Estimates for the number of vertices with an interval spectrum in proper edge colorings of some graphs

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A proper edge $t$-coloring of a graph $G$ is a coloring of edges of $G$ with colors $1, 2, ..., t$ such that each of $t$ colors is used, and adjacent edges are colored differently. The set of colors of edges incident with a vertex $x$ of $G$ is called a spectrum of $x$. A proper edge $t$-coloring of a graph $G$ is interval for its vertex $x$ if the spectrum of $x$ is an interval of integers. A proper edge $t$-coloring of a graph $G$ is persistent-interval for its vertex $x$ if the spectrum of $x$ is an interval of integers beginning from the color 1.

For graphs $G$ from some classes of graphs, we obtain estimates for the possible number of vertices for which a proper edge $t$-coloring of $G$ can be interval or persistent-interval.

1. Introduction

We consider undirected, simple, finite, connected graphs. For a graph $G$, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. For any $x \in V(G)$, $d_G(x)$ denotes the degree of the vertex $x$ in $G$. For a graph $G$, we denote by $\Delta(G)$ the maximum degree of a vertex of $G$. A function $\varphi : E(G) \to \{1, 2, \ldots, t\}$ is called a proper edge $t$-coloring of a graph $G$ if each of $t$ colors is used, and adjacent edges are colored differently. The set of all proper edge $t$-colorings of $G$ is denoted by $\alpha(G,t)$. The minimum value of $t$ for which there exists a proper edge $t$-coloring of a graph $G$ is called a chromatic index \cite{22} of $G$ and is denoted by $\chi'(G)$. Let us also define the set $\alpha(G)$ of all proper edge colorings of the graph $G$ as

\[ \alpha(G) \equiv \bigcup_{t=\chi'(G)}^{\left|E(G)\right|} \alpha(G,t). \]

If $G$ is a graph, $\varphi \in \alpha(G)$, $x \in V(G)$, then the set of colors of edges of $G$ incident with $x$ is called a spectrum of the vertex $x$ in the coloring $\varphi$ of the graph $G$ and is denoted by $S_G(x, \varphi)$.

An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element $p$ and the maximum element $q$ is denoted by $[p, q]$. An interval $D$ is called a $h$-interval if $|D| = h$. 

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For any real number $\xi$, we denote by $\lfloor \xi \rfloor$ ($\lceil \xi \rceil$) the maximum (minimum) integer which is less (greater) than or equal to $\xi$.

If $G$ is a graph, $\varphi \in \alpha(G)$, and $x \in V(G)$, then we say that $\varphi$ is interval (persistent-interval) for $x$ if $S_G(x, \varphi)$ is a $d_G(x)$-interval (a $d_G(x)$-interval with 1 as its minimum element). For an arbitrary graph $G$ and any $\varphi \in \alpha(G)$, we denote by $f_{G,i}(\varphi)$ ($f_{G,pi}(\varphi)$) the number of vertices of the graph $G$ for which $\varphi$ is interval (persistent-interval). For any graph $G$, let us set

$$\eta_i(G) \equiv \max_{\varphi \in \alpha(G)} f_{G,i}(\varphi), \quad \eta_{pi}(G) \equiv \max_{\varphi \in \alpha(G)} f_{G,pi}(\varphi).$$

A bipartite graph $G$ with bipartition $(X, Y)$ is called $(a, b)$-biregular, if $d_G(x) = a$ for any vertex $x \in X$, and $d_G(y) = b$ for any vertex $y \in Y$.

The terms and concepts that we do not define can be found in [23].

It is clear that if for any graph $G$ $\eta_{pi}(G) = |V(G)|$, then $\chi'(G) = \Delta(G)$. For a regular graph $G$, these two conditions are equivalent: $\eta_{pi}(G) = |V(G)|$ iff $\chi'(G) = \Delta(G)$. It is known [15, 19] that for a regular graph $G$, the problem of deciding whether $\chi'(G) = \Delta(G)$ or not is $NP$-complete. It means that for a regular graph $G$, the problem of deciding whether $\eta_{pi}(G) = |V(G)|$ or not is also $NP$-complete. For any tree $G$, some necessary and sufficient condition for $\eta_{pi}(G) = |V(G)|$ was obtained in [8]. In this paper, for an arbitrary regular graph $G$, we obtain a lower bound for the parameter $\eta_{pi}(G)$.

If $G$ is a graph, $R_0 \subseteq V(G)$, and the coloring $\varphi \in \alpha(G)$ is interval (persistent-interval) for any $x \in R_0$, then we say that $\varphi$ is interval (persistent-interval) on $R_0$.

$\varphi \in \alpha(G)$ is called an interval coloring of a graph $G$ if $\varphi$ is interval on $V(G)$.

We define the set $\mathfrak{N}$ as the set of all graphs for which there is an interval coloring. Clearly, for any graph $G$, $G \in \mathfrak{N}$ if and only if $\eta_i(G) = |V(G)|$.

The notion of an interval coloring was introduced in [6]. In [6, 16, 17] it is shown that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. For a regular graph $G$, these two conditions are equivalent: $G \in \mathfrak{N}$ iff $\chi'(G) = \Delta(G)$ [6, 16, 17]. Consequently, for a regular graph $G$, four conditions are equivalent: $G \in \mathfrak{N}$, $\chi'(G) = \Delta(G)$, $\eta_i(G) = |V(G)|$, $\eta_{pi}(G) = |V(G)|$. It means that for any regular graph $G$,

1. the problem of deciding whether or not $G$ has an interval coloring is $NP$-complete,
2. the problem of deciding whether $\eta_i(G) = |V(G)|$ or not is $NP$-complete.

In this paper, for an arbitrary regular graph $G$, we obtain a lower bound for the parameter $\eta_i(G)$.

We also obtain some results for bipartite graphs. The complexity of the problem of existence of an interval coloring for bipartite graphs is investigated in [3, 9, 21]. In [16] it is shown that for a bipartite graph $G$ with bipartition $(X, Y)$ and $\Delta(G) = 3$ the problem of existence of a proper edge 3-coloring which is persistent-interval on $X \cup Y$ (or even only on $Y$ [6, 16]) is $NP$-complete.

Suppose that $G$ is an arbitrary bipartite graph with bipartition $(X, Y)$. Then $\eta_i(G) \geq \max\{|X|, |Y|\}$.

Suppose that $G$ is a bipartite graph with bipartition $(X, Y)$ for which there exists a coloring $\varphi \in \alpha(G)$ persistent-interval on $Y$. Then $\eta_{pi}(G) \geq 1 + |Y|$.
Some attention is devoted to \((a, b)-biregular\) bipartite graphs \([4, 14, 13, 18]\) in the case \(b = a + 1\).

We show that if \(G\) is a \((k - 1, k)\)-biregular bipartite graph, \(k \geq 4\), then

\[
\eta_i(G) \geq \frac{k - 1}{2k - 1} \cdot |V(G)| + \left\lceil \frac{k}{2(2k - 1)} \cdot |V(G)| \right\rceil.
\]

We show that if \(G\) is a \((k - 1, k)\)-biregular bipartite graph, \(k \geq 3\), then

\[
\eta_p(G) \geq \frac{k}{2k - 1} \cdot |V(G)|.
\]

2. Results

**Theorem 1** \([17]\) If \(G\) is a regular graph with \(\chi'(G) = 1 + \Delta(G)\), then

\[
\eta_p(G) \geq \left\lceil \frac{|V(G)|}{1 + \Delta(G)} \right\rceil.
\]

**Proof.** Suppose that \(\beta \in \alpha(G, 1 + \Delta(G))\). For any \(j \in [1, 1 + \Delta(G)]\), define

\[
V_{G, \beta, j} \equiv \{x \in V(G) / j \not\in S_G(x, \beta)\}.
\]

For arbitrary integers \(j', j''\), where \(1 \leq j' < j'' \leq 1 + \Delta(G)\), we have

\[
V_{G, \beta, j'} \cap V_{G, \beta, j''} = \emptyset
\]

and

\[
\bigcup_{j=1}^{1+\Delta(G)} V_{G, \beta, j} = V(G).
\]

Hence, there exists \(j_0 \in [1, 1 + \Delta(G)]\) for which

\[
|V_{G, \beta, j_0}| \geq \left\lceil \frac{|V(G)|}{1 + \Delta(G)} \right\rceil.
\]

Set \(R_0 \equiv V_{G, \beta, j_0}\).

**Case 1** \(j_0 = 1 + \Delta(G)\).

Clearly, \(\beta\) is persistent-interval on \(R_0\).

**Case 2** \(j_0 \in [1, \Delta(G)]\).

Define a function \(\varphi : E(G) \to [1, 1 + \Delta(G)]\). For any \(e \in E(G)\), set:

\[
\varphi(e) \equiv \begin{cases} 
\beta(e), & \text{if } \beta(e) \not\in \{j_0, 1 + \Delta(G)\} \\
 j_0, & \text{if } \beta(e) = 1 + \Delta(G) \\
 1 + \Delta(G), & \text{if } \beta(e) = j_0.
\end{cases}
\]

It is not difficult to see that \(\varphi \in \alpha(G, 1 + \Delta(G))\) and \(\varphi\) is persistent-interval on \(R_0\).
Corollary 1 \cite{17} If $G$ is a cubic graph, then there exists a coloring from $\alpha(G, \chi'(G))$ which is persistent-interval for at least $\left\lceil \frac{|V(G)|}{4} \right\rceil$ vertices of $G$.

Theorem 2 \cite{17} If $G$ is a regular graph with $\chi'(G) = 1 + \Delta(G)$, then

$$\eta_i(G) \geq \left\lceil \frac{|V(G)|}{\left\lfloor 1 + \frac{\Delta(G)}{2} \right\rfloor} \right\rceil.$$  

Proof. Suppose that $\beta \in \alpha(G, 1 + \Delta(G))$. For any $j \in [1, 1 + \Delta(G)]$, define

$$V_{G, \beta, j} \equiv \{ x \in V(G) / j \notin S_G(x, \beta) \}.$$  

For arbitrary integers $j', j''$, where $1 \leq j' < j'' \leq 1 + \Delta(G)$, we have

$$V_{G, \beta, j'} \cap V_{G, \beta, j''} = \emptyset$$  

and

$$\bigcup_{j=1}^{1+\Delta(G)} V_{G, \beta, j} = V(G).$$  

For any $i \in [1, \left\lceil \frac{1 + \Delta(G)}{2} \right\rceil]$, let us define the subset $V(G, \beta, i)$ of the set $V(G)$ as follows:

$$V(G, \beta, i) \equiv \left\{ \begin{array}{ll} V_{G, \beta, 2i-1} \cup V_{G, \beta, 2i}, & \text{if } \Delta(G) \text{ is odd and } i \in [1, \frac{1 + \Delta(G)}{2}] \text{ or } \Delta(G) \text{ is even and } i \in [1, \frac{\Delta(G)}{2}], \\ V_{G, \beta, 1+\Delta(G)}, & \text{if } \Delta(G) \text{ is even and } i = 1 + \frac{\Delta(G)}{2}. \end{array} \right.$$  

For arbitrary integers $i', i''$, where $1 \leq i' < i'' \leq \left\lceil \frac{1 + \Delta(G)}{2} \right\rceil$, we have

$$V(G, \beta, i') \cap V(G, \beta, i'') = \emptyset$$  

and

$$\bigcup_{i=1}^{\left\lceil \frac{1 + \Delta(G)}{2} \right\rceil} V(G, \beta, i) = V(G).$$  

Hence, there exists $i_0 \in [1, \left\lceil \frac{1 + \Delta(G)}{2} \right\rceil]$ for which

$$|V(G, \beta, i_0)| \geq \left\lceil \frac{|V(G)|}{\left\lfloor 1 + \frac{\Delta(G)}{2} \right\rfloor} \right\rceil.$$  

Set $R_0 \equiv V(G, \beta, i_0)$.

Case 3 $i_0 = \left\lceil \frac{1 + \Delta(G)}{2} \right\rceil$.

Case 4 1.a $\Delta(G)$ is even.

Clearly, $\beta$ is interval on $R_0$.  


Corollary 2 \[ \text{If } \] 
\[ \text{Corollary 3 } \]

\[ \text{which is interval for at least } \phi \]

\[ \text{exists a coloring } \]

1.b Estimates for the number of vertices with an interval spectrum in proper edge colorings of some graphs \[ \text{Let } \]

\[ \text{Let } \]

Theorem 4 \[ \text{Suppose } \]

\[ \text{Proof. Case 7 } \]

\[ \text{Case 6 } \]

2 \[ \leq i_0 \leq \left\lfloor \frac{\Delta(G)-1}{2} \right\rfloor. \]

\[ \text{Define a function } \phi : E(G) \rightarrow [1, 1 + \Delta(G)]. \]

\[ \text{For any } e \in E(G), \text{ set:} \]

\[ \phi(e) \equiv \begin{cases} 
(\beta(e) + 1) \pmod{(1 + \Delta(G))}, & \text{if } \beta(e) \neq \Delta(G), \\
1 + \Delta(G), & \text{if } \beta(e) = \Delta(G).
\end{cases} \]

\[ \text{It is not difficult to see that } \phi \in \alpha(G, 1 + \Delta(G)) \]

and \[ \phi \] is interval on \[ R_0. \]

\[ \text{Corollary 2 } \]

\[ \text{(17) If } G \text{ is a cubic graph, then there exists a coloring from } \alpha(G, \chi'(G)) \]

which is interval for at least \[ \frac{|V(G)|}{2} \]

vertices of \[ G. \]

\[ \text{Theorem 3 } \]

\[ \text{(16, 7) Let } G \text{ be a bipartite graph with bipartition } (X, Y). \]

\[ \text{Then there exists a coloring } \phi \in \alpha(G, |E(G)|) \]

which is interval on \[ X. \]

\[ \text{Corollary 3 } \]

\[ \text{Let } G \text{ be a bipartite graph with bipartition } (X, Y). \]

\[ \text{Then } \eta_i(G) \geq \max\{|X|, |Y|\}. \]

\[ \text{Theorem 4 } \]

\[ \text{(4, 4, 7) Let } G \text{ be a bipartite graph with bipartition } (X, Y) \]

where \[ d_G(x) \leq d_G(y) \]

for each edge \[ (x, y) \in E(G) \]

with \[ x \in X \]

and \[ y \in Y. \]

\[ \text{Then there exists a coloring } \phi_0 \in \alpha(G, \Delta(G)) \]

which is persistent-interval on \[ Y. \]

\[ \text{Theorem 5 } \]

\[ \text{Suppose } G \text{ is a bipartite graph with bipartition } (X, Y), \]

and \[ \text{there exists a coloring } \phi_0 \in \alpha(G, \Delta(G)) \]

which is persistent-interval on \[ Y. \]

\[ \text{Then, for an arbitrary vertex } x_0 \in X, \]

\[ \text{there exists } \psi \in \alpha(G, \Delta(G)) \]

which is persistent-interval on \[ \{x_0\} \cup Y. \]

\[ \text{Proof. Case 7 } \]

\[ 1 \quad S_G(x_0, \phi_0) = [1, d_G(x_0)]. \]

\[ \text{In this case } \psi = \phi_0. \]

\[ \text{Case 8 } \]

\[ 2 \quad S_G(x_0, \phi_0) \neq [1, d_G(x_0)]. \]

\[ \text{Clearly, } [1, d_G(x_0)] \setminus S_G(x_0, \phi_0) \neq \emptyset, \]

\[ S_G(x_0, \phi_0) \setminus [1, d_G(x_0)] \neq \emptyset. \]

\[ \text{Since } |S_G(x_0, \phi_0)| = |[1, d_G(x_0)]| \]

\[ = d_G(x_0), \]

\[ \text{there exists } \nu_0 \in [1, d_G(x_0)] \]

\[ \text{satisfying the condition } |[1, d_G(x_0)] \setminus S_G(x_0, \phi_0)| = |S_G(x_0, \phi_0) \setminus [1, d_G(x_0)]| = \nu_0. \]

\[ \text{Now let us construct the sequence } \Theta_0, \Theta_1, \ldots, \Theta_{\nu_0} \]

\[ \text{of proper edge } \Delta(G)-\text{colorings of the graph } G, \]

\[ \text{where for any } i \in [0, \nu_0], \Theta_i \text{ is persistent-interval on } Y. \]

\[ \text{Set } \Theta_0 \equiv \phi_0. \]

\[ \text{Suppose that for some } k \in [0, \nu_0 - 1], \text{ the subsequence } \Theta_0, \Theta_1, \ldots, \Theta_k \text{ is already constructed.} \]

\[ \text{Let} \]

\[ t_k \equiv \max(S_G(x_0, \Theta_k) \setminus [1, d_G(x_0)]), \]

\[ s_k \equiv \min([1, d_G(x_0)] \setminus S_G(x_0, \Theta_k)). \]

\[ \text{Clearly, } t_k > s_k. \]

\[ \text{Consider the path } P(k) \text{ in the graph } G \text{ of maximum length with the initial vertex } x_0 \text{ whose edges are alternatively colored by the colors } t_k \text{ and } s_k. \]

\[ \text{Let } \Theta_{k+1} \text{ is obtained from } \Theta_k \text{ by interchanging the two colors } t_k \text{ and } s_k \text{ along } P(k). \]

\[ \text{It is not difficult to see that } \Theta_{\nu_0} \text{ is persistent-interval on } \{x_0\} \cup Y. \]

\[ \text{Set } \psi \equiv \Theta_{\nu_0}. \]
Corollary 4. Let $G$ be a bipartite graph with bipartition $(X, Y)$ where $d_G(x) \leq d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Let $x_0$ be an arbitrary vertex of $X$. Then there exists a coloring $\varphi_0 \in \alpha(G, \Delta(G))$ which is persistent-interval on $\{x_0\} \cup Y$.

Corollary 5. Let $G$ be a bipartite graph with bipartition $(X, Y)$ where $d_G(x) \leq d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Then $\eta_{pi}(G) \geq 1 + |Y|$.

Remark 1. Notice that the complete bipartite graph $K_{n+1,n}$ for an arbitrary positive integer $n$ satisfies the conditions of Corollary 5. It is not difficult to see that $\eta_{pi}(K_{n+1,n}) = 1 + n$. It means that the bound obtained in Corollary 5 is sharp since in this case $|Y| = n$.

Remark 2. Let $G$ be a bipartite $(k-1,k)$-biregular graph with bipartition $(X,Y)$, where $k \geq 3$. Then the numbers $\frac{|X|}{k}$, $\frac{|Y|}{k-1}$, and $\frac{|V(G)|}{2k-1}$ are integer. It follows from the equalities $\gcd(k-1,k) = 1$ and $|E(G)| = |X| \cdot (k-1) = |Y| \cdot k$.

Theorem 6. Let $G$ be a bipartite $(k-1,k)$-biregular graph, where $k \geq 4$. Then

$$\eta_i(G) \geq \frac{k-1}{2k-1} \cdot |V(G)| + \left\lfloor \frac{k}{\left\lceil \frac{k}{2} \right\rceil} \cdot (2k-1) \cdot |V(G)| \right\rfloor.$$

Proof. Suppose that $(X,Y)$ is a bipartition of $G$. Clearly, $\chi'(G) = \Delta(G) = k$. Suppose that $\beta \in \alpha(G,k)$. For any $j \in [1,k]$, define:

$V_{G,\beta,j} \equiv \{x \in X / j \notin S_G(x,\beta)\}$.

For arbitrary integers $j', j''$, where $1 \leq j' < j'' \leq k$, we have

$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \emptyset$

and

$$\bigcup_{j=1}^{k} V_{G,\beta,j} = X.$$

For any $i \in [1, \left\lceil \frac{k}{2} \right\rceil]$, let us define the subset $V(G, \beta, i)$ of the set $X$ as follows:

$V(G, \beta, i) \equiv \begin{cases} V_{G,\beta,2i-1} \cup V_{G,\beta,2i}, & \text{if } k \text{ is odd and } i \in [1, \frac{k-1}{2}] \text{ or } k \text{ is even and } i \in [1, \frac{k}{2}], \\ V_{G,\beta,k}, & \text{if } k \text{ is odd and } i = \frac{1+k}{2}. \end{cases}$

For arbitrary integers $i', i''$, where $1 \leq i' < i'' \leq \left\lceil \frac{k}{2} \right\rceil$, we have

$V(G, \beta, i') \cap V(G, \beta, i'') = \emptyset$

and

$$\bigcup_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} V(G, \beta, i) = X.$$
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Hence, there exists $i_0 \in [1, \lceil \frac{k}{2} \rceil]$ for which

$$|V(G, \beta, i_0)| \geq \left\lfloor \frac{|X|}{\frac{k}{2}} \right\rfloor.$$

Set $R_0 \equiv Y \cup V(G, \beta, i_0)$.

It is not difficult to verify that

$$|R_0| \geq \frac{k-1}{2k-1} \cdot |V(G)| + \left\lfloor \frac{k}{\frac{k}{2}} \cdot (2k-1) \cdot |V(G)| \right\rfloor.$$

Case 9 $i_0 = \lceil \frac{k}{2} \rceil$.

Case 10 $1.a$ $k$ is odd.

Clearly, $\beta$ is interval on $R_0$.

Case 11 $1.b$ $k$ is even.

Define a function $\phi : E(G) \to [1, k]$. For any $e \in E(G)$, set:

$$\phi(e) \equiv \begin{cases} (\beta(e) + 1)(\mod k), & \text{if } \beta(e) \neq k - 1, \\ k, & \text{if } \beta(e) = k - 1. \end{cases}$$

It is not difficult to see that $\phi \in \alpha(G, k)$ and $\phi$ is interval on $R_0$.

Case 12 $2. i_0 \in [1, \lceil \frac{k}{2} \rceil - 1]$.

Define a function $\phi : E(G) \to [1, k]$. For any $e \in E(G)$, set:

$$\phi(e) \equiv \begin{cases} (\beta(e) + 1 + k - 2i_0)(\mod k), & \text{if } \beta(e) \neq 2i_0 - 1, \\ k, & \text{if } \beta(e) = 2i_0 - 1. \end{cases}$$

It is not difficult to see that $\phi \in \alpha(G, k)$ and $\phi$ is interval on $R_0$.

Corollary 6 [17] Let $G$ be a bipartite $(k-1, k)$-biregular graph, where $k$ is even and $k \geq 4$. Then

$$\eta_i(G) \geq \frac{k+1}{2k-1} \cdot |V(G)|.$$  

Corollary 7 [17] Let $G$ be a bipartite $(3, 4)$-biregular graph. Then there exists a coloring from $\alpha(G, 4)$ which is interval for at least $\frac{5}{7}|V(G)|$ vertices of $G$.

Remark 3 For an arbitrary bipartite graph $G$ with $\Delta(G) \leq 3$, there exists an interval coloring of $G$ [12, 10, 17]. Consequently, if $G$ is a bipartite $(2, 3)$-biregular graph, then $\eta_i(G) = |V(G)|$.

Remark 4 Some sufficient conditions for existence of an interval coloring of a $(3, 4)$-biregular bipartite graph were obtained in [2, 5, 20].
Theorem 7 [17] Let $G$ be a bipartite $(k - 1, k)$-biregular graph, where $k \geq 3$. Then

$$\eta_{pi}(G) \geq \frac{k}{2k - 1} \cdot |V(G)|.$$ 

Proof. Suppose that $(X, Y)$ is a bipartition of $G$. Clearly, $\chi'(G) = \Delta(G) = k$. Suppose that $\beta \in \alpha(G, k)$.

For any $j \in [1, k]$, define:

$$V_{G, \beta, j} \equiv \{x \in X/j \notin S_G(x, \beta)\}.$$ 

For arbitrary integers $j', j''$, where $1 \leq j' < j'' \leq k$, we have

$$V_{G, \beta, j'} \cap V_{G, \beta, j''} = \emptyset$$

and

$$\bigcup_{j=1}^{k} V_{G, \beta, j} = X.$$ 

Hence, there exists $j_0 \in [1, k]$ for which

$$|V_{G, \beta, j_0}| \geq \frac{|X|}{k}.$$ 

Set $R_0 \equiv Y \cup V_{G, \beta, j_0}$.

It is not difficult to verify that

$$|R_0| \geq \frac{k}{2k - 1} \cdot |V(G)|.$$ 

Case 13 $j_0 = k$.

Clearly, $\beta$ is persistent-interval on $R_0$.

Case 14 $j_0 \in [1, k - 1]$.

Define a function $\varphi : E(G) \to [1, k]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} 
\beta(e), & \text{if } \beta(e) \notin \{j_0, k\} \\
$j_0$, & \text{if } \beta(e) = k \\
k, & \text{if } \beta(e) = j_0.
\end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, k)$ and $\varphi$ is persistent-interval on $R_0$.

Corollary 8 [17] Let $G$ be a bipartite $(3, 4)$-biregular graph. Then there exists a coloring from $\alpha(G, 4)$ which is persistent-interval for at least $\frac{4}{7}|V(G)|$ vertices of $G$.

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