The difference of actions between nearly lying unstable “8” and “0” like trajectories in two dimensions is computed by methods of chaotic dynamics. We found it to be equal to $pY\Theta/2$, where $\Theta$ is angle between paths at cross, $Y$ is distance between paths, and $p$ is momentum. This kind of periodic trajectories contributes $\tau^2$ (interference) term to spectral form-factor.

I. INTRODUCTION

Let’s consider two particles inside infinite potential well, for example two-dimensional billiard. What the matter do they interfere or not? In two slit experiment one can measure intensity vs number of open slits and see effect of phase difference between paths. The negative magneto-resistance of disordered systems is manifestation of interference effects destroyed by magnetic field. The interference between particles inside chaotic system suffers from lack of clear formulation, however it sounds like something that probably exist.

Physics of interference can be understood by simple methods of real space trajectories. Postponing literature review, let’s take one closed trajectory crossing itself in real space under small angle $\Theta \ll 1$. We prove that this “8” like trajectory is accompanied by “0” like trajectory, see construction below. Their actions are simply related

$$\Delta S \equiv S_0 - S_8 = \frac{pY\Theta}{2} = \frac{p^2}{2} \frac{m_{11} + m_{22} - 2m_{12}}{m_{11}m_{22} - m_{12}^2} \Theta^2 \quad (1)$$

where $m_{ij}$ are second derivatives of $S_0$ with respect to trajectory displacements near cross, and $p$ is momentum.

Figure 1 explains the choice of $\Theta$ and $Y = y_2 - y_1$. Quantum particle on “8” like trajectory (we use language of semiclassical quantum mechanics) will interfere destructively with particle on “0” like trajectory. Indeed, i) the phase difference between two wave functions is $\pi + \Delta S/\hbar$ because wave on “8”-like trajectory must be inverted twice loosing $\pi/2$ phase each time and ii) the action difference $\Delta S$ can be arbitrary small.

We arrive at subtle point. We do not mean that particles jump back and force between “8” and “0” like trajectories. We just state that $\Delta S$ can be arbitrary small. The closer trajectory approaches itself the smaller $\Delta S$ is. The spectral form-factor, for example, is very sensitive to trajectories with small phase difference. By making use of Eq. (1) and periodic orbit quantization rule we can compute the so-called interference or $\tau^2$ term of the spectral form-factor $K(\tau)$ (the diagonal term was computed by Berry, see Eq. (60) of the cited paper for definition of the spectral form-factor). First conclusion: The interference in quantum chaos means that there are infinitely many pairs of periodic trajectories with small difference of actions.

The mystery is that quantum waves do jump between nearly lying trajectories. To understand how it works let’s cut both “8” and “0” like trajectories apart from the cross, see Fig. 2. It turns out that transmission is $\sin^2(\Delta S/2\hbar)$ (the trace of density propagator must be equal to absolute value square of trace of Green function). We successfully arrive at second conclusion: The interference in quantum chaos means suppression of transmission of self-touching trajectories. This formulation of quantum interference is equivalent to that given by Khmelnitskii for disordered systems.

This introduction is followed by construction of pairs of near-lying periodic trajectories. As soon as we prove existence of such trajectories the expression for the action difference Eq. (12) can be simplified. We will use the concept of self-correlated part of periodic trajectory and derive expression useful for the operator form of the interference. The calculation of the $\tau^2$ term of $K(\tau)$ is rather straightforward, but demand counting of 8-0 pairs of trajectories of the certain period. It will be published elsewhere. We review literature and conclude at the end.
II. CONSTRUCTION OF NEAR-LYING TRAJECTORIES

We broke the action of 8-like trajectory to the two parts $S_L$ and $S_R$ and expand both on their end positions $y_1$ and $y_2$, see Fig. 1.

$$S_L = S_L^{(0)} + \frac{1}{2} m_{11}^L y_1^2 - m_{12}^L y_1 y_2 + \frac{1}{2} m_{22}^L y_2^2 + p y_2 \Theta \tag{2}$$

$$S_R = S_R^{(0)} + \frac{1}{2} m_{11}^R y_1^2 - m_{12}^R y_1 y_2 + \frac{1}{2} m_{22}^R y_2^2 - p y_2 \Theta \tag{3}$$

The sum of actions has minimum $S_8 = S_L^{(0)} + S_R^{(0)}$ reached at $y_1 = y_2 = 0$. If the particle follows left part of trajectory in other direction, see Fig. 2 then

$$S_L' = S_L^{(0)} + \frac{1}{2} m_{11}^L y_1^2 - m_{12}^L y_1 y_2 + \frac{1}{2} m_{22}^L y_2^2 + p y_1 \Theta . \tag{4}$$

The sum of actions $S'_L + S_R$ reaches it minimum at

$$\bar{Y} = \bar{p} M^{-1} \bar{\Theta} \tag{5}$$

where $\bar{Y} = (y_1, y_2)$, $\bar{\Theta} = (\Theta_1, \Theta_2)$ and

$$M = \begin{pmatrix} m_{11} & -m_{12} \\ -m_{12} & m_{22} \end{pmatrix} = \begin{pmatrix} m_{11}^L + m_{11}^R & -m_{12}^L - m_{12}^R \\ -m_{12}^L - m_{12}^R & m_{11}^L + m_{22}^R \end{pmatrix} \tag{6}$$

Here $m_{ij}$ without superscript are coefficients of expansion of 0-like trajectory action $S_0 + S_R$ in terms of $y_1$ and $y_2$.

The action of the true 0-like trajectory is

$$S_0 = S_8 - \Delta S = S_8 - \frac{p^2}{2} \bar{\Theta} M^{-1} \bar{\Theta} , \tag{7}$$

and substitution of $m_{ij}$ gives Eq. (2).

The present construction of nearly lying trajectory is valid under assumptions that i) series Eqs. (3) - (4) exist and ii) $Y$ is such small that third and higher derivatives of actions are negligible. Clearly, the result Eq. (7) does not apply to all self-crossing trajectories, but to many of them.

We leave counting of such trajectories for separate publication.

III. SOME PROPERTIES OF 8-0 PAIRS.

The previous section shows how to construct 0-like trajectory for any 8-like trajectory. The construction is valid if the expansions Eqs. (3) - (4) are valid for $y$s given by Eq. (6). This is always true for sufficiently long trajectories.

The self-touching trajectory must have “correlated” parts. Indeed in order to be able to touch itself trajectory must approach itself. In chaos convergence of trajectories (as well as divergence) must be gradual with a rate known as Lyapunov exponent. The piece where trajectory approaches itself and then leaves itself is called “correlated” part of the loop.

The calculation of Sec. II can be repeated in any crosssection of 8-0 pair, where trajectory is sufficiently close to itself. The general expression for action difference valid for any crosssection is

$$\Delta S = \frac{1}{2} (p Y_8 \Theta_8 - p Y_8 \Theta_0) . \tag{8}$$

Here we changed notations of Sec. II, $Y \rightarrow Y_0, \Theta \rightarrow \Theta_0$, and added $Y_8$ - the distance between opposite pieces of 8-like trajectory (it was zero in Sec. II) and $\Theta_0$ - the angle between pieces of 0-like orbit, see Fig. 3. Let us now make two crosssections of 8-0 pair near boundaries between correlated and un-correlated parts. Expanding action of this “common” piece of all trajectories,

$$S_c = S_c^{(0)} + \frac{1}{2} m_{11}^c Y_0^2 - m_{12}^c Y_0 Y'_0 + \frac{1}{2} m_{22}^c Y'_0^2 , \tag{9}$$

we can express action difference in terms of $Y$s:

$$\Delta S = -\frac{p m_{12}^c}{2} (Y_0 Y'_0 - Y'_0 Y_0) \approx p m_{12}^c Y_0 Y_8 \tag{10}$$

where we put $Y'_0 \approx -Y_0$. This is justified if the “uncorrelated” parts of trajectories are long enough and therefore “soft” enough (much more unstable then the correlated part).
The advantage of approximation Eq. (10) is that the \( \Delta S \) is expressed in terms of one 0-like part of the trajectory. Finally we replace \( m_{12}^c \) by the stability amplitude and arrive at
\[
\Delta S \approx p \frac{Y_0 \Theta_0^* - Y_0^* \Theta_0}{4 \cosh^2(\lambda/2)},
\]
where \( \exp \pm \lambda \) are eigenvalues of the stability matrix of the correlated part of the trajectory \( 2 \cosh(\lambda) = (m_{11}^c + m_{22}^c)/m_{12}^c \). Here \( Y_s \) and \( \Theta_s \) belong to 0-like trajectory.

IV. OPERATOR FORM OF THE INTERFERENCE.

The title of the work is divergence and interference in quantum chaos. The term divergence is clear: take two particles very close initially in the phase space, let them propagate and they will branch off exponentially fast. Let us describe trajectories of the system by \( \gamma(t) = F_1(\gamma(0)) \), where \( \gamma \) is a point on the energy shell of the phase space. The evolution of the two-particle distribution function is given by
\[
f(\gamma_1, \gamma_2; t) = \int d\gamma_3 d\gamma_4 A(\{\gamma_3\}; t) f(\gamma_3, \gamma_4; t)
\]
where
\[
A(\{\gamma_j\}; t) = \delta(\gamma_1 - F_{-t}(\gamma_1)) \delta(\gamma_2 - F_{-t}(\gamma_2))
\]
\[
= \delta(\gamma_1^* - \gamma_1^0) \delta(\gamma_2^0 - \gamma_2) \delta(\gamma_1^0 - \gamma_2^0 - \dot{\gamma}_1) \delta(\Gamma' - F_{-t}(\Gamma)).
\]

Here \( \Gamma' = (\gamma_1 + \gamma_2)/2, \Gamma = (\gamma_1 + \gamma_2)/2 \) are centers of mass, \( \gamma = \gamma_1 - \gamma_2, \gamma' = \gamma_3 - \gamma_4 \) describe relative motion of two particles. Relative coordinates are separated to components which are either parallel or perpendicular to the center-of-mass trajectory. We choose \( \gamma_1 = (Y_0, \Theta_0) \) and \( \gamma_2 = (Y_0', \Theta_0') \), see Fig. 3. The monodromy matrix \( \dot{M} \) can be expressed in terms of action derivatives:
\[
\dot{M} = \frac{1}{m_{12}^c} \left( \begin{array}{cc} m_{11}^c & m_{12}^c \\ m_{12}^c & \left( m_{12}^c \right)^2 - m_{22}^c \end{array} \right)
\]

Basically, the operator Eq. (13) is eight-leg quantum propagator. The eight coordinates were broken to four pairs and the Fourier transform with respect of their difference in each pair leads to the density propagator.

Somewhat different pairing of arguments of eight-leg propagator gives interference operator. It is called the Hikami box in theory of disordered metals. We derived it for chaotic system by making use of the method of Aleiner and Larkin. We just replaced random potential by exact scattering amplitudes. The interference operator is similar to divergence operator: it propagates center of mass along true trajectory but the relative motion is constrained in a different way:
\[
\mathcal{H}(\{\gamma_j\}; t) = -i \Delta S/\hbar (m_{12}^c)^2 \left[ \frac{A(\{\gamma_j\}; t)}{(m_{12}^c)^2} \right]_{m_{12}^c \to 0}.
\]

This is main result of the present work.

The connection between interference and divergence becomes obvious. The operator Eq. (15) is important for trajectories with \( \Delta S \lesssim \hbar \). The crucial point is that \( \Delta S \) is inversely proportional to stability of the trajectory. Therefore for any energy of the particle there are sufficiently long and unstable trajectories contributing to the interference kernel Eq. (14).

V. SUMMARY

In summary we have given definition of interference in quantum chaos. It can be seen as presence of nearly same length trajectories or as decrease of transmission of some path and therefore increase of the return probability. In this way we see that the trace formula do describes weak localization, contrary to the statement of Ref. [3].

The progress of non-perturbative field theoretical methods in quantum chaos left the problem of interference open. Indeed the interference term in these theories has exactly the same structure as in effective Lagrangian of disordered systems. In latter case interference between particles is point-like; it appears as a result of single impurity scattering. In principle, one can add weak random potential to chaotic system, derive effective Lagrangian and then put relaxation time to infinity. Interference terms slowly disappear together with random potential.

The form of the interference kernel is primarily important for building of the field theory correctly describing “interaction” of diffusion modes (called in quantum chaos Liouvillian modes). At present time it is not clear how to derive theory with interference kernel Eq. (15) in the effective action.

The developed here theory can be generalized to any dimension, however the counting of phase lost due to beam rotation becomes more complicated. In addition, the theory is not sensitive to any discrete symmetries and can be applied to modular groups. When the manuscript was in preparation author received mail about similar calculation of the action difference of 8-0 pairs undertaken recently by other people.

Comments and remarks of D. Cohen are greatly acknowledged.
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