A THEOREM OF ROE AND STRICHARTZ FOR RIEMANNIAN SYMMETRIC SPACES OF NONCOMPACT TYPE

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Abstract. Generalizing a result of Roe [14] Strichartz proved in [16] that if a doubly-infinite sequence \( \{f_k\} \) of functions on \( \mathbb{R}^n \) satisfies \( f_{k+1} = \Delta f_k \) and \( |f_k(x)| \leq M \) for all \( k = 0, \pm 1, \pm 2, \cdots \) and \( x \in \mathbb{R}^n \), then \( \Delta f_0(x) = -f_0 \). Strichartz also showed that the result fails for hyperbolic 3-space. This negative result can be indeed extended to any Riemannian symmetric space of noncompact type. Taking this into account we shall prove that for all Riemannian symmetric spaces of noncompact type the theorem actually holds true when uniform boundedness is modified suitably.

1. Introduction

Generalizing a result of Roe [14], Strichartz [16] proved the following theorem on \( \mathbb{R}^n \). (See also [11] and the references therein.)

**Theorem 1.0.1 (Strichartz).** Let \( \{f_j\}_{j \in \mathbb{Z}} \) be a doubly infinite sequence of measurable functions on \( \mathbb{R}^n \) such that for all \( j \in \mathbb{Z} \), (i) \( \|f_j\|_{L^\infty(\mathbb{R}^n)} \leq C \) for some constant \( C > 0 \) and (ii) for some \( \alpha > 0 \), \( Lf_j = \alpha f_{j+1} \) where \( L = \sum_{i=1}^{n} \partial^2_{x_i^2} \). Then \( Lf_0 = -\alpha f_0 \).

Strichartz also observed in the same paper [16] that the result holds true for Heisenberg groups \( \mathbb{H}^n \), but fails for hyperbolic 3-space. A slight generalization of the counter example given in [16] shows that in any Riemannian symmetric space \( X \) of noncompact type there is a sequence of functions \( \{f_j\} \) which satisfies the hypothesis (where the Laplace-Beltrami operator \( \Delta \) on \( X \) replaces \( L \)), but \( f_0 \) is not an eigenfunction of \( \Delta \) (see [12]). We take this negative result as our starting point. Aim of this paper is to prove an analogue of Strichartz’s result for all Riemannian symmetric spaces \( X = G/K \) of noncompact type.

From the counter example provided by Strichartz in [16] it is not difficult to perceive that the failure is influenced by the spectral properties of the Laplacian \( \Delta \) of the symmetric space. More precisely the failure is due to the difference between the \( L^2 \) and \( L^\infty \)-spectrum of \( \Delta \), which in turn depends on the exponential volume growth of the underlying manifold \( X \). This sets the task of searching for a possible analogue conducive to the structure of the space. We began our study in this direction with the rank one symmetric space in [12], where the situation was saved, substituting the \( L^\infty \)-norm by the weak \( L^2 \)-norm. However the use of weak \( L^2 \)-norm seems to be restrictive to the rank one case. The following observation can be considered as a first indication of this. We recall that the foremost examples of eigenfunctions of \( \Delta \) are the elementary spherical functions \( \phi_\lambda \) with \( \lambda \in \mathfrak{a}^\ast \). Unlike the rank one case, in general rank, these eigenfunctions do not belong to the weak \( L^2 \)-space. We also recall that on \( X \) the

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objects which correspond to \( e^{i(\lambda, x)} \) (on \( \mathbb{R}^n \)) are \( e_{\lambda, k} : x \mapsto e^{-(i\lambda + \rho)H(x^{-1}k)}, k \in K, \lambda \in \mathfrak{a}^* \). They are the basic eigenfunctions of \( \Delta \) which are constant on each horocycles \( \xi_{k,C} = \{ x \in G/K \mid H(x^{-1}k) = C \} \); but unlike their counterparts in the Euclidean case, \( e_{\lambda, k} \) are not \( L^\infty \)-functions, in particular they do not satisfy the hypothesis of Strichartz’s theorem. As in the Euclidean case, we have another set of prominent eigenfunctions namely the Poisson transforms \( P_\lambda F(x) = \int_{K/M} e_{\lambda, k}(x) F(k) dk, \lambda \in \mathfrak{a}^* \) of \( L^p \)-functions \( F \) on the boundary \( K/M \) of the space \( X \) for some \( p \geq 1 \). Taking all these into account we motivate ourselves to look for a size-estimate which accommodates a large class of eigenfunctions of \( \Delta \), including those mentioned above. One such is the so-called “Hardy-type” norm introduced in [15] and used effectively in [3]. We shall use this norm to formulate our main result, which is stated below. (For any unexplained notation see Section 2.)

**Theorem 1.0.2.** Let \( \{f_j\}_{j \in \mathbb{Z}} \) be a doubly infinite sequence of measurable functions on \( X \) such that for some real number \( \alpha \) and for all \( j \in \mathbb{Z} \):

(i) \( \Delta f_j = (\alpha^2 + |\rho|^2)f_{j+1} \),

(ii) for a fixed \( p \geq 1 \), \( \|f_j(a)\|_{L^p(K)} \leq C_p \phi_0(a) \) for all \( a \in \overline{A^*} \) and for a constant \( C_p > 0 \) depending only on \( p \).

Then \( \Delta f_0 = -(\alpha^2 + |\rho|^2)f_0 \). In particular if \( \alpha = 0, p > 1 \) then \( f_0(x) = P_0 F(x) = \int_{K} e^{-\rho H(x^{-1}k)} F(k) dk \) for some \( F \in L^p(K/M) \) and if \( \alpha = 0, p = 1 \) then \( f_0 = P_0 \mu(x) = \int_{K} e^{-\rho H(x^{-1}k)} d\mu(k) \) for some signed measure \( \mu \) on \( K/M \).

The particular case of \( p = \infty \) in the condition (ii) of the hypothesis simplifies as \( |f_j(x)| \leq C \phi_0(x) \) for all \( x \in X \) and \( j \in \mathbb{Z} \), and is close to the estimate used in Theorem 1.0.1. We take a \( \lambda \in \mathfrak{a}^* \) with \( |\lambda|^2 = \alpha^2 \). Then for any fixed \( k \in K \), the sequence of functions \( \{(-1)^j e_{\lambda, k}\}_{j \in \mathbb{Z}} \) satisfy the hypothesis with \( p = 1 \). For such a \( \lambda \) and for any \( 1 \leq p < \infty \), the sequence of Poisson transforms \( \{(−1)^j P_\lambda F\}_{j \in \mathbb{Z}} \) for any \( F \in L^p(K/M) \) also satisfy the hypothesis. (See Theorem 2.0.1 in Section 2.)

The result above may also be viewed from the following perspective. In [3] Ben sa¨ıd et. al. used the Hardy-type norm to characterize a large family of eigenfunctions of \( \Delta \) as Poisson integral of functions in \( L^P(K/M) \). However their result does not apply to the case where the eigenvalues are of the form \( |\lambda|^2 + |\rho|^2, \lambda \in \mathfrak{a}^*, \lambda \neq 0 \). This makes the situation very close to the Euclidean in the following sense. While it is well known that bounded harmonic functions on \( \mathbb{R}^n \) are constants, there is no simple characterization of bounded eigenfunctions of \( L \) on \( \mathbb{R}^n \) with nonzero real eigenvalues. Strichartz’s result on the other hand deals with the latter case. Analogously, Theorem 1.0.2 endeavors to “capture” eigenfunctions of \( \Delta \) with eigenvalues of the form mentioned above.

The space \( X = G/K \) enjoys a dichotomy as it can be viewed as a solvable Lie group, \( S = N \rtimes A \) through the Iwasawa decomposition of \( G = NAK \). The group \( S \) is amenable like \( \mathbb{R}^n \) and \( \mathbb{H}^n \), though unlike them \( S \) is nonunimodular. On \( S \), one considers a second order right \( S \)-invariant differential operator \( \mathcal{L} \) which is known as the distinguished Laplacian. Unlike \( \Delta \), the \( L^p \)-spectrum of \( \mathcal{L} \) for \( 1 \leq p < \infty \) coincides with the \( L^2 \)-spectrum (see e.g. [17]). This intrigues us to formulate a version of Theorem 1.0.2 substituting \( \Delta \) by \( \mathcal{L} \), which is the next result.
Theorem 1.0.3. Let \( \{f_j\}_{j \in \mathbb{Z}} \) be a doubly infinite sequence of measurable functions defined on \( S \). If for some real number \( \alpha > 0 \) and for some \( C > 0 \),
\[
\mathcal{L} f_j = (\alpha^2 + |p|^2) f_{j+1}, \quad |f_j(x)| \leq C\delta(x), \quad \text{for all} \ x \in S, \ \text{for all} \ j \in \mathbb{Z},
\]
where \( \delta \) is the modular function of \( S \), then \( \mathcal{L} f_0 = \alpha f_0 \).

Both of these results may be viewed as “exact” analogues of the Euclidean theorem. For Theorem 1.0.2 one can argue that any reasonable analogue of the object \( \phi_0 \) on \( \mathbb{R}^n \) is the constant function 1, while for Theorem 1.0.3 one may recall that for \( \mathbb{R}^n \) (and \( \mathbb{H}^n \)), \( \delta \equiv 1 \).

2. Notation and Preliminaries

For two positive functions \( f_1 \) and \( f_2 \) we shall write \( f_1 \asymp f_2 \) to mean there are positive constants \( C_1, C_2 \) such that \( C_1 f_1 \leq f_2 \leq C_2 f_1 \). For a measurable function \( f \) on \( \mathbb{R}^n \) we define its Euclidean Fourier transform at \( \lambda \in \mathbb{C}^n \) by
\[
\tilde{f}(\lambda) = \int_{\mathbb{R}^n} f(x)e^{-i\lambda \cdot x} \, dx,
\]
(where \( \lambda \cdot x \) is the Euclidean (real) inner product of \( \lambda \) and \( x \)), whenever the integral converges. Let \( S(\mathbb{R}^n) \) be the Schwartz space on \( \mathbb{R}^n \). Precisely \( S(\mathbb{R}^n) \) is the set of functions in \( C^\infty(\mathbb{R}^n) \) such that \( \mu_{r,s}(f) < \infty \) for all multiindex \( r \) and \( s > 0 \) where
\[
\mu_{r,s}(f) = \sup_{x \in \mathbb{R}^n} (1 + |x|)^s |D_r f(x)|.
\]
Here \( D_r = \frac{\partial^{r_1}}{\partial x_1} \cdots \frac{\partial^{r_n}}{\partial x_n} \), \( r = (r_1, \ldots, r_n) \) is a differential operator and \( |x| \) is the Euclidean norm of \( x \). The space \( S(\mathbb{R}^n) \) becomes a Frechet space with respect to the topology generated by the seminorms \( \mu_{r,s} \). The dual space of \( S(\mathbb{R}^n) \) is called the space of tempered distributions which will be denoted by \( S(\mathbb{R}^n)' \). The following facts are well known: (1) \( f \mapsto \tilde{f} \) is an isomorphism from \( S(\mathbb{R}^n) \) to itself, (2) using this isomorphism one can extend the notion of Fourier transform and derivative to \( S(\mathbb{R}^n)' \), (3) \( L^p \)-functions for \( 1 \leq p \leq \infty \) are tempered distributions.

The required preliminaries and notation related to the noncompact semisimple Lie groups and the associated symmetric spaces are standard and can be found for example in [6, 8, 9]. To make the article self-contained we shall gather only those results which will be used throughout this paper. Let \( G \) be a noncompact connected semisimple Lie group with finite centre and \( K \) be a maximal compact subgroup of \( G \). Let \( X = G/K \) be the associated Riemannian symmetric space of noncompact type. We let \( G = KAN \) denote a fixed Iwasawa decomposition of \( G \). Let \( \mathfrak{g}, \mathfrak{t}, \mathfrak{a} \) and \( \mathfrak{n} \) denote the Lie algebras of \( G, K, A \) and \( N \) respectively. Let \( \mathfrak{g}_C \) be the complexification of \( \mathfrak{g} \) and \( \mathcal{U}(\mathfrak{g}_C) \) be its universal enveloping algebra. We recall that the elements of \( \mathcal{U}(\mathfrak{g}_C) \) are identified with the left-invariant differential operators on \( G \) and there exists an anti-isomorphism \( \iota \) from \( \mathcal{U}(\mathfrak{g}_C) \) to algebra of right-invariant differential operators on \( G \). We choose and keep fixed throughout a system of positive restricted roots for the pair \((\mathfrak{g}, \mathfrak{a})\), which we denote by \( \Sigma^+ \). The multiplicity of a root \( \alpha \in \Sigma^+ \) will be denoted by \( m_\alpha \). As usual the half-sum of the elements of \( \Sigma^+ \) counted with their multiplicities will be denoted by \( \rho \). Let \( H : G \rightarrow \mathfrak{a} \) be the Iwasawa projection associated to the Iwasawa decomposition, \( G = KAN \). Then \( H \) is left \( K \)-invariant.
and right $MN$-invariant where $M$ is the centralizer of $A$ in $K$. The Weyl group of the pair $(G, A)$ will be denoted by $W$. Let $a^*$ be the real dual of $a$ and $a^*_C$ its complexification. Let $a_+$ (respectively $a^*_+$) denote the positive Weyl chamber in $a$ (respectively $a^*$). Let $A^+ = \exp a_+$ and $\overline{A^+}$ be the closure of $A^+$.

We recall that the Killing form $B$ restricted to $a$ is a positive definite inner product on $a$ and it gives a $W$-equivariant isomorphism of $a$ with $a^*$. For $\lambda \in a^*$ we denote the corresponding element in $a$ by $H_\lambda$. Let $\dim a = l$, i.e. $l$ is the rank of $X$. We will identify $a$ and $a^*$ with $\mathbb{R}^l$, as an inner product space, with the inner product on $\mathbb{R}^l$ being the pull-back of the Killing form. This inner product on $\mathbb{R}^l$ as well as on $a$, $a^*$ will be referred to as the Killing inner product and will be denoted by $\langle, \rangle$. The associated norm will be denoted by $| \cdot |$. We hope that this symbol will not be confused with the absolute value symbol. Since $\exp : a \rightarrow A$ is an isomorphism, as a group $A$ can be identified with $\mathbb{R}^l$.

For $x \in G$, we define $\sigma(x) = d(xK, K)$ where $d$ is the canonical distance function for $X = G/K$ coming from the Riemann metric on $X$ induced by the Killing form restricted to $p$. Here $g = T \oplus p$ (Cartan decomposition) and $p$ can be identified with the tangent space at $eK$ of $G/K$. The function $\sigma(x)$ is $K$-biinvariant and continuous. Note that for $x = k_1ak_2$ (polar decomposition), $k_1, k_2 \in K$, $a \in A^l$, $\sigma(x) = \sigma(a) = |\log a|$, the Killing norm of $\log a$, where $\log a$ is the unique element in $a$ such that $\exp(\log a) = a$.

On $X$ we fix the measure $dx$ which is induced by the metric we obtain from $B$. As the metric is $G$-invariant, so is $dx$. On $G$ we fix the Haar measure $dg$ satisfying

$$\int_X f(x)dx = \int_g f(g)dg,$$

for every integrable function $f$ on $X$ which we also consider as a right $K$-invariant function on $G$. While dealing with functions on $X$, we may slurry over the difference between the two measures.

Through the identification of $A$ with $\mathbb{R}^l$ we use the Lebesgue measure on $\mathbb{R}^l$ as the Haar measure $da$ on $A$. As usual on the compact group $K$ we fix the normalized Haar measure $dk$ (i.e. $\text{vol}(K) = \int_K dk = 1$). Finally we fix the Haar measure $dn$ on $N$ by the condition that

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank) dk dn da$$

holds for every integrable function $f$ on $G$.

Following integral formulae correspond to the Iwasawa and polar decompositions respectively. For any $f \in C_c^\infty(G)$,

$$\int_G f(x)dx = \int_K \int_a \int_N f(hK) e^{2\rho(H)}dn dh dk$$

and

$$\int_G f(x)dx = \int_K \int_{A^+} f(k_1 exp Hk_2) J(H) dh dk_1 dk_2,$$

where $dH$ is the Lebesgue measure of $\mathbb{R}^l$ with which $a$ is identified, $dn$, $dk$ are the normalized Haar measures of $K$, $N$ respectively and

$$J(H) = C \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha}, H \in a^+, C$$

being a normalizing constant.
For a measurable function \( f \) on \( X \) we define its \( K \)-invariant part \( K(f)(x) = \int_K f(kx)dk \). We shall call \( K \) the \( K \)-averaging operator. A function on \( X \) is \( K \)-invariant if \( f(kx) = f(x) \), for all \( x \in X, k \in K \), equivalently \( f(x) = K(f)(x) \). We note that \( \int_X K(f)(x)g(x)dx = \int_X f(x)K(g)(x)dx \) whenever the integrals on both sides converge. It follows that if \( K(f) = 0 \) and \( K(g) = g \) then \( \int_X f(x)g(x)dx = 0 \).

For \( \lambda \in \mathfrak{a}_C^* = C^l \), the elementary spherical function \( \phi_\lambda \) is given by

\[
\phi_\lambda(x) = \int_K e^{-i(\lambda + \rho)H(x^{-1}k)}dk.
\]

It is a \( K \)-biinvariant eigenfunction of the Laplace-Beltrami operator \( \Delta \): \( \Delta \phi_\lambda = -(|\lambda|^2 + |\rho|^2)\phi_\lambda \) and \( \phi_\lambda = \phi_{w,\lambda} \) for \( w \in W \). In fact, \( \phi_\lambda \) are particular examples of more general class of eigenfunctions called the Poisson transforms \( P_\lambda F \) defined in the introduction. Our main results will use certain size-estimates and characterizations of these Poisson transforms. In this regard, we quote the following particular cases of a more general result proved in [3, Proposition 3.4, Proposition 3.6].

**Theorem 2.0.1** (Ben Said et. al.). Let \( 1 \leq p \leq \infty \) be fixed and \( F \in L^p(K/M) \). Then

\[
\sup_{x \in X} \phi_0(x)^{-1} \left( \int_K |P_\lambda F(kx)|^p dk \right)^{1/p} = \|F\|_{L^p(K/M)} \text{ when } 1 \leq p < \infty \text{ and } \sup_{x \in X} \phi_0(x)^{-1}|P_\lambda F(x)|dk = \|F\|_{L^{\infty}(K/M)} \text{ when } p = \infty.
\]

Moreover, suppose that a function \( f \) on \( G/K \) satisfies \( \Delta f = -|\rho|^2 f \) and \( \sup_{x \in X} \phi_0(x)^{-1} \left( \int_K |f(kx)|^p dk \right)^{1/p} < \infty \) for some \( 1 \leq p < \infty \) or \( \sup_{x \in X} \phi_0(x)^{-1}|f(x)| < \infty \), then there exists unique \( F \in L^p(K/M) \) when \( 1 < p \leq \infty \) and a signed measure \( \mu \) when \( p = 1 \) such that \( f = P_\lambda F \) or \( f = P_\mu \).

For a suitable function \( f \) on \( G/K \) its spherical Fourier transform \( \hat{f}(\lambda) \) for \( \lambda \in C^l \) is defined by

\[
\hat{f}(\lambda) = \int_G f(x)\phi_\lambda(x)dx.
\]

It is then clear that \( \hat{f}(\lambda) = \hat{f}(w\lambda) \) for all \( w \in W \).

We now need to introduce the notion of Schwartz spaces and distributions on \( X \). The \( L^2 \)-Schwartz space \( C^2(G) \) is defined as the set of all \( C^\infty \)-functions on \( G \) such that

\[
\gamma_r,g_1,g_2(f) = \sup_{x \in G} |(tg_1)g_2 f(x)|\phi_0(x)^{-1}(1 + \sigma(x))^r < \infty,
\]

for all nonnegative integers \( r \) and \( g_1, g_2 \in U(g_C) \). Let \( C^2(G//K) \) be the set of \( K \)-biinvariant functions in \( C^2(G) \). We recall that \( (\mathbb{H}) \) the spherical Fourier transform is an isomorphism from \( C^2(G//K) \) to \( S(\mathbb{R^l})_W \), where \( S(\mathbb{R^l})_W \) is the subspace of \( W \)-invariant functions in \( S(\mathbb{R^l}) \).

The dual space of \( C^2(G//K) \) will be denoted by \( C^2(G//K)' \) and its elements will be called \( L^2 \)-tempered distributions. The translation of \( T \in C^2(G//K)' \) by an element \( y \in G \) and its convolution with a function \( g \in C^2(G//K) \) are defined in the usual way by \((\ell_y T)f = T(\ell_{y^{-1}}f)\) and \( T \ast g = (\ell_y T)(g) \), where \( \ell_y f(x) = f(y^{-1}x) \). An \( L^2 \)-tempered distribution \( T \in C^2(G//K)' \) is called \( K \)-invariant if \( (T, \psi) = (T, K(\psi)) \), \( \psi \in C^2(G//K) \). The set of \( K \)-invariant \( L^2 \)-tempered distributions on \( G/K \) will be denoted by \( C^2(G//K)' \). The heat kernel \( h_t \) for \( t > 0 \) is a \( K \)-invariant function in \( C^2(G//K) \) which is defined using the isomorphism of \( C^2(G//K) \) with \( S(\mathbb{R^l})_W \), prescribing its spherical Fourier transform \( \hat{h}_t(\lambda) = \).
Theorem 2.0.2. The Abel transform $A : C^2(G//K) \to S(\mathbb{R}^l)_W$ is a topological isomorphism.

The use of Abel transform in our proof is somewhat similar to that of [2] (see also [10]).

3. Roe-Strichartz theorem for Laplace-Beltrami operator

3.1. **Distribution-version of the Euclidean theorem.** Since the real rank of $G$ is $l$, it follows that $W$ is a subgroup of $O(l)$ as $W$ preserves the inner product induced by the Killing form. For $T \in S(\mathbb{R}^l)'_W$ and $w \in W$ we define

$$(wT)f = T(wf) \text{ for all } f \in S(\mathbb{R}^l), \text{ where } wf(x) = f(wx).$$

The $W$-invariant component $f^\#$ of $f \in S(\mathbb{R}^l)$ (respectively $T^\#$ of $T \in S(\mathbb{R}^l)'_W$) is defined as

$$f^\# = \frac{1}{|W|} \sum_{w \in W} wf, \text{ (respectively } T^\# = \frac{1}{|W|} \sum_{w \in W} wT) \text{ where } |W| \text{ denotes the cardinality of } W.$$  

A tempered distribution $T$ is called $W$-invariant if $T^\# = T$. It is easy to verify that for $T \in S(\mathbb{R}^l)'_W$ and $f \in S(\mathbb{R}^l)$, $T^\# f = T f^\#$ and in particular when $T$ is $W$-invariant $T f = T f^\#$. It is also not difficult to see that the Laplacian $L$ of $\mathbb{R}^l$ commutes with $W$-action and hence if $f \in S(\mathbb{R}^l)_W$ (respectively $T \in S(\mathbb{R}^l)'_W$) then $L f \in S(\mathbb{R}^l)'_W$ (respectively $L T \in S(\mathbb{R}^l)'_W$).

We shall first prove the following version of the Euclidean theorem. Below $L_1 = L - |\rho|^2$.

**Theorem 3.1.1.** Let $\{T_j\}$ be a doubly infinite sequence of $W$-invariant tempered distributions on $\mathbb{R}^l$ such that

(i) $L_1 T_j = z T_{j+1}$ for some $z \in \mathbb{C}$, $|z| \geq |\rho|^2$.
(ii) for all $\psi \in S(\mathbb{R}^l)'_W$, $|\langle T_j, \psi \rangle| \leq M \mu(\psi)$ for some fixed seminorm $\mu$ of $S(\mathbb{R}^l)'_W$ and $M > 0$.

Then $L_1 T_0 = -|z| T_0$.

This theorem is essentially proved in [16] [11]. For the sake of completeness, we include here only a very brief sketch of the argument.

**Proof.** Since $|z| \geq |\rho|^2$, we write $z = (\alpha^2 + |\rho|^2)e^{i\theta}$, $\alpha \geq 0, \theta \in \mathbb{R}$. For a $T \in S(\mathbb{R}^l)'$, by $\tilde{T}$ we denote its Euclidean Fourier transform. For $T_0$ to be an eigendistribution of $L_1$ with eigenvalue $-|z|$, it is
necessary that the distribution $T_0$ is supported on the sphere $\{x \in \mathbb{R}^l \mid |x| = \alpha^2 + |\rho|^2\}$. First we shall prove that. Then following exactly the steps of [11, p.210] one can show that there exists $N \geq 0$ such that $(L_1 + (\alpha^2 + |\rho|^2))^{N+1}T_0 = 0$, which will finally lead to $(L_1 + (\alpha^2 + |\rho|^2))\tilde{T}_0 = 0$ (see [11, p. 210–211] for details).

It follows from the hypothesis that

$$\tilde{T}_0 = (-1)^j e^{ij\theta} \left( \frac{\alpha^2 + |\rho|^2}{|x|^2 + |\rho|^2} \right)^j \tilde{T}_j,$$

where $x$ is a dummy variable. We take a function $\phi \in S(\mathbb{R}^l)$ such that Support $\phi \subseteq \{x \in \mathbb{R}^l \mid |x| \geq \alpha + \varepsilon\}$, for some $\varepsilon > 0$. Hence Support $\phi^\# \subseteq \{x \in \mathbb{R}^l \mid |x| \geq \alpha + \varepsilon\}$. Let $\psi$ be the Euclidean Fourier transform of the function $(\alpha^2 + |\rho|^2)^j(|x|^2 + |\rho|^2)^{-j}\phi$. Then

$$|\langle T_0, \phi \rangle| = |\langle T_j, \psi^\# \rangle| = |\langle T_j, \psi^\# \rangle| \leq M\mu(\psi^\#) \leq M\gamma_{\beta, \tau} \left[ \frac{(\alpha^2 + |\rho|^2)}{|x|^2 + |\rho|^2} \right]^j \phi^\#$$

where $\gamma_{\beta, \tau}(f) = \sup_{x \in \mathbb{R}^l}(1 + |x|)^\beta |D^\tau f(x)|$ for some positive integer $\beta$ and multi index $\tau$. It follows from the fact that $|x| \geq \alpha + \varepsilon$ on the support of $\phi$, that the right hand side of  3.1.1 goes to zero as $j \to \infty$. A similar argument taking $j \to -\infty$ will show that $\langle T_0, \phi \rangle = 0$ for all $\phi \in S(\mathbb{R}^l)$ with support of $\phi \subseteq \{x \in \mathbb{R}^l \mid |x| \leq \alpha - \varepsilon\}$. This proves that distributional support of $\tilde{T}_0$ is contained in the sphere $\{x \in \mathbb{R}^l \mid |x| = \alpha\}$.

3.2. Distribution-version of the theorem for the symmetric spaces. First we shall prove a version of the Roe-Strichartz theorem for $K$-invariant tempered distributions and then we shall generalize the result for arbitrary tempered distributions.

**Theorem 3.2.1.** If for a doubly infinite sequence $\{T_j\}$ of $K$-invariant $L^2$-tempered distributions on $X$, $\Delta T_j = zT_{j+1}$ for some $z \in \mathbb{C}$ with $|z| \geq |\rho|^2$ and for a fixed seminorm $\nu$ of $C^2(X)$, $|\langle T_j, \phi \rangle| \leq M\nu(\phi)$ for some $M > 0$ for all $\phi \in C^2(G//K)$, then $\Delta T_0 = -zT_0$.

**Proof.** Since $A : C^2(G//K) \to S(\mathbb{R}^l)_W$ is an isomorphism, its adjoint $A^* : S(\mathbb{R}^l)'_W \to C^2(G//K)'$ and $B = (A^*)^{-1} : C^2(G//K)' \to S(\mathbb{R}^l)'_W$ are isomorphisms (see [?, p. 541]).

We claim that for $T \in C^2(G//K)'$, $B(\Delta T) = L_1BT$. We note that for any $g \in C^2(G//K)$, $L_1Ag = \mathcal{A}\Delta g$. Indeed by the slice-projection theorem (see section 2) $\hat{\Delta g}(\lambda) = \hat{g}(\lambda)$ for any $\lambda \in \mathfrak{a}^*$. Therefore the Euclidean Fourier transform of $L_1Ag$ is $-(|\lambda|^2 + |\rho|^2)\hat{\Delta g}(\lambda) = -(|\lambda|^2 + |\rho|^2)\hat{g}(\lambda)$. Again by slice-projection theorem, Euclidean Fourier transform of $\mathcal{A}\Delta g$ at $\lambda$ is $\hat{\Delta g}(\lambda) = -(|\lambda|^2 + |\rho|^2)\hat{g}(\lambda)$. The assertion now follows from the injectivity of the Fourier transform. Using this we get for $g \in C^2(G//K)$ and $\mathcal{A}g = F \in S(\mathbb{R}^l)'_W$,

$$\langle L_1BT, F \rangle = \langle L_1BT, Ag \rangle = \langle BT, L_1Ag \rangle = \langle BT, \mathcal{A}\Delta g \rangle = \langle A^*BT, \Delta g \rangle = \langle T, \Delta g \rangle = \langle \Delta T, g \rangle = \langle A^*B\Delta T, g \rangle = \langle B\Delta T, Ag \rangle = \langle B\Delta T, F \rangle.$$

This shows that $L_1BT = B(\Delta T)$

The condition $\Delta T_j = zT_{j+1}$ implies $B(\Delta T_j) = zB(T_{j+1})$. Applying the identity $B(\Delta T) = L_1BT$ we have $L_1BT_j = zBT_{j+1}$.
Next we shall show that there exists a fixed seminorm $\mu$ of $S(\mathbb{R}^1)_W$ such that $|\langle BT_j, \psi \rangle| \leq M\mu(\psi)$ for all $\psi \in S(\mathbb{R}^1)_W$. Indeed, using that $A : C^2(G//K) \to S(\mathbb{R}^1)_W$ is an isomorphism, for every $\psi \in S(\mathbb{R}^1)_W$ we have a $\phi \in C^2(G//K)$ such that $A(\phi) = \psi$ and a seminorm $\mu$ on $S(\mathbb{R}^1)_W$ such that $\nu(\phi) \leq \mu(\psi)$ for all such pairs $\phi \in C^2(G//K)$ and $\psi \in S(\mathbb{R}^1)_W$. Hence,

$$|\langle BT_j, \psi \rangle| = |\langle BT_j, A\phi \rangle| = |\langle T_j, \phi \rangle| \leq M\nu(\phi) \leq M\mu(\psi).$$

Thus the sequence $\{BT_j\}$ satisfies the hypothesis of Theorem 3.1.1 and hence

$$L_1BT_0 = -|z|BT_0.$$ 

Using again the identity $B(\Delta T) = L_1BT$ we get $B(\Delta T_0) = B(-|z|T_0)$. Since $B$ is injective we have, $\Delta T_0 = -|z|T_0$. This completes the proof. \hfill \Box

Now we shall withdraw the condition that $T_j$ are $K$-invariant.

**Theorem 3.2.2.** If for a doubly infinite sequence $\{T_j\}$ of $L^2$-tempered distributions on $X$, $\Delta T_j = zT_{j+1}$ for some $z \in \mathbb{C}$ with $|z| \geq |\rho|^2$ and for a fixed seminorm $\nu$ of $C^2(X)$, $|\langle T_j, \phi \rangle| \leq M\nu(\phi)$ for some $M > 0$ for all $\phi \in C^2(X)$, then $\Delta T_0 = -|z|T_0$.

**Proof.** We need to use frequently the fact that $\Delta$ commutes with translations and the $K$-averaging operator $K$ defined in section 2. It is clear from the condition $\Delta T_j = zT_{j+1}$ that if one element of the sequence $\{T_j\}$ is zero, then every elements of the sequence is zero and we have nothing to prove. Therefore we assume that none of the $T_j$ are zero. We fix $j \in \mathbb{Z}$. We claim that there is an $x \in G$ such that $\ell_xT_j$ has nonzero $K$-invariant part. Indeed if $K(\ell_xT_j) = 0$ for all $x \in G$, then $\langle \ell_xT_j, h_t \rangle = 0$ for all $t > 0$ since the heat kernel $h_t$ is a $K$-invariant function (see section 2). That is $T_j * h_t \equiv 0$. But $T_j * h_t \to T_j$ as $t \to 0$ in the sense of distribution. Therefore $T_0 = 0$ which contradicts our assumption. We note that this also shows that if for two $L^2$-tempered distribution $T$ and $T'$, $K(\ell_xT) = K(\ell_xT')$ for all $x \in G$, then $T = T'$.

Next we claim that if $K(\ell_yT_0) \neq 0$ for some $y \in G$, then $K(\ell_yT_j) \neq 0$ for all $j \in \mathbb{Z}$. It is enough to show that if $K(\ell_yT_0) \neq 0$ then $K(\ell_yT_{-1}) \neq 0$ and $K(\ell_yT_1) \neq 0$. Indeed if $K(\ell_yT_{-1}) = 0$ then $\Delta K(\ell_yT_{-1}) = 0$ which implies $K(\ell_yT_0) = 0$ as $\Delta T_{-1} = zT_0$ for $z \neq 0$. On the other hand, if $K(\ell_yT_1) = 0$, then $\langle \ell_yT_1, \psi \rangle = 0$ for all $\psi \in C^2(G//K)$. That is $\langle \ell_y\Delta T_0, \psi \rangle = 0$ and hence $\langle \ell_yT_0, \Delta \psi \rangle = 0$. Using the characterization of the image of $C^2(G//K)$ under spherical Fourier transform (see section 2) we see that for any $\phi \in C^2(G//K)$, $\hat{\phi}(\lambda)(|\lambda|^2 + |\rho|^2)^{-1} \in S(\mathbb{R}^1)_W$. Hence $\phi$ can be written as $\phi = \Delta \psi$ for some $\psi \in C^2(G//K)$. Thus $\langle \ell_yT_0, \phi \rangle = 0$ for any $\phi \in C^2(G//K)$, i.e. $K(\ell_yT_0) = 0$.

Our aim now is to show that for any $y \in G$, the sequence $\{K(\ell_yT_j)\}$ of $K$-invariant distributions satisfies the hypothesis of Theorem 3.2.1. Since $\Delta$ commutes with the $K$-averaging operator and translations, it follows from the hypothesis $\Delta T_j = zT_{j+1}$ that $\Delta K(\ell_yT_j) = zK(\ell_yT_{j+1})$.

It now remains to show that for the seminorm $\nu$ of $C^2(G)$ given in the hypothesis of the theorem and for any $\psi_1 \in C^2(G//K)$, $|\langle K(\ell_yT_j), \psi_1 \rangle| \leq C_y\nu(\psi_1)$. First we note that for any $y \in G$ and $\psi \in C^2(G)$,
|ν(ℓ_0ψ)| ≤ C_yν(ψ), where the constant C_y depends only on y. Indeed, using φ_0(x) ≤ C_yφ_0(yx) for all x ∈ G ([3] Proposition 4.6.3., (vi)) and triangle inequality σ(yx) ≤ σ(x) + σ(y), we have,

\[ \nu(\ell_y \psi) = \sup_{x \in X} |(g_1)g_2 \psi(y^{-1}x)|φ_0(x)^{-1}(1 + \sigma(x))^L \quad (D \in U(\mathfrak{g}_\mathbb{C}), L > 0) \]

\[ = \sup_{x \in X} |(g_1)g_2 \psi(x)|φ_0(yx)^{-1}(1 + \sigma(yx))^L \]

\[ \leq C \sup_{x \in X} |(g_1)g_2 \psi(x)|φ_0(x)^{-1}(1 + \sigma(x))^L(1 + \sigma(y))^L = C_yν(ψ), \]

where L > 0 and g_1, g_2 ∈ U(\mathfrak{g}_\mathbb{C}) are fixed. Since |⟨T_j, ψ⟩| ≤ Mν(ψ) for any ψ ∈ C^2(G), it follows that for ψ_1 ∈ C^2(G//K),

\[ |⟨K(ℓ_y T_j), ψ_1⟩| = |⟨ℓ_y T_j, ψ_1⟩| = |⟨T_j, ℓ_{y^{-1}} ψ_1⟩| ≤ Mν(ℓ_{y^{-1}} ψ_1) \leq C_y Mν(ψ_1). \]

From Theorem 3.2.1 we conclude that

\[ \Delta K(ℓ_y(T_0)) = -z|K(ℓ_y(T_0))| \text{ for all } y \in G. \]

(Not that if K(ℓ_y(T_0)) = 0 for some y ∈ G, then the identity ΔK(ℓ_y(T_0)) = -z|K(ℓ_y(T_0))| is trivial.) Again appealing to the fact that Δ commutes with translations and K-averaging operator we have

\[ K(ℓ_y(ΔT_0)) = K(ℓ_y(−zT_0)) \text{ for all } y \in G. \]

This implies (see the first paragraph of the proof) that

\[ ΔT_0 = −z|T_0| \text{ which is the assertion.} \]

We define Δ_1 = (Δ + |ρ|^2). Then a step by step adaptation of the above proof yields the following, which we shall use in the last section.

**Theorem 3.2.3.** If for a doubly infinite sequence \( \{T_j\} \) of L^2-tempered distributions on X, Δ_1T_j = zT_{j+1} for some nonzero z ∈ C and for a fixed seminorm ν of C^2(X), |⟨T_j, ϕ⟩| ≤ Mν(ϕ) for some M > 0 for all ϕ ∈ C^2(X), then Δ_1T_0 = |z|T_0.

### 3.3. Proof of Theorem 1.0.2

Using Theorem 3.2.2 we shall now prove Theorem 1.0.2.

**Proof.** By Theorem 5.2.2 it suffices to show that for all j ∈ Z, f_j ∈ C^2(X)' and |⟨f_j, ϕ⟩| ≤ C_γ(ϕ) for all ϕ ∈ C^2(X), for a fixed seminorm γ of C^2(X) defined by γ(ϕ) = sup_x |ϕ(x)|(1 + σ(x))^Mφ_0^{-1}(x) with M > 0 sufficiently large. Indeed,

\[ |∫_X f_j(x)ϕ(x)dx| ≤ C ∫_{K×\mathbb{R}_+} |f_j(k \exp H)||ϕ(k \exp H)|J(H)dHdk \]

\[ = Cγ(ϕ) ∫_{\mathbb{R}_+} \left( ∫_K |f_j(k \exp H)|^p \right)^{1/p} \frac{ϕ_0(\exp H)}{(1 + |H|)^M} J(H)dHdk \]

\[ ≤ Cγ(ϕ) ∫_{\mathbb{R}_+} φ_0(\exp H)^2(1 + |H|)^{-M} J(H)dH = Cγ(ϕ), \]

Moreover when α = 0, we apply [3] Theorem 3.4. □

### 4. ROE-STRICHARTZ THEOREM FOR DISTINGUISHED LAPLACIAN

The main result of this section is an analogue of Theorem 1.0.2 for a right invariant second order differential operator which in the context of \( \mathbb{R}^l \) is nothing but the Laplace Beltrami operator of \( \mathbb{R}^l \). This is known as the *distinguished Laplacian* of X. We shall make it precise now. Let G = NAK
be the Iwasawa decomposition of $G$ and $S$ be the solvable Lie group $N \rtimes A$. We can identify the manifold $S$ with the Riemannian symmetric space $G/K$. The image of the $G$-invariant measure on $G/K$ under this identification corresponds to the left Haar measure on $S$ and the Riemannian metric on $G/K$ corresponds to a left-invariant metric on $S$. In a similar fashion we can identify functions and differential operators on $G/K$ with those on $S$. To define the distinguished Laplacian $\mathcal{L}$ we first consider the inner product $\langle X, Y \rangle = B(X, \theta Y)$ on $g$ where $B$ is the Cartan killing form and $\theta$ is a Cartan involution. With respect to the above inner product the decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is orthogonal. We choose an orthonormal basis $\{H_1, \ldots, H_l, X_1, \ldots, X_m\}$ of $\mathfrak{s}$ such that $\text{span}\{H_1, \ldots, H_l\} = \mathfrak{a}$, $\text{span}\{X_1, \ldots, X_m\} = \mathfrak{n}$ and we view these elements as right invariant vector fields in the usual way. The distinguished Laplacian $\mathcal{L}$ is defined as (see [4, p.108]),

$$\mathcal{L} = -[H_1^2 + \cdots + H_l^2 + 1/2(X_1^2 + \cdots + X_m^2)].$$

The operator $\mathcal{L}$ is essentially self adjoint on $L^2(S)$ with respect to the left Haar measure of $S$ and enjoys a special relationship with the Laplace-Beltrami operator $\Delta$ when viewed as a left-invariant operator on the solvable group $S$. This relation is explained below as it is crucial for our purpose. For a function $f$ we define $\tilde{f}(x) = f(x^{-1})$ for $x \in S$, where $x^{-1}$ is the inversion of the group $S$. We recall that $\Delta_1$ denotes the operator $- (\Delta + |\rho|^2)$. It then follows that for all $x \in S$ (see [4], p.108),

$$\delta^{1/2}(x)(\Delta_1 \delta^{1/2} \tilde{f})(x^{-1}) = \mathcal{L}f(x),$$

equivalently

$$\Delta_1(\delta^{1/2} \tilde{f})(x) = \delta^{1/2}(x)(\mathcal{L}f)(x^{-1}),$$

where we recall $\delta(a) = e^{-2\sigma(\log a)}$, for $a \in A$ and $n \in N$. It follows trivially that $\Delta_1 f = \lambda f$ for some $\lambda \in \mathbb{C}$ if and only if $\mathcal{L}(\delta^{1/2} \tilde{f}) = \lambda(\delta^{1/2} \tilde{f})$. This relation between the Laplacians yields Theorem 1.0.3 stated in the introduction.

**Proof of Theorem 1.0.3** Let $g_j = \delta^{1/2} \tilde{f}_j$ for all $j \in \mathbb{Z}$. By (4.0.1)

$$\Delta_1 g_j = \Delta_1 \delta^{1/2} \tilde{f}_j = \delta^{1/2} \tilde{L}\tilde{f}_j = \alpha \delta^{1/2} \tilde{f}_{j+1} = \alpha g_{j+1}.$$

It is also clear that $|f_j(x)| < C\delta(x)$ implies $|g_j(x)| \leq C\delta^{-1/2}(x)$. We recall that $\delta^{-1/2}(x) = e^{-\sigma(H(x^{-1}))}$ and $K(\delta^{-1/2})(x) = \phi_0(x)$. For a function $\phi \in C^2(X)$,

$$|\int_X g_j(x)\phi(x)dx| \leq C \int_X \delta(x)^{-1/2} |\phi(x)|dx \leq C\gamma(\phi) \int_X \delta(x)^{-1/2} \phi_0(x)(1 + \sigma(x))^{-M}dx = C\gamma(\phi) \int_X \phi_0(x)^2(1 + \sigma(x))^{-M}dx = C\gamma(\phi),$$

where $\gamma$ is a seminorm of $C^2(X)$ defined by $\gamma(\phi) = \sup_{x \in X} |\phi(x)|(1 + \sigma(x))^{-M} \phi_0^{-1}(x)$ for some sufficiently large $M > 0$. Thus the sequence $\{g_j\}$ satisfies the hypothesis of Theorem 5.2.8 and hence $\Delta_1 g_0 = \alpha g_0$. Using (4.0.1) again we get $\mathcal{L}f_0 = \alpha f_0$ which is the assertion. \hfill \Box

We conclude with the observation that despite the fact that the distinguished Laplacian $\mathcal{L}$ has some similarities with the usual Laplacian $L$ on $\mathbb{R}^l$ (see Introduction), a straightforward analogue of the Euclidean result of Strichartz [10] is not a possibility. Following counter example will establish this.
Counter Example: We will produce two bounded eigenfunctions $\psi_1$ and $\psi_2$ of $\mathcal{L}$ with eigenvalues $-4|\rho|^2$ and $4|\rho|^2$ respectively. We can then define $f_j = (-1)^k \psi_1 + \psi_2$, $k \in \mathbb{Z}$. It is then clear that the above sequence is uniformly bounded with $\mathcal{L}(f_j) = 4|\rho|^2((-1)^{k+1}\psi_1 + \psi_2) = 4|\rho|^2f_{j+1}$ but $f_0 = \psi_1 + \psi_2$ is not an eigenfunction of $\mathcal{L}$. We define $\psi_1 = \delta^{1/2}\phi_{2\rho}$. Since $\Delta_1(\phi_{2\rho}) = -(\Delta + |\rho|^{2}I)\phi_{2\rho} = 4|\rho|^2\phi_{2\rho}$ it follows from (4.0.1) that $\mathcal{L}\psi_1 = 4|\rho|^2\psi_1$. Since $|\phi_{2\rho}(x)| \leq C_\rho e^{-D(x)}$ and $\sigma(na) \geq |\log a|$ it follows that $|\psi_1(na)| \leq C_\rho e^{-\rho \log a} e^{-\rho |\log a|} \leq C$. Let $\psi_2$ be the constant function 1. We shall show that $\psi_2$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $-4|\rho|^2$. We define $F_{\lambda}(na) = e^{-(i\lambda + \rho)H(a^{-1}n^{-1})}$ then $\hat{F}_{2\rho}(na) = e^{\rho H(na)} = e^{\rho \log a}$ and hence $\delta^{1/2}(na) \hat{F}_{2\rho}(na) = 1$. But since $\hat{F}_{2\rho}$ is an eigenfunction of $\Delta$ with eigenvalue $3|\rho|^2$ it follows that $\Delta_1 \hat{F}_{2\rho} = -4|\rho|^2\hat{F}$. Using (4.0.1) we have

$$\mathcal{L}\psi_1 = \mathcal{L}(\delta^{1/2}\hat{F}) = \delta^{1/2}(\Delta_1 \delta^{1/2} \delta^{-1/2}\hat{F}) = -4|\rho|^2 \delta^{1/2}\hat{F} = -4|\rho|^21.$$ 

5. Concluding Remarks

1. In view of the results in [12] and in [3], it is natural to expect the following result.

Conjecture 5.0.1. Fix $q \in (1, 2)$. Let $\{f_j\}_{j \in \mathbb{N}}$ be an infinite sequence of measurable functions on $X$ such that for all $j \in \mathbb{N}$:

(i) $\Delta f_j = (4\rho^2/qq')f_{j+1}$,

(ii) for a fixed $p \geq 1$, $\|f_j(a)\|_{L_p(K)} \leq C_p \phi_{\gamma \rho}(a)$ for all $a \in \mathcal{A}$ and for a constant $C_p > 0$ depending only on $p$.

Then $\Delta f_0 = -(4\rho^2/qq')f_0$. In particular if $p > 1$ then $f_0(x) = P_{\gamma \rho}F(x)$ for some $F \in L^p(K/M)$ and if $p = 1$ then $f_0 = P_{\gamma \rho}\mu(x)$ for some signed measure $\mu$ on $K/M$.

2. A recent paper [13] studies the $L^p$-Schwartz space isomorphisms and related analysis in the context of Heckman-Opdam hypergeometric functions, which generalizes analysis of $K$-biinvariant functions on a noncompact connected semisimple Lie group with finite centre. It should be possible to prove an analogue of our result in this set-up, through similar steps.

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