Near approximations via general ordered topological spaces
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Abstract
Rough set theory is a new mathematical approach to imperfect knowledge. The notion of rough sets is generalized by using an arbitrary binary relation on attribute values in information systems, instead of the trivial equality relation. The topology induced by binary relations is used to generalize the basic rough set concepts. This paper studies near approximation via general ordered topological approximation spaces which may be viewed as a generalization of the study of near approximation from the topological view. The basic concepts of some increasing (decreasing) near approximations, increasing (decreasing) near boundary regions and increasing (decreasing) near accuracy were introduced and sufficiently illustrated. Moreover, proved results, implications and add examples.

1. Introduction
The concept of rough set has many applications in data analysis. Topology [5], one of the most important subjects in mathematics, provides mathematical tools and interesting topics in studying information systems and rough sets [2,7,8,11,12,13]. The purpose of this paper is to put a starting point for the applications of ordered topological spaces into rough set analysis. Rough set theory introduced by Pawlak in 1982, is a mathematical tool that supports the uncertainty reasoning. Rough sets generalized by many ways [3,6,9,15]. In this paper, we give a general study of $\alpha, P$ approximations, which studied in [1]. Our results in this paper became the results, which obtained before in case of taking the partially ordered relation as an equal relation.

2. Preliminaries
In this section, we give an account for the basic definitions and preliminaries to be used in the paper.

Definition 2.1[10]. A subset $A$ of $U$, where $(U, \rho)$ is a partially ordered set is said to be increasing (resp. decreasing) if for all $a \in A$ and $x \in U$ such that $a \rho x$ (resp. $x \rho a$) imply $x \in A$.

Definition 2.2[10]. A triple $(U, \tau, \rho)$ is said to be a topological ordered space, where $(U, \tau)$ is a topological space and $\rho$ is a partially order relation on $U$. 
Definition 2.3[11]. An information system is a pair \((U,A)\), where \(U\) is a non-empty finite set of objects and \(A\) is a non-empty finite set of attributes.

Definition 2.4[4]. A non-empty set \(U\) equipped with a general relation \(R\) which generate a topology \(\tau_R\) on \(U\) and a partially order relation \(\rho\) wright as \((U,\tau_R,\rho)\) is said to be general ordered topological approximation space (for short, GOTAS).

Definition 2.5[4]. Let \((U,\tau_R,\rho)\) be a GOTAS and \(A \subseteq U\). We define:

1. \(\overline{R}^{\text{Inc}}(A) = A^{\text{Inc}}\), \(A^{\text{Inc}}\) is the greatest increasing open subset of \(A\).
2. \(\overline{R}^{\text{Dec}}(A) = A^{\text{Dec}}\), \(A^{\text{Dec}}\) is the greatest decreasing open subset of \(A\).
3. \(\overline{\overline{R}}^{\text{Inc}}(A) = \overline{A}^{\text{Inc}}\), \(\overline{A}^{\text{Inc}}\) is the smallest increasing closed superset of \(A\).
4. \(\overline{\overline{R}}^{\text{Dec}}(A) = \overline{A}^{\text{Dec}}\), \(\overline{A}^{\text{Dec}}\) is the smallest decreasing closed superset of \(A\).

5. \(\alpha^{\text{Inc}} = \frac{\text{card}(R^{\text{Inc}}(A))}{\text{card}(\overline{R}^{\text{Inc}}(A))}\) (resp. \(\alpha^{\text{Dec}} = \frac{\text{card}(R^{\text{Dec}}(A))}{\text{card}(\overline{R}^{\text{Dec}}(A))}\)) and \(\alpha^{\text{Inc}}\) (resp. \(\alpha^{\text{Dec}}\)), is \(R\)– increasing (resp. decreasing) accuracy.

Definition 2.6[4]. Let \((U,\tau_R,\rho)\) be a GOTAS and \(A \subseteq U\). We define:

1. \(\underline{S}^{\text{Inc}}(A) = A \cap \overline{R}^{\text{Inc}}(\overline{R}^{\text{Inc}}(A))\), \(\underline{S}^{\text{Inc}}(A)\) is called \(R\)– inc semi lower.
2. \(\overline{S}^{\text{Inc}}(A) = A \cup R^{\text{Inc}}(\overline{R}^{\text{Inc}}(A))\), \(\overline{S}^{\text{Inc}}(A)\) is called \(R\)– inc semi upper.
3. \(\underline{S}^{\text{Dec}}(A) = A \cap \overline{R}^{\text{Dec}}(\overline{R}^{\text{Dec}}(A))\), \(\underline{S}^{\text{Dec}}(A)\) is called \(R\)– dec semi lower.
4. \(\overline{S}^{\text{Dec}}(A) = A \cup R^{\text{Dec}}(\overline{R}^{\text{Dec}}(A))\), \(\overline{S}^{\text{Dec}}(A)\) is called \(R\)– dec semi upper.

\(A\) is \(R\)– increasing (resp. decreasing) semi exact if \(\underline{S}^{\text{Inc}}(A) = \overline{S}^{\text{Inc}}(A)\) (resp. \(\underline{S}^{\text{Dec}}(A) = \overline{S}^{\text{Dec}}(A)\)), otherwise \(A\) is \(R\)– increasing (resp. decreasing) semi rough.

3. New approximations and its properties

In this section, we introduce some definitions and propositions about near approximations, near boundary regions via GOTAS, which are essential for present study.
Definition 3.1. Let \( (U, \tau_R, \rho) \) be a GOTAS and \( A \subseteq U \). We define:

1. \( \overline{\alpha}_{\text{inc}}(A) = A \cap \overline{R}_{\text{inc}}(\overline{R}_{\text{inc}}(A)) \), \( \overline{\alpha}_{\text{inc}}(A) \) is called \( R \)-increasing \( \alpha \) lower.
2. \( \overline{\alpha}_{\text{inc}}(A) = A \cup \overline{R}_{\text{inc}}(\overline{R}_{\text{inc}}(A)) \), \( \overline{\alpha}_{\text{inc}}(A) \) is called \( R \)-increasing \( \alpha \) upper.
3. \( \overline{\alpha}_{\text{dec}}(A) = A \cap \overline{R}_{\text{dec}}(\overline{R}_{\text{dec}}(A)) \), \( \overline{\alpha}_{\text{dec}}(A) \) is called \( R \)-decreasing \( \alpha \) lower.
4. \( \overline{\alpha}_{\text{dec}}(A) = A \cup \overline{R}_{\text{dec}}(\overline{R}_{\text{dec}}(A)) \), \( \overline{\alpha}_{\text{dec}}(A) \) is called \( R \)-decreasing \( \alpha \) upper.

\( A \) is \( R \)-increasing (resp. \( R \)-decreasing) \( \alpha \) exact if  \( \overline{\alpha}_{\text{inc}}(A) = \overline{\alpha}_{\text{dec}}(A) \) (resp. \( \overline{\alpha}_{\text{dec}}(A) = \overline{\alpha}_{\text{inc}}(A) \)), otherwise \( A \) is \( R \)-increasing (resp. \( R \)-decreasing) \( \alpha \) rough.

Proposition 3.2. Let \( (U, \tau_R, \rho) \) be a GOTAS and \( A, B \subseteq U \). Then

1. If \( A \subseteq B \rightarrow \overline{\alpha}_{\text{inc}}(A) \subseteq \overline{\alpha}_{\text{inc}}(B) \) ( \( A \subseteq B \rightarrow \overline{\alpha}_{\text{dec}}(A) \subseteq \overline{\alpha}_{\text{dec}}(B) \)).
2. \( \overline{\alpha}_{\text{inc}}(A \cap B) \subseteq \overline{\alpha}_{\text{dec}}(A) \cap \overline{\alpha}_{\text{dec}}(B) \) ( \( \overline{\alpha}_{\text{dec}}(A \cap B) \subseteq \overline{\alpha}_{\text{inc}}(A) \cap \overline{\alpha}_{\text{inc}}(B) \)).
3. \( \overline{\alpha}_{\text{inc}}(A \cup B) \subseteq \overline{\alpha}_{\text{dec}}(A) \cup \overline{\alpha}_{\text{dec}}(B) \) ( \( \overline{\alpha}_{\text{dec}}(A \cup B) \subseteq \overline{\alpha}_{\text{inc}}(A) \cup \overline{\alpha}_{\text{inc}}(B) \)).

Proof.

1. Omitted.
2. \( \overline{\alpha}_{\text{inc}}(A \cap B) = (A \cap B) \cap \overline{R}_{\text{inc}}(\overline{R}_{\text{inc}}(A \cap B)) \)
   \[ \subseteq (A \cap B) \cap \overline{R}_{\text{inc}}(\overline{R}_{\text{inc}}(A) \cap \overline{R}_{\text{inc}}(B)) \]
   \[ \subseteq (A \cap B) \cap \overline{R}_{\text{inc}}(\overline{R}_{\text{inc}}(A) \cap \overline{R}_{\text{inc}}(B)) \]
   \[ \subseteq (A \cap B) \cup \overline{R}_{\text{inc}}(\overline{R}_{\text{inc}}(A) \cap (\overline{R}_{\text{inc}}(B))) \]
   \[ \subseteq A \cup \overline{R}_{\text{inc}}(\overline{R}_{\text{inc}}(A) \cap \overline{R}_{\text{inc}}(B)) \]
   \[ \subseteq \overline{\alpha}_{\text{inc}}(A) \cap \overline{\alpha}_{\text{inc}}(B) \].
3. \( \overline{\alpha}_{\text{dec}}(A \cup B) = (A \cup B) \cap \overline{R}_{\text{dec}}(\overline{R}_{\text{dec}}(A \cup B)) \)
   \[ = (A \cup B) \cap \overline{R}_{\text{dec}}(\overline{R}_{\text{dec}}(A) \cup \overline{R}_{\text{dec}}(B)) \]
   \[ \supseteq (A \cup B) \cap \overline{R}_{\text{dec}}(\overline{R}_{\text{dec}}(A) \cup \overline{R}_{\text{dec}}(B)) \]
\[ (A \cup B) \cup \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cup \overline{R}^{inc} (\overline{R}^{inc} (B)) \right) \]

\[ \supseteq A \cup \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cup B \cup \overline{R}^{inc} \left( \overline{R}^{inc} (B) \right) \right) \]

\[ \supseteq \overline{\alpha}^{inc} (A) \cup \overline{\alpha}^{inc} (B) \].

One can prove the case between parentheses.

**Proposition 3.3.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

(1) \(A \subseteq B \rightarrow \alpha^{inc} (A) \subseteq \alpha^{inc} (B)\) (\(A \subseteq B \rightarrow \alpha^{dec} (A) \subseteq \alpha^{dec} (B)\)).

(2) \(\alpha^{inc} (A \cap B) \subseteq \alpha^{inc} (A) \cap \alpha^{inc} (B)\) (\(\alpha^{dec} (A \cap B) \subseteq \alpha^{dec} (A) \cap \alpha^{dec} (B)\)).

(3) \(\alpha^{inc} (A \cup B) \supseteq \alpha^{inc} (A) \cup \alpha^{inc} (B)\) (\(\alpha^{dec} (A \cup B) \supseteq \alpha^{dec} (A) \cup \alpha^{dec} (B)\)).

**Proof.**

(1) Easy.

(2) \(\alpha^{inc} (A \cap B) = (A \cap B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A \cap B) \right)\)

\[ \subseteq (A \cap B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cap \overline{R}^{inc} (B) \right) \]

\[ \subseteq (A \cap B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cap \overline{R}^{inc} (B) \right) \]

\[ \subseteq (A \cap B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cap \overline{R}^{inc} (B) \right) \]

\[ \subseteq A \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cap B \cap \overline{R}^{inc} (B) \right) \]

\[ \subseteq \alpha^{inc} (A) \cap \alpha^{inc} (B) \].

(3) \(\alpha^{inc} (A \cup B) = (A \cup B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A \cup B) \right)\)

\[ \supseteq (A \cup B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cup \overline{R}^{inc} (B) \right) \]

\[ \supseteq (A \cup B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cup \overline{R}^{inc} (B) \right) \]

\[ \supseteq (A \cup B) \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cup \overline{R}^{inc} (B) \right) \]

\[ \supseteq A \cap \overline{R}^{inc} \left( \overline{R}^{inc} (A) \cup B \cap \overline{R}^{inc} (B) \right) \]
\[ \supseteq \alpha_{\text{Inc}}(A) \cup \alpha_{\text{Inc}}(B). \]

One can prove the case between parentheses.

**Proposition 3.4.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). If \(A\) is \(R\)–increasing (resp. decreasing) exact then \(A\) is \(\alpha\)–increasing (resp. decreasing) exact.

**Proof.**

Let \(A\) be \(R\)–increasing exact. Then \(\overline{R}^\text{Inc}(A) = \overline{R}^\text{inc}(A), \overline{\alpha}^\text{Inc}(A) = \overline{\alpha}^\text{inc}(A), \alpha_{\text{Inc}}(A) = \overline{\alpha}^\text{inc}(A). \) Therefore \(\overline{\alpha}^\text{Inc}(A) = \alpha_{\text{Inc}}(A).\)

One can prove the case between parentheses.

**Definition 3.5.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then

1. \(B_{\text{adv}}(A) = \overline{\alpha}^\text{Inc}(A) - \alpha_{\text{Inc}}(A)\) (resp. \(B_{\text{addec}}(A) = \overline{\alpha}^\text{Dec}(A) - \alpha_{\text{Dec}}(A)\)), is increasing (resp. decreasing) \(\alpha\) boundary region.
2. \(\text{Pos}_{\text{adv}}(A) = \alpha_{\text{Inc}}(A)\) (resp. \(\text{Pos}_{\text{addec}}(A) = \alpha_{\text{Dec}}(A)\)), is increasing (resp. decreasing) \(\alpha\) positive region.
3. \(\text{Neg}_{\text{adv}}(A) = U - \overline{\alpha}^\text{Inc}(A)\) (resp. \(\text{Neg}_{\text{dec}}(A) = U - \overline{\alpha}^\text{Dec}(A)\)), is increasing (resp. decreasing) \(\alpha\) negative region.

**Proposition 3.6.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

1. \(\text{Neg}(A) \supseteq \text{Neg}_{\text{addec}}(A)\) (\(\text{Neg}(A) \supseteq \text{Neg}_{\text{addec}}(A)\)).
2. \(\text{Neg}_{\text{adv}}(A \cup B) \subseteq \text{Neg}_{\text{adv}}(A) \cup \text{Neg}_{\text{adv}}(B)\)

\(\left( \text{Neg}_{\text{addec}}(A \cup B) \subseteq \text{Neg}_{\text{addec}}(A) \cup \text{Neg}_{\text{addec}}(B) \right).\)
3. \(\text{Neg}_{\text{adv}}(A \cap B) \supseteq \text{Neg}_{\text{adv}}(A) \cap \text{Neg}_{\text{adv}}(B)\)

\(\left( \text{Neg}_{\text{addec}}(A \cap B) \supseteq \text{Neg}_{\text{addec}}(A) \cap \text{Neg}_{\text{addec}}(B) \right).\)

**Proof.**

1. Since \(\overline{R}(A) \subseteq \overline{R}^\text{Dec}(A)\), then \(U - \overline{R}(A) \supseteq U - \overline{R}^\text{Dec}(A)\), therefore \(\text{Neg}(A) \supseteq \text{Neg}_{\text{adv}}(A)\).
\(2\) \(\text{Neg}_{\text{adc}}(A \cup B) = U - [ (A \cup B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A \cup B) ] \)

\[= U - [ (A \cup B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cup \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq U - [ (A \cup B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq U - [ (A \cup B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq U - [ A \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cap U - B \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq \text{Neg}_{\text{adc}}(A) \cap \text{Neg}_{\text{adc}}(B). \]

\(3\) \(\text{Neg}_{\text{adc}}(A \cap B) = U - [ (A \cap B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A \cap B) ] \)

\[= U - [ (A \cap B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cap \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq U - [ (A \cap B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cap \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq U - [ (A \cap B) \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cap \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq U - [ A \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (A) \cap U - B \cup \overline{R}^{\text{Dec}} \overline{R}^{\text{Dec}} (B) ] \]

\[\subseteq \text{Neg}_{\text{adc}}(A) \cup \text{Neg}_{\text{adc}}(B). \]

One can prove the case between parentheses.

**Definition 3.7.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). We define:

1. \(P^{\text{Inc}}(A) = A \cap \overline{R}^{\text{Inc}} (\overline{R}^{\text{Inc}} (A))\), \(P^{\text{Inc}}(A)\) is called \(R\)–increasing Pre lower.
2. \(\overline{P}^{\text{Inc}}(A) = A \cup \overline{R}^{\text{Inc}} (\overline{R}^{\text{Inc}} (A))\), \(\overline{P}^{\text{Inc}}(A)\) is called \(R\)–increasing Pre upper.
3. \(P^{\text{Dec}}(A) = A \cap \overline{R}^{\text{Dec}} (\overline{R}^{\text{Dec}} (A))\), \(P^{\text{Dec}}(A)\) is called \(R\)–decreasing Pre lower.
4. \(\overline{P}^{\text{Dec}}(A) = A \cup \overline{R}^{\text{Dec}} (\overline{R}^{\text{Dec}} (A))\), \(\overline{P}^{\text{Dec}}(A)\) is called \(R\)–decreasing Pre upper.
A is $R$–increasing (resp. $R$–decreasing) Pre exact if $P_{\text{inc}}(A) = \overline{P}^{\text{inc}}(A)$ (resp. $P_{\text{dec}}(A) = \overline{P}^{\text{dec}}(A)$), otherwise A is $R$–increasing (resp. $R$–decreasing) Pre rough.

**Proposition 3.8.** Let $(U, \tau_R, \rho)$ be a GOTAS and $A, B \subseteq U$. Then

1. If $A \subseteq B \rightarrow \overline{P}^{\text{inc}}(A) \subseteq \overline{P}^{\text{inc}}(B)$ ( $A \subseteq B \rightarrow \overline{P}^{\text{dec}}(A) \subseteq \overline{P}^{\text{dec}}(B)$ ).
2. $\overline{P}^{\text{inc}}(A \cap B) \subseteq \overline{P}^{\text{inc}}(A) \cap \overline{P}^{\text{inc}}(B)$ ( $\overline{P}^{\text{dec}}(A \cap B) \subseteq \overline{P}^{\text{dec}}(A) \cap \overline{P}^{\text{dec}}(B)$ ).
3. $\overline{P}^{\text{inc}}(A \cup B) \subseteq \overline{P}^{\text{inc}}(A) \cup \overline{P}^{\text{inc}}(B)$ ( $\overline{P}^{\text{dec}}(A \cup B) \subseteq \overline{P}^{\text{dec}}(A) \cup \overline{P}^{\text{dec}}(B)$ ).

**Proof.**

1. Omitted.
2. $\overline{P}^{\text{inc}}(A \cap B) = (A \cap B) \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A \cap B))$

   $= (A \cap B) \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A) \cap R^{\text{inc}}(B))$

   $\subseteq (A \cap B) \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A)) \cap \overline{R}^{\text{inc}} (R^{\text{inc}}(B))$

   $\subseteq A \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A)) \cap B \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(B))$

   $\subseteq \overline{P}^{\text{inc}}(A) \cap \overline{P}^{\text{inc}}(B)$.

3. $\overline{P}^{\text{inc}}(A \cup B) = (A \cup B) \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A \cup B))$

   $= (A \cup B) \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A) \cup R^{\text{inc}}(B))$

   $\supseteq (A \cup B) \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A)) \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(B))$

   $\supseteq A \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(A)) \cup B \cup \overline{R}^{\text{inc}} (R^{\text{inc}}(B))$

   $\supseteq \overline{P}^{\text{inc}}(A) \cup \overline{P}^{\text{inc}}(B)$.

One can prove the case between parentheses.

**Proposition 3.9.** Let $(U, \tau_R, \rho)$ be a GOTAS and $A, B \subseteq U$. Then:

1. $A \subseteq B \rightarrow P_{\text{inc}}(A) \subseteq P_{\text{inc}}(B)$ ( $A \subseteq B \rightarrow P_{\text{dec}}(A) \subseteq P_{\text{dec}}(B)$ ).
2. $P_{\text{inc}}(A \cap B) \subseteq P_{\text{inc}}(A) \cap P_{\text{inc}}(B)$ ( $P_{\text{dec}}(A \cap B) \subseteq P_{\text{dec}}(A) \cap P_{\text{dec}}(B)$ ).
3. $P_{\text{inc}}(A \cup B) \supseteq P_{\text{inc}}(A) \cup P_{\text{inc}}(B)$ ( $P_{\text{dec}}(A \cup B) \supseteq P_{\text{dec}}(A) \cup P_{\text{dec}}(B)$ ).
Proof.

(1) Easy.

(2) \( P_{\text{inc}}(A \cap B) = (A \cap B) \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A \cap B)) \)

\[ = (A \cap B) \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A) \cap \overline{R}^{\text{inc}}(B)) \]

\[ \subseteq (A \cap B) \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A) \cap (R_{\text{inc}}(\overline{R}^{\text{inc}}(B))) \]

\[ \subseteq A \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A) \cap B \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(B))) \]

\[ \subseteq P_{\text{inc}}(A) \cap P_{\text{inc}}(B) . \]

(3) \( P_{\text{inc}}(A \cup B) = (A \cup B) \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A \cup B)) \)

\[ = (A \cup B) \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A) \cup \overline{R}^{\text{inc}}(B)) \]

\[ \supseteq (A \cup B) \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A) \cup (R_{\text{inc}}(\overline{R}^{\text{inc}}(B))) \]

\[ \supseteq A \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(A) \cup B \cap R_{\text{inc}}(\overline{R}^{\text{inc}}(B))) \]

\[ \supseteq P_{\text{inc}}(A) \cap P_{\text{inc}}(B) . \]

Proposition 3.10. Let \( (U, \tau, \rho) \) be a GOTAS and \( A, B \subseteq U \). If \( A \) is \( R \) – increasing (resp. decreasing) exact then \( A \) is \( P \) – increasing (resp. decreasing) exact.

Proof.

Let \( A \) be \( R \) – increasing exact. Then \( \overline{R}^{\text{inc}}(A) = R_{\text{inc}}(A) \), \( \overline{P}^{\text{inc}}(A) = \overline{R}^{\text{inc}}(A) \), \( P_{\text{inc}}(A) = R_{\text{inc}}(A) \). Therefore \( \overline{P}^{\text{inc}}(A) = P_{\text{inc}}(A) \).

One can prove the case between parentheses.

Proposition 3.11. Let \( (U, \tau, \rho) \) be a GOTAS and \( A, B \subseteq U \). Then we have:

(1) \( \text{Neg}(A) \supseteq \text{Neg}_{\text{inc}}(A) (\text{Neg}(A) \supseteq \text{Neg}_{\text{dec}}(A)) \).

(2) \( \text{Neg}_{\text{inc}}(A \cup B) \subseteq \text{Neg}_{\text{inc}}(A) \cup \text{Neg}_{\text{inc}}(B) \)

\[ (\text{Neg}_{\text{dec}}(A \cup B) \subseteq \text{Neg}_{\text{dec}}(A) \cup \text{Neg}_{\text{dec}}(B)) \].
(3) $\neg P_{inc}(A \cap B) \supseteq \neg P_{inc}(A) \cap \neg P_{inc}(B)$

$(\neg P_{inc}(A \cap B) \supseteq \neg P_{inc}(A) \cap \neg P_{inc}(B))$.

**Proof.**

(1) Since $U - R^{Dec}_{\neg}(A) \supseteq U - A \cup R^{Dec}_{\neg}(A)$, then $\neg(A) \supseteq \neg g_{inc}(A)$.

(2) $P_{inc}(A \cup B) = U - [(A \cup B) \cup R^{Dec}_{\neg}(A \cup B)]$

$\subseteq U - [(A \cup B) \cup R^{Dec}_{\neg}(A) \cup R^{Dec}_{\neg}(B)]$

$\subseteq U - [(A \cup B) \cup R^{Dec}_{\neg}(A) \cup R^{Dec}_{\neg}(B)]$

$\subseteq U - [A \cup R^{Dec}_{\neg}(A) \cap U - B \cup R^{Dec}_{\neg}(B)]$

$\subseteq \neg P_{inc}(A) \cap \neg P_{inc}(B)$.

(3) $P_{inc}(A \cap B) = U - [(A \cap B) \cup R^{Dec}_{\neg}(A \cap B)]$

$\supseteq U - [(A \cap B) \cup R^{Dec}_{\neg}(A) \cap R^{Dec}_{\neg}(B)]$

$\supseteq U - [(A \cap B) \cup R^{Dec}_{\neg}(A) \cap R^{Dec}_{\neg}(B)]$

$\supseteq U - [A \cup R^{Dec}_{\neg}(A) \cap B \cup R^{Dec}_{\neg}(B)]$

$\supseteq U - A \cup R^{Dec}_{\neg}(A) \cap U - B \cup R^{Dec}_{\neg}(B)]$

$\supseteq U - [P^{Dec}_{\neg}(A) \cap P^{Dec}_{\neg}(B)]$

$\supseteq \neg P_{inc}(A) \cup \neg P_{inc}(B)$.

**Proposition 3.12.** Let $(U, r, \rho)$ be a GOTAS and $A \subseteq U$. Then

$\neg P_{inc}(A) \subseteq \alpha_{inc}(A) \subseteq S_{inc}(A)$ $(R^{Dec}_{\neg}(A) \subseteq \alpha_{Dec}(A) \subseteq S_{Dec}(A))$.

**Proof.**
Let \( x \in R_{\text{inc}}(A) \). Then \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \)

(i)

Now, we have \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \). Then \( x \in A \cap R_{\text{inc}}(R_{\text{inc}}(A)) \), therefore \( x \in \alpha_{\text{inc}}(A) \). Hence \( R_{\text{inc}}(A) \subseteq \alpha_{\text{inc}}(A) \).

(1)

Since \( x \in A \cap R_{\text{inc}}(R_{\text{inc}}(A)) \), then

\[ x \in S_{\text{inc}}(A) \]  

(2)

From (1) and (2), we have \( R_{\text{inc}}(A) \subseteq \alpha_{\text{inc}}(A) \subseteq S_{\text{inc}}(A) \).

Proposition 3.13. Let \((U, \tau_{R}, \rho)\) be a GOTAS and \( A \subseteq U \). Then

\[ \alpha_{\text{inc}}(A) \subseteq P_{\text{inc}}(A) \ (\alpha_{\text{Dec}}(A) \subseteq P_{\text{Dec}}(A)). \]

Proof.

Since \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \), then \( x \in \alpha_{\text{inc}}(A) \), and then \( x \in A \cap R_{\text{inc}}(R_{\text{inc}}(A)) \), therefore \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \subseteq R_{\text{inc}}(R_{\text{inc}}(A)) \). Thus \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \), and thus \( x \in A \cap R_{\text{inc}}(R_{\text{inc}}(A)) \). Hence \( x \in P_{\text{inc}}(A) \).

Proposition 3.14. Let \((U, \tau_{R}, \rho)\) be a GOTAS and \( A \subseteq U \). Then

\[ S_{\text{inc}}(A) \subseteq \alpha_{\text{inc}}(A) \subseteq R_{\text{inc}}(A) \ (S_{\text{Dec}}(A) \subseteq \alpha_{\text{Dec}}(A) \subseteq R_{\text{Dec}}(A)). \]

Proof.

Let \( x \in S_{\text{inc}}(A) \), then \( x \in A \) or \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \). Thus

\[ x \in A \cup R_{\text{inc}}(R_{\text{inc}}(A)) \]. Hence

\[ x \in \alpha_{\text{inc}}(A) \]  

(1)

Since \( x \in A \cup R_{\text{inc}}(R_{\text{inc}}(A)) \), then

\[ x \in A \cup R_{\text{inc}}(R_{\text{inc}}(A)) \), therefore \( x \in A \cup R_{\text{inc}}(A) \). Thus

\[ x \in R_{\text{inc}}(A) \]  

(2)

From (1) and (2), we have \( S_{\text{inc}}(A) \subseteq \alpha_{\text{inc}}(A) \subseteq R_{\text{inc}}(A) \).

Definition 3.15. Let \((U, \tau_{R}, \rho)\) be a GOTAS and \( A \subseteq U \). Then:

1. \( B_{\text{inc}}(A) = \bar{P}_{\text{inc}}(A) - P_{\text{inc}}(A) \) (resp. \( B_{\text{Dec}}(A) = \bar{P}_{\text{Dec}}(A) - P_{\text{Dec}}(A) \)), is increasing (resp. decreasing) near boundary region.

2. \( Pos_{\text{inc}}(A) = P_{\text{inc}}(A) \) (resp. \( Pos_{\text{Dec}}(A) = P_{\text{Dec}}(A) \)), is increasing (resp.
decreasing) near positive region.

\[ \text{Neg}_{Inc} (A) = U - \overline{P}^{Dec} (A) \text{ (resp. } \text{Neg}_{Dec} (A) = U - \overline{P}^{Inc} (A) ), \]

is increasing (resp. decreasing) near negative region.

**Definition 3.16.** Let \((U, \tau, \rho)\) be a GOTAS and \(A\) non-empty finite subset of \(U\). Then the increasing (decreasing) near accuracy of a finite non-empty subset \(A\) of \(U\) is given by:

\[ \eta_{Inc} (A) = \frac{j_{inc} (A)}{j_{dec} (A)}, \quad j \in \{\alpha, P\}. \]

**Proposition 3.17.** Let \((U, \tau, \rho)\) be a GOTAS and \(A\) non-empty finite subset of \(U\). Then

\[ \eta_{Inc} (A) \leq \eta_{j_{inc}} (A) \text{ (} \eta_{Dec} (A) \leq \eta_{j_{dec}} (A) \text{), for all } j \in \{\alpha, P\}, \text{ where} \]

\[ \eta_{Inc} (A) = \frac{R_{inc} (A)}{R} (A) \text{ and } \eta_{Dec} (A) = \frac{R_{Dec} (A)}{R} (A). \]

**Proof.** Omitted.

**Example 3.18.** Let \(U = \{a, b, c, d\}, U / R = \{\{a\}, \{a, b\}, \{c, d\}\}\),
\(\tau_R = \{U, \phi, \{a, b\}, \{c, d\}, \{a\}, \{a, d, c\}\}, \quad \tau_R^C = \{U, \phi, \{c, d\}, \{a, b\}, \{b, c, d\}, \{b\}\}\) and
\(\rho = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, d), (a, d), (a, c), (c, d)\}\).

For \(A = \{a, c\}\), we have:

\[ R_{Dec} (A) = \{a\}, \quad \overline{R}^{Dec} (R_{Dec} (A)) = \{a, b\}, \quad \overline{R}^{Dec} (A) = U, \quad R_{Dec} (\overline{R}^{Dec} (A)) = U. \]

\[ P_{Dec} (A) = A \cap U = A, \quad \overline{P}^{Dec} (A) = \{a, b, c\}, B_{p_{Dec}} (A) = \{b\}, \quad \text{Neg}_{Inc} = \{d\}. \]

\[ \alpha_{Dec} (A) = A \cap \{a, b\} = \{a\}, \quad \alpha^{Dec} (A) = U, \quad B_{a_{Dec}} (A) = \{b, c, d\}, \quad \text{Neg}_{ad_{Inc}} = \phi. \]

**Proposition 3.19.** Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then we have

\[ B_{S_{Inc}} (A) \subseteq B_{ad_{Inc}} (A) \subseteq B_{Inc} (A) \quad (B_{S_{Dec}} (A) \subseteq B_{ad_{Dec}} (A) \subseteq B_{Dec} (A)). \]

**Proof.** Omitted.
4. Conclusion

As a step, which is rich in results up till now to generalize the generalized approximation spaces, it was the study of GOTAS which is a generalization of the study of OTAS, GAS and AS. Every GOTAS can be regarded as an OTAS if $R$ is an equivalence relation and OTAS can be regarded as an AS if $\rho$ is the equal relation. In addition, every GOTAS can be regarded as GAS if $\rho$ is the equal relation and GAS can be regarded as AS if $R$ is an equivalence relation.

References

[1] M.E. AbdEl-Monsef et al., On near open sets and near approximations, Journal of institute of Mathematics and Computer Sciences, 20(1) (2009) 99-109.

[2] D. G. Chen and W. X. Zhang, Rough sets and topological spaces, Journal of Xi’an Jiaotong University 35 (2001) 1313–1315.

[3] M.E. El-Shafei, A.M.Kozae and M.Abo-elhamayel, Rough set approximations via topological ordered spaces, Annals of fuzzy sets, fuzzy logic and fuzzy systems 2 (2013) 49-60.

[4] M.E. El-Shafei, A.M.Kozae and M.Abo-elhamayel, Semi ordered topological approximations of rough sets, Annals of fuzzy sets, fuzzy logic and fuzzy systems 2 (2013) 61-72.

[5] R. Engelking, General Topology, Polish Scientific Publishers, Warszawa (1977).

[6] M. Kondo, On the structure of generalized rough sets, Information Sciences 176 (2006) 589–600.
[7] J. Kortelainen, On the relationship between modified sets, topological spaces and rough sets, Fuzzy Sets and Systems 61 (1994) 91-95.

[8] E.F. Lashin, A.M. Kozae, A.A. Abokhadra and T Medhat, Rough set theory for topological spaces, International Journal of Approximate Reasoning 40 (2005) 35–43.

[9] T.-J. Li, Y. Yeung, W.-X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy covering, International Journal of Approximate Reasoning 48 (2008) 836–856.

[10] L. Nachbin, Topology and order van Nostrand Mathematical studies, Princeton, New Jersey (1965).

[11] Z. Pawlak, Rough sets, Theoretical Aspects of Reasoning about data, Boston: Kluwer Academic, (1991).

[12] A. S. Salama, Topological solution of missing attribute values problem in incomplete information tables, Information Sciences 180 (2010) 631–639.

[13] Q. E. Wu, T. Wang, Y. X. Huang and J. S. Li, Topology theory on rough sets, IEEE Transactions on Systems, Man and Cybernetics – Part B: Cybernetics 38 (2008) 68–77.

[14] Y.Y. Yao, Two views of the theory of rough sets in finite universes, International Journal of Approximate Reasoning 15 (1996) 291–317.

[15] W. Zhu, Topological approaches to covering rough sets, Information Sciences 177 (2007) 1499–1508.