Optimal estimation of parameters for scalar fields in expanding universe exhibiting Lorentz invariance violation

Xiaobao Liu\textsuperscript{a}, Zehua Tian\textsuperscript{b,c,d}, Jieci Wang\textsuperscript{a}, Jiliang Jing\textsuperscript{a}\textsuperscript{*}

\textsuperscript{a} Department of Physics, and Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, P. R. China
\textsuperscript{b} CAS Key Laboratory of Microscale Magnetic Resonance and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China
\textsuperscript{c} Hefei National Laboratory for Physical Sciences at the Microscale, University of Science and Technology of China, Hefei 230026, China
\textsuperscript{d} Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei 230026, China

ABSTRACT: We address the optimal estimation of quantum parameters, in the framework of local quantum estimation theory, for the massless scalar fields in the expanding Robertson-Walker universe exhibiting Lorentz invariance violation (LIV). In the case of the LIV parameter estimation during the spacetime expansion, we find that the appropriate momentum mode of field particles and larger expansion volume and rate of the universe can provide us a better precision. Moreover, in the case of the expansion volume estimation due to the effects of LIV, the optimal precision can be achieved by choosing the particles with some appropriate momentum mode and LIV parameter and the universe with a larger expansion rate. We also find that, the optimal precision for the expansion rate estimation in the presence of LIV peaks at the appropriate momentum mode and LIV parameter.

KEYWORDS: Lorentz invariance violation; expanding universe; local quantum estimation theory.

*Corresponding author. Email: jljing@hunnu.edu.cn
Contents

1. introduction 1

2. the cosmological model in the presence of Lorentz invariance violation 3

3. local quantum estimation theory 6

4. QFI for LIV parameter estimation during the spacetime expansion 7

5. QFI for expansion parameter estimation due to effects of LIV 10

6. Conclusions 14

1. introduction

The theory of general relativity plays a dominant role in the field with a very high density, where the classical physics breaks down and quantum gravity may be important to resolve the big-bang singularity which is the prediction of classical general relativity. Until now there have many different quantum gravity approaches, such as string theory [1, 2] and loop quantum gravity [3, 4]. However, several approaches to quantum gravity suggest that a microscopic structure of spacetime may lead to the Lorentz symmetry violation. In the last decades, this symmetry violation has been investigated in the context of noncommutative geometry [5, 6, 7], the extra dimensions [8, 9], and the discretization process. Besides, there has a lot of physical phenomena showing that the Lorenz symmetry may be broken [10, 11, 12, 13, 14, 15, 16]. It is well known that the standard dispersion relation represents the relationship between the energy and momentum of a particle, which is invariant under continuous Lorentz transformations. However, when the distances are as small as the Planck length, the spacetime might have a discrete structure and quantum gravity effects are dominant. This is the smallest length scale at which the standard dispersion relation should be modified by adding an extra term that violates the invariance of Lorentz symmetry [17, 18, 19, 21, 22, 23, 24, 25]. Recently, the effects of LIV have not only been studied in inflationary cosmology [26, 27, 28, 29, 30, 31], but also shown that it is relevant to the dark energy problem and baryogenesis [32, 33, 34].
On the other hand, many quantities of interest to us correspond to nonlinear functions of the density operator and thus cannot be associated with quantum observables. As a consequence, any procedure aims at evaluating the quantity of interest which is ultimately a parameter estimation problem, where the value of the quantity of interest is indirectly inferred from the measurement of one or more proper observables. Therefore, an optimization problem naturally arises, which may be properly addressed in the framework of local quantum estimation theory (QET) [35, 36, 37, 38, 39], providing a analytical tool to find the optimal measurement according to some given criterion. Relevant examples of this application of QET are given by discussions of gravitational wave detection [40, 41], measurements of entanglement [12, 13], Unruh-Hawking temperature and phase parameters for accelerated detectors [44, 45, 46, 47, 48, 49], deformed parameter for the noncommutative spacetime [50], and so on. Recently, with the help of local QET, Wang et al. have investigated the ultimate precision limits for the estimation of expansion parameters of universe [71], while such technology has not yet been applied to the quantum parameter estimation due to the effects of LIV on the expanding Robertson-Walker (RW) universe.

In this paper, we consider a massless scalar field on a two-dimensional RW expanding spacetime in the presence of LIV. It is worth noting that the effects of LIV on the massless scalar field in the expanding universe can lead to the creation of particles [30, 31]. Otherwise, the spacetime is conformally equivalent to Minkowski spacetime, and the conformally invariant massless scalar field does not produce particles in the absence of LIV. Besides, in curved spacetime, the effects of spacetime on the field are important, due to the fact that particles are created by propagation of a quantum field through an expanding universe [52]. Therefore, we apply techniques from local QET to estimate both the LIV coefficient and the parameters of the expanding universe in the two-dimensional conformally flat RW metric for massless scalar field in the presence of LIV. We will explicitly calculate the quantum Fisher information (QFI) and derive the ultimate bounds to the precision of estimated parameters. Such procedure corresponds to finding the set of measurements that allows us to get the highest precision estimation for the parameters. Moreover, we can extract the information about the history of the expanding universe in the presence of LIV by taking measurements on the states in the asymptotic future region. We find that, for the estimation of both LIV and cosmological parameters, the precision can be improved by choosing the projective measurements corresponding to eigenvectors of the particle states with appropriate LIV, cosmological and field parameters.

The paper is organized as follows. In Sec. II we introduce the quantization of massless scalar field in the RW spacetime due to the effects of LIV. In Sec. III, we briefly introduce the basic notions of local quantum estimation theory. In Sec. IV and Sec. V, we calculate the QFI in the estimation of both the LIV and cosmological parameters, to find the highest precision for the parameters estimation. Finally, a
summary of the main results of our work is present in Sec. V.

2. the cosmological model in the presence of Lorentz invariance violation

Let us start with a two-dimensional Robertson-Walker expanding spacetime with the line element

\[ ds^2 = dt^2 - a^2(t) dx^2, \]

(2.1)

where \( a(t) \) is the scale factor. By introducing the conformal time \( \eta \) which relates to cosmological time \( t \) by \( d\eta = a^{-1}(t) dt \), the metric in Eq. (2.1) takes the form

\[ ds^2 = c^2(\eta)(d\eta^2 - dx^2), \]

(2.2)

where \( c^2(\eta) = a^2(t(\eta)) \) is the conformal scale factor. As considered in the Refs. [30, 31], the generalized Lagrangian for the cosmological spacetime in the presence of LIV can be written as

\[ \mathcal{L} = \frac{1}{2} \sqrt{-g}[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - m^2 \phi^2 + \alpha^2 (D^2 \phi) + \lambda(1 - \mu^\mu \mu^\nu)], \]

(2.3)

where \( \alpha^2 \) denotes the LIV coefficient. The covariant spatial Laplacian in Eq. (2.3) is given by

\[ D^2 \phi = -D^\mu D_\mu \phi = -q^{\mu\nu} \nabla_\nu(q_\mu^\tau \nabla_\tau \phi). \]

(2.4)

where \( q_{\mu\nu} = -g_{\mu\nu} + \mu_\mu \mu_\nu \) with \( g_{\mu\nu} \) corresponding to the metric in Eq. (2.1). The vector field \( \mu_\mu \) is as a non-dynamical vector field to be specified by the conditions of the theory. The Lagrange multiplier \( \lambda \) in Eq. (2.3) constrains \( \mu^\mu \) as

\[ g_{\mu\nu} \mu^\mu \mu^\nu = 1. \]

(2.5)

Due to the Robertson-Walker metric describing a homogeneous and isotropic spacetime in Eq. (2.2), the vector field \( \mu_\mu \) is taken as to satisfy the isotropic property of cosmological metric and the Eq. (2.5). Thus, we have \( \mu^\mu = (c(\eta), 0) \). In this paper, we only consider the massless scalar field case, i.e., \( m = 0 \). By equating the variation of the action \( S = \int \mathcal{L} d^4x \) with respect to \( \phi \) to zero, the equation of motion for \( \phi \) in the metric (2.2) can be obtained as follow

\[ \Box \phi - \frac{\alpha^2}{c^2(\eta)} \partial_x^2 \phi = 0, \]

(2.6)
where $\Box := \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$ is the d’Alembertian operator. Now, we consider the conformal factor which is \cite{52, 53, 54, 55, 56, 57}

$$c^2(\eta) = 1 + \epsilon(1 + \tanh(\rho \eta)), \quad (2.7)$$

where $\epsilon$ and $\rho$ are positive real parameters corresponding to the total volume and rate of the expansion for the universe, respectively. Note that it corresponds to a flat Minkowskian spacetime asymptotically in the far past and future, i.e., $c^2(\eta) \rightarrow 1$ in the far past $\eta \rightarrow -\infty$ and $c^2(\eta) \rightarrow 1 + 2\epsilon$ in the far future $\eta \rightarrow +\infty$. We refer to the far past and future as the in and out regions, respectively. In the asymptotic in and out regions, the vacuum states and one-particle states in the presence of LIV may be well defined \cite{31}. To find the solutions to the massless scalar field equation in Eq. (2.6), we separate it into positive and negative frequency modes, i.e., $u_k(\eta, x)$ and $u_k^*(\eta, x)$. By invoking the method of separation of variables, we have the mode solution $u_k(\eta, x) = e^{ikx} \chi_k(\eta)$, such that $\chi_k(\eta)$ satisfies

$$\frac{d^2}{d\eta^2} \chi_k(\eta) + \left( k^2 - \frac{\alpha^2}{c^2(\eta)} k^4 \right) \chi_k(\eta) = 0. \quad (2.8)$$

It is easy to find that the standard dispersion is modified by adding a LIV term which violates the invariance of Lorentz symmetry due to the quantum gravity effects \cite{58}. Thus, the modified dispersion relation in the expanding spacetime is

$$\omega^2 = k^2 - \frac{\alpha^2}{c^2(\eta)} k^4. \quad (2.9)$$

It is worth mentioning that the LIV coefficient $\alpha^2$ is proportional to the Planck length $l_p$ which gives the semiclassical limit beyond the standard model. Therefore, we can recover the standard dispersion relation if we take $l_p \rightarrow 0$. On the other hand, the equation (2.8) can be solved by the hypergeometric functions. In the asymptotic past region, the normalized modes are found to be

$$u_k^{in}(\eta, x) = \frac{1}{\sqrt{4\pi \omega_{in}}} \times \exp \left( ikx - i \omega_k^+ \eta - i \frac{\omega_k^-}{\rho} \ln[(1 + 2\epsilon)e^{\rho \eta} + e^{-\rho \eta}] \right) \times F(1 + i \omega_k^-/\rho, i \omega_k^+/\rho; 1 - i \omega_{in}/\rho; z), \quad (2.10)$$
and in the asymptotic future regions, the normalized modes are

\[
u_{\text{out}}^k(\eta, x) = \frac{1}{\sqrt{4\pi \omega_{\text{in}}}} \times \exp \left( ikx - i\omega_k^+ \eta - i\frac{\omega_k^-}{\rho} \ln[(1 + 2\epsilon)e^{\rho\eta} + e^{-\rho\eta}] \right) \times F(1 + i\omega^-_k/\rho, i\omega^-_k/\rho; 1 - i\omega_{\text{out}}/\rho; z). \tag{2.11}\]

Here, \(\omega_{\text{in/out}} = \sqrt{k^2 - k^4\alpha^2/c^2}(\eta \to \pm \infty), \omega_{\pm} = \frac{1}{2}(\omega_{\text{in}} \pm \omega_{\text{out}}), z = \frac{1 + 2\epsilon}{2} \frac{1 + \tanh(\rho\eta)}{1 - \epsilon \tanh(\rho\eta)}\) and \(F\) is the ordinary hypergeometric function.

It is now possible to quantize the field and obtain the relation between the vacuum state in the far past regions and that in the far future regions. Using the inner product and the linear transformation properties of hypergeometric function, the Bogoliubov transformations associated with the \(\text{in}\) and \(\text{out}\) solutions take the form

\[
u_{\text{in}}^k(\eta, x) = \alpha_k u_{\text{out}}^k(\eta, x) + \beta_k u_{-\text{out}}^k(\eta, x), \tag{2.12}\]

where the Bogoliubov coefficients \(\alpha_k\) and \(\beta_k\) are given by

\[
\alpha_k = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(-i\omega_{\text{out}}/\rho)}{\Gamma(-i\omega_k^+/\rho)\Gamma(1 - i\omega_k^-/\rho)},
\]

\[
\beta_k = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(i\omega_{\text{out}}/\rho)}{\Gamma(-i\omega_k^-/\rho)\Gamma(1 - i\omega_k^-/\rho)}. \tag{2.13}\]

In order to find the relation between the far past and future vacuum states, we use the relationship between the operators \(b_{\text{in}}(k) = \alpha^* b_{\text{out}}^\dagger(k) - \beta^* b_{\text{out}}^\dagger(k)\). Here, the creation and annihilation operators, \(b_{\text{in/out}}(k)\) and \(b_{\text{in/out}}^\dagger(k)\), satisfy the commutation relations \([b_{\text{in}}(k), b_{\text{in}}^\dagger(k')] = \delta_{kk'}\) and \([b_{\text{out}}(k), b_{\text{out}}^\dagger(k')] = \delta_{kk'}\). According to the definition of vacuum state \(b_{\text{in}}(k)|0\rangle_{\text{in}} = 0\) and the normalization condition \(\langle 0|0\rangle_{\text{in}} = 1\), we obtain the far past vacuum state in terms of the far future bosonic Fock basis

\[
|0\rangle_{\text{in}} = \frac{1}{|\alpha_k|} \sum_{n=0}^{\infty} \left(\frac{\beta_k^*}{\alpha_k^*}\right)^n |n_{-k}\rangle_{\text{out}}. \tag{2.14}\]

It is worth noting that due to the effects of LIV, a single qubit prepared in a initial vacuum state \(|0\rangle_{\text{in}}\) is converted to an entangled state in the particle number degree of freedom in the expanding spacetime. Therefore, we can extract the information about the LIV, cosmological and field parameters which are codified in the final state.
3. local quantum estimation theory

Supposing that we have a given quantum state $\rho(\lambda)$ parametrized by an unknown parameter $\lambda$, the unknown parameter may does not correspond to a proper quantum observable and it cannot be measured directly. To get information about $\lambda$, we have to resort to indirect measurements, inferring its value by the measurements of a set of observables. That is to say, we have a parameter estimation problem. In the estimation problem we try to infer the value of a parameter $\lambda$ by measuring a different quantity $X$. Therefore, any inference strategy amounts to find an estimator, i.e., a mapping $\hat{\lambda} = \hat{\lambda}(x_1, x_2, ..., x_n)$ from the set of measurement outcomes into the space of parameters. Optimal estimators are those saturating the Cramér-Rao theorem [39] $\text{Var}(\lambda) \geq \frac{1}{MF(\lambda)}$, (3.1)

which establishes a lower bound on the variance $\text{Var}(\lambda)$ of any unbiased estimator of the parameter $\lambda$. Here, $M$ is the number of measurements and $F(\lambda)$ denotes the Fisher information (FI) which is

$$F(\lambda) = \sum_x p(x \mid \lambda) [\partial_\lambda \ln p(x \mid \lambda)]^2,$$  

where $\partial_\lambda = \frac{\partial}{\partial \lambda}$ and $p(x \mid \lambda)$ denotes the conditional probability corresponding measurement outcome $x$ with respect to a chosen positive operator valued measurement (POVM). According to the Born rule, we find $p(x \mid \lambda) = \text{Tr}[\rho(\lambda) \Pi_x]$, where $\{\Pi_x\}$ is the elements of POVM and $\rho(\lambda)$ denotes the density operator. Moreover, we can maximize the FI over all the possible quantum measurements on the quantum system. By introducing the symmetric Logarithmic derivative $L(\lambda)$ satisfying the partial differential equation $\partial_\lambda \rho(\lambda) = (\rho(\lambda) L(\lambda) + L(\lambda) \rho(\lambda))/2$, we can get that the FI $F(\lambda)$ is upper bounded by the QFI $H(\lambda)$, i.e., $F(\lambda) \leq H(\lambda) \equiv \text{Tr}[\rho(\lambda) L(\lambda)^2]$. Therefore, we have the quantum Cramér-Rao bound

$$\text{Var}(\lambda) \geq \frac{1}{MF(\lambda)} \geq \frac{1}{MH(\lambda)}, \quad (3.3)$$

for the variance of any estimator, which represents that we have the ultimate bound to precision for any quantum measurement aimed at estimating the parameter $\lambda$ for a state of the family $\rho(\lambda)$. Upon diagonalizing the density matrix as $\rho_\lambda = \sum_{i=1}^{N} p_i |\psi_i\rangle \langle \psi_i|$, with $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$, the QFI can be rephrased as

$$H(\lambda) = 2 \sum_{m,n}^{N} \frac{|\langle \psi_m | \partial_\lambda \rho_\lambda | \psi_n \rangle|^2}{p_m + p_n}.$$

\[ (3.4) \]
For a quantum state with non-full-rank density matrix, the detailed formula for the QFI is \[ H(\lambda) = \sum_n \frac{(\partial_n p_n)^2}{p_n} + 2 \sum_{n,m} \frac{(p_n - p_m)^2}{p_n + p_m} |\langle \psi_m | \partial_\lambda \psi_n \rangle|^2, \] where the sums include only the terms with \( p_n \neq 0 \) and \( p_n + p_m \neq 0 \).

4. QFI for LIV parameter estimation during the spacetime expansion

Let us begin with how precisely we can estimate the LIV parameter in an expanding spacetime, which appears in the modified dispersion relation Eq. (2.9). This is reduced to a parameter estimation problem based on the Cramér-Rao inequality \[ 39 \]. We note that the reduced density matrix corresponding to particle states of the modes \( k \) can be obtained by tracing out the modes \( -k \) in the state Eq. (2.14), which is \[ \rho_k = \frac{1}{|\alpha_k|^2} \sum_{n=0}^\infty \left( \frac{|\beta_k|}{|\alpha_k|} \right)^{2n} |n_k \rangle \langle n_k|, \] from which we can see that the information of LIV parameter is encoded in the Bogoliubov coefficients. Assuming that we are living in the spacetime which corresponds to the asymptotically flat regions of the future times, we can now estimate the LIV parameter by taking measurements on the particle state in Eq. (4.1).

In order to study the ultimate limit of precision for the LIV parameter estimation in the curved spacetime, we are looking for the optimal estimation scheme to get the largest QFI. In our paper, we consider the state of the particles as the probe. For a specific measurement process corresponding to a POVM, the Fisher information in Eq. (3.2) can be obtained in terms of the classical probabilities \( p(x | \lambda) \). However, in the quantum estimation process, we should carry out the optimization over measurement processes, i.e., maximizing the FI over all possible quantum measurements on the quantum system to obtain the QFI. Thus, the central task is to find an optimal measurement that allows us to get the QFI. It is worth noting that the eigenvector \( |\psi_m \rangle \) of the final state in Eq. (4.1) can be obtained easily. Therefore, we choose the projective measurement which is constituted by the eigenvector \( |\psi_m \rangle \) of the final state, and then the eigenvalues of the final state are the measured probabilities. Hence, according to the expression of the QFI in Eq. (3.5), we have \[ H(\alpha^2) = \sum_{n=0}^\infty \frac{(\partial_\alpha^2 \lambda_n)^2}{\lambda_n}, \]
where $\lambda_n = \gamma^n - \gamma^{n+1}$ with

$$\gamma = \frac{|\beta_k|^2}{|\alpha_k|^2} \frac{\sinh[\pi \omega_{k-}/\rho]}{\sinh[\pi \omega_+/\rho]}.$$  \hfill (4.3)

We note that the cosmological particles are created due to the effects of LIV even if the massless scalar field was in the initial vacuum state in the expanding Robertson-Walker universe. Therefore, we can exploit pairs of entangled particles which are produced in the final states of the field for our purposes. Now let us study the behavior of the precision of the LIV parameter estimation which is investigated by the QFI in Eq. (4.2).

![Figure 1: QFI in the estimation of the LIV parameter $\alpha^2$ as a function of momentum mode $k$ for the fixed values $\alpha^2 = -10$, $\epsilon = 0.99$, and three different values of expansion rate, i.e., $\rho = 0.6$ (solid line), $\rho = 0.5$ (dashed line), and $\rho = 0.4$ (dotted-dashed line).](image_url)

In Fig. 1, we plot the QFI in the estimation of the LIV parameter as a function of the momentum modes $k$ of the particles with different expansion rate $\rho$, for the fixed expansion rate $\epsilon = 0.99$ and the LIV coefficient $\alpha^2 = -10$. We can see from the plot that in the extreme case of $k \to 0$, i.e., at the large wavelengths (large spatial scales), the wavelengths reach to the classical length scale and we can not observe the effects of the LIV for the small momentum $k$ of the particles, which means that the QFI should tend to zero. On the other hand, in the extreme case of $k \to \infty$, i.e., at the small wavelengths (small spatial scales), the QFI vanishes because the system approaches to the continuous limit where the effects of discreteness are not felt by the bosons. This means that when $k \to 0$ and $k \to \infty$, the quantum parameter estimation reaches to the lowest precision. Moreover, it is worth noting that for a finite momentum modes $k$ of the particles, when the frequency of asymptotically flat regions in the past and future times satisfies

$$\omega_{in}\sinh\left(\frac{\pi}{\rho} \omega_{in}\right) = (1 + 2\epsilon)\omega_{out}\sinh\left(\frac{\pi}{\rho} \omega_{out}\right),$$  \hfill (4.4)
we get that the QFI equals to zero, which means that the QFI have two peaks as the increases of $k$. That is to say, the precision of the quantum parameter estimation have two peaks at some finite momentum modes $k$. However, we are interested in that there is a specific momentum for each mode, $k_{\text{max}}$, so that the QFI is maximum at this momentum mode, which indicates that the momentum modes $k$ have a range providing us a better precision. Besides, we find that the precision of estimation of the LIV parameter is a increasing function of the expansion rate.

![Graph](image)

**Figure 2:** The maximum QFI in the estimation of the LIV parameter $\alpha^2$ as a function of expansion volume $\epsilon$, for the fixed values $\alpha^2 = -10$ and three different values of $k$ and $\rho$, i.e., $k_{\text{max}}$ with $\rho = 0.6$ (solid line), $\rho = 0.5$ (dashed line) and $\rho = 0.4$ (dotted-dashed line).

Figure 2 shows that the maximum value of the QFI in the estimation of the LIV parameter obtained for optimal momentum mode $k$ varies with the expansion volume $\epsilon$, for the fixed LIV parameter $\alpha^2 = -10$. Three different values of expansion rate $\rho$ have been discussed in this figure. We find that as the expansion rate grows, the precision of estimation of the LIV parameter increases accordingly. Moreover, it is obvious that the maximum QFI for the privileged value of $k$ increases by increasing the expansion volume $\epsilon$. This implies that we can get the highest precision of the LIV parameter estimation by choosing the maximum expansion volume $\epsilon$.

Figure 3 shows the maximum QFI in term of the expansion rate $\rho$ for the optimal value of momentum mode $k$, with fixed value of the expansion volume $\epsilon = 0.99$. Three different values of LIV parameter $\alpha^2$ have been shown in this plot. From this figure we can deduce that the QFI grows as the expansion rate gets larger values, which implies that the highest precision of estimation can be improved by a larger expansion rate $\rho$. Moreover, the higher the LIV parameter $\alpha^2$, the bigger the QFI is, i.e., the easier it is to achieve a given precision in the estimation of LIV parameter. To sum up, we can choose the larger expansion volume and rate which are allowed for the expanding universe, as well as some suitable momentum modes of the particles to obtain the optimal strategy realizing the ultimate limit of precision in the estimation
of the LIV parameter, such that we can easier to reach the highest precision for higher LIV parameter.

\[ H(\alpha^2)_{\text{max}} \]

\[ \alpha^2 = -3 \] (solid line), \( \alpha^2 = -5 \) (dashed line), \( \alpha^2 = -10 \) (dotted-dashed line).

**Figure 3:** The maximum QFI in the estimation of the LIV parameter as a function of the expansion rate \( \rho \) for the optimal momentum mode \( k \), with fixed values \( \epsilon = 0.99 \) and three different values of LIV parameter, i.e., \( \alpha^2 = -3 \) (solid line), \( \alpha^2 = -5 \) (dashed line), \( \alpha^2 = -10 \) (dotted-dashed line).

5. QFI for expansion parameter estimation due to effects of LIV

In this section we deal with studying the effects of LIV on the estimation precision of the cosmological parameters arising in the expanding universe. Firstly, the cosmological parameter that we wish to estimate is the expansion volume \( \epsilon \), based on the quantum Cramér-Rao inequality \([39]\). It is worth mentioning that the information about the total volume \( \epsilon \) is encoded in the Bogoliubov coefficients. Hence we can calculate the operational QFI in the estimation of expansion volume \( \epsilon \) in terms of the Bogoliubov coefficients shown in Eq. (2.13). According to the reduced density matrix corresponding to the particle states of momentum \( k \) in Eq. (1.1), and assuming that now we live in the asymptotic future of the expanding spacetime in the presence of LIV, the volume of the expanding universe can be estimated by taking measurements on the particle state.

To investigate how precisely one can estimate the expansion volume in the presence of LIV, we have to find the optimal probe state preparation and interaction parameters that allow us to obtain the largest QFI. In this system, the state of the particles act as the probe. Physically, a fixed value of FI can be obtained by any set of measurement, while the QFI is equal to the largest FI optimizing over all the possible measurements. In this paper the optical projective measurement is constituted by the eigenvectors \( |\psi_m\rangle \) of the final state, and the measured probabilities are the
eigenvalues of the final state. According to the Eq. (3.5), we can obtain the QFI in the estimation of the expansion volume $\epsilon$ due to the effects of LIV, which is

$$H(\epsilon) = \sum_{n=0}^{\infty} \frac{(\partial_{\epsilon} \lambda_n)^2}{\lambda_n}, \quad (5.1)$$

where $\lambda_n = \gamma^n - \gamma^{n+1}$ is the eigenvalues of the diagonal density matrix.

In Figure 4 we plot the QFI in the estimation of expansion volume $\epsilon$ as a function of the momentum modes $k$ with three different values of expansion rate $\rho$, for fixed parameters $\epsilon = 0.99$ and $\alpha^2 = -10$. It is shown that the QFI of the probe state sensitively depends on momentum $k$ of the scalar field particles. We note that the QFI firstly increases and then starts to decrease by increasing the value of $k$, which means that the highest precision in the expansion volume estimating can be obtained at a specific momentum mode. That is to say, the precision of the estimation peaks at the optimal value of $k$. Then, we can obtain the optimal precision for the expansion volume by choosing some particles with momenta around $k_{\text{max}}$ in the quantum measuring process. Moreover, as the expansion rate $\rho$ grows, the QFI increases accordingly. Therefore, the higher precision in the estimation of expansion volume can be achieved by increasing the expansion rate.

![Figure 4: QFI in the estimation of the expanding volume as a function of the momentum modes $k$ for fixed values $\epsilon = 0.99$ and $\alpha^2 = -10$, and three values of parameter $\rho$, i.e., $\rho = 0.6$ (solid line), $\rho = 0.5$ (dashed line), $\rho = 0.4$ (dotted-dashed line), respectively.](image)

Figure 4 shows the QFI in the estimation of expansion volume $\epsilon$ with respect to the LIV parameter $\alpha^2$, with different values of momentum mode $k$ and expansion rate $\rho$. We note that there exists two different choices for $\alpha^2$ in this figure, a positive one and a negative one. For the positive values of $\alpha^2$, there is an upper bound for $k$, due to the fact that $\omega_{\text{in}}$ is an imaginary number for the value of $k$ greater than the upper bound. Thus, there is a forbidden region that we have not shown in the plot. It is obvious from the plot that the precision of the estimation of expansion volume $\epsilon$
will increase monotonically by increment in the positive value of $\alpha^2$. However, for the negative one, the $\omega_{in/out}$ is always valid. In this case, we find that the QFI increases for a while for a specific range of absolute values of $\alpha^2$ and then decreases, which means that the precision of estimation has the maximum value at a certain value of LIV parameter $\alpha_{max}^2$. Moreover, as the expansion rate $\rho$ decreases, $\alpha_{max}^2$ is shifted toward negative values for each special expansion volume. In addition, for the case $\alpha^2 = 0$, the cosmological particles cannot be created due to no difference between the asymptotically flat regions in the past and future times, i.e., $\omega_{in} = \omega_{out}$. Therefore, no one can extract any information from the expansion spacetime, which means that the parameter estimation has the lowest precision when $\alpha^2 = 0$ as expected.

**Figure 5:** QFI in the estimation of the expansion volume as a function of the LIV parameter $\alpha^2$, for fixed values $\epsilon = 0.99$ and three different values of $k$ and $\rho$, i.e., $k_{max}$ with $\rho = 0.6$ (solid line), $\rho = 0.5$ (dashed line), and $\rho = 0.4$ (dotted-dashed line). Note that the plot is not shown for some regions where $\omega_{in}$ becomes imaginary.

Figure 6 shows the maximum value of QFI at the optimal point of momentum mode $k$ as a function of expansion rate $\rho$, for the fixed value of momentum mode $k = 0.4$. Three different expansion volume $\epsilon$ have shown in this figure. For each special expansion volume $\epsilon$, the maximum QFI increases by the increment in the rapidity of the expanding universe, which means that the precision for the volume estimation can be enhanced by a larger expansion rate. Furthermore, the maximum QFI increases as the expansion volume gets smaller values, which indicates that it is easier to achieve a given precision for the expansion volume estimation.

In Fig. 7 we plot the maximum value of the QFI at the optimal value of LIV parameter $\alpha^2$ with respect to the expansion rate $\rho$, for the fixed value of the LIV parameter $\alpha^2 = -10$. Similarly, three different expansion volume $\epsilon$ have considered in Fig. 4. We find that as the expansion rate grows, the maximum QFI for the optimal value of $\alpha^2$ increases accordingly. Therefore, the precision of estimation of expansion volume is an increasing function of the expansion rate. In addition, as can
be seen, the smaller value of $\epsilon$, the bigger QFI, i.e., the easier to obtain the highest precision in the estimation process.

Similarly, we will briefly study the behaviors of the QFI in the estimation of expansion rate $\rho$ in the presence of LIV. With the help of Eqs.(19) and (20), it is easy to obtain the formula of the QFI in the estimation of the parameter $\rho$ and to find that the QFI depends on the parameters $k$, $\alpha^2$ and $\epsilon$. In Fig. 8(a), we plot the behavior of the QFI in estimation of $\rho$ as a function of momentum mode $k$ with different expansion volume $\epsilon$ and fixed parameters $\rho = 1$ and $\alpha^2 = -10$. It is shown that the QFI increases at first and then deceases by increasing the value of $k$, which means that the optimal precision for the expansion rate peaks at some finite momentum modes $k$ of particles. Moreover, to analyze the precision for the expansion rate estimation due to the effects of LIV, we show that the QFI for different
momentum mode $k$ and expansion volume $\epsilon$ is the function of the LIV parameter $\alpha^2$ in Fig. 8(b). It is worth mentioning that this plot is only shown for regions of $\alpha^2$ where $\omega_{\text{in/out}}$ is valid. We can see from the figure 8(b) that for the positive values of $\alpha^2$, the precision of the estimation of expansion rate is an increasing function of $\alpha^2$. However, for the negative values of $\alpha^2$, the QFI in the estimation of $\rho$ firstly increases and then decreases as the LIV parameter gets lesser values, which means that there is a range of LIV parameter $\alpha^2$ that provides us with the optimal precision. We also note that the QFI increases as the growth of the expansion volume $\epsilon$, which indicates that the highest precision in the estimation of expansion rate can be obtained for the larger expansion volume allowed for the expanding universe.

![Figure 8](image)

**Figure 8:** (a) QFI in the estimation of the expanding rate as a function of the momentum modes $k$ for fixed values $\rho = 1$ and $\alpha^2 = -10$, and three values of parameter $\epsilon$, i.e., $\epsilon = 0.99$ (solid line), $\epsilon = 0.50$ (dashed line), $\epsilon = 0.20$ (dotted-dashed line), respectively. (b) QFI in the estimation of the expanding rate as a function of the LIV parameter $\alpha^2$ for fixed values $\rho = 1$ and three values of $k$ and $\epsilon$, i.e., $k_{\text{max}}$ with $\epsilon = 0.99$ (solid line), $\epsilon = 0.50$ (dashed line), $\epsilon = 0.20$ (dotted-dashed line), respectively. Note that the plot is not shown for some regions where $\omega_{\text{in}}$ becomes imaginary.

### 6. Conclusions

In this paper, we have studied the quantum parameter estimation for massless scalar field on a two-dimensional asymptotically flat Robertson-Walker expanding spacetime in the presence of LIV. Based on the theory of local quantum estimation, we have derived the ultimate bounds to precision for the LIV coefficient and expansion parameters estimation, which is related to the QFI. Moreover, we have choose the probe state which is defined by particle number as an energy eigenvector to improve the precision of the estimation. It is shown that we can achieve the largest QFI by performing projective measurements on the eigenvectors of the particle states with suitable LIV and cosmological and field parameters. Two different cases, the estimation of LIV coefficient and expansion parameters, have been considered, respectively.
For the case of the LIV parameter estimation in the expanding Robertson-Walker universe, we found that there are a range of momentum modes of the field particles that provide us a better precision in the estimation process. Furthermore, we should choose the larger volume and rate of expanding universe to enhance the precision for the estimation of the LIV parameter. In particular, as the LIV parameter gets larger values, it is easier to obtain the given precision in the estimation of LIV parameter.

On the other hand, for the case of the expansion volume estimation in the presence of LIV, the QFI depends sensitively on the momentum mode of the particles and the LIV parameter, which means that we can get the optimal precision of estimation by choosing some particles with some appropriate momentum mode and LIV parameters. Moreover, the precision of the estimation can be improved by a larger rate of expanding universe. However, the lesser volume of expanding universe, the easier it is to achieve the given precision for the expansion volume estimation in the quantum measuring process. In addition, for the expansion rate estimation due to the effects of LIV, we can adjust the momentum mode of particles, LIV parameter and expansion volume allowed for the expanding universe to attain the higher precision in the estimation process.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant Nos. 11875025, 11475061 and 11675052. X. Liu thanks for the Hunan Provincial Innovation Foundation For Postgraduate under Grant No. CX2017B175. Hunan Provincial Natural Science Foundation of China under Grant No. 2018JJ1016.

References

[1] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1988), Vol. 1.

[2] J. Polchinski, *String Theory* (Cambridge University Press, New York, 1998), Vol. 1.

[3] C. Rovelli, *Quantum Gravity* (Cambridge University Press, New York, 2004).

[4] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, New York, 2007).

[5] A. Connes, *Noncommutative geometry* (Academic Press, New York, 1994).

[6] A. Connes, *A short survey of noncommutative geometry*, *J. Math. Phys.* **41**, 3832 (2000).

[7] S. Majid, *Meaning of Noncommutative Geometry and the Planck-Scale Quantum Group*, *Lect. Notes Phys.* **541**, 227 (2000).
[8] J. M. Overduin and A. Hamna, *Extra dimensions and violations of Lorentz symmetry*, arXiv:1607.04580.

[9] T. G. Rizzo, *Lorentz violation in extra dimensions*, *J. High Energy Phys.* **09**, 036 (2005); T. G. Rizzo, *Lorentz violation in warped extra dimensions*, *J. High Energy Phys.* **11**, 1 (2010).

[10] K. Greisen, *End to the Cosmic-Ray Spectrum? Phys. Rev. Lett.* **16**, 748 (1966).

[11] M. Takeda, et al., AGASA Collaboration, *Extension of the Cosmic-Ray Energy Spectrum beyond the Predicted Greisen-Zatsepin-Kuz’min Cutoff*, *Phys. Rev. Lett.* **81**, 1163 (1998).

[12] F. Krennrich, et al., *Cutoff in the TeV energy spectrum of Markarian 421 during strong flares in 2001*, *Astrophys. J.* **560**, L45 (2001).

[13] E. E. Antonov, et al., *Examination of Lorentz invariance by means of watching of extensive air showers development at super high energies*, *Pisma Zh. Eksp. Teor. Fiz.* **73**, 506 (2001).

[14] H. Sato, *Extremely High Energy and Violation of Lorentz Invariance*, arXiv:0005218.

[15] S. R. Coleman and S.L. Glashow, *High-energy tests of Lorentz invariance*, *Phys. Rev. D* **59**, 116008 (1999).

[16] Y. Zhang, X. Liu, J. Qi and H. Zhang, *Cosmological Model Independent Time Delay Method*, *JCAP* **08**, 027 (2018).

[17] G. Preparata and S. -S. Xue, *Do we live on a lattice? Fermion masses from the Planck mass*, *Phys. Lett. B* **264**, 35 (1991).

[18] S. Cacciatori, G. Preparata, S. Rovelli, I. Spagnolatti and S.-S. Xue, *On the Ground State of Quantum Gravity*, *Phys. Lett. B* **427**, 254 (1997).

[19] G. Preparata, R. Rovelli and S. -S. Xue, *Gas of wormholes: a possible ground state of quantum gravity*, *Gen. Rel. Grav.* **32**, 1859 (2000).

[20] S. -S. Xue, *Detailed discussions and calculations of quantum Regge calculus of Einstein-Cartan theory*, *Phys. Rev. D* **82**, 064039 (2010).

[21] M. Zarei, E. Bavarsad, M. Haghighat, I. Motie, R. Mohammadi and Z. Rezaei, *Generation of circular polarization of the CMB*, *Phys. Rev. D* **81**, 084035 (2010).

[22] I. Motie and S. -S. Xue, *High energy neutrino oscillation at the presence of the Lorentz Invariance Violation*, *Int. J. Mod. Phys. A* **27**, 1250104 (2012).

[23] T. Jacobson, S. Liberati and D. Mattingly, *Lorentz violation at high energy: concepts, phenomena, and astrophysical constraints*, *Ann. Phys. (N.Y.)* **321**, 150 (2006).
[24] S. Liberati and L. Maccione, *Lorentz violation: motivation and new constraints*, Annu. Rev. Nucl. Part. Sci. **59**, 245 (2009).

[25] D. Mattingly, *Modern tests of lorentz invariance*, Living Rev. Relativ. **8**, 5 (2005).

[26] R. H. Brandenberger and J. Martin, *On signatures of short distance physics in the cosmic microwave background*, Int. J. Mod. Phys. A **17**, 3663(2002); R. H. Brandenberger and J. Martin, *Back-reaction and the trans-Planckian problem of inflation reexamined*, Phys. Rev. D **71**, 023504 (2005).

[27] R. Easther, B.R. Greene, W.H. Kinney and G. Shiu, *Inflation as a probe of short distance physics*, Phys. Rev. D **64**, 103502 (2001).

[28] A. A. Starobinsky, *Robustness of the inflationary perturbation spectrum to trans-Planckian physics*, JETP Lett. **73**, 371 (2001).

[29] J. Martin and R. H. Brandenberger, *Corley-Jacobson dispersion relation and trans-Planckian inflation*, Phys. Rev. D **65**, 103514 (2002); J. Martin and R. H. Brandenberger, *Dependence of the spectra of fluctuations in inflationary cosmology on trans-Planckian physics*, Phys. Rev. D **68**, 063513 (2003).

[30] E. Khajeh, N. Khosravi and H. Salehi, *Cosmological particle creation in the presence of Lorentz violation*, Phys. Lett. B **652**, 217 (2007).

[31] H. Mohammadzadeh, M. Farahmand and M. Maleki, *Entropy production due to Lorentz invariance violation*, Phys. Rev. D **96**, 024001 (2017).

[32] O. Bertolami, *Lorentz invariance and the cosmological constant*, Class. Quantum Grav. **14**, 2785 (1997).

[33] O. Bertolami, R. Lehnert, R. Potting and A. Ribeiro, *Cosmological acceleration, varying couplings, and Lorentz breaking*, Phys. Rev. D **69**, 083513 (2004).

[34] G. L. Alberghi, R. Casadio and A. Tronconi, *Radion induced spontaneous baryogenesis*, Mod. Phys. Lett. A **22**, 339 (2007).

[35] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).

[36] A. S. Holevo, *Statistical Structure of Quantum Theory* (Springer, Berlin, 2001).

[37] S. L. Braunstein and C. M. Caves, *Statistical distance and the geometry of quantum states*, Phys. Rev. Lett. **72**, 3439 (1994).

[38] M. G. A. Pairs, *Quantum estimation for quantum technology*, Int. J. Quant. Inf. **7**, 125 (2009).

[39] H. Cramér, *Mathematical Methods of Statistics* (Princeton University, Princeton, NJ, 1946).
[40] M. Vallisneri, *Use and abuse of the Fisher information matrix in the assessment of gravitational-wave parameter-estimation prospects*, Phys. Rev. D 77, 042001 (2008).

[41] J. Aasi, et al., *Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light*, Nat. Photonics 7, 613 (2013).

[42] M. G. Genoni, P. Giorda and M. G. A. Paris, *Optimal estimation of entanglement*, Phys. Rev. A 78, 032303 (2008).

[43] G. Brida, I. P. Degiovanni, A. Florio, M. Genovese, P. Giorda, A. Meda, M. G. A. Paris, and A. P. Shurupov, *Optimal estimation of entanglement in optical qubit systems*, Phys. Rev. A 83, 052301 (2011).

[44] M. Aspachs, G. Adesso and I. Fuentes, *Optimal quantum estimation of the Unruh-Hawking effect*, Phys. Rev. Lett. 105, 151301 (2010).

[45] D. Hosler and P. Kok, *Parameter estimation using NOON states over a relativistic quantum channel*, Phys. Rev. A 88, 052112 (2013).

[46] Y. Yao, X. Xiao, L. Ge, X. G. Wang and C. P. Sun, *Quantum Fisher information in non-inertial frames*, Phys. Rev. A 89, 042336 (2014).

[47] J. Wang, Z. Tian, J. Jing and H. Fan, *Quantum metrology and estimation of Unruh effect*, Sci. Rep. 4, 07195 (2014).

[48] Z. Tian, J. Wang, J. Jing and H. Fan, *Relativistic quantum metrology in open system dynamics*, Sci. Rep. 5, 07946 (2015).

[49] Y. Yang, X. Liu, J. Wang and J. Jing, *Quantum metrology of phase for accelerated two-level atom coupled with electromagnetic field with and without boundary*, Quantum Inf. Process 17, 54 (2018).

[50] X. Liu, Z. Tian, J. Wang and J. Jing, *Relativistic motion enhanced quantum estimation of κ-deformation of spacetime*, Eur. Phys. J. C 78, 665 (2018).

[51] J. Wang, Z. Tian, J. Jing and H. Fan, *Parameter estimation for an expanding universe*, Nucl. Phys. B 892, 390 (2015).

[52] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, CUP, 1994.

[53] C. Bernard and A. Duncan, *Regularization and renormalization of quantum field theory in curved space-time*, Ann. Phys. (N.Y.) 107, 201 (1977).

[54] A. Duncan, *Explicit dimensional renormalization of quantum field theory in curved space-time*, Phys. Rev. D 17, 964 (1978).

[55] I. Fuentes, R.B. Mann, E. Martín-Martínez and S. Moradi, *Entanglement of Dirac fields in an expanding spacetime*, Phys. Rev. D 82, 045030 (2010).
[56] E. Martin-Martinez and N.C. Menicucci, Cosmological quantum entanglement, Class. Quantum Grav. 29, 224003 (2012).

[57] S. Moradi, R. Pierini and S. Mancini, Spin-particles entanglement in Robertson-Walker spacetime, Phys. Rev. D 89, 024022 (2014).

[58] G. Amelino-Camelia, Quantum-spacetime phenomenology, Living Rev. Relativ. 16, 5 (2013).