Unitarity boomerangs of quark and lepton mixing matrices

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Abstract

The most popular way to present mixing matrices of quarks (CKM) and leptons (PMNS) is the parametrization with three mixing angles and one CP-violating phase. There are two major options in this kind of parametrizations, one is the original Kobayashi-Maskawa (KM) matrix, and the other is the Chau-Keung (CK) matrix. In a new proposal by Frampton and He, a unitarity boomerang is introduced to combine two unitarity triangles, and this new presentation displays all four independent parameters of the KM parametrization in the quark sector simultaneously. In this paper, we study the relations between KM and CK parametrizations, and also consider the quark-lepton complementarity (QLC) in the KM parametrization. The unitarity boomerang is discussed in the situation of the CK parametrization for comparison with that in the KM parametrization in the quark sector. Then we extend the idea of unitarity boomerang to the lepton sector, and check the corresponding unitarity boomerangs in the two cases of parametrizations.

\textbf{Key words:} boomerang; quark mixing matrix; lepton mixing matrix; quark-lepton complementarity

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1 Introduction

Mixing of different generations of fermions is one of the most interesting issues in particle physics. To understand the mixing patterns and properties, the mixing matrix was introduced for phenomenological and theoretical studies. In the quark sector, the mixing matrix is described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{\text{CKM}}$, and in the lepton sector, it is described by the Pontecorvo-Maki-Nakawaga-Sakata (PMNS) matrix $U_{\text{PMNS}}$, 

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \quad U_{\text{PMNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}.$$

The original Kobayashi-Maskawa (KM) matrix was introduced in 1973 to accommodate CP violation in the Standard Model, by an extension of the Cabibbo’s idea of quark mixing from two generations to three generations with six quark flavors,

$$V_{\text{KM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & e^{i\delta_{\text{KM}}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & s_3 & -c_3 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta_{\text{KM}}} c_1 c_2 s_3 + s_2 c_3 e^{i\delta_{\text{KM}}} & s_1 s_2 c_3 + c_2 s_3 e^{i\delta_{\text{KM}}} c_1 s_2 s_3 - c_2 c_3 e^{i\delta_{\text{KM}}} \end{pmatrix}. \quad (1)$$

Here $s_i = \sin \theta_i$, $c_i = \cos \theta_i \ (i = 1, 2, 3)$, and $\theta_1, \theta_2, \theta_3$ are Euler angles, $\delta_{\text{KM}}$ is the CP-violating phase in the KM parametrization.

In 1984, Chau and Keung (CK) introduced a different parametrization, which has been advocated by the Particle Data Group since 1996,

$$V_{\text{CK}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{13} & s_{13} e^{-i\delta_{\text{CK}}} \\ 0 -s_{13} e^{i\delta_{\text{CK}}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$
\[
\begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\text{CK}}} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CK}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\text{CK}}} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CK}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\text{CK}}} & c_{23}c_{13}
\end{pmatrix}, \tag{2}
\]

where \( s_{ij} = \sin \theta_{ij}, \quad c_{ij} = \cos \theta_{ij} \quad (i, j = 1, 2, 3), \quad \theta_{12}, \theta_{23}, \theta_{13} \) are the rotation angles, and \( \delta_{\text{CK}} \) is the CP-violating phase in the CK parametrization.

Although Eq. (1) and Eq. (2) are drawn in the quark sector, the same form of parametrizations can be also used in the lepton sector since both CKM and PMNS matrices are unitary matrices for describing the mixing of fermions. However, if neutrinos are of Majorana type, there should be an additional diagonal matrix with two Majorana phases \( P = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1) \) multiplied to Eq. (1) and Eq. (2). In this paper, we consider the neutrinos as Dirac neutrinos, and the presentation of formalisms for Majorana neutrinos can be derived straightforwardly by including the additional phases. In the following, we use the superscripts \( Q \) and \( L \) to denote the parameters in the quark sector and the lepton sector respectively if necessary.

The experimental data of the moduli of the matrix elements are very important to understand the mixing matrices since they constitute most reliable information of the mixing patterns and properties. For quark mixing, the ranges of magnitude of the CKM matrix elements have been very well determined with [7]

\[
\begin{pmatrix}
    0.97419 \pm 0.00022 & 0.2257 \pm 0.0010 & 0.00359 \pm 0.00016 \\
    0.2256 \pm 0.0010 & 0.97334 \pm 0.00023 & 0.0415^{+0.0010}_{-0.0011} \\
    0.00874^{+0.00026}_{-0.00037} & 0.0407 \pm 0.0010 & 0.99913^{+0.000044}_{-0.000043}
\end{pmatrix}. \tag{3}
\]

For lepton mixing, the ranges for the PMNS matrix elements have been also constrained by (at 3\( \sigma \) level) [8]

\[
\begin{pmatrix}
    0.77 - 0.86 & 0.50 - 0.63 & 0.00 - 0.22 \\
    0.22 - 0.56 & 0.44 - 0.73 & 0.57 - 0.80 \\
    0.21 - 0.55 & 0.40 - 0.71 & 0.59 - 0.82
\end{pmatrix}. \tag{4}
\]

Now we have the moduli of the elements of the two mixing matrices, but it is not easy to study these elements from experimental data directly. For convenience, it is common to adopt an approximation as the basis matrix to
the lowest order. In the quark sector, a better choice is the unit matrix and/or the matrix suggested in Ref. [9]

\[
V_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad V'_0 = \begin{pmatrix}
\frac{\sqrt{2}+1}{\sqrt{6}} & \frac{\sqrt{2}-1}{\sqrt{6}} & 0 \\
-\frac{\sqrt{2}-1}{\sqrt{6}} & \frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\] (5)

The unit matrix is very simple while the later one is more close to experimental data.

In the lepton sector, it has been common to choose the bimaximal matrix [10] and/or the tri-bimaximal matrix [11] as the basis matrices

\[
U_{\text{bi}} = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
-1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2} \\
\end{pmatrix}, \quad U_{\text{tri}} = \begin{pmatrix}
2/\sqrt{6} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\
1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\
\end{pmatrix}.
\] (6)

Although the former one is not favored by present experimental data as the later one, it looks more symmetric with also possible connection with the unit basis in quark mixing [12]. The tribimaximal basis is very close to experimental data and can serve as a good approximation for lepton mixing.

Despite the fact that mixing of quarks and leptons could be treated separately, it would be interesting to find an internal expression which connects the two sectors. To this end, the quark-lepton complementarity (QLC) [13] provides a very useful relation between quark and lepton mixing angles and leads to a unified treatment of mixing in quark and lepton sectors. With the QLC, the unified parametrization of quark and lepton mixing matrices has been also discussed [9,12,14]. The QLC reads as the following equations

\[
\theta_{12}^Q + \theta_{12}^L = \frac{\pi}{4}, \quad \theta_{23}^Q + \theta_{23}^L = \frac{\pi}{4}, \quad \theta_{31}^Q \sim \theta_{31}^L \sim 0,
\] (7)

where the parameters \(\theta_{12}^{Q,L}, \theta_{23}^{Q,L}, \theta_{13}^{Q,L}\) refer to the rotation angles in the CK parametrization. Under the QLC, it is interesting to find that the unit matrix \(V_0\) in the quark sector corresponds to the bimaximal matrix \(U_{\text{bi}}\) in the lepton sector [12], which enlightened us to search for the corresponding matrix in the quark sector to the tri-bimaximal matrix in the lepton sector. That is the matrix \(V'_0\) in Eq. (5), which is drawn with the combination of the QLC and the tri-bimaximal matrix \(U_{\text{tri}}\) [9]. In other words, the basis matrix \(V'_0\) in the
quark sector corresponds to the tri-bimaximal matrix $U_{\text{tri}}$ under the QLC in the lepton sector $[14,15]$.

The unitarity of the mixing matrix imposes six vanishing combinations which can be represented in a complex plane as triangles. They are well known as the unitarity triangles, which play an important role in understanding the mixing matrices. There are three inner angles and three sides in one unitarity triangle, however, only three of them are independent. To give all four parameters in the mixing matrix, one must take another unitarity triangle into account which leads to the idea of unitarity boomerang introduced by Frampton and He $[16]$. They also suggested that the four independent parameters of the unitarity boomerang in the KM parametrization are convenient for phenomenological studies. For a systematic study of this issue, we present in Sec. II the relations between KM and CK parametrizations, and discuss the quark-lepton complementarity (QLC) under the KM parametrization. In Sec. III, we study unitarity boomerangs in the quark sector for both KM and CK parametrizations. The shapes of boomerangs in the two cases are compared, and our analysis supports the proposal by Frampton and He. In Sec. IV, we also extend the idea of unitarity boomerang to the lepton sector, and study boomerangs in both parametrizations. Finally we give a summary in Sec. V.

2 Relations between KM and CK parametrizations and the quark-lepton complementarity in the KM parametrization

Now let us check the relations between KM and CK parametrizations. Following the steps in Ref. $[5]$, we redefine the fields of the $c$ quark, $t$ quark, $s$ quark and $b$ quark by

\[ c \rightarrow ce^{i(\pi+\sigma_c)}, \quad t \rightarrow te^{i(\pi+\sigma_t)}, \quad s \rightarrow se^{i\sigma}, \quad b \rightarrow be^{i(\pi+\delta_{\text{CK}})} \]

so that

\[
V_{\text{KM}} \rightarrow V'_{\text{KM}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & -e^{i\sigma_c} & 0 \\
0 & 0 & -e^{i\sigma_t}
\end{pmatrix} V_{\text{KM}} \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -e^{-i\delta_{\text{CK}}}
\end{pmatrix}
\]

\[
V_{\text{KM}} \rightarrow V'_{\text{KM}} = \begin{pmatrix}
c_1 & s_1c_3 & s_1s_3e^{-i\delta_{\text{CK}}} \\
-s_1c_2e^{i\sigma_c}c_1c_2c_3e^{i\sigma_c} - s_2s_3e^{i(\sigma_c+\delta_{\text{KM}})} & |c_1c_2s_3 + s_2c_3e^{i\delta_{\text{KM}}}| & |c_1s_2s_3 - c_2c_3e^{i\delta_{\text{KM}}}| \\
-s_1s_2e^{i\sigma_t}c_1s_2c_3e^{i\sigma_t} + c_2s_3e^{i(\sigma_t+\delta_{\text{KM}})} & |c_1s_2s_3 - c_2c_3e^{i\delta_{\text{KM}}}| & |c_1c_2s_3 + s_2c_3e^{i\delta_{\text{KM}}}| \end{pmatrix}.
\]
where
\[ e^{i\sigma} = \frac{c_1 c_2 s_3 + s_2 c_3 e^{-i\delta_{KM}}}{|c_1 c_2 s_3 + s_2 c_3 e^{i\delta_{KM}}|} e^{i\delta_{CK}}, \quad e^{i\sigma_t} = \frac{c_1 s_2 s_3 - c_2 c_3 e^{-i\delta_{KM}}}{|c_1 s_2 s_3 - c_2 c_3 e^{i\delta_{KM}}|} e^{i\delta_{CK}}. \] (9)

To transform the KM parametrization into the CK parametrization, the relations of the parameters between the two parametrizations are derived [17]:

\[ s_{12} = s_1 c_3 (1 - s_1^2 s_3^2)^{-\frac{1}{2}}, \]
\[ s_{23} = (s_2^2 c_3^2 + c_1^2 s_2^2 s_3^2 + 2 c_1 c_2 c_3 s_2 s_3 \cos \delta_{KM})^{\frac{1}{2}} (1 - s_1^2 s_3^2)^{-\frac{1}{2}}, \]
\[ s_{13} = s_1 s_3, \]
\[ s_{23} c_{23} \sin \delta_{CK} = s_2 c_2 \sin \delta_{KM}. \] (10)

It is noted that we use the scripts “i” for quantities in the KM parametrization and “ij” for those in the CK parametrization.

With the QLC, i.e. the relations between quark and lepton mixing angles in the mode of the CK parametrization, it is interesting to consider the relations of the Euler angles between quark and lepton sectors in the mode of KM parametrization. Fortunately, we have extracted the relations between the corresponding parameters in KM and CK parametrizations in Eq. (10). Then we can see if there is any similar relations between quark and lepton mixing angles in the KM parametrization.

First, it is straightforward to consider the behavior of the mixing angles under the approximation of the basis matrices. One finds clearly that the corresponding mixing angles in the two parametrizations are the same when one only considers the basis matrices, i.e. taking the unit matrix \( V_0 \) as the basis matrix in the quark sector and the bimaximal matrix \( U_{bi} \) as the basis matrix in the lepton sector, one has

\[ \theta_1^Q = \theta_{12}^Q = 0, \quad \theta_1^L = \theta_{12}^L = \frac{\pi}{4}, \]
\[ \theta_2^Q = \theta_{23}^Q = 0, \quad \theta_2^L = \theta_{23}^L = \frac{\pi}{4}, \]
\[ \theta_3^Q = \theta_{13}^Q = 0, \quad \theta_3^L = \theta_{13}^L = 0. \]

Taking \( V_0' \) in Eq. (5) as the basis matrix in the quark sector and the the tri-bimaximal matrix \( U_{tri} \) as the basis matrix in the lepton sector, one obtains

\[ \theta_1^Q = \theta_{12}^Q = \arcsin \frac{\sqrt{2} - 1}{\sqrt{6}}, \quad \theta_1^L = \theta_{12}^L = \arcsin \frac{1}{\sqrt{3}}, \]
\[ \theta_2^Q = \theta_{23}^Q = 0, \quad \theta_2^L = \theta_{23}^L = \frac{\pi}{4}, \]
\[ \theta_3^Q = \theta_{13}^Q = 0, \quad \theta_3^L = \theta_{13}^L = 0. \]

It is consistent with the relations in Eq. (10). Anyway, the result means that the QLC relations revealed [13] in the CK parametrization as expressed in Eq. (7) are also satisfied to the lowest order in the KM parametrization.

Now let us turn to the mixing angles in the realistic mixing matrices. According to Eq. (3), we find that
\[ s_{Q1}^2 \sim O(10^{-1}), \]
\[ s_{Q2}^2 \sim O(10^{-2}) \]
and
\[ s_{Q3}^2 \sim O(10^{-2}). \]

Then we can simplify Eq. (10) as
\[ s_{Q12}^2 = s_{Q1}^2 + O(10^{-4}), \]
\[ s_{Q23}^2 = (s_{Q2}^2 + s_{Q3}^2 + 2s_{Q2}^2 s_{Q3}^2 \cos \delta_{Q_\text{KM}}^Q)^{1/2} + O(10^{-4}), \]
\[ s_{Q13}^2 = s_{Q1}^2 s_{Q3}^2, \]
\[ s_{Q23}^2 \sin \delta_{Q_\text{CK}}^Q = s_{Q2}^2 \sin \delta_{Q_\text{KM}}^Q + O(10^{-4}), \]

which has been already obtained by Chau and Keung in Ref. [5].

As to the quark sector, we find that the approximation is not so simple according to Eq. (4) in the lepton sector. Nevertheless, \(|U_{e3}| \) being small implies that \( s_3^L \) is a small parameter. Thus, we could expand the equations in Eq. (10) in powers of \( s_3^L \) and obtain
\[ s_{L12}^2 = s_1^L - \frac{1}{2}s_1^L (c_{L1}^2) (s_3^L)^2 + O((s_3^L)^4), \]
\[ s_{L23}^2 = s_2^L + c_2^L c_3^L \cos \delta_{L_\text{KM}}^L s_3^L + \frac{(c_1^L)^2}{2s_2^L} ((c_2^L)^2 - (s_2^L)^2 - 2(c_2^L)^2 \cos^2 \delta_{L_\text{KM}}^L) (s_3^L)^2 + O((s_3^L)^3), \]
\[ s_{L13}^2 = s_1^L s_3^L, \]
\[ s_{L23}^2 c_{L2}^L \sin \delta_{L_\text{CK}}^L = s_2^L c_2^L \sin \delta_{L_\text{KM}}^L. \]

From Eq. (4), we know that \( s_3^L \) is at most of order \( O(10^{-1}) \), but we do not know whether it is small enough. Combining Eq. (11) with Eq. (12), we find that:

(1) Since \( s_3^L \) is small, we always have
\[ \theta_2^Q \sim \theta_3^L \sim \theta_{13}^Q \sim \theta_{13}^L \sim 0. \]

(2) When \( s_3^L \sim O(10^{-1}) \), we have
\[ \theta_1^Q + \theta_1^L = \theta_{12}^Q + \theta_{12}^L + O(10^{-2}) \sim \frac{\pi}{4} + O(10^{-2}). \]
\[ \theta_2^Q + \theta_2^L = \theta_2^{Q23} + \theta_2^{L23} + \mathcal{O}(10^{-1}) \sim \frac{\pi}{4} + \mathcal{O}(10^{-1}). \]

(3) When \( s_3^L \sim \mathcal{O}(10^{-2}) \) or smaller, we could obtain

\[ \theta_1^Q + \theta_1^L = \theta_1^{Q12} + \theta_1^{L12} + \mathcal{O}(10^{-4}) \sim \frac{\pi}{4} + \mathcal{O}(10^{-4}). \]
\[ \theta_2^Q + \theta_2^L = \theta_2^{Q23} + \theta_2^{L23} + \mathcal{O}(10^{-2}) \sim \frac{\pi}{4} + \mathcal{O}(10^{-2}). \]

As discussed above, the QLC is still an appealing relation between quarks and leptons in the KM parametrization when \( s_3^L \sim \mathcal{O}(10^{-2}) \) or smaller. When \( s_3^L \sim \mathcal{O}(10^{-1}) \), we find that the QLC in the KM parametrization is not as good as it does in the CK parametrization. Nevertheless, the QLC with Euler angles is satisfied to the lowest order in the KM parametrization. To higher order, the relation may be corrected to some extent. More precision experimental data are needed to test the QLC in both CK and KM parametrizations. Knowledge of QLC for both cases can provide us more complete information concerning the possible connection and unification between quarks and leptons.

### 3 Unitarity boomerangs in quark sector

We know that the unitarity of the mixing matrix imposes six unitarity triangles and the areas of them are the same, equal to half of the Jarlskog invariant \( J \) \[18\] defined by

\[ \text{Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*) = J \sum_{m,\bar{n}} \epsilon_{ikm}\epsilon_{jln}, \quad (13) \]

which, as an important parameter to the mixing matrix, is phase-convention independent when measuring CP violation. Therefore, the inner angles of the unitarity triangles could be related to the parameter \( J \).

The sides of the unitarity triangles are decided by the values of the matrix elements, so they are important to find the shape of the unitarity triangles. To understand better, we should know the orders of magnitude of the matrix elements in the first step of analysis.

The Wolfenstein parametrization \[19\] displays a good hierarchy among the nine elements of the CKM matrix.
\[ V = \begin{pmatrix} 1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\rho - i \eta) \\ -\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\ A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \]  

with \( \lambda = 0.2257^{+0.0009}_{-0.0010} \) and \( A = 0.814^{+0.021}_{-0.022} \). It suggests that the CKM matrix could be simply presented as

\[
\begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix},
\]

with \( \lambda \sim 0.2 \). It is natural to see that the three sides are of the same order \( \lambda^3 \) only in two unitarity triangles arise from

\[ V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0, \quad V_{ud} V_{td}^* + V_{us} V_{ts}^* + V_{ub} V_{tb}^* = 0. \]  

While in the last four unitarity triangles, one side is \( \mathcal{O}(\lambda^2) \) or \( \mathcal{O}(\lambda^4) \) to the other two sides.

We know that one must take two unitarity triangles to give all four independent parameters in the mixing matrix. For this purpose, Frampton and He introduced a new diagram for the quark mixing matrix in Ref. [16], the unitarity boomerang. For one inner angle of a unitarity triangle, we can always find a same angle in another unitarity triangle with the Jarlskog invariant \( J \). With the common angle in overlap for the two triangles, a unitarity boomerang is then constructed. We have stated that the three sides of the same order of magnitude only exist in two unitarity triangles arising from Eq. (15). So the unitarity boomerang consisted by the two unitarity triangles arising from Eq. (15) is the most convenient one, and this is just the choice by Frampton and He [16]. Since the common angle of the two chosen unitarity triangles could be determined by the CP-violating measurement \( J \), the CP-violating phase could then be constrained.

Now we consider the unitarity boomerangs in the KM and CK parametrizations:

**Case 1.** The unitarity boomerang in the KM parametrization. As discussed in Ref. [16], the Jarlskog parameter satisfies

\[
J^Q = 2 |(V_{td})_{KM} (V_{tb}^*)_{KM}|| (V_{ud})_{KM} (V_{ub}^*)_{KM}| \sin \phi_2 \\
= 2 |(V_{ud})_{KM} (V_{td}^*)_{KM}|| (V_{ub})_{KM} (V_{tb}^*)_{KM}| \sin \phi'_2,
\]
with \( \phi_2 = \phi_2' \) as the common angle there is the diagram of unitarity boomerang, as illustrated in Fig. 1.

![Diagram of unitarity boomerang](image_url)

Fig. 1. The unitarity boomerang in the quark sector with the common angle \( \phi_2 \). The sides are: 
- \( AC = |(V_{ud})_{\text{KM}}(V_{ub})_{\text{KM}}| \), 
- \( AC' = |(V_{ub})_{\text{KM}}(V_{tb})_{\text{KM}}| \), 
- \( AB = |(V_{td})_{\text{KM}}(V_{tb})_{\text{KM}}| \), 
- \( AB' = |(V_{ud})_{\text{KM}}(V_{td})_{\text{KM}}| \), 
- \( BC = |(V_{cd})_{\text{KM}}(V_{cb})_{\text{KM}}| \) and 
- \( B'C' = |(V_{us})_{\text{KM}}(V_{ts})_{\text{KM}}| \).

According to Eq. (3), we can estimate that \( AB \sim AB', AC \sim AC', BC \sim B'C', AB \sim 2.5AC, \) and \( BC \sim 2.6AC \).

The CP-violating phase in the KM parametrization is also constrained in Ref. [16]

\[ \delta_{\text{KM}}^Q \approx \pi - \phi_2 \approx 90^\circ. \]

Case 2. The unitarity boomerang in the CK parametrization. In this case, to find the constraint of the CP-violating phase \( \delta_{\text{CK}}^Q \), we have to choose another unitarity triangle arising from

\[ V_{ud}V_{cd}^* + V_{us}V_{es}^* + V_{ub}V_{cb}^* = 0. \] (16)

We use the Jarlskog parameter expressed as

\[ J^Q = 2|(V_{cd})_{\text{CK}}(V_{cb})_{\text{CK}}|(V_{ud})_{\text{CK}}(V_{ub})_{\text{CK}}\sin \phi_3 \]
\[ = 2|(V_{td})_{\text{CK}}(V_{td}^*)_{\text{CK}}|(V_{ub})_{\text{CK}}(V_{cb})_{\text{CK}}\sin \phi_3', \]

so we have the diagram with \( \phi_3 = \phi_3' \) as the common angle, as illustrated in Fig. 2.
Fig. 2. The unitarity boomerang in the quark sector with the common angle $\phi_3$. $\phi_3 = \angle ACB = \angle DCE$. The sides are: $AC = |(V_{ud})_{CK}(V_{ub})_{CK}|$, $AB = |(V_{ud})_{CK}(V_{tb})_{CK}|$, $BC = |(V_{cd})_{CK}(V_{cb})_{CK}|$, $CD = |(V_{ub})_{CK}(V_{tb})_{CK}|$, $CE = |(V_{ud})_{CK}(V_{cd})_{CK}|$ and $DE = |(V_{us})_{CK}(V_{cs})_{CK}|$.

In Fig. 2 the unitarity triangle $ABC$ is still the same one in Fig. 1 and we find that the side $CE$ almost coincides with the side $DE$ because of the estimation $CE \sim DE \sim 60AC$ and $CD \sim 0.043AC$ according to Eq. (3).

Using experimental values for $|V_{us}| = 0.97419 \pm 0.00022$, $|V_{ub}| = 0.00359 \pm 0.00016$ and $|V_{cb}| = 0.0415_{+0.0010}^{-0.0011}$ in Eq. (3), one finds that $c_{12}^Q s_{23}^Q s_{13}^Q \ll 1$. At a few percent level, one has $(V_{cd})_{CK} = (-s_{12}^Q c_{23}^Q - c_{12}^Q s_{23}^Q s_{13}^Q e^{i \delta_{CK}^Q}) \approx -s_{12}^Q c_{23}^Q$. Then

$$\phi_3 = \arg \left( -\frac{c_{12}^Q s_{13}^Q s_{23}^Q e^{i \delta_{CK}^Q}}{(-s_{12}^Q c_{23}^Q - c_{12}^Q s_{23}^Q s_{13}^Q e^{i \delta_{CK}^Q}) s_{23}^Q c_{13}^Q} \right) \approx \arg \left( \frac{c_{12}^Q s_{13}^Q s_{23}^Q e^{i \delta_{CK}^Q}}{s_{12}^Q c_{23}^Q s_{13}^Q e^{i \delta_{CK}^Q}} \right) = \delta_{CK}^Q.$$  

The CP-violating phase $\delta_{CK}^Q$ in the CK parametrization is equal to $\phi_3$ to a good approximation. The fact that $\phi_3 = (77^{+30}_{-32})^\circ$ [7] implies $\delta_{CK}^Q \approx (77^{+30}_{-32})^\circ$.

In Ref. [16], Frampton and He also gave an example to explain how the unitarity boomerang presents the four independent parameters in the CKM matrix. They chose three sides $a$, $b$, $c$ of the unitarity triangle $ABC$ and a side $d$ of the unitarity triangle $AB'C'$ in Fig. 1 as the four parameters of the CKM matrix and obtained their expressions with the KM parameters.

$$a = |(V_{ud})_{KM}(V_{ub}^*)_{KM}| = c_1^Q s_1^Q s_3^Q;$$
$$b = |(V_{cd})_{KM}(V_{tb}^*)_{KM}| = s_1^Q c_2^Q |c_1^Q c_2^Q s_3^Q + s_2^Q c_3^Q e^{-i \delta_{KM}^Q}|;$$
$$c = |(V_{td})_{KM}(V_{cb}^*)_{KM}| = s_1^Q s_2^Q |c_1^Q s_2^Q s_3^Q - c_2^Q c_3^Q e^{-i \delta_{KM}^Q}|;$$
$$d = |(V_{ud})_{KM}(V_{td}^*)_{KM}| = c_1^Q s_1^Q s_2^Q.$$  \hspace{1cm} (17)

Considering the relations in Eq. (10), we have the expressions with the CK parameters (In fact, we can derive them from the CK parametrization directly).

$$a = |(V_{ud})_{CK}(V_{ub}^*)_{CK}| = s_{12}^Q c_{23}^Q s_{13}^Q;$$
In this section, we should realize that the unitarity boomerangs chosen are not arbitrary. In the quark sector, the unitarity triangle ABC in Fig. 1 is the most commonly used one. Since it is one of the only two unitarity triangles in which the three sides are of the same order among all the six unitarity triangles. With another unitarity triangle AB′C′ in Fig. 1 in which the three sides are of the same order, we find it more remarkable to introduce the CP-violating phase $\delta_{KM}^{L}$ in the KM parametrization. It is natural to choose the two special unitarity triangles to construct the unitarity boomerang, as Frampton and He did in Ref. [16]. And there may be some profound implications with the CP-violating phase $\delta_{KM}^{Q}$ drawn from the special unitarity boomerang. To manifest the CP-violating phase $\delta_{CK}^{Q}$ in the CK parametrization, we have to introduce a third unitarity triangle CDE in Fig. 2. Then we find that the shape of the unitarity boomerang in Fig. 1 looks much normal than that in Fig. 2. Thus, the CP-violating phase $\delta_{KM}^{Q}$ in the KM parametrization is more convenient to be constrained than $\delta_{CK}^{Q}$ in the CK parametrization with unitarity boomerang.

4 Unitarity boomerangs in lepton sector

Since both the CKM matrix for quarks and the PMNS matrix for leptons are unitary, we can extend the analysis of unitarity triangles to the PMNS matrix in correspondence to those in the quark sector. In the lepton sector, the hierarchy of the matrix elements is not so evident as that in the quark sector, however, the tri-bimaximal matrix characterizes the PMNS matrix pretty well, which means that the elements in the PMNS matrix are of the same order except $U_{e3}$. From Eq. (4), we only know that $|U_{e3}|$ is small, $|U_{e3}| \lesssim 0.2$, but we do not know whether it is small enough. We may take $|U_{e3}| \sim 0.1$ as an approximation, then the three sides are nearly of the same order in all six unitarity triangles in the lepton sector.

Corresponding to the quark sector, we consider two unitarity triangles arising from

$$U_{e1}U_{e3}^{*} + U_{\mu1}U_{\mu3}^{*} + U_{\tau1}U_{\tau3}^{*} = 0, \quad U_{e1}U_{\tau1}^{*} + U_{e2}U_{\tau2}^{*} + U_{e3}U_{\tau3}^{*} = 0.$$ (19)

The inner angles defined by the two unitarity triangles are
\[ \varphi_1 = \arg \left( \frac{-U_{\mu_1}U_{\mu_3}^*}{U_{\tau_1}U_{\tau_3}^*} \right), \]
\[ \varphi_2 = \arg \left( \frac{-U_{\mu_1}U_{\tau_3}^*}{U_{e_1}U_{e_3}^*} \right), \]
\[ \varphi_3 = \arg \left( \frac{-U_{e_1}U_{e_3}^*}{U_{\mu_1}U_{\mu_3}^*} \right), \]

(20)

and

\[ \varphi'_1 = \arg \left( \frac{-U_{e_1}U_{\tau_1}^*}{U_{e_2}U_{\tau_2}^*} \right), \]
\[ \varphi'_2 = \arg \left( \frac{-U_{e_3}U_{\tau_3}^*}{U_{e_1}U_{\tau_1}^*} \right), \]
\[ \varphi'_3 = \arg \left( \frac{-U_{e_2}U_{\tau_2}^*}{U_{e_3}U_{\tau_3}^*} \right). \]

(21)

Since the experimental data about neutrinos are not so accurate as those of quarks and we do not know the experimental data about the inner angles of the unitary triangles of the lepton mixing matrix, it is not easy to find the shape of the unitarity triangles or the unitarity boomerangs. To understand more clearly, we may take the tri-bimaximal matrix with \( |U_{e_3}| \lesssim 0.2 \) as an approximation of the PMNS matrix for example to see what we can learn from the unitarity boomerangs. However, this is only a special case, and the inner angles of the unitarity triangles cannot be determined in common cases. In this special case, we obtain

\[ \varphi_1 \lesssim 33^\circ, \quad \varphi_2 \sim \varphi_3 \gtrsim 74^\circ; \quad \varphi'_1 \lesssim 24^\circ, \quad \varphi'_2 \sim \varphi'_3 \gtrsim 78^\circ. \]

(22)

Now we attain two isosceles triangles which are built on the special case. Though the magnitude of the CP-violating phase is still unknown in the lepton sector, it may be constrained approximately. We still consider the behavior of the unitarity boomerangs in KM and CK parametrizations, respectively.

**Case 1’.** The unitarity boomerang in the KM parametrization. The Jarlskog parameter could be expressed as

\[
J^L = 2|(U_{\tau_1})_{\text{KM}}(U_{\tau_3}^*)_{\text{KM}}||(|U_{e_1})_{\text{KM}}(U_{e_3}^*)_{\text{KM}}| \sin \varphi_2
= 2|(U_{e_1})_{\text{KM}}(U_{\tau_1}^*)_{\text{KM}}||(|U_{e_3})_{\text{KM}}(U_{\tau_3}^*)_{\text{KM}}| \sin \varphi'_2,
\]

with \( \varphi_2 = \varphi'_2 \) as the common angle for the unitarity boomerang, as shown in Fig. 13.
Fig. 3. The unitarity boomerang in the lepton sector with the common angle $\varphi_2$. The sides are: $XZ = |(U_{e1})_{\text{KM}}(U^*_{e3})_{\text{KM}}|$, $XY = |(U_{\tau 1})_{\text{KM}}(U^*_{\tau 3})_{\text{KM}}|$, $YZ = |(U_{\mu 1})_{\text{KM}}(U^*_{\mu 3})_{\text{KM}}|$, $XZ' = |(U_{e3})_{\text{KM}}(U^*_{\tau 3})_{\text{KM}}|$, $XY' = |(U_{e1})_{\text{KM}}(U^*_{\tau 1})_{\text{KM}}|$, and $Y'Z' = |(U_{e2})_{\text{KM}}(U^*_{\tau 2})_{\text{KM}}|$. 

Fig. 3 is drawn under the approximation of tri-bimaximal matrix and $|U_{e3}| \sim 0.1$ as an illustration so that $XY \sim YZ$, $XY' \sim Y'Z'$, and $XY \sim 0.87XY' \sim 3.5XZ \sim 4XZ'$. 

In the KM parametrization of the PMNS matrix,

$$\varphi_2 = \arg \left( -s_L^1 s_L^2 (c_L^1 s_L^2 s_L^3 - c_L^2 c_L^3 e^{-i\delta^L_{\text{KM}}}) \right) / c_L^1 (-s_L^1 s_L^3).$$

If $|U_{e3}|$ is small enough, $s_L^3 \ll 1$, we have $c_L^1 s_L^2 s_L^3 \ll 1$, then

$$\varphi_2 \approx \arg \left( s_L^1 s_L^2 (-c_L^2 c_L^3 e^{-i\delta^L_{\text{KM}}}) / c_L^1 s_L^3 s_L^3 \right) = \pi - \delta^L_{\text{KM}}.$$

If $|U_{e3}|$ is not so small, there should be $\varphi_2 < \pi - \delta^L_{\text{KM}}$. The CP-violating phase $\delta^L_{\text{KM}}$ satisfies $\delta^L_{\text{KM}} \lesssim \pi - \varphi_2$ in the KM parametrization. So $\varphi_2 \gtrsim 74^\circ$ and implies that $\delta^L_{\text{KM}} \lesssim 106^\circ$ approximately.

**Case 2'**. The unitarity boomerang in the CK parametrization. In this case, we should introduce another unitarity triangle arising from

$$U_{e1} U^*_{\mu 1} + U_{e2} U^*_{\mu 2} + U_{e3} U^*_{\mu 3} = 0,$$

(23)

to manifest the CP-violating phase $\delta^L_{\text{CK}}$ with the CK parametrization. We present the Jarlskog parameter as

$$J^L = 2|(U_{e1})_{\text{CK}}(U^*_{e3})_{\text{CK}}||(U_{\mu 1})_{\text{CK}}(U^*_{\mu 3})_{\text{CK}}| \sin \varphi_3$$

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\[ = 2|\langle U_{e3}\rangle_{\text{CK}}(U_{\mu3}^*)_{\text{CK}}||\langle U_{e1}\rangle_{\text{CK}}(U_{\mu1}^*)_{\text{CK}}| \sin \varphi_3', \]

with \( \varphi_3 = \varphi_3' \) as the common angle for the unitarity boomerang, as shown in Fig. 4.

Fig. 4. The unitarity boomerang in the lepton sector with the common angle \( \varphi_3 \). The sides are: \( XZ = |\langle U_{e1}\rangle_{\text{CK}}(U_{e3}^*)_{\text{CK}}| \), \( XY = |\langle U_{\tau1}\rangle_{\text{CK}}(U_{\tau3}^*)_{\text{CK}}| \), \( YZ = |\langle U_{\mu1}\rangle_{\text{CK}}(U_{\mu3}^*)_{\text{CK}}| \), \( ZT = |\langle U_{e3}\rangle_{\text{CK}}(U_{\mu3}^*)_{\text{CK}}| \), \( ZW = |\langle U_{e1}\rangle_{\text{CK}}(U_{\mu1}^*)_{\text{CK}}| \) and \( TW = |\langle U_{e2}\rangle_{\text{CK}}(U_{\mu2}^*)_{\text{CK}}| \).

Here the unitarity triangle \( XYZ \) is still the same one in Fig. 3. We draw Fig. 4 under the approximation of tri-bimaximal matrix and \( |U_{e3}| \sim 0.1 \) for illustration, then \( XY \sim YZ \), \( ZW \sim TW \), and \( XY \sim 0.87ZW \sim 3.5XZ \sim 4ZT \).

The common angle \( \varphi_3 \) for the boomerang in the CK parametrization is

\[
\varphi_3 = \arg \left( -\frac{c_{12}L_{13}s_{13}e^{i\delta^L_{\text{CK}}}}{(-s_{12}L_{12}c_{13} - c_{12}L_{23}s_{13}e^{i\delta^L_{\text{CK}}})s_{23}L_{13}} \right)
\]

If \( |U_{e3}| \) is small enough, \( s_{13}^L \ll 1 \), then \( c_{12}^L s_{23}^L s_{13}^L \ll 1 \), we have

\[
\varphi_3 \approx \arg \left( \frac{c_{12}^L L_{13}^L e^{i\delta^L_{\text{CK}}}}{s_{12}^L c_{23}^L s_{23}^L c_{13}^L} \right) = \delta^L_{\text{CK}}.
\]

If \( |U_{e3}| \) is not so small, we have

\[
\varphi_3 = \arg (s_{12}^L c_{23}^L e^{i\delta^L_{\text{CK}}} + c_{12}^L s_{23}^L s_{13}^L) < \delta^L_{\text{CK}}.
\]

Then we find \( \delta^L_{\text{CK}} \gg \varphi_3 \) in the CK parametrization. As an approximation, \( \varphi_3 \approx 74^\circ \) implies that \( \delta^L_{\text{CK}} \gg 74^\circ \).
We find that the two unitarity boomerangs are more or less of similar sizes of their sides in the lepton sector in the special case, i.e. the tri-bimaximal matrix with \(|U_{e3}| \lesssim 0.2\) as an approximation of the PMNS matrix. That is because all the elements in the PMNS matrix are of the same order of magnitude except \(U_{e3}\). In this special case, we can estimate the inner angles of the unitarity triangles and give a constraint to the CP-violating phase approximately. However, we cannot get the isosceles triangles in Fig. 3 or Fig. 4 in common cases since the PMNS matrix is not the tri-bimaximal matrix and the exact value of \(|U_{e3}|\) is unknown. When \(|U_{e3}|\) is not small enough, the unitarity triangles in the unitarity boomerangs of Fig. 3 or Fig. 4 could still be nearly isosceles. But when \(|U_{e3}|\) is small enough, any deviation between the two longer sides will cause a violation of the isosceles triangle. Then the inner angles of the unitarity triangles cannot be determined and \(\varphi_2, \varphi_3, \varphi'_2, \varphi'_3\) can cover the range of 0 to \(\pi\). Nevertheless, \(\delta_{\text{KM}} \lesssim \pi - \varphi_2\) in Case 1’ and \(\delta_{\text{CK}} \gtrsim \varphi_3\) in Case 2’ still hold although there is no numerical constraint on the CP violating phases in both cases.

When discussing unitarity triangles or boomerangs in the lepton sector, we do not take the phase matrix \(P = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)\) into account. The reason is that the phase matrix \(P\) does not affect the unitarity of the PMNS matrix and \(V_{\text{PMNS}} V_{\text{PMNS}}^\dagger = V_{\text{PMNS}}^\dagger V_{\text{PMNS}} = I\) (where \(I\) is the 3 \(\times\) 3 unitary matrix) though the neutrinos are of Majorana type. For instance, the unitarity triangle \(XYZ\) arises from \(U_{e1} U_{e3}^* + U_{\mu1} U_{\mu3}^* + U_{\tau1} U_{\tau3}^* = 0\). If neutrinos are of Majorana type, the unitarity triangle \(XYZ\) will arise from \((U_{e1} U_{e3}^* + U_{\mu1} U_{\mu3}^* + U_{\tau1} U_{\tau3}^*) e^{i\alpha_1/2} = 0\). However, the phase does not take effect here and we can still consider the unitarity triangles or boomerangs with Dirac neutrinos for the PMNS matrix.

In the lepton sector, though there is not any evident difference among the six unitarity triangles since the elements of the PMNS matrix are of the same order of magnitude, we prefer to choose the three unitarity triangles arising from Eq. (19) and Eq. (23) corresponding to the quark sector because they are convenient for analyzing and comparing, especially when we wish to analyze possible relations between quarks and leptons, such as the quark-lepton complementarity discussed in Sec. II.

5 Summary

In this work, we have studied the mixing matrices of quarks and leptons for two cases of KM parametrization and CK parametrization, which express the mixing matrices with three mixing angles and one CP-violating phase. We present the transformations between the two cases of parametrizations and obtain the relations between their parameters. We also find that the quark-lepton complementarity (QLC) revealed in the CK parametrization is still
well kept in the KM parametrization when $s_3^L \sim \mathcal{O}(10^{-2})$ or smaller, i.e. the value of $|U_{e3}|$ is small enough. Then we analyse the unitarity boomerangs, a new concept proposed by Frampton and He for convenient study of the quark mixing matrix, under both the KM and CK parametrizations in the quark sector and extend the idea of unitarity boomerang to the lepton sector. With help of the unitarity boomerang, we analysed the constraints of the Dirac CP-violating phase in KM and CK parametrizations in both quark and lepton sectors. Our study is helpful for a comprehensive understanding of the mixing patterns and properties for both quarks and leptons from a unified viewpoint.

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