Multi-photon filtering

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Abstract

The purpose of this paper is to derive the filtering equations for quantum systems driven by fields in a multi-photon state. The joined dynamics of the system and filed are described by quantum stochastic differential equations. With the help of a non-Markovian embedding method, quantum filters with quadrature (homodyne) or photon counting (photodetection) measurements are derived, which are given in terms of systems of coupled equations. A two-level atom example is used to illustrate interesting features of the multi-photon quantum filtering.

Key Words multi-photon state, quantum filtering, non-Markovian embedding.

1 Introduction

When light interacts with a quantum system, system information can be transferred to the scattered light. Such information can be used to monitor the dynamics of the system, or to control it. How to extract useful information about the underlying quantum system from continuous measurement of the scattered light is a task of quantum filtering ([3]; [5]; [28]; [6]; [10]; [29]; [8]).

Quantum filters can not only be used to analyze the conditional dynamics of the system under continuous measurement, but also facilitate measurement-based quantum feedback control design ([29]; [11]; [25]; [24]). For example, in a quantum network, some output channels of a component can be measured and the resulting information is sent to another component (or to the component itself). In this case, how to physically describe the conditional dynamics of such process becomes essential.

A large portion of quantum filtering literature considers incoming light in a coherent state, not until very recently have researchers paid their attention to deriving quantum filters for non-Gaussian states, such as single photon states and superpositions of coherent states, with quadrature or photon counting measurements ([15]; [17]; [16]).

As we know, multi-photon states are very useful in quantum computing, quantum communication, and quantum cryptography ([21]; [11]; [30]). They can be used to catalyze the generation of non-classical states like coherent state superposition state ([1]). Because entangled $n$ photons can act as a collective entity acquiring a phase at a rate $n$ times faster than a single photon with the same wavelength, multi-photon states also play an important role in quantum metrology. At present, to the best of our knowledge, there are no filtering results for systems driven by continuous-mode multi-photon fields. In view of the increasing importance of multi-photon states, the purpose of this paper is to derive quantum filters for quantum systems driven by fields in multi-photon states.

In the case of multi-photon fields, especially the Fock states ([7]; [20]), it has been found that the master equation describing unconditional dynamics is a system of coupled equations ([11]), a feature of non-Markovian character. To derive the quantum filtering equations (quantum trajectories) for this class of non-Markovian master equations, there are two widely used methods. One is the Markovian embedding ([9]). It’s main idea is to construct an ancilla system.

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to combine with the system of interest to form a Markovian extended system driven by the vacuum state, then the filtering results can be obtained from the standard filter for the extended system. Essentially, it is a special kind of cascade methods. In ([15]) it has been shown how to use this approach to derive the quantum filtering equations for the single photon field. The other one is the non-Markovian embedding. In this case, an ancilla is added in parallel to the system. The ancilla, system and field are initialized in a superposition state. While standard filtering results do not apply, the quantum stochastic methods can nevertheless be applied to determine the quantum filters. Comparing with the Markovian embedding, it seems like “parallel connection”. Recently, this method has also been used to derive the quantum filter for the single photon state ([17];[16]).

Due to the complexity of multi-photon states, it appears difficult to determine a suitable ancilla system for cascaded connection. So in this paper, we adopt the non-Markovian embedding method to derive the multi-photon filters. As a by-product, we obtain the master equation for the system. A significant feature of the master equation and the quantum filter is that they are both given by a system of coupled equations, not a single equation as in the vacuum case.

It is worth mentioning that our formalism generalizes the previous results. For example, when restricted to the single-photon case, the corresponding master and filter equations reduce to those given in [15].

The paper is organized as follows. In Section 2 the continuous-mode multi-photon states are introduced, and the multi-photon quantum filtering problem is formulated. In Section 3 the master equation for a quantum system driven by a multi-photon field is derived. In Section 4, the multi-photon quantum filter for the homodyne detection case is derived with the help of the non-Markovian embedding method. The main results about the quantum filter when the direct photoncounting detection is adopted is presented in Section 5. In Section 6, an explicit example is given to present the significant performances of the multi-photon filtering equations. Some concluding remarks are given in Section 7.

2 Model Formulation

The system considered in this paper is a quantum system $S$ coupled to a quantum field $B$, as shown in Fig. 1. The system $S$ is assumed to be defined on a Hilbert space $H_S$, with an initial state denoted as $|\eta\rangle \in H_S$. The quantum field $B$ has two components, the input field $B_{in}$ and the output field $B_{out}$. The input field $B_{in}$ is described in terms of annihilation $B(t)$ and creation $B^*(t)$ operators defined on a Fock space $F$ ([22];[8]), where $^*$ is Hermitian adjoint. Unlike the previous work which mainly concerns single photon states or vacuum state, here we assume the initial state of the field is in a general $n$-photon state, denoted as $|\Phi_n\rangle$, where $n$ can be any positive integer, to be discussed shortly (Section 2.1). In Fig. 1 the interaction between the system and the field correlates them, thus measurements of the output field contains information about the system. The task of the quantum filter is to process the continuous measurement stream $Y(t)$ in order to find the best estimates $\hat{X}(t)$ of the state of the system observable $X(t)$, an operator on $H_S$.

The following of this section is divided into 4 parts to precisely formalize the multi-photon quantum filtering problem.

2.1 Multi-photon state

For a single input channel, the most general $n$-photon state in the time domain is defined as

$$|\Phi_n\rangle = \frac{1}{\sqrt{N_n}} \int_0^{+\infty} \cdots \int_0^{+\infty} \Phi(t_1, \cdots, t_n) b^*(t_1) \cdots b^*(t_n) dt_1 \cdots dt_n |0\rangle, \tag{2.1}$$

where $b^*(t)$ is the fundamental field operator, and also satisfies $dB^*(t) = \int_t^{t+dt} b^*(s) ds$. The function $\Phi(t_1, \cdots, t_n)$ is called spectral density function (SDF), which completely determines the state. Here, $N_n$ is the normalized coefficient,
Figure 1: A schematic representation of the multi-photon quantum filtering procedure in quantum optics. An optical field \( B_{\text{in}}(t) \) in a multi-photon state \( |\Phi_n\rangle \) interacts with a system initialized in a state \( |\eta\rangle \). After the interaction, the output field \( B_{\text{out}}(t) \) is continuously detected, giving rise to the measurement signal \( Y(t) \) which contains the system information. Then the output \( Y(t) \) is filtered to produce the estimate \( \hat{X}(t) \) of system operator \( X(t) \).

\[
N_n = \sum_{P \in S_n} \int_0^{+\infty} \cdots \int_0^{+\infty} \Phi^*(t_1, \cdots, t_n) \Phi(P(t_1, \cdots, t_n)) d\tau_1 \cdots d\tau_n,
\]

where \( P(t_1, \cdots, t_n) \) is a permutation of the set \( \{t_1, \cdots, t_n\} \), and \( S_n \) is the set of all permutations. It can be easily verified that \( \langle \Phi_n | \Phi_n \rangle = 1 \).

If the SDF in Eq. \((2.2)\) is factorized, that is to say, there exist functions \( \xi_i, \ i = 1, \cdots, n \) satisfying \( \Phi(t_1, \cdots, t_n) = \prod_{i=1}^{n} \xi_i(t_i) \), the \( n \)-photon state can be rewritten as

\[
|\Phi_n\rangle = \frac{1}{\sqrt{N_n}} \int_0^{+\infty} \cdots \int_0^{+\infty} \xi_1(t_1) \cdots \xi_n(t_n) b^*(t_1) \cdots b^*(t_n) d\tau_1 \cdots d\tau_n |0\rangle
\]

\[
= \frac{1}{\sqrt{N_n}} \prod_{i=1}^{n} B^*(\xi_i)|0\rangle, \tag{2.2}
\]

with \( B^*(\xi_i) = \int_0^{+\infty} \xi_i(t)b^*(t) d\tau \). We call the multi-photon state in this type as the factorized multi-photon state, which can be regarded as the \( n \)-photon state determined by the function set \( M_n = \{\xi_1, \xi_2, \cdots, \xi_n\} \). Here we distinguish functions in terms of their subscripts, so two functions with different subscripts are regarded different even if they are identical. Obviously, if all the \( \xi_i \) equals to \( \xi, i = 1, \cdots, n \), the \( n \)-photon state defined in Eq. \((2.2)\) is exactly the continuous-mode Fock state \((2.2)\). That is, the continuous-mode Fock state is a special type of the factorized multi-photon state. On the other hand, the state whose SDF can’t be factorized is called an unfactorized multi-photon state. With the help of the occupation number representation developed by \([23]\), an unfactorized multi-photon state can be written as the superposition and combination of the factorized ones. So in this paper we focus on the factorized multi-photon states.

### 2.2 Dynamics of the system

The dynamics of the quantum system is described by the quantum stochastic calculus \((19, 14, 22, 8)\), which are defined in terms of fundamental field operators \( B(t), B^*(t) \) and \( \Lambda(t) \). The non-zero Itô products for the \( n \)-photon states \((2.2)\) can be verified to be identical to those for the vacuum case, that is,

\[
dB(t)dB^*(t) = dt, \quad dB(t)d\Lambda(t) = dB(t), \quad d\Lambda(t)d\Lambda(t) = d\Lambda(t), \quad d\Lambda(t)dB^*(t) = dB^*(t). \tag{2.3}
\]

The dynamical evolution of the composite system, including the system and the field, is described by a unitary \( U(t) \) following the quantum stochastic differential equation

\[
dU(t) = \left[ (S - I)d\Lambda(t) + LdB^*(t) - L^*SdB(t) - \left( \frac{1}{2}L^*L + iH \right)dt \right] U(t), \tag{2.4}
\]
with the initial condition $U(0) = I$. Here, $H$ is a fixed self-joint operator presenting the free Hamiltonian of the system, $L$ is a system operator determining the coupling of the system to the field, and $S$ is a unitary operator describing the photon scattering phase. In this paper, we assume that the parameters $S, L, H$ are bounded operators on the system Hilbert space $H_S$.

The dynamics of the system is given in terms of the Heisenberg evolution, written as $X(t) = \tilde{j}_i(X) = U^*(t)(X \otimes I)U(t)$, for any observable $X \in H_S$. Using the quantum Itô rules in Eq. (2.3), $j_i(X)$ satisfies the flowing quantum stochastic familiar differential equations:

$$d\tilde{j}_i(X) = j_i(S^*XS-X)\,d\Lambda(t) + j_i(S^[X,L])dB(t) + j_i(L^*,X)dB(t) + j_i(L(X))dt,$$

(2.5)

where $\mathcal{L}(X) = -i[X,H] + L^*XL - \frac{1}{2}XL^*L - \frac{1}{2}L^*LX$ is the familiar Lindblad generator.

### 2.3 Measurements

Unlike in classical models, where one is able to observe the system directly, in the quantum optics case, we often use a light field to probe the system. By the interaction between the system and the field, the field is perturbed and carries away information of the system. In the Heisenberg’s picture, the field observable $Y$ at time $t$ is given by $Y(t) = U^*(t)(I \otimes Y)U(t)$, where $U(t)$ is the joint system-field unitary operation given in Eq. (2.4).

Two types of measurements are commonly used in quantum optics: homodyne detection ([18]), related to the noise quadrature as $Y^{W}(t) = U^*(t)(B(t) + B^*(t))U(t)$, and direct photodetection (photon counting) ([2]), related to the process as $Y^{A}(t) = U^*(t)\Lambda(t)U(t)$. From now on, we use the superscript $W$ and $A$ to denote the homodyne detection and the direct photodetection, respectively. Using the Itô rules, we can obtain

$$dY^{W}(t) = j_i(S^*dB^*(t) + j_i(S)dB(t) + j_i(L + L^*)dt,$$

(2.6)

$$dY^{A}(t) = dB(t) + j_i(S^*L)dB^*(t) + j_i(L^*S)dB(t) + j_i(L^*L)dt.$$

(2.7)

Intuitively, it would appear that $Y^{W}(t)$ looks like a noisy observation of $j_i(L + L^*)$, whereas $Y^{A}(t)$ is like a Poisson process whose intensity is controlled by $j_i(L^*L)$. Note that the observation processes $Y^{A}(t)$ and $Y^{W}(t)$ do obey the self-nondemolition property, i.e., $[Y^{W}(t), Y^{W}(s)] = [Y^{A}(t), Y^{A}(s)] = 0$. We denote by $\mathcal{Y}^{W}(t)$ and $\mathcal{Y}^{A}(t)$ as the commutative von Neumann algebras generated by the observation process $Y^{W}(s)$ and $Y^{A}(s), s \leq t$, respectively.

### 2.4 Filter problem

To find a least mean square estimate of a system observable $X(t) \in H_S$ in the Heisenberg picture at time $t$, given the measurement signals $Y(s)$ up to this time, we must calculate the quantum conditional expectation ([8])

$$\hat{X}(t) = \mathbb{E}_{n\mid n}[X(t)\mid \mathcal{Y}(t)],$$

(2.8)

where $\mathcal{Y}(t)$ is the commutative von Neumann algebras generated by the observations process $Y(s), s \leq t$.

The subscript in Eq. (2.8) indicates that this conditional expectation is calculated under the $n$-photon state $|\Phi_n\rangle$. This conditional expectation is well defined, since $X(t)$ is in the commutant of $\mathcal{Y}(t)$. Precisely, $[X(t), Y(s)] = 0$ for all $s \leq t$, which is known as the nondemolition property. The conditional estimate $\hat{X}(t)$ is affiliated to $\mathcal{Y}(t)$ and is characterized by the requirement that

$$\mathbb{E}_{n\mid n}[\hat{X}(t)K] = \mathbb{E}_{n\mid n}[X(t)K], \text{ for all } K \in \mathcal{Y}(t).$$

(2.9)

The multi-photon quantum filtering problem is to calculate the evolution of the conditional expectation in Eq. (2.8). Based on which measurement is adopted, the filtering problem can be further divided into the derivation of the quantum filtering equations for the homodyne measurement $\mathbb{E}_{n\mid n}[X(t)\mid \mathcal{Y}^{W}(t)]$ and the derivation for the photoncounting measurement case $\mathbb{E}_{n\mid n}[X(t)\mid \mathcal{Y}^{A}(t)]$. The calculation of the two cases are similar, thus we will precisely derive the former one, and only list the results for the latter one.
3 Multi-photon master equation

Before deriving the quantum filtering equations, we work out the dynamical equations for the unconditioned multi-photon expectations. Such equations are of fundamental importance and often called master equations. The master equations for systems driven by the field in the Fock states have previously been derived ([1]).

Denote the $n$-photon expectations $E_{n,k}(X(t)) = \langle \eta \Phi_n | X(t) | \eta \Phi_n \rangle$ as $\mu^{n,n}_t(X)$, now we derive it’s dynamical evolution. Using the equation (3.1), we can obtain

$$
d\mu^{n,n}_t(X) = \langle \eta \Phi_n | d\xi_t(X) | \eta \Phi_n \rangle = \langle \eta \Phi_n | j_t(L(X))dt | \eta \Phi_n \rangle + \langle \eta \Phi_n | j_t([L^*, X]S)dB(t) | \eta \Phi_n \rangle + \langle \eta \Phi_n | j_t(S^*X - X)d\Lambda(t) | \eta \Phi_n \rangle.
$$

The first term can be easily rewritten as $\mu^{n,n}_t(L(X))dt$. While the second term involves the action of the quantum noise increment, which should be further calculated. Precisely,

$$
dB(t) | \Phi_n \rangle = dB(t) \prod_{i=1}^n B^*(\xi_i) | 0 \rangle = \frac{1}{\sqrt{N_n}} \int_t^{t+dt} \int_0^{+\infty} \cdots \int_0^{+\infty} \xi_1(t_1) \cdots \xi_n(t_n) b(s) b^*(t_1) \cdots b^*(t_n) ds dt_1 \cdots dt_n | 0 \rangle
$$

$$
= \frac{1}{\sqrt{N_n}} \int_t^{t+dt} \int_0^{+\infty} \cdots \int_0^{+\infty} \xi_1(t_1) \cdots \xi_n(t_n) b(t_1) b^*(t_2) \cdots b^*(t_n) ds dt_1 \cdots dt_n | 0 \rangle + \cdots
$$

$$
= \frac{1}{\sqrt{N_n}} \xi_1(t) dt \int_0^{+\infty} \cdots \int_0^{+\infty} \xi_2(t_1) \cdots \xi_n(t_n) b(t_2) b^*(t_2) \cdots b^*(t_n) dt_2 \cdots dt_n | 0 \rangle + \cdots
$$

$$
\approx \sum_{k=1}^n \frac{N_{n-1,k}}{N_n} \xi_k(t) dt | \Phi_{n-1,k} \rangle,
$$

where we have used $dB(t) = \int_t^{t+dt} b(s) ds$, and the singular commutation relation $[b(s), b^*(t)] = \delta(t - s)$. The $| \Phi_{n-1,k} \rangle$, $k = 1, \cdots, n$ is defined as $| \Phi_{n-1,k} \rangle = \frac{1}{\sqrt{N_{n-1,k}}} \prod_{i=1,i\neq k}^n B^*(\xi_i) | 0 \rangle$, with $N_{n-1,k}$ being the normalized coefficient. This state can be regarded as the $(n-1)$-photon state determined by the function set $M_{n-1,k} = M_n \setminus \{\xi_k\} = \{\xi_1, \cdots, \xi_{k-1}, \xi_{k+1}, \cdots, \xi_n\}$. The $n - 1$ represents the number of photons contained in this field and $k$ reflects the function which is removed from the set $M_n$.

With the help of the notation $\mu^{n-1,n-1,k}_t(X) \triangleq \langle \eta \Phi_{n-1,k} | j_t(X) | \eta \Phi_{n-1,k} \rangle$, we can simplify the second term in Eq. (3.1) as

$$
\sum_{k=1}^n \frac{N_{n-1,k}}{N_n} \xi_k(t) \mu^{n-1,n-1,k}_t([L^*, X]S)dt.
$$

Similarly, the other two terms in the expression can be rewritten as follows,

$$
\langle \eta \Phi_n | j_t(S^*X - X)d\Lambda(t) | \eta \Phi_n \rangle = \sum_{k,l=1}^n \frac{N_{n-1,k}N_{n-1,l}}{N_n} \xi_k(t) \xi_l(t) \mu^{n-1,k,n-1,l}_t(S^*XS - X)dt,
$$

where $\mu^{n-1,k,n-1,l}_t(X) \triangleq \langle \eta \Phi_{n-1,k} | j_t(X) | \eta \Phi_{n-1,l} \rangle$ and $\mu^{n,k,n-1,l}_t(X) \triangleq \langle \eta \Phi_{n-1,k} | j_t(X) | \eta \Phi_{n-1,l} \rangle$. 


From this, we can find that the multi-photon master equation is not a single equation as the vacuum case \([13]\), it couples downwards. To solve the expectations about the \(n\)-photon state, we should derive all the terms from vacuum case to the \(n\)-photon one. Now, it is necessary to define the general \((n-k)\)-photon state. Due to the different choices of the SDF which are removed from the set \(M_n\), there are \(C_n^k\) different \((n-k)\)-photon states. To efficiently distinguish them, we adopt the following symbols \(|\Phi_{n-k,i_1\ldots i_k}\rangle\), which represents the \((n-k)\)-photon state determined by the function set \(M_{n-k,i_1\ldots i_k} \triangleq M_n \setminus \{\xi_{i_1}, \ldots, \xi_{i_k}\}\). Because one function can’t be removed twice, we assume that the \(i_m\) are different from each other, i.e., \(i_k \neq i_j, k \neq j\). Then the state can be explicitly written as:

\[
|\Phi_{n-k,i_1\ldots i_k}\rangle = \frac{1}{\sqrt{N_{n-k,i_1\ldots i_k}}} \prod_{\xi_l \in M_{n-k,i_1\ldots i_k}} B^* (\xi_l)|0\rangle,
\]

(3.2)

where \(N_{n-k,i_1\ldots i_k}\) is the corresponding normalized coefficient. This expression is widely used in the following. Once we assume

\[
n - k, i_1 \cdots i_k |k = 0 = n,
\]

(3.3)

the expression \((3.2)\) can exactly give the definition of the \(n\)-photon state \(|\Phi_n\rangle\) when \(k = 0\).

Similarly to the calculation of \(dB(t)|\Phi_n\rangle\), we can get

\[
dB(t)|\Phi_{n-k,i_1\ldots i_k}\rangle = \sum_{\xi_l \in M_{n-k,i_1\ldots i_k}} \frac{\sqrt{N_{n-k-1,i_1\ldots i_k,l}}}{\sqrt{N_{n-k,i_1\ldots i_k}}} \xi_l(t)|\Phi_{n-k-1,i_1\ldots i_k,l}\rangle dt,
\]

whenever \(k = n, N_{n-k-1,i_1\ldots i_k,l}\) is set to be 0. Then the evolution of the expectation with respect to the general state \(|\eta \Phi_{n-j,i_j\ldots i_l}|X(t)\rangle|\eta \Phi_{n-k,i_1'\ldots i_k'}(X)\rangle = \mu_t^{n-j,i_j\ldots i_l}|n-k,i_1'\ldots i_k'|X,X,S - X\rangle\) can be derived.

**Theorem 3.1** The master equation in Heisenberg form for the system, when the field is in the \(n\)-photon state \(|\Phi_n\rangle\), is given by the system of equations

\[
\mu_t^{n,n}(X) = \mu_t^{n,n}(L(X)) + \sum_{l=1}^{n-1} \sqrt{n-1,l} \xi_l(t) \mu_t^{n-1,l}([L^*, X]S) + \sum_{k=1}^{n} \sqrt{n-1,k} \xi_k(t) \mu_t^{n-1,k}([S^* [X, L]], S = X)
\]

(3.4)

more generally,

\[
\mu_t^{n-j,i_1\ldots i_1\ldots i_k}(X) = \mu_t^{n-j,i_1\ldots i_1\ldots i_k}(L(X)) + \sum_{\xi_m \in M_{n-j,i_1\ldots i_j}} \sqrt{n-j-1,i_1\ldots i_j} \xi_m(t) \mu_t^{n-j-1,i_1\ldots i_j,m,n-k,i_1'\ldots i_k'}(S^*[X, L])
\]

\[
+ \sum_{\xi_l \in M_{n-k,i_1'\ldots i_k'}} \sqrt{n-k-1,i_1'\ldots i_k'} \xi_l(t) \mu_t^{n-j,i_1\ldots i_1\ldots i_k'}([L^*, X]S) + \sum_{\xi_m \in M_{n-j,i_1\ldots i_j}} \xi_m(t) \mu_t^{n-j-1,i_1\ldots i_j,m,n-k-1,i_1'\ldots i_k'}(S^*[X, X,S - X]).
\]

(3.5)

The initial conditions are \(\mu_0^{n-j,i_1\ldots i_1\ldots i_k'/}(X) = \langle \Phi_{n-j,i_1\ldots i_1\ldots i_k'}|\Phi_{n-k,i_1'\ldots i_k'}\rangle|n, X_n\rangle\).

Clearly, once all \(\xi_i = \xi \ (i = 1, \ldots, n)\), i.e., the field is in the Fock state, the equation \((3.4)\) is exactly the one given by \([11]\) (2012).

Define the density operator \(g^{n-j,i_1\ldots i_1\ldots i_k'}(t)\) via \(\langle g^{n-j,i_1\ldots i_1\ldots i_k'}(t)|X) = \mu_t^{n-j,i_1\ldots i_1\ldots i_k'}(X), where \langle A, B \rangle = tr(A*B), we can obtain a Schrödinger form of the master equations. The operators enjoy the symmetry property \(g^{n-j,i_1\ldots i_1\ldots i_k'}(t)^* = g^{n-j,i_1'\ldots i_k'}(n-j,i_1\ldots i_j)(t)\).
Corollary 3.2 The master equation in Schrödinger form for the system when the field is in the n-photon state $|\Phi_n\rangle$ is given by the system of equations

$$
\dot{\rho}^{n,n}(t) = \mathcal{L}^{\ast} \rho^{n,n}(t) + \sum_{l=1}^{n} \frac{\sqrt{N_1^n-1}}{\sqrt{N_1^n}} \xi_l^\ast(t) [L, \rho^{n,n-1,l}(t) S^\ast] + \sum_{m=1}^{n} \frac{\sqrt{N_1^n-1,m}}{\sqrt{N_1^n}} \xi_m(t) [S \rho^{n-1,m,n}(t), L^\ast] + \sum_{m=1}^{n} \sum_{l=1}^{n} \xi_m(t) \xi_l^\ast(t) \frac{\sqrt{N_1^n-1,m}}{\sqrt{N_1^n}} \frac{\sqrt{N_1^n-1,l}}{\sqrt{N_1^n}} (S \rho^{n-1,m,n-1,l}(t) S^\ast - \rho^{n-1,m,n-1,l}(t)),
$$

(3.6)

more generally,

$$
\dot{\rho}^{n-j,i_1...i_j;n-k,i'_1...i'_k}(t) = \mathcal{L}^{\ast} \rho^{n-j,i_1...i_j;n-k,i'_1...i'_k}(t) + \sum_{\xi_l \in M_{n-k,i'_1...i'_k}} \frac{\sqrt{N_1^n-j-1,i_1...i_j,m}}{\sqrt{N_1^n-j,i_1...i_j}} \xi_l^\ast(t) [L, \rho^{n-j-1,i_1...i_j;n-k,i'_1...i'_k}(t) S^\ast] + \sum_{\xi_m \in M_{n-j,i_1...i_j}} \sum_{\xi_l \in M_{n-k,i'_1...i'_k}} \xi_m(t) \frac{\sqrt{N_1^n-j-1,i_1...i_j,m}}{\sqrt{N_1^n-j,i_1...i_j}} \frac{\sqrt{N_1^n-k-1,i'_1...i'_k}}{\sqrt{N_1^n-k,i'_1...i'_k}} (S \rho^{n-j-1,i_1...i_j;m,n-k-1,i'_1...i'_k}(t) S^\ast - \rho^{n-j-1,i_1...i_j;m,n-k-1,i'_1...i'_k}(t)),
$$

where $\mathcal{L}^{\ast} \rho = -i[H, \rho] + L_0 S^\ast L_0 - \frac{1}{2} L S^\ast L - \frac{1}{2} L^\ast S L$, is known as the Liouvillian operator. The initial conditions are $\rho^{n-j,i_1...i_j;n-k,i'_1...i'_k}(0) = (\Phi_{n-j,i_1...i_j} | \Phi_{n-k,i'_1...i'_k} \rangle \langle \phi | \eta \rangle \langle \eta |$.

It is apparent that the equations couple downward towards the vacuum master equation via the off-diagonal equations. This means that for a field in n-photon state, we should consider $2^n$ equations. From the symmetry property, the number of the independent coupled equations reduces to $\frac{2^n(2^n+1)}{2}$.

4 Multi-photon Filtering Equation for Homodyne Measurement

In this and the next sections we derive multi-photon filters for homodyne and photon-counting measurements respectively.

Noticing that the master equation derived in the preceding section is a system of coupled equations, which is a feature of the non-Markovian character, the derivation method of a multi-photon filter should be different from that of filters for coherent state. There are two widely used methods. One is the Markovian embedding (19). It’s main idea is to construct an ancilla system to combine with the system of interest to form a Markovian extended system driven by vacuum, then the filtering results can be obtained from the standard filter for the extended system. The other one is the non-Markovian embedding (17). In this case, the extended system forms a non-Markovian system, where the ancilla, system and field are initialized in a superposition state. While standard filtering results do not apply in this case, the quantum stochastic methods can nevertheless be applied to determine the quantum filters. Recently, both of these two methods have been used to derive the quantum filter for the systems driven by the fields in single photon state (15, 16). In view of the complexity of the n-photon state, it may be difficult to determine the suitable ancilla systems to realize the Markovian embedding. Thus we adopt the non-Markovian embedding.

In this section, we focus on the homodyne detection (2) and give an explicit derivation of the dynamics of the monitored system. What we want to calculate is the dynamical evolution of $\hat{X}(t) = E_{n,n} [X(t) | \Psi(t)]$. Firstly, we will introduce the non-Markovian embedding method to extend the space including the original system and the field to a larger one, then we derive the master equation and quantum filter for this extended system. Based on them, multi-photon filter for the original system can be solved.
4.1 Non-Markovian embedding

Recall that the system and field are defined on a Hilbert space \( H = H_S \otimes F \). We extend the space to a larger one \( \hat{H} = C^{2^n} \otimes H \), which includes the system, field and an ancillary \( 2^n \)-level system. An orthonormal basis for this ancillary \( C^{2^n} \) space can be chosen as \( \{ |e_{n-k,i_1 \cdots i_k}\rangle \}, \ k = 0, \cdots , n, \ 1 \leq i_1 < i_2 < \cdots < i_k \leq n \}, \) with \( |e_{n-k,i_1 \cdots i_k}\rangle \in C^{2^n} \) having only one non-zero element 1 at the \( m \)th location. If \( k = 0, \ m = 1 \), once \( k > 0 \), \( m = C_n^0 + C_n^1 + \cdots + C_n^{k-1} + r(i_1 \cdots i_k) \), where \( r(i_1 \cdots i_k) \) represents the location of the series \( i_1 \cdots i_k \) in the ordered set \( \{ i_1 \cdots i_k, \ 0 \leq i_1 < \cdots < i_k \leq n \} \).

The extended system is initialized in the superposition state

\[
|\Sigma\rangle = \alpha_n |e_n\eta\Phi_n\rangle + \alpha_0 |e_0\eta\Phi_0\rangle + \sum_{k=1}^{n-1} \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{-k,i_1,i_2 \cdots i_k} |e_{n-k,i_1,i_2 \cdots i_k}\eta\Phi_{n-k,i_1,i_2 \cdots i_k}\rangle,
\]

with \( \sum_{k=0}^{n} \sum_{i_1 < i_2 < \cdots < i_k} |\alpha_{-k,i_1,i_2 \cdots i_k}|^2 = 1 \). Here we have used the convention given in Eq. (3.3) and \(|\Phi_0\rangle\) is the vacuum state. For notational convenience, we write \( \omega_{n-j,i_1 \cdots i_j;n-k,i'_1 \cdots i'_k} = \alpha_{n-j,i_1 \cdots i_j}^{*} \alpha_{n-k,i'_1 \cdots i'_k} \).

We allow the extended system to evolve unitary according to \( I \otimes U(t) \), where \( U(t) \) is the unitary operator for the original system and field, i.e., \( \Sigma(t) = (I \otimes U(t))|\Sigma\rangle \).

The expectation with respect to the superposition state \( |\Sigma\rangle \) can be denoted as

\[
\tilde{\mu}_t(A \otimes X) = \mathbb{E}_\Sigma[A \otimes X(t)] = (\Sigma|A \otimes X(t)|\Sigma). \tag{4.1}
\]

Here, \( A \) is an operator acting on \( C^{2^n} \). This expectation is correctly normalized, i.e., \( \tilde{\mu}_t(I \otimes I) = 1 \), and most importantly, the scaled components of \( \tilde{\mu}_t(A \otimes X) \) is exactly the expectations \( \mu_t^{n-j,i_1 \cdots i_j;n-k,i'_1 \cdots i'_k} \) defined above. In further detail, for \( \omega_{n-j,i_1 \cdots i_j;n-k,i'_1 \cdots i'_k} \neq 0 \),

\[
\mu_t^{n-j,i_1 \cdots i_j;n-k,i'_1 \cdots i'_k}(X) = \frac{\tilde{\mu}_t(\langle e_{n-j,i_1 \cdots i_j}| e_{n-k,i'_1 \cdots i'_k}\rangle \otimes X)}{\omega_{n-j,i_1 \cdots i_j;n-k,i'_1 \cdots i'_k}} = \frac{\omega_{n,n}\tilde{\mu}_t(\langle e_{n-j,i_1 \cdots i_j}| e_{n-k,i'_1 \cdots i'_k}\rangle \otimes X)}{\omega_{n-j,i_1 \cdots i_j;n-k,i'_1 \cdots i'_k}} \tilde{\mu}_t\langle e_n| e_n \otimes I \rangle. \tag{4.2}
\]

4.2 Master equation for the extended system

In this subsection, we derive the master equation for the extended system driven by an \( n \)-photon state.

Similarly to the calculation of \( dB(t)|\Phi_n\rangle \), it can be easily verified that

\[
dB(t)|\Sigma\rangle = \sum_{k=0}^{n} \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{-k,i_1,i_2 \cdots i_k} dB(t)|e_{n-k,i_1,i_2 \cdots i_k}\eta\Phi_{n-k,i_1,i_2 \cdots i_k}\rangle
\]

\[
= \sum_{k=0}^{n} \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{-k,i_1,i_2 \cdots i_k} \frac{\sqrt{\pi}}{\sqrt{N_{n-k-1,i_1 \cdots i_k}}} \xi_{f}(t)|e_{n-k,i_1,i_2 \cdots i_k}\eta\Phi_{n-k-1,i_1 \cdots i_k}\rangle dt
\]

\[
= \sum_{k=0}^{n} \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{-k,i_1,i_2 \cdots i_k} \frac{\sqrt{\pi}}{\sqrt{N_{n-k-1,i_1 \cdots i_k}}} \xi_{f}(t)|e_{n-k,i_1,i_2 \cdots i_k}|\Phi_{n-k-1,i_1 \cdots i_k}\rangle dt
\]

\[
= \sum_{k=0}^{n} \sum_{i_1 < i_2 < \cdots < i_k} \sum_{i_{k+1} \in \mathcal{M}_{n-k-1,i_1 \cdots i_k-1}} \frac{\alpha_{-k,i_1,i_2 \cdots i_k}}{\sqrt{N_{n-k-1,i_1 \cdots i_k-1}}} \xi_{f}(t)|e_{n-k-1,i_1,i_2 \cdots i_k-1}|\Sigma\rangle dt
\]

\[
= \sum_{k=0}^{n} \sum_{i_1 < i_2 < \cdots < i_k} \sum_{i_{k+1} \in \mathcal{M}_{n-k-1,i_1 \cdots i_k-1}} \frac{\alpha_{-k,i_1,i_2 \cdots i_k}}{\sqrt{N_{n-k+1,i_1 \cdots i_k}}} \xi_{f}(t)|e_{n-k+1,i_1 \cdots i_k-1}|\Sigma\rangle dt,
\]

where \( \sigma_{n-k+1,i_1 \cdots i_k-1;n-k,i_1 \cdots i_k} = |e_{n-k+1,i_1 \cdots i_k-1}\rangle\langle e_{n-k,i_1 \cdots i_k}| \) acting on the ancillary system \( C^{2^n} \). The last equation is got with the help of the assumption \( N_{n-k+1,i_1 \cdots i_k-1} = 0 \) when \( k = n \).
Then we can directly get that the stochastic integrals with respect to the superposition state $|\Sigma\rangle$ are expressed in terms of non-stochastic integrals again with respect to $|\Sigma\rangle$ with the aid of the matrices $\sigma_{n-k+1,i_1\cdots i_{k-1}n-k+1,i_1\cdots i_{k-1}}$. Thus the dynamical evolution of the expectation $\hat{\mu}_t(A \otimes X)$ given by Eq. (4.1) can be expressed in a closed form, just as follows.

**Theorem 4.1** Assume $\alpha_{n-k,i_1\cdots i_k} \neq 0, k = 1, \cdots, n$. The unconditional expectation $\hat{\mu}_t(A \otimes X)$ for the extended system evolves according to

$$
\hat{\mu}_t(A \otimes X) = \hat{\mu}_t(\mathcal{G}(A \otimes X)),
$$

(4.3)

where

$$
\mathcal{G}(A \otimes X) = A \otimes L(X)
$$

$$
+ \sum_{k=1}^{n} \sum_{i_1<\cdots<i_k-1} 1 \xi_{i_1k} \in M_{n-k+1,i_1\cdots i_k-1} \xi_{ik} A \otimes X
$$

$$
+ \sum_{i=1}^{n} \sum_{i_1<\cdots<i_k-1} \xi_{i_1i_k} \in M_{n-k+1,i_1\cdots i_k-1} \xi_{ik} A \otimes X
$$

$$
+ \sum_{l=1}^{n} \sum_{i_1<\cdots<i_k-1} \xi_{i_1l} \in M_{n-k+1,i_1\cdots i_k-1} \xi_{il} A \otimes X
$$

$$
+ \sum_{m=1}^{n} \sum_{i_1<\cdots<i_k-1} \xi_{i_1m} \in M_{n-k+1,i_1\cdots i_k-1} \xi_{mi} A \otimes X
$$

$$
\frac{\alpha_{n-l+1,i_1\cdots i_k}}{\alpha_{n-l,l_1\cdots l_k}} \xi_{i_1}^*(t) \xi_{i_k}^*(t) A \otimes X
$$

$$
\frac{\alpha_{n-k+1,l_1\cdots l_k} \alpha_{n-k+1,i_1\cdots i_k}}{\alpha_{n-k+1,i_1\cdots i_k} \alpha_{n-k+1,l_1\cdots l_k}} \xi_{i_1}^*(t) \xi_{i_k}^*(t) A \otimes X
$$

$$
\frac{\alpha_{n-l+1,i_1\cdots i_k}}{\alpha_{n-l,l_1\cdots l_k}} \xi_{i_1}^*(t) \xi_{i_k}^*(t) A \otimes X
$$

$$
\frac{\alpha_{n-k+1,l_1\cdots l_k} \alpha_{n-k+1,i_1\cdots i_k}}{\alpha_{n-k+1,i_1\cdots i_k} \alpha_{n-k+1,l_1\cdots l_k}} \xi_{i_1}^*(t) \xi_{i_k}^*(t) A \otimes X
$$

$$
\frac{\alpha_{n-l+1,i_1\cdots i_k}}{\alpha_{n-l,l_1\cdots l_k}} \xi_{i_1}^*(t) \xi_{i_k}^*(t) A \otimes X
$$

$$
\frac{\alpha_{n-k+1,l_1\cdots l_k} \alpha_{n-k+1,i_1\cdots i_k}}{\alpha_{n-k+1,i_1\cdots i_k} \alpha_{n-k+1,l_1\cdots l_k}} \xi_{i_1}^*(t) \xi_{i_k}^*(t) A \otimes X
$$

$$
\frac{\alpha_{n-l+1,i_1\cdots i_k}}{\alpha_{n-l,l_1\cdots l_k}} \xi_{i_1}^*(t) \xi_{i_k}^*(t) A \otimes X
$$

It is worth noting that, by setting $A = |e_{n-j,i_1\cdots i_k}\rangle\langle e_{n-k,i_1'\cdots i_k}'|$ in Eq. (4.3) and using the relationship given in Eq. (122), the system of master equation Eq. (3.3) for $\mu_t^{n-j,i_1\cdots i_k;n-k,i_1'\cdots i_k'}(X)$ can be found.

### 4.3 Filtering equation for the extended system

The extended system provides a convenient framework for quantum filtering, since all expectations can be expressed in terms of the superposition state $|\Sigma\rangle$. Our immediate goal in this subsection is to determine the dynamical evolution for the quantum conditional expectation

$$
\mathcal{E}_t(A \otimes X) = \mathbb{E}_{\Sigma}[A \otimes X(t)|I \otimes \mathcal{W}(t)]
$$

(4.4)

The quantum conditional expectation $\mathcal{E}_t(A \otimes X) \in I \otimes \mathcal{W}(t)$ can be verified to be well defined, because $A \otimes X(t)$ commutes with the subspace $I \otimes \mathcal{W}(t)$, and is characterized by the requirement that

$$
\mathbb{E}_{\Sigma}[\mathcal{E}_t(A \otimes X)I \otimes K] = \mathbb{E}_{\Sigma}[\langle A \otimes X(t)\rangle(I \otimes K)], \text{ for all } K \in \mathcal{W}(t)
$$

(4.5)

In this case, the continuously monitored field observable with respect to the conditional expectation is $I \otimes Y^W(t)$, and the corresponding output dynamics for the extended system can be derived from Eq. (2.6) as

$$
d(I \otimes Y^W(t)) = I \otimes (L(t) + L^*(t))dt + I \otimes (S(t)dB(t) + S^*(t)dB^*(t)).
$$

**Theorem 4.2** Assume $\alpha_{n-k,i_1\cdots i_k} \neq 0, k = 1, \cdots, n$. In the case of homodyne monitoring, the conditional expectation $\mathcal{E}_t(A \otimes X)$ defined for the extended system satisfies

$$
d\mathcal{E}_t(A \otimes X) = \mathcal{E}_t(\mathcal{G}(A \otimes X))dt + \mathcal{H}_t(A \otimes X)dW(t),
$$

(4.6)
where

$$H_t(A \otimes X)$$

$$= \sum_{k=1}^{n} \sum_{i_k \in \text{M}_{n-k+1,i_1\ldots i_k}} \alpha_{n-k+1,i_1\ldots i_k-1} \xi_i(t) \tilde{\pi}_t(A \sigma_{n-k+1,i_1\ldots i_k;n-k,i_1\ldots i_k} \otimes XS)$$

$$+ \sum_{k=1}^{n} \sum_{i_k \in \text{M}_{n-k+1,i_1\ldots i_k}} \alpha_{n-k,i_1\ldots i_k-1} \xi_i(t) \tilde{\pi}_t(\sigma_{n-k+1,i_1\ldots i_k-1;n-k,i_1\ldots i_k} A \otimes S^* X)$$

$$+ \tilde{\pi}_t(A \otimes (XL+L^*X)) - \tilde{\pi}_t(A \otimes X) M(t),$$

and

$$M(t)$$

$$= \tilde{\pi}_t[I \otimes (L + L^*)]$$

$$+ \sum_{k=1}^{n} \sum_{i_k \in \text{M}_{n-k+1,i_1\ldots i_k}} \alpha_{n-k+1,i_1\ldots i_k-1} \xi_i(t) \sigma_{n-k+1,i_1\ldots i_k-1;n-k,i_1\ldots i_k} S$$

$$+ \sum_{k=1}^{n} \sum_{i_k \in \text{M}_{n-k+1,i_1\ldots i_k}} \alpha_{n-k,i_1\ldots i_k-1} \xi_i(t) \sigma_{n-k+1,i_1\ldots i_k-1;n-k,i_1\ldots i_k} S^*].$$

The process \(W(t)\) defined by \(dW(t) = dY^W(t) - M(t)dt\), is a \(I \otimes \mathcal{W}^W(t)\) Wiener process with respect to \(\Sigma\) and is called the innovation process.

PROOF. We follow the normal characteristic function method, whereby we postulate that the filter has the form

$$d\tilde{\pi}_t(A \otimes X) = F_t(A \otimes X) dt + H_t(A \otimes X) I \otimes dY^W(t),$$

where \(F_t(A \otimes X)\) and \(H_t(A \otimes X)\) are to be determined. They are all adapted to \(I \otimes \mathcal{W}^W(t)\).

Let \(f(t)\) be square integrable, and define a process \(cf(t)\) as \(cf(t) = e^{\int_0^t f(s) dY^W(s)} \tilde{\pi}_t G_t(A \otimes X)\), then it can be verified to satisfy \(dcf(t) = f(t)cf(t) dY^W(t)\), and \(cf(0) = 1\). \(I \otimes cf(t)\) is adapted to \(I \otimes \mathcal{W}^W(t)\), and the defining relation Eq. 1.6 implies that

$$E_\Sigma[A \otimes (X(t) cf(t))] = E_\Sigma[\tilde{\pi}_t(A \otimes X)(I \otimes cf(t))], \quad \text{for all } f(t).$$

By calculating the differentials of both sides, taking expectations and conditioning we obtain

$$E_\Sigma[A \otimes d(X(t) cf(t))]$$

$$= E_\Sigma[I \otimes cf(t) dX(t) + A \otimes f(t) cf(t) | X(t) dY^W(t) + dX(t) dY^W(t)]$$

$$= E_\Sigma[(I \otimes cf(t)) \tilde{\pi}_t (G_t(A \otimes X)) + (I \otimes f(t) cf(t))(\tilde{\pi}_t(A \otimes (XL + L^*X))]$$

$$+ \sum_{k=1}^{n} \sum_{i_k \in \text{M}_{n-k+1,i_1\ldots i_k}} \alpha_{n-k+1,i_1\ldots i_k-1} \xi_i(t) \tilde{\pi}_t(A \sigma_{n-k+1,i_1\ldots i_k;n-k,i_1\ldots i_k} \otimes XS)$$

$$+ \sum_{k=1}^{n} \sum_{i_k \in \text{M}_{n-k+1,i_1\ldots i_k}} \alpha_{n-k,i_1\ldots i_k-1} \xi_i(t) \tilde{\pi}_t(\sigma_{n-k+1,i_1\ldots i_k-1;n-k,i_1\ldots i_k} A \otimes S^* X)] dt,$$
Making use of the arbitrary of \( f(t) \) and \( f(t) c_f(t) \) are separately equal. Then we can solve for \( F_t(A \otimes X) \) and \( H_t(A \otimes X) \) to obtain the filtering equation.

We now prove the martingale property \( \mathbb{E}_S[I \otimes (W(t) - W(s))\mathcal{W}_W(s)] = 0 \), \( s \leq t \), that is, \( \mathbb{E}_S[I \otimes (W(t) - W(s))(I \otimes K)] = 0 \) for all \( K \in \mathcal{W}_W(s) \). Now

\[
\mathbb{E}_S[I \otimes (W(t) - W(s))(I \otimes K)] = \mathbb{E}_S[I \otimes (Y_W(t) - Y_W(s)) - \int_s^t M(r)dr)(I \otimes K)]
\]

\[
= \mathbb{E}_S[I \otimes (\int_s^t (L(r) + L^*(r))dr + \int_s^t S(r)dB(r) + \int_s^t S^*(r)dB^*(r))(I \otimes K)] - \mathbb{E}_S[I \otimes \int_s^t M(r)dr (I \otimes K)]
\]

\[
= \mathbb{E}_S[I \otimes \int_s^t (L(r) + L^*(r))dr (I \otimes K)] - \mathbb{E}_S[\int_s^t M(r)dr (I \otimes K)]
\]

\[
+ \sum_{k=1}^{n} \sum_{k_i < \cdots < k_{i-1}} \sum_{i \neq i} \left( \frac{N_{n-k,1i}}{N_{n-k+1,1i}} \alpha_{n-k+1,1i-1} \mathbb{E}[\int_s^t \sigma_{n-k,1i-1} \times n-k,1i-1 \otimes S \xi_k (r)dr(I \otimes K)]
\]

\[
+ \sum_{k=1}^{n} \sum_{k_i < \cdots < k_{i-1}} \sum_{i \neq i} \left( \frac{N_{n-k,1i}}{N_{n-k+1,1i}} \alpha_{n-k+1,1i-1} \mathbb{E}[\int_s^t \sigma_{n-k,1i-1} \times n-k,1i-1 \otimes S^* \xi_k (r)dr(I \otimes K)]
\]

\[
= \mathbb{E}_S[\int_s^t \pi_t(I \otimes (L + L^*))dr(I \otimes K)] - \mathbb{E}_S[\int_s^t M(r)dr (I \otimes K)]
\]

\[
+ \sum_{k=1}^{n} \sum_{k_i < \cdots < k_{i-1}} \sum_{i \neq i} \left( \frac{N_{n-k,1i}}{N_{n-k+1,1i}} \alpha_{n-k+1,1i-1} \mathbb{E}[\int_s^t \pi_t(\sigma_{n-k,1i-1} \times n-k,1i-1 \otimes S \xi_k (r)dr(I \otimes K)]
\]

\[
+ \sum_{k=1}^{n} \sum_{k_i < \cdots < k_{i-1}} \sum_{i \neq i} \left( \frac{N_{n-k,1i}}{N_{n-k+1,1i}} \alpha_{n-k+1,1i-1} \mathbb{E}[\int_s^t \pi_t(\sigma_{n-k,1i-1} \times n-k,1i-1 \otimes S^* \xi_k (r)dr(I \otimes K)]
\]

\[
= 0.
\]

Finally, since \( dW(t) dW(t) = dt \), Levy’s theorem implies that \( W(t) \) is a \( \mathcal{W}_W(t) \) Wiener process.

### 4.4 Filtering equation for the multi-photon state

Making use of the filtering results for the extended system obtained in subsection 4.3, we derive the quantum filter for the original system driven by the \( n \)-photon field state.
We will first define the conditional quantities \( \pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}(X) \) as
\[
I \otimes \pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}(X) = \frac{\omega_{n} \rho_{n} \pi_{n}(e_{n}) | e_{n} \rangle \langle e_{n} | \otimes X)}{\omega_{n} \rho_{n} \pi_{n}(e_{n}) | e_{n} \rangle \langle e_{n} | \otimes I},
\]
when \( \omega_{n} \rho_{n} \pi_{n}(e_{n}) | e_{n} \rangle \langle e_{n} | \neq 0 \), otherwise it can be set to 0. The \( \tilde{\pi}_{n}(A \otimes X) \) is the conditional state for the extended system given in Eq. (4.4).

This conditional quantities have significant features, that is, for all \( K \in \mathcal{W}(t) \), we have
\[
\mathbb{E}_{n} [\pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}(X)K] = \mathbb{E}_{n} [\pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}[X(t)K]].
\]
In further detail, it can be affirmed
\[
\mathbb{E}_{n} [\pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}(X)K] = \frac{1}{\omega_{n} \rho_{n}} \mathbb{E}_{\Sigma} [\langle e_{n} | \langle e_{n} | \otimes (\pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}(X)K)]
\]
\[
= \frac{1}{\omega_{n} \rho_{n}} \mathbb{E}_{\Sigma} [\tilde{\pi}_{n}(\langle e_{n} | \otimes I)(I \otimes \pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}(X)K)]
\]
\[
= \frac{1}{\omega_{n} \rho_{n}} \mathbb{E}_{\Sigma} [\tilde{\pi}_{n}(\langle e_{n} | \otimes X)(I \otimes K)]
\]
\[
= \frac{1}{\omega_{n} \rho_{n}} \mathbb{E}_{\Sigma} [(\langle e_{n} | \otimes X(t)](I \otimes K)]
\]
\[
= \mathbb{E}_{n} [\pi_{n}^{j_{1}, \ldots, j_{m}, k_{1}, \ldots, k_{l}}[X(t)K],
\]
as required.

Set \( j = k = 0 \) in the equation (4.8) and noting that \( K \in \mathcal{W}(t) \) is otherwise arbitrary, we can deduce that the \( \pi_{n}^{n,n}(X) \) is the desired conditional expectation for the n-photon field state, i.e., \( \pi_{n}^{n,n}(X) = \mathbb{E}_{n} [X(t)|\mathcal{W}(t)] \). Then using Th.4, we can easily get this conditional dynamical evolution as follows.

**Theorem 4.3** In the case of homodyne detection, the quantum filter for the conditional expectation with respect to the n-photon field is given in the Heisenberg picture by \( \hat{X}(t) = \mathbb{E}_{n} [X(t)|\mathcal{W}(t)] = \pi_{n}^{n,n}(X) \), where \( \pi_{n}^{n,n}(X) \) satisfies the system of equations
\[
d\pi_{n}^{n,n}(X) = \left[ \pi_{n}^{n,n}(\mathcal{L}(X)) + \sum_{k=1}^{n} \sqrt{N_{n-1,k}} \xi_{k}^{n,n-1,k}(L^{*}, X)S \right. + \sum_{l=1}^{n} \sqrt{N_{n-1,l}} \xi_{l}^{n,n-1,l}(S^{*}, X, L)]
\]
\[
+ \sum_{k=1}^{n} \sqrt{N_{n-1,k}} \xi_{k}^{n,n-1,k}(L^{*}, X)S \left. \right] dt + \pi_{n}^{n,n}(XL + L^{*}X)
\]
\[
+ \sum_{k=1}^{n} \sqrt{N_{n-1,k}} \xi_{k}^{n,n-1,k}(XS) + \sum_{l=1}^{n} \sqrt{N_{n-1,l}} \xi_{l}^{n,n-1,l}(S^{*}X - \pi_{n}^{n,n}(X)M(t)) \right] dW_{t}, (4.9)
\]
and more generally,

\[
\begin{align*}
&d\pi_{t}^{\alpha_{n-j,i_{j}\cdots i_{k};n-k,i'_{i_{j}\cdots i_{k}}}}(X) \\
&= \left[\pi_{t}^{\alpha_{n-j,i_{j}\cdots i_{k};n-k,i'_{i_{j}\cdots i_{k}}}}(L(X)) + \sum_{\xi_{k+1}^{i_{j}\cdots i_{k}} \in M_{n-k,i'_{i_{j}\cdots i_{k}}}} \frac{\sqrt{N_{n-1,i'_{i_{j}\cdots i_{k}}}}}{\sqrt{N_{n-k,i'_{i_{j}\cdots i_{k}}}}} \xi_{k+1}^{i_{j}\cdots i_{k}}(t)\pi_{t}^{\alpha_{n-j,i_{j}\cdots i_{k};n-k-1,i'_{i_{j}\cdots i_{k}+1}}}(L^{*},X|S) \right] dt \\
&+ \sum_{\xi_{i,j+1} \in M_{n-j,i_{j}\cdots i_{j}}} \frac{\sqrt{N_{n-j-1,i_{j}\cdots i_{j}}}}{\sqrt{N_{n-j,i_{j}\cdots i_{j}}}} \xi_{i,j+1}(t)\pi_{t}^{\alpha_{n-j-1,i_{j}\cdots i_{j};n-k-1,i'_{i_{j}\cdots i_{k}+1}}}(S^{*}X,S - X) \right] dt \\
&+ \sum_{\xi_{i,j+1} \in M_{n-j,i_{j}\cdots i_{j}}} \frac{\sqrt{N_{n-j-1,i_{j}\cdots i_{j}}}}{\sqrt{N_{n-j,i_{j}\cdots i_{j}}}} \xi_{i,j+1}(t)\pi_{t}^{\alpha_{n-j-1,i_{j}\cdots i_{j};n-k-1,i'_{i_{j}\cdots i_{k}+1}}}(S^{*}X,S - X) \right] dt.
\end{align*}
\]

Here \( M_{t} = \pi_{t}^{n,n}(L + L^{*}) + \sum_{k=1}^{n} \frac{\sqrt{N_{n-1,k}}}{\sqrt{N_{n}}} \xi_{k}(t)\pi_{t}^{n,n-1,k}(S) + \sum_{l=1}^{n} \frac{\sqrt{N_{n-1,l}}}{\sqrt{N_{n}}} \xi_{l}(t)\pi_{t}^{n,1,l,n}(S^{*}). \) The innovation process \( W_{t} \) is a \( W^{(s)}(t) \) Wiener process with respect to the \( n \)-photon state and is defined by \( dW_{t} = dY^{W}(t) - M_{t}dt. \) The initial conditions are \( \pi_{0}^{n-j,i_{j}\cdots i_{j};n-k,i'_{i_{j}\cdots i_{k}}}(X) = (\Phi_{n-j,i_{j}\cdots i_{j}}[\Phi_{n-k,i'_{i_{j}\cdots i_{k}}}])(\eta,X). \)

PROOF. Suppose first that \( \alpha_{n-k,i_{j}\cdots i_{k}} \neq 0, k = 1, \cdots, n. \) The differential equations (4.9)-(4.10) follow from the definition (4.7), the filter (4.6), for the extended system and the Itô rule, Eq. 2.3. Next, we note that the coefficients of the quantum stochastic differential equations (4.9)- (4.10), the initial conditions, and \( Y^{W}(t) \) do not depend on \( \alpha_{n-k,i_{j}\cdots i_{k}}, k = 1, \cdots, n. \) Hence, the solutions \( \pi_{t}^{n-j,i_{j}\cdots i_{j};n-k,i'_{i_{j}\cdots i_{k}}}(X) \) of this system of equations are independent of these coefficients. Therefore, the assumption \( \alpha_{n-k,i_{j}\cdots i_{k}} \neq 0, k = 1, \cdots, n \) can be removed.

We now prove that \( W_{t} \) is a \( W^{(s)}(t) \)-martingale, that is, \( \mathbb{E}_{n,n}[W_{t} - W_{s}|\mathcal{Y}(s)] = 0, s \leq t. \) To this end, let \( K \in \mathcal{Y}(s), \) then

\[
\begin{align*}
\mathbb{E}_{n,n}[W_{t} - W_{s}K] &= \mathbb{E}_{n,n}[(Y^{W}(t) - Y^{W}(s))K] - \mathbb{E}_{n,n}[\int_{s}^{t} M_{r}dK] \\
&= \mathbb{E}_{n,n}[\int_{s}^{t} (L(r) + L^{*}(r))dr + \int_{s}^{t} S(r)dB(r) + \int_{s}^{t} S^{*}(r)dB^{*}(r) - \int_{s}^{t} M_{r}dr]K] \\
&= \mathbb{E}_{n,n}[][\int_{s}^{t} \pi_{r}^{n,n}(L + L^{*})drK] + \sum_{k=1}^{n} \frac{\sqrt{N_{n-1,k}}}{\sqrt{N_{n}}} \mathbb{E}_{n,n-1,k}[\int_{s}^{t} \xi_{k}(r)S(r)drK] \\
&+ \sum_{l=1}^{n} \frac{\sqrt{N_{n-1,l}}}{\sqrt{N_{n}}} \mathbb{E}_{n-1,l,n}[\int_{s}^{t} \xi_{l}(r)S^{*}(r)drK] - \mathbb{E}_{n,n-1,k}[\int_{s}^{t} M_{r}drK].
\end{align*}
\]

Finally, since \( dW_{t}dW_{t} = dt, \) Levy’s theorem implies that \( W_{t} \) is a \( W^{W}(t) \) Wiener process.
Now, write \( \pi_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(X) = \langle \rho_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(t), X \rangle \). Then stochastic differential equations for the Schrödinger form can be immediately derived. Obviously, \( (\rho_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(t))^* = \rho_t^{n-k,i'_1\cdots i'_k; n-j,i_1\cdots i_j}(t) \).

**Corollary 4.4** In the case of homodyne detection, the quantum filter for the conditional expectation with respect to the \( n \)-photon field in the Schrödinger picture is given by

\[
\dot{\rho}_t^{n,n}(t)
= \lf L^* \rho_t^{n,n}(t) + \sum_{l=1}^{n} \sqrt{\frac{N_{n-1,l}}{N_n}} \xi_i^*(t) [L, \rho_t^{n,n-1,l}(t)] S^* \rfr + \sum_{m=1}^{n} \sqrt{\frac{N_{n-1,m}}{N_n}} \xi_i^*(t) [S \rho_t^{n-1,m:n}(t), L^*] \rfr + \sum_{m=1}^{n} \sqrt{\frac{N_{n-1,m}}{N_n}} \xi_i^*(t) [S \rho_t^{n,m:(n-1,l)}(t) \rfr + \sum_{m=1}^{n} \sqrt{\frac{N_{n-1,m}}{N_n}} \xi_i^*(t) [S \rho_t^{n,m:n}(t) \rfr - \rho_t^{n,n}(t) M_t] dW_t
\]

more generally,

\[
\dot{\rho}_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(t)
= \lf L^* \rho_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(t) + \sum_{l=1}^{n} \sqrt{\frac{N_{n-k-1,l,i'_1\cdots i'_k}}{N_n}} \xi_i^*(t) [L, \rho_t^{n-j,i_1\cdots i_j; n-k-1,i'_1\cdots i'_k}(t)] S^* \rfr + \sum_{m=1}^{n} \sqrt{\frac{N_{n-k-1,m,i'_1\cdots i'_k}}{N_n}} \xi_i^*(t) [S \rho_t^{n-j-1,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(t), L^*] \rfr + \sum_{m=1}^{n} \sqrt{\frac{N_{n-k-1,m,i'_1\cdots i'_k}}{N_n}} \xi_i^*(t) [S \rho_t^{n,j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(t) \rfr - \rho_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(t) M_t] dW_t,
\]

with

\[
M_t = \text{tr}(\rho_t^{n,n}(L + L^*)) + \sum_{k=1}^{n} \frac{\sqrt{N_{n-k-1,k}}}{N_n} \xi_i^*(t) \text{tr}(\rho_t^{n-1,k;n}(t) S) + \sum_{l=1}^{n} \frac{\sqrt{N_{n-1,l}}}{N_n} \xi_i^*(t) \text{tr}(\rho_t^{n,n-1,l}(t) S^*),
\]

and the innovation process \( \tilde{W}_t \) is defined as \( d\tilde{W}_t = dY^W(t) - M_t dt \). The initial condition is

\[
\rho_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(0) = \langle \Phi_{n-k,i'_1\cdots i'_k} \Phi_{n-j,i_1\cdots i_j} | \eta \rangle \langle \eta |.
\]

If we restrict the \( n \)-photon state as the Fock state, i.e., \( \xi_i = \xi \) for all \( i = 1, \ldots, n \), the filter equation in Th.5 can be simplified in a relatively compact form. Because all the \( \xi_i \) are the same, \( |\Phi_{n-k,i_1\cdots i_k} \rangle \) cannot be distinguished for different choices of \( i_1 \cdots i_k \), which can be denoted as \( |\Phi_{n-k} \rangle \). As a result, the superscript and the subscript can be simplified. For example, the notation \( \pi_t^{n-j,i_1\cdots i_j; n-k,i'_1\cdots i'_k}(X) \) and \( N_{n-j,i_1\cdots i_j} \) can be simplified as \( \pi_t^{n-j;n-k}(X) \) and \( N_{n-j} \), respectively.
Corollary 4.5 The quantum filter for the conditional expectation with respect to the n-photon Fock state, in the case of homodyne measurement, can be given in the Heisenberg picture by \( \hat{X}(t) = E_{n,n}[X(t)|\mathcal{W}(t)] = \pi_t^{n,n}(X) \), where \( \pi_t^{n,n}(X) \) satisfies the system of equations

\[
dt \pi_t^{n,n}(X) = \left[\pi_t^{n,n}(L(X)) + \sqrt{n} \xi(t) \pi_t^{n,n-1}([L^*, X]S) + \sqrt{n} \xi^*(t) \pi_t^{n-1,n}([S^*, X, L]) + n|\xi(t)|^2 \pi_t^{n-1,n-1}((S^* XS - X)) \right] dt \\
+ \left[ \pi_t^{n,n} (XL + L^* X) + \sqrt{n} \xi(t) \pi_t^{n,n-1}(XS) + \sqrt{n} \xi^*(t) \pi_t^{n-1,n}(S^* X - \pi_t^{n,n}(X)M_t) \right] dW_t,
\]

and more generally,

\[
dt \pi_t^{n-j,n-k}(X) = \left[ \pi_t^{n-j,n-k}(L(X)) + \sqrt{n - k} \xi(t) \pi_t^{n-j,n-k-1}([L^*, X]S) + \sqrt{n - k} \xi^*(t) \pi_t^{n-j-1,n-k}(S^*[X, L]) \right] dt \\
+ \left[ \pi_t^{n-j,n-k}(XL + L^* X) + \sqrt{n - k} \xi(t) \pi_t^{n-j-1,n-k}(S^* X) - \pi_t^{n-j,n-k}(X)M_t \right] dW_t,
\]

Here \( M_t = \pi_t^{n,n}(L + L^*) + \sqrt{n} \xi(t) \pi_t^{n,n-1}(S) + \sqrt{n} \xi^*(t) \pi_t^{n-1,n}(S^*) \). The innovation process \( W_t \) is a \( \mathcal{W}(t) \) Wiener process with respect to the n-photon state and is defined by \( dW_t = dY(t) - M_t dt \). The initial conditions are \( \pi_0^{0,0}(X) = \langle \Phi_n|X\rangle \).

5 Multi-photon Filtering Equation for Photoncounting Measurements

Using the similar way as that in the preceding section, the multi-photon quantum filter for the photoncounting monitoring case, i.e., \( \hat{X}(t) = E_{n,n}[X(t)|\mathcal{W}^A(t)] \), can also be derived. In order to avoid repetition, we omit the derivation and only list the result in the case.

Define \( \tilde{\pi}_t^A(A \otimes X) = E_{n,n}[A \otimes X(t)|\mathcal{W}^A(t)] \) for the extended system. The dynamics of this conditional expectation can be given as the following theorem.

Theorem 5.1 Assume \( \alpha_{n-k,i_1,\ldots,i_k} \neq 0, k = 1, \ldots, n \). In the case of photoncounting monitoring, the conditional expectation \( \tilde{\pi}_t^A(A \otimes X) \) defined for the extended system satisfies

\[
dt \tilde{\pi}_t^A(A \otimes X) = \tilde{\pi}_t^A(G(A \otimes X)) dt + \mathcal{H}_t^A(A \otimes X) dN(t),
\]

where

\[
\mathcal{H}_t^A(A \otimes X) = \Delta(t)^{-1} \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k} \sum_{\xi_{i_k} \in M_{n-k+1,i_1,\ldots,i_k-1}} \frac{\sqrt{N_{n-k,i_1,\ldots,i_k}}}{\sqrt{N_{n-k+1,i_1,\ldots,i_k-1}}} \alpha_{n-k+1,i_1,\ldots,i_k-1} \xi_{i_k}(t)
\]

\[
\tilde{\pi}_t^A(A\sigma_{n-k+1,i_1,\ldots,i_k-1,n-k,i_1,\ldots,i_k} \otimes L^* XS) + \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k} \sum_{\xi_{i_k} \in M_{n-k+1,i_1,\ldots,i_k-1}} \frac{\sqrt{N_{n-k,i_1,\ldots,i_k}}}{\sqrt{N_{n-k+1,i_1,\ldots,i_k-1}}} \alpha_{n-k,i_1,\ldots,i_k} \xi_{i_k}(t)
\]

\[
\xi_{i_k}(t) \tilde{\pi}_t^A(\sigma^*_{n-k+1,i_1,\ldots,i_k-1,n-k,i_1,\ldots,i_k} A \otimes S^* XL) + \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k} \sum_{\xi_{i_k} \in M_{n-k+1,i_1,\ldots,i_k-1}} \sum_{l_1 < \ldots < l_{i_k}} \sum_{\xi_{i_1} \in M_{n-l+1,i_1,\ldots,i_{i_1-1}}} \frac{\sqrt{N_{n-l,i_1,\ldots,i_{i_1-1}}}}{\sqrt{N_{n-l+1,i_1,\ldots,i_{i_1-1}}}} \alpha_{n-l+1,i_1,\ldots,i_{i_1-1}} \xi_{i_1}(t) \xi_{i_k}(t)
\]

\[
\tilde{\pi}_t^A(\sigma^*_{n-k+1,i_1,\ldots,i_k-1,n-k,i_1,\ldots,i_k} A\sigma_{n-l+1,i_1,\ldots,i_{i_1-1},n-l,i_1,\ldots,i_{i_1-1}} \otimes S^* XS) - \tilde{\pi}_t^A(A \otimes X).
\]
The innovation process \( N(t) \) is defined by \( dN(t) = dY^\Lambda(t) - \Delta(t)dt \).

The derivation of this filtering equation is similar to that in Th.4, only the choice of the \( c_f(t) \) is different. In this condition, we choose the process \( c_f(t) \) as \( c_f(t) = e^\int_0^t \Im[f(s)+1]dY^\Lambda(s) \), for any square integrable function \( f(t) \), which can be verified to satisfy \( dc_f(t) = f(t)c_f(t)dY^\Lambda(t) \), and \( c_f(0) = 1 \).

Define
\[
I \otimes \hat{\pi}_n^{\Lambda,j_1,i_1,\ldots,n-k_1,i_1,\ldots,k_n,i_n}(X) = \frac{\omega_{n,n} \hat{\pi}_n^{\Lambda}|\langle e_n^n \rangle \langle e_{n-k_1,i_1,\ldots,k_n,i_n}| \otimes X \rangle}{\omega_{n-j_1,i_1,\ldots,n-k_1,i_1,\ldots,k_n,i_n}} \hat{\pi}_n^{\Lambda}|\langle e_n^n \rangle \langle e_n^n | \otimes I \rangle. \tag{5.1}
\]

Then \( \hat{\pi}_t^{n,n}(X) \) can be verified to be the desired conditional expectation \( \mathbb{E}_{n,n}[X(t)|\mathcal{Y}^\Lambda(t)] \) for the multi-photon field state. This conditional expectation dynamically evolves according to the following theorem.

**Theorem 5.2** In the case of photon-counting monitoring, the quantum filter for the conditional expectation with respect to the \( n \)-photon field is given in the Heisenberg picture by \( \hat{X}(t) = \mathbb{E}_{n,n}[X(t)|\mathcal{Y}^\Lambda(t)] = \hat{\pi}_t^{n,n}(X) \), where \( \hat{\pi}_t^{n,n}(X) \) is given by the system of equations

\[
\begin{align*}
& d\hat{\pi}_t^{n,n}(X) = \\
& = \left[ \hat{\pi}_t^{n,n}(\mathcal{L}(X)) + \sum_{k=1}^n \frac{\sqrt{N_{n-k,1}}}{\sqrt{N_n}} \xi_k(t) \hat{\pi}_t^{n,n-1,k}(L^* X S) + \sum_{l=1}^n \frac{\sqrt{N_{n-l,1}}}{\sqrt{N_n}} \xi_l(t) \hat{\pi}_t^{n-l,1,n}(S^* X S - X) \right] dt \\
& + \left\{ \Delta_t^{-1} \left[ \hat{\pi}_t^{n,n}(L^* X L) + \sum_{k=1}^n \frac{\sqrt{N_{n-k,1}}}{\sqrt{N_n}} \xi_k(t) \hat{\pi}_t^{n,n-1,k}(L^* X S) + \sum_{l=1}^n \frac{\sqrt{N_{n-l,1}}}{\sqrt{N_n}} \xi_l(t) \hat{\pi}_t^{n-l,1,n}(S^* X L) \right] + \sum_{k=1}^n \frac{\sqrt{N_{n-k,1}}}{\sqrt{N_n}} \xi_k(t) \hat{\pi}_t^{n-1,k,n-1}(S^* X S) \right\} dN_t,
\end{align*}
\]
and generally,

\[ d\hat{\pi}^{n-j,i_1\cdots i_j;n-k,i'_{1}\cdots i'_{k}}(X) = \begin{bmatrix} \hat{\pi}^{n-j,i_1\cdots i_j;n-k,i'_{1}\cdots i'_{k}}(\mathcal{L}(X)) + \sum_{\xi'_{k+1} \in M_{n-k',i'_{k}...i'_{k}}} \frac{\sqrt{N_{n-k-1,i'_{1}\cdots i'_{k+1}}}}{\sqrt{N_{n-k,i'_{1}\cdots i'_{k}}} \xi'_{k+1}(t)\hat{\pi}^{n-j,i_1\cdots i_j;n-k-1,i'_{1}\cdots i'_{k+1}}([L^*, X]S)} + \sum_{\xi'_{j+1} \in M_{n-j,i_1\cdots i_j}} \frac{1}{\sqrt{N_{n-j,i_1\cdots i_j}}} \xi'_{j+1}(t)\hat{\pi}^{n-j-1,i_1\cdots i_j;n-k,i'_{1}\cdots i'_{k}}(S^[*][X,L]) + \sum_{\xi'_{j+i} \in M_{n-j,i_1\cdots i_j}} \sum_{\xi'_{j+i+1} \in M_{n-j,i_1\cdots i_j}} \frac{\sqrt{N_{n-k-1,i'_{1}\cdots i'_{k+1}}}}{\sqrt{N_{n-k,i'_{1}\cdots i'_{k}}} \xi'_{j+i+1}(t)\xi_{j+i+1}(t)\hat{\pi}^{n-j-1,i_1\cdots i_j;n-k-1,i'_{1}\cdots i'_{k+1}}(S^[*][XS-X])d\xi_{j+i+1} \right] } dN_{t}.

\[ \Delta_{t} \text{ is denoted as} \]

\[ \Delta_{t} = \hat{\pi}^{n,m}(L^*L) + \sum_{k=1}^{n} \frac{\sqrt{N_{n-1,k}}}{\sqrt{N_{n}}} \xi_{k}(t)\hat{\pi}^{n-n-1,k}(L^*S) + \sum_{k=1}^{n} \frac{\sqrt{N_{n-1,k}}}{\sqrt{N_{n}}} \xi^{*}_{k}(t)\hat{\pi}^{n-1,k;1}(S^[*][L]) + \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\sqrt{N_{n-1,k}}}{\sqrt{N_{n}}} \xi_{l}(t)\xi^{*}_{l}(t)\hat{\pi}^{n-1,k;1,l}(I), \]

and the innovation process \( N_{t} \) is defined by \( dN_{t} = dY^{A}(t) - \Delta_{t}dt \). The initial conditions are \( \hat{\pi}_{0}^{n-j,i_1\cdots i_j;n-k,i'_{1}\cdots i'_{k}}(X) = \langle \Phi_{n-j,i_1\cdots i_j}^{*} | \Phi_{n-k,i'_{1}\cdots i'_{k}}^{*} \rangle \eta, X\eta \rangle \).

For the Fock state, the relatively compact form can be given as follows.

**Corollary 5.3** The quantum filter for the conditional expectation with respect to the \( n \)-photon Fock state, in the case of photon-counting monitoring, is given in the Heisenberg picture by \( \hat{X}(t) = \mathbb{E}_{\pi^{n-1}}(X(t) | Y^{A}(t)) = \hat{\pi}^{n,m}(X) \), where \( \hat{\pi}^{n,m}(X) \) is given by the system of equations

\[ d\hat{\pi}^{n,m}(X) = \left[ \hat{\pi}^{n,m}(\mathcal{L}(X)) + n\sqrt{\xi}(t)\hat{\pi}^{n-1,m}(L^*S) + n\sqrt{\xi^{*}}(t)\hat{\pi}^{n-1,m}(S^[*][X,L]) + n\xi^{*}(t)\hat{\pi}^{n-1,m}(S^[*][XS-X])dt \right] + \left[ \Delta_{t}^{-1} \hat{\pi}^{n,m}(L^*XL) + n\sqrt{\xi}(t)\hat{\pi}^{n-1,m}(L^*XS) + n\sqrt{\xi^{*}}(t)\hat{\pi}^{n-1,m}(S^[*][XL]) + n\xi^{*}(t)\hat{\pi}^{n-1,m}(S^[*][XS]) \right] - \hat{\pi}^{n,m}(X) \right] dN_{t}, \]

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and generally,
\[
\frac{d\hat{n}_t^{n-j:n-k}(X)}{dt} = [\hat{n}_t^{n-j:n-k}(\mathcal{L}(X)) + \sqrt{n-k}\xi(t)\hat{n}_t^{n-j:n-k-1}(L^*X)S] + \sqrt{n-j}\xi(t)(\hat{n}_t^{n-j-1:n-k-1}(S^*X - X))dt + \{\hat{n}_t^{n-j:n-k}(X) + \Delta_t^{-1} (\hat{n}_t^{n-j:n-k}(L^*XL) + \sqrt{n-k}\xi(t)\hat{n}_t^{n-j:n-k-1}(S^*XS))\}dN_t.
\]
Here \(\Delta_t\) is denoted as
\[
\Delta_t = \hat{n}_t^{n-j:n-k}(L^*L) + \sqrt{n}\xi(t)\hat{n}_t^{n-1:n-1}(S^*L) + \sqrt{n-1}\xi(t)\hat{n}_t^{n-1:n-1}(S^*L) + n|\xi(t)|^2\hat{n}_t^{n-1:n-1}(I),
\]
and the innovation process \(N_t\) is defined by \(dN_t = dY^\lambda(t) - \Delta_t dt\). The initial conditions are
\[
\hat{n}_0^{n-j:n-k}(X) = \langle \Phi_{n-j}|\Phi_{n-k}\rangle\langle \eta,\eta \rangle.
\]

It is worth noting that, for the single photon case, Theorems 5 and 9 are exactly the one given in [17] (2012b). So, our formalism encapsulates and extends the previous results.

6 Examples

Here we apply the multi-photon quantum filters derived above to the problem of exciting a two-level atom in free space with a continuous-mode \(n\)-photon state. This problem has been considered in ([17]) for the single photon case. For a two-level atom we have \(H_S = \mathbb{C}^2\). The coupling operator is taken to be \(L = \sqrt{\kappa}\sigma_-\). Here \(\kappa > 0\) is the coupling rate (often referred to as the measurement strength) and is chosen to be \(\kappa = 1\). No scattering is assumed, i.e., \(S = I\). The atom is taken to be in the ground state \(|\eta\rangle\langle \eta |\) initially.

In ([17]) the master equation evolution of the atomic state has been studied, and the unconditional excited state population of the two-level atom is
\[
P_e(t) = \text{tr}[\rho^{n:n}(t)|e\rangle\langle e|],
\]
where \(\rho^{n:n}(t)\) is the solution to the master equations Eq. (16) and \(|e\rangle\) is the atomic excited state.
Figure 3: The excited state population of a two-level atom interacting with a general two-photon state with $t_1 = 3$, $t_2 = 7$ and $\Omega_1 = \Omega_2 = 1.46\kappa$. The thicker red line is $P_e(t)$ as calculated by the master equation. The thinner blue lines are the individual trajectories $P_c^e(t)$.

Now, we wish to calculate the conditional excited state population under homodyne detection, which can be expressed as

$$P_c^e(t) = \text{tr}[\rho^{n,n}(t)e\langle e\rangle],$$

where $\rho^{n,n}(t)$ is the solution to the filtering equations Eq. (4.11) for homodyne measurement. For the sake of presentation, we only take the case $n = 2$ into account. The spectral density function is assumed to be of a Gaussian form

$$\xi_i(t) = (\Omega_i^2/2\pi)^{1/4} \exp[-\Omega_i^2/4(t-t_i)^2], \quad i = 1, 2,$$

where $t_i$ specifies the peak arrival time and $\Omega_i$ is the frequency bandwidth of the pulse.

First, we consider the two-photon Fock state with $\xi_1(t) = \xi_2(t)$. Here we choose $t_1 = t_2 = 3$ and $\Omega_1 = \Omega_2 = 1.46\kappa$ which is known to be optimal for excitation via a single photon in a Gaussian pulse (26,27). The performance for this special state is presented in Fig. 2. For this particular SDF, there is very little spread in the trajectories for $t < 3$. After that, many of the trajectories start to decay, as evidenced by many lines below $P_c^e = 0.6$ for $t > 4$. Nevertheless, there are still a number of trajectories which continue to rise towards $P_c^e = 1$ after $t = 4$. Such behavior can’t be seen using the master equation approach.

Next we turn to the case where $\xi_1 \neq \xi_2$. If we choose $t_1 = 3$, $t_2 = 7$, and keep $\Omega_1 = \Omega_2 = 1.46\kappa$ unchanged, the conditional population is shown as Fig. 3. There are two evident peaks, one is at about $t = 4$, the other is nearly $t = 8$. During the first period $t \in [0, 4]$, the trajectories fit well with the master equation and reaches the maximum almost $P_c = 0.45$, then they decrease until $t = 6$. Then most of them begin to increase again, and decay next. In the second period, i.e., $t \in [6, 8]$, the trajectories varies strongly. Some of them can reach $P_c^e = 0.7$, while some only get $P_c^e = 0.2$. If we choose $t_1 = t_2 = 3$, $\Omega_1 = 2\kappa$ and $\Omega_2 = 4\kappa$, the conditional dynamics is presented in Fig. 4. Comparing to the Fig. 2, the maximum conditional population in this case is a little smaller, from which we can see the optimality of $\Omega$ being $1.46\kappa$.

From the above numerical simulations about 2-photon states, we can see that the multi-photon quantum filters can exhibit a variety of dynamics for various choices of spectral density functions on the input multi-photon states; and most of them are different from the single-photon case. Of course, once we consider the field containing more than 2 photons, there will be more interesting and significant phenomena.
Figure 4: The excited state population of a two-level atom interacting with a general two-photon state with $t_1 = t_2 = 3$, $\Omega_1 = 1\kappa$ and $\Omega_2 = 2\kappa$. The thicker red line is $P_e(t)$ as calculated by the master equation. The thinner blue lines are the individual trajectories $P_C^e(t)$.

7 Conclusion

We have derived quantum filters for systems driven by input fields containing $n$ photons. The master equations for this system are also presented as a by-product. By means of a two-level atom excited by a 2-photon state as a specific example, several interesting features of the $n$-photon quantum filters have been demonstrated. The formalism proposed here may be useful for measurement-based feedback control in the multi-photon setting.

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