Classifying nearest-neighbour interactions and deformations of AdS

Marius de Leeuw, Chiara Paletta, Anton Pribytok, Ana. L. Retore, and Paul Ryan

School of Mathematics & Hamilton Mathematics Institute
Trinity College Dublin
Dublin, Ireland

We classify all regular solutions of the Yang-Baxter equation of eight-vertex type. Regular solutions correspond to spin chains with nearest-neighbour interactions. We find a total of four independent solutions. Two are related to the usual six- and eight-vertex models that have $R$-matrices of difference form. We find two completely new solutions of the Yang-Baxter equation, which are manifestly of non-difference form. These new solutions contain the $S$-matrices of the AdS$_2$ and AdS$_3$ integrable models as a special case. Consequently, we can classify all possible integrable deformations of eight-vertex type of these holographic integrable systems.

INTRODUCTION

The Yang-Baxter equation (YBE) is an important equation that appears in many different areas of physics. It signals the presence of integrable structures which manifest themselves in areas ranging from condensed matter physics to holography. Famous integrable models such as the Heisenberg spin chain and the Hubbard model were important for our understanding of low-dimensional statistical and condensed matter systems. Similarly, over the last few years, exceptional progress has been made in understanding the AdS/CFT correspondence due to the discovery of integrable structures [1].

The solutions of the Yang-Baxter equation are the so-called $R$-matrices which generate the tower of conserved charges that define integrable models [2, 3]. Alternatively, they describe the two-particle scattering matrix in integrable field theories [4, 5].

Clearly, understanding and classifying the solutions of the Yang-Baxter equation is an important and open question with multidisciplinary applications. Recently we put forward a new method [6, 7] to classify regular solutions of the Yang-Baxter equation by using the so-called boost automorphism [9, 10]. Regular solutions are those whose corresponding integrable lattice models have nearest-neighbour interactions. The main idea behind this method is to use the Hamiltonian rather than the corresponding $R$-matrix as a starting point. So far, we applied this method to solutions of the YBE that were of difference form $R(u, v) = R(u - v)$. In this paper we extend our approach to the most general case.

We demonstrate our method by classifying all solutions of the Yang-Baxter equation of eight-vertex type. We find four different types of models. Two models are related to the usual six- and eight-vertex models that have $R$-matrices of difference form. However, additionally, we find two completely new solutions of the Yang-Baxter equation which are manifestly of non-difference form.

As a further application of our results, we show that the relevant $R$-matrices that appear in the lower-dimensional cases of the AdS/CFT correspondence [13–16] are indeed contained in our solutions. We can then use our results to completely classify all their integrable deformations within the aforementioned framework. In particular we show that the AdS$_2$ integrable model only admits a one-parameter deformation, while the AdS$_3$ case admits both a two-parameter elliptic deformation as well as a family of functional deformations. We postpone further details to an upcoming publication [17].

METHOD

Conserved charges Consider a general solution $R(u, v)$ of the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

(1)

where we do not assume that $R$ is of difference form i.e. $R_{ij}(u_i, u_j) \neq R_{ij}(u_i - u_j)$. Such an $R$-matrix will generate a transfer matrix corresponding to an integrable spin chain via

$$t(u, \theta) = \text{tr}_0 \left[ R_{0L}(u, \theta_L) \ldots R_{01}(u, \theta_1) \right],$$

(2)

where $L$ would be the number of sites and $\theta_i$ are physical parameters associated to each of the quantum spaces. We restrict to homogeneous spin chains in which the $\theta_i = \theta$ parameters of all the physical spaces coincide.

We will furthermore restrict to integrable models with nearest-neighbour interactions and hence we assume that $R$ is regular, i.e. $R_{ij}(u, u) = P_{ij}$ where $P_{ij}$ is the permutation operator on sites $i$ and $j$. Then, the spin chain Hamiltonian $Q_2$ has interaction range two and is given by the logarithmic derivative of the transfer matrix

$$Q_2(\theta) = \sum_i \mathcal{H}_{i,i+1}(\theta),$$

(3)

with Hamiltonian density

$$\mathcal{H}(\theta) = R^{-1}(u, \theta) \frac{d}{du} R(u, \theta) \bigg|_{u=\theta}. $$

(4)

In the special case when the $R$-matrix is of difference form, the dependence on the parameter $\theta$ drops out.
The other conserved charges of the integrable model are given by the higher derivatives of the transfer matrix. More specifically we have

\[ Q_{r+1} \sim \frac{d^r}{d\theta^r} \log t(u, \theta) \bigg|_{u=\theta}. \]  

(5)

The interaction range of \( Q_r \) is \( r \) and from the Yang-Baxter equation it is easy to show that

\[ [Q_r, Q_s] = 0. \]  

(6)

This tower of conserved charges is the defining property of an integrable system. In this paper we will construct all models with certain properties that have a tower of conserved charges coming from an \( R \)-matrix.

**Boost operator**  Instead of taking logarithmic derivatives of the transfer matrix, there is an alternative way to compute the higher conserved charges \( Q_{r=3,4} \). There is a so-called boost operator \( B[Q_2] \) such that

\[ Q_{r+1} = [B[Q_2], Q_r], \quad r > 1. \]  

(7)

The boost operator is a differential operator and depends on the coefficients of the Hamiltonian \([13]\)

\[ B[Q_2] := \partial_\theta + \sum_{n=-\infty}^{\infty} n \mathcal{H}_{n,n+1}(\theta). \]  

(8)

This expression is strictly-speaking only defined for infinite length chains, but reduces consistently to spin chains of finite length.

**Integrable Hamiltonians**  Now we consider a nearest-neighbour Hamiltonian with general entries \( h_{ij}(\theta) \) and compute the corresponding charge \( Q_3 \) by using the boost operator \([13]\). The Hamiltonian potentially corresponds to an integrable system if

\[ [Q_2, Q_3] = 0. \]  

(9)

This is a necessary condition for integrability and it takes the form of a set of coupled first order, non-linear, differential equations for the components of \( \mathcal{H} \).

**R-matrix**  In order to prove integrability we then, for each potentially integrable Hamiltonian, compute the corresponding \( R \)-matrix. Let \( \hat{R} \) be the derivative with respect to the first variable, then by expanding the YBE around the point \( u_1 = u_2 \equiv u \) to first order and by using regularity and \([14]\) we find

\[ [R_{13}R_{23}, \mathcal{H}_{12}(u)] = \hat{R}_{13}R_{23} - R_{13}\hat{R}_{23}, \]  

(10)

and with \( R' \) denoting the derivative with respect to the second variable, by expanding the YBE around the point \( u_2 = u_3 \equiv v \), we find

\[ [R_{13}R_{12}, \mathcal{H}_{23}(v)] = R_{13}R'_{12} - R'_{13}R_{12}, \]  

(11)

with \( R_{ij} = R_{ij}(u, v) \). These equations are special cases of the Sutherland equation \([15]\) and they form a set of coupled first order differential equations. Since we assume regularity and know the Hamiltonian, we see that we obtain two boundary conditions which in principle fix our solution uniquely. Subsequently, we can verify whether the solutions of the Sutherland equations satisfy the Yang-Baxter equation and formally prove integrability. Notice that this method is complete in the sense that any solution of the YBE necessarily gives an integrable Hamiltonian.

**IDENTIFICATIONS**

There are some simple ways in which different solutions of the YBE can be related to each other. In what follows we will identify models which can be mapped to each other under any of the following transformations.

**Local basis transformation**  If \( R(u, v) \) is a solution of the Yang-Baxter equation, then we can generate a different solution by defining

\[ R^{(V)}(u, v) = \left[ V(u) \otimes V(v) \right] R(u, v) \left[ V(u) \otimes V(v) \right]^{-1}. \]  

(12)

This new solution is trivially compatible with regularity. On the level of the Hamiltonian it gives rise to a new integrable Hamiltonian which takes the form

\[ \mathcal{H}^{(V)} = \left[ V \otimes V \right] \mathcal{H} \left[ V \otimes V \right]^{-1} - \left[ VV^{-1} \otimes 1 - 1 \otimes VV^{-1} \right], \]  

(13)

where everything is evaluated at \( \theta \).

**Reparametrization**  If \( R(u, v) \) is a solution, then \( R(f(u), f(v)) \) clearly is a solution of the YBE as well. This transformation affects the normalization of the Hamiltonian. We are also free to reparameterize any other functions and constants in both the \( R \)-matrix and Hamiltonian. For instance the \( R \)-matrices from \([19, 20]\) can be obtained by using a reparameterization of the usual XXX \( R \)-matrix.

**Normalization**  If \( R(u, v) \) is a solution, then for any function \( g(u, v) \) the product \( g(u, v) R(u, v) \) is a solution as well. On the level of the Hamiltonian this corresponds to rescaling and shifting \( g(\theta, \theta) \mathcal{H} + g'(\theta, \theta) \).

**Discrete transformations**  It is straightforward to see that for any solution \( R(u, v) \) of the Yang-Baxter equation, \( PR(u,v)P, R^T(u,v) \) and \( PR^T(u,v)P \) are solutions as well.

All these transformations are universal and hold for any integrable model. Moreover, they have a trivial effect on the spectrum, which means that they basically describe the same physical model.
RESULTS FOR 4 × 4

Let us apply our method to spin chains with a two-dimensional local Hilbert space. We already applied our method to R-matrices of difference form and found that in that case only Hamiltonians of eight-(or less) vertex type seem to be physical [7]. For this reason we will for the moment only consider Hamiltonians of eight-vertex type. We parametrize our Hamiltonian as

\[
\mathcal{H} = h_1 \mathbb{1} + h_2 (\sigma_z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z) + h_3 (\sigma_+ \otimes \sigma_- + h_4 \sigma_- \otimes \sigma_+ + h_5 (\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + h_6 \sigma_+ \otimes \sigma_- + h_7 \sigma_- \otimes \sigma_+ + h_8 \sigma_+ \otimes \sigma_-),
\]

where \( h_i = h_i(\theta) \) and \( \sigma_i \) are the Pauli matrices with \( \sigma^\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y) \). For future reference we define the primitive functions \( H_i(\phi) = \int_0^\phi h_i(\phi) d\phi \).

Similarly let us introduce the R-matrix as

\[
R = \begin{pmatrix}
    r_1 & 0 & 0 & r_8 \\
    0 & r_2 & r_6 & 0 \\
    0 & r_5 & r_3 & 0 \\
    r_7 & 0 & 0 & r_4
\end{pmatrix},
\]

where we suppressed the dependence on the spectral parameters.

After using our identifications, we find only four different types of integrable 4 × 4 Hamiltonians that solve the integrability condition [21] [23].

- 6-vertex A, \( h_6 \neq 0 \) and \( h_7 = h_8 = 0 \)
- 6-vertex B, \( h_6 = h_7 = h_8 = 0 \)
- 8-vertex A, \( h_6 \neq 0, h_7 \neq 0, h_8 \neq 0 \)
- 8-vertex B, \( h_6 = 0 \) and \( h_7 \neq 0, h_8 \neq 0 \).

Notice that for R-matrices of difference form, there are actually eight independent solutions [21] [23]. This means that some of these solutions are reductions of the same R-matrix of non-difference type. In particular, all 7-vertex type solutions are special cases of 8-vertex integrable models. Furthermore if a Hamiltonian is equivalent to a well-known one under identifications we will not list the corresponding R-matrix.

Let us discuss the models in more detail.

6-vertex A Setting \( h_7 = h_8 = 0 \) and plugging this Hamiltonian into (10), we see that it is satisfied if and only if the functions \( h_i(\theta) \) satisfy the following differential equations

\[
\frac{h_3}{h_6} = \frac{h_6}{h_6} + 4h_5, \quad \frac{h_4}{h_4} = \frac{h_6}{h_6} - 4h_5,
\]

provided that \( h_6 \neq 0 \). This is easily solved to give

\[
h_3 = c_3 h_6 e^{4H_5}, \quad h_4 = c_4 h_6 e^{-4H_5},
\]

where \( c_{3,4} \) are constants. The Hamiltonian is equivalent to that of the XXZ spin chain. In other words, the source of the non-difference dependence on the spectral parameters is only due to twists, basis transformations and reparameterizations of the R-matrix.

6-vertex B It is easy to see that setting \( h_6 = h_7 = h_8 = 0 \) makes the Hamiltonian satisfy \( [Q_2, Q_3] = 0 \) for any choice of \( h_1, \ldots, h_5 \). Thus, the Hamiltonian depends on five free functions. We can account for four of them by using a local basis transformation, a twist, a normalization and a reparameterization. Since there is one free function left, this model does not have an R-matrix of difference form underlying it and it is a new solution of the Yang-Baxter equation.

Without loss of generality, we choose our R-matrix normalized such that \( r_5 = 1 \) and we can set \( h_2 = 0 \). It then follows from the Sutherland equation (10) that

\[
r_6 = 1 = r_1 r_4 + r_2 r_3 \quad \text{and} \quad r_1 = \frac{r_2 + 2h_5 r_2}{h_4}, \quad r_3 = \frac{-2h_5 r_4 + r_4}{h_4},
\]

while \( r_4 \) satisfies a Riccati equation

\[
r_4 = \frac{h_4}{h_4} r_4 + r_1 \left[ h_3 h_4 - \frac{2h_5 h_4}{h_4} + 2(h_5 - 2h_5^2) \right] = 0.
\]

Now we can introduce a reparameterization of the spectral parameter

\[
u_i \mapsto x_i = 2 \int_0^{\nu_i} \frac{h_3 h_4}{h_3 h_4 - 4h_5^2} \frac{h_5}{h_4} - h_5
\]

which kills the non-derivative term in the Riccati equation. It is then straightforward to solve our system of differential equations to find

\[
r_2(x, y) = H_4(x) - H_4(y),
\]

\[
r_1(x, y) = 1 + 2 \frac{h_5(x)}{h_4(x)} r_2(x, y),
\]

\[
r_3(x, y) = \frac{4 h_5(x) h_5(y)}{h_4(x) h_4(y)} r_2(x, y) - 2 \left[ \frac{h_5(x)}{h_4(x)} \frac{h_5(y)}{h_4(y)} \right] \left[ \frac{h_3(x)}{h_4(x)} + \frac{h_3(y)}{h_4(y)} \right],
\]

\[
r_4(x, y) = 1 - 2 \frac{h_5(y)}{h_4(y)} r_2(x, y).
\]

This solution is manifestly not of difference form and it is easy to show that it satisfies the Yang-Baxter equation and the correct boundary conditions. Notice also that the form of the R-matrix depends on the type of functions \( h_i \) from the Hamiltonian. For instance, if you chose \( h_i \) to be constant, then \( R \) will be rational.
\(8\)-vertex A\n
In case \(h_6 \neq 0\), the integrability constraint gives that \(h_3 = h_4\), \(h_5 = 0\) together with the following differential equations

\[
\begin{align*}
\frac{h_3}{h_1} &= \frac{h_6}{h_6}, & \frac{h_7}{h_2} &= \frac{h_6}{h_6} + 4h_2, & \frac{h_8}{h_3} &= \frac{h_6}{h_6} - 4h_2. \quad (25)
\end{align*}
\]

These equations are easily solved to be

\[
h_3 = c_3h_6, \quad h_7 = c_7h_6 e^{4H_2}, \quad h_8 = c_8h_6 e^{-4H_2}, \quad (26)
\]

where \(c_i\) are constants. The resulting Hamiltonian is that of the XYZ spin chain under our identifications.

\(8\)-vertex B\n
In the case when \(h_6 = 0\), we find that the most general solution satisfies the differential equations

\[
\begin{align*}
\frac{\dot{h}_7}{h_7} &= 4h_2 + \frac{h_3 + h_4}{h_3 + h_4} + \frac{h_3 - h_4}{h_3 + h_4} h_5 = \frac{h_8}{h_8} + 8h_2, \quad (27)
\frac{h_5}{h_5} &= -\frac{h_3^2 - h_4^2}{4h_5} + \frac{h_3 + h_4}{h_3 + h_4} + \frac{h_3 - h_4}{h_3 + h_4} h_5. \quad (28)
\end{align*}
\]

In order to solve these equations we can introduce two new functions that simplify this set of differential equations. Define \(b_1, b_2\) such that

\[
h_3 = \pm \sqrt{\frac{b_1}{b_2}} (2h_5 + b_2), \quad h_4 = \pm \sqrt{\frac{b_1}{b_2}} (2h_5 - b_2). \quad (29)
\]

We then get a simple equation for \(b_2\) that can be solved to give

\[
b_2 = \frac{b_1}{c_2^2 e^{4B_1} + 1}, \quad B_1 = \int b_1. \quad (30)
\]

The solutions to the remaining equations are then

\[
h_7 = c_7h_5 e^{4H_2+2B_1}, \quad h_8 = c_8h_5 e^{-4H_2+2B_1}. \quad (31)
\]

We see that there are four free functions remaining and hence this model is genuinely of non-difference form.

We again normalize our \(R\)-matrix such that \(r_5 = 1\) and we use a local basis transformation to set \(a_2 = 0\). We then apply a further constant basis transformation and set \(h_7 = h_8\) which implies that \(r_7 = r_8\). Moreover, let us set the normalization of \(\mathcal{H}\) such that \(h_8 = k\), which corresponds to choosing \(h_5 = \frac{k}{2} e^{-2B_1}\). We see that \(r_5 = r_6\) and moreover obtain the following differential equation for \(r_8\)

\[
r_8^2 = k^2(r_8^2 + 1)^2 - 4r_8^2. \quad (32)
\]

This can only be solved in closed form since \(k = \frac{2c_2}{2c_2}\) is constant. The solution is

\[
r_8(u, v) = k \frac{\text{sn}(u - v, k^2) \text{cn}(u - v, k^2)}{\text{dn}(u - v, k^2)}. \quad (33)
\]

where \(\text{sn}, \text{cn}, \text{dn}\) are the usual Jacobi elliptic functions with modulus \(k^2\). The remaining entries of \(R\) can be expressed in terms of \(r_8\) and after redefining \(h_5(x) = -\frac{1}{2} \cot \eta(x)\). We find the following solution

\[
\begin{align*}
r_1 &= \frac{1}{\sin \eta(u) \sqrt{\sin \eta(v)}} \left[ \sin \eta_+ \left( \frac{\text{cn}}{\text{dn}} - \cos \eta_+ \text{sn} \right) \right], \quad (34)
\end{align*}
\]

\[
\begin{align*}
r_2 &= \frac{1}{\sin \eta(u) \sqrt{\sin \eta(v)}} \left[ \cos \eta_- \text{sn} + \sin \eta_- \frac{\text{cn}}{\text{dn}} \right]. \quad (35)
\end{align*}
\]

\[
\begin{align*}
r_3 &= \frac{1}{\sin \eta(u) \sqrt{\sin \eta(v)}} \left[ \cos \eta_- \text{sn} - \sin \eta_- \frac{\text{cn}}{\text{dn}} \right]. \quad (36)
\end{align*}
\]

\[
\begin{align*}
r_4 &= \frac{1}{\sin \eta(u) \sqrt{\sin \eta(v)}} \left[ \sin \eta_+ \frac{\text{cn}}{\text{dn}} + \cos \eta_+ \text{sn} \right]. \quad (37)
\end{align*}
\]

where \(\eta_\pm = \frac{(u \pm v)}{2}\) and all the Jacobi elliptic functions depend on the difference \(u - v\), i.e. \(\text{sn} = \text{sn}(u - v, k^2)\). This solution indeed satisfies the Yang-Baxter equation and has the correct boundary conditions. Moreover, it is easy to see that in the case where \(\eta\) is constant, it becomes of difference form and reduces to the well-known solution found in 21, 22.

The limit \(c_2 \to 0\) is interesting, and the models falling into this category include the AdS\(_2\) integrable system. However, it should be handled with certain care and is equivalent to taking \(k \to \infty\). In order to take this limit we should rescale the spectral parameters \((u, v) \to (\frac{u}{k}, \frac{v}{k})\) and make use of the well-known identities for inversions of the elliptic modulus. We also need to redefine \(B_1(\frac{u}{k}) \to B_1(u)\). We can then now safely take \(k \to \infty\) and easily find that \(R\) becomes of trigonometric type.

**DEFORMATIONS of AdS\(_{2,3}\)**

For both the AdS\(_2\) and AdS\(_3\) integrable models, the \(R\)-matrix contains separate \(4 \times 4\) blocks that need to satisfy the YBE by themselves. We demonstrate that these blocks fit into our classification. From our method, it is easy to see that it is enough to map the AdS\(_{2,3}\) Hamiltonians to the Hamiltonians that we found, rather than compare \(R\)-matrices. The AdS/CFT Hamiltonians depend on the rapidity through the variables \(x^\pm\) defined as

\[
u = \frac{1}{2} \left( x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} \right), \quad \frac{x^+}{x^-} = e^{i\phi}. \quad (38)
\]

**AdS\(_3\)** For AdS\(_3\), we see that the Hamiltonian of particles with the same chirality 13, 14 is of 6-vertex B type. We compute the Hamiltonian and identify the resulting functions with \(h_1, \ldots, h_5\). For the spin chain frame 24, the result is given by \(h_2 = 0\) and

\[
\begin{align*}
h_3(u) &= \frac{\dot{x}^-}{x^- - x^+}, & h_4(u) &= \frac{\dot{x}^+}{x^+ - x^-}, \quad (39)
\end{align*}
\]

\[
\begin{align*}
h_1 &= -\frac{1}{2}(h_3 + h_4), & h_5 &= -\frac{1}{2}h_1 \quad (40)
\end{align*}
\]

and we take the positive sign in the square root in \(h_3, h_4\). A similar expression holds for 14 up to factors of
$e^{ip/2}$ in $h_{3, 4}$ and now $h_2 \neq 0$. We now see that there are two possible types of deformations. First, we can match this model with our 6-vertex B type, which leaves us with a continuous family of deformations since we can add arbitrary functions of the spectral parameter to all of the components. This might be a reflection of the special nature of two-dimensional CFTs. Second, we can embed the Hamiltonians $[14, 24]$ in our 8-vertex B model. This gives a one-parameter elliptic deformation of the $AdS_3$ model. The embedding is given, for the spin chain frame, by

$$h_1 = \frac{1}{2} \frac{x^+ + x^-}{x^+ - x^-}, \quad b_1 = \frac{1}{2} \frac{x^+ - x^-}{x^+ - x^-}, \quad h_5 = -\frac{i}{2} h_1.$$  

Together with $h_2 = c_2 = c_7 = c_8 = 0$. This is a novel elliptic deformation. For the string frame, $c_2 \neq 0$ and $h_2 \neq 0$ and we take the positive sign of the square root in $h_{3, 4}$.

$AdS_3$ The massive sector of the $AdS_3 \times S^2 \times T^6$ string sigma model [10] is of 8-vertex B model type with the + sign in the square root in the Hamiltonian. It has $c_2 = 0$ and furthermore $c_7 = -c_8$. The non-zero components of the Hamiltonian are parameterized as

$$h_1 = \frac{1}{4} \frac{x^+ + x^-}{x^+ - x^-} \left[ \frac{x^+ + x^-}{x^+ - x^-} \right],$$

$$h_5 = \frac{1}{8} \frac{1 + e^{ip/2}}{1 - e^{ip/2}} \left[ \frac{x^+ + x^-}{x^+ - x^-} \right],$$

$$B_1 = -\frac{1}{2} \log \left[ \frac{c_6 e^{-ip/2} 1 + e^{ip/2}}{4 \left( 1 - e^{ip/2} \right)} \left( x^+ - \frac{1}{x^-} \right) \right].$$

We can conclude that this integrable model only admits a non-trivial one-parameter deformation by taking $c_2$ to be non-zero.

CONCLUSIONS & OUTLOOK

In this paper we classified all the regular solutions of the Yang-Baxter equation that are of eight-vertex type. We find 4 independent solutions of which two are new solutions. We were able to relate some $AdS/CFT$ integrable models to our new models and in this way we could find all their integrable deformations. The $AdS_3$ $R$-matrices we found correspond to the case of same chirality, while the $R$-matrices of opposite chirality are not regular and can instead be obtained up to some constants by requiring that they satisfy the YBE [17]. It is interesting that we can deform the two matrices with the same chirality independently. There are many new pressing open questions and future directions for research.

First, it would be interesting to apply our method to a wider range of physical systems. In particular the case of a four-dimensional local Hilbert space is of interest as it would contain the Hubbard model and generalizations thereof. In this way deformations of the $AdS_5$ superstring could also be studied. We plan to address some of these issues in an upcoming publication [17].

Second, it would be important to study and understand the physical and mathematical properties of the new solutions of the YBE that we have derived. For instance, one obvious direction would be computing the spectrum of the 8-vertex B model by performing the Bethe Ansatz. Similarly, it would be interesting to find out if there is a quantum algebra underlying our new solutions. It is also unclear if there are 1+1 dimensional integrable field theories whose two-body scattering matrix corresponds to our new solutions.

Third, we need to find an interpretation for the deformations for the holographic integrable models that we found. In particular, the meaning of the deformation parameters on both the String and CFT side needs to be worked out. Understanding the functional (infinite dimensional) deformation of the $AdS_3$ model should also be very fascinating.

Fourth, our method raises some further questions regarding the general structure of integrable models. So far, all solutions that we have found, imposing [19] is sufficient. This seems to support an old conjecture for integrability [23]. However, it is unclear why this is the case and attempts at proving it have failed. Moreover, we do not impose braiding unitarity, $R_{12}(u, v)R_{21}(v, u) \sim 1$, but all our solutions satisfy it nevertheless.

Lastly, it would be interesting to consider long-range deformations of our models [26, 28]. Long-range deformations of spin chains correspond to higher loop corrections in the $AdS/CFT$ correspondence.

Acknowledgements. We would like to thank S. Ekhammar, V. Korepin, V. Kazakov, O. Ohlsson Sax, L. Takhtadzhian and A. Torrielli for discussions. MdL was supported by SFI, the Royal Society and the EPSRC for funding under grants UF160578, RGF\EA\180111, RGF\EA\180167 and 18/EPSRC/3590. C.P. is supported by the grant RGF\EA\180111. A.P. is supported by the grant RGF\EA\180167. A.L.R. is supported by the grant 18/EPSRC/3590. The work of P.R. is supported in part by a Nordita Visiting PhD Fellowship and by SFI and the Royal Society grant UF160578.

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