NEW ERROR TERM FOR THE FOURTH MOMENT OF AUTOMORPHIC $L$-FUNCTIONS

OLGA BALKANOVA AND DMITRY FROLENKOV

Abstract. We improve the error term in the asymptotic formula for the twisted fourth moment of automorphic $L$-functions of prime level and weight two proved by Kowalski, Michel and Vanderkam. As a consequence, we obtain a new subconvexity bound in the level aspect and improve the lower bound on proportion of simultaneous non-vanishing.

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1. INTRODUCTION

The fourth moment of automorphic $L$-functions has been studied in [3, 7] using the large sieve inequality and $\delta$-symbol method. As an application Duke, Friendlander and Iwaniec proved the subconvexity bound in the level aspect. Another consequence – simultaneous non-vanishing – was derived by Kowalski, Michel and Vanderkam.

In this paper, we optimize several estimates of [7] and compute the explicit dependence of error terms on the smallest positive eigenvalue for the Hecke congruence subgroup. This allows us to improve the results of [3, 7] by applying the Kim-Sarnak bound.

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We borrow some notations of [3, 7]. Consider the family $H_2^*(q)$ of primitive newforms of prime level $q$ and weight 2. Every $f \in H_2^*(q)$ has a Fourier expansion

\begin{equation}
(1.1) \quad f(z) = \sum_{n \geq 1} \lambda_f(n)n^{1/2}e(nz).
\end{equation}

The associated $L$-function is defined by

\begin{equation}
(1.2) \quad L(f, s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \Re s > 1.
\end{equation}

The completed $L$-function

\begin{equation}
(1.3) \quad \Lambda(f, s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{1}{2}\right)L(f, s)
\end{equation}

can be analytically continued on the whole complex plane. It satisfies the functional equation

\begin{equation}
(1.4) \quad \Lambda(f, s) = \epsilon_f\Lambda(f, 1 - s), \quad \epsilon_f = \pm 1.
\end{equation}

We introduce the natural and harmonic averages

\begin{equation}
(1.5) \quad \sum_{f \in H_2^*(q)} \alpha_f := \sum_{f \in H_2^*(q)} \frac{\alpha_f}{|H_2^*(q)|}, \quad \sum_{f \in H_2^*(q)} \alpha_f := \sum_{f \in H_2^*(q)} \frac{\alpha_f}{4\pi \langle f, f \rangle_q},
\end{equation}

where $\langle f, f \rangle_q$ is the Petersson inner product on the space of level $q$ holomorphic modular forms.

The goal of the present paper is to improve the error term in the asymptotic formula for the twisted fourth moment

\begin{equation}
(1.6) \quad M(l) = \sum_{f \in H_2^*(q)} \lambda_f(l)|L(f, 1/2 + \mu)|^4, \quad \mu \in i\mathbb{R}.
\end{equation}

Our main result is the following.

**Theorem 1.1.** Let $q$ be a prime and $l < q$. There exists some $B > 0$ such that for any $\epsilon > 0$

\begin{equation}
(1.7) \quad M(l) = M^D(l) + M^{OD}(l) + M^{OOD}(l) + O_\epsilon \left(q^\epsilon(1 + |\mu|)^B \left(\sum_{l_{5/30}, \ldots} \right)\right),
\end{equation}

where $M^D(l)$, $M^{OD}(l)$ and $M^{OOD}(l)$ are the main terms defined by equations (17), (31) – (32) and (34) of [7].
Here
\[(1.8) \quad \theta := \sqrt{\max(0, 1/4 - \lambda_1)}\]
and \(\lambda_1 = \lambda_1(q)\) is the smallest positive eigenvalue for the Hecke congruence subgroup \(\Gamma_0(q)\). Currently the best known estimate on \(\lambda_1\) is due to Kim and Sarnak \([8]\). Accordingly, we can take \(\theta = 7/64\).

**Corollary 1.2.** Let \(q\) be a prime. For all \(\epsilon > 0\)
\[(1.9) \quad M(1) = P(\log q) + O_\epsilon \left(q^{-25/228+\epsilon}\right),\]
where \(P\) is a polynomial of degree 6 and the leading coefficient is 1/60\(\pi^2\).

This improves corollary 1.3 of \([7]\), where asymptotic formula \((1.9)\) was established with the error \(O_\epsilon \left(q^{-1/12+\epsilon}\right)\).

Note that for weight \(k > 2\) the remainder term in \((1.9)\) can be majorated by \(O_{\epsilon,k} \left(q^{-1/4+\epsilon}\right)\). This was proved in \([1]\) for the case of prime power level \(q = p^n, n > 2\).

Another consequence of theorem \([11]\) is a new subconvexity bound in the level aspect.

**Corollary 1.3.** For all \(\epsilon > 0\)
\[(1.10) \quad L(f, 1/2 + \mu) \ll_{\epsilon,\mu} q^{1/4-\delta},\]
where \(\delta = \frac{2\theta - 1}{16(8\theta - 7)}\).

Taking \(\theta = 7/64\), we obtain
\[
\delta = \frac{25}{3136} = \frac{1}{125.44}.
\]
The previously known result with \(\delta = 1/192\) was established by Duke, Friedlander and Iwaniec \([3]\).

## 2. Selberg’s eigenvalue conjecture

Let \(\Gamma\) be a congruence subgroup of modular group. Let \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots\) be the eigenvalues of the automorphic Laplacian on \(L^2(\Gamma \setminus \mathbb{H})\) induced from the Laplace operator
\[(2.1) \quad \Delta_L = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).\]
The eigenvalue \(0 < \lambda < 1/4\) is called an exceptional eigenvalue.

**Conjecture 2.1.** (Selberg, \([12]\)) The Laplacian for a congruence subgroup has no exceptional eigenvalues, i.e. \(\lambda_1 \geq 1/4\).

Below we provide several results related to conjecture \([2.1]\).
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- 1965 Selberg [12]: $\lambda_1 \geq 3/16$
- 1978 Jacquet and Gelbart [4]: $\lambda_1 > 3/16$
- 1995 Luo, Rudnick, Sarnak [10]: $\lambda_1 > 171/784$
- 1996 Iwaniec [6]: $\lambda_1 > 10/49$
- 2002 Kim, Shahidi [9]: $\lambda_1 \geq 66/289$
- 2003 Kim, Sarnak [8]: $\lambda_1 \geq 975/4096$

Using the bound of Kim-Sarnak and equation (1.8), we find
\[
(2.2) \quad \theta = \sqrt{\max (0, 1/4 - \lambda_1)} = 7/64.
\]

3. LARGE SIEVE INEQUALITY

Let $S(m, n; c)$ be the classical Kloosterman sum.

**Theorem 3.1.** (theorem 9 of [2] and lemma 9 of [11]) Let $r, s$ and $d$ be positive pairwise coprime integers with $r$ and $s$ square-free. Let $C, M, N$ be positive real numbers and $g$ be real-valued infinitely differentiable function with support in $[M, 2M] \times [N, 2N] \times [C, 2C]$ such that
\[
(3.1) \quad \left| \frac{\partial^{(j+k+l)}}{\partial m^{(j)} \partial n^{(k)} \partial c^{(l)}} g(m, n, c) \right| \leq M^{-j} N^{-k} C^{-l} \text{ for } 0 \leq j, k, l \leq 2.
\]

Let
\[
X_d := \sqrt{dMN} \frac{1}{sC\sqrt{r}}.
\]

Then for any $\epsilon > 0$ and complex sequences $a = \{a_m\}, b = \{b_n\}$ one has
\[
(3.2) \quad \sum_m a_m \sum_n b_n \sum_{(c,r)=1} g(m, n, c) S(dm\bar{r}, \pm n; sc) \ll \epsilon
\]
\[
C^\epsilon d^\theta sC\sqrt{r} \left( 1 + X_d^{-1} \right)^{2\theta} \left( 1 + X_d + \sqrt{\frac{M}{rs}} \right) \left( 1 + X_d + \sqrt{\frac{N}{rs}} \right) \times \left( \sum_{M < m \leq 2M} |a_m|^2 \right)^{1/2} \left( \sum_{N < n \leq 2N} |b_n|^2 \right)^{1/2},
\]
where $\theta$ is defined by equation (1.8).

4. ERROR TERMS

In this section, we consider the terms that give the largest contribution to the error in [7]. Our goal is to optimize the estimates of these terms and compute the exact dependence of the error on parameter $\theta$.

First, we improve bound (21) of [7].
Note that the function $F_{M,N}(m, n)$ defined on page 108 of [7] is compactly supported on $[M/2, 3M] \times [N/2, 3N]$ and
\begin{align}
F_{M,N}(x, y) \ll (1 + |\mu|)^B(MN)^{-1/2}.
\end{align}

**Lemma 4.1.** Assume that for any $\epsilon > 0$ one has $M, N \ll q^{1+\epsilon}$. Then for any $C > \sqrt{MN}$
\begin{align}
\sum_{d \in I} \frac{1}{d^{1/2}} \sum_{ab \equiv d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{c \geq C \atop \gcd(c, d)} \frac{1}{c^2} T_{M,N}(c) \ll \epsilon
\end{align}
\begin{align}
(1 + |\mu|)^B(Cq)^{\epsilon l^{1/2}} \left( \frac{\sqrt{MN}}{C} \right)^{1-2\theta}.
\end{align}

**Proof.** We split $[C, \infty)$ into dyadic intervals and take $c \in [C, 2C]$. By equation (18) of [7] we have
\begin{align}
\sum_{q \mid c} \frac{1}{c^2} T_{M,N}(c) = \sum_{m, n, \gcd(m, c), \tabular{c}{\text{Here } m \in [M/2, 3M], n \in [N/2, 3N] \text{ and } c_1 \in [C_1, 2C_1]} \text{ with } C_1 := C/q.}
\end{align}

Here $m \in [M/2, 3M]$, $n \in [N/2, 3N]$ and $c_1 \in [C_1, 2C_1]$ with $C_1 := C/q$. Let
\begin{align}
Y := \sqrt{MN}C_1 \left( \frac{\sqrt{aeMN}}{C} \right)^{-1}.
\end{align}

As a test function we choose
\begin{align}
g(m, n, c_1) := \frac{Y}{c_1} F_{M,N}(m, n) J_1 \left( \frac{4\pi \sqrt{aemn}}{c_1 q} \right).
\end{align}

It satisfies condition (3.1), and theorem 3.1 can be applied with $d = ae$, $r = 1$ and $s = q$. Hence
\begin{align}
\sum_{d \in I} \frac{1}{d^{1/2}} \sum_{ab \equiv d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{c \geq C \atop \gcd(c, d)} \frac{1}{c^2} T_{M,N}(c) \ll \epsilon
\end{align}
\begin{align}
(1 + |\mu|)^B(Cq)^{\epsilon l^{1/2}} \left( \frac{\sqrt{MN}}{C} \right)^{1-2\theta}.
\end{align}
The optimal value of $C$ can be chosen by making equal the estimate (4.2) and the first summand of equation (26) of [7], namely

$$l^{1/2} \left( \frac{\sqrt{MN}}{C} \right)^{1-2\theta} = l^{3/4} N^{1/4} C / M^{1/2} q.$$  

This gives

$$C = l^{- \frac{1}{8-8\theta}} \min \left( q^{2-2\theta} \sqrt{MN}^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}} \right).$$

After performing the dyadic summation over $M$ and $N$, we find that for any $l < q^{1/5 - 4\theta}$ the error term in lemma 4.1 is bounded by

$$O\left( q^\theta \left( 1 + |\mu| \right)^B l^{\frac{5-4\theta}{8-8\theta}} q^{-\frac{1-2\theta}{8-8\theta}} \right).$$

Now we consider two other error terms that depend on $C$. These are the errors resulting from extension of summation over $c > C$. See section 3.5 (pages 111-112) of [7].

Let

$$\eta_C(c) := \begin{cases} 1 & c \leq C \\ 0 & \text{otherwise} \end{cases}$$

**Lemma 4.2.** Let $C$ be defined by equation (4.4). For any $\epsilon > 0$

$$\sum_{M,N \ll q^{1+\epsilon}} \sum_{d = 1} d^{1/2} \sum_{ab = d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{q|c} (1 - \eta_C(c)) c^{-2T^{OD}} \ll \epsilon$$

$$\left( 1 + |\mu| \right)^B l^{\frac{5-4\theta}{8-8\theta}} q^{-\frac{1-2\theta}{8-8\theta}}.$$  

**Proof.** Consider

$$\sum_{q|c} (1 - \eta_C(c)) c^{-2T^{OD}} = -2\pi \sum_n \tau(aen) \tau(n)$$

$$\times \int_0^\infty Y_0(4\pi \sqrt{aent}) J_1(4\pi \sqrt{aent}) \sum_{q|c \atop c > C} \phi(c) F_{M,N}(c^2 t, n) dt.$$ 

Since $C^2 t < c^2 t \leq 2M$, the sum over $c$ can be estimated as follows

$$\sum_{q|c \atop c > C} \phi(c) F_{M,N}(c^2 t, n) \ll \frac{1}{\sqrt{M N}} \frac{M}{q t}.$$
Next we apply $Y_0(x) \ll \log x$ and $J_1(x) \ll x$. Then

$$\sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OD} \ll \epsilon$$

$$= (1 + |\mu|) B q^\epsilon \frac{N}{\sqrt{MN}} \int_0^{2M/C^2} t^M (ae Nt)^{1/2} dt \ll \epsilon$$

$$= (1 + |\mu|) B q^\epsilon (ae)^{1/2} \frac{MN}{qC}. $$

Finally, using (4.4), we obtain

$$\sum_{M,N \ll q^{1+\epsilon}} \sum_{d,e,l} \frac{1}{d^{1/2}} \sum_{a,b,d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OD} \ll \epsilon$$

$$= (1 + |\mu|) B q^\epsilon (ae)^{(5-4\theta)/(8-8\theta)} q^{-1/(8-8\theta)}.$$ 

**Lemma 4.3.** Let $C$ be defined by equation (4.4). For any $\epsilon > 0$

$$\sum_{M,N \ll q^{1+\epsilon}} \sum_{d,e,l} \frac{1}{d^{1/2}} \sum_{a,b,d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OOD} \ll \epsilon$$

$$= (1 + |\mu|) B q^\epsilon (ae)^{(5-4\theta)/(8-8\theta)} q^{-1/(8-8\theta)}.$$ 

**Proof.** According to [7] page 111 we have

$$\sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OOD} \ll \epsilon (1 + |\mu|) B q^\epsilon (ae)^{1/2} \frac{MN}{qC}. $$

Equation (4.4) yields the assertion. 

To sum up, the largest error terms in theorem [1.1] come from lemmas 4.1, 4.2, 4.3 and equation (26) of [7]. In particular, the error term $O_\epsilon ((1 + |\mu|)^{B l^{17/8}} q^{-1/4+\epsilon})$ is given by the second summand in (26) of [7].

5. Amplification and subconvexity

Contribution of the main terms $M^D, M^{OD}, M^{OOD}$ in [7] is bounded by

$$O_\epsilon \left( (1 + |\mu|)^{B l^{-1/2}} \right). $$
According to theorem 1.1, for \( l < q^{\frac{1}{20+120\theta}} \) we have

\[
\sum_{f \in H_{\ast}^2(q)} \frac{1}{4\pi \langle f, \bar{f} \rangle_q} \lambda_f(l)|L(f, 1/2 + \mu)|^4 \ll \epsilon, \mu
\]

\[
q^\epsilon \left( l^{-1/2} + l^{\frac{60}{36} - \frac{1}{4} \theta} q^{-\frac{1}{4} \theta} \right) .
\]

Let

\[
\Lambda_f(c) := \sum_{l \leq L \atop (l, q) = 1} c_l \lambda_f(l)
\]

be an amplifier. Then

\[
\sum_{f \in H_{\ast}^2(q)} \frac{1}{4\pi \langle f, \bar{f} \rangle_q} \Lambda_f^2(c)|L(f, 1/2 + \mu)|^4 \ll \epsilon, \mu
\]

\[
q^\epsilon \left( \|c\|_2^2 + L^{\frac{60}{36} - \frac{1}{4} \theta} q^{-\frac{1}{4} \theta} \|c\|_1^2 \right) ,
\]

where \( \|c\|_p \) denotes \( l_p \)-norm.

We choose coefficients \( c_l \) as in [3], making \( \Lambda_f(c) \) large for a particular form \( f \in H_{\ast}^2(q) \), namely

\[
c_l = \begin{cases} 
\lambda_f(l) & \text{if } l \text{ is prime } \leq L^{1/2} \\
-1 & \text{if } l \text{ is a square of a prime } \leq L^{1/2} \\
0 & \text{otherwise}.
\end{cases}
\]

Thus,

\[
\Lambda_f(c) = \sum_{l \text{ prime} \leq L^{1/2} \atop (l, q) = 1} (\lambda_f(l)^2 - \lambda_f(l^2)).
\]

Note that \( \lambda_f(l^2) = 1 \) for prime \( l \) such that \( (l, q) = 1 \). Therefore,

\[
\Lambda_f(c) \sim 2L^{1/2}(\log L)^{-1}.
\]

By Deligne’s bound

\[
\|c\|_2^2 \leq 5\Lambda_f(c) \text{ and } \|c\|_1 \leq 3\Lambda_f(c).
\]

The results of [5] imply that

\[
\frac{1}{4\pi \langle f, \bar{f} \rangle_q} \ll \frac{\log q}{q} .
\]

Taking \( L = q^{\frac{20}{20+120\theta}} \) in (5.3) and applying (5.7), (5.8), (5.9), we have

\[
L(f, 1/2 + \mu) \ll_{\epsilon, \mu} q^{1/4 - \delta}
\]
with \( \delta = \frac{2\theta - 1}{16(8\theta - 7)} \).

6. Mollification and simultaneous non-vanishing

We follow section 5.2 of [7]. In order to determine the largest admissible length of mollifier \( \Delta \), we sum the error terms in theorem 1.1 against \( l^{-1/2+\varepsilon} \) for \( l < q^{2\Delta} \). This gives

\[
q^{\frac{1-2\theta}{8-8\theta}} q^{2\Delta (\frac{1-2\theta}{8-8\theta} + 1) + \varepsilon} + q^{-1/4} q^{2\Delta/4 + \varepsilon} + q^{-\frac{1}{8-8\theta}} q^{2\Delta (\frac{1}{8-8\theta} + 1) + \varepsilon}.
\]

Therefore, the error term is negligible for any \( \Delta < \frac{1-2\theta}{2(9-10\theta)} \).

In order to change the harmonic mean into the natural average as defined by (1.5), we apply results of section 5. Accordingly, condition (82) of [7] is satisfied for any \( \Delta < \frac{1-2\theta}{4(7-8\theta)} \).

**Theorem 6.1.** Let \( M(f) \) be the mollifier defined by equation (63) of [7] with \( P(x) = x^3 \). Let \( F(\Delta) \) be defined by equation (5) of [7].

For all \( 0 < \Delta_1 < \frac{1-2\theta}{2(9-10\theta)} \) we have

\[
\sum_{f \in H^*_2(q)} h L(f, 1/2)^4 M(f)^4 = (1 + o(1)) F(\Delta_1) \left( \frac{\zeta(2)}{\log q} \right)^4.
\]

For all \( 0 < \Delta_2 < \frac{1-2\theta}{4(7-8\theta)} \) we have

\[
\sum_{f \in H^*_2(q)} n L(f, 1/2)^4 M(f)^4 = (1 + o(1)) F(\Delta_2) \left( \frac{\zeta(2)}{\log q} \right)^4.
\]

Taking \( \theta = 7/64 \), we find that \( \Delta_1 < \frac{25}{566} = \frac{1}{22.64} \) and \( \Delta_2 < \frac{25}{784} = \frac{1}{31.36} \).

This improves \( \Delta_1 < \frac{1}{37} \) and \( \Delta_2 < \frac{1}{48} \) proved in [7].

In particular, extension of admissible length of mollifier \( \Delta_2 \) gives a better lower bound on the proportion of simultaneous non-vanishing

\[
\sum_{f \in H^*_2(q)} \frac{1}{L(f, 1/2) L(f \otimes \chi, 1/2) \neq 0}
\]

where \( \chi \) is a fixed primitive character of conductor \( D \) such that \( (D, q) = 1 \). See Proposition 7.2 of [7] for the exact formulas.

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