Combinatorial computation of the motivic Poincare series

E. Gorsky

July 3, 2008

Abstract

We give the explicit algorithm computing the motivic generalization of the Poincare series of the plane curve singularity introduced by A. Campillo, F. Delgado and S. Gusein-Zade. It is done in terms of the embedded resolution of the curve. The result is a rational function depending of the parameter $q$, at $q = 1$ it coincides with the Alexander polynomial of the corresponding link. For irreducible curves we relate this invariant to the Heegard-Floer knot homologies constructed by P. Ozsvath and Z. Szabo. Many explicit examples are considered.
# Contents

1 Introduction 3

2 Poincare series and its generalization 8
   2.1 Poincare series 8
   2.2 Motivic measure 9
   2.3 Irreducible case 11
   2.4 Formula of Campillo, Delgado and Gusein-Zade 12

3 Example: nonsingular curve 14

4 Combinatorics 15
   4.1 Preliminary simplification 15
   4.2 Cancellations 19
   4.3 The algorithm 23

5 Examples 24
   5.1 One divisor 24
   5.2 Two divisors 25
   5.3 Three divisors 26

6 Symmetry and functional equations 28
   6.1 Symmetry of the motivic Poincare series 28
   6.2 Analogue of the Kapranov’s functional equation 30

7 Relation to the Heegard-Floer knot homologies 32
   7.1 Heegard-Floer homologies 32
   7.2 Matching the answers 33
   7.3 Relative $Spin^c$ structures 34
   7.4 Comparing filtered complexes 36
   7.5 Example: $A_{2n-1}$ singularities 40
1 Introduction

In the series of articles (e.g. [3], [4]) A. Campillo, F. Delgado and S. Gusein-Zade proved that the Alexander polynomial of the link of the plane curve singularity is related to the generating function arising in the purely algebraic setup.

Let \( C = \bigcup_{i=1}^{r} C_i \) be a germ of a plane curve,

\[ \gamma_i : (\mathbb{C}, 0) \to (C_i, 0) \]

are the uniformizations of its components. If \( f \in \mathcal{O} = \mathcal{O}_{C_2, 0} \) is a germ of function on \((\mathbb{C}^2, 0)\), we define

\[ v_i(f) = \text{Ord}_0 f(\gamma_i(t)) \]

and the Poincare series of the curve \( C \) is defined ([4]) as the integral with respect to the Euler characteristic

\[ P_C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdots t_r^{v_r} d\chi, \tag{1} \]

where \( \mathbb{P}\mathcal{O} \) denotes the projectivisation of \( \mathcal{O} \) as a vector space. For example, if \( C \) is irreducible, we can define the decreasing filtration

\[ \mathcal{O} \supset J_1 \supset J_2 \supset \ldots, \quad J_n = \{ f \in \mathcal{O} | v_1(f) \geq n \}, \tag{2} \]

and

\[ P_C(t) = \sum_{n=0}^{\infty} t^n \dim J_n/J_{n+1}. \tag{3} \]

Let \( \Delta^C(t_1, \ldots, t_n) \) denote the Alexander polynomial of the intersection of \( C \) with a small sphere centered at the origin. The theorem of Campillo, Delgado and Gusein-Zade says that if \( r = 1 \), then

\[ (1 - t)P_C(t) = \Delta^C(t), \tag{4} \]

and if \( r > 1 \), then

\[ P_C(t_1, \ldots, t_r) = \Delta^C(t_1, \ldots, t_r). \]

In [5] there was proposed the following natural generalization of the Poincare series. One can naturally define the motivic measure on the space of functions, and consider the following motivic integral, generalizing ([4]):

\[ P_g^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdots t_r^{v_r} d\mu. \tag{5} \]
If \( r = 1 \), we can rewrite (5) as the generalization of (3):

\[
P_g^C(t) = \sum_{n=0}^{\infty} t^n q^{\text{codim} J_n} - q^{\text{codim} J_{n+1}} \frac{1}{1 - q},
\]

so in this case one can deduce \( P_g(t) \) from \( P(t) \). If \( r \) is greater than 1, the situation becomes more complicated: the motivic Poincare series is not determined by the ordinary one and the method of its computation is more complicated. Nevertheless, the explicit algorithm is presented below (theorem 3). We also need the following

**Definition:** The reduced motivic Poincare series is the power series

\[
\overline{P}_g(t_1, \ldots, t_r) = (1 - qt_1) \cdot \ldots \cdot (1 - qt_r) \cdot P_g(t_1, \ldots, t_r).
\]

We prove that the reduced motivic Poincare series satisfies the following properties.

1. **Polynomiality.** \( \overline{P}_g(t_1, \ldots, t_n; q) \) is a polynomial in \( t_1, \ldots, t_n \) and \( q \). We give a bound for its degree on \( t_1, \ldots, t_n \).

2. **Reduction to the Alexander polynomial.** If \( n = 1 \), then

\[
\overline{P}_g(t; q = 1) = \Delta(t),
\]

where \( \Delta \) denote the Alexander polynomial of the link of the corresponding plane curve singularity. If \( n > 1 \), then

\[
\overline{P}_g(t_1, \ldots, t_n; q = 1) = \Delta(t_1, \ldots, t_n) \cdot \prod_{i=1}^{n} (1 - t_i).
\]

3. **Forgetting components.** Let \( C \) be a curve with \( n \) components, and \( C_1 \) be an irreducible curve. Then

\[
\overline{P}_g^{C \cup C_1}(t_1, \ldots, t_n, t_{n+1} = 1) = (1 - q) \overline{P}_g^C(t_1, \ldots, t_n).
\]

If \( C \) has only one component, then

\[
\overline{P}_g^C(t = 1) = 1.
\]

This property is clear from the equation (5), but seems to be curious and, for example, does not hold for the Alexander polynomial (we cannot reconstruct the Alexander polynomial of a sublink from the Alexander polynomial of a link).
4. **Symmetry.** Let $\mu_\alpha$ be the Milnor number $[2]$ of $C_\alpha$, $(C_\alpha \circ C_\beta)$ is the intersection index of $C_\alpha \circ C_\beta$, $\mu(C)$ is the Milnor number of $C$. Let

$$l_\alpha = \mu_\alpha + \sum_{\beta \neq \alpha} (C_\alpha \circ C_\beta), \quad \delta(C) = (\mu(C) + r - 1)/2.$$ 

It is known that the Alexander polynomial is symmetric in a sense that

$$\Delta(t_1^{-1}, \ldots, t_n^{-1}) = \prod t_\alpha^{-l_\alpha} \cdot \Delta(t_1, \ldots, t_n).$$

We prove the generalization of this identity, namely,

$$P_g\left(\frac{1}{qt_1}, \ldots, \frac{1}{qt_r}\right) = q^{-\delta(C)} \prod_{\alpha} t_\alpha^{-l_\alpha} \cdot P_g(t_1, \ldots, t_r).$$

In more knot-theoretic language, $\mu_\alpha$ is equal to the genus of the link of $C_\alpha$ multiplied by 2, and $(C_\alpha \circ C_\beta)$ is equal to the linking number of the corresponding link components. Another remark is the identity

$$\sum_{\alpha=1}^{r} l_\alpha = 2\delta(C).$$

5. **Relation to the knot homologies.** For irreducible curves we prove that $P_g(t)$ can be related by the simple procedure with the Poincare polynomial of the Heegard-Floer knot homologies constructed by P. Ozsvath and Z. Szabo. These homologies are the different ”categorification” of the Alexander polynomial, tightly related with the symplectic topology and Seiberg-Witten theory. Since the origins of our and their construction are quite far, the relation between them seems to be interesting. No conceptual proof for this fact is known, and we just use that both answers are determined by the Alexander polynomial in the same way.

The paper is organized in the following way. In the section 2 we recall the definition of the Poincare series of a plane curve singularity. Then we recall the definition of the motivic measure on the space of functions and give, following $[5]$, two definitions of the motivic Poincare series as a motivic integral and in terms of the multi-index filtration associated with the curve. We give the simple method of deduction of the motivic Poincare series for irreducible curves from the ordinary Poincare series. In Theorem 2 we recall the formula from $[5]$ expressing the motivic Poincare series in terms of the
embedded resolution of a curve. This formula is proved by Campillo, Delgado and Gusein-Zade using thorough analysis of the geometry of the embedded resolution of a curve.

In the section 3 we apply the Theorem 2 to a nonsingular curve and explain step-by-step the calculation of all sums involved. It turns out to be a curious exercise, and this simplest example is a toy model for the consequent combinatorial work.

The section 4 contains several steps of the simplification of Theorem 2. In the result (lemma 6) the motivic Poincare series is expressed in terms of some quantities $c_K(n)$. In lemma 5 the generating function for these quantities is explicitly written. Directly applying lemma 6, we get a lot of similar summands which cancel after all substitutions, but this cancellation is not clear from lemmas 5 and 6. For example, it is not even clear, that the answer is a polynomial.

Therefore in the rest of section 4 we discuss the analogues of the identity
\[
\sum_{n=0}^{\infty} t^n q^{n^2+n}(q^{-n} - tq) = 1
\]

arising in the nonsingular case. The result of this investigation is Theorem 3, where we formulate an explicit algorithm of calculation of the motivic Poincare series. This algorithm does not involve infinite sums, and can be successively realised as a short Mathematica program.

The answer is presented in the same manner: the motivic Poincare series is expressed in terms of some quantities $d_P(n)$, which fits into the explicitly written generating function $H_P(u)$. This function is generally more complicated, than the one from lemma 5, but in some examples (lemma 9) it is more or less compact.

Section 5 contains a bunch of explicit answers for the resolutions containing up to 3 divisors.

In the section 6 we prove the symmetry property for the motivic Poincare series (Theorem 4). It generalizes the known symmetry property for the Alexander polynomial of a link. From the viewpoint of the algebraic geometry, it is related to the Gorenstein property of the coordinate ring of a curve ([6]), and, on the other hand, to the Serre duality on the components of the exceptional divisor, which is the origin of the Kapranov’s functional equation ([10],[9]) for the motivic zeta function.

The main result of the section 7 is Theorem 6 describing the remarkable relation between the motivic Poincare series and the another deformation of the Alexander polynomial, namely, the Poincare polynomial for the Heegard-Floer knot homologies ([16],[17]). It is proved using the known algorithms.
of deduction of the Heegard-Floer homologies and motivic Poincare series from the Alexander polynomial. We also give some corollaries from this fact which seems to carry more geometry. A filtered complex of \( \mathbb{Z}[U] \)-modules analogous to the complex \( \text{CF}_L^{-}(K) \) is constructed. We also compare the motivic Poincare series with the Heegard-Floer homologies of two-component links, corresponding to the singularities of type \( A_{2n-1} \).

**Acknowledgements**

This work is partially supported by the grants RFBR-007-00593, RFBR-08-01-00110-a, NSh-709.2008.1 and the Moebius Contest fellowship for young scientists.

The author is grateful to M. Bershtein, A. Gorsky, S. Gukov, S. Gusein-Zade, G. Gusev and A. Kustarev for useful discussions and remarks. Special thanks to A. Beliakova for her impressive lecture on the Heegard-Floer homologies at the University of Zurich and to J. Rasmussen for his interest to this work.
2 Poincare series and its generalization

2.1 Poincare series

Let $C = \bigcup_{i=1}^{r} C_i$ be a reduced plane curve singularity at the origin in $\mathbb{C}^2$, and $C_i$ are its irreducible components. Let $\gamma_i : (\mathbb{C}, 0) \rightarrow (C_i, 0)$ are uniformizations of these components.

We define $r$ integer-valued functions on the space $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2, 0}$ by the formula

$$v_i(f) = \text{Ord}_0(f(\gamma_i(t)))$$

and $\mathbb{Z}^r$-indexed filtration

$$J_{\underline{v}} = \{ f \in \mathcal{O} | v_i(f) \geq v_i \}.$$ 

Note that $J_{\underline{v}}$ are also defined for negative values of $v$. This filtration is decreasing in a sense that if $\underline{v}_1 \prec \underline{v}_2$, then $J_{\underline{v}_1} \supset J_{\underline{v}_2}$. Consider the Laurent series

$$L_C(t_1, \ldots, t_r) = \sum_{\underline{v}} v_1^{v_1_1} \ldots t_r^{v_r} \cdot \dim J_{\underline{v}} / J_{\underline{v}+1}.$$ 

**Definition:**([6], [3]) We define the Poincare series of the curve $C$ by the formula

$$P_C(t_1, \ldots, t_r) = \frac{L_C(t_1, \ldots, t_r) \cdot \prod_{i=1}^{r} (t_i - 1)}{t_1 \ldots t_r - 1}.$$ 

For example, if $r = 1$, we have

$$P_C(t) = \sum_{v=0}^{\infty} t^v \cdot \dim J_v / J_{v+1}.$$ 

One can prove, that $P_C$ is always a power series. More geometric meaning of this definition is given by the following

**Proposition.**([4])

$$P_C(t_1, \ldots, t_r) = \int_{\mathcal{O}} t_1^{v_1} \ldots t_r^{v_r} d\chi.$$ 

(9)

On the other hand, we have a link of $C$ – the intersection of $C$ with a small three-dimensional sphere centered at the origin. We denote its Alexander polynomial as $\Delta_C(t_1, \ldots, t_r)$. Campillo, Delgado and Gusein-Zade in [4] proved the following
Theorem 1. If $r = 1$, then
\[ P_C(t)(1 - t) = \Delta_C(t), \quad \text{(10)} \]
and if $r > 1$, then
\[ P_C(t_1, \ldots, t_r) = \Delta_C(t_1, \ldots, t_r). \quad \text{(11)} \]

2.2 Motivic measure

Let $\mathcal{L} = \mathcal{L}_{C^2,0}$ be the space of arcs at the origin on the plane. It is the set of pairs $(x(t), y(t))$ of formal power series (without degree 0 term). Let $\mathcal{L}_n$ be the space of $n$-jets of such arcs, let $\pi_n : \mathcal{L} \to \mathcal{L}_n$ be the natural projection.

Let $K_0(Var_{\mathbb{C}})$ be the Grothendieck ring of quasiprojective complex algebraic varieties. It is generated by the isomorphism classes of complex quasiprojective algebraic varieties modulo the relations $[X] = [Y] + [X \setminus Y]$, where $Y$ is a Zariski closed subset of $X$. Multiplication is given by the formula $[X] \cdot [Y] = [X \times Y]$. Let $\mathbb{L} \in K_0(Var_{\mathbb{C}})$ be the class of the complex affine line.

The Euler characteristic provides a ring homomorphism
\[ \chi : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}. \]

Consider the ring $K_0(Var_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ with the following filtration: $F_k$ is generated by the elements of the type $[X] \cdot [\mathbb{L}^{-n}]$ with $n - \dim X \geq k$. Let $\mathcal{M}$ be the completion of the ring $K_0(Var_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ corresponding to this filtration.

On an algebra of subsets of $\mathcal{L}$ Kontsevich, and later Denef and Loeser ([7]) constructed a measure $\mu$ with values in the ring $\mathcal{M}$.

A subset $A \subset \mathcal{L}$ is said to be cylindrical if there exist $n$ and a constructible set $A_n \subset \mathcal{L}_n$ such that $A = \pi_n^{-1}(A_n)$. For the cylindrical set $A$ define
\[ \mu(A) = [A_n] \cdot \mathbb{L}^{-2n}. \]

It was proved in [7], that this measure can be extended to an additive measure on a suitable algebra of subsets in $\mathcal{L}$.

A function $f : \mathcal{L} \to G$ with values in an abelian group $G$ is called simple, if its image is countable or finite, and for every $g \in G$ the set $f^{-1}(g)$ is measurable. Using this measure, one can define in the natural way the (motivic) integral for simple functions on $\mathcal{L}$ as $\int_{\mathcal{L}} f \, d\mu = \sum_{g \in G} g \cdot \mu(f^{-1}(g))$, if the right hand side sum converges in $G \otimes \mathcal{M}$.

Note that for cylindrical sets the Euler characteristic can be well defined by the formula $\chi(A) = \chi(A_n)$. This gives a $\mathbb{Z}$-valued measure on the algebra of cylindrical sets. However, it cannot be extended to the algebra of measurable
sets. This measure provides a notion of an integral with respect to the Euler characteristic for functions on \( L \) with cylindric level sets. It is clear that for such functions
\[
\chi(\int_L f d\mu) = \int_L f d\chi.
\]

We will use some simple functions, e.g.
\[
v_x = \text{Ord}_0 x(t), \quad v_y = \text{Ord}_0 y(t)
\]
and
\[
v = \min\{v_x, v_y\},
\]
defined for an arc \( \gamma(t) = (x(t), y(t)) \).

Campillo, Delgado and Gusein-Zade \([5]\) constructed an analogous measure on the space \( \mathcal{O}_{\mathbb{C}^2,0} = \mathcal{O} \) of germs of analytic functions on the plane at the origin. Let \( j_n(\mathcal{O}) \) be the space of \( n \)-jets of functions from \( \mathcal{O} \). A subset \( A \subset \mathcal{O} \) is said to be cylindric if there exist \( n \) and a constructible set \( A_n \subset j_n(\mathcal{O}) \) such that \( A = j_n^{-1}(A_n) \). For the cylindric set \( A \) define
\[
\mu(A) = [A_n] \cdot \mathbb{L}^{-\frac{(n+1)(n+2)}{2}}.
\]

In the same way one can define the motivic integral over the space of functions.

As a direct generalisation of the equation \([9]\) Campillo, Delgado and Gusein-Zade proposed the following

**Definition**: Motivic Poincare series is the motivic integral
\[
P^C_g(t_1, \ldots, t_r) = \int_{\mathbb{P} \mathcal{O}} t_1^{v_1} \cdots t_r^{v_r} d\mu
\]
(12)

As above, this definition can be reformulated in terms of the multi-index filtration on the space of functions. Let \( q = \mathbb{L}^{-1} \) be a formal variable. Let \( h(\underline{v}) = \text{codim} J_{\underline{v}} \), and
\[
L_g(t_1, \ldots, t_r, q) = \sum_{\underline{v} \in \mathbb{Z}^r} \frac{q^{h(\underline{v})} - q^{h(\underline{v}+\underline{1})}}{1-q} \cdot t_1^{v_1} \cdots t_r^{v_r}.
\]

Then the following equation holds \([5]\):
\[
P^C_g(t_1, \ldots, t_r; q) = \frac{L^C_g(t_1, \ldots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 \cdots t_r - 1}.
\]
(13)

An example of the calculation of the motivic Poincare series for the singularities of type \( A_{2n-1} \) directly from the equation \([13]\) is presented in the section 7.5 below.
2.3 Irreducible case

If $r = 1$, the equation (13) has a very clear form. First, in this case $P^C_g(t) = L^C_g(t)$. Second, remark that

$$\text{codim} J_v = \dim \mathcal{O}/J_1 + \dim J_1/J_2 + \ldots + \dim J_{v-1}/J_v,$$

so the series $P^C_g(t)$ can be reconstructed from the series $P_C(t)$. Namely, the coefficient at $t^v$ in $P_C(t)$ vanishes, if $J_v = J_{v+1}$, and equals to 1 otherwise. Therefore we have

$$P_C(t) = 1 + t^{\sigma_1} + t^{\sigma_2} + t^{\sigma_3} + \ldots,$$

where $\sigma_i$ form the increasing sequence of integers.

This sequence has itself the nice description. The functional $v(f) = \text{Ord}_0 f(\gamma(t))$ is a valuation on the ring $\mathcal{O}$. This means that $v(fg) = v(f) + v(g)$ and $v(f + g) \geq \min(v(f), v(g))$. The set of values of $v$ is a semigroup in $\mathbb{N}$, and one can prove that this semigroup coincides with \{$\sigma_1, \sigma_2, \sigma_3, \ldots$\}. For example, for the singularity $x^p = y^q$ (its link is the torus $(p, q)$ knot) we have $x(t) = t^q, y(t) = t^p$, so the corresponding semigroup is generated by $p$ and $q$.

Now the equation (14) implies the following formula for the motivic Poincare series:

$$P^C_g(t; q) = 1 + qt^{\sigma_1} + q^2 t^{\sigma_2} + q^3 t^{\sigma_3} + \ldots.$$ 

Example. Consider the cusp $x^2 = y^3$. Its semigroup is generated by 2 and 3, the Poincare series is equal to

$$P(t) = 1 + t^2 + t^3 + t^4 + \ldots,$$

the motivic Poincare series is equal to

$$P_g(t) = 1 + qt^2 + q^2 t^3 + q^3 t^4 + \ldots.$$ 

Note that

$$P(t)(1 - t) = 1 - t + t^2,$$

what is Alexander polynomial of the trefoil knot in the intersection with a small sphere.
2.4 Formula of Campillo, Delgado and Gusein-Zade

In [5] Campillo, Delgado and Gusein-Zade gave a formula for the generalized Poincare series in terms of the resolution.

Let $\pi : (X, D) \to (\mathbb{C}^2, 0)$ be an embedded resolution where $D = \cup_{i=1}^{s} E_i$ is the exceptional divisor. Let $E_i^*$ be $E_i$ without intersection points of $E_i$ with other components of $D$, $E_i^o$ be $E_i^*$ without intersection points of $E_i$ with the components of the strict transform of our curve. Let $A = (E_i \circ E_j)$ be the intersection matrix and $M = -A^{-1}$.

Let $I_0 = \{(i, j) : i < j, E_i \cap E_j = pt\}$, $K_0 = \{1, \ldots, r\}$. For $\sigma \in I_0$, $\sigma = (i, j)$ let $i(\sigma) = i$, $j(\sigma) = j$. For $I \subset I_0$, $K \subset K_0$ let

$$\mathcal{N}_{I, K} := \{n = (n_i, n'_\sigma, n''_\sigma, \tilde{n}'_k, \tilde{n}''_k) : n_i \geq 0, i = 1 \ldots, s$$
$$n'_\sigma, n''_\sigma, \sigma \in I; \tilde{n}'_k > 0, \tilde{n}''_k > 0, k \in K\}.$$

For $n \in \mathcal{N}_{I, K}, i = 1, \ldots, s$, let

$$\hat{n}_i = n_i + \sum_{\sigma \in I : i(\sigma) = i} n'_\sigma + \sum_{\sigma \in I : j(\sigma) = i} n''_\sigma + \sum_{k \in K : i(k) = i} \tilde{n}'_k. \quad (16)$$

Let

$$F(n) = \frac{1}{2}(\sum_{i,j=1}^{s} m_{ij}\hat{n}_i\hat{n}_j + \sum_{i=1}^{s} \hat{n}_i(\sum_{j=1}^{s} m_{ij}\chi(E_j^*) + 1)) + \sum_{k \in K} \tilde{n}''_k, \quad (17)$$

$$F(\hat{n}) = \frac{1}{2}(\sum_{i,j=1}^{s} m_{ij}\hat{n}_i\hat{n}_j + \sum_{i=1}^{s} \hat{n}_i(\sum_{j=1}^{s} m_{ij}\chi(E_j^*) + 1)),$$

and

$$w(n) = \sum_{i=1}^{s} \hat{n}_i m_i, v_k(n) := w_{i(k)}(n) + \tilde{n}''_k.$$

Theorem 2 ([5])

$$P_g(t_1, \ldots, t_r, q) = \sum_{I \subset I_0, K \subset K_0} \sum_{n \in \mathcal{N}_{I, K}} q^{F(n)-\sum_{i=1}^{s} n_i - |I| - |K|} \cdot (1 - q)^{|I|+|K|} \times$$

$$\times \prod_{i=1}^{s} \left(\min(n_i, 1 - \chi(E_i^o)) \sum_{j=0}^{\min(n_i, 1 - \chi(E_i^o))} (-1)^j \binom{1 - \chi(E_i^o)}{j} q^j\right) \cdot w(n).$$
We briefly recall the sketch of the proof from [5]. Consider a function $f \in O$ and its pullback $\pi^*f$ on the space of resolution $X$. Now let $I(f)$ be the set of intersection points in $D$ such that there are components of the strict transform of $X$ passing through them, $K(f)$ is the analogous set of intersection points of strict transform of $C$ with $D$. Now $n_i(f)$ is the intersection index of the strict transform of $f$ with the smooth part of $E_i$, $n'_\sigma$ and $n''_\sigma$ are intersection indices of the component of the strict transform of $f$ passing through $\sigma$ with $E_i(\sigma)$ and $E_j(\sigma)$ respectively, $\tilde{n}'_k$ and $\tilde{n}''_k$ are intersection indices of the component passing through the point $k$ with $E_i(k)$ and corresponding component of $C$ respectively.

Given these sets and multiplicities, the value of the function $t_1^v(f) \cdot \ldots \cdot t_r^v(f)$ is equal to $t^v(\mathfrak{m})$. Every summand in Theorem 2 is equal to this value multiplied by the motivic measure of the set of functions providing such set of data.
3 Example: nonsingular curve

Let us check that for the nonsingular curve the complicated expression from Theorem 2 coincides with the expected one.

We have one divisor and one component of the strict transform of the curve. We have $I_0 = \emptyset$, $K_0 = \{1\}$. Also we have $\chi(E^0) = 1$, $\chi(E^*) = 2$, so $1 - \chi(E^0) = 0$.

1) $K = \emptyset$. In this case $F(n) = \frac{1}{2}(n^2 + 3n)$, so we have a sum

$$\sum_{n=0}^{\infty} t^n q^{\frac{n^2 + 3n}{2}} \cdot q^{-n}$$

2) $K = \{1\}$. In this case $F(n) = \frac{1}{2}(\hat{n}^2 + 3\hat{n}) + n''$, so we have a sum

$$\sum_{\hat{n}=1}^{\infty} q^{\frac{\hat{n}^2 + 3\hat{n}}{2}} t^{\hat{n}} \sum_{n=0}^{\hat{n}-1} q^{-n-1} (1 - q) \sum_{n''=1}^{\infty} q^{n''} t^{n''} = \sum_{\hat{n}=1}^{\infty} q^{\frac{\hat{n}^2 + 3\hat{n}}{2}} t^{\hat{n}} (q^{-\hat{n}} - 1) \cdot \frac{qt}{1 - qt}.$$  

Summing these two expressions, we get

$$1 + \sum_{n=1}^{\infty} t^n q^{\frac{n^2 + 3n}{2}} \left( q^{-n} + \frac{qt}{1 - qt} (q^{-n} - 1) \right) = 1 + \frac{1}{1 - qt} \sum_{n=1}^{\infty} t^n q^{\frac{n^2 + 3n}{2}} (q^{-n} - qt) =$$

$$1 + \frac{1}{1 - qt} \left( \sum_{n=1}^{\infty} t^n q^{\frac{n(n+1)}{2}} - \sum_{n=1}^{\infty} t^{n+1} q^{\frac{(n+1)(n+2)}{2}} \right).$$

In the last sum all coefficients at $t^n$ for $n \geq 2$ cancel, so we get

$$1 + \frac{tq}{1 - qt} = \frac{1}{1 - qt}.$$
4 Combinatorics

4.1 Preliminary simplification

Let

\[ P_{k,n}(q) = \sum_{j=0}^{n} (-1)^j q^j \binom{k}{j} \]

\((k\) can be negative, but \(n\) should be non-negative and integer). \(\square\)

Lemma 1

\[ [S^n(\mathbb{C}P^1 - k\{pt\})] = q^{-n} P_{k-1,n}(q). \]

Proof.

\[ \sum_{n=0}^{\infty} t^n [S^n(\mathbb{C}P^1)] = \sum_{n=0}^{\infty} t^n [\mathbb{C}P^n] = \frac{1}{(1-t)(1-Lt)}, \]

hence

\[ \sum_{n=0}^{\infty} t^n [S^n(\mathbb{C}P^1 - k\{pt\})] = \frac{(1-t)^{k-1}}{(1-Lt)} = \]

\[ \sum_{a,b} (-1)^a \binom{k-1}{a} t^a \llb b \rrb = \sum_{n=0}^{\infty} t^n \sum_{a=0}^{n} (-1)^a \binom{k-1}{a} L^{n-a} = \]

\[ \sum_{n=0}^{\infty} t^n q^{-n} P_{k-1,n}(q). \]

\(\square\)

Let

\[ f_i(I, K) = \sum_{\sigma \in I : i(\sigma) = i} 1 + \sum_{\sigma \in I : j(\sigma) = i} 1 + \sum_{k \in K : i(k) = i} 1, \]

\[ f_i(I) = \sum_{\sigma \in I : i(\sigma) = i} 1 + \sum_{\sigma \in I : j(\sigma) = i} 1. \]

Note that \( \sum_{i=1}^{s} f_i(I, K) = 2|I| + |K|, \sum_{i=1}^{s} f_i(I) = 2|I| \).

Lemma 2 Let us fix \( \hat{n}_i \). Then

\[ \sum_{n_1, n_2, \ldots, n_k} q^{-n_1-f_i(I, K)} P_{1-\chi(E_i), n_i}(q) = q^{-\hat{n}_i} P_{1-\chi(E_i), \hat{n}_i-f_i(I, K)}(q). \]
Proof. By lemma 1 we have
\[
\sum_{n_i,n'_i,n''_i,e_n'k} q^{-n_i-f_i(I,K)} P_{1-\chi(E^o_i),n_i}(q) = \sum_{n_i,n'_i,n''_i,e_n'k} q^{-f_i(I,K)} [S^{n_i}(E^o_i)].
\]

Consider a \(n_i\)-tuple of points on \(E^o_i\), intersection points \(\sigma \in I\) such that \(i(\sigma) = i\) with multiplicities \(n'_\sigma\), intersection points \(\sigma \in I\) such that \(j(\sigma) = i\) with multiplicities \(n''_\sigma\) - 1, intersection points \(k \in K\) such that \(i(k) = i\) with multiplicities \(\tilde{n}'_k\) - 1. We get the unordered \(\tilde{n}_i - f_i\)-tuple of points on \(E^o_i \cup f_i(I,K)\). Thus the sum (18) equals to
\[
q^{-f_i(I,K)} [S^{\tilde{n}_i-f_i(I,K)}(E^o_i \cup f_i(I,K))] = q^{-\tilde{n}_i} P_{1-\chi(E^o_i)-f_i(I,K),\tilde{n}_i-f_i(I,K)}(q).
\]

□

Lemma 3

\[
P_g(t_1, \ldots, t_r, q) = \sum_{I \subset I_0, K \subset K_0} \sum_{\tilde{n}_i \geq f_i(I,K)} \left[ t^{\tilde{M}_I/q} \prod_{\tilde{n}'} \right] P_{1-\chi(E^o_i)-f_i(I,K),\tilde{n}_i-f_i(I,K)}(q) \times q^{|I|}(1-q)^{|I|+|K|} \prod_{k \in K} \frac{qt_k}{1-qt_k}. \tag{19}
\]

Proof. First, remark that for every \(k\)
\[
\sum_{\tilde{n}''_k > 0} q^\tilde{n}''_k t_k = \frac{t_k q}{1-t_k q},
\]
so from now on we can forget about summation over \(\tilde{n}''_k\).

We have
\[
q^{-\sum_{i=1}^s n_i - |I| - |K|} = q^{|I|} \prod_{i=1}^s q^{-n_i-f_i(I,K)}.
\]

Therefore we can reformulate the statement of Theorem 2 in the form
\[
P_g(t_1, \ldots, t_r, q) = \sum_{I \subset I_0, K \subset K_0} q^{|I|}(1-q)^{|I|} \sum_{\tilde{n}_i \geq f_i(I,K)} t^{\tilde{M}_I/q} \prod_{\tilde{n}'} P_{1-\chi(E^o_i),n_i}(q) \times \prod_{i=1}^s \left[ \sum_{n_i,n'_i,n''_i} q^{-n_i-f_i(I,K)} P_{1-\chi(E^o_i),n_i}(q) \right].
\]

Now the equation (19) follows from the lemma 2. □
Definition: By the reduced motivic Poincare series from now on we mean

\[ P_g(t_1, \ldots, t_n) = P_g(t_1, \ldots, t_n) \cdot \prod_{j=1}^{n}(1 - t_j q). \]

To any divisor \( E_i \) we associate the factor

\[ \phi_i(I, K, \hat{n}) = P_{1 - \chi(E_i^\circ) - f_i(K, I) \hat{n} - f_i(K, I)}, \]

and let

\[ G(K, I, \hat{n}) = q^{|I|}(1 - q)^{|I|+|K|} \prod_i \phi_i(I, K, \hat{n}). \]

Lemma 4

\[ \sum u^{\hat{n}} G(K, I, \hat{n}) = q^{|I|}(1 - q)^{|I|+|K|} \prod_i \frac{u_i^{f_i(K, I)}}{1 - u_i} (1 - u_i q)^{1 - \chi(E_i^\circ) - f_i(I, K)} \tag{20} \]

Proof. We have

\[ \sum u^{\hat{n}} \phi_i(I, K, \hat{n}) = \sum_j \sum_{\hat{n} = j + f_i(K, I)}^\infty u^{\hat{n}} (-1)^j \left( 1 - \chi(E_i^\circ) - f_i(I, K) \right) q^j = \]

\[ \frac{u_i^{f_i(K, I)}}{1 - u} \sum_j (-1)^j \left( 1 - \chi(E_i^\circ) - f_i(I, K) \right) (u q)^j = \frac{u_i^{f_i(K, I)}}{1 - u} (1 - u q)^{1 - \chi(E_i^\circ) - f_i(I, K)}, \]

and

\[ \sum u^{\hat{n}} G(K, I, \hat{n}) = q^{|I|}(1 - q)^{|I|+|K|} \prod_i \frac{u_i^{f_i(K, I)}}{1 - u_i} (1 - u_i q)^{1 - \chi(E_i^\circ) - f_i(I, K)}. \]

\[ \square \]

Definition: Let

\[ c_K(n) = \sum_I \sum_{K_1 \subset K} (-1)^{|K| - |K_1|} G(K_1, I, n), \]

\[ A_K(u) = \sum_n u^n c_K(n). \]
Lemma 5

\[ A_K(u) = (-1)^{|K|} \prod_i (1-u_iq)^{|K \cap E_i|} (1-u_i)^{|K \cap E_i|-1} \prod_{\sigma} (1-qu_i(\sigma) - qu_j(\sigma) + qu_i(\sigma) u_j(\sigma)). \]

Proof.

\[ A_K(u) = \sum_I q^{|I|} (1-q)^{|I|} \sum_{K_1} (-1)^{|K|} (1-q)^{|K_1]} \sum_n u^n \prod_i \phi_i(I, K_1, n). \]

We have

\[ \sum_n u^n \prod_i \phi_i(I, K_1, n) = \prod_i \frac{u_i^{f_i(K, I)} (1-u_iq) - (q-1) u_i^{|E_i^K|} f_i(I, K)}{1-u_i}. \]

Now

\[ \sum_{K_1 \subset (K \cap E_i)} (-1)^{|K \cap E_i|} (1-q) |K \cap E_i| u_i^{f_i(K, I)} (1-u_iq)^{-1} \chi(E_i^K) - f_i(I, K) = \]

\[ \frac{1}{1-u_i} u_i^{f_i(K, I)} (1-u_iq)^{-1} \chi(E_i^K) - f_i(I, K) \times \]

\[ \sum_{K_1} (-1)^{|K \cap E_i|} (1-q) |K_1| (1-u_iq)^{|K_1| - |K \cap E_i|} (1-u_iq) |K \cap E_i| - |K_1| = \]

\[ \frac{1}{1-u_i} u_i^{f_i(K, I)} (1-u_iq)^{-1} \chi(E_i^K) - f_i(I, K) (1-q - \frac{1-u_iq}{u_i}) |K \cap E_i| = \]

\[ \frac{1}{1-u_i} (1-q) |K \cap E_i| u_i^{f_i(K, I)} (1-u_iq)^{-1} \chi(E_i^K) - f_i(I, K) (1-u_i) |K \cap E_i|. \]

Remark that \( f_i(K, I) - |K_1 \cap E_i| = f_i(I) \) and

\[ \chi(E_i^K) + f_i(K, I) = \chi(E_i^K) - |K_0 \cap E_i| + |K \cap E_i| + f_i(I), \]

so the last expression can be rewritten in a form

\[ (1-q)^{|I|} \prod_i u_i^{f_i(I)} (1-u_iq)^{-1} \chi(E_i^K) + |K \cap E_i| - f_i(I) (1-u_i) |K \cap E_i|-1. \]

Also

\[ \sum_I q^{|I|} (1-q)^{|I|} \prod_i u_i^{f_i(I)} (1-u_iq) - f_i(I) = \prod_{\sigma} (1+q(1-q)u_i(\sigma) u_j(\sigma) - u_i(\sigma) q)^{-1} (1-u_j(\sigma) q)^{-1} = \]

\[ \prod u_i (1-u_iq) \chi(E_i^K) - 2 \prod_{\sigma} (1-q u_i(\sigma) - q u_j(\sigma) + q u_i(\sigma) u_j(\sigma)). \]
Therefore

\[ A_K(u) = (-1)^{|K|} \prod_i (1 - u_iq)^{1 - \chi(E_i^*) + |K \cap E_i|} (1 - u_i)^{|K \cap E_i| - 1} \times \]

\[ \times \prod_i (1 - u_iq)^{\chi(E_i^*) - 2} \prod_{\sigma} (1 - qu_{i(\sigma)} - qu_{j(\sigma)} + qu_{i(\sigma)}u_{j(\sigma)}) = \]

\[ (-1)^{|K|} \prod_i (1 - u_iq)^{|K \cap E_i| - 1} (1 - u_i)^{|K \cap E_i| - 1} \prod_{\sigma} (1 - qu_{i(\sigma)} - qu_{j(\sigma)} + qu_{i(\sigma)}u_{j(\sigma)}). \]

\( \square \)

**Lemma 6**

\[ P_g = \sum_n t^{Mn} q^{F(n) - \sum_{n_1} \sum_K t_K q^{|K|} c_K(n)}. \]  

(21)

**Proof.** From the equation (19) we get

\[ P_g(t_1, \ldots, t_r, q) = \sum_{I \subseteq I_0, K \subseteq K_0} \sum_{\hat{n}_i \geq f_i(I, K)} t^{M\hat{n}} q^{\hat{n}} \prod_{i=1}^{s} q^{-\hat{n}_i} P_{1 - \chi(E_i^*) - f_i(I, K), \hat{n}_i - f_i(I, K)}(q) \times q^{|I| - 1} \prod_{k \in K} \frac{qt_k}{1 - qt_k} = \]

\[ \sum_{I \subseteq I_0, K \subseteq K_0} \sum_{\hat{n}_i \geq f_i(I, K)} t^{M\hat{n}} q^{\hat{n}} \prod_{i=1}^{s} q^{-\hat{n}_i} \phi_i(I, K, \hat{n}) \times q^{|I| - 1} \prod_{k \in K} \frac{qt_k}{1 - qt_k} = \]

\[ \prod_{i=1}^{n} \frac{1}{(1 - qt_i)} \sum_{\hat{n}} t^{Mn} q^{F(n) - \sum_{n_1} \sum_K t_K q^{|K|} \sum_{I \subseteq I_0} \sum_{K_1 \subseteq K} (-1)^{|K| - |K_1|} G(K_1, I, \hat{n}) = \]

\[ \prod_{i=1}^{n} \frac{1}{(1 - qt_i)} \sum_{\hat{n}} t^{Mn} q^{F(n) - \sum_{n_1} \sum_K t_K q^{|K|} c_K(\hat{n}). \]

\[ \square \]

### 4.2 Cancellations

Lemma 6 together with lemma 5 gives the concrete description of \( P_g(t) \): it is expressed in terms of some quantities \( c_K(n) \), which fits into the generating function \( A_K(u) \), which has a compact form. Nevertheless, as in the model example with a nonsingular curve, lots of summands in the sum (21) have the same power in \( t \), and for \( n \) large enough we have a huge number of cancellations.
We say that a subset $K \subset K_0$ is **proper everywhere**, if for all $i \ K \cap E_i$ is a proper subset of $K_0 \cap E_i$. We denote the set of proper everywhere subsets by $\mathcal{P}$. For any $K \subset K_0$ let $E(K)$ be the set of divisors such that for $i \in E(K)$ the set $K \cap E_i$ is empty. Sometimes we’ll write $i \in P$, if $i \notin E(P)$.

Using these notations, every subset $K \subset K_0$ can be presented (uniquely) in the following way: we fix a proper everywhere subset $P(K)$ and a set of divisors $E \subset E(P(K))$ where all intersection points with $K_0$ belong to $K$.

For a set $E$ of divisors let $\Delta(E)$ be the number of pairs of intersecting divisors from $E$. Let $\mu_i(E) = 1$, if $i \in E$ and $\mu_i(E) = 0$ otherwise.

**Lemma 7** For a proper everywhere set $P$ let

$$
\tilde{H}_P(u_1, \ldots, u_s) = \sum_{E \subset E(P)} \left( -1 \right)^{|K_0 \cap E|} \prod_{i \in E} u_i - \sum a_{ij} \mu_j \cdot q^{\Delta(E)} \prod_{i \in E} (q - u_i)^{k_i - 1} \prod_{i \notin (P \cup E)} (1 - qu_i)^{k_i - 1}
\times \prod_{\sigma} \left( 1 - q^{1 - \mu_i(\sigma)}u_i(\sigma) - q^{1 - \mu_j(\sigma)}u_j(\sigma) + q^{1 - \mu_i(\sigma)} - q^{1 - \mu_j(\sigma)}u_i(\sigma)u_j(\sigma) \right).
$$

Then the polynomial $\tilde{H}_P$ is divisible by $\prod_{i \in E(P)} (1 - u_i)$.

**Proof.** We have to prove that $\tilde{H}_P = 0$ at $u_\beta = 1$ for $\beta \in E(P)$. Suppose that $E_\beta$ is intersected by $E_{\alpha_1}, \ldots, E_{\alpha_k}$. For every set $E$ of divisors not containing $E_\beta$ let us compare the summands corresponding to $E$ and to $E \cup E_\beta$.

For $E$ at $u_\beta = 1$ we have

$$
\prod_{i \neq \beta} u_i - \sum a_{ij} \mu_j \left( -1 \right)^{|K_0 \cap E|} q^{\Delta(E)} \prod_{i \in E} (q - u_i)^{k_i - 1} \prod_{i \notin (P \cup E)} (1 - qu_i)^{k_i - 1}
\times \prod_{\sigma \notin E_\beta} \left( 1 - q^{1 - \mu_i(\sigma)}u_i(\sigma) - q^{1 - \mu_j(\sigma)}u_j(\sigma) + q^{1 - \mu_i(\sigma)} - q^{1 - \mu_j(\sigma)}u_i(\sigma)u_j(\sigma) \right) \cdot (1 - q)^k.
$$

For $E \cup E_1$ at $u_\beta = 1$ we have

$$
\prod_{j=1}^k u_{\alpha_j} \prod_{i \neq \beta} u_i - \sum a_{ij} \mu_j \left( -1 \right)^{k_\beta + |K_0 \cap E|} (q - 1)^{k_\beta - 1} \prod_{i \in E} (q - u_i)^{k_i - 1} \prod_{i \notin (E \cup P)} (1 - qu_i)^{k_i - 1}
\times \prod_{\sigma \notin E_\beta} \left( 1 - q^{1 - \mu_i(\sigma)}u_i(\sigma) - q^{1 - \mu_j(\sigma)}u_j(\sigma) + q^{1 - \mu_i(\sigma)} - q^{1 - \mu_j(\sigma)}u_i(\sigma)u_j(\sigma) \right) \cdot \prod_{j=1}^k (1 - q)^{-\alpha_j}u_{\alpha_j}.
$$

It rests to note that $\Delta(E \cup E_\beta) - \Delta(E) = \sum_{j=1}^k \mu_{\alpha_j}(E)$. □
Lemma 8

\[
\sum_{n} u^n \sum_{E \subseteq E(P)} q^{-\sum_{i \in E} n_i - \Delta(E) - \sum_{i \in E} a_i - |E|} q^{K_0 \cap E} \times c_{P \cup E}(n_i + \sum a_{ij} \mu_j(E)) =
\]

\[
(-1)^{|P|} \prod_{i \in P} \left[ (1 - qu_i)^{k_i - p_i - 1} (1 - u_i)^{p_i - 1} \right] \cdot \frac{1}{\prod_{i \in E(P)} (1 - u_i)} \tilde{H}_P(u_1, \ldots, u_k).
\]

Proof.

\[
\sum_{n} u^n \sum_{E \subseteq E(P)} q^{-\sum_{i \in E} n_i - \Delta(E) - \sum_{i \in E} a_i - |E|} q^{K_0 \cap E} \times c_{P \cup E}(n_i + \sum a_{ij} \mu_j(E)) =
\]

\[
\sum_{E \subseteq E(P)} \prod_{i} u_i^{-\sum a_{ij} \mu_j(E)} \cdot q^{|E| \mu_i(E)} \cdot q^{-\Delta(E) - \sum_{i \in E} \mu_j(E) + |K_0 \cap E| - |E|} \times \sum_{n_k} \prod_{i} \left( u_i q^{-\mu_i(E)} \right)^{n_k} \cdot c_{P \cup E}(n_k) =
\]

\[
\sum_{E \subseteq E(P)} \prod_{i} u_i^{-\sum a_{ij} \mu_j(E)} \cdot A_{P \cup E}(u_i q^{-\mu_i(E)}) q^{\Delta(E) + |K_0 \cap E| - |E|} =
\]

\[
(-1)^{|P|} \sum_{E \subseteq E(P)} \prod_{i} u_i^{-\sum a_{ij} \mu_j(E)} \cdot (-1)^{|K_0 \cap E|} q^{\Delta(E) + |K_0 \cap E| - |E|} \prod_{i \in E} \left[ (1 - qu_i)^{k_i - 1} (1 - u_i)^{p_i - 1} \right]
\]

\[
\times \prod_{i \in P} \left[ (1 - qu_i)^{k_i - p_i - 1} (1 - u_i)^{p_i - 1} \right] \prod_{i \notin (P \cup E)} \left[ (1 - qu_i)^{k_i - 1} (1 - u_i)^{-1} \right]
\]

\[
\times \prod_{\sigma} \left( 1 - q^{1-\mu_i(E)} u_{i(\sigma)} - q^{1-\mu_j(E)} u_{j(\sigma)} + q^{1-\mu_i(E)-\mu_j(E)} u_{i(\sigma)} u_{j(\sigma)} \right) =
\]

\[
(-1)^{|P|} \prod_{i \in P} \left[ (1 - qu_i)^{k_i - p_i - 1} (1 - u_i)^{p_i - 1} \right] \cdot \frac{1}{\prod_{i \in E(P)} (1 - u_i)}
\]

\[
\times \sum_{E \subseteq E(P)} (-1)^{|K_0 \cap E|} \prod_{i} u_i^{-\sum a_{ij} \mu_j(E)} \cdot q^{\Delta(E)} \prod_{i \in E} (q - u_i)^{k_i - 1} \prod_{i \notin E} (1 - qu_i)^{k_i - 1}
\]

\[
\times \prod_{\sigma} \left( 1 - q^{1-\mu_i(E)} u_{i(\sigma)} - q^{1-\mu_j(E)} u_{j(\sigma)} + q^{1-\mu_i(E)-\mu_j(E)} u_{i(\sigma)} u_{j(\sigma)} \right).
\]

\[\Box\]
Theorem 3 For a proper everywhere set $P$ define the numbers $d_P(n)$ by the equation

$$H_P(u) = \sum_n d_P(n)u^nd_P(n) = \prod_{i \in P}[(1 - qu_i)^{k_i - p_i - 1}(1 - u_i)^{p_i - 1}] \tilde{H}_P(u_1, \ldots, u_s).$$

Then

$$\overline{P}_g(t_1, \ldots, t_r) = \sum_{P \in P} (-1)^{|P|}q^{|P|}t_P \times \sum_n d_P(n)t^M n q^{F(n) - \sum n_i}.$$  

Proof. From lemma 6 we have

$$P(t) = \sum_{n_1} t^{Mn_1} q^{F(n_1) - \sum n_i} \sum_{E \subseteq E(P)} t_E q^{K_0 \cap E|E|c_P \cup E(n_1)}.$$  

Let us collect the coefficient at $t^n$. We have

$$Mn_1 + \sum \mu_j(E) = Mn, \quad n_1 = n + \sum a_{ij} \mu_j(E).$$  

and

$$(\overline{F}(n) - \sum n_i) - (\overline{F}(n_1) - \sum n_{1i}) = \frac{1}{2}[2\sum m_{ij} a_{js} \mu_j(E)$$

$$- \sum m_{ij} a_{is} \mu_s(E) a_{ji} \mu_i(E) - \sum m_{ij} \chi(E_s) a_{js} \mu_s(E) + \sum a_{ij} \mu_j(E)].$$

Remark that

$$\sum_{i \neq j} a_{ij} = 2 - \chi(E_j),$$

so we get

$$\overline{F}(n) - \sum n_i) - (\overline{F}(n_1) - \sum n_{1i}) = \sum_{i \in E} n_i + \Delta(E) + \sum_{i \in E} a_{ii} + |E|.$$  

Thus

$$\overline{P}(t) = \sum_{P \in P} q^{|P|}t_P \sum_n t^{Mn} q^{F(n) - \sum n_i} \sum_{E \subseteq E(P)} q^{\sum_{i \in E} n_i - \Delta(E) - \sum_{i \in K} a_{ii} - |E|}$$

$$\times q^{K_0 \cap E|E|c_P \cup E(n + \sum a_{ij} \mu_j(E)).}$$

Now we apply lemma 8. 

□

Corollary 1 The power series $\overline{P}_g(t_1, \ldots, t_r)$ is a polynomial.
4.3 The algorithm

If every line $E_i$ is intersected by the one component of the strict transform, any proper everywhere set should be empty. Therefore we get the following statement.

Lemma 9 Suppose that each divisor $E_i$ is intersected by exactly one component of the strict transform of the curve. Then the reduced motivic Poincare series can be computed using the following algorithm.

1. Consider the polynomial

$$A(u_1, \ldots, u_r) = \prod_{\sigma}(1 - qu_i(\sigma) - qu_j(\sigma) + qu_i(\sigma)u_j(\sigma)).$$

2. Consider the Laurent polynomial

$$\tilde{H}(u_1, \ldots, u_r) = \sum_{K \subset K_0} (-1)^{|K|}q^{\Delta(K)} \prod_{i} u_i^{-\sum a_{ij}\mu_j} \cdot A(u_1 q^{-\mu_1(K)}, \ldots, u_r q^{-\mu_r(K)}).$$

3. This polynomial is divisible by $\prod(1 - u_i)$. Let

$$H(u_1, \ldots, u_r) = \frac{\tilde{H}(u_1, \ldots, u_r)}{\prod_{i=1}^r(1 - u_i)}.$$

4. Expand this polynomial:

$$H(u_1, \ldots, u_r) = \sum d_{\mu} u^{\mu},$$

and now

$$P_g(t_1, \ldots, t_r) = \sum d_{\mu} t^{M_{\mu}} q^{F(\mu)} - \sum n_i.$$
5 Examples

5.1 One divisor

We consider the singularity

\[ x^{k_0} - y^{k_0} = 0, \]

which is geometrically a union of \( k_0 \) pairwise transversal lines. Its minimal resolution has one divisor and \( k_0 \) components of the strict transform intersecting it. For \( 0 < k < k_0 \) let the numbers \( c_k(n) \) be defined by the equation

\[
A_k(u) = \sum_{n=0}^{\infty} u^n c_k(n) = (1 - uq)^{k_0 - 1} - (1 - u)^{k_0 - 1},
\]

and for \( k = 0 \) let the numbers \( c_0(n) \) be defined by the equation

\[
A_0(u) = \sum_{n=0}^{\infty} u^n c_0(n) = \frac{(1 - uq)^{k_0} - u(u - q)^{k_0}}{1 - u}.
\]

The polynomials \( A_k(u) \) has degree \( k_0 - 2 \) for \( k > 0 \), \( A_0(u) \) has degree \( k_0 - 1 \), so we have a finite number of non-zero \( c_k(n) \).

From the Theorem 3 we conclude that

\[
\overline{P}_g(t_1, \ldots, t_{k_0}) = \sum_{K \subsetneq K_0} (-1)^{|K|} q^{|K|} t_K \sum_{n=0}^{\infty} c_{|K|}(n)(t_1 \ldots t_{k_0})^n q^{n(n+1)/2}.
\]

For example, if \( k_0 = 2 \),

\[
A_1(u) = 1, A_0(u) = \frac{1 - uq - u(u - q)}{1 - u} = 1 + u,
\]

so

\[
\overline{P}_g(t_1, t_2) = 1 - qt_1 - qt_2 + qt_1t_2.
\]

If \( k_0 = 3 \),

\[
A_1(u) = 1 - qu, A_2(u) = 1 - u, A_0(u) = 1 + (1 - 2q - q^2)u + u^2,
\]

so

\[
\overline{P}_g(t_1, t_2, t_3) = 1 - q(t_1 + t_2 + t_3) + q^2(t_1 t_2 + t_1 t_3 + t_2 t_3) + q(1 - 2q - q^2)t_1 t_2 t_3 +
q^3t_1 t_2 t_3(t_1 + t_2 + t_3) - q^3 t_1 t_2 t_3(t_1 t_2 + t_1 t_3 + t_2 t_3) + q^3 t_1^2 t_2^2 t_3^2.
\]

This answer can be rewritten as

\[
\overline{P}_g(t_1, t_2, t_3) = (1 - qt_1)(1 - qt_2)(1 - qt_3) - q^3 t_1 t_2 t_3(1 - t_1)(1 - t_2)(1 - t_3) + q(1 - q)^2 t_1 t_2 t_3.
\]
5.2 Two divisors

Suppose that the second divisor is intersected by two components of the strict transform, and the first one by one component. This corresponds to the singularity

\[ x \cdot (y - x^2) \cdot (y + x^2) = 0. \]

The matrix \( M \) is equal to

\[ M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \]

\[ \chi(E^*_1) = \chi(E^*_2) = 1, \]

so

\[ F(n_1, n_2) = \frac{1}{2}(n_1^2 + 2n_1n_2 + 2n_2^2 + 2n_1 + 3n_2). \]

If \( P = \emptyset \), we get

\[ \widetilde{H}_0(u_1, u_2) = (1 - qu_1 - qu_2 + qu_1u_2)(1 - qu_2) - (1 - u_1 - qu_2 + u_1u_2)(1 - qu_2)u_1^{-1} \]
\[ + (1 - qu_1 - u_2 + u_1u_2)(q - u_2)u_1^{-1}u_2 - q(1 - u_1 - u_2 + q^{-1}u_1u_2)(1 - qu_2)u_1 = \]
\[ \frac{1}{u_1u_2}(1 - u_1)(1 - u_2)(-u_1^3 + u_1u_2 + u_1^2u_2 - qu_2u_1^2u_2 - q^2u_1^2u_2 + qu_1^3u_2 \]
\[ + qu_2^3 + u_1u_2^2 - qu_1u_2^2 - q^2u_1u_2^2 + u_1^2u_2^2 - u_2^3), \]

if \( P \) is one point on the second divisor, we get

\[ \widetilde{H}_{pt}(u_1, u_2) = (1 - qu_1 - qu_2 + qu_1u_2) - (1 - u_1 - qu_2 + u_2)u_1^{-1} = \]
\[ \frac{1}{u_2}(1 - u_1)(u_1^2 - u_2 - u_1u_2 + qu_1u_2 - u_1u_2^2 + qu_1^2u_2). \]

Finally we get the following answer (\( t_0 \) corresponds to the first divisor):

\[ \mathcal{P}_g(t_0, t_1, t_2) = 1 - qt_0 - qt_1 + q^2t_0t_1 - qt_2 + q^2t_0t_2 + q^2t_1t_2 + qt_0t_1t_2 - q^2t_0t_1t_2 - q^3t_0t_1t_2 \]
\[ - q^2t_0^2t_1t_2 + q^3t_0^2t_1t_2 - q^2t_0t_1^2t_2 + q^3t_0t_1^2t_2 + q^2t_0t_1^2t_2 - q^3t_0t_1^2t_2 - q^4t_0t_1^2t_2 + q^4t_0^2t_1^2t_2 \]
\[ + q^4t_0^2t_1^2t_2 - q^4t_0^2t_1^2t_2 + q^4t_0^2t_1^2t_2 - q^4t_0^2t_1^2t_2 - q^4t_0^2t_1^2t_2 + q^4t_0^2t_1^2t_2. \]

This answer can be rewritten as

\[ \mathcal{P}_g(t_0, t_1, t_2) = (1 - qt_0)(1 - qt_1)(1 - qt_2) - q^4t_0t_1^2t_2(1 - t_0)(1 - t_1)(1 - t_2) \]
\[ + (1 - q)qt_0t_1t_2(1 - qt_1 - qt_2 + qt_1t_2). \]
If $q = 1$, we get the known Alexander polynomial:

$$P_g(t_0, t_1, t_2; q = 1) = (1 - t_0)(1 - t_1)(1 - t_2)(1 - t_0 t_1^2 t_2^2).$$

If $t_2 = 1$, we get the known answer for $A_1$ singularity:

$$P_g(t_0, t_1, 1) = (1 - q)(1 - qt_0 - qt_1 + qt_0 t_1).$$

If $t_0 = 1$, we get the answer for $A_3$ singularity:

$$P_g(1, t_1, t_2) = (1 - q)(1 - qt_1 - qt_2 + qt_1 t_2 + q^2 t_1 t_2 - q^2 t_1^2 t_2 - q^2 t_1 t_2^2 + q^2 t_1^2 t_2^2),$$

so

$$P_{A_3}^g(t_1, t_2) = (1 - qt_1)(1 - qt_2) + qt_1 t_2(1 - qt_1 - qt_2 + qt_1 t_2) =$$

$$(1 - qt_1)(1 - qt_2) + q^2 t_1 t_2(1 - t_1)(1 - t_2) + (1 - q)qt_1 t_2.$$

This answer coincide with the general answer for the singularities of type $A_{2n-1}$ in the section 7.5.

### 5.3 Three divisors

For simplicity we assume that each divisor is intersected by one component of the strict transform. This corresponds to the singularity

$$x \cdot y \cdot (x^2 - y^3) = 0.$$

Matrix $M$ is equal to

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix},$$

$$\chi(E_1^\bullet) = \chi(E_2^\bullet) = 1, \chi(E_3^\bullet) = 0,$$

so

$$F(n_1, n_2, n_2) = \frac{1}{2}(n_1^2 + 2n_2^2 + 6n_3^2 + 2n_1 n_2 + 4n_1 n_3 + 6n_2 n_3 + n_1 + 2n_2 + 4n_3).$$

Now

$$A(u_1, u_2, u_3) = (1 - qu_1 - qu_3 + qu_1 u_3)(1 - qu_2 - qu_3 + qu_2 u_3),$$

so

$$E(u_1, u_2, u_3) = \frac{1}{u_1 u_2 u_3^2}(u_2^3 u_3 u_1 - u_1^3 u_3^2 q + u_1^4 u_3 u_2 - u_1^2 u_2^2 u_3^2 - u_2^2 u_3^2 u_1 +$$

26
\[ u_1^4 u_2^3 u_3 - u_3^3 u_1^2 q - u_1^3 u_2 u_3^2 + u_1^3 u_2^3 u_3 + u_1^2 u_2^3 u_3 - u_3^3 q u_2 - u_1^3 u_2^2 u_3^2 - u_3^3 u_1 q - u_2^2 u_3^2 q - u_1^2 u_2 u_3^2 - u_3^2 u_1 u_2 + u_2^2 u_1^4 u_3 - u_1^2 u_2^3 q u_3 + u_2^2 u_3^2 u_1^2 q - u_1^4 u_3^2 u_2 q - u_2^3 u_3^2 u_1 q - u_2^3 u_3 u_3^2 q + u_3^3 u_1 q^2 u_2 + u_2^2 u_3^2 u_1 q^2 + u_1^3 u_3^2 u_2 q^2 - u_1^4 u_2^3 + u_1^2 u_3^3 + u_3^3 u_1 + u_3^2 u_1^2 u_2 q + u_1^3 u_3^3 + u_3^3 u_2^2 + u_3^3 u_2 + u_3^3 - u_3^4), \]

and

\[ T_g(t_1, t_2, t_3) = 1 - t_3 q + t_1^2 t_2^3 t_3^7 q^7 + t_1 t_2^2 t_3^5 q^5 + t_1 t_2 t_3^3 q^3 + t_1 t_2^2 t_3^4 q^4 - t_1^2 t_2^4 t_3^7 q^7 + t_2 t_3^2 q^2 - t_1 t_2 t_3^2 q^2 - t_1 t_2 t_3^4 q^3 - t_1^2 t_2^2 t_3^5 q^4 - t_1 t_2 t_3^2 q^2 - t_1^2 t_2^3 t_3^5 q^5 - t_1^3 t_2^2 t_3^7 q^7 - t_1^3 t_2^4 t_3^6 q^7 + t_1^2 t_2^3 t_3^5 q^4 + t_1^2 t_2^2 t_3^4 q^3 + t_1^2 t_2 t_3^3 q^4 - t_1^2 t_2^2 t_3^3 q^4 + t_1^2 t_2^4 t_3^6 q^7 + t_1^2 t_2^3 t_3^4 q^4 - t_1 t_2 t_3 q^3 + t_1 t_2^2 t_3^3 q^2 - t_2 q + t_1 t_3 q^2 - t_1 t_2 t_3^2 q^2 + t_1^3 t_2^4 t_3^7 q^7 + t_1 t_2 t_3^2 q - t_1 q - t_1^2 t_2^3 t_3^4 q^4 + t_1^3 t_2^3 t_3^6 q^7. \]

It can be rewritten as

\[ T_g(t_1, t_2, t_3) = (1 - t_1 q)(1 - t_2 q)(1 - t_3 q) - t_1^2 t_2^3 t_3^6 q^7 (1 - t_1)(1 - t_2)(1 - t_3) - t_1 t_2 t_3 q^2 (q - 1)(1 - t_1 q) - t_1^2 t_2^4 t_3^3 q^4 (q - 1)(1 - t_2)(1 - t_3) - t_1 t_2^3 t_3^2 q^2 (q - 1)(1 - t_1 q) + t_1 t_2^2 t_3^4 q^3 (q - 1)(1 - t_1). \]

In this presentation the symmetry of \( T_g \) is clear, since every line in the right hand side is invariant under the change \( t_i \rightarrow q^{-1} t_i^{-1} \).

If we set \( q = 1 \), we get

\[ T_g(t_1, t_2, t_3, q = 1) = (1 - t_1^2 t_2^3 t_3^6)(1 - t_1)(1 - t_2)(1 - t_3). \]

If we consider only singularity of type \( A_2 \), we set \( t_1 = t_2 = 1, t_3 = t \), and

\[ T_g(1, 1, t) = (1 - q)^2 (1 - t q + t^2 q), \]

so

\[ P_g(1, 1, t) = \frac{1 - t q + t^2 q}{1 - t q} = 1 + \sum_{k=2}^{\infty} t^k q^{k-1}. \]
6 Symmetry and functional equations

6.1 Symmetry of the motivic Poincare series

Lemma 10

\[ A_K\left(\frac{1}{qu_1}, \ldots, \frac{1}{qu_s}\right) = q^{1-|K|} \prod_{i=1}^s u_i^{\chi(E_i^o)} \cdot A_K(u_1, \ldots, u_s). \]

Proof.

\[ A_K\left(\frac{1}{qu}\right) = (-1)^{|K|} \prod_i (1 - \frac{1}{u_i})^{[K \cap E_i] - 1} \prod_{\sigma} (1 - \frac{1}{u_i(\sigma)} - \frac{1}{u_j(\sigma)} + \frac{1}{qu_i(\sigma)u_j(\sigma)}) = \]

\[ A_K(u) \prod_i u_i^{1-[K \cap E_i]} u_i^{1-[K \cap E_i]} q^{1-[K \cap E_i]} \prod_{\sigma} (qu_i(\sigma)u_j(\sigma))^{-1} = \]

\[ A_K(u) q^{s-[K]-|I_0|} \prod_i u_i^{-2[K \cap E_i] + \chi(E_i^o) - 2}. \]

It rests to note that \(|I_0| = s - 1\) and \(\chi(E_i^o) = \chi(E_i^*) - |K \cap E_i| \). □

Lemma 11

\[ c_K(n_1, \ldots, n_s) = q^{1-|K|+n} c_K(-\chi(E_1^o) - n_1, \ldots, -\chi(E_s^o) - n_s), \]

where \(n = \sum_{i=1}^s n_i\).

Proof.

\[ A_K\left(\frac{1}{qu_1}, \ldots, \frac{1}{qu_s}\right) = \sum_n c_K(n_1, \ldots, n_s) u^{2q-n} = q^{1-|K|} \prod_i u_i^{\chi(E_i^o)} \sum_{\bar{z}} c_{\bar{z}}(z_1, \ldots, z_s) u^{\bar{z}}. \]

We have

\[ z_i + \chi(E_i^o) = -n_i, \quad z_i = -\chi(E_i^o) - n_i. \]

□

Theorem 4 Let \(\mu_\alpha\) be the Milnor number of \(C_\alpha\), and \((C_\alpha \circ C_\beta)\) is the intersection index of \(C_\alpha \circ C_\beta\), \(\mu(C)\) is the Milnor number of \(C\). Let \(l_\alpha = \mu_\alpha + \sum_{\beta \neq \alpha} (C_\alpha \circ C_\beta)\) and \(\delta(C) = (\mu(C) + r - 1)/2\). Then

\[ \mathcal{P}_g\left(\frac{1}{qt_1}, \ldots, \frac{1}{qt_r}\right) = q^{-\delta(C)} \prod_{\alpha} t_{\alpha}^{-l_\alpha} \cdot \mathcal{P}_g(t_1, \ldots, t_r). \]

28
**Proof.** Let \( k_i = |K_0 \cap E_i| \). From lemma 6 we get

\[
\mathcal{F}_g \left( \frac{1}{q t_1}, \ldots, \frac{1}{q t_r} \right) = (t_1 \cdots t_r)^{-1} \sum_{n} t^{-M_0} q^{-\sum m_{ij} n_{ij} q F(n)} - \sum_{n_{ij}} \sum_{K} t_K c_K(n) =
\]

\[
t^{-1-M(x(E^0))} \sum_{n} t^{M(x(E^0)) - n} q^{-\sum m_{ij} n_{ij} q F(n)} - \sum_{n_{ij}} \times \sum_{K} q^{1-|K|+n} \cdot t_K \cdot c_K(-\chi(E^0) - n_i).
\]

(24)

Let \( \xi_i = -\chi(E^0_i) \), \( n_i = \xi - n \).

Then

\[
F(n) - \sum n_i = \frac{1}{2} \left[ \sum m_{ij} n_i n_j + \sum m_{ij} n_i \chi(E^*_j) - \sum n_i \right],
\]

so

\[
2[F(n_1) - \sum n_{1i} - F(n) + \sum n_i] =
\]

\[
\sum m_{ij} (\xi_i - n_i) (\xi_j - n_j) + \sum m_{ij} (\xi_i - n_i) \chi(E^*_j) - \sum (\xi_i - n_i)
\]

\[
- \sum m_{ij} n_i n_j - \sum m_{ij} n_i \chi(E^*_j) + \sum n_i =
\]

\[
-2 \sum m_{ij} (\xi_i + \chi(E^*_j)) n_j + 2 \sum n_j + 2(F(\xi) - \sum \xi_i) =
\]

\[
-2 \sum m_{ij} k_i n_j + 2 \sum n_j + 2(F(\xi) - \sum \xi_i).
\]

Thus (24) is equal to

\[
t^{-1-M} q^{-F(\xi) + \sum \xi_i} q^{1-|K_0|} \sum_{n} t^{M_{n_1}} q^{F(n_1) - \sum n_{1i}} \sum_{K} t_K q^{\sum c_K(n_1)}.
\]

To conclude we have to compute the power of \( t_\alpha \) and of \( q \).

Remark that \( \sum \xi_i = |K_0| - 2 \), so \( \sum \xi_i + 1 - |K_0| = -1 \).

Also

\[
2 F(\xi) = \sum m_{ij} k_i k_j - 2 \sum m_{ij} k_i \chi(E^*_j) + \sum m_{ij} \chi(E^*_j) \chi(E^*_j) +
\]

\[
\sum m_{ij} k_i \chi(E^*_j) - \sum m_{ij} \chi(E^*_j) \chi(E^*_j) + \sum \xi_i =
\]

\[
\sum m_{ij} k_i k_j - \sum m_{ij} k_i \chi(E^*_j) + |K_0| - 2.
\]

29
The formula of A’Campo ([1]) says that

\[ 1 - \mu = \sum m \chi(S_m) = \sum \chi(E^\circ_i) m_{ij} k_j = \sum m_{ij} (\chi(E^*_i) - k_i) k_j, \]

so

\[ 2F(\xi) = \mu - 1 + |K_0| - 2 = 2\delta - 2. \]

Thus \(-F(\xi) - 1 = -\delta.\)

Also for every \(\alpha\) one has

\[ 1 - \mu_\alpha = \sum_{j \neq i(\alpha)} m_{i(\alpha)j} \chi(E^*_j) + m_{i(\alpha),i(\alpha)} (\chi(E^*_{i(\alpha)}) - 1), \]

and for \(\beta \neq \alpha\)

\[ C_\alpha \circ C_\beta = m_{i(\alpha),i(\beta)}, \]

so

\[ \sum_{\beta \neq \alpha} C_\alpha \circ C_\beta = \sum_{j \neq i(\alpha)} m_{i(\alpha),j} k_j + m_{i(\alpha),i(\alpha)} (k_{i(\alpha)} - 1) \]

and

\[ 1 - \mu_\alpha - C_\alpha \circ C_\beta = \sum_j m_{i(\alpha),j} \chi(E^*_j). \]

\[ \square \]

**Corollary 2** The degree of the polynomial \(P_g(t_1, \ldots, t_r)\) with respect to the variable \(t_i\) is equal to \(l_i.\) The greatest monomial in it equals to \(q^{\delta(C)} \prod_{i=1}^r t_i^{l_i}.\)

### 6.2 Analogue of the Kapranov’s functional equation

Let \(C\) be a genus \(g\) curve,

\[ \zeta_C(t) = \sum_{n=0}^\infty t^n [S^n C]. \]

In [10] M. Kapranov proved that \(\zeta_C(t)\) is rational and satisfies the following functional equation:

\[ \zeta_C\left(\frac{1}{Lt}\right) = L^{1-g} t^{2-2g} \zeta_C(t). \]

For example, for \(C = \mathbb{P}^1\) one has

\[ \zeta_{\mathbb{P}^1}(t) = \frac{1}{(1-t)(1-Lt)}, \]
and the equation can be verified directly.

Kapranov’s proof is based on the Serre’s duality, so it is important that the curve is closed. Nevertheless, there exists an analogue of this equation for punctured curves, generalizing lemma 10. Since it follows directly from the Kapranov’s formula, the idea of this section is to write explicitly the function this equation can be applied to.

Let $C^o$ denote the curve $C$ without $k_0$ points, and $K \subset K_0$. Consider the following generating function:

$$F_{C^o}(K, t) = \sum_{K_1 \subset K} \sum_{n=k_1}^{\infty} (-1)^{k_1} (L - 1)^{k_1} t^n [S^{n-k_1}(C^o \cup K_1)].$$

**Proposition.**

$$F_{C^o}(K, t) = \frac{1}{L^t} = t^{2-2g-k_0} L^{1-g-k} F_{C^o}(K, t).$$  \hspace{1cm} (25)

**Proof.** First, note that

$$\sum_{n=0}^{\infty} t^n [S^n C^o] = (1 - t)^{k_0} \zeta_C(t),$$

so

$$\sum_{n=k_1}^{\infty} t^n [S^{n-k_1}(C^o \cup K_1)] = t^{k_1} (1 - t)^{k_0-k_1} \zeta_C(t).$$

Therefore

$$F_{C^o}(K, t) = \sum_{K_1 \subset K} (-1)^{k_1} (L - 1)^{k_1} t^{k_1} (1 - t)^{k_0-k_1} \zeta_C(t) =$$

$$(1 - t - (L - 1)t)^k \cdot (1 - t)^{k_0-k} \zeta_C(t) = (1 - L t)^k (1 - t)^{k_0-k} \zeta_C(t).$$

Now

$$F_{C^o}(K, \frac{1}{L^t}) = (1 - t^{-1})^{k_0-k} (1 - (L^{-1})^{k_0-k} \zeta_C(\frac{1}{L^t}) = t^{2-2g-k_0} L^{1-g-k} F_{C^o}(K, t).$$

$\square$
7 Relation to the Heegard-Floer knot homologies

7.1 Heegard-Floer homologies

In the series of articles (e.g. [16], [17], [18], [20], see also [21]) P. Ozsvath and Z. Szabo constructed new powerful knot invariants, Heegard-Floer knot (and link) homologies. To each link \( L = \bigcup_{i=1}^{r} K_i \) they assign the collection of homology groups \( \widehat{HF}_L(L, h) \), where \( d \) is an integer and \( h \) belongs to some \( r \)-dimensional lattice. Their original description was based on the constructions from the symplectic topology, later ([11], [12]) there were elaborated combinatorial models for them. All of these homologies are invariants of the link \( L \), and they have the following properties ([17], [12]).

First, they give a ”categorification” of the Alexander polynomial of \( L \): if \( r = 1 \), then
\[
\sum_{h} \chi(\widehat{HF}_L(L, h)) t^h = \Delta^s(t),
\]
where \( \Delta^s(t) = t^{-\deg \Delta/2} \Delta(t) \) is a symmetrized Alexander polynomial of \( L \). If \( r > 1 \), then
\[
\sum_{h} \chi(\widehat{HF}_L(L, h)) t^h = \prod_{i=1}^{r} \left( t_{1/2}^i - t_{-1/2}^{-i} \right) \cdot \Delta^s(t_1, \ldots, t_r).
\]

Second, they have the symmetry extending the symmetry of the Alexander polynomial:
\[
\widehat{HF}_L(L, h) \cong \widehat{HF}_L(L, -h),
\]
where \( H = \sum_{i=1}^{r} h_i \).

These properties are similar to the ones of the polynomials \( P_g(t) \), so one could be interested in comparison of these objects. It turns out, that for knots (of course, \( P_g(t) \) is defined only for the algebraic ones) this comparison can be done.

In [20] for the relatively large class of knots, containing all algebraic knots, the following statement was proved.

**Theorem 5 ([20])** Let the symmetrized Alexander polynomial has the form
\[
\Delta^s(t) = (-1)^k + \sum_{i=1}^{k} (-1)^{k-i} (t^{n_i} + t^{-n_i})
\]
for some integers $0 < n_1 < n_2 < \ldots < n_k$. Let $n_{-j} = -n_j, n_0 = 0$. For $-k \leq i \leq k$ let us introduce the numbers $\delta_i$ by the formula

$$
\delta_i = \begin{cases} 
0, & \text{if } i=k \\
\delta_{i+1} - 2(n_{i+1} - n_i) + 1, & \text{if } k-i \text{ is odd} \\
\delta_{i+1} - 1, & \text{if } k-i > 0 \text{ is even}.
\end{cases}
$$

Then $\widehat{HFL}(K, j) = 0$, if $j$ does not coincide with any $n_i$, and $\widehat{HFL}(K, n_i) = \mathbb{Z}$ belongs to the homological grading $\delta_i$.

### 7.2 Matching the answers

Consider the Poincare polynomial for the Heegard-Floer homologies:

$$
HFL(t, u) = \sum u^d t^s \dim \widehat{HFL}_{d,s}(K).
$$

It categorifies the Alexander polynomial in the sense that

$$
HFL(t, -1) = t^{-\deg \Delta/2} \Delta(t).
$$

Remark that the coefficients in $P_g(t, q)$ are always equal to 0 or to ±1. It can be proved from the equation [15].

**Theorem 6** Take $P_g(t, q)$ and let us make a following change in it: $t^\alpha q^3$ is transformed to $t^\alpha u^{-23}$, and $-t^\alpha q^3$ is transformed to $t^\alpha u^{1-23}$. We get a polynomial $\widetilde{\Delta}_g(t, u)$. Then

$$
\widetilde{\Delta}_g(t^{-1}, u) = t^{-\deg \Delta/2} HFL(t, u).
$$

**Example.** For $(3, 5)$ torus knot we have

$$
P_g(t, q) = 1 + qt^3 + q^2 t^5 + q^3 t^6 + \frac{q^4 t^8}{1 - qt},
$$

$$
\overline{P}_g(t, q) = 1 - qt + q^2 t^3 - q^2 t^4 + q^2 t^5 - q^4 t^7 + q^4 t^8,
$$

$$
\widetilde{\Delta}_g(t, q) = 1 + u^{-1} t + u^{-2} t^3 + u^{-3} t^4 + u^{-4} t^5 + u^{-7} t^7 + u^{-8} t^8,
$$

and

$$
HFL(t, u) = t^4 + u^{-1} t^3 + u^{-2} t + u^{-3} t^0 + u^{-4} t^{-1} + u^{-7} t^{-3} + u^{-8} t^{-4}.
$$
Proof. To prove (26) we match Theorem 5 with the equation (15).

In the notation of Theorem 5 the non-symmetrized Alexander polynomial equals to
\[
\Delta = \sum_{i=k}^{-k} (-1)^i t^{n_k-n_i} = \sum_{i=0}^{2k} (-1)^i t^{n_k-n_{k-i}},
\]
\[
P(t) = \frac{\Delta}{1-t} = \sum_{i=0}^{k-1} \sum_{j=n_k-n_{k-2i}}^{n_k-n_k-2i-1} t^j + \frac{t^{2n_k}}{1-t}.
\]

Note that for \(i > 0\)
\[
\delta_{k-2i} = \delta_{k-2i+1} - 1 = \delta_{k-2(i-1)} - 2(n_{k-2i+2} - n_{k-2i+1}),
\]
so
\[
P_g(t, q) = \sum_{i=0}^{k-1} \sum_{j=n_k-n_{k-2i}}^{n_k-n_k-2i-1} q^{(j-n_k+n_{k-2i})-\delta_{k-2i}/2} t^j + \frac{t^{2n_k} q^{n_k}}{1-qt},
\]
\[
\overline{P}_g(t, q) = \sum_{i=0}^{k-1} (q^{-\delta_{k-2i}/2} t^{n_k-n_{k-2i}} - q^{-\delta_{k-2i-1}/2} t^{n_k-n_{k-2i-1}}) + t^{2n_k} q^{n_k}.
\]

Now
\[
\widetilde{\Delta}_g(t, u) = \sum_{i=0}^{k-1} (u^{\delta_{k-2i}} t^{n_k-n_{k-2i}} + u^{\delta_{k-2i-1}} t^{n_k-n_{k-2i-1}}) + t^{2n_k} u^{-2n_k},
\]
\[
t^{n_k} \widetilde{\Delta}_g(t^{-1}, u) = \sum_{i=0}^{k-1} (u^{\delta_{k-2i}} t^{n_k-2i} + q^{\delta_{k-2i-1}} t^{n_k-2i-1}) + t^{2n_k} u^{-2n_k} = \sum_{i=-k}^{k} u^{i} t^{n_i} = HFL(t, u).
\]

□

7.3 Relative Spin\(c\) structures

In this paragraph we recall the notion of the relative Spin\(c\) structure from [17], based on the construction of V. Turaev ([22]).

Let \(Y\) be a closed, oriented three-manifold. We say that two nowhere vanishing vector fields \(v\) and \(v'\) are homologous, if there is a ball \(B \subset Y\) such that \(v\) and \(v'\) are homotopic (through nowhere vanishing vector fields) on the complement of \(B\). The set of equivalence classes of such vector fields can be naturally identified with the space \(\text{Spin}\(c\)(\(Y\)) of \(\text{Spin}\(c\) structures over \(Y\). In particular it is an affine space for \(H^2(Y, \mathbb{Z})\).
This notion has a straightforward generalization to the case of three-manifolds with toroidal boundary. Let \((M, \partial M)\) be a three-manifold with boundary consisting of a disjoint union of tori \(T_1 \cup \ldots \cup T_r\). The tangent bundle to torus has a canonical nowhere vanishing vector field, which is unique up to homotopy (through nowhere vanishing vector fields). Consider now nowhere vanishing vector fields on \(Y\) whose restrictions to the components of \(\partial M\) are identified with the canonical nowhere vanishing vector field on the boundary tori (in particular, it is tangent to them). We call two such fields \(v\) and \(v'\) homologous, if there is a ball \(B \subset M \setminus \partial M\) such that the restriction of \(v\) and \(v'\) to \(M \setminus B\) are homotopic. The set of homology classes of such vector fields is called the set of relative \(Spin^c\) structures, and it is an affine space for \(H^2(M, \partial M; \mathbb{Z})\). This set is denoted as \(\text{Spin}^c(M, \partial M)\).

If \(v\) is an admissible nowhere vanishing vector field, we can consider the oriented two-plane field \(v^\perp\) of orthogonal complements to vectors of \(v\). Along \(\partial M\), it has a canonical trivialisation by outward pointing vectors. Hence, there is a well-defined notion of a relative Chern class of this plane field relative to its trivialization, thought as an element of \(H^2(M, \partial M; \mathbb{Z})\). This gives a well-defined map

\[
c_1 : \text{Spin}^c(M, \partial M) \rightarrow H^2(M, \partial M; \mathbb{Z}).
\]

In our case we have \(M\) is a complement to the tubular neighborhood of the link \(L\) in the sphere \(S^3\), and \(H^2(M, \partial M; \mathbb{Z}) = \mathbb{Z}^r\). We’ll denote \(\text{Spin}^c(S^3, L) := \text{Spin}^c(M, \partial M)\). We may think of this set as generated by the nowhere vanishing vector fields having the components of \(L\) as closed orbits (with the corresponding orientation). One can also define the "filling map"

\[
G_{K_1} : \text{Spin}^c(S^3, L) \rightarrow \text{Spin}^c(S^3, L - K_1)
\]
defined by the natural continuation of vector fields to the tubular neighborhood of \(K_1\).

Now the algebraic structure of the Heegard-Floer homologies can be described in the following way ([17]). Consider the ring

\[
R = \mathbb{Z}[U_1, \ldots, U_r].
\]

For every \(r\)-component link \(L\) there exists a \(\text{Spin}^c(S^3, L)\)-filtered chain complex \(\text{CF}L^-(S^3, L)\) of \(R\)-modules, whose filtered homotopy type is an invariant of the link \(L\). The operators \(U_i\) lowers the homological grading by 2 and the filtration level by 1. The homologies of the associated graded object are denoted as \(HFL^-(S^3, L)\). If one sets \(U_1 = U_2 = \ldots = U_r = 0\), he gets a new \(\text{Spin}^c(S^3, L)\)-filtered chain complex of \(\mathbb{Z}\)-modules, which will be denoted
as $\hat{CF}(L)$. The homologies of the associated graded object are denoted as $\hat{HF}(L)$, and they are the homologies discussed above.

The filtration on the second complex is compatible with the forgetting of components (proposition 7.1 in [17]). Namely, let $M$ be the two-dimensional graded vector space with one generator in grading 0 and one in grading -1.

**Proposition.** Let $L$ be an oriented, $r$-component link in $S^3$ and distinguish the first component $K_1$. Consider the complex $\hat{CF}(L)$ viewed as a $Spin^c(S^3, L - K_1)$ filtered chain complex via the filling map $G_{K_1}$. The filtered homotopy type of this complex is identified with $\hat{CF}(L - K_1) \otimes M$.

If we forget all components of $L$, we get either the complex

$$\hat{CF}(S^3) \otimes M^{r-1},$$

where $\hat{CF}(S^3)$ has one-dimensional homologies in grading 0 or

$$\hat{CF}^-(S^3) = \mathbb{Z}[U],$$

where all $U_i$ acts by the multiplication by $U$.

This proposition is a direct analogue to the equation (8).

For the relatively large class of knots there was proved in [20], that

$$\text{rk} \ H^*(CFL^-(K)/U_1(CFL^-(K))) = 1.$$

This follows from the fact that these homologies equals to the Floer homologies of manifold $S^3_n(K)$, obtained from $S^3$ by the $n/1$ Dehn surgery along the knot $K$ for $n$ large enough. The class of knots is specified by the condition that $S^3_n(K)$ is a homology sphere with one-dimensional Floer homologies. Algebraic knots belong to this class, since the tree of resolution of a plane curve singularity gives a plumbing construction for $S^3_n(K)$, and its Floer homologies can be computed using results of [19].

### 7.4 Comparing filtered complexes

In this section we try to describe the relation between the knot filtration on the Heegard-Floer complexes and the filtration on the space of functions defined by a curve.

To be more close to the algebraic setup, we reverse all signs for filtrations and for the homological (Maslov) grading as well (so we get cohomologies). The Alexander grading is also changed to get the non-symmetrized Alexander polynomial. In another words, the Poincare polynomial of the result
cohomologies coincides with \( \tilde{\Delta}_g(t, u^{-1}) \). The operator \( U \) will now increase the homological grading by 2.

Consider a \( \mathbb{Z}_{\geq 0} \)-indexed filtration \( J_n \) by vector subspaces (with finite codimensions) on a infinite-dimensional complex vector space \( J_0 \). It induces a filtration by projective subspaces \( P J_n \) on \( P J_0 = \mathbb{CP}^\infty \):

\[
P J_0 \xrightarrow{j_1} P J_1 \xrightarrow{j_2} P J_2 \xrightarrow{j_3} \ldots,
\]

so we have a sequence of corresponding Gysin maps in cohomologies:

\[
H^*(P J_0) \xrightarrow{(j_1)_*} H^{* - 2 \text{codim} J_1} P J_1 \xrightarrow{(j_2)_*} H^{* - 2 \text{codim} J_2} P J_2 \xrightarrow{(j_3)_*} \ldots.
\]

Of course, to be correct, one should consider only ”N-jets” of these maps, i.e. consider \( J_i/J_N \), where \( N \) is large enough.

Therefore we get a \( \mathbb{Z}_{\geq 0} \)-indexed filtration

\[
F_k = (j_k)_*(H^*(P J_k))
\]

in \( H^*(\mathbb{CP}^\infty) = \mathbb{Z}[U] \), which is compatible with the multiplication by \( U \). If we also know (as for the filtration defined by the orders on the curve), that \( \text{dim } J_k/J_{k+1} \leq 1 \), we conclude that \( U \) increase the filtration level at least by 1.

The motivic Poincare series in this setup can be written as

\[
P_g(t, q) = \sum_{k,n} t^k q^{n/2} \dim H^n(F_k/F_{k+1}).
\]

The situation is very close to the Heegard-Floer complexes, but \( U \) may increase the filtration level more that by 1. To avoid this problem, we should modify the complex.

**Example.** Consider the following filtered complex \( T \): it has generators \( U^k a_0, U^k a_1 \) and \( U^k a_2 \). The homological degree of \( U^l a_j \) equals to \( 2l + j \) and its filtration level equals to \( l + j \). One can check that

\[
\sum_{k,n} t^k u^n \dim H^n(T_k/T_{k+1}) = 1 + u^2 t^2 + u^4 t^4 + \ldots
\]

(so this complex corresponds to the trefoil knot) and \( \text{rk } H^*(T_k/UT_k) = 1 \) for all \( k \). Remark that if \( \hat{T}_k = T_k/UT_{k-1} \), then

\[
\sum_{k,n} t^k u^n \dim H^n(\hat{T}_k/\hat{T}_{k+1}) = 1 + ut + u^2 t^2,
\]

37
what is the Poincare polynomial for the Heegard-Floer homologies of the

trefoil.

Let us turn to the general case. Consider the complex

\[ C_0 = F_0[U_1] + (F_0[1])[U_1] \] (27)

with the filtration

\[ C_n = \bigoplus_{k+l=n} U_1^l F_k \oplus \bigoplus_{k+l=n-1} U_1^l F_k[1] \]

and the natural action of the operator \( U_1 \) of homological degree 2. The
differential is given by the equation

\[ d(x) = U_1 \cdot x + Ux. \]

One can check that this differential preserves the filtration \( C_n \) and commutes
with \( U_1 \).

Now

\[ C_n/C_{n+1} = \bigoplus_{k+l=n} U_1^l (F_k/F_{k+1}) \oplus \bigoplus_{k+l=n-1} U_1^l (F_k/F_{k+1})[1]. \]

Since the \( U_1 \)-increasing component of the differential

\[ d_1(U_1^l x[1]) = U_1^{l+1} x \]

gives the isomorphism

\[ d_1 : U_1^l (F_k/F_{k+1}) \rightarrow U_1^{l+1} (F_k/F_{k+1}), \]

we have

\[ H^*(C_n/C_{n+1}) = F_n/F_{n+1}. \]

Also we have

\[ C_n/U_1(C_n) = F_0 \oplus F_0[1] \bigoplus_{k+l=n,l>0} U_1^l (F_k/F_{k+1}) \oplus \bigoplus_{k+l=n-1,l>0} U_1^l (F_k/F_{k+1})[1], \]

and up to the isomorphisms \( d_1 \) we have the complex \( F_0 \oplus F_0[1] \) with the
differential

\[ d_2(x[1]) = Ux, \]

so

\[ rk \ H^*(C_n/U_1(C_n)) = 1. \]
The properties of the complex $C_0$ are similar to the ones of the complex $CFL^-(K)$. More precisely, the calculations of [20] (lemma 3.1 and lemma 3.2) implies the following

**Proposition.** Suppose that a cochain complex $C$ has a filtration $C_k$, $k \geq 0$ and an injective operator $U$ of homological degree 2 acting on it such that

1) $U(C_k) \subset C_{k+1}$ and $U^{-1}(C_k) \subset C_{k-1}$ (this means that $U$ increase the level of filtration exactly by 1)

2) $H^*(C_k/U(C_k))$ has rank 1 for all $k$,

then for all $k$ the rank of $H^*(C_k/C_{k+1})$ is at most 1. Let $\{0, \sigma_1, \sigma_2, \ldots\}$ is the set of $k$ such that this rank is 1. Then

3) $H^*(C_{\sigma_k}/C_{\sigma_{k+1}})$ belongs to degree $2k$.

Let

$$Q(t, q) = \sum_{k=0}^{\infty} q^k t^{\sigma_k}, \quad \overline{Q}(t, q) = Q(t, q)(1 - qt).$$

Let us make a following change in $\overline{Q}$: $t^\alpha q^\beta$ is transformed to $t^\alpha u^{2\beta}$, and $-t^\alpha q^\beta$ is transformed to $t^\alpha u^{2\beta-1}$.

4) The result is equal to

$$\sum_{k,n} t^k u^n \dim H^n(C_k/(C_{k+1} + UC_{k-1})).$$

The last result can be reformulated as follows. Consider the complex $\widehat{C}_k = C_k/UC_{k-1}$, then the last homologies are the homologies of the associated graded object $\widehat{C}_k/\widehat{C}_{k-1}$. The multiplication by $1 - qt$ corresponds to the exact sequence

$$0 \to C_{k-1}/C_k \xrightarrow{U} C_k/C_{k+1} \to \widehat{C}_k/\widehat{C}_{k+1} \to 0.$$

As a corollary we get that the series $Q(t, 1)$ determines completely all discussed cohomologies. Since for the filtered complexes $C$ and $CFL^-$ we have $Q(t, 1) = \Delta(t)/(1-t)$ for both, we have the equality of the cohomologies of the associated graded objects and the more clear proof of the Theorem 6. For example, we get the equation

$$H^*(CFL^-(S^3)/CFL_+(S^3, K)) \cong H^*(\mathbb{P}(\mathcal{O}/J_s)),$$

which looks clearer that the Theorem 6.

**Remarks.**

1. It would be interesting to construct the analogous $\mathbb{Z}^n$-filtered complex of $\mathbb{Z}[U_1, \ldots, U_n]$ for multi-component links which would carry the information about the Poincare series of the corresponding multi-index filtration.
2. Since the order of the product of two functions on a curve is equal to 
the sum of their orders, the integers $\sigma_1, \sigma_2, \ldots$ form a semigroup. 
This may lead to some conjectural multiplicative structure on the complex $CFL^-$. 
Unfortunately, it seems that it does not preserve the homological grading.

3. It would be also interesting to compare these results with the ones of [13], [14] and [15] computing the Seiberg-Witten and Heegard-Floer invariants of the surface links.

7.5 Example: $A_{2n-1}$ singularities

Since the algorithm of computation of the (reduced) motivic Poincare series 
is quite complicated, it is useful to have a series of answers where the motivic 
Poincare series and the link homologies can be computed.

**Proposition.** Consider the singularity of type $A_{2n-1}$ given by the equation

$$y^2 = x^{2n}.$$ 

From the topological viewpoint this corresponds to the 2-component link, 
whose components are unknotted, all intersections are positive and the linking 
number of the components equals to $n$. Then

$$P_g(t_1, t_2) = 1 + qt_1t_2 + \ldots + q^{n-1}t_1^{n-1}t_2^{n-1} + \frac{q^n(1-q)t_1^n t_2^n}{(1-t_1q)(1-t_2q)}.$$ 

**Proof.** For the proof we use the equation (13). Parametrisations of the 
components are

$$(x(t_1), y(t_1)) = (t_1, t_1^n), \quad (x(t_2), y(t_2)) = (t_2, -t_2^n),$$

so

$$x^a y^b |_{C_1} = t_1^{a+bn}, \quad x^a y^b |_{C_2} = (-1)^b t_2^{a+bn}.$$ 

If $a < n$, then every function with order $a$ on $C_1$ has a form $x^a + \ldots$, so its 
order on $C_2$ is also equal to $a$.

For every $a, b \geq n$ consider the function $x^{a-n}(x^n + y) + x^{b-n}(x^n - y)$. Its restrictions on $C_1$ and $C_2$ are respectively equal to $2t_1^a$ and $2t_2^b$, therefore

$$\dim J_{a,b}/J_{a+1,b} = \dim J_{a,b}/J_{a,b+1} = 1.$$ 

The codimensions $h(v_1, v_2)$ are equal to $v_1 + v_2 - n$, if $v_1, v_2 \geq n$, to $v_2$, if $v_1 < n, v_2 \geq n$, to $v_1$, if $v_2 < n, v_1 \geq n$, and to $\max(v_1, v_2)$, if $0 \leq v_1, v_2 < n$. 
We have

$$L^{A_{2n-1}}_g(t_1, t_2, q) = \sum_{0 \leq \max(v_1, v_2) > \min(v_1, v_2) < n} t_1^{v_1} t_2^{v_2} q^{\max(v_1, v_2)} + (1+q) \sum_{v_1, v_2 = n} \sum_{v_1, v_2 < n} t_1^{v_1} t_2^{v_2} q^{v_1 + v_2 - n},$$

40
\[ L_{g}^{A_{2n-1}}(t_1-1)(t_2-1) = -1 + (1-q)t_1t_2 + \ldots + (q^{n-2} - q^{n-1})t_1^{n-1}t_2^{n-1} + q^{n-1}(1-q+q^2)t_1^n t_2^n \]
\[ + \frac{q^{n+1}t_1^{n+1}t_2(q-1)}{1-qt_1} + \frac{q^{n+1}t_1^n t_2^{n+1}(q-1)}{1-qt_2} + \frac{q^n t_1^n t_2^n (1+q)(1-q)^2}{(1-qt_1)(1-qt_2)}, \]

and

\[ p_{g}^{A_{2n-1}} = \frac{L_{g}^{A_{2n-1}}(t_1-1)(t_2-1)}{t_1 t_2 - 1} = 1 + qt_1t_2 + \ldots + q^{n-1}t_1^{n-1}t_2^{n-1} + \frac{q^{n}(1-q)t_1^n t_2^n}{(1-qt_1)(1-qt_2)}. \]

Corollary 3

\[ \overline{F}_{g}^{A_{2n-1}}(t_1, t_2) = [1 + (q + q^2)t_1t_2 + \ldots + (q^{n-1} + q^n)t_1^{n-1}t_2^{n-1} + q^n t_1^n t_2^n] \] (29)
\[ - (t_1 + t_2)(q + q^2 t_1 t_2 + \ldots + q^{n+1} t_1^n t_2^n). \]

In [17] Ozsvath and Szabo computed the Heegard-Floer homologies of the corresponding links. In their notation the answer has the following form (everywhere we write the Poincare polynomials of the corresponding complexes).

Let

\[ Y^{l}_{(d)}(t_1, t_2, u) = u^{d}(t_1^{l_1} + t_1^{l_1-1}t_2 + \ldots + t_2^{l_2}) + u^{d-1}(t_1^{l_1} + \ldots + t_2^{l_2-1}), \]

\[ B_{(d)}(t_1, t_2, u) = u^{d} + (t_1 + t_2)u^{d+1} + u^{d+2}t_1 t_2. \]

Then

\[ HFL_{A_{2n-1}}(t_1, t_2, u) = Y^{0}_{(0)}t_1^{n/2}t_2^{n/2} + Y^{1}_{(-1)}t_1^{n/2-1}t_2^{n/2-1} + \sum_{i=2}^{n} B_{(-2i)}t_1^{n/2-i}t_2^{n/2-i}. \]

Since \( Y^{0}_{(0)} = 1 \) and \( Y^{1}_{(-1)} = u^{-1}(t_1 + t_2) + u^{-2} \) one can simplify this as

\[ HFL_{A_{2n-1}}(t_1, t_2, u) = t_1^{n/2}t_2^{n/2} + (u^{-1}(t_1 + t_2) + u^{-2})t_1^{n/2-1}t_2^{n/2-1} + \sum_{i=2}^{n} (u^{-2i} + (t_1 + t_2)u^{-2i+1} + u^{-2i+2}(t_1 t_2)) t_1^{n/2-i}t_2^{n/2-i}, \]

so

\[ t_1^{n/2}t_2^{n/2} HFL_{A_{2n-1}}(t_1^{-1}, t_2^{-1}, u) = 1 + (u^{-1}(t_1 + t_2) + u^{-2}t_1 t_2) \]

41
\[ + \sum_{i=2}^{n} (u^{-2i}t_1^{i}t_2^i + (t_1 + t_2)u^{-2i+1}t_1^{i-1}t_2^{i-1} + u^{-2i+2}t_1^{i-1}t_2^{i-1}) = \\
[1 + 2u^{-2}t_1t_2 + \ldots + 2u^{-2n+2}t_1^{n-1}t_2^{n-1} + u^{-2n}t_1^{n}t_2^{n}] \\
- (t_1 + t_2)[u^{-1} + u^{-3}t_1t_2 + \ldots + u^{-2n+1}t_1^{n-1}t_2^{n-1}]. \]

The last expression is similar to (29) in analogy with the theorem 6. The author do not believe that there is a formal algorithm relating $P_g$ and $HFL$ in general, but there is a hope that the analogue of the equation (28) relating the filtration in Heegaard-Floer homologies with the filtration in the space of functions.

References

[1] N. A’Campo. La fonction zeta d’une monodromie. Comment. Math. Helv., v. 50(1975), 233-248.
[2] V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko. Singularities of differentiable maps. Vol. 2, Birkhauser, 1985.
[3] A. Campillo, F. Delgado, S. M. Gusein-Zade. The Alexander polynomial of a plane curve singularity via the ring of functions on it. Duke Math J. 117 (2003), no. 1, 125–156.
[4] A. Campillo, F. Delgado, S. M. Gusein-Zade. Integrals with respect to the Euler characteristic over spaces of functions and the Alexander polynomial. Proc. Steklov Inst. Math. 2002, no. 3 (238), 134 –147.
[5] A. Campillo, F. Delgado, S. M. Gusein-Zade. Multi-index filtrations and motivic Poincaré series. Monatshefte für Mathematik. 150 (2007), no.3, 193-210.
[6] A. Campillo, F. Delgado, K. Kiyek. Gorenstein property and symmetry for one-dimensional local Cohen-Macaulay rings. Manuscripta Math. 83(1994), no. 3-4, 405-423.
[7] J. Denef, F. Loeser. Germs of arcs on singular algebraic varieties and motivic integration. Inventiones Math. 135 (1999), no.1, 201–232.
[8] D. Eisenbud, W. Neumann. Three-dimensional link theory and invariants of plane curve singularities. Ann. of Math. Studies 110. Princeton Univ. Press, Princeton, NJ, 1985.

42
[9] F. Heinloth. A note on functional equations for zeta functions with values in Chow motives. arXiv: math.AG/0512237

[10] M. Kapranov. The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups, arXiv: math.AG/0001005

[11] C. Manolescu, P. Ozsvath, S. Sarkar. A combinatorial description of knot Floer homology. arXiv: math/0607691

[12] C. Manolescu, P. Ozsvath, Z. Szabo, D. Thurston. On combinatorial link Floer homology. arXiv:math/0610559

[13] A. Nemethi, L. Nicolaescu. Seiberg-Witten invariants and surface singularities. Geometry and Topology, 6 (2004), 269–328.

[14] A. Nemethi. On the Ozsvath-Szabo invariants of negative definite plumbed 3-manifolds. Geometry and Topology, 9 (2005), 991-1042.

[15] A. Nemethi, I. Luengo, A. Melle-Hernandes. Links and analytic invariants of superisolated singularities. Journal of Algebraic Geometry, 14 (2005), 563–565.

[16] P. Ozsvath, Z. Szabo. Holomorphic discs and knot invariants. Adv. Math., 186(1), 2004, 58–116.

[17] P. Ozsvath, Z. Szabo. Holomorphic discs and link invariants. arXiv:math/0512286

[18] P. Ozsvath, Z. Szabo. Holomorphic discs and topological invariants for closed three-manifolds. Ann. of Math. (2). 159(2004), no. 3, 1027–1158.

[19] P. Ozsvath, Z. Szabo. On the Floer homology of plumbed three-manifolds. Geometry and Topology, 7 (2003), 185–224.

[20] P. Ozsvath, Z. Szabo. On knot Floer homology and lens space surgeries. arXiv:math/0303017

[21] J. Rasmussen. Knot polynomials and knot homologies, in Geometry and Topology of Manifolds, Boden et. al. eds., Fields Institute Communications 47 (2005), 261–280.

[22] V. Turaev. Torsions of 3-manifolds, volume 4 of Geom. Topol. Monogr. Geom. Topol. Publ., Coventry, 2002.

Moscow State University, 
Department of Mathematics and Mechanics.
E. mail: gorsky@mccme.ru.