Two-Parton Contribution to the Heavy-Quark Forward-Backward Asymmetry in NNLO QCD

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Abstract

Forward-backward asymmetries, $A_{FB}^Q$, are important observables for the determination of the neutral-current couplings of heavy quarks in inclusive heavy quark production, $e^+e^- \rightarrow \gamma^*, Z^* \rightarrow Q + X$. In view of the measurement perspectives on $A_{FB}^Q$ at a future linear collider, precise predictions of $A_{FB}^Q$ are required for massive quarks. We compute the contribution of the $Q\bar{Q}$ final state to $A_{FB}^Q$ to order $\alpha_s^2$ in the QCD coupling. We provide general formulae, and we show that this contribution to $A_{FB}^Q$ is infrared-finite. We evaluate these two-parton contributions for $b$ and $c$ quarks on and near the $Z$ resonance, and for $t$ quarks above threshold. Moreover, near the $t\bar{t}$ threshold we obtain, by expanding in the heavy-quark velocity $\beta$, an expression for $A_{FB}^{t\bar{t}}$ to order $\alpha_s^2$ and NNLL in $\beta$. This quantity is equal, to this order in $\beta$, to the complete forward-backward asymmetry $A_{FB}^t$.

Key words: Electroweak measurements, Forward-backward asymmetry, Heavy quarks, Precision calculations, QCD corrections.

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1 Introduction

Forward-backward asymmetries in the production of fermions in high-energy $e^+e^-$ collisions are known to be precision observables for the determination of the respective fermionic neutral current couplings. Specifically the forward-backward asymmetry $A_{FB}^b$ of $b$ quarks, which was measured at the $Z$ resonance with an accuracy of 1.7 percent, led to a determination of the effective weak mixing angle $\sin^2\theta_{W,\text{eff}}$ of the Standard Model with a relative precision of about 1 per mille [1, 2] – notwithstanding the apparent discrepancy between this measurement and the determination of $\sin^2\theta_{W,\text{eff}}$ with similar precision from the left-right asymmetry measured by the SLD collaboration. (For a comprehensive overview, see [1].) At a future linear $e^+e^-$ collider [3], precision determinations of electroweak parameters will again involve forward-backward asymmetries. When such a collider will be operated at the $Z$ peak, accuracies of about 0.1 percent may be reached for these observables [4, 5]. Moreover, the top quark asymmetry $A_{FB}^t$ will be experimentally accessible – a crucial tool for the determination of the hitherto unexplored neutral current couplings of this quark.

The theoretical understanding of these observables, in particular those involving the heavy quarks $Q = t, b, c$, must eventually match these projected accuracies. The present theoretical description of $A_{FB}^b$ and $A_{FB}^c$ includes the fully massive next-to-leading order (NLO) electroweak [6, 7, 8] and fully massive NLO QCD [9, 10, 11] corrections. The next-to-next-to-leading order (NNLO) QCD corrections, i.e., the contributions of $\alpha^2$ to these asymmetries, were calculated so far only in the limit of massless quarks $Q$ [12, 13, 14]. To be precise, the forward-backward asymmetry of massless quarks is not computable in QCD perturbation theory, as was pointed out in [14]. It is affected in the limit $m_Q \to 0$ by logarithmic final state divergences $\sim \ln m_Q$ resulting from the contribution of the $Q\bar{Q}Q\bar{Q}$ final state. (These terms are associated with the non-perturbative $Q$ fragmentation function.) These logarithmically enhanced terms were taken into account in Ref. [14], which is the most complete calculation of $A_{FB}^b$ within QCD to date, done both with respect to the quark and the thrust axis.

In view of the future perspectives for the $b$- and $t$-quark asymmetries at a linear collider, a computation of the order $\alpha^2$ contributions to $A_{FB}^Q$ for massive quarks $Q$ is clearly desirable. The NNLO QCD corrections involve three classes of contributions: (1) the two-loop corrections to the decay of a vector boson into a heavy quark-antiquark pair; (2) the one-loop corrected matrix elements for the decay of a vector boson into a heavy quark-antiquark pair plus a gluon; (3) the tree level matrix elements for the decay of a vector boson into four partons, at least two of which being the heavy quark-antiquark pair.

In the limit of massless external quarks $Q$ the $Q\bar{Q}$ contributions to $A_{FB}^Q$ vanish up to a non-universal correction of order $\alpha^2$ due to quark triangle diagrams [12]. This is no longer the case for $m_Q \neq 0$. In this paper we determine class (1), i.e., the order $\alpha^2$ $Q\bar{Q}$ contributions to $A_{FB}^Q$, for arbitrary quark mass $m_Q$ and center-of-mass (c.m.) energy $\sqrt{s}$. For this purpose we use our recent results on the two-loop vector and axial vector vertex functions for massive quarks [20, 21, 22] which determine the amplitude of $e^+e^- \to \gamma^*, Z^* \to Q\bar{Q}$ to order $\alpha^2$ and to lowest order in the electroweak couplings. The contributions (1) from the two-parton final state, and (2) plus (3), i.e., those from the three- and four-parton final states, are separately infrared-finite, as will be discussed below. The latter can be obtained along the lines of the calculations of three-jet production involving heavy quarks [15, 16, 17]. The parity-violating part of $d\sigma(3\text{jet})$ at order $\alpha^2$, which is a necessary ingredient here, was computed in [18].
However, a full computation of $A_{FB}^{3+4\text{parton}}$ has not yet been done for massive quarks.

The paper is organized as follows. In Section 2 we set up the formulae for determining $A_Q^{FB}$, off and at the $Z$ resonance, to order $\alpha_s^2$ in terms of the symmetric and antisymmetric cross sections for inclusive production of quarks $Q$. In particular we express the $Q\bar{Q}$ contribution to $A_Q^{FB}$ by the one- and two-loop heavy-quark vector and axial vector form factors that determine the amplitude corresponding to the $Q\bar{Q}$ final state. Moreover, we show that these contributions to the respective forward-backward asymmetry, which we denote by $A_{Q\bar{Q}}^{FB}$, are infrared-finite. In Section 3 we present the NNLO $Q\bar{Q}$ cross section and the forward-backward asymmetry in the energy region $\alpha_s \ll \beta \ll 1$, where $\beta$ denotes the heavy-quark velocity. This result is applicable, for instance, to $t$ quarks in the vicinity of the $t\bar{t}$ threshold. In Section 4 we evaluate $A_{Q\bar{Q}}^{FB}$ for $b$ and $c$ quarks at the $Z$ resonance, and in the case of $t$ quarks for c. m. energies above the $t\bar{t}$ threshold till 1 TeV. We conclude in Section 5.

2 The Forward-Backward Asymmetry to Order $\alpha_s^2$

In this paper we consider the production of a heavy quark-antiquark pair in $e^+e^-$ collisions,

$$e^+e^- \rightarrow \gamma^*(q), \ Z^*(q) \rightarrow Q\bar{Q} + X,$$

(1)

where $Q = c, b, t$, to lowest order in the electroweak couplings and to second order in the QCD coupling $\alpha_s$. To this order the following final states contribute to the cross section of inclusive $Q\bar{Q}$ production, Eq. (1): the two-parton $Q\bar{Q}$ state (at Born level, to order $\alpha_s$ and to order $\alpha_s^2$), the three-parton state $Q\bar{Q}g$ (to order $\alpha_s$ and to order $\alpha_s^2$) and the four-parton states $Q\bar{Q}gg$, $Q\bar{Q}q\bar{q}$, and $Q\bar{Q}QQ\bar{Q}$ (to order $\alpha_s^2$).

The forward-backward asymmetry$^1$ $A_{FB}$ for a heavy quark $Q$ is commonly defined as the number of quarks $Q$ observed in the forward hemisphere minus the number of quarks $Q$ in the backward hemisphere, divided by the total number of observed quarks $Q$. The axis that defines the forward direction must be infrared- and collinear-safe in order that $A_{FB}$ is computable in perturbation theory. Common choices are the direction of flight of $Q$ or the thrust axis direction. The forward-backward asymmetry can also be expressed in terms of the cross section for inclusive production of quarks $Q$. We have:

$$A_{FB} = \frac{\sigma_A}{\sigma_S},$$

(2)

with the antisymmetric and symmetric cross sections $\sigma_A$ and $\sigma_S = \sigma$ defined by

$$\sigma_A = \int_0^1 \frac{d\sigma}{d\cos\vartheta} \ d\cos\vartheta - \int_{-1}^0 \frac{d\sigma}{d\cos\vartheta} \ d\cos\vartheta,$$

(3)

$$\sigma_S = \int_{-1}^1 \frac{d\sigma}{d\cos\vartheta} \ d\cos\vartheta.$$

(4)

Here $\vartheta$ is the angle between the incoming electron and the direction defining the forward hemisphere (in the $e^+e^-$ center-of-mass frame). When choosing the momentum direction of $Q$

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$^1$We drop the superscript $Q$ in the following for ease of notation.
or the thrust axis, the $Q\bar{Q}$ contribution to $A_{FB}$, which we compute in this paper, is of course the same. In the following $\vartheta = \angle(e^-, Q)$.

In analogy to its experimental measurement, $A_{FB}$ may be computed by determining the contributions from the final-state jets which, to order $\alpha_s^2$, are those of the two-, three-, and four-jet states. These contributions are separately infrared-finite, and $A_{FB}$ would not depend on the jet clustering algorithm employed when no phase space cuts are applied. Such a calculation would require a jet calculus for massive quarks at NNLO in $\alpha_s$ which is, however, not available. (For massless quarks a NNLO subtraction method was recently developed \cite{19}, cf. also references therein.) Yet, $A_{FB}$ belongs to the class of observables that can be computed at the level of unresolved partons. The two-parton and the three- plus four-parton contributions to the second-order forward-backward asymmetry are separately infrared (IR) finite, cf. \cite{12,14} and Section 2.2 below. This basic result will be exploited in the following.

To order $\alpha_s^2$ the symmetric and antisymmetric cross sections receive the following contributions from unresolved partons:

$$
\sigma_{A,S} = \sigma_{A}^{(2,0)} + \sigma_{A}^{(2,1)} + \sigma_{A}^{(2,2)} + \sigma_{A}^{(3,1)} + \sigma_{A}^{(3,2)} + \sigma_{A}^{(4,2)} + \mathcal{O}(\alpha_s^3),
$$

(5)

where the first number in the superscripts ($i,j$) denotes the number of partons in the respective final state and the second one the order of $\alpha_s$. Inserting (5) into (2) we get for $A_{FB}$ to second order in $\alpha_s$:

$$
A_{FB}(\alpha_s^2) = \frac{\sigma_{A}^{(2,0)} + \sigma_{A}^{(2,1)} + \sigma_{A}^{(2,2)} + \sigma_{A}^{(3,1)} + \sigma_{A}^{(3,2)} + \sigma_{A}^{(4,2)}}{\sigma_{S}^{(2,0)} + \sigma_{S}^{(2,1)} + \sigma_{S}^{(2,2)} + \sigma_{S}^{(3,1)} + \sigma_{S}^{(3,2)} + \sigma_{S}^{(4,2)}}.
$$

(6)

Taylor expansion of (6) with respect to $\alpha_s$ leads to

$$
A_{FB}(\alpha_s^2) = A_{FB,0} \left[ 1 + A_1 + A_2 \right],
$$

(7)

where $A_{FB,0}$ is the forward-backward asymmetry at Born level, and $A_1$ and $A_2$ are the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ contributions normalized to $A_{FB,0}$.

$$
A_{FB,0} = \frac{\sigma_{A}^{(2,0)}}{\sigma_{S}^{(2,0)}},
$$

(8)

$$
A_1 = \frac{\sigma_{A}^{(2,1)}}{\sigma_{A}^{(2,0)}} - \frac{\sigma_{S}^{(2,1)}}{\sigma_{S}^{(2,0)}} + \frac{\sigma_{A}^{(3,1)}}{\sigma_{A}^{(2,0)}} - \frac{\sigma_{S}^{(3,1)}}{\sigma_{S}^{(2,0)}},
$$

(9)

$$
A_2 = \frac{\sigma_{A}^{(2,2)}}{\sigma_{A}^{(2,0)}} - \frac{\sigma_{S}^{(2,2)}}{\sigma_{S}^{(2,0)}} + \frac{\sigma_{A}^{(3,2)}}{\sigma_{A}^{(2,0)}} - \frac{\sigma_{S}^{(3,2)}}{\sigma_{S}^{(2,0)}} + \frac{\sigma_{A}^{(4,2)}}{\sigma_{A}^{(2,0)}} - \frac{\sigma_{S}^{(4,2)}}{\sigma_{S}^{(2,0)}} \left[ \frac{\sigma_{A}^{(2,1)}}{\sigma_{A}^{(2,0)}} - \frac{\sigma_{S}^{(2,1)}}{\sigma_{S}^{(2,0)}} + \frac{\sigma_{A}^{(3,1)}}{\sigma_{A}^{(2,0)}} - \frac{\sigma_{S}^{(3,1)}}{\sigma_{S}^{(2,0)}} \right].
$$

(10)

1.2 The $Q\bar{Q}$ Contribution

It is convenient to rewrite Eq. (7) as follows:

$$
A_{FB}(\alpha_s^2) = A_{FB}^{(2p)} + A_{FB}^{(3p)} + A_{FB}^{(4p)},
$$

(11)
where the superscript \((np)\) labels the number of partons \(n\) in the final state. Collecting the two-parton term from Eqs. (8), (9), and (10) we get:

\[
A_{FB}^{(2p)} = A_{FB,0} \left[ 1 + A_1^{(2p)} + A_2^{(2p)} \right],
\]

with

\[
A_1^{(2p)} = \frac{\sigma_A^{(2,1)}}{\sigma_A^{(2,0)}} - \frac{\sigma_S^{(2,1)}}{\sigma_S^{(2,0)}},
\]

\[
A_2^{(2p)} = A_{2,2} - A_{2,1},
\]

where

\[
A_{2,2} = \frac{\sigma_A^{(2,2)}}{\sigma_A^{(2,0)}} - \frac{\sigma_S^{(2,2)}}{\sigma_S^{(2,0)}},
\]

\[
A_{2,1} = \frac{\sigma_S^{(2,1)}}{\sigma_S^{(2,0)}} A_1^{(2p)}.
\]

The remaining terms in Eqs. (9), (10) contribute to \(A^{(3p)}\) and \(A^{(4p)}\).

As already stated above, both the two-parton and the three- plus four-parton contributions \(A^{(2p)}\) and \(A^{(3p)} + A^{(4p)}\), respectively, are infrared (IR) finite, i.e., free of soft and collinear singularities. We shall show this explicitly for \(A^{(2p)}\) at the end of this section.

Next we express \(A_{FB}^{(2p)}\) in terms of the \(VQ\bar{Q}\) vertex form factors \((V = \gamma, Z)\) which determine the amplitude of the reaction \(e^+e^- \rightarrow \gamma^*, Z^* \rightarrow Q\bar{Q}\) to lowest order in the electroweak couplings and to any order in the QCD coupling; see Fig. 1. In this case the \(VQ\bar{Q}\) vertex \(\Gamma_Q^{\mu,V}\) depends,

\[
\begin{align*}
\Gamma_Q^{\mu,V} & = v_Q^V \left( F_1(s) \gamma^\mu + \frac{i}{2m_Q} F_2(s) \sigma^{\mu\nu} q_{\nu} \right) \\
& + a_Q^V \left( G_1(s) \gamma^\mu \gamma_5 + \frac{1}{2m_Q} G_2(s) \gamma_5 q^{\mu} \right),
\end{align*}
\]

Figure 1: The amplitude \(e^+e^- \rightarrow Q\bar{Q}\) in QCD.
where \( s = q^2, \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \), \( m_Q \) denotes the on-shell mass of \( Q \), and, for \( f = e, Q \),
\[
\begin{align*}
v_f^Z &= e \frac{1}{2 s_W c_W} \left( T_f^3 - 2 s_W^2 e_f \right), \\
a_f^Z &= e \frac{1}{2 s_W c_W} \left( -T_f^3 \right), \\
v_f^\gamma &= e e_f, \quad a_f^\gamma = 0.
\end{align*}
\]
Here \( e_f \) and \( T_f^3 \) denote the charge of \( f \) in units of the positron charge \( e \) and its weak isospin, respectively, and \( s_W \ (c_W) \) are the sine (cosine) of the weak mixing angle \( \theta_W \). The functions \( F_i \) and \( G_i \) denote renormalized form factors; the renormalization scheme will be specified below. Instead of using the Dirac form factor \( F_1 \) we express in the following the (anti)symmetric cross section in terms of
\[
\tilde{F}_1(s) = F_1(s) + F_2(s).
\]
Neglecting the electron mass we find for the two-parton contributions to \( \sigma_{A,S} \):
\[
\begin{align*}
\sigma_A^{(2p)} &= \frac{N_c}{8 \pi} \frac{s}{D_Z} \beta^2 a_e^Z a_Q^Z \left\{ \left[ v_e^Z v_Q^Z + \frac{1}{2} \left( 1 - \frac{m_Z^2}{s} \right) v_e^\gamma v_Q^\gamma \right] (\tilde{F}_1^* G_1 + \tilde{F}_1 G_1^*) \\
&\quad + i \frac{m_Z \Gamma_Z}{2 s} v_Q^\gamma v_e^\gamma \left( \tilde{F}_1 G_1^* - \tilde{F}_1^* G_1 \right) \right\}, \\
\sigma_S^{(2p)} &= \frac{N_c}{24 \pi} \frac{s}{D_Z} \beta \left( v_e^\gamma v_Q^\gamma \right)^2 W + \frac{N_c}{12 \pi} \frac{s}{D_Z} \beta \left( 1 - \frac{m_Z^2}{s} \right) v_e^\gamma v_Q^\gamma v_e^Z v_Q^Z W \\
&\quad + \frac{N_c}{24 \pi} \frac{s}{D_Z} \beta \left[ \left( a_e^Z \right)^2 + \left( v_e^Z \right)^2 \right] \left[ \left( v_Q^Z \right)^2 W + 2 \beta^2 \left( a_Q^Z \right)^2 (G_1 G_1^*) \right],
\end{align*}
\]
where
\[
W = (3 - \beta^2) \left( \tilde{F}_1 \tilde{F}_1^* \right) + \beta^2 \left( \tilde{F}_1^* F_2 + \tilde{F}_1 F_2^* \right) + \frac{\beta^4}{1 - \beta^2} (F_2 F_2^*),
\]
\( D_Z = [(s - m_Z^2)^2 + m_Z^2 \Gamma_Z^2] \) with \( m_Z, \Gamma_Z \) being the mass and width of the Z boson, \( \beta = \sqrt{1 - 4 m_e^2/s} \) the heavy quark velocity, and \( N_c = 3 \) the number of colors. Because we have put \( m_e = 0 \) the form factor \( G_2 \) does not contribute to Eqs. \[20\], \[21\]. The last term in \[20\], which contains \( \Gamma_Z \), is of higher order in the electroweak couplings as compared with the first term. We will neglect that term in the following.

Expanding the form factors in Eqs. \[20\], \[21\] in powers of \( \alpha_s \):
\[
\begin{align*}
\tilde{F}_1 &= 1 + \left( \frac{\alpha_s}{2 \pi} \right) \tilde{F}_1^{(1s)} + \left( \frac{\alpha_s}{2 \pi} \right)^2 \tilde{F}_1^{(2s)} + O \left( \alpha_s^3 \right), \\
F_2 &= \left( \frac{\alpha_s}{2 \pi} \right) F_2^{(1s)} + \left( \frac{\alpha_s}{2 \pi} \right)^2 F_2^{(2s)} + O \left( \alpha_s^3 \right), \\
G_1 &= 1 + \left( \frac{\alpha_s}{2 \pi} \right) G_1^{(1s)} + \left( \frac{\alpha_s}{2 \pi} \right)^2 G_1^{(2s)} + O \left( \alpha_s^3 \right),
\end{align*}
\]
leads to the expansions of the two-parton contributions as exhibited in (3), i.e.,

$$\sigma_{A,S}^{(2p)} = \sigma_{A,S}^{(2,0)} + \sigma_{A,S}^{(2,1)} + \sigma_{A,S}^{(2,2)}.$$

(25)

The contributions to $\sigma_{S}^{(2p)}$ can be further decomposed as follows:

$$\sigma_{S}^{(2,0)} = \sigma_{S}^{(2,0,\gamma)} + \sigma_{S}^{(2,0,Z)} + \sigma_{S}^{(2,0,\gamma Z)},$$

(26)

$$\sigma_{S}^{(2,1)} = \sigma_{S}^{(2,0,\gamma)} \sigma_{S}^{(2,1,\gamma)} + \sigma_{S}^{(2,0,\gamma Z)} \sigma_{S}^{(2,1,Z)} + \sigma_{S}^{(2,0,\gamma Z)} \sigma_{S}^{(2,1,\gamma Z)},$$

(27)

$$\sigma_{S}^{(2,2)} = \sigma_{S}^{(2,0,\gamma)} \sigma_{S}^{(2,2,\gamma)} + \sigma_{S}^{(2,0,Z)} \sigma_{S}^{(2,2,Z)} + \sigma_{S}^{(2,0,\gamma Z)} \sigma_{S}^{(2,2,\gamma Z)},$$

(28)

where the superscripts $\gamma$, $Z$, and $\gamma Z$ label the pure photon and $Z$ contributions, and the $\gamma - Z$ interference terms, respectively. These terms are given by

$$\sigma_{S}^{(2,0,\gamma)} = \frac{N_c}{24\pi} \frac{1}{s} \beta \left( v_{\epsilon}^\gamma v_{\epsilon}^\gamma \right)^2 (3 - \beta^2),$$

(29)

$$\sigma_{S}^{(2,1,\gamma)} = \left( \frac{\alpha_s}{2\pi} \right)^2 \left\{ 2 \text{Re} \tilde{F}_1^{(16)} + \frac{2\beta^2}{3-\beta^2} \text{Re} F_2^{(10)} \right\},$$

(30)

$$\sigma_{S}^{(2,2,\gamma)} = \left( \frac{\alpha_s}{2\pi} \right)^2 \left\{ \frac{2\beta^2}{3-\beta^2} \left[ \text{Re} F_2^{(26)} + \text{Re} \tilde{F}_1^{(16)} \text{Re} F_2^{(16)} + \pi^2 \text{Im} \tilde{F}_1^{(16)} \text{Im} F_2^{(16)} \right] \\
+ \frac{\beta^4}{(3 - \beta^2)(1 - \beta^2)} \left[ \left( \text{Re} F_2^{(16)} \right)^2 + \pi^2 \left( \text{Im} F_2^{(16)} \right)^2 \right] \\
+ \left( \text{Re} \tilde{F}_1^{(16)} \right)^2 + \pi^2 \left( \text{Im} \tilde{F}_1^{(16)} \right)^2 + 2 \text{Re} \tilde{F}_1^{(26)} \right\},$$

(31)

$$\sigma_{S}^{(2,0,Z)} = \frac{N_c}{24\pi} \frac{s}{D_Z} \beta \left[ (a_{\epsilon}^Z)^2 + (v_{\epsilon}^Z)^2 \right] \left[ 2 \left( a_{\epsilon}^Z \right)^2 \beta^2 + (v_{\epsilon}^Z)^2 (3 - \beta^2) \right],$$

(32)

$$\sigma_{S}^{(2,1,Z)} = \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{2 \left( a_{\epsilon}^Z \right)^2 \beta^2 + (3 - \beta^2)}{\left( v_{\epsilon}^Z \right)^2} \left\{ v_{\epsilon}^Z \left( 3 - \beta^2 \right) \sigma_{S}^{(2,2,\gamma)} \\
+ 4 \left( \frac{\alpha_s}{2\pi} \right)^2 (a_{\epsilon}^Z)^2 \beta^2 \text{Re} G_1^{(26)} + 2 \left( \frac{\alpha_s}{2\pi} \right)^2 (a_{\epsilon}^Z)^2 \beta^2 \left[ \text{Re} G_1^{(16)} \right)^2 + \pi^2 \left( \text{Im} G_1^{(16)} \right)^2 \right\},$$

(33)

$$\sigma_{S}^{(2,0,\gamma Z)} = \frac{N_c}{12\pi} \frac{s}{D_Z} \beta \left( 1 - \frac{m_{\gamma Z}^2}{s} \right) v_{\epsilon}^\gamma v_{\epsilon}^\gamma v_{\epsilon}^Z v_{\epsilon}^Z (3 - \beta^2),$$

(34)

$$\sigma_{S}^{(2,1,\gamma Z)} = \sigma_{S}^{(2,1,\gamma)},$$

(35)

$$\sigma_{S}^{(2,2,\gamma Z)} = \sigma_{S}^{(2,2,\gamma)},$$

(36)

where the convention $F_a = \text{Re} F_a + i\pi \text{Im} F_a$, $G_a = \text{Re} G_a + i\pi \text{Im} G_a$ ($a = 1, 2$) is used; i.e., a
factor $\pi$ is taken out of the imaginary part. The antisymmetric cross section is given by:

$$\sigma_{A}^{(2,0)} = \frac{N_c}{4\pi} s \beta^2 a_e a_q \left[ v_e^2 v_Q^2 + \frac{1}{2} \left( 1 - \frac{m_Z^2}{s} \right) v_e^2 v_Q^2 \right],$$

$$\sigma_{A}^{(2,1)} = \sigma_{A}^{(2,0)} \left( \frac{\alpha_s}{2\pi} \right) \left[ \text{Re} \tilde{F}_1^{(1f)} + \text{Re} G_1^{(1f)} \right],$$

$$\sigma_{A}^{(2,2)} = \sigma_{A}^{(2,0)} \left( \frac{\alpha_s}{2\pi} \right)^2 \left[ \text{Re} \tilde{F}_1^{(2f)} + \text{Re} G_1^{(2f)} + \pi^2 \text{Im} \tilde{F}_1^{(1f)} \text{Im} G_1^{(1f)} \right].$$

With these formulae the $Q\bar{Q}$ contribution \[\text{(12)}\] to the forward-backward asymmetry is expressed in terms of the one- and two-loop form factors Eqs. \[\text{(22)} - \text{(24)}\].

The second order term in the expansion \[\text{(21)}\] of the axial vector form factor, $G_1^{(2f)}$, receives so-called type A and type B contributions. Type A contributions are those where the $Z$ boson couples directly to the external quark $Q$ \[\text{(21)}\], while the triangle diagram contributions, summed over the quark isodoublets of the three generations, are called type B \[\text{(22)}\]. (In the terminology of \[\text{(14)}\] these correspond to universal and non-universal corrections, respectively.) Among Eqs. \[\text{(29)} - \text{(40)}\] only $\sigma_S^{(2,2,Z)}$ and $\sigma_A^{(2,2)}$ depend on $G_1^{(2f)}$. With $G_1^{(2f)} = G_1^{(2f,A)} + G_1^{(2f,B)}$ we separate in these terms the type A and B contributions:

$$\sigma_S^{(2,2,Z)} = \sigma_S^{(2,2,Z,A)} + \sigma_S^{(2,2,Z,B)}$$

with

$$\sigma_S^{(2,2,Z,A)} = \sigma_S^{(2,2,Z)} \left( G_1^{(2f)} \rightarrow G_1^{(2f,A)} \right),$$

$$\sigma_S^{(2,2,Z,B)} = \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{4}{2} \left( a_Z^2 \right)^2 \beta^2 \text{Re} G_1^{(2f,B)},$$

and

$$\sigma_A^{(2,2)} = \sigma_A^{(2,2,A)} + \sigma_A^{(2,2,B)}$$

with

$$\sigma_A^{(2,2,A)} = \sigma_A^{(2,2)} \left( G_1^{(2f)} \rightarrow G_1^{(2f,A)} \right),$$

$$\sigma_A^{(2,2,B)} = \left( \frac{\alpha_s}{2\pi} \right)^2 \sigma_A^{(2,0)} \text{Re} G_1^{(2f,B)}.$$
If one evaluates, in the case of $b\bar{b}$ and $c\bar{c}$ final states, Eq. (18) exactly at the $Z$ resonance and neglects the contributions from photon exchange, then:

$$A_2^{(2p,B)}(s = m_Z^2) = \left(\frac{\alpha_s}{2\pi}\right)^2 \Re G^{(2\ell,B)}_1 \left[\frac{(3 - \beta^2) (v_0^2)^2 - 2 (a_0'^2) \beta^2}{(3 - \beta^2) (v_0^2)^2 + 2 (a_0'^2) \beta^2}\right].$$

(49)

The type B contribution (18) to the forward-backward asymmetry is ultraviolet- and infrared-finite [22].

2.2 Infrared-Finiteness of $A^{(2p)}_{FB}$

The renormalized vector and axial vector form factors [17] were computed in [20, 21, 22] to order $\alpha_s^2$ within QCD with $N_f$ massless and one massive quark $Q$, in a renormalization scheme, which is appropriate for the case at hand: the wavefunction and the mass of $Q$ are defined in the on-shell scheme while $\alpha_s$ is defined in the MS scheme. The renormalized form factors still contain IR divergences which are regulated by dimensional regularization in $D = 4 - 2\varepsilon$ dimensions. At one loop, $F_1^{(1\ell)}$ and $G_1^{(1\ell)}$ contain $1/\varepsilon$ poles due to soft virtual gluons. At two loops, the dominant IR singularities of $F_1^{(2\ell)}$ and $G_1^{(2\ell,\ell')}$ are of order $1/\varepsilon^2$ due to soft and collinear massless partons, while $F_2^{(2\ell)}$ has only $1/\varepsilon$ poles. However, when inserting the renormalized form factors into the formulae for the order $\alpha_s$ and $\alpha_s^2$ contributions to $A^{(2p)}_{FB}$ the IR singularities cancel and we obtain a finite result. This is well-known for $A_{1F}^{(2p)}$, c.f. [11].

This cancellation of infrared poles can be seen in a straightforward manner if one takes into account the universal structure of the infrared poles in the form factors, which can be expressed in terms of one-loop and two-loop infrared singularity operators for a massive quark-antiquark pair, $I^{(1)}_{QQ}(s, \mu = m_Q, \varepsilon)$ and $I^{(2)}_{QQ}(s, \mu = m_Q, \varepsilon)$, where $\mu$ is the renormalization scale, as:

$$\begin{align*}
\text{Poles} \left(F_1^{(1\ell)}\right) &= I^{(1)}_{QQ} F_1^{(0\ell)} , \\
\text{Poles} \left(F_2^{(1\ell)}\right) &= 0 , \\
\text{Poles} \left(G_1^{(1\ell)}\right) &= I^{(1)}_{QQ} G_1^{(0\ell)} , \\
\text{Poles} \left(F_1^{(2\ell)}\right) &= I^{(2)}_{QQ} F_1^{(0\ell)} + I^{(1)}_{QQ} F_1^{(1\ell)} , \\
\text{Poles} \left(F_2^{(2\ell)}\right) &= I^{(1)}_{QQ} F_2^{(1\ell)} , \\
\text{Poles} \left(G_1^{(2\ell)}\right) &= I^{(2)}_{QQ} G_1^{(0\ell)} + I^{(1)}_{QQ} G_1^{(1\ell)} .
\end{align*}$$

(50)

This factorization of the infrared poles is a well-known feature of massless multi-loop amplitudes, where it can be derived from exponentiation [23, 24]. In massive QED, the same behaviour was observed long ago [25]. To the best of our knowledge, infrared factorization in massive QCD was established up to now only at the one-loop level [26]; the above pole structure of the form factors suggests that it holds at higher orders as well. Expressions for $I_{QQ}^{(1,2)}$ can be read off from the explicit pole structure of the form factors [20, 21]. Exploiting that

$$F_1^{(0\ell)} = G_1^{(0\ell)} = 1 ,$$

the type B contribution (18) to the forward-backward asymmetry is ultraviolet- and infrared-finite [22].
one finds immediately that the two-parton contribution (14) to the forward-backward asymmetry is infrared finite. However, it should be kept in mind that the two terms in (14) are not separately finite.

Let us illustrate how this cancellation occurs in $A_2^{(2p)}$ by considering this expression at the Z resonance, i.e., by taking into account only Z exchange contributions and neglecting the photon and $\gamma - Z$ interference terms. When inserting the form factors into Eq. (15), all the leading $1/\epsilon$ singularities cancel, but a subleading divergence remains:

$$A_{2,2} = -\left(\frac{\alpha_s}{2\pi}\right)^2 C_F^2 \frac{1}{\epsilon} \frac{4y}{3} \frac{2(a_Q^2)(1-y)^2 - 3(v_Q^2)(1+y)^2}{[(a_Q^2)(1-y)^2 + (v_Q^2)(1+y)^2 + 2(v_Q^2)y] (1-y)^2} \times [(1+y^2) \ln^2(y) + (1-y^2) \ln(y)] + A_{2,2}^{finite}$$

where $y = (1-\beta)/(1+\beta)$ and $C_F = (N_c^2 - 1)/(2N_c)$. This subleading singularity proportional to $C_F^2$ results from the real parts of the one-loop form factors, $F_1^{(1\ell)}$ and $G_1^{(1\ell)}$, and from the contribution of a set of abelian-type two-loop diagrams (gluon ladder diagram and gluon vertex diagrams with quark self-energy insertions) to the real parts of the two-loop vector and axial vector form factors. The singularity (52) is removed by the second term in (14). While

$$A_1^{(2p)} = \frac{\alpha_s}{2\pi} C_F \ln(y) \frac{2(a_Q^2)(1-y)^2 - 3(v_Q^2)(1+y)^2}{[(a_Q^2)(1-y)^2 + (v_Q^2)(1+y)^2 + 2(v_Q^2)y] (1-y)^2}$$

is finite, the first term in (16) contains a singularity,

$$\frac{\sigma_s^{(2,1)}}{\sigma_s^{(2,0)}} = -\frac{\alpha_s}{2\pi} C_F \frac{1}{\epsilon} \frac{2(1-y^2 + (1+y)^2 \ln(y))}{1-y^2} + \text{finite terms},$$

and thus the singular part of $A_{2,1}$ cancels the singularity in (52). This cancellation pattern remains also away from the Z resonance.

### 3 Cross Section and $A_{FB}$ near Threshold

In this section we analyze the cross section and $A_{FB}^{(2p)}$ for $e^+e^- \rightarrow \gamma^*, Z^* \rightarrow Q\bar{Q}$ near the production threshold. Close to threshold, where the quark velocity $\beta$ is small, the fixed order perturbative expansion in $\alpha_s$ of these and other quantities breaks down due to Coulomb singularities, which must be resummed. In addition, large logarithms in $\beta$ appear, which may be summed using the renormalization group [27, 28] applied in the framework of nonrelativistic effective field theory methods. However, in the energy region where $\alpha_s \ll \beta \ll 1$, threshold expansions of observables, i.e., expansions in $\beta$ to fixed order in $\alpha_s$ are expected to yield reliable results. In this region we derive compact formulae for the symmetric and antisymmetric $Q\bar{Q}$ cross sections at NNLO. These expressions should be useful, especially in the case of top-quark pair production, for comparison of continuum results with results obtained directly at threshold.

#### 3.1 The $Q\bar{Q}$ cross section

We start with Eqs. (25) - (28) where the $Q\bar{Q}$ cross section at NNLO, $\sigma_{NNLO} = \sigma_S^{(2p)}$, is expressed in terms of the $\gamma$, Z, and $\gamma - Z$ exchange contributions. Using these formulae we
may write
\[
\sigma_{NNLO} = \sigma_{S}^{(2,0,\gamma)} \left\{ 1 + \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}} + \sigma_{S}^{(2,1,\gamma)} \left[ 1 + \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}(1 + \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,1,\gamma)}}) \right] \right. \\
+ \sigma_{S}^{(2,2,\gamma)} \left[ 1 + \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}(1 + \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,2,\gamma)}}) \right] \right\}. 
\]

Denoting
\[
\Delta^{(0,Ax)} = \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}} \sigma_{S}^{(2,0,\gamma)} 
\]
and recalling Eqs. (30), (31) we get
\[
\sigma_{NNLO} = \sigma_{S}^{(2,0,\gamma)} \left\{ 1 + \Delta^{(0,Ax)} + \sigma_{S}^{(2,1,\gamma)} \left[ 1 + \Delta^{(0,Ax)} \right] + \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}} \left( \sigma_{S}^{(2,1,\gamma)} - \sigma_{S}^{(2,2,\gamma)} \right) \right. \\
+ \sigma_{S}^{(2,2,\gamma)} \left[ 1 + \Delta^{(0,Ax)} \right] + \frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}} \left( \sigma_{S}^{(2,2,\gamma)} - \sigma_{S}^{(2,2,\gamma)} \right) \right\}. 
\]

Putting
\[
\frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}} \left( \sigma_{S}^{(2,1,\gamma)} - \sigma_{S}^{(2,1,\gamma)} \right) = C_F \left( \frac{\alpha_s}{2\pi} \right) \Delta^{(1,Ax)} , \tag{58}
\]
\[
\frac{\sigma_{S}^{(2,0,Z)}}{\sigma_{S}^{(2,0,\gamma)}} \left( \sigma_{S}^{(2,2,\gamma)} - \sigma_{S}^{(2,2,\gamma)} \right) = C_F \left( \frac{\alpha_s}{2\pi} \right)^2 \Delta^{(2,Ax)} , \tag{59}
\]
the NNLO cross section reads:
\[
\sigma_{NNLO} = \sigma_{S}^{(2,0,\gamma)} \left\{ 1 + \Delta^{(0,Ax)} + \sigma_{S}^{(2,1,\gamma)} \left[ 1 + \Delta^{(0,Ax)} \right] + C_F \left( \frac{\alpha_s}{2\pi} \right) \Delta^{(1,Ax)} \right. \\
+ \sigma_{S}^{(2,2,\gamma)} \left[ 1 + \Delta^{(0,Ax)} \right] + C_F \left( \frac{\alpha_s}{2\pi} \right)^2 \Delta^{(2,Ax)} \right\}. \tag{60}
\]

We now expand the expression in the curly brackets of Eq. (60) up to and including terms of order \( \beta^0 \). Using the threshold expansions \[20, 21, 22\] of the one- and two-loop vector and axial vector form factors and putting the renormalization scale \( \mu = m_Q \) we obtain
\[
\sigma_{NNLO} = \sigma_{S}^{(2,0,\gamma)} \left\{ 1 + \Delta^{(0,Ax)} + C_F \left( \frac{\alpha_s}{2\pi} \right) \Delta^{(1,V_e)} \left( 1 + \Delta^{(0,Ax)} \right) \right. \\
+ \Delta^{(2,V_e)} \left( 1 + \Delta^{(0,Ax)} \right) + C_F \left( \frac{\alpha_s}{2\pi} \right)^2 \Delta^{(2,Ax)} \right\}. \tag{61}
\]

The Born cross section \( \sigma_{S}^{(2,0,\gamma)} \) is given in Eq. (29). We get for the terms \( \Delta^{(1,V_e)} \) and \( \Delta^{(2,V_e)} \) which arise from the one- and two-loop photon-exchange contributions, respectively:
\[
\Delta^{(1,V_e)} = \frac{6\zeta(2)}{\beta} - 8 + \mathcal{O}(\beta) , \tag{62}
\]
\[
\Delta^{(2,V_e)} = C_F \Delta_{A}^{(2,V_e)} + C_A \Delta_{NA}^{(2,V_e)} + N_T R \Delta_{L}^{(2,V_e)} + R \Delta_{H}^{(2,V_e)} , \tag{63}
\]
with
\[
\Delta_{A}^{(2,V_e)} = \frac{12\zeta^2(2)}{\beta^2} - \frac{48\zeta(2)}{\beta} + 24\zeta^2(2) - 4\zeta(2)\left(\frac{35}{3} - 8\ln 2 + 4\ln \beta\right) - 4\zeta(3) + 39
\]
\[+ \mathcal{O}(\bar{\beta}), \]
\[
\Delta_{N_A}^{(2,V_e)} = \frac{4\zeta(2)}{\beta}\left(\frac{31}{12} - \frac{11}{2}\ln(2\beta)\right) + 4\zeta(2)\left(\frac{179}{12} - 16\ln 2 - 6\ln \beta\right) - 26\zeta(3) - \frac{151}{9}
\]
\[+ \mathcal{O}(\beta), \]
\[
\Delta_{L}^{(2,V_e)} = \frac{4\zeta(2)}{\beta}\left(2\ln(2\beta) - \frac{5}{3}\right) + \frac{44}{9} + \mathcal{O}(\beta), \]
\[
\Delta_{H}^{(2,V_e)} = -\frac{32}{3}\zeta(2) + \frac{176}{9} + \mathcal{O}(\beta^2). \]

In the equations above, \(\zeta(2) = \pi^2/6\), \(C_A = N_c\), \(T_R = 1/2\), and \(N_f\) is the number of light quarks, which we take to be massless. For the threshold expansions of the terms \(\Delta_{i,Ax}^{(i,Ax)} (i = 0, 1, 2)\), involving \(Z\) boson exchange, we find:
\[
\Delta_{0,\text{Ax}} = \frac{s^2}{D_Z} \left\{ \frac{(a_e^2)^2 + (v^2_Z)^2}{(v^2_e v^2_Q)^2} \left[ 2 \frac{(a_e^2)^2}{1 - \frac{\beta^2}{3}} + (v^2_Z)^2 \right] + 2 \left( 1 - \frac{m^2_Z}{s} \right) \frac{v^2_Z v^2_Q}{v^2_e v^2_Q} \right\}, \quad (68)
\]
\[
\Delta_{1,\text{Ax}} = \mathcal{O}(\beta^2), \quad (69)
\]
\[
\Delta_{2,\text{Ax}} = \frac{64\zeta(2)m^4_Q(a_Q^2)^2 [(v^2_e)^2 + (a_e^2)^2]}{(v^2_Q v^2_e)^2 (4m^2_Q - m^2_Z)^2} C_F + \mathcal{O}(\beta), \quad (70)
\]

where \(\hat{v}_f, \hat{a}_f\) are defined in Eq. (18). While the multiplicative factor \(\Delta_{0,\text{Ax}}^{(0,\text{Ax})}\) is given in exact form, the higher order terms are expanded to the appropriate order in \(\beta\) such that the NNLO cross section \(\sigma_{NNLO}\) is obtained to next-to-next-to-leading logarithmic order (NNLL) in \(\beta\). We observe:

(i) To NNLL in \(\beta\) the NNLO cross section is infrared-finite, i.e., the \(Q\bar{Q}\) cross section is equal to the total production cross section to this order. This is due to the fact that close to threshold, real gluon emission is suppressed by a velocity factor with respect to the \(Q\bar{Q}\) final state. The two-loop triangle diagrams, studied in [22], do not contribute to the cross section to this order in \(\beta\).

(ii) In the expansions of \(\Delta_{1,V_e}^{(1,V_e)}\) and \(\Delta_{2,V_e}^{(2,V_e)}\), IR divergences appear at \(\mathcal{O}(\beta)\) and \(\mathcal{O}(\beta^2)\), respectively. In the expansions of \(\Delta_{1,\text{Ax}}^{(1,\text{Ax})}\) and \(\Delta_{2,\text{Ax}}^{(2,\text{Ax})}\), such divergences show up to order \(\beta^4\) and \(\beta^3\), respectively.

(iii) The threshold cross section \(\sigma_{NNLO}\) was calculated to NNLL in \(\beta\) in the above energy region, for pure vector exchange, previously in [21] (see also [30]). Putting the axial vector couplings to zero in the above expressions and comparing with [21] we find agreement.

(iv) The first and second-order \(Z\)-boson exchange terms \(\Delta_{1,Ax}^{(1,Ax)}\) and \(\Delta_{2,Ax}^{(2,Ax)}\), involving the axial vector coupling \(a_Q\), are of order \(\beta^2\) and \(\beta^0\), respectively. \(\Delta_{2,Ax}^{(2,Ax)}\) is thus relevant for an analysis aiming at NNLO accuracy at threshold (counting \(\alpha_s \sim \beta\)). The analytic result for the axial-vector contribution at \(\mathcal{O}(\alpha_s^2)\), which can be obtained to all orders in \(\beta\) from [21] [22], has not been given before in the literature. The term \(\Delta_{2,Ax}^{(2,Ax)}\) is small compared to \(\Delta_{2,V_e}^{(2,V_e)}\), see Fig. 2.

Nevertheless, at a high luminosity linear collider with polarized e\(^-\) and e\(^+\) beams one may eventually be able to disentangle the vector and axial-vector induced contributions to the \(t\bar{t}\) process.
cross section. A numerical calculation of axial vector contributions in the context of Lippmann-Schwinger equations was performed in [31], and the axial-vector contribution to the threshold cross section at next-to-next-to-leading logarithmic order was determined in [28] by a combination of numerical and analytical calculations.

For completeness we mention that a summation of all terms of the form $\alpha_s^n/\beta^n$ should be performed when the region of small heavy-quark velocities $\beta \sim \alpha_s$ is approached. The result of the resummation of these leading terms (and of part of the subleading terms) is the well-known Sommerfeld-Sakharov factor (see, for instance [29]). Subleading terms may be resummed by the renormalization group [28] in the context of effective field theory methods. In this paper we are not concerned with these summation methods, as we consider only the region $\beta \gg \alpha_s$.

### 3.2 Antisymmetric cross section and $A_{FB}$

Next we perform the threshold expansion of the second-order antisymmetric cross section $\sigma_{NNLO}^{(A)} = \sigma_A^{(2p)}$, given in Eqs. (25) and (38) - (40), in the same manner as was done above for $\sigma_{NNLO}$, using the results of [20, 21, 22]. We obtain to NNLL in $\beta$:

$$\sigma_{NNLO}^{(A)} = \sigma_A^{(2,0)} \left\{ 1 + C_F \left( \frac{\alpha_s}{2\pi} \right) \Delta^{(A,1)} + C_F \left( \frac{\alpha_s}{2\pi} \right)^2 \Delta^{(A,2)} \right\},$$

(71)

where $\sigma_A^{(2,0)}$ is given in Eq. (38) and

$$\Delta^{(A,1)} = \frac{6\zeta(2)}{\beta} - 6 + \mathcal{O}(\beta),$$

$$\Delta^{(A,2)} = C_F \Delta_A^{(A,2)} + C_A \Delta_{NA}^{(A,2)} + N_f T_R \Delta_L^{(A,2)} + T_R \left( \Delta_H^{(A,2)} + \Delta_{tr}^{(A,2)} \right),$$

(72)
with

\[
\Delta_A^{(A,2)} = \left. \frac{12\zeta^2(2)}{\beta^2} - \frac{36\zeta(2)}{\beta} + 24\zeta^2(2) - 4\zeta(2) \left( \frac{25}{6} - \frac{25}{4} \ln 2 + \frac{9}{2} \ln \beta \right) - \frac{35}{4} \zeta(3) + \frac{70}{3} \right. + \mathcal{O}(\beta),
\]

\[
\Delta_{NA}^{(A,2)} = \frac{4\zeta(2)}{\beta} \left( \frac{16}{3} - \frac{11}{2} \ln (2\beta) \right) + \frac{4\zeta(2)}{\beta} \left( \frac{67}{6} - \frac{25}{2} \ln 2 - 4 \ln \beta \right) - \frac{35}{2} \zeta(3) - 14 + \mathcal{O}(\beta),
\]

\[
\Delta_L^{(A,2)} = \frac{4\zeta(2)}{\beta} \left( 2 \ln (2\beta) - \frac{8}{3} \right) + 4 + \mathcal{O}(\beta),
\]

\[
\Delta_H^{(A,2)} = -\frac{32}{3} \zeta(2) + \frac{56}{3} + \mathcal{O}(\beta^2).
\]  

(73)

(74)

(75)

(76)

The term \(\Delta_{tr}^{(A,2)}\) in Eq. (72) is the contribution of the triangle diagrams computed in [22]. It is infrared- and ultraviolet-finite. In the case of \(t\bar{t}\) production it is given by

\[
\Delta_{tr}^{(A,2)} = \zeta(2) \left( 16 \ln 2 - \frac{23}{3} \right) - 8 \ln 2 + \frac{8}{3} \ln^2 2 + \mathcal{O}(\beta^2).
\]  

(77)

Notice that the the second-order antisymmetric \(Q\bar{Q}\) cross section (71) is infrared-finite to NNLL in \(\beta\) and is equal to the total antisymmetric cross section in this order. The terms \(\Delta^{(A,1)}\) and \(\Delta^{(A,2)}\) become infrared-divergent to order \(\beta^2\) and \(\beta\), respectively.

Finally, the second-order forward-backward asymmetry is given near threshold by

\[
A_{FB}^{QQ} = A_{FB,0} C_{FB},
\]  

where \(C_{FB}\) is the ratio of the curly bracket in Eqs. (71) and of the curly bracket in (61) divided by \((1 + \Delta^{(0,Ax)})\). To NNLL in \(\beta\) it is equal to the complete forward-backward asymmetry \(A_{FB}^{Q}\).

In Fig. 3 we have plotted the forward-backward asymmetry Eq. (78) to order \(\alpha_s^2\) for \(t\bar{t}\) production above threshold in the range \(0.2 \leq \beta \leq 0.5\), where \(C_{FB}\) is the ratio of two expressions expanded to NNLL in \(\beta\). The top mass and the other parameters were chosen as given in Eq. (79) below. A comparison with the exact second order asymmetry \(A_{FB}^{(tt)}\) will be made in Section 4.2.

### 4 Numerical Results

In this section we compute the \(Q\bar{Q}\) contributions to \(A_{FB}\) to order \(\alpha_s^2\) using the results of Section 2 and the analytic results for the one- and two-loop vector and type A and B axial vector form factors given in [20, 21, 22]. The numerical evaluation of the harmonic polylogarithms that appear in these expressions were made using the code of [34]. For \(b\) and \(c\) quarks the above formulae are evaluated at and in the vicinity of the \(Z\) resonance and for \(t\) quarks between \(2m_t < \sqrt{s} \leq 1\) TeV. The quark masses, whose values we use are given below, are defined in the on-shell scheme while \(\alpha_s\) is the QCD coupling in the \(\overline{\text{MS}}\) scheme. When calculating \(A_{FB}^{QQ}\) for \(b\) and \(c\) quarks around the \(Z\) resonance, \(\alpha_s\) is defined with respect to the effective \(N_f = 5\) flavor theory; that is, the top quark contribution to the gluon self-energy that enters the vector and
Figure 3: The forward-backward asymmetry $A_{FB}^{(tt)}$ to NNLL above the $t\bar{t}$ threshold in the range $0.2 \leq \beta \leq 0.5$ for $\mu = m_t$.

type A axial vector form factors to order $\alpha_s^2$ is absent. The forward-backward asymmetry for top quarks is computed in the six-flavor theory with the corresponding six-flavor QCD coupling determined from the five-flavor coupling at the matching point $\mu = m_t$. We use the following input values [1]:

$$
m_c = 1.5 \text{ GeV}, \quad m_b = 5 \text{ GeV}, \quad m_t = 172.7 \pm 2.9 \text{ GeV},$$
$$m_Z = 91.1875 \text{ GeV}, \quad \Gamma_Z = 2.4952 \text{ GeV},$$
$$\sin^2 \theta_W = 0.23153, \quad \alpha_s^{N_f=5}(m_Z) = 0.1187. \quad (79)$$

The value of mass of the top quark is the recent CDF and D0 average [32]. For $b$ and $c$ quarks the type B axial vector contributions are evaluated with the central value of $m_t$ given in (79).

In the following we denote $A_{FB}^{(Q\bar{Q})}$ evaluated to order $\alpha_s$ and $\alpha_s^2$, respectively, by:

$$A_{FB}^{(Q\bar{Q})}(\alpha_s) = A_{FB,0}^{(Q\bar{Q})} \left(1 + A_1^{(Q\bar{Q})}\right),$$
$$A_{FB}^{(Q\bar{Q})}(\alpha_s^2) = A_{FB,0}^{(Q\bar{Q})} \left(1 + A_1^{(Q\bar{Q})} + A_2^{(Q\bar{Q},A)} + A_2^{(Q\bar{Q},B)}\right).$$

4.1 $A_{FB}^{(Q\bar{Q})}$ for $b$ and $c$ quarks at and in the vicinity of $\sqrt{s} = m_Z$

Let us first consider the $b$ quark asymmetry. As it is to be computed for $\sqrt{s} \simeq m_Z$ we can safely neglect the masses of the $u,d,s,$ and $c$ quarks which contribute to the second order form factors. (As already mentioned above, the $t$ quark contribution to the gluon self-energy is decoupled.) The type B axial vector form factor $G_1^{(2\ell,B)}$ is non-zero due to the large mass splitting between
Table 1: The $b\bar{b}$ contributions to $A_{FB}$ for bottom quarks at $\sqrt{s} = m_Z$.

$t$ and $b$ quarks, and to very good approximation one may neglect in these triangle diagram contributions the mass of the $b$ quark. Therefore we use

$$G_{1}^{(2\ell,B)}(s) = G_{1}^{(2\ell,B)}(s, m_b = 0, m_b = 0) - G_{1}^{(2\ell,B)}(s, m_t, m_b = 0),$$

(80)

when evaluating Eq. (18), respectively (49), for the $b$ quark. The functions on the right-hand side of (80), whose second and third argument denotes the mass of the quark in the loop and the mass of the external quark, respectively, are given in [33, 22]. (We use the notation of [22].)

Putting $\beta = 1$ in Eq. (49), our result for $A_{FB}^{(2p,B)}(s = m_Z^2)$ agrees with that of [12].

Table 1 contains the values for the lowest order forward-backward asymmetry at the $Z$ resonance, together with the $b\bar{b}$ contributions to first and second order in $\alpha_s$ for three choices of the renormalization scale $\mu$. The photon and $\gamma - Z$ interference contributions, which are (on the $Z$ resonance) of higher order in the electroweak couplings, are not taken into account. Table 1 shows that the QCD corrections are dominated by the type B contributions; they are about three times as large as the order $\alpha_s$ and about nine times as large as the order $\alpha_s^2$ corrections. This is due to the fact that we are close to the chiral limit, as $m_b/m_Z \ll 1$. In this limit the order $\alpha_s$ vector and axial vector form factors $F_{1}^{(1\ell)}$, $G_{1}^{(1\ell)}$, and the order $\alpha_s^2$ vector and type A axial vector form factors $F_{1}^{(2\ell)}$, $G_{1}^{(2\ell,A)}$ become equal while the chirality-flipping form factors vanish. In this limit $A_{2}^{(QQ)}$ and $A_{2}^{(QQ,A)}$ vanish, too, as an inspection of the above formulae shows. Thus the QCD corrections are dominated by the type B contribution. As it turns out, it amounts to a correction of the lowest order asymmetry by only about one per mille.

In Fig. 4 the first and second order QCD corrections $A_{1}^{(bb)}$, $A_{2}^{(bb,A)}$, and $A_{2}^{(bb,B)}$, evaluated for $\mu = m_Z$, are shown between $88 \text{ GeV} < \sqrt{s} < 95 \text{ GeV}$. Here the contributions from photon exchange are included. Again the QCD corrections are dominated by the type B triangle diagram contributions. Varying the renormalization scale in the range $m_Z/2 \leq \mu \leq 2m_Z$ changes these numbers only by a small amount.

Next we consider the $c$ quark asymmetry for $\sqrt{s} \simeq m_Z$. To very good approximation we can neglect the masses of the $c$ and $b$ quarks in their contribution to the gluon self-energy. Here the type B axial vector form factor $G_{1}^{(2\ell,B)}$ is again determined by the large mass splitting between $t$ and $b$ quarks. In view of the convention adopted in Eq. (17), where the neutral current couplings of the external quark are factored out, we use now

$$G_{1}^{(2\ell,B)}(s) = G_{1}^{(2\ell,B)}(s, m_t, m_c = 0) - G_{1}^{(2\ell,B)}(s, m_b = 0, m_c = 0),$$

(81)

when applying Eq. (18), respectively (49), to the $c$ quark. Again we put $\beta = 1$ in these equations. Eq. (81) is equal in magnitude but opposite in sign to Eq. (80).

Table 2 contains the values for the lowest order $c$ quark forward-backward asymmetry at the $Z$ resonance – without the $\gamma$ and $\gamma - Z$ contributions –, together with the $c\bar{c}$ contributions to first
Table 2: The $c\bar{c}$ contributions to $A_{FB}$ for charm quarks at $\sqrt{s} = m_Z$.

and second order in $\alpha_s$ for three choices of the renormalization scale $\mu$. The QCD corrections are dominated again by the type B term, which in this case is about two per mille of the leading order asymmetry. The increase by a factor of about two as compared to the $b$ quark results from the fact that $|v_Z^c| < |v_Z^b|$, c.f. Eq. (49).

In Fig. 4 the first and second order QCD corrections $A_1^{(cc)}$, $A_2^{(cc,A)}$, and $A_2^{(cc,B)}$, evaluated for $\mu = m_Z$, are shown between $88 \text{ GeV} < \sqrt{s} < 95 \text{ GeV}$, including the contributions from photon exchange. For $c$ quarks, too, the QCD corrections are dominated by the type B triangle diagram contributions.

In Fig. 6 the $b\bar{b}$ and $c\bar{c}$ forward-backward asymmetries to order $\alpha_s^2$, $A_1^{(bb)} (\alpha_s^2)$ and $A_2^{(cc)} (\alpha_s^2)$ are displayed between $88 \text{ GeV} < \sqrt{s} < 95 \text{ GeV}$, using $\mu = m_Z$. Figs. 4, 5 show that the order $\alpha_s^2$ asymmetry is increased, in the case of $b$ quarks, by about one per mille and decreased, in the case of $c$ quarks, by about two per mille of the respective lowest order asymmetry in the whole energy range considered. Varying the renormalization scale in the range $m_Z/2 \leq \mu \leq 2m_Z$ changes the asymmetries shown in Fig. 6 only by a very small amount.

The complete forward-backward asymmetry to order $\alpha_s^2$ for $b$ and $c$ quarks at and in the vicinity of the $Z$ resonance is dominated by the respective three- and four-parton contributions.
Figure 5: First and second order QCD corrections $A_{1}^{(c\bar{c})}$ (dashed), $A_{2}^{(c\bar{c},A)}$ (dotted), and $A_{2}^{(c\bar{c},B)}$ (solid) for $\mu = m_Z$ in the vicinity of the $Z$ resonance.

Figure 6: $A_{V\bar{B}}^{(b\bar{b})}(\alpha_s^2)$ (solid) and $A_{V\bar{B}}^{(c\bar{c})}(\alpha_s^2)$ (dashed) for $\mu = m_Z$ in the vicinity of the $Z$ resonance.
A complete computation of these terms to order $\alpha_s^2$ has not yet been done for $m_b, m_c \neq 0$. Nevertheless, we expect that the respective results of [14], which were obtained in the massless limit, will not change dramatically.

4.2 $A_{FB}^{(t\bar{t})}$ for top quarks above threshold

Finally we compute the $t\bar{t}$ contributions to the forward-backward asymmetry in the reaction $e^+e^- \to t\bar{t}X$, sufficiently far away from the pair production threshold in order that perturbation theory in $\alpha_s$ is applicable. That is, the following results apply to events with $t$ and $\bar{t}$ velocities $\beta \gg \alpha_s$. As already stated above, $\alpha_s$ is defined in the six flavor QCD, with all quarks but the top quark taken to be massless. The value of $\alpha_s(\mu = m_t)$ is determined from the input value \[ \alpha_s \] of \[ \text{[22]} \], and values of $\alpha_s$ at other energy scales are obtained by two-loop renormalization group evolution. Most of the results below are presented for three values of the top quark mass: the present central and 1 s.d. upper and lower values 172.7 GeV, 175.6 GeV, and 169.8 GeV, respectively. The type B contribution to $A_{FB}^{(t\bar{t})}$, Eq. \[ \text{[48]} \], is computed with the two-loop axial vector form factor

$$G_1^{(2\ell,B)}(s) = G_1^{(2\ell,B)}(s, m_t, m_t) - G_1^{(2\ell,B)}(s, m_b = 0, m_t).$$

which is given in \[ \text{[22]} \].

In the following we consider the energy range $360 \text{ GeV} \leq \sqrt{s} \leq 1 \text{ TeV}$. In Fig. the leading order asymmetry $A_{FB,0}^{(t\bar{t})}$ is shown for three values of the top quark mass. In Figs. and the order $\alpha_s$ correction $A_1^{(t\bar{t})}$ is displayed for three values of the renormalization scale $\mu$ and fixed top quark mass, and for three values of $m_t$ and fixed $\mu$, respectively. The analogous cases are shown in Figs. and for the order $\alpha_s^2$ correction $A_2^{(t\bar{t},A)}$. The triangle diagram contributions

![](image-url)
Figure 8: Order $\alpha_s$ correction $A_1^{(t\bar{t})}$ for three values of the renormalization scale $\mu$, using $m_t = 172.7$ GeV.

Figure 9: Order $\alpha_s$ correction $A_1^{(t\bar{t})}$ for three values of the top quark mass, using $\mu = \sqrt{s}$. 

Figure 10: Order $\alpha_s^2$ correction $A_2^{(t\bar{t},A)}$ for three values of the renormalization scale $\mu$, using $m_t = 172.7$ GeV.

Figure 11: Order $\alpha_s^2$ correction $A_2^{(t\bar{t},A)}$ for three values of the top quark mass, using $\mu = \sqrt{s}$.
Figure 12: Order $\alpha_s^2$ correction $A_2^{(t\bar{t},B)}$ for three values of the renormalization scale $\mu$, using $m_t = 172.7$ GeV.

Figure 13: Order $\alpha_s^2$ correction $A_2^{(t\bar{t},B)}$ for three values of the top quark mass, using $\mu = \sqrt{s}$.
Figure 14: Forward-backward asymmetry $A_{FB}^{(t)}(\alpha_s^2)$ for three values of the renormalization scale $\mu$, using $m_t = 172.7$ GeV.

Figure 15: Forward-backward asymmetry $A_{FB}^{(t)}(\alpha_s^2)$ for three values of the top quark mass, using $\mu = \sqrt{s}$. 

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Figure 16: Forward-backward asymmetry to lowest, first and second order in $\alpha_s$ using $m_t = 172.7$ GeV and $\mu = \sqrt{s}$. $A_{FB,0}^{(t\bar{t})}$ (dashed), $A_{FB}^{(t\bar{t})}(\alpha_s)$ (dotted), $A_{FB}^{(t\bar{t})}(\alpha_s^2)$ (solid).

$A_{FB}^{(t\bar{t},B)}$ are given in Fig. 12 and Fig. 13 for three values of $\mu$ and $m_t$, respectively. From these figures we conclude that the two-parton QCD corrections to the lowest order asymmetry are moderate to small for $\sqrt{s} \geq 400$ GeV. At $\sqrt{s} = 400$ GeV, $A_1^{(t\bar{t})}$ is about 3.3 percent while $A_2^{(t\bar{t},A)}$ is about 2.4 percent. As expected, the relative importance of the order $\alpha_s^2$ corrections increases as the centre-of-mass energy approaches the threshold region: for $\sqrt{s} \simeq 360$ GeV, $A_2^{(t\bar{t},A)}$ is larger than $A_1^{(t\bar{t})}$, signaling that perturbation theory in $\alpha_s$ is no longer applicable. Contrary to the case of $b$ and $c$ quarks at the $Z$ resonance the two-loop type B contributions are two orders of magnitude smaller than $A_2^{(t\bar{t},A)}$.

Figs. 14 and 15 show the forward-backward asymmetry $A_{FB}^{(t\bar{t})}(\alpha_s^2)$ for three values of the renormalization scale and three values of the top quark mass, respectively. The dependence of the second order asymmetry on $\mu$ is small: changing $\mu$ from $\sqrt{s}/2$ to $2\sqrt{s}$ changes $A_{FB}^{(t\bar{t})}(\alpha_s^2)$, for fixed $m_t$, only by about 1 percent at $\sqrt{s} \gtrsim 360$ GeV, and this dependence on $\mu$ decreases with increasing c. m. energy.

In Fig. 16 the $t\bar{t}$ asymmetry is displayed to lowest, first, and second order in $\alpha_s$. This figure shows that for c. m. energies sufficiently away from threshold the QCD corrections are under control.

Finally, a comparison is made in Fig. 17 between the exact second order forward-backward asymmetry $A_{FB}^{(t\bar{t})}(\alpha_s^2)$ as given in Fig. 16 and the values obtained from the near-threshold NNLL formula Eq. (78). For $\sqrt{s} \lesssim 360$ GeV corresponding to $\beta \lesssim 0.3$ the deviation of the NNLL from the respective exact value is less than 5 percent.
Future experiments on heavy-quark production at a planned linear $e^+e^-$ collider aim at very precise measurements of the neutral-current couplings of these quarks. An important observable for this purposes is the forward-backward asymmetry $A_{FB}^Q$ in inclusive heavy quark production, $e^+e^- \rightarrow \gamma^* \rightarrow Q + X$. The projected accuracies with which $A_{FB}^Q$ can be measured in future $b$ or $t$ quark production requires also the precise determination of these observables within the Standard Model. In particular, a computation of the order $\alpha_s^2$ QCD contributions to $A_{FB}^Q$ for massive quarks is mandatory.

In view of these perspectives we have calculated the contribution of the $Q\bar{Q}$ final state to $A_{FB}^Q$ in NNLO QCD. As discussed above, and explicitly shown for the $Q\bar{Q}$ final state, the contributions of the two-parton and of the three- plus four-parton states to the second-order forward-backward asymmetry are separately infrared-finite. We have provided formulae for the symmetric and antisymmetric $Q\bar{Q}$ cross sections $\sigma_S$ and $\sigma_A$ which yield $A_{FB}^{Q\bar{Q}}$. These formulae hold for any center-of-mass energy. Specifically, in the energy region near threshold where the quark velocity $\beta$ satisfies $\alpha_s \ll \beta \ll 1$, we have expanded the order $\alpha_s$ and $\alpha_s^2$ QCD corrections to $\sigma_S$ and $\sigma_A$ to NNLL in $\beta$. To this order in $\beta$ the $Q\bar{Q}$ cross sections are equal to the corresponding total cross sections. Therefore, an (analytic) expression is obtained for the forward-backward asymmetry $A_{FB}^Q$ to order $\alpha_s^2$ and order $\beta$ near threshold.

Moreover, we have computed the two-parton forward-backward asymmetry $A_{FB}^{Q\bar{Q}}$ for $b$ and $c$ quarks on and near the $Z$-boson resonance and for $t$ quarks for center-of-mass energies $\sqrt{s}$ above threshold to 1 TeV. The two-parton asymmetry is determined by the heavy-quark vector and axial vector form factors. To order $\alpha_s^2$ the axial vector form factors receive besides type
A (universal) corrections also triangle diagram contributions resulting from the large mass splitting between $t$ and $b$ quarks. These triangle diagram terms dominate, for $\sqrt{s} \sim m_Z$, the QCD corrections from the $Q\bar{Q}$ final state to $A_{FB}^t$ and $A_{FB}^c$. This is due to the fact that here one is close to the chiral limit. However, the complete order $\alpha_s^2$ QCD corrections to these asymmetries are dominated by the contributions from the three- and four-parton final states, which were calculated so far only for massless quarks \cite{14}. For top quarks the triangle diagram contributions to $A_{FB}^t$ are negligible compared to the type A corrections. These corrections from the $t\bar{t}$ final state to the lowest order asymmetry are moderate for large $\sqrt{s}$ and increase in size towards threshold. The order $\alpha_s^2$ corrections are important, as the analysis in Section 4.2 shows.

We plan to determine in the near future also the contribution of the three- and four parton-final states to the order $\alpha_s^2$ forward-backward asymmetry for massive quarks.

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