Local Cloning of Arbitrarily Entangled Multipartite States

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We examine the perfect cloning of non-local, orthogonal states with only local operations and classical communication. We provide a complete characterisation of the states that can be cloned under these restrictions, and their relation to distinguishability. We also consider the case of catalytic cloning, which we show provides no enhancement to the set of clonable states.

Cloning and entanglement are key features of quantum information theory. The no-cloning theorem [1,2], for example, has implications for the security of quantum cryptosystems [3]. Similarly, entanglement is typically viewed as a resource that we can use to enhance various processes, such as the security of communication [4]. Significant effort has been put into developing a theory of entanglement, which is elucidated by considering the restriction to local operations and classical communication (LOCC). However, little is known about the intersection of these two ideas, such as the resource requirements for cloning a set of orthogonal states that are distributed between two or more parties. Previous works have shown that cloning maximally entangled states (MESs) is, in principle, possible [5,6], but have not broached the subject of less entangled states.

Cloning is a well-defined operation, where you start with a quantum state which is one of a set of states, \(\{|\psi_i\rangle\}\), but you do not know which. The task is to create a second copy. It is well known [7] that this can only be done perfectly for orthogonal states, i.e.

\[\langle \psi_i | \psi_j \rangle = \delta_{ij}.\]

The key is that the party (or parties) trying to perform the cloning operation must be able to distinguish between the possible states perfectly.

The situation that we wish to tackle here is when an unknown bipartite pure state from the set \(\{|\psi_i\rangle\}\) is distributed between Alice and Bob. Whether cloning can be accomplished depends on what resources they share. The case where Alice and Bob can only perform LOCC, but do not share any other resources has already been studied [8], where it was shown that, provided the states are distillable, they cannot be cloned.

If Alice and Bob share three MESs as a resource, Bob can always teleport his state to Alice, who can distinguish between the complete set of states and then they can recreate two copies of the state on the two remaining MESs. If they share two MESs then a pair of orthogonal states can always be locally distinguished [9] with projective measurements and two copies created from the MESs. For higher dimensional systems the size of the set is often larger than just two states [10].

The case that we are interested in is when Alice and Bob share only a single MES as a resource, as depicted in Fig. 1. This places the question at a fine division where there exists enough entanglement to, in principle, be able to perform the cloning operation but not sufficient that Alice and Bob can afford to make measurements on their state (which would destroy the entanglement). Previous investigations [5,6] have concentrated on restricting the \(\{|\psi_i\rangle\}\) to also being MESs. In particular, it was demonstrated in [6] that if \(|\psi_0\rangle\) is maximally entangled, the whole set of states \(\{|\psi_i\rangle\}\) must also be maximally entangled. It was further proven that the maximal size of the set of bipartite MESs that can be cloned is equal to the size of dimension of Alice’s (or Bob’s) system, \(d\). In that paper, the necessary and sufficient conditions for cloning these states were derived.

In this paper, we present an alternative interpretation of the results in [6] and extend the proof to non-maximally entangled states. As such, we entirely answer the question of which states can be cloned. We will then consider the extension to multiparties, and address the question of whether a catalyst can contribute anything to the cloning process.

We would first like to provide some motivation for which bipartite states can be cloned perfectly by only LOCC and an additional MES. It is reasonable to think that Alice and Bob should be able to distinguish between the states with measurements and only a single round of two-way classical communication (by distinguishing we mean that both Alice and Bob know which state they shared). This is, in essence, because the MES that is shared can be used to replace this single round of measurements and communication [11]. For example, it is
shown in [8] that any two orthogonal two-qubit states (with extension to higher dimensional and multipartite states) can be represented in the form
\[
|\psi_0\rangle = a |00\rangle + b |11\rangle \\
|\psi_1\rangle = c |01\rangle + d |10\rangle
\]
where \(\langle u | u\rangle = 0\), i.e. we can get to states of this form by local rotations only. To distinguish between these states, Alice and Bob, in general, have to perform two rounds of communication, i.e. Alice makes a measurement in the \([|0\rangle, |1\rangle]\) basis and communicates the result to Bob, who can choose to measure in the \([|0\rangle, |1\rangle]\) basis if Alice measured \(|0\rangle\) or in the \([|u\rangle, |u\rangle]\) basis if Alice measured \(|1\rangle\). Then Bob has to communicate his result to Alice so that they both know which state they had. In the special case where \(|u\rangle = |1\rangle\), Alice and Bob are able to distinguish the state by both performing measurements in the \([|0\rangle, |1\rangle]\) basis and then they simultaneously send the results to each other. We therefore expect to find that these states can be cloned under the restrictions of LOCC, but not pairs of states such as
\[
|\psi_0\rangle = a |00\rangle + b |11\rangle \\
|\psi_1\rangle = b |00\rangle - a |11\rangle
\]
(unless \(a = 1/\sqrt{2}\)). We also expect that to be able to perform LOCC cloning, the states should have the same Schmidt coefficients (same entanglement). This is because we need to perform a POVM on the MES in order to reduce its entanglement. This POVM will be independent of which state is being cloned, and hence we expect the coefficients to come out the same.

Before considering how to clone non-MESs we would like to return to the previously discussed results on cloning MESs. In [9] it is shown that for MESs of dimension \(d\), any set \([|\psi_i\rangle]\), of up to \(d\) states, that can be locally copied obeys the relation \(|\psi_j\rangle = (U_j \otimes 1) |\psi_0\rangle\), where
\[
U_j = \sum_{k=0}^{d-1} \omega^{jk} |k\rangle \langle k| \tag{1}
\]
and \(\omega\) is the \(d^{th}\) root of unity. Let us now reinterpret this in terms of distinguishability. By performing the basis change \(|k\rangle \rightarrow \sum_{m} \omega^{mk} |m\rangle\), we observe that \(U_j\) is equivalent, up to local unitaries, to the permutation operation \(P_{-j}\), where
\[
P_j = \sum_{j=0}^{d-1} |j + i \text{ mod } d\rangle \langle j|.
\]
Hence, measuring in this adjusted basis, Alice and Bob can distinguish these states with a single round of communication. This provides the alternative interpretation that if, and only if, Alice and Bob can distinguish between the entire set of MESs to be cloned with a single round of measurements and classical communication, they can be cloned perfectly.

Now, let us move on to the question of LOCC cloning of non-MESs with one MES as a resource. We consider, first, the case when Alice and Bob make projective measurements on the unknown state \(|\psi_0\rangle\) to determine \(i\), and then use the single MES together with a separable state to create two copies of \(|\psi_i\rangle\). To check if this is possible, we just have to verify whether the majorization condition holds [12]. If the MES that we are using as an additional resource is represented by \(|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle |i\rangle\), then the initial state \(|\phi\rangle \langle 00|\) has \(d\) non-zero Schmidt coefficients (all \(1/d\)). As a result, the target state \(|\psi_i\rangle \langle \psi_i|\) must only have \(d\) non-zero Schmidt coefficients out of a possible \(d^2\), i.e. \(|\psi_i\rangle\) must have an effective dimension \(\sqrt{d}\). We reinterpret this by saying that, in order to clone a set of states \([|\psi_i\rangle\]) of dimension \(d\) by this method, we always require a MES of dimension at least \(d^2\), or two MESs of dimension \(d\). This means that we are not able to clone with projective measurements on the unknown state. Instead of performing projective measurements, we shall now consider a more general protocol. The first state that we want to clone can always be written in its Schmidt basis,
\[
|\psi_0\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle |i\rangle
\]
We want to consider the most general possible set of operations that can result in the creation of perfect clones. To do this, we shall imagine Alice and Bob performing a whole series of POVMs, each in response to all the previous results of measurements by either party. For perfect cloning, we must achieve cloning for every single sequence of measurement results. Let us therefore pick one possible sequence of outcomes. The result of these measurements can thus be described by operators \(M \otimes N\) applied by Alice and Bob respectively. If we are to achieve cloning, the following must hold,
\[
M \otimes N \langle \psi_0 | \psi\rangle = \sqrt{\eta} \langle \psi_0 | \psi_0\rangle.
\]
This equation allows \(M\) to be written in terms of \(N\). We choose a representation for \(N\),
\[
N = U_{\lambda} \left( \sum_{i=0}^{2d-1} \beta_i |i\rangle \langle i| \right) U_{\eta},
\]
where \(U_{\lambda}\) and \(U_{\eta}\) are arbitrary unitaries over two qudits. In fact, we find that we can always write
\[
M = \left( \sum_{i=0}^{d-1} \alpha_i |i\rangle \langle i| \right) \otimes 2 \left[ \left( \sum_{i=0}^{d-1} \frac{1}{\alpha_i} |i\rangle \langle i| \right) \otimes 1 \right] \tag{1}
\]
where
\[
N' = U_{\eta} \left( \sum_{i=0}^{2d-1} \frac{1}{\beta_i} |i\rangle \langle i| \right) U_{\eta}^*.
\]
up to some constant of proportionality. The two qu-dits on which $M$ is defined are Alice’s qudits from the unknown state $|ψ_i⟩$ and the MES, $|φ⟩$. $N' \otimes N$ (or more precisely $N$, since $N'$ is completely determined by $N$) then has to be picked such that the other $|ψ_i⟩$s are also cloned. However, we can derive simple conditions from understanding the above equation. In particular, the right-most term in eqn. (1) converts the right-most terms. Recalling the previous results about MESs, we conclude that for $N$ to perform the cloning, $\left[\left(\sum_{j=0}^{d-1} \frac{1}{\sqrt{d}} |j⟩ ⟨j| \right) \otimes 1\right]|ψ_i⟩$ must, up to normalisation, be maximally entangled and distinguishable by a single round of classical communication, i.e.,

$$|ψ_i⟩ = \sum_{j=0}^{d-1} \alpha_j |j⟩ |j+i⟩ = (1 \otimes P_i)|ψ_0⟩.$$ 

Having proved the necessity of the conditions for cloning, we shall now demonstrate sufficiency by providing a protocol that clones these states using a single MES as a resource. The set of (up to) $d$ states $\{|ψ_i⟩\}$ that we want to clone can be written as

$$|ψ_i⟩ = \sum_{j=0}^{d-1} \alpha_j |j⟩ P_i |j⟩.$$ 

When both Alice and Bob perform the operation

$$U = \sum_{m=0}^{d-1} |m⟩ ⟨m| \otimes P_m,$$ (2)

the maximally entangled state is converted to

$$|φ_i⟩ = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} |l⟩ P_i |l⟩$$

as a result of the property $P_j \otimes P_j |φ_i⟩ = |φ_i⟩$. The $|φ_i⟩$s are each a different, orthogonal, state for each of the unknown states $|ψ_i⟩$. Note that, in the case of $d = 2$, $U$ is the controlled-NOT gate. We then have to convert the MES into a less entangled state. This is achieved by Alice applying the measurement operators

$$M_0 = \sum_{j=0}^{d-1} \alpha_j |j⟩ ⟨j|,$$ (3)

$$M_k = P_k M_0 P_k^†.$$ 

If Alice gets the result $k$, then both Alice and Bob apply the correction $P_k^† = P_{-k}$. This performs the required conversion $|φ_i⟩ \rightarrow |ψ_i⟩$, for all possible measurement results $k$, hence completing the cloning protocol. This protocol is easily linked to eqn. (1), because the cloning of MESs can be achieved with a unitary, so $N = N' = U$ (eqn. (2)). Further, the two POVMs that Alice performs on $|ψ_i⟩$ in eqn. (1) commute with this unitary, and hence cancel. This just leaves the measurements $M_i$ (eqn. (3)) applied to the MES.

Such a protocol extends to multiple parties in a straightforward way. For example, the tripartite case has a set of $d^2$ states that can be cloned,

$$|ψ_{ij}⟩ = \sum_{k=0}^{d-1} \alpha_k |k⟩ |P_i |k⟩ |P_j |k⟩ ,$$

provided the three parties also share a state of the form

$$|φ⟩ = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} |l⟩ |l⟩ |l⟩.$$ 

In this case, they follow exactly the same protocol as before, where the third party, Charlie, does exactly the same as Bob does. To justify that these are the only states that can be cloned, we borrow an argument from [5]. The authors describe how to distinguish between multipartite states by reducing the number of parties involved. Consider, for example, the states

$$|ψ_0⟩ = |0⟩_A |Γ_0⟩_{BC} + \cdots + |l⟩_A |Γ_l⟩_{BC}$$

$$|ψ_1⟩ = |0⟩_A |Γ^⊥_0⟩_{BC} + \cdots + |l⟩_A |Γ^⊥_l⟩_{BC}$$

where the states $|Γ⟩$ are not normalised. We can immediately see that Alice and Bob will need to be able to clone the set of bipartite states $\{|Γ_i⟩, |Γ_i^⊥⟩\}$, which simply reduces to the previous condition.

We would also like to tackle the question of catalytic cloning. In this situation, not only are we provided with a MES on which to create the clone, but some other entangled state which can be used in the protocol, but must be returned unchanged at the end of the cloning process.

$$|ψ_i⟩ |φ⟩ |C⟩ \rightarrow |ψ_i⟩ |ψ_i⟩ |C⟩$$

This state, $|C⟩$, acts as a catalyst, in much the same way as conversion between some states can only occur with the help of a catalyst [12]. The states $|ψ_i⟩$ have Schmidt coefficients $α_i^j$, and the catalyst has Schmidt coefficients $β_j$. Repeating the previous argument shows that $\{α_i^1 β_j\} = \{α_i^2 β_j\}$. It is clear that this isn’t true except in the cases that we could already clone. To see this, consider the smallest values of $α_i^1$ and $β_j$ - if $\min α_i^1 \neq \min α_i^2$, then there are values that we cannot account for. We can then progress through the hierarchy of next-largest $α_i^1$, until we find that the sets $\{α_i^1\}$ and $\{α_i^2\}$ must be the same. Hence, a catalyst cannot provide any enhancement in the cloning process.

In conclusion, we have completely characterised what can be achieved with the local cloning of non-local states. The set of clonable states is very restrictive - they must be locally distinguishable with a single round of two-way communication, and they must have the same entanglement. We have demonstrated a protocol that clones all
these states. In addition, we have ruled out the possibility of catalytic cloning.

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