Main conjectures for higher rank nearly ordinary families – I

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Abstract. In this article, we present the first half of our project on the Iwasawa theory of higher rank Galois deformations over deformations rings of arbitrary dimension. We develop a theory of Coleman maps for a very general class of coefficient rings, devise a dimension reduction procedure for locally restricted Euler systems and finally, put these into use in order to prove a divisibility in a 3-variable main conjecture for nearly ordinary families of Rankin-Selberg convolutions.

1. Introduction and set up

Our principal goal in this article is to establish a general machinery to approach the Iwasawa main conjectures for Galois representations of higher rank with coefficients in deformation rings of higher Krull dimension. This consists of two independent steps.

First, under certain technical hypotheses, we devise an inductive procedure to bound the size of a Selmer group for a Galois representation of higher rank over regular rings of higher Krull dimension with the aid of a locally restricted Euler system; see Theorem B below as a sample of our results, Theorems 3.6 and 3.21 for their most general form (see also Paragraph 1.3 for a comparison with prior related results). We note that the locally restricted Euler system machine we develop here here does not require the underlying Galois representation be $p$-ordinary.

Secondly, we construct Coleman maps for a very general class of nearly ordinary Galois deformations, with coefficients in finite extensions of the ring of power series in any number of variables.

We finally exhibit an application of the theory we have developed here, with the aid of the Beilinson-Flach Euler system of [LLZ14, KLZ17]. Our result, which we record as Theorem C below, is a divisibility in the statement of a 3-variable main conjectures for nearly ordinary deformations of Rankin-Selberg products.

1.1. A sample of results in this article. For the sake of the clarity of our exposition in the introduction, we shall phrase our main results only in the the case of Rankin-Selberg products. We first introduce our notation (which we will also rely on in the main body of this note); we refer the reader to Section 4.2.1 and Section 5 for more precise definitions of the objects that show up in the paragraph below. Fix forever a prime $p > 7$.

Let $f_1 = \sum_{n=1}^{\infty} a_n(f_1)q^n$ and $f_2 = \sum_{n=1}^{\infty} a_n(f_2)q^n$ denote two primitive Hida families of elliptic modular cusp forms with respective tame levels $N_1$ and $N_2$ and central characters $\Psi_i : (\mathbb{Z}/pN_i\mathbb{Z})^\times \rightarrow \mathbb{T}_p$ $(i = 1, 2)$, which are defined over the respective local domains $I_{f_i}$ $(i = 1, 2)$, which are both finite flat over the respective one-variable Iwasawa algebra $\mathbb{Z}_p[[\Gamma_i]]$, where $\Gamma_i \twoheadrightarrow 1 + p\mathbb{Z}_p$ for $i = 1, 2$. We write $\Psi_i = \psi^{(N_i)}_i \psi^{(p)}_i$ so that the Dirichlet character $\psi^{(N_i)}_i$ has prime-to-$p$ conductor and $\psi^{(p)}_i$ has conductor dividing $p$.

We let $\mathfrak{T}_{f_i}$ denote Hida’s big Galois representation that Hida associates to $f_i$, under mild assumptions (which we shall assume throughout) this is a free $\mathfrak{T}_r$-module of rank 2 and its
restriction $T_{f_i}|_{G_{Q_p}}$ admits a $p$-ordinary filtration $F_p^+T_{f_i} \subset T_{f_i}$, where $F_p^+T_{f_i}$ is a free direct summand.

We set $\Lambda_{\text{cyc}} := Z_p[[\Gamma_{\text{cyc}}]]$, which is called the cyclotomic Iwasawa algebra. We denote by $\Lambda_{\text{cyc}}^\xi$ (resp. $(\Lambda_{\text{cyc}}^\xi)^\gamma$) the free $\Lambda_{\text{cyc}}$-module of rank one on which the absolute Galois group $G_{Q}$ acts via the tautological character $\tilde{\chi}_{\text{cyc}} : G_{Q_p} \rightarrow \Gamma_{\text{cyc}} \rightarrow \Lambda_{\text{cyc}}^\xi$ (resp. via the inverse tautological character $\tilde{\chi}_{\text{cyc}}^{-1} : G_{Q_p} \rightarrow \Gamma_{\text{cyc}} \rightarrow \Lambda_{\text{cyc}}^\xi$, where inv is the map given by $\gamma \mapsto \gamma^{-1}$). We set $T_{\text{cyc}} = T_{f_1} \otimes_{Z_p} T_{f_2} \otimes_{Z_p} (\Lambda_{\text{cyc}}^\xi)^\gamma$, which is a free module of rank 4 over $\mathcal{R}_{\text{cyc}} := I_{f_1} \otimes_{Z_p} I_{f_2} \otimes_{Z_p} \Lambda_{\text{cyc}}$ and we consider the subquotient $F^+T_{\text{cyc}} = T_{f_1}/F_p^+T_{f_1} \otimes_{F_p^+} T_{f_2} \otimes_{Z_p} (\Lambda_{\text{cyc}}^\xi)^\gamma$. Under suitable hypotheses, it is a free $\mathcal{R}_{\text{cyc}}$-module of rank one.

A continuous homomorphism $\kappa = \kappa_1 \otimes \kappa_2 \otimes \kappa_{\text{cyc}} \in \text{Hom}(\mathcal{R}_{\text{cyc}}, \overline{Q}_p)$ is called arithmetic if it satisfies the following conditions:

(i) The homomorphism $\kappa_i$ is an arithmetic specialization of weight $w_i \geq 0$ on the ordinary Hida family $I_{f_i}$, which corresponds to a cuspidal weight $w_i = w_i + 2$ ($i = 1, 2$),

(ii) The homomorphism $\kappa_{\text{cyc}}$ is of the form $\chi_{\text{cyc}}^j \phi$ where $\chi_{\text{cyc}} : \Gamma_{\text{cyc}} \rightarrow 1 + p\mathbb{Z}_p$ is the cyclotomic character, $j$ is an integer and $\phi$ is a character of $\Gamma_{\text{cyc}}$ of finite order.

Given an arithmetic specialization $\kappa$ as above, we write $E_{\kappa} \subset \overline{Q}_p$ for the finite extension of $Q_p$ generated by the image of $\kappa$ and define $V_{\kappa} := T_{\text{cyc}} \otimes_{\kappa} E_{\kappa}$ (the specialization of $T_{\text{cyc}}$ at $\kappa$). The Galois representation $V_{\kappa}$ inherits a filtration from $T_{\text{cyc}}$ in the obvious manner. For $i = 1, 2$, we denote by $a_p(f_i(\kappa_i))$ the $U_p$-eigenvalue on the $p$-stabilized eigenform $f_i(\kappa_i)$.

We finally let $\mathcal{D}(F^+T_{\text{cyc}})$ denote the big Dieudonné module associated to the family $F^+T_{\text{cyc}}$ of local Galois representations which interpolates the de Rham Dieudonné modules of its specializations. We shall not provide its explicit form in this introduction but refer the reader to Definition 3.11 for its most general form.

**Theorem A.** There exists an $\mathcal{R}_{\text{cyc}}$-linear isomorphism

$$\text{EXP}^* : H^1(Q_p, F^+T_{\text{cyc}}) \longrightarrow \mathcal{D}(F^+T_{\text{cyc}})$$

which is characterized by the following interpolation property: For every arithmetic specialization $\kappa = \kappa_1 \otimes \kappa_2 \otimes \kappa_{\text{cyc}}$ as above with $j > w_2$ the following diagram commutes:

$$\begin{array}{ccc}
H^1(Q_p, F^+T_{\text{cyc}}) & \longrightarrow & \mathcal{D}(F^+T_{\text{cyc}}) \\
\downarrow_{\kappa} & & \downarrow_{\kappa} \\
H^1(Q_p, F^+V_{\kappa}) & \longrightarrow & \mathcal{D}(F^+V_{\kappa})
\end{array}$$

where $e_p^+ := (-1)^{j-w_2}(j-w_2)!e_p$ and $e_p = e_p(\kappa)$ is the $p$-adic multiplier given by

$$e_p = \left(1 - \frac{p^{j-w_2-1}}{a_p(f_1(\kappa_1))a_p(f_2(\kappa_2))^{-1}\Psi_2^{(N_2)}(p)}\right) \left(1 - \frac{a_p(f_1(\kappa_1))a_p(f_2(\kappa_2))^{-1}\Psi_2^{(N_2)}(p)}{p^{j-w_2}}\right)^{-1}$$

in case $F^+V_{\kappa}$ is crystalline, and

$$e_p = \left(\frac{p^{j-w_2-1}}{a_p(f_1(\kappa_1))a_p(f_2(\kappa_2))^{-1}\Psi_2^{(N_2)}(p)}\right)^n$$

when $F^+V_{\kappa}|_{I_p} \cong E_{\kappa}(\omega_2 + 1 - j)(\eta)$ with $\text{ord}_p(\text{cond}(\eta)) = n \geq 1$. 
Also, for every \( \kappa \) as above but \( j \leq w_2 \) we also have the following commutative diagram:

\[
\begin{array}{ccc}
H^1(Q_p, F^{++}T_{\text{cyc}}) & \xrightarrow{\text{EXP}^*} & D(F^{++}T_{\text{cyc}}) \\
\kappa & \downarrow & \kappa \\
H^1(Q_p, F^{++}V_{\kappa}) & \xrightarrow{e_p \times \log} & D_{\text{dr}}(F^{++}V_{\kappa})
\end{array}
\]

where \( e_p := \frac{e_p}{(w_2 - j)!} \).

We should note that our argument here goes through directly, without passing to an unramified twist as in [KLZ17]. We also refer the reader to Remark 3.15 who might be curious about absence of the Gauss sums in the portions of the interpolation formulae that concern non-crystalline specializations.

Next, we state a particular consequence on locally restricted Euler system machinery for higher dimensional coefficient rings obtained by the theory introduced in Section 3. We note that we do not require any \( p \)-ordinary hypothesis in the most general form (Theorem 3.6 below) of our results. In this portion, we shall assume that \( \mathcal{R}_{\text{cyc}} \) is isomorphic to a power series ring in three variables \( O[[X_1, X_2, X_3]] \), where \( O \) is the ring of integers of a finite extension of \( \mathbb{Q}_p \). This assumption turns out to be not so restrictive; see [FLO12] Lemma 2.7 in this regard.

Recall the subquotient \( F^{++}T_{\text{cyc}} = T_{\text{cyc}}(\Lambda^2) \). We also define the quotient \( F^{--}T_{\text{cyc}} = T_{\text{cyc}}(\Lambda^2) \). The submodules

\[
F^{++}T_{\text{cyc}} = T_{\text{cyc}}(\Lambda^2) \quad \text{and} \quad F^{--}T_{\text{cyc}} = T_{\text{cyc}}(\Lambda^2)
\]

Notice that we have a natural isomorphism \( F^{++}T_{\text{cyc}}/F^{--}T_{\text{cyc}} \xrightarrow{\sim} F^{++}T_{\text{cyc}} \).

**Theorem B** (Theorem 5.11). For all sufficiently large \( p \) and under suitable technical hypotheses (that are made precise in the main text), the following statements hold:

1. The Greenberg Selmer group \( H^1(Q_p, T'_{\text{cyc}}(1))^\vee \) attached to \( T_{\text{cyc}} \) is torsion,
2. \( \text{char} \left( H^1(Q_p, T'_{\text{cyc}}(1))^\vee \right) \supset \text{char} \left( H^1(Q_p, F^{--}T_{\text{cyc}}) / \mathcal{R}_{\text{cyc}} \cdot \text{res}_{+/f} \left( BF_{f_1, f_2} \right) \right) \)

where \( BF_{f_1, f_2} \) is the generalized Beilinson-Flach element of [KLZ17] associated to the family \( f_1 \otimes f_2 \) of Rankin-Selberg convolutions, and the map \( \text{res}_{+/f} \) is the compositum of the arrows

\[
\ker \left( H^1(Q_p, T_{\text{cyc}}) \xrightarrow{\text{res}} H^1(Q_p, F^{--}T_{\text{cyc}}) \xrightarrow{\text{res}} H^1(Q_p, F^{++}T_{\text{cyc}}) \longrightarrow H^1(Q_p, F^{++}T_{\text{cyc}}) \right).
\]

The Greenberg Selmer group and other auxiliary Selmer groups that intervene for technical reasons are defined in Section 2. We note that the \((-1)\)-eigenspace for complex conjugation acting on \( T_{\text{cyc}} \) has rank two. This is the reason why the Euler system machinery that was readily available prior to our work does not apply and this is the main motivation behind our general result (Theorem 3.6), to prove which we develop a locally restricted Euler system machinery in Section 3.

On our way to prove Theorem B, we also prove a big image result for the big Galois representation associated to a Rankin-Selberg product of two Hida families (Theorem 5.6). We believe that this result is independent interest.

Combining our construction in Theorem A with Theorem B and the reciprocity formulae in [KLZ15] Theorem 6.5.9 for Beilinson-Flach elements, we have the following result in favour of the Iwasawa main conjectures for nearly-ordinary families of Rankin-Selberg products.

**Theorem C** (Corollary 5.18). For all sufficiently large \( p \) and under certain hypothesis (see Corollary 5.15 for a precise statement) we have the following divisibility in \( \mathcal{R}_{\text{cyc}} \):

\[
\text{char}_{\mathcal{R}_{\text{cyc}}} \left( H^1_{\text{Gr}}(Q, T'_{\text{cyc}}(1))^\vee \right) \supset (H) \cdot \mathcal{R}_{\text{cyc}} L_p^\text{Hida}(f_1, f_2, 1 + j).
\]

Here, \( H \in \mathfrak{I}_{f_1} \) denotes Hida’s congruence divisor associated to \( f_1 \).
We note that the statement of this theorem involves a minimal amount of correction terms to relate the algebraic $p$-adic $L$-function to the analytic $p$-adic $L$-function, thanks to the fact that our Coleman map $\text{EXP}^*$ is surjective.

1.2. Companion article. In the second part of this project, we shall present a slight extension of our Theorem 4.14 on the construction of Coleman maps, that will cover arbitrary base fields that are unramified at all primes above $p$. Our construction will apply, for example, in the context of nearly-ordinary families of automorphic representations for $GL_n$ over CM fields. Moreover, as our construction gives rise to a collection of Coleman maps for each subquotient that appears in the $p$-ordinary filtration, we will be able to introduce a general “rank reduction principle" for higher rank Euler systems over very general coefficient rings. We will prove that this theory is non-vacuous: Building on our work in the current note, we will be able to prove that the Beilinson-Flach Euler system of $[LLZ14, KLZ17]$ lifts to a three-variable family of rank-2 Euler systems. This will provide us with the first example of a $p$-adic family of non-trivial higher rank Euler system, verifying (an extension of) a conjecture of Perrin-Riou in this context.

1.3. Related results. Before we present a detailed account of our results in full generality, we first discuss past work related to the contents of this article.

The pioneering construction of Coleman followed by Perrin-Riou’s ground-breaking work allows one to interpolate the Bloch-Kato exponential maps along the cyclotomic deformations of Galois representations. In $[Och03]$, the second named author has expanded this construction to nearly-ordinary deformations of Galois representations that are afforded by Hida families. This has been subsequently generalized in $[LO14]$ to a treatment of families of Siegel modular forms. We have two major remarks: First of all, our construction is very direct as we do not rely on elements of $p$-adic Hodge theory. Secondly, it comes out as natural consequence of our rather hands on construction that the Coleman maps we consider are indeed surjective.

Next, we discuss older results that relate to the portion of our work on the general theory of Euler systems. After Kolyvagin’s celebrated work, Kato, Perrin-Riou and Rubin developed a general machinery of Euler systems to treat Galois representations of core Selmer rank one in the sense of Mazur and Rubin $[MR01]$ and with coefficients in a DVR or their deformations of character type (i.e., for coefficient rings that arise as the universal deformation rings of characters). The second named author introduced in $[Biy09a]$ an extension of this theory to more general coefficient rings (in fact, as general as we may treat here) but still in the case when the core Selmer rank equals one. In order to handle the cases when the core Selmer rank is arbitrary, the second named author established what he called a locally restricted Euler system machinery in $[Biy09b, Biy10, Biy14]$; however, all these works allowed only the treatment of deformations of character type. In the current article, we extend all these to work with a very extensive class of deformation rings. The main difficulty we have to handle has to do with Tamagawa factors (and their effect on various control theorems), which turns out to be somewhat more notorious if we do not allow variation in the cyclotomic direction.

We finally note that a single-variable version of Theorem C (that allows variation only in the cyclotomic variable) was already proved in $[KLZ17]$ Theorem 11.6.4.

1.4. General Setup. Let us fix an odd prime $p$ throughout the paper. We fix embeddings of the algebraic closure of the field of rationals $\mathbb{Q}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ simultaneously. We also fix a system of norm compatible $p$-power roots of unity $\{\zeta^p\}$. Let $R$ be a complete local Noetherian $\mathbb{Z}_p$-algebra of mixed characteristic and finite residue field $k = R/\mathfrak{m}_R$, where we denote by $\mathfrak{m}_R$ the maximal ideal of $R$. Let $K$ be either a totally real or a CM number field. Let $\Sigma$ be a set of places of $K$ that contains all places above $p$ as well as all archimedean places. Let $K_\Sigma$ be the maximal extension of $K$ unramified outside $\Sigma$ and set $G_{K,\Sigma} := \text{Gal}(K_\Sigma/K)$. For each place $v$ of $K$, we denote by $K_v$ the completion of $K$ at $v$. We also set $G_v := \text{Gal}(\overline{K}_v/K_v)$. We denote by $\mathbb{Q}_{\text{cyc}}$ the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. We also denote by $K_{\text{cyc}}$ (resp. $K_{p,\text{cyc}}$) the composite $K\mathbb{Q}_{\text{cyc}}$ (resp. $K_p\mathbb{Q}_{\text{cyc}}$).
Let $\mathcal{T}$ be a free $R$-module of rank $d$ which is endowed with a continuous action of $G_{K, \Sigma}$. When $K$ is totally real, we shall set

$$d_+ = d_+(\mathcal{T}) = \sum_{v|\infty} \text{rank}_R H^0(K_v, \mathcal{T}) = \text{rank}_R H^0(\mathbb{R}, \text{Ind}_K^G \mathcal{T})$$

and we set $d_- = \frac{d[K : \mathbb{Q}]}{2}$ when $K$ is a CM field. In either case, we set

$$d = d_+ - d_-.$$

**Definition 1.1.** Let $R$ be a complete local Noetherian $\mathbb{Z}_p$-algebra of mixed characteristic and finite residue field. For a continuous ring homomorphism $\kappa : R \rightarrow \mathbb{Q}_p$, we denote by $E_{\kappa}$ the finite extension $\text{Frac}(\kappa(R))$ of $\mathbb{Q}_p$, and by $\mathfrak{a}_\kappa \subset E_{\kappa}$ its ring of integers. Let $\mathcal{T}$ be a free $R$-module of finite rank and let

$$S \subset \{ \kappa \in \text{Hom}(R, \mathbb{Q}_p) : R \xrightarrow{\kappa} \mathbb{Q}_p \text{ is continuous} \}$$

be the set of specializations $R \xrightarrow{\kappa} \mathbb{Q}_p$ such that the set $\{ \ker \kappa \}_{\kappa \in S} \subset \text{Spec}(R)$ is Zariski dense. Then, we call a pair $(\mathcal{T}, R, S)$ a deformation datum if it satisfies the following three properties (Geo), (Crt) and (Pan) for each prime $p$ of $K$ above $p$:

1. **(Geo)** The representation $V_{\kappa}|_{G_p} := (\mathcal{T} \otimes_{\kappa} E_{\kappa})|_{G_p}$ is de Rham as a $G_p$-representation.
2. **(Crt)** Suppose that (Geo) holds true and write $V_{\kappa} \otimes_{\kappa} E_{\kappa} \mathcal{C}_p := \bigoplus_{n \in \mathbb{Z}} C_p(n)^{m_n(p, \kappa)}$ as $G_p$-representation. We then have

$$\sum_p d_+^{(p)} [K_p : \mathbb{Q}_p] = d_+(\mathcal{T})$$

where we have set $d_+^{(p)} := \sum_{n > 0} m_n(p, \kappa)$.

3. **(Pan)** There is a direct summand $F_p^+ \mathcal{T} \subset \mathcal{T}$ (as an $R$-submodule) of rank $d_+^{(p)}$ that is stable under the $G_p$-action and we have

$$F_p^+ V_{\kappa} \otimes_{E_{\kappa}} \mathcal{C}_p \cong \bigoplus_{n > 0} C_p(n)^{m_n(p, \kappa)},$$

$$V_{\kappa}/F_p^+ V_{\kappa} \otimes_{E_{\kappa}} \mathcal{C}_p \cong \bigoplus_{n \leq 0} C_p(n)^{m_n(p, \kappa)},$$

where $F_p^+ V_{\kappa}$ is the $G_p$-stable filtration on $V_{\kappa}$ induced by $F_p^+ \mathcal{T}$.

**Remark 1.2.** The condition (Pan) is called the Panchishkin (or sometimes, Dabrowski-Panchishkin condition) in the literature and it is a generalization of the $p$-ordinary condition. Under (Pan), the condition (Crt) amounts to the requirement that for $\kappa \in S$, the representation $V_{\kappa}$ be the $p$-adic realization of a critical motive in the sense of Deligne.

Let $T_{\kappa}$ be a fixed $G_{K, \Sigma}$-stable lattice inside $V_{\kappa}$ and we set $\overline{T} := T \otimes_R \mathfrak{k}$ and call it the residual representation of $T$. We define $F_p^+ \overline{T} := F_p^+ T \otimes_R \mathfrak{k}$. We also set $T_{\kappa, \text{cyc}} := T_{\kappa} \otimes (\Lambda_{\text{cyc}})^{\vee}$ which is equipped with the diagonal action of $G_{K, \Sigma}$ and refer to it as the cyclotomic deformation of $T_{\kappa}$.

For any topological abelian group $A$, we denote the Pontrjagin dual $\text{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ by $A^{\vee}$. For a finitely generated $R$-module $M$ we denote its $R$-linear dual $\text{Hom}_R(M, R)$ by $M^R$.  

1.5. Further Notation and Hypotheses. For a prime $\lambda \notin \Sigma$, we denote by $K(\lambda)$ the maximal $p$-extension of $K$ contained in the ray class field of $K$ with module $\lambda$, by $F_{\lambda} \in \text{Fr}_K(K(\lambda)/K)$ the arithmetic Frobenius at $\lambda$. Let $\mathcal{N}_\Sigma$ be the set of square free products of primes $\lambda \notin \Sigma$. For $\eta = \lambda_1 \cdots \lambda_r \in \mathcal{N}_\Sigma$, we write $K(\eta) := K(\lambda_1) \cdots K(\lambda_r)$ for the compositum of the fields $K(\lambda_i)$ and define $K_m(\eta) := K_m K(\eta)$. We define $\Delta_\eta := \text{Gal}(K(\eta)/K) \cong \Delta_{\lambda_1} \times \cdots \times \Delta_{\lambda_r}$ and set $\Delta := \varprojlim \Delta_\eta$. We denote the compositum of all fields $K(\eta)$ as $\eta$ runs through $\mathcal{N}_\Sigma$ by $K$. We will consider the following conditions on the residual representation $\overline{T}$ of $T$:

(H.0) For every prime $p$ of $K$ above $p$, we have $H^0(K_p, \overline{T}) = 0$.

(H.0') For every prime $p$ of $K$ above $p$, we have $H^0(K_p, \overline{T}/F_p^+ \overline{T}) = 0$.

(H.2) For every prime $p$ of $K$ above $p$, we have $H^2(K_p, \overline{T}) = 0$.

(H.2+) For every prime $p$ of $K$ above $p$, we have $H^2(K_p, F_p^+ \overline{T}) = 0$.

(H.++) For some prime $p_o$ of $K$ above $p$ of degree one, there exists an $R$-module direct summand $F^{++} T$ of $T$ which is an $R$-module of rank $1 + d^{(p_o)}_+$ containing $F^{p_o}_o T$ and is stable under $G_{p_o}$-action.

(H.2++) Only under the assumption (H.++) and for a prime $p_o$ specified with the condition (H.++), we have $H^2(K_{p_o}, F^{++} \overline{T}) = 0$.

For a finite extension $E$ of $\mathbb{Q}_p$, we shall write $\mathcal{O}_E$ (or simply $\mathcal{O}$, when it is not necessary to specify $E$) for its ring of integers.

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2. Selmer structures and locally restricted Euler/Kolyvagin systems

In this section, we will describe a variety of Selmer modules that we shall work with. Our ultimate goal is to control the Greenberg Selmer modules (which are defined via the Greenberg Selmer structure $\mathcal{F}_{\text{Gr}}$ introduced below) in terms of what we call locally restricted Euler systems. The auxiliary Selmer structures we introduce here play a crucial role in our considerations for this goal.

We suppose in this section that our coefficient ring $\mathcal{R}$ is isomorphic to a power series ring in $r$ variables $\mathcal{O}[[x_1, x_2, \ldots, x_r]]$ over the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$. Let $(\mathbb{T}, \mathcal{S})$ denote a deformation datum as above.

**Definition 2.1.** For every prime $\lambda$ of $\mathcal{K}$ and any quotient $X$ of $\mathbb{T}_{\text{cyc}}$, let

$$H^1_{ur}(K_{\lambda}, Y) = \ker (H^1(K_{\lambda}, Y) \to H^1(K_{\lambda}^{ur}, Y))$$

denote the collection of unramified cohomology classes, where $Y = X$ or $X^c(1)$.

For any quotient $Y$ of $\mathbb{T}$, we write $Y_{\text{cyc}}$ for its cyclotomic deformation.

**Lemma 2.2.** For any quotient $Y$ of $\mathbb{T}$, $H^1_{ur}(K_{\lambda}, Y_{\text{cyc}}) = H^1(K_{\lambda}, Y_{\text{cyc}})$.

**Proof.** This follows from the proof of Lemma 5.3.1(ii) of [MR04].

**Proposition 2.3.** Let $R$ be a complete local Noetherian $\mathbb{Z}_p$-algebra of mixed characteristic with finite residue field $\mathbb{k} = R/\mathfrak{m}_R$. Assume that $R$ is regular. Let $X$ be a free $R$-module of finite rank with a continuous $G_p$-action satisfying $H^0(K_p, X \otimes \mathbb{k}) = H^2(K_p, X \otimes \mathbb{k}) = 0$.

Then the $R$-module $H^1(K_p, X)$ is a free $R$-module of rank $d_X := [K_p : \mathbb{Q}_p] \text{rank}_R X$.

**Proof.** First, the assumption $H^2(K_p, X \otimes \mathbb{k}) = 0$ implies that

$$H^2(K_p, Y) = 0$$

for every $R$-quotient $Y$ of $X$ thanks to Nakayama’s lemma and to the fact that the cohomological dimension of $G_p$ equals to two.

Now let us choose and fix a regular sequence $r_1, \ldots, r_l$ in $R$ such that $\mathfrak{m}_R$ is generated by $r_1, \ldots, r_l$. Note that for each $k$ satisfying $1 \leq k \leq l$, we have the exact sequence

$$0 \to X_{k-1} \xrightarrow{x_{rk}} X_{k-1} \to X_k \to 0$$

where $X_k$ stands for $X/(r_1, \ldots, r_{k-1})X$. This short exact sequence induces the long exact sequence of Galois cohomology

$$0 \to H^0(K_p, X_{k-1}) \xrightarrow{x_{rk}} H^0(K_p, X_{k-1}) \to H^0(K_p, X_k)$$

$$\to H^1(K_p, X_{k-1}) \xrightarrow{x_{rk}} H^1(K_p, X_{k-1}) \to H^1(K_p, X_k) \to 0.$$  

Here the surjectivity of the final map follows from (2.1). Notice that $X_l$ is nothing but $X \otimes \mathbb{k}$. Hence, our assumption that $H^0(K_p, X \otimes \mathbb{k}) = 0$ implies

$$H^1(K_p, X_{k-1})[r_k] = 0$$

and

$$H^1(K_p, X_{k-1}) \cong H^1(K_p, X_{k-1})$$

for every $k$.

Recall for $k = l$ that $H^1(K_p, X_l) = H^1(K_p, X \otimes \mathbb{k})$ is free of rank $d_X$ over $k = R/(r_1, \ldots, r_l)$ by our running assumptions that $H^0(K_p, X \otimes \mathbb{k}) = H^2(K_p, X \otimes \mathbb{k}) = 0$ and by Tate’s local Euler characteristic formula. By Nakayama’s lemma and the second portion of (2.3), we have

$$(R/(r_1, \ldots, r_{k-1}))^{\otimes d_X} \to H^1(K_p, X_{k-1})$$

for $k = 1, \ldots, l + 1$.

On applying the functor $(- \otimes_R R/(r_1, \ldots, r_k))$ to (2.4) and using Nakayama’s lemma and (2.3), we inductively show that the kernel of (2.4) is zero for $k = 1, \ldots, l + 1$, which completes the proof.  \qed
Definition 2.4. Let $T := T$ (resp. $T := T_{\text{cyc}}$) be a module over $R = \mathcal{R}$ (resp. $R = \mathcal{R}_{\text{cyc}}$). For every finite abelian $p$-extension $L$ of $K$, we define the semi-local cohomology group

$$H^1(L_p, F^\pm T) := \bigoplus_{p \mid l} \bigoplus_{\varepsilon \mid p} H^1(L_p, F^\varepsilon T).$$

Remark 2.6. Let $\eta$ be a natural number dividing $p$. If $\eta$ is not trivial, note that $H^1(K_p, T)$ is free of rank $d$. If $H^1(K_p, T)$ is free of rank $d_+$ (resp. of rank $d_-$) then we have

$$H^1(K_p, F^\pm T) = 0.$$

Corollary 2.5. Under the same setting as Definition 2.4, if the hypotheses $(H.0)$ and $(H.2)$ hold true, then the $R$-module $H^1(K_p, T)$ is free of rank $d$. If in addition $(H.2^+)$ (resp. $(H.0^-)$) holds true, then $H^1(K_p, F^+ T)$ (resp. $H^1(K_p, F^- T)$) is free of rank $d_+$ (resp. of rank $d_-)$.

Corollary 2.6. Let $I$ be an ideal of $R$ and set $X := T \otimes_R I$. Then the conclusions of Corollary 2.5 hold verbatim when $T$ is replaced by the $R/I$-module $X$.

Lemma 2.8. Suppose that the hypotheses $(H.0)$, $(H.2)$, $(H.0^-)$ and $(H.2^+)$ hold true. For $\eta \in \Delta_\Sigma$ and for every prime $q$ of $K$ above the prime $p$ of $K$ (see [1,3] for the notation $K(\eta)$), we have

$$H^0(K(\eta)_q, T) = H^2(K(\eta)_q, T) = 0,$$
$$H^0(K(\eta)_q, F^\pm T) = H^0(K(\eta)_q, F^\pm T) = 0.$$

Proof. Write $\Delta_{\eta, q} \subset \Delta_\eta$ for the decomposition group of $q$ and identify it with the local Galois group $\text{Gal}(K(\eta)_q/K_p)$. Let $X$ be the Galois module of finite $p$-power cardinality and such that $H^0(K_p, X) = 0$. If $\Delta_{\eta, q}$ is trivial, then $H^0(K(\eta)_q, X) = H^0(K_p, X) = 0$. If $\Delta_{\eta, q}$ is not trivial, note that $\Delta_{\eta, q}$ is a non-trivial $p$-group by the definition of $K(\eta)$. Therefore we have $\#H^0(K(\eta)_q, X) = 1$ and $\#H^0(K(\eta)_q, X)^{\Delta_{\eta, q}} \equiv 0 \pmod{p}$. By assumption the group $H^0(K_p, X) = H^0(K(\eta)_q, X)^{\Delta_{\eta, q}}$ is trivial and thus $\#H^0(K(\eta)_q, X)^{\Delta_{\eta, q}} = 1$. Hence, the cardinality $\#H^0(K(\eta)_q, X) = 0$. The proof of the lemma follows using this fact with $X = T, T^{(1)}, F^+ T$ and $F^- T$ along with our running hypothesis.

Proposition 2.9. Suppose that $(H.0)$ and $(H.2)$ hold true. Then for every $\eta, \mu \in \Delta_\Sigma$ with $\eta \mid \mu$, the corestriction map

$$H^1(K(\mu)_p, T) \rightarrow H^1(K(\eta)_p, T)$$

on the semi-local cohomology at $p$ is surjective.

Proof. By Tate local duality theorem, in order to prove the map (2.7) is surjective, it suffices to prove that restriction map

$$H^1(K(\eta)_p, T^{(1)}) \rightarrow H^1(K(\mu)_p, T^{(1)})$$

is injective. To prove this, it suffices to show that

$$H^1(K(\eta)_p, T^{(1)}) \rightarrow H^1(K(\mu)_p, T^{(1)})$$

is surjective.
is injective for any prime $p$ of $K(\eta)$ and for any prime $q$ of $K(\mu)$ over $p$. As $K(\mu)_{q}/K(\eta)_{q}$ is a succession of cyclic $p$-extensions, it suffices to prove that the map (2.9) is injective assuming that $K(\mu)_{q}/K(\eta)_{q}$ is a cyclic $p$-extension. By the inflation-restriction sequence of Galois cohomology, the kernel of (2.9) is isomorphic to the $\text{Gal}(K(\mu)_{q}/K(\eta)_{p})$-cohomology of $H^{0}(K(\mu)_{q}, \mathbf{T}^{\eta}(1))$. It follows by Lemma 2.3 that the cohomology group $H^{0}(K(\mu)_{q}, \mathbf{T}^{\eta}(1))$ vanishes and this completes the proof. □

**Proposition 2.10.** In addition to $(H.0)$ and $(H.2)$, suppose that $(H.0^-)$ and $(H.2^+)$ hold true as well. Then for every $\eta, \mu \in \mathcal{N}_{\Sigma}$, the corestriction map

$$H^{1}(K(\mu)_{p}, F^{\pm} \mathbf{T}) \rightarrow H^{1}(K(\eta)_{p}, F^{\pm} \mathbf{T})$$

on the semi-local cohomology at $p$ is surjective.

**Proof.** The additional hypothesis allows the proof of Proposition 2.9 work verbatim for $F^{\pm} \mathbf{T}$. □

**Proposition 2.11.** Let $R$ be a complete local Noetherian $\mathbb{Z}_{p}$-algebra of mixed characteristic with finite residue field $k = R/\mathfrak{m}_{R}$. Assume that $R$ is regular. If $(H.0)$ and $(H.2)$ hold true, then for every $\eta \in \mathcal{N}_{\Sigma}$, the semi-local cohomology group $H^{1}(K(\eta)_{p}, \mathbf{T})$ is a free $R[\Delta_{\eta}]$-module of rank $d$.

**Proof.** By Shapiro’s lemma on Galois cohomology, we have $H^{1}(K(\eta)_{p}, \mathbf{T}) \cong H^{1}(K_{p}, \mathbf{T} \otimes_{R} R[\Delta_{\eta}]^{\mathbb{F}})$ where $R[\Delta_{\eta}]^{\mathbb{F}}$ is a free $R[\Delta_{\eta}]$-module of rank 1 on which $\text{Gal}(\mathbb{T}/K)$ acts tautologically. By assumption, $R[\Delta_{\eta}]$ is a semi-local ring whose local components are all regular. By Proposition 2.3 the corestriction map $H^{1}(K(\eta)_{p}, \mathbf{T}) \rightarrow H^{1}(K_{p}, \mathbf{T})$ is equal to a surjective map induced by the functor $- \otimes_{R[\Delta_{\eta}]} R$. Applying the argument of the proof of Proposition 2.3 componentwise on $R[\Delta_{\eta}]$, $H^{1}(K(\eta)_{p}, \mathbf{T})$ is a free $R$-module of rank $d \cdot |\Delta_{\eta}|$. □

**Proposition 2.12.** In addition to $(H.0)$ and $(H.2)$, suppose that $(H.0^-)$ and $(H.2^+)$ hold true as well. Then for every $\eta \in \mathcal{N}_{\Sigma}$, the $R[\Delta_{\eta}]$-module $H^{1}(K(\eta)_{p}, F^{\pm} \mathbf{T})$ is free of rank $d_{\pm}$.

**Proof.** The additional hypothesis allows the proof of Proposition 2.11 work verbatim for $F^{\pm} \mathbf{T}$ and $F^{-} \mathbf{T}$. □

**Corollary 2.13.** Assuming $(H.0)$ and $(H.2)$ hold, $\lim_{\mu \in \mathcal{N}_{\Sigma}} H^{1}(K(\mu)_{p}, \mathbf{T})$ is a free $R[\Delta]$-module of rank $d$ and the natural projection map

$$\lim_{\mu \in \mathcal{N}_{\Sigma}} H^{1}(K(\mu)_{p}, \mathbf{T}) \rightarrow H^{1}(K(\eta)_{p}, \mathbf{T})$$

is surjective for every $\eta \in \mathcal{N}_{\Sigma}$ and quotient $T$ of $\mathbf{T}$.

If $(H.2^+)$ (resp. $(H.0^-)$) holds true as well, then $\lim_{\mu \in \mathcal{N}_{\Sigma}} H^{1}(K(\mu)_{p}, F^{+} \mathbf{T})$ (resp. the module $\lim_{\mu \in \mathcal{N}_{\Sigma}} H^{1}(K(\mu)_{p}, F^{-} \mathbf{T})$) is a free $R[\Delta]$-module of rank $d_{+}$ (resp. of rank $d_{-}$).

**Proof.** The first portion is immediate after Propositions 2.9 and 2.11. The second assertion follows from Propositions 2.10 and 2.12. □

**Corollary 2.14.** Suppose that the hypotheses $(H.0)$, $(H.2)$, $(H.0^-)$ and $(H.2^+)$ hold true. Then the submodule $\lim_{\mu \in \mathcal{N}_{\Sigma}} H^{1}_{F_{p}, \mathbf{T}}(K(\mu)_{p}, \mathbf{T}) \subset \lim_{\mu \in \mathcal{N}_{\Sigma}} H^{1}(K(\mu)_{p}, \mathbf{T})$ is a free direct summand of rank $d_{+}$.

If we assume in addition that $(H.++)$ holds true, there is a more natural way to define the auxiliary module $H^{1}_{+, +}(F_{p}, \mathbf{T})$. When $(H.++)$ is valid, we shall always stick to this more natural definition (given as in Definition 2.15 below) without any further warning.

**Definition 2.15.** Let $F/K$ be a finite subextension in $K/K$. If the hypotheses $(H.0)$, $(H.2)$, $(H.0^-)$, $(H.2^+)$ as well as $(H.++)$ hold true, we set

$$H^{1}_{++, +}(F_{p}, \mathbf{T}) := \left( \bigoplus_{p|\eta} H^{1}_{F_{p}, \mathbf{T}}(F_{p}, \mathbf{T}) \right) \oplus \left( \bigoplus_{p|\eta} \text{im}(H^{1}(F_{p}, F^{++p} \mathbf{T}) \rightarrow H^{1}(F_{p}, \mathbf{T})) \right)$$

where the submodule $\lim_{\mu \in \mathcal{N}_{\Sigma}} H^{1}_{F_{p}, \mathbf{T}}(K(\mu)_{p}, \mathbf{T})$ is a free direct summand of rank $d_{+}$. □
and define $\nabla_F := \left( \bigoplus_{p \mid \mathcal{P}} \text{im} \left( H^1(F_p, F^{++}^T) \to H^1(F_p, T) \right) \right)$.

By the definition of filtrations given before Theorem B and by the definitions (2.5) and (2.6), we have $H^1_{\text{Gr}}(F_p^0, T) \subset \nabla_F$.

2.1. Selmer structures and Selmer groups. We now define various Selmer structures attached to the deformation datum $(\mathcal{T}, \mathcal{S})$ and its cyclotomic deformation $(\mathbb{T}_{\text{cyc}}, \mathcal{S}_{\text{cyc}})$ and we shall rely on these structures throughout this article.

Let $\Sigma^{(p)} \subset \Sigma$ denote the set of places of $K$ that do not lie above $p$. We retain our convention from the previous section concerning our use of the symbols $\mathbb{T}$, $R$ and $L$. Let $L$ be any finite extension of $K$.

Definition 2.16. (1) The canonical Selmer structure $\mathcal{F}_{\text{can}}$ is defined via the local conditions

(i) $H^1_{\mathcal{F}_{\text{can}}}(L_\lambda, T) := \ker \left( H^1(F_\lambda, T) \to H^1(L_\lambda^\text{ur}, T) \right)$ at primes $\lambda$ of $L$ lying above those in $\Sigma^{(p)}$,

(ii) $H^1_{\mathcal{F}_{\text{can}}}(L_p, T) = H^1(L_p, T)$ at primes above $p$.

(2) Whenever the hypothesis (Pan) holds, the Greenberg Selmer structure $\mathcal{F}_{\text{Gr}}$ is defined via the local conditions

(i) $H^1_{\mathcal{F}_{\text{Gr}}}(L_\lambda, T) := H^1_{\mathcal{F}_{\text{can}}}(L_\lambda, T)$ at primes $\lambda$ of $L$ lying above those in $\Sigma^{(p)}$,

(ii) $H^1_{\mathcal{F}_{\text{Gr}}}(L_p, T) := \text{im} \left( \bigoplus_{p \mid \mathcal{P}} H^1(L_p, F^{++}_p T) \to H^1(L_p, T) \right)$ at primes above $p$.

(3) The Selmer structure $\mathcal{F}_{\lambda}$ is defined via the local conditions

(i) $H^1_{\mathcal{F}_{\lambda}}(L_\lambda, T) := H^1_{\mathcal{F}_{\text{can}}}(L_\lambda, T)$ at primes $\lambda$ of $L$ lying above those in $\Sigma^{(p)}$,

(ii) $H^1_{\mathcal{F}_{\lambda}}(L_p, T) := H^1_{\mathcal{F}_{\text{can}}}(L_p, T)$ at primes above $p$.

Remark 2.17. Notice in the situation when $T = \mathbb{T}_{\text{cyc}}$, it follows from Lemma 2.2 that $H^1_{\mathcal{F}_{\text{Gr}}}(K_\lambda, \mathbb{T}_{\text{cyc}}) = H^1(K_\lambda, \mathbb{T}_{\text{cyc}})$ for every $\lambda \in \Sigma^{(p)}$.

Definition 2.18. Given a Selmer structure $\mathcal{F}$ on $T$, we may propagate it to a quotient $X$ via Example 1.1.2 of [MR04]. We define $H^1_{\mathcal{F}}(K_\lambda, X)$ to be the image of $H^1_{\mathcal{F}}(K_\lambda, T) \subset H^1(K_\lambda, T)$ via $H^1(K_\lambda, T) \to H^1(K_\lambda, X)$. Then we define the Selmer group associated to $\mathcal{F}$ by setting $H^1_{\mathcal{F}}(K, X) := \ker \left( H^1(K_\Sigma/K, X) \to \bigoplus_{\lambda \in \Sigma} H^1(K_\lambda, X) \right)$.

We also define the dual Selmer structure $\mathcal{F}^*$ on the dual representation $X^\vee(1)$ by setting (for every place $\lambda$ of $K$)

$H^1_{\mathcal{F}^*}(K, X^\vee(1)) := H^1_{\mathcal{F}}(K, X)^\perp$,

the orthogonal complement of $H^1_{\mathcal{F}}(K, X)$ under the local Tate pairing, and similarly, the dual Selmer group $H^1_{\mathcal{F}^*}(K, X^\vee(1))$.

2.2. Locally restricted Euler systems and descend to Kolyvagin systems.

2.2.1. Existence of locally restricted Kolyvagin systems. For each prime $\lambda \notin \Sigma$, we set $P_\lambda(X) := \det \left( 1 - F_{\lambda}^{-1}X \mid T^R(1) \right)$ where $T^R = \text{Hom}(T, R)$ is the $R$-linear dual of $T$.

Definition 2.19. Suppose that we have a pair $(\mathbb{T}_{\text{cyc}}, \mathcal{K})$. A collection of elements $c := \{c_{K(\eta)}\}_{\eta \in \mathcal{N}_K}$ such that $c_{K(\mu)} \in H^1(K(\mu)\Sigma/K(\mu), T_{\text{cyc}})$ is called an Euler system if we the following conditions hold:

(ES1) for each $\eta, \eta \in \mathcal{N}_K$ we have $\text{Cor}_{K(\eta)/K(\eta)} c_{K(\eta)} = P_\lambda(F_{\lambda}^{-1}) c_{K(\eta)}$.
We denote the collection of Euler systems for the triple \((\mathbb{T}_{cyc}, \Sigma, \mathcal{K})\) by \(\text{ES}(\mathbb{T}_{cyc})\).

**Remark 2.20.** The Euler polynomials \(P_{\lambda}(X)\) that appear in the definition of an Euler system above may be altered to yield equivalent theories; see [Rub00] Chapter IX for a discussion in this regard and [LLZ14] Lemma 7.3.4 where this alteration is utilized on the Beilinson-Flach Euler system.

**Definition 2.21.** An Euler system \(c \in \text{ES}(\mathbb{T}_{cyc})\) is called locally restricted if we have
\[
\text{res}_p \left( c_F \right) \in H^1_+\left( F_p, \mathbb{T}_{cyc} \right)
\]
for every finite extension \(F\) of \(K\) contained in \(\mathcal{K}\). We denote the collection of locally restricted Euler systems by \(\text{ES}^+(\mathbb{T}_{cyc})\).

For an \((r+2)\)-tuple \(s = (s_0, s_1, \ldots, s_r, s_{r+1}) \in (\mathbb{Z}^+)^{r+2}\) and a fixed topological generator \(\gamma\) of \(\Gamma_{cyc}\), we define the ideal
\[
\mathcal{I}_s := (\pi_0^{s_0}, X_1^{s_1}, \ldots, X_r^{s_r}, (\gamma - 1)^{s_{r+1}}) \subset \mathcal{R}_{cyc}
\]
and set \(T_s := \mathbb{T}_{cyc}/\mathcal{I}_s\mathbb{T}_{cyc}\). We denote the \((r+2)\)-tuple \((1, 1, \ldots, 1)\) by \(I\). We note that \(\mathcal{I}_I = \mathfrak{m}_{\mathcal{R}_{cyc}}\) is the maximal ideal and \(T_I = \mathbb{T}\) is the residual representation. For tuples \(s\) and \(s'\) we write \(s \preceq s'\) to mean that \(s_i \leq s_{i+1}\) for \(i = 0, 1, \ldots, r + 1\).

**Definition 2.22.** We define the set of primes \(\mathcal{P}_s\) to be the set that consists of primes \(\lambda \notin \Sigma\) of \(K\) that satisfy
\[
\begin{align*}
\text{(K.1) } &\mathbb{T}/(\mathcal{I}_s\mathbb{T} + (F_{\lambda} - 1)\mathbb{T}) \text{ is a free } R/\mathcal{I}_s\text{-module of rank one,} \\
\text{(K.2) } &\mathcal{N}_{\lambda} - 1 \in \mathcal{I}_s.
\end{align*}
\]
and such that \(\mathcal{P}_s \supset \mathcal{P}_{s'}\) whenever \(s \preceq s'\). The collection \(\{\mathcal{P}_s\}\) is called the collection of Kolyvagin primes.

We shall prove (under suitable hypotheses) in Lemma 2.24 that a collection of Kolyvagin primes exists.

For a Selmer structure \(\mathcal{F}\) on \(\mathbb{T}\) and \(s\) as above, we may define the module of Kolyvagin systems on the artinian module \(\mathbb{T}_s\). We will not include its precise definition in this note and refer the reader to [MR04, Büy16]. Given a Selmer structure \(\mathcal{F}\) on \(\mathbb{T}_{cyc}\), we let \(\text{KS}(\mathbb{T}_s, \mathcal{F}, \mathcal{P}_s)\) denote the module of Kolyvagin systems for the Selmer triple \((\mathbb{T}_s, \mathcal{F}, \mathcal{P}_s)\) and set
\[
\text{KS}(\mathbb{T}, \mathcal{F}) := \lim_{\leftarrow n} \lim_{\rightarrow s} \text{KS}(\mathbb{T}_s, \mathcal{F}, \mathcal{P}_s).
\]

**Definition 2.23.** The collection \(\text{KS}(\mathbb{T}, \mathcal{F}_+)\) is called the module of locally restricted Kolyvagin systems.

2.2.2. **Existence of Kolyvagin primes.** In this section, we explain how to obtain a useful collection of Kolyvagin primes under suitable hypotheses. These hypotheses are exactly those considered in [MR04] Section 3.5 and are often satisfied. The properties we will rely on are as follows.

\[
\begin{align*}
\text{(MR1) } &\mathbb{T}\text{ is absolutely irreducible as } G_K\text{-module,} \\
\text{(MR2) There exists an element } &\tau \in G_K\text{ with the properties that} \\
&- \mathbb{T}/(\tau - 1)\mathbb{T}\text{ is a free } R\text{-module of rank one,} \\
&- \tau\text{ acts trivially on } \mathbf{\mu}_p^{\infty}. \\
\text{(MR3) } &H^0(K, \mathbb{T}) = H^0(K, \mathbb{T}^\vee(1)) = 0, \\
\text{(MR4) Either } &\text{Hom}_{\mathcal{O}_K}\left(\mathbb{T}, \mathbb{T}^\vee(1)\right) = 0\text{; or else } p > 4.
\end{align*}
\]
Note that the hypotheses (H.0) implies the first vanishing condition in (MR3).
Lemma 2.24. Suppose that the hypothesis (MR2) holds true and fix \( \tau \in G_K \) satisfying (MR2). Consider the set of primes
\[
(2.10) \quad \mathcal{P}_s := \{ \lambda \in \mathcal{N}_S : \text{Fr}_\lambda \text{ is conjugate to } \tau \text{ in } \text{Gal}(K(\mathbb{F}_s, \mu_{p^0})/K) \},
\]
where \( K(\mathbb{F}_s, \mu_{p^0}) \) is the fixed field of \( \ker(G_K \to \text{Aut}(\mathbb{F}_s \oplus \mu_{p^0})) \). Then the set \( \mathcal{P}_s \) verifies the properties (K.1) and (K.2) above, and furthermore, we have \( \mathcal{P}_s \supset \mathcal{P}_{s'} \) if \( s \leq s' \).

Proof. It is clear that the set \( \mathcal{P}_s \) verifies the properties (K.1) and (K.2). Fix a prime \( s = (s_0, s_1, \ldots, s_r, s_{r+1}) \) and note that \( K(T_{s'}, \mu_{p^0}) \supset K(T_s, \mu_{p^0}) \). Then a prime \( \lambda \in \mathcal{N}_S \) belongs to \( \mathcal{P}_{s'} \) if and only if there exists \( \sigma \in G_K \) such that
\[
\sigma \text{Fr}_\lambda \sigma^{-1} \equiv \tau \text{ within } \text{Gal}(K(T_{s'}, \mu_{p^0})/K).
\]
Hence, we have \( \sigma \text{Fr}_\lambda \sigma^{-1} \equiv \tau \) in the quotient \( \text{Gal}(K(\mathbb{F}_s, \mu_{p^0})/K) / \text{Gal}(K(T_{s'}, \mu_{p^0})/K) \) as well, and hence \( \lambda \in \mathcal{P}_s \) as claimed.

We will take the set of primes \( \mathcal{P}_s \) in (2.10) to be the set of primes \( \mathcal{P}_s \) that are required in Definition 2.22.

2.2.3. Euler systems to Kolyvagin systems map. Let \( \mathcal{F}_{\text{can}} \) denote the canonical Selmer structure on \( \mathbb{T}_{\text{cyc}} \), obtained from the Selmer structure \( \mathcal{F}_{\text{fr}} \) by relaxing the local conditions at all primes above \( p \). The following theorem is proved in [MR01, Appendix A] only when the base field \( K \) equals \( \mathbb{Q} \) and when the coefficient ring is either a discrete valuation ring or the cyclotomic Iwasawa algebra. The arguments go through for a general coefficient ring and base field to yield the following result:

Theorem 2.25 (Mazur-Rubin). Assume that the hypotheses (MR1)-(MR4) hold true and suppose for every \( \lambda \in \mathcal{P}_1 \) the homomorphisms \( \{ \text{Fr}_\lambda^r - 1 \}_{r \in \mathbb{Z}} \) are injective on \( \mathbb{T} \). Then there exists a map
\[
\Psi_{\text{MR}} : \text{ES}(\mathbb{T}_{\text{cyc}}) \longrightarrow \text{KS}(\mathbb{T}_{\text{cyc}}, \mathcal{F}_{\text{can}}, \mathcal{P}_1)
\]
such that \( c_K \in H^1(K_S/K, \mathbb{T}_{\text{cyc}}) \) coincides with \( \kappa_1 \in H^1(K_S/K, \mathbb{T}_{\text{cyc}}) \) if the Euler system \( c = \{ c_{K(\eta)} \}_{\eta \in \mathcal{N}_S} \subseteq \text{ES}(\mathbb{T}_{\text{cyc}}) \) and \( c \) maps to \( \kappa = \{ \kappa_\eta \} \in \text{KS}(\mathbb{T}_{\text{cyc}}, \mathcal{F}_{\text{can}}, \mathcal{P}_1) \) where \( \mathcal{P}_1 \) is the set of Kolyvagin primes from (2.10).

Theorem 2.26. In addition to the hypotheses of Theorem 2.25, assume that the hypotheses (H.0), (H.2), (H.0') and (H.2') hold true. Then the map \( \Psi_{\text{MR}} \) restricted to \( \text{ES}^+(\mathbb{T}_{\text{cyc}}) \subset \text{ES}(\mathbb{T}_{\text{cyc}}) \) (see Definition 2.22 for the definition of \( \text{ES}^+(\mathbb{T}_{\text{cyc}}) \)) induces a map
\[
\Psi_{\text{MR}}^+ : \text{ES}^+(\mathbb{T}_{\text{cyc}}) \longrightarrow \text{KS}(\mathbb{T}_{\text{cyc}}, \mathcal{F}_+)\).
\]
This assertion remains valid if we use the local conditions given by Definition 2.12 assuming the truth of (H.2').

Proof. Suppose \( c = \{ c_{K(\eta)} \} \in \text{ES}^+(\mathbb{T}_{\text{cyc}}) \) and we set
\[
\Psi_{\text{MR}}(c) = \{ \kappa_\eta(s) \}_{\eta \in \mathcal{N}_S} \in \text{KS}(\mathbb{T}_{\text{cyc}}, \mathcal{F}_{\text{can}}, \mathcal{P}_1),
\]
where \( \kappa_\eta(s) \in H^1(K_S/K, \mathbb{T}_s) \) for every \( (r + 2) \)-tuple \( s \). Fix \( s \) and \( \eta \in \mathcal{N}_S \) until the end of this proof. We content to prove that \( \kappa_\eta(s) \in H^1_{\mathcal{F}_{\text{can}}(\eta)}(K, \mathbb{T}_s) \), where \( \mathcal{F}_{\text{can}}(\eta) \) is the Selmer structure defined as in [MR01, Example 2.18]. However, the proof of Theorem 5.3.3 of loc. cit. shows already that \( \kappa_\eta(s) \in H^1_{\mathcal{F}_{\text{can}}(\eta)}(K, \mathbb{T}_s) \). Since \( \mathcal{F}_+ \) and \( \mathcal{F}_{\text{can}} \) determine the same local conditions away from the primes above \( p \), it suffices to show that
\[
(2.11) \quad \text{res}_p(\kappa_\eta(s)) \in H^1_{\mathcal{F}_{\text{can}}(\eta)}(K(p), \mathbb{T}_s).
\]
We recall that \( H^1_{\mathcal{F}_{\text{can}}(\eta)}(K(p), \mathbb{T}_s) \) is defined as the image of \( H^1_{\mathcal{F}_{\text{can}}(\eta)}(K(p), \mathbb{T}_{\text{cyc}}) \), which in turn is isomorphic to \( H^1_{\mathcal{F}_{\text{can}}(\eta)}(K(p), \mathbb{T}_{\text{cyc}})/T_sH^1_{\mathcal{F}_{\text{can}}(\eta)}(K(p), \mathbb{T}_{\text{cyc}}) \). In particular, it is a free \( \mathcal{F}_{\text{can}}/\mathcal{T}_s[\Delta_\eta] \)-module of rank \( 1 + d_+ \) by Proposition 2.22. We will verify the truth of (2.11) assuming the truth
of \((H.++)\), the verification in general (where the submodule \(H_{++}^1(K(\eta)_p, T_{\text{cyc}})\) is defined via Definition 2.14) is very similar. Let
\[
\{\kappa_{[K,\eta]} \in H^1(K\Sigma/K, T_s)\}_{\eta \in \mathcal{N}_s}
\]
be the collection of Kolyvagin’s derived classes (given as in [Rub00, Definition 4.4.10]) associated to the locally restricted Euler system \(c\).

It follows from Equation (33) in [MR04, Appendix A] (which explains how to relate the class \(\kappa_{[\lambda]}(s)\) to \(\kappa_{[\lambda],\eta}\) to \(\kappa_{n}\) in loc. cit.) Let \(D_{\eta}\) denote the derivative operator of Kolyvagin, as in [Rub00, Definition 4.4.1]. The class \(\kappa_{[K,\eta]}(s)\) is defined as the inverse image of \(D_{\eta}c_{K(\eta)} (\mod I_s)\) under the restriction map (which is an isomorphism since we assumed (H.3))
\[
H^1(K\Sigma/K, T_s) \longrightarrow H^1(K\Sigma/K(\eta), T_s)\Delta_{\eta}.
\]

Thence, \(\text{res}_{p}(\kappa_{[K,\eta]}(s))\) maps to \(\text{res}_{p}(D_{\eta}c_{K(\eta)}) (\mod I_s)\) under the map (which is also an isomorphism thanks to (H.2))
\[
H^1(K_p, T_s) \cong H^1(K(\eta)_p, T_s)\Delta_{\eta}.
\]

Under this map, \(H_{++}^1(K_p, T_s) \subset H^1(K_p, T_s)\) is mapped isomorphically onto the module \(H_{++}^1(K(\eta)_p, T_s)\Delta_{\eta}\). This follows from the following commutative diagram,
\[
\begin{array}{cccc}
H_{++}^1(K_p, T) / I_s H_{++}^1(K_p, T_{\text{cyc}}) & \sim & H_{++}^1(K_p, T_s)^{\epsilon} & \sim & H^1(K_p, T_s) \\
\downarrow & & \downarrow & & \downarrow \\
(H_{++}^1(K(\eta)_p, T_{\text{cyc}}) / I_s H_{++}^1(K(\eta)_p, T_{\text{cyc}}))^{\Delta_{\eta}} & \sim & H_{++}^1(K(\eta)_p, T_s)^{\Delta_{\eta}^{\epsilon}} & \sim & H^1(K(\eta)_p, T_s)^{\Delta_{\eta}}
\end{array}
\]

where the vertical isomorphism on the left follows from Proposition 2.3 and the isomorphism in the middle thanks to the one on the left.

Since \(\text{res}_{p}(\Delta_{\eta})\) is \(\Delta_{\eta}\)-equivariant, \(\text{res}_{p}(D_{\eta}c_{K(\eta)}) = D_{\eta}\text{res}_{p}(c_{K(\eta)})\). Furthermore, since \(c \in \text{ES}^+(T_{\text{cyc}})\) is locally restricted, we have \(\text{res}_{p}(c_{K(\eta)}) \in H_{++}^1(K(\eta)_p, T_{\text{cyc}})\). On the other hand we know by [Rub00, Lemma 4.4.2] that the derived class \(D_{\eta}c_{K(\eta)} (\mod I_s)\) is fixed by \(\Delta_{\eta}\), which in turn implies that
\[
\text{res}_{p}(c_{K(\eta)}) (\mod I_s) \in \left(H_{++}^1(K(\eta)_p, T_{\text{cyc}}) / I_s H_{++}^1(K(\eta)_p, T_{\text{cyc}})\right)^{\Delta_{\eta}}.
\]

This concludes the proof of the containment (2.12) and also the proof of the theorem. \(\square\)

For some of our main applications, we will utilize the Beilinson-Flach (locally restricted) Euler system of [LLZ14, KLZ13, KLZ17]). Even in cases where we do not have an Euler system at our disposal one may still prove the following result, which shows that the method developed in Section 3 is still non-vacuous for a wide variety of cases of interest.

We consider the following condition at primes in \(\Sigma(p)\):

\(H.\text{Tam}.\) For each non-archimedean place \(\lambda \in \Sigma\) that does not lie above \(p\), we have
\[
H^0(K_{\lambda}, T) = H^2(K_{\lambda}, T) = 0.
\]

**Remark 2.27.** If (H.Tam.) holds true, it follows from the local Euler characteristic formulae and Nakayama’s Lemma that we have \(H^1(K_{\lambda}, X) = 0\) for every \(\lambda \in \Sigma(p)\) and all quotients \(X\) of \(T_{\text{cyc}}\).

**Theorem 2.28.** Suppose that the hypotheses (MR1) - (MR.4), (H.2) hold true for a free \(\mathcal{R}\)-module \(T\) with continuous \(G_K\)-action. We also assume that one of the following conditions is valid.
Definition 3.1. Let $\mathcal{R}$ be an integral domain in our dimension reduction argument.

(i) The ring $\mathcal{R}$ is a discrete valuation ring and $H^0(K^\ur_\lambda, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is $p$-divisible for every $\lambda \in \Sigma^{(p)}$.

(ii) The ring $\mathcal{R}$ is isomorphic to a power series ring in $n$ variables $\Lambda^{(n)}_\mathcal{O}$ and (H.Tam) holds. Then the $\mathcal{R}_{\text{cyc}}$-module $\mathrm{KS}(\mathcal{T}_{\text{cyc}}, \mathcal{F}_+)$ is free of rank one.

See [Buy11] for the proof of (i) and [Buy16] for the proof of (ii). See also [Buy14, BL15] for its various incarnations. We remark again that the results in loc.cit. are stated only for the Selmer structure $\mathcal{F}_{\text{can}}$ and for the base field $\mathbb{Q}$, but their generalization to our set up is entirely formal.

3. Dimension reduction and locally restricted Euler systems

We retain our convention from Section 2 concerning our use of the symbols $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{L}$. Throughout Section 3, we shall assume that $\mathcal{R}$ is isomorphic to a power series ring in $n$ variables $\Lambda^{(n)}_\mathcal{O}$.

3.1. Generalities. In this subsection, we shall recall basic results from [Och05] that will be instrumental in our dimension reduction argument.

**Definition 3.1.** Let $n \geq 1$ be an integer.

1. A linear element $l$ in an $n$-variable Iwasawa algebra $\Lambda^{(n)}_\mathcal{O} = \mathcal{O}[[x_1, \ldots, x_n]]$ is a polynomial $l = a_0 + a_1 x_1 + \cdots + a_n x_n \in \Lambda^{(n)}_\mathcal{O}$ with $a_i \in \mathcal{O}$ of degree at most one such that $l$ is not divisible by $\pi_\mathcal{O}$ and is not invertible in $\Lambda^{(n)}_\mathcal{O}$. That is to say, $l$ is a polynomial of degree at most one such that $a_0$ is divisible by $\pi_\mathcal{O}$, but not all $a_i$ are divisible by $\pi_\mathcal{O}$.

2. An ideal of $\Lambda^{(n)}_\mathcal{O}$ that is generated by a linear element is called a linear ideal. We denote by

$$\mathcal{L}^{(n)}_\mathcal{O} = \left\{ (l) \subset \Lambda^{(n)}_\mathcal{O} \mid l \text{ is a linear element in } \Lambda^{(n)}_\mathcal{O} \right\}.$$ 

the set of all linear ideals of $\Lambda^{(n)}_\mathcal{O}$.

3. Let $n \geq 2$. For a torsion $\Lambda^{(n)}_\mathcal{O}$-module $M$, we denote by $\mathcal{L}^{(n)}_\mathcal{O}(M)$ a subset of $\mathcal{L}^{(n)}_\mathcal{O}$ which consists of $(l) \subset \mathcal{L}^{(n)}_\mathcal{O}$ satisfying the following conditions:

(a) The quotient $M/(l)M$ is a torsion $\Lambda^{(n)}/(l)$-module.

(b) The image of the characteristic ideal $\text{char}_{\Lambda^{(n)}_\mathcal{O}}(M) \subset \Lambda^{(n)}_\mathcal{O}$ in $\Lambda^{(n)}_\mathcal{O}/(l)$ is equal to the characteristic ideal $\text{char}_{\Lambda^{(n)}/(l)}(M/(l)M) \subset \Lambda^{(n)}/(l)$.

We have the following proposition which characterizes the characteristic ideal of a given torsion $\Lambda^{(n)}_\mathcal{O}$-module for $n \geq 2$.

**Proposition 3.2.** Let $n \geq 2$ be an integer and let $M$ and $N$ be a finitely generated torsion $\Lambda^{(n)}_\mathcal{O}$-modules. Then the following three statements are equivalent.

1. We have $\text{char}_{\Lambda^{(n)}_\mathcal{O}}(M) \supset \text{char}_{\Lambda^{(n)}_\mathcal{O}}(N)$.

2. Let $\mathcal{O}'$ be arbitrary complete discrete valuation ring which is finite flat over $\mathcal{O}$. Then, for all but finitely many $(l) \in \mathcal{L}^{(n)}_\mathcal{O}(M_{\mathcal{O}'}) \cap \mathcal{L}^{(n)}_\mathcal{O}(N_{\mathcal{O}'})$, we have the inclusion

$$\text{char}_{\Lambda^{(n)}/(l)}(M_{\mathcal{O}'}/(l)M_{\mathcal{O}'}) \supset \text{char}_{\Lambda^{(n)}/(l)}(N_{\mathcal{O}'}/(l)N_{\mathcal{O}'}).$$

3. There exists a complete discrete valuation ring $\mathcal{O}'$ which is finite flat over $\mathcal{O}$ such that we have the inclusion

$$\text{char}_{\Lambda^{(n)}/(l)}(M_{\mathcal{O}'}/(l)M_{\mathcal{O}'}) \supset \text{char}_{\Lambda^{(n)}/(l)}(N_{\mathcal{O}'}/(l)N_{\mathcal{O}'})$$

for all but finitely many $(l) \in \mathcal{L}^{(n)}_\mathcal{O}(M_{\mathcal{O}'}) \cap \mathcal{L}^{(n)}_\mathcal{O}(N_{\mathcal{O}'})$. 
A set of ideals $\mathcal{E}_\mathcal{O} = \{I_m \subset \Lambda_\mathcal{O} \mid m \in \mathbb{Z}_{\geq 1}\}$ is called Eisenstein type if $I_m = (E_m(x))$ where $E_m(x)$ is an Eisenstein polynomial of degree $m \geq 1$ in $\mathcal{O}[x]$. The following proposition provides the initial step of the induction step.

**Proposition 3.3.** Let $M$ and $N$ be finitely generated torsion $\mathcal{O}[[x]]$-modules. 

1. The following conditions are equivalent:
   (a) There exists an integer $h \geq 0$ such that $\text{char}_{\Lambda_\mathcal{O}}(M) \supset (\pi_\mathcal{O}^h)\text{char}_{\Lambda_\mathcal{O}}(N)$.
   (b) Let $\mathcal{O}'$ be arbitrary complete discrete valuation ring which is finite flat over $\mathcal{O}$. Then there exists a positive integer $c$ depending only on $M_{\mathcal{O}'}$ and $N_{\mathcal{O}'}$ such that
   $$\#(M_{\mathcal{O}'}/IM_{\mathcal{O}'}) \leq c\#(N_{\mathcal{O}'}/IN_{\mathcal{O}'})$$
   for all but finitely many $I \in \mathcal{L}_{\mathcal{O}'}$.

2. Concerning the error term $(\pi_\mathcal{O}^h)$, the following two statements are equivalent:
   (a) Let $M_{(\mathcal{O})}$ (resp. $N_{(\mathcal{O})}$) be the localization of $M$ (resp. $N$) at the prime ideal $(\pi_\mathcal{O})$. Then we have $\text{length}_{(\Lambda_\mathcal{O})(\pi_\mathcal{O})}(M_{(\mathcal{O})}) \leq \text{length}_{(\Lambda_\mathcal{O})(\pi_\mathcal{O})}(N_{(\mathcal{O})})$.
   (b) There exist a set of ideals $\mathcal{E}_\mathcal{O} = \{I_m \mid m \in \mathbb{Z}_{\geq 1}\}$ of Eisenstein type and a positive integer $c$ that depends only on $M$ and $N$ such that
   $$\#(M/I_mM) \leq c\#(N/I_mN)$$
   for all but finitely many $I_m$.

For the proof of Proposition 3.2 (resp. Proposition 3.3), we refer the reader to [Och05, Prop. 3.6] (resp. [Och05, Prop. 3.11]).

### 3.2. The Euler system bound.
Throughout Section 3.2, we assume the hypotheses (MR1) - (MR4) of Mazur and Rubin (that we have recalled in Section 2.2), (H.0), (H.0’), (H.2) and (H.2’). Recall also our convention from Section 2 that $\mathcal{T}$ stands either for $\mathcal{T}$ or $\mathcal{T}_{\text{cyc}}$ and, correspondingly, $R$ is either $\mathcal{R}$ or $\mathcal{R}_{\text{cyc}}$.

**Definition 3.4.** Let $\mathcal{R}$ be a discrete valuation ring.

1. We define the Selmer structure $\mathcal{F}_{\Sigma^+}$ on $\mathcal{T}$ determined as follows
   $$H^1_{\mathcal{F}_{\Sigma^+}}(K_v, \mathcal{T}) := \begin{cases} H^1_{\mathcal{F}}(K_v, \mathcal{T}) & \text{if } v \text{ is above } p, \\ \ker((H^1(K_v, \mathcal{T}) \rightarrow H^1(K^\text{ur}_p, \mathcal{T} \otimes \mathbb{Q}_p))) & \text{if } v \in \Sigma^{(p)}. \end{cases}$$

2. We set $\mathcal{V} := \mathcal{T} \otimes \mathbb{Q}_p$ and define
   $$\text{Tam}_{\Sigma^{(p)}}(\mathcal{V}) := \bigoplus_{\lambda \in \Sigma^{(p)}} H^1_{\mathcal{F}_{\Sigma^+}}(K_\lambda, \mathcal{T}) \bigoplus H^1_{\mathcal{F}_{\Sigma^+}}(K_\lambda, \mathcal{T}),$$
   whose order is precisely the $p$-part of the Tamagawa factors $\lambda$. Notice that since we have assumed (MR1), $G_K$-stable lattices of $\mathcal{V}$ are all isomorphic to $\mathcal{T}$. Thus, our notation for Tamagawa numbers here is justified.

The second portion of the following theorem is proved in [MR04, Büy09a, Büy10] and its remaining parts (1) and (3) follows from (2) by Poitou-Tate global duality. It constitutes the base case for our main result in this section (Theorem 3.6).

**Theorem 3.5.** Suppose that $\mathcal{R}$ is a discrete valuation ring and $c \in \text{ES}^+(\mathcal{T}_{\text{cyc}})$ is a locally restricted Euler system. Suppose that (the image of) its initial term $c_{\mathcal{F}} \in H^1(K_{\Sigma}/K, \mathcal{T})$ is non-vanishing. Then, Then, the following statements hold under the running hypotheses of §3.2.

1. The $\mathcal{R}$-module $H^1_{\mathcal{F}_{\Sigma^+}}(K, \mathcal{T}^\vee(1))^\vee$ is torsion and $H^1_{\mathcal{F}_{\Sigma^+}}(K, \mathcal{T})$ is free of rank one.
2. $\text{Fitt}(H^1_{\mathcal{F}_{\Sigma^+}}(K, \mathcal{T}^\vee(1))^\vee) \supset \text{Fitt}(H^1_{\mathcal{F}_{\Sigma^+}}(K, \mathcal{T})/Rc_K)$.
3. $\text{Fitt}(H^1_{\mathcal{F}_{\Sigma}}(K, \mathcal{T}^\vee(1))^\vee) \supset \text{Fitt}(H^1_{\mathcal{F}_{\Sigma}}(K, \mathcal{T})/Rc_K) \text{Fitt}(\text{Tam}_{\Sigma^{(p)}}(\mathcal{T})).$
In (2) and (3), Fitt stands for the initial Fitting ideal.

Proof. We first explain why (3) holds true, which is readily proved in [Büy10] and that refines the results in [MR04, Buy09a]. All references in this paragraph are to [MR04] unless otherwise stated. Under the Euler systems to Kolyvagin systems map (Theorem 3.2.4), the locally restricted Euler system with initial term \( c_K \neq 0 \) gives rise to a Kolyvagin system for the Selmer structure \( \mathcal{F}_\Sigma^+ \). The only non-trivial input here (in addition to Theorem 3.2.4) is that the derived classes verify the appropriate local conditions at the primes above \( p \). This follows from the proof of Theorem 2.2.6 above. Furthermore, the Selmer structure \( \mathcal{F}_\Sigma^+ \) verifies the hypotheses of Section 5.2 and it is easy to see that its core Selmer rank (in the sense of Definition 4.1.11) equals to one. Theorem 5.2.2 and Theorem 5.2.14 (combined with Theorem 5.2.10 (ii)) show that \( H^1_{\mathcal{F}_\Sigma^+} (K, T^\vee (1)) \) has finite cardinality and \( H^1_{\mathcal{F}_\Sigma^+} (K, T) \) is of rank one. These theorems in fact also prove the divisibility in (2).

We will first prove (3) and deduce (1) as a consequence. Consider the Poitou-Tate global duality sequence

\[
0 \to H^1_{\mathcal{F}_+} (K, T) \to H^1_{\mathcal{F}_\Sigma^+} (K, T) \to \text{Tam}_{\Sigma(p)}(T) \to H^1_{\mathcal{F}_\Sigma^+} (K, T^\vee (1))^\vee \to H^1_{\mathcal{F}_\Sigma^+} (K, T^\vee (1))^\vee \to 0
\]

We therefore conclude that

\[
\frac{\text{Fitt} \left( H^1_{\mathcal{F}_+} (K, T)/Rc_K \right)}{\text{Fitt} \left( H^1_{\mathcal{F}_\Sigma^+} (K, T^\vee (1))^\vee \right)} = \frac{\text{Fitt} \left( H^1_{\mathcal{F}_\Sigma^+} (K, T)/Rc_K \right)}{\text{Fitt} \left( H^1_{\mathcal{F}_\Sigma^+} (K, T^\vee (1))^\vee \right)}
\]

and (3) follows from (2). Moreover, the \( \mathcal{R} \)-module \( \text{Tam}_{\Sigma(p)}(T) \) is torsion as explained in [Rub00] Lemma I.3.5. As \( c_K \in H^1_{\mathcal{F}_+} (K, T) \) is non-zero and since the \( \mathcal{R} \)-module \( H^1_{\mathcal{F}_\Sigma^+} (K, T) \supset H^1_{\mathcal{F}_+} (K, T) \) has rank one by the discussion in the first paragraph of this proof, (1) follows as a consequence of (3).

Recall that our running assumption (besides those we have recorded above) in this section that \( R \) is isomorphic to a power series ring in \( n \) variables \( \Lambda^{(n)}_\mathcal{O} \).

Theorem 3.6. Suppose that the Krull dimension of \( R \) is at least two and \( c \in ES^+ (T_{\text{cyc}}) \) is a locally restricted Euler system. Suppose that (the image of) its initial term \( c_K \in H^1 (K_\Sigma/K, T) \) is non-vanishing. Then, the following statements hold under the running hypotheses of 3.3

1. \( H^1_{\mathcal{F}_+} (K, T^\vee (1))^\vee \) is \( \mathcal{R} \)-torsion.

2. When \( T = T_{\text{cyc}} \), the \( \mathcal{R} \)-module \( H^1_{\mathcal{F}_+} (K, T) \) is free of rank one over \( R \). For general \( T \), the \( \mathcal{R} \)-module \( H^1_{\mathcal{F}_+} (K, T) \) is torsion-free of generic rank one over \( R \).

3. \( \text{char} \left( H^1_{\mathcal{F}_+} (K, T^\vee (1))^\vee \right) \supset \text{char} \left( H^1_{\mathcal{F}_+} (K, T)/Rc_K \right) \).

Remark 3.7. Since we require our Euler system \( c \) to extend in the cyclotomic direction, it follows by a standard argument (see [Rub00] Proposition B.3.4 for details) that \( \text{res}_q (c_F) \in H^1_{\text{ur}} (F_q, T) \) is unramified at every place \( q \) of \( F \) which lies above a prime of \( \Sigma^{(p)} \).

The proof of this theorem is rather involved and will occupy the entire Section 3.2.1. Our argument relies on Lemmas 3.1.1, 3.1.3, 3.1.4, 3.1.6, and 3.1.17 below, which are all stated and proved within Section 3.2.1.

3.2.1. Proof of Theorem 3.6. We will proceed by induction on the Krull dimension of \( R \) (that equals \( r + 1 \) or \( r + 2 \), depending on our choice of coefficient rings among \( \mathcal{R} \) and \( \mathcal{R}_{\text{cyc}} \)) after proving the case \( r = 1 \) separately. Notice that the base case \( r = 0 \) is already settled by Theorem 3.5 above.
Remark 3.8. Since $\text{Loc}^+_p(T)$ is a finitely generated torsion-free $R$-module thanks to our running hypothesis (H.0$^{-}$) assumed at the start of Section 3.2, we have

$$(3.1) \quad \text{Loc}^+_p(T)_{\text{R-tor}} = \left( \bigoplus_{\lambda \in \Sigma^0} H^1(I_\lambda, T)_\text{Fr}_\lambda = 1 \right)_{\text{R-tor}}.$$

When $T = T_{\text{cyc}}$, we have $H^1(I_0, T_{\text{cyc}}) = 0$ and hence, $\text{Loc}^+_p(T) = \text{Loc}^+_p(T)$ is a finitely generated torsion-free $R$-module again due to the hypothesis (H.0$^{-}$).

Let $S_R$ be the set of height-one primes of $R$. We further define the exceptional set of primes $E_R$ for $T$ to be the subset of $S_R$ by setting

$$(3.2) \quad E_R = \{ \pi_O R \} \cup \{ \mathfrak{P} \in S_R : H^2(K_\Sigma/K, T)[\mathfrak{P}] \text{ is infinite} \} \cup \{ \mathfrak{P} \in S_R : \oplus_{\lambda \in \Sigma^0} H^1(K_\lambda, T)[\mathfrak{P}] \text{ is infinite} \} \cup \{ \mathfrak{P} \in S_R : \text{Tam}_{\Sigma^0}(T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}}) \neq 0 \} \cup \{ \mathfrak{P} \in S_R : \text{Tam}_{\Sigma^0}(T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}}) \neq 0 \}.$$

Here, $\pi_O \in O$ is a uniformizing element and $T_{\mathfrak{P}}$ denotes the localization of $T$ at $\mathfrak{P}$ for a height-one prime $\mathfrak{P}$ of $R$. We remark that the definition of $\text{Tam}_{\Sigma^0}(T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}})$ makes sense with Definition 3.3 since $T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}}$ is a finite dimensional $R_{\mathfrak{P}}/\mathfrak{P}$-vector space and $R_{\mathfrak{P}}/\mathfrak{P}$ is a finite extension of $\mathcal{O}_p$.

Definition 3.9. When $r = 1$ and $\mathfrak{P} \in S_R \setminus \{ \pi_O R \}$, we define the degree of $\mathfrak{P}$ to be the quantity $\text{rank}_O R/\mathfrak{P}$.

Lemma 3.10. The set $E_R$ has finite cardinality.

Proof. Using the fact that cohomology groups $H^2(K_\Sigma/K, T)$, $H^2(K_\lambda, T)$ and $H^1(I_\lambda, T)$ are finitely generated over $R$, we see that all the sets appearing in this union has finite cardinality, except for the last set $\{ \mathfrak{P} \in S_R : \text{Tam}_{\Sigma^0}(T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}}) \neq 0 \}$. The finiteness of this set is precisely the content of [Nek06] 7.6.10.10. \hfill \Box

Lemma 3.11. (1) For every height one prime $\mathfrak{P}$ of $R$, the map $T \mapsto T/\mathfrak{P}T$ induces a map

$$\text{pr}_\mathfrak{P} : \frac{H^1_{\mathfrak{P}}(K, T)}{\mathfrak{P}H^1_{\mathfrak{P}}(K, T)} \to H^1_{\mathfrak{P}}(K, T/\mathfrak{P}T).$$

If we assume that $\mathfrak{P} \notin E_R$, the kernel and the cokernel of $\text{pr}_\mathfrak{P}$ is finite and bounded independently of $\mathfrak{P}$. When $T = T_{\text{cyc}}$, the map $\text{pr}_\mathfrak{P}$ is injective.

(2) For every height one prime $\mathfrak{P} \notin E_R$, the map $T \mapsto T/\mathfrak{P}T$ induces an isomorphism

$$\text{pr}_\mathfrak{P} : H^1_{\mathfrak{P}}(K, T^\vee(1)[\mathfrak{P}]) \xrightarrow{\sim} H^1_{\mathfrak{P}}(K, T^\vee(1))/\mathfrak{P}.$$


Proof. For a height one prime $\mathfrak{P}$ of $R$, we consider the exact sequence
\[
(3.3) \quad 0 \rightarrow T \rightarrow T \rightarrow T/\mathfrak{P}T \rightarrow 0,
\]
where the first map is the multiplication of a generator of $\mathfrak{P}$. The $G_{K,\Sigma}$-cohomology of this sequence yields an injective map
\[
(3.4) \quad \frac{H^1(K_{\Sigma}/K, T)}{\mathfrak{P}H^1(K_{\Sigma}/K, T)} \hookrightarrow H^1(K_{\Sigma}/K, T/\mathfrak{P}T).
\]

Consider the following diagram with exact rows and cartesian squares,
\[
(3.5) \quad \begin{array}{cccccc}
\text{Tor}_1^R(R/\mathfrak{P}, M^+_\Sigma) & \xrightarrow{\mathfrak{P}} & H^1_{\Sigma}(K, T) & \xrightarrow{\mathfrak{P}} & H^1(K_{\Sigma}/K, T) & \xrightarrow{\text{res}_\Sigma} & \text{Loc}^+_\Sigma(T) \\
0 & \rightarrow & H^1_{\Sigma}(K/vT) & \rightarrow & H^1(K_{\Sigma}/K, T/\mathfrak{P}T) & \rightarrow & \text{Loc}^+_\Sigma(T/\mathfrak{P}T)
\end{array}
\]
where $M^+_\Sigma \subseteq \text{Loc}^+_\Sigma(T)$ is the image of $\text{res}_\Sigma : H^1(K_{\Sigma}/K, T) \rightarrow \text{Loc}^+_\Sigma(T)$ and the dotted arrow (which is our map $\text{pr}_{\mathfrak{P}}$) is induced from the rest of the diagram.

As explained in the proof of [MR04 Proposition 5.3.14], the size of the cokernel of the map $\nu$ is finite and bounded by a constant that depends only on $T$ as $\mathfrak{P}$ varies away from $\mathcal{E}_R$. In order to verify our claims concerning the kernel of $\text{pr}_{\mathfrak{P}}$, note that we have
\[
\text{Tor}_1^R(R/\mathfrak{P}, M^+_\Sigma) \hookrightarrow \left( \bigoplus_{\lambda \in \Sigma(v)} H^1(I_{\lambda}, T)^{\text{Fr}_{\lambda}=1} \right) [\mathfrak{P}]
\]
by (3.1). As the $R$-module $\left( \bigoplus_{\lambda \in \Sigma(v)} H^1(I_{\lambda}, T)^{\text{Fr}_{\lambda}=1} \right)$ is finitely generated, its maximal $\mathfrak{P}$-torsion submodule has finite order bounded independently as $\mathfrak{P}$ varies away from $\mathcal{E}_R$. The assertion in the case $T = T_{\text{cyc}}$ also follows since we have $H^1(I_{\lambda}, T_{\text{cyc}}) = 0$.

In order to control the cokernel of $\text{pr}_{\mathfrak{P}}$, it suffices to prove that the kernel of the map
\[
\nu : \frac{\text{Loc}^+_\Sigma(T)}{\mathfrak{P} \cdot \text{Loc}^+_\Sigma(T)} \rightarrow \text{Loc}^+_\Sigma(T/\mathfrak{P}T)
\]
is bounded by a constant which is independent of $\mathfrak{P}$ (as the same assertion readily holds for the cokernel of the map in the middle, which is the map (3.4) above). Consider now the following commutative diagram with exact rows.
\[
\begin{array}{cccccc}
\bigoplus_{v|p} H^1_{\Sigma}(K_v, T) & \xrightarrow{\mathfrak{P}} & \bigoplus_{v|p} H^1(K_v, T) & \xrightarrow{\mathfrak{P}} & H^1(K_{\Sigma}/K, T) & \xrightarrow{\text{res}_\Sigma} & \text{Loc}^+_\Sigma(T) \\
0 & \rightarrow & \bigoplus_{v|p} H^1_{\Sigma}(K_v, T/\mathfrak{P}T) & \rightarrow & \bigoplus_{v|p} H^1(K_v, T/\mathfrak{P}T) & \rightarrow & \text{Loc}^+_\Sigma(T/\mathfrak{P}T)
\end{array}
\]
where the surjectivity of the vertical arrow on the left follows by the definition of induced Selmer structures. By Snake Lemma, it remains to control the kernel of $\nu$, which is evidently injective. The portion of the lemma concerning $\text{pr}_{\mathfrak{P}}$ now follows.

The second portion concerning the map $\text{pr}_{\mathfrak{P}}$ is [MR04 Lemma 3.5.3], which applies since we assume (MR1) and (MR3).

For $\mathfrak{P} \notin \mathcal{E}_R$, we let $(R/\mathfrak{P})^{\text{int}}$ denote the integral closure of of $R/\mathfrak{P}$ in its field of fractions $\text{Frac}(R/\mathfrak{P})$. We set $(T/\mathfrak{P}T)^{\text{int}} := T \otimes_R (R/\mathfrak{P})^{\text{int}}$ and since we assume (MR1), observe that it is the unique $G_K$-stable $(R/\mathfrak{P})^{\text{int}}$-lattice in $T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}}$. Notice that
\[
H^1(K_v, (T/\mathfrak{P}T)^{\text{int}}) \cong H^1(K_v, T/\mathfrak{P}T) \otimes_{R/\mathfrak{P}} (R/\mathfrak{P})^{\text{int}}
\]
and it therefore follows from Proposition 2.12 (used together with the vanishing of $H^2(K_p, T)$ thanks to our running hypotheses) and Corollary 2.13 that $H^1(K_p, (T/\mathfrak{P}T)^{\text{int}})$ is a free $(R/\mathfrak{P})^{\text{int}}$-module of rank $d \cdot [K : \mathbb{Q}]$, spanned by the image of $H^1(K_p, T/\mathfrak{P}T)$ under the tautological inclusion $\iota_{\mathfrak{P}} : R/\mathfrak{P} \hookrightarrow (R/\mathfrak{P})^{\text{int}}$. We let $H^1_{++}(K_p, (T/\mathfrak{P}T)^{\text{int}}) \subset H^1(K_p, (T/\mathfrak{P}T)^{\text{int}})$ denote the submodule spanned by the image $H^1_{++}(K_p, T/\mathfrak{P}T)$ under $\iota_{\mathfrak{P}}$. Then the $(R/\mathfrak{P})^{\text{int}}$-module $H^1_{++}(K_p, (T/\mathfrak{P}T)^{\text{int}})$ is a direct summand of the $(R/\mathfrak{P})^{\text{int}}$-module $H^1(K_p, (T/\mathfrak{P}T)^{\text{int}})$ of rank $1 + d_+$. Define the Selmer structure $\mathcal{F}^0_\Sigma$ on $(T/\mathfrak{P}T)^{\text{int}}$ by setting for $\lambda \in \Sigma(\mathfrak{P})$

$$H^1_{++}(K_\lambda, (T/\mathfrak{P}T)^{\text{int}}) := \ker(H^1(K_\lambda, (T/\mathfrak{P}T)^{\text{int}}) \rightarrow H^1(K_{ur}, (T/\mathfrak{P}T)^{\text{int}} \otimes \mathbb{Q}_p))$$

and requiring the local conditions

$$H^1_{++}(K_p, (T/\mathfrak{P}T)^{\text{int}}) := H^1_{++}(K_p, T/\mathfrak{P}T)^{\text{int}}$$

at $p$.

**Remark 3.12.** As $\mathfrak{P} \notin \mathcal{E}_R$, it follows that $\text{Ram}_{\Sigma(\mathfrak{P})}(T_{\mathfrak{P}}/\mathfrak{P}T) = 0$ and hence, we have

$$H^1_{++}(K_\lambda, (T/\mathfrak{P}T)^{\text{int}}) = \ker(H^1(K_\lambda, (T/\mathfrak{P}T)^{\text{int}}) \rightarrow H^1(K_{ur}, (T/\mathfrak{P}T)^{\text{int}})) \ (\lambda \in \Sigma(\mathfrak{P})).$$

Thus the condition agrees with the unramified condition.

Notice that $\iota_{\mathfrak{P}}$ induces maps

$$\iota^\nu_{\mathfrak{P}} : H^1_{++}(K_\nu, T/\mathfrak{P}T) \rightarrow H^1_{++}(K_{\nu'}, (T/\mathfrak{P}T)^{\text{int}})$$

and for the cohomology groups on Cartier duals the maps

$$\iota^{\nu,*}_{\mathfrak{P}} : H^1_{++}(K_\nu, ((T/\mathfrak{P}T)^{\text{int}})^{\vee}(1)) \rightarrow H^1_{++}(K_{\nu'}, (T^{\vee}(1)[\mathfrak{P}]).$$

for every place $\nu \in \Sigma$.

**Lemma 3.13.** Both maps $\iota^\nu_{\mathfrak{P}}$ and $\iota^{\nu,*}_{\mathfrak{P}}$ have finite kernel and cokernel whose sizes are bounded by a constant that only depends on $\text{rank}_R T$, degree of $\mathfrak{P}$ and $[(R/\mathfrak{P})^{\text{int}} : R/\mathfrak{P}]$ as the height one prime $\mathfrak{P}$ varies away from $\mathcal{E}_R$.

**Proof.** The case that concerns the primes $\nu | p$ is obvious by the discussion above. Let us take $\nu$ to be $\lambda \in \Sigma(\mathfrak{P})$. By Remark 3.12 the map $\iota^\nu_{\mathfrak{P}}$ is identical to the composition $\varphi_2 \circ \varphi_1$ of the following maps for $\mathfrak{P} \notin \mathcal{E}_R$:

$$\varphi_1 : H^1_{++}(K_\lambda, T/\mathfrak{P}T) \hookrightarrow H^1_{ur}(K_\lambda, T/\mathfrak{P}T),$$

$$\varphi_2 : H^1_{ur}(K_\lambda, T/\mathfrak{P}T) \rightarrow H^1_{ur}(K_\lambda, (T/\mathfrak{P}T)^{\text{int}}),$$

where $H^1_{ur}(K_\lambda, (T/\mathfrak{P}T) := \ker(H^1(K_\lambda, T/\mathfrak{P}T) \rightarrow H^1(K_{ur}, T/\mathfrak{P}T))$. Hence, it suffices to prove that the kernel and the cokernel of the maps $\varphi_1$ and $\varphi_2$ are bounded only in terms of $T$ and $[(R/\mathfrak{P})^{\text{int}} : R/\mathfrak{P}]$ as $\mathfrak{P}$ varies away from $\mathcal{E}_R$.

The fact that the map $\varphi_1$ is injective follows from the definition of the induced Selmer structure $\mathcal{F}_+$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccc}
H^1_{++}(K_\lambda, T) \otimes R/\mathfrak{P} & \rightarrow & H^1(K_\lambda, T) \otimes R/\mathfrak{P} \\
\downarrow & & \downarrow \\
H^1_{ur}(K_\lambda, T/\mathfrak{P}T) & \rightarrow & H^1(K_\lambda, T/\mathfrak{P}T) \rightarrow H^1(K_{ur}, T/\mathfrak{P}T)
\end{array}$$

The left vertical map is the composition of the natural surjective map $H^1_{++}(K_\lambda, T) \otimes R/\mathfrak{P} \rightarrow H^1_{++}(K_\lambda, T/\mathfrak{P}T)$ (see Definition 2.10) and the map $\varphi_1$. Hence, we have an isomorphism $\text{coker}(\varphi_1) \cong H^2(K_\lambda, T)[\mathfrak{P}]$ by Snake Lemma. The cohomology group $H^2(K_\lambda, T)$ is isomorphic to the $G_{K_\lambda}$-covariance $T(-1)|G_{K_\lambda}$ of $T(-1)$, which is finitely generated over $R$ by local Tate duality. Hence, the size of $H^2(K_\lambda, T)[\mathfrak{P}]$ is bounded when $\mathfrak{P}$ varies.
Next, we turn our attention to \( \varphi_2 \), which is nothing but the map obtained by applying the functors \( \otimes_{R/\mathfrak{P}} H^1_{ur}(K, T/\mathfrak{P}T) \) to the injection \( R/\mathfrak{P} \hookrightarrow (R/\mathfrak{P})^\text{int} \). Hence, the kernel (resp. cokernel) of the map \( \varphi_2 \) is \( \text{Tor}_{R/\mathfrak{P}} \left( \frac{(R/\mathfrak{P})^\text{int}}{R/\mathfrak{P}}, H^1_{ur}(K, T/\mathfrak{P}T) \right) \) (resp. \( \frac{(R/\mathfrak{P})^\text{int}}{R/\mathfrak{P}} \otimes_{R/\mathfrak{P}} H^1_{ur}(K, T/\mathfrak{P}T) \)), which is bounded in the desired manner.

**Lemma 3.14.** For every height one prime \( \mathfrak{P} \subset R \), the composition \( \mathbf{T} \twoheadrightarrow \mathbf{T}/\mathfrak{P}T \hookrightarrow (\mathbf{T}/\mathfrak{P}T)^\text{int} \) induces maps

\[
\pi_{\mathfrak{P}} : H^1_{ext}(K, \mathbf{T}) / \mathfrak{P}H^1_{ext}(K, \mathbf{T}) \twoheadrightarrow H^1_{ext}(K, (\mathbf{T}/\mathfrak{P}T)^\text{int})
\]

\[
\pi^*_{\mathfrak{P}} : H^1_{ext}((\mathbf{T}/\mathfrak{P}T)^\text{int})(1) \twoheadrightarrow H^1_{ext}(K, (\mathbf{T}^\vee(1))(\mathfrak{P})).
\]

When \( \mathfrak{P} \) is not exceptional, both maps \( \pi_{\mathfrak{P}} \) and \( \pi^*_{\mathfrak{P}} \) have finite kernel and cokernel, of size bounded depending only on \( \text{rank}_R \mathbf{T} \), degree of \( \mathfrak{P} \) and \( [(R/\mathfrak{P})^\text{int} : R/\mathfrak{P}] \). Furthermore, when \( \mathbf{T} = \mathbf{T}_{cyc} \) the map \( \pi_{\mathfrak{P}} \) is injective.

**Proof.** When \( \mathbf{T} = \mathbf{T}_{cyc} \), this is nothing but [MR04, Proposition 5.3.14]. Here we extend their arguments to treat the more general set up where \( \mathbf{T} \) is not necessarily equal to \( \mathbf{T}_{cyc} \). The main difficulty arises due to Tamagawa factors and their variation in families; we may circumvent this issue thanks to Lemma 3.10 and Lemma 3.13.

As in the proof of Lemma 3.11 we start with the observation that the size of the cokernel of the map \( (3.4) \) is bounded by a constant that depends only on \( \mathbf{T} \) as \( \mathfrak{P} \) varies away from \( \mathcal{E}_R \). Likewise, the size of the cokernel of the injective map (the argument in the proof of [MR04, Proposition 5.3.14] that concerns the injectivity of this map applies)

\[
H^1(K_{cyc}/K, \mathbf{T}/\mathfrak{P}T) \twoheadrightarrow H^1(K_{cyc}/K, (\mathbf{T}/\mathfrak{P}T)^\text{int})
\]

is also bounded by a constant depending only on \( \mathbf{T} \) and \( [(R/\mathfrak{P})^\text{int} : R/\mathfrak{P}] \). Furthermore, the existence of the map \( \pi^*_{\mathfrak{P}} \) tells us that the map \( (3.6) \) restricts to an injective map

\[
H^1_{ext}(K, \mathbf{T}/\mathfrak{P}T) \twoheadrightarrow H^1_{ext}(K, (\mathbf{T}/\mathfrak{P}T)^\text{int})
\]

whose cokernel is bounded only in terms of \( \text{rank}_R \mathbf{T} \), degree of \( \mathfrak{P} \) and \( [(R/\mathfrak{P})^\text{int} : R/\mathfrak{P}] \) by our previous observations concerning the cokernel of the map \( (3.6) \) and Lemma 3.13. As the map \( \pi_{\mathfrak{P}} \) is the compositum of \( \varphi_2 \) and the map \( (3.7) \), the desired properties of \( \text{ker}(\pi_{\mathfrak{P}}) \) and \( \text{coker}(\pi_{\mathfrak{P}}) \) follows from the observations above and Lemma 3.11.

The bounds on the kernel and cokernel of the map \( \pi^*_{\mathfrak{P}} \) is obtained in a similar manner, using the second portion of Lemma 3.11 and 3.13. \( \square \)

**Proof of Theorem 3.6 for** \( r = 1 \). Since \( c_K \in H^1(K_{cyc}/K, \mathbf{T}) \) is non-zero, there are only finitely many height-one primes \( \mathfrak{Q} \subset R \) with \( c_K \in \mathfrak{Q}H^1(K_{cyc}/K, \mathbf{T}) \). Pick a prime \( \mathfrak{P} \notin \mathcal{E}_R \) such that \( c_K \notin \mathfrak{P}H^1(K_{cyc}/K, \mathbf{T}) \). Then the leading term \( \kappa_1(\mathfrak{P}) \in H^1_{ext}(K, (\mathbf{T}/\mathfrak{P}T)^\text{int}) \) of the Kolyvagin system \( \kappa(\mathfrak{P}) \) in \( K\text{S}((\mathbf{T}/\mathfrak{P}T)^\text{int}) \) obtained as the specialization mod \( \mathfrak{P} \) of the Kolyvagin system \( \Psi_{\text{MR}}(\mathfrak{e}) \in K\text{S}(\mathbf{T}, \mathcal{F}_{cyc}) \) is non-zero. Applying [MR04, Theorem 5.2.2] for \( \kappa(\mathfrak{P}) \) with the discrete valuation ring \( (R/\mathfrak{P})^\text{int} \), it follows that the module \( H^1_{ext}(K, ((\mathbf{T}/\mathfrak{P}T)^\text{int})^\vee(1)) \) has finite cardinality. Thus, by the statement of Lemma 3.11 concerning \( \pi^*_{\mathfrak{P}} \), the module \( H^1_{ext}(K, (\mathbf{T}^\vee(1))(\mathfrak{P})) \) is also finite and hence cotorsion over \( R/\mathfrak{P} \). By the structure theorem of finitely generated \( R \)-modules, \( H^1_{ext}(K, (\mathbf{T}^\vee(1))(\mathfrak{P})) \) must be cotorsion over \( R \) and this concludes the proof of (1) when \( r = 1 \).

In order to prove (2), notice that as \( c_K \in H^1_{ext}(K, \mathbf{T}) \) is non-zero and \( H^1_{ext}(K, \mathbf{T}) \subset H^1(K_{cyc}/K, \mathbf{T}) \) is torsion-free, it follows that the generic \( R \)-rank of \( H^1_{ext}(K, \mathbf{T}) \) is at least one. In the case when \( \mathbf{T} = \mathbf{T}_{cyc} \), let us choose a non-exceptional height one prime \( \mathfrak{P} \) that is generated by a linear
element. Note that $R/\mathfrak{p}$ is a discrete valuation ring and finite flat over $\mathbb{Z}_p$. By applying the second part of Theorem 3.5(1) with $T = T/\mathfrak{p}T$, we see that $H^1(K_\Sigma/K, T/\mathfrak{q}T)$ is free of rank one over $R/\mathfrak{p}$. Thanks to the injectivity of $\pi_\mathfrak{p}$, $\frac{Q H^1(K_\Sigma/K, T)}{\mathfrak{p} H^1(K_\Sigma/K, T)}$ is a non-zero $R/\mathfrak{p}$-submodule of $H^1(K_\Sigma/K, T/\mathfrak{q}T)$. Since non-trivial submodules of free modules of rank one over a discrete valuation ring are still free of rank one, we see that $H^1(K_\Sigma/K, T)$ is free of rank one over $R/\mathfrak{p}$ in this case. By Nakayama’s lemma, we conclude that $H^1(K_\Sigma/K, T)$ is free of rank one over $R$ when $T = T_{\text{cyc}}$.

We now consider the case of general $T$. As in (3.5), we have the following diagram for each $\mathfrak{p}$ generated by a linear element:

$$
\begin{array}{cccc}
\text{Tor}^R_1(R/\mathfrak{p}, M^+_{\Sigma}) & H^1_{\mathcal{F}_+}(K, T) & H^1(K_\Sigma/K, T) & \text{res}_T \mathcal{L}^+_{\Sigma}(T) \\
& \phi_\mathfrak{p} & & \phi \mathcal{L}^+_{\Sigma}(T)
\end{array}
$$

(3.8) 

Suppose on the contrary to our claim that the $R$-module $H^1_{\mathcal{F}_+}(K, T)$ has rank at least 2. By Theorem 3.5(1) and the structure theorem of $R$-modules, $\ker(\phi_\mathfrak{p})$ is not $R/\mathfrak{p}$-torsion. Thus for all but finitely many $\mathfrak{p}$ generated by linear elements, the $R/\mathfrak{p}$-module $\text{Tor}^R_1(R/\mathfrak{p}, M^+_{\Sigma}) \cong M^+_{\Sigma}[\mathfrak{p}]$ is non-torsion. This is only possible when the characteristic ideal of $M_{R,\text{tor}}$ is divisible by all but finitely many $\mathfrak{p}$, which is clearly absurd.

For the proof of (3), we make crucial use of Proposition 3.3. By making use of Theorem 3.5(2) and applying Proposition 3.3(1)(b) and Proposition 3.3(2)(b) to $M = H^1_{\mathcal{F}_+}(K, T^\vee(1))^\vee$ and $N = H^1_{\mathcal{F}_+}(K, T)/R_{\text{cyc}}$, we conclude the proof of the desired divisibility. We remark that the case when $T = T_{\text{cyc}}$ is also the subject of [MR04, Section 5.3] and our assertion is verified as part of [MR04, Theorem 5.3.10].

We remark that although the results we quote above from [MR04, Section 5.3] are stated over $\mathbb{Q}$ (or its completions $\mathbb{Q}_v$), it is a straightforward task to check that they in fact generalize to cover our desired level of generality.

**Case $r > 1$.** Suppose now that $R$ is a regular ring of dimension $r + 1 > 2$ which is isomorphic to a power series ring with coefficients in $\mathcal{O}$ in $r$ variables. Since we assume that $c_K$ is non-zero, we have $c_K \notin l H^1(K_\Sigma/K, T)$ for all but finitely many $(l) \in \mathcal{L}^{(n)}_\mathcal{O}$ in the sense of Definition 3.1. The obvious projection maps give rise to a restricted Euler system $c^{(l)} \in \text{ES}^+(T/\mathcal{T})$. The initial term $c^{(l)}_K$ is non-zero and the condition (MR2) for $T/\mathcal{T}$ holds true for all but finitely many $(l) \in \mathcal{L}^{(n)}_\mathcal{O}$. Identifying the ring $R/(l)$ with a power series ring in $r - 1$ variables (resp. in $r$-variables in case $T = T_{\text{cyc}}$), it follows by induction that

- (Ind1) The $R/(l)$-module $H^1_{\mathcal{F}_{+,l}}(K, T/(l)T^\vee(1))$ is cotorsion and $H^1_{\mathcal{F}_{+,l}}(K, T/(l)T)$ has rank one;
- (Ind2) $\text{char}_{R/(l)}(H^1_{\mathcal{F}_{+,l}}(K, (T/(l)T)^\vee(1))^{\vee}) \cap \text{char}_{R/(l)}(H^1_{\mathcal{F}_{+,l}}(K, T/(l)T)/(R/(l))c^{(l)}_K)$

for all but finitely many choices of $(l) \in \mathcal{L}^{(n)}_\mathcal{O}$. Here, $\mathcal{F}_{+,l}$ is the Selmer structure on $T/(l)T$ which is given by the local conditions determined by $\mathcal{F}_+$ at primes above $p$ and by the unramified local conditions $H^1_{\mathcal{F}_{+,l}}(K_\Sigma, T/(l)T) := H^1_{\text{un}}(K_\Sigma, T/(l)T)$ at primes $\lambda \in \Sigma^{(p)}$. Let us set the following module:

$$
Q := \frac{H^1(K_p, T)}{H^1_{\mathcal{F}_+}(K_p, T) + \text{res}_p(H^1_{\mathcal{F}_{\text{can}}}(K, T))}.
$$

(3.9)
Definition 3.15. We define the exceptional set of linear elements $\mathcal{E}_T^{(r)}$ to be the set which consists of $(l) \in L^{(r)}_O$ such that

$$(H^2(K_\Sigma/K, T) \oplus \bigoplus_{v \in \Sigma} H^2(K_v, T) \oplus \bigoplus_{\lambda \in \Sigma^G} H^1(I_{\lambda}, T)^{F_{\lambda}} \oplus Q \oplus H^1_{\mathcal{F}_{\text{can}}}(K, T^\vee(1))^\vee)[l]$$

is not a pseudo-null $R$-module. We set

$$\text{Err}_\Sigma := \oplus_{v \in \Sigma} H^2(K_v, T)_{\text{null}}$$

where we write $M_{\text{null}}$ for the maximal pseudo-null submodule of a finitely generated $R$-module $M$.

For each positive integer $r$, note that the set $\mathcal{E}_T^{(r)}$ has finite cardinality.

Lemma 3.16. The module

$$D_{\Sigma,l} := \bigoplus_{v \in \Sigma} H^1_{\mathcal{F}_{\Sigma,l}}(K_v, T/lT)$$

is isomorphic to a submodule of $\text{Err}_\Sigma[l]$ for all $l \not\in \mathcal{E}_T^{(r)}$.

Proof. We sketch a proof which is based on [Oh05] Lemma 4.1 and a simple diagram chase as in the proof of Lemma 3.11. For $v \nmid p$, observe that the submodule $H^1_{\mathcal{F}_{\Sigma,l}}(K_v, T/lT) \subset H^1_{\mathcal{F}_{\Sigma,l}}(K_v, T/lT)$ may be identified with the image of the map $\psi$ that is given as part of the commutative diagram below with exact rows:

$$
\begin{array}{ccccccccc}
H^1_{\mathcal{F}_{\Sigma,l}}(K_v, T) \otimes R/(l) & \longrightarrow & H^1(K_v, T) \otimes R/(l) & \longrightarrow & H^1(K_v, T) \otimes R/(l) & \longrightarrow & 0 \\
0 & \longrightarrow & H^1_{\text{ur}}(K_v, T/lT) & \longrightarrow & H^1(K_v, T/lT) & \longrightarrow & H^1(K_v, T/lT) & \longrightarrow & 0 \\
\end{array}
$$

By Snake Lemma, we have

$$\frac{H^1_{\mathcal{F}_{\Sigma,l}}(K_v, T/lT)}{H^1_{\mathcal{F}_{\Sigma,l}}(K_v, T/lT)} \cong \ker(\psi) \subset H^2(K_v, T)[l] \subset H^2(K_v, T)_{\text{null}}.$$

The final containment follows from the fact that $l \not\in \mathcal{E}_T^{(r)}$. A similar diagram for primes above $p$ concludes the proof. \qed

We return back to the proof of Theorem 3.6

Proof of Theorem 3.6 (1) for $r > 1$. It follows using Lemma 3.16 together with (Ind1) that the $R/(l)$-module $H^1_{\mathcal{F}_{\Sigma,l}}(K, (T/lT)^\vee(1))$ is also cotorsion for all but finitely many $(l) \in L^{(r)}_O$. Under the running hypotheses of 3.2 it is not difficult to prove the following control result

$$H^1_{\mathcal{F}_{\Sigma,l}}(K, (T/lT)^\vee(1)) \cong H^1_{\mathcal{F}_{\Sigma,l}}(K, T^\vee(1))[l]$$

(see [MR04] Lemma 3.5.3). This concludes the proof of (1) by using the structure theorem of finitely generated $R$-modules. \qed

Proof of Theorem 3.6 (2) for $r > 1$. It follows from Lemma 3.10 and (Ind1) that the $R/(l)$-module $H^1_{\mathcal{F}_{\Sigma,l}}(K, T/lT)$ is torsion-free of generic rank one for all but finitely many linear elements $l$. Furthermore, thanks to our running hypotheses (H.0) and (H.2), one may argue as in the proof of Proposition 2.3 to show that $H^1_{\mathcal{F}_{\Sigma,l}}(K, T/lT)$ is $R/(l)$-torsion free. Notice that we have an injective homomorphism

$$\frac{H^1(K_\Sigma/K, T)}{lH^1(K_\Sigma/K, T)} \hookrightarrow H^1(K_\Sigma/K, T/lT).$$
We first handle the case when $T = T_{\text{cyc}}$ and $R = \mathcal{R}_{\text{cyc}}$. Observe that the quotient $\frac{H^1(K_p, T)}{H^1_{\text{cyc}}(K_p, T)}$ is torsion free by our assumptions and it therefore follows from the exact sequence (which is deduced from the fact that $H^1(K_\lambda, T) = H^1_{\text{ur}}(K_\lambda, T)$ for $\lambda \in \Sigma^{(p)}$)

\[0 \to H^1_{\mathcal{F}_+}(K, T) \to H^1(K, T) \to \frac{H^1(K_p, T)}{H^1_{\text{cyc}}(K_p, T)}\]

that the map (3.10) induces an injection

\[\frac{H^1_{\mathcal{F}_+}(K, T)}{lH^1_{\mathcal{F}_+}(K, T)} \to H^1_{\mathcal{F}_+}(K, T/lT).\]

As $H^1_{\mathcal{F}_+}(K, T/lT)$ is a torsion-free $R/(l)$-module of rank one for all but finitely many $l$, it follows that the quotient $\frac{H^1_{\mathcal{F}_+}(K, T)}{lH^1_{\mathcal{F}_+}(K, T)}$ is either trivial or has $R/(l)$-rank one. If the former were the case, it would follow from Nakayama’s lemma that $H^1_{\mathcal{F}_+}(K, T) = 0$, contrary to the fact that this module contains the non-zero element $c_K$. We therefore conclude that the quotient $\frac{H^1_{\mathcal{F}_+}(K, T)}{lH^1_{\mathcal{F}_+}(K, T)}$ is an $R/(l)$-module of rank one, for all but finitely many $l$. We may repeat this argument to find a sequence of linear elements $\{l_i\}_{i=1}^{r-1}$ where $l_i \in \mathcal{R}_{i-1}$ for $i < r - 1$ (with the convention that $l_0 = 1$) and $l_{r-1} \in \Lambda_{\text{cyc}}$, to conclude that the quotient module $\frac{H^1_{\mathcal{F}_+}(K, T)}{(l_1, \ldots, l_{r-1})H^1_{\mathcal{F}_+}(K, T)}$ is torsion-free of rank-one over the discrete valuation ring $R/(l_1, \ldots, l_{r-1})$. Therefore the module $\frac{H^1_{\mathcal{F}_+}(K, T)}{(l_1, \ldots, l_{r-1})H^1_{\mathcal{F}_+}(K, T)}$ is cyclic over $R/(l_1, \ldots, l_{r-1})$. The proof of (2) when $T = T_{\text{cyc}}$ now follows by Nakayama’s lemma.

Next, we handle the general case. As in (3.9), we have the following diagram for each linear element $l$:

\[\begin{array}{cccc}
\text{Tor}_{\Sigma}(R/(l), M_{\Sigma}^+) & \xrightarrow{\nu_l} & H^1_{\mathcal{F}_+}(K, T) & \to H^1(K_\Sigma/K, T) \\
\downarrow & & \downarrow & \downarrow \\
0 & \to H^1_{\mathcal{F}_+}(K, T/lT) & \to H^1(K_\Sigma/K, T/lT) & \to \text{Loc}_{\Sigma}^+(T/lT)
\end{array}\]

Suppose on the contrary to our claim that the $R$-module $H^1_{\mathcal{F}_+}(K, T)$ has rank at least 2. By our induction hypothesis and the structure theorem of $R$-modules, this means that $\ker(\nu_l)$ is not $R/(l)$-torsion. The same argument as the proof of Theorem 3.6 (2) for $r = 1$, we prove that the $R$-module $H^1_{\mathcal{F}_+}(K, T)$ has rank at most one. By the existence of a non-trivial element (hence non-torsion element by our running hypothesis (H.0)) $c_K \in H^1_{\mathcal{F}_+}(K, T)$ thus implies that $H^1_{\mathcal{F}_+}(K, T)$ has rank one, as required.

We set the following $R$-modules

\[\text{Err}_+ := \left(\frac{H^2(K_\Sigma/K, T) \oplus Q \oplus H^1_{\text{can}}(K, T^\vee(1))}{\text{null}}\right)
\]

\[K_l := \text{coker} \left(\frac{H^1_{\mathcal{F}_+}(K, T)}{lH^1_{\mathcal{F}_+}(K, T)} \xrightarrow{\nu_l} H^1_{\mathcal{F}_+}(K, T/lT)\right).\]

Before we explain the proof of Theorem 3.6 (3), we give the following preliminary lemma.

**Lemma 3.17.** We have

\[\text{char}_{R/(l)}(K_l) \supset \text{char}_{R/(l)}(\text{Err}_+[l])\]
as \( l \) varies over non-exceptional linear elements. Moreover, we have
\[
\text{char}_{R/(l)}(\ker(\eta)) \supset \text{char}_{R/(l)} \left( \text{Loc}^\Sigma_{\nu}(T)_{\text{null}}[l] \right)
\]

Before going into the proof, we set the following modules
\[
C := \text{coker} \left( H^1_{\mathcal{F}_+}(K, T) \to H^1_{\mathcal{F}_{\text{can}}}(K, T) \right), \quad H^1_{\mathcal{F}_+}(K_p, T) := \frac{H^1(K_p, T)}{H^1_{\mathcal{F}_+}(K_p, T)}.
\]

**Proof.** By the definition of the Selmer structure \( \mathcal{F}_+ \), we have the following exact sequence:
\[
0 \to C \to H^1_{\mathcal{F}_+}(K, T) \to Q \to 0,
\]
where the module \( Q \) was defined in (3.9). Since the \( R \)-module \( H^1_{\mathcal{F}_+}(K, T) \) is torsion-free, we have that the following exact sequence
\[
0 \to Q[l] \to C \otimes R/(l) \to H^1_{\mathcal{F}_+}(K_p, T) \otimes R/(l) \to Q \otimes R/(l) \to 0
\]
by applying \( \otimes_R R/(l) \) to the above sequence. The definition of the Selmer structure \( \mathcal{F}_+ \) yields the following commutative diagram with exact rows:
\[
\begin{array}{ccc}
H^1_{\mathcal{F}_+}(K, T) \otimes R/(l) & \to & H^1_{\mathcal{F}_{\text{can}}}(K, T) \otimes R/(l) \to C \otimes R/(l) \to 0 \\
\downarrow f_l & & \downarrow g_l & & \downarrow h_l \\
0 & \to & H^1_{\mathcal{F}_+}(K, T/lT) & \to & H^1_{\mathcal{F}_{\text{can}}}(K, T/lT) \oplus \bigoplus_{\lambda \in \Sigma^{(p)}} H^1_{\mathcal{F}_+}(K_\lambda, T/lT) & .
\end{array}
\]
By definition we have \( \text{coker}(f_l) \cong K_l \). As for \( \text{coker}(g_l) \), we have the following claim:

**Claim.** The module \( \text{coker}(g_l) \) fits in an exact sequence
\[
H[l] \to \text{coker}(g_l) \to H^2(K_\Sigma/K, T)[l]
\]
where \( H \) isomorphic to a subquotient of \( H^1_{\mathcal{F}_{\text{can}}}(K, T_{\nu}(1))_{\text{null}}^{\nu} \). In particular,
\[
\text{char}_{R/(l)}(\text{coker}(g_l)) \supset \text{char}_{R/(l)} \left( H^1_{\mathcal{F}_{\text{can}}}(K, T_{\nu}(1))^{\nu}[l] \right) \text{char}_{R/(l)} \left( H^2(K_\Sigma/K, T)[l] \right)
\]
for every \( l \notin \Sigma^{(r)}_T \).

**Proof of Claim.** We define the modules
\[
\mathcal{R} \subset \bigoplus_{v \in \Sigma^{(p)}} H^1(K_v^{ur}, T) \quad \text{(resp. } \mathcal{R}_l \subset \bigoplus_{v \in \Sigma^{(p)}} H^1(K_v^{ur}, T/lT)\text{)}
\]
to be the image of \( H^1(K_\Sigma/K, T) \) (resp. \( H^1(K_\Sigma/K, T/lT) \)) under the map
\[
\text{res}^{(p)} : H^1(K_\Sigma/K, Y) \to \bigoplus_{v \in \Sigma^{(p)}} H^1(K_v^{ur}, Y), \ Y = T, T/lT.
\]
Consider the following commutative diagram with exact rows and columns:
\[
\begin{array}{ccc}
H^1_{\mathcal{F}_{\text{can}}}(K, T) \otimes R/(l) & \to & H^1(K_\Sigma/K, T) \otimes R/(l) \to \mathcal{R} \otimes R/(l) \to 0 \\
\downarrow g_l & & \downarrow \Xi_l \\
0 & \to & H^1_{\mathcal{F}_{\text{can}}}(K, T/lT) & \to & H^1(K_\Sigma/K, T/lT) \to \mathcal{R}_l \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
\text{coker}(g_l) & \to & H^2(K_\Sigma/K, T)[l]
\end{array}
\]
where \( \Xi_i \) is induced by the rest of this diagram. In order to conclude the proof our claim via Snake Lemma, it remains to prove that \( \ker(\Xi_i) \) is isomorphic to a subquotient of the module \( H^1_{\text{can}}(K, T^\vee(1))^\vee[l] \). By definition we have the following short exact sequence:

\[
0 \rightarrow \mathfrak{R} \rightarrow H^1(K_{\Sigma}/K, T) \rightarrow \mathcal{C} \rightarrow 0,
\]

where we set \( \mathcal{C} := \ker \left( H^1(K_{\Sigma}/K, T) \xrightarrow{\text{res}_{(p)}} \bigoplus_{v \in \Sigma(p)} H^1(K_v^\ur, T) \right) \). By applying \( \otimes R/(l) \) to this sequence, we have the connecting morphism

\[
\Omega_l : \text{Tor}^R_1(R/l, \mathcal{C}) = \mathcal{C}[l] \rightarrow \mathfrak{R} \otimes R/(l).
\]

By the definitions of the modules \( \mathfrak{R} \) and \( \mathfrak{R}_l \), we have the following commutative diagram with exact columns:

\[
\begin{array}{ccc}
\mathcal{C}[l] & \xrightarrow{\Omega_l} & \mathfrak{R} \otimes R/(l) \\
\downarrow & & \downarrow \Xi_l \\
\mathfrak{R} \otimes R/(l) & \xrightarrow{\Xi_l} & \mathfrak{R}_l \\
\bigoplus_{v \in \Sigma(p)} H^1(K_v^\ur, T) \otimes R/(l) & \xrightarrow{\Xi_l} & \bigoplus_{v \in \Sigma(p)} H^1(K_v^\ur, T/lT)
\end{array}
\]

It follows that \( \ker(\Xi_l) = \text{im}(\Omega_l) \) and our claim will be proved once we verify that \( \mathcal{C} \) is a subquotient of \( H^1_{\text{can}}(K, T^\vee(1))^\vee \). This is an immediate consequence of Poitou-Tate global duality, which identifies \( \mathcal{C} \) with \( \ker \left( H^1_{\text{can}}(K, T^\vee(1)) \rightarrow \bigoplus_{v \in \Sigma(p)} H^1(K_v^\ur, T^\vee(1)) \right) \), where

\[
H^1_{\text{str}}(K, T^\vee(1)) := \ker \left( H^1_{\text{can}}(K, T^\vee(1)) \rightarrow \bigoplus_{v \in \Sigma(p)} H^1(K_v^\ur, T^\vee(1)) \right).
\]

(Notice that \( H^1_{\text{str}}(K, T^\vee(1)) \) and \( H^1(K_{\Sigma}/K, T) \) are dual Selmer groups, in the sense of \([\text{MR04}]\).)

We resume with the proof of (3.12), which will follow (thanks to our observation that \( \text{coker}(f_l) = K_l \) and analysis of \( \text{coker}(g_l) \) above) by Snake Lemma once we check that \( Q[l] \xrightarrow{\sim} \ker(h_l) \). To see that, observe that the map \( h_l \) is the compositum of the following arrows:

\[
h_l : C \otimes R/(l) \xrightarrow{\iota_l} H^1_{\text{can}}(K_p, T) \otimes R/(l) \xrightarrow{j_l} H^1_{\text{can}}(K_p, T/lT) \rightarrow H^1_{\text{can}}(K_p, T/lT) \oplus \bigoplus_{\lambda \in \Sigma(p)} H^1_{\text{can}}(K_{\lambda}, T/lT)
\]

where the injection \( j_l \) follows form the fact that \( H^1_{\text{can}}(K_p, T) \) is \( R \)-torsion free. We conclude by the exactness of the sequence (3.14) that \( Q[l] \xrightarrow{\sim} \ker(h_l) \), as desired.

In order to prove the divisibility (3.13), notice that we have

\[
\text{char}_{R/(l)}(\ker(\nu_l)) \supset \text{char}_{R/(l)}(M_{\Sigma}^+[l]) \supset \text{char}_{R/(l)}(\text{Loc}_{\Sigma}^+(T)[l])
\]

thanks to the diagram (3.11). The proof of (3.13) follows as \( l \) is a non-exceptional linear element by choice.
Proof of Theorem 7.6 (3) for $r > 1$, based on (Ind2). The Poitou-Tate global duality yields the exact sequence

$$0 \to \frac{H_{F+}^1(K, \mathbf{T}/l \mathbf{T})}{R/(l)c^{(l)}_K} \to \frac{H_{F+}^1(K, \mathbf{T}/l \mathbf{T})}{R/(l)c^{(l)}_K} \to \mathcal{D}^+_{\Sigma,l} \to \frac{H_{F+}^1(K, (\mathbf{T}/l \mathbf{T})^\vee(1))^\vee}{H_{F+}^1(K, (\mathbf{T}/l \mathbf{T})^\vee(1))^\vee} \to 0$$

which shows that

$$(3.15) \quad \frac{\text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}/l \mathbf{T})/R/(l)c^{(l)}_K\right)}{\text{char}_{R/(l)}(H_{F+}^1(K, (\mathbf{T}/l \mathbf{T})^\vee(1))^\vee)} = \frac{\text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}/l \mathbf{T})/R/(l)c^{(l)}_K\right)}{\text{char}_{R/(l)}(H_{F+}^1(K, (\mathbf{T}/l \mathbf{T})^\vee(1))^\vee))} \cdot \text{char}_{R/(l)}\left(D^+_{\Sigma,l}\right).$$

Since we have

$$\text{char}_{R/(l)}(H_{F+}^1(K, (\mathbf{T}/l \mathbf{T})^\vee(1))^\vee) = \text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}^\vee(1))^\vee\right)$$

thanks to Lemma 3.5.2 of [MR04], we may rephrase (3.15) to read

$$(3.16) \quad \frac{\text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}/l \mathbf{T})/R/(l)c^{(l)}_K\right)}{\text{char}_{R/(l)}(H_{F+}^1(K, (\mathbf{T}/l \mathbf{T})^\vee(1))^\vee))} = \frac{\text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}/l \mathbf{T})/R/(l)c^{(l)}_K\right)}{\text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}^\vee(1))^\vee\right) \otimes R/(l))} \cdot \text{char}_{R/(l)}\left(D^+_{\Sigma,l}\right).$$

Furthermore, we have

$$\text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}/l \mathbf{T})/R/(l)c^{(l)}_K\right) \cdot \text{char}_{R/(l)}(\ker(\nu_l)) = \text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}^\vee(1))^\vee\otimes R/(l)\right) \cdot \text{char}_{R/(l)}(K_l)$$

where $\nu_l$ is as in (3.11) and $K_l$ in Lemma 3.17. Combining this with (3.16) and (Ind2), we conclude that

$$\text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}/Rc_K) \otimes R/(l)\right) \cdot \text{char}_{R/(l)}\left(D^+_{\Sigma,l}\right) \cdot \text{char}_{R/(l)}(K_l)$$

$$\subset \text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}^\vee(1))^\vee\otimes R/(l)\right) \cdot \text{char}_{R/(l)}(\ker(\nu_l))$$

$$\subset \text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}^\vee(1))^\vee\otimes R/(l)\right)$$

and hence, by Lemma 3.16 and 3.17 that

$$(3.17) \quad \text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}/Rc_K) \otimes R/(l)\right) \text{char}_{R/(l)}(\text{Err}_{\Sigma}[l]) \cdot \text{char}_{R/(l)}(\text{Err}_+[l])$$

$$\subset \text{char}_{R/(l)}\left(H_{F+}^1(K, \mathbf{T}^\vee(1))^\vee\otimes R/(l)\right)$$

for all but finitely many linear elements $l$.

Now set

$$M := H_{F+}^1(K, \mathbf{T}^\vee(1))^\vee$$

$$N := \left(H_{F+}^1(K, \mathbf{T})/Rc_K\right) \oplus \text{Err}_{\Sigma} \oplus \text{Err}_+$$
and apply Proposition 3.18 for \((l) \in L_{\mathcal{O}}^{(n)}(M_{\mathcal{O}}) \cap L_{\mathcal{O}}^{(n)}(N_{\mathcal{O}})\) for any discrete valuation ring \(\mathcal{O}'\) which is finite flat over \(\mathcal{O}\) and by using induction hypothesis for \(r - 1\). We obtain the desired conclusion noting that \(\text{char}_R(\text{Err}_\Sigma \oplus \text{Err}_+^\cdot)\) is trivial. \(\square\)

**Remark 3.18.** We note that there is Nekovář’s general descent machine developed as part of his theory of Selmer complexes in [Nek06]. When Nekovář’s theory applies, it might sometimes simplify descent arguments and yield slightly stronger results. See the proof of Proposition 3.19 below for an instance of this phenomenon.

3.2.2. **Further consequences of Theorem 3.2**. All hypotheses recorded at the start of Section 3.2 are still in effect. Moreover, we also retain our assumption that \(R \cong \Lambda^{(n)}_\mathcal{O}\) is isomorphic to a power series ring.

**Proposition 3.19.** The \(R\)-module \(H_{\mathcal{F}_G}^1(K, T^\vee(1))^\vee\) is torsion if and only if \(H_{\mathcal{F}_G}^1(K, T) = 0\).

**Proof.** By Global duality theorem, we have the following exact sequence (see Definition 2.16 for the definition of local conditions):

\[
(3.18) \quad 0 \to H_{\mathcal{F}_G}^1(K, T) \to H_{\mathcal{F}_G}^1(K, T) \to \bigoplus_{v \in \Sigma} H_{\mathcal{F}_G}^1(K_v, T) \to H_{\mathcal{F}_G}^1(K, T^\vee(1))^\vee.
\]

By Theorem 3.6, the cokernel of the last map in (3.18) is a torsion \(R\)-module. The two modules \(H_{\mathcal{F}_G}^1(K, T)\) and \(\bigoplus_{v \in \Sigma} H_{\mathcal{F}_G}^1(K_v, T)\) in the diagram (3.18) are both of generic rank one over \(R\). It now follows from (3.18) that the \(R\)-module \(H_{\mathcal{F}_G}^1(K, T)\) is torsion if and only if \(H_{\mathcal{F}_G}^1(K, T^\vee(1))^\vee\) is a torsion \(R\)-module. Since \(H^1(K_{\Sigma}/K, T)\) has no non-trivial \(R\)-torsion submodule thanks to our running assumption (MR1), we conclude our proof that \(H_{\mathcal{F}_G}^1(K, T)\) is a torsion \(R\)-module if and only if \(H_{\mathcal{F}_G}^1(K, T) = 0\). \(\square\)

**Corollary 3.20.** Let \(c \in \text{ES}^+(\mathcal{T}_{\text{cyc}})\) be an Euler system such that \(\text{res}_p^c(c_K) \neq 0\), where \(c_K \in H^1(K_{\Sigma}/K, T)\) denotes the image of its initial term. Then \(H_{\mathcal{F}_G}^1(K, T^\vee(1))^\vee\) is \(R\)-torsion.

**Proof.** Let us consider the tautological exact sequence

\[
0 \to H_{\mathcal{F}_G}^1(K, T) \to H_{\mathcal{F}_G}^1(K, T) \xrightarrow{\text{res}_p^c} H^1(K_p, T) \xrightarrow{\text{res}_p^c} H^1(K_p, T).
\]

Since \(H_{\mathcal{F}_G}^1(K, T)\) is torsion-free of rank one by Theorem 3.6 (2) and \(\text{res}_p^c(c_K) \neq 0\) by assumption, we have \(H_{\mathcal{F}_G}^1(K, T) = 0\). Our assertion follows by Proposition 3.19. \(\square\)

Recall the restricted singular quotient \(H^1_{+/f}(K_p) := \frac{H^1_{+/f}(K_p, T)}{H^1_{\mathcal{F}_G}(K_p, T)}\) and the map \(\text{res}_{+/f}\) which is given as the compositum of the arrows

\[
H^1_{\mathcal{F}_G}(K, T) \to H^1_{+/f}(K_p, T) \to H^1_{+/f}(K_p, T).
\]

**Theorem 3.21.** Let \(c \in \text{ES}^+(\mathcal{T}_{\text{cyc}})\) be an Euler system such that \(\text{res}_p^c(c_K) \neq 0\), where \(c_K \in H^1(K_{\Sigma}/K, T)\) denotes the image of its initial term. Then, we have

\[
\text{char} \left( H_{\mathcal{F}_G}^1(K, T^\vee(1))^\vee \right) \supset \text{char} \left( \frac{H^1_{+/f}(K, T)}{R\text{res}_{+/f}(c_K)} \right).
\]

**Proof.** The Poitou-Tate global duality yields (using the proof of Corollary 3.20 for the injection on the left) the exact sequence

\[
0 \to H_{\mathcal{F}_G}^1(K, T)/Rc_K \xrightarrow{\text{res}_{+/f}} H^1_{+/f}(K_p, T)/R\text{res}_{+/f}(c_K) \to H_{\mathcal{F}_G}^1(K, T^\vee(1))^\vee \to H_{\mathcal{F}_G}^1(K, T^\vee(1))^\vee \to 0.
\]
Remark 3.22. We can work with a locally restricted Kolyvagin system \( \kappa \in \text{KS}(\mathbb{T}_{\text{cyc}}, \mathcal{F}_+^\ast) \) and deduce all our conclusions in this section. Furthermore, Theorem 2.28 shows that the bounds obtained in this section via \( \kappa \) are sharp (resp. sharp after inverting \( p \)) if and only if \( \kappa \) generates the cyclic module \( \text{KS}(\mathbb{T}_{\text{cyc}}, \mathcal{F}_+^\ast) \) (resp. generates a submodule of \( \text{KS}(\mathbb{T}_{\text{cyc}}, \mathcal{F}_+^\ast) \) of finite index).

4. Coleman maps for rank-one subquotients

4.1. The setting. Let us denote by \( \Gamma_{\text{cyc}} \) the Galois group of the local cyclotomic \( \mathbb{Z}_p \)-extension of \( \mathbb{Q}_p \), which is canonically isomorphic to \( \mathbb{Z}_p[\Gamma_{\text{cyc}}] \). Let \( \mathbb{R} = 1 + p \mathbb{Z}_p \). Let \( \Gamma_1, \ldots, \Gamma_r \) be profinite groups each of which are isomorphic to \( 1 + p \mathbb{Z}_p \) via the collection of characters \( \chi_i : \Gamma_i \sim \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \) (\( i = 1, \ldots, r \)).

For each \( i \), consider the following Galois characters:

\[ \tilde{\chi}_i : \mathbb{G}_{\mathbb{Q}_p} \to \Gamma_{\text{cyc}} \xrightarrow{\chi_i^{-1} \circ \chi_{\text{cyc}}} \Gamma_i \sim \Gamma_1 \times \cdots \times \Gamma_r \sim \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \]

Let \( \mathcal{R} \) be a local domain which is finite flat over \( \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \) and let \( (\mathbb{T}, \mathcal{R}, \mathcal{S}) \) be a deformation datum in the sense of Definition 1.1 with \( K = \mathbb{Q} \). Note that we slightly relax the assumption on \( \mathcal{R} \) be regular in this portion of our article. The hypothesis (H.++) is in effect throughout this section, which we recall for the convenience of the reader:

(H.++) There exists an \( R \)-module direct summand \( F^{++} \mathbb{T} \) of \( \mathbb{T} \) which is an \( R \)-module of rank \( 1 + d_{1}^{(p)} \) containing \( F_{p}^{+} \mathbb{T} \) and is stable under \( G_{p} \)-action.

Definition 4.1. Let \( \mathcal{R} \) be a local domain which is finite flat over \( \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \) and let \( (\mathbb{T}, \mathcal{R}, \mathcal{S}) \) be a deformation datum. We will be interested in the following list of objects.

\begin{enumerate}
    \item A continuous ring homomorphism \( \kappa : \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \to \overline{\mathbb{Q}}_p \) is called arithmetic if there is an open subgroup \( U \subset \Gamma_1 \times \cdots \times \Gamma_r \) such that \( \kappa|_U \) coincides with \( \chi_1^{w_1(\kappa)} \times \cdots \times \chi_r^{w_r(\kappa)} \) for an ordered \( r \)-tuple \( (w_1(\kappa), \ldots, w_r(\kappa)) \in \mathbb{Z}^r \). When \( \kappa \) is an arithmetic specialization, we note that the set \( \{w_1(\kappa), \ldots, w_r(\kappa)\} \subset \mathbb{Z}^r \) is independent of the choice of \( U \) and the \( r \)-tuple \( (w_1(\kappa), \ldots, w_r(\kappa)) \) is called the weight of \( \kappa \).
    \item Let \( \mathcal{R} \) be a complete local Noetherian \( \mathbb{Z}_p \)-algebra which is a finite module over \( \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \). A continuous ring homomorphism \( \kappa : \mathcal{R} \to \overline{\mathbb{Q}}_p \) is called arithmetic if \( \kappa|_\mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \) is arithmetic in the sense of previous paragraph. The weight of \( \kappa \) is defined to be the weight of \( \kappa|_\mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r] \). For an arithmetic specialization \( \kappa \), we set \( V_\kappa := \mathbb{T} \otimes_\mathbb{F} \text{Frac}(\kappa(\mathcal{R})) \).
\end{enumerate}

Definition 4.2. Let \( (\mathbb{T}, \mathcal{R}, \mathcal{S}) \) be as in the previous definition. Let \( \mathcal{R}_{\text{cyc}} = \mathcal{R}[\Gamma_{\text{cyc}}] \) and we define \( \mathbb{T}_{\text{cyc}} \) to be \( \mathbb{T} \otimes_{\mathbb{Z}_p} (\Lambda^2_{\text{cyc}})^{\ast} \) where \( (\Lambda^2_{\text{cyc}})^{\ast} \) is a free \( \Lambda_{\text{cyc}} \)-module of rank one on which \( G_{\mathbb{Q}_p} \) acts via the inverse tautological character:

\[ \tilde{\chi}^{-1}_{\text{cyc}} : \mathbb{G}_{\mathbb{Q}_p} \to \Gamma_{\text{cyc}} \xrightarrow{\text{inv}} \Gamma_{\text{cyc}} \sim \Lambda^\times_{\text{cyc}} = \mathbb{Z}_p[\Gamma_{\text{cyc}}] \]

We define

\[ \mathcal{S}_{\text{cyc}} \subset \{ \kappa \in \text{Hom}(\mathcal{R}_{\text{cyc}}, \overline{\mathbb{Q}}_p) : \mathcal{R}_{\text{cyc}} \xrightarrow{\kappa} \overline{\mathbb{Q}}_p \text{ is continuous} \} \]

to be the set of specializations \( \mathcal{R}_{\text{cyc}} \to \overline{\mathbb{Q}}_p \) such that \( \kappa|_\mathcal{R} \in \mathcal{S} \). Throughout, the following objects associated to the deformation datum \( (\mathbb{T}_{\text{cyc}}, \mathcal{R}_{\text{cyc}}, \mathcal{S}_{\text{cyc}}) \) will be of interest.

\begin{enumerate}
    \item A continuous ring homomorphism \( \kappa : \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r][[\Gamma_{\text{cyc}}]] = \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r \times \Gamma_{\text{cyc}}] \to \overline{\mathbb{Q}}_p \)
\end{enumerate}
is called \textit{arithmetic} if there is an open subgroup \( U \subset \Gamma_1 \times \cdots \times \Gamma_r \times \Gamma_{\text{cyc}} \) such that \( \kappa|_U \) coincides with \( \chi^{w_1(\kappa)} \times \cdots \times \chi^{w_r(\kappa)} \times \chi^{w_{\text{cyc}}(\kappa)} \) for an ordered set of integers \( (w_1(\kappa), \ldots, w_r(\kappa), w_{\text{cyc}}(\kappa)) \) \( \in \mathbb{Z}^{r+1} \).

(2) Since \( \mathcal{R}_{\text{cyc}} \) is a complete local Noetherian \( \mathbb{Z}_p \)-algebra which is a finite module over \( \mathbb{Z}_p[\Gamma_1 \times \cdots \times \Gamma_r \times \Gamma_{\text{cyc}}] \), the notion of an \textit{arithmetic homomorphism} \( \kappa: \mathcal{R}_{\text{cyc}} \to \overline{\mathbb{Q}}_p \) is defined in the same manner as in Definition 4.1.

**Definition 4.3.** Let \( \mathcal{R} \) be a local domain which is finite flat over \( \mathbb{Z}_p[[\Gamma_1 \times \cdots \times \Gamma_r]] \) and let \( (\mathbb{T}, \mathcal{R}, \mathcal{S}) \) be a deformation. Given a strictly decreasing, \( G_{\mathbb{Q}_p} \)-stable, exhaustive filtration \( \{\text{Fil}_p^n\}_{n \in \mathbb{Z}} \) we consider the following condition:

\[(\text{Ord}) \text{ The } m\text{-th graded piece } \text{Gr}_p^m \mathbb{T} := \text{Fil}_p^m \mathbb{T}/\text{Fil}_p^{m+1} \mathbb{T} \text{ is a free } \mathcal{R}\text{-module of rank one for each } m \text{ such that } 0 \leq m \leq d-1. \text{ We have } \text{Fil}_p^m \mathbb{T} = 0 \text{ for } m < 0 \text{ and we have } \text{Fil}_p^m \mathbb{T} = \mathbb{T} \text{ for } m > 0. \text{ Moreover, the action of } G_{\mathbb{Q}_p} \text{ on } \text{Gr}_p^m \mathbb{T} \text{ is given by the character } \tilde{\chi}_1^{e_{m,1}} \cdots \tilde{\chi}_r^{e_{m,r}} \chi^{w} \chi_{\text{cyc}}^{1-a_m} \tilde{\alpha}_m \text{ where } \omega \text{ is the Teichmüller character and we have }\]
\[e_{m,i} \in \{0,1\} \text{ for } 1 \leq i \leq r, \]
\[a_m, b_m \in \mathbb{Z}, \]
\[\tilde{\alpha}_m: G_{\mathbb{Q}_p} \to \mathcal{R}^\times \text{ is a non trivial unramified character for each } m \in \{0,1,\ldots, d-1\}. \]

We say that \( \mathbb{T}_{\text{cyc}} \) verifies (Ord) if \( \mathbb{T} \) does. We note in this case that the \( G_{\mathbb{Q}_p} \)-action on \( \text{Gr}_p^m \mathbb{T}_{\text{cyc}} \) is given by the character \( \tilde{\chi}_1^{e_{m,1}} \cdots \tilde{\chi}_r^{e_{m,r}} \chi_{\text{cyc}}^{1-a_m} \chi_{\text{cyc}}^{b_m} \tilde{\alpha}_m \).

When \( (\mathbb{T}_{\text{cyc}}, \mathcal{R}_{\text{cyc}}, \mathcal{S}_{\text{cyc}}) \) is a deformation datum satisfying (Ord) and \( \kappa: \mathcal{R}_{\text{cyc}} \to \overline{\mathbb{Q}}_p \) is an arithmetic specialization of \( \mathcal{R}_{\text{cyc}} \) (not necessarily an element of \( \mathcal{S}_{\text{cyc}} \) of weight \( (w_1(\kappa), \ldots, w_r(\kappa), w_{\text{cyc}}(\kappa)) \), we set
\[c_m(\kappa) := \left( \sum_{i=1}^r w_i(\kappa) e_{m,i} \right) - w_{\text{cyc}}(\kappa) + 1 - b_m \]
\[d_m(\kappa) := -c_m(\kappa) + 1 \]
for each \( m \in \{0, 1, \ldots, d-1\} \).

**Definition 4.4.** Let \( (\mathbb{T}_{\text{cyc}}, \mathcal{R}_{\text{cyc}}, \mathcal{S}_{\text{cyc}}) \) be a deformation datum that satisfies (Ord). For each \( m \), we define \( \mathcal{S}_{\text{cyc}}^{(m)+} \) (resp. \( \mathcal{S}_{\text{cyc}}^{(m)-} \)) to be the set of continuous ring homomorphisms \( \kappa: \mathcal{R}_{\text{cyc}} \to \overline{\mathbb{Q}}_p \) such that the rank-one representation \( \text{Gr}_p^m V_\kappa := \text{Gr}_p^m \mathbb{T}_{\text{cyc}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \) is de Rham and \( d_m(\kappa) > 0 \) (resp. \( d_m(\kappa) \leq 0 \)).

**Remark 4.5.** It is not hard to see that when a filtration exists verifying (Ord) then it is necessarily unique. Assume the validity of the hypotheses (Pan), (Ord) and (H.++). It then follows that
\[F^+_p \mathbb{T} = \text{Fil}_p^{d_+} \mathbb{T}, \quad F^{++} \mathbb{T} = \text{Fil}_p^{d_+} \mathbb{T} \text{ and } \quad \text{Gr}_p^{d_+} \mathbb{T} = F^{++} \mathbb{T}/F^+_p \mathbb{T} \]
where \( d_+ := d - d_+^{(p)} \).

4.2. Examples of Galois Deformations and Admissible specializations.
4.2.1. Rankin-Selberg Convolutions. Let $f_1 = \sum_{n=1}^{\infty} a_n(f_1)q^n$ and $f_2 = \sum_{n=1}^{\infty} a_n(f_2)q^n$ be two primitive Hida families as in the introduction. Let $\Sigma$ denote the set of places that contains all rational primes dividing $pN_lN_2$ as well as the archimedean prime. Thanks to Hida and Wiles, we have a two-dimensional continuous irreducible Galois representation

$$\rho_{f_1} : G_{Q,\Sigma} \rightarrow GL_2(\text{Frac}(I_{f_1}))$$

which is characterized by the property that

$$\text{tr}(\rho_{f_1}(F \ell)) = a_\ell(f_1)$$

for every prime $\ell \nmid Np$.

Assume that both of the following hold true for $i = 1, 2$:

(A) We have a free $I_{f_i}$-module $T_{f_i}$ that realizes the Galois representation $\rho_{f_i}$.

(B) There exists a $G_{Q_p}$-stable free $I_{f_i}$-direct summand $F_{p+}T_{f_i} \subset T_{f_i}$ of rank one with which $T_{f_i}$ satisfies (Pan).

The condition (A) is true under the following hypothesis.

(F.Eis) $f_i$ is non-Eisenstein mod $p$ (in the sense that the residual representation is absolutely irreducible).

The following hypothesis together with (A) guarantees the validity of (B).

(F.Dist) $f_i$ is $p$-distinguished (in the sense that the semi-simplification of its residual representation restricted to $G_{Q_p}$ is a direct sum of distinct characters).

Let us define $T = T_{f_1} \otimes_{Z_p} T_{f_2}$, which is a free $R = I_{f_1} \otimes_{Z_p} I_{f_2}$-module of rank 4. Then $T_{cyc} = T \otimes_{Z_p} (\Lambda_{cyc}^2)$ is a free module of rank 4 over $R_{cyc} := R \otimes_{Z_p} A_{cyc}$, on which we allow $G_{Q,\Sigma}$ act diagonally. It is also easy to see that $d_+(T) = d_-(T) = 2$ and $d_+(T_{cyc}) = d_-(T_{cyc}) = 2$.

In this set up, the set

$$S_{cyc} = \{ \kappa \in \text{Hom}(R_{cyc}, \frac{\mathbb{Q}}{\mathbb{Z}}) : R_{cyc} \xrightarrow{\kappa} \frac{\mathbb{Q}}{\mathbb{Z}} \text{ is continuous} \}$$

introduced in Definition 4.1 is the set of specializations of the form $\kappa_1 \otimes \kappa_2 \otimes \kappa_{cyc}$ which are characterized by the following properties:

(i) $\kappa_i$ is an arithmetic specialization of weight $w_i \geq 0$ on the ordinary Hida family $I_{f_i}$, which corresponds to a cuspidal form of weight $k_i = w_i + 2$ ($i = 1, 2$).

(ii) $\kappa_{cyc}$ is of the form $\chi_{cyc}^{\pm} \phi$ where $j$ is an integer and $\phi$ is a character of $\Gamma_{cyc}$ of finite order.

(iii) $w_1$, $w_2$ and $j$ satisfy one of the following conditions:

(a) $w_2 + 1 \leq j < w_1 + 1$.

(b) $w_1 + 1 \leq j < w_2 + 1$.

The filtration on $T_{f_i}$ in (B) above induces a filtration $\{\text{Fil}^i_{p}T_{cyc}\}_{i \in \mathbb{Z}}$ which is characterized by the grading given as follows:

$$\text{Gr}^0_{p}T_{cyc} = (T_{f_1}/F_{p+}T_{f_1}) \otimes_{Z_p} (T_{f_2}/F_{p+}T_{f_2}) \otimes_{Z_p} (\Lambda_{cyc}^4)^i$$

$$\text{Gr}^1_{p}T_{cyc} = (T_{f_1}/F_{p+}T_{f_1}) \otimes_{Z_p} F_{p+}T_{f_2} \otimes_{Z_p} (\Lambda_{cyc}^4)^i$$

$$\text{Gr}^2_{p}T_{cyc} = F_{p+}T_{f_1} \otimes_{Z_p} (T_{f_2}/F_{p+}T_{f_2}) \otimes_{Z_p} (\Lambda_{cyc}^4)^i$$

$$\text{Gr}^3_{p}T_{cyc} = F_{p+}T_{f_1} \otimes_{Z_p} F_{p+}T_{f_2} \otimes_{Z_p} (\Lambda_{cyc}^4)^i$$

and $\text{Gr}^m_{p}T_{cyc} = 0$ for all other integers $m$. We define

$$F_{++T_{cyc}} := F_{p+}T_{f_1} \otimes_{Z_p} T_{f_2} \otimes_{Z_p} (\Lambda_{cyc}^4)^j + T_{f_1} \otimes_{Z_p} F_{p+}T_{f_2} \otimes_{Z_p} (\Lambda_{cyc}^4)^j \subset T_{cyc}$$

(4.2)

$$F_{++T_{cyc}} := F_{p+}T_{f_1} \otimes_{Z_p} T_{f_2} \otimes_{Z_p} (\Lambda_{cyc}^4)^j \subset F_{++T_{cyc}}$$

We note that $F_{++T_{cyc}}$ (resp. $F_{++T_{cyc}}$) is nothing but $\text{Fil}^2_{p}T_{cyc}$ (resp. $\text{Fil}^2_{p}T_{cyc}$) and (H.++) is valid with these choices.

Note that, as $T_{f_1} \otimes_{Z_p} T_{f_2} \otimes_{Z_p} (\Lambda_{cyc}^4)^j$ is isomorphic to $T_{f_1} \otimes_{Z_p} T_{f_2} \otimes_{Z_p} (\Lambda_{cyc}^4)^j$, the representation $T_{cyc}$ remains the same if we interchange the roles of $f_1$ and $f_2$. Similarly, it is clear that the filtration $F_{++T_{cyc}}$ remains unchanged when we interchange the roles of $f_1$ and $f_2$. However, $F_{++T_{cyc}}$ does change if we interchange $f_1$ and $f_2$. 


For each $m \in \{0, 1, 2, 3\}$, the set $\mathcal{S}^{(m),-}$ (resp. $\mathcal{S}^{(m),+}$) of Definition 4.4 is characterized in the current set up by the following conditions:

- $m = 0$: $w_1, w_2$ are arbitrary and $j < 0$ (resp. $j \geq 1$).
- $m = 1$: $w_1$ is arbitrary and $j < \omega_1 + 1$ (resp. $j \geq \omega_1 + 1$).
- $m = 2$: $w_2$ is arbitrary and $j < \omega_1 + 1$ (resp. $j \geq \omega_1 + 1$).
- $m = 3$: $j < w_1 + w_2 + 2$ (resp. $j \geq w_1 + w_2 + 2$).

Denote by $\mathcal{S}^{(1)}_{\text{cyc}}$ the subset of $\mathcal{S}_{\text{cyc}}$ that consists of all specializations for which the module $F^+_p \mathcal{T}_{\text{cyc}}$ in the statement of the condition (Pan) is chosen as in (1.2) above. It is not difficult to see that $\kappa \in \mathcal{S}^{(1)}_{\text{cyc}}$ if and only if it verifies (i), (ii), (iii)-(a). We may similarly define $\mathcal{S}^{(2)}_{\text{cyc}}$ exchanging the roles of $f_1$ and $f_2$.

4.2.2. Siegel modular forms. Our main reference in this example is [LO14] Section 2] that we summarize here after altering their notation to fit our general set up here. Let $f$ be the branch of an ordinary Hida family with tame level $N$ for the group $\text{GSp}(4)$, which is defined over a local domain $\mathbb{I}_f$ finite flat over a two-variable Iwasawa algebra $\mathbb{Z}_p[\Gamma_1 \times \Gamma_2]$. We note that the groups $\Gamma_1$ and $\Gamma_2$ correspond to the groups $G_1$ and $G_2$ in [LO14], whereas our $\mathbb{I}_f$ corresponds to $R_{\text{ord}}$ in op. cit. Attached to $f$, the work of Tilouine-Urban, Urban and Pilloni equips us (under mild technical hypothesis on the residual representation, c.f. Theorem 2.5 in [LO14]) with a free $\mathbb{I}_f$-module $\mathbb{T}_f$ of rank four on which $G_{\mathbb{Q}, \Sigma}$ acts continuously (that corresponds to $\mathbb{T}_{\text{cyc}}$ in loc. cit.). Let us define

$$\mathbb{T}_{\text{cyc}} = \mathbb{T}_f \hat{\otimes}_{\mathbb{Z}_p} (\Lambda_{\text{cyc}}^4)^e$$

which is free of rank 4 over the ring $\mathcal{R} := \mathbb{I}_f \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$ and endow it with the diagonal $G_{\mathbb{Q}, \Sigma}$ action. We note that the coefficient ring $\mathcal{R}$ here corresponds to $R_{n,\text{ord}}$ and $\mathbb{T}_{\text{cyc}}$ here to $\mathbb{T}_{\text{cyc}}^{n,\text{ord}}$ in op. cit.

The set $\mathcal{S}_{\text{cyc}}$ of Definition 1.1 in this set up is the set of specializations of the form $\lambda \otimes \kappa_{\text{cyc}}$, which are characterized by the following properties (thanks to the description of the associated local Galois representation in [LO14] Corollary 2.7] and with the choice $F^+_p \mathbb{T}_{\text{cyc}} := (F^- \mathbb{T}_{n,\text{ord}})^\mathcal{R}(1)$ where the quotient $F^- \mathbb{T}_{n,\text{ord}}$ is given as on pg. 739 of op. cit.):

(i) $\lambda$ is an arithmetic specialization of the Hida family $\mathbb{I}_f$ of weight $(w_1, w_2)$ with integers $0 < w_2 < w_1$, which corresponds to a cuspidal automorphic representation $\pi_1$ of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$.

(ii) $\kappa_{\text{cyc}}$ is of the form $\chi^j_{\text{cyc}} \phi$ where $j$ is an integer and $\phi$ is a character of $\Gamma_{\text{cyc}}$ of finite order.

(iii) $w_2 + 2 \leq j < w_1 + 3$.

It is also clear that $(\mathbb{H}, +)$ is satisfied in this case. The set $\mathcal{S}^{(m),-}$ (resp. $\mathcal{S}^{(m),+}$) for $m \in \{0, 1, 2, 3\}$ is characterized in the current set up by the following conditions:

- $m = 0$: $j < w_1 + w_2 + 4$ (resp. $j \geq w_1 + w + 2 + 4$).
- $m = 1$: $w_1$ is arbitrary and $j < w_1 + 3$ (resp. $j \geq w_1 + 3$).
- $m = 2$: $w_2$ is arbitrary and $j < w_2 + 2$ (resp. $j \geq w_2 + 2$).
- $m = 3$: $w_1, w_2$ are arbitrary and $j < 1$ (resp. $j \geq 1$).

4.3. Coleman map and its interpolation property. We retain the notation of Section 4.1. The goal of this section is to give a construction of the so called big exponential map and big dual exponential map for each graded piece of a deformation datum $(\mathbb{T}_{\text{cyc}}, \mathcal{R}_{\text{cyc}}, \mathcal{S}_{\text{cyc}})$ satisfying the condition (Ord), as stated in Section 4.1. Our main result in this section (Theorem 4.13) is presented at the end. We first introduce some notation and state various preparatory results.

Lemma 4.6. The free rank-one $\mathbb{Z}_p[[G_{\text{cyc}}]]$-module $\mathbb{D}(\mathbb{Z}_p[[G_{\text{cyc}}]]) := D_{\text{crys}}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G_{\text{cyc}}]]$ has the property that the module $\chi^j_{\text{cyc}} \phi(\mathbb{D}(\mathbb{Z}_p[[G_{\text{cyc}}]]))$ is identified with a lattice in $D_{\text{dR}}(\mathbb{Q}_p(j) \otimes \mathbb{Q}_p)$ for every character $\phi$ of $G_{\text{cyc}}$ of finite order, where $D_{\text{crys}}(\mathbb{Z}_p) := (A_{\text{crys}})^{G_{\text{crys}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the canonical lattice in $D_{\text{crys}}(\mathbb{Q}_p) = (B_{\text{crys}})^{G_{\text{crys}}}$ which is canonically identified with $\mathbb{Z}_p$. 
Proof. This is implicitly proved in [Per94] and the module $\mathcal{D}(\mathbb{Z}_p[[\zeta_p]]_{\zeta_p})$ in the statement of our lemma may be identified with a $\mathbb{Z}_p$-lattice of the module $\mathcal{D}_\infty(\mathbb{Z}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_p)$ in Perrin-Riou’s notation in the introduction of loc. cit. In more precise wording, we have

$$\mathbb{D}(\mathbb{Z}_p[[\zeta_p]]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{D}_\infty(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_p).$$

Note that in loc. cit., specialization of an element $x \in \mathcal{D}_\infty(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_p)$ via the character $\chi^j_{\zeta_p}$ is equivalent to specializing the element $x \otimes e_{1}^{j} \in \mathcal{D}_\infty(\mathbb{Q}_p(j)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_p) = \mathcal{D}_\infty(\mathbb{Q}_p) \otimes e_{1}^{j}$ via the character $\phi$, where $e_1$ is a basis of $\mathcal{D}_\text{crys}(\mathbb{Q}_p(1))$ specified by our fixed norm compatible system $\{\zeta_p^j\}$ of $p$-power roots of unity. It therefore remains to explain the definition of the specialization by $\chi^j_{\zeta_p}$ when $j = 0$. Let $n$ be the smallest natural number such that the character $\phi$ factors through the quotient $\text{Gal}(\mathbb{Q}_p(\mu_p^n)/\mathbb{Q}_p)$. Consider the projection map

$$(4.3)\quad \mathbb{D}(\mathbb{Z}_p[[\zeta_p]]) := \mathcal{D}_\text{crys}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\phi} \mathbb{Q}_p \mathbb{Q}_p(\mu_p^n)/\mathbb{Q}_p)$$

as well as the following $\text{Gal}(\mathbb{Q}_p(\mu_p^n)/\mathbb{Q}_p)$-equivariant identifications

$$(4.4)\quad \mathbb{D}_\text{dr}(\mathbb{Q}_p(\mu_p^n)/\mathbb{Q}_p)) \cong \mathbb{Q}_p(\mu_p^n) \cong \mathbb{Q}_p(\mathbb{Q}_p(\mu_p^n)/\mathbb{Q}_p)).$$

The evaluation map $\mathbb{D}_\text{dr}(\mathbb{Q}_p(\mu_p^n)/\mathbb{Q}_p))^\phi \rightarrow E_\phi(\phi)$ induces

$$(4.5)\quad \mathbb{D}_\text{dr}(\mathbb{Q}_p(\mu_p^n)/\mathbb{Q}_p)^\phi \rightarrow \mathbb{D}_\text{ur}(E_\phi(\phi)).$$

The desired specialization map

$$\phi : \mathbb{D}(\mathbb{Z}_p[[\zeta_p]]) := \mathcal{D}_\text{crys}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \rightarrow \mathbb{D}_\text{ur}(E_\phi(\phi))$$

is now defined to be the composition of the maps (1.3), (4.3) and (4.5) above.

Let us set $\mathbb{D}_\text{ur}(\mathbb{Z}_p[[\zeta_p]])^\beta := \mathbb{D}(\mathbb{Z}_p[[\zeta_p]])^\beta \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$. The main construction we shall carry out in this section relies on the following theorem, which essentially is a reformulation of the Coleman map in its most classical form (that was introduced by Coleman himself).

**Theorem 4.7.** We have a $\mathbb{Z}_p[[\zeta_p]]$-linear isomorphism

$$(\text{EXP}_\text{ur})_{\mathbb{Z}_p[[\zeta_p]]} : \mathbb{D}_{\text{ur}}(\mathbb{Z}_p[[\zeta_p]])^\beta \rightarrow H^1(\mathbb{Q}_p, \mathbb{Z}_p[[\zeta_p]]^\beta)/\mathbb{Z}_p$$

such that, for every arithmetic specialization $\kappa$ on $\mathbb{Z}_p[[\zeta_p]]$ of weight $w_{\zeta_p}(\kappa) > 0$, the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{D}_{\text{ur}}(\mathbb{Z}_p[[\zeta_p]]^\beta) & \xrightarrow{\text{EXP}_{\mathbb{Z}_p[[\zeta_p]]}^\beta} & H^1(\mathbb{Q}_p, \mathbb{Z}_p[[\zeta_p]]^\beta)/\mathbb{Z}_p \\
\kappa & \downarrow & \kappa \\
\mathbb{D}_{\text{ur}}(E_\kappa(\kappa)) & \xrightarrow{\exp_{E_\kappa(\kappa)}^\beta \circ e_p^+} & H^1(\mathbb{Q}_p, E_\kappa(\kappa)). \\
\end{array}$$

Here $e_p^+ := (-1)^{w_{\zeta_p}(\kappa)-1}(w_{\zeta_p}(\kappa)-1)!e_p^\text{ur}$ and $e_p^\text{ur} = e_p^\text{ur}(E_\kappa(\kappa))$ is the $p$-adic multiplier given by

$$e_p^\text{ur} := \begin{cases} (1 - p^{-1}\varphi^{-1})(1 - \varphi)^{-1} & \text{when } \kappa = (\chi_{\zeta_p}\omega)^{w_{\zeta_p}(\kappa)}, \\
(p^{-1}\varphi^{-1})^n & \text{when } \kappa = (\chi_{\zeta_p}\omega)^{w_{\zeta_p}(\kappa)}\phi \text{ with } \phi \text{ of conductor } p^n > 1. \end{cases}$$

Also, for every arithmetic specialization $\kappa$ on $\mathbb{Z}_p[[\zeta_p]]$ of weight $w_{\zeta_p}(\kappa) \leq 0$, the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{D}_{\text{ur}}(\mathbb{Z}_p[[\zeta_p]]^\beta) & \xrightarrow{\text{EXP}_{\mathbb{Z}_p[[\zeta_p]]}^\beta} & H^1(\mathbb{Q}_p, \mathbb{Z}_p[[\zeta_p]]^\beta)/\mathbb{Z}_p \\
\kappa & \downarrow & \kappa \\
\mathbb{D}_{\text{ur}}(E_\kappa(\kappa)) & \xrightarrow{(\exp_{E_\kappa(\kappa)}^\beta)^{-1} \circ e_p^-} & H^1(\mathbb{Q}_p, E_\kappa(\kappa)) \\
\end{array}$$
We shall apply the formal tensor product functor $\otimes$ to the isomorphism (4.6) of $R$-norm maps. Recall from [Och03, Lemma 3.3]:

\[ (\hat{\alpha}) \to \text{EX}P^\text{ur}(\mu_p) \otimes \hat{\alpha} \]

\[ \text{isomorphic by Kummer theory to the inverse limit (with respect to norm maps) of } \exp^\text{ur}(\mu_p) \otimes \hat{\alpha}. \]

On the other hand, since $R$ be a non-trivial continuous unramified character. Then, the Galois cohomology group $H^1(Q_p, Z_p[[G_{cyc}]]) \otimes_{Z_p} R(\hat{\alpha})$ is a free $R$-module of rank one over $Z_p[[G_{cyc}]].$ We do not repeat the proof of Theorem 4.7 for this reason, but simply note that the construction of the map in Theorem 4.7 follows very closely the theory of classical Coleman power series for the extension $\mathbb{Q}_p^\text{ur}(\mu_p) / \mathbb{Q}_p^\text{ur}.$ Indeed, notice that the group $H^1(Q_p, Z_p[[G_{cyc}]]) \otimes_{Z_p} \mathbb{R}(\hat{\alpha})$ is isomorphic by Kummer theory to the inverse limit (with respect to norm maps) of $\mathbb{Q}_p^\text{ur}(\mu_p) \otimes \hat{\alpha}$ where the superscript $\wedge$ stands for $p$-adic completion. With this identifications at hand, Theorem 4.7 is a reformulation of the classical theory of Coleman power series.

**Corollary 4.8.** Let $R$ be a complete Noetherian semi-local $\mathbb{Z}_p$-algebra of characteristic 0 and let $\hat{\alpha} : G_{Q_p} \to R^\times$ be a non-trivial continuous unramified character. Then, the Galois cohomology group $H^1(Q_p, Z_p[[G_{cyc}]]) \otimes_{Z_p} R(\hat{\alpha})$ is a free $R$-module of rank one over $Z_p[[G_{cyc}]].$

**Proof.** We shall apply the formal tensor product functor $(- \otimes_{Z_p} R(\hat{\alpha}))$ to the isomorphism $\exp^\text{ur}(\mu_p) / \mathbb{Q}_p$ and take the $\text{Gal}(\mathbb{Q}_p / \mathbb{Q}_p)$-invariants. We shall need the following lemma that we recall from [Och03, Lemma 3.3]:

**Lemma 4.9.** Let $R$ be a complete semi-local Noetherian $\mathbb{Z}_p$-algebra of mixed characteristic and let $M$ be a free $R$-module of finite rank $e$ that is endowed with an unramified action of $G_{Q_p}.$ Then $(M \otimes_{Z_p} \mathbb{Z}_p^\text{ur})^{\text{Gal}(\mathbb{Q}_p / \mathbb{Q}_p)}$ is a free $R$-module of rank $e.$

Recall that $\mathbb{D}(Z_p[[G_{cyc}]]) \otimes_{Z_p} R(\hat{\alpha})$ is isomorphic to $\hat{\mathbb{D}}_{\text{ur}}(\mathbb{Q}_p, Z_p[[G_{cyc}]]) \otimes_{Z_p} R(\hat{\alpha}).$ Applying Lemma 4.9 with $R = Z_p[[G_{cyc}]] \otimes_{Z_p} R$, it follows that

\[ (\mathbb{D}^\text{ur}(Z_p[[G_{cyc}]])) \otimes_{Z_p} R(\hat{\alpha}) \]

\[ \cong Z_p[[G_{cyc}]] \otimes_{Z_p} R. \]

On the other hand, since $\mathbb{R}(\hat{\alpha})^{\text{Gal}(\mathbb{Q}_p / \mathbb{Q}_p)} = 0$ by assumption we have

\[ (H^1(Q_p, Z_p[[G_{cyc}]]) / Z_p) \otimes_{Z_p} \mathbb{R}(\hat{\alpha}) \]

\[ \cong H^1(Q_p, Z_p[[G_{cyc}]]) \otimes_{Z_p} \mathbb{R}(\hat{\alpha})^{\text{Gal}(\mathbb{Q}_p / \mathbb{Q}_p)}. \]

The proof of the corollary follows combining the isomorphisms (4.6) and (4.7). \qed

For integers $a, b,$ we define $\mathbb{D}(Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2)$ by setting

\[ \mathbb{D}(Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2) := \mathbb{D}(\Lambda_{\text{cyc}}^2) \otimes_{Z_p} D_{\text{crys}}(Z_p(\omega)^a) \otimes_{Z_p} D_{\text{crys}}(Z_p(b)). \]

Here $D_{\text{crys}}(Z_p(b))$ is the $Z_p$-lattice in $D_{\text{crys}}(Q_p(b)) \subset B_{\text{dR}}^+,$ generated by $\{q_p\} \otimes t^{\otimes (-b)}$ over $Z_p,$ where $t := \log[e]$ is the well-known uniformizer of the ring of $p$-adic periods $B_{\text{dR}}^+$ of Fontaine. It is not difficult to see that $\mathbb{D}(Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2)$ is naturally a free $\Lambda_{\text{cyc}}$-module of rank one. We further define the module

\[ \mathbb{D}^\text{ur}(Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2) := \mathbb{D}(Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2) \otimes_{Z_p} \mathbb{Z}_p^\text{ur}. \]

The following is a slight generalization of Theorem 4.7.

**Theorem 4.10.** For an arbitrary pair of integers $a$ and $b$, we have an $\Lambda_{\text{cyc}}$-linear isomorphism

\[ \text{EXP}^\text{ur}(a, b)_{Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2} : \mathbb{D}^\text{ur}(Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2) \to H^1(Q_p, Z_p(\omega)^a \otimes Z_p(b) \otimes \Lambda_{\text{cyc}}^2) \]

\[ \to Z_p(\omega)^a \otimes Z_p(b) \]
such that, for every arithmetic specialization \( \kappa \) on \( \Lambda_{\text{cyc}} \) of weight \( w_{\text{cyc}}(\kappa) \geq 1 - b \), the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{D}^m (\mathbb{Z}_p(\omega^a) \otimes \mathbb{Z}_p(b) \otimes \Lambda_{\text{cyc}}^e) & \xrightarrow{\text{EXP}^m, (a, b)} & H^1(\mathbb{Q}_p^m, \mathbb{Z}_p(\omega^a) \otimes \mathbb{Z}_p(b) \otimes \Lambda_{\text{cyc}}^e) \\
\kappa & & \kappa \\
D_{dR}^m (\mathbb{Q}_p(b) \otimes E_\kappa(\omega^a \kappa)) & \xrightarrow{\exp_{\mathbb{Q}_p(b) \otimes E_\kappa(\omega^a \kappa)}^m \circ e^{m, -}_p} & H^1(\mathbb{Q}_p^m, \mathbb{Q}_p(b) \otimes E_\kappa(\omega^a \kappa))
\end{array}
\]

Here \( e^{m, +}_p := (-1)^{w_{\text{cyc}}(\kappa) - (w_{\text{cyc}}(\kappa) + b - 1)!} e^{m}_p \) and \( e^{m}_p = e^{m}_p (\mathbb{Q}_p(b) \otimes E_\kappa(\omega^a \kappa)) \) is the \( p \)-adic multiplier given by

\[
e^{m}_p := \begin{cases} 
(1 - p^{-1} \varphi^{-1})(1 - \varphi)^{-1} & \text{when } E_\kappa(\omega^a \kappa) \text{ is crystalline} \\
(1 - p^{-1} \varphi)^n & \text{when } E_\kappa(\omega^a \kappa) \cong \mathbb{Q}_p(w_{\text{cyc}}(\kappa)) \otimes E_\kappa(\phi) \text{ with } \text{ord}_p(\text{cond}(\phi)) = n \geq 1.
\end{cases}
\]

Also, for every arithmetic specialization \( \kappa \) on \( \Lambda_{\text{cyc}} \) of weight \( w_{\text{cyc}}(\kappa) < 1 - b \), the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{D}^m (\mathbb{Z}_p(\omega^a) \otimes \mathbb{Z}_p(b) \otimes \Lambda_{\text{cyc}}^e) & \xrightarrow{\text{EXP}^m, (a, b)} & H^1(\mathbb{Q}_p^m, \mathbb{Z}_p(\omega^a) \otimes \mathbb{Z}_p(b) \otimes \Lambda_{\text{cyc}}^e) \\
\kappa & & \kappa \\
D_{dR}^m (\mathbb{Q}_p(b) \otimes E_\kappa(\omega^a \kappa)) & \xrightarrow{(\exp_{\mathbb{Q}_p(b) \otimes E_\kappa(\omega^a \kappa)}^m)^{-1} \circ e^{m, -}_p} & H^1(\mathbb{Q}_p^m, \mathbb{Q}_p(b) \otimes E_\kappa(\omega^a \kappa))
\end{array}
\]

where \( e^{m, -}_p := e^{m}_p (-w_{\text{cyc}}(\kappa) - b)! \).

**Proof.** On twisting the diagrams in the statement of Theorem 4.1 (that characterizes the map \( \text{EXP}^m_{\mathbb{Z}_p[[G_{\text{cyc}}]]} \) by the character \( \chi_{\text{cyc}}(\omega)^b \)), we obtain a map \( \text{EXP}^m_{\mathbb{Z}_p(b) \otimes \mathbb{Z}_p[[G_{\text{cyc}}]]} \) that is characterized by an interpolation diagram that is the twisted version of those appear Theorem 4.1. Recall that \( \mathbb{Z}_p[[G_{\text{cyc}}]] \cong \Lambda_{\text{cyc}}[\Delta] \) where \( \Delta \) stands for \( \text{Gal}(\mathbb{Q}_p(\mu_p))/\mathbb{Q}_p \cong (\mathbb{Z}_p/p\mathbb{Z})^\times \). We define the map \( \text{EXP}^m_{\mathbb{Z}_p(b) \otimes \mathbb{Z}_p[[G_{\text{cyc}}]]} \) as the restriction of \( \text{EXP}^m_{\mathbb{Z}_p(b) \otimes \mathbb{Z}_p[[G_{\text{cyc}}]]} \) to the \( \omega^{a-b} \)-isotypic components. This map has the desired interpolation properties by construction. \( \square \)

In order to formulate a further generalization of Theorem 4.10 we introduce some notation.

**Definition 4.11.** Let \( \mathcal{R} \) be a local domain which is finite flat over \( \mathbb{Z}_p[[\Gamma_1 \times \cdots \times \Gamma_r]] \) and let \( (\mathcal{T}, \mathcal{R}, \mathcal{S}) \) be a deformation datum. Suppose that we have a strictly decreasing, \( G_{\mathbb{Q}_p} \)-stable, exhaustive filtration \( \{\text{Fil}_i \mathbb{T}\}_{i \in \mathbb{Z}} \) satisfying the conditions (Ord) of Definition 4.3. For every integer \( m \in \{0, 1, \ldots, d - 1\} \), recall that \( \text{Gr}^m_{\mathbb{T}} \mathbb{T}_{\text{cyc}} \) is the free \( \mathcal{R}_{\text{cyc}} \)-module of rank one on which \( G_{\mathbb{Q}_p} \) acts via the character \( \chi^m_{\text{cyc}} \cdot \chi_{\text{cycl}}^{-1} \cdot \chi_{\text{cycl}}^{1- \alpha m} \cdot \chi_{\text{cycl}}^{-1} \cdot \chi_{\text{cycl}}^{1- \beta m} \). We define the free \( \mathcal{R}_{\text{cyc}} \)-module \( \mathbb{D}((\text{Gr}^m_{\mathbb{T}} \mathbb{T}_{\text{cyc}})^{\mathbb{R}_{\text{cyc}}(1)}) \) of rank one as the tensor product

\[
\left( \bigotimes_{i \in A_m} \mathbb{D}(-e_{m,i}) \right)_{\mathbb{Z}_p} \otimes \mathbb{D}(1) \otimes D_{dR}(\mathbb{Z}_p(\omega^m)) \otimes D_{\text{crys}}(\mathbb{Z}_p(b_m))
\]

where \( A_m \) is the subset of \( \{0, 1, \ldots, d - 1\} \) which consists of \( i \in \{0, 1, \ldots, d - 1\} \) such that \( e_{m,i} = 1 \). We define \( \mathbb{D}(-1) \) to be is a \( \Lambda_{\text{cyc}} \)-linear dual of \( \mathbb{D}(1) \) and the completed tensor
Theorem 4.12. Let $R$ be a local domain which is finite flat over $\mathbb{Z}_p[[\Gamma_1 \times \cdots \times \Gamma_r]]$ and let $(T, R, S)$ be a deformation datum. Suppose that we have a strictly decreasing, $G_{\mathbb{Q}_p}$-stable, exhaustive filtration $\{\text{Fil}_i T\}_{i \in \mathbb{Z}}$ satisfying the condition (Ord) of Definition 4.3. Let $m$ be an integer in $\{0, 1, \ldots, d - 1\}$.

Then, we have an $R_{\text{cyc}}$-linear isomorphism

$$\text{EXP}_{(\Gamma_{\text{cyc}}^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}}^\text{ur} : D^\text{ur}((\Gamma_{\text{cyc}}^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}) \to H^1(\mathbb{Q}_p^\text{ur}, (\Gamma_{\text{cyc}}^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}/(\Gamma_{\text{cyc}}^m T)^{R(1)})$$

such that, for every $\kappa \in S_{\text{cyc}}^{(m)}$, the following diagram commutes:

$$\begin{array}{ccc}
D^\text{ur}((\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)}) & \xrightarrow{e^\text{ur}_{p}} & H^1(\mathbb{Q}_p^\text{ur}, (\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)}) \\
\kappa & \downarrow & \kappa \\
D^\text{ur}_{\text{dir}}((\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)}) & \xrightarrow{\text{EXP}_{(\Gamma_{\text{cyc}}^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}}^\text{ur}} & H^1(\mathbb{Q}_p^\text{ur}, (\Gamma_{\text{cyc}}^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})
\end{array}$$

Here $e^\text{ur}_{p,+} := (-1)^{d_m(\kappa) - 1}(d_m(\kappa) - 1)! e^\text{ur}_p$ and $e^\text{ur}_p = e^\text{ur}_p((\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)})$ is the $p$-adic multiplier given by

$$e^\text{ur}_p := \begin{cases} (1 - p^{-1} \varphi^{-1}) (1 - \varphi)^{-1} & \text{when } \Gamma_{\text{cyc}}^m V_{\kappa} \text{ is crystalline}, \\ (p^{-1} \varphi^{-1})^n & \text{when } \Gamma_{\text{cyc}}^m V_{\kappa} \mid_{T_p} \cong E_{\kappa}(c_m(\kappa))(\phi) \text{ with } \text{ord}_p(\text{cond}(\phi)) = n \geq 1. \end{cases}$$

Also, for every $\kappa \in S_{\text{cyc}}^{(m),-}$ we also have the following commutative diagram:

$$\begin{array}{ccc}
D^\text{ur}((\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)}) & \xrightarrow{\text{EXP}_{(\Gamma_{\text{cyc}}^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}}^\text{ur}} & H^1(\mathbb{Q}_p^\text{ur}, (\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)}) \\
\kappa & \downarrow & \kappa \\
D^\text{ur}_{\text{dir}}((\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)}) & \xrightarrow{\left(\text{exp}_{\Gamma_{\text{cyc}}^m V_{\kappa}}\right)^{-1} \circ e^\text{ur}_{-p}} & H^1(\mathbb{Q}_p^\text{ur}, (\Gamma_{\text{cyc}}^m V_{\kappa})^{R_{\text{cyc}}(1)})
\end{array}$$

where $e^\text{ur}_{-p} := e^\text{ur}_p(-d_m(\kappa))!$.

Proof. Recall that $R_{\text{cyc}}$ is finite flat over $\mathbb{Z}_p[[\Gamma_1 \times \cdots \times \Gamma_r \times \Gamma_{\text{cyc}}]]$ and take another set of groups $\Gamma'_1, \ldots, \Gamma'_r$ and $\Gamma'_{\text{cyc}}$ such that

(i) We have an isomorphism $\Gamma_1 \times \cdots \times \Gamma_r \times \Gamma_{\text{cyc}} \xrightarrow{-} \Gamma'_1 \times \cdots \times \Gamma'_r \times \Gamma'_{\text{cyc}}$.

(ii) The group $\Gamma'_1$ has an isomorphism $\chi'_1 : \Gamma'_1 \xrightarrow{-} 1 + p\mathbb{Z}_p$ ($i = 1, \ldots, r$) and $\Gamma'_{\text{cyc}}$ has an isomorphism $\chi'_{\text{cyc}} : \Gamma'_{\text{cyc}} \xrightarrow{-} 1 + p\mathbb{Z}_p$. 


We call such a set of groups \{\Gamma'_1, \ldots, \Gamma'_r, \Gamma'_{\text{cyc}}\} a coordinate change of \{\Gamma_1, \ldots, \Gamma_r, \Gamma_{\text{cyc}}\}. For a given coordinate change \{\Gamma'_1, \ldots, \Gamma'_r, \Gamma'_{\text{cyc}}\}, we define a Galois character \(\bar{\chi}' : G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p[[\Gamma']]^\times\) to be

\[ G_{\mathbb{Q}_p} \rightarrow \Gamma'_{\text{cyc}} \xrightarrow{(\chi'_1)^{-1} \circ \chi'_{\text{cyc}}} \Gamma' \rightarrow \mathbb{Z}_p[[\Gamma']]^\times. \]

We define a Galois character \(\bar{\chi}'_{\text{cyc}} : G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p[[\Gamma'_{\text{cyc}}]]^\times\) in the same way.

The crucial observation is that for every \(m \in \{0, 1, \ldots, d-1\}\), there exists a coordinate change such that the action of \(G_{\mathbb{Q}_p}\) on \((\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)\) is given by \(\bar{\chi}'_{\text{cyc}} \cdot \omega^m \bar{\alpha}_{m}^{-1}\).

Let us identify the cyclotomic Iwasawa algebra \(\Lambda_{\text{cyc}}\) of Theorem 4.10 and \(\mathbb{Z}_p[[\Gamma_{\text{cyc}}]]\) here. Then the commutative diagrams of Theorem 4.10 are obtained by taking base extension functor \((- \otimes_{\mathbb{Z}_p[[\Gamma_{\text{cyc}}]]} \mathcal{R}_{\text{cyc}})\) to the commutative diagrams of Theorem 4.10 and by twisting by unramified character \(\bar{\alpha}_{m}^{-1} : G_{\mathbb{Q}_p} \rightarrow \mathcal{R}^\times\). The proof follows.

The following result is the main theorem of this section, which is deduced on passing to \(\text{Gal}(\mathbb{Q}_p^ur / \mathbb{Q}_p)\)-invariants in Theorem 4.12.

**Theorem 4.13.** Let \(\mathcal{R}\) be a local domain which is finite flat over \(\mathbb{Z}_p[[\Gamma_1 \times \cdots \times \Gamma_r]]\) and let \((\mathcal{T}, \mathcal{R}, \mathcal{S})\) be a deformation datum. Suppose that we have a strictly decreasing, \(G_{\mathbb{Q}_p}\)-stable, exhaustive filtration \(\{\text{Fil}^i \mathcal{T}\}_{i \in \mathbb{Z}}\) satisfying the conditions (Ord) of Definition 4.3. Let \(m \in \{0, 1, \ldots, d-1\}\) and assume that the unramified character \(\bar{\alpha}_{m}\) is non-trivial.

Then, we have an \(\mathcal{R}_{\text{cyc}}\)-linear isomorphism

\[ \text{EXP}_{(\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)} : \mathbb{D}((\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)) \rightarrow H^1(\mathbb{Q}_p, (\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)) \]

such that, for every \(\kappa \in \mathcal{S}_{\text{cyc}}^{(m)+}\) the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{D}((\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)) & \xrightarrow{\text{EXP}_{(\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)}} & H^1(\mathbb{Q}_p, (\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)) \\
\kappa & & \kappa \\
\downarrow & & \downarrow \\
D_{\text{dR}}((\Gr^m_{p} V_{\kappa})^{\kappa(\mathcal{R}_{\text{cyc}})}(1)) & \xrightarrow{\exp_{(\Gr^m_{p} V_{\kappa})^{\kappa(\mathcal{R}_{\text{cyc}})}(1)}} & H^1(\mathbb{Q}_p, (\Gr^m_{p} V_{\kappa})^{\kappa(\mathcal{R}_{\text{cyc}})}(1))
\end{array}
\]

Here \(\epsilon^+: (-1)^{d_m(\kappa)-1}(d_m(\kappa) - 1)!\), \(\epsilon_p\) and \(\epsilon_p = \epsilon_p((\Gr^m_{p} V_{\kappa})^{\kappa(\mathcal{R}_{\text{cyc}})}(1))\) is the \(p\)-adic multiplier given by

\[
\epsilon_p := \begin{cases} 
1 - \frac{p^{d_m(\kappa)-1}}{\kappa|R(\bar{\alpha}_{m}(\text{Frob}_p))} & \text{when } \Gr^m_{p} V_{\kappa} \text{ is crystalline}, \\
\left(1 - \frac{\kappa|R(\bar{\alpha}_{m}(\text{Frob}_p))}{p^{d_m(\kappa)}}\right)^n & \text{when } \Gr^m_{p} V_{\kappa}|_{I_p} \cong E_{\kappa}(c_m(\kappa))(\phi), \\
\kappa|R(\bar{\alpha}_{m}(\text{Frob}_p)) & \text{with ord}_p(\text{cond}(\phi)) = n \geq 1.
\end{cases}
\]

Also, for every \(\kappa \in \mathcal{S}_{\text{cyc}}^{(m)-}\) we also have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{D}((\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)) & \xrightarrow{\text{EXP}_{(\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)}} & H^1(\mathbb{Q}_p, (\Gr^m_{p; \text{cyc}})^{\text{cyc}}(1)) \\
\kappa & & \kappa \\
\downarrow & & \downarrow \\
D_{\text{dR}}((\Gr^m_{p} V_{\kappa})^{\kappa(\mathcal{R}_{\text{cyc}})}(1)) & \xrightarrow{\exp_{(\Gr^m_{p} V_{\kappa})}^*} & H^1(\mathbb{Q}_p, (\Gr^m_{p} V_{\kappa})^{\kappa(\mathcal{R}_{\text{cyc}})}(1))
\end{array}
\]

where \(\epsilon^- = \frac{\epsilon_p}{(-d_m(\kappa))!}\).
Proof. We begin with a study of the $\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)$-invariants of the modules that appear in Theorem 4.12.

By definition, we have the following identification by definition:

$$\mathbb{D}((\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}) = \mathbb{D}^{\text{ur}}((\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})^{\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)}.$$  

Let us also calculate the $\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)$-invariants in the Galois cohomology side (in the diagrams of Theorem 4.12). By our requirement that $\tilde{\alpha}_m$ be non-trivial, we have

$$H^1(\mathbb{Q}_p, (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})^{\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)}.$$  

Let us calculate the right-hand side. The quotient $(\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}/m^r_{R_{\text{cyc}}}(\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}$ is a finite abelian group with continuous action of $G_{\mathbb{Q}_p}$ for any natural number $r$. Consider the restriction map

$$H^1(\mathbb{Q}_p, (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})^{\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)},$$  

By the inflation-restriction sequence, the kernel and the cokernel of this map are the modules $H^1(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p, A_{\text{r}^G_{\text{cyc}}})$ and $H^2(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p, A_{\text{r}^G_{\text{cyc}}})$ respectively, where we have set $A_r := (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}/m^r_{R_{\text{cyc}}}(\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}$.

to ease our notation. Since $\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p) \cong \mathbb{Z}$ has cohomological dimension one, we have $H^2(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p, A_{\text{r}^G_{\text{cyc}}}) = 0$ and $H^1(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p, A_{\text{r}^G_{\text{cyc}}})$ is isomorphic to the largest $\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)$-coinvariant quotient $(A_r^{G_{\mathbb{Q}_p}})_{\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)}$ of $A_r^{G_{\mathbb{Q}_p}}$. Since we have

$$(\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}^{G_{\mathbb{Q}_p}} = 0$$

thanks to our running hypothesis that the unramified character $\tilde{\alpha}_m$ (that appears in the formulation of (Ord)) is non-trivial, it follows that $\lim_{r \to} A_{\text{r}^G_{\text{cyc}}} = 0$. We therefore infer that

$$\lim_{r \to} H^1(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p, A_{\text{r}^G_{\text{cyc}}}) = 0$$

and that we have an isomorphism

$$H^1(\mathbb{Q}_p, (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})^{\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)} \cong H^1(\mathbb{Q}_p, (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})^{G_{\mathbb{Q}_p}}$$

induced by the restriction map. Taking the inverse limit with respect to $r$, we have

$$H^1(\mathbb{Q}_p, (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}) \cong H^1(\mathbb{Q}_p^\text{ur}, (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})^{G_{\mathbb{Q}_p}}$$

Let us define $\text{EXP}((\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})$ to be the map induced by $\text{EXP}^{\text{ur}}((\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})$ on the $\text{Gal}(\mathbb{Q}_p^\text{ur}/\mathbb{Q}_p)$-invariants. Thanks to (4.8), (4.9) and (4.10), we have an $R_{\text{cyc}}$-linear isomorphism $\text{EXP}((\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)} : \mathbb{D}((\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}) \to H^1(F, (\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)})$. By its very construction, the map $\text{EXP}((\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}$ verifies the desired interpolation property for every $\kappa \in S_{\text{cyc}}^{(m), +} \cup S_{\text{cyc}}^{(m), -}$.

Let us define

$$\mathbb{D}(\text{Gr}_p^m T_{\text{cyc}}) := \text{Hom}_{R_{\text{cyc}}}((\mathbb{D}(\text{Gr}_p^m T_{\text{cyc}})^{R_{\text{cyc}}(1)}), R_{\text{cyc}}).$$

The following result is an important consequence of Theorem 4.13.
Corollary 4.14. Let $\mathcal{R}$ be a local domain which is finite flat over $\mathbb{Z}_p[[\Gamma_1 \times \cdots \times \Gamma_r]]$ and let $(\mathbb{T}, \mathcal{R}, \mathcal{S})$ be a deformation datum. Suppose that we have a strictly decreasing, $G_{\mathbb{Q}_p}$-stable, exhaustive filtration $\{\text{Fil}_p^n \mathbb{T}\}_{n \in \mathbb{Z}}$ satisfying the conditions (Ord) of Definition 4.3. Let $m$ be an integer in $\{0, 1, \ldots, d - 1\}$ and assume that the unramified character $\tilde{\alpha}_m$ is non-trivial.

Then, we have an $\mathcal{R}_{\text{cyc}}$-linear isomorphism

$$\EXP^*_{(\Gr_p^m \mathbb{T}_{\text{cyc}})^{\mathcal{R}_{\text{cyc}}(1)}} : H^1(\mathbb{Q}_p, \Gr_p^m \mathbb{T}_{\text{cyc}}) \longrightarrow \mathcal{D}(\Gr_p^m \mathbb{T}_{\text{cyc}})$$

such that, for every $\kappa \in \mathcal{S}_{\text{cyc}}^{(m),+}$ the following diagram commutes:

$$\begin{array}{ccc}
H^1(\mathbb{Q}_p, \Gr_p^m \mathbb{T}_{\text{cyc}}) & \xrightarrow{\EXP^*_{(\Gr_p^m \mathbb{T}_{\text{cyc}})^{\mathcal{R}_{\text{cyc}}(1)}}} & \mathcal{D}(\Gr_p^m \mathbb{T}_{\text{cyc}}) \\
\kappa \downarrow & & \downarrow \kappa \\
H^1(\mathbb{Q}_p, \Gr_p^m V_{\kappa}) & \xrightarrow{e_p^+ \times \exp^*_{(\Gr_p^m V_{\kappa})^{\mathcal{R}_{\text{cyc}}(1)}}} & D_{\text{dR}}(\Gr_p^m V_{\kappa})
\end{array}$$

Here $e_p^+ := (-1)^d m(\kappa) - 1 (d_m(\kappa) - 1)!$ and $e_p = e_p((\Gr_p^m V_{\kappa})^{\mathcal{R}_{\text{cyc}}(1)}(1))$ is the $p$-adic multiplier given by

$$e_p := \begin{cases} 
1 - \frac{p^{d_m(\kappa) - 1}}{\kappa | \mathcal{R}((\alpha_m(Frob_p)))} & \text{when } \Gr_p^m V_{\kappa} \text{ is crystalline}, \\
\left(1 - \frac{p^{d_m(\kappa) - 1}}{\kappa | \mathcal{R}((\alpha_m(Frob_p)))}\right)^n & \text{when } \Gr_p^m V_{\kappa}|_{I_p} \cong E_r(c_m(\kappa)) \otimes \phi \\
\end{cases}$$

with $\text{ord}_p(\text{cond}(\phi)) = n \geq 1$.

Also, for every $\kappa \in \mathcal{S}_{\text{cyc}}^{(m),-}$ we also have the following commutative diagram:

$$\begin{array}{ccc}
H^1(\mathbb{Q}_p, \Gr_p^m \mathbb{T}_{\text{cyc}}) & \xrightarrow{\EXP^*_{(\Gr_p^m \mathbb{T}_{\text{cyc}})^{\mathcal{R}_{\text{cyc}}(1)}}} & \mathcal{D}(\Gr_p^m \mathbb{T}_{\text{cyc}}) \\
\kappa \downarrow & & \downarrow \kappa \\
H^1(\mathbb{Q}_p, \Gr_p^m V_{\kappa}) & \xrightarrow{e_p^+ \times \log_{(\Gr_p^m V_{\kappa})^{\mathcal{R}_{\text{cyc}}(1)}}} & D_{\text{dR}}(\Gr_p^m V_{\kappa})
\end{array}$$

where $e_p^- := \frac{e_p}{(-d_m(\kappa))!}$.

Proof. We define $\EXP^*_{(\Gr_p^m \mathbb{T}_{\text{cyc}})^{\mathcal{R}_{\text{cyc}}(1)}}$ to be $\mathcal{R}_{\text{cyc}}$-linear Kummer dual of the big exponential map $\EXP_{(\Gr_p^m \mathbb{T}_{\text{cyc}})^{\mathcal{R}_{\text{cyc}}(1)}}$. Note that we have

$$H^1(\mathbb{Q}_p, \Gr_p^m \mathbb{T}_{\text{cyc}}) \cong \text{Hom}_{\mathcal{R}_{\text{cyc}}}(H^1(\mathbb{Q}_p, (\Gr_p^m \mathbb{T}_{\text{cyc}})^{\mathcal{R}_{\text{cyc}}(1)})), \mathcal{R}_{\text{cyc}})$$

by local Tate duality theorem of Galois cohomology. Recall that, for any de Rham $p$-adic representation $V$ of $G_{\mathbb{Q}_p}$, the Kummer dual of $\exp_V$ (resp. $(\exp_V)^{-1}$) is known to be $\exp^*_V$ (resp. $\log_V$). These facts ensure that $\EXP^*_{(\Gr_p^m \mathbb{T}_{\text{cyc}})^{\mathcal{R}_{\text{cyc}}(1)}}$ satisfies the desired interpolation property and completes the proof.

Remark 4.15. (For readers who might be distressed about the absence of Gauss sum in the interpolation formula of Coleman map) In the most basic set up with the cyclotomic deformation of the $p$-adic Tate module of an ordinary elliptic curve $E$, Rubin in [Rub98, Prop. A.2] presents the following interpolation formula for any $n \geq m + 1$:

$$\chi(\text{Col}_n(z)) = \alpha^{-m} \tau(\chi) \sum_{\gamma \in G_n} \chi^{-1}(\gamma) \exp^*_{\omega_E}(\gamma^z).$$
Here $\chi$ is a nontrivial Dirichlet character of conductor $p^m$, $\tau(\chi)$ is the Gauss sum for $\chi$, and $\alpha$ is the $p$-unit root of the $p$-Euler polynomial for $E$. The group $G_n$ in the summation above is nothing but the group $\Gamma_{\text{cycl}}/\Gamma_{\text{cycl}}^{p}$ in the current article. The map $\text{Col}_{n}$ above is a map from $\mathcal{H}^{1}(\mathbb{Q}_p, \mathcal{T}_p(E) \otimes \mathbb{Z}_p[G_n]^\sharp) \cong \mathcal{H}^{1}(\mathbb{Q}_{p,n}, \mathcal{T}_p(E))$ where $\mathbb{Q}_{p,n}$ is the $n$-th layer of the local cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{p,\infty}/\mathbb{Q}_p$.

Notice also that we have the following identity by direct calculation:

\begin{equation}
\tau(\chi) \sum_{\gamma \in G_n} \chi^{-1}(\gamma) z^\gamma = \chi(z).
\end{equation}

Here, for each $z \in \mathcal{H}^{1}(\mathbb{Q}_p, \mathcal{T}_p(E) \otimes \mathbb{Z}_p[G_n]^\sharp)$, we write $\chi(z)$ for the image of $z$ under the map

\begin{equation}
\mathcal{H}^{1}(\mathbb{Q}_p, \mathcal{T}_p(E) \otimes \mathbb{Z}_p[G_n]^\sharp) \rightarrow \mathcal{H}^{1}(\mathbb{Q}_p, \mathcal{T}_p(E) \otimes \chi).
\end{equation}

for any character $\chi$ of $G_n$. It therefore follows from (4.11) and (4.12) that

$$\chi(\text{Col}_{n}(z)) = \alpha^{-m} \exp_{\omega_E}(\chi^{-1}(z)).$$

This perfectly matches up with our interpolation formulae: In the setting of [Rub98], note that the only possibility that the integer $j$ in the interpolation formulae in Theorem 4.13 can assume is the value 1. In other words, Gauss sums are not really missing in our formulae, but rather encoded in the twisted projection maps (4.13).

5. Nearly ordinary families of Rankin-Selberg convolutions

From Section 2 to Section 4, we established a general formalism of the theory and machineries to attack Iwasawa Main conjecture of general Galois deformations. In this section, we apply our results to the setting of Section 4.2.1 with help of Beilinson-Flach elements.

Until the end of Section 5, we shall work in the setting of Section 4.2.1 Throughout the section, we take the base field $K$ to be $\mathbb{Q}$ and we set $N_1$ and $N_2$ to be positive integers which are prime to $p$. We shall work with a pair of Hida families $f_i$ ($i = 1, 2$) with respective tame levels $N_i$ and central characters

$$\Psi_i : (\mathbb{Z}/pN_i\mathbb{Z})^\times \rightarrow \mathcal{O}^\times$$

by setting $\Psi_i(\ell)$ to be the eigenvalue of the diamond operator $\langle \ell \rangle$ acting on the family $f_i$. Here, $\mathcal{O}$ is the ring of integers of a finite extension $E$ of $\mathbb{Q}_p$ which contain the images of both Dirichlet characters $\Psi$. Recall also the local domain $\mathcal{R} := \mathfrak{I}_1 \otimes \mathfrak{I}_2$ and the two-dimensional $\mathcal{R}[[G_{\mathcal{O},\Sigma}]]$-representation $\mathbb{T} := \mathfrak{T}_1 \otimes \mathfrak{T}_2$.

Suppose that $p \geq 7$. In particular, the hypothesis (MR4) holds true.

5.1. Main conjectures for the nearly deformations of Rankin-Selberg products. First, we will verify that the required conditions to apply our theory holds true in a great variety of cases: The conditions (H.0), (H.2), (H.0−), (H.2+) and H.2++ are covered by Lemma 5.3 whereas (MR1) by Lemma 5.3, (MR2) by Theorem 5.6. Notice that (MR3) readily follows as a consequence of (H.0) and (H.2), whereas (MR4) also holds since we have assumed $p \geq 7$.

We will consider the following conditions for both families $f_i$ ($i = 1, 2$):

- (F.CM) $f_i$ is non-CM.
- (F.PS) For $i = 1, 2$, the ring $\mathfrak{I}_i$ is isomorphic to a power series ring in one variable with coefficients in $\mathcal{O}$.

The condition (F.PS) is expected to be valid very often; c.f. the discussion in [FO12] Lemma 2.7.

Remark 5.1. Notice that the case when $f_2$ has CM is the subject of [BL16] [Cas15] [SU14] [Wan15] and our main results in the context of Rankin-Selberg convolutions (c.f. Corollary 5.18 below) handle the case when neither of the forms have CM.
Definition 5.2. Let \( f_i = \sum a_n(f_i)q^n \in S_{k_i}(\Gamma_1(N_i)) \) \((i = 1, 2)\) be a pair of newforms of respective weights \( k_1, k_2, \) levels \( N_1, N_2. \) Let \( L_1 \) and \( L_2 \) denote the normal closure of their respective Hecke fields. We say that \( f_1 \) and \( f_2 \) are twisted conjugates to mean there exists an embedding \( \delta : L_1 \hookrightarrow \mathbb{C} \) and a Dirichlet character \( \chi_\delta \) (necessarily of conductor dividing \( 4N_1) \) such that \( \delta(a_i(f_1)) = \chi_\delta(\ell)a_i(f_2). \)

Lemma 5.3. (1) Let \( f_i \in S_{k_i}(\Gamma_1(N_i)) \) \((i = 1, 2)\) be non-CM newforms which are not twisted-conjugate to each other. Then, the residual representation \( \overline{\rho}_f \otimes \overline{\rho}_g \) (modulo \( p \)) is absolutely irreducible for every sufficiently large \( p. \)

(2) Fix a large enough \( p \) so that the conclusion of the first part holds and such that both forms \( f_i \) are \( p \)-ordinary. Let \( \tilde{f}_i \) denote the unique Hida family that admits the \( p \)-stabilization of \( f_i \) as a weight-\( k_i \) arithmetic specialization \((i = 1, 2)\). Then the condition \((\F.CM) \) holds true for \( \tilde{f}_i \) \((i = 1, 2)\) and the residual representation \( \Gamma / \mathfrak{m}_\Gamma \) is absolutely irreducible.

See [Loe17, §4.2] for the proof of Lemma 5.3.

Next, we turn our attention to the hypotheses \((\H.0), \,(\H.2), \,(\H.0^-), \,(\H.2^+), \) and \((\H.2^{++})\). To that end, we let \( \alpha_i \in \mathfrak{k} \) denote the reduction of the eigenvalue for the \( U_p \)-action on \( f_i \), modulo the maximal ideal \( \mathfrak{m}_i \) of \( \Gamma_p. \) Writing \( \overline{\rho}_i : \mathbb{Q}_\Sigma \rightarrow \GL_2(\mathfrak{k}) \) for the residual representation carried by \( \Gamma_1 \mathfrak{m}_i \Gamma_1 \), it follows that

\[
\overline{\rho}_i|_{\GL_2(\mathfrak{m})} \sim \left( \begin{array}{cc} \alpha_i^{-1} & * \\ 0 & \alpha_i \end{array} \right)
\]

where, by abuse of notation, we let \( \alpha_i \) to denote also the unramified character that assumes the value \( \alpha_i \) at the arithmetic Frobenius at \( p. \) Recall also that, the filtration given in [5.1] may be lifted to \( \Gamma_p \) thanks to our assumption that each \( f_i \) is \( p \)-distinguished. This gives rise to a 4-step filtration of \( \Gamma. \) We recall the steps \( F^++ \subset F^{++}, \) which are both direct summands of \( \Gamma \) of respective ranks 2 and 3. Recall also the subquotients \( F^{-} \subset \Gamma/F^{++} \) and \( F^++ \subset \Gamma/F^{++}. \)

Lemma 5.4.

(1) If \( \alpha_1 \alpha_2 \not\equiv 1 \mod \pi_\mathcal{O} \) then \((\H.0^-) \) holds true. If in addition

(i) either \( \overline{\Psi}_1 \overline{\Psi}_2 \) is ramified at \( p, \)

(ii) or else \( \overline{\Psi}_1 \overline{\Psi}_2(p) \not\equiv \alpha_1 \alpha_2 \mod \pi_\mathcal{O} \)

then \((\H.0) \) also holds true.

(2) Suppose

(i) either that \( \overline{\Psi}_1 \overline{\Psi}_2^{-1} \) is ramified at \( p, \)

(ii) or else \( \overline{\Psi}_1 \overline{\Psi}_2^{-1}(p) \not\equiv \alpha_1 \alpha_2 \mod \pi_\mathcal{O} \)

(iii) either that \( \overline{\Psi}_1 \) is ramified at \( p, \)

(iv) or else \( \overline{\Psi}_1^{-1}(p) \not\equiv \alpha_1^{-1} \alpha_2 \mod \pi_\mathcal{O} \)

Then \((\H.2^+) \) holds true. If in addition

(v) either \( \overline{\Psi}_2^{-1} \) is ramified at \( p, \)

(vi) or else \( \overline{\Psi}_2^{-1}(p) \not\equiv \alpha_1^{-1} \alpha_2 \mod \pi_\mathcal{O} \)

then both \((\H.2) \) and \((\H.2^{++}) \) also hold true.

Proof. This is evident thanks to the local description in [5.1] of the residual representations. □

Finally, we shall provide an explicit sufficient condition for the hypothesis \((\MR2) \) to hold true.

Lemma 5.5. Let \( f_i \in S_{k_i}(\Gamma_1(N_i)) \) \((i = 1, 2)\) be non-CM newforms of respective weights \( k_1, k_2, \) levels \( N_1, N_2 \) which are not twisted-conjugate to each other. Let \( L_i \) denote the normal closure of the Hecke field of \( f_i \) and fix an embedding of \( L_1 L_2 \) into \( \overline{\mathbb{Q}}. \)

(1) For every \( \delta \in \Gal(L_2/\mathbb{Q}) \), the set of primes \( \ell \) for which we have \( a_\ell(f_1)^2 = \ell^{k_1-k_2} \delta(a_\ell(f_2))^2 \) has zero density.
(2) For a given number field \(F/\mathbb{Q}\), there exists \(B(f_1, f_2, F) \in \mathbb{Z}^+\) such that for every prime \(p > B(f_1, f_2, F)\) and any \(\delta \in \text{Gal}(L_2/\mathbb{Q})\) we have
\[
v_p \left( a_\ell(f_1)^2 - \ell^{k_1-k_2}\delta(a_\ell(f_2))^2 \right) = 0
\]
for every prime \(\ell\) which splits completely in \(F/\mathbb{Q}\).

**Proof.** The first assertion follows from a theorem of Ramakrishnan [Ram00, Theorem A], as in [Loc17, Lemma 3.1.1]. The second assertion is an immediate consequence from the first, since there only finitely many primes at which all members of a non-zero collection of algebraic numbers have positive valuation. \(\square\)

We let \(F_0\) denote the compositum of \(L_1\) and \(L_2\). We also choose an integer \(B \geq B(f_1, f_2, F_0)\) such that for every \(p > B\), both conclusions of Lemma 5.3(ii) are valid.

**Theorem 5.6.** Let \(f_i\) be as in Lemma (5.3) and suppose \(p > B\) is a good ordinary prime for both forms. Let \(f_i\) be denote the unique Hida family that admits the \(p\)-stabilization of \(f_i\) as a weight-\(k_i\) arithmetic specialization (\(i = 1, 2\)). Suppose that the hypothesis (F.Dist) holds true. The Rankin-Selberg Galois representation \(\mathbb{T}\) satisfies (MR2) if we further assume the following conditions:

(BI.1) (i) Either \((N_1, N_2) = 1\) and there exists \(u\) such that \(\Psi_2(u) = -1\),

(ii) or the product of the reductions of two central characters \(\overline{\mathbb{T}}\Psi_2\) is non-trivial.

(BI.2) The residual representations associated to both \(f_1\) and \(f_2\) are full in the sense that they contain a conjugate of \(\text{SL}_2(\mathbb{F}_p)\).

**Remark 5.7.** Let \(f \in S_k(\Gamma_1(N), \varepsilon)\) be a non-CM newform. Then the results of Momose, Ribet, and Ghat–Gonzalez-Jimenez–Quer [GGJQ05] guarantee that for all but finitely primes \(p\), mod \(p\) representation \(\overline{\rho}_f\) associated to \(f\) is full. For \(f_i\) as in the statement of Theorem 5.6, we fix an embedding \(F_0 \hookrightarrow \overline{\mathbb{Q}}_p\) which sends both \(a_p(f_i)\) to units. Let \(\mathfrak{p}\) denote the prime of \(F_0\) induced by this embedding. By slight abuse, we shall denote the prime of \(L_i\) lying below \(\mathfrak{p}\) also by the symbol \(\mathfrak{p}\).

**Notation.** For \(f_i\) as in the statement of Theorem 5.6, we fix an embedding \(F_0 \hookrightarrow \overline{\mathbb{Q}}_p\) which sends both \(a_p(f_i)\) to units. Let \(\mathfrak{p}\) denote the prime of \(F_0\) induced by this embedding. By slight abuse, we shall denote the prime of \(L_i\) lying below \(\mathfrak{p}\) also by the symbol \(\mathfrak{p}\).

**Proof of Theorem 5.6.** Our proof builds on the work of Fischman and Loeffler; our notation in this proof is a hybrid of that used in these two articles. For \(i = 1, 2\), we let \(H_{f_i} \subset G_Q\) denote the subgroup defined at the beginning of [Fis02, §3.2] and let \(H = H_{f_1} \cap H_{f_2}\). Since \(H_{f_i}\) is of finite index in \(G_Q\), \(H\) is a subgroup of finite index in \(H_{f_1}\) and \(H_{f_2}\) and both \(\Psi_1\) and \(\Psi_2\) are trivial on \(H\). We also recall the subring \(R_{f_i} \subset \mathbb{H}_{f_i}\) defined in loc. cit. Set \(\rho := \rho_{f_1} \otimes \rho_{f_2}\). When BI.1(i) is valid, we shall prove that

\[
\left( \begin{array}{ll} 1 & 1 \\ 0 & 1 \end{array} \right), \quad \left( \begin{array}{ll} 1 & 0 \\ 0 & -1 \end{array} \right) \in \rho(G_Q(\mu_{p^\infty}))
\]

- **Step 1.** We let \(G_i^{(n)} := \rho_{f_i}(H \cap G_Q(\mu_{p^n}))\) and \(G_i^n := \bigcap_{n} G_i^{(n)} = \rho_{f_i}(H \cap G_Q(\mu_{p^n}))\).

We also set \(\mathcal{G}_n = \text{Gal}(\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q})\).

Then \(G_i^n = \text{SL}_2(R_{f_i})\).

Indeed, it follows from [Fis02, Corollary 4.11] that

\[
G_i^{(n)} \subset \{ M \in \text{GL}_2(R_{f_i}) \mid \det(M) \in (1 + p\mathbb{Z}_p)^{pn} \}
\]

(5.3) where Fischman uses the notation \(\Gamma'\) for the group \(1 + p\mathbb{Z}_p\). Note also that the quotient group \(\rho_{f_i}(H) / G_i^{(n)}\) is a cyclic group of dividing \((p-1)p^n\), being the homomorphic image of \(H/H \cap G_Q(\mu_{p^n})\) under the map induced from \(\det \circ \rho_{f_i}\). On the other hand, it also follows from [Fis02, Corollary 4.11] combined with (5.3) that

\[
\rho_{f_i}(H) / G_i^{(n)} \xrightarrow{\text{det}} (\mu_p \times (1 + p\mathbb{Z}_p)) / (1 + p\mathbb{Z}_p)^p \cong \mathbb{Z}/p^{n+1}\mathbb{Z} \times \]
and we conclude that the containment in (5.3) is an equality. We may more precisely write
\[ G_i^{(n)} = \left\{ M \in GL_2(R_f) \mid \det(M) \in \{ \Psi_i^{(p)} \tilde{\chi}_i(\gamma) \}_{\gamma \in G_\mathcal{Q}(\mu_n^p)} \right\}. \]

Since
\[ \left\{ M \in GL_2(R_f) \mid \det(M) \in \{ \Psi_i^{(p)} \tilde{\chi}_i(\gamma) \}_{\gamma \in G_\mathcal{Q}(\mu_n^p)} \right\} = \bigcup_{\gamma \in G_\mathcal{Q}(\mu_n^p)} \left( \begin{array}{cc} 1 & 0 \\ 0 & \Psi_i^{(p)} \tilde{\chi}_i(\gamma) \end{array} \right) SL_2(R_f) \]
we conclude that
\[ G_i^{(n)} = \bigcup_{\gamma \in G_\mathcal{Q}(\mu_n^p)} \left( \begin{array}{cc} 1 & 0 \\ 0 & \Psi_i^{(p)} \tilde{\chi}_i(\gamma) \end{array} \right) SL_2(R_f). \]

On considering the intersection \( \bigcap_n G_i^{(n)} \), we conclude that
\[ G_i^o = \left\{ M \in GL_2(R_f) \mid \det(M) \in \bigcap_n (1 + p\mathbb{Z}_p)^p \right\} = SL_2(R_f). \]

**Step 2.** Now set \( G^{(n)} := \rho(H \cap G_\mathcal{Q}(\mu_n^p)) \subset G_1^{(n)} \times G_2^{(n)} \) and \( G^o = \bigcap_n G^{(n)} \). We claim that \( G^o = G_1^o \times G_2^o = SL_2(R_f) \times SL_2(R_f). \)

We shall very closely follow the arguments of Loeffler in the proofs of [Loe17, Proposition 3.2.1 and Theorem 3.2.2] in order to verify this claim. To that end, we let \( U < \rho(H) \subset \rho_1(H) \times \rho_2(H) \) denote the subgroup of elements \( (M_1, M_2) \) such that \( M_i \in SL_2(R_f) \). By the discussion in the first step, notice that both natural projection maps \( U \to SL_2(R_f) \) are surjective.

We shall need the following result, which is a particular instance of Goursat’s Lemma:

**Lemma 5.8.** Let \( \mathcal{N} < SL_2(R_f) \times SL_2(R_f) \) be a closed subgroup that surjects onto each factor under the natural projection maps. Then there exists closed normal subgroups \( \mathfrak{N}_1 < SL_2(R_f) \) and \( \mathfrak{N}_2 < SL_2(R_f) \) such that the image of \( \mathcal{N} \) in \( SL_2(R_f)/\mathfrak{N}_1 \times SL_2(R_f)/\mathfrak{N}_2 \) is the graph of an isomorphism \( SL_2(R_f)/\mathfrak{N}_1 \cong SL_2(R_f)/\mathfrak{N}_2 \). Moreover, \( \mathcal{N} \) is a proper subgroup of \( SL_2(R_f) \times SL_2(R_f) \) if and only if \( \mathfrak{N}_i \) is a proper normal subgroup of \( SL_2(R_f) \).

Next, we next explain that the only maximal proper normal subgroup of \( SL_2(R_f) \) is the kernel of the projection to \( PSL_2(k) \) (where \( k \) is \( R_f/\mathfrak{m}_i \) is the residue field). Indeed, if \( \mathfrak{N} < SL_2(R_f) \) is a proper normal subgroup, then so is its image \( \overline{\mathfrak{N}} \) in \( PSL_2(k) \). However, since we have assumed that \( p > 7 \), it follows that \( PSL_2(k) \) is simple and hence we either have \( \overline{\mathfrak{N}} = \{ 1 \} \) or else \( \overline{\mathfrak{N}} = PSL_2(k) \). We shall next explain that the latter option is impossible: If \( \overline{\mathfrak{N}} = PSL_2(k) \), then since only proper non-trivial normal subgroup of \( SL_2(k) \) is \( \{ \pm 1 \} \), it follows that \( \overline{\mathfrak{N}} \) maps onto \( SL_2(k) \) (under the obvious reduction map induced from \( R_f \to k \)). This observation combined with [Fed17, Proposition 3.15] shows that \( \mathfrak{N} = SL_2(R_f) \), contrary to our assumption that \( \mathfrak{N} \) is a proper normal subgroup of \( SL_2(R_f) \), so \( \overline{\mathfrak{N}} = \{ 1 \} \). We conclude that \( \mathfrak{N} \subset \ker(SL_2(R_f) \to PSL_2(k)) \), as claimed.

Suppose now that \( U \) is a proper subgroup of \( SL_2(R_f) \times SL_2(R_f) \). By Lemma 5.8 we have isomorphisms
\[
\begin{array}{ccc}
SL_2(R_f)/\mathfrak{N}_1 & \overset{\sim}{\longrightarrow} & SL_2(R_f)/\mathfrak{N}_2 \\
\downarrow & & \downarrow \\
PSL_2(k_1) & \overset{\sim}{\longrightarrow} & PSL_2(k_2)
\end{array}
\]
(where the isomorphism on the second row is induced from the one on the first row), which in turn also induces an isomorphism \( k_1 \overset{\sim}{\rightarrow} k_2 \) and \( R_f \overset{\sim}{\rightarrow} R_{f_2} \). Henceforth, we shall identify \( k_1 \) and \( k_2 \) through this isomorphism and denote either one of them by \( k \). Since all automorphisms of \( PSL_2(k) \) arise as the compositum of a field automorphism of \( k \) and conjugation by an element of \( PSL_2(k) \), we conclude with the following:
Claim. There exists $\delta \in \text{Gal}(k/F_p)$ such that $U$ is contained in the group
\[
\{(M_1, M_2) \in \text{SL}_2(R_\ell) \times \text{SL}_2(R_\ell) \mid M_1 \mod m_1 = \delta(\pm M_2 \mod m_2)\}.
\]

Fix a choice of $\delta$ as in the claim above. Let $(A_1, A_2) \in G^{(n)}$ be any pair and set
\[
t = (A_1 \mod m_1)^{-1}\delta(A_2 \mod m_2) \in \text{GL}_2(k).
\]
Let $[t] \in \text{GL}_2(k)/\{\pm 1\}$ denote its image (and similarly, we shall also talk about $[M]$ for any $M \in \text{GL}_2(R_\ell)$). For any $(u, v) \in U$, we have the identity
\[
[u^{-1}tu] = [u^{-1}A_1^{-1}(\delta A_2)u] = [A_1^{-1}][(A_1uA_1^{-1})^{-1}(\delta(A_2vA_2)^{-1})][\delta A_2][(\delta v)^{-1}]u = [A_1^{-1}(\delta A_2)] = [t]
\]
where we have used for the third equality the fact that $U$ is a normal subgroup of $G^{(n)}$ and the Claim above. This in particular shows that $[t]$ commutes with every element of $\text{PSL}_2(k)$ and therefore that $t$ is a scalar matrix.

The fact that $t$ is a scalar matrix combined with (5.9) shows that
\[
t^2 = (\det(A_1) \mod m_1)^{-1}\delta(\det(A_2) \mod m_2),
\]
as well as that
\[
t \cdot (\text{tr}(\rho_{\ell})(\sigma) \mod m_1) \pm \delta(\text{tr}(\rho_{\ell})(\sigma) \mod m_2) = 0
\]
for every $\sigma \in H$. Let $\ell$ be any prime that splits completely in $\mathbb{Q}^H/\mathbb{Q}$. Then decomposition groups at $\ell$ are subgroups of $H$. By choosing $\sigma$ to be any lift of the Frobenius at $\ell$ to $H$ in (5.7), we conclude that
\[
t \cdot (a_\ell(f_1) \mod m_1) \pm \delta(a_\ell(f_2) \mod m_2) = 0.
\]
Let $\tau$ denote the image of $x$ in the residue field $k$. We have the following equality which takes place in $k$
\[
\frac{a_\ell(f_1)^2}{a_\ell(f_2)^2} - \ell^{k_1-k_2}\delta(a_\ell(f_2)) = a_\ell(f_1)^2 \mod m_1 - (\det(\rho_{\ell}(\text{Fr}_{\ell})) \mod m_1) \delta(\det(\rho_{\ell}(\text{Fr}_{\ell}) \mod m_2)^{-1})\delta(a_\ell(f_2) \mod m_2)^2 \mod m_1 - (\det(\rho_{\ell}(\text{Fr}_{\ell})) \mod m_1) \delta(\det(\rho_{\ell}(\text{Fr}_{\ell}) \mod m_2)^{-1})\delta(a_\ell(f_2) \mod m_2)^2 \mod m_1 = 0.
\]
Combining this with (5.9), we see for every $\ell$ as above that
\[
-\ell^{k_1-k_2}\delta(a_\ell(f_2)^2) + a_\ell(f_1)^2 = 0.
\]

Recall that $L_2$ is the normal closure of the Hecke field of $f_2$. We conclude that there exists a choice of $\tilde{\delta} \in D_P(L_2/\mathbb{Q}) := \{\tau \in \text{Gal}(L_2/\mathbb{Q}) : \tau(p) = p\}$ lifting the automorphism $\delta$ of $k$ such that
\[
v_p\left(a_\ell(f_1)^2 - \ell^{k_1-k_2}\tilde{\delta}(a_\ell(f_2)^2)\right) > 0
\]
for a set of primes $\ell$ with positive upper density. This contradicts our choice of the pair of eigenforms $f_1, f_2$ and the prime $p$ (in view of Lemma 5.5) and shows that
\[
U = \text{SL}_2(R_\ell_1) \times \text{SL}_2(R_\ell_2).
\]
It follows from the identification (5.10) combined with the explicit description (5.4) that
\[
G^{(n)} = \bigcup_{\gamma \in G_{Q(\mu_{2n})}} \left( \begin{array}{cc} 1 & 0 \\ \psi_1^{(p)}(\gamma) & 0 \end{array} \right) \cdot \left( \begin{array}{cc} 1 & 0 \\ 0 & \psi_2^{(p)}(\gamma) \end{array} \right) \text{SL}_2(R_{\ell_1}) \times \text{SL}_2(R_{\ell_2}).
\]
This concludes the proof that
\[
G^n = \bigcap_n G^{(n)} = \text{SL}_2(R_{\ell_1}) \times \text{SL}_2(R_{\ell_2}) = G_1^n \times G_2^n.
\]

- **Step 3.** There exists $\tau \in G_{Q(\mu_{2n})}$ such that $\rho(\tau) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$. 

Indeed, since we assumed that $N_1, N_2$ and $p$ are pairwise coprime, we may choose $\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}$ such that $\psi(\sigma) = 1$ for all Dirichlet characters $\psi$ of conductor $N_1$, $\Psi_2(\sigma) = -1$. In the proof of [Fis02, Theorem 4.15], Fischman has verified for each one of the Hida families $f_1, f_2$ and any $g \in G_{\mathbb{Q}(\mu_{p^\infty})}$ that

$$
\left( \begin{array}{c} \beta_i(g) \\ 0 \\ \psi_i^{(N_i)}(g) \end{array} \right) \in \rho_f(G_{\mathbb{Q}(\mu_{p^\infty})}),
$$

where we write $\beta_i(g)$ in place of Fischman’s $\alpha(g)$ when we are working with the Hida family $f_i$. Thus,

$$
\left( \begin{array}{c} \beta_1(g) \\ 0 \\ \Psi_1^{(N_1)}(g) \end{array} \right), \left( \begin{array}{c} \beta_2(g) \\ 0 \\ \Psi_2^{(N_2)}(g) \end{array} \right) \in \rho(G_{\mathbb{Q}(\mu_{p^\infty})}).
$$

Let the automorphism $\gamma$ and the Dirichlet character $\chi_\gamma$ be as in the paragraph preceding Lemma 4.14 of [Fis02], with $F = f_1$. For $\sigma$ chosen as above, we have $\chi(\sigma) = 1$ since the conductor of $\chi_\gamma$ divides $N_1$. The choice of $\beta_1(\sigma) \in \mathbb{I}_f$, which is explained in the same paragraph of op. cit. (where we remind the reader that our $\beta_1$ corresponds to Fischman’s $\alpha$), we may take $\beta_1(\sigma) = 1$. Hence,

$$
\left( \begin{array}{c} \beta_1(\sigma) \\ 0 \\ \Psi_1^{(N_1)}(\sigma) \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right).
$$

Furthermore,

$$
\left( \begin{array}{c} \beta_2(\sigma) \\ 0 \\ \Psi_2^{(N_2)}(\sigma) \end{array} \right) = \left( \begin{array}{c} \beta \\ 0 \\ -\beta^{-1} \end{array} \right).
$$

where we have set $\beta = \beta_2(\sigma)$. Combining with the main conclusion of Step 2, we deduce that

$$
\left( \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} \beta \\ 0 \\ -\beta^{-1} \end{array} \right) \right) \in \rho(G_{\mathbb{Q}(\mu_{p^\infty})}).
$$

Since we have $\left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right) \in \text{SL}_2(R_f_1)$ and $\left( \begin{array}{c} \beta^{-1} \\ 0 \\ \beta \end{array} \right) \in \text{SL}_2(R_f)$, we conclude our proof that

$$
\left( \left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ -1 \end{array} \right) \right) \in \rho(G_{\mathbb{Q}(\mu_{p^\infty})}).
$$

**Step 4.** Conclusion.

The image of the element (5.11) inside $\text{GL}_4(\mathbb{R})$ is the matrix

$$
\left( \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{array} \right)
$$

and we have verified the existence of an element in $\rho(G_{\mathbb{Q}(\mu_{p^\infty})})$ whose image has the desired shape when BI.1.(i) is valid.

When BI.1(ii) is in effect, one may similarly prove for any $x \in \mathbb{Z}_p[[X]]^\times$ (which we view as an element of $\Lambda_i := \mathbb{Z}_p[[\Gamma_i]]$ via $\chi_i$) and any $u \in (\mathbb{Z}/pN_1N_2)^\times$ with $\Psi_1(u)\Psi_2(u) \neq 1$ that

$$
\left( \left( \begin{array}{c} x^{-1} \\ 0 \\ \Psi_1(u) \end{array} \right), \left( \begin{array}{c} x \\ 0 \\ -\Psi_2(u) \end{array} \right) \right) \in \rho(G_{\mathbb{Q}(\mu_{p^\infty})}).
$$

The image of the element (5.12) inside $\text{GL}_4(\mathbb{R})$ is the diagonal matrix

$$
\text{Diag} \left( 1, x^{-2}\Psi_2(u), x^2\Psi_1(u), \Psi_1(u)\Psi_2(u) \right).
$$

On choosing $x$ in a way that

$$
x^{-2}\Psi_2(u) \neq 1 \text{ and } x^2\Psi_2(u) \neq 1 \mod (\pi_\mathcal{O}, X),
$$

we have again verified the existence of an element in $\rho(G_{\mathbb{Q}(\mu_{p^\infty})})$ whose image has the desired shape.
5.2. Beilinson-Flach elements and its relation to \( p \)-adic \( L \)-functions. \( \) Consider the following condition on the central characters of the Hida families \( f_1 \) and \( f_2 \):

\[ (H.c) \text{ The conductor of } \Psi_1, \Psi_2 \text{ is prime to } p. \]

Until the end of this article, we shall assume the validity of (H.c). For every positive integer \( n \) that is coprime to \( 6N_1N_2 \), there exists a generalized Beilinson-Flach element

\[ \text{BF}^{f_1,f_2}_n \in H^1(\mathbb{Q}(\mu_n), T_{\text{cyc}}) \]

that was constructed in [KLZ15, KLZ17], building on [LLZ14]. These elements verify an Euler system distribution relation as the tame conductors \( n \) vary. Here, we remark that the distribution relation satisfied by these elements is not the distribution relation we have asked for in Section 2.2.1, but rather a variant. However, this does not pose any problem for our purposes as we may rely on [Rub00] §9.6 to handle this minor technical issue.

**Remark 5.9.** Since we assume the validity of (H.c), we may work with the normalised collection of classes \( \{\text{BF}^{f_1,f_2}_n\} \) rather than those classes denoted by \( \{\text{BF}^{c,f}_n\} \) in [KLZ15, KLZ17]. Here, \( c \) is the “smoothing factor” of [LLZ14]. We refer the readers to [LLZ14] §6.8.1 where the authors explain how to dispose this auxiliary factor and verify the integrality of the original classes \( \text{BF}^{f_1,f_2}_n \).

The following statement is proved in [KLZ17 Proposition 8.1.7].

**Proposition 5.10.** The collection \( \text{BF} = \{\text{BF}^{f_1,f_2}_n\} \) defines an element of \( \text{ES}^+ (T_{\text{cyc}}) \) in the sense of Definition [2.21].

In concrete terms, the projection of the local class

\[ \text{res}_p(\text{BF}^{f_1,f_2}_n) \in H^1(\mathbb{Q}(\mu_n)_p, T_{\text{cyc}}) \]

to \( H^1(\mathbb{Q}(\mu_n)_p, F_p- T_{\text{cyc}}) \) is trivial and the collection \( \text{BF} \) forms a locally restricted Euler system in the sense of Definition [2.21]. This means that our Theorem 3.6 (and its corollaries) may be applied with the collection \( \text{BF} \) to yield the following result.

**Theorem 5.11.** Let \( f_i \in S_k(\Gamma_1(N_i)) \) \( (i = 1,2) \) be non-CM newforms of respective weights \( k_1, k_2 \), levels \( N_1, N_2 \) which are not twisted-conjugate to each other and suppose \( p > B \) is a good ordinary prime for both forms. Let \( f \) be the unique Hida family that admits the \( p \)-stabilization of \( f_i \) as a weight-\( k_i \) arithmetic specialization \( (i = 1,2) \).

Assume the conditions listed in Lemma 5.4 which ensure the validity of (H.0), (H.2), (H.0\( ^- \)), (H.2\( ^+ \)) and (H.2\( ^{++} \)). Assume in addition the hypotheses (H.c), (BI.1), (BI.2) and (F.Dist) as well as that the class \( \text{BF}^{f_1,f_2}_1 \in H^1(\mathbb{Q}, T_{\text{cyc}}) \) is non-trivial.

(1) The module \( H^1_{Gr} (\mathbb{Q}, T^\vee_{\text{cyc}} (1))^\vee \) is \( \mathcal{R}_{\text{cyc}} \)-torsion.

(2) The module \( H^1_{Gr} (\mathbb{Q}, T_{\text{cyc}}) \) is a torsion-free \( \mathcal{R}_{\text{cyc}} \)-module of rank one.

(3) We have

\[ \text{char}_{\mathcal{R}_{\text{cyc}}} \left( H^1_{Gr} (\mathbb{Q}, T^\vee_{\text{cyc}} (1))^\vee \right) \supset \text{char}_{\mathcal{R}_{\text{cyc}}} \left( H^1_{Gr} (\mathbb{Q}, T_{\text{cyc}})/\mathcal{R}_{\text{cyc}} \cdot \text{BF}^{f_1,f_2}_1 \right). \]

Suppose in addition that \( \text{res}_p(\text{BF}^{f_1,f_2}_1) \neq 0 \). Then,

(4) The module \( H^1_{Gr} (\mathbb{Q}, T^\vee_{\text{cyc}} (1))^\vee \) is \( \mathcal{R}_{\text{cyc}} \)-torsion.

(5) We have

\[ \text{char} \left( H^1_{Gr} (\mathbb{Q}, T^\vee_{\text{cyc}} (1))^\vee \right) \supset \text{char} \left( H^1_{Gr} (\mathbb{Q}, T_{\text{cyc}})/\mathcal{R}_{\text{cyc}} \cdot \text{res}_+ \left( \text{BF}^{f_1,f_2}_1 \right) \right) \]

where we recall that

\[ H^1_{Gr} (\mathbb{Q}, T_{\text{cyc}}) := H^1_{Gr} (\mathbb{Q}, T_{\text{cyc}})/H^1_{Gr} (\mathbb{Q}, T_{\text{cyc}}) \sim H^1 (\mathbb{Q}, F_p-T_{\text{cyc}}) \]

and the map \( \text{res}_+ \) is the compositum of the arrows

\[ H^1 (\mathbb{Q}, T_{\text{cyc}}) \xrightarrow{\text{res}_+} H^1 (\mathbb{Q}, T_{\text{cyc}}) \rightarrow H^1 (\mathbb{Q}, F_p^+ T_{\text{cyc}}). \]
Definition 5.12. Let $L_p^{\Hida}(f_1, f_2, j) \in \mathbb{I}_{f_1} \otimes \mathbb{I}_{f_2} \otimes \Lambda_{\text{cyc}}$ denote Hida’s $p$-adic $L$-function in 3-variables defined in \[Hid88\] Theorem I, where $j$ stands for the cyclotomic variable.

As in \[Hid88\], we will identify $\mathbb{I}_{f_1}$ with $\mathbb{I}_{f_2}$ by suitably extending the rings $\mathbb{I}_{f_1}$ and $\mathbb{I}_{f_2}$. Through this identification, $L_p^{\Hida}(f_1, f_2, j)$ is regarded as an element of $\frac{1}{H} \mathbb{I} \otimes \mathbb{I} \otimes \Lambda_{\text{cyc}}$, where $H := \mathbb{I}_{f_1} = \mathbb{I}_{f_2}$ and where we have set $H \subseteq \mathbb{I}_{f_1}$ to denote the congruence divisor, which is an invariant controlling the congruences between $f_1$ and other Hida families of the same tame conductor (see \[Hid88\] Section 4) for the definition and further properties of $H$). The $p$-adic $L$-function $L_p^{\Hida}(f_1, f_2, j)$ is characterized by the interpolation property that

\begin{equation}
L_p^{\Hida}(f_1(\kappa_1), f_2(\kappa_2), j) = \left( 1 - \frac{p^{j-1}}{\alpha(f_1(\kappa_1))\alpha(f_2(\kappa_2))} \right) \times \left( 1 - \frac{p^{j-1}}{\beta(f_1(\kappa_1))\beta(f_2(\kappa_2))} \right)
\times \left( 1 - \frac{p^{j-1}}{\beta(f_1(\kappa_1))} \right)^{-1} \times L(f_1(\kappa_1), f_2(\kappa_2), j)
\end{equation}

for arithmetic specializations $\kappa_1, \kappa_2$ of the families $f_1, f_2$ of respective weights $w_1 > w_2$, and integers $j \in \{w_2 + 1, w_1\}$ where

(i) $\alpha(f_1(\kappa_1)) = \kappa_1(a_p(f_1))$ and $\beta(f_1(\kappa_1)) = p^{w_1+1}\Psi_i^{(N)}(\kappa_1)(p)\kappa_1(a_p(f_1))^{-1}$;

(ii) $L(\kappa_1(f_1), \kappa_2(f_1), j)$ is the Rankin-Selberg $L$-function;

(iii) $\Omega(\kappa_1, \kappa_2, j) \in \mathbb{C}^\times$ is an appropriate complex period.

Theorem 5.13 below is the restatement of Corollary 4.14 in this particular set up for $\Gr^1_p T_{\text{cyc}} = F^{\omega} T_{\text{cyc}}$ above. We shall henceforth drop $(\Gr^1_p T_{\text{cyc}})^{\text{R}_{\text{cyc}}}(1)$ and $(\Gr^m_p V_{\kappa})^{\kappa(\text{R}_{\text{cyc}})}(1)$ from the subscripts to ease our notation, as we shall solely work with this choice until the end of this article. Notice further that we have

$d_1(\kappa) = j - \omega_2$ and $\kappa|_{\mathbb{R}}(\alpha_1(F_{\mathbb{R}})) = a_p(f_1(\kappa_1))a_p(f_2(\kappa_2))^{-1}\Psi^{(N_2)}_i(p)$

in the notation of Corollary 4.14.

Theorem 5.13. There exists an $\text{R}_{\text{cyc}}$-linear map

\[ \text{EXP}^* : H^1(\mathbb{Q}_p, F^{\omega} T_{\text{cyc}}) \longrightarrow \mathbb{D}(F^{\omega} T_{\text{cyc}}) \]

which is characterized by the following interpolation property: For every $\kappa \in \mathcal{S}_{\text{cyc}}^{(1)}$ (so that $j > w_2$) the following diagram commutes:

\[ \begin{array}{ccc}
H^1(\mathbb{Q}_p, F^{\omega} T_{\text{cyc}}) & \xrightarrow{\text{EXP}^*} & \mathbb{D}(F^{\omega} T_{\text{cyc}}) \\
\kappa \downarrow & & \kappa \downarrow \\
H^1(\mathbb{Q}_p, F^{\omega} + V_{\kappa}) & \xrightarrow{e_p \times \text{exp}^*} & D_{\text{dR}}(F^{\omega} + V_{\kappa})
\end{array} \]

Here $e_p^+ := (-1)^{j-w_2}(j - w_2)!e_p$ and $e_p = e_p((F^{\omega} + V_{\kappa})^{\kappa(\text{R}_{\text{cyc}})}(1))$ is the $p$-adic multiplier given by

\[ e_p = \left( 1 - \frac{p^{j-w_2-1}}{a_p(f_1(\kappa_1))a_p(f_2(\kappa_2))^{-1}\Psi^{(N_2)}_i(p)} \right) \left( 1 - \frac{a_p(f_1(\kappa_1))a_p(f_2(\kappa_2))^{-1}\Psi^{(N_2)}_i(p)}{p^{j-w_2}} \right)^{-1} \]

in case $\omega^{-w_2+1}\Psi^{(p)}_i(\phi_{\kappa_2} \Phi)$ is the trivial character (which is equivalent to the requirement that $F^{\omega} + V_{\kappa}$ be crystalline), and
Definition 5.15. Let us choose an element \( e_p = \left( \frac{p^{j-w-1}}{a_p(f_1(\kappa_1))a_p(f_2(\kappa_2))^{-1} \Psi_2^{(N_2)}(p)} \right)^n \) when \( F^+V_\kappa \mid_{f_p} \cong E_\kappa(\omega_2 + 1 - j)(\eta) \) with \( \text{ord}_p(\text{cond}(\eta)) = n \geq 1 \).

Also, for every \( \kappa \in S_{\text{cyc}}^{(1)} \) (so that \( j \leq w_2 \)) we also have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}^1(\mathbb{Q}_p, F^+T_{\text{cyc}}) & \xrightarrow{\text{EXPP}} & \mathcal{D}(F^+T_{\text{cyc}}) \\
\kappa & \downarrow & \kappa \\
\mathcal{H}^1(\mathbb{Q}_p, F^+V_\kappa) & \xrightarrow{e_p^{-1} \log} & D_{\text{dR}}(\text{Gr}^m_{f_p}V_\kappa)
\end{array}
\]

where \( e_p := \frac{e_p}{(w_2 - j)!} \).

Recall that we have an identification

\begin{equation}
F^+T_{\text{cyc}} = \text{Gr}^i_{p}T_{\text{cyc}} = F^+_pT_{f_1} \hat{\otimes}_{\mathbb{Z}_p} F^+_pT_{f_2} \hat{\otimes}_{\mathbb{Z}_p} (\Lambda^i_{\text{cyc}})^e
\end{equation}

and hence

\begin{equation}
\mathcal{D}(F^+T_{\text{cyc}}) = \mathcal{D}(F^+_pT_{f_1}) \otimes_{\mathbb{Z}_p} \mathcal{D}(F^+_pT_{f_2}) \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\text{cyc}}.
\end{equation}

Remark 5.14. We recall that what we call a Hida family in this article corresponds to a new, cuspidal branch of a Hida family \( F_i \) of tame level \( N_i \) in the sense of [KZ17, Remark 7.2.4]. The ring denoted by \( \Lambda_2 \) in op. cit. agrees with our \( I_{f_2} \), with \( f \) in loc. cit. chosen as \( F_2 \) and the minimal ideal \( a \) in the notation of [KZ17] corresponds to our branch \( f_1 \). Finally, the fractional ideal denoted by \( I_2 \) in [KZ17, Notation 7.7.1] agrees with the fractional ideal \( 1_H\mathbb{I}_{f_1} \) above.

Definition 5.15. Let us choose an element \( \omega_{f_2} \in \mathcal{D}(F^+_pT^*_{f_2}(1)) \) which interpolates holomorphic differential forms that correspond to arithmetic specializations of the dual Hida family \( \overline{F}_2 \) of \( f_2 \). We also take an element \( \eta_{f_1} \in \mathcal{D}(F^+_pT^*_{f_1}(1)) \) which interpolates anti-holomorphic differential forms that correspond to arithmetic specializations of the dual Hida family \( \overline{F}_1 \) of \( f_1 \).

Using the identification (5.15), consider the pairing

\[
\langle \cdot, \cdot \rangle : \mathcal{D}(F^+T_{\text{cyc}}) \times (\mathcal{D}(F^+_pT^*_{f_1}(1)) \hat{\otimes} \mathcal{D}(F^+_pT^*_{f_2}(1))) \rightarrow \mathbb{I}_{f_1} \hat{\otimes} \mathbb{I}_{f_2} \hat{\otimes} \Lambda_{\text{cyc}}.
\]

We define

\begin{equation}
L_p^{BF}(f_1, f_2, 1 + j) := \left\langle \text{EXPP} \circ \text{res}_p \left( BF_1^{f_1, f_2} \right), \eta_{f_1} \hat{\otimes} \omega_{f_2} \right\rangle \in \mathbb{I}_{f_1} \hat{\otimes} \mathbb{I}_{f_2} \hat{\otimes} \Lambda_{\text{cyc}}
\end{equation}

where \( j \) is the cyclotomic variable. We remark that, by choosing a suitable identification of \( \mathbb{I}_{f_1} \) with \( \mathbb{I}_{f_2} \), the element \( L_p^{BF}(f_1, f_2, 1 + j) \) can be regarded as an element of \( \frac{1}{H} \mathbb{I} \hat{\otimes} \mathbb{I} \hat{\otimes} \Lambda_{\text{cyc}} \) (where \( I = I_{f_1} = I_{f_2} \)), as we have explained within Definition 5.12.

The following is a very slight variant of [KZ17, Theorem 10.2.2] (and its proof follows the argument in op. cit.).

Theorem 5.16. We have the equality

\begin{equation}
R_{\text{cyc}}L_p^{BF}(f_1, f_2, 1 + j) = R_{\text{cyc}}L_p^{Hida}(f_1, f_2, 1 + j).
\end{equation}

Proof. Let us take any arithmetic point \( \kappa = (\kappa_1, \kappa_2, j) \) on \( \mathbb{I}_{f_1} \hat{\otimes} \mathbb{I}_{f_2} \hat{\otimes} \Lambda_{\text{cyc}} \) such that the weights \( w_1 \) and \( w_2 \) of \( \kappa_1 \) and \( \kappa_2 \) satisfy the condition \( 1 \leq j \leq w_2 + 1 < w_1 + 1 \).

By the definition of \( L_p^{BF}(f_1, f_2, 1 + j) \) given in (5.16), we have

\begin{equation}
\kappa(L_p^{BF}(f_1, f_2, j)) = \left\langle \kappa(\text{EXPP} \circ \text{res}_p \left( BF_1^{f_1, f_2} \right)), \eta_{f_1(\kappa_1)} \otimes \omega_{f_2(\kappa_2)} \right\rangle.
\end{equation}
By the interpolation property of the big exponential map $\text{EXP}^*$ obtained in Theorem [5.13] we have

\begin{equation}
\kappa(\text{EXP}^* \circ \text{res}_p(\text{BF}_{1}^{f_1,f_2})) = \epsilon_p^{-1} \cdot \log(\kappa(\text{res}_p(\text{BF}_{1}^{f_1,f_2}))).
\end{equation}

Furthermore, the class $L_{et}^{f_1(f_1),f_2(f_2),j}$ is related to the special value of $L_p^{\text{Hida}}$ via the identity

\begin{equation}
\left< \log(L_{et}^{f_1(f_1),f_2(f_2),j}), \eta_{f_1(f_1)} \otimes \omega_{f_2(f_2)} \right>
= (\text{Explicit Fudge Factors}) \times L_p^{\text{Hida}}(f_1(f_1),f_2(f_2),1+j)
\end{equation}

thanks to [KLZ15] Theorem 6.5.9. Note that the arithmetic points $\kappa = (\kappa_1, \kappa_2, j)$ satisfying the condition $1 \leq j \leq w_2 + 1 < w_1 + 1$ are Zariski dense in $\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_{\text{cyc}}$. Combining the equations (5.18), (5.19) and (5.20), we deduce that $\mathcal{R}_{\text{cyc}} L_p^{\text{BF}}(f_1, f_2, j) = \mathcal{R}_{\text{cyc}} L_p^{\text{Hida}}(f_1, f_2, j)$, as we have claimed.

\begin{corollary}
The class $\text{res}_p(\text{BF}_{1}^{f_1,f_2})$ is non-trivial.
\end{corollary}

\begin{proof}
Theorem 5.16 reduces the non-vanishing of the class $\text{res}_p(\text{BF}_{1}^{f_1,f_2})$ to the non-triviality of Hida’s $p$-adic Rankin-Selberg $L$-function $L_p^{\text{Hida}}(f_1, f_2, 1+j)$.

To see the non-triviality of $L_p^{\text{Hida}}(f_1, f_2, j)$, we consider the interpolation property [5.13] of $L_p^{\text{Hida}}(f_1, f_2, 1+j)$. We recall that the Euler product of the $L$-function $L(\kappa_1(f_1), \kappa_2(f_2), s)$ is absolutely convergent (and therefore non-zero) for $\text{Re}(s) > \frac{w_1+w_2+2}{2} + 1$. It is also easy to check the non-vanishing of the Euler-like factor

\[
\left( 1 - \frac{p^{j-1}}{\alpha(f_1(\kappa_1)) \alpha(f_2(\kappa_2))} \right) \left( 1 - \frac{p^{j-1}}{\alpha(f_1(\kappa_1)) \beta(f_2(\kappa_2))} \right) \left( 1 - \frac{\beta(f_1(\kappa_1)) \alpha(f_2(\kappa_2))}{p^j} \right) \left( 1 - \frac{\beta(f_1(\kappa_1))}{p^j \alpha(f_1(\kappa_1))} \right)^{-1} \left( 1 - \frac{\beta(f_1(\kappa_1))}{\alpha(f_1(\kappa_1))} \right)^{-1}.
\]

In fact, it is known that the archimedean absolute values of the eigenvalues $\alpha(f_1(\kappa_1))$ and $\beta(f_1(\kappa_1))$ belong to the range $[p^{w_1}, p^{w_1+1}]$. A similar statement holds true for $\alpha(f_2(\kappa_2))$ and $\beta(f_2(\kappa_2))$. Based on these facts, it is now easy to check that $L_p^{\text{Hida}}(f_1, f_2, j)$ has a specialization $(\kappa_1, \kappa_2, w_1)$ with $w_1 > w_2 + 2$ where the value of the function $L_p^{\text{Hida}}(f_1, f_2, j)$ is non zero. Hence, the function $L_p^{\text{Hida}}(f_1, f_2, j)$ itself is non zero.

Consequently, Theorem 5.16 implies that the class $\text{res}_p(\text{BF}_{1}^{f_1,f_2})$ is non-trivial.
\end{proof}

\begin{corollary}
Let $f_i \in \mathcal{S}_i(\Gamma_1(N_i)) (i = 1, 2)$ be non-CM newforms of respective weights $k_1, k_2$, levels $N_1, N_2$ which are not twisted-conjugate to each other and suppose $p > B$ is a good ordinary prime for both forms. Let $f_i$ be the unique Hida family that admits the $p$-stabilization of $f_i$ as a weight-$k_i$ arithmetic specialization $(i = 1, 2)$. Suppose that the conditions in Lemma 5.4 that ensure the validity of (H.0), (H.2), (H.0’), (H.2’), and (H.2++) hold. Assume also the hypotheses (H.c), (B.1), (B.2) and (F.Dist). Then,

\[
\text{char}_{\mathcal{R}_{\text{cyc}}} \left( H_{+}^{1, f_i} \left( \mathbb{Q}, T_{\text{cyc}}(1)^{\vee} \right) \right)^{\vee} \supset (H) \cdot \mathcal{R}_{\text{cyc}} L_p^{\text{Hida}}(f_1, f_2, 1+j),
\]

\begin{proof}
Under our running hypotheses we have the following chain of isomorphisms

\[
H_{+}^{1}(\mathbb{Q}_p, T_{\text{cyc}}) \sim \sim H^{1}(\mathbb{Q}_p, F^{-1}T_{\text{cyc}}) \sim \sim \mathbb{D}(F^{-1}T_{\text{cyc}}).
\]

Since $\omega_{f_2}$ is an isomorphism (see [KLZ17] Proposition 10.1.1(1)), it follows from Theorem 5.16 that

\[
\text{char}_{\mathcal{R}_{\text{cyc}}} \left( \frac{H_{+}^{1}(\mathbb{Q}_p, T_{\text{cyc}})}{\mathcal{R}_{\text{cyc}} \text{res}_{+} f_i (\text{BF}_{1}^{f_1,f_2})} \right)^{\vee} \supset (H) \cdot \mathcal{R}_{\text{cyc}} L_p^{\text{BF}}(f_1, f_2, 1+j).
\]

Corollary now follows from Theorem 5.11(3) and Corollary 5.17.
\end{proof}
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REFERENCES

[Ber03] Laurent Berger. Bloch and Kato’s exponential map: three explicit formulas. Doc. Math., J. DMV, Extra Vol.:99–129, 2003.

[BL15] Kâzım Büyükboduk and Antonio Lei. Coleman-adapted Rubin-Stark Kolyvagin systems and supersingular Iwasawa theory of CM abelian varieties. Proc. Lond. Math. Soc. (3), 111(6):1338–1378, 2015.

[BL16] Kâzım Büyükboduk and Antonio Lei. Anticyclotomic $p$-ordinary Iwasawa Theory of Elliptic Modular Forms, 2016. preprint, arXiv:1602.07508.

[Büy09a] Kâzım Büyükboduk. Kolyvagin systems of Stark units. J. Reine Angew. Math., 631:85–107, 2009.

[Büy09b] Kâzım Büyükboduk. Stark units and the main conjectures for totally real fields. Compositio Math., 145:1163–1195, 2009.

[Büy10] Kâzım Büyükboduk. On Euler systems of rank $r$ and their Kolyvagin systems. Indiana Univ. Math. J., 59(4):1277–1332, 2010.

[Büy11] Kâzım Büyükboduk. $A$-adic Kolyvagin systems. IMRN, 2011(14):3141–3206, 2011.

[Büy14] Kâzım Büyükboduk. Main conjectures for CM fields and a Yager-type theorem for Rubin-Stark elements. Int. Math. Res. Not. IMRN, (21):5832–5873, 2014.

[Büy16] Kâzım Büyükboduk. Deformations of Kolyvagin systems. Ann. Math. Qué., 40(2):251–302, 2016.

[Cas15] Francesc Castella. $p$-adic heights of Heegner points and Beilinson-Flach elements, 2015. Preprint, arXiv:1509.02761.

[Fis02] Ami Fischman. On the image of $A$-adic Galois representations. Ann. Inst. Fourier, 52(2):351–378, 2002.

[FO12] Olivier Fouquet and Tadashi Ochiai. Control theorems for Selmer groups of nearly ordinary deformations. J. Reine Angew. Math., 666:163–187, 2012.

[GGJQ05] Eknath Ghate, Enrique González-Jiménez, and Jordi Quer. On the Brauer class of modular endomorphism algebras. Int. Math. Res. Not. (12):701–723, 2005.

[Hid88] Haruzo Hida. A $p$-adic measure attached to the zeta functions associated with two elliptic modular forms. II. Ann. Inst. Fourier (Grenoble), 38(3):1–83, 1988.

[KLZ15] Guido Kings, David Loeffler, and Sarah Zerbes. Rankin–Eisenstein classes for modular forms. Ann. of Math. (2), 180(2):653–771, 2014.

[LO14] Francesco Lemma and Tadashi Ochiai. The Coleman map for Hida families of $GSp_4$. Amer. J. Math., 136(3):729–760, 2014.

[Loe17] David Loeffler. Images of adelic Galois representations for modular forms. Glasg. Math. J., 59(1):11–25, 2017.

[MR04] Barry Mazur and Karl Rubin. Kolyvagin systems. Mem. Amer. Math. Soc., 168(799):viii+96, 2004.

[Nek06] Jan Nekovář. Selmer complexes. Astérisque, (310):viii+559, 2006.

[Och03] Tadashi Ochiai. A generalization of the Coleman map for Hida deformations. Amer. J. Math., 125(4):849–892, 2003.

[Och05] Tadashi Ochiai. Euler system for Galois deformations. Ann. Inst. Fourier (Grenoble), 55(1):113–146, 2005.

[Per94] Bernadette Perrin-Riou. Théorie d’Iwasawa des représentations $p$-adiques sur un corps local. Invent. Math., 115(1):81–149, 1994.

[Ram00] D. Ramakrishnan. Recovering modular forms from squares. Appendix to “A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations” (by W. Duke and E. Kowalski). Invent. Math., 139(1):1–39, 2000.

[Rub98] Karl Rubin. Euler systems and modular elliptic curves. In Galois representations in arithmetic algebraic geometry (Durham, 1996), volume 254 of London Math. Soc. Lecture Note Ser., pages 351–367. Cambridge Univ. Press, Cambridge, 1998.

[Rub00] Karl Rubin. Euler systems, volume 147 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

[SU14] Christopher Skinner and Eric Urban. The Iwasawa main conjectures for $GL_2$. Invent. Math., 195(1):1–277, 2014.

[Wan15] Xin Wan. The Iwasawa main conjecture for Hilbert modular forms. Forum of Mathematics, Sigma, 3, 2015.