LINEAR INVARIANTS OF COMPLEX MANIFOLDS AND THEIR PLURISUBHARMONIC VARIATIONS

FUSHENG DENG, ZHIWEI WANG, LIYOU ZHANG, AND XIANGYU ZHOU

ABSTRACT. For a bounded domain $D$ and a real number $p > 0$, we denote by $A^p(D)$ the space of $L^p$ integrable holomorphic functions on $D$, equipped with the $L^p$-pseudonorm. We prove that two bounded hyperconvex domains $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ are biholomorphic (in particular $n = m$) if there is a linear isometry between $A^p(D_1)$ and $A^p(D_2)$ for some $0 < p < 2$. The same result holds for $p > 2, p \neq 2, 4, \ldots$, provided that the $p$-Bergman kernels on $D_1$ and $D_2$ are exhaustive. With this as a motivation, we show that, for all $p > 0$, the $p$-Bergman kernel on a strongly pseudoconvex domain with $C^2$ boundary or a simply connected homogeneous regular domain is exhaustive. These results show that spaces of pluricanonical sections of complex manifolds equipped with canonical pseudonorms are important invariants of complex manifolds.

The second part of the present work devotes to studying variations of these invariants. We show that the direct image sheaf of the twisted relative $m$-pluricanonical bundle associated to a holomorphic family of Stein manifolds or compact Kähler manifolds is positively curved, with respect to the canonical singular Finsler metric.

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1. Introduction

It is well known that two pseudoconvex domains (or even Stein manifolds) $\Omega_1$ and $\Omega_2$ are biholomorphic if and only if $\mathcal{O}(\Omega_1)$ and $\mathcal{O}(\Omega_2)$, the spaces of holomorphic functions, are isomorphic as $\mathbb{C}$-algebras with unit. This implies that the holomorphic structure of a pseudoconvex domain is uniquely determined by algebraic structure of the space of holomorphic functions on it. In the first part of the present work, we prove some results in a related but different direction.

To state the results, we need to recall some notions. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Let $z = (z_1, \cdots, z_n)$ be the natural holomorphic coordinates of $\mathbb{C}^n$, and let $d\lambda_n := (\frac{1}{2})^n dz_1 \wedge \cdots \wedge dz_n$ be the canonical volume form on $\Omega$. For $p > 0$, denote by $A^p(\Omega)$ the space of all holomorphic functions $\phi$ on $\Omega$ with finite $L^p$-norm

$$\|\phi\|_p := \left( \int_{\Omega} |\phi|^p d\lambda_n \right)^{1/p}.$$  

It is a standard fact that for $p \geq 1$, $A^p(\Omega)$ are separable Banach spaces, and for $0 < p < 1$, $A^p(\Omega)$ are complete separable metric spaces with respect to the metric $d(\phi_1, \phi_2) := \|\phi_1 - \phi_2\|_p^2$.

The $p$-Bergman kernel is defined as follows:

$$B_{\Omega,p}(z) := \sup_{\phi \in A^p(\Omega)} \frac{|\phi(z)|^2}{\|\phi\|_p^2}.$$  

When $p = 2$, $B_{\Omega,2}$ is the ordinary Bergman kernel. By a standard argument of Montel theorem, one can prove that $B_{\Omega,p}$ is a continuous plurisubharmonic function on $\Omega$. We say that $B_{\Omega,p}$ is exhaustive if for any real number $A$ the set $\{ z \in \Omega ||B_{\Omega,p}(z) \leq A \}$ is compact.

A bounded domain $\Omega$ is called hyperconvex if there is a plurisubharmonic function $\rho : \Omega \rightarrow [-\infty, 0)$ such that for any $c < 0$ the set $\{ z \in \Omega | \rho(z) \leq c \}$ is compact.

**Theorem 1.1.** Let $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ be bounded hyperconvex domains. Suppose that there is a $p > 0, p \neq 2, 4, 6, \cdots$, such that

1. there is a linear isometry $T : A^p(\Omega_1) \rightarrow A^p(\Omega_2)$, and
2. the $p$-Bergman kernels of $\Omega_1$ and $\Omega_2$ are exhaustive,

then $m = n$ and there exists a unique biholomorphic map $F : \Omega_1 \rightarrow \Omega_2$ such that

$$|T \phi \circ F| J_F |^{2/p} = |\phi|, \ \forall \phi \in A^p(\Omega_1),$$

where $J_F$ is the holomorphic Jacobian of $F$. If $n = 1$, the assumption of hyperconvexity can be dropped.

Throughout this paper, that $T : A^p(\Omega_1) \rightarrow A^p(\Omega_2), \phi \mapsto T \phi$ is a linear isometry means that $T$ is a linear surjective map and $\|T \phi\|_p = \|\phi\|_p$ for all $\phi \in A^p(\Omega_1)$.

In [2], we will show that for any two bounded domains $\Omega_1$ and $\Omega_2$, $dim \Omega_1 = dim \Omega_2$ provided that $A^p(\Omega_1)$ and $A^p(\Omega_2)$ are linear isometric for some $p > 0, p \neq 2, 4, \cdots$.

Theorem 1.1 implies that the exhaustion of the $p$-Bergman kernels for bounded domains is a very important property. For the case that $p < 2$, Ning-Zhang-Zhou get a complete result by showing that a bounded domain $\Omega$ is pseudoconvex if and only if its $p$-Bergman kernel is exhaustive [2], the proof of which is based on an $L^p$-variant of the Ohsawa-Takegoshi extension theorem. Therefore we get from 1.1 the following
Theorem 1.2. Let $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ be bounded hyperconvex domains. Suppose that there is a linear isometry $T : A^p(\Omega_1) \to A^p(\Omega_2)$ for some $p \in (0, 2)$, then $m = n$ and there exists a unique biholomorphic map $F : \Omega_1 \to \Omega_2$ such that

$$|T \phi \circ F| |J_F|^{2/p} = |\phi|, \forall \phi \in A^p(\Omega_1).$$

If $n = 1$, the assumption of hyperconvexity can be dropped.

It is known that bounded pseudoconvex domains with Lipschitz boundary are hyperconvex [12]. Even more, bounded pseudoconvex domains with $\alpha$-Hölder boundary for all $\alpha < 1$ are hyperconvex [1]. Recall that a bounded domain $\Omega \subset \mathbb{C}^n$ is called homogeneous regular, a concept proposed by Liu-Sun-Yau [27], if there is a constant $c \in (0, 1)$ such that for any $z \in \Omega$, there is a holomorphic injective map $f : \Omega \to \mathbb{B}^n$ with $f(z) = 0$ and $\mathbb{B}^n(c) \subset f(\Omega)$, where $\mathbb{B}^n(c) = \{z \in \mathbb{C}^n; \|z\| < c\}$. It is shown in [38] that all homogeneous regular domains are hyperconvex.

The exhaustion of $p$-Bergman kernels for $p > 2$ is a subtle problem. In the present paper, we show that $p$-Bergman kernels of simply connected homogeneous regular domains or bounded strongly pseudoconvex domains are exhaustive for all $p > 0$.

Theorem 1.3. If $\Omega \subset \mathbb{C}^n$ is a simply connected and homogeneous regular domain, then the $p$-Bergman kernel of $\Omega$ is exhaustive.

It seems interesting to consider whether the condition about simply connectedness in Theorem 1.2 can be dropped. This is the case for bounded strongly pseudoconvex domains with $C^2$ boundary, which are known to be homogeneous regular [10]. The proof is based on results on exposing boundary points of strongly pseudoconvex domains in [18][13].

Theorem 1.4. For any bounded strongly pseudoconvex domain $\Omega$ in $\mathbb{C}^n$ with $C^2$ smooth boundary, the $p$-Bergman kernel is exhaustive for all $p > 0$.

We conjecture that Theorem 1.4 holds for all bounded pseudoconvex domains with $C^2$ smooth boundary.

We have known many homogeneous regular domains, such as Teichmüller spaces of compact Riemann surfaces, bounded homogeneous domains, strongly pseudoconvex domains with $C^2$ boundary, bounded convex domains, and bounded complex convex domains (see [16], [20], [23], [31]).

By Theorem 1.1, Theorem 1.3, and Theorem 1.4 we get

Theorem 1.5. Let $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ be bounded domains which are simply connected and homogeneous regular, or strongly pseudoconvex with $C^2$ smooth boundary. Suppose that there is a linear isometry $T : A^p(\Omega_1) \to A^p(\Omega_2)$ for some $p > 0, p \neq 2, 4, \cdots$, then $m = n$ and there exists a unique biholomorphic map $F : \Omega_1 \to \Omega_2$ such that

$$|T \phi \circ F| |J_F|^{2/p} = |\phi|, \forall \phi \in A^p(\Omega_1).$$

For the case that $n = m = 1$ and $p = 1$, Theorem 1.2 was proved by Lakic and Markovic in [30]. Application of spaces of pluricanonical forms with pseudonorms to birational geometry of projective algebraic manifolds was proposed by Chi and Yau in [11] and was studied in [10][37]. These works form the motivation to Theorem 1.1.
Our method to Theorem 1.1 is partially inspired by the argument in [30]. The argument in [30] can be divided into two main parts as follows. The first part is to construct a biholomorphic map, say $F$, from a dense open subset $\Omega'_1 \subset \Omega_1$ onto a dense open subset $\Omega'_2 \subset \Omega_2$; and the second part is to show that $F$ extends to a biholomorphic map from the whole $\Omega_1$ to $\Omega_2$. The first part is based on a result of Rudin on equimeasurability [34]. The second part, namely continuing $F$ across $\Omega_1 \setminus \Omega'_1$, is more technical and involves the uniformization theorem for Riemann surfaces.

The proof of Theorem 1.1 in the present paper is also divided into two parts in the same way. The construction of $F$ in the first part is based on the method in [30]. The difference is that our starting point is an intrinsic definition of $F$, where that in [30] is a countable dense subset of $A^1(\Omega)$, from which the map $F$ is defined. The benefit of the modification is that $F$ is obviously uniquely determined by $T$ and some important properties of $F$ become clear just from the definition. The method to the second part, namely the part about extension of $F$, is totally different from that in [30]. The new observation comes from pluripotential theory. Using the assumption of exhaustion of $p$-Bergman kernels, we construct a plurisubharmonic functions say $\rho$ on $\Omega_1$ such that $\rho = -\infty$ exactly on $\Omega_1 \setminus \Omega'_1$, and hence $\Omega_1 \setminus \Omega'_1$ is a closed pluripolar set in $\Omega_1$. Then, by a basic result in pluripotential theory about extension of holomorphic functions, we can continue $F$ to a holomorphic map on the whole $\Omega_1$. The hyperconvexity property is then used to prove that $\Omega'_j = \Omega_j$, $j = 1, 2$.

Remark 1.1. (1). Let $f : \Omega_1 \to \Omega_2$ be a biholomorphic map between two bounded domains. If $\Omega_1$ is simply connected or $p = 2/m$ for some integer $m$, then $f$ induces a linear isometry $T : A^p(\Omega_2) \to A^p(\Omega_1)$ as:

$$\psi \mapsto (\psi \circ f)^{2/p}, \psi \in A^p(\Omega_2).$$

(2). When $p = 2/m$ for some integer $m$, $A^p(\Omega)$ can be defined intrinsically as the space of holomorphic sections of $\mathcal{M}_\Omega$ with finite intrinsic pseudonorm (will be defined later), where $\mathcal{M}_\Omega$ is the canonical bundle of $\Omega$.

(3). For any two bounded domains $\Omega_1$ and $\Omega_2$, $A^2(\Omega_1)$ and $A^2(\Omega_2)$ are always linear isometric since they are both separable Hilbert spaces.

The second part of the present paper devotes to the study of variation of $A^p(\Omega)$ as $\Omega$ varies. We will consider holomorphic families of more general complex manifolds.

We first consider a family of pseudoconvex domains. Let $\Omega \subset \mathbb{C}^{r+n} = \mathbb{C}_r \times \mathbb{C}^n$ be a pseudoconvex domain. Let $p : \Omega \to \mathbb{C}^r$ be the natural projection. We denote $p(\Omega)$ by $D$ and denote $p^{-1}(t)$ by $\Omega_t$ for $t \in D$. Let $\varphi$ be a plurisubharmonic function on $\Omega$ and let $m \geq 1$ be a fixed integer. For an open subset $U$ of $D$, we denote by $\mathcal{F}(U)$ the space of holomorphic functions $F$ on $p^{-1}(U)$ such that $\int_{p^{-1}(U)} |F|^{2/m} e^{-\varphi} \leq \infty$ for all compact subset $K$ of $D$. For $t \in D$, let

$$E_{m,t} = \{ F|_{\Omega_t} : F \in \mathcal{F}(U), U \subset D \text{ open and } U \ni t \}.$$ 

Then $E_{m,t}$ is a vector space with the following pseudonorm:

$$H(f) := |f|_{m} = \left( \int_{D_t} |f|^{2/m} e^{-\varphi_t} \right)^{m/2} \leq \infty,$$
where $\varphi_t = \varphi|_D$. Let $E_m = \bigsqcup_{t \in D} E_{m,t}$ be the disjoint union of all $E_{m,t}$. Then we have a natural projection $\pi : E_m \to D$ which maps elements in $E_{m,t}$ to $t$. We view $H$ as a Finsler metric on $E_m$.

In general $E_m$ is not a genuine holomorphic vector bundle over $D$. However, we can also talk about its holomorphic sections, which are the objects we are really interested in. By definition, a section $s : D \to E_m$ is a holomorphic section if it varies holomorphically with $t$, namely, the function $s(t, z) : \Omega \to \mathbb{C}$ is holomorphic with respect to the variable $t$. Note that $s(t, z)$ is automatically holomorphic on $z$ for $t$ fixed, by Hartogs theorem, $s(t, z)$ is holomorphic jointly on $t$ and $z$ and hence is a holomorphic function on $\Omega$.

Let $E_{m,t}^*$ be the dual space of $E_{m,t}$, namely the space of all complex linear functions on $E_{m,t}$. The natural projection from $E_m^* \to D$ is denoted by $\pi^*$. Note that we do not define any topology on $E_{m,t}^*$ and $E_m$. The only object we are interested in is holomorphic sections of $E_m^*$ which we are going to define. The following definition, as well as the definition of $E_m$ given above, is proposed in the recent work [17].

**Definition 1.1.** A section $\xi$ of $E_m^*$ on $D$ is holomorphic if:

1. for any local holomorphic section $s$ of $E_m$, $\langle \xi, s \rangle$ is a holomorphic function;
2. for any sequence $s_j$ of holomorphic sections of $E_m$ on $D$ such that $\int_D |s_j|_m \leq 1$, if $s_j(t, z)$ converges uniformly on compact subsets of $\Omega$ to $s(t, z)$ for some holomorphic section $s$ of $E_m$, then $\langle \xi, s_j \rangle$ converges uniformly to $\langle \xi, s \rangle$ on compact subsets of $D$.

In the same way we can define holomorphic section of $E_{m,t}^*$ on open subsets of $D$. The Finsler metric $H$ on $E_m$ induces a Finsler metric $H^*$ on $E_m^*$ (see Definition 4.2).

**Theorem 1.6.** $(E_m, H)$ has positive curvature, in the sense that for any holomorphic section $\xi : D \to E_m^*$ of $E_m^*$, the function $\psi := \log |\xi(t)|_m := \log H^*(\xi(t)) : D \to [0, +\infty)$ is plurisubharmonic on $D$.

For the case that $m = 1$, $\Omega$ is a product, and $\varphi$ is smooth up to $\overline{\Omega}$, Theorem 1.6 was proved by Berndtsson in [3]; and similar results with $m = 1$ and varying fibers was proved by Wang [36] with extra technical conditions. It is obvious from the argument that Theorem 1.6 can be generalized to general Stein families of complex manifolds equipped with pseudoeffective line bundles.

A direct corollary of Theorem 1.6 is the plurisubharmonicity of the relative $m$-Bergman kernels associated to $\Omega$ and $\varphi$, which is proved in [17] (the case that $m = 1$ is proved in [3]).

We now consider the case of a Kähler family of complex manifolds. Let $X, Y$ be Kähler manifolds of dimension $r + n$ and $r$ respectively, let $p : X \to Y$ be a proper holomorphic submersion. Let $L$ be a holomorphic line bundle over $X$, and $h$ be a singular Hermitian metric on $L$, whose curvature current is semi-positive.

Let $m > 0$ be a fixed integer. The multiplier ideal sheaf $\mathcal{I}_m(h) \subset \mathcal{O}_X$ is defined as follows. If $\varphi$ is a local weight of $h$ on some open set $U \subset X$, then the germ of $\mathcal{I}_m(h)$ at a point $p \in U$ consists of the germs of holomorphic functions $f$ at $p$ such that $|f|^{2/m} e^{-\varphi/m}$ is integrable at $p$. It is known that $\mathcal{I}(h^{1/m})$ is a coherent analytic sheaf on $X$ [8].
For \( y \in Y \) let \( X_y = p^{-1}(y) \), which is a compact submanifold of \( X \) of dimension \( n \) if \( y \) is a regular value of \( p \). Let \( K_{X/Y} \) be the relative canonical bundle on \( X \) and \( \mathcal{E}_m = p_*(mK_{X/Y} \otimes L \otimes \mathcal{I}_m(h)) \) be the direct image sheaf on \( Y \). By Grauert’s theorem, \( \mathcal{E}_m \) is a coherent analytic sheaf on \( Y \). One can choose a proper analytic subset \( A \subset Y \) such that:

1. \( p \) is submersive over \( Y \setminus A \),
2. both \( \mathcal{E}_m \) and the quotient sheaf \( p_*(mK_{X/Y} \otimes L)/\mathcal{E}_m \) are locally free on \( Y \setminus A \),
3. \( E_{m,y} \) is naturally identified with \( H^0(X_y, mK_{X_y} \otimes L|_{X_y} \otimes \mathcal{I}_m(h)|_{X_y}) \), for \( y \in Y \setminus A \).

where \( E_m \) is the vector bundle on \( Y \setminus A \) associated to \( \mathcal{E}_m \). For \( u \in E_{m,y} \), the \( m \)-norm of \( u \) is defined to be

\[
H_m(u) := \|u\|_m = \left( \int_{X_y} |u|^{m/2} h^{1/m} \right)^{m/2} \leq +\infty.
\]

Then \( H_m \) is a Finsler metric on \( E_m \).

**Theorem 1.7.** For any \( m \geq 1 \), \( H_m \) is a singular Finsler metric on the coherent analytic sheaf \( \mathcal{E}_m \) which is positively curved.

Please see Definition 4.3 for the definition of positively curved singular Finsler metrics on coherent analytic sheaves.

A consequence of Theorem 1.7 is that the \( m \)-Bergman kernel metric on the twisted relative pluricanonical line bundle \( mK_{X/Y} \otimes L \) has positive curvature current. In recent years, plurisubharmonic variation of \( m \)-Bergman kernel metrics and topics in the direction of Theorem 1.7 (in the case that \( m = 1 \)) were extensively studied by many authors (see e.g. [3][4][5][6][21][33][8][23][39][40][2][17]), in different settings, in different generality, and by different methods.

Though positivity of direct image sheaves with Hermitian metrics and plurisubharmonic variation of \( m \)-Bergman kernels have been extensively studied, it seems that the positivity of the Finsler metrics described in Theorem 1.6 and Theorem 1.7 was overlooked in literatures. Our motivation for Theorem 1.6 and Theorem 1.7 comes from Theorem 1.1 and the works of Chi and Yau as mentioned above which imply that \( E_m \) in Theorem 1.6 and \( \mathcal{E}_m \) in Theorem 1.7 are important linear invariants of the involved holomorphic families.

Our method to Theorem 1.6 and Theorem 1.7 follows from that in [21] and [23], and is based on the Ohsawa-Takegoshi type extension theorems with optimal estimate (see [7][21][8][39]) and an \( L^p \)-variant of them [6].

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2. Linear invariants of bounded pseudoconvex domains in \( C^n \)

The aim of this section is to prove Theorem 1.1. The proof is divided into three subsections.
2.1. A measure theoretic preparation. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ be holomorphic coordinates of $\mathbb{C}^n$, and let $d\lambda_n := (\frac{1}{2\pi})^n dz_1 \wedge \overline{dz}_1 \wedge \cdots \wedge dz_n \wedge \overline{dz}_n$ be the canonical volume form on $\Omega$. For $p > 0$, denote by $A^p(\Omega)$ the space of all holomorphic functions $\phi$ with finite $L^p$-norm

$$\|\phi\|_p := \left( \int_{\Omega} |\phi|^p d\lambda_n \right)^{1/p}.$$ 

It is a standard fact that for $p \geq 1$, $A^p(\Omega)$ are separable Banach spaces, and for $0 < p < 1$, $A^p(\Omega)$ are complete separable metric spaces with respect to the metric $d(\varphi_1, \varphi_2) := \|\varphi_1 - \varphi_2\|_p^p$. The following result about isometries between $L^p$ spaces, is due to Rudin.

**Lemma 2.1** ([34]). Let $\mu$ and $\nu$ be finite positive measures on two sets $M$ and $N$ respectively. Assume $0 < p < \infty$ and $p$ is not even. Let $n$ be a positive integer. If $f_i \in L^p(M, \mu)$, $g_i \in L^p(N, \nu)$ for $1 \leq i \leq n$ satisfy

$$\int_M |1 + \alpha_1 f_1 + \cdots + \alpha_n f_n|^p d\mu = \int_N |1 + \alpha_1 g_1 + \cdots + \alpha_n g_n|^p d\nu$$

for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, then $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are equimeasurable, i.e. for every bounded Borel measurable function (and for every real-valued non-negative Borel function) $u : \mathbb{C}^n \to \mathbb{C}$, we have

$$\int u(f_1, \ldots, f_n) d\mu = \int u(g_1, \ldots, g_n) d\nu.$$

Furthermore, let $I : M \to \mathbb{C}^n$ and $J : N \to \mathbb{C}^n$ be the maps $I = (f_1, \ldots, f_n)$ and $J = (g_1, \ldots, g_n)$, respectively. Then we have

$$\mu(I^{-1}(E)) = \nu(J^{-1}(E))$$

for every Borel set $E$ in $\mathbb{C}^n$.

The following lemma, observed by Markovic [30], is a direct corollary of Lemma 2.1

**Lemma 2.2.** Let $\Omega_1$ and $\Omega_2$ be two bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Suppose that $\phi_k$, $k = 0, 1, 2, \ldots, N$, $N \in \mathbb{N}$, are elements of $A^p(\Omega_1)$ and suppose that $\psi_k$, $k = 0, 1, 2, \ldots, N$ are elements of $A^p(\Omega_2)$, such that for every $N$-tuple of complex numbers $\alpha_k$, $k = 1, \ldots, N$, we have

$$\|\phi_0 + \sum_{k=1}^N \alpha_k \phi_k\|_p = \|\psi_0 + \sum_{k=1}^N \alpha_k \psi_k\|_p.$$

If neither $\phi_0$ nor $\psi_0$ is constantly zero, then for every real valued non-negative Borel function $u : \mathbb{C}^N \to \mathbb{C}$, we have

$$\int_{\Omega_1} u(\frac{\phi_1}{\phi_0}, \ldots, \frac{\phi_N}{\phi_0}) |\phi_0|^p d\lambda_n = \int_{\Omega_2} u(\frac{\psi_1}{\psi_0}, \ldots, \frac{\psi_N}{\psi_0}) |\psi_0|^p d\lambda_m.$$ 

**Proof.** Set $d\mu := |\phi_0|^p d\lambda_n$ and $d\nu := |\psi_0|^p d\lambda_m$. Then the measures $\mu$ and $\nu$ are well defined on $\Omega_1$ and $\Omega_2$, respectively. Then from the assumption, we have that for $k = 1, 2, \cdots, N$, $\frac{\phi_k}{\phi_0} \in A^p(d\mu)$ and $\frac{\psi_k}{\psi_0} \in A^p(d\nu)$, respectively, and that

$$\int_{\Omega_1} |1 + \sum_{k=1}^N \alpha_k \frac{\phi_k}{\phi_0}|^p d\mu = \int_{\Omega_2} |1 + \sum_{k=1}^N \alpha_k \frac{\psi_k}{\psi_0}|^p d\nu.$$ 


Thus Lemma 2.3 follows from Lemma 2.1. □

2.2. Definition of the map and equidimension. Let \( \Omega_1 \subset \mathbb{C}^n \) and \( \Omega_2 \subset \mathbb{C}^m \) be bounded domains and let \( p > 0, p \neq 2, 4, \ldots \). Assume there is a linear isometry \( T : A^p(\Omega_1) \to A^p(\Omega_2) \).

We define a subset \( \Omega'_1 \) of \( \Omega_1 \) as follows. For \( z \in \Omega_1 \), we say as definition that \( z \in \Omega'_1 \) if and only if there exists \( w \in \Omega_2 \) such that

\[
\phi(z)T\phi_0(w) = T\phi(w)\phi_0(z), \quad \forall \phi, \phi_0 \in A^p(\Omega_1).
\]

Similarly, we define \( \Omega'_2 \subset \Omega_2 \) by saying that \( w \in \Omega'_2 \) if and only if \( \exists z \in \Omega_1 \) such that (1) holds. Since \( \Omega_j (j = 1, 2) \) are bounded domains, \( A^p(\Omega_j) \) separates points in \( \Omega_j \). Hence for any \( z \in \Omega'_1 \) there is a unique \( w \in \Omega_2 \) such that (1) holds. Therefore we can define a map

\[
F : \Omega'_1 \to \Omega_2
\]

by setting \( F(z) = w \) if \( w \in \Omega_2 \) satisfies Equation (1). It is clear that \( F(\Omega'_1) \subset \Omega'_2 \) and

\[
F : \Omega'_1 \to \Omega_2
\]

is a bijection.

A more conceptual definition of \( \Omega'_j \) and \( F \) is as follows. For \( z \in \Omega_j, (j = 1, 2), \)

\[ L_{j,z} := \{ \phi \in A^p(\Omega_j) | \phi(z) = 0 \} \]

is a hyperplane in \( A^p(\Omega_j) \). For \( z \in \Omega_1, w \in \Omega_2 \), we say that \( z \in \Omega'_1, w \in \Omega'_2 \) and define \( F(z) = w \) if \( T(L_{1,z}) = L_{2,w} \).

From the definition, the following properties about \( \Omega'_1 \) and \( F \) is obvious.

Lemma 2.3. (1) For a sequence of points \( z_j \in \Omega'_1 \) which converges to \( z_0 \in \Omega_1 \), if \( w_j := F(z_j) \) converges to \( w_0 \in \Omega_2 \), then \( z_0 \in \Omega'_1 \), \( w_0 \in \Omega'_2 \) and \( F(z_0) = w_0 \); and

(2) For \( z \in \Omega'_1 \) and \( \phi \in A^p(\Omega), \phi(z) = 0 \) if and only if \( T\phi(F(z)) = 0 \).

To investigate more properties of \( \Omega'_j (j = 1, 2) \) and \( F \), we need to give a representation of \( F \). As in [30], this is down by choosing a countable dense subset \( \{ \phi_0, \phi_1, \cdots \} \) of it. Then \( \{ \psi_0 := T\phi_0, \psi_1 := T\phi_1, \cdots \} \) is a countable dense subset of \( A^p(\Omega_2) \). For \( N > 0 \), we define maps \( I_N, J_N \) as follows:

\[ I_N : \Omega_1 \to \mathbb{C}^N, \quad z \mapsto \frac{\phi_1(z)}{\phi_0(z)}, \frac{\phi_2(z)}{\phi_0(z)}, \cdots \]

\[ J_N : \Omega_2 \to \mathbb{C}^N, \quad z \mapsto \frac{\psi_1(z)}{\psi_0(z)}, \frac{\psi_2(z)}{\psi_0(z)}, \cdots \]

which are measurable maps on \( \Omega_1 \) and \( \Omega_2 \) respectively. We can also define \( I_{\infty} \) and \( J_{\infty} \) as

\[ I_{\infty} : \Omega_1 \to \mathbb{C}^\infty, \quad z \mapsto \frac{\phi_1(z)}{\phi_0(z)}, \frac{\phi_2(z)}{\phi_0(z)}, \cdots \]

\[ J_{\infty} : \Omega_2 \to \mathbb{C}^\infty, \quad z \mapsto \frac{\psi_1(z)}{\psi_0(z)}, \frac{\psi_2(z)}{\psi_0(z)}, \cdots \]

Then it is clear that \( \Omega'_1 \cap \phi_0^{-1}(0) = \cap_N I_N^{-1}J_N(\Omega_2) = I_{\infty}^{-1}J_{\infty}(\Omega_2) \) and \( \Omega'_2 \cap \psi_0^{-1}(0) = \cap_N J_N^{-1}I_N(\Omega_1) = J_{\infty}^{-1}I_{\infty}(\Omega_1) \). Since \( A^p(\Omega_j) \) separates points of \( \Omega_j, (j = 1, 2) \), both \( I_{\infty} \) and \( J_{\infty} \) are injective on their domains of definition. For \( z \in I_{\infty}^{-1}J_{\infty}(\Omega_2) \), it is easy to see that \( F(z) = J_{\infty}^{-1}(I_{\infty}(z)) \).

Lemma 2.4. The Lebesgue measure of \( \Omega_j \setminus \Omega'_j \) are zero, for \( j = 1, 2 \).
Proof. It suffices to prove that Lebesgue measure of $K := \Omega_1 \setminus \Omega'_1$ is zero. Set $d\mu := |\phi_0|^p d\lambda_n$ and $d\nu := |\psi_0|^p d\lambda_n$. It suffices to prove that $\mu(K) = 0$.

By Lemma 2.4, we have $\mu(I_N^{-1}(I_N(K))) = \nu(J_N^{-1}(I_N(K)))$. We have seen that $\Omega'_1 \supset \cap I_N^{-1} J_N(\Omega_2) = I_N^{-1} J_N(\Omega_2)$, so $\cap I_N^{-1} J_N(\Omega_1) = J_N^{-1} J_N(\Omega_1 \setminus \Omega'_1) = 0$. Note also that $J_N^{-1}(I_N(K))$ is decreasing with respect to $N$, and $\nu(\Omega_2) < \infty$, we have $\lim_N \nu(J_N^{-1}(I_N(K))) = 0$. Since $\mu(K) \leq \mu(I_N^{-1}(I_N(K)))$ for all $N$, we see $\mu(K) = 0$. \qed

**Proposition 2.5.** The dimensions of $\Omega_1$ and $\Omega_2$ are equal, namely $n = m$.

Proof. We first prove that $n \geq m$. Choose $\phi_0, \phi_1, \cdots$ such that $\psi_0 = 1, \psi_1 = w_1, \cdots, \psi_m = w_m$, where $(w_1, \cdots, w_m)$ are coordinates on $\mathbb{C}^m$. It is clear that $J_N^{-1}(I_N(\Omega_1)) \subset J_N^{-1}(I_N(\Omega_1))$. By Lemma 2.3 (and its proof), $J_N^{-1}(I_N(\Omega_1))$ has full measure in $\Omega_2$, hence $J_N^{-1}(I_N(\Omega_1))$ has full measure in $\Omega_2$. By the choice of $\phi_0, \phi_1, \cdots, J_m$ is just the identity map, and hence $I_m(\Omega_1)$ has positive measure in $\mathbb{C}^m$. Note that $I_m$ is a holomorphic map on $\Omega_1 \setminus \phi_0^{-1}(0)$, we have $n \geq m$. The same argument shows that $m \geq n$. So we get $m = n$. \qed

**Lemma 2.6.** $\Omega'_j$ are open subsets of $\Omega_j$, for $j = 1, 2$.

Proof. Let $z_0 \in \Omega'_1$ and $w_0 \in \Omega'_2$ such that $F(z_0) = w_0$. Choose $\phi_0, \phi_1, \cdots$ such that $\psi_0 = 1, \psi_1 = w_1, \cdots, \psi_m = w_m$. Then $J_m : \Omega_2 \to \mathbb{C}^n$ is the identity map.

By Lemma 2.3 (2), we have $\phi_0(z_0) \neq 0$. So $I_m$ is holomorphic on some neighborhood $U$ of $z_0$ in $\Omega_1$. By contracting $U$ if necessary, we may assume $I_m(U) \subset \Omega_2$.

It is obvious that $F(z) = I_m(z)$ for $z \in U \cap \Omega'_1$, which implies that

$$\phi(z) T \phi'(I_m(z)) = T \phi(I_m(z)) \phi'(z), \forall \phi, \phi' \in A^p(\Omega_1), \forall z \in U \cap \Omega'_1.$$  

By Lemma 2.4, $U \cap \Omega'_1$ is dense in $U$. By continuity, the above equality holds for all $z \in U$. By definition of $\Omega'_1$ and $F$, we have $U \subset \Omega'_1$ and $F = I_m$ on $U$. This implies $\Omega'_1$ is open. By the same argument, one can prove that $\Omega'_2$ is open. \qed

**Lemma 2.7.** For $z \in \Omega'_1$ and $\phi \in A^p(\Omega_1)$, we have

$$|T \phi(F(z))| |J_F(z)|^{2/p} = |\phi(z)|, \phi \in A^p(\Omega_1), z \in \Omega'_1,$$

where $J_F$ is the holomorphic Jacobian of $F$.

Proof. Let $K$ be any measurable subset in $\Omega_1$. Considering the function $u(t_1, \cdots, t_N) = |t_1|$ on $\mathbb{C}^N$, from Lemma 2.2, we have

$$\int_{I_N^{-1}(I_N(K))} |\phi_1|^p d\lambda_n = \int_{J_N^{-1}(I_N(K))} |\psi_1|^p d\lambda_n, \forall N.$$  

Note that $I_N^{-1}(I_N(K))$ decrease to $K$ and $J_N^{-1}(I_N(K))$ decrease to $F(K)$ as $N \to \infty$, we get

$$\int_K |\phi_1|^p d\lambda_n = \int_{F(K)} |\psi_1|^p d\lambda_n, \forall \phi \in A^p(\Omega_1).$$  

Similarly the above equality holds for all $\phi_i, i \geq 1$. Considering that $\{\phi_0, \phi_1, \cdots\}$ is dense in $A^p(\Omega_1)$, we get

$$\int_K |\phi|^p d\lambda_n = \int_{F(K)} |T \phi|^p d\lambda_n, \forall \phi \in A^p(\Omega_1).$$
On the other hand, we have

$$
\int_{F(K)} |T\phi|^p d\lambda_n = \int_K |T\phi(F(z))|^p |J_F(z)|^2.
$$

So we get

$$
\int_K |\phi(z)|^p d\lambda_n = \int_K |T\phi(F(z))|^p |J_F(z)|^2
$$

for all Borel subset $K$ in $\Omega_1$. By continuity, we have

$$
|\phi(z)|^p = |T\phi(F(z))|^p |J_F(z)|^2, \quad z \in \Omega_*.
$$

2.3. Extending $F$ to a biholomorphic map between $\Omega_1$ and $\Omega_2$. We complete the proof of Theorem 2.8. We follow the notations in 2.2. For convenience, we restate Theorem 2.8 here:

**Theorem 2.8.** Let $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ be bounded hyperconvex domains. Suppose that there is a $p > 0$, $p \neq 2, 4, 6 \cdots$, such that

1. there is a linear isometry $T : A^p(\Omega_1) \to A^p(\Omega_2)$, and
2. the $p$-Bergman kernels of $\Omega_1$ and $\Omega_2$ are exhaustive,

then $m = n$ and there exists a biholomorphic map $F : \Omega_1 \to \Omega_2$ such that

$$
|T\phi \circ F|^p |J_F|^{2/p} = |\phi|, \quad \forall \phi \in A^p(\Omega_1),
$$

where $J_F$ is the holomorphic Jacobian of $F$. If $n = 1$, the assumption of hyperconvexity can be dropped.

**Proof.** We show that $F : \Omega'_1 \to \Omega'_2$ can extend to a biholomorphic map from $\Omega_1$ to $\Omega_2$. From Proposition 2.5, we have $n = m$. The rest of the proof is divided into several lemmas.

**Lemma 2.9.** Let $S = \Omega_1 \setminus \Omega'_1$, then for any $\zeta \in S$, \( \lim_{\Omega'_1 \ni z \to \zeta} J_F(z) = 0. \)

**Proof.** We argue by contradiction. Suppose to the contrary, there exists $\{z_j\} \subset \Omega'_1$, such that $z_j \to \zeta$ and $|J_F(z_j)| \geq \varepsilon > 0$ for all $j$. Since $\Omega_2$ is bounded, we may also assume that $F(z_j) \to w \in \mathbb{C}^n$. Then either $w \in \Omega_2$ or $w \in \partial \Omega_2$.

If $w \in \Omega_2$, by Lemma 2.3, $\zeta \in \Omega'_1$. This is a contradiction.

We now have $w \in \partial \Omega_2$. Since the $p$-Bergman kernel of $\Omega_2$ is exhaustive, we have that $\lim_{j \to +\infty} K_{\Omega_2,p}(F(z_j)) = +\infty$. Thus there are $\psi_j \in A^p(\Omega_2)$, such that $\|\psi_j\|_p = 1$ and $|\psi_j(F(z_j))| \to +\infty$. Let $\varphi_j = T(\psi_j)$, then $\|\varphi_j\|_p = 1$. By Lemma 2.7, we have $|(\psi_j \circ F)||J_F|^{2/p} = |\varphi_j|$ on $\Omega'_1$. Thus $|\varphi_j(z_j)| = |\psi_j(F(z_j))||J_F(z_j)|^{2/p} \to +\infty$. This means that $K_{\Omega_1,p}(z_j) \to +\infty$, which is a contradiction, since $z_j \to \zeta \in \Omega_1$ and $K_{\Omega_1,p}$ is locally bounded on $\Omega_1$. 

We define a function $\rho : \Omega_1 \to \mathbb{R} \cup \{-\infty\}$ as $\rho(z) = \log |J_F(z)|$ for $z \in \Omega'_1$ and $\rho(z) = -\infty$ as $z \in \Omega_1 \setminus \Omega'_1$.

**Lemma 2.10.** The function $\rho$ is a plurisubharmonic function on $\Omega_1$ and $\rho^{-1}(-\infty) = \Omega_1 \setminus \Omega'_1$. 

Proof. Since $f$ is biholomorphic on $\Omega'_1$, $\rho(z) \neq -\infty$ for $z \in \Omega'_1$. By definition of $\rho$, we have $\rho^{-1}(-\infty) = \Omega'_1 \setminus \Omega'_2$. From Lemma 2.9, $\rho$ is upper semicontinuous. Since $\rho$ is plurisubharmonic on $\Omega'_1$ and, by Lemma 2.6, $\Omega_1 \setminus \Omega'_1$ is closed, it is clear that $\rho$ satisfies the mean value inequality when restricted on any complex line intersection with $\Omega_1$. Hence $\rho$ is plurisubharmonic. \qed

Recall that a subset $A$ in a complex manifold $X$ is called a pluripolar set if for any $z \in X$ there is a neighborhood $U$ of $z$ and a plurisubharmonic function $h$ on $U$ such that $A \cap U \subset h^{-1}(-\infty)$. Lemma 2.10 implies that $S := \Omega_1 \setminus \Omega'_1$ is a closed pluripolar set of $\Omega_1$.

We need the following well-known fact about continuation of holomorphic functions.

Lemma 2.11 (see e.g. [13]). Let $X$ be complex analytic manifold and let $A \subset X$ be a closed pluripolar set. Then

1. every holomorphic function on $X \setminus A$ that is locally bounded near any point in $A$ extends to a holomorphic function on $X$; and
2. every plurisubharmonic function on $X \setminus A$ that is locally bounded above near any point in $A$ extends to a plurisubharmonic function on $X$.

We have constructed a biholomorphic map $F : \Omega_1 \setminus S \to \Omega_2$. From Lemma 2.11, $F$ extends across $S$ to a holomorphic map, also denoted by $F$, from $\Omega_1$ to $\mathbb{C}^n$. We want to show that $S = \emptyset$ and $\Omega'_1 = \Omega_1$.

We argue by contradiction. If $S \neq \emptyset$, by Lemma 2.3 and note that $\Omega_1 \setminus S$ is dense in $\Omega_1$, $F(S) \subset \Omega_2 \setminus \Omega_2$.

Since $\Omega_2$ is hyperconvex, there is a plurisubharmonic function $\varphi$ on $\Omega_2$ such that $\varphi < 0$ and for any $c < 0$ the set $\{ w \in \Omega_2 | \varphi(w) \leq c \}$ is compact.

Let $\tilde{\varphi} = \varphi \circ F$, then $\tilde{\varphi}$ is a plurisubharmonic function on $\Omega_1 \setminus S$ which have $0$ as an upper bound. By Lemma 2.11, $\tilde{\varphi}$ extends to a plurisubharmonic function on $\Omega_1$ which attains its maximum on $S$. By the maximum principle of plurisubharmonic functions, $\tilde{\varphi}$ is constant. This is a contradiction. So $S = \emptyset$ and $\Omega'_1 = \Omega_1$.

By exchanging the roles of $\Omega_1$ and $\Omega_2$, we can prove that $\Omega'_2 = \Omega_2$. Hence $F : \Omega_1 \to \Omega_2$ is a biholomorphic map.

For the case that $n = 1$, it is clear that $S = \{ z \in \Omega_1 | F'(z) = 0 \}$ and hence is a discrete subset of $\Omega_1$. Note that $F : \Omega'_1 \to \Omega_2$ is injective, $F'(z) \neq 0$ for all $z \in \Omega_1$ and hence $S = \emptyset$ and $\Omega'_1 = \Omega_1$. Similarly, we have $\Omega'_2 = \Omega_2$. Hence $F$ is a biholomorphic map from $\Omega_1$ to $\Omega_2$.

The equality in Theorem 2.8 (2) follows from Lemma 2.7.

We now prove the uniqueness of $F$. If there is another biholomorphic map $G : \Omega_1 \to \Omega_2$ satisfying the conditions in Theorem 2.8. Then $G$ satisfies

$$
\phi(z)T\phi_0(G(z)) = T\phi(G(z))\phi_0(z), \forall z \in \Omega_1, \forall \phi \in A'(\Omega_2).
$$

By the definition of $F$ (see [2.2]), we have $G = F$. \qed

3. Exhaustion of $p$-Bergman kernels

Theorem 2.8 implies the importance of the exhaustion property of $p$-Bergman kernels for bounded domains. In this Section, we discuss exhaustion of $p$-Bergman kernels in various settings.

For $0 < p < 2$, Ning-Zhang-Zhou proved the following complete result.
Theorem 3.1 (32). For any bounded domain $\Omega$ in $\mathbb{C}^n$ and any $p \in (0, 2)$, $\Omega$ is pseudoconvex if and only if the $p$-Bergman kernel $B_{\Omega,p}(z)$ is exhaustive.

Proof. For the sake of completeness, we include the proof here.

Since $B_{\Omega,p}$ is a plurisubharmonic function on $\Omega$, $\Omega$ is pseudoconvex provided $B_{\Omega,p}$ is exhaustive.

We now prove that $B_{\Omega,p}$ is exhaustive if $\Omega$ is pseudoconvex. For $\zeta \in \partial D$, we need to show that $\lim_{z \to \zeta} B_{\Omega,p}(z) = +\infty$. After a unitary transform, we may assume that $\zeta = (a, 0, \cdots, 0)$. Let $L$ be the plane given by $L = \{z_2 = \cdots = z_n = 0\}$. It is clear that $f(z_1) := \frac{1}{z_1-a} \in A^p(\Omega \cap L)$. By a $L^p$ variant of the Ohsawa-Takegoshi extension theorem obtained in [6], there is $F \in A^p(\Omega)$ such that $F|_{\Omega \cap L} = f$, which implies that $\lim_{z \to \zeta} B_{\Omega,p}(z) = +\infty$. $\Box$

Recall that a bounded domain $\Omega \subset \mathbb{C}^n$ is called homogeneous regular if there is a constant $c \in (0, 1)$ such that for any $z \in \Omega$, there is a holomorphic injective map $f : \Omega \to \mathbb{B}^n$ with $f(z) = 0$ and $\mathbb{B}^n(c) \subset f(\Omega)$, where $\mathbb{B}^n(c) = \{z \in \mathbb{C}^n; ||z|| < c\}$.

Theorem 3.2. If $\Omega \subset \mathbb{C}^n$ is a simply connected and homogeneous regular domain, then the $p$-Bergman kernel of $\Omega$ is exhaustive.

To prove Theorem 3.2, we need two lemmas. The first one is about the transformation formula for $p$-Bergman kernels.

Lemma 3.3 (32). Let $\Omega_1, \Omega_2$ be simply connected domains in $\mathbb{C}^n$. Then for any biholomorphism $f : \Omega_1 \to \Omega_2$, we have $B_{\Omega_1,p}(z) = B_{\Omega_2,p}(f(z))|J_f|^2$, where $J_f$ is the holomorphic Jacobian of $f$. Furthermore, if $p = \frac{2}{n}$, then $m \in \mathbb{N}$, and there is no need to assume that the domains are simply connected.

Lemma 3.4 (Generalized Rouché’s theorem, c.f. [25]). Let $\Omega$ be a bounded domain of $\mathbb{C}^n$. Let $f$ and $g$ be two continuous mappings of $\Omega$ (the closure of $\Omega$) into $\mathbb{C}^n$ which are holomorphic in $\Omega$. Suppose that

$$||g(z)|| < ||f(z)|| \ (z \in \partial \Omega)$$

Then $f$ and $f + g$ has the same number of zeros in $\Omega$, counting multiplicities, where $|| \cdot ||$ is the standard norm in $\mathbb{C}^n$.

Proof of Theorem 3.2. By assumption, there is a positive constant $c > 0$, such that $\forall z \in \Omega$, there is a holomorphic injection $f_z : \Omega \to \mathbb{B}^n$, such that $f_z(z) = 0$, $\mathbb{B}^n(c) \subset f_z(\Omega)$.

Fix an arbitrary point $a \in \partial \Omega$. For any sequence of points $z_j \in \Omega$ that converges to $a$, we need to prove that

$$\lim_{j \to \infty} B_{\Omega,p}(z_j) = +\infty.$$  

We argue by contradiction. Suppose to the contrary, we may assume that $B_{\Omega,p}(z_j) \leq M$ for all $j$, for some positive constant $M$. For simplicity we denote by $f_j$ the map $f_{z_j}$. Since $\Omega$ is simply connected, from Lemma 3.3 we have

$$B_{\Omega,p}(z_j) = B_{f_j(\Omega,p)}(0)|J_{f_j}(z_j)|^2.$$

From the decreasing property of $p$-Bergman kernels, we have

$$B_{f_j(\Omega,p)}(0) \geq B_{\mathbb{B}^n,p}(0).$$

Thus $M \geq B_{\Omega,p}(z_j) \geq B_{\mathbb{B}^n,p}(0)|J_{f_j}(z_j)|^2$, which implies that $|J_{f_j}(z_j)| \leq M'$ for all $j$, for some constant $M' > 0$. By Montel Theorem, we may assume that $f_j \to f$, for some
and \( g_j := f^{-1}_j|_{\mathcal{E}} \rightarrow g : \mathbb{E}^n(c) \rightarrow \mathbb{C}^n \). Then \( |J_g(0)| = \lim_j |J_{g_j}(0)| \geq M^{-1} \). In particular, \( J_g(0) \neq 0 \), and hence \( g \) is a locally biholomorphic mapping at \( 0 \) with \( g(0) = \lim_j g_j(0) = \lim_j z_j = a \in \partial \Omega \). Now choose a sufficiently small neighborhood \( U \in \mathbb{E}^n(c) \) of \( 0 \), such that \( g|_U \neq a \). Then map \( g - a \) has a zero inside \( U \).

From Lemma 3.4, \( g_j - a \) has a zero inside \( U \) for all \( j \) sufficiently large, which is a contradiction since \( g_j \neq a \).

For bounded strongly pseudoconvex domains with \( C^2 \) boundary, which are homogeneous regular [16], the same statement in Theorem 3.2 holds, with the simply connectedness of \( \Omega \) dropped. The proof is based on results on exposing boundary points of strongly pseudoconvex domains in [18][14].

**Theorem 3.5.** For any bounded strongly pseudoconvex domain \( \Omega \) in \( \mathbb{C}^n \) with \( C^2 \) smooth boundary, the \( p \)-Bergman kernel is exhaustive for all \( p > 0 \).

**Proof.** Let \( a \) be an arbitrary boundary point of \( \Omega \). Let \( R > 0 \) be a sufficiently large positive number, such that \( \overline{\Omega} \subset \mathbb{B}^n(R) = \{ z \in \mathbb{C}^n : \| z \| < R \} \). Then for any (small) neighborhood \( U \) of \( a \) in \( \mathbb{C}^n \) and any \( \epsilon > 0 \), there is a holomorphic injective map \( f \) defined in some neighborhood of \( \overline{\Omega} \) such that \( f(a) = (R, 0, \ldots, 0) \), \( f(\Omega) \subset \mathbb{B}^n(R) \), and \( \| f(z) - z \| < \epsilon \) for \( z \in \Omega \setminus U \) (see [14][15]).

Let \( f(z) = (f_1(z), \ldots, f_n(z)) \). Taking \( U \) small enough, we can have \( f_1 \in A^p(\Omega) \) with \( \| f_1 \|_p \leq M \), where \( M > 0 \) is a constant that can be chosen to be independent of \( R \). Letting \( R \rightarrow +\infty \), we get \( B_{\Omega,R}(z) \rightarrow +\infty \) as \( z \rightarrow a \).

## 4. Singular Finsler metrics on coherent analytic sheaves

In this section, we recall the notions introduced in [17] of singular Finsler metrics on holomorphic vector bundles and give a definition of positively curved singular Finsler metrics on coherent analytic sheaves.

**Definition 4.1.** Let \( E \rightarrow X \) be a holomorphic vector bundle over a complex manifold \( X \). A (singular) Finsler metric \( h \) on \( E \) is a function \( h : E \rightarrow [0, +\infty] \), such that \( |v|_h^2 := h(cv) = |c|^2h(v) \) for any \( v \in E \) and \( c \in \mathbb{C} \).

In the above definition, we do not assume any regularity property of \( h \). Only when considering Griffiths positivity certain regularity is required, as shown in the following Definition 4.1.

**Definition 4.2.** For a singular Finsler metric \( h \) on \( E \), its dual Finsler metric \( h^* \) on the dual bundle \( E^* \) of \( E \) is defined as follows. For \( f \in E^*_x \), the fiber of \( E^* \) at \( x \in X \), \( |f|_h^* \) is defined to be

\[
|f|_{h^*} := \sup \{|f(v)| ; v \in E_x, |v|_h \leq 1 \} \leq +\infty.
\]

**Definition 4.3.** Let \( E \rightarrow X \) be a holomorphic vector bundle over a complex manifold \( X \). A singular Finsler metric \( h \) on \( E \) is called negatively curved (in the sense of Griffiths) if for any local holomorphic section \( s \) of \( E \) the function \( \log |s|_h^2 \) is plurisubharmonic, and is called positively curved (in the sense of Griffiths) if its dual metric \( h^* \) on \( E^* \) is negatively curved.

Let \( \mathcal{F} \) be a coherent analytic sheaf on \( X \), it is well known that \( \mathcal{F} \) is locally free on some Zariski open set \( U \) of \( X \). On \( U \), we will identify \( \mathcal{F} \) with the vector bundle associated to it.
Definition 4.4. Let $\mathcal{F}$ be a coherent analytic sheaf on a complex manifold $X$. Let $Z \subset X$ be an analytic subset of $X$ such that $\mathcal{F}|_{X \setminus Z}$ is locally free. A negatively curved singular Finsler metric $h$ on $\mathcal{F}$ is a singular Finsler metric on the holomorphic vector bundle $\mathcal{F}|_{X \setminus Z}$, such that for any local holomorphic section $s$ of $\mathcal{F}$ on any open set $U \subset X$, the function $\log |s|^2_{\mathcal{F}}$ is plurisubharmonic on $U \setminus Z$, and can be extended to a plurisubharmonic function on $U$; and a singular Finsler metric $h$ on a coherent analytic sheaf $\mathcal{F}$ is said to be positively curved if the dual metric $h^*$ on $\mathcal{F}^*$ is negatively curved.

Remark 4.1. Suppose that $\log |g|_{h^*}$ is p.s.h. on $U \setminus Z$. It is well-known that if $\text{codim}_C(Z) \geq 2$ or $\log |g|_{H^*}$ is locally bounded above near $Z$, then $\log |g|_{h^*}$ extends across $Z$ to $U$ uniquely as a p.s.h function. Definition 4.4 matches Definition 4.3 if $\mathcal{F}$ is a vector bundle.

5. Variations of Linear Invariants—For Families of Pseudoconvex Domains

In this section, we will prove Theorem 1.6.

Let $\Omega \subset \mathbb{C}^{t+n} = \mathbb{C}^t \times \mathbb{C}^n$ be a pseudoconvex domain. Let $p: \Omega \to \mathbb{C}^t$ be the natural projection. We denote $p(\Omega)$ by $D$ and denote $p^{-1}(t)$ by $\Omega_t$ for $t \in D$. Let $\varphi$ be a plurisubharmonic function on $\Omega$ and let $m \geq 1$ be a fixed integer. For an open subset $U$ of $D$, we denote by $\mathcal{F}(U)$ the space of holomorphic functions $F$ on $p^{-1}(U)$ such that $\int_{p^{-1}(K)} |F|^{2/m} e^{-\varphi} \leq \infty$ for all compact subset $K$ of $D$. For $t \in D$, let $E_{m,t} = \{ F|_{\Omega_t} : F \in \mathcal{F}(U), \ U \subset D \text{ open and } U \ni t \}$. Then $E_{m,t}$ is a vector space with the following pseudonorm:

$$H(f) := |f|_m = \left( \int_{D_t} |f|^{2/m} e^{-\varphi_t} \right)^{m/2},$$

where $\varphi_t = \varphi|_{D_t}$. Let $E_m = \bigsqcup_{t \in D} E_{m,t}$ be the disjoint union of all $E_{m,t}$. Then we have a natural projection $\pi: E_m \to D$ which maps elements in $E_{m,t}$ to $t$. We view $H$ as a Finsler metric on $E_m$.

By definition, a section $s:D \to E_m$ is a holomorphic section if it varies holomorphically with $t$, namely, the function $s(t, z): \Omega \to \mathbb{C}$ is holomorphic with respect to the variable $t$. Note that $s(t, z)$ is automatically holomorphic on $z$ for $t$ fixed, by Hartogs theorem, $s(t, z)$ is holomorphic jointly on $t$ and $z$ and hence is a holomorphic function on $\Omega$.

Let $E_{m,t}^*$ be the dual space of $E_{m,t}$, namely the space of all complex linear functions on $E_{m,t}$. Let $E_m^* = \bigsqcup_{t \in D} E_{m,t}^*$. The natural projection from $E_m^*$ to $D$ is denoted by $\pi^*$. Note that we do not define any topology on $E_{m,t}^*$ and $E_m^*$. The only object we are interested in is holomorphic sections of $E_m^*$ which we are going to define. The following definition, as well as the definition of $E_m$ given above, is proposed in the recent work [17].

Definition 5.1. A section $\xi$ of $E_m^*$ on $D$ is holomorphic if:

1. for any local holomorphic section $s$ of $E_m$, $\langle \xi, s \rangle$ is a holomorphic function;
2. for any sequence $s_j$ of holomorphic sections of $E_m$ on $D$ such that $\int_D |s_j|^m \leq 1$, if $s_j(t, z)$ converges uniformly on compact subsets of $\Omega$ to $s(t, z)$ for some
holomorphic section $s$ of $E_m$, then $\langle \xi, s_j \rangle$ converges uniformly to $\langle \xi, s \rangle$ on compact subsets of $D$.

In the same way we can define holomorphic section of $E^*_m$ on open subsets of $D$. The Finsler metric $H$ on $E_m$ induces a Finsler metric $H^*$ on $E^*_m$ (defined in the same way as in Definition 4.2).

The metric $H$ and $H^*$ have the following semicontinuity properties which are proved in [17].

**Proposition 5.1.** Assume $s$ is a holomorphic section of $E_m$, then the function $|s|_m(t) := H(s(t)) : D \to (0, +\infty]$ is lower semicontinuous; and let $\xi : D \to E^*_m$ be a holomorphic section of $E^*_m$. Then the function $|\xi|_m(t) := H^*(\xi(t)) : D \to [0, +\infty)$ is upper semicontinuous.

The following result is the main ingredient in our proof of Theorem 1.6.

**Lemma 5.2 ([17]).** Let $\Omega \subset \mathbb{C}^{n+1}$ be a pseudoconvex domain, $\Delta$ be the unit disk in $\mathbb{C}$, and $p : \Omega \to p(\Omega) \subset \Delta$ be a holomorphic projection. For $y \in \Delta$, we denote $\Omega_y := p^{-1}(y)$ by $\Omega_y$. Let $\varphi$ be a p.s.h function on $\Omega$. Let $m \geq 1$ be an integer and $y_0 \in \Delta$ such that $\varphi$ is not identically $-\infty$ on any branch of $\Omega_y$. Then for any holomorphic function $u$ on $\Omega_{y_0}$ such that $\int_{\Omega_{y_0}} |u|^{2/m} e^{-\varphi} < +\infty$, there exists a holomorphic function $U$ on $\Omega$ such that $U|_{\Omega_{y_0}} = u$ and

$$\int_{\Omega} |U|^{2/m} e^{-\varphi} \leq \pi \int_{\Omega_{y_0}} |u|^{2/m} e^{-\varphi},$$

The proof of Lemma 5.2 is a combination of the iteration method of Berndtsson-Păun [6] and the optimal Ohsawa-Takegoshi extension theorems ([7] 21).

We can now give the proof of Theorem 1.6 which we restate it here.

**Theorem 5.3.** $(E_m, H)$ has positive curvature, in the sense that for any holomorphic section $\xi : D \to E^*_m$ of $E^*_m$, the function $\psi := \log |\xi|_m(t) := \log H^*(\xi(t)) : D \to [0, +\infty)$ is plurisubharmonic on $D$.

**Proof.** Form Proposition 5.1 we already know that $\psi$ is upper semi continuous on $D$. Thus it suffices to prove that for every holomorphic mapping $\gamma : \Delta \to D$ from the unit disc $\Delta$ to $D$, the function $\psi$ satisfies the mean-value inequality

$$\psi(0) \leq \frac{1}{\pi} \int_{\Delta} (\psi \circ \gamma).$$

Without loss of generality, we may assume that $D = \Delta$ and $\gamma$ is the identity map. The above inequality holds trivially if $\psi(0) = -\infty$. We now assume that $\psi(0) \neq -\infty$. Then we can choose an element $f \in E_{m,0}$ such that $|f|_{H,0} = 1$ and $\psi(0) = \log |\xi|_{H,0} = \log |\xi(f)|$.

By Lemma 5.2, there is a holomorphic section $s \in H^0(\Delta, E_m)$, such that $s(0) = f$, $\frac{1}{\pi} \int_{\Delta} |s|_{H}^{2/m} d\mu \leq 1$
By definition of the metric $H^*$ on $E^*_m$, we have the pointwise inequality

$$|\xi|_{H^*} \geq \frac{|\xi(s)|}{|s|_H}$$

and therefore, $\psi \geq \log |\xi(s)| - \log |s|_H$. Multiplying $2/m$ both sides, we get that

$$\frac{2}{m} \psi \geq \frac{2}{m} \log |\xi(s)| - \log |s|^{2/m}_H$$

Integrating, we get

$$\frac{1}{\pi} \int_\Delta \frac{2}{m} \psi d\mu \geq \frac{1}{\pi} \int_\Delta \frac{2}{m} \log |\xi(s)| d\mu - \frac{1}{\pi} \int_\Delta \log |s|^{2/m}_H d\mu$$

Note that $\log |\xi(s)|$ is p.s.h., thus satisfies the mean-value inequality, and so the first term on the R.H.S. is at least $\frac{2}{m} \log |\xi(f)| = \frac{2}{m} \psi(0)$. Since the function $|s|^{2/m}_H$ is integrable, from Jensen’s inequality we get that

$$-\frac{1}{\pi} \int_\Delta \log |s|^{2/m}_H d\mu \geq -\log(\frac{1}{\pi} \int_\Delta |s|^{2/m}_H d\mu) \geq -\log 1 = 0.$$

In conclude, we obtain that

$$\frac{1}{\pi} \int_\Delta \frac{2}{m} \psi d\mu \geq \frac{2}{m} \psi(0),$$

which means that $\frac{2}{m} \psi$ is subharmonic, i.e. $\psi$ is subharmonic. \hfill $\square$

A consequence of Theorem 5.3 is the following

**Corollary 5.4.** For $t \in D$, let $t \to \mu_t$ be a family of complex measures on $\Omega$. Assume that $\mu_t$ has compact support along fibers in the sense that for any $t_0 \in D$, there is an open subset $U \subset D$ with $t_0 \in U$, such that $\text{supp} \mu_t \subset K (t \in U)$ for some compact subset $K \subset \Omega$. Then

$$u_t \to \int_\Omega u_t d\mu =: \mu_t(u_t)$$

defines a section of the dual bundle $E^*_m$. Suppose that the section $\mu_t$ is holomorphic in the sense that

$$t \to \mu_t((t, \cdot))$$

is holomorphic for any local holomorphic section $s$ of $E_m$. Then $\log \|\mu_t\|_{H^*}$ is plurisubharmonic on $D$.

When the measures $\mu_t$ are all Dirac delta functions, we get the following

**Corollary 5.5 (17).** For any $m \geq 1$, the function $\log K_{m,t}(z)$ is a plurisubharmonic function on $\Omega$, where $K_{m,t}(z)$ is the relative twisted $2/m$-Bergman kernel on $\Omega$ with weight $\varphi$.

**Remark 5.1.** The case of $m = 1, n = 1, \varphi = 0$ is proved by Matani and Yamaguchi [29], and that of $m = 1$ and general $n$ and $\varphi$ is proved by Berndtsson [3]. The general form is proved in [17] by applying a new characterization of p.s.h. functions and the Ohsawa-Takegoshi extension theorem.
6. Variations of Linear Invariants—For Families of Kähler Manifolds

The aim of this section is to prove Theorem 1.7. We first recall the definition of canonical pseudonorms on the space of twisted pluricanonical sections, and their semicontinuity property associated to families of compact Kähler manifolds.

Let \(X, Y\) be Kähler manifolds of dimension \(r + n\) and \(r\) respectively, let \(p : X \to Y\) be a proper holomorphic submersion. Let \(L\) be a holomorphic line bundle over \(X\), and \(h\) be a singular Hermitian metric on \(L\), whose curvature current is semi-positive.

Let \(m > 0\) be a fixed integer. The multiplier ideal sheaf \(\mathcal{I}_m(h) \subset \mathcal{O}_X\) is defined as follows. If \(\varphi\) is a local weight of \(h\) on some open set \(U \subset X\), then the germ of \(\mathcal{I}_m(h)\) at a point \(p \in U\) consists of the germs of holomorphic functions \(f\) at \(p\) such that \(|f^{2/m} e^{-\varphi/m}|\) is integrable at \(p\). It is known that \(\mathcal{I}(h^{1/m})\) is a coherent analytic sheaf on \(X\).

For \(y \in Y\) let \(X_y = p^{-1}(y)\), which is a compact submanifold of \(X\) of dimension \(n\) if \(y\) is a regular value of \(p\). Let \(K_{X/Y}\) be the relative canonical bundle on \(X\) and \(\mathcal{E}_k = p_* (mK_{X/Y} \otimes L \otimes \mathcal{I}_m(h))\) be the direct image sheaf on \(Y\). By Grauert’s theorem, \(\mathcal{E}_m\) is a coherent analytic sheaf on \(Y\). One can choose a proper analytic subset \(A \subset Y\) such that:

1. \(p\) is submersive over \(Y \setminus A\),
2. both \(\mathcal{E}_m\) and the quotient sheaf \(p_*(mK_{X/Y} \otimes L)/\mathcal{E}_m\) are locally free on \(Y \setminus A\),
3. \(E_{m,Y}\) is naturally identified with \(H^0(X_y, mK_{X_y} \otimes L|_{X_y} \otimes \mathcal{I}_m(h)|_{X_y})\), for \(y \in Y \setminus A\).

where \(E_{m}\) is the vector bundle on \(Y \setminus A\) associated to \(\mathcal{E}_m\). For \(u \in E_{m,Y}\), the \(m\)-norm of \(u\) is defined to be

\[H_m(u) := \|u\|_m = \left(\int_{X_y} |u|^m / 2 h^{1/m}\right)^{m/2} \leq +\infty.\]

Then \(H_m\) is a Finsler metric on \(E_m\).

About the semicontinuity of the metrics \(H_m\) and \(H^*_m\), we have

**Proposition 6.1** ([23]). Let \(s\) be a holomorphic section of \(E_m\). The function \(|s|_{H_m}(y) := \|s(y)\|_m : Y \to [0, +\infty]\) is lower semi-continuous. For every \(\xi \in H^0(Y, E^n_m)\), the function \(|\xi|_{H_m}(y) := H^*_m(\xi(y)) : Y \to [0, +\infty]\) is upper semi-continuous.

For the proof of Theorem 1.7 we need the following

**Theorem 6.2** ([17]). Let \(B \subset \mathbb{C}^r\) be the unit ball and let \(X\) be a Kähler manifold of dimension \(n + r\). Let \(p : X \to B\) be a holomorphic proper submersion. For \(t \in B\), denote by \(X_t\) the fiber \(p^{-1}(t)\). Let \(L\) be a holomorphic line bundle on \(X\) with a (singular) Hermitian metric \(h\) whose curvature current is positive. Let \(u \in H^0(X_0, mK_{X_0} \otimes \mathcal{I}(h)|_{X_0})\) with \(|u|_m < \infty\). Assume there exists an open subset \(U\) containing the origin and \(s_0 \in H^0(U, \mathcal{E}_m|_U)\) such that \(s_0|_{X_0} = u\), then there exists \(s \in H^0(X, mK_X \otimes L \otimes \mathcal{I}(h)) = H^0(B, mK_B \otimes \mathcal{E}_m)\) such that \(s|_{X_0} = u \land dt^{\otimes m}\) and

\[
\int_X |s|^{2/m} h^{1/m} \leq \mu(B) \int_{X_0} |u|^{m/2} h^{1/m},
\]

where \(t = (t_1, \ldots, t_r)\) is the standard coordinate on \(B\) and \(dt = dt_1 \wedge \cdots \wedge dt_r\), and \(\mu(B)\) is the volume of \(B\) with respect to the Lebesgue measure on \(B\).
The proof of this theorem is a combination of the iteration method of Berndtsson-Păun [6] and the optimal Ohsawa-Takegoshi extension theorems on weakly pseudoconvex Kähler manifolds (see [?]).

We can now give the proof of Theorem 1.7. For convenience, we restate it here.

**Theorem 6.3.** For any \( m \geq 1 \), \( H_m \) is a singular Finsler metric on the coherent analytic sheaf \( E_m \) which is positively curved.

**Proof.** The argument is similar to that in the proof of Theorem 1.6.

From Definition 4.4, we need to prove that, for every \( g \in H^0(Y, E_m^*) \), the function \( \psi = \log |g|_{H_m} \) is p.s.h. on \( Y \setminus A \), and can be extended to be a p.s.h. function on \( Y \).

From Proposition 6.1, the function \( \psi \) is upper semi-continuous on \( Y \setminus A \). From Remark 4.1, it suffices to prove that \( \psi \) is p.s.h. on \( Y \setminus A \) and locally bounded above near \( A \).

We first prove the plurisubharmonicity of \( \psi \) on \( Y \setminus A \). It suffices to prove that for every holomorphic mapping \( \gamma : \Delta \to Y \setminus A \), such that \( \psi \circ \gamma \) is not identically \( -\infty \), \( \psi \) satisfies the mean-value inequality

\[
(\psi \circ \gamma)(0) \leq \frac{1}{\pi} \int_{\Delta} (\psi \circ \gamma) d\mu.
\]

We may assume that \( Y \setminus A = \Delta \) and assume that \( \psi(0) \neq 0 \). Then there exits an element \( f \in E_{m,0} \) with \( |f|_{H_m} = 1 \) and

\[
\psi(0) = \log |g|_{H_m,0} = \log |g(f)|.
\]

By Cartan theorem B and Lemma 6.2, there is a holomorphic section \( s \in H^0(\Delta, E_m) \), such that

\[
s(0) = f, \quad \frac{1}{\pi} \int_{\Delta} |s|_{H^*_m}^{2/m} d\mu \leq 1
\]

By definition of the metric \( H^*_m \) on the dual bundle, we have the pointwise inequality

\[
|g|_{H^*_m} \geq \frac{|g(s)|}{|s|_{H_m}}
\]

and therefore, \( \psi \geq \log |g(s)| - \log |s|_{H_m} \). Multiplying \( 2/m \) both sides, we get that

\[
\frac{2}{m} \psi \geq \frac{2}{m} \log |g(s)| - \log |s|_{H_m}^{2/m}
\]

Integrating over \( \Delta \), we get

\[
\frac{1}{\pi} \int_{\Delta} \frac{2}{m} \psi d\mu \geq \frac{1}{\pi} \int_{\Delta} \frac{2}{m} \log |g(s)| d\mu - \frac{1}{\pi} \int_{\Delta} \log |s|_{H_m}^{2/m} d\mu.
\]

Since \( g(s) \) is a holomorphic function on \( \Delta \), \( \log |g(s)| \) is p.s.h. and hence satisfies the mean-value inequality. It thus suffice to prove that the second term in the above inequality is \( \leq 0 \). This is true from Jensen’s inequality which implies

\[
\frac{1}{\pi} \int_{\Delta} \log |s|_{H_m}^{2/m} d\mu \leq \log \left( \frac{1}{\pi} \int_{\Delta} |s|_{H_m}^{2/m} d\mu \right) \leq \log 1 = 0.
\]

In conclusion, we obtain that \( \frac{2}{m} \psi \) is subharmonic, so \( \psi \) is subharmonic.

We now give the proof of the boundedness of \( \psi \). Without loss of generality, we assume that \( Y = \mathbb{B}^r \) is the unit ball. For any \( y \in Y \setminus A \) such that \( |g(y)| \neq 0 \) (otherwise there is nothing to prove), there is \( a \in E_{m,y} \) such that \( |a| = 1 \) and
\[ \langle g(y), a \rangle = \| g(y) \|. \] By Cartan theorem B and Lemma 6.2 there exists a holomorphic section \( s \) of \( E_m \) on \( Y \) such that \( s(y) = a \) and
\[ \int_Y |s|^{2/m}_{H_m} \leq C, \]
where \( C \) is a constant independent of \( y \) and \( a \). Let
\[ S = \{ f \in H^0(Y, E_m); \int_Y |f|^{2/m}_{H_m} \leq C \}. \]

Since the metric \( h \) on \( L \) is lower semicontinuous, by the mean value inequality and Montel theorem, \( S \) is a normal family, namely, any sequence in \( S \) has a subsequence that converges uniformly on compact subsets of \( Y \). Note that if \( s_j \) is a subsequence of \( S \) that converges uniformly on compact subsets of \( Y \), then the sequence of holomorphic functions \( \{u, s_j\} \) converges on compact subsets of \( Y \), and hence is uniformly bounded on compact sets of \( Y \). So \( \{(u, s); s \in S\} \) is uniformly bounded on compact sets of \( Y \).

We no complete the proof of the theorem \( 6.3 \). \( \square \)

We can consider the pull back \( \tilde{E}_m := p^* E_m \), which is a coherent analytic sheaf on \( X \setminus \Gamma^{-1}(A) \). The Finsler metric \( H_m \) on \( E_m \) naturally induces a metric \( \tilde{H}_m \) on \( \tilde{E}_m \) which is clearly positively curved in the sense of Definition 6.3. The evaluation map from \( \tilde{E}_m|_{X \setminus \Gamma^{-1}(A)} \) to \( (mK_{X/Y} \otimes L)|_{X \setminus \Gamma^{-1}(A)} \) induces a hermitian metric (the quotient metric) on \( (mK_{X/Y} \otimes L)|_{X \setminus \Gamma^{-1}(A)} \), which is called the \( m \)-Bergman kernel metric on \( (mK_{X/Y} \otimes L)|_{X \setminus \Gamma^{-1}(A)} \). From Theorem 6.3 we get

**Corollary 6.4.** The relative \( m \)-Bergman kernel metric defined above on \( (mK_{X/Y} \otimes L)|_{X \setminus \Gamma^{-1}(A)} \) has nonnegative curvature current, and can be extended to a singular Hermitian metric on \( mK_{X/Y} \otimes L \) whose curvature current is nonnegative.

**6.1. Application to Teichmüller metric.** In this section, we apply Theorem 6.3 to show that the Teichmüller metric on Teichmüller spaces are negatively curved. For basic knowledge about Teichmüller spaces please see [24] and [35].

Let \( C \) be a compact Riemann surface of genus \( g \geq 2 \). Given a quasiconformal mapping \( f : C \rightarrow S \), the pair \( (S, f) \) defined to be a marked Riemannian surface. Two marked Riemannian surfaces \( (S_1, f_1) \) and \( (S_2, f_2) \) are called equivalent if \( f_2 \circ f_1^{-1} : S_1 \rightarrow S_2 \) is homotopic to a conformal mapping \( c : S_1 \rightarrow S_2 \). Denote by \([S, f]\) the equivalence class of \((S, f)\). The Teichmüller space \( T = T(C) \) is the set of equivalence classes of all marked Riemann surfaces \([S, f]\).

It is well-known that \( T \) carries a natural complex structure and there is a holomorphic family \( p : X \rightarrow T \) of compact Riemann surfaces such that for each \( t \in T \) the fiber \( X_t := p^{-1}(t) \) belongs to the equivalence class that is represented by \( t \).

The cotangent space \( T^*_t T \) of \( T \) at \( t \) can be identified with the space \( H^0(t, 2K_{X_t}) \) of holomorphic quadratic differentials on \( X_t \). There is a natural Finsler metric on \( T \) which is called the Teichmüller metric on \( T \). The norm on \( T^*_t T \) dual to the norm on \( T_t T \) associated to the Teichmüller metric is given by
\[ \| s \|_1 := \int_S (s \wedge \pi)^{1/2}. \]

Now we consider the coheret sheaves \( E_m := p_* (mK_{X/T}) \) equipped with the canonical Finsler metric \( H_m \) introduced in Section 6. As explained above, \( E_2 \) can
be identified with the sheaf associated to the cotangent bundle of $T$ and $H_2$ is dual to the Teichmüller metric on the tangent bundle of $T$.

As a direct corollary of Theorem 6.3, we have the following

**Corollary 6.5.** $H_m$ is a positively curved Finsler metric on $E_m$.

When $m = 2$, we have

**Theorem 6.6.** The Teichmüller metric on $T$ is a negatively curved Finsler metric.

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FUSHENG DENG: SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, P. R. CHINA
E-mail address: fshdeng@ucas.ac.cn

ZHWEI WANG: SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, BEIJING, 100875, P. R. CHINA
E-mail address: zhiwei@bnu.edu.cn

LIYOU ZHANG: SCHOOL OF MATHEMATICAL SCIENCES, CAPITAL NORMAL UNIVERSITY, BEIJING, 100048, P. R. CHINA
E-mail address: zhangly@cnu.edu.cn

XIANGYU ZHOU: INSTITUTE OF MATHEMATICS, AMSS, AND HUA LOO-KENG KEY LABORATORY OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA
E-mail address: xyzhou@math.ac.cn