Contributions of electromagnetic and strong anomalies to the $\eta (\eta') \rightarrow \gamma \gamma$ decays

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Abstract. We study contributions of electromagnetic and strong axial anomalies in the radiative decays of $\eta$ and $\eta'$. Applying a dispersive approach, we derive the anomaly sum rule for a singlet axial current where both electromagnetic and strong parts of the axial exist. A low energy theorem is generalized for a case of mixing states and is applied to evaluate a subtraction part of the strong anomaly. We found a relatively small contribution of the strong anomaly to the two-photon decay amplitudes of the $\eta$ and $\eta'$ mesons.

1. Introduction.
An important property of QCD is the presence of an axial (chiral) anomaly – the violation of the axial symmetry of the classical Lagrangian in the quantum theory. A two-photon decay of the $\pi^0$ is a well-known example of the process that happens mainly due to the axial anomaly. In fact, it is the pion decay problem that led to the discovery of the axial anomaly [1, 2] and triggered further studies that revealed a tremendous role of the anomalies in the quantum field theory. Precise measurements of the pion decay width [3, 4] provide a valuable test for theoretical approaches at low energies (e.g. Chiral perturbation theory) as the usual perturbative QCD methods encounter difficulties in this region due to the confinement. Two-photon decays of the $\eta$ and $\eta'$ mesons is another important example of the processes which are governed by the axial anomaly and are closely related to the problems of mixing and chiral symmetry breaking.

Besides the case of real photons, axial anomaly plays an important role in the processes involving off-shell photons. In particular, the transitions $\gamma \gamma \rightarrow \pi^0 (\eta, \eta')$ are known to be connected with the axial anomaly. One of the methods to study these processes is based on the anomaly sum rules (ASRs) [6, 7] – the result of the dispersive treatment of the axial anomaly [5]. The anomaly sum rules approach was developed for the study of the $\pi^0$ [8] as well as the $\eta$ and $\eta'$ [9, 10, 11, 12] transition form factors in the space-like and time-like regions of the photon virtuality. Recent experimental progress in measurements of the $\pi^0, \eta, \eta'$ transition form factors has led to extensive theoretical studies [13].

The $\eta$ and $\eta'$ mesons, which we deal in this work with, manifest significant mixing, i.e. they are not pure eigenstates of the 8th (octet) component of the octet of axial currents $J^{(8)}_{\mu5} = \frac{1}{\sqrt{6}}(u\gamma_\mu \gamma_5 u + d\gamma_\mu \gamma_5 d - 2s\gamma_\mu \gamma_5 s)$ and singlet axial current $J^{(0)}_{\mu5} = \frac{1}{\sqrt{3}}(u\gamma_\mu \gamma_5 u + d\gamma_\mu \gamma_5 d + s\gamma_\mu \gamma_5 s)$ respectively. This fact is reflected in the non-vanishing off-diagonal terms of the mixing matrix.
formed by the decay constants \( f_M^i \) \((i = 8, 0; M = \eta, \eta')\),

\[
\langle 0 | j^{(i)}_{\mu5}(0) | M(p) \rangle = ip_{\mu}f^i_M. \tag{1}
\]

The 8th component of the octet of axial currents acquires an electromagnetic anomaly term

\[
\partial^{\mu} J^{(8)}_{\mu5} = \frac{2i}{\sqrt{6}} (m_u \pi_5 u + m_d \pi_5 d - 2m_s \pi_5 s) + \frac{e^2}{8\pi^2} C^{(8)} N_c \tilde{F}\tilde{F}, \tag{2}
\]

while the singlet axial current additionally acquires a strong (gluon) anomaly term

\[
\partial^{\mu} J^{(0)}_{\mu5} = \frac{2i}{\sqrt{3}} (m_u \pi_5 u + m_d \pi_5 d + m_s \pi_5 s) + \frac{e^2}{8\pi^2} C^{(0)} N_c \tilde{F}\tilde{F} + \frac{\sqrt{3}\alpha_s}{4\pi} G\tilde{G}, \tag{3}
\]

where \( F \) and \( G \) are electromagnetic and gluon strength tensors respectively, \( \tilde{F} \) and \( \tilde{G} \) are their duals, and charge factors \( C^{(i)} \) are

\[
C^{(8)} = \frac{1}{\sqrt{6}} (\epsilon_u^2 + \epsilon_d^2 - 2\epsilon_s^2) = \frac{1}{3\sqrt{6}},
\]

\[
C^{(0)} = \frac{1}{\sqrt{3}} (\epsilon_u^2 + \epsilon_d^2 + \epsilon_s^2) = \frac{2}{3\sqrt{3}}. \tag{4}
\]

The distinctive feature of the octet of axial currents – absence of the strong component of the axial anomaly – significantly simplifies the study of the anomaly-related processes corresponding to with these currents, leading to exact ASRs in the cases of the 3rd and the 8th components. On the other hand, \( \eta \) and \( \eta' \) two-photon processes are related to the singlet axial current as well. This gives us a good opportunity to study the role of electromagnetic and strong anomalies in these processes. In order to do this, basing on a dispersive representation of axial anomaly, we develop the ASR for the singlet axial current.

2. Dispersive approach to axial anomaly for the singlet axial current

Let us outline the derivation of the anomaly sum in the singlet channel of the axial current. Consider the triangle graph amplitude, composed of the axial current \( J_{\alpha5} \) with momentum \( p = k + q \) and two vector currents with momenta \( k \) and \( q \)

\[
\int d^4x d^4y e^{i(kx+iy)} \langle 0 | T \{ J_{\alpha5}(0) J_\mu(x) J_\nu(y) \} | 0 \rangle = e^2 T_{\alpha\mu\nu}(k, q). \tag{5}
\]

This amplitude can be decomposed [14] (see also [15, 16]) as

\[
T_{\alpha\mu\nu}(k, q) = F_1 \varepsilon_{\alpha\mu\nu\rho} k^\rho + F_2 \varepsilon_{\alpha\mu\nu\rho} q^\rho + F_3 k_\nu \varepsilon_{\alpha\mu\rho\sigma} k^\sigma q^\rho + F_4 q_\nu \varepsilon_{\alpha\mu\rho\sigma} k^\sigma + F_5 k_\mu \varepsilon_{\alpha\nu\rho\sigma} k^\sigma q^\rho + F_6 q_\mu \varepsilon_{\alpha\nu\rho\sigma} k^\sigma q^\rho, \tag{6}
\]

where the coefficients \( F_j = F_j(p^2, k^2, q^2; m^2) \), \( j = 1, \ldots, 6 \) are the Lorentz invariant amplitudes constrained by current conservation and Bose symmetry. Note that the latter includes the interchange \( \mu \leftrightarrow \nu, k \leftrightarrow q \) in the tensor structures and \( k^2 \leftrightarrow q^2 \) in the arguments of the scalar functions \( F_j \).

An anomalous axial Ward identity for \( T_{\alpha\mu\nu}(k, q) \) for the singlet axial current \( J^{(0)}_{\mu5}(p) \) and photons \( \gamma(k, \epsilon(k)), \gamma(q, \epsilon(q)) \) reads
\[ p_\alpha T^{\alpha\mu} = 2mG\epsilon^{\mu\nu\rho\sigma}k_\rho q_\sigma + \frac{C_0 N_c}{2\pi^2} \epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma + N(p^2, q^2, k^2)\epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma, \]  

(7)

where

\[ \langle 0 | \frac{1}{\sqrt{3}} \sum_q m_q \bar{q} \gamma_5 q | \gamma\gamma \rangle = 2mG\epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma, \]  

(8)

\[ \langle 0 | \frac{\sqrt{3} \alpha_s}{4\pi} G\tilde{G} | \gamma\gamma \rangle = e^2 N(p^2, k^2, q^2)\epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma \epsilon^{(k)}_\sigma(q), \]  

(9)

\[ \langle 0 | F \bar{F} | \gamma\gamma \rangle = 2e\epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma \epsilon^{(k)}_\sigma(q). \]  

(10)

We have introduced here the form factors \( G, F \), and while the transition \( F \bar{F} \rightarrow \gamma\gamma \) is point-like up to QED corrections.

In the kinematical configuration with one real photon \((k^2 = 0)\) which we consider in the rest of this section, the above anomalous Ward identity can be rewritten in terms of form factors \( F_3, G, N \) as follows \((N(p^2, q^2) \equiv N(p^2, q^2, k^2 = 0)), \)

\[ (q^2 - p^2)F_3 - q^2 F_4 = 2mG + \frac{C_0 N_c}{2\pi^2} + N(p^2, q^2). \]  

(11)

We can write the form factors \( G, F_3, F_4 \) as dispersive integrals without subtractions: in the case of isovector and octet channels (free from gluon anomaly) it can be shown explicitly [7]. For the singlet current it can be shown using dimensional arguments. At the same time, one cannot claim the existence of unsubtracted dispersion integral for the form factor \( N \). We rewrite it in the form with one subtraction,

\[ N(p^2, q^2) = N(0, q^2) + p^2 R(p^2, q^2), \]  

(12)

where the new form factor \( R \) can be written as an unsubtracted dispersive integral. By taking the imaginary part of (11) w.r.t. \( p^2 \) \((s \text{ in the complex plane})\), dividing the obtained equation by \((s - p^2)\) and integrating it over \( s \in [0, +\infty) \), using the dispersive relations for the form factors \( F_3, F_4, G, R \), in the end we arrive at

\[ (q^2 - p^2)F_3 - \frac{1}{\pi} \int_0^\infty ImF_3 ds - q^2 F_4 = 2mG + p^2 R + \frac{1}{\pi} \int_0^\infty ImR ds. \]  

(13)

Comparing (13) with (11) we can write down the anomaly sum rule for the singlet current:

\[ \frac{1}{\pi} \int_0^\infty ImF_3 ds = \frac{C_0 N_c}{2\pi^2} + N(0, q^2) - \frac{1}{\pi} \int_0^\infty ImR(s, q^2) ds. \]  

(14)

Hereafter, we limit our consideration to the case of real photons \((k^2 = q^2 = 0)\).

3. Low-energy theorem generalized for mixing states

The form factor \( N \) in the ASR (14) represents the strong anomaly and is related to the matrix element \( \langle 0 | GG | \gamma\gamma \rangle \). Rigorous QCD calculation of it is not known yet because of difficulties of confinement. Despite this, it is possible to estimate it in the limit \( p^4 = 0 \). The idea (see, e.g., [17]) as follows. Supposing that there are no massless particles in the singlet channel in the chiral limit (i.e. no \( \eta \) meson contribution in the singlet channel), as the \( \eta' \) meson remains massive, one must get \( \lim_{p \to 0} p^4 \langle 0 | J_{\mu\bar{\nu}}(p) | \gamma\gamma \rangle = 0 \). This corresponds to \( \langle 0 | \partial_\mu J_{\mu\bar{\nu}} | \gamma\gamma \rangle = 0 \), and therefore, one immediately relates the matrix elements \( \langle 0 | GG | \gamma\gamma \rangle \) and \( \langle 0 | F \bar{F} | \gamma\gamma \rangle \) in the considered limits using the expressions for the divergence of the singlet axial current in the chiral limit \((m_\eta = 0)\). In
reality, the $\eta$ meson has a significant contribution to the singlet channel (because of the $\eta - \eta'$ mixing) spoiling this low-energy theorem.

Nevertheless, we can follow the line of reasoning of the theorem for a specifically constructed current with no contribution of the states yielding the poles in the chiral limit (namely, $\eta$ in our approximation). Requiring $\langle 0 | J_{\mu 5}^{(s)} | \eta \rangle = 0$ for the linear combination of $J_{\mu 5}^{(8)}$ and $J_{\mu 5}^{(0)}$ we obtain the current that is suitable for the theorem,

$$ J_{\mu 5}^{(s)} = b (j_{\mu 5}^{(8)} - f_{\eta}^{8} j_{\mu 5}^{(0)}) , \quad (15) $$

where $b$ is an (arbitrary) constant, and the decay constants $f_{M}^i$ are defined as the currents’ projections onto meson states ($i = 8, 0; M = \eta, \eta'$),

$$ \langle 0 | J_{\mu 5}^{(i)} | M(p) \rangle = ip_{\mu} f_{M}^{i} . \quad (16) $$

So, for this current we can conclude that even in the chiral limit $\lim_{p^\mu \to 0} \langle 0 | \partial_{\mu} J_{\mu 5}^{(s)} | p \gamma \gamma \rangle = 0$, and therefore, using (2) and (3) in the chiral limit, at $p^\mu = 0$ we obtain the following relation between the matrix elements of $G\tilde{G}$ and $FF$:

$$ \langle 0 | \sqrt{3} \alpha_s 4\pi G\tilde{G} | \gamma \gamma \rangle = \frac{N_c}{f_{\eta}^{8}} (f_{\eta}^{0} C^{(8)} - f_{\eta}^{8} C^{(0)}) \langle 0 | \frac{\alpha_s}{2\pi} F\tilde{F} | \gamma \gamma \rangle . \quad (17) $$

This yields the value of the form factor $N$ (9),

$$ N(0, 0, 0) = \frac{N_c}{2\pi^{2} f_{\eta}^{8}} (f_{\eta}^{0} C^{(8)} - f_{\eta}^{8} C^{(0)}) . \quad (18) $$

4. Two-photon decays of $\eta$ and $\eta'$ and analysis of the ASR

In order to draw the conclusions for the processes of two-photon decays of the $\eta$ and $\eta'$, let us saturate the l.h.s. of the ASR (14) with resonances according to the quark-hadron duality. The first (lowest) contributions are given by the $\eta$ and $\eta'$, while the rest (higher) states are represented by the integral with a lower limit $s_0$,

$$ f_{\eta}^{0} A_{\eta} + f_{\eta'}^{0} A_{\eta'} = \frac{1}{2\pi^{2}} N_c C_0 + B_0 + B_1 . \quad (19) $$

where, for the sake of brevity, we introduced notations

$$ B_0 \equiv N(0, 0, 0) , \quad B_1 \equiv - \frac{1}{\pi} \int_{s_0}^{\infty} \text{Im} R(s) ds - \frac{1}{\pi} \int_{s_0}^{\infty} \text{Im} F_3 ds . \quad (20) $$

The decay amplitudes of the two-photon decays of the $\eta$ and $\eta'$ mesons $A_{M} (M = \eta, \eta')$ can be expressed in terms of their decay widths

$$ A_{M} = \sqrt{\frac{64\pi^{2} m_{\gamma}^{2} - 2\gamma}{e^{4} m_{\gamma}^{2}}} . \quad (21) $$

Let us have a closer look at the obtained ASR (19). The term $B_0$ stands for the subtraction constant in the dispersion representation of the gluon anomaly and we evaluated it from the the low energy theorem (18).

The term $B_1$ consists of the integral representing the spectral part of the gluon anomaly and the term covering higher resonances. The value of the lower limit $s_0$ ("continuum threshold") in
Strong anomaly term contributions for different sets of meson decay constants

| Set                        | \( f_\eta^8 \) | \( f_{\eta'}^8 \) | \( \frac{1}{\Delta} \) | \( B_0 \)      | \( B_1 \)      | \( B_0 + B_1 \) |
|----------------------------|----------------|-----------------|----------------|----------------|----------------|-----------------|
| [11], free analysis        | 1.11           | -0.42           | 0.16           | 1.04           | -5.55×10^{-2} | 4.91×10^{-2}   | -0.64×10^{-2}   |
| [11], OS mix. sch.         | 0.85           | -0.22           | 0.20           | 0.81           | -5.36×10^{-2} | 3.84×10^{-2}   | -1.53×10^{-2}   |
| [11], QF mix. sch.         | 1.38           | -0.63           | 0.18           | 1.35           | -5.58×10^{-2} | 6.39×10^{-2}   | 0.81×10^{-2}    |
| [18], free analysis        | 1.39           | -0.59           | 0.054          | 1.29           | -5.77×10^{-2} | 5.86×10^{-2}   | 0.095×10^{-2}   |
| [19], QF mix. sch.         | 1.17           | -0.46           | 0.19           | 1.15           | -5.51×10^{-2} | 5.47×10^{-2}   | -0.047×10^{-2}  |

The last integral of \( B_1 \) should range between the masses squared of the last taken into account resonance (\( \eta' \)) and the first resonance included into the integral term, \( s_0 \geq 1 \text{ GeV}^2 \). This results in a \( \alpha_s^2 \) suppression of this integral comparing to the first integral term in \( B_1 \) as the form factor \( F_3 \) is described by a triangle graph (no \( \alpha_s \) corrections) plus diagrams with additional boxes (\( \propto \alpha_s^2 \) for the first box term).

By making use of the ASR for the 8th component of the axial current [11]

\[
f_\eta^8 A_\eta + f_{\eta'}^8 A_{\eta'} = \frac{1}{2\pi^2} N_c C^{(8)},
\]

we can express the two-photon decay amplitudes,

\[
A_\eta = \frac{1}{\Delta} \left( N_c \frac{2\pi^2}{2\pi^2} (C^{(8)} f_{\eta'}^0 - C^{(0)} f_{\eta'}^8) - (B_0 + B_1) f_{\eta'}^8 \right),
\]

\[
A_{\eta'} = \frac{1}{\Delta} \left( N_c \frac{2\pi^2}{2\pi^2} (C^{(0)} f_\eta^8 - C^{(8)} f_\eta^0) + (B_0 + B_1) f_\eta^8 \right),
\]

where \( \Delta = f_\eta^8 f_{\eta'}^0 - f_\eta^0 f_{\eta'}^8 \).

In order to do a numerical analysis of the ASR (19), we use the experimental values of the two-photon decay widths of the \( \eta \) and \( \eta' \) mesons and the decay constants values \( f_M^{(0)} \). We employ the values of decay constants estimated in different analyses basing on the octet-singlet (OS) mixing scheme [11], quark-flavor (QF) mixing scheme [11, 19] or mixing-scheme-free analyses [11, 18].

The \( B_0 + B_1 \) term can be evaluated directly from Eq. (19). The low energy theorem additionally gives estimation for \( B_0 \) (18), so we can separately evaluate \( B_0 \) and \( B_1 \). The results are shown in the Table 1. These results demonstrate that \( B_0 \) and \( B_1 \) enter the ASR with opposite signs and almost cancel each other, giving only a small total contribution to the two-photon decay widths of the \( \eta \) and \( \eta' \). The contribution of the gluon anomaly and higher order resonances (expressed by the \( B_0 + B_1 \) term) to the two-photon decay amplitudes appears to be rather small numerically in comparison with the contribution of the electromagnetic anomaly.
5. Conclusions

Using the dispersive approach to axial anomaly in the singlet current, we have obtained the sum rule with the electromagnetic and strong anomaly contributions. The strong anomaly contribution consists of the spectral part (originated from $p^2$-dependent term) and the subtraction constant (independent of $p^2$).

The gluon matrix element $\langle 0|G^a G^a|\gamma\gamma\rangle$ is related to the electromagnetic amplitude $\langle 0|\not{F}\not{F}|\gamma\gamma\rangle$ in the chiral limit at $p^2 = 0$ by means of the low energy theorem, which we generalized for the case of mixing states. It gives us an estimation for the subtraction constant of the gluon anomaly contribution in the dispersive form of the axial anomaly.

Combining the low-energy theorem and the anomaly sum rule in the singlet channel we determined separately the subtraction and the spectral parts of the gluon anomaly. The spectral part of gluon anomaly is found to be significant: it is of the order of electromagnetic anomaly contribution. However, it is almost canceled out by the subtraction term, resulting in overall small contribution of gluon anomaly to the $\eta(\eta') \rightarrow \gamma\gamma$ decays.

The contributions of gluon and photon anomalies in the $\eta(\eta') \rightarrow \gamma\gamma$ decays have been evaluated using the anomaly sum rule for the singlet axial current. We found a relatively small contribution of the gluon anomaly part.

As an outlook, let us note that ASR also allows one to fix the sign of pseudoscalar meson pole contributions to the vector meson electroproduction amplitudes reflected in spin asymmetry, discussed at this conference [20].

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