Simulation of two spin-\( s \) singlet correlations for all \( s \) involving spin measurements

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Abstract

In a recent paper [A. Ahanj et al., quant-ph/0603053], we gave a classical protocol to simulate quantum correlations corresponding to the spin \( s \) singlet state for the infinite sequence of spins satisfying \( 2s + 1 = 2^n \). In the present paper, we have generalized this result by giving a classical protocol to exactly simulate quantum correlations implied by the spin-\( s \) singlet state corresponding to all integer as well as half-integer spin values \( s \). The class of measurements we consider here are only those corresponding to spin observables, as has been done in the above-mentioned paper. The required amount of communication is found to be \( \lceil \log_2(s + 1) \rceil \) in the worst case scenario, where \( \lceil x \rceil \) is the least integer greater than or equal to \( x \).

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1 Introduction

It is well known that quantum correlations implied by an entangled quantum state of a bipartite quantum system cannot be produced classically, i.e., using only the local and realistic properties of the subsystems, without any communication between the two subsystems [1]. By quantum correlations we mean the statistical correlations between the outputs of measurements independently carried out on each of the two entangled parts. Naturally, the question arises as to the minimum amount of classical communication (number of cbits) necessary to simulate the quantum correlations of an entangled bipartite system. This amount of communication quantifies the nonlocality of the entangled bipartite quantum system. It also helps us gauge [2] the amount of information hidden in the entangled quantum system itself in some sense, the amount of information that must

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be space-like transmitted, in a local hidden variable model, in order for nature to account for the excess quantum correlations.

In this scenario, Alice and Bob try and output $\alpha$ and $\beta$ respectively, through a classical protocol, with the same probability distribution as if they shared the bipartite entangled system and each measured his or her part of the system according to a given random Von Neumann measurement. As we have mentioned above, such a protocol must involve communication between Alice and Bob, who generally share finite or infinite number of random variables. The amount of communication is quantified either as the average number of cbits $\overline{C}(P)$ over the directions along which the spin components are measured (average or expected communication) or the worst case communication, which is the maximum amount of communication $C_w(P)$ exchanged between Alice and Bob in any particular execution of the protocol. The third method is asymptotic communication i.e., the limit $\lim_{n \to \infty} \overline{C}(P^n)$ where $P^n$ is the probability distribution obtained when $n$ runs of the protocol carried out in parallel i.e., when the parties receive $n$ inputs and produce $n$ outputs in one go. Note that, naively, Alice can just tell Bob the direction of her measurement to get an exact classical simulation, but this corresponds to an infinite amount of communication. The question whether a simulation can be done with finite amount of communication was raised independently by Maudlin [4], Brassard, Cleve and Tapp [5], and Steiner [6]. Brassard, Cleve and Tapp used the worst case communication cost while Steiner used the average. Steiner’s model is weaker as the amount of communication in the worst case can be unbounded although such cases occur with zero probability. Brassard, Cleve and Tapp gave a protocol to simulate entanglement in a singlet state (i.e., the EPR pair) using eight cbits of communication. Csiirk [7] has improved it where one requires six bits of communication. Toner and Bacon [8] gave a protocol to simulate two-qubit singlet state entanglement using only one cbit of communication.

Until now, an exact classical simulation of quantum correlations, for all possible projective measurements, is accomplished only for spin $s = 1/2$ singlet state, requiring 1 cbit of classical communication [8]. It is important to know how does the amount of this classical communication change with the change in the value of the spin $s$, in order to quantify the advantage offered by quantum communication over the classical one. Further, this communication cost quantifies, in terms of classical resources, the variation of the nonlocal character of quantum correlations with spin values. In our earlier paper [9], it was shown that only $\log_2(2s + 1)$ bits of communication is needed, in the worst case scenario, to simulate the measurement correlation of two spin-$s$ singlet state for performing only measurement of spin observables on each site, where $s$ is a half-integer spin satisfying $2s + 1 = 2^n$. Thus these spin values do not include any integer spin as well as all half-integer spins. In the present paper we give a classical protocol to simulate the measurement correlation in a singlet state of two spin-$s$ systems, for all the integer as well as half-integer values of $s$, considering only (as above) measurement of spin observables (i.e., measurement of observables of the form $\hat{a} \hat{A}$ where $\hat{a}$ is any unit vector in $\mathbb{R}^3$ and $\hat{A} = (\Lambda_x, \Lambda_y, \Lambda_z)$ with each $\Lambda_i$ being a $(2s + 1) \times (2s + 1)$ traceless Hermitian matrix and
the all three together form the $SU(2)$ algebra). We show that, using $\lceil \log_2(s+1) \rceil$ bits of classical communication, one can simulate the above-mentioned measurement correlation.

We will describe measurement correlations in two spin-s singlet state in section 2. Before describing our general simulation scheme, we will explain the scheme with few examples in section 3. In section 4, we will describe our general simulation scheme. We will draw our conclusion in section 5.

## 2 Singlet state correlation

The singlet state $|\psi^-_s\rangle_{AB}$ of two spin-$s$ particles $A$ and $B$ is the eigenstate corresponding to the eigenvalue 0 of the total spin observable of these two spin systems, namely the state

$$|\psi^-_s\rangle_{AB} = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^{s} (-1)^{s-m} |m\rangle_A \otimes |-m\rangle_B,$$

where $|-s\rangle$, $|-s+1\rangle$, ..., $|s-1\rangle$, $|s\rangle$ are eigenstates of the spin observable of each of the individual spin-$s$ system. Thus $|\psi^-_s\rangle_{AB}$ is a maximally entangled state of the bipartite system $A + B$, described by the Hilbert space $\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$.

We will consider here measurement of ‘spin observables’, namely the observables of the form $\hat{a}.J$ on each individual spin-$s$ system, where $\hat{a}$ is an arbitrary unit vector in $\mathbb{R}^3$ and $J = (J_x, J_y, J_z)$ (see ref. [9] for a discussion on the choice of measurement observables). For the $(2s+1) \times (2s+1)$ matrix representations of the spin observables $J_x$, $J_y$, and $J_z$, please see page 191 - 192 of ref. [11]. $J$ matrices satisfy the $SU(2)$ algebra, namely $[J_x, J_y] = iJ_z$, $[J_y, J_z] = iJ_x$, $[J_z, J_x] = iJ_y$. The eigenvalues of $\hat{a}.J$ are $-s$, $-s+1$, ..., $s-1$, $s$ for all $\hat{a} \in \mathbb{R}^3$. The quantum correlations $\langle \psi^-_s | \hat{a}.J \otimes \hat{b}.J | \psi^-_s \rangle$ (which we will denote here as $\langle \alpha \beta \rangle$, where $\alpha$ runs through all the eigenvalues of $\hat{a}.J$ and $\beta$ runs through all the eigenvalues of $\hat{b}.J$) is given by

$$\langle \psi^-_s | \hat{a}.J \otimes \hat{b}.J | \psi^-_s \rangle = \langle \alpha \beta \rangle = -\frac{1}{3} s(s+1) \hat{a} \cdot \hat{b},$$

where $\hat{a}$ and $\hat{b}$ are the unit vectors specifying the directions along which the spin components are measured by Alice and Bob respectively (see section 6-6 of page 179 in [12]). Note that, by virtue of being a singlet state , $\langle \alpha \rangle = 0 = \langle \beta \rangle$ irrespective of directions $\hat{a}$ and $\hat{b}$.

Let us now come to our protocol. In the simulation of the measurement of the observable $\hat{a}.J$ (where $\hat{a} \in \mathbb{R}^3$ is the supplied direction of measurement), Alice will have to reproduce the $2s+1$ number of outcomes $\alpha = s, s-1, \ldots, -s+1, -s$ with equal probability. Similarly, Bob will have to reproduce the $2s+1$ number of outcomes $\beta = s, s-1, \ldots, -s+1, -s$ with equal probability. We will describe our protocol for the simulation by first giving the ones for smaller values of the spin and then by giving the protocol for general value of the spin.
Before describing the simulation scheme, we mention here a few mathematical results which will be frequently needed during our discussion of the simulation scheme. Consider the unit sphere in three-dimensional Euclidean space: \( S_2 = \{ |\mathbf{r}| = 1 : \mathbf{r} \in \mathbb{R}^3 \} \). Let \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\mu}_1, \hat{\mu}_2, \hat{\nu}_1, \hat{\nu}_2 \) be (mutually) independent but uniformly distributed random variables on \( S_2 \). Let \( \hat{a} \) and \( \hat{b} \) be given any two elements from \( S_2 \). Also \( \hat{z} \) be the unit vector along the \( z \)-axis of the rectangular Cartesian co-ordinate axes \( x, y \) and \( z \) – the associated reference frame. Let us define:

\[
c_k = \text{Sgn}(\hat{a} \cdot \hat{\lambda}_k) \, \text{Sgn}(\hat{a} \cdot \hat{\mu}_k) \quad (k = 1, 2),
\]

\[
f_k = \text{Sgn}(\hat{z} \cdot \hat{\nu}_k + p_k) \quad (p_k \in (0, 1)),
\]

where \( \text{Sgn} : \mathbb{R} \to \{ +1, -1 \} \) is the function defined as \( \text{Sgn}(x) = 1 \) if \( x \geq 0 \) and \( \text{Sgn}(x) = -1 \) if \( x < 0 \). One can show that (see ref. [8] for the derivations):

\[
\text{Prob}(\text{Sgn}(\hat{a} \cdot \hat{\lambda}_k) = \pm 1) = \frac{1}{2}, \quad (\text{for } k = 1, 2),
\]

and hence

\[
\langle \text{Sgn}(\hat{a} \cdot \hat{\lambda}_k) \rangle = 0 \quad (\text{for } k = 1, 2).
\]

\[
\text{Prob}(\text{Sgn}[(\hat{\nu}_k + c_k \hat{\mu}_k)] = \pm 1) = \frac{1}{2}, \quad (\text{for } k = 1, 2),
\]

and hence

\[
\langle \text{Sgn}[(\hat{\nu}_k + c_k \hat{\mu}_k)] \rangle = 0 \quad (\text{for } k = 1, 2).
\]

\[
\langle \text{Sgn}(\hat{a} \cdot \hat{\lambda}_k) \times \text{Sgn}[(\hat{\nu}_k + c_k \hat{\mu}_k)] \rangle = \delta_{kl} \, \langle \hat{a} \cdot \hat{b} \rangle \quad (\text{for } k, l = 1, 2).
\]

Also we have (taking \( \hat{\nu}_k = (\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, \cos \theta_k) \))

\[
\text{Prob}(f_k = +1) = \frac{1}{4\pi} \int_{\phi_k = 0}^{2\pi} \int_{\theta_k = 0}^{\cos^{-1}(-p_k)} \sin \theta_k d\theta_k d\phi_k = \frac{1 + p_k}{2} \quad (\text{for } k = 1, 2),
\]

and hence

\[
\text{Prob}(f_k = -1) = \frac{1 - p_k}{2} \quad (\text{for } k = 1, 2).
\]

So

\[
\langle f_k \rangle = p_k \quad (\text{for } k = 1, 2).
\]

Moreover, as \( f_k^2 \) will always have the value +1, therefore

\[
\langle f_k^2 \rangle = 1 \quad (\text{for } k = 1, 2).
\]

Consequently

\[
\langle (1 + f_k)^2 \rangle = 2 (1 + p_k) \quad (\text{for } k = 1, 2)
\]

and (as \( \hat{\nu}_1 \) and \( \hat{\nu}_2 \) are independent random variables)

\[
\langle (1 + f_1)(1 + f_2) \rangle = \langle (1 + f_1)^2 \rangle \times \langle (1 + f_2)^2 \rangle = 4 (1 + p_1)(1 + p_2).
\]
Again, as $\hat{\lambda}_1$, $\hat{\mu}_2$, $\hat{\nu}_1$, $\hat{\nu}_2$ are independent random variables, therefore
\begin{equation}
\langle (1 + f_k)^2 \times \text{Sgn}(\hat{a}.\hat{\lambda}_t) \times \text{Sgn}(b. (\hat{\lambda}_m + c_l\hat{\mu}_m)) \rangle = 2(1 + p_k) \delta_{lm} (\hat{a}.b),
\end{equation}
and
\begin{equation}
\langle (1 + f_1)^2 (1 + f_2)^2 \times \text{Sgn}(\hat{a}.\hat{\lambda}_t) \times \text{Sgn}(b. (\hat{\lambda}_t + c_l\hat{\mu}_t)) \rangle = 4(1 + p_1)(1 + p_2) \delta_{kl} (\hat{a}.b).
\end{equation}

3 Examples

For each value $s$ of the spin, we can always find a positive integer $n$ such that $2^{n-1} < s + 1 \leq 2^n$. We show here below that the above-mentioned simulation can be done with just $n$ bits of communication if $s$ is such that $2^{n-1} < s + 1 \leq 2^n$. To give a clear picture, let us first describe our protocol for few lower values of $s$, and after that, the general protocol will be given. To start with, Alice and Bob fix a common reference frame (with rectangular Cartesian co-ordinate axes $x$, $y$, and $z$) for them.

Example 1: $2^{1-1} < s + 1 \leq 2^1$. Thus the allowed values of $s$ are $1/2$ and $1$.

Case (1.1) $s = 1/2$:
Alice and Bob a priori share two independent and uniformly distributed random variables $\hat{\lambda}_{1/2}$, $\hat{\mu}_{1/2} \in S_2$. Given the measurement direction $\hat{a} \in S_2$, Alice calculates her output as $\alpha = -(1/2)\text{Sgn}(\hat{a}.\hat{\lambda}_{1/2}) \equiv -\alpha(1/2)$ (say). She also sends the bit value $c_{1/2} = \text{Sgn}(\hat{a}.\hat{\lambda}_{1/2}) \text{Sgn}(\hat{a}.\hat{\mu}_{1/2})$ to Bob by classical communication. After receiving this bit value and using the supplied measurement direction $\hat{b} \in S_2$, Bob now calculates his output as $\beta = (1/2)\text{Sgn}(\hat{b}.(\hat{\lambda}_{1/2} + c_{1/2}\hat{\mu}_{1/2})) \equiv \beta(1/2)$ (say). It is known that (see equations (11) - (17)) for the two spin-1/2 singlet state $|\psi_{1/2}⟩$, $\alpha, \beta \in \{±1/2, -1/2\}$, $\text{Prob}(\alpha = ±1/2) = \text{Prob}(\beta = ±1/2) = 1/2$ (and so $⟨\alpha⟩ = ⟨\beta⟩ = 0$), and $⟨\alpha\beta⟩ = -(1/3)(1/2)(1/2 + 1)\hat{a}.\hat{b} = ⟨\alpha\beta⟩_{QM}$. Thus the total number of cbits required (we denote it by $n_c$), for simulating the measurement correlation in the worst case scenario, is one and the total number of shared random variable is two: $\lambda_{1/2}$ and $\mu_{1/2}$. Thus here $n_\lambda \equiv$ the total number of $\hat{\lambda}$’s $= 1$ and $n_\mu \equiv$ the total number of $\hat{\mu}$’s $= 1$.

Case (1.2) $s = 1$:
Alice and Bob a priori share three independent and uniformly distributed random variables $\hat{\lambda}_1$, $\hat{\mu}_1$, $\hat{\nu}_1 \in S_2$. Given the measurement direction $\hat{a} \in S_2$, Alice calculates her output as $\alpha = -((1 + f_1)/2)\text{Sgn}(\hat{a}.\hat{\lambda}_1) \equiv -\alpha(1)$ (say). She also sends the bit value $c_1 = \text{Sgn}(\hat{a}.\hat{\lambda}_1) \text{Sgn}(\hat{a}.\hat{\mu}_1)$ to Bob by classical communication. After receiving this bit value and using the supplied measurement direction $\hat{b} \in S_2$, Bob now calculates his output as $\beta = ((1 + f_1)/2)\text{Sgn}(\hat{b}.(\hat{\lambda}_1 + c_1\hat{\mu}_1)) \equiv \beta(1)$ (say), where $f_1 = \text{Sgn}(\hat{\lambda}.\hat{\nu}_1 + 1/3)$ and $c_1 = \text{Sgn}(\hat{a}.\hat{\lambda}_1) \text{Sgn}(\hat{a}.\hat{\mu}_1)$. Now, by equations (8) - (11), we have $\text{Prob}(f_1 = +1) = 2/3$, $\text{Prob}(f_1 = -1) = 1/3$ and $⟨f_1⟩ = 1/3$. Thus we see that (using equations (3), (5),...
the probability distribution of $f_1$, and the fact that $\hat{\lambda}_1$, $\hat{\mu}_1$, $\hat{\nu}_1$ are independent random variables) $\alpha, \beta \in \{+1, 0, -1\}$ and $\text{Prob}(\alpha = j) = \text{Prob}(\beta = k) = 1/3$ for all $j, k \in \{+1, 0, -1\}$. Also we have (using equation (13)) $\langle \alpha \beta \rangle = -(1/3) \times 1 \times (1 + 1)\hat{a}.\hat{b} = \langle \alpha \beta \rangle_{QM}$. Thus here $n_c = 1$, $n_\lambda = 1$, $n_\mu = 1$, $n_\nu \equiv \text{the total number of } \hat{\nu}'s = 1$.

**Example 2:** $2^{2-1} < s + 1 \leq 2^2$. Thus the allowed values of $s$ are $3/2$, $2$, $5/2$, and $3$.

**Case (2.1) $s = 3/2$:**

Alice and Bob *a priori* share four independent and uniformly distributed random variables $\hat{\lambda}_{1/2}$, $\hat{\lambda}_{3/2}$, $\hat{\mu}_{1/2}$, $\hat{\mu}_{3/2} \in S_2$. Given the measurement direction $\hat{a} \in S_2$, Alice calculates her output as $\alpha = -[\text{Sgn}(\hat{a}.\hat{\lambda}_{3/2} + \alpha(1/2))] \equiv -\alpha(3/2)$ (say), where $\alpha(1/2)$ involves $\hat{\lambda}_{1/2}$ and is described in (1.1) above. She also sends the two bit values $c_k = \text{Sgn}(\hat{a}.\hat{\lambda}_k)$ $\text{Sgn}(\hat{a}.\hat{\mu}_k)$ (for $k = 1/2, 3/2$) to Bob by classical communication. After receiving these two bit values and using the supplied measurement direction $\hat{b} \in S_2$, Bob now calculates his output as $\beta = \text{Sgn}[\hat{b}.(\hat{\lambda}_{3/2} + c_3/2\hat{\mu}_{3/2})] + \beta(1/2) \equiv \beta(3/2)$ (say), where $\beta(1/2)$ involves $\hat{\lambda}_{1/2}$, $\hat{\mu}_{1/2}$ and is described in (1.1) above. Using equations (3) and (5), and using the fact that $\lambda_{1/2}$, $\hat{\lambda}_{3/2}$, $\hat{\mu}_{1/2}$, $\hat{\mu}_{3/2}$ are independent and uniformly distributed random variables on $S_2$, we have $\text{Prob}(\alpha = j) = \text{Prob}(\beta = k) = 1/4$ for all $j, k \in \{-3/2, +1/2, -1/2, -3/2\}$. Also, by using equation (7), we have $\langle \alpha \beta \rangle = -(1/3) \times (3/2) \times (3/2 + 1)\hat{a}.\hat{b} = \langle \alpha \beta \rangle_{QM}$. Thus here $n_c = 2$, $n_\lambda = 2$, $n_\mu = 2$ and $n_\nu = 0$.

**Case (2.2) $s = 2$:**

Alice and Bob *a priori* share five independent and uniformly distributed random variables $\hat{\lambda}_{1/2}$, $\hat{\lambda}_2$, $\hat{\mu}_{1/2}$, $\hat{\mu}_2$, $\hat{\nu}_2 \in S_2$. Given the measurement direction $\hat{a} \in S_2$, Alice calculates her output as $\alpha = -((1 + f_2)/2)[(3/2)\text{Sgn}(\hat{a}.\hat{\lambda}_2) + \alpha(1/2)] \equiv -\alpha(2)$ (say), where $\alpha(1/2)$ involves $\hat{\lambda}_{1/2}$ and is described in (1.1) above. She also sends the two bit values $c_k = \text{Sgn}(\hat{a}.\hat{\lambda}_k)$ $\text{Sgn}(\hat{a}.\hat{\mu}_k)$ (for $k = 1/2, 2$) to Bob by classical communication. After receiving these two bit values and using the supplied measurement direction $\hat{b} \in S_2$, Bob now calculates his output as $\beta = ((1 + f_2)/2)[(3/2)\text{Sgn}[\hat{b}.(\hat{\lambda}_2 + c_2\hat{\mu}_2)] + \beta(1/2)] \equiv \beta(2)$ (say), where $\beta(1/2)$ involves $\hat{\lambda}_{1/2}$, $\hat{\mu}_{1/2}$ and is described in (1.1) above. Here $f_2 = \text{Sgn}(\hat{\nu}_2, \hat{\nu}_2 + 3/5)$. By using equations (8) - (10), we see that $\text{Prob}(f_2 = +1) = 4/5$, $\text{Prob}(f_2 = -1) = 1/5$ and $\langle f_2 \rangle = 3/5$. Using these facts and the fact that $\lambda_{1/2}$, $\hat{\lambda}_2$, $\hat{\mu}_{1/2}$, $\hat{\mu}_2$, $\hat{\nu}_2$ are independent and uniformly distributed random variables on $S_2$, we have $\text{Prob}(\alpha = j) = \text{Prob}(\beta = k) = 1/5$ for all $j, k \in \{+2, +1, 0, -1, -2\}$. Also, by using equation (13) $\langle \alpha \beta \rangle = -(1/3) \times 2 \times (2 + 1)\hat{a}.\hat{b} = \langle \alpha \beta \rangle_{QM}$. Thus here $n_c = 2$, $n_\lambda = 2$, $n_\mu = 2$ and $n_\nu = 1$.

**Case (2.3) $s = 5/2$:**

Alice and Bob *a priori* share five independent and uniformly distributed random variables $\hat{\lambda}_1$, $\hat{\lambda}_{5/2}$, $\hat{\mu}_1$, $\hat{\mu}_{5/2}$, $\hat{\nu}_1 \in S_2$. Given the measurement direction $\hat{a} \in S_2$, Alice calculates her output as $\alpha = -[(3/2)\text{Sgn}(\hat{a}.\hat{\lambda}_{5/2}) + \alpha(1)] \equiv -\alpha(5/2)$ (say), where $\alpha(1)$ involves $\hat{\lambda}_1$, $\hat{\nu}_1$ and is described in (1.2) above. She also sends the two bit values $c_k = \text{Sgn}(\hat{a}.\hat{\lambda}_k)$ $\text{Sgn}(\hat{a}.\hat{\mu}_k)$ (for $k = 1, 5/2$) to Bob by classical communication. After receiving these two bit val-
ues and using the supplied measurement direction \( \hat{b} \in S_2 \), Bob now calculates his output as \( \beta = (3/2) \text{Sgn}[\hat{b} \cdot (\hat{\lambda}_{5/2} + c_{5/2} \hat{\mu}_{5/2})] + \beta(1) \equiv \beta(5/2) \) (say), where \( \beta(1) \) involves \( \hat{\lambda}_1, \hat{\mu}_1, \hat{\nu}_1 \) and is described in (1.2) above. Using the fact that \( \hat{\lambda}_1, \hat{\lambda}_{5/2}, \hat{\mu}_3, \hat{\mu}_{5/2}, \hat{\nu}_1 \) are independent and uniformly distributed random variables on \( S_2 \) and the discussions in (1.2) above, we have \( \text{Prob}(\alpha = j) = \text{Prob}(\beta = k) = 1/6 \) for all \( j, k \in \{+5/2, -5/2\} \). Also, by using equation (13), \( \langle \alpha \beta \rangle = - \frac{1}{3} \times (5/2) \times (5/2 + 1) \hat{a} \cdot \hat{b} = \langle \alpha \beta \rangle_{QM} \). Thus here \( n_c = 2, n_\lambda = 2, n_\mu = 2 \) and \( n_\nu = 1 \).

**Case (2.4) \( s = 3 \):**

Alice and Bob \( a \) \( priori \) share six independent and uniformly distributed random variables \( \hat{\lambda}_1, \hat{\lambda}_3, \hat{\mu}_1, \hat{\mu}_3, \hat{\nu}_1, \hat{\nu}_3 \in S_2 \). Given the measurement direction \( \hat{a} \in S_2 \), Alice calculates her output as \( \alpha = -((1 + f_3)/2)[2 \text{Sgn}(\hat{a} \cdot \hat{\lambda}_3) + \alpha(1)] \equiv -\alpha(3) \) (say), where \( \alpha(1) \) involves \( \hat{\lambda}_1, \hat{\nu}_1 \) and is described in (1.2) above. Here \( f_3 = \text{Sgn}(\hat{z} \cdot \hat{v}_3 + 5/7) \). She also sends the two bit values \( c_k = \text{Sgn}(\hat{a} \cdot \hat{\lambda}_k) \text{Sgn}(\hat{a} \cdot \hat{\mu}_k) \) (for \( k = 1, 3 \)) to Bob by classical communication. After receiving these two bit values and using the supplied measurement direction \( \hat{b} \in S_2 \), Bob now calculates his output as \( \beta = ((1 + f_3)/2)[2 \text{Sgn}(\hat{b} \cdot (\hat{\lambda}_3 + c_{3} \hat{\mu}_3)) + \beta(1)] \equiv \beta(3) \) (say), where \( \beta(1) \) involves \( \hat{\lambda}_1, \hat{\mu}_1, \hat{\nu}_1 \) and is described in (1.2) above. Using the fact that \( \hat{\lambda}_1, \hat{\lambda}_3, \hat{\mu}_1, \hat{\mu}_3, \hat{\nu}_1, \hat{\nu}_3 \) are independent and uniformly distributed random variables on \( S_2 \), equations (3) and (5), and the discussions in (1.2) above, we have \( \text{Prob}(\alpha = j) = \text{Prob}(\beta = k) = 1/7 \) for all \( j, k \in \{+3, -3\} \). Also, by using equations (13) and (14), we have \( \langle \alpha \beta \rangle = -((1/3) \times 3 \times (3 + 1)) \hat{a} \cdot \hat{b} = \langle \alpha \beta \rangle_{QM} \). Thus here \( n_c = 2, n_\lambda = 2, n_\mu = 2 \) and \( n_\nu = 2 \).

### 4 General simulation scheme

Let us now describe the protocol for general \( s \). One can always find out uniquely a positive integer \( n \) such that \( 2^{n-1} < s + 1 \leq 2^n \). Equivalently, given the dimension \( d = 2s + 1 \) of the Hilbert space, one can always find out a unique positive integer \( n \) such that \( 2^n - 1 < d \leq 2^{n+1} - 1 \). Let \( d = a_0 2^n + a_1 2^{n-1} + \ldots + a_n 2^0 \equiv a_0 a_1 \ldots a_n \) be the binary representation of \( d \) (where \( a_0, a_1, \ldots, a_n \in \{0, 1\} \)). So we must have \( a_0 \neq 0 \). Before describing the general simulation scheme, using the help of the above-mentioned examples, let us describe below the scheme pictorially (see Figure 1) in terms of binary representation of the dimension of the individual spin system. The simulation scheme, we have described in ref. [9] for the simulation of the measurement correlation in two spin-\( s \) singlet state, where \( 2s + 1 = 2^n \), corresponds to the upper most chain

\[
2^1 = 10 \rightarrow 2^2 = 100 \rightarrow 2^3 = 1000 \rightarrow \ldots \rightarrow 2^{n-1} = 1000 \ldots 00 \rightarrow 2^n = 1000 \ldots 000
\]

in Figure 1. In other words, when \( 2s + 1 = 2^n \), given the measurement directions \( \hat{a} \), Alice will calculate her output \( -\alpha \left( \frac{2^n - 1}{2} \right) \equiv -\alpha \left( \frac{1000 \ldots 000 - 1}{2} \right) \) as:

\[
-\alpha \left( \frac{1000 \ldots 000 - 1}{2} \right) = - \left[ \left( \frac{1000 \ldots 000 - 1}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a} \cdot \hat{\lambda}_{1000 \ldots 000} \right) + \alpha \left( \frac{1000 \ldots 000 - 1}{2} \right) \right]
\]
= - \left[ \left( \frac{1000 \ldots 000-1}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a} \hat{\lambda}_{\frac{1000 \ldots 000-1}{2}} \right) + \left( \frac{1000 \ldots 000-1}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a} \hat{\lambda}_{\frac{1000 \ldots 000-1}{2}} \right) + \right. \\
\left. \alpha \left( \frac{1000 \ldots 0 - 1}{2} \right) \right] \\
\ldots \\
= - \left[ \left( \frac{1000 \ldots 000-1}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a} \hat{\lambda}_{\frac{1000 \ldots 000-1}{2}} \right) + \left( \frac{1000 \ldots 000-1}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a} \hat{\lambda}_{\frac{1000 \ldots 000-1}{2}} \right) + \right. \\
\left. \ldots + \left( \frac{10-1}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a} \hat{\lambda}_{\frac{10}{2}} \right) \right] \\
= - \frac{1}{2} \sum_{k=1}^{n} 2^{n-k} \text{Sgn} \left( \hat{a} \hat{\eta}_k \right),

where \( \hat{\eta}_k = \hat{\lambda}_{\frac{k}{2}} \). Similarly for Bob. We have generalized below this scheme to arbitrary value of \( s \) (see equations (15) - (20)).
Figure 1: The paths (mentioned by concatenated arrows from left to right) of simulation for each integer and half-integer spins $s$ such that $2^{n-1} < s + 1 \leq 2^n$.
To describe the general simulation, we consider the following two cases:

**s is a half-integer spin:**

Over and above the \( n - 1 \) number of \( \hat{\lambda} \)'s, \( n - 1 \) number of \( \hat{\mu} \)'s and \((a_1 + a_2 + \ldots + a_{n-1})\) number of \( \hat{\nu} \)'s appeared in the expression for \( \alpha \left( \frac{a_0 a_1 \ldots a_{n-1}}{2} \right) \) and \( \beta \left( \frac{a_0 a_1 \ldots a_{n-1}}{2} \right) \), Alice and Bob share the random variables \( \hat{\alpha} \)'s and \( \hat{\nu} \)'s by \( S \), Alice calculates her output as

\[
\alpha = -\left[ \left( \frac{a_0 a_1 \ldots a_{n-1}}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a} \cdot \hat{\lambda} \right) + \alpha \left( \frac{a_0 a_1 \ldots a_{n-1} - 1}{2} \right) \right] 
\equiv -\alpha \left( \frac{a_0 a_1 \ldots a_n - 1}{2} \right),
\]

and she sends the \( n \) cbits

\[
c_k = \text{Sgn}(\hat{a} \cdot \hat{\lambda}_k) \text{Sgn} \left( \hat{a} \cdot \hat{\mu}_k \right),
\]

to Bob where \( k = \frac{a_0 a_1 \ldots a_{n-1}}{2}, \frac{a_0 a_1 \ldots a_{n-1} - 1}{2}, \ldots, \frac{a_0 a_1 - 1}{2} \). After receiving these \( n \) cbits and using his measurement direction \( \hat{b}_2 \), Bob calculates his output as

\[
\beta = \left[ \left( \frac{a_0 a_1 \ldots a_{n-1}}{2} + \frac{1}{2} \right) \text{Sgn} \left[ \hat{b} \cdot \left( \hat{\lambda} + c \hat{\mu} \hat{\nu} \right) \right] \right] 
+ \beta \left( \frac{a_0 a_1 \ldots a_{n-1} - 1}{2} \right) \equiv \beta \left( \frac{a_0 a_1 \ldots a_n - 1}{2} \right).
\]

Let \( L = a_1 + a_2 + \ldots + a_n \) and let \( i_1, i_2, \ldots, i_L \) be all those elements from \( \{1, 2, \ldots, n\} \) such that \( i_1 < i_2 < \ldots < i_L \) and \( a_{i_1} = a_{i_2} = \ldots = a_{i_L} = 1 \). It is then easy to see that

\[
S_\lambda = \left\{ \hat{\lambda}, \hat{\lambda}_{a_0 a_1 \ldots a_{n-1}} \right\}, \\
S_\mu = \left\{ \hat{\mu}, \hat{\mu}_{a_0 a_1 \ldots a_{n-1}} \right\}, \\
S_\nu = \left\{ \hat{\nu}, \hat{\nu}_{a_0 a_1 \ldots a_{i_L - 1}} \right\}.
\]

**s is an integer spin:**

Over and above the \( n - 1 \) number of \( \hat{\lambda} \)'s, \( n - 1 \) number of \( \hat{\mu} \)'s and \((a_1 + a_2 + \ldots + a_{n-1})\) number of \( \hat{\nu} \)'s appeared in the expression for \( \alpha \left( \frac{a_0 a_1 \ldots a_{n-1}}{2} \right) \) and \( \beta \left( \frac{a_0 a_1 \ldots a_{n-1}}{2} \right) \), Alice
and Bob share the random variables $\hat{\lambda}_{a,a_1\ldots a_{n-1}}$ and $\hat{\mu}_{a,a_1\ldots a_{n-1}}$, where, it has been assumed that all these $2n + (a_1 + a_2 + \ldots + a_{n-1})$ number of random variables are independent and uniformly distributed on $S_2$. Given the measurement direction $\hat{a} \in S_2$, Alice calculates her output as

$$\alpha = -\left(1 + f_{a,a_1\ldots a_{n-1}} \right) \left[ \left( \frac{a_0a_1\ldots a_{n-1}}{2} + \frac{1}{2} \right) \text{Sgn} \left( \hat{a}, \hat{\lambda}_{a,a_1\ldots a_{n-1}} \right) + \alpha \left( \frac{a_0a_1\ldots a_{n-1} - 1}{2} \right) \right]$$

$$\equiv -\alpha \left( \frac{a_0a_1\ldots a_{n-1} - 1}{2} \right),$$

and she sends the $n$ cbits

$$c_k = \text{Sgn}(\hat{a}, \hat{\lambda}_k) \ \text{Sgn}(\hat{a}, \hat{\mu}_k),$$

(19)

to Bob where $k = \frac{a_0a_1\ldots a_{n-1}}{2}, \frac{a_0a_1\ldots a_{n-1}}{2}, \ldots, \frac{a_0a_1\ldots a_{n-1}}{2}$. After receiving these $n$ cbits and using his measurement direction $\hat{b} \in S_2$, Bob calculates his output as

$$\beta = \left(1 + f_{a,a_1\ldots a_{n-1}} \right) \left[ \left( \frac{a_0a_1\ldots a_{n-1}}{2} + \frac{1}{2} \right) \text{Sgn} \left[ \hat{b}, \left( \hat{\lambda}_{a_0a_1\ldots a_{n-1}} + c_{a_0a_1\ldots a_{n-1}} \hat{\mu}_{a_0a_1\ldots a_{n-1}} \right) \right] \right]$$

$$+ \beta \left( \frac{a_0a_1\ldots a_{n-1} - 1}{2} \right) \equiv \beta \left( \frac{a_0a_1\ldots a_{n-1} - 1}{2} \right).$$

(20)

Here

$$f_{a,a_1\ldots a_{n-1}} = \text{Sgn} \left( \hat{\nu}, \hat{\nu}_{a_0a_1\ldots a_{n-1}} + \frac{a_0a_1\ldots a_{n-1}}{2} \right).$$

(21)

Let $L = a_1 + a_2 + \ldots + a_n$ and let $i_1, i_2, \ldots, i_L$ be elements from $\{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \ldots < i_L$ and $a_{i_1} = a_{i_2} = \ldots = a_{i_L} = 1$. It is then easy to see that

$$S_\lambda = \left\{ \hat{\lambda}_{a_0a_1\ldots a_{n-1}}, \hat{\lambda}_{a_2a_3\ldots a_{n-1}}, \ldots, \hat{\lambda}_{a_{n-1}a_1\ldots a_{n-2}} \right\},$$

$$S_\mu = \left\{ \hat{\mu}_{a_0a_1\ldots a_{n-1}}, \hat{\mu}_{a_2a_3\ldots a_{n-1}}, \ldots, \hat{\mu}_{a_{n-1}a_1\ldots a_{n-2}} \right\},$$

$$S_\nu = \left\{ \hat{\nu}_{a_0a_1\ldots a_{n-1}}, \hat{\nu}_{a_2a_3\ldots a_{n-1}}, \ldots, \hat{\nu}_{a_{n-1}a_1\ldots a_{n-2}} \right\}.$$

The way we have defined $\alpha \left( \frac{a_0a_1\ldots a_{n-1}}{2} \right)$ as well as $\beta \left( \frac{a_0a_1\ldots a_{n-1}}{2} \right)$ (see examples (1.1) - (2.4) as well as equations (15), (17), (18) and (20)), one can show recursively that

$$\text{Prob} \left( \alpha \left( \frac{a_0a_1\ldots a_{n-1}}{2} \right) = j \right) = \text{Prob} \left( \beta \left( \frac{a_0a_1\ldots a_{n-1}}{2} \right) = k \right) = \frac{1}{a_0a_1\ldots a_n}.$$
for \( j, k \in \{ (a_0 a_1 \ldots a_n - 1)/2, (a_0 a_1 \ldots a_n - 3)/2, \ldots, -(a_0 a_1 \ldots a_n - 1)/2 \} \) and also

\[
\langle \alpha \beta \rangle = \langle -\alpha \left( \frac{a_0 a_1 \ldots a_n - 1}{2} \right) \times \beta \left( \frac{a_0 a_1 \ldots a_n - 1}{2} \right) \rangle = \frac{-1}{3} \times \frac{a_0 a_1 \ldots a_n - 1}{2} \times \left( \frac{a_0 a_1 \ldots a_n - 1}{2} + 1 \right) (\hat{a}, \hat{b}) = \langle \alpha \beta \rangle_{QM}
\]

Thus we see that for any given value of the spin \( s \) (integer or half-integer) for which \( 2^n - 1 < d = 2s + 1 \leq 2^{n+1} - 1 \) (hence \( d \) has the binary representation \( d = a_0 a_1 \ldots a_n \) where \( a_0, a_1, \ldots, a_n \in \{0, 1\} \) and \( a_0 \neq 0 \)), Alice and Bob can simulate, in the worst case scenario, the measurement correlation in the two spin-\( s \) singlet state \( |\psi^-_s\rangle \) for performing measurement of arbitrary spin observables by using only \( n = \lceil \log_2(s + 1) \rceil \) bits of communication if they \textit{a priori} share \( 2n + (a_1 + a_2 + \ldots a_n) \) number of independent and uniformly distributed random variables on \( S_2 \).

For any maximally entangled state \( |\psi_{max}\rangle \) of two spin-\( s \) systems, we know that there exists a \((2s+1) \times (2s+1)\) unitary matrix \( U \) such that \( |\psi_{max}\rangle = (U \times I)|\psi^-_s\rangle \). Our protocol works equally well for those two spin-\( s \) maximally entangled state \( |\psi_{max}\rangle \) for each of which the above-mentioned unitary matrix \( U \) induces a rotation in \( \mathbb{R}^3 \), as in those cases, both Alice and Bob can perform the protocol for the spin-\( s \) singlet state \( |\psi^-_s\rangle \) for the rotated input vectors \( \hat{a} \) and \( \hat{b} \) and, hence, they will achieve their goal.

5 Conclusion

Our result provides the amount of classical communication in the worst case scenario if we consider only measurement of spin observables on both sides of a two spin-\( s \) singlet state for all the values of \( s \) – just \( n = \lceil \log_2(s + 1) \rceil \) bits of communication from Alice to Bob is sufficient. Thus, in our simulation protocol, the required amount of classical communication is increased only by one cbit if dimension of the individual spin system becomes double. In other words, the amount of classical communication, in our simulation scheme, is equal to the maximum number of qubit(s) one can accommodate within the Hilbert space dimension of the individual spin system.

It should be noted that if we consider most general projective measurements on both the sides of a maximally entangled state of two qudits, with \( d = 2^n \), it is known that (see [3]) Alice would require at least of the order of \( 2^n \) bits of communication to be sent to Bob, in the worst case scenario when \( n \) is large enough. But for general \( d \), \( \log_2 d \) can be shown to be a lower bound on the average amount of classical communication that one would require to simulate the maximally entangled correlation of two qudits considering most general type of projective measurements [13]. So, in the worst case scenario, one would require at least \( \log_2 d \) number of bits of communication for simulating measurement correlation of the two-qudit maximally entangled state, where the measurement can be
arbitrary but projection type. If one can show that $\log_2 d$ is again a lower bound for considering measurement of spin observables only (which we believe to be true), our simulation scheme will turn out to be optimal.

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**References**

[1] J. S. Bell, *Physics* (Long Island City, N.Y) **1**, 195 (1964).

[2] A. A. Méthot, *Eur. Phys. Journal D*, **29**, 445 (2004).

[3] S. Pironio, *Phys. Rev. A* **68**, 062102 (2003).

[4] T. Maudlin, in PSA 1992, Volume 1, edited by D.Hull, M.Forbes, and K. Okruhlik (Philosophy of Science Association,East Lansing, 1992), pp. 404 - 417.

[5] G. Brassard, R. R. Cleve, and A. Tapp, *Phys. Rev. Lett. * **83**, 1874 (1999).

[6] M. Steiner, *Phys. Lett. A* **270**, 239 (2000).

[7] J. A. Csirik, *Phys. Rev. A* **66** 014302 (2002).

[8] B. F. Toner and D. Bacon, *Phys. Rev. Lett.* **91**, 187904 (2003).

[9] A. Ahanj, P. S. Joag, and S. Ghosh, [quant-ph/0603053](http://arxiv.org/abs/quant-ph/0603053) (to be published in *Phys. Lett. A*).

[10] J. Schlienz and G. Mahler, *Phys. Rev. A* **52**, 4396 (1995).

[11] J. J. Sakurai, “Modern Quantum Mechanics” (revised edition) (Addison-Wesley, 1999).

[12] A. Peres, “Quantum Theory: Concepts and Methods”,(Kluwer Academic Publishers, 1993).

[13] J. Barrett, A. Kent, and S. Pironio, *Phys, Rev. Lett. * **97**, 170409 (2006).