JIMWLK evolution in the Gaussian approximation

Edmond Iancu
Institut de Physique Théorique de Saclay

with D.N. Triantafyllopoulos, arXiv:1109.0302, 1112.1104 [hep-ph]
Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)
Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)

... it appeared to be so complicated that any solution to it seemed to be out of reach !
Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ... Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00) ... it appeared to be so complicated that any solution to it seemed to be out of reach!

Yet, only a few years later (following Blaizot, Iancu, Weigert, 02), Rummukainen and Weigert presented the first numerical solution (03).

Nowadays, we have several ‘codes’ available: Lappi, Schenke & collabs
Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ... 

*Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)*

... it appeared to be so complicated that any solution to it seemed to be out of reach!

Yet, only a few years later (following *Blaizot, Iancu, Weigert, 02*), Rummukainen and Weigert presented the first numerical solution (03).

Nowadays, we have several ‘codes’ available: *Lappi, Schenke & collabs*

This has been completed by various ‘mean field studies’

- solutions to the Balitksy–Kovchegov (BK) equation (large $N_c$)
- Gaussian Ansatz for the CGC weight function

*Iancu, Itakura, McLerran (02), Kovchegov, Kuokkanen, Rummukainen, Weigert (09)*
Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)

... it appeared to be so complicated that any solution to it seemed to be out of reach!

Yet, only a few years later (following Blaizot, Iancu, Weigert, 02), Rummukainen and Weigert presented the first numerical solution (03).

Nowadays, we have several ‘codes’ available: Lappi, Schenke & collabs

This has been completed by various ‘mean field studies’

- solutions to the Balitksy–Kovchegov (BK) equation (large $N_c$)
- Gaussian Ansatz for the CGC weight function

Iancu, Itakura, McLerran (02), Kovchegov, Kuokkanen, Rummukainen, Weigert (09)

For quite some time, these efforts were restricted to the dipole amplitude (a 2–point function generalizing the gluon distribution)
Directly relevant to the phenomenology ...

- deep inelastic scattering
- single inclusive particle production in $p+A$

... and also easier to compute.
Directly relevant to the phenomenology ...
  - deep inelastic scattering
  - single inclusive particle production in p+A

... and also easier to compute.

Recently, phenomenology started to be more demanding:
  multi–particle correlations at RHIC

... thus requiring the study of higher $n$–point correlations ($n \geq 4$).
Directly relevant to the phenomenology ...

- deep inelastic scattering
- single inclusive particle production in p+A

... and also easier to compute.

Recently, phenomenology started to be more demanding: multi–particle correlations at RHIC

... thus requiring the study of higher $n$–point correlations ($n \geq 4$).

The first numerical calculation of 4-p and 6-p functions for special configurations (Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan, 11)

... came with a big surprise: numerics is very well reproduced by high–energy extrapolations of the McLerran–Venugopalan model!
• Directly relevant to the phenomenology ...
  • deep inelastic scattering
  • single inclusive particle production in p+A
• ... and also easier to compute.

• Recently, phenomenology started to be more demanding:
  multi–particle correlations at RHIC
• ... thus requiring the study of higher $n$–point correlations ($n \geq 4$).

• The first numerical calculation of 4-p and 6-p functions for special configurations (*Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan, 11*)
• ... came with a big surprise: numerics is very well reproduced by high–energy extrapolations of the McLerran–Venugopalan model !

• A Gaussian approximation : information only about the 2–p function !
Is that just numerical coincidence restricted to special configurations?
Is that just **numerical coincidence** restricted to special configurations?

Or rather is a **generic feature** of the B–JIMWLK evolution that one could further exploit?
Is that just numerical coincidence restricted to special configurations?

Or rather is a generic feature of the B–JIMWLK evolution that one could further exploit?

Previous studies of the Gaussian approximation did not address its validity for higher $n$–point correlations

No a priori reason to expect it should work!

- complicated, non–linear, evolution
- infinite hierarchy of equations coupling $n$–p functions with arbitrary $n$
Is that just numerical coincidence restricted to special configurations?

Or rather is a generic feature of the B–JIMWLK evolution that one could further exploit?

Previous studies of the Gaussian approximation did not address its validity for higher $n$–point correlations

No *a priori* reason to expect it should work!

- complicated, non–linear, evolution
- infinite hierarchy of equations coupling $n$–p functions with arbitrary $n$

And yet it works! *(E.I., Triantafyllopoulos, 2011)*

- a meaningful piecewise approximation, which is correct both in the dilute (BFKL) and the dense (saturation) regimes
- smooth interpolation between the two limiting regimes
- good agreement with numerics ... whenever the latter exists

Analytic solutions which should greatly facilitate phenomenology
Di–hadron azimuthal correlations

Typical final state: a pair of jets back–to–back in the transverse plane

Particle distribution as a function of the azimuthal angle:

a peak at $\Delta \Phi = 180^\circ$
The colliding partons carry longitudinal momentum fractions

\[ x_1 = \frac{|p_a| e^{y_a} + |p_b| e^{y_b}}{\sqrt{s}}, \quad x_2 = \frac{|p_a| e^{-y_a} + |p_b| e^{-y_b}}{\sqrt{s}} \]

Forward rapidities: \( y_a \sim y_b \) are both positive and large

\[ \Rightarrow x_1 \sim \mathcal{O}(1) \text{ and } x_2 \ll 1 \] (‘dense–dilute scattering’)

One may be able to probe saturation effects in the target

These effects are enhanced for a nuclear target
Di–hadron correlations at RHIC: $p+p$ vs. $d+Au$

$\begin{align*}
\mathbf{k}_1 &\quad \mathbf{k}_2 \\
\eta_1 &\quad \eta_2
\end{align*}$

$d+Au$: the ‘away jet’ gets smeared out $\Longrightarrow$ saturation in Au
Di–hadron correlations at RHIC: p+p vs. d+Au

\( p+p \rightarrow \pi^0\pi^0 + X, \sqrt{s} = 200 \text{ GeV} \)

\( p_{T,S} > 2 \text{ GeV/c}, 1 \text{ GeV/c} < p_{T,S} < p_{T,L} \)

\( \langle \eta_s \rangle = 3.2, \langle \eta_l \rangle = 3.1 \)

\( \Delta \phi = 0 \) (near side)

\( \Delta \phi = \pi \) (away side)

\( \Delta \phi \)

(k1, k2) \( \rightarrow \) (k1, k2)

(Albacete and Marquet, 2010, PRL)

- d+Au: the ‘away jet’ gets smeared out \( \Rightarrow \) saturation in Au
The produced quark and gluon undergo multiple scattering.

Broadening of their transverse momentum distribution: important if \( p_\perp \sim Q_s(x_2, A) \) ... in agreement with the data!

Eikonal approximation \( \Rightarrow \) Wilson lines:

\[
V_{x}^\dagger \equiv \text{P exp} \left[ i g \int dx^- A_a^+(x^-, x) T^a \right]
\]

\( \Rightarrow \) two WL’s per parton (direct amplitude + the c.c. amplitude)
Higher–point correlations of the Wilson lines

- Quark–gluon pair production: the color trace of a product of 4 Wilson lines (2 fundamental, 2 adjoint)

- Equivalently (after using Fierz identity): 6 fundamental Wilson lines
  \[
  \left\langle \frac{1}{N_c} \text{tr}(V_{x_1} V_{x_2} V_{x_3} V_{x_4}) \frac{1}{N_c} \text{tr}(V_{x_4} V_{x_3}) \right\rangle_Y \equiv \left\langle \hat{Q}_{x_1 x_2 x_3 x_4} \hat{S}_{x_4 x_3} \right\rangle_Y
  \]

- Expectation value of a 2–trace operator: quadrupole \times dipole
Higher–point correlations of the Wilson lines

- Quark–gluon pair production: the color trace of a product of 4 Wilson lines (2 fundamental, 2 adjoint)

- Equivalently (after using Fierz identity): 6 fundamental Wilson lines

\[
\left\langle \frac{1}{N_c} \text{tr}(V_{x_1} V_{x_2} V_{x_3} V_{x_4}) \right. \frac{1}{N_c} \text{tr}(V_{x_4} V_{x_3}) \left. \right\rangle_Y \equiv \left\langle \hat{Q}_{x_1 x_2 x_3 x_4} \hat{S}_{x_4 x_3} \right\rangle_Y
\]

- Expectation value of a 2–trace operator: quadrupole \times dipole

- The target dynamics is encoded in the CGC average:

\[
\langle \hat{O} \rangle_Y \equiv \int D\alpha \mathcal{O}[\alpha] W_Y[\alpha], \quad \alpha_a \equiv A_a^+(x^-, x), \quad Y \equiv \ln \frac{1}{x_2}
\]

- The CGC weight function \( W_Y[\alpha] \) obeys JIMWLK equation

high–energy evolution [leading log \( \ln(1/x) \)] of the multigluon correlations for the case of a dense target
JIMWLK Hamiltonian

- Renormalization group equation for the CGC weight function $W_Y[\alpha]$:
  \[
  \frac{\partial}{\partial Y} W_Y[\alpha] = H W_Y[\alpha]
  \]

\[
H = -\frac{1}{16\pi^3} \int_{uvz} M_{uvz} \left( 1 + \tilde{V}^*_u \tilde{V}_v - \tilde{V}^*_u \tilde{V}_z - \tilde{V}^*_z \tilde{V}_v \right)^{ab} \frac{\delta}{\delta \alpha^a_u} \frac{\delta}{\delta \alpha^b_v}
\]

- Dipole kernel: $M_{uvz} \equiv \frac{(u-v)^2}{(u-z)^2(z-v)^2}$

- Functional derivatives: ‘creation operators’ for the emission of a new gluon at small $x$

- (Adjoint) Wilson lines: multiple scattering between the newly emitted gluon and the color field created by the previous ones with $x' \gg x$

- N.B.: The first 2 terms within $H$ (‘virtual’) and the last 2 ones (‘real’) will play different roles in what follows
Balitsky–JIMWLK hierarchy

- Infinite hierarchy of coupled evolution equations for the $n$–point functions of the Wilson lines (Balitsky, 1996)

$$\frac{\partial \langle \hat{O} \rangle_Y}{\partial Y} = \int D\alpha \mathcal{O}[\alpha] \frac{\partial}{\partial Y} W_Y[\alpha] = \langle H\hat{O} \rangle_Y$$

- Functional derivatives act on the color field at the largest value of $x^-$:

$$\frac{\delta}{\delta \alpha^a_{uu}} V^\dagger_{x} = ig\delta_{uu} t^a V^\dagger_{x}$$

... i.e. at the end point of the Wilson lines

- Generators of color rotations ‘on the left’ (or ‘left Lie derivatives’):
  - each evolution step adds a new layer of field at a larger value of $x^-$:

$$V^\dagger_n(x) \rightarrow V^\dagger_{n+1}(x) = \exp[ig\epsilon \alpha_{n+1}(x)] V^\dagger_n(x)$$

- We shall later return to this point (longitudinal structure of the target)
Dipole evolution (1)

- Observables involving $2n$ Wilson lines are coupled to those with $2n+2$

- Dipole $S$–matrix: 
  \[
  \hat{S}_{x_1 x_2} = \frac{1}{N_c} \text{tr}(V_{x_1}^\dagger V_{x_2})
  \]

  \[
  H_{\text{virt}} \hat{S}_{x_1 x_2} = -\frac{\bar{\alpha}}{2\pi} \left(1 - \frac{1}{N_c^2}\right) \int_z M_{x_1 x_2 z} \hat{S}_{x_1 x_2}
  \]

  \[
  H_{\text{real}} \hat{S}_{x_1 x_2} = \frac{\bar{\alpha}}{2\pi} \int_z M_{x_1 x_2 z} \left(\hat{S}_{x_1 z} \hat{S}_{z x_2} - \frac{1}{N_c^2} \hat{S}_{x_1 x_2}\right)
  \]

- The $1/N_c^2$ corrections cancel between ‘real’ and ‘virtual’ contributions

  \[
  \frac{\partial \langle \hat{S}_{x_1 x_2} \rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int_z M_{x_1 x_2 z} \langle \hat{S}_{x_1 z} \hat{S}_{z x_2} - \hat{S}_{x_1 x_2} \rangle_Y
  \]

- Physical interpretation: projectile (dipole) evolution
Dipole evolution (2)

- Use the rapidity increment \((Y \rightarrow Y + dY)\) to boost the dipole.
- The dipole ‘evolves’ by emitting a small–\(x\) gluon.
- ‘Real’ term: quark-antiquark-gluon system interacts with the target.
- ‘Virtual’ term: the emitted gluon does not interact with the target.

At large \(N_c\), this system looks like two dipoles.

The probability for the dipole not to evolve.
Quadrupole evolution (1)

\[ \hat{Q}_{x_1 x_2 x_3 x_4} = \frac{1}{N_c} \text{tr}(V_{x_1}^\dagger V_{x_2} V_{x_3}^\dagger V_{x_4}) \]

\[
\frac{\partial \langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{4\pi} \int_z \left[ (M_{x_1 x_2 z} + M_{x_1 x_4 z} - M_{x_2 x_4 z}) \langle \hat{S}_{x_1 z} \hat{Q}_{z x_2 x_3 x_4} \rangle_Y \\
+ M_{x_1 x_2 z} + M_{x_2 x_3 z} - M_{x_1 x_3 z}) \langle \hat{S}_{z x_2} \hat{Q}_{x_1 x_3 x_4} \rangle_Y \\
- (M_{x_1 x_2 z} + M_{x_3 x_4 z} + M_{x_1 x_4 z} + M_{x_2 x_3 z}) \langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y \\
- (M_{x_1 x_2 z} + M_{x_3 x_4 z} - M_{x_1 x_3 z} - M_{x_2 x_4 z}) \langle \hat{S}_{z x_2} \hat{S}_{x_3 x_4} \rangle_Y \\
- (M_{x_1 x_4 z} + M_{x_2 x_3 z} - M_{x_1 x_3 z} - M_{x_2 x_4 z}) \langle \hat{S}_{x_3 x_2} \hat{S}_{x_1 x_4} \rangle_Y \right]
\]
More complicated, but the same structural properties as for the dipole:

- Real terms \((2n + 2 = 6 \text{ WL's})\): \(\langle \hat{S}_{x_1 z} \hat{Q}_{z x_2 x_3 x_4} \rangle_Y\)
- Virtual terms \((2n = 4 \text{ WL's})\): \(\langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y, \langle \hat{S}_{x_1 x_4} \hat{S}_{x_3 x_2} \rangle_Y\)

- \(1/N_c^2\) corrections have cancelled between ‘real’ and ‘virtual’
- Single–trace couples to double–trace under the evolution
Multi–trace expectation values of WL’s factorize into single–trace ones

\[
\left\langle \frac{1}{N_c} \text{tr}(V_{x_1} V_{x_2} \cdots) \frac{1}{N_c} \text{tr}(V_{y_1} V_{y_2}) \right\rangle_Y \simeq \left\langle \frac{1}{N_c} \text{tr}(V_{x_1} V_{x_2} \cdots) \right\rangle_Y \left\langle \frac{1}{N_c} \text{tr}(V_{y_1} V_{y_2}) \right\rangle_Y
\]

B–JIMWLK hierarchy boils down to closed equations

Dipole: \( \langle \hat{S}_{x_1 z} \hat{S}_{z x_2} \rangle \simeq \langle \hat{S}_{x_1 z} \rangle \langle \hat{S}_{z x_2} \rangle \implies \text{Balitsky–Kovchegov (BK)} \)

\[
\frac{\partial \langle \hat{S}_{x_1 x_2} \rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int_z M_{x_1 x_2 z} \left[ \langle \hat{S}_{x_1 z} \rangle_Y \langle \hat{S}_{z x_2} \rangle_Y - \langle \hat{S}_{x_1 x_2} \rangle_Y \right]
\]

Closed, non–linear equation for \( \langle \hat{S}_{x_1 x_2} \rangle_Y \), studied at length.

Saturation momentum : unitarity limit for the dipole scattering

\( \langle \hat{S}(r) \rangle_Y \sim \mathcal{O}(1) \quad \text{when} \quad 1/r \sim Q_s(Y) \propto e^{\lambda Y} \)
The limit of a large number of colors: \( N_c \to \infty \)

- **Quadrupole:**
  \[ \langle \hat{S}_{x_1 z} \hat{Q}_{x_2 x_3 x_4} \rangle_Y \simeq \langle \hat{S}_{x_1 z} \rangle_Y \langle \hat{Q}_{x_2 x_3 x_4} \rangle_Y \]
  \[
  \frac{\partial \langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{4\pi} \int_z \left[ (\mathcal{M}_{x_1 x_2 z} + \cdots) \langle \hat{S}_{x_1 z} \rangle_Y \langle \hat{Q}_{x_2 x_3 x_4} \rangle_Y \right. \\
  + \ldots \ldots \right. \\
  - (\mathcal{M}_{x_1 x_2 z} + \cdots) \langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y \\
  - (\mathcal{M}_{x_1 x_2 z} + \cdots) \langle \hat{S}_{x_1 x_2} \rangle_Y \langle \hat{S}_{x_3 x_4} \rangle_Y \right].
  
- An equation for \( \langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y \) with \( \langle \hat{S}_{x_1 x_2} \rangle_Y \) acting as a source.
- Numerical solution still complicated (due to real terms)
  - non–linear terms
  - transverse non–locality (integral over \( z \))

In practice it is easier to solve the full JIMWLK equation (finite \( N_c \)) using its reformulation as a (functional) Langevin equation

*Blaizot, E.I., Weigert, 2002* cf. talk by T. Lappi
Towards a Gaussian approximation

- The prototype for it: the McLerran–Venugopalan model

\[ W_{MV}[\rho] = \exp \left[ -\frac{1}{2} \int dx^- \int d^2 x \frac{\rho^a(x^-, x) \rho^a(x^-, x)}{\lambda(x^-)} \right] \]

- Large nucleus \((A \gg 1)\), not so small \(x\):
  - ‘color sources’ = independent valence quarks
- \(\rho_a(x^-, x)\) color charge density:
  \[-\nabla_\perp^2 \alpha_a = \rho_a\]
- Often used as an initial condition for JIMWLK at \(Y_0 \sim 4\)
- Could a Gaussian be a reasonable approximation also at \(Y \gg Y_0\) ?
  - high energy evolution introduces correlations among the color sources
  - non-linear effects ⇒ coupled equations for \(n\)-point functions of WL’s

Yet... there is impressive agreement between numerical solutions to JIMWLK and simple extrapolations of the MV model!

(Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan 2011)
In the dilute regime \((k_\perp \gg Q_s(Y)\) or \(|\mathbf{x}_i - \mathbf{x}_j| \ll 1/Q_s(Y)\)), the correlations refer to the BFKL evolution of the 2–point function:

\[
\langle \hat{S}_{\mathbf{x}_1 \mathbf{x}_2} \rangle_Y \simeq 1 - \frac{g^2}{4N_c} \langle (\alpha_{\mathbf{x}_1}^a - \alpha_{\mathbf{x}_2}^a)^2 \rangle_Y \equiv 1 - \langle \hat{T}_{\mathbf{x}_1 \mathbf{x}_2} \rangle_Y
\]

\[
1 - \langle \hat{Q}_{\mathbf{x}_1 \mathbf{x}_2} \mathbf{x}_3 \mathbf{x}_4 \rangle_Y \simeq \langle \hat{T}_{\mathbf{x}_1 \mathbf{x}_2} - \hat{T}_{\mathbf{x}_1 \mathbf{x}_3} + \hat{T}_{\mathbf{x}_1 \mathbf{x}_4} + \hat{T}_{\mathbf{x}_2 \mathbf{x}_3} - \hat{T}_{\mathbf{x}_2 \mathbf{x}_4} + \hat{T}_{\mathbf{x}_3 \mathbf{x}_4} \rangle_Y
\]

\[
\langle \hat{T}_{\mathbf{x}_1 \mathbf{x}_2} \rangle_Y \text{ (dipole scattering amplitude) obeys the BFKL equation :}
\]

\[
\frac{\partial \langle \hat{T}_{\mathbf{x}_1 \mathbf{x}_2} \rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int z \mathcal{M}_{\mathbf{x}_1 \mathbf{x}_2 z} \langle \hat{T}_{\mathbf{x}_1 z} + \hat{T}_{\mathbf{z} \mathbf{x}_2} - \hat{T}_{\mathbf{x}_1 \mathbf{x}_2} \rangle_Y
\]

A 2–point function can always be encoded in a Gaussian!
Some encouraging arguments (2)

- Saturation regime: \( k_\perp \ll Q_s(Y) \) or \( |x_i - x_j| \gg 1/Q_s(Y) \)

\[ \rightarrow \text{‘keep only the first term (no WL’s) in } H_{\text{JIMWLK}} \]

\[
H = -\frac{1}{16\pi^3} \int_{uvwz} M_{uvwz} \left( 1 + \tilde{V}_u^\dagger \tilde{V}_v - \tilde{V}_u^\dagger \tilde{V}_z - \tilde{V}_z^\dagger \tilde{V}_v \right)^{ab} \frac{\delta}{\delta \alpha_u^a} \frac{\delta}{\delta \alpha_v^b}
\]

‘Random phase approximation’ (E.I. & McLerran, 2001)

\[
H_{\text{RPA}} \simeq -\frac{1}{8\pi^2} \int_{uv} \ln \left[ (u - v)^2 Q_s^2(Y) \right] \frac{\delta}{\delta \alpha_u^a} \frac{\delta}{\delta \alpha_v^a}
\]

- Free diffusion ... obviously consistent with a Gaussian weight function!

- Qualitatively right, but a bit naive though!

- The first two terms within \( H_{\text{JIMWLK}} \) act on the same footing!

  together, they generate the ‘virtual’ terms in the B-JIMWLK equations
On the importance of the virtual terms

\[ H_{\text{virt}} = -\frac{1}{16\pi^3} \int_{uvz} \mathcal{M}_{uvz} \left( 1 + \bar{V}_u \bar{V}_v \right)^{ab} \frac{\delta}{\delta \alpha_u^a} \frac{\delta}{\delta \alpha_v^b} \]

- The virtual terms dominate the evolution deeply at saturation
  - surprising at the first sight: the non–linear effects are encoded precisely in the real terms
  - even less obvious at finite \( N_c \): real and virtual term seem to receive \( 1/N_c^2 \) corrections of the same order

- One can promote \( H_{\text{virt}} \) into a mean field approximation to \( H_{\text{JIMWLK}} \) which is valid both in the dense and the dilute regimes!

- Is this consistent with a Gaussian weight function \( W_Y[\alpha] \)?
  \( H_{\text{virt}} \) is still non–linear to all orders in the field \( \alpha_a \) ...
Virtual terms dominate deeply at saturation

- They control the approach towards the ‘black disk limit’:
  \[ \langle \hat{S} \rangle_{Y} \to 0, \langle \hat{Q} \rangle_{Y} \to 0, \text{ etc.} \]

- Easier to understand at large \( N_c \); e.g. for the dipole (BK equation)
  \[
  \frac{\partial \langle \hat{S}_{x_1x_2} \rangle_{Y}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int_{z} \mathcal{M}_{x_1x_2z} \left[ \langle \hat{S}_{x_1z} \rangle_{Y} \langle \hat{S}_{zx_2} \rangle_{Y} - \langle \hat{S}_{x_1x_2} \rangle_{Y} \right]
  \]

- Deeply at saturation: \[ \langle \hat{S} \rangle_{Y} \langle \hat{S} \rangle_{Y} \ll \langle \hat{S} \rangle_{Y} \ll 1 \]
  \[
  \frac{\partial \langle \hat{S}(r) \rangle_{Y}}{\partial Y} \simeq -\bar{\alpha} \ln[r^2Q_s^2(Y)] \langle \hat{S}(r) \rangle_{Y}
  \]

- A Sudakov factor: the probability for the dipole not to evolve.

- The conclusion persists at finite \( N_c \), for the same physical reason:
  
  *the dipole (quadrupole, etc) has more chances to survive its scattering off the CGC if it remains simple!*
Virtual terms can encode BFKL too...

... provided one generalizes the kernel in the Hamiltonian:

\[ H_{\text{MFA}} = -\frac{1}{2} \int_{uv} \gamma_Y(u,v) (1 + \tilde{V}_u^\dagger \tilde{V}_v) \frac{\delta}{\delta \alpha_u^a} \frac{\delta}{\delta \alpha_v^b} \]

Mean–field evolution of the dipole:

\[ \partial \langle \hat{S}_{x_1x_2} \rangle_Y \frac{\partial}{\partial Y} = \langle H_{\text{MFA}} \hat{S}_{x_1x_2} \rangle_Y = -2g^2 C_F \gamma_Y(x_1, x_2) \langle \hat{S}_{x_1x_2} \rangle_Y \]

Weak scattering (BFKL): \[ \langle \hat{S} \rangle_Y = 1 - \langle \hat{T} \rangle_Y \text{ with } \langle \hat{T} \rangle_Y \ll 1 \]

\[ \partial \langle \hat{T}_{x_1x_2} \rangle_Y \frac{\partial}{\partial Y} = 2g^2 C_F \gamma_Y(x_1, x_2) \]

Use this equation, with the l.h.s. estimated at the BFKL level, as the definition of \( \gamma_Y(x_1, x_2) \) for \( |x_1 - x_2| \ll 1/Q_s(Y) \)
The Mean Field Approximation

... is defined by the following Hamiltonian:

$$H_{\text{MFA}} = -\frac{1}{2} \int_{uv} \gamma_Y(u, v) \left( 1 + \tilde{V}_u \tilde{V}_v \right)^{ab} \frac{\delta}{\delta \alpha^a_u} \frac{\delta}{\delta \alpha^b_v}$$

... where the kernel $\gamma_Y(u, v)$ is uniquely defined

- in the dilute regime at $|u - v| \ll 1/Q_s(Y)$ (BFKL)
- in the dense regime at $|u - v| \gg 1/Q_s(Y)$

The transition region around $|u - v| \sim 1/Q_s(Y)$ goes beyond the accuracy of the MFA $\Rightarrow$ any smooth interpolation is equally good

In practice: trade the kernel for the dipole $S$–matrix:

$$\gamma_Y(u, v) = -\frac{1}{2g^2C_F} \frac{\partial \ln \langle \hat{S}_{uv} \rangle_Y}{\partial Y}$$
... is defined by the following Hamiltonian:

\[ H_{\text{MFA}} = -\frac{1}{2} \int_{uv} \gamma_Y(u, v) (1 + \tilde{V}_u \tilde{V}_v)^{ab} \frac{\delta}{\delta \alpha_u^a} \frac{\delta}{\delta \alpha_v^b} \]

... where the kernel \( \gamma_Y(u, v) \) is uniquely defined

- in the dilute regime at \( |u - v| \ll 1/Q_s(Y) \) (BFKL)
- in the dense regime at \( |u - v| \gg 1/Q_s(Y) \)

The transition region around \( |u - v| \sim 1/Q_s(Y) \) goes beyond the accuracy of the MFA \( \Rightarrow \) any smooth interpolation is equally good

The kernel is independent of \( N_c \) \( \Rightarrow \) can be inferred from the solution to the BK equation (large \( N_c \)) ... and then used at finite \( N_c \):

\[ \gamma_Y(u, v) = -\frac{1}{g^2 N_c} \frac{\partial \ln(\hat{S}_{uv}^{\text{BK}})_Y}{\partial Y} \]

\textit{N.B. this yields the same kernel as Heribert’s ‘Gaussian truncation’}
Evolution equations in the MFA

- Obtained by keeping only the virtual terms in the respective B–JIMWLK equations and replacing the kernel according to

\[
\frac{1}{8\pi^3} \int_z \mathcal{M}_{uvz} \rightarrow \gamma_Y(u, v)
\]

- Considerably simpler than the original equations:
  - linear
  - local in transverse coordinates
  - coupled, but closed, systems: they couple only \(n\)-point functions with the same value of \(n\) (e.g. \(\langle \hat{Q} \rangle_Y\) with \(\langle \hat{S} \hat{S} \rangle_Y\))

- The equations can be solved analytically.

- The solutions becomes especially simple if
  - the kernel is separable: \(\gamma_Y(u, v) = h_1(Y) g(u, v) + h_2(Y)\)
  - at large \(N_c\) (any kernel)
  - for special configurations of the external points in the transverse space
The mean–field equations allow one to compute the \( n \)-point functions of the WL’s with \( n \geq 4 \) in terms of the dipole \( S \) matrix \( \langle \hat{S} \rangle_Y \) \((n = 2)\)

For a separable kernel, the \( Y \)-dependence in the final results enters exclusively via \( \langle \hat{S} \rangle_Y \)

\( \triangleright \) separability is a good approximation, in both dense and dilute limits

In that case, the functional form of the solutions is formally the same as in the MV model!

This is rewarding: it explains the numerical findings in \texttt{arXiv:1108.4764} \((\text{Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan 2011})\)

... but it also rises a puzzle: it strongly suggests that the mean field approximation has an underlying Gaussian structure

How is that possible?
The Gaussian CGC weight function

\[ H_{\text{MFA}} = -\frac{1}{2} \int_{uv} \gamma_Y(u, v) \left( 1 + \widetilde{V}_u \widetilde{V}_v \right)^{ab} \frac{\delta}{\delta \alpha_u^a} \frac{\delta}{\delta \alpha_v^b} \]

- The functional derivatives act as generators of color rotations:

\[
\frac{\delta}{\delta \alpha_u^a} V_x^\dagger = i g \delta_{xu} t^a V_x^\dagger \quad \widetilde{V}_u^{ab} \frac{\delta}{\delta \alpha_u^b} V_x^\dagger = i g \delta_{xu} V_x^\dagger t^a, 
\]

- ... both on the left and on the right

\[ H_{\text{MFA}} = -\frac{1}{2} \int_{uv} \gamma_Y(u, v) \left( \frac{\delta}{\delta \alpha_L^a} \frac{\delta}{\delta \alpha_L^a} + \frac{\delta}{\delta \alpha_R^a} \frac{\delta}{\delta \alpha_R^a} \right) \]

- This is free diffusion ... but simultaneously ‘towards the left’ (increasing \( x^- \)) and ‘towards the right’ (decreasing \( x^- \))

- With increasing \( Y \), the target color field expands symmetrically in \( x^- \) around the light–cone (\( x^- = 0 \))

- The CGC weight function in the MFA is a Gaussian symmetric in \( x^- \)
Longitudinal structure of the CGC

\[ W_Y[\alpha] = \mathcal{N}_Y \exp \left\{ -\frac{1}{2} \int_{-x_M(Y)}^{x_M(Y)} dx^- \int_{x_1 x_2} \frac{\alpha_a(x^-, x_1) \alpha_a(x^-, x_2)}{\gamma(x^-, x_1, x_2)} \right\} \]

- \( x_M(Y) = x_0^+ \exp(Y - Y_0) \)

- valence quarks
- small \( x \) gluons
- even smaller \( x \) gluons
The mirror symmetry

- This has observable consequences: \( \langle \hat{Q} x_1 x_2 x_3 x_4 \rangle_Y = \langle \hat{Q} x_1 x_4 x_3 x_2 \rangle_Y \)

- Time reversal symmetry for the projectile (with 'time' = \( x^- \)).

- Similar identities hold for the higher \( n \)–point functions.

- An exact symmetry of the JIMWLK equation.
Applications to special configurations

- Di–hadron correlations: quadrupole × dipole — line configuration

\[ \hat{S}_6 x_1 x_2 x_3 x_4 = \frac{N_c^2}{N_c^2 - 1} \hat{Q} x_1 x_2 x_3 x_4 \hat{S} x_4 x_3 - \frac{1}{N_c^2 - 1} \hat{S} x_1 x_2 \]

- Our full MFA result cannot be distinguished from the numerical solution to JIMWLK (Dumitru et al, 2011)
A versatile configuration

- \( \langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y \) with \( r_{13} = r_{14} \) and \( r_{23} = r_{24} \) & arbitrary \( r_{12} \) and \( r_{34} \)

One finds exact factorization: \( \langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y = \langle \hat{S}_{x_1 x_2} \rangle_Y \langle \hat{S}_{x_3 x_4} \rangle_Y \)

Natural when \( r_{12}, r_{34} \ll r_{14}, r_{23} \) ... but remarkable in general.

\[
\langle \hat{S}_{x_1 x_2 x_3 x_4} \rangle_Y = \langle \hat{S}_{x_1 x_2} \rangle_Y \left( \langle \hat{S}_{x_3 x_4} \rangle_Y \right)^{\frac{2N_c^2}{N_c^2 - 1}} \approx \langle \hat{S}_{x_1 x_2} \rangle_Y \left[ \langle \hat{S}_{x_3 x_4} \rangle_Y \right]^2
\]
A versatile configuration

- $\langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y$ with $r_{13} = r_{14}$ and $r_{23} = r_{24}$ & arbitrary $r_{12}$ and $r_{34}$

One finds exact factorization: $\langle \hat{Q}_{x_1 x_2 x_3 x_4} \rangle_Y = \langle \hat{S}_{x_1 x_2} \rangle_Y \langle \hat{S}_{x_3 x_4} \rangle_Y$

Natural when $r_{12}, r_{34} \ll r_{14}, r_{23} \ldots$ but remarkable in general.

$\langle \hat{S}_6 x_1 x_2 x_3 x_4 \rangle_Y = \langle \hat{S}_{x_1 x_2} \rangle_Y \left[ \langle \hat{S}_{x_3 x_4} \rangle_Y \right]^2 \frac{2N_c^2}{N_c^2 - 1} \simeq \langle \hat{S}_{x_1 x_2} \rangle_Y \left[ \langle \hat{S}_{x_3 x_4} \rangle_Y \right]^2$
THANK YOU!