NON-UNIQUENESS OF WEAK SOLUTIONS TO 3D GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS

YACHUN LI, ZIRONG ZENG, AND DENG ZHANG

Abstract. We prove the non-uniqueness of weak solutions to 3D generalized magneto-hydrodynamic (MHD for short) equations, for the exponents of viscosity and resistivity less than 5/4, which coincides exactly with the Lions exponent for the well-posedness of 3D hyperviscous Navier-Stokes equations. The constructed weak solutions do not conserve the magnetic helicity and can be close to the given smooth, divergence-free and mean-free velocity and magnetic fields. Furthermore, we prove that the weak solutions constructed by Beekie-Buckmaster-Vicol [2] for the ideal MHD can be obtained as a strong vanishing viscosity and resistivity limit of a sequence of weak solutions to the generalized MHD. This shows that, in contrast to the weak ideal limits, Taylor’s conjecture does not hold along the vanishing viscosity and resistivity limits of weak solutions to the generalized MHD. Unlike the intermittent flows in the context of the NSE [13] and the ideal MHD [2], one of the novelties of our proof is the construction of a new class of velocity and magnetic flows, which feature the refined intermittency in both the space and time. Two types of temporal correctors adapted to the structure of MHD are also introduced to balance the high spatial and temporal oscillations.

1. Introduction and main results

1.1. Introduction. In this paper we consider the three-dimensional generalized MHD equations on the torus $\mathbb{T}^3 := [−\pi, \pi]^3$,

$$\begin{align*}
\partial_t u + \nu_1 (-\Delta)^{\alpha_1} u + (u \cdot \nabla) u - (B \cdot \nabla) B + \nabla P &= 0, \\
\partial_t B + \nu_2 (-\Delta)^{\alpha_2} B + (u \cdot \nabla) B - (B \cdot \nabla) u &= 0, \\
\text{div} u &= 0, \\
\text{div} B &= 0,
\end{align*}$$

(1.1)

where $\alpha_1, \alpha_2 \in (0, 5/4)$, $u = (u_1, u_2, u_3)^\top (t, x) \in \mathbb{R}^3$, $B = (B_1, B_2, B_3)^\top (t, x) \in \mathbb{R}^3$ and $P = P(t, x) \in \mathbb{R}$ correspond to the velocity field, magnetic field and pressure of the fluid, respectively. The fractional Laplacian $(-\Delta)^{\alpha_i}$, $i = 1, 2$, are defined via the Fourier transform on the flat torus

$$\mathcal{F}((-\Delta)^{\alpha_i} u)(\xi) = |\xi|^{2\alpha_i} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3,$$

and $\nu_1, \nu_2 \geq 0$ are the viscous and resistive coefficients, respectively.

Equation (1.1) arises from several canonical hydrodynamic models. One typical model is the viscous and resistive MHD system (i.e., $\alpha_1 = \alpha_2 = 1$), which is a classical macroscopic model to describe the motion of conductive fluid in an electromagnetic field (cf. [4, 5, 27, 60]). In the case where $B \equiv 0$, (1.1) reduces to the hyperviscous Navier-Stokes equations

$$\begin{align*}
\partial_t u + \nu (-\Delta)^{\alpha} u + (u \cdot \nabla) u + \nabla P &= 0, \\
\text{div} u &= 0,
\end{align*}$$

(1.2)

In particular, if $\alpha = 1$, (1.2) is the incompressible Navier-Stokes equations (NSE for short).

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Another important model is the ideal MHD system, where the viscous and resistive coefficients vanish, namely,

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - (B \cdot \nabla) B + \nabla P &= 0, \\
\partial_t B + (u \cdot \nabla) B - (B \cdot \nabla) u &= 0, \\
\text{div} u &= 0, \quad \text{div} B = 0.
\end{aligned}
\]  

The incompressible ideal MHD system is the classical macroscopic model coupling the Maxwell equations to the evolution of an electrically conducting incompressible fluid (cf. \cite{27}). In the case \( B \equiv 0 \), (1.3) reduces to the Euler equations.

The well-posedness problem of NSE and MHD has been extensively studied in the literature. For the initial data with finite energy, the existence of global weak solution \( u \) to NSE was first proved by Leray \cite{48} in 1934 and later by Hopf \cite{40} in 1951 in bounded domains, which satisfies \( u \in C_{\text{weak}}([0, +\infty); L^2(\Omega)) \cap L^p([0, +\infty); H^1(\Omega)), \Omega = \mathbb{R}^3 \) or \( T^3 \), and obeys the following energy inequality

\[
\|u(t)\|^2_{L^2} + 2\nu \int_{t_0}^{t} \|
abla u(s)\|^2_{L^2} \, ds \leq \|u(t_0)\|^2_{L^2} \tag{1.4}
\]

for any \( t > 0 \) and a.e. \( t_0 \geq 0 \). The existence of Leray-Hopf solutions to generalized MHD equations was proved by Wu \cite{66}.

The uniqueness of Leray-Hopf solutions is considered as one of the most important issues in the mathematical fluid mechanics. The well known Ladyzhenskaya-Prodi-Serrin condition states that, the Leray-Hopf solution to NSE is regular and thus unique in the space \( BMO^{-1} \) was proved by Koch-Tataru \cite{47}. Moreover, Lions \cite{51} obtained the well-posedness for the hyperviscous NSE \cite{2} when \( \alpha \geq 5/4 \). Concerning the generalized MHD system \cite{11}, the uniqueness of global classical solutions was proved by Wu \cite{66} under the condition that \( \alpha_1, \alpha_2 \geq 1/2 + n/4 \), which is exactly the Lions exponent for the hyperviscous NSE in dimension three \( n = 3 \). Moreover, Zhou \cite{68} proved that, when \( \alpha_1 = \alpha_2 \), the solutions are smooth and unique in \( L^p_t L^2_x \) with \( 2\alpha_1/\gamma + 3/p \leq 2\alpha_1 - 1 \), which reduces to the Ladyzhenskaya-Prodi-Serrin condition for the NSE when \( B = 0 \) and \( \alpha_1 = 1 \). See also \cite{50, 50, 50, 50, 50, 50, 50, 50} and references therein.

In the last decade, the non-uniqueness of weak solutions to various hydrodynamic models has been proved successfully by convex integration methods. Originating from the work by Nash \cite{58}, the convex integration scheme was applied in the pioneering paper by De Lellis-Székelyhidi \cite{27} in 2009 to the 3D Euler equations, where bounded solutions with compact support in space-time were obtained (see also \cite{29, 30}). Afterwards, a remarkable progress is the resolution of the famous Onsager conjecture for the 3D Euler equations by Isett \cite{41}, and by Buckmaster-De Lellis-Székelyhidi-Vicol \cite{11} for dissipative solutions. We also refer to \cite{17, 21, 54} for Euler equations, \cite{9, 19, 56, 57} for transport equations, and \cite{12} for SQG equations. See also the surveys \cite{14, 15, 31} on convex integration methods.

A recent break-through was obtained by Buckmaster-Vicol \cite{13} for the 3D incompressible NSE, to which the non-uniqueness of weak solutions was proved by an \( L^2_x \)-based intermittent convex integration scheme. Luo-Titi \cite{53} then proved the non-uniqueness results for the 3D hyperviscous NSE \cite{12}, whenever \( \alpha \) is less than the Lions exponent, i.e., \( \alpha \in (1, 5/4) \). Moreover, Luo-Qu \cite{52} proved the non-uniqueness for the 2D hyperviscous NSE with \( \alpha \in (0, 1) \). The non-uniqueness results for the stationary NSE were proved by Cheskidov-Luo \cite{18, 55}. We also refer to the works by Jia-Šverák \cite{42, 43} for another method for the non-uniqueness issue of Leray-Hopf solutions in \( L^p_t L^3_x \) under a
certain assumption for the linearized Navier-Stokes operator, and Colombo-De Lellis-De Rosa\cite{22} and De Rosa\cite{32} for the non-uniqueness of Leray weak solutions to hypoviscous NSE with small viscosity.

Regarding the ideal MHD system \((1.3)\), it possesses a number of global invariants:

- The total energy: \(E(t) = \frac{1}{2} \int_{T^3} |u(t, x)|^2 + |B(t, x)|^2 \, dx\);
- The cross helicity: \(\mathcal{H}_{\omega,B}(t) = \int_{T^3} u(t, x) \cdot B(t, x) \, dx\);
- The magnetic helicity: \(\mathcal{H}_{B,B}(t) := \int_{T^3} A(t, x) \cdot B(t, x) \, dx\).

Here \(A\) is a mean-free periodic vector field satisfying \(\text{curl} A = B\).

In analogy with the Onsager conjecture for Euler equations, it was conjectured in \cite{15} that, \(L^3_{t,x}\) (resp. \(L^3_{t,x}B^0_{3,\infty,x}\)) is the critical space for the conservation of the magnetic helicity. More precisely, any weak solutions belonging to \(L^3_{t,x}\) (resp. \(L^3_{t,x}B^0_{3,\infty,x}\)) conserve the magnetic helicity (the rigidity part), while below the threshold there exist weak solutions violating the magnetic helicity conservation (the flexible part). Moreover, the space \(C^{1/3}_{t,x}\) (resp. \(L^3_{t,x}B^{1/3}_{3,\infty,x}\)) was conjectured to be the threshold for the conservation of the total energy and the cross helicity (see Remark 1.6 (ii) below).

These conjectures indeed lie in the general scope of the important issue to identify the critical/threshold regularity for the conservation laws. We refer to Klainerman \cite{46} for the general formulations of different thresholds for supercritical Hamiltonian evolution equations, including the 3D MHD equations.

On the rigidity side, the magnetic helicity conservation for the 3D ideal MHD was proved in \cite{1, 34, 44} and \cite{16}, respectively, for the critical spaces \(L^3_{t,x}\) and \(L^3_{t,x}B^\alpha_{3,\infty,x}\) with \(\alpha > 0\). The corresponding rigidity results for the total energy conservation were proved in \cite{16}. On the flexible side, weak solutions with non-trivial energy and vanishing magnetic helicity were first given in \cite{8}, by embedding the ideal MHD into the 2+1D Euler flow via a symmetry assumption. Wild solutions with compact support in space-time were first constructed by Faraco-Lindberg-Székelyhidi \cite{36}, via the \(L^\infty_{t,x}\) convex integration scheme. The constructed solutions in \cite{36} violate the energy conservation, while the magnetic helicity is still conserved and thus vanishes due to the compact support in time of solutions.

The delicate point, as observed by Beekie-Buckmaster-Vicol \cite{2}, is that the magnetic helicity is conserved under much milder regularity conditions. As a matter of fact, the magnetic helicity is commonly expected in the plasma physics to be conserved in the infinite conductivity limit, known as Taylor’s conjecture \((63, 64)\). This conjecture was proved by Faraco and Lindberg \cite{35} under the weak ideal limits, namely, the weak limits of Leray solutions to MHD \((1.1)\) when \(\alpha_i = 1, i = 1, 2\). For general weak solutions, Beekie-Buckmaster-Vicol \cite{2} first constructed weak solutions in \(C_tL^2_x\) violating the magnetic helicity conservation. Unlike in \cite{36}, the proof of \cite{2} relies on the intermittent convex integration scheme, in which the intermittent shear flows are constructed to respect the geometry of MHD. See also Dai \cite{25} for the Hall MHD system where the Hall nonlinearity takes the dominant effect, and Feireisl-Li \cite{37} for the ill-posedness of the MHD system where the fluid is compressible, inviscid and magnetically resistive.

The applicability of convex integration scheme to MHD equations, as pointed out in \cite{15}, is at an early stage.

The main purpose of this paper is to address the non-uniqueness problem of weak solutions for (generalized) MHD equations. Motivated by the above progress for the NSE and the ideal MHD, it is natural to ask whether the non-uniqueness of weak solutions...
holds for the MHD equations, or, more generally, for the generalized MHD system (1.1) with \( \alpha_i \) below the Lions exponent. If so, would there be some relationship between the non-uniqueness of solutions to the ideal MHD and generalized MHD? The major difficulty here, as in the context of NSE [13], is to beat the viscosity and resistivity \(-\Delta)^{\alpha_i}, i = 1, 2\), which are greater for larger exponents \( \alpha_i \).

We prove the non-uniqueness of weak solutions to the generalized MHD equations (1.1), for the exponents \( \alpha_1, \alpha_2 \) less than the sharp Lions exponent \( 5/4 \). The constructed weak solutions can be close to the given divergence-free, mean-free velocity and magnetic fields. In particular, the weak solutions live in the space \( L_t^s W_x^{s,p} \) with \( 0 \leq s < 2/p + 1/\gamma - 3/2, 1 \leq p, \gamma < +\infty \), and do not conserve the magnetic helicity. Thus, this provides more examples of weak solutions with the above regularity \( L_t^s W_x^{s,p} \) for the flexible part of the Onsager-type conjecture for the ideal MHD in [15].

Our second result is concerned with the vanishing viscosity and resistivity limits of weak solutions to the generalized MHD. As pointed out in [2, 15], the weak solutions to the ideal MHD (1.3) with non-trivial magnetic helicity cannot be obtained as the weak ideal limits of MHD Leray-Hopf solutions. Instead, we prove that the weak solutions constructed in [2] can be obtained as a strong vanishing viscosity and resistivity limit of a sequence of weak solutions to the more general MHD (1.1), where the exponents \( \alpha_i \) can be even larger than 1. An interesting outcome is that, together with the non-conservative results in [2], this shows that Taylor’s conjecture does not hold along the vanishing viscosity and resistivity limits of general weak solutions, in contrast to the weak ideal limits of Leray weak solutions.

Our proof is based on the intermittent convex integration method. Unlike in the context of the NSE [13], because of the strong coupling between the magnetic and velocity fields, the intermittent magnetic flows are also required to respect the geometry of MHD.

One new ingredient of our proof is the construction of the velocity and magnetic flows with the refined spatial intermittency, which are designed specifically to concentrate on small spatial cuboids and, as the intermittent Mikado flows in [26], have mutually disjoint supports. This particular structure enables us to gain an additional almost 1D spatial intermittency than that in the ideal MHD case [2], and thus to control the fractional viscosity and resistivity \(-\Delta)^{\alpha_i} \) with \( \alpha_i \in [0, 1) \), \( i = 1, 2 \).

Another main novelty is the construction of the \( L_t^2 \)-based temporal intermittent flows adapted to the structure of MHD, which enables us to control the stronger viscosity and resistivity \(-\Delta)^{\alpha_i}, \alpha_i \geq 1, i = 1, 2 \). We introduce the high temporal oscillations in the velocity and magnetic flows. Rather than the pointwise analysis in time, we measure the solutions on average, namely, in the space \( L_t^1 L_x^2 \), which is in spirit close to the works [13, 19]. In particular, this new temporal intermittency gains an extra almost 1D intermittency, which enables us to pass beyond the border line \( \alpha_i = 1 \) and, more interestingly, even to achieve all the exponents \( \alpha_i \in [1, 5/4) \) up to the sharp Lions exponent.

Let us also mention that, besides the incompressibility correctors in [2], two new types of temporal correctors are also introduced for the velocity and magnetic perturbations, in order to balance both the high spatial and temporal oscillations arising from the corresponding concentration functions. It would be interesting to note that, the new velocity flows and magnetic flows indeed exhibit a quite compatible algebraic structure in the convex integration scheme.

**Notations.** For \( p \in [1, +\infty] \) and \( s \in \mathbb{R} \), we use the following short notations
\[
L^p_x := L^p_x(\mathbb{T}^3), \quad H^s_x := H^s_x(\mathbb{T}^3), \quad W^{s,p}_x := W^{s,p}_x(\mathbb{T}^3).
\]
The mean of \( u \in L^1(\mathbb{T}^n) \) is given by \( \int_{\mathbb{T}^n} udx = |\mathbb{T}^n|^{-1} \int_{\mathbb{T}^n} u dx \), where \(| \cdot |\) denotes the Lebesgue measure. For any \( p, \gamma \in [1, +\infty] \), \( L^p_{\gamma}(\mathbb{T}; L^p(\mathbb{T}^3)) \) denotes the usual Banach space \( L^p_{\gamma}(\mathbb{T}; L^p(\mathbb{T}^3)) \). Moreover, given any Banach space \( X \), \( C(\mathbb{T}; X) \) denotes the space of continuous functions from \( \mathbb{T} \) to \( X \), equipped with the norm \( \| u \|_{C_tX} := \sup_{t \in \mathbb{T}} \| u(t) \|_X \). In particular, we write \( L^p_t := L^p_t L^p_x \) and \( C_{t,x} := C_tC_x \) for simplicity. Let

\[
\| u \|_{W^{m,p}_{t,x}} := \sum_{0 \leq m + |\zeta| \leq N} \| \partial^m_t \nabla^\zeta u \|_{L^p_{t,x}}, \quad \| u \|_{C^{m}_{t,x}} := \sum_{0 \leq m + |\zeta| \leq N} \| \partial^m_t \nabla^\zeta u \|_{C_{t,x}},
\]

where \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \) is the multi-index and \( \nabla^\zeta := \partial_{x_1}^{\zeta_1} \partial_{x_2}^{\zeta_2} \partial_{x_3}^{\zeta_3} \). For any \( A \subseteq \mathbb{T} \), set

\[
N_{\varepsilon_s}(A) := \{ t \in \mathbb{T} : \exists s \in A, \ s.t. |t - s| \leq \varepsilon_s \}.
\]

We also adapt the notations from [49]. Let \( u, v \) be two vector fields, the corresponding second order tensor product is defined by

\[ u \otimes v := (u_jv_j)_{1 \leq i,j \leq 3}. \]

For any second-order tensor \( A = (a_{ij})_{1 \leq i,j \leq 3} \), set

\[ \text{div} A := \left( \sum_{j=1}^{3} \partial_{x_j} a_{1j}, \sum_{j=1}^{3} \partial_{x_j} a_{2j}, \sum_{j=1}^{3} \partial_{x_j} a_{3j} \right)^\top. \]

The right product of a vector field \( v = (v_1, v_2, v_3)^\top \) to a second-order tensor \( A = (a_{ij})_{1 \leq i,j \leq 3} \) is defined by

\[ Av := \left( \sum_{j=1}^{3} a_{1j} v_j, \sum_{j=1}^{3} a_{2j} v_j, \sum_{j=1}^{3} a_{3j} v_j \right)^\top. \]

In particular, for any scalar function \( f \) and second-order tensor \( A = (a_{ij})_{1 \leq i,j \leq 3} \), one has the Leibniz rule

\[ \text{div}(fA) = f\text{div}A + A\nabla f. \]

We use the notation \( a \lesssim b \), which means that \( a \leq Cb \) for some constant \( C > 0 \).

1.2. **Formulation of main results.** To begin with, let us formulate precisely the definition of weak solutions to (1.1).

**Definition 1.1.** (Weak solution). We say that \( (u, B) \in L^2(\mathbb{T}; L^2(\mathbb{T}^3)) \) is a weak solution to the generalized MHD system (1.1) if

- For all \( t \in \mathbb{T} \), \( (u(t, \cdot), B(t, \cdot)) \) are divergence free in the sense of distributions and have zero spatial mean.
- Equation (1.1) holds in the sense of distributions, i.e., for any divergence-free test functions \( \varphi \in C^\infty_0(\mathbb{T} \times \mathbb{T}^3) \),

\[
\int_{\mathbb{T}^3} \partial_t \varphi \cdot u - \nu_1 (\Delta) a_1 \varphi \cdot u + \nabla \varphi : (u \otimes u - B \otimes B) dx = 0,
\]

\[
\int_{\mathbb{T}^3} \partial_t \varphi \cdot B - \nu_2 (\Delta) a_2 \varphi \cdot B + \nabla \varphi : (B \otimes u - u \otimes B) dx = 0.
\]

Here, for any \( 3 \times 3 \) matrices \( A = (A_{ij}) \) and \( S = (S_{ij}) \), \( A : S = \sum_{i,j=1}^{3} A_{ij} S_{ij} \).

For simplicity, we consider the solutions on the temporal interval \( \mathbb{T} \) to be consistent with the spatial torus \( \mathbb{T}^3 \). The arguments of this paper can be also extended to any bounded temporal intervals.

The first main result of this paper is formulated below.
Theorem 1.2. Let $\alpha_1, \alpha_2 \in [0, 5/4)$, $(\tilde{u}, \tilde{B})$ be any smooth, divergence-free and mean-free vector fields on $T \times T^3$, and $(s, p, \gamma) \in A$ where
\[
A := \left\{ (s, p, \gamma) \in \left[ 0, \frac{3}{2} \right) \times [1, +\infty) \times [1, +\infty) : 0 \leq s < \frac{2}{p} + \frac{1}{\gamma} - \frac{3}{2} \right\}. \tag{1.5}
\]
(See Figure 1. for the admissible set $A$ of regularities when $s = 0$.)

Then, there exists $\beta' \in (0, 1)$, such that for any given $\varepsilon_*>0$, there exist a velocity field $u$ and a magnetic field $B$ such that the following holds:

(i) Weak solution: $(u, B)$ is a weak solution to (1.1) in the sense of Definition 1.1 with zero spatial mean.

(ii) Regularity: $u, B \in H^{\beta'}_{t,x} \cap L^\gamma_t W^{s,p}_x$.

(iii) Temporal support: $\text{supp}_t u \cup \text{supp}_t B \subseteq N_{\varepsilon_*} (\text{supp}_t \tilde{u} \cup \text{supp}_t \tilde{B})$.

(iv) Small deviations on average: $\|u - \tilde{u}\|_{L^1_t L^2_x} \leq \varepsilon_*$, $\|B - \tilde{B}\|_{L^1_t L^2_x} \leq \varepsilon_*$.

(v) Small deviations of magnetic helicity: $\|H_{B,B} - H_{\tilde{B},\tilde{B}}\|_{L_1^t} \leq \varepsilon_*$.

Figure 1.
(The red point is the conjectured regularity corresponding to $L^3_{t,x}$, the blue region is the admissible set $A$ of regularities when $s = 0$.)

As a consequence, Theorem 1.2 gives the following result concerning the non-uniqueness of weak solutions and the non-conservation of magnetic helicity.

Corollary 1.3. For any $\alpha_1, \alpha_2 \in [0, 5/4)$ and any $(s, p, \gamma) \in A$, where the regularity admissible set $A$ is as in (1.5), there exist infinitely many weak solutions to (1.1) with the same data at time zero, which live in the space $H^{\beta'}_{t,x} \cap L^\gamma_t W^{s,p}_x$ for some $\beta' > 0$ and do not conserve the magnetic helicity.

Remark 1.4.

(i). The non-uniqueness results for the ideal MHD (1.3) were proved by Faraco-Lindberg-Székelyhidi \[36\] for weak solutions with compact support in space-time, and by Beekie-Buckmaster-Vicol \[2\] for weak solutions with non-trivial magnetic helicity.

To the best of our knowledge, Corollary 1.3 provides the first non-uniqueness result for the weak solutions to generalized MHD equations (1.1) with $\alpha_i \in (0, 5/4)$, $i = 1, 2$. The value $5/4$ actually coincides with the Lions exponent for the well-posedness of the hyperviscous NSE \[51\], and with the exponent $1/2 + n/4$ for the well-posedness of 3D generalized MHD \[66\].
(ii). By the Onsager-type conjecture for the magnetic helicity in [15], it is expected that \( L_t^{3} \) is the threshold for the magnetic helicity conservation for the ideal MHD. In view of Corollary 1.3 we see that the magnetic helicity conservation fails for the weak solutions in the space \( H_{t,x}^{\beta'} \cap L_t^7 W_x^{s,p} \), where \( \beta' \) is a small positive constant and \( (s, p, \gamma) \in A \) with \( A \) as in Theorem 1.2. In particular, Corollary 1.3 provides more examples in the spaces \( L_t^{3} L_x^{12/7} \) and \( L_t^{6/5} L_x^{2} \) for the flexible part of the conjecture in [15]. It would be very interesting to fill the spatial gap from \( L_t^{12/7} \) to \( L_t^{3} \) or the temporal gap from \( L_t^{6/5} \) to \( L_t^{3} \).

(iii). In the higher dimensions \( d \geq 4 \), one may prove the non-uniqueness of weak solutions in \( C_t L_x^{2} \) for the viscous and resistive MHD (i.e., \( \alpha_s = 1 \) in (1.1), \( i = 1, 2 \)). The solutions can be continuous in time. This is possible because, in high dimensions \( d \geq 4 \), there is more freedom to choose the oscillation directions in the velocity and magnetic flows, which permits to gain the 3D spatial intermittency to control the Laplacian \((-\Delta)\).

(iv). We expect that the proof of this paper applies to the non-uniqueness in law for the stochastic generalized MHD equations driven by Wiener processes, which will be done in a forthcoming work. We would like to refer to [6, 20, 38, 39] for the recent developments in the stochastic contexts.

Our next result is concerned with the vanishing viscosity and resistivity limits of the weak solutions to generalized MHD equations.

**Theorem 1.5.** Let \( \alpha_1, \alpha_2 \in (0, 5/4) \) and \( (u, B) \in H_{t,x}^{\beta} \times H_{t,x}^{\beta} \) be any mean-free weak solution to the ideal MHD (1.3), where \( \beta > 0 \). Then, there exist \( \beta' \in (0, \beta) \) and a sequence of weak solutions \( (u^{(\nu_n)}, B^{(\nu_n)}) \in H_{t,x}^{\beta'} \times H_{t,x}^{\beta'} \) to the generalized MHD (1.1), where \( \nu_n = (\nu_{1,n}, \nu_{2,n}) \) and \( \nu_{1,n}, \nu_{2,n} \) are the viscosity and resistivity coefficients, respectively, such that as \( \nu_n \to 0 \),

\[
 u^{(\nu_n)} \to u, \quad B^{(\nu_n)} \to B \quad \text{strongly in } H_{t,x}^{\beta'}.
\]

**Remark 1.6.**

(i). By Theorem 1.5, the set of the accumulation points of weak solutions to the generalized MHD equations, in the \( H_{t,x}^{\beta'} \) topology, contains the weak solutions to the ideal MHD equations. This extends the vanishing viscosity result in the context of the NSE [13] to the generalized MHD equations.

Let us mention that, the Sobolev space \( H_{t,x}^{\beta} \) is less regular than \( C_{t,x}^{\beta} \) used in the vanishing viscosity limit result for the NSE in [13]. In order to get the temporal regularity of solutions, we use the Slobodetskii-type norm of \( H_{t,x}^{\beta} \) to gain the regularity on average, which enables to prove the vanishing viscosity limit in the Sobolev space \( H_{t,x}^{\beta'} \) with \( \beta' < \beta \). See the proof of (6.35) and (6.36) below.

(ii). Another interesting outcome of Theorem 1.5 is related to the weak solutions constructed in [2] and to the Taylor conjecture concerning the conservation of magnetic helicity.

The Taylor conjecture is mathematically formulated in the concept of weak ideal limits, i.e., the weak limits of Leray weak solutions, and has been recently proved by Faraco-Lindberg [35]. The crucial ingredient of the proof is the regularity of Leray approximating solutions. Hence, the weak solutions constructed in [2] with non-trivial magnetic helicity cannot be obtained as the weak ideal limits of Leray-Hopf solutions to the MHD (i.e., (1.1) with \( \alpha_i = 1 \), \( i = 1, 2 \)).

Let us mention that, the weak solutions constructed in [2] has the regularity \( H_{t,x}^{\beta} \) for some \( \beta > 0 \) (see Remark 6.1 below). Hence, by virtue of Theorem 1.5 we note that, in contrast to the weak ideal limits, the weak solutions in [2] can be obtained as the...
vanishing viscosity and resistivity limits of those to more general MHD equations \((1.1)\), where the exponents \(\alpha_i\) can be even larger than one. In particular, this yields that Taloy’s conjecture does not hold along the vanishing viscosity and resistivity limits of the weak solutions to the (generalized) MHD equations.

### 1.3. Outline of the proof.

Our strategy of proof is based on the intermittent convex integration scheme, inspired by the works \([2, 13, 19]\).

More precisely, for each integer \(q \in \mathbb{N}\), we consider the following relaxation system

\[
\begin{aligned}
\partial_t u_q + \nu_1 (-\Delta)^{\alpha_1} u_q + \text{div} (u_q \otimes u_q - B_q \otimes B_q) + \nabla P_q &= \text{div} \hat{R}_q^u, \\
\partial_t B_q + \nu_2 (-\Delta)^{\alpha_2} B_q + \text{div} (B_q \otimes u_q - u_q \otimes B_q) &= \text{div} \hat{R}_q^B,
\end{aligned}
\]

where the Reynolds stress \(\hat{R}_q^u\) is a symmetric traceless 3 \(\times\) 3 matrix, and the magnetic stress \(\hat{R}_q^B\) is a skew-symmetric 3 \(\times\) 3 matrix.

The quantitative estimates of the relaxation solutions \((u_q, B_q, \hat{R}_q^u, \hat{R}_q^B)\) are specified in the main iteration result below.

More precisely, let \(\alpha := \max\{\alpha_1, \alpha_2\}\), \(\alpha_* := \max\{2\alpha - 1, 0\}\). Take \(\varepsilon \in \mathbb{Q}_+\) sufficiently small such that

\[
\varepsilon \leq \frac{3 - 2\alpha_*}{20},
\]

and

\[
s + \frac{3}{2} \frac{2}{p} - \frac{1}{\gamma} + 8\varepsilon < 0.
\]

For \(q \in \mathbb{N}\), the frequency parameter \(\lambda_q\) and the amplitude parameter \(\delta_{q+2}\) are defined by

\[
\lambda_q = a^{(\text{br})}, \quad \delta_{q+2} = \lambda_{q+2}^{-2\beta}.
\]

Here, \(a \in 5\mathbb{N}\) is a large integer of multiple 5 such that \(a^e \in 5\mathbb{N}\), \(b \in 2\mathbb{N}\) is a large integer of multiple 2 such that \(\lambda_{q+1}^{2e} = \lambda_q^{-2b} \in \mathbb{N}\) and

\[
b > \frac{1000}{\varepsilon}, \quad 0 < \beta < \frac{1}{100b^2}.
\]

Let \(q \in \mathbb{N}\), we assume the following inductive estimates of the relaxation solutions to \((1.7)\) at level \(q\):

\[
\begin{aligned}
&\|u_q\|_{C_i^1, x} \leq \lambda_q^7, \quad \|B_q\|_{C_i^1, x} \leq \lambda_q^7, \\
&\|\hat{R}_q^u\|_{C_i^1, x} \leq \lambda_q^{14}, \quad \|\hat{R}_q^B\|_{C_i^1, x} \leq \lambda_q^{14}, \\
&\|\hat{R}_q^u\|_{L_i^1, x} \leq \delta_{q+1}, \quad \|\hat{R}_q^B\|_{L_i^1, x} \leq \delta_{q+1}.
\end{aligned}
\]

The main iteration result is contained in the following theorem.

**Theorem 1.7 (Main iteration).** Let \(\alpha_1, \alpha_2 \in [0, 5/4)\). Then, there exist \(\beta \in (0, 1)\), \(M > 0\) large enough and \(a_0 = a_0(\beta, M)\), such that for any integer \(a \geq a_0\), the following holds:

Suppose that \((u_q, B_q, \hat{R}_q^u, \hat{R}_q^B)\) solves \((1.7)\) and satisfies \((1.12)-(1.14)\). Then, there exists \((u_{q+1}, B_{q+1}, \hat{R}_{q+1}^u, \hat{R}_{q+1}^B)\) solving \((1.7)\) and satisfying \((1.12)-(1.14)\) with \(q + 1\) replacing \(q\). In addition, we have

\[
\begin{aligned}
\|u_{q+1} - u_q\|_{L_i^2, x} &\leq M\delta_{q+1}^\frac{1}{2}, \quad \|B_{q+1} - B_q\|_{L_i^2, x} \leq M\delta_{q+1}^\frac{1}{2}, \\
\|u_{q+1} - u_q\|_{L_i^1, L_i^2} &\leq \delta_{q+2}^\frac{1}{2}, \quad \|B_{q+1} - B_q\|_{L_i^1, L_i^2} \leq \delta_{q+2}^\frac{1}{2}.
\end{aligned}
\]
and
\[
\supp_i u_{q+1} \cup \supp_i B_{q+1} \subseteq N_{\frac{1}{\lambda_r^{q+2}}} \left( \supp_i u_q \cup \supp_i B_q \cup \supp_i \hat{R}_q^u \cup \supp_i \hat{R}_q^B \right),
\]
(1.17)
\[
\supp_i \hat{R}_q^{u+1} \cup \supp_i \hat{R}_q^{B+1} \subseteq N_{\frac{1}{\lambda_r^{q+2}}} \left( \supp_i u_q \cup \supp_i B_q \cup \supp_i \hat{R}_q^u \cup \supp_i \hat{R}_q^B \right).
\]
(1.18)

The heart of the proof of Theorem 1.7 is to construct suitable perturbations
\[ w_{q+1} \simeq u_{q+1} - u_q, \quad d_{q+1} \simeq B_{q+1} - B_q, \]
module small mollification errors, such that the corresponding nonlinear effects decrease the amplitudes of the Reynolds and magnetic stresses, in order to fulfill the above iteration procedure.

1.4. New ingredients of proof. One major difference between the NSE and the ideal MHD equations is that, because of the strong coupling of the velocity and magnetic fields, besides the intermittent velocity flows \{W_{(k)}\}, the intermittent magnetic flows \{D_{(k)}\} shall also be introduced in order to respect the geometry of MHD. This forces to limit the oscillation directions, and thus the intermittent velocity and magnetic flows do not have the 3D spatial intermittency as in the NSE case \[13\].

In the context of the ideal MHD \[1.3\], the key intermittent shear velocity flows and the intermittent shear magnetic flows were introduced of the form
\[ W_{(k)} = \phi_{(k)} k_1, \quad D_{(k)} = \phi_{(k)} k_2, \]
(1.19)
where \{\phi_{(k)}\} are the spatial concentration functions with one oscillation direction \(k\), orthogonal to the other two directions \(k_1, k_2\). The shear flows have the almost 1D spatial intermittency, i.e.,
\[ \|W_{(k)}\|_{C^i L_2} + \|D_{(k)}\|_{C^i L_2} \lesssim \lambda^{-\frac{1}{2} + \varepsilon}, \]
(1.20)
which permits to control the fractional viscosity and resistivity \((-\Delta)^{\alpha_i}\) with \(\alpha_i \in [0, 1/2], i = 1, 2\).

One of the novelties of our proof is the construction of a new class of intermittent velocity and magnetic flows, which feature the refined intermittency in both the space and time and thus enable us to control the stronger viscosity and resistivity \((-\Delta)^{\alpha_i}\) with \(\alpha_i \geq 1/2, i = 1, 2\).

(i) Spatial intermittent building blocks. We construct the refined intermittent velocity and magnetic flows
\[ W_{(k)} = \psi_{(k)} \phi_{(k)} k_1, \quad D_{(k)} = \psi_{(k)} \phi_{(k)} k_2, \]
(1.21)
which are \((T/\lambda r_{\perp})^3\)-periodic and concentrate on smaller cuboids with volume \(\lambda^{-3} r_{\perp}^{3} r_{\parallel}\) in each of the cubes with side length \(2\pi/(\lambda r_{\perp})\). Here, \(\psi_{(k)}\) and \(\phi_{(k)}\) are suitable concentration functions (see \[3.8\] below), \(r_{\parallel}\) and \(r_{\perp}\) are the small constants to parameterize the spatial concentration of the building blocks, as in the intermittent jets for the NSE \[14\]. The refined flows enable to gain the almost 2D-intermittency, i.e.,
\[ \|W_{(k)}\|_{C^i L_2} + \|D_{(k)}\|_{C^i L_2} \lesssim \lambda^{-1 + \varepsilon}, \]
(1.22)
and thus enables to control the fractional viscosity and resistivity \((-\Delta)^{\alpha_i}\) with \(\alpha_i \in [1/2, 1], i = 1, 2\).

Another nice feature of the refined intermittent velocity and magnetic flows is that the spatial supports are mutually disjoint, as the Mikado flows in \[20\], which in particular simplifies the control of the interactions between different flows.
Let us also mention that, unlike the one oscillation direction $k$ in (1.19) in the ideal MHD case, the building blocks $W(k)$ and $D(k)$ in (1.21) oscillate in two orthogonal directions $k_1$ and $k$. This new oscillation direction $k_1$ causes extra high spatial oscillations, which will be balanced by the introduction of a new temporal parameter $\mu$ in the flows and the temporal correctors $w^{(t)}_{q+1}$ and $d^{(t)}_{q+1}$. See (4.37a), (4.37b) and the important identities (4.38) and (4.39) below.

It might be tempting to introduce one more oscillation direction $k_2$ in the building blocks, i.e., with the oscillation directions $(k, k_1, k_2)$ as in the NSE context [13, 14]. However, this causes even more high spatial oscillations, which seem not to be balanced by the temporal parameter $\mu$.

(ii) **Temporal intermittent building blocks.** The next major difficulty is to pass beyond the border line $\alpha_i = 1, \ i = 1, 2$.

The key idea here is to exploit the new intermittency from the temporal oscillations. Thus, rather than performing the pointwise analysis in time, we measure the Reynolds and magnetic stresses on average, i.e., in the space $L^1_{t,x}$.

Let us mention that, this is in spirit close to the works [13, 18]. In [18] the temporal intermittency is crucial to achieve the non-uniqueness of weak solutions for transport equations at the critical space regularity. In the NSE context [13], the key ingredient is the $L^2_x$-based spatial intermittency, instead of the previous pointwise analysis in space in the Euler settings.

We introduce the temporal concentration function $g(\tau)$ in the principal part of perturbations $(w^{(p)}_{q+1}, d^{(p)}_{q+1})$, and control the oscillation errors in the space $L^1_t$. Here, $\tau$ is the constant to parameterize the temporal concentration. In particular, this permits to gain an additional almost 1D intermittency, i.e.,

$$\|g(\tau)\|_{L^1_t} \lesssim \lambda^{-\frac{1}{2}+\varepsilon}. \quad (1.23)$$

Quite interestingly, the gained temporal intermittency enables to control the viscosity and resistivity $(-\Delta)^{\alpha_i}$ for all the exponents $\alpha_i$ less than $5/4, \ i = 1, 2$, that is, exactly the Lions exponents.

Let us also mention that, extra high temporal oscillations arise due to the presence of the temporal concentration function $g(\varepsilon)$. In order to balance these high oscillations, another new type of temporal correctors $w^{(o)}_{q+1}, d^{(o)}_{q+1}$ will be introduced, respectively, for the velocity and magnetic flows. See (4.40a), (4.40b) and the important identities (4.41) and (4.42) below.

The remainder of this paper is organized as follows. We first regularize the relaxation solutions to (1.7) by the standard mollification procedure in Section 2. Section 3 is devoted to the construction of the crucial intermittent velocity and magnetic flows, which feature the intermittency in both the space and the time. Then, in Section 4 we construct the velocity and magnetic perturbations, consisting of the principal parts to decrease the Reynolds and magnetic stresses, the incompressibility correctors and two types of temporal correctors. Several key algebraic identities and analytic estimates will also be given, which lead to the verification of the inductive estimates for the velocity and magnetic fields. Then, the inductive estimates of Reynolds and magnetic stresses are verified in Section 5. Consequently, the main results are proved in Section 6.

### 2. Mollification

In order to avoid the loss of derivatives, we mollify the velocity and magnetic fields. Let $\phi_\varepsilon$ and $\varphi_\varepsilon$ be the mollifiers on $\mathbb{T}^3$ and $\mathbb{T}$, respectively, $\varepsilon > 0$, and supp $\varphi_\varepsilon \subseteq (-\varepsilon, \varepsilon)$. 

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The mollifications of \((u_q, B_q, \hat{R}_q^u, \hat{R}_q^B)\) in space and time are defined by

\[
\begin{align*}
  u_t &:= (u_q \ast_x \phi_t) \ast_t \varphi_t, \quad B_t := (B_q \ast_x \phi_t) \ast_t \varphi_t, \\
  \hat{R}_t^u &:= (\hat{R}_q^u \ast_x \phi_t) \ast_t \varphi_t, \quad \hat{R}_t^B := (\hat{R}_q^B \ast_x \phi_t) \ast_t \varphi_t,
\end{align*}
\]  

(2.1)

where the scale of mollification is parameterized by

\[
\ell := \lambda_q^{-20}.
\]

(2.2)

Then, by equation (1.7), \((u_t, B_t, \hat{R}_t^u, \hat{R}_t^B)\) satisfies

\[
\begin{align*}
  \partial_t u_t + \nu_1(-\Delta)^{\alpha_1} u_t + \text{div}(u_t \otimes u_t - B_t \otimes B_t) + \nabla P_t &= \text{div}(\hat{R}_t^u + \hat{R}_t^u_{\text{com}}), \\
  \partial_t B_t + \nu_2(-\Delta)^{\alpha_2} B_t + \text{div}(B_t \otimes u_t - u_t \otimes B_t) &= \text{div}(\hat{R}_t^B + \hat{R}_t^B_{\text{com}}),
\end{align*}
\]

(2.3)

where the traceless symmetric commutator stress \(\hat{R}_{\text{com}}^u\) and the skew-symmetric commutator stress \(\hat{R}_{\text{com}}^B\) are of form

\[
\begin{align*}
  \hat{R}_{\text{com}}^u &:= u_t \otimes u_t - B_t \otimes B_t - (u_q \otimes u_q - B_q \otimes B_q) \ast_x \phi_t \ast_t \varphi_t, \\
  \hat{R}_{\text{com}}^B &:= B_t \otimes u_t - u_t \otimes B_t - (B_q \otimes u_q - u_q \otimes B_q) \ast_x \phi_t \ast_t \varphi_t,
\end{align*}
\]

(2.4)

and the pressure \(P_t\) is given by

\[
P_t := (P_q \ast_x \phi_t) \ast_t \varphi_t - |u_t|^2 + |B_t|^2 + (|u_q|^2 - |B_q|^2) \ast_x \phi_t \ast_t \varphi_t.
\]

(2.5)

By the standard mollification estimates and the inductive estimates \((1.12)-(1.14)\), for any \(N \in \mathbb{N}_+\),

\[
\|u_t\|_{C^{N+1}_{t,x}} + \|B_t\|_{C^{N+1}_{t,x}} \lesssim \ell^{-N+1}(\|u_q\|_{C^1_{t,x}} + \|B_q\|_{C^1_{t,x}}) \lesssim \ell^{-N+1} \lambda_q^7 \lesssim \ell^{-N},
\]

(2.6)

\[
\|\hat{R}_t^u\|_{C^{N+1}_{t,x}} + \|\hat{R}_t^B\|_{C^{N+1}_{t,x}} \lesssim \ell^{-N+1}(\|\hat{R}_{\text{com}}^u\|_{C^1_{t,x}} + \|\hat{R}_{\text{com}}^B\|_{C^1_{t,x}}) \lesssim \ell^{-N+1} \lambda_q^{14} \lesssim \ell^{-N},
\]

(2.7)

\[
\|\hat{R}_t^u\|_{L^1_t L^p_x} \lesssim \delta_{q+1}, \quad \|\hat{R}_t^B\|_{L^1_t L^p_x} \lesssim \delta_{q+1}.
\]

(2.8)

Moreover, by the inductive estimate \((1.12)\) and the double commutator estimate (see, e.g., \[55\] Proposition B.1, [24] Lemma 1, [23]), for \(1 < p < +\infty\),

\[
\|\hat{R}_{\text{com}}^u\|_{L^1_t L^p_x} \lesssim \|\hat{R}_{\text{com}}^u\|_{C^1_{t,x}} \lesssim \ell((\|u_q\|^2_{C^1_{t,x}} + \|B_q\|^2_{C^1_{t,x}}) \lesssim \ell \lambda_q^{14},
\]

(2.9)

and, similarly,

\[
\|\hat{R}_{\text{com}}^B\|_{L^1_t L^p_x} \lesssim \|\hat{R}_{\text{com}}^B\|_{C^1_{t,x}} \lesssim \ell \lambda_q^{14}.
\]

(2.10)

3. INTERMITTENT VELOCITY AND MAGNETIC FLOWS

This section contains the crucial intermittent velocity and magnetic flows, which are indeed the fundamental building blocks in the convex integration scheme.

We set the parameters \(r_\perp, r||, \lambda, \mu, \tau\) and \(\sigma\) as follows

\[
r_\perp := \lambda^{-14-2\varepsilon}_q, \quad r|| := \lambda^{-1+6\varepsilon}_q, \quad \lambda := \lambda_q^{+1}, \quad \mu := \lambda^{2-6\varepsilon}_q, \quad \tau := \lambda^{1-6\varepsilon}_q, \quad \sigma := \lambda^{2\varepsilon}_q,
\]

(3.1)

where \(\varepsilon\) is the small constant satisfying \((1.8)\).
3.1. Geometric lemmas. To begin with, let us first recall two geometric lemmas in [2].

**Lemma 3.1. (First Geometric Lemma, [2, Lemma 4.1])** There exists a set \( \Lambda_B \subset S^2 \cap \mathbb{Q}^3 \) that consists of vectors \( k \) with associated orthonormal bases \((k, k_1, k_2)\), \( \varepsilon_B > 0 \), and smooth positive functions \( \gamma(k) : B_{\varepsilon_B}(0) \to \mathbb{R} \), where \( B_{\varepsilon_B}(0) \) is the ball of radius \( \varepsilon_B \) centered at 0 in the space of \( 3 \times 3 \) skew-symmetric matrices, such that for \( A \in B_{\varepsilon_B}(0) \) we have the following identity:

\[
A = \sum_{k \in \Lambda_B} \gamma_{(k)}^2(A) (k_2 \otimes k_1 - k_1 \otimes k_2).
\]

**Lemma 3.2. (Second Geometric Lemma, [2, Lemma 4.2])** There exists a set \( \Lambda_u \subset S^2 \cap \mathbb{Q}^3 \) that consists of vectors \( k \) with associated orthonormal bases \((k, k_1, k_2)\), \( \varepsilon_u > 0 \), and smooth positive functions \( \gamma(k) : B_{\varepsilon_u}(\text{Id}) \to \mathbb{R} \), where \( B_{\varepsilon_u}(\text{Id}) \) is the ball of radius \( \varepsilon_u \) centered at the identity in the space of \( 3 \times 3 \) symmetric matrices, such that for \( S \in B_{\varepsilon_u}(\text{Id}) \) we have the following identity:

\[
S = \sum_{k \in \Lambda_u} \gamma_{(k)}^2(S) k_1 \otimes k_1.
\]

Furthermore, we may choose \( \Lambda_u \) such that \( \Lambda_B \cap \Lambda_u = \emptyset \).

As pointed out in [2], there exists \( N_A \in \mathbb{N} \) such that

\[
\{N_A k, N_A k_1, N_A k_2\} \subset N_A S^2 \cap \mathbb{Z}^3.
\]

For instance, \( N_A = 65 \) suffices. We also denote by \( M_\ast \) the geometric constant such that

\[
\sum_{k \in \Lambda_u} \|\gamma(k)\|_{C^4(B_{\varepsilon_u}(\text{Id}))} + \sum_{k \in \Lambda_B} \|\gamma(k)\|_{C^4(B_{\varepsilon_B}(0))} \leq M_\ast.
\]

This parameter is universal and will be used later in the estimates of the size of perturbations.

3.2. Spatial building blocks. Let \( \Phi : \mathbb{R} \to \mathbb{R} \) be a smooth cut-off function supported on the interval \([-1, 1]\). We normalize \( \Phi \) such that \( \phi := -\frac{d^2}{dx^2} \Phi \) satisfies

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \phi^2(x) dx = 1.
\]

Moreover, let \( \psi : \mathbb{R} \to \mathbb{R} \) be a smooth and mean zero function, supported on the interval \([-1, 1]\), satisfying

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \psi^2(x) dx = 1.
\]

The corresponding rescaled cut-off functions are defined by

\[
\phi_{r_\perp}(x) := r_\perp^{-\frac{1}{2}} \phi \left( \frac{x}{r_\perp} \right), \quad \Phi_{r_\perp}(x) := r_\perp^{-\frac{1}{2}} \Phi \left( \frac{x}{r_\perp} \right), \quad \psi_{r_\perp}(x) := r_\perp^{-\frac{1}{2}} \psi \left( \frac{x}{r_\perp} \right).
\]

Note that, \( \phi_{r_\perp} = -r_\perp^2 \frac{d^2}{dx^2} \Phi_{r_\perp} \), and \( \phi_{r_\perp}, \psi_{r_\perp} \) are supported in the ball of radius \( r_\perp \) and \( r_\parallel \), respectively, in \( \mathbb{R} \). By an abuse of notation, we periodize \( \phi_{r_\perp}, \Phi_{r_\perp} \) and \( \psi_{r_\perp} \) so that they are treated as periodic functions defined on \( \mathbb{T} \).

The intermittent velocity flows are defined by

\[
W(k) := \psi_{r_\parallel}(\lambda r_\perp N_A (k_1 \cdot (x - \alpha(k)) + \mu t)) \phi_{r_\perp}(\lambda r_\perp N_A k_1 \cdot (x - \alpha(k))) k_1, \quad k \in \Lambda_u \cup \Lambda_B,
\]

and the intermittent magnetic flows are defined by

\[
D(k) := \psi_{r_\parallel}(\lambda r_\perp N_A (k_1 \cdot (x - \alpha(k)) + \mu t)) \phi_{r_\perp}(\lambda r_\perp N_A k_1 \cdot (x - \alpha(k))) k_2, \quad k \in \Lambda_B.
\]
Here, $N_\Lambda$ is given by (3.4) above, $(k, k_1, k_2)$ are the orthonormal bases in $\mathbb{R}^3$ in Lemmas 3.1 and 3.2. The parameters $r_\parallel$ and $r_\perp$ parameterize the concentration of the flows, and $\mu$ is the temporal oscillation parameter.

Moreover, the vectors $\alpha(k) \in \mathbb{R}^3$, $k \in \Lambda_u \cup \Lambda_B$, are shifts to ensure that the intermittent flows $\{W(k), D(k)\}$ have mutually disjoint supports. The existence of vectors $\alpha(k)$ is analogous to that of Mikado flows in [14, 26]. In fact, $\{W(k), D(k)\}$ are $(T/(\lambda r_\perp))^3$-periodic, supported on the cuboids with length $\sim 1/\lambda r_\parallel$, width $\sim r_\parallel/(\lambda r_\perp)$ and height $\sim 1/\lambda$.

Because the width and height are much smaller than the length, each cuboid can be viewed as a straight line. As there are only a finite number of wavevectors in $\Lambda_B$ and $\Lambda_u$, one may choose the shifts $\{\alpha(k)\}$ such that these straight lines do not intersect. (See the intermittent flows in Figure 2.)

\[ \begin{align*}
\phi(k)(x) &:= \phi_{r_\perp}(\lambda r_\perp N_\Lambda k \cdot (x - \alpha(k))), \\
\psi(k_1)(x) &:= \psi_{r_\parallel}(\lambda r_\perp N_\Lambda (k_1 \cdot (x - \alpha(k)) + \mu t)), \\
\end{align*} \]

(3.8)

and thus

\[ \begin{align*}
W(k) &= \psi(k_1) \phi(k_1) k_1, \quad k \in \Lambda_u \cup \Lambda_B, \\
D(k) &= \psi(k_1) \phi(k_2) k_2, \quad k \in \Lambda_B. \\
\end{align*} \]

(3.9)

Note that, $W(k)$ and $D(k)$ are mean zero on $\mathbb{T}^3$. Moreover, since

\[ \int_{\mathbb{T}^3} \phi(k)^2(x) dx = 1, \quad \text{and} \quad \int_{\mathbb{T}^3} \psi(k_1)^2(x) dx = 1, \]

(3.10)

it follows that

\[ \begin{align*}
\int_{\mathbb{T}^3} W(k) \otimes W(k) dx &= k_1 \otimes k_1, \\
\int_{\mathbb{T}^3} D(k) \otimes D(k) dx &= k_2 \otimes k_2, \\
\int_{\mathbb{T}^3} W(k) \otimes D(k) dx &= k_1 \otimes k_2, \\
\int_{\mathbb{T}^3} D(k) \otimes W(k) dx &= k_2 \otimes k_1. \\
\end{align*} \]

(3.11) (3.12)

It is worth noting that, the building blocks in [2] have one oscillation direction $k$, orthogonal to the direction vectors $k_1$ and $k_2$, and have the 1D-intermittency. In order to control the strong viscosity and resistivity $(-\Delta)^{\alpha_i}$ when $\alpha_i \geq 1/2$, $i = 1, 2$, the idea here to explore more intermittency, inspired by the work [13] in the NSE context, is to introduce the new concentration function $\psi(k_1)$ in (3.9). Two new parameters $r_\perp$ (the rescaling parameter in the direction vertical to the flow) and $r_\parallel$ (the rescaling parameter...
in the direction parallel to the flow) parameterize the concentration of the intermittent flows $W(k)$ and $D(k)$. In particular, one has

$$\sum_{k \in \Lambda} \|W(k)\|_{L^1} + \sum_{k \in \Lambda} \|D(k)\|_{L^1} \lesssim r^{1+\frac{1}{2}} = \lambda^{-1+\varepsilon},$$

which gives the almost 2D intermittency and thus permits to control the fractional viscosity and resistivity $(-\Delta)^{\alpha}$ in (1.1) for any $\alpha_1, \alpha_2 \in [0, 1]$.

Furthermore, the new parameter $\mu$ permits to control the temporal correctors later to balance the high spatial oscillations, arising from the concentration function $\psi(k)$. More precisely, using the orthogonality

$$\nabla \phi(k) \cdot k_1 = \nabla \phi(k) \cdot k_2 = 0,
\nabla \psi(k_1) \cdot k = \nabla \psi(k_1) \cdot k_2 = 0,$$

we have the important algebraic identities

$$\text{div}(W(k) \otimes W(k)) = 2(W(k) \cdot \nabla \psi(k_1)) \phi(k) k_1 = \frac{1}{\mu} \partial_t \left( \psi^2(k_1) \phi^2(k) k_1 \right),$$
$$\text{div}(D(k) \otimes D(k)) = 2(D(k) \cdot \nabla \psi(k_1)) \phi(k) k_2 = 0,$$

and

$$\text{div}(D(k) \otimes W(k)) = (\text{div} W(k)) D(k) + (W(k) \cdot \nabla) D(k) = \frac{1}{\mu} \partial_t \left( \psi^2(k_1) \phi^2(k_2) k_1 \right),$$
$$\text{div}(W(k) \otimes D(k)) = (\text{div} D(k)) W(k) + (D(k) \cdot \nabla) W(k) = 0.$$

Thus, when combining with the temporal correctors, module a pressure term, the time derivative can be moved onto the low oscillating amplitudes (see (4.38), (4.39) below).

Since the intermittent flow $W(k)$ is not divergence-free, we also need the corrector

$$\tilde{W}^c(k) := \frac{1}{\lambda^2 N^2} \nabla \psi(k_1) \times \text{curl}(\Phi(k) k_1).$$

Then, straightforward computations show that

$$W(k) + \tilde{W}^c(k) = \text{curl} \text{curl} \left( \frac{1}{\lambda^2 N^2} \psi(k_1) \Phi(k) k_1 \right) = \text{curl} \text{curl} W^c(k),$$

where $W^c(k)$ is given by

$$W^c(k) := \frac{1}{\lambda^2 N^2} \psi(k_1) \Phi(k) k_1.$$  

Thus, it follows that

$$\text{div}(W(k) + \tilde{W}^c(k)) = 0.$$  

Regarding the magnetic flows, we introduce the correctors

$$D^c_k := \frac{1}{\lambda^2 N^2} \psi(k_1) \Phi(k) k_2, \quad \tilde{D}^c_k := -\frac{1}{\lambda^2 N^2} \Delta \psi(k_1) \Phi(k) k_2, \quad k \in \Lambda.$$

It holds that

$$D(k) + \tilde{D}^c_k = \text{curl} \text{curl} D^c_k.$$

Lemma 3.3 below contains the key estimates of the intermittent velocity and magnetic flows.
Lemma 3.3 (Estimates of spatial intermittency). For $p \in [1, +\infty]$, $N, M \in \mathbb{N}$, we have
\[
\|\nabla^N \partial_t^M \psi(k_1)\|_{C_1 L^p_{x}} \lesssim r_{\perp}^{\frac{1}{p} - \frac{1}{2}} \left( \frac{r_{\perp} \lambda}{r_{\parallel}} \right)^N \left( \frac{r_{\perp} \lambda \mu}{r_{\parallel}} \right)^M, \tag{3.23}
\]
and
\[
\|\nabla^N \phi(k)\|_{L^p_{x}} + \|\nabla^N \Phi(k)\|_{L^p_{x}} \lesssim r_{\perp}^{\frac{1}{p} - \frac{1}{2}} \lambda^N, \tag{3.24}
\]
where the implicit constants are independent of $r_{\perp}$, $r_{\parallel}$, $\lambda$ and $\mu$. In particular, we have
\[
\|\nabla^N \partial_t^M W(k)\|_{C_1 L^p_{x}} + \frac{r_{\parallel}}{r_{\perp}} \left\| \nabla^N \partial_t^M W_{\gamma}(k) \right\|_{C_1 L^p_{x}} + \lambda^2 \left\| \nabla^N \partial_t^M W_{\gamma}(k) \right\|_{C_1 L^p_{x}} \lesssim r_{\perp}^{\frac{1}{p} - \frac{1}{2}} r_{\parallel}^{\frac{1}{2}} \lambda^N, \quad k \in \Lambda_u \cup \Lambda_B, \tag{3.25}
\]
and
\[
\|\nabla^N \partial_t^M D_{\gamma}(k)\|_{C_1 L^p_{x}} + \frac{r_{\parallel}}{r_{\perp}} \left\| \nabla^N \partial_t^M D_{\gamma}(k) \right\|_{C_1 L^p_{x}} + \lambda^2 \left\| \nabla^N \partial_t^M D_{\gamma}(k) \right\|_{C_1 L^p_{x}} \lesssim r_{\perp}^{\frac{1}{p} - \frac{1}{2}} r_{\parallel}^{\frac{1}{2}} \lambda^N, \quad k \in \Lambda_B. \tag{3.26}
\]

Proof. By the definitions of $\psi(k_1), \phi(k)$ and $\Phi(k)$, the $L^\infty$-norms can be bounded by
\[
\|\nabla^N \partial_t^M \psi(k_1)\|_{C_1 L^\infty_{x}} \lesssim \|\psi(k_1)\|_{C_1 L^\infty_{x}} \lesssim \left( \frac{r_{\perp} \lambda}{r_{\parallel}} \right)^N \left( \frac{r_{\perp} \lambda \mu}{r_{\parallel}} \right)^M, \tag{3.27}
\]
and
\[
\|\nabla^N \phi(k)\|_{L^\infty_{x}} + \|\nabla^N \Phi(k)\|_{L^\infty_{x}} \lesssim \lambda^N r_{\perp}^{-\frac{1}{2}}. \tag{3.28}
\]

In order to control the $L^p$-norm, $1 \leq p < +\infty$, let us estimate the volume of the support of $\psi(k_1)$ below. As $\psi(k_1)$ is $(\mathbb{T} / (\lambda r_{\perp})^3)$-periodic, it suffices to estimate the volume of the support of $\psi(k_1)$ on the cubes with side length $2\pi / (\lambda r_{\perp})$ and then multiply the resulting estimate by $(\lambda r_{\perp})^3$. One may let the oscillation parameter $\mu = 0$ and the shifts $\alpha(k) = 0$, $k \in \Lambda_u \cup \Lambda_B$, without changing the volume of the support of $\psi(k_1)$.

In each one of these cubes, $\psi(k_1)$ is supported on at most $2N\Lambda$ thickened planes with thickness $\sim r_{\parallel} / (\lambda r_{\perp})$, which are parallel to each other and perpendicular to the direction $k_1$. Hence, the volume of the support of $\psi(k_1)$ on $\mathbb{T}^3$ can be bounded by
\[
|\operatorname{supp} \psi(k_1)| \lesssim (\lambda r_{\perp})^{-2} \frac{r_{\parallel}}{\lambda r_{\perp}} (\lambda r_{\perp})^3 \lesssim r_{\parallel}. \tag{3.29}
\]
This along with (3.27) yields that
\[
\|\nabla^N \partial_t^M \psi(k_1)\|_{C_1 L^p_{x}} \lesssim |\operatorname{supp} \psi(k_1)|^{\frac{1}{p}} \left\| \nabla^N \partial_t^M \psi(k_1) \right\|_{C_1 L^{p}_{x}} \lesssim r_{\parallel}^{-\frac{1}{p}} \left( \frac{r_{\perp} \lambda}{r_{\parallel}} \right)^N \left( \frac{r_{\perp} \lambda \mu}{r_{\parallel}} \right)^M. \tag{3.30}
\]
Since there are only finite wavevectors in $\Lambda_u$ and $\Lambda_B$, the implicit constant can be taken independent of $k_1$ by taking the maximum over the wavevector sets $\Lambda_u$ and $\Lambda_B$. Thus, (3.23) is verified.

Estimate (3.24) can be proved similarly. Actually, in each of the cubic with length $2\pi / (\lambda r_{\perp})$, $\phi(k)$ is supported on at most $2N\Lambda$ thickened planes with thickness $\sim \lambda^{-1}$, perpendicular to the direction $k$. Hence,
\[
|\operatorname{supp} \phi(k)| \lesssim (\lambda r_{\perp})^{-2} \lambda^{-1} (\lambda r_{\perp})^3 \lesssim r_{\perp},
\]
which along with (3.28) yields that
\[
\|\nabla^N \phi(k)\|_{L^p_{x}} \lesssim |\operatorname{supp} \phi(k)|^{\frac{1}{p}} \left\| \nabla^N \phi(k) \right\|_{L^{p}_{x}} \lesssim r_{\perp}^{-\frac{1}{p}} \lambda N r_{\perp}^{-\frac{1}{2}} \lesssim r_{\perp}^{-\frac{1}{2}} \lambda^N.
\]
Thus, (3.24) is verified.
Finally, estimates (3.25) and (3.26) follow from (3.23), (3.24) and Fubini’s theorem. □

3.3. Temporal building blocks. As mentioned in Section 1, the previous construction of intermittent velocity and magnetic flows cannot provide us with enough intermittency to handle the stronger viscosity and resistivity \((-\Delta)^{\alpha_i}\) when \(\alpha_i \in [1, 5/4], i = 1, 2\). The key idea is to exploit more intermittency from temporal oscillations. Two new parameters \(\tau\) and \(\sigma\) will be introduced to parameterize the concentrations of the temporal function.

More precisely, let \(g \in C_c^\infty(T)\) be any cut-off function such that
\[
\int_T g^2(t) dt = 1.
\]
Let \(\tau \in \mathbb{N}_+\) and rescale the cut-off function \(g\) by
\[
g_{\tau}(t) = \tau^{\frac{1}{2}} g(\tau t). \quad (3.31)
\]
Then, we periodize \(g_{\tau}\) such that the resulting functions (by an abuse of notion, still denoted by \(g_{\tau}\)) are \(T\)-periodic functions.

In order to construct the temporal correctors \(w_{q+1}^{(o)}\) and \(d_{q+1}^{(o)}\) (see (4.40a)-(4.40b)) to balance the high temporal oscillations arising from the concentration function \(g_{(\tau)}\), we also need the function \(h_{\tau} : T \rightarrow \mathbb{R}\), defined by
\[
h_{\tau}(t) := \int_{-\pi}^{\pi} (g_{\tau}^2(s) - 1) \, ds. \quad (3.32)
\]
Set
\[
g_{(\tau)} := g_{\tau}(\sigma t), \quad h_{(\tau)}(t) := h_{\tau}(\sigma t). \quad (3.33)
\]
Note that, \(h_{(\tau)}\) has the uniform \(L_\infty\)-bound
\[
\|h_{(\tau)}\|_{L_\infty(T)} \leq 1. \quad (3.34)
\]
Moreover, we have the following temporal intermittent estimates of \(g_{(\tau)}\).

Lemma 3.4 (Estimates of temporal intermittency). For \(\gamma \in [1, +\infty], M \in \mathbb{N}\), we have
\[
\left\| \partial_t^M g_{(\tau)} \right\|_{L_\gamma(T)} \lesssim \sigma^M \tau^{M+\frac{1}{2}-\frac{1}{\gamma}}, \quad (3.35)
\]
where the implicit constants are independent of \(\tau\) and \(\sigma\).

Proof. As in the proof of Lemma 3.3, we first estimate the \(L_\infty\)-norm of \(g_{(\tau)}\). By definition,
\[
\left\| \partial_t^M g_{(\tau)} \right\|_{L_\infty(T)} \lesssim \sigma^M \tau^{M+\frac{1}{2}}. \quad (3.36)
\]
Next, we give a bound on the volume of the support of \(g_{(\tau)}\) on \(T\). Since \(g_{(\tau)}\) is \(T/\sigma\)-periodic, it is sufficient to obtain the estimate on the interval \([-\pi/\sigma, \pi/\sigma]\) and then multiply the result by \(\sigma\). Note that, in the interval \([-\pi/\sigma, \pi/\sigma]\), \(g_{(\tau)}\) is supported on an interval with length \(\sim (\sigma \tau)^{-1}\), and thus the volume of the support of \(g_{(\tau)}\) on \(T\) is bounded by
\[
|\text{supp } g_{(\tau)}| \lesssim (\sigma \tau)^{-1} \sigma \lesssim \tau^{-1}. \quad (3.37)
\]
Combining (3.36) and (3.37) together we obtain
\[
\left\| \partial_t^M g_{(\tau)} \right\|_{L_\gamma(T)} \lesssim |\text{supp } g_{(\tau)}|^{\frac{1}{2}} \left\| \partial_t^M g_{(\tau)} \right\|_{L_\infty(T)} \lesssim \sigma^M \tau^{M+\frac{1}{2}-\frac{1}{\gamma}}. \quad (3.38)
\]
Therefore, estimate (3.35) is proved. □
4. Velocity and Magnetic Perturbations

This section is devoted to the construction of the key velocity and magnetic perturbations, including the principal parts, the incompressibility correctors, the spatial and temporal correctors. To begin with, let us first define suitable amplitudes of perturbations, which are the key to decrease the effects of old velocity and magnetic stresses.

4.1. Amplitudes.

4.1.1. The magnetic amplitudes. Let \( \chi : [0, +\infty) \to \mathbb{R} \) be a smooth cut-off function such that

\[
\chi(z) = \begin{cases} 
1, & 0 \leq z \leq 1, \\
, & z \geq 2,
\end{cases}
\]

and

\[
\frac{1}{2} z \leq \chi(z) \leq 2 z \quad \text{for} \quad z \in (1, 2).
\]

Set

\[
\rho_B(t, x) := 2\varepsilon_B^{-1}\delta_{q+1}\chi\left(\frac{|\hat{R}^B(t, x)|}{\delta_{q+1}}\right),
\]

where \( \varepsilon_B \) is the small radius in the geometric Lemma 3.1. Then, by (4.1), (4.2) and (4.3),

\[
\frac{\hat{R}^B}{\rho_B} = \frac{\hat{R}^B}{2\varepsilon_B^{-1}\delta_{q+1}\chi(\delta_{q+1}|\hat{R}^B|)} \leq \varepsilon_B,
\]

and for any \( p \in [1, +\infty] \),

\[
\rho_B \geq \varepsilon_B^{-1}\delta_{q+1},
\]

\[
\|\rho_B\|_{L^p_t} \leq 8\varepsilon_B^{-1}\left(16\pi^4\frac{1}{2}\delta_{q+1} + \|\hat{R}^B\|_{L^p_t}\right).
\]

Furthermore, by the standard Hölder estimate (see \[10\] (130)), the iterative estimate (4.13) and (4.5), for \( 1 \leq N \leq 4 \),

\[
\|\rho_B\|_{C^1_{t,x}} \lesssim \ell^{-1}, \quad \|\rho_B\|_{C^2_{t,x}} \lesssim \ell^{-N}\delta_{q+1},
\]

\[
\|\rho_B^{1/2}\|_{C^1_{t,x}} \lesssim \ell^{-1}, \quad \|\rho_B^{1/2}\|_{C^2_{t,x}} \lesssim \ell^{-N}\delta_{q+1},
\]

\[
\|\rho_B^{-1}\|_{C^1_{t,x}} \lesssim \delta_{q+1}, \quad \|\rho_B^{-1}\|_{C^2_{t,x}} \lesssim \ell^{-N}\delta_{q+1}.
\]

Moreover, we choose the smooth temporal cut-off function \( f_B \) such that

\begin{itemize}
  \item \( 0 \leq f_B \leq 1, f_B \equiv 1 \) on \( \supp_t \hat{R}^B \);
  \item \( \supp f_B \subseteq N_t(\supp_t \hat{R}^B) \);
  \item \( \|f_B\|_{C^N_t} \lesssim \ell^{-N}, 1 \leq N \leq 4 \).
\end{itemize}

The amplitudes of the magnetic perturbations are defined by

\[
a_{(k)}(t, x) := a_{k,B}(t, x) = \rho_B^{1/2}(t, x) f_B(t) \gamma_{(k)}\left(\frac{-\hat{R}^B(t, x)}{\rho_B(t, x)}\right), \quad k \in \Lambda_B,
\]

where \( \gamma_{(k)} \) is the smooth function in the geometric Lemma 3.1.

Note that, by virtue of the geometric Lemma 3.1, the identity (3.12) and the expression (4.10) of \( a_{(k)} \), the algebraic identity holds:

\[
\sum_{k \in \Lambda_B} a_{(k)}^2 g_{(7)}^2(D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)})
\]

\[
= -\hat{R}^B + \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(7)}^2 D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}
\]
Lemma 4.1 (Estimates of magnetic amplitudes). For $1 \leq N \leq 4$, $k \in \Lambda_B$, we have
\[
\|a(k)\|_{L^2_{t,x}} \lesssim \delta_{k+1}^{\frac{3}{2}},
\]
\[
\|a(k)\|_{C^1_{t,x}} \lesssim \ell^{-1}, \quad \|a(k)\|_{C^N_{t,x}} \lesssim \ell^{-4N}.
\]

Proof. To obtain the $L^2_{t,x}$-estimate of $a(k)$, $k \in \Lambda_B$, we use (1.14) and (4.6) to bound
\[
\|a(k)\|_{L^2_{t,x}} \leq \|\rhoB\|_{L^1_{t,x}} \frac{1}{\ell} \left|\gamma(k)\right| \|C_{(B,\rhoB(0))}\|_{C^1_{t,x}} \|FB\|_{C^N_{t,x}}
\]
\[
\lesssim M_0 \left(8e^{-1} \right)^{\frac{1}{2}} \left(\frac{16\pi^4}{2} \delta_{k+1} + \|\bar{R}_\ell\|_{L^1_{t,x}}\right)^{\frac{1}{2}}
\]
\[
\lesssim M_0 \left(8e^{-1} \right)^{\frac{1}{2}} \left(\frac{16\pi^4}{2} + 1 \right)^{\frac{1}{2}} \delta_{k+1},
\]
which yields (4.12).

Moreover, for $N \geq 1$, by (4.8) and the standard Hölder estimate (cf. [10] (129)),
\[
\|a(k)\|_{C^N_{t,x}} \lesssim \sum_{0 \leq N_1 + N_2 + N_3 \leq N} \|\rhoB\|_{C^1_{t,x}} \|\gamma(k)\| \left|C_{(B,\rhoB(0))}\right| \|FB\|_{C^N_{t,x}}
\]
\[
\lesssim \sum_{0 \leq N_1 + N_2 + N_3 \leq N} \ell^{-N_1-N_2-N_3} \left(1 + \|\rhoB^{-1} \bar{R}_\ell\|_{C^1_{t,x}} + \|\rhoB^{-1} \bar{R}_\ell\|_{C^N_{t,x}}^2\right).
\]
Since by (1.13), (2.7) and (4.9),
\[
\|\rhoB^{-1} \bar{R}_\ell\|_{C^N_{t,x}} \lesssim \sum_{0 \leq N_1 + N_2 \leq N_2} \|\rhoB^{-1}\|_{C^1_{t,x}} \|\bar{R}_\ell\|_{C^N_{t,x}}^2
\]
\[
\lesssim \sum_{0 \leq N_1 + N_2 \leq N_2} \ell^{-N_1-N_2} \lesssim \ell^{-N_2-1},
\]
we get
\[
\|a(k)\|_{C^N_{t,x}} \lesssim \sum_{0 \leq N_1 + N_2 + N_3 \leq N} \ell^{-N_1-N_2-N_3} \left(\ell^{-N_2-1} + \ell^{-2N_2}\right) \lesssim \ell^{-4N}.
\]

The $C^1_{t,x}$-estimate of $a(k)$ can be bounded by, via (4.12),
\[
\|a(k)\|_{C^1_{t,x}} \lesssim \frac{1}{\ell} \|\rhoB\|_{C^1_{t,x}} \|FB\|_{C^1_{t,x}} \|\gamma(k)\| \|C_{(B,\rhoB(0))}\| \lesssim \ell^{-1}.
\]

Therefore, the estimates in (4.13) are verified. The proof is complete. \hfill \Box

4.1.2. The velocity amplitudes. Below we define the velocity amplitudes. Unlike in the previous magnetic case, because of the strong coupling between the velocity and magnetic fields, a new matrix $\hat{G}^B$ is needed in order to maintain the cancellation between the perturbations and the old stresses.
\[
\hat{G}^B := \sum_{k \in \Lambda_B} a^2(k) \int_{\mathbb{T}^3} W(k) \otimes W(k) - D(k) \otimes D(k) dx.
\]

In view of estimates (4.12) and (4.13), we have that for $N \geq 1$,
\[
\|\hat{G}^B\|_{C^1_{t,x}} \lesssim \ell^{-2}, \quad \|\hat{G}^B\|_{C^N_{t,x}} \lesssim \ell^{-4N-1}, \quad \|\hat{G}^B\|_{L^1_{t,x}} \lesssim \delta_{q+1}.
\]
Set
\[ \rho_u(t, x) := 2\varepsilon_u^{-1} \delta_{q+1} \chi \left( \frac{| \dot{R}_t^u(t, x) + \dot{G}^B(t, x)|}{\delta_{q+1}} \right). \] (4.20)

Then, by (4.1) and (4.2),
\[ \left| \dot{R}_t^u + \dot{G}^B \right| \leq \varepsilon_u. \] (4.21)

We choose the smooth temporal cut-off function \( f_u \) such that
- \( 0 \leq f_u \leq 1 \) on \( \text{supp} \dot{R}_t^u \cup \text{supp} \dot{G}^B \);
- \( \text{supp} f_u \subseteq N_{1}(\text{supp} \dot{R}_t^u \cup \text{supp} \dot{G}^B) \subseteq N_{2}(\text{supp} \dot{R}_t^u \cup \text{supp} \dot{R}_t^B) ;
- \| f_u \|_{C^N} \leq \ell^{-N}, 1 \leq N \leq 4.

The velocity amplitudes are defined by
\[ a_{(k)}(t, x) := a_{k,u}(t, x) = \rho_u^2 f_u(t) \gamma_{(k)} \left( \text{Id} - \frac{\dot{R}_t^u(t, x) + \dot{G}^B(t, x)}{\rho_u(t, x)} \right), \quad k \in \Lambda_u. \] (4.22)

In view of the normalization (3.11), the geometric Lemma 3.2 and the expression (4.22) of \( a_{(k)} \), we get the following algebraic identity:
\[ \sum_{k \in \Lambda_u} a_{(k)}^2 g_{(r)}^2 W_{(k)} \otimes W_{(k)} = \rho_u f_u^2 \text{Id} - \dot{R}_t^u - \dot{G}^B + \sum_{k \in \Lambda_u} a_{(k)}^2 g_{(r)}^2 \mathbb{P} \neq 0 (W_{(k)} \otimes W_{(k)}) \]
\[ + \sum_{k \in \Lambda_u} a_{(k)}^2 \left( g_{(r)}^2 - 1 \right) \int_{T^3} W_{(k)} \otimes W_{(k)} dx. \] (4.23)

Moreover, similarly to (4.5)-(4.9), we have for any \( p \in [1, +\infty) \),
\[ \rho_u \geq \varepsilon_u^{-1} \delta_{q+1}, \] (4.24)
\[ \| \rho_u \|_{L^p_{t,x}} \leq 8 \varepsilon_u^{-1} \left( (16\pi^2)^{\frac{3}{2}} \delta_{q+1} + \| \dot{R}_t^u + \dot{G}^B \|_{L^p_{t,x}} \right), \] (4.25)
and for \( 1 \leq N \leq 4 \),
\[ \| \rho_u \|_{C^1_{t,x}} \ll \ell^{-2}, \quad \| \rho_u \|_{C^N_{t,x}} \ll \ell^{-N} \delta_{q+1}^{-N+1}, \] (4.26)
\[ \| \rho_u^{1/2} \|_{C^1_{t,x}} \ll \ell^{-1}, \quad \| \rho_u^{1/2} \|_{C^N_{t,x}} \ll \ell^{-N} \delta_{q+1}^{-2N}, \] (4.27)
\[ \| \rho_u^{-1} \|_{C^1_{t,x}} \ll \delta_{q+1}^{-1}, \quad \| \rho_u^{-1} \|_{C^N_{t,x}} \ll \ell^{-N} \delta_{q+1}^{-2N}. \] (4.28)

Arguing as in the proof of Lemma 4.1, using the standard Hölder estimate (cf. [10, (129)]) and estimates (4.14), (4.19), (4.24)-(4.28) we have the following analytic estimates for the velocity amplitude functions.

**Lemma 4.2 (Estimates of velocity amplitudes).** For \( 1 \leq N \leq 4 \), \( k \in \Lambda_u \), we have
\[ \| a_{(k)} \|_{L^2_{t,x}} \ll \delta_{q+1}^\frac{3}{2}, \] (4.29)
\[ \| a_{(k)} \|_{C^1_{t,x}} \ll \ell^{-1}, \quad \| a_{(k)} \|_{C^N_{t,x}} \ll \ell^{-N}. \] (4.30)

### 4.2. Principal parts of perturbations
We define the principal parts \( u_{q+1}^{(p)} \) and \( d_{q+1}^{(p)} \), respectively, of the velocity and magnetic perturbations by
\[ u_{q+1}^{(p)} := \sum_{k \in \Lambda_u \cup \Lambda_B} a_{(k)} g_{(r)} W_{(k)}, \] (4.31a)
\[ d_{q+1}^{(p)} := \sum_{k \in \Lambda_B} a_{(k)} g_{(r)} D_{(k)}. \] (4.31b)
By the orthogonality $W_{(k)} \otimes D_{(k^\prime)} = 0$, $k \neq k^\prime$, and the algebraic identity (4.11),
\[
\begin{align*}
d_{q+1}^{(p)} & \otimes w_{q+1}^{(p)} + w_{q+1}^{(p)} \otimes d_{q+1}^{(p)} + \hat{R}_\ell^B \\
& = \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(\ell)}^2 (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) + \hat{R}_\ell^R \\
& = \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(\ell)}^2 \mathbb{P} \neq 0 (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \\
& \quad + \sum_{k \in \Lambda_B} \sum_{\ell = 1}^2 a_{(k)}^2 (g_{(\ell)}^2 - 1) \int_{\mathbb{T}^3} D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)} \, dx. \quad (4.32)
\end{align*}
\]

Moreover, for the nonlinearity in the velocity equation, by (4.18) and (4.23),
\[
\begin{align*}
w_{q+1}^{(p)} & \otimes w_{q+1}^{(p)} - d_{q+1}^{(p)} \otimes d_{q+1}^{(p)} + \hat{R}_\ell^{u} \\
& = \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(\ell)}^2 W_{(k)} \otimes W_{(k)} + \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(\ell)}^2 (W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)}) + \hat{R}_\ell^{R} \\
& = \rho_u a_{(k)}^2 Id + \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(\ell)}^2 \mathbb{P} \neq 0 (W_{(k)} \otimes W_{(k)}) \\
& \quad + \sum_{k \in \Lambda_B} \sum_{\ell = 1}^2 a_{(k)}^2 (g_{(\ell)}^2 - 1) \int_{\mathbb{T}^3} W_{(k)} \otimes W_{(k)} \, dx \\
& \quad + \sum_{k \in \Lambda_B} \sum_{\ell = 1}^2 a_{(k)}^2 (g_{(\ell)}^2 - 1) \int_{\mathbb{T}^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} \, dx. \quad (4.33)
\end{align*}
\]

**Remark 4.3.** The key fact here is that, module the high frequency part, the nonlinearities of the principal parts cancel with the old magnetic stresses $\hat{R}_\ell^B$ and $\hat{R}_\ell^R$, which enables to decrease the amplitudes of the old stresses and to yield the inductive estimate (1.14).

### 4.3. Incompressibility correctors

Because the amplitude functions $\{a_{(k)}, k \in \Lambda_u \cup \Lambda_B\}$ depend on the space, the principal parts of perturbations are not divergence free. This leads to the introduction of the incompressibility correctors
\[
\begin{align*}
w_{q+1}^{(c)} & := \sum_{k \in \Lambda_u \cup \Lambda_B} g_{(\ell)} \left( \nabla a_{(k)} \times W_{(k)}^c + \nabla a_{(k)} \times \nabla W_{(k)}^c + a_{(k)} \nabla W_{(k)}^c \right), \quad (4.34a) \\
d_{q+1}^{(c)} & := \sum_{k \in \Lambda_B} g_{(\ell)} \left( \nabla a_{(k)} \times D_{(k)}^c + \nabla a_{(k)} \times \nabla D_{(k)}^c + a_{(k)} \nabla D_{(k)}^c \right), \quad (4.34b)
\end{align*}
\]

where $W_{(k)}^c$ and $\nabla W_{(k)}^c$ are given by (3.19) and (3.17), respectively, and $D_{(k)}^c$ and $\nabla D_{(k)}^c$ are as in (3.21). Then, it follows from (3.18) and (3.22) that
\[
\begin{align*}
w_{q+1}^{(p)} + w_{q+1}^{(c)} & = \text{curl curl} \left( \sum_{k \in \Lambda_u \cup \Lambda_B} a_{(k)} g_{(\ell)} W_{(k)}^c \right), \quad (4.35a) \\
d_{q+1}^{(p)} + d_{q+1}^{(c)} & = \text{curl curl} \left( \sum_{k \in \Lambda_B} a_{(k)} g_{(\ell)} D_{(k)}^c \right). \quad (4.35b)
\end{align*}
\]

In particular,
\[
\text{div} (w_{q+1}^{(p)} + w_{q+1}^{(c)}) = \text{div} (d_{q+1}^{(p)} + d_{q+1}^{(c)}) = 0, \quad (4.36)
\]

which justifies the definition of the incompressibility correctors.
4.4. Temporal correctors. Two new types of temporal correctors will be introduced in order to balance the high spatial and temporal oscillations in (4.32) and (4.33).

4.4.1. Temporal correctors to balance spatial oscillations. In order to balance the high spatial oscillations in (4.32) and (4.33) caused by the spatial concentration function \( \psi(k_1) \), we introduce the temporal correctors \( w_{q+1}^{(t)} \) and \( d_{q+1}^{(t)} \), defined by

\[
    w_{q+1}^{(t)} := -\mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_H \mathbb{P}_{\neq 0} \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_1)} \right),
\]

\[
    d_{q+1}^{(t)} := -\mu^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_H \mathbb{P}_{\neq 0} \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_2)} \right),
\]

where \( \mathbb{P}_H \) denotes the Helmholtz-Leray projector, i.e., \( \mathbb{P}_H = \text{Id} - \nabla \Delta^{-1} \text{div} \).

Then, by the Leibniz rule, (3.15) and (3.16),

\[
    \partial_t w_{q+1}^{(t)} + \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_{\neq 0} \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_1)} \right)
    = -\mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_H \mathbb{P}_{\neq 0} \partial_t \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_1)} \right)
    + \mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_{\neq 0} \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k)} \phi_{(k_1)} \right)
    = (\nabla \Delta^{-1} \text{div}) \mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_{\neq 0} \partial_t \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_1)} \right),
\]

and

\[
    \partial_t d_{q+1}^{(t)} + \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_2)} \right)
    = (\nabla \Delta^{-1} \text{div}) \mu^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \partial_t \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_2)} \phi_{(k_1)} \right)
    - \mu^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \partial_t \left( a_k^2 \partial_2^2 \psi_{(k_1)}^2 \phi_{(k)}^2 \phi_{(k_2)} \phi_{(k_1)} \right).
\]

Remark 4.4. The important fact here is that, the first terms on the R.H.S. of (4.38) and (4.39) can be treated as the pressures and can be removed by using the Helmholtz-Leray projector, while the remaining terms are of low spatial oscillations and thus can be controlled by the large parameter \( \mu \). See (5.39) and (5.51) below.

4.4.2. Temporal correctors to balance temporal oscillations. Another new type of temporal correctors is introduced below, in order to balance the high temporal oscillations in (4.32) and (4.33) caused by the temporal concentration function \( g(t) \),

\[
    w_{q+1}^{(o)} := -\sigma^{-1} \sum_{k \in \Lambda_u} \mathbb{P}_H \mathbb{P}_{\neq 0} \left( \mathbb{h} \int_{T^3} W_k \otimes W_k \text{d}x \nabla \left( a_k^2 \mathbb{h} \right) \right)
    - \sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_H \mathbb{P}_{\neq 0} \left( \mathbb{h} \int_{T^3} W_k \otimes W_k - D_k \otimes D_k \text{d}x \nabla \left( a_k^2 \mathbb{h} \right) \right),
\]

\[
    d_{q+1}^{(o)} := -\sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_H \mathbb{P}_{\neq 0} \left( \mathbb{h} \int_{T^3} D_k \otimes W_k - W_k \otimes W_k \text{d}x \nabla \left( a_k^2 \mathbb{h} \right) \right),
\]
where we recall that \( h_{(\tau)} \) is given by (3.32).

Then, by the Leibniz rule and the fact that \( \frac{d}{dt}h_{(\tau)} = \sigma(g_{(\tau)}^2 - 1) \),

\[
\partial_{t}w_{q+1}^{(o)} + \sum_{k \in \Lambda_{u}} \mathbb{P}_{\neq 0} \left( (g_{(\tau)}^2 - 1) \int_{T^3} W_{(k)} \otimes W_{(k)} dx \nabla(a_{(k)}^2) \right) \\
+ \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \left( (g_{(\tau)}^2 - 1) \int_{T^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} dx \nabla(a_{(k)}^2) \right) \\
= \left( \nabla \Delta^{-1} \text{div} \right) \sigma^{-1} \sum_{k \in \Lambda_{u}} \mathbb{P}_{\neq 0} \partial_{t} \left( h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} dx \nabla(a_{(k)}^2) \right) \\
+ \left( \nabla \Delta^{-1} \text{div} \right) \sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \partial_{t} \left( h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} dx \nabla(a_{(k)}^2) \right) \\
- \sigma^{-1} \sum_{k \in \Lambda_{u}} \mathbb{P}_{\neq 0} \left( h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} dx \partial_{t} \nabla(a_{(k)}^2) \right) \\
- \sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \left( h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} dx \partial_{t} \nabla(a_{(k)}^2) \right),
\]

and

\[
\partial_{t}q_{q+1}^{(o)} + \sum_{k \in \Lambda_{u}} \mathbb{P}_{\neq 0} \left( (g_{(\tau)}^2 - 1) \int_{T^3} D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)} dx \nabla(a_{(k)}^2) \right) \\
= \left( \nabla \Delta^{-1} \text{div} \right) \sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \partial_{t} \left( h_{(\tau)} \int_{T^3} D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)} dx \nabla(a_{(k)}^2) \right) \\
- \sigma^{-1} \sum_{k \in \Lambda_{u}} \mathbb{P}_{\neq 0} \left( h_{(\tau)} \int_{T^3} D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)} dx \partial_{t} \nabla(a_{(k)}^2) \right).
\]

\[\text{Remark 4.5.}\] As in the previous spatial oscillation case in Remark 4.4, the first two terms on the R.H.S. of (4.41) are the harmless pressures, and the remaining terms are of low temporal oscillations which can be controlled by the large parameter \( \sigma \). Similar structure also appear in the formula (4.42).

\[\text{4.5. Velocity and magnetic perturbations.}\] We are now in position to define the velocity and magnetic perturbations \( w_{q+1} \) and \( d_{q+1} \) at level \( q + 1 \):

\[
w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(o)},
\]

\[
d_{q+1} := d_{q+1}^{(p)} + d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}.
\]

Then, the velocity and magnetic fields at level \( q + 1 \) are defined by

\[u_{q+1} := u_{\ell} + w_{q+1},\]

\[B_{q+1} := B_{\ell} + d_{q+1}.
\]

The estimates of velocity and magnetic perturbations are summarized in Lemma 4.6.

\[\text{Lemma 4.6 (Estimates of perturbations).}\] For any \( \rho \in (1, \infty), \gamma \in [1, \infty] \) and integer \( 0 \leq N \leq 4 \), the following estimates hold:

\[
\| \nabla^N w_{q+1}^{(p)} \|_{L^2_t L^\gamma_{x}} + \| \nabla^N d_{q+1}^{(p)} \|_{L^2_t L^\gamma_{x}} \lesssim \ell^{-1} \lambda^N \tau_{1/2}^{\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}},
\]

\[
\| \nabla^N w_{q+1}^{(c)} \|_{L^2_t L^\gamma_{x}} + \| \nabla^N d_{q+1}^{(c)} \|_{L^2_t L^\gamma_{x}} \lesssim \ell^{-1} \lambda^N \tau_{1/2}^{\frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2}},
\]
\[\|\nabla^N w_{q+1}^{(t)} \|_{L^7 L^\infty} + \|\nabla^N d_{q+1}^{(t)} \|_{L^7 L^\infty} \lesssim \ell^{-2}\lambda^N \mu^{-1} r_\perp^{\frac{3}{2} - 1} r_\parallel^{\frac{3}{2} - 1} \tau^{-1 - \gamma}, \tag{4.47}\]

\[\|\nabla^N w_{q+1}^{(c)} \|_{L^7 L^\infty} \lesssim \ell^{-8N - 10\sigma - 1}, \quad \|\nabla^N d_{q+1}^{(c)} \|_{L^7 L^\infty} \lesssim \ell^{-4N - 6\sigma - 1}, \tag{4.48}\]

\[\|w_{q+1}^{(p)}\|_{C_{1,x}^{N_1}} + \|w_{q+1}^{(c)}\|_{C_{1,x}^{N_1}} + \|w_{q+1}^{(o)}\|_{C_{1,x}^{N_1}} + \|w_{q+1}^{(o)}\|_{C_{1,x}^{N_1}} \lesssim \lambda^{\frac{5N}{2} + 3}, \tag{4.49}\]

\[\|d_{q+1}^{(p)}\|_{C_{1,x}^{N_1}} + \|d_{q+1}^{(c)}\|_{C_{1,x}^{N_1}} + \|d_{q+1}^{(o)}\|_{C_{1,x}^{N_1}} + \|d_{q+1}^{(o)}\|_{C_{1,x}^{N_1}} \lesssim \lambda^{\frac{5N}{2} + 3}. \tag{4.50}\]

**Proof.** By Lemmas 3.3, 3.4, 4.1 and 4.2 and \(\ell^{-6N_1} \lesssim \lambda^{N_1}\), for any \(\rho \in (1, +\infty)\),

\[\|\nabla^N u_{q+1}^{(p)}\|_{L^7 L^\infty} + \|\nabla^N d_{q+1}^{(p)}\|_{L^7 L^\infty} \lesssim \sum_{k \in \Lambda_0, \Lambda_B} \sum_{N_1 + N_2 = N} \|a(k)\|_{C_{1,x}^{N_1}} \|g(\tau)\|_{L^7} \|\nabla^N W_{(k)}\|_{C_{1,x} L^\infty} \right) L^7 L^\infty \right) \right. \right]

\[\lesssim \sum_{k \in \Lambda_0, \Lambda_B} g(\tau) \|\nabla^N \left( \text{curl} (\nabla a(k) \times W_{(k)}) + \nabla a(k) \times \text{curl} W_{(k)} + a(k) \overline{W}_{(k)} \right) \right) \right. \right] \right. \right]

\[\lesssim \sum_{k \in \Lambda_0, \Lambda_B} \sum_{N_1 + N_2 = N} \left( \|a(k)\|_{C_{1,x}^{N_1 + 2}} \|\nabla^N W_{(k)}\|_{C_{1,x} L^\infty} + \|a(k)\|_{C_{1,x}^{N_1}} \|\nabla^N \overline{W}_{(k)}\|_{C_{1,x} L^\infty} \right) \right. \right]

\[\lesssim \left( \ell^{-8N_1 - 17} \rho_\parallel^{\frac{1}{2} - \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} \lambda^{N_2 - 1} + \ell^{-8N_1 - 17} \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} \lambda^{N_2} \right) \tau^{-\frac{1}{2} - \gamma} \right. \right]

\[\lesssim \ell^{-1}\lambda^N \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{1}{2}} \lambda^{N_2} \tau^{-\frac{1}{2} - \frac{1}{2}}, \tag{4.51} \]

which verifies (4.45).

Moreover, using (4.11), (3.25), (4.34a) and Lemmas 4.1 and 4.2 we have

\[\|\nabla^N u_{q+1}^{(c)}\|_{L^7 L^\infty} \lesssim \sum_{k \in \Lambda_0, \Lambda_B} \sum_{N_1 + N_2 = N} \left( \|a(k)\|_{C_{1,x}^{N_1 + 2}} \|\nabla^N W_{(k)}\|_{C_{1,x} L^\infty} + \|a(k)\|_{C_{1,x}^{N_1}} \|\nabla^N \overline{W}_{(k)}\|_{C_{1,x} L^\infty} \right) \right. \right]

\[\lesssim \left( \ell^{-8N_1 - 17} \rho_\parallel^{\frac{1}{2} - \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} \lambda^{N_2 - 1} + \ell^{-8N_1 - 17} \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} \lambda^{N_2} \right) \tau^{-\frac{1}{2} - \gamma} \right. \right]

\[\lesssim \ell^{-1}\lambda^N \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{1}{2}} \lambda^{N_2} \tau^{-\frac{1}{2} - \frac{1}{2}}, \tag{4.52} \]

Similarly, by (3.26), (4.13) and (4.34b),

\[\|\nabla^N d_{q+1}^{(c)}\|_{L^7 L^\infty} \lesssim \sum_{k \in \Lambda_0, \Lambda_B} \sum_{N_1 + N_2 = N} \left( \|a(k)\|_{C_{1,x}^{N_1 + 2}} \|\nabla^N D_{(k)}\|_{C_{1,x} L^\infty} + \|a(k)\|_{C_{1,x}^{N_1}} \|\nabla^N \overline{D}_{(k)}\|_{C_{1,x} L^\infty} \right) \right. \right]

\[\lesssim \left( \ell^{-4N_1 - 9} \rho_\parallel^{\frac{1}{2} - \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} + \ell^{-4N_1 - 9} \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} \right) \tau^{-\frac{1}{2} - \frac{1}{2}} \right. \right]

\[\lesssim \ell^{-1}\lambda^N \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{1}{2}} \lambda^{N_2} \tau^{-\frac{1}{2} - \frac{1}{2}}, \tag{4.53} \]

Thus, we obtain (4.46).

Regarding the temporal correctors, by (4.37a), Lemmas 3.3, 3.4, 4.1 and 4.2 and the boundedness of operators \(P_{\neq 0}\) and \(P_{H}\) in \(L^p\),

\[\|\nabla^N w_{q+1}^{(t)}\|_{L^7 L^\infty} \lesssim \sum_{k \in \Lambda_0, \Lambda_B} \sum_{N_1 + N_2 = N} \left( \|a(k)\|_{C_{1,x}^{N_1 + 2}} \|\nabla^N D_{(k)}\|_{C_{1,x} L^\infty} + \|a(k)\|_{C_{1,x}^{N_1}} \|\nabla^N \overline{D}_{(k)}\|_{C_{1,x} L^\infty} \right) \right. \right]

\[\lesssim \left( \ell^{-4N_1 - 9} \rho_\parallel^{\frac{1}{2} - \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} + \ell^{-4N_1 - 9} \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{3}{2}} \right) \tau^{-\frac{1}{2} - \frac{1}{2}} \right. \right]

\[\lesssim \ell^{-1}\lambda^N \rho_\parallel^{\frac{1}{2} + \frac{1}{2}} \rho_\perp^{-\frac{1}{2} - \frac{1}{2}} \lambda^{N_2} \tau^{-\frac{1}{2} - \frac{1}{2}}, \tag{4.54} \]
where we also used (1.11) and (3.1) in the last step.

which yields (4.47).

Concerning the estimates of temporal correctors \( w_{q+1}^{(o)} \) and \( d_{q+1}^{(o)} \), by (3.34), Lemmas 4.1 and 4.2,

\[
\| \nabla^N d_{q+1}^{(o)} \|_{L^p_t L^r_x} \lesssim \epsilon^{-2} \lambda^N \mu^{-1} r_+^{-\frac{1}{2}} r_-^{-\frac{1}{2}} \tau^{-\frac{1}{2}},
\]

which remains to prove the \( C^N \)-estimates (4.49) and (4.50) of perturbations. By Lemmas 3.3, 3.4, 4.1 and 4.2,

\[
\| w_{q+1}^{(p)} \|_{C^{N}_{t,x}} \lesssim \sum_{k \in \Lambda_B} \| a(k) \|_{C^{N}_{t,x}} \sum_{0 \leq N_1 + N_2 \leq N} \| g(\tau) \|_{C^{N}_{t,x}} \| W(k) \|_{C^{N}_{t,x}}
\]

\[
\lesssim \sum_{0 \leq N_1 + N_2 \leq N} \sum_{N_2 \geq N_1 + 2 N_2 \geq N_2} \epsilon^{-8 N - 1} \lambda^{N_1} \tau^{-N_1 + \frac{1}{2}} r_+^{-\frac{1}{2}} r_-^{-\frac{1}{2}} \lambda^{N_2} \left( \frac{r_+ \lambda}{r_-} \right)^{N_2},
\]

where we also used (1.11) and (3.1) in the last step.

Similarly, we have

\[
\| w_{q+1}^{(c)} \|_{C^{N}_{t,x}} \lesssim \sum_{k \in \Lambda_B} \| g(\tau) \left( \text{curl}(\nabla a(k) \times W^c(k)) + \nabla a(k) \times \text{curl} W^c(k) + a(k) \tilde{W}^c(k) \right) \|_{C^{N}_{t,x}}
\]

\[
\lesssim \sum_{k \in \Lambda_B} \| a(k) \|_{C^{N+2}_{t,x}} \sum_{0 \leq N_1 + N_2 \leq N} \| g(\tau) \|_{C^{N_1}_{t,x}} \left( \| W^c(k) \|_{C^{N_2}_{t,x}} + \| \nabla W^c(k) \|_{C^{N_2}_{t,x}} + \| \tilde{W}^c(k) \|_{C^{N_2}_{t,x}} \right)
\]

\[
\lesssim \sum_{0 \leq N_1 + N_2 \leq N} \epsilon^{-8 N - 1} \lambda^{N_1} \tau^{-N_1 + \frac{1}{2}} r_+^{-\frac{1}{2}} r_-^{-\frac{1}{2}} \left( \frac{r_+ \lambda}{r_-} \right)^{N_2} \left( \epsilon^{-16} \lambda^{-1} + \epsilon^{-16} \frac{r_+}{r_-} \right)
\]

Moreover, by Sobolev’s embedding \( W^{1,6}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3) \) and the boundedness of operators \( \mathbb{P}_H \mathbb{P} \neq 0 \) in the space \( L^6_\epsilon \),

\[
\| w_{q+1}^{(t)} \|_{C^{N}_{t,x}} \lesssim \mu^{-1} \sum_{k \in \Lambda_B} \| a(k) \|_{C^{N}_{t,x}} \| g(\tau) \|_{C^{N}_{t,x}} \| \phi(\tau) \|_{C^{N}_{t,x}} \| W^c(k) \|_{C^{N+1.6}_{t,x}}
\]

\[
\lesssim \mu^{-1} \sum_{k \in \Lambda_B} \sum_{0 \leq N_1 + N_2 \leq N + 1} \| a(k) \|_{C^{N}_{t,x}} \| g(\tau) \|_{C^{N}_{t,x}} \| \phi(\tau) \|_{C^{N}_{t,x}} \| W^c(k) \|_{C^{N+1.6}_{t,x}}
\]

\[
\lesssim \frac{1}{3} \lambda^{\frac{N+3}{2}},
\]
where we also used Lemmas 3.3, 3.4, 4.1 and 4.2 in the last step.

Finally, arguing as above we get
\[ \|w_{q+1}^{(o)}\|_{C_{t,x}^N} \lesssim \sigma^{-1} \sum_{k \in \mathbb{A}_u \cup \mathbb{A}_B} \|h(\tau) \nabla(a_k^2)\|_{C_{t,x}^N W_x^{N+1.6}} \]
\[ \lesssim \sigma^{-1} \sum_{k \in \mathbb{A}_u \cup \mathbb{A}_B} \|h(\tau)\|_{C_{t,x}^{N+1}} \|\nabla(a_k^2)\|_{C_{t,x}^{N+1}} \]
\[ \lesssim \sigma^{N+1} \ell^{-8(N+1)-2} \lambda^2N \lambda^{1-8\epsilon}(N+1)-2 \lambda^{5N-\epsilon} \lesssim \lambda^{N+1}, \quad (4.61) \]
where the last two steps were due to (1.11) and (3.1).

Thus, combining (4.58)–(4.61) altogether and using (3.1) we conclude that
\[ \|w_{q+1}^{(p)}\|_{C_{t,x}^N} + \|w_{q+1}^{(c)}\|_{C_{t,x}^N} + \|w_{q+1}^{(l)}\|_{C_{t,x}^N} + \|w_{q+1}^{(o)}\|_{C_{t,x}^N} \leq \frac{1}{2} \lambda^{\frac{5N}{2}+3}. \quad (4.62) \]
The \(C^N\)-estimates of the magnetic perturbations in (4.50) can be proved similarly. \( \square \)

4.6. Verification of inductive estimates (1.12), (1.15) and (1.16). We are now in position to verify the inductive estimates (1.12), (1.15) and (1.16) for the velocity and magnetic fields.

Note that, the previous estimate (4.45) cannot yield the decay properties required by the inductive estimates (1.15) and (1.16). An important ingredient here is the \(L^p\) principal parts.

Below we verify the iterative estimates for \(u_{q+1}\) and \(B_{q+1}\). For the velocity perturbation \(w_{q+1}\), using (4.43a), (4.64) and Lemma 4.6 we have
\[ \|w_{q+1}\|_{L_{t,x}^2} \lesssim \|w_{q+1}^{(p)}\|_{L_{t,x}^2} + \|w_{q+1}^{(c)}\|_{L_{t,x}^2} + \|w_{q+1}^{(l)}\|_{L_{t,x}^2} + \|w_{q+1}^{(o)}\|_{L_{t,x}^2} \]
\[ \lesssim \frac{1}{2} r_{q+1}^{-1} + \epsilon^{-2} \mu^{-1} r_{q+1}^{-\frac{1}{2}} r_{q+1}^{-\frac{3}{2}} \tau^{-\frac{1}{2}} + \epsilon^{-10} \sigma^{-1} \lesssim \delta_{q+1}^{-1}, \quad (4.66) \]
and
\[ \|w_{q+1}\|_{L_{t,x}^1 L_{t}^2} \lesssim \|w_{q+1}^{(p)}\|_{L_{t,x}^1 L_{t}^2} + \|w_{q+1}^{(c)}\|_{L_{t,x}^1 L_{t}^2} + \|w_{q+1}^{(l)}\|_{L_{t,x}^1 L_{t}^2} + \|w_{q+1}^{(o)}\|_{L_{t,x}^1 L_{t}^2} \]
\[ \lesssim \ell^{-1}T^{-\frac{1}{2}} + \ell^{-1}r_{\parallel}^{-1}T^{-\frac{1}{2}} + \ell^{-2}\mu^{-1}r_{\perp}^{-\frac{1}{2}}r_{\parallel}^{-\frac{1}{2}} + \ell^{-10}\sigma^{-1} \lesssim \lambda_{q+1}^{-\varepsilon}, \]  

(4.67)

where we also used (1.11) in the last steps.

Then, in view of (1.12), (4.49), (4.44a) and the standard mollification estimates, we get for a sufficiently large,

\[ \|u_{q+1}\|_{C_{t,x}^{1}} \lesssim \|u_{\ell}\|_{C_{t,x}^{1}} + \|w_{q+1}\|_{C_{t,x}^{1}} \lesssim \lambda_{q}^{\frac{1}{2}} + \lambda_{q+1}^{\frac{1}{2}} \lesssim \lambda_{q+1}^{\varepsilon}, \]  

(4.68)

\[ \|u_{q}-u_{q+1}\|_{L_{t,x}^{2}} \lesssim \|u_{q}-u_{\ell}\|_{L_{t,x}^{2}} + \|u_{\ell}-u_{q+1}\|_{L_{t,x}^{2}} \lesssim \ell\|u_{q}\|_{L_{t,x}^{2}} + \delta_{q+1}^{\frac{1}{2}} \leq M\delta_{q+1}^{\frac{1}{2}}, \]  

(4.69)

and

\[ \|u_{q}-u_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} \lesssim \|u_{q}-u_{\ell}\|_{C_{t,x}^{\varepsilon}} + \|w_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} \lesssim \ell\lambda_{q}^{\varepsilon} = \delta_{q+2}^{\frac{1}{2}}, \]  

(4.70)

where we also used the inequalities \( \lambda_{q}^{\varepsilon} \ll \delta_{q+1}^{\frac{1}{2}} \) and \( \ell\lambda_{q}^{\varepsilon} = \delta_{q+2}^{\frac{1}{2}} \), due to (1.11) and \( M \) is a fixed large universal constant.

Similarly, for the magnetic perturbation \( d_{q+1} \), by (4.43b), (4.65) and Lemma 4.6,

\[ \|d_{q+1}\|_{L_{t,x}^{2}} \lesssim \|d_{q+1}\|_{L_{t,x}^{2}} + \|d_{q+1}\|_{L_{t,x}^{2}} + \|d_{q+1}\|_{L_{t,x}^{2}} + \|d_{q+1}\|_{L_{t,x}^{2}} \lesssim \delta_{q+1}^{\frac{1}{2}} + \ell^{-1}r_{\perp}^{-1}r_{\parallel}^{-\frac{1}{2}} + \ell^{-2}\mu^{-1}r_{\perp}^{-\frac{1}{2}}r_{\parallel}^{-\frac{1}{2}} + \ell^{-6}\sigma^{-1} \lesssim \delta_{q+1}^{\frac{1}{2}}, \]  

(4.71)

and

\[ \|d_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} \lesssim \|d_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} + \|d_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} + \|d_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} + \|d_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} \lesssim \ell^{-1}r_{\perp}^{-\frac{1}{2}} + \ell^{-1}r_{\perp}^{-1}r_{\parallel}^{-\frac{1}{2}} + \ell^{-2}\mu^{-1}r_{\perp}^{-\frac{1}{2}}r_{\parallel}^{-\frac{1}{2}} + \ell^{-6}\sigma^{-1} \lesssim \lambda_{q+1}^{-\varepsilon}, \]  

(4.72)

which along with (1.12), (4.44b) and Lemma 4.6 yield that

\[ \|B_{q+1}\|_{C_{t,x}^{1}} \lesssim \|B_{q}\|_{C_{t,x}^{1}} + \|w_{q+1}\|_{C_{t,x}^{1}} \lesssim \lambda_{q}^{\frac{1}{2}} + \lambda_{q+1}^{\frac{1}{2}} \lesssim \lambda_{q+1}^{\varepsilon}, \]  

(4.73)

\[ \|B_{q}-B_{q+1}\|_{L_{t,x}^{2}} \lesssim \|B_{q}-B_{\ell}\|_{C_{t,x}^{\varepsilon}} + \|d_{q+1}\|_{L_{t,x}^{2}} \lesssim \ell\|B_{q}\|_{C_{t,x}^{\varepsilon}} + \delta_{q+1}^{\frac{1}{2}} \leq M\delta_{q+1}^{\frac{1}{2}}, \]  

(4.74)

and

\[ \|B_{q}-B_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} \lesssim \|B_{q}-B_{\ell}\|_{C_{t,x}^{\varepsilon}} + \|d_{q+1}\|_{L_{t,x}^{1}L_{t}^{2}} \lesssim \ell\lambda_{q}^{\varepsilon} + \lambda_{q+1}^{\varepsilon} \lesssim \delta_{q+2}^{\frac{1}{2}}. \]  

(4.75)

Therefore, the inductive estimates in (1.12), (1.15) and (1.16) are verified.

5. REYNOLDS AND MAGNETIC STRESSES

The main purpose of this section is to determine the Reynolds and magnetic stresses and to prove the main iteration in Theorem 1.7.

The important roles here are played by the inverse-divergence operators \( \mathcal{R}^{u} \) and \( \mathcal{R}^{B} \), defined by

\[ (\mathcal{R}^{u} v)^{kl} := \partial_{k}\Delta^{-1}v^{l} + \partial_{l}\Delta^{-1}v^{k} - \frac{1}{2}(\delta_{kl} + \partial_{k}\partial_{l}\Delta^{-1})\text{div}\Delta^{-1}v, \]  

(5.1)

\[ (\mathcal{R}^{B} f)_{ij} := \varepsilon_{ijk}(-\Delta)^{-1}(\text{curl}f)_{k}, \]  

(5.2)

where \( \int_{\mathbb{R}^{3}}vdx = 0, \text{div}f = 0, \) and \( \varepsilon_{ijk} \) is the Levi-Civita tensor, \( i, j, k, l \in \{1, 2, 3\} \).
The operator $\mathcal{R}^u$ returns symmetric and trace-free matrices, while the operator $\mathcal{R}^B$ returns skew-symmetric matrices. Moreover, one has the algebraic identities

$$\text{div}\mathcal{R}^u(v) = v, \quad \text{div}\mathcal{R}^B(f) = f.$$ 

Both $|\nabla|\mathcal{R}^u$ and $|\nabla|\mathcal{R}^B$ are Calderon-Zygmund operators and thus they are bounded in the spaces $L^p$, $1 < p < +\infty$. See [2 30] for more details.

5.1. **Decomposition of magnetic stress.** Using (1.7) with $q+1$ replacing $q$, (2.3), (4.43), and (4.44) we derive the equation for the magnetic stress $\hat{R}_{q+1}^B$:

$$\text{div}\hat{R}_{q+1}^B = \hat{R}_{lin}^B + \hat{R}_{osc}^B + \hat{R}_{cor}^B + \hat{R}_{com}^B,$$

where

$$\hat{R}_{lin}^B := \mathcal{R}^B \left( \partial_t (d_{q+1}^{(p)} + d_{q+1}^{(c)}) + \nu_2 (-\Delta)^{\alpha_2} d_{q+1} \right) + \mathcal{R}^B \rho \mathcal{R}^B (-\Delta)^{\alpha_2} d_{q+1} + \mathcal{R}^B P H \text{div} (d_{q+1} \otimes u_\ell - u_\ell \otimes d_{q+1} + B_\ell \otimes w_{q+1} - w_{q+1} \otimes B_\ell),$$

the oscillation error

$$\hat{R}_{osc}^B := \sum_{k \in \Lambda_B} \mathcal{R}^B P H P \neq 0 (g_{(r)}^2 \psi_{(k)}^2 0 (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \nabla (a_{(k)}^2))\$$

$$- \mu^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^B P H P \neq 0 (\partial_i (a_{(k)}^2 \psi_{(k)}^2) \nabla (a_{(k)}))\$$

$$- \sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^B P H P \neq 0 \left( h(r) \int_{\mathbb{R}^3} D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)} dx \partial_i \nabla (a_{(k)}^2) \right),$$

the corrector error

$$\hat{R}_{cor}^B := \mathcal{R}^B P H \text{div} \left( d_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(o)}) - (w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(o)}) \otimes d_{q+1} \right)$$

$$+ (d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}) \otimes w_{q+1} - w_{q+1}^{(p)} \otimes (d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}),$$

and the commutator error

$$\hat{R}_{com}^B = \mathcal{R}^B P H \text{div} (B_\ell \otimes u_\ell - u_\ell \otimes B_\ell - (B_q \otimes u_q - u_q \otimes B_q) *_x \phi_\ell *_x \varphi_\ell).$$
Remark 5.1. Since
\[ \mathbb{P}_H \mathbb{P}_H = \mathbb{P}_H = \mathbb{P}_H \mathbb{P}_H, \quad \mathbb{P}_H \text{div} = \mathbb{P}_H \text{div}, \]  
by (4.36) and the definitions (4.37b) and (4.40b),
\[ \text{div}(d^{(p)}_{q+1} + d^{(c)}_{q+1}) = 0, \quad \text{div}(-\Delta)^2 d_{q+1} = 0. \]  
Taking into account \( \text{div} \mathbb{P}_H = 0 \) we see that the terms in (5.5)-(5.8) are in the domain of the operator \( \mathcal{R}^B \).

By (5.10),
\[ d^{(p)}_{q+1} + d^{(c)}_{q+1} = \mathbb{P}_H (d^{(p)}_{q+1} + d^{(c)}_{q+1}), \quad (-\Delta)^2 d_{q+1} = \mathbb{P}_H (-\Delta)^2 d_{q+1}. \]

We also note that, in contrast to the nonlinearities in the velocity equation, the nonlinear terms in the magnetic equation is skew symmetric, which, in particular, yields that
\[ \text{div}(\text{div} \hat{R}^B_{q+1}) = 0. \]
Thus, by virtue of (5.11) and (5.12), we obtain
\[ \text{div} \hat{R}^B_{q+1} = \mathbb{P}_H \text{div} \hat{R}^B_{q+1}. \]

Then, using the algebraic identities \( \text{div} \mathcal{R}^B = \text{Id} \), (4.32), (4.39) and (4.42), together with the facts that \( \text{div} = \text{div} \mathbb{P}_H = \mathbb{P}_H \text{div} \), we obtain that \( \hat{R}^B_{q+1} \) satisfies the relaxation magnetic equation in (1.7) and the following identity holds
\[ \hat{R}^B_{q+1} = \mathcal{R}^B \mathbb{P}_H \text{div} \hat{R}^B_{q+1}. \]

5.2. Decomposition of Reynolds stress. Concerning the Reynolds stress we compute
\[ \text{div} \hat{R}^u_{q+1} - \nabla P_{q+1} = \partial_t (w^{(p)}_{q+1} + w^{(c)}_{q+1}) + \nu_1 (-\Delta)^\alpha \text{div} (u_\ell \otimes w_{q+1} + w_{q+1} \otimes u_\ell - B_\ell \otimes d_{q+1} - d_{q+1} \otimes B_\ell) \]
\[ + \text{div}(w^{(p)}_{q+1} \otimes w^{(p)}_{q+1} - d^{(p)}_{q+1} \otimes d^{(p)}_{q+1} + \hat{R}^u) + \partial_t w^{(t)}_{q+1} + \partial_t w^{(o)}_{q+1} \]
\[ + \text{div}(w^{(c)}_{q+1} + w^{(t)}_{q+1} + w^{(o)}_{q+1} \otimes w_{q+1} + w^{(p)}_{q+1} \otimes (w^{(c)}_{q+1} + w^{(t)}_{q+1} + w^{(o)}_{q+1}) \]
\[ - (d^{(c)}_{q+1} + d^{(t)}_{q+1} + d^{(o)}_{q+1}) \otimes d_{q+1} - d^{(p)}_{q+1} \otimes (d^{(c)}_{q+1} + d^{(t)}_{q+1} + d^{(o)}_{q+1}) \]
\[ + \text{div} (u_\ell \otimes u_\ell - B_\ell \otimes B_\ell - (u_\ell \otimes u_\ell - B_\ell \otimes B_\ell) *_x \phi_\ell *_l \varphi_\ell) - \nabla P_\ell. \]  

Then, using the inverse divergence operator \( \mathcal{R}^u \) we define the Reynolds stress by
\[ \hat{R}^u_{q+1} := \hat{R}^u_{\text{lin}} + \hat{R}^u_{\text{osc}} + \hat{R}^u_{\text{cor}} + \hat{R}^u_{\text{com}}, \]
where
\[ \hat{R}^u_{\text{lin}} := \mathcal{R}^u \left( \partial_t (w^{(p)}_{q+1} + w^{(c)}_{q+1}) + \nu_1 \mathcal{R}^u (-\Delta)^\alpha w_{q+1} \right) \]
\[ + \mathcal{R}^u \mathbb{P}_H \text{div} (u_\ell \otimes w_{q+1} + w_{q+1} \otimes u_\ell - B_\ell \otimes d_{q+1} - d_{q+1} \otimes B_\ell), \]
the oscillation error
\[ \tilde{R}^u_{\text{osc}} := \sum_{k \in \Lambda_u} \mathcal{R}^u_{\mathcal{P}H} \mathcal{P} \neq 0 \left( g^2_{(r)} \mathcal{P} \neq 0 (W_{(k)} \otimes W_{(k)}) \nabla (a^2_{(k)}) \right) \]
\[ + \sum_{k \in \Lambda_B} \mathcal{R}^u_{\mathcal{P}H} \mathcal{P} \neq 0 \left( g^2_{(r)} \mathcal{P} \neq 0 (W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)}) \nabla (a^2_{(k)}) \right) \]
\[ - \mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathcal{R}^u_{\mathcal{P}H} \mathcal{P} \neq 0 \left( \partial_t (a^2_{(k)} g_{(r)}(x) \psi^2_{(k)}(x) \phi^2_{(k)}(x)) \right) \]
\[ - \sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^u_{\mathcal{P}H} \mathcal{P} \neq 0 \left( h_{(r)} \int_{T^3} W_{(k)} \otimes W_{(k)} dx \partial_t \nabla (a^2_{(k)}) \right) \), \quad (5.18) \]
the corrector error
\[ \tilde{R}^u_{\text{cor}} := \mathcal{R}^u_{\mathcal{P}H} \text{div} \left( u^{(p)}_{q+1} \hat{\otimes} (w^{(c)}_{q+1} + w^{(o)}_{q+1}) + (w^{(c)}_{q+1} + w^{(o)}_{q+1}) \hat{\otimes} w_{q+1} \right) \]
\[- d^{(p)}_{q+1} \hat{\otimes} (d^{(c)}_{q+1} + d^{(o)}_{q+1}) - (d^{(c)}_{q+1} + d^{(o)}_{q+1}) \) \), \quad (5.19) \]
and the commutator error
\[ \tilde{R}^u_{\text{com}} := \mathcal{R}^u_{\mathcal{P}H} \text{div} \left( B \hat{\otimes} u_{\ell} - u_{\ell} \hat{\otimes} B \hat{\otimes} B_q \hat{\otimes} u_q - u_q \hat{\otimes} B \right) \), \quad (5.20) \]
Remark 5.2. We infer from (5.9) that the above terms in (5.17)-(5.20) are in the domain of the operator \( \mathcal{R}^u \).

As in (5.10), by (4.36), the definitions (4.37a) and (4.40a), and div \( \mathcal{P} H = 0 \),
\[ \text{div}(u^{(p)}_{q+1} + w^{(c)}_{q+1}) = 0, \quad \text{div}(w^{(o)}_{q+1} + w^{(o)}_{q+1}) = 0, \quad (5.21) \]
which yield that
\[ u^{(p)}_{q+1} + w^{(c)}_{q+1} = \mathcal{P} H (u^{(p)}_{q+1} + w^{(c)}_{q+1}), \quad (-\Delta)^{a_1} w_{q+1} = \mathcal{P} H (-\Delta)^{a_1} w_{q+1}. \quad (5.22) \]
Then, using the algebraic identities div \( \mathcal{R}^u = \text{Id} \), (4.33), (4.38) and (4.41) we obtain
\[ \text{div} \tilde{R}^u_{q+1} = \mathcal{P} H \left( \partial_t u_q + \nu (-\Delta)^{a_1} u_q + \text{div}(u_q \otimes u_q - B_q \otimes B_q) \right), \quad (5.23) \]
which yields that \( \tilde{R}^u_{q+1} \) satisfies the relaxation velocity equation in (1.7).
Moreover, since \( \mathcal{P}^2 = \mathcal{P} H \), it also holds that
\[ \tilde{R}^u_{q+1} = \mathcal{R}^u_{\mathcal{P}H} \text{div} \tilde{R}^u_{q+1}. \quad (5.24) \]
5.3. Estimates of magnetic stress. The purpose of this subsection is to estimate the magnetic stress \( \tilde{R}_{q+1} \) given by (5.4) at level \( q + 1 \).
Since the Calderón-Zygmund operators are bounded in the space \( L^p_\mu \) for any \( 1 < p < +\infty \), we choose
\[ p := \frac{2 - 8\varepsilon}{2 - 9\varepsilon} \in (1, 2), \]
where \( \varepsilon \) is given by (1.8). In particular,
\[ (1 - 4\varepsilon)(1 - \frac{1}{p}) = \frac{\varepsilon}{2}, \quad (5.25) \]
which, via (3.1), yields that
\[ r^{\frac{p}{2} - 1} r^{\frac{p}{2} - 1} = \lambda^p, \quad r^{\frac{p}{2} - 1} r^{\frac{p}{2} - 1} = \lambda^{-1 + 5\varepsilon}. \quad (5.26) \]
Let us estimate the four parts in the decomposition of \( \tilde{R}_{q+1}^B \) separately.

**Linear error.** Note that, by (4.35b),
\[
\| R^B \partial_t (d^{(p)}_{q+1} + d^{(c)}_{q+1}) \|_{L^1_t L^p_x} \\
\lesssim \sum_{k \in \Delta_B} \| R^B \text{curl} \partial_t (g(\tau) a(k) D^{e}_k) \|_{L^1_t L^p_x} \\
\lesssim \sum_{k \in \Delta_B} \| \text{curl} \partial_t (g(\tau) a(k) D^{e}_k) \|_{L^1_t L^p_x} \\
\lesssim \sum_{k \in \Delta_B} \left( \| g(\tau) \|_{L^1_t} \| a(k) \|_{C^{2} \times} \| D^{e}_k \|_{C^{1} W^{1, p}_{x}^{-}} + \| a(k) \|_{C^{1} \times} \| \partial_t D^{e}_k \|_{C^{1} W^{1, p}_{x}^{-}} \right) \\
+ \| \partial_t g(\tau) \|_{L^1_t} \| a(k) \|_{C^{1} \times} \| D^{e}_k \|_{C^{1} W^{1, p}_{x}^{-}} \right). \tag{5.27}
\]

Then, by Lemmas 3.3 and 4.1 (1.8), (3.1) and (5.26),
\[
\| R^B \partial_t (d^{(p)}_{q+1} + d^{(c)}_{q+1}) \|_{L^1_t L^p_x} \\
\lesssim T^{-\frac{1}{2}} (\ell^{-8} r^{\frac{1}{2}} r^{\frac{1}{2}} - \ell^{-4} r^{\frac{1}{2}} + \sigma T^{\frac{1}{2}} r^{\frac{1}{2}} + \mu) + \ell^{-6} r^{\frac{1}{2}} + \ell^{-6} r^{\frac{1}{2}} + \ell^{-6} r^{\frac{1}{2}} - \ell^{-6} \lambda - 2 \varepsilon \lesssim \ell^{-6} r^{\frac{1}{2}} - \ell^{-6} \lambda - 2 \varepsilon. \tag{5.28}
\]

The control of the fractional viscosity \(-\Delta)^{\alpha_2}\) requires both the temporal and spatial intermittency of the magnetic flows. More precisely, by Lemma 4.6 for \( \alpha_2 \in [0, 1/2] \),
\[
\| R^B (-\Delta)^{\alpha_2} d_{q+1} \|_{L^1_t L^p_x} \lesssim \| d_{q+1} \|_{L^1_t L^p_x} \\
\lesssim \ell^{-1} r^{\frac{3}{2} - 2} r^{\frac{1}{2}} r^{\frac{1}{2}} - r^{\frac{1}{2}} - \ell^{-6} \sigma^{-1} \\
\lesssim \ell^{-6} r^{-2} \varepsilon - \ell^{6} r^{-2} \varepsilon. \tag{5.29}
\]

For the stronger viscosity \(-\Delta)^{\alpha_2}\) with \( \alpha_2 \in (1/2, 5/4) \), by (4.43b),
\[
\| R^B (-\Delta)^{\alpha_2} d_{q+1} \|_{L^1_t L^p_x} \lesssim \| R^B (-\Delta)^{\alpha_2} d^{(p)}_{q+1} \|_{L^1_t L^p_x} + \| R^B (-\Delta)^{\alpha_2} d^{(c)}_{q+1} \|_{L^1_t L^p_x} \\
+ \| R^B (-\Delta)^{\alpha_2} d^{(o)}_{q+1} \|_{L^1_t L^p_x} + \| R^B (-\Delta)^{\alpha_2} d^{(o)}_{q+1} \|_{L^1_t L^p_x}. \tag{5.30}
\]

In order to control the R.H.S. of (5.30), using the interpolation inequality (cf. 7) and (4.45),
\[
\| R^B (-\Delta)^{\alpha_2} d^{(p)}_{q+1} \|_{L^1_t L^p_x} \lesssim \| \nabla \|^{2 \alpha_2 - 1} d^{(p)}_{q+1} \|_{L^1_t L^p_x} \\
\lesssim \| d^{(p)}_{q+1} \|_{L^1_t L^p_x}^{\frac{2 \alpha_2 - 1}{2 \alpha_2 - 1}} d^{(p)}_{q+1} \|_{L^1_t W^{2, p}_{x}} \\
\lesssim \ell^{-1} \lambda^{2 \alpha_2 - 1} r^{\frac{1}{2}} r^{\frac{1}{2}} - r^{\frac{1}{2}} - r^{\frac{1}{2}}. \tag{5.31}
\]

Similarly, by Lemma 4.6,
\[
\| R^B (-\Delta)^{\alpha_2} d^{(c)}_{q+1} \|_{L^1_t L^p_x} \lesssim \ell^{-1} \lambda^{2 \alpha_2 - 1} r^{\frac{1}{2}} r^{\frac{1}{2}} - r^{\frac{1}{2}} - r^{\frac{1}{2}}. \tag{5.32}
\]
\[
\| R^B (-\Delta)^{\alpha_2} d^{(o)}_{q+1} \|_{L^1_t L^p_x} \lesssim \ell^{-2} \lambda^{2 \alpha_2 - 1} \mu^{-1} r^{\frac{1}{2}} r^{\frac{1}{2}} - r^{\frac{1}{2}}. \tag{5.33}
\]
\[
\| R^B (-\Delta)^{\alpha_2} d^{(o)}_{q+1} \|_{L^1_t L^p_x} \lesssim \ell^{-1} \sigma^{-1}. \tag{5.34}
\]

Hence, we conclude from (5.30)-(5.34) and the fact that \( 2 \alpha_2 - 1 \leq 3/2 - 10 \varepsilon \) that
\[
\| R^B (-\Delta)^{\alpha_2} d_{q+1} \|_{L^1_t L^p_x} \lesssim \ell^{-1} \lambda^{2 \alpha_2 - 1} r^{\frac{1}{2}} r^{\frac{1}{2}} - r^{\frac{1}{2}} - r^{\frac{1}{2}} + \ell^{-1} \lambda^{2 \alpha_2 - 1} r^{\frac{1}{2}} r^{\frac{1}{2}} - r^{\frac{1}{2}} - r^{\frac{1}{2}} \]
\[
+ \ell^{-2} \lambda^{2 \alpha_2 - 1} \mu^{-1} r^{\frac{1}{2}} r^{\frac{1}{2}} - r^{\frac{1}{2}} + \ell^{-1} \sigma^{-1}.
\]
\begin{align}
\lesssim \ell^{-1} \lambda^{-\varepsilon}.
\end{align}

Regarding the nonlinear terms in (5.5), estimating as in (5.29) we have
\begin{align}
\| \mathcal{R}^B P_H \text{div} \left( d_{q \perp} \otimes u_t - u_t \otimes d_{q \perp} + B_t \otimes w_{q \perp} - w_{q \perp} \otimes B_t \right) \|_{L^1_t L^2_x} \\
\lesssim \| d_{q \perp} \otimes u_t - u_t \otimes d_{q \perp} + B_t \otimes w_{q \perp} - w_{q \perp} \otimes B_t \|_{L^1_t L^2_x} \\
\lesssim \| u_t \|_{C_{t,x}} \| d_{q \perp} \|_{L^1_t L^2_x} + \| B_t \|_{C_{t,x}} \| w_{q \perp} \|_{L^1_t L^2_x} \\
\lesssim \ell^{-\frac{1}{2}} \| \tau^{-\frac{1}{2}} - \frac{1}{2} \tau^{-\frac{1}{2}} + \varepsilon^{-6} \sigma^{-1} + \varepsilon^{-10} \sigma^{-1} \| \lesssim \ell^{-10} \lambda^{-2\varepsilon}.
\end{align}

Therefore, combining (5.28), (5.29), (5.35), (5.36) together and using (1.11) and the fact that $2\alpha_2 - 1 \leq 3/2 - 10\varepsilon$ we arrive at
\begin{align}
\| \hat{R}^B_{\text{osc}} \|_{L^1_t L^2_x} \lesssim \varepsilon^{-8} \lambda^{-2\varepsilon} + \varepsilon^{-1} \lambda^{-\varepsilon} + \varepsilon^{-10} \lambda^{-2\varepsilon} \lesssim \varepsilon^{-1} \lambda^{-\varepsilon}.
\end{align}

**Oscillation error.** In order to treat the delicate magnetic oscillations, we decompose
\begin{align}
\hat{R}^B_{\text{osc}} = \hat{R}^B_{\text{osc,1}} + \hat{R}^B_{\text{osc,2}} + \hat{R}^B_{\text{osc,3}},
\end{align}
where $\hat{R}^B_{\text{osc,1}}$ contains the low-high spatial oscillations
\begin{align}
\hat{R}^B_{\text{osc,1}} := \sum_{k \in \Lambda_B} \mathcal{R}^B P_H P_{\neq 0} \left( g^2 \right) P_{\neq 0} (D(k) \otimes W(k) - W(k) \otimes D(k)) \nabla (a^2(k)) ,
\end{align}
$\hat{R}^B_{\text{osc,2}}$ contains the high temporal oscillation
\begin{align}
\hat{R}^B_{\text{osc,2}} := -\mu^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^B P_H P_{\neq 0} \left( \partial_t (a^2(k) g^2) \psi^2(k) \phi^2(k) \right) ,
\end{align}
and $\hat{R}^B_{\text{osc,3}}$ is of low frequency
\begin{align}
\hat{R}^B_{\text{osc,3}} := -\sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^B P_H P_{\neq 0} \left( h(r) \int_{\mathbb{T}^3} D(k) \otimes W(k) - W(k) \otimes D(k) dx \partial_t \nabla (a^2(k)) \right) .
\end{align}

In order to estimate $\hat{R}^B_{\text{osc,1}}$, the key fact is that, the velocity and magnetic flows are of high oscillations
\begin{align}
P_{\neq 0} (D(k) \otimes D(k) - W(k) \otimes W(k)) = P_{\geq \lambda r \perp} (D(k) \otimes D(k) - W(k) \otimes W(k)),
\end{align}
while the amplitude function $a(k)$ is slowly varying. Hence, intuitively, the frequency of $\hat{R}^B_{\text{osc,1}}$ concentrates at the high mode, and thus the inverse-divergence operator $\mathcal{R}^B$ permits to gain a small factor $(\lambda r \perp)^{-1}$. This is the content of Lemma 5.3 below.

**Lemma 5.3** (53, Lemma 6; see also 13, Lemma B.1). Let $a \in C^2 (\mathbb{T}^3)$. For all $1 < p < +\infty$ we have
\begin{align}
\| | \nabla |^{-1} P_{\neq 0} (a P_{\geq k} f) \|_{L^p (\mathbb{T}^3)} \lesssim k^{-1} \| \nabla a \|_{L^\infty (\mathbb{T}^3)} \| f \|_{L^p (\mathbb{T}^3)},
\end{align}
holds for any smooth function $f \in L^p (\mathbb{T}^3)$.

We apply Lemma 5.3 with $a = \nabla (a^2(k))$ and $f = \psi^2(k) \phi^2(k)$ to get
\begin{align}
\| \hat{R}^B_{\text{osc,1}} \|_{L^1_t L^2_x} \lesssim \sum_{k \in \Lambda_B} \| g^2 \|_{L^2_x} \| | \nabla |^{-1} P_{\neq 0} (P_{\geq \lambda r \perp} (D(k) \otimes W(k) - W(k) \otimes D(k)) \nabla (a^2(k))) \|_{C_t L^2_x} \\
\lesssim \sum_{k \in \Lambda_B} \lambda^{-1} \rho^{-1} \| \nabla^3 (a^2(k)) \|_{C_{t,x}} \| \psi^2(k) \phi^2(k) \|_{C_t L^2_x} \\
\lesssim \sum_{k \in \Lambda_B} \ell^{-14} \lambda^{-1} \rho^{-1} \| \psi^2(k) \|_{C_t L^2_x} \| \phi^2(k) \|_{C_t L^2_x}.
\end{align}
where we also used Lemmas 3.3 and 4.1 in the third step.

Regarding $\hat{R}_{\text{osc},2}$, as mentioned in Remark 4.4, the high temporal oscillations arising from $g_{(\tau)}$ can be balanced by the large parameter $\mu$, namely, we have

$$\|\hat{R}_{\text{osc},2}\|_{L_t^1 L_x^p} \lesssim \mu^{-1} \sum_{k \in \Lambda_b} \|\nabla^{-1} \Psi_{H_0 \neq 0} (\partial_t (a_{(k)}^2 g_{(\tau)}) \psi_{(k)}^2 \phi_{(k)}^2)\|_{L_t^1 L_x^p}$$

$$\lesssim \mu^{-1} \sum_{k \in \Lambda_b} \left(\|\partial_t (a_{(k)}^2)\|_{C_t \times x} \|g_{(\tau)}^2\|_{L_t^1} + \|a_{(k)}\|_{C_t \times x} \|\partial_t (g_{(\tau)}^2)\|_{L_t^1}\right) \|\psi_{(k)}\|_{L_t^1 L_x^p}^2 \|\phi_{(k)}\|_{L_t^1 L_x^p}^2$$

$$\lesssim \left(\epsilon^{-4} + \epsilon^{-2} \tau \sigma \mu^{-1} r_{\perp}^{1/4} r_{\parallel}^{-1/4}\right) \lesssim \epsilon^{-4} \tau \sigma \mu^{-1} r_{\perp}^{1/4} r_{\parallel}^{-1/4}.$$  \hspace{1cm} (5.39)

Finally, the low frequency part $\hat{R}_{\text{osc},3}$ can be estimated easily by (3.34) and (4.13),

$$\|\hat{R}_{\text{osc},3}\|_{L_t^1 L_x^p} \lesssim \sigma^{-1} \sum_{k \in \Lambda_B} \|h_{(\tau)} (k_2 \otimes k_1 - k_1 \otimes k_2) \partial_t \nabla (a_{(k)}^2)\|_{L_t^1 L_x^p}$$

$$\lesssim \sigma^{-1} \sum_{k \in \Lambda_B} \|h_{(\tau)} c_1 \left(\|a_{(k)}\|_{C_t \times x} \|a_{(k)}\|_{C_t \times x}^2 + \|a_{(k)}\|_{C_t \times x}^2\right)$$

$$\lesssim \epsilon^{-9} \sigma^{-1}.$$ \hspace{1cm} (5.40)

Therefore, putting estimates (5.38)-(5.40) altogether and using (3.1), (5.26) we obtain

$$\|\hat{R}_{\text{osc}}\|_{L_t^1 L_x^p} \lesssim \|\hat{R}_{\text{cor}}\|_{L_t^1 L_x^p} \lesssim (d_{q+1}^{(p)} \otimes (u_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(o)}) - (u_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(o)}) \otimes d_{q+1}$$

$$+ (d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}) \otimes w_{q+1} - w_{q+1}^{(p)} \otimes (d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)})\|_{L_t^1 L_x^p}$$

$$\lesssim \|w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(o)}\|_{L_t^1 L_x^p} + \|w_{q+1}^{(p)}\|_{L_t^1 L_x^p} + \|d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}\|_{L_t^1 L_x^p}$$

$$\lesssim \left(\epsilon^{-1} r_{\perp}^{1/2} r_{\parallel}^{-1/2} + \epsilon^{-2} \mu^{-1} r_{\perp}^{1/2} r_{\parallel}^{-1/2} \tau \sigma^{-1}\right) \lesssim \epsilon^{-20} \lambda^{-2} \sigma^{-1}.$$ \hspace{1cm} (5.42)

where the last step is due to the fact that $-\epsilon/2 + 2 \eta - 8 \eta \beta \leq 0$.

**Commutator error.** This error can be bounded easily by using (2.10):

$$\|\hat{R}_{\text{com}}\|_{L_t^1 L_x^p} \lesssim \|B \otimes \epsilon - u_{\ell} \otimes B - (B q \otimes u_{q} - u_{q} \otimes B q) \ast x \phi_{\ell} \ast x \psi_{\ell}\|_{L_t^1 L_x^p}$$

$$\lesssim \epsilon^{-14}.$$ \hspace{1cm} (5.43)

**5.4. Estimates of Reynolds stress.** This subsection treats the Reynolds stress $\hat{R}_{q+1}^{u}$ given by (5.16).
Linear error. By the $L^p$-boundedness of $\mathcal{R}^u$, (4.35) and Lemmas 3.4, 4.1, 4.6
\[
\left\| \mathcal{R}^u \partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)}) \right\|_{L^1_t L^p_x} \lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \left\| \mathcal{R}^u \text{curl} \text{curl} \partial_t (g_{(r)} a_{(k)} W_k^c) \right\|_{L^1_t L^p_x}
\lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \left\| \text{curl} \partial_t (g_{(r)} a_{(k)} W_k^c) \right\|_{L^1_t L^p_x}
\lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \left( \left\| g_{(r)} \right\|_{L^1_k} \left( \left\| a_{(k)} \right\|_{C^2_{r,x}} \left\| W_k^c \right\|_{C^1_t W^1_x}^p + \left\| a_{(k)} \right\|_{C^1_{r,x}} \left\| \partial_t W_k^c \right\|_{C^1_t W^1_x}^p \right) \right.
\left. + \left\| \partial_t g_{(r)} \right\|_{L^1_k} \left( \left\| a_{(k)} \right\|_{C^1_{r,x}} \left\| W_k^c \right\|_{C^1_t W^1_x}^p \right) \right.
\lesssim \tau^{-\frac{1}{2}} (\ell^{-6} r^\frac{1}{2} \frac{1}{2} - \frac{1}{2} \lambda^{-1} + \ell^{-8} r^\frac{1}{2} + \frac{1}{2} \frac{1}{2} - \frac{3}{2} \mu) + \sigma \tau \frac{1}{2} \ell^{-8} r^\frac{1}{2} + \frac{1}{2} \frac{1}{2} - \frac{5}{2} \lambda^{-1}
\lesssim \ell^{-16} \lambda^{-2\varepsilon}.
\]

Regarding the viscosity term $(-\Delta)^{\alpha_1}$ in (5.17), arguing as in the proof of (5.29) and (5.35), the spatial and temporal intermittencies yield that
\[
\left\| \mathcal{R}^u (-\Delta)^{\alpha_1} w_{q+1} \right\|_{L^1_t L^p_x} \lesssim \ell^{-1} \lambda^{-\varepsilon}.
\]
Moreover, similarly to (5.36),
\[
\left\| \mathcal{R}^u P^H \text{div} \left( u_t \otimes w_{q+1} + w_{q+1} \otimes u_t - d_{q+1} \otimes B_t - B_t \otimes d_{q+1} \right) \right\|_{L^1_t L^p_x} \lesssim \ell^{-10} \lambda^{-2\varepsilon}.
\]
Thus, combining the above estimates together we arrive at
\[
\left\| \tilde{\mathcal{R}}^u_{\text{osc}} \right\|_{L^1_t L^p_x} \lesssim \ell^{-16} \lambda^{-2\varepsilon} + \ell^{-1} \lambda^{-\varepsilon} + \ell^{-10} \lambda^{-2\varepsilon} \lesssim \ell^{-1} \lambda^{-\varepsilon}.
\]

Oscillation error. For the velocity oscillation $\hat{\mathcal{R}}^u_{\text{osc}}$ using (5.18) we decompose
\[
\hat{\mathcal{R}}^u_{\text{osc}} = \hat{\mathcal{R}}^u_{\text{osc},1} + \hat{\mathcal{R}}^u_{\text{osc},2} + \hat{\mathcal{R}}^u_{\text{osc},3},
\]
where $\hat{\mathcal{R}}^u_{\text{osc},1}$ is the low-high spatial frequency part
\[
\hat{\mathcal{R}}^u_{\text{osc},1} := \sum_{k \in \Lambda_u} \mathcal{R}^u \mathbb{P} \neq 0 \left( g_{(r)}^2 \mathbb{P} \neq 0 (W_k \otimes W_k) \nabla (a_{(k)}^2) \right)
+ \sum_{k \in \Lambda_B} \mathcal{R}^u \mathbb{P} \neq 0 \left( g_{(r)}^2 \mathbb{P} \neq 0 (W_k \otimes W_k) - D_k \otimes D_k \nabla (a_{(k)}^2) \right),
\]
$\hat{\mathcal{R}}^u_{\text{osc},2}$ contains the high temporal oscillations
\[
\hat{\mathcal{R}}^u_{\text{osc},2} := -\mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathcal{R}^u \mathbb{P} \neq 0 \left( \partial_t (a_{(k)}^2 g_{(r)}^2) \psi_{(k)}^2 \phi_{(k)}^2 \right),
\]
and $\hat{\mathcal{R}}^u_{\text{osc},3}$ is of low frequency
\[
\hat{\mathcal{R}}^u_{\text{osc},3} := -\sigma^{-1} \sum_{k \in \Lambda_u} \mathcal{R}^u \mathbb{P} \neq 0 \left( h_{(r)} \int_{T^3} W_k \otimes W_k dx \nabla (a_{(k)}^2) \right)
- \sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^u \mathbb{P} \neq 0 \left( h_{(r)} \int_{T^3} W_k \otimes W_k - D_k \otimes D_k dx \nabla (a_{(k)}^2) \right).
\]
Let us treat $\hat{\mathcal{R}}^u_{\text{osc},1}$, $\hat{\mathcal{R}}^u_{\text{osc},2}$ and $\hat{\mathcal{R}}^u_{\text{osc},3}$ separately. By Lemmas 3.3, 3.4, 4.1, 4.2 and 5.3
\[
\left\| \hat{\mathcal{R}}^u_{\text{osc},1} \right\|_{L^1_t L^p_x} \lesssim \sum_{k \in \Lambda_u} \left\| g_{(r)}^2 \right\|_{L^1_k} \left\| \nabla^{-1} \mathbb{P} \neq 0 \left( \mathbb{P} \neq 0 (W_k \otimes W_k) \nabla (a_{(k)}^2) \right) \right\|_{L^1_t L^p_x}.
\]
Commutator error.

Using (2.9) one has

\[ \sum_{k \in \Lambda_B} \left\| g_{(r)} \right\|_{L^1_t} \left\| \nabla |^{-1} \mathcal{P}_{\neq 0} \left( \mathcal{P}_{\neq 0} (W_k \otimes W_k - D_{(k)} \otimes D_{(k)}) \nabla (a^2_{(k)}) \right) \right\|_{L^1_t L^p_x} \]

\[ \lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \left( \lambda r_{\perp} \right)^{-1} \left\| \nabla^3 (a^2_{(k)}) \right\|_{C^1_{t,x}} \left\| \psi^2_{(k)} \phi^2_{(k)} \right\|_{C^1 L^p_x} \]

\[ \lesssim \ell^{-26} \lambda^{-1} r_{\perp}^{-\frac{1}{2}} r_{\parallel}^{\frac{1}{2}}. \quad (5.50) \]

Regarding the temporal oscillation term \( \hat{R}_{\text{osc,2}}^u \), we estimate

\[ \left\| \hat{R}_{\text{osc,2}}^u \right\|_{L^1_t L^p_x} \lesssim \mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \left( \left\| \partial_t (a^2_{(k)}) \right\|_{C_{t,x}} + \left\| g_{(r)} \right\|_{L^1_t} + \left\| a_{(k)} \right\|^2_{C^1_{t,x}} \right) \left\| \psi_{(k)} \right\|^2_{L^2_p} \]

\[ \lesssim (\ell^{-10} + \ell^{-2} \tau \sigma) \mu^{-1} r_{\perp}^{-\frac{1}{2}} r_{\parallel}^{-\frac{1}{2}}. \quad (5.51) \]

Concerning the low frequency part \( \hat{R}_{\text{osc,3}}^u \), it holds that

\[ \left\| \hat{R}_{\text{osc,3}}^u \right\|_{L^1_t L^p_x} \lesssim \sigma^{-1} \sum_{k \in \Lambda_u} \left\| \nabla |^{-1} \mathcal{P}_{\neq 0} \left( h_{(r)} \int_{\mathbb{T}^3} W_k \otimes W_k \partial_t \nabla (a^2_{(k)}) \right) \right\|_{L^1_t L^p_x} \]

\[ + \sigma^{-1} \sum_{k \in \Lambda_B} \left\| \nabla |^{-1} \mathcal{P}_{\neq 0} \left( h_{(r)} \int_{\mathbb{T}^3} W_k \otimes W_k - D_{(k)} \otimes D_{(k)} \partial_t \nabla (a^2_{(k)}) \right) \right\|_{L^1_t L^p_x} \]

\[ \lesssim \sigma^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \left( \left\| a_{(k)} \right\|_{C^1_{t,x}} \left\| a_{(k)} \right\|^2_{C^1_{t,x}} \right). \]

\[ \lesssim \ell^{-17} \sigma^{-1}. \quad (5.52) \]

Therefore, putting estimates (5.50)-(5.52) altogether we arrive at

\[ \left\| \hat{R}_{\text{osc}}^u \right\|_{L^1_t L^p_x} \lesssim \ell^{-26} \lambda^{-1} r_{\perp}^{-\frac{1}{2}} r_{\parallel}^{\frac{1}{2}} + \ell^{-10} \tau \sigma \mu^{-1} r_{\perp}^{-\frac{1}{2}} r_{\parallel}^{-\frac{1}{2}} + \ell^{-17} \sigma^{-1} \]

\[ \lesssim \ell^{-26} \lambda^{-\varepsilon}. \quad (5.53) \]

Corrector error. Take \( p_1, p_2 \in (1, \infty) \) as in the proof of (5.42). Applying Lemma 4.6 and using (4.64) and (4.65), we have, similarly to (5.42),

\[ \left\| \hat{R}_{\text{cor}}^u \right\|_{L^1_t L^p_x} \lesssim \left\| w_{q+1}^{(c)} + w_{q+1}^{(l)} \right\|_{L^2_x} \left( \left\| w_{q+1}^{(p)} \right\|_{L^2_{t,x}} + \left\| w_{q+1} \right\|_{L^2_{t,x}} \right) \]

\[ + (\left\| d_{q+1}^{(c)} \right\|_{L^2_{t,x}} + \left\| d_{q+1}^{(l)} \right\|_{L^2_{t,x}}) \left\| d_{q+1}^{(o)} \right\|_{L^2_{t,x}} \]

\[ \lesssim \ell^{-20} \lambda^{-\frac{2}{3}}. \quad (5.54) \]

Commutator error. Using (2.9) one has

\[ \left\| \hat{R}_{\text{com}}^u \right\|_{L^1_t L^p_x} \lesssim \left\| u_{q} \otimes u_{q} - B_{\ell} \otimes B_{\ell} - (u_{q} \otimes u_{q} - B_{\ell} \otimes B_{\ell})^{*} \right\|_{L^1_t L^p_x} \]

\[ \lesssim \ell \lambda^{14}. \quad (5.55) \]

5.5. Proof of main iteration in Theorem 1.7. The inductive estimates (1.12), (1.15) and (1.16) of the velocity and magnetic fields have been verified in Subsection 4.6.

Hence, it remains to verify the estimates (1.13) and (1.14) for the stresses and to verify the inductive inclusions (1.17) and (1.18) for the temporal supports.

(i) Verification of the inductive estimates (1.13) and (1.14). Regarding the inductive estimates in (1.13), by the identity (5.24), Sobolev’s embedding \( W_{1,6}^{1,6} \hookrightarrow L_{x}^{\infty} \),

\[ \left\| R_{q+1}^{a} \right\|_{C_{t} C_{x}^{2}} \lesssim \left\| \mathcal{R}^{a} \mathcal{P}_{H} (\text{div} \hat{R}_{q+1}^{a}) \right\|_{C_{x} W_{2,6}^{1,6}}. \]
Similarly, where the last step is due to (1.10) and (1.11).

and, similarly, Verification of the inductive inclusions

Moreover, combining (5.46), (5.53), (5.54) and (5.55) we obtain

Thus, the estimates in (1.13) are verified.

and, similarly,

Thus, the estimates in (1.13) are verified.

Moreover, combining (5.46), (5.53), (5.54) and (5.55) we obtain

Combining (5.37), (5.41), (5.42) and (5.43) altogether we also get

where the last step is due to (1.10) and (1.11).

Therefore, (5.56) and (5.57) together verify the estimates in (1.14).

(ii) Verification of the inductive inclusions (1.17) and (1.18). By definitions,

\begin{align}
\text{supp}_t w_{q+1} &\subseteq \bigcup_{k \in \Lambda_\omega \cup \Lambda_B} \text{supp}_t a_{(k)} \subseteq N_{3\delta}(\text{supp}_t \hat{R}_q^u \cup \text{supp}_t \hat{R}_q^B), \\
\text{supp}_t d_{q+1} &\subseteq \bigcup_{k \in \Lambda_B} \text{supp}_t a_{(k)} \subseteq N_{3\delta}(\text{supp}_t \hat{R}_q^u \cup \text{supp}_t \hat{R}_q^B),
\end{align}
which yield that
\[
\begin{align*}
\text{supp}_t u_{q+1} &\subseteq \text{supp}_t u_{\ell} \cup \text{supp}_t w_{q+1} \subseteq N_{\ell}\left(\text{supp}_t u_q \cup \text{supp}_t \hat{R}_u \cup \text{supp}_t \hat{R}_B\right), \\
\text{supp}_t B_{q+1} &\subseteq \text{supp}_t B_{\ell} \cup \text{supp}_t d_{q+1} \subseteq N_{\ell}\left(\text{supp}_t B_q \cup \text{supp}_t \hat{R}_u \cup \text{supp}_t \hat{R}_B\right).
\end{align*}
\]

Thus, taking into account $3\ell \ll \delta_{q+1}^2$ we verify the inductive inclusion in (1.17).

Moreover, by (5.16) and (5.4),
\[
\begin{align*}
\text{supp}_t \hat{R}_u &\subseteq \bigcup_{k \in \Lambda_u \cup \Lambda_B} \text{supp}_t a_{(k)} \cup \text{supp}_t u_{\ell} \cup \text{supp}_t B_{\ell}, \\
\text{supp}_t \hat{R}_B &\subseteq \bigcup_{k \in \Lambda_u \cup \Lambda_B} \text{supp}_t a_{(k)} \cup \text{supp}_t u_{\ell} \cup \text{supp}_t B_{\ell}.
\end{align*}
\]

In view of (2.1) and (5.58a), we arrive at
\[
\text{supp}_t \hat{R}_u \cup \text{supp}_t \hat{R}_B \subseteq N_{\ell}\left(\text{supp}_t u_q \cup \text{supp}_t B_q \cup \text{supp}_t \hat{R}_u \cup \text{supp}_t \hat{R}_B\right). 
\]

Since $\ell \ll \delta_{q+2}$ due to (1.11), we then verify the inductive inclusion (1.18).

Therefore, the proof of Theorem 1.7 is complete.

6. PROOF OF MAIN THEOREMS

6.1. Proof of Theorem 1.2

We prove the statements (i)-(v) in Theorem 1.2 below.

(i). In the initial step $q = 0$, we take $u_0 = \bar{u}$ and $B_0 = \bar{B}$ and define $P_0$, $\hat{R}_u$ and $\hat{R}_B$ in the relaxation equation (1.7) by
\[
\begin{align*}
P_0 &= -\frac{1}{3}(|u_0|^2 - |B_0|^2), \\
\hat{R}_u &= R_u^*(\partial_t u_0 + \nu_1(-\Delta)^{\alpha_1}u_0) + u_0 \odot u_0 - B_0 \odot B_0, \\
\hat{R}_B &= R_B^*(\partial_t B_0 + \nu_2(-\Delta)^{\alpha_2}B_0) + B_0 \odot u_0 - u_0 \odot B_0.
\end{align*}
\]

Below we choose $a$, $M$ sufficiently large and set
\[
\delta_1 := \max\{\|\hat{R}_u\|_{L^1_{t,x}}, \|\hat{R}_B\|_{L^1_{t,x}}\}.
\]

Then, (1.12)-(1.14) are satisfied at level $q = 0$. Thus, by Theorem 1.7, there exists a sequence of solutions $(u_{q+1}, B_{q+1}, \hat{R}_u, \hat{R}_B)$ to (1.7) obeying estimates (1.12)-(1.14) for all $q \geq 0$.

Then, by the interpolation inequality and (1.10), (1.12) and (1.15), for any $\beta' \in (0, \frac{1}{7-\beta})$,
\[
\begin{align*}
&\sum_{q \geq 0}\|u_{q+1} - u_q\|_{H^\beta_{t,x}} + \sum_{q \geq 0}\|B_{q+1} - B_q\|_{H^{1-\beta}_{t,x}} \\
\leq &\sum_{q \geq 0}\|u_{q+1} - u_q\|_{L^1_{t,x}}^{1-\beta'}\|u_q\|_{L^\beta'_{t,x}} + \sum_{q \geq 0}\|B_{q+1} - B_q\|_{L^1_{t,x}}^{1-\beta'}\|B_q\|_{H^{1-\beta'}_{t,x}} \\
\lesssim &\sum_{q \geq 1}M^{1-\beta'}\delta_{q+1}\lambda_{q+1}^{1-\beta'} \lesssim M^{1-\beta'}\delta_1^{1-\beta'}\lambda_1^{1-\beta'} + \sum_{q \geq 1}M^{1-\beta'}\lambda_{q+1}^{-(1-\beta')q+1} \lesssim \infty,
\end{align*}
\]

where the last step is due to the fact that $-\beta(1-\beta') + 7\beta' < 0$.

Thus, $\{(u_q, B_q)\}_{q \geq 0}$ is a Cauchy sequence in $H^{\beta}_{t,x}$ and $\lim_{q \to +\infty}(u_q, B_q) = (u, B)$ for some $u, B \in H^{\beta}_{t,x}$. Taking into account $\lim_{q \to +\infty}\hat{R}_u = \lim_{q \to +\infty}\hat{R}_B = 0$ in $L^1_{t,x}$ we consequently conclude that $(u, B)$ is a weak solution to (1.1) in the sense of Definition 1.1.
(ii). Concerning the regularity of \((u, B)\), we first claim that for all \(q \geq 0\),
\[
\|u_q\|_{C^{14}_t, x} \leq \lambda_{q, 14}, \quad \|B_q\|_{C^{14}_t, x} \leq \lambda_{q, 14}.
\]
To this end, for \(a\) sufficiently large, (6.5) holds at level \(q = 0\). For \(q \geq 1\), assuming that (6.5) is correct at level \(q\), we apply (4.44), (4.49) and (4.50) to get
\[
\|u_{q+1}\|_{C^{14}_t, x} \leq \|u_{\ell}\|_{C^{14}_t, x} + \|w_{q+1}\|_{C^{14}_t, x} \leq \lambda_{q, 14} + \frac{1}{2}\lambda_{q+1, 14} \leq \lambda_{q+1, 14},
\]
\[
\|B_{q+1}\|_{C^{14}_t, x} \leq \|B_{\ell}\|_{C^{14}_t, x} + \|d_{q+1}\|_{C^{14}_t, x} \leq \lambda_{q, 14} + \frac{1}{2}\lambda_{q+1, 14} \leq \lambda_{q+1, 14},
\]
which yields (6.5) by inductive arguments, as claimed.

Therefore, for any \((s, p, \gamma) \in A\), using (1.11), (3.1), (6.5) and Lemma 4.6 we have
\[
\sum_{q \geq 0} \|u_{q+1} - u_q\|_{L^s_t W^p_x} \lesssim \sum_{q \geq 0} \|u_{\ell} - u_q\|_{L^s_t W^p_x} + \sum_{q \geq 0} \|w_{q+1}\|_{L^s_t W^p_x}
\]
\[
\lesssim \sum_{q \geq 0} (\ell^{ls_{q+1}} + \ell^{-1}\lambda_{q+1, 14} + \frac{1}{2}\epsilon + \frac{1}{2}\frac{1}{\epsilon + \frac{1}{2}} + \ell^{-26}\sigma^{-1})
\]
\[
\lesssim \sum_{q \geq 0} (\lambda_{q, 14} + \ell^{-1}\lambda_{q+1, 14} + \frac{1}{2}\epsilon + \frac{1}{2}\frac{1}{\epsilon + \frac{1}{2}} + \ell^{-26}\lambda_{q+1, 14})
\]
\[
\lesssim \sum_{q \geq 0} (\lambda_{q, 6} + \lambda_{q, 14} + \frac{1}{2}\epsilon + \frac{1}{2}\frac{1}{\epsilon + \frac{1}{2}} + \lambda_{q+1, 14}).
\]

Analogous arguments also yield that
\[
\sum_{q \geq 0} \|B_{q+1} - B_q\|_{L^s_t W^p_x} \lesssim \sum_{q \geq 0} (\lambda_{q, 6} + \lambda_{q, 14} + \frac{1}{2}\epsilon + \frac{1}{2}\frac{1}{\epsilon + \frac{1}{2}} + \lambda_{q+1, 14}).
\]

Taking into account (1.9) we have
\[
s + \frac{3}{2} - \frac{2}{p} - \frac{1}{\gamma} + \epsilon \left(\frac{8}{p} + \frac{6}{\gamma} - 6\right) \leq s + \frac{3}{2} - \frac{2}{p} - \frac{1}{\gamma} + 6\epsilon < 0,
\]
which yields that
\[
\sum_{q \geq 0} \|u_{q+1} - u_q\|_{L^s_t W^p_x} + \sum_{q \geq 0} \|B_{q+1} - B_q\|_{L^s_t W^p_x} < \infty.
\]

Therefore, \(\{(u_q, B_q)\}_{q \geq 0}\) is a Cauchy sequence in \(L^s_t W^p_x \times L^s_t W^p_x\) for any \((s, p, \gamma) \in A\). Using the uniqueness of weak limits we then conclude that \(u, B \in H^s_{\ell, x} \cap L^s_t W^p_x\). Thus, the regularity statement (ii) is proved.

(iii). Concerning the temporal support, let
\[
K_q := \text{supp}_t u_q \cup \text{supp}_t B_q \cup \text{supp}_t \hat{R}^u_q \cup \text{supp}_t \hat{R}^B_q, \quad q \geq 1.
\]
By (6.2) and (6.3),
\[
\text{supp}_t \hat{R}^u_0 \cup \text{supp}_t \hat{R}^B_0 \subseteq K_0 := \text{supp}_t u_0 \cup \text{supp}_t B_0.
\]
Moreover, by (1.17) and (1.18),
\[
K_{q+1} \subseteq N_{\frac{1}{8} \delta_{q+2}} K_q \subseteq \cdots \subseteq N_{\sum_{j=2}^{q+2} \delta_{j}} K_0,
\]
which along with the fact that \(\sum_{q \geq 0} \delta_{q+2}^{1/2} \leq \varepsilon_*\) yields that
\[
\text{supp}_t u \cup \text{supp}_t B \subseteq \bigcup_{q \geq 0} K_q \subseteq N_{\varepsilon_*}(\text{supp}_t \tilde{u} \cup \text{supp}_t \tilde{B}),
\]
thereby yielding the temporal support statement (iii).

(iv). Using (1.10) and (1.16) we get that for a sufficiently large (depending on \( \varepsilon_* \)),
\[
\|u - \tilde{u}\|_{L^1_tL^2_x} + \|B - \tilde{B}\|_{L^1_tL^2_x} \\
\leq \sum_{q \geq 0} \|u_{q+1} - u_q\|_{L^1_tL^2_x} + \sum_{q \geq 0} \|B_{q+1} - B_q\|_{L^1_tL^2_x} \\
\leq 2 \sum_{q \geq 0} \frac{\delta_{q+2}^2}{\alpha} \leq 2 \sum_{q \geq 2} \alpha^{-\beta q} = \frac{2\alpha^{-2\beta} \varepsilon_*}{1 - \alpha^{-\beta}} \leq \varepsilon_*,
\]
(6.13)
which yields the small deviation statement (iv).

(v). It remains to prove the small deviations of magnetic helicity. Note that
\[
\|H_{B,B} - H_{\tilde{B},\tilde{B}}\|_{L^1_t} \leq \|B - B_0\|_{L^1_tL^2_x} \|A_0\|_{C^1_tL^2_x} + \|A - A_0\|_{L^1_tL^2_x} \|B\|_{L^2_{t,x}},
\]
(6.14)
where \( A \) and \( A_0 \) are the potential fields corresponding to \( B \) and \( B_0 \), respectively.

Since \( B_q \) is divergence free, by the Biot-Savart law,
\[
A_q := \text{curl}(-\Delta)^{-1} B_q, \quad q \geq 0,
\]
(6.15)
which yields that
\[
\|A_0\|_{C^1_tL^2_x} = \|\text{curl}(-\Delta)^{-1} B_0\|_{C^1_tL^2_x} \lesssim 1.
\]
(6.16)
Moreover, by the inductive estimate (1.16),
\[
\|B - B_0\|_{L^1_tL^2_x} \leq \sum_{q \geq 0} \|B_{q+1} - B_q\|_{L^1_tL^2_x} \leq \sum_{q \geq 0} \delta_{q+2}^\frac{3}{2} \lesssim \alpha^{-2\beta q}.
\]
(6.17)
Regarding the \( L^2 \) estimate of \( B \), since \( \|B_0\|_{L^2_{t,x}} \lesssim 1 \) and
\[
\|B - B_0\|_{L^2_{t,x}} \leq \sum_{q \geq 0} \|B_{q+1} - B_q\|_{L^2_{t,x}} \leq \sum_{q \geq 0} M \delta_{q+1}^\frac{3}{2} \lesssim 1,
\]
we infer that
\[
\|B\|_{L^2_{t,x}} \leq \|B_0\|_{L^2_{t,x}} + \|B - B_0\|_{L^2_{t,x}} \lesssim 1.
\]
(6.18)
Furthermore, we have, via (6.15),
\[
\|A - A_0\|_{L^2_{t,x}} \leq \sum_{q \geq 0} \|A_{q+1} - A_q\|_{L^2_{t,x}} \leq \sum_{q \geq 0} \|\text{curl}(-\Delta)^{-1}(B_{q+1} - B_q)\|_{L^2_{t,x}} \\
\leq \sum_{q \geq 0} \|\text{curl}(-\Delta)^{-1}(B_\ell - B_q)\|_{L^2_{t,x}} + \sum_{q \geq 0} \|\text{curl}(-\Delta)^{-1} d_{q+1}\|_{L^2_{t,x}}.
\]
(6.19)
Since \( B_\ell - B_q \) is mean free, by the Poincaré inequality,
\[
\sum_{q \geq 0} \|\text{curl}(-\Delta)^{-1}(B_\ell - B_q)\|_{L^2_{t,x}} \lesssim \sum_{q \geq 0} \|B_\ell - B_q\|_{L^2_{t,x}} \lesssim \sum_{q \geq 0} \|B_q\|_{C^1_{t,x}} \lesssim \sum_{q \geq 0} \lambda_{q}^{-13} \lesssim a^{-13} + (a^{13\beta} - 1)^{-1} \lesssim a^{-13}.
\]
(6.20)
Moreover, by (4.35b), (4.43b) and Lemmas 3.3, 3.4, 4.1 and 4.6
\[
\|\text{curl}(-\Delta)^{-1} d_{q+1}\|_{L^2_{t,x}} \leq \sum_{k \in \Lambda N} \|\text{curl}(a(k) g(\tau) D_{(k)}^\ell)\|_{L^2_{t,x}} + \|d_{q+1}^{(f)}\|_{L^2_{t,x}} + \|d_{q+1}^{(o)}\|_{L^2_{t,x}} \\
\leq \sum_{k \in \Lambda N} \|g(\tau)\|_{L^2_{t}} \|a(k) D_{(k)}^\ell\|_{C^1_tL^2_x} + \|d_{q+1}^{(f)}\|_{L^2_{t,x}} + \|d_{q+1}^{(o)}\|_{L^2_{t,x}} \\
\lesssim \sum_{k \in \Lambda N} \|g(\tau)\|_{L^2_{t}} |a|_{C^1_tL^2_x} \|D_{(k)}^\ell\|_{C^1_tW^1_{x,2}} + \|d_{q+1}^{(f)}\|_{L^2_{t,x}} + \|d_{q+1}^{(o)}\|_{L^2_{t,x}}.
\]
\[ \lesssim \ell^{-4} \lambda_{q+1}^{-1} + \ell^{-2} \mu^{-1} \tau^{-1} r_\perp^{-1/2} r_\parallel^{-1} + \ell^{-6} \sigma^{-1} \lesssim \lambda_{q+1}^{-2}. \]  

(6.21)

Thus, we conclude from (6.19)-(6.21) that

\[ \|A - A_0\|_{L^2} \lesssim a^{-13} + \sum_{q \geq 0} \lambda_{q+1}^{-2} \lesssim a^{-13} + \sum_{q \geq 1} (a^{-2} b)^{-q} \lesssim a^{-13} + a^{-2} b. \]  

(6.22)

Therefore, plugging (6.16), (6.17), (6.18) and (6.22) into (6.14) and using (1.11) we obtain that for \( a \) large enough

\[ \|H_{\tilde{B},B}(t) - H_{\tilde{B},B}(t)\|_{L^1} \lesssim (a^{-2} b + a^{-13} + a^{-2} b) \leq \varepsilon_s. \]  

(6.23)

Consequently, the proof of Theorem 1.2 is complete.

### 6.2. Proof of Corollary 1.3

For any \( m \in \mathbb{N}_+ \), we choose the incompressible, mean-free fields \( \tilde{u}_m \) and \( \tilde{B}_m \), defined by

\[
\tilde{u}_m := m \Psi(t)(\sin x_3, 0, 0)^\top, \quad \tilde{B}_m := m \Psi(t)(\sin x_3, \cos x_3, 0)^\top,
\]

(6.24a, 6.24b)

where \( \Psi : \mathbb{T} \to \mathbb{R} \) is any smooth cut-off function supported on the interval \([1/4, 5/4]\), such that \( \Psi(t) = 1 \) for \( 1/2 \leq t \leq 1 \) and \( 0 \leq \Psi(t) \leq 1 \) for all \( t \in \mathbb{T} \).

Then, for \( \varepsilon_s \leq 1/100 \), by Theorem 1.2 there exist \( u_m, B_m \in H^\beta_{(s,p)} \cap L^1_t W^\gamma_{x,p} \), where \( \beta > 0 \) and \((s,p,\gamma) \in \mathcal{A} \) with \( \mathcal{A} \) given by (1.5), such that \((u_m, B_m)\) solves (1.1) and satisfies the properties (ii)-(v) in Theorem 1.2.

Straightforward computations show that

\[ \|\tilde{u}_m\|_{L^1(1/2,1;L^2)} = m \pi^{1/2}, \quad \|\tilde{B}_m\|_{L^1(1/2,1;L^2)} = \frac{m}{2} (2\pi)^{3/2}. \]

Then, in view of the small deviations on average, i.e.,

\[ \|u_m - \tilde{u}_m\|_{L^1(1/2,1;L^2)} \leq \varepsilon_s, \quad \|B_m - \tilde{B}_m\|_{L^1(1/2,1;L^2)} \leq \varepsilon_s, \]

we infer that for any \( m \neq m' \),

\[ \|u_m\|_{L^1(1/2,1;L^2)} \neq \|u_{m'}\|_{L^1(1/2,1;L^2)}, \quad \|B_m\|_{L^1(1/2,1;L^2)} \neq \|B_{m'}\|_{L^1(1/2,1;L^2)}. \]  

(6.25)

In particular, this yields that

\[ (u_m, B_m) \neq (u_{m'}, B_{m'}), \quad \forall m \neq m'. \]

But since zero is not contained in the support of \( \Psi \), by the temporal support result in Theorem 1.2 (iii), the temporal support of \((u_m, B_m)\) is contained in \([1/8, 11/8]\), and so

\[ u_m(0) = B_m(0) = 0, \quad \forall m \geq 1. \]  

(6.26)

Thus, we obtain infinitely many different weak solutions to (1.1) with the same initial data at time zero.

Concerning the magnetic helicity, on one hand, we have from (6.26) that

\[ H_{B_m,B_m}(0) = 0, \quad \forall m \geq 1. \]  

(6.27)

On the other hand, the vector potential \( A_m \) corresponding to the magnetic field \( B_m \) can be computed explicitly by

\[ A_m(t, x) := m \Psi(t)(\sin x_3, \cos x_3, 0). \]

This yields that, for the magnetic helicity of \((u_m, B_m)\),

\[ H_{B_m,B_m}(t) = m^2 \Psi(t)^2(2\pi)^3, \]
and so
\[ \|p H_{B_m,B_m}\|_{L^1_t(\frac{1}{2},1)} = \frac{m^2}{2}(2\pi)^3. \]

Taking into account \( \varepsilon_* \leq 1/100 \) and the small deviation of the magnetic helicity in Theorem 1.2 (v) we lead to
\[ \|p H_{B_m,B_m}\|_{L^1_t(\frac{1}{2},1)} > 0. \] (6.28)

Thus, we conclude from (6.27) and (6.28) that the magnetic helicity \( H_{B_m,B_m} \) is not conserved, \( m \geq 1 \). Therefore, the proof is complete. \( \square \)

6.3. Proof of Theorem 1.5. Let \( \{\phi_\varepsilon\}_{\varepsilon>0} \) and \( \{\varphi_\varepsilon\}_{\varepsilon>0} \) be two families of standard compactly support mollifiers on \( T^3 \) and \( T \), respectively. Set
\[ u_n := (u \ast_x \phi_\varepsilon^{-1}) \ast_t \varphi_\varepsilon^{-1}, \quad B_n := (u \ast_x \phi_\varepsilon^{-1}) \ast_t \varphi_\varepsilon^{-1}. \] (6.29)

Since \((u, B)\) is a weak solution to the ideal MHD system, we infer that there exists a mean-free function \( P_n \) such that
\[ \begin{cases} 
\partial_t u_n + \lambda_n^{-2\alpha_1}(-\Delta)^{\alpha_1} u_n + \text{div}(u_n \otimes u_n - B_n \otimes B_n) + \nabla P_n = \text{div} R_n^u, \\
\partial_t B_n + \lambda_n^{-2\alpha_2}(-\Delta)^{\alpha_2} B_n + \text{div}(B_n \otimes u_n - u_n \otimes B_n) = \text{div} R_n^B,
\end{cases} \] (6.30)

where the stresses \( R_n^u \) and \( R_n^B \) are given by
\[ \begin{align*}
R_n^u &= (u_n \otimes u_n - B_n \otimes B_n) - ((u \otimes u - B \otimes B) \ast_x \phi_\varepsilon^{-1}) \ast_t \varphi_\varepsilon^{-1} \\
&\quad + \lambda_n^{-2\alpha_1} \mathcal{R}^u(-\Delta)^{\alpha_1} u_n, \\
R_n^B &= (B_n \otimes u_n - u_n \otimes B_n) - ((B \otimes u - u \otimes B) \ast_x \phi_\varepsilon^{-1}) \ast_t \varphi_\varepsilon^{-1} \\
&\quad + \lambda_n^{-2\alpha_2} \mathcal{R}^B(-\Delta)^{\alpha_2} B_n,
\end{align*} \] (6.31)

and
\[ P_n := P \ast_x \phi_\varepsilon^{-1} \ast_t \varphi_\varepsilon^{-1} - \|u_n\|^2 + \|B_n\|^2 + (\|u\|^2 - \|B\|^2) \ast_x \phi_\varepsilon^{-1} \ast_t \varphi_\varepsilon^{-1}. \]

Let
\[ \nu_1 := \nu_{1,n} := \lambda_n^{-2\alpha_1}, \quad \nu_2 := \nu_{2,n} := \lambda_n^{-2\alpha_2}, \quad \nu_n := (\nu_{1,n}, \nu_{2,n}) \]
and
\[ \tilde{M} := \max\{\|u\|_{H^{\tilde{b}}_{t,x}}, \|B\|_{H^{\tilde{b}}_{t,x}}\}. \]

Claim: For \( a \) sufficiently large, \((u_n, B_n, R_n^u, R_n^B)\) satisfy the iterative estimates (1.12)-(1.14) at level \( q = n \)(\( \geq 1 \)).

To this end, let us start with the most delicate estimates in (1.14). Applying the Minkowski inequality we have
\[ \|u - u_n\|_{L^2_{t,x}} \]
\[ \lesssim \left\| \int_T \int_{T^3} |u(t,x) - u(t-s,x-y)| \phi_\varepsilon^{-1}(y) \varphi_\varepsilon^{-1}(s)dyds \right\|_{L^2_{t,x}} \]
\[ \lesssim \left\| \int_T \int_{T^3} \frac{|u(t,x) - u(t-s,x-y)|}{(|s| + |y|)^{2+\beta}} (|s| + |y|)^{2+\beta} \phi_\varepsilon^{-1}(y) \varphi_\varepsilon^{-1}(s)dyds \right\|_{L^2_{t,x}} \]
\[ \lesssim \left\| \int_T \int_{T^3} \frac{|u(t,x) - u(t-s,x-y)|}{(|s| + |y|)^{2+\beta}} (|s| + |y|)^{2+\beta} \phi_\varepsilon^{-1}(y) \varphi_\varepsilon^{-1}(s)dyds \right\|_{L^2_{t,x}} \].
Since the Slobodetskii-type norm can be bounded by (see, e.g., [3, Proposition 1.4])
\[ \left\| \frac{u(t, x) - u(t - s, x - y)}{(|s| + |y|)^{2+\beta}} \right\|_{L_{t,s}^1 L_{x,y}^2} \lesssim \|u\|_{H_{t,x}^\beta} \tag{6.33} \]
and
\[ \|(|s| + |y|)^{2+\beta} \phi_{\lambda_n^{-1}} \varphi_{\lambda_n^{-1}} \|_{L_{t,s}^2} \lesssim \lambda_n^{-\beta} \|(|s| + |y|)^{2+\beta} \phi \varphi \|_{L_{t,s}^\infty} \lesssim \lambda_n^{-\beta}, \tag{6.34} \]
we obtain that
\[ \|u - u_n\|_{L_{t,s}^2} \lesssim \lambda_n^{-\beta} \|u\|_{H_{t,x}^\beta} \lesssim \lambda_n^{-\beta} \hat{M}. \tag{6.35} \]
Moreover, we note that, if \( \delta_{s,y} u(t, x) := u(t, x) - u(t - s, x - y) \),
\[ \|u_n \otimes u_n - ((u \otimes u) *_x \phi_{\lambda_n^{-1}}) *_t \varphi_{\lambda_n^{-1}}\|_{L_{t,s}^1} \lesssim \lambda_n^{-\beta} \|u\|_{H_{t,x}^\beta} \tag{6.36} \]
Estimating as in (5.29) and (5.31) we have
\[
\|\lambda_n^{-2\alpha_1} \mathcal{R}^u (-\Delta)^{\alpha_1} u_n\|_{L_{t,s}^1} \lesssim \lambda_n^{-2\alpha_1} \|\mathcal{R}^u (-\Delta)^{\alpha_1} u_n\|_{L_{t,s}^2} \lesssim \lambda_n^{-2\alpha_1} \left( \|u_n\|_{L_{t,s}^4} + \|u_n\|_{L_{t,s}^2} \right)
\]
Hence, we conclude from (6.31), (6.36), (6.37) and (6.38) that
\[
\|\tilde{R}_n\|_{H_{t,s}^\beta} \lesssim \left( \lambda_n^{-2\alpha_1} + \lambda_n^{-1} \right) \tilde{M} + \lambda_n^{-2\beta} \tilde{M}^2. \tag{6.39} \]
Analogous arguments also yield that
\[
\|\tilde{R}_n\|_{H_{t,s}^\beta} \lesssim \left( \lambda_n^{-2\alpha_2} + \lambda_n^{-1} \right) \tilde{M} + \lambda_n^{-2\beta} \tilde{M}^2. \tag{6.40} \]
Thus, estimate (1.14) is verified at level \( q = n \) for \( \beta \) small enough, where \( \beta > 0 \) is as in the proof of Theorem 1.2
Moreover, using the Sobolev embedding \( H_{t,s}^{3, \infty} \hookrightarrow L_{t,s}^\infty \) we have
\[
\|u_n\|_{C_{t,s}^1} + \|B_n\|_{C_{t,s}^1} \lesssim \|u_n\|_{H_{t,s}^4} + \|B_n\|_{H_{t,s}^4} \lesssim \lambda_n^4 \left( \|u\|_{L_{t,s}^2} + \|B\|_{L_{t,s}^2} \right) \lesssim \lambda_n \tilde{M}, \tag{6.41} \]
which yields \((1.13)\) at level \(q = n\).

Regarding estimate \((1.12)\), by \((6.31)\) and the Sobolev embedding \(W^{5,1}_{t,x} \hookrightarrow L^\infty_{t,x}\),

\[
\|\hat{R}_n^{\alpha}\|_{C_{t,x}^1} \leq \|u_n \otimes u_n - (u \otimes u) \ast_x \phi_{\lambda_n-1} \ast_t \varphi_{\lambda_n-1}\|_{W^{5,1}_{t,x}} + \|B_n \otimes B_n - (B \otimes B) \ast_x \phi_{\lambda_n-1} \ast_t \varphi_{\lambda_n-1}\|_{W^{5,1}_{t,x}} + \lambda_n^{-2\alpha_1} \|R\|_{(\Delta)^{\alpha_1} u_n\|_{W^{5,1}_{t,x}}} \\
= J_1 + J_2 + J_3. \quad (6.42)
\]

Note that

\[
J_1 \leq \sum_{0 \leq N_0 + N \leq 6} \|\partial_t^{N_0} \nabla^N (u_n \otimes u_n)\|_{L^1_{t,x}} + \|\partial_t^{N_0} \nabla^N ((u \otimes u) \ast_x \phi_{\lambda_n-1} \ast_t \varphi_{\lambda_n-1})\|_{L^1_{t,x}} \leq \sum_{0 \leq N_0 + N \leq 6} \|\partial_t^{N_0} \nabla^N u_n\|_{L^1_{t,x}} \leq \lambda_n^6 \|\nabla u\|_{L^2_{t,x}} \lesssim \lambda_n^6 \tilde{M}^2. \quad (6.43)
\]

Similarly,

\[
J_2 \lesssim \lambda_n^6 \|B\|_{L^2_{t,x}} \lesssim \lambda_n^6 \tilde{M}^2. \quad (6.44)
\]

Moreover, if \(\alpha_1 \in (0, 1/2]\), we have

\[
J_3 \lesssim \lambda_n^{-2\alpha_1} \|u_n\|_{H^6_{t,x}} \lesssim \lambda_n^{6-2\alpha_1} \|u\|_{L^2_{t,x}} \lesssim \lambda_n^{6-2\alpha_1} \tilde{M}. \quad (6.45)
\]

If \(\alpha_1 \in (1/2, 5/4]\), by the interpolation inequality,

\[
J_3 \lesssim \lambda_n^{-2\alpha_1} \|\nabla^{2\alpha_1-1} u_n\|_{H^6_{t,x}} \lesssim \lambda_n^{-2\alpha_1} \|\nabla^{1/2\alpha_1} u_n\|_{L^{\frac{\alpha_1+2}{\alpha_1}}_{t,x}} \|\nabla^{\frac{5+2\alpha_1}{\alpha_1}} u_n\|_{L^\infty_{t,x}} \lesssim \lambda_n^5 \tilde{M}. \quad (6.46)
\]

Thus, we conclude from \((6.43)\)-\((6.46)\) that

\[
\|\hat{R}_n^{\alpha}\|_{C_{t,x}^1} \lesssim \lambda_n^6 (\tilde{M} + \tilde{M}^2). \quad (6.47)
\]

Similarly, we have

\[
\|\hat{R}_n^{B}\|_{C_{t,x}^1} \lesssim \lambda_n^6 (\tilde{M} + \tilde{M}^2). \quad (6.48)
\]

Thus, taking \(a\) sufficiently large we prove estimate \((1.12)\) at level \(q = n\), as claimed.

Now, we apply Theorem 1.7 to obtain a sequence of solutions \((u_{n,q}, B_{n,q}, \hat{R}_{n,q}^{\alpha}, \hat{R}_{n,q}^{B})_{q \geq n}\) satisfying \((1.12)-(1.14)\). Letting \(q \to +\infty\) we obtain a weak solution \((u^{(\infty)}, B^{(\infty)})\) in \(H_{t,x}^{\beta'} \times H_{t,x}^{\beta'}\) to \((1.1)\) for some \(\beta' \in (0, \beta/(7 + \beta))\).

Moreover, as in \((6.4)\), it holds that for \(n \geq 1\),

\[
\|u^{(\infty)} - u_n\|_{H_{t,x}^{\beta'}} + \|B^{(\infty)} - B_n\|_{H_{t,x}^{\beta'}} \leq C \sum_{q=n}^{\infty} \lambda_n^{-\beta(1-\beta')} \lambda_{q+1}^{\beta'} \leq \frac{1}{2n}. \quad (6.49)
\]

Taking \(\beta'\) smaller such that \(0 < \beta' < \min\{\tilde{\beta}, \beta/(7 + \beta)\}\) and using \((6.35)\) we obtain

\[
\|u - u_n\|_{H_{t,x}^{\beta'}} + \|B - B_n\|_{H_{t,x}^{\beta'}} \leq \|u - u_n\|_{L^2_{t,x}}^{\frac{\beta'}{\beta}} \|u - u_n\|_{H_{t,x}^{\beta'}}^{\frac{\beta'}{\beta}} + \|B - B_n\|_{L^2_{t,x}}^{\frac{\beta'}{\beta}} \|B - B_n\|_{H_{t,x}^{\beta'}}^{\frac{\beta'}{\beta}} \lesssim \lambda_n^{-\beta(1-\beta')} \tilde{M} \leq \frac{1}{2n}. \quad (6.50)
\]

Thus, we conclude from \((6.49)\) and \((6.50)\) that for \(n \geq 1:\)

\[
\|u^{(\infty)} - u\|_{H_{t,x}^{\beta'}} \leq \|u^{(\infty)} - u_n\|_{H_{t,x}^{\beta'}} + \|u - u_n\|_{H_{t,x}^{\beta'}} \leq \frac{1}{n},
\]
Acknowledgment

It follows from the uniqueness of strong limits that

\[ \| B^{(\nu_n)} - B \|_{H^\beta_{t,x}} \leq \| B^{(\nu_n)} - B_n \|_{H^\beta_{t,x}} + \| B - B_n \|_{H^\beta_{t,x}} \leq \frac{1}{n}, \]

which converge to zero, thereby yielding (1.6).

Therefore, the proof of Theorem 1.5 is complete. \(\square\)

Remark 6.1. We close this section with the remark that, the weak solutions constructed in [2] to the ideal MHD equations (1.3) is in the space \(H^{\tilde{\beta}}_{t,x}\) for some \(\tilde{\beta} > 0\).

Though the solutions constructed in [2] are on the time interval \([0, T]\), with slight modifications the proof in [2] also applies to the time interval \(T\) below.

The constructed solution \((u, B) \in C_t H^{\tilde{\beta}} \times C_t H^\beta_x\) in [2], where \(\tilde{\beta} > 0\), can be approximated by the relaxation solutions \((u_\nu, B_\nu, \tilde{R}^{\nu}_{\beta}, \tilde{R}^{\nu}_{\beta})\) to (1.7) with \(\nu_1 = \nu_2 = 0\), which satisfy the inductive estimates that for any \(q \geq 0\) (see [2, (2.3),(2.4),(2.5)])

\[ \| u_q \|_{C_t L^q} \leq 1 - \delta_{q+1}^\frac{1}{2}, \quad \| B_q \|_{C_t L^2} \leq 1 - \delta_{q+1}^\frac{1}{2}, \]
\[ \| u_q \|_{C_t^1 L^2} \leq \lambda_{q+1}, \quad \| B_q \|_{C_t^1 L^2} \leq \lambda_{q+1}, \]
\[ \| \tilde{R}^{\nu}_{\beta} \|_{C_t L^1} \leq c_u \delta_{q+1}, \quad \| \tilde{R}^{\nu}_{\beta} \|_{C_t L^1} \leq c_B \delta_{q+1}, \]
\[ \| u_{q+1} - u_q \|_{C_t L^2} \leq \delta_{q+1}^\frac{1}{2}, \quad \| B_{q+1} - B_q \|_{C_t L^2} \leq \delta_{q+1}^\frac{1}{2}, \]

where \(\delta_{q+1} = \lambda_{q+1}^{-2\beta}, \quad \tilde{\beta} \in (0, \beta/(2 + \beta))\), \(\beta\) is a small positive constant, and \(c_u, c_B > 0\).

Then, the interpolation inequality yields that

\[ \sum_{q \geq 0} \| u_{q+1} - u_q \|_{H^\beta_{t,x}} + \sum_{q \geq 0} \| B_{q+1} - B_q \|_{H^\beta_{t,x}} \]
\[ \leq \sum_{q \geq 0} \| u_{q+1} - u_q \|_{L^1_{t,x}}^{1-\beta} \| u_{q+1} - u_q \|_{H^\beta_{t,x}} + \sum_{q \geq 0} \| B_{q+1} - B_q \|_{L^1_{t,x}}^{1-\beta} \| B_{q+1} - B_q \|_{H^\beta_{t,x}} \]
\[ \leq \sum_{q \geq 0} \| u_{q+1} - u_q \|_{L^1_{t,x}}^{1-\beta} \| u_{q+1} - u_q \|_{C_t^1 L^2} + \sum_{q \geq 0} \| B_{q+1} - B_q \|_{C_t^1 L^2} \]
\[ \lesssim \sum_{q \geq 0} \delta_{q+1}^\frac{1+\beta}{2} \lambda_{q+1}^2 < \infty, \]

which yields that \(\{(u_q, B_q)\}_{q \geq 0}\) also converges in the Sobolev space \(H^\beta_{t,x} \times H^\beta_{t,x}\), and thus, it follows from the uniqueness of strong limits that \((u, B) \in H^\beta_{t,x} \times H^\beta_{t,x}\).

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