Counting rational curves on K3 surfaces

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Version 0

Introduction

The aim of these notes is to explain the remarkable formula found by Yau and Zaslow [Y-Z] to express the number of rational curves on a K3 surface. Projective K3 surfaces fall into countably many families \((\mathcal{F}_g)_{g \geq 1}\); a surface in \(\mathcal{F}_g\) admits a \(g\)-dimensional linear system of curves of genus \(g\). A naïve count of constants suggests that such a system will contain a positive number, say \(n(g)\), of rational (highly singular) curves. The formula is

\[
\sum_{g \geq 0} n(g)q^g = \frac{q}{\Delta(q)},
\]

where \(\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}\) is the well-known modular form of weight 12, and we put by convention \(n(0) = 1\).

To explain the idea in a nutshell, take the case \(g = 1\). We are thus looking at K3 surfaces with an elliptic fibration \(f : S \to \mathbb{P}^1\), and we are asking for the number of singular fibres. The (topological) Euler-Poincaré characteristic of a fibre \(C_t\) is 0 if \(C_t\) is smooth, 1 if it is a rational curve with one node, 2 if it has a cusp, etc. From the standard properties of the Euler-Poincaré characteristic, we get

\[
e(S) = \sum_t e(C_t);
\]

hence \(n(1) = e(S) = 24\), and this number counts nodal rational curves with multiplicity 1, cuspidal rational curves with multiplicity 2, etc.

The idea of Yau and Zaslow is to generalize this approach to any genus. Let \(S\) be a K3 surface with a \(g\)-dimensional linear system \(\Pi\) of curves of genus \(g\). The role of \(f\) will be played by the morphism \(\overline{\mathcal{J}C} \to \Pi\) whose fibre over a point \(t \in \Pi\) is the compactified Jacobian \(\overline{JC_t}\). To apply the same method, we would like to prove the following facts:

1) The Euler-Poincaré characteristic \(e(\overline{JC})\) is the coefficient of \(q^g\) in the Taylor expansion of \(q/\Delta(q)\).

2) \(e(\overline{JC_t}) = 0\) if \(C_t\) is not rational.

3) \(e(\overline{JC_t}) = 1\) if \(C_t\) is a rational curve with nodes as only singularities. Moreover \(e(\overline{JC_t})\) is positive when \(C_t\) is rational, and can be computed in terms of the singularities of \(C_t\).

4) For a generic K3 surface \(S\) in \(\mathcal{F}_g\), all rational curves in \(\Pi\) are nodal.

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The first statement is proved in §1, by comparing with the Euler-Poincaré characteristic of the Hilbert scheme \( S^{[g]} \) which has been computed by Göttsche. The assertion 2) is proved in §2. The situation for 3) is less satisfactory: though I can express \( e(\bar{JC}) \), for a rational curve \( C \), in terms of a local invariant of the singularities of \( C \), and compute this local invariant in a number of cases, at this moment I am not able to prove that it is always positive. Finally 4) seems to be still open, despite recent progress by Xi Chen.

The outcome (see Cor. 2.3) is that the coefficient of \( q^g \) in \( q/\Delta(q) \) counts the rational curves in \( \Pi \) with a certain multiplicity, which is 1 for a nodal curve and can be computed explicitly in many cases; the two missing points are the positivity of this multiplicity, and the fact that only nodal curves occur on a generic surface in \( F_g \).

1. The compactified relative Jacobian

   (1.1) Let \( X \) be a complex variety; we denote by \( e(X) \) its Euler-Poincaré characteristic, defined by \( e(X) = \sum_p (-1)^p \dim \mathbb{Q} H^p_c(X, \mathbb{Q}) \). Recall that this invariant is additive, i.e. satisfies \( e(X) = e(U) + e(X \smallsetminus U) \) whenever \( U \) is an open subset of \( X \).

   (1.2) We consider a projective K3 surface \( S \) with a complete linear system \( (C_t)_{t \in \Pi} \) of curves of genus \( g \geq 1 \) (so \( \Pi \) is a projective space of dimension \( g \)). We will assume that all the curves \( C_t \) are integral (i.e. irreducible and reduced). This is a simplifying assumption, which can probably be removed at the cost of various technical complications. It is of course satisfied if the class of \( C_t \) generates \( \text{Pic}(S) \).

   Let \( C \to \Pi \) be the morphism with fibre \( C_t \) over \( t \in \Pi \). For each integer \( d \in \mathbb{Z} \), we denote by \( \bar{J}C = \bigcoprod_{d \in \mathbb{Z}} \bar{J}^dC \) the compactified Picard scheme of this family. \( \bar{J}^dC \) is a projective variety of dimension \( 2g \), which parameterizes pairs \( (C_t, \mathcal{L}) \) where \( t \in \Pi \) and \( \mathcal{L} \) is a torsion free, rank 1 coherent sheaf on \( C_t \) of degree \( d \) (i.e. with \( \chi(\mathcal{L}) = d + 1 - g \)). According to Mukai ([M], example 0.5), \( \bar{J}^dC \) can be viewed as a connected component of the moduli space of simple sheaves on \( S \), and therefore is smooth, and admits a (holomorphic) symplectic structure.

   The simplest symplectic varieties associated to the K3 surface \( S \) are the Hilbert schemes \( S^{[d]} \), which parameterize finite subschemes of length \( d \) of \( S \). The birational comparison of the symplectic varieties \( \bar{J}^dC \), for various values of \( d \), with \( S^{[g]} \) is an interesting problem, about which not much seems to be known. There is one easy case:

**Proposition 1.3.** – *The compactified Jacobian \( \bar{J}^gC \) is birationally isomorphic to \( S^{[g]} \).*
Proof: Let U be the open subset of $\overline{\mathcal{J}}^gC$ consisting of pairs $(C_t, L)$ where $L$ is invertible and $\dim H^0(C_t, L) = 1$. To such a pair corresponds a unique effective Cartier divisor $D$ on $C_t$ of degree $g$, which can be viewed as a length $g$ subscheme of $S$; since $\dim H^0(C_t, \mathcal{O}_{C_t}(D)) = 1$ it is contained in a unique curve of $\Pi$, namely $C_t$. This provides an isomorphism between U and the open subset $V$ of $S^g$ parameterizing finite subschemes of $S$ contained in a unique curve of $\Pi$ and defining a Cartier divisor in this curve. ■

Corollary 1.4. Write $\frac{q}{\Delta(q)} = \sum_{g \geq 0} e(g) q^g$. Then $e(\overline{\mathcal{J}}^gC) = e(g)$.

Proof: We can either use a recent result of Batyrev and Kontsevich [?] saying that two birationally equivalent projective Calabi-Yau manifolds have the same Betti numbers, or a more precise result of Huybrechts [H]: two birationally equivalent projective symplectic manifolds which are isomorphic in codimension 2 are diffeomorphic (note that the open subsets $U$ and $V$ appearing in the above proof have complements of codimension $\geq 2$). It remains to apply Göttsche’s formula $e(S^g) = e(g)$ [G]. ■

2. The compactified Jacobian of a non-rational curve

Let $C$ be an integral curve. By a rank 1 sheaf on $C$ I will mean a torsion free, rank 1 coherent sheaf. The rank 1 sheaves $L$ on $C$ of degree $d$ are parameterized by the compactified Jacobian $\overline{\mathcal{J}}^dC$. If $L$ is an invertible sheaf of degree $d$ on $C$, the map $L \mapsto L \otimes L$ is an isomorphism of $\overline{\mathcal{J}}C$ onto $\overline{\mathcal{J}}^dC$, so we can restrict our study to degree 0 sheaves.

Let $L \in \overline{\mathcal{J}}C$; the endomorphism ring of $L$ is an $\mathcal{O}_C$-subalgebra of the sheaf of rational functions on $C$. It is finitely generated as an $\mathcal{O}_C$-module, hence contained in $\mathcal{O}_C$. It is thus of the form $\mathcal{O}_{C'}$, where $f : C' \to C$ is some partial normalization of $C$. The sheaf $L$ is a $\mathcal{O}_{C'}$-module, which amounts to say that it is the direct image of a rank 1 sheaf $L'$ on $C'$.

Lemma 2.1. Let $L \in \mathcal{J}C$. Then $L \otimes L$ is isomorphic to $L$ if and only if $f^*L$ is trivial.

Proof: The sheaf $L \otimes L$ is isomorphic to $f_*(L' \otimes f^*L)$, hence to $L$ if $f^*L$ is trivial. On the other hand we have

$$\text{Hom}_{\mathcal{O}_C}(L, L \otimes L) \cong \mathcal{E}nd_{\mathcal{O}_C}(L) \otimes_{\mathcal{O}_C} L \cong f_*\mathcal{O}_{C'} \otimes L \cong f_*f^*L,$$

so if $f^*L$ is non-trivial, the space $\text{Hom}(L, L \otimes L)$ is zero, and $L \otimes L$ cannot be isomorphic to $L$. ■
Proposition 2.2. Let $C$ be an integral curve whose normalization $\tilde{C}$ has genus $\geq 1$. Then $e(\tilde{J}^d C) = 0$.

Proof: We have an exact sequence

$$0 \to G \to JC \to J\tilde{C} \to 0,$$

where $G$ is a product of additive and multiplicative groups. In particular, $G$ is a divisible group, hence this exact sequence splits as a sequence of abelian groups. For each integer $n$, we can therefore find a subgroup of order $n$ in $JC$ which maps injectively into $J\tilde{C}$. By Lemma 2.1, this group acts freely on $\bar{JC}$, which implies that $n$ divides $e(\bar{JC})$; since this holds for any $n$ the Proposition follows. ■

Corollary 2.3. Write $\frac{q}{\Delta(q)} = \sum_{g \geq 0} e(g) q^g$; let $\Pi_{rat} \subset \Pi$ be the (finite) subset of rational curves. Then $e(g) = \sum_{t \in \Pi_{rat}} e(\bar{JC}_t)$.

Proof: We first make a general observation: let $f : X \to Y$ be a surjective morphism of complex algebraic varieties whose fibres have Euler characteristic 0; then $e(X) = 0$. This is well known (and easy) if $f$ is a locally trivial fibration; the general case follows using (1.1), because there exists a stratification of $Y$ such that $f$ is locally trivial above each stratum $[V]$.

The set $\Pi_{rat}$ is finite because otherwise it would contain a curve, so $S$ would be ruled. Consider the morphism $p : \bar{J}^g C \to \Pi$ above $\Pi - \Pi_{rat}$; by the above remark, we have $e(p^{-1}(\Pi - \Pi_{rat})) = 0$, hence the result using again (1.1). ■

In other words, $e(g)$ counts the number of rational curves with multiplicity, the multiplicity of a curve $C$ being $e(\bar{JC})$. In the next two sections we will try to show that this is indeed a reasonable notion of multiplicity (with only partial success, as explained in the introduction).

3. The compactified Jacobian of a rational curve

Lemma 3.1. Let $f : C' \to C$ be a partial normalization of $C$. The morphism $f_* : \bar{JC}' \to \bar{JC}$ is a closed embedding.

Proof: Let $\mathcal{L}, \mathcal{M}$ be two rank 1 sheaves on $C'$. We claim that any $\mathcal{O}_C$-homomorphism $u : f_*\mathcal{L} \to f_*\mathcal{M}$ is actually $f_*\mathcal{O}_{C'}$-linear. Let $U$ be a Zariski open subset of $C$, $\varphi \in \Gamma(U, f_*\mathcal{O}_{C'})$, $s \in \Gamma(U, f_*\mathcal{L})$; the rational function $\varphi$ can be written as $a/b$, with $a, b \in \Gamma(U, \mathcal{O}_C)$ and $b \neq 0$. Then the element $u(\varphi s) - \varphi u(s)$ of $\Gamma(U, f_*\mathcal{M})$ is killed by $b$, hence is zero since $f_*\mathcal{M}$ is torsion-free.
Therefore if \( f_* \mathcal{L} \) and \( f_* \mathcal{M} \) are isomorphic as \( \mathcal{O}_C \)-modules, they are also isomorphic as \( f_* \mathcal{O}_{C'} \)-modules, which means that \( \mathcal{L} \) and \( \mathcal{M} \) are isomorphic: this proves the injectivity of \( f_* \) (which would be enough for our purpose). Now if \( S \) is any base scheme, the same argument applies to sheaves \( \mathcal{L} \), \( \mathcal{M} \) on \( C \times S \), flat over \( S \), whose restrictions to each fibre \( C \times \{s\} \) are torsion free rank 1 (observe that a local section \( b \) of \( \mathcal{O}_C \) is \( \mathcal{M} \)-regular because it is on each fibre, and \( \mathcal{M} \) is flat over \( S \)). This proves that \( f_* \) is a monomorphism; since it is proper, it is a closed embedding.

(3.2) Recall that the curve \( C \) is said to be unibranch if its normalization \( \tilde{C} \to C \) is a homeomorphism. Any curve \( C \) admits a unibranch partial normalization \( \tilde{\pi} : \tilde{C} \to C \) which is minimal, in the sense that any unibranch partial normalization \( C' \to C \) factors through \( \tilde{\pi} \). To see this, let \( \mathcal{C} \) be the conductor of \( C \), and let \( \tilde{\Sigma} \) be the inverse image in \( \tilde{C} \) of the singular locus \( \Sigma \in C \). The finite-dimensional \( k \)-algebra \( A := \mathcal{O}_{\tilde{C}} / \mathcal{O}_C \) is a product of local rings \( (A_x)_{x \in \tilde{\Sigma}} \); let \( (e_x)_{x \in \tilde{\Sigma}} \) be the corresponding idempotent elements of \( A \). A sheaf of algebras \( \mathcal{O}_{C'} \) with \( \mathcal{O}_C \subset \mathcal{O}_{C'} \subset \mathcal{O}_{\tilde{C}} \) is unibranch if and only if \( \mathcal{O}_{C'} / \mathcal{O}_C \) contains each \( e_x \), or equivalently \( \mathcal{O}_{C'} \) contains the classes \( e_x + \mathcal{C} \) for each \( x \in \tilde{\Sigma} \); clearly there is a smallest such algebra, namely the algebra \( \mathcal{O}_{\tilde{C}} \) generated by \( \mathcal{O}_C \) and the classes \( e_x + \mathcal{C} \). The completion of the local ring of \( \tilde{C} \) at a point \( y \) is the image of \( \hat{\mathcal{O}}_{\tilde{C}, \tilde{\pi}(y)} \) in \( \hat{\mathcal{O}}_{\tilde{C}, y} \).

**Proposition 3.3.** — With the above notation, \( e(\bar{J} C) = e(\bar{J} \tilde{C}) \).

**Proof:** In view of Prop. 2.2, we may suppose that \( \tilde{C} \) is rational. As before we denote by \( \Sigma \) the singular locus of \( C \), and by \( \tilde{\Sigma} \) its inverse image in \( \tilde{C} \). The cohomology exact sequence associated to the short exact sequence

\[
1 \to \mathcal{O}_C^* \longrightarrow \mathcal{O}_C^* / \mathcal{O}_C^* \longrightarrow \mathcal{O}_C^* / \mathcal{O}_C^* \to 1
\]

provides a bijective homomorphism (actually an isomorphism of algebraic groups) \( \mathcal{O}_C^* / \mathcal{O}_C^* \to JC \).

The evaluation maps \( \mathcal{O}_C^* \to (\mathbb{C}^*)^{\tilde{\Sigma}} \) and \( \mathcal{O}_C^* \to (\mathbb{C}^*)^{\Sigma} \) give rise to a surjective homomorphism \( \mathcal{O}_C^* / \mathcal{O}_C^* \to (\mathbb{C}^*)^{\tilde{\Sigma}} / (\mathbb{C}^*)^{\Sigma} \); its kernel is unipotent, i.e. isomorphic to a vector space. If \( n \) is any integer \( \geq \text{Card}(\tilde{\Sigma}) \), it follows that we can find a section \( \varphi \) of \( \mathcal{O}_C^* \) in a neighborhood of \( \tilde{\Sigma} \) such that the numbers \( \varphi(x) \) for \( x \in \tilde{\Sigma} \) are all distinct, but \( \varphi^n \) belongs to \( \mathcal{O}_C \). Let \( L \) be the line bundle on \( JC \) associated to the class of \( \varphi \) in \( \mathcal{O}_C^* / \mathcal{O}_C^* \).

Let \( U \) be the complement of \( \tilde{\pi}_*(\bar{J} \tilde{C}) \) in \( \bar{J} C \); according to 1.1 and Lemma 3.1, our assertion is equivalent to \( e(U) = 0 \). We claim that the line bundle \( L \) acts freely on \( U \); since the order of \( L \) in \( JC \) is finite and arbitrary large, this will finish
the proof. Let \( \mathcal{L} \in \mathcal{U} \), and let \( C' \) be the partial normalization of \( C \) such that \( \text{End}(\mathcal{L}) = \mathcal{O}_{C'} \); by definition of \( \mathcal{U} \), \( C' \) is not unibranch, hence there are two points of \( \tilde{\Sigma} \) mapping to the same point of \( C' \); this implies that the function \( \varphi \) does not belong to \( \mathcal{O}_{C'}^* \). From the commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{O}}_C^*/\mathcal{O}_C^* & \sim & \mathcal{J}C \\
\downarrow & & \downarrow \\
\mathcal{O}_C^*/\mathcal{O}_{C'}^* & \sim & \mathcal{J}C'
\end{array}
\]

we conclude that the pull back of \( \mathcal{L} \) to \( \mathcal{J}C' \) is non-trivial; by Lemma 2.1 this implies that \( \mathcal{L} \otimes \mathcal{L} \) is not isomorphic to \( \mathcal{L} \).

\textbf{Corollary 3.4.} – For a rational nodal curve \( C \), we have \( e(\mathcal{J}C) = 1 \).

\textbf{Remark 3.5.} – Consider a rational curve \( C \) whose singularities are all of type \( A_{2l-1} \), i.e. locally defined by an equation \( u^2 - v^{2l} = 0 \). Locally around such a singularity, the curve \( C \) is the union of two smooth branches with a high order contact, so by 3.3 \( e(\mathcal{J}C) \) is equal to 1. The fact that some highly singular curves count with multiplicity one looks rather surprising. The case \( g = 2 \) provides a (modest) confirmation: the surface \( S \) is a double covering of \( \mathbb{P}^2 \) branched along a sextic curve \( B \); the curves \( C_t \) are the inverse images of the lines in \( \mathbb{P}^2 \), and they become rational when the line is bitangent to \( B \). We get an \( A_3 \)-singularity when the line has a contact of order 4; thus our assertion in this case follows from the (certainly classical) fact that a line with a fourth order contact counts as a simple bitangent.

(3.6) Prop. 3.3 reduces the computation of the invariant \( e(\mathcal{J}C) \) to the case of a unibranch (rational) curve. To understand this invariant we will use a construction of Rego ([R], see also [G-P]). For each \( x \in C \), we put \( \delta_x = \dim \mathcal{O}_{\mathcal{C},x}/\mathcal{O}_{C,x} \) and we denote by \( \mathcal{C} \) the ideal \( \mathcal{O}_C(-\sum x(2\delta_x) x) \); it is contained in the conductor of \( C \) (but the inclusion is strict unless \( C \) is Gorenstein).

For \( x \in C \), we denote by \( A_x \) and \( \tilde{A}_x \) the finite dimensional algebras \( \mathcal{O}_{C,x}/\mathcal{C}_x \) and \( \mathcal{O}_{\mathcal{C},x}/\mathcal{C}_x \). Let \( \mathcal{G}(\delta_x, \tilde{A}_x) \) be the Grassmannian of codimension \( \delta_x \) subspaces of \( \tilde{A}_x \), and \( \mathcal{G}_x \) the closed subvariety of \( \mathcal{G}(\delta_x, \tilde{A}_x) \) consisting of sub-\( A_x \)-modules. We can also view \( \mathcal{G}_x \) as parameterizing the sub-\( \mathcal{O}_{C,x} \)-modules \( \mathcal{L}_x \) of codimension \( \delta_x \) in \( \mathcal{O}_{\mathcal{C},x} \), because any such sub-module contains \( \mathcal{C}_x \) ([G-P], lemma 1.1 (iv)). Since
\(O_{\widetilde{C}}/\mathcal{C}\) is a skyscraper sheaf with fibre \(\widetilde{A}_x\) at \(x\), the product \(\prod_{x \in \Sigma} G_x\) parameterizes sub-\(O_{\mathcal{C}}\)-modules \(\mathcal{L} \subset O_{\widetilde{C}}\) such that \(\dim O_{\widetilde{C},x}/\mathcal{L}_x = \delta_x\) for all \(x\). This implies \(\chi(O_{\mathcal{C}}/\mathcal{L}) = \sum x \delta_x = \chi(O_{\mathcal{C}}/O_{\mathcal{C}})\), hence \(\mathcal{L} \in \bar{\mathcal{J}C}\). We have thus defined a morphism \(e: \prod_{x \in \Sigma} G_x \to \bar{\mathcal{J}C}\).

**Proposition 3.7.**— The map \(e\) is a homeomorphism.

Note that \(e\) is not an isomorphism, already when \(\mathcal{C}\) is a rational curve with one ordinary cusp \(s\): the Grassmannian \(G_s\) is isomorphic to \(\mathbf{P}^1\), while \(\bar{\mathcal{J}C}\) is isomorphic to \(\mathcal{C}\).

Since we are dealing with compact varieties, it suffices to prove that \(e\) is bijective.

**Injectivity.** Let \(\mathcal{L}, \mathcal{M}\) be two sub-\(O_{\mathcal{C}}\)-modules of \(O_{\widetilde{C}}\) containing \(\mathcal{C}\). If \(\mathcal{L}\) and \(\mathcal{M}\) give the same element in \(\bar{\mathcal{J}C}\), there exists a rational function \(\varphi\) on \(\widetilde{C}\) such that \(\mathcal{M} = \varphi \mathcal{L}\). But the equalities \(\dim O_{\mathcal{C},x}/\mathcal{M}_x = \dim O_{\widetilde{C},x}/\mathcal{L}_x = \dim \varphi_x O_{\mathcal{C},x}/\mathcal{M}_x\) imply \(\varphi_x O_{\mathcal{C},x} = O_{\mathcal{C},x}\) for all \(x\), which means that \(\varphi\) is constant.

**Surjectivity.** Let \(f: \widetilde{C} \to C\) be the normalization morphism, and \(\mathcal{L} \in \bar{\mathcal{J}C}\). Let us denote by \(\mathcal{\tilde{L}}\) the line bundle on \(\widetilde{C}\) quotient of \(f^* \mathcal{L}\) by its torsion subsheaf. We claim that its degree is \(\leq 0\): we have an exact sequence

\[
0 \to \mathcal{L} \longrightarrow f_* \mathcal{\tilde{L}} \longrightarrow \mathcal{T} \to 0
\]

where \(\mathcal{T}\) is a skyscrapersheaf supported on the singular locus of \(\mathcal{C}\), such that \(\dim \mathcal{T}_x \leq \delta_x\) for all \(x \in \mathcal{C}\) ([G-P], lemma 1.1); this implies \(\chi(\mathcal{\tilde{L}}) - \chi(\mathcal{L}) \leq \chi(O_{\mathcal{C}}) - \chi(O_{\mathcal{C}})\), from which the required inequality follows. Since \(\widetilde{C}\) is rational, it follows that \(\mathcal{\tilde{L}}^{-1}\) admits a global section whose zero set is contained in \(\Sigma\).

Because of the canonical isomorphisms

\[
\text{Hom}_{O_{\mathcal{C}}}(\mathcal{L}, O_{\widetilde{C}}) \cong \text{Hom}_{O_{\widetilde{C}}}(f^* \mathcal{L}, O_{\widetilde{C}}) \cong \text{Hom}_{O_{\mathcal{C}}}(\mathcal{\tilde{L}}, O_{\widetilde{C}})
\]

we conclude that there exists a homomorphism \(i: \mathcal{L} \to O_{\widetilde{C}}\) which is bijective outside \(\Sigma\). Put \(n_x = \dim O_{\widetilde{C},x}/i(\mathcal{L}_x)\) for each \(x \in \Sigma\). Since

\[
\sum_{x \in \Sigma} n_x = \dim O_{\widetilde{C}}/i(\mathcal{L}) = \chi(O_{\widetilde{C}}) - \chi(\mathcal{L}) = g = \sum_{x \in \Sigma} \delta_x
\]

there exists a rational function \(\varphi\) on \(\widetilde{C}\) with divisor \(\sum x (\delta_x - n_x) x\). Replacing \(\mathcal{L}\) by \(\varphi \mathcal{L}\), we may assume \(n_x = \delta_x\) for all \(x\), which means that \(\mathcal{L}\) belongs to the image of \(e\).

The variety \(G_x\) depends only on the local ring \(O\) of \(C\) at \(x\) (even only on its completion); we will also denote it by \(G_O\). Recall that \(G_O\) parameterizes the sub-\(O\)-modules \(L\) of the normalization \(\widetilde{O}\) of \(O\) with \(\dim \widetilde{O}/L = \dim \widetilde{O}/O\). We put \(e(x) = e(G_x)\) (or \(e(O) = e(G_O)\)). The above Proposition gives:
Proposition 3.8. — Let $C$ be a rational unibranch curve; then $e(\overline{JC}) = \prod_{x \in C} \varepsilon(x)$. ■

Of course $\varepsilon(x)$ is equal to 1 for a smooth point, so we could as well consider the product over the singular locus $\Sigma$ of $C$. Note that in view of Prop. 3.3, we may define $\varepsilon(x)$ for a non-unibranch singularity by taking the product of the $\varepsilon$-invariants of each branch; Prop. 3.8 remains valid.

4. Examples

(4.1) **Singularities with $\mathbb{C}^*$-action**

Assume that the local, unibranch ring $\mathcal{O}$ admits a $\mathbb{C}^*$-action. This action extends to its completion, so we will assume that $\mathcal{O}$ is complete. The $\mathbb{C}^*$-action also extends to the normalization $\tilde{\mathcal{O}}$ of $\mathcal{O}$, and there exists a local coordinate $t \in \tilde{\mathcal{O}}$ such that the line $Ct$ is preserved (this is because the pro-algebraic group $\text{Aut}(\tilde{\mathcal{O}})$ is an extension of $\mathbb{C}^*$ by a pro-unipotent group, hence all subgroups of $\text{Aut}(\tilde{\mathcal{O}})$ isomorphic to $\mathbb{C}^*$ are conjugate). It follows that the graded subring $\mathcal{O}$ is associated to a semi-group $\Gamma \subset \mathbb{N}$, i.e. $\mathcal{O}$ is the ring $\mathbb{C}[[\Gamma]]$ of formal series $\sum_{\gamma \in \Gamma} a_\gamma t^\gamma$.

The $\mathbb{C}^*$-actions on $\mathcal{O}$ and $\tilde{\mathcal{O}}$ give rise to a $\mathbb{C}^*$-action on $G_\mathcal{O}$. The fixed points of this action are the submodules of $\tilde{\mathcal{O}}$ which are graded, that is of the form $\mathbb{C}[[\Delta]]$, where $\Delta$ is a subset of $\mathbb{N}$; the condition $\dim \tilde{\mathcal{O}}/\mathbb{C}[[\Delta]] = \dim \tilde{\mathcal{O}}/\mathcal{O}$ means $\text{Card}(\mathbb{N} - \Delta) = \text{Card}(\mathbb{N} - \Gamma)$, and the condition that $\mathbb{C}[[\Delta]]$ is a $\mathcal{O}$-module means $\Gamma + \Delta \subset \Delta$. The first condition already implies that there are only finitely many such fixed points. According to [B], the number of these fixed points is equal to $e(G_\mathcal{O})$. We conclude:

**Proposition 4.2.** — Let $\Gamma \subset \mathbb{N}$ be a semi-group with finite complement. The number $\varepsilon(\mathbb{C}[[\Gamma]])$ is equal to the number of subsets $\Delta \subset \mathbb{N}$ such that $\Gamma + \Delta \subset \Delta$ and $\text{Card}(\mathbb{N} - \Delta) = \text{Card}(\mathbb{N} - \Gamma)$.

I do not know whether there exists a closed formula computing this number, say in terms of a minimal set of generators of $\Gamma$. This turns out to be the case in the situation we were originally interested in, namely planar singularities. The semi-group $\Gamma$ is then generated by two coprime integers $p$ and $q$, which means that the local ring $\mathcal{O}$ is of the form $\mathbb{C}[[u, v]]/(u^p - v^q)$.

**Proposition 4.3.** — Let $p, q$ be two coprime integers. Then

$$\varepsilon(\mathbb{C}[[u, v]]/(u^p - v^q)) = \frac{1}{p+q} \binom{p+q}{p}.$$
Proof: The following proof has been shown to me by P. Colmez.

(4.3.1) We first observe that if a subset \( \Delta \) satisfies \( \Gamma + \Delta \subset \Delta \), all its translates \( n + \Delta \) \((n \in \mathbb{Z})\) contained in \( \mathbb{N} \) have the same property; moreover, among all these translates, there is exactly one with \( \text{Card}(\mathbb{N} - \Delta) = \text{Card}(\mathbb{N} - \Gamma) \). Thus the number we want to compute is the cardinal of the set \( \mathcal{D} \) of subsets \( \Delta \subset \mathbb{N} \) such that \( \Gamma + \Delta \subset \Delta \), modulo the identification of a subset and its translates.

(4.3.2) For such a subset \( \Delta \), let us introduce the generating function
\[
F_{\Delta}(T) = \sum_{\delta \in \Delta} T^\delta \in \mathbb{Z}[[T]].
\]
Since \( p + \Delta \subset \Delta \), we can write, in a unique way,
\[
\Delta = \bigcup_{i=1}^{p} (a(i) + p\mathbb{N}) \ ; \text{then} \ (1 - T^p) F_{\Delta}(T) = \sum_{i=1}^{p} T^{a(i)}. \]
Writing similarly \( \Delta = \bigcup_{j=1}^{q} (b(j) + q\mathbb{N}) \), we get \( (1 - T^q) F_{\Delta}(T) = \sum_{j=1}^{q} T^{b(j)} \). Put \( a(j) = b(j - p) + p \) for \( p + 1 \leq j \leq p + q \); the equality \( (1 - T^p) \sum_{j=p+1}^{p+q} T^{a(j)-p} = (1 - T^q) \sum_{i=1}^{p} T^{a(i)} \) reads
\[
(4.3 \ a) \sum_{i=1}^{p+q} T^{a(i)} = \sum_{i=1}^{p} T^{a(i)+q} + \sum_{j=p+1}^{p+q} T^{a(j)-p}.
\]

Conversely, given a function \( a : [1, p + q] \rightarrow \mathbb{N} \) satisfying (4.3 \( a \)), the set \( \Delta = \bigcup_{i=1}^{p} (a(i) + p\mathbb{N}) \) is equal to \( \bigcup_{j=p+1}^{p+q} (a(j) - p + q\mathbb{N}) \), and therefore satisfies \( \Gamma + \Delta \subset \Delta \) (note that (4.3 \( a \)) implies that the classes (mod. \( p \)) of the \( a(i) \)'s, for \( 1 \leq i \leq p \), are all distinct).

The equality (4.3 \( a \)) means that there exists a permutation \( \sigma \in \mathfrak{S}_{p+q} \) such that \( a(\sigma i) \) is equal to \( a(i) + q \) if \( i \leq p \) and to \( a(i) - p \) if \( i > p \). This implies that \( a(\sigma^m(i)) \) is of the form \( a(i) + \alpha q - \beta p \), with \( \alpha, \beta \in \mathbb{N} \) and \( \alpha + \beta = m \); since \( p \) and \( q \) are coprime, it follows that \( \sigma \) is of order \( p + q \), i.e. is a circular permutation. It also follows that the numbers \( a(i) \) are all distinct; hence the permutation \( \sigma \) is uniquely determined. Let \( \tau \) be a permutation such that \( \tau \sigma \tau^{-1} \) is the permutation \( i \mapsto i + 1 \) (mod. \( p + q \)), and let \( S_\Delta = \tau([1,p]) \). Replacing \( a \) by \( a \circ \tau^{-1} \), our function \( a \) satisfies
\[
(4.3 \ b) \quad a(i + 1) = \begin{cases} a(i) + q & \text{if } i \in S_\Delta, \\ a(i) - p & \text{if } i \notin S_\Delta. \end{cases}
\]

Since \( \tau \) is determined up to right multiplication by a power of \( \sigma \), the set \( S_\Delta \subset [1, p + q] \) is well determined up to a translation (mod. \( p + q \)). Note that replacing \( \Delta \) by \( n + \Delta \) amounts to add the constant value \( n \) to the function \( a \), hence does not change \( S_\Delta \).
Conversely, let us start from a subset $S \subset [1, p + q]$ with $p$ elements. We define inductively a function $a_S$ on $[1, p + q]$ by the relations (4.3 b), giving to $a_S(1)$ an arbitrary value, large enough so that $a_S$ takes its values in $\mathbb{N}$. By construction the function $a_S$ satisfies (4.3 b), so by (4.3.2) the subset $\Delta_S = \bigcup_{s \in S} (a_S(s) + p\mathbb{N})$ satisfies $\Gamma + \Delta_S \subset \Delta_S$.

An easy computation gives $a_{S+1}(i + 1) = a_S(i)$ and therefore $\Delta_{S+1} = \Delta_S$. Let $\mathcal{S}$ be the set of subsets of $[1, p + q]$ with $p$ elements, modulo translation; the maps $\Delta \mapsto S_\Delta$ from $\mathcal{D}$ to $\mathcal{S}$ and $S \mapsto \Delta_S$ from $\mathcal{S}$ to $\mathcal{D}$ are inverse of each other. Since $\text{Card}(\mathcal{S}) = \frac{1}{p + q} \binom{p + q}{p}$, the Proposition follows.

(4.4) Simple singularities

We now consider the case where the singularities of $C$ are simple, i.e. of $A, D, E$ type. The local ring of such a singularity has only finitely many isomorphism classes of torsion free rank 1 modules, and this property characterizes these singularities among all plane curves singularities [G-K].

**Proposition 4.5.** Let $\mathcal{O}$ be the local ring of a simple singularity. Then $\varepsilon(\mathcal{O})$ is the number of isomorphism classes of torsion free rank 1 $\mathcal{O}$-modules. It is given by:

- $\varepsilon(\mathcal{O}) = l + 1$ if $\mathcal{O}$ is of type $A_{2l}$;
- $\varepsilon(\mathcal{O}) = 1$ if $\mathcal{O}$ is of type $A_{2l+1}$;
- $\varepsilon(\mathcal{O}) = 1$ if $\mathcal{O}$ is of type $D_{2l}$ ($l \geq 2$);
- $\varepsilon(\mathcal{O}) = l$ if $\mathcal{O}$ is of type $D_{2l+1}$ ($l \geq 2$);
- $\varepsilon(\mathcal{O}) = 5$ if $\mathcal{O}$ is of type $E_6$;
- $\varepsilon(\mathcal{O}) = 2$ if $\mathcal{O}$ is of type $E_7$;
- $\varepsilon(\mathcal{O}) = 7$ if $\mathcal{O}$ is of type $E_8$.

**Proof:** Let $C$ be a rational curve with only one simple singularity, with local ring $\mathcal{O}$; the action of $J_C$ on $\bar{J}_C$ has finitely many orbits, corresponding to the different isomorphism classes of rank 1 $\mathcal{O}$-modules. Since each orbit is an affine space, its Euler characteristic is 1, hence by (1.1) $\varepsilon(\mathcal{O}) = e(\bar{J}_C)$ is equal to the number of these orbits.

If $\mathcal{O}$ is unibranch, its completion is of the form $\mathbb{C}[[u, v]]/(u^p - v^q)$, with $p = 2$, $q = 2l + 1$ for the type $A_{2l}$, $p = 3$, $q = 4$ for the type $E_6$ and $p = 3$, $q = 5$ for the type $E_8$; in these cases the result follows from 4.3. We have already observed that $\varepsilon = 1$ for a $A_{2l+1}$ singularity (Remark 3.5). A $D_l$ singularity is the union of a $A_{l-3}$ branch and a transversal smooth branch, hence the result by 3.3. Finally an $E_7$ singularity is the union of an ordinary cusp and its tangent, hence has $\varepsilon = 2$. 


Remark 4.6.− Let $\mathcal{D}$ be the set of graded sub-$\mathcal{O}$-modules $L \subset \tilde{\mathcal{O}}$ with $\dim \tilde{\mathcal{O}}/L = \dim \tilde{\mathcal{O}}/\mathcal{O}$. Two modules $L$, $M$ in $\mathcal{D}$ are isomorphic if and only if $M = t^n L$ for some $n \in \mathbb{Z}$, but the dimension condition forces $n = 1$. It follows that each torsion free rank 1 $\mathcal{O}$-module is isomorphic to exactly one element of $\mathcal{D}$. It is quite easy that way to write down the list of isomorphism classes of rank 1 $\mathcal{O}$-modules (which is of course well-known, see e.g. [G-K]). For instance if $\mathcal{O}$ is of type $\text{E}_8$, we get the following modules (with the notation of 4.1):
$\mathcal{O}$, $\mathcal{O}t + \mathcal{O}t^8$, $\mathcal{O}t^2 + \mathcal{O}t^6$, $\mathcal{O}t^2 + \mathcal{O}t^4$, $\mathcal{O}t^3 + \mathcal{O}t^4$, $\mathcal{O}t^3 + \mathcal{O}t^5 + \mathcal{O}t^7$, $\tilde{\mathcal{O}}t^4$.

REFERENCES

[B] A. Bialynicki-Birula: On fixed point schemes of actions of multiplicative and additive groups. Topology 12, 99-103 (1973).

[G] L. Göttsche: The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann. 86, 193-207 (1990).

[G-K] G.-M. Greuel, H. Knörrer: Einfache Kurvensingularitäten und torsion-freie Moduln. Math. Ann. 270, 417-425 (1985).

[G-P] G.-M. Greuel, G. Pfister: Moduli spaces for torsion free modules on curve singularities, I. J. Algebraic Geometry 2, 81-135 (1993).

[H] D. Huybrechts: Birational symplectic manifolds and their deformations. Preprint alg-geom/9601013.

[M] S. Mukai: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. math. 77, 101-116 (1984).

[R] C. J. Rego: The compactified Jacobian. Ann. scient. Éc. Norm. Sup. 13, 211-223 (1980).

[V] J.-L. Verdier: Stratifications de Whitney et théorème de Bertini-Sard. Invent. math. 36, 295-312 (1976).

[Y-Z] S.-T. Yau, E. Zaslow: BPS states, string duality, and nodal curves on K3. Preprint hep-th/9512121.

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