Cowen-Douglas tuples and fiber dimensions

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Abstract. Let $T \in L(X)^n$ be a Cowen-Douglas system on a Banach space $X$. We use functional representations of $T$ to associate with each $T$-invariant subspace $Y \subseteq X$ an integer called the fiber dimension $fd(Y)$ of $Y$. Among other results we prove a limit formula for the fiber dimension, show that it is invariant under suitable changes of $Y$ and deduce a dimension formula for pairs of homogeneous invariant subspaces of graded Cowen-Douglas tuples on Hilbert spaces.

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1 Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $\mathcal{H} \subseteq \mathcal{O}(\Omega, \mathbb{C}^N)$ be a functional Hilbert space of $\mathbb{C}^N$-valued analytic functions on $\Omega$. The number

$$fd(\mathcal{H}) = \max_{\lambda \in \Omega} \dim \mathcal{H}_\lambda,$$

where $\mathcal{H}_\lambda = \{ f(\lambda); f \in \mathcal{H} \}$, is usually referred to as the fiber dimension of $\mathcal{H}$. Results going back to Cowen and Douglas [6], Curto and Salinas [7] show that each Cowen-Douglas operator tuple $T \in L(H)^n$ on a Hilbert space $H$ is locally uniformly equivalent to the tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(H)^n$ of multiplication operators with the coordinate functions on a suitable analytic functional Hilbert space $\mathcal{H}$. In the present note we use corresponding model theorems for Cowen-Douglas operator tuples $T \in L(X)^n$ on Banach spaces to associate with each $T$-invariant subspace $Y \subseteq X$ an integer $fd(Y)$ called the fiber dimension of $Y$. We thus extend results proved by L. Chen, G. Cheng and X. Fang in [3] for single Cowen-Douglas operators on Hilbert spaces to the case of commuting operator systems on Banach spaces.

By definition a commuting tuple $T = (T_1, \ldots, T_n) \in L(X)^n$ of bounded operators on a Banach space $X$ is a weak dual Cowen-Douglas tuple of rank $N \in \mathbb{N}$ on $\Omega$ if

$$\dim X / \sum_{i=1}^n (\lambda_i - T_i)X = N$$
for each point $\lambda \in \Omega$. We call $T$ a dual Cowen-Douglas tuple if in addition

$$\bigcap_{\lambda \in \Omega} \sum_{i=1}^{n} (\lambda_i - T_i)X = \{0\}.$$ 

We show that weak dual Cowen-Douglas tuples $T \in L(X)^n$ admit local representations as multiplication tuples $M_z \in L(\hat{X})^n$ on suitable functional Banach spaces $\hat{X}$ and prove that dual Cowen-Douglas tuples can be characterized as those commuting tuples $T \in L(X)^n$ that are locally jointly similar to a multiplication tuple $M_z \in L(\hat{X})^n$ on a divisible holomorphic model space $\hat{X}$. We use the functional representations of weak dual Cowen-Douglas tuples $T \in L(X)^n$ to associate with each linear $T$-invariant subspace $Y \subseteq X$ an integer $fd(Y)$ called the fiber dimension of $Y$.

Based on the observation that the fiber dimension $fd(Y)$ of a closed $T$-invariant subspace $Y \in \text{Lat}(T)$ is closely related to the Samuel multiplicity of the quotient tuple $S = T/Y \in L(X/Y)^n$ on $\Omega$ we show that the fiber dimension of $Y \in \text{Lat}(T)$ can be calculated by a limit formula

$$fd(Y) = n! \lim_{k \to \infty} \frac{\dim(Y + M_k(T-\lambda)/M_k(T-\lambda))}{k^n} \quad (\lambda \in \Omega),$$

where $M_k(T-\lambda) = \sum_{|\alpha| = k} (T-\lambda)^{\alpha}X$. Furthermore, we show how to calculate the fiber dimension using the sheaf model of $T$ on $\Omega$. We deduce that the fiber dimension is invariant against suitable changes of $Y$ and we show that the fiber dimension for graded dual Cowen-Douglas tuples $T \in L(H)^n$ on Hilbert spaces satisfies the dimension formula

$$fd(Y_1 \vee Y_2) + fd(Y_1 \cap Y_2) = fd(Y_1) + fd(Y_2)$$

for any pair of homogeneous invariant subspaces $Y_1, Y_2 \in \text{Lat}(T)$. The proof is based on an idea from [4] (see also [3]) where a corresponding result is proved for analytic functional Hilbert spaces given by a complete Nevanlinna-Pick kernel.

2 Fiber dimension for invariant subspaces

In the following, let $\Omega \subseteq \mathbb{C}^n$ be a domain, that is, a connected open set in $\mathbb{C}^n$. Let $D$ be a finite-dimensional vector space and let $M \subseteq \mathcal{O}(\Omega, D)$ be a $\mathbb{C}[z]$-submodule. We denote the point evaluations on $M$ by $\epsilon_\lambda : M \to D, f \mapsto f(\lambda) \quad (\lambda \in \Omega)$.

For $\lambda \in \Omega$, the range of $\epsilon_\lambda$ is a linear subspace $M_\lambda = \{ f(\lambda); f \in M \} \subseteq D$. \n
**Definition 2.1.** The number
\[
fd(M) = \max_{z \in \Omega} \dim M_z
\]
is called the fiber dimension of \( M \). A point \( z_0 \in \Omega \) with \( \dim M_{z_0} = fd(M) \) is called a maximal point of \( M \).

For any \( \mathbb{C}[z] \)-submodule \( M \subseteq \mathcal{O}(\Omega, D) \) and any point \( \lambda \in \Omega \), we have
\[
\sum_{i=1}^{n} (\lambda_i - M_{z_i})M \subseteq \ker \epsilon_{\lambda}.
\]

Under the condition that the codimension of \( \sum_{i=1}^{n} (\lambda_i - M_{z_i})M \) is constant on \( \Omega \), the question whether equality holds here is closely related to corresponding properties of the fiber dimension of \( M \).

**Lemma 2.2.** Consider a \( \mathbb{C}[z] \)-submodule \( M \subseteq \mathcal{O}(\Omega, D) \) such that there is an integer \( N \) with
\[
\dim M/\sum_{i=1}^{n} (\lambda_i - M_{z_i})M \equiv N
\]
for all \( \lambda \in \Omega \). Then \( fd(M) \leq N \). If \( fd(M) < N \), then
\[
\sum_{i=1}^{n} (\lambda_i - M_{z_i})M \nsubseteq \ker \epsilon_{\lambda}
\]
for all \( \lambda \in \Omega \). If \( fd(M) = N \), then there is a proper analytic set \( A \subseteq \Omega \) with
\[
\Omega \setminus A \subseteq \{ \lambda \in \Omega; \dim M_{\lambda} = N \} = \{ \lambda \in \Omega; \sum_{i=1}^{n} (\lambda_i - M_{z_i})M = \ker \epsilon_{\lambda} \}.
\]

**Proof.** Since the maps
\[
M/\sum_{i=1}^{n} (\lambda_i - M_{z_i})M \rightarrow M/\ker \epsilon_{\lambda} \cong \text{Im} \epsilon_{\lambda}, [m] \mapsto [m]
\]
are surjective for \( \lambda \in \Omega \), it follows that \( fd(M) \leq N \) and that
\[
\{ \lambda \in \Omega; \dim M_{\lambda} = N \} = \{ \lambda \in \Omega; \sum_{i=1}^{n} (\lambda_i - M_{z_i})M = \ker \epsilon_{\lambda} \}.
\]
Hence, if \( fd(M) < N \), then \( \sum_{i=1}^{n} (\lambda_i - M_{z_i})M \nsubseteq \ker \epsilon_{\lambda} \) for all \( \lambda \in \Omega \). A standard argument (cf. Lemma 1.4 in [8] and its proof) shows that there is a proper analytic set \( A \subseteq \Omega \) such that
\[
\Omega \setminus A \subseteq \{ \lambda \in \Omega; \dim M_{\lambda} = \text{fd}(M) \}.
\]
This observation completes the proof.
In [3] a fiber dimension was defined for invariant subspaces of dual Cowen-Douglas operators on Hilbert spaces. In the following we extend this definition to the case of weak dual Cowen-Douglas tuples on Banach spaces (see Definition 2.3).

Let $X$ be a Banach space and let $T = (T_1, \ldots, T_n) \in L(X)^n$ be a commuting tuple of bounded operators on $X$. For $z \in \mathbb{C}^n$, we use the notation $z - T$ both for the commuting tuple $z - T = (z_1 - T_1, \ldots, z_n - T_n)$ and for the row operator $z - T : X^n \to X$, $(x_i)_{i=1}^n \mapsto \sum_{i=1}^n (z_i - T_i)x_i$.

With this notation, we have
\[ \sum_{i=1}^n (z_i - T_i)X = \text{Im}(z - T). \]

**Definition 2.3.** Let $T \in L(X)^n$ be a commuting tuple of bounded operators on $X$ and let $\Omega \subseteq \mathbb{C}^n$ be a fixed domain. We call $T$ a weak dual Cowen-Douglas tuple of rank $N \in \mathbb{N}$ on $\Omega$ if
\[ \dim(X/\sum_{i=1}^n (z_i - T_i)X) = N \]
for all $z \in \Omega$. If in addition the condition
\[ \bigcap_{z \in \Omega} \text{Im}(z - T) = \{0\} \]
holds, then $T$ is called a dual Cowen-Douglas tuple of rank $N$ on $\Omega$.

If $X = H$ is a Hilbert space, then a tuple $T \in L(H)^n$ is a dual Cowen-Douglas tuple on $\Omega$ if and only if the adjoint $T^* = (T_1^*, \ldots, T_n^*)$ is a tuple of class $B_n(\Omega^*)$ on the complex conjugate domain $\Omega^* = \{\overline{z} : z \in \Omega\}$ in the sense of Curto and Salinas [7]. One can show (Theorem 4.12 in [17]) that, for a weak dual Cowen-Douglas tuple $T \in L(X)^n$ on a domain $\Omega \subseteq \mathbb{C}^n$, the identity
\[ \bigcap_{z \in \Omega} \text{Im}(z - T) = \bigcap_{k=0}^{\infty} \sum_{|\alpha| = k} (\lambda - T)^\alpha X \]
holds for every point $\lambda \in \Omega$. In particular, if $T \in L(X)^n$ is a dual Cowen-Douglas tuple on $\Omega$, then it is a dual Cowen-Douglas tuple on each smaller domain $\emptyset \neq \Omega_0 \subseteq \Omega$.

**Definition 2.4.** Let $\Omega \subseteq \mathbb{C}^n$ be open. A holomorphic model space of rank $N$ over $\Omega$ is a Banach space $\hat{X} \subseteq \mathcal{O}(\Omega, D)$ such that $D$ is an $N$-dimensional complex vector space and
\begin{enumerate}
  \item[(i)] $M_z \in L(\hat{X})^n$,
  \item[(ii)] for each $\lambda \in \Omega$, the point evaluation $\epsilon_\lambda : \hat{X} \to D, \hat{x} \mapsto \hat{x}(\lambda)$, is continuous and surjective.
\end{enumerate}
A holomorphic model space $\hat{X}$ on $\Omega$ is called divisible if in addition, for $\hat{x} \in \hat{X}$ and $\lambda \in \Omega$ with $\hat{x}(\lambda) = 0$, there are functions $\hat{y}_1, \ldots, \hat{y}_n \in \hat{X}$ with

$$\hat{x} = \sum_{i=1}^n (\lambda_i - M_z) \hat{y}_i.$$ 

The multiplication tuple $M_z$ on a divisible holomorphic model space $\hat{X} \subseteq O(\Omega, D)$ is easily seen to be a dual Cowen-Douglas tuple of rank $N = \dim D$ on $\Omega$.

In the following let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank $N$ on a fixed domain $\Omega \subseteq \mathbb{C}^n$. We extend a notion introduced in [3] to our setting.

**Definition 2.5.** Let $\emptyset \neq \Omega_0 \subseteq \Omega$ be a connected open subset. A CF-representation of $T$ on $\Omega_0$ is a $\mathbb{C}[z]$-module homomorphism

$$\rho : X \to O(\Omega_0, D)$$

with a finite-dimensional complex vector space $D$ such that

(i) $\ker \rho = \bigcap_{z \in \Omega} (z - T)X^n$,

(ii) the submodule $\hat{X} = \rho X \subseteq O(\Omega_0, D)$ satisfies

$$\text{fd}(\hat{X}) = \dim \hat{X} / \sum_{i=1}^n (\lambda_i - M_z) \hat{X}$$

for all $\lambda \in \Omega_0$.

Let $O(\Omega_0, D)$ be equipped with its canonical Fréchet space topology. Our first aim is to show that weak dual Cowen-Douglas tuples possess sufficiently many CF-representations that are continuous and satisfy certain additional properties.

**Theorem 2.6.** Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank $N$ on $\Omega$. Then, for each point $\lambda_0 \in \Omega$, there is a CF-representation $\rho : X \to O(\Omega_0, D)$ of $T$ on a connected open neighbourhood $\Omega_0 \subseteq \Omega$ of $\lambda_0$ such that

(i) $\rho : X \to O(\Omega_0, D)$ is continuous,

(ii) $\hat{X} = \rho(X)$ equipped with the norm $\|\rho(X)\| = \|x + \ker \rho\|$ is a divisible holomorphic model space of rank $N$ on $\Omega_0$.

**Proof.** Let $\lambda_0 \in \Omega$ be arbitrary. Choose a linear subspace $D \subseteq X$ such that

$$X = (\lambda_0 - T)X^n \oplus D.$$
Then \( \dim D = N \). The analytically parametrized complex
\[
T(z) : X^n \oplus D \to X, \quad ((x_i)_{i=1}^n, y) \mapsto \sum_{i=1}^n (z_i - T_i)x_i + y
\]
of bounded operators between Banach spaces is onto at \( z = \lambda_0 \). By Lemma 2.1.5 in [11] there is an open polydisc \( \Omega_0 \subseteq \Omega \) such that the induced map
\[
O(\Omega_0, X^n \oplus D) \to O(\Omega_0, X), \quad ((g_i)_{i=1}^n, h) \mapsto \sum_{i=1}^n (z_i - T_i)g_i + h
\]
is onto. In particular, for each \( z \in \Omega_0 \), the linear map
\[
D \to X/ \sum_{i=1}^n (z_i - T_i)X, \quad x \mapsto [x]
\]
is surjective between \( N \)-dimensional complex vector space. Hence these maps are isomorphisms. But then, for each \( x \in X \) and \( z \in \Omega_0 \), there is a unique vector \( x(z) \in D \) with \( x - x(z) \in \sum_{i=1}^n (z_i - T_i)X \). By construction, for each \( x \in X \), the mapping \( \Omega_0 \to D, z \mapsto x(z) \), is analytic. The induced mapping
\[
\rho : X \to O(\Omega_0, D), \quad x \mapsto x(\cdot)
\]
is linear with
\[
\ker \rho = \bigcap_{z \in \Omega_0} \sum_{i=1}^n (z_i - T_i)X = \bigcap_{z \in \Omega} \sum_{i=1}^n (z_i - T_i)X.
\]
For \( x \in X \), \( z \in \Omega_0 \) and \( j = 1, \ldots, n \),
\[
T_jx - z_jx(z) = T_j(x - x(z)) - (z_j - T_j)x(z) \in \sum_{i=1}^n (z_i - T_i)X.
\]
Hence \( \rho \) is a \( \mathbb{C}[z] \)-module homomorphism. Equipped with the norm \( \|\rho(x)\| = \|x + \ker \rho\| \), the space \( \hat{X} = \rho(X) \) is a Banach space and \( M_z \in L(\hat{X})^n \) is a commuting tuple of bounded operators on \( \hat{X} \). By definition
\[
\rho(x) \equiv x \quad \text{for } x \in D.
\]
Hence the point evaluations \( \epsilon_z : \hat{X} \to D (z \in \Omega_0) \) are surjective. Since the mappings
\[
q_z : D \to X/ \sum_{i=1}^n (z_i - T_i)X, \quad x \mapsto [x] \quad (z \in \Omega_0)
\]
are topological isomorphisms and since the compositions
\[
X \to X/ \sum_{i=1}^n (z_i - T_i)X, \quad x \mapsto q_z(\epsilon_z(\rho(x))) = [x]
\]
are continuous, it follows that the point evaluations \( \epsilon_z : \hat{X} \to D \) \((z \in \Omega_0)\) are continuous. Thus we have shown that \( \hat{X} \subseteq \mathcal{O}(\Omega_0, D) \) with the norm induced by \( \rho \) is a holomorphic model space.

To see that \( \hat{X} \) is divisible, fix a vector \( x \in X \) and a point \( \lambda \in \Omega_0 \) such that \( x(\lambda) = 0 \). Then there are vectors \( x_1, ..., x_n \in X \) with
\[
    x = \sum_{i=1}^{n}(\lambda_i - T_i)x_i.
\]
Hence
\[
    \rho(x) = \sum_{i=1}^{n}(\lambda_i - z_i)\rho(x_i) \in \sum_{i=1}^{n}(\lambda_i - M_{z_i})\hat{X}.
\]

To conclude the proof, it suffices to observe that
\[
    \dim(\hat{X}/\sum_{i=1}^{n}(\lambda_i - M_{z_i})\hat{X}) = \dim(\hat{X}/\ker \epsilon_{\lambda}) = \dim(\text{Im} \epsilon_{\lambda}) = \dim D = N
\]
for all \( z \in \Omega_0 \).

Note that, for a dual Cowen-Douglas tuple \( T \in L(X)^n \) on a Banach space \( X \), the mappings \( \rho : X \to \hat{X} \subseteq \mathcal{O}(\Omega_0, D) \) constructed in the previous proof are isometric joint similarities between \( T \in L(X)^n \) and the tuples \( M_z \in L(\hat{X})^n \) on the divisible holomorphic model space \( \hat{X} \subseteq \mathcal{O}(\Omega_0, D) \).

**Corollary 2.7.** Let \( T \in L(X)^n \) be a commuting tuple on a complex Banach space and let \( \Omega \subseteq \mathbb{C}^n \) be a domain. The tuple \( T \) is a dual Cowen-Douglas tuple of rank \( N \) on \( \Omega \) if and only if, for each point \( \lambda \in \Omega \), there exist a connected open neighbourhood \( \Omega_0 \subseteq \Omega \) of \( \lambda \) and a joint similarity between \( T \) and the multiplication tuple \( M_z \in L(\hat{X})^n \) on a divisible holomorphic model space \( \hat{X} \) of rank \( N \) on \( \Omega_0 \).

**Proof.** The necessity of the stated condition follows from Theorem 2.6 and the subsequent remarks. Since the tuple \( M_z \in L(\hat{X})^n \) on a divisible holomorphic model space of rank \( N \) is a dual Cowen-Douglas tuple of rank \( N \), and since the same is true for every tuple similar to \( M_z \), also the sufficiency is clear.

The preceding result should be compared with Corollary 4.39 in [17], where a characterization of dual Cowen-Douglas tuple on suitable admissible domains in \( \mathbb{C}^n \) is obtained.

There is a canonical way to associate with each weak dual Cowen-Douglas tuple of rank \( N \) on \( \Omega \subseteq \mathbb{C}^n \) a dual Cowen-Douglas tuple of rank \( N \).

**Corollary 2.8.** Let \( T \in L(X)^n \) be a weak dual Cowen-Douglas tuple of rank \( N \) on a domain \( \Omega \subseteq \mathbb{C}^n \). Then the quotient tuple
\[
    T^{CD} = T/\bigcap_{z \in \Omega} \sum_{i=1}^{n}(z_i - T_i)X
\]
defines a dual Cowen-Douglas tuple of rank \( N \) on \( \Omega \).
Proof. Let \( z_0 \in \Omega \) be arbitrary. Choose a CF-representation \( \rho : X \to \mathcal{O}(\Omega_0, D) \) as in Theorem 2.6. Then \( \tilde{X} = \rho(X) \subseteq \mathcal{O}(\Omega_0, D) \) is a divisible holomorphic model space of rank \( N \) on \( \Omega_0 \). Since 

\[
\ker \rho = \bigcap_{z \in \Omega} \sum_{i=1}^{n} (z_i - T_i)X,
\]

the mapping \( \rho \) induces a similarity between \( T^{\text{CD}} \) and \( M_{z} \in L(\tilde{X})^n \). By Corollary 2.7 the tuple \( T^{\text{CD}} \) is a dual Cowen-Douglas tuple of rank \( N \) on \( \Omega \).

As before, let \( T \in L(X)^n \) be a weak dual Cowen-Douglas tuple of rank \( N \) on a domain \( \Omega \subseteq \mathbb{C}^n \). We denote by \( \text{Lat}(T) \) the set of closed subspaces \( Y \subseteq X \) which are invariant under each component \( T_i \) of \( T \). Our next aim is to show that, for \( Y \in \text{Lat}(T) \), the fiber dimension of \( Y \) can be defined as 

\[
\text{fd}(Y) = \text{fd}(\rho(Y)),
\]

where \( \rho \) is an arbitrary CF-representation of \( T \). We have of course to show that the number \( \text{fd}(\rho(Y)) \) is independent of the chosen CF-representation \( \rho \). In the first step, we use an argument from [3] to show that \( \text{fd}(\rho_1(Y)) = \text{fd}(\rho_2(Y)) \) for each pair of CF-representations \( \rho_1, \rho_2 \) over domains \( \Omega_1, \Omega_2 \subseteq \Omega \) with non-trivial intersection.

**Lemma 2.9.** Let \( \Omega_1, \Omega_2 \subseteq \mathbb{C}^n \) be domains with \( \Omega_1 \cap \Omega_2 \neq \emptyset \) and let \( M_i \subseteq \mathcal{O}(\Omega_i, D_i) \) be \( \mathbb{C}[z] \)-submodules with finite-dimensional complex vector spaces \( D_i \) such that 

\[
\text{fd}(M_i) = \dim M_i/(\lambda - M_z)M^n_i \quad (i = 1, 2, \lambda \in \Omega_i).
\]

Suppose that there is a \( \mathbb{C}[z] \)-module isomorphism \( U : M_1 \to M_2 \). Then, for any submodule \( M \subseteq M_1 \), we have 

\[
\text{fd}(M) = \text{fd}(UM).
\]

**Proof.** Using Lemma 1.4 in [8] and elementary properties of analytic sets, we can choose a proper analytic subset \( A \subseteq \Omega_1 \cap \Omega_2 \) such that each point \( \lambda \in (\Omega_1 \cap \Omega_2)\setminus S \) is a maximal point for \( M, M_1 \) and \( UM \). Fix such a point \( \lambda \).

If \( f, g \in M \) are functions with \( f(\lambda) = g(\lambda) \), then by Lemma 2.2 applied to \( M_1 \), there are functions \( h_1, ..., h_n \in M_1 \) such that 

\[
f - g = \sum_{i=1}^{n} (\lambda_i - M_{z_i})h_i.
\]

But then also 

\[
U(f - g) = \sum_{i=1}^{n} (\lambda_i - M_{z_i})Uh_i.
\]

Hence we obtain a well-defined surjective linear map \( U_\lambda : M_\lambda \to (UM)_\lambda \) by setting 

\[
U_\lambda x = (Uf)(\lambda) \text{ if } f \in M \text{ with } f(\lambda) = x.
\]
It follows that $\text{fd}(M) = \dim M_\lambda \geq \dim(UM)_\lambda = \text{fd}(UM)$. By applying the same argument to $U^{-1}$ and $UM$ instead of $U$ and $M$ we find that also $\text{fd}(UM) \geq \text{fd}(M)$.

If $\rho_i : X \to \mathcal{O}(\Omega_i, D_i)$ ($i = 1, 2$) are CF-representations on domains $\Omega_i \subseteq \Omega$ with non-trivial intersection $\Omega_1 \cap \Omega_2 \neq \emptyset$, then the submodules $M_i = \rho_i X \subseteq \mathcal{O}(\Omega_i, D_i)$ are canonically isomorphic

$$M_1 \cong X / \ker \rho_1 = X / \ker \rho_2 \cong M_2$$

as $\mathbb{C}\langle z \rangle$-modules. As an application of the previous result one obtains that

$$\text{fd}(\rho_1 Y) = \text{fd}(\rho_2 Y)$$

for each linear subspace $Y \subseteq X$ which is invariant for $T$.

**Theorem 2.10.** Let $\rho_i : X \to \mathcal{O}(\Omega_i, D_i)$ ($i = 1, 2$) be CF-representations of $T$ on domains $\Omega_i \subseteq \Omega$. Then

$$\text{fd}(\rho_1 Y) = \text{fd}(\rho_2 Y)$$

for each linear subspace $Y \subseteq X$ which is invariant for $T$.

**Proof.** Since $\Omega$ is connected, we can choose a continuous path $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) \in \Omega_1$ and $\gamma(1) \in \Omega_2$. By Theorem 2.6 there is a family $(\rho_z)_{z \in \text{Im} \gamma}$ of CF-representations $\rho_z : X \to \mathcal{O}(\Omega_z, D_z)$ of $T$ on connected open neighbourhoods $\Omega_z \subseteq \Omega$ of the points $z \in \text{Im} \gamma$ such that $\rho_{\gamma(0)} = \rho_1$ and $\rho_{\gamma(1)} = \rho_2$. Using the fact that there is a positive number $\delta > 0$ such that each set $A \subseteq [0, 1]$ of diameter less than $\delta$ is completely contained in one of the sets $\gamma^{-1}(\Omega_z)$ (see e.g. Lemma 3.7.2 in [15]), one can choose a sequence of points $z_1 = \gamma(0), z_2, \ldots, z_n = \gamma(1)$ in $\text{Im} \gamma$ such that $\Omega_{z_i} \cap \Omega_{z_{i+1}} \neq \emptyset$ for $i = 1, \ldots, n-1$. Let $Y \subseteq X$ be a linear $T$-invariant subspace. By the remarks following Lemma 2.9 we obtain that

$$\text{fd}(\rho_1 Y) = \text{fd}(\rho_{z_2} Y) = \ldots = \text{fd}(\rho_2 Y)$$

as was to be shown.

Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank $N$ on a domain $\Omega \subseteq \mathbb{C}^n$ and let $Y \subseteq X$ be a linear subspace that is invariant for $T$. In view of Theorem 2.10 we can define the fiber dimension of $Y$ by

$$\text{fd}(Y) = \text{fd}(\rho Y),$$

where $\rho : X \to \mathcal{O}(\Omega_0, D)$ is an arbitrary CF-representation of $T$. We shall mainly be interested in the fiber dimension of closed invariant subspaces $Y \in \text{Lat}(T)$, but the reader should observe that the definition makes perfect sense for linear $T$-invariant subspaces $Y \subseteq X$. Since by Theorem 2.6 there
are always continuous CF-representations \( \rho : X \to \mathcal{O}(\Omega_0, D) \) and since in this case the inclusions
\[
\varepsilon_\lambda(\rho(Y)) \subseteq \overline{\varepsilon_\lambda(\rho(Y))} = \varepsilon_\lambda(\rho(Y))
\]
hold for all \( \lambda \in \Omega_0 \), it follows that \( \text{fd}(Y) = \text{fd}(\overline{Y}) \) for each linear \( T \)-invariant subspace \( Y \subseteq X \).

It follows from Theorem 2.6 that \( \text{fd}(X) = N \). In general, the fiber dimension \( \text{fd}(Y) \) of a linear \( T \)-invariant subspace \( Y \subseteq X \) is an integer in \( \{0, \ldots, N\} \) which depends on \( Y \) in a monotone way. Obviously, \( \text{fd}(Y) = 0 \) if and only if \( Y \subseteq \ker \rho = \bigcap_{z \in \Omega} (z - T)X^n \).

We conclude this section with an alternative characterization of CF-representations.

**Corollary 2.11.** Let \( T \in \mathcal{L}(X)^n \) be a weak dual Cowen-Douglas tuple of rank \( N \) on a domain \( \Omega \subseteq \mathbb{C}^n \), and let \( \rho : X \to \mathcal{O}(\Omega_0, D) \) be a \( \mathbb{C}[z] \)-module homomorphism on a domain \( \emptyset \neq \Omega_0 \subseteq \Omega \) with a finite-dimensional vector space \( D \) such that
\[
\ker \rho = \bigcap_{z \in \Omega} (z - T)X^n.
\]
Then \( \rho \) is a CF-representation of \( T \) if and only if \( \text{fd}(\rho X) = N \).

**Proof.** Suppose that \( \text{fd}(\rho X) = N \). Define \( \hat{X} = \rho(X) \). Since the maps
\[
X/(\lambda - T)X^n \to \hat{X}/(\lambda - M_z)\hat{X}^n, \ [x] \mapsto [\rho x]
\]
and
\[
\hat{X}/(\lambda - M_z)\hat{X}^n \to \hat{X}_\lambda, \ [f] \mapsto f(\lambda)
\]
are surjective for each \( \lambda \in \Omega_0 \), it follows that
\[
\dim \hat{X}/(\lambda - M_z)\hat{X}^n \leq N
\]
for all \( \lambda \in \Omega_0 \) and that equality holds on \( \Omega_0 \setminus A \) with a suitable proper analytic subset \( A \subseteq \Omega_0 \). Equipped with the norm \( \|\rho(x)\| = \|x + \ker \rho\| \), the space \( \hat{X} \) is a Banach space and \( M_z \in \mathcal{L}(\hat{X})^n \) is a commuting tuple of bounded operators on \( \hat{X} \). A result of Kaballo (Satz 1.5 in [13]) shows that the set
\[
\{\lambda \in \Omega_0; \dim \hat{X}/(\lambda - M_z)\hat{X}^n > \min_{\mu \in \Omega_0} \dim \hat{X}/(\mu - M_z)\hat{X}^n\}
\]
is a proper analytic subset of \( \Omega_0 \). Combining these results we find that
\[
\dim \hat{X}/(\lambda - M_z)\hat{X}^n = N
\]
for all \( \lambda \in \Omega_0 \). Hence \( \rho \) is a CF-representation of \( T \).

Conversely, if \( \rho \) is a CF-representation of \( T \), then \( \text{fd}(\rho X) = N \) by the remarks preceding the corollary.

\[\square\]
3 A limit formula for the fiber dimension

Let $\Omega \subseteq \mathbb{C}^n$ be a domain with $0 \in \Omega$ and let $D$ be a finite-dimensional complex vector space. For $k \in \mathbb{N}$, let us consider the mapping $T_k : \mathcal{O}(\Omega, D) \to \mathcal{O}(\Omega, D)$ which associates with each function $f \in \mathcal{O}(\Omega, D)$ its $k$-th Taylor polynomial, that is,

$$T_k(f)(z) = \sum_{|\alpha| \leq k} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha.$$ 

In [8] (Lemma 1.4) it was shown that, for a given $\mathbb{C}[z]$-submodule, there is a proper analytic subset $A \subseteq \Omega$ such that

$$\dim M_z = \max_{w \in \Omega} \dim M_w = n! \lim_{k \to \infty} \frac{\dim T_k(M)}{k^n}$$

holds for all $z \in \Omega \setminus A$.

Based on this observation, we will deduce a similar limit formula for the fiber dimension of invariant subspaces of weak Cowen-Douglas tuples on $\Omega$.

Given a commuting tuple $T \in L(X)^n$ of bounded operators on a Banach space $X$, we write

$$K^\bullet(T, X) : 0 \longrightarrow \Lambda^0(X) \overset{\delta_0}{\longrightarrow} \Lambda^1(X) \overset{\delta_1}{\longrightarrow} \cdots \overset{\delta_{n-1}}{\longrightarrow} \Lambda^n(X) \longrightarrow 0$$

for the Koszul complex of $T$ (cf. Section 2.2 in [11]). For $i = 0, ..., n$, let

$$H^i(T, X) = \ker(\delta_i)/\text{Im}(\delta_{i-1})$$

be the $i$-th cohomology group of $K^\bullet(T, X)$. There is a canonical isomorphism $H^n(T, X) \cong X/\sum_{i=1}^n T_iX$ of complex vector spaces.

In the following, given a commuting operator tuple $T \in L(X)^n$ and an invariant subspace $Y \in \text{Lat}(T)$, we denote by

$$R = T|_Y \in L(Y)^n, S = T/Y \in L(Z)^n$$

the restriction of $T$ to $Y$ and the quotient of $T$ modulo $Y$ on $Z = X/Y$. The inclusion $i : X \to Y$ and the quotient map $q : X \to Z$ induce a short exact sequence of complexes

$$0 \longrightarrow K^\bullet(z - R, Y) \overset{i}{\longrightarrow} K^\bullet(z - T, X) \overset{q}{\longrightarrow} K^\bullet(z - S, Z) \longrightarrow 0.$$ 

It is a standard fact from homological algebra that there are connecting homomorphisms $d^i_1 : H^i(z - S, Z) \to H^{i+1}(z - R, Y)$ ($i = 0, ..., n - 1$) such that the induced sequence of cohomology spaces...
is exact again. In particular, we obtain

\[ \text{Im}(d_{n-1}^n) = \ker(H^n(\lambda - R, Y) \to Y) \]

\[ = (Y \cap (\lambda - T)X^n)/(\lambda - R)Y^n. \]

**Lemma 3.1.** Let \( T \in L(X)^n \) be a weak dual Cowen-Douglas tuple of rank \( N \) on a domain \( \Omega \subseteq \mathbb{C}^n \) and let \( Y \in \text{Lat}(T) \) be a closed invariant subspace of \( T \). Then there is a proper analytic subset \( A \subseteq \Omega \) such that

\[ \dim H^n(\lambda - S, Z) = N - \text{fd}(Y) \]

for all \( \lambda \in \Omega \setminus A \).

**Proof.** Choose a CF-representation \( \rho : X \to \mathcal{O}(\Omega_0, D) \) of \( T \) on some domain \( \Omega_0 \subseteq \Omega \) as in Theorem 2.6. Let \( Y \in \text{Lat}(T) \) be arbitrary. Define \( \hat{X} = \rho(X) \) and \( \hat{Y} = \rho(Y) \). Since the compositions

\[ Y^n \xrightarrow{\lambda - R} Y \xrightarrow{\rho} \mathcal{O}(\Omega_0, D) \xrightarrow{\rho} D \]  

\( (\lambda \in D) \)

are zero, we obtain well-defined surjective linear maps

\[ \delta_\lambda : H^n(\lambda - R, Y) \to \hat{Y}, \quad [y] \mapsto \rho(y)(\lambda). \]

Obviously, for each \( \lambda \in \Omega \), the inclusion

\[ \text{Im}(d_{n-1}^n) = (Y \cap (\lambda - T)X^n)/(\lambda - R)Y^n \subseteq \ker \delta_\lambda \]

holds. To see that also the reverse inclusion holds, fix an element \( y \in Y \) with \( \rho(y)(\lambda) = 0 \). Since \( \hat{X} \) is a divisible holomorphic model space, there are vectors \( x_1, \ldots, x_n \in \hat{X} \) with

\[ \rho(y) = \sum_{i=1}^n (\lambda_i - M_{z_i}) \rho(x_i) = \rho(\sum_{i=1}^n (\lambda_i - T_i)x_i). \]
But then
\[ y - \sum_{i=1}^{n} (\lambda_i - T_i)x_i \in \bigcap_{z \in \Omega} (z - T)X^n \]
and hence \( y \in Y \cap (\lambda - T)X^n \). Thus, for each \( \lambda \in \Omega \), we obtain an exact sequence
\[ H^{n-1}(\lambda - S, Z) \stackrel{\delta_{\lambda}}{\longrightarrow} H^n(\lambda - R, Y) \stackrel{\delta_{\lambda}}{\longrightarrow} \hat{Y}_\lambda \rightarrow 0. \]
Using the exactness of these sequences and of the long exact cohomology sequences explained in the section leading to Lemma 2.1, we find that
\[ \dim H^n(\lambda - S, Z) = \dim H^n(\lambda - T, X) - \dim H^n(\lambda - R, Y)/\delta_{\lambda} H^{n-1}(\lambda - S, Z) \]
for all \( \lambda \in \Omega \). Hence the assertion follows. \( \square \)

By the cited result of Kaballo (Satz 1.5 in [13]), in the setting of Lemma 2.1, the set
\[ \{ \lambda \in \Omega; \dim H^n(\lambda - S, Z) > \min_{\mu \in \Omega} \dim H^n(\mu - S, Z) \} \]
is an analytic subset of \( \Omega \). It is well known that the minimum occurring here can be interpreted as a suitable Samuel multiplicity of the tuples \( S - \mu \) for \( \mu \in \Omega \). Let us recall the necessary details.

For simplicity, we only consider the case where \( \Omega \) is a domain in \( \mathbb{C}^n \) with \( 0 \in \Omega \). For an arbitrary tuple \( T \in L(X)^n \) of bounded operators on a Banach space \( X \) with
\[ \dim H^n(T, X) < \infty, \]
all the spaces \( M_k(T) = \sum_{|\alpha|=k} T^\alpha X \) \((k \in \mathbb{N})\) are finite codimensional in \( X \) and the limit
\[ c(T) = n! \lim_{k \to \infty} \frac{\dim X/M_k(T)}{k^n} \]
extists. This number is referred to as the Samuel multiplicity of \( T \). For each domain \( \Omega \subseteq \mathbb{C}^n \) with \( 0 \in \Omega \) and \( \dim H^n(\lambda - T, X) < \infty \) for all \( \lambda \in \Omega \), there is a proper analytic subset \( A \subseteq \Omega \) such that
\[ c(T) = \dim H^n(\lambda - T, X) < \dim H^n(\mu - T, X) \]
for all \( \lambda \in \Omega \setminus A \) and \( \mu \in A \) (see Corollary 3.6 in [9]). In particular, if \( S \in L(Z)^n \) is as in Lemma 2.1 and \( 0 \in \Omega \), then the formula
\[ c(S) = N - \text{fd}(Y) \]
holds. Hence the following result from [8] allows us to deduce the announced limit formula for the fiber dimension.
Lemma 3.2. (Lemma 1.6 in [8]) Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space $X$, let $Y \in \text{Lat}(T)$ be a closed invariant subspace and let $S = T/Y \in L(Z)^n$ be the induced quotient tuple on $Z = X/Y$. Suppose that

$$\dim H^n(T, X) < \infty.$$ 

Then the Samuel multiplicities of $T$ and $S$ satisfy the relation

$$c(S) = c(T) - n! \lim_{k \to \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n}.$$ 

As a direct application we obtain a corresponding formula for the fiber dimension.

Corollary 3.3. Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank $N$ on a domain $\Omega \subseteq \mathbb{C}^n$ with $0 \in \Omega$, and let $Y \in \text{Lat}(T)$ be a closed invariant subspace for $T$. Then the formula

$$\text{fd}(Y) = n! \lim_{k \to \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n}$$

holds.

Proof. It suffices to observe that in the setting of Corollary 3.3 the identity $c(T) = N$ holds and then to compare the formula from Lemma 3.2 with the formula

$$c(S) = N - \text{fd}(Y)$$

deduced in the section leading to Lemma 3.2.

For weak dual Cowen-Douglas tuples $T \in L(X)^n$ on general domains $\Omega \subseteq \mathbb{C}^n$ (not necessarily containing 0), the above formula for $\text{fd}(Y)$ remains true if on the right-hand side the spaces $M_k(T)$ are replaced by the spaces $M_k(T - \lambda_0)$ with $\lambda_0 \in \Omega$ arbitrary. This follows by an elementary translation argument.

If in Corollary 3.3 the space $X$ is a Hilbert space and if we write $P_k$ for the orthogonal projections onto the subspaces $M_k(T)^\perp$, then there are canonical vector space isomorphisms

$$(Y + M_k(T))/M_k(T) \to P_kY, \ [y] \mapsto P_kY.$$ 

Thus the resulting formula

$$\text{fd}(Y) = n! \lim_{k \to \infty} \frac{\dim(P_kY)}{k^n}$$

extends Theorem 19 in [3].

In the final result of this section we show that the fiber dimension $\text{fd}(Y)$ is invariant under sufficiently small changes of the space $Y$. For given invariant subspaces $Y_1, Y_2 \in \text{Lat}(T)$ with $Y_1 \subseteq Y_2$, we write $\sigma(T, Y_2/Y_1)$ for the Taylor spectrum of the quotient tuple induced by $T$ on $Y_2/Y_1$. 

14
Corollary 3.4. Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank $N$ on a domain $\Omega \subseteq \mathbb{C}^n$. Suppose that $Y_1, Y_2 \in \text{Lat}(T)$ are closed $T$-invariant subspaces with $Y_1 \subseteq Y_2$ and $\Omega \cap (\mathbb{C}^n \setminus \sigma(T, Y_2/Y_1)) \neq \emptyset$. Then $\text{fd}(Y_1) = \text{fd}(Y_2)$.

Proof. By Lemma 2.1 there is a point $\lambda \in \Omega \cap (\mathbb{C}^n \setminus \sigma(T, Y_1/Y_2))$ with
\[
\dim H^n(\lambda - T/Y_i, X/Y_i) = N - \text{fd}(Y_i)
\]
for $i = 1, 2$. Using the long exact cohomology sequences induced by the canonical exact sequence
\[
0 \rightarrow Y_2/Y_1 \rightarrow Y/Y_1 \rightarrow Y/Y_2 \rightarrow 0
\]
one finds that the $n$-th cohomology spaces of $\lambda - T/Y_1$ and $\lambda - T/Y_2$ are isomorphic. Hence we obtain that $\text{fd}(Y_1) = \text{fd}(Y_2)$. 

To make the above proof work, it suffices that there is a point in $\Omega$ which is not contained in the right spectrum of the quotient tuple induced by $T$ on $Y_2/Y_1$ (cf. Section 2.6 in [11]). The hypotheses of Corollary 2.4 are satisfied for instance if $\dim(Y_2/Y_1) < \infty$. Thus Corollary 2.4 can be seen as an extension of Proposition 2.5 in [3].

4 Analytic Samuel multiplicity

We briefly indicate an alternative way to calculate fiber dimensions which extends a corresponding idea from [3]. Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space $X$ and let $\Omega \subseteq \mathbb{C}^n$ be a domain such that
\[
\dim H^n(\lambda - T, X) < \infty
\]
for all $\lambda \in \Omega$. For simplicity, we again assume that $0 \in \Omega$. By Corollary 2.2 in [9] the quotient sheaf
\[
\mathcal{H}_T = \mathcal{O}_X^X/(z - T)\mathcal{O}_X^X
\]
of the sheaf of all analytic $X$-valued functions on $\Omega$ is a coherent analytic sheaf on $\Omega$. Let $Y \in \text{Lat}(T)$ be a closed invariant subspace for $T$. As before denote by $R = T|_Y \in L(Y)^n$ the restriction of $T$ and by $S = T/Y \in L(Z)^n$ the quotient tuple induced by $T$ on $Z = X/Y$. Let $i : Y \rightarrow X$ and $q : X \rightarrow Z$ be the inclusion and quotient map, respectively. Then
\[
0 \rightarrow K^\bullet(z - R, \mathcal{O}_\Omega^Y) \xrightarrow{i} K^\bullet(z - T, \mathcal{O}_\Omega^X) \xrightarrow{q} K^\bullet(z - S, \mathcal{O}_\Omega^Z) \rightarrow 0
\]
is a short exact sequence of complexes of analytic sheaves on $\Omega$. Passing to stalks and using the induced long exact cohomology sequences, one finds
that the upper horizontal in the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_R & \xrightarrow{i} & \mathcal{H}_T \\
\pi_Y & \downarrow & \quad \pi_X \\
\mathcal{O}^Y_{\Omega} & \xrightarrow{i} & \mathcal{O}^X_{\Omega}
\end{array}
\]

is an exact sequence of analytic sheaves. Here \(\pi_Y\) and \(\pi_X\) denote the canonical quotient maps. The sheaf \(\mathcal{M} = \pi_X(i\mathcal{O}^Y_{\Omega})\) is the kernel of the surjective sheaf homomorphism

\(\mathcal{H}_T \xrightarrow{q} \mathcal{H}_S\).

Since \(\mathcal{H}_T\) and \(\mathcal{H}_S\) are coherent, also the sheaf \(\mathcal{M}\) is a coherent analytic sheaf on \(\Omega\) (Satz 26.13 in [14]). Hence

\[
0 \to \mathcal{M}_0 \xrightarrow{i} \mathcal{H}_{T,0} \xrightarrow{q} \mathcal{H}_{S,0} \to 0
\]

is an exact sequence of Noetherian \(\mathcal{O}_0\)-modules. For a Noetherian \(\mathcal{O}_0\)-module \(E\), let us denote by \(e_{\mathcal{O}_0}(E)\) its analytic Samuel multiplicity, that is, the multiplicity of \(E\) with respect to the multiplicity system \((z_1, ..., z_n)\) on \(E\) (see Section 7.4 in [16]). Since the analytic Samuel multiplicity is additive with respect to short exact sequences of Noetherian \(\mathcal{O}_0\)-modules (Theorem 7.5 in [16]), it follows that

\[
e_{\mathcal{O}_0}(\mathcal{H}_{T,0}) = e_{\mathcal{O}_0}(\mathcal{M}_0) + e_{\mathcal{O}_0}(\mathcal{H}_{S,0}).
\]

By Corollary 4.1 in [9] the analytic Samuel multiplicities \(e_{\mathcal{O}_0}(\mathcal{H}_{T,0})\) and \(e_{\mathcal{O}_0}(\mathcal{H}_{S,0})\) coincide with the Samuel multiplicities \(c(T)\) and \(c(S)\) as defined in Section 2. Thus we obtain the identity

\[
c(T) = e_{\mathcal{O}_0}(\mathcal{M}_0) + c(S).
\]

By Theorem 8.5 in [16] the analytic Samuel multiplicity \(e_{\mathcal{O}_0}(\mathcal{M}_0)\) can also be calculated as the Euler characteristic \(\chi(K^* (z, \mathcal{M}_0))\) of the Koszul complex of the multiplication operators with \(z_1, ..., z_n\) on \(\mathcal{M}_0\).

Summarizing we obtain the following result.

**Theorem 4.1.** Let \(T \in L(X)^n\) be a weak dual Cowen-Douglas tuple on a domain \(\Omega \subseteq \mathbb{C}^n\) with \(0 \in \Omega\) and let \(Y \in \text{Lat}(T)\) be a closed invariant subspace for \(T\). Then with the notation from above, the fiber dimension of \(Y\) can be calculated as

\[
\text{fd}(Y) = n! \lim_{k \to \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n} = e_{\mathcal{O}_0}(\mathcal{M}_0).
\]
5 A lattice formula for the fiber dimension

Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple of rank $N$ on a domain $\Omega \subseteq \mathbb{C}^n$ and let $Y_1, Y_2 \in \text{Lat}(T)$ be closed invariant subspaces. A natural problem studied in [3] is to find conditions under which the dimension formula

$$\text{fd}(Y_1) + \text{fd}(Y_2) = \text{fd}(Y_1 \vee Y_2) + \text{fd}(Y_1 \cap Y_2)$$

holds. Note that, for a dual Cowen-Douglas tuple of rank 1, the validity of this formula for all closed invariant subspaces $Y_1, Y_2$ is equivalent to the condition that any two non-zero closed invariant subspaces $Y_1, Y_2$ have a non-trivial intersection. As observed in [3] elementary linear algebra can be used to obtain at least an inequality.

**Lemma 5.1.** Let $T \in L(X)^n$ be a weak dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^n$ and let $Y_1, Y_2 \subseteq X$ be linear $T$-invariant subspaces. Then the inequality

$$\text{fd}(Y_1) + \text{fd}(Y_2) \geq \text{fd}(Y_1 + Y_2) + \text{fd}(Y_1 \cap Y_2)$$

holds.

**Proof.** Let $\rho : X \to \mathcal{O}(\Omega_0, D)$ be a CF-representation of $T$ on a domain $\Omega_0 \subseteq \Omega$. It suffices to observe that, for each point $\lambda \in \Omega_0$, the estimate

$$\dim \epsilon_\lambda \rho(Y_1 + Y_1) = \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim (\epsilon_\lambda \rho(Y_1) \cap \epsilon_\lambda \rho(Y_2))$$

$$\leq \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim \epsilon_\lambda \rho(Y_1 \cap Y_2)$$

holds and then to choose $\lambda$ as a common maximal point for the submodules $\rho(Y_1 + Y_2)$, $\rho(Y_1)$, $\rho(Y_2)$ and $\rho(Y_1 \cap Y_2)$.

Note that, for closed invariant subspaces $Y_1, Y_2 \in \text{Lat}(T)$, the inequality in 5.1 can be rewritten as

$$\text{fd}(Y_1) + \text{fd}(Y_2) \geq \text{fd}(Y_1 \vee Y_2) + \text{fd}(Y_1 \cap Y_2).$$

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $D$ be an $N$-dimensional complex vector space. We shall say that a function $f \in \mathcal{O}(\Omega, D)$ has coefficients in a given subalgebra $A \subseteq \mathcal{O}(\Omega)$ if the coordinate functions of $f$ with respect to some, or equivalently, every basis of $D$ belong to $A$. Let $M \subseteq \mathcal{O}(\Omega, D)$ be a $\mathbb{C}[z]$-submodule. We shall say that $A$ is dense in $M$ if every function $f \in M$ is the pointwise limit of a sequence $(f_k)_{k \in \mathbb{N}}$ of functions in $M$ such that each $f_k$ has coordinate functions in $A$.

**Theorem 5.2.** Let $M_1, M_2 \subseteq \mathcal{O}(\Omega, D)$ be $\mathbb{C}[z]$-submodules such that $A$ is dense in $M_1$ and in $M_2$ and such that $AM_i \subseteq M_i$ for $i = 1, 2$. Then we have

$$\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2).$$

17
Proof. Exactly as in the proof of Lemma 4.1 it follows that

\[ \text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) \leq \text{fd}(M_1) + \text{fd}(M_2). \]

To prove the reverse inequality it suffices to check that the arguments used in [4] to prove the corresponding result for invariant subspaces of analytic functional Hilbert spaces \( H(K) \) given by a complete Nevanlinna-Pick kernel on a domain in \( \mathbb{C} \) remain valid. For the convenience of the reader, we indicate the main ideas.

Define \( M = M_1 + M_2 \) and choose a point \( \lambda \in \Omega \) which is maximal with respect to \( M_1, M_2 \) and \( M \). Define \( E = (M_1)_\lambda \cap (M_2)_\lambda \) and choose direct complements \( E_1 \) of \( E \) in \( (M_1)_\lambda \) and \( E_2 \) of \( E \) in \( (M_2)_\lambda \). Fix bases \( (e_1, ..., e_{d_1}) \) of \( E_1 \), \( (e_{d_1+1}, ..., e_{d_1+d_2}) \) for \( E_2 \) and \( (e_{d_1+d_2+1}, ..., e_{d_1+d_2+d'}) \) for \( E \), where \( d_1, d_2, d' \geq 0 \) are non-negative integers. Set \( d = d_1 + d_2 + d' \). An elementary argument shows that \( (e_1, ..., e_d) \) is a basis of \( M_\lambda \). Let us complete this basis to a basis \( B = (e_1, ..., e_d, e_{d+1}, ..., e_N) \) of \( D \). Since \( \text{fd}(M_1) + \text{fd}(M_2) - \text{fd}(M) = d' \), we have to show that

\[ \text{fd}(M_1 \cap M_2) \geq d'. \]

We may of course assume that \( d' \neq 0 \). Since \( A \) is dense in \( M \), there are functions \( h_1, ..., h_d \in M \) with

\[ h_i(\lambda) = e_i \quad (i = 1, ..., d) \]

such that each \( h_i \) has coefficients in \( A \). Write

\[ h_i = \sum_{j=1}^{N} h_{ij} e_j \quad (i = 1, ..., d). \]

Then \( \theta = (h_{ij})_{1 \leq i, j \leq d} \) is a \( (d \times d) \)-matrix with entries in \( A \) such that \( \theta(\lambda) = E_d \) is the unit matrix. By basic linear algebra there is a \( (d \times d) \)-matrix \( (A_{ij}) \) with entries in \( A \) such that \( (A_{ij}) \theta = \text{diag}(\text{det} \theta) \) is the \( (d \times d) \)-diagonal matrix with all diagonal terms equal to \( \text{det}(\theta) \). Then

\[ (A_{ij})_{1 \leq i, j \leq d} (h_{ij})_{1 \leq i, j \leq d} = (\text{diag}(\text{det} \theta), (g_{ij})), \]

where \( (g_{ij}) \) is a suitable matrix with entries in \( A \). We define functions \( H_1, ..., H_d \in M \) by setting

\[ H_i = \det(\theta)e_i + \sum_{j=1}^{N-d} g_{ij} e_{d+j} = \sum_{j=1}^{N} (\sum_{\nu=1}^{d} A_{i\nu} h_{\nu j}) e_j = \sum_{\nu=1}^{d} A_{i\nu} h_{\nu}. \]

By construction \( H_i(\lambda) = e_i \) and \( (H_1(z), ..., H_d(z)) \) is a basis of \( M_z \) for every point \( z \in \Omega \) with \( \text{det}(\theta(z)) \neq 0 \). If \( f = f_1 e_1 + \ldots + f_N e_N \in M \) is arbitrary,
then at each point \( z \in \Omega \) which is not contained in the zero set \( Z(\det(\theta)) \) of the analytic function \( \det(\theta) \in \mathcal{O}(\Omega) \), the function \( f \) can be written as a linear combination

\[
f(z) = \lambda_1(z, f)H_1(z) + ... + \lambda_d(z, f)H_d(z).
\]

Using the definition of the functions \( H_i \), we find that

\[
f_1 = \lambda_1(\cdot, f)\det(\theta), ..., f_d = \lambda_d(\cdot, f)\det(\theta).
\]

Hence, for \( j = d + 1, ..., N \) and \( z \in \Omega \setminus Z(\det(\theta)) \), we obtain that

\[
f_j(z) = \lambda_1(z, f)g_{1,j-d}(z) + ... + \lambda_d(z, f)g_{d,j-d}(z)
\]

\[
= \frac{g_{1,j-d}(z)}{\det(\theta)(z)}f_1(z) + ... + \frac{g_{d,j-d}(z)}{\det(\theta)(z)}f_d(z).
\]

In particular, each function \( f = f_1e_1 + ... + f_Ne_N \in M \) is uniquely determined by its first \( d \) coordinate functions \((f_1, ..., f_d)\).

Since \( A \) is dense in \( M_1 \) and in \( M_2 \), we can choose functions \( F_1, ..., F_{d_1+d'} \in M_1 \) and \( G_1, ..., G_{d_2+d'} \in M_2 \) with coefficients in \( A \) such that

\[
(F_1(\lambda))_{i=1,...,d_1+d'} = (e_1, ..., e_{d_1}, e_{d_1+d_2+1}, ..., e_{d_1+d_2+d'}
\]

and

\[
(G_i(\lambda))_{i=1,...,d_2+d'} = (e_{d_1+1}, ..., e_{d_1+d_2+d'}).
\]

Write the first \( d \) coordinate functions of each of the functions

\[
F_1, ..., F_{d_1}, G_1, ..., G_{d_2}, F_{d_1+1}, ..., F_{d_1+d'}, G_{d_2+1}, ..., G_{d_2+d'}
\]

with respect to the basis \((e_1, ..., e_N)\) of \( D \) as column vectors and arrange these column vectors to a matrix \( \Delta \) in the indicated order. Then \( \Delta \) is a \((d \times (d + d'))\)-matrix with entries in \( A \). Write \( \Delta = (\Delta_0, \Delta_1) \) where \( \Delta_0 \) is the \((d \times d')\)-matrix consisting of the first \( d \) columns of \( \Delta \) and \( \Delta_1 \) is the \((d \times d')\)-matrix consisting of the last \( d' \) columns of \( \Delta \).

By construction we have \( \det(\Delta_0(\lambda)) = 1 \). On \( \Omega \setminus Z(\det(\Delta_0)) \), we can write

\[
(\det(\Delta_0)\Delta_0^{-1}\Delta = (\text{diag}(\det(\Delta_0)), \Gamma),
\]

where \( \text{diag}(\det(\Delta_0)) \) is the \((d \times d)\)-diagonal matrix with all diagonal terms equal to \( \det(\Delta_0) \) and \( \Gamma = (\gamma_{ij}) \) is a \((d \times d')\)-matrix with entries in \( A \). The column vectors

\[
r_j = (\gamma_{1j}, ..., \gamma_{dj}, 0, ..., 0, -\det(\Delta_0), 0, ..., 0)^t \quad (j = 1, ..., d')
\]

where \( -\det(\Delta_0) \) is the entry in the \((d+j)\)-th position, satisfy the equations

\[
(\det(\Delta_0)\Delta_0^{-1}\Delta r_j = ((\det(\Delta_0))\gamma_{ij} - (\det(\Delta_0))\gamma_{ij})_{i=1}^d = 0
\]

19
on $\Omega \setminus Z(\det \Delta_0)$. Hence $\Delta r_j = 0$ for $j = 1, \ldots, d'$, or equivalently, for each $j = 1, \ldots, d$, the first $d$ coordinate functions of

$$
\gamma_1 F_1 + \ldots + \gamma_{d,j} F_{d,1} + \gamma_{d+1,d+1,j} F_{d,1+1} + \ldots + \gamma_{d_1+d_2+d',j} F_{d_1+d'}
$$

with respect to $(e_1, \ldots, e_N)$ coincide with those of

$$(\det \Delta_0) G_{d_2+j} - \gamma_{d_1+1,j} G_1 - \ldots - \gamma_{d_1+d_2,j} G_{d_2}.$$

Since, for each $j$, both functions belong to $M$, they coincide. But then these functions belong to $M_1 \cap M_2$. Since the vectors $G_{i}(\lambda) = e_{d_1+i}$ $(i = 1, \ldots, d_2 + d')$ are linearly independent and since $\det(\Delta_0(\lambda)) = 1$, it follows that $\text{fd}(M_1 \cap M_2) = \dim(M_1 \cap M_2) \lambda \geq d'$.

Suppose for the moment that $\Omega \subseteq \mathbb{C}^n$ is a Runge domain. Since by the Oka-Weil approximation theorem, the polynomials are dense in $\mathcal{O}(\Omega)$ with respect to the Fréchet space topology of uniform convergence on compact subsets, each $\mathbb{C}[z]$-submodule $M \subseteq \mathcal{O}(\Omega, D)$ which is closed with respect to the Fréchet space topology of $\mathcal{O}(\Omega, D)$ is automatically an $\mathcal{O}(\Omega)$-submodule. Hence we obtain the following consequence of Theorem 5.2.

**Corollary 5.3.** Let $\Omega \subseteq \mathbb{C}^n$ be a Runge domain and let $D$ be a finite-dimensional complex vector space. Then the fiber dimension formula

$$
\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2)
$$

holds for each pair of closed $\mathbb{C}[z]$-submodules $M_1, M_2$ of the Fréchet space $\mathcal{O}(\Omega, D)$.

Suppose that $T \in L(X)^n$ is a dual Cowen-Douglas tuple of rank $N$ on a domain $\Omega \subseteq \mathbb{C}^n$. Choose a CF-representation

$$
\rho : X \to \mathcal{O}(\Omega_0, D)
$$

of $T$ as in the proof of Theorem 2.4. Let $M \in \text{Lat}(T)$ be an invariant subspace of $T$ such that each vector $m \in M$ is the limit of a sequence of vectors in

$$
M \cap \text{span}\{T^\alpha x; \alpha \in \mathbb{N}^n \text{ and } x \in D\}.
$$

Then $\rho(M) \subseteq \mathcal{O}(\Omega_0, D)$ is a $\mathbb{C}[z]$-submodule in which the polynomials are dense in the sense explained in the section leading to Theorem 5.2. Hence, for any two invariant subspaces $M_1, M_2 \in \text{Lat}(T)$ of this type, the fiber dimension formula

$$
\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(\rho(M_1) + \rho(M_2)) + \text{fd}(\rho(M_1) \cap \rho(M_2))
$$

$$
= \text{fd}(\rho(M_1)) + \text{fd}(\rho(M_2)) = \text{fd}(M_1) + \text{fd}(M_2)
$$

20
holds. The above density condition on $M$ is trivially fulfilled for every closed $T$-invariant subspace $M$ which is generated by a subset of $D$. But there are other situations to which this observation applies.

Recall that a commuting tuple $T \in L(H)^n$ of bounded operators on a complex Hilbert space $H$ is called graded if $H = \bigoplus_{k=0}^\infty H_k$ is the orthogonal sum of closed subspaces $H_k \subseteq H$ such that $\dim H_0 < \infty$ and

(i) $T_j H_k \subseteq H_{k+1}$ \hspace{1cm} (k \geq 0, j = 1, ..., n),

(ii) $\sum_{j=1}^n T_j H \subseteq H$ is closed,

(iii) $\bigvee_{\alpha \in \mathbb{N}^n} T^\alpha H_0 = H$.

It is elementary to show (Lemma 2.4 in [10]) that under these hypotheses the identities

$$\sum_{|\alpha| = k} T^\alpha H = \bigoplus_{j=k}^\infty H_j \quad \text{and} \quad \sum_{|\alpha| = k} T^\alpha H_0 = H_k$$

hold for all integers $k \geq 0$. By definition a closed invariant subspace $M \in \text{Lat}(T)$ of a graded tuple $T \in L(H)^n$ is said to be homogeneous if

$$M = \bigoplus_{k=0}^\infty M \cap H_k.$$

**Corollary 5.4.** Let $T \in L(H)^n$ be a graded dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^n$. Then the fiber dimension formula

$$\text{fd}(M_1 + M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2)$$

holds for any pair of homogeneous invariant subspaces $M_1, M_2 \in \text{Lat}(T)$.

**Proof.** By the remarks preceding the corollary

$$H = \left( \sum_{j=1}^n T_j H \right) \oplus H_0.$$

Hence in the proof of Theorem 2.6 we can choose $D = H_0$. Let $\rho : H \to \mathcal{O}(\Omega_0, H_0)$ be a CF-representation of $T$ as constructed in the proof of Theorem 2.6. Let $M \in \text{Lat}(T)$ be a homogeneous invariant subspace for $T$. Then each element $m \in M$ can be written as a sum $m = \sum_{k=0}^\infty m_k$ with

$$m_k \in M \cap \sum_{|\alpha| = k} T^\alpha H_0 \hspace{1cm} (k \in \mathbb{N}).$$

Hence the assertion follows from the remarks preceding Corollary 5.4. \qed
Typical examples of graded dual Cowen-Douglas tuples are multiplication tuples $M_z = (M_{z_1}, ..., M_{z_n}) \in L(H)^n$ with the coordinate functions on analytic functional Hilbert spaces $H = H(K_f, \mathbb{C}^N)$ given by a reproducing kernel

$$K_f : B_r(a) \times B_r(a) \to L(\mathbb{C}^n), K_f(z, w) = f((z, w))1_{\mathbb{C}^N},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a one-variable power series with radius of convergence $R = r^2 > 0$ such that $a_0 = 1, a_n > 0$ for all $n$ and

$$0 < \inf_{n \in \mathbb{N}} \frac{a_n}{a_{n+1}} \leq \sup_{n \in \mathbb{N}} \frac{a_n}{a_{n+1}} < \infty$$

(see [12] or [17]). In this case $H$ is the orthogonal sum

$$H = \bigoplus_{k=0}^{\infty} H_k \otimes \mathbb{C}^N$$

of the subspaces consisting of all homogeneous $\mathbb{C}^N$-valued polynomials of degree $k$ and every invariant subspace $M = \bigvee_{i=1}^{r} \mathbb{C}[z]p_i \in \text{Lat}(M_z)$

generated by a finite set of homogeneous polynomials $p_i \in H_k \otimes \mathbb{C}^N$ is homogeneous. This class of examples contains the Drury-Arveson space, the Hardy space and the weighted Bergman spaces on the unit ball.

Let $H = H(K) \subseteq \mathcal{O}(\Omega)$ be an analytic functional Hilbert space on a domain $\Omega \subseteq \mathbb{C}^n$, or equivalently, a functional Hilbert space given by an analytic reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$. Let $D$ be a finite-dimensional complex Hilbert space. Then the $D$-valued functional Hilbert space $H(K_D) \subseteq \mathcal{O}(\Omega, D)$ given by the kernel

$$K_D : \Omega \times \Omega \to L(D), K_D(z, w) = K(z, w)1_D$$

can be identified with the Hilbert space tensor product $H(K) \otimes D$. Let us denote by $M(H) = \{ \varphi : \Omega \to \mathbb{C}; \varphi H \subseteq H \}$ the multiplier algebra of $H$.

**Corollary 5.5.** Suppose that $H = H(K)$ contains all constant functions and that $z_1, ..., z_n \in M(H)$.

(a) For any pair of closed subspaces $M_1, M_2 \subseteq H(K_D)$ such that $M(H)M_i \subseteq M_i$ for $i = 1, 2$ and such that $M(H)$ is dense in $M_1$ and $M_2$, the fiber dimension formula

$$\text{fd}(M_1 \vee M_2) + \text{fd}(M_1 \cap M_2) = \text{fd}(M_1) + \text{fd}(M_2)$$

holds.
(b) If in addition $K$ is a complete Nevanlinna-Pick kernel, that is, $K$ has no zeros and also the mapping $1 - \frac{1}{K}$ is positive definite, then the fiber dimension formula holds for all closed subspaces $M_1, M_2 \subseteq H(K_D)$ which are invariant for $M(H)$.

Proof. Part (a) is a direct consequence of Theorem 5.2. If $K$ is a complete Nevanlinna-Pick kernel, then the Beurling-Lax-Halmos theorem for Nevanlinna-Pick spaces proved by McCullough and Trent (see Theorem 8.67 in [1] or Theorem 3.3.8 in [2]) implies that $M(H)$ is dense in every closed subspace $M \subseteq H(K_D)$ which is invariant for $M(H)$.

Note that the condition that $M(H)$ is dense in a subspace $M \subseteq H(K_D)$ is satisfied for every closed $M(H)$-invariant subspace $M \subseteq H(K_D)$ that is generated by an arbitrary family of functions $f_i : \Omega \to D$ $(i \in I)$ with coefficients in $M(H)$. Part (b) for domains $\Omega \subseteq \mathbb{C}$ was proved in [3]. The proof in the multivariable case is the same.

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