The Hartle-Hawking-Israel state on stationary black hole spacetimes

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Abstract. We consider a free quantized Klein-Gordon field in a spacetime \((M, g)\) containing a stationary black hole, more precisely a spacetime with a stationary bifurcate Killing horizon in the sense of Kay and Wald. We prove the existence of the Hartle-Hawking-Israel ground state, which is a pure state on the whole spacetime whose restriction to the exterior of the black hole is a thermal state at Hawking temperature \(T_H\).

We show that the HHI state is a Hadamard state and is the unique Hadamard extension of the above thermal state to the whole spacetime. We construct the HHI state by Wick rotation in Killing time coordinates, using the notion of the Calderón projector for elliptic boundary value problems.

1. Introduction

In this paper we consider a free quantized Klein-Gordon field in a spacetime \((M, g)\) containing a stationary black hole. It was conjectured by Hartle and Hawking [HH] and Israel [I] that a free Klein-Gordon quantum field admits a ground state \(\omega_{\text{HHI}}\), called the Hartle-Hawking-Israel state, whose restriction to the exterior region of the black hole is a thermal state at the Hawking temperature \(T_H = \kappa (2\pi)^{-1}\), where \(\kappa\) is the surface gravity of the black hole.

The physical motivation was that the stationary black hole spacetime \((M, g)\) describes the final state of the collapse of a massive object, and that the quantum Klein-Gordon field will eventually settle down to the ground state \(\omega_{\text{HHI}}\). The fact that \(\omega_{\text{HHI}}\) is a thermal state at Hawking temperature in the exterior of the black hole is then viewed as a justification of the Hawking radiation.

The first construction of the HHI state in the double wedge region of the Kruskal spacetime is due to Kay [K3]. This construction was valid for any temperature, the resulting state being an example of a double KMS state.

Later on Kay and Wald [KW] addressed the question of the extendability of the HHI state from the double wedge region to the black hole interior. In particular they introduced the definition of spacetimes with a bifurcate Killing horizon and gave a first rigorous definition of the notion of Hadamard states.

They proved that some subalgebra of the free field algebra admits at most one quasi-free state which is both invariant under the Killing isometries and Hadamard near the blackhole horizon. If such a state exists, Kay and Wald proved moreover that it is a thermal state at the Hawking temperature in the exterior region.

The first global construction of the HHI state in the whole spacetime is due to Sanders [S1], who considered spacetimes with a static bifurcate Killing horizon, ie such that the Killing vector field \(V\) is static in the exterior region. Sanders proved
in [S1] the existence of the HHI state and showed that it is a pure Hadamard state. The proof in [S1] relied on the Wick rotation in the Killing time coordinates, which was also the basis for the heuristic arguments in [HHI1] and which we will also use in this paper.

In [G] we gave another proof of the Hadamard property of the HHI state in the situation considered in [S1], by combining the Wick rotation with a tool which is familiar in elliptic boundary value problems, namely the Calderón projectors, see [1.2.2]. The use of Calderón projectors allows to construct the HHI state directly on a Killing surface $\Sigma$ and avoids to consider its behavior near the Killing horizon. In collaboration with Michal Wrochna, we have recently used Calderón projectors in [GW2] to construct analytic Hadamard states on general analytic spacetimes.

In the present paper we consider the more general stationary case, and give a construction of the HHI state for spacetimes with a stationary bifurcate Killing horizon.

1.1. Results. We now present more in detail the result of this paper.

1.1.1. Bifurcate Killing horizons. Let $(M, g)$ a globally hyperbolic spacetime with a complete Killing vector field $V$. $(M, g)$ admits a bifurcate Killing horizon [KW], if the bifurcation surface $B = \{x \in M : V(x) = 0\}$ is a compact, connected, orientable submanifold of codimension 2 and if there exists a Cauchy surface $\Sigma$ containing $B$. $M$ splits then into four globally hyperbolic regions, the right/left wedges $M^+, M^-$ and the future/past cones $F, P$, each invariant under the flow of $V$.

The Killing horizon is then $H = \partial(F \cup P)$. An important object related with the Killing horizon is its surface gravity $\kappa$, which is a scalar, constant over all of $H$.

One also assumes the existence of a wedge reflection $R : M \rightarrow M$ which is an isometry of $(M^+ \cup U \cup M^+, g)$, where $U$ is a neighborhood of $B$ in $M$, such that $R \circ R = 1d$, $R = 1d$ on $B$, $R$ reverses the time orientation and $R^*V = V$. In concrete situations, the left wedge $M^-$ is actually constructed by reflection of the right wedge $M^+$, so the existence of a wedge reflection does not seem to be such a strong hypothesis.

The bifurcate Killing horizon $H$ is stationary resp. static if $V$ is time-like on $\Sigma \setminus B$, resp. orthogonal to $\Sigma \setminus B$. For technical reasons, we require $V$ to be uniformly time-like near infinity on $\Sigma$, see Subsect. 2.3. This condition is imposed only far away from the bifurcation surface $B$ and will hold for example if $(M, g)$ is asymptotically flat near spatial infinity.

We consider on $(M, g)$ a free quantum Klein-Gordon field associated to the Klein-Gordon equation

$$-\Box_g \phi(x) + m(x)\phi(x) = 0,$$

where $m \in C^\infty(M, \mathbb{R})$ is invariant under $V$ and $R$. We assume that $m(x) \geq m_0^2 > 0$ ie the Klein-Gordon field is massive.

1.1.2. The double $\beta$-KMS state. Since $(M^+, g, V)$ is a stationary spacetime, there exists (see [S2]) for any $\beta > 0$ a thermal state $\omega_\beta$ at temperature $\beta^{-1}$ with respect to the group of Killing isometries of $(M^+, g)$ generated by $V$.

The wedge reflection $R : M^+ \rightarrow M^-$ allows to extend $\omega_\beta$ to the double $\beta$-KMS state $\omega_D$ on $M^+ \cup M^-$. This extension exists for any $\beta > 0$ and is a pure state in $M^+ \cup M^-$. We prove in this paper the following theorem.

**Theorem 1.1.** Let $(M, g, V)$ be a globally hyperbolic spacetime with a stationary bifurcate Killing horizon and a wedge reflection. Let $P = -\Box_g + V^2$ a Klein-Gordon operator invariant under the Killing vector field $V$ and the wedge reflection $R$. Assume moreover that conditions (H) in Subsect. 2.3 are satisfied.
Then there exists a state $\omega_{\text{HHI}}$ for $P$ in $(M, g)$ called the Hartle-Hawking-Israel state such that:

1. $\omega_{\text{HHI}}$ is a pure Hadamard state in $M$,
2. the restriction of $\omega_{\text{HHI}}$ to $M^+ \cup M^-$ is the double $\beta$-KMS state $\omega_{\beta}$ at Hawking temperature $T_{\text{HHI}} = \kappa(2\pi)^{-1}$ where $\kappa$ is the surface gravity of the horizon,
3. $\omega_{\text{HHI}}$ is the unique extension of $\omega_{\beta}$ such that its spacetime covariances $\Lambda^\pm$ map $C_0^\infty(M)$ into $C_0^\infty(M)$ continuously. In particular it is the unique Hadamard extension of $\omega_{\beta}$.

Thm. [1.1] will be proved in Sect. [1]

1.2. Main ideas of the construction. We now outline the construction of the HHI state $\omega_{\text{HHI}}$. We look for $\omega_{\text{HHI}}$ as an extension to $M$ of the double $\beta$-KMS state $\omega_{\beta}$ on $M^- \cup M^+$, where $\beta^{-1} = \kappa(2\pi)^{-1}$ is the Hawking temperature. The first step consists in understanding in sufficient details the $\beta$-KMS state in $M^+$.

Writing the metric $g$ in $M^+$ using the Killing time coordinate associated to $V$ and $\Sigma$, $M^+$ is identified with $\mathbb{R} \times \Sigma^+$ and the metric $g$ becomes

$$g = -N^2(y)dt^2 + h_{ij}(y)(dy^i + w^i(y)dt)(dy^j + w^j(y)dt),$$

where $N$ is the lapse function, $w$ the shift vector field, $h$ the induced metric on $\Sigma$. The Killing field $V$ is simply $\frac{\partial}{\partial t}$. The fact that $V$ is time-like in $M^+$ is equivalent to the inequality $N^2(y) > w^i(y)h_{ij}(y)w^j(y)$ for $y \in \Sigma^+$.

The Klein-Gordon operator $P$ associated to $g$ can be written as:

$$P = P = (\partial_t + w^*)N^{-2}(\partial_t - w) + h_0,$$

where $w = w^i \cdot P_{yi}$ and $h_0 = \nabla^* h^{-1} \nabla + m$ is an elliptic operator on $\Sigma$.

1.2.1. The Wick rotation. The Wick rotation consists in replacing $t$ by $i$ and produces the complex metric

$$g^{\text{eucl}} = N^2(y)ds^2 + h_{ij}(y)(dy^i + iw^i(y)ds)(dy^j + iw^j(y)ds).$$

In the static case considered in [S1] [G] $w$ vanishes and $g^{\text{eucl}}$ is Riemannian. The fact that $g^{\text{eucl}}$ is now a complex metric causes several new difficulties. Performing the same transformation on $P$ yields the Wick rotated operator

$$K = -(\partial_s + iw^*)N^{-2}(\partial_s + iw) + h_0.$$

There are several different linear operators that can be associated to the formal expression $K$. The first one consists in working on $L^2(\mathbb{R} \times \Sigma^+)$, using the sesquilinear form

$$Q_\infty(u, u) = \|N^{-1}\partial_s u\|^2 + (u/hu) - i(N^{-1}\partial_s u|N^{-1}wu) - i(N^{-1}wu|N^{-1}\partial_s u),$$

where $h = h_0 - w^*N^{-2}w$, with $\text{Dom}Q_\infty = C_0^\infty(\mathbb{R} \times \Sigma)$. Another possibility is to work on $L^2(S_\beta \times \Sigma^+)$ where $S_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}]$ is the circle of length $\beta$. The sesquilinear form $Q_\beta$ has the same expression as $Q_\infty$ but the domain is now $\text{Dom}Q_\beta = C_0^\infty(S_\beta \times \Sigma)$, which corresponds to imposing $\beta$-periodic boundary conditions on $K$.

Since we have assumed that $V$ is uniformly time-like near infinity, see Subsect. [2.4] one can show that the sesquilinear forms $Q_\infty, Q_\beta$ are closeable and sectorial and hence generate injective linear operators $K_\infty, K_\beta$. Their inverses $K_\infty^{-1}, K_\beta^{-1}$ are then well defined between abstract Sobolev spaces, using the Lax-Milgram theorem.
1.2.2. Calderón projectors. Let $\Omega_\infty = ]0, +\infty[ \times \Sigma^+, \Omega_\beta = [0, \frac{\beta}{2}] \times \Sigma^+$ and $\nu$ the exterior unit normal for $g^{\text{encl}}$ to $\partial \Omega_\beta$, $\beta \in [0, +\infty]$. Note that $\nu$ is a complex vector field, but its imaginary part is tangent to $\partial \Omega_\beta$.

For $u \in \mathcal{C}_0^\infty(\Omega_\beta)$ such that $K_\beta u = 0$ in $\Omega_\beta$, the trace $\gamma_\beta u$ of $u$ on $\partial \Omega_\beta$ defined as

$$\gamma_\beta u = \left( u|_{\partial \Omega_\beta}, \frac{\partial u}{\partial \nu}|_{\partial \Omega_\beta} \right)$$

is not arbitrary, because $K_\beta$ is an elliptic operator. Instead $\gamma_\beta u$ belongs to the range of a projector $c_\beta^\pm$, called the Calderón projector associated to $\Omega_\beta$. The same construction with $\Omega_\beta$ replaced by its complement produces the complementary Calderón projector $c_\beta^\mp$, with $c_\beta^+ + c_\beta^- = 1$.

The projectors $c_\beta^\pm$ can be explicitly expressed in terms of the inverse $K_\beta^{-1}$, see Subsect. 8.7.

1.2.3. Vacuum and double $\beta$-KMS states. If $\beta = \infty$, the boundary $\partial \Omega_\infty$ equals $\Sigma^+$, and one can try to construct a state in $\mathcal{M}^+$ by defining its covariances on $\Sigma^+$ as

$$\lambda^\pm_\beta = \pm q \circ c_\infty^\pm,$$

where $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the charge defining the symplectic structure on the space $C_0^\infty(\Sigma^+; \mathbb{C}^2)$ of Cauchy data on $\Sigma^+$. It turns out that $\lambda^\pm_\beta$ are actually the covariances of the vacuum state $\omega_\text{vac}$ in $\mathcal{M}^+$.

Of course the study of the vacuum state $\omega_\text{vac}$, corresponding to $\beta = \infty$, is not necessary for the construction of the HHI state, but gives a nice introduction to the more complicated case $\beta < \infty$.

If $\beta < \infty$, the boundary $\partial \Omega_\beta$ has two components, both isomorphic to $\Sigma^+$. The state $\omega_\text{D}$ obtained similarly from the Calderón projectors $c_\beta^\pm$ is now the double $\beta$-KMS state $\omega_\text{D}$ in $\mathcal{M}_0^+ \cup \mathcal{M}^+$, modulo the identification of $\Sigma^+$ with $\Sigma^-$ by the wedge reflection.

The proof of these facts takes up a large part of the paper. First of all we reduce ourselves to the situation $N(y) = 1$ by considering $P = NP\bar{N}$ and $\bar{K}_\beta = NK_\beta\bar{N}$, the last identity taking a rather transparent form if we use the framework of sesquilinear forms, see Subsect. 8.6. The covariances of $\omega_\text{vac}$, $\omega_\text{D}$ for the Klein-Gordon operator $P$ can similarly be deduced from those of the analogous states $\bar{\omega}_\text{vac}$, $\bar{\omega}_\text{D}$ for $\bar{P}$.

The operator $\bar{P}$ can be written as $(\partial_t + \bar{w}^*)(\partial_t - \bar{w}) + \bar{h}_0$, and the computations of $\bar{\omega}_\text{vac}$, $\bar{\omega}_\text{D}$ can be done by reducing the Klein-Gordon equation $\bar{P}\phi = 0$ to a first order system $\partial_t f - iHf = 0$, see Sects. 4-5. This system is an example of a stable symplectic dynamics, which is studied in Sects. 4-5.

1.2.4. The surface gravity and the extended Euclidean metric. All the constructions up to now are valid for any value of the inverse temperature $\beta$. The metrics $g$ and $g^{\text{encl}}$ are degenerate at the bifurcation surface $\mathcal{B} = \partial \Sigma^+$.

If $\beta = (2\pi)\kappa^{-1}$, ie if $\beta^{-1}$ equals the Hawking temperature $\kappa(2\pi)^{-1}$, where $\kappa$ is the surface gravity of the horizon, one can show that $(\Sigma_\beta \times \Sigma^+, g^{\text{encl}})$ has a unique extension $(M^{\text{ext}}_\beta, g^{\text{ext}}_\beta)$, which corresponds exactly to passing from polar to cartesian coordinates in the plane.

1.2.5. The Hartle-Hawking-Israel state. The open set $]0, \frac{\beta}{2}[ \times \Sigma^+$ extends as an open set $\Omega_\text{ext}$ with boundary isomorphic to the full Cauchy surface $\Sigma$. The Wick rotated operator $K_\beta$ extends as an elliptic operator $K_\text{ext}$ acting on $M^{\text{ext}}_\beta$, and one can consider the Calderón projectors $c_\text{ext}^\pm$ associated to $K_\text{ext}$ and $\Omega_\text{ext}$.
One defines the covariances on $\Sigma$
\[
\lambda_{\text{HHI}}^\pm = \pm q \circ c_{\text{ext}}^\pm,
\]
and one can rather easily show that $\lambda_{\text{HHI}}^\pm$ are the covariances of a pure quasi-free state $\omega_{\text{HHI}}$ defined on the whole of $M$. One uses that the restriction of $\lambda_{\text{HHI}}^\pm$ to $C_0^\infty(\Sigma \setminus B)$ are precisely the covariances of the double $\beta$-KMS state $\omega_D$, and some continuity properties of Calderón projectors and density results in Sobolev spaces, see Subsect. 9.2.

One can also prove that the HHI state $\omega_{\text{HHI}}$ is a Hadamard state, by an argument already used in [G] in the static case, relying on the fact that the covariances of any Hadamard state on $\Sigma$ are matrices of pseudodifferential operators.

1.3. Notations. We now collect some notation.

We set $\langle \lambda \rangle = (1 + \lambda^2)^{\frac{1}{2}}$ for $\lambda \in \mathbb{R}$.

We write $A \in B$ if $A$ is relatively compact in $B$.

If $X,Y$ are sets and $f : X \to Y$ we write $f : X \to Y$ if $f$ is bijective. If $X,Y$ are equipped with topologies, we write $f : X \to Y$ if the map is continuous, and $f : X \to Y$ if it is a homeomorphism.

1.3.1. Duals and antiduals. Let $X$ be a real vector space. Its dual will be denoted by $X^\ast$. Let $Y$ be a complex vector space. We denote by $Y_\mathbb{R}$ its real form, ie $Y$ as a vector space over $\mathbb{R}$. We denote by $Y^\mathbb{R}$ its dual, ie the space of $\mathbb{C}$–linear forms on $Y$ and by $Y^\ast$ its anti-dua1, ie the space of $\mathbb{C}$–antilinear forms on $Y$.

We denote by $\overline{Y}$ the conjugate vector space to $Y$, ie $\overline{Y} = Y_\mathbb{R}$ as a $\mathbb{R}$–vector space, equipped with the complex structure $-i$, if $i \in L(Y_\mathbb{R})$ is the complex structure of $Y$. The identity map $Id : Y \to \overline{Y}$ will be denoted by $y \mapsto \overline{y}$, ie $\overline{y}$ equals $y$ but considered as an element of $\overline{Y}$.

If $Y$ is a Hilbert space, then $\overline{Y}$ inherits also a Hilbert space structure by
\[
(\overline{g_1}, \overline{g_2})_{\overline{Y}} := (g_1 | g_2)_Y.
\]

By definition we have $Y^\ast = \overline{Y^\mathbb{R}}$. Note that we have a $\mathbb{C}$–linear identification $\overline{Y^\mathbb{R}} \sim \overline{Y^\ast}$ defined as follows: if $y \in \overline{Y}$ and $w \in Y^\mathbb{R}$ then
\[
\overline{w} \cdot y := \overline{w} \cdot \overline{y}
\]
This identifies $\overline{w} \in \overline{Y^\mathbb{R}}$ with an element of $\overline{Y^\ast}$. Similarly we have a $\mathbb{C}$–linear identification $\overline{Y}^\ast \sim \overline{Y^\mathbb{R}}$.

1.3.2. Linear operators. If $X_i$, $i = 1,2$ are real or complex vector spaces and $a \in L(X_1, X_2)$ we denote by $a^\ast \in L(X_2^\ast, X_1^\ast)$ its transpose. If $Y_i$, $i = 1,2$ are complex vector spaces we denote by $a^* \in L(Y_2^\ast, Y_1^\ast)$ its adjoint, and by $\overline{a} \in L(\overline{Y}_1, \overline{Y}_2)$ its conjugate, defined by $\overline{w} \cdot \overline{y} = \overline{w}^a \cdot \overline{y}$. With the above identifications we have $a^* = \overline{a^T} = \overline{a^T}$.

1.3.3. Bilinear and sesquilinear forms. If $X$ is a real or complex vector space, a bilinear form on $X$ is given by $a \in L(X, X^\ast)$, its action on a couple $(x_1, x_2)$ is denoted by $x_1 a x_2$. We denote by $L_{\mathbb{R}/\mathbb{C}}(X, X^\ast)$ the symmetric/antisymmetric forms on $X$. $a$ is non-degenerate if Ker$a = \{0\}$. An antisymmetric, non-degenerate form $\sigma$ is called a symplectic form on $X$.

Similarly if $Y$ is a complex vector space, a sesquilinear form on $Y$ is given by $a \in L(Y, Y^\ast)$, its action on a couple $(y_1, y_2)$ is denoted by $\overline{g_1} a g_2$, the last notation being a reminder that $Y^\ast = \overline{Y}$. We denote by $L_{\mathbb{R}/\mathbb{C}}(Y, Y^\ast)$ the Hermitian/antiHermitian forms on $Y$. Non-degenerate forms are defined as in the real case. An antiHermitian, non-degenerate form $\sigma$ is called a (complex) symplectic form on $Y$. 

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If $a \in L(Y, Y^*)$ then $\overline{a} \in L(Y, \overline{Y}^*)$ and with the above identifications we have $(\overline{a} | \overline{y}_2) = (y_1 | ay_2)$ for $y_1, y_2 \in Y$.

1.3.4. **Linear operators on Hilbert spaces.** The domain of a closed, densely defined operator $a$ on a Hilbert space $H$ will be denoted by $\text{Dom}a$, equipped with the graph norm, its spectrum by $\sigma(a)$ and its resolvent set by $\text{res}(a)$. We will similarly denote by $\text{Dom}Q$ the domain of a sesquilinear form $Q$. If $Q$ is closeable we denote by $Q^{\infty}$ its closure.

If $a$ is selfadjoint on $H$, we write $a > 0$ if $a \geq 0$ and $\text{Kera} = \{0\}$. If $a > 0$ and $s \in \mathbb{R}$ we denote by $a^s H$ the completion of $\text{Dom}a^{-s}$ for the norm $\|a^{-s}u\|_H$.

Equipped with the scalar product $(u|v)_s = (a^{-s}u|a^{-s}v)_H$, it is a Hilbert space. The spaces $a^s H$ and $a^{-s} H$ form a dual pair for the duality pairing $(u|v)_s = (a^{-s}u|a^s v)_H$.

We define similarly the spaces $(a)^s H$ for any selfadjoint operator on $H$. We have $(a)^{-s} H = \text{Dom}|a|^s$ for $s > 0$. We have $(a)^{-s} H \subset c H \subset (a)^s H$ for $s \geq 0$ and $(a)^s : c H = a^s H$ if $0 \notin \sigma(a)$.

If $a_1, a_2$ are selfadjoint on $H$ with $a_1, a_2 > 0$ we write $a_1 \lesssim a_2$ if $\text{Dom}a_1^{\frac{1}{2}} \supset \text{Dom}a_2^{\frac{1}{2}}$ and $a_1 \lesssim ca_2$ on $\text{Dom}a_2^{\frac{1}{2}}$ for some $c > 0$. We write $a_1 \sim a_2$ if $a_1 \lesssim a_2$ and $a_2 \lesssim a_1$.

If $a_1 \sim a_2$ the Kato-Heinz theorem implies that $a_2^{-1} \sim a_1^{-1}$ and that $a_1^2 H = a_2^2 H$ as Banach spaces for $s \in [-\frac{1}{2}, \frac{1}{2}]$.

1.3.5. **Quadratic forms.** Similarly if $q_1, q_2$ are two positive quadratic forms with $q_i(u, u) = 0 \Rightarrow u = 0$, we write $q_1 \lesssim q_2$ if $\text{Dom}q_1 \supset \text{Dom}q_2$ and $q_1 \lesssim cq_2$ on $\text{Dom}q_2$ and we write $q_1 \sim q_2$ if $q_1 \lesssim q_2$ and $q_2 \lesssim q_1$.

1.3.6. **Differential operators on manifolds.** If $X$ is a smooth manifold and $a, b$ are differential operators on $X$ the composition $ab$ is denoted by $ab$. If $a$ is a differential operator on $X$ and $u \in C^\infty(X)$, then $au$ denotes the composition of $a$ with the operator of multiplication by $u$, while $(au) \in C^\infty(X)$ denotes the image of $u$ under $a$.

1.3.7. **Spaces of distributions.** Let $X$ a smooth manifold. Fixing a smooth density we identify distributions and distributional densities on $X$. If $\Omega \subset X$ is an open set with smooth boundary and $F(X) \subset \mathcal{D}'(X)$ is a vector space, we denote by $\overline{F}(\Omega) \subset \mathcal{D}'(\Omega)$ the space of restrictions of elements of $F(X)$ to $\Omega$.

Any $u \in \overline{F}(\Omega)$ has a unique extension $cu \in \mathcal{D}'(X)$ with $\text{supp} cu \subset \overline{\Omega}$.

We denote by $\delta_a \in \mathcal{D}'(\mathbb{R})$ the Dirac distribution at $a \in \mathbb{R}$.

2. **Spacetimes with a stationary bifurcate Killing horizon**

In this section we recall the definition of spacetimes with stationary Killing horizons, following [KW] [ST]. We express various natural objects, like the lapse function, shift vector field and induced Riemannian metric in Gaussian coordinates near the bifurcation surface.

We then consider the \textit{Wick rotated metric} $g^{\text{eucl}}$, obtained by the Wick rotation $t \rightarrow$ is in the Killing time $t$, and show that if $s$ belongs to the circle $S_{(2\pi)^{-1}}$ of length $(2\pi)^{-1}$, for $\kappa$ the surface gravity of the horizon, $g^{\text{eucl}}$ has a smooth extension up the the bifurcation surface $\mathcal{B}$. This fundamental fact, already known for static horizons, see [ST] Sect. 2.2] lies at the basis of the construction of the HHI state in later sections.

2.1. **Bifurcate Killing horizons.**

\begin{definition}
A spacetime with a bifurcate Killing horizon is a triple $(M, g, V)$ such that:
\begin{enumerate}
\item $(M, g)$ is a globally hyperbolic spacetime,
\end{enumerate}
\end{definition}
(2) \( V \) is a smooth, complete Killing vector field on \((M, g)\),
(3) \( B := \{ x \in M : V(x) = 0 \} \) is a compact, connected, orientable submanifold of codimension 2, called the bifurcation surface,
(4) there exists a smooth, space-like Cauchy hypersurface \( \Sigma \) with \( B \subset \Sigma \).

If \( n \) is the future directed normal vector field to \( \Sigma \), one defines the lapse function \( N \in C^\infty(\Sigma) \) and shift vector field \( w \), which is a smooth vector field tangent to \( \Sigma \), by
\[
V = Nn + w \text{ on } \Sigma,
\]
ie
\[
N := -V \cdot gn, \quad w := V - Nn \text{ on } \Sigma.
\]

Let us denote by \( y \) the elements of \( \Sigma \). The Cauchy surface \( \Sigma \) is then decomposed
\[
\Sigma = \Sigma^- \cup B \cup \Sigma^+, \quad \Sigma^\pm := \{ y \in \Sigma : \pm N(y) > 0 \},
\]
ie \( V \) is future/past directed over \( \Sigma^\pm \).

The spacetime \( M \) splits as
\[
M = M^+ \cup M^- \cup \mathcal{F} \cup \mathcal{P},
\]
where the future cone \( \mathcal{F} := I^+(B) \), the past cone \( \mathcal{P} := I^-(B) \), the right/left wedges \( M^\pm := D(\Sigma^\pm) \), are all globally hyperbolic when equipped with \( g \).

The future cone \( \mathcal{F} \) may be a black hole. The bifurcate Killing horizon is then
\[
\mathcal{H} := \partial \mathcal{F} \cup \partial \mathcal{P}.
\]
The Killing vector field \( V \) is tangent to \( \mathcal{H} \). In Figure 1 below the vector field \( V \) is represented by arrows.

![Figure 1](image_url)

**Definition 2.2.** A triple \((M, g, V)\) as in Def. 2.1 is called a spacetime with a stationary, resp. static bifurcate Killing horizon if \( V \) is time-like on \( \Sigma \setminus B \), resp. \( g \)-orthogonal to \( \Sigma \setminus B \).

2.2. **Wedge reflection.** Additionally one assumes the existence of a wedge reflection, see [S1] Def. 2.6.

**Definition 2.3.** A wedge reflection \( R \) for a spacetime \((M, g, V)\) with a stationary Killing horizon is a diffeomorphism \( R : M^- \cup U \cup M^+ \to M^- \cup U \cup M^+ \), where \( U \) is a neighborhood of \( B \) in \( M \) such that:
1. \( R \) is an isometry of \((M^- \cup U \cup M^+, g)\) which reverses the time orientation,
2. \( R \circ R = Id, R = Id \) on \( B \),
3. \( R^*V = V \).
2.2.1. \textit{Weak wedge reflection.} It is known, see [S1, Prop. 2.7] that if \( R \) is a wedge reflection, one can find a Cauchy surface \( \Sigma \) as in Def. 2.1 such that \( R : \Sigma \xrightarrow{\sim} \Sigma \). The map \( r := R|_{\Sigma} \) is called a \textit{weak wedge reflection}. If the Riemannian metric \( h \) is the restriction of \( g \) to \( \Sigma \), one has:

(1) \( r \) is an isometry of \( (\Sigma, h) \) with \( r \circ r = \text{Id} \),
(2) \( r = \text{Id} \) on \( B \),
(3) \( r^* N = -N, \ r^* w = w \).

By (3) above we have \( r : \Sigma^+ \xrightarrow{\sim} \Sigma^- \).

2.3. \textit{Klein-Gordon operators.} We fix a real function \( m \in C^\infty(M) \). As in [S1] we assume that \( m \) is stationary w.r.t. the Killing vector field \( V \) and invariant under the wedge reflection, ie:

\[
V^a \nabla_a m(x) = 0, \ m \circ R(x) = m(x), \ x \in M^+ \cup M^- \cup U.
\]

We also assume that

\[
m(x) \geq m_0^2 > 0, \ x \in M,
\]

ie we consider only \textit{massive} Klein-Gordon fields. The \textit{Klein-Gordon operator} is

\[
P = -\square_g + m.
\]

2.4. \textbf{Conditions near infinity on} \( \Sigma \). It will be necessary, in order to control various energy spaces in Sect. 8, to impose conditions on the Killing vector field \( V \) near infinity on \( \Sigma \).

\[
\exists \ U \ \text{neighborhood of} \ B \ \text{in} \ \Sigma \ \text{such that}:
\]

(\text{H1}) \( V + \delta w \) is time-like on \( \Sigma \setminus U \) for some \( \delta > 0 \),
(\text{H2}) \( N^{-2} w^i (\nabla^i N), \ N^{-1} \nabla^i w^i \) are bounded on \( \Sigma \setminus U \).

(\text{H1}) means that \( V \) is \textit{uniformly time-like} near infinity on \( \Sigma \). Conditions (\text{H}) are clearly satisfied if \( (M, g) \) is for example asymptotic to the Kerr spacetime, near spatial infinity.

2.5. \textbf{The surface gravity}. The surface gravity is defined by:

\[
\kappa^2 = -\frac{1}{2} (\nabla^b g) V^a (\nabla_b V_a)|_B, \ \kappa > 0.
\]

It is a fundamental fact, see [KW Sect. 2], that \( \kappa \) is constant on \( B \) and actually on the whole horizon \( \mathcal{H} \).

For \( \omega \in B \) let \( n_\omega \in T_\omega \Sigma \) the unit normal to \( B \) for \( h \) pointing towards \( \Sigma^+ \). We introduce Gaussian normal coordinates to \( B \) in \( (\Sigma, h) \) by:

\[
\chi : \quad [-\delta, \delta] \times B \to \Sigma
\]

\[
\chi : \quad (u, \omega) \mapsto \exp^h_{\omega}(un)
\]

which is a smooth diffeomorphism from \([-\delta, \delta] \times B\) to a relatively compact neighborhood \( U \) of \( B \) in \( \Sigma \). In the next proposition we express \( h, N, w \) and the wedge reflection \( r \) in the local coordinates \((u, \omega)\) on \( U \). We recall that the elements of \( \Sigma \) are denoted by \( y \).

\textbf{Proposition 2.4.} On \( U \) one has:

\[
r(u, \omega) = (-u, \omega),
\]
and
\[ h_{ij}(y)dy^i dy^j = du^2 + k_{\alpha\beta}(u, \omega) d\omega^\alpha d\omega^\beta, \]
\[ w^i(y) \partial_{y^i} = w^0(u, \omega) \partial_u + w^\alpha(u, \omega) \partial_{y^\alpha}, \]
\[ N(y) = N(u, \omega), \]
\[ m(y) = m(u, \omega), \]

where \( k_{\alpha\beta}(u, \omega) d\omega^\alpha d\omega^\beta \) is a smooth, \( u \)-dependent Riemannian metric on \( B \) with:
\[ N(u, \omega) = u(\kappa + u^2 d(u^2, \omega)), \]
\[ w^0(u, \omega) = u^0 b(u^2, \omega), \]
\[ w^\alpha(u, \omega) = u^2 c^\alpha(u^2, \omega), \]
\[ k_{\alpha\beta}(u, \omega) = d_{\alpha\beta}(u^2, \omega), \]
\[ m(u, \omega) = n(u^2, \omega), \]

for smooth functions \( b, d, n, c^\alpha, d_{\alpha\beta} \) and \( \epsilon \in \mathbb{R} \times B \rightarrow \mathbb{R} \) with
\[ n(0, \omega) \geq c > 0, \quad c^{-1} \mathbb{I} \leq [d_{\alpha\beta}(0, \omega)] \leq c \mathbb{I}, \quad \text{for some } c > 0. \]

The proof of Prop. 2.4 is given in Appendix A.1.

2.6. The metric in \( M^+ \). Let us denote by \( \Phi_t \) the flow of the Killing vector field \( V \). We identify \( \mathbb{R} \times \Sigma^+ \) with \( M^+ \) by
\[ \chi: \mathbb{R} \times \Sigma^+ \ni (t, y) \mapsto \Phi_t(y) \in M^+. \]

We have \( \chi^* V = \partial / \partial t \) and
\[ \chi^* g = -N^2(y) dt^2 + (dy^i + w^i(y) dt) h_{ij}(y)(dy^j + w^j(y) dt) = -N^2(y) dt^2 + w_i(y) dy^i dt + w_j(y) dt dy^j + h_{ij}(y) (dy^i dy^j), \]
for \( v^2(y) = (N^2(y) - w^i(y) h_{ij}(y) w^j(y)) \). Note that The fact that \( V \) is time-like in \( M^+ \) is equivalent to
\[ N^2(y) > w^i(y) h_{ij}(y) w^j(y), \]
\[ y \in \Sigma^+. \]
The unit normal vector field to the foliation \( \Sigma_t = \{ t \} \times \Sigma \) is
\[ n = N^{-1} (\partial / \partial t - \omega), \]
Denoting \( \chi^* g \) on \( \mathbb{R} \times \Sigma^+ \) simply by \( g \), we have \( |g| = N^2 |h| \) and
\[ g^{-1} = -N^{-2} \partial_t^2 + N^{-2} (w^i \partial_{y^i} \partial_t + w^i \partial_{y^i} \partial_{y^i}) + (h^{ij} - N^{-2} w^i w^j) \partial_{y^i} \partial_{y^j}. \]

Since the potential \( m \) is invariant under the Killing vector field, we have \( m = m(y) \).

2.7. The Wick rotated metric.

2.7.1. Complex metrics. If \( X \) is a smooth manifold, we denote by \( T^d_q(X) \) the space of smooth, real \( (p, q) \) tensors on \( X \) and by \( CT^d_q(X) \) its complexification. An element \( k = k_{ab}(x) dx^a dx^b \) of \( CT^d_q(X) \) which is symmetric and non-degenerate will be called a complex metric on \( X \).
2.7.2. The Wick rotated metric. We denote by $S_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}]$ with endpoints identified the circle of length $\beta$ and

$$M^{eucl} := S_\beta \times \Sigma^+,$$

with variables $(s, y)$. Replacing $t$ by $is$ we obtain the complex metric on $M^{eucl}$:

$$g^{eucl} = N^2(y) ds^2 + (dy^i + i\omega^i(y) ds) h_{jk}(y)(dy^j + i\omega^j(y) ds + i\omega^j(y) ds dy^j + h_{jk}(y) dy^j dy^k).$$

We embed $\Sigma \setminus B$ into $M^{cucl} = S_\beta \times \Sigma^+$ by the map

$$\tilde{\psi}: y \mapsto (0, y) \text{ for } y \in \Sigma^+, \quad \left(\beta, r(y)\right) \text{ for } y \in \Sigma^-,$$

where $r : \Sigma \to \Sigma$ is the weak wedge reflection.

2.8. The smooth extension.

**Proposition 2.5.** Assume that $\beta = (2\pi)^{-1}$. Then there exists a smooth manifold $M^{cucl}_{ext}$ equipped with a smooth complex metric $g^{cucl}_{ext}$ and

1. a smooth embedding $\psi : \Sigma \to M^{cucl}_{ext}$,
2. a smooth isometric embedding $\chi : (M^{cucl}_{ext}, g^{cucl}_{ext}) \to (M^{cucl}_{ext} \setminus B_{ext}, g^{cucl}_{ext})$, where $B_{ext} = \psi(B)$,
3. an open set $\Omega_{ext}$ such that $\partial\Omega_{ext} = \psi(\Sigma)$ and $\chi : [0, \frac{\beta}{2}] \times \Sigma^+ \sim \Omega_{ext} \setminus B_{ext}$,
4. a smooth function $m_{ext} : M^{cucl}_{ext} \to \mathbb{R}$ with $m_{ext} \geq m_0 > 0$,

such that:

$$\psi|\Sigma \setminus B = \chi \circ \tilde{\psi}, \quad \chi^* m_{ext} = m|_{M^{cucl}}.$$

The proof of Prop. 2.5 is given in Appendix [A.2]

3. **Free Klein-Gordon fields**

In this section we briefly recall some well-known background material on free quantum Klein-Gordon fields on globally hyperbolic spacetimes. We follow the presentation in [GW1, Sect. 2] based on **charged fields**.

3.1. **Charged CCR algebra.**
3.1.1. Charged bosonic fields. Let $\mathcal{Y}$ a complex vector space and $q \in L_h(\mathcal{Y}, \mathcal{Y}^*)$ a non degenerate Hermitian form on $\mathcal{Y}$.

The $\text{CCR}^*$-algebra $\text{CCR}(\mathcal{Y}, q)$ is the complex $*$-algebra generated by symbols $\mathbb{I}, \psi(y), \psi^*(y), y \in \mathcal{Y}$ and the relations:

$\psi(y_1 + \lambda y_2) = \psi(y_1) + \overline{\lambda} \psi(y_2)$, $y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}$,

$\psi^*(y_1 + \lambda y_2) = \psi^*(y_1) + \lambda \psi^*(y_2)$, $y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}$,

$[\psi(y_1), \psi(y_2)] = [\psi^*(y_1), \psi^*(y_2)] = 0$, $[\psi(y_1), \psi^*(y_2)] = \overline{y}_1 \cdot qy_2 \mathbb{I}$, $y_1, y_2 \in \mathcal{Y}$.

A state $\omega$ on $\text{CCR}(\mathcal{Y}, q)$ is (gauge invariant) quasi-free if

$$\omega(\prod_{i=1}^{p} \psi(y_i) \prod_{j=1}^{q} \psi^*(y_j)) = \begin{cases} 0 & \text{if } p \neq q, \\ \sum_{\sigma \in S} \prod_{i=1}^{p} \omega(\psi(y_i)\psi^*(y_{\sigma(i)})) & \text{if } p = q. \end{cases}$$

There is no loss of generality to restrict oneself to charged fields and gauge invariant states, see eg the discussion in [GW1, Sect. 2]. It is convenient to associate to $\omega$ its (complex) covariances $\lambda^\pm \in L_h(\mathcal{Y}, \mathcal{Y}^*)$ defined by:

$$\omega(\psi(y_1)\psi^*(y_2)) = \overline{y}_1 \cdot \lambda^+ y_2, \quad y_1, y_2 \in \mathcal{Y}.$$  

The following results are well-known, see eg [DG, Sect. 17.1] or [GW1, Sect. 2] for Prop. 3.1 and [GOW, Prop. 7.1] for Prop. 3.2.

**Proposition 3.1.** Two Hermitian forms $\lambda^\pm \in L_h(\mathcal{Y}, \mathcal{Y}^*)$ are the covariances of a quasi-free state $\omega$ on $\text{CCR}(\mathcal{Y}, q)$ iff

$$\lambda^\pm \geq 0, \quad \lambda^+ - \lambda^- = q.$$  

**Proposition 3.2.** Let $\mathcal{Y}_\omega$ be the completion of $\mathcal{Y}$ for the Hilbertian scalar product $\lambda^+ + \lambda^-$. Then the state $\omega$ on $\text{CCR}(\mathcal{Y}, q)$ is pure iff there exist linear operators $c^\pm \in L(\mathcal{Y}_\omega)$ such that

$$c^+ c^- = \mathbb{I}, \quad (c^\pm)^2 = c^\pm,$$

$$(\text{ie } c^\pm \text{ is a pair of complementary projections})$$ and $\lambda^\pm = \pm q \circ c^\pm.$

3.2. Free Klein-Gordon fields. Let $P = -\Box_g + m(x), \quad m \in C^\infty(M, \mathbb{R})$ a Klein-Gordon operator on a globally hyperbolic spacetime $(M, g)$ (we use the convention $(1, n-1)$ for the Lorentzian signature). Let $G_{\text{ret/adv}}$ be the retarded/advanced inverses of $P$ and $G := G_{\text{ret}} - G_{\text{adv}}$. We apply the above framework to

$$\mathcal{Y} = \frac{C^\infty_0(M)}{PC^\infty_0(M)}, \quad [u] \cdot q[u] = i(u|Gu)_M,$$

where $(u|v)_M = \int_M u(x)v(x) Vol_g$. Denoting by $\text{Sol}_{uc}(P)$ the space of smooth space-compact solutions of $P \phi = 0$, it is well known that

$$[G] : \frac{C^\infty_0(M)}{PC^\infty_0(M)}, \quad i([-G]_M) \ni [u] \mapsto Gu \in (\text{Sol}_{uc}(P), q)$$

is unitary for

$$\overline{\phi}_1 \cdot q\phi_2 := i \int \Sigma (\nabla_{\mu} \overline{\phi}_1 \phi_2 - \overline{\phi}_1 \nabla_{\mu} \phi_2) n^\mu d\sigma,$$

where $\Sigma$ is any spacelike Cauchy hypersurface, $n^\mu$ is the future directed unit normal vector field to $\Sigma$ and $d\sigma$ the induced surface density. Setting

$$q : C^\infty_{uc}(M) \ni \phi \mapsto \left( \langle \phi^{[\Sigma]} | i^{-1} n^\mu \partial_\mu \phi^{[\Sigma]} \rangle \right) = f \in C^\infty_{uc}(\Sigma; \mathbb{C}^2)$$
the map

\[
\frac{C^\infty_0(M)}{PC^\infty_0(M)} \ni i(|G\rangle) \ni [u] \mapsto gGu \in (C^\infty_0(\Sigma; C^2), q)
\]
is unitary for

\[
(3.3) \quad \mathcal{J} \cdot qf := \int_{\Sigma} \mathcal{J}_f f_0 + \mathcal{J}_0 f_1 d\sigma, \quad f = \left(\begin{array}{c} f_0 \\ f_1 \end{array}\right).
\]

In the sequel the *-algebra $\text{CCR}(\mathcal{V}, q)$ where $(\mathcal{V}, q)$ is any of the above equivalent Hermitian spaces will be denoted by $\text{CCR}(P)$.

### 3.3. Quasi-free states

One restricts attention to quasi-free states on $\text{CCR}(P)$ whose covariances are given by distributions on $M \times M$, i.e., such that there exists $\Lambda^\pm \in \mathcal{D}'(M \times M)$ with

\[
(3.4) \quad \omega(\psi([u_1])\psi^*([u_2])) = (u_1|\Lambda^+ u_2)_M, \quad \omega(\psi^*([u_2])\psi([u_1])) = (u_1|\Lambda^- u_2)_M, \quad u_1, u_2 \in C^\infty_0(M).
\]

In the sequel the distributions $\Lambda^\pm \in \mathcal{D}'(M \times M)$ will be called the *spacetime covariances* of the state $\omega$.

In (3.4) we identify distributions on $M$ with distributional densities using the density $dV_\partial$ and use the notation $(u|\varphi)_M, u \in C^\infty_0(M), \varphi \in \mathcal{D}'(M)$ for the duality bracket. We have then

\[
(3.5) \quad P(x, \partial_x)\Lambda^\pm(x, x') = P(x', \partial_x)\Lambda^\pm(x, x') = 0,
\]

\[
\Lambda^+(x, x') - \Lambda^-(x, x') = iG(x, x').
\]

### 3.4. Cauchy surface covariances

Using $(C^\infty_0(\Sigma; C^2), q)$ instead of $(\frac{C^\infty_0(M)}{PC^\infty_0(M)}, i(|G\rangle)_M)$ one can associate to a quasi-free state its *Cauchy surface covariances* $\lambda^\pm$ defined by:

\[
(3.6) \quad \Lambda^\pm =: (gG)^*\lambda^\pm (gG).
\]

Using the canonical scalar product $(f|f)_\Sigma := \int_{\Sigma} \mathcal{J}_f f_0 + \mathcal{J}_0 f_1 d\sigma_{\Sigma}$ we identify $\lambda^\pm$ with operators, still denoted by $\lambda^\pm : C^\infty_0(\Sigma; C^2) \to D'(\Sigma; C^2)$.

### 3.5. Hadamard states

A quasi-free state is called a *Hadamard state*, (see [R] for the neutral case and [GW] for the complex case) if

\[
(3.7) \quad \text{WF}(\Lambda^\pm)' \subset N^\pm \times N^\pm,
\]

where $\text{WF}(\Lambda)'$ denotes the ‘primed’ wavefront set of $\Lambda$, i.e., $S' := \{(x, \xi), (x', -\xi') : (x, \xi), (x', \xi') \in S\}$ for $S \subset T^*M \times T^*M$, and $N^\pm$ are the two connected components (positive/negative energy shell) of the characteristic manifold:

\[
(3.8) \quad N^\pm := \{(x, \xi) \in T^*M \setminus o : \xi_\mu g^{\mu\nu}(x)\xi_\nu = 0\}.
\]

We recall that $T^*X \setminus o$ denotes the cotangent bundle of $X$ with the zero section removed.

Large classes of Hadamard states were constructed in terms of their Cauchy surface covariances in [GW], [GOW] using pseudodifferential calculus on $\Sigma$, see below for a short summary.
3.6. Pseudodifferential operators. We briefly recall the notion of (classical) pseudodifferential operators on a manifold, referring to [SH] Sect. 4.3 for details.

For \( m \in \mathbb{R} \) we denote by \( \Psi^m(\mathbb{R}^d) \) the space of classical pseudodifferential operators on \( \mathbb{R}^d \), associated with poly-homogeneous symbols of order \( m \), see eg [SH] Sect. 3.7.

Let \( X \) be a smooth, \( d \)-dimensional manifold. Let \( U \subset X \) a precompact chart open set and \( \psi : U \to \tilde{U} \) a chart diffeomorphism, where \( \tilde{U} \subset \mathbb{R}^d \) is precompact, open. We denote by \( \psi^* : C^\infty(\tilde{U}) \to C^\infty(U) \) the map \( \psi^* u(x) := u \circ \psi(x) \).

**Definition 3.3.** A linear continuous map \( A : C^\infty_0(X) \to C^\infty(X) \) belongs to \( \Psi^m(X) \) if the following condition holds:

(C) Let \( U \subset X \) be precompact open, \( \psi : U \to \tilde{U} \) a chart diffeomorphism, \( \chi_1, \chi_2 \in C^\infty_0(U) \) and \( \tilde{x}_i = \chi_i \circ \psi^{-1} \). Then there exists \( \tilde{A} \in \Psi^m(\mathbb{R}^d) \) such that

\[
(3.9) \quad (\psi^*)^{-1} \chi_1 A \chi_2 \psi^* = \tilde{\chi}_1 \tilde{A} \tilde{\chi}_2.
\]

Elements of \( \Psi^m(X) \) are called (classical) pseudodifferential operators of order \( m \) on \( X \).

The subspace of \( \Psi^m(X) \) of pseudodifferential operators with properly supported kernels is denoted by \( \Psi^m_c(X) \).

Note that if \( \Psi^\infty_c(M) := \bigcup_{m \in \mathbb{R}} \Psi^m_c(X) \), then \( \Psi^\infty_c(X) \) is an algebra, but \( \Psi^\infty(X) \) is not, since without the proper support condition, pseudodifferential operators cannot in general be composed.

To \( A \in \Psi^m(X) \) one can associate its principal symbol \( \sigma_{pr}(A) \in C^\infty(T^*M \setminus 0) \), which is homogeneous of degree \( m \) in the fiber variable \( \xi \) in \( T^*M \), in \( \{ |\xi| \geq 1 \} \). \( A \) is called elliptic in \( \Psi^m(X) \) at \( (x_0, \xi_0) \in T^*X \setminus 0 \) if \( \sigma_{pr}(A)(x_0, \xi_0) \neq 0 \).

If \( A \in \Psi^m(X) \) there exists (many) \( \tilde{A}_c \in \Psi^m_c(X) \) such that \( A - \tilde{A}_c \) has a smooth kernel.

3.7. The Cauchy surface covariances of Hadamard states. We now state a result which follows directly from a construction of Hadamard states in [GW1] Subsect. 8.2.

**Theorem 3.4.** Let \( \omega \) be any Hadamard state for the free Klein-Gordon field on \( (M, g) \) and \( \Sigma \) a smooth space-like Cauchy surface. Then its Cauchy surface covariances \( \lambda^\pm \) are \( 2 \times 2 \) matrices with entries in \( \Psi^\infty(\Sigma) \).

We refer the reader to [G] Thm. 3.2 for the proof.

4. Green operators and Calderón projectors

In this section we collect some formulas expressing the Green operators, ie inverses for abstract operators of the form \( \partial_s + b \), where \( s \) belongs either to \( \mathbb{R} \) or to the circle \( S_\beta \). We also compute various Calderón projectors. The formulas in this section will be used later in Sect. 5 to express Calderón projectors for second order elliptic operators obtained from abstract Klein-Gordon operators by Wick rotation.

4.1. Green operators and Calderón projectors. Let \( b \) a selfadjoint operator on a Hilbert space \( \mathfrak{h} \) with Ker\(b = \{0\} \). We recall that \( S_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}] \) is the circle of length \( \beta \). For \( 0 < \beta \leq \infty \) we set

\[
(4.10) \quad \mathfrak{h}_\beta = L^2(S_\beta) \otimes \mathfrak{h}, \quad \text{for} \quad \beta < \infty, \quad \mathfrak{h}_\infty = L^2(\mathbb{R}) \otimes \mathfrak{h}.
\]

The operator \( \partial_s \) is anti-selfadjoint on \( \mathfrak{h}_\beta \) with its natural domain. Denoting still by \( b \) the extension of \( b \) to \( \mathfrak{h}_\beta \) we see that \( B_\beta = \partial_s + b \) with domain Dom\(\partial_s \cap \text{Dom}b \) is normal.
4.1. Green operators. If $0 \in \sigma(b)$ then $0 \in \sigma(B_\beta)$ but we can still make sense out of $B_\beta^{-1}$ as

$$B_\beta^{-1} : (-\partial^2_x + b^2)^\frac{1}{2} \mathfrak{h}_\beta \to \mathfrak{h}_\beta.$$ 

A straightforward computation shows that:

(4.11) \hspace{1cm} B_\infty^{-1} f(s) = \int_{\mathbb{R}} G_\infty(s - s') f(s') ds', \ f \in C^0_0(\mathbb{R}; \mathfrak{h}), 

for

(4.12) \hspace{1cm} G_\infty(s) := e^{-sb} (\mathbb{1}_{\mathbb{R}^+}(s) \mathbb{1}_{\mathbb{R}^+}(b) - \mathbb{1}_{\mathbb{R}^-}(s) \mathbb{1}_{\mathbb{R}^-}(b)). 

Similarly for $\beta < \infty$, we have

(4.13) \hspace{1cm} B_\beta^{-1} f(s) = \int_{\mathbb{S}_\beta} G_\beta(s - s') f(s') ds', \ f \in C^0(\mathbb{S}_\beta; \mathfrak{h}), 

for $G_\beta(s)$ defined as follows: we set

$$G_\beta(s) := e^{-sb} (\mathbb{1}_{\mathbb{R}^+}(s)(1 - e^{-\beta b})^{-1} - \mathbb{1}_{\mathbb{R}^-}(s)(1 - e^{\beta b})^{-1}), \ s \in [-\beta, \beta].$$

(4.14) \hspace{1cm} G_\beta(s) = e^{-sb} (\mathbb{1}_{\mathbb{R}^+}(s)(1 - e^{-\beta b})^{-1} - \mathbb{1}_{\mathbb{R}^-}(s)(1 - e^{\beta b})^{-1}), \ s \in [-\beta, \beta]. 

4.1.2. Calderón projectors for $B_\infty$. We set $I^\pm = [-\beta, \beta]$. In the sequel we use the notation recalled in [1.3.7]. If $F \in C^0(I^\pm; \mathfrak{h})$ satisfies $(\partial_s + b)F = 0$ in $I^\pm$ we set

$$\Gamma^\pm_\infty F = F(0^\pm) = \lim_{s \to 0^\pm} F(s).$$

Denoting by $i^\pm_\infty F$ the extension of $F$ by 0 in $\mathbb{R} \setminus I^\pm$ we have

$$(\partial_s + b)i^\pm_\infty F = \pm \delta_0(s) \otimes \Gamma^\pm_\infty F,$$

hence $i^\pm_\infty F = \pm B_\infty^{-1}\delta_0(s) \otimes f$ for $f = \Gamma^\pm_\infty F$. This implies formally that

$$f = \pm \Gamma^\pm_\infty \circ B_\infty^{-1}(\delta_0(s) \otimes f)$$

if $f = \Gamma^\pm_\infty F$ for $F$ solving $(\partial_s + b)F = 0$ in $I^\pm_\infty$. This motivates the following definition:

**Definition 4.1.** The Calderón projectors $C^\pm_\infty \in B(\mathfrak{h})$ are:

(4.15) \hspace{1cm} C^\pm_\infty f = \pm \Gamma^\pm_\infty \circ B_\infty^{-1}(\delta_0(s) \otimes f), \ f \in \mathfrak{h}.

**Proposition 4.2.** We have:

(4.16) \hspace{1cm} C^\pm_\infty = \mathbb{1}_{\mathbb{R}^\pm}(b).

It follows that $C^\pm_\infty$ are bounded projections on $\mathfrak{h}$ with $C^\pm_\infty + C^-_\infty = \mathbb{1}$.

**Proof.** We approximate $\delta_0(\cdot)$ by a sequence $n\chi(n)$ where $\chi \in C^0_0(\mathbb{R})$ with $\int \chi(s) ds = 1$ and see from (4.12) that $C^\pm_\infty$ are well defined and (4.16) follows directly from (4.12). \Box
4.1.3. Calderón projectors for $B_{\beta}$. For $\beta < \infty$ we set $I_{\beta}^{\pm} = \pm|0, \frac{a}{2}|$. If $F \in \mathcal{C}^0(I_{\beta}^{\pm}, \mathfrak{h})$ satisfies $(\partial_s + b)F = 0$ in $I_{\beta}^{\pm}$ we set:

$$
\Gamma_{\beta}^{\pm} F := F(0^+) \oplus F(\frac{\beta}{2}) =: \Gamma_{\beta}^{(0)} F \oplus \Gamma_{\beta}^{(\frac{\beta}{2})} F;
$$

$$
\Gamma_{\beta}^{-} F := F(0^-) \oplus F(\frac{\beta}{2}) =: \Gamma_{\beta}^{(0)} F \oplus \Gamma_{\beta}^{(\frac{\beta}{2})} F.
$$

Denoting by $i_{\beta}^{\pm} F$ the extension of $F$ by $0$ in $S_{\beta} \setminus I_{\beta}^{\pm}$, we have

$$(\partial_s + b)i_{\beta}^{\pm} F = \pm(\delta_0(s) \otimes \Gamma_{\beta}^{(0)} F - \delta_\beta F) \otimes \Gamma_{\beta}^{(\frac{\beta}{2})} F),$$

which as before leads to the following definition:

**Definition 4.3.** The Calderón projectors $C_{\beta}^{\pm} \in B(\mathfrak{h} \oplus \mathfrak{h})$ are:

$$(4.18) \quad C_{\beta}^{\pm} f := \pm \Gamma_{\beta}^{\pm} \circ B_{\beta}^{-1}(\delta_0(s) \otimes f(0) - \delta_\beta(s) \otimes f(\frac{\beta}{2})), \quad f = f(0) \oplus f(\frac{\beta}{2}) \in \mathfrak{h} \oplus \mathfrak{h}.$$

**Proposition 4.4.** We have:

$$(4.19) \quad C_{\beta}^{\pm} = \begin{pmatrix}
(1 - e^{-\beta b})^{-1} & (1 - e^{\beta b})^{-1} e^{\frac{\beta}{2} b} \\
(1 - e^{-\beta b})^{-1} e^{-\frac{\beta}{2} b} & (1 - e^{\beta b})^{-1}
\end{pmatrix},
$$

On $\mathbb{1}_I(b) \oplus \mathbb{1}_I(b)$ for any $I \in \mathbb{R}^+$ one has:

$$C_{\beta}^{+} C_{\beta}^{+} = C_{\beta}^{\pm}, \quad C_{\beta}^{+} + C_{\beta}^{-} = \mathbb{1}.$$

Note that if $0 \in \sigma(b)$ then $C_{\beta}^{\pm}$ are unbounded on $\mathfrak{h} \oplus \mathfrak{h}$.

**Proof.** The proof of [4.19] is a routine computation, using [4.14]. The second statement is checked using the identity $(1 - a)^{-1} + (1 - a^{-1})^{-1} = 1$ for $a = e^{-\beta b}$. \(

5. Vacua and KMS states for stable symplectic dynamics

In this section we recall well-known formulas for the covariances of the vacuum and KMS states associated to a symplectic flow on a symplectic space. The symplectic flow has to be *stable*, i.e., generated by a positive classical energy. In concrete situations the symplectic flow is generated by a time-like Killing vector field. We also recall the definition of the double KMS state, due to Kay [K1, K2], which is related to the Araki-Woods representation of a KMS state.

The new result of this section is that the covariances of the vacuum and double KMS states can be expressed by the Calderón projectors introduced in Sect. 4. Note that only the double KMS states will be important for the construction of the HHI state later on. Nevertheless the case of vacuum state is simpler and we include it for pedagogical reasons.

5.1. Weakly stable symplectic dynamics. We describe now a framework for symplectic dynamics, which can be found in [DG] Sect. 18.2.1], called there a *weakly stable symplectic dynamics*.

Let $(\mathcal{Y}, q)$ a Hermitian space and $E \in L_2(\mathcal{Y}, \mathcal{Y}^*)$ with $E > 0$, the function $\mathcal{Y} \ni y \mapsto \mathcal{Y} E y$ being the classical energy. The energy space $\mathcal{Y}_{\text{en}}$ is the completion of $\mathcal{Y}$ for the scalar product $(y_1, y_2)_{\text{en}} = \mathcal{Y} E y_2$ and is a complex Hilbert space.

Let $r_t = e^{ibt}$ be a strongly continuous unitary group on $\mathcal{Y}_{\text{en}}$ with selfadjoint generator $b$. We assume that $r_t : \mathcal{Y} \to \mathcal{Y}, \mathcal{Y} \subset \text{Dom} b$, Ker$b = \{0\}$ and:

$$
(5.1) \quad \mathcal{Y} E y_2 = \mathcal{Y} q y_2, \quad y_1, y_2 \in \mathcal{Y}.
$$
The meaning of (5.1) is that \( \{r_t\}_{t \in \mathbb{R}} \) is the symplectic evolution group associated to the classical energy \( \overline{\rho} \cdot E \cdot \gamma \) and the symplectic form \( \sigma = i^{-1} q \).

5.1.1. Dynamical Hilbert space. It is convenient, in connection with the quantization of the symplectic flow \( \{r_t\}_{t \in \mathbb{R}} \), to introduce the dynamical Hilbert space
\[
\mathcal{Y}_{\text{dyn}} := |b|^{\frac{1}{2}} \mathcal{Y}_{\text{en}},
\]
see [DG Subsect. 18.2.1], equipped with the scalar product \( (y_1|y_2)_{\text{dyn}} = (y_1|b|^{-1}y_2)_{\text{en}} \). The group \( \{r_t\}_{t \in \mathbb{R}} \) extends obviously as a unitary group on \( \mathcal{Y}_{\text{dyn}} \). If we denote the generator of \( r_t \) on \( \mathcal{Y}_{\text{en/dyn}} \) by \( b_{\text{en/dyn}} \) then \( b_{\text{en}} = b \) and \( b_{\text{dyn}} = |b|^{\frac{1}{2}} b_{\text{en}} |b|^{-\frac{1}{2}} \) since \( |b|^{-\frac{1}{2}} : \mathcal{Y}_{\text{dyn}} \rightarrow \mathcal{Y}_{\text{en}} \) is unitary. Therefore we will denote both generators by the same letter \( b \).

Moreover from (5.1) we obtain that:
\[
(5.2) \quad \overline{\gamma}_1 \cdot q y_2 = (y_1|\text{sgn}(b)y_2)_{\mathcal{Y}_{\text{dyn}}},
\]
so \( q \) is a bounded sesquilinear form on \( \mathcal{Y}_{\text{dyn}} \), but in general not on \( \mathcal{Y}_{\text{en}} \), unless \( 0 \not\in \sigma(b) \).

5.2. Vacuum state. We now recall the definition of the vacuum state \( \omega_{\text{vac}} \) associated to the dynamics \( \{r_t\}_{t \in \mathbb{R}} \).

Definition 5.1. The vacuum state \( \omega_{\text{vac}} \) is defined by the covariances:
\[
(5.3) \quad \overline{\gamma}_1 \cdot \lambda_{\text{vac}}^\pm y_2 = (y_1|\mathbb{I}_R \pm (b)y_2)_{\mathcal{Y}_{\text{dyn}}}.
\]

From (5.2) we obtain that:
\[
(5.4) \quad c_{\text{vac}}^- := \pm \lambda_{\text{vac}}^+ \circ q^{-1} = \mathbb{I}_R \pm (b).
\]

It follows from Def. 4.1 that:
\[
(5.5) \quad c_{\text{vac}}^+ = C_\infty^\pm,
\]
where the Calderón projectors \( C_\infty^\pm \) are defined in Def. 4.1 for \( \mathfrak{h} = \mathcal{Y}_{\text{dyn}} \).

5.3. KMS state. Let us now define the \( \beta \)-KMS state \( \omega_\beta \) associated to the dynamics \( \{r_t\}_{t \in \mathbb{R}} \).

Definition 5.2. The \( \beta \)-KMS state \( \omega_\beta \) is defined by the covariances:
\[
(5.6) \quad \overline{\gamma}_1 \cdot \lambda_\beta^+_\gamma y_2 = \overline{\gamma}_1 \cdot q (1 - e^{-\beta b})^{-1} y_2, \\
\overline{\gamma}_1 \cdot \lambda_\beta^- y_2 = \overline{\gamma}_1 \cdot q (e^{\beta b} - 1)^{-1} y_2.
\]

5.4. Double \( \beta \)-KMS states. The double \( \beta \)-KMS state see [K1, K2] can easily be related to the Araki-Woods representation of \( \omega_\beta \), see eg [DG Subsect. 17.1.5], that we first briefly recall. In the sequel \( \mathcal{Y}_{\text{dyn}} \) will be simply denoted by \( \mathcal{Y} \).

5.4.1. Araki-Woods representation. Let us denote by \( \mathcal{Z} \) the space \( \mathcal{Y}_R \) equipped with the complex structure
\[
j := i \circ \text{sgn}(b)
\]
and the scalar product:
\[
(5.7) \quad (z_1|z_2)_\mathcal{Z} := (y_1+y_2)_R + (y_2-|y_1-)_\mathcal{Y},
\]
for \( y_{\pm} := \mathbb{I}_R \pm (b)y \) and \( z = y \) (considered as an element of \( \mathcal{Z} \)). \( \mathcal{Z} \) is a Hilbert space equal to \( \mathcal{Y}_+ \oplus \overline{\mathcal{Y}}_- \) for \( \mathcal{Y}_\pm = \mathbb{I}_R \pm (b) \mathcal{Y} \). Note that since \( [b,j] = 0, b \) induces a selfadjoint operator on \( \mathcal{Z} \), still denoted by \( b \). We set
\[
\rho := (e^{\beta|b|} - 1)^{-1},
\]
which is a selfadjoint operator on \( \mathcal{Z} \).
We also introduce the Hilbert space $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ and the bosonic Fock space $\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, see eg [DG, Subsect. 3.3.1]. For $(z_1, \overline{z_2}) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$ we denote by $a^{(z)}(z_1, \overline{z_2})$ the Fock creation/annihilation operators acting on $\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})$.

The left/right Araki-Woods creation/annihilation operators are defined by:

$$a^*_t(z) = a^*((1 + \rho)\frac{1}{2} z, 0) + a(0, \rho^{1/2} z),$$

$$a_t(z) = a((1 + \rho)\frac{1}{2} z, 0) + a^*(0, \rho^{1/2} z),$$

$$(5.8)$$

$$a^*_t(\overline{z}) = a((\rho^{1/2} z, 0) + a^*(0, (1 + \rho)^{1/2} \overline{z}),$$

$$a_t(\overline{z}) = a^*((\rho^{1/2} z, 0) + a(0, (1 + \rho)^{1/2} \overline{z}).$$

One has:

$$[a_t(z_1), a^*_t(z_2)] = (z_1|z_2)\mathbb{I}, \quad [a_t(\overline{z}_1), a^*_t(\overline{z}_2)] = (\overline{z}_1|\overline{z}_2)\mathbb{I},$$

all other commutators being equal to 0. Setting $z_\pm = y_\pm$ for $y \in \mathcal{Y}$ we set

$$\psi_t^*(y) := a_t^*(z_+) + a_t(z_-), \quad \psi_t(y) := a_t(z_+) + a_t^*(z_-),$$

$$(5.9)$$

$$\psi_t^*(y) := a_t^*(\overline{z}_-) + a_t(\overline{z}_+), \quad \psi_t(y) := a_t(\overline{z}_-) + a_t^*(\overline{z}_+),$$

An easy computation shows that

$$\psi_t^*(y_1), \psi_t^*(y_2)] = \overline{y}_1 y_2, \quad [\psi_t(y_1), \psi_t^*(y_2)] = -\overline{y}_1 y_2,$$

all other commutators being equal to 0. Moreover $\mathcal{Y} \ni y \mapsto \psi_t^*(y)$ is $\mathbb{C}$-linear.

This means that $\mathcal{Y} \ni y \mapsto \psi_t^*(y)$ induces two commuting representations of $\text{CCR}(\mathcal{Y}, \pm q)$.

From (5.9) we obtain that:

$$(\Omega|\psi_t(y_1)\psi_t^*(y_2)\Omega)_{\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})} = \overline{y}_1 y_2,$$

$$(\Omega|\psi_t^*(y_2)\psi_t(y_1)\Omega)_{\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})} = \overline{y}_1 y_2,$$

$$(\Omega|\psi_t(y_2)\psi_t(y_1)\Omega)_{\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})} = (\Omega|\psi_t^*(y_2)\psi_t^*(y_1)\Omega)_{\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})} = 0,$$

where $\Omega$ is the vacuum vector in $\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. If $\pi_{AW,1}$ is the representation of $\text{CCR}(\mathcal{Y}, q)$ defined by $\pi_{AW,1}(\psi^*)(y) = \psi_t^*(y)$, then $(\pi_{AW,1}, \Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}}), \Omega)$ is the GNS representation associated to the $\beta$-KMS state $\omega_\beta$.

5.4.2. The double $\beta$-KMS state. To define the double $\beta$-KMS state associated to $\omega_\beta$ we set

$$(\mathcal{X}, Q) := (\mathcal{Y} \oplus \mathcal{Y}, q \oplus -q).$$

Recalling that $\sigma = i^{-1}q$, this corresponds to add to the real symplectic space $(\mathcal{Y}, \mathcal{R})$ its anti-symplectic copy $(\mathcal{Y}, -\mathcal{R})$. From (5.10) we see that $\mathcal{X} \ni x \mapsto \Psi^*_A(x)$ for

$$\Psi^*_A(x) := \psi_t^*(y) + \psi^*(y'), \quad x = (y, y') \in \mathcal{X}$$

induces a representation of $\text{CCR}(\mathcal{X}, Q)$.

Definition 5.3. The double $\beta$-KMS state $\omega_\beta$ is the quasi-free state on $\text{CCR}(\mathcal{X}, Q)$ defined by

$$\omega_\beta(\Psi^*(x_1)) := (\Omega|\Psi^*_A(x_1)\psi^*_A(x_2)\Omega)_{\Gamma_s(\mathcal{Z} \oplus \overline{\mathcal{Z}})}, \quad x_1, x_2 \in \mathcal{X}.$$

Proposition 5.4. $\omega_\beta$ is a pure, gauge invariant quasi-free state on $\text{CCR}(\mathcal{X}, Q)$. If $\lambda^+_{\beta}$ are the covariances of $\omega_\beta$ we have

$$\Pi_1 \lambda^+_{\beta} x_2 = \pm \Pi_1 QC^+_{\beta} x_2, \quad x_1, x_2 \in \mathcal{X},$$

where $C^+_{\beta}$ is the Calderón projectors for $B_{\beta}$ defined in Def. 4.3.
Remark 5.5. Let us denote \((\mathcal{V}, q)\) by \((\mathcal{V}_1, q_1)\) and let \((\mathcal{V}_2, q_2)\) another Hermitian space with \(1 : (\mathcal{V}_2, q_2) \to (\mathcal{V}_1, -q_1)\) unitary. Then \(\omega_d\) induces a quasi-free state on \(\text{CCR}(\mathcal{A}_1 \oplus \mathcal{A}_2, q_1 \oplus q_2)\). Its covariances are

\[
\left( \begin{array}{cc} 1 & 0 \\ 0 & I^* \end{array} \right) \lambda^\pm \left( \begin{array}{cc} 1 & 0 \\ 0 & I \end{array} \right) = \pm \left( \begin{array}{cc} q_1 & 0 \\ 0 & q_2 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & I^{-1} \end{array} \right) \left( \begin{array}{cc} q_1 & 0 \\ 0 & q_2 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & I \end{array} \right).
\]

Proof of Prop. 5.4. We obtain from (5.9), (5.8):

\[
\Psi_{AW}(x)(\Omega) = a^*_r(z_+)\Omega + a_l(z_-)\Omega + a^*_r(\tau_-)\Omega + a_l(\tau_+)\Omega
\]

(5.12)

\[
= ((\rho + 1)^{1/2}z_+ + \rho^{1/2}z'_+, \rho^{1/2} \tau_+ + (\rho + 1)^{1/2} \tau_-),
\]

\[
\Psi_{AW}(x)\Omega = a_l(z_-)\Omega + a^*_r(z_-)\Omega + a_l(\tau_-)\Omega + a^*_r(\tau_+)\Omega
\]

(\rho + 1)^{1/2}z_- + \rho^{1/2}z'_-, \rho^{1/2} \tau_+ + (\rho + 1)^{1/2} \tau_+),

as elements of \(\mathcal{Z} \oplus \overline{\mathcal{Z}}\). From (5.12) we immediately obtain that

\[
\omega_d(\Psi_{AW}(x_1)\Psi_{AW}(x_2)) = \omega_d(\Psi_{AW}(x_1)^*\Psi_{AW}(x_2)) = 0,
\]

ie \(\omega_d\) is gauge invariant for the complex structure \(i \oplus i\) of \(X\). We have next

\[
\omega_d(\Psi_{AW}(x_1)^*\Psi_{AW}(x_2)) = (\Psi_{AW}(x_1)\Omega|\Psi_{AW}(x_2)\Omega)_{\mathcal{Z} \otimes \overline{\mathcal{Z}}}
\]

\[
= ((\rho + 1)^{1/2}y_1 + \rho^{1/2}y'_1, (\rho + 1)^{1/2}y_2 + \rho^{1/2}y'_2) + ((\rho + 1)^{1/2}y_3 - \rho^{1/2}y'_3, (\rho + 1)^{1/2}y_4 - \rho^{1/2}y'_4)\]

If \(\lambda^+_d = Q \circ C^+_d\), where \(Q = q \oplus -q\), we obtain from (5.2) that:

\[
C^+_d = \begin{pmatrix}
(\rho + 1)^{1/2}1_+ - \rho 1_- & -\rho^{1/2}(\rho + 1)^{1/2}1_+ + \rho^{1/2}(\rho + 1)^{1/2}1_-\\
\rho^{1/2}(\rho + 1)^{1/2}1_+ - \rho^{1/2}(\rho + 1)^{1/2}1_- & -\rho 1_+ + (\rho + 1) 1_-
\end{pmatrix},
\]

for \(1_\pm = 1_{\mathbb{R}^\pm}(b)\). We compute:

\[
(1 + \rho) 1_+ - \rho 1_-
\]

\[
= (1 - e^{-\beta b})^{-1} 1_+ - e^{\beta b}(1 - e^{\beta b})^{-1} 1_- = (1 - e^{-\beta b})^{-1};
\]

\[
-\rho^{1/2}(1 + \rho)^{1/2}1_+ + \rho^{1/2}(1 + \rho)^{1/2}1_-
\]

\[
= -e^{-\beta b/2}(1 - e^{-\beta b})^{-1} + e^{\beta b/2}(1 - e^{\beta b})^{-1} = e^{\beta b/2}(1 - e^{-\beta b})^{-1};
\]

\[
\rho^{1/2}(1 + \rho)^{1/2}1_+ - \rho^{1/2}(1 + \rho)^{1/2}1_-
\]

\[
= -e^{\beta b/2}(1 - e^{\beta b})^{-1} - e^{-\beta b/2}(1 - e^{-\beta b})^{-1};
\]

\[
-\rho 1_+ + (\rho + 1) 1_-
\]

\[
= -e^{-\beta b}(1 - e^{-\beta b})^{-1} 1_+ + (1 - e^{\beta b})^{-1} 1_- = (1 - e^{\beta b})^{-1}.
\]

Therefore \(C^+_d = C^+_d\). Since \(C^+_d + C^-_d = 1\), we have also \(C^-_d = C^-_d\).

To see that \(\omega_d\) is pure, we have to check that the representation of the Weyl algebra \(\text{CCR}_{\text{Weyl}}(X, Q)\) associated to \(\Psi_{AW}(x), x \in X\) is irreducible. This follows from the definition (5.11) of \(\Psi_{AW}(x)\) and statements (5), (7) in [DQ] Thm. 17.24. \(\Box\)

6. Abstract Klein-Gordon equations

In this section we collect some results about abstract Klein-Gordon equations of the form

\[
(\partial_t + \tilde{\omega}^*)(\partial_t - \tilde{\omega})\tilde{\phi} + \tilde{h}_0\tilde{\phi} = 0,
\]

where \(\tilde{\omega} : \mathbb{R} \to \tilde{H}, \tilde{H}\) is some Hilbert space and \(\tilde{h}_0, \tilde{\omega}\) are linear operators on \(\tilde{H}\). Such Klein-Gordon equations arise from stationary metrics on a spacetime \(M = \mathbb{R} \times S\),
with Killing vector field equal to $\frac{\partial}{\partial t}$, when $\tilde{w}$ represent the shift vector field and the lapse function is equal to 1. The case of general stationary Klein-Gordon operators will be considered later in Sect. 5.

We will also consider the Wick rotated operator $\tilde{K}_\beta$ obtained by setting $t = is$, where $s$ belongs either to $\mathbb{R}$ or to $S_\beta$. Using sesquilinear form techniques we give a rigorous meaning to its inverse $\tilde{K}_\beta^{-1}$ and relate it to the Green operators in Sect. 4.

6.1. Hypotheses. We will assume the following hypotheses:

1) $\tilde{h}_0$ is selfadjoint on $\tilde{H}$ and $\tilde{h}_0 > 0$,

\begin{equation}
(6.2) \quad \tilde{\phi}[\tilde{h}_0]^{-\frac{1}{2}}, \tilde{\phi}^*[\tilde{h}_0]^{-\frac{1}{2}} \in B(\tilde{H}),
\end{equation}

2) if $\tilde{h} := \tilde{h}_0 - \tilde{\phi}^* \tilde{\phi}$ then $\tilde{h} \sim \tilde{h}_0$.

We can rewrite (6.1) as

\begin{equation}
(6.3) \quad \partial_t^2 \tilde{\phi} - 2i\tilde{k}\partial_t \tilde{\phi} + \tilde{h} \tilde{\phi} = 0,
\end{equation}

where $\tilde{k} = (2\tilde{t})^{-1}(\tilde{w} - \tilde{\phi}^*)$, which was considered in [GCH] in a more general situation.

6.2. Quadratic pencils. One associates to (6.3) the quadratic pencil

\[ p(z) = z(2\tilde{k} - z) + \tilde{h} = (iz + \tilde{\phi}^*)(iz - \tilde{\phi}) + \tilde{h}_0 \in B((\tilde{h}_0)^{-\frac{1}{2}} \tilde{H}, (\tilde{h}_0)^{-\frac{1}{2}} \tilde{H}), z \in \mathbb{C}, \]

obtained by replacing $\partial_t$ by $iz$, and denotes by $\rho(\tilde{h}, \tilde{k})$ the set of $z \in \mathbb{C}$ such that $p(z) : (\tilde{h}_0)^{-\frac{1}{2}} \tilde{H} \rightarrow (\tilde{h}_0)^{-\frac{1}{2}} \tilde{H}$. Since $\tilde{h} > 0$ it follows from [GCH] Prop. 2.3 that \( \{ z : |\text{Im} z| \geq |\text{Re} z| + c_0 \} \subset \rho(\tilde{h}, \tilde{k}) \) for some $c_0 > 0$.

6.3. First order system. Setting

\begin{equation}
(6.4) \quad \tilde{f}(t) = \tilde{w} \tilde{\phi} := \left( \begin{array}{c} \tilde{\phi}(t) \\
\text{i}^{-1}(\partial_t - \tilde{\phi}) \tilde{\phi}(t) \end{array} \right) = \left( \begin{array}{c} \tilde{f}_0(t) \\
\text{i} \tilde{\phi} \end{array} \right),
\end{equation}

(6.1) is formally rewritten as

\begin{equation}
(6.5) \quad \partial_t \tilde{f} = i\tilde{H} \tilde{f}, \quad \tilde{H} = \left( \begin{array}{cc} -i\tilde{\phi} & \mathbb{I} \\
\text{i} \tilde{\phi} \tilde{h}_0 & -i\tilde{\phi}^* \end{array} \right).
\end{equation}

The conserved energy is

\begin{equation}
(6.6) \quad \tilde{J} \cdot \hat{E} \tilde{f} = ||\tilde{f}_0 - i\tilde{\phi} \tilde{f}_0||^2 + (\tilde{f}_0 | \tilde{h} \tilde{f}_0),
\end{equation}

which is positive definite by (6.2). The Hilbert space associated to $\tilde{E}$ will be denoted by $\tilde{E}$. It equals $\tilde{h}_0^{-\frac{1}{2}} \tilde{H} \oplus \tilde{H}$ as a topological vector space. We set also $\tilde{E}^* := \tilde{H} \oplus \tilde{h}_0^\frac{1}{2} \tilde{H}$.

The following proposition will be proved in Subsect. 6.5.

**Proposition 6.1.** The operator $\tilde{H} = \left( \begin{array}{cc} -i\tilde{\phi} & \mathbb{I} \\
\text{i} \tilde{\phi} \tilde{h}_0 & -i\tilde{\phi}^* \end{array} \right)$ is bounded from $\tilde{E}$ to $\tilde{E}^*$. It induces on $\tilde{E}$ the operator $\hat{H}$ defined by

\[ \text{Dom} \hat{H} = \{ \tilde{f} \in \tilde{E} : \hat{H} \tilde{f} \in \tilde{E} \cap \tilde{E}^* \}. \]

$\hat{H}$ is a densely defined selfadjoint operator on $\tilde{E}$ with $\text{res}(\hat{H}) = \rho(\tilde{h}, \tilde{k})$.

Note that $(\tilde{E}, \tilde{E}^*)$ form a non degenerate dual pair for the charge

\begin{equation}
(6.7) \quad \tilde{J} \cdot \hat{q} \tilde{f} = (\tilde{f}_0 | \tilde{f}_1)_{\hat{H}} + (\tilde{f}_0 | \tilde{f}_1^*)_{\hat{H}}, \quad \tilde{f} \in \tilde{E}, \quad \tilde{f}' \in \tilde{E}^*,
\end{equation}

and one has

\[ \tilde{J} \cdot \hat{E} \tilde{f} = \tilde{J} \cdot \hat{q} \hat{H} \tilde{f}, \tilde{f} \in \tilde{E}. \]
6.4. The Wick rotated operator. Setting formally \( t = i \) we obtain the formal expression
\[
\tilde{K} = -(\partial_s + i\tilde{w}^\ast)(\partial_s - i\tilde{w}) + \tilde{h}_0.
\]
To give a meaning to (6.8), we will use sesquilinear forms techniques. Let us set as in Sect. 4.1 for \( 0 < \beta \leq \infty \):
\[
\tilde{H}_{\beta} = L^2(S_{\beta}) \otimes \tilde{H}, \quad \text{for} \beta < \infty, \quad \tilde{H}_{\infty} = L^2(\mathbb{R}) \otimes \tilde{H}.
\]
We consider the sesquilinear form associated to (6.10)
\[
\tilde{\beta}(u, u) = \|\partial_s u\|_{\tilde{H}_{\beta}}^2 + (u|\tilde{h}_0)_{\tilde{H}_{\beta}} - i(\partial_s u|\tilde{w}u)_{\tilde{H}_{\beta}} - i(\tilde{w}u|\partial_s u)_{\tilde{H}_{\beta}},
\]
with domain \( \text{Dom} \tilde{\beta} = (-(\partial_s^2 + \tilde{h}_0) + 1/2)\tilde{H}_{\beta} \), where \( \partial_s \) is equipped with its natural domain on \( \tilde{H}_{\beta} \). From hypotheses (6.2) we obtain that
\[
\text{Re} \tilde{\beta}(u, u) \sim \|\partial_s u\|_{\tilde{H}_{\beta}}^2 + (u|\tilde{h}_0 u)_{\tilde{H}_{\beta}}, \quad |\text{Im} \tilde{\beta}(u, u)| \leq C\text{Re} \tilde{\beta}(u, u),
\]
hence \( \tilde{\beta} \) is a closed sectorial form. By Lax-Milgram theorem \( \tilde{\beta} \) induces a boundedly invertible operator
\[
\tilde{K}_{\beta} : (-(\partial_s^2 + \tilde{h}_0) - 1/2)\tilde{H}_{\beta} \sim (-(\partial_s^2 + \tilde{h}_0) + 1/2)\tilde{H}_{\beta}.
\]
We can apply the results of Subsect. 4.1 setting \( b = \tilde{\mathcal{E}}, b = \tilde{H} \), see (4.10) for the notation used, and obtain an operator
\[
\partial_s + \tilde{H} : \tilde{\mathcal{E}} \sim (-(\partial_s^2 + \tilde{H})^2)\tilde{E}_{\beta}.
\]
The relation between \( \tilde{K}_{\beta}^{-1} \) and \( \partial_s + \tilde{H} \) is given by the following proposition. Below we denote by \( \pi_i \) the maps \( \pi_i \tilde{f} = \tilde{f}_i \) for \( \tilde{f} = \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \end{pmatrix} \).

**Proposition 6.2.** One has
\[
\tilde{K}_{\beta}^{-1} = \pi_0(\partial_s + \tilde{H})^{-1}\pi_1^\ast.
\]

6.5. Proofs of Props. 6.1 and 6.2

6.5.1. Preparations. We will prove Props. 6.1, 6.2 using results in [GCH]. There the form (6.3) of the Klein-Gordon equation is used and instead of (6.4) one sets:
\[
g := \begin{pmatrix} \phi \\ i\tilde{w}\partial_s \phi \end{pmatrix},
\]
(6.12) is formally rewritten as
\[
\partial_t g = i\tilde{H}g, \quad \tilde{H} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 2\tilde{k} \end{pmatrix}.
\]
The conserved energy is
\[
\tilde{g} \cdot \tilde{E} g = \|g_1\|^2 + (g_0|\tilde{h}_0 g_0).
\]
The Hilbert space \( \tilde{\mathcal{E}} \) naturally associated to \( \tilde{E} \) equals again \( \tilde{h}_0^{1/2} \tilde{H} \oplus \tilde{H} \).

If \( \tilde{f} \) is given by (6.4) and \( g \) by (6.12) one has
\[
\tilde{f} = U^\ast g \quad \text{for} \quad U = \begin{pmatrix} \mathbb{I} & 0 \\ i\tilde{w} & \mathbb{I} \end{pmatrix},
\]
and
\[
U : \tilde{\mathcal{E}} \sim \tilde{\mathcal{E}}, \quad (U g|U g)_\tilde{\mathcal{E}} = (g|g)_\tilde{\mathcal{E}}.
\]
Formally one has \( \tilde{H} = U \tilde{H} U^{-1} \), and since \( U : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \) is unitary, Prop. 6.1 follows if we prove the analogous result for \( \tilde{H} \). One sets then
\[
\tilde{\mathcal{E}}^\ast := \tilde{H} \oplus \tilde{h}_0^{1/2} \tilde{H},
\]
which forms again a dual pair with $\hat{E}$ for
\[
\mathfrak{g} q^i = (g_1 + i\hat{w}g_0|g_0^i)_\hat{H} + (g_0|g_0^i + i\hat{w}g_0^i), \quad g \in \hat{E}, g^i \in \hat{E}^*.
\]
We have of course $\hat{q} = U^* q U$.

6.5.2. Proof of Prop. 6.1 The matrix $\hat{H}$ induces a bounded operator $\hat{H} : \hat{E} \to \hat{E}^*$. One denotes by $\hat{H}$ the linear operator induced by $\hat{H}$ on $\hat{E}$. Its domain is
\[
\text{Dom}\hat{H} = \{ g \in \hat{E} : \hat{H}g \in \hat{E} \cap \hat{E}^* \}.
\]
Note that although in general $\hat{E}$ is not included in $\hat{E}^*$, the intersection $\hat{E} \cap \hat{E}^*$ is well defined. In fact the intersections $\hat{H} \cap \hat{h}_0^{-\frac{1}{2}}\hat{H}$ and $\hat{H} \cap \hat{h}_0^{\frac{1}{2}}\hat{H}$ are well defined, as follows easily from the spectral theorem. In concrete applications, where $\hat{h}_0, \hat{w}$ are differential operators on some manifold $N$, one can also consider these intersections inside $\mathcal{D}(N)$.

It follows then from [GGH] Prop. 5.8, Thm. 5.9, and the fact that there exists $z \neq 0$ in $\rho(\hat{h}, \hat{k})$, that $\hat{H}$ with the domain above is a densely defined selfadjoint operator on $\hat{E}$ with $\text{res}(\hat{H}) = \rho(\hat{h}, \hat{k})$. Setting
\[
\tilde{H} := U\hat{H}U^{-1}
\]
completes the proof of Prop. 6.1.

6.5.3. Proof of Prop. 6.2 One can express the resolvent $(\hat{H} - z)^{-1}$ using $p(z)$ as follows: if $z \in \rho(\hat{h}, \hat{k})$ then:
\[
(\hat{H} - z)^{-1} = p(z)^{-1} \begin{pmatrix} z - 2\hat{k} & 0 \\ \hat{h} & z \end{pmatrix} \in B(\hat{E}, \hat{E}).
\]
Note that (6.14) is different from the formula found in [GGH] Prop. 5.8, because weaker assumptions on $\hat{h}, \hat{k}$ were used there. In our case using that $\tilde{k}|\hat{h}_0|^{-\frac{1}{2}} \in B(\hat{H})$ one deduces from [GGH] Lemma 2.2 that
\[
p(z) : \hat{H} + |\hat{h}_0|^{\frac{1}{2}}\hat{H} \to (\hat{h}_0)^{-\frac{1}{2}}\hat{H} \subset \hat{H} \cap |\hat{h}_0|^{-\frac{1}{2}}\hat{H}, \quad z \in \rho(\hat{h}, \hat{k}).
\]
Using this fact it is straightforward to show that the rhs in (6.14) maps $\hat{E}$ into itself.

In general we have $0 \notin \rho(\hat{h}, \hat{k})$ hence $0 \in \sigma(\hat{H})$ but $\hat{H}^{-1}$ is well defined as
\[
\hat{H}^{-1} = \begin{pmatrix} -2\hat{k}^{-1} & 0 \\ \hat{h} & -\hat{k}^{-1} \end{pmatrix} = B(\hat{E}, |\hat{h}|^{-1}\hat{H} \oplus \hat{H})
\]
which corresponds to (6.14) for $z = 0$.

We have $\text{Ker}\hat{H} = \{ 0 \}$ since $\hat{H}g = 0$ implies $g_1 = 0, \hat{h}g_0 = 0$ and $\hat{h}$ is injective. Therefore we can apply the results of Subsect. 4.1 to construct $(\partial_\beta + \hat{H})^{-1}$ for $b = \hat{H}, \beta = \hat{E}$. As before we introduce the Hilbert spaces $\hat{H}_\beta$ and $\hat{E}_\beta$ for $\beta \in [0, \infty]$.

Using Fourier transform in $s$ either on $\mathbb{R}$ for $\beta = \infty$ or on $\mathbb{S}_\beta$ for $\beta < \infty$ we can express $(\partial_\beta + \hat{H})^{-1}$ using (6.14), replacing $z$ by $-\partial_\beta$. We claim that for $\tilde{K}_\beta$ defined in Subsect. 6.3 we have
\[
\tilde{K}_\beta^{-1} = \pi_0(\partial_\beta + \hat{H})^{-1} \pi_1^*,
\]
which will prove Prop. 6.2 since $\pi_0 U^{-1} = \pi_0$ and $U\pi_1^* = \pi_1$.

Let us prove (6.16). We have:
\[
(-\partial_\beta^2 + \hat{h}_0)\frac{1}{2}\hat{H}_\beta = [\partial_\beta, \hat{H}_\beta] + \hat{h}_0^\frac{1}{2}\hat{H}_\beta,
\]
\[
(\partial_\beta^2 + \hat{H}^2)z\hat{E}_\beta = [\partial_\beta, \hat{E}_\beta] + \hat{H}\hat{E}_\beta.
\]
If \( v = \partial_s u \in \partial\beta |\mathcal{H}_\beta | \) then \( \pi_1^* v = \partial_s \pi_1^* u \in \partial_s \mathcal{E}_\beta \). Similarly if \( v = \hat{h}_0^\beta u \in |\mathcal{H}_0|^{\beta} \mathcal{H}_\beta | \) then \( \pi_1^* v = \hat{H} \left( \hat{h}_0^{-1} \hat{h}_0^{\beta} \right) u \in \hat{H} \mathcal{E}_\beta \). In conclusion we have shown that

\[
\pi_1^* ( -\partial_s^2 + \hat{h}_0 )^{\beta} \mathcal{H}_\beta \rightarrow (\partial_s^2 + \hat{H}^2)^{\beta} \mathcal{E}_\beta \text{ continuously.}
\]

Next if \( g \in \mathcal{E}_\beta \) and \( (\partial_s + \hat{H}) g = \left( \begin{array}{c} 0 \\ v \end{array} \right) \) for \( v \in (-\partial_s^2 + \hat{h}_0)^{\beta} \mathcal{H}_\beta \) we have

\[
\partial_s g_0 + g_1 = 0, \quad \hat{K} g_0 = v,
\]

hence \( \partial_s g_0 \in \mathcal{H}_\beta \), \( \hat{h}_0^\beta g_0 \in \mathcal{H}_\beta \) and \( \hat{K} g_0 = v \), which shows that \( \pi_0 (\partial_s + \hat{H})^{-1} \pi_1^* v = \hat{K}_s^{-1} v \). This completes the proof of (6.16). \( \Box \)

7. Vacua and KMS states for abstract Klein-Gordon equations

In this section we consider vacuum and KMS states for abstract, time-independent Klein-Gordon equations, which can be reduced to the framework of Sect. 6. We will show that the covariances of the vacuum and double \( \beta \)-KMS states can be expressed by the Calderón projectors defined in Sect. 5.

7.1. Vacua and KMS states. Let us consider an abstract Klein-Gordon equation

\[
(\partial_t + \bar{w}^*) (\partial_t - \bar{w}) \tilde{\phi} + \hat{h}_0 \tilde{\phi} = 0,
\]

as in Sect. 6 where \( \tilde{\phi} : \mathbb{R} \rightarrow \mathcal{H} \) and \( \mathcal{H} \) is a Hilbert space. We denote by

\[
\tilde{P} = (\partial_t + \bar{w}^*) (\partial_t - \bar{w}) + \hat{h}_0
\]

the corresponding Klein-Gordon operator. In the sequel we use the notation introduced in Subsect. 5.1.

The assumptions corresponding to those in Subsect. 5.1 are as follows:

We assume that there exists a dense subspace \( \mathcal{D} \subset \mathcal{H} \) and set

\[
\tilde{\mathcal{Y}} := \mathcal{D} \oplus \tilde{\mathcal{D}}, \quad \tilde{f}^* \tilde{\phi} := (f_1 | \tilde{f}_0) + (\tilde{f}_0 | f_1), \quad \tilde{f} = \left( \begin{array}{c} \tilde{f}_0 \\ f_1 \end{array} \right) \in \tilde{\mathcal{Y}}.
\]

We fix linear operators \( \tilde{h}_0, \tilde{w}, \tilde{w}^* \) on \( \mathcal{H} \) with domain \( \tilde{\mathcal{D}} \) such that:

\[
(\hat{u} | \tilde{h}_0 u) \geq 0, \quad (\bar{w}^* u | v) = (u | \bar{w} v), \quad u, v \in \tilde{\mathcal{D}},
\]

\[
\|w u\|^2 \leq (1 - \delta) (u | \tilde{h}_0 u), \quad \|\bar{w}^* u\|^2 \leq c (u | \tilde{h}_0 u), \quad u \in \tilde{\mathcal{D}} \text{ for } c > 0, 0 < \delta < 1.
\]

Setting \( \tilde{\omega}_0 (u, u) = (u | \tilde{h}_0 u) \) with \( \tilde{\omega}_0 = \tilde{\mathcal{D}} \), it follows that \( \tilde{\omega}_0 \) is closeable and we still denote by \( \tilde{h}_0 \) the operator associated to \( \tilde{\omega}_0 \), i.e. the Friedrichs extension of \( \tilde{h}_0 \) on \( \mathcal{D} \). We assume that \( \text{Ker} \tilde{h}_0 = \{ 0 \} \) and deduce then from (7.2) that hypotheses (6.2) are satisfied by \( \tilde{h}_0, \tilde{w}, \tilde{w}^* \). By construction \( \tilde{\mathcal{D}} \) is dense in \( \tilde{h}_0^{\beta} \mathcal{H} \).

We set then

\[
\tilde{f}^* \tilde{\mathcal{E}} \tilde{f} = (\tilde{f}^* \tilde{f})_\mathcal{E} = \| f_1 - i \tilde{w} \tilde{f}_0 \|^2 + (\tilde{f}_0 | \tilde{h}_0 \tilde{f}_0), \quad \tilde{f} \in \tilde{\mathcal{Y}},
\]

and by the density of \( \mathcal{D} \) in \( \tilde{h}_0^{\beta} \mathcal{H} \) we obtain that \( \tilde{\mathcal{Y}}_{\text{en}} = \tilde{\mathcal{E}} \). Setting then

\[
b = \tilde{H} = \left( \begin{array}{cc} -i \tilde{w} & \mathbb{I} \\ \tilde{h}_0 & i \tilde{w}^* \end{array} \right),
\]

where \( \tilde{H} \) is defined as a selfadjoint operator on \( \tilde{\mathcal{E}} \) by Prop. 6.1 we see that the identity (5.1) follows from (7.1), (7.3).

We can then apply Subsects. 5.2, 5.3 and define the vacuum state \( \tilde{\omega}_{\text{vac}} \), the \( \beta \)-KMS state \( \tilde{\omega}_\beta \) and the double \( \beta \)-KMS state \( \tilde{\omega}_d \) associated to the symplectic dynamics \( r_t = e^{i t \tilde{h}} \).

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7.2. The Calderón projectors. In Subsect. 6.4 we defined the Wick rotated operators
\[ \tilde{K}_\beta = -(\partial_s + i\tilde{\omega}^*)(\partial_s - i\tilde{\omega}) + \tilde{h}_0 \]
and the Hilbert spaces \( \tilde{\mathcal{H}}_\beta \) for \( 0 < \beta \leq \infty \) defined in (6.9). We now define Calderón projectors for \( \tilde{K}_\beta \), which are similar to the Calderón projectors for the operators \( B_\beta = \partial_s + \tilde{H} \), acting on the Hilbert spaces \( \tilde{\mathcal{E}}_\beta \), defined in 4.1.2 and 4.1.3.

7.2.1. Calderón projectors for \( \tilde{K}_\infty \). We follow the construction and notation in 4.1.2 in particular \( I^\pm_\infty = \pm [0, +\infty[ \) and \( i^\pm_\infty \) is the extension by 0 in \( \mathbb{R} \setminus I^\pm_\infty \). If \( \tilde{u} \in C^0(\mathbb{R}; \tilde{\mathcal{H}}) \) we denote by \( \tilde{\gamma}_\infty \tilde{u} \) the trace of \( \tilde{u} \) at \( s = 0 \):
\[ \tilde{\gamma}_\infty \tilde{u} = \left( \begin{array}{c} u(0) \\ - (\partial_s - i\tilde{\omega}) \tilde{u}(0) \end{array} \right), \]
whose formal adjoint \( \tilde{\gamma}_\infty^* \) is given by:
\[ \tilde{\gamma}_\infty^* \tilde{g} = \delta_0(s) \otimes \tilde{g}_1 + \delta_0(s) \otimes (\tilde{g}_0 - i\tilde{\omega}^* \tilde{g}_1). \]
If \( \tilde{u} \in \overline{C^0(I^\pm_\infty; \tilde{\mathcal{H}})} \) satisfies \( \tilde{K} \tilde{u} = 0 \) in \( I^\pm_\infty \) we set
\[ \tilde{\gamma}^\pm_\infty \tilde{u} = \left( \begin{array}{c} \tilde{u}(0^+) \\ - (\partial_s - i\tilde{\omega}) \tilde{u}(0^+) \end{array} \right) = \left( \begin{array}{c} \tilde{g}_0 \\ \tilde{g}_1 \end{array} \right), \]
We have formally
\[ \tilde{K}_\infty i^\pm_\infty \tilde{u}(s) = \pm (\delta_0(s) \otimes \tilde{g}_0 + \delta_0(s) \otimes (\tilde{g}_1 - i\tilde{\omega}^* \tilde{g}_0)) \]
for
\[ \tilde{S} = \begin{pmatrix} 2i\tilde{\omega}^* & - \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \]
It follows that \( i^\pm_\infty \tilde{u} = \mp \tilde{K}^{-1} \tilde{\gamma}^\pm_\infty \tilde{S} \tilde{g} \) for \( \tilde{g} = \tilde{\gamma}^\pm_\infty \tilde{u} \). This implies formally that
\[ \tilde{g} = \mp \tilde{\gamma}^\pm_\infty \tilde{K}^{-1} \tilde{\gamma}^* \tilde{S} \tilde{g} \]
if \( \tilde{g} = \tilde{\gamma}^\pm_\infty \tilde{u} \) for \( \tilde{u} \) solving \( \tilde{K} \tilde{u} = 0 \) in \( I^\pm_\infty \). This leads to the following definition.

**Definition 7.1.** The Calderón projectors \( \tilde{c}^\pm_\infty \) are
\[ (7.4) \quad \tilde{c}^\pm_\infty = \mp \tilde{\gamma}^\pm_\infty \tilde{K}^{-1} \tilde{\gamma}^* \tilde{S}. \]

**Proposition 7.2.** We have:
\[ \tilde{c}^\pm_\infty = C^\pm_\infty, \]
where \( C^\pm_\infty \) are the Calderón projectors for \( B_\infty = \partial_s + b \), with \( b = \tilde{H} \), \( \eta = \tilde{E} \), defined in Def. 4.1.

**Proof.** We prove only the + case. Let \( F \in \overline{C^0(I^\pm_\infty; \tilde{\mathcal{E}})} \) with \( (\partial_s + \tilde{H}) F(s) = 0 \) in \( I^\pm_\infty \). If \( \tilde{u}(s) = F_0(s) \) we have \( \tilde{K} \tilde{u}(s) = 0 \) and \( F_1(s) = -(\partial_s - i\tilde{\omega}) \tilde{u}(s) \) in \( I^\pm_\infty \) which implies that \( \Gamma^\pm_\infty F = \tilde{\gamma}^\pm_\infty \tilde{u} \). We have \( (\partial_s + \tilde{H}) i^\pm_\infty F(s) = \delta_0(s) \otimes \tilde{g} \) for \( \tilde{g} = \Gamma^\pm_\infty F \) which implies that
\[ \tilde{K} i^\pm_\infty F_0(s) = -\delta_0(s) \otimes \tilde{g}_0 + \delta_0(s) \otimes (\tilde{g}_1 - i\tilde{\omega}^* \tilde{g}_0). \]
This implies using the relation between \( \tilde{K}^{-1} \) and \( (\partial_s + \tilde{H})^{-1} \) in Prop. 6.2 that:
\[ C^\pm_\infty \tilde{g} = \Gamma^\pm_\infty (\partial_s + \tilde{H})^{-1} (\delta_0 \otimes \tilde{g}) = \tilde{\gamma}^\pm_\infty \tilde{K}^{-1} (-\delta_0(s) \otimes \tilde{g}_0 + \delta_0(s) \otimes (\tilde{g}_1 - i\tilde{\omega}^* \tilde{g}_0)) = \tilde{c}^\pm_\infty \tilde{g}. \]
From Subsect. 5.2, we obtain the following result, expressing the covariances of the vacuum state \( \tilde{\omega}_{\text{vac}} \) for \( P \) in terms of the Calderón projectors \( \tilde{c}^\pm_\infty \) for the Wick rotated operator \( \tilde{K}_\infty \).
Proposition 7.3. The covariances of the vacuum state $\omega_{\text{vac}}$ are equal to:

$$\lambda_{\text{vac}}^\pm = \pm \hat{\varphi} \circ \tilde{c}_\infty^\pm.$$

7.2.2. Calderón projectors for $\tilde{K}_\beta$. We follow now the construction and notation in [1,1,3] in particular $I^\pm_\beta = \pm [0, \frac{\beta}{2}]$ and $i^\pm_\beta$ is the extension by 0 in $S_\beta \setminus I^\pm_\beta$. If $\tilde{u} \in C^\infty(S_\beta; \tilde{H})$ we denote by $\tilde{\gamma}_\beta \tilde{u}$ the vector obtained from its traces at $s = 0$ and $s = \frac{\beta}{2}$:

$$\tilde{\gamma}_\beta \tilde{u} = \tilde{\gamma}_\beta^{(0)} \tilde{u} \oplus \tilde{\gamma}_\beta^{(\frac{\beta}{2})} \tilde{u},$$

for

$$\tilde{\gamma}_\beta^{(0)} \tilde{u} = \left( \tilde{u}(0), - (\partial_s - i \tilde{\omega}) \tilde{u}(0) \right), \quad \tilde{\gamma}_\beta^{(\frac{\beta}{2})} \tilde{u} = \left( \tilde{u}(\frac{\beta}{2}), (\partial_s - i \tilde{\omega}) \tilde{u}(\frac{\beta}{2}) \right).$$

Note the change of sign in the second component of $\tilde{\gamma}^{(\frac{\beta}{2})} \tilde{u}$, which corresponds to choosing the exterior normal derivative to $I^{\pm}_\beta$. We have

$$\tilde{\gamma}_\beta^+ = \tilde{\gamma}_\beta^{(0)*} + \tilde{\gamma}_\beta^{(\frac{\beta}{2})*},$$

for

$$\tilde{\gamma}_\beta^{(0)*} f^{(0)} = \delta_0'(s) \otimes f_1^{(0)} + \delta_0(s) \otimes (f_0^{(0)} - i \tilde{\omega}^* f_1^{(0)}),$$

$$\tilde{\gamma}_\beta^{(\frac{\beta}{2})*} f^{(\frac{\beta}{2})} = \delta_\frac{\beta}{2}'(s) \otimes f_1^{(\frac{\beta}{2})} + \delta_\frac{\beta}{2}(s) \otimes (f_0^{(\frac{\beta}{2})} + i \tilde{\omega}^* f_1^{(\frac{\beta}{2})}).$$

If $\tilde{u} \in C^\infty(I^{\pm}_\beta; \tilde{H})$ satisfies $\tilde{K} \tilde{u} = 0$ in $I^{\pm}_\beta$ we set:

$$\tilde{\gamma}_\beta^+ \tilde{u} = \tilde{\gamma}_\beta^{(0)\pm} \tilde{u} \oplus \tilde{\gamma}_\beta^{(\frac{\beta}{2})\pm} \tilde{u},$$

for

$$\tilde{\gamma}_\beta^{(0)\pm} \tilde{u} = \left( \tilde{u}(0^\pm), - (\partial_s - i \tilde{\omega}) \tilde{u}(0^\pm) \right), \quad \tilde{\gamma}_\beta^{(\frac{\beta}{2})\pm} \tilde{u} = \left( \tilde{u}(\pm \frac{\beta}{2}), (\partial_s - i \tilde{\omega}) \tilde{u}(\pm \frac{\beta}{2}) \right).$$

The same computation as before shows that if

$$\tilde{\gamma}_\beta^{(s)} \tilde{u} = \tilde{g}^{(s)} = \left( \frac{g_0^{(s)}}{g_1^{(s)}} \right), \text{ for } s = 0, \frac{\beta}{2},$$

one has:

$$\tilde{K}^{1\pm}_\beta \tilde{u} = \pm \left( \delta_0'(s) \otimes g_0^{(0)} + \delta_0(s) \otimes (g_1^{(0)} - i \tilde{\omega}^* g_0^{(0)}) \right)$$

$$\pm \left( \delta_\frac{\beta}{2}'(s) \otimes g_0^{(\frac{\beta}{2})} + \delta_\frac{\beta}{2}(s) \otimes (g_1^{(\frac{\beta}{2})} + i \tilde{\omega}^* g_0^{(\frac{\beta}{2})}) \right),$$

for

$$\tilde{S}^{(0)} = \left( \begin{array}{cc} 2i \tilde{\omega}^* & - \mathbb{I} \\ \mathbb{I} & 0 \end{array} \right), \quad \tilde{S}^{(\frac{\beta}{2})} = \left( \begin{array}{cc} - 2i \tilde{\omega}^* & - \mathbb{I} \\ \mathbb{I} & 0 \end{array} \right).$$

Again this leads to the following definition.

Definition 7.4. The Calderón projectors $\tilde{c}_\beta^\pm$ are

$$\tilde{c}_\beta^\pm = \mp \gamma_\beta^\pm \tilde{K}^{-1} \gamma_\beta^* (\tilde{S}^{(0)} \pi^{(0)} + \tilde{S}^{(\frac{\beta}{2})} \pi^{(\frac{\beta}{2})}),$$

where $\pi^{(0)} \frac{\beta}{2} \tilde{g} = \tilde{g}^{(0)} \frac{\beta}{2}$. 

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Proposition 7.5. On has
\[ \tilde{c}^+_\beta = (\mathbb{1} \oplus T) \circ C^+_\beta \circ (\mathbb{1} \oplus T)^{-1}, \]
where \( C^+_\beta \) are the Calderón projectors for \( B_\beta = \partial_s + b \), with \( b = \tilde{H}, \beta = \tilde{E} \), defined in Def. 4.3, and \( T = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \).

**Proof.** We prove only the + case. Let \( F \in C^0(I^+_\beta; \tilde{E}) \) with \( (\partial_s + \tilde{H})F(s) = 0 \) in \( I^+_\beta \). If \( \tilde{u}(s) = F_0(s) \) we have \( \tilde{K}\tilde{u}(s) = 0 \) and \( F_1(s) = -\partial_s - i\tilde{w}\tilde{u}(s) \) in \( I^+_\beta \). This implies that
\[ \tilde{\gamma}_\beta(0) + \tilde{u} = \Gamma^{(0)}_\beta F, \quad \tilde{\gamma}_\beta(\tilde{\beta}) + \tilde{u} = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \Gamma^{(\tilde{\beta})}_\beta F, \]
where \( \Gamma^{(0)}_\beta, \Gamma^{(\tilde{\beta})}_\beta \) are defined in (4.17). Setting
\[ \tilde{g} = \tilde{c}^+_\beta \tilde{u}, \quad f = \Gamma^+_\beta F, \]
we can rewrite this identity as
\[ (\ref{eq7.7}) \quad \tilde{g} = (\mathbb{1} \oplus T)f, \quad \text{for } T = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \]
Next we have
\[ (\partial_s + \tilde{H})\tilde{c}^+_\beta = \delta_0(s) \otimes f^{(0)} - \delta_\beta(s) \otimes f^{(\tilde{\beta})}, \]
where \( f = f^{(0)} \oplus f^{(\tilde{\beta})} \), which implies that:
\[ \tilde{K}\tilde{c}^+_\beta = \left( -\delta_0(s) \otimes f^{(0)}_1 + \delta_\beta(s) \otimes (f^{(0)}_1 - i\tilde{w}^* f^{(0)\dagger}_0) \right) \]
\[ + \left( \delta_\beta(s) \otimes f^{(\tilde{\beta})}_1 + \delta_\beta(s) \otimes (-f^{(\tilde{\beta})}_1 + i\tilde{w}^* f^{(\tilde{\beta})\dagger}_0) \right). \]
If we compare this with the first line in (7.5) and use also the relation between \( \tilde{K}\) and \( (\partial_s + \tilde{H})^{-1} \) in Prop. 6.2 we obtain that \( \tilde{c}^+_\beta = (\mathbb{1} \oplus T) \circ C^+_\beta \circ (\mathbb{1} \oplus T)^{-1} \) as claimed. \( \square \)

As in Prop. 7.3 we can using Subsect. 5.4 express the covariances of the double \( \beta \)-KMS state \( \tilde{\omega}_\beta \) for \( \tilde{P} \) in terms of the Calderón projectors \( \tilde{c}^+_\beta \) for the Wick rotated operator \( \tilde{K}_\beta \).

Proposition 7.6. The covariances of the double \( \beta \)-KMS state for \( \tilde{P} \) are equal to
\[ \tilde{\lambda}^\pm_\beta = \pm \tilde{Q} \circ (\mathbb{1} \oplus T)^{-1} \tilde{c}^+_\beta (\mathbb{1} \oplus T), \quad \text{for } \tilde{Q} = \tilde{q} \oplus -\tilde{q}. \]

8. Klein-Gordon equations on stationary spacetimes

In this section we consider Klein-Gordon equations on stationary spacetimes. If the lapse function \( N \) associated to the Killing vector field \( w \) is equal to 1, one can directly reduce oneself to the situation of Sect. 7. In general one has to replace the Klein-Gordon operator \( P \) by \( \tilde{P} = NPN \), which has the same purpose as a conformal transformation.

As an application we consider the Klein-Gordon operator \( P \) in \( M^+ \) and express the covariances of the double \( \beta \)-KMS state in \( M^- \cup M^+ \) using the Calderón projectors for the elliptic operator \( K_\beta \) obtained from \( P \) by Wick rotation in the Killing time coordinate \( t \).

8.1. Klein-Gordon equations on stationary spacetimes.
8.1.1. Stationary metrics. Let \((S, h)\) a Riemannian manifold, \(N \in C^\infty(S), N > 0\) and \(w^i\) a vector field on \(S\). Let us denote by \(y\) the elements of \(S\). We define the Lorentzian metric \(g\) on \(M = \mathbb{R} \times S\):
\[
g = -N^2(y)dt^2 + h_{ij}(y)(dy^i + w^i(y)dt)(dy^j + w^j(y)dt).
\]
We assume that \(\{0\} \times S\) is a Cauchy surface for \((M, g)\). Such spacetimes are called standard stationary spacetimes in the terminology of [S2].

The vector field \(\frac{\partial}{\partial t}\) is Killing for \(g\) and is time-like iff
\[
N^2(y) > w^i(y)h_{ij}(y)w^j(y), \quad y \in S.
\]
We will need later to impose the following stronger condition:

**Definition 8.1.** The Killing vector field \(\frac{\partial}{\partial t}\) is uniformly time-like if there exists \(0 < \delta < 1\) such that:
\[
(1 - \delta)N^2(y) \geq w^i(y)h_{ij}(y)w^j(y), \quad x \in S.
\]

We have:
\[
|g| = N^2|h|, n = N^{-1}(\frac{\partial}{\partial t} - w),
\]
where \(n\) is the future directed unit normal to the foliation \(S_t = \{t\} \times S\).

8.1.2. Stationary Klein-Gordon operators. We consider a stationary Klein-Gordon operator on \((M, g)\):
\[
P = -\Box_g + m(y), \quad m \in C^\infty(S; \mathbb{R}).
\]
We will always assume that
\[
m(y) \geq m_0^2, \quad m_0 > 0,
\]
and the Klein-Gordon equation is massive. Setting
\[
h_0 := \nabla^* h^{-1} \nabla + m, \quad w := w^i \partial_i,
\]
we have
\[
P = (\partial_t + w^*)N^{-2}(\partial_t - w) + h_0,
\]
where in [8.5], [8.6] the adjoints are computed with respect to the scalar product
\[
(u|v)_M = \int_M \bar{v} e N|h|^\frac{1}{2} dt dy.
\]

8.1.3. Hilbert spaces. We denote by \(L^2(M)\) the Hilbert space associated to the scalar product \((\cdot|\cdot)_M\) and by \(\mathcal{H} = L^2(S,h|\frac{1}{2} dy)\) the Hilbert space associated to the scalar product
\[
(u|v)_\mathcal{H} = \int_S \bar{v} e N|h|^\frac{1}{2} dy.
\]
We will also need the Hilbert space \(\tilde{\mathcal{H}} = L^2(S,N|h|^\frac{1}{2} dy)\) associated to the scalar product
\[
(u|v)_{\tilde{\mathcal{H}}} = \int_S \bar{v} e N|h|^\frac{1}{2} dy,
\]
so that \(L^2(M) = L^2(\mathbb{R},dt;\tilde{\mathcal{H}})\).
8.1.4. An operator inequality. The inequality in Lemma 8.2 below is understood as an operator inequality on $\tilde{H}$.

**Lemma 8.2.** Assume that $\frac{\partial}{\partial t}$ is uniformly time-like. Then

$$ (1 - \delta)h_0 \geq w^* N^{-2} w \text{ on } C_0^\infty(S). $$

**Proof.** Let $\mathcal{X}$ a real vector space, $k \in L_a(\mathcal{X}, \mathcal{X}^*)$ strictly positive and $c \in \mathcal{X}$. Then for $\gamma = kc \in \mathcal{X}^*$ and $\xi \in \mathbb{C} \mathcal{X}^*$ we have

$$ (\xi - (\xi|c)\gamma) \cdot k^{-1}(\xi - (\xi|c)\gamma) $$

$$ = \xi \cdot k^{-1} \xi - 2\text{Re}((\xi|c)\gamma \cdot k^{-1} \xi) + |(\xi|c)|^2 \gamma \cdot k^{-1} \gamma $$

$$ = \xi \cdot k^{-1} \xi - (2 - c \cdot kc)|\xi|c|^2, $$

hence

$$ k^{-1} - |c\rangle\langle c| \geq (1 - c \cdot kc)|c\rangle\langle c|. $$

Replacing $k$ by $(1 - \delta)^{-1} k$ shows that if $(1 - \delta) \geq c \cdot kc$ we have

$$(7.8) \quad (1 - \delta)k^{-1} \geq |c\rangle\langle c|. $$

For $u \in C_0^\infty(S)$ we write

$$ (u|((1 - \delta)h_0 - w^* N^{-2} w)u)_\tilde{H} $$

$$ = \int_S \partial_y \pi((1 - \delta)h_0^j(y) - w^j(y) N^{-2} w^i(y)) \partial_y u(y) N|y|^{1/2} dy. $$

Applying (7.8) under the integral sign for $k = h_0^j(y)$, $c = N^{-1}(y)w^i(y)$ we obtain the lemma. \(\square\)

8.2. Selfadjoint operators. In the rest of this section we will assume that $\frac{\partial}{\partial t}$ is uniformly time-like.

Let $q_0(u, u) = (u|h_0 u)_\tilde{H}$ with $\text{Dom}q_0 = C_0^\infty(S)$. The form $q_0$ is closeable and we denote still denote by $h_0$ the selfadjoint operator on $\tilde{H}$ associated to $q_0^\sharp$, i.e the Friedrichs extension of $h_0$ on $C_0^\infty(S)$. We have:

$$ h_0 : h_0^\frac{1}{2} \tilde{H} \to h_0^\frac{1}{2} \tilde{H}. $$

Note that $h_0^\frac{1}{2} \tilde{H} \subset \tilde{H}$ since $h_0 \geq m_0^2$. We set also

$$(8.8) \quad \tilde{q}_0(u, u) = q_0(Nu, Nu)_\tilde{H}, \quad \text{Dom} \tilde{q}_0 = C_0^\infty(S), $$

and denote by $\tilde{h}_0$ the selfadjoint operator on $\tilde{H}$ associated to $\tilde{q}_0$, which formally equals $Nh_0 N$. From (8.8) we obtain that

$$(8.9) \quad N : \tilde{h}_0^\frac{1}{2} \tilde{H} \to h_0^\frac{1}{2} \tilde{H}, \quad N : h_0^\frac{1}{2} \tilde{H} \to \tilde{h}_0^\frac{1}{2} \tilde{H}, $$

and we have:

$$(8.10) \quad \tilde{h}_0 = Nh_0 N \text{ as an identity in } B(\tilde{h}_0^\frac{1}{2} \tilde{H}, \tilde{h}_0^\frac{1}{2} \tilde{H}). $$

We also set

$$ \tilde{w} = N^{-1} w N = N^{-1} w^i \cdot \partial_y N, $$

$$(8.11) \quad \tilde{w}^* = N w^* N^{-1} = -|y|^{-\frac{1}{2}} \partial_y \cdot w^i |y|^{1/2}, $$

with domain $C_0^\infty(S)$.

Let us introduce the assumption

$$(8.12) \quad N^{-2} w^i \cdot (\nabla^h_i N), \quad N^{-1} \nabla^h_i w^i \text{ are bounded on } S. $$

**Lemma 8.3.** Assume (8.12). Then $\tilde{h}_0, \tilde{w}, \tilde{w}^*$ satisfy the conditions (7.2) for $\tilde{D} = C_0^\infty(S)$.\(\square\)
Proof. We have seen in Lemma [8.2] that \( w^* N^{-2} w \leq (1 - \delta) h_0 \) on \( C_0^\infty(S) \), which implies \( \tilde{w}^* \tilde{w} \leq (1 - \delta) \tilde{h}_0 \) on \( C_0^\infty(S) \). Let \( (y^1, \ldots, y^d) \) be local coordinates on \( S \). We have
\[
\tilde{w} = w^1 \partial_{y^1} + N^{-1} w^i \partial_{y^i} N, \quad \tilde{w}^* = -w^1 \partial_{y^1} - |h|^{-\frac{1}{2}} (\partial_{y^i} w^i |h|^{\frac{1}{2}}) = -w^1 \partial_{y^1} - \nabla^h w^i.
\]
Condition (8.12) implies that \( \tilde{w}^* = -w + r \), where \( r \in C^\infty(S) \) and \( N^{-1} \) is bounded on \( \tilde{H} \). The inequality \( \tilde{w} \bar{w}^* \leq C \tilde{h}_0 \) follows from \( \tilde{w}^* \tilde{w} \leq (1 - \delta) \tilde{h}_0 \) and \( m_0^2 N^2 \leq \tilde{h}_0 \). \( \Box \)

8.3. Associated first order system. We set:
\[
\frac{\partial}{\partial t} \phi = \left( i^{-1} N^{-1} (\partial_t - w) \phi(t) \right) = f = \left( f_0 \ f_1 \right) \tag{8.13}
\]
and rewrite \( P \phi = 0 \) as:
\[
N^{-1} \partial_t f = i H f, \quad H = \begin{pmatrix} -i N^{-1} w & I \\ \frac{1}{h_0} & i w^* N^{-1} \end{pmatrix}, \quad f \in C_0^\infty(S; \mathbb{C}^2). \tag{8.14}
\]
The conserved energy is
\[
\tilde{\mathcal{J}} \cdot E f = \| f_1 - i N^{-1} w f_0 \|^2_{\tilde{H}} + (f_0 | h f_0)_{\tilde{H}}, \quad h = h_0 - w^* N^{-2} w,
\]
and the conserved charge is
\[
\tilde{\mathcal{J}} \cdot q f = (f_1 | f_0)_{\tilde{H}} + (f_0 | f_1)_{\tilde{H}}.
\]
The energy space \( \tilde{E} \) associated to \( E \) equals \( h_0^{\frac{1}{2}} \tilde{H} \oplus \tilde{H} \) as topological vector spaces.

8.4. Reduction. We now introduce the Klein-Gordon operator
\[
\tilde{P} = N P N = (\partial_t + \tilde{w}^*) (\partial_t - \tilde{w}) + \tilde{h}_0,
\]
which is of the form considered in Sects. [6, 7]. The operators \( \tilde{\mathcal{G}}, \tilde{H} \), the energy \( \tilde{E} \) and charge \( \tilde{q} \) are defined as in Subsect. [8.3]
\[
\tilde{\mathcal{G}} \tilde{\phi} = \left( i^{-1} (\partial_t - \tilde{w}) \phi(t) \right), \quad \tilde{H} = \begin{pmatrix} -i \tilde{w} & I \\ \frac{1}{\tilde{h}_0} & i \tilde{w}^* \end{pmatrix},
\]
\[
\tilde{\mathcal{J}} \tilde{E} \tilde{f} = \| \tilde{f}_1 - i \tilde{w} \tilde{f}_0 \|^2_{\tilde{H}} + (\tilde{f}_0 | h \tilde{f}_0)_{\tilde{H}}, \quad \tilde{\mathcal{J}} \tilde{q} \tilde{f} = (\tilde{f}_1 | \tilde{f}_0)_{\tilde{H}} + (\tilde{f}_0 | \tilde{f}_1)_{\tilde{H}}, \quad \tilde{f} \in C_0^\infty(S; \mathbb{C}^2).
\]
Setting
\[
Z := \begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix}, \quad Z' := \begin{pmatrix} I & 0 \\ 0 & N^{-1} \end{pmatrix},
\]
we have:
\[
\frac{\partial}{\partial t} N = Z \tilde{\mathcal{G}}, \quad N^{-1} \partial_t - i H = Z' (\partial_t - i \tilde{H}) Z^{-1}, \tag{8.17}
\]
\[
Z^* E Z = \tilde{E}, \quad Z^* q Z = \tilde{q} \text{ on } C_0^\infty(S; \mathbb{C}^2). \tag{8.18}
\]
We saw that the energy space \( \tilde{E} \) associated to \( \tilde{E} \) equals \( h_0^{\frac{1}{2}} \tilde{H} \oplus \tilde{H} \), and from (8.9) we obtain that:
\[
Z : \tilde{E} \xrightarrow{\sim} \mathcal{E}.
\]
8.5. Vacuum and KMS states. In Subsect. 7.1 we defined vacuum and \( \beta \)-KMS states for \( \tilde{P} \). We obtain the corresponding vacuum and \( \beta \)-KMS states for \( P \) by conjugation by the map \( \tilde{Z} \).

**Definition 8.4.** We define the vacuum state \( \omega_{\text{vac}} \), the \( \beta \)-KMS state \( \omega_\beta \) and the double \( \beta \)-KMS state \( \omega_\beta^d \) by their covariances:

\[
\lambda_{\text{vac}}^\pm = (Z^{-1})^* \tilde{\lambda}_\text{vac}^\pm Z^{-1}, \quad \lambda_\beta^\pm = (Z^{-1})^* \tilde{\lambda}_\beta^\pm Z^{-1}, \\
\lambda_\beta^d = (Z^{-1} \oplus Z^{-1})^* \tilde{\lambda}_\beta^d (Z^{-1} \oplus Z^{-1}),
\]

where the covariances \( \lambda_{\text{vac}}^\pm, \lambda_\beta^\pm \) and \( \lambda_\beta^d \) are defined in Defs. 5.4, 5.2 and Prop. 5.4 for \( b = \tilde{H} \).

8.6. The Wick rotated operator.

8.6.1. The Wick rotated metric. Let us denote by \( \mathbf{k} \) the complex metric on \( \mathbb{R} \times S \) obtained from \( \mathbf{g} \) by the substitution \( t = i s \). We have:

\[
\mathbf{k} = N^2(y) ds^2 + h_{ij}(y)(dy^i + iw^i(y)ds)(dy^j + iw^j(y)ds),
\]

Using that \( \frac{\partial}{\partial t} \) is uniformly time-like we obtain that there exists \( C > 0 \) such that

\[
|\tilde{\eta}| \leq C \tilde{\eta} \text{Re}(\eta), \quad y \in S, \eta \in \mathcal{C}T_y S,
\]

Moreover we have

\[
|\mathbf{k}|(y) = \text{real valued and } |\mathbf{k}|^{1/2}(y) = N(y)|\mathbf{h}|^{1/2}(y).
\]

With the terminology in Def. 9.1 this means that the complex metric \( \mathbf{k} \) is uniformly sectorial.

If \( \Omega = ]0, +\infty[ \times S \), the outer unit normal vector field to \( \Omega \) for \( \mathbf{k} \), see 9.1.2 is

\[
\nu = -N^{-1}(\frac{\partial}{\partial s} - iw),
\]

while if \( \Omega = ]0, \frac{\beta}{2}[ \times S \) it equals

\[
\nu^{(0/\frac{\beta}{2})} = \pi N^{-1}(\frac{\partial}{\partial s} - iw) \text{ on } \{0/\frac{\beta}{2}\} \times S.
\]

The real vectors \( \text{Im} \nu, \text{Im} \nu^{(0/\frac{\beta}{2})} \) are tangent to \( S \), i.e. condition (9.5) below is satisfied.

8.6.2. The Wick rotated operator. We consider now the Wick rotated operator \( K \) obtained from \( P \) by the substitution \( t = is \). We have:

\[
K = -\Delta_k + m(y) = -\partial_s + iw^* N^{-2}(\partial_s - iw) + h_0,
\]

acting on the Hilbert spaces \( \mathcal{H}_\beta \) for \( 0 < \beta \leq \infty \) defined in (6.9). We refer the reader to 9.1.1 fo the Laplacian \( \Delta_k \) associated to \( \mathbf{k} \). We recall that

\[
\tilde{\mathcal{H}}_\beta = L^2(S_\beta \times S, N(y)|\mathbf{h}|^{1/2}(y)dyds), \quad 0 < \beta < \infty,
\]

\[
\mathcal{H}_\infty = L^2(\mathbb{R} \times S, N(y)|\mathbf{h}|^{1/2}(y)dyds).
\]

It follows from Lemma 8.2 that if \( h = h_0 - w^* N^{-2}w \) we have:

\[
h \sim h_0, \quad w^* N^{-2}w \lesssim h, \quad \text{on } C_0^\infty(\mathbb{R} \times S),
\]

where we use the scalar product of \( \tilde{\mathcal{H}}_\beta \) in the operator inequalities. We have:

\[
(u|Ku)_{\tilde{\mathcal{H}}_\beta} = \|N^{-1}\partial_s u\|^2_{\tilde{\mathcal{H}}_\beta} + (u|h_0)_{\tilde{\mathcal{H}}_\beta} - i(N^{-1}\partial_s u|N^{-1}wu)_{\tilde{\mathcal{H}}_\beta} - i(N^{-1}wu|N^{-1}\partial_s u)_{\tilde{\mathcal{H}}_\beta}, \quad u \in C_0^\infty(\mathbb{R} \times S).
\]
The sesquilinear form associated to the realization $K_\beta$ of $K$ is
\[
Q_\beta(u, u) = \|N^{-1} \partial_u \tilde{u}\|_{\tilde{H}_\beta}^2 + (u|hu)_{\tilde{H}_\beta} - i(N^{-1} \partial_u u|N^{-1} w u)_{\tilde{H}_\beta} - i(N^{-1} w u|N^{-1} \partial_u u)_{\tilde{H}_\beta},
\]
with domain $\text{Dom}Q_\beta = (K_0)^{-1/2} \tilde{H}_\beta$ and $K_0 = -N^{-2} \partial^2_x + h_0$ with its natural domain on $\tilde{H}_\beta$. From (8.9) we obtain that
\[
N^{-1} : \text{Dom}Q_\beta \xrightarrow{\sim} \text{Dom}\tilde{Q}_\beta, \quad Q_\beta(Nu, Nu) = \tilde{Q}_\beta(u, u),
\]
where $\tilde{Q}_\beta$ is defined in Subsect. 6.4. It follows that $Q_\beta$ is a closed sectorial form and we denote as before by $K_\beta$:
\[
K_\beta : K_0^{-1/2} \tilde{H}_\beta \xrightarrow{\sim} K_0^{1/2} \tilde{H}_\beta
\]
the induced operator. We have
\[
N : K_0^{-1/2} \tilde{H}_\beta \xrightarrow{\sim} K_0^{-1/2} \tilde{H}_\beta, \quad N : K_0^{1/2} \tilde{H}_\beta \rightarrow K_0^{1/2} \tilde{H}_\beta
\]
where $K_\beta$ is the operator defined in Subsect. 6.4.

8.7. **Calderón projectors.** We now define the Calderón projectors for $K_\beta$ and relate them to those for $K_\beta$ defined in Subsect. 7.2. We use the notation $I_\beta^\pm$, $\tilde{\beta}^\pm$, $\gamma_\beta^\pm$, $\delta_\beta^\pm$ introduced in Subsect. 7.2.

8.7.1. **Calderón projectors for $K_\infty$.** If $u \in \overline{C(0, T; \tilde{H})}$ the trace $\gamma_\infty u$ of $u$ on $s = 0$ is
\[
\gamma_\infty u = \begin{pmatrix}
    u(0) \\
    \frac{\alpha}{2} u(0)
\end{pmatrix},
\]
where $Z$ is defined in (8.10). We denote by $\gamma^\pm_\infty$ the formal adjoint of $\gamma_\infty$ from $L^2(\mathbb{R} \times S; |h|^{1/2} ds dy)$ to $L^2(S, |h|^{1/2} ds dy; \mathbb{C}^2) = \mathcal{H} \otimes \mathbb{C}^2$. We have:
\[
\gamma^\pm_\infty g = \delta^g_\infty(s) \otimes N^{-2} g_1 + \delta^g_\infty(s) \otimes (N^{-1} g_0 - iw^* N^{-2} g_1).
\]
If $u(s) \in \overline{C(0, T; \tilde{H})}$ satisfies $Ku = 0$ in $I^\pm_\infty$ we set
\[
\gamma^\pm_\infty u = \begin{pmatrix}
    u(0^\pm) \\
    -N^{-1}(\partial_s - iw) u(0^\pm)
\end{pmatrix},
\]
so that
\[
\gamma^\pm_\infty N = Z \gamma^\pm_\infty.
\]
Setting $g = \gamma_\infty u$ and $u = N \tilde{u}$, $\tilde{g} = \gamma^\pm_\infty \tilde{u}$, $Z^{-1} g$, we obtain from 7.2.1 that
\[
\tilde{g} = \mp \gamma^\pm_\infty K_\infty^{-1} \gamma^\pm_\infty \tilde{S} \tilde{g},
\]
hence using $\gamma^\pm_\infty N = Z \gamma^\pm_\infty$ and $K_\infty = NK_\infty N$,
\[
g = \mp \gamma^\pm_\infty K_\infty^{-1} N^{-1} \gamma^\pm_\infty \tilde{S} Z^{-1} g,
\]
where $\tilde{S} = \begin{pmatrix}
    2i \tilde{w}^* & -\mathbb{I} \\
    \mathbb{I} & 0
\end{pmatrix}$. A tedious computation shows that
\[
N^{-1} N^{-1} \gamma^\pm_\infty \tilde{S} Z^{-1} = \gamma^\pm_\infty S, \quad S := \begin{pmatrix}
    2i N w^* N^{-2} & -\mathbb{I} \\
    \mathbb{I} & 0
\end{pmatrix}.
\]
Note that the imaginary part of $\nu$ equals $N^{-1} w$ and its adjoint on $L^2(S, |h|^{1/2} ds dy)$ equals $Nw^* N^{-2}$ (recall that $w^*$ is the adjoint of $w$ for the scalar product of $L^2(S, N|h|^{1/2} ds dy)$).
This leads to the following definition.

**Definition 8.5.** The Calderón projectors $c^\pm_{\infty}$ for $K_\infty$ are:

$$c^\pm_{\infty} := \pm i \nu_{\infty}^{-1} \gamma^* S.$$

**Proposition 8.6.** The covariances of the vacuum state $\omega_{\text{vac}}$ are equal to:

$$\lambda^\pm_{\text{vac}} = \pm q \circ c^\pm_{\infty}.$$

**Proof.** This follows from the identities:

1. $\lambda^\pm_{\text{vac}} = Z^* \lambda^\pm_{\text{vac}} Z$, $\tilde{q} = Z^* q Z$,
2. $\lambda^\pm_{\text{vac}} = \pm \tilde{q}^\pm_{\infty}$, $c^\pm_{\infty} = \tilde{Z} v^\pm_{\infty} Z^{-1}$.

The identities in i) are obvious, the first identity in ii) is shown in Prop. 7.3, the second follows from the computations before Def. 8.5. \( \square \)

**8.7.2. Calderón projectors for $K_\beta$.** If $u \in \mathcal{C}^0(\mathcal{S}_\beta; \tilde{H})$ we denote by $\gamma_{\beta} u$ the vector obtained from its traces at $s = 0$ and $s = \frac{\beta}{2}$:

$$\gamma_{\beta} u = \gamma^{(0)}_{\beta} u \oplus \gamma^{(0)}_{\beta} u,$$

for

$$\gamma^{(0)}_{\beta} u = \left( \begin{array}{c} u(0) \\ - N^{-1}(\partial_s - iw) u(0) \end{array} \right), \quad \gamma^{(0)}_{\beta} u = \left( \begin{array}{c} u(0) \\ - N^{-1}(\partial_s - iw) u(\frac{\beta}{2}) \end{array} \right).$$

see (8.22), and we have:

$$\gamma_{\beta} N = (Z \oplus Z) \gamma_{\beta},$$

where $Z$ is defined in (8.16). Again we denote by $\gamma^*_{\beta}$ the formal adjoint of $\gamma_{\beta}$ from $L^2(\mathbb{R} \times S; N|h|^{\frac{1}{2}} dydz) = \mathcal{H}_\infty$ to $L^2(S, |h|^{\frac{1}{2}} dy; \mathbb{C}^2) \oplus L^2(S, |h|^{\frac{1}{2}} dy; \mathbb{C}^2)$. We have:

$$\gamma^*_{\beta} = \gamma^{(0)*}_{\beta} + \gamma^{(\frac{1}{2})*}_{\beta},$$

for

$$\gamma^{(0)*}_{\beta} g^{(0)} = \delta_0^{(s)}(s) \otimes N^{-2} g_1^{(0)} + \delta_0^{(s)}(s) \otimes (N^{-1} g_0^{(0)} - i w^* N^{-2} g_1^{(0)}),$$

$$\gamma^{(\frac{1}{2})*} g^{(0)} = - \delta_{\frac{1}{2}}^{(s)}(s) \otimes N^{-2} g_{1}^{(0)} + \delta_{\frac{1}{2}}^{(s)}(s) \otimes (N^{-1} g_0^{(0)} + i w^* N^{-2} g_{1}^{(0)}).$$

If $u \in \mathcal{C}^0(\mathcal{I}_{\beta}; \tilde{H})$ satisfies $K u = 0$ in $\mathcal{I}_{\beta}$ we set:

$$\gamma^\pm_{\beta} u = \gamma^{(0)\pm}_{\beta} u \oplus \gamma^{(\frac{1}{2})\pm}_{\beta} u,$$

for

$$\gamma^{(0)\pm}_{\beta} u = \left( \begin{array}{c} u(0, \pm) \\ - N^{-1}(\partial_s - iw) u(0, \pm) \end{array} \right), \quad \gamma^{(\frac{1}{2})\pm}_{\beta} u = \left( \begin{array}{c} u(\mp \frac{\beta}{2}) \\ - N^{-1}(\partial_s - iw) u(\mp \frac{\beta}{2}) \end{array} \right).$$

Setting $g = \gamma^\pm_{\beta} u$ and $u = N\tilde{u}$, $\tilde{g} = \gamma^\pm_{\beta} \tilde{u} = (Z \oplus Z)^{-1} g$, we obtain that

$$\tilde{g} = \mp \gamma^\pm_{\beta} \tilde{K}_{\beta}^{-1} (\gamma^{(0)*}_{\beta} \tilde{S}^{(0)} \tilde{g}^{(0)} + \gamma^{(\frac{1}{2})*}_{\beta} \tilde{S}^{(\frac{1}{2})} \tilde{g}^{(\frac{1}{2})}),$$

where $\tilde{S}^{(0)}$, $\tilde{S}^{(\frac{1}{2})}$ are defined in (7.6). The same computation as in (8.7.1) gives that

$$g = \mp \gamma^\pm_{\beta} K_{\beta}^{-1} (\gamma^{(0)*} S^{(0)} g^{(0)} + \gamma^{(\frac{1}{2})*} S^{(\frac{1}{2})} g^{(\frac{1}{2})}),$$

for

$$S^{(0)} := \left( \begin{array}{cc} 2 i N w^* N^{-2} & - \mathbb{I} \\ \mathbb{I} & 0 \end{array} \right), \quad S^{(\frac{1}{2})} := \left( \begin{array}{cc} -2 i N w^* N^{-2} & - \mathbb{I} \\ \mathbb{I} & 0 \end{array} \right).$$

Again this leads to the following definition.
Definition 8.7. The Calderón projectors \( e^\pm_\beta \) for \( K_\beta \) are:
\[
e^\pm_\beta := \mp \beta^{-1} (e_{(0)} (0,0) g + e_{(0)} (0,0) g),
\]
where \( e_{(0)} = g^{0,0} \).

Using now Prop. 7.6 the same argument as in 8.7.1 gives the following proposition.

Proposition 8.8. The covariances of the double \( \beta \)-KMS state \( \omega_d \) are equal to:
\[
\lambda^\pm_d = \pm Q \circ (\mathbb{1} \oplus T)^{-1} e^\pm_\beta (\mathbb{1} \oplus T), \quad Q = q \oplus q,
\]
where \( T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \).

8.8. The double \( \beta \)-KMS state in \( \mathcal{M}^+ \cup \mathcal{M}^- \). We now apply the computations of the previous subsections to \( S = \Sigma^+ \). In fact if \( \phi_t \) is the flow of the Killing vector field \( V \), the map
\[
\chi : \mathbb{R} \times \Sigma^+ \ni (t,y) \mapsto \phi_t(y) \in \mathcal{M}^+
\]
is a diffeomorphism such that \( \chi^* g \) is as in Subsect. 8.1.

We first claim that \( \frac{\partial}{\partial t} \) is uniformly time-like and (8.12) holds on \( \Sigma^+ \).

Proposition 8.9. Assume that hypothesis (H) holds. Then \( \frac{\partial}{\partial t} \) is uniformly time-like and (8.12) holds on \( \Sigma^+ \).

Proof. We first check that \( \frac{\partial}{\partial t} \) is uniformly time-like and that (8.12) holds on \( \Sigma^+ \setminus U \), where \( U \) is any small neighborhood of \( B \) in \( \Sigma^+ \), by hypotheses (H). To check the conditions on \( U \), we use Prop. 2.4. Recalling that \((u,\omega)\) are Gaussian normal coordinates to \( B \) in \((\Sigma,\mathbf{h})\), we obtain
\[
w \cdot \mathbf{h} w \in O(a^4), w \cdot (\nabla N) \in O(u^3), (\nabla \cdot w) \in O(u^2), w \cdot (\mathbf{h} \cdot w) \in O(u^2),
\]
from which our claim follows, since \( N(y) = \kappa u + O(a^3) \).

8.8.1. The double \( \beta \)-KMS state in \( \mathcal{M}^+ \cup \mathcal{M}^- \). Let us now define the double \( \beta \)-KMS state in \( \mathcal{M}^+ \cup \mathcal{M}^- \).

The wedge reflection \( R \) is an isometric involution from \((\mathcal{M}^-,g)\) to \((\mathcal{M}^+,g)\). It induces on \( \Sigma \) the weak wedge reflection \( r \), which equals the identity on \( B \) and maps \( \Sigma^- \) bijectively on \( \Sigma^+ \).

\( R \) reverses the time orientation, hence induces a unitary involution:
\[
\mathcal{R} : \left( \frac{C^\infty_0 (\mathcal{M}^-)}{PC^\infty_0 (\mathcal{M}^-)}, iG \right) \ni [u] \mapsto [u \circ R] \in \left( \frac{C^\infty_0 (\mathcal{M}^+)}{PC^\infty_0 (\mathcal{M}^+)}, -iG \right).
\]

In a more familiar language, \( \mathcal{R} \) is anti-symplectic. Since
\[
\rho_{\Sigma^\pm} \circ G : \left( \frac{C^\infty_0 (\mathcal{M}^\pm)}{PC^\infty_0 (\mathcal{M}^\pm)} \right) \xrightarrow{\mathcal{R}} \left( \frac{C^\infty_0 (\Sigma^\pm)}{PC^\infty_0 (\Sigma^\pm)}, q \right)
\]
is unitary, \( \mathcal{R} \) induces the unitary involution
\[
\mathcal{R}_\Sigma : \left( \frac{C^\infty_0 (\Sigma^-)}{PC^\infty_0 (\Sigma^-)}, q \right) \xrightarrow{\mathcal{R}_\Sigma} \left( \frac{C^\infty_0 (\Sigma^+)}{PC^\infty_0 (\Sigma^+)}, -q \right).
\]
The following expression for \( \mathcal{R}_\Sigma \) follows from the fact that \( R \) reverses the time orientation.

Lemma 8.10. One has
\[
\mathcal{R}_\Sigma f = Tr^* f,
\]
where \( T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( r^* f(y) = f(r(y)) \).
We have defined in Subsect. 8.3 the double $\beta$-KMS state $\omega_d$ through its Cauchy surface covariances $\lambda^\pm_d$. The associated Hermitian space is

$$C^\infty_0(\Sigma^+; \mathbb{C}^2) \oplus C^\infty_0(\Sigma^+; \mathbb{C}^2), -q).$$

$(\mathcal{M}^+ \cup \mathcal{M}^-, g)$ is a (disconnected) globally hyperbolic spacetime with Cauchy surface $\Sigma^+ \cup \Sigma^-$ and we denote a Cauchy data on $\Sigma^+ \cup \Sigma^-$ as

$$f = f^+ \oplus f^-, f^\pm \in C^\infty_0(\Sigma^\pm; \mathbb{C}^2).$$

Using Remark 5.5 we obtain from $\omega_d$ a pure, quasi-free state $\omega_D$ in $\mathcal{M}^+ \cup \mathcal{M}^-$ as follows:

**Definition 8.11.** The double $\beta$-KMS state $\omega_D$ in $\mathcal{M}^+ \cup \mathcal{M}^-$ is defined by the Cauchy surface covariances:

$$\mathcal{F} \cdot \lambda^\pm_D f := \text{Re}((\mathbb{1} \oplus \mathcal{R}_\Sigma) \mathcal{F} \cdot \lambda^\pm_D(\mathbb{1} \oplus \mathcal{R}_\Sigma)f), f = f^+ \oplus f^- \in C^\infty_0(\Sigma^+ \cup \Sigma^-; \mathbb{C}^2).$$

From Prop. 8.8 and Lemma 8.10 we obtain the following expression for $\lambda^\pm_D$.

**Proposition 8.12.** One has:

$$\lambda^\pm_D = \pm Q \circ (\mathbb{1} \oplus r^*)^{-1} e^\pm_\beta(\mathbb{1} \oplus r^*),$$

where $e^\pm_\beta$ are the Calderón projectors for $K_\beta$ defined in Def. 8.7 and $Q = q \oplus q$.

9. **The HHI state**

In this section we construct the HHI state $\omega_{HHI}$ in $M$ and prove that it is a pure Hadamard state, extending the double $\beta$-KMS state $\omega_D$ in $\mathcal{M}^+ \cup \mathcal{M}^-$ for $\beta = (2\pi)\kappa^{-1}$. We use the expression of $\omega_D$ by Calderón projectors for the Wick rotated operator $K_\beta$, see Subsect. 8.8.

Since $K_\beta$ is a Laplace operator for the complex metric $g_{\text{eucl}}$ on $M_{\text{eucl}} = S_\beta \times \Sigma^+$, one can if $\beta = (2\pi)\kappa^{-1}$ extend it to a Laplace operator $K_{\text{ext}}$ on the smooth extension $(M^\text{eucl}_{\text{ext}}, g^\text{eucl}_{\text{ext}})$.

The boundary of the open set $\Omega_{\text{ext}}$, extending $\Omega_\beta := \{0, \frac{\pi}{2}\} \times \Sigma^+$ is diffeomorphic to the full Cauchy surface $\Sigma$, and we can use the Calderón projectors for $K_{\text{ext}}$, $\Omega_{\text{ext}}$ to define a pair of covariances $\lambda^\pm_{HHI}$. The fact that they define a pure state is actually quite easy, using some standard continuity properties of the Calderón projectors and density results in Sobolev spaces. The proof of the Hadamard property of $\omega_{HHI}$ relies also on an easy argument using pseudodifferential calculus, taken from [5].

9.1. **Laplacians for complex metrics.** We recall that complex metrics on a manifold $X$ are defined in [2,7.1]

**Definition 9.1.** A complex metric $k$ on a manifold $X$ is called uniformly sectorial if

1. there exists $C > 0$ such that

$$\text{Im}((\overline{\sigma}^0 k_{ab}(x)v^b)) \leq C \text{Re}(\overline{\sigma}^0 k_{ab}(x)v^b), \quad \forall x \in X, \quad v \in \mathbb{C}T_x X;$$

2. $|k(x)| \geq \text{det}(k_{ab}(x)) > 0 \forall x \in X$.

Note that if $k$ is uniformly sectorial, then

$$\text{Im}(\overline{\xi}^0 k_{ab}(x)\xi_b) \leq C \text{Re}(\overline{\xi}^0 k_{ab}(x)\xi_b) \quad \forall x \in X, \quad \xi \in \mathbb{C}T^*_x X,$$

ie $k^{-1}$ is also uniformly sectorial. In fact if $\xi = kv$ we have $\overline{\xi} \cdot k^{-1} \xi = k\overline{v} \cdot v = \overline{\sigma} \cdot \overline{k v}$ and [9.2] follows from [9.1].
9.1.1. Laplacians for complex metrics. If $k$ is a complex metric on $X$, one defines the Christoffel symbols:

$$\Gamma^c_{ab} := \frac{1}{2} k^{cd} (\partial_a k_{cd} + \partial_b k_{ad} - \partial_d k_{ab}),$$

the covariant derivative:

$$\nabla^{(k)}_a T^b = \partial_a T^b + \Gamma^b_{ac} T^c,$$

and the Laplacian associated to $k$, acting on $C^\infty_0 (X)$:

$$\Delta_k := \nabla^{(k)}_a k^{ab} \nabla^{(k)}_b$$

as for real metrics.

For $m \in C^\infty (X, \mathbb{R})$, we set:

$$K := -\Delta_k + m,$$

and equip $C^\infty_0 (X)$ with the scalar product:

$$(u|v) := \int_X \pi e|k|^\frac{1}{2} dx.$$

**Proposition 9.2.** Assume that $k$ is uniformly sectorial and that $m_0^2 \leq m(x)$ for $m_0 > 0$. Let

$$Q(u, u) = (u|Ku), \text{Dom} Q = C^\infty_0 (X).$$

Then $Q$ is closeable, the domain $\text{Dom} Q^c$ of its closure $Q^c$ is the space $H^1_k (X)$ equal to the completion of $C^\infty_0 (X)$ for the norm

$$\|u\|^2 = \int_X (\overline{\partial_a u} \text{Re} k^{ab} \partial_b u + m(x) \overline{u} u)|k|^{\frac{1}{2}} dx.$$

Moreover $Q^c$ is sectorial and induces an isomorphism:

$$K^c : H^1_k (X) \cong H^1_k (X)^*,$$

with $K^c = K$ on $C^\infty_0 (X)$.

**Proof.** We have

$$\nabla^{(k)}_a T^a = |k|^{-\frac{1}{2}} \partial_a (|k|^{\frac{1}{2}} T^a),$$

which is proved in Subsect. A.3. Therefore $K = -|k|^{-\frac{1}{2}} \partial_a k^{ab} |k|^{\frac{1}{2}} \partial_a + m$ and

$$Q(u, u) = (u|Ku) = \int_X (\overline{\partial_a u} k^{ab} \partial_b u + m(x) \overline{u} u)|k|^{\frac{1}{2}} dx.$$

Using (9.2) under the integral sign, we obtain $|\text{Im} Q(u, u)| \leq C \text{Re} Q(u, u)$ and that $Q$ is closeable. The domain of its closure $Q^c$ equals $H^1_k (X)$. The statement about $K^c$ follows from the Lax-Milgram theorem. $\square$

9.1.2. Outer unit normal. Let $\Omega \subset X$ with a smooth boundary $\partial \Omega$ denoted by $\Sigma$ in the sequel. We set

$$\Omega^+ := \Omega, \ \Omega^- := X \setminus \Omega^c.$$

We can define the outer unit normal vector field to $\Sigma$, denoted by $n \in CT X$ by the following conditions:

i) $n(x) \cdot k(x)v = 0, \ \forall v \in T_x \Sigma$,

ii) $n(x) \cdot k(x)n(x) = 1$,

iii) $\text{Re} n(x)$ is outwards pointing.

If $\Omega$ is locally equal to $\{ f > 0 \}$ for $f \in C^\infty (X, \mathbb{R})$ with $df \neq 0$ on $\{ f = 0 \}$, we have:

$$n^a = \frac{-k^{ab} \nabla_b f}{(\nabla_a k^{ab} \nabla_b f)^{\frac{1}{2}}}.$$
where in the denominator we take the usual determination of $z^\frac{1}{2}$.

We also assume the following condition:
\[(9.5)\]
$$\text{Im}n(x) \in T_z\Sigma, \ x \in \Sigma,$$
which is equivalent to $\nabla_a f b^{\alpha b} b f \in \mathbb{R}$ on $\Sigma$, if $\Omega = \{ f > 0 \}$.

The volume form $d\text{Vol}_k = |k|^\frac{1}{2} dx^1 \wedge \cdots \wedge dx^n$ associated to $k$ is real, as is the associated density $d\mu_k = |d\text{Vol}_k| = |k|^\frac{1}{2} dx$. It is easy to see from (9.5) that the induced density $d\sigma_h = |d\text{Vol}_h|$ associated to the induced metric $h$ on $\Sigma$ is also real valued.

9.1.3. Trace operators. For $u \in C^\infty(\Omega)$ we set:
\[\gamma u := \left( \frac{u|_\Sigma}{\partial_n u|_\Sigma} \right) \in C^\infty(\Sigma; \mathbb{C}^2).\]
We denote by $\gamma^*$ the formal adjoint of
\[\gamma : L^2(\Omega, d\mu_k) \to L^2(\Sigma, d\sigma_h) \otimes \mathbb{C}^2.\]
We have
\[\gamma^* f = (d\mu_k)^{-1} f_0 d\Sigma + (d\mu_k)^{-1} (n^\alpha \partial_\alpha)^* f_1 d\Sigma,\]
where if $g \in C^\infty(\Sigma)$, $gd\Sigma$ is the distributional density defined as
\[\langle u | gd\Sigma \rangle = \int_\Sigma \pi_g d\sigma_h, \ u \in C^\infty_0(\Omega),\]
and
\[\langle u | (n^\alpha \partial_\alpha)^* gd\Sigma \rangle = \langle n^\alpha \partial_\alpha u | gd\Sigma \rangle.\]
Similarly for $u \in C^\infty(\Omega^\pm)$ we set:
\[\gamma^\pm u := \left( \frac{u|_\Sigma}{\partial_n u|_\Sigma} \right),\]
where the trace is taken from $\Omega^\pm$.

In the rest of this subsection, we assume that $k$ is uniformly sectorial and that (9.5) holds.

9.1.4. Calderón projectors.

**Definition 9.3.** The Calderón projectors $c^\pm$ associated to $(K, \Omega)$ are defined as
\[c^\pm := \mp \gamma^\pm \circ K^{-1} \circ \gamma^* \circ S,\]
where
\[S = \begin{pmatrix} 2b^* & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},\]
b = $\text{Im}n a \nabla_a$ and $b^*$ is the adjoint of $b$ in $L^2(\Sigma, d\sigma_h)$.

Note that the operator $S$ is well defined on $C^\infty_0(\Sigma; \mathbb{C}^2)$, since $\text{Im}n$ is tangent to $\Sigma$.

It is not a priori clear that $c^\pm$ are well defined, since even for $f \in C^\infty_0(\Sigma; \mathbb{C}^2)$, $\gamma^* S f$ does not belong to $H^1_k(\Sigma)^*$. To show that $c^\pm$ make sense, one can apply the following proposition. We denote by $H^s_c(\Sigma)$ resp. $H^s_{c, \text{loc}}(\Sigma)$ for $s \in \mathbb{R}$, the compactly supported, resp. local Sobolev spaces on $\Sigma$ and set:
\[(9.6)\]
\[H^s_{c, \text{loc}}(\Sigma) = H^{s-\frac{1}{2}}_{c, \text{loc}}(\Sigma) \oplus H^{s-\frac{1}{2}}_{c, \text{loc}}(\Sigma),\]
\[H^s_{c, \text{loc}}(\Sigma) = H^{s+\frac{1}{2}}_{c, \text{loc}}(\Sigma) \oplus H^{s+\frac{1}{2}}_{c, \text{loc}}(\Sigma), \ s \in \mathbb{R}.\]

**Proposition 9.4.** (1) $c^\pm : H^s_c(\Sigma) \to H^s_{c, \text{loc}}(\Sigma)$ continuously for any $s \in \mathbb{R}$,
(2) $c^\pm$ are $2 \times 2$ matrices with entries in $\Psi^\infty(\Sigma)$.
Proof. The differential operator $K$ is elliptic, since its principal symbol equals $\xi \cdot k^{-1}(x) \xi$. $K$ admits hence a properly supported parametrix $Q \in \Psi^{-2}_c(\Sigma)$, and $K^{-1} - Q$ is a smoothing operator, ie has a smooth distributional kernel.

It suffices hence to check the proposition with $K^{-1}$ replaced by $Q$ in the definition of $e^Q$. From the topology of $\mathcal{H}^s_{c/loc}(\Sigma)$, we see that we can assume that $\Sigma$ is compact, and since $Q$ is properly supported, that $\Sigma$ is compact, which is the situation considered in [Gr, Sect. 11.1].

A neighborhood $V$ of $\Sigma$ in $X$ is then diffeomorphic to $[-\delta, \delta] \times \Sigma$, and one can use coordinates $(s, y)$ on $[-\delta, \delta] \times \Sigma$. In [Gr, Sect. 11.1] the trace operator is defined as

$$\tilde{\gamma} u = \left( \begin{array}{c} u(0, y) \\ i^{-1} \partial_s u(0, y) \end{array} \right).$$

Clearly we have $\gamma = L \circ \tilde{\gamma}$, where $L = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ and $r$ is a first order differential operator on $\Sigma$. This implies that $L : H^s_{c/loc}(\Sigma) \to H^s_{c/loc}(\Sigma)$. The Calderón projectors $\tilde{C}^{\pm}$ in [Gr] Sect. 11.1 are equal to $L^{-1} \circ c^{\pm} \circ L$, and [Gr] Prop. 11.7 implies that $\tilde{C}^{\pm} : H^s(\Sigma) \to H^s_{loc}(\Sigma)$ for all $s \in \mathbb{R}$, which implies (1).

Property (2) is a standard fact, see [Gr] Sect. 11.1. $\Box$

9.2. Construction of the HHI state.

9.2.1. The Laplacian on $M^{\text{ext}}$. We now apply the above framework to $(X, k) = (M^{\text{ext}}, g^{\text{ext}})$, the smooth extension of $(M, g)$ constructed in Prop. 2.5, for $\beta = (2\pi)\kappa^{-1}$. We assume that hypothesis $(H)$ in Subsect. 2.4 holds. By Prop. 8.9 the Wick rotated metric $g^{\text{ext}}$ satisfies the conditions in 8.6.1, ie is uniformly sectorial. By continuity the same is true of its extension $g^{\text{ext}}_{\text{ext}}$. We denote by

$$K_{\text{ext}} = \Delta_{g^{\text{ext}}_{\text{ext}}} + m_{\text{ext}},$$

the associated Laplacian. We choose the open set $\Omega_{\text{ext}} \subset M^{\text{ext}}_{\text{ext}}$, whose boundary $\partial \Omega_{\text{ext}}$ is diffeomorphic to $\Sigma$, see Prop. 2.5. We saw in 8.6.1 that if $\nu$ is the unit outer normal to $[0, \frac{1}{2}] \times \Sigma^+$, then $\text{Im} \nu$ is tangent to $\partial([0, \frac{1}{2}] \times \Sigma^+)$. Again by continuity, the same is true of the unit outer normal to $\Omega_{\text{ext}}$, ie condition 9.5 is satisfied. Therefore we can apply the results of Subsect. 9.1 to $K_{\text{ext}}$ and $\Omega_{\text{ext}}$.

We need one more result, which states that $K_{\text{ext}}$ is the unique extension of $K(2\pi)_{\kappa^{-1}}$ to $L^2(M^{\text{ext}})$.

Proposition 9.5. Let $U : C^\infty_0(M^{\text{ext}}) \to C^\infty_0(M^{\text{ext}}_{\text{ext}} \setminus B_{\text{ext}})$ defined by:

$$U u = u \circ \chi^{-1}.$$

Then $U$ extends as a unitary operator

$$U : L^2(M^{\text{ext}}) \to L^2(M^{\text{ext}}_{\text{ext}} \setminus g^{\text{ext}}_{\text{ext}}) \frac{1}{\kappa} dx, dx,$$

with $K_{\text{ext}} = U K(2\pi)_{\kappa^{-1}} U^*$. $\Box$

Proof. $U$ clearly extends as a unitary operator. Let us check the second statement.

As a differential operator, $K(2\pi)_{\kappa^{-1}}$ equals $-\Delta_{g^{\text{ext}}} + m$. As an unbounded operator, $K(2\pi)_{\kappa^{-1}}$ is defined in 8.6.2 using the sesquilinear form $Q(2\pi)_{\kappa^{-1}}$, while $K_{\text{ext}}$ is defined with the sesquilinear form $Q_{\text{ext}}$ for $k = g^{\text{ext}}_{\text{ext}}$ and $m = m_{\text{ext}}$, see Prop. 9.2, $Q(2\pi)_{\kappa^{-1}}$ is the closure of its restriction to $C^\infty_0(M)$, while $Q_{\text{ext}}$ is the closure of its restriction to $C^\infty_0(M^{\text{ext}}_{\text{ext}})$. Taking into account the isometry $\chi : M^{\text{ext}} \to M^{\text{ext}}_{\text{ext}} \setminus B_{\text{ext}}$, it suffices to check that $C^\infty_0(M^{\text{ext}}_{\text{ext}} \setminus B_{\text{ext}})$ is a form core for $Q_{\text{ext}}$, ie that this space is dense in the space $H^1_{\text{ext}}(X)$ for $(X, k) = (M^{\text{ext}}_{\text{ext}} \setminus g^{\text{ext}}_{\text{ext}})$, see Prop. 9.2.
Using the coordinates \((X, Y, \omega)\) near \(B_{\text{ext}} \sim \{0\} \times B\), this follows from the fact that \(C_0^\infty(\mathbb{R}^2 \setminus \{0\})\) is dense in \(H^1(\mathbb{R}^2)\), see eg. [A, Thm. 3.23]. □

9.2.2. The HHI state. Let us denote by \(c^\pm_{\text{ext}}\) the Calderón projectors for \((K_{\text{ext}}, \Omega_{\text{ext}})\), defined as in Def. 9.3.

The following theorem is a slightly more precise version of Thm. 1.1.

**Theorem 9.6.** (1) \(\lambda_{\text{HHI}}^\pm = \pm q \circ c^\pm_{\text{ext}}\) are the Cauchy surface covariances of a pure quasi-free state \(\omega_{\text{HHI}}\) for \(P\) in \(M\), called the HHI state.

(2) The restriction of \(\omega_{\text{HHI}}\) to \(\mathcal{M}^+ \cup \mathcal{M}^-\) is the double \(\beta\)-KMS state \(\omega_D\) for \(\beta = (2\pi)\kappa^{-1}\).

(3) \(\omega_{\text{HHI}}\) is a Hadamard state in \(M\).

(4) Let \(\omega\) a quasi-free state for \(P\) in \(M\) whose restriction to \(\mathcal{M}^+ \cup \mathcal{M}^-\) equals \(\omega_D\) and such that its space-time covariances map \(C_0^\infty(\Sigma)\) into \(C_c^\infty(M)\). Then \(\omega = \omega_{\text{HHI}}\).

Note that it follows from (4) above that \(\omega_{\text{HHI}}\) is the unique Hadamard state in \(M\) whose restriction to \(\mathcal{M}^+ \cup \mathcal{M}^-\) equals \(\omega_D\).

**Proof.** We first prove (2). We note that the map \((\mathbb{I} \oplus r^*)\) in Prop. 8.12 corresponds to the embedding of \(C_0^\infty(\Sigma^+ \cup \Sigma^-; C^2)\) into \(C_0^\infty(\Sigma \setminus B)\) obtained from \(\psi : \Sigma \to M_{\text{ext}}\) in Prop. 2.5. The exterior normal to \(\Omega_{\text{ext}}\) is the image under \(\chi\) of the exterior normal to \([0, \pi\kappa^{-1}] \times \Sigma^+\) for defined in (8.22). Therefore using also Prop. 9.5 we obtain that

\[
(\mathbb{I} \oplus r^*)^{-1}c_{\pm_{(2\pi/\kappa)}}(\mathbb{I} \oplus r^*) = c^\pm_{\text{ext}},
\]

on \(C_0^\infty(\Sigma^+ \cup \Sigma^-)\). This implies (2).

Let us now prove (1). Let us denote by \(h_{\text{ext}}\) the metric induced by \(g_{\text{ext}}\) on \(\Sigma\) and use the scalar product of \(L^2(\Sigma, |h_{\text{ext}}|^{1/2}dy) \otimes C^2\) to identify sesquilinear forms with operators, so that \(q = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)\).

We recall that the spaces \(\mathcal{H}_{\text{loc}}^\pm(\Sigma)\), \(\mathcal{H}_{\text{loc}}^\pm(\Sigma)\) are defined in (9.6). We note that \(q : \mathcal{H}_{\text{loc}}^\pm(\Sigma) \to \mathcal{H}_{\text{loc}}^{-q, \pm}(\Sigma)\), and that \(\mathcal{H}_{\text{loc}}^\pm(\Sigma), \mathcal{H}_{\text{loc}}^{-q, \pm}(\Sigma)\) form a dual pair for the above scalar product. It follows then from Prop. 9.4 that \(\lambda^\pm, q\) are continuous sesquilinear forms for the topology of \(\mathcal{H}_{\text{loc}}^\pm(\Sigma) \cong H^\pm_{\text{loc}}(\Sigma) \oplus H^\pm_{\text{loc}}(\Sigma)\). Now it is a well-known fact that since \(B \subset \Sigma\) is of codimension 1, \(C_0^\infty(\Sigma \setminus B)\) is dense in \(H^\pm_{\text{loc}}(\Sigma)\).

By (2) and the fact that \(\omega_D\) is a state, we obtain that

\[
\lambda_{\text{HHI}}^\pm \geq 0, \quad \lambda_{\text{HHI}}^+ - \lambda_{\text{HHI}}^- = q,
\]

on \(C_0^\infty(\Sigma \setminus B; C^2)\). By the continuity and density result shown above, this extends to \(C_0^\infty(\Sigma; C^2)\), which proves that \(\lambda_{\text{HHI}}^\pm\) are the Cauchy surface covariances of a state \(\omega_{\text{HHI}}\).

Let us now prove that \(\omega_{\text{HHI}}\) is pure. Let us set for simplicity of notation \(\mathcal{Y} = C_0^\infty(\Sigma; C^2), Y_0 = C_0^\infty(\Sigma \setminus B; C^2)\) and denote by \(\mathcal{Y}^{\text{cpl}}, Y_0^{\text{cpl}}\) the completion of \(\mathcal{Y}, Y_0\) for the norm \(\|f\|_2^2 = \mathcal{T}(\lambda_{\text{HHI}}^+ + \lambda_{\text{HHI}}^-)f\). The density and continuity result above shows that \(\mathcal{Y}^{\text{cpl}} = Y_0^{\text{cpl}}\). The purity of \(\omega_{\text{HHI}}\) follows then from the purity of \(\omega_D\) and Prop. 3.2.

Let us now prove (3). By Thm. 3.4 there exists a reference Hadamard state \(\omega_{\text{ref}}\) for \(P\) in \(M\) whose Cauchy surface covariances on \(\Sigma\) \(\lambda_{\text{ref}}^\pm\) are \(2 \times 2\) matrices with entries in \(\Psi^\infty(\Sigma)\). By Prop. 9.4 the same is true for \(\lambda_{\text{HHI}}^\pm\). The restriction of \(\omega_{\text{HHI}}\) to \(\mathcal{M}^+\) is a Hadamard state for \(P\), since it is a \((2\pi)\kappa^{-1}\)-KMS state for a time-like, complete Killing vector field. The restriction of \(\omega_{\text{HHI}}\) to \(\mathcal{M}^-\) is also a Hadamard state for \(P\).
In fact by Prop. 8.12, its Cauchy surface covariances on $\Sigma^-$ are the images of those of $\omega_D$ on $\Sigma^+$ by the weak wedge reflection $r$. Since $r^* h = h$, $r^* N = -N$ and $r^* w = w$, see (2.2.1) the expression (8.6) of $P$ in $\mathbb{R} \times \Sigma^-$ shows that the restriction of $\omega_D$ to $\mathcal{M}^-$ is also a Hadamard state.

This implies that the restriction of $\omega_{\text{HHI}}$ to $\mathcal{M}^+ \cup \mathcal{M}^-$ is a Hadamard state. The same is true of the restriction of the reference Hadamard state $\omega_{\text{ref}}$ to $\mathcal{M}^+ \cup \mathcal{M}^-$. Passing to Cauchy surface covariances on $\Sigma^+ \cup \Sigma^-$, this implies that if $\chi \in C^\infty_0(\Sigma^\pm)$, then

$$\chi \circ (\lambda^\pm_{\text{HHI}} - \lambda^\pm_{\text{ref}}) \circ \chi$$ is a smoothing operator on $\Sigma$.

We claim that this implies that $\lambda^\pm_{\text{HHI}} - \lambda^\pm_{\text{ref}}$ is smoothing, which will imply that $\omega_{\text{HHI}}$ is a Hadamard state.

If fact let $a$ be one of the entries of $\lambda^\pm_{\text{HHI}} - \lambda^\pm_{\text{ref}}$, which is a scalar pseudodifferential operator belonging to $\Psi^m(\Sigma)$ for some $m \in \mathbb{R}$. We know that $\chi \circ a \circ \chi$ is smoothing for any $\chi \in C^\infty_0(\Sigma \setminus B)$. Then its principal symbol $\sigma_{pr}(a)$ vanishes on $T^* (\Sigma \setminus B)$ hence on $T^* \Sigma$ by continuity, so $a \in \Psi^{m-1}(\Sigma)$. Iterating this argument we obtain that $a$ is smoothing, which completes the proof of (3).

The proof of (4) is identical to [G, Prop. 7.4]. □

APPENDIX A.

A.1. Proof of Prop. 2.4. Since $r$ is an isometry of $(\Sigma, h)$, $r_B = Id$ and $r : \Sigma^+ \to \Sigma^-$ we obtain (2.4). The first identity in (2.5) follows from the fact that $(u, \omega)$ are normal Gaussian coordinates to $B$ for $h$, the other are tautologies.

We obtain from (2.4) and 2.2.1 that $v, w^0$ are odd in $u$, $w^\alpha$, $k_{\alpha\beta}$ are even in $u$ with $w^\alpha(0, \omega) = 0$. The function $m$ is even in $u$ by invariance under $r$. We now use Killing’s equation

(A.1) $\nabla_a V_b + \nabla_b V_a = 0$,

noting that since $V = 0$ on $B$ we have

(A.2) $\nabla_a V_b = \partial_b V_a$ on $B$.

If we work in Gaussian normal coordinates to $\Sigma$ for $g$, so that

$$g = -dt^2 + h_{ij}(t, y) dy^i dy^j$$

$V = -N(t, y) \partial_t + w^0(t, y) \partial_u + w^\alpha(t, y) \partial_{\omega^\alpha}$

and $y = (u, \omega)$, we obtain from (A.1), (A.2) that:

$$\partial_a V_0(0, \omega) = 0 \Rightarrow \partial_a w^0(0, \omega) = 0.$$

Summarizing we have:

$$N(u, \omega) = u a(u^2, \omega),$$

(A.3) $w^0(u, \omega) = u^3 b(u^2, \omega), \quad w^\alpha(u, \omega) = u^2 c^\alpha(u^2, \omega),$

for smooth functions $a, b, c^\alpha, d_{\alpha\beta} : - \epsilon, \epsilon [\times B] \to \mathbb{R}$ with

$$n(0, \omega) \geq c > 0, c^{-1} \ll = [d_{\alpha\beta}(0, \omega)] \leq c \ll,$$ for some $c > 0$.

To complete the proof of the proposition it remains to show that $\kappa = a(0, \omega)$.

To do this we reexpress the surface gravity $\kappa$. By [SL] Lemma 2.5 we have:

$$\kappa^2 = (h^{ij} \partial_i N \partial_j N)|_B - \frac{1}{2} (h^{ij} h^{kl} \nabla_i w^1 \nabla_j w^c)|_B,$$

which using (A.3) gives $\kappa = a(0, \omega).$ □
A.2. Proof of Prop. [2.5] We recall that we defined the coordinates \((u, \omega) \in [-\delta, \delta] \times \mathcal{B}\) on a small neighborhood \(U\) of \(\mathcal{B}\) in \(\Sigma\). \(U \cap \Sigma^+\) is diffeomorphic to \([0, \delta] \times \mathcal{B}\) using the coordinates \((u, \omega)\). If 
\[
X = u \cos(\kappa s), \quad Y = u \sin(\kappa s),
\]
we have:
\[
du = u^{-1}(XdX + YdY), \quad ds = \kappa^{-1} u^{-2}(XdY - YdX).
\]
By Prop. [2.4] we obtain:
\[
k_{\alpha\beta}(u, \omega) du^\alpha du^\beta = d_{\alpha\beta}(X^2 + Y^2, \omega) du^\alpha du^\beta,
\]
\[
i\omega_{\alpha}(u, \omega) du^\alpha ds = i\kappa^{-1} b_{\alpha}(X^2 + Y^2, \omega)(XdY - YdX) du^\alpha,
\]
\[
i\omega_0(u, \omega) duds = i\kappa^{-1} b_0(X^2 + Y^2, \omega)(XdX + YdY)(XdX + YdY),
\]
\[
u^2(u, \omega) ds^2 + du^2 = u^2 \kappa^2 (1 + u^2 d(u^2, \omega)) \kappa^{-2} u^{-4}(XdY - YdX)^2 + u^{-2}(XdX + YdY)^2
\]
\[
= dX^2 + dY^2 + d(X^2 + Y^2, \omega)(XdX - YdY)^2.
\]
Let us denote by \(B_2(0, \delta) = \{(X, Y) \in \mathbb{R}^2 : X^2 + Y^2 \leq \delta^2\}\) the open disk of center 0 and radius \(\delta\) in \(\mathbb{R}^2\). If \(\beta = (2\pi)^{-1}\), then \((u, \kappa s) \in [0, \delta] \times \mathbb{S}_{2\pi}\) are polar coordinates on \(B_2(0, \delta) \setminus \{0\}\). The expression [2.10] for \(g^{\text{ext}}\) and the estimates above show that \(g^{\text{ext}}\) extends as a smooth complex metric on \(B_2(0, \delta) \otimes \mathcal{B}\).

We then construct \(M^{\text{ext}}\) by gluing \(B_2(0, \delta) \times \mathcal{B}\) with \(M^{\text{ext}} = \mathbb{S}_\beta \times \Sigma^+\) over \(\{(X, Y) \in \mathbb{R}^2 : \frac{1}{2} \delta^2 < X^2 + Y^2 < \delta^2\}\) \(\times \mathcal{B}\) using the map:
\[
S_{\beta} \times [0, \delta] \times \mathcal{B} \to B_2(0, \delta) \times \mathcal{B}
\]
\[(s, u, \omega) \mapsto (u \cos(\kappa s), u \sin(\kappa s), \omega).
\]
The complex metric \(g^{\text{ext}}\) defined on \(S_{\beta} \times \Sigma^+\) extends to a smooth complex metric \(g^{\text{ext}}\) on \(M^{\text{ext}}\) By Prop. [2.4] we have \(m = n(X^2 + Y^2, \omega)\), hence \(m\) extends as a smooth function on \(M^{\text{ext}}\).

Let us now embed \(\Sigma\) isometrically into \(M^{\text{ext}}\). In the coordinates \((u, \omega)\) on \(\Sigma\) near \(\mathcal{B}\) the embedding \(\psi\) becomes
\[
(u, \omega) \mapsto \begin{cases} (0, u, \omega) & \text{for } 0 < u < \delta, \\ \left(\frac{\delta}{2}, -u, \omega\right) & \text{for } -\delta < u < 0, \end{cases}
\]
which smoothly extends to \(u = 0\), the image of \(\Sigma\) under this extension being locally equal to \(\{Y = 0\}\).

The open set \(\Omega^{\text{ext}}\) is obtained by gluing \(\{Y > 0\}\) with \([0, \frac{\delta}{2}] \times \Sigma^+\) using the map [A.4]. This completes the proof.

A.3. Proof of [4.3]. A mechanical computation gives:
\[
\sum_i \nabla_i T^i = \sum_i \partial_i T^i + \frac{1}{2} \sum_{i, k, l} k^{ii}(\partial_i k_{kl} + \partial_k k_{il} - \partial_k k_{il}) T^k
\]
\[
= \sum_i \partial_i T^i + \frac{1}{2} \sum_{i, k, l} k^{ii}\partial_k k_{il} T^k =: I,
\]
using that \(k^{ii} = k^{ii}, \ k_{kl} = k_{kl}\). Next
\[
\sum_i \lambda_i (|k|^{-\frac{1}{2}} \partial_i (|k|^{-\frac{1}{2}} T^i) = \sum_i \partial_i T^i + \frac{1}{2} \sum_i |k|^{-1} \partial_i |k| T^i =: II.
\]
Since
\[
\det A(t)^{-1} \frac{d}{dt} \det A(t) = \text{Tr}(A(t)^{-1} \frac{d}{dt} A(t)),
\]
we get that \(\partial_i |k| = |k| \text{Tr}(k^{-1} \partial_i k)\). Next we compute:
\[
(k^{-1} \partial_i k)_{il} = \sum_l k^{il} \partial_l k_{lk}, \quad \text{Tr}(k^{-1} \partial_i k) = \sum_{k, l} k^{kl} \partial_l k_{lk}.
\]
which shows that $I = II$. □

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