Extremal generalized quantum measurements

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Abstract. A measurement on a section $K$ of the set of states of a finite dimensional C*-algebra is defined as an affine map from $K$ to a probability simplex. Special cases of such sections are used in description of quantum networks, in particular quantum channels. Measurements on a section correspond to equivalence classes of so-called generalized POVMs, which are called quantum testers in the case of networks. We find extremality conditions for measurements on $K$ and characterize generalized POVMs such that the corresponding measurement is extremal. These results are applied to the set of channels. We find explicit extremality conditions for two outcome measurements on qubit channels and give an example of an extremal qubit 1-tester such that the corresponding measurement is not extremal.

Keywords: section of the state space, measurement, generalized POVM, quantum channel, quantum 1-tester, extremality

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1 Introduction

The motivation for the present work comes from recent papers [6, 9], see also [15], where a general framework for description of quantum networks was developed in terms of positive matrices, also called quantum combs, satisfying a set of linear constraints. This description has been useful for some important applications, see e.g. [2, 3, 4, 5, 11, 16]. In particular, measurements on quantum networks are performed by a special kind of networks, which are represented by so-called quantum testers, [7, 21]. The extreme points of the set of testers were characterized in [14].

The set of all combs corresponding to a given type of a network forms (a multiple of) a section of the state space, which is an intersection of the set of all positive matrices with unit trace and a linear subspace. Motivated by this application, general sections of the state space were studied in [18]. A measurement on a section, or a generalized measurement, was defined as an affine map from the section to the probability simplex over the set of outcomes. It was proved that measurements are given by so-called generalized POVMs (positive
operator valued measures). In the case of quantum combs, the corresponding
generalized POVMs are exactly the quantum testers.

Since the set of generalized measurements is convex and compact, and since
figures of merit for optimization of such measurements are usually convex, it
is useful to determine the extreme points. Extremal generalized POVMs were
characterized in [19], but for general sections, in particular for quantum combs,
there may exist many generalized POVMs describing the same measurement.
This defines an equivalence relation on generalized POVMs, such that general-
ized measurements correspond precisely to the equivalence classes. This means
that an extremal generalized POVM does not necessarily give an extremal mea-
surement and, on the other hand, an extremal generalized measurement can
have non-extremal generalized POVMs in its equivalence class.

The aim of the present paper is to determine extremal generalized mea-
surements and to characterize generalized POVMs such that the corre-
sponding measurement is extremal. For this, we need to describe the largest support
projections for generalized POVMs in the same equivalence class. We also give
necessary and sufficient conditions on the support projections such that the gen-
eralized POVM is unique in its equivalence class. These results are then applied
to the simplest case of a network consisting of a quantum channel with qubit
input and output. Moreover, we find an example of an extremal quantum tester
such that the corresponding generalized measurement is not extremal.

2 Notations and preliminaries

Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $\mathcal{A}$ be a $C^*$-subalgebra in the
algebra $B(\mathcal{H})$ of all linear operators on $\mathcal{H}$. The identity in $\mathcal{A}$ will be denoted by
$I_\mathcal{A}$ and we fix the trace $\text{Tr}_\mathcal{A}$ on $\mathcal{A}$ to be the restriction of the trace $\text{Tr}_\mathcal{H}$ in $B(\mathcal{H})$,
we omit the subscript $\mathcal{A}$ if no confusion is possible. We denote by $\mathcal{A}^h$ the set of
all self-adjoint elements in $\mathcal{A}$ and by $\mathcal{A}^+$ the convex cone of positive elements in
$\mathcal{A}$. If $S \subset \mathcal{A}$ is an arbitrary subset, we use the notation $S^+ = S \cap \mathcal{A}^+$. If $p \in \mathcal{A}$
is a projection, we denote the compressed algebra $p\mathcal{A}p$ by $\mathcal{A}_p$. For $a \in \mathcal{A}^+$, the
projection onto the support of $a$ will be denoted by $s(a)$.

The dual space $\mathcal{A}^*$ is usually identified with $\mathcal{A}$, with duality given by $\langle a, b \rangle = \text{Tr}(a^*b)$. The functional determined by $a \in \mathcal{A}$ is hermitian if and only if $a \in \mathcal{A}^h$
and positive if and only if $a \in \mathcal{A}^+$. Positive unital functionals are called states
and are identified with density operators, that is, elements $\rho \in \mathcal{A}^+$ with $\text{Tr} \rho = 1$.
We denote the set of states by $\mathcal{S}(\mathcal{A})$. If $\mathcal{A} = B(\mathcal{H})$, we use the notation $I_\mathcal{H}$,
$\text{Tr}_\mathcal{H}$, $\mathcal{S}(\mathcal{H})$ etc., with obvious meaning.

Let $\mathcal{B}$ be another (finite dimensional) $C^*$-algebra, then $\text{Tr}_\mathcal{A}^{\mathcal{A} \otimes \mathcal{B}}$, or just $\text{Tr}_\mathcal{A}$,
will denote the partial trace on the tensor product $\mathcal{A} \otimes \mathcal{B}$, determined by $\text{Tr}_\mathcal{A}(a \otimes
b) = \text{Tr}(a)b$. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a linear map, then $T$ is positive if $T(\mathcal{A}^+) \subset \mathcal{B}^+$
and completely positive if the map

$$T \otimes \text{id}_\mathcal{L} : \mathcal{A} \otimes B(\mathcal{L}) \rightarrow \mathcal{B} \otimes B(\mathcal{L})$$
is positive for all finite dimensional Hilbert spaces $\mathcal{L}$. A channel $T : \mathcal{A} \rightarrow \mathcal{B}$ is a completely positive and trace preserving map.

Any linear map $T : \mathcal{A} \rightarrow \mathcal{B}$ is represented by a unique operator $X_T \in \mathcal{B} \otimes \mathcal{A}$, called the Choi matrix of $T$. This can be obtained as follows: If $\mathcal{A} = B(H)$, then we have
\[ X_T = T \otimes id_H (|\psi_H \rangle \langle \psi_H|), \quad |\psi_H \rangle = \sum_i |i \otimes i \rangle \in \mathcal{H} \otimes \mathcal{H} \]
where $|i\rangle$ denotes an orthonormal basis in $\mathcal{H}$. If $\mathcal{A} = \bigoplus_n B(H_n)$, then there are maps $T_n : B(H_n) \rightarrow \mathcal{B}$ such that $T(a) = T_n(a)$ for $a \in B(H_n)$, and $X_T = \bigoplus_n X_{T_n}$. The Choi matrix is positive if and only if $T$ is a completely positive map, and $T$ preserves trace if and only if $\text{Tr}_B X_T = I_A$. In this way, the set of quantum channels $\mathcal{A} \rightarrow \mathcal{B}$ is identified with the set
\[ \mathcal{C}(\mathcal{A}, \mathcal{B}) := \{ X \in \mathcal{B} \otimes \mathcal{A}, \text{Tr}_B X = I_A \}. \]

In particular, put $\mathcal{A} = B(H)$ and $\mathcal{B} = \mathbb{C}^m$ and let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a channel. Then $T$ restricts to an affine map from $\mathcal{G}(\mathcal{H})$ to the probability simplex over the set $\{1, \ldots, m\}$. Such maps are called measurements on $B(\mathcal{H})$. The Choi matrix of $T$ has the form $X_T = \sum_i |i\rangle \langle i| \otimes X_i$, with $X_i \in B(\mathcal{H})^+$ and $\sum_i X_i = I_H$. We have $T(\rho)(i) = \text{Tr} X_i^* \rho$, where $a^*$ denotes the transpose of $a$.

The relation $T \leftrightarrow X_T$ gives a one-to-one correspondence between measurements and positive operator valued measures (POVMs).

### 3 Generalized measurements and generalized POVMs

We will fix the following notations throughout the paper: $K$ will denote a closed convex subset of $\mathcal{G}(\mathcal{A})$, $Q := \bigcup_{\lambda \geq 0} \lambda K \subseteq \mathcal{A}^+$ the closed convex cone and $J := Q - Q \subseteq \mathcal{A}^h$ the real vector subspace generated by $K$. Then $Q$ satisfies $Q \cap -Q = \{0\}$, so that $Q$ defines a partial order in $J$.

The dual space $J^*$ of $J$ can be identified with the quotient of $\mathcal{A}^h$, $J^* \equiv \mathcal{A}^h / K^\perp$, where $K^\perp = J^\perp = \{ x \in \mathcal{A}^h, \text{Tr} ax = 0, \ a \in K \}$. Let $\pi : \mathcal{A}^h \rightarrow J^*$ be the quotient map, $\pi(a) = a + K^\perp$. Then the duality of $J$ and $J^*$ is given by
\[ \langle \pi(a), x \rangle = \text{Tr} ax, \quad a \in \mathcal{A}^h, \ x \in J. \]

A linear functional on $J$ is positive if its value is positive on every element of $Q$. The set of all positive functionals is the dual cone $Q^*$. This is a closed convex cone in $J^*$ and $Q^{**} = Q$. Since $Q \subseteq \mathcal{A}^+$, we always have $\pi(\mathcal{A}^+) \subseteq Q^*$.

**Theorem 1** [13] $Q^* = \pi(\mathcal{A}^+)$ if and only if $K = J \cap \mathcal{G}(\mathcal{A})$. 
Definition 1 A subset \( K \subseteq \mathcal{G}(A) \) satisfying \( K = J \cap \mathcal{G}(A) \) will be called a section of \( \mathcal{G}(A) \).

We will next show some important examples of sections.

Example 1 (Quantum channels) Let \( K = (\text{Tr} I_A)^{-1} \mathcal{C}(A,B) \), then it is not difficult to see that \( K \) is a section of \( \mathcal{G}(B \otimes A) \). In this case, \( J = \{ X \in (B \otimes A)^h : \text{Tr} B X = tI_A, t \in \mathbb{R} \} \) and \( K^\perp = I_B \otimes \{ I_A \}^\perp \); note that \( \{ I_A \}^\perp \) is the set of elements in \( A^h \) with zero trace.

Example 2 (Quantum supermaps) Quantum supermaps were introduced in [8] as completely positive maps that map the set \( \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) \) into \( \mathcal{C}(K_0, K_1) \). The following definition was used in [18].

Let \( B_0, B_1, \ldots \) be a sequence of finite dimensional \( C^* \) algebras. We denote by \( \mathcal{C}(B_0, B_1, \ldots, B_n) \) the set of Choi matrices of completely positive maps \( B_{n-1} \otimes \cdots \otimes B_0 \rightarrow B_n \) such that \( \mathcal{C}(B_0, \ldots, B_{n-1}) \) is mapped into \( \mathcal{G}(B_n) \). If \( n \) is odd, then \( \mathcal{C}(B_0, \ldots, B_n) \) corresponds to the set of (conditional) quantum combs, which are defined as Choi matrices of completely positive maps \( B_{n-1} \otimes \cdots \otimes B_1 \rightarrow B_n \otimes B_0 \), such that \( \mathcal{C}(B_1, \ldots, B_{n-1}) \) is mapped into \( \mathcal{C}(B_0, B_n) \). Quantum combs are used for representation of general quantum networks, [6] [9] [15]. The most general form of a conditional comb was introduced in [10].

As it was shown in [9], quantum supermaps are quantum channels whose Choi matrices satisfy a set of linear constraints. This implies that \( \mathcal{C}(B_0, \ldots, B_n) \) is again (a multiple of) a section of a state space.

Next we define a generalized quantum measurement with values in a finite set \( U \). Let \( \mathcal{P}(U) \) denote the probability simplex over \( U \).

Definition 2 Let \( K \subseteq \mathcal{G}(A) \) be a closed convex set. A generalized measurement on \( K \) with values in a (finite) set \( U \) is an affine map \( K \rightarrow \mathcal{P}(U) \). The set of all generalized measurements on \( K \) will be denoted by \( \mathcal{M}(K,U) \).

It is easy to see that any measurement \( m : K \rightarrow \mathcal{P}(U) \) is given by elements \( m_u \in Q^* \), \( m_u(\rho) := m(\rho)(u), \rho \in K, u \in U \), and we must have \( \sum_{u \in U} m_u = \pi(I) \). Let \( M_u \in A^h \) be such that \( m(M_u) = m_u \), then we have \( \sum_{u \in U} M_u \in \pi(I) = I + K^\perp \). Conversely, it is clear that any collection of positive operators satisfying this condition defines a generalized measurement.

Definition 3 [15] A generalized POVM (with respect to \( K \)) is a collection of positive operators \( \{ M_u, u \in U \} \) such that \( \sum_u M_u \in I + K^\perp \). The set of all generalized POVMs will be denoted by \( \mathcal{M}_K(A,U) \).

From now on, we will always assume that \( K \) is a section of \( \mathcal{G}(A) \). By Theorem [10] generalized measurements in this case correspond precisely to equivalence classes of generalized POVMs. If \( M = \{ M_u, u \in U \} \in \mathcal{M}_K(A,U) \), the corresponding measurement is \( \pi(M) := \{ \pi(M_u), u \in U \} \).
Example 3 (Quantum testers) Since $K = \text{Tr} \left( (I_A)^{-1} C(A,B) \right)$ is a section, any generalized measurement on the set of quantum channels is given by a generalized POVM with respect to $K$, multiplied by $\text{Tr} \left( (I_A)^{-1} \right)$. We obtain a collection $\{M_u\}$ of positive operators such that $\sum_u M_u = I_B \otimes \sigma$ for some $\sigma \in \mathcal{G}(A)$. Such objects were studied in [7, 9, 21] and called quantum 1-testers, or PPOVMs in [21]. We will denote the set of all quantum 1-testers by $\mathcal{C}(B, \ldots, B_n, \mathcal{C}(U))$ and hence are quantum supermaps themselves.

Let $M \in \mathcal{M}_K(A, U)$. The following decomposition was used for quantum testers in [7] and in [18, 19] for generalized POVMs: Let $\sum_u M_u = c$ and let $p = s(c)$. Let us define $\chi_c : A \to A$ by $a \mapsto c^{1/2} a c^{1/2}$. Then $\chi_c$ is completely positive and preserves trace on $A$. Moreover, restricted to the compressed algebra $A_p$, $\chi_c$ has an inverse $\chi_c^{-1} = \chi_c^{-1}$. Put $\Lambda_u = \chi_c^{-1} (M_u)$ (note that $s(M_u) \leq p$, so that $M_u \in A_p$). Then $\Lambda = \{\Lambda_u, u \in U\}$ is a POVM on $A_p$, such that

$$\text{Tr} M_u \rho = \text{Tr} \Lambda_u \chi_c(\rho), \quad u \in U, \ \rho \in K.$$ 

This decomposition will be written as $M = \Lambda \circ \chi_c$.

For quantum 1-testers, this has the following form [7, 21]: Suppose that $A = B(\mathcal{H})$ and let $M = \{M_u\}$, $M_u \in B \otimes B(\mathcal{H})$ be a quantum 1-tester, with $\sum_u M_u = I_B \otimes \sigma$, $\sigma \in \mathcal{G}(\mathcal{H})$. Let $\chi = \chi_{I_B \otimes \sigma}$ and let $q = s(\sigma)$ and $\mathcal{L} = q \mathcal{H}$, so that $M = \Lambda \circ \chi$ with a POVM $\Lambda = \{\Lambda_u, u \in U\}$ on $B \otimes B(\mathcal{L})$. Then for a channel $T : A \to B$ with Choi matrix $X_T$,

$$\chi(X_T) = (T \otimes id_{\mathcal{L}})(\xi)$$

where $\xi := (I_{\mathcal{H}} \otimes \sigma^{1/2}) |\psi_{\mathcal{H}}\rangle \langle \psi_{\mathcal{H}}| (I_{\mathcal{H}} \otimes \sigma^{1/2})$ is a pure state with $\text{Tr}_{\mathcal{L}} \xi = \sigma^T$. Hence the tester $M$ has an implementation $\text{Tr} M_u X_T = \text{Tr} \Lambda_u (T \otimes id_{\mathcal{L}})(\xi)$.

4 Extremality conditions

Since $K$ contains an element with largest support, we may always assume that $K$ contains an element of full rank, by restriction to a compressed algebra. In this case, it follows from [18, Proposition 6] that $\mathcal{M}_K(A, U)$ is compact, and obviously also convex. The extreme points of the set of testers were obtained in [15] and for generalized POVMs in [19]. The proposition below summarizes some of the results.

Let $M = \Lambda \circ \chi_c$ with $p := s(c)$. Let $K_c := \chi_c(J) \cap \mathcal{G}(A)$. It is clear that the POVM $\Lambda$ is a generalized POVM with respect to any section, hence in particular, $\Lambda \in \mathcal{M}_{K_c}(A_p, U)$.

Proposition 1 [15]

(i) Let $p_u = s(M_u)$. Then $M$ is extremal in $\mathcal{M}_K(A, U)$ if and only if for any collection of operators $D_u \in A_p^\perp$, $\sum_u D_u \in J \perp$ implies that $D_u = 0$ for all $u \in U$. 

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(ii) $M$ is extremal in $\mathcal{M}_K(\mathcal{A}, U)$ if and only if $\Lambda$ is extremal in $\mathcal{M}_{K_u}(\mathcal{A}_p, U)$.

(iii) Let $M$ be a quantum 1-tester, $M \in T(\mathcal{H}, K, U)$, and let $c = I_K \otimes \sigma$ for $\sigma \in \mathcal{S}(\mathcal{H})$, $s(\sigma) = q$. Then $M$ is extremal in $T(\mathcal{H}, K, U)$ if and only if $\text{Tr}(q)^{-1}M$ is extremal in $T(q\mathcal{H}, K, U)$.

(iv) If $|U| = 2$ in (iii), so that $M$ is a 1-tester with 2 outcomes, then $M$ is extremal if and only if $\Lambda = (\Lambda_1, \Lambda_2)$ with $\Lambda_1$ a projection not commuting with any projection of the form $I_K \otimes e$ with $e \neq 0, q$.

Remark 1 Note that if $K = \mathcal{S}(\mathcal{A})$, hence in the case of ordinary POVMs, the condition (i) becomes weak independence of the supports of $M_u, u \in U$. This extremality condition for POVMs was obtained by Arveson [1] in a very general infinite dimensional setting. In finite dimensions, this condition was proved by a perturbation method in [12].

Example 4 For $\dim(\mathcal{H}) = 2$, all extreme points in $T(\mathcal{H}, K, \{0, 1\})$ can be characterized as follows [14] [19]: Let $M = \Lambda \circ \chi_{I \otimes \sigma}$. If $\text{rank}(\sigma) = 1$, then $M$ is extremal if and only if $M$ is a PVM. If $\text{rank}(\sigma) = 2$, then $M$ is extremal if and only if $\Lambda$ is a PVM and $\Lambda_0$ (and hence also $\Lambda_1$) is not of the form

$$\Lambda_0 = e \otimes |\psi\rangle\langle\psi| + f \otimes |\psi^\perp\rangle\langle\psi^\perp|,$$

where $\psi, \psi^\perp \in \mathcal{H}$ are orthogonal unit vectors and $e, f$ are projections on $K$.

We now turn to extremality conditions for generalized measurements. Since the set $\mathcal{M}(K, U)$ is the image of $\mathcal{M}_K(\mathcal{A}, U)$ under the linear map $\pi$, it must be convex and compact as well. We will now characterize the extreme points. First, let $a$ be any element in $Q^*$, so that $a = \pi(a)$ for some $a \in A^+$. Consider the set $(a + K^\perp)^+$ of all positive elements in the equivalence class of $a$. Since this is a closed convex subset in $A^+$, it contains some element $b$ with largest support. Let us denote $s(a) := s(b)$.

Theorem 2 Let $m \in \mathcal{M}(K, U)$ and let $s(m_u) = s_u, u \in U$. Then $m$ is extremal if and only if for any collection $\{x_u \in \pi(A_{h_u}^+), u \in U\}$, $\sum_u x_u = \pi(0)$ implies that $x_u = \pi(0)$ for all $u \in U$.

Proof. The proof uses the standard perturbation method of convex analysis. So let us suppose that $m$ is extremal in $\mathcal{M}(K, U)$ and let $\{x_u \in \pi(A_{h_u}^+), u \in U\}$ be such that $\sum_u x_u = \pi(0)$. Let $M_u$ be such that $\pi(M_u) = m_u$ and $s(M_u) = s_u$; and choose any $X_u \in A_{h_u}$ such that $\pi(X_u) = x_u$. Then there is some $s > 0$ satisfying $M_{\pm,u} := M_u \pm sX_u \in A^+$, for all $u \in U$. Hence $m_{\pm,u} := \pi(M_{\pm,u}) \in Q^*$, moreover, $\sum_u m_{\pm,u} = \sum_u m_u = \pi(I)$. Since $m = \frac{1}{2}(m_+ + m_-)$ and $m$ is extremal, this implies that $m_+ = m_-$, hence $2sX_u = M_{+,u} - M_{-,u} \in J^\perp$, so that $x_u = \pi(0)$ for all $u \in U$.

Conversely, suppose the condition is satisfied and let $m = \frac{1}{2}(m_1 + m_2)$, $m_i$ be generalized POVMs such that $\pi(M^i) = m^i, i = 1, 2$, then $\pi(\frac{1}{2}(M^1 + M^2)) = m$, therefore $s(M^i) \leq s(M^1_u + M^2_u) \leq s(m_u)$, so that $M^i_u \in A_{h_u}^+$. 

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Hence \( x_u = \pi(M_u^1 - M_u^2) \in \pi(A_{\pi_u}^b) \), moreover, \( \sum_u x_u = \pi(0) \). It follows that \( x_u = \pi(0) \) for all \( u \) and hence \( m^1 = \pi(M^1) = \pi(M^2) = m^2 \).

\[ \square \]

**Corollary 1** Let \( m \in \mathcal{M}(K, U) \) be extremal and let \( s(m_u) = s_u \). Then

\[ \sum_u \dim(s_u J s_u) \leq \dim(J). \]

**Proof.** Let us denote \( J_U^* = \oplus_{u \in U} J_u^* \) and let \( L = \{ x \in J_U^*; \sum_u x_u = \pi(0) \} \). Then the extremality condition in Theorem 2 has the form

\[ \oplus_{u \in U} \pi(A_{s_u}^h) \cap L = \{ \pi(0)_U \}, \]

where \( \pi(0)_U \) is the zero element in \( J_U^* \). By taking orthogonal complements, we obtain that

\[ (\oplus_{u \in U} \pi(A_{s_u}^h)) \perp \perp L^\perp = J_U^*, \]

which implies that

\[ \sum_u \dim(\pi(A_{s_u}^h)) = \dim(\oplus_u \pi(A_{s_u}^h)) \leq \dim(L^\perp). \]

Note that \( s_u J s_u \) is a subspace in \( A_{s_u}^h \) and it is easy to see that \( (s_u J s_u)^\perp \cap A_{s_u}^h = J^\perp \cap A_{s_u}^h \). As before, we may identify the dual space with the quotient space \( (s_u J s_u)^\perp A_{s_u}^h |_{J^\perp \cap A_{s_u}^h} \). Let \( \pi_u \) be the quotient map, then for elements \( x, y \in A_{s_u}^h, \pi(x) = \pi(y) \) if and only if \( \pi_u(x) = \pi_u(y) \). It follows that

\[ \dim(\pi(A_{s_u}^h)) = \dim(\pi_u(A_{s_u}^h)) = \dim((s_u J s_u)^\perp) = \dim(s_u J s_u). \]

Moreover, it is easy to check that \( L^\perp = \{ y \in J_U := \oplus_u J, y_u = y_v, u, v \in U \} \equiv J, \)
so that \( \dim(L^\perp) = \dim(J) \).

Putting this together, we obtain the statement.

\[ \square \]

We now characterize generalized POVMs corresponding to an extremal measurement. For \( a \in A^+ \), let \( s_K(a) := s(\pi(a)) \), then \( s_K(a) \) is the largest support of an element in \( (a + K^\perp)^+ \). We call \( s_K(a) \) the \( K \)-support of \( a \). The next statement follows directly from Theorem 2 (compare with Proposition 1 (i)).

**Theorem 3** Let \( M \in \mathcal{M}(A, U) \) and let \( s_u = s_K(M_u), u \in U \). Then \( \pi(M) \) is extremal in \( \mathcal{M}(K, U) \) if and only if for any \( \{ D_u \in A_{s_u}^h, u \in U \} \), \( \sum_u D_u \in K^\perp \) implies that \( D_u \in K^\perp \) for all \( u \in U \).

To make the above characterization more useful, we need to describe the \( K \)-supports of positive elements in \( A \).

**Proposition 2** Let \( a \in A^+ \). Then \( s(a) = s_K(a) \) if and only if there is an element \( b \in Q \) such that \( s(a) = I - s(b) \).
Proof. Let \( p = s(a) \). Suppose \( p = s_K(a) \). Note that we have
\[
\{ x \in \mathcal{A}^h, \exists t > 0, a + tx \in \mathcal{A}^+ \} = \mathcal{A}^+ + \mathcal{A}^h_p
\] (2)

Let now \( x \in (\mathcal{A}^+ + \mathcal{A}^h_p) \cap K^\perp \), then (2) implies that there is some \( t > 0 \) such that \( a_0 := a + tx \in (a + K^\perp)^+ \subseteq \mathcal{A}^+_p \), since \( p = s_K(a) \). Hence \( x = t^{-1}(a_0 - a) \in \mathcal{A}^h_p \), so that \((\mathcal{A}^+ + \mathcal{A}^h_p) \cap K^\perp \subseteq \mathcal{A}^h_p \cap K^\perp \). Since the converse inclusion is clear, we have
\[
(\mathcal{A}^+ + \mathcal{A}^h_p) \cap K^\perp = \mathcal{A}^h_p \cap K^\perp
\]
Applying the duality * of the convex cones to this equality, we get ([20 Corollary 11.4.2])
\[
cl((\mathcal{A}^+ + \mathcal{A}^h_p)^* + J) = cl(\mathcal{A}^h_{I-p} + J).
\]
Since \( \mathcal{A}^h_{I-p} + J \) is an affine subspace, we may remove the closure operator and we get from \((\mathcal{A}^+ + \mathcal{A}^h_p)^* = \mathcal{A}^+ \cap \mathcal{A}^h_{I-p} = \mathcal{A}^h_{I-p} \) that
\[
\mathcal{A}^h_{I-p} + J = \mathcal{A}^h_{I-p} + J
\]
In particular, \( \mathcal{A}^h_{I-p} \subseteq \mathcal{A}^h_{I-p} + J \). Let \( c \in \mathcal{A}^+ \), \( s(c) = I - p \), then there are some \( d \in \mathcal{A}^h_{I-p} \) and \( x \in J \) such that \(-c = d + x \). But then \( b := c + d \in Q \), \( s(b) = I - p \).

Conversely, let \( b \in Q \), \( s(b) = I - p \), then
\[
\mathcal{A}^+_p = \{ b \}^\perp \cap \mathcal{A}^+ \supseteq (\mathcal{A}^+_p + K^\perp)^+ \supseteq \mathcal{A}^+_p
\]
so that \( \mathcal{A}^+_p = (\mathcal{A}^+_p + K^\perp)^+ \). In particular, \((a + K^\perp)^+ \subseteq \mathcal{A}^+_p \), so that \( p = s_K(a) \).
\[\square\]

Let us denote
\[
\mathcal{P}_K(A) := \{ I - s(b), b \in Q \}.
\]
Note that for any subset \( \mathcal{P} \subset \mathcal{P}_K(A) \), we have \( \bigwedge \mathcal{P} \in \mathcal{P}_K(A) \), so that \( \mathcal{P}_K(A) \) is a \( \wedge \)-complete semilattice. Indeed, let \( R \subset Q \) be the set of elements such that \( \mathcal{P} = \{ I - s(a), a \in R \} \), then there is some \( b \in b \) in the closed convex hull of \( R \), such that \( s(b) = \bigvee \{ s(a), a \in R \} \) so that \( I - s(b) = \bigwedge \{ I - s(a), a \in R \} = \bigwedge \mathcal{P} \). Since \( b \in Q \), we have \( I - s(b) \in \mathcal{P}_K \).

Proposition 3 Let \( a \in \mathcal{A}^+ \). Then \( s_K(a) = \bigwedge \{ s \in \mathcal{P}_K(A), s(a) \leq s \} \).

Proof. By Proposition 2 \( s_K(a) \in \mathcal{P}_K(A) \) and \( s(a) \leq s_K(a) \) by definition. Let \( s' \) be another such projection, with \( 1 - s' = s(b'), b' \in Q \). Then \( \text{Tr} b'd = \text{Tr} b'a = 0 \) for all \( d \in (a + K^\perp)^+ \). This implies \( s(d) \leq 1 - s(b') = s' \), so that \( s_K(a) \leq s' \).
\[\square\]

Remark 2 Let \( p \in \mathcal{A} \) be a projection, then it is easy to see that the set \( \{ a \in Q^*, s(a) \leq p \} \) is a face of \( Q^* \). Conversely, any face of \( Q^* \) has this form: if
$F \subset Q^*$ is a face, then $\pi^{-1}(F) \cap \mathcal{A}^+$ is a face of $\mathcal{A}^+$, hence $\pi^{-1}(F) \cap \mathcal{A}^+ = \mathcal{A}_p^+$ for some projection $p$. Consequently, $F = \pi(\mathcal{A}_p^+) = \{a \in Q^*, s(a) \leq p\}$. By Proposition 2, there is a 1-1 correspondence between faces of $Q^*$ and $\mathcal{P}_K(\mathcal{A})$.

Similarly, faces of $\mathcal{M}(K, U)$ are the sets $\{m \in \mathcal{M}(K, U), s(m_u) \leq p_u, u \in U\}$ for some projections $p_u$ and there is a 1-1 correspondence between faces of $\mathcal{M}(K, U)$ and $U$-tuples $\{p_u, u \in U\}$, $p_u \in \mathcal{P}_K$. In particular, $\mathcal{M}_p(K, U) := \{m \in \mathcal{M}(K, U), s(m_u) \leq p, u \in U\}$ is a face of $\mathcal{M}(K, U)$, for any projection $p$.

Suppose that $M \in \mathcal{M}_K(\mathcal{A}, U)$ has the decomposition $M = \Lambda \circ \chi_c$, with $p = s(c)$. We now relate extremality of the measurement given by $\Lambda$ to extremality of the measurement given by $\Lambda$, cf. Proposition 5 (ii). Let $\Lambda : \mathcal{M}_p(K, U) \to \mathcal{M}_p(K, U)$ be the corresponding quotient map $\pi_\Lambda : \mathcal{A}_p^+ \to \mathcal{A}_p^+|_{\chi_c(J)^+}$. Note that we have $p = \vee_u s(M_u) \leq \vee_u s(K_u)$.

**Theorem 4** Suppose that $p = \vee_u s(K_u)$. Then $\pi(M)$ is extremal in $\mathcal{M}(K, U)$ if and only if $\pi_\Lambda(\Lambda)$ is extremal in $\mathcal{M}(\Lambda, U)$.

**Proof.** Let $a \in \mathcal{A}^+$ be any element such that $s(K)(a) \leq p$. Then $(a + K^+)^+ \subset \mathcal{A}_p^+$, so that $(a + K^+)^+ = (a + (K^+ \cap \mathcal{A}_p^+))^+$ and it is easy to check that $\chi_c^{-1}(K^+ \cap \mathcal{A}_p^+) = K_c^+ \cap \mathcal{A}_p^+$. It follows that $\chi_c^{-1}$ maps $(a + K^+)^+$ onto $(\chi_c^{-1}(a) + K_c^+)^+$ and hence $a \mapsto \pi_\Lambda(\chi_c^{-1}(a))$ defines an invertible affine map from the face $\{a \in Q^*, s(a) \leq p\}$ onto $Q^*_c$.

Moreover, let $d \in (I + K^+) \cap \mathcal{A}_p^+$ and let $\rho_c \in K_c$, so that $\rho_c$ is a state of the form $\rho_c = \chi_c(x)$ for some $x \in J$ and we have $1 = \text{Tr} \rho_c = \text{Tr} xc = \text{Tr} x$ (since $c \in I + K^+$). Then

$$\text{Tr} \chi_c^{-1}(d) \rho_c = \text{Tr} dpxp = \text{Tr} dx = \text{Tr} x = 1,$$

hence $\chi_c^{-1}(d) \in (p + K^+)^+$. By a similar argument, we can see that $\chi_c(p + K^+)^+ = (I + K^+) \cap \mathcal{A}_p^+$.

From this, one can see that $\chi_c^{-1}$ defines an affine invertible map from the face $\mathcal{M}_p(K, U)$ onto $\mathcal{M}(K_c, U)$, see Remark 2. Since $\pi(M) \in \mathcal{M}_p(K, U)$ and $\chi_c^{-1}(\pi(M)) = \pi_\Lambda(\Lambda)$, the statement follows.

\[\square\]

### 4.1 Equivalence of generalized POVMs

In this paragraph, we deal with the question whether a given generalized POVM $M$ is the unique element in its equivalence class. For this, it is enough to characterize the situation when $(a + K^+)^+ = \{a\}$ for $a \in \mathcal{A}^+$.

**Lemma 1** Let $a \in \mathcal{A}^+$, then $(a + K^+)^+$ is convex and compact.

**Proof.** It is enough to show that $(a + K^+)^+$ is bounded in some norm. Let $\rho \in K$ be of full rank, then for all $b \in (a + K^+)^+$, $\|b^{1/2} \rho b^{1/2}\|_1 = \text{Tr} \rho b = \text{Tr} \rho a =: t$. Hence $\|b\|_1 \leq \|\rho^{-1}\|t$ and $(a + K^+)^+$ is bounded.

\[\square\]
Lemma 2 Let \( a \in \mathcal{A}^+ \), \( s = s(a) \). Then \( a \) is extremal in \((a + K^\perp)^+\) if and only if \( \dim(s(J_s)) = \dim(A^+_{\mathcal{B}}) \).

Proof. By applying the perturbation method, it is easily seen that \( a \) is extremal in \((a + K^\perp)^+\) if and only if \( K^\perp \cap A^+_{\mathcal{B}} = \{0\} \). Since \( K^\perp \cap A^+_{\mathcal{B}} = (s(J_s))^\perp \cap A^+_{\mathcal{B}} \), this is equivalent with \( \dim(s(J_s)) = \dim(A^+_{\mathcal{B}}) \).

\[ \Box \]

Proposition 4 Let \( a \in \mathcal{A}^+ \), \( s = s(a) \). Then \( a \) is the unique positive element in its equivalence class if and only if \( s \in \mathcal{P}_K(\mathcal{A}) \) and \( \dim(s(J_s)) = \dim(A^+_{\mathcal{B}}) \).

Proof. The conditions say that \( a \) is an extreme point in \((a + K^\perp)^+\) such that \( s(a) \) contains the supports of all other elements in \((a + K^\perp)^+\). This happens if and only if \( a \) is the unique point in this set.

\[ \Box \]

5 Extremal measurements on qubit channels

We now apply the results of the previous section to the set \( \mathcal{T}(\mathcal{H}, \mathcal{K}, U) \) of quantum 1-testers for finite dimensional Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \). Let \( c = I_K \otimes \sigma \), \( \sigma \in \mathcal{S}(\mathcal{H}) \) and let \( J \) be as in Example I. Then

\[ K_c = \chi_c(J) \cap \mathcal{S}(\mathcal{K} \otimes \mathcal{H}) = \{ \rho \in \mathcal{S}(\mathcal{K} \otimes \mathcal{H}), \text{Tr}_K \rho = \sigma \}. \]

Note that for \( \sigma = \text{dim}(\mathcal{H})^{-1}I_\mathcal{H} \), \( K_c = K = \text{dim}(\mathcal{H})^{-1}C(\mathcal{H}, \mathcal{K}) \). Let \( p \neq I \) be a projection on \( \mathcal{K} \otimes \mathcal{H} \). Then one can see from the definition that \( p \in \mathcal{P}_K(\mathcal{K} \otimes \mathcal{H}) \) if and only if there are one-dimensional projections \( p_i = |\phi_i\rangle\langle\phi_i|, i = 1, \ldots, k \) such that \( 1 - p = \vee_i p_i \) and the convex hull \( \text{co}\{\text{Tr}_K p_1, \ldots, \text{Tr}_K p_k\} \) contains \( \sigma \). In particular, if \( 1 - p = |\phi\rangle\langle\phi| \), then \( p \in \mathcal{P}_K(\mathcal{K} \otimes \mathcal{H}) \) if and only if \( |\phi\rangle\langle\phi| \) is maximally entangled.

While it is not easy to describe the sets \((a + K^\perp)^+\), we can establish the following simple facts.

Lemma 3 Let \( a \in B(\mathcal{K} \otimes \mathcal{H})^+ \) and let \( c = I_K \otimes \sigma \) with \( \sigma \in \mathcal{S}(\mathcal{H}) \) of full rank.

(i) If \( \text{rank}(a) < \text{dim}(\mathcal{K}) \) then \((a + K^\perp)^+ = \{a\} \).

(ii) If \( \text{rank}(a) < 2\text{dim}(\mathcal{K}) \) then \( a \) is an extreme point in \((a + K^\perp)^+\).

(iii) If \( \text{rank}(a)^2 > \text{dim}(\mathcal{H})^2 \dim(\mathcal{K})^2 - \text{dim}(\mathcal{H})^2 + 1 \), then \((a + K^\perp)^+ \neq \{a\} \).

Proof. It is clear that \( K^\perp = I_K \otimes \{\sigma\}^\perp \).

(i) Suppose \( b \in (a + K^\perp) \), \( b \geq 0 \) and \( b \neq a \), then there is some nonzero \( y \in \{\sigma\}^\perp \), such that \( b = a + I_K \otimes y \). Let \( y = y_+ - y_- \) be the decomposition of \( y \) into its positive and negative part, that is, \( y_+ \in B(\mathcal{H})^+ \) and \( s(y_+)s(y_-) = 0 \). Then we have

\[ I_K \otimes y_- \leq a + I_K \otimes y_+ \]
Since we have positive elements on both sides, this implies that

\[ I_K \otimes s(y_-) \leq s(a + I_K \otimes y_+) = s(a) \vee (I_K \otimes s(y_+)) \]

and since \( s(y_+) \) and \( s(y_-) \) are orthogonal projections, we must have \( \text{rank}(I_K \otimes s(y_-)) \leq \text{rank}(s(a)) \). It follows that

\[ \text{rank}(a) = \text{rank}(s(a)) \geq \text{rank}(I_K \otimes s(y_-)) \geq \dim(K). \]

Hence \( \text{rank}(a) < \dim(K) \) implies \( (a + K^+) = \{a\} \).

(ii) Let \( s = s(a) \) and let \( 0 \neq z \in B(s(K \otimes H))^h \cap K^+ \), then \( z = I_K \otimes y \) for \( y \in \{\sigma\}^\perp \). Since \( y \neq 0 \), \( \text{rank}(y) \) must be at least 2. Hence \( \text{rank}(z) \geq 2 \dim(K) \) and we must have \( \text{rank}(a) = \text{rank}(s) \geq \text{rank}(z) \). Hence \( \text{rank}(a) < 2 \dim(K) \) implies that \( B(s(K \otimes H))^h \cap K^+ = \{0\} \). By the proof of Lemma 2 \( a \) is then an extreme point in \( (a + K^+) \).

(iii) Let \( J_c = \chi_c(J) \), then \( J_c \) is the real linear span of \( K_c \). By Proposition 4 \( (a + K^+) = \{a\} \) implies

\[ \text{rank}(a)^2 = \dim(B(s(K \otimes H))^h) = \dim(sJ_c \sigma) \leq \dim(J) \]

and \( \dim(J) = \dim(B(K \otimes H)^h) - \dim(J^\perp) = \dim(H)^2 \dim(K)^2 - \dim(H)^2 + 1 \).

Lemma 4 Suppose \( \dim(H) = \dim(K) = 2 \) in the previous lemma. Then \( (a + K^+) \neq \{a\} \) if and only if \( s_{K,c}(a) = I \).

Proof. Let \( b \in (a + K^+) \) be such that \( s(b) = s_{K,c}(a) \). If \( \text{rank}(b) < 4 \) then by Lemma 2 (ii), \( b \) is an extreme point in \( (b + K^+) \), so that \( (a + K^+) = (b + K^+) \) has exactly one element. The converse follows by Lemma 3 (iii).

We can now characterize extremal generalized measurements for the set of qubit channels.

Proposition 5 Let \( \dim(H) = \dim(K) = 2, M \in \mathcal{T}(H,K,U) \). Then \( \pi(M) \) is extremal if and only if \( M \) is extremal in \( \mathcal{T}(H,K,U) \) and \( s(M_u) \in \mathcal{P}_K(K \otimes H) \) for all \( u \).

Proof. Suppose that \( \pi(M) \) is extremal. Then Theorem 4 implies that \( s_K(M_u) \) cannot be equal to \( I_K \otimes H \) for any \( u \). By Lemma 4 this implies \( s_{K+c}(M_u) = \{M_u\} \). It follows that \( M \) is an extremal 1-tester and \( s(M_u) = s_K(M_u) \in \mathcal{P}_K(K \otimes H) \) for all \( u \).

Conversely, extremality of \( M \) and the fact that \( s(M_u) = s_K(M_u) \) imply that \( M \) is unique in its equivalence class and hence the corresponding measurement must be extremal as well.

We will next characterize extremality of \( \pi(M) \) in terms of the implementing POVM. So let \( M = \Lambda \circ \chi, \chi = \chi_{I \otimes \sigma} \), be the decomposition of \( M \) and let \( q = s(\sigma) \).
**Corollary 2** \( \pi(M) \) is an extremal measurement on qubit channels if and only if \( \text{Tr} (q)^{-1} \Lambda \) is extremal in \( T(q)H, K, U \) and \( s(\Lambda_u) \in P_{K_c}(K \otimes qH) \) for all \( u \in U \).

**Proof.** Suppose first that \( \sigma = |\phi\rangle\langle \phi| \) for some \( \phi \in H \). Then \( M_u = N_u \otimes |\phi\rangle\langle \phi| = \Lambda_u \) for some POVM \( N_u \) on \( B(K) \) and \( K_c = \mathcal{S}(K \otimes |\phi\rangle) \), so that the assertion follows by Proposition 5.

If \( \text{rank}(\sigma) = 2 \), then the assertion follows by Theorem 4, Proposition 1 (iii) and Lemma 4 similarly as in the proof of Proposition 5.

The next Example shows an extremal qubit 1-tester, such that the corresponding measurement is not extremal.

**Example 5** Let us apply the above results to the case of two outcomes. Let \( M = (M_1, M_2) \) be a qubit 1-tester with \( M_1 + M_2 = I \otimes \sigma \), where \( \text{rank}(\sigma) = 2 \) and let \( M = \Lambda \circ \chi \). Then by Corollary 2 and Example 4 \( \pi(M) \) is extremal if and only if \( \Lambda_1, \Lambda_2 \in P_{K_c}(K \otimes H) \) and \( \Lambda_1 \) is not of the form 4. In particular, suppose that \( \Lambda_1 = |\phi\rangle\langle \phi| \), then \( \pi(M) \) is extremal if and only if \( \text{Tr}_K(|\phi\rangle\langle \phi|) = \sigma \).

Indeed, this means \( \Lambda_2 \in P_{K_c}(K \otimes H) \) and \( \Lambda_1 \in P_{K_c}(K \otimes H) \) by Lemma 3 (i).

Since \( \sigma \) has full rank, \( \phi \) is not a product vector and hence \( M \) is an extremal 1-tester, by Example 4. The converse is clear. In particular, if \( \sigma = 1/2I_H \), then \( M \) is extremal if and only if the vector \( \phi \) is maximally entangled.

On the other hand, if \( \phi \) is not a product vector but also not maximally entangled, then \( M \) is an extremal qubit 1-tester such that \( \pi(M) \) is not extremal.

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