Polarization statistical properties of the emission from the single mode Vertical-Cavity Surface-Emitting Lasers with the equally living laser levels

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The full quantum-statistical theory of the Vertical-Cavity Surface-Emitting Laser (VCSEL) in the form of the Langevin equations is constructed for arbitrary relations between the frequency parameters. The same theoretical treatment as in Ref. [1,2] are used. For detailed analysis the theory is applied for lasers with equally living laser levels and on this basis the analytical expressions for the spectral densities of the Stokes parameter fluctuations are obtained in the explicit dependence on the physical phenomena, including the spin-flip and the optical anisotropy. It is demonstrated the arbitrary distribution of electrons between the sub-levels under pumping does not restrict a possibility to achieve the noise reduction below the quantum limit. Under comparison with phenomenological treatment Ref. [3] it is shown this approach turns out to be not quite satisfied.
I. INTRODUCTION

In the last years there has been an increased interest to the polarization properties of the VCSELs. This interest is motivated in the first line by the potential applications of this type of lasers in the high-rate optical communications Ref. [4]. But there is also more fundamental reason for understanding of the polarization behavior in VCSELs, namely, a possibility of generating the intensity-squeezed light using the sub-Poissonian pump of the active medium Ref. [5, 6]. To date, squeezing in VCSELs has been demonstrated experimentally for both the single-mode operation and in the multi-transverse-mode regime Ref. [7].

Now in the different scientific groups two models of laser are discussed in the main. First of them is with the shortly living upper level and the other - with the equaled lifetimes for both the levels. The known spin-flip theory [8] was elaborated for the first time just for the latter system. Nevertheless a lot of it is suitable for the first one too.

As for statistical aspects the laser with the equally living levels was studied only phenomenologically Ref. [3] in distinguish from the other system which has been studied in details within the limits of the quantum electrodynamics Ref. [1, 2]. Our main goal here to develop not less qualitative theory for the system with the equally living levels. This will allow us not only to write the different correct analytical expressions, but also to estimate the possibilities of the phenomenological treatment Ref. [3].

The paper is organized as follows. In Sec. II the basic equations of the theory will be given in the adiabatical approximation. In Sec. III, IV the fluctuations of the Stokes parameters will be introduced into consideration, be written the respective linearized equations and their solutions. In Sec. V the comparison of our results with the phenomenological ones will be made. At last, in Sec. VI the possibilities for the polarization squeezing observation will be discussed.

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II. THE LANGEVIN EQUATIONS FOR THE TWO-LEVEL LASERS WITH TWICE DEGENERATED LEVELS

The well known spin-flip VCSEL theory Ref. [8] is based on the representation about the semiconductor medium as a two-level system with a twice degeneration of the levels (see fig. 1). One pair of sub-levels is connected by wave with the $\sigma_+$-polarization and the other - with the $\sigma_-$ one. The upper "atomic" states $|a\rangle$ relax with the rate $\gamma_a$, and the lower - with the rate $\gamma_b$. For the description of overturns of the electron spins (the spin-flip) we introduce into consideration some incoherent processes $|a_+\rangle \rightarrow |a_-, b_+\rangle$ and back with the respective rates $\gamma^{(a,b)}_c$.

We will not mention all the details, how the main equations are produced. They can be found in the previous work Ref. [1, 2, 9]. The main items are as follows. At first, the equations for the Hermitian operators of the field amplitudes with the $\sigma_\pm$-polarizations, the populations of the four sub-levels and the polarizations of the two actual transitions are constructed on the basis of the full quantum theory for field and matter. The obtained equations (the Langevin-Heizenberg ones), first, are operator and, second, contain the operator sources (the inhomogeneous terms) in distinguish from the respective dynamical theory. Usually for writing the correct correlation functions one uses so-called Einstein relationships Ref. [10].

Next, to make the mathematical situation much simpler, one passes to c-number representation: all the normally ordered operators can be converted onto respective c-number functions Ref. [9, 10]. As a result the equations, called the Langevin’s equations, read:

\[
\dot{a}_\pm = -\kappa a_\pm - (\kappa_a + \omega_P) a_\mp + gP_\pm, \tag{2.1}
\]

\[
\dot{P}_\pm = - (\gamma_\perp + i\Delta) P_\pm + g(N_{a_\pm} - N_{b_\pm})a_\mp + F_{P_\pm}(t), \tag{2.2}
\]

\[
\dot{N}_{a_\pm} = \mu_a - \gamma_a N_{a_\pm} - \gamma^{(a)}_c(N_{a_\pm} - N_{a_\mp}) - g(a^{*}_\pm P_\pm + a_{\pm} P^{*}_\pm) + F_{a_\pm}(t), \tag{2.3}
\]

\[
\dot{N}_{b_\pm} = \mu_b - \gamma_b N_{b_\pm} - \gamma^{(b)}_c(N_{b_\pm} - N_{b_\mp}) + g(a^{*}_\pm P_\pm + a_{\pm} P^{*}_\pm) + F_{b_\pm}(t). \tag{2.4}
\]

Here $a_\pm$ are the $\sigma_\pm$ complex amplitudes. The atomic c-number variables are represented by the polarizations $P_\pm$ and the populations $N_{a_\pm}$ and $N_{b_\pm}$. One can see the theory turns out to be very complicated mathematically, because generally speaking there is a system of the 12 differential equations relative to the 12 variables.

The frequency coefficients have the physical senses: $\kappa$ is the spectral width of the cavity
laser mode, $g$ is the atom-field coupling constant, $\mu_{a,b}$ is the mean rate of the incoherent pump to the upper, lower laser level, $\gamma_{a,b}$ is the mentioned above constant of decay of the upper, lower level, $\gamma_c$ is the spin-flip rate, $\gamma_\perp$ is the rate of the transverse atomic relaxation, $\Delta = \omega - \nu$ - detuning of the laser frequency $\nu$ from the frequency of the laser transition $\omega$, the coefficients $\kappa_a$ and $\omega_p$ present the linear dichroizm and linear birefringence, connected with the optical anisotropy of the semiconductor crystal.

As for the stochastic sources in the Langevin equations $F_{b\pm}, F_{a\pm}, F_{P\pm}$, their properties are given by the following non-zero correlation functions:

$$F_{a\pm}(t)F_{a\pm}(t') = \left[\left(\gamma_a + \gamma_c^{(a)}\right) N_{a\pm} + \mu_a + \gamma_c^{(a)} N_{a\mp} - g \left(a^\mp_{a\pm} P_a + a_{a\pm} P^\mp_a\right) \right. - p_a \mu_a / 2 \left. \right] \delta(t - t'), \quad (2.5)$$

$$F_{b\pm}(t)F_{b\mp}(t') = \left[\left(\gamma_b + \gamma_c^{(b)}\right) N_{b\pm} + \mu_b + \gamma_c^{(b)} N_{b\mp} - g \left(a^\mp_{b\pm} P_b + a_{b\pm} P^\mp_b\right) \right. - p_b \mu_b / 2 \left. \right] \delta(t - t'), \quad (2.7)$$

$$F_{a\pm}(t)F_{b\mp}(t') = g \left(a_{a\pm}^* P_b + a_{b\pm}^* P_a\right) \delta(t - t'), \quad (2.9)$$

$$F_{P\pm}(t)F_{P\pm}(t') = \left[\left(2 \gamma_\perp - \gamma_a - \gamma_c^{(a)}\right) N_{a\pm} + \mu_a + \gamma_c^{(a)} N_{a\mp} \right) \delta(t - t'), \quad (2.10)$$

$$F_{P\pm}(t)F_{P\pm}(t') = 2g \left(a_{a\pm}^* P_a \right) \delta(t - t'), \quad (2.11)$$

$$F_{P\pm}(t)F_{P\pm}(t') = \left(\gamma_b + \gamma_c^{(b)}\right) \left(P^\pm_a\right) \delta(t - t'). \quad (2.12)$$

Here the parameters $p_{a,b}$ determine a statistical aspect of the pump: the Poissonian (quite random) pump takes a place with $p_{a,b} = 0$, the sub-Poissonian (strictly regular) - with $p_{a,b} = 1$. Under the calculation of these formulas it was proposed that distribution of the electrons between sub-levels turns out to be random even with $p_{a,b} = 1$.

The Langevin equations (2.1)-(2.4) with the correlation functions (2.5)-(2.12) are suitable, generally speaking, for description of any two-level laser, because there is no any restriction on the relationship between the different frequency parameters. Further we will make the area of our interest narrower adopting our theory for the VCSELs with the equally living laser levels.
As mentioned in Sec. I the two VCSEL models was developed. In one of them, which is studied in the San Miguel’s group (including the pioneer work Ref. [8]), the lifetimes of both the laser levels were equaled. In the other model (the Giacobino’s group Ref. [1, 2]) to the contrary the lifetime of the lower level was much less than the lifetime of the upper one. As stated above we want to apply our theory for the first model and further we will put: $\gamma_a = \gamma_b \equiv \gamma$ and $\gamma_c(a) = \gamma_c(b) \equiv \gamma_c$. Besides it is suggested the incoherent pump takes a place only to the upper level: $\mu_a \equiv \mu$ ($\mu_b = 0$).

To compare our calculations with the ones, presented in Ref. [3], we introduce new functions $D$ and $d$ instead of the populations $N_{a\pm, b\pm}$:

$$D = \frac{1}{2} \left[ (N_{a+} + N_{a-}) - (N_{b+} + N_{b-}) \right]$$

(2.13)

$$d = \frac{1}{2} \left[ (N_{a+} - N_{a-}) - (N_{b+} - N_{b-}) \right]$$

(2.14)

and put that the transverse relaxation constant is the highest between the others and we have a right to apply the adiabatical approximation. In the approximation the equations read:

$$\dot{a}_\pm = -\kappa a_\pm - (\kappa_a + \omega_p) a_\mp + c (1 - i\alpha) (D \pm d) a_\pm + \xi_\pm$$

(2.15)

$$\dot{D} = \mu - \gamma D - 2cD \left( |a_+|^2 + |a_-|^2 \right) - 2cd \left( |a_+|^2 - |a_-|^2 \right) + \xi_D$$

(2.16)

$$\dot{d} = -\gamma_s d - 2cD \left( |a_+|^2 - |a_-|^2 \right) - 2cd \left( |a_+|^2 + |a_-|^2 \right) + \xi_d$$

(2.17)

Here

$$c = \frac{g^2}{\gamma_\perp (1 + \alpha^2)}, \quad \alpha = \frac{\nu - \omega}{\gamma_\perp}, \quad \gamma_\perp = \gamma + 2\gamma_c.$$  

(2.18)

The stochastic sources in the equations (2.15)-(2.17) are some linear combinations of the initial ones. The respective expressions can be found in Appendix A.

III. FLUCTUATIONS OF THE STOKES PARAMETERS AND LINEARIZATION OF EQUATIONS

The usual approach to non-linear equations is to try linearizing them relative to some small parameter(s). In the cases of statistical theories of laser systems these small parameters are introduced as additions in the exact solutions to the stationary solutions.
We will study here only one stationary regime of generation with the linearly polarized emission. Certainly, everywhere further we must imply the set of the physical parameters which is able to ensure a stability of this solution Ref. [11].

Following to Ref. [3, 8] the stationary semi-classical solutions can be written in the form:

\[ a_{\pm, st.} = Q e^{-i\Delta x t}, \quad \Delta_x = \omega_p + \alpha \kappa_x, \quad \kappa_x = \kappa + \kappa_a \] (3.1)

Here the value \( Q \) is expressed via the pump parameter \( r = \mu / \mu_{th.} \) (\( \mu_{th.} \) - the threshold pump rate) as \( 2Q^2 = r - 1 \).

The equality between the circular components means we have the linearly polarized along the x-axes non-zero solution. Really, because the Cartesian field components are expressed via the circular ones in the form:

\[ a_x = \frac{1}{\sqrt{2}} (a_+ + a_-), \quad a_y = \frac{1}{i\sqrt{2}} (a_+ - a_-) \] (3.2)

\[ a_x = \sqrt{2}Q, \quad a_y = 0. \]

The respective stationary solutions for the active medium are:

\[ P_{\pm, st.} = \frac{c}{g} (1 - i\alpha) D_{st.}, \quad Q, \quad D_{st.} = \frac{\mu}{\gamma + 4cQ^2}, \quad d_{st.} = 0 \] (3.3)

According to (2.10) in the absence of the stochastic source the condition of the stationary lasing (\( \dot{D} = 0 \)) in the explicit form is: \( D_{st.} = \kappa_x / c \). This provides us with a direct connection of the mean radiation power \( Q \) with the mean pump rate of the medium \( \mu \):

\[ \frac{\mu}{\gamma + 4cQ^2} = \frac{\kappa_x}{c} \] (3.4)

We have discussed the stationary solutions of the problem and now the exact solutions with taking all the fluctuations into account read:

\[ a_{\pm} = \left( Q + \delta a_{\pm}(t) \right) e^{-i\Delta x t}, \quad D = D_{st.} + \delta D(t), \quad d = d_{st.} + \delta d(t) \] (3.5)

For the VCSELs we focus on the polarization effects. That is why it is convenient to introduce into consideration the Stokes parameters instead of the complex field amplitudes. They are expressed via the Cartesian field components as:

\[ S_0 = |a_x|^2 + |a_y|^2, \quad S_1 = |a_x|^2 - |a_y|^2, \quad S_2 = a_x^* a_y + a_x a_y^*, \quad S_3 = i(a_x^* a_y - a_x a_y^*) \] (3.6)
Respectively in our case of the linearly polarized regime their fluctuations are:

\[ \delta S_1 = \sqrt{2}Q(\delta a_x^* + \delta a_x), \quad \delta S_2 = \sqrt{2}Q(\delta a_y^* + \delta a_y), \quad \delta S_3 = i\sqrt{2}Q(\delta a_y^* - \delta a_y) \quad (3.7) \]

(\( \delta S_0 = \delta S_1 \)) and the linearized system of equations reads:

\[ \delta \dot{S}_1 = \gamma(r - 1)\delta D + \xi_{S_1} \quad (3.8) \]
\[ \delta \dot{D} = -\gamma r\delta D - 2\kappa_x\delta S_1 + \xi_D \quad (3.9) \]

and

\[ \delta \dot{S}_2 = 2\kappa_a\delta S_2 - 2\omega_p\delta S_3 - \alpha \gamma(r - 1)\delta d + \xi_{S_2} \quad (3.10) \]
\[ \delta \dot{S}_3 = 2\kappa_a\delta S_3 + 2\omega_p\delta S_2 - \gamma(r - 1)\delta d + \xi_{S_3} \quad (3.11) \]
\[ \delta \dot{d} = -\Gamma_s\delta d + 2\kappa_x\delta S_3 + \xi_d, \quad \Gamma_s = \gamma_s + \gamma(r - 1) = 2\gamma_c + \gamma_r \quad (3.12) \]

One can see there are two independent systems of linear differential equations. The relation between the stochastic sources \( \xi_{s_1,2,3}, \xi_D, \xi_d \) and the initial ones can be found in Appendix A.

**IV. THE SPECTRAL DENSITIES OF FLUCTUATIONS OF THE STOKES PARAMETERS**

Introducing the Fourier image of the function \( F(t) \) as

\[ F_\Omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t) e^{i\Omega t} \, dt, \quad F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F_\Omega e^{-i\Omega t} \, d\Omega \quad (4.1) \]

we can rewrite the equations (3.8)-(3.12) as algebraic ones:

\[ -i\Omega \delta S_{1,\Omega} = \gamma(r - 1) \delta D_{\Omega} + \xi_{S_1,\Omega} \quad (4.2) \]
\[ (-i\Omega + \gamma r) \delta D_{\Omega} = -2\kappa_x \delta S_{1,\Omega} + \xi_{D,\Omega} \quad (4.3) \]

and

\[ -(i\Omega + 2\kappa_a) \delta S_{2,\Omega} = -2\omega_p \delta S_{3,\Omega} - \alpha \gamma(r - 1) \delta d + \xi_{S_2,\Omega} \quad (4.4) \]
\[ -(i\Omega + 2\kappa_a) \delta S_{3,\Omega} = 2\omega_p \delta S_{2,\Omega} - \gamma(r - 1) \delta d + \xi_{S_3,\Omega} \quad (4.5) \]
\[ (-i\Omega + \Gamma_s) \delta d_{\Omega} = 2\kappa_x \delta S_{3,\Omega} + \xi_d_{\Omega} \quad (4.6) \]
Now relatively simply to solve it, and the interesting solutions expressed via the stochastic sources in the explicit form read:

\[
\begin{align*}
\delta S_{1,\Omega} &= -\left[ \gamma(r-1) \xi_{D,\Omega} + (\gamma r - i\Omega) \xi_{S_1,\Omega} \right] \left[ \Omega^2 + i\Omega \gamma r - 2\gamma \kappa_x (r-1) \right]^{-1} \\
\delta S_{2,\Omega} &= 1/2 \left[ \gamma(r-1) \left( \omega_p + \alpha \kappa_a + i\Omega \alpha/2 \right) \xi_{d,\Omega} - (\omega_p (\Gamma_s - i\Omega) + \alpha \gamma \kappa_x (r-1)) \xi_{S_2,\Omega} + \\
&+ (- (\Gamma_s - i\Omega) (\kappa_a + i\Omega/2) + \gamma \kappa_x (r-1)) \xi_{S_2,\Omega} \right] \times \\
&\left[ \left( \omega_p^2 + \kappa_a^2 + i\Omega \kappa_a - \Omega^2/4 \right) (\Gamma_s - i\Omega) - \gamma (r-1) \kappa_x (\kappa_a - \alpha \omega_p + i\Omega/2) \right]^{-1} \\
\delta S_{3,\Omega} &= 1/2 \left[ -\gamma(r-1) \left( \alpha \omega_p - \kappa_a - i\Omega/2 \right) \xi_{d,\Omega} + \omega_p (\Gamma_s - i\Omega) \xi_{S_2,\Omega} - \\
&- (\Gamma_s - i\Omega) (\kappa_a + i\Omega/2) \xi_{S_3,\Omega} \right] \times \\
&\left[ \left( \omega_p^2 + \kappa_a^2 + i\Omega \kappa_a - \Omega^2/4 \right) (\Gamma_s - i\Omega) - \gamma (r-1) \kappa_x (\kappa_a - \alpha \omega_p + i\Omega/2) \right]^{-1}
\end{align*}
\]

So-called spectral densities of the Stokes parameter fluctuations \((\delta S_i \delta S_k)_{\Omega}\) take an important part under physical discussion. They are defined under writing the spectral correlation functions:

\[
\overline{\delta S_{i,\Omega} \delta S_{k,\Omega'}} = (\delta S_i \delta S_k)_{\Omega} \delta(\Omega + \Omega')
\]

To get the wished spectral densities we must make the last preliminary step and write the respective spectral densities for the stochastic sources. According to Appendix A and (3.1)-(3.4) they are given as:

\[
\begin{align*}
(\xi_{S_1}^2)_{\Omega} &= (\xi_{S_2}^2)_{\Omega} = (\xi_{S_3}^2)_{\Omega} = -(\xi_D \xi_{S_1})_{\Omega} = (\xi_d \xi_{S_3})_{\Omega} = \kappa_x (r^2 - 1) \gamma/c \\
(\xi_D^2)_{\Omega} &= \kappa_x r (r - p/2) \gamma/c \\
(\xi_d^2)_{\Omega} &= \kappa_x r (\gamma s/\gamma + r - 1) \gamma/c
\end{align*}
\]

We define the non-zero spectral densities of the stochastic sources in the same way as for the Stokes parameters (4.10):

\[
\begin{align*}
\overline{\xi_{S_1,\Omega} \xi_{S_3,\Omega'}} &= (\xi_{S_1}^2)_{\Omega} \delta(\Omega + \Omega') \\
\overline{\xi_{D,\Omega} \xi_{D,d,\Omega'}} &= (\xi_{D}^2)_{\Omega} \delta(\Omega + \Omega') \\
\overline{\xi_{D,\Omega} \xi_{S_1,\Omega'}} &= (\xi_D \xi_{S_1})_{\Omega} \delta(\Omega + \Omega') \\
\overline{\xi_{d,\Omega} \xi_{S_3,\Omega'}} &= (\xi_d \xi_{S_3})_{\Omega} \delta(\Omega + \Omega')
\end{align*}
\]
Now it is not difficult to get:

\[
(\delta S_1^2)_{\Omega} = n_x\kappa_x \left(2\Omega^2 (r + 1) + \gamma^2 r (p - pr + 4)\right)/\lambda_{1,\Omega} \tag{4.18}
\]

\[
(\delta S_2^2)_{\Omega} = 2n_x\kappa_x \left(\Omega^4/4 (r + 1) + a_2\Omega^2 + b_2\right)/\lambda_{\Omega} \tag{4.19}
\]

\[
(\delta S_3^2)_{\Omega} = 2n_x\kappa_x \left(\Omega^4/4 (r + 1) + a_3\Omega^2 + b_3\right)/\lambda_{\Omega} \tag{4.20}
\]

\[
(\delta S_2\delta S_3)_{\Omega} = 2n_x\kappa_x \gamma(r - 1) \left(a_{23}\Omega^2/4 + b_{23}\right)/\lambda_{\Omega} \tag{4.21}
\]

where the coefficients are defined as:

\[
a_2 = (\omega_p^2 + \kappa_a^2 + \Gamma_s^2/4)(r + 1) + \gamma(r - 1)\left(\alpha^2 r\Gamma_s/4 + (\alpha\omega_p - \kappa_a)(r + 1)\right)
\]

\[
b_2 = \gamma_a^2(\omega_p^2 + \kappa_a^2)(r + 1) + \gamma_s\gamma(r - 1)\left[r(\omega_p + \alpha\kappa_a)^2 + 2\kappa(\alpha\omega_p - \kappa_a)(r + 1)\right] +
+ \gamma^2(r - 1)^2\left[\kappa^2(r + 1)(\alpha^2 + 1) - (\omega_p + \alpha\kappa_a)^2\right]
\]

\[
a_3 = (\omega_p^2 + \kappa_a^2 + \gamma_a^2/4)(r + 1) + (\alpha\omega_p - \Gamma_s/2)(r - 1)(r + 1)\gamma + \gamma(r - 1)r\Gamma_s/4
\]

\[
b_3 = \Gamma_s^2(r + 1)(\kappa_a^2 + \omega_p^2) + \gamma(r - 1)\Gamma_s\left[r(\alpha^2 \omega_p^2 - \kappa_a^2) + 2\kappa_a(\alpha\omega_p - \kappa_a)\right]
\]

\[
a_{23} = \alpha\left[\Gamma_s + 2(r + 1)(\kappa - \kappa_a)\right]
\]

\[
b_{23} = \gamma_s \left[r(\omega_p + \alpha\kappa_a)(\alpha\omega_p - \kappa_a) + (r + 1)\left(\omega_p(\alpha\omega_p - \kappa_a) + \kappa(\omega_p + \alpha\kappa_a)\right)\right] +
+ \gamma(r - 1)\left[\omega_p(\alpha\kappa_a)(\alpha\omega_p - \kappa_a) + (r + 1)(\alpha^2 + 1)\kappa\omega_p\right]
\]

\[
\lambda_{1,\Omega} = \left(\Omega^2 - 2\gamma\kappa_x(r - 1)\right)^2 + \Omega^2\gamma^2 r^2
\]

\[
\lambda_{\Omega} = \Omega^2\left[-\Omega^2/4 + \kappa_a^2 + \omega_p^2 - \kappa_a\gamma_s + \gamma(r - 1)(\kappa - \kappa_a)/2\right]^2 + \left[\Omega^2(\kappa_a - \Gamma_s/4) +
+ \gamma(r - 1)\kappa_x(\alpha\omega_p - \kappa_a) + \Gamma_s(\kappa_a^2 + \omega_p^2)\right]^2
\]

**V. COMPARISON WITH RESULTS OF THE PHENOMENOLOGICAL CALCULATIONS**

As mentioned above the phenomenological calculations were made in Ref. [3] and the respective curves were drawn as a result of some numerical analysis. Here we are going to compare some of these curves with our respective ones and then to estimate the phenomenological approach on this basis.
In the fig. 2 and 4 are presented the spectral dependences, which we have copied simply from the cited Ref. [3]. The curves have drawn with the following set of parameters: $\gamma = 1 GHz$, $\kappa = 300 GHz$, $\omega_p = 1 GHz$, $\alpha = -3$, $\gamma_s = 100 GHz$, $r = 1.04$ in absence of the linear dichroism $\kappa_a = 0$.

According Ref. [3] the curves on Fig. 2 are the frequency dependences of the full spectral power of the laser emission (the curve with single maximum) and - of the circularly polarized components (the curve with two maxima). These match to our $(\delta S_1^2)_{\Omega}$ and $(\delta S_1^2)_{\Omega} + (\delta S_2^2)_{\Omega}$, which are constructed with help of the formulas (4.18)-(4.19) and presented in fig. 3.

As is seen our and obtained phenomenologically curves turn out to be qualitatively alike especially as the positions of maxima on the frequency axes are exactly the same. At the same time one can see the levels of noises in our approach turn out to be incomparably lower than in the phenomenological one (please, take into account, the vertical axes on all the pictures are chosen in the logarithm scale).

The frequency dependences for the normalized cross-correlation

$$C_{+-} = \frac{(\delta S_1^2)_{\Omega} - (\delta S_2^2)_{\Omega}}{(\delta S_1^2)_{\Omega} + (\delta S_2^2)_{\Omega}} \quad (5.1)$$

are presented again respectively for phenomenological (fig. 4) and our approaches (fig. 5). Fixing here some qualitative (and quantitative - relative to the position of maxima) similarity, nevertheless one can see some serious differences take a place. For example, the right side of the graphics after maximum exhibits the absence of correlation in the phenomenological calculation and the appreciable anti-correlation about $-1/2$ according to our formulas.

It is easy to understand why the positions of the maxima are the same along the x-axes in different approaches. It connects only with so-called relaxation oscillations and that is why this is perfectly independent of the choice of stochastic sources.

As for quantitative differences we see the two reasons, why they take a place. First, under phenomenological introducing the sources to get the respective correlation functions one must use some additional physical considerations. Usually, especially under conditions of non-linear field-matter interaction, it is not a simple problem and requires a special attention. Regrettably this side of the question has quite fallen out of discussion in Ref. [3], and certainly it provides us with a possibility for doubts relative to the obtained correlation functions. At any rate we have to fix our formulas for the correlation functions of the stochastic sources are quite different.
Second, one of the reasons of our differences is very clear. In Ref. \[3\] the Langevin’s equations were produced under introducing the sources into not the complete system of the dynamical equations but into the simpler system, where already two variables \(D\) and \(d\) appear instead of four ones \(N_{a\pm}\) and \(N_{b\pm}\). We remember use of the simpler system of equations turned out to be possible in the dynamical theory of the VCSELs with the equaled relaxation constants of both the levels. But in the statistical theory it leads to the losses of the important sources, which depend on the populations of the sub-levels \(N_{a\pm}\) and \(N_{b\pm}\). We think it is main reason why the correlation functions in Ref. \[3\] are proportional only to the population differences of the kind of \(N_{a\pm} - N_{b\pm}\), what can not be correct.

VI. POLARIZATION SQUEEZING IN THE VCSEL EMISSION

In Ref. \[2\] we have discussed in details how to observe the Stokes parameter fluctuations in experiments with two photodetectors. Choosing the available geometry of the experiment we are able to select the wished signal. The most important cases read:

\[
\frac{(\delta i^-_2)_{\Omega}}{\langle i_+ \rangle} = 1 + 2\kappa/n_x (\delta S^2_{1})_{\Omega}, \quad \varphi = 0; \quad (6.1)
\]

\[
\frac{(\delta i^-_2)_{\Omega}}{\langle i_+ \rangle} = 1 + 2\kappa/n_x (\delta S^2_{2})_{\Omega}, \quad \varphi = \pi/4, \quad \theta = 0; \quad (6.2)
\]

\[
\frac{(\delta i^-_2)_{\Omega}}{\langle i_+ \rangle} = 1 + 2\kappa/n_x (\delta S^2_{3})_{\Omega}, \quad \varphi = \pi/4, \quad \theta = \pi/2; \quad (6.3)
\]

and

\[
\varphi = \pi/4, \quad \theta = \pi/4:
\]

\[
\frac{(\delta i^-_2)_{\Omega}}{\langle i_+ \rangle} = 1 + \kappa/n_x \left[ (\delta S^2_{1})_{\Omega} + (\delta S^2_{2})_{\Omega} + 2(\delta S_{2}\delta S_{3})_{\Omega} \right], \quad (6.4)
\]

\[
\varphi = \pi/8, \quad \theta = 0:
\]

\[
\frac{(\delta i^-_2)_{\Omega}}{\langle i_+ \rangle} = 1 + \kappa/n_x \left[ (\delta S^2_{1})_{\Omega} + (\delta S^2_{3})_{\Omega} + 2(\delta S_{1}\delta S_{2})_{\Omega} \right], \quad (6.5)
\]

\[
\varphi = \pi/8, \quad \theta = \pi/2:
\]

\[
\frac{(\delta i^-_2)_{\Omega}}{\langle i_+ \rangle} = 1 + \kappa/n_x \left[ (\delta S^2_{1})_{\Omega} + (\delta S^2_{3})_{\Omega} + 2(\delta S_{1}\delta S_{3})_{\Omega} \right]. \quad (6.6)
\]

Here \(\varphi\) is the angle between the direction of the linear polarization of the VCSEL emission and the polarization beam-splitter axis, \(\theta\) is additional phase shift, introducing by the phase plates between the orthogonal field components.

Our theory elaborated in the previous sections gives the possibilities to study any signals. But here we will discuss only the first of them \(6.1\), connected with the polarization squeez-
ing and that is why having the principal character for quantum optics. As for the others to our mind it is interesting to consider them together with the respective experimental date for comparison.

In semiconductor lasers (including the VCSELs) it is easy achieved the regularity in the pump, that, as we remember, leads to the essential intensity noise reduction below quantum limit for two level lasers without any degeneration. The degeneration of the laser levels introduces to system some additional random process, namely electrons under pumping are distributed between the sub-levels quite accidentally. This turned out to be inessential for the lasers with the shortly living lower level, because there in the linearly polarized emission only the full population of both the upper laser sub-levels \( N_{a+} + N_{a-} \) plays the role for squeezing. And just it does not fluctuate under the regular pump.

At the same time in our case the situation appears more complicated for understanding because now the value \( (N_{a+} + N_{a-} - N_{b+} - N_{b-}) \) produces squeezing.

To make some conclusion it is enough to watch only over the point \( \Omega = 0 \) in the spectrum (6.1). It means we want to watch over a depth of the noise reduction below the quantum limit. Putting \( p = 1 \) (the regular pump) and \( \kappa_a = 0 \) (without the dichroizm) we obtain that

\[
(\delta i^2)_{\Omega=0} / \langle i_+ \rangle = 1 + \frac{1}{2} \frac{5 - r}{r - 1} \frac{r}{r - 1} \frac{r \gg 1}{2} \frac{1}{2}
\]

(6.7)

Comparing it with the respective formula for the laser without the level degeneration [5]:

\[
(\delta i^2)_{\Omega=0} / \langle i_+ \rangle = 1 + \frac{1}{2} \frac{5 - r}{r - 1} \frac{1}{2} \frac{r \gg 1}{2}
\]

(6.8)

one can see there is an additional factor in (6.7) equaled to \( r / (r - 1) \). It is clear this plays an important role for a small amount of the pump parameter \( r \) (especially just near the threshold with \( r - 1 \ll 1 \)). At the same time, squeezing is able to appear only with the high enough pump parameter, namely with \( r > 5 \). Then the additional factor is already about one and hence turns out to be quite inessential. So our general conclusion is the degeneration of the laser levels does not leads to some additional difficulties in the production of squeezing.

Also it is interesting to compare two kind of the VCSELs with the shortly living lower level (the case in Ref. [2]) and with the equally living levels. The respective formula in Ref. [2] is:

\[
(\delta i^2)_{\Omega=0} / \langle i_+ \rangle = 1 + 3 \frac{r}{r - 1} \frac{r}{r - 1} \frac{r \gg 1}{0}
\]

(6.9)
For example, if we choose \( r = 6 \), then according to the last formula we have the noise level is 0.28 of the quantum limit. At the same time for our case here it is only 0.84. One can see the reduction of noises below the quantum limit is more effective for the VCSEL with the shortly living lower laser level.

\[ \text{VII. CONCLUSION} \]

To conclude we would like to say once more which concrete targets have been achieved in this work. First of all, the full quantum-statistical theory of the VCSEL with the equally living laser levels has been built. Thereby now we have a possibility to consider the problem on the same level of understanding as for the VCSEL with the shortly living lower level.

On this base the spectral densities of the Stokes parameter fluctuations have been written under taking into account the optical anisotropy of the semi-conductor crystal (the linear dichroizm and the linear birefringence) and the spin-flip.

We have been discussed the experimental situation in which it is possible to observe any spectral densities of the Stokes parameter fluctuations and also their correlations. Under discussion our main attention was devoted to the problem of polarization squeezing and the role of the degeneration of the laser levels. We concluded that the role of the random distribution of the electrons between the sub-levels under pumping is inessential for the shot noise reduction.

\[ \text{Acknowledgments} \]

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APPENDIX A: STOCHASTIC SOURCES IN THE LINEARIZED LANGEVIN EQUATIONS

The basic equations of the theory (2.1)-(2.4) have the stochastic sources as inhomogeneous terms. The properties of them are specified by the non-zero correlation functions (2.5)-(2.12). The case of our interest is $\mu_a \equiv \mu$, $\mu_b = 0$ and $\gamma_a = \gamma_b \equiv \gamma$ and $\gamma_c^{(a)} = \gamma_c^{(b)} \equiv \gamma_c$. We can rewrite the equations in the form which they is given in Ref. [3] in:

\[ \dot{a}_\pm = -\kappa a_\pm - (\kappa_a + i\omega_p) a_\mp + g P_\pm, \quad (A1) \]
\[ \dot{P}_\pm = -(\gamma_\perp + i\Delta) P_\pm + g (D \pm d) a_\pm + F_{P_\pm} (t), \quad (A2) \]
\[ \dot{D} = \mu - \gamma D - g \left( a^*_+ P_+ + a^*_+ P^*_+ + a^*_- P_- + a^*_- P^*_- \right) + F_D (t), \quad (A3) \]
\[ \dot{d} = -\gamma_s d - g \left( a^*_+ P_+ + a^*_+ P^*_+ - a^*_- P_- - a^*_- P^*_- \right) + F_d (t), \quad \gamma_s = \gamma + 2\gamma_c. \quad (A4) \]

Here instead of the populations $N_{a\pm}$ and $N_{b\pm}$ the new variables are introduced:

\[ D = \frac{1}{2} \left[ (N_{a+} + N_{a-}) - (N_{b+} + N_{b-}) \right] \quad (A5) \]
\[ d = \frac{1}{2} \left[ (N_{a+} - N_{a-}) - (N_{b+} - N_{b-}) \right] \quad (A6) \]

Respectively instead of the initial sources $F_{a\pm}$ and $F_{b\pm}$ the new ones are created:

\[ F_D = \frac{1}{2} (F_{a+} + F_{a-} - F_{b+} - F_{b-}) \quad (A7) \]
\[ F_d = \frac{1}{2} (F_{a+} - F_{a-} - F_{b+} + F_{b-}) \quad (A8) \]

The correspondent non-zero correlation functions read:

\[ \overline{F_D(t)F_D(t')} = \frac{1}{4} \left[ \gamma \left( N_{a+} + N_{a-} + N_{b+} + N_{b-} \right) + 2\mu (1 - p) \right. \]
\[ \left. -4g \left( a^*_+ P_+ + a^*_+ P^*_+ + a^*_- P_- + a^*_- P^*_- \right) \right] \delta(t - t'). \quad (A9) \]
\[ \overline{F_d(t)F_d(t')} = \frac{1}{4} \left[ (\gamma + 4\gamma_c) \left( N_{a+} + N_{a-} + N_{b+} + N_{b-} \right) + 2\mu - \right. \]
\[ \left. -4g \left( a^*_+ P_+ + a^*_+ P^*_+ + a^*_- P_- + a^*_- P^*_- \right) \right] \delta(t - t'). \quad (A10) \]
\[ \overline{F_D(t)F_{P_\pm}(t')} = \overline{F_d(t)F_{P_\pm}(t')} = \frac{1}{4} \left[ \gamma \left( N_{a+} - N_{a-} + N_{b+} - N_{b-} \right) - \right. \]
\[ \left. -4g \left( a^*_+ P_+ + a^*_+ P^*_+ - a^*_- P_- - a^*_- P^*_- \right) \right] \delta(t - t'). \quad (A11) \]
\[ \overline{F_D(t)F_{P_\pm}(t')} = -\frac{1}{2} (\gamma + \gamma_c) P_\pm \delta(t - t'). \quad (A12) \]
\[ \overline{F_d(t)F_{P_\pm}(t')} = +\frac{1}{2} (\gamma + \gamma_c) P_\pm \delta(t - t'). \quad (A13) \]
\[
\overline{F}_{P\pm}(t)F_{P\pm}(t') = \left[ (2\gamma_\perp - \gamma) \overline{N}_{a\pm} - \gamma_c \left( \overline{N}_{a\pm} - \overline{N}_{a\mp} \right) \right] \delta(t - t'),
\]
\[
F_{P\pm}(t)F_{P\mp}(t') = 2g \left( a_\pm P_\pm \right) \delta(t - t'),
\]

(A14) \hspace{1cm} (A15)

Under the adiabatical approximation \( \dot{P}_\pm = 0 \), and we can get:

\[
P_\pm = \frac{1}{\gamma_\perp + i\Delta} \left[ g \left( D \pm d \right) a_\pm + F_{P\pm} \right]
\]

(A16)

Taking this into account we can write our basic equations in the adiabatical approximation (2.13)-(2.17). There the sources are:

\[
\xi_\pm = \frac{c}{g} (1 - i\alpha) F_{P\pm}
\]
\[
\xi_D = F_D - \left( a_+^* \xi_+ + a_-^* \xi_- + a_+ \xi_+^* + a_- \xi_-^* \right)
\]
\[
\xi_d = F_d - \left( a_+^* \xi_+ - a_- \xi_- + a_+ \xi_+^* - a_- \xi_-^* \right)
\]

(A17) \hspace{1cm} (A18) \hspace{1cm} (A19)

The linearization of the equations relative to the fluctuations near the stationary semi-classical solutions leads to the equations (3.18)-(3.12) which the fluctuations of the Stokes parameters are introduced in and correspondently the new sources read:

\[
\xi_{S1} = \sqrt{2}Q (\xi_x e^{i\Delta_x t} + \xi_x^* e^{-i\Delta_x t})
\]
\[
\xi_{S2} = \sqrt{2}Q (\xi_y e^{i\Delta_y t} + \xi_y^* e^{-i\Delta_y t})
\]
\[
\xi_{S3} = \sqrt{2}Q (\xi_x^* e^{-i\Delta_x t} - \xi_y e^{i\Delta_y t})
\]
\[
\xi_D = F_D - \sqrt{2}Q (\xi_x + \xi_x^*)
\]
\[
\xi_d = F_d - \sqrt{2}Q (\xi_y - \xi_y^*)
\]

(A20) \hspace{1cm} (A21) \hspace{1cm} (A22) \hspace{1cm} (A23) \hspace{1cm} (A24)

where

\[
\xi_x = \frac{1}{\sqrt{2}} (\xi_+ + \xi_-) = \frac{c}{\sqrt{2}g} (1 - i\alpha) (F_{P+} + F_{P-})
\]
\[
\xi_y = \frac{1}{\sqrt{2}} (\xi_+ - \xi_-) = \frac{c}{\sqrt{2}g} (1 - i\alpha) (F_{P+} - F_{P-})
\]

(A25) \hspace{1cm} (A26)

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FIG. 1: Four-level configuration of the VCSEL medium

\begin{equation}
\kappa = 300 \text{GHz}, \quad \omega_p = 1 \text{GHz}, \quad \alpha = -3, \quad \gamma = 1 \text{GHz}, \quad \gamma_s = 100 \text{GHz} \quad \text{and} \quad \kappa_a = 0, \quad r = 1.04
\end{equation}

FIG. 2: The full spectral power (fragment of the figure 1) with the following set of parameters:
FIG. 3: The full spectral power (numerical calculation in our approach) with the following set of parameters: \( \kappa = 300 \text{GHz}, \ \omega_p = 1 \text{GHz}, \ \alpha = -3, \ \gamma = 1 \text{GHz}, \ \gamma_s = 100 \text{GHz} \) and \( \kappa_a = 0, \ r = 1.04 \)
FIG. 4: The frequency dependence of the correlations between the circularly polarized field components (fragment of the figure 1 [3]) with the following set of parameters: $\kappa = 300\text{GHz}$, $\omega_p = 1\text{GHz}$, $\alpha = -3$, $\gamma = 1\text{GHz}$, $\gamma_s = 100\text{GHz}$ and $\kappa_a = 0$, $\tau = 1.04$. 
FIG. 5: The frequency dependence of the correlations between the circularly polarized field components (numerical calculation in our approach) with the following set of parameters: $\kappa = 300 \text{GHz}$, $\omega_p = 1 \text{GHz}$, $\alpha = -3$, $\gamma = 1 \text{GHz}$, $\gamma_s = 100 \text{GHz}$ and $\kappa_a = 0$, $r = 1.04$. 