Evolution of non-compact hypersurfaces by inverse mean curvature

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Abstract

We study the evolution of complete non-compact convex hypersurfaces in $\mathbb{R}^{n+1}$ by the inverse mean curvature flow. We establish the long time existence of solutions and provide the characterization of the maximal time of existence in terms of the tangent cone at infinity of the initial hypersurface. Our proof is based on an a’priori pointwise estimate on the mean curvature of the solution from below in terms of the aperture of a supporting cone at infinity. The strict convexity of convex solutions is shown by means of viscosity solutions. Our methods also give an alternative proof of the result by Huisken and Ilmanen in [24] on compact start-shaped solutions, based on maximum principle argument.

1 Introduction

A one-parameter family of immersions $F : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth complete solution to the inverse mean curvature flow (IMCF) in $\mathbb{R}^{n+1}$ if each $M_t := F(\cdot, t)(M^n)$ is a smooth strictly mean convex complete hypersurface satisfying

$$\frac{\partial}{\partial t} F(p, t) = H^{-1}(p, t) \nu(p, t)$$

where $H(p, t) > 0$ and $\nu(p, t)$ denote the mean curvature and exterior unit normal of $M_t$.

This flow has been extensively studied in the case of compact hypersurfaces. Gerhardt [16] and Urbas [29] showed that for smooth star-shaped compact initial hypersurface of strictly positive mean curvature, there is a unique smooth solution for all times $t > 0$. Moreover, the solution approaches to a homothetically expanding sphere as $t \to \infty$.

For non-starshaped initial data it is well known that singularities may develop (See [21] [28]). This happens when the mean curvature vanishes in some regions which makes the classical flow undefined. However, in [21, 22] Huisken and Ilmanen developed a level set approach to weak variational solutions of the flow which allows the solutions to jump outwards in possible regions where $H = 0$. Using the weak formulation, they gave the first proof of the Riemannian Penrose inequality in General Relativity. One key observation in [22] was the fact the Hawking mass of a 2d-surface in a 3-manifold of nonnegative scalar curvature is monotone under the weak flow, which was first discovered for classical solutions by Geroch [19]. Note that the Riemannian Penrose inequality was shown in more general settings by Bray [2] and Bray-Lee [3] by different methods. Using similar techniques, the IMCF has also been used to show geometric inequalities in various settings. For instance, see [20, 5] for Minkowski type inequalities, [25] for Penrose inequalities and [27, 12, 15] for Alexandrov-Fenchel type inequalities among other results. Note another important application of the flow by Bray and Neves in [1] as well.
In [24] Huisken and Ilmanen studied the higher regularity of solutions to the IMCF, for compact star-shaped weakly mean convex initial data of class $C^1$. Using star-shapedness and the ultra-fast diffusion character of the flow, they derive a bound from above on $H^{-1}$ for $t > 0$ which is independent of the initial curvature assumption. This follows by a Stampacchia iteration argument and utilizes the Michael-Simon Sobolev inequality. The $C^\infty$ regularity of solutions for $t > 0$ easily follows from the bound on $H^{-1}$. The estimate in [24] is local in time, but necessarily global in space as it depends on the area of the initial hypersurface $M_0$ and uses global integration on $M_t$. As a consequence of the techniques in [24] cannot be applied directly to the non-compact setting. Let us also note that the works [26] and [30] provide similar estimates on $H^{-1}$ for compact star-shaped solutions of the IMCF in some negatively curved ambient spaces. Their methods are similar to those in [24], however the proofs are simpler as a Sobolev type inequality and iteration techniques are unnecessary due to the negative curvature of the ambient space which helps to create good negative terms in the evolution equation of $H^{-1}$.

This work addresses the long time existence of non-compact smooth convex solutions $M_t$ to the IMCF embedded in Euclidean space $\mathbb{R}^{n+1}$. While extrinsic geometric flows have been extensively studied in the case of compact hypersurfaces, much remains to be investigated for non-compact cases. The important works by K. Ecker and G. Huisken [13, 14] address the evolution of entire graphs by mean curvature flow and establish a surprising result: existence for all $0 < t < +\infty$ with the only assumption that the initial data $M_0$ is a locally Lipschitz entire graph and no assumption of the growth at infinity of $M_0$. This result is based on priori estimates which are localized in space. In addition, the main local bound on the second fundamental form $|A|^2$ of $M_t$ is achieved without any bound assumption on $|A|^2$ on $M_0$. An open question between experts in the field has been whether the techniques of Ecker and Huisken in [13, 14] can be extended to the fully-nonlinear setting, in particular on entire convex graphs evolving by the $\alpha$-Gauss curvature flow (powers $K^\alpha$ of the Gaussian curvature) and the inverse mean curvature flow. Note that Gauss curvature flow is an example of degenerate diffusion while the inverse mean curvature flow is the opposite, an example of ultra-fast diffusion.

In [8] the second author, jointly with Kyeongsu Choi, Lami Kim and Kiahm Lee, established the long time existence of the $\alpha$-Gauss curvature flow on any strictly convex complete non-compact hypersurface and for any $\alpha > 0$. They showed that similar estimates as in [13, 14] which are localized in space can be obtained for this flow, however the methods are more involved due to the degenerate and fully-nonlinear character of the Monge-Amperé type of equation involved. However, such localized results are not expected to hold for the inverse mean curvature flow where the ultra-fast diffusion tends to cause instant propagation from spatial infinity. In fact, one sees certain similarities between the latter two flows and the well known quasilinear models of diffusion on $\mathbb{R}^n$

$$u_t = \text{div}(u^{m-1}\nabla u).$$

(1.2)

Exponents $m > 1$ correspond to degenerate diffusion while exponents $m < 0$ to ultra-fast diffusion. We will see in the sequel that under the IMCF the mean curvature $H$ satisfies an equation which is similar to (1.2) with $m = -1$. Our goal in this work is to study this phenomenon and establish the long time existence of complete non-compact convex hypersurfaces, the analogue of the results in [13, 14] and [8].

We will next state our main result in this work. The following observation motivates the formulation of our theorem.

We will next state our main result in this work. The following observation motivates the formulation of our theorem.
Example 1.1 (Conical solutions of IMCF). For a solution of the IMCF $\Gamma_t$ in $\mathbb{S}^n$, the family of cones generated by $\Gamma_t$

$$\mathcal{C}\Gamma_t := \{rx \in \mathbb{R}^{n+1} : r \geq 0, x \in \Gamma_t\}$$

is a solution of the IMCF in $\mathbb{R}^{n+1}$ which is smooth except from the origin. When $\Gamma_0^{n-1} \subset \mathbb{S}^n$ is a smooth strictly convex hypersurface, the results of Gerhardt [18] and Makowski-Scheuer [27] show that there exists a unique solution $\Gamma_t \subset \mathbb{S}^n$ of the IMCF in $\mathbb{S}^n$ with initial data $\Gamma_0^{n-1}$, which exists for time $t \in [0, T)$ with $T < \infty$ and converges to an equator, as $t \to T$. Moreover one can explicitly compute using the exponential growth of area with respect to time that

$$T = \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0|.$$ 

From Example 1.1 and the ultra-fast diffusive character of the equation, it is reasonable to guess that for a general convex non-compact solution with initial data $M_0$, its existence time is governed by the asymptotics at infinity. For a non-compact convex set $\hat{M}_0$ and the associated hypersurface $M_0 = \partial \hat{M}_0$, we recall the definition of the blow-down, so called the tangent cone at infinity.

Definition 1.1 (Tangent cone at infinity). Let $\hat{M}_0 \subset \mathbb{R}^{n+1}$ be a non-compact closed convex set. For a point $p \in \hat{M}_0$, we denote the tangent cone of $\hat{M}_0$ at infinity by

$$\hat{C}_0 := \cap_{\lambda > 0} \lambda(\hat{M}_0 - p).$$

We also define $C_0 := \partial \hat{C}_0$, $\hat{\Gamma}_0 := \hat{C}_0 \cap \mathbb{S}^n$, $\Gamma_0 := C_0 \cap \mathbb{S}^n$. The definition is independent of $p \in \hat{M}_0$. We say $\hat{C}_0$, $C_0$ the tangent cone of $\hat{M}_0$ and $M_0 = \partial \hat{M}_0$ at infinity, respectively. We say $\hat{\Gamma}_0$ and $\Gamma_0$ the link of $\hat{C}_0$ and $C_0$, but we will also often call them as the tangent cone at infinity.

Our main result below establishes the long time existence of the IMCF for any complete non-compact convex $C^1_1 \text{loc}$ initial data $M_0$ and determines its maximum time of existence $T$ of the solution in terms of the size of the tangent cone at infinity $\Gamma_0$.

Theorem 1.2. For $n \geq 2$, let $M_0^n = \partial \hat{M}_0$ be a convex non-compact embedded $C^1_1 \text{loc}$ hypersurface in $\mathbb{R}^{n+1}$. Then, there is a smooth convex solution of the IMCF, say $\{M_t\}_{t \in (0, T)}$, which converges to
\( M_0 \) locally uniformly as \( t \to 0 \). The time of existence is given in terms of the link of tangent cone of \( \hat{M}_0 \) at infinity, say \( \hat{\Gamma}_0 \subset S^n \), by

\[
T = \ln |S^{n-1}| - \ln P(\hat{\Gamma}_0) \in [0, \infty].
\]

(1.3)

Here, \(| \cdot | := H^{n-1}(\cdot) \) and \( P(\hat{\Gamma}) := \) the perimeter of a convex set \( \hat{\Gamma} \) in \( S^n \). The solution is strictly convex when \( \hat{\Gamma}_0 \subset S^{n-1} \) is compactly included in an open hemisphere.

**Remark 1.2.** Under our assumption of \( M_0 \), \( \hat{\Gamma}_0 \) can be an arbitrary convex set in \( S^n \). For a convex set \( \hat{\Gamma}_0 \subset S^n \) and \( \Gamma_0 = \partial \hat{\Gamma}_0 \), note that

\[
P(\hat{\Gamma}_0) = \begin{cases} |\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has non-empty interior in } S^n \\ 2|\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has empty interior in } S^n. \end{cases}
\]

Moreover if \( M_t \) evolves by IMCF then its tangent cone at infinity \( \Gamma_t \), evolves by IMCF on \( S^n \) in some generalized sense and becomes flat as \( t \to T \). See Remark [4.1] for this.

Finally, formula (1.3) says \( T = 0 \) when \( P(\hat{\Gamma}_0) = |S^{n-1}| \). In [6], it was shown that for a convex set \( \hat{\Gamma}_0 \subset S^n \) if \( P(\hat{\Gamma}_0) = |S^{n-1}| \) then \( \hat{\Gamma}_0 \) is either a hemisphere or a wedge

\[
\hat{W}_{\theta_0} = S^n \cap \{(r \sin \theta, r \cos \theta) : \theta \in [0, \theta_0], \text{ and } r > 0 \} \times \mathbb{R}^{n-1}
\]

for some \( \theta_0 \in [0, \pi] \), up to an isometry of \( S^n \). According to the formula, \( T = \infty \) when \( P(\hat{\Gamma}_0) = 0 \), which happens when the cone degenerates and it is lower dimensional.

**Remark 1.3.** Let us emphasize that Theorem 1.2 allow \( H = 0 \) on a possibly non-compact region of \( M_0 \) and in that case \( H > 0 \) instantly for \( t > 0 \) provided \( T = T(M_0) > 0 \). This is possible due to our main apriori estimate Theorem 1.4. Note that the similar phenomenon was observed for solutions to the Cauchy problem on \( \mathbb{R}^n \) of the ultra-fast diffusion equation (1.2) with \( m < 0 \). (See Remark 4.2 for the details.)

Next, we show \( T = T(M_0) \) in Theorem 1.2 is the maximal time of existence. The following theorem holds not only for the solutions of our constructions, but applies to arbitrary solutions.

**Theorem 1.3.** Let \( M_0 = \partial \hat{M}_0 \) satisfy the same assumptions as in Theorem 1.2 and \( T = T(M_0) \in [0, \infty] \) be given by the formula (1.3). If \( T < \infty \), then no smooth solution \( M_t \), which locally uniformly converges to \( M_0 \) as \( t \to 0^+ \), can be defined beyond \( t > T(M_0) \). In particular, this implies non-existence of a smooth solution when \( T(M_0) = 0 \).

Non-compact solutions of the IMCF in \( \mathbb{R}^{n+1} \) were first considered by the second author and G. Huisken in [11], where they established the existence and uniqueness of a smooth solution to the IMCF, under the assumption that the initial hypersurface \( M_0 \) is an entire \( C^2 \) graph, \( x_{n+1} = u_0(x') \) with \( H > 0 \), in the following two cases:

1. \( M_0 \) has super linear growth at infinity and it is strictly star-shaped, that is \( H(F-x_0, \nu) \geq \delta > 0 \) holds, for some \( x_0 \in \mathbb{R}^{n+1} \);
2. \( M_0 \) a convex graph satisfying \( 0 < c_0 \leq H(F-x_0, e_{n+1}) \leq C_0 < +\infty \), for some \( x_0 \in \mathbb{R}^{n+1} \) and lies between two round cones of the same aperture, that is

\[
\alpha_0 |x'| \leq u_0(x') \leq \alpha_0 |x'| + k, \quad \alpha_0 > 0, k > 0.
\]

(1.4)
In the first case, a unique smooth solution exists up to time $T = +\infty$, while in the second case a unique smooth convex solution $M_t$ exists for $t \in [0, T]$ where $T = T(\alpha_0) > 0$ is the exact time when an evolving cone solution of the IMCF $\{x_{n+1} = \alpha(t)|x'|\}$, with $\alpha(0) = \alpha_0$ becomes flat (i.e. $\alpha(t) \to 0$). In the latter case, the solution $M_t$ lies between two evolving round cones and becomes flat as $t \to T$. To derive a local lower bound of $H$ up to $t < T$, a parabolic Moser’s iteration argument was used in [11] along with a variant of Hardy’s inequality, which plays a similar role as the Micheal-Simon Sobolev inequality in [24].

Theorem 1.2 and the results in [11] show that convex surfaces with linear growth at infinity have critical behavior in the sense that in this case the maximal time of existence is finite and it depends on the behavior at infinity of the initial data. However, while the techniques in [11] only treat this critical linear case under the condition (1.4), Theorem 1.2 allows any behavior at infinity. Moreover, the techniques in [11] require to assume that $H$ is globally controlled from below a initial time, namely that $H \langle F - x_0, e_n \rangle \geq \delta > 0$ in the case of super-linear growth and $H \langle F - x_0, e_{n+1} \rangle \geq c > 0$ in the case of linear growth.

In this work we depart from the techniques in [11] and [24] and establish a priori $L^\infty$ bound on $H^{-1}$ which is local in time. In this attempt, we develop a new method based on the maximum principle rather than the integration methods used in [11] and [24]. Our key estimate is the following bound on $H^{-1}$ which roughly says that one has a global bound on $(H|F|)^{-1}$ as long as a nontrivial convex cone is supporting our surface from outside.

**Theorem 1.4.** Let $F : M^n \times [0, T] \to \mathbb{R}^{n+1}$, $n \geq 2$, $T > 0$, be a smooth convex closed solution of the IMCF and suppose there is $\theta_1 \in (0, \pi/2)$ for which

$$\langle F, e_{n+1} \rangle \geq \sin \theta_1 |F| \quad \text{on} \quad M^n \times [0, T].$$

Then

$$\frac{1}{H|F|} \leq C \left(1 + \frac{1}{t^{1/2}}\right) \quad \text{on} \quad M^n \times [0, T]$$

for a constant $C = C(\theta_1) > 0$.

Let us note that the assumption that $M_t$ is a closed hypersurface will only be used to apply maximum principle and will not affect the application of the estimate in proving of our main non-compact result, Theorem 1.2, as we will approximate non-compact solutions by closed ones. Also, let us emphasize that our bound is independent on an initial upper bound on $(H|F|)^{-1}$. This will allow non-compact initial data to have flat regions where $H = 0$. In addition to the non-compact results stated above, our new methods lead to an equivalent estimate of the result by Ilmanen and Huisken, Theorem 1.1 in [24], for compact, star-shaped (not necessarily convex) solutions. This is included in Theorem A.5 in the appendix. In fact, one expects that similar estimates as in Theorem A.5 can be possibly derived for the IMCF in other ambient spaces, including some positively curved spaces or asymptotically flat spaces, using this new method and this generalize the results of [24, 26, 30]. See in [23] for a consequence of such an estimate when this is shown in asymptotically flat ambient spaces.

**Remark 1.4.** Recently, the first author and P.-K. Hung in [6] addressed the IMCF on convex solutions allowing singularities on $M_0$. Using our main estimate Theorem 1.4 as a key ingredient, it was shown in [6] that the limiting tangent cone after blowing-up at a singularity evolves by the IMCF. As a corollary of this, one can consider an arbitrary non-compact convex hypersurface $M_0$ in Theorem 1.2 and obtain the following necessary and sufficient condition for existence of a smooth
solution: for an arbitrary non-compact convex \( M_0 \) with \( T(M_0) > 0 \), there is a smooth solution if and only if \( M_0 \) has density one everywhere, i.e. \( \Theta_0(p) = \lim_{r \to 0} \frac{|B_r(p) \cap M_0|}{\omega_n r^n} = 1 \) for all \( p \in M_0 \). See [6] for more details.

A brief outline of this paper is as follows: In Section 2, we introduce basic notation, evolution equations of basic geometric quantities, and prove some identities which will be useful in the upcoming sections. Section 3 is devoted to the proof of our main a priori estimate Theorem 1.4. Only assuming that the solution stays above a round cone, the estimate shows a uniform bound of \( (H|F|)^{-1} \), for \( t > 0 \), which is independent of the initial bound. In Section 4, we prove the long time existence theorem of non-compact convex solution via an approximation argument that uses a priori estimates in Section 3. In Appendix A.1, we prove the convexity of solution is preserved and show the solution become strictly convex immediately for \( t > 0 \) unless the lowest principle curvature \( \lambda_1 \) is zero everywhere initially. This will be shown for the solutions of the IMCF in a space of constant sectional curvature as this adds no difficulty in the proof but could be useful in other application. Finally in Appendix A.2, we give an alternative proof of a priori \( H^{-1} \) estimate shown in [24] using our maximum principle argument. This is to show how star-shapedness condition can also be incorporated in our method.

2 Preliminaries and Notation

Let \( \nabla := \nabla^{g(t)} \) and \( \Delta := \Delta^{g(t)} \) denote the connection and Laplacian on \( M^n \) with respect to the induced metric \( g_{ij}(t) = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \rangle \). Recall that on a local system of coordinates \( \{ x^i \} \) on \( M^n \),

\[
\frac{\partial^2 F}{\partial x^i \partial x^j} = -h_{ij} \nu + \Gamma^k_{ij} \frac{\partial F}{\partial x^k} \quad \text{and} \quad \left\langle \frac{\partial F}{\partial x^j}, \frac{\partial \nu}{\partial x^i} \right\rangle = h_{ij}
\tag{2.1}
\]

where \( \nu \) denotes the exterior unit normal. We also define the operator

\[
\Box := \left( \partial_t - \frac{1}{H^2} \Delta \right)
\]

and use it frequently as this is the linearized operator of the IMCF.

Note that the IMCF or generally curvature flows of homogeneous degree \(-1\), have the following scaling property which can be directly checked and will be frequently used:

**Lemma 2.1** (Scaling of IMCF). If \( M_t^\mu \subset \mathbb{R}^{n+1} \) is a solution of the IMCF, then \( \tilde{M}_t^\mu = \lambda M_t^\mu \) is again a solution for \( \lambda > 0 \).

**Lemma 2.2** (Huisken, Ilmanen [24]). Any smooth solution of the IMCF \((1.1)\) in \( \mathbb{R}^{n+1} \) satisfies

1. \( \partial_t g_{ij} = \frac{2}{H} h_{ij} \)
2. \( \partial_t d\mu = d\mu, \) where \( d\mu \) is the volume form induced from \( g_{ij} \)
3. \( \partial_t \nu = -\nabla H^{-1} = \frac{1}{H^2} \nabla H \)
4. \( (\partial_t - \frac{1}{H^2} \Delta) h_{ij} = -\frac{2}{H^2} \nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij} \)
5. \( \partial_t H = \nabla_i \left( \frac{1}{H^2} \nabla_i H \right) - \frac{|A|^2}{H^2} = \frac{1}{H^2} \Delta H - \frac{2}{H^2} |\nabla H|^2 - \frac{|A|^2}{H^2} \)
(6) \((\partial_t - \frac{1}{H^2} \Delta) H^{-1} = \frac{|A|^2}{H^2} H^{-1}\)

(7) \((\partial_t - \frac{1}{H^2} \Delta) \langle F - x_0, \nu \rangle = \frac{|A|^2}{H^2} \langle F - x_0, \nu \rangle.\)

Remark 2.1. If the ambient space is not \(\mathbb{R}^{n+1}\), then the evolution equations of \(g_{ij}, d\mu,\) and \(\nu\) remain the same as in \(\mathbb{R}^{n+1}\), but the evolution of curvature \(h_{ij}\) is different and complicated. On a space form of sectional curvature \(K\), the formula hugely simplifies becoming

\[
\partial_t h_{ij} = \frac{1}{H^2} \Delta h_{ij} + \frac{|A|^2}{H^2} h_{ij} - \frac{2}{H^3} \nabla_i H \nabla_j H - \frac{nK h_{ij}}{H^2} \tag{2.2}
\]

(See Chapter 2 in [17,]). In this paper we will mostly focus on the flow in Euclidean space and we will only use (2.2) in Appendix A.1.

Using Lemma 2.2, one can easily deduce the following formulas.

Lemma 2.3. For a fixed vector \(\omega\) in \(\mathbb{R}^{n+1}\), the smooth solutions of the IMCF (1.1) in \(\mathbb{R}^{n+1}\) satisfy

(1) \((\partial_t - \frac{1}{H^2} \Delta) |F - x_0|^2 = -\frac{2n}{H^2} + \frac{4}{H} \langle F - x_0, \nu \rangle\)

(2) \((\partial_t - \frac{1}{H^2} \Delta) \langle \omega, \nu \rangle = \frac{|A|^2}{H^2} \langle \omega, \nu \rangle\)

(3) \((\partial_t - \frac{1}{H^2} \Delta) \langle \omega, F - x_0 \rangle = \frac{2}{H} \langle \omega, \nu \rangle.\)

Proof. By (2.1) we have

\[\Delta F = g^{ij} (\partial^2_{ij} F - \Gamma^k_{ij} F_k) = g^{ij} (-h_{ij} \nu + \Gamma^k_{ij} F_k - \Gamma^k_{ij} F_k) = -H \nu.\]

which combined with \(\partial_t F = H^{-1} \nu\) implies (3). Next,

\[\Delta |F - x_0|^2 = 2\langle \Delta F, F - x_0 \rangle + 2\langle \nabla F, \nabla F \rangle = 2H \langle \nu, F - x_0 \rangle + 2n\]

implies (1). Finally,

\[\Delta \nu = g^{ij} (\partial^2_{ij} \nu - \Gamma^k_{ij} \partial_k \nu) = g^{ij} (\partial_j (h^k_{ij} F_k) - \Gamma^k_{ij} h^l_{kl} F_l)\]

\[= g^{ij} ((\partial_j h^k_{ij}) F_k - h^k_{ijk} \nu + \Gamma^l_{jk} h^k_{il} F_l - \Gamma^l_{ik} h^k_{jl} F_l)\]

\[= -|A|^2 \nu + g^{ij} \nabla_j h^k_{ij} F_k = -|A|^2 \nu + \nabla H\]

where we used the Codazzi identity in the last equation. This implies (2). \(\square\)

The following simple lemma, which commonly appears in Pogorelov type computations, will be useful in the sequel when we compute the evolution of products.

Lemma 2.4. For any \(C^2\) functions \(f_i(p, t), i = 1, \ldots, m,\) denote

\[w := f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}.\]

Then on the region where \(w \neq 0\), we have

\[
(\partial_t - \frac{1}{H^2} \Delta) \ln |w| = \frac{(\partial_t - H^{-2} \Delta) w}{w} + \frac{1}{H^2} \frac{\nabla |w|^2}{w^2} = \sum_{i=1}^m \alpha_i \left( \frac{(\partial_t - H^{-2} \Delta) f_i}{f_i} + \frac{1}{H^2} \frac{\nabla f_i^2}{f_i^2} \right). \tag{2.3}
\]
Proof. The lemma simply follows from
\[
(\partial_t - \frac{1}{H^2} \Delta) \log |f| = \frac{(\partial_t - H^{-2} \Delta)f}{f} + \frac{1}{H^2} \frac{\nabla f}{f^2}.
\] (2.4)

Next two lemmas are straightforward computations and we leave their proofs for readers.

Lemma 2.5. For any two $C^2$ functions $f, g$ defined on $M^n \times (0, T)$ and any $C^2$ function $\psi : \mathbb{R} \to \mathbb{R},$
\[
\Box (fg) = (\Box f) g + f(\Box g) - \frac{2}{H^2} \langle \nabla f, \nabla g \rangle
\]
and
\[
\Box \psi(f) = \psi'(f) \Box \psi(f) - \frac{\psi''(f)}{H^2} |\nabla f|^2
\]
where $\Box := (\partial_t - H^{-2} \Delta).$

Lemma 2.6. If a $C^2$ function $f$ is defined on a solution $M_t$ of the IMCF and satisfies
\[
\left( \frac{\partial}{\partial t} - \frac{1}{H^2} \Delta \right) f = \frac{|A|^2}{H^2} f
\]
then for any fixed $\beta \in \mathbb{R}$ we have
\[
\left( \frac{\partial}{\partial t} - \frac{1}{H^2} \Delta \right) f^\beta = \beta \frac{|A|^2}{H^2} f^\beta - \frac{\beta(\beta - 1)}{\beta^2} \frac{\nabla f^\beta}{H^2} |\nabla|^2 f^\beta.
\]
For instance, $H^{-1}, \langle \omega, \nu \rangle$ and $\langle F - x_0, \nu \rangle$ are examples of such a function $f.$

We finish with the following local estimate which is an easy consequence of Proposition 2.11 in [11]. Here $B_r$ denotes an extrinsic ball of radius $r > 0$ in $\mathbb{R}^{n+1}.$

Proposition 2.7 (Proposition 2.11 [11]). For a solution $M_t, t \in [0, T]$ of the IMCF, there is a constant $C_n > 0$ such that
\[
\sup_{M_t \cap B_r} H \leq C_n \max \left( \sup_{M_0 \cap B_2r} H, r^{-1} \right).
\]

3 $L^\infty$ bound of $\left( H|F| \right)^{-1}$

In this section, we give the proof of Theorem 1.4 which gives the main a priori estimate on which our main existence result Theorem 1.2 is based upon. Let us first introduce some standard notation. We consider spherical coordinates with respect to the origin in $\mathbb{R}^{n+1},$ namely
\[
x = (x_1, \ldots, x_{n+1}) = (r \omega \sin \theta, r \cos \theta) \quad \text{with } r \geq 0, \, \omega \in S^{n-1}, \, \text{and } \theta \in [0, \pi]
\]
which are smoothly well-defined except from the origin or $x_{n+1}$-axis. We will also denote by $\bar{\nabla}$ and $\nabla$ metric-induced connections on $(\mathbb{R}^{n+1}, g_{euc})$ and $(M^n, F^*g_{euc})$ respectively. Before the proof, we need the evolution equation of the important quantity $\theta,$ defined in the ambient space as follows:
Definition 3.1. We define

$$\theta : \mathbb{R}^{n+1} \setminus \{0\} \to [0, \pi] \quad \text{by} \quad \theta(x) := \arccos \left( \frac{x, e_{n+1}}{|x|} \right)$$  \hspace{1cm} (3.1)

and

$$r : \mathbb{R}^{n+1} \to [0, \infty) \quad \text{by} \quad r(x) := |x|.$$  
Moreover, we define smooth unit orthogonal vector fields

$$e_\theta(x) = e_\theta(x', x_{n+1}) := \frac{1}{|x|} \frac{\partial}{\partial \theta} = \left( \frac{x' \cos \theta}{|x| \sin \theta}, -\sin \theta \right) \quad \text{on} \quad \mathbb{R}^{n+1} \setminus \{x_{n+1} = 0\}$$

and

$$e_r(x) := \frac{\partial}{\partial r} = \frac{x}{|x|} \quad \text{on} \quad \mathbb{R}^{n+1} \setminus \{0\}.$$  

Though \( \theta \) is not smooth at the points on the \( x_{n+1} \)-axis, note that \( \theta^2, \cos \theta, \) and \( \sec \theta \) are all smooth on \( \{x_{n+1} > 0\} \).

Lemma 3.2. On the region \( \{\theta \neq 0, \pi\} \cap \{|x| \neq 0\} \),

$$(\partial_t - \frac{1}{H^2} \Delta) \theta = -\frac{1}{H^2 r^2} \left( n - |\nabla r|^2 \right) + \frac{1}{H^2} |\nabla \theta|^2 + \frac{2}{H^2} \left( \nabla_r \cdot \nabla \theta \right) + \frac{2}{H} \langle \nu, \nabla \theta \rangle.$$  

Proof. Consider a spherical coordinate chart

\( (r, \theta, (w^\alpha)_{\alpha=1 \ldots n-1}) \) with \( r > 0, \ \theta \in (0, \pi) \), \( (w^\alpha) \in S^{n-1} \)

around a point \( \{\theta \neq 0, \pi\} \cap \{|x| \neq 0\} \) in \( \mathbb{R}^{n+1} \), where \( w^\alpha \) is a coordinate chart of \( S^{n-1} \). On this chart,

$$g_{\text{euc}} = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \sigma_{\alpha\beta} dw^\alpha dw^\beta.$$  \hspace{1cm} (3.2)

Also note that

$$\nabla \theta = \frac{1}{r^2} \frac{\partial}{\partial \theta} = \frac{1}{r} e_\theta \quad \text{and} \quad \nabla r = \frac{\partial}{\partial r} = e_r \quad \text{on} \quad (\mathbb{R}^{n+1}, g_{\text{euc}}).$$  \hspace{1cm} (3.3)

At a given \( p \in M^n \) with \( \{\theta \neq 0, \pi\} \cap \{|x| \neq 0\} \), let us choose a geodesic normal coordinate of \( M^n \), say \( \{y^i\}_{i=1}^n \). In this coordinate at this point,

$$\Delta \theta = \partial_i \partial_i \theta = \frac{\partial}{\partial y^i} d\theta \left( \frac{\partial}{\partial y^i} \right) = \frac{\partial}{\partial y^i} \left( \frac{1}{r^2} \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial y^i} \right)$$

$$= -\frac{2}{r^2} \langle \frac{\partial}{\partial y^i}, e_\theta \rangle \langle \frac{\partial}{\partial y^i}, e_r \rangle + \frac{1}{r^2} \langle \nabla \partial_i \partial_\theta, \frac{\partial}{\partial y^i} \rangle + \langle \frac{\partial}{\partial \theta}, -h_{ii} \nu \rangle.$$  

Since \( \left( \frac{\partial}{\partial y^i} \right)_{i=1}^n, \nu \) constitutes an orthonormal basis of \( T_{F(p)} \mathbb{R}^{n+1} \),

$$\langle \frac{\partial}{\partial y^i}, e_\theta \rangle \langle \frac{\partial}{\partial y^i}, e_r \rangle + \langle \nu, e_r \rangle \langle \nu, e_\theta \rangle = \langle e_r, e_\theta \rangle = 0.$$  \hspace{1cm} (3.4)

Therefore,

$$\Delta \theta = -\frac{H}{r} \langle \nu, e_\theta \rangle + \frac{2}{r^2} \langle \nu, e_r \rangle \langle \nu, e_\theta \rangle + \frac{1}{r^2} \sum_i \langle \nabla \partial_i \partial_\theta, \frac{\partial}{\partial y^i} \rangle.$$  \hspace{1cm} (3.5)
Claim 3.1.
\[ \sum_{i=1}^{n} \langle \nabla \partial_i, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y^i} \rangle = \frac{\cos \theta}{\sin \theta} \left( n - (1 - \langle \nu, e_r \rangle^2) - (1 - \langle \nu, e_\theta \rangle^2) \right). \]  
(3.6)

Proof of Claim 3.1. By computing the Christoffel symbols from the metric (3.2), we get:
\begin{align*}
\nabla_{\partial} \frac{\partial}{\partial \theta} = -r \frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \nabla_{\partial r} \frac{\partial}{\partial \theta} = \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta},
\end{align*}
(3.7)
Suppose \( \partial_i = \partial_{y^i} = a_\theta \partial_\theta + a_r \partial_r + \sum_{\alpha} a_\alpha \partial_{w^\alpha} \). Then \( \nabla_{\partial_i} \frac{\partial}{\partial \theta} = -r a_\theta \partial_r + a_\alpha \partial_\theta + \sum_{\alpha} a_\alpha \frac{\cos \theta}{\sin \theta} \partial_{w^\alpha} \) and hence
\[ \langle \nabla_{\partial_i}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y^i} \rangle = -r a_\theta a_r + ra_\alpha a_\beta + \sum_{\alpha} a_\alpha \frac{\cos \theta}{\sin \theta} \partial_{w^\alpha} \]

The claim follows by summing this over \( i \).

Now \( \partial_t \theta = d\theta(\partial_t F) = \frac{1}{H}(\nu, \text{grad } \theta) = \frac{\langle \nu, e_\theta \rangle}{r H} \). (3.5) and (3.6) imply
\[ (\partial_t - \frac{1}{H^2} \Delta) \theta = 2 \frac{(\nu, e_\theta)}{r H} - \frac{1}{(r H)^2} \left[ \frac{\cos \theta}{\sin \theta} \left[ n - (1 - \langle \nu, e_r \rangle^2) - (1 - \langle \nu, e_\theta \rangle^2) \right] + 2 \langle \nu, e_r \rangle \langle \nu, e_\theta \rangle \right]. \]

Hence, the lemma follows by using (3.3) and the orthonormality of \( \left( \frac{\partial}{\partial y^i} \right)_{i=1}^n, \nu \) in the equation above.

Proof of Theorem 1.4. Using the definition (3.1), our condition (1.5) can be written as \( \theta(p, t) \leq \pi/2 - \theta_1 \). Setting \( c := \frac{\pi - \theta_1}{\pi - 2\theta_1} > 1 \), we have \( c \theta \leq \frac{\pi}{2} - \frac{\theta_1}{2} < \frac{\pi}{2} \) and \( \sec(c \theta) \leq 2 \sec \theta \) for \( \theta = \theta(p, t) \) on \( t \in [0, T] \).

By lemma 2.5
\[ \square c \theta = c \sec(c \theta) \tan(c \theta) \square \theta - \frac{1}{H^2} c^2 [\sec(c \theta) \tan^2(c \theta) + \sec^3(c \theta)] |\nabla \theta|^2 \]
\[ = \sec(c \theta) \left[ c \tan(c \theta) \square \theta - \frac{c^2}{H^2} (2 \tan^2(c \theta) + 1) |\nabla \theta|^2 \right]. \]

After defining \( \varphi := \sec(c \theta) \), Lemma 3.2 and \( \nabla \varphi = c \tan(c \theta) \nabla \theta \) imply
\[ \square \varphi = -\frac{c}{H^2 r^2} \frac{\tan(c \theta)}{\tan \theta} (n - |\nabla r|^2 - r^2 |\nabla \theta|^2) + \frac{2}{H^2} \left( \frac{\nabla r}{r}, \frac{\nabla \varphi}{\varphi} \right) + \frac{2}{H} \langle \nu, \frac{\nabla \varphi}{\varphi} \rangle \]
\begin{align*}
&= -\frac{2}{H^2} |\nabla \varphi|^2 - \frac{1}{H^2} c^2 |\nabla \theta|^2 \\
&= \left( \text{since } n - |\nabla r|^2 - r^2 |\nabla \theta|^2 = n - 2 \geq 0 \text{ and } \frac{\tan(c \theta)}{\tan \theta} \geq c \right) \\
&\leq -\frac{c^2}{H^2 r^2} (n - |\nabla r|^2) + \frac{2}{H^2} |\nabla \varphi|^2 + \frac{2}{H^2} \left( \frac{\nabla r}{r}, \frac{\nabla \varphi}{\varphi} \right) + \frac{2}{H} \langle \nu, \frac{\nabla \varphi}{\varphi} \rangle.
\end{align*}
Note that this inequality holds on \( \{x_{n+1} > 0\} \), where our solution \( M_t \) is located. Let \( w := \frac{\sec(c \theta) t}{Hr} = \varphi \psi r^{-1} t \) where \( \psi := H^{-1} \). Then by Lemma 2.4 and the previous inequality

\[
\Box w - \frac{1}{H^2} \frac{\nabla w^2}{w^2} = \left( \frac{\Box \varphi}{\varphi} + \frac{1}{H^2} \frac{\nabla \varphi^2}{\varphi^2} \right) + \left( \frac{|A|^2}{H^2} + \frac{1}{H^2} \frac{\nabla |\psi|^2}{\psi^2} \right) - \frac{1}{2} \left( \frac{\Box r^2}{r^2} + \frac{1}{H^2} \frac{\nabla r^2}{r^4} \right) + \frac{1}{t} \tag{3.8} \]

\[
\leq \left[ \frac{|A|^2}{H^2} + \frac{1}{t} + \frac{2}{H} \langle \nu, \frac{\nabla \varphi}{r} \rangle - \frac{2}{H} \langle \nu, \frac{\nabla r}{r} \rangle \right] \nabla r^2 - \frac{2}{r^2} - c^2 \frac{n - |\nabla r|^2}{r^2} - \frac{|\nabla \varphi|^2}{\varphi^2} + 2 \left( \frac{\nabla r}{r}, \frac{\nabla \varphi}{r} \right) \]

\[
= : (1) + (2). \]

Suppose a nonzero maximum of \( w(p, t) \) on \( M^n \times [0, t_1] \) is achieved at \( (p_0, t_0) \) with \( t_0 \in (0, t_1] \). At this point,

\[
0 = \nabla w = \frac{\nabla |\psi|^2}{\psi^2} + \frac{\nabla \varphi}{\varphi} - \frac{\nabla r}{r}
\]

and therefore

\[
\frac{|\nabla |\psi|^2|^2}{\psi^2} = \frac{|\nabla r|^2}{r^2} - \frac{\nabla \varphi}{\varphi} = \frac{|\nabla r|^2}{r^2} + \frac{|\nabla \varphi|^2}{\varphi^2} - 2 \left( \frac{\nabla r}{r}, \frac{\nabla \varphi}{r} \right). \]

At the maximum point, by plugging this into (2) in (3.8), \( (2) = -(c^2 - 1) \frac{n - |\nabla r|^2}{H^2 r^2} \). Therefore at the maximum point,

\[
0 \leq (1) - (c^2 - 1) \frac{n - |\nabla r|^2}{H^2 r^2}. \]

Let us estimate terms in (1). Note that by our choice of \( c > 1 \),

\[
|\frac{\nabla \varphi}{\psi}| = |c \tan(c \theta) \nabla \theta| \leq c \frac{\tan(c \theta)}{r} \leq \frac{1}{r} \sin(c \theta) \sec(c \theta) \leq \frac{2}{r \sin \theta} \leq \frac{1}{r} = C \]

for some \( C = C(\theta_1) \). Next, \( \frac{|A|^2}{H^2} \leq 1 \) from convexity and \( |\nabla r| \leq 1 \) imply at \( (p_0, t_0) \),

\[
0 \leq -(n - |\nabla r|^2) \frac{c - 1}{H^2 r^2} + \frac{C}{Hr} + 1 + \frac{1}{t_0}.
\]

\[
\leq - \frac{c - 1}{H^2 r^2} + \frac{C}{Hr} + 1 + \frac{1}{t_0} \quad \text{(since } |\nabla r|^2 \leq 1, n \geq 2) \]

\[
\leq - \frac{c - 1}{2H^2 r^2} + \frac{C}{2(c - 1)} + \frac{1}{t_0}. \]

Note that \( 0 < t_0 \leq t_1 \) and \( 1 \leq \varphi \leq C \) on \( M \times [0, t_1] \). Multiplication of \( (\varphi(p_0, t_0) t_0)^2 \) implies

\[
w^2(p_0, t_0) = \left( \frac{\varphi t_0}{Hr} \right)^2 \leq C(t_1 + 1). \]

On any other point \( p \in M \) at \( t = t_1 \),

\[
\frac{1}{H^2 r^2}(p, t_1) = \left( \frac{w(p, t_1)}{t_1 \varphi(p, t_1)} \right)^2 \leq \frac{w^2(p_0, t_0)}{t_1^2 \varphi^2(p, t_1)} \leq C \left( 1 + \frac{1}{t_1} \right). \]

We used \( \varphi \geq 1 \) in the last inequality. This finishes the proof of Theorem 1.4. \( \Box \)
Remark 3.1. If we define \( \tilde{w} := \varphi \psi r^{-1} \) and follow the rest similarly, we get an estimate which includes the initial bound
\[
\frac{1}{H|F|} \leq C \max \left( \sup_{M_0} \frac{1}{H|F|}, 1 \right).
\]

4 Long time existence of non-compact solutions

In this section we will give the proof of our main results in this work concerning the long time existence of non-compact solutions of the IMCF, Theorem 1.2 and Theorem 1.3 stated in the introduction. The proof of Theorem 1.2 is based on our main a priori estimate, Theorem 1.4 which provides an estimate of \((H |F|)^{-1}\) from above, in terms of the angle \(\theta\) of a supporting cone from outside. Since, this estimate holds for compact surfaces, we will first construct a family of compact convex approximating solutions \(M_{i,t} = \partial M_{i,t}\) which is monotone in \(i\). The results in [16] and [29] guarantee the existence of each compact expanding solution \(M_{i,t}\), for all \(t \in (0, +\infty)\). However, we will see that the limit \(M_t := \lim_{i \to +\infty} M_{i,t}\) is non-trivial only up to time \(T = T(M_0)\). In fact, the proof of Theorem 1.3 in this section shows that the limit \(M_t\) must a hyperplane in \(\mathbb{R}^{n+1}\) for \(t > T\), i.e. \(\bigcup_i M_{i,t} = \mathbb{R}^{n+1}\). Here is where the connection between our non-compact solution \(M_t\) in Euclidean space and solutions on the sphere is revealed. Recall the notation \(\Gamma_0 := C_0 \cap S^n\) of the link of the tangent cone \(C_0\) of \(M_0\) at infinity. For each time \(T - \delta < T(M_0)\), we are going to find smooth strictly convex \(\Gamma_0^\delta \subset S^n\) such that \(\tilde{\Gamma_0} \subset \Gamma_0^\delta\) and \(T_{\delta} := \ln |S^n| - \ln |\Gamma_0^\delta| > T - \delta\). In view of the results in [27] and [18], also described in Example 1.1, for each such \(\Gamma_0^\delta\) there is a smooth IMCF solution \(\Gamma_t^\delta \subset S^n\) which exists up to time \(T'\) and we can make use of \(C_{T_t}^\delta\) as an outer barrier for \(M_{i,t}\). Indeed, moving its vertex far away from \(M_0\) initially, we can make \(C_{T_t}^\delta\) (after an initial translation) to contain \(M_{i,t}\) up to time \(T_{\delta}\), implying that each \(M_{i,t}\) satisfies condition (1.5) in Theorem 1.4 up to time \(T - \delta\) for a uniform \(\theta_1 > 0\). Theorem 1.4 then leads to an upper bound on \((|F| H)^{-1}\) implying that the IMCF on \(M_{i,t}\) is locally uniformly parabolic and the rest is straightforward. We begin with Theorem 1.2

Proof of Theorem 1.2. Let \(M_0 = \partial \hat{M}_0\) satisfy the assumptions of our theorem and let \(C_0, \hat{C}_0\) be the tangent cones at infinity of \(M_0\), \(\bar{M}_0\) respectively and \(\Gamma_0 = C_0 \cap S^{n-1}, \hat{\Gamma}_0 = \hat{C}_0 \cap S^{n-1}\) their links. Assume that \(T\) given by (1.3) satisfies \(T > 0\), as there is nothing to prove for the case \(T = 0\). Note that, if \(\hat{M}_0\) contains an infinite straight line, then \(\hat{M}_0\) splits off in the direction of the line by an elementary convexity argument. By repeating this, \(\hat{M}_0 = \hat{N}_0 \times \mathbb{R}^k\) for some \(k \geq 0\) and we can assume \(\hat{N}_0\) does not contain any infinite lines. Also,
\[
T(M_0) = \ln |S^n| - \ln P(\hat{\Gamma}_{0,M_0}) = \ln |S^{n-k}| - \ln P(\hat{\Gamma}_{0,N_0}) = T(N_0).
\]
Moreover, \(0 \leq k \leq n - 2\) since \(k = n - 1\) or \(n\) imply \(T(N_0) = T(M_0) = 0\). In conclusion, it suffices to show the existence of solution for \(N_0^{n-k} = \partial \hat{N}_0 \subset \mathbb{R}^{n-k+1}\). Hence, we may assume, without loss of generality, that \(M_0\) does not contain any straight lines. In this case, the link of the tangent cone at infinity \(\hat{\Gamma_0}\) does not contain any antipodal points and is compactly contained in an open hemisphere
\[
H(v_0) := \{ p \in S^n : \langle p, v_0 \rangle > 0 \}
\]
for some \(v_0 \in S^n\) (see Lemma 3.8 [27]).

Next, we create a sequence of strictly monotone convex compact hypersurfaces \(M_{i,0}\) which approximate \(M_0\) from inside as follows: let \(\Sigma_{i,0}\) be the compact hypersurface \(\Sigma_{i,0} := \partial [B_i(0) \cap M_0]\) where \(B_i(0)\) denotes a all of radius \(i\) in \(\mathbb{R}^{n+1}\). To smoothen out each \(\Sigma_{i,0}\) at the intersection of
Figure 2: Approximation of $M_0$

$\partial B_i(0)$ and $M_0$ we let $\Sigma_{i,s}$, $s > 0$, be the mean curvature flow (MCF) running from $\Sigma_{i,0}$. For a positive decreasing sequence $s_i \to 0$, let $M_{i,0} := \Sigma_{i,s_i}$. Then, $\{M_{i,0}\}$ satisfies the desired properties and $M_{i,0} \to M_0$ locally uniformly on compact sets. Moreover, $M_0 \in C^{1,1}_{\text{loc}}$ implies the mean curvature $H$ of $M_{i,0}$ is uniformly bounded on every extrinsic ball of finite radius. By the results in [29] and [18], for each $M_{i,0}$ there exists a unique smooth solution of the IMCF, $M_{i,t} = \partial \hat{M}_{i,t}$ for $t \in [0, \infty)$. $M_{i,t}$ are strictly convex (see Remark A.1) and strictly monotone increasing in $i$ by the comparison principle. By Proposition 2.7, the mean curvature $H$ is locally uniformly bounded for $M_{i,t}$, i.e. given $R > 0$, there is $M > 0$ such that

$$0 < H \leq M \quad \text{on} \quad B_R(0) \cap M_{i,t} \quad \text{for all} \quad i \quad \text{and} \quad t \geq 0. \quad (4.1)$$

We define our solution by

$$M_t = \partial \hat{M}_t \quad \text{with} \quad \hat{M}_t := \bigcup_{i=1}^{\infty} \hat{M}_{i,t} \quad \text{for} \quad t \in [0, \infty).$$

This is convex by definition and it remains to prove that $M_t$ is (nontrivial) strictly convex smooth solution of the flow for $t \in (0, T(M_0))$ and converges to $M_0$ locally uniformly as $t \to 0+$. We will need the following simple observation.

**Claim 4.1.** Let $\hat{\Gamma}_0 \subset \mathbb{S}^n$ be a convex set which is compactly contained in an open hemisphere $H(v_0)$. Then there is a family of smooth, strictly convex hypersurfaces $\{\Gamma^\epsilon_0\}_{\epsilon > 0}$ in $\mathbb{S}^n$ with $\partial \hat{\Gamma}^\epsilon_0 = \Gamma^\epsilon_0$ which are also contained in $H(v_0)$, strictly monotone decreasing in the sense that

$$\hat{\Gamma}^\epsilon_1 \subset \subset \hat{\Gamma}^\epsilon_2 \quad \text{for} \quad 0 < \epsilon_1 < \epsilon_2$$

and $\bigcap_{\epsilon > 0} \hat{\Gamma}^\epsilon_0 = \hat{\Gamma}_0$. For such a sequence, $|\Gamma^\epsilon_0| = P(\hat{\Gamma}^\epsilon_0) \to P(\hat{\Gamma}_0)$.

**Proof of Claim 4.1.** If $\hat{\Gamma}_0$ is a single point, we may choose $\Gamma^\epsilon_0$ to be concentric geodesic spheres in $\mathbb{S}^n$. Thus we may assume that $\hat{\Gamma}_0$ is a closed convex set in an open hemisphere which is not a point. Define the dual of $\hat{\Gamma}_0$ by

$$\hat{\Gamma'}_0 := \{v \in \mathbb{S}^n \mid \langle v, w \rangle \leq 0 \quad \text{for all} \quad w \in \hat{\Gamma}_0\}.$$

Then, it can be easily checked that $\hat{\Gamma'}_0$ is contained in a closed hemisphere. The fact that $\hat{\Gamma}_0$ lies in $\text{int}H(v_0)$ implies $\hat{\Gamma'}_0$ has non-empty interior. i.e. $\partial \hat{\Gamma'}_0$ is a convex hypersurface. We may run mean curvature flow $\Gamma'_{0,s}$ starting at $\Gamma'_{0,s} = \partial \hat{\Gamma'}_0$ for a short time $s \in (0, s')$. $\{\Gamma'_{0,s}\}$ are smooth, strictly
convex and monotone decreasing unless \( \hat{\Gamma}_0 \) is a hemisphere (which isn’t the case as \( \hat{\Gamma}_0 \) is not a point). We define \( \hat{\Gamma}_0' = (\hat{\Gamma}_0', \varepsilon) \). Then, it is known (see Chapters 9, 10 of [17] and also in [18]) that \( \Gamma_0' = \partial \Gamma_0' \) is the image of \( \Gamma_0' \), under the Gauss map and \( \{ \Gamma_0' \} \) are smooth, strictly convex, and strictly monotone decreasing in \( \varepsilon \). Since \( \Gamma_0' \) converges to \( \Gamma_0' \) as \( \varepsilon \to 0 \) from inside, \( \Gamma_0' \) converges to \( \Gamma_0 \) from outside. It follows that \( |\Gamma_0'| = P(\hat{\Gamma}_0') \setminus P(\hat{\Gamma}_0) \).

Now fix \( t_0 \in (0, T) \) an arbitrary time. By the claim, we may find a small \( \varepsilon_0 > 0 \) such that \( T_0^\varepsilon := \ln |\mathbb{S}^{n-1}| - \ln |\Gamma_0'| > t_0 \). Since \( \hat{\Gamma}_0 \) is contained in the interior of \( \hat{\Gamma}_0' \), we may find a vector \( v_0' \in \mathbb{R}^{n+1} \) such that \( M_{i,0} \subset M_0 \subset \mathcal{C}'_{\varepsilon_0} + v_0' \). Theorem 1.4 [27] guarantees the existence of a smooth strictly convex IMCF solution \( \Gamma_0^\varepsilon \) in \( \mathbb{S}^n \) with initial data \( \Gamma_0^\varepsilon \), for \( t \in [0, T_0^\varepsilon) \). Then by the comparison principle \( \hat{M}_{i,t} \subset \mathcal{C}'_{\varepsilon_0} + v_0' \) for \( t \in [0, T_0^\varepsilon) \). Since \( \Gamma_0^\varepsilon \) is a strictly convex solution which converges to an equator, we may find a direction \( \omega_0 \in \mathbb{S}^n \) and small \( \delta_0 > 0 \) such that

\[
\langle F - v_0', \omega_0 \rangle \geq (\sin \delta_0) |F - v_0'| \quad \text{for } t \in [0, t_0] \text{ on } M_{i,t}.
\]

By Theorem 1.4, we have uniform bounds of \((|H|F|)^{-1}\) for \( M_{i,t} \) on \([t_0/2, t_0]\). The conical barrier \( C\mathcal{C}'_{\varepsilon_0} + v_0' \) also shows \( M_i \) is nonempty for \( t \in [0, t_0] \).

Let us choose an arbitrary point \( x_0 \in M_{t_0} \). By the previous argument, we have uniform bounds of \( H \) and \( H^{-1} \) on \( M_{i,t} \cap B_1(x_0) \) for \( t \in [t_0/2, t_0] \). Since \( M_{t_0} \) is convex, there is a supporting hyperplane at \( x_0 \) and after an isometry, we may assume \( x_0 = 0 \) and the hyperplane is \( \{ x_{n+1} = 0 \} \) and \( M_{i,t} \) are located in \( \{ x_{n+1} \geq 0 \} \) for \( t \leq t_0 \).

**Claim 4.2.** Let \( D_{r_0} = \{ x' \in \mathbb{R}^n : |x'| \leq r_0 \} \). Then there is small \( r_0 > 0 \) and \( \tau_0 > 0 \) such that for large \( i \geq i_0 \), \( M_{i,t} \cap (D_{r_0} \times [-\tau_0, \tau_0]) \) can be written as graphs \( x_{n+1} = u^{(i)}(x', t) \) on \( D_{r_0} \times [t_0 - \tau_0, t_0] \) with uniformly bounded \( C^1 \) norm.

**Proof of Claim 4.2.** Assume \( H, H^{-1} \leq M \) on \( B_1(0) \cap M_{i,t} \) for \( t \in [t_0/2, t_0] \). By the bound of \( H^{-1} \) and \( x_0 = 0 \in M_{t_0} \), for every \( r \in (0, 1/2) \), there is \( \tau > 0 \) such that \( M_{i,t} \cap (D_r \times [-\tau, \tau]) \neq \phi \) for large \( i \geq i_0 \) and \( t \in [t_0 - \tau, t_0] \) (for instance, we can choose \( \tau = \min \left( \frac{1}{2}, \frac{1}{2t_0} \right) \)). Meanwhile, \( H \leq M \) implies that every point on \( M_{i,t} \) has a inscribed ball of radius \( M^{-1} \). Since \( M_{i,t} \) lies in \( \{ x_{n+1} \geq 0 \} \), by choosing \( r \) small enough compared to \( M^{-1} \), those points in \( M_{i,t} \cap (D_r \times [-\tau, \tau]) \) should have uniformly bounded gradient in terms of \( r \) and \( M \). Now, we can choose smaller \( r_0 \) (if needed) to make \( M_{i,t} \cap (D_r \times [-\tau, \tau]) \) a one sheeted graph over \( D_r \). We also choose \( \tau_0 = \tau(r_0) \).

Since \( M_{i,t} \) are solutions to IMCF, the graphs \( u(x', t) = u^{(i)}(x', t) \) evolve by the fully nonlinear parabolic equation

\[
\partial_t u = -\frac{(1 + |Du|^2)^{1/2}}{H} = -(1 + |Du|^2)^{1/2} \left[ \text{div} \left( \frac{Du}{1 + |Du|^2} \right) \right]^{-1}
\]

and the equation is uniformly parabolic if \( |Du|, H, H^{-1} \) are bounded. Therefore, our estimates above show that \( u^{(i)} \) are solutions of a uniformly parabolic equation on \( D_{r_0} \times [t_0 - \tau_0, t_0] \) and moreover they are uniformly bounded, since \( |u^{(i)}| \leq r_0 \). Standard parabolic regularity theory implies the smooth subsequential convergence \( u^{(i)} \to u \) on \( D_{r_0/2} \times [t_0 - \tau_0/2, t_0] \). Since \( M_{i,t} \) are monotone in \( i \), this proves that \( x_{n+1} = u(x', t) \) is a smooth graphical parametrization of \( M_t \), i.e. \( M_t \) is a smooth solution of the IMCF for \( t \in (0, T) \). In addition, the locally uniform convergence of \( M_t \) to \( M_0 \), as \( t \to 0 \), follows from the bound in Theorem 1.4 as \( t^{-1/2} \) is integrable around \( t = 0 \).
It remains to check the strict convexity of $M_{t_0}$, for any $t_0 \in (0,T)$. Note that

$$\int_{M_{t_0}} \lambda_1 \ldots \lambda_n \, d\mu = \int_{M_{t_0}} K \, d\mu = \mathcal{H}^n(\nu|M_{t_0}) = \mathcal{H}^n((\hat{\Gamma}_{t_0})').$$

Here $(\Gamma_{t_0})'$ is the dual of the tangent cone of $\hat{M}_{t_0}$ at infinity. On the other hand, $\hat{\Gamma}_{t_0} \subset \hat{\Gamma}_{t_0}^{\epsilon}$ implies $(\hat{\Gamma}_{t_0}^{\epsilon})' \subset (\hat{\Gamma}_{t_0})'$. $\hat{\Gamma}_{t_0}^{\epsilon}$ is compactly contained in an open hemisphere and hence $(\hat{\Gamma}_{t_0}^{\epsilon})'$ has nonempty interior. This shows that $\mathcal{H}^n((\hat{\Gamma}_{t_0}^{\epsilon})') \geq \mathcal{H}^n((\hat{\Gamma}_{t_0})') > 0$. By Corollary A.4 in our Appendix, $M_t$ is strictly convex for $t \in (0,t_0)$ and this finishes the proof.

The following simple observation says that our constructed solution in Theorem 1.2 is the smallest of all solutions with initial data $M_0$.

**Lemma 4.1.** Let $N_t = \partial \hat{N}_t$ be a smooth solution of the IMCF and $M_t = \partial \hat{M}_t$ be a convex solution obtained from Theorem 1.2. If $\hat{M}_0 \subset \hat{N}_0$, then $\hat{M}_t \subset \hat{N}_t$ as long as both solutions exist.

**Proof.** Note that the approximating sequence of convex closed hypersurfaces $M_{t,0}$ in Theorem 1.2 was strictly monotone. i.e $\hat{M}_{i,0} \subset \hat{M}_{j,0}$ if $j > i$. This implies $\hat{M}_{i,0} \subset \hat{N}_0$. By classical comparison principle between compact and non-compact solutions, $\hat{M}_{i,t} \subset \hat{N}_t$ and hence $\hat{M}_t \subset \hat{N}_t$.

We next show that the comparison principle also holds between a non-compact solution and a conical solution which is inserted inside.

**Lemma 4.2.** Let $\Gamma_0 = \partial \hat{\Gamma}_0 \subset S^n$ be a smooth strictly convex hypersurface in $S^n$ and $\Gamma_t$ be the unique solution of the IMCF by Theorem 1.4 in [27]. Suppose that $N_t := \partial \hat{N}_t$ is a smooth complete non-compact solution of the IMCF which converges to $\hat{N}_0$ locally uniformly as $t \to 0$. If $\hat{\Gamma}_0 \subset \hat{N}_0$, then $\hat{\Gamma}_t \subset \hat{N}_t$ as long as the solution exists.

**Proof.** Since $\hat{\Gamma}_0$ is singular at the origin, we first smoothen it inside the ball $B_{1/2}(0)$, creating a smooth hypersurface $\hat{M}_0 = \partial \hat{\Gamma}_0 \subset \hat{\Gamma}_0$ such that $\hat{M}_0 = \hat{\Gamma}_0$ outside of $B_{1/2}(0)$. Theorem 1.2 shows the existence of the smallest smooth solution $M_t$, for $t \in (0, \ln |S^{n-1}| - \ln |\Gamma_0|)$ with initial data $M_0$. For every $\epsilon \in (0,1)$, $M_0 \subset \hat{\Gamma}_0 \subset \epsilon^{-1} \hat{N}_0$ implies $M_t \subset \epsilon^{-1} \hat{N}_t$ by Lemma 4.1 i.e. $\epsilon M_t \subset \hat{N}_t$. We want to argue that $\epsilon M_t$ converges to $\hat{\Gamma}_t$, as $\epsilon \to 0$, and conclude that $\hat{\Gamma}_t \subset \hat{N}_t$.

From our construction we have $H(|F| + 1) \leq C$ for some $C > 0$ on $M_0$. By Proposition 2.7 $H|F| \leq C$ on $M_t$ for some larger $C > 0$. Next, since $\hat{\Gamma}_t$ works as a conical barrier outside, Theorem 1.4 (Remark 3.1) can be applied to the approximating compact solutions of $M_t$ to conclude that $H|F| \leq C_\delta$ on $M_t$ for $t \in [0, T(M_0) - \delta)$. This implies $M_t \backslash B_1(0)$ satisfies $c_\delta \leq H|F| \leq C$ for $t \in [0, T - \delta]$ and hence the same bound holds for $\epsilon M_t$ with $\epsilon < 1$. Using these bounds (following a similar argument of the proof of Theorem 1.2) it is easy to pass a smooth blow-down limit $\epsilon \to 0$ outside of $B_1(0)$ and get a solution of IMCF. On the other hand, convexity implies $\epsilon M_t$ converges to a cone as $\epsilon \to 0$ at each $t$ and in particular $\epsilon M_0$ converges to $\hat{\Gamma}_0$. Since $\hat{\Gamma}_t$ is the unique solution of the IMCF from $\hat{\Gamma}_0$, the previous observation implies $\epsilon M_t$ converges to $\hat{\Gamma}_t$. This proves $\hat{\Gamma}_t \subset \hat{N}_t$.

We shall show Theorem 1.3, which shows the solution obtained from Theorem 1.2 has the maximal time of existence. We the next barrier lemma which shows that if $\hat{M}_0$ contains a round cylinder $\hat{D}_{R,\epsilon} = B_{R}(0) \times (-\epsilon, \epsilon)$ of radius $R > 0$ and small height $\epsilon \leq R/10$, then $\hat{M}_1$ contains a whole $(n+1)$-ball $B_{c_0R/2}(0)$ of radius $c_0 t R$, for some $c_0$ depending only on dimension $n$. 

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Lemma 4.3. Let \( \hat{D}_{R,\epsilon} = B_R^n(0) \times (-\epsilon, \epsilon) \subset \mathbb{R}^{n+1} \) be a round cylinder of radius \( R > 0 \) and small height \( \epsilon \in (0, R/10) \). If \( \hat{D}_{R,\epsilon} \subset M_0 \), then there is small \( c_n > 0 \) such that \( B_{c_n R_1}(0) \subset \hat{M}_t \) for \( t \in [0, c_n] \).

**Proof.** By smoothing the edges of \( D_{R,\epsilon} \) (outside the ball \( B_{R/2}^{n+1}(0) \)) we get a smooth pancake like convex hypersurface \( \Sigma_{R,\epsilon} \) which coincides with \( D_{R,\epsilon} \) on \( B_{R/2}^{n+1}(0) \). We can further assume that \( \Sigma_{R,\epsilon} \) has the same symmetry of \( D_{R,\epsilon} \), i.e. it has \( O(n) \) rotational symmetry and reflection symmetry with respect to \( \{x_{n+1} = 0\} \). Then, the IMCF solution \( \Sigma_{R,\epsilon}(t) \) starting at \( \Sigma_{R,\epsilon} \) has two points \((0, \epsilon + c(t))\) and \((0, -\epsilon - c(t))\) for each \( t > 0 \) and their normal vectors are \( e_{n+1} \) and \( -e_{n+1} \), respectively. In view of Lemma 2.7 \( c'(t) > cR \) as long as \( \epsilon + c(t) < R/2 \). Since \( \Sigma_{R,\epsilon}(t) \) contains these two points and the disk \( B_{R/2}^n \times \{0\} \), the convexity implies that \( \hat{M}_t \) includes the desired ball.

**Proof of Theorem 1.3.** Suppose there is a smooth solution on \( t \in (0, T + \tau] \) for some \( \tau > 0 \). We will show \( \hat{M}_{T+\tau} \) contains \( B_R(0) \) for all \( R > 0 \), which is a contradiction. The same notations in Theorem 1.2 will be used. After a translation, we assume \( 0 \in \hat{M}_0 \).

**Case 1.** Suppose \( \hat{\Gamma}_0 \) has nonempty interior in \( S^n \).

Using an approximation by the mean curvature flow, we may find a smooth strictly convex hypersurface \( \Gamma'_0 \subset \subset \hat{\Gamma}_0 \) with \( T' = \ln |S^{n-1}| - \ln |\Gamma'_0| \in (T, T + \tau/2) \). Let \( \{\Gamma'_t\}_{t \in [0, T')} \) be the unique smooth solution of the IMCF in \( S^n \) which converges to an equator as \( t \to T' \). By Lemma 4.2, \( \hat{M}_t \) contains \( \hat{\Gamma}_t \) for \( t \in [0, T') \) and thus \( \hat{M}_t \) contains a half space at \( t = T' \). WLOG let’s assume the half space is \( \{x_{n+1} \geq 0\} \). For each \( r > 0 \), \( \partial B_{re^{n+1}}(re_{n+1}) \) is a solution of the IMCF and is contained in \( \hat{M}_{T'+t} \) by avoidance principle. Note \( B_{t(e^{n+1} - 1)}(0) \subset B_{t e^{n+1}}(re_{n+1}) \). By choosing \( r > 0 \) arbitrary large, we get \( \hat{M}_{T+\tau} \) contains arbitrary large ball centered at the origin.

**Case 2.** Suppose \( \hat{\Gamma}_0 \) has empty interior in \( S^n \).

After splitting out \( \mathbb{R} \) factors, we may also assume \( \hat{\Gamma}_0 \) is compactly contained in an open hemisphere. Let’s define \( \hat{\Gamma}_t \) for small \( t > 0 \) to be the tangent cone of \( \hat{M}_t \) at infinity. Since \( \hat{M}_t \) is monotone, \( \hat{\Gamma}_t \) is monotone increasing convex set in \( S^n \). Since convex set in a hemisphere is outer area/perimeter minimizing, \( P(\hat{\Gamma}_t) = |\Gamma_t| \geq 2|\Gamma_0| = P(\hat{\Gamma}_0) \) and it is increasing. If we show \( \hat{\Gamma}_t \) has non-empty interior for \( t > 0 \) and \( |\Gamma_t| = P(\hat{\Gamma}_t) \to 2|\Gamma_0| = P(\hat{\Gamma}_0) \) as \( t \to 0 \). Then Case 1 applied to \( \hat{M}_t \) and \( \hat{\Gamma}_t \) for sufficiently small \( \tau > 0 \) and the fact \( P(\hat{\Gamma}_t) \) is monotone increasing imply a contradiction.
Note $P(\hat{\Gamma}_0) > 0$ since $T < \infty$. When $\hat{\Gamma}_0$ has empty interior, it can be checked that $\hat{\Gamma}_0 = \Gamma_0$ and is a $n - 1$ dimensional convex set in some (totally geodesic) equator $S^{n-1} \subset S^n$ with non-empty interior in $S^{n-1}$. Let’s say $e_{n+1} \in \hat{\Gamma}_0 \subset S^{n-1}$ and $B^{2r}_n(e_{n+1})$, $n$-dim geodesic ball of radius $2r$ in $S^{n-1}$, is contained in $\hat{\Gamma}_0$. For $M_0$, this implies that there is a small $\epsilon_0 > 0$ such that at each point $(0, he_{n+1}) \in \hat{M}_0$ for $h \geq h_0 > 0$, a thick disk $B^r_{\epsilon} \times (-\epsilon, \epsilon)$ centered at $(0, he_{n+1})$ could be inserted in $\hat{M}_0$ after a rotation. This implies $\hat{M}_t$ has $B^{n+1}_{\epsilon}((0, he_{n+1})$ for small $t > 0$. This proves $\hat{\Gamma}_t$ has a ball centered at $e_{n+1}$ for $t > 0$ and hence has non-empty interior.

Remark 4.1. Theorems 1.2 and 1.3 indicate that it is likely true that the tangent cone $\Gamma_t \subset S^n$ of our solution $M_t$ at infinity also evolves by the IMCF in $S^n$. Indeed, if one inserts cones which approximate $C \Gamma_0$ from inside and outside and use the solutions with initial data those cones as barriers, it is not hard to see that the assertion is true when $\Gamma_0$ produces a unique classical solution of IMCF on $S^n$ for $t > 0$. However, for general Lipschitz $\Gamma_0$, there might be no classical solution of the IMCF and we can only conclude that $\Gamma_t$ satisfies the IMCF in some generalized limit sense. In fact, this is how a weak solution is defined in the upcoming work of the first author with P.K. Hung in \cite{[6]} and it turns out that $\Gamma_t$ will then be a weak solution of IMCF on $S^n$ in the sense of \cite{[6]}.

Remark 4.2 (The connection with ultra-fast diffusion on $\mathbb{R}^n$). In \cite{[10, 9]}, the second author and M. del Pino studied the Cauchy problem of ultra-fast diffusion equations $u_t = \nabla \cdot (u^{m-1} \nabla u)$ on $\mathbb{R}^n$ for $m < -1$. In an attempt to find the fastest possible decay of initial data $u_0$ which guarantees a solvability of the equation on $t \in (0, T)$, some partial necessary or sufficient conditions had been found. As pointed earlier, the evolution of $H$ in the IMCF is similar to the ultra-fast diffusion equation of $m = -1$ and it shares similar features. Let us first summarize some of results when $m = -1$ from \cite{[10, 9]}. First, there is $C(n)$ so that if the Cauchy problem $u_t = \nabla \cdot (u^{-2} \nabla u)$ with $u(x, 0) = u_0(x) \geq 0$ has a solution for $t \in (0, T)$, then

$$
\limsup_{R \to \infty} \frac{1}{R^{n-1}} \int_{B_R} u_0 dx \geq C T^{1/2}.
$$

There exist, however, some $u_0(x) \geq 0$ such that

$$
\lim_{R \to \infty} \frac{1}{R^{n-1}} \int_{B_R} u_0 = C > 0
$$
but for which no solution exist with initial data $u_0$, for any $T > 0$. Such solutions are characterized by a non-radial structure at spatial infinity. Indeed, for initial data which is bounded from below near infinity by positive radial functions there is a necessary and sufficient for existence as follows: there is an explicit constant $E^* > 0$ such that if the problem has a solution for $t \in (0, T)$, then

$$\limsup_{R \to \infty} \left[ R \left( \int_0^R \frac{ds}{w_n s^{n-1}} \int_{B_s} u_0 \, dx \right) \right] \geq E^* T^{1/2}.$$  

Moreover, if $u_0$ is radially symmetric and locally bounded,

$$\liminf_{R \to \infty} \left[ R \left( \int_0^R \frac{ds}{w_n s^{n-1}} \int_{B_s} u_0 \, dx \right) \right] \geq E^* T^{1/2}$$

guarantees an existence of a solution on $\mathbb{R}^n \times (0, T)$. For non-radial $u_0$, there is a similar condition in Theorem 1.3. [9]. Every result mentioned here is in some sense sharp when explicit solutions $v^T(x, t) = \sqrt{2(n-1)(T-t)} + |x|$ are considered. These results explain partial conditions for non-existence and existence of solutions, but a complete description was missing. For the convex IMCF, however, Theorem 1.2 and 1.3 depict a fairly complete picture. This was possible by the geometric estimate Theorem 1.4. Roughly, the estimate shows that $H \geq c |x|^{-1}$ as long as $t > 0$ and $t < T$. Note that this lower bound has the same decay of $v^T(x, t)$ above. Instead of the integral operators used in [9], the asymptotic geometry of $M_0$ is used to provide the lower bound on $H$ in Theorem 1.4. It would be interesting to see if a similar idea could be implemented in the theory of ultra-fast diffusion equation (1.2), with $m < 0$.

**A Appendix**

**A.1 Strict convexity of solutions in space form**

Throughout this section, we assume that $F : M^n \times (0, T) \to (N^{n+1}, \bar{g})$ is a complete smooth convex solution of the IMCF, where $(N^{n+1}, \bar{g})$ is a space form of sectional curvature $K \in \mathbb{R}$, in particular which includes Euclidean space, the sphere, or hyperbolic space. As before, $\nu$ denotes the unit outward normal, $H$ the mean curvature and $h_{ij}$ the second fundamental form.

Suppose we have an (incomplete) smooth convex solution of the IMCF with $H > 0$ on an open set $\Omega \subset M$ for $t \in (0, T)$. Our aim is to prove Theorem A.3, a strong minimum principle on $\lambda_1$. However by looking at the evolution of the second fundamental form $h_{ij}$ given in (2.2), it is not clear that the convexity is preserved. To do so we need to use a viscosity solution argument and we need the following lemma shown from [4].

**Lemma A.1** (Lemma 5 in Section 4 [4]). Suppose that $\phi$ is a smooth function such that $\lambda_1 \geq \phi$ everywhere and $\lambda_1 = \phi$ at $x = \bar{p} \in \Omega$. Let us choose an orthonormal frame so that

$$h_{ij} = \lambda_i \delta_{ij} \text{ at } \bar{p} \in \Omega \quad \text{with } \lambda_1 = \lambda_2 = \ldots = \lambda_{\mu} < \lambda_{\mu+1} \leq \ldots \leq \lambda_n.$$  

We denote $\mu \geq 1$ by the multiplicity of $\lambda_1$. Then at $\bar{p}$, $\nabla_i h_{kl} = \delta_{kl} \nabla_i \phi$ for $1 \leq k, l \leq \mu$. Moreover,

$$\nabla_i \nabla_i \phi \leq \nabla_i \nabla_i h_{11} - 2 \sum_{j>\mu} (\lambda_j - \lambda_1)^{-1} (\nabla_i h_{1j})^2.$$
Proposition A.2. For $n \geq 1$, let $F : \Omega \times (0, T) \rightarrow (N^{n+1}, \bar{g})$ be a smooth convex solution of the IMCF where $(N, \bar{g})$ is a space form. Let $\lambda_1$ denote the lowest eigenvalue of $h^i_j$. Then $u := \lambda_1/H$ is a viscosity supersolution of the equation
\[
\frac{\partial}{\partial t} u - \frac{1}{H^2} \Delta u + \frac{1}{H^3} (V, \nabla u) + \left( \frac{W}{H^4} \right) u \geq 0
\]  
(A.1)
where $V$ is a vector field, and $W$ is a scalar function such that
\[
|W|, |V| \leq C(|\nabla H|, n) \quad \text{at each point.}
\]

Proof. Using equation (2.2) in Remark 2.1, we can easily compute the evolution equation of $h^i_j/H$:
\[
(\partial_t - \frac{1}{H^2} \Delta) \frac{h^i_j}{H} = 2\lambda_1 \frac{h^i_j}{H} - 2 \frac{h^{ik}h_{kj}}{H^2} + \frac{2}{H^4} (\nabla_m H \nabla^m h^i_j - \nabla^i H \nabla_j H).
\]  
(A.2)
We will use this equation and the Lemma above to the proposition. Suppose a smooth function of space time, namely $\phi/H$, touches $\lambda_1/H$ from below at $(\bar{p}, \bar{t})$. At time $t$ around $\bar{p}$, let us fix a time independent frame $\{e_i\}$ using the metric $g(t)$ as in Lemma A.1.
Since $\phi \leq \lambda_1 \leq h^1_1$ and they coincide at $(\bar{p}, \bar{t})$, $\partial_t \phi \geq \partial_1 h^1_1$ at $(\bar{p}, \bar{t})$. At this point $(\bar{p}, \bar{t})$ with the frame $\{e_i\}$, we use Lemma A.1, equation (A.2), and the Codazzi identity $\nabla_i h_{jk} = \nabla_j h_{ik}$ to obtain
\[
\square \frac{\phi}{H} \geq \partial_t h^1_1 - \frac{1}{H^2} \Delta \frac{h^1_1}{H} + \frac{2}{H^3} \sum_{i \geq 1, j > \mu} (\lambda_j - \lambda_1)^{-1} |\nabla_i h_{1j}|^2
\]  
(A.3)
\[
= \square \frac{h^1_1}{H} + \frac{2}{H^3} \sum_{i \geq 1, j > \mu} (\lambda_j - \lambda_1)^{-1} |\nabla_i h_{1j}|^2
\]  
\[
= 2 \sum_j \lambda_j^2 - 2 \lambda_1 \sum_j \lambda_j \lambda_1 + \frac{2}{H^4} \left[ \nabla_m H \nabla^m h_{11} - |\nabla_1 H|^2 + H \sum_{i \geq 1, j > \mu} \frac{|\nabla_i h_{1j}|^2}{\lambda_j - \lambda_1} \right] \geq \frac{2}{H^4} \left[ \nabla_m H \nabla^m \phi - |\nabla_1 H|^2 + H \sum_{i \geq 1, j > \mu} \frac{|\nabla_i h_{1j}|^2}{\lambda_j - \lambda_1} \right].
\]
In the last line, we used $\lambda_1 \sum_j \lambda_j \leq \sum_j \lambda_j^2 = |A|^2$ which is true for $H > 0$.
Next, note that
\[
\nabla_1 H \nabla_1 H = \sum_{i,j} \nabla_1 h_{ii} \nabla_1 h_{jj} = 2\mu \nabla_1 H \nabla_1 \phi - \mu^2 |\nabla_1 \phi|^2 + \sum_{i,j \mu, j > \mu} \nabla_1 h_{ii} \nabla_1 h_{jj}.
\]  
(A.4)
Since $H \nabla_1 \phi = \nabla \phi - \frac{\phi}{H} \nabla H$, we have the following for each fixed unit direction $e_m$
\[
\nabla_m H \nabla_m \phi = H \nabla_m H \nabla_m \frac{\phi}{H} + \frac{\phi}{H} |\nabla_m H|^2.
\]  
(A.5)
We first plug (A.5) with $m = 1$ into (A.4) and then plug that into the last line of (A.3) to obtain
\[
\square \frac{\phi}{H} \geq \frac{2}{H^4} \sum_{m > 1} |\nabla_m H|^2 \frac{\phi}{H} + \frac{2(1 - 2\mu)}{H^4} |\nabla_1 H|^2 \frac{\phi}{H}
\]  
\[
+ \frac{2}{H^3} \sum_{m > 1} \nabla_m H \nabla_m \frac{\phi}{H} + \frac{2(1 - \mu)}{H^3} \nabla_1 H \nabla_1 \frac{\phi}{H} + \frac{2\mu^2}{H^3} |\nabla_1 \phi|^2
\]  
(A.6)
\[
+ \frac{2}{H^4} \left[ H \sum_{i \geq 1, j > \mu} (\lambda_j - \lambda_1)^{-1} |\nabla_i h_{1j}|^2 - \sum_{i > \mu, j > \mu} \nabla_1 h_{ii} \nabla_1 h_{jj} \right].
\]
We now use the convexity, \( \lambda_1 \geq 0 \), in the proof of the following claim.

**Claim A.1.** \( H \sum_{i \geq 1, j > \mu} (\lambda_j - \lambda_1)^{-1} |\nabla_1 h_{ij}|^2 - \sum_{i > \mu, j > \mu} \nabla_1 h_{ii} \nabla_1 h_{jj} \geq 0 \) on \( \{ \lambda_1 \geq 0 \} \).

Assuming that the claim is true, then by taking away the good term \( \frac{2 \mu^2 |\nabla_1 \phi|^2}{H^3} \) in (A.6), we easily conclude that (A.1) holds by choosing a vector filed \( V \) and a scalar function \( W \) as a function of \( \nabla H \) accordingly. Thus it remains to show the claim.

**Proof of Claim A.1.** Since \( \lambda_1 \geq 0 \), \( H = \sum_{i \geq 1} \lambda_i \geq \sum_{i > \mu} \lambda_i \), the claim follows by:

\[
H \sum_{i \geq 1, j > \mu} (\lambda_j - \lambda_1)^{-1} |\nabla_1 h_{ij}|^2 \geq \sum_{i \geq \mu} \lambda_i \sum_{j > \mu} (\lambda_j - \lambda_1)^{-1} |\nabla_1 h_{ii}|^2 = \sum_{i > \mu, j > \mu} \lambda_j \lambda_i^{-1} |\nabla_1 h_{ii}|^2
\]

\[
= \sum_{i > \mu, j > \mu} \frac{\lambda_j \lambda_i^{-1} |\nabla_1 h_{ii}|^2 + \lambda_i \lambda_j^{-1} |\nabla_1 h_{jj}|^2}{2} \quad (A.7)
\]

\[
\geq \sum_{i > \mu, j > \mu} |\nabla_1 h_{ii} \nabla_1 h_{jj}|
\]

\[\square\]

Now, let \( M_t \subset N^{n+1} \) be a smooth complete convex solution for \( t > 0 \), which could be either compact or non-compact. One expects \( M_t \) to be strictly convex, that is to have \( \lambda_1 > 0 \) for \( t > 0 \). Indeed, this follows easily by Proposition A.2 and the strong minimum principle for nonnegative supersolutions which is a consequence of the weak Harnack inequality for viscosity solutions of (locally) uniformly parabolic equations.

**Theorem A.3.** Suppose \( F : M^n \times (0, T) \to (N^{n+1}, \bar{g}) \) is a smooth convex solution of the IMCF with \( H > 0 \) where \((N^{n+1}, \bar{g})\) is a space form. If \( \lambda_1(p_0, t_0) = 0 \) at some \( (p_0, t_0) \) with \( 0 < t_0 < T \), then \( \lambda_1 = 0 \) on \( M^n \times (0, t_0) \).

**Proof.** Since solution is smooth, \( |H|, |\nabla H|, \) and \( |H^{-1}| = |\partial_t F| \) are locally bounded. Therefore, \( \lambda_1 \) is a nonnegative supersolution of equation \( (A.1) \) which is locally uniformly parabolic with bounded coefficients. We can apply strong minimum principle on a sequence \( \{ \Omega_k \} \) of expanding domains containing \( (p_0, t_0) \) such that \( M^n = \cup_k \Omega_k \) and conclude that the theorem holds. \[\square\]

**Corollary A.4.** Let \( F : M^n \times (0, T) \to \mathbb{R}^{n+1} \) be a smooth convex complete solution of the IMCF. If \( \mathcal{H}^n(\nu[M_{t_0}]) > 0 \) at \( t_0 \in (0, T) \), then the solution is strictly convex for \( (0, t_0) \).

**Proof.** If it is not, \( \lambda_1 \equiv 0 \) for all \( M^n \times (0, t_0) \). In particular, \( \mathcal{H}^n(\nu[M_{t_0}]) = \int_{M} K(\cdot, t_0) d\mu = 0. \) This contradicts and proves the assertion. \[\square\]

**Remark A.1** (Strict convexity of our solutions). The theorem and corollary above do not exactly explain why convexity is preserved along the IMCF since they both assume the convexity of the solution. First of all, assume \( M^0 \subset N^{n+1} \) is a smooth compact solution for \( t \in [0, T] \), where \( M_0 \) is smooth and strictly convex. Then by considering the first time when \( \lambda_1 \) becomes zero, Theorem A.3 implies that \( M_t \) is strictly convex for all time. i.e. the strict convexity is preserved for compact solutions. We observe next that all solutions, including non-compact ones, which appear in this paper are obtained as a locally smooth limit of strictly convex solutions. Thus, they are convex. Therefore one can apply Theorem A.3 and Corollary A.4 and conclude that they are strictly convex.
A.2 Speed estimate for closed star-shaped solutions

The goal is this section is to give an alternative proof of Theorem 1.1 in [24] which will be based on the maximum principle. The theorem holds in any dimension \( n \geq 1 \).

**Theorem A.5** (Theorem 1.1 in [24]). Let \( F : M^n \times [0,T] \rightarrow \mathbb{R}^{n+1} \) be a smooth closed star-shaped solution of (1.1) such that \( M_0 := F_0(M^n) \) satisfies

\[
0 < R_1 \leq \langle F, \nu \rangle \leq R_2.
\]

Then, there is a constant \( C_n > 0 \) depending only on \( n \) such that

\[
1 \leq C_n \left( \frac{R_2}{R_1} \right) \left( 1 + \frac{1}{t^{1/2}} \right) R_2 e^\frac{t}{n}
\]

holds everywhere on \( M^n \times [0,T] \).

**Proof.** Since \( M_0 \) satisfies (A.8), by Proposition 1.3 in [24], we have

\[
R_1 \leq R_1 e^\frac{t}{n} \leq \langle F, \nu \rangle \leq |F| \leq R_2 e^\frac{t}{n}
\]

for all \( 0 < t < +\infty \). Let us denote \( w := \langle F, \nu \rangle^{-1} \) and we will consider a function

\[
Q := \frac{\varphi^{1-\epsilon}(w) e^{\gamma \langle F \rangle^2}}{H}
\]

for some function \( \varphi := \varphi(w) \), constants \( \gamma > 0 \) and \( \epsilon \in (0,1) \) which will be chosen shortly.

Direct computation shows that

\[
(\partial_t - \frac{1}{H^2} \Delta) \ln e^{\langle F \rangle^2} = (\partial_t - \frac{1}{H^2} \Delta) |F|^2 = -\frac{2n}{H^2} + \frac{4}{H} w.
\]

Moreover, by (7) in Lemma 2.2 and Lemma 2.6 with \( \beta = -1 \),

\[
(\partial_t - \frac{1}{H^2} \Delta) w = -\frac{|A|^2}{H^2} w - \frac{2}{wH^2} |\nabla w|^2
\]

and hence, on \( \{ \varphi \neq 0 \} \),

\[
(\partial_t - \frac{1}{H^2} \Delta) \ln \varphi = \frac{\varphi}{\varphi} (\partial_t - \frac{H^{-2} \Delta}{\varphi}) \varphi + \frac{1}{H^2} |\nabla \varphi|^2 = -\frac{|A|^2}{H^2} \varphi' w - \frac{|\nabla w|^2}{H^2} \left( 2 \frac{\varphi'}{w \varphi} + \frac{\varphi''}{\varphi^2} - \frac{\varphi'^2}{\varphi^2} \right).
\]

Inspired by the choice of \( \varphi \) in the well known interior curvature estimate by Ecker and Huisken in [14] (see also [7]), we define

\[
\varphi(s) := \left( \frac{s}{2R_1^{-1} - s} \right).
\]

For this \( \varphi := \varphi(w) \), under the notation \( \varphi' = \varphi'(w) \) and \( \varphi'' = \varphi''(w) \), a direct computation yields

\[
\frac{\varphi' w}{\varphi} = -\left( \frac{2}{2 - wR_1} \right) \quad \text{and} \quad 2 \frac{\varphi'}{w \varphi} + \frac{\varphi''}{\varphi^2} - \frac{\varphi'^2}{\varphi^2} = \frac{\varphi'^2}{\varphi^2}.
\]
Lemma 2.4 and the computations above imply
\[
(\partial_t - \frac{1}{H^2} \Delta) \ln Q = \left[ \frac{|A|^2}{H^2} + \frac{1}{H^2} \frac{1}{H^2 - 2} \right] + \gamma \left[ \frac{4}{H w} - \frac{2n}{H^2} \right] - (1 - \epsilon) \left[ \frac{|A|^2}{H^2} \frac{\phi w}{\phi} + \frac{1}{H^2} \frac{|\phi|^2}{\phi^2} \right]
\]
\[
= - \left( \frac{w R_1 - 2 \epsilon}{2 - w R_1} \right) \frac{|A|^2}{H^2} + \left( \frac{\gamma}{4} \frac{4}{H w} - \gamma \frac{2n - 4 \epsilon^{-1} \gamma |F|^2 |\nabla F|^2}{H^2} \right)
- \frac{1}{H^2} \left[ (1 - \epsilon) \frac{|\nabla \phi|^2}{\phi^2} - \frac{1}{H^2 - 2} + \epsilon^{-1} \frac{\gamma |\nabla e^{F}|^2}{e^{F}|F|^2} \right].
\]
(A.12)

Note that we have added and subtracted the term \( \frac{\epsilon^{-1} \gamma |F|^2}{H^2} \) in the last equality. At a nonzero critical point of \( Q \),
\[
0 = \frac{\nabla Q}{Q} = (1 - \epsilon) \frac{\nabla \phi}{\phi} + \frac{\nabla e^{F}|F|^2}{e^{F}|F|^2} + \frac{|\nabla H^{-1}|^2}{H^{-2}},
\]
and thus
\[
\left| \frac{\nabla H^{-1}|^2}{H^{-2}} \right| = \left| (1 - \epsilon) \frac{\nabla \phi}{\phi} + \frac{\nabla e^{F}|F|^2}{e^{F}|F|^2} \right|^2 = (1 - \epsilon)^2 \left| \frac{\nabla \phi}{\phi} \right|^2 + 2(1 - \epsilon) \gamma \left( \left< \frac{\nabla \phi}{\phi}, \frac{\nabla e^{F}|F|^2}{e^{F}|F|^2} \right> + \gamma^2 \frac{\nabla e^{F}|F|^2}{e^{F}|F|^2} \right)
\leq \left( (1 - \epsilon)^2 + \epsilon(1 - \epsilon) \right) \left| \frac{\nabla \phi}{\phi} \right|^2 + (1 + \frac{1 - \epsilon}{\epsilon}) \gamma^2 \frac{\nabla e^{F}|F|^2}{e^{F}|F|^2} \right|^2
\leq (1 - \epsilon) \left| \frac{\nabla \phi}{\phi} \right|^2 + \epsilon^{-1} \frac{\gamma^2 |\nabla e^{F}|^2}{e^{F}|F|^2}.
\]

For a given \( T > 0 \), note that \( \frac{R_1}{2 R_2 e^{\pi}} \leq w R_1 \leq 1 \). It remains to choose \( \epsilon \) and \( \gamma \). The choice \( \epsilon := \frac{R_1}{2 R_2 e^{\pi}} \) makes the first term on RHS of the second equality in (A.12) nonpositive. Next, choose \( \gamma := \frac{\epsilon}{4n (R_2 e^{\pi})^2} > 0 \) so that \( 4 \epsilon^{-1} \gamma |F|^2 \leq n \) on \( M_t \) for \( t \in [0, T] \). Combining the choices and estimates, at a nonzero spatial critical point of \( Q \),
\[
(\partial_t - \frac{1}{H^2} \Delta) \ln Q = \frac{(\partial_t - H^{-2} \Delta)Q}{Q} + \frac{\nabla Q|^2}{Q^2} \leq \gamma \left( -\frac{n}{H^2} + \frac{4}{H w} \right).
\]
(A.13)

We will now apply the maximum principle on \( \hat{Q} := t Q \). Suppose that nonzero maximum of \( \hat{Q} \) on \( M^n \times [0, T] \) occurs at the point \((p_0, t_0)\), which necessarily implies \( t_0 > 0 \). At this point, (A.13) implies
\[
0 \leq (\partial_t - \frac{1}{H^2} \Delta) \ln \hat{Q} \leq \gamma \left( -\frac{n}{H^2} + \frac{4}{H w} \right) + \frac{1}{t_0} \leq \gamma \left( -\frac{n}{2H^2} + \frac{8}{n} R_2 e^{2 \pi} \right) + \frac{1}{t_0}
\]
(A.14)

where the second inequality comes from
\[
\frac{4}{H w} \leq \frac{8}{n w^2} + \frac{n}{2H^2} \leq \frac{8}{n} R_2 e^{2 \pi} + \frac{n}{2H^2}.
\]
The rest is a standard argument shown in the proof of Theorem 3.1. By the choices of \( \epsilon, \gamma \), bounds (A.10) and
\[
\frac{R_1}{2 R_2 e^{T/n}} \leq \phi((R_2 e^{T/n} - 1)) \leq \phi(w) \leq \phi(R_1^{-1}) = 1,
\]
22
we proceed and obtain, for every \((p,t) \in M^n \times (0,T]\),

\[
\frac{1}{H^2}(p,t) \leq C_n \left(\frac{R_2}{R_1} e^{\frac{t}{2}}\right)^{2-\epsilon} \left(1 + \frac{1}{t}\right).
\]  

(A.15)

Now for time \(t > 1\), we can always apply this estimate starting at time \(t - 1\). Inequality \((A.10)\) implies that the ratio between star-shapedness bounds from above and below remains unchanged over time. This way we can replace \((R_2/R_1)^{2-\epsilon}\) in the above estimate by \((R_2/R_1)^{2-\epsilon}\) after possibly enlarging the constant \(C_n\). Since \((R_2/R_1)^{2-\epsilon} \leq (R_2/R_1)^2\), the theorem follows. \(\square\)

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