STOCHASTIC MAXIMUM PRINCIPLE FOR EQUATIONS WITH DELAY:  
THE NON-CONVEX CASE  

GIUSEPPINA GUATTERI AND FEDERICA MASIERO

Abstract. In this paper we develop necessary conditions for optimality, in the form of the stochastic Pontryagin maximum principle, for controlled equation with pointwise delay in the state and with control dependent noise, in the general case of controls $u \in U$ with $U$ not necessarily convex. The maximum principle is formulated by means of a first and second order adjoint BSDEs. We also outline how to deal with control problems with pointwise delay both in the state and in the control.

1. Introduction

In this paper we consider a controlled stochastic differential equation in $\mathbb{R}^n$ with pointwise delay in the state:

$$
\begin{align*}
  dX(t) &= b(t, X(t), X(t-d), u(t)) \, dt + \sigma(t, X(t), X(t-d), u(t)) \, dW(t), \quad t \in [0,T], \\
  X_0 &= x(\theta), \quad \theta \in [-d,0].
\end{align*}
$$

(1)

Associated to this controlled state equation we consider a cost functional to be minimized over all admissible controls:

$$
J(u) := \mathbb{E} \int_0^T \ell(t, X(t), X(t-d), u(t)) \, dt + \mathbb{E}(h(X(T)))
$$

(2)

In the case of control processes taking values in a convex set $U$, the problem has already been treated see e.g. [4], see also [6] where the linear quadratic case is considered and [5] where, for problems with only delay in the state, the authors prove also existence of optimal controls.

In all the aforementioned papers the stochastic maximum principle for stochastic control problems with delay is formulated by means of anticipated BSDEs, that have been introduced in [17] and generalized in [18]. We also cite [12], and the references therein, where a more general dependence on the past is considered, and the problem is solved by means of anticipated BSDEs, and the paper [14], where the authors formulate the stochastic maximum principle for a problem with delay in the state by functional analysis methods.

In the present paper, we consider the general case of control dependent noise when the control processes take values in a set $U$ not necessarily convex. A similar problem has been considered in [16] for a state equation without delay, see also [19], and it is solved using the so called spike variation method, consisting in perturbing the optimal control only in a measurable set $E_\varepsilon \subset [0,T]$ and considering the variation of the state with respect to this perturbation of the control. We restrict to the case when no delay appears in the control, mainly for the sake of simplicity, but we outline in some remarks how to handle the case with pointwise delay in the control in the state equation (1).

In [16] the maximum principle has been formulated by means of two backward stochastic differential equations (BSDEs in the following): towards the case of $U$ convex, besides the (first order) adjoint
BSDE, which is the adjoint equation of the variation of the state, in [16], the authors introduce a second order adjoint BSDE, which is the adjoint equation of the equation for the square of variation of the state. We notice that without delay, the equation for the square of the variation of the state has itself the structure of a linear equation, with other terms that can be handled. In the case with delay, the square of the variation does not solve an equation written in a closed form since a mixed term with present and past appears, see comments in Section 3. The point is to deal with this "extra" term.

We finally mention that control problems with delay both in the state and in the control can be treated by the dynamic programming principle. We quote here [9] and [10], see also [11] for the case of pointwise delay in the control, where a control problem with delay both in the state and in the control, only for additive noise, is treated by solving the associated HJB equation: the optimal cost and the optimal controls are both characterized. We also mention [2], where a general control problem with delay in the state and in the control is considered also under partial observation: the problem is solved by a randomized dynamic programming approach and so the optimal cost is characterized, on the contrary there is no characterization of the optimal controls, and this characterization is just what we are able to give treating the problem by means of the stochastic maximum principle.

The paper is organized as follows. In section 2 we present the problem and we formulate the first order adjoint equation, which is an anticipated BSDE, in Section 3 we consider the equation for the square of variation of the state we show how to recover, in some sense, a linear structure, and in Section 4 we formulate the second order adjoint equation, which again is an anticipated BSDE, Finally in section 5 we state and prove the maximum principle.

2. Assumptions and preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and let \(W(t)\) be a \(m\) dimensional brownian motion. We endow \((\Omega, \mathcal{F}, \mathbb{P})\) with the natural filtration \((\mathcal{F}_t)_{t \geq 0}\) generated by \(W\) and augmented in the usual way with the family of \(\mathbb{P}\)-null sets of \(\mathcal{F}\). A finite time horizon \(T > 0\) is fixed and we denote the norm in \(\mathbb{R}^n\) by \(|\cdot|\) and the inner product by \((\cdot, \cdot)\).

We introduce the following spaces for any \(p \geq 1\) and for every \(0 \leq r \leq s \leq T\):

- \(L^p(\Omega, \mathcal{F}_s; \mathbb{R}^n)\) the set of \(\mathcal{F}_s\) measurable random vectors in \(\mathbb{R}^n\);
- \(L^p_{\mathcal{P}}(\Omega \times [r, s]; \mathbb{R}^n)\), the set of all \((\mathcal{F}_t)\)-progressively measurable processes with values in \(\mathbb{R}^n\) such that
  \[ \|X\|_{L^p_{\mathcal{P}}(\Omega \times [r, s]; \mathbb{R}^n)} = \left( \mathbb{E} \int_r^s |X(t)|^p \, dt \right)^{1/p} < \infty; \]
- \(L^p_\mathcal{P}(\Omega; C([r, s]; \mathbb{R}^n))\), the set of all \((\mathcal{F}_t)\)-progressively measurable processes with values in \(\mathbb{R}^n\) such that
  \[ \|X\|_{L^p_\mathcal{P}(\Omega; C([r, s]; \mathbb{R}^n))} = \left( \mathbb{E} \sup_{t \in [r, s]} |X(t)|^p \right)^{1/p} < \infty. \]

2.1. Formulation of the control problem. Let \(U \subset \mathbb{R}^k\) be a non empty set. By admissible control we mean any process \(u : \Omega \times [0, T] \rightarrow U\) that is \((\mathcal{F}_t)\)-progressive measurable and we denote by \(\mathcal{U}_{ad}\) the set of admissible controls.

As already noticed, we do not make any convexity assumptions on the space \(U\): this allows e.g. to consider the case of \(U = 0, 1\), or more generally of \(U\) any discrete space.

We study the following past-dependent state equation

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\ dX(t) = b(t, X(t), X(t-d), u(t)) \, dt + \sigma(t, X(t), X(t-d), u(t)) \, dW(t), \\
X_0 = x(\theta), \quad \theta \in [-d, 0].
\end{array}
\right.
\end{aligned}
\]

where \(X(t) \in \mathbb{R}^n\), while \(x \in C([-d, 0]; \mathbb{R}^n)\), so \(X_0\) is an element of \(C([-d, 0]; \mathbb{R}^n)\), in general we will use the notation \(X_t(\theta) = X(t + \theta), \theta \in [-d, 0]\), \(t \in [0, T]\) to denote the whole trajectory.

We make the following assumptions on the coefficients:
Here the coefficients are such that are continuously differentiable with bounded derivatives with respect to \((x, y)\) up to second order and there exist positive constants \(M\) and \(L\), such that for \(\psi = b, \sigma:\)
\[
|\psi(t, x, y, u) - \psi(t, x, y, u')| \leq L|u - u'|, \forall t \in [0, T], u, u' \in U.
\]
\[
|\psi(t, 0, 0, u)| \leq M \quad \forall t \in [0, T], u \in U.
\]

We associate the following cost functional
\[
J(u) := \mathbb{E} \int_0^T \ell(t, X(t), X(t - d), u(t)) \, dt + \mathbb{E}(h(X(T)))
\]
where we assume:

A.2 The functions \(\ell\) and \(h:\)
\[
\ell(t, x, y, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R},
\]
\[
h(x) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}
\]
are continuously differentiable with bounded derivatives with respect to \((x, y)\) and there exist positive constants \(M\) and \(L\), such that:
\[
|\ell(t, x, y, u) - \ell(t, x, y, u')| \leq L|u - u'|, \forall t \in [0, T], u, u' \in U.
\]
\[
|\ell(t, 0, 0, u)| \leq M \quad \forall t \in [0, T], u \in U.
\]

Our control problem consists in minimizing over all admissible controls the cost functional given in \(J\).

It is well known that, see e.g. \([15]\), that equation \((\mathbf{3})\) admits a unique solution, as stated in the following Theorem.

**Theorem 1.** Under A.1 for every \(x \in C([-d, 0]; \mathbb{R}^n)\) and every \(u \in \mathcal{U}\) one has that there exists a unique solution \(X\) of \((\mathbf{3})\) and there exists a constant \(c > 0\) such that:
\[
\mathbb{E} \sup_{t \in [-d, T]} |X(t)|^2 \leq c(1 + |x|_{C([-d, 0]; \mathbb{R}^n)}^2)
\]

**Remark 1.** We present here the control problem related to a delay equation like equation \((\mathbf{3})\), with pointwise delay in the control and we outline how to handle such problems.

Let us consider a state equation with delay both in the state and in the control:
\[
\left\{ \begin{array}{l}
\frac{dX(t)}{dt} = b(t, X(t), X(t - d), u(t), u(t - d)) \, dt + \sigma(t, X(t), X(t - d), u(t), u(t - d)) \, W(t), \quad t \in [0, T], \\
X_0 = x(\theta), \quad \theta \in [-d, 0].
\end{array} \right.
\]

Here the coefficients are such that
\[
b(t, x, y, u, v) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^n,
\]
\[
\sigma(t, x, y, u, v) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^{n \times m}
\]
are continuously differentiable with bounded derivatives with respect to \(x\) and \(y\), and there exist positive constants \(M\) and \(L\), such that for \(\psi = b, \sigma:\)
\[
|\psi(t, x, y, u, v) - \psi(t, x, y, u', v')| \leq L(|u - u'| + |v - v'|), \quad \forall t \in [0, T], x, y \in \mathbb{R}^n, u, u', v, v' \in U.
\]
\[
|\psi(t, 0, 0, u, v)| \leq M \quad \forall t \in [0, T], u, v \in U.
\]

The maximum principle for such a problem has been studied in \([2]\) in the case of \(U\) convex, and it is shown that delay in the control does not affect the adjoint equation. In the present paper we
consider the case when \( U \), the space of the controls, is not convex: we need to consider also a second adjoint equation. This further adjoint equation turns out to be the adjoint equation related to the square of the variation of the state. We build this adjoint equation as the adjoint equation of a new equation with delay in the state and in the control, so again the presence of the delay in the control does not affect our equation, see Section 7 remark 4.

**Remark 2.** We briefly present some models we are able to treat with our techniques.

The first one is an optimal consumption problem in a stochastic growth model. The state equation is given by

\[
\begin{align*}
\frac{dx(t)}{dt} = & \left[\rho x(t) + (\mu - r)x(t) - c(t)\right]dt + \sigma x(t)dB(t), \quad t > 0 \\
x(0) = & x_0 > 0
\end{align*}
\]

where \( c(t) \) is the consumption rate, the control, the state \( x(t) \) is the capital stock, the process \( \alpha(t) \) is another control, representing the investment in "risky" asset.

The aim is to maximize the functional \( (U \text{ is a given utility function}) \)

\[
E [U (x(T))],
\]

over the set of the admissible strategies, that here has to keep the wealth \( x(\cdot) \) positive.

Let us assume that \( u \) is an optimal control and \( X \) the corresponding optimal trajectory, we introduce the spike variation: let \( \varepsilon > 0 \) and \( E_\varepsilon \subset [0, T] \), such that \( \lambda(E_\varepsilon) = \varepsilon \), where \( \lambda \) is the Lebesgue measure. Let \( v \in U \) we set

\[
u^\varepsilon(t) = \begin{cases}
u(t) & t \in [0, T] \setminus E_\varepsilon \\
v & t \in E_\varepsilon
\end{cases}
\]

We will denote by \( X^\varepsilon \) the solution of \( [1] \) corresponding to \( u^\varepsilon \). For simplicity we’ll use the following notation

\[
l^\varepsilon(t) := l(t, X^\varepsilon(t), X^\varepsilon(t - d), u^\varepsilon(t)), \\
l(t) := l(t, X(t), X(t - d), u(t)) \\
\]

\[
l_\rho(t) = l_\rho(t, X(t), X(t - d), u(t)) \\
\delta l(t) = l(t, X(t), X(t - d), u^\varepsilon(t)) - l(t, X(t), X(t - d), u(t))
\]

where \( l = b^i, \sigma^{ij}, \ell \) and \( \rho = x, y, u, i : 1, \ldots, n \) and \( j = 1, \ldots, m \).

We introduce the first variation equation:

\[
\begin{align*}
\frac{dY^\varepsilon(t)}{dt} = & \frac{b_{x}(t)Y^\varepsilon(t)}{dt} + \frac{b_{y}(t)Y^\varepsilon(t - d)}{dt} + \frac{\sigma_{x}(t)Y^\varepsilon(t)}{dt} dW(t) + \frac{\sigma_{y}(t)Y^\varepsilon(t - d)}{dt} dW(t) \\
& + \delta b(t) dt + \delta \sigma(t) dW(t), \quad t \in [0, T], \\
Y_0 = & 0, \quad \theta \in [-d, 0].
\end{align*}
\]

where \( b_\rho(t) = (b^i_\rho(t))_{i:1,\ldots,n} \) and \( \sigma_\rho(t) = (\sigma^{ij}_\rho(t))_{i:1,\ldots,n; j:1,\ldots,m} \), \( \rho = x, y \).

We introduce also the second variation:

\[
\begin{align*}
\frac{dZ^\varepsilon(t)}{dt} = & \frac{b_{x}(t)Z^\varepsilon(t)}{dt} + \frac{b_{y}(t)Z^\varepsilon(t - d)}{dt} + \frac{1}{2} b_{x}^2(t)(Y^\varepsilon(t))^2 dt + \frac{1}{2} b_{y}^2(t)(Y^\varepsilon(t - d))^2 dt \\
& + b_{xy}(t)Y^\varepsilon(t)Y^\varepsilon(t - d) dt + \frac{1}{2} \sigma_{x}^2(t)Z^\varepsilon(t)dW(t) + \frac{1}{2} \sigma_{y}^2(t)Z^\varepsilon(t - d)dW(t) \\
& + \frac{1}{2} \sigma_{xy}(t)(Y^\varepsilon(t))^2 + \sigma_{xy}(t)(Y^\varepsilon(t - d))^2 + 2 \sigma_{xy}(t)Y^\varepsilon(t)Y^\varepsilon(t - d)dW(t) \\
& + \delta \sigma x(t)Y^\varepsilon(t)I_{E^\varepsilon}(t) dW(t) + \delta \sigma y(t)Y^\varepsilon(t)I_{E^\varepsilon}(t) dW(t), \quad t \in [0, T],
\end{align*}
\]

\[
Z_0 = 0, \quad \theta \in [-d, 0].
\]
where the term \( b_{\rho}(t)vw = (\tr[b_{\rho}(t)vw^*])_{i:1,...,n} \) and \( \sigma_{\rho}(t)vw = (\tr[\sigma_{\rho}^j(t)vw^*])_{i:1,...,n,j:1,...,m} \) for any \( v, w \in \mathbb{R}^n \) and \( \rho, \eta = x, y \). We will use such notation in sequel also for \( \ell \) and \( h \).

We have that the following holds:

**Theorem 2.** Under assumptions A.1 - A.2 equations \((14)\) and \((17)\) admit unique solutions, given respectively by \( Y^\varepsilon \in L^2_T(\Omega; C([-d, T]; \mathbb{R}^n)) \) and \( Z^\varepsilon \in L^2_T(\Omega; C([-d, T]; \mathbb{R}^n)) \). Moreover, we have

\[
\mathbb{E} \sup_{t \in [-d, T]} |X(t) - X^\varepsilon(t)|^2 = O(\varepsilon),
\]

\[
\mathbb{E} \sup_{t \in [-d, T]} |Y^\varepsilon(t)|^2 = O(\varepsilon),
\]

\[
\mathbb{E} \sup_{t \in [-d, T]} |Z^\varepsilon(t)|^2 = O(\varepsilon),
\]

\[
\mathbb{E} \sup_{t \in [-d, T]} |X^\varepsilon(t) - X(t) - Y^\varepsilon(t)|^2 = O(\varepsilon^2),
\]

\[
\mathbb{E} \sup_{t \in [-d, T]} |X^\varepsilon(t) - X(t) - Y^\varepsilon(t) - Z^\varepsilon(t)|^2 = o(\varepsilon^2),
\]

Moreover the following expansion of the cost functional holds true

\[
J(u^\varepsilon) = J(u) + \mathbb{E}\langle h_x(X(T)), Y^\varepsilon(T) + Z^\varepsilon(T) \rangle + \frac{1}{2} \mathbb{E}h_{xx}(X(T))(Y^\varepsilon(T))^2
\]

\[
+ \mathbb{E} \int_0^T [\ell_x(t)Y^\varepsilon(t) + \ell_y(t)Y^\varepsilon(t-d)] \, dt
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^T [\ell_{xx}(t)(Y^\varepsilon(t))^2 + \ell_{yy}(t)(Y^\varepsilon(t-d))^2 + 2\ell_{xy}(t)Y^\varepsilon(t)Y^\varepsilon(t-d)] \, dt.
\]

\[
+ \mathbb{E} \int_0^T \delta \ell(t) \, dt + o(\varepsilon)
\]

**Proof.** All the statements follows essentially from [10, Theorem 4.4], see also [3], where the same pointwise delay is considered. We only sketch [22].

Let’s call \( \rho^\varepsilon(t) = X^\varepsilon(t) - X(t) - Y^\varepsilon(t) - Z^\varepsilon(t) \), then one has that

\[
\rho^\varepsilon(t) = 0, \quad t \in [-d, 0]
\]

\[
\rho^\varepsilon(t) = \int_0^t b_x(s)\rho^\varepsilon(s) \, ds + \int_0^t b_y(s)\rho^\varepsilon(s-d) \, ds + \int_0^t \sigma_x(s)\rho^\varepsilon(s) \, dW(s)
\]

\[
+ \int_0^t \sigma_y(s)\rho^\varepsilon(s-d) \, dW(s) + \int_0^t N^\varepsilon(s) \, ds + \int_0^t R^\varepsilon(s) \, dW(s), \quad t \in [0, T].
\]

In order to describe \( N^\varepsilon \) and \( R^\varepsilon \) properly, we introduce also:

\[
l_{\rho}^\varepsilon(t) = l_{\rho}(t, X(t), X(t-d), u^\varepsilon(t))
\]

\[
l_{\rho}^\varepsilon(t) = \int_0^1 2\theta l_{\rho,\eta}(t, X(t) + \theta(X^\varepsilon(t) - X(t)), X(t-d) + \theta(X^\varepsilon(t-d) - X(t-d)), u^\varepsilon(t)) \, d\theta
\]

for \( l = b, \sigma \) and \( \rho, \eta = x, y \), then

\[
l(X^\varepsilon(t), X^\varepsilon(t-d), u^\varepsilon(t)) - l(X(t), X(t-d), u^\varepsilon(t)) = l_{\rho}^\varepsilon(t)\xi^\varepsilon(t) + l_{\rho}^\varepsilon(t)\xi^\varepsilon(t-d) + \tilde{l}_{\rho}^\varepsilon(t)\xi^\varepsilon(t)^2
\]

\[
+ \tilde{l}_{\rho}^\varepsilon(t)\xi^\varepsilon(t-d)^2 + 2\tilde{l}_{\rho}^\varepsilon(t)\xi^\varepsilon(t)\xi^\varepsilon(t-d).
\]

where \( \xi^\varepsilon(t) = X^\varepsilon(t) - X(t) \).
Then we can write $N^\epsilon$ and $R^\epsilon$ in the following way:

\[
N^\epsilon(s) = \frac{1}{2} \tilde{b}_{xx}^\epsilon(s) - b_{xx}(X(s), X(s - d), u^\epsilon(s))(X^\epsilon(s) - X(s))^2 \\
+ \frac{1}{2} \tilde{b}_{xy}^\epsilon(t) - b_{xy}(X(t), X(t - d), u^\epsilon(t))(X^\epsilon(s) - X(s))(X^\epsilon(s - d) - X(s - d)) \\
+ \frac{1}{2} \tilde{b}_{yy}^\epsilon(s) - b_{yy}(X(s), X(s - d), u^\epsilon(s))(X^\epsilon(s - d) - X(s - d))^2 \\
+ \delta b_x(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s)) + \delta b_y(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s)) \\
+ \frac{1}{2} \delta b_{xx}(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s))^2 + \frac{1}{2} \delta b_{yy}(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s)) \\
+ \frac{1}{2} \delta b_{xy}(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s))(X^\epsilon(s) - X(s)) \\
+ \frac{1}{2} b_{xx}(s)[(\xi^\epsilon(s))^2 - (Y^\epsilon(s))^2] + \frac{1}{2} b_{yy}(s)[(\xi^\epsilon(s) - d)^2 - (Y^\epsilon(s) - d)^2] \\
+ b_{xy}(s)[\xi^\epsilon(s)\xi^\epsilon(s - d) - Y^\epsilon(s)Y^\epsilon(s - d)]
\]

\[
R^\epsilon(s) = \frac{1}{2} \tilde{\sigma}_{xx}^\epsilon(s) - \sigma_{xx}(X(s), X(s - d), u^\epsilon(s))(X^\epsilon(s) - X(s))^2 \\
+ \frac{1}{2} \tilde{\sigma}_{xy}^\epsilon(t) - \sigma_{xy}(X(t), X(t - d), u^\epsilon(t))(X^\epsilon(s) - X(s))(X^\epsilon(s - d) - X(s - d)) \\
+ \frac{1}{2} \tilde{\sigma}_{yy}^\epsilon(s) - \sigma_{yy}(X(s), X(s - d), u^\epsilon(s))(X^\epsilon(s - d) - X(s - d))^2 \\
+ \frac{1}{2} \sigma_x(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s)) + \frac{1}{2} \sigma_y(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s)) \\
+ \frac{1}{2} \sigma_{xx}(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s))^2 + \frac{1}{2} \sigma_{yy}(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s)) \\
+ \frac{1}{2} \sigma_{xy}(s) I_{E_\epsilon}(s)(X^\epsilon(s) - X(s))(X^\epsilon(s) - X(s)) \\
+ \frac{1}{2} \sigma_{xy}(s)[(\xi^\epsilon(s))^2 - (Y^\epsilon(s))^2] + \frac{1}{2} \sigma_{yy}(s)[(\xi^\epsilon(s) - d)^2 - (Y^\epsilon(s) - d)^2] \\
+ \sigma_{xy}(s)[\xi^\epsilon(s)\xi^\epsilon(s - d) - Y^\epsilon(s)Y^\epsilon(s - d)]
\]

Thanks to \cite{18}, \cite{19}, \cite{20} and \cite{21}, hypothesis (A.2) and noticing that all terms vanish as $s < d$ one has that

\[
\mathbb{E} \int_0^T |N^\epsilon(s)| \, ds = o(\epsilon)
\]  

(25)

and

\[
\mathbb{E} \int_0^T |R^\epsilon(s)| \, ds = o(\epsilon^2)
\]  

(26)

Thus by a standard Gromwall argument we prove \cite{22}. Then \cite{23} follows from \cite{18}, \cite{19}, \cite{20}, \cite{21} and \cite{22}. \hfill \Box

2.2. First order adjoint equation. In this subsection we will introduce the first order adjoint equation involved in the stochastic maximum principle. It is well known in the literature, we refer e.g. to \cite{4} where the pointwise delay case is studied. Such an adjoint equation turns out to be an anticipated backward stochastic differential equation, see \cite{17} and \cite{18}. The first order adjoint equation is given by the following ABSDE

\[
- dp(t) = \left[ b_x^\epsilon(t, x(t), x(t - d), u(t)) p(t) + \sigma_x^\epsilon(t, x(t), x(t - d), u(t + d)) q(t) \right] dt \\
+ \mathbb{E}^f_t \left[ b_y^\epsilon(t + d, x(t + d), x(t), u(t + d)) p(t + d) + \sigma_y^\epsilon(t + d, x(t + d), x(t), u(t + d)) q(t + d) \right] dt \\
+ \ell_x(t, x(t), x(t - d), u(t)) + \mathbb{E}^f_t \left[ \ell_y(t + d, x(t + d), x(t), u(t + d)) \right] dt - q(t) \, dW_t \\
p(T) = - h_x(\bar{x}(T)), \ \ p(t) = 0 \ \ t \in (T, T + d), \ \ q(t) = 0 \ \ t \in [T, T + d].
\]

Then, by \cite{17}, Theorem 4.2] we have:
Theorem 3. The anticipated equation (27) admits a unique solution \((p, q)\) that is a couple of processes such that \(p \in L^2_F(\Omega; C([0, T + d]; \mathbb{R}^n))\) and \(q \in L^2_F(\Omega \times [0, T + d]; \mathbb{R}^{n \times m})\).

Proof. The driver \(f\) introduced in [17, Theorem 4.2], takes the following form in our case:

\[
\begin{align*}
&f(t, y, z, \xi_{t+d}, \eta_{t+d}) = b^*_y(t, \bar{x}(t), \bar{x}(t-d), \bar{u}(t)) y + \sigma^*_y(t, \bar{x}(t), \bar{x}(t-d), \bar{u}(t)) z + \ell_x(t, \bar{x}(t), \bar{x}(t-d), \bar{u}(t)) \\
&+ \mathbb{E}^{\mathcal{F}_t} [b^*_q(t+d, \bar{x}(t+d), \bar{x}(t), \bar{u}(t+d)) \xi(t+d)] + \mathbb{E}^{\mathcal{F}_t} [\sigma^*_q(t+d, \bar{x}(t+d), \bar{x}(t), \bar{u}(t+d)) \eta(t+d)] \\
&+ \mathbb{E}^{\mathcal{F}_t} [\varepsilon_y(t+d, \bar{x}(t+d), \bar{x}(t), \bar{u}(t+d))]
\end{align*}
\]

Hence from (A.1) and (A.2) it is straightforward to deduce (H1) and (H2) required in [17, Theorem 4.2].

Making use of this first adjoint equation, it turns out that the expansion of the cost in formula (23) can be rewritten in terms of \((p, q)\).

Proposition 1. Under hypotheses (A.1) – (A.2), the following expansion for the cost holds true:

\[
J(u) - J(u^\varepsilon) = \mathbb{E} \int_0^T (-\delta(t) + \langle p(t), \delta\beta(t) \rangle + q(t)^T \delta\sigma(t)) I_{E_u}(t) \, dt
\]

\[
- \frac{1}{2} \mathbb{E} [b_{xx}(X(T))(Y^\varepsilon(T))^2] - \frac{1}{2} \mathbb{E} \int_0^T \left[ \ell_{xx}(t)(Y^\varepsilon(t))^2 + \ell_{yy}(t)(Y^\varepsilon(t-d))^2 + 2\ell_{xy}(t)(Y^\varepsilon(t)Y^\varepsilon(t-d)) \right] \, dt
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle p(t), b_{xx}(Y^\varepsilon(t))^2 + b_{yy}(Y^\varepsilon(t-d))^2 + 2b_{xy}(Y^\varepsilon(t)Y^\varepsilon(t-d)) \rangle \right] \, dt
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^T \sum_{j=1}^m \left[ \langle q^j(t), \sigma^j_{xx}(Y^\varepsilon(t))^2 + \sigma^j_{yy}(Y^\varepsilon(t-d))^2 + 2\sigma^j_{xy}(Y^\varepsilon(t)Y^\varepsilon(t-d)) \rangle \right] \, dt
\]

\[
+ o(\varepsilon)
\]

Proof. Notice the notation \(q(t)^* \delta\sigma(t)\) that is for \(\text{tr}[q(t)^* \delta\sigma(t)]\). This expansion follows from the computation of \(d(p(t), Y^\varepsilon(t))\) and \(d(p(t), Z^\varepsilon(t))\). These differentials follow immediately from the Itô formula, we only have to pay attention to the terms involving the delay and the conditional expectation. These last terms cancel only once the expectation is calculated. For clearness we show an example where, along the calculations, these difference appears:

\[
\mathbb{E} \int_0^T \langle \mathbb{E}^{\mathcal{F}_t}(b_y(t+d)p(t+d), Y^\varepsilon(t)) \rangle \, dt - \int_0^T \langle b_y(t)p(t), Y^\varepsilon(t-d) \rangle \, dt.
\]

It is clear that, by a change of variable, thanks to the final condition for \(p\) and the initial condition for \(Y^\varepsilon\):

\[
\mathbb{E} \int_0^T \langle \mathbb{E}^{\mathcal{F}_t}(b_y(t+d)p(t+d), Y^\varepsilon(t)) \rangle \, dt = \int_0^{T+d} \mathbb{E} \langle b_y(t)p(t), Y^\varepsilon(t-d) \rangle \, dt
\]

\[
= \int_0^d \mathbb{E} \langle b_y(t)p(t), Y^\varepsilon(t-d) \rangle \, dt + \int_0^T \mathbb{E} \langle b_y(t)p(t), Y^\varepsilon(t-d) \rangle \, dt
\]

thus

\[
\mathbb{E} \left[ \int_0^T \langle \mathbb{E}^{\mathcal{F}_t}(b_y(t+d)p(t+d), Y^\varepsilon(t)) \rangle \, dt - \int_0^T \langle b_y(t)p(t), Y^\varepsilon(t-d) \rangle \, dt \right] = 0
\]

Remark 3. We notice that we are presenting our results in the case of one pointwise delay, nothing would change if we consider \(n + 1\), with \(n > 0\), delay times: \(d_0 = d > d_1 > ... > d_n > 0\). The first
adjoint would have naturally an extension, and the expansion of the cost \( \text{(28)} \) would present terms

\[
\begin{pmatrix}
Y^\varepsilon(t) \\
Y^\varepsilon(t-d_n) \\
\vdots \\
Y^\varepsilon(t-d_1) \\
Y^\varepsilon(t-d)
\end{pmatrix}
\]

that can be reduced to a quadratic form for the vector

\[ Y \]

3. The square of the variation of the state

We come back to the case of one pointwise delay.

We make the assumption that \( T > d \), otherwise the problem reduces to the non delay case, indeed in the equation of the first variation of the state, the term \( Y^\varepsilon(t-d) \) would be identically equal to 0 if \( t \leq T < d \).

Following the idea of \([16]\), see also \([19]\), we write down the equation satisfied by \( S^\varepsilon(t) := Y^\varepsilon(t)(Y^\varepsilon(t))^* \) in \( \mathbb{R}^{n \times n} \). It turns out that the equation satisfied by \( Y^\varepsilon(t)(Y^\varepsilon(t))^* \) is no enough to solve our problem. Indeed

\[
\begin{aligned}
edS^\varepsilon(t) &= dY^\varepsilon(t)(Y^\varepsilon(t))^* = Y^\varepsilon(t)d(Y^\varepsilon(t))^* + dY^\varepsilon(t)(Y^\varepsilon(t))^* \\
&+ \sum_{j=1}^{m} \left( \sigma_x^2(t) Y^\varepsilon(t)(Y^\varepsilon(t))^* \delta\sigma^2(t) + \sigma_y^2(t) Y^\varepsilon(t-d)(Y^\varepsilon(t))^* \delta\sigma^2(t) \right) \\
&+ \left( \delta\sigma^2(t) I_{E^\varepsilon}(t)(Y^\varepsilon(t))^* \delta\sigma^2(t) + \delta\sigma^2(t) Y^\varepsilon(t-d)(Y^\varepsilon(t))^* \delta\sigma^2(t) \right) dt \\
&+ \left( \delta\sigma^2(t) I_{E^\varepsilon}(t)(\delta\sigma^2(t))^* \right) dt \\
&+ S^\varepsilon(\theta) = 0, \quad \theta \in [-d,0].
\end{aligned}
\]

We rewrite this equation putting in evidence only the terms which are quadratic in \( Y^\varepsilon(t) \) and \( Y^\varepsilon(t-d) \). It turns out that this equation cannot be written as an equation even with delay, for the process \( S^\varepsilon(t) \) since, even if \( T \leq 2d \), the term \( Y^\varepsilon(t)(Y^\varepsilon(t-d))^* \) appears, and this term cannot be written in terms of \( Y^\varepsilon(t)(Y^\varepsilon(t))^* \) and of \( Y^\varepsilon(t-d)(Y^\varepsilon(t-d))^* \).

So we introduce the process \( S^\varepsilon_1(t) := Y^\varepsilon(t)(Y^\varepsilon(t-d))^* \). Equation \( \text{(29)} \) can be rewritten as

\[
\begin{aligned}
dS^\varepsilon(t) &= \left( b_x^\varepsilon(t) S^\varepsilon_1(t) + S^\varepsilon(t)b_x^\varepsilon(t) + S^\varepsilon(t)b_y^\varepsilon(t) + b_y^\varepsilon(t)(S^\varepsilon_1(t))^* \right) dt \\
&+ \sum_{j=1}^{m} \left( \sigma_x^2(t) S^\varepsilon(t)(S^\varepsilon_1(t))^* + \sigma_y^2(t) S^\varepsilon(t) + \sigma_y(t)(S^\varepsilon_1(t))^* \right) dW^j(t) \\
&+ \left( \sigma_x(t) S^\varepsilon(t)(S^\varepsilon_1(t))^* + \sigma_y(t)(S^\varepsilon_1(t))^* \sigma_x(t) \right) dt \\
&S^\varepsilon(\theta) = 0, \quad \theta \in [-d,0].
\end{aligned}
\]

where \( R_1(t) \) and \( R_2(t) \) are \( o(\varepsilon^2) \) and can be neglected. We write the Ito differential of the process \( S^\varepsilon_1(t) \). At first, using equation \( \text{(16)} \) and taking into account that \( Y^\varepsilon \) is not null only after positive times, we get

\[
\begin{aligned}
dY^\varepsilon(t-d) &= b_x(t-d)Y^\varepsilon(t-d) dt + b_y(t-d)Y^\varepsilon(t-d) dt + \sigma_x(t-d)Y^\varepsilon(t-d) dW(t) + \sigma_y(t-d)Y^\varepsilon(t-d) dW(t) + \delta\sigma(t-d) I_{E^\varepsilon}(t-d) dW(t), \quad t \in (d,T], \\
Y^\varepsilon(s-d) &= 0, \quad s \in [-d,d].
\end{aligned}
\]
We notice that if $T \leq 2d$, the term $Y^\varepsilon(t-2d)$ is identically equal to 0 and equation (31) becomes
\[ dY^\varepsilon(t-d) = b_x(t-d)Y^\varepsilon(t-d) dt + \sigma_x(t-d)Y^\varepsilon(t-d) dW(t) + \delta\sigma(t-d)I_{E^\varepsilon}(t-d) dW(t), \quad t \in (d,T], \]
where
\[ Y^\varepsilon(s) = 0, \quad s \in [-d,d]. \]

So, if $T \leq 2d$, setting $S^\varepsilon_1(t) := Y^\varepsilon(t)(Y^\varepsilon(t-d))^*$ we get
\[
\begin{align*}
\left\{ \begin{array}{l}
dS^\varepsilon_1(t) = dY^\varepsilon(t)(Y^\varepsilon(t-d))^* = (b_x(t)Y^\varepsilon(t)(Y^\varepsilon(t-d))^* + b_y(t)Y^\varepsilon(t-d)(Y^\varepsilon(t-d))^*) dt \\
+ Y^\varepsilon(t)(Y^\varepsilon(t-d))^*b_x(t-d) dt + \sum_{j=1}^m \left( \sigma^j_x(t)Y^\varepsilon(t)(Y^\varepsilon(t-d))^* + \sigma^j_y(t)Y^\varepsilon(t-d)(Y^\varepsilon(t-d))^* \right) dW^j(t) \\
+ \sum_{j=1}^m \left( Y^\varepsilon(t)(Y^\varepsilon(t-d))^*(\sigma^j_x(t-d))^* + Y^\varepsilon(t)(\delta\sigma^j(t-d))^*I_{E^\varepsilon}(t-d) \right) dW^j(t) \\
+ \sum_{j=1}^m \left( \sigma^j_x(t)Y^\varepsilon(t)(Y^\varepsilon(t-d))^*(\sigma^j_x(t-d))^* + \sigma^j_y(t)Y^\varepsilon(t-d)(Y^\varepsilon(t-d))^*(\sigma^j_x(t-d))^* \right) dW^j(t) \\
+ \delta\sigma^j(t)I_{E^\varepsilon}(t)(\sigma^j(t-d))^*I_{E^\varepsilon}(t-d) dt, \quad t \in (d,T]
\end{array} \right.
\end{align*}
\]

We notice that the term $\delta\sigma^j(t)I_{E^\varepsilon(t-d)}(\sigma^j(t-d))^*I_{E^\varepsilon}(t-d)$ is identically equal to 0 if $\varepsilon$ is small enough, e.g. $\varepsilon < \frac{d}{2}$, in fact in such a case $I_{E^\varepsilon(t-d)}I_{E^\varepsilon}(t-d) \equiv 0$. Writing this equation in terms of $S^\varepsilon$ and of $S^\varepsilon_1$, we get
\[
\begin{align*}
\left\{ \begin{array}{l}
dS^\varepsilon_1(t) = (b_x(t)S^\varepsilon_1(t) + b_y(t)S^\varepsilon(t-d) + S^\varepsilon_1(t)b_x(t-d)) dt \\
+ \sum_{j=1}^m \left( \sigma^j_x(t)S^\varepsilon_1(t) + \sigma^j_y(t)S^\varepsilon(t-d) + S^\varepsilon_1(t)\sigma^j_x(t-d))^* \right) dW^j(t) \\
+ \sum_{j=1}^m \left( \sigma^j_x(t)S^\varepsilon_1(t)(\sigma^j_x(t-d))^* + \sigma^j_y(t)S^\varepsilon(t-d)(\sigma^j_x(t-d))^* \right) dt \\
+ R^1_{S^\varepsilon_1}(t) dt + R^2_{S^\varepsilon_1}(t) dW(t), \quad t \in (d,T]
\end{array} \right.
\end{align*}
\]

where $R^1_{S^\varepsilon_1}(t)$ and $R^2_{S^\varepsilon_1}(t)$ are as in (30).

If $T > 2d$, the process $Y^\varepsilon(t-2d)$ is not identically equal to 0 for $t > 2d$, and so it cannot be deleted in equation (31), and so in equation for $S^\varepsilon_1(t)$ there are some terms containing $Y^\varepsilon(t)(Y^\varepsilon(t-2d))^*$ which cannot be expressed as linear combination of $S^\varepsilon(t)$ and of $S^\varepsilon_1(t)$, so one should write an equation for $S^\varepsilon_2(t) := Y^\varepsilon(t)(Y^\varepsilon(t-2d))^*$.

**Remark 4.** Here and in the following Section we will consider the case of one pointwise delay and with time horizon $T$. Our procedure clearly apply to problems with a finite number of pointwise delays: for the sake of clarity we present the case of one pointwise delay.

Moreover since our procedure applies to the case of a finite number of delays, it applies to equations that can be written in the following way:
\[
\begin{align*}
\left\{ \begin{array}{l}
dX(t) = b(t, \int_{-d}^0 X(t + \theta)\mu^n(d\theta), u(t)) dt + \sigma(t, \int_{-d}^0 X(t + \theta)\mu^n(d\theta), u(t)) dW(t), \quad t \in [0,T], \\
X_0 = x(\theta), \quad \theta \in [-d,0],
\end{array} \right.
\end{align*}
\]

where $\mu^n$ is a discrete measure, and in particular $\mu^n$ is a combination of $n$ discrete measures. For example, if $n = 2$ and if
\[ \mu^n = \delta_0 + \delta_{-d} \]
we recover the case presented in equation (3). The next step is to consider a state equation of the following form

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{dX(t)}{dt} &= b(t, \int_{-d}^{0} X(t + \theta)\mu(d\theta), u(t)) dt + \sigma(t, \int_{-d}^{0} X(t + \theta)\mu(d\theta), u(t)) dW(t), & t \in [0, T], \\
X_0 &= x(\theta), & \theta \in [-d, 0],
\end{array}
\right.
\end{align*}
\tag{36}
\]

where \( \mu \) is a general finite regular measure. It is well known, see e.g. \([3]\), that, denoted by \( \mathcal{M} \) the set of regular probability measures on the bounded and closed interval \([-d, 0]\), then \( \text{Extr}\mathcal{M} = \{\delta_x : x \in [-d, 0]\} \). Moreover, by the Krein-Milman Theorem it turns out that \( \mathcal{M} = \sigma\text{Extr}\mathcal{M} = \sigma\{\delta_x : x \in [-d, 0]\} \), and so any probability measure in \( \mathcal{M} \) can be approximated by a linear convex combination of \( \delta_x, x \in [-d, 0] \). So let us denote by \( (\mu^n) \) the approximating sequence of discrete measures which are linear convex combinations of \( \delta_x, x \in [-d, 0] \). The process \( X^n \), solution of equation (36), can be approximated by the process \( X^n \), solution of

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{dX^n(t)}{dt} &= b(t, \int_{-d}^{0} X^n(t + \theta)\mu^n(d\theta), u(t)) dt + \sigma(t, \int_{-d}^{0} X^n(t + \theta)\mu^n(d\theta), u(t)) dW(t), & t \in [0, T], \\
X^n_0 &= x(\theta), & \theta \in [-d, 0],
\end{array}
\right.
\end{align*}
\tag{37}
\]

For this problem we will be able to formulate a stochastic maximum principle in terms of \( (p^n, q^n) \) solution of a first order adjoint ABSDE like (27), which will turn out to approximate a first order adjoint BSDE, see \([13]\) for details on the approximation of BSDEs. Regarding the second order approximating ABSDE, instead we are not able to write down the equation for the limit of the approximating sequence. We also notice that, when the second order adjoint equation is not available, because either we do not know the structure of the equation or we are not able to solve the equation, one could try to formulate the stochastic maximum principle in a more abstract way, see e.g. \([13]\) where the process \( P \), which here is the solution to the second order adjoint BSDE, is introduced as an operator connected to the cost functional.

4. Second order adjoint

We introduce now the second adjoint that is a system of equations for matrix valued processes \((P, Q)\) and \((P_1, Q_1)\):

\[
\begin{align*}
-dP(t) &= \{P(t)b_x(t) + b^*_x(t)P(t) + \mathbb{E}^{\mathcal{F}}b^*_x(t + d)P_1(t + d)\} dt + H_{x,x}(t) dt + \mathbb{E}^{\mathcal{F}}(H_{x,x}(t + d)) dt \\
&+ \sum_{j=1}^{m} \left[ \mathbb{E}^{\mathcal{F}}(\sigma_{y}^j(t + d)) P(t + d)\sigma_{y}^j(t) + \sigma_{x}^j(t) P(t)\sigma_{x}^j(t) + \mathbb{E}^{\mathcal{F}}(\sigma_{y}^j(t + d)) P_1(t + d)\sigma_{x}^j(t) \right] dt \\
&+ \sum_{j=1}^{n} \left[ \sigma_{x}^j(t)Q_1^j(t) + Q^j(t)\sigma_{x}^j(t) + \mathbb{E}^{\mathcal{F}}\sigma_{x}^j(t + d)Q_1(t + d) \right] dt + \sum_{j=1}^{d} Q_1^j(t) dW^j(t) \\
P(T) &= -h_{xx}(X(T)), & P(t) = 0 & t \in [T, T + d], & Q(t) = 0 & t \in [T, T + d], \\
-dP_1(t) &= \{b^*_x(t)P_1(t) + P_1(t)b_x(t - d) + P^*(t)b_y(t) + P(t)b_y(t)\} dt + 2H_{x,y}(t) dt \\
&+ \sum_{j=1}^{m} \left[ \sigma_{x}^j(t)P_1(t)\sigma_{x}^j(t - d) + \sigma_{y}^j(t)P(t)\sigma_{y}^j(t) + \sigma_{x}^j(t)P(t)\sigma_{x}^j(t) \right] dt \\
&+ \sum_{j=1}^{n} \left[ Q_1^j(t)\sigma_{x}^j(t - d) + \sigma_{x}^j(t)Q_1^j(t) + Q^j(t)\sigma_{y}^j(t) + Q_1(t)\sigma_{x}^j(t) Q_1(t) \right] dt + \sum_{j=1}^{d} Q_1^j(t) dW^j(t) \\
P_1(\theta) &= 0, & \theta \in [T, T + d], & Q_1(t) = 0 & t \in [T, T + d]
\end{align*}
\tag{38}
\]

where

\[
H(t, x, y, u, p, q) = pb(t, x, y, u) + \text{tr}[q^*\sigma(t, x, y, u)] - \ell(t, x, y, u)
\tag{39}
\]

is the hamiltonian function and for brevity we denote

\[
H(t) = H(t, X(t), X(t - d), u(t), p(t), q(t)).
\]

Then, by \([17, \text{Theorem 4.2}]\) we have, as for theorem \([3]\)
The anticipated system (29) admits a unique solution that is two coupled of processes \((P, Q), (P_1, Q_1)\) such that \(P, P_1 \in L^2_T(\Omega; C([0, T + d]; \mathbb{R}^{n \times n}))\) and \(Q, Q_1 \in L^2_T(\Omega \times [0, T + d]; \mathbb{R}^{n \times n})\).  

**Proof.** Notice that the anticipated system (29) is a linear anticipated BSDE for the processes \((P, P_1) \in L^2_T(\Omega; C([0, T + d]; \mathbb{R}^{2n \times n}))\) and \((Q, Q_1) \in L^2_T(\Omega \times [0, T + d]; \mathbb{R}^{2n \times n})\). So it admits a unique solution by [17].

**Remark 5.** In the case of delay in the control it turns out that equations (30) and (34) remains unchanged in their structure. The presence of the delay in the controls affects the term of order less than or equal to one, so also the second order adjoint equation does not change.

Thus we have following reduction for the cost expansion:

**Proposition 2.** Under hypotheses (A.1) – (A.2) the following expansion for the cost holds true:

\[
J(u) - J(u^\varepsilon) = \mathbb{E} \int_0^T \left( -\delta(t) + (p(t), \delta b(t)) + q(t)^* \delta \sigma(t) + \frac{1}{2} \sum_{j=1}^m \delta \sigma^j(t)^* P(t) \delta \sigma^j(t) I_{E_\varepsilon}(t) \right) dt + o(\varepsilon).  
\]

(40)

**Proof.** From Proposition 1 taking into account the definitions of \(H\), and the processes \(S^\varepsilon\) and \(S^\varepsilon_1\) we get:

\[
J(u) - J(u^\varepsilon) = \mathbb{E} \int_0^T \left( -\delta(t) + (p(t), \delta b(t)) + q(t)^* \delta \sigma(t) \right) I_{E_\varepsilon}(t) dt 
- \frac{1}{2} \mathbb{E} \left[ h_{xx}(X(T)) S^\varepsilon(T) \right] + \frac{1}{2} \mathbb{E} \int_0^T \left[ H_{xx}(t) S^\varepsilon(t) + H_{yy}(t) S^\varepsilon(t - d) + 2 H_{xy}(t) S^\varepsilon_1(t) \right] dt + o(\varepsilon)  
\]

Hence it is just a matter to calculate the Itô formula to \(\text{tr}[S^\varepsilon(t)P(t)] + \text{tr}[S^\varepsilon_1(t)P_1(t)]\), as before we will drop the symbol for the trace to simplify the notation.

We obtain the following identity, taking into account the time change as in Proposition 1

\[
\mathbb{E} P(T) S^\varepsilon(T) + \mathbb{E} P_1(T) S^\varepsilon_1(T) - \mathbb{E} P(0) S^\varepsilon(0) + \mathbb{E} P_1(0) S^\varepsilon_1(0) = 
\int_0^T \sum_{j=1}^m \left[ \delta \sigma^j(t)^* P(t) \delta \sigma^j(t) I_{E_\varepsilon}(t) \right] dt - \int_0^T \left[ H_{xx}(t) S^\varepsilon(t) + H_{yy}(t) S^\varepsilon(t - d) + 2 H_{xy}(t) S^\varepsilon_1(t) \right] dt  
\]

From which, plugging the initial and final conditions we have

\[
\int_0^T \sum_{j=1}^m \text{Tr}[\delta \sigma^j(t)]^* P(t) \delta \sigma^j(t) I_{E_\varepsilon}(t) dt = 
- \mathbb{E} h_{xx}(X(T)) S^\varepsilon(T) - \int_0^T \left[ H_{xx}(t) S^\varepsilon(t) + H_{yy}(t) S^\varepsilon(t - d) + 2 H_{xy}(t) S^\varepsilon_1(t) \right] dt  
\]

Thus we obtain the desired expansion for the cost.

\[
\ □
\]

5. **Maximum principle**

Now we are able to prove a version of the Stochastic Maximum Principle, in its necessary form, for the control problem with state equation and cost functional given by (3) and (4), respectively. We recall that the expansion of the cost is given in (24) and it is rewritten in (40) in terms of the pair of processes \((p, q)\) solution of the first adjoint equation (27) and of \((P, Q), (P_1, Q_1)\) solution of the system of second order adjoint equations (38).

Let us state the stochastic maximum principle which is written in terms of the Hamiltonian function \(H\) defined in (39).

**Theorem 5.** Let Assumptions (A.1) and (A.2) be satisfied and suppose that \((X, u)\) is an optimal pair for the control problem, and let us consider \(u^\varepsilon\) defined in (17). There exist pair of processes \((p, q) \in L^2_T(\Omega; C([0, T + d]; \mathbb{R}^n)) \times q \in L^2_T(\Omega \times [0, T + d]; \mathbb{R}^{n \times m})\) which are the solution to the
first order adjoint equation \( P(t_t, X(t), X(t-d), u(t), p(t), q(t)) - H(t, X(t), X(t-d), v, p(t), q(t)) \)

\[
= \inf_{v \in U} \{ H(t, X(t), X(t-d), v, p(t), q(t)) + \frac{1}{2} \text{Tr}[(\sigma^j(t, X(t), X(t-d), v, p(t), q(t)))^*P(t)\sigma^j(t, X(t), X(t-d), v, p(t), q(t))] \}
\]

Equivalently, a.e. \( t \in [0, T], \mathbb{P} - \text{a.s.} \),

\[
H(t, X(t), X(t-d), u(t), p(t), q(t)) + \tfrac{1}{2} \text{Tr}[(\sigma^j(t, X(t), X(t-d), u(t), p(t), q(t)))^*P(t)\sigma^j(t, X(t), X(t-d), u(t), p(t), q(t))] = \inf_{v \in U} \{ H(t, X(t), X(t-d), v, p(t), q(t)) + \tfrac{1}{2} \text{Tr}[(\sigma^j(t, X(t), X(t-d), v, p(t), q(t)))^*P(t)\sigma^j(t, X(t), X(t-d), v, p(t), q(t))] \}.
\]

**Proof.** By \( (40) \) we get that

\[
0 \geq J(u) - J(u^e) = \mathbb{E} \int_0^T \left( -\delta t + (p(t), \delta b(t)) + q(t)^\ast \delta \sigma(t) + \sum_{j=1}^m \text{Tr}[\delta \sigma^j(t)] P(t) \delta \sigma^j(t) \right) I_{E_\varepsilon(t)} dt + o(\varepsilon)
\]

Notice that the right hand side is not equal to 0 only for \( t \in E_\varepsilon \). So if we divide both by \( \varepsilon \) and we let \( \varepsilon \to 0 \) we get

\[
0 \geq H(t, X(t), X(t-d), u(t), p(t), q(t)) - H(t, X(t), X(t-d), v, p(t), q(t)) + \frac{1}{2} \text{Tr}[(\sigma^j(t))^\ast P(t)\sigma^j(t)] \forall v \in U, \text{a.e. } t \in [0, T], \mathbb{P} - \text{a.s., and this concludes the proof.} \]

**References**

[1] E. Augeraud-Veron, M. Bambi, F. Gozzi, *Solving internal habit formation models through dynamic programming in infinite dimension*. J. Optim. Theory Appl. 173 (2017), no. 2, 584–611.

[2] E. Bandini, A. Cosso, M. Fuhrman, H. Pham, *Randomization method and backward SDEs for optimal control of partially observed path-dependent stochastic systems*, [arXiv:1511.09274](https://arxiv.org/abs/1511.09274) to appear in Annals of Applied Probability.

[3] B. Bruder, H. Pham, *Impulse control problem on finite horizon with execution delay*. Stochastic Process. Appl. 119 (2009), no. 5, 1436–1469.

[4] L. Chen, Z. Wu, *Maximum principle for the stochastic optimal control problem with delay and application*. Automatica J. IFAC 46 (2010), no. 6, 1074–1080.

[5] L. Chen, Z. Wu, *A type of general forward-backward stochastic differential equations and applications*. Chin. Ann. Math. Ser. B 32 (2011), no. 2, 279–292.

[6] L. Chen, Z. Wu, Z. Yu, *Delayed stochastic linear-quadratic control problem and related applications*. J. Appl. Math. 2012, 22 pp.
[7] G. Constantinides, *Habit formation: a resolution of the equity premium puzzle*. J. Polit. Econ. 98 (3) (1990), 519–543.

[8] Y. Eidelman, V. Milman, A. Tsolomitis, *Functional analysis*, Graduate Studies in Mathematics, 66, American Mathematical Society, Providence, RI, 2004.

[9] F. Gozzi, F. Masiero *Stochastic Optimal Control with Delay in the Control, I: solving the HJB equation through partial smoothing*. SIAM J. Control Optim. 55 (2017), no. 5, 2981–3012.

[10] F. Gozzi, F. Masiero *Stochastic Optimal Control with Delay in the Control, II: Verification Theorem and Optimal Feedback Controls*. SIAM J. Control Optim. 55 (2017), no. 5, 3013–3038.

[11] F. Gozzi, F. Masiero *Generalized Partial Derivatives, Partial Smoothing and Stochastic Control Problems with Unbounded Control Operators*, working paper.

[12] G. Guatteri, F. Masiero, C. Orrieri, *Stochastic maximum principle for SPDEs with delay*, Stochastic Process. Appl. 127 (2017), no. 7, 2396-2427.

[13] M. Fuhrman, Y. Hu, G. Tessitore, *Stochastic maximum principle for optimal control of SPDEs*, Appl. Math. Optim. 68 (2013), no. 2, 181–217.

[14] Y. Hu, S. Peng, *Maximum principle for optimal control of stochastic system of functional type*, Stoch. Anal. Appl. 14 (3) (1996) 283-301

[15] S.-E. A. Mohammed, *Stochastic differential systems with memory: theory, examples and applications*, Stochastic analysis and related topics, VI (Geilo, 1996), Progr. Probab., 42, 1-77, Birkhäuser Boston, MA, 1998.

[16] S. Peng, *A general stochastic maximum principle for optimal control problems*. SIAM J. Control Optim. 28 (1990), no. 4, 966-979.

[17] S. Peng, Z. Yang, *Anticipated backward stochastic differential equations*, Ann. Probab. 37, (2009), no. 3, pp. 877–902.

[18] Z. Yang, R.J. Elliott, *Some properties of generalized anticipated backward stochastic differential equations*. Electron. Commun. Probab. 18 (2013), no. 63, 10 pp.

[19] J. Yong, X.Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York: Springer-Verlag, (1999).

(G. Guatteri) Dipartimento di Matematica, Politecnico di Milano. via Bonardi 9, 20133 Milano, Italia

E-mail address: giuseppina.guatteri@polimi.it

(F. Masiero) Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca. via Cozzi 55, 20125 Milano, Italia

E-mail address: federica.masiero@unimib.it