THE FOURTH MOMENT OF DIRICHLET L-FUNCTIONS FOR
THE RATIONAL FUNCTION FIELD

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Abstract. We study the moments of the Dirichlet L-function when defined
over the polynomial ring over finite fields. We find an asymptotic formula to
the fourth moment of the central value of Dirichlet L functions in this context.
We also find a lower bound to the 2kth moment of these L-functions.

1. Introduction

This work deals with function field analogues of recent studies concerning
moments of central values \( L(\frac{1}{2}, \chi) \) of Dirichlet L-functions, where it has been conjectured that as \( \chi \) varies over all (primitive) Dirichlet characters modulo \( Q \), the 2k-th
moment of \( L(\frac{1}{2}, \chi) \) is asymptotically equal to

\[ C_k Q (\log Q)^{k^2}, \quad Q \to \infty \]

for a positive constant \( C_k \). An exact form for \( C_k \) was conjectured by Keating and
Snaith \cite{14} using Random Matrix Theory. The similarity between the statistics
of the zeros of the Riemann zeta function and the eigenvalues of random unitary
matrices chosen uniformly with respect to Haar measure was first observed by Mont-
gomery and Dyson \cite{9, 16, 17}. Keating and Snaith \cite{14} introduced a random matrix
model for the study of L-functions. They suggested that the value distribution of
the L-functions on the critical line is related to the characteristic polynomials of
random unitary matrices. Earlier, Katz and Sarnak \cite{13} conjectured that statistics
of low lying zeros of families of L-functions coincide with the distribution of low-
lying eigenvalues of the classical compact groups. They divided the L-functions into
symmetry types families and found an appropriate conjecture to each one. Since the
family of all Dirichlet L-functions is believed to be unitary (see the recent results
of Katz \cite{12}), Keating and Snaith conjectured an exact formula for the constant \( C_k \)
using moments of characteristic polynomials of random unitary matrices, for which
Conjecture \cite{14} below is the equivalent for the ring of polynomials.

We work with the finite field \( \mathbb{F}_q \), where \( p \) a prime, \( q = p^n \) its power, and the poly-
nomial ring \( \mathbb{A} = \mathbb{F}_q[x] \) over it. For a nonzero polynomial \( Q \in \mathbb{A} \) set \( |Q| = q^{\deg Q} \).
For a Dirichlet character \( \chi \) modulo \( Q \) denote by \( L(s, \chi) \) the associated Dirichlet
L-function (see e.g. \cite{19}).

We first formulate a direct analog in the polynomial ring of the Keating-Snaith
Conjecture.

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Conjecture 1.1. For polynomial $Q$ and $k \in \mathbb{N}$

$$
\frac{1}{\phi^*(Q)} \sum^*_{\chi \pmod{Q}} \left| L \left( \frac{1}{2}, \chi \right) \right|^{2k} \sim a(k) \frac{G^2(k+1)}{G(2k+1)} \prod_{P|Q} \left( \sum_{m \geq 0} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} (\deg Q)^{k^2}
$$

as $\deg Q \to \infty$, when $G(z)$ is the Barnes G-function, $d_k(N)$ is the number of ways to represent $N$ as a product of $k$ factors when $k \in \mathbb{N}$, and $\phi^*(Q)$ is the number of primitive characters modulo $Q$, and $\sum^*$ denotes summation over all primitive characters modulo $Q$. Also

$$a(k) = \prod_P \left( 1 - \frac{1}{|P|} \right)^{k^2} \sum_{m \geq 0} \frac{d_k(P^m)}{|P|^m}.$$

In this paper, the main proofs are under the assumption that $Q$ is an irreducible polynomial, and the summations are over all nontrivial characters. For an irreducible polynomial all nontrivial characters are primitive. Thus, these summations are over all primitive characters. We will calculate the first and second moment of $L(\frac{1}{2}, \chi)$, and obtain an immediate conclusion about their non-vanishing. Our main result concerns the 4-th moment, and is analogous to the result by Heath-Brown [10]:

Theorem 1.2. For all irreducible $Q \in \mathcal{A}$

$$
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 = \frac{q-1}{12q} (\deg Q)^4 + O \left( (\deg Q)^3 \right).
$$

A direct calculation will show

$$a_2 = \zeta(2)^{-1} = 1 - \frac{1}{q}, \quad \frac{G^2(3)}{G(5)} = \frac{1}{12}$$

$$\prod_{P|Q} \left( \sum_{m \geq 0} \frac{d_2(P^m)^2}{|P|^m} \right)^{-1} = 1 + O \left( \frac{1}{|Q|} \right).$$

Hence our Theorem 1.2 is consistent with Conjecture 1.1.

For $k = 2$, the formula

$$
\frac{1}{\phi^*(Q)} \sum^*_{\chi \pmod{Q}} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 \sim \frac{1}{2\pi^2} \prod_{p|Q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} (\log Q)^4
$$

for almost all large $Q$, was proved by Heath-Brown [10], in 1981. In 2005, K. Soundararajan, [23], improved this result by showing that it holds for all large $Q$. Later on Young [26] proved that for a prime modulus $Q$ the main term in the fourth moment is a polynomial in $\log Q$, with a power saving

$$
\frac{1}{\phi^*(Q)} \sum^*_{\chi \pmod{Q}} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 = \sum_{i=0}^{4} c_i (\log Q)^i + O \left( Q^{-\delta} \right),
$$

for certain computable absolute constants $c_i$. For $k > 2$ there are no proven asymptotic results.

Our second result is an analog of the general lower bound of Rudnick and Soundararajan [20], for the $2k$-th moment
Theorem 1.3. Let $k$ be a fixed natural number. Then for all irreducible polynomial $Q$, with a sufficiently large degree

$$\sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^{2k} \gg_k |Q|^{(\deg Q)k^2}.$$ 

Note that a lower bound was recently proved over the integers for rational $k \in (0, 1)$ by V. Chandee and X. Li, $[5]$, and for $1 < k \in \mathbb{R}$, large $Q$ (not necessarily prime) by Radziwiłł and Soundararajan, $[18]$.

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2. An Outline of the Proof of Theorem 1.2

One of the differences between the Dirichlet L function and its function field analog $L(s, \chi)$, where $\chi$ is a character modulo $Q$, is that in the latter case, the L-function is a polynomial in $q^{-s}$ of degree $D := \deg Q - 1$. This will be recalled later on. When evaluating the fourth moment, we work with the squared L function, which brings us to a polynomial of degree $2D$. By using the functional equation we can reduce our L function into a function with a smaller degree, $D$ (section 7). This will simplify the calculation. Now, the mean value of the summation over all characters mod $Q$ is

$$\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 \sim \frac{4}{\phi(Q)} \sum_{\chi \neq \chi_0} \sum_{\deg AB, \deg CD \leq D} \chi(AC) \overline{\chi}(BD) |ABCD|^{1/2}$$

when the remainder term is bounded by $O(D^3)$. The orthogonality relation for characters mod $Q$ implies that only the terms $AC \equiv BD$ remain. We split this sum into diagonal and off-diagonal terms

$$\sum_{AC \equiv BD \pmod{Q}, \deg AB, \deg CD \leq D} |ABCD|^{-1/2} = \sum_{AC \equiv BD \pmod{Q}, \deg AB, \deg CD \leq D} |ABCD|^{-1/2} + \sum_{AC \equiv BD \pmod{Q}, AC \neq BD \pmod{Q}, \deg AB, \deg CD \leq D} |ABCD|^{-1/2}.$$ 

We can calculate the diagonal term

$$\sum_{AC = BD \pmod{Q}, \deg AB, \deg CD \leq D} |ABCD|^{-1/2} = \frac{q-1}{48q} D^4 + O(D^3)$$

by comparison of series coefficients. To estimate the off-diagonal term, we divide the terms $\deg AB, \deg CD \leq D$ into dyadic blocks. We will use a bound for divisor sums over arithmetic progressions

$$\sum_{\deg N \leq x, \ N \equiv A \pmod{Q}} d(N) \ll \frac{q^2x}{\phi(Q)}.$$
which is the polynomial analog for a Theorem of P. Shiu [22] that we will prove in §. For each block we have

\[
\sum_{Z_1-2 \leq \deg AB \leq Z_1} 1 \ll \frac{q^{Z_1+Z_2} (Z_1Z_2)^3}{|Q|},
\]

Therefore, the off-diagonal term is bounded by \( O(D^3) \).

The full version of the above proof is in sections §. Along the way, we shall require function field analogues of results in sieve theory (section §).

3. Background

A survey of number theory over function fields can be found in [19].

3.1. The Ring of Polynomials Over a Finite Field.

Notation. \( A \) will denote the polynomial ring over \( \mathbb{F} \). We will use upper-case letters such as \( A, B, C, D, N, M \) to denote monic polynomials in \( A \), the letter \( P \) to denote an irreducible monic polynomial in \( A \), and lower case letters such as \( x, y, z \) to denote integers.

In all that follow \( \mathbb{F} \) will denote a finite field with \( q \) elements, when \( q = p^n \) for some prime integer \( p \) and \( n \geq 1 \).

Every element \( N \in A \) has the form

\[ N = a_nT^n + a_{n-1}T^{n-1} + \ldots + a_0 \]

if \( a_n \neq 0 \) we say that \( N \) has degree \( n \), notationally \( \deg N = n \). If \( a_n = 1 \) we say that \( N \) is a monic polynomial. We will only work with monic polynomials.

If \( 0 \not= N \in A \), \( A/N \mathbb{A} \) is a finite ring with \( q^{\deg N} \) elements, \(|4|\). The absolute value of a polynomial \( N \in A \) is defined by \( |N| = q^{\deg N} \) if \( N \not= 0 \), or otherwise by \(|0| = 0\).

We denote by \( \phi \) the Euler totient function for polynomial:

\[
\phi(N) = \# \{ M \in A/N A \mid (M, N) = 1 \} = |N| \prod_{P \mid N} \left( 1 - \frac{1}{|P|} \right).
\]

We denote by \( \pi(n) \) the number of monic irreducible polynomials of degree \( n \). The prime polynomial theorem says that

\[
\sum_{\deg N = n} \Lambda(N) = q^n, \quad \Lambda(N) = \begin{cases} 
\deg N; & \text{if } N = P^k, \ P \text{ prime, } k \geq 1 \\
0; & \text{else}
\end{cases}
\]

Therefore we have

\[
n\pi(n) \leq \sum_{d|n} d\pi(d) = q^n,
\]

and in particular \( \pi(n) \leq \frac{q^n}{n} \). Also, the prime polynomial theorem says that \( \pi(n) = \frac{q^n}{n} + O\left( \frac{q^{n/2}}{n} \right) \) (by the Möbius inversion formula).
In section §6 we will use the function $d(N)$ to denote the divisor function for polynomials $d(N) = \sum_{d|N} 1$, the sum is over monic polynomials. It can be calculated, that for an integer $x \geq 1$,

$$\sum_{\text{deg } N \leq x} d(N) = \frac{(x+1)q^{x+1} - q^{x+1} - 1}{q-1}$$

thus we have the bound

$$\sum_{\text{deg } N \leq x} d(N) \ll xq^x.$$

This bound was also proved in the integer ring setting in [2].

We let $p_+(N)$ and $p_-(N)$ denote the greatest and least degree of prime factor of $N$ respectively, while $\Omega(N)$ is the number of prime factors of $N$, taking into account the multiplicativity.

**Definition.** For $x, y, z \geq 0$, and $A, K$ monic polynomials we define

$$\Psi(x, z) = \sum_{\text{deg } N \leq x \atop p_+(N) \leq z} 1,$$

and satisfies

$$\Psi(x, z) = q^x \rho\left(\frac{x}{z}\right) + O\left(q^x\right)$$

where $\rho$ is the Dickman function, for which it is known that $\rho(u) \approx u - u^\beta$, [8].

In section §5 we will use:

**Definition.** The Möbius function for polynomials, $\mu$, is defined by

$$\mu(N) = \begin{cases} (-1)^k; & \text{if } N = P_1 \cdots P_k \text{ for } 1 \leq i < j \leq k : P_i \neq P_j \\ 0; & \text{else} \end{cases}$$

and satisfies

$$\sum_{D|N} \mu(D) = \begin{cases} 1; & N = 1 \\ 0; & \text{else} \end{cases}$$

3.2. **Dirichlet L-Functions.** Let $Q \in \mathcal{A}$, and denote $\mathcal{D} = deg Q - 1$. We will mostly assume that $Q$ is irreducible.

**Definition.** A Dirichlet character modulo $Q$ is a function $\chi$ from $\mathcal{A} \rightarrow \mathbb{C}$ such that

1. $\chi(A + BQ) = \chi(A)$ for all $A, B \in \mathcal{A}$.
2. $\chi(A) \chi(B) = \chi(AB)$ for all $A, B \in \mathcal{A}$.
3. $\chi(A) \neq 0$ if and only if $(A, Q) = 1$.

A character $\chi$ modulo $Q$ is called primitive if its minimal period is $Q$ (for a prime $Q$ all characters are primitive).

Dirichlet characters satisfy the following orthogonal relations: if $\chi, \psi$ Dirichlet characters modulo $Q$ and $A, B \in \mathcal{A}$ are relatively prime to $Q$, then
(1) \[ \sum_{A \in A} \chi (A) \bar{\psi} (A) = \phi (Q) \delta (\chi, \psi). \]

(2) \[ \sum_{\chi \in \hat{A}} \chi (A) \bar{\chi} (B) = \phi (Q) \delta (A, B). \]

The Dirichlet L function is defined for \( \Re (s) > 1 \) by

\[ L (s, \chi) = \sum_{N \text{ monic}} \frac{\chi (N)}{|N|^s}. \]

For each \( \chi \) non-trivial Dirichlet character modulo \( Q \) we may denote \( u = q^{-s} \) and rewrite \( L (s, \chi) \) as

\[ L^* (u, \chi) = \sum_{N \text{ monic}} \chi (N) u^{\deg N} = \sum_{n=0}^{\infty} L_n (\chi) u^n \]

when \( L_n (\chi) = \sum_{\deg N = n} \chi (N) \). Now, if \( m > \mathcal{D} \), for each monic \( M \in A \) such that \( \deg M = m \), we can write uniquely \( M = SQ + R \) when \( \deg R \leq \mathcal{D} \) or \( R = 0 \). Since \( \chi \) is periodic modulo \( Q \)

\[ \sum_{\deg M = m} \frac{\chi (M)}{|M|^s} = q^{-ms} \sum_{\deg R \leq \mathcal{D}} \chi (R) = 0. \]

Hence, we have

\[ L^* (u, \chi) = \sum_{n=0}^{\mathcal{D}} L_n (\chi) u^n, \]

i.e. \( L^* (u, \chi) \) is a polynomial in \( u \) of degree \( \mathcal{D} \).

**Definition.** \( \chi \) is an “even” character if \( \chi (cN) = \chi (N) \) for all \( c \in \mathbb{F}^*, N \in \mathbb{F}[x] \).

If we define

\[ \lambda_\chi = \begin{cases} 1, & \chi \text{ "even"} \\ 0, & \text{else} \end{cases} \]

\[ \Lambda (u, \chi) := (1 - \lambda_\chi u)^{-1} L^* (u, \chi) \]

then by [19], the “completed” L-function \( \Lambda (u, \chi) \) is a polynomial in \( u \) of degree \( \mathcal{D} \) if \( \chi \) is an “even” character, and of degree \( \mathcal{D} - 1 \) otherwise, which satisfies the functional equation

\[ \Lambda (u, \chi) = \epsilon (\chi) \left( q^{1/2} u \right)^{\mathcal{D} - \lambda_\chi} \Lambda \left( \frac{1}{qu}, \bar{\chi} \right) \]

when \( |\epsilon (\chi)| = 1 \). There is a simple proof for the case that \( \chi \) is not “even” in [15], and a complete proof in [19].

### 4. The First and Second Moments

During the study of the Dirichlet L-functions moments for polynomials basic results are the first and second moment. In this section the direct calculations of the first and second moments are brought for completeness. Also, these results give an immediate conclusion about the non-vanishing of \( L \left( \frac{1}{2}, \chi \right) \). In the number field case, it is believed that \( L \left( \frac{1}{2}, \chi \right) \) does not vanish. It has been proved that a positive proportion of the L-values are non-zero (with the latest record showing 34%), [3][4][11].

By denoting

\[ X (Q) = \left\{ \chi \not\equiv \chi_0 \text{ mod } Q \mid L \left( \frac{1}{2}, \chi \right) \neq 0 \right\} \]
for an irreducible \( Q \in \mathcal{A} \), we can deduce (see \( \text{4.1} \))
\[
\frac{|X(Q)|}{|\{\chi \neq \chi_0 \mod Q\}|} \gg \frac{1}{D}.
\]
This bound implies non-vanishing in a non empty set, but one of decreasing size. The possibility of improving this bound to a positive proportion seems intriguing.

**Proposition 4.1.** Let \( Q \in \mathcal{A} \) be an irreducible polynomial. The mean value of the Dirichlet \( L \)-function is
\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} L \left( \frac{1}{2}, \chi \right) = 1 + O \left( q^{-n/2} \right).
\]

**Proof.** We have
\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} L \left( \frac{1}{2}, \chi \right) = \frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \sum_{\deg N < \mathcal{D}} \frac{\chi(N)}{|N|^{1/2}}
\]
we know
\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \chi(N) = \begin{cases} 1 - \frac{1}{\phi(Q)} & N = 1 \mod (Q) \\ -\frac{1}{\phi(Q)} & \text{else} \end{cases}
\]
since \( \deg N < \mathcal{D} \) we have \( N = 1 \mod (Q) \iff N = 1 \). Hence
\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} L \left( \frac{1}{2}, \chi \right) = 1 - \frac{1}{\phi(Q)} - \frac{1}{\phi(Q)} \sum_{n=1}^{D-1} q^{-n/2} q^n
\]
we get the desired result. \( \square \)

**Proposition 4.2.** Let \( Q \in \mathcal{A} \) be an irreducible polynomial. The second moment of the Dirichlet \( L \)-function is
\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^2 = \mathcal{D} + O(1).
\]

**Proof.** We have
\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^2 = \frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \sum_{\deg N, \deg M < \mathcal{D}} \frac{\chi(N) \bar{\chi}(M)}{|NM|^2}
\]
we know
\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \chi(N) \bar{\chi}(M) = \begin{cases} 1 - \frac{1}{\phi(Q)} & N = M \mod (Q) \\ -\frac{1}{\phi(Q)} & \text{else} \end{cases}
\]
since \( \deg N, \deg M < D \) we have \( N \equiv M \pmod{Q} \iff N = M \). Hence

\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^2 = \left( 1 - \frac{1}{\phi(Q)} \right) \sum_{\substack{N=M \\text{deg} N, \deg M < D}} |NM|^{-1/2} \sum_{\substack{NM \neq M \\text{deg} N, \deg M < D}} |NM|^{-1/2}
\]

\[
= \sum_{\deg N < D} |N|^{-1} - \frac{1}{\phi(Q)} \sum_{\deg N, \deg M < D} |NM|^{-1/2}
\]

\[
= \sum_{n=0}^{D-1} q^{-n} q^n - \frac{1}{\phi(Q)} \left( \sum_{n=0}^{D-1} q^{-n/2} q^n \right)^2
\]

\[
= \mathcal{D} - \frac{1}{\phi(Q)} \left( \frac{q^{D/2} - 1}{q^{D/2} - 1} \right)^2
\]

and since \( \phi(Q) = |Q| - 1 = q^D - 1 \), we get

\[
= \mathcal{D} - \frac{1}{q^{D/2} - 1} \left( q^{D/2} - 1 \right)^2 = \mathcal{D} + O(1).
\]

This completes the proof. \( \square \)

**Corollary 4.3.** Let \( Q \in \mathcal{A} \) be an irreducible polynomial, then

\[ |X(Q)| \gg \frac{q^D}{\mathcal{D}}. \]

Since \( |\{ \chi \neq \chi_0 \pmod{Q} \}| = q^D - 1 \), we get

\[ (4.1) \quad \frac{|X(Q)|}{|\{ \chi \neq \chi_0 \pmod{Q} \}|} \gg \frac{1}{\mathcal{D}}. \]

**Proof.** By the Cauchy-Schwarz inequality we have

\[
\sum_{\chi \in X(Q)} \left| L \left( \frac{1}{2}, \chi \right) \right| \leq \sqrt{\sum_{\chi \in X(Q)} 1^2} \sqrt{\sum_{\chi \in X(Q)} \left| L \left( \frac{1}{2}, \chi \right) \right|^2},
\]

hence

\[
|X(Q)| \geq \left( \frac{\sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^2}{\sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^2} \right),
\]

and by Propositions 4.14.2, we arrive at

\[ |X(Q)| \gg \frac{(q^D)^2}{q^D D} = \frac{q^D}{D}. \]
5. The Selberg Sieve for Polynomials

Sieve methods are of general techniques designed to estimate the size of sifted sets of integers. The Eratosthenes sieve was the first method to estimate this size, and around 1946 Atle Selberg introduced a new method for finding upper bounds to the sieve estimate, [21]. This method gives much better bounds than Eratosthenes sieve. The polynomial analog to Selberg sieve was proved by Webb [25]. In this section we present the polynomial analog to the Selberg sieve and prove a special case, similar to the one found in [6]. We will use this case in section 6, where we will prove a Theorem we need for one of our main results, the asymptotic formula of the fourth moment of Dirichlet L-functions in the polynomial ring.

Let \( x \geq 1 \), and let \( a_N \) be a non-negative real number for each \( N \) monic polynomial, which holds

\[
\sum_{\deg N \leq x} a_N = A < \infty.
\]

For each square free polynomial \( D \) let

\[
A_D := \sum_{\deg M \leq x - \deg D} a_{MD} = A\alpha(D) + r(D),
\]

where \( \alpha \) is a multiplicative function with \( 0 \leq \alpha(D) \leq 1 \) for each \( D \). Also define

\[
A(D) := \sum_{(N,D)=1} a_N.
\]

The Selberg sieve:

**Theorem 5.1.** For each \( z \geq 1 \), and \( \mathcal{D} \) a square free polynomial, we have

\[
A(D) \leq \frac{A}{S(D,z)} + R(D,z)
\]

where \( S, R \) are defined by

\[
S(D,z) = \sum_{D \mid \mathcal{D}, \deg D \leq z} \prod_{P \mid D} \frac{\alpha(P)}{1 - \alpha(P)},
\]

\[
R(D,z) = \sum_{D \mid \mathcal{D}, \deg D \leq 2z} 3^{\omega(D)} |r(D)|
\]

and \( \omega(D) = \sum_{P \mid d} 1 \).

5.1. A Special Case. In our examination in section 6 we have a particular case in which we know the following

\[
\alpha(D) = \frac{1}{|D|}, \quad P_z = \prod_{\deg P \leq z} P, \quad r(D) = O(1).
\]

By Selberg sieve, according to the conditions above, we find

\[
A(P_z) \leq \frac{A}{z} + O\left(\frac{q^2z}{z}\right).
\]

**Proposition 5.2.** For \( x \geq 1 \), \( K \) polynomial, we can define

\[
H_K(x) := \sum_{\deg D < x, (D,K)=1} \frac{\mu^2(D)}{\phi(D)}
\]
and then

\[ H_K(x) \geq \prod_{P | K} \left( 1 - \frac{1}{|P|} \right) x. \]

**Proof.** For \( \deg K \geq 1 \) we have

\[
H_1(x) = \sum_{\deg D < x} \frac{\mu^2(D)}{\phi(D)} = \sum_{L | K} \sum_{\deg D < x, (D,K)=L} \frac{\mu^2(D)}{\phi(D)} \\
= \sum_{L | K} \sum_{\deg H < x - \deg L, (H,K)=1} \frac{\mu^2(LH)}{\phi(LH)} \\
= \sum_{L | K} \frac{\mu^2(L)}{\phi(L)} \sum_{\deg H < x - \deg L, (H,K)=1} \frac{\mu^2(H)}{\phi(H)} \\
= \sum_{L | K} \frac{\mu^2(L)}{\phi(L)} H_K(x - \deg L) \\
\leq \left( \sum_{L | K} \frac{\mu^2(L)}{\phi(L)} \right) H_K(x).
\]

Now, since \( \frac{\mu^2}{\phi} \) is a multiplicative function we know

\[
\sum_{L | K} \frac{\mu^2(L)}{\phi(L)} = \prod_{P | K} \left( 1 + \frac{1}{|P| - 1} \right) = \prod_{P | K} \left( 1 - \frac{1}{|P|} \right)^{-1} = \frac{|K|}{\phi(K)}.
\]

We also have

\[
H_1(x) = \sum_{\deg D < x} \frac{\mu^2(D)}{|D|} \prod_{P | D} \left( 1 - \frac{1}{|P|} \right)^{-1} \\
= \sum_{\deg D < x} \frac{\mu^2(D)}{|D|} \prod_{P | D} \left( 1 + \frac{1}{|P|} + \frac{1}{|P|^2} + \cdots \right),
\]

if we define \( \kappa(N) \) to be the maximal square free divisor of \( N \), we get

\[
= \sum_{\deg \kappa(N) < x} \frac{1}{|N|} \geq \sum_{\deg |N| < x} \frac{1}{|N|} = x.
\]

\( \Box \)

**Corollary 5.3.** For \( \alpha(D) = \frac{1}{|D|} \), \( P_z = \prod_{\deg P \leq z} P \) and \( z \geq 1 \), we have

\[ S(P_z, z) \geq z. \]
Proof. For \( y \geq 1 \)

\[
S(P_z, y) = \sum_{D|P_z} \prod_{P|D} \frac{1}{1 - \frac{1}{|P|}} = \sum_{D|P_z} \frac{1}{\phi(D)}
\]

\[
= \sum_{(D, P_z) = \phi(D)\prod_{P|D} \left(1 - \frac{1}{|P|}\right)} \prod_{P|D} (1 - \frac{1}{|P|}) (y + 1)
\]

\[
\geq \prod_{\deg P \leq z} \left(1 - \frac{1}{|P|}\right) y
\]

when \( P^c_z = \prod_{\deg P \leq z} |P| = \prod_{\deg P \leq y} |P| \). If we take \( y = z \) we get the Corollary. \( \square \)

Proposition 5.4. If \( r(D) = O(1) \) for each square free polynomial \( d \), then

\[
R(D, z) = O(q^{2z}).
\]

Proof. By definition

\[
R(\emptyset, y) = \sum_{D_1, D_2 | \emptyset} r([D_1, D_2]) |\lambda_{D_1} \lambda_{D_2}| \leq \sum_{D_1, D_2 | \emptyset} r([D_1, D_2])
\]

\[
\ll \sum_{D_1, D_2 | \emptyset} 1 \leq \sum_{\deg D_1 \leq z} \sum_{\deg D_2 \leq z} 1 = (q^{z+1})^2.
\]

\( \square \)

6. Divisor Sums

In 1980 P. Shiu proved that for arithmetic functions \( f \) which satisfy similar properties as the divisor function for integers, \( 0 < \alpha < \frac{1}{2}, 0 < \beta < \frac{1}{2} \) and \( a, k \) are integers satisfying \( 0 < a < k, (a, k) = 1 \), then, as \( x \to \infty \),

\[
\sum_{x-y<n\leq x \atop n \equiv a \pmod{k}} f(n) \ll \frac{y}{\phi(k)} \frac{1}{\log x} \exp \left( \sum_{p \leq x, p|k} \frac{\sum_{p \leq x} f(p)}{p} \right),
\]

uniformly in \( a, k \) and \( y \) provided that \( k < y^{1-\alpha}, x^\beta < y \leq x \).

Here we give a polynomial analog to a special case of the above, to be used it in section 5.

Theorem 6.1. Let \( 0 < \alpha < 1 \), and let \( A, K \) be monic polynomials, \( K \) prime, satisfying \( (A, K) = 1 \). Then, as \( x \to \infty \)

\[
\sum_{\deg N \leq x \atop N \equiv A \pmod{K}} d(N) \ll \frac{q^x x}{\phi(K)}
\]

uniformly in \( A \) and \( K \) provided that \( \deg K < (1-\alpha)x \).
Note that since it is known that

\[ \sum_{\deg N \leq x} d(N) \ll q^{x}, \quad \frac{\# \{ N : \deg N \leq x, N \equiv A \pmod{K} \}}{\# \{ N : \deg N \leq x \}} \approx \frac{1}{\phi(K)}, \]

we expect to find an approximation of the above form.

To prove this Theorem we shall require the following Lemmas:

6.1. Preliminary Lemmas.

Lemma 6.2. For all sufficiently large \( x \)

\[ \Psi (x, \log q x + \log \log q x) \leq q^{\frac{3x}{(\log q x)^{1/2}}}. \]

Proof. Let \( 0 \leq \delta \leq 1 \). For all large \( y \) we have

\[
\sum_{P \text{ prime} \atop \deg P \leq y} \frac{1}{\deg P} = \sum_{P \text{ prime} \atop \deg P \leq \delta y} \frac{1}{\deg P} + \sum_{P \text{ prime} \atop \delta y \leq \deg P \leq y} \frac{1}{\deg P} \\
\leq q^{\delta y} + (\delta y)^{-2} \sum_{P \text{ prime} \atop \delta y \leq \deg P \leq y} \deg p \\
= q^{\delta y} + (\delta y)^{-2} \sum_{\delta y \leq n \leq y} n \pi (n),
\]

by prime polynomials theorem we know \( n \pi (n) \leq q^n \) (as discussed in subsection 3.1), so we arrive at

\[
\leq q^{\delta y} + (\delta y)^{-2} \sum_{\delta y \leq n \leq y} q^n \\
= q^{\delta y} + (\delta y)^{-2} q^y q^{[\delta y]-1} 1 - 1/q \\
\leq q^{\delta y} + (2 + \epsilon) \frac{q^y}{(\delta y)^2},
\]

for \( \epsilon > 0 \). By setting \( \delta = \sqrt{25/26}, \ \epsilon = 1/12 \), we get

\[
\sum_{P \text{ prime} \atop \deg P \leq y} \frac{1}{\deg P} \leq 2 \frac{1}{6} q^y. \]

We now use Rankin’s method. Let \( \delta > 0 \). We have, for large \( y \),
\[
\Psi(x, y) = \sum_{\deg N \leq x, p_+(N) \leq y} \sum_{\deg N \leq x} \frac{1}{|N|^{\delta}} \\
\leq q^{\delta x} \sum_{\deg N \leq x, p_+(N) \leq y} \frac{1}{|N|^{\delta}} \\
= q^{\delta x} \prod_{\deg P \leq y} \left( 1 + \frac{1}{|P|^{\delta}} + \frac{1}{|P|^{2\delta}} + \ldots \right) \\
= q^{\delta x} \prod_{\deg P \leq y} \left( 1 + \frac{1}{|P|^{\delta} - 1} \right) \\
\leq q^{\delta x} \exp \left( \sum_{\deg P \leq y} \frac{1}{|P|^{\delta}} \right) \leq q^{\delta x} \exp \left( \frac{1}{\delta} \sum_{\deg P \leq y} \frac{1}{\deg P} \right) \\
\leq q^{\delta x} \exp \left( \frac{13 q^y}{6 \delta y^2} \right) \leq q^{\delta x + 13/6 \frac{q^y}{y^2}}.
\]

When \( y = \log_q x + \log_q \log_q x \) and set \( \delta = \frac{3}{2} \log_q^{-1/2} x \), the result follows. \(\square\)

**Lemma 6.3.** Let \( A \) and \( K \) satisfy \( \deg A < \deg K \), \( (A, K) = 1 \). Assume that \( K \) is a prime polynomial which satisfies \( \deg K < x \) and \( z \geq 2 \). Then

\[
\Phi(x, z; K, A) \leq \frac{q^x}{\phi(K) z} + q^{2z}.
\]

**Proof.** We use the Selberg sieve method (section §5) to prove this Lemma. We know

\[
B := \sum_{\deg N \leq x, N \equiv A \pmod{K}} 1 = \frac{q^x}{\phi(K)} < \infty,
\]

also, for each polynomial \( D \)

\[
A_D := \sum_{\deg N \leq x, N \equiv A \pmod{K}, N \equiv 0 \pmod{D}} 1 = B \alpha(D) + r(D),
\]

when \( \alpha(D) = \frac{1}{|D|} \) and \( r(D) = O(1) \). Since \( \deg N \leq x \) and \( K \) prime we know \( (N, K) = 1 \) and can estimate

\[
\# \{ N : \deg N \leq x, N \equiv A \pmod{K}, D \mid N \} = \frac{q^x}{\phi(K) |D|} + O(1),
\]

\( \alpha \) is a multiplicative function with \( 0 \leq \alpha(D) \leq 1 \) for each \( D \). Let

\[
P_z := \prod_{\deg P \leq z} P,
\]

from the polynomial analog to the Selberg sieve, we know
\[ \Phi(x, z; K, A) = \sum_{\deg N \leq x, \ N \equiv A \pmod{K}, \ \rho_{-}(N) > z} 1 = \sum_{\deg N \leq x, \ N \equiv A \pmod{K}, \ (N, P_{z}) = 1} 1 \leq \frac{B}{S(P_{z}, z)} + R(P_{z}, z). \]

By the special case, Corollary 5.3, we get
\[
S(P_{z}, z) = \sum_{D | P_{z}, \deg D \leq z} \prod_{P_{|D}} \frac{1}{|P| - 1} \geq z
\]
since for all \( \deg D \leq z, \ D | P_{z} \), and by Proposition 5.3 we have
\[
R(P_{z}, z) = \sum_{D | P_{z}, \deg D \leq 2z} 3^{\omega(D)} |r(D)| \ll q^{2z}.
\]

\[\square\]

\textbf{Lemma 6.4.} For all integers \( x > 0 \)
\[
\sum_{\deg N \leq x} \frac{d(N)}{|N|} = \frac{(x + 1)(x + 2)}{2}.
\]

\textit{Proof.} We have
\[
\sum_{\deg N \leq x} \frac{d(N)}{|N|} = \sum_{\deg A \leq x} \sum_{\deg B \leq x - \deg A} \frac{1}{|AB|} = \sum_{n=0}^{x} \sum_{\deg A = n} \frac{1}{|A|} \sum_{m=0}^{x-n} \sum_{\deg B = m} \frac{1}{|B|} = \sum_{n=0}^{x} \sum_{m=0}^{x-n} 1 = \sum_{n=0}^{x} x - n + 1 = \frac{(x + 1)(x + 2)}{2}.
\]

\[\square\]

\textbf{Lemma 6.5.} As \( z \to \infty \)
\[
\sum_{\deg N \geq \frac{z}{r}, \ p_{+}(N) \leq \frac{z}{\log q} r} \frac{d(N)}{|N|} \ll z^{2} q^{-r \log r} r
\]
provided that \( 1 \leq r \leq \frac{z}{\log q} \).

\textit{Proof.} Again we use Rankin’s method. Let \( \frac{4}{3} \leq \delta \leq 1 \). We have
Therefore, we have

\[
\sum_{\deg N \geq x} \frac{d(N)}{|N|} \leq q^{(\delta - 1)x} \sum_{\deg N \geq x} \frac{d(N)}{|N|^\delta} \leq q^{(\delta - 1)x} \sum_{N \text{ monic}} \frac{d(N)}{|N|^\delta} = q^{(\delta - 1)x} \prod_{\deg P \leq y} \left( \sum_{l=0}^{\infty} \frac{d(P^l)}{|P|^{\delta l}} \right) \ll q^{(\delta - 1)x} \exp \left( 2 \sum_{\deg P \leq y} \frac{1}{|P|^{\delta}} \right).
\]

Now, \( \frac{1}{|P|^\delta} = \frac{1}{|P|} + \frac{1}{|P|^{1 - \delta}} \). By the Taylor series for the exponential function

\[
\sum_{\deg P \leq y} \frac{1}{|P|} \left( |P|^{1 - \delta} - 1 \right) \leq \sum_{\deg P \leq y} \frac{1}{|P|} \sum_{n=1}^{\infty} \left( (1 - \delta) \ln |P| \right)^n n!
\]

\[
\leq \sum_{n=1}^{\infty} \frac{(1 - \delta)^n}{n!} \ln^n q \sum_{\deg P \leq y} \frac{\deg P}{|P|}
\]

\[
\leq 2 \sum_{n=1}^{\infty} \frac{(1 - \delta)^n}{n!} \left( (1 - \delta) \ln q \right)^n
\]

\[
\leq 2 \exp \left( (1 - \delta) \ln q \right) = 2q^{(1 - \delta)y}
\]

when the third inequality is due to PNT, since \( n \pi(n) \leq q^n \) we get

\[
\sum_{\deg P \leq y} \frac{\deg P}{|P|} = \sum_{n \leq y} \frac{n}{q^n} \pi(n) \leq \sum_{n \leq y} 1 = y + 1 \leq 2y.
\]

Therefore, we have

\[
\sum_{\deg N \geq x} \frac{d(N)}{|N|} \ll q^{(\delta - 1)x} \exp \left( 2 \sum_{\deg P \leq y} \frac{1}{|P|} + 2q^{(1 - \delta)y} \right)
\]

\[
\ll q^{(\delta - 1)x} \exp \left( 2 \log y + 4q^{(1 - \delta)y} \right).
\]

Now put \( x = \frac{\delta}{4}, y = \frac{\delta}{4} \) and set \( \delta = 1 - \frac{r \log_2 r}{4z} \). Note that if \( 1 \leq r \leq \frac{x}{\log x} \) then \( r \log_r r < z \) and so \( \frac{3}{4} < \delta \leq 1 \).

\[
\sum_{\deg N \geq x} \frac{d(N)}{|N|} \ll q^{-\frac{r \log_2 r}{4z} \frac{3}{2} z^2} \exp \left( 4q^{\frac{r \log_2 r}{4z} \frac{3}{2}} \right) = z^2 q^{-\frac{3}{8} r \log_2 r} e^{4z^2}
\]

and the result follows. \( \square \)

6.2. **Proof of Theorem 1.2** Let \( K \) as required above, and put \( z = \frac{\delta}{4} \).

For each \( N \) satisfying \( \deg N \leq x \), \( N \equiv A \mod K \) we express \( N \) in the form

\[
N = P_1^{x_1} \cdots P_j^{x_j} P_{j+1}^{x_{j+1}} \cdots P_1^{x_1} = B_N D_N, \quad (\deg P_1 \leq \deg P_2 \leq \cdots \leq \deg P_l)
\]

where \( B_n \) is chosen so that

\[
\deg B_N \leq z < \deg B_N P_j^{x_j+1}
\]
We divide the set of such polynomials into the following classes:

I \quad p_-(D_N) > \frac{z}{2}

II \quad p_-(D_N) \leq \frac{z}{2}, \deg B_N \leq \frac{z}{2}

III \quad p_-(D_N) \leq \log_q z + \log_q \log_q z, \deg B_N > \frac{z}{2}

IV \quad \log_q z + \log_q \log_q z < p_-(D_N) \leq \frac{z}{2}, \deg B_N > \frac{z}{2}.

First we have

$$\sum_{N \in I} d(N) = \sum_{N \in I} d(B_N) d(D_N) \leq \sum_{\deg B \leq z} d(B) \sum_{\deg N \leq x, \atop N \equiv A \pmod{K}, \atop N \equiv 0 \pmod{B}, \atop p_-(D) > \frac{z}{2}} d\left(\frac{N}{B}\right)$$

$$= \sum_{\deg B \leq z} d(B) \sum_{\deg D \leq x - \deg B, \atop D \equiv A' \pmod{K}, \atop p_-(D) > \frac{z}{2}} d(D)$$

where \(A' \equiv A \bar{B}, \bar{B} \equiv 1 \pmod{K}\). By definition we know \(p_-(D) > \frac{z}{2} = \frac{\sqrt{q^*}}{z}\), so that \(\Omega(D) \leq \frac{\sqrt{q^*}}{p_-(D)} < \frac{\sqrt{q}}{z}\), therefore, \(d(D) = 2^{\Omega(D)} \leq 2^{\frac{1}{2}z}\). Hence, we have

$$\sum_{N \in I} d(N) \ll \sum_{\deg B \leq z \atop (B,K)=1} d(B) \sum_{\deg D \leq x - \deg B, \atop D \equiv A' \pmod{K}, \atop p_-(D) > \frac{z}{2}} 1.$$ 

Since \(|K| < q^{(1-\alpha)x}\) and \(|B| \leq q^* < q^{\alpha x}\), so that \(|KB| < x\), it follows from Lemma 6.3 that \(\Phi(x - \deg B, \frac{z}{2}; K, A') \leq \frac{2q^x}{\phi(K)z} + q^z\), and accordingly

$$\sum_{N \in I} d(N) \ll \left(\frac{2q^x}{\phi(K)z} + q^z\right) \sum_{\deg B \leq z \atop (B,K)=1} d(B) \frac{1}{|B|}.$$ 

From Lemma 6.4 we arrive at

$$\sum_{N \in I} d(N) \ll \left(\frac{2q^x}{\phi(K)z} + q^z\right) z^2.$$ 

Next, to each \(N \in \Pi\), there are corresponding \(P\) and \(s\) such that \(P^s \parallel N\), \(\deg P \leq \frac{z}{2}\) and \(\deg P^s > \frac{z}{2}\) (for example, \(P_{j+1}\) will hold). Let \(s_p\) denote the least positive integer \(s\) satisfying \(\deg P^s > \frac{z}{2}\) so that \(s_p \geq 2\) and hence \(|P|^{-s_p} \leq \min\left(q^{-\frac{z}{2}}, |P|^{-2}\right)\). Thus

$$\sum_{\deg P \leq \frac{z}{2}} \frac{1}{|P|^{-s_p}} \leq \sum_{\deg P \leq \frac{z}{2}} q^{-\frac{z}{2}} + \sum_{\deg P > \frac{z}{2}} |P|^{-2} \ll q^{-\frac{z}{2}}.$$ 

It now follows that
\[
\sum_{N \equiv A \pmod{K}, N \equiv 0 \pmod{P}} \sum_{\deg P \leq \frac{z^2}{4}} 1 = \sum_{N \equiv A \pmod{K}, N \equiv 0 \pmod{P}} \left( \frac{q^x}{|K| |P^*|} + O(1) \right) \ll \frac{q^x}{|K|} q^{-\frac{x}{4}} + q^{\frac{x}{4}}.
\]

Suppose next that \(N \in III\). \(B\) exists such that \(B \mid N, \frac{x}{2} < \deg B \leq z\), and \(p_+(B) < \log_q z + \log_q \log_q z\) (for example, \(B_N\)). Consequently we have

\[
\sum_{N \in III} 1 \leq \sum_{\frac{x}{2} < \deg B \leq z} \sum_{p_+(B) < \log_q x + \log_q \log_q x} \sum_{N \equiv A \pmod{K}, N \equiv 0 \pmod{B}} 1
\]

\[
= \sum_{\frac{x}{2} < \deg B \leq z} \left( \frac{q^x}{|K| |B|} + O(1) \right)
\]

\[
\leq \frac{q^x}{|K|} q^{-\frac{x}{4}} \Psi(z, \log_q z + \log_q \log_q z) + O(q^x)
\]

\[
\ll \frac{q^x}{|K|} q^{-\frac{x}{4}} + q^x
\]

by Lemma 6.2. Since \(\deg K < (1 - \alpha)x\) we have, by the definition of \(z\),

\[
q^x < q^{\alpha x} q^{-\frac{x}{4}} < \frac{q^x}{|K|} q^{-\frac{x}{4}}.
\]

And so, out of the last two inequality

\[
\sum_{N \in I} 1 + \sum_{N \in III} 1 \ll \frac{q^x}{|K|} q^{-\frac{x}{4}}.
\]

We have (by the properties of the divisor function and the definition of \(z\))

\[
d(N) \ll |N|^{\frac{x}{4}} \leq q^{\frac{x^2}{4}} = q^{\frac{x}{4}}.\]

Consequently,

\[
(6.2) \quad \sum_{N \in I} d(N) + \sum_{N \in III} d(N) \ll \frac{q^x}{|K|} q^{-\frac{x}{4}}.
\]

Lastly we address the class IV. We have

\[
\sum_{N \in IV} d(N) = \sum_{N \in IV} d(B_N) d(D_N) \leq \sum_{\frac{x}{2} < \deg B \leq z} \sum_{N \equiv A \pmod{K}} \sum_{B_N = B, p_-(D_N) > p_+(B)} \sum_{\log_q x + \log_q \log_q z < p_-(D_N) \leq \frac{x}{2}} d(D_N).
\]

Let us denote \(r_0 := \left\lfloor \frac{x}{\log_q x + \log_q \log_q x} \right\rfloor\), so that \(\log_q z + \log_q \log_q z > \frac{x}{r_0 + 1}\). Let \(2 < r < r_0\) and consider these \(N\) for which \(\frac{x}{r_0 + 1} < p_-(D_N) \leq \frac{x}{2}\). For such \(N\), we have \(p_+(B_N) = p_+(B) < p_-(D_N) < \frac{x}{2}\) and moreover, as before,
\[ \Omega(D_N) \leq \frac{x}{p_-(D_N)} \leq \frac{(r + 1)x}{z} = \frac{10(r + 1)}{\alpha} < \frac{20r}{\alpha} \]

so that \( d(D_N) \leq 2^{\Omega(D_N)} \leq \gamma^r \) where \( \gamma = \frac{20}{\alpha} \). It follows that

\[
\sum_{N \in IV} d(N) \leq \sum_{2 \leq r \leq r_0} \gamma^r \sum_{\frac{\phi}{2} < \deg B \leq \frac{z}{r}} d(B) \sum_{\deg N \leq x} 1 \\
\leq \sum_{2 \leq r \leq r_0} \gamma^r \sum_{\frac{\phi}{2} < \deg B \leq \frac{z}{r}} d(B) \Phi \left( x - \deg B, \frac{z}{r} + 1 ; K, A' \right)
\]

where \( A' \equiv AB, BB \equiv 1 \pmod{K} \). By Lemma 6.3 we have

\[
\Phi \left( x - \deg B, \frac{z}{r} + 1 ; K, A' \right) \leq \frac{q^x (r + 1)}{\phi(K) |B| z} + q^{\frac{2z}{r}}
\]

and therefore

\[
\sum_{N \in IV} d(N) \leq \left( \frac{q^x}{\phi(K) z} + q^{2z} \right) \sum_{2 \leq r \leq r_0} (r + 1) \gamma^r \sum_{\frac{\phi}{2} < \deg B \leq \frac{z}{r}} \frac{d(B)}{|B|}.
\]

By the definition of \( r_0 \) we see that we can apply Lemma 6.5 to the inner sum, giving

\[
(6.3) \quad \sum_{N \in IV} d(N) \ll \left( \frac{q^x}{\phi(K) z} + q^{2z} \right) z^2 \sum_{2 \leq r \leq r_0} (r + 1) \gamma^r q^{-\frac{10}{2} \log_2 r} \ll \frac{q^{3x}}{\phi(K) z} + z^2 q^{2z}.
\]

For \( |K| < x^{1-\alpha} \) we have, by the definition of \( z \),

\[
q^{2z} \ll \frac{|K|}{\phi(K) z} q^{3z} \ll \frac{q^{(1-\alpha)z} q^{3z}}{\phi(K) z} \ll \frac{q^z}{\phi(K) z}.
\]

In respect to 6.1, 6.2 and 6.3 we receive the desired result.

7. Expressing \( L(s, \chi) \) As a Short Sum

One of our main goal is to find an asymptotic formula to the fourth moment of the Dirichlet L-function over the polynomial ring. In this section we reduce this sum into a polynomial of degree \( D \) to simplify the fourth moment problem. We have

\[
|L^* (u, \chi)|^2 = \sum_{\deg N, \deg M \leq D} \chi(N) \bar{\chi}(M) u^{\deg N + \deg M} = \sum_{n=0}^{2D} u^n A_n
\]
when \( A_n := \sum_{\deg N M = n} \chi(N) \bar{\chi}(M) \). One can consider this reduction as the polynomial analogue of the approximate functional equation for the Riemann zeta function \([24]\). Also, there is a similar analogue for the quadratic Dirichlet L–function, \([1]\).

Note, that in our case the formula is an identity rather than an approximation.

**Proposition 7.1.** For \( \mathcal{D} \geq 1 \), \( \chi \) primitive character modulo \( Q \), we have

\[
\left| L\left( \frac{1}{2}, \chi \right) \right|^2 = 2 \sum_{\deg N M < \mathcal{D}} \frac{\chi(N) \bar{\chi}(M)}{|NM|^{1/2}} + \pi(\chi)
\]

when

\[
\pi(\chi) := \begin{cases} 
2q^{-\mathcal{D}/2}(A_{\mathcal{D}-3} - A_{\mathcal{D}-2} + A_{\mathcal{D}-1}) & \chi \text{ "even"} \\
q^{-\mathcal{D}/2}A_{\mathcal{D}} & \text{else}.
\end{cases}
\]

**Proof.** We divide to two cases:

First, if \( \chi \) is "even" then the functional equation gives

\[
|\Lambda(u, \chi)|^2 = \left( q^{1/2}u \right)^{2(\mathcal{D}-1)} |\Lambda\left( \frac{1}{qu} \right)|^2.
\]

Also \( |\Lambda(u, \chi)|^2 \) is a polynomial in \( u \) of degree \( \mathcal{D} - 1 \), and we can write

\[
|\Lambda(u, \chi)|^2 = \sum_{n=0}^{2(\mathcal{D}-1)} u^n B_n
\]

(note that if \( u \) is real, by properties of complex functions \( |\Lambda(u, \bar{\chi})|^2 = |\Lambda(u, \chi)|^2 \), it can also be represented this way). Now, by the functional equation

\[
\sum_{m=0}^{2(\mathcal{D}-1)} u^m B_m = \left( q^{1/2}u \right)^{2(\mathcal{D}-1)} \sum_{n=0}^{2(\mathcal{D}-1)} \left( \frac{1}{qu} \right)^n B_n = \sum_{n=0}^{2(\mathcal{D}-1)} q^{\mathcal{D}-1-n} u^{2(\mathcal{D}-1)-n} B_n
\]

and by writing \( m = 2(\mathcal{D} - 1) - n \), we get

\[
= \sum_{m=0}^{2(\mathcal{D}-1)} u^m q^{m-\mathcal{D}+1} B_{2(\mathcal{D}-1)-m}.
\]

We can conclude

\[(7.1) \quad B_m = q^{m-\mathcal{D}+1} B_{2(\mathcal{D}-1)-m} \Rightarrow B_{2(\mathcal{D}-1)-m} = q^{\mathcal{D}-1-m} B_m.
\]

Now,

\[
\left| L\left( \frac{1}{2}, \chi \right) \right|^2 = \left( 1 - q^{-1/2} \right)^2 \sum_{n=0}^{2(\mathcal{D}-1)} q^{-n^2/2} B_n
\]

\[
= \left( 1 - q^{-1/2} \right)^2 \left[ \sum_{n=0}^{\mathcal{D}-1} q^{-n^2/2} B_n + \sum_{n=\mathcal{D}}^{2(\mathcal{D}-1)} q^{-n/2} B_n \right],
\]

as shown earlier, \( m = 2(\mathcal{D} - 1) - n \).
\[ \begin{aligned}
&= \left(1 - q^{-1/2}\right)^2 \left[ \sum_{n=0}^{D-1} q^{-n/2} B_n + \sum_{m=0}^{D-2} q^{-1/2[2(D-1)-m]} B_{2(D-1)-m} \right], \\
\text{by equation 7.1 we get}
&= \left(1 - q^{-1/2}\right)^2 \left[ \sum_{n=0}^{D-1} q^{-n/2} B_n + \sum_{m=0}^{D-2} q^{-m/2} B_m \right] \\
&= \left(1 - q^{-1/2}\right)^2 \left[ 2 \sum_{n=0}^{D-2} q^{-n/2} B_n - 2 \sum_{n=1}^{D-1} q^{-n/2} B_{n-1} + \sum_{n=2}^{D} q^{-n/2} B_{n-2} \right] \\
&\quad + B_{D-1} \left(q^{-D/2} - 2q^{-D/4} + q^{-(D+1)/2}\right) \\
&= 2[B_0 + q^{-1/2} (B_1 - 2B_0) + \sum_{n=2}^{D-2} q^{-1/2n} (B_n - 2B_{n-1} + B_{n-2}) \\
&\quad + q^{-3(1/2)} (-2B_{D-2} + B_{D-3}) + q^{-3/2} B_{D-2}] \\
&\quad + B_{D-1} \left(q^{-D/2} - 2q^{-D/4} + q^{-(D+1)/2}\right).
\end{aligned}\]

Since \( |L\left(\frac{1}{2}, \chi\right)|^2 = (1 - q^{-1/2})^2 |\Lambda (q^{-1/2}, \chi)|^2 \),
\[
\sum_{n=0}^{2D} q^{-n/2} A_n = \left(1 - q^{-1/2}\right)^2 \sum_{n=0}^{2(D-1)} q^{-n/2} B_n \\
= B_0 + q^{-1/2} (B_1 - 2B_0) + \sum_{n=2}^{2(D-1)} q^{-n/2} (B_n - 2B_{n-1} + B_{n-2}) \\
\quad + q^{-2(D-1)/2} (-2B_{2D-2} + B_{2D-3}) + q^{-D} B_{2D-2}.
\]

We conclude
\[
A_n = \begin{cases} 
B_0, & n = 0 \\
B_1 - 2B_0, & n = 1 \\
B_n - 2B_{n-1} + B_{n-2}, & 2 \leq n \leq 2D - 2 \\
-2B_{2D-2} + B_{2D-3}, & n = 2D - 1 \\
B_{2D-2}, & n = 2D 
\end{cases}
\]

and
\[
B_n = \begin{cases} 
A_0, & n = 0 \\
A_1 + 2A_0, & n = 1 \\
A_{n-1} + \sum_{k=1}^{n} A_k, & 2 \leq n \leq 2D - 2 \\
A_{2D-1} + 2A_{2D}, & n = 2D - 1 \\
A_{2D}, & n = 2D - 2 
\end{cases}
\]
which yields

\[
\left| L\left(\frac{1}{2}, \chi\right) \right|^2 = 2\left[ A_0 + q^{-1/2} A_1 + \sum_{n=2}^{D-2} q^{-n/2} A_n + q^{-\left(D-1\right)/2} (A_{D-1} - B_{D-1}) \right. \\
+ q^{-D/2} B_{D-2}] + \left( q^{-\left(D-1\right)/2} - 2q^{-D/2} + q^{-\left(D+1\right)/2} \right) B_{D-1} \\
+ 2 \sum_{n=0}^{D-1} q^{-n/2} A_n + 2q^{-D/2} B_{D-2} - (q^{-\left(D-1\right)/2} + 2q^{-D/2} \\
- q^{-\left(D+1\right)/2}) B_{D-1} \\
+ 2 \sum_{n=0}^{D-1} q^{-n/2} A_n + 2q^{-D/2} (A_{D-3} - A_{D-2} + A_{D-1}) \\
- \left( q^{-\left(D-1\right)/2} - q^{-\left(D+1\right)/2} \right) \left( A_{D-2} + \sum_{k=0}^{D-1} A_k \right) \\
+ 2 \sum_{n=0}^{D-1} q^{-n/2} A_n + 2q^{-D/2} (A_{D-3} - A_{D-2} + A_{D-1}) \\
- \left( q^{-\left(D-1\right)/2} - q^{-\left(D+1\right)/2} \right) A_{D-2},
\]

when the last equality holds according to

\[
\sum_{k=1}^{D-1} A_k = \sum_{\deg N < D} \chi(N) = 0.
\]

This concludes the Theorem for the case \( \chi \) is “even”. Now, if \( \chi \) is not “even” the functional equation gives

\[
\left| L^* (u, \chi) \right| = \left( q^{1/2} u \right)^{2D} \left| L^* \left( \frac{1}{qu}, \chi \right) \right|^2
\]

(since then \( \Lambda (u, \chi) = L^* (u, \chi) \)). So

\[
\sum_{m=0}^{2D} u^m A_m = q^D u \sum_{n=0}^{2D} \left( \frac{1}{qu} \right)^n A_n = \sum_{n=0}^{2D} u^{\left(2D-n\right)} q^{D-n} A_n
\]

and by denoting \( m = 2D - n \) we get

\[
= \sum_{m=0}^{2D} u^m q^{m-D} A_{2D-m}.
\]

We conclude

\[
A_m = q^{m-D} A_{2D-m} \Rightarrow A_{2D-m} = q^{D-m} A_m.
\]

(7.2)

\[
A_m = q^{m-D} A_{2D-m} \Rightarrow A_{2D-m} = q^{D-m} A_m.
\]

Now,
\[ \left| L \left( \frac{1}{2}, \chi \right) \right|^2 = \sum_{n=0}^{2D} q^{-n/2} A_n = \sum_{n=0}^{D} q^{-n/2} A_n + \sum_{n=D+1}^{2D} q^{-n/2} A_n \]

once more, we denote \( m = 2D - n \)

\[ \begin{align*}
= & \sum_{n=0}^{D} q^{-n/2} A_n + \sum_{m=0}^{D-1} q^{-(2D-m)/2} A_{2D-m} \\
= & 2 \sum_{n=0}^{D-1} q^{-n/2} A_n + q^{-D/2} A_D
\end{align*} \]

and this conclude the Theorem for the case \( \chi \) is not \textit{even}. \( \square \)

**Remark.** We haven’t used the fact that \( Q \) is irreducible, therefore the Theorem applies to any \( Q \in \mathcal{A} \). However, only if \( Q \) is an irreducible polynomial, then every \( \chi \neq \chi_0 \) is a primitive character. Hence, only then we can conclude

\[ \sum_{\chi \not\equiv \chi_0}^{\ast} \left| L \left( \frac{1}{2}, \chi \right) \right|^2 = \sum_{\chi \not\equiv \chi_0}^{\ast} 2 \left( \sum_{\deg N M < D} \frac{\chi(N) \bar{\chi}(M)}{|NM|^{1/2}} + \pi(\chi) \right). \]

### 8. The Fourth Moment

In \cite{23}, K. Soundararajan proved that for all large \( q \)

\[ \sum_{\chi \not\equiv \chi_0}^{\ast} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 = \frac{\varphi^*(q)}{2\pi^2} \prod_{p \mid q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} (\log q)^4 + O \left( q (\log q)^{7/2} \right) \]

Here \( \sum^{\ast} \) denotes summation over primitive characters \( \chi \) (mod \( q \)), \( \varphi^*(q) \) denotes the number of primitive characters modulo \( q \), and \( \omega(q) \) denotes the number of distinct prime factors of \( q \).

We will show the polynomial analog of this Theorem for a prime polynomial \( Q \).

**Theorem 8.1.** Let \( Q \) be a prime polynomial, \( \deg Q - 1 = D \). For all large \( D \geq 1 \) we have

\[ \frac{1}{\varphi(Q)} \sum_{\chi \not\equiv \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 = \frac{q - 1}{12q} D^4 + O \left( D^3 \right). \]

This main Theorem will follow from the following Lemmas:
8.1. Preliminary Lemmas.

Lemma 8.2. For all integers \( x \geq 0 \)

\[
\sum_{\deg N \leq x} \frac{2^\omega(N)}{|N|} = \frac{(q-1)x^2 + (3q+1)x + 2q}{2q}.
\]

Proof. For \( \Re(s) > 1 \) we can define

\[
F(s) := \sum_{N \text{ monic}} \frac{2^\omega(N)}{|N|^{s+1}} = \prod_P \left( 1 + \sum_{k=1}^\infty \frac{2^\omega(p^k)}{|p|^{(s+1)k}} \right)
\]

\[
= \prod_P \left( 1 + 2 \sum_{k=1}^\infty \frac{|P|^{-(s+1)k}}{|P|} \right) = \prod_P \left( \frac{1 + |P|^{-(s+1)}}{1 - |P|^{-(s+1)}} \right)
\]

\[
= \frac{\prod_P (1 - |P|^{-2(s+1)})}{\prod_P (1 - |P|^{-(s+1)})^2} = \frac{\zeta(s+1)^2}{\zeta(2s+2)}
\]

where the first equality result of the multiplicativity of the function \( \frac{2^\omega(N)}{|N|^{s+1}} \). Now, we look at \( F \) as a function of \( T := q^{-s} \), and define \( A_n := q^{-n} \sum_{\deg N = n} 2^\omega(N) \), so

\[
F(T) = \sum_{k=0}^\infty A_k T^k = \frac{(1-q^{-1}T^2)}{(1-T)^2}
\]

Since

\[
\frac{1}{(1-T)^2} = \left( \sum_{k=0}^\infty T^k \right)^2 = \sum_{k=0}^\infty (k+1) T^k
\]

we have

\[
F(T) = \left( 1 - q^{-1}T^2 \right) \sum_{k=0}^\infty (k+1) T^k = \sum_{k=0}^\infty (k+1) T^k - \sum_{k=0}^\infty q^{-1} (k+1) T^{k+2}
\]

\[
= 1 + 2T + \sum_{k=2}^\infty \left( k+1 - \frac{k-1}{q} \right) T^k = 1 + \sum_{k=1}^\infty \left( k+1 - \frac{k-1}{q} \right) T^k,
\]

therefore

\[
A_k = \begin{cases} 
1; & k = 0 \\
\frac{k+1 - \frac{k-1}{q}}{q}; & k \geq 1,
\end{cases}
\]

hence
\[ \sum_{\deg N \leq x} \frac{2^{\omega(N)}}{|N|} = \sum_{k=0}^{x} A_k \]
\[ = 1 + \sum_{k=1}^{x} \left( k + 1 - \frac{k-1}{q} \right) \]
\[ = \frac{(q-1)x^2 + (3q+1)x + 2q}{2q}. \]

\[ \Box \]

**Lemma 8.3.** We have
\[ \sum_{\substack{AC=BD \\deg AB, \deg CD \leq \mathfrak{D}}} |ABCD|^{-1/2} = \frac{q-1}{48q} \mathfrak{D}^4 + O(\mathfrak{D}^3). \]

**Proof.** For \( AC = BD \) we can write \( A = UR, B = US, C = VS, D = VR \), where \( R \) and \( S \) are coprime. We put \( N = RS \), and note that given \( N \) there are \( 2^{\omega(N)} \) ways of writing it as \( RS \) with \( R \) and \( S \) coprime. Note also that \( AB = U^2 N \) and \( CD = V^2 N \). So

\[ \sum_{\substack{AC=BD \\deg AB, \deg CD \leq \mathfrak{D}}} |ABCD|^{-1/2} = \sum_{\deg N \leq \mathfrak{D}} \frac{2^{\omega(N)}}{|N|} \left( \sum_{\deg U < 1/2 (\mathfrak{D} - \deg N)} \frac{1}{|U|} \right)^2 \]
\[ = \sum_{\deg N \leq \mathfrak{D}} \frac{2^{\omega(N)}}{|N|} \left( \frac{\mathfrak{D} - \deg N}{2} \right)^2 \]
\[ = \sum_{\deg N \leq \mathfrak{D}} \frac{2^{\omega(N)}}{|N|} \left( \frac{\mathfrak{D} - \deg N}{2} \right)^2 + O(1) \]
\[ = \frac{1}{4} \sum_{\deg N \leq \mathfrak{D}} \frac{2^{\omega(N)}}{|N|} (\mathfrak{D} - \deg N)^2 + O \left( \sum_{\deg N \leq \mathfrak{D}} \frac{2^{\omega(N)}}{|N|} \right), \]

and by Lemma 8.2 we arrive at

\[ = \frac{1}{4} \sum_{\deg N \leq \mathfrak{D}} \frac{2^{\omega(N)}}{|N|} (\mathfrak{D} - \deg N)^2 + O(\mathfrak{D}^2). \]

Now, as seen in Lemma 8.2.
\[
\sum_{\deg N < \mathcal{D}} \frac{2^{\omega(N)}}{|N|} (\mathcal{D} - \deg N)^2 = \sum_{k=0}^{\mathcal{D}-1} A_k (\mathcal{D} - k)^2
\]
\[
= \mathcal{D}^2 + \sum_{k=1}^{\mathcal{D}-1} \left( k + 1 - \frac{k - 1}{q} \right) (\mathcal{D} - k)^2
\]
\[
= \mathcal{D}^2 + \frac{(q-1) \mathcal{D}^4 + 4 (q+1) \mathcal{D}^3 - (7q+5) \mathcal{D}^2 + 2 (q+1) \mathcal{D}}{12q}
\]
\[
= \frac{(q-1) \mathcal{D}^4 + 4 (q+1) \mathcal{D}^3 + 5 (q-1) \mathcal{D}^2 + 2 (q+1) \mathcal{D}}{12q},
\]
which completes the proof.

Lemma 8.4. We have
\[
\sum_{\substack{AC=BD \\
\mathcal{D}-3 \leq \deg AB \leq \mathcal{D} \\
\deg CD \leq \mathcal{D}}} |ABCD|^{-1/2} \leq \frac{(q-1) \mathcal{D}^3}{2q} + O(\mathcal{D}^2).
\]

Proof. As seen in Lemma 8.2 and 8.3 we get
\[
\sum_{\substack{AC=BD \\
\mathcal{D}-3 \leq \deg AB \leq \mathcal{D} \\
\deg CD \leq \mathcal{D}}} |ABCD|^{-1/2} = \sum_{\deg N \leq \mathcal{D}} \frac{2^{\omega(N)}}{|N|} \left( \sum_{\mathcal{D}-3 \leq \deg N \leq \deg U \leq \mathcal{D}-\deg N} \frac{1}{|U|} \right)
\]
\[
\cdot \left( \sum_{\deg V \leq \deg N} \frac{1}{|V|} \right)
\]
\[
\leq 2 \sum_{\deg N \leq \mathcal{D}} \frac{2^{\omega(N)}}{|N|} \left[ \frac{\mathcal{D} - \deg N}{2} + 1 \right]
\]
\[
\leq \sum_{k=0}^{\mathcal{D}} A_k (\mathcal{D} - k + 3)
\]
\[
= \frac{(q-1) \mathcal{D}^3}{2q} + O(\mathcal{D}^2).
\]

Lemma 8.5. Let \(Q\) be an irreducible monic polynomial, and let \(Z_1, Z_2\) be positive integers, \(2 < Z_1, Z_2 \leq \mathcal{D}\), then
\[
\sum_{\substack{Z_1 - 2 \leq \deg AB \leq Z_1 \\\nZ_2 - 2 \leq \deg CD \leq Z_2 \\\nAC \equiv BD \pmod{Q} \\\nAC \neq BD}} 1 \ll \frac{q^{Z_1+Z_2} (Z_1 Z_2)^3}{|Q|} + \mathcal{D}.
\]
Proof. If $Z_1 + Z_2 \leq \mathcal{D}$ we get $\deg ABCD \leq \mathcal{D}$ so $AC \equiv BD \pmod{Q}$ iff $AC = BD$ so the sum is zero. Hence we can assume $Z_1 + Z_2 > \mathcal{D}$. By symmetry we may just focus on the terms with $\deg AC > \deg BD$. Denote $U = BD$ and $W = AC$. Note that since $\deg AC \geq \deg Q = \mathcal{D} + 1$ and $\deg U = \deg BD \leq Z_1 + Z_2$ we also have $\mathcal{D} < \deg W = \deg AC \leq Z_1 + Z_2 - \deg U$. Hence

$$\sum_{\deg U \leq Z_1 + Z_2 - \mathcal{D}} \sum_{\substack{D < \deg W \leq Z_1 + Z_2 - \deg U \\ W \equiv U \pmod{Q}}} d(D)(U) \sum_{\substack{D < \deg W \leq Z_1 + Z_2 - \deg U \\ W \equiv U \pmod{Q}}} d(D)(W).$$

Now, we divide this sum into the following:

$$I = \sum_{\deg U \leq Z_1 + Z_2 - \mathcal{D}} \sum_{\deg W = \mathcal{D} + 1} d(U) d(W)$$

$$\sum_{\deg U \leq Z_1 + Z_2 - \mathcal{D}} \sum_{\mathcal{D} + 2 \leq \deg W \leq Z_1 + Z_2 - \deg U} d(U) d(W).$$

First, since $U \neq W$ and $\deg W > \deg U$ we know

$$I = \sum_{\deg U = 1} d(U) \sum_{\deg W = \mathcal{D} + 1} d(W)$$

$$= \sum_{\deg U = 1} d(U + Q),$$

when the second equation is due to the fact $W$ is a monic polynomial. We can use the bound $d(N) \ll \deg N$ and obtain

$$\ll q \cdot \mathcal{D} \ll \mathcal{D}.$$

In the second sum, for $\alpha = \frac{1}{20}$ we have $|W|^{(1-\alpha)} \geq q^{\mathcal{D} + 2 (1 - \alpha)} > q^{\mathcal{D} + 1 / 2 + 1 - \alpha} \geq |Q|$, so we can use the bound $\sum_{\deg N \leq x} d(N) \ll \frac{q^{x^2}}{x^{1/4}}$, Theorem 6.1 which yields

$$\ll \sum_{\deg U \leq Z_1 + Z_2 - \mathcal{D}} d(U) q^{Z_1 + Z_2} (Z_1 + Z_2)^3 \ll \frac{q^{Z_1 + Z_2} (Z_1 + Z_2)^3}{|Q|}$$

by Lemma 6.4. This completes the proof.

Corollary 8.6. We have

$$\sum_{\substack{AC \equiv BD \\ AC \neq BD \leq \mathcal{D}}} |ABCD|^{-1/2} \ll \mathcal{D}^3.$$

Proof. To estimate this sum we divide the terms $\deg AB, \deg CD \leq \mathcal{D}$ into dyadic blocks. Consider the block $Z_1 - 2 \leq \deg AB < Z_1$, and $Z_2 - 2 \leq \deg CD < Z_2$. By Lemma 5.5 the contribution of each block to the sum is

$$\ll q^{-1/2(Z_1 + Z_2)} \left[ \frac{q^{Z_1 + Z_2}}{|Q|} (Z_1 + Z_2)^3 + \mathcal{D} \right] \ll \frac{q^{1/2(Z_1 + Z_2)}}{|Q|} \mathcal{D}^3$$

Summing over all such dyadic blocks we obtain that
\[
\sum_{\begin{subarray}{c}
AC \equiv BD \\
AC \neq BD \\
deg AB,\deg CD \leq D
\end{subarray}} |ABCD|^{-1/2} \ll \frac{q^D}{|Q|} D^3 \ll D^3.
\]

This proves the Corollary.

8.2. **Proof of the Main Theorem.** By the previous section, Proposition 7.1, we have

\[
\left| L \left( \frac{1}{2}, \chi \right) \right|^2 = 2 \sum_{\deg AB \leq D} \frac{\chi(A) \bar{\chi}(B)}{|AB|^{1/2}} + \pi(\chi)
\]

when

\[
\pi(\chi) := \begin{cases}
2q^{-\delta/2} (A_{D-3} - A_{D-2} + A_{D-1}) \\
- (q^{-(\delta-1)/2} - q^{-(\delta+1)/2}) A_{D-2}; & \chi \text{ "even"} \\
q^{-\delta/2} A_{D}; & \text{else}
\end{cases}
\]

therefore, we get

\[
\frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 = \frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \left[ 4 \sum_{\deg AB \leq D} \frac{\chi(AC) \bar{\chi}(BD)}{|ABCD|^{1/2}} + 4\pi(\chi) \sum_{\deg AB \leq D} \frac{\chi(A) \bar{\chi}(B)}{|AB|^{1/2}} + \pi(\chi)^2 \right]
\]

and we can divide this sum into the following:

- **I** := \( \frac{4}{\phi(Q)} \sum_{\chi \neq \chi_0} \sum_{\deg AB, \deg CD \leq D} \frac{\chi(AC) \bar{\chi}(BD)}{|ABCD|^{1/2}} \)
- **II** := \( \frac{4}{\phi(Q)} \sum_{\chi \neq \chi_0} \pi(\chi) \sum_{\deg AB \leq D} \frac{\chi(A) \bar{\chi}(B)}{|AB|^{1/2}} \)
- **III** := \( \frac{1}{\phi(Q)} \sum_{\chi \neq \chi_0} \pi(\chi)^2 \).

First, since

\[
\sum_{\chi \neq \chi_0} \chi(AC) \bar{\chi}(BD) = \begin{cases}
\phi(Q) - 1, & AC \equiv BD \pmod{Q} \\
-1, & \text{else}
\end{cases}
\]

and
\[
\frac{1}{\phi(Q)} \sum_{\deg AB,\deg CD < D} |ABCD|^{-1/2} = \frac{1}{\phi(Q)} \left( \sum_{\deg AB < D} |AB|^{-1/2} \right)^2 \\
\leq \frac{1}{\phi(Q)} \left( \sum_{n=0}^{D-1} q^{-n/2}q^n \sum_{m=0}^{D-1-n} q^{-m/2}q^m \right)^2 \\
\ll \frac{1}{|Q|} q^D \ll D^2
\]

we obtain

\[
\frac{1}{\phi(Q)} \sum_{\deg AB,\deg CD < D} \sum_{\chi \neq \chi_0} \chi(AC) \overline{\chi}(BD) |ABCD|^{1/2} = \sum_{AC \equiv BD} |ABCD|^{-1/2} \\
- \frac{1}{\phi(Q)} \sum_{\deg AB,\deg CD < D} |ABCD|^{-1/2} \\
= \sum_{AC \equiv BD} |ABCD|^{-1/2} \\
+ \sum_{AC \equiv BD, AC \neq BD} |ABCD|^{-1/2} + O(D^2).
\]

Hence in the first case we have

\[
I = 4 \left[ \sum_{AC \equiv BD, \deg AB, \deg CD < D} |ABCD|^{-1/2} + \sum_{AC \equiv BD, AC \neq BD, \deg AB, \deg CD < D} |ABCD|^{-1/2} + O(D^2) \right]
\]

From Lemma 8.3 and Corollary to Lemma 8.5 we conclude

\[
I = \frac{q-1}{12q} D^4 + O(D^3).
\]

Next, accordingly, from \(\sum_{\chi \neq \chi_0} \pi(\chi) \ll \phi(Q) \sum_{A=B} |AB|^{-1/2}\) we have

\[
II \ll 4 \left[ \sum_{AC \equiv BD, D-3 \leq \deg AB \leq D} |ABCD|^{-1/2} + \sum_{AC \equiv BD, AC \neq BD, D-3 \leq \deg AB \leq D} |ABCD|^{-1/2} + O(D^2) \right]
\]

From Lemma 8.4 and Corollary 8.6 we have

\[
II \ll D^3
\]
Lastly, from \( \sum_{\chi \neq \chi_0} \pi(\chi) \ll \phi(Q) \sum_{A=B}^{D \leq \deg AB \leq D} |AB|^{-1/2} \) we also receive
\[
III \ll \Pi \ll D^3
\]
This proves the Theorem.

9. Lower Bound

Here we will show the second main result of this paper, an analog of the general lower bound of Rudnick and Soundararajan [20], for the \( 2k \)-th moment

**Theorem 9.1.** Let \( k \) be a fixed natural number. Then for all irreducible polynomial \( Q \), with a sufficiently large degree
\[
\sum_{\chi \mod Q} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \gg_k |Q| (\deg Q)^k^2.
\]

We shall require the following Lemmas in order to prove the Theorem above:

**Lemma 9.2.** For \( k \in \mathbb{N} \)
\[
\sum_{\deg N \geq 0} \frac{d_k(N)^2}{|N|^{s}} = \zeta(s)^k \prod_p \left( 1 + \frac{k^2}{|P|^s} \right)
\]
when
\[
p_k(x) = (1 - x)^{(k-1)^2} \sum_{n=0}^{k-1} \binom{k-1}{n}^2 x^n.
\]

**Proof.** We have
\[
\sum_{\deg N \geq 0} \frac{d_k(N)^2}{|N|^{s}} = \prod_p \left( \sum_{l=0}^{\infty} \frac{(k+l-1)^2}{|P|^l} \right) - \zeta(s)^k \prod_p \left( 1 + \frac{k^2}{|P|^s} \right)
\]
If we denote
\[
f_k(P^{-s}) = \sum_{l=0}^{\infty} \frac{(k+l-1)^2}{|P|^{ls}} = 1 + \frac{k^2}{|P|^s} + \ldots,
\]
we have
\[
\left( 1 - \frac{k^2}{|P|^s} \right)^2 f_k(P^{-s}) = \left( 1 - \frac{k^2}{|P|^s} + \ldots \right) \left( 1 + \frac{k^2}{|P|^s} + \ldots \right) = 1 + O\left( |P|^{-2s} \right)
\]
hence
\[
\sum_{\deg N \geq 0} \frac{d_k(N)^2}{|N|^{s}} = \zeta(s)^k \prod_p \left( 1 + \frac{k^2}{|P|^s} \right)
\]
when
\[ p_k(x) = (1 - x)^k \left( \sum_{l=0}^{\infty} \binom{k + l - 1}{k - 1}^2 x^l \right) \]
\[ = (1 - x)^{(k-1)^2} (1 - x)^{2k-1} \left( \sum_{l=0}^{\infty} \binom{k + l - 1}{k - 1}^2 x^l \right) \]
\[ = (1 - x)^{(k-1)^2} \left( \sum_{n=0}^{2k-1} \binom{2k-1}{n} (-x)^n \right) \left( \sum_{l=0}^{\infty} \binom{k + l - 1}{k - 1}^2 x^l \right) \]
for any \( m \leq k - 1 \) the coefficient of \( x^m \) in the sum is
\[ \sum_{d \leq m} (-1)^d \binom{2k-1}{d} \binom{k + m - d - 1}{m - d}^2 = \binom{k - 1}{m}^2, \]
otherwise it is 0. \qed

**Corollary 9.3.** We have
\[ \sum_{\deg N \geq 0} \frac{d_k(N)^2}{|N|^s} = \zeta(s) k^2 c(s) \]
\[ = \frac{c(s)}{(1 - q^{-s+1})^k^2} \]
when \( c(s) = \prod_p p_k(|P|^{-s}) \) is an analytic function when \( \Re(s) > 1/2 \).

**Lemma 9.4.** For fixed \( k \in \mathbb{N} \)
\[ \sum_{\deg N \leq y} \frac{d_k(N)^2}{|N|^s} \sim c_k y^k. \]

**Proof.** For \( k \geq 1 \), define
\[ A_n := q^{-n} \sum_{\deg N = n} d_k(N)^2. \]
We will show that
\[ (9.1) \quad A_n = h_{k^2-1}(n) + O(q^{-\delta n}) = c_{k^2-1} n^{k^2-1} + O_k \left( n^{k^2-2} \right) \]
where \( h_l(x) \) is a polynomial of degree \( l \) in \( x \), with leading coefficient \( c_{k^2} > 0 \), and \( 0 < \delta < 1/2 \). Consequently, we find that
\[ \sum_{\deg N \leq y} \frac{d_k(N)^2}{|N|^s} = c_k y^{k^2} + O(y^{k^2-1}) \]
with \( c_k > 0 \).

If we look at the functions in Corollary 9.3 as functions of \( T := q^{-s} \), we get
\[ F(T) := \sum_{n \geq 0} A_n T^n = \frac{C(T)}{(1 - T)^k^2}, \]
where $C(T) = \prod_p p_k \left( |P|^{-1} T^{\deg P} \right)$ is an analytic function when $T < q^{1/2}$. Since the product is convergent (hence nonzero) in $T < q^{1/2}$, by Cauchy’s integral formula we get

$$A_n = \frac{1}{2\pi i} \oint_{|T|=q^{1/2}} \frac{F(T)}{T^{n+1}} dT,$$

where the contour of integration is a small circle about the origin, containing no singularity of the integrand except $T = 0$, traversed anti-clockwise. Expanding the contour beyond the pole of $F(T)$ at $T = 1$ gives

$$A_n = -\text{Res}_{T=1} \frac{C(T)}{(1 - T)^{k^2} T^{n+1}} + \frac{1}{2\pi i} \oint_{|T|=q^\delta} \frac{F(T)}{T^{n+1}} dT,$$

where the contour of integration is the circle $|T| = q^\delta$ traversed anti-clockwise, and $0 < \delta < 1/2$. We have

$$-\text{Res}_{T=1} \frac{C(T)}{(1 - T)^{k^2} T^{n+1}} = -\frac{(-1)^{k^2}}{(k^2 - 1)!} \frac{d^{k-1}}{dT^{k-1}} \left. \frac{C(T)}{T^{n+1}} \right|_{T=1}.$$

By Leibnitz’s rule, taking into account that the $r$-th derivative of $1/T^{n+1}$ at $T = 1$ is $(-1)^r (n+1) \cdots (n+1-(r-1))$,

$$-\frac{(-1)^{k^2}}{(k^2 - 1)!} \frac{d^{k-1}}{dT^{k-1}} C(T) \bigg|_{T=1} = \frac{(-1)^{k^2} - 1}{(k^2 - 1)!} \sum_{r=0}^{k^2-1} \binom{k^2-1}{r} C^{(k^2-1-r)}(1) \cdot (-1)^r (n+1) \cdots (n+1-(r-1))$$

$$= \sum_{r=0}^{k^2-1} \binom{n+1}{r} \frac{(-1)^{k^2-1-r}}{(k^2 - 1-r)!} C^{(k^2-1-r)}(1) \equiv h_{k^2-1}(n)$$

is a polynomial of degree $k^2-1$ in $n$, whose leading coefficient comes from the term $r = k^2 - 1$:

$$h_{k^2-1}(n) = \frac{C(1)}{(k^2 - 1)!} n^{k^2-1} + O\left(n^{k^2-2}\right).$$

Here $C(1) = \prod_p p_k \left( |P|^{-2} \right)$ with

$$p_k(x) = (1 - x)^{(k-1)/2} \sum_{n=0}^{k-1} \binom{k-1}{n} x^n$$

so clearly $C(1) > 0$. Finally, we bound the integral over the contour $|T| = q^\delta$ by

$$\left| \frac{1}{2\pi i} \oint_{|T|=q^\delta} \frac{F(T)}{T^{n+1}} dT \right| \leq q^{-n \delta} \max_{|T|=q^\delta} |F(T)|$$

which proves 9.1.3 and hence the Theorem.

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**Theorem 9.5.** Let $k$ be a fixed natural number. Then for all irreducible polynomial $Q$, with a big enough degree

$$\chi \sum_{\chi \mod Q, \chi \neq \chi_0} \left| L \left( \frac{1}{2}, \chi \right) \right|^{2k} \gg_k |Q| (\deg Q)^{k^2}.$$
Proof. Let \( x := \frac{D}{2k} \), \( A(\chi) := \sum_{\deg N \leq x} \frac{\chi(N)}{|N|} \), and set

\[
S_1 := \sum_{\chi \not\equiv \chi_0 \pmod{Q}} L\left(\frac{1}{2}, \chi \right) A(\chi)^{k-1} \overline{A(\chi)}^k, \quad S_2 := \sum_{\chi \not\equiv \chi_0 \pmod{Q}} |A(\chi)|^{2k}.
\]

By the Triangle and Hölder’s inequalities we have

\[
\left| \sum_{\chi \not\equiv \chi_0 \pmod{Q}} L\left(\frac{1}{2}, \chi \right) A(\chi)^{k-1} \overline{A(\chi)}^k \right| \leq \left( \sum_{\chi \not\equiv \chi_0 \pmod{Q}} \left| L\left(\frac{1}{2}, \chi \right) \right|^2 \right)^{1/2} \ \cdot \ \left( \sum_{\chi \not\equiv \chi_0 \pmod{Q}} |A(\chi)|^{2k} \right)^{1/2k-1}.
\]

therefore

\[
\sum_{\chi \not\equiv \chi_0 \pmod{Q}} \left| L\left(\frac{1}{2}, \chi \right) \right|^{2k} \geq \frac{|S_1|^{2k}}{S_2^{2k-1}}.
\]

Hence, if we show \( S_2 \ll |Q|(\deg Q)^{k^2} \ll S_1 \), the Theorem will follow.

First we evaluate \( S_2 \). Since \( A(\chi_0) \ll q^{1/2} \) we have

\[
S_2 = \sum_{\chi \pmod{Q}} |A(\chi)|^{2k} + O\left(q^{k/2}\right) = \sum_{\deg M, \deg N \leq kx} \frac{d_k(M, x) d_k(N, x)}{|MN|^{1/2}} \sum_{\chi \pmod{Q}} \chi(M) \overline{\chi(N)} + O\left(q^{k/2}\right).
\]

Since \( kx = \frac{D}{2} < D \) the orthogonality relation for characters mod \( Q \) gives that only the diagonal term \( M = N \) survives. Thus,

\[
S_2 = \phi(Q) \sum_{\deg N \leq kx} \frac{d_k(N, x)^2}{|N|} + O(q^{k/2}) \leq \phi(Q) \sum_{\deg N \leq kx} \frac{d_k(N)^2}{|N|} + O\left(q^{k/2}\right),
\]

since \( d_k(N, x) \leq d_k(N) \). Due to Lemma 9.4 we get \( S_2 \ll |Q|(\deg Q)^{k^2} \), as claimed.
Now we evaluate $S_1$. We have
\[
S_1 = \sum_{\chi \bmod Q, \deg N \leq D} \frac{\chi(N)^k}{|N|^{\frac{1}{2}}} \frac{A(\chi)^{k-1}}{A(\chi)^x} \sum_{\chi \neq \chi_0} \frac{\chi(N)^k}{|N|^{\frac{1}{2}}} A(\chi)^{k-1} A(\chi)^x + O \left( q^{\frac{3D}{2}} \right)
\]
\[
= \sum_{\chi \bmod Q, \deg N \leq D} \frac{\chi(N)^k}{|N|^{\frac{1}{2}}} A(\chi)^{k-1} A(\chi)^x + O \left( q^{\frac{3D}{2}} \right)
\]
when the first equality holds according to
\[
\sum_{\deg N \leq D} \frac{1}{|N|^{\frac{1}{2}}} A(\chi_0)^{2k-1} \ll q^{\frac{D}{2}} q^{(2k-1)^{\frac{1}{2}}} < \ll \frac{q^{\frac{3D}{2}}}{D}
\]
and the second equality holds by using the orthogonality relation for characters. Since $d_{k-1}(A,x) d_k(B,x) / |ABN|^{\frac{1}{2}} \geq 0$ we can write
\[
S_1 \geq \phi(Q) \sum_{\deg B \leq kx} \sum_{\deg A \leq (k-1)x} \sum_{\deg N \leq D} \sum_{AN=B} d_{k-1}(A,x) d_k(B,x) / |ABN|^{\frac{1}{2}} + O \left( \frac{q^D}{D} \right).
\]
Since
\[
\sum_{\deg A \leq (k-1)x, \deg N \leq D} d_{k-1}(A,x) \geq \sum_{\deg A \leq (k-1)x, \deg N \leq x} d_{k-1}(A,x) = d_k(B,x)
\]
and $d_k(B,x) = d_k(B)$ for $\deg B \leq x$, we deduce that
\[
S_1 \geq \phi(Q) \sum_{\deg B \leq x} \frac{d_k(B)^2}{|B|} + O \left( \frac{q^D}{D} \right) \gg |Q| (\deg Q)^k.
\]
This proves the Theorem. \qed

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