Classical non-linear operators
in Grand Lebesgue Spaces.

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Abstract.

We study in this short report the boundedness of classical non-linear operators: Nemytskii, Urysohn, Hammerstein acting from one Grand Lebesgue Space to another one, and deduce some its upper norm estimates.

We bring also some examples to illustrate the exactness of our estimates.

Key words and phrases. Measure and measurable space, Nemytskii, Urysohn, Hammerstein and other operators, boundedness, ordinary and operators norm, estimating factorization, power operator, degree, ordinary and mixing (anisotropic) Lebesgue - Riesz and Grand Lebesgue Spaces (GLS), estimate, Hölder’s inequality, examples, measurable functions, generating function.

1 Statement of problem. Notations. Definitions.

Let $(X = \{x\}, \mathcal{M}, \mu)$; $(Y = \{y\}, \mathcal{N}, \nu)$; be two measurable spaces equipped with two non-trivial measures correspondingly $(\mu, \nu)$. Let also $U = U[g](x)$, where
\[ f(x) := K[g(\cdot)](x) = K[g(\cdot), x, \cdot], \]  
\[ f : X \to R, \quad g : Y \to R \]

be an *operator*, not necessary to be linear.

*It is presumed that all the considered functions are measurable.*

We intent in this short report to estimate the norms of these operators as an operator acting between two Grand Lebesgue Spaces (GLS).

We will consider the following three important types of nonlinear operators: Nemytskii, Urysohn and Hammerstein.

**Definitions of operators.**

**Definition 1.** Nemytskii operator, see [8], [29], [33], [43]:

\[ N[g](x) \overset{\text{def}}{=} n(x, g(x)). \]  
\[ (2) \]

**Definition 2.** Urysohn’s operator, see [27], pp. 423 - 429, [28], [43]:

\[ U[g](x) \overset{\text{def}}{=} \int_Y u(x, y, g(y)) \, \nu(dy). \]  
\[ (3) \]

**Definition 3.** Hammerstein’s operator, [16], [25], [26], [29]:

\[ H[g](x) \overset{\text{def}}{=} \int_Y h(x, y) \, w(y, g(y)) \, \nu(dy). \]  
\[ (4) \]

Here all the introduced functions \( n(x, y), u(x, y, z), h(x, y), w(x, y) \) are measurable and numerical valued.

These operators appear in particular in the theory of non-linear Integral and Partial Differential Equations (PDE), see e.g. [10] - [13], [41], [42], and reference therein.

**A brief excursus in the theory of Grand Lebesgue Spaces (GLS).**

Let \((V = \{v\}, \mathcal{D}, \gamma)\) be again measurable space equipped with non-trivial sigma finite measure \( \gamma \). Let also \((a, b) = \text{const}, \ 1 \leq a < b \leq \infty \) and let \( \psi = \psi(p), \ a < p < b \) be certain strictly positive numerical valued function; which is named as ordinary *generating function*. By definition, the Grand Lebesgue Space
(GLS) \( G_\psi [V, \psi; a, b] = G_\psi \) consists on all the numerical valued measurable functions \( f : V \to \mathbb{R} \) having the following finite norm

\[
\| f \|_{G_\psi} = \| f \|_{G_\psi [V, \psi; a, b]} \overset{\text{def}}{=} \sup_{p \in (a, b)} \left\{ \frac{\| f \|_{p, V}}{\psi(p)} \right\}.
\]

(5)

Hereafter \( \| f \|_{p, V} \) denotes the usually Lebesgue - Riesz norm:

\[
\| f \|_{p, V} \overset{\text{def}}{=} \left[ \int_{V} | f(v) |^p \gamma(\nu) \, dv \right]^{1/p}, \ 1 \leq p < \infty.
\]

(6)

Notation: \( (a, b) \overset{\text{def}}{=} \text{supp} \, \psi \).

Define formally \( \psi(p) = \infty \), when \( p \notin \text{supp} \, \psi \).

**Definition of the natural generating function.** Let \( f = f(v) \) be measurable numerical valued function such that

\[
\exists (a, b), \ 1 \leq a < b \leq \infty \Rightarrow \forall p \in (a, b) \ | f |_{p, V} < \infty.
\]

The natural generating function \( \psi[f](p) \) is defined as follows:

\[
\psi[f](p) \overset{\text{def}}{=} | f |_{p, V}, \ p \in (a, b).
\]

Evidently, \( \| f \|_{G_\psi[f]} = 1 \).

The general theory of these spaces is represented in many works, see e.g. [1], [3], [4], [9], [15], [18], [19], [20], [21], [22], [23], [24], [30], [35], [36], [37], [38] etc. In particular, these spaces are complete, Banach functional and rearrangement invariant.

It is known, see e.g. [39], that in the case when \( a = 1, b = \infty \) the space \( G_\psi_{1, \infty} \) coincides with appropriate exponential Orlicz space. The belonging of some function \( h = h(v) \) is closely related with its tail behavior

\[
T_h(t) := \gamma\{ \ v : \ | h(v) | \geq t \}, \ t \to \infty.
\]

2 **Main result: Nemytskii operator.**

We consider here the Nemytskii’s operator (2). Let us impose the following condition of factorization estimating on the leading function \( n = n(x, y) \):

\[
\exists \beta = \text{const} \in [1, \infty), \ \exists \phi = \phi(x), \ x \in X, \Rightarrow | n(x, y) | \leq \phi(x) \cdot | y |^\beta,
\]

(7)

\( \phi : X \to \mathbb{R} \). Assume further that in (2) [and in (7)]
\[ g(\cdot) \in G\psi, \ \phi(\cdot) \in G\nu, \] (8)

for suitable generating functions \( \psi, \nu \); so that

\[ ||g||_{p,X} \leq C_1 \psi(p), \ ||\phi||_{q,X} \leq C_2 \nu(q), \]

where \( C_1 = ||g||_{G\psi}, \ C_2 = ||\phi||_{G\nu} \).

Of course, one can choose as such functions \( \psi, \nu \) the natural ones: \( \psi(p) = ||g||_{p,X}, \ \nu(q) = ||\phi||_{q,X} \), if they there exist. Then \( C_1 = C_2 = 1 \).

Following, if we introduce the so-called power operator of the degree \( \beta : P_\beta[g](x) = |g(x)|^\beta \), where as before \( \beta = \text{const} \geq 1 \), then

\[ ||P_\beta[g]||_{p,X} = ||g^\beta||_{p,X} \leq C_1^\beta \cdot \psi^\beta(\beta p). \]

We have:

\[ |f(x)| \leq |\phi(x)| \cdot |g(x)|^\beta. \]

One can apply Hölder’s inequality:

\[ ||f||_{r,X} \leq ||\phi||_{q,X} \cdot ||g^\beta||_{p,X}, \] (9)

where \((p, q, r)\) are arbitrary numbers such that \( p, q, r \geq 1 \) and moreover

\[ \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \]

or equally

\[ q = \frac{pr}{p-r}, \ 1 < r < p. \]

Introduce the following auxiliary functions

\[ W_{a}[\psi, \nu; \beta](p, r) = W_{a}(p, r) \stackrel{def}{=} \nu \left( \frac{pr}{p-r} \right) \times \psi^\beta(\beta p), \] (10)

\[ W[\psi, \nu; \beta](r) = W(r) \stackrel{def}{=} \inf_{p>r} W_{a}(p, r). \] (11)

It follows from the inequality (9)

\[ ||f||_{r,X} \leq C_1^\beta \ C_2 \ W_{a}(p, r), \ p > r. \] (12)

To summarize: we have from (12) after optimization over \( p; \ p \in (r, \infty) \):

**Theorem 2.1.** Suppose that for some non-trivial segment \( r \in (c, d) : 1 \leq c < d \leq \infty \) the function \( W[\psi, \nu; \beta](r) \) is finite. Then we propose in (2) under our notations
\[ ||f||GW[\psi, \nu; \beta] \leq ||\phi||G\nu \times \{ ||g||G\psi \}^\beta. \] (13)

**Example 2.1.** Let us show by means of bringing examples the exactness of proposition (13), yet for arbitrary positive value \( \beta \). Suppose \( \mu(X) = 1, \phi(x) = 1 \), so that the natural function for the \( \phi(\cdot) \) is equal to 1:

\[ \nu(q) = ||\phi||_{q,X} = 1, \ q \in [1, \infty). \]

Put also \( \psi(p) = ||g||_p, 1 \leq p < \infty \), i.e. \( \psi(\cdot) \) is the natural generating function for \( g(\cdot) \). Choose for definiteness as a function \( g = g(x) \) certain positive measurable bounded function having support with finite measure.

Notice that this function \( p \to \psi(p), 1 \leq p < \infty \) is monotonically increasing.

On the other words,

\[ f(x) = g^\beta(x), \ x \in X. \]

Let us investigate both the sides of the assertion of theorem 2.1 (13) in this case. We calculate the left hand side taking into account the monotonicity and equality \( \nu(q) = 1 \) at the value \( p = r + 0 \)

\[ W_a[\psi, \nu; \beta](r + 0, r) = W_a(r + 0, r) = W(r) = \psi^\beta(\beta r), \]

We have at the same time for the right hand side of (13) choosing also \( p = r + 0 \), so that \( q = +\infty : \)

\[ ||g^\beta||_r = \psi^\beta(\beta r), \ 1 \leq r < \infty, \]

which coincides with the right-hand side (in this example).

3 Main result. Urysohn’s operator.

We impose the following condition on the "kernel" \( u = u(x, y, z) \) in the definition of the Urysohn’s operator (3):

\[ \exists u_0 = u_0(x, y) \in R, \ x \in X, \ y \in Y, \]

\[ |u(x, y, z)| \leq u_0(x, y) |z|^\beta, \ \exists \beta = \text{const} \geq 1. \] (14)

Denote for at the same values \( (p, q, r) \) as before

\[ \kappa(q, r) \overset{def}{=} \{ ||u_0||_{q,Y} \} \{ ||u_0||_{r,X} \} \]

here \( q = pr/(p-r), \ p > r > 1. \)
Assume as above that for some non-trivial generating function \( \psi(\cdot) \)
\[ g(\cdot) \in G\psi. \]

We have by virtue of Hölder’s inequality for these values of parameters \((p, q, r)\)
\[
||U[g]||_{r,X} \leq \left[ ||g||_{G\psi}^{\beta} \times \left[ \psi^{\beta}(\beta p) \kappa(pr/(p-r), r) \right] \right].
\] (16)

Introduce the following generation function
\[
\theta(r) \overset{\text{def}}{=} \inf_{p \in (r, \infty)} \left\{ \psi^{\beta}(\beta p) \times \kappa \left( \frac{pr}{p-r}, r \right) \right\}.
\]

We conclude under our notations

**Theorem 3.1.**
\[
||U[g]||_{G\theta} \leq \left[ ||g||_{G\psi} \right]^{\beta}.
\] (17)

**Remark 3.1.** Unimprovability. The equality in the proposition of theorem 3.1 (17) for all the positive values \( \beta \) is attained if for instance \( \nu(Y) = 1 = \mu(X) \), \( g(y) = C = \text{const} \in (0, \infty) \), \( u_0(x,y) = 1 \), \( \kappa(q, r) = 1 \), \( \psi(p) = 1 \); then both the sides in (17) are equal to \( C^{\beta} \).

### 4 Main result. Hammerstein’s operator.

The Hammerstein’s operator, which we will rewrite as
\[
H[g](x) \overset{\text{def}}{=} \int_Y h(x, y) n(y, g(y)) \nu(dy),
\] (18)
may be represented as a superposition of Nemytskii operator and Urysohn’s one. Therefore, we retain all the notations and restrictions imposed above on the functions \( n(\cdot, \cdot) \).

We have consequently by means of Hölder’s inequality
\[
||H[g]||_{r,X} \leq || h ||_{q,Y} \times ||n(\cdot, g(\cdot))||_{r,Y},
\]
where as above \( 1/r = 1/q + 1/p, \ p, q, r > 1, \ p > r. \)

The functional
\[
h(\cdot, \cdot) \rightarrow || h ||_{q,Y} ||_{r,X} = || h ||_{q,Y,r,X}
\] (19)
is named as ordinary mixed, or anisotropic Lebesgue - Riesz $L_{q,r}$ norm of the function $h(\cdot, \cdot)$, see [6], [7], chapters 1,2. It is a multivariate generalization of the classical Lebesgue - Riesz norms.

The correspondent multidimensional Grand Lebesgue Mixed (Anisotropic) norm $||h(\cdot, \cdot)||^\tau$ is defined very similar to the one - dimensional case

$$||h(\cdot, \cdot)||^\tau \equiv \sup_{(q,r)} \left\{ \frac{||h||_{q,r}}{\tau(q,r)} \right\}. \quad (20)$$

Evidently, the space $G^\tau$ relative the norm $|| \cdot ||^\tau$ is complete bi-rearrangement invariant Banach functional space. Herewith, for all the admissible values $(q, r)$, i.e. for which $\tau(q, r) < \infty$,

$$||h||_{q,r} \leq ||h(\cdot, \cdot)||G^\tau \cdot \tau(q, r). \quad (21)$$

Further,

$$||n(\cdot, g(\cdot))||_{p,Y} \leq ||\phi||_{s,Y} \cdot ||g^\beta||_{t,Y}, \quad 1/p = 1/r + 1/t, \quad s = tp/(t - p), \quad t > p.$$ 

Thus,

$$||H[g]||_{r,X} \leq v(r; p, t), \quad (22)$$

where

$$v(r; p, t) := ||h||_{q,Y} ||_{r,X} \times ||\phi||_{s,Y} \times ||g^\beta||_{t,Y}. \quad (23)$$

Introduce the following domain on the positive quadrant $D(r) = \{ p, t \}$, dependent on the real number (parameter) $r; r > 1$ : where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{s} + \frac{1}{t} = \frac{1}{p}, \quad p, q, r, s, t > 1, \quad p > r, \quad t > p, \quad (24)$$

and

$$q = pr/(p - r), \quad p > r; s = tp/(t - p), \quad t > p.$$ 

Denote also

$$\Delta(r) := \inf_{(p,t) \in D(r)} v(r; p, t). \quad (25)$$

Notice that if

$$h(\cdot, \cdot) \in G^\tau, \quad \phi(\cdot) \in G^\nu, \quad g(\cdot) \in G^\psi,$$ 

then

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\( v(r; p, t) \leq \|h\|G_\tau \|\phi\|G_\nu [\|g\|G_\psi]^{\beta} \times \tau(q, r)\nu(s) \psi^\beta(\beta t). \) (27)

We obtained actually the following proposition.

**Theorem 4.1.** We conclude in our notations \( H[g] \in G\Delta \) and moreover

\[ \|H[g]\|G\Delta \leq 1. \] (28)

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