Long-wavelength iteration scheme and scalar-tensor gravity

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Abstract

Inhomogeneous and anisotropic cosmologies are modeled within the framework of scalar-tensor gravity theories. The inhomogeneities are calculated to third-order in the so-called long-wavelength iteration scheme. We write the solutions for general scalar coupling and discuss what happens to the third-order terms when the scalar-tensor solution approaches at first-order the general relativistic one. We work out in some detail the case of Brans-Dicke coupling and determine the conditions for which the anisotropy and inhomogeneity decay as time increases. The matter is taken to be that of perfect fluid with a barotropic equation of state.

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I. INTRODUCTION

Recently, Damour and Nordvedt [1] have shown quite generally that General Relativity is an attractor of a large class of scalar-tensor gravity theories in the context of cosmology (see also [2]). Earlier, Bekenstein and Meisels [3] had shown this to be true for Bekenstein’s variable mass theory [4]. In both studies only Friedmann-Robertson-Walker spacetimes were considered. We shall here extend these results to anisotropic and inhomogeneous universes using the long-wavelength iteration scheme. A general barotropic equation of state is used so that universes driven by ordinary as well as “inflationary” matter can be considered.

The long-wavelength iteration scheme is an approach that allows for a generic description of cosmological inhomogeneities and anisotropies that goes beyond the linear regime. This approach was introduced by Lifschitz and Khalatnikov [5] and Tomita [6]. Recently various groups have employed what amounts to the same scheme: Salopek, Stewart and collaborators (see [7] and references therein) use it in their analysis of the Hamilton-Jacobi equation for general relativity; Comer, Deruelle, Langlois and collaborators (see [8, 9] and references therein, as well as [10] and [11]) discuss among other things how the equation of state for matter affects the growth or decay of cosmological inhomogeneities (see also [12–14]); finally Soda et al. [15] solve the Hamilton-Jacobi equation in Brans-Dicke theory when matter is pressureless. We shall here extend the Soda et al. result to fluids with pressure and to other couplings, using the approach developed in [8] and [9].

The essential point of the long-wavelength iteration scheme is that spatial gradients must be small in the following sense: Choose a synchronous gauge where the metric takes the form

\[ ds^2 = -dt^2 + \gamma_{ij}(t, x^k)dx^idx^j . \]  

(1)

Let \( L \) be the characteristic comoving length on which the spatial metric \( \gamma_{ij} \) varies spatially:

\[ \partial_i \gamma_{ik} \sim L^{-1} \gamma_{jk} . \]

The Hubble time \( H^{-1} \) (where \( H \equiv \dot{a}/a, a^2 \equiv \gamma^{1/3}, \gamma \) being the determinant of \( \gamma_{ij} \) and an overdot signifying \( \partial/\partial t \)) is the characteristic time on which the metric varies temporally:

\[ \dot{\gamma}_{ij} \sim H \gamma_{ij} . \]

The long-wavelength approximation, accurate when \( L \gg R_H \) where \( R_H \equiv (aH)^{-1} \) is the local (comoving) Hubble radius, consists in neglecting all spatial gradients in the Einstein equations. However, the Hubble radius, in general, depends on time and eventually the condition \( L \gg R_H \) can be violated. To get more accurate (and “longer-lived”) results, \( \dot{\gamma}_{ij} \) is expanded in terms of ever increasing spatial gradients of an arbitrary “seed” metric. Since each term in the expansion must be a tensor, the gradients of the “seed” enter via all possible contractions and derivatives (leaving two free indices) of the Riemann (or equivalently in 3-dimensions, Ricci) tensor formed from the “seed”. The coefficients of the gradient terms entering the expansion are time-dependent and are determined order by order. For more discussion, see [8, 9].

The only difficult part about implementing the iteration scheme is finding exact solutions at the very first iteration. Fortunately Barrow and Mimoso [16] have given an algorithmic way of generating such solutions for scalar-tensor gravity theories. We shall adopt their techniques and extend them to the third-order. To get an explicit solution however the scalar-tensor coupling must be specified. We shall work out in some detail the Brans-Dicke coupling, consider the class studied by Damour and Nordvedt [1] and comment on those chosen by Barrow and Mimoso [16] and Serna and Alimi [17].

In Sec. II we give the action for scalar-tensor gravity in the so-called “Jordan frame” and write the field equations that follow in a synchronous gauge. In Sec. III we show how to solve
them in general in the long-wavelength approximation following Barrow and Mimoso \cite{16} and particularize to the specific couplings mentioned above. In Sec. IV the second iteration equations are written down and their solutions explored in the late time limit. Finally the corresponding calculations in the “Einstein frame” are outlined in an Appendix.

The conclusion will be that, as in general relativity, inhomogeneities decay only if matter is inflationary. However there is an additional condition on the scalar-tensor coupling which is it must be close enough to its general relativistic value (see eq. (42) in the text).

II. THE FIELD EQUATIONS

In the “Jordan frame” the action for scalar-tensor gravity is

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi R - \phi^{-1} \omega(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] + S_M. \] (2)

The function \( \omega(\phi) \) is an a priori arbitrary function which couples the scalar field \( \phi \) to the metric \( g_{\mu\nu} \). \( S_M \) is the action for the matter which will be taken to be a perfect fluid. Its stress-energy tensor is

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \] (3)

where \( u_\mu \) is its 4-velocity (\( u^\mu u_\mu = -1 \)) and where the energy density \( \rho \) and pressure \( p \) are chosen to be related by a barotropic equation of state of the form

\[ p = (\Gamma - 1)\rho \] (4)

where \( \Gamma \) is a constant (\( 0 \leq \Gamma \leq 2 \)).

The field equations which follow from extremising eq. (2) are

\[ R^\mu_\nu = \frac{8\pi}{\phi^2} \left( T^\mu_\nu - \frac{1}{2} T \delta^\mu_\nu \right) + \frac{1}{\phi^2} \left( \phi_\nu^{\mu\rho} + \frac{1}{2} \phi_\sigma^{\rho\nu} \delta^\mu_\rho \right) + \frac{\omega}{\phi^2} \partial_\nu \phi \partial^\mu \phi \] (5)

and

\[ \Omega^{1/2}(\Omega^{1/2} \partial^\sigma \phi)_{,\sigma} = 8\pi T \] (6)

where \( \Omega \equiv 2\omega + 3 \) (\( \Omega \) is assumed to be positive) and \( T \) is the trace of the stress-energy tensor.

In the synchronous gauge (see eq. (1)), letting \( \gamma^{ij} \) (which raises spatial indices) be the inverse of \( \gamma_{ij} \) (which lowers spatial indices), defining \( \dot{K}_{ij} \equiv \dot{\gamma}_{ij} \), \( \dot{K} \equiv K_i^i \), the field equations (3-4) become

\[ \frac{1}{2} \dot{K} + \frac{1}{4} K_i^j K_j^i = \frac{8\pi}{\phi^2} \left( 1 - \frac{3}{2} \Gamma - \Gamma u^i u_i \right) \rho - \frac{1}{2\phi} \left( \frac{3}{2} \dot{\phi} - \dot{\phi} \dot{\phi}^{ii} \right) - \frac{\omega}{\phi^2} \dot{\phi}^2, \] (7)

\[ - \frac{1}{2}(K_i^j - K^j_i)_{,ij} = \frac{8\pi}{\phi} \sqrt{1 + w} u_j \rho u_i + \frac{1}{2\phi} (K_i^j \phi_{,ij} - 2\dot{\phi}_{,ij}) - \frac{\omega}{\phi^2} \dot{\phi} \phi_{,ij}, \] (8)
\[ (3) R_{ij}^i + \frac{1}{2\sqrt{\gamma}} \frac{\partial}{\partial t} (\sqrt{\gamma} K_{ij}^i) = \frac{8\pi}{\phi} \left[ \Gamma u^j u_i + \left( 1 - \frac{\Gamma}{2} \right) \delta_i^j \right] \rho + \frac{\omega}{\phi^2} \phi_i \phi^i_j + \frac{1}{2\phi} \left[ 2\phi^{ij}_i - K_{ij}^i \phi - \left( \ddot{\phi} + \frac{1}{2} K \dot{\phi} - \phi^{[k]}_i \delta_i^j \right) \right], \]  

(9)

\[- \frac{\partial}{\partial t} (\Omega^{1/2} \dot{\phi}) - \frac{1}{2} K \Omega^{1/2} \dot{\phi} + (\Omega^{1/2} \phi^i_i)_i = \frac{8\pi}{\Omega^{1/2}} (3\Gamma - 4) \rho \]  

(10)

where \((3) R_{ij}\) is the 3-Ricci tensor formed out of \(\gamma_{ij}\), a slash means a covariant derivative with respect to \(\gamma_{ij}\) and \(\gamma \equiv \text{det}(\gamma_{ij})\). Eqs. (9-10) are 11 equations for the 11 unknowns \(\rho, u_i, \phi\) and \(\gamma_{ij}\).

III. THE LONG-WAVELENGTH FIELD EQUATIONS AND THEIR SOLUTIONS

In the long-wavelength approximation all spatial gradients in eqs. (7-10) are ignored. Eq. (8) then simply implies that

\[ u_i = 0. \]  

(11)

The trace-free part of eq. (9) reduces to

\[ \frac{\partial}{\partial t} \left[ \sqrt{\gamma} \phi \left( K_{ij}^i - \frac{1}{3} K \delta_i^j \right) \right] = 0 \]  

(12)

which yields

\[ K_{ij}^i = \frac{\dot{A}}{A} \delta_{ij} + \frac{1}{A^{3/2} \phi} S_{ij}^i \]  

(13)

where we have defined \(A \equiv a^2 = \gamma^{1/3}\) so that \(K = 3\dot{A}/A\), and where the so-called anisotropy matrix \(S_{ij}^i\) is time-independent and traceless. Eq. (13) can be integrated to yield the 3-metric \(\gamma_{ij}\) (see [8,9] for details). The final result is

\[ \gamma_{ij} = e^a_i \gamma_{ab} e^b_j \]  

(14)

where the \(e^a_i, a = 1, 2, 3\), are the components of the 3 eigenvectors of \(S_{ij}^i\) and \(\gamma_{ab} = A \text{ Diag} [\lambda_1 e^{r_1 \tau}, \lambda_2 e^{r_2 \tau}, \lambda_3 e^{r_3 \tau}]\) with \(r_a\) the eigenvalues of \(S_{ij}^i\) (with \(r_1 + r_2 + r_3 = 0\) because \(S_{ij}^i\) is traceless), \(\lambda_a\) three functions of space such that \(\lambda_1 \lambda_2 \lambda_3 = 1\), and \(d\tau/dt = (A^{3/2} \phi)^{-1}\). When \(S_{ij}^i = 0\) the result simplifies into

\[ \gamma_{ij} = A h_{ij} \]  

(15)

where \(h_{ij}(x^k)\) is an arbitrary “seed” metric.

The solution for \(\rho\) is most easily obtained from the conservation law \(u^\nu T^\mu_{\nu \mu} = 0\) which follows from the field equations (5-6) and boils down, at lowest order in the gradients, to

\( \dot{\rho} = 3\Gamma H \rho = 0 \) (with \(H \equiv \dot{A}/2A\)) and thus
\[
\rho = \frac{3M_1}{8\pi}A^{-3\Gamma/2}
\]  

(16)

where \(M_1\) is a function of space only. Note that eq. (16) holds whether \(S^i_j\) is zero or not.

What remains is to solve for \(A\) and \(\phi\). They are given by the traces of eq. (9), eq. (7) and eq. (10), neglecting all gradients and using eqs. (13) and (16). (One of the equations of course is redundant and can be used as a consistency check.) When \(S^i_j = 0\) these equations boil down to the Friedmann equations. Barrow and Mimoso [16] have shown how to reduce the problem to a quadrature when \(\Omega(t)\) rather than \(\Omega(\phi)\) is specified. For \(S^i_j \neq 0\) Barrow and Mimoso’s procedure can still be followed the only difference being that their constraint equation acquires an additional anisotropy term. We therefore refer the reader to [3] for details. The final results are

\[
\ln \phi = \int_0^\eta \frac{\eta}{g(\eta)} d\eta
\]

(17)

and

\[
A^{\frac{3(2-\Gamma)}{2}} = M_1(4 - 3\Gamma)\frac{g(\eta)}{\phi}
\]

(18)

where \(\eta\) is related to the time \(t\) by

\[
d\eta/dt = A^{\frac{3(1-\Gamma)}{2}}\sqrt{3/\Omega},
\]

(19)

and where \(g(\eta)\) is the function which characterizes the coupling through the relation

\[
\Omega = \frac{4 - 3\Gamma}{3(2 - \Gamma)^2} \frac{[2g'-(3\Gamma-4)\eta]^2}{4g-(3\Gamma-4)\eta^2 + S^i_jS^j_i/[6M_1^2(4 - 3\Gamma)]}
\]

(20)

with \(g' \equiv dg/d\eta\). Integration of this equation for \(g\) yields a constant of integration, which can be used to parametrize the space of solutions. The anisotropy matrix only shows up in eq. (20). However, as one can see from this equation, the anisotropy will be negligible at late times.

**A. Brans-Dicke coupling**

This case corresponds to \(\Omega = \text{const}\). The Friedmann solutions (\(S^i_j = 0\) in eq. (21)) have been extensively studied (see, e.g., [3,18–21] and references therein). From eq. (21) and the constancy of \(\Omega\) we obtain the “generating” function \(g\):

\[
4(4 - 3\Gamma)g(\eta) = [3\Omega(2 - \Gamma)^2 - (4 - 3\Gamma)^2]\eta^2 + \lambda \eta + \frac{\chi^2}{12\Omega(2 - \Gamma)^2} - \frac{S^i_jS^j_i}{6M_1^2}
\]

(21)

where \(\lambda\) is a function of space which parameterizes the solutions. It is then an elementary exercise to integrate eqs. (17) and (18). Only the relationship between \(\eta\) and \(t\) [eq. (19)] remains non trivial as it involves a non elementary function of the type \(\int d\eta(\eta - \eta_+)^\alpha(\eta - \eta_-)^\beta\) where \(\eta_\pm\) are the roots of \(g(\eta) = 0\). At late time (or at all times if \(\lambda = S^i_j = 0\)) the solution takes the familiar form:
\[ \phi \propto t^q, \quad q = \frac{4(4 - 3\Gamma)}{(2 - 3\Gamma)(4 - 3\Gamma) + 3\Gamma(2 - \Gamma)\Omega}, \]  
(22)

\[ A \propto t^{2h}, \quad h = \frac{2[\Omega(2 - \Gamma) - (4 - 3\Gamma)]}{(2 - 3\Gamma)(4 - 3\Gamma) + 3\Gamma(2 - \Gamma)\Omega}. \]  
(23)

(When \( \Gamma = 1 \), eqs. (22,23) agree with the solution discussed by Weinberg [21].) As \( A \) must remain positive we must have that \( (4 - 3\Gamma)g(\eta) \geq 0 \) (see eq. (18)) and therefore

\[ \Omega \geq \frac{(4 - 3\Gamma)^2}{3(2 - \Gamma)^2}. \]  
(24)

(See [22] for more discussion on the vanishing of anisotropy in Brans-Dicke theory.)

**B. Dynamical couplings**

Various coupling functions \( \Omega(\phi) \) have been considered in the literature (see e.g. [17] and references therein). To be viable the space of the corresponding cosmological solutions must exhibit a basin of attraction towards general relativity. This means that when, say, \( \phi \to 1 \) then \( \Omega \to \infty \) (with \( \frac{d\Omega}{d\phi}/\Omega^3 \to 0 \) [23]). Many different behaviours of course are possible, for example

\[ \Omega \simeq \frac{\alpha}{\ln \phi} \]  
(25)

which is the one considered by Damour and Nordvedt [1] and also by Barrow and Mimoso [16] as it is the late-time behaviour of all solutions generated by a function \( g \propto \eta^n \) with \( n > 2 \). As follows from eqs. (17-19) we have indeed [16]

\[ H \equiv \dot{A}/2A \simeq \frac{2}{3\Gamma t}, \]  
(26)

\[ P \equiv \frac{\dot{\phi}}{\phi} \simeq \text{const} \left( t^{-2(n-2)(2-\Gamma)/n} \right)^{-1}, \]  
(27)

\[ \Omega \simeq \frac{4 - 3\Gamma}{3(2 - \Gamma)^2} \frac{n^2}{(2 - n) \ln \phi}. \]  
(28)

The alternative behaviour

\[ \Omega \simeq \frac{\lambda}{(\phi - 1)^\varepsilon} \]  
(29)

with \( \frac{1}{2} < \varepsilon < 2 \) was studied by Serna and Alimi [17] who analyzed numerically the solutions.

An important point to note is that it is not very meaningful to look for solutions (either analytically [16] or numerically [17]) for \( \Omega \) behaving as eq. (25) or eq. (29) at all times as the “true” function \( \Omega \) may be significantly different from eq. (25) or eq. (29) away from the basin of attraction.

**IV. THE THIRD-ORDER SOLUTIONS AND THEIR APPROACH TO GENERAL RELATIVITY**

The first-order being under control we now construct the third-order solutions of the long-wavelength scheme describing large-scale inhomogeneities. We shall set \( S^i_j = 0 \) which in any case is negligible at late time.
In accordance with the method presented in [8] the third-order expansions for the 3-metric, the Brans-Dicke field, the energy density and the three-velocity are taken to be

\[ \gamma_{ij} = A(t) \left[ h_{ij} + f_2(t) R_{ij} + \frac{1}{3} (g_2(t) - f_2(t)) Rh_{ij} \right], \]  

(30)

\[ \phi = \phi(t) \left[ 1 + \phi_2(t) R \right], \quad \rho = \rho(t) + \rho_2(t) R, \quad u_i = u_3(t) \nabla_i R, \]  

(31)

where \( R_{ij} \) is the Ricci tensor formed from the seed \( h_{ij} \) and \( R \) the corresponding Ricci scalar and where \( A(t), \phi(t) \) and \( \rho(t) \) are the (Friedmann) first-order solutions of the previous section.

The procedure is to insert the above expansion in eqs. (7)–(10) and keep only terms with two spatial derivatives. Gathering all “like” terms, i.e., the coefficients of \( R^2 \) and \( R \), gives the following ordinary differential equations for \( f_2, g_2, \) and \( \phi_2 \):

\[ \ddot{f}_2 + (3H + P) \dot{f}_2 = -\frac{2}{A}, \]  

(32)

\[ \ddot{g}_2 + (2H + \frac{P}{2}) \dot{g}_2 = \Omega \left( \frac{2 - 3\Gamma}{4 - 3\Gamma} \right) X - 3\ddot{\phi}_2 + (2\Omega P + 3H) \dot{\phi}_2 + P\ddot{\phi}_2, \]  

(33)

\[ \ddot{g}_2 + (6H + \frac{5P}{2}) \dot{g}_2 + \frac{2}{A} = 3\Omega \left( \frac{2 - \Gamma}{4 - 3\Gamma} \right) X - 3[\ddot{\phi}_2 + (2P + 5H) \dot{\phi}_2], \]  

(34)

with

\[ X \equiv \ddot{\phi}_2 + \left( 2P + 3H + \frac{\Omega}{\Omega} \right) \dot{\phi}_2 + \frac{\dot{\Omega}}{2\Omega P} \left( \dot{P} + 2P^2 + PH + \frac{P\ddot{\Omega}}{\Omega} \right) \phi_2 + \frac{1}{2} P \dot{\phi}_2. \]  

(35)

Eq. (32) can be integrated rightaway. Unfortunately the solutions to the other two equations are not so easily forthcoming. However the interesting regime is when \( \Omega \to \infty \) (with \( (d\Omega/d\phi)/\Omega^3 \to 0 \) for \( t \) big) which is the limit where the first-order solutions approach their general relativistic limit. Now from eqs. (17)–(18) it follows that \( 3(2 - \Gamma)H/P = 2(\dot{g} - \eta)/\eta \) and hence diverges as \( \eta \to \infty \) when \( \Omega \to \infty \), that is when \( g \) increases faster than \( \eta^2 \). In this limit eqs. (33–34) decouple, yielding

\[ \ddot{g}_2 + 3\Gamma H \dot{g}_2 \simeq \frac{2 - 3\Gamma}{2A}. \]  

(36)

As for eq. (32) it reduces to

\[ \ddot{f}_2 + 3H \dot{f}_2 \simeq -2/A. \]  

(37)

Eqs. (36–37) are the same as those obtained by Comer et al. (eq. 2.15 in [8]) in their study of inhomogeneities in general relativity. Thus, \( f_2 \) and \( g_2 \) ultimately have the same behaviour as the general relativity solutions discussed in [8]. Unfortunately, it is difficult to make precise statements about \( \phi_2 \) (and hence \( \rho_2 \) and \( u_3 \)) even in the limit of diverging \( \Omega \). We shall therefore turn to particular coupling functions.
A. Brans-Dicke coupling

The late time behaviour of all Friedmann solutions in the Brans-Dicke theory of gravity is given by eqs. (22-23). Eqs. (32-35) for the evolution of inhomogeneities can then be easily integrated yielding

\[ f_2 = \alpha_1 + \alpha_2 t^{1-3h-q} - \frac{t^{2(1-h)}}{(1-h)(h+q+1)}, \]  

(38)

\[ g_2 = \beta_1 + \frac{\beta_2}{t} + \frac{\Omega(2-3\Gamma) - 3(4-3\Gamma) t^{2(1-h)}}{4\Omega(1-h)(3-2h)}, \]  

(39)

\[ \phi_2 = \gamma_1 + \gamma_2 t^{1-3h-q} + \frac{4-3\Gamma}{3[\Omega(2-\Gamma) - (4-3\Gamma)]} \frac{\beta_2}{t} + \frac{(4-3\Gamma)t^{2(1-h)}}{4\Omega(3-2h)(1-h)} \left[ \frac{(4-3\Gamma)^2 - \Omega(3\Gamma^2 - 12\Gamma + 4)}{(4-3\Gamma)^2 + \Omega(2-\Gamma)(2+3\Gamma)} \right], \]  

(40)

where \(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\) are integration constants. One can check that in the limit \(\Omega \to \infty\) the results for \(f_2, g_2\) agree with those of Comer et al. [8]. Furthermore, when \(\Gamma = 1\), corresponding to a dust fluid, all three agree with the solution of Soda et al. [15].

The growth or decay of \(f_2, g_2\) and \(\phi_2\) depends on the signs of the exponents in eqs. (38-40). Recalling the condition of positivity of \(A\) (eq. (24)) one finds that \((f_2, g_2, \phi_2)\) all decay as time increases if and only if

\[ \Gamma < 2/3, \]  

(41)

that is if matter is of an inflationary type (the same condition for decay holds in general relativity, see [3]), and if

\[ \Omega > \frac{(4-3\Gamma)^2}{(2-\Gamma)(2-3\Gamma)}. \]  

(42)

This condition on \(\Omega\) is not demanding if \(p = -\rho\) (\(\Gamma = 0\)) but becomes all the more stringent as \(\Gamma \to 2/3\).

As for the three-velocity its time-dependence is obtained from

\[ \frac{8\pi \Gamma \rho}{\phi} u_3 = \frac{1}{3} \dot{g}_2 - \frac{1}{12} \dot{f}_2 - \left[ H + P \left( \frac{1-\Gamma}{2} \right) \right] + \dot{\phi}_2, \]  

(43)

which yields \(u_3 \propto t^{1+q+h(3\Gamma-2)} \to t^{3-4/3\Gamma}\) when \(\Omega \to \infty\) in accordance with the general relativity results [8], unless \(\Gamma = 1\) (dust) in which case \(u_3 = 0\) (since dust can be comoving in a synchronous gauge). Finally the time behaviour of the energy density correction \(\rho_2\) is given by \(\rho_2 \propto t^{q+h(3\Gamma-2)} \to t^{2(1-2/3\Gamma)}\), again in agreement with the general relativity result.
B. Dynamical coupling: an example

Let us consider as an example the dynamical coupling generated by the function \( g = E \eta^n \). The late time behaviour of the long-wavelength, Friedmann solution, is given in eqs. (26-28). As already mentioned, as \( \Omega \to \infty \) in this limit, \( P/H \to 0 \) so that the functions \( f_2 \) and \( g_2 \) are governed by the same equation as in general relativity, eqs. (33-37), the general solutions of which are

\[
f_2 = \alpha_1 + \alpha_2 t^{-(2-\Gamma)/\Gamma} - \frac{9 \Gamma^2}{9 \Gamma^2 - 4} t^{2-4/3 \Gamma} \tag{44}
\]

and

\[
g_2 = \beta_1 + \beta_2 t^{-1} - \frac{9 \Gamma^2}{4(9 \Gamma - 4)} t^{2-4/3 \Gamma} \tag{45}
\]

As for the equation for \( \phi_2 \) [eqs. (33-35)] it can be integrated and yields

\[
\phi_2 = \gamma_1 t^{-(n-2)(2-\Gamma)/3n} + \gamma_2 t^{-2(2-\Gamma)/3n} - \frac{(4 - 3 \Gamma)[2(2 - \Gamma) - n]}{6(2 - \Gamma)[(2 \Gamma - 3)n + 2(2 - \Gamma)]} t^{2(2-\Gamma)(n-2)/\Gamma_n} + \frac{9 \Gamma^2 n(4 - 3 \Gamma)[2 \Gamma(2 - \Gamma) + n(3 \Gamma - 2)]}{4(9 \Gamma - 4)(2 - \Gamma)(3 \Gamma - 2)[(9 \Gamma - 10)n + 12(2 - \Gamma)]} t^{2(2-\Gamma)(n-3)(2-\Gamma)/\Gamma} \tag{46}
\]

All terms decay if \( \Gamma < 2/3 \), that is for inflationary matter. However \( \phi_2 \) can still decay if \( \frac{2}{3} < \Gamma < \frac{2(2n-3)}{3(n-1)} < 4/3 \).

V. APPENDIX: LONG-WAVELENGTH SCHEME IN THE EINSTEIN FRAME

In the Jordan frame, the action is given by

\[
S = \frac{1}{16 \pi} \int d^4 x \sqrt{- \tilde{g}} \left[ \phi \tilde{R} - \phi^{-1} \omega(\phi) \tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right] + S_m . \tag{47}
\]

In this appendix, contrarily to the main text, all the physical quantities will be labelled by a tilde, whereas the non-physical quantities will remain bare. Upon using a change of the metric and of the fields defined by

\[
\tilde{g}_{\mu \nu} = e^{2 \alpha(\varphi)} g_{\mu \nu} , \tag{48}
\]

\[
\phi^{-1} = e^{2 \alpha(\varphi)} \tag{49}
\]

and

\[
\alpha^2 = (2 \omega + 3)^{-1} , \tag{50}
\]

with \( \alpha(\varphi) = \partial a/\partial \varphi \), one can reexpress the above action in the form

\[
S = \frac{1}{16 \pi} \int d^4 x \sqrt{- g} \left[ R - 2 g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right] + S_M [\psi_m, e^{2 \alpha(\varphi)} g_{\mu \nu}] . \tag{51}
\]
This form is said to correspond to the “Einstein frame” because the curvature term in this new action is the same as in the Einstein-Hilbert action for general relativity. The advantage of this formulation is that most of the equations can be obtained almost directly from the equations of [8] by combining the results for a perfect fluid and for a scalar field.

The Brans-Dicke theory corresponds to a “linear” coupling in the sense that

$$a(\varphi) = \alpha\varphi$$  \hspace{1cm} (52)

Then

$$\phi = e^{-2\alpha\varphi}$$  \hspace{1cm} (53)

The field equations in the Einstein frame read

$$R_{\mu\nu} = 2\partial_{\mu}\varphi\partial_{\nu}\varphi + \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$$  \hspace{1cm} (54)

and

$$\nabla_{\mu}\nabla^{\mu}\varphi = -\frac{\kappa}{2}\alpha(\varphi)T$$  \hspace{1cm} (55)

with $\kappa \equiv 8\pi$. In the case of a perfect fluid, this gives

$$R_{\mu\nu} = 2\partial_{\mu}\varphi\partial_{\nu}\varphi + \kappa(\rho + p)u_{\mu}u_{\nu} + \frac{\kappa}{2}(\rho - p)g_{\mu\nu}$$  \hspace{1cm} (56)

$$\nabla_{\mu}\nabla^{\mu}\varphi = -\frac{\kappa}{2}\alpha(\varphi)(3p - \rho)$$  \hspace{1cm} (57)

One must however be aware that the $\rho$ and $p$ that appear here, defined by $T_{\mu\nu} = (\rho + p)u^\mu u^\nu + pg_{\mu\nu}$, with $g_{\mu\nu}u^{\mu}u^{\nu} = -1$, are not the physical energy density and pressure, but are related to the latter by $\rho = e^{4\alpha}\bar{\rho}$, $p = e^{4\alpha}\bar{p}$.

In a synchronous gauge (adopting all the notations of the main text but for the unphysical metric)

$$^{(3)}R^j_i + \frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t} \left(\sqrt{\gamma}K^j_i\right) = \frac{\kappa}{2}\rho \left[2\Gamma u^j u_i + \delta^j_i(2 - \Gamma)\right] + 2\partial_i\varphi \partial^j\varphi$$  \hspace{1cm} (58)

$$\frac{1}{2}\dot{K} + \frac{1}{4}K^{ij}K^j_i = \kappa\rho \left[1 - \frac{3}{2}\Gamma - \Gamma u_k u^k\right] - 2\dot{\varphi}^2$$  \hspace{1cm} (59)

$$-\frac{1}{2} \left(K^j_{i;j} - K_{ij}\right) = \kappa\rho\Gamma u_i \sqrt{1 + u^k u_k} - 2\dot{\varphi}\partial_i\varphi$$  \hspace{1cm} (60)

and

$$\ddot{\varphi} + \frac{1}{2}K\dot{\varphi} - \nabla_i \nabla^i \varphi = \frac{\kappa}{2}\alpha(3p - \rho)$$  \hspace{1cm} (61)

At first-order the trace of eq. (58) yields
Using this equation, eq. (59) at first-order then gives the following equation for $A$:

$$
\frac{\partial}{\partial t} \left( \frac{A^{3/2} \dot{A}}{A} \right) + \frac{3\Gamma}{4} \left( \frac{A^{2}}{A} \right)^{2} + \frac{2 - \Gamma}{8} S_{i}^{j} S_{j}^{i} A^{-3} + (2 - \Gamma) \dot{\phi}^{2} = 0 .
$$

(63)

In contrast with pure general relativity, we now have an additional term coming from the scalar field. One thus needs an extra equation to close the system and solve for $A$. This is of course given by eq. (61), which reads at first-order

$$
\ddot{\phi} + 3H \dot{\phi} = \frac{\kappa}{2} \alpha (3\Gamma - 4) \rho .
$$

(64)

Combining with eq. (62), one finds

$$
\frac{\partial}{\partial t} \left( \frac{A^{3/2} \dot{A}}{A} \right) = 2(2 - \Gamma) \alpha (3\Gamma - 4) \frac{\partial}{\partial t} \left( \frac{A^{3/2} \dot{\phi}}{A} \right) .
$$

(65)

Let us now consider the Brans-Dicke case, i.e., when $\alpha$ is a constant. Then the previous equation can be integrated immediately, so as to give

$$
\frac{\dot{A}}{A} - \frac{2(2 - \Gamma)}{\alpha (3\Gamma - 4)} \dot{\phi} = \frac{C}{A^{3/2}} .
$$

(66)

In the simplest case $C = 0$ and $S_{i}^{j} S_{j}^{i} = 0$, one finds

$$
\frac{\dot{A}}{A} = 2\lambda t^{-1} , \quad \lambda = \frac{2(2 - \Gamma)}{3\Gamma(2 - \Gamma) + \alpha^{2}(3\Gamma - 4)^{2}} ,
$$

(67)

$$
\dot{\phi} = \xi t^{-1} , \quad \xi = \frac{2\alpha (3\Gamma - 4)}{3\Gamma(2 - \Gamma) + \alpha^{2}(3\Gamma - 4)^{2}} .
$$

(68)

Let us make explicit the relation between this result and eqs. (22-23) in the main body of this article. The relation between the “Jordan time” $\tilde{t}$ and the “Einstein time” $t$ is given by

$$
\tilde{t} \sim t^{\alpha \xi + 1} .
$$

(69)

Hence

$$
\tilde{A} \sim t^{-\frac{2\alpha \xi + 2\lambda}{\alpha \xi + 1}} ,
$$

(70)

which agrees with eq. (22), using $\alpha^{2} \equiv \Omega^{-1}$. Finally,

$$
\phi \sim \tilde{\phi}^{-\frac{2\alpha \xi}{\alpha \xi + 1}} ,
$$

(71)

which agrees with eq. (21).
The third-order is obtained by taking into account the terms neglected previously, in particular the Ricci tensor of the three-metric. It is consistent with the approximation to take the Ricci tensor of \((1)\gamma_{ij}\) in place of \((3)R^i_j\), and convenient to write it in terms of the Ricci tensor (denoted \(R_{ij}\)) of \(h_{ij}\), seen as the components of a metric, to be called “seed” metric. We will thus look for corrections to the metric of the form

\[
(3)\gamma_{ij} = A(t) \left\{ f_2(t) R_{ij} + \frac{1}{3} [g_2(t) - f_2(t)] R h_{ij} \right\},
\]

where \(R^i_j \equiv h^{ik} R_{kj}\) and \(R \equiv R^i_i\) and corrections to the matter of the form

\[
\varphi = \varphi(t) + \varphi_2(t) R, \quad \rho = \rho(t) + \rho_2(t) R.
\]

We substitute \(\gamma_{ij} = (1)\gamma_{ij} + (3)\gamma_{ij}\) in the Einstein equation and keep the terms with two gradients. The traceless part of eq. (58) gives an equation for \(f_2\),

\[
\ddot{f}_2 + \frac{3}{2} \dot{A} f_2 = -\frac{2}{A}.
\]

Eq. (59) then yields

\[
\kappa\rho_2 = \frac{1}{2 - 3\Gamma} (\ddot{\varphi}_2 + 2H\dot{\varphi}_2 + 8\dot{\varphi}_1\dot{\varphi}_2).
\]

Using this relation, the trace part of eq. (58) gives an equation involving \(g_2\) and \(\varphi_2\):

\[
\ddot{g}_2 + \frac{3}{2} \frac{\dot{A}}{A} \dot{g}_2 + 6(2 - \Gamma) \dot{\varphi}_2 = \frac{2 - 3\Gamma}{2A}.
\]

Eq. (61) provides, once more after use of eq. (75), another such equation:

\[
\ddot{\varphi}_2 + 3H\dot{\varphi}_2 + \frac{1}{2} \ddot{\varphi}_2 = \frac{3\Gamma - 4}{2 - 3\Gamma} \left[ \frac{1}{2} \ddot{g}_2 + H\dot{g}_2 + 4\dot{\varphi}_2 \right].
\]

In the case of Brans-Dicke the system of equations can be solved explicitly. All the particular solutions for \(f_2\), \(g_2\) and \(\varphi_2\) behave like \(t^{2-2\lambda}\). After transformation of the time according to eq. (69), one sees that this is in agreement with the results in the “Jordan” frame.

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