BIHARMONIC $\delta(r)$-IDEAL HYPERSURFACES IN EUCLIDEAN SPACES ARE MINIMAL

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Abstract. A submanifold $M^n$ of a Euclidean space $\mathbb{E}^N$ is called biharmonic if $\Delta \vec{H} = 0$, where $\vec{H}$ is the mean curvature vector of $M^n$. A well known conjecture of B.Y. Chen states that the only biharmonic submanifolds of Euclidean spaces are the minimal ones. Ideal submanifolds were introduced by Chen as those which receive the least possible tension at each point. In this paper we prove that every $\delta(r)$-ideal biharmonic hypersurfaces in the Euclidean space $\mathbb{E}^{n+1}$ ($n \geq 3$) is minimal. In this way we generalize a recent result of B. Y. Chen and M. I. Munteanu. In particular, we show that every $\delta(r)$-ideal biconservative hypersurface in Euclidean space $\mathbb{E}^{n+1}$ for $n \geq 3$ must be of constant mean curvature.

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1. Introduction

In the middle of 1980’s B.Y. Chen initiated the investigation of properties of submanifolds in Euclidean spaces $\mathbb{E}^N$. For an historical overview we refer to his books [3] and [6]. Among several important problems that Chen had raised that time was the following well-known conjecture [4]:

The only biharmonic submanifolds of Euclidean spaces are the minimal submanifolds.

A biharmonic submanifold $M$ is defined by the equation $\Delta \vec{H} = 0$, where $\Delta$ and $\vec{H}$ denote respectively the Laplace operator and the mean curvature vector field of $M$. It is well known that $M$ is biharmonic if and only if the immersion $x : (M, g) \to \mathbb{E}^N$ is a biharmonic map.

The conjecture was originally proved for surfaces in $\mathbb{E}^3$ by B.Y. Chen in [4] and for certain submanifolds in $\mathbb{E}^n$ (including one dimensional) by I. Dimitrić in [13]. An alternative approach was proposed by T. Hasanis and T. Vlachos in [22] who proved the conjecture for hypersurfaces in Euclidean 4-spaces. Since then several researchers have made significant contributions towards proving it, such as F. Defever [12], K. Akatagawa and S. Maeta [1], Y. Fu [10], [17], R. Shankar and A. Sharfuddin [21], and more recently in B.Y. Chen [9], and N. Koiso, H. Urakawa [24].

In contrast to its simple statement, the conjecture has turned quite endure to several attempts for its proof. Therefore, it was natural for researchers to impose some natural assumptions. The most usual one, and in fact the most successful into confirming the conjecture for several cases, was to assume that the submanifold is a hypersurface in the Euclidean space. In this case one usually makes some extra assumption about the number of distinct eigenvalues of the shape operator, or about the scalar curvature.

Another natural assumption for the hypersurface $M$ is to be an ideal (or $\delta(r)$-ideal) hypersurface. We give the formal definition in Section 2. Such hypersurfaces
were introduced by B.Y. Chen via the concept of $\delta$-invariants, in his investigation to define “nice immersed submanifolds” as those which receive the least possible tension at each point. We refer to [6] and [7] for a deeper motivation.

In the work [8] B.Y. Chen and M. I. Munteanu proved that $\delta(2)$-ideal and $\delta(3)$-ideal biharmonic hypersurfaces of a Euclidean space is minimal. In the present work we extend this result by proving the following:

**Theorem 1.1.** Every $\delta(r)$-ideal oriented biharmonic hypersurface with at most $r + 1$ distinct principal curvatures in the Euclidean spaces $E^{n+1}$ ($n \geq 3$), is minimal.

Closely related to the concept of biharmonic submanifolds is the concept of bi-conservative submanifolds. These were introduced by R. Caddeo et al. in [2] and are submanifolds with conservative stress-energy tensor.

For the case of hypersurfaces $M^n$ in $E^{n+1}$ it can be shown that the biconservativity condition is equivalent to the equation $2\mathcal{A}(\text{grad}H) + nH\text{grad}H = 0$, where $\mathcal{A}$ is the shape operator of $M^n$ and $H$ the mean curvature. This equation is one of the two equations which are equivalent to the condition of biharmonicity, $\Delta H = 0$ (cf. (2.8), (2.9)). Therefore, a biharmonic hypersurface is biconservative.

Biconservative hypersurfaces had appeared in the literature under the name $H$-hypersurfaces ([22]). They have attracted recently the interest of several researchers (e.g. [10], [11], [14], [15], [20], [25], [26], [27], [28]). From the proof of Theorem 1.1 we also obtain the following result:

**Proposition 1.2.** Every $\delta(r)$-ideal oriented biconservative hypersurface with at most $r + 1$ distinct principal curvatures in Euclidean spaces $E^{n+1}$ ($n \geq 3$), has constant mean curvature.

We briefly present the central idea of the proof of Theorem 1.1 which is simpler than the method used in [8] and [11]. Using that $M^n$ is a $\delta(r)$ ideal hypersurface its shape operator has a simpler form. Since $M^n$ is biharmonic in particular it is biconservative, hence we use the corresponding equation to simplify the connection forms by using Codazzi equation and Gauss equation. Then we see that the definition of mean curvature provides us an equation showing the relation between eigenvalues of the shape operator and mean curvature $H$. This equation plays a very important role in the proof. By differentiating this equation two or more times we obtain polynomial equations showing relations among the eigenvalues, connection forms and mean curvature $H$. Then using a standard argument involving the resultant of two polynomials as defined in Lemma 2.2, we are able to eliminate all the eigenvalues as well as the connection forms one by one, to obtain an algebraic polynomial equation in $H$ with constant coefficients which implies that $H$ must be constant. By taking into account the second equation that comes from the biharmonicity assumption, we prove that $H$ is zero.

2. Preliminaries

Let $(M^n, g)$ be an oriented hypersurface isometrically immersed in Euclidean space $(E^{n+1}, \bar{g})$, that is $g$ is the induced metric by the immersion that defines the hypersurface. Let $\bar{\nabla}$ and $\nabla$ denote the linear connections on $E^{n+1}$ and $M$ respectively. Then the Gauss and Weingarten formulae are given by

\begin{equation}
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall \ X, Y \in \Gamma(TM),
\end{equation}
(2.2) \[ \nabla_X \xi = -A_\xi X, \]
where \( \xi \) be the unit normal vector to \( M \), \( h \) is the second fundamental form and \( A \) is the shape operator. It is well known that the second fundamental form \( h \) and shape operator \( A \) are related by

(2.3) \[ \bar{g}(h(X, Y), \xi) = g(A_\xi X, Y). \]

The mean curvature is given by

(2.4) \[ H = \frac{1}{n} \text{trace} A, \]
and the mean curvature vector \( \vec{H} = H\xi \) is a well defined normal vector field to \( M^n \) in \( \mathbb{E}^{n+1} \). The Gauss and Codazzi equations are given by

(2.5) \[ R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY, \]
(2.6) \[ (\nabla_X A)Y = (\nabla_Y A)X \]
respectively, where \( R \) is the curvature tensor and

(2.7) \[ (\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y) \]
for all \( X, Y, Z \in \Gamma(TM) \).

The hypersurface \( M^n \) is called biharmonic if

\[ \Delta \vec{H} = 0. \]

By identifying the tangential and normal parts in the above equation, it is known (3) that it is equivalent to the system

(2.8) \[ 2A(\text{grad} H) + nH\text{grad} H = 0, \]
(2.9) \[ \Delta H + H\text{trace}(A^2) = 0, \]
where \( \Delta \) is the Laplace operator (our sign convention is such that \( \Delta f = -f'' \) when \( f \) is a function of one real variable).

Next, we recall the concept of \( \delta \)-invariants and \( \delta \)-ideal hypersurfaces. We refer to [6] and [5] for more details.

For a Riemannian manifold \( M^n \) with \( n \geq 3 \) and an integer \( r \in [2, n-1] \), let \( \tau(p) \) be the scalar curvature at \( p \in M^n \) and let \( \tau(L^r) \) be the scalar curvature of a linear subspace \( L^r \) of dimension \( r \geq 3 \) of the tangent space \( T_p(M) \). The \( \delta \)-invariant \( \delta(r) \) of \( M^n \) at \( p \) is defined as

(2.10) \[ \delta(r)(p) = \tau(p) - \inf_{L^r} \tau(L^r). \]

For any \( n \)-dimensional submanifold \( M^n \) in a Euclidean space \( \mathbb{E}^m \) and for an integer \( r = 2, \ldots, n-1 \), Chen proved the following universal sharp inequality

(2.11) \[ \delta(r)(p) \leq \frac{n^2(n-r)}{2(n-r+1)} H^2(p), \]
where \( H^2 = \langle \vec{H}, \vec{H} \rangle \) is the squared mean curvature.

**Definition 2.1.** A submanifold \( M^n \) in \( \mathbb{E}^m \) is called \( \delta(r) \)-ideal if equality in (2.11) is satisfied identically.
We will need the following result.

**Theorem 2.1.** ([6, Theorem 13.7]) Let $M^n$ be a hypersurface in the Euclidean spaces $\mathbb{E}^{n+1}$. Then for any integer $r = 2, \ldots, n-1$ it is

$$
\delta(r) \leq \frac{n^2(n-r)}{2(n-r+1)}H^2.
$$

Equality holds at a point $p$ if and only if there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ at $p$ such that the shape operator is given by

$$
\mathcal{A} = \begin{pmatrix}
D_r & 0 \\
0 & u_r I_{n-r}
\end{pmatrix},
$$

where $D_r = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $u_r = \lambda_1 + \lambda_2 + \cdots + \lambda_r$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the principal curvature functions of $M^n$ at $p$. If this happens at every point, we call $M^n$ a $\delta(r)$-ideal hypersurface in $\mathbb{E}^{n+1}$.

Finally, the following algebraic lemma will be useful to our study.

**Lemma 2.2.** ([23, Theorem 4.4, pp. 58–59]) Let $D$ be a unique factorization domain, and let $f(X) = a_0 X^m + a_1 X^{m-1} + \cdots + a_m$, $g(X) = b_0 X^n + b_1 X^{n-1} + \cdots + b_n$ be two polynomials in $D[X]$. Assume that the leading coefficients $a_0$ and $b_0$ of $f(X)$ and $g(X)$ are not both zero. Then $f(X)$ and $g(X)$ have a non constant common factor if and only if the resultant $\mathcal{R}(f, g)$ of $f$ and $g$ is zero, where

$$
\mathcal{R}(f, g) = \begin{vmatrix}
a_0 & a_1 & a_2 & \cdots & a_m \\
a_0 & a_1 & \cdots & \cdots & a_m \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & b_1 & b_2 & \cdots & b_n \\
b_0 & b_1 & \cdots & \cdots & b_n \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
b_0 & b_1 & b_2 & \cdots & b_n
\end{vmatrix}.
$$

Here there are $n$ rows of “$a$” entries and $m$ rows of “$b$” entries.

3. $\delta(r)$-ideal biharmonic hypersurfaces in $\mathbb{E}^{n+1}$

In the present section we will prove Theorem 1.1.

**Proof of Theorem 1.1** Let $M^n$ be an oriented $\delta(r)$-ideal biharmonic hypersurface in $\mathbb{E}^{n+1}(n > 2)$. From Theorem 2.1 the shape operator (2.13) of $M^n$ with respect to some orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ can be expressed as

$$
\mathcal{A}(e_i) = \lambda_i e_i, \quad i = 1, 2, \ldots, n,
$$

where $\lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_r$, for $i = r+1, \ldots, n$. Since we will need to differentiate the principal curvature functions of $\mathcal{A}$ we need to know that these are smooth (at least at some connected component). To this end, we use an argument given in [19, Section 3, lines 3-10]. The set $M_A$ of all points of $M^n$, at which the number of distinct eigenvalues of the shape operator $\mathcal{A}$ (i.e. the principal curvatures) is locally constant, is open and dense in $M^n$. Therefore, we can work only on the connected component of $M_A$ consisting of points where the number of principal curvatures is at most $r + 1$. On that connected component, the principal curvature functions of $\mathcal{A}$ are always smooth.
Claim: The mean curvature $H$ of $M^n$ is constant.

Assume the contrary and we will end up into contradiction. Then there exists an open connected subset $U$ of $M$ with $\operatorname{grad} pH \neq 0$, for all $p \in U$. From (2.8) it is easy to see that $\operatorname{grad} H$ is an eigenvector of the shape operator $A$ with corresponding principal curvature $-\frac{nH}{2}$.

Without lose of generality we choose $e_1$ in the direction of $\operatorname{grad} H$, which gives $\lambda_1 = -\frac{nH}{2}$. We express $\operatorname{grad} H$ as

$$\operatorname{grad} H = \sum_{i=1}^{n} e_i(H)e_i. \quad (3.2)$$

As we have taken $e_1$ parallel to $\operatorname{grad} H$, it is

$$e_1(H) \neq 0, \quad e_i(H) = 0, \quad i = 2, \ldots, n. \quad (3.3)$$

We express

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k, \quad i, j = 1, 2, \ldots, n. \quad (3.4)$$

Using (3.4) and the compatibility conditions $(\nabla_{e_k} g)(e_i, e_i) = 0$, $(\nabla_{e_k} g)(e_i, e_j) = 0$, we obtain

$$\omega_{ki}^j = 0, \quad \omega_{kj}^i + \omega_{ji}^k = 0, \quad (3.5)$$

for $i \neq j$, and $i, j, k = 1, 2, \ldots, n$.

We set $\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = \lambda$ and we consider the following cases:

**Case A.** $\lambda_i \neq \lambda, \quad i = 2, 3, \ldots, r$.

Taking $X = e_i, Y = e_j, (i \neq j)$ in (2.7) and using (3.1), (3.4), we get

$$(\nabla_{e_i} A) e_j = e_i(\lambda_j)e_j + \sum_{k=1}^{n} \omega_{ij}^k e_k(\lambda_j - \lambda_k).$$

Putting the value of $(\nabla_{e_i} A) e_j$ in (2.6), we find

$$e_i(\lambda_j)e_j + \sum_{k=1}^{n} \omega_{ij}^k e_k(\lambda_j - \lambda_k) = e_j(\lambda_i)e_i + \sum_{k=1}^{n} \omega_{ji}^k e_k(\lambda_i - \lambda_k),$$

whereby taking inner product with $e_j$ and $e_k$, we obtain

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ij}^j = (\lambda_j - \lambda_i)\omega_{ji}^i, \quad (3.6)$$

$$(\lambda_j - \lambda_k)\omega_{ij}^k = (\lambda_i - \lambda_k)\omega_{ji}^i, \quad (3.7)$$

respectively, for distinct $i, j, k = 1, 2, \ldots, n$.

Using (3.3), (3.4) and the fact that $[e_i, e_j](H) = 0 = \nabla_{e_i} e_j(H) - \nabla_{e_j} e_i(H) = \omega_{ij}^1 e_1(H) - \omega_{ji}^1 e_1(H)$, for $i \neq j$ and $i, j = 2, \ldots, n$, we find

$$\omega_{ij}^1 = \omega_{ji}^1. \quad (3.8)$$

Using (2.4), (2.13) and $\lambda_1 = -\frac{nH}{2}$, we obtain

$$\sum_{i=2}^{r} \lambda_i = \frac{n(n - r + 3)}{2(n - r + 1)} H, \quad (3.9)$$

and $\lambda = \frac{nH}{n-r+1}$. 

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Therefore, using \((3.5)\) and \((3.9)\), we obtain
\[
e_i(\lambda_i) \neq 0, \quad e_j(\lambda_i) = 0,
\]
for \(i = 1, 2, \ldots, n\) and \(j = 2, 3, 4, \ldots, n\).

Now, it can be seen that \(\lambda_1\) can never be equal to \(\lambda_i\) \((i = 2, 3, \ldots, r)\) and \(\lambda\). Indeed, if \(\lambda_1 = \lambda_i\) for some \(i\), then from \((3.6)\), we find that
\[
e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega^{j}_{11} = 0, \quad j = 2, 3, \ldots, r,
\]
which contradicts the first expression of \((3.10)\). Similarly, if \(\lambda_1 = \lambda\) we get a contradiction.

Putting \(i \neq 1, j = 1, r + 1, \ldots, n\) in \((3.6)\) and using \((3.10)\) and \((3.5)\), we find
\[
\omega^1_{11} = \omega^1_{ii} = \omega^1_{iA} = \omega^1_{1A} = 0, \quad i = 1, 2, A = r + 1, \ldots, n.
\]

Putting \(k = 1, i \neq j\), and \(i, j = 2, 3, \ldots, n\) in \((3.7)\), and using \((3.5)\), we get
\[
\omega^1_{ij} = \omega^1_{ji} = \omega^1_{ii} = \omega^1_{iA} = \omega^1_{iA} = \omega^1_{iA} = 0, \quad A = r + 1, \ldots, n.
\]

Now, putting \(i = 1, 2, \ldots, r, k = r + 1, \ldots, n\) and \(j = r + 1, \ldots, n\) \((j \neq k)\) in \((3.7)\), and using \((3.5)\), we get
\[
\omega^1_{B1} = \omega^1_{AB} = \omega^1_{AB} = \omega^1_{AB} = 0, \quad i = 1, 2, \ldots, r
\]
where \(A \neq B\) and \(A, B = r + 1, \ldots, n\).

Now, evaluating \(g(R(e_1, e_i)e_1, e_i)\), using \((3.12)\sim(3.14)\) and Gauss equation \((2.5)\), we find the following:

- For \(X = e_1, Y = e_i, Z = e_1, W = e_i\),

\[
e_1(\omega^1_{ii}) - (\omega^1_{ii})^2 = -\frac{nH}{2}\lambda_i, \quad i = 2, 3, \ldots, n.
\]

Now, using \(\lambda_1 = -\frac{nH}{2}, \lambda = \frac{nH}{n-r+1}\), and \((3.6)\) for \(i = 1\) and \(j = r + 1, \ldots, n\), we get
\[
2e_1(H) = (n - r + 1)H\omega^1_{AA}, \quad A = r + 1, \ldots, n.
\]

Now, differentiating \((3.9)\) along \(e_1\) two times alternatively by using \((3.15)\) and \((3.16)\), we obtain
\[
\sum_{i=2}^{r} (2\lambda_i + nH)\omega^1_{ii} = \frac{n(n-r+3)}{2}H\omega^1_{AA},
\]
\[
\sum_{i=2}^{r} \left[2(2\lambda_i + nH)(\omega^1_{ii})^2 + \frac{n(n-r+1)}{2}H\omega^1_{ii}\omega^1_{AA} - \frac{nH}{2}\lambda_i(2\lambda_i + nH)\right] = \frac{n(n-r+3)^2}{4}H(\omega^1_{AA})^2 - \frac{n^3(n-r+3)}{4(n-r+1)}H^3,
\]
for \(A = r + 1, \ldots, n\).

Eliminating \(\lambda_2\) from \((3.17)\) and \((3.18)\) by using \((3.9)\), we obtain
\[
\left[\frac{2n(n-r+2)}{n-r+1}H + 2\sum_{i=3}^{r} \lambda_i\right]\omega^1_{22} + \sum_{i=3}^{r} (2\lambda_i + nH)\omega^1_{ii} = \frac{n(n-r+3)}{2}H\omega^1_{AA},
\]
\[
\left[\frac{4n(n-r+2)}{n-r+1}H - 4\sum_{i=3}^{r} \lambda_i\right](\omega^1_{22})^2 - nH\left[\frac{n(n-r+2)}{2(n-r+1)}H - \sum_{i=3}^{r} \lambda_i\right] = \frac{n(n-r+2)}{2(n-r+1)}H

\sum_{i=3}^{r} \lambda_i + \sum_{i=3}^{r} \left[2(2\lambda_i + nH)(\omega^1_{ii})^2 + \frac{n(n-r+1)}{2}H\omega^1_{ii}\omega^1_{AA} - \frac{nH}{2}\lambda_i(2\lambda_i + nH)\right] = \frac{n(n-r+3)^2}{4}H(\omega^1_{AA})^2 - \frac{n^3(n-r+3)}{4(n-r+1)}H^3,
\]
respectively.
We consider (3.19), (3.20) as polynomials of $\omega^1_2$ with coefficients in polynomial ring $R_1[H, \lambda_3, \lambda_4, \ldots, \lambda_r, \omega^1_{33}, \omega^1_{44}, \ldots, \omega^1_{rr}, \omega^1_{AA}]$ over real field $\mathbb{R}$. Since equations (3.19), (3.20) have a common root $\omega^1_2$, Lemma 2.2 implies that the resultant of their coefficients is equal to zero, which gives another polynomial equation defined as

$$g_1(H, \lambda_3, \lambda_4, \ldots, \lambda_r, \omega^1_{33}, \omega^1_{44}, \ldots, \omega^1_{rr}, \omega^1_{AA}) = 0. \tag{3.21}$$

Again differentiating (3.21) along $e_1$ two times alternatively and using (3.15) and (3.16), we obtain two polynomial equations defined as

$$g_2(H, \lambda_3, \lambda_4, \ldots, \lambda_r, \omega^1_{33}, \omega^1_{44}, \ldots, \omega^1_{rr}, \omega^1_{AA}) = 0, \tag{3.22}$$

$$g_3(H, \lambda_3, \lambda_4, \ldots, \lambda_r, \omega^1_{33}, \omega^1_{44}, \ldots, \omega^1_{rr}, \omega^1_{AA}) = 0. \tag{3.23}$$

We consider (3.21), (3.22) and (3.21), (3.23) as polynomials of $\omega^1_3$ with coefficients in polynomial ring $R_2[H, \lambda_3, \lambda_5, \ldots, \lambda_r, \omega^1_{33}, \omega^1_{44}, \ldots, \omega^1_{AA}]$ over real field $\mathbb{R}$. Also, Equations (3.21), (3.22) and (3.21), (3.23) have a common root $\omega^1_3$ and Lemma 2.2 implies that the resultant of their coefficients is equal to zero, which gives polynomial equations

$$g_4(H, \lambda_3, \lambda_5, \ldots, \lambda_r, \omega^1_{33}, \omega^1_{44}, \ldots, \omega^1_{rr}, \omega^1_{AA}) = 0, \tag{3.24}$$

$$g_5(H, \lambda_3, \lambda_5, \ldots, \lambda_r, \omega^1_{33}, \omega^1_{44}, \ldots, \omega^1_{rr}, \omega^1_{AA}) = 0. \tag{3.25}$$

Similarly, we can eliminate $\omega^1_{33}$ from (3.22), (3.25) by considering $\omega^1_{33}$ as a common root of (3.21), (3.25) and by using Lemma 2.2 we obtain another polynomial equation

$$g_6(H, \lambda_3, \lambda_5, \ldots, \lambda_r, \omega^1_{44}, \omega^1_{55}, \ldots, \omega^1_{rr}, \omega^1_{AA}) = 0. \tag{3.26}$$

Proceeding in the same way, we will be able to eliminate $\lambda_4, \omega^1_{44}, \lambda_5, \omega^1_{55}, \ldots, \lambda_r, \omega^1_{rr}, \omega^1_{AA}$ and obtain a polynomial equation in $H$ with constant coefficients, which implies that $H$ must be a constant.

**Case B.** $\lambda_i = \lambda_j$, for some $i, j = 2, 3, \ldots, r$.

For simplicity we will prove it for $i = 2, j = 3$ and the other cases can be obtained similarly. Then for $\lambda_2 = \lambda_3$ (3.9) reduces to

$$2\lambda_3 + \sum_{i=4}^r \lambda_i = \frac{n(n-r+3)}{2(n-r+1)}H. \tag{3.27}$$

By differentiating (3.27) two times along $e_1$ and using (3.15) and (3.16), we obtain polynomial equations in $\lambda_i, \omega^1_2$ and $H$. As in the above case, by using Lemma 2.2 we will be able to find a polynomial equation in $H$ with constant coefficients which implies that $H$ must be a constant.

**Case C.** $\lambda_i = \lambda$ for some $i = 1, 2, \ldots, r$.

In a similar way with Case B we obtain that $H$ must be constant, and this concludes the proof of the claim.

Now, since $H$ is constant it follows from (2.9) that $\text{tr} \{ A^2 \} = 0$, which implies that $H = 0$, and this concludes the proof of Theorem 1.1. \qed

Proposition 1.2 now follows from the above proof. Since $M^n$ is a $\delta(r)$-ideal biconservative hypersurface, equation (2.8) is satisfied and we proved that this implies that $H$ is constant.
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References

[1] Akutagawa K. and Maeta S.: Biharmonic properly immersed submanifolds in Euclidean spaces, Geom. Dedicata 164(1) (2013) 351–355.
[2] Caddeo R., Montaldo S., Oniciuc C. and Piu P.: Surfaces in three dimensional space forms with divergence-free stress-bienergy tensor, Ann. Mat. Pura Appl. 193(2) (2014) 529–550.
[3] Chen B. Y.: Total Mean Curvature and Submanifolds of Finite Type World Scientific, Singapore 1984, 2nd Edition 2014.
[4] Chen B. Y.: Some open problems and conjectures on submanifolds of finite type, Soochow J. Math. 17 (1991) 169–188.
[5] Chen B. Y.: Some new obstruction to minimal and Lagrangian isometric immersions, Japan. J. Math. 26(1) (2000) 105-127.
[6] Chen B. Y.: Pseudo-Riemannian Geometry, δ-Invariants and Applications, World Scientific, Hackensack, NJ, 2011.
[7] Chen B. Y.: A tour through δ-invariants: From Nash’s embedding theorem to ideal immersions, best ways of living and beyond, Publ. Inst. Math. 94(108) (2013) 67–80.
[8] Chen B. Y. and Munteanu M. I.: Biharmonic ideal hypersurfaces in Euclidean spaces, Differential Geom. Appl. 31(1) (2013) 1–16.
[9] Chen B. Y.: Chen’s biharmonic conjecture and submanifolds with parallel normalized mean curvature vector, Mathematics 7(8) 710 (2019).
[10] Deepika: On biconservative Lorentz hypersurface with non diagonal shape operator, Mediterr. J. Math. 14 (2017), article:127.
[11] Deepika and Arvanitoyeorgos A.: Biconservative ideal hypersurfaces in Euclidean spaces, J. Math. Anal. Appl. 458 (2) (2018) 1147–1165.
[12] Defever F.: Hypersurfaces of $\mathbb{E}^4$ with harmonic mean curvature vector, Math. Nachr. 196 (1998) 61–69.
[13] Dimitrič I.: Submanifolds of $E^n$ with harmonic mean curvature vector, Bull. Inst. Math. Acad. Sinica 20(1) (1992), 53–65.
[14] Fetcu D., Oniciuc C. and Pinheiro A. L.: CMC biconservative hypersurfaces in $S^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, J. Math. Anal. Appl. 425 (2015) 588–609.
[15] Fu Y.: On biconservative surfaces in Minkowski 3-Space, J. Geom. Phys. 66 (2013) 71–79.
[16] Fu Y.: Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space, J. Geom. Phys. 75 (2014) 113–119.
[17] Fu Y.: Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean space, Tohoku Math. J. 67(3) (2015) 465–479.
[18] Fu Y.: Explicit classification of biconservative surfaces in Lorentz 3-space forms, Ann. Mat. Pura Appl. 194(3) (2015) 805–822.
[19] Fu Y., Hong M-C.: Biharmonic hypersurfaces with constant scalar curvature in spaceforms, Pacific J. Math. 294(2) 2018 329–350.
[20] Gupta R. S.: On biharmonic hypersurfaces in Euclidean space of arbitrary dimension, Glasgow Math. J. 57(3) (2015), 633-642.
BIHARMONIC $\delta(r)$-IDEAL HYPERSURFACES IN EUCLIDEAN SPACES ARE MINIMAL

[21] Gupta R. S., Sharfuddin A.: Biharmonic hypersurfaces in Euclidean space $\mathbb{E}^5$, J. Geom. 107(3) (2016) 685–705.
[22] Hasanis Th. and Vlachos Th.: Hypersurfaces in $E^4$ with harmonic mean curvature vector field, Math. Nachr. 172 (1995) 145–169.
[23] Kendig K.: Elementary Algebraic Geometry, GTM 44, Springer-Verlag, 1977.
[24] N. Koiso and H. Urakawa: Biharmonic submanifolds in a Riemannian manifold, Osaka J. Math. 55(2) (2018) 325–346.
[25] Manfio F., Turgay N.C. and Upadhyay A.: Biconservative submanifolds in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, J. Geom. Anal. (in press).
[26] Montaldo S., Oniciuc C. and Ratto A.: Proper biconservative immersions into the Euclidean space, Ann. Mat. Pura Appl. 195(2) (2016) 403–422.
[27] Montaldo S., Oniciuc C. and Ratto A.: Biconservative surfaces, J. Geom. Anal. 26(1) (2016) 313–329.
[28] Turgay N. C.: $H$-hypersurfaces with three distinct principal curvatures in the Euclidean spaces, Ann. Mat. Pura Appl. 194 (6) (2015) 1795–1807.

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