Stability of neutral pantograph stochastic differential equations with generalized decay rate

Mingxuan Shen, Xue Gong and Yingjuan Yang

School of Mathematics-Physics and Finance, Anhui Polytechnic University, Wuhu, Anhui, People’s Republic of China

ABSTRACT
In this paper, we investigate the stability of highly nonlinear hybrid neutral pantograph stochastic differential equations (NPSDEs) with general decay rate. By applying the method of the Lyapunov function, the $p$th moment and almost sure stability with general decay rate of solution for NPSDEs are derived. Finally, an example is presented to show the effectiveness of the proposed methods.

1. Introduction

Hybrid stochastic differential delay equations (SDDEs) have been widely used to model many practical systems in science and industry, where the systems may experience abrupt changes in their structure and parameters. One of the important issues in the study of hybrid SDDEs is the analysis of stability and boundedness (see, e.g. Hu et al., 2021; Li et al., 2018; Mao, 1999, 2002, 2007; Wu et al., 2018). Most of the papers in this area only considered stability problems where the underlying systems were either linear or nonlinear with the linear growth condition (see e.g. Mao, 1991; Mao & Yuan, 2006; Qi et al., 2021). However, many hybrid SDDEs do not satisfy this linear growth condition (namely they are highly nonlinear). Recently, some significant results have been done for highly nonlinear stochastic delay systems. For instance, Fei et al. (2017) explored delay dependent stability of hybrid SDDEs under the polynomial growth condition, while Fei et al. (2018) gave the structured robust stability criteria for highly nonlinear hybrid SDDEs.

A pantograph differential equation was used to investigate how an electric current is collected by the pantograph of an electric locomotive. Pantograph stochastic differential equations (PSDEs) as a class of special SDDE which have unbounded delay have been frequently applied in many practical areas, including electrodynamics and the collection of current by the pantograph of an electric locomotive. In recent years, the stability of PSDEs without linear growth condition has also attracted the interest of many researchers. For example, Hu et al. (2013) obtained polynomial stability of a class nonlinear hybrid PSDE. Shen, Mei, et al. (2019) explored the structured robust stability of highly nonlinear hybrid PSDEs. You et al. (2015) investigated the exponential stability of solutions for highly nonlinear hybrid PSDEs which experience abrupt changes in their parameters. On the other hand, many practical stochastic systems may not only depend on present and past states but also involve derivatives with delays. The neutral stochastic differential equations (NSDEs) have been developed to cope with such a situation, a great deal of results about the stability for NSDEs have been reported in the literature (see e.g. Shen, Fei, et al., 2019; Shen, Fei, Fei, et al., 2020; Shen et al., 2018; Song & Shen, 2013; Wu et al., 2010; Zhang et al., 2019). However, to the best of our knowledge, there are only few results about the stability for NPSDEs. For example, Shen, Fei, Mao, et al. (2020) investigated the exponential stability of NPSDEs by both Lyapunov functional and M-matrix method. Liu and Deng (2018) discussed $p$th moment exponential stability of highly nonlinear NPSDEs driven by Lévy noise.

In fact, the speed of the solution decays to zero is different. However, these stability concepts can be generalized to the stability with general decay rate (see, e.g. Li & Deng, 2017; Song & Shen, 2013; Wu et al., 2010). Consider a hybrid pantograph stochastic differential equation

$$d[x(t) − D(x(θt), r(t), t)] = f(x(t), x(θt), r(t), t) dt + g(x(t), x(θt), r(t), t) dB(t),$$

$$0 < θ < 1,$$  

(1)
where $B(t)$ is a scalar Brownian motion, $r(t)$ is a right-continuous Markov. Mao et al. (2019) have studied almost sure stability of NPSDEs (1) with Markov switching. However, the results in Mao et al. (2019) may have some limitations in application. We now state a theorem of Mao et al. (2019) to show the limitations.

**Theorem 1.1:** There exist function $V(x, i, t) \in C^2_1(R^n \times S \times R_+, R_+)$, and positive constants $\gamma, c_1, c_2, a_i, (i = 1, \ldots, 4)$ such that

\[
\begin{align*}
    c_1|x|^2 &\leq V(x, i, t) \leq c_2|x|^2, \\
    LV(x, y, i, t) &\leq -\alpha_1|x|^2 + \alpha_2 \theta \psi^{-\epsilon}((1 - \theta)t)|y|^2 \\
    &\quad - \alpha_3|x|^\gamma + \alpha_4 \theta \psi^{-\epsilon}((1 - \theta)t)|y|^\gamma,
\end{align*}
\]

where $c_2 > c_1, \alpha_1 > \alpha_2 \geq 0, \alpha_3 > \alpha_4 \geq 0$. Then the solution of the NPSDE (1) has the property that

\[
\lim sup_{t \to \infty} \frac{\log |x(t)|}{\log \psi(t)} < -\frac{\eta}{2} \text{ a.s.}
\]

where $\eta$ is a sufficient small positive constant.

However, Theorem 3.4 in Mao et al. (2019) requires not only the function $V(x, i, t)$ has the same degree for each $i \in S$ but also that the diffusion operator $LV(x, y, i, t)$ be bounded by a polynomial with the same degree for every $i \in S$ (the notation will be explained in Section 2). These restrictive requirements make Theorem 3.4 to be applied to hybrid NPSDEs which have different nonlinear structures in different modes. To explain more clearly, let us, for example, consider a scalar hybrid NPSDE

\[
d[x(t) - D(x(\theta(t)), r(t), t)] = f(x(t), x(\theta(t)), r(t), t) dt \\
+ g(x(t), x(\theta(t)), r(t), t) dB(t), \quad 0 < \theta < 1,
\]

where $B(t)$ is a scalar Brownian motion, $r(t)$ is a right-continuous Markov chain on the state space $S = \{1, 2\}$ with the generator

\[
\Gamma = \begin{pmatrix}
-1 & 1 \\
8 & -8
\end{pmatrix},
\]

while

\[
\begin{align*}
\theta &= 0.5, \quad D(y, 1, t) = 0.2 \exp(-0.25t)y, \\
D(y, 2, t) &= 0.2 \exp(-0.25t)y, \\
f(x, y, 1, t) &= -3x - 4x^3, \quad f(x, y, 2, t) = -0.5x, \\
g(x, y, 1, t) &= 0.5 \exp(-0.25t)y^2, \\
g(x, y, 2, t) &= 0.5 \exp(-0.25t)y.
\end{align*}
\]

If we define the Lyapunov function as

\[
V(x, i, t) = \begin{cases}
\theta_1 x^2 & \text{if } i = 1, \\
\theta_2 x^2 & \text{if } i = 2,
\end{cases}
\]

where $\theta_1, \theta_2$ are positive numbers.

It is easy to show

\[
\begin{align*}
LV(x, y, 1, t) &\leq (-4\theta_1 + \theta_2)x^2 \\
&\quad - 4\theta_1 x^4 + 0.25\theta_1 \exp(-0.5t)y^4, \\
LV(x, y, 2, t) &\leq (-9\theta_2 + 8\theta_1)x^2 \\
&\quad - 0.25\theta_2 \exp(-0.5t)y^2.
\end{align*}
\]

No matter what parameters $\theta_1$ and $\theta_2$ we choose, we cannot get the form as condition (2) has. This problem prevents Theorem 1.1 from being used.

Due to this NPSDE has different nonlinear structures in different modes, it is natural to use different types of Lyapunov functions in different modes. For example, let us define the Lyapunov function as

\[
V(x, i, t) = \begin{cases}
x^2 & \text{if } i = 1, \\
2x^2 + x^4 & \text{if } i = 2,
\end{cases}
\]

for $(x, t) \in R \times R_+$. Noting that $e^{-at} \geq e^{-bt}$, if $a \leq b$, thus, by the inequality $a^\beta b^{1-\beta} \leq \beta a + (1 - \beta)b$, we can obtain

\[
\begin{align*}
LV(x, y, 1, t) &\leq -4.2x^2 + 0.84 \exp(-0.5t)y^2 \\
&\quad - 5x^4 + 0.92 \exp(-0.5t)y^4 \\
LV(x, y, 2, t) &\leq -8.2x^2 + 1.5 \exp(-0.5t)y^2 \\
&\quad - 7.4x^4 + 1.8 \exp(-0.5t)y^4.
\end{align*}
\]

Thus, we have

\[
\begin{align*}
LV(x, y, i, t) &\leq -4.2x^2 + 1.5 \exp(-0.5t)y^2 \\
&\quad - 5x^4 + 1.8 \exp(-0.5t)y^4 \\
&\leq -2.1(2x^2 + 2x^4) \\
&\quad + 0.9 \exp(-0.5t)(2y^2 + 2y^4).
\end{align*}
\]

Let $U_1(x) = x^2, U_2(x) = 2x^2 + 2x^4$, then we have

\[
U_1(x) \leq V(x, i, t) \leq U_2(x)
\]

and

\[
LV(x, y, i, t) \leq -2.1 U_2(x) + 1.8\theta \exp(-0.5t)U_2(\theta x).
\]

In this paper, we will establish the stability results under these weaker conditions.

The key contributions of our paper are highlighted below:
This paper investigates stability for highly nonlinear hybrid NPSDEs with general decay rate. A significant amount of new mathematics has been developed to deal with the difficulties due to the neutral term.

In the paper of Mao et al. (2019), there are some restrictive conditions which may exclude many nonlinear NPSDEs. In this paper, we will loosen this restrictive condition to cover a much wider class NPSDEs.

The stability criteria can be applied to a class of hybrid NPSDEs which have the same structures in different modes and a class of hybrid NPSDEs which have different structures in different modes.

To develop our theory, we will introduce some necessary notations in Section 2. Our main results on stability with general decay rate will be discussed in Section 3. We will present an example in Section 4 to illustrate our theory. We will finally conclude our paper in Section 5.

2. Notation

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e., it is increasing and right continuous while \(\mathcal{F}_0\) contains all \(P\)-null sets). Let \(\mathcal{B}(t) = (B_1(t), \ldots, B_m(t))^T\) be an \(m\)-dimensional Brownian motion defined on the probability space. Let \(r(t)\) be a continuous Markov chain on the probability space taking values in a finite state space \(S = \{1, 2, \ldots, N\}\) with generator \(\Gamma = (\gamma_{ij})_{N \times N}\) given by

\[
P[r(t + \Delta) = j | r(t) = i] = \begin{cases} 
\gamma_{ij} \Delta + o(\Delta) & \text{if } i \neq j, \\
1 + \gamma_{ii} \Delta + o(\Delta) & \text{if } i = j,
\end{cases}
\]

where \(\Delta > 0\) and \(\gamma_{ij} \geq 0\) is the transition rate from \(i\) to \(j\) if \(i \neq j\) while \(\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}\). We assume that the Markov chain \(r(\cdot)\) is independent of the Brownian motion \(\mathcal{B}(\cdot)\). We also denote by \(|x|\) the Euclidean norm for \(x \in \mathbb{R}^n\). If \(A\) is a vector or matrix, its transpose is denoted by \(A^T\). If \(A\) is a matrix, its trace norm is denoted by \(|A| = \sqrt{\text{trace}(A^T A)}\).

Consider an \(n\)-dimensional hybrid NPSDE

\[
d[x(t) - D(x(\theta t), r(t), t)] = f(x(t), x(\theta t), r(t), t) \, dt + g(x(t), x(\theta t), r(t), t) \, dB(t),
\]

\[
0 < \theta < 1,
\]

on \(t \geq 0\) with initial date \(x(0) = x_0, r(0) = r_0\), where the coefficients \(f: \mathbb{R}^n \times \mathbb{R}^n \times S \times R_+ \to \mathbb{R}^n\) and \(g: \mathbb{R}^n \times \mathbb{R}^n \times S \times R_+ \to \mathbb{R}^{n \times m}\) are Borel measurable functions.

For the purpose of stability with general decay rate, we need the following definition of \(\psi\)-type function.

**Definition 2.1 (Song & Shen, 2013; Wu et al., 2010):**

The function \(\psi: [-\tau, \infty) \to (0, \infty)\) is said to be \(\psi\)-type function if it satisfies the following conditions:

(i) It is continuous and nondecreasing in \(R\) and differentiable in \(R_+\);

(ii) \(\psi(0) = 1\) and \(\psi(\infty) = \infty\) and \(r = \sup_{t \geq 0} \psi'(t)/\psi(t) < \infty\);

(iii) \(\psi(t + s) \leq \psi(t)\psi(s)\), for any \(t, s \geq 0\).

It is obvious that the functions \(\psi(t) = e^t\) and \(\psi(t) = 1 + t\) are all \(\psi\)-type functions since they satisfy the three conditions of Definition 2.1.

As a standing hypothesis, we assume the coefficients are locally Lipschitz continuous.

**Assumption 2.2 (Mao, 1999, 2007):** For each integer \(h \geq 1\) there is a positive constant \(K_h\) such that

\[
|f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)|^2 + |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)|^2 \\
\leq K_h(|x - \bar{x}|^2 + |y - \bar{y}|^2)
\]

for those \(x, y, \bar{x}, \bar{y} \in \mathbb{R}^n\) with \(|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq h\) and all \((t, i) \in \mathbb{R}_+ \times S\).

**Assumption 2.3 (Mao et al., 2019):** Assume that there is a constant \(\lambda \in (0, 1)\) such that

\[
|D(u, i, t) - D(v, i, t)| \leq \lambda \psi^{-1}((1 - \theta) t)|u - v|
\]

for all \(u, v \in \mathbb{R}^n\) and \(D(0, i, t) = 0\), where \(\psi^{-1}\) denotes the reciprocal of \(\psi\).

For \(V \in C^{2,1}((\mathbb{R}^n \times S) \times \mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+), \) define an operator \(LV: \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}\) by

\[
LV(x, \bar{x}, y, \bar{y}, i, t) = V_t(x, i, t) + V_x(x, i, t)f(x, y, i, t)
\]

\[
+ \frac{1}{2} \text{trace}[g^T(x, y, i, t)Vxx(x, i, t)g(x, y, i, t)]
\]

\[
+ \sum_{j=1}^{N} \gamma_{ij} V(x, j, t),
\]

where

\[
V_t(x, i, t) = \frac{\partial V(x, i, t)}{\partial t},
\]

\[
V_x(x, i, t) = \left(\frac{\partial V(x, i, t)}{\partial x_1}, \ldots, \frac{\partial V(x, i, t)}{\partial x_n}\right),
\]

\[
V_{xx}(x, i, t) = \left(\frac{\partial^2 V(x, i, t)}{\partial x_k \partial x_l}\right)_{n \times n}.
\]
3. Stability of PNSDEs with general decay rate

With the notations and assumptions introduced in above, the following theorem shows the existence and uniqueness of the global solution. The theorem forms the foundation of this paper.

**Theorem 3.1:** Let Assumptions 2.2 and 2.3 hold. Assume there are the functions $V_1, V_2 \in C(R^0 \times R_+; R_+)$ and $V \in C^{2,1}(R^0 \times S \times R_+)$, as well as nonnegative constants $K, K_j, p_j (1 \leq j \leq L)$ for some integer $L$ such that

$$
\lim_{|x| \to 0} \inf_{0 \leq t < \infty} V_1(x, t) = \infty,
$$

$$
V_1(x, t) \leq V(x, i, t) \leq V_2(x, t), \quad \forall (x, i, t) \in R^0 \times S \times R_+
$$

and

$$
LV(x, y, i, t) \leq K + \sum_{j=1}^{L} K_j |\theta| |y|^{p_j} - |x|^{p_j},
$$

then for any given initial date, there is a unique global solution $x(t)$ to the NPSDE (4).

**Proof:** Fix the initial data $x_0 \in R^0$ and $r_0 \in S$ arbitrarily. Define $z(t) = x(t) - D(x(\theta t), r(t), t)$, then we have $|z(0)| \leq |x_0| + |D(x_0, r_0, 0)| \leq (1 + k)|x_0|$. Let $k_0 > 0$ be a sufficiently large integer such that $(1 + k)|x_0| < k_0$. For each integer $k > k_0$, define the stopping time

$$
\sigma_k = \inf\{t > 0 : |x(t)| \geq k\},
$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). It is easy to see that $\sigma_k$ is increasing as $k \to \infty$ and $\lim_{k \to \infty} \sigma_k = \infty$ a.s. By the generalized Itô formula (see, e.g. Mao & Yuan, 2006), we obtain that

$$
EV(z(t \wedge \sigma_k), r(t \wedge \sigma_k), t \wedge \sigma_k)
= V(z(0), r(0), 0)
+ E \int_0^{t \wedge \sigma_k} LV(x(s), x(\theta s), r(s), s) ds
\leq V(z(0), r(0), 0) + Kt
+ \sum_{j=1}^{L} K_j E \int_0^{t \wedge \sigma_k} |\theta| |x(\theta s)|^{p_j} - |x(s)|^{p_j} ds.
$$

Define $v_k = \inf\{|z| \geq (1 - \lambda)k|t| \geq 0 \} V_1(x, t)$. By Assumption 2.3, we have

$$
|x(\sigma_k) - D(x(\theta \sigma_k), r(\sigma_k), \sigma_k)|/|\sigma_k| \leq t
\geq (k - |D(x(\sigma_k), r(\sigma_k), \sigma_k)|)|/|\sigma_k| \geq (k - \lambda)|x(\sigma_k)|/|\sigma_k| \geq (1 - \lambda)k|\sigma_k|.
$$

By the definition of $\sigma_k$, we can obtain

$$
V(z(0), r(0), 0) + Kt
\geq E[|V_1(z \wedge \sigma_k), t \wedge \sigma_k|/|\sigma_k|]
\geq E \left[ \inf_{|z| \geq (1 - \lambda)k|t| \geq 0} V_1(x, t)|/|\sigma_k| \right]
= v_k P(\sigma_k \leq t).
$$

Noting that $v_k \to \infty$ as $k \to \infty$. Letting $k \to \infty$, we have that $P(\sigma \leq t) = 0$, which implies $\sigma > t$ a.s. Letting $t \to \infty$, we can get $\sigma \to \infty$ a.s, which implies that there exists a global solution $x(t)$ to system (4).

Before giving our results, we state the following lemma which plays a crucial role in this paper.

**Lemma 3.2:** Define the quasi polynomial function $G(x) = a_m|x|^m + \cdots + a_2|x|^2$, $x \in R^0$, where $|x|$ is the Euclidean norm of $x$, $a_m > 0$, $a_i \geq 0, i = 1, \ldots, m - 1$ and $m \geq 2$ are positive integers. Let Assumption 2.3 hold, we have the following conclusion:

$$
\int_0^T G(x(t)) - D(x(\theta t), r(t), t) dt
\leq \lambda \int_0^T \psi^{-1}((1 - \theta t)G(x(\theta t))) dt
+ (1 - \lambda)^{1-m} \int_0^T G(x(t)) dt, \quad \forall T > 0.
$$

**Proof:** For $p \geq 2$, we apply the inequality

$$
(u + v)^p \leq (1 + \sigma)^{p-1}(u^p + \sigma^{1-p}v^p)
\forall u, v \geq 0, \sigma > 0,
$$

we can have

$$
|x(t) - D(x(\theta t), r(t), t)|^p
\leq (1 + \sigma)^{p-1}(|x(t)|^p + \sigma^{1-p}|D(x(\theta t), r(t), t)|^p).
$$

Setting $\sigma = \frac{\lambda}{1 - \lambda}$, we derive

$$
|x(t) - D(x(\theta t), r(t), t)|^p
\leq (1 - \lambda)^{1-p}|x(t)|^p + \lambda^{1-p}|D(x(\theta t), r(t), t)|^p
\leq (1 - \lambda)^{1-p}|x(t)|^p + \lambda\psi^{-1}((1 - \theta t)|x(\theta t)|^p
\leq (1 - \lambda)^{1-p}|x(t)|^p + \lambda\psi^{-1}((1 - \theta t)|x(\theta t)|^p.
$$
which, together with (6) and $0 < \lambda < 1$, shows that
\[
\int_0^T G(x(t) - D(x(\theta t), r(t), t)) \, dt
\]
\[
= \int_0^T \left( a_m|x(t) - D(x(\theta t), r(t), t)|^m + \cdots + a_2|x(t) - D(x(\theta t), r(t), t)|^2 \right) \, dt
\]
\[
\leq (1 - \lambda)^{1-m} \int_0^T (a_m|x(t)|^m + \cdots + a_2|x(t)|^2) \, dt
\]
\[
+ \lambda \int_0^T \psi^{-1}((1 - \theta)t)(a_m|x(\theta t)|^m + \cdots + a_2|x(\theta t)|^2) \, dt
\]
\[
= \lambda \int_0^T \psi^{-1}((1 - \theta)t)G(x(\theta t)) \, dt
\]
\[
+ (1 - \lambda)^{1-m} \int_0^T G(x(t)) \, dt. \tag{7}
\]
Hence, the proof is complete. \[\blacksquare\]

The key technique used in this paper is the method of the Lyapunov function. To study the stability of the NPSDE (4), we need to impose the following assumption which gives the our Lyapunov function.

**Assumption 3.3:** There exists function $V(x, i, t) \in C^{1,1}(\mathbb{R}^n \times S \times R_+, R_+)$, $U_1, U_2$ are nonnegative coefficient quasi polynomial functions, and positive constants $\alpha_i, (i = 1, \ldots, 4)$ such that
\[
0 \leq \alpha_2 \leq \alpha_1, \quad \alpha_4 < \alpha_3,
\]
\[
U_1(x) \leq V(x, i, t) \leq U_2(x), \quad V(x, i, t) \in \mathbb{R}^n \times S \times R_+, \quad \forall (x, i, t) \in \mathbb{R}^n \times S \times R_+, \tag{8}
\]
\[
LV(x, y, i, t) \leq -\alpha_1 U_1(x) + \alpha_2 \theta \psi^{-1}((1 - \theta)t)|y|^2
\]
\[
- \alpha_3 U_2(x) + \alpha_4 \theta \psi^{-1}((1 - \theta)t)U_2(y). \tag{9}
\]
We can finally establish our theory on the stability of NPSDEs with a generalized decay rate.

**Theorem 3.4:** Let Assumptions 2.2, 2.3 and 3.3 hold, then for any given initial date, the solution of the NPSDE (4) has the properties that
\[
\int_0^\infty EU_2(x(t)) \, dt < \infty, \tag{10}
\]
\[
\limsup_{t \to \infty} \frac{\log EU_1(x(t) - D(x(\theta t), r(t), t))}{\log \psi(t)} < 0. \tag{11}
\]

**Proof:** The condition (9) is stronger than Assumption 5, thus, there is a unique global solution for Equation (4). By the generalized Itô formula (see, e.g. Mao & Yuan, 2006), we obtain that
\[
EV(x(t), r(t), t) = V(x(0), r(0), 0)
\]
\[
+ E \int_0^t LV(x(s), x(\theta s), r(s), s) \, ds.
\]
By condition (8) and (9) and $\psi^{-1}((1 - \theta)t) \leq 1$, we can obtain that
\[
EV(x(t), r(t), t) \leq V(x(0), r(0), 0) - \alpha_1 E \int_0^t U_1(x(s)) \, ds
\]
\[
+ \alpha_2 \theta E \int_0^t U_1(x(\theta s)) \, ds
\]
\[
- \alpha_3 E \int_0^t U_2(x(s)) \, ds
\]
\[
+ \alpha_4 \theta E \int_0^t U_2(x(\theta s)) \, ds.
\]
Noting that
\[
\theta E \int_0^t U_1(x(\theta s)) \, ds \leq E \int_0^t U_1(x(s)) \, ds,
\]
\[
\theta E \int_0^t U_2(x(s)) \, ds \leq E \int_0^t U_2(x(s)) \, ds.
\]
For $\alpha_1 \geq \alpha_2, \alpha_3 > \alpha_4$ we can obtain
\[
(\alpha_3 - \alpha_4) E \int_0^t U_2(x(s)) \, ds \leq V(x(0), r(0), 0).
\]
Letting $t \to \infty$, we get
\[
E \int_0^\infty U_2(x(s)) \, ds \leq \frac{V(x(0), r(0), 0)}{\alpha_3 - \alpha_4}.
\]
Using the well-known Fubini theorem to obtain
\[
\int_0^\infty EU_2(x(s)) \, ds \leq \frac{V(x(0), r(0), 0)}{\alpha_3 - \alpha_4},
\]
which implies the assertion (10). By the generalized Itô formula, we have
\[
E[\psi^\varepsilon(t)V(x(t), r(t), t)]
\]
\[
= V(x(0), r(0), 0)
\]
\[
+ \varepsilon \int_0^t \frac{\psi^\varepsilon(s)}{\psi(s)} \psi^\varepsilon(s)V(x(s), r(s), s) \, ds
\]
\[
+ \varepsilon \int_0^t \psi^\varepsilon(s)LV(x(s), x(\theta s), r(s), s) \, ds.
\]
By Lemma 3.2, we have
\[
E \int_0^t \psi'(s) \psi^e(s) V(x(s), s) \, ds \\
\leq E \int_0^t \varepsilon \psi'(s) U_2(x(s)) \, ds \\
\leq \lambda \varepsilon t E \int_0^t \psi^{-1}((1 - \theta)s) \psi^e(s) U_2(x(\theta s)) \, ds \\
+ C_0 \varepsilon \int_0^t \psi'(s) U_2(x(s)) \, ds,
\]
where \( C_0 \) depends on the highest power of \( U_2 \). Thus, we can obtain
\[
E[\psi^e(t) V(x(t), r(t), t)] \\
\leq V(x(0), r(0), 0) - \alpha_1 E \int_0^t \psi^e(s) U_1(x(s)) \, ds \\
+ \alpha_2 \theta E \int_0^t \psi^e(s) \psi^{-1}((1 - \theta)s) U_1(x(\theta s)) \, ds \\
- (\alpha_3 - \varepsilon r C_0) E \int_0^t \psi^e(s) U_2(x(s)) \, ds \\
+ (\alpha_4 + \varepsilon \theta r) E \int_0^t \psi^e(s) \psi^{-1}((1 - \theta)s) U_2(x(\theta s)) \, ds.
\]
By Definition 2.1, we have that \( \psi^e(t) \psi^{-1}((1 - \theta)t) \leq \psi^e(\theta t), \) thus,
\[
E[\psi^e(t) V(x(t), r(t), t)] \leq V(x(0), r(0), 0) \\
- (\alpha_1 - \alpha_2) E \int_0^t \psi^e(s) U_1(x(s)) \, ds \\
- (\alpha_3 - \alpha_4 - \varepsilon r C_0 - \varepsilon \theta r) E \int_0^t \psi^e(s) U_2(x(s)) \, ds.
\]
Now we choose \( \varepsilon \) sufficiently small such that \( \alpha_3 \geq \alpha_4 + \varepsilon r C_0 + \varepsilon \theta r, \) then by (8), we can deduce that
\[
\psi^e(t) EU_1(x(s) - D(x(\theta t), r(t), t)) \\
\leq V(x(0), r(0), 0), \quad \forall t \geq 0, \quad (12)
\]
which implies assertion (11).

**Proof:** Noting that
\[
|x(t)|^p \leq (1 - \lambda)^{-p} |x(t) - D(x(\theta t), r(t), t)|^p \\
+ \lambda^{-p} |D(x(\theta t), r(t), t)|^p \\
\leq (1 - \lambda)^{-p} |x(t) - D(x(\theta t), r(t), t)|^p \\
+ \lambda^{-1}((1 - \theta)t) |\psi(\theta t)|^p \\
\leq (1 - \lambda)^{-p} |x(t) - D(x(\theta t), r(t), t)|^p + \lambda |\psi(\theta t)|^p.
\]
Then, for any \( T > 0 \), we have
\[
\sup_{0 \leq t \leq T} E|x(t)|^p \\
\leq (1 - \lambda)^{-p} \sup_{0 \leq t \leq T} E|x(t) - D(x(\theta t), r(t), t)|^p \\
+ \sup_{0 \leq t \leq T} \lambda E|\psi(\theta t)|^p \\
\leq (1 - \lambda)^{-p} \sup_{0 \leq t \leq T} E|x(t) - D(x(\theta t), r(t), t)|^p \\
+ \sup_{0 \leq t \leq T} \lambda E|x(t)|^p,
\]
which means
\[
(1 - \lambda)^{-p} \sup_{0 \leq t \leq T} E|x(t)|^p \\
\leq \sup_{0 \leq t \leq T} E|x(t) - D(x(\theta t), r(t), t)|^p.
\]
This together with (12), we can get
\[
\sup_{0 \leq t < \infty} \psi^e(t) E|x(t)|^p \leq C_1,
\]
where \( C_1 \) is a positive constant. Letting \( T \to \infty \), we have
\[
\sup_{0 \leq t < \infty} \psi^e(t) E|x(t)|^p \leq C_1, \quad (13)
\]
that is
\[
\limsup_{t \to \infty} \frac{\log E|x(t)|^p}{\log \psi(t)} \leq -\varepsilon.
\]
By the generalized Itô formula, we have
\[
\psi^e(t) V(x(t), r(t), t) \\
= V(x(0), r(0), 0) \\
+ \int_0^t \psi'(s) V(x(s), r(s), s) \, ds \\
+ \int_0^t \psi^e(t) LV(x(s), x(\theta s), r(s), s) \, ds + M_1(t),
\]
where \( M_1(t) \) is a local martingale with the initial value \( M_1(0) = 0 \). Applying the same argument on deriving (13),
we have
\[ \sup_{0 \leq t < \infty} \psi^\varepsilon(t) |x(t)|^p \leq C_1 + M_2(t), \]
where \( M_2(t) \) is a local martingale. Applying the nonnegative semi-martingale convergence theorem, we get
\[ \psi^\varepsilon(t) |x(t)|^p < \infty \quad \text{a.s.} \]
Therefore, there is a finite positive random variable \( \xi \) such that
\[ \psi^\varepsilon(t) |x(t)|^p \leq \xi \quad \text{a.s.} \]
which implies that
\[ \limsup_{t \to \infty} \frac{\log |x(t)|}{\log \psi(t)} < 0 \quad \text{a.s.} \]
which completes the proof. \( \square \)

4. An example

In this section we will discuss an example to illustrate our theory. Although our example is scalar hybrid NPSDEs, it will illustrate our theory fully.

**Example 4.1:** Consider a scalar hybrid NPSDE
\[
d[x(t) - D(x(\theta t), r(t), t)] = f(x(t), x(\theta t), r(t), t) \, dt \\
+ g(x(\theta t), r(t), t) \, dB(t),
\]
where \( B(t) \) is a scalar Brownian motion, \( r(t) \) is a Markov chain on the state space \( S = \{1, 2\} \) with its generator
\[
\Gamma = \begin{pmatrix}
-1 & 1 \\
8 & -8
\end{pmatrix}.
\]
The corresponding initial values are given by \( x(0) = x_0 \in \mathbb{R}, r(0) = i_0 \in S \), moreover, \( \theta, D, f \) and \( g \) are defined by (3).

By Theorem 3.4 and Corollary 3.5, we can conclude that the solution of the NPSDE (14) has the properties that
\[
\limsup_{t \to \infty} \frac{\log E|x(t)|^2}{t} < 0
\]
and
\[
\limsup_{t \to \infty} \frac{\log |x(t)|}{t} < 0 \quad \text{a.s.}
\]
Moreover, the solution also obeys
\[
\int_0^\infty Ex^4(t) \, dt < \infty \quad \text{and} \quad \int_0^\infty x^4(t) \, dt < \infty.
\]
This shows that solution of Equation (14) is almost surely exponentially stable. Figure 1 illustrates the sample path of the NPSDE (14) while Figure 2 illustrates the moment characteristic of \( \int_0^t x^4(s) \, ds \) for the solution \( x(t) \) of the NPSDE (14).

5. Conclusion

In this paper we have established the general decay rate stability criteria for hybrid NPSDEs by removing the linear growth condition. Our results not only can be applied to a class of hybrid NPSDEs which have the same structures in different modes but also can be applied to a class of hybrid NPSDEs which have different structures in different modes.

Disclosure statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
**Funding**

This work was supported by the Natural Science Foundation of University of Anhui Province under grant KJ2020A0367 and KJ2020A0368, the Startup Foundation for Introduction Talent of Anhui Polytechnic University under grant 2020YQQ066 and 2021YQQ056, the Pre-research Foundation of National Natural Science Foundation of China of Anhui Polytechnic University under grant Xjky08201906.

**ORCID**

Mingxuan Shen http://orcid.org/0000-0002-2197-7289

**References**

Fei, W., Hu, L., Mao, X., & Shen, M. (2017). Delay dependent stability of highly nonlinear hybrid stochastic systems. *Automatica*, 82, 165–170. https://doi.org/10.1016/j.automatica.2017.04.050

Fei, W., Hu, L., Mao, X., & Shen, M. (2018). Structured robust stability and boundedness of nonlinear hybrid delay systems. *SIAM Journal on Control and Optimization*, 56(4), 2662–2689. https://doi.org/10.1137/17M1146981

Hu, L., Mao, X., & Shen, Y. (2013). Stability and boundedness of nonlinear hybrid stochastic differential delay equations. *Systems & Control Letters*, 62(2), 178–187. https://doi.org/10.1016/j.sysconle.2012.11.009

Hu, Z., Ma, L., Wang, B., Zou, L., & Bo, Y. (2021). Finite-time consensus control for heterogeneous mixed-order nonlinear stochastic multi-agent systems. *Systems Science & Control Engineering*, 9(1), 405–416. https://doi.org/10.1080/21642583.2021.1914238

Li, K., Suo, J., & Shen, B. (2018). Mean-square exponential stability for the hysteretic Hopfield neural networks with stochastic disturbances. *Systems Science & Control Engineering*, 6(1), 547–553. https://doi.org/10.1080/21642583.2018.1552899

Li, M., & Deng, F. (2017). Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with Lévy noise. *Nonlinear Analysis: Hybrid Systems*, 24, 171–185. https://doi.org/10.1016/j.nahs.2017.01.001

Liu, L., & Deng, F. (2018). 8th moment exponential stability of highly nonlinear neutral pantograph stochastic differential equations driven by Lévy noise. *Applied Mathematics Letters*, 86, 313–319. https://doi.org/10.1016/j.aml.2018.07.003

Mao, W., Hu, L., & Mao, X. (2019). Almost sure stability with general decay rate of neutral stochastic pantograph equations with Markovian switching. *Electronic Journal of Qualitative Theory of Differential Equations*, 52, 1–17. https://doi.org/10.14232/ejqtde.2019.1.52

Mao, X. (1991). *Stability of stochastic differential equations with respect to semimartingales*. Longman Scientific and Technical.

Mao, X. (1999). Stability of stochastic differential equations with Markovian switching. *Stochastic Processes and Their Applications*, 79(1), 45–67. https://doi.org/10.1016/S0304-4149(98)00070-2

Mao, X. (2002). Exponential stability of stochastic delay interval systems with Markovian switching. *IEEE Transactions on Automatic Control*, 47(10), 1604–1612. https://doi.org/10.1109/TAC.2002.803529

Mao, X. (2007). *Stochastic differential equations and applications* (2nd ed.). Horwood Publishing.

Mao, X., & Yuan, C. (2006). *Stochastic differential equations with Markovian switching*. Imperial College Press.

Qi, W., Ju, H. P., Zong, G., Cao, J., & Cheng, J. (2021). Synchronization for quantized semi-Markov switching neural networks in a finite time. *IEEE Transactions on Neural Networks and Learning Systems*, 32(3), 1264–1275. https://doi.org/10.1109/TNNLS.5962385

Shen, M., Fei, C., Fei, W., & Mao, X. (2019). Boundedness and stability of highly nonlinear hybrid neutral stochastic systems with multiple delays. *Science China Information Sciences*, 62, Article 202205. https://doi.org/10.1007/s11432-018-9755-7

Shen, M., Fei, C., Fei, W., & Mao, X. (2020). Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations. *Systems & Control Letters*, 137, Article 104645. https://doi.org/10.1016/j.sysconle.2020.104645

Shen, M., Fei, W., Mao, X., & Deng, S. (2020). Exponential stability of highly nonlinear neutral pantograph stochastic differential equations. *Asian Journal of Control*, 22(1), 436–448. https://doi.org/10.1002/asjc.v22.1

Shen, M., Fei, W., Mao, X., & Liang, Y. (2018). Stability of highly nonlinear neutral stochastic differential delay equations. *Systems & Control Letters*, 115, 1–8. https://doi.org/10.1016/j.sysconle.2018.02.013

Shen, M., Mei, C., & Deng, S. (2019). Analysis on structured stability of highly nonlinear pantograph stochastic differential equations. *Systems Science & Control Engineering*, 7(3), 54–64. https://doi.org/10.1080/21642583.2019.1651679

Song, Y., & Shen, Y. (2013). New criteria on asymptotic behavior of neutral stochastic functional differential equations. *Automatica*, 49(2), 626–632. https://doi.org/10.1016/j.automatica.2012.11.045

Wu, F., Hu, S., & Huang, C. (2010). Robustness of general decay stability of nonlinear neutral stochastic functional differential equations with infinite delay. *Systems & Control Letters*, 59(3–4), 195–202. https://doi.org/10.1016/j.sysconle.2010.01.004

Wu, X., Tang, Y., Cao, J., & Mao, X. (2018). Stability analysis for continuous-time switched systems with stochastic switching signals. *IEEE Transactions on Automatic Control*, 63(9), 3083–3094. https://doi.org/10.1109/TAC.9

You, S., Mao, W., Mao, X., & Hu, L. (2015). Analysis on exponential stability of hybrid pantograph stochastic differential equations with highly nonlinear coefficients. *Applied Mathematics and Computation*, 263(4), 73–83. https://doi.org/10.1016/j.amc.2015.04.022

Zhang, H., Cao, J., & Xiong, L. (2019). New stochastic stability criteria for nonlinear neutral Markovian jump systems. *International Journal of Control, Automation and Systems*, 17(3), 630–638. https://doi.org/10.1007/s12555-018-0442-x