DECOMPOSITIONS OF BINOMIAL IDEALS IN MACAULAY 2

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Abstract. The package Binomials contains implementations of specialized algorithms for binomial ideals, including primary decomposition into binomial ideals. The current implementation works in characteristic zero. Primary decomposition is restricted to binomial ideals with trivial coefficients to avoid computations over the algebraic numbers. The basic ideas of the algorithms go back to Eisenbud and Sturmfels’ seminal paper on the subject. Two recent improvements of the algorithms are discussed and examples are presented.

1. Binomial ideals

Let $S = k[x_1, \ldots, x_n]$ denote the standard polynomial ring over a field $k$. A binomial ideal $I \subseteq S$ is an ideal generated by binomials $x^u - \lambda x^v$, where $u, v \in \mathbb{N}^n$ are exponent vectors and $\lambda \in k$ is a coefficient. Monomials are also considered binomials. Assumptions on $k$ will be forced upon us when computing primary decompositions. The ideal $\langle x^3 - 1 \rangle$ has no primary decomposition into binomial ideals when $k$ does not contain a third root of unity. Interest in binomial ideals is due to the frequency with which they arise in applications. To name one, in algebraic statistics one is interested in primary decompositions of conditional independence ideals whose components describe various combinatorial ways in which a set of conditional independence statements can be realized [3, 5]. Because the minimal primes of binomial ideals are toric ideals [2], binomial conditional independence models are unions of exponential families. In particular they are unirational. Knowledge of a primary decomposition also gives a piecewise parameterization of such models.

The new Macaulay 2 [4] package Binomials offers specialized implementations of primary decomposition, radical computations and minimal and associated primes. The starting point for this implementation was Section 9 in Eisenbud and Sturmfels’ foundational paper [2], but various improvements have been discovered and implemented. Binomials is the fastest and often only way to compute large primary decompositions of binomial ideals.

Example 1.

```plaintext
i1 : needsPackage "Binomials"
i2 : R = QQ[x,y]
i3 : I = ideal (x^2-x*y, x*y-y^2)
i4 : binomialPrimaryDecomposition I
[...]
2
o4 = {ideal(x - y), ideal (x, y )}
```

A binomial primary decomposition starts with a cellular decomposition. Recall that a binomial ideal $I \subseteq S$ is cellular if in $S/I$ every monomial is either regular (i.e. a nonzerodivisor) or nilpotent. The implemented algorithm to compute a cellular decomposition is discussed
Since cellular decomposition is independent of \( k \), it can serve as a first approximation of primary decomposition over any field.

In this paper we focus on decomposing a cellular binomial ideal further. To this end, assume that \( I \) is \( J \)-cellular for some \( J \subseteq [n] \), that is, the variables with indices in \( J \) are regular, while the variables with indices in \( \mathcal{J} := [n] \setminus J \) are nilpotent.

2. Computing associated primes

If \( k \) is algebraically closed, then the associated primes of a binomial ideal are guaranteed to be binomial. Since computer algebra system usually don’t implement algebraically closed fields, the input binomial ideals are restricted to be generated by unital binomials \( x^u - x^v \).

In this case the binomial primary decomposition together with the associated primes exist over a cyclotomic extension of \( \mathbb{Q} \). If necessary, Binomials will construct this extension and return its result over a different ring.

Example 2.

```plaintext
i1 : R = QQ[x]
i2 : I = ideal(x^3-1)
i3 : BPD I
 [...] 
o3 = {ideal(x - 1), ideal(x - \text{ww}), ideal(x + \text{ww} + 1)}
```

In the following discussion we will assume \( k \) to be algebraically closed and of characteristic zero. Let \( I \subseteq S \) be a \( J \)-cellular binomial ideal. Let \( I_{\rho,J} + m_J \) be an associated prime of \( I \), then there exists a monomial \( m \in k[\mathcal{J}] \), and a character \( \tau \) on \( \mathbb{Z}^J \) whose saturation is \( \sigma \), such that

\[
(I : m) \cap k[\mathcal{J}] = I_{\tau}.
\]

It can be seen that the converse also holds. Every associated prime of any occurring lattice ideal is associated to \( I \). Theorem 3 shows that a sub-problem in the computation of associated primes is to determine the set of lattice ideals of the form \((I : m) \cap k[\mathcal{J}]\).

Definition 4 ([7]). A lattice \( L \subseteq \mathbb{Z}^J \) is potentially associated to \( I \) if there exists a witness monomial \( m \in k[\mathcal{J}] \) such that \((I : m) \cap k[\mathcal{J}] = I_{\rho,J} \) for some character \( \rho : L \rightarrow k^* \).

Properly defining the set of associated lattices, which is contained in the set of potentially associated lattices, requires care and is one of the topics of [7]. For computational purposes the few lattices that are potentially associated, but not associated, play a minor role. They
will eventually yield redundant primary decompositions, a problem that has to be handled in any case, since cellular decomposition introduces redundancy on a large scale. The lattice ideals \( (I : m) \cap k[J] \) are partially ordered by inclusion, and so are their lattices.

**Definition 5.** A potentially associated lattice is called *embedded*, if it properly contains the lattice of \( I \cap k[J] \).

A first algorithm to find potentially associated lattices would examine all ideals \( (I : m) \) where \( m \) is a nonzero monomial in \( k[J]/(I \cap k[J]) \). By cellularity of \( I \) there are only finitely many such monomials and this search will terminate. The associated primes algorithm in *Binomials* instead uses a random search. The set of monomials to be examined can be very large compared to relatively few potentially associated lattices. The design goal in Algorithm 1 is to compute as few colon ideals \( (I : m) \) as possible. If a monomial \( m \) divides a monomial \( n \), then \( (I : m) \subseteq (I : n) \) and containment also holds for the potentially associated lattices. Due to this fact we can exclude large posets of monomials if we find two monomials with the same potentially associated lattice.

**Algorithm 1.**

**Input:** A \( J \)-cellular binomial ideal \( I \).

**Output:** The potentially associated lattices of \( I \)

1. Compute the lattice ideal \( I \cap k[J] \).
2. Initialize a list of known potentially associated lattices and witnesses containing only the pair \( (I \cap k[J], 1) \).
3. Initialize a todo-list with all monomials in a \( k \)-basis of \( k[J]/(I \cap k[J]) \).
4. Iterate the following until the todo-list is empty
   - Choose and remove a random monomial \( m \) from the todo-list. Compute the lattice ideal \( (I : m) \cap k[J] \) and check if its lattice is already on the list of potentially associated lattices.
     - If yes, then add \( m \) as a new witness for that lattice, remove from the todo-list every monomial between existing witnesses and \( m \).
     - If no, then add \( (I \cap k[J], m) \) to the list of potentially associated lattices.

To save space and time, the implementation in *Binomials* does not save all the witness monomials. If \( m, n \) are both witnesses for the same potentially associated lattice and \( m | n \), then only \( m \) needs to saved.

Given the set of potentially associated lattices, determining the associated primes is easy. It consists of saturating characters and will not be discussed here. The necessary cyclotomic extensions are handled in a separate package *Cyclotomic*, published together with *Binomials*.

3. **Computing minimal primary components**

Let \( I \subseteq S \) be a \( J \)-cellular binomial ideal, and \( P = I_{\rho,J} + m_J \) one of its associated primes. Eisenbud and Sturmfels show that any primary component of \( I \) over \( P \) contains \( I_{\rho,J} \). In fact, \( I + I_{\rho,J} \) has \( P \) as its unique minimal prime and a primary component is computed by removing all embedded primary components from \( I + I_{\rho,J} \). This is the content of [2, Theorem 7.1]. Let \( \text{Hull}(I) \) denote the intersection of the minimal primary components of a binomial ideal \( I \). If \( I \) is cellular, then \( \text{Hull}(I) \) is binomial. Computing \( \text{Hull} \) of a binomial ideal is a cumbersome procedure. One way, described in [2], is to successively identify binomials \( b \) such that \( (I : b) \)
is a binomial ideal strictly containing $I$. This approach is slow. Here we will use a similar strategy like in Algorithm 1. Denote $M_{\text{emb}}(I)$ the monomial ideal generated by all witnesses of embedded lattices of $I$. Then [1] Theorem 3.2 implies the following simplification.

**Proposition 6.** If $I$ is $J$-cellular and has exactly one minimal prime, then

$$\text{Hull}(I) = I + M_{\text{emb}}(I)$$

In particular Hull$(I)$ is binomial.

To compute the minimal primary component of $I$ over $P = I_{\rho,J} + m_J$ one computes Hull$(I + I_{\rho,J})$ [2, 8]. The monomial ideal $M_{\text{emb}}(I + I_{\rho,J})$ is determined essentially by Algorithm 1. It is in fact simpler, since only minimal generators of $M_{\text{emb}}(I)$ need to be computed. In most cases only a small fraction of the standard monomials needs to be examined.

**Example 7.** This example demonstrates how a component of high multiplicity leads to many monomials to be examined.

```plaintext
i1 : R = QQ[a,b]
i2 : I = ideal (a^10000 * (b-1))
i3 : BPD I
    10000
o3 = {ideal(b - 1), ideal(a )}
```

In this case the poset of nilpotent monomials is totally ordered and a typical run of Algorithm 1 would only compute $\lceil \log_2(10000) \rceil = 14$ lattice ideals. Since the structure of the poset of embedded associated lattices can be complicated it is not known if there are better search algorithms than random search.

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**References**

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