Quantum Noise Filtering via Cross-Correlations

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Abstract

Motivated by successful classical models for noise reduction, we suggest a quantum technique for filtering noise out of quantum states. The purpose of this paper is twofold: presenting a simple construction of quantum cross-correlations between two wave-functions, and presenting a scheme for a quantum noise filtering. We follow a well-known scheme in classical communication theory that attenuates random noise, and show that one can build a quantum analog by using non-trace-preserving operators. By this we introduce a classically motivated signal processing scheme to quantum information theory, which can help reducing quantum noise, and particularly, phase flip noise.

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I. INTRODUCTION AND MOTIVATION

In classical communication theory, the use of cross-correlation and autocorrelation functions is very common. There is a family of classical algorithms for the retrieval of information below noise. These are based on cross-correlations and use some type of referential wave function to detect the presence of a signal, or its shape [1], [2]. To name just a few, there is the phase sensitive detector that uses a synchronous referential wave form; the boxcar detector that correlates a repetitive waveform with a pulse function as a gating function; the matched filter that detects the presence of a signal with known shape but unknown amplitude, and the more general correlator; the lock-in amplifier; the integrator which is a low-pass filter, etc. Cross-correlations are also used in spectrum analysis or for estimating the level of randomness, etc. see [3] - [7]. Here we wish to present a quantum analog for the classical correlator, that is, a quantum noise filter that utilizes cross-correlations to attenuate the noise.

The power of cross-correlation stems from a deep connection between the correlation and the energy or power spectrum density [8]. In classical theory this is known as the Wiener-Khinchin theorem [1]. Similar relations are also true in quantum optics [9].

Quantum correlations are well-known in quantum optics following the work of Glauber [10]. The single photon interference in a two-slit experiment can be described using the first-order cross-correlation Glauber function $G^{(1)}(r_1,t_1,r_2,t_2)$. Moreover, the intensity-intensity correlation or the Hanbury-Brown-Twiss effect [11] are explained via the second-order Glauber function $G^{(2)}(r_1,t_1,r_2,t_2)$. Cross-correlations are also used in photon detection, an example could be the heterodyne or monodyne detection schemes [9]. Both are the quantum (optical) analogues of the classical schemes for the detection of weak radio frequency signals. For other uses of cross-correlations see for example [12], [13].

The paper is organized as follows. We start (section II) with a few essential preliminaries which will be used later on for the construction of the proposed method. The heart of the paper lies in section III. First we describe a simple way to construct correlation integrals. Given a discrete pure wave function with density matrix $\mathcal{S}$ and a ‘reference’ (see below) discrete pure wave function $\mathcal{S}_0$ we present a non trace-preserving operator $\hat{E}_{\mathcal{S}_0}$ such that $\hat{E}_{\mathcal{S}_0}(\mathcal{S})$ is a density matrix with the correlations $\langle \mathcal{S}, \mathcal{S}_0 \rangle$ of $\mathcal{S}_0$ and $\mathcal{S}$ as its coefficients. We use von-Neuman measurement theory [14] and some techniques used in weak measurement theory [15] [16] to construct $\mathcal{E}$ (also discussed in the appendix).

Next we analyze the case of a quantum signal with noise; let $\rho = \rho_0 S + (1 - \rho_0)N$ where $N = \sum_i E_i S E_i^\dagger$ and $\sum E_i^\dagger E_i = 1$. We prove that our operator $\mathcal{E}$ increases the fidelity between $\rho$ and $\mathcal{S}$, to be more precise:

$$F(\mathcal{E}(\rho), \mathcal{E}(\mathcal{S})) \geq F(\rho, \mathcal{S})$$

We also show that the increase in fidelity is paid out by the number of postselections (final projective measurements) in the construction of $\mathcal{E}_{\mathcal{S}_0}(\mathcal{S})$. We therefore reduce the amount of noise and pay with the number of measurements. The operator $\hat{E}_{\mathcal{S}_0}$ is the quantum parallel of a classical correlator or low-pass filter integrator. Another way to look at it, is as a lock-in amplifier (recently discussed in [17]).

In section IV we discuss the results and suggest a few future research directions.
II. PRELIMINARIES

In this section we outline the scope of our problem and define the correlator which will be later used for noise filtering.

Let $|\phi\rangle = \sum_{k=1}^{N} \phi(k)|k\rangle$ and $|\psi\rangle = \sum_{k=1}^{N} \psi(k)|k\rangle$ be two pure state vectors. We will also use the notation: $|\phi(k)\rangle = \phi(k)|k\rangle$

**Definition:** Define the correlation coefficient as:

$$C(|\phi\rangle, |\psi\rangle) = \frac{1}{N} \sum_{i=1}^{N} |\langle\phi(k-i)|\psi(k)\rangle_{k}|^2, \quad (1)$$

where $\langle\phi(k-i)|\psi(k)\rangle_{k}$ is the $k$ integral making the scalar product.

The following lemma is a simple consequence of the above definition:

**Lemma:** $0 \leq C(|\phi\rangle, |\psi\rangle) \leq 1$.

Consider the following density matrix describing a signal with some noise:

$$\rho = p|\phi\rangle\langle\phi| + (1-p)E|\phi\rangle\langle\phi|E^\dagger, \quad (2)$$

where $E^\dagger E = 1$. Let $S$ denote the signal $|\phi\rangle\langle\phi|$ and $N$ the noise $E|\phi\rangle\langle\phi|E^\dagger$, then we can extend the above definition of correlation coefficient to the densities:

$$C(S, N) = \frac{1}{N} \sum_{i=1}^{N} |\langle\phi(k-i)|E|\phi(k)\rangle_{k}|^2. \quad (3)$$

**Example:** For a phase flip type of noise $E = \frac{1}{N} \sum_{i=1}^{N} Z^{(i)}$ and a fixed amplitude signal $|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |k\rangle$ it is easy to see that:

$$C(S, N) = 0$$

$$C(S, S) = 1.$$

Therefore, if we could somehow correlate $\rho$ with $S$ we will eventually get rid of the noise. In other words, we are looking for a quantum counterpart of the classical scheme for filtering noise by correlation integrals.

Classically, if $S$ is a signal, $N$ is some random amplitude noise and $\langle S + N, S + N \rangle$ is their autocorrelation, then:

$$\langle S + N, S + N \rangle = \langle S, S \rangle + \langle S, N \rangle +$$

$$+ \langle N, S \rangle + \langle N, N \rangle = \langle S, S \rangle.$$

It is well-known that the spectrum of $\langle S, S \rangle$ is very close to that of $S$, and therefore we can analyze $S$ by analyzing $\langle S, S \rangle$, regardless of the noise $N$ [6]. This scheme is not practical as a quantum process since it demands the cloning of $S + N$ [13]. Therefore we will use a similar scheme:

$$\langle S_0, S + N \rangle = \langle S_0, S \rangle + \langle S_0, N \rangle,$$

where $S_0$ is some known referential function.

To construct the correlation functions we will present a non-trace-preserving operator $\mathcal{E} = \mathcal{E}_{|\phi\rangle}$ where $|\phi_0\rangle\langle\phi_0|$ is some referential signal $S_0$. Applying $\mathcal{E}$ to $\rho$ we have:

$$\mathcal{E}_{|\phi\rangle}(\rho) = qE_{|\phi\rangle}(S) + (1-q)E_{|\phi\rangle}(N), \quad (4)$$

where $E_{|\phi\rangle}(S)$ (res. $E_{|\phi\rangle}(N)$) is a density matrix with the correlations $\langle \phi_0(k-i), \phi(k) \rangle$ (res. $\langle \phi_0(k-i)|E|\phi(k)\rangle$) as its coefficients. Hence the correspondence of quantum to classical signals is as follows:

$$E_{|\phi\rangle}(\rho) \sim \langle S_0, S \rangle$$

$$E_{|\phi\rangle}(N) \sim \langle S_0, N \rangle$$

$$E_{|\phi\rangle}(\rho) \sim \langle S_0, S + N \rangle.$$

The probability $q$ (res. $1-q$) is a functions of $p$ (res. $1-p$)) and the correlation coefficient $C(S_0, S)$ (res. $C(S_0, N)$). In terms of fidelity measure we will further show that:

$$F(\mathcal{E}_{|\phi\rangle}(\rho), \mathcal{E}_{|\phi\rangle}(S)) \geq F(\rho, S), \quad (6)$$

and therefore $\mathcal{E}_{|\phi\rangle}(\rho)$ can be looked at as a rotation in the direction of the signal and away from the noise.

III. THE CONSTRUCTION OF CORRELATIONS AND AUTOCORRELATIONS

In this section we will provide a general argument showing how to construct correlations and autocorrelations between wave-functions. We will follow the general scheme presented below:
Given a system $Q$ described by the density matrix $\rho = \langle \phi | \phi \rangle$ of the pure state $|\phi\rangle$, and a system $A$ described by $|\psi \rangle\langle \psi | = \frac{1}{N} \sum_{i,j} |i\rangle \langle j |$, we will couple the two systems by a unitary operator that will entangle them (see the Appendix for further details). Thus we will get:

$$U \rho U^\dagger = \frac{1}{N} \sum_{i,k,j,l} |\phi(k-i)\rangle \langle i | \langle j | \langle \phi(l-j)| \rangle.$$

Next we will post-select the referential function $|\phi_0\rangle$ using the measurement operator $M = |\phi_0\rangle \langle \phi_0 |$. This will leave the system $A$ as a density matrix with the cross-correlations $\langle \phi_0(k) | \phi(k-i) \rangle_k$ as matrix coefficients. Explicitly:

$$\mathcal{E}[\phi, \phi] = \frac{1}{N} \sum_{i,j} \langle \phi_0(k) | \phi(k-i) \rangle_k \langle \phi(k-j) | \phi_0(k) \rangle_k |i\rangle \langle j |.$$

Note the following trace formula:

$$tr(\mathcal{E}[\phi, \phi]) = \frac{1}{N} \sum_{i} |\langle \phi_0(k) | \phi(k-i) \rangle_k|^2 = C(|\phi_0\rangle, |\phi\rangle).$$

In other words, the probability to get the post-selection $|\phi_0\rangle \langle \phi_0 |$ is the average correlation between the reference state $|\phi_0\rangle$ and the original signal $|\phi\rangle$. If the correlation between the reference and the signal is high, then the probability to post-select the reference is also high.

We will now show that the operator $\mathcal{E} = \mathcal{E}[\phi_0]$ increases the fidelity $F$.

**Theorem:** $F(\mathcal{E}[\rho], \mathcal{E}(S)) \geq F(\rho, S)$.

Since $\mathcal{E}(\rho)$ is non-trace-preserving we have to normalize it by its trace. If we use the reference $|\phi_0\rangle$ then $\mathcal{E} = \mathcal{E}[\phi_0]$ and:

$$\frac{\mathcal{E}[\phi_0](\rho)}{tr\mathcal{E}[\phi_0](\rho)} = p \frac{\mathcal{E}[\phi_0](|\phi\rangle \langle \phi |)}{tr\mathcal{E}[\phi_0](\rho)} + (1 - p) \frac{\mathcal{E}[\phi_0](|E\rangle \langle E |)}{tr\mathcal{E}[\phi_0](\rho)}.$$

Next we normalize $\mathcal{E}[\phi_0](|\phi\rangle \langle \phi |)$ and $\mathcal{E}[\phi_0](|E\rangle \langle E |)$ to get:

$$\frac{\mathcal{E}[\phi_0](\rho)}{tr\mathcal{E}[\phi_0](\rho)} = p \frac{tr\mathcal{E}[\phi_0](|\phi\rangle \langle \phi |)}{tr\mathcal{E}[\phi_0](\rho)} \frac{\mathcal{E}[\phi_0](|\phi\rangle \langle \phi |)}{tr\mathcal{E}[\phi_0](\rho)} + (1 - p) \frac{tr\mathcal{E}[\phi_0](|E\rangle \langle E |)}{tr\mathcal{E}[\phi_0](\rho)} \frac{\mathcal{E}[\phi_0](|E\rangle \langle E |)}{tr\mathcal{E}[\phi_0](\rho)}.$$

However by the above trace formula (9):

$$\mathcal{E}[\phi_0](\rho) \approx \sqrt{p} \frac{C(|\phi_0\rangle, |\phi\rangle)}{tr\mathcal{E}[\phi_0](\rho)}.$$

We can also compute $F(\rho, S)$ directly:

$$F(\rho, S) = \sqrt{p + (1 - p) |E\rangle \langle E |^2}.$$

Whenever $C(|\phi_0\rangle, |\phi\rangle)$ is close to 1 (by choosing the right referential function) and $C(|\phi_0\rangle, |E\rangle)$ is close to 0 (this will depend on the type of noise) we can guarantee that:

$$F(\rho, S) \approx \sqrt{p},$$

and

$$F(\mathcal{E}[\rho], \mathcal{E}(S)) \approx \sqrt{\frac{p}{tr\mathcal{E}[\phi_0](\rho)}}.$$  

**Corollary:** The increase in fidelity due to the filter is proportional to:

$$\frac{1}{\sqrt{tr\mathcal{E}[\phi_0](\rho)}}.$$  

Fig.1. Schematic illustration of the correlator
It is important to note that filtering the noise has a cost in terms of the number of post-selection trials. This is the content of the above corollary. We will need several applications of the protocol until post-selection is achieved. This argument is similar to the one employed in signal amplification protocols using weak measurement methods [20, 21].

Example: For \( |\phi \rangle = |\phi_0 \rangle = \frac{1}{\sqrt{N}} \sum_i |i \rangle \) and \( E = \frac{1}{\pi} \sum_i Z^{(i)} \) a phase flip noise as above:

\[
F(\rho, S) = \sqrt{p}
\]

\[
F(\mathcal{E}(\rho), \mathcal{E}(S)) = 1,
\]

and the increase in fidelity is exactly \( \frac{1}{\sqrt{p}} \).

IV. DISCUSSION

Motivated by the theory of cross-correlations in classical physics and its successful applications such as filters, correlators, integrators, etc., we extended its methods to the theory of quantum information processing.

We have constructed a simple method for creating cross-correlations using a quantum measurement scheme followed by post-selection. The cross-correlation integrals (for each of the lags) are represented as the amplitudes of the output vector of the quantum filter, whereas the average cross correlation over all lags corresponds to the post-selection probability. This immediately suggests the construction of a quantum noise filter. We have shown that such a filter can be defined by the use of a non-trace-preserving operator.

For the protocol to work we need to be sure that the correlation of the reference function with the original signal is high, and the type of noise is such that the correlation of the noise with the signal is low. These conditions are similar to those in the classical counterpart of lock-in amplifier.

We applied the proposed method to the case of phase flip channel in the special case there the input signal was constant. The phase flip channel is strictly related to the phase damping channel—one of the most subtle and important processes in the study of quantum information. The use of cross-correlation led to a significant reduction of this noise.

We believe that the above scheme (or a modification of it) could be generalized to different wave-functions and noise patterns. Moreover, this suggests the use of other classical signal processing techniques in quantum information theory.

V. APPENDIX

We shall describe the entanglement process made by \( U \) in details, first for a continuous, and then for an almost discrete, quantum pointers. Consider a system \( Q \) described by a the density matrix \( |\eta\rangle\langle\eta| \). Let \( V_Q \) be its Hilbert space and let \( |\eta\rangle \) be described by a real continuous wave-function:

\[
\eta(x) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} e^{-\frac{x^2}{2\sigma^2}}
\]

(18)

We couple \( Q \) to another system \( A \) with \( N \) eigenvectors \( |a_j\rangle \) such that \( \hat{A}|a_j\rangle = a_j|a_j\rangle \). The eigenvectors \( |a_j\rangle \) will be used to shift the argument of the function \( \eta(x) \). Consider also the state vector

\[
|\psi\rangle = \frac{1}{\sqrt{N}} \sum_j |a_j\rangle
\]

in the system \( A \). We will now couple the two systems by the von Neumann interaction Hamiltonian [15, 16]:

\[
\hat{H} = \hat{H}_{\text{int}} = g(t)\hat{A}\hat{P},
\]

(19)

where \( [\hat{X}, \hat{P}] = i\hbar \) and the coupling function \( g(t) \) satisfies:

\[
\int_0^T g(t) dt = 1,
\]

during the coupling time \( T \). We shall start with the vector:

\[
|\Psi\rangle = |\psi\rangle|\Phi\rangle,
\]

in the tensor space of the two systems. Applying the time evolution operator we get:

\[
e^{-i\hat{A}\hat{P}/\hbar}|\Psi\rangle.
\]

It is easy to see that on the subspace \( |a_j\rangle V_Q \) the Hamiltonian \( \hat{H} \) takes \( \hat{X} \) (i.e. \( I \cdot \hat{X} \)) to \( \hat{X} + ia_j \), since by the time \( T \) the coupling is already done, we have (Heisenberg equation):

\[
\hat{X}(T) - \hat{X}(0) = \int_0^T dt \frac{\partial \hat{X}}{\partial t} = \int_0^T \frac{i\hbar}{\hbar} [\hat{H}, \hat{X}] dt = a_j,
\]

(20)
and therefore this Hamiltonian induces a transformation of the operator $\hat{X}$. The corresponding transformation of the coordinates of the wave function is (see [22] section 8.4):

$$e^{-i\hat{A}\hat{P}/\hbar}\psi(x) = \frac{1}{\sqrt{N}} \sum_j |a_j\rangle\eta(x-a_j). \quad (21)$$

We now examine the case of almost discrete pointer by taking:

$$|k\rangle = \left(\frac{1}{2\pi\epsilon^2}\right)^{1/4} e^{-\frac{(x-k)^2}{4\epsilon^2}},$$

where $\epsilon$ is small enough so that $|k\rangle$ is a sharp Gaussian centered around $x = k$ (this will help us defining an almost discrete function in the variable $k$). Hence,

$$e^{-i\hat{A}\hat{P}/\hbar}|a_j\rangle|k\rangle = |k - a_j\rangle.$$

Therefore, for the state vector $|\phi\rangle = \sum \phi(k)|k\rangle$ we have:

$$e^{-i\hat{A}\hat{P}/\hbar}|a_j\rangle|\phi\rangle = \sum \phi(k)|k - a_j\rangle.$$

We choose

$$\hat{A} = \sum_{l=1}^{n} 2^{l-1} \left(1 - \frac{\sigma^2}{2}\right)^{(l)}.$$

Let $j = i_n, \ldots, i_1$ be the binary decomposition of $j$:

$$j = \sum_{l=1}^{n} i_l 2^{l-1},$$

then

$$\hat{A}|i_n \otimes i_{n-1} \otimes \cdots \otimes i_1\rangle = j|i_n \otimes i_{n-1} \otimes \cdots \otimes i_1\rangle.$$

Therefore we can write $|a_j\rangle = |i_n \otimes i_{n-1} \otimes \cdots \otimes i_1\rangle$, $a_j = j$, and thus:

$$e^{-i\hat{A}\hat{P}/\hbar}|a_j\rangle|\phi\rangle = \sum \phi(k)|k - j\rangle.$$

The above formula is the entanglement we need for the correlation integrals in the main text.

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VII. REFERENCES

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