Abstract

We give a direct derivation of a proposal of Nekrasov-Shatashvili concerning the quantization conditions of the Toda chain. The quantization conditions are formulated in terms of solutions to a nonlinear integral equation similar to the ones coming from the thermodynamic Bethe ansatz. This is equivalent to extremizing a certain function called Yang’s potential. It is shown that the Nekrasov-Shatashvili formulation of the quantization conditions follows from the solution theory of the Baxter equation, suggesting that this way of formulating the quantization conditions should indeed be applicable to large classes of quantized algebraically integrable models.

1 Introduction

The $N$-body quantum mechanical Hamiltonian

$$H = \sum_{\ell=1}^{N} \frac{p_{\ell}^2}{2} + (\kappa g^2)^{\frac{\hbar}{2}} e^{x_{N} - x_{1}} + \sum_{k=2}^{N} g^{2h} e^{x_{k-1} - x_{k}}.$$  \hspace{1cm} (1.1)$$

is known as the quantum Toda chain. Above, $p_{\ell}$ and $x_{k}$ are quantum observables satisfying the canonical commutation relations $[p_{\ell}, x_{k}] = i\hbar \delta_{\ell,k}$, $g$ and $\kappa$ are coupling constant. When $\kappa = 1$ one deals with the so-called closed Toda chain and, when $\kappa = 0$, with the open Toda chain.

This model appears to be a prototype for an interesting class of integrable models called algebraically integrable models. It was introduced and solved, on the classical level, by Toda [To] in 1967. Then, in 1977, Olshanetsky and Perelomov [OP] constructed the set of $N$ commuting and independent integrals of motion for the closed chain, thus proving the so-called quantum integrability of the model. In 1980-81, Gutzwiller [Gu] was able to build explicitly the eigenfunctions and write down the quantization conditions for small numbers of particles
\( N = 2, 3, 4 \). In particular he expressed the eigenfunctions of the closed chain with \( N \)-sites as a linear combination of the eigenfunctions of the open chain with \( N - 1 \) particles.

A particularly successful approach to the solution of the quantum Toda chain was initiated by Sklyanin. In 1985, Sklyanin applied the quantum inverse scattering method (QISM) to the study of the Toda chain. This led to the development of the so-called quantum separation of variables method. In this novel framework, he was able to obtain the Baxter equations for the model, from which Gutzwiller’s quantization conditions can be obtained. The Baxter equation was later re-derived by Pasquier and Gaudin [GP] with the help of an explicit construction of the so-called Q-operator, similar to the method developed by Baxter for the solution of the eight vertex model [Ba].

In '99 Kharchev and Lebedev [KL] constructed the multiple integral representations for the eigenfunctions of the closed \( N \)-particle Toda chain. Their construction can be seen as a generalization of Gutzwiller’s solution allowing one to express the eigenfunctions of the closed \( N \)-particle Toda chain in terms of those of the open chain with \( N - 1 \) particles for all \( N \). In '09, An [An] completed the picture by proving rigorously that Gutzwiller’s quantization conditions are necessary and sufficient for obtaining a state in the spectrum.

However, the form of the quantization conditions obtained in the above-mentioned works appears to be rather involved. Recently Nekrasov and Shatashvili proposed in [NS] that the quantization conditions for the Toda chain can be reformulated in terms of the solutions to a nonlinear integral equation (NLIE) similar to the equations originating in the thermodynamic Bethe ansatz method. With the help of the solutions to the relevant nonlinear integral equation, Nekrasov and Shatashvili defined a function \( W \) whose critical points are in a one-to-one correspondence with the simultaneous eigenstates of the conserved quantities. This formulation not only seems to be in some respects more efficient than the previous one, it also indicates an amazing universality of the form the quantization conditions may take in integrable models.

The proposal of [NS] was based on rather indirect arguments coming from the study of supersymmetric gauge theories. It seems desirable to derive the proposal more directly from the integrable structure of the model. Our main aim in this note is to give such a derivation. It is obtained from the solution theory of the Baxter equation. In other integrable models there are known connections between the Baxter equation and nonlinear integral equations that look similar to the one that appears here, see e.g. [BLZ, Za, Te]. However, the precise form of the NLIE depends heavily on the analytic properties that the relevant solutions of the Baxter equation must have in the different models. In the present case of a particle system we encounter an interesting new feature: the quantization conditions are not formulated as equations on the zeros of the solutions of the Baxter equation, but instead, they are equations on the poles of the so-called quantum Wronskian formed from two linearly independent solutions of the Bax-
ter equation. The positions \( \delta = (\delta_1, \ldots, \delta_N) \) of these poles are the variables that the Yang’s potential \( W = W(\delta) \) depends on.

It is worth stressing that the Baxter equation or generalizations thereof have a good chance to figure as a universal tool for the study of the spectrum of quantum integrable models. Our method of derivation strongly indicates that similar formulations of the quantization conditions should exist for large classes of quantized algebraically integrable models.

This article is organized as follows. We first recall how the Separation of Variables method reduces the problem to find the eigenstates of the Toda chain to the problem to find a certain set of solutions to the Baxter functional equation specified by strong conditions on the analyticity and the asymptotics of its elements. Then, in Section 3 we explain how Gutzwiller’s quantization conditions can be reformulated in terms of the solutions to a certain NLIE and in terms of the Yang’s potential \( W(\delta) \). The derivation of this reformulation is sketched. The proofs of our claims are presented in the Appendices. In Appendix A we establish the relevant properties of Gutzwiller’s basis of fundamental solutions to the T-Q equation. Then in Appendix B we prove the existence and uniqueness of solutions to the NLIE introduced in Section 3. In Appendix C we give rigorous proofs of the main results presented in Section 3.

Acknowledgements. The authors gratefully acknowledge support from the EC by the Marie Curie Excellence Grant MEXT-CT-2006-042695.

We are happy to dedicate this paper to T. Miwa on the occasion of his 60th birthday.

2 Separation of variables approach to the Toda chain

2.1 Integrability of the Toda chain

The integrability of the Toda chain follows from the existence of Lax matrices

\[
L_n (\lambda) = \begin{pmatrix}
\lambda - p_n & g^h e^{-x_n} \\
-g^h e^{x_n} & 0 \\
\end{pmatrix} \quad [x_n, p_n] = i ,
\]

satisfying a Yang-Baxter equation with a rational, six-vertex type, R-matrix. Thus, the set of \( \kappa \)-twisted monodromy matrices

\[
M (\lambda) = \begin{pmatrix}
1 & 0 \\
0 & \kappa^h \\
\end{pmatrix} L_N (\lambda) \ldots L_1 (\lambda) .
\]

allows one to build the transfer matrix \( T (\lambda) = \text{tr} [M (\lambda)] \) which is the generating function of the set of \( N \) commuting Hamiltonians associated with the Toda chain

\[
T (\lambda) = \lambda^N + \sum_{k=0}^{N-1} (-1)^k \lambda^{N-k} H_k .
\]
The first two Hamiltonians read
\[
H_1 = \sum_{k=1}^{N} p_k = P \quad H_2 = \frac{P^2}{2} - \left\{ \sum_{\ell=1}^{N} \frac{P\ell^2}{2} + g^2 t^h \kappa^h e^x e^{-x} + \sum_{k=2}^{N} g^2 \kappa^h e^x e^{-x} \right\} . \tag{2.4}
\]

Any eigenvector of the transfer matrix defines a polynomial \( t(\lambda) = \prod_{k=1}^{N} (\lambda - \tau_k) \). The \( N \) commuting Hamiltonians are self-adjoint, hence the set \( \{\tau\} \) is necessarily self conjugated: \( \{\tau_k\} = \{\bar{\tau}_k\} \).

### 2.2 Separation of variables

The Separation of Variables (SOV) method was developed for the Toda chain in [SK, GP, KL, An]. The main results of these works may be summarized as follows:

The wave-functions \( \Psi_t(x) = (x_1, \ldots, x_N) \) of any eigenstate to the transfer matrix \( T(\lambda) \) with eigenvalue \( t(\lambda) \) can be represented by means of an integral transformation of the form
\[
\Psi_t(x) = \int_{\mathbb{R}}^{} d\mu(\gamma) \Phi_t(\gamma) \Xi_P(\gamma|x) , \tag{2.5}
\]
where integration is over vectors \( \gamma = (\gamma_1, \ldots, \gamma_{N-1}) \in \mathbb{R}^{N-1} \) with respect to a measure \( d\mu(\gamma) \) first found in [Sk], \( \Xi_P(\gamma|x) \) is an integral kernel for which the explicit expression can be found in [KL], \( P \) is the eigenvalue of the center of mass momentum \( P \) in the state \( \Psi_t \), and \( \Phi_t(\gamma) \) is the wave-function in the so-called SOV-representation. The key feature of the SOV representation is that \( \Phi_t(\gamma) \) takes a factorized form
\[
\Phi_t(\gamma) = \prod_{k=1}^{N-1} q_t(\gamma_k) . \tag{2.6}
\]

The integral transformation (2.5) is constructed in such a way that the eigenvalue equation for the family of operators \( T(\lambda) \) is equivalent to the fact that the function \( q_t(y) \) which represents the state \( \Psi_t \) via (2.5) and (2.6) satisfies the so-called Baxter equation,
\[
t(\lambda)q_t(\lambda) = i^N g^N q_t(\lambda + i\hbar) + \kappa^h (-i)^N g^N q_t(\lambda - i\hbar) . \tag{2.7}
\]

The integral transformation (2.5) can be inverted to express \( \Phi_t(\gamma) \) in terms of \( \Psi_t(x) \). In this way it becomes possible to find the necessary and sufficient conditions that \( q_t(y) \) has to satisfy in order to represent an eigenstate of \( T(\lambda) \) via (2.5) and (2.6). The conditions are [KL, An]

(i) \( t(\lambda) \) is a polynomial of the form \( t(\lambda) = \prod_{k=1}^{N} (\lambda - \tau_k) \), with \( \{\tau_k\} = \{\bar{\tau}_k\} \).

(ii) \( q(\lambda) \) is entire and has asymptotic behavior \( |q(\lambda)| = O\left(e^{-N\pi |\Re(\lambda)|} |\lambda|^1 \prod_{j=1}^{2N} (\Re(\lambda) - \hbar)\right) \).

Above, the \( O \) symbol is uniform in the strip \( \{z : |\Im(z)| \leq \hbar/2\} \). This reduces the problem to construct all the eigenfunctions and finding the complete spectrum of the Toda chain to finding the set \( S \) of all solutions \( (t(\lambda), q_t(\lambda)) \) to the Baxter equation (2.7) that satisfy the conditions (i) and (ii) above.


3 Quantization conditions

It turns out that the Baxter equation (2.7) admits solutions within the class $S$ described above only for a discrete set of choices for the polynomial $t(\lambda)$. This is what expresses the quantization of the spectrum of $T(\lambda)$ within the SOV-framework. Our first aim in this section will be to outline how to reformulate the resulting conditions on $t(\lambda)$ more concretely, following the approaches initiated by Gutzwiller and Pasquier-Gaudin.

It gives useful insight to divide the problem to construct and classify the solutions to the Baxter equation (2.7) which satisfy (i) and (ii) into in two steps. In the first step, one weakens the analytic requirements (ii) on $q(\lambda)$ slightly by allowing $q(\lambda)$ to have a certain number of poles. In this case, it will be possible to find two linearly independent solutions $q^\pm_t(\lambda)$ to (2.7) for arbitrary $t(\lambda)$ satisfying (i). In the second step, one constructs the solution $q(\lambda)$ satisfying (i) and (ii) in the form

$$q(\lambda) = P_+ q^+_t(\lambda) + P_- q^-_t(\lambda),$$

where $P_\pm$ are constants. The requirement that $q(\lambda)$ is entire means that the poles of $q^\pm_t(\lambda)$ must cancel each other in (3.1) which is only possible if $t(\lambda)$ is fine-tuned in a suitable way. This is the origin of the quantization of the spectrum of $T(\lambda)$.

3.1 Gutzwiller’s formulation of the quantization conditions

It turns out that there is a canonical minimal choice for the set of poles of $q^\pm_t(\lambda)$ that one needs to allow. One needs to allow $N$ poles $\delta_1, \ldots, \delta_N$ whose positions are determined by the choice of $t(\lambda)$. More precisely, out of $t(\lambda)$ one constructs the so-called Hill determinant

$$\mathcal{H}(\lambda) = \det \begin{bmatrix} \cdots & \rho^h & \cdots & \cdots & \cdots \\ \cdots & t(\lambda-i\hbar) & \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots & \cdots & \cdots \\ \cdots & \frac{\rho^h}{t(\lambda)} & \cdots & \cdots & \cdots \\ \cdots & \frac{\rho^h}{t(\lambda)} & \cdots & \cdots & \cdots \end{bmatrix},$$

where $\rho := \kappa g^{2N}$. It can be shown that $\mathcal{H}(\lambda)$ admits the representation

$$\mathcal{H}(\lambda) = \prod_{a=1}^{N} \left( \frac{\sinh \frac{\pi}{\hbar} (\lambda - \delta_a)}{\sinh \frac{\pi}{\hbar} (\lambda - \tau_a)} \right),$$

where one chooses $|\Im (\delta_k)| < \hbar/2$. This defines $\delta_1, \ldots, \delta_N$ in terms of $t(\lambda)$. 

\[5\]
Let us then, instead of \( S \) consider the class \( S' \) of solutions to (2.7) which satisfy the conditions (i) and (ii)',

(ii)' \( q(\lambda) \) is meromorphic with set of poles contained in \( \{\delta_1, \ldots, \delta_N\} \) and it has an asymptotic behavior 
\[
|q(\lambda)| = O\left( e^{-\frac{N\pi}{2\hbar}|\Re(\lambda)|} |\lambda|^{\frac{2(3|\Im(\lambda)|-\hbar)}{2N}} \right).
\]

The Baxter equation (2.7) has two linearly independent solutions \( q^\pm(\lambda) \) within \( S' \) for arbitrary \( t(\lambda) \). One possible construction of the solutions \( q^\pm(\lambda) \) goes back to Gutzwiller’s work on the Toda chain. They may be defined as follows:

\[
q^\pm(\lambda) = \frac{Q^\pm(\lambda)}{\prod_{a=1}^N \{ e^{-\frac{N\pi}{2\hbar} \sinh \frac{\pi}{h}(\lambda - \delta_a)}\}}, \tag{3.4}
\]

where

\[
Q^+_t(\lambda) = \left( \kappa g^N \right)^{-i\lambda} K_+(\lambda) e^{-N\frac{\pi}{2} \lambda} \prod_{k=1}^N h^{-\frac{\pi}{h}} \Gamma(1 - i(\lambda - \tau_k)/\hbar), \quad Q^-_t(\lambda) = \frac{g^{iN\lambda} K_- (\lambda) e^{-N\frac{\pi}{2} \lambda}}{\prod_{k=1}^N h^{\frac{\pi}{h}} \Gamma(1 + i(\lambda - \tau_k)/\hbar)}. \tag{3.5}
\]

with \( K_\pm(\lambda) \) being half-infinite determinants:

\[
K_+(\lambda) = \det \begin{bmatrix}
1 & \mathbf{t}^{-1}(\lambda + i\hbar) & 0 & \cdots \\
\rho^\hbar & 1 & \mathbf{t}^{-1}(\lambda + 2i\hbar) & 0 & \cdots \\
\mathbf{t}(\lambda + 2i\hbar) & \mathbf{t}(\lambda + 2i\hbar) & \cdots & \cdots & \cdots \\
0 & \ddots & \ddots & \ddots & \cdots & \cdots \\
\end{bmatrix} \tag{3.6}
\]

and \( K_-(\lambda) = K_+(\overline{\lambda}) \) (recall that \( \{\tau_k\} = \{\overline{\tau_k}\} \)). For the reader’s convenience we have included a self-contained proof that \( q^\pm(t) \in S' \) in Appendix A. It’s worth noting that \( Q^\pm_t \) are linearly independent entire functions whose Wronskian can be evaluated explicitly, cf. Lemma[1]

\[
W \left[ Q^+_t, Q^-_t \right](\lambda) = \kappa^{-i\lambda} g^{-N\hbar} e^{-2N\frac{\pi}{h} \lambda} \cdot \prod_{a=1}^N \left\{ \frac{\hbar}{i\pi} \sinh \frac{\pi}{h} (\lambda - \tau_k) \cdot \mathcal{H}(\lambda) \right\}, \tag{3.7}
\]

It can be shown that the most general solution \( q(\lambda) \in S' \) to the Baxter equation (2.7) may be represented in the form (3.1). The additional requirement that \( q(\lambda) \) should be entire implies

\[
Q^+_t(\delta_a) - \zeta Q^-_t(\delta_a) = 0, \quad \text{for} \quad a = 1, \ldots, N \quad \text{and some} \quad \zeta \in \mathbb{C}, \quad |\zeta| = 1, \tag{3.8}
\]

to be supplemented by the condition that \( \sum_{k=1}^N \delta_k = P \mathbf{[Gu, GP]} \). This formulation of the quantization conditions looks fairly involved. It may be considered as a highly transcendental system of equations on the parameters \( \tau_k \) determining \( t(\lambda) \), in which both \( Q^\pm_t(\lambda) \) and \( \delta_1, \ldots, \delta_N \) have to be constructed from \( t(\lambda) \) by means of (3.5) and (3.2), (3.3) respectively.
3.2 Reformulation in terms of solutions to a nonlinear integral equation

One may note that the set of parameters in $\delta = (\delta_1, \ldots, \delta_N)$ is just as big as the set of parameters in $\tau = (\tau_1, \ldots, \tau_N)$ characterizing the polynomial $t(\lambda)$ appearing on the left hand side of the Baxter equation. The form of the quantization conditions \((3.8)\) suggests that it may be useful to formulate these conditions directly in terms of the parameters $\delta = (\delta_1, \ldots, \delta_N)$ with $t(\lambda) = t(\lambda|\tau(\delta))$ being determined in terms of $\delta$ by inverting the relation $\delta = \delta(\tau)$. A more convenient representation of the quantization conditions \((3.8)\) would then be obtained if one was able to construct the functions $Q^\pm(\lambda)$ more directly as functions of the parameters $\delta = (\delta_1, \ldots, \delta_N)$. In the following, for a given polynomial $\vartheta(\lambda) = \prod_{k=1}^{N}(\lambda - \delta_k)$ with complex conjugated roots, we will construct functions $Q^\pm(\lambda)$. These will be shown to yield solutions the Baxter equation \((2.7)\) via \((3.4)\), with $t_\delta(\lambda)$ being a polynomial whose coefficients depend on the parameters $\delta$.

The functions $Q^\pm(\lambda)$ will be build out of the solutions $Y_\delta(\lambda)$ to the following NLIE,

$$\log Y_\delta(\lambda) = \int_\mathbb{R} d\mu \, K(\lambda - \mu) \ln \left( 1 + \frac{\rho^h Y_\delta(\mu)}{\vartheta(\mu - i\hbar/2)|^2} \right),$$

(3.9)

where

$$K(\lambda) = \frac{\hbar}{\pi (\lambda^2 + \hbar^2)}. \quad (3.10)$$

It will be shown in Appendix \[B\] that the solutions $Y_\delta(\lambda)$ to \((3.9)\) are unique, and that they exist for all tuples $\delta = (\delta_1, \ldots, \delta_N)$ of zeros of Hill determinants $\mathcal{H}(\lambda)$ constructed from polynomials $t(\lambda)$ whose zeroes $t_k$ satisfy $|\Im(t_k)| < \hbar/2$. The function $Y_\delta(\lambda)$ is meromorphic, with its poles accumulating in the direction $|\arg(\lambda)| = \pi/2$ and such that $Y_\delta \to 1$ if $\lambda \to \infty$ for $\lambda$ uniformly away from its set of poles. The properties of $Y_\delta$ allow one to define two auxiliary functions:

$$\ln v_\uparrow(\lambda) = - \int_\mathbb{R} \frac{d\mu}{2\pi} \frac{1}{\lambda - \mu + i\hbar/2} \left( 1 + \frac{\rho^h Y_\delta(\mu)}{\vartheta(\mu - i\hbar/2) \vartheta(\mu + i\hbar/2)} \right),$$

$$\ln v_\downarrow(\lambda - i\hbar) = \int_\mathbb{R} \frac{d\mu}{2\pi} \frac{1}{\lambda - \mu - i\hbar/2} \left( 1 + \frac{\rho^h Y_\delta(\mu)}{\vartheta(\mu - i\hbar/2) \vartheta(\mu + i\hbar/2)} \right). \quad (3.11)$$

Out of $v_\uparrow(\lambda)$ and $v_\downarrow(\lambda - i\hbar)$, we may then construct

$$Q^+(\lambda) = \left( \frac{\kappa g^N e^{-iN\lambda}}{\prod_{k=1}^{N} (1 - i(\lambda - \delta_k))} \right) \prod_{k=1}^{N} \Gamma(1 - i(\lambda - \delta_k) / \hbar), \quad Q^-(\lambda) = \left( \frac{\kappa g^N e^{-iN\lambda}}{\prod_{k=1}^{N} (1 + i(\lambda - \delta_k))} \right) \prod_{k=1}^{N} \Gamma(1 + i(\lambda - \delta_k) / \hbar). \quad (3.12)$$

It is shown in Appendix \[C\] that the functions $Q^\pm(\lambda)$ are entire (cf. Lemma \[4\]). It is also shown there that the functions $q^\pm(\lambda)$ defined from $Q^\pm(\lambda)$ by relations like \((3.4)\) are solutions to the Baxter equation \((2.7)\) which have the right asymptotic behavior to be contained in $\mathcal{S}'$. 


One may therefore take the construction of $Q_{\delta}^\pm(\lambda)$ from the solutions of the nonlinear integral equation (3.9) as a replacement for the construction based on Gutzwiller’s solutions. The quantization conditions (3.8) may now be rewritten in terms of the integral equation (3.9) as a replacement for the construction based on Gutzwiller’s solutions. The equation comes about, the key observation is that the two functions

\[ Q_{\delta}(\lambda) = \kappa^{-i\lambda} \left( \frac{\hbar e^{-\frac{2\pi \lambda}{i\pi g^N}}}{2\pi} \prod_{k=1}^{N} \sinh \left( \frac{\pi}{\hbar} (\lambda - \delta_k) \right) \right). \]  

(3.14)

The quantum Wronskian equation allows one to show that $Q_{\delta}^\pm(\lambda)$ satisfy a Baxter-type equation

\[ t_{\delta}(\lambda)Q_{\delta}^\pm(\lambda) = i^{-N} g^{Nh} Q_{\delta}^\pm(\lambda + i\hbar) + \kappa^{N}(i)^N g^{Nh} Q_{\delta}^\pm(\lambda - i\hbar), \]

(3.15)

where the polynomial $t_{\delta}(\lambda)$ is defined by

\[ t_{\delta}(\lambda) = \left( i\kappa g^{N} \right)^{\hbar} \frac{Q_{\delta}^+ (\lambda - i\hbar) Q_{\delta}^- (\lambda + i\hbar) - Q_{\delta}^+ (\lambda + i\hbar) Q_{\delta}^- (\lambda - i\hbar)}{Q_{\delta}^+ (\lambda) Q_{\delta}^- (\lambda + i\hbar) - Q_{\delta}^+ (\lambda + i\hbar) Q_{\delta}^- (\lambda)}. \]

(3.16)

On the one hand, the Wronskian relation (3.14) allows one to show that $t_{\delta}(\lambda)$ and $Q_{\delta}^\pm(\lambda)$ are related by the Baxter equation (3.15). On the other hand, it also ensures that the residues of the possible poles of $t_{\delta}$ (3.16) vanish. Then, the polynomiality of $t_{\delta}$ is a consequence of the asymptotic behavior of the functions $Q_{\delta}^\pm$. The details of the arguments are found in Appendix C.

In order to see how the quantum Wronskian relation is connected to the NLIE (3.9), let us, starting from a solution $Y_{\delta}(\lambda)$ to (3.9), introduce two functions $v_{\uparrow}(\lambda)$ and $v_{\downarrow}(\lambda)$ via (3.11). On the one hand, noting that the kernel $K(\lambda)$ defined in (3.10) can be written as

\[ K(\lambda) = \frac{1}{2\pi i} \left( \frac{1}{\lambda - i\hbar} - \frac{1}{\lambda + i\hbar} \right). \]
It is easy to see that (3.9) implies
\[ \ln Y_\delta (\lambda) = \ln (v_\uparrow (\lambda + i\hbar/2)) + \ln (v_\downarrow (\lambda - 3i\hbar/2)) \] (3.17)

On the other hand, note that
\[
\ln \left[ v_\uparrow \left( \lambda - \frac{i\hbar}{2} + i0 \right) \right] + \ln \left[ v_\downarrow \left( \lambda - \frac{i\hbar}{2} - i0 \right) \right] =
\]
\[
\left( \int_{\mathbb{R}+i0} - \int_{\mathbb{R}-i0} \right) \frac{d\mu}{2i\pi} \frac{1}{\lambda - \mu} \left( 1 + \frac{\rho^b Y_\delta (\mu)}{\vartheta (\mu - i\hbar/2) \vartheta (\mu + i\hbar/2)} \right)
\]
\[
= 1 + \frac{\rho^b Y_\delta (\lambda)}{\vartheta (\lambda - i\hbar/2) \vartheta (\lambda + i\hbar/2)} .
\] (3.18)

Thus, using that \( v_\uparrow/\downarrow \) are meromorphic on \( \mathbb{C} \), we are able to continue the obtained relation everywhere on \( \mathbb{C} \), leading to the functional relation
\[ v_\uparrow (\lambda) v_\downarrow (\lambda) = 1 + \frac{\rho^b}{\vartheta (\lambda - i\hbar/2) \vartheta (\lambda + i\hbar/2)} v_\uparrow (\lambda + i\hbar) v_\downarrow (\lambda - i\hbar) . \] (3.19)

Rewriting this in terms of \( Q_\delta^\pm \) by means of (3.12) yields the quantum Wronskian equation (3.14).

At the moment we don’t have a direct proof that a solution to (3.9) exists for all choices of \( \vartheta (\lambda) \). We are able, however, to prove that all functions \( Y_\delta (\lambda) \) that can be constructed from Gutzwillers solutions are in fact solutions to (3.9). This implies that all functions \( Y_\delta (\lambda) \) needed for the formulation of the quantization conditions (3.13) can be obtained in this way.

Let us finally note that there is a more direct way (cf. Proposition 5) to reconstruct the Newton polynomials in the zeroes \( \{ \tau_k \} \) of \( t_\delta \) from the solution \( Y_\delta \) to (2.7):
\[ \sum_{p=1}^N \tau_p^k = \sum_{p=1}^N \delta_p^k - k \int_{\mathbb{R}} \frac{d\tau}{2i\pi} \left\{ (\tau + i\hbar/2)^{k-1} - (\tau - i\hbar/2)^{k-1} \right\} \]
\[ \times \ln \left( 1 + \frac{\rho^b Y_\delta (\tau)}{\vartheta (\tau - i\hbar/2)^{k-1}} \right) . \] (3.20)

As one may reconstruct the eigenvalues \( h_k \) of the conserved quantities \( H_k \) from the \( \sum_{p=1}^N \tau_p^k \), this essentially amounts to a reconstruction of the \( h_k \).

### 3.4 Definition of Yang’s potential

It is interesting to notice that the quantization conditions (3.13) characterize the extrema of a certain function \( W (\delta) \) called Yang’s potential in \( [NS] \). This Yang’s potential is defined as \( W (\delta) = W^{\text{inst}} (\delta) + W^{\text{pert}} (\delta), \) where
\[ W^{\text{pert}} (\delta) = i \sum_{k=1}^N \frac{\delta_k^2}{2} \ln \left( \frac{h^{N/h}}{\rho} \right) - \ln \zeta \sum_{k=1}^N \delta_k + \sum_{j,k=1}^N \varpi (\delta_k - \delta_j) - 2i\pi \sum_{k=1}^N n_k , \] (3.21)
where \( z' (\lambda) = \ln \Gamma (1 + i \lambda / \hbar) \), \( n_k \) are some integers parameterizing the eigenstate, and

\[
\mathcal{W}^{\text{inst}} (\delta) = -i \int_{\mathbb{R}} \left\{ \frac{\ln Y_\delta (\mu)}{2} \ln \left( 1 + \frac{\rho^h Y_\delta (\mu)}{|\vartheta (\mu + i \hbar / 2)|^2} \right) + \text{Li}_2 \left( \frac{-\rho^h Y_\delta (\mu)}{|\vartheta (\mu + i \hbar / 2)|^2} \right) \right\} \frac{d\mu}{2i\pi},
\]

where

\[
\text{Li}_2 (z) = \int_{z}^{0} \frac{\ln (1 - t)}{t} dt.
\]

It seems worth emphasizing that, in our treatment, the reformulation of the quantization conditions in terms of Yang’s potential was obtained from the solution theory of the Baxter equation (2.7). This may be seen as a hint towards a more direct understanding of the claim in [NS] that the quantization conditions for large classes of quantized algebraically integrable models can be formulated in this way. The claim should follow quite generally from the solution theory of the Baxter equation.

A Properties of Gutzwiller’s solutions

A.1 Analytic properties of Gutzwiller’s solution

The explicit construction of a set of two linearly independent entire solutions \( Q^\pm_t \) of (2.7) with arbitrary monic polynomial \( t (\lambda) \) goes back to Gutzwiller [Gu]. Prior to writing down these two solutions, recall the definition of the Wronskian of two solutions \( q_1 \) and \( q_2 \) defined by

\[
W [q_1, q_2] (\lambda) = q_1 (\lambda) q_2 (\lambda + i \hbar) - q_2 (\lambda) q_1 (\lambda + i \hbar).
\]

It is straightforward to see, using (2.7), that \( W [q_1, q_2] \) is \( i \hbar \) quasi-periodic:

\[
W [q_1, q_2] (\lambda + i \hbar) = (-1)^N \kappa^h W [q_1, q_2] (\lambda).
\]

**Proposition 1.** Let \( t (\lambda) = \prod_{k=1}^{N} (\lambda - \tau_k) \) be a monic polynomial of degree \( N \) with roots appearing in complex-conjugate pairs \( \{ \tau_k \} = \{ \overline{\tau_k} \} \). Then, the two functions below are entire solutions to the Baxter equation (3.13),

\[
Q^+_t (\lambda) = \frac{(\kappa g)^N e^{-N \frac{\pi}{\hbar} \lambda}}{\prod_{k=1}^{N} \hbar^{-i \frac{\pi}{\hbar} \Gamma(1 - i(\lambda - \tau_k) / \hbar)}}, \quad Q^-_t (\lambda) = \frac{g^{iN \lambda} e^{-N \frac{\pi}{\hbar} \lambda}}{\prod_{k=1}^{N} \hbar^{i \frac{\pi}{\hbar} \Gamma(1 + i(\lambda - \tau_k) / \hbar)}},
\]
Here $K_\pm (\lambda)$ correspond to the unique meromorphic solutions to difference equations

\[ K_+ (\lambda - i \hbar) = K_+ (\lambda) - \frac{\rho^h K_- (\lambda + i \hbar)}{t (\lambda) t (\lambda + i \hbar)}, \quad (A.4) \]

\[ K_- (\lambda + i \hbar) = K_- (\lambda) - \frac{\rho^h K_+ (\lambda - i \hbar)}{t (\lambda) t (\lambda - i \hbar)}. \quad (A.5) \]

that go to 1, when $\lambda \to \infty$ uniformly away from their set of poles. The solutions to these recurrence relations are given explicitly by the determinant formula (3.6) and its complex conjugate.

These two linearly independent solutions are entire and possess the asymptotic behavior

\[ |Q_\pm (\lambda)| = e^{-N \frac{\text{Re}(\lambda)}{\hbar}} O \left( e^{N \frac{\text{Re}(\lambda)}{\hbar}} |\lambda| \frac{N}{2} (\pm 2 \text{Im}(\lambda) - \hbar) \right) \quad \text{for} \quad \text{Re}(\lambda) \to \pm \infty, \quad (A.6) \]

and the $O$ is uniform in every bounded strip of $\mathbb{C}$.

**Proof.** The only non-trivial part concerns the asymptotic behavior of $K_\pm$. It follows as a corollary of Lemma 3. □

**Lemma 1.** The functions $Q_\pm^t$ are linearly independent and their Wronskian is expressed in terms of the Hill determinant (3.2) by (3.7). The latter is closely related to $K_+$ and $K_-

\[ \mathcal{H} (\lambda) = K_+ (\lambda) K_- (\lambda + i \hbar) - \rho^h \frac{K_+ (\lambda + i \hbar) K_- (\lambda)}{t (\lambda) t (\lambda + i \hbar)}. \quad (A.7) \]

Its zeroes form complex conjugated pairs $\{ \delta_k \} = \{ \overline{\tau_k} \}$, belong to the fundamental strip $\{ z : |\text{Im}(z)| < \hbar/2 \}$ and fulfill $\sum_{p=1}^N \tau_p = \sum_{p=1}^N \delta_p$.

**Proof.** The fact that $Q_\pm^t$ are linearly independent is a consequence of the fact that their Wronskian does not vanish identically. The explicit expression for this Wronskian follows after some algebra.

As we have assumed that the set $\{ \tau_k \}$ is self-conjugated, it follows from the determinant representation for $\mathcal{H}$ that $\mathcal{H} (\lambda) = \mathcal{H} (\lambda)$, i.e., the set $\{ \delta_k \}$ is self-conjugated. In its turn, this implies a particular relation between the set of $\delta$’s and $\tau$’s. Namely, computing the $\text{Re}(\lambda) \to + \infty$ asymptotics of $\mathcal{H}$ yields

\[ \sum_{p=1}^N \tau_p = \sum_{p=1}^N \delta_p + in \hbar, \quad \text{for some} \quad n \in \mathbb{N}. \quad (A.8) \]

However, as $\sum \tau_k \in \mathbb{R}$ and $\sum \delta_k \in \mathbb{R}$, the only possibility is $n = 0$. □

We are now in position to prove the

**Lemma 2.** Let $q$ be any meromorphic solution to (2.7). Then, there exists two meromorphic $i \hbar$-periodic functions $P_\pm (\lambda)$ such that

\[ q (\lambda) = P_+ (\lambda) Q_\uparrow^t (\lambda) + P_- (\lambda) Q_-^t (\lambda). \quad (A.9) \]
Proof. Let \( q \) be any meromorphic solution to Baxter’s T-Q equation (2.7). Then consider
\[
\tilde{q} (\lambda) = q (\lambda) - \frac{W [q, Q^{-}_t] (\lambda)}{W [Q^{-}_t, Q^{-}_t] (\lambda)} \cdot Q^{+}_t (\lambda) + \frac{W [q, Q^{+}_t] (\lambda)}{W [Q^{+}_t, Q^{+}_t] (\lambda)} \cdot Q^{-}_t (\lambda).
\]  
(A.10)

The ratio of two Wronskian being \( i \hbar \) periodic, one gets that, by construction
\[
W [\tilde{q}, Q^{+}_t] (\lambda) = W [\tilde{q}, Q^{-}_t] (\lambda) = 0.
\]  
(A.11)

This leads to the system of equations for \( \tilde{q} (\lambda) \):
\[
\begin{pmatrix}
Q^{+}_t (\lambda) & Q^{+}_t (\lambda + i\hbar) \\
Q^{-}_t (\lambda) & Q^{-}_t (\lambda + i\hbar)
\end{pmatrix}
\begin{pmatrix}
-\tilde{q} (\lambda + i\hbar) \\
\tilde{q} (\lambda)
\end{pmatrix} = 0.
\]  
(A.12)

Given any fixed \( \lambda \), there exist non-trivial solutions to (A.12) if only if the determinant of the matrix defining the system vanishes, \( i.e. W [Q^{+}_t, Q^{-}_t] (\lambda) = 0 \). However, it follows from (3.7) that \( W [Q^{+}_t, Q^{-}_t] (\lambda) \) is an entire function that is non-identically zero. Therefore, it can only vanish at isolated points. Hence, we get that \( \tilde{q} (\lambda) \neq 0 \) only at an at most countable set. As \( \tilde{q} (\lambda) \) is meromorphic on \( \mathbb{C} \), \( \tilde{q} = 0 \).

We now provide a rough characterization of the set of zeroes of \( Q^{\pm}_t \). As the \( \Gamma \) function has no zeroes on \( \mathbb{C} \), the only zeroes of \( Q^{\pm}_t \) are those of \( K^{\pm}_t (\lambda) \).

Proposition 2. Assume that \( |\Im (\tau_k)| < h/2 \) and that the set \( \{ \tau_k \} \) is invariant under complex conjugation. Then, the set of zeroes of \( K^{+}_+ (\lambda - i\hbar/2) \) belongs to the half-plane \( \{ z \in \mathbb{C} : \Im (z) < -h/2 \} \) and
\[
\frac{|K^{+}_+ (\lambda - i\hbar/2)|^2}{\mathcal{H}(\lambda - i\hbar/2)} > 1, \quad \text{for } \lambda \in \mathbb{R}.
\]  
(A.13)

A similar statement holds for \( K^{-}_- (\lambda + i\hbar/2) \), namely the set of zeroes of \( K^{-}_- (\lambda + i\hbar/2) \) lies in the half-plane \( \{ z \in \mathbb{C} : \Im (z) > +h/2 \} \) and \( K^{-}_- (\lambda + i\hbar/2) \) does not vanish on \( \mathbb{R} \).

Proof. It follows from the determinant representations that \( K^{+}_+ (\lambda - i\hbar/2) \) has poles at \( \tau_k - i \frac{(2n + 1)\hbar}{2}, n \in \mathbb{N} \), in particular they all belong to the half-plane \( \{ z \in \mathbb{C} : \Im (z) < -h/2 \} \). Also, since the zeroes and poles of the Hill determinant are self-conjugated,
\[
\mathcal{H}(\lambda - i\hbar/2) = \prod_{k=1}^{N} \frac{\cosh \frac{\pi}{\hbar} (\lambda - \delta_k)}{\cosh \frac{\pi}{\hbar} (\lambda - \delta_k)} > 0, \quad \forall \lambda \in \mathbb{R}.
\]  
(A.14)

As the set \( \{ \tau_k \} \) is self-conjugate, it is easy to see that
\[
\overline{K^{+}_+ (\lambda)} = K^{-}_- (\overline{\lambda}) \quad \text{and} \quad t (\lambda - i\hbar/2) t (\lambda + i\hbar/2) = |t (\lambda - i\hbar/2)|^2,
\]  
(A.15)
This allows us to rewrite (A.7) in the form

\[
\frac{|K_+ (\lambda - i\hbar/2)|^2}{\mathcal{H}(\lambda - i\hbar/2)} = 1 + \frac{\rho^h}{\mathcal{H}(\lambda - i\hbar/2)} \left| \frac{K_+ (\lambda + i\hbar/2)}{\mathfrak{t}(\lambda - i\hbar/2)} \right|^2. \tag{A.16}
\]

It follows from (A.14) and (A.16) that there exists a \( c > 0 \) such that \( |K_+ (\lambda - i\hbar/2)| > c \) for \( \lambda \in \mathbb{R} \). Hence, \( K_+ (\lambda - i\hbar/2) \) has no zeroes on \( \mathbb{R} \). As \( K_+ (\lambda - i\hbar/2) \) has manifestly no poles in the half-plane \( \{ z : \Im (z) > -\hbar/2 \} \), one has that

\[
f (\rho) = \int_{\mathbb{R} - i\hbar/2} \frac{d\tau}{2i\pi} \frac{K'_+ (\tau)}{K_+ (\tau)} = \# \left\{ z \in \mathbb{C} : \Im (z) > -\frac{\hbar}{2} \text{ and } K_+ (z) = 0 \right\}. \tag{A.17}
\]

Here, we remind that \( \rho \) is the deformation parameter appearing in (3.6). We also specify that the function \( f (\rho) \) is well defined as \( |K_+|_{\mathbb{R} - i\hbar/2} > c \) and the ratio \( K'_+ / K_+ \) decays at least as \( \lambda^{-(2N+1)} \) at infinity, uniformly in \( \rho \), cf lemma 3. Thus, applying the dominated convergence theorem we obtain that \( f (\rho) \) is continuous in \( \rho \). As it is integer valued, it is constant. The value of this constant is fixed from \( f (0) = 0 \) (as then \( K_+ = 1 \)). This shows that \( K_+ (\lambda) \) has all of its zeros lying below the line \( \mathbb{R} - i\hbar/2 \).

**A.2 Bounds for \( K_+ \)**

**Lemma 3.** Let \( \{\tau_k\} = \{\overline{\tau}_k\} \) and \( |\Im (\tau_k)| < \hbar/2 \), then \( K_\pm \to 1 \) for \( \lambda \to \infty \) uniformly away from its set of poles and

\[
\frac{K'_+}{K_+} (\lambda) = O \left( \lambda^{-(2N+1)} \right), \tag{A.18}
\]

where the \( O \) is uniform as long as \( \rho \) belongs to some fixed compact subset of \( \mathbb{C} \).

**Proof.** As the T-Q equations can be solved explicitly when \( N = 1 \) in terms of Bessel functions, it is enough to consider the case \( N \geq 2 \). We focus on \( K_+ \) as behavior of \( K_- \) follows by complex conjugation.

\( K_+ \) admits the discrete Fredholm series representation:

\[
K_+ (\lambda) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{h_1, \ldots, h_n \in \mathbb{N}} \det_n [M_{h_ah_b} (\lambda)], \tag{A.19}
\]

where we have

\[
M_{ab} (\lambda) = \frac{\delta_{a,b+1}}{\mathfrak{t}(\lambda - i\hbar)} + \frac{\delta_{a,b-1}\rho^h}{\mathfrak{t}(\lambda - i\hbar)}. \tag{A.20}
\]
Then, by Hadamard’s inequality
\[
\left| \sum_{h_1, \ldots, h_n \in \mathbb{N}} \det_n [M_{h_a h_b} (\lambda)] \right| \leq \sum_{h_1, \ldots, h_n \in \mathbb{N}} \left\{ \sum_{a=1}^n \left| M_{h_a h_b} \right|^2 \right\}^{\frac{1}{2}} \leq \prod_{a=1}^n \left\{ \sum_{a, b=1}^n \left| M_{h_a h_b} \right| \right\} \leq u^n (\lambda) \left( 1 + |\rho|^h \right)^n . \tag{A.21}
\]
There we have set
\[
u (\lambda) = \sum_{k=1}^{+\infty} |t (\lambda - ikh)|^{-1} \leq \left( \frac{2}{\hbar} \right)^N \sum_{k=1}^{+\infty} k^{-N} \quad \text{and} \quad \nu (\lambda) = O \left( \lambda^{-\frac{1}{2}} \right) . \tag{A.22}
\]
Hence, we get that
\[
|K_+ (\lambda) - 1| \leq e^{\nu (\lambda) (1 + |\rho|^h)} - 1 , \tag{A.23}
\]
and thus \( K_+ \to 1 \) for \( \lambda \to \infty \) uniformly away from its set of poles. It remains to provide the stronger estimates for its decay at infinity.

Termwise differentiation of the series leads to
\[
|K'_+ (\lambda)| \leq \sum_{n \geq 1} \frac{n}{n!} \left( u (\lambda) (1 + |\rho|^h)^{n-1} \right) \tilde{u} (\lambda) (1 + |\rho|^h) \leq \tilde{u} (\lambda) (1 + |\rho|^h) e^{\nu (\lambda) (1 + |\rho|^h)} ,
\]
where \( \tilde{u} (\lambda) = \sum_{n \geq 1} |t' / t^2 (\lambda - in\hbar)| = O (|\lambda|^{-1}) \). One then takes the derivative of the Hill’s determinant relation (A.7) at \( \lambda - i\hbar/2 \). This leads to
\[
K'_+ (\lambda) = \mathcal{H}' (\lambda) + \partial_\lambda \left\{ \rho^h \frac{K_+ (\lambda + i\hbar) K_- (\lambda)}{t (\lambda) t (\lambda + i\hbar) K_- (\lambda + i\hbar)} \right\} . \tag{A.24}
\]
The uniform estimates in \( \rho \) that we have established combined with the fact that \( \mathcal{H}' (\lambda) = O (\lambda^{-\infty}) \) uniformly in \( \rho \) lead to the desired form of the estimates, with a \( O \) that is uniform as long as \( \rho \) belongs to some compact subset of \( \mathbb{C} \).

\[\square\]

**B Existence and uniqueness of solutions to the nonlinear integral equation**

In this Appendix we prove the existence and uniqueness of solutions to the TBA non-linear integral equation (3.15). Let \( \vartheta (\lambda) = \prod_{k=1}^N (\lambda - \delta_k) \) have its zeroes given by the \( N \) zeroes of the Hill determinant built out of \( t (\lambda) \). Then set
\[
Y_t (\lambda) = \frac{K_+ (\lambda + i\hbar/2) K_- (\lambda - i\hbar/2)}{\mathcal{H} (\lambda - i\hbar/2)} , \quad \vartheta (\lambda - i\hbar/2) \vartheta (\lambda + i\hbar/2) . \tag{B.1}
\]
It is a straightforward consequence of Lemma 3 that \( Y_t \) is a meromorphic function whose poles accumulate in the direction \( |\arg(\lambda)| = \pi/2 \) and such that \( Y_t \to 1 \) for \( \lambda \to \infty \) uniformly away from the set of its poles.

**Proposition 3.** The function \( \ln Y_t \) defined in (B.1) is continuous, positive and bounded on \( \mathbb{R} \). It is the unique solution in this class to the non-linear integral equation (3.9).

**Proof.** We first prove the uniqueness of solutions. Let \( \| \cdot \|_\infty \) stand for the sup norm on bounded and continuous functions on \( \mathbb{R} \). We set \( \mathcal{F} = \{ f \in C^0(\mathbb{R}) : f \geq 0 \text{ and } \|f\|_\infty < +\infty \} \). Then we define the operator \( L \) on \( \mathcal{F} \) by

\[
L[f](\lambda) = \int \limits_\mathbb{R} d\mu \, K(\lambda - \mu) \ln \left( 1 + \frac{\rho e^{f(\mu)}}{\vartheta(\lambda - i\hbar/2)^2} \right).
\]

(B.2)

The mapping \( L \) stabilizes \( \mathcal{F} \). Indeed,

\[
|L[f](\lambda)| \leq \int \limits_\mathbb{R} d\mu \, K(\lambda - \mu) \frac{\rho e^{f(\mu)}}{\vartheta(\lambda - i\hbar/2)^2} \leq \ln \left( 1 + \rho e^{\|f\|_\infty J^{-1}} \right),
\]

(B.3)

where \( J = \inf_{\lambda \in \mathbb{R}} |\vartheta(\lambda - i\hbar/2)|^2 > 0 \), due to \( |\Im(\delta_k)| < \hbar/2 \).

Any solution to the NLIE appears as a fixed point of \( L \) in \( \mathcal{F} \). We shall now prove that \( L \) can have at most one fixed point. This settles the question of uniqueness of solutions to (3.9). This part goes as in [FKS]. Let \( f, g \in \mathcal{F} \), then

\[
|L[f] - L[g](\lambda)| = \left| \int_0^1 dt \int \mathbb{R} d\tau \, K(\lambda - \tau) \frac{\rho e^{g(\tau)+t(f-g)(\tau)}}{|\vartheta(\tau + i\hbar/2)|^2 + \rho e^{g(\tau)+t(f-g)(\tau)}} (f - g)(\tau) \right|
\]

\[
\leq \frac{\rho e^{\max(\|g\|_\infty,\|f\|_\infty)}}{J + \rho e^{\max(\|g\|_\infty,\|f\|_\infty)}} \|f - g\|_\infty < \|f - g\|_\infty.
\]

(B.4)

Hence, \( L \) admits a unique fixed point.

A direct proof of existence of the solutions to (3.9) is possible if \( \rho^h/J < 1 \). In this case it is easily seen that \( L[f](\lambda) \) is a bounded mapping in the sense that it stabilizes all balls in \( \mathcal{F} \) of radius \( R \geq -\ln \left( 1 - \rho^h/J \right) \). In such a case, (B.4) implies that \( L[f] \) is a contractive map on a Banach space. It thus admits a unique fixed point.

However, it is always possible to construct a solution to (3.9) in terms of the the half-infinite determinants \( K_\pm \). Recall that \( \{\tau_k\} \) and hence \( \{\delta_k\} \) are invariant under complex conjugation.
Let
\[ v^\uparrow (\lambda) = K^+ (\lambda) \prod_{k=1}^N \frac{\Gamma(1 - i (\lambda - \delta_k) / \hbar)}{\Gamma(1 - i (\lambda + \tau_k) / \hbar)} \]  
\[ v^\downarrow (\lambda) = K^- (\lambda + i \hbar) \prod_{k=1}^N \frac{\Gamma(i (\lambda - \delta_k) / \hbar)}{\Gamma(i (\lambda - \tau_k) / \hbar)} . \]  
(B.5)  
(B.6)

It follows from Proposition 2 that \( v^\uparrow / v^\downarrow (\lambda - i \hbar / 2) \) are holomorphic and non-vanishing in \( \mathbb{H}_+/− \). Moreover, as \( \sum \delta_k = \sum \tau_k \), we get that \( v^\uparrow / v^\downarrow (\lambda - i \hbar / 2) = 1 + O (\lambda^{-1}) \) in their respective domains of holomorphy. Also, due to the Hill determinant identity (A.7)
\[ |K^+(\lambda - i \hbar / 2)|^2 / \mathcal{H}(\lambda) = v^\uparrow (\lambda - i \hbar / 2) v^\downarrow (\lambda - i \hbar / 2) = 1 + \frac{\rho^h Y_t(\lambda)}{|\vartheta (\lambda - i \hbar / 2)|^2} \]  
(B.7)

We agree upon choosing a determination of \( v^\uparrow / v^\downarrow (\lambda - i \hbar / 2) \) such that
\[
\ln v^\uparrow / v^\downarrow (\lambda - i \hbar / 2) \xrightarrow[\lambda \to +\infty]{} 0 \implies \ln [v^\uparrow v^\downarrow] (\lambda - i \hbar / 2) = \ln v^\uparrow (\lambda - i \hbar / 2) + \ln v^\downarrow (\lambda - i \hbar / 2) .
\]

Thus, for \( \lambda \in \mathbb{R} \), by computing the residues in the upper or lower half-plane and using the decay properties of the integrand at infinity, one sees that \( v^\uparrow \) and \( v^\downarrow \) are recovered from \( Y_t(\lambda) \) via the integral representations (3.11). Hence, with the same choice of branches of logarithm as before (the one that goes to 0 when \( \Re (\lambda) \) goes to +\( \infty \)) we get, on the one hand, that
\[
\ln [v^\uparrow (\lambda + i \hbar / 2) v^\downarrow (\lambda - 3i \hbar / 2)] = \int_\mathbb{R} d\tau K(\lambda - \tau) \ln \left( 1 + \frac{\rho^h Y_t(\tau)}{|\vartheta (\tau - i \hbar / 2)|^2} \right) .
\]  
(B.8)

On the other hand, it is straightforward to check that \( v^\uparrow (\lambda + i \hbar / 2) v^\downarrow (\lambda - 3i \hbar / 2) = Y(\lambda) \).
This proves the existence of the relevant set of solutions to (3.9). \( \square \)

C Baxter Equation and quantization conditions from TBA

We first prove basic properties of the functions \( Q^\pm_\delta \). Then we derive the T-Q equation generated by \( Q^\pm_\delta \) and finally obtain the quantization conditons.

C.1 Analytic properties of \( Q^\pm_\delta \)

Lemma 4. The functions \( Q^\pm_\delta \) defined in (3.12) are entire and have the asymptotic behavior
\[
|Q^\pm_\delta| = e^{-N\lambda/2} \cdot O \left( e^{N\frac{\hbar}{2}} |\Re(\lambda)| |\lambda|^{N/2(\pm 2 \Im(\lambda) - \hbar)} \right) \quad \Re(\lambda) \to \pm \infty \]  
(C.1)

where the O symbol is uniform in \( \{ z : |\Im(z)| \leq \hbar / 2 \} \).
Proof. It is readily seen from the asymptotic behavior in the strip \( \{ z : |\Im (z)| < h \} \) of the solution \( Y_\delta \) to (3.9), that \( v_\uparrow (\lambda) \to 1 \) and \( v_\uparrow (\lambda - ih) \to 1 \) when \( \Re (\lambda) \to \pm \infty \) in the strip \( \{ z : |\Im (z)| \leq h/2 \} \). Then a straightforward computation leads to (C.1). We assume that all the \( \delta \)'s are distinct and, if necessary, take the limit of coinciding \( \delta \)'s at the end of the calculation.

It remains to prove that \( Q_\delta^\pm \) are entire. For this, we show that the products of \( \Gamma \)-functions cancel the poles of \( v_\downarrow/4 \). We need to construct a meromorphic continuation to \( \mathbb{C} \) of \( v_\downarrow/4 \) starting from the strip \( \mathcal{B}^{(1)} \), with \( \mathcal{B}^{(n)} = \{ z : |\Im (z)| < nh \} \). It follows from the very form of the NLIE (3.9) that the solution \( Y_\delta \) is holomorphic in \( \mathcal{B}^{(1)} \). Let us introduce the notation

\[
V_\delta (\mu) = 1 + \frac{\rho^h Y_\delta (\mu)}{|\vartheta (\mu - i h/2)|^2}.
\]  

The only singularities of \( V_\delta'(\mu)/V_\delta(\mu) \) in \( \mathcal{B}^{(1)} \) correspond to the zeroes of \( V_\delta \) and to its poles. The latter are located at \( \mu = \delta_k \pm i h/2, k = 1, \ldots, N \). Hence, as \( V_\delta'(\mu)/V_\delta(\mu) \) is decaying sufficiently fast at infinity, one gets

\[
\frac{V_\delta'(\tau)}{V_\delta(\tau)} = \sum_{z \in \mathcal{B}^{(n)}_1} \frac{n_z}{\lambda - z - i h} + \sum_{z \in \mathcal{B}^{(n)}_2} \frac{n_z}{\lambda - z + i h}
\]

\[
- \sum_{p=1}^{n} \sum_{k=1}^{N} \frac{1}{\lambda - \delta_k - i (2p + 1) h/2} - \sum_{p=1}^{n} \sum_{k=1}^{N} \frac{1}{\lambda - \delta_k + i (2p + 1) h/2}
\]

\[
+ \int_{\mathbb{R}+inh-\epsilon} \frac{d\mu}{2i\pi} \frac{V_\delta'(\mu)/V_\delta(\mu)}{\lambda - \mu - i h} - \int_{\mathbb{R}-inh+\epsilon} \frac{d\mu}{2i\pi} \frac{V_\delta'(\mu)/V_\delta(\mu)}{\lambda - \mu + i h}.
\]  

Above, we have denoted by \( n_z \) the multiplicity of a zero \( z \) of \( V_\delta \) and \( \mathcal{B}^{(n)}_1, \mathcal{B}^{(n)}_2 \) stands for \( \mathcal{B}^{(n)} \cap \mathbb{H}_+, \mathcal{B}^{(n)} \cap \mathbb{H}_- \). It thus follows that \( V_\delta \) has simple poles at \( \delta_k + i h/2 (2n + 1), n \in \mathbb{Z} \). Exactly the same reasoning as before shows that \( v_\downarrow (\lambda - ih) \) has its only simple poles at \( \delta_k - inh, k = 1, \ldots, N \) and \( n \in \mathbb{N}^+ \). Similarly, \( v_\uparrow (\lambda) \) has its only simple poles at \( \delta_k + inh, k = 1, \ldots, N \) and \( n \in \mathbb{N} \). Therefore, these poles are canceled out by the zeroes of the \( \Gamma \)-functions and \( Q_\delta^\pm \) are both entire.

\[\Box\]

### C.2 Baxter equation

**Proposition 4.** The polynomial \( t_\delta (\lambda) \) given in (3.16) is a monic real valued polynomial of degree \( N \). It is such that the functions \( Q_\delta^\pm \) solve the Baxter equation

\[
t_\delta (\lambda) Q_\delta^\pm (\lambda) = i^N g^{Nh} Q_\delta^\pm (\lambda + ih) + \kappa^h (-i)^N g^{Nh} Q_\delta^\pm (\lambda - ih),
\]  

Finally, given any monic real valued polynomial of degree \( N \) with roots in the strip \( \{ z : |\Im (z)| < h/2 \} \), it is always possible to find a complex-conjugation invariant set of parameters \( \{ \delta_k \} \) such that \( t_\delta \) equals to this polynomial.
Proof. We first prove that $Q_\delta^+$ satisfies (C.4) with $t_\delta$ being given by (3.16).

\[
\kappa^{-\lambda} e^{-\frac{2\pi N}{\hbar} \lambda} \prod_{p=1}^{N} \left\{ \frac{\hbar}{\pi} \sinh \frac{\pi}{\hbar} (\lambda - \delta_p) \right\} \cdot t_\delta (\lambda) Q_\delta^+ (\lambda)
\]

\[
= \left( \kappa g^2 N \right)^h Q_\delta^+ (\lambda - i\hbar) \left\{ \kappa^{-\lambda} \left( \frac{\hbar e^{-\frac{2\pi N}{\hbar} \lambda}}{i\pi g^h} \right)^N \prod_{k=1}^{N} \sinh \frac{\pi}{\hbar} (\lambda - \delta_k) + Q_\delta^+ (\lambda + i\hbar) Q_\delta^- (\lambda) \right\} \nonumber
\]

\[
- \left( \kappa g^2 N \right)^h Q_\delta^+ (\lambda + i\hbar) \left\{ Q_\delta^+ (\lambda - i\hbar) Q_\delta^- (\lambda) - \kappa^{-\lambda-h} \left( \frac{i\hbar e^{-\frac{2\pi N}{\hbar} \lambda}}{\pi g^h} \right)^N \prod_{k=1}^{N} \sinh \frac{\pi}{\hbar} (\lambda - \delta_k) \right\} \nonumber
\]

\[
= \kappa^{-\lambda} e^{-\frac{2\pi N}{\hbar} \lambda} \prod_{p=1}^{N} \left\{ \frac{\hbar}{\pi} \sinh \frac{\pi}{\hbar} (\lambda - \delta_p) \right\} \left\{ g^N h^N Q_\delta^+ (\lambda + i\hbar) + \kappa^h g^N (-i)^N Q_\delta^+ (\lambda - i\hbar) \right\} .
\]

In the intermediate steps, we have used the definition of $t_\delta$ and the q-Wronskian equation (3.14).

Now we show that $t_\delta$, as defined in (3.16), is indeed a monic real valued polynomial of degree $N$. We first assume that the $\delta_k$’s are pairwise distinct. The case when several $\delta_k$’s coincide follows by taking the limit in the final formulae. As $Q_\delta^\pm$ are both entire, we get that the only potential poles of $t_\delta (\lambda)$ are located at $\lambda = \delta_k + i n \hbar$, $n \in \mathbb{Z}$. The set of zeroes of $v_{1/4}$ differs necessarily from its set of poles (as follows readily from (C.3) and similar representations for $v_{1/4}$). Hence, $Q_\delta^\pm (\delta_k + in \hbar) \neq 0$, for $k = 1, \ldots, N$ and $n \in \mathbb{Z}$. Therefore, it follows from the Wronskian relation (3.14), that

\[
\frac{Q_\delta^+ (\delta_k)}{Q_\delta^- (\delta_k)} = \frac{Q_\delta^+ (\delta_k + in \hbar)}{Q_\delta^- (\delta_k + in \hbar)} \quad \text{for} \quad k \in [1 \ldots N] \quad \text{and} \quad n \in \mathbb{N} .
\]

This implies that the possible poles of the expression in (3.16) get canceled. It follows that $t_\delta$ is entire. It remains to control its asymptotic behavior. We may express $t_\delta$ in terms of $v_{1/4}$,

\[
t_\delta (\lambda) = v_\tau (\lambda - i\hbar) v_\tau (\lambda) \prod_{a=1}^{N} \left( \lambda - \delta_a \right) - \frac{(\kappa g^2 N)^{2h} v_\tau (\lambda + i\hbar) v_\tau (\lambda - 2i\hbar)}{\prod_{a=1}^{N} \left( \lambda - \delta_a \right) \left( \hbar^2 + (\lambda - \delta_a)^2 \right)} .
\]

Due to the asymptotic behavior of $Y_\delta$ at $\infty$, one can deform the integration contour in the definition $v_{1/4}$ so as to obtain its asymptotic behavior in the whole plane $\lambda \to \infty$, for $\lambda$ uniformly away from the set of poles of $v_{1/4}$.

As we have that $v_{1/4} \to 1$ when $\lambda \to \infty$, we get that $t_\delta \simeq \lambda^N$ when $\lambda \to \infty$. Hence, $t_\delta$ is a monic polynomial of degree $N$. It is real valued for $\lambda \in \mathbb{R}$ as for such $\lambda$’s, $V_\delta (\lambda) \in \mathbb{R}$, what implies that $v_\tau (\lambda - i\hbar) = v_\tau (\lambda)$, i.e $t_\delta (\lambda) = t_\delta (\overline{\lambda})$.

The fact that, in this way, one is able to generate any monic polynomial with roots in the strip $\{ z : |\text{Re}(z)| < \hbar/2 \}$ follows from the uniqueness of solutions to the TBA-NLIE and the construction of the function $v_{1/4}$ in terms of determinants, as given in (B.5)-(B.6).
C.3 The quantization conditions

We now prove that the quantization conditions for the model (conditions on the zeroes of the polynomial \( t_\delta (\lambda) \) for (2.7) to have entire solutions with a prescribed decay as given in point (ii)) can be written down in a TBA-like form. Moreover, as opposed to the Gutzwiller form of the quantization conditions, the ones that will follow only involve one set of parameters. Namely, the zeroes \( \{ \delta_k \} \) of the Hill determinant associated with \( t_\delta (\lambda) \) given in (3.16). We show that under certain reasonable assumptions, it is possible to reconstruct the Newton polynomials in the zeroes of \( t_\delta \) and hence the spectrum of the model. This proves the Nekrasov-Shatashvili conjecture [NS]. We first reconstruct the zeroes of \( t_\delta \).

**Proposition 5.** Let \( t_\delta (\lambda) = \prod_{p=1}^N (\lambda - \tau_a) \) be a polynomial whose zeroes \( \tau_k \) lie in the strip \( \{ z \in \mathbb{C} : |\Im (z)| < \hbar/2 \} \). If \( \{ \delta_k \} \) is the associated set of zeroes of the Hill determinant and \( Y_\delta \) the unique solution to the NLIE (3.9), then the Newton polynomials \( E_k = \sum_{p=1}^N \tau_p \) in the zeroes of \( t_\delta \) are reconstructed by means of formula (3.20) above. The convergence of these integrals is part of the conclusion.

**Proof.** Due to the uniqueness of solutions to the NLIE (3.9), one has that the solution \( Y_\delta \) can be expressed, as in (B.1) in terms of \( K_{\pm}, \mathcal{H} \). The latter determinants are parameterized by the zeroes \( \{ \tau_k \} \) of \( t_\delta \), and the parameters \( \{ \delta_k \} \) appearing in the NLIE (3.9) coincide with the set of zeroes of the Hill determinant. By invoking the continuity of the logarithm on \( \mathbb{R} \) and its decay at infinity, we get

\[
k \int_{\mathbb{R}} \frac{d\mu}{2i\pi} \left\{ \left( \mu + i\hbar/2 \right)^{k-1} - \left( \mu - i\hbar/2 \right)^{k-1} \right\} \ln \left( 1 + \frac{\rho \hbar Y_\delta (\mu)}{|\vartheta (\mu - i\hbar/2)|^2} \right) = - \int \frac{d\mu}{2i\pi} \left\{ (\mu + i\hbar)^k - \mu^k \right\} \left[ \frac{K_+'}{K_+} (\mu) + \frac{K_-'}{K_-} (\mu + i\hbar) - \frac{\mathcal{H}'}{\mathcal{H}} (\mu) \right]
\]

\[
= \int \frac{d\mu}{2i\pi} \mu^k \frac{\mathcal{H}'}{\mathcal{H}} (\mu) = \sum_{p=1}^N (\tau_p^k - \delta_p^k) . \tag{C.8}
\]

In the intermediate steps, we have used the quick decay at infinity of the integrand

\[
\frac{K_+'}{K_+} (\lambda) = O \left( \lambda^{-2N-1} \right) \quad \text{and} \quad \frac{\mathcal{H}'}{\mathcal{H}} (\lambda) = O \left( \lambda^{-\infty} \right) . \tag{C.9}
\]

This allows us to split the integral in three and compute the parts involving \( K_+ \), resp. \( K_- \), by the residues in the upper/lower half plane (thus giving 0). Hence, the only part that gives a non-trivial contribution is the contour integral involving \( \mathcal{H}'/\mathcal{H} \). The only poles that contribute to the
result are located at the zeroes $\delta_k$ of the Hill determinant (they have residue +1) and at the poles $\tau_k$ of the Hill determinant (they have residue -1) that are located in the strip $|\Im (z)| < \hbar/2$.  

This result offers a direct way to recover the spectum of the model from a solution to the TBA equation \ref{eq:3.9}. It remains to derive the set of quantization conditions on the parameters $\delta_k$.

**Theorem 1.** There exists a unique entire solution $q$ to the T-Q equation \ref{eq:2.7} whose asymptotic behavior is as stated in (ii) if and only if the parameters $\{\delta_k\}$ appearing in the TBA NLIE \ref{eq:3.9} satisfy to the quantization conditions given in \ref{eq:3.13}.

**Remark 1.** The solvability of the quantization conditions, the occurrence of complex solutions ($\Im (\delta_k) \neq 0$, $\delta_k \in \{z : |\Im (z)| < \hbar/2\}$, the uniqueness of solutions for a given choice of integers $n_k \in \mathbb{Z}$ are all open questions.

**Proof.** According to lemma \ref{lem:2} any meromorphic solution $q$ to the T-Q equation takes the form

\[
q (\lambda) = \frac{W [q, Q_\delta^+] (\lambda)}{W [Q_\delta^+, Q_\delta^-] (\lambda)} \cdot Q_\delta^+ (\lambda) - \frac{W [q, Q_\delta^-] (\lambda)}{W [Q_\delta^+, Q_\delta^-] (\lambda)} \cdot Q_\delta^- (\lambda).
\]  

(C.10)

Recall that the Wronskian $W [Q_\delta^+, Q_\delta^-] (\lambda)$ is given by \ref{eq:3.14}. It is possible to compute the Wronskians $W [q, Q_\delta^+] (\lambda)$ by using the asymptotic behavior of $q$ and $Q_\delta^\pm$. Due to their $i\hbar$ quasi-periodicity, these Wronskians take the form $W [q, Q_\delta^+] (\lambda) = e^{-N\pi^2 \kappa} e^{-i\lambda w_\pm (\lambda)}$, where $w_\pm (\lambda)$ are entire $i\hbar$-periodic functions. However, using the asymptotic behavior of $q$ and $Q_\delta^\pm$ we get that $w_\pm (\lambda)$ are bounded at infinity in the strip $|\Im (\lambda)| \leq \hbar/2$, and hence on $\mathbb{C}$. They are thus constant. This proves the uniqueness of solutions for a given choice of $\tau_k$’s and hence $\delta_k$’s. Indeed, up to a normalization constant, any solution $q$ satisfying to the requirements stated in point (ii), is of the form

\[
q (\lambda) = e^{\frac{Nz\lambda}{\hbar}} \frac{Q_\delta^+ (\lambda) - \zeta Q_\delta^- (\lambda)}{\prod_{k=1}^N \sinh \frac{\pi}{\hbar} (\lambda - \delta_k)}
\]  

(C.11)

As the solution $q$ is entire, it has a vanishing residue at $\lambda = \delta_k$, $k = 1, \ldots, N$. Therefore, the quantization conditions for the Toda chain appear as the set of $N - 1$ conditions that $q (\lambda)$ has a vanishing residue at $\delta_k$, $k = 1, \ldots, N$ supplemented with the $N^{th}$ quantization condition for the overall momentum : $\sum_{p=1}^N \tau_p = \sum_{p=1}^N \delta_p = P$. Note that it follows from $W [Q_\delta^+, Q_\delta^-] (\delta_k + in\hbar) = 0$ that if $q$ has a vanishing residue at a $\delta_k$ then it also has a vanishing residue at $\delta_k + in\hbar$, $n \in \mathbb{Z}$. Therefore, there is indeed only a finite number $N$ of constraints of the parameters $\delta$. The explicit form of these quantization conditions is then indeed as given in \ref{eq:3.13}.

Conversely, if the quantization conditions are satisfied, then by taking $q$ as in \ref{eq:C.11}, one obtains an entire solution with the desired asymptotics.
References

[An] D. An, "Complete set of Eigenfunctions of the quantum Toda chain.", Lett. Math. Phys. 87 (2009), 209–223.

[Ba] R. J. Baxter, Exactly solved models in statistical mechanics, Academic Press, 1984.

[BLZ] V. Bazhanov, S. Lukyanov, A.B. Zamolodchikov, "Integrable Structure of Conformal Field Theory II. Q-operator and DDV equation." Comm. Math. Phys. 190, (1997), 247-278.

[FKS] A. Fring, C. Korff, and B.J. Schulz, "The ultraviolet Behaviour of Integrable Quantum Field Theories, Affine Toda Field Theory.", Nucl.Phys. B 549 (1999), 579–612.

[GP] M. Gaudin and V. Pasquier, "The periodic Toda chain and a matrix generalization of the Bessel function recursion relations.", J. Phys. A: Math. Gen 25 (1992), 5243–5252.

[Gu] M. C. Gutzwiller, "The quantum mechanical Toda lattice.", Ann. Phys. 124 (1980), 347–381; "The quantum mechanical Toda lattice II.", Ann. Phys. 133 (1981), 304–331.

[KL] S. Kharchev and D. Lebedev, "Integral representation for the eigenfunctions of a quantum periodic Toda chain.", Lett. Math. Phys. 50 (1999), 53–77; "Integral representations for the eigenfunctions of quantum open and periodic Toda chains from QISM formalism". J. Phys. A34 (2001) 2247-2258

[NS] S. L. Shatashvili N. A. Nekrasov, "Quantization of Integrable Systems and Four Dimensional Gauge Theories.", Preprint [arXiv:0908.4052] (hep-th)

[OP] M.A. Olshanetsky and A.M. Perelomov, "Quantum completely integrable systems connected with semi-simple Lie algebras."., Lett. Math. Phys. 2 (1977), 7–13.

[Sk] E.K. Sklyanin, "The quantum Toda chain.", Lect. Notes in Phys. 226 (1985), 196–233.

[Te] J. Teschner, "On the spectrum of the Sinh-Gordon model in finite volume." Nucl. Phys. B799, (2008), 403-429.

[To] W. Toda, "Wave propagation in anharmonic lattices.", J. Phys. Soc. Jap. 23 (1967), 501–506.

[WW] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, Cambridge University Press, 1927.

[Za] Al.B. Zamolodchikov “On the Thermodynamic Bethe Ansatz Equation in Sinh-Gordon Model”. J. Phys. A39, (2006), 12863-12887.