Semidirect product of CCR and CAR algebras and asymptotic states in quantum electrodynamics

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Abstract

A $C^*$-algebra containing the CCR and CAR algebras as its subalgebras and naturally described as the semidirect product of these algebras is discussed. A particular example of this structure is considered as a model for the algebra of asymptotic fields in quantum electrodynamics, in which Gauss’ law is respected. The appearance in this algebra of a phase variable related to electromagnetic potential leads to the universal charge quantization. Translationally covariant representations of this algebra with energy-momentum spectrum in the future lightcone are investigated. It is shown that vacuum representations are necessarily nonregular with respect to total electromagnetic field. However, a class of translationally covariant, irreducible representations is constructed explicitly, which remain as close as possible to the vacuum, but are regular at the same time. The spectrum of energy-momentum fills the whole future lightcone, but there are no vectors with energy-momentum lying on a mass hyperboloid or in the origin.

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I. Introduction and summary

In this paper we continue the study of the algebra of asymptotic fields in quantum electrodynamics, in the framework developed earlier in [1] (and motivated by the classical asymptotic structure discussed in [2]). However, the present work is self-contained: the main results of Ref.[2] are recalled and the construction of Ref.[1] is restated here in a modified form.

The existence of, and the algebraic relations between the asymptotic ("in" or "out") observables and fields in quantum electrodynamics is a question of great physical interest. In the asymptotic limit, on the one hand, the details and full complication of the dynamics should lose their importance. On the other hand, the consequences of Gauss' law and of the long-range character of the electromagnetic interaction must survive. Suppose a closed algebra of asymptotic fields in quantum electrodynamics may be constructed. This should imply, at least, that (some) states over this algebra are approached in the asymptotic limit by the expectation values of actual fields in physical states. Therefore, the infrared and charged structure of the full theory should be encoded in the asymptotic algebra, and physical insight into this structure may be gained by investigating representations of this algebra.

Investigations into the infrared structure and asymptotic fields of electrodynamics have a long history, see Refs.[3], [4] and [5] for a review. They have led to the discovery of such structures and effects as superselection sectors of the local observables, the infraparticle problem, and spontaneous breaking of the Lorentz symmetry. However, a clear formulation of a closed algebraic structure of asymptotic fields has not been achieved, although there do exist various partial answers to this problem, with varying balance of mathematical rigour on the one hand and physical concreteness on the other (the asymptotic dynamics of Kulish and Faddeev [6] and Zwanziger [7], the dressed electron states of Fröhlich [8], the asymptotic electromagnetic fields [9] and particle weights [10] of Buchholz). The difficulties are twofold. First, we do not have a complete, mathematically sound formulation of QED. Second, the complete asymptotic separation of matter and electromagnetic field may not be expected. These difficulties are not of purely technical nature. The physical factor playing the decisive role in the infrared structure of electrodynamics is the presence of constraints, the Gauss' law. It is not clear at all, in our opinion, to what degree the experience gained by the quantization of simpler, unconstrained and short-range interactions may be taken over to the formulation of quantum electrodynamics. In fact, the analysis of Refs.[1] and [3] shows that the usual canonical quantization on hypersurfaces of constant, finite time leads to the asymptotic evolution which in the limit takes the states out of the space in which the theory is defined for finite times. This may be understood as an indication that this method of obtaining a quantum theory may not be best suited for electrodynamics. Also, the localization properties of observables which form the base of the axiomatic algebraic approach to quantum field theory problems still need justification in the case of electrodynamics, and may seem somewhat artificial in the version considered there (localization in
spacelike cones).

The above remarks should not be understood as an attempt to question a priori the standard wisdom on the subject. We would like, rather, to indicate that the problem of the algebraic structure of quantum electrodynamics may still be validly regarded as open. This view is also supported by the failure of the standard electrodynamics to explain such striking and universal physical fact as the observed spectrum of charge [11].

In the present work we follow the approach of Refs. [1] and [2] to formulate and investigate a concrete C*-algebra of asymptotic fields in electrodynamics. The basic idea of the approach consists of interchanging the order of “quantization” and “asymptotic limit”. (Quantization of free electromagnetic field along these lines has been discussed in Refs. [12] and [13]. Our approach develops the ideas of Ref. [13] and extends the program to include charged fields.) In Ref. [2] the asymptotic structure of the classical field electrodynamics (the Maxwell-Dirac system) has been discussed. In Ref. [1] then a class of models of the asymptotic algebra has been obtained by the quantization of this classical structure by the correspondence principle. The Gauss’ law is naturally implemented on the algebraical level. Now we develop the approach further in the following ways.

(i) We restrict attention to the technically simplest of the models introduced in Ref. [1], the one in which only the Coulomb field is undetachable from a particle (the possibility (b) of Sec.VI in Ref. [1]). While not rejecting other possibilities at this stage, we observe that this model is also the one following most naturally by the quantization of the classical structure. In this case the construction of Ref. [1] may be simplified to yield the asymptotic algebra in the form of a particular instant of a semidirect product of the CCR (canonical commutation relations) and CAR (canonical anticommutation relations) algebras. The definition and the discussion of some mathematical aspects of this structure comprise the self-contained Section II of the paper.

(ii) In the course of quantization we bring into play a new factor, which is present at the classical level in Ref. [2], but has not been properly taken into account in the quantization procedure in Ref. [1]. It has been shown in Ref. [2] that one of the asymptotic variables has a natural interpretation as a phase variable. The quantization of the classical structure which properly respects the phase character of this variable leads unambiguously to the quantization of the physical charge spectrum in units of elementary charge. As the phase variable is connected with the free electromagnetic potential, this quantization law is universal: it has to be respected by any carriers of charge. (For similar reasoning, but with a different identification of the phase variable, see the works by Staruszkiewicz [11].) The classical asymptotic structure, its quantization and the resulting algebra are discussed in Sec. III, including the action of the Poincaré group on the algebra and the identification of observables.

(iii) Section IV contains some general results on physically admissible representations of the asymptotic algebra. It is argued that the representations should be regular with respect to all Weyl operators, as otherwise the Coulomb field part of
the total field is lost. In representations satisfying Borchers’ criterion (the spectrum of energy-momentum in the future lightcone) the Gauss constraint, which is hidden in the commutation relations on the abstract algebraic level, is shown to be recovered in the functional form.

(iv) More special representations are investigated in Sec.\(\text{VI}\) with the use of technical tools discussed in Sec.\(\text{V}\) (some technical material is shifted to the Appendix). All vacuum representations are shown to be nonregular with respect to the Weyl operators with infrared-singular test functions, which is a handicap as explained earlier. Also, superselection sectors with respect to regular operators are distinguished by the free field spacelike asymptotic, which is not what one would expect in a physical state \(\text{[14]}\). There exists a Poincaré-invariant vacuum on the field algebra (sectors transform into each other under Lorentz transformations), whose GNS representation space contains charge-one vector states with the energy-momentum on the mass hyperboloid (no infraparticle problem in this vacuum). A class of representations satisfying Borchers’ criterion is constructed, which remain as close as possible to this representation, but are regular at the same time. The nonexistence of charged states on the mass hyperboloid follows here from nonexistence of a vacuum vector state. It is not known at present whether there do exist regular Poincaré covariant representations satisfying Borchers’ criterion (standard arguments \(\text{[13, 14]}\) do not apply here).

II. Semidirect product of CCR and CAR algebras

In this section we discuss a generalization of the direct product of CCR and CAR algebras\(\text{[1]}\). We also identify a class of representations of the resulting \(C^*\)-algebra.

Suppose we are given the following constructs:

(i) a complex Hilbert space \(\mathcal{K}\) and the \(\ast\)-algebra \(\ast\text{CAR}\) generated by the elements \(a(f)\) depending antilinearly on \(f \in \mathcal{K}\) and by the identity \(E\), according to the CAR relations (eqs. (2.2) below);

(ii) an Abelian group \(X\) (with the additive notation of the group multiplication) equipped with a symplectic form \(X \times X \ni (x, y) \rightarrow \{x, y\} \in \mathbb{R}\);

(iii) a representation of the group \(X\) in the automorphism group of the \(\ast\)-algebra \(\ast\text{CAR}\)

\[
\forall x \rightarrow \beta_x \in \text{Aut}^\ast \text{CAR}\, , \quad \beta_x \beta_y = \beta_{x+y} .
\]

We consider the \(\ast\)-algebra \(\mathcal{B}\) generated by the elements \(a(f)\) \((f \in \mathcal{K})\), \(W(x)\) \((x \in X)\) and the identity \(E\) according to the relations:

\[
W(x)^\ast = W(-x) , \quad W(0) = E , \quad W(x)W(y) = e^{-\frac{i}{2}\{x, y\}}W(x+y) , \quad (2.1)
\]

\(1\) The resulting algebra and some of the statements in the present section may be obtained, as pointed out to me by H. Grundling, by the application of the general theory of twisted crossed products of \(C^*\)-algebras by groups \(\text{[16]}\).
\[ [a(f), a(g)]_+ = 0, \quad [a(f), a(g)^*]_+ = (f, g)E, \quad (2.2) \]
\[ W(x)C = \beta_x(C)W(x) \quad \forall C \in \text{CAR}. \quad (2.3) \]

With the use of (2.3) every element of \( B \) may be given the form \( \sum_{i=1}^{n} C_i W(x_i) \), where \( C_i \in \text{CAR} \).

**Proposition 2.1**

(i) If \( x_i \neq x_k \) for \( i \neq k \) and \( C_i \neq 0 \) in \( \text{CAR} \) then \( \sum_{i=1}^{n} C_i W(x_i) \neq 0 \). Every nonzero element of \( B \) is uniquely represented in this way.

(ii) The \( \ast \)-algebra \( B \) has a faithful representation by bounded operators in a Hilbert space.

Proof. Let \( \pi_f \) be a representation of \( \text{CAR} \) in a Hilbert space \( H \) (e.g. the Fock representation). Let \( \pi(W(y)) \) and \( \pi(C) \) be operators acting in the direct sum Hilbert space \( \bigoplus_{x \in X} H_x, H_x = H \), defined by

\[ [\pi(W(y))\psi]_x = e^{\frac{i}{2} \{ x, y \}} \psi_{x-y}, \quad [\pi(C)\psi]_x = \pi_f(\beta_x C)\psi_x, \]

where \( \psi = \bigoplus \psi_x \). One easily shows that these bounded operators satisfy the relations (2.1, 2.3) and part (i) of the proposition. This implies part (ii).

This representation defines a \( C^* \)-norm on the algebra \( B \). All \( C^* \)-norms on \( B \) are jointly bounded, as for each such norm \( p \) there is \( p(W(x)) = 1 \) and \( p(a(f)) = \| f \|_K \). Therefore, the set of \( C^* \)-norms contains the maximal element defined by \( \| A \| := \sup_{p} p(A) \quad [17] \).

**Definition 2.1** The field algebra \( (F, \| \|) \) is the \( C^* \)-completion of the \( \ast \)-algebra \( B \) in the norm \( \| \| \).

**Remarks.**

(i) The elements \( a(f) \) (resp. \( W(x) \)) generate a \( C^* \)-subalgebra of \( F \) which will be called the CAR (resp. the CCR, or the Weyl) algebra in this article.

(ii) The construction would be slightly more general if the CAR algebra with its automorphisms instead of \( \text{CAR} \) and \( \beta_x \) were used. This, however, would not be convenient for our purposes.

**Corollary 2.2**

(i) Every Hilbert space representation of \( B \) is given in terms of bounded operators and extends to a representation of \( F \).

(ii) Every positive linear functional on \( B \) defines via the GNS construction a Hilbert space representation of \( B \) and extends to a positive linear functional on \( F \).

(iii) Every \( \ast \)-automorphism of \( B \) extends to a \( \ast \)-automorphism of \( F \).
We omit a simple proof. The corollary establishes the 1:1 correspondence between the positive functionals and representations of \( \mathcal{F} \) on the one hand, and of \( \mathcal{B} \) on the other hand. It also gives a class of automorphisms of \( \mathcal{F} \) in simple terms. We shall repeatedly (and tacitly, mostly) take advantage of these simplifications in what follows.

An important class of automorphisms of \( \mathcal{B} \) is given as follows. Let \( T \) be a symplectic additive mapping in \( X \):

\[
T(x + y) = Tx + Ty, \quad \{Tx, Ty\} = \{x, y\}
\]

Let, further, \( \tau \) be an automorphism of \( ^*\text{CAR} \) satisfying for each \( x \in X \)

\[
\tau \beta_x = \beta_{Tx} \tau \tag{2.5}
\]

Define \( \tau(W(x)) = W(Tx) \). Then \( \tau \) extends to an automorphism of \( \mathcal{B} \) (and \( \mathcal{F} \)). This is a Bogoliubov transformation when restricted to the Weyl algebra.

In the following sections the constructions outlined above will be needed in a special case. Both \( \beta_x \) and \( \tau \) will be Bogoliubov transformations given by

\[
\beta_x a(f) = a(S_x f), \quad \tau a(f) = a(Rf),
\]

where \( S_x \) is a unitary representation of \( X \) in \( \mathcal{K} \) and \( R \) is a unitary operator in \( \mathcal{K} \). The condition (2.5) now takes the form

\[
RS_x = S_{Tx} R. \tag{2.7}
\]

In the remaining part of this section we introduce a particular class of representations of \( \mathcal{B} \). We pose, namely, the following question. Suppose \( \omega_f \) and \( \omega_r \) are states on the CAR and the Weyl algebra, respectively. When does \( \omega_f (CW(x)) = \omega_f (C) \omega_r (W(x)) \) define a state on \( \mathcal{F} \)? The hermiticity of \( \omega \) implies that \( (\omega_f(\beta_x C) - \omega_f(C)) \omega_r (W(x)) = 0 \) for every \( C \in \text{CAR} \) and \( x \in X \). This condition is satisfied, in particular, in each of the following two cases:

\[
\omega_f(\beta_x C) = \omega_f(C) \text{ for all } x \in X, \; C \in \text{CAR}, \tag{2.8}
\]

\[
\omega_r (W(x)) = 0 \text{ if } \beta_x \neq \text{id}. \tag{2.9}
\]

One easily shows that in each of these cases \( \omega \) is also positive on \( \mathcal{B} \), hence it is a state on \( \mathcal{F} \). We generalize this construction and write down the explicit prescription for the resulting representation in terms of \( \pi_r \) and \( \pi_f \) in the following statement.

**Proposition 2.3** Suppose \( \pi_f \) and \( \pi_r \) are representations of the CAR algebra and the Weyl algebra in Hilbert spaces \( \mathcal{H}_f \) and \( \mathcal{H}_r \), respectively. Define operators \( \pi(A) \), acting in the space \( \mathcal{H}_f \otimes \mathcal{H}_r \), in the following two cases:

(i) \( \pi_f \) has a cyclic vector \( \Omega_f \), such that the expectation values in that state satisfy (2.3); then

\[
\pi(C) := \pi_f(C) \otimes 1_r, \quad \pi(W(x)) [\pi_f(B)\Omega_f \otimes \varphi] := \pi_f(\beta_x B)\Omega_f \otimes \pi_r(W(x)) \varphi. \tag{2.10}
\]
(ii) $\pi_r$ has a cyclic vector $\Omega_r$, such that the expectation values in that state satisfy (2.3); then

$$\pi(W(x)) := 1_f \otimes \pi_r(W(x)),$$

$$\pi(C)[\psi \otimes \pi_r(W(y)) \Omega_r] := \pi_f(\beta_{-y} C) \psi \otimes \pi_r(W(y)) \Omega_r; \quad (2.11)$$

Then $\pi$ defines a representation of the algebra $B$ (and $F$) in each of the two cases.

Proof. Part (i) is immediately verified if one observes that $\pi(W(x)) = U_x \otimes \pi_r(W(x))$, where $U_x$ canonically implement $\beta_x$ in the representation generated by $\omega_f$. To prove (ii) one has to show that $\pi(C)$ in (2.11) is well defined as a linear operator. Once this is established, the verification of the algebraic properties is a simple exercise, which we omit. Let $\sum_{i=1}^n \psi_i \otimes \pi_r(W(x_i)) \Omega_r = 0$. Without loss of generality $\beta_{x_i} = \beta_{x_k}$ $(i, k = 1, \ldots, n)$, as otherwise the vectors are orthogonal. Then $\sum_i \pi_f(\beta_{-x_i} C) \psi_i \otimes \pi_r(W(x_i)) \Omega_r = \pi_f(\beta_{-x_1} C) \otimes 1_r \sum_{i=1}^n \psi_i \otimes \pi_r(W(x_i)) \Omega_r = 0$, which justifies the definition of $\pi(C)$. $\blacksquare$

The following result will be needed later.

**Proposition 2.4** Let $\tau$ be an automorphism of $F$ satisfying (2.4, 2.5), and let $\pi$ be a representation of $F$ of the type (2.10). The following two conditions are equivalent.

(i) The symmetry $\tau$ is implementable in the representation $\pi$ by a unitary operator leaving the subspace $\Omega_f \otimes \mathcal{H}_r$ invariant:

$$\pi(\tau A) = U \pi(A) U^* \quad \forall A \in F, \quad (2.12)$$

$$U(\Omega_f \otimes \mathcal{H}_r) \subset \Omega_f \otimes \mathcal{H}_r. \quad (2.13)$$

(ii) The state $\omega_f$ is invariant under $\tau$

$$\omega_f(\tau C) = \omega_f(C) \quad \forall C \in \text{CAR}, \quad (2.14)$$

and $\tau$ is implementable in the representation $\pi_r$

$$\pi_r(\tau A) = U_r \pi_r(A) U_r^* \quad \forall A \in \text{CCR}. \quad (2.15)$$

If these conditions are satisfied, there is a 1:1 correspondence between operators $U$ and $U_r$ given by

$$U = U_f \otimes U_r, \quad \text{where} \quad U_f \pi_f(C) \Omega_f := \pi_f(\tau C) \Omega_f. \quad (2.16)$$

If, moreover, $G \ni g \rightarrow \tau_g \in \text{Aut} F$ is a symmetry group then $U(g)$ is a representation of $G$ iff $U_r(g)$ is.

Proof. (i) $\Rightarrow$ (ii). If $U$ satisfies (2.13), then the equation $U(\Omega_f \otimes \varphi) = \Omega_f \otimes U_r \varphi$ defines a unitary operator $U_r$. Then, by (2.10) and (2.12), $\Omega_f \otimes U_r \pi_r(W(x)) \varphi = U \pi(W(x))(\Omega_f \otimes \varphi) = \Omega_f \otimes \pi_r(\tau W(x)) U_r \varphi$, which is equivalent to (2.13). Moreover, again by (2.10) and (2.12), $U(\pi_f(C) \Omega_f \otimes \varphi) =$
$U\pi(C)(\Omega_f \otimes \varphi) = \pi_f(\tau C)\Omega_f \otimes U_r\varphi$, hence $U$ has the form (2.16). Finally, 
$\omega_f(\tau C) = (\Omega_f, \pi_f(\tau C)\Omega_f) = \omega_f(C)$.

(ii) $\Rightarrow$ (i). Choose $U$ as in (2.16). Then (2.12) for $A \in \text{CAR}$ is obvious from (2.10). Moreover, by (2.10) and (2.5),
\begin{align*}
U\pi(W(x))U^* & (\pi_f(C)\Omega_f \otimes \varphi) = \pi_f(\beta x \tau^{-1} C)\Omega_f \otimes \pi_r(W(x))U^*\varphi \\
& = \pi_f(\beta T x C)\Omega_f \otimes \pi_r(W(T x))\varphi = \pi(W(T x)) (\pi_f(C)\Omega_f \otimes \varphi).
\end{align*}

If $\tau_g$ is a symmetry group then $U_f(g)$ is a representation of $G$, which implies the last equivalence of the proposition. \hfill \Box

A similar proposition holds for the representations of the type (2.11).

**III. The asymptotic field algebra**

We turn in this section to the proper task of this article, the investigation of an asymptotic algebra of fields for charged particles in interaction with an electromagnetic field, as outlined in the Introduction. For this purpose it is necessary to review the asymptotic structure of the classical field electrodynamics (the Maxwell-Dirac system) discussed in Ref.[2] (see also Ref.[13]). This paper contains, more precisely, rigorous results for both external field problems and supplies plausibility arguments for the persistence of the resulting structures in the fully interacting theory. (For recent rigorous results on the dynamics of the classical Maxwell-Dirac system obtained by the adaptation of a modified Dollard method [18] see Ref.[19]; our approach is different.) Here we only briefly sketch the results without bothering about regularity assumptions or the exact sense of limits, with the purpose of merely identifying the asymptotic variables. For more details we refer the reader to [2]. On the other hand it should be clear that quantization by the correspondence principle is a heuristic procedure itself, so there is no need for excessive formalization of this point.

Due to the difference in propagation of the electromagnetic field on the one hand, and massive fields on the other, the natural asymptotic directions are also different in the two cases, lightlike in the first and timelike in the second case, respectively. Consider the electromagnetic field first. Let $l_a$ be a null, future-pointing vector ($a$, $b$, etc. are spacetime indices). Then the leading asymptotic term in this direction of the electromagnetic potential in a Lorentz gauge $A_a(x)$ is given by
\begin{align*}
A_a(x + Rl) \sim \frac{1}{R} V_a(x \cdot l, l) \quad \text{for} \quad R \to \infty, \quad (3.1)
\end{align*}

where $x$ is any spacetime point, $V_a(s, l)$ is a real, spacetime-vector-valued function of a real variable $s$ and a null vector $l$, and $x \cdot l$ denotes the scalar product with signature $(+, -, -, -)$. Different gauges yield $V$’s differing by the transformation $V_a(s, l) \rightarrow V_a(s, l) + l_a \alpha(s, l)$, which will also be referred to as gauge transformation, but determine the same electromagnetic field asymptotic
\begin{align*}
F_{ab}(x + Rl) \sim \frac{1}{R} \left( l_b \dot{V}_a(x \cdot l, l) - l_a \dot{V}_b(x \cdot l, l) \right) \quad \text{for} \quad R \to \infty, \quad (3.2)
\end{align*}
where the dot over $V$ denotes differentiation with respect to $s$. The functions $V_a(s,l)$ are homogeneous of degree $-1$

\[ V_a(\mu s, \mu l) = \mu^{-1}V_a(s,l), \quad \mu > 0, \]  

satisfy

\[ l \cdot V(s,l) = Q, \]  

where $Q$ is the total charge of the system as measured by the integrated electric flux in spatial infinity, and tend to the limit functions $V_a(\pm \infty, l)$ for $s \to \pm \infty$. The homogeneity implies that $V_a(s,l)$ is determined by its values for $l$'s on a manifold cutting each null direction once. In each Minkowski frame, if $l$'s are scaled to satisfy $t \cdot l = 1$ (where $t$ is the timelike basis vector of the frame), then $V_a(s,l)$ falls off componentwise for $|s| \to \infty$ at least as $|s|^{-(1+\epsilon)}$, for some $\epsilon > 0$.

Differentiations with respect to independent variables in the null vector (i.e., differentiations in directions tangent to the lightcone) are conveniently carried out with the use of the operator

\[ L_{ab} := l_a \partial_b - l_b \partial_a, \partial_a := \partial/\partial l^a. \]  

With this notation the limit functions $V_a(\pm \infty, l)$ satisfy the differential condition

\[ L_{[ab}V_{c]}(\pm \infty, l) = 0. \]  

The limit function $V_a(-\infty, l)$ has a clear physical meaning. For any spacelike vector $y$ there is

\[ \lim_{R \to \infty} R^2 F_{ab}(x + Ry) = \frac{1}{2\pi} \int (l_a V_b(-\infty, l) - l_b V_a(-\infty, l)) \delta'(y \cdot l) \, d^2 l, \]  

where $\delta'(\cdot)$ is the derivative of the Dirac delta-function and $d^2 l$ is the Lorentz-invariant measure on the set of null directions applicable to functions homogeneous in $l$ of degree $-2$. (If $l$'s are scaled to $t \cdot l = 1$, then $d^2 l$ is the spherical angle measure on the unit sphere in the hyperplane orthogonal to $t$.) Therefore, $l \wedge V(-\infty, l)$ is responsible for the long-range part of the electromagnetic field. The physical content of the property (3.5) of $V(-\infty, l)$ is that the long-range field is of purely electrical type. The physical meaning of the limit function $V(+\infty, l)$ and its property (3.5) will become clear in the sequel.

Consider now the timelike asymptotic of the Dirac field, in the sense of asymptotic behaviour for $\lambda \to \infty$ of $\psi(\lambda v)$, where $v$ lies on the future part of the unit hyperboloid. In the Dirac equation choose the electromagnetic potential in a local gauge $A_{tr}$, related (locally) to a Lorentz potential $A$ by $A_{tr}^a(x) = A_a(x) - \nabla_a S(x)$, with the condition that for $x^2 \to +\infty$, $x^a > 0$ the leading term of $S(x)$ is $\ln \sqrt{x^2} \cdot A$. Then the leading asymptotic term of the Dirac field in this gauge, $\psi_{tr}(x) = e^{-ieS(x)}\psi(x)$, is

\[ \psi_{tr}(\lambda v) \sim -i\lambda^{-3/2}e^{-i(m\lambda + \pi/4)\gamma \cdot v}f(v) \]  

for $\lambda \to \infty$, where $\gamma^a$ are the Dirac matrices. The function $f(v)$ is a $C^4$-valued function with finite norm squared $\int \overline{f(v)} \gamma \cdot v f(v) \, d\mu(v)$, where bar denotes the usual Dirac
conjugation and $d\mu(v)$ is the invariant measure $d^3v/v^0$. The merit of the above asymptotics lies in its simplicity and form-independence of the potential, as long as the latter lies in the distinguished class. This does not contradict the modified Dollard asymptotic dynamics [18, 19], as here the asymptotic sequence of hyperboloids rather than spacelike hyperplanes is considered, and a special class of gauges is used.

The asymptotic variables $V_a(s, l)$ and $f(v)$ have well-defined transformation properties under the action of the Poincaré group. The element $(x, A)$ of its universal covering group $(x$ is a spacetime vector and $A \in SL(2, \mathbb{C})$) transforms the Lorentz (covariant) gauge fields by $A_a(y) \to \Lambda(A)_a^b A_b(A^{-1}(y - x))$, $\psi(y) \to S(A)\psi(\Lambda^{-1}(y - x))$, where $S(A)$ is the bispinor representation and $\Lambda(A)$ is the representation of $SL(2, \mathbb{C})$ in $\mathcal{L}^+_\gamma$ (with the notation $(\Lambda y)^a = \Lambda^a_b y^b$). The field $\psi^\tau(y)$ transforms noncovariantly, but the noncovariant phase factors are shown to cancel out in the limit. The asymptotics transform then by the representations of the Poincaré group

$$[T_{x, A}V]_a(s, l) = \Lambda(A)_a^b V_b(s - x \cdot l, \Lambda^{-1}l),$$

$$[R_{x, A}f](v) = e^{imx \cdot v} S(A)f(\Lambda^{-1}v).$$

Let us introduce the following structures on the space of asymptotic variables: the symplectic form

$$\{V_1, V_2\} = \frac{1}{4\pi} \int (V_1 \cdot V_2 - V_2 \cdot V_1)(s, l) \, ds \, dl,$$

and the scalar product

$$(f_1, f_2) = \int \overline{f_1(v)}\gamma \cdot v f_2(v) d\mu(v),$$

(both of them are well defined). Then $T_{x, A}$ is a symplectic transformation, and $R_{x, A}$ a unitary one:

$$\{T_{x, A}V_1, T_{x, A}V_2\} = \{V_1, V_2\}, \quad (R_{x, A}f_1, R_{x, A}f_2) = (f_1, f_2).$$

The generators of these transformations defined by

$$T_{x, A} - 1 \approx x^a r_a - \frac{1}{2} \omega^{ab} n_{ab}, \quad R_{x, A} - 1 \approx ix^a p_a - \frac{i}{2} \omega^{ab} m_{ab},$$

for infinitesimal $x^a$ and $\omega^{ab}$, where $\Lambda^a_b \approx g^a_b + \omega^a_b$, are

$$(r_a V)_c(s, l) = -l_a \dot{V}_c(s, l), \quad (n_{ab} V)_c(s, l) = -L_{ab} V_c(s, l) - g_{ca} V_b(s, l) + g_{cb} V_a(s, l),$$

$$(p_a f)(v) = mv_\gamma \cdot v f(v), \quad (m_{ab} f)(v) = \left(\psi_\gamma - v_\gamma \delta_a - \frac{i}{2} \left[\gamma_\gamma, \gamma_a\right]\right) f(v),$$

where $\delta_a$ is the derivative tangent to the hyperboloid, $\delta_a f(v) := (\nabla_a - x_a x^c \nabla_c f(x))_{x = y}^\prime$, and on the r.h.side any extension of $f(v)$ to the local neighborhood of the hyperboloid is used.
The discussion of Ref. [2] suggests that the asymptotic variables \( V(s, l) \) and \( f(v) \) form a causally complete set, in the sense that they determine the state of the system at any spacetime point (this has not been proved in the present approach, but cf. Ref. [19]). The total energy-momentum and angular momentum are shown to be sums of two terms, the first one describing the respective quantity going out in timelike directions and expressed in terms of \( f(v) \) only, and the second one describing the respective quantity going out in lightlike directions and expressed in terms of \( V_a(s, l) \) only. (For angular momentum this is, actually, the natural and well-defined way for extending the definition of this quantity to the infrared-singular case; the standard integral over a Cauchy surface, as resulting from Noether theorem, is ill defined.) Explicitly, they may be put into the form

\[
P_a = (f, p_a f) + \frac{1}{2} \{V, r_a V\},
\]

\[
M_{ab} = (f, m_{ab} f) + \frac{1}{2} \{V, n_{ab} V\}.
\]

(The first terms in these formulae are those of Eqs.(5.15) and (5.16) in Ref.[2], while the second ones are the tensor forms of the expressions (3.13) and (3.14) in the same reference.) Note that the local gauge freedom of the Dirac field is lost in the asymptotic limit as defined in the present approach – a change of phase of \( f(v) \) by a nonconstant function of \( v \) spoils the form of the angular momentum. The electromagnetic terms are separately gauge invariant, although the symplectic form (3.10) depends on gauges of \( V(s, l) \). The conditions of relativistic quantization for the quantum variables \( V^q \) and \( f^q \) corresponding to the classical ones are [20]

\[
[P_a, V^q] = i(r_a V^q)_c, \quad [M_{ab}, V^q] = i(n_{ab} V^q)_c, \]

\[
[P_a, f^q] = -p_a f^q, \quad [M_{ab}, f^q] = -m_{ab} f^q,
\]

where in the generators also the respective quantum variables should be substituted. Suppose, that the variables \( V^q \) and \( f^q \) commute, \( [V^q, f^q] = 0 \) and all fundamental (anti)commutators are c-numbers. This assumption remains in concord with the standard wisdom on canonical quantization in local gauges (on which the derivation of both variables is based) and fixes the quantization rules uniquely:

\[
[[V_1, V^q], [V_2, V^q]] = i\{V_1, V_2\},
\]

\[
[(f_1, f^q), (f_2, f^q)]_+ = 0, \quad [(f_1, f^q), (f_2, f^q)]^*_+ = (f_1, f_2).
\]

The above quantization relations must be considered as merely the first step towards our aim, as up to now we have not taken into account the constraints between the asymptotic variables. To remedy this deficiency we return to the discussion of the classical structure. The symplectic form (3.10) is invariant under the constant gauge transformation \( V_a(s, l) \rightarrow V_a(s, l) + l_a \alpha(l) \). One shows that with the appropriate choice of this gauge there is

\[
V_a(+\infty, l) = \int n(v) V_a^c(v, l) d\mu(v),
\]
where \( n(v) = \overline{f(v)} \gamma \cdot v f(v) \) is the asymptotic density of particles moving with velocity \( v \) and \( V_a^e(v,l) = ev_a/v \cdot l \) is the null asymptotic (3.1) of the Lorentz potential of the Coulomb field surrounding a particle with charge \( e \) moving with constant velocity \( v \). Therefore, the above relation is the implementation of the Gauss constraint on the space of classical asymptotic variables. The relation (3.5) for this limit function is now seen to be satisfied identically. Let, furthermore, \( A_{\text{adv}}(x) \) be the advanced field of the sources. It turns out that the asymptotic (3.1) of this field is given by \( V_{\text{adv}}^a(s,l) = V_a(\infty,l) \). Hence, the asymptotic of the free outgoing field potential, standardly defined by \( A_{\text{out}} = A - A_{\text{adv}} \), is determined by \( V_{\text{out}}^a(s,l) = V_a(s,l) - V_a(\infty,l) \). The “out” field is recovered from its asymptotic by the formula

\[
A_{\text{out}}^b(x) = -\frac{1}{2\pi} \int \dot{V}_{\text{out}}^b(x \cdot l, l) d^2 l. \tag{3.19}
\]

The connection of this formula with the Fourier representation

\[
A_{\text{out}}^b(x) = \frac{1}{\pi} \int a_b(k) \delta(k^2) \epsilon(k^0) e^{-i\omega k} d^4 k \tag{3.20}
\]

is supplied by the relation \( a_b(\omega l) = -\omega^{-1} \dot{V}_{\text{out}}^b(\omega,l) \), where the following one-dimensional Fourier transformation has been introduced

\[
\check{h}(\omega, l) = \frac{1}{2\pi} \int h(s,l)e^{i\omega s} ds. \tag{3.21}
\]

The limit function \( V_{\text{out}}^a(-\infty, l) = -2\pi \dot{V}_{\text{out}}^a(0, l) \) describes the long-range (infrared-singular) part of \( A_{\text{out}} \); the limit function at \( s \rightarrow \infty \) vanishes, and the charge in formula (3.4) is zero.

The infrared characteristic \( V_{\text{out}}^a(-\infty, l) \) has a simple representation, to become of importance below. Eqs (3.3), (3.4) (with \( Q = 0 \), and (3.5) satisfied by \( V_{\text{out}}^a(-\infty, l) \) imply that there exists a homogeneous of degree 0 function \( \Phi(l) \) such that

\[
l_a V_{\text{out}}^b(-\infty, l) - l_b V_{\text{out}}^a(-\infty, l) = L_{ab} \Phi(l). \tag{3.22}
\]

The function \( \Phi(l) \) is determined up to an additive constant, but one of the solutions is distinguished by being determined linearly and Lorentz-covariantly by \( V_{\text{out}}^a(-\infty, l) \):

\[
\Phi_{V_{\text{out}}}^b(l) = \frac{1}{4\pi} \int l \cdot V_{\text{out}}(-\infty, l') l' \cdot d^2 l'. \tag{3.23}
\]

(This explicit formula appears here for the first time; it may be obtained by a technique similar to that used in Appendix to prove (A.8).) This new variable transforms by an addition of a constant with the gauge transformation of the potential: if \( V_{\text{out}}^a(s,l) \rightarrow V_{\text{out}}^a(s,l) + \alpha(s,l) l_a \), then \( \Phi_{V_{\text{out}}}^b(l) \rightarrow \Phi_{V_{\text{out}}}^b(l) + \frac{1}{4\pi} \int \alpha(-\infty, l') d^2 l' \).

The solution (3.23) is the only one which satisfies (as shown by a simple calculation)

\[
\int \frac{\Phi_{V_{\text{out}}}^b(l)}{(v \cdot l)^2} d^2 l = \int \frac{v \cdot V_{\text{out}}(-\infty, l)}{v \cdot l} d^2 l \tag{3.24}
\]
for any velocity $v$.

Next, we want to determine the outgoing Dirac field which may be regarded as independent of $V^{\text{out}}$ from the point of view of Poincaré generators. To this end put $V(s, l) = V^{\text{out}}(s, l) + V(+\infty, l)$ into (3.13) and (3.14). One finds $\frac{1}{2}\{V, r_a V\} = \frac{1}{2}\{V^{\text{out}}, r_a V^{\text{out}}\}$, but $\frac{1}{2}\{V, n_{ab} V\} = \frac{1}{2}\{V^{\text{out}}, n_{ab} V^{\text{out}}\} + \{V^{\text{out}}, n_{ab} V(+\infty, .)\}$. Substituting (3.18) for $V(+\infty, l)$ and using the identity $(n_{ab} V^e)_c(v, l) = (v_a \delta_b - v_b \delta_a) V^e_c(v, l)$ we obtain $\{V^{\text{out}}, n_{ab} V(+\infty, .)\} = \int n(v) (v_a \delta_b - v_b \delta_a) i\{V^e(v, .), V^{\text{out}}\} d\mu(v)$. Finally, introducing a new variable

$$g(v) = e^{i\{V^e(v, .), V^{\text{out}}\}} f(v)$$

we bring the generators to the form

$$P_a = (g, p_a g) + \frac{1}{2}\{V^{\text{out}}, r_a V^{\text{out}}\}, \quad (3.26)$$

$$M_{ab} = (g, m_{ab} g) + \frac{1}{2}\{V^{\text{out}}, n_{ab} V^{\text{out}}\}.$$  

(3.27)

Now, define the free outgoing Dirac field by

$$\psi^{\text{out}}(x) = \left(\frac{m}{2\pi}\right)^{3/2} \int e^{-i m x \cdot v} \gamma_\nu v_\nu g(v) d\mu(v), \quad (3.28)$$

which is a special, concise form of the Fourier representation and which implies the asymptotic of the form (3.7) with $f(v)$ replaced by $g(v)$. Then the generators (3.26) and (3.27) turn out to be the sums of the conserved quantities for free fields $\psi^{\text{out}}(x)$ and $F^{\text{out}}_{ab}(x)$. Therefore, $\psi^{\text{out}}(x)$ should be interpreted as the field describing free particles together with their Coulomb fields. We have seen that the new separation of variables (3.26,3.27) forced the explicit appearance of a gauge dependent quantity $\{V^e(v, .), V^{\text{out}}\}$, but only as a phase transformation. With the use of (3.24) the phase factor in (3.28) takes the form

$$e^{i\{V^e(v, .), V^{\text{out}}\}} = \exp\left(\frac{ie}{4\pi} \int \frac{\Phi_{V^{\text{out}}}(l)}{(v \cdot l)^2} d^2 l\right), \quad (3.29)$$

and this is the only way in which a gauge-dependent quantity appears in the classical asymptotic structure. It is natural and economic, therefore, to assume, that the additive constant in $e\Phi_{V^{\text{out}}}(l)$ is a phase variable. Consequently, we put into one class gauges $V_1^{\text{out}}$ and $V_2^{\text{out}}$ such that $l \land V_1^{\text{out}}(s, l) = l \land V_2^{\text{out}}(s, l)$ and $\Phi_{V_1^{\text{out}}}(l) - \Phi_{V_2^{\text{out}}}(l) = n 2\pi / e$, $n \in \mathbb{Z}$.

With the above knowledge of the classical structure we can now return to the problem of taking into account the Gauss constraint on the quantum level. The form (3.18) of this constraint is not suited for the translation to an abstract algebraic level. However, the physical interpretation prompts an indirect solution. Instead of either the pair $(V, f)$ or $(V^{\text{out}}, g)$ it is natural to work with the pair of variables having the direct physical meaning: the asymptotic total electromagnetic field $V$ and the asymptotic field of charged particles, with their
Coulomb fields included, \( g \). The commutation relations (3.16) and (3.17) will be now reformulated in terms of these variables with the use of the relation

\[ g(v) = e^{i\{V^e(v,\cdot),V\}} f(v), \]  

(3.30)

which is equivalent to (3.25) by \( \{V^e(v,\cdot),V(+\infty,\cdot)\} = 0 \). Two circumstances have to be taken into account. First, on the classical level the Gauss constraint has been completely solved, so for the electromagnetic test field asymptotic the free field part only should be taken. Second, the quantization has to be consistent with our identifying the gauges differing by \( n^2 \pi \) in \( e^\Phi(l) \). This problem is solved, as is easily seen from (3.16) and (3.17), by using the electromagnetic variable in the form

\[ e^{-i\{V^1_{\text{out}},V^q\}} e^{-i\{V^2_{\text{out}},V^q\}} = e^{-i\frac{1}{2}\{V^1_{\text{out}},V^1_{\text{out}}\}} e^{-i\{V^1_{\text{out}} + V^2_{\text{out}},V^q\}}, \]  

(3.31)

the relations (3.17) remain true for \( g^q \), but now the two variables do not commute:

\[ e^{-i\{V^1_{\text{out}},V^q\}} g^q(v) = e^{\frac{i}{2}\{V^1_{\text{out}},V^e(v,\cdot)\}} g^q e^{-i\{V^1_{\text{out}},V^q\}}. \]  

(3.32)

The last relation has an obvious physical interpretation: the element \( g^q(v) \) beside its fermionic role, annihilates the Coulomb field with the asymptotic \( V^e(v,l) \). This is, clearly, the implementation of Gauss’ law on the quantum level. Note, also, that by (3.29) the relation is indeed consistent with our identification of gauge classes. Observe, furthermore, that the element \( e^{-i\{V^1_{\text{out}},V^q\}} \) creates the field with the asymptotic \( V^1_{\text{out}}(s,l) \). The use of only free fields as test fields reflects the fact, that the Coulomb field is fastened to a particle, which is a neat confirmation of the consistency of our scheme. Nevertheless, the element \( e^{-i\{V^1_{\text{out}},V^q\}} \) is a functional of the total field \( V^q \), as assumed in the construction. This is seen from (3.32), and also from

\[ \{V^1_{\text{out}},V\} = \{V^1_{\text{out}},V^1_{\text{out}}\} - \frac{1}{4\pi} \int V^1_{\text{out}}(-\infty,l) V(+\infty,l) d^2l. \]  

(3.33)

It is clear from both arguments, that in order to “catch” the whole field, it is absolutely necessary that all free test fields are admitted, also those infrared-singular, for which \( V^1_{\text{out}}(-\infty,l) \neq 0 \). This fact is to become of crucial importance for the interpretation of further results.

The quantum structure thus obtained will be now given an unobjectionable algebraic form, formulated in terms of elements heuristically identified by

\[ W(V) = e^{-i\{V,V^q\}} , \quad B(g) = (g,g^q), \]

where from now on all the test fields \( V(s,l) \) are free fields, so we omit the superscript “out”. We have to specify the scope of the test functions. Let \( \mathcal{K} \) be the Hilbert space of (equivalence classes of) \( \mathbb{C}^4 \)-valued functions \( g(v) \) on the hyperboloid \( v^2 = 1, v^0 > 0 \) with the scalar product (3.14). Let \( \mathcal{V} \) be the linear space of homogeneous of degree \(-1\) (Eq.(3.3)) functions \( V_a(s,l) \), infinitely differentiable
in both variables outside \( l = 0 \) (differentiations with respect to \( l \) in the sense of the action of the operator \( L_{ab} \)) and satisfying the conditions

\[
\begin{align*}
Ve(s,l) &= 0, & k = 0, 1, \ldots, \\
L_{bc}V_{a}(s,l) &= 0, & \|L_{bc}V_{a}(s,l)\| < \frac{\text{const.}(k)}{(t-l)^{2(1 + |s|/t-l)^{1+\epsilon}}}, \quad k = 0, 1, \ldots, \\
V_{a}(+\infty,l) &= 0, \\
L_{[ab}V_{c]}(-\infty,l) &= 0,
\end{align*}
\]  

(3.34)

where the second condition holds for some \((V- \text{ and } k-\text{dependent}) \epsilon > 0\) and for an arbitrarily chosen unit timelike, future-pointing vector \( t \); the bounds are then true for any other such vector (with some other constants \( \text{const.}(k) \)). These bounds guarantee the existence of the limit functions as infinitely differentiable, homogeneous functions of degree \(-1\). Let \( L \) be the Abelian additive group of elements \((V)\) defined as

\[
(V) = (l \wedge V(s,l), \Phi(V)(l) \mod 2\pi/e),
\]

(3.38)

where \( V \in \mathcal{V} \) and \( \Phi(V) \) is defined by (3.23). In other words, \( L \) is the quotient of the additive group \( \mathcal{V} \) through the equivalence relation \( \sim \), \( L = \mathcal{V}/\sim \), where

\[
V_1 \sim V_2 \text{ iff } V_2(s,l) - V_1(s,l) = l \alpha(s,l) \quad \text{and} \quad \frac{1}{4\pi} \int \alpha(-\infty,l) d^2l = n \frac{2\pi}{e}.
\]

(3.39)

The group \( L \) inherits from \( \mathcal{V} \) the symplectic form (3.10). Denote, also, for later use, \( \lambda_0 := \{ V \in \mathcal{V} | l \wedge V(-\infty,l) = 0 \} \) and \( L_0 := \lambda_0/\sim \).

The symplectic group \( L \) and the Hilbert space \( \mathcal{K} \) supply the test fields for the elements \( W(V), \quad (V) \in L \) (the parenthesis in the symbol \((V)\) appearing as an argument of \( W \) or of the symplectic form will be omitted) and \( B(f), \quad f \in \mathcal{K} \) which generate a particular \(*\)-algebra \( \mathcal{B} \) of Sec.[7] according to the relations

\[
W(V)^* = W(-V), \quad W(0) = E, \quad W(V_1)W(V_2) = e^{-i\frac{1}{2}\{V_1, V_2\}}W(V_1 + V_2),
\]

(3.40)

\[
[B(g_1), B(g_2)]_+ = 0, \quad [B(g_1), B(g_2)^*]_+ = (g_1, g_2)E,
\]

(3.41)

\[
W(V)B(g) = \beta_{\Phi}(B(g))W(V),
\]

(3.42)

where

\[
\beta_{\Phi}(B(g)) = B(S_{\Phi}g), \quad (S_{\Phi}g)(v) = \exp \left( i \frac{e}{4\pi} \int \frac{\Phi(l)}{(v \cdot l)^2} d^2l \right) g(v).
\]

(3.43)

**Definition 3.1** The asymptotic field algebra is the \( C^*\)-algebra \((\mathcal{F}, ||.||)\) obtained from the above \(*\)-algebra \( \mathcal{B} \) according to Definition [2.4].
Let us consider the role of elements \( W((0, c \mod 2\pi/e)) \), which form an Abelian one-parameter group \( W((0, c_1 \mod 2\pi/e))W((0, c_2 \mod 2\pi/e)) = W((0, c_1 + c_2 \mod 2\pi/e)) \). The relations (3.4) and (3.33) suggest that in any representation in which \( \pi(W((0, c \mod 2\pi/e))) \) are strongly continuous and written as \( e^{icQ_\pi} \), the operator \( Q_\pi \) has the interpretation of the charge operator. This interpretation is confirmed by the action of the automorphism \( \gamma_c \) of \( \mathcal{F} \),

\[
A \rightarrow \gamma_e(A) := W((0, c \mod 2\pi/e))AW((0, c \mod 2\pi/e))^* \tag{3.44}
\]
on the basic elements:

\[
\gamma_e(W(V)) = W(V), \quad \gamma_e(B(g)) = e^{-iec}B(g).
\]

Now, as \( 2\pi/e = 0(\mod 2\pi/e) \), there is \( e^{i2\pi Q_\pi/e} = 1 \), which implies that the spectrum of charge is contained in the set \( \{ne|n \in \mathbb{Z}\} \). The variable \( e\Phi(l) \) is connected with the free electromagnetic field, so bringing into play other carriers of charge should respect the phase character of the additive constant in this function. This means that the assumption in the following corollary is well founded.

**Corollary 3.1** If a \( C^* \)-algebra of asymptotic fields contains the subalgebra generated by elements \( W((0, c \mod 2\pi/e)) \), then the charge is quantized in units of \( e \).

In the following definition particular elements of \( \mathcal{F} \) are distinguished as observables in the obvious way.

**Definition 3.2** The algebra of observables \( \mathcal{A} \) is the \( C^* \)-subalgebra of \( \mathcal{F} \) of elements invariant under the gauge transformation (3.44).

One has to stress at this point that all the Weyl elements are therefore (functions of) observables. Denying the elements with infrared-singular test functions \( V(\infty, l) \neq 0 \) the status of observables one would deprive the total electromagnetic field of its Coulomb part, as discussed earlier. We shall return to this important point when discussing representations of our algebra. Also, all elements \( B(f)^*B(g) \) are in \( \mathcal{A} \).

The restricted Poincaré group (or rather its covering group) is represented in the group of automorphisms of the field algebra \( \mathcal{F} \). One easily shows that the operators (3.8) and (3.9) (the variable \( g(v) \) undergoes the same transformations as \( f(v) \)) satisfy the consistency condition (2.7), namely

\[
R_{x,A}S_\Phi = S_{T_{x,A}\Phi}R_{x,A},
\]

where the transformation \( [T_{x,A}\Phi](l) = \Phi(A^{-1}l) \) is implied by (3.8) and (3.23). Therefore, the action of the Poincaré group on \( \mathcal{F} \) may be consistently defined by

\[
\alpha_{x,A}(W(V)) = W(T_{x,A}V), \quad \alpha_{x,A}(B(g)) = B(R_{x,A}g).
\]
We end this section with the demonstration that for free test fields the symplectic structure discussed above is an extension of the structure used in more traditional algebraic formulations. To see this we find the connection with the work of Roepstorff [21]. This author uses the electromagnetic test fields of the form

\[ F_{ab}(x) = 4\pi \int D(x-y)(\nabla_a \varphi_b(y) - \nabla_b \varphi_a(y))d^4y, \]

where \( D(x) \) is the Pauli-Jordan function and \( \varphi_a(x) = \nabla^b \varphi_{ab}(x) \) for some antisymmetric test function of compact support \( \varphi_{ab}(x) \). With the use of representation \( D(x) = -\frac{i}{8\pi^2} \int \delta'(x\cdot l) d^2l \) a Lorentz gauge potential for this field takes the form (3.19), with

\[ V_a(s,l) = \int \delta(s-y\cdot l)\varphi_a(y)d^4y. \quad (3.45) \]

Substituting two functions of this form in (3.10) one has

\[ \{V_1, V_2\} = 4\pi \int \varphi_1^a(x)D(x-y)\varphi_2a(y)d^4xd^4y. \quad (3.46) \]

The r.h.side is the symplectic form used by Roepstorff (up to multiplicative constants due to electromagnetic conventions). However, the space of test fields is smaller in this formulation. It is obvious from (3.45) that \( V_a(-\infty, l) = 0 \), so all these fields are infrared-regular (the spacelike asymptotic of \( F_{ab} \) has no \( 1/r^2 \) term). In fact, even a stronger regularity property holds. There is \( V_a(s,l) = \hat{J}_a(s,l) \), where \( J_a(s,l) \) is a smooth function vanishing outside a compact region (for a fixed scaling \( t\cdot l = 1 \), given by \( J_a(s,l) = \int \delta(s-y\cdot l)l^b\varphi_{ab}(y)d^4y \). It will prove convenient to reformulate the above formulas in the Fourier-transformed version. It is shown in Sec.V below that

\[ \{\hat{V}_1, \hat{V}_2\} = iP\int \hat{V}_1(\omega,l)\hat{\varphi}_2(\omega)d\omega d^2l, \quad (3.47) \]

where \( P \) denotes the principal value. If \( \hat{V}_a(\omega,l) \) vanishes for \( \omega \to 0 \) sufficiently fast (e.g., as \( |\omega|^{\epsilon} \)), then the principal value sign may be omitted. This is true, in particular, for \( V \) given by (3.45). In this case \( \hat{V}_a(\omega,l) = \hat{\varphi}_a(\omega l) \), where \( \hat{\varphi}(p) = \frac{1}{2\pi} \int \varphi(x)e^{ip\cdot x}d^4x \), and then \( \hat{V}_a(\omega,l) = -i\omega \hat{\varphi}_a(\omega l) \). On the other hand, the last equation shows that our general field satisfies the condition of Roepstorff’s space \( L_1 \).

**IV. Existence of charge and energy-momentum, and the regularity of representations**

In the present section we investigate the consequences of putting some physical restrictions on representations. The following definitions, the second of which is standard [2], will simplify the formulation of propositions.

**Definition 4.1**

(i) A representation \( \pi \) of \( \mathcal{F} \) will be called a charge-representation of \( \mathcal{F} \) iff the
one-parameter group $R \ni c \to \pi(W((0,c \mod 2\pi/e)))$ is strongly continuous.

(ii) A representation $\pi$ of the algebra $F$ acting in the Hilbert space $H$ is said to satisfy Borchers’ criterion iff there exists a unitary, strongly continuous representation of the translation group $U(x)$ acting in $H$, with the spectrum contained in the closed forward lightcone, $\text{Spec}\,U(x) \subset \overline{V_+}$, and implementing translations of all $A \in F$:

$$\pi(\alpha_x A) = U(x)\pi(A)U(-x). \tag{4.1}$$

(iii) A representation $\pi$ of the algebra $F$ will be said to satisfy Borchers’ criterion with respect to $A$ (w.r.t. observables) iff the restriction of $\pi$ to the subalgebra $A$ satisfies Borchers’ criterion.

The defining properties of (i) and (iii) are necessary requirements for a physically admissible representation. We analyze their implications.

**Proposition 4.1** If $\pi$ is a charge-representation of $F$ then it has one of the following properties (or is a direct sum of these three types):

(i) The charge takes on all the values $ke$, $k \in \mathbb{Z}$, and each charge eigenspace is cyclic.

(ii) (resp. (iii)) The subspace of vectors satisfying $\pi(B(g))\psi = 0 \ \forall g \in \mathcal{K}$ (resp. $\pi(B(g)^*)\psi = 0 \ \forall g \in \mathcal{K}$) is cyclic.

Proof. Given $\pi(F)$ on $H$ let $H_k \subset H$ be the subspace of all charge eigenvectors to the eigenvalue $ke$ for a given $k \in \mathbb{Z}$, and let $H_k' = [\pi(F)H_k]$ (the closed linear subspace spanned by vectors in $\pi(F)H_k$). Then $H = H_k' \oplus H_k^\perp$, where both subspaces are invariant. Moreover, if we split $H_k'^\perp = H_{k+}^\perp \oplus H_{k-}^\perp$, where the spectrum of charge goes from $(k+1)e$ upwards on $H_{k+}$ and from $(k-1)e$ downwards on $H_{k-}$, then both subspaces are separately invariant. This occurs because the generating elements $\pi(W(V)), \pi(B(g))$ and $\pi(B(g)^*)$ carry charge 0 or $\pm e$, so they cannot match the gap between $H_{k+}$ and $H_{k-}$.

The representations $\pi(F)|_{H_{k+}'}$ and $\pi(F)|_{H_{k-}'}$ have the properties (ii) and (iii) respectively. To see this, let $H_{k+}'$ be the subspace of $H_{k+}'$ of vectors satisfying the equation in (ii). Let $\psi$ be an element of the invariant subspace $H_{k+}' \cap [\pi(F)H_{k+}''\perp]$, with a bounded spectral content of charge (these vectors form a dense subspace as $e^{ieQx} \in \pi(F)$). If $\psi \neq 0$ then there exists such $f \in \mathcal{K}$ that $\pi(B(f))\psi \neq 0$. Continuing in this way one can lower the charge spectral content of the vector unlimittedly. This contradicts the charge content of $H_{k+}'$. A similar proof holds for $\pi(F)|_{H_{k-}'}$.

The rest of the proof is simple inductive reasoning. Let the set of integers $\mathbb{Z}$ be organized into a sequence $\{k_n\}$. Suppose that for $\pi(F)$ on $H(n)$ the charge eigensubspaces $H_{k_1}, \ldots, H_{k_n}$ are cyclic. Take the next charge eigenspace $H_{k_{n+1}}$ and decompose $H(n)$ according to the above prescription: $H(n) = H_{k_{n+1}'} \oplus H_{k_{n+1}'}' \oplus H_{k_{n+1}''\perp}$. The representations $\pi(F)|_{H_{k_{n+1}'}'}$ satisfy the properties (ii) and (iii) respectively. By construction $H_{k_{n+1}'}$ is cyclic for $\pi(F)$ on $H(n+1) := H_{k_{n+1}'}$. As each
Proposition 4.2

The charge spectrum consists of all values $ke$, $k \in \mathbb{Z}$, and eigenspace to each charge value is cyclic for a charge-representation of $\mathcal{F}$ satisfying Borchers’ criterion with respect to observables.

The proof of the proposition will be based on the following observation.

Lemma 4.3

If the representation $\pi$ of $\mathcal{F}$ has the property (ii) of Proposition 4.1, then it is unitarily equivalent to a representation of type (2.10), where $\pi_f$ is the Fock representation based on the cyclic vector $\Omega_f$ satisfying $\pi_f(B(g))\Omega_f = 0 \quad \forall g \in \mathcal{K}$.

An analogous result holds for representations satisfying the property (iii) of Prop. 4.1. (In that case $\pi_f$ is the Fock representation based on the cyclic vector satisfying $\pi_f(B(g)^*)\Omega_f = 0 \quad \forall g \in \mathcal{K}$. We use the term Fock representation in the wider sense, referring to any of the representations differing by a Bogoliubov transformation from the one appearing in the lemma.)

Proof. Let the representation $\pi$ act in $\mathcal{H}$ and denote the respective cyclic subspace by $\mathcal{H}_\pi$. If $C \in \mathcal{CAR}$ then it may be represented as $C = \omega_F(C)E + C'$, where $\omega_F$ is the Fock state and $C'$ is a sum of elements having $B(f)$ on the right and/or $B(g)^*$ on the left. Therefore for $\varphi, \psi \in \mathcal{H}_\pi$ there is $(\varphi, \pi(C)\psi) = \omega_F(C)(\varphi, \psi)$. $\mathcal{H}_\pi$ is invariant under $\pi(W(V))$, hence the vectors $\sum_{k=1}^N \pi(C_k)\varphi_k$, $\varphi_k \in \mathcal{H}_\pi$, are dense in $\mathcal{H}$. Let $\pi_r$ be the restriction of $\pi$ to the Weyl algebra and to the space $\mathcal{H}_\pi$. We map $\mathcal{H}$ onto $\mathcal{H}_F \otimes \mathcal{H}_r$ ($\mathcal{H}_F$ is the Hilbert space of the Fock representation $\pi_F$) by the rule $\sum_k \pi(C_k)\varphi_k \rightarrow \sum_k \pi_F(C_k)\Omega_F \otimes \varphi_k$. It is now easy to show this is a unitary map providing the claimed equivalence of representations.

Proof of Prop. 4.2. In view of the result of Prop. 4.1 one has to show that a representation of the type described in the lemma cannot be a subrepresentation of a representation satisfying the assumptions of Prop. 4.2. By a general theorem by Borchers [22] (see also [3]) the representation $U(x)$ may be chosen to lie in $\pi(\mathcal{A})''$, hence it suffices to show that for the representation of the lemma itself there is no $U(x)$ in $\pi(\mathcal{A})''$ implementing translations of observables and satisfying the spectral condition. Suppose the converse is true. Then for any $C \in \mathcal{CAR} \cap \mathcal{A}$ there is $U(x)\pi(C)U(-x) = \pi(\alpha_x C) = \pi_F(\alpha_x C) \otimes 1_r = [U_F(x) \otimes 1_r][\pi_F(C) \otimes 1_r][U_F(-x) \otimes 1_r]$, where $U_F(x)$ is the unitary representation of translations in the Fock representation. This means that $U(x) [U_F(-x) \otimes 1_r] \equiv R(x)$ commutes with $\pi_F(C) \otimes 1_r$ for all $C \in \mathcal{CAR} \cap \mathcal{A}$. The
representation $\pi_F(CAR \cap \mathcal{A})$ acts irreducibly on each of the subspaces $\mathcal{H}_n \subset \mathcal{H}_F$ spanned by all vectors $\pi_F(B(g_1)^* \cdots B(g_n)^*)\Omega_F$, and all spaces $\mathcal{H}_n \otimes \mathcal{H}_r$ are invariant with respect to $R(x)$ (as they are invariant with respect to $\pi(\mathcal{A})$). Hence, $U(x)\big|_{\mathcal{H}_n \otimes \mathcal{H}_r} = U_F(x)\big|_{\mathcal{H}_n} \otimes U_{r,n}(x)$, where $U_{r,n}$ is a strongly continuous representation of translations in $\mathcal{H}_r$. However, the Fock representation $\pi_F$ appearing in this construction is not “the right” Fock representation of the free Dirac field ($B(g)$ contains both positive and negative frequencies), and the energy spectrum of $U_F(x)\big|_{\mathcal{H}_n}$ is not bounded from below for $n \geq 1$, which contradicts the assumption.

The existence of a charge operator is a necessary, but by far not a sufficient condition for the operators $\pi(W(V))$ to have a clear physical interpretation. We recall that, as explained in the previous section, in our algebra all these Weyl operators should be understood as observables, more precisely, as exponentials of (unbounded) observable electromagnetic field operators. The test function set $L$ is an Abelian group rather than a vector space, so a direct multiplication of $V$ by a parameter is not possible. However, it is sufficient to find a map $R \ni \lambda \to (V^\lambda) \in L$, such that $(V^0) = (0)$, $(V^1) = (V)$, $(V^\lambda) + (V^\nu) = (V^{\lambda+\nu})$ and $-(V^\lambda) = (V^{-\lambda})$. Then $W(V^\lambda)$ is a one-parameter group and the above condition on the representation may be formulated as the strong continuity of $\pi(W(V^\lambda))$ in $\lambda$. Let $(V) = (l \wedge V, \Phi \bmod 2\pi/e)$. Then it is easily shown, that for each choice of the representant $\Phi$ from the class $\Phi \bmod 2\pi/e$ the map

$$R \ni \lambda \to (V^\lambda)_\Phi := (l \wedge \lambda V, \lambda\Phi \bmod 2\pi/e) \quad (4.2)$$

satisfies the listed requirements.

**Definition 4.2** Representation $\pi$ of the algebra $\mathcal{F}$ (or of a subalgebra of $\mathcal{F}$) will be called regular iff all one-parameter groups $R \ni \lambda \to \pi\left(W((V^\lambda)_\Phi)\right)$ are strongly continuous.

Remarks.

(i) If $\pi$ is regular then it is a charge representation. This follows by the special choice $(V) = (0, e \bmod 2\pi/e)$.

(ii) The generators of the groups $\pi\left(W((V^\lambda)_\Phi')\right)$ and $\pi\left(W((V^\lambda)_\Phi)\right)$, where $\Phi' \in \Phi \bmod 2\pi/e$, differ by a multiple of the charge operator. This follows from $W((V^\lambda)_{\Phi'}) = W((V^\lambda)_\Phi)W((0, \lambda 2k\pi/e \bmod 2\pi/e))$ for $\Phi' = \Phi + 2k\pi/e$.

We are now prepared to partly characterize the representations satisfying the condition of Definition [4.2](ii). For the positive energy Fock representation $\pi_F$ of the algebra CAR let us denote $\pi_F(B(g)) = \int g(v)\gamma \cdot vb(v)d\mu(v)$ and let : : : denote the standard normal ordering.

**Theorem 4.4**

(i) A representation $\pi$ of the algebra $\mathcal{F}$ satisfies Borchers’ criterion if, and only if, it is unitarily equivalent to a representation of type $[2,10]$, where $\pi_f = \pi_F$ is the positive energy Fock representation of CAR and $\pi_r$ is a representation of the
Weyl algebra satisfying Borchers’ criterion.
For representations of this type the Gauss constraint in the form (3.13) is recovered in the von Neumann algebra \(\pi(A)\)^:

\[
\pi(W(V)) = \exp \left( -i \int :b(v)\gamma \cdot v b(v)\{V, V^e(v,.)\} d\mu(v) \right) \otimes \pi_r(W(V)). \tag{4.3}
\]

(ii) \(\pi\) is irreducible iff \(\pi_r\) is irreducible.
(iii) \(\pi\) is regular iff \(\pi_r\) is regular.

Remark. For \(\pi(\mathcal{F})\) in the form given by (i) the operator \(1 \otimes \pi_r(W(V))\) has the interpretation of the free field exponent, and one sees once more the essentiality of the regularity assumption.

Proof. If the unitary equivalence is proved, then (4.3) follows by simple calculation. This formula then implies (ii) by irreducibility of the Fock representation (and the identity \(\mathcal{L}(\mathcal{H}_F) \otimes \pi_r(\text{CCR})\) = \(\mathbb{C}\mathbb{1}_F \otimes \pi_r(\text{CCR})\)) and (iii) by \(\{(V^\lambda)_\phi, V^e(v,.)\} = -\lambda e \frac{c}{4\pi} \int \frac{\Phi(l)}{(v \cdot l)^2} d^3l\). The equivalence of representations is proved by adapting to the present case the idea of Ref.\[24\]. Let \(g \in \mathcal{K}\). The positive and negative frequency parts of \(g\) are easily extracted from \(g\) by \(g_\pm = P_\pm g\), where \(P_\pm\) are projection operators in \(\mathcal{K}\) defined by \((P_\pm g)(v) = P_\pm (v)g(v)\), \(P_\pm (v) = \frac{1}{2}(1 \pm \gamma \cdot v)\). Let \(\pi\) satisfy Borchers’ criterion. From translational covariance one shows in standard way that \(\pi(B(g_+))\) and \(\pi(B(g_-))\) (resp. \(\pi(B(g_+))^*\) and \(\pi(B(g_-))\)) lower (resp. raise) the energy content of a vector by at least \(m\) (the mass of the fermion). Let \(\mathcal{H}_r\) be the subspace of the representation space formed by all vectors \(\psi\) satisfying \(\pi(B(g_+))\psi = \pi(B(g_-))^* \psi = 0 \forall g \in \mathcal{K}\). The subspace \(\mathcal{H}_r\) is invariant under \(U\) and the subspaces \([\pi(\mathcal{F})\mathcal{H}_r]\) and \([\pi(\mathcal{F})\mathcal{H}_r]\) are invariant both under \(\pi\) and \(U\). Let \(\psi \neq 0\) lie in the dense subspace of finite energy vectors in \([\pi(\mathcal{F})\mathcal{H}_r]\). Then for some \(g \in \mathcal{K}\) there is \(\pi(B(g_+))\psi \neq 0\) or \(\pi(B(g_-))^* \psi \neq 0\). By recurrence, this gives the way for reaching negative energies, which contradicts the assumption. Hence \(\mathcal{H}_r\) is cyclic for \(\pi(\mathcal{F})\). The unitary equivalence to a representation of type (2.10) follows now as in the proof of Lemma 4.3. However, the Fock representation appearing in the construction is in the present case “the right”, positive energy Fock representation. The use of Proposition 2.4 finishes the proof. From this proposition also the “if” statement of (i) follows easily.

Our next objectives are the characterization of vacuum states on our algebra, and the construction of a class of physically meaningful regular representations. The first problem is solved by the adaptation to the present situation of standard analytical methods. We shall have to discuss some properties of the symplectic space of test functions. For \(V \in \mathcal{V}\) let us denote by \([V]\) the respective field test function, i.e. the equivalence class of \(V\) with respect to the equivalence

\[
\hat{V}' \approx \hat{V} \text{ iff } l \wedge \hat{V}' = l \wedge \hat{V}. \tag{4.4}
\]

The vector space of these classes will be denoted by \(L^f\), and its subspace of classes \([\hat{V}]\) with \(l \wedge V(-\infty, l) = 0\) by \(L^f_0\). The space \(L^f\) inherits the symplectic form from
\( V, \{[\hat{V}_1], [\hat{V}_2]\} = \{\hat{V}_1, \hat{V}_2\} \). Now, the analytical properties of this space needed for the characterization of vacuum states will also play role for its extension \( \hat{L}' \) to be used in the construction of regular representations. Therefore, in the next section we introduce this auxiliary symplectic space and formulate the necessary properties in this wider setting.

V. Extension and analytical properties of symplectic structure

The first step towards extension of the symplectic space of test fields \( L' \) will be generalizing the fall-off condition (3.33) for \( k = 0 \). We do it first for scalar functions. Consider the real Hilbert space \( L^2_{\epsilon,t} \) of (equivalence classes of) real, measurable functions \( f(s,l) \), homogeneous of degree \(-2\), with finite norm

\[
\|f\|_{\epsilon,t}^2 = \int f^2(\tau t \cdot l)(|\tau| + 1)^{1+\epsilon}(t \cdot l)^2 \, d\tau \, dl,
\]

where \( \epsilon > 0 \) and \( t \) is a unit future-pointing vector. If \( \tilde{t} \) is another future-pointing vector and \( c_{\epsilon,t} \equiv \tilde{t} \cdot t + \sqrt{(\tilde{t} \cdot t)^2} - 1 \) then for every null vector \( l \) there is \( c_{\epsilon,t}^{-1} \leq \tilde{t} l / tl \leq c_{\epsilon,t} \). Using these bounds one shows that \( c_{\epsilon,t}^{-2-\epsilon}\|f\|_{\epsilon,t}^2 \leq \|f\|_{\epsilon,t}^2 \leq c_{\epsilon,t}^{2+\epsilon}\|f\|_{\epsilon,t}^2 \). Therefore \( L^2_{\epsilon,t} \) does not depend on \( t \) when considered as a linear topological space.

Components of smooth functions satisfying (3.33) obviously lie in this space.

If \( f \in L^2_{\epsilon,t} \) then for \( \delta < \epsilon \)

\[
\left( \int |f(\tau t \cdot l)(|\tau| + 1)^{\frac{1}{2}} |f(st \cdot l)(|s| + 1)^{\frac{1}{2}} (t \cdot l)^2 \, d\tau \, ds \, dl \right)^{\frac{1}{2}} \leq \frac{2}{\epsilon - \delta} \|f\|_{\epsilon,t}^2.
\]

Consequently, the integral \( \int |f(\tau t \cdot l)(|\tau| + 1)^{\frac{1}{2}} (t \cdot l)^2 \, d\tau \, dl \) is finite \( d^2l \)-almost everywhere and as a function of \( l \) (homogeneous of degree \(-1\)) is square integrable with respect to \( d^2l \). In particular, the Fourier transform of \( f(s,l) \) with respect to \( s \) is defined by the integral (3.27), satisfies

\[
\int |\tilde{f} \left( \frac{\omega}{t \cdot l}, l \right)|^2 \, d^3l \leq \frac{1}{2\pi^2 \epsilon} \|f\|_{\epsilon,t}^2,
\]

and

\[
\int |\tilde{f} \left( \frac{\omega}{t \cdot l}, l \right)|^2 \, d\omega = \frac{1}{2\pi} \int f^2(\tau t \cdot l) \, d\tau (t \cdot l)^2.
\]

The function \( \tilde{f}(\omega/t \cdot l, l) \) depends continuously on \( \omega \), both pointwise in \( l \) and as an element of the Hilbert space \( L^2(d^2l) \). In fact, even stronger conditions are satisfied. With the use of the bound \( |e^{ix} - 1| < 2|x|^\alpha \) for \( 0 < \alpha < 1 \), one finds for any \( \delta \in \langle 0, \min\{2, \epsilon\} \rangle \)

\[
\left| \tilde{f} \left( \frac{\omega'}{t \cdot l}, l \right) - \tilde{f} \left( \frac{\omega}{t \cdot l}, l \right) \right| \leq \frac{1}{\pi} |\omega' - \omega|^{\frac{1}{2}} \int |\tau|^{\frac{1}{2}} |f(\tau t \cdot l)| \, d\tau \, t \cdot l,
\]

\[
\int |\tilde{f} \left( \frac{\omega}{t \cdot l}, l \right)|^2 \, d\omega = \frac{1}{2\pi} \int f^2(\tau t \cdot l) \, d\tau \, (t \cdot l)^2.
\]
and then
\[ \int \left| \tilde{f} \left( \frac{\omega'}{t \cdot l}, l \right) - \tilde{f} \left( \frac{\omega}{t \cdot l}, l \right) \right|^2 \, d^2 l \leq \frac{2 \| f \|_{\epsilon, l}^2 \| \omega' - \omega \|^4}{(\epsilon - \delta) \pi^2}. \] (5.5)

This implies, with the use of (5.2),
\[ \left| \int \left| \tilde{f} \left( \frac{\omega'}{t \cdot l}, l \right) \right|^2 \, d^2 l - \int \left| \tilde{f} \left( \frac{\omega}{t \cdot l}, l \right) \right|^2 \, d^2 l \right| \leq \frac{2 \pi^2 \sqrt{\epsilon (\epsilon - \delta)}}{\| f \|_{\epsilon, l}^2 \| \omega' - \omega \|^4}. \] (5.6)

Next, we derive some integral identities. Let \( \tilde{K}_{\alpha, \beta}(\omega) = \omega^{-1} \chi_{\alpha, \beta}(\omega) \), where \( \chi_{\alpha, \beta} \) is the characteristic function of the set \( \{-\beta, -\alpha\} \cup \{\alpha, \beta\} \), and let \( f, g \in L^2_{\epsilon, l} \).

For almost all \( l \) the function \( \frac{1}{(2\pi)^2} f(\tau, l) \tilde{K}_{\alpha, \beta}(\omega) e^{-i\omega(\tau - s)} g(s, l) \) is absolutely integrable with respect to \( d\tau \, ds \, d\omega \), so the iterated integrals are equal, i.e.,
\[ \frac{1}{(2\pi)^2} \int f(\tau, l) \tilde{K}_{\alpha, \beta}(\tau - s) g(s, l) \, d\tau \, ds = \int_{|\omega| \in \{\alpha, \beta\}} \tilde{f}(\omega, l) \tilde{g}(\omega, l) \, d\omega. \] (5.7)

\( K_{\alpha, \beta}(s) \) is uniformly bounded and \( \lim_{\alpha \to 0, \beta \to \infty} K_{\alpha, \beta}(s) = -i\pi \epsilon(s) \), so
\[ \frac{1}{4\pi} \int f(\tau, l) \epsilon(\tau - s) g(s, l) \, d\tau \, ds = i \int \tilde{f}(\omega, l) \tilde{g}(\omega, l) \, d\omega, \] (5.8)

where \( P \) denotes the principal value operation. Integrating over \( d^2 l \) we obtain
\[ \frac{1}{4\pi} \int d^2 l \int f(\tau, l) \epsilon(\tau - s) g(s, l) \, d\tau \, ds = i \int d^2 l \, P \int \tilde{f}(\omega, l) \tilde{g}(\omega, l) \, d\omega. \] (5.9)

Another form of the Fourier representation of this integral will be useful. We integrate \( f(\tau' t \cdot l, l) \tilde{K}_{\alpha, \beta}(\omega) e^{-i\omega(\tau' - s')} g(s' t \cdot l, l)(t \cdot l)^2 \) over \( ds' \, d\tau' \, d^2 l \, d\omega \), and then calculate the limits (in the same order as before). Changing the variables \( \tau', s' \) back to \( \tau, s \) we have
\[ \frac{1}{4\pi} \int d^2 l \int f(\tau, l) \epsilon(\tau - s) g(s, l) \, d\tau \, ds = i \int \frac{d\omega'}{\omega'} \int \tilde{f} \left( \frac{\omega'}{t \cdot l}, l \right) \tilde{g} \left( \frac{\omega'}{t \cdot l}, l \right) d^2 l. \] (5.10)

Consider now the linear space of vector functions \( u_a(\tau, l) \) such that \( lu(\tau, l) = 0 \) and each component of \( u_a \) (in any Minkowski frame) is an element of the space \( L^2_{\epsilon, l} \). Divide this space into classes with respect to the equivalence relation (1.4):
\[ [u_a] = u_a(\tau, l) \mod \alpha(s, l) l_a. \] We denote this factor space by \( \mathcal{T} \). For decreasing \( \epsilon \) the spaces \( \mathcal{T}_\epsilon \) form an increasing family of vector spaces. Their union \( \mathcal{T} := \bigcup_{\epsilon > 0} \mathcal{T}_\epsilon \) is therefore again a vector space. We now list some properties of \( \mathcal{T} \) and of some structures on it. Simple proofs based on the preceding discussion are omitted.

\( \mathcal{T} \) becomes a symplectic space with the form
\[ \{[u_1], [u_2]\} = \frac{1}{4\pi} \int d^2 l \int u_{1a}(\tau, l) \epsilon(\tau - s) u_{2a}(s, l) \, d\tau \, ds = -i \int \frac{d\omega'}{\omega'} \int \left[ -\bar{u}_{1a} \left( \frac{\omega'}{t \cdot l}, l \right) \bar{u}_{2a} \left( \frac{\omega'}{t \cdot l}, l \right) \right] d^2 l. \] (5.11)
The form is nondegenerate on \( \mathcal{T} \). Consider, further, the linear space of real measurable vector functions \( f_a(l) \) on the cone, homogeneous of degree \( -1 \), orthogonal to \( l \), \( l \cdot f(l) = 0 \), and such that each component is an element of the Hilbert space \( L^2(d^2l) \). Divide this space into equivalence classes \( [f_a](l) := f_a(l) \mod \beta(l) l \). This factor space is a Hilbert space, denoted by \( \mathcal{H}_0 \), with the scalar product

\[
([f_1], [f_2])_0 = \int [-f_{1a}(l)f^a_{2}(l)] \, d^2l .
\] (5.12)

Now, one easily shows with the use of (5.2) that if \( [u] \in \mathcal{T} \), then \( [\hat{u}(0, .)] \in \mathcal{H}_0 \). The map

\[
\mathcal{T} \ni [u] \to p([u]) = [\hat{u}(0, .)] \in \mathcal{H}_0
\] (5.13)

is linear and onto, \( p(\mathcal{T}) = \mathcal{H}_0 \) (for a given \( [f] \in \mathcal{H}_0 \) put \( u_a(s, l) = f_a(l)(t \cdot l)^{-1} h(s/t \cdot l) \), with \( h \) of compact support and \( \hat{h}(0) = 1 \); then \( p([u]) = [f] \)). If at least one of the pair of functions \( [u_i] \in \mathcal{T}, i = 1, 2 \), satisfies \( p([u_i]) = 0 \), then, by \( (5.4) \), the following integral is well defined

\[
F ([u_1], [u_2]) = \int_{\omega \geq 0} \left[ -\overline{\hat{u}_{1a}(\omega, l)} \hat{u}^a_{2}(\omega, l) \right] \frac{d\omega}{\omega} \, d^2l ,
\] (5.14)

and the symplectic form may be expressed by

\[
\{[u_1], [u_2]\} = 2 \text{Im} F ([u_1], [u_2]) ,
\] (5.15)

where the hermiticity of \( F \) has been used:

\[
\overline{F ([u_1], [u_2])} = F ([u_2], [u_1]) .
\] (5.16)

For almost all \( l \) the integral \( U_a(s, l) = -\int s^{-\infty} u_a(\tau, l) \, d\tau \) is well defined for all \( s \), and for almost all \( s \) there is \( \frac{\partial}{\partial s} U_a(s, l) = u_a(s, l) \). It is now easy to see that the functions \( U_a(s, l) \) generalize the test functions \( V_a(s, l) \), and the symplectic form \( (5.11) \) extends the form \( (5.10) \). The limit values at \( -\infty \) may be expressed by \( U_a(-\infty, l) = -2\pi \hat{u}_a(0, l) \). The only essential property of \( V_a \) which has not been taken into account yet is \( (3.22) \), \( l_{[a} \dot{V}_{b]}(0, l) = -2\pi l_{[a} \partial_{b]} \Phi(l) \). In order to formulate its generalization for \( U_a \) consider the Hilbert subspace \( \mathcal{H}_{IR} \) of \( \mathcal{H}_0 \) obtained as the closure of the linear subspace of elements \( [\partial_a \phi] \), where \( \phi \) is a real smooth homogeneous function of degree 0 (different extensions of \( \phi(l) \) outside the cone yield \( \partial_a \phi(l) \) on the cone in one class \( [\partial_a \phi] \)),

\[
\mathcal{H}_{IR} := \{ [\partial_a \phi] \in \mathcal{H}_0 | \phi \in C^\infty \}^{\mathcal{H}_0} .
\] (5.17)

For \( \phi \in C^\infty \) the class \( [\partial_a \phi] \) determines \( \phi \) uniquely up to a constant. With the notation \( [\phi] \equiv \phi \mod \text{const} \), the map \( [\partial_a \phi] \to [\phi] \) is linear and injective. We show in the Appendix that this map extends canonically to an injective map \( j \) of \( \mathcal{H}_{IR} \) into the space of classes \( [\phi] \) with \( \phi \) square-integrable with respect to any (and all) of the measures \( d^2l/(tl)^2 \). In this way every element of \( \mathcal{H}_{IR} \) corresponds
The extended symplectic space is now chosen as the subspace of $T$ given by $\mathcal{L} := p^{-1}(H_{\text{IR}})$, with $p$ defined in (5.13), and equipped with the symplectic form (5.14). The elements of $\mathcal{L}$ satisfying $p([u]) = 0$ form a linear subspace, denoted by $\mathcal{L}_0$. These structures extend the field test functions spaces $L^f \subset \mathcal{L}^f$ and $L^f_0 = L^f \cap \mathcal{L}_0$.

The following properties of the form $F(.,.)$ will be needed for the characterization of vacuum states.

**Lemma 5.1**
(i) For each pair $[u_i] \in \mathcal{L}^f$, $i = 1, 2$, the function $x \rightarrow F([u_1], [T_xu_2 - u_2])$ is the boundary value for $\text{Im} z = 0$ of the function

$$
\mathbb{R}^4 + iV_+ \ni z \rightarrow F([u_1], [T_xu_2 - u_2]) := -\int_{\omega \geq 0} (e^{i\omega l \cdot z} - 1)\bar{u}_1 \cdot u_2(\omega, l) \frac{d\omega}{\omega} d^2 l,
$$

which is continuous on its domain and analytic on $\mathbb{R}^4 + iV_+$.

The following bounds (ii)–(iv) hold on the whole domain $z = x + iy \in \mathbb{R}^4 + iV_+$.

(ii) For any fixed, unit, future-pointing vector $t$

$$
\left| e^{F([u_1], [T_xu_2 - u_2])} \right| \leq C(t, [u_1], [u_2]) \times \left( 1 + y^0 + |\vec{y}| \right)^2 + \left( |x^0| + |\vec{x}| \right)^2 \left( \frac{1}{2} \int \bar{u}_1 \cdot u_2(0, l) \theta(\bar{u}_1 \cdot u_2(0, l)) d^2 l \right),
$$

where $y^0 \equiv y \cdot t$, $|\vec{y}| \equiv \sqrt{(y \cdot t)^2 - y^2}$ (and the same for $x$), and $\theta(\cdot)$ is the Heaviside step function.

(iii) If $\bar{u}_1 \cdot u_2(0, l) \leq 0$ $d^2 l$-almost everywhere, then

$$
\left| e^{F([u_1], [T_xu_2 - u_2])} \right| \leq C(t, [u_1], [u_2]) \times \left( 1 + y^0 + |\vec{y}| \right)^2 + \theta(x \cdot x) \left( |x^0| - |\vec{x}| \right)^2 \left( -\frac{1}{2} \int |\bar{u}_1 \cdot u_2(0, l)| d^2 l \right).
$$

(iv) If $\bar{u}_1 \cdot u_2(0, l) \geq 0$ $d^2 l$-almost everywhere, then

$$
\left| e^{-F([u_1], [T_xu_2 - u_2])} \right| \leq C(t, [u_1], [u_2]).
$$

(v) If $p([u_1 + u_2]) = 0$, then for $x \in \mathbb{R}^4$

$$
F([u_1 + T_xu_2], [u_1 + T_xu_2]) = F([u_1], [T_xu_2 - u_2]) + F([T_xu_2 - u_2], [u_1]) + F([u_1 + u_2], [u_1 + u_2]).
$$

(vi) The Fourier representation (5.14) of $F(.,.)$ is the usual one-photon scalar product when restricted to $\mathcal{L}_0$, which yields a dense subspace of the one-photon Hilbert space, and

$$
F([T_xu_1], [T_xu_2]) = F([u_1], [u_2]).
$$
Proof. If \( z \in \mathbb{R}^4 + iV_+ \), then for any \( k \in \mathbb{R}^4 \) the function \( C \ni z \to F([u_1], [T_z u_2 - u_2]) \) is analytic in some neighbourhood of \( \xi = 0 \), which implies (i) by Hartog’s theorem. Properties (ii)–(iv) follow easily from the bound

\[
|F([u_1], [T_z u_2 - u_2]) - \int \bar{u}_1 \cdot \bar{u}_2(0,l) \ln \left| 1 - \frac{i z \cdot l}{t \cdot l} \right| d^2l| \leq \text{const}(t, [u_1], [u_2]),
\]

and the inequalities \( 1 \leq (1 + y^0 - |\bar{y}|)^2 + \theta(x \cdot x) |x^0| - |\bar{x}|)^2 \leq |1 - iz \cdot l/t \cdot l|^2 \leq (1 + y^0 + |\bar{y}|)^2 + (|x^0| + |\bar{x}|)^2 \). To prove the bound we split \( F \) into two parts

\[
F([u_1], [T_z u_2 - u_2]) = - \int_{\omega \geq 0} (e^{i \omega t \cdot z} - 1) \left[ \bar{u}_1 \cdot \bar{u}_2(\omega, l) - \bar{u}_1 \cdot \bar{u}_2(0, l)e^{-\omega t \cdot l} \right] \frac{d\omega}{\omega} d^2l
\]

\[- \int \bar{u}_1 \cdot \bar{u}_2(0,l) \int_0^\infty \left( \exp \left( i \omega \frac{z \cdot l}{t \cdot l} \right) - 1 \right) e^{-\omega' \frac{d\omega'}{\omega'}} d^2l.\]

The first term is absolutely bounded by

\[
2 \int \left| \bar{u}_1 \cdot \bar{u}_2(\omega, l) - \bar{u}_1 \cdot \bar{u}_2(0, l)e^{-\omega t \cdot l} \right| \frac{d\omega}{\omega} d^2l < \infty.
\]

The second term is equal to \( \int \bar{u}_1 \cdot \bar{u}_2(0, l) \ln (1 - iz \cdot l/t \cdot l) d^2l \). The imaginary part of \( \ln (1 - iz \cdot l/t \cdot l) \) yields a term bounded in \( z \), which ends the proof of the bound. Property (v) is easily proved by straightforward calculation in the special case \( u_1 = -u_2 \), and then the general case follows from \( u_1 + T_z u_2 = (u_1 + u_2) + (T_z u_2 - u_2) \). Statement (vi) follows from our discussion of the relation of the present formulation with the traditional one, ending Sec. [III].

VI. Vacuum versus regular, positive energy representations

Now we can take up the study of physical representations of the asymptotic algebra. First of all, vacuum states have to be characterized. We denote by \( \mathcal{F}_0 \) the subalgebra of \( \mathcal{F} \) generated by CAR and by elements \( W(V) \) with \( (V) \in L_0 \).

**Theorem 6.1** (i) If a cyclic representation \( \pi \) of the algebra \( \mathcal{F} \) satisfies Borchers’ criterion with respect to the Weyl algebra of the electromagnetic field, and \( U(x) \Omega = \Omega \) for some choice of the pertinent representation of translations \( U(x) \) and of the cyclic vector \( \Omega \), then for each \( C \in \text{CAR} \)

\[
(\Omega, \pi(CW(V))\Omega) = 0 \quad \text{if} \quad l \wedge V(-\infty, l) \neq 0.
\]

(ii) A cyclic representation \( \pi \) of the algebra \( \mathcal{F} \) satisfies Borchers’ criterion and \( U(x) \Omega = \Omega \) for some choice of the pertinent representation of translations \( U(x) \) and of the cyclic vector \( \Omega \) if, and only if, there is \( \Omega, \pi(CW(V))\Omega) = \omega_F(C)\omega_r(W(V)) \), where \( \omega_F \) is the positive energy Fock state on CAR and \( \omega_r \) is
the state on CCR given by

\[ \omega_r(W(V)) = \begin{cases} 
0, & \text{if } l \wedge V(-\infty, l) \neq 0, \\
 f(V)e^{-\frac{1}{2}F([\hat{V}], [\hat{V}])}, & \text{if } l \wedge V(-\infty, l) = 0, 
\end{cases} \quad (6.2) \]

where \( f : L_0 \to \mathbb{C} \) is a function of positive type, satisfying the condition

\[ f(V_1 + (T_x - 1)V_2) = f(V_1) \quad \forall (V_1) \in L_0, (V_2) \in L, x \in M. \quad (6.3) \]

Proof. (i) Using the algebraic relations (3.40) and (3.42), and the relation (5.15) one finds

\[ e^{F([\hat{V}_1], [T_x\hat{V}_2 - \hat{V}_2])}CW(V_1)W(T_xV_2) = e^{-i\{V_1, V_2\}e^{F([\hat{V}_1], [T_x\hat{V}_2 - \hat{V}_2])}}W(T_xV_2)\beta_{-\Phi_4}(C)W(V_1). \]

It follows now from the invariance of \( \Omega \) under \( U(x) \) and from the spectral properties of \( U(x) \) that the value of \( \omega(.) := (\Omega, \pi(.)\Omega) \) on the l.h.s. of this identity is the boundary value for \( \text{Im}_z = 0 \) of the function

\[ \mathbb{R}^4 + i\mathbb{V}_+ \ni z \to e^{F([\hat{V}_1], [T_x\hat{V}_2 - \hat{V}_2])} (\Omega, \pi(CW(V_1))U(z)\pi(W(V_2))\Omega), \]

continuous on its domain and analytic inside. By Lemma 5.1(ii) this function is polynomially bounded. The expectation value of the r.h. side of the above identity has similar properties in the time-reflected region \( \mathbb{R}^4 - i\mathbb{V}_+ \). By the edge of the wedge theorem there is an open region containing \( \mathbb{R}^4 + i\mathbb{V}_+ \cup \mathbb{V}_- \) and a function analytic in this region which is the analytic continuation of both these functions. This function is polynomially bounded on \( \mathbb{R}^4 + i\mathbb{V}_+ \cup \mathbb{V}_- \), hence it is a polynomial. To prove this implication let \( h(z) \) be a function with the stated properties, and choose a basis of Minkowski vector space consisting of four unit future-pointing vectors \( t_i \), \( i = 1, \ldots, 4 \). For fixed real \( \alpha^2, \alpha^3, \alpha^4 \) the function \( C \ni \xi \to h(\xi t_1 + \alpha^2 t_2 + \alpha^3 t_3 + \alpha^4 t_4) \) is analytic and polynomially bounded, hence by Cauchy inequality for analytic functions it is a polynomial in \( \xi \). Therefore, by repeated use of the implication, \( h(x) \) is a polynomial, hence \( h(z) \) is the same polynomial for complex \( z \). We have thus proved that

\[ e^{F([\hat{V}_1], [T_x\hat{V}_2 - \hat{V}_2])}\omega(CW(V_1))W(T_xV_2) = P_{C,V_1,V_2}(x), \quad (6.4) \]

where \( P_{C,V_1,V_2} \) is a polynomial. If \( p([\hat{V}_1]) = p([\hat{V}_2]) \neq 0 \), then by Lemma 5.1(iii) \( P_{C,V_1,V_2}(x) = 0 \) (take \( |\vec{x}| = \text{const. and } |x^0| \to +\infty \)). For \( x = 0 \), \( [\hat{V}_1] = [\hat{V}_2] = \frac{1}{2}[V] \), \( p([\hat{V}]) \neq 0 \) we get \( \omega(CW(V)) = 0 \), which ends the proof if (i).

(ii) If \( U(x) \) implement translations and \( U(x)\Omega = \Omega \) for cyclic \( \Omega \), then writing each \( C \in \text{CAR} \) in the form \( C = \omega_F(C)E + C' \), where \( \omega_F \) is the positive energy Fock state and \( C' \) is a sum of elements having \( B(g_+), B(g_-)^* \) on the right and/or
$B(g_+^*)$, $B(g_-)$ on the left, one finds that $\omega(CW(V)) = \omega_F(C)\omega(W(V))$. By (3.4), (5.14), and Lemma 5.1(v), for $p(V_1 + V_2) = 0$ Eq. (6.4) is now equivalent to

$$ f(V_1 + T_x V_2) = P'_{V_1, V_2}(x), \quad (6.5) $$

where $P'_{V_1, V_2}$ is a polynomial and $f$ is a function on $L_0$ defined by

$$ f(V) = \omega(W(V))e^{\frac{i}{2}F([\hat{V}], [\hat{V}])}. \quad (6.6) $$

By Lemma 5.1(vi) $f(V_1 + T_x V_2)$ is absolutely bounded as a function of $x$ for fixed $(V_1), (V_2) \in L_0$. Hence, by (6.5),

$$ f(V_1 + T_x V_2) = f(V_1 + V_2) \text{ for } (V_1), (V_2) \in L_0. \quad (6.7) $$

The positivity of $\omega(A^*A)$ for $A = \sum_{i=1}^n \alpha_i W(V_i)$, for $(V_i) \in L_0$ and all sequences of complex numbers $\{\alpha_i\}_{i=1}^n$, is equivalent to the condition

$$ \sum_{i,k=1}^n \beta_i \beta_k e^{F([\hat{V}_i], [\hat{V}_k])} f(V_k - V_i) \geq 0 \text{ for all sequences of complex numbers } \{\beta_i\}_{i=1}^n. $$

Following [25] we replace all $V_i$ in this condition by $\frac{1}{N} \sum_{k=0}^{N-1} T_x^k V_i$ and take the limit $N \to \infty$. Then by (6.7), Lemma 5.1(vi), and the ergodic theorem one has $\sum_{i,k=1}^n \beta_i \beta_k f(V_k - V_i) \geq 0$, hence $f$ is of positive type (the remaining conditions, $f(0) = 1$ and $f(V) = f(-V)$, are obviously satisfied). Consequently, $f$ is bounded and, by (6.3), (6.7) is satisfied for all $(V_1), (V_2) \in L$ with $(V_1 + V_2) \in L_0$. This is equivalent to (6.3), which ends the proof that the state $\omega$ has the form given in (ii).

Conversely, let $\omega_r$ be a linear functional of the form given by (ii). Then it is a state (one uses the fact that if $A$ and $B$ are two Hermitian positive matrices $n \times n$, then the matrix $C$ defined by $C_{ik} = A_{ik}B_{ik}$ is also positive). Translations are implemented in $\pi_r$ by the canonically defined representation of translations $U_r(x)$ ($\omega_r$ is translationally invariant). This representation is strongly continuous and satisfies the spectrum condition. To show this, it is sufficient to demonstrate that for any pair $(V_1), (V_2) \in L$ the function $x \mapsto \omega_r(W(V_1)W(T_x V_2))$ is continuous, and its Fourier transform has support in $\mathbf{V}_z$. If $(V_1 + V_2) \notin L_0$, then this function vanishes identically. For $(V_1 + V_2) \in L_0$ we have by (5.13) and Lemma 5.1(v)

$$ \omega_r(W(V_1)W(T_x V_2)) = e^{-\frac{i}{2} \{V_1, V_2\} - \frac{i}{2} \{V_1, T_x V_2 - V_2\}} \omega_r(W(V_1 + T_x V_2)) $$

$$ = \omega_r(W(V_1)W(V_2))e^{-F([\hat{V}_1], [\hat{T}_x \hat{V}_2 - \hat{V}_2])}. $$

The function $\mathbf{R}^4 + i\mathbf{V}_+ \ni z \mapsto e^{-F([u_1], [T_z u_2 - u_2])}$ is analytic and, by Lemma 5.1(iv), bounded on its domain. Therefore it is the Laplace transform of a distribution with support in $\mathbf{V}_+$. Now, the Fock state $\omega_F$ on CAR satisfies (2.8), so the state on $\mathcal{F}$ defined by $\omega(CW(V)) = \omega_F(C)\omega_r(W(V))$ generates a representation unitarily equivalent to a representation of the type (2.10). Translations are implemented in this representation by $U_F(x) \otimes U_r(x)$, where $U_F(x)$ is the representation canonically correlated to $\omega_F$, which ends the proof. □
The representation space $\mathcal{H}$ of a vacuum representation, as implied by the general result (i) of the above theorem, is easily seen to be the uncountable direct sum of the space $[\pi(F_0)\Omega]$ and spaces derived from it by the action of operators $\pi(W(V))$; if $l \wedge V_1(-\infty, l) = l \wedge V_2(-\infty, l)$ the respective spaces are equal, in other case they are orthogonal. In consequence, vacuum representations are nonregular with respect to the Weyl operators with infrared-singular test functions ($l \wedge V(-\infty, l) \neq 0$). Now, the derivation of our asymptotic algebra has led unambiguously to its interpretation, as explained in Sec.III and IV. From the point of view of this interpretation the above structure does not seem a physically justified idealization, for two reasons:

(i) The infrared singular Weyl operators, being degraded to operators intertwining between different representations of $F_0$, no longer describe the electromagnetic field observable. However, the regular Weyl operators, as discussed earlier, are functions of the free outgoing field only, and the Coulomb field is lost. If the vacuum is of the form (ii) of the theorem, one can separately define it by the first term in (4.3), but this is done “by hand”, and the information on the unique way in which the Coulomb field and the “out” field add to form the total field is lost.

(ii) The superselection sectors with respect to regular observables $A_0 := F_0 \cap A$ are labeled by the spacelike asymptotic of the free field (and, of course, by total charge, if it exists), so that even the standard wisdom does not apply here. The spacelike asymptotic of electromagnetic field according to Buchholz [14] yields in the subspace $[\pi(W(V))\pi(F_0)\Omega]$ the field (3.6), which is a free field for free field asymptotic $V$. Moreover, $A_0$ should not be interpreted as the algebra of local observables: creation or annihilation of a charged particle together with its Coulomb field is a nonlocal operation, so $B(g)^*B(f) \in A_0$ is a nonlocal observable.

Consider the particular, Poincaré-invariant vacuum state as given by Theorem 6.1(ii) with $f \equiv 1$. In this representation there is no infraparticle problem: all one-particle states $\pi(B(g_+)^*\Omega), \pi(B(g_-))\Omega$ have energy-momentum on the mass hyperboloid. We want to construct representations which remain as close in their structure to this vacuum representation, but which are regular at the same time. The obvious idea how to do it, is to try to integrate the superselection sectors of this vacuum into a direct integral Hilbert space. As the representation is of the form determined by Theorem 4.4, it is sufficient to confine attention to the electromagnetic part of the representation, $\pi_r$. However, for measure-theoretical reasons one has to extend the scope of sectors which are to be integrated. It is now that the extension of the symplectic space introduced in Sec.V will be needed. This extension allows us to consider the Weyl algebra CCR generated uniquely (due to nondegeneracy of the symplectic form [27]) by elements $\hat{W}([u]), [u] \in \hat{L}_f^f$, 

$$\hat{W}([u_1])\hat{W}([u_2]) = e^{-\frac{i}{2}\{[u_1], [u_2]\}}\hat{W}([u_1 + u_2]).$$

The elements $\hat{W}([u])$ with $[u]$ from the subspace $\hat{L}_0^f$ generate a $C^*$-subalgebra, which we denote CCR$_0$. The Poincaré transformations act on the algebra CCR...
by
\[ \alpha_{x,A} \hat{W}([u]) = \hat{W}([T_x A u]) \]
\[ (T_x A u)_a(s,l) = \Lambda(A)_a \frac{b}{u_b(s - x \cdot l, \Lambda^{-1} l))}. \]

The Poincaré invariant vacuum state is easily obtained:
\[ \hat{\omega}(\hat{W}([u])) = \begin{cases} 
0, & \text{if } p([u]) \neq 0, \\
\frac{i}{2} F([u], [u]), & \text{if } p([u]) = 0.
\end{cases} \]

The representation space \( \hat{H} \) of the representation \( \hat{\pi} \) canonically obtained from \( \hat{\omega} \) is the uncountable direct sum \( \hat{H} = \bigoplus_{[f] \in \mathcal{H}_{IR}} \hat{H}_{[f]} \), where \( \hat{H}_{[f]} \) is the subspace spanned by the vectors \( \hat{\pi}(\hat{W}([u])) \hat{\Omega} \) with \( p([u]) = [f] \). The restriction of the representation \( \hat{\pi} \) to the subalgebra \( \hat{\mathcal{CR}_0} \) and to the subspace \( \hat{H}_{[f]} \) is a coherent state representation \( [21] \), \( \hat{\pi}_{[f]}(\hat{\mathcal{CR}_0}) := \hat{\pi}(\hat{\mathcal{CR}_0})|_{\hat{H}_{[f]}} \). Two representations \( \hat{\pi}_{[f_i]}, \)
\[ i = 1,2, \]
are disjoint for \( [f_1] \neq [f_2] \) \( [21] \). In particular, \( \hat{\pi}_{[0]}(\hat{\mathcal{CR}_0}) \) is the (positive energy) Fock representation. Each subspace \( \hat{H}_{[f]} \) is invariant under the action of translations \( \hat{U}(x) \), thus \( \hat{U}(x) = \bigoplus_{[f] \in \mathcal{H}_{IR}} \hat{U}_{[f]}(x) \), each component representation being a positive energy representation, \( \text{Spec} \hat{U}_{[f]}(x) \subset \mathbb{V}_+ \).

Suppose we are given a cylindrical \( \sigma \)-additive measure \( \mu \) on the Hilbert space \( \mathcal{H}_{IR} \) \( [5.17], [28] \). Let \( \psi([f]) \) vary over the set of all measurable functions \( \mathcal{H}_{IR} \rightarrow \mathcal{H}_{[0]} \) (i.e., the set of functions for which \( (\varphi, \psi([f]))_{\mathcal{H}_{[0]}} \) is measurable for all \( \varphi \in \mathcal{H}_{[0]} \) \( [23] \)). Let further \( h(s,l) \) vary over the set of smooth homogeneous functions of degree \( -1 \), satisfying for some \( \epsilon > 0 \) and all \( k = 0, 1, \ldots \) the bounds
\[ |L_{b_1 c_1} \ldots L_{b_k c_k} h(s,l)| \leq \frac{\text{const} \langle k \rangle}{(|s| + 1)^{1+\epsilon}} \quad (6.8) \]
(for any gauge \( t l = 1 \)) and such that \( \hat{h}(0,l) = 1 \). Then \( [fh] \in \hat{L}' \) and \( p([fh]) = [f] \) for any \( [f] \in \mathcal{H}_{IR} \). Consider the set \( \Gamma \) of all functions of the form
\[ \mathcal{H}_{IR} \ni [f] \rightarrow \Psi([f]) = \hat{\pi}(\hat{W}([fh]))\psi([f]) \in \hat{H}_{[f]} \quad (6.9) \]

**Lemma 6.2** For any fixed \( h \) every function in \( \Gamma \) has a unique representation \( (6.3) \).

Proof. If \( h' \) is another function satisfying the same conditions as \( h \) then
\[ \pi'(\hat{W}([fh'])\psi([f]) = \pi(\hat{W}([fh]))e^{\frac{i}{2} \{[fh],[fh']\}} \pi_{[0]}(\hat{W}([fh' - h]))\psi([f]) \]
It has to be shown that if \( \psi([f]) \) is measurable, then \( \psi'([f]) = e^{\frac{i}{2} \{[fh],[fg]\}} \pi_{[0]}(\hat{W}([fg]))\psi([f]) \), where \( g \equiv h' - h \), is also measurable. It is
sufficient to show that \((e_i, \psi'([f]))\) are measurable for a basis \(\{e_i\}\) of \(\mathcal{H}_{[0]}\). If one chooses for \(\{e_i\}\) the finite “particle” number basis, then these products have the following form: 
\[
e^{\frac{i}{2}}\{[fh], [fg]\} - \frac{1}{2} F([fg], [fg]) \sum_{k=1}^{\infty} (e_k, \psi([f])) C_k([f]),
\]
where \(C_k([f])\) are linear combinations of products of expressions \(F([fg], [\chi])\) and their complex conjugations, where \(\chi \in \mathcal{L}_d\) are profiles of the photons in the basis vectors. Now, the expressions \(F([fg], [\chi]), \{[fh], [fg]\}\), and \(F([fg], [fg])\) are easily seen to be absolutely bounded as functions of \([f]\), the first by const.\(||[f]||_{\mathcal{H}_{IR}}\), the other two by const.\(||[f]||_{\mathcal{H}_{IR}}^2\). Hence, the first may be written as \(([f], [k]), [k] \in \mathcal{H}_{IR},\) and the other two as \(([f], B[f])_{\mathcal{H}_{IR}},\) where \(B\) is a bounded operator on \(\mathcal{H}_{IR}\). Both expressions are measurable functions of \([f]\), which ends the proof.

The set \(\Gamma\) thus has the natural structure of a vector space, and the pair \(\{\mathcal{H}_{[f]} [f] \in \mathcal{H}_{IR}, \Gamma\}\) is easily seen to form a \(\mu\)-measurable family of Hilbert spaces \([\mathcal{H}]\). This family determines the direct integral Hilbert space 
\[
\mathcal{H} = \int_{\oplus} \mathcal{H}_{[f]} d\mu([f]),
\]
with its elements denoted by \(\Psi = \int_{\oplus} \Psi([f]) d\mu([f])\). It is necessary for our purposes to assume that \(\mu\) is quasi-invariant with respect to translations by smooth elements of \(\mathcal{H}_{IR}\). This set, more exactly the set of elements in \(\mathcal{H}_{IR}\) having smooth representants, will be denoted by \(\mathcal{C}_{\mathcal{H}_{IR}}^\infty\). Hence, we demand that 
\[
\mu_{[k]}(B) := \mu(B - [k]) = 0 \iff \mu(B) = 0 \quad (6.10)
\]
(This was the reason for the need to extend \(L^d\) to \(\hat{L}^d\), and consequently \(\mathcal{C}_{\mathcal{H}_{IR}}^\infty\) to \(\mathcal{H}_{IR}\): there are no measures on function spaces quasi-invariant with respect to all translations \([\mathcal{H}]\).) Then, by Radon–Nikodym theorem, 
\[
d\mu_{[k]}([f]) = \left(d\mu_{[k]}/d\mu\right) ([f]) d\mu([f]), \quad \left(d\mu_{[k]}/d\mu\right) ([f]) \quad \text{and} \quad \left((d\mu_{[k]}/d\mu) ([f])\right)^{-1}
\]
are integrable, non-negative functions.

Formula (6.11) may be used to define, for each fixed \(h\), an isomorphism of the space \(\mathcal{H}\) with the tensor product space \(L^2(\mathcal{H}_{IR}, \mu) \otimes \mathcal{H}_{[0]}\). Namely, we put by definition 
\[
\mathcal{U}_h : L^2(\mathcal{H}_{IR}, \mu) \otimes \mathcal{H}_{[0]} \to \mathcal{H}, \quad \chi \otimes \psi \to \Psi = \int_{\oplus} \Psi([f]) d\mu([f]), \quad (6.11)
\]
This is a noncanonical isomorphism, as it depends on the choice of the function \(h\), which has no intrinsic meaning. The use of the isomorphism will be restricted to technical purposes only.

We now define in \(\mathcal{H}\) a new representation of CCR. Note that if \(V\) is a test function of \(W(V)\), then by (3.33) \(\tilde{V}(\omega, l)\) is a smooth function of \(l\) for each \(\omega\). Therefore \(p([\tilde{V}])\) is a smooth element of \(\mathcal{H}_{IR}\) (has a smooth representant \(\tilde{V}(0, l)\)).
Theorem 6.3  The linear operators on $\mathcal{H}$ introduced by
\[
[\pi_r(W(V))\Psi](\hat{f}) := \left(\frac{d\mu_{\pi_r}[\hat{V}]}{d\mu}(\hat{f})\right)^{\frac{1}{2}} \hat{\pi}(\hat{W}(\hat{V}))\Psi(\hat{f} - p(\hat{V})),
\]
(6.12)
define a representation $\pi_r$ of CCR satisfying Borchers’ criterion. The unitary representation of translations defined by
\[
[U_r(x)\Psi](\hat{f}) := U_{[f]}(x)\Psi(\hat{f})
\]
(6.13)
implements translations in the representation $\pi_r$ and satisfies $\text{Spec}U(x) = \mathbb{V}_+$.  

Proof. A straightforward calculation shows that $\pi_r$ and $U_r$ satisfy the algebraic conditions of representations and that $U_r(x)$ implement translations in the representation $\pi_r$. From the strong continuity and spectral properties of each of $U_{[f]}(x)$ it follows that also $U_r(x)$ is strongly continuous and $\text{Spec}U_r(x) \subset \mathbb{V}_+$. The proof will be now completed by showing that $\text{Spec}(x)U_{[0]} \subset \text{Spec}U_r(x)$ ($U_{[0]}$ is the standard transformation of the Fock representation, hence $\text{Spec}(x)U_{[0]} = \mathbb{V}_+$). Take $\Psi = U_h(\chi \otimes \psi)$, with $\|\chi\|_{L^2(H_{IR,\mu})} = 1$, $\|\psi\|_{\tilde{H}_{[0]}} = 1$. Then
\[
(\Psi, U_r(x)\Psi) = \int |\chi(f)|^2 e^{\frac{i}{2} \{[fh], [fT_xh]\}} (\psi, \hat{\pi}_{[0]}(\hat{W}(\{f(T_xh - h)\}))\hat{U}_{[0]}(x)\psi) \, d\mu([f]).
\]
Let $\Psi_k = U_{h_k}(\chi \otimes \psi) \ (k = 1, 2 \ldots)$, with $h_k(s, l) = k^{-1}h(k^{-1}s, l)$. Then
\[
\lim_{k \to \infty} F([fh_k], [f(T_xh_k - h_k)]) = 0, \quad \lim_{k \to \infty} F([f(T_xh_k - h), [f(T_xh_k - h_k)]) = 0
\]
and $w-\lim_{k \to \infty} \hat{\pi}_{[0]}(\hat{W}(\{f(T_xh_k - h_k)\})) = 1$. By Lebesgue’s theorem
\[
\lim_{k \to \infty} (\Psi_k, U_r(x)\Psi_k) = (\psi, U_{[0]}(x)\psi).
\]
Hence $\text{Spec}U_{[0]}(x) \subset \text{Spec}U_r(x)$.  

Further properties of the representation $\pi_r$ are discussed after finding its unitarily equivalent form $U^{-1}_h \pi_r U_h$ by (6.14). Let $V_a(s, l)$ be any test function of $W(V)$. The integral
\[
-\frac{1}{4\pi} \int h(s, l)\epsilon(s - \tau)V_a(\tau, l) \, ds \, d\tau = -i P \int \hat{h}(\omega, l)\hat{V}_a(\omega, l) \frac{d\omega}{\omega}
\]
is a real smooth vector function orthogonal to $l$, and a representant of an element of $\mathcal{H}_0$ (defined before (5.12)). The orthogonal projection of this element to the subspace $H_{IR}$ of $\mathcal{H}_0$ will be denoted by $r_h(\hat{V})$. We show in the Appendix that the orthogonal projection to $H_{IR}$ of a smooth element of $\mathcal{H}_0$ is smooth, hence $r_h(\hat{V}) \in C_{IR}^\infty$. Let ccr be the Weyl algebra over the vector space $C_{IR}^\infty \oplus C_{IR}^\infty$ with the symplectic form $\{[g_1] \oplus [k_1], [g_2] \oplus [k_2]\}_{IR} := ([g_1, [k_2])_{H_{IR}} - ([k_1], [g_2])_{H_{IR}}$. This algebra is generated by elements $w([g] \oplus [k])$ satisfying
\[
\begin{align}
w([g_1] \oplus [k_1]) w([g_2] \oplus [k_2]) &= e^{-\frac{4}{2} \{[g_1] \oplus [k_1], [g_2] \oplus [k_2]\}_{IR}} w([g_1 + g_2] \oplus [k_1 + k_2]), \\
w([g] \oplus [k])^* &= w(-(g) \oplus (k)).
\end{align}
\]
(6.14)
Theorem 6.4
(i) The following definition determines a representation \( \pi_\mu \) of the algebra \( \text{CCR} \) on the Hilbert space \( L^2(\mathcal{H}_{IR}, \mu) \)

\[
(\pi_\mu(w([g] \oplus [k])) \chi)([f]) := 
\left( \frac{d\mu_{[g]}}{d\mu}([f]) \right)^{\frac{1}{2}} e^{i([f] - \frac{1}{2}[g], [k])_{\mathcal{H}_{IR}}} \chi([f - g]).
\]

(6.15)

The vector \( \Omega_\mu([f]) = 1 \) is cyclic in \( L^2(\mathcal{H}_{IR}, \mu) \) for the representation \( \pi_\mu \) restricted to the subalgebra generated by elements \( w([0] \oplus [k]) \).

(ii) The representation \( \pi_r \) goes over under the unitary transformation \( U_h \) of representation space to

\[
U_h^{-1} \pi_r(W(V)) U_h = \pi_\mu(w(p(\dot{V}) \oplus r_h([\dot{V}]))) \otimes \pi_{[0]}(\tilde{W}([\dot{V} - \tilde{V}(0, .)h])).
\]

(6.16)

(iii) \( (U_h^{-1} \pi_r(\text{CCR}) U_h)^n = \pi_\mu(\text{CCR})^n \otimes \mathcal{L}(\mathcal{H}_{[0]}) \) (the von Neumann tensor product), where \( \mathcal{L}(\mathcal{H}_{[0]}) \) is the algebra of bounded operators on \( \mathcal{H}_{[0]} \). Therefore, \( \pi_r \) is irreducible iff \( \pi_\mu \) is irreducible.

(iv) \( \pi_r \) is regular iff \( \pi_\mu \) is regular.

Proof.
(i) The algebraic properties needed for \( \pi_\mu \) to be a representation of \( \text{CCR} \) are checked by direct calculation. For any \( \chi \in L^2(\mathcal{H}_{IR}, \mu) \) there is

\[
(\chi, \pi_\mu(w([0] \oplus [k]))) \Omega_\mu) = \int \chi([f]) e^{i([f], [k])} d\mu([f]),
\]

If this integral vanishes for all smooth \([k]\), then by continuity in \([k]\) it vanishes for all \([k] \in \mathcal{H}_{IR} \) and then \( \chi = 0 \), which shows that \( \Omega_\mu \) is cyclic.

(ii) A direct calculation gives

\[
[\pi_r(W(V)) U_h (\chi \otimes \psi)]([f]) = \pi_r(\tilde{W}([f h])) \left( \frac{d\mu_{p(\dot{V})}}{d\mu}([f]) \right)^{\frac{1}{2}} \times 
\]

\[
e^{i \{([f - \frac{1}{2} \tilde{V}(0, .)h], [\dot{V}]) \chi([f] - p([\dot{V}])) \pi_{[0]}(\tilde{W}([\dot{V} - \tilde{V}(0, .)h])) \psi}.
\]

Denote \( f' \equiv f - \frac{1}{2} \tilde{V}(0, .) \). The symplectic form in the exponent of the last formula is evaluated by

\[
\{[f'h], [\dot{V}]\} = - \int f'_a(l) \left( -i P \int \tilde{h}(\omega, l) \tilde{V}^a(\omega, l) \frac{d\omega}{\omega} \right) d^2l
\]

\[
= ([f'], r_h([\dot{V}]))_{\mathcal{H}_{IR}} = ([f] - \frac{1}{2} P([\dot{V}]), r_h([\dot{V}]))_{\mathcal{H}_{IR}}.
\]

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Transforming back with $U_h^{-1}$ one obtains (6.16).

(iii) Let $(V') \in L_0$. Further, let $f_a(l)$ be a smooth function representing an element $[f] \in \mathcal{H}_{IR}$ and put $V_a(s,l) = f_a(l)h(s,l) + \dot{V}_a(s,l)$. Then $(V) \in L$, $p([V]) = [f]$, and

$$U_h^{-1}\pi_r(W(V))U_h = \pi_\mu([f] \oplus r_h([\dot{V}'])) \otimes \hat{\pi}_{[0]}(\hat{W}([\dot{V}'])) .$$

(6.17)

The functions $\dot{V}'s$ are dense in the Hilbert space of test functions of the Fock representation, which is irreducible. Therefore, to prove the equality in (iii) it is sufficient to show that all operators $\pi_\mu([f] \oplus [g]) \otimes 1_{[0]}$ are in $(U_h^{-1}\pi_r(\text{CCR})U_h)^\prime$. The second statement of (iii) then follows from $(U_h^{-1}\pi_r(\text{CCR})U_h)^\prime = \pi_\mu(\text{CCR}) \otimes \textbf{C}1_{[0]}$ (see 2.3).

To fill the missing step consider for $\beta \in (0, \epsilon)$ a one-parameter family of functions $N_\beta(\omega,l) = \tilde{\kappa}_\beta(\omega t \cdot l)h(\omega,l)$, where $\tilde{\kappa}_\beta(\omega) = ie^{-|\omega|}\beta \text{sgn}(\omega)$. Then $N_\beta(s,l)$ are real homogeneous functions of degree $-1$ given by

$$N_\beta(s,l) = \frac{1}{2\pi t \cdot l} \int \kappa_\beta \left( \frac{s - \tau}{t \cdot l} \right) h(\tau,l) d\tau ,$$

with

$$\kappa_\beta(s) = 2\Gamma(1 + \beta)(s^2 + 1)^{-(1 + \beta)/2} \sin((1 + \beta)\arctan s).$$

With the use of bounds (6.8) one shows that $N_\beta(s,l)$ are smooth and also satisfy bounds of the form (6.3), with $\epsilon$ replaced by $\beta$. Consider the smooth homogeneous function of degree $0$ given by the integral

$$c_\beta(l) = -\frac{1}{4\pi} \int h(\tau,l)e(\tau - s)N_\beta(s,l) d\tau ds = 2 \int_0^\infty \left| h(l) \right| e^{-|\omega|\beta - 1} d\omega' ,$$

the last equality by (5.8). From the bound of the form (5.4) satisfied by $h$ and the condition $h(0,l) = 1$ we know that there is a positive $u$, such that $\left| h(\omega' / t \cdot l) \right|^2 e^{-\omega'} > \frac{1}{2}$ for $\omega' \in (0,u)$. On the other hand, the function $\left| h(l) \right|^2$ is bounded by a constant from above. Hence, $(u^3 / \beta) < c_\beta(l) < \text{const} \Gamma(\beta)$. Therefore, the new auxiliary function defined by $n_\beta(s,l) = N_\beta(s,l)/c_\beta(l)$ is smooth, has all the properties listed above for $N_\beta(s,l)$, and in addition satisfies

$$-\frac{1}{4\pi} \int h(\tau,l)e(\tau - s)n_\beta(s,l) d\tau ds = 1 .$$

(6.18)

Now, choose a smooth element $[g]$ in $\mathcal{H}_{IR}$ and put $\dot{V}'_{\beta a}(s,l) = g_a(l)n_\beta(s,l)$. These $\dot{V}'_{\beta a}$ satisfy the conditions for a test function $V'$ in (6.17) and, by (6.18), yield $r_h([\dot{V}'_{\beta}]) = [g]$. For $\dot{V}'_{\beta a}(s,l) = f_a(l)h(s,l) + \dot{V}'_{\beta a}(s,l)$ Eq. (6.17) reads

$$U_h^{-1}\pi_r(W(V))U_h = \pi_\mu([f] \oplus [g]) \otimes \hat{\pi}_{[0]}(\hat{W}([\dot{V}'_{\beta}])) .$$

(6.19)

By a straightforward calculation one obtains

$$F([\dot{V}'_{\beta},[\dot{V}'_{\beta}]) = \int d^2l \frac{-g^2(l)}{c_\beta^2(l)} \int_0^\infty \left| h(l) \right|^2 e^{-2\omega'\beta^2 - 1} d\omega'$$

$$< \int (-g^2(l)) \frac{c_\beta^2(l)}{2c_\beta^2(l)} d^2l < \text{const.} \frac{\beta^2 \Gamma(2\beta)}{u^{2\beta}} \int (-g^2(l)) d^2l = \text{const.} \frac{\Gamma(1 + 2\beta)}{u^{2\beta}} .$$

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Hence \( \lim_{\beta \to 0} F([[\dot{V}_\beta]], [[\dot{V}_\beta]]) = 0 \), and then \( w \lim_{\beta \to 0} \hat{\pi}_{[0]}(\hat{W}([[\dot{V}_\beta]])) = 1 \). Then, by (6.19),
\[ w \lim_{\beta \to 0} \mathcal{U}^{-1}_h \pi_r (W(V_\beta)) \mathcal{U}_h = \pi_\mu (w([f] \oplus [g])) \otimes 1, \]
which ends the proof of (iii).

(iv) This is obvious by regularity of the Fock representation and by
\[ \mathcal{U}^{-1}_h \pi_r (W((\lambda V)_\Phi)) \mathcal{U}_h = \pi_\mu (w(\lambda(p([\dot{V}]) \oplus r_h([\dot{V}])))) \otimes \hat{\pi}_{[0]}(\hat{W}(\lambda[\dot{V} - \tilde{\dot{V}}(0,.,h)])) \).

Remarks. (i) Theorems 4.4, 6.3, and 6.4 together characterize a class of regular, irreducible representations of the algebra \( \mathcal{F} \) satisfying Borchers’ criterion. When restricted to \( \text{CCR} \cap \mathcal{F}_0 \) the representations decompose into direct integral of coherent state representations and in this respect resemble the scattering representations considered in Ref. [15]. However, here the infrared clouds are independent of the charged particles (they are there even if there are no such particles present). In particular, the arguments of this reference for the Lorentz symmetry breaking do not apply here.

(ii) The vacuum vector is replaced here by “infravacua”, and states with finite charged particle number are obtained by the action of creation operators on any such state. (The “infravacua” are not of the KPR type [29], which does not lead to coherent states.) There are no vectors with the energy-momentum on mass hyperboloid, but the arguments of Ref. [14] do not apply here either: the asymptotic of electromagnetic field according to Buchholz catches only the free field part, and does not characterize states by classical distribution of electric flux.

(iii) The operators \( \pi(W(V)) \) with \( l \wedge V(-\infty, l) \neq 0 \) representing the exponentials of total electromagnetic field do not commute with \( \pi(B(g)) \), which reflects the fact that creation or annihilation of a charged particle together with its Coulomb field is a nonlocal operation. They do not commute with nonlocal observables \( \pi(B(g)^*B(f)) \) either.

A particular representation in the class thus characterized is given whenever a measure \( \mu \) is chosen, such that the condition (6.10) is satisfied and the representation \( \pi_\mu \) is regular and irreducible. Explicit characterization of such measures may be given in the subclass of Gaussian measures. For any positive, trace-class operator \( B \) in the Hilbert space \( \mathcal{H}_{IR} \) the characteristic function
\[ \int e^{i([f],[g])} d\mu_B([f]) = e^{-\frac{1}{2}([g],B[g])} \]
defines a cylindrical, \( \sigma \)-additive measure, a Gaussian measure with covariance \( B \) [30]. The following proposition is obtained by the application of general standard results.

**Proposition 6.5**

(i) A Gaussian measure with covariance \( B \) satisfies the quasi-invariance condition (6.11) iff
\[ \mathcal{C}^\infty_{IR} \subset B^{\frac{1}{2}} \mathcal{H}_{IR} \] (6.20)
(hence \( \text{Ker}B^\frac{1}{2}\IR = \{0\} \), as \( C^\infty_{\IR} / \text{Ker}B^\frac{1}{2}\IR = \mathcal{H}_{\IR} \)). Then the representation \( \pi_\mu \) is regular and it is unitarily equivalent to the GNS representation of the quasi-free state

\[
\omega_\mu(w([g] \oplus [k])) = e^{-\frac{1}{2}s([g] \oplus [k], [g] \oplus [k])},
\]

(6.21)

where \( s \) is a bilinear, positive definite form on \( C^\infty_{\IR} \oplus C^\infty_{\IR} \) satisfying the defining condition of a quasi-free state

\[
|([g_1] \oplus [k_1], [g_2] \oplus [k_2])_{\IR}|^2 \leq s([g_1] \oplus [k_1], [g_1] \oplus [k_1])s([g_2] \oplus [k_2], [g_2] \oplus [k_2]) \quad (6.22)
\]

given explicitly by

\[
s([g_1] \oplus [k_1], [g_2] \oplus [k_2]) = \frac{1}{2} \left( B^{-\frac{1}{2}}[g_1], B^{-\frac{1}{2}}[g_2] \right) + 2 \left( B^{\frac{1}{2}}[k_1], B^{\frac{1}{2}}[k_2] \right). \quad (6.23)
\]

(ii) If (6.20) is satisfied, then \( \pi_\mu \) is irreducible iff \( B^{-\frac{1}{2}}C^\infty_{\IR} = \mathcal{H}_{\IR} \). The state \( \omega_\mu \) is a quasi-free state, which always yields a Fock representation. Otherwise \( \pi_\mu \) is non-factor.

Proof. (i) A Gaussian measure on a Hilbert space \( \mathcal{H}_{\IR} \) and with covariance \( B \) is equivalent to its translation by an element \( [g] \in \mathcal{H}_{\IR} \) if, and only if, \( [g] \in B^{\frac{1}{2}}\mathcal{H}_{\IR} \) (see [31]), which proves the first statement of (i). If this is the case, then

\[
\frac{d \mu_{[g]}}{d \mu}([f]) = e^{-\frac{1}{2}\|B^{-\frac{1}{2}}[g]\|^2 + \langle [f], B^{-1}[g]\rangle},
\]

where \( \langle [f], B^{-1}[g]\rangle \) is to be understood in the sense of a measurable linear functional on \( \mathcal{H}_{\IR} \) [31]. As the vector \( \Omega_\mu \) is cyclic for \( \pi_\mu \), this representation is unitarily equivalent to the GNS representation obtained from the state \( \omega_\mu(A) := (\Omega_\mu, \pi_\mu(A)\Omega_\mu) \). The relations (6.21)–(6.23) are now verified by calculation. By the results of Ref. [31], \( \omega_\mu \) is a quasi-free state, which always yields a regular representation.

(ii) By the results of [31] the state \( \omega_\mu \) is primary iff the extension \( \{.,.\}^s_{\IR} \) of the symplectic form \( \{.,.\}_{\IR} \) to the completion of \( C^\infty_{\IR} \oplus C^\infty_{\IR} \) with respect to \( s \) is nondegenerate. This completed space is a real Hilbert space \( \mathcal{H}_s \). In our case \( \mathcal{H}_s = \mathcal{H}_- \oplus \mathcal{H}_+ \), where \( \mathcal{H}_- \) (resp. \( \mathcal{H}_+ \)) is the completion of \( C^\infty_{\IR} \) with respect to the norm \( \|\|_{\mathcal{H}_-} := \|B^{-\frac{1}{2}}[f]\| \) (resp. \( \|\|_{\mathcal{H}_+} := \|B^{\frac{1}{2}}[f]\| \)), and \( s \) is extended to \( (x_1 \oplus y_1, x_2 \oplus y_2) = \frac{1}{2}(x_1,x_2) - 2(y_1,y_2) \). The linear operator \( U_- \) (resp. \( U_+ \)) on \( C^\infty_{\IR} \) defined by \( U_-[f] := B^{-\frac{1}{2}}[f] \) (resp. \( U_+[f] := B^{\frac{1}{2}}[f] \)) maps \( C^\infty_{\IR} \) as a subspace of \( \mathcal{H}_- \) (resp. of \( \mathcal{H}_+ \)) isometrically into \( \mathcal{H}_{\IR} \). By continuity \( U_{\pm} \) extend to isometric operators \( U_{\pm} : \mathcal{H}_{\pm} \to \mathcal{H}_{\IR} \). By restricting arguments in the following equation to smooth elements one sees that the extension of \( \{.,.\}_{\IR} \) to \( \{.,.\}^s_{\IR} \) is given by \( \{x_1 \oplus y_1, x_2 \oplus y_2\}^s_{\IR} = (U_-x_1, U_+y_2)_{\IR} - (U_-x_2, U_+y_1)_{\IR} \). This form is nondegenerate iff \( U_-\mathcal{H}_- = U_+\mathcal{H}_+ \). However, \( U_+\mathcal{H}_+ = B^{\frac{1}{2}}C^\infty_{\IR} = \mathcal{H}_{\IR} \) (as \( \text{Ker}B^{\frac{1}{2}} = \{0\} \)). Therefore, \( \omega_\mu \) is primary iff \( U_-\mathcal{H}_- \equiv B^{-\frac{1}{2}}C^\infty_{\IR} = \mathcal{H}_{\IR} \). If this is the case, then it is checked by direct calculation that \( (x_1 \oplus y_1, x_2 \oplus y_2)_s = \{x_1 \oplus y_1, A(x_2 \oplus y_2)\}^s_{\IR} \).
where $A(x \oplus y) := \left(-2U^{-1}U_+ y\right) \oplus \left(\frac{1}{2}U_+^{-1}U_- x\right)$. The operator $A$ satisfies the equation $A^2 = -1$, which is a necessary and sufficient condition for $\omega_\mu$ to be pure. \[\square\]

Concrete examples of trace-class operators $B$ satisfying the conditions of Proposition 6.3 (i) and (ii) are most easily constructed in the unitarily equivalent version of the space $H_{1R}$, the Hilbert space $H_{\partial}$ discussed in the Appendix. Let $\tilde{B}_\alpha = \left[\left((t \cdot l)^2 \partial^2\right)^{-1}\right]^{1+\alpha}$, $\alpha > 0$, where $\left((t \cdot l)^2 \partial^2\right)^{-1}$ is the positive operator on $H_{\partial}$ defined in Appendix (Eq. (A.10) and the following discussion). It follows from the spectral properties of this operator that each of the operators $\tilde{B}_\alpha$ may serve as an example of the transformed covariance operator.

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Appendix: Homogeneous functions on the light-cone

In this Appendix we briefly discuss some structures and operations on spaces of homogeneous functions on the future lightcone. Let, first, $\phi(l)$, $\phi_1(l)$ and $\phi_2(l)$ be smooth ($C^\infty$ in the sense of differentiation by $L_{ab} = l_a \partial_b - l_b \partial_a$) functions, homogeneous of degree 0. Take, for the sake of differentiation, extensions of these functions which remain homogeneous in some neighbourhood of the cone. Straightforward calculation then gives on the cone

$$L_a^b L_{cb} \phi = *L_c^b L_{ab} \phi = -L_{ac} \phi + l_a l_c \partial^2 \phi,$$

(A.1)

$$*L_a^b L_{cb} \phi = -L_c^b *L_{ab} \phi = -*L_{ac} \phi,$$

(A.2)

$$L_a^b \phi_1 L_{cb} \phi_2 = *L_c^b \phi_1 *L_{ab} \phi_2 = l_a l_c \partial \phi_1 \cdot \partial \phi_2,$$

(A.3)

where $*L_{ab} = \frac{1}{2} \epsilon_{abcd} L^{cd}$ and $\partial^2 = \partial_a \partial^a$. As the action of $L_{ab}$ is extension-independent (this is the tangent derivative), these formulae give extension-independent meaning of $\partial^2 \phi$ and $\partial \phi_1 \cdot \partial \phi_2$, which in this form were calculated for a homogeneous (but otherwise arbitrary) extensions. Contracting (A.1) and (A.3) with $t^a t^c$, where $t$ is any unit future-pointing vector, one obtains

$$\partial^2 \phi = (t \cdot l)^{-2} *L_0^b L_{0b} \phi,$$

(A.4)

$$\partial \phi_1 \cdot \partial \phi_2 = (t \cdot l)^{-2} *L_0^b \phi_1 *L_{0b} \phi_2,$$

(A.5)

where $*L_{0b} = t^a L_{ab}$. If $\psi$ is any differentiable function, homogeneous of degree $-2$, then

$$\int L_{ab} \psi(l) \, d^2 l = 0.$$

(A.6)
Using this identity one obtains from (A.4) and (A.5) by integration by parts (and taking into account \( *L_{0b}t \cdot l = 0 \))

\[
\int \phi_1 \partial^2 \phi_2 \, d^2 l = - \int \partial \phi_1 \cdot \partial \phi_2 \, d^2 l.
\]  

(A.7)

Therefore, \( \int \phi \partial^2 \phi \, d^2 l \geq 0 \) and \( \int \phi \partial^2 \phi \, d^2 l = 0 \) iff \( \phi = \text{const.} \). Thus \( \partial^2 \phi = 0 \) iff \( \phi = \text{const.} \). This positivity of \( \partial^2 \) is also seen from the identity (A.4), which says that \( \partial^2 \), when applied to a homogeneous function of degree 0, is the “orbital angular momentum squared” in each Minkowski frame.

The action of \( \partial^2 \) may be explicitly reversed. For each smooth, homogeneous of degree 0 function \( \phi \) the function \( \psi = \partial^2 \phi \) is smooth, homogeneous of degree \(-2\), and satisfies \( \int \psi \, d^2 l = 0 \). Conversely, if \( \psi \) is any function with these properties, then the formula

\[
\phi_t(l) = -\frac{1}{4\pi} \int \ln \frac{l' \cdot l'}{t' \cdot l'} \psi(l') \, d^2 l'
\]

(A.8)
gives the unique such smooth function that \( \partial^2 \phi_t = \psi \), and with the additive constant chosen such that

\[
\int \frac{\phi_t(l)}{(t' \cdot l')^2} \, d^2 l = 0.
\]

(A.9)

Smoothness of \( \phi_t(l) \) is proved by showing that for \( \epsilon \downarrow 0 \) the functions

\[
\phi_{\epsilon \phi}(l) = -\frac{1}{4\pi} \int \ln \frac{l' \cdot l' + \epsilon t' \cdot l'}{t' \cdot l'} \psi(l') \, d^2 l'
\]

converge uniformly to \( \phi_t(l) \), and \( L_{ab} \phi_{\epsilon \phi}(l) \) converge uniformly to

\[-\frac{1}{4\pi} \int \ln \frac{l' \cdot l' + \epsilon t' \cdot l'}{t' \cdot l'} L_{ab} \psi(l') \, d^2 l' + \frac{1}{4\pi} \int \frac{l'_a t_b - l'_b t_a}{t' \cdot l'} \psi(l') \, d^2 l'.
\]

Then it remains to show that

\[-\frac{1}{4\pi (t' \cdot l')^2} \int \ln \frac{l' \cdot l' + \epsilon t' \cdot l'}{t' \cdot l'} * L_{bb}^t L_{0b}^t \psi(l') \, d^2 l' = \frac{2\epsilon + \epsilon^2}{4\pi} \int \left( \frac{t' \cdot l'}{l' \cdot l' + \epsilon t' \cdot l'} \right)^2 \psi(l') \, d^2 l',
\]

converges to \( \psi(l) \) pointwise, which is an easy exercise.

Consider now the linear space \([C^\infty] \) of equivalence classes \([\phi] = \{ \phi' | \phi' = \phi + \text{const.} \}\) of smooth, homogeneous of degree 0, functions \( \phi \). On this space the operators \([\phi] \rightarrow [(t \cdot l)^2 \partial^2 \phi] \) and \([\phi] \rightarrow [((t \cdot l)^2 \partial^2)^{-1} \phi] \), where \( \phi_t \) is the representant satisfying (A.9), and

\[
((t \cdot l)^2 \partial^2)^{-1} \phi_t(l) = -\frac{1}{4\pi} \int \ln \frac{l' \cdot l'}{t' \cdot l'} \frac{\phi_t(l')}{(t' \cdot l')^2} \, d^2 l',
\]

(A.10)

are bijective, mutually inverse maps. Moreover, it is easily seen that the operator \(((t \cdot l)^2 \partial^2)^{-1} \) is bounded with respect to the norm of the scalar product on \([C^\infty] \) defined by

\[
([\phi_1], [\phi_2])_{t} = \int \frac{\phi_1(t \cdot l) \phi_2(t \cdot l)}{(t \cdot l)^2} \, d^2 l,
\]

(A.11)
that is
\[ \left\| \left( (t^2 \partial^2)^{-1} \phi \right) \right\|_t \leq c\left\| \phi \right\|_t . \tag{A.12} \]
The expression \( [A.7] \) introduces another scalar product on \( [C^\infty] \)
\[ ([\phi_1], [\phi_2])_{\partial^2} = \int \phi_1 \partial^2 \phi_2(l) \, dl . \tag{A.13} \]
By \( [A.11] \)–\( [A.13] \) we have
\[ \left\| \phi \right\|_{\partial^2}^2 = ([\phi], [(t^2 \partial^2)^{-1} \phi])_{\partial^2} \leq \left\| \phi \right\|_{\partial^2} \left\| \left( (t^2 \partial^2)^{-1} \phi \right) \right\|_{\partial^2} \]
\[ = \left\| \phi \right\|_{\partial^2} \sqrt{([\phi], [(t^2 \partial^2)^{-1} \phi])_t} \leq \sqrt{C}\left\| \phi \right\|_{\partial^2} \left\| \phi \right\|_t , \]

hence
\[ \left\| \phi \right\|_t \leq \sqrt{C}\left\| \phi \right\|_{\partial^2} , \tag{A.14} \]
and
\[ \left\| \left( (t^2 \partial^2)^{-1} \phi \right) \right\|_{\partial^2} \leq c\left\| \phi \right\|_{\partial^2} , \tag{A.15} \]
that is, the operator \( [A.10] \) is bounded with respect to this norm as well. Furthermore, for any sequence \( [\phi_n] \in [C^\infty] \) there is
\[ \left\| \phi_n \right\|_{\partial^2}^2 \leq \left\| [\phi_n - \phi_m] \right\|_{\partial^2}^2 + 2([\phi_n], [\phi_m])_{\partial^2} \]
\[ \leq \left\| [\phi_n - \phi_m] \right\|_{\partial^2}^2 + 2\left\| [\phi_n] \right\|_t \left\| (t^2 \partial^2 \phi_m) \right\|_t . \]
Suppose \( \lim_{m,n \to \infty} \left\| [\phi_m - \phi_n] \right\|_{\partial^2} = 0 \) and \( \lim_{n \to \infty} \left\| [\phi_n] \right\|_t = 0 \). For \( \epsilon > 0 \) let \( \left\| [\phi_m - \phi_n] \right\| < \epsilon \) for all \( m, n \geq N \). Put \( m = N \) and let \( N' \geq N \) be such that for all \( n \geq N' \) there is \( 2\left\| [\phi_n] \right\|_t \left\| (t^2 \partial^2 \phi_N) \right\| < \epsilon^2 \). Then for all \( n \geq N' \) there is \( \left\| [\phi_n] \right\|_{\partial^2} < 2\epsilon^2 \), hence
\( \lim_{n \to \infty} \left\| [\phi_n] \right\|_{\partial^2} = 0 \). Summing up, the norm \( \left\| \cdot \right\|_{\partial^2} \) is stronger than \( \left\| \cdot \right\|_t \) and these norms are compatible \( [12] \). This implies that the Hilbert space \( H_{\partial^2} \) obtained from \( [C^\infty] \) by completion with respect to \( \left\| \cdot \right\|_{\partial^2} \) is (may be canonically identified with) a subspace of the completion of \([C^\infty] \) with respect to \( \left\| \cdot \right\|_t \). The latter space is the Hilbert space \( L^2_t \) of equivalence classes of homogeneous of degree 0, measurable functions modulo constant for which \( \left\| [\phi] \right\|_t < \infty \). This space does not depend on \( t \) when considered as a vector space. All elements of \( H_{\partial^2} \) are therefore equivalence classes \( [\phi] \) of such functions.

The operator \( [A.10] \) extends to a bounded, self-adjoint, positive operator on \( H_{\partial^2} \). In view of the comment following \( [A.7] \) this is a compact operator with eigenvalues \( (j(j + 1))^{-1} \) of multiplicity \( 2j + 1 \), where \( j = 1, 2, \ldots \), and eigenspaces \( H_j \) contained in \([C^\infty] \).

With the use of the Hilbert space \( H_{\partial^2} \) the operation of projecting \( H_0 \) to \( H_{IR} \) needed in Sec.V may be described more explicitly. Let, first, \( f_a(l) \) be a smooth function, homogeneous of degree \(-1\) and orthogonal to \( l^a \), which means that \( [f_a] \) is a smooth element of \( H_0 \). For every such function there are smooth, homogeneous of degree 0 functions \( \phi(l) \) and \( \psi(l) \), unique up to additive constants, such that
\[ l_a f_b(l) - l_b f_a(l) = L_{ab} \phi(l) - \ast L_{ab} \psi(l) . \tag{A.16} \]
This is shown most easily with the use of the spinor formalism, which we do not intend to discuss here and refer the reader to [33], and to [2] for application to related problems. Within this formalism it is a simple result that
\[ o^A f_{\alpha A}(l) = \frac{\partial}{\partial \alpha^A}\chi(l), \]
where \( o^A \) is the spinor of the null vector \( l^a \) and \( \chi(l) \) is a smooth complex function, homogeneous of degree 0, determined by this equation up to an additive constant. The equivalent form of this equation in the tensor language is
\[ -(l_a f_b - l_b f_a) = -L_{ab} \phi, \]
where \( -G_{ab} \) for an antisymmetric tensor \( G_{ab} \) denotes its antiselfdual part \( -G_{ab} = \frac{1}{2}(G_{ab} + iG_{ab}) \). Solving this equation for \( l_a f_b - l_b f_a \) one obtains (A.16), with \( \phi = \text{Re} \chi \) and \( \psi = \text{Im} \chi \). Now, let \( f_1 \) and \( f_2 \) be two such smooth functions represented as in (A.16). Then by (A.16) and the first equality in (A.3)
\[ f_1 \cdot f_2(l) = \left( t \cdot l \right)^{-2} t e^c (l_a f_{1b} - l_b f_{1a})(l_c f_{2b} - l_b f_{2c}) \]
\[ = \left( t \cdot l \right)^{-2} \left( *L_{0b} \phi_1 *L_0^b \phi_2 + *L_{0b} \psi_1 *L_0^b \psi_2 - L_{0b} \psi_1 L_0^b \phi_2 - L_{0b} \phi_1 *L_0^b \psi_2 \right). \]
Now integrate this identity with respect to \( d^2l \), integrate one \( *L \) in each term by parts and use (A.1) and (A.2) to obtain
\[ ([f_1], [f_2])_0 = ([\phi_1], [\phi_2])_{g^2} + ([\psi_1], [\psi_2])_{g^2}. \quad \text{(A.17)} \]
Therefore, the map \((\phi, \psi) \to f\) given by (A.16) for smooth elements extends to a unitary map \( \mathcal{H}_{g^2} \oplus \mathcal{H}_{g^2} \to \mathcal{H}_0 \). The space \( \mathcal{H}_{IR} \) is the image of the subspace \( \mathcal{H}_{g^2} \oplus 0 \) in this map.

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