A multilayered plate theory with transverse shear and normal warping functions

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Abstract

A multilayered plate theory which takes into account transverse shear and normal stretching is presented. The theory is based on a seven-unknowns kinematic field with five warping functions. Four warping functions are related to the transverse shear behaviour, the fifth is related to the normal stretching. The warping functions are issued from exact three-dimensional solutions. They are related to the variations of transverse shear and normal stresses computed at specific points for a simply supported bending problem. Reddy, Cho–Parmerter and (a modified version of) Beakou–Touratier theories have been retained for comparisons. Extended versions of these theories, able to manage the normal stretching, are also considered. All these theories can be emulated by the kinematic field of the present model thanks to the adaptation of the five warping functions. Results of all these theories are confronted and compared to analytical solutions, for the bending of simply supported plates. Various plates are considered, with special focus on very low length-to-thickness ratios: an isotropic plate, two homogeneous orthotropic plates with ply orientation of 0 and 5 degrees, a [0/90] sandwich panel and a [−45/0/45/90], composite plate. Results show that models are more accurate if their kinematic fields (i) depend on all material properties (not only the transverse shear stiffnesses) (ii) depend on the length-to-thickness ratio (iii) present a coupling between the x and y directions.

Keywords: Plate theory, warping function, normal stretching, laminate, multilayered, composite, sandwich, vibration

1. Introduction

Plate theories have been enhanced in order to model structures becoming more and more complex over the years. For thin homogeneous plates, works of Cauchy [1], Kirchhoff [2] and Love [3] have lead to the so-called Love–Kirchhoff theory, which does not take into account the transverse shear. For moderately thick homogeneous plates, transverse shear must appear in the formulation. Authors like Reissner, Hencky, Bolle, Uflyand, Hildebrand and Mindlin [4–9] have proposed to integrate the shear phenomenon into their formulation. In particular, Reissner made assumptions on stresses, hence shear stresses have a parabolic distribution and the normal stress is considered in his model which is derived from a complementary energy. The other authors made assumptions on displacements. For example, Hencky and Bolle considered a linear variation of the displacements u and v with constant transverse shear strains, but no normal strain. Mindlin developed the dynamic version of Hencky’s theory. Mindlin’s and Reissner’s theories are often associated but it is incorrect as demonstrated in reference [10]. The Hencky–Mindlin theory tends to overestimate the transverse shear stiffnesses. Hence, a shear correction factor has been proposed, generally fixed to the value of 5/6 for static studies of homogeneous plates. Except works such as Lévy’s memoir [11], Hildebrand’s second-order theory [8] and Vlasov’s and Murthy’s third-order theories [12, 13], higher-order theories have mainly been proposed for inhomogeneous structures, and hence are presented below. The models presented in the 1877 Lévy’s memoir were ahead of their time, with a displacement field which depends on through the sum of a polynomial of degree 3 and a sine function.

Laminated composite plates, including sandwich panels, are an important class of structures, widely used in the industrial field. The mechanical behaviour of such structures is difficult to model in the general case, because of their heterogeneous nature. Although previous works on heterogeneous plates and/or sandwiches have been done by authors like Lekhnitskii [14], Reissner [15, 16] among others, the classical Lamination Theory, which is the multilayer extension of the Love–Kirchhoff theory is generally attributed to Stravsky [17], Reissner and Stavsky [18], and Dong & al. [19]. The first order shear Deformation laminated plate Theory (FoSDT) based on the Hencky–Mindlin theory is attributed to Yang & al. [20] and Whitney [21, 22]. The use of shear correction factors is here mandatory because the overestimation of the FoSDT for shear stiffnesses is even worse for laminated and sandwich structures than for homogeneous ones. Although everybody agree for the value of 5/6 for the shear correction factor for homogeneous plates in static studies, values for the 3 needed correction factors of general laminates may take different values depending on the method used to calculate them [23–25] and it has been proved that they depend on the wavelength [26].

It is possible to model laminated plates with a layerwise approach, but this leads to a number of unknowns which depends on the number of layers, which may be large. This class of layerwise (LW) theories is opposed to the class of equivalent single layer (ESL) theories which contains all the previously cited theories. In the ESL class, theories have a more or less great number of unknowns, which but does not depend on the number of layers.

In this ESL framework, the complex behaviour of laminated structures has pushed researchers into proposing higher order plate theories. These plate theories are characterized by the use a displacement field with a higher order (higher than one) dependence on the normal coordinate . Except the work of Whitney [27] which presents a second order theory, the well known and commonly used higher order theories are of order 3 like the Levinson’s [28] and Reddy’s [29] ones. Other higher order theories have been proposed in following works, which differ on the unknowns which are considered, the order of the developments, etc. [30]. Non polynomial theories have also been proposed, characterized by the use of trigonometric, hyperbolic or such similar functions of to model the displacement field [31–33]. They are often
considered and classified as higher order theories.

Some a priori LW models can reduce to ESL models with the help of assumptions between the fields in each layer. Zig-Zag (ZZ) models enter in this category. Early works of Lekhnitskii [34] and Ambartsumyan [35] have been classified as such by Carrera [36] who shows also that other authors have integrated the multilayer structure in their model [21, 37, 38] in a very similar manner. The main idea of ZZ models is to let in-plane displacements vary with $z$ according to the superposition of a zig-zag law to a global law – cubic for example. With these models, shear stresses can satisfy both continuity at interfaces and null (or prescribed) values at the top and bottom faces of the plate.

In reference [39], authors present a plate model with four warping functions (WFs) $\phi_{g}(z)$ which embed the transverse shear behaviour into the displacement field. These functions are issued, for each laminate sequence, from 3D elasticity solutions. An important conclusion of this paper is that, as all "without-normal-stretching" models use reduced stiffnesses according to the generalized plane stress assumption, the comparison of these models with an exact 3D solution has no sense for very low length-to-thickness ratios like 2 or 4. The results provided by these models can however be compared to an exact solution for virtual laminates which have been highly stiffened in the $z$ direction, see the paper [39] for more details. The corollary is that, to compare models with 3D solutions at very low length-to-thickness ratios, models must integrate the normal stretching behaviour.

Plate theories taking into account the normal stretching behaviour have been proposed since a long time [11, 16] and in many other works as it can be seen in recent reviews on the subject [40–42]. However, some angle-ply, layers. All the quantities are related to unknown functions defined at the $z = 0$ middle plane, and which are marked with the superscript $0$. In the following, Greek subscripts take values 1 or 2 and Latin subscripts take values 1, 2 or 3. Einstein’s summation convention is used for subscripts only. The comma used as a subscript index means the partial derivative with respect to the following indices.

2.2. Displacement, strain and stress fields

The kinematic assumptions of the theory are

$$\begin{align*}
\left\{ 
\begin{array}{l}
u_\alpha(x, y, z) = \nu_\alpha^0(x, y) - 2w_\alpha^0(x, y) + \varphi_{g}(z)w_\alpha^0(x, y) \\
u_3(x, y, z) = w_3^0(x, y) + 2\varepsilon_{33}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y)
\end{array}
\right. 
\end{align*}
$$

where $\nu_\alpha^0(x, y)$, $w_3^0(x, y)$ are the in-plane displacements and the deflection evaluated at $z = 0$, $\gamma_{\alpha\beta}^0(x, y)$ are the engineering transverse shear strains evaluated at $z = 0^+$, $\varepsilon_{33}^0(x, y)$ and $\kappa_{33}^0(x, y)$ are the values of the first and second derivatives of $w(x, y, z)$ with respect to $z$ at $z = 0^+$, and $\varphi_{g}(z)$ and $\varphi_{g3}(z)$ are the five WFs, which are continuous. The unknowns which are evaluated at $z = 0^+$ correspond to quantities which may be discontinuous if an interface between different materials is located at the middle plane $z = 0$. The above definitions imply $\varphi_{g}(0) = 0$, $\varphi_{g3}(0) = \varphi_{g3}'(0) = 0$ and $\varphi_{g3}'(0^+) = 1$. The associated strain field is derived from equation (1):

$$\begin{align*}
\varepsilon_{\alpha\beta}(x, y, z) &= \varepsilon_{\alpha\beta}^0(x, y) - 2\varepsilon_{\alpha\beta}^0(x, y) + \frac{1}{2}(\varphi_{g}(z)\gamma_{\alpha\beta}^0(x, y) \\
&+ \varphi_{g3}(z)\kappa_{33}^0(x, y)) \\
\varepsilon_{33}(x, y, z) &= \frac{1}{2}(\varphi_{g}(z)\gamma_{33}^0(x, y) + \varepsilon_{33}^0(x, y)) \\
&+ \varphi_{g3}(z)\kappa_{33}^0(x, y) \\
\varepsilon_{33}(x, y, z) &= \varepsilon_{33}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y)
\end{align*}
$$

In addition to the aforementioned conditions, equation (2b) shows that the WFs must also verify

$$\begin{align*}
\sigma_{\alpha\beta}(x, y, z) &= C_{\alpha\beta\gamma\delta}(z)\varepsilon_{\gamma\delta}^0(x, y) - 2\sigma_{\alpha\beta}^0(x, y) + \varphi_{g}(z)\gamma_{\alpha\beta}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y) \\
&+ C_{\alpha33\gamma\delta}(z)(\varepsilon_{33}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y)) \\
\sigma_{33}(x, y, z) &= C_{33\gamma\delta}(z)\varepsilon_{\gamma\delta}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y) + C_{3333}(z)(\varepsilon_{33}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y)) \\
&+ C_{3333}(z)(\varepsilon_{33}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y))
\end{align*}
$$

2.3. Strain energy, generalized forces and strains

Let us now consider the strain energy surface density:

$$J = \frac{1}{2} \int_{-h/2}^{h/2} \sigma_{ij} \varepsilon_{ij} \, dz = \frac{1}{2} \int_{-h/2}^{h/2} \left( \varepsilon_{\alpha\beta}(x, y, z) \varepsilon_{\alpha\beta}^0(x, y) + 2\sigma_{\alpha\beta}(x, y, z) + 2\varepsilon_{33}(x, y, z) \right) \sigma_{\alpha\beta} \, dz$$

$$= \frac{1}{2} \int_{-h/2}^{h/2} \left( \varepsilon_{\alpha\beta}^0(x, y) - 2\varepsilon_{\alpha\beta}^0(x, y) + \varphi_{g}(z)\gamma_{\alpha\beta}^0(x, y) \right) \sigma_{\alpha\beta}^0 \\
+ \left( \varphi_{g}(z)\gamma_{\alpha\beta}^0(x, y) + \varepsilon_{\alpha\beta}^0(x, y) \right) \sigma_{\alpha\beta}^0 \\
+ \left( \varepsilon_{33}^0(x, y) + \varphi_{g3}(z)\kappa_{33}^0(x, y) \right) \sigma_{33}^0 \, dz$$

It can also be written

$$J = \frac{1}{2} \left[ \varepsilon_{\alpha\beta}^0 N_{\alpha\beta} + \kappa_{33}^0 M_{33} + \gamma_{\alpha\beta}^0 P_{\alpha\beta} + \gamma_{33}^0 N_{33} \right]$$
where $\kappa_{0}^{\alpha} = -\tilde{\nu}^{\alpha}$ and $\kappa_{0}^{0} = 0$, naturally introducing the 18 following quantities which are the generalized forces, each associated with a corresponding generalized displacement in the strain energy formula (5). They are then set, by type, into vectors

$$\mathbf{N} = \begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ N_{33} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} M_{11} \\ M_{22} \\ M_{33} \\ M_{12} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} P_{11} \\ P_{22} \\ P_{12} \\ P_{33} \end{bmatrix}$$

and the same is done for the generalized strains:

$$\mathbf{E} = \begin{bmatrix} e_{11}^{0} \\ e_{22}^{0} \\ e_{33}^{0} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} k_{11}^{0} \\ k_{22}^{0} \\ k_{33}^{0} \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} \gamma_{11}^{0} \\ \gamma_{22}^{0} \\ \gamma_{33}^{0} \end{bmatrix}, \quad \mathbf{\gamma} = \begin{bmatrix} \gamma_{11}^{0} \\ \gamma_{22}^{0} \\ \gamma_{33}^{0} \end{bmatrix}$$

2.4. Laminate behaviour

Generalized forces are linked with the generalized strains by the 12 × 12 and 6 × 6 following stiffness matrices

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} & \mathbf{\mathcal{N}} \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} \mathbf{H} & \mathbf{I} & \mathbf{J} & \mathbf{\mathcal{M}} \end{bmatrix}$$

with the following definitions

$$[A_{ijkl}] = \int_{-h/2}^{h/2} C_{ijkl}(z)\,dz$$

2.5. Kinetic energy

With the help of the displacement field expressions (1), the kinetic energy surface density $E_{k}(x, y)$ of the structure can be written

$$E_{k}(x, y) = \frac{1}{2} \int_{\Omega} \rho u_{x}(x, y, z) u_{x}(x, y, z) \,dz$$

Note that the $U_{00}$ and $V_{00}$ are antisymmetric tensors but $W_{00}$ is symmetric. Then, there are 17 independent mass coefficients to consider.

2.6. Laminate equations of motion

Let us recall the equilibrium conditions within a solid. Without loss of generality, body forces are neglected here, and the previous convention on indices is kept:

$$\sigma_{\alpha\beta} + \sigma_{\alpha33} = \rho \ddot{u}_{\alpha}$$

Integrating the equations of equilibrium (14) over the thickness with the help of formulas (1), (6) and (13) leads to

$$\mathbf{N}_{00} + \sigma_{00}(z) \int_{-h/2}^{h/2} = \mathbf{R}_{\mathcal{Q}_{0}}, \quad \mathbf{Q}_{\mathcal{Q}_{0}} + \sigma_{33}(z) \int_{-h/2}^{h/2} = \mathbf{R}_{\mathcal{Q}_{0}}$$

where the $\mathcal{Q}_{0}$ are the classical shear forces. In order to get more equations, weighted integrals over the thickness of equation (14a) with weight functions $z$ and $\varphi_{33}(z)$, and of equation (14b) with weight functions $z$ and $\varphi_{33}(z)$ computed. It gives six more equations:

Taking into account symmetries, this leads to (in the order which their appear in formula (10)) $6 + 6 + 6 + 12 + 12 + 10 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 95$ independent stiffnesses in the more general case.
3. Warping functions issued from an exact 3D solution

The set of WFs is issued from a 3D analytical solution of the bending of a simply supported rectangular plate submitted to a bi-sine load (see section 5.1 for details). For dynamic studies, the static solution is replaced by the response of the plate to a bi-sine load at a given frequency. Let us define three points of the middle plane $A (\xi, 0)$, $B (0, \zeta)$ and $C (\xi, \zeta)$, where $a$ and $b$ are the side lengths of the plate. The two-step procedure is described in the following.

3.1. Computation of the $\psi_{33}(z)$ warping function

The through-the-thickness variations of the normal strain $\varepsilon_{33}$ and its $z$-derivative $\varepsilon'_{33}$, are computed at the centre of the plate $C$. Then, according to the nature of the strain field, equation (2c), the derivative $\delta_3$, is restricted to the search of the first natural frequency. For laminates which are not of cross-ply nor anti-symmetrical angle-ply types, the simply supported boundary condition is replaced by a globally simply supported condition. In this case, the plate could have a non-null deflection on its edges with respect of an antisymmetry with the opposite edge, and the first vibration mode splits into two modes, see reference [46] for more details.

The Fourier series is limited to one term, hence the generalized displacement field is

$$\mathbf{u} = \begin{pmatrix} u_1 \\
\omega_m \sin(\xi x) \sin(\eta y) + \omega_m \cos(\xi x) \cos(\eta y) \\
\omega_m \sin(\xi x) \cos(\eta y) + \omega_m \cos(\xi x) \sin(\eta y) \\
\gamma_{13} \cos(\xi x) \sin(\eta y) + \gamma_{13} \sin(\xi x) \cos(\eta y) \\
\gamma_{23} \cos(\xi x) \sin(\eta y) + \gamma_{23} \sin(\xi x) \cos(\eta y) \\
\varepsilon_{33} \sin(\xi x) \sin(\eta y) + \varepsilon_{33} \cos(\xi x) \cos(\eta y) \\
\kappa_{33} \sin(\xi x) \sin(\eta y) + \kappa_{33} \cos(\xi x) \cos(\eta y) \end{pmatrix}$$

with

$$\xi = \frac{mn}{a} \quad \text{and} \quad \eta = \frac{mn}{b}$$

where $m$ and $n$ are wavenumbers, set to 1 in this study. Considering formula (29), the motion equations of section 2.6 give a stiffness and a mass matrix, respectively $[K]$ and $[M]$, related to the vector $[\mathbf{U}] = [u_1, \omega_m, \gamma_{13}, \gamma_{23}, \varepsilon_{33}, \kappa_{33}]$. The static case is treated solving the linear system $[K][\mathbf{U}] = [\mathbf{F}]$, where $[\mathbf{F}]$ is a force vector containing $-q''$, $-q'''$ and $-q''''$ for its third, sixth and seventh components. Solving the dynamic case consists in researching the generalized eigenvalues for matrices $[K]$ and $[M]$.

4. Solving method by a Navier-like procedure

A Navier-like procedure is implemented to solve both static and dynamic problems for a simply supported plate. The dynamic study is restricted to the search of the first natural frequency. For laminates which are not of cross-ply nor anti-symmetrical angle-ply types, the simply supported boundary condition is replaced by a globally simply supported condition. In this case, the plate could have a non-null deflection on its edges with respect of an antisymmetry with the opposite edge, and the first vibration mode splits into two modes, see reference [46] for more details.

The Fourier series is limited to one term, hence the generalized displacement field is

$$\mathbf{u} = \begin{pmatrix} u_1 \\
\omega_m \sin(\xi x) \sin(\eta y) + \omega_m \cos(\xi x) \cos(\eta y) \\
\omega_m \sin(\xi x) \cos(\eta y) + \omega_m \cos(\xi x) \sin(\eta y) \\
\gamma_{13} \cos(\xi x) \sin(\eta y) + \gamma_{13} \sin(\xi x) \cos(\eta y) \\
\gamma_{23} \cos(\xi x) \sin(\eta y) + \gamma_{23} \sin(\xi x) \cos(\eta y) \\
\varepsilon_{33} \sin(\xi x) \sin(\eta y) + \varepsilon_{33} \cos(\xi x) \cos(\eta y) \\
\kappa_{33} \sin(\xi x) \sin(\eta y) + \kappa_{33} \cos(\xi x) \cos(\eta y) \end{pmatrix}$$

with

$$\xi = \frac{mn}{a} \quad \text{and} \quad \eta = \frac{mn}{b}$$

where $m$ and $n$ are wavenumbers, set to 1 in this study. Considering formula (29), the motion equations of section 2.6 give a stiffness and a mass matrix, respectively $[K]$ and $[M]$, related to the vector $[\mathbf{U}] = [u_1, \omega_m, \gamma_{13}, \gamma_{23}, \varepsilon_{33}, \kappa_{33}]$. The static case is treated solving the linear system $[K][\mathbf{U}] = [\mathbf{F}]$, where $[\mathbf{F}]$ is a force vector containing $-q''$, $-q'''$ and $-q''''$ for its third, sixth and seventh components. Solving the dynamic case consists in researching the generalized eigenvalues for matrices $[K]$ and $[M]$.

5. Reference models retained for comparisons

The model presented in section 2, which uses five WFs issued from transverse shear and normal stresses of analytical solutions, will be denoted 3D-SWF. The results obtained with the 3D-SWF model are compared to those obtained with different models issued from the literature and with the exact analytical solution. These reference models are presented below.
5.1. Exact solution (Exa)

Each studied case is solved by a state-space method described in reference [46]. This method is a generalization of general lamination sequences of existing methods for cross-ply and antisymmetric angle-ply lamination sequences. In this work, the exact solution is used to obtain deflections, stresses and natural frequencies taken as reference for comparisons, but it is also used to create sets of WFs for both 3D-4WF and 3D-5WF models, as explained in section 3. The corresponding solution is denoted Exa in the following text and in tables.

5.2. Models without normal deformation

Some more or less classical models have been chosen for comparison matters. They offer the advantage to be easily simulated with the present model when appropriate sets of WFs are selected:

- ToSDT, Third-order Shear Deformation Theory: often called Reddy’s third order theory, verifies that transverse shear stresses are null at the top and bottom faces of the plate. It is simulated using the following third order theory, verifies that transverse shear stresses are null at the present model when appropriate sets of WFs are selected:

- ToZZ-4, Third-order Zig-Zag model with 4 WFs: This formulation, presented in references [38, 47] consists in superimposing a cubic displacement field, which permits the transverse shear stresses to be null at the top and bottom faces of the laminate, to a zig-zag displacement field issued from the continuity of the transverse shear stresses at layer interfaces. The corresponding WFs, which are polynomials of third order in \( \zeta \), are not detailed. Indeed, as their computation involves the resolution of a system of equations, it is difficult to give here an explicit form.

- SIZZ-4, Sine Zig-Zag model with 4 WFs: This model, inspired from the Beakou-Touratier model [48], verifies the continuity of transverse stresses is issued from reference [39].

5.3. Models with normal deformation

The three first models described in the precedent section, have been extended in order to take into account a normal deformation. This has been done considering a modified displacement field including equation (1b), with the following choice of \( \varphi_{33}(z) \):

\[
\varphi_{33}(z) = \frac{z^2}{2}
\]  

(33)

6. Numerical results

This section proposes the study of five laminate configurations including an isotropic plate, two single layer orthotropic plates and a sandwich panel. Three materials are involved, an isotropic material for the isotropic plate, an orthotrophic composite material for all laminates and an honeycomb-type material for the core of the sandwich panel. All the properties are given in table 1. For all computations, the loading is divided into two equal parts which are applied to the top and bottom faces.

Deflections \( w \), first natural frequencies \( \omega \) and stresses \( \sigma_{33} \) are nondimensionalized using the following formulas

\[
w' = 100 \frac{E_i h^3}{(-q)a^2}, \quad \omega' = a^2 \frac{\sqrt{E_i}}{E_i} h, \quad \sigma_{33}' = 10 \frac{h}{(-q)a} \sigma_{33}
\]

(34)

where \( E_{ii}' \) and \( \rho_{ii}' \) are taken as values of the core material for the sandwich and as values of the corresponding material for other cases.

6.1. Isotropic plate

For this first study, the WFs of different models do not strongly differ, it is the reason why they are not plotted. Such comparisons are let for the following examples. It can be seen in table 2 that for \( a/h = 2 \), all “extended” models give better results for the deflection than the original model they are issued from. The results on transverse stresses are less good but quite comparable. Replacing the Poisson’s coefficient value of 0.25 by 0.35 leads to a change in the ranking of the ZZ models. Table 2 also shows that results of models without normal deformation are better if \( a/h \) takes higher values.

6.2. Square [0] composite plate

In table 3, results show an inverse tendency than for the previous case: all “extended” models show less good results for the deflection than original models, except the 3D-5WF model. All models, except the 3D-4WF and 3D-5WF models which have material-sensitive WFs, have the same WFs for an orthotropic single layer plate than for an isotropic single layer one. It can be seen that small differences in the kinematic assumptions of models can have great influence in results. The 3D-4WF and 3D-5WF models have different WFs for the \( x \) and \( y \) directions, \( \varphi_{11}(x) \neq \varphi_{22}(y) \), as it can be seen in figure 1. This is due to different shear/longitudinal modulus ratios in the \( x \) and \( y \) directions. For all the other models \( \varphi_{11}(x) = \varphi_{22}(y) \). The WFs of classical models are not presented because they are the same than those of the corresponding extended models. Those of the 3D-4WF have not been presented for clarity. Further, even if the differences on the WFs are small, strong differences can be observed when stresses are computed, as shown in figure 2. Due to their formulation, the ToZZ-Z and the ToSDT coincide for a one layer plate, and the SIZZ-5 model do not strongly differ from the two previous. These three models give transverse shear stresses that differ from the exact solution, especially in the \( x \) direction. This shows that the \( z=2 \sqrt{\frac{h}{6}} \) function, and also the sine function of model SiZZ-5, are not able to fit the behaviour of material with a shear/longitudinal modulus ratio of 0.02 that differs strongly from the isotropic case. On the contrary, as \( G_{13}/E_2 = 0.2 \), value closer
to the previous isotropic ratio of 0.4, the $\sigma_{22}$ stress is better fitted by these models. This may also explain why, unlike the previous case, the “extended” models give worse deflection values than the corresponding originals ones. The 3D-5WF model, with its material sensitive formulation, predicts the good values for transverse stresses.

6.3. Square [5] composite plate

When used to study cross-ply multilayered plates, the ToZZ-5 and SiZZ-5 zig-zag models have non null $\varphi_{13}$ and $\varphi_{23}$ functions. However, for an angle-ply single layer plate, these zig-zag models do not lead to coupling between the $x$ and $y$ directions, which can be seen as a limitation. Indeed, as we can see in figure 4, only the 3D-5WF model have non null $\varphi_{12}$ and $\varphi_{21}$ functions. It is probably the reason why the

![Figure 1: Transverse shear WFs of the [0] single ply composite plate with $a/h = 2$ for each model.](image)

![Figure 2: Nondimensionalized transverse shear stresses of the [0] single ply composite plate with $a/h = 2$ for each model.](image)

![Figure 3: Normal WF and nondimensionalized normal stress for the [0] single ply composite plate with $a/h = 2$ for each model.](image)

![Figure 4: Nondimensionalized transverse shear stresses of the [0] single ply composite plate with $a/h = 2$ for each model.](image)

![Table 2: Comparison between the different models for the rectangular [iso] isotropic plate with various length-to-thickness ratios.](table)

![Table 3: Comparison between the different models for the rectangular [iso] isotropic plate with various length-to-thickness ratios.](table)
formation, all the models present similar WFs, included the \( \sigma \) in figure 10 which are similar, we can see in table 6 that, due to its

| Model | \( w^* \) | \( \sigma_{11}(B) \) | \( \sigma_{21}(A) \) | \( \sigma_{12}(B) \) | \( \sigma_{22}(B) \) | \( \sigma_{13}(A) \) | \( \omega^* \) |
|-------|---------|----------------|----------------|----------------|----------------|----------------|---------|
| ToSDT | 4.5068  | 0.3330        | 0.3330         | 0.3330         | 0.3330         | 0.3330         | 0.3330   |
| ToZZ-4| 4.5068  | 0.3330        | 0.3330         | 0.3330         | 0.3330         | 0.3330         | 0.3330   |
| SiZZ-4| 4.5068  | 0.3330        | 0.3330         | 0.3330         | 0.3330         | 0.3330         | 0.3330   |
| Exact | 4.5068  | 0.3330        | 0.3330         | 0.3330         | 0.3330         | 0.3330         | 0.3330   |

Table 4: Comparison between the different models for the rectangular [5] composite plate with \( a/h = 2 \).
deformation. The considered zig-zag theories (ToZZ5 and SiZZ5) are not material-dependent when they are used for a single layer plate. The study of the [0] laminate shows that the ToZZ5 and SiZZ5 theories cannot adapt themselves to the different shear/longitudinal modulus ratio in the x and y directions, in other words $\varphi_{11}(z) = \varphi_{22}(z)$.

The study of the [5] laminate shows that, as the zig-zag mechanism is inoperative on a one-layer laminate, both ToZZ5 and SiZZ5 theories give null $\varphi_{12}(z)$ or $\varphi_{21}(z)$ WFs, while they are scheduled to propose non null $\varphi_{13}(z)$ or $\varphi_{23}(z)$ functions on multilayered angle-ply laminates. On the contrary, the present model has non cross WFs which better describes the reality.

The studies of the [0/c/0] sandwich panel and of the [−45/0/45/90], laminate, show that, as expected, ZZ theories differentiate themselves from the ToSDT when multilayered structures are considered. This is particularly evident when the transverse stresses of the [−45/0/45/90], plate are computed. The two considered ZZ models (ToZZ5 and SiZZ5) have four shear WFs which permit them to consider a kinematic field that respect the transverse stress continuity at each interface for angle-ply structures.

All considered theories could have given better results if higher length-to-thickness ratios had been considered. Low length-to-thickness ratios (2, 4) can be considered as unrealistic, but for dynamic analysis, the effective length to thickness ratio depends on the wavelength, then can reach such low values. Comparisons show that the present theory gives better results than other tested theories, for all considered lamination sequences, and for all length-to-thickness ratios. However, this result has to be seen in the special context of this study. Plate problems solved in this study are simply supported problems with bi-sine loading. The WFs used for the present theory are issued from an exact 3D solution of this particular bending problem. Their shapes depend

![Figure 4: Transverse shear WFs of the [5] single ply composite plate with $a/h = 2$ for each model.](image)

![Figure 5: Nondimensionalized transverse shear stresses of the [5] single ply composite plate with $a/h = 2$ for each model.](image)

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Figure 6: Transverse shear WFs of the [0\%/c\%/0\%] sandwich plate with $a/h = 4$ for each model.

Figure 7: Nondimensionalized transverse shear stresses of the [0\%/c\%/0\%] sandwich plate with $a/h = 2$ for each model.

Figure 8: Normal WF and nondimensionalized normal stress for the [0\%/c\%/0\%] sandwich plate with $a/h = 2$ for each model. By definition, the exact $\varphi_{33}$ coincides with the 3D-5WF one, hence it is not plotted.
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