Common subspaces of $L_p$-spaces

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Abstract. For $n \geq 2$, $p < 2$ and $q > 2$, does there exist an $n$-dimensional Banach space different from Hilbert spaces which is isometric to subspaces of both $L_p$ and $L_q$? Generalizing the construction from the paper ”Zonoids whose polars are zonoids” by R.Schneider we give examples of such spaces. Moreover, for any compact subset $Q$ of $(0, \infty) \setminus \{2k, k \in \mathbb{N}\}$, we can construct a space isometric to subspaces of $L_q$ for all $q \in Q$ simultaneously.

AMS classification: Primary 46B04, Secondary 46E30
Key words: isometries, positive definite functions, spherical harmonics.
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1. Introduction.

This work started with the following question: For given $n \geq 2$, $p \in (0, 2)$ and $q > 2$, does there exist an $n$-dimensional Banach space which is different from Hilbert spaces and which is isometric to subspaces of both $L_p$ and $L_q$?

It is a well-known fact first noticed by P. Levy that Hilbert spaces are isometric to subspaces of $L_q$ for all $q > 0$. On the other hand, it was proved in [4] that, for $n \geq 3$, $q > 2$, $p > 0$, the function $\exp(-\|x\|_q^p)$ is not positive definite where $\|x\|_q = (|x_1|^q + \ldots + |x_n|^q)^{1/q}$. (This result gave an answer to a question posed by I.J. Schoenberg [10] in 1938.) In 1966, J. Bretagnolle, D. Dacunha-Castelle and J.L. Krivine [1] proved that, for $0 < p < 2$, a space $(E, \| \cdot \|)$ is isometric to a subspace of $L_p$ if and only if the function $\exp(-\|x\|_p^p)$ is positive definite. Thus, in the language of isometries, the above mentioned result from [4] means that, for every $n \geq 3$, $q > 2$, $p \in (0, 2)$, the space $l_n^q$ is not isometric to a subspace of $L_p$. (For $p \geq 1$, this fact was first proved in [2]) The initial purpose of this work was to find a non-Hilbertian subspace $(E, \| \cdot \|)$ of $L_q$ with $q > 2$ of the dimension at least 3 such that the function $\exp(-\|x\|_p^p)$ is positive definite. The latter problem is equivalent to that at the beginning of the paper.

We prove, however, a more general fact: For every $n \geq 2$ and every compact subset $Q$ of $(0, \infty) \setminus \{2k, k \in N\}$, there exists an $n$-dimensional Banach space different from Hilbert spaces which is isometric to subspaces of $L_q$ for all $q \in Q$ simultaneously.

In 1975, R. Schneider [9] proved that there exist non-trivial zonoids whose polars are zonoids or, in other words, there exist non-Hilbertian Banach spaces $X$ such that $X$ and $X^*$ are isometric to subspaces of $L_1$. It turns out that Schneider’s construction of special subspaces of $L_1$ can be extended to all numbers $q > 0$ which are not even integers and in this way we obtain our main result.
2. Some properties of spherical harmonics.

We start with some properties of spherical harmonics (see [6] for details).

Let $P_m$ denote the space of spherical harmonics of degree $m$ on the unit sphere $\Omega_n$ in $\mathbb{R}^n$. Remind that spherical harmonics of degree $m$ are restrictions to the sphere of harmonic homogeneous polynomials of degree $m$. We consider spherical harmonics as functions from the space $L_2(\Omega_n)$. Any two spherical harmonics of different degrees are orthogonal in $L_2(\Omega_n)$ [6, p.2]

The dimension $N(n, m)$ of the space $P_m$ can easily be calculated [6, p.4]:

\begin{equation}
N(n, m) = \frac{(2m + n - 2)\Gamma(n + m - 2)}{\Gamma(m + 1)\Gamma(n - 1)}
\end{equation}

Let $\{Y_{mj} : j = 1, ..., N(n, m)\}$ be an orthonormal basis of the space $P_m$. By the Addition Theorem [6, p.9], for every $x \in \Omega_n$,

\begin{equation}
\sum_{j=1}^{N(n,m)} Y_{mj}^2(x) = \frac{N(n, m)}{\omega_n}
\end{equation}

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface of the sphere $\Omega_n$.

Linear combinations of functions $Y_{mj}$ are dense in the space $L_2(\Omega_n)$ [6, p.43]. Therefore, if $F$ is a continuous function on $\Omega_n$ and $(F, Y_{mj}) = 0$ for every $m = 0, 1, 2, ...$ and every $j = 1, ..., N(n, m)$ then $F \equiv 0$ on $\Omega_n$. Here $(F, Y)$ stands for the scalar product in $L_2(\Omega_n)$.

Let $\Delta$ be the Laplace-Beltrami operator on the sphere $\Omega_n$. Then for every $Y_m \in P_m$ we have [6, p.39]

\begin{equation}
\Delta Y_m + m(m + n - 2)Y_m \equiv 0
\end{equation}

An immediate consequence of (3) (and a well-known fact) is that $\Delta$ is a symmetric operator and we can apply Green’s formula: for every function $H$ from the class $C^{2r}, r \in N$ of functions on $\Omega_n$ having continuous partial derivatives of order $2r$ and for every $Y_m \in P_m, m \geq 1$,

\begin{equation}
(-m(m + n - 2))^r(H, Y_m) = (H, \Delta^r Y_m) = (\Delta^r H, Y_m)
\end{equation}

We also need the Funk-Henke formula [6, p.20]: for every $Y_m \in P_m$, every continuous function $f$ on $[-1, 1]$ and every $x \in \Omega_n$,

\begin{equation}
\int_{\Omega_n} f(\langle x, \xi \rangle) Y_m(\xi) d\xi = \lambda_m Y_m(x)
\end{equation}

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^n$ and

\begin{equation}
\lambda_m = \frac{(-1)^m \pi^{(n-1)/2}}{2^{m-1} \Gamma(m + (n-1)/2)} \int_{-1}^{1} f(t) \frac{d^m}{dt^m}(1 - t^2)^{m+(n-3)/2} dt
\end{equation}

Let us calculate $\lambda_m$ in the case where $f(t) = |t|^q, q > 0$. 

2
**Lemma 1.** If $q > 0, q \neq 2k, k \in \mathbb{N}$ and $f(t) = |t|^q$ then

$$
\lambda_m = \frac{\pi^{n/2-1} \Gamma(q+1) \sin(\pi(m-q)/2) \Gamma((m-q)/2)}{2^{q-1} \Gamma((m+n+q)/2)}
$$

**Proof:** Assume first that $q > m$ and calculate the integral from (6) by parts $m$ times. Then use the formula $\int_1^{-1} t^{2\alpha-1}(1-t^2)^{\beta-1}dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ and formulas for $\Gamma$-function: $\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2)/\pi^{1/2}$ and $\Gamma(1-x)\Gamma(x) = \pi/\sin(\pi x)$. We get (7) for $q > m$. Note that both sides of (7) are analytic functions of $q$ in the domain $\Re q > 0, q \neq 2k, k \in \mathbb{N}$. Because of the uniqueness of analytic extension, (7) holds for every $q$ from this domain. We are done.\[]

**3. Main result.**

Let $X$ be an $n$-dimensional subspace of $L_q = L_q([0,1])$ with $q > 0$. Let $f_1, ..., f_n$ be a basis in $X$ and $\mu$ be the joint distribution of the functions $f_1, ..., f_n$ with respect to Lebesgue measure ($\mu$ is a finite measure on $\mathbb{R}^n$). Then, for every $x \in \mathbb{R}^n$,

$$
\|x\|^q = \| \sum_{k=1}^{n} x_k f_k \|^q = \int_0^1 |\sum_{k=1}^{n} x_k f_k(t)|^q dt = \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^q d\mu(\xi) = \int_{\Omega_n} |\langle x, \xi \rangle|^q d\nu(\xi)
$$

where $\nu$ is the projection of $\mu$ to the sphere. (For every Borel subset $A$ of $\Omega_n$, $\nu(A) = \int_{\{\xi, \mu(\xi) \in R\}} \|x\|^q d\mu(\xi)$) The representation (11) of the norm is usually called the Levy representation. It is clear now that a norm in an $n$-dimensional Banach space admits the Levy representation with a probability measure on the sphere if and only if this space is isometric to a subspace of $L_q$. (Given the Levy representation we can choose functions $f_1, ..., f_n$ on $[0,1]$ with the joint distribution $\nu$ and define an isometry by $x \rightarrow \sum_{k=1}^{n} x_k f_k, x \in R^n$)

If we replace the measure $\nu$ by an arbitrary continuous (not necessarily non-negative) function on the sphere then a representation similar to the Levy representation is possible for a large class of Banach spaces (see [5] for the Levy representation with distributions instead of measures; such representation is possible for any Banach space and any $q$ which is not an even integer) This is an idea going back to W.Blaschke that any smooth enough function on the sphere can be represented in the form (11) with a continuous function instead of a measure on the sphere. However, W.Blaschke and then R.Schneider [8] restricted themselves to the case $q = 1$ which is particularly important in the theory of convex bodies. The following theorem is an extension of R.Schneider’s results from [8, p.77] and [9, p.367] to all positive numbers $q$ which are not even integers

**Theorem 1.** Let $q > 0, q \neq 2k, k \in \mathbb{N}$ and let $H$ be an even function of the class $C^{2r}$ on $\Omega_n$ where $r \in \mathbb{N}$ and $2r > n+q$. Then there exists a continuous function $b_H$ on the sphere
Ωₙ such that, for every \( x \in \Omegaₙ \),

\[
H(x) = \int_{\Omegaₙ} |\langle x, \xi \rangle|^q b_H(\xi) d\xi
\]

Besides that, there exist constants \( K(q) \) and \( L(q) \) depending on \( n \) and \( q \) only such that, for every \( x \in \Omegaₙ \),

\[
|b_H(x)| \leq K(q)\|H\|_{L_2(\Omegaₙ)} + L(q)\|\Delta^{2r}H\|_{L_2(\Omegaₙ)}
\]

**Proof:** Define a function \( b_H \) on \( \Omegaₙ \) by

\[
b_H(x) = \sum_{m=0}^{\infty} \lambda_m^{-1} \sum_{j=1}^{N(n,m)} (N, Y_{mj}) Y_{mj}(x)
\]

Since \( H \) is an even function and \( Y_{mj} \) are odd functions if \( m \) is odd the sum is, actually, taken over even integers \( m \) only.

Let us prove that the series in the right-hand side of (10) converges uniformly on \( \Omegaₙ \).

By the Cauchy-Schwartz inequality, (2) and the fact that \( Y_{mj} \) form an orthonormal basis in \( P_m \), we get

\[
|\sum_{j=1}^{N(n,m)} (\Delta^r N, Y_{mj}) Y_{mj}(x)| \leq \left( \sum_{j=1}^{N(n,m)} (\Delta^r N, Y_{mj})^2 \right)^{1/2} \left( \sum_{j=1}^{N(n,m)} Y_{mj}^2(x) \right)^{1/2}
\]

\[
\leq \|\Delta^r H\|_{L_2(\Omegaₙ)} \left( \frac{N(n,m)}{\omega_n} \right)^{1/2}
\]

It follows from (4) and the latter inequality that

\[
|b_H(x)| \leq |\lambda_0^{-1}(H, Y_0)Y_0(x)| + \sum_{m=2;2|m}^{\infty} \lambda_m^{-1} (\frac{-1}{m(m+n-2)})^r \left( \sum_{j=1}^{N(n,m)} (\Delta^r H, Y_{mj}) Y_{mj}(x) \right) \leq
\]

\[
|\lambda_0|^{-1} \omega_n^{-1/2} \|H\|_{L_2(\Omegaₙ)} + \sum_{m=2;2|m}^{\infty} \lambda_m^{-1} m^{-2r} \left( \frac{N(n,m)}{\omega_n} \right)^{1/2} \|\Delta^r H\|_{L_2(\Omegaₙ)}
\]

Let us show that the series \( \sum_{m=2;2|m}^{\infty} \lambda_m^{-1} m^{-2r} (N(n,m)/\omega_n)^{1/2} \) converges. In fact, it follows from (1) that \( N(n, m) = O(m^{n-2}) \) and it is an easy consequence of (7) and the Stirling formula that \( \lambda_m^{-1} = O(m^{(n+2q)/2}) \). Since \( 2r > n + q = (n + 2q)/2 + (n - 1)/2 + 1 \) we get \( \lambda_m^{-1} m^{-2r} (N(n,m)/\omega_n)^{1/2} = o(m^{n-\epsilon}) \) for some \( \epsilon > 0 \), and the series is convergent. We denote the sum of this series by \( L(q) \) and put \( K(q) = |\lambda_0|^{-1} \omega_n^{-1/2} \), so we get (9).
We have proved that the series in (10) converges uniformly and defines a continuous function on \( \Omega_n \). It follows from (5) and the fact that all functions \( Y_{mj} \) are orthogonal that 
\[
(H, Y_{mj}) = (\int_{\Omega_n} |\langle x, \xi \rangle|^q b_H(\xi) d\xi, Y_{mj}(x))
\]
for every \( m = 0, 1, 2, \ldots \) and \( j = 1, \ldots, N(n, m) \). Hence, the function \( b_H \) satisfies (8). 

Let \( X \) be an \( n \)-dimensional Banach space, \( q > 0, q \neq 2k, k \in N \). Let \( c(q) = \Gamma((n + q)/2)/(2\Gamma((q + 1)/2)\pi^{(n-1)/2}) \) be a constant such that 
\[
1 = c(q) \int_{\Omega_n} |\langle x, \xi \rangle|^q d\xi \quad \text{for every } x \in \Omega_n.
\]
(The latter integral does not depend on the choice of \( x \in \Omega_n \); it means that the norm of the space \( l_2^n \) admits the Levy representation with the uniform measure on the sphere and the space \( l_2^n \) is isometric to a subspace of \( L_q \) for every \( q \).

Denote by \( H(x), x \in \Omega_n \) the restriction of the function \( \|x\|^q \) to the sphere \( \Omega_n \). Assume that the function \( H \) belongs to the class \( C^{2r} \) on \( \Omega_n \) where \( 2r > n + q, r \in N \). Let \( b_H \) be the function corresponding to \( H \) by Theorem 1.

**Lemma 2.** If the number \( K(q)\|H - 1\|_{L_2(\Omega_n)} + L(q)\|\Delta^r H\|_{L_2(\Omega_n)} \) is less than \( c(q) \) then the space \( X \) is isometric to a subspace of \( L_q \).

**Proof:** By (8) and definition of the number \( c(q) \),
\[
H(x) - 1 = \int_{\Omega_n} |\langle x, \xi \rangle|^q (b_H(\xi) - c(q)) d\xi
\]
for every \( x \in \Omega_n \). By (9), \( |b_H(x) - c(q)| < c(q) \) for every \( x \in \Omega_n \). It means that the function \( b_H \) is positive on the sphere. The equality (8) means that the space \( X \) admits the Levy representation with a non-negative measure and, by the reasoning at the beginning of Section 3, \( X \) is isometric to a subspace of \( L_q \).

Now we are able to prove the main result of this paper. Let us only note that, for every function \( f \) of the class \( C^2 \) on the sphere \( \Omega_n \) and for a small enough number \( \lambda \), the function \( N(x) = 1 + \lambda f(x), x \in \Omega_n \) is the restriction to the sphere of some norm in \( R^n \). This is an easy consequence of the following one-dimensional fact: If \( a, b \in R, g \) is a convex function on \( [a, b] \) with \( g'' > \delta > 0 \) on \( [a, b] \) for some \( \delta \) and \( h \in C^2[a, b] \) then functions \( g + \lambda h \) have positive second derivatives on \( [a, b] \) for sufficiently small \( \lambda \)'s, and, hence, are convex on \( [a, b] \).

**Theorem 2.** Let \( Q \) be a compact subset of \((0, \infty) \setminus \{2k, k \in N\} \). Then there exists a Banach space different from Hilbert spaces which is isometric to a subspace of \( L_q \) for every \( q \in Q \).

**Proof:** Let \( f \) be any infinitely differentiable function on \( \Omega_n \) and fix a number \( r \in N \) so that \( 2r > n + q \) for every \( q \in Q \). Choose a sufficiently small number \( \lambda \) such that the function \( N(x) = 1 + \lambda f(x), x \in \Omega_n \) is the restriction to the sphere of some norm in \( R^n \) (see the
remark before Theorem 2) and such that, for every $q \in Q$, the function $H(x) = (N(x))^q$ satisfies the condition of Lemma 2. The possibility of such choice of $\lambda$ follows from the facts that $K(q), L(q)$ and $c(q)$ are continuous functions of $q$ on the set $Q$ and that $\|H - 1\|_{L_2(\Omega_n)}$ and $\|\Delta^{2r} H\|_{L_2(\Omega_n)}$ tend to zero uniformly with respect to $q \in Q$ as $\lambda$ tends to zero. Now we can apply Lemma 2 to complete the proof.

Finally, let us consider the case where $q$ is an even integer. It is easy to see that, for any fixed number $2k, k \in \mathbb{N}, k > 1$, we can make the space $X$ constructed in Theorem 2 isometric to a subspace of $L_{2k}$. In fact, let $N(x) = (1 + \lambda(x_1^{2k} + ... + x_n^{2k}))^{1/4}$. For sufficiently small numbers $\lambda$, $N$ is the restriction to the sphere of some norm in $\mathbb{R}^n$ and the corresponding space $X$ is isometric to a subspace of $L_q$ for every $q \in Q$. On the other hand, $X$ is isometric to a subspace of $L_{2k}$ because the norm admits the Levy representation with a measure on the sphere:

$$1 + \lambda(x_1^{2k} + ... + x_n^{2k}) = \int_{\Omega_n} |\langle x, \xi \rangle|^{2k}(c(2k)d\xi + \lambda d\delta_1(\xi) + ... + \lambda d\delta_n(\xi))$$

where $\delta_i$ is a unit mass at the point $\xi \in \mathbb{R}_n$ with $\xi_i = 1, \xi_j = 0, j \neq i$.

Let us show that one can not make the space $X$ isometric to subspaces of $L_{2p}$ and $L_{2q}$ if $p, q \in \mathbb{N}$ and do not have common factors. In fact, if $(X, \|\cdot\|)$ is such a space then, for every $x \in \mathbb{R}_n$,

$$\|x\|^{4pq} = (\int_{\Omega_n} |\langle x, \xi \rangle|^{2p}d\mu(\xi))^{2q} = (\int_{\Omega_n} |\langle x, \xi \rangle|^{2q}d\nu(\xi))^{2p}$$

for some measures $\mu, \nu$ on $\Omega_n$. The functions in the latter equality are polynomials and, since the polynomial ring has the unique factorization property, we conclude that $\|x\|^2$ is a homogeneous polynomial of the second order and $X$ is a Hilbert space.

The situation is not clear if $p$ and $q$ have common factors. One can find some interesting results on Banach spaces with polynomial norms and on the structure of subspaces of $L_{2k}, k \in \mathbb{N}$ in the paper [7].

Acknowledgements. I wish to thank Prof. Nigel Kalton for valuable remarks and helpful discussions during the work on this problem. I am grateful to Prof. Hermann König for bringing the paper [9] to my attention.
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