Towards the understanding of $Z_c(3900)$ from lattice QCD

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Within the framework of three-channel Ross-Shaw effective range theory, we derive the constraints among different parameters of the theory in the case of a narrow resonance close to the threshold of the third channel, which is relevant for the resonance-like structure $Z_c(3900)$. The usage of these constraint relations, together with the multi-channel Lüscher formula in lattice QCD calculations are also discussed and the strategies are outlined.

I. INTRODUCTION

In the past decade, exotic hadronic resonance-like structures, known as XYZ particles, have been discovered by various experiments, with $Z_c(3900)$ being a typical example [1, 2]. The exotic structures have been discovered in both charm and bottom sectors which necessarily bear a four valence quark structure $Qq'Qq$ with $Q$ being a heavy-flavor quark while $q$ and $q'$ being two different light flavored ones. They also tend to appear close to the threshold of two heavy mesons with valence structure $Qq$ and $q'Q$. The physical nature of these structures have been contemplated and discussed in many phenomenological studies. For a recent review on these matters, see e.g. Ref. [4, 5]. Despite many studies, the nature of $Z_c(3900)$ remains unclear. It is therefore highly desirable that non-perturbative studies like lattice QCD could provide some useful information.

Contrary to the phenomenological studies, lattice studies on these states remain relatively scarce. A lattice study was performed by S. Prelovsek et al. who investigated the energy levels of the two charmed meson system in the channel where $Z_c$ appearing in a finite volume [6]. They used quite a number of operators, including two-meson operators in the channel of $J/\psi\pi$, $D\bar{D}$ etc. and even tetraquark operators. However, no indication of extra new energy levels apart from the almost free scattering states of the two mesons. Taking $D\bar{D}^*$ as the main relevant channel, which is also supported by experimental facts, CLQCD utilized single-channel Lüscher scattering formalism [7, 11] to tackle the problem within single-channel approximation. They found slightly repulsive interaction between the two charmed mesons [12, 13], making them unlikely to form bound states. A similar study using staggered quarks also finds no clue for the existence of the state [14].

On the other hand, HALQCD studied the problem using the so-called HALQCD approach [15] which is different from Lüscher’s adopted by the other groups mentioned above. An effective potential is first extracted from lattice data which is then substituted into the Schrödinger-like equation to solve for the scattering. They claimed that $Z_c(3900)$ can be reproduced and it is a structure formed due to strong cross-channel interactions among three channels, $J/\psi\pi$, $\eta_\rho$ and $D\bar{D}^*$, see Ref. [16, 17] and references therein. This scenario will be referred to as the HALQCD scenario in the following.

Recently, in order to clarify this mismatch of the two types of approaches, CLQCD performed a two-channel lattice study using the two-channel Ross-Shaw effective range expansion [18]. They took the two channels $J/\psi\pi$ and $D\bar{D}^*$ that are most strongly coupled to $Z_c(3900)$. It is found that, in this two-channel approach, the parameters of the Ross-Shaw matrix do not seem to support the HALQCD scenario. The parameters turn out to be large and the Ross-Shaw $M$ matrix is far from singular, which is required for a resonance close to the threshold. However, since only two channels are studied, it is still not a direct comparison with the HALQCD approach in which three channels have been studied. In this paper, we move one step further to close this gap. We take exactly the same three channels as HALQCD did, namely $J/\psi\pi$, $\eta_\rho$ and $D\bar{D}^*$. We utilize the Ross-Shaw effective range theory [19, 20] for the above mentioned three channels and derive the constraint relations among the parameters of Ross-Shaw matrix $M$ (6 real parameters in total), assuming that there is a resonance close to the threshold of the third channel. Similar constraint relations in the two-channel case have been discussed in detail by Ross and Shaw long time ago, see e.g. Ref. [20]. However, to our knowledge, the corresponding constraint relations in the three-channel case is still lacking, which will be established in this paper. These constraint relations can be further checked in future lattice simulations, the strategy of which will also be outlined in this paper.

The difficulty with the multi-channel Lüscher approach is two-fold which we briefly outline below:

- First, with the number of channels $n$ increasing, the number of unknown functions entering the $S$-matrix also increases rapidly. For example, in the case of two-channels, there are 3 functions while in the case of three channels, 6 functions are needed.

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to describe the full $S$-matrix. On the other hand, Lüscher formula only offers a single relation among these functions at a particular energy level, which is extracted from lattice simulation. Therefore, one needs to parameterize these unknown functions of energy in terms of constant parameters. Here, one could use the so-called $K$-matrix parameterization or the multi-channel effective range expansion advocated by Ross and Shaw, which is in fact a special case of the former. We will also take this choice in this paper.

- Second, the number of constant parameters needed to parameterize the $S$-matrix also grows quadratically fast when the number of channels $n$ is increased. Therefore, one should refrain to include too many channels. Based on the experimental facts and also hints from the HALQCD study, we focus on the three-channel Lüscher approach in this paper. To be more specific, we will single out three most relevant channels for $Z_c(3900)$: $J/ψ\pi$, $η_cρ$ and $D\bar{D}^*$, the first being the discovery channel for $Z_c(3900)$ and the second and the third have been shown to be dominant channels that couple to $Z_c(3900)$ by BESIII experiments [21]. Similar to the single-channel effective range expansion which is characterized by two real parameters, namely the scattering length $a_0$ and the effective range $r_0$, in a three-channel situation, one needs 9 real parameters to describe the so-called Ross-Shaw matrix $M$: 6 for the scattering length matrix and 3 for the effective range parameters.

This paper is organized as follows. In Section II, we briefly review the Ross-Shaw effective range expansion that is needed to parameterize $S$-matrix elements. In section III within the zero-range approximation of Ross-Shaw theory, we derive the constraint conditions that need to be satisfied in order to have a narrow resonance behavior close to the third threshold. These conditions are derived first in the limit where the coupling of the first two channels are switched off and then generalized to the case where it is turned on. In section IV we briefly outline the strategies of the lattice computations and discuss how the constraints derived in Sec. III can be tested. In Section V we will conclude with some general remarks.

II. THE ROSS-SHAW EFFECTIVE RANGE THEORY

In this section, we briefly recapitulate the Ross-Shaw effective range theory which is a generalization of usual effective range expansion to multi-channels. As already mentioned in the previous section, in order to utilize the multi-channel Lüscher formula, it is crucial to have a parameterization of the $S$-matrix elements in terms of constants instead of functions of the energy and the multi-channel effective range expansion developed by Ross and Shaw [19, 20] serves this purpose.

In the single-channel case, this theory is just the well-known effective range expansion for low-energy elastic scattering,

$$ k \cot \delta(k) = \frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \cdots, \quad (1) $$

where $\cdots$ designates higher order terms in $k^2$ that vanish in the limit of $k^2 \to 0$. Therefore, in low-energy elastic scattering, the scattering length $a_0$ and the effective range $r_0$ completely characterize the scattering process. Ross-Shaw theory simply generalize the above theory to the case of multi-channels. For that purpose, they define a matrix $M$ via

$$ M = k^{1/2} \cdot K^{-1} \cdot k^{1/2}, \quad (2) $$

where $k$ and $K$ are both matrices in channel space. The matrix $k$ is the kinematic matrix which is a diagonal matrix given by

$$ k = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad (3) $$

and $k_1$, $k_2$ and $k_3$ are related to the scattering energy $E$. The matrix $K$ is the $K$-matrix in scattering theory whose relation with the $S$-matrix is given by [4]

$$ S = \frac{1 + iK}{1 - iK}. \quad (4) $$

Another useful formal expression for the matrix $K$ is

$$ K = \tan \delta, \quad (5) $$

where both sides are interpreted as matrices in channel space. From the above expressions, it is easily seen that $K^{-1}$ that appears in Eq. (2) is simply the matrix $\cot \delta$ and without cross-channel coupling, the $M$-matrix is also diagonal with entries $M \sim \text{Diag}(k_1 \cot \delta_1, k_2 \cot \delta_2, k_3 \cot \delta_3)$. Thus, it is indeed a generalization of the single channel case in Eq. (1). In their original paper, Ross and Shaw showed that the $M$-matrix as function of energy $E$ can be Taylor expanded around some reference energy $E_0$ as,

$$ M_{ij}(E) = M_{ij}(E_0) + \frac{1}{2} R_i \delta_{ij} [k_i^2(E) - k_i^2(E_0)] + \cdots, \quad (6) $$

where we have explicitly written out the channel indices $i$ and $j$. The matrix $M_{ij}(E_0) \equiv M_{ij}^{(0)}$ is a real symmetric matrix in channel space that we will call the inverse scattering length matrix; $R \equiv \text{Diag}(R_1, R_2, R_3)$ is a diagonal matrix which we shall call the effective range matrix. $k_i^2$ are the entries for the kinematic matrix defined in Eq. [3].

1 $K$-matrix is hermitian so that $S$-matrix is unitary.
Therefore, for three channels, there are altogether 9 parameters to describe the scattering close to some energy \( E_0 \): 6 in the inverse scattering length matrix \( M'(0) \) and 3 in the effective range matrix \( R \). One could further reduce the number of parameters to 6 by neglecting terms associated with effective ranges. This is called the zero-range approximation \([19]\). For convenience, we usually take \( E_0 \) to be the threshold of the third channel, i.e. that of \( DD^* \).

It is understood that Ross-Shaw parameterization in Eq. (6) is equivalent to the so-called \( K \)-matrix parameterization with three poles. In this \( K \)-matrix representation, assuming there are altogether \( n \) open channels, the \( n \times n \) \( K \)-matrix is parameterized as,

\[
K(E) = k^{1/2} \left( \sum_{\alpha=1}^{n} \frac{\gamma_\alpha \otimes \gamma_\alpha^T}{E - E_\alpha} \right) k^{1/2},
\]

where \( k \) is the kinematic matrix analogue of Eq. (4), the label \( \alpha = 1, 2, \cdots, n \) designates the channels and each \( \gamma_\alpha \) is a \( 1 \times n \) real constant matrix (an \( n \)-component vector). It is shown in Ref. \([20]\) that this is equivalent to the effective range expansion \([6]\) but the parameters are more flexible. In particular, \( K \)-matrix parameterization contains \( (n^2 + n) \) real parameters: \( n^2 \) for \( n \) copies of \( \gamma_\alpha \)'s and another \( n \) for the \( E_\alpha \)'s while an \( n \)-channel Ross-Shaw parameterization has \( n(n+1)/2 + n \) real parameters, \( n(n - 1)/2 \) parameters less than the most general \( K \)-matrix given in Eq. (7). In this paper, we will focus on the case of \( n = 3 \) only.

### III. RESONANCE SCENARIO IN ROSS-SHAW THEORY

In this section, we investigate the possibility of a narrow peak just close to the threshold of the third channel. In particular, this will be studied within the framework of three-channel Ross-Shaw theory. It turns out that this requirement will implement some constraints among the different parameters in Ross-Shaw theory.

It is convenient to inspect the resonance scenario using the so-called \( T \)-matrix which is continuous across the threshold. Formally, it is related to the \( K \)-matrix via,

\[
K^{-1} = T^{-1} + i,
\]

or equivalently as \( T = K(1 - iK)^{-1} \). The relation between the \( S \)-matrix and the \( T \)-matrix is given by,

\[
S = 1 + 2iT,
\]

where both \( S \) and \( T \) now are \( 3 \times 3 \) matrices in channel space. Since the scattering cross section \( \sigma_{ij} \) is essentially proportional to \( |T_{ij}|^2 \), the so-called elastic cross section in a particular channel \( i \) is given by,

\[
\sigma_{ii} = \frac{4\pi}{k_i^2} |T_{ii}|^2.
\]

Therefore, if we denote,

\[
w_{ii} = \frac{T_{ii}}{k_i} = \frac{1}{\alpha_i(E) - i\beta_i(E)},
\]

with \( \alpha_i \) and \( \beta_i \) being real functions of the energy, then the elastic cross section in channel \( i \) reads,

\[
\sigma_{ii} = \frac{4\pi}{\alpha_i^2 + \beta_i^2} |w_{ii}|^2.
\]

Normally, the imaginary part of \( w_{ii} \), namely \( \beta_i(E) \), is a positive, smooth function of the energy in the energy region to be studied. In fact, if there were no coupling among different channels, we have \( \beta_i = k_i \). The real part (i.e. \( \alpha_i \)), however, could develop a zero in the corresponding energy range, which then leads to a resonance peak structure. To be more specific, a resonance peak happens when \( \alpha_i(E) = 0 \) and the half-width positions for this peak can be obtained by the condition \( \alpha_i(E)/\beta_i(E) = \pm 1 \), respectively.

To be more specific, the \( T \)-matrix in channel space looks like,

\[
T = k^{1/2} (M - ik)^{-1} k^{1/2},
\]

Therefore, if we define the matrix \( w \) in channel space as,

\[
w = (M - ik)^{-1},
\]

the elements of which will be denoted by \( w_{ij} \), then the following expression for \( T_{11} \) can be obtained,

\[
w_{11} \equiv \frac{T_{11}}{k_1} = \frac{1}{D} \left| \begin{array}{ccc} M_{22} - ik_2 & M_{23} & M_{33} - ik_3 \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} - ik_3 \end{array} \right|,
\]

where \( D \) is the determinant of the \( 3 \times 3 \) matrix,

\[
\begin{align*}
D &= \left| \begin{array}{ccc} M_{11} - ik_1 & M_{12} & M_{13} \\ M_{12} & M_{22} - ik_2 & M_{23} \\ M_{13} & M_{23} & M_{33} - ik_3 \end{array} \right|.
\end{align*}
\]

Similar expressions are obtained for \( w_{22} \) and \( w_{33} \). We get the following expression for \( w_{ii} \) with \( i = 1, 2, 3 \),
In the above formulae, below a specific threshold, the corresponding momentum becomes pure imaginary. For example, below the threshold of the third channel, we have \(-i\kappa_3 = \kappa_3\) with \(\kappa_3\) being a positive real number.

On the other hand, it is known from BESIII experiments [1, 24] that, close to the threshold of the third channel, all three elastic channels show resonant peaks. If we assume that these three peaks correspond to a single resonance structure, constraint equations can be obtained from Eq. (17). In the following, using Eq. (17), we will derive these equations that needs to be satisfied among the parameters. The corresponding conditions in the two-channel case has been studied long time ago by Ross and Shaw, e.g. Refs [19, 20]. However, to our knowledge, the case of three channels has not been studied explicitly which will be done within this paper.

\[ w_{11}^{-1} = \alpha_1 - i\beta_1 = M_{11} - i\kappa_1 - M_{12} \]

\[ w_{22}^{-1} = \alpha_2 - i\beta_2 = M_{22} - i\kappa_2 - M_{12} \]

\[ w_{33}^{-1} = \alpha_3 - i\beta_3 = M_{33} - i\kappa_3 + M_{12} \]

Here, \(m_{J/\psi}, m_\pi,\) etc. are the masses of the corresponding mesons and \(k_i\)'s with \(i = 1, 2, 3\) being the scattering momenta in various channels. Now, viewing the \(k_i^2\)'s, \(i = 1, 2, 3\) as complex variables that are related to each other by Eq. (19), one can solve Eq. (18) in some Riemann sheet to yield the pole position for the complex \(k_i^2\)'s. This pole then manifests itself as peaks in elastic cross sections in all three channels. Therefore, in the limit of \(M_{13} = 0\), the so-called HALQCD scenario is fully represented by Eq. (18) in Ross-Shaw theory.

To search for such solutions, we utilize the following notations. We assume that \(\kappa_3 \equiv z\) is small in magnitude. Thus, we have,

\[ \delta E \equiv E - (m_{D^*} + m_D) = \frac{z^2}{2\mu_{DD^*}}, \]

with \(\mu_{DD^*}\) being the reduced mass of \(D^*\) and \(D\). Similarly, \(k_1\) and \(k_2\) will assume their values at the third threshold, namely \(k_3^{(0)}\) and \(k_2^{(0)}\), plus small corrections that are linear in \(z^2\).

\[ \delta k_1 = \frac{z^2}{2\mu_{DD^*}}(v_\pi^{(0)} + v_\rho^{(0)}) = \gamma_1 z^2, \]

\[ \delta k_2 = \frac{z^2}{2\mu_{DD^*}}(v_\pi^{(0)} + v_\rho^{(0)}) = \gamma_2 z^2, \]

where \(v_\pi^{(0)}, v_\rho^{(0)}\) are the speed of the corresponding mesons at the threshold. To be specific, we have, \(v_\pi^{(0)} = k_3^{(0)}/E_{J/\psi}(k_3^{(0)})\), etc. Therefore, the solution \(z_0\), where all \(w_{ii}\) diverge satisfy the following equation,

\[ M_{33} - iz_0 = \frac{M_{23}^2}{M_{11} - i\kappa_1^{(0)} - i\gamma_1 z_0} + \frac{M_{23}^2}{M_{22} - i\kappa_2^{(0)} - i\gamma_2 z_0}. \]
measured in some reasonable unit. A convenient choice is to use a unit system in which \( k_1^{(0)} = 1 \) adopted in Ref. [18]. In such a system, every quantity in Eq. (22) becomes dimensionless and we are searching for \(|z_0| \ll 1\) in the complex plane.

Now, note that the l.h.s of Eq. (22) is linear in \( z_0 \) while the r.h.s depends on \( z_0^2 \), therefore, we could write the solution \( z_0 \) as,

\[
z_0 = z_0^{(1)} + z_0^{(2)} + \cdots,
\]

where \( z_0^{(i)} \) for different \( i \) designates different orders of \( z_0 \), all of which are small, but the higher the index \( i \) is, the even smaller the \( z_0^{(i)} \) becomes. Taylor-expanding both sides of Eq. (22), order by order, we obtain the following equations,

\[
i z_0^{(1)} = \varepsilon = M_{33} - \frac{M_{13}^2}{M_{11} - i k_1^{(0)}} - \frac{M_{23}^2}{M_{22} - i k_2^{(0)}} \equiv \varepsilon
\]
\[
z_0^{(2)} = \left[ \frac{M_{13}^2 \gamma_1}{(M_{11} - i k_1^{(0)})^2} + \frac{M_{23}^2 \gamma_2}{(M_{22} - i k_2^{(0)})^2} \right] \varepsilon^2
\]
\[
z_0^{(3)} = 2i \left[ \frac{M_{13}^2 \gamma_1}{(M_{11} - i k_1^{(0)})^2} + \frac{M_{23}^2 \gamma_2}{(M_{22} - i k_2^{(0)})^2} \right] \varepsilon^3
\]
\[
z_0^{(4)} = \cdots
\]

It is seen that the leading order equation (23) demands that the quantity \( \varepsilon \) thus defined needs to be a complex number that is small in magnitude. Otherwise, there is no consistent small \( z \) solution for Eq. (22). This implies that both the real part and the imaginary part has to be small. If we denote

\[
\varepsilon = \varepsilon_1 - i \varepsilon_2,
\]

with both \( \varepsilon_1 \) and \( \varepsilon_2 \) being real, it is easy to work out the explicit expressions. It is also found that, the imaginary part parameter \( \varepsilon_2 > 0 \) at the threshold of the third channel. The sign of \( \varepsilon_1 \), however, is not definite, depending on other parameters. In order for them to be small, we have,

\[
\begin{align*}
M_{33} - \frac{M_{13}^2 M_{11}}{M_{11}^2 + (k_1^{(0)})^2} - \frac{M_{23}^2 M_{22}}{M_{22}^2 + (k_2^{(0)})^2} & \ll 1, \\
\frac{M_{13}^2 k_1^{(0)}}{M_{11}^2 + (k_1^{(0)})^2} + \frac{M_{23}^2 k_2^{(0)}}{M_{22}^2 + (k_2^{(0)})^2} & \ll 1.
\end{align*}
\]

To leading order, the solution of the pole reads,

\[
\varepsilon_0^{(1)} = -i \varepsilon = -\varepsilon_2 - i \varepsilon_1,
\]

which points out the approximate location of the pole position in the complex plane. To be more precise, the location is given by,

\[
z_0 = -\varepsilon_2 - i \varepsilon_1 + z^{(2)} + z^{(3)} + \cdots,
\]

where \( z^{(2)} \) and \( z^{(3)} \) are given by Eq. (24) and Eq. (25). More iterates can be obtained if necessary.

We can now work out the elastic scattering cross sections close to the threshold of the third channel. These are given by Eqs. (17) by taking \( M_{12} = 0 \). Taking e.g. the first channel, we have,

\[
w_{11}^{-1} = \frac{M_{13}^2}{M_{33} - i k_3 - \frac{M_{23}^2}{M_{22} - i k_2}},
\]

where \( k_i \)’s take real or pure imaginary values, depending on whether it is above or below the thresholds. Since the \( k_i \)’s are related to the total energy via Eq. (14), we know that the r.h.s vanishes when the \( k_i \)’s take complex values at \( k_3 = z_0 \):

\[
M_{11} - i k_1(z_0) = \frac{M_{13}^2}{M_{33} - i k_3(z_0) - \frac{M_{23}^2}{M_{22} - i k_2(z_0)}},
\]

which is consistent with Eq. (22). Since the pole position is rather close to the third threshold, we expect that, large cross sections will be observed. Therefore, we introduce the function

\[
w_{ii}^{-1} = F_i(z),
\]

where in \( F_i(z) \) the \( k_i \)’s are viewed as complex functions of \( z \), which we still take as the complex \( k_3 = z \). We know that the function \( F_i(z) \) has a zero at the location \( z_0 \) which is given by Eq. (23), and that \( z_0 \) is close to the origin. Therefore, we may expand,

\[
F_i(z) = F_i(z_0) + F_i'(z_0)(z - z_0) + \cdots \approx F_i'(z_0)(z - z_0),
\]

where we have utilized the condition \( F_i(z_0) = 0 \) and \( F_i'(z_0) \approx F_i'(0) \) since \( z_0 \) is rather close to the origin. Thus,
the elastic cross section in channel $i$ reads,

$$\sigma_{ii} = \frac{4\pi}{|F_i(z)|^2} = \frac{4\pi}{|F_i'(0)|^2|z - z_0|^2},$$  \hspace{1cm} (35)$$

which exhibits a typical resonance behavior. Here, it is understood that $z$ takes real or pure imaginary values, depending on whether it is above or below the third threshold. To be more explicit, if we take only the first approximation for $z_0$, we have the following cross sections for above and below the third threshold,

$$\sigma_{ii} = \begin{cases} 
\frac{4\pi}{|F_i'(0)|^2|(k_3 + \varepsilon_2)^2 + \varepsilon_1^2|}, & \text{if } z > z_0 \\
\frac{4\pi}{|F_i'(0)|^2|(k_3 + \varepsilon_1)^2 + \varepsilon_2^2|}, & \text{if } z < z_0
\end{cases} \hspace{1cm} (36)$$

where the first/second line is for above/below the threshold, with $k_3 = z = i\kappa_3$, $\kappa_3 > 0$ in the second case. Since we have $\varepsilon_2 > 0$, so the peak above the third threshold must be in the tail region. If $\varepsilon_1 < 0$, then we could see a full peak just below the threshold. If $\varepsilon_1 > 0$, however, a cusp will show up exactly at the threshold.

B. Resonance scenario in Ross-Shaw theory: general case

Here we would like to go beyond the approximation of $M_{12} = 0$. We will show below that, the above results in fact hold in the most general case of three-channel scattering.

$\eta \xi_0^{(1)} = \varepsilon = \frac{D(0)}{\Delta_{33}(0)} = M_{33} + M_{14}$

$$\begin{vmatrix}
M_{12} & M_{22} - ik_2^{(0)} \\
M_{13} & M_{23} \\
M_{11} - ik_1^{(0)} & M_{12} \\
M_{11} - ik_1^{(0)} & M_{12} \\
M_{22} - ik_2^{(0)} & M_{23}
\end{vmatrix} - \begin{vmatrix}
M_{11} - ik_1^{(0)} & M_{12} \\
M_{13} & M_{23} \\
M_{12} & M_{22} - ik_2^{(0)}
\end{vmatrix} = 0$$  \hspace{1cm} (40)

It is easy to verify that, in the limit of $M_{12} = 0$, this reproduces the previous result in Eq. (23). The discussions about elastic scattering cross section remains unchanged. The only thing that needs to be modified is the explicit expression for the solution $\eta_0$ to various orders, which, to the first order, is now shown in Eq. (40) instead of Eq. (23). Again, higher order expressions can be obtained easily if necessary.

IV. MULTI-CHANNEL LÜSCHER FORMULA AND THE STRATEGY FOR LATTICE COMPUTATIONS

In this section, we briefly outline the strategies for a lattice calculation within the multi-channel Lüscher approach for three channels. As we have mentioned in Sec. 1 in the case of three-channels, one first needs a parameterization for the $S$-matrix in terms of functions, and furthermore in terms of the Ross-Shaw parameters.

The most general form of $S$-matrix for three channels, assuming time reversal symmetry, was first given by Waldenstrom in 1974 and it looks like the following (22),

$$S = \begin{pmatrix}
\eta_1 e^{2i\delta_1} & iX_{12} e^{i(\delta_{12})} & iX_{13} e^{i(\delta_{13})} \\
iX_{12} e^{i(\delta_{12})} & \eta_2 e^{2i\delta_2} & iX_{23} e^{i(\delta_{23})} \\
iX_{13} e^{i(\delta_{13})} & iX_{23} e^{i(\delta_{23})} & \eta_3 e^{2i\delta_3}
\end{pmatrix},$$  \hspace{1cm} (41)$$

where $\delta_1$, $\delta_2$ and $\delta_3$ are scattering phases in channel 1, 2 and 3 respectively and $\eta_i \in [0, 1], i = 1, 2, 3$ are called the inelasticity parameters for each channel, all of which are functions of the energy. The other parameters $X_{ij}$ and $\delta_{ij}$ are related to the $\delta_i$ and $\eta_i$ in a
complicated manner hence also functions of the energy. Interested reader can consult Ref. [22] for details. These 6 functions of energy are then parameterized within Ross-Shaw theory in terms of 9 real parameters: 6 for the scattering length matrix $M$, 3 for the effective ranges. Note that $S$ matrix is related to the $T$ matrix via $S = 1 + 2iT$ while the latter is further related to the Ross-Shaw $M$ matrix via Eq. (12).

The multi-channel L"uscher formula has many forms. The most convenient one is the one that is directly related to the Ross-Shaw $M$-matrix,

$$\det \left[ M - B^{(P)} \right] = 0 ,$$

where the matrix $B^{(P)}$, called the box function by Colin Morningstar et al. [23], is a complicated but computable function involving modified zeta-functions that can be obtained from the energy eigenvalues in a finite box. The label $P$ designates the total three-momentum of the two-particle system so that it applies to also moving frames. The corresponding constraint equations that are derived in previous section needs to be boosted accordingly using an appropriate Lorentz transformation. The explicit expression for the box function reads:

$$\langle J'm,J'L'S'\alpha'|B^{(P)}|Jm,J'L'Sa \rangle = -i\delta_{aa'}\delta_{SS'}(u_a)^{L+L'+1} W_{m_L}^{(Pa)}(k_i^2) \langle J'm,J'L'|L'm_L,Sa \rangle \langle Jm,J'L'|Lm_L,Sa \rangle .$$

Here, $J$, $m_J$, $L$ and $S$ corresponds to total angular momentum quantum number, the third component of total angular momentum, orbital angular momentum and spin quantum number of the two-particle state. The index $a$ designates other quantum numbers, e.g. channel or isospin etc. The function $W_{m_L}^{(Pa)}(k_i^2)$ involves zeta-functions and the arguments $k_i^2$ with $i = 1, 2, 3$ represents the momenta in the corresponding channels which are related to the energy via Eq. (12).

For a given set of parameters in Ross-Shaw matrix $M$, the multi-channel L"uscher formula can be viewed as a equation for the energy eigenvalues that enters the equation via the box function $B^{(P)}$. Therefore, when solved numerically it yields a set of energy eigenvalues in the finite box. These energy levels can be compared with the real energy levels obtained from the lattice simulations. This comparison in turn yields an estimate for various $M_{ij}$’s in the Ross-Shaw matrix, as illustrated in Ref. [18]. On the other hand, as we have obtained the conditions that need to be satisfied by these parameters in order to have a resonance peak close to the threshold of the third channel, c.f. Eq. (28), one can directly check if the lattice extracted parameters really support such a scenario or not, as was already done in the two-channel case in Ref. [18].

It is interesting to note that, in the general Ross-Shaw theory, namely Eq. (6) can be utilized to any energy region. In particular, if we investigate only the region close to the threshold, it is good enough to use the zero range expansion. This sets all the effective ranges to zero, leaving us with only 6 parameters. In other words, if we focus on the energy region very close to the threshold, zero range approximation is always valid. Of course, by utilizing the multi-channel L"uscher formula, other energy levels that are somewhat distant from the threshold enter the game (via fitting of $M_{ij}$’s), therefore there could be some deviations from the zero-range approximation. Still, extraction of the $M_{ij}$’s and check whether they satisfy the constraints as outlined in Eq. (28) offers a crucial test. This comparison will hopefully clarify, or at least shed some light on the differences from two different approaches so far: the HALQCD approach and the conventional L"uscher approach. In fact, one could try to arrange a situation where as many as possible energy levels are close to the third threshold. In such a case, one could utilize the zero range approximation without any problem as long as one drops the energy levels that are too distant from the threshold.

V. CONCLUSIONS

To shed more light on the nature of the resonance-like structure $Z_c(3900)$, lattice studies have been performed over the years. However, some puzzles still remain. The existing lattice studies fall into two categories: the ones using L"uscher’s approach and the ones using the HALQCD approach. The results from these two types of approaches are not consistent with each other as they should. This discrepancy needs to be clarified.

In this paper, we study the problem using the three-channel Ross-Shaw theory, which is the generalization of the effective range expansion. We have obtained the constraint conditions that needs to be satisfied by various parameters of the theory in order to have a narrow resonance close to the threshold of the third channel, a scenario that $Z_c(3900)$ realizes. We have pointed out that, combined with the multi-channel L"uscher formula, a real lattice computation could be performed which will yield the results for these parameters and furthermore, one could check if these constraint relations are supported by the lattice results or not. We have also outlined the strategies of such a lattice simulations on how to extract these parameters in a more reliable fashion. Currently, we are working on the simulations details along the lines that are described here and we hope to report the results.
soon.

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