Elasticity $\mathcal{M}$-tensors and the Strong Ellipticity Condition

Weiyang Ding\textsuperscript{*}  Jinjie Liu\textsuperscript{†}  Liqun Qi\textsuperscript{‡}  Hong Yan\textsuperscript{§}

June 6, 2017

Abstract

In this paper, we propose a class of tensors satisfying the strong ellipticity condition. The elasticity $\mathcal{M}$-tensor is defined with respect to the $\mathcal{M}$-eigenvalues of elasticity tensors. We prove that any nonsingular elasticity $\mathcal{M}$-tensor satisfies the strong ellipticity condition by employing a Perron-Frobenius-type theorem for $\mathcal{M}$-spectral radii of nonnegative elasticity tensors. We also establish other equivalent definitions of nonsingular elasticity $\mathcal{M}$-tensors.

Key words. Elasticity tensor, strong ellipticity, M-positive definite, $\mathcal{M}$-tensor, nonnegative tensor.

AMS subject classifications. 74B20, 74B10, 15A18, 15A69, 15A99.

1 Introduction

The strong ellipticity condition is essential in the theory of elasticity, since it guarantees the existence of solutions of basic boundary-value problems of elastostatics and thus ensures an elastic material to satisfy some mechanical properties. Thus to identify whether the strong ellipticity holds or not for a given material is an important problem in mechanics \cite{9}. Knowles and Sternberg \cite{13, 14} proposed necessary and sufficient conditions for strong ellipticity of the equations governing finite plane equilibrium deformations of a compressible hyperelastic solid. Their works were further extended by Simpson and Spector \cite{20} to the spatial case using the representation theorem for copositive matrices. Rosakis \cite{19} and Wang and Aron \cite{22} also established some reformulations. Furthermore, Walton and Wilber

\textsuperscript{*}Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong. Email: weiyang.ding@gmail.com. This author’s work was partially supported by the Hong Kong Research Grants Council (Grant No. C1007-15G).

\textsuperscript{†}Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong. Email: liujinjie 1990@163.com.

\textsuperscript{‡}Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong. Email: liqun.qi@polyu.edu.hk. This author’s work was partially supported by the Hong Kong Research Grants Council (Grant No. PolyU 501913, 15302114, 15300715, 15301716 and C1007-15G).

\textsuperscript{§}Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong. Email: h.yan@cityu.edu.hk. This author’s work was partially supported by the Hong Kong Research Grants Council (Grant No. C1007-15G).
provided sufficient conditions for strong ellipticity of a general class of anisotropic hyperelastic materials, which require the first partial derivatives of the reduced-stored energy function to satisfy several simple inequalities and the second partial derivatives to satisfy a convexity condition. Chirita, Danescu, and Ciarletta [4] and Zubov and Rudev [28] gave sufficient and necessary conditions for the strong ellipticity of certain classes of anisotropic linearly elastic materials. Gourgiotis and Bigoni [8] investigated the strong ellipticity of materials with extreme mechanical anisotropy.

Qi, Dai, and Han [17] also proved a necessary and sufficient condition of the strong ellipticity by introducing M-eigenvalues for ellipticity tensors and showing that the strong ellipticity holds if and only if all the M-eigenvalues of the ellipticity tensor is positive. A practical power method for computing the largest M-eigenvalue of any ellipticity tensor was proposed by Wang, Qi, and Zhang [23] and may also be applied to the verification of the strong ellipticity. Very recently, Huang and Qi [12] generalized the M-eigenvalues of fourth-order ellipticity tensors and related algorithms to higher order cases. Another type of “eigenvalues” for ellipticity tensors called singular values was defined by Chang, Qi, and Zhou, and the positivity of all the singular values of the ellipticity tensor is also a necessary and sufficient condition for the strong ellipticity. Han, Dai, and Qi [10] linked the strong ellipticity condition to the rank-one positive definiteness of three second-order tensors, three fourth-order tensors, and a sixth-order tensor. Ding, Qi, and Yan [6] proposed easily checkable sufficient conditions for the strong ellipticity condition.

Symmetric M-tensors, also called the Stieljes matrices, are an important class of positive semidefinite matrices used in many disciplines in science and engineering, such as linear systems of equations, numerical solutions of partially differential equations, the Markov chains, the queueing theory, and the graph theory [1]. Zhang, Qi, and Zhou [27] introduced higher order M-tensors and showed that an even-order symmetric M-tensor is positive semidefinite. Ding, Qi, and Wei [5] proposed several equivalent definitions of nonsingular M-tensors. Note that the nonsingular M-tensor is also called the strong M-tensor in some literatures. Ding and Wei [7] proved that there exists a unique positive solution of any polynomial system of equations whose coefficient tensor is a nonsingular M-tensor and the right-hand side is a positive vector. They proposed an iterative algorithm for solving such systems. Furthermore, Xie, Jin, and Wei [24] and Li, Xie, and Xu [15] also proposed other numerical methods.

Actually, the above M-structure is defined with respect to the tensor eigenvalues introduced by Qi [16]. According to [15, Chapter 2], a tensor whose M-eigenvalues are all positive (or nonnegative) is said to be M-positive (semi)definite. We shall define the M-tensors with respect to the M-eigenvalues, which will be shown to be M-positive semidefinite. Subsequently, we can find a large class of tensors satisfying the strong ellipticity condition.

The rest of the paper is organized as follows. We briefly introduce the strong ellipticity condition and its relationship with several types of positive definiteness in Section 2. In Section 3, we investigate the M-spectral radius of nonnegative elasticity tensors as a preparation for defining the elasticity M-tensors. Then we introduce the elasticity M-tensors and the nonsingular elasticity M-tensors in Section 4, and prove their M-positive (semi)definiteness and propose other equivalent definitions for nonsingular elasticity M-tensors. Finally, conclusion remarks will be drawn in Section 5.
2 Strong ellipticity and positive definiteness

The tensor of elastic moduli for a linearly elastic material represented in a Cartesian coordinate system is a fourth-order three-dimensional tensor \( \mathbf{A} = (a_{ijkl}) \in \mathbb{R}^{3 \times 3 \times 3 \times 3} \) which is invariant under the following permutations of indices

(1) \[ a_{ijkl} = a_{jikl} = a_{ijlk}. \]

We use \( \mathbb{E}_{4,n} \) to denote the set of all fourth-order \( n \)-dimensional tensors satisfying (1), where \( \mathbb{E}_{4,3} \) is exactly the set of all elasticity tensors. The strong ellipticity condition (SE-condition) for a tensor in \( \mathbb{E}_{4,n} \) is stated by

(2) \[ \mathbf{A} \mathbf{x}^2 \mathbf{y}^2 := \sum_{i,j,k,l=1}^{n} a_{ijkl} x_i x_j y_k y_l > 0 \]

for any nonzero vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \).

The SE-condition equivalently requires that the optimal value of the following minimization problem is positive:

(3) \[ \min_{\mathbf{x}, \mathbf{y}} \mathbf{A} \mathbf{x}^2 \mathbf{y}^2, \quad \text{s.t.} \quad \mathbf{x}^\top \mathbf{x} = 1, \mathbf{y}^\top \mathbf{y} = 1. \]

The KKT condition of the minimization problem (3) can be written as

(4) \[ \begin{cases} \mathbf{A} \mathbf{x} \mathbf{y}^2 = \lambda \mathbf{x}, \\ \mathbf{A} \mathbf{x}^2 \mathbf{y} = \lambda \mathbf{y}, \\ \mathbf{x}^\top \mathbf{x} = 1, \mathbf{y}^\top \mathbf{y} = 1, \end{cases} \]

where \( (\mathbf{A} \mathbf{x} \mathbf{y}^2)_i := \sum_{k,l=1}^{n} a_{ijkl} x_i y_k y_l \) and \( (\mathbf{A} \mathbf{x}^2 \mathbf{y})_i := \sum_{i,j,k=1}^{n} a_{ijkl} x_i x_j y_k \). In this formulation, Qi, Dai, and Han [17] defined the scalar \( \lambda \in \mathbb{R} \) and two vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) as an \( M \)-eigenvalue and a pair of corresponding \( M \)-eigenvectors of \( \mathbf{A} \), respectively. Thus, we also call a tensor satisfying the SE-condition to be \( M \)-positive definite (M-PD) [18]. Similarly, a tensor \( \mathbf{A} \in \mathbb{E}_{4,n} \) is said to be \( M \)-positive semidefinite (M-PSD) [18] if \( \mathbf{A} \mathbf{x}^2 \mathbf{y}^2 \geq 0 \) for any vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \). The following theorem reveals that the \( M \)-positive definiteness is equivalent to the positivity of a tensor’s \( M \)-eigenvalues.

**Theorem 2.1** ([17]). A tensor in \( \mathbb{E}_{4,n} \) is \( M \)-positive definite if and only if all of its \( M \)-eigenvalues are positive; A tensor in \( \mathbb{E}_{4,n} \) is \( M \)-positive semidefinite if and only if all of its \( M \)-eigenvalues are nonnegative.

We define a special tensor \( \mathbf{E} \in \mathbb{E}_{4,n} \) by

\[ e_{ijkl} = \begin{cases} 1, & \text{if } i = j \text{ and } k = l, \\ 0, & \text{otherwise}, \end{cases} \]

which serves as an identity element in \( \mathbb{E}_{4,n} \). We may call it the identity tensor in this paper. It can be verified that \( \mathbf{E} \mathbf{x} \mathbf{y}^2 = \mathbf{x}(\mathbf{y}^\top \mathbf{y}), \mathbf{E} \mathbf{x}^2 \mathbf{y} = (\mathbf{x}^\top \mathbf{x}) \mathbf{y}, \) and \( \mathbf{E} \mathbf{x}^2 \mathbf{y}^2 = (\mathbf{x}^\top \mathbf{x})(\mathbf{y}^\top \mathbf{y}). \) Hence, we may also define \( M \)-eigenvalues by

(5) \[ \begin{cases} \mathbf{A} \mathbf{x} \mathbf{y}^2 = \lambda \mathbf{E} \mathbf{x} \mathbf{y}^2, \\ \mathbf{A} \mathbf{x}^2 \mathbf{y} = \lambda \mathbf{E} \mathbf{x}^2 \mathbf{y}. \end{cases} \]
Comparing (4) and (5), we can see that if the triplet \((\lambda, x, y)\) satisfies (4) then \((\lambda, \alpha x, \beta y)\) satisfies (5) for any nonzero real scalar \(\alpha, \beta\). We note that (5) is exactly the KKT condition of the following minimization problem:

\[
\min \mathcal{A} x^2 y^2, \\
\text{s.t.} \quad (x^\top x) (y^\top y) = 1,
\]

whose optimal value being positive also guarantees the SE-condition. The following proposition is an observation from the definition of the identity tensor.

**Proposition 2.2.** Let \(\mathcal{A} \in \mathbb{R}_{+}^{n^2}\). Suppose that \(\mathcal{B} = \alpha(\mathcal{A} + \beta \mathcal{E})\), where \(\alpha, \beta\) are two real scalars. Then \(\mu\) is an \(M\)-eigenvalue of \(\mathcal{B}\) if and only if \(\mu = \alpha (\lambda + \beta)\) and \(\lambda\) is an \(M\)-eigenvalue of \(\mathcal{A}\). Furthermore, \(\lambda\) and \(\mu\) correspond to the same \(M\)-eigenvectors.

There are two common ways to unfold a tensor in \(\mathbb{E}_{n, n}\) into an \(n^2\)-by-\(n^2\) matrix:

\[
(1) \quad A_x = \begin{bmatrix}
\mathcal{A}(1, 1, 1) & \mathcal{A}(1, 1, 2) & \cdots & \mathcal{A}(1, 1, n) \\
\mathcal{A}(1, 1, 1) & \mathcal{A}(1, 2, 2) & \cdots & \mathcal{A}(1, 2, n) \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{A}(1, n, 1) & \mathcal{A}(1, n, 2) & \cdots & \mathcal{A}(1, n, n)
\end{bmatrix} \in \mathbb{R}^{n^2 \times n^2},
\]

\[
(2) \quad A_y = \begin{bmatrix}
\mathcal{A}(1, 1, 1) & \mathcal{A}(1, 2, 1) & \cdots & \mathcal{A}(1, 1, n) \\
\mathcal{A}(1, 1, 2) & \mathcal{A}(2, 2, 2) & \cdots & \mathcal{A}(1, n, n) \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{A}(n, 1, 1) & \mathcal{A}(n, 2, 1) & \cdots & \mathcal{A}(n, n, n)
\end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}.
\]

Note that \(A_x\) and \(A_y\) are permutation similar to each other, i.e., there is a permutation matrix \(P\) such that \(A_x = P^\top A_y P\). Then \(\mathcal{A}\) is \(M\)-PD if \(A_x\) (or equivalently \(A_y\)) is PSD. This can be proved by noticing that

\[
\mathcal{A} x^2 y^2 = (y \otimes x)^\top A_x (y \otimes x) = (x \otimes y)^\top A_y (x \otimes y),
\]

where \(\otimes\) denotes the Kronecker product [11]. Thus we call \(\mathcal{A}\) S-positive (semi)definite if \(A_x\) or \(A_y\) is positive (semi)definite, and call the eigenvalues of \(A_x\) or \(A_y\) the S-eigenvalues of \(\mathcal{A}\). The S-positive definiteness is a sufficient condition for the \(M\)-positive definiteness, but the converse is not true. Some counter examples were given in [6, 17].

## 3 Nonnegative elasticity tensors

We have a well-known theory about nonnegative matrices called the Perron-Frobenius theorem [11], which states that the spectral radius of any nonnegative matrix is an eigenvalue with a nonnegative eigenvector and the eigenvector is positive and unique if the matrix is irreducible. In the past decades, the Perron-Frobenius theorem has been extended to higher order tensors by Chang, Pearson, and Zhang [3] and Yang and Yang [25, 26]. One may refer to [18, Chapter 3] for a whole picture of the nonnegative tensor theory. We will also obtain similar results for nonnegative elasticity tensors in this section.

From the discussions in Section 2, we have variational forms of the extremal \(M\)-eigenvalues. Let \(\mathcal{B} \in \mathbb{E}_{n, n}\). Denote \(\lambda_{\text{max}}(\mathcal{B})\) and \(\lambda_{\text{min}}(\mathcal{B})\) as the maximal and the minimal \(M\)-eigenvalues of \(\mathcal{B}\), respectively. Then

\[
\lambda_{\text{max}}(\mathcal{B}) = \max \{ \mathcal{B} x^2 y^2 : x, y \in \mathbb{R}^n, x^\top x = y^\top y = 1 \},
\]

\[
\lambda_{\text{min}}(\mathcal{B}) = \min \{ \mathcal{B} x^2 y^2 : x, y \in \mathbb{R}^n, x^\top x = y^\top y = 1 \}.
\]
The maximal absolute value of all the M-eigenvalues is called the M-spectral radius of a tensor in $\mathbb{E}_{4,n}$, denoted by $\rho_M(\mathcal{T})$. Apparently, the M-spectral radius is equal to the greater one of the absolute values of the maximal and the minimal M-eigenvalues. The following theorem reveals that $\rho_M(\mathcal{T}) = \lambda_{\max}(\mathcal{T})$ when $\mathcal{T} \in \mathbb{E}_{4,n}$ is nonnegative.

**Theorem 3.1.** The M-spectral radius of any nonnegative tensor in $\mathbb{E}_{4,n}$ is exactly its greatest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.

**Proof.** It enough to show that $\lambda_{\max}(\mathcal{T}) \geq |\lambda_{\min}(\mathcal{T})|$ for proving the first statement. For convenience, denote $\lambda_1$ and $\lambda_2$ as the maximal and the minimal M-eigenvalues of $\mathcal{T}$ respectively, and $(x_1, y_1)$ and $(x_2, y_2)$ are the corresponding M-eigenvectors. By (6), we know that $\lambda_1 = \mathcal{T}x_1^2y_1^2$ and $\lambda_2 = \mathcal{T}x_2^2y_2^2$. Then employing the nonnegativity of the entries of $\mathcal{T}$, we have $|\lambda_2| = |\mathcal{T}x_2^2y_2^2| \leq \mathcal{T}|x_2|^2|y_2|^2 \leq \lambda_1$. Thus, $(|x_1|, |y_1|)$ is also a pair of M-eigenvectors corresponding to $\lambda_1$, which is nonnegative.

Theorem 3.1 can be regarded as the weak Perron-Frobenius theorem for the tensors in $\mathbb{E}_{4,n}$. Combining Theorem 3.1 and (6), we have the following corollary, which shrinks the feasible domain in (6).

**Corollary 3.2.** Let $\mathcal{T} \in \mathbb{E}_{4,n}$. If $\mathcal{T}$ is nonnegative, then $\rho(\mathcal{T}) = \max \{ \mathcal{T}x^2y^2 : x, y \in \mathbb{R}^n_+, x^T x = y^T y = 1 \}$.

Chang, Qi, and Zhou [2] also studied the strong ellipticity for nonnegative elasticity tensors. They introduced the singular values of a tensor $\mathcal{T} \in \mathbb{E}_{4,n}$ as

$$\begin{align*}
\mathcal{T}xy^2 &= \sigma x^{[3]}, \\
\mathcal{T}x^2y &= \sigma y^{[3]},
\end{align*}$$

and they also investigate the Perron-Frobenius theorem for the singular values. Nevertheless, it is hard to find an identity tensor similar to the tensor $\mathcal{E}$ in our case, thus we may not be able to define a kind of $\mathcal{M}$-tensors with respect to their singular values.

4 Elasticity $\mathcal{M}$-tensors

Recall that the identity tensor $\mathcal{E}$ is defined by $e_{iikk} = 1$ and other entries being zero. Let $\mathcal{A} \in \mathbb{E}_{4,n}$. Accordingly, we call the entries $a_{iikk}$ ($i, k = 1, 2, \ldots, n$) diagonal, and other entries are called off-diagonal. Obviously, the diagonal entries of an M-PD tensor must be positive, and the ones of an M-PSD tensor must be nonnegative. It is worth noting that the diagonal entries of an elasticity tensor lies also on the diagonal of its unfolding matrix.

A tensor in $\mathbb{E}_{4,n}$ is called an elasticity $\mathcal{Z}$-tensor if all its off-diagonal entries are non-positive. If $\mathcal{A} \in \mathbb{E}_{4,n}$ is an elasticity $\mathcal{E}$-tensor, then we can always write it as $\mathcal{A} = s\mathcal{E} - \mathcal{T}$, where $\mathcal{T}$ is a nonnegative tensor in $\mathbb{E}_{4,n}$. Such partition of an elasticity $\mathcal{E}$-tensor is not
unique. If a tensor $\mathcal{A} \in \mathbb{E}_{4,n}$ can be written as $\mathcal{A} = s\mathcal{E} - \mathcal{B}$ satisfying that $\mathcal{B} \in \mathbb{E}_{4,n}$ is nonnegative and $s \geq \rho_M(\mathcal{B})$, then we call $\mathcal{A}$ an elasticity $\mathcal{M}$-tensor. Furthermore, if $s > \rho_M(\mathcal{B})$, then we call $\mathcal{A}$ a nonsingular elasticity $\mathcal{M}$-tensor.

**Theorem 4.1.** Let $\mathcal{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathcal{Z}$-tensor. Then $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor if and only if $\alpha > \rho_M(\alpha\mathcal{E} - \mathcal{A})$, where $\alpha = \max \{a_{ikk} : i, k = 1, 2, \ldots, n\}$.

**Proof.** The “if” part is obvious by the partition $\mathcal{A} = \alpha\mathcal{E} - (\alpha\mathcal{E} - \mathcal{A})$. Thus we focus on the “only if” part. If $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor, then it can be written as $\mathcal{A} = s\mathcal{E} - \mathcal{B}$ satisfying that $\mathcal{B} \in \mathbb{E}_{4,n}$ is nonnegative and $s > \rho_M(\mathcal{B})$. Denote $\beta = \min \{b_{ikk} : i, k = 1, 2, \ldots, n\}$, then $\alpha = s - \beta$. Moreover, we can also write $\alpha\mathcal{E} - \mathcal{A} = \mathcal{B} - \beta\mathcal{E}$, thus $\rho_M((\alpha\mathcal{E} - \mathcal{A}) - \beta\mathcal{E}) = \rho_M(\mathcal{B} - \beta\mathcal{E})$. Therefore, $s > \rho_M(\mathcal{B})$ implies that $\alpha > \rho_M(\alpha\mathcal{E} - \mathcal{A})$. □

The above theorem is a simple but useful observation. We can utilize this theorem to prove the following proposition, which reveals that any elasticity $\mathcal{M}$-tensor is the limit of a series of nonsingular elasticity $\mathcal{M}$-tensors. Hence, we may omit the proofs of following results for general elasticity $\mathcal{M}$-tensors, since it can be verified by taking limits of the results for nonsingular elasticity $\mathcal{M}$-tensors.

**Proposition 4.2.** $\mathcal{A} \in \mathbb{E}_{4,n}$ is an elasticity $\mathcal{M}$-tensor if and only if $\mathcal{A} + t\mathcal{E}$ is a nonsingular elasticity $\mathcal{M}$-tensor for any $t > 0$.

**Proof.** Since $\mathcal{A}$ is an elasticity $\mathcal{M}$-tensor, there exists a nonnegative elasticity tensor $\mathcal{B}$ and $s \geq \rho_M(\mathcal{B})$ such that $\mathcal{A} = s\mathcal{E} - \mathcal{B}$. Then for any $t > 0$, we have $\mathcal{A} + t\mathcal{E} = (s + t)\mathcal{E} - \mathcal{B}$. Clearly, $s + t > \rho(\mathcal{B})$, which implies that $\mathcal{A} + t\mathcal{E}$ is a nonsingular elasticity $\mathcal{M}$-tensor.

Conversely, if $\mathcal{A} + t\mathcal{E}$ is a nonsingular elasticity $\mathcal{M}$-tensor for any $t > 0$, then by the previous theorem we have $\alpha_t > \rho_M(\alpha_t\mathcal{E} - (\mathcal{A} + t\mathcal{E}))$, where $\alpha_t$ is the greatest diagonal entry of $\mathcal{A} + t\mathcal{E}$. Denote $\alpha$ as the greatest diagonal entry of $\mathcal{A}$. Then $\alpha_t = \alpha + t$, thus $\alpha + t > \rho_M(\alpha\mathcal{E} - \mathcal{A})$ for any $t > 0$. When $t$ approaches $0$, it can be concluded that $\alpha \geq \rho_M(\alpha\mathcal{E} - \mathcal{A})$, which implies that $\mathcal{A}$ is an elasticity $\mathcal{M}$-tensor. □

It is well known that a symmetric nonsingular $\mathbb{M}$-matrix is positive definite [1]. The same statement was also proved for symmetric nonsingular $\mathcal{M}$-tensors in [27]. Moreover, we shall show that a nonsingular elasticity $\mathcal{M}$-tensor is $\mathbb{M}$-positive definite thus satisfies the strong ellipticity condition. In this spirit, we find a class of structured tensors that satisfies the strong ellipticity condition.

**Theorem 4.3.** Let $\mathcal{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathcal{Z}$-tensor. Then $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor if and only if $\mathcal{A}$ is $\mathbb{M}$-positive definite; and $\mathcal{A}$ is an elasticity $\mathcal{M}$-tensor if and only if $\mathcal{A}$ is $\mathbb{M}$-positive semidefinite.

**Proof.** Denote $\mathcal{A} = s\mathcal{E} - \mathcal{B}$, where $\mathcal{B}$ is nonnegative.

If $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor, then $s > \rho_M(\mathcal{B})$. By [6], we have $s > \langle \mathcal{B}x, y \rangle^2$ for all $x^T x = y^T y = 1$. Recall that $\mathcal{E}x^2y^2 = (x^T x)(y^T y)$. Then $s\mathcal{E}x^2y^2 > \langle \mathcal{B}x, y \rangle^2$, which is equivalent to $\mathcal{A}x^2y^2 > 0$ for all $x^T x = y^T y = 1$. Therefore $\mathcal{A}$ is $\mathbb{M}$-positive definite.

On the other hand, suppose that $\mathcal{A}$ is $\mathbb{M}$-positive definite, i.e., $\mathcal{A}x^2y^2 > 0$ for all $x^T x = y^T y = 1$. Then similarly we have $s = s\mathcal{E}x^2y^2 > \langle \mathcal{B}x, y \rangle^2$ for all $x^T x = y^T y = 1$. We know from [6] that $s > \rho_M(\mathcal{B})$, i.e., $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor.

The result for general elasticity $\mathcal{M}$-tensors can be proved similarly. □

Recall that the S-eigenvalues of a tensor in $\mathbb{E}_{4,n}$ are defined by the eigenvalues of its unfolding matrices $\mathbf{A}_x$ and $\mathbf{A}_y$. Of course, we can also define $\mathcal{M}$-tensors with respect to
S-eigenvalues, which coincides with those tensors $\mathcal{A}$ whose unfolding matrices $A_x$ and $A_y$ are M-matrices. In this case, $\mathcal{A}$ is also M-positive semidefinite since $A_x$ and $A_y$ are positive semidefinite matrices. However, the converse may not hold as shown by the following example.

**Example 4.1.** Consider the case $n = 2$. Let $\mathcal{A} \in \mathbb{E}_{4,2}$ be defined by

$$
\begin{align*}
    a_{1111} &= 13, & a_{1122} &= 2, & a_{2211} &= 2, \\
    a_{2222} &= 12, & a_{1112} &= -2, & a_{1211} &= -2, \\
    a_{1212} &= -4, & a_{1222} &= -1, & a_{2212} &= -1.
\end{align*}
$$

By our calculations with Mathematica, $\mathcal{A}$ has six M-eigenvalues: 13.4163, 12.1118, 11.2036, 6.1778, 0.2442, and 0.1964. The minimal M-eigenvalue of $\mathcal{A}$ is 0.1964, which is positive. Thus $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor by Theorem 4.3. Nonetheless, the unfolding matrices of $\mathcal{A}$ are

$$
A_x = A_y = \begin{bmatrix}
13 & -2 & -2 & -4 \\
-2 & 2 & -4 & -1 \\
-2 & -4 & 2 & -1 \\
-4 & -1 & -1 & 12
\end{bmatrix},
$$

with four eigenvalues: $-2.8331, 6.0000, 9.2221,$ and $16.6110$. There is a negative eigenvalue, which implies that $A_x$ and $A_y$ are not positive semidefinite and thus not M-matrices.

We now provide some equivalent definitions of nonsingular elasticity $\mathcal{M}$-tensors, which serve as verification conditions. First, we need to introduce several notations for convenience. Let $\mathcal{A} \in \mathbb{E}_{4,n}$ and $x, y \in \mathbb{R}^n$. We define two $n$-by-$n$ matrices $(\mathcal{A}x^2) \in \mathbb{R}^{n \times n}$ and $(\mathcal{A} \cdot y^2) \in \mathbb{R}^{n \times n}$ by

$$
\begin{align*}
(\mathcal{A}x^2)_{kl} &= \sum_{i,j=1}^n a_{ijkl}x_ix_j, & k, l = 1, 2, \ldots, n, \\
(\mathcal{A} \cdot y^2)_{ij} &= \sum_{k,l=1}^n a_{ijkl}y_ky_l, & i, j = 1, 2, \ldots, n.
\end{align*}
$$

We note that

$$
(\mathcal{A}x^2.) = 
\begin{bmatrix}
x^\top \mathcal{A}(\cdot, ; 1, 1)x & x^\top \mathcal{A}(\cdot, ; 1, 2)x & \cdots & x^\top \mathcal{A}(\cdot, ; 1, n)x \\
x^\top \mathcal{A}(\cdot, ; 2, 1)x & x^\top \mathcal{A}(\cdot, ; 2, 2)x & \cdots & x^\top \mathcal{A}(\cdot, ; 2, n)x \\
\vdots & \vdots & \ddots & \vdots \\
x^\top \mathcal{A}(\cdot, ; n, 1)x & x^\top \mathcal{A}(\cdot, ; n, 2)x & \cdots & x^\top \mathcal{A}(\cdot, ; n, n)x
\end{bmatrix},
$$

and there is a similar representation for $(\mathcal{A} \cdot y^2)$. Furthermore, it is straightforward to verify that

$$
\begin{align*}
\mathcal{A}x^2y^2 &= y^\top (\mathcal{A}x^2. )y = x^\top (\mathcal{A} \cdot y^2)x, \\
\mathcal{A}x^2y &= (\mathcal{A}x^2.)y, & \mathcal{A}xy^2 &= (\mathcal{A} \cdot y^2)x.
\end{align*}
$$

The next theorem shows that these two matrices admit the same structures with the original elasticity tensor.

**Theorem 4.4.** Let $\mathcal{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathcal{M}$-tensor. Then $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor if and only if $(\mathcal{A}x^2.)$ is a nonsingular M-matrix for each $x \geq 0$; $\mathcal{A}$ is an elasticity $\mathcal{M}$-tensor if and only if $(\mathcal{A}x^2.)$ is an M-matrix for each $x \geq 0$. 

Proof. Suppose that $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor. Then we know by (7) that $(\mathcal{A}^2)$ is positive definite for each $x \in \mathbb{R}^n$ since $\mathcal{A}$ is M-positive definite. Another simple observation is that $(\mathcal{A}^2)$ is a Z-matrix for each $x \geq 0$ when $\mathcal{A}$ is an elasticity $\mathcal{Z}$-tensor. Thus $(\mathcal{A}^2)$ is a positive definite Z-matrix for each $x \geq 0$. From the equivalent definitions of nonsingular M-matrices [11], it can be concluded that $(\mathcal{A}^2)$ is a nonsingular M-matrix for each $x \geq 0$.

Conversely, if $(\mathcal{A}^2)$ is a nonsingular M-matrix for each $x \geq 0$, then $(\mathcal{A}^2)$ is always positive definite. That is, $\mathcal{A}^2 y^2 = y^\top (\mathcal{A}^2) y > 0$ for each $x \geq 0$ and $y \in \mathbb{R}^n$. Write $\mathcal{A} = s \mathcal{E} - \mathcal{B}$, where $\mathcal{B}$ is nonnegative. Then $s > \mathcal{B} x^2 y^2$ for each $x, y \geq 0$ satisfying $x^\top x = y^\top y = 1$. Hence, Corollary [3.2] tells that $s > \rho_M(\mathcal{B})$, i.e., $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor. \hfill \square

Similarly, we have a parallel result for $(\mathcal{A} \cdot y^2)$.

**Theorem 4.5.** Let $\mathcal{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathcal{Z}$-tensor. Then $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor if and only if $(\mathcal{A} \cdot y^2)$ is a nonsingular M-matrix for each $y \geq 0$; $\mathcal{A}$ is an elasticity $\mathcal{M}$-tensor if and only if $(\mathcal{A} \cdot y^2)$ is an M-matrix for each $y \geq 0$.

There is a well-known equivalent definition for nonsingular M-matrices called semi-positivity. That is, a Z-matrix $\mathcal{A}$ is a nonsingular M-matrix if and only if there exists a positive (or equivalently nonnegative) vector $x$ such that $\mathcal{A} x$ is also a positive vector. Ding, Qi, and Wei [5] proved that this also holds for nonsingular $\mathcal{M}$-tensors. The semi-positivity is essential to verify whether a tensor is a nonsingular $\mathcal{M}$-tensor and is also important for solving the polynomial systems of equations with $\mathcal{M}$-tensors [7]. Combining the semi-positivity of nonsingular M-matrices and Theorems 4.4 and 4.5, we have the following equivalent definitions for nonsingular elasticity $\mathcal{M}$-tensors immediately.

**Theorem 4.6.** Let $\mathcal{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathcal{Z}$-tensor. The following conditions are equivalent:

1. $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor;
2. For each $x \geq 0$, there exists $y > 0$ such that $\mathcal{A} x^2 y > 0$;
3. For each $x \geq 0$, there exists $y \geq 0$ such that $\mathcal{A} x^2 y > 0$;
4. For each $y \geq 0$, there exists $x > 0$ such that $\mathcal{A} x y^2 > 0$;
5. For each $y \geq 0$, there exists $x \geq 0$ such that $\mathcal{A} x y^2 > 0$.

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called strictly diagonally dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n.$$  

Condition (2) in Theorem 4.6 states that for each nonnegative vector $x$, there exists a positive vector $y$ such that $(\mathcal{A} x^2) y = \mathcal{A} x^2 y > 0$. Denote a diagonal matrix $\mathbf{D}$ with $d_{ii} = y_i$ for $i = 1, 2, \ldots, n$ and $\tilde{\mathbf{A}} := (\mathcal{A} x^2) \mathbf{D}$. When $\mathcal{A}$ be an elasticity $\mathcal{Z}$-tensor, the matrix $\tilde{\mathbf{A}}$ is also a Z-matrix. Thus we have

$$\tilde{a}_{ii} = \sum_{j \neq i} [\tilde{a}_{ij}] = \tilde{a}_{ii} + \sum_{j \neq i} \tilde{a}_{ij} = (\mathcal{A} x^2 y)_i > 0, \quad i = 1, 2, \ldots, n,$$

which implies that $\tilde{\mathbf{A}}$ is strictly diagonally dominant. Applying the above discussion, we can prove the following corollary of Theorem 4.6.
Corollary 4.7. Let $\mathcal{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathcal{Z}$-tensor. The following conditions are equivalent:

1. $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor;
2. For each $x \geq 0$, there exists a positive diagonal matrix $D$ such that $D(\mathcal{A}x^2)D$ is strictly diagonally dominant;
3. For each $y \geq 0$, there exists a positive diagonal matrix $D$ such that $D(\mathcal{A} \cdot y^2)D$ is strictly diagonally dominant.

5 Conclusions

Combining Theorems 4.1, 4.3–4.6 and Corollary 4.7, we summarize the equivalent definitions for nonsingular elasticity $\mathcal{M}$-tensors given in this paper.

Theorem 5.1. Let $\mathcal{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathcal{Z}$-tensor. The following conditions are equivalent:

1. $\mathcal{A}$ is a nonsingular elasticity $\mathcal{M}$-tensor;
2. $\mathcal{A}$ is $M$-positive definite, i.e., $\mathcal{A}x^2y^2 > 0$ for all nonzero $x, y \in \mathbb{R}^n$;
3. All the $M$-eigenvalues of $\mathcal{A}$ are positive;
4. $\alpha > \rho_M(\alpha \mathcal{M} - \mathcal{A})$, where $\alpha = \max \{a_{iik} : i, k = 1, 2, \ldots, n\}$;
5. For each $x \geq 0$, $(\mathcal{A}x^2)$ is a nonsingular $M$-matrix;
6. For each $x \geq 0$, there exists $y > 0$ such that $\mathcal{A}x^2y > 0$;
7. For each $x \geq 0$, there exists $y \geq 0$ such that $\mathcal{A}x^2y > 0$;
8. For each $x \geq 0$, there exists a positive diagonal matrix $D$ such that $D(\mathcal{A}x^2)D$ is strictly diagonally dominant;
9. For each $y \geq 0$, $(\mathcal{A} \cdot y^2)$ is a nonsingular $M$-matrix;
10. For each $y \geq 0$, there exists $x > 0$ such that $\mathcal{A}xy^2 > 0$;
11. For each $y \geq 0$, there exists $x \geq 0$ such that $\mathcal{A}xy^2 > 0$;
12. For each $y \geq 0$, there exists a positive diagonal matrix $D$ such that $D(\mathcal{A} \cdot y^2)D$ is strictly diagonally dominant.

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