Generalized Mattig’s relation in Brans-Dicke-Rastall gravity

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The Geodesic Deviation Equation is being studied in Brans-Dicke-Rastall gravity. We briefly discuss the Brans-Dicke-Rastall gravity and then construct GDE for FLRW metric. In this way, the obtained geodesic deviation equation will correspond to the Brans-Dicke-Rastall gravity. Eventually, we solve numerically the null vector GDE to obtain from Mattig relation, the deviation vector η(z) and observer area distance r0(z) and compare the results with ΛCDM model.

I. INTRODUCTION

One of the main differences between general relativity (and its alternatives) and Newtonian theory is that the effects of matter distribution as well as gravitation are encoded in geometry of space-time. In these contexts, the space-time is being curved due to the presence of matter fields and can be described by a pair (\(\mathcal{M}, g\)) where \(\mathcal{M}\) and \(g\) correspond to a \(d\)-dimensional manifold and the metric tensor on it respectively, while the curvature can be represented by the Riemann tensor \(\mathbf{R}\) [1]. Therefore, it is important to look for a relation which associates the curvature of space-time to a physically measurable quantity. This aim is reachable via Geodesic Deviation Equation (GDE) [2–4] in which the Riemann tensor is connected to the relative acceleration between two close test particles [5, 6]. In fact, this relative acceleration is produced by a kind of tidal force which is the result of the tendency of two neighbor free falling particles to approach or recede from one another under influence of a space-dependent gravitational field [2–6]. Furthermore, in addition to elegant description and interesting visualization that GDE presents about space-time structure, various crucial solutions such as Raychaudhuri equation and Mattig relation can be found through the solution of GDE for timelike, null and spacelike geodesic congruences. One can also obtain the relative acceleration of two neighbor geodesics by using GDE [5, 6].

Besides general relativity (GR) which shows relative successes in various field strength regimes [7], there are various alternative theories which extend GR from different points of view. Some of these modifications to GR are constructed by adding a scalar field degree of freedom to it which are called scalar-tensor gravities [8]. Among scalar-tensor gravities, Brans-Dicke (BD) theory [9] is one of the most impressive and physically viable modifications to GR. The motivations of this theory that are Mach’s principle and Dirac’s large number hypothesis are encoded by employing a scalar field \(\phi\) which is proportional to the inverse of gravitational constant \(G\) and coupled nonminimally to gravitation [10]. There is also a coupling constant \(\omega\) in BD theory. This theory reproduces GR in the limit \(\omega \to \infty\) and \(\phi = constant\) [11–13]. Moreover, this theory is equivalence to some other alternatives of GR in particular \(f(R)\) gravities that provide us a good tool to gain better insights about these theories [14]. Other important property of BD theory is that it produces simple expanding solutions [15] for scalar field \(\phi(t)\) and scale factor \(a(t)\) which are more compatible with solar system experiments [16–18]. There are also many works on interesting physical aspects of the BD theory [19].

As we pointed out before, various geometrical theories which are alternatives to GR have been put forward since its beginning for explaining the gravitational phenomenon [9, 20–26]. The conservation law \((T^{\mu\nu},_{\nu} = 0)\) is one of them which does not hold true in a curved space-time in some of these theories. Among these theories, the steady-state model is the pioneer non-conservative theory of gravity [27, 28] which has been developed by following some ideas already presented by Jordan [29]. Following this idea, Rastall has introduced alternative version of non-conservative theory of gravity which suggests that gravitational equations can be obtained via modification of conservation laws [25, 26]. Also, the breaking of the weak and null energy conditions for the average value of the EMT of a quantum field in curved space-time was first shown in [30] for weak gravitational fields and in [31] for strong gravitational

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fields in cosmology. Hence, Rastall’s idea leads to the classical formulation of the quantum phenomenon because the violation of the energy-momentum conservation is linked with curvature. The idea of Rastall has also been employed to extend BD gravity and give rise to Brans-Dicke-Rastall (BDR) gravity in which the conservation law is no longer respected [32, 33].

In the present work, our main goal is to study the GDE in the metric context of Brans-Dicke-Rastall gravity. We also numerically solve the Mattig relation which helps in measuring the cosmological distances.

II. BRIEF REVIEW ON BRANS-DICKE-RASTALL THEORY

A. Rastall theory

As we mentioned in introduction, Rastall has introduced alternative version of non-conservative theory of gravity which proposes that equations of gravitation can be obtained by modification of conservation laws as follow [25, 26]

\[ T_{\mu\nu}^{\text{;\mu}} = \kappa R_{\mu\nu}^{\text{;\mu}}, \] (1)

where \( \kappa, T_{\mu\nu} \) and \( R_{\mu\nu} \) indicate the energy-momentum tensor, a coupling constant and Ricci scalar curvature, respectively. In view of weak field limit, the usual expressions can be recovered. We can also write the above equation in terms of trace energy-momentum (\( T \)) as follows

\[ T_{\mu\nu}^{\text{;\mu}} = \bar{\kappa} T_{\mu\mu}, \] (2)

where \( \bar{\kappa} \) is a new constant. In addition, Rastall's modification to Einstein equations takes the form

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left( T_{\mu\nu} - \frac{\lambda_{\text{Ras}} - 1}{2} g_{\mu\nu} T \right), \] (3)

\[ T_{\mu\nu}^{\text{;\mu}} = \frac{\lambda_{\text{Ras}} - 1}{2} T_{\mu\nu}, \] (4)

where \( c = 1 \) and \( \lambda_{\text{Ras}} \) appears as Rastall’s parameter and for \( \lambda_{\text{Ras}} = 1 \), we can obtain GR. There also exists a possibility of obtaining the above equations from Lagrangian formulation [39]. As \( T = 0 \), \( R = 0 \) for a radiative fluid which implies that the cosmological evolution during the radiative phase is the same as in the standard cosmological scenario. At the same time, a single fluid inflationary model as described by a cosmological constant is the same as it would be in GR case. Hence, Rastall cosmologies seem like a curious departure from standard cosmological model and from the beginning of the matter dominated phase [40–43].

B. Brans-Dicke gravity

The action of BD theory (where scalar field and matter field are non-minimally coupled) is displayed below [44]

\[ S = \int d^4x \sqrt{-g} \left( \phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + L_m \right), \] (5)

where \( \phi, V(\phi) \) and \( \omega \) represents the BD scalar field, potential and dimensionless BD parameter, respectively. The variation of action (5) with respect to metric tensor yields the following gravitational field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2} \left( \phi \mu \phi_\nu - \frac{1}{2} g_{\mu\nu} \phi^\alpha \phi_\alpha \right) + \frac{1}{\phi} [\phi_{\mu\nu} - g_{\mu\nu} \Box \phi] - g_{\mu\nu} \frac{V(\phi)}{2\phi}, \] (6)

while \( T_{\mu\nu} \) (stress-energy tensor) can be defined as follows

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}, \] (7)

where \( u^\mu \) represents the four-vector velocity of the fluid which satisfies \( u^\mu u_\mu = -1 \). Also \( \rho \) and \( p \) corresponds to the energy density and pressure of various kinds of matters. One can describe the various eras of the universe through equation of state parameter such as radiation, dust and dark energy etc. The Klein-Gordon equation (or the wave equation) for the scalar field has the form

\[ \Box \phi = \frac{T}{3 + 2\omega} + \frac{1}{2\omega + 3} (\phi V_{,\phi} - 2V), \] (8)
The Bianchi identities lead to

\[ R = \frac{1}{3} T - \frac{\omega}{\phi^2} \phi_{\rho \phi} + 3 \Box \phi - 2 V(\phi) \]  

(9)

With the aid of this expression equation (6) can be rewritten as

\[ R_{\mu \nu} = \frac{1}{\phi} T_{\mu \nu} + \frac{\omega}{\phi^2} \left( \phi_{\mu \phi_{\nu}} - \frac{1}{2} g_{\mu \nu} \phi_{\alpha} \phi_{\alpha} \right) + \frac{1}{\phi} \left[ \phi_{\mu \phi_{\nu}} - g_{\mu \nu} \Box \phi \right] - g_{\mu \nu} \frac{V(\phi)}{2 \phi} - \frac{1}{2} g_{\mu \nu} \left( \frac{1}{\phi} T - \frac{\omega}{\phi^2} \phi_{\rho \phi} - 3 \Box \phi - 2 V(\phi) \right) \]

(10)

In the following, we will consider the potential \( V(\phi) \) null as in [32, 33].

C. Brans-Dicke-Rastall gravity

Let us generalize Rastall’s version of the field equations to the Brans-Dicke case. Following the original formulation in the context of GR, a minimal modification implies:

\[ R_{\mu \nu} - \frac{\lambda}{2} g_{\mu \nu} R = \frac{8 \pi}{\phi} T_{\mu \nu} + \frac{\omega}{\phi^2} \left( \phi_{\mu \phi_{\nu}} - \frac{1}{2} g_{\mu \nu} \phi_{\alpha} \phi_{\alpha} \right) + \frac{1}{\phi} \left( \phi_{\mu \phi_{\nu}} - g_{\mu \nu} \Box \phi \right). \]

(11)

It is important to remark that even if the structure of the right hand side is the same as in the BD gravity, the whole equation (11) can be derived from a Lagrangian only when \( \lambda = 1 \).

The trace of these “Einsteinian equations” reads:

\[ R = \frac{1}{1 - 2 \lambda} \left\{ \frac{8 \pi}{\phi} T - \frac{\omega}{\phi^2} \phi_{\rho \phi} - 3 \Box \phi \right\}. \]

(12)

With the aid of this expression equation (11) can be rewritten as

\[ R_{\mu \nu} = \frac{8 \pi}{\phi} \left( T_{\mu \nu} - \frac{1 - \lambda}{2(1 - 2 \lambda)} g_{\mu \nu} T \right) + \frac{\omega}{\phi^2} \left( \phi_{\mu \phi_{\nu}} + \frac{\lambda}{2(1 - 2 \lambda)} g_{\mu \nu} \phi_{\alpha} \phi_{\alpha} \right) + \frac{1}{\phi} \left( \phi_{\mu \phi_{\nu}} + \frac{(1 + \lambda)}{2(1 - 2 \lambda)} g_{\mu \nu} \Box \phi \right) + \frac{g_{\mu \nu}}{2(1 - 2 \lambda)} \left\{ \frac{8 \pi}{\phi} T - \frac{\omega}{\phi^2} \phi_{\rho \phi} - 3 \Box \phi \right\}. \]

(13)

The Bianchi identities lead to

\[ \Box \phi = \frac{8 \pi \lambda}{3 \lambda - 2(1 - 2 \lambda) \omega} T - \frac{\omega(1 - \lambda)}{3 \lambda - 2(1 - 2 \lambda) \omega} \phi_{\rho} \phi_{,\rho}. \]

(14)

The complete set of equations is:

\[ T^{\mu \nu} :_\mu = \frac{(1 - \lambda) \phi}{16 \pi} R^\nu, \]

(15)

\[ R_{\mu \nu} = \frac{8 \pi}{\phi} \left( T_{\mu \nu} - \frac{1 - \lambda}{2(1 - 2 \lambda)} g_{\mu \nu} T \right) + \frac{\omega}{\phi^2} \left( \phi_{\mu \phi_{\nu}} + \frac{\lambda}{2(1 - 2 \lambda)} g_{\mu \nu} \phi_{\alpha} \phi_{\alpha} \right) + \frac{1}{\phi} \left( \phi_{\mu \phi_{\nu}} + \frac{(1 + \lambda)}{2(1 - 2 \lambda)} g_{\mu \nu} \Box \phi \right) + \frac{g_{\mu \nu}}{2(1 - 2 \lambda)} \left\{ \frac{8 \pi}{\phi} T - \frac{\omega}{\phi^2} \phi_{\rho \phi} - 3 \Box \phi \right\}, \]

(16)

\[ \Box \phi = \frac{8 \pi \lambda}{3 \lambda - 2(1 - 2 \lambda) \omega} T - \frac{\omega(1 - \lambda)}{3 \lambda - 2(1 - 2 \lambda) \omega} \phi_{\rho} \phi_{,\rho}. \]

(17)
FIG. 1: Brans-Dicke-Rastall theory

Through Fig. (1), we see how the BDR theory was constructed. We can also note the passage from one theory to another by changing the parameters $\phi$ and $\lambda$. Thus, when we take $\lambda = 1$ in Rastall theory or when we take $\phi \sim 1/G$ in BD theory or when we take $\phi \sim 1/G$ and $\lambda = 1$ in Brans-Dicke-Rastall theory, the GR is recovered.

III. GEODESIC DEVIATION EQUATION

Here, we will discuss the GDE by following [4–6]. Assuming that $\gamma_0$ and $\gamma_1$ are two neighboring geodesics along with affine parameter $\nu$. There exists a family of interpolating geodesics $s$ which can be described as $x^\alpha(\nu, s)$ (Fig. 2). Consequently, the tangent to the geodesic is the vector field $V^\alpha = \frac{dx^\alpha}{d\nu}$, while $\eta^\alpha = \frac{ds^\alpha}{ds}$ is the tangent vector field to the family $s$. However, the acceleration for this vector field can be defined as follows [5, 6]

$$\frac{D^2 \eta^\alpha}{D\nu^2} = -R^\alpha_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta, \quad (18)$$

which is also called GDE. In this equation, $\frac{D}{D\nu}$ indicates the covariant derivative along the curve. The Riemann tensor can be written as [5, 45]

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma} R_{\delta\beta} - g_{\alpha\delta} R_{\gamma\beta} + g_{\beta\delta} R_{\gamma\alpha} - g_{\beta\gamma} R_{\delta\alpha}) - \frac{R}{6} (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}), \quad (19)$$

where $C_{\alpha\beta\gamma\delta}$ appears as the the Weyl tensor.

Next we defines the Friedman-Lamaître-Robertson-Walker (FLRW) universe as follows

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \quad (20)$$

where $a(t)$ represents the scale factor and $k$ corresponds to spatial curvature of the universe. For this metric, Weyl tensor $C_{\alpha\beta\gamma\delta}$ turns out to be zero. Through this paper we use the sign convention $(-,+,+,+)$ and geometrical units with $c = 1$. However, the trace of energy-momentum tensor becomes

$$T = 3p - \rho. \quad (21)$$

The Einstein field equations (with cosmological constant) in GR are

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}. \quad (22)$$

Hence, Ricci scalar $R$ and Ricci tensor $R_{\alpha\beta}$ with the help of above equation can be written as

$$R = \kappa (\rho - 3p) + 4\Lambda, \quad (23)$$

$$R_{\alpha\beta} = \kappa (\rho + p) u_{\alpha} u_{\beta} + \frac{1}{2} [\kappa (\rho - p) + 2\Lambda] g_{\alpha\beta}. \quad (24)$$
With the help of these expressions, the right side of (18) turns out to be [4]

\[ R_{\beta\gamma\delta}^\alpha V_\beta \eta^\gamma V^\delta = \left[ \frac{1}{3} (\kappa \rho + \Lambda) \epsilon + \frac{1}{2} \kappa (\rho + p) E^2 \right] \eta^\alpha, \]  

(25)

where \( \epsilon = V^\alpha V_\alpha \) and \( E = -V_\alpha u^\alpha \). This equation is called Pirani equation [3]. The detailed study about GDE and some solutions for spacelike, timelike and null congruences has given in [4]. Also, crucial results about cosmological distances has given in [45]. This equation is well known [3] which specify the spacelike components of the geodesic deviation vector \( \eta^\mu \) that describes the distance between two infinitesimally close particles in free fall. In our study, we generalized these results for BDR metric formalism.

IV. GDE IN BRANS-DICKE-RASTALL GRAVITY

The expression (19) in Brans-Dicke-Rastall gravity field equations can be written as

\[
\begin{align*}
R_{\alpha\beta\gamma\delta} &= C_{\alpha\beta\gamma\delta} \\
&+ \frac{8\pi}{\phi} \left( T_{\delta\beta} g_{\alpha\gamma} - T_{\gamma\beta} g_{\alpha\delta} + T_{\gamma\alpha} g_{\beta\delta} - T_{\delta\alpha} g_{\beta\gamma} \right) + \left( \frac{8\pi \lambda T}{2\phi(1-2\lambda)} + \frac{\omega(\lambda-1)\phi_\beta\phi_\delta}{2\phi^2(1-2\lambda)} \right) \\
&\times \left( g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta} \right) + \frac{\omega}{\phi^2} \left( g_{\alpha\gamma} D\Delta_{\delta\beta} - g_{\delta\gamma} D\Delta_{\alpha\beta} + g_{\beta\delta} D\Delta_{\gamma\alpha} - g_{\beta\gamma} D\Delta_{\delta\alpha} \right) \phi \\
&+ \frac{1}{\phi} \left( g_{\alpha\gamma} D\epsilon_{\delta\beta} - g_{\alpha\delta} D\epsilon_{\gamma\beta} + g_{\beta\delta} D\epsilon_{\gamma\alpha} - g_{\beta\gamma} D\epsilon_{\delta\alpha} \right) \phi \\
&+ \frac{\lambda - 2}{2\phi(1-2\lambda)} \left( g_{\alpha\gamma} D\epsilon_{\epsilon\delta\beta} - g_{\alpha\delta} D\epsilon_{\epsilon\gamma\beta} + g_{\beta\delta} D\epsilon_{\epsilon\gamma\alpha} - g_{\beta\gamma} D\epsilon_{\epsilon\delta\alpha} \right) \phi \\
&+ \frac{\omega(\lambda-1)}{2\phi^2(1-2\lambda)} \left( g_{\alpha\gamma} D\Delta_{\delta\beta} - g_{\alpha\delta} D\Delta_{\gamma\beta} + g_{\beta\delta} D\Delta_{\gamma\alpha} - g_{\beta\gamma} D\Delta_{\delta\alpha} \right) \phi \\
&- \frac{1}{6(1-2\lambda)} \left[ \frac{8\pi}{\phi} T - \frac{\omega}{\phi^2} \phi_\beta\phi_\delta^\rho - 3 \Box \phi \right] \left( g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta} \right).
\end{align*}
\]  

(26)

where we defined the operators

\[
\begin{align*}
D\epsilon_{\alpha\beta\gamma\delta} &\equiv g_{\alpha\beta} \phi_{\gamma\delta} \\
D\epsilon_{\alpha\beta}\phi_{\gamma} &\equiv g_{\alpha\beta} \phi_{\gamma}\phi_{\delta} \\
D\epsilon_{\alpha\beta}\phi_{\gamma} &\equiv g_{\alpha\beta} \phi_{\gamma}\phi_{\delta} \\
D\Delta_{\alpha\beta}\phi &\equiv g_{\alpha\beta} \phi_{\gamma}\phi_{\delta}.
\end{align*}
\]
By rising the first index and contracting with $V^\beta \eta V^\delta$ of Riemann tensor, the GDE turns out to be

$$R_{\beta\gamma\delta}^{\alpha} V^\beta \eta V^\delta = C_{\gamma\delta}^{\alpha} V^\beta \eta V^\delta$$

$$+ \frac{8\pi}{\phi} T \left[ (T_{\beta\gamma} \delta^\alpha_{\gamma} - T_{\gamma\beta} \delta^\alpha_{\gamma} + T_{\gamma}^\alpha \rho_{\beta} - T_{\delta}^\alpha \rho_{\gamma}) V^\beta \eta V^\delta \right]$$

$$+ \frac{8\pi \lambda T}{2\phi^2 (1 - 2\lambda)} \left( \frac{\omega(\lambda - 1) \phi_{\rho}\phi_{\rho}}{2\phi^2 (1 - 2\lambda)} \right) \left( \delta^\alpha_{\delta \beta} - \delta^\alpha_{\delta \gamma} \right) V^\beta \eta V^\delta$$

$$+ \frac{\omega}{\phi^2} \left( \delta_{\gamma}^\alpha D_{\delta \beta} - \delta_{\beta}^\alpha D_{\delta \gamma} + g_{\beta \delta} D_{\delta \gamma} - g_{\beta \gamma} D_{\delta \alpha} \right) \phi V^\beta \eta V^\delta$$

$$+ \frac{\lambda - 2}{2\phi^2 (1 - 2\lambda)} \left( \delta^\alpha_{\gamma} D_{\delta \beta} - \delta^\alpha_{\delta} D_{\gamma \beta} + g_{\beta \delta} D_{\delta \gamma} - g_{\beta \gamma} D_{\delta \alpha} \right) \phi V^\beta \eta V^\delta$$

$$\lambda (1 - \frac{1}{2\phi^2 (1 - 2\lambda)}) \left( \delta^\alpha_{\gamma} D_{\delta \beta} - \delta^\alpha_{\delta} D_{\gamma \beta} + g_{\beta \delta} D_{\delta \gamma} - g_{\beta \gamma} D_{\delta \alpha} \right) \phi V^\beta \eta V^\delta$$

$$- \frac{1}{6(1 - 2\lambda)} \left( \frac{8\pi}{\phi^2} T - \frac{\omega}{\phi^2} \phi_{\rho}\phi_{\rho} - 3 \frac{\square \phi}{\phi} \right) \left( \delta^\alpha_{\delta \beta} - \delta^\alpha_{\delta \gamma} \right) V^\beta \eta V^\delta. \quad (27)$$

**A. GDE for FLRW universe**

Here, we find the GDE in Brans-Dicke-Rastall gravity for FLRW metric and we also compare our results with GR in the limiting case where $f(R) = R - 2\Lambda$. For FLRW metric, we have

$$R = \frac{1}{1 - 2\lambda} \left\{ \frac{8\pi}{\phi} (3p - \rho) - \frac{\omega}{\phi^2} \phi_{\rho}\phi_{\rho} - 3 \frac{\square \phi}{\phi} \right\}. \quad (28)$$

With the help of above expressions, the Riemann tensor becomes

$$R_{\alpha\beta\gamma\delta} = \frac{8\pi}{\phi} \left[ (p + p) \left( u_{\delta} u_{\beta} g_{\alpha\gamma} - u_{\gamma} u_{\beta} g_{\alpha\delta} + u_{\gamma} u_{\alpha} g_{\beta\delta} - u_{\delta} u_{\alpha} g_{\beta\gamma} \right) \right]$$

$$+ \left[ \frac{(8\pi \lambda (3p - \rho)}{2\phi (1 - 2\lambda)} + \frac{\omega(\lambda - 1) \phi_{\rho}\phi_{\rho}}{2\phi^2 (1 - 2\lambda)} \right] - \frac{1}{6(1 - 2\lambda)} \left\{ \frac{8\pi}{\phi} (3p - \rho) - \frac{\omega}{\phi^2} \phi_{\rho}\phi_{\rho} - 3 \frac{\square \phi}{\phi} \right\} \left( g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\beta\gamma} \right)$$

$$+ \frac{\omega}{\phi^2} \left( g_{\alpha\gamma} D_{\delta \beta} - g_{\alpha\delta} D_{\beta \gamma} + g_{\beta \delta} D_{\gamma \alpha} - g_{\beta \gamma} D_{\delta \alpha} \right) \phi$$

$$+ \frac{\lambda - 2}{2\phi^2 (1 - 2\lambda)} \left( g_{\alpha\gamma} D_{\delta \beta} - g_{\alpha\delta} D_{\beta \gamma} + g_{\beta \delta} D_{\gamma \alpha} - g_{\beta \gamma} D_{\delta \alpha} \right) \phi$$

$$+ \frac{(\lambda - 1) \omega}{2\phi^2 (1 - 2\lambda)} \left( g_{\alpha\gamma} D_{\delta \beta} - g_{\alpha\delta} D_{\beta \gamma} + g_{\beta \delta} D_{\gamma \alpha} - g_{\beta \gamma} D_{\delta \alpha} \right) \phi. \quad (30)$$
For normalized vector field $V^\alpha$ we have $V^\alpha V_\alpha = \epsilon$ and

$$R_{\alpha\beta\gamma\delta} V^\beta V^\delta = \frac{8\pi}{\phi} \left[ (\rho + p) (u_\beta V^\beta)^2 - 2(u_\beta V^\beta) V_{(\alpha} u_{\gamma)} + \epsilon u_\alpha u_\gamma \right] + \left[ \frac{8\pi \lambda (3\rho - \rho)}{2\phi(1 - 2\lambda)} + \omega (\lambda - 1) \frac{\phi_\rho \phi_\rho}{2\phi^2 (1 - 2\lambda)} \right] - \frac{1}{6(1 - 2\lambda)} \left\{ \frac{8\pi}{\phi} (3\rho - \rho) - \frac{\omega}{\phi^2} \phi_\rho \phi_\rho - 3 \frac{\Box \phi}{\phi} \right\} \left( \epsilon g_{\alpha\gamma} - V_\alpha V_\gamma \right)$$

By rising the first index and contracting with $\eta^\gamma$, the Riemann tensor gives

$$R_{\beta\gamma\delta}^\alpha V^\beta \eta^\gamma V^\delta = \frac{8\pi}{\phi} \left[ (\rho + p) ((u_\beta V^\beta)^2 \eta^\alpha - (u_\beta V^\beta) V_{(\alpha} \eta_{\gamma)} - (u_\beta V^\beta) u^\alpha V^\gamma + \epsilon u_\alpha u_\gamma \eta^\gamma \right] + \left[ \frac{8\pi \lambda (3\rho - \rho)}{2\phi(1 - 2\lambda)} + \omega (\lambda - 1) \frac{\phi_\rho \phi_\rho}{2\phi^2 (1 - 2\lambda)} \right] - \frac{1}{6(1 - 2\lambda)} \left\{ \frac{8\pi}{\phi} (3\rho - \rho) - \frac{\omega}{\phi^2} \phi_\rho \phi_\rho - 3 \frac{\Box \phi}{\phi} \right\} \left( \epsilon g_{\beta\gamma} - \eta_{\beta} V^\gamma \right) \times (\eta^\gamma - V^\gamma (V_\gamma \eta^\gamma)) + \frac{\omega}{\phi^2} \left[ (\delta_\beta^\alpha D\eta^\delta - \delta_\delta^\beta D\eta^\alpha + g_{\beta\delta} D\eta^\alpha - g_{\beta\gamma} D\eta^\alpha) \phi \right] V^\beta V^\delta \eta^\gamma$$

where $E = -V_\alpha u^\alpha, \eta_\alpha u^\alpha = \eta_\alpha V^\alpha = 0$ [4]. Also, the above expression reduces to

$$R_{\beta\gamma\delta}^\alpha V^\beta \eta^\gamma V^\delta = \frac{8\pi}{\phi} \left[ (\rho + p) \epsilon^2 \right] + \left[ \frac{8\pi \lambda (3\rho - \rho)}{2\phi(1 - 2\lambda)} + \omega (\lambda - 1) \frac{\phi_\rho \phi_\rho}{2\phi^2 (1 - 2\lambda)} \right] - \frac{1}{6(1 - 2\lambda)} \left\{ \frac{8\pi}{\phi} (3\rho - \rho) - \frac{\omega}{\phi^2} \phi_\rho \phi_\rho - 3 \frac{\Box \phi}{\phi} \right\} \left( \epsilon g_{\beta\gamma} - \eta_{\beta} V^\gamma \right) \times (\eta^\gamma - V^\gamma (V_\gamma \eta^\gamma)) + \frac{\lambda - 2}{2\phi(1 - 2\lambda)} \left[ (\delta_\beta^\alpha D\eta^\delta - \delta_\delta^\beta D\eta^\alpha + g_{\beta\delta} D\eta^\alpha - g_{\beta\gamma} D\eta^\alpha) \phi \right] V^\beta V^\delta \eta^\gamma$$

where $H = \frac{\dot{a}}{\dot{a}}$ is the Hubble parameter and $\phi$ is only a function of time. Hence, the non-vanishing operators are

$$\Box \phi = \ddot{\phi}, \quad D\in_{00} = \dot{\phi}^2, \quad D\in_{00} \equiv \ddot{\phi}, \quad D\in_{00} = \ddot{\phi}, \quad D\Delta_{00} = \dot{\phi}^2$$

(34)
By make using of (34), $R^a_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta$ becomes

$$R^a_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \left[ \frac{\omega\dot{\phi}^2 + \phi(8\pi(p + \rho) - H\dot{\phi} + \ddot{\phi})}{2\phi^2} ight]$$

$$+ \epsilon \left( \frac{(-2 + 3\lambda)\omega^2 + \phi(8\pi(3\rho(-1 + \lambda) + (-1 + 3\lambda)p) - 3H(1 + \lambda)\dot{\phi} + 33(-1 + \lambda)\ddot{\phi})}{6(-1 + 2\lambda)\phi^2} \right) \eta^a. \quad (35)$$

This equation is the generalization of the Pirani equation for the BDR metric formalism. Note that when $\phi \sim 1/G$ and $\lambda = 1$, the previous equation reduces to (25). Finally, we can write the GDE (18) in BDR gravity as the following form

$$\frac{D^2\eta^\alpha}{Dt^2} = \left[ \frac{\omega\dot{\phi}^2 + \phi(8\pi(p + \rho) - H\dot{\phi} + \ddot{\phi})}{2\phi^2} ight]$$

$$+ \epsilon \left( \frac{(-2 + 3\lambda)\omega^2 + \phi(8\pi(3\rho(-1 + \lambda) + (-1 + 3\lambda)p) - 3H(1 + \lambda)\dot{\phi} + 33(-1 + \lambda)\ddot{\phi})}{6(-1 + 2\lambda)\phi^2} \right) \eta^a. \quad (36)$$

Thus the GDE produces variation only in the magnitude of the deviation vector $\eta^a$, which also occurs in GR. This result is expected FLRW universe. The GDE produces variation in the direction of the deviation vector in case of anisotropic universes (such as Bianchi I) [46].

B. GDE for fundamental observers

As, $V^\alpha$ appears as a four-velocity of the fluid $u^\alpha$. For temporal geodesics, we have $\epsilon = -1$ and for normalized vector field, we have $E = 1$. For these conditions, Eq.(35) gives

$$R^a_{\beta\gamma\delta}u^\beta\eta^\gamma u^\delta = \left[ \frac{(-1 + 3\lambda)\omega\phi^2 + \phi(8\pi(-2\rho + 3\lambda(p + \rho)) - 3H(-2 + \lambda)\dot{\phi} + 3\lambda\ddot{\phi})}{6(-1 + 2\lambda)\phi^2} \right] \eta^a. \quad (37)$$

If the deviation vector is $\eta_a = \ell e_a$, isotropy implies

$$\frac{De^\alpha}{Dt} = 0, \quad (38)$$

and

$$\frac{D^2\eta^\alpha}{Dt^2} = \frac{d^2\ell}{dt^2}e^\alpha, \quad (39)$$

Using this result in GDE (18) and (37) gives

$$\frac{d^2\ell}{dt^2} = \left[ \frac{(-1 + 3\lambda)\omega\phi^2 + \phi(8\pi(-2\rho + 3\lambda(p + \rho)) - 3H(-2 + \lambda)\dot{\phi} + 3\lambda\ddot{\phi})}{6(-1 + 2\lambda)\phi^2} \right] \ell. \quad (40)$$

For $\ell = a(t)$, we have

$$\frac{\dot{a}}{a} = \left[ \frac{(-1 + 3\lambda)\omega\phi^2 + \phi(8\pi(-2\rho + 3\lambda(p + \rho)) - 3H(-2 + \lambda)\dot{\phi} + 3\lambda\ddot{\phi})}{6(-1 + 2\lambda)\phi^2} \right] \frac{\dot{\phi}}{\phi}. \quad (41)$$

This is a particular case of the generalized Raychaudhuri equation given in [47]. In standard form of the modified Friedmann equations, the Raychaudhuri equation gives [14]

$$3H^2 = \frac{8\pi}{\phi} \left\{ \frac{(1 - 3\lambda)p}{2(1 - 2\lambda)} + \frac{3p(1 - \lambda)}{2(1 - 2\lambda)} \right\} + \omega \left[ \frac{2 - 3\lambda}{2(1 - 2\lambda)} \right] \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{3(1 - \lambda)}{2(1 - 2\lambda)} \frac{\dot{\phi}}{\phi}, \quad (42)$$

$$2\dot{H} + 3H^2 = -\frac{8\pi}{\phi} \left\{ \frac{\rho(1 - \lambda) - (1 + \lambda)p}{2(1 - 2\lambda)} \right\} + \omega \lambda \left[ \frac{2 - 3\lambda}{2(1 - 2\lambda)} \right] \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{1 + \lambda}{2(1 - 2\lambda)} \frac{\dot{\phi}}{\phi} + \frac{5 - \lambda}{2(1 - 2\lambda)} \frac{\dot{\phi}}{\phi} \quad (43)$$
C. GDE for null vector fields

Here, we assume the GDE for null vector fields past directed for which \( V^\alpha = k^\alpha \) and \( k_\alpha k^\alpha = 0 \). Under these conditions, Eq. (35) becomes

\[
R^\alpha_{\beta\gamma\delta} k^\beta \eta^\gamma k^\delta = \frac{\omega \dot{\phi}^2 + \phi (8\pi (p + \rho) - H\dot{\phi} + \ddot{\phi}) E^2 \eta^\alpha}{2\phi^2},
\]  

(44)

which can be described as the Ricci focusing in \( f(R) \) gravity. By choosing \( \eta^\alpha = \eta e^\alpha, e^\alpha e^\alpha = 1, e^\alpha u_\alpha = e^\alpha k^\alpha = 0 \) and aligned base parallel propagated

\[
\frac{D\eta}{D\nu} = k^\beta \nabla_{\beta} e^\alpha = 0,
\]

the GDE (36) turns out to be

\[
\frac{d^2 \eta}{d\nu^2} = -\frac{\omega \dot{\phi}^2 + \phi (8\pi (p + \rho) - H\dot{\phi} + \ddot{\phi}) E^2 \eta}{2\phi^2}.
\]

(45)

In the case of GR discussed in [4], all families of past-directed null geodesics experience focusing, provided \( \kappa (\rho + p) > 0 \), and for a fluid with equation of state \( p = -\rho \) (cosmological constant) there is no influence in the focusing [4]. From (45) the focusing condition for BDR gravity is

\[
\frac{\omega \dot{\phi}^2 + \phi (8\pi (p + \rho) - H\dot{\phi} + \ddot{\phi}) E^2 \eta}{2\phi^2} > 0.
\]

(46)

Eq. (45) can be written in terms of redshift parameter \( z \) with the help of following differential operators as

\[
\frac{d}{d\nu} = \frac{dz}{d\nu} \frac{d}{dz},
\]

(47)

\[
\frac{d^2}{d\nu^2} = \frac{dz}{d\nu} \frac{d}{dz} \left( \frac{d}{d\nu} \right) = \left( \frac{dz}{d\nu} \right)^{-2} \left[ -\left( \frac{dz}{d\nu} \right)^{-1} \frac{d^2\nu}{dz^2} \frac{d}{dz} + \frac{d^2}{dz^2} \right].
\]

(48)

For null geodesics, we have

\[
(1 + z) = \frac{a_0}{a} = \frac{E}{E_0} \rightarrow \frac{dz}{d\nu} = \frac{1}{E_0 H(1 + z)^2} \frac{d\nu}{dz} = \frac{1}{E_0 H(1 + z)^2}
\]

(50)

and

\[
\frac{d^2\nu}{dz^2} = -\frac{1}{E_0 H(1 + z)^3} \left[ \frac{1}{H(1 + z)} \frac{dH}{dz} + 2 \right].
\]

(51)

Also, we can write

\[
\frac{dH}{dz} = \frac{d\nu}{dz} \frac{dt}{d\nu} \frac{dH}{dt} = -\frac{1}{H(1 + z)} \frac{dH}{dt}.
\]

(52)

Here, minus sign is due to past directed geodesic, when \( z \) increases, \( \nu \) decreases. Also, \( \frac{dH}{d\nu} = E_0 (1 + z) \). The derivative of Hubble parameter \( H \) also yields

\[
\dot{H} = \frac{dH}{dt} = \frac{\dot{a}}{a} - H^2.
\]

(53)

The Raychaudhuri equation (41) yields

\[
\dot{H} = \left[ \frac{(-1 + 3\lambda)\omega \dot{\phi}^2 + \phi (8\pi (-2\rho + 3\lambda (p + \rho)) - 3H (-2 + \lambda)\dot{\phi} + 3\lambda \ddot{\phi})}{6(1 - 2\lambda)\phi^2} \right] - H^2.
\]

(54)
\[
\frac{d^2 \nu}{d z^2} = -\frac{3}{E_0 H (1+z)^3} \left[ 1 + \frac{1}{3 H^2} \left( \frac{(-1+3\lambda)\omega \phi^2 + \phi(8\pi(-2\rho + 3\lambda(p + \rho)) - 3H(-2 + \lambda)\phi + 3\lambda\dot{\phi}}{6(-1 + 2\lambda)\phi^2} \right) \right]. 
\] 

Finally, the operator \( \frac{d^2 \eta}{d \nu^2} \) is

\[
\frac{d^2 \eta}{d \nu^2} = (EH(1+z))^2 \left\{ \frac{d^2 \eta}{d \nu^2} + \frac{3}{(1+z)} \left[ 1 + \frac{1}{3 H^2} \left( \frac{(-1+3\lambda)\omega \phi^2 + \phi(8\pi(-2\rho + 3\lambda(p + \rho)) - 3H(-2 + \lambda)\phi + 3\lambda\dot{\phi}}{6(-1 + 2\lambda)\phi^2} \right) \right] \frac{d \eta}{d \nu} \right\},
\]

and the GDE (45) reduces to

\[
\frac{d^2 \eta}{d z^2} + \frac{3}{(1+z)} \left[ 18H^2(-1+2\lambda)\phi^2 + (-1+3\lambda)\omega \phi^2 + \phi(8\pi(-2\rho + 3\lambda(p + \rho)) - 3H(-2 + \lambda)\phi + 3\lambda\dot{\phi}) \right] \frac{d \eta}{d z} + \frac{\omega \dot{\phi}^2 + \phi(8\pi(p + \rho) - H\phi + \dot{\phi})}{2\phi^2 H^2(1+z)^2} \eta = 0.
\]

This equation is so complicated and is difficult to solve analytically. In order to solve it, it is possible to rewrite as a differential equation containing only the unknown \( H(z) \) and \( f(z) \) and their derivatives with respect to \( z \). By taking the contributions of radiation dominated era, then the energy density \( \rho \) and the pressure \( p \) turns out to be

\[
\frac{8\pi}{\phi} \rho = H_0^2 \Omega_{m0}(1+z)^3 + 3H_0^2 \Omega_{r0}(1+z)^4, \quad \frac{8\pi}{\phi} p = H_0^2 \Omega_{r0}(1+z)^4,
\]

where \( p = p_r = \frac{1}{3} \rho_r \). The GDE takes the following form

\[
\frac{d^2 \eta}{d z^2} + \mathcal{P}(H, R, z) \frac{d \eta}{d z} + Q(H, R, z) \eta = 0,
\]

with

\[
\mathcal{P}(H, R, z) = \frac{3}{(1+z)} \left\{ 18H_0^2 \left\{ \left( \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \right) \left[ \frac{1-3\lambda}{2(1-2\lambda)} \right] + \Omega_{DE} \right\} (-1 + 2\lambda)\phi^2 + (-1 + 3\lambda)\omega \phi^2 \right. \\
\left. + \frac{2\phi^2 H_0^2}{18H_0^2} \left\{ \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \right\} \left[ \frac{1-3\lambda}{2(1-2\lambda)} \right] + \Omega_{DE} \right\} (-1 + 2\lambda)\phi^2 \\
- \frac{3H(-2 + \lambda)\phi + 3\lambda\dot{\phi}}{18H_0^2} \left\{ \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \right\} \left[ \frac{1-3\lambda}{2(1-2\lambda)} \right] + \Omega_{DE} \right\} (-1 + 2\lambda)\phi^2 \\
\right. \\
- \frac{3H(-2 + \lambda)\phi + 3\lambda\dot{\phi}}{2\phi^2 H_0^2} \left\{ \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \right\} \left[ \frac{1-3\lambda}{2(1-2\lambda)} \right] + \Omega_{DE} \right\} (1+z)^2 \\
\right. \\
\left. Q(H, R, z) = \frac{\omega \dot{\phi}^2 + \left( 3H_0^2 \Omega_{m0}(1+z)^3 + 4H_0^2 \Omega_{r0}(1+z)^4 \right) - H\phi \dot{\phi} + \phi \ddot{\phi}}{2\phi^2 H_0^2} \left\{ \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \right\} \left[ \frac{1-3\lambda}{2(1-2\lambda)} \right] + \Omega_{DE} \right\} (1+z)^2 \\
\right.
\]

and \( H \) given by the modified field equations (42)

\[
H^2 = H_0^2 \left\{ \left( \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \right) \left[ \frac{1-3\lambda}{2(1-2\lambda)} \right] + \Omega_{DE} \right\}, 
\]

where

\[
\Omega_{DE} = \frac{1}{3H_0^2} \left\{ \Lambda + \omega \left[ \frac{2 - 3\lambda}{2(1-2\lambda)} \right] \left( \frac{\dot{\phi}}{\phi} \right)^2 + \left[ \frac{3(1-\lambda)}{2(1-2\lambda)} \right] \phi + \left[ \frac{3(1+\lambda)}{2(1-2\lambda)} \right] H \frac{\dot{\phi}}{\phi} + \left[ \frac{3(1-\lambda)}{2(1-2\lambda)} \right] p \right\}. 
\]
At present, we will verify the consistency of our results with those found in GR by taking the special case $\lambda \sim 1$, $\omega \sim \infty$ and $\phi \sim \frac{1}{c}$. Therefore $\Omega_{DE}$ in Eq.(63) reduces to

$$\Omega_{DE} = \frac{\Lambda}{3H_0^2} \equiv \Omega_\Lambda,$$  
(64)

which allows us to rewrite the first Friedmann equation in GR as following

$$H^2 = H_0^2[\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_\Lambda].$$  
(65)

Thus, the expressions $P$ and $Q$ become

$$P(z) = \frac{\frac{7}{2}\Omega_{m0}(1 + z)^3 + 4\Omega_{r0}(1 + z)^4 + 2\Omega_\Lambda}{(1 + z)[\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_\Lambda]},$$  
(66)

$$Q(z) = \frac{3\Omega_{m0}(1 + z) + 4\Omega_{r0}(1 + z)^2}{2(\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_\Lambda)}.$$  
(67)

Ultimately, the GDE for null vector fields becomes

$$\frac{d^2\eta}{dz^2} + \frac{\frac{7}{2}\Omega_{m0}(1 + z)^3 + 4\Omega_{r0}(1 + z)^4 + 2\Omega_\Lambda}{(1 + z)[\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_\Lambda]} \frac{d\eta}{dz} + \frac{3\Omega_{m0}(1 + z) + 4\Omega_{r0}(1 + z)^2}{2(\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_\Lambda)} \eta = 0.$$  
(68)

we note that to obtain the GDE for null vector fields in GR [48], we have used $\Omega_{r0} + \Omega_{m0} = 1$ which leads to

$$\frac{d^2\eta}{dz^2} + \frac{\frac{7}{2}\Omega_{m0}(1 + z)^3 + 4\Omega_{r0}(1 + z)^4}{(1 + z)[\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4]} \frac{d\eta}{dz} + \frac{3\Omega_{m0}(1 + z) + 4\Omega_{r0}(1 + z)^2}{2(\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4)} \eta = 0.$$  
(69)

### D. Mattig’s Equation

Mattig’s equation is one of the most important formulae in observational cosmology and extragalactic astronomy which gives relation between radial coordinate and redshift of a given source. It depends on the cosmological model which is being used and is needed to calculate luminosity distance in terms of redshift. In BDR, the Mattig’s equation can be interpreted by using Eq.(59) as follows:

$$r_0(z)_{BDR} = \sqrt{\left| \frac{dA_0(z)}{d\Omega} \right| = \left| \frac{\eta_{BDR}(z')}{d\eta_{BDR}(z')/dz'} \right|_{z'=0}},$$  
(70)

where $A_0$ is the area of the object and also $\Omega$ is the solid angle. This equation (70) could be interpreted as a generalization of the Mattig relation in BDR gravity because $\eta(z)$ is numerical solution which provide from (59). The equation (70) can also be used to generate the observer area distance $r_0(z)$ and the deviation vector $\eta(z)$. Analytical expression for the observable area distance for GR with no cosmological constant can be found in [48], whereas for more general scenarios numerical integration is usually required. Equipped with the $d/d\ell = E_0^{-1}(1 + z)^{-1}d/d\nu = H(1 + z)d/dz$ and setting the deviation at $z = 0$ to zero, clearly we have

$$r_0(z)_{BDR} = \left| \frac{\eta_{BDR}(z)}{H(0)d\eta_{BDR}(z')/dz'} \right|_{z'=0},$$  
(71)

where $H(0)$ is the evaluated modified Friedmann equation (64) at $z = 0$. In the next section, the numerical analysis will also be performed in order to find the deviation vector $\eta(z)$ and observer area distance $r_0(z)$ and compare the results of BDR Jordan frame with those of $\Lambda$CDM model.
E. Numerically solution of GDE for null vector fields in BDR gravity

In order to solve numerically the null vector GDE in BDR gravity, we consider the model \( \phi = \phi_0 t^{\sigma} \) [49] where \( \sigma \) is constant parameter and \( \phi_0 \) is positive quantity representing the present day values of the corresponding quantities. In agreement with this model, the equations (60), (61), (63) and (59) can be rewritten as follow

\[
\frac{d^2 \eta}{dz^2} + P \frac{d\eta}{dz} + Q \eta = 0, \quad (72)
\]

\[
\Omega_{DE} = \frac{1}{3H_0^2} \left\{ \Lambda + \omega \left[ \frac{2 - 3\lambda}{2(1 - 2\lambda)} \right] \left( \frac{d\phi}{\phi} \right)^2 + \left[ \frac{3(1 - \lambda)}{2(1 - 2\lambda)} \phi + \frac{3(1 + \lambda)}{2(1 - 2\lambda)} H \frac{d\phi}{\phi} + \frac{3(1 - \lambda)}{2(1 - 2\lambda)} \phi \right] \right\}, \quad (73)
\]

\[
P(H, \frac{dH}{dz}, z) = \frac{3}{(1 + z)} \left\{ \frac{18H_0^2}{\left[ \frac{18H_0^2}{\left( \Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 \right) + \Omega_{DE} \right]} \left( -1 + 2\lambda \right) \phi^2 + (-1 + 3\lambda) \omega \phi^2 \right\} \left( -1 + 2\lambda \phi^2 \right) \right\} \left( -1 + 2\lambda \phi^2 \right), \quad (74)
\]

\[
Q(H, \frac{dH}{dz}, z) = \frac{\omega \phi^2 + \left( 3H_0^2 \Omega_{m0}(1 + z)^3 + 4H_0^2 \Omega_{r0}(1 + z)^4 \right) - H \phi \dot{\phi} + \phi \ddot{\phi}}{2\phi^2 H_0^2 \left( \frac{\Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4}{2(1 - 2\lambda)} + \Omega_{DE} \right)} (1 + z)^2, \quad (75)
\]

where \( \dot{\phi} = \phi_0 \sigma t^{\sigma-1} \) and \( \ddot{\phi} = \phi_0 \sigma (\sigma - 1) t^{\sigma-2} \).

FIG. 3: The graphs shows the deviation vector magnitude \( \eta(z) \) (left panel) and observer area distance \( r_0(z) \) (right panel) for null vector field GDE with FLRW background as functions of redshift. The graphs are plotted for \( H_0 = 80Km/s/Mpc, \Omega_{m0} = 0.3, \Omega_{r0} = \Omega_{k0} = 0, \Lambda = 1.7 \times 10^{-121} \) and we imposed in equation (57) the initial conditions \( \eta(z = 0) = 0 \) and \( \eta'(z = 0) = 1 \).

We numerically solve the differential equations (69) (\( \Lambda CDM \)) and (72) (BDR) by plotting the deviation vector magnitude \( \eta \) and observer area distance \( r_0 \) as functions of redshift \( z \) (Fig. 3). For physical reasons, we will choose the
values of $\lambda$ as in [32, 33]. In each panel of (Fig. 3), we observe that the evolution of the deviation $\eta(z)$ and observer area distance $r_0(z)$ show similar behavior to those of $\Lambda$CDM. Within this model, we can see as the $\lambda \geq 1$ increases and when one goes to the highest values of the redshift ($z \leq 0.5$), the deviation $\eta(z)$ and observer area distance $r_0(z)$ decouple each of the model $\Lambda$CDM but still keeps the same place, while for the low values of the redshifts, i.e., for the current day, the BDR model reproduce exactly $\Lambda$CDM. We can conclude that the results are similar to $\Lambda$CDM for all the cases, which means that the above BDR model can be considered remain phenomenologically viable and tested with observational data. We can also observe for small value of redshift $0 < z < 0.5$, the magnitude of deviation vector $\eta$ presents the same behavior that of $\Lambda$CDM model ($\lambda = 1$). Also, the same behavior was observed for area distance $r_0$ at the same level. However, when considering large values of redshift $z > 0.5$, we find a gap between the BDR model ($\lambda \neq 1$) and the $\Lambda$CDM model. This indicates that the BDR model predicts a strong acceleration than $\Lambda$CDM model for large values of redshift.

V. CONCLUSIONS

In this paper, we have considered GDE in BDR gravity and calculated the Ricci tensor and Ricci scalar with the modified field equations in BDR gravity. Then, in FLRW universe, corresponding GDE for BDR gravity is obtained. We restricted our attention to extract the GDE for two special cases, namely the fundamental observers and past directed null vector fields. In these two cases, we have found the Raychaudhuri equation, GDE for null vector fields and the diametral angular distance differential for BDR gravity. We have also numerically computed the geodesic deviation ($\eta(z)$) and the area distance ($r_0(z)$) from Mattig’s relation for BDR gravity models and compared our results with those of $\Lambda$CDM model (Fig.3).

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