Linearized Polynomials, Galois Groups and Symmetric Power Modules

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Abstract

We investigate some Galois groups of linearized polynomials over fields such as \(\mathbb{F}_q(t)\). The space of roots of such a polynomial is a module for its Galois group. We present a realization of the symmetric powers of this module, as a subspace of the splitting field of another linearized polynomial.

1 Introduction

Let \(p\) be a prime number, and let \(q = p^a\) where \(a\) is a positive integer. Let \(\mathbb{F}_q\) denote the finite field with \(q\) elements. Let \(F\) be a field of characteristic \(p\), and assume that \(F\) contains \(\mathbb{F}_q\). A \(q\)-linearized polynomial over \(F\) is a polynomial of the form

\[
L = a_0x + a_1x^q + a_2x^{q^2} + \cdots + a_nx^{q^n} \in F[x].
\]

If \(a_n \neq 0\) we say that \(n\) is the \(q\)-degree of \(f\). We will usually say \(q\)-polynomial instead of \(q\)-linearized polynomial.

The set of roots of a \(q\)-polynomial \(L\) forms an \(\mathbb{F}_q\)-vector space, which is contained in a splitting field of \(L\). We make this statement more precise in the following simple lemma, which also serves as an introduction to the topics of this paper.

Lemma 1. Let \(F\) be a field of prime characteristic \(p\) that contains \(\mathbb{F}_q\). Let \(L\) be a \(q\)-polynomial of \(q\)-degree \(n\) in \(F[x]\), with

\[
L = a_0x + a_1x^q + a_2x^{q^2} + \cdots + a_nx^{q^n}.
\]
Let $E$ be a splitting field for $L$ over $F$ and let $V$ be the set of roots of $L$ in $E$. Let $G$ be the Galois group of $E$ over $F$. Suppose that $a_0 \neq 0$. Then $V$ is an $F_q$-vector space of dimension $n$ and $G$ is naturally a subgroup of $GL(n,q)$.

**Proof.** The derivative $L'(x) = a_0$ and is thus a nonzero constant under the hypothesis above. It follows that $L$ has no repeated roots and thus $|V| = q^n$. Let $\alpha$ and $\beta$ be elements of $V$. Then $L(\alpha) = L(\beta) = 0$. Since

$$(\alpha + \beta)^{q^i} = \alpha^{q^i} + \beta^{q^i}$$

for all $i \geq 0$, it is clear that $L(\alpha + \beta) = 0$, and hence $\alpha + \beta \in V$.

Let $\lambda$ be an element of $F_q$. Since $\lambda^{q} = \lambda$, we have $(\lambda \alpha)^{q^i} = \lambda \alpha^{q^i}$. It follows that $L(\lambda \alpha) = \lambda L(\alpha) = 0$ and hence $\lambda \alpha \in V$. These arguments show that $V$ is a vector space over $F_q$. Furthermore, since $|V| = q^n$, $V$ has dimension $n$ over $F_q$.

Let $\sigma$ be an element of $G$. By definition of Galois group action, $G$ permutes the roots of $L$ and hence maps $V$ into itself. In addition, since $G$ is a group of field automorphisms of $E$, we have

$$\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta)$$

for all $\alpha$ and $\beta$ in $V$.

Now let $\lambda$ be an element of $F_q$. Since $F_q$ is assumed to be a subfield of $F$ and $G$ fixes $F$ elementwise, we have $\sigma(\lambda) = \lambda$. Then, again by definition of Galois group action,

$$\sigma(\lambda \alpha) = \sigma(\lambda) \sigma(\alpha) = \lambda \sigma(\alpha).$$

This shows that the action of $G$ on $V$ is $F_q$-linear and thus $G$ may be considered to be a subgroup of $GL(n,q)$.

We call $V$ the ($F_q$-vector) space of roots of $L$. The proof shows that $V$ is an $FG$-module.

We remark that the Galois group of a generic $q$-polynomial is $GL(n,q)$, as shown for example in [8] when proving a theorem of Dickson.

We recall the definition of the symmetric powers. Let $V$ be any vector space over a field $F$. The $r$-th symmetric power of $V$, $Sym^r(V)$, is another vector
space constructed as the quotient space of \( V^{\otimes r} \) by the subspace generated by all 
\[ v_1 \otimes \cdots \otimes v_r - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \] 
where \( \sigma \in S_r \). If \( e_1, e_2, \ldots, e_r \) is a basis for \( V \) then 
\[ e_1^{k_1} e_2^{k_2} \cdots e_r^{k_r} \] 
where \( k_1 + \cdots + k_n = r \) is a basis for \( Sym^r(V) \), where we denote the operation as ordinary multiplication. The \( r \)-th symmetric power can thus be identified with the space of all homogeneous polynomials of degree \( r \). The symmetric powers play an important role in representation theory.

For the remainder of this section we shall outline the structure of this paper. First, Section 2 presents a background discussion of \( q \)-polynomials, with a focus on the relevant issues for this article.

Let \( L(x) \) be a monic \( q \)-linearized polynomial with the same hypotheses as Lemma 1. Let \( M(x) = L(x)/x \). Then it is easy to see that \( M(x) \) is a monic polynomial in \( x^{q-1} \), say \( M(x) = P(x^{q-1}) \). The polynomial \( P(x) \) is monic of degree \((q^n - 1)/(q - 1)\) and is called the projective polynomial associated to \( L \). If \( M \) is irreducible over \( F \) then so is \( P \).

We will show in Section 3 that any polynomial divides a linearized polynomial. Applying this lemma to \( P(x) \), let \( L_P(x) \) be the linearized polynomial of minimal degree that is divisible by \( P(x) \). We will show in Section 3 that \( L_P \) has degree \( q^d \) where \( d \) is the dimension of the \( F_q \) span of the roots of \( P \). So we have a construction that starts with \( L \), then constructs \( P \), and then another linearized polynomial \( L_P \).

We wish to compare the Galois groups of \( L \) and \( P \), which we denote by \( G_L \) and \( G_P \) respectively. Let \( K_L \) be a splitting field for \( L \). Let \( K_P \) be a splitting field for \( P \), which is the same as a splitting field for \( L_P \) (so \( P \) and \( L_P \) have the same Galois group). If \( \alpha \in K_L \) is a nonzero root of \( L \), then \( P(\alpha^{q-1}) = 0 \) and so \( L_P(\alpha^{q-1}) = 0 \). Each \( \alpha \) and all its \( F_q \)-multiples give rise to the same root of \( P \). Conversely, if \( \beta \) is a root of \( P \) then the roots of \( x^{q-1} - \beta \) are roots of \( L \). This implies that \( K_P \subseteq K_L \) and that \( K_P \) contains all the \((q - 1)\)-th powers of the roots of \( L \). This also implies that \( G_L = Gal(L) \) has a normal subgroup \( N = Gal(K_L : K_P) \) such that \( G_L/N \cong G_P \), and that \( N \) will be a subgroup of a cyclic group of order \( q - 1 \).

The result we wish to present is that the symmetric powers are easily visible in this setting. To summarize our results, let \( V_L \) denote the space of roots of \( L \), a \( G_L \)-module, and let \( V_P \) denote the space of roots of \( L_P \), a \( G_P \)-module. By the construction, since the \((q - 1)\)-th powers of the roots of \( L \) are roots of \( L_P \), one might intuitively expect the \((q - 1)\)-th symmetric power of \( V_L \) to be related to \( V_P \). We will
see that in order to make this precise, the following idea is important.

Let \( \alpha_1, \ldots, \alpha_n \) be a basis for \( V_L \). The linear independence of the elements \( \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n} \) will be of crucial importance when trying to find symmetric powers. Thus, we will study the evaluation mapping \( x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \mapsto \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n} \) from the space of all homogeneous polynomials in \( n \) variables, having degree \( r \) and coefficients in \( F \), to \( E \). In order to realize a copy of the \( r \)-th symmetric power in \( E \), it is necessary and sufficient that this evaluation map be injective.

Sections 4-7 of the paper present the details of the above summary. Section 4 begins the discussion of the injectivity of evaluation maps. Next in Section 5 we present the homomorphism from some symmetric powers to \( E \), and explain the importance of injectivity. Section 7 has a detailed discussion of the injectivity of the evaluation map, and proves the injectivity of the evaluation map in some cases. Section 7.3 discusses the particular case of \( q \)-degree 2. In Section 6 we talk about \( V_L \) as a module for the Galois group, and discuss the irreducibility of the symmetric power modules.

2 Linearized Polynomials Background

We present two simple results which we will need later. The first is a converse to Lemma 1. The proof is essentially the same as Theorem 3.52 in [7], which goes back to Dickson.

**Lemma 2.** Let \( V \) be a finite dimensional vector space over \( \mathbb{F}_q \), which is contained in a field extension \( E \) of \( \mathbb{F}_q \). Then the polynomial \( \prod_{v \in V} (x - v) \) is a \( q \)-linearized polynomial.

**Proof.** Let \( \alpha_1, \ldots, \alpha_n \in E \) be a basis for \( V \). Consider the polynomial in \( E[x] \)

\[
D(x) := \det \begin{bmatrix}
\alpha_1 & \alpha_1^q & \alpha_1^{q^2} & \cdots & \alpha_1^{q^n} \\
\alpha_2 & \alpha_2^q & \alpha_2^{q^2} & \cdots & \alpha_2^{q^n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_n & \alpha_n^q & \alpha_n^{q^2} & \cdots & \alpha_n^{q^n} \\
x & x^q & x^{q^2} & \cdots & x^{q^n}
\end{bmatrix}
\]

which will clearly be a \( q \)-polynomial in \( x \), of \( q \)-degree at most \( n \). We claim that the roots of \( D(x) \) are precisely the elements of \( V \), from which it follows that \( D(x) \) is a scalar multiple of \( \prod_{v \in V} (x - v) \).
The claim follows by observing that each $\alpha_i$ is a root of $D(x)$, and since $D(x)$ is a $q$-polynomial, all $\mathbb{F}_q$-linear combinations of the $\alpha_i$ are roots of $D(x)$ by Lemma 1. Since $V$ has $q^n$ elements, and $D(x)$ has degree at most $q^n$, it follows that $D(x)$ has degree exactly $q^n$ and the proof is complete.

We next consider a slight variation on the theme of $q$-polynomials. Let $s \geq 1$ be an integer and let

$$L(x) = a_0x + a_1x^{q^s} + a_2x^{q^{2s}} + \cdots + a_m x^{q^{ms}} \in F[x],$$

where we assume that $a_m \neq 0$. Then $L$ is a $q$-polynomial of $q$-degree $ms$ and it is also a $q^s$-polynomial. We will not always require that $\mathbb{F}_{q^s}$ is a subfield of $F$ when $s > 1$.

The following result generalizes Lemma 1.

**Lemma 3.** Let $F$ be a field of prime characteristic $p$ that contains $\mathbb{F}_q$. Let $L$ be a $q^s$-polynomial of $q^s$-degree $m$ in $F[x]$. Assume that the coefficient of $x$ in $L$ is nonzero. Let $E$ be a splitting field for $L$ over $F$ and let $V$ be the set of roots of $L$ in $E$. Let $G$ be the Galois group of $E$ over $F$.

1. The field $\mathbb{F}_{q^s}$ is a subfield of $E$.
2. $V$ is an $m$-dimensional vector space over $\mathbb{F}_{q^s}$.
3. $G$ acts on $V$ as a group of automorphisms that are semilinear with respect to the group of $\mathbb{F}_q$-automorphisms of $\mathbb{F}_{q^s}$ induced by the $q$-th power map.
4. $G$ contains a normal subgroup $H$, say, such that $H$ is a subgroup of $GL(m, q^s)$ and $G/H$ is cyclic of order dividing $s$.

**Proof.** It is clear from the proof of Lemma 1 that if $\alpha$ and $\beta$ are in $V$, so also is $\alpha + \beta$. Working in the algebraic closure of $F$, let $\lambda$ be an element of $\mathbb{F}_{q^s}$. Then $\lambda^{q^s} = \lambda$ and it follows that

$$L(\lambda \alpha) = \lambda L(\alpha) = 0$$

for all $\alpha \in V$. Thus $\lambda \alpha \in V$. Since $E$ contains $(\lambda \alpha)\alpha^{-1} = \lambda$, it follows that $\lambda \in E$ and thus $E$ contains a copy of $\mathbb{F}_{q^s}$. Since $L$ has $q^{ms}$ different roots under the hypothesis above, $V$ has dimension $m$ over $\mathbb{F}_{q^s}$. 

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The copy of $\mathbb{F}_{q^s}$ contained in $E$ is a normal subfield, since it is the splitting field of $x^{q^s} - x$ over $\mathbb{F}_q$ (and $F$ contains $\mathbb{F}_q$). It follows that $G$ maps $\mathbb{F}_{q^s}$ into itself and induces a subgroup of $\mathbb{F}_q$-automorphisms of the field.

Following the proof of Lemma 1, $G$ maps $V$ into itself and satisfies
\[ \sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta) \]
for all $\sigma \in G$ and all $\alpha$ and $\beta$ in $V$. Let $\lambda$ be an element of $\mathbb{F}_{q^s}$ and $\sigma$ be an element of $G$. Then we have
\[ \sigma(\lambda \alpha) = \sigma(\lambda)\sigma(\alpha) \]
for all $\alpha$ in $V$. This implies that $G$ acts semilinearly on $V$ with respect to the group of automorphisms it induces of $\mathbb{F}_{q^s}$.

Finally, let $H$ be the subgroup of $G$ that acts trivially on $\mathbb{F}_{q^s}$. Standard Galois theory shows that $H$ is normal in $G$ and $G/H$ is isomorphic to a subgroup of $\mathbb{F}_q$-automorphisms of $\mathbb{F}_{q^s}$. Since the Galois group of $\mathbb{F}_{q^s}$ over $\mathbb{F}_q$ is cyclic of order $s$, the quotient $G/H$ is cyclic of order dividing $s$.

We remark that Lemma 3 is really only of interest when $\mathbb{F}_{q^s}$ is not a subfield of $F$, since otherwise the result is a repetition of Lemma 1.

## 3 Serre’s Linearization Trick

In this paper we will sometimes assume that $L(x)/x$ is irreducible, which is the generic case, and will usually coincide with the Galois group being $GL(n, q)$. In this section we will approach from a different direction, and we will construct $q$-polynomials $L$ such that $L(x)/x$ is not irreducible. One can hope for more exotic Galois groups in such cases, which is indeed the purpose of our approach.

Given any polynomial $f \in F[x]$, it is useful to find a $q$-polynomial in $F[x]$ that is divisible by $f$. The following Lemma is well known, and can be found for example in Kedlaya [5]. We give two proofs here.

**Lemma 4.** Let $F$ be a field of characteristic $p$ that contains $\mathbb{F}_q$. For any polynomial
Let $f \in F[x]$ of degree $m$, there exists $L \in F[x]$ which is divisible by $f$ and has the form

$$L(x) = \sum_{i=0}^{d} b_i x^{q^i}$$

where $d \leq m$.

**Proof.** First proof. Consider the $m + 1$ elements $x^{q^i} \mod f$, $0 \leq i \leq m$, of the $m$-dimensional $F$-vector space $F[x]/(f)$. There must be a nontrivial linear dependence relation between these elements, say

$$\sum_{i=0}^{m} b_i (x^{q^i} \mod f) = 0.$$  

This implies $(\sum_{i=0}^{m} b_i x^{q^i}) \mod f = 0$, and therefore $L(x) = \sum_{i=0}^{m} b_i x^{q^i}$ is the required polynomial.

Second proof (this is essentially the proof in [5]). Let $E$ be a splitting field for $f$ and let $\beta_1, \ldots, \beta_m$ be the roots of $f$ in $E$, which are distinct by hypothesis. Let $V$ be the $F_q$-span of $\beta_1, \ldots, \beta_m$. The dimension of $V$ is at most $m$. By Lemma 2 the polynomial $\prod_{v \in V} (x - v)$ is a $q$-linearized polynomial, and it will clearly be divisible by $f$. This polynomial has coefficients in $F$ because the coefficients are fixed by the Galois group. 

There may be a linear relation of smaller degree, which would result in an additive polynomial $L$ of smaller $q$-degree than $m$. Abhyankar and Yie [11] say that $f$ linearizes at $d$ if there exists a $q$-linearized polynomial $L$ of $q$-degree $d$ such that $f$ divides $L$. Lemma 4 says that $f$ will linearize at an integer $d$ with $d \leq m$. If $f$ linearizes at $d$ where $d$ is significantly smaller than $m$, then useful information can be obtained about the Galois group of $f$, because the Galois group of $f$ is (almost always) equal to the Galois group of $L$, which is isomorphic to a subgroup of $GL(d, q)$. They refer to this as Serre’s linearization trick.

Our next lemma focuses on the smallest $d$ such that $f$ linearizes at $d$.

**Lemma 5.** Let $F$ be a field of characteristic $p$ that contains $\mathbb{F}_q$. The minimal $q$-degree of a $q$-linearized polynomial $L \in F[x]$ which is divisible by $f$ is equal to the dimension of the $\mathbb{F}_q$-span of the roots of $f$ in a splitting field.
Proof. Let \( f \) be a polynomial of degree \( m \), with coefficients in the field \( F \). Let \( d \) be the smallest positive integer such that there exists a \( q \)-linearized polynomial \( L \in F[x] \) of \( q \)-degree \( d \) which is divisible by \( f \). By Lemma 4, \( d \leq m \).

Let \( E \) be a splitting field for \( f \). The set of all \( F_\mathbb{F}_q \)-linear combinations of the roots of \( f \) is an \( F_\mathbb{F}_q \)-vector space \( V \subseteq E \). Let \( L' \) be the monic polynomial whose roots are precisely the elements of \( V \). Then \( L' \) is a linearized polynomial by Lemma 2 and \( f \) divides \( L' \) because its roots are a subset of the roots of \( L' \). Let \( d' \) be the \( q \)-degree of \( L' \). Then \( d' \geq d \) by definition of \( d \). If \( d' > d \) then there would exist a linearized polynomial \( L \) of degree \( d' \) that contains the roots of \( f \) among its roots. Since \( L \) is linearized, it also contains every linear combination of the roots of \( f \) among its roots. Then \( L' \) would divide \( L \), which contradicts \( d' > d \). Therefore \( d' = d \).

Example: (from [2]) Consider the polynomial \( f = x^{24} + x + t \) over \( \mathbb{F}_2(t) \). Using Magma we find that \( f \) linearizes at 12 and we find the following linearized polynomial divisible by \( f \):

\[
L(x) = x^{4096} + (t^{24} + t)x^{2048} + t^{128}x^{1024} + (t^{88} + t^{65})x^{512} + t^{16}x^{32} + \]
\[
t^9x^{16} + (t^{40} + t^{17})x^8 + x^2 + (t^{24} + t)x.
\]

The degrees of the irreducible factors of \( L \) are 1, 24, 276, 1771, 2024.

Remark: An affine polynomial is a polynomial of the form \( L(x) + c \) where \( L \) is a linearized polynomial and \( c \in F \). It is possible that \( f \) divides an affine polynomial of smaller \( q \)-degree than the minimal positive integer at which \( f \) linearizes. In the example above, \( f \) minimally linearizes at 12, however there exists an affine polynomial of 2-degree 11 which is divisible by \( f \). It is the product of \( f \) and the degree 2024 irreducible factor.

4 Constructions related to the Space of Roots

We continue with the notation and themes of the previous section. Let \( L \) be a \( q \)-polynomial of \( q \)-degree \( n \) in \( F[x] \). We assume that \( L \) has \( q^n \) different roots. These form the space of roots \( V \), which is an \( n \)-dimensional vector space over \( \mathbb{F}_q \). The Galois group \( G \) of \( L \) acts \( \mathbb{F}_q \)-linearly on \( V \), under the assumption that \( F \) contains \( \mathbb{F}_q \).
Associated to the vector space $V$ are such vector spaces as the symmetric powers of $V$, which we wish to consider in this section. Let $x_1, \ldots, x_n$ be $n$ algebraically independent indeterminates over $\mathbb{F}_q$ and let $A(\mathbb{F}_q) = \mathbb{F}_q[x_1, \ldots, x_n]$ be the ring of polynomials in the $x_i$. As is well known, the general linear group $GL(n, q)$ acts on $A(\mathbb{F}_q)$, and $A(\mathbb{F}_q)$ is an $\mathbb{F}_q$-module for the group. We may explain this idea as follows.

We identify $GL(n, q)$ with the group of invertible $n \times n$ matrices over $\mathbb{F}_q$. Given an element $g$ in $GL(n, q)$, write $g$ as an $n \times n$ matrix $\left( \begin{array}{cccc} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{array} \right)$, $1 \leq i, j \leq n$. Then we set

$$gx_i = a_{i1}x_1 + \cdots + a_{in}x_n$$

for $1 \leq i \leq n$. This linear action is extended to powers of the $x_i$, so that, for example, $g$ sends $x_i^r$ to $(gx_i)^r$ for each positive integer $r$. The action is extended to monomials in the $x_i$ in the obvious way. Finally we extend the action to polynomials: given $P \in A(\mathbb{F}_q)$ and $g$ in $GL(n, q)$, we define $P^g$ by

$$P^g(x_1, \ldots, x_n) = P(gx_1, \ldots, gx_n).$$

Clearly, $P^g$ is a polynomial of the same degree as $P$.

**Definition** For each positive integer $r$, let $H_{n,r}(\mathbb{F}_q)$ denote the subspace of $A(\mathbb{F}_q)$ consisting of all homogeneous polynomials of degree $r$ in the $x_i$, along with the zero polynomial.

As is well known, $H_{n,r}(\mathbb{F}_q)$ is a $GL(n, q)$-submodule of $A$ (in other words, $GL(n, q)$ maps the space of homogeneous polynomials of degree $r$ into itself). Of course, these concepts hold for any field, not just $\mathbb{F}_q$, and we will work with corresponding spaces of polynomials defined over $F$ in the next section. We note the dimension formula

$$\dim_{\mathbb{F}_q} H_{n,r}(\mathbb{F}_q) = \binom{n+r-1}{r},$$

where the expression on the right is the binomial coefficient.

As before, let $E$ denote a splitting field for $L$ over $F$. Let $v_1, \ldots, v_n \in E$ be an $\mathbb{F}_q$-basis for $V$. Consider the mapping $\epsilon_r : H_{n,r}(\mathbb{F}_q) \longrightarrow E$ defined by

$$\epsilon_r P(x_1, \ldots, x_n) = P(v_1, \ldots, v_n).$$

Thus $\epsilon_r$ evaluates a homogeneous polynomial of degree $r$ on the basis of $V$ and hence determines an element of $E$. Note that $\epsilon_r$ is an $\mathbb{F}_q$-linear mapping. Let $\epsilon_r(V)$ denote
the $\mathbb{F}_q$-subspace of $E$ spanned by the image of $\epsilon_r$. As we shall show in the next lemma, this subspace is independent of the choice of basis and thus the notation $\epsilon_r(V)$ makes sense.

The following lemma is surely well known but we include a proof for definiteness.

**Lemma 6.** Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be two ordered bases of $V$ and let $\epsilon_r$ and $\epsilon'_r$ be the corresponding maps from $H_{n,r}(\mathbb{F}_q)$ to $E$ determined by the bases. Then the $\mathbb{F}_q$-subspaces of $E$ spanned by the images of $\epsilon_r$ and $\epsilon'_r$ are identical. Thus we can speak unambiguously of the subspace $\epsilon_r(V)$ contained in $E$.

**Proof.** As we have two $\mathbb{F}_q$-bases of $V$, there is a unique $g \in GL(n, q)$ with $gv_i = w_i$ for $1 \leq i \leq n$. Given $P \in H_{n,r}(\mathbb{F}_q)$, we gave

$$P^g(v_1, \ldots, v_n) = P(w_1, \ldots, w_n).$$

Since $g$ is invertible, $P^g$ runs over $H_{n,r}(\mathbb{F}_q)$ as $P$ runs over $H_{n,r}(\mathbb{F}_q)$. It follows that the images of $\epsilon_r$ and $\epsilon'_r$ span the same subspace of $E$. \qed

The question we wish to raise is the following. Given a $q$-polynomial $L$ and space of roots $V$ of $L$, for what values of $r$ is $\epsilon_r$ injective? Because, if $\epsilon_r$ is injective, then we can identify $\epsilon_r(V)$ as the $r$-th symmetric power of $V$, where we recall that the $r$-th symmetric power of a vector space $V$ is the subspace of the symmetric algebra of $V$ consisting of all degree $r$ elements under the tensor product. We will show later by a trivial argument (Theorem 14) that if $F$ is the finite field $\mathbb{F}_q$, $\epsilon_r$ is not injective if $r \geq q + 1$ but it is injective if $r < q$. We note on the other hand that if $F$ is infinite, the possibility exists that $\epsilon_r$ is injective for all $r$. We shall see some examples of this.

## 5 Projective Polynomials and Symmetric Powers

We recall the setup from earlier sections, except that we will assume $q = p$ in this section.

Let $F$ be a field of prime characteristic $p$. Let $L$ be a $p$-polynomial of $p$-degree $n$ in $F[x]$. Let $E$ be a splitting field for $L$ over $F$ and let $V_L$ be the set of roots of $L$ in $E$. Let $G$ be the Galois group of $E$ over $F$. We assume that the roots of $L$ are distinct. Let $M(x) = L(x)/x$. Then it is easy to see that $M(x)$ is a monic polynomial in
\[x^{p-1}, \text{ say } M(x) = P(x^{p-1}). \text{ The polynomial } P(x) \text{ is monic of degree } (p^n - 1)/(p - 1) \text{ and is called the projective polynomial associated to } L.\]

**Lemma 7.** Let \( L_P \) be the linearized polynomial of smallest degree that is divisible by \( P \), as in Lemma 2. Let \( \alpha_1, \ldots, \alpha_n \) be a basis for \( V_L \). Then the elements \( \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n} \) where \( k_1 + \cdots + k_n = p - 1 \) (and the \( k_j \) are nonnegative integers) are roots of \( L_P \).

**Proof.** Given notation above, let \( \alpha_1, \ldots, \alpha_n \) be the roots of \( L \) in \( E \). We know that if \( \beta \) is any root of \( L \) then \( \beta^{p-1} \) is a root of \( P \), so the elements \( (i_1 \alpha_1 + i_2 \alpha_2 + \cdots + i_n \alpha_n)^{p-1} \) are roots of \( P \), for any \( i_1, \ldots, i_n \in \mathbb{F}_p \). We claim that each \( \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n} \) where \( k_1 + \cdots + k_n = p - 1 \) (and the \( k_j \) are nonnegative integers) can be written as an \( \mathbb{F}_p \)-linear combination of the elements \( (i_1 \alpha_1 + i_2 \alpha_2 + \cdots + i_n \alpha_n)^{p-1} \), and therefore they are roots of \( L_P \).

By the Multinomial Theorem we write
\[
(i_1 \alpha_1 + i_2 \alpha_2 + \cdots + i_n \alpha_n)^{p-1} = \sum_{k_1+\cdots+k_n=p-1} \binom{p-1}{k_1, \ldots, k_n} i_1^{k_1} \cdots i_n^{k_n} \alpha_1^{k_1} \cdots \alpha_n^{k_n}
\]
which gives us a system of linear equations
\[
\begin{pmatrix}
\vdots \\
(i_1 \alpha_1 + \cdots + i_n \alpha_n)^{p-1} \\
\vdots
\end{pmatrix} = \begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix} \begin{pmatrix}
\alpha_1^{k_1} \\
\vdots \\
\alpha_n^{k_n}
\end{pmatrix}
\]
where the coefficient matrix has rows labelled by \( (i_1, \ldots, i_n) \) and columns labelled \( (k_1, \ldots, k_n) \). In the column labelled \( (k_1, \ldots, k_n) \) every term contains \( \binom{p-1}{k_1, \ldots, k_n} \) so this nonzero factor may be taken out. The remaining matrix is
\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix} \begin{pmatrix}
\alpha_1^{k_1} \\
\vdots \\
\alpha_n^{k_n}
\end{pmatrix}
\]
which is a non-square van der Monde matrix, having \( p^n \) rows and \( \binom{n+p-2}{p-1} \) columns. Removing \( p^n - \binom{n+p-2}{p-1} \) rows at the bottom leaves a square van der Monde matrix, which is invertible, and this proves the claim. \( \square \)

We remark that the same argument works when \( p - 1 \) is replaced by any divisor \( r \) of \( p - 1 \), i.e., where \( P \) is defined by \( L(x)/x = P(x^r) \). An important point is that the
multinomial coefficients \( \binom{r}{k_1, \ldots, k_n} \) are nonzero when \( 1 \leq r \leq p - 1 \). This is one reason why we assume that \( q = p \); extending this lemma from \( p \) to \( q = p^a \) is not immediate. A generalization seems to be possible, it will involve the \( p \)-adic representation of divisors of \( q - 1 \), and twisted tensor products using a Frobenius action.

**Corollary 8.** Let \( F \) be a field of prime characteristic \( p \). Let \( L \) be a \( p \)-polynomial of \( p \)-degree \( n \) in \( F[x] \) with no repeated roots. Let \( E \) be a splitting field for \( L \) over \( F \) and let \( V_L \) be the set of roots of \( L \) in \( E \). Let \( r \) be a divisor of \( p - 1 \). Let \( P(x) \) be defined by \( L(x)/x = P(x') \). Let \( L_P \) be the linearized polynomial of smallest degree that is divisible by \( P \), as in Lemma 5. Then the space of roots of \( L_P \) is a homomorphic image of the \( r \)-th symmetric power of \( V_L \).

**Proof.** From Lemma 7 the evaluation map \( \epsilon_r \) maps surjectively, but perhaps not injectively, into the space of roots of \( L_P \).

**Example:** let us choose a linearized polynomial \( L \) of \( p \)-degree 2 such that the necessary hypotheses hold. Let \( V_L \) be the space of roots of \( L \), with basis \( \alpha \) and \( \beta \). The \( (p - 1) \)-th symmetric power of \( V_L \) has dimension \( p \). By Lemma 7 the space of roots of \( L_P \), which is spanned by \( \alpha^i \beta^{p-1-i} \), \( 0 \leq i \leq p - 1 \), is a homomorphic image of the \( (p - 1) \)-th symmetric power of \( V_L \).

In any case that \( \epsilon_r \) is injective, we may conclude that the space of roots of \( L_P \) is isomorphic to the \( r \)-th symmetric power of \( V_L \). For this reason, the next sections investigate the injectivity of \( \epsilon_r \).

### 6 Representation Theory and Modules

In this section we discuss one hypothesis that will guarantee the injectivity of \( \epsilon_r \).

Recall the notation of the evaluation mapping \( \epsilon_r \) from the space \( H_{n,r}(\mathbb{F}_q) \) of homogeneous polynomials of degree \( r \) into the splitting field \( E \) of some \( q \)-polynomial \( L \) of \( q \)-degree \( n \) in \( F[x] \). Similarly we let \( \eta_r \) denote the evaluation mapping from \( H_{n,r}(F) \longrightarrow E \).

Suppose that \( L \) has no repeated roots and \( G \) is the Galois group of \( E \) over \( F \). Then \( G \) acts on the \( \mathbb{F}_q \)-space \( H_{n,r}(\mathbb{F}_q) \) and on the \( F \)-space \( H_{n,r}(F) \). (We only use the
fact that $G$ acts linearly on $V$ and on $F$.) It is easy to see that $\ker \epsilon_r$ is an $F_q G$-submodule of $H_{n,r}(F_q)$ and similarly, $\ker \eta_r$ is an $FG$-submodule of $H_{n,r}(F)$. If we know enough about the actions of $G$ on the two spaces of polynomials, we may be able to deduce something about the respective kernels. In particular, if $H_{n,r}(F_q)$ is an irreducible module, then $\ker \epsilon_r$ is trivial. This is one way to guarantee an injective $\epsilon_r$, the irreducibility of the module. We state this as a Corollary.

**Corollary 9.** Let $F$ be a field of prime characteristic $p$. Let $L$ be a $p$-polynomial of $p$-degree $n$ in $F[x]$ with no repeated roots. Let $E$ be a splitting field for $L$ over $F$ and let $V_L$ be the set of roots of $L$ in $E$. Let $G$ be the Galois group of $E$ over $F$. Let $r$ be a divisor of $p - 1$. Suppose that the $r$-th symmetric power of $V_L$ is an irreducible $FG$-module. Let $P(x)$ be defined by $L(x)/x = P(x^r)$. Let $L_P$ be the linearized polynomial of smallest degree that is divisible by $P$, as in Lemma 5. Then the $r$-th symmetric power of $V_L$ is isomorphic to the space of roots of $L_P$.

**Proof.** Since the kernel of $\epsilon_r$ is an $FG$-submodule of the $r$-th symmetric power of $V_L$, and this module is irreducible, the Corollary follows from Corollary 8.

In the cases that the Galois group is $GL(n, q)$ or $SL(n, q)$, it follows from a theorem of Doty [4] that $H_{n,r}(F_q)$ is an irreducible module for $1 \leq r \leq p - 1$, or equivalently, that the $r$-th symmetric power is an irreducible module for $1 \leq r \leq p - 1$.

As a sample application of this Corollary, we take the example of a $p$-linearized polynomial over $F_p(t)$ of $p$-degree $n$ with Galois group $GL(n, p)$. This is the generic case, and an explicit example is $L(x) = x^n + x^p + tx$ (due to Abhyankar). Then $V_L$ has dimension $n$ over $F_p$. The $(p - 1)$-th symmetric power of $V_L$ has dimension $\binom{n+p-2}{p-1}$. We let $L_P$ be the linearized polynomial of smallest degree that is divisible by $P$, as in Lemma 5. By Doty’s theorem we know that the $(p - 1)$-th symmetric power of $V_L$ is an irreducible $GL(n, p)$-module. By Corollary 9 the $(p - 1)$-th symmetric power of $V_L$ is isomorphic to the space of roots of $L_P$.

As an example, let us take $n = 2$ and $F = F_p(t)$. Choose a linearized polynomial $L$ of $p$-degree 2 having Galois group $GL = SL(2, p)$ or $GL(2, p)$. Let $V_L$ be the natural 2-dimensional module for $G_L$, i.e., the space of roots of $L$. Let $\alpha$ and $\beta$ be a basis. The $(p - 1)$-th symmetric power of $V_L$ has dimension $p$. By the theorem of Doty, this is an irreducible module for $G_L$. By Lemma 7 and Theorem 20, we can realize

\footnote{using a theorem of Steinberg, which states that an irreducible module over the algebraic closure of $F_q$ remains irreducible upon restriction to $F_q$}
this module inside the splitting field of $L$ as the space of roots of $L_P$, which has a basis $\alpha^i\beta^{p-1-i}$, $0 \leq i \leq p - 1$.

7 Injectivity

We note the following simple principle relating to non-injectivity of $\epsilon_r$.

**Lemma 10.** Let $L$ be a $q$-polynomial in $F[x]$ and suppose that $L$ has no repeated roots in its splitting field $E$ over $F$. Suppose that $\epsilon_r : H_{n, r}(F_q) \rightarrow E$ is not injective. Then $\epsilon_t : H_{n, t}(F_q) \rightarrow E$ is not injective for all $t \geq r$.

*Proof.* Suppose that the homogeneous polynomial $P$ of degree $r$ vanishes when evaluated on a given basis of the space of roots. Then if we set $Q = x_{t-r}P$, $Q$ is homogeneous of degree $t$ and it also vanishes on the basis. \hfill $\square$

It seems reasonable when investigating the question posed above to restrict attention to the case that $L(x)/x$ is irreducible in $F[x]$. Note then that this hypothesis ensures that $L$ has no repeated roots. We begin by examining what must be the easiest case, when $F = F_q$.

7.1 Finite Fields

**Lemma 11.** Let $L$ be a $q$-polynomial in $F_q[x]$ of $q$-degree $n$, such that $L(x)/x$ is irreducible in $F_q[x]$. Let $\alpha \neq 0$ be a root of $L$ in a splitting field over $F_q$. Then $\alpha$, $\alpha^q$, $\ldots$, $\alpha^{q^{n-1}}$ are linearly independent over $F_q$ and are a basis for the space of roots of $L$.

*Proof.* We note that $\alpha^{q^i}$ is a root of $L$ for all $i$, because $L$ has coefficients in $F_q$. Suppose that we have a linear dependence relation

$$
\lambda_0 \alpha + \lambda_1 \alpha^q + \cdots + \lambda_{n-1} \alpha^{q^{n-1}},
$$

where the $\lambda_i$ are in $F_q$. Then, unless all the $\lambda_i$ are zero, $\alpha$ is the root of a nonzero polynomial of degree at most $q^{n-1}$ over $F_q$, and this contradicts the assumption that
the minimal polynomial of $\alpha$ has degree $q^n - 1$. It follows that the $\lambda_i$ are all zero and we have proved that the $n$ powers of $\alpha$ are a basis of the space of roots.

The use of this special basis of $V$ enables us to prove that $\epsilon_r$ is a monomorphism for many values of $r$.

**Theorem 12.** Let $L$ be a $q$-polynomial in $\mathbb{F}_q[x]$ of $q$-degree $n$, such that $L(x)/x$ is irreducible in $\mathbb{F}_q[x]$. Let $V$ be the space of roots of $L$ in a splitting field $E$ of $L$ over $\mathbb{F}_q$ ($E$ is isomorphic to $\mathbb{F}_{q^d}$, where $d = q^n - 1$). Then $\epsilon_r : H_{n,r}(\mathbb{F}_q) \rightarrow E$ is injective for $1 \leq r \leq q - 1$.

**Proof.** Let $\alpha$ be a nonzero root of $L$ in $E$. We use the basis of $V$ described in Lemma 11 to study $\epsilon_r$. We find that $\epsilon_r(V)$ consists of $\mathbb{F}_q$-linear combinations of powers $\alpha^t$, where we have

$$t = r_0 + r_1 q + \cdots + r_{n-1} q^{n-1},$$

and the $r_i$ are nonnegative integers that satisfy $r_0 + r_1 + \cdots + r_{n-1} = r$.

Now given the hypothesis that $r \leq q - 1$, we have

$$r_0 + r_1 q + \cdots + r_{n-1} q^{n-1} \leq (q - 1)q^{n-1} < q^n - 1.$$

Thus the existence of any nontrivial dependence relation among the powers of $\alpha$ occurring in $\epsilon_r(V)$ implies that $\alpha$ is a root of a nonzero polynomial in $\mathbb{F}_q[x]$ of degree less than $q^n - 1$. This is a contradiction and we have established the desired result. \hfill \Box

As we shall see later, it easy to show that $\epsilon_q$ is also injective but $\epsilon_{q+1}$ is not. The interest of restricting to values of $r$ at most $q - 1$ is suggested by our next result, where we give a more precise description of the way in which the subspaces $\epsilon_r(V)$ are embedded in $E$.

**Corollary 13.** Using the notation and hypotheses of Theorem 12, and taking $\epsilon_0(V)$ to be the one-dimensional subspace spanned by 1, the splitting field $E$ contains the direct sum

$$\epsilon_0(V) \oplus \epsilon_1(V) \oplus \cdots \oplus \epsilon_{q-1}(V)$$

of dimension

$$\frac{q(q + 1) \cdots (q + n - 1)}{n!}.$$
Proof. We have noted that \( \epsilon_r(V) \) is spanned by powers of \( \alpha \) where the exponents of \( \alpha \) have the form

\[
t = r_0 + r_1q + \cdots + r_{n-1}q^{n-1},
\]

and the \( r_i \) are nonnegative integers that satisfy \( r_0 + r_1 + \cdots + r_{n-1} = r \). The integer \( t \) is expressed as a \( q \)-adic integer, which we will say has weight \( r \). The expression is unique: its representation as a \( q \)-adic integer has exactly one weight. The number of such integers of weight \( r \) is \( \dim H_{n,r}(\mathbb{F}_q) \).

If the sum of the subspaces \( \epsilon_i(V) \) is not direct, we must have a dependence of the form

\[
\lambda_0v_0 + \lambda_1v_1 + \cdots + \lambda_{q-1}v_{q-1},
\]

where the \( \lambda_i \) are in \( \mathbb{F}_q \) and the \( v_i \) in \( \epsilon_i(V) \). Each \( v_i \) is an \( \mathbb{F}_q \)-combination of powers of \( \alpha \), where the exponents have weight \( i \). Thus the dependence involves \( \dim H_{n,0} + \dim H_{n,1} + \cdots + \dim H_{n,q-1} \) different powers of \( \alpha \). Since the powers \( \alpha^i, 0 \leq i < q^n - 1 \), are linearly independent, this is clearly impossible and we deduce that the sum is direct.

Finally, the dimension of the direct sum is

\[
\sum_{i=0}^{q-1} \binom{n+i-1}{i} = \frac{q(q+1) \cdots (q+n-1)}{n!}.
\]

As we remarked earlier, the following result holds in the finite field case.

**Theorem 14.** Assume the hypotheses of Theorem 12 and suppose that \( n \geq 2 \). Then the mapping \( \epsilon_r \) is not injective for \( r \geq q + 1 \).

Proof. We use the basis \( v_1 = \alpha, v_2 = \alpha^q, \ldots, v_n = \alpha^{q^{n-1}} \) of the space of roots of \( L \) and assume first that \( n \geq 3 \). Lemma 10 shows that it suffices to prove the result when \( r = q + 1 \). Consider the homogeneous polynomial

\[
P(x_1, \ldots, x_n) = x_2^{q+1} - x_1^q x_3.
\]

of degree \( q + 1 \). It is clear that when \( P \) is evaluated on the given basis, it vanishes.

Now suppose that \( n = 2 \). In this case, we can assume that \( L = x^{q^2} + ax^q + b \) for suitable \( a \) and \( b \) in \( \mathbb{F}_q \). Consider the polynomial

\[
P(x_1, x_2) = x_2^{q+1} + ax_1^q x_2 + bx_1^{q+1}
\]
in $\mathbb{F}_q[x]$. We have

$$P(\alpha, \alpha^q) = \alpha^{q(q+1)} + a\alpha^{2q} + b\alpha^{q+1} = \alpha^q(\alpha^q + a\alpha + b) = 0.$$ 

Thus $P$ vanishes on the basis and we deduce that $\epsilon_{q+1}$ is not injective in this case also.

We shall describe next one circumstance where we cannot expect to find any analogue of Theorem 12. This relates to the subject of $q^s$-polynomials.

**Theorem 15.** Suppose that $s > 1$ is an integer and the field $\mathbb{F}_{q^s}$ is not contained in $F$. Let $L$ be a $q^s$-polynomial in $F[x]$ with no repeated roots in its splitting field $E$ over $F$. Then the mapping $\epsilon_s$ into $E$ is not injective.

**Proof.** Let $\alpha$ be a nonzero root of $L$ in $E$ and let $\omega$ be an element of $\mathbb{F}_{q^s}$ not in $\mathbb{F}_q$ (recall that $\mathbb{F}_{q^s}$ is contained in $E$ by Lemma 3). Let $\beta = \omega \alpha$. Then $\alpha$ and $\beta$ are roots of $L$ that are linearly independent over $\mathbb{F}_q$.

The powers $1, \alpha, \alpha^2, \ldots, \alpha^s$ are linearly dependent over $\mathbb{F}_q$, since $\mathbb{F}_{q^s}$ has dimension $s$ over $\mathbb{F}_q$. It follows that $\alpha^s$, $\alpha^{s-1} \beta$, $\ldots$, $\beta^s$ are linearly dependent over $\mathbb{F}_q$, say

$$\lambda_0 \alpha^s + \lambda_1 \alpha^{s-1} \beta + \cdots + \lambda_s \beta^s = 0,$$

where the $\lambda_i \in \mathbb{F}_q$ and are not all zero.

Consider the homogeneous polynomial $P$ of degree $s$ in the variables $x_1, \ldots, x_n$ (where $n$ is the $q$-degree of $L$) given by

$$P(x_1, \ldots, x_n) = \lambda_0 x_1^s + \lambda_1 x_1^{s-1} x_2 + \cdots + \lambda_s x_2^s,$$

all other terms being 0. We extend the linear independent roots $\alpha$ and $\beta$ to a basis $v_1 = \alpha$, $v_2 = \beta$, $\ldots$, $v_n$ of the space of roots of $L$. Then we find that

$$\epsilon_s P(x_1, \ldots, x_n) = P(v_1, \ldots, v_n) = 0.$$

Since $P$ is a nonzero polynomial in $H_{n,s}(\mathbb{F}_q)$ that is in the kernel of $\epsilon_s$, we see that $\epsilon_s$ is not injective.

It may be observed that in the previous theorem, the space of roots of $L$ is more naturally a vector space over $\mathbb{F}_{q^s}$ than one over $\mathbb{F}_q$, and we might expect it to be
easier to obtain dependencies when we work over $\mathbb{F}_q$, rather than $\mathbb{F}_{q^s}$. Nonetheless, we feel that the theorem serves to indicate that caution is necessary when we try to generalize Theorem 12 to other situations.

In this connection the following theorem related to Theorem 15 is of interest, although the proof we use is unrelated to the ideas developed so far.

**Theorem 16.** Let $L$ be a $q$-polynomial in $\mathbb{F}_q[x]$ such that $L(x)/x$ is irreducible over $\mathbb{F}_q$. Then $L$ is not a $q^s$-polynomial for any $s > 1$.

**Proof.** Let the $q$-degree of $L$ be $n$ and let

$$L = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x,$$

where $a_n \neq 0$. The polynomial $\ell(x)$ in $\mathbb{F}_q[x]$ defined by

$$\ell(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

is called the conventional $q$-associate of $L$.

Given that $L(x)/x$ is irreducible, Theorem 3.63 of [7] implies that $\ell(x)$ is irreducible and any root $\alpha$ of $\ell(x)$ in $\mathbb{F}_{q^n}$ has multiplicative order $q^n - 1$. Suppose now that $L$ is a $q^s$-polynomial, where $s > 1$. Then $n = ms$ for some integer $m$ and $\ell$ must be a polynomial in $x^s$, say $\ell(x) = g(x^s)$ where $g$ has degree $m$. Now we have $\ell(\alpha) = 0 = g(\alpha^s)$. This shows that $\alpha^s$ is a root of $g$.

Since $\ell$ is irreducible of degree $m$, its roots have order dividing $q^m - 1$. Thus $\alpha^{s(q^m - 1)} = 1$. On the other hand, Theorem 3.63 of [7] shows that $\alpha$ has order $q^{ms} - 1$. Thus $q^{ms} - 1$ divides $s(q^m - 1)$. This is clearly impossible if $s > 1$. Thus $L$ is not a $q^s$-polynomial.

This theorem only applies to finite fields.

### 7.2 Arbitrary Fields, Dimension 2

We now move to the case where the field of coefficients $F$ is an arbitrary field of characteristic $p$. We wish to examine the case of two linearly independent roots $\alpha$ and $\beta$ of a $q$-polynomial. We present two theorems, one concerning linear dependence
of $\alpha^d$, $\alpha^{d-1}\beta$, $\ldots$, $\beta^d$ over $\mathbb{F}_q$, and the other concerning linear dependence over $F$. These results will be applied in the next section to give us more special cases in our investigations of the injectivity of the evaluation maps.

**Theorem 17.** Let $L$ be a monic $q$-polynomial of $q$-degree $n$ in $F[x]$ with the following properties:

1. $L(x)/x$ is irreducible over $F$.
2. $L$ is not a $q^s$-polynomial for any integer $s > 1$.
3. $L$ has the form
   \[ x^{q^n} + a_{n-k}x^{q^{n-k}} + \cdots + a_0x, \]
   where $a_{n-k} \neq 0$, i.e. either $k = 1$ and $a_{n-1} \neq 0$, or $k > 1$ and $a_{n-1} = \cdots = a_{n-k+1} = 0$.
4. $V$ is the space of roots of $L$ in a splitting field $E$ of $L$ over $F$.

Let $\alpha$ and $\beta$ be elements of $V$ linearly independent over $\mathbb{F}_q$. Suppose that there exists a positive integer $d$ such that $\alpha^d$, $\alpha^{d-1}\beta$, $\ldots$, $\beta^d$ are linearly dependent over $\mathbb{F}_q$ (such $d$ may not exist). Then $d \geq q^k + 1$.

**Proof.** Let $m$ be the smallest such $d$ and let $\lambda_0$, $\ldots$, $\lambda_m$ be elements of $\mathbb{F}_q$, not all 0, with

\[ \lambda_0\alpha^m + \lambda_1\alpha^{m-1}\beta + \cdots + \lambda_m\beta^m = 0. \]

We claim that $\lambda_0 \neq 0$. For if $\lambda_0 = 0$, we may divide the resulting equality by $\beta$ to obtain

\[ \lambda_1\alpha^{m-1} + \cdots + \lambda_m\beta^{m-1} = 0, \]

and this contradicts the minimality of $m$. Thus $\lambda_0 \neq 0$ and likewise $\lambda_m \neq 0$. Note also that $m > 1$, since we are assuming that $\alpha$ and $\beta$ are linearly independent over $\mathbb{F}_q$.

We now see that if we set $\gamma = \alpha/\beta$, $\gamma$ is a root of a polynomial of degree $m$ over $\mathbb{F}_q$. The minimality of $m$ again shows that this polynomial is irreducible over $\mathbb{F}_q$. We deduce that $\gamma \in \mathbb{F}_{q^m}$ and hence $\gamma^{q^m} = \gamma$. 

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Since $\alpha = \gamma/\beta$ and $\beta$ are roots of $L$, we have

$$\beta q^n + a_{n-k} \beta^{q^{n-k}} + \cdots + a_0 \beta = 0$$

and likewise

$$\gamma q^n \beta + \gamma q^{n-k} a_{n-k} \beta^{q^{n-k}} + \cdots + \gamma a_0 \beta = 0.$$ We divide by $\gamma q^n$ and subtract the resulting two monic equations involving powers of $\beta$ to obtain

$$a_{n-k}(\gamma^{(q^{n-k}-q^n)} - 1)\beta^{q^{n-k}} + \cdots + a_0(\gamma^{1-q^n} - 1)\beta = 0.$$ We see that $\beta$ is a root of a $q$-polynomial of $q$-degree at most $n - k$ over the field $F(\gamma)$.

Suppose if possible that this polynomial is zero. It follows that

$$a_{n-i}(\gamma^{(q^{n-i}-q^n)} - 1) = 0$$

for $k \leq i \leq n$. It is certainly true that $a_0 \neq 0$ under our hypothesis that $L(x)/x$ is irreducible. Then the $i = n$ case yields $\gamma q^n = \gamma$. Thus since we know that $\gamma \in \mathbb{F}_{q^m}$ and no smaller field $\mathbb{F}_{q^t}$, we deduce that $m$ divides $n$.

Consider now an equation

$$a_{n-i}(\gamma^{(q^{n-i}-q^n)} - 1) = 0$$

and suppose that $a_{n-i} \neq 0$. Then we have

$$\gamma q^n = \gamma q^{n-i}.$$ The argument just applied above shows that $m$ divides $n - i$. This implies that $L$ is a $q^m$-polynomial, where $m > 1$, contrary to hypothesis. We deduce that $\beta$ is indeed a root of a nonzero $q$-polynomial of $q$-degree at most $n - k$ over $F(\gamma)$.

We obtain the inequality

$$[F(\gamma, \beta) : F(\gamma)] \leq q^{n-k} - 1.$$ Since $\mathbb{F}_q$ is assumed to be a subfield of $F$, and since the minimal polynomial of $\gamma$ over $\mathbb{F}_q$ has degree $m$, we have

$$[F(\gamma) : F] \leq m.$$
Thus we obtain

\[ [F(\gamma, \beta) : F] \leq m(q^n - k - 1). \]

Since \( F(\beta) \) is a subfield of \( F(\gamma, \beta) \), we obtain the inequality

\[ [F(\gamma, \beta) : F] = [F(\gamma, \beta) : F(\beta)][F(\beta) : F] = [F(\gamma, \beta) : F(\beta)](q^n - 1). \]

Hence the inequality

\[ [F(\gamma, \beta) : F(\beta)](q^n - 1) \leq m(q^n - k - 1) \]

holds. We deduce that

\[ m \geq (q^n - 1)/(q^n - k - 1). \]

Since \( m \) is an integer, we obtain that \( m \geq q^k + 1 \). Thus the \( d \) in the statement of the theorem is at least \( q^k + 1 \). \( \square \)

So far we studied the evaluation mapping \( \epsilon_r \) from the space \( H_{n,r}(\mathbb{F}_q) \) of homogeneous polynomials of degree \( r \) into the splitting field \( E \) of some \( q \)-polynomial \( L \) of \( q \)-degree \( n \) in \( F[x] \). It seems reasonable to use the field \( \mathbb{F}_q \) for coefficients because the space of roots of \( L \) is a vector space over \( \mathbb{F}_q \). However, we can also study evaluations when we replace \( H_{n,r}(\mathbb{F}_q) \) by \( H_{n,r}(F) \) but still evaluate on a basis of \( V \). We shall let \( \eta_r \) denote the evaluation mapping from \( H_{n,r}(F) \rightarrow E \). The image of \( \eta_r \) is an \( F \)-subspace of \( E \). Since the dimension of \( H_{n,r}(F) \) over \( F \) increases monotonically as \( r \) increases, whereas \( E \) has finite dimension over \( F \), \( \eta_r \) is not injective for almost all \( r \). This suggests the problem of finding or estimating the smallest \( r \) for which \( \eta_r \) is not injective.

The next Theorem is similar to Theorem 17, but is not the same, because the next Theorem assumes that \( \alpha^d, \alpha^{d-1}\beta, \ldots, \beta^d \) are linearly dependent over \( F \), whereas the previous Theorem assumed linear dependence over \( \mathbb{F}_q \).

**Theorem 18.** Let \( L \) be a monic \( q \)-polynomial of \( q \)-degree \( n \) in \( F[x] \) with the following properties:

1. \( L(x)/x \) is irreducible over \( F \).
2. \( L \) is not a \( q^s \)-polynomial for any integer \( s > 1 \).
3. $L$ has the form

$$x^n + a_{n-k}x^{q^{n-k}} + \cdots + a_0x,$$

where $a_{n-k} \neq 0$ (thus $a_{n-1} = \cdots a_{n-k+1} = 0$ if $k > 1$).

4. $V$ is the space of roots of $L$ in a splitting field $E$ of $L$ over $F$.

Let $\alpha$ and $\beta$ be elements of $V$ linearly independent over $\mathbb{F}_q$. Let $d$ be a positive integer such that $\alpha^d$, $\alpha^{d-1}\beta$, $\ldots$, $\beta^d$ are linearly dependent over $F$ (such $d$ will certainly exist). Then $d \geq q^k + 1$.

**Proof.** The proof is very similar to that of Theorem 17, so we sketch the details.

Let $m$ be the smallest $d$ such that $\alpha^d$, $\alpha^{d-1}\beta$, $\ldots$, $\beta^d$ are linearly dependent over $F$. We set $\gamma = \alpha/\beta$ and show that $\gamma$ is a root of an irreducible polynomial of degree $m$ over $F$, as in the proof of Theorem 17. Then we obtain that

$$a_{n-k}(\gamma^{q^{n-k}-q^n} - 1)\beta^{q^{n-k}} + \cdots + a_0(\gamma^{1-q^n} - 1)\beta = 0.$$ 

This shows that $\beta$ is a root of a $q$-polynomial of $q$-degree at most $n - k$ over the field $F(\gamma)$.

We consider the case that this polynomial is zero. It follows that

$$a_{n-i}(\gamma^{q^{n-i}-q^n} - 1) = 0$$

for $k \leq i \leq n$. It is certainly true that $a_0 \neq 0$ under our hypothesis that $L(x)/x$ is irreducible. We must then have $\gamma^{q^n} = \gamma$. This implies that $\gamma \in \mathbb{F}_{q^n}$. Suppose that $\gamma \in \mathbb{F}_{q^t}$ and no smaller field $\mathbb{F}_{q^s}$, where $t < m$. Then $m$ divides $n$. Note that $m > 1$, since $\alpha$ and $\beta$ are linearly independent over $\mathbb{F}_q$.

Consider now an equation

$$a_{n-i}(\gamma^{q^{n-i}-q^n} - 1) = 0$$

and suppose that $a_{n-i} \neq 0$. Then we have

$$\gamma^{q^n} = \gamma^{q^{n-i}}.$$ 

It must then be the case that $m$ divides $n - i$. This implies that $L$ is a $q^n$-polynomial and hence $m = 1$, which we know not to be the case. It follows that $\beta$ is indeed a root of a nonzero $q$-polynomial of $q$-degree at most $n - k$ over $F(\gamma)$. 

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We obtain the inequality
\[ [F(\gamma, \beta) : F(\gamma)] \leq q^{n-k} - 1. \]

We also have
\[ [F(\gamma) : F] = m, \]
since the minimal polynomial of \( \gamma \) over \( F \) has degree \( m \). The rest of the proof follows as before. \( \square \)

### 7.3 \( q \)-Degree 2

As we shall show, the theorems in the previous section may be applied effectively to investigate the space of roots of a \( q \)-polynomial of \( q \)-degree 2 but we need to make a hypothesis concerning the field \( F \), for without some assumption, the results we have in mind are not necessarily true.

We are always assuming that \( F \) contains \( \mathbb{F}_q \). Our new hypothesis is that the largest finite subfield of \( F \) is \( \mathbb{F}_q \) itself. This means that the only elements of \( F \) that are algebraic over \( \mathbb{F}_q \) are the elements of \( \mathbb{F}_q \), or in other words, \( F \) is a regular extension of \( \mathbb{F}_q \). A standard example of such a regular field \( F \) is the function field \( \mathbb{F}_q(t) \) in a single variable \( t \) over \( \mathbb{F}_q \).

In the regular extension case, the following result is true.

**Lemma 19.** Suppose that the field is a regular extension of \( \mathbb{F}_q \). Let \( f \) be an irreducible polynomial in \( \mathbb{F}_q[x] \). Then considered as a polynomial in \( F[x] \), \( f \) remains irreducible.

A proof may be found in [6], Lemma 4.10, p.366.

We recall here that if \( L \) is a \( q \)-polynomial of \( q \)-degree 2 with no repeated roots, its Galois group is a subgroup of \( GL(2, q) \), by Lemma [1]. The group \( GL(2, q) \) contains a normal subgroup \( SL(2, q) \), the special linear group, consisting of the elements of determinant 1. It is well known that if \( q \) is a power of 2 greater than 2, then \( SL(2, q) \) is a simple group. However, \( SL(2, 2) \) is isomorphic to the symmetric group \( S_3 \) and is anomalous as it has a normal subgroup of index 2.
If \( q \) is odd and greater than 3, \( SL(2, q) \) contains no proper normal subgroups with abelian quotient (in other words, the group is perfect). \( SL(2, 3) \) contains a normal subgroup of order 8 with cyclic quotient of order 3.

We may now proceed to our main theorem relating to \( q \)-polynomials of \( q \)-degree 2.

**Theorem 20.** Let \( L \) be a \( q \)-polynomial of \( q \)-degree 2 in \( F[x] \). Suppose that the following hold.

1. \( L(x)/x \) is irreducible over \( F \).
2. \( q > 2 \).
3. \( F \) is a regular extension of \( \mathbb{F}_q \).
4. \( E \) is the splitting field for \( L \) over \( F \).
5. The Galois group \( G \) of \( E \) over \( F \) contains \( SL(2, q) \).

Then the evaluation mapping \( \epsilon_r \) from the space \( H_{2,r}(\mathbb{F}_q) \) of homogeneous polynomials of degree \( r \) in two variables into \( E \) is injective for all \( r \geq 1 \).

**Proof.** We first show that \( L \) is not a \( q^2 \)-polynomial. For if \( L \) is such a polynomial, the space \( V \) of roots is a one-dimensional vector space over \( \mathbb{F}_{q^2} \) and the Galois group acts semilinearly on the vector space (see Lemma 3). Since the Galois group of \( \mathbb{F}_{q^2} \) over \( \mathbb{F}_q \) has order 2, \( G \) has order dividing \( 2(q^2 - 1) \). Since we are assuming that \( G \) contains \( SL(2, q) \) whose order is \( q(q^2 - 1) \), we must have the inequality \( q(q^2 - 1) \leq 2(q^2 - 1) \) and hence \( q = 2 \), a possibility we have excluded. It follows that \( L \) is not a \( q^2 \)-polynomial under our hypothesis.

Let \( \alpha \) and \( \beta \) be elements of the space of roots of \( L \) that are linearly independent over \( \mathbb{F}_q \). Let \( \epsilon_m \) be the evaluation mapping from \( H_{2,m}(\mathbb{F}_q) \). Suppose that \( \epsilon_m \) is not injective, and \( m \) is chosen minimal with this property. Then \( \alpha^m, \alpha^{m-1}\beta, \ldots, \beta^m \) are linearly dependent over \( \mathbb{F}_q \) and it follows from Theorem 17 that \( m \geq q + 1 \).

We have proved in Theorem 17 that \( \gamma = \alpha/\beta \) is a root of an irreducible polynomial \( f \) of degree \( m \) in \( \mathbb{F}_q[x] \). Since \( F \) is regular over \( \mathbb{F}_q \), Lemma 19 implies that \( f \) is also irreducible over \( F \) and hence \( [F(\gamma) : F] = m \). Now \( F(\gamma) \) is a normal subfield of \( E \), since it is the splitting field for \( f \) over \( F \). The Galois group \( G \) thus maps \( F(\gamma) \) into
itself and induces the Galois group of $F(\gamma)$ over $F$ by its action. The Galois group is cyclic of order $m$, since it is isomorphic to the Galois group of $\mathbb{F}_q(\gamma)$ over $\mathbb{F}_q$. Let $H$ be the subgroup of $G$ that acts trivially on $F(\gamma)$. Elementary Galois theory tells us that $H$ is normal in $G$ and $G/H$ is cyclic of order $m$.

Let $S$ be a subgroup of $G$ isomorphic to $SL(2, q)$. Then $SH$ is a subgroup of $G$ and the quotient

$$SH/H \cong S/S \cap H$$

is a subgroup of $G/H$ and is thus cyclic. Since we remarked before the proof that $SL(2, q)$ is perfect for $q > 3$, we deduce that $S$ is contained in $H$ when $q > 3$.

To finish the proof, assume $q > 3$ and consider $G$ as a subgroup of $GL(2, q)$. Then $|G : H| = m$ is a divisor of $|GL(2, q) : H|$, and since $H$ contains $SL(2, q)$, $m$ is then a divisor of $|GL(2, q) : SL(2, q)| = q - 1$. This is impossible, as we already know that $m \geq q + 1$. In the case that $q = 3$, our assumption is that $G$ is either $SL(2, 3)$ or $GL(2, 3)$. Since neither of these groups has an abelian quotient of order greater than 3, the theorem holds in this case also.

We remark that the theorem also holds in the excluded case $q = 2$ provided that $L$ is not a 4-polynomial.

The next theorem is similar to Theorem 20 however it does not assume that $F$ is a regular extension, and it does not assume that the Galois group contains $SL(2, q)$. The conclusion this time is about injectivity of evaluation maps $\eta_r$ over $F$, which we know will fail when $r$ is sufficiently large, whereas Theorem 20 is about the evaluation maps $\epsilon_r$ over $\mathbb{F}_q$, which can be injective for all $r$.

**Theorem 21.** Let $L$ be a $q$-polynomial of $q$-degree 2 in $F[x]$. Suppose that the following hold.

1. $L(x)/x$ is irreducible over $F$.
2. $L$ is not of the form $x^2 + bx$ for some nonzero $b \in F$.
3. $E$ is the splitting field for $L$ over $F$.

Then the evaluation mapping $\eta_r$ from the space $H_{2r}(F)$ into $E$ is injective if $r < [E : F]/(q - 1)$.
Proof. Let $\alpha$ and $\beta$ be elements of the space of roots of $L$ that are linearly independent over $\mathbb{F}_q$. We claim that $E = F(\alpha, \beta)$. This follows from the fact that $E$ is generated over $F$ by the roots of $L$ and these roots are $\mathbb{F}_q$-combinations of $\alpha$ and $\beta$. Likewise, if we set $\gamma = \alpha/\beta$, we have $E = F(\alpha, \gamma)$.

Let $m$ be the smallest positive integer such that the evaluation mapping $\eta_m$ is not injective. Then, as we showed in the proof of Theorem [18], $[F(\gamma) : F] = m$. Furthermore, the same proof shows that $[F(\alpha, \gamma) : F(\gamma)] \leq q - 1$.

We now assemble the parts to show that

$$[E : F] = [F(\alpha, \gamma) : F] = [F(\alpha, \gamma) : F(\gamma)][F(\gamma) : F] \leq m(q - 1)$$

and this yields the desired inequality for $m$. \qed

Based on the results so far, we make the following conjecture.

**Conjecture 1.** Let $L = L(x)$ be a $q$-polynomial in $F[x]$ of $q$-degree $n$, where $F$ is a field of characteristic $p$ that contains $\mathbb{F}_q$. Assume that $L$ is not a $q^s$-polynomial for $s > 1$. Assume that $L(x)/x$ is irreducible over $F$. If $1 \leq r \leq q - 1$ then $\epsilon_r$ is injective.

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