Confining Phase of $N = 1$ $Sp(N_c)$ Gauge Theories via M Theory Fivebrane

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Abstract

The moduli space of vacua for the confining phase of $N = 1$ $Sp(N_c)$ supersymmetric gauge theories in four dimensions is studied by M theory fivebrane. We construct M theory fivebrane configuration corresponding to the perturbation of superpotential in which the power of adjoint field is related to the number of NS'5 branes in type IIA brane configuration. We interpret the dyon vacuum expectation values in field theory results as the brane geometry and the comparison with meson vevs will turn out that the low energy effective superpotential with enhanced gauge group $SU(2)$ is exact.

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1 Introduction

We have seen how string/M theory can be exploited to study non-perturbative dynamics of low energy supersymmetric gauge theories in various dimensions. One of the main motivations is to understand the D brane dynamics where the gauge theory is realized on the worldvolume of D branes. This work was initiated by Hanany and Witten [1] where the mirror symmetry of $N = 4$ gauge theory in three dimensions was described by changing the position of the NS5 branes (See, for example, [2]). As one changes the relative orientation of the two NS5 branes [3] while keeping their common four spacetime dimensions intact, the $N = 2$ supersymmetry is broken to $N = 1$ supersymmetry [4]. By analyzing this brane configuration they [4] described and checked a stringy derivation of Seiberg’s duality for $N = 1$ supersymmetric $SU(N_c)$ gauge theory with $N_f$ flavors. This result was generalized to the brane configurations with orientifolds which explain $N = 1$ supersymmetric theories with gauge group $SO(N_c)$ or $Sp(N_c)$ [5] (See also [6] for a relevant geometrical approach).

Both the D4 branes and NS5 branes used in type IIA string theory originate from the fivebrane [1] of M theory. That is, D4 brane is an M theory fivebrane wrapped over $S^1$ and NS5 brane is the one on $R^{10} \times S^1$. In order to insert D6 branes one studies a multiple Taub-NUT space [8] whose metric is complete and smooth. The singularities are removed in eleven dimensions where the brane configuration becomes smooth, the D4 branes and NS5 branes being the unique M theory fivebrane and the D6 branes being the Kaluza-Klein monopoles. The property of $N = 2$ supersymmetry in four dimensions requires that the worldvolume of M theory fivebrane is $R^{1,3} \times \Sigma$ where $\Sigma$ is uniquely identified with the curves that occur to the solutions for Coulomb branch of the four dimensional field theory. Further generalizations of this configuration with orientifolds were studied in [9]. The exact low energy description of $N = 1$ supersymmetric $SU(N_c)$ gauge theories with $N_f$ flavors in four dimensions have been found in [10] (See also [11, 12] for theories with orientifolds). This approach has been developed further and used to study the moduli space of vacua of confining phase of $N = 1$ supersymmetric $SU(N_c)$ gauge theories in four dimensions [13]. In terms of brane configuration of IIA string theory, this was done by taking multiples of NS'5 branes rather than a single NS'5 brane. In field theory, we regard this as taking the superpotential $\Delta W = \sum_{k=2}^{N_c} \mu_k \text{Tr}(\Phi^k)$. This perturbation lifts the non singular locus of the $N = 2$ Coulomb branch while at singular locus there exist massless dyons that can condense due to the perturbation.

In the present work we extend the results of [13, 14] to $N = 1$ supersymmetric theories with gauge group $Sp(N_c)$ and also generalize the previous work [11] which dealt with a single NS'5 brane in the sense that we are considering multiple copies of NS'5 branes. We will describe how the field theory analysis [15] obtained in the low energy superpotential gives rise to the geometrical structure in $(v, t, w)$ space. The minimal
form for the effective superpotential obtained by “integrating in” is not exact \[16\], in general, for several massless dyons. Note that the intersecting branes in string/M theory have been studied to obtain much information about supersymmetric gauge theories with different gauge groups and in various dimensions \[17\].

2 Field Theory Analysis

- **N = 2 Theory**

Let us consider $N = 2$ supersymmetric $Sp(N_c)$ gauge theory with matter in the $2N_c$ dimensional representation of the gauge group $Sp(N_c)$. In terms of $N = 1$ superfields, $N = 2$ vector multiplet consists of a field strength chiral multiplet $W_{a}^{i}$ and a scalar chiral multiplet $\Phi_{a}^{i}$, both in the adjoint representation. The quark hypermultiplets are made of a chiral multiplet $Q_{i}^{a}$ which couples to the Yang-Mills fields where $i = 1, \cdots, 2N_f$ are flavor indices and $a = 1, \cdots, 2N_c$ are color indices. The $N = 2$ superpotential takes the form,

$$W_{N=2} = \sqrt{2}Q_{a}^{i}\Phi_{b}^{i}J^{bc}Q_{c}^{i} + \sqrt{2}m_{ij}Q_{a}^{i}J^{ab}Q_{b}^{j},$$

(2.1)

where $J_{ab}$ is the symplectic metric $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_{N_c \times N_c}$ used to raise and lower $Sp(N_c)$ color indices ($1_{N_c \times N_c}$ is the $N_c \times N_c$ identity matrix) and $m_{ij}$ is an antisymmetric quark mass matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \cdots, m_{N_f})$. Classically, the global symmetries are the flavor symmetry $O(2N_f)$ when there are no quark masses, in addition to $U(1)_R \times SU(2)_R$ chiral R-symmetry. The theory is asymptotically free for the region $N_f < 2N_c + 2$ and generates dynamically a strong coupling scale $\Lambda_{N=2}$. The instanton factor is proportional to $\Lambda_{N=2}^{2N_c + 2 - N_f}$. Then the $U(1)_R$ symmetry is anomalous and is broken down to a discrete $Z_{2N_c + 2 - N_f}$ symmetry by instantons. The $N_c$ complex dimensional moduli space of vacua contains the Coulomb and Higgs branches. The Coulomb branch is parameterized by the gauge invariant order parameters

$$u_{2k} = < \text{Tr}(\phi^{2k}) >, \quad k = 1, \cdots, N_c,$$

(2.2)

where $\phi$ is the scalar field in $N = 2$ chiral multiplet. Up to a gauge transformation $\phi$ can be diagonalized to a complex matrix, $< \phi > = \text{diag}(A_1, \cdots, A_{N_c})$ where $A_i = (a_i^0 \ 0_{N_c})$. At a generic point the vevs of $\phi$ breaks the $Sp(N_c)$ gauge symmetry to $U(1)^{N_c}$ and the dynamics of the theory is that of an Abelian Coulomb phase. The Wilsonian effective Lagrangian in the low energy can be made of the multiplets of $A_i$ and $W_i$ where $i = 1, 2, \cdots, N_c$. If $k$ $a_i$’s are equal and nonzero then there exists an enhanced $SU(k)$ gauge symmetry. When they are also zero, an enhanced $Sp(k)$ gauge symmetry appears. The quantum moduli space is described by a family of hyperelliptic spectral curves \[18\] with
associated meromorphic one forms,

\[
y^2 = \left( v^2 C_{2N_c}(v^2) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i \right)^2 - \Lambda_{N=2}^{4N_c+4-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2),
\]

where \( C_{2N_c}(v^2) \) is a degree \( 2N_c \) polynomial in \( v \) with coefficients depending on the moduli \( u_{2k} \) appearing in (2.3) and \( m_i (i = 1, 2, \cdots, N_f) \) is the mass of quark \( i \).

- Breaking \( N = 2 \) to \( N = 1 \) (Pure Yang-Mills Theory)

We are interested in a microscopic \( N = 1 \) theory mainly in a phase with a single confined photon coupled to the light dyon hypermultiplet while the photons for the rest are free. By taking a tree level superpotential perturbation \( \Delta W \) of (2.3) made out of the adjoint fields in the vector multiplets to the \( N = 2 \) superpotential (2.1), the \( N = 2 \) supersymmetry can be broken to \( N = 1 \) supersymmetry. That is,

\[
W = W_{N=2} + \Delta W, \quad \Delta W = \sum_{k=1}^{N_c-1} \mu_{2k} \text{Tr}(\Phi^{2k}) + \mu_{2N_c} s_{2N_c},
\]

where \( \Phi \) is the adjoint \( N = 1 \) superfield in the \( N = 2 \) vector multiplet and Note that the \( \mu_{2N_c} \) term is not associated with \( u_{2N_c} \) but \( s_{2N_c} \) which is proportional to the sum of \( u_{2N_c} \) and the polynomials of other \( u_{2k} (k < N_c) \) according to the recurrence relation.

Then microscopic \( N = 1 \) \( Sp(N_c) \) gauge theory is obtained from \( N = 2 \) \( Sp(N_c) \) Yang-Mills theory perturbed by \( \Delta W \). Let us first study \( N = 1 \) pure \( Sp(N_c) \) Yang-Mills theory with tree level superpotential (2.4). Near the singular points where dyons become massless, the macroscopic superpotential of the theory is given by

\[
W = \sqrt{2} \sum_{i=1}^{N_c-1} M_i A_i M_i + \sum_{k=1}^{N_c-1} \mu_{2k} U_{2k} + \mu_{2N_c} S_{2N_c}.
\]

We denote by \( A_i \) the \( N = 2 \) chiral superfield of \( U(1) \) gauge multiplets, by \( M_i \) those of \( N = 2 \) dyon hypermultiplets, by \( U_{2k} \) the chiral superfields corresponding to \( \text{Tr}(\Phi^{2k}) \) (and by \( S_{2k} \) the chiral superfields which are related to \( U_{2k} \), in the low energy theory. The vevs of the lowest components of \( A_i, M_i, U_{2k}, S_{2N_c} \) are written as \( a_i, m_i, u_{2k}, s_{2N_c} \) respectively. Recall that \( N = 2 \) configuration is invariant under the group \( U(1)_R \) and \( SU(2)_R \) corresponding to the chiral R-symmetry of the field theory. However, in \( N = 1 \) theory \( SU(2)_R \) is broken to \( U(1)_J \). The equations of motion obtained by varying the

\footnote{Note that the polynomial \( C_{2N_c}(v^2) \) is an even function of \( v \) which will be identified with a complex coordinate \( (x^4, x^5) \) directions in next section and is given by \( C_{2N_c}(v^2) = v^{2N_c} + \sum_{i=1}^{N_c} s_{2i} v^{2(N_c-i)} = \prod_{k=1}^{N_c}(v^2 - a_k^2) \) where \( s_{2k} \) and \( u_{2k} \) are related each other by so-called Newton’s formula \( 2k s_{2k} + \sum_{i=1}^{k} s_{2k-2i} u_{2i} = 0, (k = 1, 2, \cdots, N_c) \) with \( s_0 = 1 \). From this recurrence relation, we obtain \( \partial s_{2j}/\partial u_{2k} = -s_{2(j-k)}/2k \) for \( j \geq k \).}

\footnote{Our \( \mu_{2k} \) is the same as their \( g_{2k}/2k \) in [13].}
superpotential with respect to each field read as follows. At a generic point in the
moduli space, no massless fields occur \((a_i \neq 0 \text{ for } i = 1, \cdots, N_c - 1)\) which implies
\(m_{i,dy} = 0\) and so \(\mu_{2k}\) and \(\mu_{2N_c}\) vanish. Then we obtain the moduli space of vacua of
\(N = 2\) theory.

On the other hand, we are considering a singular point in the moduli space where
\(l\) mutually local dyons are massless. This means that \(l\) one cycles shrink to zero. The
right hand side of (2.3) becomes,

\[
y^2 = \left( v^2 C_{2N_c} (v^2) + \Lambda_{N=2}^{2N_c+2-N_f} \right)^2 - \Lambda_{N=2}^{4N_c+4-2N_f} = \prod_{i=1}^{l} \left( v^2 - p_i^2 \right)^2 \prod_{j=1}^{2N_c+2-2l} \left( v^2 - q_j^2 \right), \tag{2.6}
\]

with \(p_i\) and \(q_j\) distinct. A point in the \(N = 2\) moduli space of vacua is characterized by
\(p_i\) and \(q_j\). The degeneracy of this curve is checked by explicitly evaluating both \(y^2\) and
\(\partial y^2/\partial v^2\) at the point \(v = \pm p_i\), leading to vanish. Since \(a_i = 0\) for \(i = 1, \cdots, l\) and \(a_i \neq 0\)
for \(i = l + 1, \cdots, N_c - 1\), we get \(m_{i,dy} = 0\), \((i = l + 1, \cdots, N_c - 1)\) while \(m_{i,dy}(i = 1, \cdots, l)\)
are not constrained. We will see how the vevs \(m_{i,dy}\) originate from the information
about \(N = 2\) moduli space of vacua which is encoded by \(p_i\) and \(q_j\). We assume that the
matrix \(\partial a_i/\partial u_{2k}\) is nondegenerate. In order to calculate \(\partial a_i/\partial u_{2k}\), we need the relation
between \(\partial a_i/\partial s_{2k}\) and the period integral on a basis of holomorphic one forms on the
curve, \(\partial a_i/\partial s_{2k} = \oint_{a_i} v^2 (N_c-k)dv/y\). We can express the generating function\footnote{\(\mu_{2k} = -\sqrt{2} \sum_{i=1}^{N_c-1} m_{i,dy}^2 \partial a_i/\partial u_{2k}, \mu_{2N_c} = -\sqrt{2} \sum_{i=1}^{N_c-1} m_{i,dy}^2 \partial a_i/\partial s_{2N_c}\) and \(a_i m_{i,dy} = 0\).}
for the \(\mu_{2k}\) in terms of \(\omega_i\) as follows:

\[
\sum_{k=1}^{N_c} 2k \mu_{2k} v^{2(k-1)} = \sum_{k=-\infty}^{N_c} \sum_{i=1}^{N_c} \sum_{j=1}^{N_c} v^{2(k-1)} s_{2(j-k)}^2 p_i^{2(N_c-j)} \omega_i
\]

\[
= \sum_{k=-\infty}^{N_c} \sum_{i=1}^{N_c} \sum_{j=1}^{N_c} v^{2(k-1)} s_{2(j-k)}^2 p_i^{2(N_c-j)} \omega_i + \mathcal{O}(v^2)
\]

\[
= \sum_{i=1}^{l} \frac{C_{2N_c} (v^2)}{(v^2 - p_i^2)} \omega_i + \mathcal{O}(v^2). \tag{2.7}
\]

Therefore we can find the parameter \(\mu_{2k}\) by reading off the right hand side of (2.7).
This result determines whether a point in the \(N = 2\) moduli space of vacua classified by
the set of \(p_i\) and \(q_j\) in (2.6) remains as an \(N = 1\) vacuum after the perturbation,
for given a set of perturbation parameters \(\mu_{2k}\) and \(\mu_{2N_c}\). We will see in section 3 that
this corresponds to one of the boundary conditions on a complex coordinate in \((x^8, x^9)\)
directions as \( v \) goes to infinity. In order to make the comparison with the brane picture, it is very useful to define the polynomial \( H(v^2) \) of degree \( 2l - 4 \) by

\[
\sum_{i=1}^{l} \frac{\omega_i}{v^2(v^2 - p_i^2)} = \frac{2H(v^2)}{\prod_{i=1}^{l}(v^2 - p_i^2)}.
\]

At a given point \( p_i \) and \( q_j \) in the \( N = 2 \) moduli space of vacua, \( H(v^2) \) determines the dyon vevs, in other words,

\[
m^2_{i,dy} = \sqrt{2}p_i^2H(p_i^2)\prod_{m}(p_i^2 - q_m^2)^{1/2},
\]

which will be described in terms of the geometric brane picture in next section. Therefore, all the vevs of dyons \( m_{i,dy}(i = 1, \cdots, l) \) are found as well as \( m_{i,dy}(i = l+1, \cdots, N_c-1) \) which are zero.

- The Meson Vevs

Let us discuss the vevs of the meson field along the singular locus of the Coulomb branch. This is due to the nonperturbative effects of \( N = 1 \) theory and obviously was zero before the perturbation (2.4). We will see the property of exactness in field theory analysis in the context of M theory fivebrane in section 4. Equivalently, the exactness of superpotential for any values of the parameters is to assume \( W_\Delta = 0 \). We will follow the method presented in [19]. Let us consider the vacuum where one massless dyon exists with unbroken \( SU(2) \times U(1) \)

\[
J\Phi^{cl} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(a_1, a_2, a_3, \cdots, a_{N_c-1}),
\]

and the chiral multiplet \( Q = 0 \) and \( J \) is the symplectic metric as before. These eigenvalues of \( \Phi \) can be obtained by differentiating the superpotential (2.4) with respect to \( \Phi \) and setting the chiral multiplet \( Q = 0 \). The vacua with classical \( SU(2) \times U(1)^{N_c-1} \) group are those with two eigenvalues equal to \( a_1 \) and the rest given by \( a_2, a_3, \cdots, a_{N_c-1} \). It is known from [15] that, if using \( s_{2N_c} \) in the superpotential perturbation rather than \( u_{2N_c} \), the degenerate eigenvalue of \( \Phi \) is obtained to be

\[
a_1^2 = \frac{(N_c-1)\mu_{2(N_c-1)}}{N_c\mu_{2N_c}}.
\]

We will see in section 3 that the asymptotic behavior of a complex coordinate in \((x^8, x^9)\) directions for large \( v \) determines this degenerate eigenvalue by using the condition for generating function of \( \mu_{2k} \) (2.7). The scale matching condition between the high energy \( Sp(N_c) \) scale \( \Lambda_{N=2} \) and the low energy \( SU(2) \) scale \( \Lambda_{SU(2),N_f} \) is related each other. After integrating out \( SU(2) \) quarks we obtain the scale matching between \( \Lambda_{N=2} \) and \( \Lambda_{SU(2)} \).
for pure $N = 1$ SU(2) gauge theory. That is,

$$
\Lambda_{SU(2)}^6 = \left( \frac{2N_c^2\mu_2^2}{(N_c - 1)\mu_{2(N_c-1)}} \right)^2 \Lambda_{N=2}^{4(N_c+1)-2N_f} \det(a_1^2 - m^2),
$$

(2.12)

where matrix $a_1$ means $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \otimes a_1$ and matrix $m$ being $\left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \otimes \text{diag}(m_1, \ldots, m_{N_f})$. Then the full exact low energy effective superpotential is given by

$$
W_L = \sum_{k=1}^{N_c-1} \mu_{2k} \text{Tr}(\Phi_{cl}^{2k}) + \mu_{2N_c} s_{2N_c}^{cl} \pm 2\Lambda_{SU(2)}^3,
$$

(2.13)

where the last term is generated by gaugino condensation in the low energy SU(2) theory (the sign reflects the vacuum degeneracy). In terms of the original $N = 2$ scale, we can write it as

$$
W = \sum_{k=1}^{N_c-1} \mu_{2k} \text{Tr}(\Phi_{cl}^{2k}) + \mu_{2N_c} s_{2N_c}^{cl} \pm 4N_c^2\mu_2^{2N_c} \Lambda_{N=2}^{2(N_c+1)-N_f} \det(a_1^2 - m^2)^{1/2}.
$$

(2.14)

Therefore, one obtains the vevs of meson $M_i = Q_a^i J^{ab} Q_b^i$ which gives

$$
M_i = \frac{\partial W}{\partial m_i^2} = \pm \frac{4N_c^2\mu_2^{2N_c}}{\sqrt{2(a_1^2 - m_i^2)}} \Lambda_{N=2}^{2(N_c+1)-N_f} \det(a_1^2 - m^2)^{1/2},
$$

(2.15)

where $a_1$ is given by (2.11). We will see in section 3 that the finite value of a complex coordinate in $(x^8, x^9)$ directions corresponds to the above vevs of meson when $v \to \pm m_i$ and the other complex coordinate related to $(x^6, x^{10})$ directions vanishes.

### 3 Brane Configuration from M Theory

- **Type IIA Brane Configuration**

We study the theory with the superpotential perturbation $\Delta W$ by analyzing M theory fivebranes. Let us first describe them in the type IIA brane configuration. Following the procedure of [4], the brane configuration in $N = 2$ theory consists of three kind of branes: the two parallel NS5 branes extend in the directions $(x^0, x^1, x^2, x^3, x^4, x^5)$, the D4 branes are stretched between two NS5 branes and extend over $(x^0, x^1, x^2, x^3)$ and are finite in the direction of $x^6$, and the D6 branes extend in the directions $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$. In order to study symplectic gauge groups, we consider an O4 orientifold which is parallel to the D4 branes in order to keep the supersymmetry and is not of finite extent in $x^6$ direction. The D4 branes is the only brane which is not intersected by this O4 orientifold. The orientifold gives a spacetime
reflection as \((x^4, x^5, x^7, x^8, x^9) \rightarrow (-x^4, -x^5, -x^7, -x^8, -x^9)\), in addition to the gauging of worldsheet parity \(\Omega\). The fixed points of the spacetime symmetry define this O4 planes. Each object which does not lie at the fixed points, must have its mirror. Thus NS5 branes have a mirror in \((x^4, x^5)\) directions and D6 branes do a mirror in \((x^7, x^8, x^9)\) directions. In order to realize the \(N = 1\) theory with a perturbation \((2.4)\) we can think of a single NS5 brane and multiple copies of NS5 branes which are orthogonal to a NS5 brane with worldvolume, \((x^0, x^1, x^2, x^3, x^8, x^9)\) and between them there exist D4 branes intersecting D6 branes. The number of NS5 branes is \(N_c - 1\) by identifying the power of adjoint field appearing in the superpotential \((2.4)\). The brane description for \(N = 1\) theory with \(\mu_{2N_c} = 0\) has been studied in the paper of [11] in type IIA brane configuration. In this case, all the couplings, \(\mu_{2k}\) can be regarded as tending uniformly to infinity. On the other hand, in M theory configuration there will be no such restrictions.

• M Theory Fivebrane Configuration

Let us describe how the above brane configuration can be embedded in M theory in terms of a single M theory fivebrane whose worldvolume is \(\mathbb{R}^{1,3} \times \Sigma\) where \(\Sigma\) is identified with Seiberg-Witten curves [13] that determine the solutions to Coulomb branch of the field theory. As usual, we write \(s = (x^6 + ix^{10})/R, t = e^{-s}\) where \(x^{10}\) is the eleventh coordinate of M theory which is compactified on a circle of radius \(R\). Then the curve \(\Sigma\), describing \(N = 2\) \(Sp(N_c)\) gauge theory with \(N_f\) flavors, is given [9] by an equation in \((v, t)\) space

\[
t^2 - 2 \left(v^2 C_{2N_c}(v^2, u_{2k}) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i \right) t + \Lambda_{N=2}^{4N_c+4-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2) = 0. \tag{3.1}
\]

It is easy to check that this description is the same as \((2.3)\) under the identification \(t = y + v^2 C_{2N_c}(v^2, u_{2k}) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i\). By adding \(\Delta W\) which corresponds to the adjoint chiral multiplet, the \(N = 2\) supersymmetry will be broken to \(N = 1\). To describe the corresponding brane configuration in M theory, we introduce complex coordinates

\[
v = x^4 + ix^5, \quad w = x^8 + ix^9. \tag{3.2}
\]

To match the superpotential perturbation \(\Delta W (2.4)\), we propose the following boundary conditions

\[
w^2 \rightarrow \sum_{k=2}^{N_c} 2k \mu_{2k} v^{2(k-1)} \quad \text{as} \quad v \rightarrow \infty, \quad t \sim \Lambda_{N=2}^{2(2N_c+2-N_f)} v^{2N_f-2N_c-2},
\]

\[
w \rightarrow 0 \quad \text{as} \quad v \rightarrow \infty, \quad t \sim v^{2N_c+2}. \tag{3.3}
\]

After deformation, \(SU(2)_{7,8,9}\) is broken to \(U(1)_{8,9}\) if \(\mu_{2k}\) has the charges \((4 - 4k; 4)\) under \(U(1)_{4,5} \times U(1)_{8,9}\). When we consider now only \(k = 2\), we obtain that \(w^2 \sim \mu_4 v^2\) as \(v \rightarrow \infty\) which is the same as the relation \(w \rightarrow \mu v\) in [11] after we identify \(\mu_4\) with \(\mu^2\).
This identification comes also from the consideration of $U(1)_{1,5}$ and $U(1)_{8,9}$ charges of $\mu$ and $\mu_4$. After perturbation, only the singular part of the $N = 2$ Coulomb branch with $l$ or less mutually local massless dyons remains in the moduli space of vacua. Let us construct the M theory fivebrane configuration satisfying the above boundary conditions and assume that $w^2$ is a rational function of $v^2$ and $t$. Our result is really similar to the case of [13, 14] and we will follow their notations. We write $w^2$ as follows

$$w^2(t, v^2) = \frac{a(v^2)t + b(v^2)}{c(v^2)t + d(v^2)}, \quad (3.4)$$

where $a, b, c, d$ are arbitrary polynomials of $v^2$ and $t$ satisfying (3.1). Now we calculate the following two quantities. Since $w^2$ has no poles for finite value of $v^2$, $w^2(t_+(v^2), v^2) \pm w^2(t_-(v^2), v^2)$ also does not have poles which leads to arbitrary polynomials $H(v^2)$ and $N(v^2)$ given by

$$\frac{acG + adC + bcC + bd}{c^2G + 2cdC + d^2} = N, \quad \frac{(ad - bc)S}{(c^2G + 2cdC + d^2)} = H. \quad (3.5)$$

It will turn out that the function $H(v^2)$ is exactly the same as the one (2.8) defined in field theory analysis. By making a shift of $a \rightarrow a + Nc, b \rightarrow b + Nd$ due to the arbitrariness of the polynomials $a$ and $b$ and combining all the information for $b$ and $d$ (See, for details, [13, 14]), we get the most general rational function $w^2$ which has no poles for finite value of $v^2$,

$$w^2 = N + \frac{at + cHST - aC}{ct - cC + aS/H}, \quad (3.6)$$

where $N, a, c, H$ are arbitrary polynomials. As we choose two $w^2$'s, each of them possessing different polynomials $a$ and $c$ and subtract them, then the numerator of it will be proportional to $t^2 - 2Ct + G$ which vanishes according to (3.1). This means $w^2$ does not depend on $a$ and $c$. Therefore, when $c = 0$, the form of $w^2$ is very simple. The general solution for $w^2$ is

$$w^2 = N(v^2) + \frac{H(v^2)}{\prod_{i=1}^l (v^2 - p_i^2)} \left( t - \left( v^2C_{2N_i}(v^2, u_{2k}) + \Lambda_{N=2}^{N_i+2-N_f} \prod_{i=1}^N m_i \right) \right), \quad (3.7)$$

where $H(v^2)$ and $N(v^2)$ are arbitrary polynomials of $v^2$. Now we want to impose the boundary conditions on $w^2$ in the above general solution. From the relation,

$$w^2(t_\pm(v^2), v^2) = N \pm H\sqrt{T}, \quad (3.8)$$

\[\text{Using the two solutions of } t, \text{ denoted by } t_+ \text{ and } t_- \text{ satisfying (3.3), } w^2(t_+(v^2), v^2) + w^2(t_-(v^2), v^2) = (2acG + 2adC + 2bcC + 2bd)/(c^2G + 2cdC + d^2) \text{ and } w^2(t_+(v^2), v^2) - w^2(t_-(v^2), v^2) = 2(ad - bc)S\sqrt{T}/(c^2G + 2cdC + d^2) \text{ where we define } C \equiv v^2C_{2N_i}(v^2, u_{2k}) + \Lambda_{N=2}^{N_i+2-N_f} \prod_{i=1}^N m_i, \text{ and } G \equiv \Lambda_{N=2}^{N_i-4-2N_f} \prod_{i=1}^N (v^2 - m_i^2) \text{ implying that } C^2(v^2) - G(v^2) = S(v^2)T(v^2) \text{ where } S(v^2) = \prod_{i=1}^N (v^2 - p_i^2), T(v^2) = \prod_{j=1}^{2N_i+2-2l}(v^2 - q_j^2) \text{ with all } p_i \text{ and } q_j \text{ 's different.} \]
when we impose the boundary condition $w \to 0$ for $v \to \infty$, $t = t_1(v) \sim v^{2N_c+2}$ the polynomial $N(v^2)$ is determined as follows,

$$N(v^2) = \left[ H(v^2) \sqrt{T(v^2)} \right]_+ = \left[ H(v^2) \prod_{j=1}^{2N_c+2-l} (v^2 - q_j^2)^{1/2} \right]_+, \quad (3.9)$$

where $\left[ H(v^2) \sqrt{T(v^2)} \right]_+$ means only nonnegative power of $v^2$ when we expand around $v = \infty$. The other boundary condition tells that $w^2$ behaves as $w^2 \to \sum_{k=1}^{N_c} 2k\mu_{2k}v^{2(k-1)}$ from (3.3). Then by expanding $w^2$ in powers of $v^2$ we can identify $H(v^2)$ with parameter $\mu_{2k}$. Using $T^{1/2} = (t - (v^2 C_{2N_c}(v^2, u_{2k}) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i)/S$ and $t = 2(v^2 C_{2N_c} + \cdots)$ from (3.1) we get

$$w^2 = \left[ 2H(v^2) \sqrt{T(v^2)} \right]_+ + \mathcal{O}(v^{-2})$$

$$= \frac{2H(v^2)}{\prod_{i=1}^{l}(v^2 - p_i^2)} \left( v^2 C_{2N_c}(v^2, u_{2k}) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i \right) + \mathcal{O}(v^{-2})$$

$$= \sum_{i=1}^{l} \frac{C_{2N_c}(v^2) \omega_i}{(v^2 - p_i^2)} + \mathcal{O}(v^{-2}) = \sum_{k=1}^{N_c} 2k\mu_{2k}v^{2(k-1)}, \quad (3.10)$$

where we used the definition of $H$ in (2.8) and the generating function of $\mu_{2k}$ in (2.7). From this result one can find the explicit form of $H(v^2)$ in terms of $\mu_{2k}$ by comparing both sides in the above relation. This is an explanation for field theory results of (2.7) and (2.8) which determine the $N = 1$ moduli space of vacua after the perturbation, from the point of view of M theory fivebrane. It reproduces the equations which determine the vevs of massless dyons along the singular locus. The dyon vevs $m_{i,dy}^2$, given by (2.9) $m_{i,dy}^2 = \sqrt{2p_i^2 H(p_i^2) \sqrt{T(p_i^2)}}$, are nothing but the difference between the two finite values of $v^2 w^2$. This can be seen by taking $v = \pm p_i$ in (3.8). The $N = 2$ curve of (3.1) and (2.6) contains double points at $v = \pm p_i$ and $t = C(p_i^2)$. The perturbation $\Delta W$ of (2.4) splits these into separate points in $(v, t, w)$ space and the difference in $v^2 w^2$ between these points becomes the dyon vevs. This is a geometric interpretation of dyon vevs in M theory fivebrane configuration. By noting that $w^2$ satisfies $w^4 - 2N w^2 + N^2 - TH^2 = 0$, and restricting the form of $N, T$ and $H$ like as $N \sim c_1 v^2 + c_2, T \sim c_3 v^8 + c_4 v^6 + c_5 v^4 + c_6 v^2 + c_7, H \sim \frac{c_8}{v^2}$, it leads to $w^4 + (c_9 + c_{10} v^2) w^2 + c_{11} = 0$ for some constants $c_i (i = 1, \cdots, 11)$. Then we can solve for $v^2$ in terms of $w^2$ to reproduce the result of [11]. As all the couplings $\mu_{2k}$ are becoming very large, $H(v^2)$ and $N(v^2)$ go to infinity. The term of $N^2 - TH^2$ goes to zero as we take the limit of $\Lambda_{N=2} \to 0$. This tells us that $w^2$ becomes $(N^2 - TH^2)/2N$ and as $N(v^2)$ goes to zero, $w^2 \to \infty$ showing the findings in [4].
4 The Meson Vevs in M Theory

We continue to study for the meson vevs from the singularity structure of $N = 2$ Riemann surface. The vevs of meson will depend on the moduli structure of $N = 2$ Coulomb branch (See, for example, [4.4]). Also, the finite values of $w^2$ can be determined fully by using the property of boundary conditions of $w^2$ when $v$ goes to be very large. Let us consider the case of finite $w^2$ at $t = 0, v = \pm m_i$ and we want to compare with the meson vevs we have studied in (2.13). At a point where there exists a single massless dyon (in other words, by putting $l = 1$) and recalling the definition of $T(v^2)$, we have for Yang-Mills with matter

$$
\left( v^2 C_{2N_c}(v^2) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i \right)^2 - \Lambda_{N=2}^{4N_c+4-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2) = (v^2 - p_1^2)^2 T(v^2), \quad (4.1)
$$

and the function $w^2$ according to (3.3) reads

$$
w^2 = \left[ \frac{h}{v^2} \sqrt{T(v^2)} \right]_{+} \pm \frac{h}{v^2} \sqrt{T(v^2)}, \quad (4.2)
$$

where in this case $l = 1$ means that the polynomial $v^2 H(v^2)$ has the degree of zero and we denote it by a constant $h \mathbb{H}$. Thus as $v \rightarrow \pm m_i$ the finite value of $w^2$, denoted by $w_i^2$ can be written as

$$
w_i^2 = w^2(v^2 \rightarrow m_i^2) = h \tilde{C}(m_i^2) \pm \frac{h}{m_i^2} \sqrt{T(m_i^2)}. \quad (4.3)
$$

From (4.1), the relation $\sqrt{T(m_i^2)/m_i^2} = C(m_i^2)/m_i^2 (m_i^2 - p_1^2) + \mathcal{O}(m_i^{-4})$ holds and the decomposition of $C$ yields $\sqrt{T(m_i^2)/m_i^2} = C(p_1^2)/p_1^2 (m_i^2 - p_1^2) + \tilde{C}(m_i^2)$. By plugging this value into (4.3) and taking the minus sign which corresponds to $t \rightarrow 0$, we end up with

$$
w_i^2 = \frac{h}{p_1^2} \frac{C(p_1^2)}{(p_1^2 - m_i^2)} = h \frac{\Lambda_{N=2}^{2N_c+2-N_f} \det(p_1^2 - m_i^2)^{1/2}}{(p_1^2 - m_i^2)}, \quad (4.4)
$$

where we evaluated $C(p_1^2)$ from (4.1) at $v^2 = p_1^2$. In the above expression we need to know the values of $h$ and $p_1$. The boundary condition for $w^2$ for large $v$ leads to

$$
w^2 \sim 2h \frac{C_{2N_c}(v^2)}{v^2 - p_1^2} \sim 2hv^2(2N_c-1) + 2hp_1^2 v^2(2N_c-2) + \cdots, \quad (4.5)
$$

** From (4.1) we see for $N_f < 2N_c + 2$, $\sqrt{T(v^2)/v^2} = C(v^2)/v^2 (v^2 - p_1^2) + \mathcal{O}(v^{-4})$ and we decompose $C$ as $C(v^2)/v^2 = C(p_1^2)/p_1^2 + (v^2 - p_1^2) \tilde{C}(v^2)$ for some polynomial $v^2 \tilde{C}(v^2)$ of degree $2N_c$. This means that the coefficients of $\tilde{C}(v^2)$ can be fixed from the explicit form of the polynomial $C(v^2)$. The part with nonnegative powers of $v^2$ in $\sqrt{T(v^2)/v^2}$ becomes $\tilde{C}(v^2)$ as follows $\sqrt{T(v^2)/v^2} = C(v^2) + \mathcal{O}(v^{-2}) \rightarrow \left[ \frac{1}{\sqrt{T(v^2)/v^2}} \right]_+ = \tilde{C}(v^2)$.**
which should be equal to $\sum_{k=1}^{N_c} 2k\mu_{2k}v^{2(k-1)}$. Now we can read off the values of $h$ and $p_1$ by comparing both sides,

$$h = N_c\mu_{2N_c}, \quad p_1^2 = \frac{(N_c - 1)\mu_{2(N_c-1)}}{N_c\mu_{2N_c}}.$$  \hspace{1cm} (4.6)

Finally, the finite value for $w^2$ can be written as

$$w_i^2 = \frac{N_c^2\mu_{2N_c}^2}{(N_c - 1)\mu_{2(N_c-1)}} \Lambda_{N=2}^{N_c+2-N_r} \frac{\det(a_i^2 - m^2)^{1/2}}{(a_i^2 - m_i^2)},$$  \hspace{1cm} (4.7)

which is exactly, up to constant, the same expression for meson vevs (2.13) obtained from field theory analysis in the low energy superpotential (2.13). This illustrates the fact that at vacua with enhanced gauge group $SU(2)$ the effective superpotential by integrating in method with the assumption of $W_\Delta = 0$ is really exact.

5 Discussions

It is straightforward to deal with Yang-Mills theory with massless matter. When some of branch points of (2.3) collide as one changes the moduli, the Riemann surface degenerates and gives a singularity in the theory corresponding to an additional massless field. By redefining $y$ we get the $2r + 1$ branch points of Riemann surface and one may expect that an unbroken $Sp(r + 1)$ gauge symmetry.

It is easy to generalize the case of several massless dyons. The classical moduli space is given by several eigenvalues of $a_i$’s. After integrating out the adjoint fields in each $SU(r_i)$, we obtain the scale matching condition between the high energy scale and the low energy scale. Then the meson vevs can be written as the differentiation of the low energy effective superpotential with respect to $m_i^2$ in field theory approach. On the other hand, in the M theory fivebrane configuration we can proceed the method done in a single massless dyon. That is, $\sqrt{T(v^2)/v^2}$ should be expressed for the several dyons and the decomposition of $C$ will be given also. One realizes that a mismatch is found between field theory results and brane configuration results with $W_\Delta = 0$. This tells us that the minimal form for the effective superpotential obtained by integrating in method is not exact. That is, $W_\Delta$ is not zero for the enhanced gauge group $SU(r), r > 2$. It will be interesting to find the corrections in the future.

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