Extensions of Finite Abelian Groups

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Abstract

We study group extensions of Finite Abelian Groups using matrices. We also prove a Theorem for equivalence of extensions using matrices.

Introduction

Presentation of Abelian Groups

Every Abelian Group \((A, +)\) can be considered as a \(\mathbb{Z}\)-module under the ring action:

\[
\mathbb{Z} \times A \rightarrow A
\]

\[
(n, x) \mapsto nx = x + x + \ldots + x\text{ (}\text{ntimes}\text{)}
\]

\[
(-n, x) \mapsto -(nx) = -(x + x + \ldots + x)\text{ (}\text{ntimes}\text{)}
\]

We call \(x^d = (x_1, x_2, \ldots, x_n), x_i \in A\) a generating set of \(A\) if every element in \(A\) can be written as \(a = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\) for some \(\alpha_i \in \mathbb{Z}\) under the ring action defined above.

A relation in \(A\) is an equation of the form

\[
\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0
\]

If we have have a set of ‘m’ relations

\[
\alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n = 0
\]

\[
\alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n = 0
\]

\[
\vdots
\]

\[
\alpha_{n1} x_1 + \alpha_{n2} x_2 + \ldots + \alpha_{nn} x_n = 0
\]
We can write that down as a matrix equation \( \alpha \vec{x} = 0 \) or

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nm}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= 0
\]

A complete set of relations of a group \( A \) is one from which every other relation of the group can be derived by a linear combination. A 'Presentation of an Abelian Group' is a generating set along with a complete set of relations.

**Finite Abelian Groups and their Sylow Subgroups**

Let \( A \) be a Finite Abelian Groups of order \( n \) where \( n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n} \) then \( A \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_n} \), where each \( \mathbb{Z}_{p_i} \) is a Sylow subgroup associated with that prime. Therefore, we study abelian p-groups.

**The Extension Problem for abelian p-groups**

Given two p-groups \( G_{\lambda} \) and \( G_{\mu} \) of type \( \lambda \) and type \( \mu \) respectively, we describe all groups \( E \) containing \( G_{\lambda} \) such that \( E/G_{\lambda} \cong G_{\mu} \).

i.e. we have a Short Exact Sequence \( 0 \rightarrow G_{\lambda} \rightarrow E \rightarrow G_{\mu} \rightarrow 0 \)

By the structure theorem for finite abelian groups, \( G_{\lambda} \) has a presentation with generating set \( y = \langle y_1, y_2, \ldots, y_l \rangle \) of \( l \) elements and relations \( p^{\lambda_i} y_i = 0 \forall i \) from 1 to \( l \) and for some integers \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0 \)

Similarly \( G_{\mu} \) is generated by \( m \) elements \( \langle x_1, x_2, \ldots, x_m \rangle \) with the relations \( p^{\mu_j} x_j = 0 \forall j \) from 1 to \( m \)

Now choose \( x_1^\sim, x_2^\sim, x_3^\sim, \ldots, x_m^\sim \in E \) such that \( \phi(x_i^\sim) = x_i \forall j \) from 1 to \( m \)

But from the above relations, we have \( \phi(p^{\mu_j} x_j^\sim) = p^{\mu_j} \phi(x_j^\sim) = \phi(x_j) = 0 \Rightarrow p^{\mu_j} x_j^\sim \in ker(\phi) = G_{\lambda} \) (Definition of Short Exact Sequence)

Since every element of \( G_{\lambda} \) can be written as some \( a \vec{y} \) we have \( p^{\mu_j} x_j^\sim = a_{j1} y_1 + a_{j2} y_2 + \ldots + a_{jl} y_l \forall j \) from 1 to \( m \)

Combining the two sets of relations, we have in hand altogether \( l + m \) relations. Putting these relations together as a matrix, we would get a pre-
sentation for \( E \)
\[
\begin{pmatrix}
 p^\lambda_1 \\
p^\lambda_2 \\
\vdots \\
a_{11} & a_{12} & \ldots & a_{1l} & p^{\mu_1} \\
a_{21} & a_{22} & \ldots & a_{2l} & p^{\mu_2} \\
\vdots \\
a_{21} & a_{22} & \ldots & a_{ml} & \ldots & \ldots & p^{\mu_m}
\end{pmatrix}
\begin{pmatrix}
 y_1 \\
y_2 \\
\vdots \\
y_l \\
x_1 \sim \\
x_2 \sim \\
\vdots \\
x_m \sim
\end{pmatrix}
\] = 0

Or as blocks they would be
\[
\begin{pmatrix}
p^\lambda \\A \ p^{\mu}
\end{pmatrix}
_{m+l\times m+l}
\begin{pmatrix}
y \\x \sim
\end{pmatrix}
= 0
\]

One is now interested in classifying extensions upto isomorphisms
\[
\begin{array}{c}
0 \rightarrow G_{\lambda} \xrightarrow{\psi_1} E_1 \xrightarrow{\phi_1} G_{\mu} \rightarrow 0 \\
0 \rightarrow G'_{\lambda} \xrightarrow{\psi_2} E_2 \xrightarrow{\phi_2} G''_{\mu} \rightarrow 0
\end{array}
\]

If there exist isomorphisms \( f \) and \( g \) and a homomorphism \( h \) such that the above diagram commutes, then the extension in the first row is said to be equivalent to the extension in the second row. It then follows that \( h \) is also an isomorphism.

\[
G'_{\lambda} = \langle y'_i \rangle
\]

where \( i \) is from 1 to \( l \)

\[
G''_{\mu} = \langle x'_j \rangle
\]

where \( j \) is from 1 to \( m \) (We have assumed that isomorphic groups have equal number of generators and relations. In rigor, one should look at isomorphic groups through Tietze Transformations.)

We can now describe isomorphisms \( f, g \) and \( h \) as matrices acting on basis (vectors).

\[
f(\langle y_1, y_2, \ldots, y_l \rangle) = \langle y'_1, y'_2, \ldots, y'_l \rangle [F]
\]

\[
g(\langle x_1, x_2, \ldots, x_m \rangle) = \langle x'_1, x'_2, \ldots, x'_m \rangle [G]
\]

\[
h(\langle y_1, \ldots, y_l, x_1 \sim, \ldots, x_m \sim \rangle) = \langle y_1, \ldots, y'_l, x'_1 \sim, \ldots, x'_m \sim \rangle [H]
\]

Here \([F]\), \([G]\) and \([H]\) are the matrices of automorphisms. We write \([H]\) as
\[
\begin{bmatrix}
h_{11} & h_{12} \\h_{21} & h_{22}
\end{bmatrix}
\]

where \( h_{11} \) is \( l \times l \) and \( h_{22} \) is \( m \times m \).

Also the matrices are invertible and preserve the structure of the groups, i.e
\[ F \in GL_l(\mathbb{Z}) \cap p^{-\lambda}GL_l(\mathbb{Z})p^\lambda \]
\[ G \in GL_m(\mathbb{Z}) \cap p^{-\mu}GL_m(\mathbb{Z})p^\mu \]
\[ H \in GL_{l+m}(\mathbb{Z}) \cap p^{-(\lambda+\mu)}GL_{l+m}(\mathbb{Z})p^{(\lambda+\mu)} \]

Using the fact that the above diagram commutes, we have
\[ \psi_2(f(k)) = h(\psi_1(k)) \forall k \in G_\lambda \]
\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix} \Rightarrow h_{11} = F \text{ and } h_{21} = 0 \]
\[ \phi_2(h(l)) = g(\phi_1(l)) \forall l \in E_1 \]
\[ \begin{bmatrix} F & h_{12} \\ 0 & h_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} G \end{bmatrix} \]
\[ \Rightarrow \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix} \]
\[ \Rightarrow [H] = \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix} \]

Now, looking at the presentations for \( E_1 \) and \( E_2 \)
\[ E_1 : \begin{bmatrix} y & x^\sim \end{bmatrix} \begin{bmatrix} A_1 & p^\lambda \\ p^\mu & 0 \end{bmatrix} = 0 \]
\[ E_1 : \begin{bmatrix} y' & x'^\sim \end{bmatrix} \begin{bmatrix} A_2 & p^\lambda \\ p^\mu & 0 \end{bmatrix} = 0 \]

Note that the matrices have been transposed for convenience.

Using the isomorphism condition, i.e. \( h(0) = 0 \), we have
\[ \begin{bmatrix} y & x^\sim \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} A_1 & p^\lambda \\ p^\mu & 0 \end{bmatrix} = 0_{E_2} \]
\[ \begin{bmatrix} y & x^\sim \end{bmatrix} \begin{bmatrix} FA_1 & Fp^\lambda \\ Gp^\mu & 0 \end{bmatrix} = 0_{E_2} \]

But given that we have a complete set of relations in \( E_2 \), these matrices must be related by a column operation (Note, we had transposed the matrices and hence instead of row operations, we now must consider column operations).
\[ \begin{bmatrix} A_2 & p^\lambda \\ p^\mu & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} FA_1 & Fp^\lambda \\ Gp^\mu & 0 \end{bmatrix} \]

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\( \Rightarrow p^\mu \alpha = Gp^\mu \) or \( \alpha = p^{-\mu} Gp^\mu \)
\( \Rightarrow A_2 p^{-\mu} Gp^\mu + p^\lambda \gamma = F A_1 \)
\( \Rightarrow F^{-1} A_2 p^{-\mu} Gp^\mu + F^{-1} p^\lambda \gamma = A_1 \)
Now, given the conditions on \( F \) and \( G \), we re-write the above equation as
\[
F^{-1} A_2 G' + F^{-1} p^\lambda \gamma = A_1
\]
But, \( A_1 \) is taken modulo \( p^\lambda \Rightarrow F' A_2 G' = A_1 \) where \( G' = p^{-\mu} Gp^\mu \) and \( F' = F^{-1} \). Thus we have the following Theorem.

**Theorem**  The extensions associated to two finite abelian groups \( E_1 \) and \( E_2 \) are equivalent if and only if there exist matrices \( F \in GL_l(\mathbb{Z}) \cap p^\lambda GL_l(\mathbb{Z}) p^{-\lambda} \) and \( G \in GL_m(\mathbb{Z}) \cap p^{-\mu} GL_m(\mathbb{Z}) p^\mu \) such that \( F A_2 G = A_1 \) where the \((i,j)\) entry of each matrix is taken modulo \( p^{\min(\lambda_i, \mu_j)} \).

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**References**
[1] Garrett Birkhoff. Subgroups of Abelian Groups Proc. London Math. Soc. 38:385-401, 1935.
[2] David S. Dummit and Richard M. Foote Abstract Algebra Third Edition John Wiley and Sons inc. 2004.