On geometry of a special class of solutions to generalised WDVV equations

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Abstract. A special class of solutions to the generalised WDVV equations related to a finite set of covectors is considered. We describe the geometric conditions (∨-conditions) on such a set which are necessary and sufficient for the corresponding function to satisfy the generalised WDVV equations. These conditions are satisfied for all Coxeter systems but there are also other examples discovered in the theory of the generalised Calogero-Moser systems. As a result some new solutions for the generalized WDVV equations are found.

Introduction.

The WDVV (Witten-Dijgraaf-Verlinde-Verlinde) equations have been introduced first in topological field theory as some associativity conditions [1, 2]. Dubrovin has found an elegant geometric axiomatisation of the these equations introducing a notion of Frobenius manifold (see [3]).

What we will discuss in this paper are their generalised versions appeared in the Seiberg-Witten theory [4, 5]. The generalised WDVV equations are the following overdetermined system of nonlinear partial differential equations:

\[ F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad i, j, k = 1, \ldots, n, \quad (1) \]
where $F_m$ is the $n \times n$ matrix constructed from the third partial derivatives of the unknown function $F = F(x^1, \ldots, x^n)$:

$$(F_m)_{pq} = \frac{\partial^3 F}{\partial x^m \partial x^p \partial x^q},$$  \hspace{1cm} (2)$$

In this form these equations have been presented by A. Marshakov, A. Mironov and A. Morozov, who showed that the Seiberg-Witten prepotential in $N = 2$ four-dimensional supersymmetric gauge theories satisfies this system [3].

We will consider the following special class of the solutions to (1):

$$F^{\mathcal{A}} = \sum_{\alpha \in \mathcal{A}} (\alpha, x)^2 \log (\alpha, x)^2,$$  \hspace{1cm} (3)$$

where $\mathcal{A}$ be a finite set of noncollinear vectors $\alpha$ in $\mathbb{R}^n$. It is known to be a solution of the generalised WDVV equations in case when $\mathcal{A}$ is a root system (see [6] for the classical root systems and [7] for the general case). In the paper [8] it was observed that the same is true for any Coxeter configuration and has been found the general geometric conditions on $\mathcal{A}$, which guarantee that (3) satisfies the generalised WDVV equations (the so-called $\vee$-conditions). We describe these conditions in the first section below.

It turned out (see [8]) that these conditions are satisfied not only for the root systems but also for their deformations discovered by O. Chalykh, M. Feigin and the author in the theory of the generalised Calogero-Moser systems [3, 9, 10].

The corresponding families of the solutions to WDVV equations have the form

$$F = \sum_{i<j} (x_i - x_j)^2 \log (x_i - x_j)^2 + \frac{1}{m} \sum_{i=1}^n x_i^2 \log x_i^2$$  \hspace{1cm} (4)$$

with an arbitrary real value of the parameter $m$ and

$$F = k \sum_{i<j} [(x_i + x_j)^2 \log (x_i + x_j)^2 + (x_i - x_j)^2 \log (x_i - x_j)^2] +$$

$$+ \sum_{i=1}^n [(x_i + x_{n+1})^2 \log (x_i + x_{n+1})^2 + (x_i - x_{n+1})^2 \log (x_i - x_{n+1})^2] +$$

$$+ 4m \sum_{i=1}^n x_i^2 \log x_i^2 + 4l x_{n+1}^2 \log x_{n+1}^2,$$  \hspace{1cm} (5)$$

where the real parameters $k, m, l$ satisfy the only relation

$$k(2l + 1) = 2m + 1.$$  \hspace{1cm} (6)$$

When $m = 1$ the formula (3) gives the well-known solution to WDVV equations, corresponding to the leading perturbative approximation to the exact Seiberg-Witten prepotential for the gauge group $SU(n + 1)$ (see [3]). For the general $m$ it corresponds to the deformation $A_n(m)$ of the root system $A_n$ related to the Lie algebra $su(n + 1)$ (see [3] and below). Corresponding solution (3) has been found first in [3] (see the formula (3.15) and the calculations after
that) although the fact that it is related to the configurations with non-Coxeter geometry seems to be realised only in [8].

The second family of the solutions to WDVV equations (5) has been found in [8]. When $k = m = l = 1$ they correspond to the root system $C_{n+1}$, in the general case - to its deformation $C_{n+1}(m, l)$ (see [11] and below).

In this paper, which is an extended version of [8], we present also a new family of solutions which is related to a configuration discovered in the theory of integrable Schrödinger operators by Yu.Berest and M.Yakimov [12]. It has a form $F(x_1, \ldots, x_k, y_1, \ldots y_l)$,

$$F = \sum_{i<j} (x_i - x_j)^2 \log (x_i - x_j)^2 + \sum_{p<q} \mu^2 (y_p - y_q)^2 \log (y_p - y_q)^2 + \sum_{i=1}^{k} \sum_{p=1}^{l} \mu (x_i - y_p)^2 \log (\mu (x_i - y_p)^2),$$

for any integer $k$ and $l$ and arbitrary parameter $\mu$.

The fact that all the families of the configurations discovered so far in the theory of multidimensional integrable Schrödinger operators satisfy the $\vee$-conditions seems to be remarkable and calls for better understanding.

1 $\vee$-systems and a particular class of solutions to WDVV equations.

It is known [5, 13] that WDVV equations (1), (2) are equivalent to the equations

$$F_i G^{-1} F_j = F_j G^{-1} F_i, \quad i, j = 1, \ldots , n,$$

where $G = \sum_{k=1}^{n} \eta^k F_k$ is any particular invertible linear combination of $F_i$ with the coefficients, which may depend on $x$. Introducing the matrices $\hat{F}_i = G^{-1} F_i$ one can rewrite (8) as the commutativity relations

$$[\hat{F}_i, \hat{F}_j] = 0, \quad i, j = 1, \ldots , n,$$

We will consider the following particular class of the solutions to these equations.

Let $V$ be a real linear vector space of dimension $n$, $V^*$ be its dual space consisting of the linear functions on $V$ (covectors), $\mathfrak{A}$ be a finite set of noncollinear covectors $\alpha \in V^*$.

Consider the following function on $V$:

$$F^\mathfrak{A} = \sum_{\alpha \in \mathfrak{A}} (\alpha, x)^2 \log (\alpha, x)^2,$$

where $(\alpha, x) = \alpha(x)$ is the value of covector $\alpha \in V^*$ on a vector $x \in V$. For any basis $e_1, \ldots , e_n$ we have the corresponding coordinates $x^1, \ldots , x^n$ in $V$ and the
matrices $F_i$ defined according to (2). In a more invariant form for any vector $a \in V$ one can define the matrix

$$F_a = \sum_{i=1}^{n} a^i F_i.$$  

By a straightforward calculation one can check that $F_a$ is the matrix of the following bilinear form on $V$

$$F^A_a = \sum_{\alpha \in A} \frac{(\alpha, a)}{(\alpha, x)} \alpha \otimes \alpha,$$

where $\alpha \otimes \beta(u, v) = \alpha(u)\beta(v)$ for any $u, v \in V$ and $\alpha, \beta \in V^*.$

Another simple check shows that $G^A_a$ which is defined as as $F^A_x$, i.e.

$$G^A_a = \sum_{i=1}^{n} x^i F_i$$

is actually the matrix of the bilinear form

$$G^A_a = \sum_{\alpha \in A} \alpha \otimes \alpha,$$  

which does not depend on $x$.

We will assume that the covectors $\alpha \in A$ generate $V^*$, in this case the form $G^A_a$ is non-degenerate. This means that the natural linear mapping $\varphi_A : V \to V^*$ defined by the formula

$$(\varphi_A(u), v) = G^A_a(u, v), \ u, v \in V$$

is invertible. We will denote $\varphi_A^{-1}(\alpha), \ \alpha \in V^*$ as $\alpha^\vee$. By definition

$$\sum_{\alpha \in A} \alpha^\vee \otimes \alpha = Id$$

as an operator in $V^*$ or equivalently

$$(\alpha, v) = \sum_{\beta \in A} (\alpha, \beta^\vee)(\beta, v).$$  

(12)

for any $\alpha \in V^*, v \in V$. Now according to (3) the WDVV equations (1,2) for the function $F^A_a$ can be rewritten as

$$[F^A_a, F^A_b] = 0$$  

(13)

for any $a, b \in V$, where the operators $\hat{F}^A_a$ are defined as

$$\hat{F}^A_a = \sum_{\alpha \in A} \frac{(\alpha, a)}{(\alpha, x)} \alpha^\vee \otimes \alpha.$$  

(14)
A simple calculation shows that \((\ref{eq:13})\) can be rewritten as

\[
\sum_{\alpha \neq \beta, \alpha, \beta \in \mathfrak{A}} \frac{G^\mathfrak{A}(\alpha^\vee, \beta^\vee)B_{\alpha,\beta}(a,b)}{(\alpha, x)(\beta, x)} \alpha \land \beta \equiv 0,
\]

where

\[
\alpha \land \beta = \alpha \otimes \beta - \beta \otimes \alpha
\]

and

\[
B_{\alpha,\beta}(a,b) = \alpha \land (a, b) = \alpha(a)\beta(b) - \alpha(b)\beta(a).
\]

Thus the WDVV equations for the function \((\ref{eq:10})\) are equivalent to the conditions \((\ref{eq:15})\) to be satisfied for any \(x, a, b \in V\).

Notice that WDVV equations \((\ref{eq:1}, \ref{eq:2})\) and, therefore, the conditions \((\ref{eq:15})\) are obviously satisfied for any two-dimensional configuration \(\mathfrak{A}\). This fact and the structure of the relation \((\ref{eq:15})\) motivate the following notion of the \(\vee\)-systems \(\cite{8}\).

Remind first that for a pair of bilinear forms \(F\) and \(G\) on the vector space \(V\) one can define an eigenvector \(e\) as the kernel of the bilinear form \((F - \lambda G)(v, x) = 0\) for any \(v \in V\). When \(G\) is non-degenerate \(e\) is the eigenvector of the corresponding operator \(\tilde{F} = G^{-1}F\):

\[
\tilde{F}(e) = G^{-1}F(e) = \lambda e.
\]

Now let \(\mathfrak{A}\) be as above any finite set of non-collinear covectors \(\alpha \in V^*\), \(G = G^\mathfrak{A}\) be the corresponding bilinear form \((\ref{eq:11})\), which is assumed to be non-degenerate, \(\alpha^\vee\) are defined by \((\ref{eq:12})\). Define now for any two-dimensional plane \(\Pi \subset V^*\) a form

\[
G^\mathfrak{A}_\Pi(x, y) = \sum_{\alpha \in \Pi \cap \mathfrak{A}} (\alpha, x)(\alpha, y).
\]

**Definition.** We will say that \(\mathfrak{A}\) satisfies the \(\vee\)-conditions if for any plane \(\Pi \in V^*\) the vectors \(\alpha^\vee, \alpha \in \Pi \cap \mathfrak{A}\) are the eigenvectors of the pair of the forms \(G^\mathfrak{A}\) and \(G^\mathfrak{A}_\Pi\). In this case we will call \(\mathfrak{A}\) as \(\vee\)-system.

The \(\vee\)-conditions can be written explicitly as

\[
\sum_{\beta \in \Pi \cap \mathfrak{A}} \beta(\alpha^\vee)\beta^\vee = \lambda \alpha^\vee,
\]

for any \(\alpha \in \Pi \cap \mathfrak{A}\) and some \(\lambda\), which may depend on \(\Pi\) and \(\alpha\).

If the plane \(\Pi\) contains no more that one vector from \(\mathfrak{A}\) then this condition is obviously satisfied, so the \(\vee\)-conditions should be checked only for a finite number of planes \(\Pi\).

If the plane \(\Pi\) contains only two covectors \(\alpha\) and \(\beta\) from \(\mathfrak{A}\) then the condition \((\ref{eq:17})\) means that \(\alpha^\vee\) and \(\beta^\vee\) are orthogonal with respect to the form \(G^\mathfrak{A}\):

\[
\beta(\alpha^\vee) = G^\mathfrak{A}(\alpha^\vee, \beta^\vee) = 0.
\]
If the plane $\Pi$ contains more than two covectors from $\mathfrak{A}$ this condition means that $G^\mathfrak{A}$ and $G^\mathfrak{A}_\Pi$ restricted to the plane $\Pi^\vee \subset V$ are proportional:

$$G^\mathfrak{A}_\Pi|_{\Pi^\vee} = \lambda(\Pi) \ G^\mathfrak{A}|_{\Pi^\vee}$$

(18)

**Theorem 1.** A function (14) satisfies the WDVV equations (1) if and only if the configuration $\mathfrak{A}$ is a $\vee$-system

**Proof.** We have shown above that the function (14) satisfies the generalised WDVV equation iff the relations (15) are satisfied. Rewriting these relations as

$$\sum_{\beta \neq \alpha, \beta \in \Pi \cap \mathfrak{A}} G^\mathfrak{A}(\alpha^\vee, \beta^\vee) B_{\alpha,\beta}(a,b) \alpha \wedge \beta|_{(\alpha,x)} = 0$$

(19)

for any $\alpha \in \mathfrak{A}$ and any two-dimensional plane $\Pi$ containing $\alpha$ (cf. [11]). The last conditions are equivalent to

$$\sum_{\beta \neq \alpha, \beta \in \Pi \cap \mathfrak{A}} G^\mathfrak{A}(\alpha^\vee, \beta^\vee) B_{\alpha,\beta}(a,b) = 0$$

(19)

for any $\alpha \in \mathfrak{A}$ and $\Pi$ such that $\alpha \in \Pi$.

We would like to show that these relations are actually equivalent to the $\vee$-conditions. If $\Pi$ contains only two covectors $\alpha$ and $\beta$ from $\mathfrak{A}$ then it is obvious since in this case both of these relations are simply saying that $G^\mathfrak{A}(\alpha^\vee, \beta^\vee) = 0$.

Assume now that $\Pi$ contains more than two covectors. We should show that in this case $G^\mathfrak{A}$ is proportional $G^\mathfrak{A}_\Pi$ after the restriction to $\Pi^\vee$. First of all the relations (14) are obviously satisfied if we replace $G^\mathfrak{A}$ by $G^\mathfrak{A}_\Pi$:

$$\sum_{\beta \neq \alpha, \beta \in \Pi \cap \mathfrak{A}} G^\mathfrak{A}_\Pi(\alpha^\vee, \beta^\vee) B_{\alpha,\beta}(a,b) = 0$$

(20)

for any $\alpha \in \mathfrak{A}$ and any plane $\Pi$ containing $\alpha$. This follows for example from the fact that in two dimensions any function satisfies the generalised WDVV equations, but can be easily checked in a straightforward way as well. In particular this immediately implies that the $\vee$-conditions are sufficient for (14) to satisfy the generalised WDVV equations.

To prove that they are also necessary for this let’s suppose that this is not the case, i.e. $G^\mathfrak{A}$ is not proportional $G^\mathfrak{A}_\Pi$ on $\Pi^\vee$. Then we can find such a constant $c$ that the restriction of the form $G^\mathfrak{A} - cG^\mathfrak{A}_\Pi$ onto $\Pi^\vee$ has a rank 1:

$$G^\mathfrak{A} - cG^\mathfrak{A}|_{\Pi^\vee} = \epsilon \gamma \otimes \gamma|_{\Pi^\vee}, \epsilon = +1or -1.$$  

for some $\gamma \in V^*$. Without loss of generality we can assume that $(\gamma, \alpha^\vee) \geq 0$ for all $\alpha \in \mathfrak{A}$. Let $\alpha^\vee_0$ be ”the very right” vector from $\mathfrak{A}^\vee$ in the half-plane
\((\gamma, v) \geq 0, v \in \Pi^\vee\) such that \(B_{\alpha_0, \beta}(a, b) = \alpha_0 \wedge \beta(a, b)\) has the same sign for all \(\beta \in \mathfrak{A}\). Now put in the relations (19) and (20) \(\alpha = \alpha_0\) and subtract from the first relation the second one multiplied by \(c\):

\[
\sum_{\beta \not= \alpha_0, \beta \in \Pi \cap \mathfrak{A}} (G^G - cG^\mathfrak{A})(\alpha_0^\vee, \beta^\vee)B_{\alpha_0, \beta}(a, b) = \epsilon(\gamma, \alpha_0^\vee) \sum_{\beta \not= \alpha_0, \beta \in \Pi \cap \mathfrak{A}} (\gamma, \beta^\vee)B_{\alpha_0, \beta}(a, b) = 0.
\]

Since due to the choice of \(\gamma\) and \(\alpha_0\) all the summands have the same sign this is possible only if all of them are zero, i.e. \((\gamma, \beta^\vee) = 0\) for all \(\beta \in \Pi \cap \mathfrak{A}\) different from \(\alpha_0\). But this contradicts to the assumption that we have more than two noncollinear covectors from \(\mathfrak{A}\) belonging to \(\Pi\). This completes the proof of the Theorem 1.

**Remark.** It is easy to see from (13) that WDVV equations are equivalent to the commutativity of the following differential operators of the Knizhnik-Zamolodchikov type

\[
\nabla_a = \partial_a - \sum_{\alpha \in \mathfrak{A}} \frac{(\alpha, a)}{(\alpha, x)} \alpha^\vee \otimes \alpha.
\]

As it follows from the Theorem 1 these operators commute and therefore define a flat connection on \(V\) if and only if the configuration \(\mathfrak{A}\) is a \(\vee\)-system.

### 2 Examples of \(\vee\)-systems and new solutions to generalised WDVV equations.

Let \(V\) be now Euclidean vector space with a scalar product \((\cdot, \cdot)\), and \(G\) be any irreducible finite group generated by orthogonal reflections with respect to some hyperplanes (Coxeter groups [14]). Let \(\mathcal{R}\) be a set of normal vectors to the reflection hyperplanes of \(G\). We will not fix the length of the normals but assume that \(\mathcal{R}\) is invariant under the natural action of \(G\) and contains exactly two normal vectors for any such hyperplane. Let us choose from each such pair of vectors one of them and form the system \(\mathcal{R}^+\):

\[
\mathcal{R} = \mathcal{R}^+ \cup (-\mathcal{R}^+).
\]

Usually \(\mathcal{R}^+\) is chosen simply by taking from \(\mathcal{R}\) vectors which are positive with respect to some linear form on \(V\). We will call a system \(\mathcal{R}^+\) as *Coxeter system* and the vectors from \(\mathcal{R}^+\) as *roots*.

**Theorem 2.** Any Coxeter system \(\mathcal{R}^+\) is a \(\vee\)-system.

Proof is very simple. First of all the form (11) in this case is proportional to the euclidean structure on \(V\) because it is invariant under \(G\) and \(G\) is irreducible. By the same reason this is true for the form \(G\Pi\) if the plane \(\Pi\) contains more than two roots from \(\mathcal{R}^+\). When \(\Pi\) contains only two roots they must be orthogonal and therefore satisfy \(\vee\)-conditions.

**Corollary.** For any Coxeter system \(\mathcal{R}^+\) the function

\[
F = \sum_{\alpha \in \mathcal{R}^+} (\alpha, x)^2 \log (\alpha, x)^2
\]

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satisfy WDVV equations (1), (2).

When the Coxeter system is a root system of some semisimple Lie algebra this result has been proven in [6, 7]. Notice that even when \( G \) is a Weyl group our formula (22) in general gives more solutions since we have not fixed the length of the roots.

It is remarkable that the ∨-conditions are also satisfied for the following deformations of the root systems discovered in the theory of the generalised Calogero-Moser systems in [9, 10, 11].

To show this let us make first the following remark. One can consider the class of functions related to a formally more general situation when the covectors \( \alpha \) have also some prescribed multiplicities \( \mu_\alpha \)

\[
F^{(3, \mu)} = \sum_{\alpha \in \tilde{\Gamma}} \mu_\alpha (\alpha, x)^2 \log (\alpha, x)^2. \tag{23}
\]

But it is easy to see that this actually will give no new solutions because \( F^{(3, \mu)} = F^{3} + \text{quadratic terms} \), where \( \tilde{\Gamma} \) consists of covectors \( \sqrt{\mu_\alpha} \alpha \).

The following configurations \( A_n(m) \) and \( C_{n+1}(m, l) \) have been introduced in [9, 10, 11]. They consist of the following vectors in \( \mathbb{R}^{n+1} \):

\[
A_n(m) = \begin{cases} 
  e_i - e_j, & 1 \leq i < j \leq n, \text{ with multiplicity } m, \\
  e_i - \sqrt{m} e_{n+1}, & i = 1, \ldots, n \text{ with multiplicity } 1,
\end{cases}
\]

and

\[
C_{n+1}(m, l) = \begin{cases} 
  e_i \pm e_j, & 1 \leq i < j \leq n, \text{ with multiplicity } k, \\
  2e_i, & i = 1, \ldots, n \text{ with multiplicity } m, \\
  e_i \pm \sqrt{k} e_{n+1}, & i = 1, \ldots, n \text{ with multiplicity } 1, \\
  2\sqrt{k} e_{n+1} \text{ with multiplicity } l,
\end{cases}
\]

where \( k = \frac{2m+1}{2l+1} \).

When all the multiplicities are integer the corresponding generalisation of Calogero-Moser system is algebraically integrable, but usual integrability holds for any value of multiplicities (see [9, 10, 11]).

Notice that when \( m = 1 \) the first configuration coincides with the classical root system of type \( A_n \) and when \( k = m = l = 1 \) the second configuration is the root system of type \( C_{n+1} \). So these families can be considered as the special deformations of these roots systems.

One can easily check that the corresponding sets

\[
\tilde{A}_n(m) = \begin{cases} 
  \sqrt{m} (e_i - e_j), & 1 \leq i < j \leq n, \\
  e_i - \sqrt{m} e_{n+1}, & i = 1, \ldots, n
\end{cases}
\]
\[ C_n+1(m, l) = \begin{cases} \sqrt{k} e_i \pm \sqrt{k} e_j, & 1 \leq i < j \leq n \\ 2\sqrt{m} e_i, & i = 1, \ldots, n \\ e_i \pm \sqrt{k} e_{n+1}, & i = 1, \ldots, n \\ 2\sqrt{k} e_{n+1}, & \end{cases} \]

with \( k = \frac{2m+1}{2l+1} \) satisfy the ∨-conditions. To write down the corresponding solution in a more simple form it is suitable to make the following linear transformation:

\[ \tilde{A}_n(m) = \begin{cases} e_i - e_j, & 1 \leq i < j \leq n, \\ \frac{1}{\sqrt{m}} e_i, & i = 1, \ldots, n \end{cases} \]

and

\[ \tilde{C}_n+1(m, l) = \begin{cases} \sqrt{k}(e_i \pm e_j), & 1 \leq i < j \leq n, \\ 2\sqrt{m} e_i, & i = 1, \ldots, n \\ e_i \pm e_{n+1}, & i = 1, \ldots, n \\ 2\sqrt{e_{n+1}}, & \end{cases} \]

where again \( k = \frac{2m+1}{2l+1} \).

Now the corresponding functions \( F \) have the form (4), (5) written in the Introduction.

The third family of solutions (7) is related to the configuration which I would denote as \( A_k \ast A_l(\mu) \)

\[ A_k \ast A_l(\mu) = \begin{cases} e_i - e_j, & 1 \leq i < j \leq k, \\ \mu(f_p - f_q), & 1 \leq p < q \leq l, \\ \mu e_i - f_p, & i = 1, \ldots, k, p = 1, \ldots, l \end{cases} \]

where \( e_i \) and \( f_p \) are the basic vectors in \( \mathbb{R}^k \) and \( \mathbb{R}' \) correspondingly. This configuration has been discovered by Yu.Berest and M.Yakimov who were looking for a special "isomonodromial deformation" of the Calogero-Moser problem related to a direct sum of the root systems of types \( A_k \) and \( A_l \). When the parameter \( \mu = 1 \) it coincides with the standard \( A_{k+l+1} \) root system. Again the fact that this family of configurations satisfies the ∨-conditions can be checked in a straightforward way.

As a corollary we have the following

**Theorem 3.** The functions \( F \) given by the formulas (4), (5), (7) satisfy the generalised WDVV equations.
It is easy to see that these solutions are really different, i.e. they are not equivalent under a linear change of variables, which is a symmetry of the generalised WDVV equations. Indeed the bilinear form $G^A$ is determined by the configuration $A$ in an invariant way and thus induces an invariant Euclidean structure on $V$ and $V^*$. In particular, all the angles between the covectors of the configuration are invariant under any linear transformation applied to the configuration. A simple calculation shows that these angles depend on the parameters of the families and the only case when these configurations have the same geometry is when $m = 1$ in the first family and $\mu = \pm 1$ in the third one. In this case we have simply the root system of type $A_N$.

3 Concluding remarks.

At the moment there are no satisfactory explanations why the deformed root systems arisen in the theory of the generalised Calogero-Moser problems turned out to be $\vee$-systems. It may be that it is a common geometrical property of all the so-called locus configurations [11]. In this connection I’d like to mention that $\vee$-systems can be naturally defined in a complex vector space. Their classification seems to be very interesting and important problem.

Another very interesting problem is the investigation of the corresponding almost Frobenius structures related to these systems (see [3, 15]). Dubrovin discovered some very interesting duality in the Coxeter case with the Frobenius structures on the spaces of orbits of Coxeter groups [15, 16]. The natural question is whether this can be generalised for the non-Coxeter $\vee$-systems and what are the corresponding dual Frobenius structures.

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