The orbifold fundamental group of Persson-Noether-Horikawa surfaces.

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This article is dedicated to the memory of Boris Moishezon.

1 Introduction.

Among the minimal surfaces of general type, the Noether surfaces are those for which the Noether inequality $K^2 \geq 2p_g - 4$ is an equality ($K^2$ is the self intersection of a canonical divisor, $p_g$ is the dimension of the space of holomorphic 2-forms).

These surfaces were described by Noether ([No]) and more recently by Horikawa ([Ho]) who proved that if $8 \mid K^2$ then there are two distinct deformation types, namely the Noether-Horikawa surfaces of connected type (for short, N-H surfaces of type C), and those of non connected type (for short, of type N). This notation refers to the fact that, the canonical map being a double covering of a rational ruled surface, for type C the branch locus is connected, whereas for type N it is not connected.

In particular Horikawa proved that the intersection forms are both of the same parity (in fact, both odd) if and only if $16 \mid K^2$.

From M. Freedman’s theorem ([Fr]) follows that if $16 \mid K^2$ type N and type C provide two orientedly homeomorphic compact 4-manifolds.

Horikawa posed the question whether type N and type C provide two orientedly diffeomorphic compact 4-manifolds.

It looked like a natural problem to try to see whether the two differentiable structures could be distinguished by means of the invariants introduced by S. Donaldson in [Do].

In the case of type C we have been able ([Ca]) to calculate the constant Donaldson invariants (corresponding to zero-dimensional moduli spaces) using some singular canonical models of these surfaces with very many singularities, and an approach introduced by P. Kronheimer ([Kr]) for the case of the Kummer surfaces. The number we obtained, namely $2^{2k}$ when $K^2 = 8k$, is the leading term of the Donaldson series (see [K-M]), which was later fully calculated by Fintushel and Stern in the case of N-H surfaces of type C via the technique of
rational blow-downs ($[\mathbb{P}^2\mathbb{S}]$).
The Donaldson series for $N$-H surfaces of type $N$ has not yet, to our knowledge, been calculated; although, after the Seiberg-Witten theory ($[W]$) has been introduced, and after Pidstrigach and Tyurin ($[P-T]$) have announced the equality between Kronheimer-Mrowka and Seiberg-Witten classes, the two series should be equal.

Our original aim was to extend the application of the Kronheimer theory to the case of $N$-H surfaces of type $N$ using a very singular model constructed by Ulf Persson ($[\text{Per}]$), describing its orbifold fundamental group, its representations into $SO(3)$, and then trying to see which of those have virtual dimension zero. In this article we consider the singular $N$-H surfaces of type $N$ with maximal Picard number constructed by Persson, henceforth called Persson-Noether-Horikawa surfaces ($P$-$N$-$H$ for short), and we determine their orbifold fundamental group.

This is our main result:

**Theorem.** The orbifold fundamental group of the $P$-$N$-$H$ surfaces is

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2$$

if $16 \mid K^2$,

$$\mathbb{Z}_4 \oplus \mathbb{Z}_4$$

in the other case where $8 \mid K^2$ but $16$ does not divide $K^2$.

It follows immediately that we have, for $16 \mid K^2$, only six nontrivial classes of orbifold $SO(3)$-representations, and a result which we do not prove here is that we do not get anyone of virtual dimension zero.

This is not surprising in view of ($[P-T]$), since if Kronheimer’s approach would have worked, we would have had only a finite number of constant Donaldson invariants.

On the other hand, the algebro-geometric technique of studying canonical models with many rational double points produces on the smooth model configurations of $(-2)$-projective lines (spheres) whose tubular neighborhood has a unique holomorphic structure and, in particular, a unique compatible $C^\infty$ structure. In this way one produces a decomposition of the 4-manifold in geometric pieces, one of which is the nonsingular part of the singular canonical model.

From this point of view, the calculation of the orbifold fundamental group leads to a better understanding of the differentiable structures of the smooth model. Since our proof is rather involved technically we would like to give a brief geometrical “explanation” of our result.

Persson’s construction starts with a plane nodal cubic $C$ meeting a conic $Q$ at only one point $P$. Moreover, $C$ and $Q$ have two common tangents $L_{-1}$ and $L_1$ which meet in a point $O$ collinear with $P$ and the node of $C$.

Blowing up $O$ we get a $\mathbb{P}^1$-bundle $f': \mathbb{P}F_1 \to \mathbb{P}^1$ with a section $\Sigma_\infty$, a bisection $Q'$ and a 3-section $C'$ ($'$ denoting the proper transform under the blow up). A cyclic cover of order $2k+2$ branched on $L'_{-1}$ and $L'_1$ yields a new $\mathbb{P}^1$-bundle
f'' : \mathbb{P}^{2k+2} \rightarrow \mathbb{P}^1 \text{ with a section } \Sigma'' \text{ disjoint from a 3-section } C'' \text{ and two sections } Q''_1, Q''_2 \text{ (the inverse image } Q'' \text{ splits into two components).} 

The curve \( B = C'' \cup Q''_1 \cup Q''_2 \cup \Sigma'' \) has many singular points, and our canonical model \( X_{2k+2} \) is the double cover of \( \mathbb{P}^{2k+2} \) branched on \( B \). By construction \( X_{2k+2} \) has a genus 2 fibration onto \( \mathbb{P}^1 \), whence the orbifold fundamental group \( \pi_1(X_{2k+2}) \), \( X_{2k+2} \) being the nonsingular part of \( X_{2k+2} \), is a quotient of \( \pi_1(F) \), where \( F \) is a fixed genus 2 fibre.

\( F \) being a double cover of \( \mathbb{P}^1 \) branched in six points \( P_0 = \mathbb{P}^1 \cap \Sigma'' \), \( P_1 = \mathbb{P}^1 \cap Q''_1 \), \( P_2 = \mathbb{P}^1 \cap Q''_2 \), \( \{ P_3, P_4, P_5 \} = \mathbb{P}^1 \cap C'' \), \( \pi_1(F) \) is the subgroup of a free product \( \mathcal{F}_5(2) \) of five copies of \( \mathbb{Z}_2 \), given by words of even length.

\( \mathcal{F}_5(2) \) is generated by elements \( \varepsilon_1, \ldots, \varepsilon_6 \) such that \( \varepsilon_1 \cdots \varepsilon_6 = 1 \) (\( \varepsilon_i \) corresponds to a loop in \( \mathbb{P}^1 \) around the point \( P_{i-1} \)).

The first main point (we must be rather vague here, else we must give the full proof) is that, since curve \( C'' \) is irreducible, when the fibre \( F \) moves around, \( \varepsilon_1, \varepsilon_5, \varepsilon_6 \) become identified.

Thus we only have \( \varepsilon_1, \ldots, \varepsilon_4 \) with \( \varepsilon_1 \cdots \varepsilon_4 = 1 \), and therefore we have "proved" that our group is abelian, being a quotient of the fundamental group \( \Gamma \) of a curve of genus 1 obtained as the double cover of \( \mathbb{P}^1 \) branched in four points. More precisely, \( \Gamma \) is an abelian group with generators \( \varepsilon_1 \varepsilon_2, \varepsilon_1 \varepsilon_3 \).

We must still take into account the fact that, when the fibre \( F \) moves towards a singular point (corresponding to points of intersection \( C'' \cap Q''_1, C'' \cap Q''_2, Q''_1 \cap Q''_2 \)), further relations are introduced. These relations are hard to control globally but if we look locally around these points of intersection, and accordingly take a new basis \( \varepsilon'_1, \ldots, \varepsilon'_4 \), the situation becomes simpler.

In fact, the local equation of the double cover is \( z^2 = y^2 - x^{2c} \), where \( c = 6 \) or \( c = k + 1 \), and \( x \) is the pullback of a local coordinate on \( \mathbb{P}^1 \), so that the corresponding local braid yields the relation \( (\varepsilon'_j \varepsilon'_i)^c = (\varepsilon'_i \varepsilon'_j)^c \). In turn, using \( (\varepsilon'_i)^2 = 1 \), we obtain the relation \( (\varepsilon'_i \varepsilon'_j)^{2c} = 1 \).

That's how one shows that the two generators of the abelian group have period 2 or 4.

The paper is organized as follows:

In section two we take up Persson's construction using explicit equations showing that the surface is defined over a real quadratic field.

In the third section we describe the five steps leading to a presentation of our fundamental group in terms of the braid monodromy of the plane curve \( D = C \cup Q \).

Finally, in section four we apply combinatorial group theory arguments in order to give the main result concerning the orbifold fundamental group.

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2 Persson’s configuration.

In this section we will provide explicit equations for the configuration constructed by Ulf Persson in [Per]. This is the configuration formed by a smooth conic \( Q \) and a nodal cubic \( C \) intersecting in only one point \( P \) which is smooth for \( C \). Moreover \( Q \) and \( C \) have two common tangents \( L_1 \) and \( L_{-1} \) meeting in a point \( O \) lying on the line joining \( P \) and the node of \( C \).

Let \( Q \subset \mathbb{C} \mathbb{P}^2 \) be the conic \( \{ (x, y, z) \in \mathbb{C} \mathbb{P}^2 \mid x^2 + 2zy + z^2 = 0 \} \).

Since
\[
 x^2 + 2zy + z^2 = (x+z)^2 + 2z(y-x) = (x-z)^2 + 2z(y+x)
\]
\( Q \) is tangent to the lines \( L_1 = \{ x-y=0 \} \) and \( L_{-1} = \{ x+y=0 \} \).

The tangency points are:
\[
x-y=x+z=0 \Rightarrow (1, 1, -1)
\]
\[
x+y=-x+z=0 \Rightarrow (1, -1, 1).
\]

Note that \( Q \) is also tangent to the line \( z=0 \) at the point \( (0, 1, 0) = P \).

We want to find an irreducible nodal cubic \( C \) such that \( C \cdot Q = 6P \) and such that \( C \) is tangent to the lines \( x= \pm y \) in points different from those of \( Q \).

Let \( C \) be a cubic s.t. \( P \in C \) and \( C \cdot Q = 6P \). Note that if \( C \) were reducible, then the previous condition would imply that \( z=0 \) is a component of \( C \).

We then have \( \text{div}(C) = \text{div}(z^3) \mod Q \), so \( C = z^3 + Q L \) with \( L \) a linear form, and thus
\[
 C = z^3 + (x^2 + 2zy + z^2)(ax + by + cz).
\]

Since we want \( C \) to be tangent to the two lines \( L_1 \) and \( L_{-1} \) we obtain that the following homogeneous polynomials in \( (x, z) \)
\[
z^3 + (x+z)^2((a+b)(x+z) + z(c-a-b)) \quad (2.1)
\]
\[
z^3 + (x-z)^2((a-b)(x-z) + z(c+a-b)) \quad (2.2)
\]

must have a double root.

Set \( \zeta = (\frac{1}{x+z}) \) and \( \hat{\zeta} = (\frac{1}{x-z}) \) and rewrite \( 2.1 \) \( 2.2 \) as:
\[
 \zeta^3 + \zeta(c-a-b) + (a+b) = 0 \quad \hat{\zeta}^3 + \hat{\zeta}(c+a-b) + (a-b) = 0.
\]

We recall that if \( \zeta \) is a double root of \( z^3 + pz + q = 0 \) then
\[
 3\zeta^2 + p = 0 \quad \text{whence} \quad \frac{2}{3} \zeta + q = 0
\]
and this implies that
\[
 \zeta = -\frac{3p}{2q} \quad \text{thus} \quad 27q^2 + 4p^3 = 0.
\]
Therefore we have a double root of $\zeta$ if and only if

$$\exists A : \zeta = A, \ q = 2A^3, \ p = -3A^2, \ \text{i.e.} \ \begin{cases} a+b = 2A^3 \\ c-(a+b) = -3A^2. \end{cases}$$

Similarly if we set $\hat{\zeta} = -B$ we have

$$\begin{cases} a-b = -2B^3 \\ c-(b-a) = -3B^2 \end{cases}$$

and so

$$\begin{align*}
a &= A^3 - B^3 \\
b &= A^3 + B^3 \\
c &= 2A^3 - 3A^2 = 2B^3 - 3B^2.
\end{align*}$$

Then $A$ and $B$ must satisfy $2(A^3 - B^3) = 3(A^2 - B^2)$. Recall that (we make no distinction between a curve and its equation)

$$C = z^3 + (x^2 + 2zy + z^2)((A^3 - B^3)x + (A^3 + B^3)y + (2A^3 - 3A^2)z)$$

while $x-y=0$ is tangent to $C$ at the point where

$$\zeta = \frac{z}{x+z} = A.$$

Therefore the tangency point is $(1-A, 1-A, A)$.

Similarly $x+y=0$ is tangent to $C$ at the point $(B-1, 1-B, B)$.

Let us now search for a cubic $C$ with a singular point on the line $x=0$, as in Persson’s construction. Since $\frac{\partial C}{\partial x}$ on the line $x=0$ equals $aQ$ and the singular point is different from $P$ it follows that $a=0$. Whence $A^3 - B^3 = A^2 - B^2 = 0$ and so $A = B$.

If $A = B$ then $C$ contains only the monomial $x^2$ as a polynomial in $x$, so the involution $x \mapsto -x$ leaves the curve $C$ invariant. From this we deduce that a singular point of $C$ must have its $x$ coordinate equal to 0 and $C$ has then a singularity on the line $x=0$ if and only if

$$z^3 + (z^2 + 2zy + z^2)((A^3 - B^3)x + (A^3 + B^3)y + (2A^3 - 3A^2)z)$$

has a double root.

Remembering that it can’t be $A = B = 0$, the double root cannot be $z=0$ and we can write the above as

$$z(z^2 + (z+2y)(2A^3y + (2A^3 - 3A^2)z)).$$

So we must check that

$$z^2(1+2A^3 - 3A^2) + 2zy(A^3 + 2A^3 - 3A^2) + 4A^3y^2 = 0$$

$$= z^2(1+2A^3 - 3A^2) + 2zy3A^2(A-1) + 4A^3y^2.$$
has a double root. This is the case when

$$9A^4(A-1)^2=4A^3(1+2A^3-3A^2)$$ \text{i.e.}

$$9A^6-18A^5+9A^4=4A^3+8A^6-12A^5.$$ 

Upon dividing by $A^3\neq 0$ we get

$$A^3-6A^2+9A-4=0.$$ 

Observe that 1 is a root of this equation, but if $A=1$ then the singular point is $(0,0,1)$ and coincides with the point of tangency of $x+y=0$ so this root has to be discarded. Since

$$A^3-6A^2+9A-4=(A-1)(A^2-5A+4)=(A-1)^2(A-4)$$

the other possible root is then $A=4$, and in this case we have $B=A=4$, $a=0$, $b=8\cdot4^2$, $c=5\cdot4^2$. Then

$$C = z^3+4^2(x^2+2yz+z^2)(8y+5z)$$

The tangency points are $(-3,-3,4)$ and $(3,-3,4)$, while for the singular point we have $x=0$ and a double root of

$$z^2+4^2(2y+z)(8y+5z)=0 \iff 81z^2+4^218zy+4^4y=0 \iff 9z+4^2y=0$$

so the singular point is $(0,9,-16)$. With this choice of $A$ and $B$, $C$ is irreducible (since $z=0$ is not a component of $C$).

We want to find the lines through $(0,0,1)$ and tangent to $C$. Let $A=B=\lambda$ and consider more generally the 1-parameter family of curves:

$$C_\lambda = z^3+(x^2+2yz+z^2)(2\lambda^3y+(2\lambda^3-3\lambda^2)z)=0.$$ 

The tangency points on the two fixed lines $x+y=0$, $x-y=0$ are, as we know, $(1-\lambda,1-\lambda,\lambda)$ and $(\lambda-1,1-\lambda,\lambda)$. Rewriting the last equation in powers of $z$ we obtain:

$$z^3(1+2\lambda^3-3\lambda^2)+z^26y\lambda(\lambda-1)+z\lambda^2(4\lambda y^2+(2\lambda-3)x^2)+2\lambda^3x^2y=0.$$ 

Since we know what happens for $\lambda=0$, we can divide by $\lambda^3$, set $w=\frac{x}{\lambda}$ and obtain:

$$w^3(1+2\lambda^3-3\lambda^2)+w^26y\lambda(\lambda-1)+w(4\lambda y^2+(2\lambda-3)x^2)+2x^2y=0.$$ 

We let now $\Delta$ be the discriminant of $C_\lambda$ with respect to the variable $w$, and using a standard formula for $\Delta$, we find a degree 6 equation in $x$ and $y$ which
is divisible by \( x^2(x^2-y^2) \).

Remembering that the discriminant of \( a_0 x^3 + a_1 x^2 + a_2 x + a_3 \) is:

\[
\Delta = a_1^2 a_2^2 - 4 a_0 a_2^3 - 4 a_1^3 a_3 - 27 a_0^2 a_3^2 + 18 a_0 a_1 a_2 a_3
\]

and applying this formula for simplicity when \( \lambda = 4 \), we obtain:

\[
y^2 z^6 3^4 (16 y^2 + 5 x^2)^2 - 2^2 3^4 (16 y^2 + 5 x^2)^3 - \\
-2^{12} 3^6 x^2 y^4 - 2^2 3^{11} x^4 y^2 + 2^5 3^8 (16 y^2 + 5 x^2) x^2 y^2
\]

and factoring this binary form we get:

\[
x^2 (x^2 - y^2) 2^2 3^4 (27 y^2 - 5 x^2).
\]

So we have that the tangent lines to \( C \) passing through \((0,0,1)\) are \( x = \pm y \), \( x = \pm \sqrt{\frac{128}{125}} y \) while \( x = 0 \) passes through the node of \( C \). We denote by \( L_0 \) the line \( x = 0 \) and by \( L_+, L_- \) the two lines \( x = \sqrt{\frac{128}{125}} y \), \( x = -\sqrt{\frac{128}{125}} y \) respectively.

In order to find the tangency point on the lines \( L_+, L_- \) we by symmetry may restrict to the line \( L_+ \). Writing \( x = 2^{3/2} a \), \( y = 5 \sqrt{5} a \) we have that

\[
z^3 + 2^4 (27 a^2 + 10 \sqrt{5} a z + z^2) (40 \sqrt{5} a + 5 z) = 0
\]

(2.3)

has a double root. Since for its derivative we have

\[
3 z^2 + 2^4 (10 \sqrt{5} a + 2 z) (40 \sqrt{5} a + 5 z) + 2^4 5 (27 a^2 + 10 \sqrt{5} a z + z^2) = 0
\]

\[
(15 + \frac{3}{16}) z^2 + 180 \sqrt{5} a z + 2640 a^2 = 0
\]

\[
a = \frac{-90 \sqrt{5} \pm \sqrt{90^2 5 - 2640 (15 + \frac{3}{16})}}{2640} = \sqrt{5} \frac{-30 \pm 3}{880}.
\]

Thus \( \frac{z}{x} = \frac{-25(30+3)}{880}, \frac{z}{x} = \frac{-8 \sqrt{10}(30+3)}{880} \) and the point of tangency is one of the points \((-33 \cdot 8 \sqrt{10}, -25 \cdot 33, 880), (-27 \cdot 8 \sqrt{10}, -25 \cdot 27, 880)\).

Upon substituting these values in the polynomial \( 2.3 \) we find that the correct choice is \((-24 \sqrt{10}, -75, 80)\).

By symmetry the point \((24 \sqrt{10}, -75, 80)\) is the tangency point of the line \( L_- \). Let us write

\[
C = 4^2 (8 y + 5 z) x^2 + z (16 y + 9 z)^2 = 0
\]

and let us set \( u = 16 y + 9 z \). We have:

\[
C = z u^2 + 8 x^2 (u + z) = 0
\]
In these coordinates the singular point of $C$ is $(0, 0, 1)$, so the tangents at the singular point are given by:

$$8x^2 + u^2 = 0$$

whence they are complex and we have an isolated point.

In order to draw $C$, let’s compute its flexes. Using the coordinates $x$, $u$, and $z$ the Hessian matrix is:

$$
\begin{pmatrix}
16(u + z) & 16x & 16x \\
16x & 2z & 2u \\
16x & 2u & 0
\end{pmatrix}
$$

The Hessian curve is then given by the determinant of

$$
\begin{pmatrix}
(u + z) & 0 & x \\
0 & z - 2u & u \\
8x & u & 0
\end{pmatrix}
$$

which equals

$$-(u + z)u^2 - 8x^2(z - 2u) = 0.$$

Eliminating $8x^2$ from the two equations we get

$$(u + z)^2u^2 - zu^2(z - 2u) = 0$$

so either $u = 0$, and this implies either $x = 0$ (the singular point) or $z = 0$ that gives the point $(1, 0, 0)$, or

$$(u + z)^2 - z(z - 2u) = u^2 + 4uz = 0$$

that gives ($u \neq 0$) $u = -4z$, that is $z = -1$, $u = 4$, $y = \frac{13}{10}$, $x = \pm \sqrt{2/3}$.

For these points $\frac{x}{y} = \pm \sqrt{2/3 \frac{16}{13}}$. 

8
3 Fundamental groups.

In this section we are going to describe the five steps leading to the determination of the orbifold fundamental group of the Persson’s surfaces.

Step 1.
Let \( I F_1 \) be the blow up of \( \mathbb{P}^2 \) at the point \((0, 0, 1)\) and let \( \Sigma_\infty \) be the exceptional divisor.

We consider the fibre bundle \( I F_1 \rightarrow \mathbb{P}^1 \) and its restriction \( f \)

\[
\mathbb{P}^1 \setminus (C \cup Q \cup \Sigma_\infty \cup L_1 \cup L_{-1} \cup L_+ \cup L_- \cup L_0) = \tilde{I}F_1
\]

\[
f \downarrow
\]

\[
\mathbb{P}^1 \setminus \{P_1, P_{-1}, P_+, P-, P_0\} = \mathbb{P}^1 \setminus \{5 \text{ pts.}\}.
\]

\( f \) is again a fibre bundle and we have a corresponding homotopy exact sequence of fundamental groups

\[
1 \rightarrow \mathcal{F}_5 \rightarrow \tilde{\Pi} \rightarrow \mathcal{F}_4 \rightarrow 1
\]

where \( \mathcal{F}_k \) denotes the free group with \( k \) generators and \( \tilde{\Pi} = \pi_1(\tilde{I}F_1) \).

Here we choose a small positive real number \( \varepsilon \) and \( x=\varepsilon, y=1 \) as base point on \( \mathbb{P}^1 \setminus \{5 \text{ pts.}\} \) and \( x=-\varepsilon, y=1, z=-4\sqrt{-1} \) as base point on \( I F_1 \).

We let \( \delta_1, \ldots, \delta_5 \) be a natural geometric basis of the free group

\[
\mathcal{F}_5 = \pi_1(f^{-1}(\text{base pt.})) = \pi_1(L_\varepsilon \setminus (C \cup Q \cup \Sigma_\infty))
\]

where the five points \( L_\varepsilon \cap C, L_\varepsilon \cap Q \) are ordered by lexicographic order on \( \text{Re(} \frac{z}{y} \text{)}, \text{Im(} \frac{z}{y} \text{)} \).
\( \mathcal{F}_5 \) is generated by the five geometric paths \( \gamma_i' \) around the five critical values described in figure \( \square \) and whose product is the identity. For these elements we choose lifts to \( \mathbb{F}_1 \) using a \( C^\infty \) section of a tubular neighborhood of \( \Sigma_\infty \) meeting \( \Sigma_\infty \) just in the point \( \infty (y=0) \) with intersection number equal to \(-1\). Therefore such lifts give paths \( \gamma_i \) such that

\[
\prod \gamma_i = \prod \delta_i
\]

and more specifically

\[
\gamma_{+1}\gamma_{0}\gamma_{-1} = \delta_1 \cdots \delta_5 = \gamma_{-1}\gamma_{1}\gamma_{-}.\]

We have that, indeed, \( \tilde{\Pi} \) occurs as a semidirect product described by the relations

\[
\gamma_{-1}^{-1}\delta_i \gamma_j = (\delta_i)\beta_j
\]

where the \( \beta_j \)'s are suitable braids in

\[
\mathfrak{B}_5 = \langle \sigma_1, \ldots, \sigma_4 | \sigma_i \sigma_j = \sigma_j \sigma_i \forall 1 \leq i < j \leq 5 \rangle
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \forall 1 \leq i < 4 \rangle
\]

the braid group on 5 strings which acts on the right on the free group \( \mathcal{F}_5 \) by the formulae

\[
(\delta_h)\sigma_k = \delta_h \text{ if } h \neq k, k+1
\]

\[
(\delta_k)\sigma_k = \delta_{k+1}
\]

\[
(\delta_{k+1})\sigma_k = \delta_{k+1}^{-1}\delta_k \delta_{k+1}.
\]

The braids \( \beta_j \) are constructed by following the motion of the five points of the intersection of \( f'^{-1}(P) \) with \( C \cup Q \) while \( P \) goes along \( \gamma_j' \). With our choice of the \( \gamma' \)'s we have, as the reader can easily verify,

\[
\beta_0 = \sigma_4^2 \sigma_2^2
\]

\[
\beta_1 = \sigma_1^{-1} \sigma_2 \sigma_3 \sigma_1 \sigma_2^{-1} \sigma_1
\]

\[
\beta_{-1} = \sigma_4^{-6} \sigma_2^{-1} \beta_1 \sigma_2^6
\]

\[
\beta_+ = \sigma_1^{-2} \sigma_2 \sigma_3 \sigma_4 \sigma_3^{-1} \sigma_2^{-1} \sigma_1^2
\]

\[
\beta_- = \sigma_4^{-6} \sigma_2^{-1} \beta_+ \sigma_2 \sigma_4^6.
\]

**Step 2.**

By taking \( \sqrt{\frac{x-y}{x+y}} \) we have a new fibre bundle \( \mathbb{F}_2 \xrightarrow{g'} \mathbb{P}^1 \) obtained by base change. Under this base change the inverse image \( Q' \) of the conic \( Q \) splits into two sections of \( g' \) which we will denote by \( Q'_1 \) and \( Q'_2 \). Again, by restriction we have a fibre bundle \( g \)

\[
\hat{\mathbb{F}}_2 = \mathbb{F}_2 \setminus (C' \cup Q'_1 \cup Q'_2 \cup \Sigma'_\infty \cup \{8 \text{ fibres}\}) \xrightarrow{g} \mathbb{P}^1 \setminus \{8 \text{ pts.}\}.
\]
Correspondingly we get an exact sequence

\[ 1 \rightarrow \mathcal{F}_3 = <\delta_1, \ldots, \delta_5> \rightarrow \hat{\Pi} \rightarrow \mathcal{F}_7 = <\gamma_0, \gamma_-, \gamma_0, \gamma_+, \gamma_1^2> \rightarrow 1 \]

where \( \bar{\gamma}_i = \gamma_i^1 = \gamma_1 \gamma_i \gamma_1^{-1} \) and \( \hat{\Pi} = \pi_1(\hat{\mathcal{F}}_2) \).

The fact that \( \mathcal{F}_7 \) has seven generators as above follows since the double cover of \( \mathbb{P}^1 \setminus \{5 \text{ pts.} \} \) corresponds to the homomorphism \( \mathcal{F}_4 \rightarrow \mathbb{Z}_2 \) sending \( \gamma'_1, \gamma'_- \rightarrow 1 \), and \( \gamma_0, \gamma', \gamma'_- \rightarrow 0 \).

If we want to keep track of the eight critical values, we can also use \( (\gamma_2^-)_1 \gamma_1 \) as a generator. In fact

\[ (\delta_1 \cdots \delta_5)^2 = (\gamma + \gamma_1 \gamma_0 \gamma^-_1)(\gamma^1 + \gamma_1 \gamma_0 \gamma^-) \]

thus

\[ \gamma + \gamma_0^2 \gamma_1^2 = (\gamma_1^2 \gamma_0 \gamma^-_0) \]

The geometric meaning of the above formula is related to the fact that \( (\Sigma_\infty^{'})^2 = -2 \), and more precisely to the fact that the new generators of \( \mathcal{F}_7 \) lie in a \( C^\infty \) section meeting \( \Sigma_\infty' \) in one point with intersection number \( (-2) \), and not meeting the other curves.

A presentation of \( \hat{\Pi} \) is thus given by

\[ <\delta_1, \ldots, \delta_5, \gamma_0, \gamma_-, \gamma_0, \gamma_+, \bar{\gamma}_0, \bar{\gamma}_-, \bar{\gamma}^+_+ | \gamma_0^{-1} \delta_i \gamma_0 = (\delta_i) \beta_0 \]

\[ \vdots \]

\[ \bar{\gamma}_0^{-1} \delta_i \bar{\gamma}_0 = (\delta_i) \beta_1 \beta_0 \beta_1^{-1} \]

\[ \vdots \]

\[ \Gamma \delta_i \Gamma^{-1} = (\delta_i) \beta_1^2 \]

**Step 3.**

The fundamental group

\[ \Pi' = \pi_1(\mathbb{F}_2 \setminus (C' \cup Q_1' \cup Q_2' \cup \Sigma_\infty' \cup L_1' \cup L_2')) \]

is a quotient of \( \hat{\Pi} \). The presentation of \( \Pi' \) is readily accomplished simply by introducing in the above presentation the further relations

\[ \gamma_0 = \gamma_- = \gamma_+ = \bar{\gamma}_0 = \bar{\gamma}_- = \bar{\gamma}_+ = 1. \]

Then \( \Pi' \) is presented as

\[ <\delta_1, \ldots, \delta_5, \Gamma | \delta_i = (\delta_i) \beta_0 \quad \delta_1 = (\delta_i) \beta_0 \quad \delta_i = (\delta_i) \beta_1 \]

\[ \delta_i = (\delta_i) \beta_1 \beta_0 \beta_1^{-1} \quad \delta_i = (\delta_i) \beta_1 \beta_0 \beta_1^{-1} \]

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\[ \delta_i = (\delta_i)\beta_1\beta_2^{-1} \quad \Gamma\delta_i\Gamma^{-1} = (\delta_i)\beta_2^{-1} \]

**Remark:** with the new relations we get, setting \( \Gamma_{-1} = (\gamma_{-1}^{2})^{71}, \)

\[ \Gamma_{-1}\Gamma = (\delta_1 \cdots \delta_5)^2 \]

**Step 4’.**

We denote by \( X^#_2 \) the non singular part of the double cover \( X_2 \) of \( \mathbb{F}_2 \) (branched over \( C' \cup Q'_1 \cup Q'_2 \cup \Sigma'_\infty \)) and by \( Z^#_2 \) the complement in \( X^#_2 \) of \( L'_1, L'_0, \) the respective inverse images of \( L'_1, L'_0. \)

We finally let \( Y^#_2 \) be the double cover of \( \mathbb{F}_2 \setminus (C' \cup Q'_1 \cup Q'_2 \cup \Sigma'_\infty \cup L'_1 \cup L'_{-1}). \)

Thus \( Y^#_2 \subset Z^#_2 \subset X^#_2. \)

Clearly \( \pi_1(Y^#_2) = \ker(\Pi' \longrightarrow \mathbb{Z}/2), \) where \( \delta, \rightarrow \overline{1} \) and \( \Gamma, \rightarrow \overline{0}, \) is generated by \( \Gamma, \sigma = \delta_1\Gamma\delta_1^{-1}, A_i = \delta_i\delta_1 \) (\( i=1, \ldots, 5 \)) and \( B_j = \delta_1\delta_1^{-1} \) (\( j=2, \ldots, 5, \)).

To find the relations we apply the Reidemeister-Shreier rewriting process to the relations \( R_\alpha, \) of \( \Pi' \) and to the relations \( \delta_1 R_\alpha\delta_1^{-1}. \)

**Step 4”.**

Clearly, \( \pi_1(Y^#_2) \) maps onto \( \pi_1(Z^#_2) \) surjectively with kernel normally generated by \( \delta_1^{i_2} = B_iA_i \) (\( i=2, \ldots, 5 \)), and \( \delta_1 \cdots \delta_5 \), thus \( \pi_1(Z^#_2) \) is generated by \( A_2, \ldots, A_5, \Gamma \) and has for relations the relations coming from the rewriting of \( R_\alpha, \delta_1 R_\alpha\delta_1^{-1}, \) and the rewriting of \( \delta_1 \cdots \delta_5 \) \( \Gamma = 1, \) i.e. \( A_2A_5^{-1}A_4A_5^{-1}A_2^{-1}A_3A_4^{-1}A_5 = 1. \)

**Remark:** This relation says that the four generators \( A_2, \ldots, A_5 \) are the generators of \( \pi_1(\text{fibre}) = \pi_1(\text{genus 2 curve}). \)

**Step 5.**

Let \( m=k+1 \) and consider \( X^#_{2m} \), the non singular part of the \( m \)-fold cyclic cover of \( X_2 \) totally branched over \( L'_1 \) and \( L'_2. \)

To find a presentation of \( X^#_{2m} \) we first need a presentation of the kernel of the map \( \pi_1(Z^#_2) \longrightarrow \mathbb{Z}/m \) such that \( A_i \longrightarrow 0 \) and \( \Gamma, \sigma \longrightarrow 1, \) and then we add the relations \( \Gamma^m = \sigma^m = 1. \)

Applying the Reidemeister-Shreier method, we find that the kernel is generated by \( \Gamma^m, \Gamma A_j\Gamma^{-1} \) for \( i=1, \ldots, m-1 \) and \( j=2, \ldots, 5, \) by \( \Gamma^i\sigma\Gamma^{-i} \) for \( i=1, \ldots, m-2 \) and \( \Gamma^{m-1}\sigma; \) it has for relations the rewriting in the new generators of the relations \( R'_\alpha, \) of \( \pi_1(Z^#_2) \) and the rewriting of \( \Gamma^i R'_\alpha \Gamma^{-i} \) for \( i=1, \ldots, m-1. \)

**4 Calculations.**

**Step 3.**

We have

\[
\begin{align*}
\beta_0 &= \sigma_1^2\sigma_2^2 \\
\beta_1 &= \sigma_1^{-1}\sigma_2\sigma_3\sigma_1\sigma_2^{-1}\sigma_1 \\
\beta_{-1} &= \sigma_4^{-1}\sigma_2^{-1}\beta_1\sigma_2\sigma_4
\end{align*}
\]
Thus, the relations $\delta_i=(\delta_i)\beta_0$ are equivalent to the two relations

\[
(\delta_4\delta_5)^6 = (\delta_5\delta_4)^6
\]  
(4.1)

\[
\delta_2\delta_3 = \delta_3\delta_2.
\]  
(4.2)

The relations $\delta_i=(\delta_i)\beta_+$ amount to

\[
\delta_5 = \delta_2^{-1}\delta_1^{-1}\delta_2\delta_1\delta_2.
\]  
(4.3)

In fact, here and in the sequel, we use the following argument: $\beta_+$ is a conjugate $\sigma\sigma_4\sigma^{-1}$ of the braid $\sigma_4$ and the braid $\sigma_4$ yields the relation $\delta_4=\delta_5$. Therefore, if we set $\delta'_4=(\delta_4)\sigma^{-1}$, $\delta'_5=(\delta_5)\sigma^{-1}$, we get the relation $\delta'_4=\delta'_5$. By our particular choice of $\sigma$

\[
\begin{align*}
\delta_5' &= \delta_5 \\
\delta_4' &= (\delta_4)\sigma_3^{-1}\sigma_2^{-1}\sigma_1 \\
&= (\delta_3)\sigma_2^{-1}\sigma_1 \\
&= (\delta_2)\sigma_2^2 \\
&= (\delta_2^{-1}\delta_1\delta_2)\sigma_1 \\
&= \delta_2^{-1}\delta_1\delta_2\delta_1\delta_2
\end{align*}
\]

Similarly, the relations $\delta_i=(\delta_i)\beta_-$ are equivalent to the relation

\[
\delta_3^{-1}\delta_1^{-1}\delta_3\delta_1\delta_3 = (\delta_4\delta_5)^{-3}\delta_5(\delta_4\delta_5)^3.
\]  
(4.4)

We write down, for convenience of the reader, the action of the braid $\beta_1^{-1}$, since the new relations $\delta_i=(\delta_i)\beta_1\beta_j\beta_1^{-1}$ will be obtained from the relations equivalent to $\delta_i=(\delta_i)\beta_j$ simply by applying the automorphism $\beta_1^{-1}$.

\[
\begin{align*}
(\delta_1)\beta_1^{-1} &= \delta_1\delta_2\delta_3\delta_2^{-1}\delta_1^{-1}\delta_2^{-1}\delta_1\delta_2\delta_4\delta_2^{-1}\delta_1^{-1}\delta_2\delta_1\delta_2\delta_3^{-1}\delta_2^{-1}\delta_1^{-1} \\
(\delta_2)\beta_1^{-1} &= \delta_1\delta_2\delta_3\delta_2^{-1}\delta_1^{-1} \\
(\delta_3)\beta_1^{-1} &= \delta_2^{-1}\delta_1\delta_2\delta_4^{-1}\delta_2^{-1}\delta_1^{-1}\delta_2\delta_1\delta_2\delta_4\delta_2^{-1}\delta_1^{-1}\delta_2 \\
(\delta_4)\beta_1^{-1} &= \delta_2^{-1}\delta_1\delta_2 \\
(\delta_5)\beta_1^{-1} &= \delta_5
\end{align*}
\]

Thus, the relations $\delta_i=(\delta_i)\beta_1\beta_0\beta_1^{-1}$ are equivalent to the relations

\[
(\delta_1\delta_2)^6 = (\delta_2\delta_1)^6
\]  
(4.5)

\[
\delta_5\delta_3\delta_5^{-1}\delta_4^{-1}\delta_5\delta_4 = \delta_4^{-1}\delta_3\delta_4\delta_5\delta_3\delta_5^{-1}
\]  
(4.6)
We take as Shreier set for the left cosets of the kernel the set 

\[ S_0 = 1, S_1 = \delta_1, \] 

Finally we have to write the relations \( \Gamma \delta_1 \Gamma^{-1} = (\delta_1) \beta_1^2 \), i.e.

\[
\begin{align*}
\Gamma \delta_1 \Gamma^{-1} &= \Gamma \delta_2 \Gamma^{-1} \delta_2^{-1} \delta_1 \delta_2 \delta_4 \Gamma \delta_2^{-1} \Gamma^{-1} \\
\Gamma \delta_4 \Gamma^{-1} &= \delta_4^{-1} \delta_2 \delta_3 \delta_5 \delta_3^{-1} \delta_2^{-1} \Gamma^{-1} \\
\Gamma \delta_4 \Gamma^{-1} &= \delta_4^{-1} \delta_2 \delta_3 \delta_5 \delta_3^{-1} \delta_2^{-1} \Gamma^{-1} \\
\Gamma \delta_5 \Gamma^{-1} &= \delta_5 
\end{align*}
\]

\textbf{Step 4.}

We take as Shreier set for the left cosets of the kernel the set \( \{ S_0 = 1, S_1 = \delta_1 \} \), so applying the Reidemeister-Shreier method we get the generators \( \Delta = \delta_1^2, \Gamma, \sigma = \delta_1 \delta_1^{-1}, A_i = \delta_i \delta_1 \) and \( B_i = \delta_i \delta_1^{-1} \) for \( i = 2, 3, 4, 5 \). For the relations we must
rewrite the relations 4.1,...,4.13 and their conjugate by $\delta_1$ in terms of the new generators. The rewriting process goes as follows (cf. [MKS], pages 86-98):

$$S_0\delta_1 = S_1$$
$$S_0\delta_i = B_iS_1$$
$$S_0\Gamma = \Gamma S_0$$

for $i=2,3,4,5$

We want to show that it suffices to rewrite only the relations 4.1,...,4.13.

Observe that all our relations can be written in the form $W \delta_i W^{-1} = \delta_k$ for a suitable word $W$. Assume that $\Gamma$ doesn’t appear in the relation and do the rewriting after moding out by the relations

$$\Delta = B_iA_i = 1. \quad (4.14)$$

Since $S_0\delta_i = A_i^{-1}S_1$ and also $S_0\delta_i^{-1} = A_i^{-1}S_1$, if we write $W = \prod_{\lambda=1}^{h} \delta_{j_\lambda}^{\pm 1}$, the rewriting of $W\delta_i W^{-1} \delta_k^{-1}$ is given by

$$A_{j_1}^{-1} A_{j_2} \cdots A_{i}^{\pm 1} \cdots A_{j_1}^{-1} A_k$$

(note that $A_1=1$). The rewriting of the same relation conjugated by $\delta_1$ yields instead

$$A_{j_1} A_{j_2}^{-1} \cdots A_{i}^{\mp 1} \cdots A_{j_1} A_k^{-1}.$$  

We get thus two relations of respective form $UA_k = 1$, $U^{-1}A_k^{-1} = 1$, which are obviously equivalent.

If instead $\Gamma$ appears in the relation, we have one of the 4.9,...,4.13 which are of the form $\Gamma \delta_i \Gamma^{-1} = W\delta_i W^{-1}$ where we can in fact assume that $\Gamma$ doesn’t appear in the word $W$.

The rewriting of $\Gamma \delta_i \Gamma^{-1} W\delta_i^{-1} W^{-1}$ yields, again a relation of the form

$$\Gamma A_i^{-1} \sigma^{-1} U^{-1} = 1,$$

whereas the rewriting of the conjugate by $\delta_1$ gives a relation

$$\sigma A_i \Gamma^{-1} U = 1,$$

which is an equivalent relation.

For convenience of notation we shall keep the generators $B_i = A_i^{-1}$.

To calculate $\pi_1(Z_2^\#)$ we must add the rewriting of $(\prod_{i=1}^{5} \delta_i)^2 = 1$ which gives

$$A_2B_3A_4B_5B_2A_3B_4A_5 = 1.$$  

We have thus that $\pi_1(Z_2^\#)$ is generated by $A_2$, $A_3$, $A_4$, $A_5$, $\Gamma$ and $\sigma$ and has the following set of relations

$$(B_4A_5)^6 = (B_3A_4)^3 \quad (4.15)$$
\[
B_3 A_2 = B_2 A_3 \quad (4.16) \\
B_5 = B_2^3 \quad (4.17) \\
B_3^3 = (B_5 A_4)^3 B_5 \quad (4.18) \\
A_2^{12} = 1 \quad (4.19) \\
B_5 A_3 B_3 A_4 B_5 A_4 = B_4 A_5 B_4 A_5 B_3 A_5 \quad (4.20) \\
B_5 = B_2^2 A_4 B_5 A_3 B_5 A_4 B_2^2 \quad (4.21) \\
B_3 B_2 B_3 = (B_5 A_4)^4 B_5 \quad (4.22) \\
\sigma A_2^2 \Gamma^{-1} = A_4 B_2^2 A_4 \quad (4.23) \\
\Gamma B_2 \sigma^{-1} = B_2^2 A_3 B_5 A_3 B_2^2 \quad (4.24) \\
\Gamma B_3 \sigma^{-1} = B_4 A_2^2 B_4 A_2^2 B_3 A_5 B_3 A_5 B_3 A_2^2 B_4 A_2^2 B_4 \quad (4.25) \\
\Gamma B_4 \sigma^{-1} = B_4 A_2^2 B_4 A_2^2 B_4 \quad (4.26) \\
\Gamma B_5 \sigma^{-1} = B_5 \quad (4.27) \\
A_2 B_3 A_4 B_5 B_2 A_3 B_4 A_5 = 1 \quad (4.28)
\]

where \( B_i = A_i^{-1}. \)

Let’s reduce this presentation. Using \ref{4.17} relation \ref{4.21} becomes

\[
B_4 A_2 B_4 = B_5 A_3 B_5 \quad (4.29)
\]

and with this \ref{4.20} becomes

\[
B_2^3 = 1 \quad (4.29)
\]

which implies \ref{4.19} and changes \ref{4.17} into

\[
B_5 = A_2. \quad (4.29)
\]

Moreover, using \ref{4.22} and the last equation, relation \ref{4.18} gives

\[
(A_2 A_4)^2 = A_3 A_2 B_3^2 \quad (4.29)
\]

and with this, using also \ref{4.16} \ref{4.22} becomes

\[
B_3 B_2 = A_2 A_3 \quad (4.30)
\]

thus transforming \ref{4.29} into

\[
B_3 = B_4 A_2 B_4 \quad (4.30)
\]

which allows us to delete the generator \( A_3. \) Upon substituting the expressions of \( A_5 \) and \( A_3 \) into \ref{4.28} and \ref{4.30} we have

\[
A_2 A_4 = A_4 A_2 \quad (4.30)
\]
\[ A_4^4 = 1. \]

We can then see that the relations 4.16,...,4.28 are equivalent to the following

\[ A_5 = A_2^{-1} \quad A_3 = A_2^{-1}A_4^2 \]
\[ A_2^4 = A_4^1 = 1 \quad A_2A_4 = A_4A_2 \]
\[ \sigma A_2^2\Gamma^{-1} = A_2^2A_4^2 \]
\[ \Gamma A_2^{-1} = A_2^{-1}\sigma \]
\[ \Gamma A_4^2 = A_2A_4^3\sigma \]
\[ \Gamma A_4^{-1} = A_4\sigma \]
\[ \Gamma A_2^1 = A_2\sigma. \]

**Step 5.**

We take as Shreier set for the left cosets of the kernel the set

\[ \{R_i=\Gamma^i \mid i=0,1,\ldots,m-1\} \]

and we apply the Reidemeister-Shreier method.

The generators are \( \hat{\Gamma} = \Gamma^m, A_{2,i} = \Gamma^i A_2 \Gamma^{-i}, A_{4,i} = \Gamma^i A_4 \Gamma^{-i} \) for \( i=0,\ldots,m-1 \)
\( \sigma_i = \Gamma^i \sigma \Gamma^{-(i+1)} \) for \( i=0,\ldots,m-2 \) and \( \sigma_{m-1} = \Gamma^{m-1}\sigma \).

For the rewriting process we have

\[ R_iA_j = A_{j,i}R_i \quad \text{for } j=2,4 \text{ } i=0,\ldots,m-1 \]
\[ R_i\Gamma = R_{i+1} \quad \text{for } i=0,\ldots,m-2 \]
\[ R_{m-1}\Gamma = \hat{\Gamma}R_0 \]
\[ R_i\sigma = \sigma_iR_{i+1} \quad \text{for } i=0,\ldots,m-2 \]
\[ R_{m-1}\sigma = \sigma_{m-1}R_0. \]

Thus, taking indices \( i \text{ (mod } m) \) and adding (as we must) the relation \( \hat{\Gamma} = 1 \), we obtain the relations

\[ A_{2,i}^2 = A_{4,i}^2 = 1 \]
\[ A_{2,i}A_{4,i} = A_{4,i}A_{2,i} \]
\[ \sigma_i A_{2,i+1}^2 = A_{2,i}^2A_{4,i}^2 \]
\[ A_{2,i+1}^{-1} = A_{2,i}^{-1}\sigma_i \]
\[ A_{2,i+1}A_{4,i+1}^2 = A_{2,i}A_{4,i}^2\sigma_i \]
\[ A_{4,i+1} = A_{4,i}\sigma_i \]
\[ A_{2,i+1} = A_{2,i}\sigma_i. \]
To simplify this presentation we write

\[ \sigma_i = A_{2,i}^2 A_{4,i}^2 A_{2,i+1}^2 = A_{2,i} A_{2,i+1}^{-1} = A_{4,i}^2 A_{2,i+1} A_{4,i+1}^2 = A_{4,i} A_{4,i+1}^{-1} = A_{2,i} A_{2,i+1} \]

From the last and the second equations we get

\[ A_{2,i}^2 = A_{2,i+1}^2 = A_{2,0}^2 \]

and from the first one, remembering that \( A_{2,i} \) commutes with \( A_{4,i} \) and that \( A_{2,0} = 1 \),

\[ \sigma_i = A_{2,i}^2. \]

The fourth equation then gives

\[ A_{4,i} = A_{4,i+1} = A_{4,0} \]

which makes the last and the third relations equivalent. These two cancellation relations enable us to delete all the generators \( \sigma_j \) and \( A_{4,i} \) for \( i = 1, \ldots, m - 1 \).

We may rewrite the five relations above as

\[
\begin{align*}
\sigma_i &= A_{1,0}^2 \\
A_{4,i} &= A_{4,0} \\
A_{2,0}^2 &= A_{2,i}^{-1} A_{2,i+1} \\
A_{4,0}^2 &= A_{2,i} A_{2,i+1}^{-1}.
\end{align*}
\]

Clearly the last two equations are equivalent and give

\[ A_{2,2i} = A_{2,0} \quad A_{2,2i+1} = A_{2,0} A_{4,0}. \quad (4.31) \]

Moreover, if we add the relation \( \sigma^m = 1 \), that in the generators of \( \pi_1(X_{2m}^\#) \) reads out as \( \sigma_0 \sigma_1 \cdots \sigma_{m-1} = 1 \), we get \( A_{2,0}^2 = 1 \), i.e., if \( m \) is odd, \( A_{2,0}^2 = 1 \), while if \( m \) is even we have no new relations. Observe that this is in accordance with the fact that in \([3.3]\) the index is cyclic \( \text{mod}(m) \).

Summing up, we have a commutative group with only two generators, namely \( a = A_{2,0} \) and \( b = A_{4,0} \), such that \( a^4 = 1 \) and \( b^4 = 1 \) if \( m \) is even, \( b^2 = 1 \) if \( m \) is odd, i.e.

\[ \pi_1(X_{2k+2}^\#) = \mathbb{Z}_4 \times \mathbb{Z}_4 \]

if \( k \) is odd and

\[ \pi_1(X_{2k+2}^\#) = \mathbb{Z}_4 \times \mathbb{Z}_2 \]

if \( k \) is even.
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