GENUINE WHITTAKER FUNCTOR AND THE QUANTUM AFFINE SCHUR–WEYL DUALITY

FAN GAO, NADYA GUREVICH, AND EDMUND KARASIEWICZ

Abstract. We explicate relations among the Gelfand–Graev modules for central covers, the Euler–Poincaré polynomial of the Arnold–Brieskorn manifold, and the quantum affine Schur–Weyl duality. These three objects and their relations are dictated by a permutation representation of the Weyl group.

Contents

1. Introduction 1
2. The Whittaker functor $F_{Wh}$ 4
3. An Euler–Poincaré polynomial and stability 6
4. Quantum affine Schur–Weyl duality and $F_{Wh}$ 9
5. Some remarks 19
References 19

1. Introduction

It might be appropriate to retitle the paper as “A short tale of a permutation representation of the Weyl group”, since the main content concerns objects and their relations as depicted in the following diagram:

![Diagram](image)

where

(i) $W \curvearrowright Y/Y_{Q,n}$ denotes the permutation representation

\[ \sigma_{\mathcal{X}} : W \rightarrow \text{Perm}(\mathcal{X}_{Q,n}) \]

of the Weyl group $W$ of a root datum $\Phi$ acting on $\mathcal{X} := \mathcal{X}_{Q,n} := Y/Y_{Q,n}$ with $Y$ the cocharacter lattice and $Y_{Q,n} \subset Y$ a certain sublattice (see §2 for details);

(ii) $\text{EP}(\mathcal{M}_{AB}, X)$ denotes the Euler–Poincaré polynomial of the Arnold–Brieskorn manifold $\mathcal{M}_{AB}$ (see [Arn69, Bri73]), the complement of the full Coxeter hyperplane arrangement of the root datum $\Phi$;

2010 Mathematics Subject Classification. Primary 11F70; Secondary 22E50, 20G42.

Key words and phrases. covering groups, Iwahori–Hecke algebras, Gelfand–Graev representations, Coxeter hyperplane arrangement, quantum affine Schur–Weyl duality.
(iii) \((\text{Ind}_{\mu_n \times U}^G \iota \times \psi)^I =: \mathcal{V}\) is the Iwahori-component of the Gelfand–Graev representation of an \( n \)-fold central cover \( \overline{G} \) of a connected reductive group \( G \) with root datum \( \Phi \), see [GGK];

(iv) \( \mathcal{F}_{\text{SW}} : \mathcal{M}_L(\mathcal{H}(S^\text{aff})) \to \mathcal{M}_L(U_q(\mathfrak{sl}(n))) \) is the quantum affine Schur–Weyl duality functor of left modules, defined for the affine Hecke algebra \( \mathcal{H}(S^\text{aff}) \) of \( \text{GL}_r \) and the quantum affine group \( U_q(\mathfrak{sl}(n)) \), see [CP96, DDF12, Ant].

Now we elaborate on our main results relating the above objects. Note, some notation is taken from [GGK]. We provide precise references when needed.

Henceforth, we fix a \( p \)-adic field \( F \) with prime ideal \( \mathfrak{p}_F \) and ring of integers \( O_F \). We have \( G := G(F) \) for a smooth connected split reductive group scheme \( G \) over \( O_F \) with root datum \( \Phi \). Assume that \( F^\times \) contains the full group \( \mu_n \) of \( n \)-th roots of unity and \( \text{gcd}(p, n) = 1 \). For a maximal torus \( T \subset G \) let

\[
Y := \text{Hom}(G_n, T)
\]
denote the cocharacter lattice. Let \( W \) be the Weyl group of \( (G, T) \). Partially depending on a \( W \)-invariant quadratic form

\[
Q: Y \to \mathbb{Z},
\]
one has a central extension

\[
\mu_n \hookrightarrow \overline{G}(n) \xrightarrow{\text{pr}} G.
\]

For any \( H \subset G \), let \( \overline{H} = \text{pr}^{-1}(H) \). In particular, \( \overline{G} = \overline{G}(n) \). For every root \( \alpha \), we define

\[
n_\alpha := \frac{n}{\text{gcd}(n, Q(\alpha^\vee))} \in \mathbb{N},
\]

which appears in (iv) and only depends on the length of \( \alpha \).

Let \( U^- \subset B^- = TU^- \) be the opposite unipotent radical of a fixed Borel subgroup \( B = TU \). Let

\[
\psi: U^- \to \mathbb{C}^\times
\]
be a non-degenerate character of conductor \( \mathfrak{p}_F \), i.e., \( \psi \circ \alpha^\vee: F \to \mathbb{C} \) is of conductor \( \mathfrak{p}_F \) for every simple root \( \alpha \). The cover \( \overline{G} \) splits canonically over \( U \) and \( U^- \), which are thus viewed as subgroups of \( \overline{G} \).

We consider only \( \iota \)-genuine representations of \( \overline{G} \), where \( \mu_n \) acts via a faithful character \( \iota: \mu_n \to \mathbb{C}^\times \). Define \( \overline{K} := G(O_F) \), and let \( I := B \cap K \) be the Iwahori subgroup. We fix a splitting \( s_K: K \to \overline{G} \) and identify \( K \) and \( I \) with their image in \( \overline{G} \). This gives

\[
\mathcal{H}_I := C_{c, I}(\overline{G}/I),
\]
the \( \iota \)-genuine Iwahori–Hecke algebra. By Borel and Casselman, the map

\[
\pi \mapsto \pi^I
\]
gives a bijection between irreducible Iwahori-spherical representations of \( \overline{G} \) and simple modules over \( \mathcal{H}_I \).

In [GGK], we determined the structure of \( \mathcal{V} = (\text{Ind}_{\mu_n \times U}^G \iota \times \psi)^I \) as a right \( \mathcal{H}_I \)-module for \( \mathcal{D} \)-splitting covers (see Definition 2.1). In particular, the permutation representation \( \sigma_{\mathcal{D}} \) in (i) underlies this structure. We explain this in \( \S 2 \), which gives the relation \( f_{02} \) in (1.1). Note, in this paper we use \( \mathcal{V} \), while we used \( \mathcal{V}' \) in [GGK] for the same module.

For any algebra \( A \), we denote by \( \mathcal{M}_L(A) \), \( \mathcal{M}_R(A) \) the category of left and right modules over \( A \) respectively. Arising from the Gelfand–Graev module \( \mathcal{V} \) is the Whittaker functor

\[
\mathcal{F}_{\text{Wh}}: \mathcal{M}_R(\mathcal{H}_I) \to \mathcal{M}_R(\text{End}_{\mathcal{H}_I}(\mathcal{V})),
\]
given by $\pi \mapsto \text{Hom}_{\mathcal{H}_I}(\mathcal{V}, \pi^\vee)$, where $\pi^\vee$ denotes the contragredient of $\pi$. Naturally,

$$\text{Wh}_\psi(\pi) := \text{Hom}_{\mathcal{H}_I}(\mathcal{V}, \pi^\vee)$$

is called the Whittaker space of $\pi$. We show in Proposition 2.3 that if $n$ is stable (see Definition 2.2), then the functor $\mathcal{F}_{\text{Wh}}$ gives an equivalence of categories, and in particular $\mathcal{H}_I$ and $\text{End}_{\mathcal{H}_I}(\mathcal{V})$ are Morita equivalent. For Kazhdan–Patterson covers and Savin covers of $\text{GL}_r$, stability is equivalent to the inequality $n_\alpha \geq r$, see Example 2.4. Such results provide a partial understanding of the affine $q$-Schur algebra $\text{End}_{\mathcal{H}_I}(\mathcal{V})$. In the linear case, Bushnell–Henniart determine the structure of $\text{End}_{\mathcal{H}_I}(\mathcal{V})$ for general Bernstein classes [BH03, Theorem 4.3].

In §3, we determine the stability of $n$ for covers $\mathcal{G}^{(m)}_{(n)}$ of semisimple simply-connected $G$. To this end, we must compute the Whittaker dimension of the theta representation

$$\dim \text{Wh}_\psi(\Theta(\mathcal{G}^{(m)}_{(n)})).$$

Using Sommers’ computation of the character of the permutation representation $\sigma_x$ [Som97], we show that this dimension is equal to

$$|W|^{-1} \cdot n^r \cdot \text{EP}(\mathcal{M}_{AB}, -n^{-1}) = |W|^{-1} \cdot \prod_{j=1}^r (n - m_j),$$

where the $m_j$'s are the exponents of the Weyl group $W$. This dimension is also related to the characteristic polynomial of the Coxeter hyperplane arrangement by results of Orlik–Solomon [OS83]. We also give an analogous formula for the Steinberg representation $\text{St}(\mathcal{G}^{(m)}_{(n)})$ in terms of $\text{EP}(\mathcal{M}_{AB}, X)$. For details, see Theorem 3.1. This yields the link $f_{01}$.

Moreover, by an alternative characterization of stability, Theorem 3.1 enables us to determine when $\mathcal{H}_I \subseteq \mathcal{V}$ as $\mathcal{H}_I$-modules. This provides the connection $f_{12}$ in (1.1).

Recall that the space $\text{Wh}_\psi(\pi)$ is naturally a $Z(G)$-module, where $Z(G)$ denotes the center of $G$. Part of the goal of our paper is to show that for type A groups it is naturally a $U_q(\mathfrak{sl}(n))$-module via relating $\mathcal{F}_{\text{Wh}}$ to $\mathcal{F}_{\text{SW}}$. In §4, we recall the quantum affine Schur–Weyl functor $\mathcal{F}_{\text{SW}}$ for $\text{GL}_r$, which arises from the vector space $\mathcal{V}_{\text{SW}} \simeq (\mathbb{C}[Z])^{\otimes r}$ endowed with commuting actions

$$U_q(\hat{\mathfrak{sl}}(m)) \triangleleft \mathcal{V}_{\text{SW}} \triangleleft \mathcal{H}(S^\text{aff}_r)$$

for general $m, r \in \mathbb{N}$. We highlight that the above action of $\mathcal{H}(S^\text{aff}_r)$ on $\mathcal{V}_{\text{SW}}$ depends on $m$ sensitively, while the action of $U_q(\mathfrak{sl}(n))$ is a “standard” one (see §4.1). The presence of $\sigma_x$ in describing this action of $\mathcal{H}(S^\text{aff}_r)$ on $\mathcal{V}_{\text{SW}}$ is clear by a simple translation of terminology from that in [Gre99, DDF12, Ant] for example, see the proof of Theorem 4.2. This gives the link $f_{03}$.

For covers of $\mathcal{G}^{(m)}_{(n)}$ of type (C1) (see §2), one has

$$\mathcal{H}(S^\text{aff}_r) \simeq \mathcal{H}_I$$

and we make such an identification. Fix a certain character $\varepsilon_{\mathcal{H}_I} : \mathcal{H}_I \to \mathbb{C}$ such that $\varepsilon_{\mathcal{H}_I}|_{\mathcal{H}_W}$ equals the sign character $\varepsilon_{\mathcal{H}_W}$ of $\mathcal{H}_W$ (see §4.3). Our first result in §4 is Theorem 4.2, which contains two related statements. First,

$$\mathcal{V} \simeq \mathcal{V}_{\text{SW}} \otimes_{\mathbb{C}} \varepsilon_{\mathcal{H}_I}$$

as right $\mathcal{H}_I$-modules. Since $\mathcal{V}_{\text{SW}}$ is a $(U_q(\mathfrak{sl}(n)), \mathcal{H}(S^\text{aff}_r))$-bimodule, $\mathcal{V}$ inherits the left $U_q(\mathfrak{sl}(n))$-action from $\mathcal{V}_{\text{SW}}$, which might be referred to as a process of quantization. Second, this gives us a precise relation between $\mathcal{F}_{\text{SW}}$ and $\mathcal{F}_{\text{Wh}}.$
Following this, we use $\mathcal{F}_{SW}$ to identify the local scattering matrix associated with intertwining operators on genuine principal series $\mathcal{I}(\chi_s)$, $s = (s_1, ..., s_r) \in \mathbb{C}^r$ of $\overline{GL}_r^n$ with an $R$-matrix, up to a flipping map. See Theorem 4.6 and the subsequent discussion.

This result was already proved in [BBB19] by showing that the $R$-matrix (suitably twisted) gives a solution to the Yang–Baxter equation of metaplectic ice model, and it coincides with the local scattering matrix; thus, a natural map was defined from $\text{Wh}_\psi(\mathcal{I}(\chi_s)^I)$ to the quantum space

$$V(n_a s_1) \otimes ... \otimes V(n_a s_r),$$

where $V(u)$ is the shifted standard evaluation $U_q(\hat{\mathfrak{sl}}(n_a))$-module associated with any $u \in \mathbb{C}$, see Definition 4.1. The map is equivariant with respect to intertwiners and homomorphisms associated with $R$-matrices.

Here we give a reproof of [BBB19, Theorem 1]. Indeed, the functor $\mathcal{F}_{SW}$ provides a natural substitute for the above map. Also, the functorial and monoidal properties of $\mathcal{F}_{SW}$, coupled with its relation with $\mathcal{F}_{Wh}$ (Theorem 4.2), imply that the local scattering matrix represents a $U_q(\hat{\mathfrak{sl}}(n_a))$-homomorphism. This gives Theorem 4.6. The above results, especially the aforementioned “quantization”, give the connection $f_{23}$ in (1.1).

In view of the relation between $\mathcal{F}_{SW}$ and $\mathcal{F}_{Wh}$, the polynomial $\text{EP}(\mathcal{M}_{AB}, X)$ then determines the faithfulness of $\mathcal{F}_{SW}$, see Remark 4.3. This gives the connection $f_{13}$, which is in effect the composite of $f_{12}$ and $f_{23}$.

In the last section §5, we add several remarks regarding existing generalizations of $\mathcal{F}_{SW}$ to general Cartan types in the literature. Inevitably, we are not able to give exhaustive references on those deep topics, but can only content ourselves with a brief mention which pertains to our discussions above.

1.1. Acknowledgement. We would like to thank Valentin Buciumas and Dennis Gaitsgory for some communications on an earlier version of the paper. We are especially grateful to Buciumas for very helpful comments and clarifications on the earlier work relevant to the topics discussed here. The work of F. G. is partially supported by NSFC–12171422.

2. The Whittaker functor $\mathcal{F}_{Wh}$

In this section, let $\overline{G} = G^{(n)}$ be a Brylinski–Deligne cover of $G$. Recall the bilinear form $B_Q(y, z) := Q(y + z) - Q(y) - Q(z)$ associated with $Q$ and the $W$-stable sublattice

$$Y_{Q,n} = \{ y \in Y : B_Q(y, z) \in n\mathbb{Z} \text{ for all } z \in Y \}. $$

The Weyl group $W$ acts naturally on $X_{Q,n} = Y/Y_{Q,n}$.

**Definition 2.1.** A covering group $\overline{G}^{(n)}$ is $\mathcal{X}$-splitting if the $W$-equivariant quotient map

$$f : Y \twoheadrightarrow X_{Q,n}$$

has a $W$-equivariant section.

Consider $\text{GL}_r$ with lattice $Y$ given with the standard $\mathbb{Z}$-basis $\{e_i : 1 \leq i \leq r\}$. A Brylinski–Deligne cover $\overline{GL}_r^{(n)}$ of $\text{GL}_r$ is associated with $p, q \in \mathbb{Z}$ such that

$$B(e_i, e_j) = \begin{cases} 2p & \text{if } i = j, \\ q & \text{if } i \neq j. \end{cases}$$

We have $Q(\alpha^\vee) = 2p - q$. The Kazhdan–Patterson covers are those with $Q(\alpha^\vee) = -1$, where $p$ corresponds to the parameter $c$ in [KP84]. Meanwhile, Savin’s “nice” cover is the one with $p = -1, q = 0$. These covers are $\mathcal{X}$-splitting by [GGK, Example 6.5].
The following two classes of Brylinski–Deligne covers, which are also \( \mathcal{X} \)-splitting covers, are our main interest in this paper.

(C1) A Brylinski–Deligne cover \( GL_r^{(n)} \) satisfying

\[ n|q \]

is called of type (C1). Such covers satisfy the block commutativity, i.e., for partition \( r = (r_1, \ldots, r_k) \) of \( r \) with associated standard Levi subgroup \( M_r = GL_{r_1} \times \cdots \times GL_{r_k} \), one has an isomorphism of groups

\[ \left( \prod_i GL_r(n_i) \right)/H \to M_r, \]

where \( H = \{ (\zeta_i)_i : \prod_i \zeta_i = 1 \} \). Moreover, for such covers

\[ Y_{Q,n} = n_\alpha Y \text{ with } n_\alpha = n/\gcd(2p, n). \]

These covers are \( \mathcal{X} \)-splitting by [GGK, Example 6.5].

As concrete examples, recall that the Savin covers are Brylinski–Deligne covers with \( (p, q) = (-1, 0) \), and thus are of type (C1). On the other hand, for Kazhdan–Patterson covers of type (C1) \( n \) must be odd.

(C2) Let \( G \) be semisimple and simply-connected. In [GGK, Corollary 6.4] we show that all oasitic covers (see Definition 6.1 of loc. cit.) of \( G \) are \( \mathcal{X} \)-splitting. And in the tables of [GGK, §6.1], we explicitly list the \( n \)'s such that an \( n \)-fold cover of such \( G \) is oasitic.

Now assume \( \overline{G} \) is an \( \mathcal{X} \)-splitting cover. For every \( W \)-orbit \( O_y \subset X_{Q,n} \), one may view \( O_y \subset Y \) such that

\[ W_y := \text{Stab}_W(y, Y) = \text{Stab}_W(y, \mathcal{X}) \]

is a parabolic Weyl subgroup. Let

\[ \mathcal{H}_{W_y} \subset H \subset H_I \]

be the subalgebra associated with \( W_y \) and \( \varepsilon_{W_y} \) be the sign character of \( \mathcal{H}_{W_y} \). In [GGK, Theorem 5.18] we show that

\[ V = \bigoplus_{O_y \subset X_{Q,n}} \varepsilon_{W_y} \otimes \mathcal{H}_{W_y} H_I \]

as a right \( H_I \)-module. We write

\[ V_{O_y} := \varepsilon_{W_y} \otimes \mathcal{H}_{W_y} H_I. \]

Consider the Whittaker functor

\[ F_{Wh} : M_R(\mathcal{H}_I) \to M_R(\text{End}_{H_I}(V)), \]

of right modules defined by

\[ F_{Wh}(\pi) := Wh_\psi(\pi) := \text{Hom}_{H_I}(V, \pi^\vee). \]

The action of \( a \in \text{End}_{H_I}(V) \) on \( f \in \text{Hom}_{H_I}(V, \pi^\vee) \) is given by

\[ (f \cdot a) = f \circ a. \]

**Definition 2.2.** Let \( \overline{G} \) be an \( \mathcal{X} \)-splitting \( n \)-fold cover. The degree \( n \) and also \( \overline{G} \) are called stable if the following equivalent conditions are satisfied:

(i) there is a free \( W \)-orbit in \( \mathcal{X}_{Q,n} \);

(ii) \( H_I \) is a direct summand of \( V \) as \( H_I \)-modules;

(iii) every theta representation \( \Theta(\overline{G}, \chi) \) is generic, i.e., \( F_{Wh}(\Theta(\overline{G}, \chi)) \neq 0. \)
Here a theta representation $\Theta(G, \chi)$ is the Langlands quotient of the genuine principal series $I(\chi)$ associated with an “exceptional” genuine central character $\chi$ of $Z(T)$, see [KP84,Gao17]. By (2.1), we have $O_y \subset \mathcal{Q}_{n,n}$ is a free $W$-orbit if and only if $\mathcal{V}_{O_y} \simeq \mathcal{H}_I$, and this immediately gives the equivalence between (i) and (ii). The equivalence of (i) and (iii) is discussed in the proof of [GGK, Theorem 8.8]. It is easy to see that if $n$ is big enough, then it is stable since Definition 2.2 (i) is satisfied then.

**Proposition 2.3.** Let $\mathcal{G}^{(n)}$ be an $\mathcal{X}$-splitting cover. If $n$ is stable, then $\mathcal{V}$ is a progenerator of $\mathcal{H}_I$-mod. In this case, the functor $\mathcal{F}_{\text{Wh}}$ gives an equivalence of categories, and thus $\mathcal{H}_I$ and $\text{End}_{\mathcal{H}_I}(\mathcal{V})$ are Morita-equivalent.

**Proof.** The $\mathcal{H}_I$-module $\mathcal{V}$ is a progenerator if $\mathcal{V}$ is projective, finitely generated, and is a generator (see [Roc09, §1.5.1]). The module $\mathcal{V}$ is projective by [GGK, Lemma 8.6]. Since $\mathcal{V}$ is projective, $\mathcal{V}$ is a generator if and only if for any nonzero $\sigma \in \mathcal{M}_R(\mathcal{H}_I)$, the space
\[
\text{Hom}_{\mathcal{H}_I}(\mathcal{V}, \sigma) \neq 0.
\]
Since $n$ is stable, this follows from Definition 2.2 (ii). Clearly, $\mathcal{V}$ is finitely-generated in view of its structure given in (2.1).

Since $\mathcal{V}$ is a progenerator, it follows from general category theory (see [Roc09, §1.5.1]) that $\mathcal{F}_{\text{Wh}}$ gives an equivalence of categories. This completes the proof. $\square$

**Example 2.4.** Let $GL_r$ be any $\mathcal{X}$-splitting cover, for example of Kazhdan–Patterson or Savin type, or of type (C1). It then follows from Lemma 3.1 and Theorem 3.7 of [GT22] that $n$ is stable if and only if $n_\alpha \geq r$. In fact, for general $\mathcal{G}$, the results in [GT22, §3.1–3.5, Theorem 3.7] already give a lower bound for stable $n$.

**Remark 2.5.** If $\mathcal{G}^{(n)}$ is a stable $\mathcal{X}$-splitting cover such that $\mathcal{H}_I \simeq \mathcal{H}(G, I)$, then the Morita equivalence between $\mathcal{H}_I$ and $\text{End}_{\mathcal{H}_I}(\mathcal{V})$ also follows from the work of Vigneras [Vig03]. For every subset $J \subset \Delta$ of the simple roots $\Delta$, let $\mathbf{v}_J \in \mathcal{H}(G, I)$ (resp. $\mathbf{v}_J^+$) denote the idempotent corresponding to $\mathbb{E}_{\mathcal{H}_I(J)}$ (resp. 1). This gives
\[
\mathcal{S}_e := \text{End}_{\mathcal{H}(G, I)} \left( \bigoplus_{J \subset \Delta} \mathbf{v}_J \mathcal{H}(G, I) \right).
\]
Its analogue $\mathcal{S}_1$ was studied in [Vig03, MS19]. For stable $n$, the algebra $\text{End}_{\mathcal{H}_I}(\mathcal{V})$ and $\mathcal{S}_e$ are Morita equivalent and thus might be both called affine $q$-Schur algebras, as is the case for $\mathcal{S}_1$ in [Vig03, MS19]. However, we note that they are not identical since $\mathcal{V}_{O_y}$, which corresponds to certain $\mathbf{v}_J \mathcal{H}(G, I)$, has multiplicities in $\mathcal{V}$ in general; that is, it is possible that $J_y = J_{y'}$ for $O_y \neq O_{y'}$. Such multiplicities are given in Sommers’ work [Som97] for oasitic covers of semisimple simply-connected $G$, which we will use in §3.

In type A, such multiplicities are also important in relating $\mathcal{F}_{\text{Wh}}$ to the quantum affine Schur–Weyl functor, see §4 below.

### 3. An Euler–Poincaré polynomial and stability

In this section, we consider exclusively a semisimple simply-connected $G$ and an oasitic cover $\mathcal{G}^{(n)}$ of $G$, such that for any short coroot $\alpha^\vee$,
\[
Q(\alpha^\vee) = -1.
\]

The unramified genuine exceptional characters $\chi$ form a torsor over $Z(\mathcal{G}^{(n)})$, which is trivial in this case; thus, there is a unique theta representation $\Theta(\mathcal{G}^{(n)})$, the unique irreducible quotient of $I(\chi)$. The unique irreducible subrepresentation $\text{St}(\mathcal{G}^{(n)})$ of $I(\chi)$ is the analogue of the Steinberg representation.
Using work of Sommers [Som97], we determine \( \dim \text{Wh}_\psi(\Theta(G^{(n)})) \) explicitly in terms of the Euler–Poincaré polynomial of the Coxeter hyperplane arrangement of \( G \). This gives the stability condition for \( n \) as well.

For semisimple simply-connected \( G \), one has \( Y = Y^{sc} \), which is \( \mathbb{Z} \)-spanned by the simple coroots. For each root \( \alpha \in \Phi \), consider the hyperplane
\[
H_\alpha := \{ v \in Y \otimes \mathbb{R} : \langle v, \alpha \rangle = 0 \}.
\]
We have the set
\[
\mathcal{A} := \{ H_\alpha : \alpha \in \Phi \}
\]
of the full Coxeter hyperplane arrangements in \( Y \otimes \mathbb{R} \). Let \( \mathcal{L} = \mathcal{L}(\mathcal{A}) \) be the set of intersections of hyperplanes in \( \mathcal{A} \). We consider
\[
Y \otimes \mathbb{R} \in \mathcal{L}
\]
by taking the empty intersection of elements in \( \mathcal{A} \). Elements in \( \mathcal{L} \) are ordered, and we write \( x < y \) if \( y \subset x \). We also use \( x \leq y \) to represent \( x < y \) or \( x = y \). The Mobius function
\[
\mu : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{Z}
\]
is uniquely defined by requiring \( \mu(x, x) = 1 \),
\[
\sum_{z : x \leq z \leq y} \mu(z, y) = 0 \quad \text{if} \quad x < y,
\]
and \( \mu(x, y) = 0 \) otherwise. The characteristic polynomial of \( \mathcal{L} \) is thus given by
\[
\omega(\mathcal{L}, X) := \sum_{x \in \mathcal{L}} \mu(Y \otimes \mathbb{R}, x) \cdot X^{\dim x}.
\]

On the other hand, consider the Arnold–Brieskorn complex manifold
\[
\mathcal{M}_{AB} := Y \otimes \mathbb{C} - \bigcup_{\alpha \in \Phi} H_\alpha \otimes \mathbb{C}
\]
and its Euler–Poincaré polynomial
\[
\text{EP}(\mathcal{M}_{AB}, X) := \sum_{i \geq 0} \dim H^i(\mathcal{M}_{AB}, \mathbb{C}) \cdot X^i.
\]

**Theorem 3.1.** Let \( G^{(n)} \) be an oasitic cover of a semisimple simply-connected \( G \) such that \( Q(\alpha^\vee) = -1 \) for any long root \( \alpha \). Then one has
\[
\dim \text{Wh}_\psi(\Theta(G^{(n)})) = \frac{\omega(\mathcal{L}, n)}{|W|} = \frac{n^r \cdot \text{EP}(\mathcal{M}_{AB}, -n^{-1})}{|W|} = |W|^{-1} \cdot \prod_{j=1}^{r} (n - m_j),
\]
where \( m_j, 1 \leq j \leq r \) are the exponents of the Weyl group \( W \). Similarly,
\[
\dim \text{Wh}_\psi(\text{St}(G^{(n)})) = \frac{(-1)^r \cdot \omega(\mathcal{L}, -n)}{|W|} = \frac{n^r \cdot \text{EP}(\mathcal{M}_{AB}, n^{-1})}{|W|} = |W|^{-1} \cdot \prod_{j=1}^{r} (n + m_j).
\]

**Proof.** Before we begin, we note that we cite results from [GGK] that include the assumption \( Q(\alpha^\vee) = 1 \), for short coroots. When \( G \) is semisimple, these results also hold when \( Q(\alpha^\vee) = -1 \), for short coroots.

First, note that we have
\[
\omega(\mathcal{L}, X) = X^r \cdot \text{EP}(\mathcal{M}_{AB}, -X^{-1}) = \prod_{j=1}^{r} (X - m_j) \in \mathbb{C}[X],
\]
where the first equality was a classical result of Orlik–Solomon [OS83] and the second equality already follows from work of Brieskorn [Bri73, Théorème 6].

It follows from [GGK, Theorem 8.8] that
\[ \dim \text{Wh}_\psi(\Theta(G^{(n)})) = \langle \varepsilon_W, \sigma_X \rangle_W = \# \{ \text{free } W\text{-orbits in } \mathcal{X}_{Q,n} \}. \]

Let \( P_1, P_2, \ldots, P_k \subset W \) be the conjugacy classes of parabolic subgroups of \( W \) such that \( P_1 = W, P_k = \{1\} \).

Note that \( X_{Q,n} = Y_{sc}/nY_{sc} \). Since \( G^{(n)} \) is an oasitic cover (see [GGK, §6.1]), we see that \( n \) is “very good” in the terminology of [Som97, Definition 3.5]. Following this, we have from [Som97, Lemma 4.2] that
\[ \sigma_X = \bigoplus_{i=1}^k m_i(n) \cdot \text{Ind}^W_{P_i}(\mathds{1}_{P_i}) \]
for well-defined \( m_i(n) \in \mathbb{Z}_{\geq 0} \). Furthermore, it is shown in [Som97, Proposition 5.1] that
\[ m_k(n) = \omega(L, n)/|W|. \]

This gives
\[ \langle \varepsilon_W, \sigma_X \rangle_W = \sum_{i=1}^k m_i(n) \cdot \langle \varepsilon_{P_i}, \mathds{1}_{P_i} \rangle = m_k(n) = \frac{\omega(L, n)}{|W|}. \]

Now we consider the Steinberg representation \( \text{St}(G^{(n)}) \). Again by [GGK, Theorem 8.8],
\[ \dim \text{Wh}_\psi(\text{St}(G^{(n)})) = \langle \mathds{1}_W, \sigma_X \rangle_W = \# \{ \text{all } W\text{-orbits in } \mathcal{X}_{Q,n} \}. \]

For every \( w \in W \), let \( s(w) \) denote the least number of reflections whose product is \( w \) and consider
\[ d(w) := \dim(Y \otimes R)^w, \]
the dimension of the set of fixed points of \( w \) in \( Y \otimes R \). One has \( d(w) = r - s(w) \). Let \( \chi_{\sigma_X} \) denote the character of \( \sigma_X \). It was shown in [Som97, Proposition 3.9] that
\[ \chi_{\sigma_X}(w) = n^{d(w)} \]
for every \( w \in W \). We also note that
\[ \varepsilon_W(w) = (-1)^{s(w)}. \]

Thus,
\[ \dim \text{Wh}_\psi(\text{St}(G^{(n)})) = \frac{1}{|W|} \sum_{w \in W} n^{d(w)} = \frac{(-1)^r}{|W|} \sum_{w \in W} \varepsilon_W(w) \cdot (-n)^{d(w)} = \frac{(-1)^r}{|W|} \cdot \omega(L, -n) = |W|^{-1} \cdot \prod_{j=1}^r (n + m_j). \]

This completes the proof. \( \square \)
The above recovers those explicit formulas for $\overline{\text{Sp}}_2$ and $\overline{G}_2$ computed in [Gao17, §8.2–8.3]. Applying $n = 1$ in (3.2), since $\text{St}(G)$ is generic with unique Whittaker model, we get the classical Weyl order formula $|W| = \prod_{j=1}^r (1 + m_j)$, see [Sol66, Corollary 2.3].

Now assume that $m_1 \leq m_2 \leq \ldots \leq m_r$.

**Corollary 3.2.** Let $\overline{G}^{(n)}$ be as in Theorem 3.1. Then $n$ is stable if and only if $n > m_r$, in which case

$$\mathcal{F}_{\text{Wh}} : \mathcal{M}_R(\mathcal{H}_I) \longrightarrow \mathcal{M}_R(\text{End}_{\mathcal{H}_I}(V))$$

gives an equivalence of categories.

**Proof.** If $n > m_r$, then $n$ is stable by Theorem 3.1. Conversely, the result follows by comparing [GGK, Tables 1, 2], with the exponents of the associated Weyl group. We include the details for a group of type $E_8$. The other cases are similar.

By [GGK, Table 2], if $G$ is an oasitic cover of type $E_8$, then $n$ is not divisible by 2, 3, or 5. The exponents of the $E_8$ Weyl group are 1, 7, 11, 13, 17, 19, 23, 29 (e.g. see [Bou02]). These are exactly the positive integers less than 30 that are not stable by 2, 3, or 5. Since $n$ is stable it follows that $n > 29$, as desired.

**Remark 3.3.** We note that Theorem 3.1 is compatible with Example 2.4, since $m_r = r$ for type $A_r$ groups. For $\overline{\text{GL}}_r^{(n)}$, it is possible to compute $\chi_{\sigma_x}$ similarly as in [Som97] and thus obtain a formula for $\dim \text{Wh}_\psi(\Theta(\overline{\text{GL}}_r, \chi))$ and $\dim \text{Wh}_\psi(\text{St}(\overline{\text{GL}}_r, \chi))$, where $\chi$ is an exceptional character. Alternatively, one can argue as follows in a special case. For this purpose, we consider a Kazhdan–Patterson cover $\overline{\text{GL}}_r^{(n)}$ satisfying $\gcd(n, r) = 1$. The covering subgroup $\overline{\text{SL}}_r^{(n)}$ is thus an oasitic cover. In this case, the pair $(\overline{\text{GL}}_r^{(n)}, \overline{\text{SL}}_r^{(n)})$ is an isotypic pair in the sense of [GSSb, Definition 2.23]. In particular, one has

$$\dim \text{Wh}_\psi(\text{St}(\overline{\text{GL}}_r^{(n)}, \chi)) = |Y/(Y^{sc} + Y_{Q, n})| \cdot \dim \text{Wh}_\psi(\text{St}(\overline{\text{SL}}_r^{(n)}, \chi_0))$$

for the unique exceptional character $\chi_0$ for $\overline{\text{SL}}_r^{(n)}$ obtained from the restriction of $\chi$. In fact, (3.3) holds for any irreducible constituent (in particular, $\Theta(\overline{\text{GL}}_r, \chi)$ here as well) of a regular unramified principal series of $\overline{\text{GL}}_r$, see [GSSb, §4.3.1]. Combining (3.3) and (3.2), we get

$$\dim \text{Wh}_\psi(\text{St}(\overline{\text{GL}}_r, \chi)) = \frac{n}{|Y_{Q, n}/nY|} \cdot \frac{1}{r!} \prod_{j=1}^{r-1} (n + j) = \frac{1}{|Y_{Q, n}/nY|} \cdot \left( r + n - 1 \right),$$

where $|Y_{Q, n}/nY| = \gcd(n, 2rp + r - 1)$. We note that this formula can also be obtained from [Zou, Theorem 4.7], which gives the Whittaker dimensions of general square integrable representations of the Kazhdan–Patterson covers.

4. Quantum Affine Schur–Weyl Duality and $\mathcal{F}_{\text{Wh}}$

In this section, we consider exclusively a cover $\overline{G} = \overline{\text{GL}}_r$ of type (C1). The goal is to show that

$$\mathcal{F}_{\text{Wh}} : \mathcal{M}_R(\mathcal{H}_I) \longrightarrow \mathcal{M}_R(\text{End}_{\mathcal{H}_I}(V))$$

gives rise to the quantum affine Schur–Weyl functor

$$\mathcal{F}_{\text{SW}} : \mathcal{M}_L(\mathcal{H}(S_r^{\text{aff}})) \rightarrow \mathcal{M}_L(U_q(\mathfrak{sl}(n_a)))$$

valued in the category of $U_q(\mathfrak{sl}(n_a))$-modules. Here

$$S_r^{\text{aff}} = Y \rtimes S_r = \mathbb{Z}^r \rtimes S_r$$
is the extended affine Weyl group of $GL_r$. A first link between $\mathcal{F}_{Wh}$ and $\mathcal{F}_{SW}$ is that one has naturally

\[(4.1) \quad \mathcal{H}_I \simeq \mathcal{H}(S_r^{\text{aff}})\]

for a type (C1) cover of $GL_r$, see [Sav88, McN12, GG18].

Let $\mathbb{C}[\mathbb{Z}]$ be the $\mathbb{C}$-vector space with a basis given by $\{v_i : i \in \mathbb{Z}\}$. Let

\[\mathcal{V}_{SW} := (\mathbb{C}[\mathbb{Z}])^{\otimes r}.\]

The functor $\mathcal{F}_{SW}$ arises from commuting $(U_q(\hat{\mathfrak{sl}}(n_m)), \mathcal{H}(S_r^{\text{aff}}))$-actions on the left and right hand side of $\mathcal{V}_{SW}$. We briefly describe the actions below, where some details are referred to [CP96, Ant].

4.1. **Left $U_q(\hat{\mathfrak{sl}}(m))$-action on $\mathcal{V}_{SW}$**. We first recall the quantum affine group $U_q(\hat{\mathfrak{sl}}(m))$ for any $m \in \mathbb{N}$ given as follows. First, let $\alpha^\dagger$ be the highest root of the root system of type $A$. Write

\[\alpha_0 = -\alpha^\dagger.\]

Let

\[\hat{\mathcal{C}}_A := \{[\langle \alpha_i, \alpha_j^\vee \rangle]_0 \leq i, j \leq m-1\}\]

be the generalized Cartan matrix of type $A_r$. For convenience, we write

\[a_{i,j} = \langle \alpha_i, \alpha_j^\vee \rangle.\]

Now, $U_q(\hat{\mathfrak{sl}}(m))$ is the quantized affine Lie algebra associated with $\hat{\mathcal{C}}_A$. More precisely, it is the $\mathbb{C}$-algebra with generators

\[\{E_i, F_i, K_i^{\pm 1} : 0 \leq i \leq m-1\}\]

and relations (see [CP96, §2.4] and [Ant, §2.5])

(R1) $K_i^{-1} K_i^{\pm 1} = 1 = K_i^{-1} K_i$ and $K_i K_j = K_j K_i$,

(R2) $K_i E_j = q^{a_{i,j}} E_j K_i$ and $K_i F_j = q^{-a_{i,j}} F_j K_i$,

(R3) $[E_i, F_j] = \delta_{i,j} \cdot (K_i - K_i^{-1})/(q - q^{-1})$,

(R4)

\[
\begin{align*}
E_i^2 E_{i+1} E_i + E_{i+1} E_i^2 &= (q + q^{-1}) E_i E_{i+1} E_i \\
F_i^2 F_{i+1} F_i + F_{i+1} F_i^2 &= (q + q^{-1}) F_i F_{i+1} E_i \\
E_i E_j &= E_j E_i \quad \text{if } i - j \neq 0, 1, m-1 \\
F_i F_j &= F_j F_i \quad \text{if } i - j \neq 0, 1, m-1.
\end{align*}
\]

The computation of $i \pm 1$ and $i - j$ above are all valued in $\{0, 1, ..., m-1\}$, the fixed set of representatives of $\mathbb{Z}/m\mathbb{Z}$.

In terms of the basis $\{v_i : i \in \mathbb{Z}\}$ of $\mathbb{C}[\mathbb{Z}]$, one has an action of $U_q(\hat{\mathfrak{sl}}(m))$ on $\mathbb{C}[\mathbb{Z}]$ given by the following:

\[
\begin{align*}
E_i \cdot v_j &= \delta_{i+1,j} \cdot v_{j-1} \\
F_i \cdot v_j &= \delta_{i,j} \cdot v_{j+1} \\
K_i \cdot v_j &= q^{\delta_{i,j} - \delta_{i+1,j}} \cdot v_j.
\end{align*}
\]

Again, the computation of $i + 1$ and $j \pm 1$ in the subscripts of the $\delta$-functions above are valued in the set $\{0, 1, ..., m-1\}$ of representatives for $\mathbb{Z}/m\mathbb{Z}$.

To have an action of $U_q(\hat{\mathfrak{sl}}(m))$ on $\mathcal{V}_{SW} = (\mathbb{C}[\mathbb{Z}])^{\otimes r}$, we note that the quantum group $U_q(\hat{\mathfrak{sl}}(m))$ has a comultiplication map

\[
\Delta : U_q(\hat{\mathfrak{sl}}(m)) \longrightarrow U_q(\hat{\mathfrak{sl}}(m)) \otimes_\mathbb{C} U_q(\hat{\mathfrak{sl}}(m))
\]
Recall the standard evaluation representation given by
\[ V/mY \]
Thus, a basis of Bernstein presentation, is in the notation of Example 12.3.17. Then it follows from loc. cit. that
\[ V(s) = \mathbb{C}[\mathbb{Z}]/S_{m,s} \] is a set of representatives of \( \mathbb{C}[\mathbb{Z}] \) associated with \( s \).

Here \( V(s) \) is exactly the \( U_q(\mathfrak{sl}(m)) \)-module \( V(q^s) \) in [CP96, §2.4], for \( a = q^s \) there.

Recall the standard evaluation representation \( V_{e_v}(b) := (V_q^{(s)})_{q^v}, b \in \mathbb{C} \), where the latter is in the notation of [CP95, Example 12.3.17]. Then it follows from loc. cit. that
\[ V(s) = V_{e_v}(2/(m + 1) + s) \], whence called the shifted evaluation representation.

**Definition 4.1.** For every \( s \in \mathbb{C} \), the subspace
\[ S_{m,s} := \text{Span}_\mathbb{C} \{ v_i - q^sv_{i+m} : i \in \mathbb{Z} \} \subset \mathbb{C}[\mathbb{Z}] \]
is \( U_q(\mathfrak{sl}(m)) \)-stable, and we call \( V(s) := \mathbb{C}[\mathbb{Z}]/S_{m,s} \) the shifted evaluation representation of \( U_q(\mathfrak{sl}(m)) \) associated with \( s \).

Clearly,
\[ q \mathcal{B}_m := \left\{ \sum_{j=1}^r i_j \cdot e_j : 0 \leq i_j \leq m - 1 \right\} \]
is a set of representatives of \( Y/mY \). For every integer \( i_j \in [0, m - 1] \) we write
\[ \tilde{v}_{i_j} := \tilde{v}(i_j \cdot e_j). \]
Thus, a basis of \( \mathbb{C}[Y/mY] = (\mathbb{C}^m)^{\otimes r} \) is given by
\[ \mathcal{B}_m = \{ \tilde{v}_{i_1} \otimes \tilde{v}_{i_2} \otimes \ldots \otimes \tilde{v}_{i_r} : 0 \leq i_j \leq m - 1 \text{ for every } j \}. \]

One has the \( \mathbb{C} \)-vector space isomorphism
\[ \tau_m : (\mathbb{C}^m)^{\otimes r} \otimes \mathbb{C}[Y] \rightarrow \mathcal{V}_{SW} \]
given by
\[ (\tilde{v}_{i_1} \otimes \ldots \otimes \tilde{v}_{i_r}) \otimes (e_1^{k_1} \cdot \ldots \cdot e_r^{k_r}) \mapsto v_{i_1-k_{1m}} \otimes \ldots \otimes v_{i_r-k_{rm}}. \]

Let \( (\gamma, \mathbb{C}[Y/mY]) \) be any \( \mathcal{H}_W \)-module afforded on the space \( \mathbb{C}[Y/mY] \). From the Bernstein presentation,
\[ \mathcal{H}(S_{r}^{\text{aff}}) = \mathcal{H}_W \otimes \mathbb{C} \mathbb{C}[Y]. \]
Thus as \( \mathbb{C} \)-vector spaces
\[ \mathcal{V}_{SW} \simeq (\mathbb{C}^m)^{\otimes r} \otimes \mathbb{C}[Y] \simeq (\mathbb{C}^m)^{\otimes r} \otimes \gamma, \mathcal{H}_W \mathcal{H}(S_{r}^{\text{aff}}). \]

This gives a right action of \( \mathcal{H}(S_{r}^{\text{aff}}) \) on \( \mathcal{V}_{SW} \) depending on the choice of \( \gamma \).

If \( \gamma \) is chosen arbitrarily, then the left \( U_q(\mathfrak{sl}(m)) \)-action and right \( \mathcal{H}(S_{r}^{\text{aff}}) \)-action on \( \mathcal{V}_{SW} \) may not commute. To obtain a commuting action, \( \gamma \) should be “quantized” from
the permutation action of \( S_r \) on \((\mathbb{C}^m)^{\otimes r}\). We describe this special action denoted by \( \gamma_m^\natural \) as follows.

Following notations of [Gre99, DDF12, Ant], let
\[
\Lambda(m, r)
\]
denote the set of all \( \lambda := (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{Z}_{\geq 0}^m \) satisfying
\[
\sum_{i=1}^{m} \lambda_i = r.
\]
For each \( \lambda \in \Lambda(m, r) \), consider
\[
y_{\lambda} = (1, 1, ..., 1, 2, 2, ..., m, m, ..., m) \in (\mathbb{Z}/m\mathbb{Z})^r.
\]
It is clear that the stabilizer subgroup of \( y_{\lambda} \) in \( W = S_r \) is
\[
W_{\lambda} := W_{y_{\lambda}} = S_{\lambda_1} \times \cdots \times S_{\lambda_m}.
\]
Then one has a \( \mathbb{C}\)-vector isomorphism
\[
(4.3) \bigoplus_{\lambda \in \Lambda(m, r)} 1 \otimes \mathcal{H}_{W_{\lambda}} \mathcal{H} \longrightarrow (\mathbb{C}^m)^{\otimes r}.
\]
This gives the sought after \( \mathcal{H}_W \)-module
\[
(\gamma_m^\natural, (\mathbb{C}^m)^{\otimes r})
\]
with \( \mathcal{H}_W \)-action transported via (4.3). We also obtain the right action
\[
\mathcal{Y}_{SW} \curvearrowright \mathcal{H}(S_{aff})
\]
arising from \( \gamma_m^\natural \).

It is known (see [Ant, Corollary 2.55]) that the above left and right actions on \( \mathcal{Y}_{SW} \) commute, which we write as
\[
(4.4) \quad U_q(\hat{\mathfrak{sl}}(m)) \curvearrowright \mathcal{Y}_{SW} \curvearrowright \mathcal{H}(S_{aff}).
\]
This commuting action gives rise to the so-called quantum affine Schur–Weyl functor
\[
\mathcal{F}_{SW} : \mathcal{M}_L(\mathcal{H}(S_{aff})) \longrightarrow \mathcal{M}_L(U_q(\hat{\mathfrak{sl}}(m)))
\]
given by
\[
\pi \mapsto \mathcal{Y}_{SW} \otimes_{\mathcal{H}(S_{aff})} \pi =: \mathcal{F}_{SW}(\pi).
\]
The functor \( \mathcal{F}_{SW} \) can be viewed as the composite \( \mathcal{F}_\varphi \circ \mathcal{F}_1 \), where
\[
\mathcal{F}_1 : \mathcal{M}_L(\mathcal{H}(S_{aff})) \longrightarrow \mathcal{M}_L(\text{End}_{\mathcal{H}(S_{aff})}(\mathcal{Y}_{SW})), \quad \pi \mapsto \mathcal{Y}_{SW} \otimes_{\mathcal{H}(S_{aff})} \pi.
\]
On the other hand, (4.4) also gives an algebra homomorphism
\[
\varphi : U_q(\hat{\mathfrak{sl}}(m)) \longrightarrow \text{End}_{\mathcal{H}(S_{aff})}(\mathcal{Y}_{SW}),
\]
which is surjective if \( m > r \), see [DG07, Theorem 3.2.1] or [DDF12, Theorem 3.8.3]. Here \( \varphi \) induces
\[
\mathcal{F}_{\varphi} : \mathcal{M}_L(\text{End}_{\mathcal{H}(S_{aff})}(\mathcal{Y}_{SW})) \longrightarrow \mathcal{M}_L(U_q(\hat{\mathfrak{sl}}(m)));
\]
It is clear that \( \mathcal{F}_{SW} = \mathcal{F}_{\varphi} \circ \mathcal{F}_1 \).
4.3. Comparison of $\mathcal{F}_{\text{Wh}}$ and $\mathcal{F}_{\text{SW}}$. We first revert to the discussion of the Gelfand–Graev module $\mathcal{V}$ for a cover $\text{GL}_r^{(n)}$ of type (C1). We write
\[ V_{\mathcal{X}} := \bigoplus_{\mathcal{O}_y \subset \mathcal{X}_{Q,n}} \varepsilon_{W_y} \otimes_{\mathcal{H}_{W_y}} \mathcal{H}_W, \]
which is a right $\mathcal{H}_W$-module, then it is clear from (2.1) that
\[ \mathcal{V} \simeq V_{\mathcal{X}} \otimes_{\mathcal{H}_W} \mathcal{H}_I. \]
For these $\text{GL}_r^{(n)}$, one has $Y_{Q,n} = n_{\alpha}Y$. We set $\mathcal{H}_I \simeq \mathcal{H}_W \otimes_{\mathbb{C}[Y_{Q,n}]} \mathbb{C}$-vector spaces and define the modified extended affine Weyl group to be
\[ \tilde{W}_{\text{ex}} := Y_{Q,n} \rtimes W. \]
Recall that there is the modified coroot lattice
\[ Y_{Q,n}^{sc} = n_{\alpha} \cdot Y_{Q,n} \quad \subset \quad Y_{Q,n} \]
which gives the modified affine Weyl group
\[ \tilde{W}_{\text{aff}} := Y_{Q,n}^{sc} \rtimes W \subset \tilde{W}_{\text{ex}}. \]
Associated with $\tilde{W}_{\text{aff}}$ is a subalgebra $\mathcal{H}_{\tilde{W}_{\text{aff}}} \subset \mathcal{H}_I$. In fact, there is a subgroup $\Omega \subset \tilde{W}_{\text{ex}}$ such that
\[ \mathcal{H}_I \simeq \mathcal{H}_{\tilde{W}_{\text{aff}}} \otimes_{\mathbb{C}} \mathbb{C}[\Omega], \]
where the twisted algebra multiplication of the right hand side is given in [IM65, Proposition 3.8] or [Sol21, §1.4]. Let
\[ \varepsilon_{\mathcal{H}_I} : \mathcal{H}_I \longrightarrow \mathbb{C} \]
be the unique character which extends the sign character $\varepsilon_{\tilde{W}_{\text{aff}}} \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{H}_{\tilde{W}_{\text{aff}}}]$ and also
\[ \varepsilon_{\mathcal{H}_I|\mathbb{C}[Y_{Q,n}]} = \mathbb{C}_{t_-}, \quad \text{with} \quad t_- = (q^{-1})/2, q^{-3}/2, \ldots, q^{-(r-1)/2}, \]
see [Sol21, §2.3].

For any rings $A, B$ and a functor
\[ \mathcal{F} : \mathcal{M}_L(A) \longrightarrow \mathcal{M}_L(B) \]
of left modules over these rings, we denote by
\[ \mathcal{F}^o : \mathcal{M}_R(A^o) \longrightarrow \mathcal{M}_R(B^o) \]
the naturally associated functor of right modules over the opposite rings $A^o, B^o$ of $A$ and $B$. That is, for any $\sigma \in \mathcal{M}_L(A)$ one has
\[ \mathcal{F}^o(\sigma^o) := \mathcal{F}(\sigma)^o, \]
where $(-)^o$ denotes the corresponding module over the opposite ring.

Let $\mathcal{H}$ be one of the Hecke algebras $\mathcal{H}(S_{\text{aff}}), \mathcal{H}_{\tilde{W}_{\text{aff}}},$ or $\mathcal{H}_I$. There is a lattice $Z$ such that $\mathcal{H} \simeq \mathcal{H}_W \otimes_{\mathbb{C}[Z]}$ as vector spaces. The algebra $\mathcal{H}$ is isomorphic to its opposite algebra in view of the star involution
\[ (\ast) : \mathcal{H} \longrightarrow \mathcal{H}, \quad T_w^* = T_{w^{-1}}, \theta_z^* = \theta_z \]
for any $T_w \in \mathcal{H}_W$ and $\theta_z \in \mathbb{C}[Z]$ (see [BC, §2.3]). This gives an isomorphism of algebras
\[ I^* : \mathcal{H}^o \longrightarrow \mathcal{H} \]
given by $I^*(x) = x^*$. Thus, if $\mathcal{A} = \mathcal{H}$ is an affine Hecke algebra, then using $I^*$ we will view $\mathcal{F}$ and $\mathcal{F}^o$ both to be defined from the category of (left and right) modules over $\mathcal{H}$.
Theorem 4.2. Let $\overline{G} = \overline{\text{GL}_r^{(n)}}$ be a cover of $\text{GL}_r$ of type (C1). Then one has an isomorphism

$$V_\mathcal{X} \simeq \gamma_{n_a}^V \otimes_{\mathbb{C}} \varepsilon_W$$

of $\mathcal{H}_W$-modules and thus

$$\mathcal{V} \simeq \mathcal{V}_\mathcal{W} \otimes_{\mathbb{C}} \varepsilon_{\mathcal{H}_I}$$

as $\mathcal{H}_I$-modules, where we make the identification $\mathcal{H}_I \simeq \mathcal{H}(S^\text{aff}_r)$ as in (4.1). Moreover,

$$(4.5) \quad \mathcal{F}_{SW}^o(\pi \otimes_{\mathbb{C}} \varepsilon_{\mathcal{H}_I}) = \mathcal{F}_V^o \circ \mathcal{F}_{\text{Wh}}(\pi^V) \text{ and } \mathcal{F}_{SW}^o(\pi) = \mathcal{F}_V^o \circ \mathcal{F}_{\text{Wh}}(\pi^V \otimes_{\mathbb{C}} \varepsilon_{\mathcal{H}_I})$$

every $\pi \in \mathcal{M}_R(\mathcal{H}_I)$.

Proof. We have

$$\gamma_{n_a}^V \otimes_{\mathbb{C}} \varepsilon_W = \bigoplus_{\lambda \in \Lambda(n_a,r)} \varepsilon_{W_\lambda} \otimes_{\mathcal{H}_W} \mathcal{H}_W \text{ and } V_\mathcal{X} = \bigoplus_{\mathcal{O}_y \subset \mathcal{X}_{Q,n}} \varepsilon_{W_y} \otimes_{\mathcal{H}_{W_y}} \mathcal{H}_W.$$ 

Since $\mathcal{X}_{Q,n} = Y/n_a Y$, it is easy to see that the following map is a well-defined bijection.

$$\Lambda(n_a, r) \longrightarrow \{W\text{-orbits in } \mathcal{X}_{Q,n}\}, \quad \lambda \mapsto W(y_\lambda)$$

This immediately implies $V_\mathcal{X} \simeq \gamma_{n_a}^V \otimes_{\mathbb{C}} \varepsilon_W$ and also $\mathcal{V} \simeq \mathcal{V}_\mathcal{W} \otimes \varepsilon_{\mathcal{H}_I}$.

Recall that we view $\mathcal{F}_{SW}^o$ as defined on $\mathcal{M}_R(\mathcal{H}_I)$ via $I^*$. For any $\pi \in \mathcal{M}_R(\mathcal{H}_I)$ we have

$$\mathcal{F}_{SW}^o(\pi) = (\mathcal{V}_\mathcal{W} \otimes \mathcal{H}_I, \pi^V)^o = \pi \otimes_{\mathcal{H}_I} \mathcal{V}_\mathcal{W}^o,$$

where $\mathcal{V}_\mathcal{W}^o$ is the $(\mathcal{H}_I, (\text{End}_{\mathcal{H}_I}(\mathcal{V}_\mathcal{W}^o))$-module opposite to $\mathcal{V}_\mathcal{W}$. For every $\sigma \in \mathcal{M}_R(\mathcal{H}_I)$ we set

$$\sigma^* := \text{Hom}_{\mathcal{H}_I}(\sigma, \mathcal{H}_I),$$

which is also an $(\mathcal{H}_I, (\text{End}_{\mathcal{H}_I}(\sigma)))$-bimodule. Now

$$\mathcal{F}_{\text{Wh}}(\pi) = \text{Hom}_{\mathcal{H}_I}(\mathcal{V}, \pi^V) = \pi^V \otimes_{\mathcal{H}_I} \mathcal{V}^*$$

where the last equality follows from the fact that $\mathcal{V}$ is finite-generated projective $\mathcal{H}_I$-module, see [Bou98, Page 271]. Note

$$\mathcal{V}^* = \bigoplus_{\mathcal{O}_y \subset \mathcal{X}_{Q,n}} \text{Hom}_{\mathcal{H}_I}(\varepsilon_{W_y} \otimes_{\mathcal{H}_{W_y}} \mathcal{H}_I, \mathcal{H}_I) = \bigoplus_{\mathcal{O}_y \subset \mathcal{X}_{Q,n}} \text{Hom}_{\mathcal{H}_{W_y}}(\varepsilon_{W_y}, \mathcal{H}_I).$$

Again, since $\varepsilon_{W_y}$ is finitely-generated projective $\mathcal{H}_{W_y}$-module, we have

$$\text{Hom}_{\mathcal{H}_{W_y}}(\varepsilon_{W_y}, \mathcal{H}_I) = \mathcal{H}_I \otimes_{\mathcal{H}_{W_y}} \varepsilon_{W_y}^*,$$

where $\varepsilon_{W_y}^* = \text{Hom}_{\mathcal{H}_{W_y}}(\varepsilon_{W_y}, \mathcal{H}_{W_y})$ as left $\mathcal{H}_{W_y}$-module. Since the algebra $\mathcal{H}_{W_y}$ is semisimple and contains $\varepsilon_{W_y}$ with multiplicity one, we have $\text{dim}_{\mathbb{C}} \varepsilon_{W_y}^* = 1$. Moreover,

$$\varepsilon_{W_y} \simeq \varepsilon_{W_y}^* \simeq \varepsilon_{W_y}^*,$$

as $\mathcal{H}_{W_y}$-modules, where the middle denotes the dual representation. Hence,

$$(4.6) \quad \mathcal{V}^* \simeq \mathcal{V}^o \simeq \bigoplus_{\mathcal{O}_y \subset \mathcal{X}_{Q,n}} \mathcal{H}_I \otimes_{\mathcal{H}_{W_y}} \varepsilon_{W_y}, \quad \mathcal{V}_\mathcal{W}^o \simeq \mathcal{V}_\mathcal{W}^o \simeq \bigoplus_{\mathcal{O}_y \subset \mathcal{X}_{Q,n}} \mathcal{H}_I \otimes_{\mathcal{H}_{W_y}} 1_{W_y}$$

as left $\mathcal{H}_I$-modules.

The above also gives that

$$(\text{End}_{\mathcal{H}_I}(\mathcal{V}_\mathcal{W}^o))^o \simeq (\text{End}_{\mathcal{H}_I}(\mathcal{V}^o) \circ (\mathcal{V} \otimes_{\mathcal{H}_I} \mathcal{V}^*)^o \simeq (\mathcal{V}^o) \otimes_{\mathcal{H}_I} \mathcal{V}^o \simeq \text{End}_{\mathcal{H}_I}(\mathcal{V} = \text{End}_{\mathcal{H}_I}(\mathcal{V}_\mathcal{W}),$$

where the fourth isomorphism follows from (4.6) and the rest are canonical. With this identification, one can also check easily that $\mathcal{V}_\mathcal{W} \simeq \mathcal{V}_\mathcal{W}^o$ as right $\text{End}_{\mathcal{H}_I}(\mathcal{V}_\mathcal{W})$-modules. Thus, for every $\pi \in \mathcal{M}_R(\mathcal{H}_I)$ we have

$$\mathcal{F}_{\text{Wh}}(\pi) = (\pi^V \otimes_{\mathcal{H}_I} \varepsilon_{\mathcal{H}_I}) \otimes_{\mathcal{H}_I} \mathcal{V}_\mathcal{W}^o = \mathcal{F}_{SW}^o(\pi^V \otimes_{\mathcal{H}_I} \varepsilon_{\mathcal{H}_I}).$$
as right \( \text{End}_{\mathcal{H}_I}(\mathcal{V}_W) \)-modules. Since \( \mathcal{F}_{SW} = \mathcal{F}_\varphi \circ \mathcal{F}_1 \), the first equality (4.5) follows, which also gives the second equality.

It is immediate from Theorem 4.2 that for every Iwahori-spherical representation \( \pi \) one has
\[
\dim \text{Wh}_\varphi(\pi^\vee) = \dim \mathcal{F}_{SW}^o(\pi^I \otimes \varepsilon_{\mathcal{H}_I}).
\]
Moreover, if \( n_\alpha > r \), then \( \mathcal{F}_{SW} \) is a faithful functor (see [Ant, Theorem 2.60]) and this gives that \( \dim \text{Wh}_\varphi(\pi) > 0 \) for every \( \pi \) in this case. We mention that the above equality and the faithfulness for \( n_\alpha > r \) were first observed by V. Buciumas. In fact, he already noticed that one can relate Whittaker models of such \( \pi \) to quantum affine representations by using the work of Chari–Pressley [CP96].

Remark 4.3. We may consider \( U_q(\hat{\mathfrak{g}}(n_\alpha)) \) instead of \( U_q(\hat{\mathfrak{gl}}(n_\alpha)) \). In this case, one can define the natural left \( U_q(\hat{\mathfrak{gl}}(n_\alpha)) \) action on \( \mathcal{V}_W \) which still commutes with the right \( \mathcal{H}_I \)-action. It then gives rise to the quantum affine Schur–Weyl functor
\[
\mathcal{F}_{SW}^o : \mathcal{M}_R(\mathcal{H}_I) \longrightarrow \mathcal{M}_R(U_q(\hat{\mathfrak{gl}}(n_\alpha))),
\]
which is faithful if \( n_\alpha \geq r \). This faithfulness can be obtained as follows. First, \( \varphi' : U_q(\hat{\mathfrak{g}}(n_\alpha)) \to \text{End}_{\mathcal{H}_I}(\mathcal{V}_W) \) is surjective if \( n_\alpha \geq 2 \), see [DDF12, Theorem 3.8.1]. On the other hand, if \( n_\alpha \geq r \), then \( \mathcal{F}_{\text{Wh}} \) gives an equivalence of categories from Proposition 2.3 and Example 2.4. Since \( \mathcal{F}_{SW}^o(\pi \otimes \varepsilon_{\mathcal{H}_I}) = \mathcal{F}_{\varphi'} \circ \mathcal{F}_{\text{Wh}}(\pi^\vee) \) as in (4.5) still holds, we see that \( \mathcal{F}_{SW}^o \) is faithful for \( n_\alpha \geq r \).

4.4. Local scattering matrices and R-matrices. Henceforth, we write
\[
\mathcal{F}_{\text{Wh}}^o = \mathcal{F}_\varphi \circ \mathcal{F}_{\text{Wh}} : \mathcal{M}_R(\mathcal{H}_I) \longrightarrow \mathcal{M}_R(U_q(\hat{\mathfrak{gl}}(n_\alpha)))
\]
for simplicity of notations.

First, we consider the principal series representation of \( \text{GL}_r^{(n)} \) of type (C1). We have
\[
\mathcal{T} \simeq (T_1 \times \ldots \times T_r)/H,
\]
where \( H = \{ (\zeta_j) : \prod_j \zeta_j = 1 \} \) and \( T_j \subset \mathcal{T} \) is the covering torus associated to \( Y_j := \mathbb{Z}e_j \).

One has \( Z(T_j) = (T_j)^{\mu_n} \), which splits naturally over \( (T_j)^{\mu_n} \) by the section \( s \) in the notation of [GGK, §2.3]. Thus,
\[
Z(T_j) \simeq s \times (T_j)^{\mu_n}
\]
and this gives the Weyl-invariant unramified genuine character \( \omega_{s,j} \) of \( Z(T_j) \) trivial on \( (T_j)^{\mu_n} \). Consider the character \( \chi_{s,j} : T_j \to \mathbb{C}^\times \) given by \( \chi_{s,j}(a^s) = |a|^{s_j} \). This gives
\[
i(\chi_{s,j}) := i(\omega_{s,j}) \otimes \chi_{s,j} \in \text{Irr}(T_j),
\]
where \( i(\omega_{s,j}) \) is uniquely determined by \( \omega_{s,j} \) by the Stone-von Neumann theorem. For \( s = (s_1, \ldots, s_2) \in \mathbb{C}^r \), we have
\[
i(\chi_s) := \otimes_j i(\chi_{s,j}) \in \text{Irr}(\mathcal{T})
\]
and thus the principal series representation \( \Pi(\chi_s) := \text{Ind}_{R}(\mathcal{T})(i(\chi_s)) \). We can identify naturally \( \Pi(\chi_s)^\vee \simeq \Pi(\chi_{-s}) \).

Since
\[
\mathcal{F}_{SW}^o : \mathcal{M}_R(\mathcal{H}_I) \longrightarrow \mathcal{M}_R(U_q(\hat{\mathfrak{gl}}(n_\alpha)))
\]
is a monoidal functor (see [CP96, Proposition 4.8]), one has
\[
(4.7) \quad \mathcal{F}_{SW}^o(\Pi(\chi_s)^I) = \mathcal{F}_{SW}^o(i(\chi_{s_1})^{I_1}) \otimes \ldots \otimes \mathcal{F}_{SW}^o(i(\chi_{s_r})^{I_r}),
\]
where
\[
I_j = I \cap T_j = T_j(O_F).
\]
In fact, we can directly verify (4.7) as follows.

**Proposition 4.4.** For every $1 \leq j \leq r$ one has

\[(4.8) \quad \mathcal{F}_\text{SW}^0(i(\chi_{s_j})^I_J) \simeq V(n_{\alpha s_j}),\]

where $V(n_{\alpha s_j})$ is the shifted evaluation representation in Definition 4.1. Moreover,

\[(4.9) \quad \mathcal{F}_\text{SW}^r(\Pi(\chi_s)^I_J) \simeq V(n_{\alpha s_1}) \otimes \ldots \otimes V(n_{\alpha s_j}) \otimes \ldots \otimes V(n_{\alpha s_r}).\]

**Proof.** Write $Y_j = \mathbb{Z}e_j, Y_{Q,n,j} = \mathbb{Z}n_{\alpha}e_j$ and $\mathcal{Z}_{Q,n,j} = Y_j/Y_{Q,n,j}$. Note that

\[\mathcal{F}_\text{SW}^r(i(\chi_{s_j})^I_J) = i(\chi_{s_j})^I_J \otimes \mathcal{H}_{I_j} \mathcal{F}_\text{SW}^r,\]

where $\mathcal{H}_{I_j} = \mathbb{C}[Y_{Q,n,j}] = \mathbb{C}[\mathbb{Z}n_{\alpha}e_j]$ and $\mathcal{Y}_\text{SW}^r = \mathcal{H}_{I_j} \otimes \mathbb{C}[\mathcal{Z}_{Q,n,j}]$. Also, in this case

\[i(\chi_{s_j})^I_J = \mathbb{C}_{s_j},\]

the one-dimensional unramified subspace of $i(\chi_{s_j})$ affording the character

\[\chi_{s_j} : \mathbb{C}[Y_{Q,n,j}] \rightarrow \mathbb{C}.\]

It is then easy to check that $\mathcal{F}_\text{SW}^r(i(\chi_{s_j})^I_J) \simeq V(n_{\alpha s_j}).$

For the second isomorphism, we note $\Pi(\chi_s)^I = i(\chi_s)^I \otimes \mathcal{H}_T \mathcal{F}_\text{SW}^r$ and thus

\[\mathcal{F}_\text{SW}^r(\Pi(\chi_s)^I) = \Pi(\chi_s)^I \otimes \mathcal{H}_T \mathcal{F}_\text{SW}^r\]

\[\simeq i(\chi_s)^I \otimes \mathcal{H}_T \mathcal{F}_\text{SW}^r(\mathcal{Y}_\text{SW}^r|\mathcal{Q}_{n,j})\]

\[\simeq \bigotimes_{j=1}^r i(\chi_{s_j})^I_J \otimes \mathcal{H}_{I_j} \otimes \mathcal{C}[\mathcal{Z}_{Q,n,j}]\]

\[\simeq \bigotimes_{j=1}^r \mathcal{F}_\text{SW}^r(i(\chi_{s_j})^I_J).\]

This gives (4.7). Coupled with (4.8), we obtain (4.9). \qed

**Corollary 4.5.** The unramified principal series $\Pi(\chi_s)$ is reducible if and only if $q = q^{n_{\alpha}(s_i-s_j)}$ for some $i \neq j$.

**Proof.** This follows immediately from combining Proposition 4.4 and [CP96, §4.8 Corollary]. In fact, more directly, since $\mathcal{H}_T \simeq \mathcal{H}(S^\text{aff}_{\ell})$ for $\mathbf{GL}_r$ of type (C1), the points of reducibility of $\Pi(\chi_s)$ are exactly those of its Shimura lifted principal series $\Pi(\chi_s)$ of $\mathbf{GL}_r$ modulo an $n_{\alpha}$-scaling, where $\chi_s$ is a linear character of $T \subset \mathbf{GL}_r$ corresponding to $\chi_s$. \qed

In view of the Theorem 4.2, the isomorphism in (4.7) or (4.9) is an incarnation of Rodier’s heredity of Whittaker models [Rod73]. We make this precise as follows. First, one has a natural $\mathbb{C}$-isomorphism

\[\eta(\chi_s) : \text{Wh}_\psi(\Pi(\chi_s)^I \otimes \varepsilon_{\mathcal{H}_T}) \rightarrow \text{Wh}_\psi(\Pi(\chi_s)^I).\]

Indeed, one has

\[\text{Wh}_\psi(\Pi(\chi_s)^I \otimes \varepsilon_{\mathcal{H}_T}) = \text{Hom}_{\mathcal{H}_T}(V, \Pi(\chi_s)^I \otimes \varepsilon_{\mathcal{H}_T}) = \text{Hom}_{\mathcal{H}_W}(V_{W'}, \Pi(\chi_s)^I \otimes \varepsilon_W).\]

As $\mathcal{H}_W$-modules, $\Pi(\chi_s)^I \simeq \mathcal{H}_W$ for any principal series. Thus, one has a natural identification $\Pi(\chi_s)^I \otimes \varepsilon_W \simeq \Pi(\chi_s)^I$ as $\mathcal{H}_W$-modules and

\[\text{Wh}_\psi(\Pi(\chi_s)^I \otimes \varepsilon_{\mathcal{H}_T}) = \text{Hom}_{\mathcal{H}_W}(V_{W'}, \Pi(\chi_s)^I) \simeq \text{Wh}_\psi(\Pi(\chi_s)^I).\]

This gives the above map $\eta(\chi_s)$. For every $j$ one has the natural $\mathbb{C}$-isomorphisms

\[(4.10) \quad \mathbb{C}[\mathbb{Z}e_j/\mathbb{Z}n_{\alpha}e_j] \simeq \mathcal{F}_\text{Wh}^e(i(\chi_{s_j})^I_J) \simeq \mathcal{F}_\text{SW}^0(i(\chi_{s_j})^I_J) \simeq V(n_{\alpha s_j}).\]
Also, the same argument as in the proof of Proposition 4.4 gives the \( \mathbb{C} \)-isomorphisms

\[
\text{Wh}_\psi(\mathbb{I}(\chi_{-s})^f) \cong \mathbb{I}(\chi_s)^f \otimes_{\mathcal{H}_t} V^* \\
\cong \bigotimes_{j=1}^r (i(\chi_{-s})^j)^T \otimes_{\mathcal{H}_{ij}} (\mathcal{H}_{ij} \otimes \mathbb{C}[\mathcal{X}_{Q,n,j}])
\]

(4.11)

\[
\cong \bigotimes_{j=1}^r \mathcal{F}^\varphi_{\text{Wh}}(i(\chi_{-s})^j) \cong \bigotimes_{j=1}^r V(n_\alpha s_j),
\]

which can be referred to as the Rodier’s heredity. It is easy to see that the diagram

\[
\bigotimes_{j=1}^r V(n_\alpha s_j) \\
\xrightarrow{n(\chi_s)} \mathcal{F}^\varphi_{\text{SW}}(\mathbb{I}(\chi_s)^f) \xrightarrow{\varphi(w_\beta)\ast} \mathcal{F}^\varphi_{\text{Wh}}(\mathbb{I}(\chi_{-s})^f)
\]

(4.12)

of vector spaces commutes. Here the left and right slanted arrows are given by (4.9) and (4.11) respectively.

For every simple root \( \beta = \beta_k \) with \( 1 \leq k \leq r - 1 \), we have the simple reflection \( w_\beta = (k, k+1) \in S_r \) associated with \( \beta \). Consider the intertwining map \( T(w_\beta) : \mathbb{I}(\chi_s)^f \rightarrow \mathbb{I}(w_\beta \chi_s)^f \) as in McNamara [McN12, Equation (7.2)], where they are written \( T_{w_\beta} \) passing to \( I \)-fixed vectors gives an \( \mathcal{H}_I \)-module map \( \mathbb{I}(\chi_s)^f \rightarrow \mathbb{I}(w_\beta \chi_s)^f \), which we still write as \( T(w_\beta) \). Note \( w_\beta \chi_s = \chi_{w_\beta(s)} \) and further \( w_\beta \) acts on \( s = (s_1, ..., s_j, ..., s_r) \) via that on the index \( j \). Consider the map

\[
T^\varphi(w_\beta) : \mathbb{I}(w_\beta \chi_{-s})^f \rightarrow \mathbb{I}(\chi_{-s})^f.
\]

We then have the following commutative diagram arising from (4.12):

\[
\bigotimes_{j=1}^r V(n_\alpha s_j) \\
\xrightarrow{n(\chi_s)} \mathcal{F}^\varphi_{\text{SW}}(\mathbb{I}(\chi_s)^f) \xrightarrow{\varphi(w_\beta)\ast} \mathcal{F}^\varphi_{\text{Wh}}(\mathbb{I}(\chi_{-s})^f) \\
\downarrow \mathcal{F}^\varphi_{\text{SW}}(T(w_\beta)) \\
\bigotimes_{j=1}^r V(n_\alpha s_{w_\beta(j)}) \\
\xrightarrow{T^\varphi(w_\beta)\ast} \mathcal{F}^\varphi_{\text{Wh}}(\mathbb{I}(w_\beta \chi_{-s})^f).
\]

(4.13)

Here, the front square diagram, and in particular the two maps \( \mathcal{F}^\varphi_{\text{SW}}(T(w_\beta)) \) and \( T^\varphi(w_\beta)^\ast \) are induced from \( T(w_\beta) \) and \( T^\varphi(w_\beta) \). The map \( \mathfrak{F}(w_\beta)^\ast \) naturally arises from the front square in view of (4.12). In fact, by the functorial and monoidal property of \( \mathcal{F}^\varphi_{\text{SW}} \), one has (noting that \( w_\beta = (k, k+1) \in S_r \))

\[
\mathfrak{F}(w_\beta)^\ast = \text{id} \otimes ... \otimes \mathfrak{F}(w_\beta) \otimes ... \otimes \text{id}
\]

where

\[
\mathfrak{F}(w_\beta) : V(n_\alpha s_k) \otimes V(n_\alpha s_{k+1}) \rightarrow V(n_\alpha s_{k+1}) \otimes V(n_\alpha s_k)
\]

is a \( U_q(\mathfrak{sl}(n_\alpha)) \)-homomorphism.
It is clear from (4.13) that $\mathfrak{F}(w_\beta)^*$ can be represented by any matrix form of $T^\sharp(w_\beta)^*$. We write

$$s^* = -w_\beta(s) = (-s_1, -s_2, ..., -s_{k+1}, -s_k, ..., -s_r) \in \mathbb{C}^r.$$  

The operator $T^\sharp(w_\beta)^*$ is represented by a so-called local scattering square matrix

$$[S(y', y; w_\beta, \chi_{s^*})]_{y', y \in \mathfrak{R}_{n_\alpha}},$$

of size $|\mathcal{X}_{Q,n}|$, where $\mathfrak{R}_{n_\alpha} \subset Y$ is the set of representatives of $\mathcal{X}_{Q,n}$ in (4.2). This matrix was first studied by Kazhdan–Patterson [KP84] for $\text{GL}_r$ and then generalized by McNamara [McN16] for general $\mathcal{G}$. More precisely, for every $y = \sum_j y_j e_j \in \mathfrak{R}_{n_\alpha}$ with $y_j \in [0, n_\alpha - 1]$, one has

$$\lambda_{y,s}^{−s} \in \text{Wh}_V(\Pi(\chi_{−s})),$$

the naturally associated element via (4.10) and (4.11). The matrix $[S(y', y; w_\beta, \chi_{s^*})]$ is then by definition the one satisfying

$$(4.14) \quad T^\sharp(w_\beta)^*(\lambda_{y,s}^{−s}) = \sum_{y \in \mathfrak{R}_{n_\alpha}} S(y', y; w_\beta, \chi_{s^*}) \cdot \lambda_{y,s}^{−s}. $$

Note that we also have $u_{y_j}^{s_j} \in V(n_\alpha s_j)$ for every $y_j$ as above, using (4.10). Hence, for $y = \sum_j y_j e_j \in \mathfrak{R}_{n_\alpha}$, one has

$$u_{y,s}^* := u_{y_1}^{s_1} \otimes ... \otimes u_{y_r}^{s_r} \in V(n_\alpha s_1) \otimes ... \otimes V(n_\alpha s_r),$$

which corresponds to $\lambda_{y,s}^{−s}$ in the right top slanted arrow in (4.13). Similar correspondence holds for $\lambda_{y,s}^{−s}$ and $u_{y,s}^{−s*}$. The commutativity of (4.13) and the discussion above immediately give the following.

**Theorem 4.6.** Let the domain and codomain of the $U_q(\hat{\mathfrak{sl}}(n_\alpha))$-homomorphism $\mathfrak{F}(w_\beta)^*$ be endowed with the bases $\{u_{y,s}^* : y \in \mathfrak{R}_{n_\alpha}\}$ and $\{u_{y,s}^{−s} : y \in \mathfrak{R}_{n_\alpha}\}$ respectively. Then $\mathfrak{F}(w_\beta)^*$ is represented by the local scattering matrix $[S(y', y; w_\beta, \chi_{s^*})]_{y', y \in \mathfrak{R}_{n_\alpha}}$.

Consider the matrix

$$[\mathcal{M}(y', y; w_\beta, \chi_{s^*})] := [S(w_\beta(y'), y; w_\beta, \chi_{s^*})]_{y, y' \in \mathfrak{R}_{n_\alpha}}$$

and let

$$R(w_\beta, \chi_{s^*}) \in \text{End}_\mathbb{C}(V(n_\alpha s_1) \otimes ... \otimes V(n_\alpha s_r))$$

be the $\mathbb{C}$-endomorphism represented by $[\mathcal{M}(y', y; w_\beta, \chi_{s^*})]_{y, y' \in \mathfrak{R}_{n_\alpha}}$ with respect to the basis $\{u_{y,s}^* : y \in \mathfrak{R}_{n_\alpha}\}$. It is then easy to see that

$$\mathfrak{F}(w_\beta)^* = \tau(w_\beta) \circ R(w_\beta, \chi_{s^*})$$

where $\tau(w_\beta)$ is the obvious $\mathbb{C}$-homomorphism induced from the flipping map

$$V(n_\alpha s_k) \otimes V(n_\alpha s_{k+1}) \longrightarrow V(n_\alpha s_{k+1}) \otimes V(n_\alpha s_k), \quad v \otimes w \mapsto w \otimes v.$$

As a consequence of (4.13), the $\mathfrak{F}(w_\beta)^*$-s satisfy the braid relation. Thus, the operators $R(w_\beta, \chi_{s^*})$ solve the quantum Yang–Baxter equation, and the representing matrices $[\mathcal{M}(y', y; w_\beta, \chi_{s^*})]$ can be properly called $R$-matrices. One can check easily that it agrees with the one given in [BBB19, Page 105]. We also note that the matrix $[\mathcal{M}(y', y; w_\beta, \chi_{s^*})]$ is exactly a “local coefficients matrix” associated with $\Pi(\chi_{s^*})$ and $w_\beta$ as in [GSSa, §3.2].
5. Some remarks

The functor $\mathcal{F}_{Wh}$ is defined for any root system type, while

$$\mathcal{F}_{SW} : \mathcal{M}_L(\mathcal{H}(S^r)) \to \mathcal{M}_L(U_q(\mathfrak{sl}(n_\alpha)))$$

is for type $A$ only. It is a natural question to ask for a generalization of $\mathcal{F}_{SW}$, if we replace
the domain or codomain of $\mathcal{F}_{SW}$ by more general root system types.

First, if one considers $\mathcal{M}_R(U_q(\hat{\mathfrak{g}}))$ for a general semisimple Lie algebra $\mathfrak{g}$, then it was shown by [KKK18, Fu20] that there exists a natural functor

$$\mathcal{F}^g_{SW} : \bigoplus_\beta \mathcal{M}_L(R^I(\beta)) \to \mathcal{M}_L(U_q(\hat{\mathfrak{g}})),$$

where $R^I(\beta)$ is certain quiver algebra given by Khovanov–Lauda [KL09] and Rouquier [Rou]. Many results are established regarding $\mathcal{F}^g_{SW}$, especially those pertaining to the categorification of the two sides of $\mathcal{F}^g_{SW}$.

On the other hand, instead of $\mathcal{H}(S^r)$ in $\mathcal{F}_{SW}$ above, if one considers affine Hecke algebras of general Cartan type, for example of type $B$ or $C$, then the recent work of W.-Q. Wang, H.-C. Bao and their collaborators [BKLW18, BWW18, FLL+20] have made advances in the framework of quantum symmetric pairs. In particular, a Schur–Weyl type duality between affine Hecke algebras of type $B_r$ or $C_r$ and certain coideal subalgebra of $U_q(\mathfrak{gl}(k))$ is established. We refer the reader to loc. cit. and references therein for details.

In the geometric setting, the work of Gaitsgory and Lysenko [Gai08, GL18, Lys17] represents some significant progress towards understanding certain Whittaker category in the metaplectic setting and also its relation to the quantum groups in the framework of the quantum Langlands program (see [Gai]). It will be very interesting to see what would be the analogue of such results in the classical $p$-adic context, and conversely what role the quantum affine Schur–Weyl functor plays in the formulation of some geometric results.

References

[Ant] Jonas Antor, Affine versions of Schur–Weyl duality, Master’s Thesis, 2020, available at https://www.math.uni-bonn.de/ag/stroppel/Masterarbeit_Antor-4-1.pdf.

[Arn69] V. I. Arnol’d, The cohomology ring of the group of dyed braids, Mat. Zametki 5 (1969), 227–231 (Russian). MR242196

[BKLW18] Huanchen Bao, Jonathan Kujawa, Yiqiang Li, and Weiqiang Wang, Geometric Schur duality of classical type, Transform. Groups 23 (2018), no. 2, 329–389, DOI 10.1007/s00031-017-9447-4. MR3805209

[BWW18] Huanchen Bao, Weiqiang Wang, and Hideya Watanabe, Multiparameter quantum Schur duality of type $B$, Proc. Amer. Math. Soc. 146 (2018), no. 8, 3203–3216, DOI 10.1090/proc/13749. MR3803649

[BC] Dan Barbasch and Dan Ciubotaru, Star operations for affine Hecke algebras, preprint, available at https://arxiv.org/abs/1504.04361v1.

[Bou98] Nicolas Bourbaki, Algebra I. Chapters 1–3, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998. Translated from the French; Reprint of the 1989 English translation [ MR0979982 (90d:00002)]. MR1727844

[Bou02] ———, Lie groups and Lie algebras. Chapters 4–6, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[Bri73] Egbert Bröskorn, Sur les groupes de tresses [d’après V. I. Arnol’d], Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, Springer, Berlin, 1973, pp. 21–44. Lecture Notes in Math., Vol. 317 (French). MR0422674

[BBB19] Ben Brubaker, Valentin Buciumas, and Daniel Bump, A Yang-Baxter equation for metaplectic ice, Comm. Number Theory and Physics 13 (2019), no. 1, 101–148, DOI 10.4310/CNTP.2019.v13.n1.a4.
[BBBF18] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Solomon Friedberg, Hecke modules from metaplectic ice, Selecta Math. (N.S.) 24 (2018), no. 3, 2523–2570, DOI 10.1007/s00029-017-0372-0. MR3816510

[BJH03] Colin J. Bushnell and Guy Henniart, Generalized Whittaker models and the Bernstein center, Amer. J. Math. 125 (2003), no. 3, 513–547. MR1981032

[CP95] Vyjayanthi Chari and Andrew Pressley, A guide to quantum groups, Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1994 original. MR1358358

[CP96] ——. Quantum affine algebras and affine Hecke algebras, Pacific J. Math. 174 (1996), no. 2, 295–326. MR1405590

[DDF12] Bangming Deng, Jie Du, and Qiang Fu, A double Hall algebra approach to affine quantum Schur-Weyl theory, London Mathematical Society Lecture Note Series, vol. 401, Cambridge University Press, Cambridge, 2012. MR3113018

[DG07] S. R. Doty and R. M. Green, Presenting affine $q$-Schur algebras, Math. Z. 256 (2007), no. 2, 311–345, DOI 10.1007/s00209-006-0076-1. MR2289877

[FLL+20] Zhaobing Fan, Chun-Ju Lai, Yiqiang Li, Li Luo, Weiqiang Wang, and Hideya Watanabe, Quantum Schur duality of affine type C with three parameters, Math. Res. Lett. 27 (2020), no. 1, 79–114. MR4088809

[Fuj20] Ryo Fujita, Geometric realization of Dynkin quiver type quantum affine Schur-Weyl duality, Int. Math. Res. Not. IMRN 22 (2020), 8353–8386, DOI 10.1093/imrn/rny226. MR4216691

[Gai08] Dennis Gaitsgory, Twisted Whittaker model and factorizable sheaves, Selecta Math. (N.S.) 13 (2008), no. 4, 617–659, DOI 10.1007/s00029-008-0053-0. MR2403306

[Gai] ——. Quantum Langlands Correspondence, available at https://arXiv.org/abs/1601.05279.

[GL18] Dennis Gaitsgory and Sergey Lysenko, Parameters and duality for the metaplectic geometric Langlands theory, Selecta Math. (N.S.) 24 (2018), no. 1, 227–301, DOI 10.1007/s00029-017-0360-4. MR3769731

[GG18] Wee Teck Gan and Fan Gao, The Langlands-Weissman program for Brylinski-Deligne extensions, Astérisque 398 (2018), 187–275 (English, with English and French summaries). L-groups and the Langlands program for covering groups. MR3802419

[Gao17] Fan Gao, Distinguished theta representations for certain covering groups, Pacific J. Math. 290 (2017), no. 2, 333–379, DOI 10.2140/pjm.2017.290.333.

[GGK] Fan Gao, Nadya Gurevich, and Edmund Karasiewicz, Genuine pro-p Iwahori–Hecke algebras, Gelfand–Graev representations, and some applications, preprint (2022, 63 pages), available at https://arxiv.org/abs/2204.13053.

[GSSa] Fan Gao, Freydoon Shahidi, and Dani Szpruch, Local coefficients and gamma factors for principal series of covering groups, Memoirs of the AMS (2019, accepted), available at https://arxiv.org/abs/1902.02686.

[GSSb] ——. Restrictions, L-parameters, and local coefficients for genuine representations, preprint (2021, submitted), available at https://arxiv.org/abs/2102.08859.

[GT22] Fan Gao and Wan-Yu Tsai, On the wavefront sets associated with theta representations, Math. Z. 301 (2022), no. 1, 1–40, DOI 10.1007/s00209-021-02894-5. MR4405642

[Gre99] R. M. Green, The affine $q$-Schur algebra, J. Algebra 215 (1999), no. 2, 379–411, DOI 10.1006/jabr.1998.7753. MR1686197

[IM65] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of $p$-adic Chevalley groups, Inst. Hautes Études Sci. Publ. Math. 25 (1965), 5–48. MR185016

[KKK18] Seok-Jin Kang, Masaki Kashiwara, and Myungho Kim, Symmetric quiver Hecke algebras and $R$-matrices of quantum affine algebras, Invent. Math. 211 (2018), no. 2, 591–685, DOI 10.1007/s00222-017-0754-0. MR3748315

[KP84] D. A. Kazhdan and S. J. Patterson, Metaplectic forms, Inst. Hautes Études Sci. Publ. Math. 59 (1984), 35–142. MR743816

[KL09] Mikhail Khovanov and Aaron D. Lauda, A diagrammatic approach to categorification of quantum groups. I, Represent. Theory 13 (2009), 309–347, DOI 10.1090/S1088-4165-09-00346-X. MR2525917

[Lys17] Sergey Lysenko, Twisted Whittaker models for metaplectic groups, Geom. Funct. Anal. 27 (2017), no. 2, 289–372, DOI 10.1007/s00039-017-0403-1. MR3626614

[McN12] Peter J. McNamara, Principal series representations of metaplectic groups over local fields, Multiple Dirichlet series, L-functions and automorphic forms, Progr. Math., vol. 300,
Birkhäuser/Springer, New York, 2012, pp. 299–327, DOI 10.1007/978-0-8176-8334-413.

MR2963537

[McN16] The metaplectic Casselman-Shalika formula, Trans. Amer. Math. Soc. 368 (2016), no. 4, 2913–2937, DOI 10.1090/tran/6977. MR3449262

[MS19] The metaplectic Casselman-Shalika formula, Trans. Amer. Math. Soc. 368 (2016), no. 4, 2913–2937, DOI 10.1090/tran/6597. MR3449262

[OS83] Peter Orlik and Louis Solomon, Coxeter arrangements, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 269–291. MR713255

[Roc09] Alan Roche, The Bernstein decomposition and the Bernstein centre, Ottawa lectures on admissible representations of reductive $p$-adic groups, Fields Inst. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 2009, pp. 3–52, DOI 10.1090/fim/026/01. MR2508719

[Rod73] François Rodier, Whittaker models for admissible representations of reductive $p$-adic split groups, Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 425–430. MR0354942

[Rou] Raphael Rouquier, 2-Kac-Moody algebras, available at https://arxiv.org/abs/0812.5023.

[Sav88] Gordan Savin, Local Shimura correspondence, Math. Ann. 280 (1988), no. 2, 185–190, DOI 10.1007/BF01456050. MR929534

[Sol21] Maarten Solleveld, Affine Hecke algebras and their representations, Indag. Math. (N.S.) 32 (2021), no. 5, 1005–1082, DOI 10.1016/j.indag.2021.01.005. MR4310011

[Sol66] Louis Solomon, The orders of the finite Chevalley groups, J. Algebra 3 (1966), 376–393, DOI 10.1016/0021-8693(66)90097-X. MR199275

[Som97] Eric Sommers, A family of affine Weyl group representations, Transform. Groups 2 (1997), no. 4, 375–390, DOI 10.1007/BF01234541. MR1486037

[Vig03] Marie-France Vignéras, Schur algebras of reductive $p$-adic groups. I, Duke Math. J. 116 (2003), no. 1, 35–75, DOI 10.1215/S0012-7094-03-11612-9. MR1950479

[Zou] Jiandi Zou, Metaplectic correspondence and applications, preprint, available at https://arxiv.org/abs/2209.05852v1.

F. Gao: School of Mathematical Sciences, Zhejiang University, 866 Yuhangtang Road, Hangzhou, China 310058
Email address: gaofan@zju.edu.cn

N. Gurevich: Department of Mathematics, Ben Gurion University of the Negev, Be’er Sheva, Israel 8410501
Email address: ngur@math.bgu.ac.il

E. Karasiewicz: Department of Mathematics, University of Utah, Salt Lake City, USA 84112
Email address: karasiewicz@math.utah.edu