Stable minimal hypersurfaces in $\mathbb{R}^{N+1+\ell}$ with singular set an arbitrary closed $K \subset \{0\} \times \mathbb{R}^\ell$

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0 Introduction

With respect to a $C^\infty$ metric which is close to the standard Euclidean metric on $\mathbb{R}^{N+1+\ell}$, where $N \geq 7$ and $\ell \geq 1$ are given, we here construct a class of embedded $(N+\ell)$-dimensional hypersurfaces (without boundary) which are minimal and strictly stable, and which have singular set equal to an arbitrary preassigned closed subset $K \subset \{0\} \times \mathbb{R}^\ell$. A precise statement of the theorem is given in § 1 below, and includes examples in the lowest dimension possible for embedded stable minimal hypersurfaces with non-isolated singularities—which is dimension 8 in $\mathbb{R}^9$.

Thus the question is settled, with a strong affirmative, as to whether there can be "gaps" (as in [Sim95]) or even fractional dimensional parts in the singular set. Such questions, for both stable and unstable minimal submanifolds, remain open in all dimensions in the case of real analytic metrics and in particular for the standard Euclidean metric.

Whether or not there can be examples like those established here in the case of low dimensional submanifolds which are minimal with respect to smooth or real analytic metrics also remains largely an open question. In this direction, Zhenhua Liu [Liu20] has recently constructed examples of 3-dimensional minimizers (in higher codimension) which have singular set consisting of the union of an arbitrary number of arcs.

The methods used in the present paper are primarily PDE methods, utilizing solutions and supersolutions of the symmetric minimal surface equation (SME) and an implicit function theorem argument in combination with a Liouville-type theorem (from [Sim21]) for stable minimal hypersurfaces which lie on one side of a cylindrical hypercone. The SME is ideal for these constructions, since it admits a rich class of singular solutions while at the same time, as discussed in [24] having nice continuity and Lipschitz estimates, and it can also be conveniently modified to handle the class of smooth ambient metrics introduced here. Additionally the method enables us to obtain a rather precise description of the shape of the singular examples—see Theorem 1.1 and Remark 0.1 below.

The proof of the main theorem, including the selection of appropriate metrics but deferring the proof of strict stability, is given in § 3 below, contingent on having a suitable family of solutions of the SME. In § 4 the existence of a such a family is established, using preliminaries established in § 1.1. The strict stability of the examples obtained in § 3 is discussed at the conclusion of § 4 (see 6.3).

1 Notation and Statement of Main Theorem

For $N \in \{1,2,\ldots\}$, $Z \in \mathbb{R}^N$ and $\rho > 0$ we let

$$B^N_\rho(Z) = \{ X \in \mathbb{R}^N : |X-Z| \leq \rho \}, \quad \bar{B}^N_\rho(Z) = \{ X \in \mathbb{R}^N : |X-Z| < \rho \},$$

sometimes written $B_\rho(Z), \bar{B}_\rho(Z)$ when no confusion is likely to arise, and

$$B^1_1 = B^1_1(0), \quad \bar{B}^1_1 = \bar{B}^1_1(0).$$

$\mu_j$ (sometimes written $\mu$ if no confusion is likely to arise) will denote $j$-dimensional Hausdorff measure on $\mathbb{R}^N$.

Let $M$ be a smooth embedded hypersurface in an open subset $U \subset \mathbb{R}^{N+1}$, meaning that $M \subset U$ is non-empty and for each $X \in M$ there is $\rho > 0$ with $\bar{B}^{N+1}_\rho(X) \cap M = \psi(V)$ for some smooth proper rank $N$ injective map $\psi$ from an open set $V \subset \mathbb{R}^N$ into $\mathbb{R}^{N+1}$.

For such $M$ we let $\text{reg} M$ be the relatively open subset of $U \cap M$ (the closure of $M$ in $U$) consisting of all points $X \in U \cap M$ such that, for some $\sigma > 0$, $\bar{B}^{\sigma+1}_\rho(X) \cap M$ is a smooth embedded hypersurface, and we let

$$\text{sing} M = U \cap M \setminus \text{reg} M.$$

We shall always assume

$$\text{reg} M = M$$

and $\text{sing} M = U \cap M \setminus M$, since otherwise we could work with reg $M$ instead of $M$.

Henceforth $n \geq 3, m \geq 2, n + m \geq 8, \ell \geq 1$, and points in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell$ will be denoted $(x, \xi, y)$.

The main theorem is then as follows—a more explicit version of this theorem, with good information about the shape of the singular examples, is given later in Theorem 1.1 and Remark 0.3 below.

1.1 Theorem. Let $K$ be an arbitrary closed subset of $\mathbb{R}^\ell$. Then for each $\tau \in (0, 1)$ there is a $C^\infty(\mathbb{R}^{n+m+\ell})$ function $f = f(x, y)$ with $\sup |f| < \tau$ and $\sup |D^j f| < C\tau^j$, $j \geq 1$, $C = C(n, m, \ell, \tau)$, and a smooth oriented embedded hypersurface $M \subset \mathbb{R}^{n+m+\ell}$ which is minimal and strictly stable with respect to the metric $g$ on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell$ defined by

$$g_{(x, \xi, y)} = \sum_{j=1}^n dx_i^2 + f(x, y) \sum_{j=1}^m \xi_j^2 \sum_{k=1}^\ell dy_k^2, \quad (x, \xi, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell,$$

and which has

$$\text{sing} M = \{0\} \times \{0\} \times K.$$

Note: By saying that $M$ is strictly stable we mean that there is a constant $\lambda = \lambda(M) > 0$ such that

$$\int_M \sigma(M, \zeta) \leq \int_M |\nabla \zeta|^2 \zeta^2 d\mu_g \geq \lambda \int_M |\nabla \zeta|^2 \zeta^2 d\mu_g(x, \xi, y).$$
for all \( \zeta \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell) \), where \( |A_M| |\nabla^M \zeta| \) denote respectively the length of the second fundamental form and length of the gradient of \( \zeta \) on the submanifold \( M \) relative to the metric \( g \) for \( \mathbb{R}^{n+m+\ell} \), and \( \mu_\delta \) is \((n+m-1+\ell)\)-dimensional Hausdorff measure with respect to the metric \( g \). The left side of (2.2) is the second variation of \( \delta_{ij} \), at least up to terms \( E \) involving derivatives of \( f \), which satisfy \( |E| \leq \delta \in \int \frac{1}{M} |x^{−2}\zeta|^2 \, dt \), \( C = C(n, m, \ell) \), where \( M = \{ (x, \xi, y) + \epsilon \xi : (x, \xi, y) \in \{ (x, \xi, y) \} \} \) with \( \nu \) a smooth unit normal for \( M \), so indeed the inequality (2.2) is a strict stability condition on \( M \) with respect to the metric \( g \) provided \( \tau \) is small enough.

## 2 The Symmetric Minimal Surface Equation (SME)

The Symmetric Minimal Surface Equation (SME) on a connected open \( \Omega \subset \mathbb{R}^N \), for positive \( u \in C^2(\Omega) \), is

2.1 \( M_0(u) = 0 \),

with

2.2 \( M_0(u) = \sum_{i=1}^N D_i \left( \frac{D_i u}{\sqrt{1+|Du|^2}} \right) - \frac{m-1}{\sqrt{1+|Du|^2}} u, \quad m \in \{2, 3, \ldots\}, \quad N \geq 2. \)

Equivalently (2.1) can be written \( M(u) = 0 \), where

2.3 \( M(u) = \sum_{i,j=1}^N (\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}) D_i D_j u - \frac{m-1}{u} (\Delta u - \sum_{i,j=1}^N \frac{D_i u D_j u}{|Du|^2} D_i D_j u - \frac{m-1}{u}). \)

Subsequently we shall apply the discussion of this section to the case when \( N = n+\ell \), so \( u = u(x, y) \) with \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^\ell \).

The left side of (2.2) is just the mean curvature operator in \( \mathbb{R}^N \) so the equation expresses the fact that the graph \( G(u) \) of \( u \) is a hypersurface in \( \mathbb{R}^{n+1} \) with mean curvature \((m-1)\epsilon_{N+1} \nu/u \), where \( \nu = (-Du, 1)/\sqrt{1+|Du|^2} \) is the upward pointing unit normal of \( G(u) \).

More important for our present application is that the SME on a domain \( \Omega \subset \mathbb{R}^N \) actually expresses the fact that the symmetric graph \( SG(u) \subset \Omega \times \mathbb{R}^m \), defined by

\[
SG(u) = \{ (x, \xi) \in \Omega \times \mathbb{R}^m : |\xi| = u(x) \},
\]

is a minimal (i.e. zero mean curvature) hypersurface in \( \Omega \times \mathbb{R}^m \). This is checked as follows: Let \( \tau_1, \ldots, \tau_N \) be the standard orthonormal basis \( e_1, \ldots, e_N \) for \( \mathbb{R}^N \) and \( \tau_{N+1}, \ldots, \tau_{N+m-1} \) a locally defined orthonormal basis of the tangent space of \( S^{m-1} \), and let \( U : \Omega \times S^{m-1} \to \mathbb{R}^{n+m} \) be defined by

\[
U(x, \omega) = (x, u(x) \omega).
\]

Then \( U \) is \( C^\infty \) and injective, and \( U(\Omega \times S^{m-1}) = SG(u) \), so by the area formula

\[
\mu_{N+m-1}(SG(u)) = \int \int_{S^{m-1}} \sqrt{\text{det} P} \, d\omega dx,
\]

where \( P = (p_{ij}) = (D_i U \cdot D_j U) \), so \( p_{ij} = D_{ij}(x, u(x) \omega) \cdot D_{ij}(x, u(x) \omega) = \delta_{ij} + D_i u D_j u \) for \( i, j = 1, \ldots, N \) and \( p_{ij} = D_{ij}(x, u(x) \omega) \cdot D_{ij}(x, u(x) \omega) = u^2 \delta_{ij} \) for \( i, j = 1, \ldots, N \) and \( p_{ij} = p_{ji} = 0 \) for \( i = 1, \ldots, N \) and \( j = N+1, \ldots, N+m-1 \). Hence

2.4

\[
\mu_{N+m-1}(SG(u)) = \mu_{m-1}(S^{m-1}) \int_{\Omega} \sqrt{1+|Du|^2} u^m \, dx, \quad u \in C^1(\Omega), \quad u > 0.
\]

But on the other hand one can directly compute that the SME is the Euler-Lagrange equation for the functional on the right and so the SME expresses the fact that \( SG(u) \) is a stationary point for the area functional \( \mu_{N+m-1}(SG(u)) \), and hence solutions of the SME have minimal symmetric graphs as claimed.

Being a solution of the SME is a “geometrically scale invariant” property: Thus if \( G = \text{graph} u \) is the graph of a solution \( u \) of the SME then any homothety of \( G \) is also the graph of a solution, or, equivalently, with \( t^{-1} \Omega = \{ t^{-1} x : x \in \Omega \} \),

2.5

If \( u(x) \) satisfies the SME on \( \Omega \subset \mathbb{R}^N \) and \( t > 0 \) then

\[
t^{-1} u(tx) \text{ also satisfies the SME on } t^{-1} \Omega.
\]

If \( u \geq 0 \) is continuous on \( \Omega \) we say that \( u \) is a singular solution of the SME on \( \Omega \) if \( u^{-1}(\{0\}) \neq \emptyset \) and \( u \) is locally the uniform limit of smooth positive solutions of the SME on \( \Omega \).

An example of a singular solution of the SME is

\[
u(x) = \alpha_0 |x|, \quad \text{where } \alpha_0 = \sqrt{\frac{m-1}{n-1}}.
\]

Observe that in this case the symmetric graph \( SG(u) \) is the minimal cone \( \{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m : |\xi|^2/(m-1) = |\xi|^2/(m-1) \} \). For a discussion of the main properties of singular and regular solutions of the SME we refer to [FS20]. The main results in [FS20] include a gradient estimate for both singular and regular solutions, but here we shall only need the more standard gradient estimate from [Sim76, Theorem 1], which includes (see [Sim76, Example 4.1]) the result that if \( u \) is a \( C^2(B^N_\rho) \) solution of the prescribed mean curvature equation

\[
\sum_{i=1}^N D_i(D_i u/\sqrt{1+|Du|^2}) = H,
\]

where \( |H| \leq b \sqrt{1+|Du|^2} \) and \( |u| \leq M \) on \( B^N_\rho \), then \( |Du| \) is bounded in \( B^N_{\rho/2} \) in terms of \( N, \rho \) and \( M/\rho \). In particular this applies to the SME on the ball \( B^N_{\rho+\ell} \) provided there are constants \( M > L > 0 \) with \( L \leq u \leq M \), in which case we have the above hypotheses with \( b = (m-1)/L \), so

2.6

\[
\sup_{B^N_{\rho/2}} |Du| \leq C, \quad C = C(n, m, L, M/\rho, L/\rho).
\]

If \( u_1, u_2 \) are positive \( C^2 \) functions on a domain \( \Omega \subset \mathbb{R}^N \) and \( M_0 \) is as in (2.2) then

2.7

\[
M_0(u_1) - M_0(u_2) = M(u_1 - u_2) + (1 + |Du_1|^2)^{-1/2} \frac{m-1}{u_1 u_2} (u_1 - u_2),
\]

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where $L$ is the divergence form elliptic operator with smooth coefficients defined by
\[ L(v) = \sum_{i,j=1}^N D_i(a_{ij} D_j v) + \sum_{j=1}^N b_j D_j v, \]
where $(a_{ij})$ is the $C^\infty$ positive definite matrix given by
\[ a_{ij} = \int_0^1 \left( 1 + |D(u_1 + t(u_2 - u_1))| \right)^{-1/2} \times \left( \delta_{ij} - \frac{D_i(u_1 + t(u_2 - u_1)) D_j(u_1 + t(u_2 - u_1))}{1 + |D(u_1 + t(u_2 - u_1))|^2} \right) dt, \]
and
\[ b_j = (m-1)u_2^{-1} \left( 1 + |Du_1|^2 \right)^{1/2} \times \left( \left( 1 + |Du_1|^2 \right)^{1/2} + (1 + |Du_2|^2)^{1/2} \right) D_j(u_1 + u_2). \]
Thus if $u_1, u_2$ are solutions of the SME then
\[ L(u_1 - u_2) + (1 + |Du_1|^2)^{-1/2} m^{-1} \frac{u_1 - u_2}{u_1 - u_2} = 0. \]
Also, if $u_1, u_2$ are in $C^2(\overline{\Omega})$ and $M(u_1) \leq M(u_2)$, then
\[ L(u_1 - u_2) \leq -(1 + |Du_1|^2)^{-1/2} m^{-1} \frac{u_1 - u_2}{u_1 - u_2}, \]
where $L$ is as in (2.8) so by the classical maximum principle $u_1 - u_2$ cannot have a zero minimum in $\Omega$ unless $u_1 = u_2$ in $\Omega$, because (2.10) says $L(u_1 - u_2) \leq 0$ in $\Omega$ in case $u_1 \geq u_2$.

Using the above fact we can establish the following:

**2.11 Lemma.** Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $u > 0$ and with $u$ satisfying the SME $M(u) = 0$ (i.e. (2.3) on $\Omega$).

(i) If $\{s_t\}_{t \in [0,1]}$ is a continuous family of positive $C^2(\Omega) \cap C^0(\overline{\Omega})$ supersolutions of the SME (i.e. $M(s_t) \leq 0$) with $s_t > u$ on $\partial \Omega$ for each $t \in (0,1)$ and $s_1 \geq u$ everywhere in $\Omega$, then $u \leq s_0$ everywhere in $\Omega$.

(ii) If $\{s_t\}_{t \in [0,1]}$ is a continuous family of positive $C^2(\Omega) \cap C^0(\overline{\Omega})$ subsolutions of the SME (i.e. $M(s_t) \geq 0$) with $s_t < u$ on $\partial \Omega$ for each $t \in (0,1)$ and $u \geq s_t$ everywhere in $\Omega$ for some $t \in (0,1)$, then $u \geq s_0$ everywhere in $\Omega$.

**Proof:** We prove (i); the proof of (ii) is similar. Suppose on the contrary that $u > s_0$ at some point of $\Omega$ and pick the smallest $t \in (0,1]$ with $s_t \geq u$ in $\Omega$. Then $s_t - u$ has a zero minimum in $\Omega$ and $M(s_t) - M(u) \leq 0$, and by the discussion preceding the lemma this is impossible unless $s_t = u$ in $\Omega$, which contradicts the assumption that $s_t > u$ on $\partial \Omega$. \( \square \)

We also need to discuss second variation of the symmetric area functional $F$. By definition of $M(u)$, the first variation $\frac{d}{dt} F(u + t \zeta)|_{t=0}$, assuming we are looking at positive functions $u \in C^2(\Omega)$ with $\Omega \subset \mathbb{R}^N$, is given by
\[ \frac{d}{dt} F(u + t \zeta)|_{t=0} = - \int_\Omega V^{-1} M(u) \zeta \, dx, \quad \zeta \in C_0^1(\Omega), \]
where $M$ is as in (2.3) and $V = \sqrt{1 + |Du|^2}$. If $M(u) = 0$ then we can compute the second variation
\[ \frac{d^2}{dt^2} F(u + t \zeta)|_{t=0} = - \int_\Omega \zeta \frac{d}{dt} M(u) \, dx, \]
and, after some calculation,
\[ 2.12 \quad L_u(\psi) = D_i(V u_{m-1} g^{ij} D_j (V^{-1} \psi)) + (u_{m-1} V) (|A_{G(u)}|^2 + \frac{m-1}{V u^2}) (V^{-1} \psi), \]
where $g^{ij} = \delta_{ij} - \nu_i u_j$, $\nu_i = V^{-1} D_i u$, and
\[ |A_{G(u)}|^2 = V^{-2} \sum_{i,j,p,q} g^{ij} g^{pq} u_{ip} u_{jq}, \quad u_{ij} = D_i D_j u, \]
is the squared length of the second fundamental form of
\[ G(u) = \text{graph } u = \{(x, z) \in \mathbb{R}^N \times \mathbb{R} : z = u(x)\}. \]

Notice that the equation (2.12) can be thought of as a linear operator applied to $V^{-1} \psi$ (rather than to $\psi$), and in that case the coefficient of the degree zero term is $(u_{m-1} V) (|A_{G(u)}|^2 + \frac{m-1}{V u^2})$, which one can check is just the volume element $u_{m-1} V$ times the squared length $|A_{G(u)}|^2$ of the second fundamental form of the symmetric graph $SG(u)$. Also the remaining terms (i.e. the first and second order terms) are in fact just $u_{m-1} V$ times the Laplace-Beltrami operator $\Delta_{SG(u)} (V^{-1} \psi)$ of the symmetric graph $SG(u)$, written in terms of the local coordinates $x \in \Omega$ and valid for functions $\psi$ which are also written in the terms of the local variables $x \in \Omega$.

So (2.12) can alternatively be written in the more compact form
\[ 2.13 \quad (u_{m-1} V)^{-1} L_u(\psi) = \Delta_{SG} (V^{-1} \psi) + |A_{G(u)}|^2 (V^{-1} \psi). \]

Finally we need the following SME regularity results for solutions which are bounded below by the cylindrical solution $\alpha_0(x)$. Here
\[ 2.14 \quad S = \{ u \in C^2(\tilde{B}_1^{n+\ell}) : u \text{ satisfies the SME and } u(x, y) - \alpha_0(x) > 0 \ \forall (x, y) \in \tilde{B}_1^{n+\ell} \}. \]

**2.15 Lemma.** For each $\kappa_0 \in (0, \frac{1}{2})$ and $\theta \in [\frac{1}{2}, 1)$, there is $p = p(n, m, \ell, \kappa_0, \theta) > 1$ and $\eta = \eta(n, m, \ell, \kappa_0, \theta) \in (0, \frac{1}{2})$ such that if $\eta \in S$ with $u(0,0) < \eta$, then
\[ \sup \{ (x, y) \in B_{1/2}(0,0) \times \mathbb{R}^N : |x| > \kappa_0 |y| \} \left( |x|^{-1} (u(x, y) - \alpha_0(x)) + |D(u(x, y) - \alpha_0(x))| + |x| |D^2(u(x, y) - \alpha_0(x))| \right) \leq C \left( n, m, \ell, \theta, \kappa_0 \right). \]

where $C = C(n, m, \ell, \kappa_0)$. \( \square \)

**Proof:** Let $\kappa_0 \in (0, \frac{1}{2})$ be given. We first claim that there are $p = p(n, m, \ell, \theta, \kappa_0) > 1$ and $\eta = \eta(n, m, \ell, \kappa_0, \theta) < \frac{1}{2}$ such that, for each $u \in S$ with $u(0,0) < \eta$ and each $t \in [pu(0,0), \frac{1}{2}]$,
\[ \sup \{ (x, y) \in B_{1/2}(0,0) \times \mathbb{R}^N : |x| < \eta \} \left( 1 + \theta t, |x| > \kappa_0 |y| / 2 \right) \left( u - \alpha_0(x) \right) < \kappa_0^2 t. \]

(1)
If this fails then there are sequences $\eta_k \to 0$ with $p_k \to \infty$, $u_k \in S$ with $u_k(0,0) < \eta_k$, and $t_k \in [p_k u_k(0,0), 1]$ such that
\begin{equation}
\sup \{(x,y) \in B_{1+(\eta)}/z_k : t_k/5 < |x| < \frac{1}{2} (1+\theta) t_k, |x| > \kappa_0 |y|/2 \} |u_k - \alpha_0||x|| \geq \kappa_0^4 t_k.
\end{equation}

Let $\tilde{u}_k(x,y) = t_k^{-1} u_k(t_k x, t_k y)$ for $(x,y) \in \tilde{B}_{1/t_k}$, and $M_k = SG(\tilde{u}_k)$. Then $\tilde{u}_k(0,0) = t_k^{-1} u_k(0,0) \leq p_k^{-1} \to 0$ and, by [FS20] Lemma 2.3, the $(n+m)$-dimensional Hausdorff measure of $M_k$ is locally bounded in $(\{x, \xi, y : |(x,y)| < T \}$, $T = \lim \inf_{k \to \infty} t_k^{-1} \in [1, \infty)$ so by the Allard compactness theorem there is a subsequence $(\kappa)$ (still denoted $k$) such that $M_k$ converges in the varifold sense locally in $(\{x, \xi, y : |(x,y)| < T \}$ to a stationary integer multiplicity varifold $V$ with support of $V$ equal to a closed set $M \subset \overline{C}_+ \times \mathbb{R}^\ell$, $0 \in M$, and $T = \lim t_k^{-1} \in (1, \infty)$. By virtue of the maximum principle of Solomon and White [SW98] we then have either $M \cap C = \emptyset$ or $\cap \{x, \xi, y : |(x,y)| < 1 \} \subset M$, and, because $M_k = SG(\tilde{u}_k)$ (i.e. a symmetric graph over $B_{t_k^{-1}}$), the latter case gives
\begin{equation}
M \cap \{x, \xi, y : |(x,y)| < T \} = \emptyset \cap \{x, \xi, y : |(x,y)| < T \}.
\end{equation}

On the other hand if $M \cap C = \emptyset$ then we would have $0 \in M \cap \cap \subset \{0 \} \times \mathbb{R}^\ell$, which contradicts the maximum principle of Ilmanen [Ilm96]. So indeed we always have (3). In particular $M_k$ converges to $\emptyset$ in the distance sense locally in $(\{x, \xi, y : |(x,y)| < T \}$, and hence $\tilde{u}_k(x,y) - \alpha_0 |x| \to 0$ uniformly for $(x,y) \in \frac{1}{2} (1+\theta) T$ with $\frac{1}{2} \leq |x| < \frac{1}{2} (1+\theta)$, $|x| \geq \kappa_0 |y|/2$, which contradicts (2) for sufficiently large $k$.

Observe next that, in view of (1) and the gradient estimate 2.6 elliptic regularity estimates (in balls of radius $(1-\theta)\kappa_0^{-1/6}$), using the equation 2.9 for the difference $u - \alpha_0 |x|$, imply
\begin{equation}
\sup \{(x,y) \in B_{\alpha_0 t/2} < |x| < \alpha_0 t, |x| > \kappa_0 |y| \} |D(u - \alpha_0 |x|)| < C \kappa_0^2 (x,y) \in B_{\alpha_0 t/2} < |x| < \alpha_0 t, |x| > \kappa_0 |y| \}
\end{equation}
for each $t \in [p(0,0), 1]$, where $C = C(n, m, \ell, \theta)$. Hence this lemma is proved for $\kappa_0 = \kappa_0(n, m, \ell, \theta)$ small enough, and the lemma is then trivially true (with the same $\eta, p$ any larger $\kappa_0$).

3 Alternate Version and Proof of the Main Theorem

With $K$ an arbitrary closed non-empty subset of $\mathbb{R}^\ell$, let $U = \mathbb{R}^\ell \setminus K$. The following is a more explicit version of the main theorem.

3.1 Theorem (Main Theorem.) For each $\tau \in (0, \frac{4}{3}]$ there is a $C^{\infty} (\mathbb{R}^n \times \mathbb{R}^\ell)$ function $f = f(x,y)$ with $f = 1$ on $\mathbb{R}^n \times K$, $\sup |f - 1| < \tau$, $\sup |Df| < C\tau$ for each $j = 1, 2, \ldots$, with $C = C(n, m, \ell, j)$, and a non-negative Lipschitz function $u = u(r,y)$ $(r = |x|)$ on $\mathbb{R}^n + \ell$ with $u(r,y) = \alpha_0 r$ on $\mathbb{R}^n \times K$, $u$ positive and $C\infty$ on $\mathbb{R}^n + \ell \setminus \mathbb{R}^n \times K$, sup $\text{dist}^{-1}((x,y), [0] \times K) u(r,y) - \alpha_0 r < \infty$ for each $j = 1, 2, \ldots$,

and $SG(u)$ (the symmetric graph of $u$) is minimal and strictly stable with respect to the metric
\begin{equation}
g_{(x, \xi, y)} = \sum_{\kappa=1}^n dx^2 + f(x,y) \sum_{j=1}^m dy^2 + \sum_{k=1}^\ell dy_k^2, \quad (x, \xi, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell.
\end{equation}

Let $\tau_0 \in (0, \frac{4}{3})$ (where $\tau_0$ will be chosen later, depending only on $n, m, \ell$) and let $h \in C^\infty (\mathbb{R}^\ell)$ satisfy
\begin{equation}
\frac{h > 0 \text{ on } U, h = 0 \text{ on } K = \mathbb{R}^\ell \setminus U, |D_h| + |D^2_h| < \tau_0 \text{ on } \mathbb{R}^\ell},
\end{equation}
\begin{equation}
dist^{-1}(|D_h U, |D^2_h| h) \leq C\tau_0 \text{ for each } k, j, r = 0, 1, 2, \ldots,
\end{equation}
where $C = C(j, k)$. It is of course standard that such functions $h$ exist.

For the proof of 3.3 we shall need the following theorem, which guarantees, for each $\tau$ sufficiently small, the existence of a positive smooth solution $u_\tau(r, y) (r = |x|)$ of the SME on
\begin{equation}
\Omega = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^\ell : y \in U, |x| < h^2(y)\}
\end{equation}
with $u_\tau(r, y) - \alpha_0 r > 0 (\alpha_0 = (\frac{m-1}{m})^{1/2})$ on $\Omega \setminus \{0 \times K\}$ and $u_\tau - \alpha_0 r$ vanishing to infinite order on approach to $(0, y) \in \{0 \times \partial U\}$, and $|D_h| u_\tau$ small.

3.4 Theorem. Let $\delta > 0$. There is $\tau_0 = \tau_0(\delta, n, m, \ell, h)$ such that, with $h$ as in 3.2 and $\Omega$ as in 3.3, for each $\tau \in (0, \tau_0]$ there is a $u_\tau = u_\tau(r, y) \in C^{\infty}(\Omega)$ with $u_\tau$ a positive solution of the SME on $\Omega$.

\begin{equation}
\begin{aligned}
\{D_h u_\tau(y) < \delta, |D_h u_\tau(y)| \leq 2\alpha_0 \forall y \in U, r < h^2(y),
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\alpha_0 r < u_\tau(r, y) < \alpha_0 r + C\tau h^2(y), \forall y \in U, r < h^2(y), j \geq 0,
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
C = C(j, \ell, m, n, h), \text{ and } M = SG(u_\tau) \text{ satisfies the strict stability inequality} |h_2| \Omega \text{ with } \lambda = \lambda(n, m) > 0.
\end{aligned}
\end{equation}

3.5 Remark: Since $u_\tau - \alpha_0 r$ satisfies the linear elliptic equation 2.9 on $(x, y) \in \mathbb{R}^n \times \mathbb{R}^\ell : \frac{1}{2} h^2(y) \leq |x| < h^2(y)$, by using (1) together with standard interior estimates for such equations we have
\begin{equation}
\sup_{B_{\rho_0/2}(y_0, y_0)} |D^k(u_\tau - \alpha_0 r)| \leq C \rho_0^{-k} \sup_{B_{\rho_0/2}(y_0, y_0)} (u_\tau - \alpha_0 r),
\end{equation}
where $y_0 \in U$, $\rho_0 = \frac{1}{2} h^2(y_0)$. Using this together with estimate (1) we then have
\begin{equation}
\sup_{\frac{1}{2} h^2(y) < r < \frac{1}{2} h^2(y)} (h(y))^{-\frac{1}{2}} |D^k(u_\tau(r, y) - \alpha_0 |x|)| \leq C\tau, \forall j, k = 1, 2, \ldots,
\end{equation}
$C = C(\ell, m, n, j, k, h, h)$. Theorem 3.4 the proof of which will be given in [7] enables us to construct the relevant class of metrics on $\mathbb{R}^n + m + \ell$, which we now discuss.
Take any positive \( f = f(x,y) \in C^\infty(\mathbb{R}^{n+1}) \) and define a smooth metric 

\[
g = \sum_{i=1}^{n} dx_i^2 + f(x,y) \sum_{j=1}^{m} d\xi_j^2 + \sum_{k=1}^{\ell} dy_k^2
\]
on \( \mathbb{R}^{n+m+\ell} = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \); i.e.

\[
g((x,\xi,y),(v,\chi,p),(w,\zeta,q)) = v \cdot w + f(x,y) \chi \cdot \zeta + p \cdot q,
\]

\( v, w \in \mathbb{R}^n, \chi, \zeta \in \mathbb{R}^m, p, q \in \mathbb{R}^\ell \), \((x,\xi,y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \), where \( v \cdot w, \chi \cdot \zeta, \) and \( p \cdot q \) denote the usual inner products on \( \mathbb{R}^n, \mathbb{R}^m, \) and \( \mathbb{R}^\ell \) respectively.

Applying the area formula as in the discussion of \([\ref{2}]\) with \( N = n + \ell \), except that now we use the metric \( g \) for \( \mathbb{R}^{n+m+\ell} \) rather than the standard metric, we have

\[
\mu_g(SG(u)) = \mu_{m-1}(S^{m-1}) \int_{\Omega} \sqrt{1 + f(Du)^2} f^{(m-1)/2} u^{-m} \, dx dy
\]

for any positive \( C^2 \) function \( u \) on a domain \( \Omega \subset \mathbb{R}^{n+\ell} \), where \( \mu_g \) denotes \((n+m+\ell-1)\)-dimensional Hausdorff measure on \( \mathbb{R}^{n+m+\ell} \) with respect to the metric \( g \) for \( \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \). Thus the Euler-Lagrange equation for the functional

\[
\int_{\Omega} \sqrt{1 + f(Du)^2} f^{(m-1)/2} u^{-m} \, dx dy
\]
is equivalent to the statement that the symmetric graph \( SG(u) \) is a minimal (zero mean curvature) hypersurface relative to the metric \( g \) for \( \mathbb{R}^{n+m+\ell} \).

Let

\[
\frac{1}{2} (m + f |Du|^2) \Delta f \cdot Du = -f \sum_{i,j=1}^{n+\ell} (\delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}) D_i D_j u + \frac{m-1}{u},
\]

where we use the notation \((x,y) = (x_1, \ldots, x_{n+\ell}) \) (i.e. \( x_{n+j} = y_j \)). In case \( u = u(r,y) \), which we assume below, we can take \( f = f(r,y) \) with equation

\[
\frac{1}{2} (m + f |Du|^2) \Delta f \cdot Du = -f \left( \frac{Q(u)}{1 + f |Du|^2} \right) + \frac{m-1}{u},
\]

where \( D = (D_r, D_y) = (D_r, D_{y_1}, \ldots, D_{y_l}) \) and

\[
Q(u) = u_r^2 u_{rr} + \sum_{j=1}^{l} u_j u_{ry} u_{y_j} + 2 u_r \sum_{j=1}^{l} u_j u_{y_j}.
\]

Let \( \zeta : \mathbb{R} \to [0,1] \) be a \( C^\infty \) function with \( \zeta(t) = 1 \) for \( t \leq \frac{1}{2} \), \( \zeta(t) = 0 \) for \( t \geq 1 \), and \( |D_t^k \zeta(t)| \leq c_k, k = 1, 2, \ldots \). With \( \tau \in (0, 1] \) fixed, \( u_\tau \) as in Theorem \([\ref{3.1}]\) with \( u_\tau = u_\tau (r,y) \) and \( h \) as in \([\ref{3.2}]\) let

\[
\zeta(r/h^2(y)) u_\tau (r,y) + (1 - \zeta(r/h^2(y))) \alpha_0 r, \quad (r,y) \in [0, \infty) \times U
\]

\( \alpha_0 r \), \((r,y) \in [0, \infty) \times K \).

Then

\[
u \in C^\infty([0, \infty) \times (\mathbb{R}^\ell \setminus K) \) and \( u(r,y) - \alpha_0 r = \zeta(r/h^2(y))(u_\tau (r,y) - \alpha_0 r), y \in U, \) and, by \([\ref{3.3}]\) and \([\ref{3.5}]\) we can choose \( \tau_0 = \tau_0(n,m) \in (0, \frac{1}{m}] \) such that

\[
\begin{cases}
M(u) = 0, & 0 \leq r < \frac{1}{m} h^2(y), \ u(r,y) = \alpha_0 r, \ 0 < h^2(y) \leq r < \infty, \\
|D(u(r,y) - \alpha_0 r)| < \tau h^2(y) < \frac{1}{2}, & 0 < \frac{1}{m} h^2(y) < r, \ (D = (D_r, D_y))
\end{cases}
\]

\[
|D^k(u(r,y) - \alpha_0 r)| \leq c_k \tau h^2(y), \quad 0 < \frac{1}{m} h^2(y) < r, \ \forall j, k.
\]

The following lemma describes what we need subsequently concerning existence and regularity of solutions \( f \) of the first order PDE \([\ref{3.8}]\).

**3.11 Lemma.** There is \( \tau_0 = \tau_0(n,m,\ell) \in (0, \frac{1}{m}] \) such that if \( h \) is as in \([\ref{3.2}]\) and if \( u = u(r,y) \) is as in \([\ref{3.10}]\) then there is a \( C^\infty((0, \infty) \times U : r > \frac{1}{m} h^2(y)) \) solution \( f \) of \([\ref{3.8}]\) with

\[
\begin{cases}
f = 1 \text{ everywhere on } \{(r,y) : \frac{1}{m} h^2(y) \leq r \leq \frac{1}{m} h^2(y), y \in U, \}
|f(r,y) - 1| + |r^k D^k f(r,y)| \leq C \tau h^2(y), \quad \frac{1}{m} h^2(y) \leq r, y \in U, \ j, k = 1, 2, \ldots,
\end{cases}
\]

where \( C = C(j, k) \).

**Proof:** \([\ref{3.8}]\) is a non-degenerate quasilinear first order PDE for the function \( f \) at points where \( Du = (D_r u, D_y u) \neq 0 \), and if \( f \) is a local solution of the equation in a ball \( B_r(0,y_0) \), then, with \( f_\sigma(r,y) = f((0,y_0) + (\sigma, \sigma y)) \) for \( |(r,y)| < 1 \) (i.e. translation of \( y \) and scaling of \( r,y \)),

\[
(1) \quad f_\sigma \text{ satisfies } \[\ref{3.8}\] \text{ on the unit ball } B_1(0) \text{ provided we replace } u \text{ by the geometrically rescaled function } \sigma^{-1} u((0,y_0) + (\sigma, \sigma y)).
\]

(This scaling behavior is of course to be expected, given the geometric context leading to \([\ref{3.3}]\).

With \( z = 1 - f_\sigma \), \([\ref{3.8}]\) can be written in the form

\[
(2) \quad A(u, z) Du : Dz = M(u) - z E(u, z),
\]

where \( M \) is as in \([\ref{2.3}]\) and

\[
A(u, z) = \frac{1}{2} (m + (1 + |Du|^2 - z |Du|^2) - 1) \frac{Q(u)}{1 + |Du|^2} + (1 + |Du|^2 - z |Du|^2) Q(u) - (1 + |Du|^2 - z |Du|^2) (1 + |Du|^2).
\]

Also, since \( M(\alpha_0 r) = 0, A(\alpha_0 r, z) = \frac{1}{2} (m + (1 + \alpha_0^2 - \alpha_0^2 z)^{-1} \alpha_0 z + a(r,y,z) \cdot Dz = -(n-1)z/r + zb(r,y,z) + c(r,y)
\]

\[
(3) \quad \frac{1}{2} (m + (1 + \alpha_0^2 - \alpha_0^2 z)^{-1} \alpha_0 z + a(r,y,z) \cdot Dz = -(n-1)z/r + zb(r,y,z) + c(r,y)
\]
where
\[ a(r, y, z) = \mathcal{A}(u, z)Du - \mathcal{A}(\alpha_0 r, z)D(\alpha_0 r), \]
\[ b(r, y, z) = \mathcal{E}(u, z) - \mathcal{E}(\alpha_0 r, z), \]
\[ c(r, y) = M(u) = M(u) - \mathcal{M}(\alpha_0 r), \]
and so by \ref{3.10} (4)
\[ \sup_{\{r, h^2(y) < r < h^2(\eta), \ |z| < 1/2 \}} (r^k |D_{r,y,z} a| + r^{k+1} |D_{r,y,z} b| + r^{k+1} |D_{r,y,z} c|) \leq C \tau h^j(y), \ j, k \geq 0, \]
where \( C = C(j, k, n, m, \ell) \). In particular there is \( \tau_0 = \tau_0(n, m, \ell) \) such that
\[ |a| < \frac{1}{\tau} \text{ provided } \tau \in (0, \tau_0) \text{ for as long as } |z| < 1. \]

We first aim to get local solutions of \ref{3} with initial value 0 on the hypersurface \( \Sigma = \{(h^2(y), y) : y \in U \} \).

In view of \ref{1}, it is convenient to discuss this in a rescaled setting. In fact, for given \( y_0 \in U \), we take the translation/rescaling \( (r, y) \to \rho^{-1}(r, y - y_0) \) with \( \rho = \frac{1}{4} h^2(y_0) \), and in the rescaled setting we claim, with \( \tau_0 = \tau_0(n, m, \ell) > 0 \) sufficiently small (and independent of \( y_0 \)) and assuming also \ref{4} that we can find a \( C^\infty \) solution of the local initial value problem
\[ \begin{aligned}
&\left(\frac{1}{4}(m + (1 + \alpha_0^2 - \alpha_0^2 z^{-1}))\alpha_0^2z + a(r, y, z) \cdot Dz
\right.
= -(n-1)z/r + z b(r, y, z) + c(r, y), \quad \Psi(y) \leq r < 4 \Psi(y), \ |y| \leq 4,
& \left. z(\Psi(y), y) = 0, \quad |y| \leq 4, \right.
\end{aligned} \]
where \( \Psi(y) = \frac{1}{2} h^2(y_0 + \rho y)/\rho \) with \( \rho = \frac{1}{4} h^2(y_0) \), so that by \ref{3.2}
\[ \Psi(0) = 1, \quad \sup_{|y| < 4} |D^k \Psi(y)| \leq C \tau_0, \ k = 1, 2, \ldots, \ C = C(n, m, \ell, k). \]

Recall that the Lagrange procedure ("method of characteristics") guarantees local solvability in \( C^\infty \) of first order equations in \( \mathbb{R}^N \) of the form \( \sum_{i=1}^N a_i(x, z)D_i z = c(x, z) \) (\( a_i, c \in C^\infty \)) with zero initial data on the hypersurface \( \Sigma \):
\[ \Sigma = \{(\Psi(y), y) : y \in V \}, \]
where \( V \) is open in \( \mathbb{R}^{N-1}, \Psi \in C^\infty(U) \), and \( a(\Psi, \eta) \cdot (-D\Psi, 1) \neq 0 \).

Notice that geometrically this latter condition requires \( a \) to not be tangent to \( \Sigma \) at each point of \( \Sigma \).

The method involves first solving the ODE system
\[ \begin{aligned}
&\frac{\partial}{\partial t} X(t, \eta) = a(X(t, \eta), Z(t, \eta))
&\frac{\partial}{\partial t} Z(t, \eta) = c(X(t, \eta), Z(t, \eta)),
\end{aligned} \]
subject to the initial condition
\[ X(0, \eta) = (\Psi(\eta), \eta), \ Z(0, \eta) = 0, \ \eta \in U. \]

Then one proves that for each \( \eta \in U \), and suitable \( \rho = \rho(y_0, a, c) > 0 \), the map \( X : (t, \eta) \in [0, \rho] \times B_p^{N-1}(y_0) \to X(t, \eta) \in \mathbb{R}^N \) is a diffeomorphism onto some open neighborhood \( W \) of \( (\Psi(y_0), y_0) \) in \( \mathbb{R}^N \), and then \( z \) is defined in \( W \) by \( z = Z \circ X^{-1} \). One can then check that \( z \) satisfies the PDE in \( W \) with \( z = 0 \) on \( W \cap \Sigma \).

In the present case \ref{6} we have \( N = 1 + \ell \) and \( X = (R, Y) \), with points in \( \mathbb{R}^{1+\ell} \) denoted \((r, y), r > 0, \) and \( \Psi(\eta) \) as in \ref{7} and the ODE system is
\[ \begin{aligned}
&\frac{\partial}{\partial t} R(t, \eta) = \frac{1}{4} (m + (1 + \alpha_0^2 - \alpha_0^2 Z^{-1}) \alpha_0 + a_1(R, Y, Z)
&\frac{\partial}{\partial t} Y(t, \eta) = \tilde{a}(R, Y, Z) \quad (\tilde{a} = (a_2, \ldots, a_{\ell+1}))
&\frac{\partial}{\partial t} Z(t, \eta) = -(n-1)R^{-1} + b(R, Y, Z))Z + c(R, Y),
\end{aligned} \]
subject to the initial conditions
\[ R(0, \eta) = \Psi(\eta), \ Y(0, \eta) = \eta, \ Z(0, \eta) = 0, \ |\eta| < 5. \]

We first claim that if \( P \in \{2, 3, \ldots\} \) and \( \tau \leq \tau_0 = \tau_0(P, n, m, \ell) \) small enough, then the solution \((R(t, \eta), Y(t, \eta), Z(t, \eta))\) exists for \((t, \eta) \in [0, P] \times B_p^\ell \). To prove this claim, first note that by \ref{4} the equation for \( R \) ensures that \( D_t R > 0 \) and then the initial condition for \( R \) ensures that
\[ R(t, \eta) \geq \Psi(\eta) \quad (1 - C \tau > \frac{1}{2}) \quad \text{for } (t, \eta) \in [0, P] \times B_p^\ell, \]
provided \( \tau_0 = \tau_0(P, n, m, \ell) \) is small enough. Then the equation for \( Z \), together with \ref{4} says \( |D_z Z| \leq 2n|Z| + \tau_0 \leq 2n(|Z| + \tau_0) \), and hence \( e^{-2nt}(|Z| + \tau_0) \) is decreasing, so
\[ |Z(t, \eta)| \leq C \tau_0, \ (t, \eta) \in [0, P] \times B_p^\ell. \]

Then by differentiating the equation for \( Z \) with respect to \( \eta_1 \), integrating with respect to \( t \) and using the initial condition \( Z(0, \eta) = 0 \) (hence \( D_{\eta_1} Z(0, \eta) = 0 \)) we see that also
\[ |D_{\eta_1} Z(t, \eta)| \leq C \tau_0. \]

So now by using the equations for \((R, Y)\) directly
\[ |D_t (R, Y) - (c_0, 0, \ldots, 0)| \leq C \tau_0, \ c_0 = \frac{1}{4} \alpha_0 (m + (1 + \alpha_0^2)^{-1}), \]
and by integrating with respect to \( t \),
\[ (R, Y)(t, \eta) = (c_0 t + \Psi(\eta), \eta) + E(t, \eta), \]
where \( E(0, \eta) = 0 \) and \(|E| + |D_t E| \leq C_\eta_0 \). Also by first differentiating the \((R, Y)\) equations with respect to \( \eta \) and then integrating with respect to \( t \), we prove that \(|D_t E| \leq C_\eta_0\), so in fact \(|E| + |D_t E| < C_\eta_0\). So, taking \( \varrho = 5 \), \([11]\) shows that

\[
(R, Y)(t, \eta) = (c_0(t, \eta) + \tilde{E}(t, \eta)),
\]

with \(|\tilde{E}(t, \eta)| + |D_{t, \eta} \tilde{E}(t, \eta)| \leq C_\eta_0\), so, with \( \tau_0 = \tau_0(m, n, \ell) > 0 \) small enough, \((R, Y)\) is a \(C^1\) diffeomorphism

\[
\Phi : [0, 5] \times B^3_{\xi} \to \mathbb{W} \triangleright \{(r, y) : y \in B^3_{\xi}, \; \Psi(y) \leq r \leq 4\Psi(y)\},
\]

and hence \( z = Z \circ \Phi^{-1} \mid \{(r, y) : y \in B^3_{\xi}, \; \Psi(y) \leq r \leq 4\Psi(y)\} \) is the required solution of \([3]\) on \( \{(r, y) : y \in B^3_{\xi}(y_0), \; \Psi(y) \leq r \leq 4\Psi(y)\} \) with \( z = 0 \) on the hypersurface \( \{(\Psi(y), y) : y \in B^3_{\xi}\} \). Also, because \( M(u) = 0 \) in \( \{(r, y) : y \in U \) and \( \frac{1}{2}h^2(y) < r < \frac{1}{4}h^2(y)\)\), this solution \( z \) vanishes identically in the region \( \{(r, y) : |y| < 4, \; \Psi(y) \leq r \leq 2\Psi(y)\} \).

Next note that, with

\[
X_k = D^k_{\eta}(R, Y, Z) \quad \text{and} \quad X_0 = (R, Y, Z),
\]

we can successively differentiate in \([8]\) to give

\[
D_t X_k = F_k(t, \eta) + G_k(t, \eta) X_k,
\]

for \( k \geq 1 \) with \( F_k, G_k \) smooth functions and

\[
|G_k| \leq C_0, \quad C_0 = C_0(n, m, \ell), \quad |F_k| \leq C, \quad C = C(n, m, \ell, k),
\]

where the second inequality is subject to the inductive assumption that for \( k \geq 1 \) we already have bounds \( |X_j| \leq C_k \) for \( j = 0, \ldots, k-1 \). Then by subdividing the interval \([0, P] = \sum_{j=1}^N \{(j-1)/N, j/N\}\), and by integration in \([14]\) with respect to \( s \in (0, 1/N) \), we obtain

\[
\sup_{(t, \eta) \in [(j-1)/N, j/N] \times B^3_{\xi}} |X_k(t, \eta)| \leq \sup_{(j-1)/N, \eta \in B^3_{\xi}} |X_k(t, \eta)| + C + N^{-1}C_0 \sup_{(t, \eta) \in [(j-1)/N, j/N] \times B^3_{\xi}} |X_k(t, \eta)|,
\]

where \( C = C(n, m, \ell, k) \). Hence choosing \( N = N(n, m, \ell) > 2C_0 \) we have

\[
\sup_{(t, \eta) \in [(j-1)/N, j/N] \times B^3_{\xi}} |X_k(t, \eta)| \leq 2\sup_{(j-1)/N, \eta \in B^3_{\xi}} |X_k(t, \eta)| + 2C.
\]

In case \( j = 1 \) we can use the initial data \( X_0(0, \eta) = (\Psi(\eta), \eta) \), and so \([15]\) gives

\[
\sup_{(t, \eta) \in [0, 1/N] \times B^3_{\xi}} |X_k(t, \eta)| \leq C, \quad C = C(k, n, m, \ell).
\]

For \( j \geq 2 \) and with \( N = N(k, n, m, \ell) > 2C_0 \), \([15]\) gives

\[
\sup_{(t, \eta) \in [(j-1)/N, j/N] \times B^3_{\xi}} |X_k(t, \eta)| \leq 2\sup_{(t, \eta) \in [(j-2)/N, (j-1)/N] \times B^3_{\xi}} |X_k(t, \eta)| + 2C.
\]

and so

\[
\sup_{(t, \eta) \in [0, 0] \times B^3_{\xi}} |X_k(t, \eta)| \leq C, \quad C = C(k, n, m, \ell).
\]

Now it follows that

\[
\sup_{(t, \eta) \in [0, \rho] \times B^3_{\xi}} |D^k_{\eta} \Phi^{-1}(r, y)| \leq C, \quad C = C(n, m, \ell, j, k), \quad j, k = 0, 1, 2, \ldots,
\]

because \( j = 0 \) holds by \([17]\) and then the case \( j = 1 \) of \([18]\) is true by \([14]\) and finally the case \( j \geq 2 \) of \([18]\) is proved by induction on \( j \) by applying \( D^j_{\eta} \) to each side of \([14]\). So \([18]\) is proved for all \( j, k \).

Thus \( \Phi = (R, Y) \) in \([13]\) is actually a \(C^\infty\) diffeomorphism with

\[
|D^k_{\eta, \rho} \Phi^{-1}(r, y)| \leq C, \quad y \in B^3_{\xi}, \quad \Psi(y) \leq r \leq 4\Psi(y), \quad C = C(n, m, \ell, k),
\]

and in particular \( C \) does not depend on \( y_0 \).

In view of \([18]\) with \( Z_k(t, \eta) = D^k_{\eta} Z(t, \eta) \) and \( Z_0 = Z \), we can take \( k \) derivatives with respect to the \( \eta \) variables in the equation for \( Z \) to give

\[
D_t Z_k = F_k(t, \eta) + D^k_{\eta} (c(R(t, \eta), Y(t, \eta))),
\]

for \( k \geq 1 \), where \(|F_k| \leq C \sum_{j=0}^k |D^j_{\eta, \rho} (c(R(t, \eta), Y(t, \eta)))| \) subject to the inductive assumption, \(|Z_k| \leq C \sum_{j=0}^k |D^j_{\eta, \rho} (c(R(t, \eta), Y(t, \eta)))| \) for \( j = 0, \ldots, k-1 \), and then arguing inductively as in the proof of \([17], [18]\) (except that here the argument is slightly simpler because \( Z_k \) has initial data zero by virtue of the fact that \( Z(t, \eta) = 0 \) for all sufficiently small \( t \)), because \( c(r, y) = 0 \) for \( \Psi(y) \leq r \leq 2\Psi(y) \) to give

\[
\sup_{(t, \eta) \in [0, \rho] \times B^3_{\xi}} |D^k_{\eta, \rho} Z(t, \eta)| \leq C \sum_{j=0}^k |D^j_{\eta, \rho} (c(R(t, \eta), Y(t, \eta)))| \leq C \tau h^j(y_0)
\]

by \([4]\) \( i, j, k = 0, 1, 2, \ldots \), where \( C = C(n, m, \ell, i, j, k) \) and in particular \( C \) does not depend on \( y_0 \).

Thus \( z = Z \circ \Phi^{-1} \) is the required solution of \( [0] \) on \( \{(r, y) : \Psi(y) \leq r \leq 4\Psi(y), \; |y| \leq 4\} \), changing the scale back to the original (i.e. \( r, y \to (0, y_0) + \rho(r, y) \) with \( \rho = \frac{1}{4}h^2(y_0) \)), and using the uniqueness theorem for solutions of the initial value problem for first order quasilinear PDE, we finally have a smooth solution \( z \) on \( \{(r, y) : y \in U, \; \frac{1}{4}h^2(y) \leq r \leq h^2(y)\} \) with \( z \) identically zero on \( \frac{1}{4}h^2(y) \leq r \leq \frac{1}{2}h^2(y) \). Also, since \( \frac{1}{4}h^2(y_0) \leq h^2(y) \leq 2h^2(y_0) \) for \( |y-y_0| < h^2(y_0) \) (provided \( \tau_0 = \tau_0(n, m, \ell) \) is chosen small enough), \( z \) satisfies

\[
|r^k D^k_{\eta, \rho} z(r, y)| \leq C \tau h^j(y), \quad C = C(j, k, n, m, \ell), \quad j, k = 0, 1, 2, \ldots,
\]

for all \( y \in U \) and \( \frac{1}{4}h^2(y) \leq r \leq h^2(y) \) by \([19]\) and \([20]\).

For \( r \geq h^2(y) \) (where \( u(x, y) = \alpha_0 r \)) the equation \([3]\) is just the ODE

\[
(m + (1 + \alpha_0^2 - \alpha_0^2)z^{-1})z_r = -2z(n-1)/r,
\]
and by integration (22) is equivalent to
\[(1 + \alpha_0^2 - \alpha_0^2 z(r, y))^{-\beta_1} z(r, y) = \frac{(1 + \alpha_0^2 - \alpha_0^2 z(r, y))^{-\beta_1} z(r, y)}{(1 + \alpha_0^2 - \alpha_0^2 z(r, y))^{-\beta_1} z(r, y)}
\] So in particular
\[
(23) \quad (1 + \alpha_0^2 - \alpha_0^2 z(r, y))^{-\beta_1} z(r, y) = \frac{1}{(1 + \alpha_0^2 - \alpha_0^2 z(r, y))^{-\beta_1} z(r, y)}, \quad r \geq h^2(y).
\]
Thus \( f = 1 - z \) is defined and smooth on the entire region \( \{(r, y) : y \in U, r \geq \frac{1}{4} h^2(y) \} \) with the required properties, including (21) and the fact that \( f \) is identically 1 in the strip \( \frac{1}{4} h^2(y) \leq r \leq \frac{1}{2} h^2(y), y \in U. \)

**Proof of the Main Theorem 3.1:** In view of the above lemma, we can extend \( f \) to be \( C^\infty \) on \( \mathbb{R}^n \times \mathbb{R}^k \) by taking \( f = 1 \) on \( \mathbb{R}^n \times \mathcal{K} \), and, with \( u \) as in (3.9) \( u \) positive and \( C^\infty \) on \( \mathbb{R}^n \times \mathbb{R}^k \ \setminus \{(0) \times \mathcal{K}\} \), \( u \) Lipschitz on all of \( \mathbb{R}^n \times \mathbb{R}^k \), \( u = \alpha_0 r \) on \( \mathbb{R}^n \times \mathcal{K} \), and \( u - \alpha_0 r \) vanishes to infinite order on approach to the set \( \{0 \} \times \mathcal{K} \).

Also, by the continuity of (3.8) holds on all of \( \mathbb{R}^n \times \mathbb{R}^k \ \setminus \{(0) \times \mathcal{K}\} \), so the symmetric graph \( M = SG(u) \) is a minimal hypersurface with respect to the metric \( \sum dx_i^2 + f(x, y) \sum dt_i^2 + \sum dt_j^2 \) and \( M = \{0\} \times \mathcal{K} \).

This completes the proof of the main theorem, except for the proof of the existence result of Theorem 3.4 which will be given in [7] and the proof of the strict stability of \( M \), which will be established in 5.5 at the conclusion of [6]. □

## 4 Radially Symmetric Solutions of the SME

To facilitate the construction of a suitable family of solutions of the SME of the type specified in Theorem 4.3 of the previous section, we first need to consider the special solutions \( u(x, y) = \varphi(r) \) \((r = |x|)\)—i.e. solutions of the SME which are expressible as a function of the variable \( r = |x| \), or in other words solutions \( \varphi(r) \) which satisfy the Euler-Lagrange equation of the area functional

\[
\mathcal{F}(u) = \int_0^1 \sqrt{1 + (u'(r))^2} u'^{n-1}r^{n-1} \, dr.
\]

In this case the Euler-Lagrange equation is the ODE

\[
\frac{\varphi''}{1 + (\varphi')^2} + \frac{(n - 1)}{r} \varphi' = \frac{(m - 1)}{\varphi}
\]

One such solution, although singular at \( r = 0 \), is

\[
\varphi_0 = \alpha_0 r, \quad \alpha_0 = \frac{m}{m - 1}.
\]

Notice in this case that the symmetric graph \( SG(\varphi_0) \) is just the minimal cone \( \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m : (n - 1)\xi^2 = (m - 1)|x|^2\} \). We use the notation

\[
C_0 = SG(\varphi_0), \quad C = C_0 \times \mathbb{R}^k.
\]

Notice that the solution \( \varphi_0 \) has an isolated singularity when viewed as a function of \( x \in \mathbb{R}^n \), but as a function of \((x, y) \in \mathbb{R}^{n+k}\) the singular set is the entire subspace \( \{0\} \times \mathbb{R}^k \).

We know from ODE theory that there is a unique solution \( \varphi \) of (4.2) subject to the initial conditions

\[
4.5 \quad \lim_{r \to 0} \varphi(r) = 1, \quad \lim_{r \to 0} \varphi'(r) = 0
\]
on a maximal interval \((0, r_0)\), where \( 0 < r_0 \leq \infty \). By differentiating the equation,

\[
\varphi'' + (1 + (\varphi')^2) \frac{n - 1}{r} \varphi' \geq (1 + (\varphi')^2)((n - 1)r^{-2} - (m - 1)\varphi^{-2})\varphi' > 0
\]
at points \( r \) where \( \varphi > \alpha_0 r \) and \( \varphi' > 0 \), which says

\[
(r^{n-1}e^{A(r)}\varphi'')' > 0, \quad A(r) = (n - 1) \int_1^r (\varphi'(t))^2 t^{-1} \, dt
\]
at such points. So \( r^{n-1}e^{A(r)}\varphi'' \) is strictly increasing at points \( r \) where \( \varphi > \alpha_0 r \) and \( \varphi' > 0 \) and in particular \( \varphi'' > 0 \) and \( \varphi' > 0 \) on any interval \((0, \rho)\) where \( \varphi > \alpha_0 r \).

Next notice that the equation for \( \varphi \) can be written

\[
(\varphi - \alpha_0 r)'' + (1 + (\varphi')^2) \frac{(n - 1)}{r} (\varphi - \alpha_0 r)' = (m - 1)(1 + (\varphi')^2)\left(\frac{1}{\varphi} - \frac{1}{\alpha_0 r}\right),
\]

which is

\[
4.6 \quad (\varphi - \alpha_0 r)'' + (1 + (\varphi')^2) \frac{(n - 1)}{r} (\varphi - \alpha_0 r)' + \frac{m}{\alpha_0 r} \varphi(1 + (\varphi')^2)(\varphi - \alpha_0 r) = 0,
\]
hence

\[
(\alpha_0 r)^{-1}(\varphi - \alpha_0 r)'' < 0
\]
at points where \( \varphi > \alpha_0 r \), so \( e^{A(r)}(\varphi - \alpha_0 r)'' \) is strictly decreasing, hence \( < 0 \) since it vanishes as \( r \downarrow 0 \), on any interval \((0, \rho)\) where \( \varphi > \alpha_0 r \). In particular

\[
\varphi' < \alpha_0, \quad \text{and hence } \varphi(r) < \alpha_0 r + 1
\]
on any interval \((0, \rho)\) where \( \varphi > \alpha_0 r \). Thus on any such interval \((0, \rho)\) we have

\[
4.7 \quad \varphi''(r) > 0, \quad 0 < \varphi'(r) < \alpha_0, \quad \alpha_0 r < \varphi(r) < \alpha_0 r + 1, \quad \text{and } \varphi(r) - r\varphi'(r) > 0.
\]

Now according to [HS85, Theorem 2.1] there is a smooth complete area minimizing hypersurface \( S \subset U_+ = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m : |\xi| > \alpha_0 |x|\} \), and the homotheties \( \{tS\}_{t>0} \) foliate all of \( U_+ \). Then if \( \varphi(\rho) = \alpha_0 p \) for some \( p \in (0, r_0) \) we could choose a homothety \( tS \) of \( S \) which lies on one side of \( SG(\varphi) \) and makes contact at some point in \( U_+ \), which contradicts the maximum principle. So in fact \( \varphi(\rho) \geq \alpha_0 p \) for all \( r \in (0, r_0) \) and (4.7) holds on the whole maximal interval \((0, r_0)\) and in particular \( 1 < \varphi(r) < \alpha_0 r + 1 \) and \( 0 < \varphi'(r) < \alpha_0 \) on \((0, r_0)\). So \( r_0 = \infty \) by the ODE extension theorem.
Now, since \( \varphi - r \varphi' > 0 \), we see that graph \( \varphi \) intersects every ray \( \{ t, st : t > 0 \} \) with \( s > \alpha_0 \) transversely in a single point, and so the homotheties \( t\varphi_S(\varphi) = \varphi_S(\varphi_t) \), where \( \varphi_S(\varphi) = t\varphi(\varphi/t) \) foliate all of \( U_+ \). Thus \( \varphi_S(\varphi) \) is minimizing, hence the uniqueness part of [HS55] Theorem 2.1 is applicable, giving \( \varphi_S(\varphi) = \varphi \).

Also, the calibration argument of Lawson [Law72] shows that \( \varphi_0 = \alpha_0 \varphi \) strictly minimizes the area functional (4.1) in the sense that there is a fixed constant \( C > 0 \) such that

\[
F(u) \geq F(\varphi_0) + C\rho^{n+m-1}
\]

whenever \( \rho \in (0, \frac{1}{\varphi_0}] \) and \( u : [0,1] \to [0,\infty) \) is \( C^1 \) with \( u(r) - \alpha_0 \varphi \geq 0, u(r) > \rho \) for each \( r \in (0,1) \), and \( (u(r) - \alpha_0 \varphi)|_{r=1} = 0 \). Hence [HS55] Theorem 3.2 is applicable, giving \( \varphi(\varphi) < \alpha_0 \varphi \sim \kappa r^\gamma \) as \( r \to \infty \) for some \( \kappa > 0 \), where

\[
\gamma = -(n + m - 3)/2 + \sqrt{(n + m - 3)^2/2 - (m + n - 2)}.
\]

5 Families of SME Supersolutions

Let \( \eta > 0 \) and

\[
\bar{n} - 1 = (n - 1)/(1 + \eta), \quad \bar{m} - 1 = (m - 1)/(1 + \eta).
\]

We assume for the remainder of the discussion that

\[
\eta = \eta(m, n) \in (0, \frac{1}{2}]
\]

sufficiently small to ensure that we can select \( \bar{\varphi} \in C^\infty(0,\infty) \), in accordance with the discussion of Remark 4.12 to satisfy 4.12(1) and all the conditions 4.9 with \( \alpha_0 = \sqrt{\bar{m}-1}/\sqrt{\bar{m}-1}, \) with \( \bar{\varphi} \) in place of \( \varphi \), and with \( \bar{\gamma} \) in place of \( \gamma \), where

\[
\bar{\gamma} = -\bar{m}+\bar{n}-3 + \sqrt{(\bar{m}+\bar{n}-3)^2 - (\bar{m} + \bar{n} - 2)}.
\]

Notice that then

\[
\bar{\gamma} < -\bar{m}+\bar{n}-3 + (\bar{m}+\bar{n}-3)^2 - (m + n - 2))^{1/2} = \gamma,
\]

so the solution \( \bar{\varphi}(r) \) of Remark 4.12 decays to \( \alpha_0 r \) faster than the solution of 4.2 as \( r \to \infty \), and, with a constant \( c > 0 \) such that \( 1 + c > |\bar{\gamma}|/\gamma \),

\[
\varphi_{\alpha_0}(r) = \bar{\varphi}(r) \leq \bar{\varphi}(r), \quad \text{for all } r \leq 1, \quad \varepsilon \in (0, \frac{1}{2}].
\]

Also (1 + \eta)^{-1}(1 + \eta) \bar{\varphi}' + (\bar{m} - 1) \bar{\varphi}' - \bar{m} - 1 = 0,

so

\[
\varphi_{\alpha_0}(r) = \bar{\varphi}(r) \leq \bar{\varphi}(r), \quad \text{for all } r \leq 1, \quad \varepsilon \in (0, \frac{1}{2}].
\]

The following lemma we prove the existence of a certain family of supersolutions of the SME. Here \( h \) is as in 4.3 and, for \( \varepsilon, \tau \in (0, \tau_0) \) and \( t \geq 0 \), we let

\[
h_c = (\varepsilon^{1/4} + h)^2, \quad \psi_{1, \tau, \varepsilon}(y) = t + \tau e^{-h_c^{1/2}} (\varepsilon^{1/4} + h)^{-1},
\]

and

\[
\Omega_c = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^\ell : |x| < h_c(y) \}.
\]

Note that then \( \Omega_0 = \lim_{c \to 0} \Omega_c = \Omega \) as defined in 3.3.

5.7 Lemma (A Family of Supersolutions.) With \( h_c, \psi_{1, \tau, \varepsilon}, \Omega_c \) as above, let

\[
S_{t, \tau, \varepsilon}(x, y) = \psi_{1, \tau, \varepsilon}(y) \eta(|x|/\psi_{1, \tau, \varepsilon}(y)).
\]

Then there is \( \tau_0 = \tau_0(m, n, \ell) \in (0, \frac{1}{2}] \) such that

\[
\mathcal{M}(S_{t, \tau, \varepsilon}) < 0 \text{ on } \Omega_c, \quad \forall t \geq 0, \quad \tau, \varepsilon \in (0, \tau_0).
\]
5.8 Remark. Note that
\[ 0 < \phi(t(|x|)) - \alpha_0 |x| \leq S_{t,\tau,\epsilon}(x, y) \leq C |x| \leq C t + C \tau h^j \]
\( \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^\ell, j \geq 1, t > 0, \) where \( C = C(n, m, \ell) \) and \( C_j = C(n, m, \ell, j) \), because \( t < \psi_{t,\tau,\epsilon}(y) \leq t + C(j, n, m, \ell) \tau h^j \) for each \( j = 0, 1, \ldots \) and each \( t \geq 0 \) by (1.10) and definition (5.5).

Proof of (5.7) Let \( \psi \in C^\infty_c(\mathbb{R}^n) \) with \( 0 < \psi \leq 1 \) and let
\[ S(x, y) = s(r, y) = \psi(y) \phi(r/\psi(y)), \quad r = |x|, \ y \in \mathbb{R}^\ell. \]
Then, with \( M \) as in (2.5), \( s_y = D_y s = (D_{y_1} s, \ldots, D_{y_m} s) \), and \( s_y y = (s_y y_1, \ldots, s_y y_m) = (D_y D_y s) \).

(2)
\[ M(S) = s_{rr} \ + \ \frac{n-1}{s} s_{r} \ \ - \ \frac{m-1}{s} \ \Delta_y s \ \ - \ s_{y y}^2 \ \sum_{i=1}^{\ell} s_{y_i} s_{y_j} + 2 \epsilon \sum_{i=1}^{\ell} s_{y_i} s_{y_j} \]
\[ = \frac{1 + \epsilon |s|^2}{1 + s^2 + |s|^2} s_{rr} + \frac{n-1}{s} s_{r} \ \ - \ \frac{m-1}{s} \ \Delta_y s \ \ - \ \sum_{i=1}^{\ell} s_{y_i} s_{y_j} + 2 s_{y y} \sum_{i=1}^{\ell} s_{y_i} s_{y_j}. \]

Since \( s_r = \phi'(r/\psi) \) and \( s_{rr} = \psi^{-1} \phi''(r/\psi) \), we have by (5.4)
\[ \left( \frac{n-1}{s} \right) s_r \ \ - \ \frac{m-1}{s} \ \Delta_y s \ \ - \ \frac{1 + \epsilon |s|^2}{1 + s^2 + |s|^2} s_{rr} + \frac{n-1}{s} s_{r} \ \ - \ \frac{m-1}{s} \ \Delta_y s \ \ - \ \sum_{i=1}^{\ell} s_{y_i} s_{y_j} + 2 s_{y y} \sum_{i=1}^{\ell} s_{y_i} s_{y_j} \]
so (2) gives
\[ (3) \quad M(S) = \left( \frac{1 + \epsilon |s|^2}{1 + s^2 + |s|^2} \right) s_{rr} \ \ - \ \frac{1 + \epsilon |s|^2}{1 + s^2 + |s|^2} s_{r} \ \ - \ \frac{1 + \epsilon |s|^2}{1 + s^2 + |s|^2} \Delta_y s \ \ - \ \sum_{i=1}^{\ell} s_{y_i} s_{y_j} + 2 s_{y y} \sum_{i=1}^{\ell} s_{y_i} s_{y_j} \]
so
\[ (4) \quad (1 + s_{y y}^2) M(S) \leq \eta + |s_y|^2 s_{rr} + \epsilon (1 + s_{y y}^2 + |s_y|^2) |D_y^2 s| + 2 s_{y y} |s_y|^2 |s_{y y}| \]
Now \( s_r = \phi'(r/\psi), s_{y y} = \phi(y/\psi) \), \( s_r = \psi^{-1} \phi''(r/\psi), s_{y y} = \psi^{-1} \phi''(r/\psi), \) and \( s_{y y y} = \phi(y/\psi) \), \( s_{y y y} = \phi(y/\psi) \), \( s_{y y y} = \phi(y/\psi) \), \( s_{y y y} = \phi(y/\psi) \), so we use the notation
\[ \Phi(t) = \phi(t) - t \phi'(t), \quad t \geq 0, \]
so (4) gives
\[ (5) \quad (1 + s_{y y}^2) \psi(y) M(S) \leq \eta + \Phi^2(|s_y|^2) \phi''(r/\psi) + 2 \alpha_0 r |\psi^{-1}| \psi |\phi''(r/\psi) + \epsilon (1 + \alpha_0^2 + 2 \Phi^2(|s_y|^2) \phi''(r/\psi) + (\Phi(\psi/\psi) |D_y^2 \psi| + r^2 \psi^{-2} |\psi| \phi''(r/\psi)). \]
By (4.9) and (4.12) there is a constant \( b = b(\mathbb{C}_0) \) such that
\[ 0 < \Phi(t) \leq b(1 + t^2) \phi''(t) \quad \forall t \geq 0, \quad \text{and} \quad \Phi(t) \to 0 \quad \text{as} \quad t \to \infty, \]
so since \( \phi' \leq \alpha_0 \) and \( M = \sup_{t>0} \Phi(t) < \infty \), (5) gives
\[ (6) \quad (1 + s_{y y}^2) \psi M(S) \leq \phi''(r/\psi)(-\eta + \epsilon) \]
where
\[ (7) \quad |\epsilon| \leq M^2 |\psi_y|^2 + 2 \alpha_0 r |\psi^{-1}| |\psi| \]
\[ + \epsilon (1 + \alpha_0^2 + 2 \Phi^2(|s_y|^2) \phi''(r/\psi) + r^2 \psi^{-2} |\psi| \phi''(r/\psi)^2) \]
\[ = M^2 |\psi_y|^2 + \epsilon (1 + \alpha_0^2 + 2 \Phi^2(|s_y|^2) \phi''(r/\psi) + r^2 \psi^{-2} |\psi| \phi''(r/\psi)^2) \]
Then the negative exponential factor in \( D\psi_{t,\tau,\epsilon} = D\psi_{t,\tau,\epsilon} \) ensures \( M^2 |D\psi_{t,\tau,\epsilon}|^2 < \eta/8 < 1 \) and also \( \ell (1 + \alpha_0^2 + 2 \Phi^2(|s_y|^2) \phi''(r/\psi) + r^2 \psi^{-2} |\psi| \phi''(r/\psi)^2) \)
for suitable \( \tau_0 = \tau_0(n, m, \ell) \).

Also \( |D\psi_{t,\tau,\epsilon}| \leq h^{-2} \psi_{t,\tau,\epsilon} |D\psi_{t,\tau,\epsilon}| \leq 2 \psi_{t,\tau,\epsilon} (h^{-2} |D\psi_{t,\tau,\epsilon}| + h^{-1} |D^2 \psi_{t,\tau,\epsilon}|) \), hence (8) gives
\[ (9) \quad |\epsilon| \leq \frac{\eta}{2} + 2 \alpha_0 r h^{-2} |D\psi_{t,\tau,\epsilon}| + \ell (2 + \alpha_0^2) (2 \phi''(r/\psi) |D^2 \psi_{t,\tau,\epsilon}| + r^2 \psi^{-2} |\psi| \phi''(r/\psi)^2) \]
\[ \leq \frac{\eta}{2} + 2 \alpha_0 r h^{-1} |D\psi_{t,\tau,\epsilon}| + 2 \ell (2 + \alpha_0^2) (2 r h^{-2} |D^2 \psi_{t,\tau,\epsilon}| + h^{-1} |D^2 \psi_{t,\tau,\epsilon}|) \]
So on \( \Omega_{\epsilon} = \{(x, y) : |x| < h_{\epsilon}(y)\} \)
\[ |\epsilon| \leq \frac{\eta}{2} + 2 \alpha_0 r |D\psi_{t,\tau,\epsilon}| + 2 \ell (2 + \alpha_0^2) (1 + b) (|D^2 \psi_{t,\tau,\epsilon}| + |D^2 \psi_{t,\tau,\epsilon}|)^2 \]
\[ \leq \frac{\eta}{2} + 2 \ell (2 + \alpha_0^2) (1 + b) (|D^2 \psi_{t,\tau,\epsilon}| + |D^2 \psi_{t,\tau,\epsilon}|)^2 \leq \frac{\eta}{2} \]
by (3.2) for suitable \( \tau_0 = \tau_0(n, m, \ell) > 0 \), so \( M(S_{t,\tau,\epsilon}) < 0 \) on \( \Omega_{\epsilon} \) by (6). \( \square \)

For later reference observe that, with \( S_{t,\tau,\epsilon} = S_{t,\tau,\epsilon} \) (i.e. \( S_{t,\tau,\epsilon} \) with \( t = \epsilon \)) we have, by (4.10) and the definition (5.5) with suitable \( C = C(n, m) \),
\[ 5.9 \quad h_{\epsilon}^{-1}(y) S_{t,\tau,\epsilon}(0, y) \leq Ch_{\epsilon}^{-1}(y) \psi_{t,\tau,\epsilon}(y) \leq Ch_{\epsilon}^{-1}(y) (\epsilon + (1 + e^{-1/4} + h(y))) \]
\[ \leq Ch_{\epsilon}^{-1}(y) (\epsilon + h(y)) \leq Ch_{\epsilon}(y) \leq C(\epsilon^{1/4} + \epsilon)^2 \quad \forall y \in U. \]

6 Solutions \( u \) of the SME with Small \( D_y u \)

In this section we establish some conditions for a good \( C^2 \) approximation of the \( u = \) const. slices of \( u \), plus stability consequences, in case \( u \) is a solution of the SME satisfying a \( |D_y u| \) smallness condition.
We shall need the following consequence of the Liouville-type result of [Sim21 Corollary 1]:

6.1 Lemma. There is $\delta_0 = \delta_0(n, m, \ell) > 0$ such that if $u = u(x, y) \in C^2(\mathbb{R}^{n+\ell})$ is a positive solution of the SME with $u(x, y) > \alpha_0 |x|$ everywhere on $\mathbb{R}^{n+\ell}$ and $|D_y u| \leq \delta_0$, then $u(x, y) = \varphi_r(x)$ for some $r > 0$. (In particular $u(x, y)$ is independent of $y$.)

Proof: By Lemma 2.15(i) the rescaled functions $u_R(x, y) = R^{-1} u(Rx, Ry)$ have gradient bound independent of $R$, for all sufficiently large $R$, in each ball $B_{R_0}$, $R_0 > 1$, and by Lemma 2.15(ii) with $\kappa_0 \downarrow 0$, as $R \to \infty$ the $u_R$ converge to $\alpha_0 |x|$. So $M = SG(u)$ has $C$, with multiplicity 1, as its tangent cone at $\infty$, and hence $M = SG(u)$ satisfies the hypotheses of Corollary 1 of [Sim21]. Thus $u(x, y) = u_0(x)$ for some positive function $u_0$ with $u_0(x) > \alpha_0 |x|$ and with $u_0$ satisfying the SME on $\mathbb{R}^n$. But then $u_0 = \varphi_r$ for some $r > 0$ by [Sim21, Lem. 7.7]. □

6.2 Corollary ($C^2$ approximation.) Let $\kappa_0 \in (0, \delta_0]$ with $\delta_0 = \delta_0(n, m, \ell)$ as in 6.1 and $\theta \in [1, 2)$ be arbitrary. There is $\eta = \eta(n, m, \ell, \theta, \kappa_0) \in (0, \frac{\delta_0}{4})$ such that if $u \in S \setminus \{0\}$ with $|D_y u(x, y)| \leq \kappa_0$ on $B_{R+1}^0$ and $u(0, 0) < \eta$, then, with $\tau = u(0, 0)$ and $u_\tau(x, y) = \tau^{-1} u(\tau x, \tau y)$ for $(x, y) \in B^1_{R+1}$, for $y = 0$ and $|x| < \tau^{-1} \theta$; in particular $|D_y u(x, 0)| < \kappa_0$.

6.3 Remark: In view of the above corollary we thus have a fairly precise picture of the shape of the examples in the main theorem 3.1 in that for each $y_0 \in \mathbb{R}^{\ell}$ the slice $M \cap \{(x, y): y = y_0\}$ of the singular example $M = SG(u)$ is $SG(\alpha_0 \tau)$ if $y_0$ is in $K$ while if $y_0 \in u(\mathbb{R}^{\ell} \setminus K)$ the slice, after rescaling, is $C^2$ close to $SG(\tau)$.

Proof of 6.2. If the lemma fails then there are sequences $\eta_k \downarrow 0$ and solutions $u_k \in S$ with $|D_y u_k| \leq \delta_0$ and $u_k(0, 0) < \eta_k$, yet such that the conclusion ($\dagger$) fails for $u_k$ with $\eta_k$ in place of $\eta$. By Lemma 2.15 for each $\kappa_0 > 0$ and $\theta \in [1, 2)$ there is a $p = p(n, m, \ell, \theta, \kappa_0) > 1$ such that

$$\sup_{\{(x, y) \in B_{\tau, \theta}: |x| > p \eta_k(0, 0), |x| > |y|\}} |\varphi^{-1}(u_k(x, y) - \alpha_0 |x|)| + |D(u_k(x, y) - \alpha_0 |x|)| + |D^2(u_k(x, y) - \alpha_0 |x|)| \leq \frac{1}{\tau} \kappa_0.$$

Let $\tau_k = u_k(0, 0)$ and $\tilde{u}_k(x, y) = \tau_k^{-1} u_k(\tau_k x, \tau_k y)$ for $(x, y) \in B_{\tau_k}^1$, and ($\dagger$) says

$$\sup_{\{(x, y) \in B_{\tau, \theta}: |x| > p \eta_k(0, 0), |x| > |y|\}} |\varphi^{-1}(\tilde{u}_k(x, y) - \alpha_0 |x|)| + |D(\tilde{u}_k(x, y) - \alpha_0 |x|)| + |D^2(\tilde{u}_k(x, y) - \alpha_0 |x|)| \leq \frac{1}{2} \kappa_0.$$

By 4.3

$$\left|\varphi(x) - \alpha_0 |x|\right| + |D_x \varphi(\varphi(x) - \alpha_0 |x|)| + |D^2_x \varphi(\varphi(x) - \alpha_0 |x|)| \leq C R_0^{-\gamma |x|}.$$

Hence we may choose $R_0 = R_0(\ell, m, \kappa_0)$ large enough to ensure $CR_0^{-\gamma |x|} < \frac{1}{4} \kappa_0$ we have from ($\dagger$)

$$\sup_{\{(x, y) \in B_{\tau, \theta}: |x| > p \eta_k(0, 0), |x| > |y|\}} \left( |\varphi^{-1}(u_k(x, y) - \alpha_0 |x|) - D(u_k(x, y) - \alpha_0 |x|)| + |D^2(u_k(x, y) - \alpha_0 |x|)| \right) \leq \kappa_0.$$
Since $f$ is smoothly as close to 1 as we wish, this also gives the required strict stability with respect to the metric $\sum dx_i^2 + \tilde{f}(x,y) \sum dx_j^2 + \sum dy_k^2$.

7 Proof of Theorem 3.4

Let $h_e, S_{\tau,e}, \Omega_e$ be as in 5.5, 5.6 define

$7.1$ $S_{\tau,e} = S_{\tau,e}$ (i.e. $S_{\tau,e}$ as in 5.5, 5.7) with $t = \varepsilon$,

and, for any given $\alpha \in (0,1)$,

$C_{\alpha,\varepsilon} = \{ u \in C^2(\Omega_e) : D^2u \text{ is H"older continuous with exponent } \alpha \text{ on } \Omega_e \}$

$7.2$ Lemma. Let $\delta \in (0,\delta_0)$ with $\delta_0 = \delta_0(n, m, \ell)$ as in Lemma 8.1. Then there is $\tau_0 = \tau_0(n, m, \ell, \delta) \in (0,\frac{1}{2})$ such that for each $\varepsilon, \tau \in (0,\tau_0)$ there is a positive solution $u_{e,\tau} \in C_{\alpha,\varepsilon} \Omega_e$ of the SWE with $u_{e,\tau} = S_{\tau,e}$ on $\partial \Omega_e$ ($S_{\tau,e}$ is as in 3.4) and


\[
\begin{aligned}
&\left\{ \begin{array}{ll}
0 < u_{e,\tau}(x,y) - \alpha_0|x| \leq S_{\tau,e} - \alpha_0|x| & ( \leq C(\varepsilon + \tau h^j(y))) \\
|\partial y u_{e,\tau}| \leq \delta_0, & |\partial u_{e,\tau}| \leq C, \quad C = C(n, m, \ell, t)
\end{array} \right. \\
&\text{for all } (x,y) \in \Omega_e.
\end{aligned}
\]

Proof of 7.2 Let $\zeta : \mathbb{R} \rightarrow [0,1]$ be a $C^\infty$ function with $\zeta(t) = 1$ for $t \leq \frac{1}{2}$, $\zeta(t) = 0$ for $t \geq 1$, and $|D^j \zeta| \leq C_j$, $j = 1, 2, \ldots$, and, for each $q = 1, 2, \ldots$, let

$7.1$ $\zeta_q(t) = \zeta(q^{-1}t)$, $t \in \mathbb{R}$.

Let $Z = \{0, \pm 1, \pm 2, \ldots\}$, let $h$ be as in 3.2a and, for $\varepsilon \in (0,\tau_0)$, define $\tilde{h}, \bar{h}$, $\hat{h}$ on all of $\mathbb{R}^\ell$ by

$7.2$ $\tilde{h}(y + qz) = \zeta_q(|y|)h(y)$, $|y_j| \leq q, j = 1, \ldots, \ell$, $z = (z_1, \ldots, z_{\ell}) \in \mathbb{Z}^\ell$

and (as in 5.5)

$\tilde{h}_e = (\varepsilon^{1/4} + \tilde{h})^2$

Then $\tilde{h}, \bar{h}$, $\hat{h}$ agree with $h, \bar{h}$ on $\{y : |y_j| < q/2\}$ and are $q$-periodic in each of the variables $y_j$, $j = 1, \ldots, \ell$, and we let

$\Omega_e = \{(x,y) \in \mathbb{R}^{n+\ell} : |x| < \tilde{h}_e^2(y)\}$

Then $qz + \Omega_e = \Omega_e$ for each $z \in \mathbb{Z}^\ell$, and, using 5.2a and modifying the choice of constants C in 5.3a if necessary, we have

$7.3$ $\left\{ \begin{array}{l}
0 < \tilde{h}_e \leq \tau_0, \\
|D\tilde{h}| + |D^2\tilde{h}| < \tau_0 \\
|\text{dist}^{-3}((\partial U, y))D^k\tilde{h}(y)| \leq C_0 \tau_0 \text{ for each } k, j, k = 0, 1, 2, \ldots, C = C(j,k),
\end{array} \right.$

where $U = \{y : \tilde{h}(h(y) > 0\}$ and $\tilde{h} = h$ on $B_{q/2}$. We note in particular that the constants $C$ above do not depend on $q$.

The supersolutions $S_{\tau,e}$ of 3.4 (with $h$ in place of $h$) are then also periodic in $y_j$ variables, and, for each $R > 0$, $S_{\tau,e}|B_{2R}$ is independent of $q$ for all $q > 2R$.

Let

$7.4$ $C_{q,\alpha,\varepsilon} = \{ u \in C_{\alpha,\varepsilon} : u(x,y + qz) = u(x,y) \forall z \in \mathbb{Z}^\ell, (x,y) \in \Omega_e \}$

and, for each $\sigma \in [0,1]$, with the requirement that $u \subset C_{\alpha,\varepsilon} \Omega_e$, consider the boundary value problem

$7.5$ $\left\{ \begin{array}{l}
Mu = 0 \text{ on } \Omega_e \\
u = (1 - \sigma) \varphi_{\varepsilon^{1+\alpha}} + \sigma S_{\tau,e} \text{ on } \partial \Omega_e.
\end{array} \right.$

The function $u = \varphi_{\varepsilon^{1+\alpha}} < S_{\varepsilon,t}$ is a suitable solution in case $\sigma = 0$. Since $SG(\varphi_{\varepsilon^{1+\alpha}})$ is strictly stable by 3.11 we have, in particular, by 2.12 and 2.13 that

$0$ is not an eigenvalue of the linearized operator $L_{\varphi_{\varepsilon^{1+\alpha}}}$. So, working in the space $C_{\alpha,\varepsilon} \Omega_e$, for small enough $\sigma \in (0,1)$ the implicit function theorem guarantees a suitable solution $u = u_{\varepsilon,t} \quad \varepsilon, \tau, \alpha, q$ such that $u$ is $q$-periodic in each variable $y_j$. Since $\varphi_{\varepsilon^{1+\alpha}}$ is independent of the $y$ variables then we have

$7.6$ $|D_y u_{\varepsilon,t} \varepsilon, \tau, \alpha, q| \leq \delta_0$

for small enough $\sigma = \sigma(q, n, m, \ell, \delta_0) \in (0,1)$. In fact, for any $\sigma \in (0,1)$ such that a solution $u_{\varepsilon,t} \varepsilon, \tau, \alpha, q$ exists,

$7.7$ $\varphi_{\varepsilon^{1+\alpha}} < u_{\varepsilon,t} \varepsilon, \tau, \alpha, q < S_{\varepsilon,t} < \alpha_0|y| < S_{\varepsilon,t} < \alpha_0|y| \text{ on } \Omega_e,

by virtue of 3.11(i) with $s_1 = S_{\varepsilon,t}$ for $t > \varepsilon$ and $s_t = S_{\varepsilon,t}$ in case $t \in [0,\varepsilon]$, and 3.11(ii) with $s_t = \varphi(\varphi_{\varepsilon^{1+\alpha}})$; we also use the classical maximum principle here to get the strict inequalities in 7.7.

Let $M = M_{\varepsilon,t} \varepsilon, \tau, \alpha, q = SG(u_{\varepsilon,t} \varepsilon, \tau, \alpha, q)$, and suppose that $\zeta \in C^1(\mathbb{R}^{n+m+\ell})$ is $q$-periodic in each of the variables $y_j$, $j = 1, \ldots, \ell$, and that $|\zeta|\{y : |y_j| < q, j = 1, \ldots, \ell\}$ has compact support. If $\delta_0 = \delta_0(n, m, \ell, \lambda)$ is chosen appropriately, we can (after a translation of the $y$ variable) apply Remark 6.3 exactly as done to derive 5.3 except that now we only integrate over $|y_j| \leq q$. This gives

$7.8$ $\frac{1}{\lambda} \int_{M^{(q)}} |x|^{-2} \zeta^2(x, \xi, y) d\mu(x, \xi, y) \leq \int_{M^{(q)}} (|D_M \zeta|^2 + |A_M|^2 \zeta^2) d\mu$

$\forall \zeta \in C^1(\mathbb{R}^{n+m} \{y : |y_j| < q, j = 1, \ldots, \ell\})$, where $M^{(q)} = \{x, \xi, y \in M : |y_j| < q, j = 1, \ldots, \ell\}$. We emphasise that this is valid for any $\sigma \in (0,1)$ such that a solution $u = u_{\varepsilon,t} \varepsilon, \tau, \alpha, q$ of 7.5 exists and satisfies 7.7 and 7.8 with $\delta_0 = \delta_0(n, m, \ell, \lambda)$ small enough.

Now let

$7.9$ $\sigma_0 = \sup \{t \in [0,1] : u_{\varepsilon,t} \varepsilon, \tau, \alpha, q \subset C_{\alpha,\varepsilon} \Omega_e \}$

and has the properties $\sup |D_y u_{\varepsilon,t} \varepsilon, \tau, \alpha, q| < 2\alpha_0$ 6.6 and 7.7 $\forall \sigma \in (0, t)$. 

\[ \text{L. Simon} \]
Take any sequence $\sigma_k \in (0, \sigma_0]$ with $\sigma_k \uparrow \sigma_0$ and let $u_k = u_{\sigma_k, \tau, q}$. Again applying part (ii) of Lemma 2.11 with $s_1 = \varphi(1-\epsilon)x_1$, $s_2 = \varphi(1-\epsilon)x_2$, we conclude $u_k \geq \varphi_{1-\epsilon} \circ \Omega_2$, so the estimate 2.6 is applicable and in combination with standard quasilinear estimates gives a fixed bound on the $C^{2,\alpha}$ norm of $u_k$, independent of $k$. So a subsequence of $u_k$ converges in $C^2$ to a positive solution $u_{\tau, \tau, q} \geq \varphi_{1-\epsilon} \circ \Omega_2$ of the SME satisfying
\begin{equation}
\max |Du_{\tau, \tau, q}| \leq \delta_0, \quad \max |Du| \leq 2\alpha_0,
\end{equation}
and also (7), where the strict inequality on the right of (7) is a consequence of the maximum principle, and then of course the strict inequality on the right of (7) holds on $\Omega_2$ in case $\sigma_0 < 1$.

Next we want to apply Lemma 6.2 to check that we have strict inequality in both the inequalities in (10) To do this let $y_0 \in \mathbb{R}^d$ be arbitrary and let $\rho_0 = h_{\epsilon}(y_0)$. Then Lemma 6.2 is applicable, with $\theta = 1/2$, to the function $u_{\rho_0}(r, y) = \rho_0^{-1}u_{\tau, \tau, q}(\rho_0 r, y + \rho_0 y)\tilde{B}_{\rho_0^{-1}\epsilon}$ with $\omega_0 = 1/2 \min(\delta_0, \alpha_0)$, provided we can check that $u_{\rho_0}(0, 0) < \eta$, with $\eta = \eta(n, m, \ell, \theta, \omega_0)$ as in Lemma 6.2 with $\theta = 1/2$ and $\omega_0 = 1/2 \min(\delta_0, \alpha_0)$. In terms of $u_{\tau, \tau, q}$, the requirement $u_{\rho_0}(0, 0) < \eta$ is $h_{\epsilon}(y_0)^{-1}u_{\tau, \tau, q}(0, y_0) < \eta$, and by construction $u_{\tau, \tau, q}(r, y) \leq S_{\tau, \tau}(r, y)$, and hence by (5.3), we have $h_{\epsilon}(y_0)^{-1}u_{\tau, \tau, q}(0, y_0) \leq C(\varepsilon^{1/4} + \tau_0)^2$ with $C = C(n, m)$, so, with $\varepsilon \leq \tau_0$ and $\tau_0 = \tau_0(n, m, \ell)$ small enough, we do have $h_{\epsilon}(y_0)^{-1}u_{\tau, \tau, q}(0, y_0) < \eta$ and hence Lemma 6.2 applies to give in particular that, with suitable $\lambda = \lambda(y_0)$,
\begin{equation}
|D(u_{\tau, \tau, q}(r, y) - \varphi_\lambda(r))| \leq \frac{1}{2} \min(\delta_0, \alpha_0) \quad \forall r < h_{\epsilon}(y_0)/2.
\end{equation}

On the other hand we have by (7) that $u_{\tau, \tau, q}(r, y) - \alpha_0 r \leq S_{\tau, \tau}(r, y) - \alpha_0 r \leq C(\varepsilon^{1/4} + \tau_0)^2$ and so by elliptic interior and boundary estimates for the equation 2.9 applied to the difference $u_{\tau, \tau, q} - \alpha_0 r$, we have $|D(u_{\tau, \tau, q}(r, y) - \alpha_0 r)| \leq C(\varepsilon^{1/4} + \tau_0)^2$ and so for $\varepsilon \leq \tau_0$ small enough $\tau_0 = \tau_0(n, m, \ell)$ we also have (11) for $r \in [h_{\epsilon}(y_0)/2, h_{\epsilon}(y_0)]$. Thus, since $\varphi_\lambda(r) < \alpha_0$ and $|D(u_{\tau, \tau, q}(r, y) - \alpha_0 r)| \leq C(\varepsilon^{1/4} + \tau_0)^2$, strict inequality holds in both the inequalities in (10)

But then if $\sigma_0 < 1$, since the strict stability (8) ensures that 0 is not an eigenvalue of $C_{u_{\tau, \tau, q}}$ (by 3.12), we could repeat the above implicit function argument to contradict the definition of $\sigma_0$ in (9). So $\sigma_0 = 0$ and, by (10) and (7), $u_{\tau, \tau, q}$ is a solution family satisfying the bounds (4) with $\tilde{h}$ (depending on $q$) in place of $h$ and with $C$ independent of $e, q$. Also, for given fixed $R > 0$, $S_{\tau, \tau} B_R$ remains fixed, independent of $q$, for $q > 2R$.

So we can let $q \to \infty$, and again using quasilinear estimates, we can pass to a subsequence which gives $u_{\tau, \tau, q} \to u_{\tau, \tau}$ in $C^2$, and $u_{\tau, \tau}$ satisfies the bounds (4), and also the strict stability (8) for all $\zeta \in C^1_c(\mathbb{R}^{n+m+\ell})$, where $\lambda = \lambda(n, m) > 0$. □

Now, with $h$ as in 3.2 let

\[ S_{\tau}(x, y) = \psi(y)\varphi(|x|/\psi(y)) = \lim_{\varepsilon \downarrow 0} S_{\tau, \varepsilon}(x, y), \] where $\psi(y) = \tau e^{-h^{-1}(y)}$, $y \in U$.

Then by letting $\varepsilon \downarrow 0$ in (7.2) and using the gradient estimate $|Du_{\tau, \tau}| \leq 2\alpha_0$ (true by construction of $u_{\tau, \tau}$) we obtain a family of solutions $u_{\tau}$ with $|Du_{\tau}| \leq 2\alpha_0$, $u = S_{\tau}$ on $\partial \Omega$, and

\[ 7.4 \quad 0 \leq u_{\tau} - \alpha_0 |x| \leq S_{\tau} \leq C\sigma h^j(y) \] on $\Omega = \{(x, y) : |x| < h^2(y)\}$, $j = 1, 2, \ldots$, where $C = C(n, m, \ell, j)$. To complete the proof we just have to prove positivity of $u_{\tau}$—i.e. strict inequality in the inequality on the left of (7.4). For this the argument is exactly as in the proof of Lemma 2.15 utilizing the maximum principles of SW89 and Ilm96. This completes the proof of Theorem 5.4.

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