Flips in symmetric separated set-systems

Vladimir I. Danilov * Alexander V. Karzanov †
Gleb A. Koshevoy ‡

Abstract

For a positive integer \( n \), a collection \( S \) of subsets of \( [n] = \{1, \ldots, n\} \) is called symmetric if \( X \in S \) implies \( X^* \in S \), where \( X^* := \{ i \in [n]: n - i + 1 \notin X \} \) (the involution * was introduced by Karpman). Leclerc and Zelevinsky showed that the set of maximal strongly (resp. weakly) separated collections in \( 2^{[n]} \) is connected via flips, or mutations, “in the presence of six (resp. four) witnesses”.

We give a symmetric analog of those results, by showing that each maximal symmetric strongly (weakly) separated collection in \( 2^{[n]} \) can be obtained from any other one by a series of special symmetric local transformations, so-called symmetric flips. Also we establish the connectedness via symmetric flips for the class of maximal symmetric r-separated collections in \( 2^{[n]} \) when \( n, r \) are even (where sets \( A, B \subseteq [n] \) are called r-separated if there are no elements \( i_0 < i_1 < \cdots < i_{r+1} \) in \([n]\) which alternate in \( A \setminus B \) and \( B \setminus A \)). This is related to a symmetric version of higher Bruhat orders.

These results are obtained as consequences of our study of related geometric objects: symmetric rhombus and combined tilings and symmetric cubillages.

Keywords: strong separation, weak separation, chord separation, higher separation, purity phenomenon, rhombus tiling, combined tiling, cubillage, higher Bruhat order

1 Introduction

We fix a positive integer \( n \) and interpret the elements of the set \([n] := \{1, 2, \ldots, n\}\) as colors. For each \( i \in [n] \), the color \( n - i + 1 \) is regarded as complementary to \( i \) and denoted as \( i^\circ \). Following Karpman [11], for \( X \subseteq [n] \), define the set

\[ X^* := \{ i \in [n]: i^\circ \notin X \}. \tag{1.1} \]
One can see that \((X^*)^* = X\); so the relation * gives an involution on the set \(2^{[n]}\) of all subsets of \([n]\). We say that the sets \(X\) and \(X^*\) are *-symmetric, or, simply, symmetric, to each other. When \(X\) coincides with \(X^*\), it is called self-symmetric. Accordingly, a collection \(S \subseteq 2^{[n]}\) is called symmetric if \(X \in S\) implies \(X^* \in S\).

Recently, extending a result in [11], it was proved in [4] that symmetric strongly, weakly and chord separated collections \(S\) in \(2^{[n]}\) possess the property of purity, which means that if \(S\) is maximal by inclusion (among all collections of the given type), then it is maximal by size. This matches the purity behavior for usual strongly, weakly and chord separated collections in \(2^{[n]}\) (proved in [12, 1, 8], respectively). Recall that

- Sets \(A, B \subseteq [n]\) are called strongly separated (from each other) if there are no three elements \(i < j < k\) of \([n]\) such that one of \(A - B\) and \(B - A\) contains \(i, k\), and the other contains \(j\).
- Sets \(A, B \subseteq [n]\) are called chord separated (or 2-separated) if there are no elements \(i < j < k < \ell\) of \([n]\) such that one of \(A - B\) and \(B - A\) contains \(i, k\), and the other contains \(j, \ell\).
- Sets \(A, B \subseteq [n]\) are called weakly separated if they are chord separated and the additional condition holds: if \(A\) surrounds \(B\) and \(B - A \neq \emptyset\) then \(|A| \leq |B|\), and if \(B\) surrounds \(A\) and \(A - B \neq \emptyset\) then \(|B| \leq |A|\).

Accordingly, a collection \(A \subseteq 2^{[n]}\) of subsets of \([n]\) is called strongly (chord, weakly) separated if any two members of \(A\) are strongly (resp. chord, weakly) separated.

(Hereinafter, for sets \(A, B \subseteq [n]\), \(|A|\) is the size (the number of elements) of \(A\); \(A - B\) denotes the set difference \(\{i : A \ni i \notin B\}\); we write \(A < B\) if the maximal element \(\max(A)\) of \(A\) is smaller than the minimal element \(\min(B)\) of \(B\), letting \(\min(\emptyset) := -\infty\); and we say that \(A\) surrounds \(B\) if \(\min(A - B) < \min(B - A)\) and \(\max(A - B) > \max(B - A)\).)

For brevity, in what follows we refer to strongly, weakly, and chord separated collections as \(s\)-, \(w\)-, and \(c\)-collections, respectively. The sets (classes) of maximal collections among those in \(2^{[n]}\) are denoted by \(S_n\), \(W_n\), and \(C_n\), respectively. Their *-symmetric counterparts are denoted by \(\text{sym-}S_n\), \(\text{sym-}W_n\), and \(\text{sym-}C_n\), respectively.

It is well-known that each of \(S_n\), \(W_n\), \(C_n\) is extended to a poset with a unique minimal and a unique maximal elements; in particular, such posets are connected. Here the poset structures arise when the collections forming these classes are linked by binary relations, or “edges”, where each edge corresponds to a local transformation, called a flip (or mutation), which turns one collection into a “neighboring” collection in this class. See [12, 8] for details.

Leclerc and Zelevinsky [12] revealed the poset structures on \(S_n\) and \(W_n\) (establishing flips “in the presence of six and four witnesses”, respectively), and the poset structure on \(C_n\) was demonstrated by Galashin [8].

In this paper, we introduce local transformations, called symmetric flips, on the members of \(\text{sym-}S_n\), \(\text{sym-}W_n\), and \(\text{sym-}C_n\) (with \(n\) even in the last case), and show that such flips endow these classes with the structure of connected graphs (though not
necessarily posets). In other words, for any two maximal symmetric collections in each class, we can obtain one from the other by a series of flips within this class.

Extending some of these results, we further introduce and study symmetric flips for more sophisticated classes, namely, ones formed by size-maximal symmetric strongly $r$-separated collections in $2^{[n]}$. Here for an integer $r \geq 1$, sets $A, B \subseteq [n]$ are called $r$-separated if there are no elements $i_0 < i_1 < \cdots < i_{r+1}$ of $[n]$ alternating in $A - B$ and $B - A$. Accordingly, a collection $\mathcal{S} \subseteq 2^{[n]}$ is called strongly $r$-separated if any two of its members are such. Usually, when speaking of such sets and collections, the adjective “strong” will be omitted for brevity. (So $s$- and $c$-collections are just 1- and 2-separated ones, respectively.)

An essential part of our study will be focused on the case when both $n$ and $r$ are even, yielding a generalization for symmetric $c$-collections with $n$ even. We show that in this case the symmetric flip structure forms a connected poset. Moreover, we explain that this poset is associated with a structure on $\binom{[n]}{r+1}$ that can be interpreted as a symmetric version of higher Bruhat orders, which is related to type $C$ parameterized by $(n, r+1)$ (where $\binom{[n]}{m}$ is formed by the $m$-element subsets of $[n]$). (Recall that the notion of higher Bruhat orders was introduced by Manin and Schechtman [13] as a generalization of the classical weak Bruhat order on permutations.) A wider discussion on higher Bruhat orders of types $B$ and $C$ will appear in the forthcoming paper [6].

As is shown in Galashin and Postnikov [9], the purity behavior does not continue to hold when $\min\{r, n - r\} \geq 3; \text{ namely, there exist maximal by inclusion } r\text{-separated collection in } 2^{[n]} \text{ which are not maximal by size. A similar behavior takes place for symmetric } r\text{-separated collections as well.}

In light of this, we write $s_{n,r}$ for the maximal possible size $|\mathcal{S}|$ among all (not necessarily symmetric) $r$-separated collections $\mathcal{S}$ in $2^{[n]}$, and refer to such collections as size-maximal. The class of $r$-separated collections in $2^{[n]}$ which are size-maximal and simultaneously $s$-symmetric is denoted as $\text{sym-S}_{n,r}$. (Note that a priori $\text{sym-S}_{n,r}$ may be empty; e.g. this happens when $n$ is odd and $r = 1$.) When both $n, r$ are even, we show that $\text{sym-S}_{n,r} \neq \emptyset$, introduce symmetric flips for the members of $\text{sym-S}_{n,r}$ and establish a poset structure on $\text{sym-S}_{n,r}$ (as mentioned above).

In order to obtain the above-mentioned results, we essentially attract geometric methods. We are based on the nice bijections between (i) maximal $s$-collections and rhombus tilings (see [12, Th. 1.6] where the language of pseudo-line arrangements, dual to rhombus tilings, is used), (ii) maximal $w$-collections and combined tilings (see [2]), and (iii) between size-maximal $r$-separated collections and fine zonotopal tilings, or cubillages, on cyclic zonotopes of dimension $r + 1$. In these cases, the sets in each collection $\mathcal{S}$ as above are encoded the vertices of the corresponding geometric object. (For various aspects of cubillages, see survey [3].)

It turns out that similar relationships between set-systems and geometric objects take place in the symmetric versions as well (for symmetric $s$, $w$, and $c$-cases, see [4]). Using such correspondences, we study symmetric tilings and cubillages, introduce symmetric flips on them, and examine the obtained flip graph in each case.

This paper is organized as follows. Section 2 reviews basic facts on rhombus and combined tilings needed to us. Section 3 is devoted to flips in symmetric $s$-collections.
(yielding Theorem 3.4), and Subsection 4.1 to flips in symmetric w-collections (yielding Corollary 4.2), both dealing with an even number \( n \) of colors. The case of symmetric s- and w-collections with \( n \) odd is considered in Subsection 4.2. Both Sections 3 and 4 are well illustrated to facilitate understanding.

Then we proceed to a study of the flip structure for symmetric \( r \)-separated collections in \( 2^n \) and symmetric \( d \)-dimensional cubillages on the cyclic zonotope \( Z(n, d = r + 1) \). Section 5 gives necessary definitions and reviews known results on important ingredients of cubillages that will be used later, in particular, membranes (subcomplexes of a cubillage that admit bijective projections to cubillages of the previous dimension). In Section 6 we show that for \( n, r \) even, the symmetric flip structure forms a poset with one minimal and one maximal elements (see Theorem 6.3 and Corollary 6.4). This leads to a symmetric higher Bruhat order (of type C parameterized by \( (n, d = r + 1) \)), which is discussed in Section 7 and summarized in Theorem 7.1. The main part of the paper finishes with Section 8 which presents assertions of a general character, showing that a symmetric cubillage contains a symmetric membrane (Theorem 8.1), and that any symmetric cubillage can be lifted as a membrane in a symmetric cubillage of the next dimension (Theorem 8.2). As a consequence, any cyclic zonotope \( Z(n, d) \) with \( n \) even has at least one symmetric cubillage (Corollary 8.3).

Three additional settings are discussed in appendixes to this paper. Appendix A considers symmetric \( r \)-separated collections in \( 2^n \) when \( n \) is even but \( r \) is odd, while Appendix B deals with \( n \) odd and \( r \) even. In its turn, Appendix C is devoted to symmetric weakly \( r \)-separated set-systems. In each of these three cases, we specify the definitions of appropriate symmetric flips and raise conjectures whose validity would imply the connectedness of the corresponding flip graphs.

## 2 Preliminaries

In this section we recall (or specify) the definitions of zonogon, rhombus and combined tilings and flips in them, and review some basic properties of these objects that we will use later on.

As before, let \( n \in \mathbb{Z}_{>0} \). An interval in \([n]\) is a set of the form \( \{a, a+1, \ldots, b\} \subseteq [n], \) denoted as \([a..b]\) (so \([n] = [1..n]\)). For disjoint subsets \( A \) and \( \{a, \ldots, b\} \) of \([n]\), we will use the abbreviated notation \( Aa \ldots b \) for \( A \cup \{a, \ldots, b\} \), and write \( A - c \) for \( A - \{c\} \) when \( c \in A \).

**Zonogon.** Let \( \Xi \) be a set of \( n \) vectors \( \xi_i = (x_i, y_i) \in \mathbb{R}^2 \) such that

\[
(2.1) \quad x_1 < \cdots < x_n \text{ and } y_i = 1 - \delta_i, \ i = 1, \ldots, n,
\]

where each \( \delta_i \) is a sufficiently small positive real. For our purposes, we additionally assume that

\[
(2.2) \quad \text{ (i) } \Xi \text{ satisfies the strict concavity condition: for any } i < j < k, \text{ there exist } \lambda, \lambda' \in \mathbb{R}_{>0} \text{ such that } \lambda + \lambda' > 1 \text{ and } \xi_j = \lambda \xi_i + \lambda' \xi_k; \text{ and (ii) the vectors in } \Xi \text{ are } \mathbb{Z}_2\text{-independent (i.e., all 0,1-combinations of these vectors are different).}
\]
The zonogon generated by \( \Xi \) is the Minkowski sum of segments \([0, \xi_i] \):

\[
Z = Z(\Xi) := \{ \lambda_1 \xi_1 + \cdots + \lambda_n \xi_n : \lambda_i \in \mathbb{R}, 0 \leq \lambda_i \leq 1, i = 1, \ldots, n \}.
\]

The choice of \( \Xi \) is usually not important to us (subject to (2.1), (2.2)), and we may denote \( Z \) as \( Z(n, 2) \). Each subset \( X \subset [n] \) is identified with the point \( \sum_{i \in X} \xi_i \) in \( Z \); so, due to (2.2)(ii), different subsets are identified with different points.

Besides \( \xi_1, \ldots, \xi_n \), we will often use the vectors \( \epsilon_{ij} := \xi_j - \xi_i \) for \( 1 \leq i < j \leq n \).

**Rhombus and combined tilings.** A tiling of these sorts is a subdivision of the zonogon \( Z = Z(\Xi) \) into convex polygons specified below and called tiles. Any two intersecting tiles share a common vertex or edge, and each edge of the boundary of \( Z \) belongs to exactly one tile. We associate to a tiling \( T \) the planar graph \( (V_T, E_T) \) whose vertex set \( V_T \) and edge set \( E_T \) are formed by the vertices and edges occurring in tiles. Each vertex is (a point identified with) a subset of \([n]\). And each edge is a line segment viewed as a parallel translation of either \( \xi_i \) or \( \epsilon_{ij} \) for some \( i < j \). In the former case, it is called an edge of type or color \( i \), or an \( i \)-edge, and in the latter case, an edge of type \( ij \), or an \( ij \)-edge. These edges are directed accordingly. In particular, the left (right) boundary of \( T \) forms the directed path \((v_0, e_1, v_1, \ldots, e_n, v_n)\) in which each vertex \( v_i \) represents the interval \([i]\) (resp. the interval \([n + 1 - i, n]\)).

In a rhombus tiling, each tile \( \tau \) is a parallelogram with edges of types \( i \) and \( j \). For brevity we refer to \( \tau \) as a rhombus, or an \( ij \)-rhombus (where \( i < j \)), and denote it as \( \diamond = \diamond(X|ij) \), where \( X \) is its bottommost vertex, or simply the bottom of \( \diamond \).

In a combined tiling, or a combi for short, there are three sorts of tiles, namely, \( \Delta \)-tiles, \( \nabla \)-tiles, and lenses, as illustrated in the picture below.

![Diagram of rhombus and combined tilings](image.png)

I. A \( \Delta \)-tile (\( \nabla \)-tile) is a triangle with vertices \( A, B, C \subset [n] \) and edges \( (B, A), (C, A), (B, C) \) (resp. \( (A, C), (A, B), (B, C) \)) of types \( j, i \) and \( ij \), respectively, where \( i < j \). We denote this tile as \( \Delta(A|BC) \) (resp. \( \nabla(A|BC) \)).

II. In a lens \( \lambda \), the boundary is formed by two directed paths \( U_\lambda \) and \( L_\lambda \), with at least two edges in each, having the same beginning (or left) vertex \( \ell_\lambda \) and the same end (right) vertex \( r_\lambda \). The upper boundary \( U_\lambda = (v_0, e_1, v_1, \ldots, e_p, v_p) \) is such that \( v_0 = \ell_\lambda \), \( v_p = r_\lambda \), and \( v_k = X_{ik} \) for \( k = 0, \ldots, p \), where \( p \geq 2, X \subset [n] \) and \( i_0 < i_1 < \cdots < i_p \) (so \( k \)-th edge \( e_k \) is of type \( i_{k-1}i_k \)). And the lower boundary \( L_\lambda = (u_0, e_0', u_1, \ldots, e_q', u_q) \) is such that \( u_0 = \ell_\lambda \), \( u_q = r_\lambda \), and \( u_m = Y - j_m \) for \( m = 0, \ldots, q \), where \( q \geq 2, Y \subset [n] \) and \( j_0 > j_1 > \cdots > j_q \) (so \( m \)-th edge \( e_m' \) is of type \( j_mj_{m-1} \)). Then \( Y = X_{i_0j_0} = X_{i_pj_q} \), implying \( i_0 = j_0 \) and \( i_p = j_q \). Note that \( X \) as well as \( Y \) need not be a vertex in a combi. Due to (2.2)(i), \( \lambda \) is a convex polygon of which vertices are exactly the vertices of \( U_\lambda \cup L_\lambda \).
Note that any rhombus tiling turns into a combi without lenses by splitting each $ij$-rhombus $\Diamond$ into two “semi-rhombi” $\Delta$ and $\nabla$ by drawing the edge of type $ij$ connecting the left and right vertices.

Two properties of tilings are of a fundamental character:

(2.3) \[12\] the map $T \mapsto V_T =: S$ gives a bijection between the set of rhombus tilings $T$ on $Z(n, 2)$ and the set of maximal $s$-collections $S$ in $2^n$;

(2.4) \[2\] the map $K \mapsto V_K =: W$ gives a bijection between the set of combies $K$ on $Z(n, 2)$ and the set of maximal $w$-collections $W$ in $2^n$.

(Note that in \[12\] the bijection exhibited in (2.3) is given in equivalent terms to rhombus tilings, namely, via the commutation classes of pseudo-line arrangements.)

**Flips in set-systems and tilings.** Leclerc and Zelevinsky \[12\] established two important facts on mutations, or flips, in strongly and weakly separated collections:

(2.5) Let $i < j < k$ be three elements of $[n]$ and let $X \subseteq [n] - \{i, j, k\}$.

(i) If an $s$-collection $S \subseteq 2^n$ contains a set $U \in \{Xj, Xik\}$ and the six sets (“witnesses”) $X, Xi, Xk, Xij, Xjk, Xijk$, then replacing in $S$ the set $U$ by the other member of $\{Xj, Xik\}$, we again obtain an $s$-collection.

(ii) If a $w$-collection $W \subseteq 2^n$ contains a set $U \in \{Xj, Xik\}$ and the four sets (“witnesses”) $Xi, Xk, Xij, Xjk$, then replacing in $W$ the set $U$ by the other member of $\{Xj, Xik\}$, we again obtain a $w$-collection.

Following \[12\], in case (i), we call the replacement $Xj$ by $Xik$ (resp. $Xik$ by $Xj$) a raising (resp. lowering) flip in $S$ “in the presence of six witnesses” (where the adjectives are justified by the inequality $|Xj| < |Xik|$). A similar transformation in case (ii) is called a raising (resp. lowering) flip in $W$ “in the presence of four witnesses”. Let $\Gamma_n^s$ ($\Gamma_n^w$) be the directed graph whose vertex set is formed by the maximal $s$-collections (resp. $w$-collections) in $2^n$ and whose edges are formed by the pairs $(A, B)$ of collections where $B$ is obtained by one raising flip from $A$. Since $\sum(|B| : B \in B) = \sum(|A| : A \in A) + 1$, both graphs $\Gamma_n^s$ and $\Gamma_n^w$ are acyclic (have no directed cycles). So they determine posets on the sets (classes) $S_n$ and $W_n$, denoted as $PS_n$ and $PW_n$, respectively. These posets are connected; moreover,

(2.6) both $PS_n$ and $PW_n$ have unique minimal and maximal elements, the former consisting of all intervals, and the latter of all co-intervals in $[n]$.

(where a co-interval is the complement of an interval in $[n]$). Relying on (2.3) and (2.4), one can express the above flips in a geometric form, as follows. Let $T$ be a rhombus or combined tiling on $Z(n, 2)$ and suppose that for some $i < j < k$ and $X \subseteq [n] - \{i, j, k\}$, $T$ contains five vertices $Xi, Xk, Xij, Xjk$ and $U \in \{Xj, Xik\}$. Then (2.3), (2.4), (2.5) imply the existence of a tiling $T'$ with the vertex set $V_T$, equal to $(V_T - \{U\}) \cup \{U'\}$, where $\{U, U'\} = \{Xj, Xik\}$. It is known (cf. \[2\], Prop. 3.2) that if a tiling has vertices of the form $X'$ and $X'i'$, then it has the $i'$-edge $(X', X'i')$. Therefore, assuming for definiteness that $U = Xj$,
• $T$ contains the quadruple $Q$ of edges $(X_i, X_{ij}), (X_k, X_{jk}), (X_j, X_{ij}), (X_j, X_{jk})$, whereas $T'$ the quadruple $Q'$ of edges $(X_i, X_{ij}), (X_k, X_{jk}), (X_i, X_{ik}), (X_k, X_{ik})$.

By a natural visualization, one says that the quadruple $Q$ forms an $M$-configuration, and $Q'$ a $W$-configuration. Accordingly, the transformation $T \rightsquigarrow T'$ is called a raising, or $M$-to-$W$, flip, while $T' \rightsquigarrow T$ a lowering, or $W$-to-$M$, flip on tilings. A simple fact is that $Q$ and $Q'$ are extendable to larger structures, namely:

(2.7) if a rhombus tiling $T$ contains a six-tuple of vertices as in (2.5)(i), then $T$ contains six edges forming a hexagon on these vertices, denoted as $H = H(X|ijk)$; furthermore, there is one extra vertex in the interior of $H$, namely, either $X_j$ or $X_{ik}$; and in the former (latter) case, $H$ is subdivided into three rhombuses as illustrated in the left (resp. right) fragment in Fig. 1.

Figure 1: Flips in rhombus tilings. M- and W-configurations are drawn in bold.

So each M-to-W or W-to-M flip within a hexagon in a rhombus tiling produces another rhombus tiling. This gives rise to a poset on the rhombus tilings in $Z(n, 2)$, which is isomorphic to $PS_n$ as above.

In case of a combi $K$, the flips involving $i, j, k, X$ as above, though changing only one vertex of $K$, can change edges and tiles in a neighborhood of the corresponding W- or M-configuration in a several possible ways; all of them are described in [2, Sect. 3.3]. One possible way is illustrated in Fig. 2.

Figure 2: An example of lowering and raising flips in a combi. Here $\lambda, \lambda', \mu$ are lenses.

**Symmetric tilings and the middle line.** Define $m := \lfloor n/2 \rfloor$; then $n = 2m$ if $n$ is even, and $n = 2m + 1$ if $n$ is odd. An obvious property of the $*$-symmetry is that

(2.8) any $A \subseteq [n]$ satisfies $|A| + |A^*| = n$. 

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Indeed, * is the composition of two involutions on $2^{|n|}$, of which one is defined by $A \mapsto \overline{A} := [n] - A$, and the other by $A \mapsto A^\circ := \{i^\circ : i \in A\}$. Then (2.8) follows from the equalities $|A| + |\overline{A}| = n$ and $|A| = |A^\circ|$.

Next, in order to handle symmetric collections, it is convenient to assume that the set $\Xi$ of generating vectors $\xi_i = (x_i, y_i)$ is symmetric w.r.t. the vertical line $VL := \{(x, y) \in \mathbb{R}^2 : x = 0\}$ through the origin, i.e.,

$$x_i = -x_i^\circ \quad \text{and} \quad y_i = y_i^\circ, \quad i = 1, \ldots, n. \quad (2.9)$$

This implies that the zonogon $Z := Z(\Xi)$ is self-mirror-reflected w.r.t. $VL$.

The even and odd cases of $n$ differ in essential details and will be considered separately. In the rest of this section we assume that $n$ is even, $n = 2m$. Then (2.10) the zonogon $Z$ is self-mirror-reflected w.r.t. the horizontal line

$$ML := \{(x, y) \in \mathbb{R}^2 : y = y_1 + \cdots + y_m\},$$

called the middle line; we denote $y_1 + \cdots + y_m$ by $y^{ML}$. From (1.1) with $n$ even it follows that for $X \subseteq [n]$ and $i \in [n]$, if $i, i^\circ \in X$ then $i, i^\circ \notin X^*$, and if $i \in X \neq i^\circ$ then $i \in X^* \neq i^\circ$. This implies the following nice property of the middle line $ML$:

(2.11) for any $A \subseteq [n]$, the sets $A$ and $A^\circ$ are mirror-reflected, or symmetric, to each other w.r.t. $ML$, i.e., their corresponding points $(x_A, y_A)$ and $(x_{A^\circ}, y_{A^\circ})$ in $\mathbb{R}^2$ satisfy $x_A = x_{A^\circ}$ and $y_A - y^{ML} = y_{A^\circ} - y^{ML}$; in particular, a set $A \subset [n]$ (regarded as a point in $\mathbb{R}^2$) lies on $ML$ if and only if $A$ is self-symmetric: $A = A^\circ$.

**Definition.** A rhombus or combined tiling $T$ on $Z$ is called symmetric if the collection (vertex set) $V_T$ is symmetric.

Since a rhombus or combined tiling is determined by its vertex set (in view of (2.3), (2.4)), $T$ itself is symmetric, i.e., for any tile of $T$, its mirror-reflected tile w.r.t. $ML$ belongs to $T$ as well.

One more useful property of $ML$ shown in [4] (for $n$ even) is as follows:

(2.12) For a symmetric combi $K$ on $Z$, let $V_{K,ML} = (v_0, v_1, \ldots, v_r)$ be the sequence of vertices of $K$ lying on $ML$ from left to right. Then $r = m$ and there is a permutation $\sigma$ on $[m]$ such that for $i = 1, \ldots, m$, $v_i - v_{i-1}$ is congruent to the vector $e_{\sigma(i)\sigma(i^\circ)}$, and the line segment $[v_{i-1}, v_i]$ is either an edge of $K$ or the middle section of some self-symmetric lens $\lambda$ in $K$ (i.e., $v_{i-1} = f_\lambda$ and $v_i = r_\lambda$). The vertices $v_0$ and $v_m$ represent the intervals $[m]$ and $[(m + 1)..n]$, respectively.

We associate to $K$ the permutation $\sigma = \sigma_K$ on $[m]$ as in (2.12) and say that the combi $K$ is agreeable with $\sigma$, and similarly for $V_K$ and for $v_0, \ldots, v_m$.

**Remark 1.** We can split $K$ into the lower half-combi $K^{low}$ and the upper half-combi $K^{up}$, the “parts” of $K$ lying in the halves $Z^{low}$ and $Z^{up}$ of $Z$ formed by the points $(x, y)$ with $y \leq y^{ML}$ and $y \geq y^{ML}$, respectively. Here if $ML$ cuts the interior of a lens
\( \lambda \), then \( K^{\text{low}} \) acquires the lower semi-lens \( \lambda^{\text{low}} \), and \( K^{\uparrow} \) the upper semi-lens \( \lambda^{\uparrow} \) (the parts of \( \lambda \) below and above \([\ell\lambda, r\lambda] \), respectively). Then \( K^{\uparrow} \) is symmetric to \( K^{\text{low}} \).

Due to this, a symmetric combi is determined by its lower half-combi. More precisely, fix a permutation \( \sigma \) on \([m] \) and consider a half-combi \( K' \) on \( Z^{\text{low}} \) consisting of \( \Delta \)- and \( \nabla \)-triangles, lenses and lower semi-lenses at level \( m \) so that the edges lying on \( ML \) be of types \( \vec{i} \vec{i} \) and follow in the order \( \sigma \). Combining \( K' \) and its symmetric half-combi and removing redundant edges lying on \( ML \), we obtain a correct symmetric combi.

Finally, since a rhombus tiling \( T \) is in fact a particular case of combies (up to cutting each rhombus \( \Diamond \) into a pair of \( \Delta \)- and \( \nabla \)-tiles), we can consider the lower and upper half-rhombus-tilings \( T^{\text{low}} \) and \( T^{\uparrow} \). Here each tile of \( T^{\text{low}} \) (\( T^{\uparrow} \)) is either an entire rhombus or a \( \nabla \)-tile (resp. \( \Delta \)-tile) having one edge on \( ML \).

### 3 Flips in symmetric s-collections

As mentioned above, the cases of \( n \) even and odd differ essentially and are considered separately. This section deals with symmetric strongly separated collections in \( 2^{[n]} \) when the number \( n \) of colors is even, \( n = 2m \). The case of symmetric s-collections with \( n \) odd will be considered in Sect. 4.2 simultaneously with w-collections.

Consider a maximal symmetric s-collection \( S \) in \( 2^{[n]} \) and the corresponding rhombus tiling \( T \) with \( V_T = S \) (cf. (2.3)). By (2.12), there are exactly \( m+1 \) sets \( A_0, A_1, \ldots, A_m \) in \( S \) lying on the middle line \( ML \) and ordered from left to right: they are agreeable with a permutation \( \sigma \) on \([m] \), namely, for \( i = 1, \ldots, m \), \( A_i \) is obtained from \( A_{i-1} \) by replacing the element \( \sigma(i) \) by \((\sigma(i))^\circ\).

Let us fix a permutation \( \sigma \) and define \( \text{sym-}S_n(\sigma) \) to be the set of collections in \( \text{sym-}S_n \) agreeable with \( \sigma \). We first are going to define appropriate flips within \( \text{sym-}S_n(\sigma) \), and then will introduce flips which connect certain representatives of \( \text{sym-}S_n(\sigma) \) and \( \text{sym-}S_n(\sigma') \) when permutations \( \sigma \) and \( \sigma' \) differ by one transposition.

Flips in \( \text{sym-}S_n(\sigma) \) are constructed in a natural way relying on (2.5)(i). Here, acting in terms of rhombus tilings \( T \) with \( V_T \in \text{sym-}S_n(\sigma) \), we choose a hexagon \( H \) of \( T \) lying in the lower half \( T^{\text{low}} \) and simultaneously make a flip within \( H \) and its symmetric flip in the symmetric hexagon \( H^* \) lying in \( T^{\uparrow} \).

For a vertex \( v \) of a tiling \( T \), denote the set of (directed) edges entering \( v \) by \( \delta^\text{in}(v) = \delta^\text{in}_T(v) \). Also let \(|v|\) denote the level of \( v \) (i.e., the number of elements when \( v \) is regarded as a subset of \([n]\)). We rely on the next two lemmas.

**Lemma 3.1** Let \( v \) be a vertex of \( T \) such that (a) \(|\delta^\text{in}(v)| \geq 3 \), and (b) \(|v| \) is minimum subject to (a). Then \( v \) is the top vertex of a hexagon \( H(X \mid i < j < k) \) having the W-configuration (i.e., formed by the rhombuses \( \Diamond(X|ik), \Diamond(Xi|jk), \Diamond(Xk|ij) \)).

**Proof** (The method of proof is, in fact, well-known in the literature.) Let \( e, e', e'' \) be three consecutive edges (from left to right) in \( \delta^\text{in}(v) \) and let they have colors \( k > j > i \), respectively. Then \( v \) is the top vertex of the rhombus of \( T \) containing \( e, e' \), say, \( \Diamond' = \Diamond(X'|jk), \) and the rhombus containing \( e', e'' \), say, \( \Diamond'' = \Diamond(X''|ij) \). (So
any vertex $v$ of vertices $v_i$ two consecutive vertices at level $i = 1$ of vertices $v_i$.

By the minimality of $v$, we have $|\delta P^m(v')| < 3$; therefore, the edges $q'$ and $q''$ belong to the same rhombus $\diamondsuit = \diamondsuit(X|ik)$ (where $X = X' - i = X'' - k$). Combining $\diamondsuit, \diamondsuit', \diamondsuit''$, we obtain the desired hexagon with the W-configuration.

It follows that if we apply to a symmetric tiling agreeable with $\sigma$ a sequence of lowering flips involving hexagons $H$ with the top vertex at level $\leq m$ (and simultaneously make raising flips in the corresponding symmetric hexagons above the middle line) as long as possible, we eventually obtain a tiling $T$ with $V_T \in \text{sym}\mathbf{S}_n(\sigma)$ such that

\begin{equation}
(3.1) \text{any vertex } v \text{ of } T \text{ with } |v| \leq m \text{ satisfies } |\delta P^m(v)| \leq 2.
\end{equation}

**Definition.** A flip which makes a W-transformation below $ML$ and its symmetric M-transformation above $ML$, is called a double (hexagonal) flip agreeable with $\sigma$.

**Lemma 3.2** For each permutation $\sigma$ on $[n]$, there exists exactly one tiling $T$ with $V_T \in \text{sym}\mathbf{S}_n(\sigma)$ for which (3.1) is valid.

**Proof** First of all we check that $\text{sym}\mathbf{S}_n(\sigma)$ is nonempty. To see this, let $A_0, A_1, \ldots, A_m$ be the sequence of points (subsets of $[n]$) on $ML$ agreeable with $\sigma$. Then $A_0 = [m]$, and for each $i = 1, \ldots, m$,

$$A_i = ([m] - \{\sigma(1), \ldots, \sigma(i)\}) \cup \{\sigma(1)^\circ, \ldots, \sigma(i)^\circ\}.$$  

This implies $(A_i - A_j) < (A_j - A_i)$ for any $i < j$ (in view of $a < b^\circ$ for any $a, b \in [m]$). Therefore, the collection $\mathcal{A} := \{A_0, \ldots, A_m\}$ is strongly separated. Moreover, one can check that the collection $\mathcal{A}'$ obtained by adding to $\mathcal{A}$ the sets $A_{i-1} \cup A_i$ and $A_{i-1} \cap A_i$ for all $i \in [m]$ is strongly separated as well.

Let $\diamondsuit_i$ be the $\sigma(i)\sigma(i^\circ)$-rhombus with the left vertex $A_{i-1}$ and right vertex $A_i$, $i = 1, \ldots, m$. The union of these rhombuses has the vertex set just $\mathcal{A}'$. Since $\mathcal{A}'$ is strongly separated, there exists a rhombus tiling $T$ with $V_T \supset A'$. This $T$ must contain the rhombuses $\diamondsuit_1, \ldots, \diamondsuit_m$. Replacing in $T$ the half-tiling $T^{\uparrow}$ by the one symmetric to $T'^\downarrow$, we obtain a symmetric tiling on $Z(n, 2)$ agreeable with $\sigma$, as required.

Since $\text{sym}\mathbf{S}_n(\sigma)$ is nonempty, it contains a tiling $T$ obeying (3.1). The uniqueness of such a $T$ follows from the observation that for each $i = m, m - 1, \ldots, 1$, the set $V^i$ of vertices $v$ at level $i$ in $T$ determines the next set $V^{i-1}$. (Indeed, if $\diamondsuit, \diamondsuit'$ are two consecutive rhombuses in which the right vertex $v$ of the former coincides with the left vertex of the latter and lies at level $i$, then (3.1) implies that the bottoms of $\diamondsuit, \diamondsuit'$ form two consecutive vertices at level $i - 1$.)

Denoting the tiling as in this lemma by $T^\text{min}_n(\sigma)$, we obtain the following

**Corollary 3.3** Let $\Gamma^\text{sym}_n(\sigma)$ be the directed graph with the vertex set $\text{sym}\mathbf{S}_n(\sigma)$ whose edges are the pairs $(T, T')$ where the tiling $T$ is obtained by a double hexagonal flip from $T'$ which is lowering below $ML$. Then $\Gamma^\text{sym}_n(\sigma)$ is connected and $T^\text{min}_n(\sigma)$ is its unique minimal (zero-indegree) vertex.
Hence $\Gamma_{n}^{\text{sym}}(\sigma)$ determines a poset, denoted as $\text{PS}_{n}^{\text{sym}}(\sigma)$, in which $T_{n}^{\text{min}}(\sigma)$ is the unique minimal element. Figure 3 illustrates all possible lower half-tilings for $n = 6$ and $\sigma = 123, 132, 312$. Here the ones appeared from minimal tilings $T_{n}^{\text{min}}(\sigma)$ are labeled $A$ (for $\sigma = 123$), $B$ (for $\sigma = 132$), and $C$ (for $\sigma = 312$).

![Figure 3: Case $n = 6$. The configurations (lower half-tilings) for $\sigma = 123$ (namely, A, A2, A3), $\sigma = 132$ (namely, B), and $\sigma = 312$ (namely, C, C2, C3).](image)

The graphs $\Gamma_{n}^{\text{sym}}(\sigma)$ for different $\sigma$’s are disjoint, and in order to connect them we need to introduce one more sort of flips. Consider two permutations $\sigma, \sigma'$ on $[m]$ that differ by one transposition: there is $i$ such that $\sigma(i) = \sigma'(i + 1)$, $\sigma(i + 1) = \sigma'(i)$, and $\sigma(j) = \sigma'(j)$ for $j \neq i, i + 1$. Let for definiteness $\sigma(i) < \sigma(i + 1)$.

Let $A_0, A_1, \ldots, A_m$ be, as before, the corresponding sequence of vertices (subsets of $[n]$) in the middle line for tilings agreeable with $\sigma$, and $A'_0, A'_1, \ldots, A'_m$ a similar sequence for $\sigma'$. Then $A_i \neq A'_i$ and $A_j = A'_j$ for all $j \neq i$. For brevity we further write $\alpha$ for $\sigma(i)$, and $\beta$ for $\sigma(i + 1)$.

Form the auxiliary “zonogon” $\Omega$ with the origin at $A_{i-1} \cap A_{i+1}$ and the generating vectors $\xi_\alpha, \xi_\beta, \xi_{\beta^\circ}, \xi_{\alpha^\circ}$. This $\Omega$ is self-symmetric w.r.t. $ML$ and contains $A_{i-1}, A_{i+1}$ on the boundary and $A_i, A'_i$ in the interior. There are two self-symmetric tilings $\Pi_\sigma, \Pi_{\sigma'}$ on $\Omega$ formed by six rhombuses (of types $\alpha\alpha^\circ, \beta\beta^\circ, \alpha\beta^\circ, \alpha\beta^\circ, \beta\alpha^\circ, \beta^\circ\alpha^\circ$), where $\Pi_\sigma$ contains the vertex $A_i$, and $\Pi_{\sigma'}$ the vertex $A'_i$; they are illustrated in Fig. 4.
The collection consisting of rhombuses $\diamond_1, \ldots, \diamond_{i-1}, \diamond_{i+2}, \ldots, \diamond_m$ as in the proof of Lemma 3.2 plus the ones in $\Pi_\sigma$ can be extended to a tiling $T$ in $\text{sym-}S_n(\sigma)$ (this is shown using the fact that the lower boundary of this collection forms a path from $[m]$ to $[(m+1)..n]$ consisting of $n$ edges of different types). Replacing in $T$ the fragment $\Pi_\sigma$ by $\Pi_{\sigma'}$, we obtain a tiling $T'$ in $\text{sym-}S_n(\sigma')$, and vice versa.

**Definition.** The transformation $T \sim T'$ or $T' \sim T$ as above is called a big (or barrel) flip turning the shape $\Pi_\sigma$ into $\Pi_{\sigma'}$, or conversely. (A transformation of this sort appeared in [7, Sect. 6] under the name of an octagon-flip.)

Now combine the above directed graphs $\Gamma_n^{\text{sym}}(\sigma)$ (over all permutations $\sigma$) into one mixed graph $\Gamma_\text{sym}^{\text{sym}}$ by adding undirected edges to connect each pair of tilings of which one is obtained by a big flip from the other. Figure 5 illustrates the graph $\Gamma_6^{\text{sym}}$. Here $\text{sym-}S_6$ consists of 14 symmetric tilings, seven of them are illustrated in Fig. 3, and the other ones (labeled with primes) are their dual (agreeable with the permutations 321, 231, 213).

Summing up the above constructions and reasonings, we obtain the following result; in terms of flips in symmetric tilings it was shown in [7, Sect. 6] (based on a general fact on reduced words for finite Coxeter groups).

**Theorem 3.4** Any two maximal symmetric s-collections in $2^{[n]}$ with $n$ even can be linked by a series of flips of two sorts: double hexagonal flips (acting within the same “block” $\text{sym-}S_n(\sigma)$) and big flips (linking collections agreeable with neighboring $\sigma, \sigma'$).

Figure 4: big flips swapping fragments $\Pi_\sigma$ and $\Pi_{\sigma'}$. 

Figure 5: The flip graph for $n = 6$. 
Therefore, $\Gamma_n^{sym}$ is connected.

4 Flips in symmetric w-collections

The first part of this section considers maximal symmetric weakly separated collections in $2^{[n]}$ when $n$ is even, while the second one deals with the case of $n$ odd.

4.1 Maximal symmetric w-collections in $2^{[n]}$ with $n$ even. Consider such a collection $\mathcal{W}$ and the symmetric combi $K$ with $V_K = \mathcal{W}$ on the zonogon $Z = Z(n, 2)$ (existing by reasonings in Remark 1). Since any combi without lenses is equivalent to a rhombus tiling, in which case symmetric flips are described in the previous section, we may assume that the set of lenses in $K$ is nonempty.

The symmetry of $K$ w.r.t. the middle line $ML$ of $Z$ (subject to (2.11)) implies that $K$ has at least one lens $\lambda$ whose lower boundary $L_\lambda$ is contained in the lower half-combi $K^{low}$. (In this case, either $\lambda$ lies entirely in $K^{low}$, or $\lambda$ is self-symmetric and its vertices $\ell_\lambda$ and $r_\lambda$ lie on $ML$.) It is known that the directed graph whose vertices are the lenses of a combi and whose edges are the pairs $(\lambda, \lambda')$ such that the upper boundary $U_\lambda$ and the lower boundary $L_\nu$ share an edge is acyclic (see [2 Sec. 3]). This implies that

(4.1) for an arbitrary combi $K'$ on $Z(n', 2)$ and $h \in [n']$, if $K'$ has at least one lens of level $h$, then there is a lens $\lambda$ of this level such that each edge in $L_\lambda$ is shared with a $\nabla$-tile, not with another lens (where the level of a lens is the size $|A|$ of any of its vertices (regarded as subsets of $[n']$)).

Returning to a symmetric combi $K$ on $Z = Z(n, 2)$, take a lens $\lambda$ of level $\leq m$ in $K$ satisfying (4.1) (where $n = 2m$). Choose two consecutive edges $e = (A, B)$ and $e' = (B, C)$ in $L_\lambda$, and let $\nabla = \nabla(D|AB)$ and $\nabla' = \nabla(D'|BC)$ be the $\nabla$-tiles of $K$ containing these edges. By the definition of a lens, there are colors $k > j > i$ such that $Ak = Bj = Ci$ (which is the upper root of $\lambda$). Then the edge $e$ has type $jk$, and $e'$ type $ij$. It follows that the edges $(D, A), (D, B), (D', B), (D', C)$ have colors $j, k, i, j$, respectively, and therefore they form a $W$-configuration in $K$.

This leads to the corresponding symmetric flip in $K$ and $\mathcal{W}$. In terms of $\mathcal{W}$, denoting $D \cap D'$ by $X$, one can see that $D = Xi, D' = Xk, A = Xij, B = Xik, C = Xjk$. Then the symmetric weak flip determined by $\lambda, e, e'$ consists of the usual lowering (weak) flip $Xik \rightsquigarrow Xj$ (in the “presence of four witnesses” $Xi, Xk, Xij, Xjk$) and simultaneously of the raising (weak) flip $(Xik)^* \rightsquigarrow (Xj)^*$. This combined flip transforms $\mathcal{W}$ into a symmetric collection $\mathcal{W}'$.

In terms of $K$, the raising flip handles the lens $\lambda^*$ symmetric to $\lambda$, the edges $e^*$ and $e'^*$ in $U_{\lambda^*}$, and the $\Delta$-tiles $\Delta = \Delta(D^*|A'B^*)$ and $\Delta' = \Delta(D'^*|B'C^*)$ whose edges $(A^*, D^*), (B^*, D^*), (B'^*, D'^*), (C^*, D'^*)$ form an $M$-configuration. The lens $\lambda^*$ coincides with $\lambda$ if the level $h$ of $\lambda$ equals $m$, while $\lambda^*$ lies in $K^{up}$ if $h < m$. As is seen from the description of flips in [2 Sec. 3], the former flip makes a local transformation within $K^{low}$ (a particular case is illustrated in Fig. 2), while the latter flip does so within $K^{up}$. These flips act independently and result in a correct symmetric combi $K'$ with $V_{K'} = \mathcal{W}'$. Hence $\mathcal{W}'$ is weakly separated.
The above double flip preserves the vertex structure on $ML$ and decreases by 1 the total size of vertices of the current combi that are contained in $Z_{low}$. If the new combi $K'$ still has a lens, one can repeat the procedure, and so on. The process terminates when all lenses in the current combi vanish. Then we can conclude with the following

**Proposition 4.1** Let $\mathcal{W}$ be a maximal $w$-collection agreeable with a permutation $\sigma$ on $[m]$. Then one can apply to $\mathcal{W}$ a series of symmetric double (weak) flips as described above, preserving $\sigma$ during the process, so as to eventually obtain an $s$-collection in $\text{sym-}S_n(\sigma)$.

Combining this with Theorem 3.4, we come to the following

**Corollary 4.2** Any two maximal symmetric $w$-collections in $2^{[n]}$ with $n$ even can be linked by a series of flips of two sorts: double hexagonal or weak flips (keeping permutations on $[m]$) and big flips (linking $s$-collections agreeable with neighboring $\sigma, \sigma'$).

### 4.2 When the number $n$ of colors is odd.

Let $n$ be odd, $n = 2m + 1$. We first consider a maximal symmetric $w$-collection $C$ in $2^{[n]}$ and introduce symmetric flips for it, whereas flips for $s$-collections will be obtained as a by-product in the end of this section. It should be noted that the size of such a $C$ is strictly less than that of a maximal non-symmetric collection in $2^{[n]}$, i.e., $|C| < s_{n,1}$ (where $s_{n,r}$ is defined in Sect. 1); see [4]. We rely on the description of flips in the even case, with $2m$ colors, in Sect. 3, and on a relationship between even and odd cases established in [4, Sec. 4].

Consider a maximal symmetric $w$-collection $C \in \text{sym-}W_n$. The color $m + 1$ is self-symmetric: $(m + 1)^\circ = m + 1$, and from definition (1.1) it follows that for any $X \subseteq [n]$, exactly one of $X, X^*$ contains the element $m + 1$. In particular, $C$ has no self-symmetric sets. Partition $C$ as $C' \sqcup C''$, where

$$C' := \{X \in C: m + 1 \notin X\} \quad \text{and} \quad C'' := \{X \in C: m + 1 \in X\}. \quad (4.2)$$

Using the fact that $C$ is weakly separated, one can show that

$$|X| \leq m \quad \text{for all} \quad X \in C' \quad \text{and} \quad |X| \geq m + 1 \quad \text{for all} \quad X \in C'' \quad (4.3)$$

(cf. [4, Exp. (4.1)]). Define

$$\mathcal{W}' := C', \quad \mathcal{W}'' := \{X - \{m + 1\}: X \in C''\}, \quad \text{and} \quad \mathcal{W} := \mathcal{W}' \cup \mathcal{W}''. \quad (4.4)$$

In what follows we denote the $2m$-element set $[n] - \{m + 1\}$ as $[n]^-\circ$, keeping the definition of complementary colors: for $i \in [n]^-\circ$, $i^\circ = n + 1 - i$, and keeping the definition of set symmetry (1.1). In view of (4.3), the symmetric collections in $2^{[n]^-\circ}$ are, in fact, equivalent to those in $2^{[2m]}$, and instead of the set $\text{sym-}W_{2m}$ of maximal symmetric $w$-collections in $2^{[2m]}$, we may deal with the equivalent set for $2^{[n]^-\circ}$, denoted as $\text{sym-}W([n]^-\circ)$. The following relation (shown in [4, Sec. 3]) is valid:

$$\text{(4.5) the map } C \rightarrow \mathcal{W} \text{ (where } \mathcal{W} \text{ is obtained from } C \text{ by (4.2), (4.4)) gives a bijection between } \text{sym-}W_n \text{ and } \text{sym-}W([n]^-\circ).}$$
As a generalization of the usual strong separation, [9] introduced a concept of (strong)\n5 Strongly r-separated collections and cubillages\nMore precisely, we associate to each $C \in \text{sym-W}_n$ the corresponding permutation $\sigma$ \nonumber\nby the vertical segment congruent \nonumber\nBy (2.9)). The shifted half-combi, denoted as \nonumber\nMore precisely, we associate to each $C \in \text{sym-W}_n$ the corresponding permutation $\sigma$ \nonumber\nFinally, let $C$ be strongly separated. One easily checks that so is $\mathcal{W}$. Symmetric \nonumber\nBased on the above observations, we have a natural correspondence between flips \nonumber\nUsing this, we assign flips for $\text{sym-W}_n$ as natural analogs of flips for $\text{sym-W}_{2m}$. \nonumber\nIn other words, the sequence $\mathcal{M} := ([m] = A_0, A_1, \ldots, A_m = [(m + 2) \ldots n])$ of subsets (vertices) on the middle line $ML$ of the zonogon $Z(2m, 2)$ generated by the vectors $\xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_n$ determines two sequences in $C$, both “agreeable with $\sigma$”; namely, $\mathcal{M}$ and its symmetric (in $Z(n, 2)$) sequence \nonumber\nGeometrically, we cut the combi $K$ along the middle line $ML$, preserve the lower \nonumber\nBased on the above observations, we have a natural correspondence between flips \nonumber\nAs to a big flip in a strongly separated collection $\mathcal{W}$, which replaces a vertex $A_i$ \nonumber\n5 Strongly r-separated collections and cubillages\nAs a generalization of the usual strong separation, [9] introduced a concept of (strong)\n$r$-separation for subsets of $[n]$. Recall that for $r \in \mathbb{Z}_{>0}$, sets $A, B \subseteq [n]$ are called
strongly $r$-separated if there are no elements $i_0 < i_1 < \ldots < i_{r+1}$ of $[n]$ that alternate in $A - B$ and $B - A$. Accordingly, a collection $A \subseteq 2^{[n]}$ is called strongly $r$-separated if any two of its members are such.

In particular, the strong separated sets are strong 1-separated, and the chord separated ones are 2-separated. In what follows (except for Appendix C) we will deal with merely strong, not weak, $r$-separation, omitting the adjective “strong” everywhere.

In Sect. 4 we will consider symmetric $r$-separated collections in $2^{[n]}$ when $n$ and $r$ are even, and describe the flip structure for the classes of size-maximal collections $S$ among these, i.e., with $|S|$ equal to $s_{n,r}$ defined in Sect. 1. (The case with $n$ even and $r$ odd will be discussed in Appendix A while that with $n$ odd and $r$ even in Appendix B.) These collections (in each case) are representable, which means that they can be represented by the vertex sets of cubillages on a cyclic zonotope of dimension $r + 1$. The purpose of this section is to give necessary definitions and review some known results on cubillages that will be used later. Some facts that we quote here can be found in [3].

5.1 Cyclic zonotope and cubillages. Let $n, d$ be integers with $n \geq d > 1$. A cyclic configuration of size $n$ in $\mathbb{R}^d$ is meant to be an ordered set $\Xi$ of $n$ vectors $\xi_i = (\xi_i(1), \ldots, \xi_i(d)) \in \mathbb{R}^d$, $i = 1, \ldots, n$, satisfying

\begin{equation}
(5.1) \quad \text{(a) } \xi_i(1) = 1 \text{ for each } i, \text{ and}
\end{equation}

\begin{equation}
\text{(b) for the } d \times n \text{ matrix } A \text{ formed by } \xi_1, \ldots, \xi_n \text{ as columns (in this order), any flag minor of } A \text{ is positive.}
\end{equation}

(A typical sample of such a $\Xi$ is generated by the Veronese curve; namely, take reals $t_1 < t_2 < \cdots < t_n$ and assign $\xi_i := \xi(t_i)$, where $\xi(t) = (1, t, t^2, \ldots, t^{d-1})$.)

Definitions. The zonotope $Z = Z(\Xi)$ generated by $\Xi$ is the Minkowski sum of line segments $[0, \xi_1], \ldots, [0, \xi_n]$. A cubillage (called also a “fine zonotopal tiling” in the literature) is a subdivision $Q$ of $Z$ into $d$-dimensional parallelotopes such that any two either are disjoint or share a face, and each face of the boundary of $Z$ is contained in some of these parallelotopes. For brevity, we refer to these parallelotopes as cubes.

When $n, d$ are fixed, the choice of one or another cyclic configuration $\Xi$ (subject to (5.1)) does not matter in essence, and we unify notation $Z(n, d)$ for $Z(\Xi)$, referring to it as the cyclic zonotope of dimension $d$ having $n$ colors.

Each subset $X \subseteq [n]$ naturally corresponds to the point $\sum_{i \in X} \xi_i$ in $Z(n, d)$, and the cardinality $|X|$ is called the height or level of this subset/point. (W.l.o.g., we usually assume that all combinations of vectors $\xi_i$ with coefficients 0,1 are different.)

Depending on the context, we may think of a cubillage $Q$ on $Z(n, d)$ in two ways: either as a set of $d$-dimensional cubes (and write $C \in Q$ for a cube $C$) or as a polyhedral complex. The 0-, 1-, and $(d-1)$-dimensional faces of $Q$ are called vertices, edges, and facets, respectively. By the subset-to-point correspondence, each vertex is identified with a subset of $[n]$. In turn, each edge $e$ is a parallel translation of some segment $[0, \xi_e]$; we say that $e$ has color $i$, or is an $i$-edge. When needed, $e$ is regarded as a directed edge (according to the direction of $\xi_i$). The set of vertices of $Q$ is denoted by $V_Q$. A face $F$ of $Q$ can be denoted as $(X \mid T)$, where $X \subset [n]$ is the bottommost vertex,
or simply the bottom, and \( T \subset [n] \) is the set of colors of edges, or the type, of \( F \) (note that \( X \cap T = \emptyset \) always holds). An important correspondence shown by Galashin and Postnikov is that

\[(5.2) \text{the map } Q \mapsto V_Q \text{ gives a bijection between the set of cubillages on } Z(n, d) \text{ and the set of size-maximal } (d - 1)\text{-separated collections in } 2^{[n]} \text{ (where the vertices are regarded as subsets of } [n]).\]

In particular, every cubillage \( Q \) can be uniquely restored from its vertex set \( V_Q \). An explicit construction is based on the following property (see, e.g., \([3]\)):

\[(5.3) \text{ for } X, T \subset [n] \text{ with } X \cap T = \emptyset, \text{ if a cubillage } Q \text{ contains the vertices } X \cup S \text{ for all } S \subseteq T, \text{ then } Q \text{ has the face } (X \mid T).\]

### 5.2 Membranes and capsids

Certain subcomplexes in a cubillage are of importance to us. To define them, let \( \pi \) be the projection \( \mathbb{R}^{d+1} \to \mathbb{R}^d \) given by \( x = (x(1), \ldots, x(d+1)) \mapsto (x(1), \ldots, x(d)) =: \pi(x) \) for \( x \in \mathbb{R}^{d+1} \). From (5.1) it follows that if a set \( \Xi \) of vectors \( \xi_1, \ldots, \xi_n \) forms a cyclic configuration in \( \mathbb{R}^{d+1} \), then the set \( \Xi \) of their projections \( \xi_i := \pi(\xi_i), \ i = 1, \ldots, n, \) is a cyclic configuration in \( \mathbb{R}^d \). So \( Z(\Xi) = \pi(Z(\Xi')) \), and we may liberally say that \( \pi \) projects the zonotope \( Z(n, d+1) \) onto \( Z(n, d) \).

For a closed subset \( U \) of points in \( Z(n, d+1) \), let \( U^{\text{fr}} \) (resp. \( U^{\text{rear}} \)) denote the subset of \( U \) “seen” in the direction of the last, \( (d+1)\text{-th} \), coordinate vector \( e_{d+1} \) (resp. \( -e_{d+1} \)), i.e., formed by the points \( x \in U \) such that there is no \( y \in U \) with \( \pi(y) = \pi(x) \) and \( y(d+1) < x(d+1) \) (resp. \( y(d+1) > x(d+1) \)). We call \( U^{\text{fr}} \) (resp. \( U^{\text{rear}} \)) the front (resp. rear) side of \( U \).

**Definition.** A membrane of a cubillage \( Q' \) on \( Z(n, d+1) \) is a subcomplex \( M \) of \( Q' \) such that \( \pi \) homeomorphically projects \( M \) (regarded as a subset of \( \mathbb{R}^{d+1} \)) on \( Z(n, d) \).

Then each facet of \( Q' \) occurring in \( M \) is projected to a cube of dimension \( d \) in \( Z(n, d) \) and these cubes constitute a cubillage \( Q \) on \( Z(n, d) \), denoted as \( \pi(M) \) as well.

Sometimes it is useful to deal with a membrane \( M \) in the zonotope \( Z' = Z(n, d+1) \) without specifying a cubillage on \( Z' \) to which \( M \) belongs. In this case, \( M \) is meant to be a \( d \)-dimensional polyhedral complex lying in \( Z' \) whose vertex set consists of subsets of \( [n] \) (regarded as points) and corresponds to the vertex set of some cubillage \( Q \) on \( Z = Z(n, d) \). Equivalently, the projection \( \pi \) establishes an isomorphism between \( M \) and \( Q \). We call such an \( M \) an (abstract) membrane in \( Z' \) and denote as \( M_Q \). Both notions of membranes are “consistent” since (see, e.g. \([3]\))

\[(5.4) \text{ for any membrane } m \text{ in } Z', \text{ there exists a cubillage on } Z' \text{ containing } M.\]

Two s-membranes in \( Z' \) are of an especial interest. These are the front side \( Z'^{\text{fr}} \) and the rear side \( Z'^{\text{rear}} \) of \( Z' \). Their projections \( \pi(Z'^{\text{fr}}) \) and \( \pi(Z'^{\text{rear}}) \) (regarded as complexes) are called the standard and anti-standard cubillages on \( Z(n, d) \), respectively. Such cubillages in dimension \( d = 2 \) (viz. rhombus tilings) with \( n = 4 \) are drawn in Fig. 6.
Figure 6: left: standard tiling; right: anti-standard tiling

In particular, if \( n = d + 1 \), then \( Z' \) is nothing else than the \((d+1)\)-dimensional cube \((\emptyset \mid [d+1])\), and there are exactly two membranes in \( Z' \), namely, \( Z'_\text{fr} \) and \( Z'_\text{rear} \).

Definitions. Let \( C = (X \mid T) \) be a \((d+1)\)-dimensional cube in \( Z' = Z(n, d+1) \) whose type \( T \) consists of elements \( p_1 < \cdots < p_{d+1} \) of \([n]\). Following terminology in [3, Sec. 8], the image \( \pi(C) \) of \( C \), which is denoted as \((X \mid T)\) as well, is called the capsid in \( Z = Z(n,d) \) with the bottom \( X \) and type \( T \). Since \( C'_\text{fr} \) and \( C'_\text{rear} \) are the only membranes in \( C \), there are exactly two possible cubillages in the capsid \( D = \pi(C) \), which are just \( \pi(C'_\text{fr}) \) and \( \pi(C'_\text{rear}) \). The former (latter) looks like the standard (resp. anti-standard) cubillages (using terminology as above), and we say that \( D \) has the standard filling (resp. the anti-standard filling), and denote it as \( D^\text{st} \) (resp. \( D^\text{ant} \)).

The fillings of \( \mathcal{D} \) are formed by the following cubes (cf., e.g. [5, Exp. (A.1)]):

\[
(5.5) \quad \begin{align*}
(\text{i}) & \quad \mathcal{D}^\text{st} \text{ consists of the cubes } F_i = (X \mid T - p_i) \text{ and } G_j = (Xp_j \mid T - p_j), \text{ where } d - i \text{ is odd (i.e., } i = d + 1, d - 1, \ldots \text{) and } d - j \text{ is even (} j = d, d - 2, \ldots \text{));} \\
(\text{ii}) & \quad \mathcal{D}^\text{ant} \text{ consists of the cubes } F_i = (X \mid T - p_i) \text{ and } G_j = (Xp_j \mid T - p_j), \text{ where } d - i \text{ is even and } d - j \text{ is odd.}
\end{align*}
\]

(As before, for disjoint subsets \( A \) and \( \{a, \ldots, b\} \) of \([n]\), we use the abbreviated notation \( Aa \ldots b \) for \( A \cup \{a, \ldots, b\} \), and write \( A - c \) for \( A - \{c\} \) when \( c \in A \).)

Also one can check that (see [3, Prop. 8.1] or [5, Exp. (3.2)])

\[
(5.6) \quad \text{for } \mathcal{D} \text{ as above, there is a unique vertex in the interior of } \mathcal{D}^\text{st}, \text{ namely, } X \cup \{p_i : d - i \text{ even}\}, \text{ denoted as } I^\text{st}_{X,T}, \text{ and a unique vertex in the interior of } \mathcal{D}^\text{ant}, \text{ namely, } X \cup \{p_i : d - i \text{ odd}\}, \text{ denoted as } I^\text{ant}_{X,T}.
\]

Definition. Suppose that a cubillage \( Q \) on \( Z(n, d) \) contains a capsid \( \mathcal{D} \) as above having the standard (anti-standard) filling. Then the replacement of \( \mathcal{D}^\text{st} \) by \( \mathcal{D}^\text{ant} \) (resp. \( \mathcal{D}^\text{ant} \) by \( \mathcal{D}^\text{st} \)) is called the raising flip (resp. lowering flip) in \( Q \) using \( \mathcal{D} \). Such flips are denoted as \( \mathcal{D}^\text{st} \sim \mathcal{D}^\text{ant} \) and \( \mathcal{D}^\text{ant} \sim \mathcal{D}^\text{st} \).

(A similar mutation in a fine zonotopal tiling was introduced for \( d = 3 \) in [8, Sec. 3]). The resulting set of cubes is again a cubillage on \( Z \). Let \( Q_{n,d} \) denote the set of all cubillages on \( Z(n,d) \). The following property is of importance (cf. [3, Th. D.1]):
(5.7) for \( n, d \) arbitrary, the directed graph \( \Gamma_{n,d} \) whose vertex set is \( Q_{n,d} \) and whose edges are the pairs \((Q, Q')\) such that \( Q' \) is obtained from \( Q \) by a raising flip using some capsid is acyclic and has unique minimal (zero-indegree) and maximal (zero-outdegree) vertices, which are the standard cubillage \( Q_{n,d}^{st} \) and the anti-standard cubillage \( Q_{n,d}^{ant} \) on \( Z(n, d) \), respectively.

As a consequence, any two cubillages on \( Z(n, d) \) can be connected by a series of capsid flips, and \( \Gamma_{n,d} \) determines a poset with the minimal element \( Q_{n,d}^{st} \) and the maximal element \( Q_{n,d}^{ant} \).

### 5.3 Partial order on cubes and inversions.

For two cubes \( C, C' \) of a cubillage \( Q \), if the rear side \( C_{\text{rear}} \) of \( C \) and the front side \( C_{\text{fr}} \) of \( C' \) share a facet, we say that \( C' \) immediately precedes \( C' \). A known fact if that

(5.8) the directed graph whose vertices are the cubes of a cubillage \( Q \) and whose edges are the pairs \((C, C')\) such that \( C \) immediately precedes \( C' \) is acyclic.

This determines a partial order on the set of cubes of \( Q \), called in [3] the natural order on \( Q \); we denote it as \( Q, \prec \) or \( \prec_Q \).

We also will use the fact that the restriction of \( \prec_Q \) to a capsid gives a linear order. More precisely, the following is valid (cf. [3, Prop. 10.1]):

(5.9) For a capsid \( \mathfrak{D} = (X \mid T = (p_1 < p_2 < \cdots < p_{d+1})) \) in \( Q \) and for \( i = 1, \ldots, d + 1 \), let \( C_i \) denote the cube in the filling of \( \mathfrak{D} \) having the type \( T - p_i \) (which exists and unique by (5.5)). Then

(i) \( C_{d+1} \prec C_d \prec \cdots \prec C_1 \) if \( \mathfrak{D} \) has the standard filling \( \mathfrak{D}^{st} \), and

(ii) \( C_1 \prec C_2 \prec \cdots \prec C_{d+1} \) if \( \mathfrak{D} \) has the anti-standard filling \( \mathfrak{D}^{ant} \).

Next, the set \( \mathcal{M}(Q') \) of membranes of a cubillage \( Q' \) on \( Z' = Z(n, d + 1) \) forms a distributive lattice. To see this, let us associate with a membrane \( M \in \mathcal{M}(Q') \) the set of cubes of \( Q' \), denoted as \( Q'(M) \), lying between \( Z'_{\text{fr}} \) and \( M \) (i.e., before \( M \), in a sense). One easily shows that if \( C, C' \) are cubes in \( Q' \) such that \( C \) immediately precedes \( C' \) and \( C' \in Q'(M) \), then \( C \in Q'(M) \) as well. This implies that \( Q'(M) \) forms an ideal of the natural order \( \prec_{Q'} \). Conversely, any ideal of \( \prec_{Q'} \) is representable as \( Q'(M) \) for some membrane \( M \) of \( Q' \). It follows that

(5.10) the set \( \mathcal{M}(Q') \) of membranes of a cubillage \( Q' \) on \( Z' = Z(n, d + 1) \) is a distributive lattice in which for membranes \( M, M' \in \mathcal{M}(Q') \), the w-membranes \( M \land M' \) and \( M \lor M' \) satisfy \( Q'(M \land M') = Q'(M) \cap Q'(M') \) and \( Q'(M \lor M') = Q'(M) \cup Q'(M') \); the minimal and maximal elements of this lattice are \( Z'^{fr} \) and \( Z'^{rear} \), respectively.

Another important known fact is that the set of types \( T \) of the cubes \( C = (X \mid T) \) in the ideal \( Q'(M) \) does not depend on \( Q' \), in the sense that any two cubillages on \( Z' \) containing the same membrane \( M \) have equal sets of types of cubes before \( M \) (see, e.g., [10, 13]). This set of types is denoted as \( \text{Inv}(M) \) and called the set of inversions of \( M \). We also say that this is the set of inversions of the cubillage \( Q = \pi(M) \) on \( Z(n, d) \) and use notation \( \text{Inv}(Q) \) for \( \text{Inv}(M) \).
5.4 Symmetric cubillages. A cubillage $Q$ on $Z = Z(n, d)$ is called symmetric if its vertex set $V_Q$ is symmetric. By \((\ref{5.2})\), such cubillages are bijective to symmetric (strongly) $r$-separated collections $S$ with $r = d - 1$ and $|S| = s_{n,r}$ (earlier we have called such collections size-maximal and representable and denoted their set by $\text{sym-S}_{n,d}$).

In fact, the $\ast$-symmetry on the vertices of $Q$ is extended in a natural way to the faces (edges, facets, cubes) of $Q$. More precisely, if $F = (X \mid T)$ is a face of $Q$ with $T = (i_1 < \cdots < i_r)$, where $r \leq d$, then $F$ has the vertex set $V_F = \{X S = X \cup S: S \subseteq T\}$. By the symmetry of $V_Q$, $Q$ contains the collection of vertices (subsets of $[n]$) symmetric to those in $V_F$; this collection is viewed as $\{(XT)^\ast \cup S': S' \subseteq T^\circ\}$, where we write $T^\circ$ for $(i_1^\circ < \cdots < i_r^\circ)$. (This follows from the identity $(XS)^\ast = (XT)^\ast \cup (T^\circ - S^\circ)$ for any $S \subseteq T$, which is valid when $X \cap T = \emptyset$; a verification of the identity is straightforward and we leave it to the reader as an exercise.) Then, by \((\ref{5.3})\), $Q$ contains the face $((XT)^\ast \mid T^\circ)$, which is regarded as symmetric to $F$ and denoted as $F^\ast$. (When $|T| = d$, we obtain a cube $C$ of $Q$ and its symmetric cube $C^\ast$.)

Strictly speaking, the above construction gives a symmetric “combinatorial-cubic” complex embedded in $Z(n, d)$. However, when needed, we can use a “purely geometric” definition, as follows. Define the generating vectors $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$ as in \((\ref{5.1})\) in a “symmetrized Veronese form”, by

\[(\ref{5.11}) \quad \xi_i = (1, t_i, t_i^2, \ldots, t_i^{d-1}); \quad t_1 < \cdots < t_n; \quad \text{and} \quad t_{i^\circ} = -t_i, \quad \text{where} \quad i^\circ := n + 1 - i.\]

(Also we assume that all 0,1-combinations of these vectors are different.) In particular, for $i = 1, \ldots, [n/2]$, we have $t_i < 0$, $t_{i^\circ} > 0$, and $j$-th coordinate of $\xi_{i^\circ}$ is $(-t_i)^{j-1}$. The zonotope $Z = Z(\Xi = (\xi_1, \ldots, \xi_n))$ has the center at the point $\zeta_Z := \frac{1}{2} (\xi_1 + \cdots + \xi_n)$ and admits two involutions:

(a) the central symmetry $\nu$ w.r.t. $\zeta_Z$, which sends $x$ to $\nu(x) = 2\zeta_Z - x$, and

(b) the symmetry $\mu$ w.r.t. the subspace $\{x \in \mathbb{R}^d: x(p) = 0 \text{ if } p \text{ is even}\}$, which sends $x \in Z$ to $\mu(x) = (x(1), -x(2), \ldots, (-1)^{d-1}x(j), \ldots)$. The composition $\sigma := \mu \circ \nu = \nu \circ \mu$ is again an involution on $Z$, and one can check that the linear map $\sigma$ gives the desired symmetry $X \mapsto X^\ast$ on the subsets $X \subseteq [n]$ (regarded as points $\sum (\xi_i: i \in X)$). Moreover, $\sigma$ is orthonormal, and for each cube $C = (X \mid T)$ in $Z$, $\sigma(C)$ is congruent (up to reversing) to $C$, giving the cube $C^\ast$.

Using this geometric setting, let us demonstrate the following useful fact.

**Lemma 5.1** (i) If $d$ is even, then the front side $Z^{fr}$ of $Z = Z(n, d)$ is self-symmetric, and similarly for the rear side $Z^{rear}$.

(ii) If $d$ is odd, then $Z^{fr}$ and $Z^{rear}$ are symmetric to each other.

**Proof** For a point $x$ in $Z$, consider the points $u = \nu(x)$, $y = \mu(x)$, and $z = \sigma(x)$. Since $\nu$ is the central symmetry on $Z$, $x$ lies on $Z^{fr}$ if and only if $u$ lies on $Z^{rear}$, and similarly, $y \in Z^{fr}$ if and only if $z \in Z^{rear}$. Let $x \in Z^{fr}$. It suffices to show that

(*) if $d$ is even, then $y \in Z^{rear}$ (implying $z \in Z^{fr}$);

(**) if $d$ is odd, then $y \in Z^{fr}$ (implying $z \in Z^{rear}$).

(Then (*) gives (i) in the lemma (concerning $Z^{fr}$), and (**))
To show (⋆) and (⋆⋆), represent \( x \) as \( \sum (\lambda_i x_i : i \in X) \), where \( X \subseteq [n] \) and \( \lambda > 0 \). Then \( y = \sum (\lambda_i x_i : i \in X) \). Comparing the last coordinates of \( x \) and \( y \), we observe that \( y(d) = -x(d) \) if \( d \) is even, and \( y(d) = x(d) \) if \( d \) is odd (since \( \xi_i(d) = t_i^{d-1}, \xi_i(d) = t_i^{d-1} \), and \( t_i = -t_i \)).

For \( d \) even, suppose that \( y \notin Z^{\text{max}} \). Then there is a point \( y' \) in \( Z \) such that \( y'(p) = y(p) \) for \( p = 1, \ldots, d-1 \), and \( y'(d) < y(d) \). Taking the point \( x' = \mu(y') \), we obtain \( x'(p) = x(p) \) for \( p = 1, \ldots, d-1 \), and \( x'(d) = -y'(d) > -y(d) = x(d) \), contrary to the fact that \( x \in Z^\ell \).

When \( d \) is odd, we argue in a similar way. \[\]

Considering the projection \( \pi \) on \( Z(n, d) \), we obtain from this lemma that

\[
(5.12) \quad \text{(i) for } d' \text{ odd, each of the standard and anti-standard cubillages on } Z(n, d') \text{ is self-symmetric;}

\text{(ii) for } d' \text{ even, the standard and anti-standard cubillages on } Z(n, d') \text{ are symmetric to each other.}
\]

6 Flips in symmetric strongly \( r \)-separated collections and symmetric \((r + 1)\)-dimensional cubillages when \( r \) is even.

This section deals with size-maximal (viz. representable) symmetric \( r \)-separated collections in \( 2^n \) when both \( n \) and \( r \) are even. It turns out that in this case the flip structure has a nice property: it forms a poset with a unique minimal and a unique maximal elements, as we show in Theorem 6.3 and Corollary 6.4. Since any maximal chord separated (viz. strongly 2-separated) collection in \( 2^n \) is representable, due to [8], we will obtain a nice characterization of the flip structure for symmetric c-collections with \( n \) even.

Let \( S \subseteq 2^n \) be an \( r \)-separated collection which is size-maximal and symmetric (such an \( S \) exists by Corollary 5.3 below). Let \( d := r + 1 \); then \( d \) is odd. By (5.2), there exists a cubillage \( Q \) on \( Z(n, d) \) such that \( V_Q = S \); this \( Q \) is symmetric.

First of all, using terminology and notation as in Sect. 5.2, consider a capsid \( \mathcal{O} = (X \mid T) \) in \( Q \); let \( T = (p_1 < \cdots < p_{d+1}) \). Taking the cubes symmetric to those occurring in \( \mathcal{O} \), we obtain a capsid in \( Q \) as well, denoted as \( \mathcal{O}^* \). We need two lemmas.

**Lemma 6.1** If \( \mathcal{O} \) has the standard filling, then so does \( \mathcal{O}^* \), and similarly when the filling of \( \mathcal{O} \) is anti-standard.

**Proof** Since \( \mathcal{O} \) has the top vertex \( XT \) and its edges are of colors \( p_1 < \cdots < p_{d+1} \), \( \mathcal{O}^* \) has the bottom \( (XT)^* \) and its edges have the colors \( p_{d+1}^* < \cdots < p_1^* \). Hence \( \mathcal{O}^* = ((XT)^* \mid T^o) \), where \( T^o = \{ p_{d+1}^*, \ldots, p_1^* \} \). Let us examine cubes in \( \mathcal{O} \) and \( \mathcal{O}^* \), using notation as in (5.5).

We say that a cube \( C = (X' \mid T') \) in a filling of \( \mathcal{O} \) is lower (upper) if \( X' = X \) (resp. \( X'T' = XT \)), i.e., it is of the form \( F_i = (X \mid T - p_i) \) for some \( i \) (resp. of the form

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\[ G_j = (Xp_j \mid T - p_j) \text{ for some } j \]. When \( d - i \) or \( d - j \) is odd (even) we say that the cube \( C \) is odd (resp. even). And similarly for the capsid \( \mathfrak{D}^* \).

One can check that for a lower cube \( F_i = (X \mid T - p_i) \) in \( \mathfrak{D} \), its symmetric cube \( F_i^\ast \) in \( \mathfrak{D}^* \) is viewed as \( ((XT)^* p_i \mid T^\circ - p_i^\circ) \). Therefore, \( F_i^\ast \) is upper in \( \mathfrak{D}^* \). Since \( p_i^\circ \) is \((d + 2 - i)\)-th element in the ordered \( T^\circ \) and \( d \) is odd, if \( F_i \) is odd (even) in \( \mathfrak{D} \), then \( F_i^\ast \) is even (resp. odd) in \( \mathfrak{D}^* \). As to an upper cube \( G_j = (Xp_j \mid T - p_j) \) in \( \mathfrak{D} \), its symmetric cube \( G_j^\ast \) in \( \mathfrak{D}^* \) is viewed as \( ((XT)^* \mid T^\circ - p_j^\circ) = (X' \mid T' - p_{d+2-j}) \), implying that \( G_j^\ast \) is lower, and if \( G_j \) is odd (even), then \( G_j^\ast \) is even (odd).

Now suppose that \( \mathfrak{D} \) has the standard filling \( \mathfrak{D}^\ast \). Then, by (5.5)(i), each lower cube in it is odd, and each upper cube is even. From the above analysis it follows that each upper cube in the filling of \( \mathfrak{D}^\ast \) is even, while each lower cube is odd. This means that \( \mathfrak{D}^* \) has the standard filling \( \mathfrak{D}^\ast \), as required.

By similar reasonings, when \( \mathfrak{D} \) has the anti-standard filling, so does \( \mathfrak{D}^* \) as well. 

**Lemma 6.2** Either \( \mathfrak{D} = \mathfrak{D}^* \), or \( \mathfrak{D} \) and \( \mathfrak{D}^* \) have no cube of \( Q \) in common.

**Proof** We use the following

**Claim** For arbitrary integers \( n > d \geq 2 \), let \( \mathfrak{D} = (X \mid T) \) be a capsid in a cubillage \( Q \) on \( Z(n,d) \). Let \( \mathcal{N} \) be a nonempty set of cubes contained in one of \( \mathfrak{D}^\ast, \mathfrak{D}^{\text{ant}} \) such that the union \( \Omega \) of these cubes is convex and different from the whole \( \mathfrak{D} \). Then \( \mathcal{N} \) consists of exactly one cube.

**Proof of the Claim.** Let for definiteness \( \mathcal{N} \) is contained in \( \mathfrak{D}^\ast \) (when \( \mathcal{N} \subset \mathfrak{D}^{\text{ant}} \), the argument is similar). Let \( \mathcal{N}_1, (\mathcal{N}_2) \) be the set of lower (resp. upper) cubes in \( \mathcal{N} \), i.e., of the form \( F_i \) (resp. \( G_j \)) in (5.5)(i).

Suppose that \( \mathcal{N}_1 \) contains two different cubes \( C' = (X \mid T') \) and \( C'' = (X \mid T'') \). Then \( T' \cup T'' = T \), and by the convexity of \( \Omega \) and \( \mathfrak{D} \), \( \mathcal{N}_1 \) coincides with the set \( \mathcal{D}_1 \) of all lower cubes in \( \mathfrak{D}^\ast \). Similarly, if \( |\mathcal{N}_2| \geq 2 \), then \( \mathcal{N}_2 \) is the set \( \mathcal{D}_2 \) of upper cubes in \( \mathfrak{D}^\ast \). Hence at least one of \( |\mathcal{N}_1|, |\mathcal{N}_2| \) is 0 or 1 (for otherwise \( \Omega = \mathfrak{D} \)).

Note that any cube \( F_i = (X \mid T - p_i) \) in \( \mathcal{D}_1 \) and any cube \( G_j = (Xp_j \mid T - p_j) \) in \( \mathcal{D}_2 \) share a facet in common, namely, \( (Xp_j | T - \{p_i, p_j\}) \), where \( T = (p_1 < \cdots < p_{d+1}) \). (In fact, any two cubes in a capsid share a facet, but this is not needed to us.) This implies that if \( |\mathcal{N}_k| \geq 2 \) for some \( k \in \{1, 2\} \), then any cube \( C \) in \( \mathcal{D}_{3-k} \) has at least two facets shared with cubes in \( \mathcal{N}_k \), and by the convexity of \( \Omega \), \( C \) must belong to \( \mathcal{N} \).

It remains to consider the case \( |\mathcal{N}_1| = |\mathcal{N}_2| = 1 \). Let \( F_i \) be the cube in \( \mathcal{N}_1 \), and \( G_j \) the cube in \( \mathcal{N}_2 \). Let \( R \) be their common facet (of type \( T - \{i, j\} \)). Then the cube \( F_i \) is the Minkowsky sum of \( R \) and the segment \([0, -\xi]\), while \( G_j \) is the sum of \( R \) and \([0, \xi]\). Since \( \Omega \) is convex, it contains the convex hull of \( F_i \cup G_j \), and by evident geometric reasons, the latter is strictly larger than \( F_i \cup G_j \) itself (taking into account that \( \xi_i \) and \( \xi_j \) non-colinear). This contradiction implies \( |\mathcal{N}| = 1 \), as required.

**Remark 2.** Using a method from [3, Sec. 4]), one can obtain a sharper property, namely: if the union of some cubes of a cubillage on a zonotope \( Z \) forms a convex region \( \Omega \), then \( \Omega \) is representable as a subzonotope in \( Z \) (we omit a proof here).
Return to the proof of the lemma. Suppose that \( \Omega := \mathcal{D} \cap \mathcal{D}^* \) is different from \( \mathcal{D} \) and contains a cube of \( Q \). Since \( \Omega \) is convex, it consists of exactly one cube \( C = (X \mid T) \), by the Claim. Moreover, \( C \) is self-symmetric, \( C = C^* \). But then the color set \( T \) must be self-symmetric as well. This is impossible since \(|T| = d\) is odd and \( n \) is even.

Based on the above lemmas, we define the desired flips in a symmetric cubillage \( Q \) on \( Z(n, d) \) when \( n \) is even and \( d \) is odd as follows. Note that a capsid \( \mathcal{D} = (X \mid T) \) in \( Q \) is self-symmetric (i.e., \( \mathcal{D} = \mathcal{D}^* \)) if and only if \( X = (XT)^* \) and \( T = T^o \).

**Definition.** For a capsid \( \mathcal{D} \) in \( Q \), the symmetric raising (lowering) flip using \( \mathcal{D} \) consists in the single flip \( \mathcal{D}^{st} \sim \mathcal{D}^{ant} \) (resp. \( \mathcal{D}^{ant} \sim \mathcal{D}^{st} \)) when \( \mathcal{D} \) is self-symmetric, and the pair of raising (resp. lowering) flips, one occurring in \( \mathcal{D} \) and the other in \( \mathcal{D}^* \), when \( \mathcal{D} \neq \mathcal{D}^* \). We also call such a transformation in \( Q \) a central flip in the former case, and a double flip in the latter case.

Clearly such a flip makes again a symmetric cubillage on \( Z(n, d) \). Moreover, whenever \( Q \) has a capsid with the standard (anti-standard) filling, we are always able to make a symmetric raising (resp. lowering) flip in \( Q \). Let \( \text{sym-}Q_{n,d} \) denote the set of symmetric cubillages on \( Z(n, d) \). Relying on (5.7), we can conclude with the following

**Theorem 6.3** For \( n \) even and \( d \) odd, the directed graph \( \Gamma_{\text{sym}}^{n,d} \) whose vertex set is \( \text{sym-}Q_{n,d} \) and whose edges are the pairs \((Q', Q)\) such that \( Q' \) is obtained from \( Q \) by one symmetric (central or double) raising flip is acyclic and has unique minimal (zero-indegree) and maximal (zero-outdegree) vertices, which are the standard cubillage \( Q_{n,d}^{st} \) and the anti-standard cubillage \( Q_{n,d}^{ant} \) on \( Z(n, d) \), respectively.

(Note that by the minimality of \( Q_{n,d}^{st} \), any capsid in it has the standard filling. Similarly, by the maximality of \( Q_{n,d}^{ant} \), any capsid in it has the anti-standard filling. Each of these two cubillages is symmetric, by (5.12).)

As a consequence, we obtain a description of symmetric flips in the set \( \text{sym-}S_{n,r} \) of representable symmetric \( r \)-separated collections in \( 2^{[n]} \) with \( n, r \) even and \( r \geq 2 \). More precisely, for \( S \in \text{sym-}S_{n,r} \), consider the symmetric cubillage \( Q \) on \( Z(n, d = r + 1) \) with \( V_Q = S \) and a capsid \( \mathcal{D} = (X \mid T) \) in \( Q \). The set \( \{XS : S \subseteq T\} \) of (boundary) vertices of \( \mathcal{D} \) forms a subcollection in \( S \), denoted as \( \mathcal{N}_{X,T} \). If \( \mathcal{D} \) has the standard (anti-standard) filling, then, besides \( \mathcal{N}_{X,T} \), \( \mathcal{D} \) contains one more member of \( S \), namely, the set corresponding to the vertex \( I_{X,T}^{st} \) (resp. \( I_{X,T}^{ant} \)) occurring in the interior of \( \mathcal{D} \), as indicated in (5.10). The above observations and results give rise to the following definition and corollary from Theorem 6.3.

**Definition.** Suppose that \( S \in \text{sym-}S_{n,r} \) includes the subcollection \( \mathcal{N}_{X,T} \) for some \( X,T \subset [n] \) with \( X \cap T = \emptyset \) and \(|T| = r + 2 \). Then (by the size-maximality) \( S \) must contain the set either \( I_{X,T}^{st} \) or \( I_{X,T}^{ant} \). The symmetric raising (lowering) flip w.r.t. \((X,T)\) consists of the single replacement of \( I_{X,T}^{st} \) by \( I_{X,T}^{ant} \) (resp. \( I_{X,T}^{ant} \) by \( I_{X,T}^{st} \)) when \( \mathcal{N}_{X,T} \) is self-symmetric, and the pair of symmetric replacements \( I_{X,T}^{st} \sim I_{X,T}^{ant} \) and \( I_{(XT)^*,T^o}^{st} \sim I_{(XT)^*,T^o}^{ant} \) (resp. \( I_{X,T}^{ant} \sim I_{X,T}^{st} \) and \( I_{(XT)^*,T^o}^{ant} \sim I_{(XT)^*,T^o}^{st} \)) otherwise.
Corollary 6.4 For $n, r$ even, the directed graph with the vertex set $\mathsf{sym-S}_{n,r}$ whose edges are the pairs $(S, S')$ such that $S'$ is obtained from $S$ by a symmetric raising flip as in the above definition is acyclic and determines a poset on $\mathsf{sym-S}_{n,r}$ having unique minimal and maximal elements, which are the vertex sets of the standard and anti-standard cubillages on $Z(n, r+1)$, respectively.

It is known that for $r$ even, the vertex set of $Q^\text{st}_{n,r+1}$ consists of all $k$-intervals with $k \leq r/2$ in $\lbrack n \rbrack$ and all $(r/2+1)$-intervals containing the element 1, while the vertex set of $Q^\text{nt}_{n,r+1}$ consists of all $k$-intervals with $k \leq r/2$ and all $(r/2+1)$-intervals containing $n$; cf., e.g. [3] Prop. 16.3]. Here a $k$-interval is meant to be the union of $k$ intervals, but not $k'$ intervals with $k' < k$.

In particular, it follows that the set $\mathsf{sym-C}_n$ of maximal symmetric chord separated collections in $2^{[n]}$ with $n$ even forms a poset with unique minimal and maximal elements, which are formed by all intervals in $\lbrack n \rbrack$ and all 2-intervals containing 1 (in the former case) and $n$ (in the latter case).

7 A relation to higher Bruhat orders

Manin and Schechtman [13] introduced higher Bruhat orders as a generalization of the notion of a weak Bruhat order (being a partial order on the symmetric group via inversions). Recall some definitions from [13].

Definitions. Consider integers $n > d > 0$. A set $P \in \binom{[n]}{d+1}$ whose elements are ordered lexicographically is called a packet (of size $d+1$ in $\lbrack n \rbrack$). The family $F(P)$ is meant to be the ordered collection of all $d$-element subsets of $P$ where these subsets follow in the lexicographic order (i.e., if $P = (i_1 < \cdots < i_{d+1})$, then $F(P)$ is $(P-i_{d+1}, P-i_d, \ldots, P-i_1)$. A linear (total) order $\rho$ on all $d$-element subsets of $\lbrack n \rbrack$ is called admissible if its restriction to each packet $P$ is either lexicographic or anti-lexicographic, i.e., $\rho_P$ is either identical or reverse to the order on $F(P)$. The set of admissible orders is denoted by $A(n, d)$. Two orders $\rho, \rho' \in A(n, d)$ are called elementarily equivalent if they differ by interchanging two neighbors and these neighbors are not contained in the same packet. The quotient of $A(n, d)$ by the equivalence relation is denoted by $B(n, d)$, and the natural projection of $A(n, d)$ to $B(n, d)$ by $\pi$. The set $\text{Inv}(\rho)$ of inversions of $\rho \in A(n, d)$ consists of the packets $P \in \binom{[n]}{d+1}$ for which $F(P)$ has the reverse order in $\rho$. For $r = \pi(\rho) \in B(n, d)$, we write $\text{Inv}(r)$ for $\text{Inv}(\rho)$ (this is correct since $\text{Inv}(\rho) = \text{Inv}(\rho')$ when $\pi(\rho) = \pi(\rho')$). This provides a partial order $\prec$ on $A(n, d)$ or on $B(n, d)$ (the latter is called the Bruhat order for $(n, d)$); here for $r, r' \in B(n, d)$, we write $r \prec r'$ if there is a sequence $r = r_1, r_2, \ldots, r_N = r'$ such that for each $i = 2, \ldots, N$, $\text{Inv}(r_{i-1}) \subset \text{Inv}(r_i)$ and $|\text{Inv}(r_i) - \text{Inv}(r_{i-1})| = 1$.

(In particular, $A(n, 1)$ is the set of linear orders $\sigma$ on $\lbrack n \rbrack$, $\text{Inv}(\sigma)$ is the set of pairs $i < j$ with $\sigma(i) > \sigma(j)$ (inversions), and $(A(n, 1), \prec)$ turns into the weak Bruhat order on the symmetric group $S_n$.)

Kapranov and Voevodsky [10] and Ziegler [14] gave a nice geometric interpretation of higher Bruhat orders. More precisely, the following is true (using terminology and notation from Sect. 5).
There is a bijection $\tau$ of $B(n,d)$ to the set $Q_{n,d}$ of cubillages on $Z(n,d)$ or, equivalently, to the set of (abstract) membranes $M$ in $Z(n,d+1)$. Here $r \in B(n,d)$ is mapped to $M = \tau(r)$ if $\text{Inv}(r)$ is equal to $\text{Inv}(M)$ (the set of types of cubes in $Q(M)$ (i.e., lying before $M$) for any cubillage on $Z(n,d+1)$ containing $M$). Under this correspondence, $r < r'$ holds if and only if the membranes $\tau(r)$ and $\tau(r')$ belong to the same cubillage $Q$ on $Z(n,d+1)$ and the former is obtained from the latter by a series of lowering flips using cubes of $Q$.

Now we are going to define a sort of symmetric Bruhat orders as follows.

**Definitions.** For a packet $P = (p_1 \prec \cdots \prec p_{d+1})$ in $[n]$, define its symmetric packet to be $P^s = (p^s_{d+1} \prec \cdots \prec p^s_1)$. Accordingly, the family $\mathcal{F}(P^s)$ is regarded as symmetric to $\mathcal{F}(P)$. We say that a linear order $\rho$ on $[n]$ is $s$-admissible if for each packet $P \in \binom{[n]}{d+1}$, the restrictions of $\rho$ to (the $d$-element subsets of) $P$ and $P^s$ are either both lexicographic or both anti-lexicographic. The set of $s$-admissible orders is denoted by $A^s(n,d)$. Orders $\rho, \rho' \in A^s(n,d)$ are called *elementarily equivalent* if they differ by interchanging two neighbors not contained in the same packet and, simultaneously, by interchanging its symmetric neighbors. The quotient of $A^s(n,d)$ by the equivalence relation is denoted by $B^s(n,d)$, and the natural projection of $A^s(n,d)$ to $B^s(n,d)$ by $\pi^s$. This gives a symmetric set $\text{Inv}(\rho) = \text{Inv}(r)$ of (pairs of) inversions, where $\rho \in A^s(n,d)$ and $r = \pi^s(\rho)$. The partial order $\prec^s$ on $A^s(n,d)$ or on $B^s(n,d)$ is defined accordingly.

So for $r, r' \in B^s(n,d)$, we write $r \prec r'$ if there is a sequence $r = r_1, \ldots, r_N = r'$ such that for each $i$, $\text{Inv}(r_i)$ is obtained from $\text{Inv}(r_{i-1})$ by adding either one self-symmetric inversion or a pair of symmetric ones. We call $(A^s(n,d), \prec^s)$ (or $(B^s(n,d), \prec^s)$ the symmetric Bruhat order for $(n,d)$.

(Note that, acting in a somewhat similar fashion, we could attempt to formally define a sort of *skew-symmetric* Bruhat orders. Here one should exclude from consideration the self-symmetric packets and think of admissible orders as those linear orders on $\binom{[n]}{d}$ whose restriction to each packet $P$ is lexicographic if and only if the restriction to $P^s$ is anti-lexicographic. But this stuff is beyond our paper.)

When $n$ is odd (even), $B^s(n,d)$ may be interpreted as the Bruhat order of *type B* (resp. *type C*) for $(n,d)$. For an extensive discussion on Bruhat orders of type B and C and their implementations, see [6].

Our constructions and results in Sect. 6 lead to the following

**Theorem 7.1** When $n$ is even and $d$ is odd, there is a bijection $\sigma$ between $B^s(n,d)$ (of type C) and the set $\text{sym} \cdot Q_{n,d}$ of symmetric cubillages on $Z(n,d)$, or, equivalently, the set of (abstract) symmetric membranes $M$ in $Z(n,d+1)$. Here $r \in B^s(n,d)$ corresponds to $M = \sigma(r)$ if $\text{Inv}(r) = \text{Inv}(M)$. Under this correspondence, $r \prec r'$ holds if and only if $\sigma(r)$ and $\sigma(r')$ belong to the same symmetric cubillage on $Z(n,d+1)$ and the former membrane is obtained from the latter by a series of symmetric (double or central) lowering flips.
8 Interrelations between symmetric cubillages and membranes

We know that each zonotope \( Z(n, d) \) contains a cubillage (in particular, the standard and anti-standard ones), and that any cubillage in \( Z(n, d) \) is lifted as a membrane in some cubillage on \( Z(n, d + 1) \) (by (5.4)). This section gives symmetric versions of those properties when the number \( n \) of colors is even (regarding the \(*\)-symmetry).

**Theorem 8.1** For \( n \) even, let \( Q \) be a symmetric cubillage on \( Z(n, d) \). Then \( Q \) contains a symmetric membrane.

**Theorem 8.2** For \( n \) even, let \( Q \) be a symmetric cubillage on \( Z(n, d) \). Then there exists a symmetric cubillage \( Q' \) on \( Z(n, d + 1) \) and a membrane \( M \) of \( Q' \) isomorphic to \( Q \), i.e., \( M = M_Q \).

**Proof of Theorem 8.1.** When \( d \) is even, the theorem follows from assertion (i) in Lemma 7.1. Now assume that \( d \) is odd, and let \( Q \) be a symmetric cubillage on \( Z = Z(n, d) \). To obtain the desired symmetric membrane in \( Q \), we construct, step by step, a sequence of pairs of symmetric membranes \( M, M' \) starting with the pair \( Z^{fr}, Z^{rear} \) (which are symmetric to each other by Lemma 5.1(ii)).

For a membrane \( M \), consider the set \( Q(M) \) of cubes of \( Q \) between \( Z^{fr} \) and \( M \) (see Sect. 5.3). We assume that the current membranes \( M \) and \( M' \) satisfy \( Q(M) \subseteq Q(M') \).

If \( Q(M) = Q(M') \), then \( M = M' \), and we are done. Now assume that \( Q(M') \) strictly includes \( Q(M) \). Among the cubes in \( Q(M') - Q(M) \), choose a minimal cube \( C \) (w.r.t. the natural order \( \prec_Q \)). Then \( C^{fr} \) is entirely contained in \( M \). Replacing in \( M \) the disc \( C^{fr} \) by \( C^{rear} \), we obtain membrane \( M' \) of \( Q \) with \( Q(M') = Q(M) \cup \{C\} \).

Since \( Q \) is symmetric, it has the cube \( C^* \) symmetric to \( C \). By the symmetry of \( M \) and \( M' \), the side \( C^{rear} \) (which is symmetric to \( C^{fr} \)) is entirely contained in \( M' \), and \( C^* \) belongs to \( Q(M') - Q(M) \). The oddness of \( d \) implies that the cubes \( C \) and \( C^* \) are different (for if \( C = (X | T) \), then \( C^* = ((XT)^* | T^o) \), and \( T = T^o \) is impossible since \( d \) is odd and \( n \) is even).

Thus, replacing in \( M' \) the side \( C^{rear} \) by \( C^{fr} \), we obtain the membrane \( M'' \) symmetric to \( M' \), and moreover, \( Q(M') \subseteq Q(M'') \) is valid. Also the gap between \( M' \) and \( M'' \) becomes smaller. Repeating the procedure, we eventually obtain a membrane coinciding with its symmetric one, as required.

**Corollary 8.3** For any \( d \leq n \) with \( n \) even, the zonotope \( Z(n, d) \) has a symmetric cubillage. Therefore, the set \( \text{sym-S}_{n,d-1} \) of symmetric \((d - 1)\)-separated collections of size \( s_{n,d-1} \) in \( 2^{[n]} \) is nonempty.

This is shown by induction on \( d \) (by decreasing \( d \)). The zonotope \( Z(n, n) \) (which is a single cube) is symmetric. Suppose that for \( d < n \), the zonotope \( Z = Z(n, d + 1) \) has a symmetric cubillage \( Q \). By Theorem 8.1 \( Q \) contains a symmetric membrane \( M \). Then the projection \( \pi(M) \) is a symmetric cubillage on \( Z(n, d) \).

**Proof of Theorem 8.2.** Take the (abstract) membrane \( M = M_Q \) in \( Z' = Z(n, d + 1) \). This \( M \) is self-symmetric.
If \( d + 1 \) is odd, then in order to construct the desired symmetric cubillage on \( Z' \) containing \( M \), we first choose an arbitrary cubillage \( R \) on \( Z' \) containing \( M \). Take the set \( R(M) \) of cubes of \( R \) between \( Z'^{\text{fr}} \) and \( M \). Since \( M \) is self-symmetric and \( Z'^{\text{rear}} \) is symmetric to \( Z'^{\text{fr}} \), by Lemma 5.1(ii) (with \( d + 1 \) instead of \( d \)), the cubes symmetric to those in \( R(M) \) should be disposed in the region between \( M \) and \( Z'^{\text{rear}} \), and moreover, they give a proper subdivision of this region. Hence \( R(M) \cup \{ C^* \colon C \in R(M) \} \) is a symmetric cubillage containing \( M \), as required.

Now let \( d + 1 \) be even. Then \( d \) is odd, and using Corollary 6.4 we can construct a sequence of central or double lowering flips in \( Q \) so as to reach the standard cubillage \( Q_{n,d}^{\text{st}} \) on \( Z = Z(n,d) \). Each central flip uses a self-symmetric capsid \( \mathfrak{D} \) with the anti-standard filling \( \mathfrak{D}^{\text{st}} \); this corresponds to a self-symmetric cube \( C \) in \( Z' \) and the flip \( \mathfrak{D}^{\text{ant}} \sim \mathfrak{D}^{\text{st}} \) determines the flip \( C'^{\text{rear}} \sim C'^{\text{fr}} \) in \( Z' \). In its turn, each double flip uses a pair of innerly disjoint symmetric capsids \( \mathfrak{D}, \mathfrak{D}^{\text{st}} \), both with the anti-standard fillings. They correspond to (different) symmetric cubes \( C, C^* \), and the flip \( (\mathfrak{D}^{\text{ant}}, \mathfrak{D}^{\text{ant}}) \sim (\mathfrak{D}^{\text{st}}, \mathfrak{D}^{\text{st}}) \) determines the double lowering flip \( (C'^{\text{rear}}, C'^{\text{rear}}) \sim (C'^{\text{fr}}, C'^{\text{fr}}) \) in \( Z' \).

Thus, the above sequence of flips in \( Z \) starting with \( Q \) and terminating with \( Q_{n,d}^{\text{st}} \) determines a symmetric set of cubes, denoted as \( Q^- \), which subdivide the region between \( Z'^{\text{fr}} \) and \( M \). Acting similarly with symmetric raising (central or double) flips in \( Z \) starting with \( Q \) and ending with \( Q_{n,d}^{\text{ant}} \), we can construct a symmetric set \( Q^+ \) of cubes filling the region between \( M \) and \( Z'^{\text{rear}} \). Now \( Q^- \cup Q^+ \) is the desired symmetric cubillage in \( Z' \) containing \( M \), as required.

9 Concluding Remarks

Besides the \( * \)-symmetry (given in (1.1)), one can deal with the \( \omega \)-symmetry on \( 2^{[n]} \), also called the color exchanging symmetry, which sends \( X \subseteq [n] \) to \( X^\circ := \{ i \in [n] \colon i^\circ \in X \} \). There are interesting interrelations between both sorts of symmetry. One of them is that for a \( * \)-symmetric cubillage \( Q \) on \( Z(n,d) \), each cube \( C = (X \upharpoonright T) \) is symmetric to the cube \( C^* \) viewed as \( ((XT)^* \upharpoonright T^\circ) \) (see Sect. 5.4); so under the \( * \)-symmetry on \( Q \), the types of cubes obey the \( \circ \)-symmetry relation.

When \( n \) is even, \( n = 2m \), there is a bijection \( \omega \) on \( 2^{[n]} \) that turns \( * \)-symmetric sets into \( \circ \)-symmetric ones. More precisely, for \( X \subseteq [n] \), let

\[
X_- := X \cap [m] \quad \text{and} \quad X_+ := X \cap [m+1..2m].
\]

For \( i \in [2m] \), define

\[
i^\circ := \begin{cases} m - i + 1 & \text{if } i \leq m, \\ 2m - i + 1 & \text{if } m < i \leq 2m, \end{cases}
\]  

and accordingly extend this to subsets \( X \subseteq [2m] \) by setting \( X^\circ := \{ i^\circ \colon i \in X \} \). Note that \( \circ \) and \( \hat{\circ} \) commute: \( (i^\circ)^\circ = (i^\hat{\circ})^\circ \).

Now \( X \mapsto Y = \omega(X) \) is defined by

\[
Y_- = [m] - (X_-)^\circ = ([m] - X_-)^\circ \quad \text{and} \quad Y_+ = (X_+)^\circ.
\]
Lemma 9.1 Let $X \subseteq [n]$ and $Y = \omega(X)$. Then $\omega(X^*) = Y^*$.

Proof Let $i \in [m]$. We examine four cases of $\{i, i^\circ\}$ relative to $X$.

Case 1: $i \in X \not\ni i^\circ$. Then $i \in X^* \not\ni i^\circ$. Applying (9.2) to $X$ and to $X^*$, we have $i^\circ \not\in \omega(X)$ and $(i^\circ)^\circ \not\in \omega(X^*)$, and similarly $i^\circ, (i^\circ)^\circ \not\in \omega(X^*)$.

Case 2: $i \not\in X \ni i^\circ$. Then $i \not\in X^* \ni i^\circ$. It follows that $i^\circ \in \omega(X)$ and $(i^\circ)^\circ \in \omega(X)$, and similarly $i^\circ, (i^\circ)^\circ \ni \omega(X^*)$.

Case 3: $i, i^\circ \in X$. Then $i, i^\circ \not\ni X^*$. It follows that $i^\circ \in \omega(X) \ni (i^\circ)^\circ$ and that $i^\circ \in \omega(X^*) \not\ni (i^\circ)^\circ$.

Case 4: $i, i^\circ \not\in X$. Then $i, i^\circ \in X^*$, yielding $i^\circ \in \omega(X) \not\ni (i^\circ)^\circ$ and $i^\circ \not\in \omega(X^*) \ni (i^\circ)^\circ$.

In all cases, $\omega(X)$ and $\omega(X^*)$ are $\circ$-symmetric within the pair $\{i^\circ, (i^\circ)^\circ\}$, whence the result follows.

Thus, $\omega$ maps $*$-symmetric collections into $\circ$-symmetric ones. One can see that the converse holds as well. The next lemma involves separation relations.

Lemma 9.2 Let $n = 2m$ be even (as before) and $r$ odd. Let $A, B \subseteq [n]$ be $r$-separated. Then $C := \omega(A)$ and $D := \omega(B)$ be $r$-separated as well. Conversely, if $C, D$ are $r$-separated, then so are $A, B$.

Proof Sets $X, Y \subseteq [n]$ are said to be $k$-intertwined if $k$ is the minimal number such that there are elements $i_1 < \cdots < i_k$ of $[n]$ that alternate in $X - Y$ and $Y - X$; we denote this $k$ as $\iota(X, Y)$.

One can see that $\iota(A_-, B_-) + \iota(A_+, B_+)$ is equal to either $\iota(A, B)$ or $\iota(A, B) + 1$. Also one can see that

$$
\iota(C_-, D_-) = \iota([m] - A^\circ, [m] - B^\circ) = \iota(A^\circ, B^\circ) = \iota(A_-, B_-);
\iota(C_+, D_+) = \iota(A^\circ_+, B^\circ_+) = \iota(A_+, B_+); \quad \text{and} \quad \iota(C, D) \leq \iota(C_-, D_-) + \iota(C_+, D_+).
$$

This implies that if $\iota(A, B) \leq r$ or if $\iota(A, B) = r + 1 = \iota(A_-, B_-) + \iota(A_+, B_+)$, then $\iota(C, D) \leq r + 1$, and therefore $C, D$ are $r$-separated.

So assume that $\iota(A, B) = r + 1$ and $\iota(A_-, B_-) + \iota(A_+, B_+) = \iota(A, B) + 1$. The latter equality is possible only if both $p := \max(A_- \Delta B_-)$ and $q := \min(A_+ \Delta B_+)$ belong to the same set among $A - B$ and $B - A$; let for definiteness $p, q \in A - B$.

The transformations $A_- \mapsto [m] - A_-$ and $B_- \mapsto [m] - B_-$ swaps the alternating pieces of $A_--B_-$ and $B_- - A_-$. This implies that $\max(([m] - A_-) \Delta ([m] - B_-))$ is equal to $p$ and that $p \in [m] - B_-$. Then

$$
p^\circ = \min(([m] - A_-)^\circ \Delta ([m] - B_-)^\circ) = \min(C_- \Delta D_-) \in D_- - C_-
$$

(since $p \in [m] - B_-$ implies $p^\circ \in ([m] - B_-)^\circ$).

At the same time, under the transformations $A_+ \mapsto A^\circ_+$ and $B_+ \mapsto B^\circ_+$, the minimal element $q$ of $A_+ \Delta B_+$ maps to the maximal element $q^\circ$ of $A^\circ_+ \Delta B^\circ_+ = C_+ \Delta D_+$, and this $q^\circ$ belongs to $A^\circ_+ - B^\circ_+ = C_+ - D_+$ (since $q \in A_+ - B_+$).

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Thus, \( \min(C \triangle D) = p^\circ \in D - C \) while \( \max(C \triangle D) = q^\circ \in C - D \), implying that \( \mathcal{t}(C, D) \) is even. Since \( r \) is odd and \( \mathcal{t}(C, D) \leq \mathcal{t}(A, B) + 1 = r + 2 \), we can conclude that \( \mathcal{t}(C, D) = r + 1 \), and therefore \( C, D \) are \( r \)-separated.

The converse assertion is shown by reversing the above reasonings.

The above lemmas imply the following

**Corollary 9.3** For \( n \) even and \( r \) odd, if \( \mathcal{S} \) is a \( * \)-symmetric \( r \)-separated collection in \( 2^n \), then \( \omega(\mathcal{S}) := \{ \omega(X) : X \in \mathcal{S} \} \) is a \( o \)-symmetric \( r \)-separated collection, and vice versa.

This gives a bijection between the max-size \( * \)-symmetric and \( o \)-symmetric collections in \( 2^n \), leading to a one-to-one correspondence between \( * \)-symmetric and \( o \)-symmetric cubillages on \( Z = Z(n, d = r + 1) \) (when both \( n, d \) are even).

More precisely, the correspondence \( Q \mapsto Q' =: \omega(Q) \) of such cubillages (where \( Q \) is \( * \)-symmetric and \( Q' \) is \( o \)-symmetric) is given via the relation \( V_{Q'} = \omega(V_Q) \) on their spectra, the sets of vertices regarded as collections in \( 2^n \). (Here we use the facts that \( \omega(V_Q) \) is a max-size \((d-1)\)-separated collection and that each max-size \((d-1)\)-separated collection in \( 2^n \) forms a spectrum of a cubillage on \( Z \).)

One can check that for vertices \( A, B \) of \( Q \), \( |A \triangle B| = 1 \) implies \( |\omega(A) \triangle \omega(B)| = 1 \), and vice versa. Equivalently, vertices \( A, B \) of \( Q \) are connected by edge if and only if so are the vertices \( \omega(A) \) and \( \omega(B) \) of \( Q' \). This is extended to the faces of \( Q \) and \( Q' \), namely, if \( F = (X \mid T) \) is a face of \( Q \), then \( (\omega(X) \mid T^\circ) \) is a face of \( Q' \), denoted as \( \omega(F) \), and conversely, for a face \( (X' \mid T') \) of \( Q' \), \( (\omega^{-1}(X') \mid T'^\circ) \) is a face of \( Q \).

Therefore, the \( * \)- and \( o \)-symmetric cubillages are, in fact, represented by the same complex (by ignoring the directions of edges). The correspondence \( Q \mapsto Q' \) has a nice visualization when \( d = 2 \); namely, \( Q' \) is mirror-reflected to \( Q \) w.r.t. the SW-to-NE line (at angle of \( 45^\circ \)) through the center of \( Z(n, 2) \).

In general, the bijection on the cubes is extended, in a natural way, to the capsids of \( Q \) and \( Q' \). However, the fillings of corresponding capsids may be different: if a capsid \( \mathcal{D} \) of \( Q \) has the standard filling, say, then the capsid \( \omega(\mathcal{D}) \) of \( Q' \) may have any of the two possible fillings.

One can see that for a cube \( C = (X \mid T) \) in \( Z \), its \( o \)-symmetric cube \( C^o \) is viewed as \( (X^o \mid T^o) \). This easily implies that \( 9.3 \) for a capsid \( \mathcal{D} \) in \( Z \), both \( \mathcal{D} \) and its \( o \)-symmetric capsid \( \mathcal{D}^o \) have fillings of the same type: both are either standard or anti-standard.

When all capsids of \( Q' \) have standard (resp. anti-standard) fillings, we just deal with the standard (resp. anti-standard) cubillages on \( Z \), which give important special cases of \( o \)-symmetric cubillages.

From a viewpoint of capsids, the difference between the \( * \)- and \( o \)-symmetric settings is impressive for \( d = 2 \) where, in a \( * \)-symmetric cubillage, any pair of symmetric capsids \( \mathcal{D} \) and \( \mathcal{D}^* \) have different fillings (whereas the fillings of their symmetric counterparts \( \omega(\mathcal{D}) \) and \( \omega(\mathcal{D}^*) \) are similar).
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Appendix 1: Symmetric $r$-separated collections and $(r + 1)$-dimensional cubillages when $r$ is odd.

In this additional section we assume, as before, that the number $n$ of colors is even, and are going to explore the flip structure in the set $\text{sym-S}_{n,r}$ of size-maximal (viz. representable) symmetric $r$-separated collections in $2^n$ when $r$ is odd. An attempt to show the connectedness of this structure looks more intricate than in the case of $r$ even considered in Sect. 6. Again, we essentially attract a machinery of cubillages.

Consider a collection $\mathcal{S} \in \text{sym-S}_{n,r}$ and the corresponding symmetric cubillage $Q$ with $V_Q = \mathcal{S}$ on $Z = Z(n,d)$, where $d := r + 1$ is even. We may assume that $d \neq n$ (since $\text{sym-S}_{n,n-1} = \{2^{2n}\}$). Usual flips in cubillages are performed by using capsids (see Sect. 5.2), and we are going to apply them to construct symmetric transformations.

So let $\mathcal{D} = (X | T)$ be a capsid with $T = \{p_1 < \cdots < p_d\}$ in $Q$ and assume for definiteness that it has the standard filling $\mathcal{D}^{st}$. This filling is formed by $[d/2]$ lower cubes $F_i$ (with $d - i$ odd) and $[d/2]$ upper cubes $G_j$ (with $d - j$ even), namely:

$$F_i = (X | T - p_i) \quad \text{for } i = 1, 3, \ldots, d + 1, \quad \text{and} \quad G_j = (Xp_j | T - p_j) \quad \text{for } j = 2, 4, \ldots, d$$

(cf. [5.5](i)). Then the capsid $\mathcal{D}^*$ symmetric to $\mathcal{D}$ (existing since $Q$ is symmetric) has the anti-standard filling formed by the $[d/2]$ upper cubes

$$F_i^* = ((XT)^*p_\i^\circ | T^0 - p_\i^\circ) =: \mathcal{G}_{d+2-i}^\prime = (X'p_\i^\circ | T^0 - p_\i^\circ), \quad i = 1, 3, \ldots, d + 1,$$

(relabeling $(XT)^*$ as $X'$) and $[d/2]$ lower cubes

$$G_j^* = ((XT)^* | T^0 - p_j^\circ) =: \mathcal{F}_{d+2-j}^\prime = (X'| T^0 - p_j^\circ), \quad j = 2, 4, \ldots, d.$$  

It follows that $\mathcal{D} \neq \mathcal{D}^*$. Moreover, since the Claim in the proof of Lemma 6.2 is valid for any $n$ and $d < n$, only two cases are possible:

(A.1) $\mathcal{D}$ and $\mathcal{D}^*$ share either (a) no cube, or (b) exactly one cube.

In case (a), we can make the double (symmetric) flip in $Q$ using $\mathcal{D}, \mathcal{D}^*$, by replacing the filling $\mathcal{D}^{st}$ by $\mathcal{D}^{ant}$, and $\mathcal{D}^{ant}$ by $\mathcal{D}^{st}$. This results in another symmetric cubillage on $Z$. (Note that, in contrast to flips in Sect. 6, the double flip is now viewed as "undirected" since it is raising in $\mathcal{D}$ but lowering in $\mathcal{D}^*$, as it was demonstrated for the special case $d = r + 1 = 2$ described in Sect. 6.)

Now suppose that we are in case (b) of (A.1). Then $\mathcal{D} \cap \mathcal{D}^*$ is a self-symmetric cube $C = (\hat{X} | \hat{T})$ whose type (color set) $\hat{T}$ is symmetric and occurs in both $T$ and $T^0$. Then $\hat{T} = T - p_k = T^0 - p_k^\circ$ for some $k \in [d + 1]$, and either

(i) $d - k$ is odd, $C = F_k = \mathcal{G}_{d+2-k}^\prime$ and $X = X'p_k^\circ$, or

(ii) $d - k$ is even, $C = G_k = \mathcal{F}_{d+2-k}^\prime$ and $X' = Xp_k$.

We call $X'$ in case (i) and $X$ in case (ii) the bottom of $\mathcal{D} \cup \mathcal{D}^*$, denoted by $\tilde{X}$, and denote $T \cup T^0$ by $\tilde{T}$. Then $\tilde{T}$ is symmetric and $|\tilde{T}| = d + 2$ (which is even).

The important special case arises when the union of $\mathcal{D}, \mathcal{D}^*$ and some extra cubes of $Q$ forms a subzonogon $\mathcal{B}$ with the bottom $\tilde{X}$ and the color set $\tilde{T}$. In other words,
is isomorphic to \(Z(\{\xi_n: \alpha \in T\}) \simeq Z(d + 2, d)\). We call \(\mathcal{B}\) a barrel of \(Q\) (or a barrel in \(Z\) compatible with \(Q\)). Also we refer to the set of cubes of \(Q\) occurring in \(\mathcal{B}\) as the filling of \(\mathcal{B}\), and the cube \(C\) as its central cube.

**Remark 3.** Since \(d\) and \(d + 2\) are even, the filling of \(\mathcal{B}\) forms a symmetric subcubillage of \(Q\). By Theorem [5.2], any symmetric cubillage on \(Z(d + 2, d)\) is the projection of some membrane in some symmetric cubillage \(Q'\) on \(Z(d + 2, d + 1)\). The latter zonotope is, in fact, a capsid; it admits only two cubillages: the standard and anti-standard ones, say, \(Q_1, Q_2\) (see Sect. 5.2). Both of them are self-symmetric (as being the projections of the front and rear sides of the cube \(Z(d + 2, d + 2)\); cf. Lemma 5.1). By Theorem 8.1 each of \(Q_1, Q_2\) has a symmetric membrane, and one can conclude from (5.9) (with \(d + 2\) instead of \(d + 1\) that each \(Q_i\) has exactly one symmetric membrane, namely, the one dividing the sequence in (i) or (ii) of \([5.9]\) half-to-half. Hence the barrel \(\mathcal{B}\) admits two symmetric fillings, one coming from a membrane of \(Q_1\), and another from \(Q_2\).

**Example.** Let \(d = 2\) and \(n = 4\). Then \([n] = \{1, 2, 3, 4\}\), \(1^o = 4\) and \(2^0 = 3\). The zonotope (zonogon) \(Z(4, 2)\) is the simplest barrel \(\mathcal{B}\). It has two symmetric fillings (tilings) \(B_1\) and \(B_2\). Here \(B_1\) is the projection of the (unique) symmetric membrane \(M_1\) of the standard filling \(\mathcal{D}^s\) of the capsid (zonotope) \(\mathcal{D} = Z(4, 3)\), while \(B_2\) is the projection of the symmetric membrane \(M_2\) in \(\mathcal{D}^\text{ant}\). The cubillage \(\mathcal{D}^s\) consists of the cubes \(F_1 < G_3 < F_2 < G_1\), having types 123, 124, 134, respectively, and \(M_1\) divides these cubes half-to-half. This gives four facets of \(M_1\), namely: \(H_1 := F_3 \cap G_1\), \(H_2 := G_3 \cap F_2\), \(H_3 := F_4 \cap F_2\), \(H_4 := G_3 \cap G_1\). These facets have types 23, 14, 13, 24, respectively, and their projections generate four cubes (rhombuses) in \(B_1\); the first two rhombuses have symmetric types 23 and 14 and are ordered as \(\pi(H_1) < \pi(H_2)\) (by the natural partial order in \(B_1\)). Also \(M_1\) has two more facets, contained in the boundary of \(\mathcal{D}\) and having types 12 and 34. Altogether, we obtain 4+2=6 rhombuses in \(B_1\). On the other hand, one can check that the tiling \(B_2\) (generated by \(M_2\) in \(\mathcal{D}^\text{ant}\) with the cubes \(F_1 < G_2 < F_3 < G_4\) is formed by another six-tuple of rhombuses \(\pi(H'_i)\), in which the rhombuses \(\pi(H'_1), \pi(H'_2)\) having symmetric types 23 and 14 (respectively) are ordered as \(\pi(H'_1) > \pi(H'_2)\). Such barrel fillings are illustrated in Fig. 4.

**Definition.** We call the replacement of one symmetric filling of a barrel \(\mathcal{B}\) of \(Q\) by the other one a big (or barrel) flip in \(Q\) using \(\mathcal{B}\), borrowing terminology from Sect. 3.

Let \(\Gamma_{n,r}^s\) denote the undirected graph whose vertex set is \(\text{sym-S}_{n,r}\) and whose edges are related to the pairs of symmetric cubillages on \(Z(n, d = r + 1)\) where one is obtained from the other by either a double flip or a big flip.

One can see that \(\Gamma_{n,r}^s\) is nothing else than the graph \(\Gamma_{n,r}^s\) (up to discarding the orientation of edges) constructed in Sect. 3 (see Theorem 3.4). The big flips in it are just those as illustrated in Fig. 3 (cf. Example above).

We study the connected components of \(\Gamma_{n,r}^s\), trying to show that the entire graph is connected, as follows. Let \(\text{sym-Q}_{n',d'}\) denote the set of symmetric cubillages, and \(\text{sym-M}_{n',d'}\) the set of (abstract) symmetric membranes in \(Z(n', d')\).

We know that the cubillages \(Q \in \text{sym-Q}_{n,d}\) one-to-one correspond to the membranes \(M \in \text{sym-M}_{n,d+1}\) (this bijection is given by \(M \mapsto Q = \pi(M)\) and \(Q \mapsto M = M_Q\)); so we may concentrate on handling such membranes and related flips on
them. For $M \in \text{sym-}M_{n,d+1}$, let $K(M)$ denote the set of symmetric cubillages on $Z' = Z(n, d+1)$ that contain $M$, and for $K \in \text{sym-}Q_{n,d+1}$, let $\mathcal{M}(K)$ denote the set of symmetric membranes contained in $K$. We say that $K$ and $M$ are agreeable if $K \in K(M)$ (and $M \in \mathcal{M}(K)$). We introduce the following binary relation on $\text{sym-}M_{n,d+1}$.

**Definition.** Two membranes $M, M' \in \text{sym-}M_{n,d+1}$ are called equivalent if there is a sequence $M = M_0, M_1, \ldots, M_N = M'$ of membranes in $\text{sym-}M_{n,d+1}$ and a sequence $K_1, \ldots, K_N$ of cubillages in $\text{sym-}Q_{n,d+1}$ such that for each $i = 1, \ldots, N$, both $M_{i-1}, M_i$ belong to $\mathcal{M}(K_i)$ (and $K_j, K_j+1 \in \mathcal{K}(M_j)$, $1 \leq j \leq N - 1$).

The equivalence relation is transitive and a maximal set of equivalent membranes is called an orbit. So $\text{sym-}M_{n,d+1}$ is partitioned into a number of orbits.

**Lemma A.1** Let $M, M'$ belong to the same orbit. Then $\pi(M)$ and $\pi(M')$ are connected by a series of symmetric double flips.

**Proof** It suffices to assume that $M, M'$ belong to the same set $\mathcal{M}(K), K \in \text{sym-}Q_{n,d+1}$. As in Sect. 5.3, we associate with $M$ the set $K(M)$ of cubes of $K$ lying between $Z^\text{fr}$ and $M$, and similarly for $M'$. From the symmetry of $M, M'$ it follows that if $C \in K(M') - K(M)$, then $C^* \in K(M) - K(M')$, and if $C^\text{fr} \subset M$, then $C^*\text{rear} \subset M$. Then, using the natural order on the cubes of $K$, one can find a sequence of $k := |K(M') - K(M)|$ cubes $C_1, \ldots, C_k$ and a sequence of $k+1$ membranes $M = M_0, M_1, \ldots, M_k = M'$ in $\mathcal{M}(K)$ such that for each $i$,

$$K(M_i) - K(M_{i-1}) = \{C_i\} \quad \text{and} \quad K(M_{i-1}) - K(M_i) = \{C_i^*\}.$$ 

Then $M_i$ is obtained from $M_{i-1}$ by a symmetric cubic flip using $C_i, C_i^*$, namely, by replacing the front side (disc) $C_i^\text{fr}$ by the rear side $C_i^\text{rear}$, and simultaneously, by replacing $C_i^*\text{rear}$ by $C_i^*\text{fr}$. Therefore, the cubillage $Q_i := \pi(M_i)$ on $Z$ is obtained from $Q_{i-1} := \pi(M_{i-1})$ by a symmetric double flip using the capsids formed by the projections by $\pi$ of the corresponding sides of $C_i$ and $C_i^*$.

(In fact, the orbits of $\text{sym-}M_{n,d+1}$ are analogous to blocks in Sect. 3 where a block is the set of rhombus tilings agreeable with a fixed permutation on $\{|m|\}$.)

Thus, we obtain the connectedness within each orbit, and now we come to the problem of connecting different orbits. We try to do this by attracting special membranes and capsids and making “barrel flips” in the projection of such membranes.

More precisely, suppose that some cubillage $K \in \text{sym-}Q_{n,d+1}$ contains a self-symmetric capsid $\mathcal{D}$, and let $M$ be a membrane in $\mathcal{M}(K)$. By the symmetry of both $M$ and $\mathcal{D}$, the membrane $M$ must split $\mathcal{D}$ into two symmetric halves (each containing $(d+2)/2$ cubes). Let $Q = \pi(M)$; then $Q$ contains $\frac{1}{2}(d+2)^2$ cubes coming from the facets in the interior $I$ of $M \cap \mathcal{D}$ (since any two cubes in a filling of $\mathcal{D}$ share a facet).

Assume, in addition, that besides the facets lying in $I$, $M$ contains a set $J$ of facets of the boundary of $\mathcal{D}$ so that the following property holds: $M$ goes through the whole rim $\mathcal{D}^\text{fr} \cap \mathcal{D}^\text{rear}$ of $\mathcal{D}$; we call such an $M$ perfect w.r.t. $\mathcal{D}$. Then $I \cup J$ must have at least $\binom{d+2}{d}$ facets, and $\pi(I \cup J)$ is nothing else than a barrel $\mathcal{B}$ in $Q$. So we can make
the big flip in $Q$ using $\mathfrak{B}$ (as described above). The resulting symmetric cubillage $Q'$ coincides with $Q$ outside $\mathfrak{B}$, and therefore, the updated membrane $M' = M_Q$ coincides with $M$ outside $\mathfrak{D}$. Moreover, $M'$ must be agreeable with the cubillage $K'$ obtained from $K$ by the flip using $\mathfrak{D}$.

Based on the above observations, we can argue as follows. Define $\text{sym-}D$ to be the set of all (abstract) self-symmetric capsids $\mathfrak{D} = (X \mid T)$ in $Z' = Z(n, d + 1)$ (i.e., $X, T \subset [n], X \cap T = \emptyset, \mid T \mid = d + 2, X = (XT)^*$ and $T = T^\circ$). Also we associate with each orbit $\mathcal{O}$ in $\text{sym-M}_{n,d+1}$ the set $\mathcal{K}(\mathcal{O}) := \cup(\mathcal{K}(M): M \in \mathcal{O})$, called the train of $\mathcal{O}$, and finish with the following conjecture.

(C1): Let $n, d$ be even. Then:

(i) for each “central” capsid $\mathfrak{D} \in \text{sym-D}$, there exists a cubillage $K \in \text{sym-Q}_{n,d+1}$ containing $\mathfrak{D}^\text{ant}$ and a membrane $M \in \mathcal{M}(K)$ that is perfect w.r.t. $\mathfrak{D}$; and

(ii) for each orbit $\mathcal{O}$ in $\text{sym-M}_{n,d+1}$, the train $\mathcal{K}(\mathcal{O})$ contains either the standard cubillage on $Z'$, or a cubillage $K$ for which there are $\mathfrak{D} \in \text{sym-D}$ and $M \in \mathcal{M}(K)$ such that $K$ contains $\mathfrak{D}^\text{ant}$ and $M$ is perfect w.r.t. $\mathfrak{D}$.

One can realize that (C1) gives rise to a method that, starting with an arbitrary membrane in $\text{sym-M}_{n,d+1}$ along with a cubillage agreeable with $M$, updates, step by step, current membranes and cubillages (by making double (cubic) flips on membranes preserving current cubillages or lowering central (capsid) flips properly updating both the current cubillage and membrane) so as to eventually reach the orbit whose train contains the standard cubillage on $Z'$. As a consequence, the validity of (C1) would provide the desired property: for $n, d$ even, any two symmetric cubillages on $Z(n, d)$ could be connected by a series of symmetric double or barrel flips, yielding the connectedness of $\text{sym-S}_{n,d-1}$ by symmetric flips.

B Appendix 2: Symmetric $r$-separation in $2^{[n]}$ when $n$ is odd.

Earlier we have described flip structures on $\text{sym-S}_{n,r}$ in $2^{[n]}$ when $n$ is even. In this additional section we consider $\text{sym-S}_{n,r}$ when $n$ is odd. Note that if, in addition, $r$ is odd, then this class may be empty; we have seen this in Sect 1.2 for $r = 1$, and suspect that a similar behavior takes place for any odd $r$. (Recall that $\text{sym-S}_{n,r}$ consists of those size-maximal $r$-separated collections $\mathcal{S}$ in $2^{[n]}$ (i.e., satisfying $\mid \mathcal{S} \mid = s_{n,r}$; see Sect. 1.2) that are symmetric.)

In what follows we assume that $r$ is even (while $n$ is odd). First of all we have to explain that in this case the set $\text{sym-S}_{n,r}$ is nonempty. Equivalently, for $d := r + 1$, the set $\text{sym-Q}_{n,d}$ of symmetric cubillages on $Z(n, d)$ is nonempty (since $Q \mapsto V_Q$ gives a bijection between $Q_{n,d}$ and $S_{n,r}$). This is stated in Corollary 1.2 below and can be shown by using certain operations for cubillages on $Z(n, d)$ and $Z(n - 1, d)$, as follows.

It is convenient to assume that the set $\Xi$ of generating vectors of the zonotope $Z = Z(\Xi) \simeq Z(n, d = r + 1)$ is given in a symmetrized form. Let $n = 2m + 1$. We
relabel the colors in \([n]\) as \(-m, \ldots, -1, 0, 1, \ldots, m\); then for each \(i\), the symmetric color \(i^0\) is \(-i\). When the generating vectors are given as in (5.11), \(\xi_0\) turns into the first unit base vector \((1, 0, \ldots, 0)\).

Let \(Z' \simeq Z(n-1, d)\) be the zonotope generated by \(\Xi - \{\xi_0\}\). Consider a symmetric cubillage \(Q'\) on \(Z'\) (existing since \(\text{sym-}S_{n-1, r} \neq \emptyset\), by Corollary [8.3]).

**Definitions.** Let \(\pi^0\) denote the projection of \(\mathbb{R}^d\) to \(\mathbb{R}^{d-1}\) given by \(x = (x(1), \ldots, x(d)) \mapsto (x(2), \ldots, x(d))\) (where the coordinates of \(\mathbb{R}^{d-1}\) are labeled \(2, \ldots, d\)). A \((d-1)\)-dimensional subcomplex \(M\) of \(Q'\) is called a 0-membrane if \(\pi^0\) homeomorphically maps \(M\) (regarded as a subset of \(\mathbb{R}^d\)) onto \(Z^0 := \pi^0(Z')\). Such an \(M\) subdivides \(Z'\) into two closed regions \(Z'^{\text{low}}(M)\) and \(Z'^{\text{up}}(M)\) formed by the points below and above \(M\) (in the direction of \(\xi_0\)), respectively; so \(Z'^{\text{low}}(M) \cap Z'^{\text{up}}(M) = M\). Accordingly, \(Q'^{\text{low}}(M)\) and \(Q'^{\text{up}}(M)\) denote the subcubillages of \(Q'\) occurring in \(Z'^{\text{low}}(M)\) and \(Z'^{\text{up}}(M)\), respectively.

There is a nice correspondence between cubillages and 0-membranes. It involves two operations. The 0-expansion operation is applied to a pair consisting of a cubillage on \(Z'\) and a 0-membrane \(M\) in \(Q'\) and acts as follows:

(\(\text{EXP}\)): Move the set (subcubillage) \(Q'^{\text{up}}(M)\) upward by \(\xi_0\), keeping \(Q'^{\text{low}}(M)\), and fill the gap between \(Q'^{\text{low}}(M)\) and \(Q'^{\text{up}}(M) + \xi_0\) by cubes, each being the Minkowsky sum of \(F\) and \([0, \xi_0]\), where \(F\) runs over the set of facets in \(M\).

As a result, we obtain a cubillage on \(Z = Z(n, d)\), called the 0-expansion of \(Q'\) using \(M\) and denoted as \(Q(Q', M)\).

Conversely, let \(Q\) be a cubillage on \(Z\) and let \(\Pi_0\) be the set of cubes \(C = (X | T)\) whose type \(T\) contains color 0; this \(\Pi_0\) is called the 0-pie in \(Q\) (adapting terminology in [3]). The 0-contraction operation applied to \(Q\) acts as follows:

(\(\text{CON}\)): Shrink each cube \(C = (X | T) \in \Pi_0\) to its “lower” facet \(F = (X | T - \{0\})\), and for each cube \(\tilde{C} = (\tilde{X} | \tilde{T})\) of \(Q\) whose bottom \(\tilde{X}\) contains color 0, move \(\tilde{C}\) by \(-\xi_0\), forming the cube \((\tilde{X} - \{0\} | \tilde{T})\) (preserving the remaining cubes of \(Q\)).

One shows that the resulting set of cubes forms a cubillage \(Q'\) on \(Z'\), and the set of facets obtained by shrinking the cubes of \(\Pi_0\) forms a 0-membrane \(M\) in \(Q'\); we call \((Q', M)\) the 0-contraction of \(Q\). The 0-expansion operation applied to \((Q', M)\) returns \(Q\). This leads to the following relation (cf. [3]):

(\(B.1\)) the correspondence \((Q', M) \mapsto Q(Q', M)\) gives a bijection between the set of pairs \((Q', M)\), where \(Q'\) is a cubillage on \(Z(n-1, d)\) and \(M\) is a 0-membrane in \(Q'\), and the set of cubillages on \(Z(n, d)\).

Returning to symmetric settings as before, one can realize that (\(\text{EXP}\)) applied to \(Q' \in \text{sym-}Q_{n-1, d}\) and a symmetric 0-membrane \(M\) in \(Q'\) produces a symmetric cubillage on \(Z(n, d)\), and conversely, (\(\text{CON}\)) applied to \(Q \in \text{sym-}Q_{n, d}\) produces a symmetric cubillage on \(Z(n-1, d)\) and a symmetric 0-membrane in \(Q'\). This yields a symmetric counterpart of (\(B.1\)), namely:
the correspondence \((Q', M) \mapsto Q(Q', M)\) gives a bijection between the set of pairs \((Q' \in \text{sym-}Q_{n-1,d}, M)\), where \(M\) is a symmetric 0-membrane in \(Q'\), and the set \(\text{sym-}Q_{n,d}\).

By Corollary B.2 the set \(\text{sym-}Q_{n-1,d}\) is nonempty. A similar fact for \((n, d)\) is provided by (B.2) and the following

**Lemma B.1** Let \(n, d\) be odd and let \(Q'\) be a symmetric cubillage on \(Z' = Z(n - 1, d)\). Then \(Q'\) contains a symmetric 0-membrane.

**Proof** Such a membrane \(M\) is constructed by a method similar to that in the proof of Theorem 8.1.

More precisely, let \(Z^{\text{up}}' (Z^{\text{low}}')\) denote the upper (resp. lower) side of the boundary of \(Z'\) (which is formed by the points \(x \in Z'\) such that there is no \(y \in Z'\) with \(\pi^0(y) = \pi^0(x)\) and \(y(1) > x(1)\) (resp. \(y(1) < x(1)\)). Both \(Z^{\text{up}}'\) and \(Z^{\text{low}}'\) are 0-membranes in \(Q'\), and moreover, they are symmetric to each other (to see the latter, one can use the fact that \(Z^\text{fr}'\) and \(Z^\text{rear}'\) are symmetric to each other, in view of Lemma 5.1).

Starting with \((Z^{\text{up}}', Z^{\text{low}}')\), we construct, step by step, a sequence of pairs of symmetric 0-membranes \((M, M^*)\) in \(Q'\) such that \(Q'(M) \supseteq Q'(M^*)\), where \(Q'(M^*)\) denotes that set of cubes of \(Q'\) lying below a 0-membrane \(M^*\). When \(Q'(M) = Q'(M^*)\), the current \(M\) coincides with \(M^*\), and we are done.

So assume that \(Q'(M)\) strictly includes \(Q'(M^*)\). To construct the next pair of 0-membranes, let us say that in \(Q'\) a cube \(C\) immediately 0-precedes a cube \(C'\) if \(C^{\text{up}} \cap C'^{\text{low}}\) is a facet. Like the natural order on the cubes of a cubillage (defined in Sect. 5.3), one shows that the relation of immediately 0-preceding is free of directed cycles, and therefore it determines a partial order on \(Q'\), denoted as \(\prec_0\). Note that \(C \prec_0 C'\) implies \((C^*)^\text{low} \prec_0 C^*\). Take a maximal w.r.t. \(\prec_0\) cube \(C\) in \(Q'(M) - Q'(M^*)\). Then \(C^{\text{up}}\) is entirely contained in \(M\). By the symmetry, \((C^*)^{\text{low}} \subset M^*\). Moreover, the cubes \(C\) and \(C^*\) are different (since \(d\) is odd and the colors in \(Q'\) are partitioned into symmetric pairs). Now replacing \(C^{\text{up}}\) by \(C^{\text{low}}\) in \(M\), and \((C^*)^{\text{low}}\) by \((C^*)^{\text{up}}\) in \(M^*\), we obtain a pair \((M', M^*)\) of symmetric 0-membranes for which \(Q'(M') \supseteq Q'(M^*)\) and the gap \(Q'(M') - Q'(M^*)\) becomes smaller. This yields the result.

**Corollary B.2** For \(n, d\) odd, the set \(\text{sym-}Q_{n,d}\) is nonempty.

Now we are going to devise symmetric flips in \(\text{sym-}Q_{n,d}\) (and \(\text{sym-}S_{n,r}\)). Consider a symmetric cubillage \(Q\) on \(Z = Z(n, d)\) and suppose that it contains a capsid \(D = (X \mid T)\) having the anti-standard filling \(D^\text{ant}\). Using the fact that \(d\) is odd and arguing as in the proof of Lemma 6.1, we observe that the symmetric capsid \(D^* = ((XT)^* \mid T^*)\) has the anti-standard filling as well. Two cases are possible.

**Case 1**: \(D\) and \(D^*\) have no cube in common. Then we can apply to \(Q\) the double lowering flip, by making the replacements \(D^\text{ant} \rightsquigarrow D^\text{st}\) and \(D^* \text{ant} \rightsquigarrow D^* \text{st}\). This results in another symmetric cubillage on \(Z\).

**Case 2**: \(D\) and \(D^*\) share a cube \(C = (X' \mid T')\). Note that \(D \neq D^*\) (for otherwise \(D\) must contain an edge of color 0 and the 0-contraction operation transforms \(D\) into a
Lemma 6.2). Hence \( C = C' \), \( D \) is self-symmetric and \( T' = T \cap T^0 \). Since \( |T'| = d \) is odd, \( T' \) is of the form \( \{0\} \cup \{p_1, \ldots, p_t \} \cup \{-p_1, \ldots, -p_t \} \), where \( t = (d-1)/2 \) and \( 0 < p_1 < \cdots < p_t \). Also \( T = T' \cup \{ j \} \) and \( T^0 = T' \cup \{ j^0 = -j \} \) for some \( j \neq 0 \). One may assume that \( X \) is the lowest vertex of \( D \cap D^* \) (then \( |X| + 1 = |X'| = |(XT)^*| - 1 \)).

An important special case arises when the union of \( D, D^* \) and some extra cubes of \( Q \) forms a symmetric subzonotope in \( Z \). It has the bottom \( X \), the color set \( T^\cup := T \cup T^0 = T' \cup \{ j, j^0 \} \) and is isomorphic to \( Z(\{ \xi_i : i \in T^\cup \}) \cong Z(d + 2, d) \). Using terminology as in Sect. A, we call \( B \) a barrel in \( Q \), and refer to the set of \( (d^2 + 2) \) cubes of \( Q \) occurring in it as the filling of \( B \) (where \( 2d + 1 \) cubes belong to \( D \cup D^* \)).

**Definition.** Let \( D, D^*, B \) be as above. The lowering barrel flip in \( Q \) w.r.t. \( B \) replaces the filling of \( B \) by the corresponding color-symmetric filling. More precisely, each face \( (X \cup Y \mid S) \) of \( Q \) within \( B \) turns into the face \( (X \cup Y^0 \mid S^0) \) (as though making the mirror-reflection w.r.t. the corresponding hyperplane through \( X \)).

Under this flip, we obtain again a symmetric cubillage on \( Z \). Here the capsid \( \mathcal{D} = (X \mid T^0) \) (having anti-standard filling in \( Q \)) turns into the color-symmetric capsid \( (X \mid T^0) \) with the standard filling, and similarly the capsid \( \mathcal{D}^* = ((XT)^* \mid T^0) \) turns into \( ((XT)^* \mid T) \) with the standard filling either.

Thus, flips of both sorts decrease the number of capsids with the anti-standard filling. Let \( \mathcal{D}^+(Q) \) denote the set of such capsids in \( Q \). We conjecture the following.

\((C2)\) Let \( n, d \) be odd and let \( Q \in \text{sym-Q}_{n,d} \) be such that \( \mathcal{D}^+(Q) \neq \emptyset \). Then there exists a capsid \( \mathcal{D} \in \mathcal{D}^+(Q) \) such that either \( \mathcal{D}, \mathcal{D}^* \) have no cube in common, or \( \mathcal{D}, \mathcal{D}^* \) share a cube and \( \mathcal{D} \cup \mathcal{D}^* \) is extended to a barrel in \( Q \) (so \( Q \) admits a lowering double flip in the former case, and a lowering barrel flip in the latter case).

In light of reasonings above, the validity of \((C2)\) would imply the following result: for \( n, d \) odd, any cubillage in \( \text{sym-Q}_{n,d} \) can be connected by a series of symmetric lowering (double or barrel) flips to the standard cubillage on \( Z(n, d) \) (which is symmetric), yielding the connectedness of \( \text{sym-S}_{n,d-1} \) via symmetric flips.

**Remark 4.** For \( d \) odd, a symmetric cubillage \( Q \) on \( Z(d + 2, d) \) can be lifted as a symmetric abstract membrane \( M \) in the zonotope \( Z' = Z(d + 2, d + 1) \). However, \( M \) cannot be extended to a symmetric cubillage on \( Z' \). Indeed, \( Z' \) has exactly two cubillages, standard and anti-standard ones. They are projections of the front and rear sides of the cube \( C = Z(n + 2, n + 2) \), but neither \( C^\text{fr} \) nor \( C^\text{rear} \) is symmetric.

C Appendix 3: Symmetric weak \( r \)-separation.

In Sect. A we explained how to devise symmetric flips in maximal symmetric weakly separated collections (or \( w \)-collections) in \( 2^n \). The notion of weak separation is generalized in [3] to any odd integer \( r > 0 \), where
(C.1) sets $A, B \subseteq [n]$ are called weakly $r$-separated if they are strongly $(r+1)$-separated and, in addition: if there are elements $i_1 < i_2 < \cdots < i_{r+2}$ of $[n]$ alternating in $A - B$ and $B - A$, then $|A| \leq |B|$ when $A$ surrounds $B$ (equivalently, $i_1, i_{r+2} \in A - B$, and $|B| \leq |A|$ when $B$ surrounds $A$.

Accordingly, a collection $W \subseteq 2^{[n]}$ is called weakly separated if any two sets in it are such. When $r = 1$, this turns into the notion of w-collection. An important fact shown in [5, Th. 1.1] is that (for $r$ odd) the maximal possible sizes of weakly and strongly $r$-separated collections in $2^{[n]}$ are the same, denoted as $s_{n,r}$. (Note that to introduce and study the concept of weak $r$-separation when $r$ is even is a more sophisticated task; see a discussion in [5, Appendix B]).

In what follows we assume that the number $n$ of colors is even (while $r$ is odd) and denote the set (class) of symmetric weakly $r$-separated collections $W \subseteq 2^{[n]}$ whose size $|W|$ is equal to $s_{n,r}$ by $\text{sym-W}_{n,r}$. (It should be noted that when $n$ is odd, the maximal size of a symmetric weakly $r$-separated collection in $2^{[n]}$ need not be equal to $s_{n,r}$; this is seen already for $r = 1$ in Sect. [12.2. This case is omitted here.)

Our approach to devise flips symmetric in $\text{sym-W}_{n,r}$ is based on the following result (which in turn is a generalization of a result in [12, Th. 1.7] on flips “in the presence of four witnesses” for usual w-collections).

**Theorem C.1 (see [5])** For $r$ odd (and $n$ arbitrary) and for $r' := (r+1)/2$, let $P = (p_1, \ldots, p_r)$ and $Q = (q_0, \ldots, q_r)$ consist of elements of $[n]$ such that $q_0 < p_1 < q_1 < \ldots < p_r < q_r$, and let $X \subseteq [n] - (P \cup Q)$. Define the sets of neighbors (or “witnesses”) of $P, Q$ to be $N(P, Q) := N_{P,Q}^+ \cup N_{P,Q}^-$, where

$$N_{P,Q}^+ := \{Pq : q \in Q\} \cup \{(P-p)q : p \in P, q \in Q\};$$

$$N_{P,Q}^- := \{Q-q : q \in Q\} \cup \{(Q-q)p : p \in P, q \in Q\}.$$  

Suppose that a weakly $r$-separated collection $W \subseteq 2^{[n]}$ contains the set $XP = X \cup P$ (resp. $XQ$) and the sets $XS$ for all $S \in N_{P,Q}^-$. Then the collection obtained from $W$ by replacing $XP$ by $XQ$ (resp. replacing $XQ$ by $XP$) is again weakly $r$-separated.

(In fact, Theorem 1.2 in [5] gives a sharper assertion, but this is not needed to us.) Note that $\{P, Q\}$ is the only pair in $N_{P,Q}^+ := N_{P,Q} \cup \{P, Q\}$ which is not weakly $r$-separated.

We refer to the replacement $XP \sim XQ$ (resp. $XQ \sim XP$) as the raising (resp. lowering) flip using the gadget $G = G_{X,P,Q} := (X \upharpoonright N_{P,Q}^+)$. Also we say that $G$ has the root $X$, type $T := P \cup Q$, height $|XP| = |X| + r'$, lower layer $L_{X,T}^{\text{low}} := \{XS : S \in N_{P,Q}^-, |S| = r'\}$, and upper layer $L_{X,T}^{\text{up}} := \{XS : S \in N_{P,Q}^+, |S| = r + 1\}$.

Now let $W$ be symmetric and suppose that it contains the gadget $G$ as above. Then $W$ contains the gadget $G^*$ symmetric to $G$. One can check that $G^*$ has the root $(XT)^*$ and type $T^*$; the latter is partitioned into the alternating sets $P^*$ (of size $r'$) and $Q^*$ (of size $r' + 1$). The lower layer $L_{(XT)^*, T^*}^{\text{low}}$ of $G^*$ is symmetric to $L_{X,T}^{\text{low}}$, and the upper layer $L_{(XT)^*, T^*}^{\text{up}}$ to $L_{X,T}^{\text{up}}$. Then (in view of (2.8))
the height $|(XT)^*| + r'$ of $G^*$ is equal to $n - 1 + (|X| + r')$, the set $(XT)^*P^\circ$ is symmetric to $XQ$, and $(XT)^*Q^\circ$ is symmetric to $XP$.

**Definition.** The symmetric flip in $W$ using a gadget $G = G_{X,P,Q}$ (and its symmetric gadget $G^*$) consists of the raising flip w.r.t. one, and the lowering flip w.r.t. the other of $G$ and $G^*$, say, $XP \leadsto XQ$ and $(XT)^*Q^\circ \leadsto (XT)^*P^\circ$. A reasonable question is whether these two flips are compatible. This is so, and the resulting double flip produces again a symmetric weakly $r$-separated collection, if the gadgets $G$ and $G^*$ (regarded as subcollections in $W$) do not meet. The latter is guaranteed when the difference of their heights $h_G := |X| + r$ and $h_{G^*} := |(XT)^*| + r'$ is greater than or equal to 2.

Suppose that $\Delta := |h_G - h_{G^*}| < 2$. Since $n$ is even, (C.2) implies that $\Delta = 1$. Then, w.l.o.g., we may assume that $h_{G^*} = h_G + 1$. In other words, the heights of the upper layer of $G$ and the lower layer of $G^*$ are equal. Nevertheless, in this case, if the transformation consists of the raising flip in $G$ and the lowering flip in $G^*$, then no conflict can arise (since the former flip preserves both layers of $G^*$).

On the other hand, the lowering flip $XQ \leadsto XP$ in $G$ may affect $G^*$. This happens if $XQ$ belongs to the lower layer of $G^*$, in which case $XQ$ disappears, the lower layer of $G^*$ decreases, and we cannot appeal to Theorem C.1 with $G^*$. We conjecture that

(C3): If $h_{G^*} = h_G + 1$, then the set $XQ$ does not belong to $L_{(XT)^*P^\circ}^{\text{low}}$.

Subject to the validity of (C3), it is reasonable to raise the next conjecture:

(C4): For $r$ odd and $n$ even, any two collections in sym-$W_{n,r}$ can be connected by a series of symmetric flips using gadgets as above.

**Remark 5.** We know (cf. (5.2)) that any size-maximal strongly $r$-separated collection in $2^{[n]}$ is representable, i.e., it is viewed as the vertex set of a cubillage on $Z(n, r + 1)$ or, equivalently, of a (strong) membrane of some cubillage on $Z(n, r + 2)$. In contrast, it is open at present, whether any size-maximal weakly $r$-separated collection in $2^{[n]}$ is representable, in the sense that it forms the vertex set of a weak membrane in the fragmentation of some cubillage on $Z(n, r + 2)$ (for definitions, see [5, Sec. 6], where also the above open question is stated as a conjecture). In light of this, one can simplify verification of (C3) and weaken (C4), by restricting ourselves by the (sub)class of representable collections in sym-$W_{n,r}$, this would enable us to use a geometric interpretation of gadgets (which are associated with some “central fragments” of cubes of odd dimensions).