Minimax prediction of random processes with stationary increments from observations with stationary noise

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Abstract: We deal with the problem of mean square optimal estimation of linear functionals which depend on the unknown values of a random process with stationary increments based on observations of the process with noise, where the noise process is a stationary process. Formulas for calculating values of the mean square errors and the spectral characteristics of the optimal linear estimates of the functionals are derived under the condition of spectral certainty, where the spectral densities of the processes are exactly known. In the case of spectral uncertainty, where the spectral densities of the processes are not exactly known while a class of admissible spectral densities is given, relations that determine the least favorable spectral densities and the minimax robust spectral characteristics are proposed.

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1. Introduction

Traditional methods of finding solutions to problems of estimation of unobserved values of a random process based on a set of available observations of this process, or observations of the process with a noise process, are developed under the condition of spectral certainty, where the spectral densities of the processes are exactly known. Methods of solution of these problems, which are known as interpolation, extrapolation, and filtering of stochastic processes, were developed for stationary stochastic processes by A.N. Kolmogorov, N. Wiener, and A.M. Yaglom (see selected works by Kolmogorov (1992), books by Wiener (1966), Yaglom (1987a, 1987b), Rozanov (1967). Stationary stochastic processes and sequences admit some generalizations, which are properly described in books by Yaglom (1987a, 1987b). Random processes with stationary nth increments are among such generalizations. These processes were introduced in papers by Pinsker and Yaglom (1954), Yaglom (1955, 1957), and Pinsker (1955). In the indicated papers, the authors described the spectral representation of the stationary increment process and the canonical factorization of the spectral density, solved the extrapolation problem, and proposed some examples.

Traditional methods of finding solutions to extrapolation, interpolation, and filtering problems may be employed under the basic assumption that the spectral densities of the considered random processes are exactly known. In practice, however, the developed methods are not applicable since the complete information on the spectral structure of the processes is not available in most cases. To solve the problem, the parametric or nonparametric estimates of the unknown spectral densities are found or these densities are selected by other reasoning. Then, the classical estimation method is applied, provided that the estimated or selected densities are the true ones. However, as was shown by Vastola and Poor (1983) with the help of concrete examples, this method can result in significant increase of the value of the error of estimate. This is a reason to search estimates which are optimal for all densities from a certain class of the admissible spectral densities. The introduced estimates are called minimax robust since they minimize the maximum of the mean square errors for all spectral densities from a set of admissible spectral densities simultaneously. The paper by Grenander (1957) should be marked as the first one where the minimax approach to extrapolation problem for stationary processes was proposed. Franke and Poor (1984) and Franke (1985) investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for various classes of admissible densities. A survey of results in minimax (robust) methods of data processing can be found in the paper by Kassam and Poor (1985).

A wide range of results in minimax robust extrapolation, interpolation, and filtering of random processes and sequences belong to Moklyachuk (2000, 2001, 2008a). Later, Moklyachuk and Masyutka (2011 – 2012) developed the minimax technique of estimation for vector-valued stationary processes and sequences. Dubovets’ka, Masyutka, and Moklyachuk (2012) investigated the problem of minimax robust interpolation for another generalization of stationary processes—periodically correlated processes. In the further papers, Dubovets’ka and Moklyachuk (2013a, 2013b, 2014a, 2014b) investigated the minimax robust extrapolation, interpolation, and filtering problems for periodically correlated processes and sequences.

The minimax robust extrapolation, interpolation, and filtering problems for stochastic sequences with nth stationary increments were investigated by Luz and Moklyachuk (2012., 2013a, 2013b, 2014a, 2014b, 2015a, 2015b, 2015c); Moklyachuk and Luz (2013). In particular, the minimax robust extrapolation problem based on observations with and without noise for such sequences is investigated in papers by Luz and Moklyachuk (2015b), Moklyachuk and Luz (2013). Same estimation problems for random processes with stationary increments with continuous time are investigated in articles by Luz and Moklyachuk (2014a, 2015a, 2015b).

In this article, we deal with the problem of the mean square optimal estimation of the linear functionals $A^t = \int_0^t a(t) \xi(t) dt$ and $A^t = \int_0^t a(t) \xi(t) dt$ which depend on the unknown values of a random process $\xi(t)$ with stationary nth increments from observations of the process $\xi(t) + \eta(t)$ at points $t < 0$, where $\eta(t)$ is an uncorrelated with $\xi(t)$ stationary process. The case of spectral certainty
as well as the case of spectral uncertainty are considered. Formulas for calculating values of the mean square errors and the spectral characteristics of the optimal linear estimates of the functionals are derived under the condition of spectral certainty, where the spectral densities of the processes are exactly known. In the case of spectral uncertainty, where the spectral densities of the processes are not exactly known while a class of admissible spectral densities is given, relations that determine the least favorable spectral densities and the minimax spectral characteristics are derived for some classes of spectral densities.

2. Stationary random increment process. Spectral representation

In this section, we present basic definitions and spectral properties of random processes with stationary increment. For more details, see the book by Yaglom (1987a, 1987b).

Definition 2.1 For a given random process \( \xi(t), t \in \mathbb{R}, \) the process

\[
\xi^{(n)}(t, \tau) = (1 - \mathcal{B}_n) \xi(t) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \xi(t - i\tau),
\]

where \( \mathcal{B}_n \) is a backward shift operator with a step \( \tau \in \mathbb{R}, \) such that \( \mathcal{B}_n \xi(t) = \xi(t - \tau) \) is called the random \( n \)th increment with step \( \tau \in \mathbb{R} \) generated by the random process \( \xi(t). \)

Definition 2.2 The random \( n \)th increment process \( \xi^{(n)}(t, \tau) \) generated by a random process \( \xi(t), t \in \mathbb{R}, \) is in wide sense stationary, if the mathematical expectations

\[
E\xi^{(n)}(t_0, \tau_0) = c^{(n)}(\tau_0),
\]

\[
E\xi^{(n)}(t_0 + t, \tau_1, \tau_2) = D^{(n)}(t, \tau_1, \tau_2)
\]

exist for all \( t_0, \tau_1, \tau_2 \) and do not depend on \( t_0. \) The function \( c^{(n)}(\tau) \) is called the mean value of the \( n \)th increment and the function \( D^{(n)}(t, \tau_1, \tau_2) \) is called the structural function of the stationary \( n \)th increment (or the structural function of \( n \)th order of the random process \( \xi(t), t \in \mathbb{R} \)).

The random process \( \xi(t), t \in \mathbb{R}, \) which determines the stationary \( n \)th increment process \( \xi^{(n)}(t, \tau) \) by formula (1) is called the process with stationary \( n \)th increments.

The following theorem describes representations of the mean value and the structural function of the random stationary \( n \)th increment process \( \xi^{(n)}(t, \tau). \)

**Theorem 2.1** The mean value \( c^{(n)}(\tau) \) and the structural function \( D^{(n)}(t, \tau_1, \tau_2) \) of the random stationary \( n \)th increment process \( \xi^{(n)}(t, \tau) \) can be represented in the following forms:

\[
c^{(n)}(\tau) = c \xi^n,
\]

\[
D^{(n)}(t, \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{it\lambda}(1 - e^{-i\tau_1 \lambda})(1 - e^{-i\tau_2 \lambda})^p \frac{(1 + \lambda^2)^p}{\lambda^{2n}} dF(\lambda),
\]

where \( c \) is a constant and \( F(\lambda) \) is a left-continuous nondecreasing bounded function, such that \( F(-\infty) = 0. \) The constant \( c \) and the function \( F(\lambda) \) are determined uniquely by the increment process \( \xi^{(n)}(t, \tau). \)

The representation (3) of the structural function \( D^{(n)}(t, \tau_1, \tau_2) \) and the Karhunen theorem (see Karhunen, 1947) allow us to write the following spectral representation of the stationary \( n \)th increment process \( \xi^{(n)}(t, \tau): \)

\[
\xi^{(n)}(t, \tau) = \int_{-\infty}^{\infty} e^{it\lambda}(1 - e^{-i\tau \lambda})^p \frac{(1 + \lambda^2)^p}{(i\lambda)^n} dZ^{(n)}(\lambda),
\]

where \( Z^{(n)}(\lambda) \) is a random process with uncorrelated increments on \( \mathbb{R} \) connected with the spectral function \( F(\lambda) \) from representation (3) by the relation
E|Z^m(t_2) - Z^m(t_1)|^2 = F(t_2) - F(t_1) < \infty \quad \text{for all } t_2 > t_1, t_1 \in \mathbb{R}, t_2 \in \mathbb{R}. \tag{5}

3. The Hilbert space projection method of extrapolation

Consider a random process \( \xi(t), t \in \mathbb{R}, \) which generates a stationary random increment process \( \xi^m(t, r) \) with the absolutely continuous spectral function \( F(\lambda) \) and the spectral density function \( f(\lambda) \). Let \( \eta(t), t \in \mathbb{R}, \) be another random process which is stationary and uncorrelated with \( \xi(t) \). Suppose that the process \( \eta(t) \) has absolutely continuous spectral function \( G(\lambda) \) and the spectral density \( g(\lambda) \).

Without loss of generality, we can assume that the increment step \( \tau > 0 \) and both processes \( \xi^m(t, r) \) and \( \eta(t) \) have zero mean values: \( E\xi^m(t, \tau) = 0, E\eta(t) = 0 \).

The main purpose of this paper is to find optimal, in the mean square sense, linear estimates of the functionals

\[
A_\xi = \int_0^\infty a(t)\xi(t)dt, \quad A_\tau \xi = \int_0^\tau a(t)\xi(t)dt
\]

which depend on the unknown values of the random process \( \xi(t) \) at time \( t \geq 0 \) based on observations of the process \( \xi(t) = \xi(t) + \eta(t) \) at time \( t < 0 \).

For further analysis, we need to make the following assumptions. Let the function \( a(t), t \geq 0, \) which determines the functionals \( A_\xi, A_\tau \xi, \) and the linear transformation \( D' \), being defined below, satisfy the conditions

\[
\int_0^\infty |a(t)|dt < \infty, \quad \int_0^\infty t|a(t)|^2dt < \infty, \quad \tag{6}
\]

and

\[
\int_0^\infty |D'a(t)|dt < \infty, \quad \int_0^\infty t|D'a(t)|^2dt < \infty. \quad \tag{7}
\]

Suppose also that the spectral densities \( f(\lambda) \) and \( g(\lambda) \) satisfy the minimality condition

\[
\int_{-\infty}^{\infty} \frac{|\gamma(\lambda)|^2e^{i\lambda}2^n}{|1 - e^{i\lambda}2^\sigma(1 + \lambda^2)^n [(1 + \lambda^2)^nf(\lambda) + \lambda^2ng(\lambda)]}d\lambda < \infty, \quad \tag{8}
\]

for some function \( \gamma(\lambda) \) of the form \( = \int_0^\infty a(t)e^{i\lambda}dt \). Assumption (8) guarantees that the mean square errors of estimates of the considered functionals are greater than 0.

Following the classical estimation theory developed for stationary processes, it is reasonable to apply the method proposed by Kolmogorov (see selected works by Kolmogorov [1992]), where the estimate is a projection of an element of the Hilbert space \( H = L_2(\Omega, \mathfrak{F}, P) \) of the random variables \( \gamma \) with zero mean value, \( E\gamma = 0, \) and finite variance, \( E|\gamma|^2 < \infty \) on a subspace of the space \( H = L_2(\Omega, \mathfrak{F}, P) \). The inner product in the space \( H = L_2(\Omega, \mathfrak{F}, P) \) is defined as \( \langle \gamma_1; \gamma_2 \rangle = E\gamma_1\bar{\gamma}_2 \). Since we have no observations of the process \( \xi(t) \) to take as initial values, the issue is that both functionals \( A_\xi \) and \( A_\tau \xi \) have infinite variance. Thus, we need to derive other objects from the space \( H = L_2(\Omega, \mathfrak{F}, P) \) to proceed with the Hilbert space projection method.

Consider a representation of the functional \( A_\xi \) in the form

\[
A_\xi = A_\xi - A_\eta,
\]

where

\[
A_\xi = \int_0^\infty a(t)\xi(t)dt, \quad A_\eta = \int_0^\infty a(t)\eta(t)dt.
\]
Under the condition (6), the functional \( A_\eta \) has finite variance and, hence, it belongs to the space \( H = L_2(\Omega, \mathcal{F}, P) \). A representation of the functional \( A_\zeta \) is described in the following lemma.

**Lemma 3.1** The linear functional \( A_\zeta \) admits a representation

\[
A_\zeta = B_\zeta - V_\zeta,
\]

where

\[
B_\zeta = \int_0^\infty b_\zeta(t)\xi^n(t, \tau)\,dt, \quad V_\zeta = \int_{-\infty}^0 v_\zeta(t)\xi(t)\,dt,
\]

\[
v_\zeta(t) = \sum_{n=0}^\infty (-1)^j \binom{n}{j} b_j(t + lr), \quad t \in [-rn; 0),
\]

\[
b_j(t) = \sum_{k=0}^{\infty} a(t + rk)d(k) = D^\prime a(t), \quad t \geq 0,
\]

\(|x|^\prime\) denotes the least integer number among numbers that are greater than or equal to \( x \), coefficients \( \{d(k): k \geq 0\} \) are determined from the relation

\[
\sum_{k=0}^{\infty} d(k)x^k = \left( \sum_{j=0}^{\infty} x^j \right)^n,
\]

\( D^\prime \) is a linear transformation of a function \( x(t), t \geq 0 \), defined by the formula

\[
D^\prime x(t) = \sum_{k=0}^{\infty} x(t + rk)d(k).
\]

**Corollary 3.1** The linear functional \( A_\ell_\zeta \) admits a representation

\[
A_\ell_\zeta = B_\ell_\zeta - V_\ell_\zeta,
\]

where

\[
B_\ell_\zeta = \int_0^T b_\ell_\zeta(t)\xi^n(t, \tau)\,dt, \quad V_\ell_\zeta = \int_{-\infty}^0 v_\ell_\zeta(t)\xi(t)\,dt,
\]

\[
v_\ell_\zeta(t) = \sum_{n=0}^{\min\{\left|\frac{t}{\tau}\right|, n\}} (-1)^j \binom{n}{j} b_j(t + lr), \quad t \in [-rn; 0),
\]

\[
b_{\ell_\zeta}(t) = \sum_{k=0}^{\left|\frac{t}{\tau}\right|} a(t + rk)d(k) = D^\prime a(t), \quad t \in [0; T],
\]

\( D^\prime_\ell \) is a linear transformation of an arbitrary function \( x(t), t \in [0; T] \), defined by the formula

\[
D^\prime_\ell x(t) = \sum_{k=0}^{\left|\frac{t}{\tau}\right|} x(t + rk)d(k).
\]

Under the condition (7), the functional \( B_\zeta \) from Lemma 3.1 belongs to the space \( H = L_2(\Omega, \mathcal{F}, P) \), while the functional \( V_\zeta \) is observed and can be considered as an initial value. Thus, Lemma 3.1 implies the following representation of the functional \( A_\zeta \):

\[
A_\zeta = A_\eta - A_\eta = B_\zeta - A_\eta - V_\zeta = H_\xi - V_\zeta,
\]
where the functional $H_\xi = B \zeta - A \eta$ belongs to the space $H = L_2(\Omega, S, P)$ and the Hilbert space projection method can be applied. Since the functional $V_\zeta$ depends on the observations $\zeta(t)$, $-n < t < 0$, the following relations hold true for the estimates $A_\xi, \hat{H}_\xi$ and the mean square errors $\Delta(f, g, \hat{A}_\xi, \Delta(f, g, \hat{H}_\xi)$:

$$\hat{A}_\xi = \hat{H}_\xi - V_\zeta,$$

$$\Delta(f, g, \hat{A}_\xi): = E|A_\xi - \hat{A}_\xi|^2 = E|H_\xi - V_\zeta - \hat{H}_\xi + V_\zeta|^2 = E|H_\xi - \hat{H}_\xi|^2 = \Delta(f, g, \hat{H}_\xi).$$

(13)

Therefore, the problem is reduced to finding the optimal mean square estimate $\hat{H}_\xi$ of the functional $H_\xi$.

The next step is to describe the spectral structure of the functional $H_\xi$. The stationary random process $\eta(t)$ admits the spectral representation (see Gikhman & Skorokhod, 2004).

$$\eta(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ_\eta(\lambda),$$

where $Z_\eta(\lambda)$ is a random process with uncorrelated increments on $\mathbb{R}$ which correspond to the spectral function $G(\lambda)$. Taking into account (4), the spectral representation of the random process $\zeta^{(n)}(t, r)$ can be described by the formulas

$$\zeta^{(n)}(t, r) = \int_{-\infty}^{\infty} e^{i\lambda t} (1 - e^{i\lambda r})^n \frac{(1 + i\lambda)^n}{(i\lambda)^n} dZ_n^\eta(\lambda) + \int_{-\infty}^{\infty} e^{i\lambda t} (1 - e^{i\lambda r})^n \frac{(1 + i\lambda)^n}{(i\lambda)^n} dZ_n^\eta(\lambda),$$

where

$$dZ_n^\eta(\lambda) = (i\lambda)^n (1 + i\lambda)^{-n} dZ_n(\lambda), \quad \lambda \in \mathbb{R}.$$

One can easily conclude that the spectral density $p(\lambda)$ of the random process $\zeta(t)$ is the following:

$$p(\lambda) = f(\lambda) + \frac{1}{(1 + \lambda^2)^n} g(\lambda).$$

The functional $H_\xi$ admits the spectral representation

$$H_\xi = \int_{-\infty}^{\infty} B_\xi(\lambda) (1 - e^{-i\lambda t})^n \frac{(1 + i\lambda)^n}{(i\lambda)^n} dZ_n^\eta(\lambda) - \int_{-\infty}^{\infty} A(\lambda) dZ_n^\eta(\lambda),$$

where

$$B_\xi(\lambda) = \int_{0}^{\infty} b_\xi(t) e^{i\lambda t} dt = \int_{0}^{\infty} (D^\alpha a)(t) e^{i\lambda t} dt, \quad A(\lambda) = \int_{0}^{\infty} a(t) e^{i\lambda t} dt.$$
\[ e^{i\lambda t}(1 - e^{-i\lambda t})^{n}(1 + i\lambda)^{n}(i\lambda)^{-n} \]

of the space \( L^{2}(p) \) to the vector \( \xi^{(m)}(t, \tau) + \eta^{(m)}(t, \tau) \) of the space \( H^{0}((\xi^{(m)} + \eta^{(m)})) \) may be extended to a linear isometry between the above spaces. The following relation holds true:

\[
E_{\xi^{(m)}}(t_{1}, t_{2}) = \int_{-\infty}^{\infty} e^{i\lambda t_{1}}(1 - e^{-i\lambda t_{1}})^{n}(1 - e^{i\lambda t_{1}})^{n}(1 + \lambda^{2})^{n} \rho(\lambda) d\lambda.
\] (15)

Every linear estimate \( \hat{A}(\xi) \) of the functional \( A(\xi) \) admits the representation

\[
\hat{A}(\xi) = \int_{-\infty}^{\infty} h_{\lambda}(\xi) dZ_{\xi^{0}+\rho(\lambda)}(\lambda) - \int_{-\infty}^{0} v_{\lambda}(\xi(t) + \eta(t)) dt,
\] (16)

where \( h_{\lambda}(\xi) \) is the spectral characteristic of the estimate \( \hat{R}(\xi) \). We can find the estimate \( \hat{R}(\xi) \) as a projection of the element \( H_{\xi} \) of the space \( H \) on the subspace \( H^{0}((\xi^{(m)} + \eta^{(m)})) \). This projection is characterized by two conditions:

1. \( H_{\xi} \in H^{0}((\xi^{(m)} + \eta^{(m)}) \)

2. \( (H_{\xi} - \hat{R}(\xi)) \perp H^{0}((\xi^{(m)} + \eta^{(m)}) \)

Condition (2) and property (15) imply the following relations which hold true for every \( t < 0 \):

\[
E(H_{\xi} - \hat{R}(\xi))(\xi^{(m)}(t, \tau) + \eta^{(m)}(t, \tau)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ B_{\lambda}(1 - e^{-i\lambda t})^{n} - A(\lambda) - \frac{(i\lambda)^{n}h_{\lambda}(\lambda)}{(1 + i\lambda)^{n}} \right] e^{-i\lambda t}(1 - e^{i\lambda t})^{n}(1 + \lambda^{2})^{n} \rho(\lambda) d\lambda
\ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ B_{\lambda}(1 - e^{-i\lambda t})^{n} - A(\lambda) - \frac{(i\lambda)^{n}h_{\lambda}(\lambda)}{(1 + i\lambda)^{n}} \right] e^{-i\lambda t}(1 - e^{i\lambda t})^{n}(1 + \lambda^{2})^{n} \rho(\lambda) d\lambda = 0.
\]

Let us define for \( \lambda \in \mathbb{R} \) the function

\[
C_{\lambda}(\lambda) = \left[ B_{\lambda}(1 - e^{-i\lambda t})^{n} - A(\lambda) - \frac{(i\lambda)^{n}h_{\lambda}(\lambda)}{(1 + i\lambda)^{n}} \right] \frac{(1 + \lambda^{2})^{n}}{\lambda^{2n}} \rho(\lambda) - \frac{(-i\lambda)^{n}C_{\lambda}(\lambda)}{(1 - e^{i\lambda t})^{n}(1 - i\lambda)^{n}} p(\lambda),
\]

and its Fourier transform

\[
c_{\lambda}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{\lambda}(\lambda)e^{-i\lambda t} d\lambda, \quad t \in \mathbb{R}.
\]

We have \( c_{\lambda}(t) = 0 \) for \( t < 0 \), hence

\[
C_{\lambda}(\lambda) = \int_{0}^{\infty} c_{\lambda}(t)e^{i\lambda t} dt,
\]

which allows us to construct the representation of the spectral characteristic

\[
h_{\lambda}(\lambda) = B_{\lambda}(1 - e^{-i\lambda t})^{n}(1 + i\lambda)^{n} - A(\lambda) \frac{(-i\lambda)^{n}C_{\lambda}(\lambda)}{(1 - e^{i\lambda t})^{n}(1 - i\lambda)^{n}} p(\lambda) - \frac{(-i\lambda)^{n}C_{\lambda}(\lambda)}{(1 - e^{i\lambda t})^{n}(1 - i\lambda)^{n}} p(\lambda).
\]

It follows from the condition 1) that the spectral characteristic \( h_{\lambda}(\lambda) \) admits the representation

\[
h_{\lambda}(\lambda) = h(\lambda)(1 - e^{-i\lambda t})^{n}(1 + i\lambda)^{n},
\]

\[
h(\lambda) = \int_{-\infty}^{0} s(t)e^{i\lambda t} dt, \quad s(t) \in L_{2}.
\]
which leads to the following relations holding true for every \( s \geq 0 \):

\[
\int_{-\infty}^{\infty} B_s(\lambda) - \frac{A(\lambda)(1 - e^{-i\lambda s} - i\lambda g(\lambda))}{(1 + \lambda^2)^n p(\lambda)} - \frac{|1 - e^{-|\lambda| s} - \lambda^2 n C_s(\lambda)|}{(1 + \lambda^2)^n p(\lambda)} e^{-i\lambda s} d\lambda = 0.
\]  

Relation (17) can be represented in terms of linear operators in the space \( L_2[0; \infty) \). Let us define the operators

\[
(T_s x)(t) = \frac{1}{2\pi} \int_{0}^{\infty} x(t) \int_{-\infty}^{\infty} e^{i(t-s)} \frac{\lambda^2 g(\lambda)}{|1 - e^{i\lambda(s-t)} + \lambda^2 y p(\lambda)|} d\lambda dt, \; s \in [0; \infty),
\]

\[
(P_s y)(t) = \frac{1}{2\pi} \int_{0}^{\infty} y(t) \int_{-\infty}^{\infty} e^{i(t-s)} \frac{\lambda^2 n}{|1 - e^{i\lambda(s-t)} + \lambda^2 y p(\lambda)|} d\lambda dt, \; s \in [0; \infty),
\]

\[
(Q_s z)(t) = \frac{1}{2\pi} \int_{0}^{\infty} z(t) \int_{-\infty}^{\infty} e^{i(t-s)} \frac{f(\lambda)(\lambda^2 + \lambda^2 y p(\lambda))}{p(\lambda)} d\lambda dt, \; s \in [0; \infty),
\]

where \( x(t), y(t), z(t) \in L_2[0; \infty) \). The introduced operators allow us to represent relations (17) in the form

\[
b_s(s) - (T_s a_s)(s) = (P_s c_s)(s), \; s \geq 0,
\]

where

\[
a_s(t) = \min_{n \geq 1} \left\{ \frac{1}{n!} \right\} \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} a(t - i\lambda), \; t \geq 0.
\]  

Then, under the condition that the linear operator \( P_s \) is invertible, the function \( c_s(t) \), \( t \geq 0 \), can be found by the formula

\[
c_s(t) = (P_s^{-1} D_s a - P_s^{-1} T_s a_s)(t), \; t \geq 0.
\]

Consequently, the spectral characteristic \( h_s(\lambda) \) of the optimal estimate \( \hat{H}_s \) of the functional \( H_\xi \) is calculated by the formula

\[
h_s(\lambda) = B_s(\lambda) \left(1 - \frac{e^{-i\lambda s} - i\lambda g(\lambda)}{(1 + i\lambda)^n} \right) - \frac{A(\lambda)(1 + i\lambda)^n (-i\lambda)^n g(\lambda)}{(1 + \lambda^2)^n f(\lambda) + \lambda^2 g(\lambda)}
\]

\[
- \frac{(1 + i\lambda)^n (-i\lambda)^n C_s(\lambda)}{(1 - e^{i\lambda s})(1 + \lambda^2)^n f(\lambda) + \lambda^2 g(\lambda)},
\]

where

\[
C_s(\lambda) = \int_{0}^{\infty} (P_s^{-1} D_s a - P_s^{-1} T_s a_s)(t) e^{i\lambda s} dt.
\]

The value of mean square error is calculated by the formula

\[
\Delta(f, g; \hat{H}_\xi) = \Delta(f, g; \hat{H}_\xi) = E\|H_\xi - \hat{H_\xi}\|^2
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|A(\lambda)(1 - e^{i\lambda s} - i\lambda g(\lambda)) - \lambda^2 n C_s(\lambda)|^2}{|1 - e^{i\lambda s} - \lambda^2 n C_s(\lambda)|^2} g(\lambda) d\lambda
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|A(\lambda)(1 - e^{i\lambda s} - i\lambda g(\lambda)) + \lambda^2 n C_s(\lambda)|^2}{|1 - e^{i\lambda s} - \lambda^2 n C_s(\lambda)|^2} f(\lambda) d\lambda
\]

\[
= \langle D_s a - T_s a_s, P_s^{-1} D_s a - P_s^{-1} T_s a_s \rangle + \langle Q_s a, a \rangle.
\]
The obtained results can be summarized in the following theorem.

**Theorem 3.1** Let \( \zeta(t), t \in \mathbb{R}, \) be a random process with stationary \( n \)th increment process \( \xi^n(t, r) \) and let \( \eta(t), t \in \mathbb{R}, \) be an uncorrelated with \( \zeta(t) \) stationary random process. Suppose that the spectral densities \( f(\lambda) \) and \( g(\lambda) \) of the random processes \( \zeta(t) \) and \( \eta(t) \) satisfy the minimality condition (8) and the function \( a(t), t \geq 0, \) satisfies conditions (6) and (7). Suppose also that the linear operator \( F \) is invertible. The optimal estimate \( \hat{A}_\xi \) of the functional \( A_\xi \) based on observations \( \zeta(t) + \eta(t) \) at time \( t < 0 \) is calculated by formula (16). The spectral characteristic \( h_\xi(\lambda) \) and the value of mean square error \( \Delta(f, g; \hat{A}_\xi) \) of the estimate \( \hat{A}_\xi \) can be calculated by formulas (19) and (20), respectively.

Remark 3.1 The spectral characteristic \( h_\xi(\lambda) \) determined by formula (19) can be presented in the form \( h_\xi(\lambda) = h^1_\xi(\lambda) - h^2_\xi(\lambda), \) where

\[
h^1_\xi(\lambda) = B_\xi(\lambda) \frac{(1 - e^{-ix})(1 + i\lambda)^n}{i\lambda^n} \quad \frac{(1 + i\lambda)^n \int_0^1 (F^{-1}D' a)(t)e^{-itd} dt}{(1 - e^{it\lambda})(1 + i\lambda)^n f(\lambda)},
\]

\[
h^2_\xi(\lambda) = -\frac{A(\lambda)(1 + i\lambda)^n (-i\lambda)^n g(\lambda)}{(1 + i\lambda)^n f(\lambda) + i\lambda^2 g(\lambda)} \quad \frac{(1 + i\lambda)^n \int_0^1 (P_z, a)(t)e^{-itd} dt}{(1 - e^{it\lambda})(1 + i\lambda)^n f(\lambda) + i\lambda^2 g(\lambda)}.
\]

The functions \( h^1_\xi(\lambda) \) and \( h^2_\xi(\lambda) \) are the spectral characteristics of the mean square optimal estimates \( \hat{B}_\xi \) and \( \hat{A}_\eta \) of the functionals \( B \xi \) and \( A_\eta \), respectively, based on observations \( \zeta(t) + \eta(t) \) at time \( t < 0.\)

In the case of observations without noise, we have the following corollary.

**Corollary 3.2** Let \( \zeta(t), t \in \mathbb{R}, \) be a random process with stationary \( n \)th increment process \( \xi^n(t, r) \) and suppose that the spectral density \( f(\lambda) \) of the random processes \( \zeta(t) \) satisfies the minimality condition (8) with \( g(\lambda) = 0 \) and the function \( a(t), t \geq 0, \) satisfies conditions (6) and (7). Suppose also that the linear operator \( F \) defined below is invertible. The optimal linear estimate \( \hat{A}_\xi \) of the functional \( A_\xi \) which depends on the unknown values \( \xi(t), t \geq 0, \) of the random process \( \zeta(t), \) based on observations of the process \( \zeta(t), t < 0, \) is calculated by the formula

\[
\hat{A}_\xi = \int_{-\infty}^{\infty} h_\xi(\lambda) dZ^\xi(\lambda) - \int_{-\infty}^{0} v(t)\xi(t)dt.
\]  

The spectral characteristic \( h_\xi(\lambda) \) and the mean square error \( \Delta(f; \hat{A}_\xi) \) of the optimal estimate \( \hat{A}_\xi \) of the functional \( A_\xi \) are calculated by the formulas

\[
h_\xi(\lambda) = B_\xi(\lambda) \frac{(1 - e^{-ix})(1 + i\lambda)^n}{i\lambda^n} \quad \frac{(1 + i\lambda)^n \int_0^1 (F^{-1}D' a)(t)e^{-itd} dt}{(1 - e^{it\lambda})(1 + i\lambda)^n f(\lambda)},
\]

\[
\Delta(f; \hat{A}_\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda^{2n} \int_0^1 (F^{-1}D' a)(t)e^{-itd} dt}{\left|1 - e^{-it\lambda}\right|^{2n}(1 + i\lambda)^n f(\lambda)} d\lambda = (F^{-1}D' a, D' a),
\]

where \( F \) is the linear operator in the space \( L_2[0; \infty) \) determined by the formula

\[
(F, y)(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(t) \int_{-\infty}^{\infty} e^{i(t-s)\lambda} \frac{\lambda^{2n}}{\left|1 - e^{-it\lambda}\right|^{2n}(1 + i\lambda)^n f(\lambda)} d\lambda dt, s \in [0; \infty).
\]

Remark 3.2 In Corollary 3.2, we provide formulas for calculating the optimal linear estimate \( \hat{A}_\xi \) of the functional \( A_\xi \) and the value of the mean square error \( \Delta(f; \hat{A}_\xi) \) of the estimate \( \hat{A}_\xi \) based on observations of the process \( \xi(t) \) at time \( t < 0 \) using the Fourier transform of the function \( \frac{\lambda^{2n}}{\left|1 - e^{-it\lambda}\right|^{2n}(1 + i\lambda)^n f(\lambda)} \). In the article by Luz and Moklyachuk (2014c), the same problem is considered. However, a solution is
derived in terms of the function \( \varphi(t), t \geq 0 \), which is determined by the canonical factorization of the function

\[
|1 - e^{-it\lambda}|^2 |1 + x^2|^n f(\lambda) = \left| \int_0^\infty \varphi(t)e^{-it\lambda}dt \right|^2.
\]

Theorem (3.1) can be used to obtain the optimal estimate \( \hat{A}_\tau \xi \) of the functional \( A_\tau \xi \) which depends on the unknown values \( \xi(t), 0 \leq t \leq T \), of the random process \( \xi(t) \), based on observations of the process \( \xi(t) + \eta(t) \) at time \( t < 0 \). To derive the corresponding formulas, let us put \( a(t) = 0 \) if \( t > T \).

We get that the spectral characteristic \( h_{\tau r}(\lambda) \) of the optimal estimate

\[
\hat{A}_\tau \xi = \int_{-\infty}^\infty h_{\tau r}(\lambda)dZ_{e^{\omega t}(\lambda)} - \int_{-\infty}^0 v_{\tau r}(t)(\xi(t) + \eta(t))dt,
\]

is calculated by the formula

\[
h_{\tau r}(\lambda) = B_r^f(\lambda) \frac{(1 - e^{-it\lambda})^p(1 + i\lambda)^n}{(i\lambda)^n} - \frac{A_r(\lambda)(1 + i\lambda)^n(-i\lambda)^n g(\lambda)}{(1 + \lambda^2)^n f(\lambda) + \lambda^2 n g(\lambda)}
\]

\[
\times \frac{1}{(1 - e^{it\lambda})(1 + \lambda^2)^n C_r^f(\lambda)}
\]

\[
B_r^f(\lambda) = \int_0^T b_{\tau r}(t)e^{it\lambda}dt = \int_0^T (D_1^r a_r)(t)e^{it\lambda}dt,
\]

\[
A_r(\lambda) = \int_0^\infty a(t)e^{it\lambda}dt,
\]

\[
C_r^f(\lambda) = \int_0^\infty (P_1^r D_1^r a_r - P_1^r T_1^r a_r)(t)e^{it\lambda}dt,
\]

where the linear operator \( T_r^f \) in the space \( L_2[0; \infty) \) is determined by the formula

\[
(T_r^f x)(s) = \frac{1}{2\pi} \int_0^\infty x(t) \int_{-\infty}^\infty e^{-i(s+t\lambda)} \frac{\lambda^{2n} g(\lambda)}{|1 - e^{i(s+t\lambda)}|^{2n}(1 + \lambda^2)^n p(\lambda)}d\lambda dt, s \in [0; \infty),
\]

the function \( a_{\tau r}(t), t \in [0; T + \tau n] \) is calculated by formula

\[
a_{\tau r}(t) = \min \left\{ \left\lfloor \frac{t}{\tau} \right\rfloor \right\} n, t \in [0; T + \tau n].
\]

The mean square error of the optimal estimate \( \hat{A}_\tau \xi \) is calculated by the formula

\[
\Delta(f, g; \hat{A}_\tau \xi) = \Delta(f, g; \hat{H}_\tau \xi) = E|H_\tau \xi - \hat{H}_\tau \xi|^2
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^\infty \left| \frac{A_r(\lambda)(1 - e^{i\lambda t})(1 + \lambda^2)^n f(\lambda) - \lambda^{2n} C_r^f(\lambda)}{|1 - e^{i\lambda t}(1 + \lambda^2)^n p(\lambda)|^2} g(\lambda) \right| d\lambda
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^\infty \left| \frac{A_r(\lambda)(1 - e^{i\lambda t})(-i\lambda)^n g(\lambda) + (-i\lambda)^n C_r^f(\lambda)}{|1 - e^{i\lambda t}(1 + \lambda^2)^n p(\lambda)|^2} f(\lambda) \right| d\lambda
\]

\[
= \langle D_r^f a_r - T_r^f a_r, P_1^r D_r^f a_r - P_1^r T_r^f a_r \rangle + \langle Q_r a_r, a_r \rangle,
\]

where the linear operator \( Q_r \) in the space \( L_2[0; \infty) \) is determined by the formula
\[
(Q_x)(s) = \frac{1}{2\pi} \int_0^T z(t) \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \frac{(1 + \lambda^2)^n f(\lambda)g(\lambda)}{(1 + \lambda^2)^n f(\lambda) + \lambda^{2n}g(\lambda)} \, d\lambda dt, \ s \in [0;\infty),
\]
and the function \(a_f(t), t \in [0;T]\), is determined as \(a_f(t) = a(t)\).

The described results can be summarized in the following theorem.

**Theorem 3.2** Let \(\xi(t), t \in \mathbb{R}\), be a random process with stationary \(n\)th increment process \(\xi^n(t, \tau)\) and let \(\eta(t), t \in \mathbb{R}\), be an uncorrelated with \(\xi(t)\) stationary random process. Suppose that the spectral densities \(f(\lambda)\) and \(g(\lambda)\) of the random processes \(\xi(t)\) and \(\eta(t)\) satisfy the minimality condition (8) and the function \(a(t), 0 \leq t \leq T\), satisfies conditions (6) and (7). Suppose also that the linear operator \(P\), is invertible. The optimal linear estimate \(A_{\xi, \xi}(\tau)\) of the functional \(A_{\xi, \xi}\) based on observations of the process \(\xi(t) + \eta(t)\) at time \(t < 0\) is calculated by formula (24). The spectral characteristic \(h_{\xi, \xi}(\lambda)\) and the value of mean square error \(\Delta(f, g; A_{\xi, \xi})\) of the optimal estimate \(A_{\xi, \xi}\) are calculated by formulas (25) and (26), respectively.

**4. Minimax robust method of extrapolation**

The values of the mean square errors and the spectral characteristics of the optimal estimates of the functionals \(A_{\xi, \xi}\) and \(A_{\eta, \eta}\) based on observations of the process \(\xi(t) + \eta(t)\) or observations of the process \(\xi(t)\) without noise can be calculated by formulas (20), (23), (26) and (19), (22), and (25), respectively, in the case where the spectral densities \(f(\lambda)\) and \(g(\lambda)\) of the random processes \(\xi(t)\) and \(\eta(t)\) are exactly known. In the case where the spectral densities are not exactly known while sets \(D = D_f \times D_g\) or \(D = D_f\) of admissible spectral densities are given, the minimax robust method of estimation of the functionals which depend on the unknown values of the random process with stationary increments can be applied. The method consists in determining an estimate which minimizes the value of the mean square error for all spectral densities from the given class \(D = D_f \times D_g\) or \(D = D_f\), simultaneously. The following definitions formalize the proposed method.

**Definition 4.1** For a given class of spectral densities, \(D = D_f \times D_g\) spectral densities \(f_0(\lambda) \in D_f\), \(g_0(\lambda) \in D_g\) are called the least favorable in the class \(D\) for the optimal linear extrapolation of the functional \(A_{\xi, \xi}\) if the following relation holds true

\[
\Delta(f^0, g^0) = \Delta(h(f^0, g^0); f^0, g^0) = \max_{(f, g) \in D_f \times D_g} \Delta(h; f, g).
\]

**Definition 4.2** For a given class of spectral densities \(D = D_f \times D_g\), the spectral characteristic \(h^0(\lambda)\) of the optimal linear estimate of the functional \(A_{\xi, \xi}\) is called minimax robust if there are satisfied conditions:

\[
h^0(\lambda) \in H^0 = \bigcap_{(f, g) \in D_f \times D_g} \mathbb{L}^0_2(p(\lambda)),
\]

\[
\min_{h \in H^0} \max_{(f, g) \in D_f \times D_g} \Delta(h; f, g) = \max_{(f, g) \in D_f \times D_g} \Delta(h^0; f, g).
\]

Let us now formulate lemmas which follow from the introduced definitions and formulas (20) and (23) derived in the previous section.

**Lemma 4.1** The spectral densities \(f^0(\lambda) \in D_f\) and \(g^0(\lambda) \in D_g\) which satisfy the minimality condition (8) are the least favorable in the class \(D\) for the optimal linear extrapolation of the functional \(A_{\xi, \xi}\) based on observations of the random process \(\xi(t) + \eta(t)\) at time \(t < 0\) if linear operators \(P_{f^0}, T_{f^0}, Q_{f^0}\), determined by the Fourier transform of the functions

\[
\frac{\lambda^{2n}[1 - e^{i\lambda t} + e^{i\lambda t}]}{(1 + \lambda^2)^n f(\lambda) + \lambda^{2n}g(\lambda)}, \quad \frac{\lambda^{2n}[1 - e^{i\lambda t} + e^{i\lambda t}]}{(1 + \lambda^2)^n f(\lambda) + \lambda^{2n}g(\lambda)},
\]

determine a solution of the constrain optimization problem.
\[
\max_{f,g \in D_t} \left( (D' a - T_a, P_f^{-1} D' a - P_f^{-1} T_a) + (Q a, a) \right) \\
= \langle D' a - T_0 a, (P_0^{-1})^{-1} D' a - (P_0^{-1})^{-1} T_0 a \rangle + (Q_0 a, a). \tag{27}
\]

The minimax robust spectral characteristic \( h^0 = h_r(f^0, g^0) \) can be found by formula (19) if \( h_r(f^0, g^0) \in H_D \).

The corresponding result holds true in the case where observations of the process \( \xi(t) \) at time \( t < 0 \) are available.

**LEMMA 4.2** The spectral density \( f^0 \in D_t \) satisfying the minimality condition

\[
\int_{-\infty}^{\infty} \frac{|\tilde{r}(\lambda)|^2 \lambda^{2n}}{1 - e^{i \xi} |\lambda|^{2n} (1 + \lambda^2)^n (f(\lambda))^{-1}} d\lambda < \infty
\tag{28}
\]

is the least favorable in the class \( D_t \) for the optimal linear extrapolation of the functional \( A_\xi \) based on observations of the process \( \xi(t) \) at time \( t < 0 \) if the linear operator \( F_0 \) defined by the Fourier transformation of the function

\[
\lambda^{2n} |1 - e^{i \xi} |\lambda|^{2n} (1 + \lambda^2)^n (f(\lambda))^{-1}
\]

determines a solution to the constrain optimization problem

\[
\max_{f \in D_t} \langle F_0^{-1} D' a, D' a \rangle = \langle (P_0^{-1})^{-1} D' a, D' a \rangle. \tag{29}
\]

The minimax robust spectral characteristic \( h^0 = h_r(f^0) \) is calculated by formula (22) under the condition \( h_r(f^0) \in H_D \).

The least favorable spectral densities can be found directly using the definition or applying the proposed lemmas. However, there is an approach which gives us a possibility to simplify the optimization problem using the following property of the function \( \Delta(h; f, g) \). This function has a saddle point on the set \( H_D \times D \), which is formed by the minimax robust spectral characteristic \( h^0 \) and a pair \( (f^0, g^0) \) of the least favorable spectral densities. The saddle point inequalities

\[
\Delta(h; f^0, g^0) \geq \Delta(h^0; f^0, g^0) \geq \Delta(h^0; f, g) \quad \forall f \in D_t, \forall g \in D_g, \forall h \in H_D
\]

hold true if \( h^0 = h_r(f^0, g^0), h_r(f^0, g^0) \in H_D \) and the pair \( (f^0, g^0) \) determines a solution of the constrain optimization problem

\[
\widetilde{\Delta}(f, g) = -\Delta(h_r(f^0, g^0); f, g) \rightarrow \inf, \quad (f, g) \in D,
\]

where

\[
\Delta(h_r(f^0, g^0); f, g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|A(\lambda)| (1 - e^{i \xi}) f(\lambda) e^{i \xi} (\lambda) - \lambda^2 C(\lambda) |^2}{1 - e^{i \xi} |\lambda|^{2n} (1 + \lambda^2)^n (f(\lambda))^{-1}} g(\lambda) d\lambda
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|A(\lambda)| (1 - e^{i \xi}) f(\lambda) e^{i \xi} (\lambda) + (\lambda^2 C(\lambda) |^2}{1 - e^{i \xi} |\lambda|^{2n} (1 + \lambda^2)^n (f(\lambda))^{-1}} f(\lambda) d\lambda,
\]

\[
C(\lambda) = \int_{0}^{\infty} ((P_0^{-1})^{-1} D' a - (P_0^{-1})^{-1} T_0 a) (t) e^{i \xi} dt.
\]

In the case of estimating the functional \( A_\xi \) based on the observations \( \xi(t), t < 0 \), we have the following constrain optimization problem
\[ \tilde{\Delta}(f) = -\Delta(h, (f^0); f) \to \inf, \quad f \in D, \]

where

\[
\Delta(h, (f^0); f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^{2n} \left| \int_0^\infty ((F^0_\lambda)^{-1} D^0_n a)(t) e^{i\lambda t} dt \right|^2 f(\lambda) d\lambda. 
\]

Using the indicator functions \( \delta(f, g|D_1 \times D_2) \) and \( \delta(f, g|D_p) \) of the sets \( D_1 \times D_2 \) and \( D_p \), the indicated constraint optimization problems can be presented as unconditional optimization problems

\[
\Delta_0(f, g) = \tilde{\Delta}(f, g) + \delta(f, g|D_1 \times D_2) \to \inf, \\
\Delta_0(f) = \tilde{\Delta}(f) + \delta(f|D_1) \to \inf
\]

respectively. In this case, solutions \((f^0, g^0)\) and \( f^0 \) are characterized by the conditions \( 0 \in \partial \Delta_0(f^0, g^0) \) and \( 0 \in \partial \Delta_0(f^0) \) which are necessary and sufficient conditions that the pair \((f^0, g^0)\) belongs to the set of minima of the convex functional \( \Delta_0(f, g) \) and the function \( f^0 \) belongs to the set of minima of the convex functional \( \Delta_0(f) \). By \( \partial \Delta_0(f^0, g^0) \) and \( \partial \Delta_0(f^0) \), we denote subdifferentials of the functionals \( \Delta_0(f, g) \) and \( \Delta_0(f) \) at point \((f, g) = (f^0, g^0)\) and \( f^0 \), respectively (see books by Ioffe & Tihomirov, 1979, Moklyachuk, 2008a, Pshenichnyi, 1971, Rockafellar, 1997)).

5. Least favorable densities in the class \( D^0_1 \times D^0_2 \)

In this section, we consider the problem of minimax robust extrapolation of the functional \( A_\xi \) based on observations of the process \( \xi(t) + \eta(t) \) at time \( t < 0 \) on the set of admissible spectral densities \( D = D^0_1 \times D^0_2 \) where

\[
D^0_1 = \left\{ (f(\lambda)) \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) d\lambda \leq P_1 \right\}, \\
D^0_2 = \left\{ (g(\lambda)) \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\lambda) d\lambda \leq P_2 \right\}.
\]

Let us suppose that the spectral densities \( f^0 \in D^0_1 \) and \( g^0 \in D^0_2 \) and the functions

\[
A_{1, f}(f^0, g^0) = \frac{\tilde{A}(\lambda)(1 - e^{i\lambda n}(-i \lambda)^n g^0(\lambda) + (-i \lambda)^n C^0_n(\lambda))}{|1 - e^{i\lambda n}(1 + \lambda^2)^n/2 g^0(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^2)^n} g^0(\lambda)|}, \quad (30)
\]

\[
A_{2, g}(f^0, g^0) = \frac{\tilde{A}(\lambda)(1 - e^{i\lambda n}(1 + \lambda^2)^n f^0(\lambda) - \lambda^{2n} C^0_n(\lambda))}{|1 - e^{i\lambda n}(1 + \lambda^2)^n f^0(\lambda) + \frac{\lambda^{2n}}{(1 + \lambda^2)^n} g^0(\lambda)|}, \quad (31)
\]

are bounded. These conditions ensure that the functional \( \Delta(h, (f^0, g^0); f, g) \) is continuous and bounded in the space \( L_1 \times L_1 \). Condition \( 0 \in \partial \Delta_0(f^0, g^0) \) implies the spectral densities \( f^0 \in D^0_1 \), \( g^0 \in D^0_2 \) satisfy the equalities

\[
\begin{align*}
\tilde{A}(\lambda)(1 - e^{i\lambda n}(1 + \lambda^2)^n f^0(\lambda) - \lambda^{2n} C^0_n(\lambda)) &= a_1 \tilde{A}(\lambda)(1 - e^{i\lambda n}(1 + \lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda)), \quad (32) \\
\tilde{A}(\lambda)(1 - e^{i\lambda n}(-i \lambda)^n g^0(\lambda) + (-i \lambda)^n C^0_n(\lambda)) &= a_2 \tilde{A}(\lambda)(1 - e^{i\lambda n}(-i \lambda)^n g^0(\lambda) + (-i \lambda)^n C^0_n(\lambda)), \quad (33)
\end{align*}
\]

where the constants \( a_1 \geq 0, a_2 \geq 0, \) and \( a_1 \neq 0 \) if

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} f^0(\lambda) d\lambda = P_1,
\]

\( a_2 \neq 0 \) if

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} g^0(\lambda) d\lambda = P_2.
\]

We can summarize the obtained results in the following theorem.
THEOREM 5.1 Let the spectral densities $f_0(\lambda) \in D^0_f$ and $g_0(\lambda) \in D^0_g$ satisfy condition (8) and let the functions $h_{i,j}(f_0^i, g_0^j)$ and $h_{i,j}(f_0^i, g_0^j)$ determined by formulas (30) and (31) be bounded. The spectral densities $f_0(\lambda)$ and $g_0(\lambda)$ are the least favorable in the class $D = D^0_f \times D^0_g$ for the optimal linear extrapolations of the functional $A^\xi$ if they satisfy equations (32) and (33) and determine a solution of the optimization problem (27). The function $h_{i,j}(f_0^i, g_0^j)$ calculated by formula (19) is the minimax robust spectral characteristic of the optimal estimate of the functional $A^\xi$.

THEOREM 5.2 Let the spectral density $f(\lambda)$ be known, let the spectral density $g(\lambda) \in D^0_g$ and let the spectral densities $f(\lambda)$, $g(\lambda)$ satisfy the minimality condition (8). Suppose also that the function $h_{i,j}(f, g)$ determined by formula (31) is bounded. The spectral density

$$
    g_0(\lambda) = \max \left\{ 0, f_0(\lambda) - (1 + \lambda^2)\lambda^{-2n} f(\lambda) \right\},
$$

$$
    f_1(\lambda) = \frac{A(\lambda)(1 - e^{ix})^n(1 + \lambda^2)^n f(\lambda) - \lambda^{2n} C(\lambda)}{a_1 |1 - e^{ix}|^n \lambda^{2n}},
$$

(34)

is the least favorable in the class $D^0_g$ for the optimal linear extrapolation of the functional $A^\xi$ if the functions $f(\lambda) + (1 + \lambda^2)^{-n} \lambda^2 n g(\lambda) g(\lambda)$ determine a solution of the optimization problem (27). The function $h_{i,j}(f, g)$ calculated by formula (19) is the minimax robust spectral characteristic of the optimal estimate of the functional $A^\xi$.

THEOREM 5.3 Let the spectral density $g(\lambda)$ be known, let the spectral density $f_0(\lambda) \in D^0_f$ and let the spectral densities $f_0(\lambda)$, $g(\lambda)$ satisfy the minimality condition (8). Suppose also that the function $h_{i,j}(f, g)$ determined by formula (30) is bounded. The spectral density

$$
    f_0(\lambda) = \max \left\{ 0, g_0(\lambda) - (1 + \lambda^2)^{-n} \lambda^{2n} g(\lambda) \right\},
$$

$$
    g_1(\lambda) = \frac{A(\lambda)(1 - e^{ix})^n(-i\lambda)^n g(\lambda) + (-i\lambda)^n C_0(\lambda)}{a_2 |1 - e^{ix}|^n(1 + \lambda^2)^{n/2}},
$$

(35)

is the least favorable in the class $D^0_f$ for the optimal linear extrapolation of the functional $A^\xi$ if the function $f_0(\lambda) + (1 + \lambda^2)^{-n} \lambda^2 n g(\lambda)$ determines a solution of the optimization problem (27). The function $h_{i,j}(f_0^i, g_0^j)$ calculated by formula (19), is the minimax robust spectral characteristic of the optimal estimate of the functional $A^\xi$.

In the case of estimating the functional $A^\xi$ based on the observations of the process $\xi(t)$ at time $t < 0$ without noise, we can formulate the following theorem.

THEOREM 5.4 Suppose that the spectral density $f_0(\lambda) \in D^0_f$ satisfies condition (28). The spectral density

$$
    f_0(\lambda) = \frac{|\lambda|^n \int_0^\infty ((\lambda^0)^{-1} D^* w(t)) e^{ix} dt}{a_1 |1 - e^{ix}|^n(1 + \lambda^2)^{n/2}}
$$

is the least favorable in the class $D = D^0_f$ for the optimal linear extrapolations of the functional $A^\xi$ based on observations of the process $\xi(t)$ at time $t < 0$ if it determines a solution of the optimization problem (29). The function $h_{i,j}(f_0^i, g_0^j)$ calculated by formula (22) is the minimax robust spectral characteristic of the optimal estimate of the functional $A^\xi$. 

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6. Least favorable densities in the class \( D = D^\nu \times D^\varrho \)

Let us consider the problem of minimax robust extrapolation of the functional \( A_\xi \) based on observations of the process \( \xi(t) + \eta(t) \) at time \( t < 0 \) on the set of admissible spectral densities \( D = D^\nu \times D^\varrho \), where

\[
D^\nu = \left\{ f(\lambda) | V(\lambda) \leq f(\lambda) \leq u(\lambda), \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) d\lambda = \mathcal{P}_1 \right\},
\]

\[
D^\varrho = \left\{ g(\lambda) | g(\lambda) = (1 - \varepsilon)g_1(\lambda) + \varepsilon W(\lambda), \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda) d\lambda = \mathcal{P}_2 \right\}.
\]

Here, the spectral densities \( u(\lambda), V(\lambda), g_1(\lambda) \) are supposed to be known and the spectral densities \( u(\lambda), V(\lambda) \) are assumed to be bounded.

Using the condition \( 0 \in \partial \Delta_p(f^0, g^0) \), we obtain the following equalities determining the spectral densities: \( f^0 \in D^\nu, g^0 \in D^\varrho \):

\[
\begin{align*}
A(\lambda)(1 - e^{i\xi^0(1 + \lambda^2)}f^0(\lambda) - \lambda^{2\eta}C^0(\lambda)) &= |1 - e^{i\xi}f^0(\lambda) + \lambda^{2\eta}g^0(\lambda)| \gamma_1(\lambda) + \gamma_2(\lambda) \\
+ \alpha_1, \quad A(\lambda)(1 - e^{i\xi^0(-i\lambda)^\eta}g^0(\lambda) + (-i\lambda)^\eta C^0(\lambda)) &= |1 - e^{i\xi}\eta f(\lambda) + \lambda^{2\eta}g^0(\lambda)|\beta(\lambda) + \alpha_2,
\end{align*}
\]

where the function \( \gamma_1(\lambda) \leq 0 \) and \( \gamma_2(\lambda) = 0 \) if \( f^0(\lambda) \geq v(\lambda) \); the function \( \gamma_2(\lambda) \geq 0 \) and \( \gamma_2(\lambda) = 0 \) if \( f^0(\lambda) \leq u(\lambda) \); and the function \( \beta(\lambda) \leq 0 \) and \( \beta(\lambda) = 0 \) if \( g^0(\lambda) \geq (1 - \varepsilon)g_1(\lambda) \).

**Theorem 6.1.** Let the spectral densities \( f^0(\lambda) \in D^\nu \) and \( g^0(\lambda) \in D^\varrho \) satisfy the minimality condition (8) and let the functions \( h_{f, g}(f^0, g^0) \) and \( h_{\varnothing, f}(f^0, g^0) \) determined by formulas (30) and (31) be bounded. The spectral densities \( f^0(\lambda) \) and \( g^0(\lambda) \) determined by equations (36) and (37) are the least favorable in the class \( D = D^\nu \times D^\varrho \) for the optimal linear extrapolations of the functional \( A_\xi \) if they determine a solution of the optimization problem (27). The function \( h_{f, g}(f^0, g^0) \) calculated by formula (19) is the minimax robust spectral characteristic of the optimal estimate of the functional \( A_\xi \).

**Theorem 6.2.** Let the spectral density \( f(\lambda) \) be known, let the spectral density \( g^0(\lambda) \in D^\varrho \) and let the spectral densities \( f(\lambda), g^0(\lambda) \) satisfy the minimality condition (8). Suppose also that the function \( h_{f, g}(f^0, g^0) \) determined by formula (31) is bounded. The spectral density

\[
g^0(\lambda) = \max \left\{ (1 - \varepsilon)g_2(\lambda), f^0(\lambda) - (1 + \lambda^2)^\eta \lambda^{2\eta}f(\lambda) \right\},
\]

where the function \( f_2(\lambda) \) is defined by formula (34), is the least favorable in the class \( D^\nu \) for the optimal linear extrapolation of the functional \( A_\xi \) if the functions \( f(\lambda) + (1 + \lambda^2)^{-\eta} \lambda^{2\eta}g^0(\lambda) \) determine a solution of the optimization problem (27). The function \( h_{f, g}(f^0, g^0) \) calculated by formula (19) is the minimax robust spectral characteristic of the optimal estimate of the functional \( A_\xi \).

**Theorem 6.3.** Let the spectral density \( g(\lambda) \) be known, let the spectral density \( f^0(\lambda) \in D^\nu \) and let the spectral densities \( f^0(\lambda), g(\lambda) \) satisfy the minimality condition (8). Suppose also that the function \( h_{f, g}(f^0, g) \) determined by formula (30) is bounded. The spectral density

\[
f^0(\lambda) = \min \left\{ u(\lambda), \max \left\{ v(\lambda), g^0(\lambda) - (1 + \lambda^2)^{-\eta} \lambda^{2\eta}g(\lambda) \right\} \right\},
\]

where the function \( g^0(\lambda) \) is defined by formula (35), is the least favorable in the class \( D^\nu \) for the optimal linear extrapolation of the functional \( A_\xi \) if the function \( f^0(\lambda) + (1 + \lambda^2)^{-\eta} \lambda^{2\eta}g^0(\lambda) \) determines a solution to optimization problem (27). The function \( h_{f, g}(f^0, g) \) calculated by formula (19) is the minimax robust spectral characteristic of the optimal estimate of the functional \( A_\xi \).
In the case of estimating the functional $A_0 \xi$ based on the observations of the process $\xi(t)$ at time $t < 0$ without noise, we can formulate the following theorem.

**THEOREM 6.4** Suppose that the spectral density $f_0(\lambda) \in D_0$ satisfies condition (28). The spectral density

$$f_0(\lambda) = \min\left\{ u(\lambda), \max\left\{ v(\lambda), \frac{|A|^2}{a} \right\} \right\}$$

is the least favorable in the class $D = D_0$ for the optimal linear extrapolations of the functional $A_0 \xi$ based on observations of the process $\xi(t)$ at time $t < 0$ if it determines a solution of the optimization problem (29). The function $h_1(f_0)$ calculated by formula (22) is the minimax robust spectral characteristic of the optimal estimate of the functional $A_0 \xi$.

7. Conclusions
In this paper, we present results of investigating of the problem of optimal linear estimation of the functional $A_0 \xi = \int_0^\infty a(t)\xi(t)dt$ and $A_0 \xi = \int_0^\infty a(t)\xi(t)dt$ which depend on the unknown values of a random process $\xi(t)$ with $n$th stationary increments based on observations of the process $\xi(t) + \eta(t)$ at time $t < 0$. In the case where the spectral densities of the processes are known, we found formulas for calculating the values of the mean square error and the spectral characteristics of the estimates of the functionals $A_0 \xi$ and $A_0 \xi$. In the case where the spectral densities are not exactly known, but a set of admissible spectral densities was available, we applied the minimax robust method to derive relations which determine the least favorable spectral densities from the given set and the minimax robust spectral characteristics.

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References
Dubovets’ka, I. I., Masyutka, O. Yu., & Moklyachuk, M. P. (2012). Interpolation of periodically correlated stochastic sequences. Theory of Probability and Mathematical Statistics, 84, 43–56.

Dubovets’ka, I. I. & Moklyachuk, M. P. (2013b). Filtration of linear functionals of periodically correlated sequences. Theory of Probability and Mathematical Statistics, 86, 51–64.

Dubovets’ka, I. I. & Moklyachuk, M. P. (2013b). Minimax estimation problem for periodically correlated stochastic processes. Journal of Mathematics and System Science, 2(1), 26–30.

Dubovets’ka, I. I. & Moklyachuk, M. P. (2014a). Extrapolation of periodically correlated processes from observations with noise. Theory of Probability and Mathematical Statistics, 88, 67–83.

Dubovets’ka, I. I., & Moklyachuk, M. P. (2014b). On minimax estimation problems for periodically correlated stochastic processes. Contemporary Mathematics and Statistics, 2, 123–150.

Franke, J. (1986). Minimax robust prediction of discrete time series. J. Wahrsc. Verw. Gebiete, 68, 337–346.

Franke, J., & Poor, H. V. (1984). Minimax-robust filtering and finite-length robust predictors. Robust and nonlinear time series analysis, Lecture notes in statistics (Vol. 26, pp. 87–126). Heidelberg, Springer-Verlag.

Gikhman, I. I., & Skorokhod, A. V. (2004). The theory of stochastic processes. I. Berlin: Springer.

Golichenko, I. I., & Moklyachuk, M. P. (2015). Estimates of functionals of periodically correlated processes. Kyiv: NVP “Interservis”.

Grenander, U. (1957). A prediction problem in game theory. Arkiv för Matematik, 3, 371–379.

Ioffe, A. D., & Tihomirov, V. M. (1979). Theory of extremal problems (p. 460). Amsterdam, North-Holland Publishing Company.

Karhunen, K. (1947). Uber lineare Methoden in der Wahrscheinlichkeitsrechnung. Annalen Academiae Scientiarum Fennicae. Series A I. Mathematica, 37, 3–79.

Kassam, S. A., & Poor, H. V. (1988). Robust techniques for signal processing: A survey. Proceedings of the IEEE, 76, 433–481.

Kolmogorov, A. N. (1992). Selected works of A. N. Kolmogorov. Volume II: Probability theory and mathematical statistics. Edited by A. N. Shiryaev. Dordrecht etc.: Kluwer Academic Publishers.

Luz, M. M., & Moklyachuk, M. P. (2012). Interpolation of functionals of stochastic sequences with stationary increments from observations with noise. Prykladna Statystyka. Aktuarna ta Finansova Matematyka, 2, 131–148.
Luz, M. M., & Moklyachuk, M. P. (2013a). Minimax-robust estimation technique for stationary stochastic processes (pp. 296). Saarbrücken, LAP LAMBERT Academic Publishing.

Pinsker, M. S., & Yaglom, A. M. (1956). On linear extrapolation of random processes with nth stationary increments. Doklady Akademii Nauk SSSR, 94, 385–388.

Pinsker, M. S. (1953). The theory of curves with nth stationary increments in Hilbert spaces. Izvestiya Akademii Nauk SSSR. Ser. Mat., 19, 319–344.

Pshenichnyi, B. N. (1971). Necessary conditions for an extremum. Pure and Applied mathematics 4 (Vol. XVIII, p. 230). New York: Marcel Dekker, Inc.

Rockafellar, R. T. (1997). Convex Analysis (p. 451). Princeton, NJ: Princeton University Press.

Rozanov, Y. A. (1967). Stationary stochastic processes. San Francisco, CA: Holden-Day.

Vostolo, K. S., & Poor, H. V. (1998). An analysis of the effects of spectral uncertainty on Wiener filtering. Automatica, 28, 289–293.

Wiener, N. (1966). Extrapolation, interpolation, and smoothing of stationary time series. With engineering applications. Massachusetts: The M. I. T. Press, Massachusetts Institute of Technology.

Yaglom, A. M. (1955). Correlation theory of stationary and related random functions. Supplementary notes and references (Vol. 2, p. 258). Springer Series in Statistics, New York (NY): Springer-Verlag.

Yaglom, A. M. (1957). Some classes of random fields in n-dimensional space related with random stationary processes. Teor. Veroyatn. Primen., 2, 289–338.

Yaglom, A. M. (1967a). Correlation theory of stationary and related random functions. Basic results (Vol. 1, p. 526). Springer series in statistics, New York (NY): Springer-Verlag.

Yaglom, A. M. (1987b). Correlation theory of stationary and related random functions. Supplementary notes and references (Vol. 2, p. 258). Springer Series in Statistics, New York (NY): Springer-Verlag.