Iterative Method for Positive Definite Solution of a Class of Nonlinear Matrix Equation

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Abstract. In this paper, we study the matrix equation $X + A^* (R + B^* X B) A = Q$ ($0 < t \leq 1$), where $A$, $B$, $R$, $Q$ are matrices of appropriate size and $R$, $Q$ are both positive definite matrices. Based on the fixed point theorem, we suggest four iterative algorithms for solving this equation and prove that the suggested iterative algorithms are convergent with proper conditions. What’s more, the conditions for the existence of the positive definite solution are given. The convergence analysis of the suggested algorithms is established. Some numerical examples are presented to illustrate the convergence behaviour of the various algorithms.

1. Introduction

Riccati equation represents a significant subclass of control systems that received much attention during the past several decades, which have wide applications in many areas and are needed to solve in control theory, stability analysis and system theory[1-4]. In this article, we consider the matrix equation derived from the Riccati equation

$$X + A^* (R + B^* X B) A = Q,$$

where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{m \times n}$, $R \in \mathbb{C}^{m \times m}$, $Q \in \mathbb{C}^{n \times n}$ and $Q$, $R$ are both Hermitian positive definite matrices, $X \in \mathbb{C}^{n \times n}$ is an unknown matrix. Meanwhile, Hermitian positive definite solution plays a key role in the control systems. Indeed, the following discrete-time algebraic Riccati equation

$$X + (F^* X B + A) (R + B^* X B)^{-1} (F^* X B + A) = Q + F^* X F,$$

can be converted to (1) by setting $F=0$, where $Q$ is a Hermitian positive definite matrix.

In addition, (1) can be simplified to

$$X + A^* X^{-1} A = Q,$$

replacing $B=I$, $R=0$, $t=1$. Therefore, (1) can also be viewed as a special case of the discrete-time algebraic Riccati equation and a generalization of (2). As we all know, (2) has many applications in control theory, ladder networks, dynamic programming, engineering calculation, stochastic filtering, applied statistics and so on [5-8]. What’s more, this class of matrix equations have been studied by several authors. For instance, Zhang Kaiyuan[9] presented a double iterative algorithm for solving the positive definite solution of the matrix equation

$$X + X^{-1} A = Q,$$

Meng and Kim[10] showed that $X^* = A + M (B + X^{-1}) M^*$ has a unique positive definite solution when $M$ is an arbitrary matrix, $A$ and $B$ are Hermitian positive semidefinite matrices. Liu Panpan[11] obtained the equivalent form of the matrix equation

$$X = Q - A^* (I \otimes X - C)^{-1} A$$

and proposed Newton’s iteration.
method together with deriving the existence conditions of the solutions. In Reference [12], Li lei investigated the nonlinear matrix equations \( X + A^rX^{-q}A = Q \) (0 < q ≤ 1) and \( X - A^rX^{-q}A = Q \) (q > 1). Zhang Huamin[13] proposed a new quasi gradient-based inversion free iterative algorithm for solving the nonlinear matrix equation \( X + A^rX^{-q}A = I \). Ran and Reurings[14] gave some results for the general matrix equations \( X + A^rF(X)A = Q \), where \( F \) is a map from the set of all Hermitian positive semidefinite matrices into . However, to the best of our knowledge, there were few works involved in a solution of the nonlinear (1) with \( t \in (0, 1] \).

The following notations are used throughout this article, the notation \( 0(0) \geq M \geq 0 \) means that \( M \) is positive definite (positive semidefinite), and \( 0(0) > N \geq 0 \) is used as a different notation for \( 0(0) > N - M \geq 0 \). we write \( A^* \), \( A^{-1} \) to denote the conjugate transpose, the inverse of the matrix \( A \).

2. Conditions for the existence of positive definite solutions of (1)

In this section, we first give some preliminary results which play a fundamental role in this paper. Next, we will discuss some properties of (1) and obtain the conditions for the existence of a positive definite solution to (1).

Lemma 2.1 (see [15]). If \( 0 > B > 0 \) (or \( 0 \geq B > 0 \)), then \( 0 > A > B > 0 \) (or \( 0 \geq A > B > 0 \)) for all \( \alpha \in (0,1] \), and \( 0 > A > B > 0 \) (or \( 0 > A > B > 0 \)) for all \( \alpha \in [-1,0) \).

Lemma 2.2 (see [16]). If \( C \) and \( P \) are Hermitian matrices of the same order with \( 0 > P \), then \( 1 > (C + P)^{-1} \geq 2C \).

Theorem 2.1 If \( A, B \) are invertible and (1) has a positive definite solution \( X \), then we have the following inequality.

\[
B^r[(AQ^{-1}A)^{-1/2} - R]B^{-1} < X < Q - A^r(R + B^rQB)^{-1}A.
\]

Proof. Since \( X \) is a positive definite solution of (1), then \( X \leq Q \), \( A^r(R + B^rXB)^{-1}A < Q \), thus \((R + B^rXB)^{-1} < A^{-1}Q^{-1}A^{-1} \) and \((R + B^rXB)^{1/2} > A^{-1}A^{1/2} \), therefore \( X > B^r[(A^{-1}A)^{-1/2} - R]B^{-1} \).

On the other hand, by \( X = Q - A^r(R + B^rXB)^{-1}A < Q \), we obtain \( X < Q - A^r(R + B^rQB)^{-1}A \).

Theorem 2.2 Suppose \( Q - A^rR^{-1}A > 0 \), then (1) has a positive definite solution.

Proof. Let \( F(X) = Q - A^r(R + B^rXB)^{-1}A \) and \( \Omega = \{ X : Q - A^rR^{-1}A \leq X \leq Q \} \). Obviously, \( \Omega \) is convex, closed and bounded set and set \( f(X) \) is continuous on \( \Omega \). For any \( X \in \Omega \), we have \( X > 0 \) and \( B^rXB > 0 \), then

\[
F(X) \geq Q - A^r(R + B^rXB)^{-1}A \geq Q - A^rR^{-1}A,
\]

\[
F(X) = Q - A^r(R + B^rXB)^{-1}A < Q.
\]

Combing (4) and (5), we have \( F(X) \in \Omega \), which implies that \( F(X) \subseteq \Omega \). By Brouwer’s fixed point theorem, \( F(X) \) has a fixed point above \( \Omega \) which is a solution of (1).

Theorem 2.3 Suppose \( A \) is a full rank square matrix and (1) has a positive definite solution \( X_+ \), then

(a) \( X_+ < Q \),

(b) \( A^r(R + B^rQB)^{-1}A < Q \).

Proof. (a) Since \( X_+ \) is a positive definite solution of (1), we have \( X_+ > 0 \) and \( R + B^rXB > 0 \), thus \( A^r(R + B^rXB)^{-1}A > 0 \), then

\[
X_+ = Q - A^r(R + B^rXB)^{-1}A < Q.
\]

(b) Since \( X_+ < Q \), we have \( (R + B^rXB)^{-1} > (R + B^rQB)^{-1} \), then
$$A'(R + B'QB)^{-1}A < A'(R + B'X_iB)^{-1}A = Q - X_i < Q.$$ 

3. Iterative algorithms

In this section, we propose four iterative algorithms to solve (1) and derive a sufficient condition for the convergence of these iterative methods. Firstly, we propose the following fixed point iteration algorithm to solve (1).

**Algorithm 3.1.1**

$$
\begin{align*}
X_0 &= Q, \\
X_{k+1} &= Q - A'(R + B'X_kB)^{-1}A.
\end{align*}
$$

**Theorem 3.1** If (1) has a positive definite solution and sequence \( \{X_i\} \) is determined by Algorithm 3.1.1, then \( \{X_i\} \) is monotone decreasing and convergent to the maximal solution \( X_0 \) and \( X_i \to X_0 \) \((k \to \infty)\).

**Proof.** Initially, we will prove that \( X_0 \geq X_1 \geq \cdots \geq X_s \geq X_0 \). Since \( X_s \) is the solution of (1), i.e. \( X_s = Q - A'(R + B'X_sB)^{-1}A \), then \( X_0 = Q \geq X_1 \), also \( X_1 = Q - A'(R + B'X_0B)^{-1}A < Q = X_0 \) and \( X_1 = Q - A'(R + B'X_0B)^{-1}A < Q - A'(R + B'X_kB)^{-1}A = X_0 \), i.e. \( X_0 \geq X_1 \geq X_s \). That means that the inequalities are true for \( k=0,1 \). Assume that the above inequalities are true for \( k=s \), we will prove inequalities at \( k=s+1 \). Then \( X_{s+1} = Q - A'(R + B'X_sB)^{-1}A < Q - A'(R + B'X_kB)^{-1}A = X_k \) and \( X_{s+1} = Q - A'(R + B'X_0B)^{-1}A > Q - A'(R + B'X_kB)^{-1}A = X_0 \). We have \( X_s \geq X_{s+1} \geq X_s \). Therefore, \( Q = X_0 \geq X_1 \geq \cdots \geq X_k \geq X_s \) are true for all \( k \). It follows that sequence \( \{X_i\} \) is monotone decreasing and bounded from below by the zero matrix. So, (1) has a positive definite solution.

Suppose (1) has a positive definite solution \( X_i \), according to Theorem 2.3, we have \( X_i \in [0,Q] \). Therefore, we can set up a new iterative algorithm as follows.

**Algorithm 3.1.2**

$$
\begin{align*}
X_0 &= \omega Q, \quad \omega \in (0,1), \\
X_{k+1} &= Q - A'(R + B'X_kB)^{-1}A.
\end{align*}
$$

**Theorem 3.2** Assume that (1) has a positive definite solution and \( (1 - \omega)Q > A'(R + B'Q\omega QB)^{-1}A \), then Algorithm 3.1.2 defines a monotonically increasing matrix sequence \( \{X_i\} \) converging the minimal positive solution \( X_0 \).

**Proof.** Initially, we prove that \( X_0 \leq X_1 \leq \cdots \leq X_k \leq X_0 \leq \cdots \leq Q \). Since \( X_0 \) is a positive definite solution of (1), i.e. \( X_0 = Q - A'(R + B'X_0B)^{-1}A \). By Algorithm 3.1.2 and known conditions, we get \( X_k \leq Q, \quad k = 0, 1, 2, \ldots \). And \( X_1 = Q - A'(R + B'X_0B)^{-1}A > \omega Q \), thus \( X_0 \leq X_1 \leq Q \). That means that the inequalities are true for \( n = 0, 1 \). So, we suppose that \( X_k \leq X_k \leq X_0 \), then \( X_{k+1} = Q - A'(R + B'X_kB)^{-1}A > Q - A'(R + B'X_kB)^{-1}A = X_k \), i.e. \( X_k \leq X_{k+1} \). Therefore, \( X_0 \leq X_1 \leq \cdots \leq X_k \leq X_0 \leq \cdots \leq Q \) are true for all \( k \), we have now proved that iterative sequence \( \{X_i\} \) is monotonically increasing with an upper bound, and \( X_i \) \((k \to \infty) \) exist. As \( X_0 = Q - A'(R + B'X_0B)^{-1}A \) and each \( X_k \) \( < X_0 \), then \( X = X_0 \). Then theorem is proved.

Note that when \( \omega > 1 \), sequence \( \{X_i\} \) is monotonically decreasing with a lower bound. The proof is similar to that of Theorem 3.2. Here, according to the interval of the solution, we just consider \( 0 < \omega < 1 \).

In [16], Zhan proposed an algorithm to find the maximal positive definite solution of (2) with \( Q = I \). This algorithm avoids matrix inversion for every, and each iteration requires only matrix
multiplication, which greatly simplifies the operation process. In recent years, this algorithm has been widely used to solve nonlinear matrix equations, and highly accurate results have been obtained. Next, based on Zhan’s idea, we propose an inversion free iterative algorithm that avoids matrix inversion for each iteration.

Algorithm 3.1.3

\[
\begin{cases}
X_0 = Q, 
Y_0 = (R + B'QB)^{-1}, \\
X_{k+1} = Q - A'Y_k'A, \quad k = 0, 1, 2, \ldots, \\
Y_{k+1} = Y_k[2I_k - (R + B'X_kB)Y_k].
\end{cases}
\]

Theorem 3.3 If (1) has a positive definite solution and two sequences \( \{X_k\} \) and \( \{Y_k\} \) are determined by Algorithm 3.1.3, then \( \{X_k\} \) is monotone decreasing and converges to the maximal solution \( X_\ast \), and \( \{Y_k\} \) is monotone increasing converges to \( (R + B'X_\ast B)^{-1} \).

Proof. First, we will prove that
\[
Q = X_0 \geq X_1 \geq \cdots \geq X_k \geq X_\ast. \quad (6)
\]

Since \( X_\ast \) is a positive definite solution of (1), i.e. \( X_\ast = Q - A'(R + B'X\ast B)'A \), then \( X_\ast \leq Q \) and
\[
(R + B'X_\ast B)^{-1} \geq (R + B'QB)^{-1}. \quad (7)
\]

By Algorithm 3.1.3, we have
\[
\begin{align*}
X_1 &= Q - A'(R + B'QB)'A > Q - A'(R + B'X_\ast B)'A = X_\ast, \\
Y_1 &= 2Y_0 - Y_0(R + B'QBY_0) = 2Y_0 - Y_0Y_0^{-1}Y_0 = Y_0, \\
Y_1 &= 2Y_0 - Y_0(R + B'QB)Y_0 < (R + B'QB)^{-1} < (R + B'X_\ast B)^{-1}.
\end{align*}
\]

Thus, we have \( Q = X_0 \geq X_1 \geq X_\ast \) and \( Y_0 \leq Y_1 \leq (R + B'X_\ast B)^{-1} \). Assume that \( Q = X_{k-1} \geq X_k \geq X_\ast \) and \( Y_{k-1} \leq Y_k \leq (R + B'X_\ast B)^{-1} \). Concerning the two sequences \( \{X_k\} \) and \( \{Y_k\} \), we have
\[
\begin{align*}
X_{k+1} &= Q - A'Y_k'A < Q - A'Y_{k-1}'A = X_k, \\
Y_{k+1} - Y_k &= Y_k - Y_k(R + B'X_kB)Y_k = Y_k[Y_k^{-1} - (R + B'X_kB)]Y_k > 0.
\end{align*}
\]

Inspired by the above algorithm, we propose a new inversion free variant of the basic fixed-point iteration method for solving (1). Set \( Y = (R + B'XB)^{-1} \), therefore, \( Y = (R + B'XB)^{-1} \), and
\[
B^{-1}(Y^{-1} - R)B^{-1} + A'Y'A = Q. \quad (8)
\]

premultiplying by \( B'^{-1} \), and postmultiplying by \( B^{-1} \), we have
\[
(Y^{-1} - R) + B'AY'A = B'^{-1}QB, \quad (9)
\]

premultiplying and postmultiplying by \( Y \), we get
\[
Y = 2Y - Y(R + B'^{-1}QB - B'^{-1}AY'A)Y. \quad (10)
\]

(8) and (10) have the same solution. Therefore, we can propose the following iteration.

Algorithm 3.1.4

\[
\begin{cases}
Y_0 = (R + B'QB)^{-1}, \\
Y_{k+1} = 2Y_k - Y_k(R + B'^{-1}QB - B'^{-1}AY_k'AB)Y_k, \\
X_{k+1} = B^{-1}(Y_{k+1}^{-1} - R)B^{-1}.
\end{cases}
\]

In the sequel, it is denoted that Algorithm 3.1.4 can obtain the maximal solution of (1). Moreover, it is proved that the generated sequence by Algorithm 3.1.4 is linearly convergent.

Theorem 3.4 If \( B \) is nonsingular and (8) has a positive definite solution \( Y_\ast \), the \( \{Y_k\} \) is given by Algorithm 3.1.4, then \( \{Y_k\} \) is monotone increasing convergence to the minimal \( Y_\ast \) and \( Y_k \to Y_\ast (k \to \infty) \).
**Proof.** By induction, we show that \(Y_0 \leq Y_1 \leq \cdots \leq Y_k \leq Y_*\). Since \(Y_*\) is a positive definite solution of (8), that is \(Y_*^{-1} = R + B^*QB - B^*A^*Y_*^tAB\). Also, \(Y_0 = (R + B^*QB)^{-1}\), then

\[
Y_0^{-1} = (R + B^*QB)
\]

so

\[
Y_i = 2Y_0 - Y_0(Y_0^{-1} - B^*A^*Y_0^tAB)Y_0 = Y_0 + Y_0B^*A^*Y_0^tABY_0 > Y_0.
\]

By Lemma 2.2, we have

\[
Y_i = 2Y_0 - Y_0(R + B^*QB - B^*A^*Y_0^tAB)Y_0 \leq (R + B^*QB - B^*A^*Y_0^tAB)^{-1} = Y_*.\]

Thus \(0 < Y_0 \leq Y_i \leq Y_*\). Assume that \(0 < Y_{k-1} \leq Y_k \), therefore,

\[
Y_{k+1} = 2Y_k - Y_k(Y_k^{-1} - B^*A^*Y_k^tAB)Y_k = (R + B^*QB - B^*A^*Y_k^tAB)^{-1} = Y_*.\]

For convenience, set \(Z_k = R + B^*QB - B^*A^*Y_k^tAB\), then

\[
Y_{k+1} - Y_k = 2Y_k - Y_kZ_kY_k - Y_k = Y_kY_k = Y_k(Y_k^{-1} - Z_k)Y_k,
\]

and \(0 < Z_k = R + B^*QB - B^*A^*Y_k^tAB < R + B^*QB - B^*A^*Y_0^tAB = Z_0\). From (9), we have \(Z_k = Y_k^{-1} > 0\), thus \(Z_{k+1} < Z_k^{-1}\), by Algorithm 3.1.4 and Lemma 2.2, we have \(Y_k = 2Y_{k-1} - Y_{k-1}Z_{k-1}Y_{k-1} \leq Z_k^{-1} \leq Z_k\), thus \(Y_k^{-1} \geq Z_k\). By (11) and \(Y_k > 0\), \(Y_k^{-1} \geq Z_k\), we have \(Y_{k+1} = Y_k^{-1} = Y_k(Y_k^{-1} - Z_k)Y_k > 0\). Hence, \(0 < Y_k \leq Y_{k+1} \leq Y_*\), this completes the induction. From the above process, we know easily that \(\{Y_k\}\) is a monotone decreasing sequence and bounded from above \(Y_*\). Thus \(\lim_{k \to \infty} Y_k = Y_*\) exist, taking limits in Algorithm 3.1.4, thus \(X_* = B^*(Y_*^{-1} - R)B^{t}\) is a positive definite solution to (1).

4. **Numerical examples**

In this section, some numerical examples are used to illustrate the theoretical results and the effectiveness of the iterative algorithm. The software environment of algorithm implementation in MATLAB R2020a, and the PC is Intel(R) Core (TM) i7-8750h CPU @ 2.20ghz.

**Note:**
- **Iter:** The number of required iterations,
- **Error:** Norm of the residual
- **CPU:** The CPU time spend
- **MM:** The number of matrix-matrix products.

We use the practical stopping criterion

\[
R_k = \left\| X_k + A^*(R + B^*X_kB)^{-1} - I \right\| \leq 1.0 \times 10^{-10}, k \geq 1.
\]

**Example 4.1.** Consider (1) with \(n=s=5, t=0.5\),

\[
A = \begin{bmatrix} 2 & i & \cdots & \cdots & -i \\ -i & 2 & \cdots & \cdots & i \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ -i & \cdots & \cdots & 2 \\ 4 & 1.3i & \cdots & \cdots & -1.3i \\ -1.3i & 4 & \cdots & \cdots & 1.3i \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ -1.3i & \cdots & \cdots & 1.3i & 4 \\ \end{bmatrix}, \quad B = \begin{bmatrix} 12 & -2i \\ 2i & 12 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2i & 12 \\ \end{bmatrix},
\]

\[
R = \begin{bmatrix} 10 & 1.5i \\ -1.5i & 10 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1.5i & 10 \\ \end{bmatrix}, \quad Q = \begin{bmatrix} 10 & 1.5i \\ -1.5i & 10 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1.5i & 10 \\ \end{bmatrix}.
\]

The four iterative algorithms obtain the same maximal positive definite solution...
When \( t = 0.3 \), \( n = 100, 500, 1000 \). Using four iterative algorithms in (1), the numerical results are presented in Table 1.

### Table 1 Results of Example 4.1

| Algorithm | \( n \) | Iter | Error     | CPU | MM |
|-----------|-------|------|-----------|-----|----|
| 3.1.1     | 100   | 8    | 2.5345e-11| 0.13| 32 |
|           | 500   | 8    | 2.5509e-11| 3.93| 32 |
|           | 1000  | 8    | 2.5684e-11| 26.00| 32 |
| \( \omega = 0.8 \) | 100 | 7    | 7.8780e-11| 0.11| 28 |
| 3.1.2     | 1000  | 7    | 7.9170e-11| 22.83| 28 |
| 3.1.3     | 500   | 16   | 8.8164e-11| 0.22| 96 |
|           | 1000  | 16   | 8.8443e-11| 52.04| 96 |
| 3.1.4     | 100   | 5    | 7.5193e-11| 0.08| 30 |
|           | 500   | 5    | 7.5193e-11| 2.38| 30 |
|           | 1000  | 5    | 7.5193e-11| 15.67| 30 |

### Example 4.2
Consider (1) with \( t = 0.7 \) and the matrix \( A, B, R, Q \) have the same value as in example 4.1. Similarly, the results are presented in Table 2 as follows.

### Table 2 Results of Example 4.2

| Algorithm | \( n \) | Iter | Error     | CPU | MM |
|-----------|-------|------|-----------|-----|----|
| 3.1.1     | 100   | 4    | 4.1844e-12| 0.07| 16 |
|           | 500   | 4    | 4.1934e-12| 1.82| 16 |
|           | 1000  | 4    | 4.2108e-12| 12.54| 16 |
| \( \omega = 0.8 \) | 100 | 4   | 8.5573e-11| 0.06| 16 |
| 3.1.2     | 1000  | 4    | 8.5593e-11| 1.92| 16 |
| 3.1.3     | 100   | 8    | 1.0057e-11| 0.12| 48 |
|           | 500   | 8    | 1.5570e-11| 3.87| 48 |
|           | 1000  | 8    | 1.5570e-11| 26.18| 48 |
| 3.1.4     | 100   | 2    | 4.1844e-12| 0.04| 12 |
|           | 500   | 2    | 4.1934e-12| 0.92| 12 |
|           | 1000  | 2    | 4.2108e-12| 6.27| 12 |

In Table 1 and Table 2, by comparing four iterative algorithms are proposed in this paper and considering the numerical results, we concluded that Algorithm 3.1.2 and Algorithm 3.1.4 are very powerful and efficient in finding the numerical solutions for (1), and it provides highly accurate results in a lower number of iterations, computational cost, the number of matrix multiplications and CPU time spent as compared to other methods. But when \( B \) is singular, the Algorithm 3.1.4 is invalid. The remaining three iterative algorithms are applicable in the case of \( n \neq s \). Moreover, the iterative Algorithm 3.1.2 can solve the Hermitian positive definite solution of the nonlinear matrix (1) more effectively. When the \( t \) is small, Algorithm 3.1.2 has the least number of matrix multiplication.
5. Conclusions
In this study, we discussed a nonlinear matrix equation, it is a special Riccati equation derived from the control system. An improved fixed point iteration algorithm is proposed by introducing parameters according to the interval of the solution. By considering the inversion free variant of the fixed-point iteration method has been analyzed and a novel iterative algorithm was thoroughly investigated. Considering the numerical results, we concluded that the new iterative algorithms are very powerful and efficient in finding the numerical solutions for a wide class of nonlinear matrix equations.

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