1 Introduction

In this report I have tried to describe the main streams of research presented at the workshop. Occasionally I have described some contributions which were presented in poster form, whenever they were directly related to the subjects presented in the talks. This paper is, essentially, a patchwork of various people description of their work, which I have edited in an attempt to obtain some uniformity in style, and to produce a report of finite length. Much to my regret, I had to cut or shorten a lot of material because of length constraints. I wish to thank all those who have sent me a contribution.

2 The Riemannian Penrose Inequality

In 1972, Penrose [89] gave a heuristic argument to establish the geometric inequality

\[ m_{ADM}(M) \geq \sqrt{|N|/16\pi} \]  

(2.1)

for a time-reversible initial data set \((M, g)\), where \(|N|\) is the area of a marginally trapped surface bounding \(M\). We refer to (2.1) as the Riemannian Penrose inequality, because throughout this section it is assumed that the initial data have vanishing extrinsic curvature, \(K_{ij} = 0\). Penrose’s argument relies on knowledge of the future evolution of \(M\), including the Hawking area monotonicity of the (suitably differentiable) event horizon. Therefore, Penrose suggested that (2.1) can be seen as a Riemannian test for the standard picture of black hole formation, particularly the Weak Cosmic Censorship Hypothesis. Recently Huisken and
Ilmanen have managed to establish an appropriately understood version of the inequality \( (2.1) \). This is without doubt one of the major achievements in mathematical general relativity in recent years, and the method of proof deserves a short description. Huisken and Ilmanen’s starting point is the argument of Geroch [57] and Jang–Wald [72], which proceeds as follows: Let \( H \) denote the mean curvature and \( \nu \) the outward unit normal of a 2-surface \( N \) as a submanifold of \( M \). Let \( (N_t)_{t \geq 0} \) be a family of surfaces in \( M \) evolving outward with speed \( 1/H \), that is,

\[
\frac{\partial x}{\partial t} = \frac{\nu(x)}{H(x)}, \quad x \in N_t, \quad t \geq 0,
\]

with \( N_0 = N \). Then Geroch shows that as long as things go smoothly, the Hawking mass

\[
m_H(N_t) := \frac{1}{64 \pi^{3/2}} |N_t|^{1/2} \left( 16\pi - \int_{N_t} H^2 \right)
\]

is monotone nondecreasing during the flow. If \( N_t \) becomes a large, round sphere in the limit, then \( m_H(N_t) \to m_{ADM}(M) \). This calculation proves \((2.1)\), provided the surfaces remain smooth. Now one does not expect these surfaces to remain smooth in general, and Huisken and Ilmanen manage to handle that difficulty. A key idea is to require that each \( N_t \) minimizes area among all surfaces surrounding the “past history” \( \cup_{s < t} N_s \). The effect is that \( N_t \) must sometimes jump in order to maintain this condition. This is implemented through a minimization principle for a function \( u \) whose level sets form the flow. An approximation scheme is used, and convergence is proven using methods of geometric measure theory. The final theorem is stated as follows:

**Theorem 2.1 (Huisken & Ilmanen [64, 65])** Assume that \( M \) is a complete, connected 3-manifold which has nonnegative scalar curvature and which is asymptotically flat:

\[
|g_{ij} - \delta_{ij}| \leq C/|x|, \quad |g_{ij,k}| \leq C/|x|^2, \quad R_{ij} \geq -C g_{ij}/|x|^2,
\]

for some flat metric \( \delta_{ij} \) near infinity. Suppose that \( \partial M \) is a compact minimal surface and that \( M \) contains no other compact minimal surfaces. Then \((2.1)\) holds for each connected component \( N \) of \( \partial M \). Moreover equality holds if and only if \( M \) is one-half of the spatial Schwarzschild manifold.

As mentioned previously, this theorem assumes that the initial data set has vanishing extrinsic curvature\(^1\) so that the general case remains open.

A result which comes very close to that of Huisken and Ilmanen has been very recently obtained by Bray: In [26, 27] Bray considers the case where \((M^3, g)\) has only one outermost minimal sphere \( \Sigma_0 \). He assumes that for each volume

\(^1\)Actually for all purposes of this section it would suffice to assume that the initial data surface is maximal and that the extrinsic curvature vanishes at the minimal surface.
V > 0, if one or more area minimizers exist for V, then at least one of these area minimizers for the volume V has exactly one component. Under this condition he shows that (2.1) holds.

Another noteworthy new result is that of Herzlich, who establishes an inequality, somewhat similar to (2.1), which relates the mass, the area, and a function theoretic quantity σ:

**Theorem 2.2 (Herzlich [62])** Let (M, g) be a 3-dimensional asymptotically flat Riemannian manifold with a compact, connected, (inner) boundary ∂M that is a minimal (topological) 2-sphere. Suppose also that the scalar curvature of (M, g) is nonnegative. Then its mass m, if defined, satisfies

\[ m \geq \frac{1}{2} \frac{\sigma}{1 + \sigma} \sqrt{\frac{\text{Area}(\partial M)}{\pi}} \]

where σ is a dimensionless quantity defined as

\[ \sigma = \sqrt{\frac{\text{Area}(\partial M)}{\pi}} \inf_{f \in C^\infty_0, f \neq 0} \frac{\|df\|^2_{L^2(M)}}{\|f\|^2_{L^2(\partial M)}}. \]

Moreover, equality is achieved if and only if (M, g) is a spacelike Schwarzschild metric of mass \( \frac{1}{4} \sqrt{\text{Area}(\partial M)/\pi} \).

It would be of interest to study the properties of σ, in particular to find out whether σ can be large. It has been pointed out by P. Tod (private communication) that σ tends to zero for Reissner–Norström black holes when the extreme limit is approached.

The above results relate the ADM mass to the area of a single component of the minimal surface. When the outermost minimal surface is not connected one can expect that the inequality can be improved. A naive way would be to write

\[ m \geq \sum_i \sqrt{A_i/16\pi}. \]

We note that this inequality becomes an equality for Majumdar–Papapetrou black holes. H. Bray conjectures (private communication) that this is the correct generalization; in his thesis he proves the following, weaker inequality, under a restrictive condition:

\[ m \geq \left( \sum_i \left( \frac{A_i}{16\pi} \right)^\frac{3}{2} \right)^\frac{1}{3}. \] (2.3)

In order to establish (2.3), Bray introduces the following functional:

\[ F(V) = \inf \{ \sum_i \text{Area}(\Sigma_i)^{\frac{3}{2}} \mid \{\Sigma_i\} \text{ contain a volume } V \text{ outside the horizons} \} \]
where the $\{\Sigma_i\}$ are the boundaries of the components of some 3-dimensional open region in $M^3$. Here one assumes that and $\bigcup_i \Sigma_i$ is in the closure $M^3$ of that component of $M^3 - \Sigma_0$ which contains the asymptotically flat end. Moreover in the definition of $F$ one requires that $\bigcup_i \Sigma_i$ is in the homology class of $\tilde{M}^3$ which contains both a large sphere at infinity and the union of the horizons. Bray’s condition reads as follows: For each $V > 0$, if one or more sets of surfaces minimize $F$ for the volume $V$, then at least one of these sets is pairwise disjoint, that is, $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$.

**Theorem 2.3 (Bray [26, 27])** Suppose $(M^3, g)$ is complete, has nonnegative scalar curvature, contains outermost minimal spheres with surface areas $\{A_i\}$, is asymptotically flat with total mass $m$, and satisfies the above described condition. Then (2.3) holds.

### 3 Initial data

A standard approach to constructing initial data for the Einstein equations is the so-called conformal method [68]. Except for some special situations with symmetries, the only alternative (more or less) systematic method of obtaining those data seems to be via the *thin sandwich problem*. This problem is known to be ill posed in general [37], but is known to be solvable under some conditions [13]. One advantage thereof is that the restriction of constant extrinsic curvature, which is standard in the conformal approach, does not arise. Another is that one has complete control of the initial data metric. In his contribution to this problem D. Giulini considers Einstein gravity on $\Sigma \times \mathbb{R}$, where $\Sigma$ is a compact Riemannian manifold, coupled to a gauge field (with compact gauge group $G$ and trivial bundle), and $n$ real scalar fields carrying some representation of $G$. It is assumed that there are no couplings to 2nd (or higher) derivatives of the metric in the matter Lagrangian. Let $\Phi_A$ collectively denote all fields. In this setup the thin sandwich problem consists of prescribing $\Psi := \{\Phi_A, \dot{\Phi}_A\}$ and trying to solve the scalar constraint, the vector constraints, and the Gauss constraints for the lapse $\alpha$, the shift $\beta$, and the Lie–algebra valued gauge functions $\lambda$. Giulini assumes that I.) the generalized De Witt metric $G^{AB}$ satisfies $G^{AB} \gamma_A \gamma_B < 0$, and II.) the potential energy of all fields $U$ is strictly positive. This allows one to algebraically solve for $\alpha$ and leads to a system $F(X, \Psi) = 0$ of non-linear PDE’s for $X := (\beta, \lambda)$, with $F$ — a smooth map between appropriate Sobolev spaces. Let $X'$ be a solution for given $\Psi$ which satisfies I and II. Under conditions I.) and II.) Giulini proves that 1.) $\partial_X F(X', \Psi)$ is a linear, self-adjoint, elliptic operator (from the space of $X$’s to itself), and 2.) has trivial kernel iff $S$: $D_{\beta, \lambda} \Phi_A = \alpha \dot{\Phi}_A$, the symmetry-equation projected to $\Sigma$, implies $\alpha = \beta = \lambda = 0$. Hence, if $S$ has just trivial solutions, the implicit function theorem implies that $F(X, \Psi') = 0$ can be uniquely solved in terms of $X(\Psi')$ for any $\Psi'$ in a neighborhood of $\Psi$. This
result generalizes the results of Bartnik and Fodor \[13\] to the presence of matter fields of the kind mentioned above.

The talk by S. Husa was concerned with application of conformal compactification and conformal symmetries to the (numerical) construction and analysis of asymptotically flat (AF) vacuum initial data within the conformal approach. Instead of picking an AF metric as ‘input’ for the Lichnerowicz equation, Beig and Husa choose a conformally compactified representative of the conformal equivalence class. This leads to a convenient treatment of asymptotic regions, provides a simple construction of (multi-)black hole initial data, and allows one to exploit conformal symmetries. In particular the $U(1) \times U(1)$ conformal symmetry of the physical metric is used by Beig and Husa to derive a class of exact solutions to the momentum constraint, and to decompose the time symmetric conformal factor in a double Fourier series on the group orbits \[14\]. The solutions are then given in terms of a countable family of uncoupled ODE’s on the orbit space. The existence of positive solutions is obtained by computing the sign of the first eigenvalue of the conformal Laplacian, which in this case becomes an ODE problem. The authors have carried out a numerical analysis (including the existence and properties of apparent horizons) for (i) Brill waves (plus black holes), (ii) initial data containing a marginally outer trapped torus \[66\], and (iii) time-asymmetric initial data, where the extrinsic curvature is obtained as an exact solution for non-conformally flat geometries with conformal symmetry.

4 The evolution question

4.1 Local aspects

For hyperbolic partial differential equations two classes of problems are usually well posed: the initial value problem, also called the Cauchy problem, and the initial – boundary value problem. While for the Einstein equations the local aspects of the former problem are well understood (cf., e.g., \[93\]), the first general result about the latter is a theorem of Friedrich and Nagy \[53\] announced during the workshop. Now there are various ways of stating the problem, and here we will only describe one of the results obtained, the reader is referred to \[53\] for more general results.

Consider, then, a space-like hypersurface $\Sigma$ and a time-like hypersurface $T$ in a vacuum space–time $(M, g)$, which intersect along a space–like two dimensional surface $S = \Sigma \cap T$. One can then ask the question, which data need to be given on $\Sigma$ and $T$ so that one can reconstruct (by solving Einstein’s equation) $g$ on an appropriate neighborhood $M' \subset M$ of a given point $p \in S$, with $M'$ — one-sided with respect to $\Sigma$ and $T$ (“1/2 sided”). To answer this question, the authors perform the following construction: Let $x^3$ be a coordinate on $M'$ such that $T_c = \{x^3 = c = const. \geq 0\}$ is time-like, with $T = T_0$. Let, next, $e_k$ be a
smooth orthonormal frame field on \( M' \) with the property that the time-like vector field \( e_0 \) is orthogonal to \( \Sigma \cap T_c \) and satisfies \( e_0(x^3) = 0 \). The field \( e_3 \) is chosen to be orthogonal to \( T_c \), while the fields \( e_A, A = 1, 2 \), are tangent to \( \Sigma \cap T_c \) and Fermi propagated in the direction of \( e_0 \) with respect to the intrinsic connection \( D \) induced on \( T_c \). The coordinates \( x^0, x^1, x^2 \) are chosen so that \( \{ x^0 = 0 \} = \Sigma \), \( e_0(x^\mu) = \delta^\mu_0 \). Let \( \chi(x^0, x^1, x^2, c) \) denote the trace of the second fundamental form of \( T_c \). On \( T \) one finally defines

\[
C_{AB} = C_{\mu\nu\lambda\rho}^e e_0^\mu e_A^\nu e_0^\lambda e_B^\rho, \quad c_{AB} = C_{AB} - \frac{1}{2} \delta_{AB} \delta^{CD} C_{CD},
\]

where \( C_{\mu\nu\lambda\rho} \) is the Weyl tensor. Then one of the results proved in [53] reads as follows:

**Theorem 4.1 (Friedrich, Nagy, 1997)** Let \( (h_{\alpha\beta}, k_{\alpha\beta}) \) be the standard Cauchy data induced on \( \Sigma \) by \( g \), let \( \chi(x^0, x^1, x^2, 0) \) be the mean extrinsic curvature of \( T \) and let \( c_{AB} \) be related to the Weyl tensor as described above. Let further \( \Gamma_A(x^\mu) = g(e_A, D_{e_0} e_0) \) and \( \chi(x^\mu), x^3 > 0 \) be some given “gauge source functions”. Then for any point \( p \in S \) there exists a “1/2 sided” neighborhood \( M' \) of \( p \) on which the data determine a unique, smooth solution to Einstein’s field equations. In that solution the data assume the geometrical interpretation given above.

We emphasize that the result above has been presented as a reconstruction problem. Friedrich and Nagy also consider the question, how to construct space–times “from scratch”: in that case standard corner conditions arise [53].

In the workshop T. Velden gave a talk in which she points out a gap in the analysis of the Cauchy problem for Einstein equations with non-rotating dust as a source. She emphasized that the question of propagation of constraints was not properly handled previously, and presented an analysis where this problem is taken care of. She also described a local existence theorem of the Cauchy problem assuming analyticity of the data. It is likely that the analyticity hypothesis can be gotten rid of by extracting a symmetric hyperbolic system out of the system of equations at hand. This can most probably be done by repeating the analysis carried out by H. Friedrich in a related context [52]. In that last paper H. Friedrich extracts a well posed system of equations out of the Einstein equations with ideal fluid as a source, *without* assuming that the density of the fluid is bounded away from zero. This is particularly remarkable, as no such procedure is known in general for the Euler equations considered by themselves, *i.e.* when not coupled to general relativity.

To close this section, let us note that there has been quite a lot of activity in the last few years concerning the question, how to extract a well posed system of dynamical equations out of the Einstein equations [1,21,51,54,94]. In particular A. Anderson has submitted an abstract to this workshop describing his construction, in collaboration with Y. Choquet–Bruhat and J. York, of a new first order symmetric hyperbolic system for the evolution part of the Cauchy problem of general relativity [36].
4.2 A semi–global result

There are only very few global or semi–global existence results concerning solutions of the Cauchy problem in general relativity (see [93] for an exhaustive list). So far the only ones which concerned space–times without any symmetries were the Christodoulou–Klainerman theorem of stability of Minkowski space–time [38], the semi–global stability results of Friedrich (cf., e.g. [50] and references therein), and the semi–global existence results concerning the Robinson–Trautman space–times [39]. This short list has been recently extended by Andersson and Moncrief with a semi–global existence result concerning the following class of spatially compact space–times: Let $M$ be a compact 3–manifold of hyperbolic type and let $g_0$ be the standard hyperbolic metric on $M$ with sectional curvature $-1$. The couple $(M, g_0)$ is called rigid if there are no trace free, divergence free symmetric 2–tensors $u_{ij}$ satisfying $u_i^i = 0$, $\nabla^j u_{ji} = 0$, $\nabla_k u_{ij} - \nabla_j u_{ik} = 0$. The class of rigid hyperbolic 3–manifolds is nonempty.

On $\bar{M} = M \times \{ t > 0 \}$ define the metric $\bar{g}_0 = -dt^2 + t^2 g_0$. Then $(\bar{M}, \bar{g}_0)$ is a flat globally hyperbolic spacetime (its universal cover is the $\kappa = -1$ Friedmann – Robertson – Walker vacuum spacetime). Given a spacelike hypersurface $M$ of a spacetime $(\bar{M}, \bar{g})$, let $g_{ij}, T^a, k_{ij} = -\nabla_i T_j$ be the metric, future normal and second fundamental form on $M$, where $\bar{\nabla}$ is the covariant derivative on $\bar{M}$. With these conventions, the induced initial data for the Einstein equations on $M = \{ t = 1 \}$ in the standard spacetime $(\bar{M}, \bar{g}_0)$ are $(g_0, -g_0)$. Rigidity of $(M, g_0)$ is equivalent to rigidity of $(\bar{M}, \bar{g}_0)$ in the moduli space of flat spacetime structures. (Note that Mostow rigidity does not apply in the case of flat spacetime structures.)

Andersson and Moncrief consider the 3+1 vacuum Einstein evolution equations, with constant mean curvature time gauge $tr_k = \tau$ and spatial gauge fixing given by the affine conformal slice (see [18]) $S_\tau$. Note that with the orientation used, the mean curvature $\tau = tr_g k$ satisfies $\tau < 0$ and $\tau \nearrow 0$ corresponds to infinite expansion (and proper time $t \nearrow \infty$).

Theorem 4.2 (Andersson & Moncrief [7]) Let $M$ be rigid, and let $(g, k) \in S_{-3}$ be vacuum data for Einstein equations, sufficiently close in $H^4 \times H^3$ to standard data $(g_0, -g_0)$. Then (1) Global existence in the expanding direction, $\tau \nearrow 0$, holds for the 3+1 vacuum Einstein evolution equations, gauge fixed as above, with initial data $(g, k)$. (2) The maximal globally hyperbolic vacuum extension $(\bar{M}, \bar{g})$ is geodesically complete in the expanding direction.

In contrast to the asymptotically flat spacetimes considered by Christodoulou–Klainerman [38], the maximal globally hyperbolic vacuum developments of data considered by Andersson and Moncrief are not causally geodesically complete to the past. This follows from the singularity theorems of Hawking and Penrose. No other properties of the space–time in the contracting direction are known (existence of curvature singularities? Cauchy horizons?).
The proof of Theorem 4.2 makes use of a Bel–Robinson type energy function \( E \) defined with respect to the the Weyl fields \( W^{(0)}_{abcd} = \bar{R}_{abcd} \), where \( \bar{R} \) is the Riemann tensor of \( (\bar{M}, \bar{g}) \), \( W^{(1)}_{abcd} = T^f \bar{\nabla}_f W_{abcd} \) and \( W^{(2)}_{abcd} = T^f T^g \bar{\nabla}_f \bar{\nabla}_g W_{abcd} \). See [38] for background and a different approach to defining higher order Bel–Robinson energies. The use of an energy function to control the gauge fixed evolution used in the proof of Theorem 4.2 is closely related to the method used to prove global existence in CMC time for 2+1 gravity [8].

The Andersson–Moncrief theorem shows once again that the Bel–Robinson tensor is an object which deserves attention — recall that this tensor has already been used by Christodoulou and Klainerman in their stability theorem [38]. During the workshop J. Senovilla has presented a simple proof, obtained using the Bel–Robinson tensor, of the following fact [24]: For any conformally vacuum spacetime (with or without cosmological constant), if the Weyl tensor vanishes on any closed achronal set \( \Sigma \), then it vanishes on its domain of dependence \( D(\Sigma) \). While this result is well known, at least in vacuum, the previous proofs made use of rather more involved considerations.

5 The nature of singularities

5.1 Numerical experiments

One of the main challenges of mathematical general relativity is the description of the generic singularities that arise during evolution via the Einstein equations out of appropriately regular initial data. This problem seems to be completely out of reach of the analytical tools which are currently at our disposal, and our best hope today to get some real insight about that question is to carry out reliable numerical experiments. For vacuum Einstein equations two types of behavior have been known to occur, when a singular boundary (whatever this means) of a spatially compact space–time is approached: a “velocity dominated” behavior (AVTD) and a “Mixmaster” behavior. The latter case is expected to correspond to curvature singularities. The AVTD behaviour includes both curvature singularities, Cauchy horizons, and “topological singularities” — those arise when “what would have been a Cauchy horizon” is quotiented–out by an ergodic action of a isometry group [71, 84], and leads to space–times which are inextendible with the curvature remaining bounded. A new kind of behavior has been recently observed by Breitenlohner – Lavrelashvili – Maison [28] and, independently by Gal’tsov – Donets – Zotov [45]. While the studies of those authors are in principle concerned with black hole space–times, one can spatially compactify the space–time (“below the event horizon”) to obtain models with

\[2\] While D. Gal’tsov reported on his studies in the workshop, G. Lavrelashvili was kind to contribute to this report a description of his results with Breitenlohner and Maison. I wish to thank him for this, as well as for many clarifying comments concerning his work.
$S^2 \times U(1)$ topology and $SO(3) \times U(1)$ isometry group. The numerical analysis of the interior geometry of static, spherically symmetric black holes of the Einstein-Yang-Mills-Higgs theory shows the following $[28, 45]$:

First, within the set of initial data considered, generically no inner (Cauchy) horizon is formed inside the non-Abelian black holes. This is then consistent with the (strong) cosmic censorship hypothesis.

Next, the generic black hole solution of the EYM theory has an oscillatory behavior inside the horizon $[28, 45]$. As one performs numerical integration starting at the horizon and integrates towards $r = 0$ one observes a sudden steep rise of a derivative of the $SU(2)$ gauge field amplitude, $W'$, and a subsequent exponential growth of the mass function $m(r)$ (parametrizing the $g^{rr}$-component of the metric via $g^{rr} = 1 - 2m(r)/r$). Within a short interval of $r$ the mass function reaches a plateau and stays constant in some range of $r$’s until it starts to decrease again. When the solution comes close to an inner horizon, the same inflationary process repeats itself with an even more violent next “explosion”. In the black hole context this behavior seems to be related to the “mass inflation” phenomenon observed for linear perturbations of the Reissner-Nordström black holes $[22]$. In fact, the results of $[28, 45]$ can be thought of as giving a non–linear counterpart of that effect. It should, however, be emphasized that the dynamics here is essentially different, because in the linearized case considered in $[22]$ no plateaux occur. An apparently related “oscillatory mass inflation” has also been observed in $[88]$, in a semi–phenomenological model with two null fluids.

By a suitable fine tuning of the initial data at the horizon it is possible to obtain solutions with a different (non–generic) behavior, see $[28, 45]$ for details. In $[28, 45]$ one can also find a simplified dynamical system which seems to provide a qualitative understanding of the behavior of the generic solutions. In particular one can derive a “plateau – to – plateau formula” $[28]$ which (in the simplified model) relates quantities at one plateau (before the “explosion”) to those on the next plateau (after the “explosion”).

In order to study the model dependence of these results the theory with an additional Higgs field was also investigated $[28]$. It was found that after adding the Higgs field no more oscillations occur in the asymptotic behavior inside the horizon. This change in the behavior of the generic solution can presumably be understood by a change of the character of fixed points of the corresponding simplified dynamical system. In that system the addition of a Higgs fields makes the focal point disappear, and the asymptotic behavior becomes governed by a stable attractor $[28]$.

The main conclusions are, that no inner (Cauchy) horizon are formed inside non-Abelian black holes in the generic case, instead one obtains a “spacelike” singularity at $r = 0$. Without a Higgs field, $i.e.$ for the EYM theory, one obtains a kind of “mass inflation” that repeats itself in cycles of ever more violent growth. This behavior near the singularity does not have a counterpart in the dynamics of spatially homogeneous vacuum cosmological models. With the Higgs field no
such cycles occur in the asymptotic behavior.

The next numerical study which has been reported on in this workshop concerns space–times with $U(1)$ or $U(1) \times U(1)$ symmetry. Recall that Grubišić and Moncrief \cite{59,60} have used formal asymptotics expansions as a tool to understand the asymptotic approach to the singularity in spatially inhomogeneous cosmologies. They have found that solutions in the form of formal series could be obtained with the following property: When approaching the singularity the formal solutions approach a solution of the equations obtained by dropping, in Einstein’s equations, terms containing spatial derivatives. This is the AVTD behaviour alluded to above. (It has been recently shown by Kichenassamy and Rendall that the Grubišić–Moncrief formal series are convergent in some cases \cite{73}.) Self–consistent formal solutions with an AVTD singularity \cite{69} at (say) $\tau = \infty$ are obtained if the terms neglected in the truncated equations are exponentially small, when evaluated using the AVTD solution. For example, in vacuum Bianchi IX (Mixmaster) \cite{15,85} space–times, the substitution of the Kasner solution into the minisuperspace potential always yields exponential growth in one of the terms, which is not consistent with an AVTD behavior. On the other hand, in the (plane-symmetric, vacuum) Gowdy cosmologies on $T^3 \times \mathbb{R}$ \cite{58}, nonlinear terms in the wave equations allow the (formal) AVTD solution provided an “asymptotic velocity $v^\prime$” satisfies $0 < v(\theta) < 1$. Numerical simulations performed by Berger, Garfinkle and Moncrief show how $v(\theta)$ outside the allowed range is driven into the allowed range by these same nonlinear terms \cite{18,19}. Gowdy models with a modified topology generalized to include a magnetic field become the spatially inhomogeneous generalization of magnetic Bianchi VI$_0$ homogeneous solutions that are known to display Mixmaster dynamics \cite{17,81}. The extra nonlinear terms from the magnetic field cause AVTD behavior to be inconsistent for essentially all values of $v(\theta)$. The Mixmaster approach to the singularity one then expects has been observed numerically \cite{101}. It should be emphasized that this is a first numerical observation of Mixmaster behavior in a non–homogeneous space–time. A further generalization to $U(1)$ symmetric cosmologies on $T^3 \times \mathbb{R}$ \cite{20,86} shows that the AVTD solution is consistent for polarized models but inconsistent for generic ones. This leads one to expect a Mixmaster–like singularity at each spatial point. Numerical simulations provide strong support for an AVTD singularity in the polarized case \cite{19}. In the generic case, numerical results are suggestive but numerical errors associated with the failure to preserve the Hamiltonian constraint prevent strong conclusions \cite{16,19}.

5.2 Other studies

The singularity theorems of Hawking and Penrose predict geodesic incompleteness, and many authors identify this feature with the existence of a singularity. While this attitude is justified in many cases, situations occur in which one could question the validity of this conclusion. In his talk, C. Clarke summarized the
philosophy of regarding as singular only those points in the space-time manifold (with a metric that was not necessarily continuous) at which the propagation of test-fields was disrupted. More precisely, he proposes to use uniqueness and existence results for the wave equation in a neighborhood of a point to classify points as “regular” or “singular”. Such an approach allows one to class as regular some of those points which would be classed as singular in terms of the completeness of unique geodesics. He announced a theorem concerning space-times for which 1) $g_{ij}$ and $g^{ij}$ are continuous with square integrable weak derivatives, and which 2) have a point $p$ such that $g_{ij}$ is in $C^1(M \setminus J^+(p))$. Moreover he requires that, in a coordinate neighborhood of $p$, the integrals $I_\gamma(a) := \int_0^a |\Gamma^i_{jk}(\gamma(s))|^2 ds$ and $J_\gamma(a) := \int_0^a |R_{ijkl}(\gamma(s))| ds$ are bounded by positive functions $M(a), N(a)$ tending to zero with $a$, for all curves $\gamma$ whose tangent vector had components lying in some fixed cone $C$. Under those conditions, he claimed, the wave equation has unique solutions, in $H^1(S_t)$ for a slicing by spacelike hypersurfaces $S_t$, in a neighbourhood of $p$ for $C^2$ data on a partial Cauchy surface to past of, and sufficiently close to, $p$.

He suggested that this theorem might be applicable to shell crossing singularities, which would become regular points on this proposed definition. If this were the case, and if one reformulates the cosmic censorship question according to his proposal, then such singularities would stop being counter-examples to cosmic censorship, without the special pleading of dismissing the dust matter as unphysical. He also claimed that a similar treatment could be applied to cosmic strings.

Since the pioneering work of Belinski, Lifschitz and Khalatnikov already mentioned above [15], a tool which has been often used when trying to understand the nature of singularities is that of formal expansions (cf. also [59,60]). On one hand, both 1) the numerical results of Berger and collaborators described above and 2) the analytical results of [10, 59, 73] give strong support to the validity of such an approach in some situations. On the other hand, it is not clear that the Mixmaster, or the “plateau to plateau” behavior of Breitenlohner et al. — Gal’tsov et al. described in Section 5.1 for the Einstein–Yang–Mills equations, are compatible with any formal expansions framework. Whatever the status of such expansions in general, it is of interest to find if the evolution equations for some other gravitating systems are (perhaps formally) compatible with this idea. L. Burko has used such an approach in the context of spherical charged black holes perturbed nonlinearly by a self-gravitating, minimally-coupled, massless scalar field. As in the Einstein–Yang–Mills–Higgs case, this model can be spatially compactified under the horizon to give a cosmological space–time. The numerical simulations described in [25, 32] suggest that in this case the Cauchy horizon turns into a “null weak singularity” which “focuses” monotonically to $r = 0$ at late times, where the singularity “becomes spacelike”[3]. Burko [33, 34]

3All the terms in inverted commas here have an obvious meaning in the coordinate system
examines a formal series–expansion solution for the metric functions and for the scalar field near \( r = 0 \) under the simplifying assumption of homogeneity. He finds that such solutions are self–consistent, and \textit{generic} in the sense that the solution depends on the expected number of free parameters. The properties of the formal solutions are similar to those found in the fully-nonlinear (and inhomogeneous) numerical simulations.

Cauchy horizons are objects which are closely related to singularities in many situations, so a study of those complements naturally that of singularities. T. Helliwell and D. Konkowski have presented a poster in which they describe some progress in their program to study the stability of Cauchy horizons \[78\]: In \[79\] they study vacuum plane wave spacetimes and assert, among others, that there exists a relationship between the stability of the Cauchy horizon and the behaviour of test fields in the space–time, provided there is no Weyl–tensor singularity in the spacetime.

\section{Global techniques}

In his talk describing joint work with L. Andersson and R. Howard, G. Galloway discussed some results about cosmological time functions, defined as follows: Let \((M, g)\) be a time oriented Lorentzian manifold. The \textit{cosmological time function} of \(M\) is the function \(\tau : M \to (0, \infty]\) defined by: \(\tau(q) = \sup_{p<q} d(p, q)\), where \(d\) is the Lorentzian distance function. In the cosmological setting, \(\tau\) has a simple physical interpretation: \(\tau(q)\) is the maximum proper time from \(q\) back to the initial singularity. In general, \(\tau\) need not be well-behaved. For example, even if \(M\) is globally hyperbolic and \(\tau\) is finite valued, \(\tau\) need not be continuous (\[5,100\]). Define \(\tau\) to be \textit{regular} provided it satisfies: \(\tau(q) < \infty\) for all \(q\) and \(\tau \to 0\) along every past inextendible causal curve. In \[5\] some consequences of regularity are obtained. It is shown, for example, that if \(\tau\) is regular then (1) \(M\) is necessarily globally hyperbolic, (2) \(\tau\) is a time function in the usual sense, \(i.e.,\) \(\tau\) is continuous and strictly increasing along future directed causal curves, (3) \(\tau\) is \textit{semi-convex}, and hence has first and second derivatives almost everywhere and (4) every point of \(M\) can be connected to the initial singularity by a timelike geodesic ray that realizes the distance to the singularity. See \[5\] for further results.

It should be mentioned that Wald and Yip \[100\] introduced the cosmological time function (or rather its time dual, which they referred to as the “maximum lifetime function”) quite some time ago in order to study the existence of synchronous coordinates in a neighborhood of a spacelike singularity. More recently, Andersson and Howard \[4\] have made use of the cosmological time function to prove some results concerning the rigidity of Robertson-Walker and related spacetimes.

\begin{footnotesize}
used. I have used inverted commas because I am not aware of any standard meaning of those notions in general.
\end{footnotesize}
In his talk, E. Woolgar reported on work in progress with G. Galloway concerning the generalization of the topological censorship theorem \cite{49,55} to spacetimes with more general asymptotic structures than the usual asymptotic flatness. Galloway and Woolgar \cite{56} make no assumptions on the geometry near infinity, and therefore allow timelike, spacelike, and null scris. In particular, they show that $\pi_1$ of the domain of outer communications is a subgroup of $\pi_1$ of scri. For this one needs to assume a null energy condition and the null generic condition, although there are some reasons to believe that the latter assumption may be unnecessary. It is also assumed that no compact set of spacetime contains all of scri within its past, and that there are no naked singularities (in the sense that there is no incomplete null geodesic whose past is contained within the past of a point of scri).

We note that black hole horizons with non-zero genus in locally anti-de Sitter spacetime have been the subject of much recent attention \cite{2,3,29,63}. The work by Galloway and Woolgar sheds light on the compatibility of those solutions with the notion of topological censorship.

P. Aichelburg and F. Schein contributed to the workshop an interesting new time machine: They have constructed a static axisymmetric wormhole from the gravitational field of two charged shells which are kept in equilibrium by their electromagnetic repulsion. The interior of the wormhole is a Reissner-Nordström black hole matching the two shells. The wormhole is one-way traversable, \textit{i.e.}, one throat lies in the (local) future of the other and connects to the same asymptotic region. Moreover, the shells of matter can be chosen to satisfy the energy conditions. The solution is an “eternal time machine”: every point in the Majumdar-Papapetrou region lies on a closed timelike curve.

In his talk F. Stahl, described some results he obtained in his studies of the Schmidt metric, and made some remarks on his intended further studies. Recall that the Schmidt metric $G$ is a Riemannian metric on the frame bundle $LM$ of a manifold $M$ with connection, related to the generalized affine parameter length of curves in $M$ \cite{97}. Stahl has shown \cite{98} that geodesics of $G$ project down to geodesics on $M$. He hopes to be able to find a relationship between the conjugate points in $(LM, G)$ and $M$, and to explore further the properties of singularities by studying the structure of $(LM, G)$ near a $b$-boundary point.

7 Stationary solutions

There have been many attempts to construct models of stationary stars using the Einstein equations with a perfect fluid. The results obtained so far include the famous disk models of Meinel and Neugebauer (\textit{cf.} \cite{87} and references therein), and the slowly rotating solutions of Heilig \cite{61}. It is, nevertheless, clear that much more research is needed before we get a good understanding of the problems involved. In his talk M. Mars described his work with J. Senovilla \cite{82} in which
one discusses how to construct the exterior gravitational field created by a given (arbitrary) distribution of stationary and axisymmetric matter. The matching conditions \[83\] between the known interior and the unknown vacuum exterior introduce two new, essential parameters in the interior metric which are closely related with the state of motion of the body as seen from infinity. For each value of these parameters, the exterior matching boundary is uniquely determined. Furthermore, the matching conditions provide the boundary conditions for the exterior problem. In terms of the Ernst potential \[E \equiv e^{2U} + i \Omega \] [47], the values of \(U\) and the normal derivatives of \(E\) are fixed on the boundary and the values of \(\Omega\) are fixed on the boundary up to an arbitrary, non-trivial, additive constant. Since the equations to be solved are elliptic of second order the problem is not well-posed, so that not every interior metric can be interpreted as an isolated rotating body. Mars and Senovilla prove uniqueness of the exterior field using, in a first step, the well-known Mazur–Bunting identity [31] for harmonic maps between manifolds to show that the exterior field is unique for each value of the undetermined additive constant in \(\Omega\) on the matching hypersurface. The second step uses the high degree of symmetry of the target manifold of the harmonic map in order to construct a three-parameter conserved current which is then exploited to show that only one of the additive constants in \(\Omega\) is possible, thus completing the proof. Mars also described some related results by Weinstein which, while mostly concerned with stationary black holes, also do apply to the problem at hand (see \[102\] and references therein).

While the results of Mars and Senovilla concerned uniqueness, U. Schaudt, in collaboration with H. Pfister, has proved both existence and uniqueness in a “small data” context. (The qualification “small” here does not refer to some abstract small number, but to a precise explicit bound given by the authors, which might actually be considered as large in some physical contexts.) In \[95,96\] it was shown that the Dirichlet problem for the vacuum region outside a ball, as well as for a ball inside the matter, has a unique regular solution if the absolute values of the appropriate boundary data are limited by an appropriately understood “radius” of the ball. The results obtained include an existence and uniqueness–proof for the exterior solutions in a range of parameters which includes all known white dwarf–stars. The method of proof (which is a fixed point argument in appropriately chosen spaces) and the results have connections with a numerical solution technique for rotating stars [23]. A proof of K. Thorne’s “hoop–conjecture” [99] is also given under, however, a rather restrictive set of hypotheses.

A completely different approach to the construction of stationary stars is that using Riemann–Hilbert techniques, discussed by C. Klein in his talk. Those techniques provide an important tool in the context of integrable equations since they can be used to generate solutions with a free function. If this function can be determined from a boundary or initial value problem, the Riemann–Hilbert problem (RHP) is equivalent to the solution of a linear integral equation. In the case of the Ernst equation, Klein and Richter prove two theorems: 1. The RHP
for the Ernst equation (see [77]) with analytic ‘jump data’ is gauge equivalent to a scalar problem on a four–sheeted Riemann surface. For rational jump data, it can be solved explicitly in terms of hyperelliptic theta functions. The obtained (normalized) solutions are regular except at a contour in the meridian plane that corresponds to the contour of the RHP. They are in general asymptotically flat, have a regular axis of symmetry, and can have ergoregions and an ultrarelativistic limit, see [76]. These theorems suggest that it might be possible to solve boundary value problems for the stationary axisymmetric vacuum Einstein equations by directly identifying the branch points of the Riemann surface and the free function of the RHP from the boundary data.

8 Distributions and their generalizations

Singular null hypersurfaces on a spacetime manifold, i.e. null hypersurfaces across which the metric tensor is only $C^0$, can be used as models for the propagation of an impulsive lightlike signal (e.g. a burst of neutrinos together with an impulsive gravitational wave). Using a formalism developed by Barrabès and Israel [12], C. Barrabès, G. Bressange and P. Hogan have shown that in the general case a lightlike shell coexists with an impulsive gravitational wave on such a hypersurface. They have analyzed in detail two examples illustrating this phenomenon: The first concerns abrupt changes of the multipole moments in the axisymmetric Weyl solutions, and can be used as a model for a supernova [11]. The second describes jumps in the mass and angular momentum in Kerr spacetime, and is supposed to model pulsar glitches [10].

While the above work concerned distributions within a well posed mathematical framework, it is often the case that in general relativistic considerations one encounters products of distributions, leading to various problems as far as the mathematical meaning of the resulting expressions is concerned. Steinbauer has been studying those issues in the context of a study of geodesics in space–times with impulsive pp–waves. Here the main problem to deal with are the products “$\theta \delta$”, “$\theta^2 \delta$” and “$\delta^2$”, which arise due to the presence of the Dirac-$\delta$-“function” in the metric. He regularizes those terms by a natural procedure which corresponds to the physical idea of viewing the impulsive wave as the limiting case of a sandwich wave. He shows that in this way one can end up with regularisation independent distributional solutions, without recourse to ad-hoc strategies which had been used in previous analyses of this problem.

In his talk J. Wilson has described how Colombeau’s theory of generalised functions [43], which gives a mathematical framework in which products of distributions are well defined objects, can be used to calculate the curvature of metrics which are too singular to be handled by the standard distribution theory. Detailed calculations have been performed for static cosmic strings in flat space-time [42] and for dynamical cosmic strings on curved backgrounds [103].

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In particular it was shown that the mass per unit length of a radiating cosmic string, whose metric takes the form

\[ ds^2 = e^{2\gamma(t-r)}(-dt^2 + dr^2) + r^2d\varphi^2 + dz^2, \]

is \(2\pi(1 - e^{-\gamma(t)})\), which agrees with the mass at null infinity. In this context it should be pointed out that Colombeau’s theory itself is not invariant under \(C^\infty\) diffeomorphisms, so one must exercise some caution in choosing the coordinate system when applying it to a covariant physical theory. In his talk J. Wilson expressed the hope to be able to construct such a diffeomorphically invariant theory of generalised functions, basing on a recent paper by Colombeau and Meril [44].

It should be pointed out that R. Mansouri and K. Nozari have presented a poster describing their work on a distributional approach to the change of signature of spacetime metric, based again on Colombeau’s generalized functions. They claim to obtain an Einstein equation for dynamics of signature changing spacetime with a nonvanishing distributional stress-energy tensor supported at the signature–change hypersurface.

Let us finally note that V. Pelykh has contributed to this workshop an abstract describing an approach to handle Einstein equations with distributional sources.

9 The energy of the gravitational field

A very old problem, which still attracts some attention, is that of energy of the gravitational field. In his talk J. Nester reported on his attempts, with C.C. Chang and C.M. Chen, to construct gravitational Hamiltonians for finite regions. (A very similar approach has already been advocated by Kijowski, see [43, 76] and references therein). Those Hamiltonians consist of boundary integrals, the numerical value of which can be thought of being the quasi–local energy contained in the volume enclosed. The variation of the Hamiltonian leads to surface terms which determine what needs to be held fixed on the boundary if one wishes to obtain a Hamiltonian system. There isn’t therefore only one energy because there are many boundary condition which one might wish to impose, each with its distinct boundary term. Nester and his collaborators examine specific cases, identifying the appropriate boundary conditions. They suggest that certain principles (good limiting values, covariance [35]) can be used to restrict the possibilities. Nester emphasizes that one can give a “respectable” quasilocal interpretation to the pseudo–tensors, and claims that superpotentials associated with pseudotensors form legitimate Hamiltonian boundary terms. It would be of interest to make a detailed Hamiltonian analysis, along the lines presented by Kijowski in [76] and also by Nester, of those boundary conditions which lead to a well posed Cauchy problem, such as e.g. the Friedrich–Nagy conditions described in Section 4.4.
A different approach was described in a poster by Petrov, who advocates the use of a background metric to define the energy–momentum tensor of the gravitational field, and hence the local and quasi–local energy. In a joint work with Narlikar they assert that one can give a distributional meaning to this tensor calculated for the Schwarzschild and the Reissner-Nordström solutions with respect to an appropriately chosen background.

Yet another approach to the energy question was described by Yoon, who proposes a derivation based on the Hamiltonian formalism of general relativity in the (2,2)–foliation of spacetime. Here one uses Kaluza–Klein type variables as the configuration variables. The analysis is carried out in the Newman–Unti gauge, which partially fixes the spacetime diffeomorphisms. An “advanced time” is used as the time coordinate on which the Hamiltonian formalism is based. The four Einstein’s constraints are written in the canonical variables, and are found to be the first–class constraints generating the residual spacetime diffeomorphisms that survive after the Newman–Unti gauge is chosen. By integrating one of the constraint equations over a closed spacelike 2–surface one obtains an integral equation which relates the rate of change of a certain integral over the 2–surface to a flux integral across that 2–surface. It is suggested to define the quasi–local gravitational energy of a region enclosed by the 2–surface via the former integral. The latter integral becomes then the gravitational energy flux integral crossing the 2–surface. The proposed quasi–local integrals reproduce the Bondi energy and Bondi flux integral at null infinity.

Let us also mention that N. Dahdich has presented a poster in which he analyzes the properties of the Brown–York quasilocal energy in spherically symmetric black hole space–times.

It is clear that all those approaches lead to a plethora of definitions of energy, while one would like to have a unique one. In a poster by J. Jezierski, M. McCallum and this author a uniqueness theorem for the Trautman–Bondi mass was presented, which can be used to single out this energy among the competitors.

10 Miscellaneous

In his talk F. Andersson presented some new results concerning the Lanczos potential, due to himself and to S.B. Edgar and A. Höglund. Recall that the original proof of the existence of that potential by Lanczos was flawed, and the first complete proof was given by F. Bampi and G. Caviglia (see also [67]). F. Andersson described an alternative and considerably simpler proof of the existence of that potential, both in a spinor formalism and in tensor formalism [4,46]. In the new proof an explicit restriction to four dimensions arises, but the signature can be arbitrary. In dimensions higher than four one can, using symbolic computer algebra, derive an integrability condition for the Lanczos equations which does not appear to be identically satisfied. It would be of interest to determine
whether this is a real obstruction, or whether this is simply a new identity which relates the objects at hand.

M. Iriondo was supposed to present some results, in collaboration with E. Leguizamón and O. Reula, concerning the Newtonian limit of general relativity on asymptotically null foliations. Unfortunately she could not attend the conference, and N. O’Murchadha was kind enough to give a replacement talk at very short notice. He presented the results of his joint studies with J. Guven concerning the question, “How Big Can a Spherically Symmetric Static Object Be?”.

It is regrettable that some further speakers could not attend the meeting. For instance, A. Rendall was supposed to present his results in [91,92] concerning foliations by spacelike hypersurfaces in space–times with two–dimensional symmetry groups. He proves in particular that space–times containing such foliations have crushing singularities for a class of matter models. J. Isenberg was supposed to talk about some new results obtained with A. Rendall in [70]: in this paper they show that there exist maximal globally hyperbolic solutions of the Einstein–dust equations which admit a constant mean curvature Cauchy surface, but do not admit a constant mean curvature foliation. This is a sharpening of a previous observation of Rendall [91].

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