Functional equations from generating functions: a novel approach to deriving identities for the Bernstein basis functions

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Abstract

The main aim of this paper is to provide a novel approach to deriving identities for the Bernstein polynomials using functional equations. We derive various functional equations and differential equations using generating functions. Applying these equations, we give new proofs for some standard identities for the Bernstein basis functions, including formulas for sums, alternating sums, recursion, subdivision, degree raising, differentiation and a formula for the monomials in terms of the Bernstein basis functions. We also derive many new identities for the Bernstein basis functions based on this approach. Moreover, by applying the Laplace transform to the generating functions for the Bernstein basis functions, we obtain some interesting series representations for the Bernstein basis functions.

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1. Introduction

The Bernstein polynomials have many applications: in approximations of functions, in statistics, in numerical analysis, in $p$-adic analysis and in the solution of differential equations. It is also well-known that in Computer Aided Geometric Design polynomials are often expressed in terms of the Bernstein basis functions. These polynomials are called Bezier curves and surfaces.

Many of the known identities for the Bernstein basis functions are currently derived in an ad hoc fashion, using either the binomial theorem, the binomial distribution, tricky algebraic manipulations or blossoming. The main purpose of this work is to construct novel functional equations for the Bernstein polynomials. Using these functional equations and Laplace transform, we develop a novel approach both to standard and to new identities for the
Bernstein polynomials. Thus these polynomial identities are just the residue of a much more powerful system of functional equations.

The remainder of this study is organized as follows: We find several functional equations and differential equations for the Bernstein basis functions using generating functions. From these equations, many properties of the Bernstein basis functions are then derived. For instance, we give a new proof of the recursive definition of the Bernstein basis functions as well as a novel derivation for the two term formula for the derivatives of the $n$th degree Bernstein basis functions. Using functional equations, we give new derivations for the sum and alternating sum of the the Bernstein basis functions and a formula for the monomials in terms of the Bernstein basis functions. We also derive identities corresponding to the degree elevation and subdivision formulas for Bezier curves. We prove many new identities for the Bernstein basis functions. Finally, we give some applications of the Laplace transform to the generating functions for the Bernstein basis functions. We obtain interesting series representations for the Bernstein basis functions. We also give some remarks and observations related to the Fourier transform and complex generating functions for the Bernstein basis functions.

2. Generating Functions

The Bernstein polynomials and related polynomials have been studied and defined in many different ways, for examples by $q$-series, complex functions, $p$-adic Volkenborn integrals and many algorithms. In this section, we provide novel generating functions for the Bernstein basis functions.

The Bernstein basis functions $B^n_k(x)$ are defined as follows:

**Definition 1.**

$$B^n_k(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad (2.1)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$k = 0, 1, \ldots, n$ cf. [1]-[13].

Generating functions for the Bernstein basis functions can be defined as follows:

**Definition 2.**

$$f_{B,k}(x, t) = \sum_{n=0}^{\infty} B^n_k(x) \frac{t^n}{n!}. \quad (2.2)$$

Note that there is one generating function for each value of $k$.

**Theorem 1.**

$$f_{B,k}(x, t) = \frac{t^k x^k e^{(1-x)t}}{k!}. \quad (2.3)$$
Proof. By substituting (2.1) into the right hand side of (2.2), we get
\[
\sum_{n=0}^{\infty} B_n^k(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \binom{n}{k} x^k (1 - x)^{n-k} \right) \frac{t^n}{n!}.
\]
Therefore
\[
\sum_{n=0}^{\infty} B_n^k(x) \frac{t^n}{n!} = (xt)^k \sum_{n=k}^{\infty} \frac{(1 - x)^{n-k} t^{n-k}}{(n-k)!}.
\]
The right hand side of the above equation is a Taylor series for \(e^{(1-x)t}\), thus we arrive at the desired result.

We give some alternative forms of the generating functions in (2.2) as follows:
\[
\sum_{n=0}^{\infty} B_n^k(x) \frac{t^n}{n!} e^{xt} = \frac{t^k x^k e^t}{k!},
\]
\[
\sum_{n=0}^{\infty} B_n^k(x) \frac{t^n}{n!} e^{-t} = \frac{t^k x^k e^{-xt}}{k!},
\]
and
\[
\sum_{n=0}^{\infty} B_n^k(x) \frac{t^n}{n!} e^{(x-1)t} = \frac{t^k x^k}{k!}.
\]
By using the above alternative forms we derive some new identities for the Bernstein basis functions.

Remark 1. If we replace \(x\) by \(\frac{x-a}{b-a}\) in (2.3), where \(a < b\), then
\[
\frac{t^k (\frac{x-a}{b-a})^k e^{(\frac{x-a}{b-a})t}}{k!} = \sum_{n=0}^{\infty} B_n^k(x, a, b) \frac{t^n}{n!},
\]
where \(B_n^k(x, a, b)\) denotes the generalized Bernstein basis function defined by:
\[
B_n^k(x, a, b) = \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^m}
\]
cf. [1].

A Bernstein polynomial \(\mathcal{P}(x)\) is a polynomial represented in the Bernstein basis functions:
\[
\mathcal{P}(x) = \sum_{k=0}^{n} c_k^a B_k^a(x)
\]
cf. [4], Simsek [14]-[15], Simsek et al. [13] and Acikgoz et al. [1] also studied on the generating function for Bernstein basis function.

3. Identities for the Bernstein basis functions

In this section, we use the generating functions for the Bernstein basis functions to derive a family of functional equations. Using these equations, we derive a collection of identities for the Bernstein basis functions.
3.1. **Sums and Alternating sums.** From (2.3), we get the following functional equations:

\[
\sum_{k=0}^{\infty} f_{B,k}(x, t) = e^t
\]  

(3.1)

and

\[
\sum_{k=0}^{\infty} (-1)^k f_{B,k}(x, t) = e^{(1-2x)t}.
\]

(3.2)

**Theorem 2.** *(Sum of the Bernstein basis functions)*

\[
\sum_{k=0}^{n} B^n_k(x) = 1.
\]

*Proof.* From (3.1), one finds that

\[
\sum_{k=0}^{\infty} f_{B,k}(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}.
\]

(3.3)

By combining (2.2) and (3.3), we get

\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} B^n_k(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!}.
\]

Comparing the coefficients of \(\frac{t^n}{n!}\) on the both sides of the above equation, we arrive at the desired result. \(\square\)

**Theorem 3.** *(Alternating sum of the Bernstein basis functions)*

\[
\sum_{k=0}^{n} (-1)^k B^n_k(x) = (1 - 2x)^n.
\]

*Proof.* By combining (3.2) and (3.3), we obtain

\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-1)^k B^n_k(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(1 - 2x)^n t^n}{n!}.
\]

Comparing the coefficients of \(\frac{t^n}{n!}\) on the both sides of the above equation, we arrive at the desired result. \(\square\)

**Remark 2.** Goldman [6] - [5, Chapter 5, pages 299-306] derived the formula for the alternating sum of the Bernstein basis functions algebraically.

3.2. **Subdivision.** From (2.3), we have the following functional equation:

\[
f_{B,j}(xy, t) = f_{B,j}(x, ty) e^{t(1-y)}.
\]

(3.4)

From this functional equation, we get the following identity which is the basis for subdivision of Bezier curves cf. ([4], [5], [6], [14]).
Theorem 4.

\[ B^n_j(xy) = \sum_{k=j}^{n} B^n_j(x) B^n_k(y). \]

Proof. By equations (2.3) and (3.4)

\[ \sum_{n=j}^{\infty} B^n_j(xy) \frac{t^n}{n!} = \left( \sum_{n=0}^{\infty} B^n_j(x) y^n \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{(1-y)^n t^n}{n!} \right). \]

Therefore

\[ \sum_{n=j}^{\infty} B^n_j(xy) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=j}^{n} B^n_j(x) y^n \frac{t^n}{k! (n-k)!} \right) \right) t^n. \]

Substituting (2.1) into the above equation, we arrive at the desired result. \(\square\)

Remark 3. Theorem 4 is a bit tricky to prove with algebraic manipulations. Goldman [6, Chapter 5, pages 299-306] proved this identity algebraically. He also proved the following related subdivision identities:

\[ B^n_j((1-y)x + y) = \sum_{k=0}^{j} B^n_{j-k}(x) B^n_k(y), \]

and

\[ B^n_j((1-y)x + yz) = \sum_{k=0}^{n} \left( \sum_{p+q=j} B^n_{p-k}(x) B^n_q(z) \right) B^n_k(y). \]

For additional identities, see cf. [6, Chapter 5, pages 299-306].

3.3. Formula for the monomials in terms of the Bernstein basis functions. Multiplying both sides of (2.3) by \(\binom{k}{l}\), we get

\[ \binom{k}{l} \frac{(xt)^k}{k!} e^{t(1-x)} = \binom{k}{l} \sum_{n=0}^{\infty} B^n_k(x) \frac{t^n}{n!}. \]

Summing both sides of the above equation over \(k\), we obtain the following functional equation, which is used to derive a formula for the monomials in terms of the Bernstein basis functions:

\[ \frac{x^l t^l}{l!} e^t = \sum_{k=0}^{\infty} \binom{k}{l} f_{B,k}(x, t). \quad (3.5) \]

Theorem 5.

\[ \binom{n}{l} x^l = \sum_{k=l}^{n} \binom{k}{l} B^n_k(x) \]
Proof. Combining (2.2) and (3.5), we get
\begin{equation}
\frac{x^l}{l!} \sum_{n=0}^{\infty} \frac{t^{n+l}}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{k}{l} B_k^n(x) \right) \frac{t^n}{n!}.
\end{equation}

Therefore
\begin{equation}
\sum_{n=0}^{\infty} \left( \binom{n}{l} x^l \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{k}{l} B_k^n(x) \right) \frac{t^n}{n!}.
\end{equation}

Comparing the coefficients of \( \frac{t^n}{n!} \) on the both sides of the above equation, we arrive at the desired result. \( \square \)

3.4. Differentiating the Bernstein basis functions. In this section we give higher order derivatives of the Bernstein basis functions. We begin by observing that
\begin{equation}
f_{B,k}(x, t) = g_k(t, x)h(t, x),
\end{equation}
where
\begin{equation}
g_k(t, x) = \frac{t^k x^k}{k!}
\end{equation}
and
\begin{equation}
h(t, x) = e^{(1-x)t}.
\end{equation}

Using Leibnitz’s formula for the \( l \)th derivative, with respect to \( x \), we obtain the following higher order partial differential equation:
\begin{equation}
\frac{\partial^l f_{B,k}(x, t)}{\partial x^l} = \sum_{j=0}^{l} \left( \binom{l}{j} \frac{\partial^j g_k(t, x)}{\partial x^j} \right) \left( \frac{\partial^{l-j} h(t, x)}{\partial x^{l-j}} \right).
\end{equation}

From this equation, we arrive at the following theorem:

**Theorem 6.**
\begin{equation}
\frac{\partial^l f_{B,k}(x, t)}{\partial x^l} = \sum_{j=0}^{l} \left( \binom{l}{j} \right) (-1)^{l-j} t^j f_{B,k-j}(x, t).
\end{equation}

**Proof.** Formula (3.8) follows immediately from (3.7). \( \square \)

Applying Theorem 6, we obtain a new derivation for the higher order derivatives of the Bernstein basis functions.

**Theorem 7.**
\begin{equation}
\frac{d^l B_k^n(x)}{dx^l} = \frac{n!}{(n-l)!} \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} B_{k-j}^{n-l}(x).
\end{equation}

**Proof.** By substituting the right hand side of (2.2) into (3.8), we get
\begin{equation}
\sum_{n=0}^{\infty} \left( \frac{d^l B_k^n(x)}{dx^l} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} B_{k-j}^n(x) \right) \frac{t^{n+l}}{n!}.
\end{equation}
Therefore
\[
\sum_{n=0}^{\infty} \left( \frac{d^r B^n_k(x)}{dx^l} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} \frac{n!}{l!} B_{k-j}^{n-l}(x) \right) \frac{t^n}{n!}
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) on the both sides of the above equation, we arrive at the desired result. \( \square \)

Substituting \( l = 1 \) into (3.9), we arrive at the following standard corollary:

**Corollary 1.**
\[
\frac{d}{dx} B^n_k(x) = n \left( B^{n-1}_{k-1}(x) - B^{n-1}_k(x) \right).
\]
(cf. [1] - [13].

### 3.5. Recurrence Relation

In the previous section we computed the derivative of (3.6) with respect to \( x \) to derive a derivative formula for the Bernstein basis functions. In this section we are going to differentiate (3.6) with respect to \( t \) to derive a recurrence relation for the Bernstein basis functions.

Using Leibnitz’s formula for the \( v \)th derivative, with respect to \( t \), we obtain the following higher order partial differential equation:
\[
\partial^v f_{B,k}(x, t) \over \partial t^v = \sum_{j=0}^{v} \binom{v}{j} \left( \partial^j g_k(t, x) \over \partial t^j \right) \left( \partial^{v-j} h(t, x) \over \partial t^{v-j} \right).
\]
(3.10)

From the above equation, we have the following theorem:

**Theorem 8.**
\[
\frac{\partial^v f_{B,k}(x, t)}{\partial t^v} = \sum_{j=0}^{v} B^v_j(x) f_{B,k-j}(x, t).
\]
(3.11)

**Proof.** Formula (3.11) follows immediately from (3.10). \( \square \)

Using definition (2.3) and (2.1) in Theorem 8, we obtain a recurrence relation for the Bernstein basis functions:

**Theorem 9.**
\[
B^n_k(x) = \sum_{j=0}^{v} B^v_j(x) B^{n-v}_{k-j}(x).
\]
(3.12)

**Proof.** By substituting the right hand side of (2.2) into (3.11), we get
\[
\frac{\partial^v}{\partial t^v} \left( \sum_{n=0}^{\infty} B^n_k(x) \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{v} B^v_j(x) B^{n}_{k-j}(x) \right) \frac{t^n}{n!}.
\]
Therefore
\[
\sum_{n=0}^{\infty} B^n_k(x) \frac{t^{n-v}}{(n-v)!} = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{v} B^v_j(x) B^{n}_{k-j}(x) \right) \frac{t^n}{n!}.
\]
From the above equation, we get
\[
\sum_{n=v}^{\infty} B^v_k(x) \frac{t^{n-v}}{(n-v)!} = \sum_{n=v}^{\infty} \left( \sum_{j=0}^{v} B^v_j(x) B^{n-v}_{k-j}(x) \right) \frac{t^{n-v}}{(n-v)!}.
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) on the both sides of the above equation, we arrive at the desired result. \(\square\)

**Remark 4.** Setting \( v = 1 \) in (3.12), one obtains the standard recurrence
\[
B^n_k(x) = (1 - x)B^{n-1}_k(x) + xB^{n-1}_{k-1}(x).
\]

### 3.6. Degree raising.
In this section we present a functional equation which we apply to provide a new proof of the degree raising formula for the Bernstein polynomials.

From (2.3), we obtain the following functional equation:
\[
(xt)^d f_{\mathbb{B}, k}(x, t) = \frac{(k + d)!}{k!} f_{\mathbb{B}, k+d}(x, t).
\]
Therefore
\[
x^d B^n_k(x) = \frac{n!(k + d)!}{k!(n + d)!} B^{n+d}_{k+d}(x).
\]
Substituting \( d = 1 \) into the above equation, we have
\[
xB^n_k(x) = \frac{k + 1}{n + 1} B^{n+1}_{k+1}(x).
\]
The above relation can also be proved by (2.1) cf. (4, 5, 6).

From (2.3), we also get the following functional equation:
\[
(xt)^{-d} f_{\mathbb{B}, k}(x, t) = \frac{(k - d)!}{k!} f_{\mathbb{B}, k-d}(x, t).
\]
Therefore
\[
(1 - x)^d B^n_k(x) = \frac{n!(n + d - k)!}{(n + d)!(n - k)!} B^{n+d}_{k}(x).
\]
Substituting \( d = 1 \), we have
\[
(1 - x)B^n_k(x) = \frac{(n + 1 - k)}{(n + 1)} B^{n+1}_{k}(x).
\]
Adding (3.14) and (3.15), we get the standard degree elevation formula
\[
B^n_k(x) = \frac{1}{n + 1} \left( (k + 1) B^{n+1}_{k+1}(x) + (n + 1 - k) B^{n+1}_{k}(x) \right).
\]
4. New Identities

In this section, using alternative forms of the generating functions, functional equations and Laplace transform, we give many new identities for the Bernstein basis functions.

Using (2.3), we obtain the following functional equations:

\[ f_{\mathbb{B},k_1}(x,t) f_{\mathbb{B},k_2}(x,t) = \binom{k_1 + k_2}{k_1} \frac{1}{2^{k_1+k_2}} f_{\mathbb{B},k_1+k_2}(x,2t), \quad (4.1) \]

and

\[ f_{\mathbb{B},k}(x,t) f_{\mathbb{B},k}(y,-t) = \frac{(-xyt)^k}{(k!)^2} e^{t(y-x)}. \quad (4.2) \]

**Theorem 10.**

\[ B_{n_1+n_2}^{n_1+n_2}(x) = \frac{2^{n_1+n_2-k_1-k_2}(k_1+k_2)!}{(k_1+k_2)!} \sum_{j=0}^{n} \binom{n}{j} B_j^{n-j}(x) B_k^{n-j}(x). \]

**Proof.** By substituting the right hand side of (2.2) into (4.1), we get

\[ \sum_{n=0}^{\infty} B_{k_1}^{n}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} B_{k_2}^{n}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{n_1+n_2}^{n_1+n_2}(x) \frac{2^{n-k_1-k_2}(k_1+k_2)!t^n}{n!}. \]

Therefore

\[ \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} B_j^{n-j}(x) B_k^{n-j}(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{n_1+n_2}^{n_1+n_2}(x) \frac{2^{n-k_1-k_2}(k_1+k_2)!t^n}{n!}. \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on the both sides of the above equation, we arrive at the desired result. \( \square \)

**Theorem 11.**

\[ (-xy)^k (y-x)^{n-2k} = \frac{(k!)^2}{(n-k)!} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} B_j^k(x) B_k^{n-j}(y) \]

where

\[ (n)_k = n(n-1)\ldots(n-2k+1), \]

and \( (n)_0 = 1. \)

**Proof.** Combining (2.2) and (4.2), we get

\[ \left( \sum_{n=0}^{\infty} B_{k}^{n}(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} (-1)^n B_{k}^{n}(y) \frac{t^n}{n!} \right) = \frac{(-xy)^k}{(k!)^2} \sum_{n=0}^{\infty} \frac{(y-x)^n t^{n+2k}}{n!}. \]

From the above equation, we obtain

\[ \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} B_j^k(x) B_k^{n-j}(y) \right) \frac{t^n}{n!} = \frac{(-xy)^k}{(k!)^2} \sum_{n=0}^{\infty} \frac{(n)_2k (y-x)^{n-2k} t^n}{n!}. \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on the both sides of the above equation, we arrive at the desired result. \( \square \)
Theorem 12. Let $x \neq 0$. For all positive integers $k$ and $n$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} x^{j-k} B_k^{n-j}(x) = \binom{n}{k}.$$ 

Proof. By using (2.4), we obtain

$$\sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = \frac{t^{k+x}}{k!} \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$ 

Therefore

$$\sum_{n=k}^{\infty} \sum_{j=0}^{n-k} \binom{k}{j} x^j B_k^{n-j}(x) \frac{t^n}{n!} = x^k \sum_{n=k}^{\infty} \binom{n}{k} \frac{t^n}{n!}.$$ 

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result.

Theorem 13. For all positive integers $k$ and $n$, we have

$$\sum_{j=0}^{n-k} (-1)^j \binom{n}{j} B_k^{n-j}(x) = (-1)^{n-k} \binom{n}{k} x^n.$$ 

Proof. By using (2.3), we get

$$\sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} = \frac{t^k x^k}{k!} \sum_{n=0}^{\infty} (-1)^n x^n \frac{t^n}{n!}.$$ 

Therefore

$$\sum_{n=k}^{\infty} \sum_{j=0}^{n-k} (-1)^j \binom{k}{j} B_k^{n-j}(x) \frac{t^n}{n!} = \sum_{n=k}^{\infty} \binom{n}{k} (-1)^{n-k} x^n \frac{t^n}{n!}.$$ 

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result.

Theorem 14. For all positive integers $k$ and $n$, we have

$$\sum_{j=0}^{n-k} (-1)^j \binom{n}{j} (1-x)^j B_k^{n-j}(x) = \begin{cases} 
  x^k, & \text{for } x = k, \\
  0, & \text{for } x \neq k.
\end{cases}$$ 

Proof. Proof of Theorem 14 is same as that of Theorem 12. So we omit it.

5. Applications of the Laplace transform to the generating functions for the Bernstein basis functions

In this section, we give some applications of the Laplace transform to the generating functions for the Bernstein basis functions. We obtain interesting series representations for the Bernstein basis functions.
Theorem 15. Let \( x \neq 0 \). For all positive integer \( k \), we have
\[
\sum_{n=0}^{\infty} xB_n^k(x) = 1.
\]

Proof. Integrate equation (2.5) (by parts) with respect to \( t \) from 0 to \( \infty \), we have
\[
\sum_{n=0}^{\infty} B_n^k(x) \frac{1}{n!} \int_0^\infty t^n e^{-t} dt \frac{x^k}{k!} \int_0^\infty t^k e^{-xt} dt.
\]
If we appropriately use the case
\[
x > 0
\]
of the following Laplace transform of the function \( f(t) = t^k \):
\[
\mathcal{L}(t^k) = \frac{k!}{x^{k+1}},
\]
on the both sides of (5.1), we find that
\[
\sum_{n=0}^{\infty} B_n^k(x) = \frac{1}{x}.
\]
From the above equation, we arrive at the desired result. \( \square \)

Theorem 16. Let \( x \neq 0 \). For all positive integer \( k \), we have
\[
\sum_{n=0}^{\infty} (-1)^n B_n^k(x) x^{n+1} = (-1)^k x^k.
\]

Proof. Proof of Theorem 16 is same as that of Theorem 15. That is if we replace \( t \) by \( -t \) in equation (2.4) and integrate by parts with respect to \( t \) from 0 to \( \infty \) and using Laplace transform of the function \( f(t) = t^n \), then we arrive at the desired result. \( \square \)

6. Further Remarks and Observations

Fourier series of the Bernstein polynomials has been studied, without generating functions, by Izumi et al. [8]. They investigated many properties of the Fejer mean of the Fourier series of these polynomials. Fourier transform of the Bernstein polynomials has also been given, without generating functions, by Chui et al. [3]. By replacing \( t \) by \( it \) in (2.4)-(2.6), one may give applications of the Fourier transform to the \textit{complex} generating functions for the Bernstein basis functions.

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REFERENCES

[1] M. Acikgoz and S. Araci, On generating function of the Bernstein polynomials, Numerical Analysis and Applied Mathematics, Amer. Inst. Phys. Conf. Proc. CP1281 (2010), 1141-1143.
[2] S. N. Bernstein, Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités, Comm. Soc. Math. Charkow Sér., 2 t. (1912-1913), 13, 1-2.
[3] C. K. Chui, T.-X. He, and Q. Jiang, Fourier transform of Bernstein-Bézier polynomials, preprint; http://www.iwu.edu/~the/YLMAA11029_Final.pdf.
[4] R. Goldman, An Integrated Introduction to Computer Graphics and Geometric Modeling, CRC Press, Taylor and Francis, New York, 2009.
[5] R. Goldman, Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling, Morgan Kaufmann Publishers, Academic Press, San Diego, 2002.
[6] R. Goldman, Identities for the Univariate and Bivariate Bernstein Basis Functions, Graphics Gems V, edited by Alan Paeth, Academic Press (1995), 149-162.
[7] Lorentz, G. G. (1986) Bernstein Polynomials, Chelsea Pub. Comp. New York.
[8] S.-I. Izumi, M. Satō and S. Uchiyama, Fourier series. XII. Bernstein polynomials, Proc. Japan Acad. 33 (1957), 67-69.
[9] M.-S. Kim, T. Kim, B. Lee, and C.-S. Ryoo, Some Identities of Bernoulli Numbers and Polynomials Associated with Bernstein Polynomials, Advances in Difference Equations, vol. 2010, Article ID 305018, 7 pages, 2010.
[10] G. M. Phillips, Bernstein polynomials based on the q-integers, Ann. Numer Math. 4 (1997), 511-518.
[11] G. M. Phillips, Interpolation and approximation by polynomials, CMS Books in Mathematics/ Ouvrages de Mathématiques de la SMC, 14. Springer-Verlag, New York, 2003.
[12] G.M. Phillips, On generalized Bernstein polynomials, In: D.F. Griffits and G.A. Watson, Editors, Numerical Analysis: A.R. Mitchell 75th Birthday Volume, World Science, Singapore (1996), 263-269.
[13] Y. Simsek and M. Acikgoz, A new generating function of (q-) Bernstein-type polynomials and their interpolation function, Abstr. Appl. Anal., 2010, Article ID 769095 (2010), 12 pages.
[14] Y. Simsek, Interpolation function of generalized q–Bernstein-type basis polynomials and applications, Curves and Surfaces 2011, LNCS 6920, (2011), 647-662, Springer-Verlag Berlin Heidelberg 2011.
[15] Y. Simsek, Construction a new generating function of Bernstein type polynomials, Appl. Math. Comput. 218 (2011), 1072-1076.