ON THE DISTRIBUTION OF RANK AND CRANK STATISTIC FOR INTEGER PARTITIONS

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Abstract. Let $M(m,n)$ be the number of partitions of $n$ with crank $m$. We prove that Dyson’s crank distribution conjecture valid if and only if $|m| = o(n^{3/4})$ as $n$ tends to infinity. We also establish some asymptotic formulas for the finite differences of Garvan’s k-rank $N_k(n,m)$ respect to $m$ with $m \gg \sqrt{n \log n}$.

1. Introduction and statement of results

A partition of a positive integer $n$ is a sequence of non-increasing positive integers whose sum equals $n$. Let $p(n)$ be the number of integer partitions of $n$. In 1944, F.J. Dyson [1] introduced the rank statistic for integer partitions. The rank of a partition was introduced to explain Ramanujan’s famous partition congruences with modulus 5 and 7. If $N(m,n)$ denotes the number of partitions of $n$ with rank $m$, then it is well known that

$$
\sum_{n \geq 0} N(m,n) q^n = \left( \prod_{r \geq 1} (1 - q^r)^{-1} \right) \sum_{n \geq 1} (-1)^{n-1} q^{n(3n-1)/2+|m|n} (1 - q^n).
$$

However the rank fails to explain Ramanujan’s congruence modulo 11. Therefore Dyson conjectured the existence of another statistic which he called the crank which would explain Ramanujan’s partition congruence modulo 11. The crank was found by G.E. Andrews and F.G. Garvan [2, 3]. Let $M(m,n)$ be the number of partitions of $n$ with crank $m$, then

$$
\sum_{n \geq 0} M(m,n) q^n = \left( \prod_{r \geq 1} (1 - q^r)^{-1} \right) \sum_{n \geq 1} (-1)^{n-1} q^{n(3n-1)/2+|m|n} (1 - q^n).
$$

In [4], K. Bringmann and J. Dousse prove the following asymptotic behavior for $M(m,n)$, which conjectured by F.J. Dyson [5]:

**Theorem 1.1.** If $|m| \leq \frac{1}{\pi} \sqrt{\frac{2}{5}} \log n$, we have

$$
M(m,n) = \frac{\pi}{4 \sqrt{6n}} \text{sech}^2 \left( \frac{\pi m}{2 \sqrt{6n}} \right) p(n) \left( 1 + O \left( \frac{|m|^{1/3} + 1}{n^{1/4}} \right) \right)
$$

holds for $n$ tends to infinity.\(^a\)

Interestingly, J. Dousse and M.H. Mertens [6] prove that the same asymptotic formula (1.1) holds for $N(m,n)$.

F.J. Dyson [5] also asked the question about the precise range of $m$ in (1.1), which holds and about the error term. In this paper, we give a more complete answer for these questions.

\(^a\)In [4], the $O$-term of (1.1) is $O \left( |m|^{1/3} n^{-1/4} \right)$, the case of $m = 0$ was missed.
We will focus on the case of \( m \gg \sqrt{n} \) with \( n \) is sufficiently large. We have the following result for Dyson’s conjecture.

**Theorem 1.2.** Let \( M(m,n) \) be the number of partitions of \( n \) with rank \( m \). Then Dyson’s crank distribution conjecture (1.1) valid if and only if \( |m| = o(n^{3/4}) \) as \( n \) tends to infinity. More precise, we have as \( n \to \infty \),

\[
M(m,n) = \frac{\pi}{4\sqrt{6}n} \text{sech}^2 \left( \frac{\pi m}{2\sqrt{6}n} \right) p(n) \left( 1 + O \left( e^{-\frac{\pi|m|}{2\sqrt{6}n} + \frac{m^2}{n^{3/2}}} \right) \right)
\]

holds for \( \sqrt{n} \ll |m| \ll n^{3/4} \).

**Remark 1.1.** The error term of Theorem 1.2 is more precisely then Theorem 1.1 when \( |m| > \frac{1}{\pi \sqrt{6}} (\log n - 4 \log \log n) \) as \( n \to \infty \).

We will consider more general problem. Let \( N_k(m,n) \) be the number of partitions of \( n \) into at least \((k-1)\) successive Duree squares with \( k \)-rank equal to \( m \in \mathbb{Z} \) and \( k \in \mathbb{N} \). F.G. Garvan [7] proved

\[
(1.2) \quad \sum_{n \geq 0} N_k(m,n)q^n = \left( \prod_{r \geq 1} (1-q^r)^{-1} \right) \sum_{n \geq 1} (-1)^{n-1} q^{n((2k-1)n-1)/2+|m|n} (1-q^n).
\]

Clearly, \( M(m,n) = N_k(1,m,n) \) and \( N(m,n) = N_0(m,n) \). Theorem 1.2 follows from Theorem 1.1 and the following more general asymptotic result for \( N_k(m,n) \).

**Theorem 1.3.** Let \( p(n) \) be the number of integer partitions of \( n \). Also let \( p(-\ell) = 0 \) for \( \ell \in \mathbb{N} \). We have if \( |m| \geq (n+3)/2 - 2k \) then

\[
N_k(m,n) = F_k(1; |m|, n),
\]

where \( F_k(1; |m|, n) = p(n-(|m|+k)+1) - p(n-(|m|+k)) \). If \( n/2 > |m| \gg \sqrt{n} \) then

\[
N_k(m,n) = F_k(1; |m|, n) \left( 1 + O_k \left( e^{-\frac{\pi|m|}{\sqrt{6}n} + \frac{\pi
\sqrt{n/6}}{k}} \right) \right)
\]

holds for \( n \to \infty \).

**Remark 1.2.** From Theorem 1.3 and the following Proposition 2.3, we can determine more precisely asymptotic expansion of \( N_k(m,n) \) for all \( |m| < n \) with \( n - |m| \) tends to infinity. We note that the case of \( m \) be fixed has been considered by R. Mao [8], K. Bringmann and J. Manschot [9], and B. Kim, E. Kim and J. Seo [10].

**Remark 1.3.** The asymptotic behavior of \( N_k(m,n) \) for \( m \gg \sqrt{n \log n} \) has been studied by D. Parry and R.C. Rhoades [11], recently. Their main result is:

Given constants \( C_1, C_2 > 0 \) and \( \delta \in [0, 1/4] \), one has, as \( n \to \infty \) and \( C_1 n^{1/2} \leq m + k \leq C_2 n^{3/4 - \delta} \),

\[
N_k(m,n) \sim \frac{\pi}{\sqrt{6n}} \left( e^{-\frac{\pi(m+k)}{2\sqrt{6}n}} + e^{-\frac{\pi(m+k)}{2\sqrt{6}n}} \right)^{-2} p(n).
\]

Combining Theorem 1.1, Theorem 1.2 and Theorem 1.3 we find that the asymptotic formula (1.3) is false for \( \delta = 0 \), and hence the main theorem of [11] is incorrect. See Lemma 3.1 of the following Section 3 for detail.
By straightforward calculation, we obtain the following asymptotic result for the finite differences of $N_k(n, m)$ respect to $m$ with $m \gg \sqrt{n} \log n$.

**Corollary 1.4.** Let $m, n, r, k \in \mathbb{N}$ and $r, k$ be fixed. If $m \geq c \sqrt{n} \log n$ with $c > \left(\frac{r+1}{2\pi}\right)^{1/6}$ be fixed and $n - m \to \infty$, then

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} N_k(m + j, n) \sim \left(\frac{\pi}{\sqrt{6(n-m)}}\right)^{r+1} p(n-m).$$

In particular,

$$N_k(m, n) - N_k(m + 1, n) \sim \frac{\pi^2}{\sqrt{6(n-m)}} p(n-m)$$

holds for $m \geq c_1 \sqrt{n} \log n$ with $c_1 > \sqrt{6}/\pi$ be fixed and $n - m \to \infty$.

**Remark 1.4.** We can improve (1.4) and give the size of the error term by Theorem 1.3. We note that the monotonicity properties of $N(m, n)$ has been investigated by S.H. Chan and R. Mao [12].

The proof of Theorem 1.3 is based on the generating function (1.2) of $N_k(m, n)$ and the Hardy-Ramanujan asymptotic formula for $p(n)$. We present the relation between $N_k(m, n)$ and $p(n)$, and some basic properties of $p(n)$ in Section 2. In Section 3, we prove the main results of this paper.

2. Preliminaries

2.1. **Basic properties for Garvan $k$-ranks.** A key ingredient of our asymptotic results is find the connection $N_k(m, n)$ with $p(n)$. Note that $N_k(m, n) = N_k(-m, n)$, we just need consider the case of $m \geq 0$. We prove the following representation for $N_k(m, n)$.

**Proposition 2.1.** Let $N_k(m, n)$ be defined as (1.2), $n \in \mathbb{N}$ and $m \in \mathbb{Z}_{\geq 0}$. We have

$$N_k(m, n) = \sum_{\ell \geq 1} (-1)^{\ell-1} F_k(\ell; m, n),$$

where

$$F_k(\ell; m, n) = p \left( n - m\ell - \frac{(2k-1)\ell^2 - \ell}{2} \right) - p \left( n - m\ell - \frac{(2k-1)\ell^2 + \ell}{2} \right)$$

and we define $p(r) = 0$ for $r < 0$.

**Proof.** Let $k \in \mathbb{N}$ and $m \in \mathbb{Z}_{\geq 0}$. From

$$\prod_{r \geq 1} (1 - q^r)^{-1} = \sum_{r \in \mathbb{Z}} p(r) q^r$$

and

$$\sum_{n \geq 0} N_k(m, n) q^n = \left( \prod_{r \geq 1} (1 - q^r)^{-1} \right) \sum_{\ell \geq 1} (-1)^{\ell-1} q^{\ell((2k-1)\ell-1)/2+m\ell}(1 - q^\ell)$$
we have
\[
\sum_{n \geq 0} N_k(m, n)q^n = \sum_{\ell \geq 1} \sum_{r \in \mathbb{Z}} (-1)^{\ell-1} p(r)q^{r + \ell((2\ell-1)/2 + m\ell}(1 - q^\ell)
\]
\[
= \sum_{n \geq 1} q^n \left( \sum_{\ell \geq 1, \ell \not\equiv 0 \pmod{2}} \sum_{r \in \mathbb{Z}} - \sum_{\ell \geq 1, \ell \equiv 0 \pmod{2}} \sum_{r \in \mathbb{Z}} \right) (-1)^{\ell-1} p(r).
\]
Then, comparing the coefficients of \(q\) on both sides complete the proof of the proposition. \(\square\)

2.2. Hardy–Ramanujan for \(p(n)\) and its applications. We need the following Hardy–Ramanujan asymptotic result for \(p(n)\), which could be find in [13].

**Lemma 2.2.** We have for \(n \in \mathbb{N}\),
\[
p(n) - \hat{p}(n) \ll n^{-1} e^{B\sqrt{n}/2},
\]
where \(B = 2\pi/\sqrt{6}\) and
\[
\hat{p}(n) = \frac{e^{B\sqrt{n-1/24}}}{4\sqrt{3(n-1/24)}} \left(1 - \frac{1}{B(n-1/24)^{1/2}}\right)
\]
(throughout, \(B\) and \(\hat{p}(n)\) be defined as above). In particular,
\[
p(n) \ll n^{-1} e^{B\sqrt{n}}.
\]

Next, we shall prove an useful estimate as follows.

**Proposition 2.3.** Letting \(n, k \in \mathbb{N}\), \(n\) be sufficiently large and denoting by
\[
\hat{F}_k(\ell; m, n) = \hat{p} \left( n - m\ell - \frac{(2k-1)\ell^2 - \ell}{2} \right) - \hat{p} \left( n - m\ell - \frac{(2k-1)\ell^2 + \ell}{2} \right)
\]
for \(m\ell + (k - 1/2)\ell^2 \leq n/2\). We have for nonnegative integer \(m \leq n/3\),
\[
N_k(m, n) = I_k(m, n) + E_k(m, n)
\]
with
\[
I_k(m, n) = \sum_{\ell \geq 1} \sum_{m\ell + (k-1/2)\ell^2 \leq n/2} (-1)^{\ell-1} \hat{F}_k(\ell; m, n)
\]
and
\[
E_k(m, n) \ll e^{B\sqrt{3n/5}}.
\]

**Proof.** From Proposition 2.1 and Lemma 2.2, as \(n\) is sufficiently large we estimate that
\[
E_k(m, n) = N_k(m, n) - I_k(m, n)
\]
\[
\ll \sum_{\ell \geq 1} \sum_{m\ell + (k-1/2)\ell^2 > n/2} |F_k(\ell; m, n)| + \sum_{\ell \geq 1} \sum_{m\ell + (k-1/2)\ell^2 \leq n/2} |F_k(\ell; m, n) - \hat{F}_k(\ell; m, n)|
\]
\[
\ll \sqrt{np([3n/5])} + \sum_{\ell \geq 1} \sum_{m\ell + (k-1/2)\ell^2 \leq n/2} n^{-1} e^{B\sqrt{3n}/2} \ll e^{B\sqrt{3n/5}},
\]

\(\square\)
where $\lfloor \cdot \rfloor$ is the greatest integer function. This finishes the proof of the proposition.

Remark 2.4. We remark that Proposition 2.3 transform the estimate of the asymptotic of $N_k(m, n)$ into an usual problem, about estimate the asymptotic of a finite series with "operable" summands. This allow us determine the further term of the asymptotic of $N_k(m, n)$ by straightforward calculation. However, it is should point out that such calculation of the further term for $m \ll \sqrt{n \log n}$ is difficult due to the complicated summands.

To prove the results of this paper, we also need the following approximation for $\hat{p}(n + x)$ with $x = O(\sqrt{n})$.

**Lemma 2.5.** Let $x = O(\sqrt{n})$ and $n$ be sufficiently large, we have

$$\hat{p}(n + x) - \left(1 + \frac{Bx}{2\sqrt{n}}\right)\hat{p}(n) \ll \frac{1 + |x| + |x|^2}{n}\hat{p}(n).$$

**Proof.** It is clear that

$$\frac{\hat{p}(n + x)}{\hat{p}(n)} = e^{B(\sqrt{n+x-1/24} - \sqrt{n-1/24}) \left(1 + O\left(\frac{|x|}{n}\right)\right)} = e^{B\sqrt{n}\left(\frac{1}{24} + O\left(\frac{|x|^2 + 1}{n^2}\right)\right) \left(1 + O\left(\frac{|x|}{n}\right)\right)} = \left(1 + \frac{Bx}{2\sqrt{n}} + O\left(\frac{|x|^2 + |x| + 1}{n}\right)\right) \left(1 + O\left(\frac{|x|}{n}\right)\right)$$

by generalized binomial theorem. Therefore,

$$\frac{\hat{p}(n + x)}{\hat{p}(n)} - 1 - \frac{Bx}{2\sqrt{n}} \ll \frac{1 + |x| + |x|^2}{n},$$

which completes the proof of the lemma.

\[\square\]

3. The proof of the results of this paper

We prove Theorem 1.3 in Subsection 3.1, Theorem 1.2 in Subsection 3.2 and Corollary 1.4 in the last subsection of this section.

3.1. The proof of the main theorem.

3.1.1. The case of $m \gg n$. Let $m \geq 0$ and $n \geq k - 1$. From Proposition 2.1, it is easy to prove that

$$N_k(m, n) = F_k(1; m, n)$$

for $m \geq (n + 3)/2 - 2k$. Also, if $n/2 > m \geq n/5$ with $n$ is sufficiently large, then

$$N_k(m, n) - F_k(1; m, n) \ll \sum_{2 \leq \ell \leq 5} p(n - m\ell) \ll p(n - 2m) \ll e^{B\sqrt{3n/5}}$$

by Lemma 2.2.
3.1.2. The case of $m \ll n$. Let $n$ be sufficiently large. If $\sqrt{n} \ll m < n/5$, then

$$N_k(m, n) = F_k(1; m, n) + (I_k(m, n) - F_k(1; m, n) + E_k(m, n))$$

by Proposition 2.3. Further, we estimate that

$$\Sigma_{m,N} := I_k(m, n) - F_k(1; m, n) + E_k(m, n)$$

$$\ll \sum_{\ell \geq 2} \left| \hat{F}_k(\ell; m, n) \right| + e^{B\sqrt{3n/5}}.$$  

Notice that $n/m = O(\sqrt{n})$ and Lemma 2.5, it is clear that

$$\hat{F}_k(\ell; m, n) \ll \frac{\ell}{\sqrt{n-m\ell - ((2k-1)\ell^2 + \ell)/2}} \hat{p}(n - m\ell) \ll \frac{\ell}{n^{3/2}} e^{B\sqrt{n-m\ell}}$$

$$\ll \ell \frac{e^{B\sqrt{n-2m}}}{n^{3/2}} e^{B(\sqrt{n-m\ell - \sqrt{n-2m}})} \ll \frac{e^{B\sqrt{n-2m}}}{n^{3/2}} \left( \ell e^{-\frac{m(n-2m)}{2\sqrt{n}}} + e^{B\sqrt{3n/5}} \right)$$

for $2 \leq \ell \leq n/(2m)$. Thus we obtain that

$$\Sigma_{m,N} \ll \sum_{\ell \geq 2} \frac{e^{B\sqrt{n-2m}}}{n^{3/2}} \left( \ell e^{-\frac{m(n-2m)}{2\sqrt{n}}} + e^{B\sqrt{3n/5}} \right) \ll \frac{e^{B\sqrt{n-2m}}}{n^{3/2}} + e^{B\sqrt{3n/5}}$$

for $m \gg \sqrt{n}$. Hence for $\sqrt{n} \ll m < n/5$, we have

$$N_k(m, n) - F_k(1; m, n) \ll e^{B\sqrt{n-2m}} \frac{1}{n^{3/2}} + e^{B\sqrt{3n/5}}.$$  

Further, from Lemma 2.5 we have

$$F_k(1; m, n) = p(n - (m + k) + 1) - p(n - (m + k))$$

$$= \frac{B}{2} p(n - m) \left( 1 + O_k \left( \frac{1}{\sqrt{n-m}} \right) \right) + e^{B\sqrt{n-m}}$$

$$\approx \frac{B}{8\sqrt{3}} (n - m \sqrt{n-m})^{3/2} \left( 1 + O_k \left( \frac{1}{\sqrt{n-m}} \right) \right)$$

for $n-m \to +\infty$. Therefore, for $\sqrt{n} \ll m < n/5$, using (3.3) and (3.4) it is easy to find that

$$\frac{N_k(m, n) - F_k(1; m, n)}{F_k(1; m, n)} \ll e^{-\frac{Bm}{2\sqrt{n}}} + e^{-\frac{B\sqrt{m}}{10}} = e^{-\frac{m}{\sqrt{6n}}} + e^{-\frac{m}{5n}}.$$  

Combing (3.1), (3.2) and (3.5), we get the proof of the Theorem 1.3.
3.2. The proof of Theorem 1.2. From Theorem 1.3, it is clear that Theorem 1.2 follows the following lemma by substitute $B = 2\pi/\sqrt{6}$.

Lemma 3.1. Let $0 \leq m \leq n$. Then as $n \to \infty$,

$$p(n-m+1) - p(n-m) \sim \frac{B}{8\sqrt{n}} \text{sech}^2 \left( \frac{Bm}{4\sqrt{n}} \right) \quad (3.6)$$

if and only if $1/m = o(n^{-1/2})$ and $m = o(n^{3/4})$. Moreover, we have

$$\frac{p(n-m+1) - p(n-m)}{p(n)} = \frac{B}{8\sqrt{n}} \text{sech}^2 \left( \frac{Bm}{4\sqrt{n}} \right) \left( 1 + O \left( e^{-\frac{Bm}{4\sqrt{n}} + \frac{m^2}{n^{3/2}}} \right) \right).$$

Proof. For $n \geq m \geq 8n/9$, we have

$$\frac{p(n-m+1) - p(n-m)}{p(n)} \ll \frac{e^{B\sqrt{n/9}}}{n^{-1}eB\sqrt{n}} = ne^{-\frac{2B}{\sqrt{n}}}$$

by Lemma 2.2 and

$$\frac{B}{8\sqrt{n}} \text{sech}^2 \left( \frac{Bm}{4\sqrt{n}} \right) \gg n^{-1} e^{-\frac{Bm}{4\sqrt{n}}} \gg n^{-1} e^{-\frac{B}{\sqrt{n}}}$$

by the definition of sech$(x)$. Therefore, if (3.6) valid then $m < 8n/9$. If $0 \leq m \leq 8n/9$ then

$$\frac{p(n-m+1) - p(n-m)}{p(n)} = \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \frac{B e^{-\frac{Bm}{4\sqrt{n}}}}{2\sqrt{n}} \left( 1 - \frac{m}{n} \right)^{-3/2}$$

by Lemma 2.2 and (3.4), and note that

$$\frac{B}{8\sqrt{n}} \text{sech}^2 \left( \frac{Bm}{4\sqrt{n}} \right) = \frac{B}{2\sqrt{n}} e^{-\frac{Bm}{4\sqrt{n}}} \frac{1}{\left( 1 + e^{-\frac{Bm}{4\sqrt{n}}} \right)^2}.$$ 

Hence clearly, if (3.6) valid then $m = o(n)$. Let $0 \leq m = o(n)$, from (3.7) we have

$$\frac{p(n-m+1) - p(n-m)}{p(n)} = \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \frac{B e^{-\frac{Bm}{4\sqrt{n}} - \frac{m^2}{4n}}}{2\sqrt{n}} \left( 1 + o(1) \right).$$

by generalized binomial theorem. Combining (3.8) and (3.9) we immediately obtain that (3.6) valid if and only if $1/m = o(n^{-1/2})$ and $m = o(n^{3/4})$. Moreover, we obtain that

$$\frac{p(n-m+1) - p(n-m)}{p(n)} = \frac{B}{8\sqrt{n}} \text{sech}^2 \left( \frac{Bm}{4\sqrt{n}} \right) \left( 1 + O \left( e^{-\frac{Bm}{4\sqrt{n}} + \frac{m^2}{n^{3/2}}} \right) \right).$$

3.3. The proof of the Corollary 1.4. First of all, let $k \in \mathbb{N}$ be fixed, $m \gg \sqrt{n}$ and $n - m$ tends to infinity. We have

$$N_k(m, n) = F_k(1; m, n) \left( 1 + O_k \left( e^{-\frac{\pi m}{\sqrt{6m}} + e^{-\frac{\pi \sqrt{6}}{4}}} \right) \right)$$

$$= p(n - (m + k) + 1) - p(n - (m + k)) + R_k(m, n)$$

with

$$R_k(m, n) \ll_k \left( e^{-\frac{\pi m}{\sqrt{6m}} + e^{-\frac{\pi \sqrt{6}}{4}}} \right) p(n - m)$$

(3.10)
by Theorem 1.3. Now, from [14, Equ. (1.2)] we have for \( r \in \mathbb{Z}_{\geq 0} \) be fixed and as \( N \to \infty \)

\[
\Delta_{N}^{r} p(N) = \left( \frac{\pi}{\sqrt{6N}} \right)^{r} p(N) \left( 1 + O \left( \frac{1}{\sqrt{N}} \right) \right),
\]

where for a function \( f(x) \),

\[
\Delta_{x}^{0} f(x) := f(x), \quad \Delta_{x} f(x) := f(x) - f(x - 1)
\]

and

\[
\Delta_{x}^{\ell} p(x) := \Delta_{x} \left( \Delta_{x}^{\ell-1} p(x) \right) \quad \text{for} \quad \ell \in \mathbb{N}
\]
is the backward difference. Hence clearly,

\[
(-1)^{r} \Delta_{m}^{r} N_{k}(m + r, n) = \sum_{j=0}^{r} (-1)^{j} \binom{r}{j} N_{k}(m + j, n).
\]

By (3.10) we obtain that

\[
\Delta_{m}^{r} N_{k}(m + r, n) = - \Delta_{m}^{r+1} p(n - ((m + r) + k))
\]

\[
+ O_{k,r} \left( e^{-\frac{n}{\sqrt{6}}} + e^{-\frac{\pi\sqrt{6n}}{6}} \right) \sum_{j=0}^{r+1} p(n - m + j) \right).
\]

Moreover, we have

\[
\Delta_{m}^{r+1} p(n - ((m + r) + k)) = (-1)^{r+1} \Delta_{j}^{r+1} p(j) \bigg|_{j=n-m-k+1}.
\]

Using (3.11)–(3.14) and Lemma 2.2, it is clear that

\[
\left( \frac{\sqrt{6(n-m)}}{\pi} \right)^{r+1} \frac{(-1)^{r} \Delta_{m}^{r} N_{k}(m + r, n)}{p(n-m)} - 1 \ll_{k,r} \frac{1}{\sqrt{n-m}} + \frac{(n-m)^{r+1}}{e^{\frac{\pi n}{\sqrt{6}}}}
\]

\[
\ll \frac{1}{\sqrt{n-m}} + n^{\frac{r+1}{2}} e^{-\frac{\pi n}{\sqrt{6}}}
\]

and the proof of the corollary follows.

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