\partial} \) and \( \partial \) operators are defined by

\[
 \bar{\partial} u = \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} - \frac{1}{i} \frac{\partial u}{\partial y}, \\
 \partial u = \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} + \frac{1}{i} \frac{\partial u}{\partial y}.
\]

We will consider the equation

(1.1) \( (1 - |z|^2) \bar{\partial} u = f \)

and try to determine, for certain weights \( w \), when there exists a constant \( C \) such that it can always be solved with \( \|u\|_{p,w} \leq C \|f\|_{p,w} \). The above mentioned conjecture deals with \( L^p_Z \), which is just the case \( w = e^{\bar{z}k} \). The main result of this article is the following validation of that conjecture.

**Theorem A.** A sequence Z in the unit disk is interpolating for A^p, p ≥ 1, if and only if it is separated in the hyperbolic metric and there is a constant C such that for every \( f \in L^p_Z \), there is a solution of equation (1.1) satisfying \( \|u\|_{p,Z} \leq C \|f\|_{p,Z} \).
In section 2 we present the background on $k_Z$ and other topics, and include some of the basic lemmas needed for proving the main results. In section 3 we prove one direction of the main result, that the existence of solutions of the $\bar{\partial}$-equation with the given bounds implies interpolation. Then, in section 4, we prove the converse. In section 5 we show how the main result may be modified to provide a version that holds for $p < 1$. All this relies only on simple properties of interpolating sequences and rather easy properties of the connection between zero sequences and $k_Z$ set out in [3] and [4]. In fact, this author believes the result itself is not as important as the variety of techniques used in proving it.

In the final two sections we explore the connection between K. Seip’s criterion for interpolation (a certain uniform upper bound on the density of the sequence $Z$) and a recent criterion by J. Ortega-Cerdà for the solution of (1.1) in weighted $L^p$ with certain types of weights. In section 6 we show that Seip’s criterion translates to an upper bound on certain averages of $\partial k_Z$ over disks of a fixed pseudo-hyperbolic radius. In section 7, we observe that Ortega-Cerdà’s criterion (for interpolation) and this form of Seip’s criterion (for interpolation) are remarkably similar. While Ortega-Cerdà’s criterion is stated in a context that, on the face of it, excludes $e^{k_Z}$ as the weight, we can show that the weight is equivalent one that does satisfy Ortega-Cerdà’s criterion. This produces an alternate proof of the main result, showing that Seip’s criterion also implies solutions of the $\bar{\partial}$ equation with bounds in $L^p_Z$.

Many of the auxiliary results we will prove are already known for $A^p$, and in some cases much stronger results are known. However, we will use the weaker versions and provide proofs because, (i) the proofs are new; (ii) they rely only on quite simple properties of zero/interpolating sequences; and (iii) they extend to weighted Bergman spaces where the stronger results do not. In fact, the above Theorem remains valid (appropriately modified) for all the usual weighted spaces $A^{p,\alpha}$, $\alpha > -1$. We will always try to make it clear when we are reproving a known result and to state what settings our proof extends to. In most cases the proof will extend almost without change to the weighted spaces.

I would like to gratefully acknowledge the help of J. Ortega-Cerdà, who showed me the methods of section 7 after reading a preliminary version of this paper.

2. Background and auxiliary results

Let $\phi(z)$ denote a real-valued function on $\mathbb{D}$ and denote by $L^p_\phi$ the space $L^p(e^{p\phi})$, that is

$$L^p_\phi = \left\{ f : \int_{\mathbb{D}} |f(z)e^{\phi(z)}|^p \, dA(z) < \infty \right\}.$$ 

The norm $\|f\|_{p,\phi}$ will denote the $p$-th root of the integral in the above definition. We will always assume that $e^{p\phi}$ is locally integrable, so that all continuous functions with compact support in $\mathbb{D}$ belong to $L^p_\phi$. The space $L^p_Z$ is just $L^p_\phi$ with $\phi(z) = k_Z(z)$.

We will say that we can solve the $\bar{\partial}$-equation with bounds in $L^p_\phi$ if there is a constant $C > 0$ such that equation (1.1) has, for all $f \in L^p_\phi$, a solution $u \in L^p_\phi$ satisfying $\|u\|_{p,\phi} \leq C\|f\|_{p,\phi}$. We understand the equation to hold either in the sense of distributions, or in the sense that it holds classically for all $f$ which are continuous with compact support. The hypotheses on $\phi$ make such functions dense in $L^p_\phi$. 

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Recall that the pseudo-hyperbolic metric \( \psi \) is defined by
\[
\psi(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|
\]
For \( a \in \mathbb{D} \) we denote by \( M_a \) the Möbius transformation
\[
M_a(z) = \frac{a - z}{1 - \overline{a}z}.
\]
Of course, the metric \( \psi \) is invariant under Möbius transformations, and is locally equivalent to the Euclidean metric in \( \mathbb{D} \).
Define \( D(z, r) \) to be the ball in the metric \( \psi \) with center \( z \) and radius \( r \). Note that \( D(z, r) \) is the Euclidean disk centered at \( z(1 - r^2)/(1 - r^2|z|^2) \) with radius \( r(1 - |z|^2)/(1 - r^2|z|^2) \).
We will have need of the following lemma which provides a test, often called the Schur criterion, for the boundedness in \( L^p \) of integral operators.

**Lemma 2.1.** (Schur criterion) Let \( K(z, w) \) be a measurable kernel on measure spaces \((Z, \nu) \times (W, \mu)\) and let \( p > 1 \). If there exist measurable functions \( h_1(w) > 0 \) and \( h_2(z) > 0 \), and constants \( C_1 \) nd \( C_2 \) such that
\[
(2.1) \quad \int |K(z, w)|h_1(w)^{p'} \, d\mu(w) \leq C_1 h_2(z)^{p'} \quad w \in W
\]
and
\[
(2.2) \quad \int |K(z, w)|h_2(z)^p \, d\nu(z) \leq C_2 h_1(z)^p \quad z \in Z
\]
where \( p' = p/(p - 1) \), then the operator \( T \) defined by \( Tf(z) = \int K(z, w)f(w) \, d\mu(w) \) is bounded from \( L^p(W, \mu) \) to \( L^p(Z, \nu) \) and \( \|T\| \leq C_1^{1/p'} C_2^{1/p} \). Moreover, \( T \) will be bounded from \( L^1(W, \mu) \) to \( L^1(Z, \nu) \), with \( \|T\| \leq C \) if
\[
\sup_{w \in W} \int |K(z, w)| \, d\nu(z) \leq C.
\]
What we will mostly need is the following consequence.

**Lemma 2.2.** If \( p \geq 1 \), the kernels
\[
K(z, w) = \frac{(1 - |z|^2)^a(1 - |w|^2)^b}{|1 - \overline{w}z|^{a+b+2}}
\]
\[
B(z, w) = \frac{(1 - |z|^2)^a(1 - |w|^2)^b}{|z - w| |1 - \overline{w}z|^{a+b+1}}
\]
define bounded operators on \( L^p(dA) \) \( (\text{integration with respect to } w) \) provided \( a > -1/p \) and \( b > -1/p' \). In case \( p = 1 \) the condition on \( b \) means \( b > 0 \).

**Proof.** It is well known (see for example [10]) that if \(-1 < \beta < M\) then
\[
(2.3) \quad \int \frac{(1 - |w|^2)^\beta}{|1 - \overline{w}z|^{M+2}} \, dA(w) \leq C(1 - |z|^2)^{\beta-M}.
\]
We observe that there is a similar estimate
\[
(2.4) \quad \int \frac{(1 - |w|^2)^\beta}{|z - w| |1 - \overline{w}z|^{M+1}} \, dA(w) \leq C(1 - |z|^2)^{\beta-M}.
\]
This can be seen by writing the integral as the sum of two integrals, one over the set where
\(|\psi(z, w)| \geq \eta\) (for some 0 < \(\eta < 1\)) and one over the complement. The first
integral can be estimated as at most \(1/\eta\) times the integral in (2.3). In the second
integral we can use the fact that, for \(w \in D(z, \eta)\), we have
\[
(1 - |z|^2)/C_\eta \leq 1 - |w|^2 \leq C_\eta(1 - |z|^2) \quad \text{and} \quad |1 - \bar{w}z| > (1 - |z|^2)/C_\eta
\]
for some constant \(C_\eta\) depending only on \(\eta\). This gives
\[
\int_{D(z, \eta)} \frac{(1 - |w|^2)^\beta}{|z - w| |1 - \bar{w}z|^{M+1}} dA(w) \leq \left[ C_\eta(1 - |z|^2)\right]^{\beta-M-1} \int_{D(z, \eta)} \frac{1}{|z - w|} dA(w).
\]
Finally, the integral on the right side above is equivalent to the Euclidean radius of
\(D(z, \eta)\), which is bounded by \(C_\eta(1 - |z|^2)\).

Let \(p > 1\). In Lemma 2.1, let \(h_1(z) = (1 - |z|^2)^{-\alpha}\) and let \(h_2 = h_1\). Letting
\(M = a + b\), apply inequality (2.3) or (2.4) twice, once with \(\beta = b - p'\alpha\) and once with
\(\beta = a - p\alpha\), then both (2.1) and (2.2) will be satisfied if we can find \(\alpha\) satisfying the
following simultaneous inequalities.
\[
-1 < b - \alpha p' < a + b \\
-1 < a - \alpha p < a + b
\]
These are equivalent to
\[
-a/p' < \alpha < (1 + b)/p' \\
-b/p < \alpha < (1 + a)/p
\]
The existence of \(\alpha\) satisfying all four inequalities is equivalent to
\[
\max\{-a/p', -b/p\} < \min\{(1 + b)/p', (1 + a)/p\}.
\]
This gives four inequalities, two of which are equivalent to each other and implied by
the other two, reducing the requirement to
\[
-a/p' < 1/p + a/p \\
-b/p < 1/p' + b/p'
\]
which are equivalent to the two requirements in the statement of this lemma.

The \(p = 1\) case is immediate from (2.3) or (2.4) with \(\beta = b\) and \(M = a + b\). \(\square\) \(\square\)

We need some estimates for the maximum size of an analytic function at the origin,
given a bound on its norm plus the condition that it vanish exactly on an interpolating
sequence \(Z\).

A sequence is said to be separated in the pseudo-hyperbolic metric if there is a
positive lower bound on \(\psi(z, w)\) for \(z \neq w\) in \(Z\). The phrase uniformly discrete is
also used with the same meaning. The largest lower bound will be referred to as the
separation constant.

Now if \(Z\) is an interpolating sequence then it is a zero sequence for \(A^p\) and it is also
separated in the pseudo-hyperbolic metric. Either condition forces \(\sum_{a \in Z} (1 - |a|^2)^2 < \infty\). Define for \(a \in \mathbb{D}\),
\[
E_a(z) = \begin{cases} 
\frac{\bar{a} - a - z}{|a| (1 - \bar{a}z)} \exp \left[ 1 - \bar{a} \frac{a - z}{1 - \bar{a}z} - (1 - |a|^2)/2 \right] & a \neq 0 \\
\frac{z e^{1/2}}{1} & a = 0
\end{cases}
\]
and then let $\Psi_Z(z) = \prod_{a \in Z} E_a(z)$. This is almost the same function as in [3], but it has been normalized by dividing each factor by $|a| \exp[(1 - |a|^2)/2]$, and an oversight corrected so that $0 \in Z$ is properly accounted for. This function $\Psi_Z$ is analytic in $\mathbb{D}$ and vanishes only on $Z$. The odd form of the factor for zeros at the origin is for consistency with the other factors and also so that (2.8) below holds.

Now let
\begin{equation}
(2.6) \quad \sigma_Z(z) = \prod_{a \in Z} \left| \frac{a - z}{1 - az} \right| \exp \left[ \frac{1}{2} \left( 1 - \frac{|a - z|^2}{1 - az} \right) \right].
\end{equation}

Here no special provision needs to be made for $a = 0 \in Z$. What was shown in [3] is that there is a constant $C_p$ depending only on $p$ such that if $f \in A^p$ vanishes on $Z$ then
\begin{equation}
(2.7) \quad \|f/\sigma_Z\|_p \leq C_p \|f\|_p.
\end{equation}

We remark here the rather surprising fact that $C_p$ does not depend on $Z$. This follows from the proof in [3], although that paper does not mention it. Since
\begin{equation}
(2.8) \quad |\Psi_Z| = \sigma_Z e^{kZ},
\end{equation}
it follows that division by $\Psi_Z$ takes the subspace $I_Z^p$ of functions in $A^p$ that vanish on $Z$ into $A_Z^p$ boundedly, with a bound independent of $Z$. Moreover, since $\sigma_Z \leq 1$, it also follows that multiplication by $\Psi_Z$ is a bounded from $A_Z^p$ onto $I_Z^p$, with norm at most 1. These results are actually the easiest part of [3], requiring only an integration of Jensen’s formula followed by the inequality of geometric and arithmetic means, followed by the Schur criterion (Lemma 2.1 above).

Now observe the simple fact that if $W \subset Z$ (taking multiplicities into account) then $A_Z^p \subset A_W^p$, with the inclusion map bounded by 1 in norm. This is because the weights $\exp kZ$ increase with $Z$. The composition of these three operators: division by $\Psi_Z$, this inclusion, and multiplication by $\Psi_W$ is therefore a bounded map from $I_Z^p$ into $I_W^p$, with a bound that depends only on $p$. This operator is obviously the same as division by $\Psi_{Z\setminus W}$. This makes the following lemma almost obvious.

**Lemma 2.3.** If $f \in A^p$ satisfies $\|f\|_p = 1$, $|f(0)| \geq \delta$ and $W \subset Z(f)$, then there is a function $g \in A^p$ such that $W = Z(g)$, satisfying $\|g\|_p = 1$ and $|g(0)| > \delta/C_p$, where $C_p$ is a positive constant depending only on $p$.

We note that this lemma actually holds with $C_p = 1$ on $A^p$ by [1]. While our result is weaker, it suffices for our needs. Moreover, our result extends without change to the weighted case $A^{p,\alpha}$ with a constant $C_{p,\alpha}$ depending only on $p$ and $\alpha$ and in fact to all the spaces covered in [3], with the constant depending only on the space.

**Proof.** Apply the preceding discussion with $Z$ being equal to the zero set of $f$, producing $h = f/\Psi_Z$ where $Z' = Z \setminus W$. Then $\|h\|_p \leq C_p$ with a constant $C_p$ depending only on $p$. Now
\[
\log |\Psi_{Z'}(0)| = \sum_{a \in Z'} \log |a| + \frac{1}{2} (1 - |a|^2)
\]
is easily seen to be negative, so $|\Psi_{Z'}(0)| < 1$. Therefore $|h(0)| > |f(0)| > \delta$ and so $g = h/\|h\|_p$ has the required properties. \(\square\)
The interpolation constant for an interpolating sequence \( Z \) is the least number \( M \) such that if \( (c_a : a \in Z) \) is a sequence in the unit ball of \( l_\infty^p \) then there exists a function \( f \in A^p \) such that \( f(a) = c_a \) for all \( a \in Z \) and \( \|f\|_p \leq M \). The existence of \( M \) is a consequence of the open mapping theorem since the operation of evaluation on \( Z \) is bounded from \( A^p \) onto \( l_\infty^p \).

The following seems to be well-known, but we include a proof for completeness, and because it is more elementary than some proofs that the author has seen, and because a published proof was hard to find.

**Lemma 2.4.** Let \( Z \) be an interpolation sequence for \( A^p \) with interpolation constant \( M \). Let \( 0 < \eta < 1 \), and let \( a_0 \) be a point of \( \mathbb{D} \) satisfying \( \psi(a_0, a) > \eta \) for all \( a \in Z \). Then \( Z \cup \{a_0\} \) is also an interpolating sequence for \( A^p \) with interpolation constant less than \( C/\eta \), where \( C \) depends only on \( M \) and \( p \).

**Proof.** Let \( Z \) be interpolating and let \( Z' = Z \cup \{a_0\} \). Let \( c \in l_\infty^p \) with \( \|c\|_{p, Z'} = 1 \). We seek a function \( f \) in \( A^p \) with \( f(a) = c_a \) for all \( a \in Z' \), together with an estimate of its norm. Since a Möbius transform of an interpolating sequence is also interpolating with the same interpolation constant we may assume \( a_0 = 0 \) and therefore \( |a| > \eta \) for all \( a \in Z \). First obtain a function \( g \in A^p \) with \( g(a) = (c_a - c_0)/a \) for \( a \in Z \). The norm of \( ((c_a - c_0) : a \in Z') \) is easily estimated in terms \( p \) and \( \sum (1 - |a|^2)^2 \), and the latter depends only on the separation constant, which can be seen to depend only on \( M \) and \( p \). Therefore, the norm of \( ((c_a - c_0)/a) \) is at most \( C'/\eta \), and so such a \( g \) can be found with \( \|g\|_p \leq MC'/\eta \). Then let \( f(z) = zg(z) + c_0 \) to get the required interpolation with norm at most \( MC'/\eta + 1 \leq C/\eta \).

Combining this lemma with the previous one we obtain the result we will later need.

**Corollary 2.5.** Let \( Z \) be an interpolating sequence for \( A^p \) with interpolation constant \( M \), let \( 0 < \eta < 1/2 \) and let \( W = Z \setminus D(0, \eta) \). There is a constant \( \delta > 0 \) depending only on \( p \), \( M \) and \( \eta \) such that there exists a function \( f \in A^p \) with \( \|f\|_p \leq 1 \), \( \mathcal{Z}(f) = W \) and \( |f(0)| > \delta \).

**Proof.** The sequence \( W \) will be interpolating with constant at most \( M \), the sequence \( W' = \{0\} \cup W \) will be interpolating with constant at most \( C/\eta \), where \( C \) depends only on \( M \) and \( p \). Select a function \( g \in A^p \) with at most this norm that is 1 at the origin and 0 on the rest of \( W' \). Let \( h = g/\|g\|_p \) so that \( |h(0)| \geq \eta/C \). Replace \( h \), if necessary, by a function \( f \) that vanishes only on \( W \). This can be done, by Lemma 2.3, with a function \( f \) satisfying \( \|f\|_p = 1 \) and \( |f(0)| \geq |h(0)|/C_p \). This proves the corollary with \( \delta \geq \eta/(C_p C) \).

The following stability result for interpolating sequences was proved in 2:

**Proposition 2.6.** If \( Z = \{a_n\} \) is an interpolating sequence for \( A^p \) with interpolation constant \( M \) then there exists a constant \( \delta > 0 \) depending only on \( M \) and \( p \) such that any sequence \( W = \{b_n\} \) satisfying \( \psi(a_n, b_n) < \delta \) is also interpolating for \( A^p \) with interpolation constant less than \( 2M \).

Actually, the constant for the new sequence \( W \) can be taken to be arbitrarily close to \( M \) if \( \delta \) is made sufficiently small, but we will not need that fact.
Another necessary result, the following was mostly proved in [41]. If $0 < \lambda < 1$ and $a \in \mathbb{D}$ is nonzero, let $p_\lambda(a)$ denote the point with the same argument as $a$ satisfying $1 - |p_\lambda(a)|^2 = \lambda(1 - |a|^2)$. Let $p_\lambda(0) = 0$.

**Proposition 2.7.** Suppose $0 \notin \mathcal{Z}$ and the sequence $\mathcal{Z} \cup \{0\}$ is separated in the pseudo-hyperbolic metric with separation constant $\eta$. Let $0 < \lambda < 1$ and let $p_\lambda(\mathcal{Z})$ denote the set of all $p_\lambda(a)$ for $a \in \mathcal{Z}$. If $\mathcal{Z}$ is a zero sequence for $A^p$ then $p_\lambda(\mathcal{Z})$ is a zero set for $A^{p/\lambda}$. Moreover, if $f \in A^p$ with $\mathcal{Z} = \mathcal{Z}(f)$, then there exists an $h \in A^{p/\lambda}$ and a constant $C > 0$ depending only on $\lambda$ and $\eta$ such that $\|h\|_{p/\lambda} \leq \|f\|_p$ and $\|h(0)\| > |f(0)|^{\lambda}/C$.

The last part of this above proposition was not mentioned in [41], but follows from its method of proof. For completeness, we outline the proof here.

**Proof.** For simplicity of notation, let $\mathcal{Z}' = p_\lambda(\mathcal{Z})$ and for each $a \in \mathcal{Z}$ let $a' = p_\lambda(a)$. Given an $f \in A^p$ with zero sequence $\mathcal{Z}$, it follows from [41] that $g = f/\Psi_\mathcal{Z}$ is a nowhere zero function in $A^p_{\mathcal{Z}}$. In [41], the hypotheses were shown to imply that the following sum converges to a harmonic function

$$u(z) = \sum_{a \in \mathcal{Z}} (1 - |a|^2) \left[ \frac{1 - |a'|^2|z|^2}{1 - \overline{a'}z^2} - \frac{1 - |a|^2|z|^2}{1 - \overline{a}z^2} \right].$$

It was shown, moreover, that

$$k_{\mathcal{Z}'}(z) - \lambda k_\mathcal{Z}(z) - u(z)$$

is bounded, with the bound $c$ depending only on $\lambda$ and the separation constant of $\mathcal{Z}$. Let $v$ be harmonic in $\mathbb{D}$ with $u + iv$ analytic and put $h_1 = g^\lambda e^{-u - iv}$, then

$$\|h_1e^{k_{\mathcal{Z}'}z}\|^p_{p/\lambda} \leq c^\lambda \|ge^{k_{\mathcal{Z}}z}\|^p.$$  

Since $u(0) = 0$, we see that $|h_1(0)| = |g(0)|^\lambda$. Now let $h = h_1\Psi_{\mathcal{Z}'}$. Then we have

$$\|h\|_{p/\lambda} \leq \|h_1\|_{p/\lambda, \mathcal{Z}'} \leq C\|g\|_{p, \mathcal{Z}} \leq C'\|f\|,$$

and

$$\|h(0)\| = |f(0)|^\lambda \frac{|\Psi_{\mathcal{Z}'}(0)|}{|\Psi_\mathcal{Z}(0)|^\lambda}.$$  

Normalizing $h$ to have the same norm as $f$ gives the required result, since $|\Psi_{\mathcal{Z}}(0)|$ and $|\Psi_{\mathcal{Z}'}(0)|$ can be estimated in terms of the separation constant $\eta$. \qed

**Proposition 2.8.** Let $\mathcal{Z}$ be any zero set for $A^p$, $p > 0$, and suppose $0 \notin \mathcal{Z}$. Then there exists a solution $f^*$ of the extremal problem: maximize $|f(0)|$ for $f \in A^p$ subject to $\|f\|_p \leq 1$ and $\mathcal{Z}(f) = \mathcal{Z}$. This solution satisfies

$$\int |f^*|^p u \, dA = u(0)$$

for all bounded harmonic $u$ on $\mathbb{D}$. As a consequence of (2.9), there is a constant $C$ such that

$$|f^*(z)|^p(1 - |z|^2) \leq C' \quad \text{for all } z \in \mathbb{D},$$

and $g = f^*/\Psi_\mathcal{Z}$ satisfies a similar growth condition

$$|g(z)|^p e^{pk_{\mathcal{Z}}(z)}(1 - |z|^2) \leq C' \quad \text{for all } z \in \mathbb{D}. $$
Note: In [1] a similar extremal problem is considered, but the constraint is weakened to require only $\mathcal{Z} \subset \mathcal{Z}(f)$. Then the solution is proved to satisfy $\mathcal{Z}(f^*) = \mathcal{Z}$. Here we maximize only over the set of functions already having the given zero set. This has the advantage of producing a result that remains valid in the weighted spaces $A^{\rho,\alpha}$.

Proof. The solution exists by the following normal families argument. We observe that the unit ball of $A^p$ is a normal family. Let $B$ be the supremum of $|f(0)|$ over the unit ball of $A^p$. Let $f_n$ be a sequence in the unit ball with $|f_n(0)| \to B$ and let $f^*$ be any normal limit point of the sequence. Clearly $|f^*(0)| = B$ and $\|f^*\|_p \leq 1$. If the norm were not equal to 1 we could increase $|f^*(0)|$ by dividing $f^*$ by its norm, contradicting the definition of $B$.

The dual problem (minimize $\|f\|_p$ subject to $|f(0)| = B$) has the same solutions. Let $u$ be any bounded harmonic function on $\mathbb{D}$ and $v$ its harmonic conjugate. Then for all real $t$, $f^* e^{t(u-u(0)+iv)}$ are among the candidates for this minimization, and so

$$\int |f^* e^{t(u-u(0)+iv)}|^p dA$$

has a minimum at $t = 0$. Setting the derivative at $t = 0$ to zero gives

$$\int |f^*|^p (u - u(0)) dA = 0$$

which implies (2.9).

To get (2.10) we apply (2.9) to a sequence of bounded harmonic functions converging uniformly on compact sets to the Poisson kernel $P_\zeta(z) = (1 - |z|^2)/|\zeta - z|^2$, $|\zeta| = 1$, to get, after Fatou’s lemma:

$$\int |f^*|^p P_\zeta dA \leq 1.$$

Let $z \in \mathbb{D}$ with $z/|z| = \zeta$ and consider the restriction of the above integral to the Euclidean disk $D_z = \{w : |w - z| < (1 - |z|)/2\}$. Then the above inequality combined with a routine estimate of $P_\zeta$ on $D_z$ gives:

$$\frac{1}{|D_z|} \int_{D_z} |f^*|^p \leq \frac{C}{1 - |z|^2},$$

where $|D_z|$ denotes the area of $D_z$.

Finally, we only have to observe that, by subharmonicity, $|f^*(z)|^p$ is less than the above average. This gives inequality (2.10). Thus, $f^*$ belongs to one of the growth spaces $A(\infty, \infty, 1/p)$ considered in [3]. The results of that paper show that inequality (2.11) follows from (2.10).

Note: The inequality (2.11) (for some zero-free $g \in A^p_{\infty}$) is the one we will need later. One could obtain it directly by maximizing $|g(0)|$ among zero-free functions with norm 1 in $A^p_{\infty}$, and making use of the fact that $\log |g| + k_z$ is subharmonic.

In the case of weighted spaces $A^{\rho,\alpha}$, we find that $|f^*(z)|^p (1 - |z|^2)^{\alpha + 1}$ is bounded. We also note that we can interpolate: $f^*(z) (1 - |z|^2)^{1/p}$ belongs to both $L^\infty$ and $L^p$ for the measure $(1 - |z|^2)^{-1}$ and therefore it belongs simultaneously to $L^q((1 - |z|^2)^{-1}dm(z))$ for all $q \geq p$. That is, if $\mathcal{Z}$ is an $A^p$ zero set, it is also an $A^{q/p-1}$ zero set for all
3. Interpolating a Given Sequence

Here we prove the first part of the main theorem.

**Theorem 3.1.** Assume that \( \mathcal{Z} \) is separated in the pseudo-hyperbolic metric and that we can solve the \( \bar{\partial} \)-equation with bounds in \( L^p_\mathcal{Z} \), \( p \geq 1 \). Then \( \mathcal{Z} \) is an interpolating sequence for \( A^p \).

**Proof.** Let \((a : a \in \mathcal{Z})\) be a sequence in \( l^p_\mathcal{Z} \), and let \( D_a = D(a, 2\eta) \), \( a \in \mathcal{Z} \), be disjoint disks with equal pseudo-hyperbolic radius. Let \( D'_a = D(a, \eta) \). Since the Euclidean radius of \( D_a \) is on the order of \( C(1 - |a|)\eta \), we can find \( C^1 \) functions \( \beta_a \) with support in \( D_a \) satisfying

\[
0 \leq \beta_a \leq 1
\]

\[
\beta_a(z) = 1 \text{ for } z \in D'_a
\]

\[
|\nabla \beta_a(z)|(1 - |a|) \leq C
\]

It is clear that

\[
g(z) = \sum_{a \in \mathcal{Z}} c_a \beta_a(z)
\]

satisfies the interpolation \( g(a) = c_a \), and moreover it belongs to \( L^p \) with norm \( \|g\|_p \leq C\|c\|_{p, \mathcal{Z}} \). We correct \( g \) by putting \( f = g - u\Psi_Z \) and try to determine \( u \) so that \( f \) is analytic. This requires \( \bar{\partial}u = \bar{\partial}g/\Psi_Z \) or

\[
(1 - |z|^2)\bar{\partial}u = \frac{(1 - |z|^2)\bar{\partial}g}{\Psi_Z}
\]

We will show momentarily that the right hand side of this belongs to \( L^p_\mathcal{Z} \), with norm at most \( C\|c\|_{p, \mathcal{Z}} \). Assuming this for the moment, we then have a solution \( u \) belonging to the same space with a similar norm estimate. Because \( |\Psi_Z| \leq e^{kz} \), this means \( u\Psi_Z \) and therefore \( f = g + u\Psi_Z \) belongs to \( L^p \). Moreover, it is analytic and satisfies \( f(a) = c_a \), as required.

Since \( |\Psi_Z| = \sigma_Z e^{kz} \), we need only estimate the quotient \( (1 - |z|^2)\bar{\partial}g/\sigma_Z \). Now \( \bar{\partial}g \) is zero inside each \( D'_a \) as well as outside the union of the \( D_a \). Moreover, in each annular region \( D_a \setminus D'_a \) it is bounded by \( Cc_a/(1 - |a|^2) \). It therefore suffices to show that \( \sigma_Z(z) \geq \delta > 0 \) for \( z \) in each \( D_a \setminus D'_a \). We do this by estimating from above

\[
-2 \log \sigma_Z(z) = \sum_{a \in \mathcal{Z}} \log \left| \frac{a - z}{1 - \bar{a}z} \right|^{-2} - \left( 1 - \left| \frac{a - z}{1 - \bar{a}z} \right|^2 \right).
\]

Now, if \( 1 > x > \eta > 0 \),

\[
\log 1/x = \sum_{n=1}^\infty \frac{(1-x)^n}{n} = 1 - x + (1-x)^2 \sum_{k=0}^\infty \frac{(1-\eta)^k}{k+2} \leq 1 - x + C\eta(1-x)^2
\]

so, with \( x = |(a - z)/(1 - \bar{a}z)|^2 \), we have

\[
-2 \log \sigma_Z(z) \leq C\eta \sum_{a \in \mathcal{Z}} \left( 1 - \left| \frac{a - z}{1 - \bar{a}z} \right|^2 \right)^2
\]
as long as \(|a - z|/|1 - az| > \eta\) for all \(a \in Z\), which is true if \(z \in D_\lambda \setminus D'_\lambda\). This last sum is finite, with a bound that can be estimated solely in terms of the separation constant of the sequence
\[
\left\{ \frac{a - z}{1 - az} : a \in Z \right\}.
\]
But that separation constant is the same as that of \(Z\), so the sum has an upper bound independent of \(z\).

\[
\square
\]

4. A solution operator for the \(\bar{\partial}\)-equation

We will construct a kernel for the solution of the \(\bar{\partial}\)-equation, (1.1). Our method will be to construct local solutions and patch them together with the aid of a family of analytic functions \(g_a, a \in \mathbb{D}\), with special properties. The properties we need are: \(|g_a(z)e^{kz(z)}|p(1 - |M_a(z)|^2)^{1 - \epsilon} \leq C\), and \(|g_a(z)e^{kz(z)}| > \delta > 0\) in \(D(a, \eta)\) for some \(C\), \(\epsilon\), \(\delta\) and \(\eta\) independent of \(a\). We obtain this by using the results of section 2. These results imply that any sequence \(Z\) that is interpolating for \(A^p\) is a separated zero sequence for \(A^q\), some \(q > p\), and moreover we have norm estimates on functions that vanish on \(Z\) but are bounded below at 0.

We will first do the construction of \(g_a\) for \(a = 0\) and then invoke invariance of the hypotheses under Möbius transformations of \(Z\) to obtain such a function for each \(a \in \mathbb{D}\).

A rough outline of the construction of \(g_0\) is the following. Take the interpolating sequence \(Z\), assume it stays away from the origin, and perturb each \(a\) to \(a'\) which has the same argument as \(a\) but \(1 - |a'|^2\) is increased by a factor \(1/\lambda\) with \(\lambda < 1\). Make \(\lambda\) so close enough to 1 that the resulting sequence is still interpolating for \(A^p\) (Proposition 2.6). Then perturb it back to where it was and it becomes a zero sequence for \(A^q\), \(q = p/\lambda\), with the special properties of Proposition 2.7, namely the existence of a function with norm 1 with zeros only on \(Z\) having a lower bound at the origin depending on \(p\) and the interpolation constant of \(Z\). Finally we apply proposition 2.8, possibly increasing the value at the origin and forcing the appropriate growth estimates, namely \(|g(z)/\Psi(z)|^{p/\lambda}e^{kz(z)}(1 - |z|^2) \leq C\). Then \(g_0 = (g/\Psi)\) will satisfy the appropriate growth with \(\epsilon = 1 - \lambda\).

Now for the details.

Lemma 4.1. Let \(0 < p < \infty\) and suppose that \(Z\) is an interpolating sequence for \(A^p\). There exist positive constants \(C\), \(\delta\), \(\eta\) and \(\epsilon\) depending only on \(p\) and the interpolation constant \(M\) of \(Z\), and a family of analytic functions \(\{g_a : a \in \mathbb{D}\}\) such that \(|g_a e^{kz(z)}| > \delta\) in \(D(a, \eta)\) and \(|g_a(z)e^{kz(z)}|p(1 - |M_a(z)|^2)^{1 - \epsilon} \leq C\) for all \(z \in \mathbb{D}\).

Proof. Let us first assume that \(D(0, r)\) is disjoint from \(Z\). If we choose \(\lambda \in (0, 1)\) close enough to 1 then \(Z = p\lambda (Z')\) for some sequence \(Z'\) in \(\mathbb{D}\). For each \(a \in Z\) let \(a' \in Z'\) satisfy \(\arg a' = \arg a\) and \(1 - |a|^2 = \lambda(1 - |a'|^2)\), that is \(a = p\lambda(a')\). It is straightforward to see that \(\sup_{a \in Z} \psi(a, a')\) tends to 0 as \(\lambda\) tends to 1. Therefore, by Proposition 2.6 \(Z'\) will be an interpolating sequence for \(A^p\) if \(\lambda\) is sufficiently close to 1 (\(\lambda\) depending only on \(r\), \(p\) and \(M\)), with an interpolation constant close to \(M\).

By Corollary 2.5 there is a \(\delta_1 > 0\) depending only on \(r\), \(M\) and \(p\), and a function \(f\) in the unit ball of \(A^p\) which has \(Z(f) = Z'\) and \(|f(0)| > \delta_1\). Then by Proposition 2.7 we
can find a function $h$ in the unit ball of $A^{p/\lambda}$ with $\mathcal{Z}(h) = \mathcal{Z}$ and $|h(0)| > \delta_2 = \delta_1^\lambda/C_1$ with $C_1$ depending only on $\lambda$ and the separation constant of $\mathcal{Z}'$ (which ultimately depends only on $p$ and $M$).

By Proposition 2.8 with $p$ replaced by $p/\lambda$, we obtain a zero-free function $f^*$ satisfying $|f^*(0)| > \delta_2$ and the growth condition $|f^*(0)|^{p/\lambda}(1 - |z|^2) < C_2$. This growth condition imposes a bound on the absolute value of the derivative of $f^*$ on compact sets, and so we have $|f^*(z)| > \delta_3 = \delta_2/2$ in some neighborhood $D(0, \eta)$ of the origin. By the second part of Proposition 2.8 we obtain $g_0$ satisfying

$$
(4.1) \quad |g_0(z)e^{kz(z)}|^{p/\lambda}(1 - |z|^2) < C_3, \quad z \in \mathbb{D}
$$

and

$$
(4.2) \quad |g_0(z)e^{kz(z)}| > \delta_2, \quad z \in D(0, \eta).
$$

The second inequality follows because $|g_0|e^{kz} = |f^*|/\sigma$, and $\sigma(z) < 1$ for all $z \in \mathbb{D}$.

Now let $\mathcal{Z}$ be arbitrary and let $r$ be the separation constant of $\mathcal{Z}$. Then $D(0, r)$ contains at most one point of $\mathcal{Z}$. If there is such a point and we let it be $b$, apply the above argument to $\mathcal{Z} \setminus \{b\}$ to obtain $g_0$. This $g_0$ satisfies the stated conditions for $\mathcal{Z} \setminus \{b\}$, but it also satisfies them for the full sequence $\mathcal{Z}$ with $C_3$ and $\delta_2$ replaced by slightly different $C_4$ and $\delta$. This is because the weight functions for the different sequences differ only by the single factor $\exp(\frac{\av{b}^2}{2}(1 - |b|^2)^2)$, which is bounded above and away from 0, the bounds depending only on $r$.

To get $g_a$ for arbitrary $a$ consider the interpolating sequence $\mathcal{Z}_a = M_a(\mathcal{Z})$ obtained by subjecting $\mathcal{Z}$ to the Möbius transformation $M_a$. This has the same interpolation constant as $\mathcal{Z}$. Moreover, the function $k_{\mathcal{Z}_a}$ differs from $k_{\mathcal{Z} \circ M_a}$ by a harmonic function. This follows from the easily calculated formula

$$
(4.3) \quad (1 - |z|^2)^2\partial\bar{\partial}k_{\mathcal{Z}_a} = \sum_{b \in \mathcal{Z}} (1 - |M_b(z)|^2)^2 = \sum_{b \in \mathcal{Z}} (1 - \psi(z, b)^2)^2,
$$

and the Möbius invariance of both sides. Therefore, if we obtain a $g_0$ as above for the sequence $\mathcal{Z}_a$ we have

$$
(4.4) \quad |g_0e^{k_{\mathcal{Z}_a}}| = |g_0|e^{u}e^{k_{\mathcal{Z} \circ M_a}}.
$$

for some harmonic function $u$. If we now let $h = u + iv$ be analytic, and let $g_a = g_0(M_a) \exp h(M_a)$, then from (4.1) and (4.2) for $\mathcal{Z}_a$ we get for $g_a$ the estimates

$$
(4.5) \quad g_a(z)e^{kz(z)} > \delta > 0, \quad z \in D(a, \eta)
$$

and

$$
(4.6) \quad |g_ae^{kz}|^p(1 - |M_a(z)|^2)^{\lambda} \leq C_4^\lambda.
$$

Which is what was required, with $\epsilon = 1 - \lambda$ and $C = C_4^\lambda$. \hfill \square

Now we are ready to construct a solution kernel for the $\bar{\partial}$-equation on the weighted space $L^p_\mathcal{Z}$. 
Theorem 4.2. If \( Z \) is an interpolating sequence for \( A^p \), \( p \geq 1 \), then there exists a covering of \( \mathbb{D} \) by disks \( D(a_j, \eta) \), a partition of unity \( \beta_j \) subordinate to this covering, functions \( g_{a_j} \) as in Lemma 4.1, and a positive number \( m \) such that

\[
(4.7) \quad u(z) = \frac{1}{\pi} \sum_{j=1}^{\infty} g_{a_j}(z) \int_{\mathbb{D}} \frac{\beta_j(w) f(w)}{g_{a_j}(w) (z-w)(1-\bar{w}z)^m} dA(w)
\]
satisfies the equation \((1-|z|^2)\bar{\partial}u = f\) and the estimate \( \|u\|_{p,\mathbb{D}} \leq C\|f\|_{p,\mathbb{D}} \) for some constant \( C \) independent of \( f \).

Proof. We obtain in the usual way a covering of \( \mathbb{D} \) by disks \( D_j = D(a_j, \eta) \) which also satisfies a finite overlap condition \( \sup_{x \in \mathbb{D}} \sum f_j(z) < \infty \) by taking a maximal sequence \( a_j \) such that \( D(a_j, \eta/2) \) is disjoint.

We observe that if \( m > 0 \) then each individual term \( u_j \) in the sum defining \( u \) satisfies \((1-|z|^2)\bar{\partial}u_j = \beta_j f\). It is enough to know that

\[
(4.8) \quad v(z) = \frac{1}{\pi} \int \phi(w) \frac{(1-|w|^2)^m}{(z-w)(1-\bar{w}z)^m} dA(w)
\]
satisfies \( \bar{\partial}v = \phi \) for any compactly supported \( \phi \). This follows from the well-known property of the kernel \((z-w)^{-1}\) and the fact that this kernel differs from that one by a function analytic in \( z \).

Now we want to prove the estimates. Multiplying both sides by \( e^{kz(z)} \), and rewriting the integrals in terms of \( f_1(w) = f(w) e^{kz(w)} \in L^p \), it is enough to show that the kernel

\[
(4.9) \quad g_{a_j}(z) e^{kz(z)} \leq C(1-|M_{a_j}(z)|^2)^{\epsilon-1} = C \frac{|1-\bar{a}_j z|^{2-2\epsilon}}{(1-|a_j|^2)^{1-\epsilon}(1-|z|^2)^{1-\epsilon}}
\]

for some positive \( \epsilon \) and \( C \). Moreover, for \( w \) in the support of \( \beta_j \) we have \( 1-|a_j|^2 \geq c(1-|w|^2) \) and \( |1-\bar{a}_j z| \leq C|1-\bar{w}z| \) for all \( z \in \mathbb{D} \). Using these estimates in (4.8), we need only prove estimates for the kernel

\[
(4.10) \quad \sum_j \frac{|1-\bar{a}_j z|^{2-2\epsilon}/p \beta_j(w)}{(1-|a_j|^2)^{1-\epsilon}/p(1-|z|^2)^{1-\epsilon}/p} \frac{(1-|w|^2)^m-1}{|1-\bar{w}z|^m} \leq C \sum_j \frac{\beta_j(w) (1-|z|^2)^{\epsilon-1}/p(1-|w|^2)^{m-1+(\epsilon-1)/p}}{|z-w||1-\bar{w}z|^{m+2(\epsilon-1)/p}}
\]

Now we invoke Lemma 2.2 with \( a = (\epsilon-1)/p \) and \( b = m-1 + (\epsilon-1)/p \). Clearly \( a > -1/p \) and we will also have \( b > -1/p' \) provided we choose \( m \geq 2/p \). In particular \( m = 2 \) suffices for all \( p \geq 1 \).
Finally, we need to show that \( u \) satisfies the \( \overline{\partial} \)-equation. Fix \( z \in \mathbb{D} \) and note that only finitely many \( \beta_j \) have support intersecting \( D(z, \eta) \). Based on the above estimates, it is straightforward to see that in this disk, the sum over the remaining terms is a convergent sum of functions analytic in \( D(z, \eta) \) and so \( u \) differs from an analytic function by a finite sum \( \sum u_j \) with each term satisfying \( (1 - |z|^2)\overline{\partial}u_j(z) = \beta_j(z)f(z) \). Thus, \( (1 - |z|^2)\overline{\partial}u(z) = \sum \beta_j(z)f(z) = f(z) \), as required. \( \square \)

**Remark 4.3.** While the functions \( g_a \) used in constructing this solution operator were zero-free, there is really no need for this in the proof. We needed the precursors of \( 4.3 \) to this observation in section 7. The need here was the local lower bound and a global upper bound estimate. We return to this observation in section 7.

5. When \( p < 1 \)

When \( p < 1 \), the proof of Theorem 5.1 cannot hold as it stands because the integrals defining \( u \) need not exist if \( f \) is only assumed to be in \( L^p \). However, it is possible to replace \( L^p_Z \) with a space that is locally like \( L^1_Z \) but globally like \( L^p_Z \) and which contains \( A^p_Z \).

To this end we fix a number \( R \) in the interval \((0, 1)\) and let \( D_z \) denote the pseudo-hyperbolic disk with center \( z \) and pseudohyperbolic radius \( R \). For a function \( f \) with \(|f|^q \) locally integrable, we define the local means, \( m_q(f) \) by

\[
m_q(f, z) = \left( \frac{1}{|D_z|} \int_{D_z} |f(w)|^q \, dA(w) \right)^{1/q}.
\]

Then let \( L^p_q \) denote the set of measurable functions \( f \) such that \( m_q(f) \) belongs to \( L^p \) and let \( L^p_{q,Z} \) denote the functions \( f \) such that \( fe^{kz} \) belongs to \( L^p_q \). Let \( A^p_q \) and \( A^p_{q,Z} \) denote the subspaces consisting of all analytic functions in the respective space. The space \( A^p_q \) is actually the same as \( A^p \). Moreover, as long as \( Z \) is a separated sequence, \( A^p_{q,Z} \) is the same as \( A^p_q \), though we will not need this fact. We will use \( \|f\|_{p,q,Z} \) to denote the \( L^p(dA) \) norm of \( m_q(f) \) and \( \|f\|_{p,q,Z} \) will denote the \( L^p \) norm of \( m_q(f)e^{kz} \).

**Theorem 5.1.** If \( p < 1 \) and \( Z \) is separated in the pseudo-hyperbolic metric, then the following are equivalent:

(a) \( Z \) is an interpolating sequence for \( A^p \).

(b) For every \( q \geq 1 \) there is a constant \( C \) such that the \( \overline{\partial} \)-equation (1.1) has for any \( f \in L^p_{q,Z} \) a solution \( u \) satisfying \( \|u\|_{p,Z} \leq C\|f\|_{p,q,Z} \).

(c) There exists a \( q \geq 1 \) and a constant \( C \) such that the \( \overline{\partial} \)-equation (1.1) has for any \( f \in L^p_{q,Z} \) a solution \( u \) satisfying \( \|u\|_{p,Z} \leq C\|f\|_{p,q,Z} \).

**Proof.** Suppose (a) holds, then the function \( (1 - |z|^2)g(z)/\Psi(z) \) constructed in Theorem 3.1 also belongs to \( L^p_{q,Z} \) so the identical proof and the fact that \( A^p_q = A^p \) gives us (a).

Now suppose \( Z \) is an interpolating sequence for \( A^p \) with \( p < 1 \), and let \( q \geq 1 \) be arbitrary and let \( f \in L^p_{q,Z} \).

The existence of the family \( g_a \) used in Theorem 4.2 was shown for every \( p > 0 \). Moreover, the integrals making up the solution operator constructed there are defined because \( fe^{kz} \) is locally in \( L^q \) with \( q \geq 1 \). It therefore suffices to show that it satisfies
the appropriate bounds. To this end we only need the appropriate generalization of Lemma 2.2 which is Lemma 5.3 below.

When estimating bounds on the types of integral operators we encounter here, a sort of meta-theorem is that best results are obtained for $p > 1$ using Schur methods, that is, using Lemma 2.1 or the techniques of its proof; while for $p < 1$, best results seem to come from discretizing the integrals involved and applying the inequality between the $p$th power of a sum and the sum of $p$th powers. In this case, where $p < 1$ but $q \geq 1$, we will need to use both.

First we need a lemma on discretizing the $L^p_q$ norm.

**Lemma 5.2.** Let $\{z_k\}$ be a sequence in $\mathbb{D}$ with separation constant at least $R/2$ and such that the disks $D(z_k, R/2)$ cover $\mathbb{D}$. Then for any $0 < p < \infty$ there are constants $c_1$ and $C_2$ such that

\[
(5.1) \quad c_1 \sum_k (1 - |z_k|^2)^p m_q(f, z_k)^p \leq \int m_q(f)^p \, dA \leq C_2 \sum_k (1 - |z_k|^2)^p m_q(f, z_k)^p
\]

Moreover, the space $L^p_q$ is independent of the radius $R$ used in its definition.

**Proof.** Let us temporarily show the dependence of $m_q$ on $R$ (and suppress the dependence on $q$) by writing $m_R$ in place of $m_q$. Let $R_- < R < R^+$ be defined as follows: $R^+$ is the radius of the disk formed by taking the union of $D(z, R)$ over all $z \in D(0, R)$, in formula: $R^+ = 2R/(1 + R^2)$. And $R_-$ is chosen so that $R = 2R_-/(1 + R^2)$. It follows that $R_+ > R/2$.

Since the disks $D(z_k, R/4)$ are disjoint while the disks $D(z_k, R/2)$ cover $\mathbb{D}$, we can write the integral above as a sum of integrals over disjoint sets $E_k$ satisfying $D(z_k, R/4) \subset E_k \subset D(z_k, R/2)$. Note that if $z \in E_k$ then $E_k \subset D(z, R) \subset D(z_k, R^+)$. Therefore, on $E_k$ we have $m_R(f, z)^p \leq C_R m_{R^+}(f, z_k)^p$. Integrating this inequality over $E_k$ and summing on $k$ gives the second inequality in (5.1) with $m_{R^+}$ used for $m_q$ on the right side.

We also have, if $z \in E_k$, then $D(z_k, R_-)$ is contained in $D(z, R)$, so $m_{R_-}(f, z_k)^p \leq C_R m_R(f, z)^p$ for all $z \in E_k$. Integrating this over $E_k$ and summing gives the other inequality, but for $m_{R_-}$ in place of $m_q$ on the left side.

Finally, each disk $D(z_k, R^+)$ is covered by the finite number (independent of $k$) of $D(z_j, R_-)$ that intersect it, and so $\sum (1 - |z_k|^2)^p m_{R^+}(f, z_k)^p \leq C \sum (1 - |z_k|^2)^p m_{R_-}(f, z_k)^p$, which shows that the flanking sums are equivalent.

Repeating the process indefinitely with a new $R = R^+$ or $R = R^-$ shows that the discrete sums and the integrals are equivalent norms for all $R$.

**Lemma 5.3.** If $p < 1$ and $q \geq 1$, then the kernels

\[
K(z, w) = \frac{(1 - |z|^2)^a (1 - |w|^2)^b}{|1 - \overline{w}z|^{a+b+2}}
\]

\[
B(z, w) = \frac{(1 - |z|^2)^a (1 - |w|^2)^b}{|z - w| |1 - \overline{w}z|^{a+b+1}}
\]

define bounded operators on $L^p_q(dA)$ (integration with respect to $w$) provided $a > -1/p$, $b > 2/p - 1/q - 1$ and $a + b + 2 > q/p$. 
Proof. Since $B(z, w) \geq K(z, w)$, it suffices to prove this for the kernel $B$ only. Let $R$ be some convenient radius, and select a sequence $\{z_k\}$ as in Lemma 2.2. Let $D_k$ denote $D(z_k, R)$ and recall that $1 - |w|^2$ is equivalent to $1 - |z_k|^2$ when $w \in D_k$, and $|1 - \bar{w}z|$ is equivalent to $|1 - z_k z|$, and $|D_k|$ is equivalent to $(1 - |z_k|^2)^2$. Therefore,

\[
Bf(z) = \int B(z, w)f(w)\,dA(w)
\]

\[
\leq \sum_k \int_{D_k} K(z, w)|f(w)| \left| \frac{1 - \bar{w}z}{z - w} \right| dA(w)
\]

\[
\leq \sum_k K(z, z_k)(1 - |z_k|^2)^2 \frac{1}{|D_k|} \int_{D_k} |f(w)| \left| \frac{1 - \bar{w}z}{z - w} \right| dA(w)
\]

\[
= \sum_k K(z, z_k)(1 - |z_k|^2)^2b_k(z)
\]

where, for notational convenience, we have put

\[
b_k(z) = \frac{1}{|D_k|} \int_{D_k} |f(w)| \left| \frac{1 - \bar{w}z}{z - w} \right| dA(w).
\]

Applying Hölder’s inequality to the sum, we get

(5.2)

\[ Bf(z)^q \leq \left( \sum K(z, z_k)(1 - |z_k|^2)^{2+\alpha q}b_k(z)^q \right)^{\frac{q}{\alpha q'}} \left( \sum K(z, z_k)(1 - |z_k|^2)^{2-\alpha q} \right)^{\frac{q}{\alpha q'}} \]

The second sum is less than a constant times $(1 - |z|^2)^{-\alpha q'}$ by the discrete version of inequality (2.3). (See [4] Lemma 3 for a general version.) This inequality requires $\alpha$ to be chosen so that $1 < b + 2 - \alpha q' < a + b + 2$. Putting it in the above and integrating the result over $D_j$ with respect to $z$ we get

(5.3)

\[ m_q(Bf, z_j)^q \leq \sum_k \frac{(1 - |z_j|^2)^{a-\alpha q}(1 - |z_k|^2)^{2+\alpha q}}{|1 - \bar{z}_k z_j|^{a+b+2}}m_q(b_k, z_j)^q. \]

When the distance from $D_j$ to $D_k$ is greater than $R$ the factor multiplying $|f(w)|$ in the definition of $b_k(z)$ is at most $1/R$, so $b_k(z) \leq C|D_k|^{-2}\int_{D_k} |f|\,dA \leq Cm_q(f, z_k)$, and therefore $m_q(b_k, z_j) \leq Cm_q(f, z_k)$.

On the other hand when the distance from $D_j$ to $D_k$ is less than $R$ and $z \in D_j$, then $|1 - \bar{w}z|$ is equivalent to $1 - |z_k|^2$ and so $b_k(z)$ is bounded by a constant times the convolution of $|f|\chi_{D_k}$ with $(1 - |z_k|^2)^{-1}|z|^{-1}\chi_{S_k}$, where $S_k$ is a disk centered at 0 with radius equal to the Euclidean diameter of $D_j \cup D_k$, which is $C(1 - |z_k|^2)$. Using Young’s inequality, we have

(5.4)

\[ m_q(b_k, z_j) \leq \frac{C}{|D_j|^{1/q}} \| f\chi_{D_k}\|_q (1 - |z_k|^2)^{-1}|z|^{-1}\chi_{S_k}\|_1 \leq Cm_q(f, z_k). \]

Putting these estimates for $m_q(b_k, z_j)$ into (5.3) we arrive at

(5.5)

\[ m_q(Bf, z_j)^q \leq \sum_k \frac{(1 - |z_j|^2)^{a-\alpha q}(1 - |z_k|^2)^{2+\alpha q}}{|1 - \bar{z}_k z_j|^{a+b+2}}m_q(f, z_k)^q. \]
Now we need to raise this to the $p/q$ power and bring the power inside the summation

\[
m_q(Bf, z_j)^p \leq C \sum_k \frac{(1 - |z_j|^2)(a-\alpha q)p/q}{|1 - \bar{z}_k z_j|^{(a+b+2)p/q}} m_q(f, z_k)^p,
\]

and, finally, multiply by $(1 - |z_j|^2)^2$ and sum on $j$.

\[
\|Bf\|_{p,q}^p \leq C \sum_k \sum_j (1 - |z_j|^2)^2 m_q(Bf, z_j)^p
\leq C \sum_k \sum_j \frac{(1 - |z_j|^2)(a-\alpha q)p/q+2(1 - |z_k|^2)(2+\alpha q)p/q}{|1 - \bar{z}_k z_j|^{(a+b+2)p/q}} m_q(f, z_k)^p
\leq C \sum_k \sum_j (1 - |z_k|^2)^2 m_q(f, z_k)^p
\leq C \|f\|_{p,q}^p
\]

Where the next-to-last inequality again uses the discrete version of (2.3) and requires $\alpha$ to be chosen satisfying $1 < (a - \alpha q)p/q + 2 < (a + b + 2)p/q$.

Thus, everything depends on the existence of a real number $\alpha$ satisfying

\[
1 < b + 2 - \alpha q' < a + b + 2,
\]

and

\[
1 < (a - \alpha q)p/q + 2 < (a + b + 2)p/q,
\]

or equivalently

\[
\frac{a}{q'} < \alpha < \frac{b + 1}{q'}
\]

and

\[
\frac{2}{p} - \frac{(b + 2)}{q} < \alpha < \frac{a}{q} + \frac{1}{p}.
\]

Such an $\alpha$ will exist if each left side is less than both right sides, leading to the following four requirements:

\[
\begin{align*}
0 &< a + b + 1 \\
0 &< a + \frac{1}{p} \\
\frac{q}{p} &< a + b + 2 \\
\frac{2}{p} &< b + \frac{1}{q} + 1
\end{align*}
\]

The last three of these are the hypotheses of this lemma, and the first one is implied by the third, so the lemma is proved. □ □
Returning to Theorem 5.1, the arguments of Theorem 4.2 lead to the requirement that the kernel
\[
(1 - |z|^2)^{(e-1)/p}(1 - |w|^2)^{m-1+(e-1)/p}
\]
\[
|z - w| |1 - \bar{w}z|^{m+2(e-1)/p}
\]
define a bounded operator. In the present context we require that \((e-1)/p > -1/p\), which is clear, and also that \(m+1+(e-1)/p > 2/p+1/q\) and \(m+1+2(e-1)/p > q/p\), both of which can be made to hold by choosing \(m\) sufficiently large.

6. Seip’s density condition

Here we show that Seip’s density condition (7) for interpolation is equivalent to a condition on \(k_Z\). First we recall the definitions and the condition.

Let \(Z\) be a sequence in \(D\) separated in the hyperbolic metric. For \(b \in D\) let \(Z_b = M_b(Z)\). We define \(D^+(Z, r)\) by
\[
D^+(Z, r) = \frac{\sup_{b \in D} \sum_{a \in Z_b, |a| < r} \log \frac{1}{|a|}}{\log \frac{1}{1 - r}}.
\]
then the upper uniform density of \(Z\) is \(D^+(Z) = \limsup_{r \to 1} D^+(Z, r)\). Given that the denominator is unbounded, we may replace the numerator by any quantity that differs from the given one by a constant independent of \(b\) and \(r\), we choose to use
\[
D^+(Z) = \limsup_{r \to 1} \frac{\sum_{a \in Z, |a| < r} 1 - |a|^2}{2} \frac{1}{\log \frac{1}{1 - r^2}}.
\]
Then Seip’s density condition for interpolation in \(A^p\) is \(D^+(Z) < 1/p\).

We now show that the numerator can be replaced by
\[
(6.1) \quad \frac{1}{2\pi} \int_0^{2\pi} k_Z(re^{i\theta}) \, d\theta = \frac{r^2}{2} \sum_{a \in Z} \frac{(1 - |a|^2)^2}{1 - |a|^2 r^2}.
\]
Let us define the alternative density
\[
(6.2) \quad S^+(Z) = \limsup_{r \to 1} \frac{\int_0^{2\pi} k_{M_b(Z)}(re^{i\theta}) \, d\theta}{\log \frac{1}{1 - r^2}}.
\]
Obviously this remains unchanged if we replace the \(r\) with \(r_\epsilon\) where \(1 - r_\epsilon^2 = \epsilon(1 - r^2)\). But since \(\log(1-r_\epsilon^2) = \epsilon + \log(1-r^2)\), we get the same \(\limsup\) using \(r_\epsilon\) in the numerator and just \(r\) in the denominator. Let us now estimate the difference in the numerators.
\[
\sum_{|a| < r} \frac{1 - |a|^2}{2} - \frac{1}{2\pi} \int_0^{2\pi} k_Z(r_\epsilon e^{i\theta}) \, d\theta.
\]
Clearly it equals
\[(6.3) \quad \sum_{|a|<r} \frac{1-|a|^2}{2} \left(1 - \frac{(1 - |a|^2)r^2_\epsilon}{1 - |a|^2r^2_\epsilon} \right) - \frac{r^2_\epsilon}{2} \sum_{|a|\geq r} \frac{(1 - |a|^2)^2}{1 - |a|^2r^2_\epsilon} \]
The first sum simplifies to
\[\sum_{|a|<r} \frac{1-|a|^2}{2} \left(1 - \frac{(1 - |a|^2)r^2_\epsilon}{1 - |a|^2r^2_\epsilon} \right)\]
and since
\[\frac{(1 - r^2_\epsilon)}{1 - |a|^2r^2_\epsilon} \leq \frac{(1 - r^2_\epsilon)}{1 - r^2} \leq \frac{(1 - r^2_\epsilon)}{1 - r^2} = \epsilon\]
when \( |a| < r \), we have
\[(6.4) \quad (1 - \epsilon) \sum_{|a|<r} \frac{1-|a|^2}{2} \leq \sum_{|a|<r} \frac{1-|a|^2}{2} \leq C \frac{1-r^2}{1-r^2_\epsilon} \leq \frac{C}{\epsilon}.\]
As for the second sum in (6.3), we first observe that \( \sum_{|a|\geq r} (1 - |a|^2)^2 \) is dominated by the area of the annulus \( r < |z| < 1 \) up to a constant that depends only on the separation constant of the sequence so
\[(6.5) \quad \frac{r^2_\epsilon}{2} \sum_{|a|\geq r} \frac{(1 - |a|^2)^2}{1 - |a|^2r^2_\epsilon} \leq \sum_{|a|\geq r} \frac{(1 - |a|^2)^2}{1 - r^2_\epsilon} \leq C \frac{1-r^2}{1-r^2_\epsilon} \leq \frac{C}{\epsilon}.\]
These estimates continue to hold for if we replace \( \mathcal{Z} \) with \( M_b(\mathcal{Z}) \) and so they hold if we take the supremum over \( b \). Combining (6.4) and (6.5) with (6.3) (for all \( M_b(\mathcal{Z}) \)), we see that we have \( (1 - \epsilon)D^+(\mathcal{Z}) \leq S^+(\mathcal{Z}) \leq D^+(\mathcal{Z}) \). Since \( \epsilon \) is arbitrary, the two densities are the same.

Let us restate the criterion in terms of \( S^+ \) but in a form not using sup or lim sup.

**Proposition 6.1.** In order that \( \mathcal{Z} \) be an interpolating sequence for \( A^p \) it is necessary and sufficient that there exist an \( \epsilon > 0 \) and \( r^* < 1 \) such that for all \( w \in \mathbb{D} \) and all \( r \) in the interval \( (r^*, 1) \) we have
\[(6.6) \quad \frac{p}{2\pi} \int_0^{2\pi} k_{M_w(z)}(re^{i\theta}) \, d\theta < (1 - \epsilon) \log \frac{1}{1 - r^2}.\]
We now show that this condition can be replaced with one involving the Laplacian of \( k_\mathcal{Z} \) (easy) and also that it is sufficient to have the inequality for a single value of \( r \) (not as easy).

**Corollary 6.2.** In order that \( \mathcal{Z} \) be an interpolating sequence for \( A^p \) it is necessary and sufficient that it be separated in the pseudo-hyperbolic metric and there exist an \( \epsilon > 0 \) and an \( r^* \) with \( 0 < r^* < 1 \) such that for all \( w \in \mathbb{D} \) have
\[(6.7) \quad p \int_{|z|<r^*} \partial \bar{\partial} k_{M_w(z)}(z) \log \frac{r^2}{|z|^2} \, dA(z) < (1 - \epsilon) \int_{|z|<r^*} \frac{1}{(1 - |z|^2)^2} \log \frac{r^2}{|z|^2} \, dA(z).\]

**Proof.** The equivalence of inequality (6.6) (for a fixed \( r = r^* \)) and (6.7) is just a matter of Green’s formula. What we need to do is show that the (6.7) for a single value of \( r^* \) implies it for all \( r \) sufficiently close to 1.
We replace \( k_{M_w(z)} \) with \( k_Z \circ M_w \), which has the same Laplacian, and then change variables, rewriting \((6.7)\) as

\[
p \int_{D(w,r_s)} \partial \bar{\partial} k_Z(z) \log \frac{r_s^2}{|M_w(z)|^2} \, dA(z)
< (1 - \epsilon) \int_{D(w,r_s)} \frac{1}{(1 - |z|^2)^2} \log \frac{r_s^2}{|M_w(z)|^2} \, dA(z).
\]

Now we let \( r_s < R < 1 \) and then integrate the above inequality with respect to \( \log(R^2/|w|^2) \, d\lambda(w) \) over \( D(0,R) \), where \( d\lambda(w) = dA(w)/(1 - |w|^2)^2 \) is the Möbius invariant measure on \( \mathbb{D} \). After exchanging order of integration, we get

\[
\begin{aligned}
(6.8) \quad \frac{p}{\pi} \int_{\mathbb{D}} & \int_{\mathbb{D}} \chi_{D(z,r_s) \cap D(0,R)}(z) \log \frac{R^2}{|w|^2} \log \frac{r_s^2}{|M_z(w)|^2} \, d\lambda(w) \partial \bar{\partial} k_Z(z) \, dA(z) \\
& < (1 - \epsilon) \int_{\mathbb{D}} \int_{\mathbb{D}} \chi_{D(z,r_s) \cap D(0,R)}(z) \log \frac{R^2}{|w|^2} \log \frac{r_s^2}{|M_z(w)|^2} \, d\lambda(w) \frac{dA(z)}{(1 - |z|^2)^2}.
\end{aligned}
\]

We remark for later use, that \( \partial \bar{\partial} k_Z \) is bounded by a constant (depending only on the separation constant) times \((1 - |z|^2)^{-2}\), which follows from the formula \((4.3)\).

We split the outermost integrals above into three parts:

(I) The integral over those \( z \) with \( |z| \geq r_s \) and \( D(z,r_s) \subset D(0,R) \). Then \( D(z,r_s) \cap D(0,R) = D(z,r_s) \) and \( \log(R^2/|w|^2) \) is harmonic on \( D(z,r_s) \).

(II) The integral over \( |z| < r_s \).

(III) The integral over those \( z \) such that \( D(z,r_s) \) meets \( D(0,R) \) but is not contained in it.

In the integral (I), the inner integral on both sides of inequality \((6.8)\) is

\[
\int_{D(z,r_s)} \log \frac{R^2}{|w|^2} \, d\lambda(w) = \int_{D(0,r_s)} \log \frac{R^2}{|M_z(w)|^2} \, d\lambda(w) = C \log \frac{R^2}{|z|^2}
\]

where \( C \) depends only on \( r_s \).

The integral (II) (on both sides of the inequality) is easily estimated to be bounded by a constant depending only on \( r_s \) and the separation constant of \( Z \).

Finally, for the integral (III), the inner integral is dominated by \( C(1 - R^2) \). And the function \( \partial \bar{\partial} k_Z \) is less than \( C/(1 - |z|^2)^2 \) and the set in question is \( R_1 < |z| < R_2 \) with \( 1 - R_2^2 \sim (1 - R^2) \), so that integral (for both the left and right side) is dominated by a constant depending only on \( r_s \) and the separation constant of \( Z \).

Putting these estimates into \((6.8)\), we obtain a constant, depending only on \( r_s \) and the separation constant of \( Z \) such that

\[
p \int_{|z|<R} \partial \bar{\partial} k_Z(z) \log \frac{R^2}{|z|^2} \, dA(z) < C + (1 - \epsilon) \int_{|z|<R} \frac{1}{(1 - |z|^2)^2} \log \frac{R^2}{|z|^2} \, dA(z).
\]

Since the integral on the right grows like \( \log(1/(1 - R^2)) \) (in fact it is a constant multiple of it), we can reduce \( \epsilon \) slightly and omit the \( C \) for sufficiently large \( R \). Finally, we get

\[
p \int_{|z|<R} \partial \bar{\partial} k_{M_w(z)}(z) \log \frac{R^2}{|z|^2} \, dA(z) < (1 - \epsilon) \int_{|z|<R} \frac{1}{(1 - |z|^2)^2} \log \frac{R^2}{|z|^2} \, dA(z).
\]

for all $M_b$ and all sufficiently large $R$ because the estimates depended only on the $r_*$ and the separation constant of $Z$. □ □

For reference below, we can rewrite (6.10) as

\[
\int_{D(w)} (1 - p(1 - |z|^2)^2 \partial \bar{\partial} k_z(z)) \log \frac{R^2}{|M_w(z)|^2} d\lambda(z) > \epsilon \int_{D(w)} \log \frac{R^2}{|M_w(z)|^2} d\lambda(z).
\]

where $D(w) = D(w, r_*)$.

7. Connection with other weighted $\bar{\partial}$-estimates

Here we show that a recent result of J. Ortega-Cerdà ([5]) on weighted $\bar{\partial}$-estimates, together with Seip’s criterion, implies that the $\bar{\partial}$-equation has solutions with bounds in $L^p_Z$. Ortega-Cerdà’s result deals with the spaces $L^p(\omega dA)$ where $\omega(z) = e^{-\phi(z)}/(1 - |z|^2)$. If we want to see the spaces $L^p_Z$ into this context, we must put

\[
\phi(z) = \log \frac{1}{1 - |z|^2} - pk_z(z).
\]

The main result in [5] required that $\phi$ be subharmonic, which need not be the case for (7.1) (and cannot be for $p > 1$). So $\partial \bar{\partial} \phi$ need not be a positive measure. Another requirement was that the measure $\partial \bar{\partial} \phi$ be locally doubling. Finally, the main requirement was that there exists $\epsilon > 0$ and $0 < r_* < 1$ such that for all $b \in \mathbb{D}$

\[
\int_{D(b, r_*)} \partial \bar{\partial} \phi(z) dA(z) \geq \epsilon
\]

The conclusion of his result is that the $\bar{\partial}$ equation always has solutions with bounds in the $L^p(\omega dA)$ norm.

We have seen (6.10) that Seip’s condition is equivalent to the following, where $\phi$ is given by (7.1): there exists $\epsilon > 0$ and $0 < r_* < 1$ such that for all $b \in \mathbb{D}$

\[
\int_{D(b, r_*)} \partial \bar{\partial} \phi(z) \log \frac{r_*^2}{|M_b(z)|^2} dA(z) \geq \epsilon
\]

The similarity between (7.2) and (7.3) is striking. The $\phi$ in the latter inequality need not be subharmonic, but if it were subharmonic, it can be shown that then these two conditions would be equivalent, and Seip’s criterion plus Ortega-Cerdà’s result would then imply Theorem 4.2 in section 4.

Now we show (thanks to J. Ortega-Cerdà for pointing me in the right direction) that $\phi$ can be modified in such a way that the difference between $\phi$ and its modification is bounded, and such that the new function satisfies the hypotheses of Ortega-Cerdà’s theorem.

**Proposition 7.1.** Given $\phi$ as above and $r_*$ as in (6.10), define

\[
\phi_*(w) = \frac{1}{\pi \log[1/(1 - r_*^2)]} \int_{D(w, r_*)} \phi(z) \log \frac{r_*^2}{|M_w(z)|^2} d\lambda(z)
\]

If $Z$ is an interpolating sequence for $A^p$ then $\phi_*$ is subharmonic and satisfies $1 \geq (1 - |z|^2)^2 \partial \bar{\partial} \phi_*(z) > \epsilon$ for some $\epsilon > 0$. Moreover $\phi_* - \phi$ is bounded.
Proof. The definition of $\phi_*$ is the invariant convolution $\phi * g$ of $\phi$ with the rotationally invariant function
\[
g(z) = \frac{1}{\pi \log[1/(1 - r^2)]} \chi_{D(a,r)} \log(r^2/|z|^2)
\]
(see M. Stoll [2]). While it does not appear to be explicitly stated in [2], it follows from the discussion there (page 34–37) that when one of the functions is rotationally invariant, convolution commutes with the invariant Laplacian. That is, if $\tilde{\Delta} f(z) = (1 - |z|^2)^2 \partial \bar{\partial} f(z)$, then $\Delta(\phi * g) = (\Delta \phi) * g$.

The factor preceding the integral in the definition of $\phi_*$ is a normalization:
\[
\pi \log[1/(1 - r^2)] = \int_{D(w,r)} \phi(z) \log(r^2/|M_w(z)|^2) d\lambda(z).
\]
This follows from the Möbius invariance of the right hand side and a simple calculation for $w = 0$. Moreover, the left hand side of (6.10) is, after a normalization, equal to $(\tilde{\Delta} \phi) * g = \tilde{\Delta} \phi_*$. Therefore, (6.10) is equivalent to $\tilde{\Delta} \phi_* > \epsilon$, which is one of the conclusions we wanted. The upper bound $\tilde{\Delta} \phi_* (z) \leq 1$ is immediate from the fact that $\tilde{\Delta} \phi = 1 - p \tilde{\Delta} k_2 < 1$.

The fact that interpolating sequences are separated implies that $(1 - |z|^2)^2 \nabla k_2 (z)$ is bounded. This is also trivially true of $\log[1/(1 - |z|^2)]$, and therefore $(1 - |z|^2)^2 \nabla \phi (z)$ is bounded. This means that there is a constant $C$ such that $|\phi(w) - \phi(z)| < C$ whenever $z \in D(w, r_0)$. Integrating in the $z$ variable we get $|\phi(w) - \phi_*(w)| < C$, as required.

The inequalities we have just seen on $\tilde{\Delta} \phi_*$ trivially imply that the measure $d\mu(z) = \tilde{\Delta} \phi_*(z) d\lambda(z)$ is locally doubling. Moreover, $\mu(D(z, r)) \geq \epsilon \lambda(D(z, r))$ so all of Ortega-Cerdá’s conditions are satisfied. Thus, we get another proof of the main result for $p \geq 1$. The case $p < 1$ (in a form such as Theorem 5.11) is a routine extension.

There is another connection between the results obtained here and those of Ortega-Cerdà. For an arbitrary $\phi$ satisfying the hypotheses in [3], a family of analytic functions $g_a$, satisfying inequalities similar to those required in the proof of Theorem 1.2 (namely $|g_a(z)|^p \omega(z)$ should be bounded below independent of $a$ in a suitable neighborhood $D(z, \eta)$ of $a$, and should grow at worst like $(1 - |M_a(z)|^2)^{\epsilon - 1}$) would be enough to obtain the appropriate weighted $L^p$ estimates. Again, one need only construct one such function and obtain the whole family using a Möbius invariance argument.

In [5] a much more special function is constructed: a single function $f$ with a specified sequence of zeros, satisfying both upper and lower bounds on $f(z)e^{-\phi}$. The same techniques used in [5] to construct $f$ can be somewhat simplified to construct instead the $g_a$. Unlike the $g_a$ in Theorem 1.2 which are zero-free, these would necessarily have many zeros. But as we’ve pointed out, $g_a$ only needs to be free of zeros near $a$.

The details of the construction of such a family $\{g_a\}$ are so similar to those of the construction of $f$ in [5] as not to be worth repeating here.

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