Efficient Isomorphism for $S_d$-Graphs and $T$-Graphs

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Abstract
An $H$-graph is one representable as the intersection graph of connected subgraphs of a suitable subdivision of a fixed graph $H$, introduced by Biró et al. (Discrete Mathematics 100:267–279, 1992). An $H$-graph is proper if the representing subgraphs of $H$ can be chosen incomparable by the inclusion. In this paper, we focus on the isomorphism problem for $S_d$-graphs and $T$-graphs, where $S_d$ is the star with $d$ rays and $T$ is an arbitrary fixed tree. Answering an open problem of Chaplick et al. (2016, personal communication), we provide an FPT-time algorithm for testing isomorphism and computing the automorphism group of $S_d$-graphs when parameterized by $d$, which involves the classical group-computing machinery by Furst et al. (in Proceedings of 11th southeastern conference on combinatorics, graph theory, and computing, congressum numerantium 3, 1980). We also show that the isomorphism problem of $S_d$-graphs is at least as hard as the isomorphism problem of posets of bounded width, for which no efficient combinatorial-only algorithm is known to date. Then we extend our approach to an XP-time algorithm for isomorphism of $T$-graphs when parameterized by the size of $T$. Lastly, we contribute an FPT-time combinatorial algorithm for isomorphism testing in the special case of proper $S_d$- and $T$-graphs.

Keywords Intersection graph · Isomorphism testing · Chordal graph · $H$-graph · Parameterized complexity
1 Introduction

A graph is a pair $G = (V, E)$ where $V = V(G)$ is the finite vertex set and $E = E(G)$ is the edge set—a set of unordered pairs of vertices. A subdivision of an edge $\{u, v\}$ of a graph $G$ is the operation of replacing $\{u, v\}$ with a new vertex $x$ and two new edges $\{u, x\}$ and $\{x, v\}$. Two graphs $G_1$ and $G_2$ are called isomorphic and denoted by $G_1 \cong G_2$, if there exists a bijection $f : V(G_1) \rightarrow V(G_2)$, called an isomorphism, such that $\{u, v\} \in E(G_1)$ if and only if $\{f(u), f(v)\} \in E(G_2)$ for all $\{u, v\} \subseteq V(G_1)$.

The graph isomorphism problem is to determine whether the two given graphs are isomorphic. It is in a sense a quite special problem in computer science; on one hand, under some widely-believed complexity-theoretic assumptions, it can be shown that graph isomorphism is not an NP-hard problem, while on the other hand, a polynomial-time algorithm for graph isomorphism is still elusive (and not everybody expects existence of such an algorithm). It has actually defined its own complexity class GI of the problems which are reducible in polynomial time to graph isomorphism. The current state of the art is a quasi-polynomial algorithm of Babai [5]. Nevertheless, the problem has been shown to be solvable efficiently for various natural graph classes such as trees, planar and permutation graphs [2, 13, 20] and for parameterized classes such as those listed below.

A (decision) problem with a parameter $k$ belongs to the class $\text{FPT}$ if it can be solved in time $f(k) \cdot n^{O(1)}$ where $f$ is a computable function and $n$ is the size of the input instance. Similarly, a decision problem with a parameter $k$ belongs to $\text{XP}$ if it can be solved in time $f(k) \cdot n^g(k)$ where $f$ and $g$ are two computable functions and $n$ is the size of the input. Even though the graph isomorphism problem has not been studied as intensively as other “classical” graph problems in the parameterized setting, some of the well-known parameterizations yielding to $\text{FPT}$- and $\text{XP}$-time algorithms are the maximum degree [24], eigenvalue multiplicity [14], genus [21, 26, 27], tree-depth [8] and tree-width [23].

Now, let us briefly introduce the graph classes which are the subject of our research. The intersection graph $G$ of a finite collection of sets $\{S_1, \ldots, S_n\}$ is a simple undirected graph in which each set $S_i$ is associated with a vertex $v_i \in V(G)$ and each pair $v_i, v_j$ of vertices is joined by an edge if and only if the corresponding sets have a non-empty intersection, i.e. $\{v_i, v_j\} \in E(G) \iff S_i \cap S_j \neq \emptyset$. The relevant special collections of sets are described below.

A graph is chordal if every cycle of length more than three induces a chord. This can be defined as the intersection graph of subtrees of some suitable tree [19]. Chordal graphs have linearly many maximal cliques which can be listed in polynomial time [28]. Deciding the isomorphism of chordal graphs is a $\text{GI-complete}$ problem [30]. This means that testing whether two chordal graphs are isomorphic is polynomial-time equivalent to the graph isomorphism problem in the general case.

A graph $G$ is an interval graph if it is the intersection graph for a set of intervals on the real line, and interval graphs form a subclass of chordal graphs. The isomorphism problem for interval graphs can be solved in linear time [7].

Split graphs are chordal graphs whose vertex set can be partitioned into a clique and an independent set. They present a special case of intersection graphs of substars.
of a suitable subdivided star, and the isomorphism problem for split graphs is also \textit{GI-complete} \cite{12}.

For a fixed graph \( H \), an \( \textit{H-graph} \) is the intersection graph of connected subgraphs of a suitable subdivision \( H' \) of the graph \( H \) \cite{6}. Such an intersection representation is also called an \( \textit{H-representation} \). They generalize all mentioned intersection graphs as follows. Interval graphs are \( K_2 \)-graphs, chordal graphs are the union of \( T \)-graphs where \( T \) ranges over all trees, and split graphs are contained in the union of \( S_d \)-graphs where \( d \) ranges over all positive integers. Every \( S_d \)-graph is chordal, but not every \( S_d \)-graph is a split graph. Various optimization problems such as maximum clique and minimum dominating set on \( H \)-graphs (for particular graphs \( H \)) have been shown to be solvable in polynomial and/or \textit{FPT}-time \cite{10,11,16}.

If a graph \( G \) has an \( \textit{H-representation} \), i.e., an intersection representation by subgraphs of a subdivision \( H' \) of \( H \), such that these subgraphs of \( H' \) are pairwise incomparable by inclusion (no one is a subgraph of another), then we speak about a \textit{proper representation} and \( G \) is called a \textit{proper H-graph}. Obviously, a proper \( H \)-graph is also an \( H \)-graph, but the converse is far from being true. For instance, the class of proper \( K_2 \)-graphs coincides with the class of unit interval graphs.

1.1 Organization of the Paper

- In Sect. 2, we give the definitions and basic properties of \( S_d \)- and \( T \)-graphs in closer detail.
- In Sect. 3, we consider \( S_d \)-graphs with bounded maximum clique size at most \( p \) and give, as a warm-up exercise, a simple combinatorial \textit{FPT}-time isomorphism algorithm parameterized only by \( p \) (Theorem 3.1).
- In Sect. 4, we prove that the \( S_d \)-graph isomorphism problem includes isomorphism testing of posets of width \( d \) (Theorem 4.2) which can be solved using the group-based approach by Furst et al. \cite{17} via Babai \cite{4} (but not known to have a combinatorial algorithm).
- In Sect. 5, as our main result, we combine the case of posets of bounded width with a specific adaptation of the general group-computing approach by Furst et al. \cite{17} to obtain an \textit{FPT}-time isomorphism algorithm for \( S_d \)-graphs (Theorem 5.10).
- In Sect. 6, we extend our result on \( S_d \)-graph isomorphism to \( T \)-graph isomorphism, and obtain an \textit{XP}-time isomorphism algorithm for \( T \)-graphs. This algorithm actually applies to all chordal graphs parameterized by their leafage.
- In Sect. 7, we focus on the isomorphism problem for proper \( S_d \)- and \( T \)-graphs, and show that their isomorphism can be tested in \textit{FPT}-time by a combinatorial algorithm.

We remark that all our algorithms expect only (abstract) graphs on the input, i.e., they do not require an intersection representation of the graph to be given.
2 \( S_d \)-Graphs and \( T \)-Graphs

We first introduce necessary details about representations of \( S_d \)- and \( T \)-graphs. For basic poset terms related to \( S_d \)- and \( T \)-representations, we refer the readers to [15]. We just recall that the width of a poset is the maximum size of its antichain, and this number is equal to the minimum number of chains covering all poset elements.

An \( S_d \)-graph \( G \) is the intersection graph of connected substars of a suitable subdivision \( S' \) of the star \( S_d \). In such a representation, every ray of \( S' \) defines an induced interval subgraph of \( G \), and the central node of \( S' \) defines a clique \( C \) of \( G \). We may always straightforwardly modify the representation such that \( C \) is a maximal (by inclusion) clique of \( G \). For further reference, we call a maximal clique \( C \) of \( G \) a central clique of \( G \) if there is an \( S_d \)-representation of \( G \) in which the central node defines \( C \).

Given a graph \( G \) and a maximal clique \( C \) of \( G \), let \( \mathcal{X} = \{X_1, X_2, \ldots, X_c\} \) denote the set of connected components of \( G - C \). For each connected component \( X_i \in \mathcal{X} \), let \( NC(X_i) \) be the set of neighbors of \( X_i \) in \( C \) which is called the attachment of \( X_i \).

A connected component together with its attachment edges is called a bridge of \( C \) in \( G \). Chaplick et al. [10] obtained a characterization of \( S_d \)-graphs yielding to a polynomial time recognition algorithm for arbitrary \( d \). This useful characterization is based on the following partial order \( P \) on the connected components of \( G - C \) which we will call the central poset of \( G \) on the clique \( C \).

Since each \( G[C \cup X_j] \) induces an interval subgraph, the neighborhoods of the vertices of \( X_i \) in \( C \) form a chain by inclusion. The upper attachment of \( X_i \), denoted by \( NC^U(X_i) \), is the maximum neighborhood in \( C \) among the vertices of \( X_i \), and it is \( NC(X_i) = NC^U(X_i) \). Analogously, the lower attachment, denoted by \( NC^L(X_i) \), is the minimum neighborhood in \( C \) among the vertices of \( X_i \). Note that the upper and lower attachments are well-defined since they are unique for each \( X_i \) in \( G \). After the attachment (i.e., the lower and upper one) of each connected component of \( G - C \) is determined, the partial order \( P \) is constructed by comparing the attachments of each pair of connected components. A pair \( (X_i, X_j) \) of components is comparable in \( P \), denoted by \( X_i \preceq P X_j \), if \( NC^U(X_i) \subseteq NC^L(X_j) \) holds, and incomparable if neither of \( X_i \preceq P X_j \), \( X_j \preceq P X_i \) holds. We will also speak about comparable/incomparable attachments (in \( C \)) of the components.

Naturally, for every chain \( X_1 \preceq P \cdots \preceq P X_k \), the induced subgraph \( G[C \cup X_k \cup \cdots \cup X_1] \) is also an interval graph. In addition, distinct connected components \( X_i \) and \( X_j \) of \( G - C \) are called equivalent and treated as one bridge of \( C \) when \( NC^U(X_i) = NC^L(X_i) = NC^U(X_j) = NC^L(X_j) \) holds.

The mentioned characterization simply reads:

**Proposition 2.1** (Chaplick et al. [10]) A graph \( G \) is an \( S_d \)-graph if and only if there exists a maximal clique \( C \) of \( G \) such that, for \( \mathcal{X} \) and \( P \) as above,

- for all \( X_i \in \mathcal{X} \), the induced subgraph \( G[C \cup X_i] \) is an interval graph, and
- the central poset \( P \) (of \( G \) on \( C \)) can be covered by at most \( d \) chains.

In Fig. 1a, we demonstrate an \( S_d \)-graph with its central clique \( C \) colored orange, and the connected components \( X_1, \ldots, X_6 \) of \( G - C \) colored differently. In Fig. 1b, we see the partial order \( P \) on the connected components of \( G - C \) with respect to their...
Fig. 1  a An $S_d$-graph $G$ with its central clique $C = \{1, 2, 3, 4\}$ and the connected components of $G - C$. b The partial order $P$ on the connected components, with their upper and lower attachment sets shown next to the components. c An $S_d$-representation of $G$ with $C$ in the center

attachments. The three tuples of components $(X_1, X_2, X_3), (X_4, X_5), (X_6)$ form a chain cover of $P$. Therefore, $G$ is an $S_d$-graph for $d = 3$. In Fig. 1c, the corresponding $S_d$-representation of $G$ is given where the connected components are placed on the rays of a subdivision of $S_3$ according to this chain cover, and $C$ is placed in the center.

If $G$ is not connected, then the characterization in Proposition 2.1 says that all components of $G$ not containing the central clique $C$ are interval graphs, and their attachments are empty (hence they are treated as one bridge in our setting, and can be placed together at the end of any edge of $S_d$).

Note that the intersection representation of an $S_d$-graph is not uniquely determined by a central maximal clique. For instance, the tuples $(X_1, X_6), (X_2, X_3), (X_4, X_5)$ also form a chain cover of the partial order given in Fig. 1b, and it leads to another $S_d$-representation. In particular, there can be $d^{\Omega(n)}$ many distinct chain covers of a poset of width $d$ on $n$ elements when the depth of $P$ is not bounded. This means that the isomorphism problem of $S_d$-graphs cannot be simply solved by comparing the rays using the linear time isomorphism test for interval graphs [7] for suitable pairs of maximal cliques.

We actually have an easy observation:

**Proposition 2.2** (originally proved in [30]) The isomorphism problem of proper $S_d$-graphs (and hence of all $S_d$-graphs) with $d$ on the input is GI-complete.

**Proof** Let $G_1$ and $G_2$ be two arbitrary graphs on the same number of at least 4 vertices. We construct $G'_i, i = 1, 2$, as follows: subdivide every edge with a new vertex, and then make a clique on the original vertex set $V(G_i)$. Then, $G_1 \simeq G_2$ if and only if
Fig. 2 a A $T$-graph $G$ with its two disjoint maximal cliques $C_1 = \{1, 2, 3, 4\}$ and $C_2 = \{5, 6, 7\}$. b A $T$-representation of $G$ with $C_1$ and $C_2$ placed on the branching nodes

$G'_1 \cong G'_2$; this is easy since any isomorphism of $G'_1$ and $G'_2$ must map $V(G_1)$ to $V(G_2)$ due to vertex degrees. Since each $G'_i$ is an $S_d$-graph for $d = |E(G_i)|$, with the central clique on $V(G_i)$, solving their isomorphism would solve also the isomorphism of $G_1$ and $G_2$. Furthermore, $G'_i$ is a proper $S_{d'}$-graph for $d' = |E(G_i)| + |V(G_i)|$, where the additional $|V(G_i)|$ rays of $S_{d'}$ are used to make the subgraphs representing the central clique pairwise incomparable.

More generally, a $T$-graph $G$ is the intersection graph of connected subtrees of a suitable subdivision $T'$ of a fixed tree $T$. In such a representation, the path between any pair of branching nodes of $T'$ (i.e., nodes of degree at least 3) defines an induced interval subgraph of $G$, and each branching node of $T'$ defines a clique of $G$ which can again be assumed maximal in $G$. $S_d$-graphs are $T$-graphs for a tree $T$ with only one branching node, and $T$-graphs are also chordal. While $S_{d'}$-graphs can be recognized in polynomial time for an arbitrary $d$ [10], the recognition problem for $T$-graphs is NP-complete when $T$ is on the input [22].

In [10], Chaplick et al. proved that a graph $G$ is a $T$-graph for some fixed tree $T$ if and only if there exists a set of maximal cliques $C_1, \ldots, C_k$ of $G$ placed on the branching nodes $b_1, \ldots, b_k$ of $T$ such that, for each $b_i$, the induced subgraph formed by the maximal clique $C_i$ placed on $b_i$ and a selection of the connected components of $G - (C_1 \cup \cdots \cup C_k)$ is an $S_{d'}$-graph with $d = \text{deg}(b_i)$ and with additional restrictions detailed in [10]. Using this characterization, they showed that $T$-graphs can be recognized in $\text{XP}$-time with respect to $|V(T)|$ by checking all assignments of maximal cliques to the branching nodes of $T$ by brute-force.

In Fig. 2a, we see a $T$-graph $G$ whose disjoint maximal cliques $C_1$ and $C_2$ placed on the branching nodes of $T$ are colored orange and yellow, respectively, and the
connected components $X_1, \ldots, X_7$ of $G - (C_1 \cup C_2)$ are colored differently. All $X_1, X_2$ and $X_3$ have non-empty attachments in $C_1$, thus they are placed on incident edges to $b_1$. Analogously, all $X_3, X_4, X_5, X_6$ and $X_7$ have non-empty attachments in $C_2$, thus they are placed on incident edges to $b_2$ (for $X_3$, its is the edge $b_1b_2$). For two $S_d$-instances with the centers $C_1$ and $C_2$, let $P_1$ and $P_2$ denote the partial orders on the connected components. Since $X_3$ has attachments in both $C_1$ and $C_2$, i.e. $(N_{C_1 \cup C_2}(X_3) \cap C_1) \setminus C_2 \neq \emptyset$ and $(N_{C_1 \cup C_2}(X_3) \cap C_2) \setminus C_1 \neq \emptyset$, it is the common component for these $S_d$-instances and appears in both $P_1$ and $P_2$. One of the chain covers of $P_1$ is $(X_1), (X_2), (X_3)$, and one of the chain covers of $P_2$ is $(X_3), (X_4, X_5), (X_6, X_7)$. In Fig. 2b, the corresponding $T$-representation of $G$ is given where the connected components are placed on the edges of $T$ according to these chain covers.

Let the leafage of a chordal graph $G$ be defined as the smallest number of leaves of a tree $T$ such that $G$ is a $T$-graph. We remark that the leafage can be equivalently defined only from the clique graph of $G$ [25].

Due to being a superclass of $S_d$-graphs, the isomorphism problem for $T$-graphs and proper $T$-graphs is also GI-complete by Proposition 2.2.

3 $S_d$-Graph Isomorphism Parameterized by the Clique Size

We start with an easy case and show that $S_d$-graph isomorphism can be solved in FPT-time when the maximal clique size of given $S_d$-graphs are bounded by a parameter $p$. We emphasize that the complexity of this case does not depend on the width $d$ and we do not need the $S_d$-representations.

Recall that $S_d$-graphs are chordal. Therefore, they have linearly many maximal cliques which can be listed in linear time [28]. For an $S_d$-graph $G$ with a central clique $C$, the induced subgraph $G[C \cup X_i]$ is an interval graph for each connected component $X_i$ of $G - C$, and the isomorphism of interval graphs can be tested in linear time [7]. Given two $S_d$-graphs $G$ and $H$, we hence compute their collections of maximal cliques $S$ and $T$, respectively. For each pair of equal-size cliques $C \in S$ and $D \in T$ such that all connected components in $G - C$ and $H - D$ are interval graphs, we can efficiently compare each pair of connected components of $G - C$ and $H - D$ to interval graph isomorphism. If these components can be perfectly matched with respect to the isomorphism, we only need to guarantee a consistent bijection map between the cliques $C$ and $D$ to correctly conclude that $G$ and $H$ are isomorphic. When the maximal clique size of given $G$ and $H$ is bounded by a parameter $p$, we can simply loop through all $p!$ possible bijections between the cliques $C$ and $D$. The fine details are shown in Algorithm 1. The complexity analysis of Algorithm 1. Let $n$ be the number of vertices of both $G$ and $H$. The maximal clique collections $S$ and $T$ of them (each of length $\leq n$) are found using simplicial vertex elimination in time $O(pn)$. Comparing the cardinalities of the members of $S$ and $T$ takes $O(n \log n)$ time. Finding the connected components of $G - C$ (also of $H - D$) takes linear time and testing for interval graphs is also in linear time [7], and so a suitable clique $C \in S$ can be fixed in time $O(n^3)$.

Then we loop through $D \in T$ and all labelings of $D$, in the worst case, which means at most $p! \cdot n$ iterations. Comparing the interval graphs $G[C \cup X_i]$ and $H[D \cup Y_j]$
Algorithm 1 Isomorphism test for $S_d$-graphs with bounded clique size $(G,H)$

**Require:** Given two $S_d$-graphs $G$ and $H$ (their representations not required), the parameters $d$ of $S_d$ and $p$ bounding the maximal clique size of $G$ and $H$.

**Ensure:** Result of the isomorphism test between $G$ and $H$.

1: Find the maximal clique collections $S$ and $T$ of the chordal graphs $G$ and $H$; 
2: if $|S| \neq |T|$, or the cardinalities of members of $S$ and $T$ do not match then
3: return “$G$ and $H$ are not isomorphic”;
4: repeat for each maximal clique $C \in S$:
5: Find the connected components $X_1, X_2, \ldots, X_k$ of $G - C$, assuming the equivalent connected components are joined into single bridge(s) of $C$;
6: until all $G[C \cup X_i], 1 \leq i \leq k$, are interval graphs;
7: Fix any bijective labeling $C \rightarrow \{1, 2, \ldots, |C|\}$ on the vertices of $C$;
8: for each $D \in T$ with $|C| = |D|$ do
9: Find the connected components $Y_1, Y_2, \ldots, Y_l$ of $H - D$, assuming the equivalent connected components are joined into single bridge(s) of $D$;
10: if $k = l$, and all $H[D \cup Y_j], 1 \leq j \leq k$, are interval graphs then
11: for each bijective $D \rightarrow \{1, 2, \ldots, |D|\}$ on the vertices of $D$ do
12: for each pair $X_i$ and $Y_j$, $1 \leq i, j \leq k$ do
13: Compare the interval graphs $G[C \cup X_i]$ and $H[D \cup Y_j]$ to isomorphism, respecting the labels of $C$ and $D$;
14: Mark isomorphic pairs of them as “symmetric”;
15: $\mathcal{Y} \leftarrow \{Y_1, Y_2, \ldots, Y_k\}$;
16: for each $X_i, 1 \leq i \leq k$ do
17: Find greedily $Y_j \in \mathcal{Y}$ symmetric to $X_i$, and delete $Y_j$ from $\mathcal{Y}$;
18: if successful for all $X_i, 1 \leq i \leq k$ then
19: return “$G$ and $H$ are isomorphic”;
20: return “$G$ and $H$ are not isomorphic”;

Comparing all the pairs of graphs $G[C \cup X_i]$ and $H[D \cup Y_j]$ takes altogether $O(pn^2)$. Checking for a perfect matching of symmetric (i.e., isomorphic) pairs among them is then trivially in time $O(n^2)$.

Thus, the overall complexity of this algorithm is

$$O(n^3 + p! \cdot n \cdot (pn^2 + n^2)) = O(p! \cdot pn^3)$$

which belongs to $\text{FPT}$ with respect to $p$. 

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Theorem 3.1 Algorithm 1 correctly decides whether two $S_d$-graphs are isomorphic in FPT-time parameterized by the maximal clique size $p$.

Proof Assume that Algorithm 1 returns “$G$ and $H$ are isomorphic”. Since the fixed labelings of $C$ and $D$ are respected when comparing each pair $G[C \cup X_i]$ and $H[D \cup Y_j]$ to interval graph isomorphism, we indeed have that $G \simeq H$. Note that this conclusion holds true regardless of whether the fixed clique $C$ is the central clique in some $S_d$-representation of $G$.

Let, on the other hand, $G$ and $H$ be two isomorphic $S_d$-graphs with cliques of size $\leq p$, and $f$ be their isomorphism. Since $G$ has an $S_d$-graph representation, an admissible clique $C$ must be found on line 6 of the algorithm (as the central clique of the representation, if not before). Then, for the fixed labeling of $C$ and the image of this labeling in $D$ under $f$, the isomorphism tests between $G[C \cup X_i]$ and $H[D \cup Y_j]$ where $Y_j = f(X_i)$ on line 13 must succeed. Moreover, since isomorphism is an equivalence relation, the subsequent greedy pairing of symmetric components succeeds as well. Hence the algorithm returns “$G$ and $H$ are isomorphic”, as desired. \hfill \square

4 Reduction to $S_d$-Graph Isomorphism from Posets of Width $d$

In Sect. 3, we focused on an easy case of $S_d$-graph isomorphism which can be tested in FPT-time parameterized by the maximal clique size $p$; by greedily trying all $p!$ labelings of the (suitably chosen) central cliques. In general, the maximal clique size can grow up to $\Omega(n)$, and thus we now cannot afford to try all their possible labelings. As suggested by Proposition 2.2, we must take advantage of the parameter $d$, that is, of the underlying structure of $S_d$-representations of $G$ and $H$ (as we will see, we do not need a particular representation given for that).

According to the characterization of Chaplick et al. from Proposition 2.1, it looks useful to study the isomorphism of the central posets of $G$ and $H$ (on their chosen central cliques) in order to decide isomorphism of $G$ and $H$ themselves. Before we do this in Sect. 5, we show that it is indeed necessary to be able to solve isomorphism of posets of width $d$; by giving the following polynomial-time reduction and subsequent Theorem 4.2.

Given a poset $P$ of width $d$ on $n$ elements, we construct an $S_d$-graph $G$ from $P$ (in polynomial time) as follows:

1. Model $P$ by the set inclusion between the sets $M_1, \ldots, M_n$, where each $M_i$ consists of all comparable elements with $i$ from the lower levels and itself. Formally, $M_i = \{ j \in P : j \preceq_P i \}$ for all $i \in P$.
2. Take the union $M = \bigcup_{i \in P} M_i$ and form the central clique $C$ of size $|M| + 2$ by adding $|M|$ vertices corresponding to the elements of $M$, two dummy vertices, and all edges on $C$. Note that the two dummy vertices are added to $C$ to mark $C$ as the unique maximum-size clique of $G$.
3. For each $M_i$, add a new vertex $v_i$ to $G$ adjacent exactly to the subset of vertices in $C$ corresponding to $M_i$.

Figure 3 is an illustration of this reduction. In Fig. 3a, we have a poset $P$ of width $d = 3$ whose elements are 1, 2, 3, 4, 5, 6, 7, 8 and 9. Using the above construction,
we model the poset \( P \) by the set inclusion with sets \( M_1, \ldots, M_9 \) as in Fig. 3b, and then we construct the graph \( G \) in Fig. 3c, where the cyan set consists of all vertices of the unique maximum clique \( C \) (assuming that there is an edge between each pair of vertices in this set). Each connected component of \( G - C \) is a single vertex denoted by \( v_1, \ldots, v_9 \) and adjacent to exactly the vertices from \( M_1, \ldots, M_9 \), respectively.

**Lemma 4.1** The graph \( G \) constructed from a poset \( P \) of width \( d \) using the above reduction is an \( S_d \)-graph.

**Proof** \( C \) is the maximal (and also maximum) clique of \( G \), represented in the branching node of a subdivision of \( S_d \). A chain cover of \( P \) of size \( \leq d \) determines an ordered distribution of the vertices \( V(G - C) = \{v_1, \ldots, v_n\} \) to the \( d \) rays of the representation, such that their adjacent vertices in \( C \) form a chain by inclusion on each ray. Hence this arrangement is realizable as interval graphs on the \( d \) rays of a subdivision of \( S_d \). \( \Box \)

**Theorem 4.2** The isomorphism problem of (colored) posets of width \( d \) reduces in polynomial time to the isomorphism problem of \( S_d \)-graphs.

**Proof** Let \( P \) and \( Q \) with \( n \) elements be two posets of width \( d \), and \( G \) and \( H \) be the \( S_d \)-graphs (cf. Lemma 4.1) formed from \( P \) and \( Q \), respectively, by the construction described above.

Assume that \( P \) and \( Q \) are isomorphic under a bijection \( f : P \rightarrow Q \). Then, \( f \) directly defines a bijection from the central clique \( C \) to \( D \), mapping the dummy vertices of \( C \) to those of \( D \) in any order, and a bijection from \( V(G - C) = \{v_1, \ldots, v_n\} \) to \( V(H - D) = \{w_1, \ldots, w_n\} \). The composition of these two mappings from \( C \) to \( D \) and \( G - C \) to \( H - D \) is clearly a graph isomorphism between \( G \) and \( H \).
Assume that $G$ and $H$ are isomorphic under a bijection $g : V(G) \rightarrow V(H)$. Since $C \subseteq V(G)$ and $D \subseteq V(H)$ are the unique maximum cliques, we have that $g(C) = D$. Therefore, a restriction of $g$ is a bijection from $V(G - C) = \{v_1, \ldots, v_n\}$ to $V(H - D) = \{w_1, \ldots, w_n\}$. With respect to the construction of $G$ and $H$, this restriction induces a bijection between the sets $\{M_1, \ldots, M_n\}$ of $P$ (as represented by $v_1, \ldots, v_n$) and the sets $\{N_1, \ldots, N_n\}$ of $Q$ (as represented by $w_1, \ldots, w_n$) and, in turn, a bijection $g' : P \rightarrow Q$. If $a \preceq_P b$, then $M_a \subseteq M_b$ and the neighborhood of $v_a$ in $G$ is included in the neighborhood of $v_b$. Then the neighborhood of $g(v_a)$ in $H$ is included in the neighborhood of $g(v_b)$ and, regarding to the induced bijection $g'$, $N_{g'(a)} \subseteq N_{g'(b)}$ and $g(a) \preceq_Q g'g(b)$. The converse implication holds the same way, and so the posets $P$ and $Q$ are isomorphic.

Lastly, we remark that if the input posets $P$ and $Q$ are given with colored elements, and we are looking for color-preserving isomorphism, we can use the same approach. Let, say, the given distinct colors be denoted by $c_1, \ldots, c_r$, where $r \in O(n)$. We then use the above reduction with the following changes: (i) If $j$ is a poset element of color $c_j$, then we represent $M_j$ in $G$ not by a single vertex $v_j$, but by a copy of the clique $K_i$. Therefore, only the mappings between the vertices contained in equal sized cliques are allowed corresponding to the color-preserving isomorphisms. (ii) The number of dummy vertices in this case is $r + 1$ in order to mark $C$ of size $n + r + 1$ as the unique maximum clique of $G$.

\[\Box\]

5 Isomorphism of $S_d$-Graphs in General, Parameterized by $d$

When $d$ is a part of the input, the isomorphism problem for $S_d$-graphs is GI-complete (Proposition 2.2), and we proved that $S_d$-graph isomorphism can be solved in FPT-time parameterized by the maximal clique size (Theorem 3.1). In this section, we consider $S_d$-graphs without bounding the clique size, and give an FPT-time algorithm solving their isomorphism parameterized by $d$.

We first recall the notion of the automorphism group which is closely related to the graph isomorphism problem. An automorphism is an isomorphism of a graph $G$ to itself, and the automorphism group of $G$ is the group $\text{Aut}(G)$ of all automorphisms of $G$. There exists an isomorphism from $G$ to $H$ if and only if the automorphism group of the disjoint union $G \uplus H$ contains a permutation exchanging the vertex sets of $G$ and $H$. In fact, assuming connectivity of the graphs $G$ and $H$, it is enough to look for a permutation mapping some vertex of $G$ to a vertex of $H$, and only among generators of the automorphism group. For further details regarding the automorphism groups, see e.g., [17].

The elements of any poset $R$ can be partitioned into levels $L_i \subseteq R$ where $i \geq 1$; $L_1$ is formed by the minimal elements of $R$, and $L_{i+1}$ is inductively formed by the minimal elements of $R \setminus (L_1 \cup \ldots \cup L_i)$.

Consider now two $S_d$-graphs $G$ and $H$. As forwarded in Sect. 4, we will approach the isomorphism problem for $G$ and $H$ via the automorphism group of the union of the underlying colored central posets of bounded width, whose elements are colored regarding the isomorphism types of their interval bridges in $G$ and $H$. However, we will also need to ensure that an automorphism on the central posets is indeed
consistent with some permutation on the union of the central cliques. In contrast to the exhaustive approach used in Sect. 3, we will utilize an involved group-computing approach respecting the attachment sets of bridges on $C \cup D$ (so-called cardinality Venn diagram on $C \cup D$).

We first give the following overview of our approach, and then we discuss it in more detail (cf. Algorithms 2 and 3).

**Procedure 5.1** Given two $S_d$-graphs $G$ and $H$ on $n$ vertices with central cliques $C \subseteq G$ and $D \subseteq H$ such that $|C| = |D|$, let $\mathcal{X} = \{X_1, \ldots, X_k\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$ be the sets of connected components in $G - C$ and $H - D$, respectively. Assume (as in Algorithm 1) that the equivalent connected components are joined into single bridge(s) of $C$ and $D$, respectively. Let $K = G \cup H$ be the disjoint union of our graphs. For $Z \in \mathcal{X} \cup \mathcal{Y}$, let $K(Z)$ denote the induced subgraph $G[C \cup Z]$ if $Z \in \mathcal{X}$, and the induced subgraph $H[D \cup Z]$ otherwise. Call an isomorphism of $K(Z)$ to $K(Z')$ respectful if it maps $V(K(Z)) \cap (C \cup D)$ to $V(K(Z')) \cap (C \cup D)$. We test the existence of an isomorphism between $G$ and $H$ mapping $C$ to $D$ as follows:

1. Construct the central posets $P$ and $Q$ on $\mathcal{X}$ and $\mathcal{Y}$, respectively, as described in Sect. 2. Make the disjoint union $R = P \cup Q$, and compute the levels of $R$. For each pair of components $Z, Z' \in \mathcal{X} \cup \mathcal{Y}$ on the same level of $R$, compare $K(Z)$ and $K(Z')$ to respectful isomorphism of interval graphs, and color $Z$ and $Z'$ with respect to the isomorphism type. That is, $Z$ and $Z'$ receive the same color if and only if $Z$ and $Z'$ are on the same level and $K(Z) \simeq K(Z')$ with a respectful isomorphism.

2. Respecting the colors by the previous step, compute the color-preserving automorphism group $\Gamma$ of $R$. We use Theorem 5.2 and Corollary 5.3 here.

3. Compute the subgroup $\Gamma'$ of $\Gamma$, consisting of those automorphisms $\rho$ of $R$ for which there exists a permutation $f_\rho$ of the set $C \cup D$ such that the following holds: for every component $Z \in \mathcal{X} \cup \mathcal{Y}$, there is a respectful isomorphism from $K(Z)$ to $K(\rho(Z))$ whose restriction to the intersection with $C \cup D$ equals the respective restriction of $f_\rho$. We use Lemma 5.4, and Theorem 5.6 with Corollary 5.8 here.

4. If $P$ and $Q$ are swapped in some automorphism from $\Gamma'$, then return that $G$ and $H$ are isomorphic.

If the above procedure does not say that $G$ and $H$ are isomorphic for any pair of maximal cliques $C \subseteq G$ and $D \subseteq H$, we return that $G$ and $H$ are non-isomorphic.

In step 1 of Procedure 5.1, we construct the central posets $P$ and $Q$, the disjoint union $R = P \cup Q$, and its levels. Then, we compute colors on $R$ using the interval graph isomorphism algorithm of [7]. All these are easily executed in polynomial time. We will show in detail how to achieve an FPT-time implementation of the other steps in the following subsections.

### 5.1 Computing the Automorphism Group of a Poset of Width $d$

Note that poset isomorphism problem is GI-complete [29]. However, posets of width $d$ are special and can be handled using the following classical concept of bounded color multiplicity.
A \textit{d-bounded color multiplicity graph} is a graph $G$ whose vertex set is arbitrarily partitioned into $k$ color classes $V(G) = V_1 \cup \ldots \cup V_k$ such that $V_i \cap V_j = \emptyset$ for all $1 \leq i < j \leq k$. The number $k$ of colors is arbitrary, but for all $1 \leq i \leq k$, the cardinality $|V_i|$, called the multiplicity of $V_i$, is at most $d$. We apply the following classical result in our approach.

\textbf{Theorem 5.2} (Babai [4], with Furst et al. [18]) \textit{The color-preserving automorphism group (i.e., the generators of it) of a d-bounded color multiplicity graph can be determined in FPT-time parameterized by d.}

Consider a poset $R$ of width $\leq d$ and the levels $L_1, \ldots, L_k$ of $R$, where $|L_i| \leq d$ for $1 \leq i \leq k$. By having the levels of $R$ as color classes, we can ignore the edge directions since the colors will directly correspond to the levels (from a vertex of the color corresponding to the lower level to the vertex of the color corresponding to the higher level). Therefore, a poset of width $d$ is a $d$-bounded multiplicity graph. Note that any automorphism of $R$ preserves its levels and this corresponds to the color-preserving automorphisms of a bounded multiplicity graph. Hence we have the following corollary:

\textbf{Corollary 5.3} ([4, 18]) \textit{The automorphism group of a poset $R$ of width $d$ can be determined in FPT-time parameterized by $d$.}

By Corollary 5.3, we can finish step 2 of Procedure 5.1 in FPT-time parameterized by $d$. However, this is not all. Having isomorphic central posets only ensures a kind of structural equivalence between the chosen central cliques of our graphs $G$ and $H$, regardless of the existence of a bijection between the elements of the central cliques.

In Fig. 4, as an illustration example, we have two isomorphic colored posets $P$ and $Q$ obtained from graphs $G$ and $H$ with the central clique $C = D = \{1, 2, \ldots, 8\}$ (shaded gray), such that the components of $G - C$ (of $H - C$) are 9 singleton vertices having attachments in exactly the listed vertices of $C$ (of $D$). The colors are fully determined by the cardinality of the shown attachments. In step 2, $P$ and $Q$ can be swapped respecting inclusion and the colors. However, the components of $G - C$ with blue attachments in $P$ (in Fig. 4, it is the second level marked with an arrow from the right) have a common neighbor (3) while the analogous two components of $H - D$ with blue attachments in $Q$ have no common neighbor. Therefore, the cardinalities of the intersections of blue attachments in $P$ and in $Q$ differ, which means that there is no bijection between $C$ and $D$ which maps these attachments to each other, and the corresponding graphs $G$ and $H$ are hence not isomorphic.

We next move to step 3 of Procedure 5.1, and show, in Lemma 5.4, that it suffices to ensure that problems with cardinalities of the intersections of attachment sets like the one in the example do not happen, to claim that $G$ and $H$ are indeed isomorphic.

\section{5.2 Checking the Consistency on the Central Cliques}

Recall the automorphism group $\Gamma$ of $R$ from step 2 of Procedure 5.1. We say that an automorphism $\varrho \in \Gamma$ is \textit{consistent on the central cliques} $C \cup D$ if $\varrho$ satisfies the condition in step 3 of Procedure 5.1, that is, if
there exists a permutation \( f_\varrho \) of \( C \cup D \) such that, for every component \( Z \in \mathcal{X} \cup \mathcal{Y} \), there is a respectful isomorphism from \( K(Z) \) to \( K(\varrho(Z)) \) whose restriction to \( C \cup D \) equals the respective restriction of \( f_\varrho \).

While we explicitly processed all bijections between the central cliques \( C \) and \( D \) in the simple approach given in Sect. 3, we consider no explicit permutations on \( C \cup D \) in this general case. Instead, we will indirectly check for an existence of a permutation \( f_\varrho \) on \( C \cup D \) witnessing consistency of \( \varrho \in \Gamma \) on the central cliques.

Observe that, for a component \( Z \in \mathcal{X} \cup \mathcal{Y} \) (of \( K-(C \cup D) = (G-C) \cup (H-D) \)), the \( a \geq 1 \) distinct neighborhoods (attachment sets) of vertices of \( Z \) in \( C \cup D \) form a sequence \( N_1(Z), \ldots, N_a(Z) \subseteq C \cup D \) ordered by the strict inclusion \( N^U_{C \cup D}(Z) = N_1(Z) \subset N_2(Z) \subset \cdots \subset N_a(Z) = N^U_{C \cup D}(Z) \), and we denote this whole family by \( N_{C \cup D}(Z) := \{N_1(Z), \ldots, N_a(Z)\} \).

We call the multiset of sets \( \mathcal{U} := \bigcup_{Z \in \mathcal{X} \cup \mathcal{Y}} N_{C \cup D}(Z) \) the attachment collection of \( \mathcal{X} \cup \mathcal{Y} \) in \( C \cup D \) of our graph \( K \) (\( \mathcal{U} \) is a multiset since the same attachment set may occur several times in \( \mathcal{U} \) if the occurrences come from distinct bridges of \( C \cup D \) which are incomparable in \( R \)). Recall the notation \( K(Z) \) from Procedure 5.1. If an automorphism \( \varrho \) of \( R \) maps \( Z \) to \( Z' = \varrho(Z) \), then, in particular, \( K(Z) \) is respectfully isomorphic to \( K(Z') \). While there may exist different respectful isomorphisms from \( K(Z) \) to \( K(Z') \), they all define (because of the strict inclusion order) the same unique mapping of the attachment sets from \( N_{C \cup D}(Z) \) to \( N_{C \cup D}(Z') \). So, the automorphism \( \varrho \) of \( R \) induces a unique corresponding permutation on \( \mathcal{U} \), denoted here by \( \tilde{\varrho} \).

![Fig. 4 Two isomorphic colored posets a P and b Q obtained from non-isomorphic S_d-graphs a’ G and b’ H with the central cliques shaded gray](image-url)
For a set family $U$, we call a cardinality Venn diagram of $U$ the vector $(\ell_{U(U_1)} : \emptyset \neq U_1 \subseteq U)$ such that $\ell_{U(U_1)} := |L_{U(U_1)}|$ where $L_{U(U_1)} = \bigcap_{A \in U_1} A \setminus \bigcup_{B \in U \setminus U_1} B$. That is, we record the cardinality of every internal cell of the Venn diagram of $U$. Let $\tilde{\varrho}(U_1) = \{\tilde{\varrho}(A) : A \in U_1\}$ for $U_1 \subseteq U$. See a brief illustration in Fig. 5.

It is reasonably easy to see that an automorphism $\varrho$ of $R$ is consistent on the central cliques $C \cup D$ of $K$, if and only if the corresponding permutation $\tilde{\varrho}$ on the attachment collection $U$ of $\mathcal{X} \cup \mathcal{Y}$ preserves the values of all cells of the cardinality Venn diagram of $U$. This is precisely formulated as follows:

**Lemma 5.4** Let $K = G \cup H$ and $C$, $D$, the sets $\mathcal{X}$ and $\mathcal{Y}$, and the posets $P$, $Q$ and $R$ (on the ground set $\mathcal{X} \cup \mathcal{Y}$) be as in Procedure 5.1. Let $U$ be the attachment collection of $\mathcal{X} \cup \mathcal{Y}$ in $C \cup D$, and assume an automorphism $\varrho$ of $R$ and the corresponding permutation $\tilde{\varrho}$ of $U$. There is an automorphism $f$ of the graph $K$ such that $f(C \cup D) = C \cup D$ and, for every component $Z \in \mathcal{X} \cup \mathcal{Y}$, $f$ maps $V(Z)$ to $V(\varrho(Z))$, if and only if the cardinality Venn diagrams of $U$ and of $\tilde{\varrho}(U)$ are the same, meaning that $\ell_{U(U_1)} = \ell_{U(\tilde{\varrho}(U_1))}$ for all $\emptyset \neq U_1 \subseteq U$.

**Proof** ⇒ Suppose that there exists an automorphism $f$ of $K$ such that $f(C \cup D) = C \cup D$ and, for every component $Z \in \mathcal{X} \cup \mathcal{Y}$, $f$ maps the vertices of $Z$ to the vertices of $\varrho(Z)$. Then, each pair $K(Z)$ and $K(\varrho(Z))$ are isomorphic interval graphs, and if there are $a$ distinct neighborhoods $N_1(Z), \ldots, N_a(Z)$ of vertices of $Z$ in $C \cup D$, then there are $a$ neighborhoods $N_1(\varrho(Z)), \ldots, N_a(\varrho(Z))$ of vertices of $\varrho(Z)$ in $C \cup D$ determined by the restriction of $f$ to $C \cup D$. Since $f$ is an automorphism of $K$, indeed, $\tilde{\varrho}(N_i(Z)) = N_i(\varrho(Z))$ for $1 \leq i \leq a$. Moreover, by our assumption, if $v \in N_i(Z)$, then $f(v) \in N_i(\varrho(Z))$, and vice versa. Now consider arbitrary $\emptyset \neq U_1 \subseteq U$. By the
previous; if
\[
v \in \bigcap_{A \in \mathcal{U}_1} A \setminus \bigcup_{B \in \mathcal{U} \setminus \mathcal{U}_1} B,
\]
then (since \(\tilde{\varrho}\) is a permutation of \(\mathcal{U}\))
\[
f(v) \in \bigcap_{A \in \mathcal{U}_1} \tilde{\varrho}(A) \setminus \bigcup_{B \in \mathcal{U} \setminus \mathcal{U}_1} \tilde{\varrho}(B) = \bigcap_{A \in \tilde{\varrho}(\mathcal{U}_1)} A \setminus \bigcup_{B \in \mathcal{U} \setminus \tilde{\varrho}(\mathcal{U}_1)} B,
\]
and vice versa. Consequently, \(\ell_{\mathcal{U}, \mathcal{U}_1} = \ell_{\mathcal{U}, \tilde{\varrho}(\mathcal{U}_1)}\) for all \(\emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U}\).

\(\Leftarrow\) We start with a simple claim: if, for some \(Z \in \mathcal{X} \cup \mathcal{Y}\), \(f\) is a permutation of \(C \cup D\) which set-wise stabilizes all sets in \(\mathcal{N}_{C \cup D}(Z)\), then \(f\) extended with the identity on \(Z\) is a respectful automorphism of \(K(Z)\). This is trivially true for edges of \(K(Z)\) having both ends either in \(Z\) or in \(C \cup D\). For \(v \in C \cup D\) and \(w \in Z\), we have \(vw \in E(K(Z)) \iff f(v)w \in E(K(Z))\) since otherwise the neighborhood set of \(w\) in \(C \cup D\) would not be stabilized by \(f\).

Now suppose that \(\ell_{\mathcal{U}, \mathcal{U}_1} = \ell_{\mathcal{U}, \tilde{\varrho}(\mathcal{U}_1)}\) holds for our automorphism \(\varrho\) of \(R\) and all \(\emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U}\). Then, in particular, for every such \(\mathcal{U}_1\) there exists
\[
a bijection from \bigcap_{A \in \mathcal{U}_1} A \setminus \bigcup_{B \in \mathcal{U} \setminus \mathcal{U}_1} B \text{ to } \bigcap_{A \in \tilde{\varrho}(\mathcal{U}_1)} A \setminus \bigcup_{B \in \mathcal{U} \setminus \tilde{\varrho}(\mathcal{U}_1)} B.
\]
The composition of these bijections results in a permutation \(f_0\) of \(C \cup D\). For every \(Z \in \mathcal{X} \cup \mathcal{Y}\), since \(\varrho\) is an automorphism of \(R\), there is a respectful isomorphism \(f_Z^0\) from \(K(Z)\) to \(K(\varrho(Z))\). Let \(f_Z'\) be the restriction of \(f_Z\) to \(C \cup D\) and let \(f_0 := f_Z^{-1} \circ f_0\) extended with the identity map on \(Z\). Then \(f_0\) set-wise stabilizes all sets in \(\mathcal{N}_{C \cup D}(Z)\) by the definition of \(\tilde{\varrho}\) and \(f_0\). Consequently, \(f_0\) is a respectful automorphism of \(K(Z)\) and so \(f_Z' := f_Z \circ f_0\) is a respectful isomorphism from \(K(Z)\) to \(K(\varrho(Z))\) which coincides with \(f_0\) on \(C \cup D\).

Finally, since the members of \(\mathcal{X} \cup \mathcal{Y}\) are pairwise disjoint sets, the composition of \(f_Z'\) over all \(Z \in \mathcal{X} \cup \mathcal{Y}\) is well-defined and it is hence an automorphism of the graph \(K\) satisfying the desired properties.

At first sight, the condition of Lemma 5.4 may not seem efficient since \(\mathcal{U}\) has up to \(2n\) attachment sets, and so an exponential number of Venn diagram cells. Though, only at most \(2n\) of the cells may be nonempty since the ground set of \(\mathcal{U}\) is of cardinality \(|C \cup D| \leq 2n\), and so we can handle the situation as follows:

**Lemma 5.5** For any \(\mathcal{U}' \subseteq \mathcal{U}\) such that \(\tilde{\varrho}(\mathcal{U}') = \mathcal{U}'\), one can in \(O(n^2)\) time test whether the equalities \(\ell_{\mathcal{U}, \mathcal{U}_1} = \ell_{\mathcal{U}', \tilde{\varrho}(\mathcal{U}_1)}\) hold for all \(\emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U}'\).

**Proof** We loop through all vertices \(w\) of \(C \cup D\), and for each \(w\) we record in \(O(n)\) time to which of the sets in \(\mathcal{U}'\) this \(w\) belongs to. Summing the obtained records at the end precisely gives the \(O(n)\) nonzero values \(\ell_{\mathcal{U}, \mathcal{U}_1}\) over \(\emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U}'\).

We analogously compute the \(O(n)\) nonzero values \(\ell_{\mathcal{U}', \tilde{\varrho}(\mathcal{U}_1)}\) over \(\emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U}'\), and then compare the two sets of values with respect to each \(\mathcal{U}_1\) and matching \(\tilde{\varrho}(\mathcal{U}_1)\).

\(\Box\)
5.3 Computing the Subgroup of Consistent Poset Automorphisms

Knowing how to efficiently test whether an automorphism \( \varrho \in \Gamma \) in step 3 of Procedure 5.1 is consistent (Lemmas 5.4 and 5.5), we would like to finish a computation of the subgroup \( \Gamma' \subseteq \Gamma \). This, however, cannot be done directly by processing all members of the group \( \Gamma \) which can be exponentially large. Instead, inspired by famous Babai’s “tower-of-groups” procedure (cf. [4] and Theorem 5.2), we iteratively compute a chain of subgroups \( \Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_h = \Gamma' \) leading to the result. Here, by “computing a group” we mean to output a set of its generators.

We will show that this step can also be executed in \( \text{FPT} \)-time. There are two important ingredients making this computation work. First, we look for a manageable combinatorially defined “gradual refinement” of the condition tested by Lemma 5.4. Our intention is to define \( \Gamma_i \) as the subgroup of \( \Gamma \) respecting the \( i \)-th step of this refinement. By manageable we mean that the ratio of orders (sizes) of consequent groups \( \Gamma_i \) and \( \Gamma_{i+1} \) in the chain is always bounded and, at the same time, that the number of refinement steps \( (h) \) is not too big. Second, having such manageable refinement steps, we then stepwise apply another classical result (as illustrated below):

**Theorem 5.6** (Furst et al. [17, Cor. 1]) Let \( \Pi \) be a permutation group given by its generators, and \( \Pi_1 \) be any subgroup of \( \Pi \) such that one can test in polynomial time whether \( \pi \in \Pi_1 \) for any \( \pi \in \Pi \) (membership test). If the ratio \( |\Pi|/|\Pi_1| \) is bounded by a function of a parameter \( d \), then a set of generators of \( \Pi_1 \) can be computed in \( \text{FPT} \)-time (with respect to \( d \)).

To illustrate the typical use and the strength of Theorem 5.6, we show an outline of how it can be used to design an algorithm (known as Babai’s tower-of-groups procedure [4]) proving Theorem 5.2:

1. Let \( G \) be a graph on \( n \) vertices, and \( V(G) = V_1 \cup \ldots \cup V_k \) be a partition of its vertex set into \( k \) color classes such that \( |V_i| \leq d \). Let \( \Pi_0 \) be the group of all permutations \( \pi \) on \( V(G) \) which satisfy \( \pi(V_j) = V_j \) for all \( 1 \leq j \leq k \) (i.e., \( \Pi_0 \) is a product of the symmetric groups of the color classes).
2. For \( i = 1, 2, \ldots, \binom{k}{2} \), let \( (a, b) \) be the \( i \)-th pair in the following sequence of pairs: \( (1, 2) , (1, 3) , \ldots , (1, k) , (2, 3) , (2, 4) , \ldots , (2, k) , (3, 4) , \ldots , (k-1, k) \) (the order of which is not really important). Denote by \( E_i \) the set of edges of the induced subgraph \( G[V_a \cup V_b] \), and by \( \Pi_i \) the subgroup of \( \Pi_{i-1} \) consisting of all permutations \( \pi \in \Pi_{i-1} \) such that \( \pi \) induces an automorphism of the graph with the vertex set \( V(G) \) and the edge set \( E_1 \cup E_2 \cup \ldots \cup E_i \). Then \( |\Pi_{i-1}|/|\Pi_i| \leq |V_a|! \cdot |V_b|! \leq (d!)^2 \), and so we may use Theorem 5.6 to compute \( \Pi_i \) from \( \Pi_{i-1} \).
3. Finally, \( \Pi_k \) is the color-preserving automorphism group of \( G \), as desired.

In a nutshell, the outlined procedure stepwise removes those permutations of \( \Pi_0 \) which violate the edge subsets between the parts \( V_a \) and \( V_b \), ranging over all pairs \( i = (a, b) \). To detail an analogous procedure in our case, we need to closely analyze what happens if an automorphism \( \varrho \in \Gamma \) does not pass the cardinality Venn diagram test described in Lemma 5.4. This is not nearly as simple as in case of the procedure for Theorem 5.2, since the cardinality Venn diagram involves attachment sets from...
all levels of the poset at once. Fortunately, thanks to considering posets of bounded width, we can prove that every violation of the consistency check from Lemma 5.4 is witnessed by a subcollection of at most \( d \) sets of \( \mathcal{U} \), i.e., such failure involves only at most \( d \) levels of the posets which is manageable.

**Lemma 5.7** Let \( \mathcal{U} \) be a set family and \( \bar{\varrho} \) be a permutation of \( \mathcal{U} \). If there exists \( \mathcal{U}_1 \) such that \( \emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U} \) and \( \ell_{\mathcal{U}, \mathcal{U}_1} \neq \ell_{\mathcal{U}, \bar{\varrho}(\mathcal{U}_1)} \), then there exist \( \mathcal{U}_2, \mathcal{U}_3 \subseteq \mathcal{U} \) such that \( |\mathcal{U}_2| \leq 2 \) or \( \mathcal{U}_2 \) is an antichain in the inclusion, \( \emptyset \neq \mathcal{U}_3 \subseteq \mathcal{U}_2 \) and \( \ell_{\mathcal{U}_2, \mathcal{U}_3} \neq \ell_{\mathcal{U}_2, \bar{\varrho}(\mathcal{U}_3)} \).

In our case, \( \mathcal{U}_2 \) is an antichain of attachment sets of one of \( G \) or \( H \), and hence \( |\mathcal{U}_2| \leq d \) follows from the assumption of considering \( S_d \)-graphs.

**Proof** (of Lemma 5.7) Notice that \( \mathcal{U} \) itself is not Venn-good, and \( \mathcal{U}_1 \) is a witness. Choose \( \mathcal{U}_2 \subseteq \mathcal{U} \) such that \( \mathcal{U}_2 \) is not Venn-good and it is minimal such by inclusion, and assume (for a contradiction) that there are \( A_1, A_2 \in \mathcal{U}_2 \) such that \( A_1 \subseteq A_2 \). If \( \bar{\varrho}(A_1) \nsubseteq \bar{\varrho}(A_2) \), then already \( \mathcal{U}_2 := \{ A_1, A_2 \} \) is not Venn-good (with a witness \( \{ A_1 \} \)), and so let \( \bar{\varrho}(A_1) \subseteq \bar{\varrho}(A_2) \).

Let \( \mathcal{U}_3 \) be a witness of \( \mathcal{U}_2 \) not being Venn-good, and for \( j = 2, 3 \) denote: \( \mathcal{U}_j^0 := \mathcal{U}_j \setminus \{ A_1, A_2 \}, \mathcal{U}_j^1 := (\mathcal{U}_j \cup \{ A_1 \}) \setminus \{ A_2 \}, \mathcal{U}_j^2 := (\mathcal{U}_j \cup \{ A_2 \}) \setminus \{ A_1 \} \). By our minimality assumption, all three subfamilies \( \mathcal{U}_2^0, \mathcal{U}_2^1 \) and \( \mathcal{U}_2^2 \) are Venn-good. We first easily derive

\[
\ell_{\mathcal{U}_2, \mathcal{U}_3^0} = \ell_{\mathcal{U}_2 \setminus \{ A_1, A_2 \}, \mathcal{U}_3^0} = \ell_{\mathcal{U}_2 \setminus \{ A_1, A_2 \}, \bar{\varrho}(\mathcal{U}_1)} = \ell_{\bar{\varrho}(\mathcal{U}_2), \bar{\varrho}(\mathcal{U}_3^0)},
\]

\[
\ell_{\mathcal{U}_2, \mathcal{U}_3^1} = 0 = 0 = \ell_{\bar{\varrho}(\mathcal{U}_2), \bar{\varrho}(\mathcal{U}_3^0)},
\]

\[
\ell_{\mathcal{U}_2, \mathcal{U}_3^2} = \ell_{\mathcal{U}_2 \setminus \{ A_2 \}, \mathcal{U}_3^2 \setminus \{ A_2 \}} = \ell_{\mathcal{U}_2 \setminus \{ A_2 \}, \bar{\varrho}(\mathcal{U}_1)} = \ell_{\bar{\varrho}(\mathcal{U}_2), \bar{\varrho}(\mathcal{U}_3^2)}.
\]

Then, using trivial \( \ell_{\mathcal{U}_0^0, \mathcal{U}_0^0} = \ell_{\mathcal{U}_2, \mathcal{U}_3^0} + \ell_{\mathcal{U}_2, \mathcal{U}_3^1} + \ell_{\mathcal{U}_2, \mathcal{U}_3^2} + \ell_{\mathcal{U}_2, \mathcal{U}_3^3} \) and its counterpart under \( \bar{\varrho} \), we conclude

\[
\ell_{\mathcal{U}_2, \mathcal{U}_3^3} = \ell_{\mathcal{U}_2, \mathcal{U}_3^0} - \ell_{\mathcal{U}_2, \mathcal{U}_3^2} = \ell_{\bar{\varrho}(\mathcal{U}_2), \bar{\varrho}(\mathcal{U}_3^0)} - \ell_{\bar{\varrho}(\mathcal{U}_2), \bar{\varrho}(\mathcal{U}_3^2)} = \ell_{\bar{\varrho}(\mathcal{U}_2), \bar{\varrho}(\mathcal{U}_3^3)}.
\]

However, \( \mathcal{U}_3 \in \{ \mathcal{U}_3^0, \mathcal{U}_3^1, \mathcal{U}_3^2, \mathcal{U}_3^3 \} \), and so one of the latter four equalities contradicts the assumption that \( \mathcal{U}_3 \) witnessed \( \mathcal{U}_2 \) not being Venn-good. \( \square \)

**Corollary 5.8** Let a poset \( R \), its automorphism \( \varrho \), attachment collection \( \mathcal{U} \) and permutation \( \bar{\varrho} \) of \( \mathcal{U} \) be as in Lemma 5.4. We have that \( \mathcal{U} \) is Venn-good (wrt. \( \bar{\varrho} \)), if and only if every \( \mathcal{U}' \) is Venn-good, where \( \mathcal{U}' \subseteq \mathcal{U} \) is the subcollection of attachment sets of the union of some (any) \( \ell \) levels of the poset \( R \). \( \square \)
Algorithm 2 One step of computation of the subgroup $\Gamma' \subseteq \Gamma$

**Require:**
- the attachment collection $\mathcal{U}$ of the graph $K$;
- the colored poset $R$ of $K$ with levels $L_1, L_2, \ldots, L_k$;
- a subgroup $\Gamma_{i-1}$ (via a generator set) of the full automorphism group of $R$.

**Ensure:**
- either a certificate that $\mathcal{U}$ is Venn-good for every member of $\Gamma_{i-1}$; or
- a subgroup $\Gamma_{i} \subset \Gamma_{i-1}$ (via a generator set) such that, for the attachment collection $\mathcal{T} \subseteq \mathcal{U}$ of some $d$-tuple of levels of $R$, $\mathcal{T}$ is Venn-good precisely for every member of $\Gamma_{i}$ (and not for members of $\Gamma_{i-1}\setminus \Gamma_{i}$).

1: $M_2 \leftarrow \emptyset$
2: repeat for $a := 1, 2, \ldots, d$ :
3: repeat for $b := 1, 2, \ldots, k+1$ :
4: if $b > k$ then return "$\mathcal{U}$ is Venn-good for all of $\Gamma_{i-1}$";
5: $M_1 \leftarrow (L_1 \cup L_2 \cup \cdots \cup L_b) \cup M_2$;
6: $U_1$ ← the subcollection of attachment sets of $M_1$, $U_1 \subseteq \mathcal{U}$;
7: until $U_1$ is not Venn-good (cf. Lemma 5.5) for some generator of $\Gamma_{i-1}$;
8: $j_a \leftarrow b$;
9: $M_2 \leftarrow L_{j_1} \cup L_{j_2} \cup \cdots \cup L_{j_a}$;
10: until $j_a = 1$ or $a = d$;
11: $U_2$ ← the subcollection of attachment sets of $M_2$, $U_2 \subseteq \mathcal{U}$;
12: Call the algorithm of Theorem 5.6 to compute the subgroup $\Gamma_{i} \subseteq \Gamma_{i-1}$, such that the membership test of $\varrho \in \Gamma_{i}$ checks whether $\mathcal{U}_2$ is Venn-good for $\varrho$ (cf. Lemma 5.5);
13: return $\Gamma_{i}$

Corollary 5.8 shows a clear road to computing the subgroup $\Gamma' \subseteq \Gamma$ in step 3 of Procedure 5.1. In every refinement step of a chain $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_h = \Gamma'$, the following constraint is imposed: the subcollection of attachment sets of some $d$-tuple of levels of the poset $R$ is Venn-good for every member of the next subgroup. All these steps are manageable; the ratio $|\Gamma_{i+1}|/|\Gamma_i|$ is bounded from above by the maximum number of subpermutations of $\Gamma_i$ on the respective $d$ levels (as proved below), which is $\leq (2d)^d$. Since the height of $R$ is $\Theta(n)$, though, we cannot afford to check all $d$-tuples of levels this way. Fortunately, it is also not necessary by the following argument.

By Lagrange’s group theorem, $|\Gamma_{i+1}|$ divides $|\Gamma_i|$, and so either $\Gamma_{i+1} = \Gamma_i$ or $|\Gamma_{i+1}| \leq \frac{1}{2} |\Gamma_i|$. Hence the number of strict refinement steps in our chain of subgroups is $h \leq \log |\Gamma|$. Since $|\Gamma| \leq (2d)^d$, we get $h = O(nd \log d)$. Furthermore, the $d$-tuples of levels giving our $h$ strict refinement steps can be, one at each step, computed by Algorithm 2.

**Lemma 5.9** Computation of the subgroup $\Gamma' \subseteq \Gamma$ can be accomplished in $\textbf{FPT}$-time with respect to the fixed parameter $d$ by iterated calls to Algorithm 2, starting from $\Gamma_0 = \Gamma$.

**Proof** We give this proof by analyzing a call to Algorithm 2. If $\mathcal{U}$ is Venn-good for every generator of $\Gamma_{i-1}$ (which is equivalent to that of every member of $\Gamma_{i-1}$), then...
we find this already in the first iteration of \( a = 1 \), on line 4. Hence we may further assume that \( \mathcal{U} \) is not Venn-good.

Let \( k \geq j'_1 > j'_2 > \cdots > j'_c \geq 1 \) be an index sequence of length \( c \leq d \) such that the subcollection of attachment sets of \( L_{j'_1} \cup L_{j'_2} \cup \cdots \cup L_{j'_c} \) is not Venn-good for some generator of \( \Gamma_{i-1} \) (such a sequence must exist by Corollary 5.8), and the vector \((j'_1, j'_2, \ldots, j'_c)\) is lexicographically minimal of these properties. Then one can straightforwardly verify that \((j'_1, j'_2, \ldots, j'_c)\) is a prefix of (or equal to) the vector \((j_1, j_2, \ldots, j_a)\) computed by Algorithm 2. Consequently, the collection \( \mathcal{U}_2 \) of Algorithm 2 is not Venn-good for some generator of \( \Gamma_{i-1} \).

Next, we verify the fulfillment of the assumptions of Theorem 5.6. Generators of \( \Gamma_{i-1} = \Pi \) have been given to Algorithm 2. The ratio \(|\Pi|/|\Pi_1|\) where \( \Pi_1 = \Gamma_i \) in our case, can be bounded as follows (despite we do not know \( \Gamma_i \) yet): \( |\Gamma_{i-1}|/|\Pi_i| \) equals the number of distinct cosets of the subgroup \( \Gamma_i \) in \( \Gamma_{i-1} \). If we consider two automorphisms \( \alpha, \beta \in \Gamma_{i-1} \) which are equal when restricted to \( M_2 \), then the automorphism \( \alpha^{-1}\beta \) determines a permutation of \( \mathcal{U} \) which is identical on \( \mathcal{U}_2 \) (so it is Venn-good), and hence \( \alpha^{-1}\beta \in \Gamma_i \). The latter means that \( \alpha \) and \( \beta \) belong to the same coset of \( \Gamma_i \), and consequently, the number of distinct cosets is at most the number of distinct subpermutations on \( M_2 \) possibly induced by \( \Gamma_{i-1} \), that is at most \((2d)^d\) (so it is Venn-good). Therefore, we can finish this step in \( \text{FPT} \)-time with respect to \( d \).

Finally, as argued already, there can be at most \( O(nd \log d) \) calls to Algorithm 2 altogether (and the last one certifies that \( \mathcal{U} \) is already Venn-good).

\( \square \)

### 5.4 The Complete Algorithm

At last we review a detailed implementation of Procedure 5.1 as a pseudocode in Algorithm 3 listing all steps of the full isomorphism testing procedure.

**Theorem 5.10** The isomorphism problem of \( S_d \)-graphs can be solved in \( \text{FPT} \)-time with respect to the fixed parameter \( d \) by Algorithm 3.

**Proof** We apply Procedure 5.1. This process runs \( O(n) \) iterations of choices of \( C \) and \( D \); we first greedily find any valid central clique \( C \) of an \( S_d \)-representation of \( G \), and then iterate all maximal cliques \( D \subseteq H \). In each iteration, we routinely compute in polynomial time the sets of components \( X \) and \( Y \), and the central posets \( P \) and \( Q \) on them and \( R = P \cup Q \). If any of \( P, Q \) has width greater than \( d \), then we reject this iteration. Similarly, we reject the iteration if any of the graphs \( K(Z) \) for \( Z \in X \cup Y \) is not interval [7]. Otherwise, we compute colors on \( R \) such that \( Z, Z' \in X \cup Y \) receive the same color if they are on the same level of \( R \) and \( K(Z) \cong K(Z') \) with a respectiful isomorphism. For the latter we use the isomorphism algorithm of [7] with coloring of \( C \cup D \). This finishes step 1.

Step 2—computing the automorphism group \( \Gamma \) of \( R \), is done by Corollary 5.3.

Step 3—finding the subgroup \( \Gamma' \subseteq \Gamma \), is accomplished by an iterated application of Theorem 5.6; the refinement steps are defined by \( d \)-tuples of levels of \( R \) according to Corollary 5.8 and the above outlined procedure for finding them, and there are \( O(nd \log d) \) such steps. The membership test used in Theorem 5.6 is provided by Lemma 5.5.
Algorithm 3 Isomorphism test for general $S_d$-graphs $(G,H)$

Require: Given two $S_d$-graphs $G$ and $H$ (their representations not required), and the parameter $d$ of $S_d$.

Ensure: Result of the isomorphism test between $G$ and $H$.

1: Find the maximal clique collections $S$ and $T$ of the chordal graphs $G$ and $H$;
2: if $|S| \neq |T|$, or the cardinalities of members of $S$ and $T$ do not match then
3: return “$G$ and $H$ are not isomorphic”;
4: $K \leftarrow G \uplus H$ (disjoint union);
5: repeat for each maximal clique $C \in S$:
6: Find the connected components $X_1, X_2, \ldots, X_k$ of $G - C$, assuming the equivalent connected components are joined into single bridge(s) of $C$;
7: $P \leftarrow$ the partial order on the connected components $X_1, X_2, \ldots, X_k$ of $G - C$ determined by their attachments;
8: until all $G[C \cup X_i], 1 \leq i \leq k$, are interval graphs, and $P$ is of width $\leq d$;
9: for each $D \in T$ with $|C| = |D|$ do
10: Find the connected components $Y_1, Y_2, \ldots, Y_l$ of $H - D$, assuming the equivalent connected components are joined into single bridge(s) of $D$;
11: $Q \leftarrow$ the partial order on the connected components $Y_1, Y_2, \ldots, Y_l$ of $H - D$;
12: if $k = l$, all $H[D \cup Y_i], 1 \leq i \leq k$, are interval graphs, and $Q$ is of width $\leq d$ then
13: $R \leftarrow P \uplus Q$ (disjoint union);
14: Determine the levels $L_1, L_2, \ldots, L_k$ of the poset $R$;
15: for $i := 1, 2, \ldots, k$ do
16: for each pair of components $Z, Z' \in L_i$ do
17: Compare the interval graphs $K(Z)$ and $K(Z')$ to respectful isomorphism (recall the notation $K(Z)$ from Procedure 5.1);
18: Having computed in the previous step the isomorphism equivalence classes $L_i^1, \ldots, L_i^{c_i}$ of $L_i$, give the element $Z \in L_i$ of poset $R$ color $(i, j)$ iff $Z \in L_i^j$;
19: $\Gamma \leftarrow$ the color-preserving automorphism group of $R$, computed by Theorem 5.2;
20: $\Gamma' \leftarrow$ the subgroup of $\Gamma$ consisting of those automorphisms $\varrho$ of $R$ which pass the test of Lemma 5.4; computed by Algorithm 2;
21: if $P$ and $Q$ are swapped in some of the generators of $\Gamma'$ then
22: return “$G$ and $H$ are isomorphic”;
23: return “$G$ and $H$ are not isomorphic”;

Finally, we straightforwardly check in step 4 on the generators of $\Gamma'$ whether some of them maps an element of $P$ to an element of $Q$.

If any iteration of Procedure 5.1 succeeds in step 4, then, by Lemma 5.4, there exists an automorphism of the graph $K = G \uplus H$ which moreover swaps $G$ and $H$. Then $G \simeq H$. Conversely, assume $G \simeq H$. Since $G$ is an $S_d$-graph, we find a maximal central clique $C \subseteq G$, and since all maximal cliques $D \subseteq H$ are tried, we get into an
iteration with \( D \) being the isomorphic image of \( C \). Then the posets \( P \) and \( Q \) (wrt. \( C \), \( D \)) must be isomorphic respecting their colors, which follows from \( G \simeq H \). Therefore, there exists an automorphism \( \varrho \in \Gamma' \), and hence also a generator of \( \Gamma' \), swapping \( P \) and \( Q \) (and preserving the computed cardinality Venn diagram by Lemma 5.4). Some iteration hence succeeds and returns that \( G \simeq H \).

Remark 5.11 We do not explicitly state the runtime in Theorem 5.10 partly since it is not really useful and since neither [17] which we use states explicit runtime. Here we briefly remark that Procedure 5.1 loops \( O(n) \) times with suitable \( C \) and different choices of \( D \), analogously to the procedure of Theorem 3.1, and this initial setup of the procedure altogether takes time \( O(d^2 n^3) \). Then we have to account for \( O(n) \) calls to steps 2 and 3 of Procedure 5.1, that is, \( O(n) \) computations of the subgroups \( \Gamma \) and \( \Gamma' \). Each time this part is dominated by step 3 which performs \( O(nd \log d) \) calls to the algorithm of Theorem 5.6 [17] in order to compute \( \Gamma' \) from \( \Gamma \). Reading the fine details of [17], and adjusting it (the “sift table”) to our setting, leads to an estimate of \( O(n^3) \cdot d!^{O(d)} \) for each of these calls. After summarizing, we get the total estimate of \( O(n^5) \cdot d!^{O(d)} \).

Remark 5.12 Theorem 5.10 easily extends to colored graphs;

– on line 18 of Algorithm 3 we can test for colored isomorphism of interval graphs as well, and
– in the calls to Algorithm 2 we simply test the cardinality Venn diagrams separately in each color class.

This may also be a useful consequence.

Corollary 5.13 The automorphism group \( \text{Aut}(G) \) of an \( S_d \)-graph \( G \) can be computed in \( \text{FPT} \)-time with respect to the fixed parameter \( d \).

Before we turn to the proof, we remark that the temptingly easy way to this corollary—to take just the part of Algorithm 3 dealing with the graph \( G \) and the automorphisms of the poset \( P \)—would give us only the subgroup of the automorphism group of \( G \) set-wise stabilizing \( C \). This may not be complete \( \text{Aut}(G) \). On the other hand, \( \text{Aut}(G) \) can also be computed via a standard reduction to the colored isomorphism test as in Remark 5.12, but we prefer to give the following straightforward modification of our algorithm.

Proof We run slightly modified Algorithm 3 on two copies of the graph \( G \) (i.e., \( H = G \)). On line 21, we denote the computed group by \( \Gamma'_D = \Gamma' \) (since it depends on the choice of \( D \) in the loop). The actual modification comes on line 23; over all choices \( D \) such that \( \Gamma'_D \) swaps \( P \) and \( Q \), we record the following permutations \( \varrho^* \) of \( V(K) \) for every generator \( \varrho \) of \( \Gamma'_D \):

a) The permutation \( \varrho^* \) is such that for every \( Z \in \mathcal{X} \cup \mathcal{Y} \) we let \( \varrho^* \) on \( Z \) be any isomorphism from \( K(Z) \) to \( K(\varrho(Z)) \) (e.g., on line 18 of the algorithm).

b) On \( C \cup D \), we let the restriction of \( \varrho^* \) be any subpermutation respecting the mapping \( \tilde{\varrho} \) on the attachment collection of \( \mathcal{X} \cup \mathcal{Y} \) in \( C \cup D \) (cf. Lemma 5.4). This is a correct definition; recall that the attachments of \( Z \) and of \( \varrho(Z) \) are invariant on the choice of isomorphism in (a).
Let $\Delta_1$ be the permutation group generated on $V(K)$ by the permutations $\varphi^*$ for all generators $\varphi$ of $\Gamma'_{D'}$ and over all matching choices of $D$. Let $\Delta_2$ be the group obtained by adding to $\Delta_1$ the generators of the automorphism groups of the interval graphs $G[C \cup X]$ for $X \in \mathcal{X}$, computed as in [7]. Let $\Delta_3$ be the group obtained by adding to $\Delta_2$ the generators of the symmetric groups on subsets of $C$ which are the cells of the attachment collection $\mathcal{U}$ of $\mathcal{X}$ in $C$. Finally, let $\Delta$ be the restriction of $\Delta_3$ to $V(G)$.

We claim $\text{Aut}(G) = \Delta$. Indeed, every member of $\Delta$ is an automorphism of $G$ which follows from the analysis of Algorithm 3. On the other hand, any automorphism $g \in \text{Aut}(G)$ is “discovered” by Algorithm 3 as a member of $\Gamma'_{D'}$ for $D = g(C)$, modulo the concrete choices of isomorphisms between the interval components and of a permutation of $C \cup D$ which respects the attachment collection $\mathcal{U}$ of $\mathcal{X}$ in $C$, and $g$ thus is contained in the constructed group $\Delta$.

### 6 An XP-Time Isomorphism Approach for $T$-Graphs

We now show that the approach from Sect. 5 can be extended to solve even the isomorphism problem of $T$-graphs. However, this relatively easy extension comes at a price—the more general algorithm runs only in XP-time.

Before getting to the extension, we note that there is another possible approach to $T$-graph isomorphism, to which we will return in the next section. Given two $T$-graphs $G$ and $H$ on $n$ vertices, we may consider all possible assignments of maximal cliques of $G$ and $H$ to the branching nodes of $T$. This can be achieved in $n^{O(T)}$ iterations. In each iteration, we use the assignment to define $S_d$-graph isomorphism subproblems “around” each branching node of $T$, and apply the algorithm from Sect. 5. However, this seemingly straightforward procedure has some rather deep technical problems when considering general $T$-graphs, and that is why we prefer another recursive approach outlined next.

Our actual approach combines the building blocks of Algorithm 3 with a special recursive routine which, in a nutshell, takes care of the non-interval pieces which occur as bridges of the chosen clique(s). It is described in the next procedure.

**Procedure 6.1** Given two $T$-graphs $G$ and $H$ on $n$ vertices, we compute the automorphism group $\text{Aut}(G \uplus H)$ if $G$ and $H$ are isomorphic, or output that $G$ and $H$ are not isomorphic.

First, fix a maximal clique $C \subseteq G$ such that every component of $G - C$ is a $T'$-graph for some strict subtree $T' \subset T$. Then apply the following procedure to each maximal clique $D \subseteq H$ such that $|C| = |D|$: 

1. Identify the set $\mathcal{X}$ of all components $Z$ of $G - C$ such that $K(Z)$ is an interval graph, and the set $\mathcal{Y}$ of analogous components of $H - D$. If $|\mathcal{X}| = |\mathcal{Y}|$, then let $A_1, \ldots, A_a$ and $B_1, \ldots, B_b$ be the remaining (i.e., non-interval) components of $G - C$ and $H - D$, respectively. Note that the numbers $a, b$ of the latter components are bounded as $a, b \leq |V(T)|$. If $a = b$, proceed with the next steps.
2. As in Algorithm 3, construct the posets $P$ and $Q$ formed by the interval components $\mathcal{X}$ and $\mathcal{Y}$, respectively, color their nodes by respectful isomorphism of the compo-
nents, and compute the color-preserving automorphism group \( \Gamma_0 \) of \( R = P \cup Q \).
(Recall that the group \( \Gamma_0 \) may not yet preserve the cardinality Venn diagram of the attachment collection of \( \mathcal{X} \cup \mathcal{Y} \) in \( C \cup D \).
If \( \Gamma_0 \) contains a generator swapping \( P \) and \( Q \), proceed with the next steps.
3. Loop through all bijections \( m \) from \( \mathcal{A} = \{A_1, \ldots, A_a\} \) to \( \mathcal{B} = \{B_1, \ldots, B_a\} \):

(a) For each matching pair of subgraphs \( G[C \cup A_i] \) and \( H[D \cup B_{m(i)}] \), \( i = 1, \ldots, a \), compute recursively the automorphism group

\[
\Gamma_i := \text{Aut}(G[C \cup A_i] \cup H[D \cup B_{m(i)}])
\]
set-wise stabilizing \( C \cup D \). If the graphs \( G[C \cup A_i] \) and \( H[D \cup B_{m(i)}] \) are not isomorphic this way, stop this iteration and continue with the next bijection \( m \).
(b) Define the group \( \Gamma \) as the direct product of the following groups
– of \( \Gamma_0 \) on \( \mathcal{X} \cup \mathcal{Y} \) (step 2), and
– for \( i = 1, \ldots, a \), of \( \Gamma_i \) restricted to \( A_i \cup B_{m(i)} \) (step 3a).

See Remark 6.2 for an explanation of the purpose of this step.
(c) Let \( \mathcal{U} \) be the attachment collection of \( \mathcal{X} \cup \mathcal{Y} \) in \( C \cup D \), and \( \mathcal{V}_i \) be the attachment collection of \( A_i \cup B_{m(i)} \) in \( C \cup D \). Let \( \mathcal{V} = \mathcal{U} \cup \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_a \) (recall that this is a multiset). Define \( \tilde{\Gamma} \) as the group of permutations of \( \mathcal{V} \) generated by the action of \( \Gamma \) on the attachment collection \( \mathcal{V} \).
(d) Using analogy of Algorithm 2 (see details in the proof of Theorem 6.4), compute the subgroup \( \Gamma'' \subseteq \Gamma \) of those members of \( \Gamma \) which are consistent on \( C \cup D \). Precisely, \( \Gamma'' \) is the maximum subgroup of \( \Gamma \) such that the related group \( \tilde{\Gamma}'' \), defined as in (3c), preserves the cardinality Venn diagram of \( \mathcal{V} \). If no member of \( \Gamma'' \) swaps \( G - C \) with \( H - D \), stop this iteration and go to the next \( m \).
Otherwise, let \( \Gamma'' \) be denoted by \( \Gamma''_m \) w.r.t. the current bijection \( m \).
4. If none of the bijections \( m \) in step 3 successfully computed the group \( \Gamma''_m \), then the result of the current choice of the clique \( D \subseteq H \) is void. Otherwise, let \( \Gamma''_D \) be the group generated by all \( \Gamma''_m \) over such “successful” \( m \) for the current clique \( D \).
This is a sound definition of \( \Gamma''_D \) since every member of each \( \Gamma''_m \) is actually a permutation of the set \( \mathcal{X} \cup \mathcal{Y} \cup \bigcup_{i=1}^{a}(A_i \cup B_{m(i)}) \) (cf. step 3b).
5. At last, we use the same construction on the components of \( \mathcal{X} \cup \mathcal{Y} \) as in Corollary 5.13 to turn \( \Gamma''_D \) into an automorphism subgroup (not complete yet) acting on the vertex set \( \mathcal{V}(G \cup H) \). Let this group be denoted by \( \Delta_D \).

At the end, the desired automorphism group \( \text{Aut}(G \cup H) \) is generated by the union of the generators of the non-void groups \( \Delta_D \) computed in the procedure.

Remark 6.2 To fully understand Procedure 6.1, it is important to notice why the group \( \Gamma \) in step (3b) is such seemingly over-complicated; the group acts on the interval components (as whole) of \( \mathcal{X} \cup \mathcal{Y} \), while it simultaneously acts on the individual vertices of the non-interval components.

The reason for such handling is that, as we know from Sect. 5, the attachments of a single component from \( \mathcal{X} \cup \mathcal{Y} \) in \( C \cup D \) form a chain by the inclusion (and
so they are invariant on automorphism of the component), but the attachments of a single non-interval component from \( \{A_i, B_i : i = 1, \ldots, a\} \) in \( C \cup D \) may be “shuffled” by an automorphism of the component (and hence we need to “track” the mapping of individual vertices of that component throughout the algorithm). On the other hand, detailed tracking of the attachments of the non-interval components can be done efficiently since the number of them is bounded from above by \( |V(T)| \).

**Remark 6.3** Procedure 6.1 implicitly tests for representability as \( T \)- and \( T' \)-graphs; this can be done using the \( \text{XP} \)-time algorithm of [10]. On the other hand, we do not explicitly use intersection representations of the graphs, and we only need two properties implied by \( T \)-representability: that the number of non-interval components and the recursion depth is bounded in \( |V(T)| \), and that there can be only bounded number of pairwise incomparable attachments in the collection \( \mathcal{V} \). We may thus just assume \( T \)- or \( T' \)-representability of our graphs without checking, and throw away the computation branches in which we detect violation of the implied properties.

**Theorem 6.4** The isomorphism problem of \( T \)-graphs can be solved in \( \text{XP} \)-time with respect to the fixed parameter \( |V(T)| \) by Procedure 6.1.

**Proof** The run of Procedure 6.1, without the recursive calls from it, takes \( \text{FPT} \)-time with the size of \( T \) as the parameter. This follows analogously to the proof of Theorem 5.10; in particular, since the maximum number of incomparable neighborhoods (attachment sets) of all vertices of \((G - C) \cup (H - D) \) in \( C \cup D \) is clearly in \( \Theta(|V(T)|) \) for \( T \)-graphs. Though, the depth of the recursion may be of order up to \( \Theta(|V(T)|) \); every recursive call from Procedure 6.1 assumes \( T' \)-graphs for some \( T' \) strictly smaller than original \( T \). Hence the overall \( \text{XP} \)-time of order \( n^{O(|V(T)|)} \). (We remark, without a proof, that the exponent can be improved to \( O(diam(T) + \log |V(T)|) \) by a suitable choice of \( C \) and management of the recursion, but such small improvement is not worth the effort, as we explain in the conclusion section.)

It can be easily seen that, as in Sect. 5, every generator \( \gamma \) of the computed groups \( \Delta_D \) in Procedure 6.1 really is an automorphism of the graph \( G \cup H \). Indeed, \( \gamma \) consists of a permutation of the set \((X \cup Y) \cup \bigcup_{i=1, \ldots, a} (A_i \cup B_{m(i)})\) (step 4 of Procedure 6.1) for some bijection \( m \) from step 3, and a composition of respectful automorphisms of the individual components of \( X \cup Y \) (step 5). Moreover, since \( \gamma \in \Gamma'_m \), the restriction of \( \gamma \) to \( G[C \cup A_i] \cup H[D \cup B_{m(i)}] \) is an automorphism of that subgraph set-wise stabilizing \( C \cup D \), for each \( i = 1, \ldots, a \). And since the group \( \Gamma'_m \) consists only of permutations which are consistent on \( C \cup D \), \( \gamma \) preserves also the edges of \( G \cup H \) which have one end on \( C \cup D \) and the other end in any component of \((G - C) \cup (H - D) \).

Henceforth, if there is a generator of \( \Delta_D \) swapping \( G \) and \( H \), we correctly conclude that \( G \) and \( H \) are isomorphic.

Conversely, if \( G \) and \( H \) are two isomorphic \( T \)-graphs, then there exists a maximal clique \( C \subseteq G \), as assumed by Procedure 6.1, and the isomorphic image \( D \subseteq H \) of \( C \). The interval components of \( G - C \) are mapped by isomorphism to the interval components of \( H - D \), which defines an automorphism of the poset \( R \) (cf. \( \Gamma_0 \)). The non-interval components of \( G - C \) are mapped bijectively to the non-interval components of \( H - D \), defining the bijection \( m \) (step 3). The recursive automorphism groups on these non-interval components matched by \( m \) are computed correctly by
induction. Since these particular automorphisms in the procedure come from an actual automorphism of \(G \cup H\) swapping \(G\) and \(H\), they are consistent on \(C \cup D\) and the original automorphism thus is a member of the computed group \(\Gamma_m\).

Lastly, we justify step 3d; that Algorithm 2 can be used in the setting of Procedure 6.1. By the proof of Lemma 5.9 it suffices to distribute the vertices of \(A_i \cup B_{m(i)}\) adjacent to \(C \cup D\), for each \(i = 1, \ldots, a\), into bounded-size “levels” (clustered by exactly same neighborhood) which are invariant upon respectful automorphisms. For instance, we can easily define such levels by equal cardinalities of the neighborhoods of \(A_i \cup B_{m(i)}\) in \(C \cup D\), and this finishes the proof. \(\Box\)

Since the algorithm of Theorem 6.4 does not care about the shape of the tree \(T\) (and degree-2 nodes of \(T\) are irrelevant), we actually get the following more general conclusion:

**Corollary 6.5** The isomorphism problem of chordal graphs of leafage at most \(\ell\) can be solved in \(XP\)-time with respect to the fixed parameter \(\ell\).

### 7 Isomorphism of Proper \(T\)-Graphs in FPT-Time

As noted in the introduction, proper \(T\)-graphs present a significantly more restrictive graph class than general \(T\)-graphs. Recently, Chaplick et al. [9] have shown that proper \(T\)-graphs can be recognized in \(FPT\)-time parameterized by the size of \(T\) (which is not yet known for general \(T\)-graphs). Here we show that also the isomorphism problem of proper \(T\)-graphs is notably easier than that of \(S_d\)-graphs and \(T\)-graphs in general. Though, the isomorphism problem stays GI-complete for proper \(S_d\)-graphs with \(d\) on the input—recall Proposition 2.2, and so we cannot realistically hope for polynomial-time algorithms here.

In the case of \(T = S_d\), the algorithm is really simple and purely combinatorial:

**Theorem 7.1** The isomorphism of proper \(S_d\)-graphs can be tested in \(FPT\)-time parameterized by \(d\).

**Proof** This is based on the following observation:

If \(C \subseteq G\) is the central clique of a proper \(S_d\)-representation of \(G\) and \(X_1, \ldots, X_k\) are the connected components of \(G - C\) which have nonempty attachment in \(C\), then \(k \leq d\). Indeed, if \(k > d\), then some two components, say \(X_i, X_j\), would be represented on the same ray of a subdivision \(S'_d\) of \(S_d\) such that \(X_i\) is closer to the center than \(X_j\). If \(X_j\) is adjacent to \(v \in C\), then the representation of \(v\) in \(S'_d\) actually contains the whole representation of \(X_i\), which is a contradiction to \(G\) being a proper \(S_d\)-graph.

Assume we get two proper \(S_d\)-graphs \(G\) and \(H\) on \(n\) vertices. Analogously to the previous algorithms, we hence fix a central maximal clique \(C \subseteq G\), determine \(X_1, \ldots, X_k\) which are the connected components of \(G - C\) which have nonempty attachment in \(C\), and denote by \(X_0\) the possible connected components of \(G\) disjoint from \(C\) (or set \(X_0 = \emptyset\)). We loop through all maximal cliques \(D \subseteq H\) such that \(|D| = |C|\), and similarly determine components \(Y_1, \ldots, Y_l\) and \(Y_0\) of \(H - D\). If
$k = l$, we compare $X_0$ and $Y_0$ to (proper) interval graph isomorphism [7]. Then we loop through all bijections $\varphi : \{1, \ldots, k\} \to \{1, \ldots, k\}$, compare the graphs $G[C \cup X_i]$ and $H[D \cup Y_{\varphi(i)}], i = 1, \ldots, k$, to interval graph isomorphism, and compare their attachment collections in $C$ and in $D$ using the test provided by Lemmas 5.4 and 5.5. (Recall that the attachment collections are automorphism-invariant.) If we ever succeed, then we mark $G$ and $H$ as isomorphic.

The overall approach takes $O(k!n^3) \leq O(d!n^3)$ time which is in FPT with respect to $d$. The correctness follows by the same arguments as in Sect. 5.

We now move onto proper $T$-graphs. As with $S_d$-graphs, we may always restrict to $T$-representations of graphs $G$ (in a subdivision $T'$ of $T$) such that the cliques represented at the branching nodes of $T'$ are maximal cliques of $G$, and we call such cliques the branching cliques of this representation. Informally, our idea is to use the finding of Chaplick et al. [9] that those maximal cliques of a proper $T$-graph $G$ which occur as the branching cliques of any $T$-representation of $G$ are somehow special, and that their number, modulo an easy canonical adjustment, is bounded. We will call such cliques (after the adjustment) the rich cliques of $G$ here. Then it will not be difficult to try all mappings between the rich cliques in FPT-time, as mentioned already in Sect. 6, and then answer the isomorphism problem similarly as in the proof of Theorem 7.1.

The full definition of the terms, following [9], is quite technical and we skip the details here since they are not important for our proof. For our purpose, it is enough to accept the notion of a chain in a proper $T$-graph $G$, which is (without further unnecessary details that can be found in [9]) a collection $\mathcal{Y}$ of maximal cliques of $G$ such that $\mathcal{Y}$ is partitioned into a sequence $(C_1, \ldots, C_s)$ of the inner cliques and into two terminals which are nonempty subcollections $T_1, T_2 \subset \mathcal{Y}$ of (remaining) cliques. Then we use the following:

**Proposition 7.2** (Chaplick et al. [9]) Assume $G$ is a connected proper $T$-graph. Then the following sets are unique in $G$ (meaning that they appear the same in every proper $T$-representation of $G$): the set $\mathcal{H}(G)$ of the chains in $G$, and the set $\mathcal{S}(G)$ of the remaining maximal cliques of $G$ not contained in the chains of $\mathcal{H}(G)$. Furthermore:

(a) The inner cliques of every chain of $\mathcal{H}(G)$ form a path in any proper $T$-representation of $G$.

(b) If a terminal $T_1$ of a chain in $\mathcal{H}(G)$ is not a singleton clique, then $T_1$ is the set of the inner cliques of some (other) chain in $\mathcal{H}(G)$, and

(c) If an inner clique $C_i$ of a chain is a branching clique in some proper $T$-representation of $G$, then $C_i$ is contained in a terminal, say $T_1$, of some (other) chain in $\mathcal{H}(G)$, and any (other) clique $C_j \in T_1$ may replace $C_i$ in a proper $T$-representation of $G$.

The sets $\mathcal{H}(G)$ and $\mathcal{S}(G)$ can be computed in FPT-time parameterized by $|T|$, and their cardinalities are in $O(|T|^2)$.

We immediately get:

**Corollary 7.3** If $G$ is a connected proper $T$-graph, then every branching clique appearing in some proper $T$-representation of $G$ is contained in the set $X := \{H, G\}$.\end{equation}
Note that, although the cardinality of the set $S(G)$ and the number of chains in $\mathcal{H}(G)$ (and hence also the number of terminals) is bounded with respect to $T$ by Proposition 7.2, the number of potential branching cliques in $X$ is not bounded since the terminals may have arbitrary cardinality. Fortunately, Proposition 7.2(c) allows us to select just two representative cliques out of every non-singleton terminal as follows:

**Lemma 7.4** If $G$ is a connected proper $T$-graph, then there is a special isomorphism-invariant set $R \subseteq X$ of maximal cliques of $G$ such that there exists a $T$-representation of $G$ in which every branching clique belongs to $R$. This set $R$ can be found in $\text{FPT}$-time parameterized by $|T|$ and its cardinality is in $O(|T|^2)$.

**Proof** We compute the set $R$ of $G$ algorithmically as follows:

1. Using the Algorithm of [9], identify the set $\mathcal{H}(G)$ of the chains in $G$, and $S(G)$ of remaining maximal cliques of $G$ in $\text{FPT}$-time parameterized by $|T|$.
2. Initially set $R := S(G)$.
3. For each terminal of every chain in $\mathcal{H}(G)$ which is a single clique, add it to $R$.
4. For every pair of chains in $\mathcal{H}(G)$; if a non-singleton terminal of one is a set of the inner cliques of the other chain, ordered as the sequence $(C_1, \ldots, C_s)$, then add $C_1$ and $C_s$ to $R$.

The runtime is in $\text{FPT}$ only because of the Algorithm of [9], while the rest of the computation of $R$ is clearly in polynomial time. The cardinalities $|\mathcal{H}(G)| = O(|T|^2)$ and $|S(G)| = O(|T|^2)$ are by Proposition 7.2, and for every chain in $\mathcal{H}(G)$, at most 4 cliques are added to $R$. Finally, Proposition 7.2 also guarantees that the set $R$ is invariant on the choice of a particular proper $T$-representation of $G$, hence isomorphism-invariant. \(\square\)

We call the maximal cliques appearing in the isomorphism-invariant set $R$ obtained by the algorithm given in the proof of Lemma 7.4 the **rich cliques** of a connected proper $T$-graph $G$, and give our fully combinatorial algorithm:

**Theorem 7.5** The isomorphism of proper $T$-graphs can be tested in $\text{FPT}$-time with respect to the size of $T$.

**Proof** We are given two proper $T$-graphs $G$ and $H$ on $n$ vertices. Let $V_b(T)$ denote the set of branching nodes of $T$ (which will be the set of branching nodes of any subdivision of $T$, too). For simplicity, we assume

- that both $G$ and $H$ are connected, as otherwise we can test the isomorphism between each pair of components separately, and
- that $G$ and $H$ are not proper $T_1$-graphs for any $T_1 \subseteq T$, since we can exhaustively try the coming algorithm for all $T_1 \subseteq T$ before trying with $T$.

Imagine now that $G \simeq H$. Then there exist proper $T$-representations of $G$ and $H$ in the same subdivision $T'$ of $T$, such that the branching cliques of $G$ are mapped to the branching cliques of $H$ in a chosen isomorphism $f$ of $G$ and $H$. Let these
branching cliques be $C_1, \ldots, C_m \subseteq G$ and $D_1, \ldots, D_m \subseteq H$, where $m = |V_b(T)|$ and $f(C_i) = D_i$. By Lemma 7.4, we may also assume that these cliques are among the rich cliques of $G$ and of $H$. We define the graphs $G_0 := G - (C_1 \cup \ldots \cup C_m)$ and $H_0 := G - (D_1 \cup \ldots \cup D_m)$. Then, since $C_1, \ldots, C_m$ are the branching cliques in an intersection representation of $G$ in $T'$, we have that every connected component $G_1 \subseteq G_0$ is adjacent to at most two cliques $(C_i, C_j)$ among $C_1, \ldots, C_m$, and we denote by $G_1^+ := G_1 \cup C_i \cup C_j$ which is always an interval graph. We do the same for $H_0$.

On the other hand, every branching clique $C_i$ is adjacent to at most $\ell$ components of $G_0$, where $\ell$ is the number of leaves of $T$. If this was not true, then there would be two intervals in $T'$ (indeed intervals since all branching cliques have been subtracted from $G_0$) representing two vertices of distinct components adjacent to $C_i$, and so one lying on a path in $T'$ from the branching node of $C_i$ to the other. That would violate that our representation is proper. In particular, the number of connected components of $G_0$ and of $H_0$ is in $O(|T|^2)$.

Since $f$ is an isomorphism from $G$ to $H$, there is a pairing of the components of $G_0$ and $H_0$ such that for $G_1 \subseteq G_0$ and $H_1 \subseteq H_0$, we have $f(G_1) = H_1$, and this can be verified as an interval graph isomorphism. The attachment collections of the components in the branching cliques can then be compared (between $G$ and $H$) in polynomial time the same way as in the proof of Theorem 7.1. Altogether, we can verify an isomorphism from $G$ to $H$ in polynomial time once we “guess” the right matching selections of branching cliques in $G$ and $H$.

Our algorithm hence first identifies the sets of all rich cliques $C$ and $D$ of $G$ and $H$, respectively, using Lemma 7.4. Then we loop through all assignments $f : V_b(T) \to C$ and $g : V_b(T) \to D$ of rich cliques to the branching nodes, and for each assignment we check whether the graph $G_1^+$ (resp., $H_1^+$) over all components of $G_0$ (resp., of $H_0$) as above is an interval graph. If this test succeeds, then we check whether there is an isomorphism from $G$ to $H$ respecting the assignment $f$ and $g$, as described above. If no isomorphism is found in any of the rounds, then we conclude that $G \not\cong H$.

As for the correctness, if the isomorphism test ever succeeds, then obviously $G \cong H$. If $G \cong H$, then, as argued above, there exist “matching” proper $T$-representations of $G$ and $H$ with their rich cliques as the branching cliques, this assignment of the rich cliques is among the tested ones, and hence the outcome will be that $G \cong H$. □

8 Conclusions

In this paper, we have focused on the graph isomorphism problem on $S_d$-graphs and $T$-graphs. We have shown that $S_d$-graph isomorphism problem includes testing for the isomorphism of posets of bounded width which can be solved in FPT-time using a classical group-based approach by Furst, Hopcroft and Luks [17] via Babai [4]. We have given an FPT-time algorithm to test the isomorphism of $S_d$-graphs with the parameter $d$, which also builds on the mentioned classical group-based approach.

Since it does not seem easy to solve the isomorphism problem of posets of bounded width in a combinatorial (or algebra-free) way, and we are not aware of any published result in this direction, the related question of an existence of a purely combinatorial
FPT-time algorithm for $S_d$-graph isomorphism remains open (due to Theorem 4.2). We have only shown a partial answer to this open question in the case of more restricted proper $S_d$-graphs and $T$-graphs in Sect. 7.

As a natural extension of our FPT-time algorithm for $S_d$-graph isomorphism, we have shown that the isomorphism problem for $T$-graphs can be solved in XP-time with respect to the size of $T$. In the direct approach we have taken here, the jump from FPT- to XP-time seems unavoidable. However, in subsequent very recent papers independently Arvind et al. [3] and these authors [1] have found different ways to address the isomorphism problem for $T$-graphs in FPT-time. The approach of [1] builds on the core algebraic tools of Sect. 5, combined with a tricky canonical decomposition of $T$-graphs, while [3] use a different approach combining classical Weisfeiler-Leman algorithm with similar algebraic tools applied to hypergraphs.

Chaplick et al. also showed that the graph isomorphism problem for $H$-graphs is GI-complete when $H$ contains a double triangle as a minor [10]. However, it is an open problem what is the complexity of the $H$-graph isomorphism problem when $H$ contains one cycle, and we would like to consider this problem in a future research.

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