A Linearly Convergent Conditional Gradient Algorithm with Applications to Online and Stochastic Optimization

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Abstract

Linear optimization is many times algorithmically simpler than non-linear convex optimization. Linear optimization over matroid polytopes, matching polytopes and path polytopes are example of problems for which we have simple and efficient combinatorial algorithms, but whose non-linear convex counterpart is harder and admit significantly less efficient algorithms. This motivates the computational model of convex optimization, including the offline, online and stochastic settings, using a linear optimization oracle. In this computational model we give several new results that improve over the previous state of the art. Our main result is a novel conditional gradient algorithm for smooth and strongly convex optimization over polyhedral sets that performs only a single linear optimization step over the domain on each iteration and enjoys a linear convergence rate. This gives an exponential improvement in convergence rate over previous results.

Based on this new conditional gradient algorithm we give the first algorithms for online convex optimization over polyhedral sets that perform only a single linear optimization step over the domain while having optimal regret guarantees, answering an open question of [20, 16]. Our online algorithms also imply conditional gradient algorithms for non-smooth and stochastic convex optimization with the same convergence rates as projected (sub)gradient methods.

1 Introduction

First-order optimization methods, such as (sub)gradient-descent methods [31, 25, 26] and conditional-gradient methods [9, 7, 5, 12, 17], are often the method of choice for coping with very large scale optimization tasks. While theoretically attaining inferior convergence rate compared to other efficient optimization algorithms (e.g. interior point methods [27]), modern optimization problems are often so large that
using second-order information or other super-linear operations becomes practically infeasible.

The computational bottleneck of gradient descent methods is usually the computation of orthogonal projections onto the convex domain. This is also the case with proximal methods [26]. Computing such projections is very efficient for simple domains such as the euclidean ball, the hypercube and the simplex but much more involved for more complicated domains, making these methods impractical for such problems in high-dimensional settings.

On the other hand, for many convex sets of interest, optimizing a linear objective over the domain could be done by a very efficient and simple combinatorial algorithm. Prominent examples for this phenomena are the matroid polytope for which there is a simple greedy algorithm for linear optimization, and the flow polytope (convex hull of all \(s - t\) paths in a directed acyclic graph) for which linear optimization amounts to finding a minimum-weight path [28]. Other important examples include the set of rotations for which linear optimization is very efficient using Wahba’s algorithm [32], and the bounded PSD cone, for which linear optimization amounts to a leading eigenvector computation whereas projections require SVD decompositions.

This phenomena motivates the study of optimization algorithms that require only linear optimization steps over the domain and in particular algorithms that require only a constant number of such steps per iteration.

The main contribution of this work is a conditional gradient algorithm for offline smooth and strongly convex optimization over polyhedral sets that requires only a single linear optimization step over the domain on each iteration and enjoys a linear convergence rate, an exponential improvement over previous results in this setting.

| Setting                        | Previous | This paper |
|--------------------------------|----------|------------|
| Offline, smooth and strongly convex | \(t^{-1}\) [17] | \(e^{-O(t)}\) |
| Offline, non-smooth and convex | \(t^{-1/3}\) [16] | \(t^{-1/2}\) |
| Offline, non-smooth and strongly convex | \(t^{-1/3}\) [16] | \(\log(t)/t\) |
| Stochastic, non-smooth and convex | \(t^{-1/3}\) [16] | \(t^{-1/2}\) |
| Stochastic, non-smooth and strongly convex | \(t^{-1/3}\) [16] | \(\log t/t\) |
| Online, convex losses          | \(T^{3/4}\) [16] | \(\sqrt{T}\) |
| Online, strongly convex losses | \(T^{3/4}\) [16] | \(\log T\) |

Table 1: Comparison of linear oracle-based methods for optimization over polytopes in various settings. In the offline and stochastic settings we give the approximation error after \(t\) linear optimization steps over the domain, omitting constants. In the online setting we give the order of the regret after \(T\) rounds.

We also consider the setting of online convex optimization [33, 29, 13, 21]. In this setting, a decision maker is iteratively required to choose a point in a fixed convex decision set. After choosing his point, an adversary chooses some convex function and the decision maker incurs a loss that is equal to the function evaluated at the point chosen. In this adversarial setting there is no hope to play as well as
an optimal offline algorithm that has the benefit of hindsight. Instead the standard benchmark is an optimal naive offline algorithm that has the benefit of hindsight but that must play the same fixed point on each round. The difference between the cumulative loss of the decision maker and that of of this offline benchmark is known as regret. Based on our new linearly converging conditional gradient algorithm, we give algorithms for online convex optimization over polyhedral sets that perform only a single linear optimization step over the domain on each iteration while having optimal regret guarantees in terms of the game length, answering an open question of [20, 16]. Using existing techniques we give an extension of this algorithm to the partial information setting which obtains the best known regret bound for this setting.

Finally, our online algorithms also imply conditional gradient-like algorithms for offline non-smooth convex optimization and stochastic convex optimization that enjoys the same convergence rates as projected (sub)gradient methods, but replacing the projection step of (sub)gradient methods with a single linear optimization step, again improving over the previous state of the art in this setting.

Our results are summarized in table 1.

1.1 Related work

Offline smooth optimization  Conditional gradient methods for offline minimization of convex and smooth functions date back to the work of Frank and Wolfe [9] which presented a method for smooth convex optimization over polyhedral sets whose iteration complexity amounts to a single linear optimization step over the convex domain. More recent works of Clarkson [5], Hazan [12] and Jaggi [17] consider the conditional gradient method for the cases of smooth convex optimization over the simplex, semidefinite cone and arbitrary convex and compact sets respectively. Albeit its relatively slow convergence rates - additive error of the order $1/t$ after $t$ iterations, the benefit of the method is two folded: i) its computational simplicity - each iteration is comprised of optimizing a linear objective over the set and ii) it is known to produce sparse solutions (for the simplex this mean only a few non zeros entries, for the semidefinite cone this means that the solution has low rank). Due to these two proprieties, conditional gradient methods have attracted much attention in the machine learning community in recent years, see [19, 22, 18, 6, 13, 23, 2].

It is known that in general the convergence rate $1/t$ is also optimal for this method without further assumptions, as shown in [5, 12, 17]. In case the objective function is both smooth and strongly convex, there exist extensions of the basic method which achieve faster rates under various assumptions. One such extension of the conditional-gradient algorithm with linear convergence rate was presented by Migdalas [24], however the algorithm requires to solve a regularized linear problem on each iteration which is computationally equivalent to computing projections.

\footnote{Since we use only one linear optimization step per iteration, our regret bounds suffer from a certain blow-up in the form of a small polynomial in the dimension and certain quantities of the polytope, however these additional factors are independent of the game length and are all polynomial in the input representation for standard combinatorial optimization problems, i.e. matroid polytopes, matching polytopes.}
case the convex set is a polytope, GuéLat and Marcotte \cite{10} has shown that the algorithm of Frank and Wolfe \cite{9} converges in linear rate assuming that the optimal point in the polytope is bounded away from the boundary. The convergence rate is proportional to a quadratic of the distance of the optimal point from the boundary. We note that in case the optimum lies in the interior of the convex domain then the problem is in fact an unconstrained convex optimization problem and solvable via much more efficient methods. GuéLat and Marcotte \cite{10} also gave an improved algorithm based on the concept of ”away steps” with a linear convergence rate that holds under weaker conditions, however this linear rate still depends on the location of the optimum with respect to the boundary of the set which may result in an arbitrarily bad convergence rate. Beck and Taboule \cite{3} gave a linearly converging conditional gradient algorithm for solving convex linear systems, but as in \cite{10}, their convergence rate depends on the distance of the optimum from the boundary of the set. Here we emphasise that in this work we do not make any assumptions on the location of the optimum in the convex domain and our convergence rates are independent of it.

Online, stochastic and non-smooth optimization  The two closest works to ours are those of Kalai and Vempala\cite{20} and Hazan and Kale \cite{16}, both present projection-free algorithms for online convex optimization in which the only optimization carried out by the algorithms on each iteration is the minimization of a single linear objective over the decision set. \cite{20} gives a random algorithm for the online setting in the special case in which all loss function are linear, also known as online linear optimization. In this setting their algorithm achieves regret of $O(\sqrt{T})$ which is optimal \cite{1}. On iteration $t$ their algorithm plays a point in the decision set that minimizes the cumulative loss on all previous iterations plus a vector whose entries are independent random variables. The work of \cite{16} introduces algorithms for stochastic and online optimization which are based on ideas similar to ours - using the conditional gradient update step to approximate the steps a meta-algorithm for online convex optimization known as Regularized Follow the Leader (RFTL) \cite{13, 29}. For stochastic optimization, in case that all loss functions are smooth they achieve an optimal convergence rate of $1/\sqrt{T}$, however for non-smooth stochastic optimization they only get convergence rate of $T^{-1/3}$ and for the full adversarial setting of online convex optimization they get suboptimal regret that scales like $T^{3/4}$.

Also relevant to our work is the very recent work of Harchaoui, Juditsky and Nemirovski \cite{11} who give methods for i) minimizing a norm over the intersection of a cone and the level set of a convex smooth function and ii) minimizing the sum of a convex smooth function and a multiple of a norm over a cone. Their algorithms are extensions of the conditional gradient method that assume the availability of a stronger oracle that can minimize a linear objective over the intersection of the cone and a unit ball induced by the norm of interest. They present several problems of interest for which such an oracle could be implemented very efficiently, however in general such an oracle could be computationally much less efficient than the linear oracle required by standard conditional gradient methods.
1.2 Paper Structure

The rest of the paper is organized as follows. In section 2 we give preliminaries, including notations and definitions that will be used throughout this work, overview of the conditional gradient method and describe the settings of online convex optimization and stochastic optimization. In section 3 we give an informal statement of the results presented in this work. In section 4 we present our main result - a new linearly convergent conditional gradient algorithm for offline smooth and strongly convex optimization over polyhedral sets. In section 5 we present and analyse our main new algorithmic machinery which we refer to as a local linear oracle. In section 6 we present and analyse our algorithms for online and stochastic optimization and finally in section 7 we discuss lower bound for the problem of minimizing a smooth and strongly convex function using only linear optimization steps - showing that the oracle complexity of our new algorithm presented in section 4 is nearly optimal.

2 Preliminaries

Given two vectors \( x, y \) we write \( x \geq y \) if every entry of \( x \) is greater or equal to the corresponding entry in \( y \). We denote \( B_r(x) \) the euclidean ball of radius \( r \) centred at \( x \). We denote by \( \| x \| \) the \( l_2 \) norm of the vector \( x \) and by \( \| A \| \) the spectral norm of the matrix \( A \), that is \( \| A \| = \max_{x \in \mathbb{B}} \| Ax \| \). Given a matrix \( A \) we denote by \( A(i) \) the vector that corresponds to the \( i \)th row of \( A \).

**Definition 1.** We say that a function \( f(x) : \mathbb{R}^n \to \mathbb{R} \) is Lipschitz with parameter \( L \) over the set \( \mathcal{K} \subset \mathbb{R}^n \) if for all \( x, y \in \mathcal{K} \) it holds that,

\[
|f(x) - f(y)| \leq L\|x - y\|
\]

**Definition 2.** We say that a function \( f(x) : \mathbb{R}^n \to \mathbb{R} \) is \( \beta \)-smooth over the set \( \mathcal{K} \) if for all \( x, y \in \mathcal{K} \) it holds that,

\[
f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \beta\|x - y\|^2
\]

**Definition 3.** We say that a function \( f(x) : \mathbb{R}^n \to \mathbb{R} \) is \( \sigma \)-strongly convex over the set \( \mathcal{K} \) if for all \( x, y \in \mathcal{K} \) it holds that,

\[
f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \sigma\|x - y\|^2
\]

The above definition together with first order optimality conditions imply that for a \( \sigma \)-strongly convex \( f \), if \( x^* = \arg\min_{x \in \mathcal{K}} f(x) \), then for all \( x \in \mathcal{K} \)

\[
f(x) - f(x^*) \geq \sigma\|x - x^*\|^2
\]

Note that a sufficient condition for a twice-differential function \( f \) to be \( \beta \)-smooth and \( \sigma \)-strongly convex over a domain \( \mathcal{K} \) is that,

\[
\forall x \in \mathcal{K} : \quad \beta I \succeq \nabla^2 f(x) \succeq \sigma I
\]
Given a polytope $\mathcal{P} = \{ x \in \mathbb{R}^n \mid A_1 x = b_1, A_2 x \leq b_2 \}$, $A_2$ is $m \times n$, let $\mathcal{V}$ denote the set of vertices of $\mathcal{P}$ and let $N = |\mathcal{V}|$. We assume that $\mathcal{P}$ is bounded and we denote $D(\mathcal{P}) = \max_{x,y \in \mathcal{P}} \| x - y \|$.

We denote $\xi(\mathcal{P}) = \min_{v \in \mathcal{V}} \min \{ b_2(j) - A_2(j)v : j \in [m], A_2(j)v < b_2(j) \}$. Let $r(A_2)$ denote the row rank of the matrix $A_2$. Let $\mathcal{A}(\mathcal{P})$ denote the set of all $r(A_2) \times n$ matrices whose rows are linearly independent vectors chosen from the rows of $A_2$ and denote $\psi(\mathcal{P}) = \max_{M \in \mathcal{A}(\mathcal{P})} \| M \|$. Finally denote $\mu(\mathcal{P}) = \frac{\psi(\mathcal{P}) D(\mathcal{P})}{\xi(\mathcal{P})}$.

Henceforth we shall use the shorthand notation of $D, \xi, \psi, \mu$ when the polytope at hand is clear from the context.

Throughout this work we will assume that we have access to an oracle for minimizing a linear objective over $\mathcal{P}$. That is we are given a procedure $\mathcal{O}_\mathcal{P} : \mathcal{V} \rightarrow \mathbb{R}$ such that for all $c \in \mathbb{R}^n$, $\mathcal{O}_\mathcal{P}(c) \in \arg \min_{v \in \mathcal{V}} c^\top v$.

### 2.1 The Conditional Gradient Method and Local Linear Oracles

The conditional gradient method is a simple algorithm for minimizing a smooth convex function $f$ over a convex set $\mathcal{P}$ - which in this work we assume to be a polytope. The appeal of the method is that it is a first order interior point method - the iterates always lie inside the convex set and thus no projections are needed and the update step on each iteration simply requires to minimize a linear objective over the set. The basic algorithm is given below.

**Algorithm 1** Conditional Gradient

1. Let $x_1$ be an arbitrary point in $\mathcal{P}$.
2. for $t = 1, \ldots$ do
   3. $p_t \leftarrow \mathcal{O}_\mathcal{P}(\nabla f(x_t))$.
   4. $x_{t+1} \leftarrow x_t + \alpha_t (p_t - x_t)$ for $\alpha_t \in (0, 1)$.
3. end for

Let $x^* = \arg \min_{x \in \mathcal{K}} f(x)$. The convergence of algorithm 1 is due to the following simple observation.

\[
\begin{aligned}
f(x_{t+1}) - f(x^*) &= f(x_t + \alpha_t(p_t - x_t)) - f(x^*) \\
&\leq f(x_t) - f(x^*) + \alpha_t(p_t - x_t)^\top \nabla f(x_t) + \alpha_t^2 \beta \| p_t - x_t \|^2 \quad \text{\textit{\beta-smoothness of $f$}} \\
&\leq f(x_t) - f(x^*) + \alpha_t(x^* - x_t)^\top \nabla f(x_t) + \alpha_t^2 \beta \| p_t - x_t \|^2 \quad \text{\textit{\beta-optimality of $p_t$}} \\
&\leq f(x_t) - f(x^*) + \alpha_t(f(x^*) - f(x_t)) + \alpha_t^2 \beta \| p_t - x_t \|^2 \quad \text{\textit{convexity of $f$}} \\
&\leq (1 - \alpha_t)(f(x_t) - f(x^*)) + \alpha_t^2 \beta D^2
\end{aligned}
\]

The relatively slow convergence of the conditional gradient algorithm is due to the term $\| p_t - x_t \|$ in the above analysis, that may remain as large as the diameter of $\mathcal{P}$ while the term $f(x_t) - f(x^*)$ keeps on shrinking, that forces us to choose values of $\alpha_t$ that decrease like $\frac{1}{t}$. Notice that if $f$ is $\sigma$-strongly convex for some $\sigma > 0$ then knowing that for some
iteration $t$ it holds that $f(x_t) - f(x^*) \leq \epsilon$ implies that $\|x_t - x^*\|^2 \leq \frac{\epsilon}{\sigma}$. Thus when choosing $p_t$, denoting $r = \sqrt{\epsilon/\sigma}$, it is enough to consider points that lie in the intersection set $\mathcal{P} \cap B_r(x_t)$. In this case the term $\|p_t - x_t\|^2$ will be of the same magnitude as $f(x_t) - f(x^*)$ (or even smaller) and as observable in (1), linear convergence may be attainable. However solving the problem $\min_{p \in \mathcal{P} \cap B_r(x_t)} p^\top \nabla f(x_t)$ is much more difficult than solving the original linear problem $\min_{p \in \mathcal{P}} p^\top \nabla f(x_t)$ and is not straight-forward solvable using linear optimization over the original set alone.

To overcome the problem of solving the linear problem in the intersection $\mathcal{P} \cap B_r(x_t)$ we introduce the following definition which is a primary ingredient of our work.

**Definition 4 (Local Linear Oracle).** We say that a procedure $A(x, r, c)$, $x \in \mathcal{P}$, $r \in \mathbb{R}^+$, $c \in \mathbb{R}^n$ is a Local Linear Oracle for the polytope $\mathcal{P}$ with parameter $\rho$, if $A(x, r, c)$ returns a point $p \in \mathcal{P}$ such that:

1. $\forall y \in B(x, r) \cap \mathcal{P}$ it holds that $c^\top y \geq c^\top p$.
2. $\|x - p\| \leq \rho \cdot r$.

The local linear oracle (LLO) relaxes the problem $\min_{p \in \mathcal{P} \cap B_r(x_t)} p^\top \nabla f(x_t)$ by solving the linear problem on a larger set, but one that still has a diameter that is not much larger than $\sqrt{f(x_t) - f(x^*)}$. Our main contribution is showing that for a polytope $\mathcal{P}$ a local linear oracle can be constructed such that the parameter $\rho$ depends only on dimension $n$ and the quantity $\mu(\mathcal{P})$. Moreover the construction requires only a single call to the oracle $\mathcal{O}_\mathcal{P}$.

### 2.2 Online convex optimization and its application to stochastic and offline optimization

In the problem of online convex optimization (OCO) [33, 14, 13], a decision maker is iteratively required to choose a point $x_t \in \mathcal{K}$ where $\mathcal{K}$ is a fixed convex set. After choosing the point $x_t$, a convex loss function $f_t(x)$ is chosen and the decision maker incurs loss $f_t(x_t)$. The emphasis in this model is that the loss function at time $t$ may be chosen completely arbitrarily and even in an adversarial manner given the current and past decisions of the decision maker. In the **full information** setting after suffering the loss, the decision maker gets full knowledge of the function $f_t$. In the **partial information** setting (bandit) the decision maker only learns the value $f_t(x_t)$ and does not gain any other knowledge about $f_t$.

The standard goal in such games is to have overall loss which is not much larger than that of the best fixed point in $\mathcal{P}$ in hindsight. Formally the goal is to minimize a quantity known has **regret** which is given by,

$$\text{regret}_T = \sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x)$$

In certain cases, such as in the bandit setting, the decision maker must use randomness in order to make his decisions. In this case we consider only the expected
regret, where the expectation is taken over the randomness in the algorithm of the decision maker.

In the full information setting and for general convex losses the optimal regret bound attainable scales like \( \sqrt{T} \) \([4]\) where \( T \) is the length of the game. In the case that all loss functions are strongly convex the optimal regret bound attainable scales like \( \log(T) \) \([15]\).

A simple algorithm that attains optimal regret of \( O(\sqrt{T}) \) for general convex losses is known as the Regularized Follows The Leader algorithm (RFTL) \([13]\). On time \( t \) the algorithm predicts according to the following rule.

\[
x_t \leftarrow \min_{x \in \mathcal{K}} \left\{ \eta \sum_{t=1}^{t-1} \nabla f_\tau(x_\tau)^	op x + \mathcal{R}(x) \right\}
\]

Where \( \eta \) is a parameter known as the learning rate and \( \mathcal{R} \) is a strongly convex function known as the regularization. From an offline optimization point of view, achieving low regret is thus equivalent to solving a strongly-convex quadratic minimization problem on every iteration. In fact, choosing \( \mathcal{R}(x) = \|x\|^2 \), we get that the problem in (2) is equivalent to computing a euclidean projection to the domain. In case of strongly-convex losses a slight variant of (2), which also takes the form of minimizing a smooth and strongly convex quadratic function, guarantees optimal \( O(\log(T)) \) regret.

In the partial information setting the RFTL rule (2) with the algorithmic conversion of the bandit problem to that of the full information problem, established in \([8]\), yields an algorithm with regret \( O(T^{3/4}) \), which is the best to date.

Our algorithms for online optimization are based on iteratively approximating the RFTL objective in (2) using our new linearly convergent CG algorithm for smooth and strongly convex optimization, thus replacing the projection step in (2) (in case \( \mathcal{R}(x) = \|x\|^2 \)) with a single linear optimization step over the domain.

**Remark 1.** The update rule in (2) uses the gradients of the loss functions which are denoted by \( \nabla f_\tau \). In fact it is not required to assume that the loss functions are differentiable everywhere in the domain and it suffices to assume that the loss functions only have a sub-gradient everywhere in the domain, making the algorithm suitable also for non-smooth settings. Throughout this work we do not differentiate between these two cases and the notation \( \nabla f(x) \) should be understood as a gradient of \( f \) in the point \( x \) in case \( f \) is differentiable in \( x \) and as a sub-gradient of \( f \) in the point \( x \) in case \( f \) only has a sub-gradient in this point.

### 2.2.1 Stochastic optimization

In stochastic optimization the goal is to minimize a convex function \( F(x) \) over the convex set \( \mathcal{P} \) where we assume that there exists a distribution \( \mathcal{D} \) over a set of convex functions such that \( F = \mathbb{E}_{f \sim \mathcal{D}}[f] \). In this setting we don’t have direct access to the function \( F \), instead we assume to have a random oracle for \( F \) that we can query, which returns a function \( f \) that is sampled according to the distribution \( \mathcal{D} \) independently of previous samples. Thus if the oracle returns a function \( f \) it holds that \( \mathbb{E}[f(x)] = F(x) \) for all \( x \in \mathcal{P} \).
The general setting of online convex optimization is strictly harder than that of stochastic optimization, in the sense that stochastic optimization could be simulated in the setting of online convex optimization - on each iteration \( t \) the loss function revealed \( f_t \) is produced by a query to the oracle of \( F \). In this case denote by \( \mathcal{R}_T \) the regret of an algorithm for online convex optimization after \( T \) iterations. That is,

\[
\sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^{T} f_t(x) = \mathcal{R}_T
\]

Denoting \( x^* = \arg \min_{x \in \mathcal{P}} F(x) \) we thus in particular have,

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \mathcal{R}_T
\]

Taking expectation over the randomness of the oracle for \( F \) and dividing by \( T \) we have,

\[
\frac{1}{T} \sum_{t=1}^{T} F(x_t) - F(x^*) \leq \frac{\mathcal{R}_T}{T}
\]

Denoting \( \bar{x} = \frac{1}{T} \sum_{t=1}^{T} x_t \) we have by convexity of \( F \) that,

\[
F(\bar{x}) - F(x^*) \leq \frac{\mathcal{R}_T}{T}
\]

Thus the same regret rates that are attainable for online convex optimization hold as convergence rates, or sample complexity, for stochastic optimization.

### 2.2.2 Non-smooth optimization

As in stochastic optimization (see subsection 2.2.1), an algorithm for OCO also implies an algorithm for offline non-smooth optimization. Thus a conditional gradient algorithm for OCO implies a conditional gradient algorithm for non-smooth optimization, in contrast to previous conditional gradient algorithms for offline optimization that are suitable for smooth optimization only.

### 3 Our Results

In all of our results we assume that we perform optimization (either offline or online) over a polytope \( \mathcal{P} \) and that we have access to an oracle \( \mathcal{O}_\mathcal{P} \) that given a linear objective \( c \in \mathbb{R}^n \) returns a vertex of \( \mathcal{P} - v \in \mathcal{V} \) that minimizes \( c \) over \( \mathcal{P} \).
Offline smooth and strongly-convex optimization  Given a $\beta$-smooth, $\sigma$-strongly convex function $f(x)$ we present an iterative algorithm that after $t$ iterations returns a point $x_{t+1} \in \mathcal{P}$ such that

$$f(x_{t+1}) - f(x^*) \leq C \exp\left(-\frac{\sigma}{4\beta n\mu^2} t\right)$$

where $x^* = \arg\min_{x \in \mathcal{P}} f(x)$ and $C = f(x_1) - f(x^*)$. The algorithm makes a total of $t$ calls to the linear oracle of $\mathcal{P}$.

As we show in section 7, this rate is nearly optimal in certain settings.

Online optimization  In the online setting we present conditional gradient algorithms that require only a single call to the linear optimization oracle of $\mathcal{P}$ on each iteration of the game. In the following we let $G$ denote an upper bound on the $l_2$ norm of the (sub)gradients observed by the algorithms. Our online results are the following,

1. An algorithm for OCO with arbitrary convex loss functions whose regret after $T$ rounds is

$$\sum_{t=1}^T f_t(x_t) - \min_{x^* \in \mathcal{P}} \sum_{t=1}^T f_t(x^*) = O\left(GD\sqrt{n\mu T}\right)$$

This bound is optimal in terms of $T$ [4].

2. An algorithm for OCO with $H$-strongly convex loss functions whose regret after $T$ rounds is

$$\sum_{t=1}^T f_t(x_t) - \min_{x^* \in \mathcal{P}} \sum_{t=1}^T f_t(x^*) = O\left((G + HD)n\mu^2 \log(T)\right)$$

This bound is also optimal in terms of $T$ [15].

3. A randomized algorithm for the partial information setting whose expected regret after $T$ rounds is

$$\mathbb{E}\left[\sum_{t=1}^T f_t(y_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^T f_t(x)\right] = O\left(\frac{\sqrt{n^{3/2}CL\mu R}}{\sqrt{T}} T^{3/4}\right)$$

Here we assume w.l.o.g. that $\mathcal{P}$ contains the origin and that $r\mathbb{B} \subseteq \mathcal{P} \subseteq R\mathbb{B}$. We assume further that $f$ is $L$-Lipschitz and $|f_t(x)| \leq C$ for all $x \in \mathcal{P}$. This bound matches the current state-of-the-art in this setting in terms of $T$ [8].

Stochastic and non-smooth optimization  Applying our online algorithms to the stochastic setting as specified in subsection 2.2.1 yields algorithms that given a function $F(x) = \mathbb{E}_{f \sim \mathcal{D}}[f(x)]$, after viewing $t$ i.i.d samples from the distribution $\mathcal{D}$ and after $t$ calls to the linear oracle $\mathcal{O}_\mathcal{P}$ return a point $x_t$ such that
1. If $D$ is a distribution over arbitrary convex functions then

$$F(x_t) - \min_{x^* \in P} F(x^*) = O\left(\frac{GD \sqrt{n\mu}}{\sqrt{t}}\right)$$

2. If $D$ is a distribution over $H$-strongly convex functions then

$$F(x_t) - \min_{x^* \in P} F(x^*) = O\left(\frac{(G + HD)n\mu^2 \log(t)}{Ht}\right)$$

Here again $G$ denotes an upper bound on the $l_2$ norm of the (sub)gradients of the functions $f$ sampled from the distribution $D$.

Since stochastic optimization generalizes offline optimization, the above rates hold also for non-smooth convex and strongly-convex optimization.

4 A Linearly Convergent CG Algorithm for Smooth and Strongly Convex Optimization

In this section we present and analyse an algorithm for the following offline optimization problem

$$\min_{x \in P} f(x)$$

where we assume that $f$ is $\beta$-smooth and $\sigma$-strongly convex and $P$ is a polytope.

We assume that we have a LLO oracle for $P - A(x, r, c)$. In section 5 we show that given an oracle for linear minimization over $P$, such a LLO oracle could be constructed.

Our algorithm for problem (3) is given below,

Algorithm 2

1: Input: $A(x, r, c)$ - LLO with parameter $\rho$.
2: Let $x_1$ be an arbitrary point in $V$ and let $C \geq f(x_1) - f(x^*)$.
3: Let $\alpha = \frac{\sigma}{2\beta \rho}$.
4: for $t = 1 \ldots$ do
5: $r_t \leftarrow \min\{\sqrt{C}e^{-\alpha^2(t-1)}, D\}$.
6: $p_t \leftarrow A(x_t, r_t, \nabla f(x_t))$.
7: $x_{t+1} \leftarrow x_t + \alpha(p_t - x_t)$.
8: end for

Theorem 1. After $t \geq 1$ iterations algorithm 2 has made $t$ calls to the linear oracle $O_P$ and the point $x_{t+1} \in P$ satisfies

$$f(x_{t+1}) - f(x^*) \leq C \exp\left(-\frac{\sigma}{4\beta n\mu^2 t}\right)$$

where $x^* = \arg\min_{x \in P} f(x)$ and $C \geq f(x_1) - f(x^*)$. 
We now turn to analyse the convergence rate of algorithm \[2\]. The following lemma is of general interest and will be used also in the section on online optimization.

**Lemma 1.** Assume that \(f(x)\) is \(\beta\)-smooth. Let \(x^* = \arg\min_{x \in \mathcal{P}} f(x)\) and assume that on iteration \(t\) it holds that \(\|x_t - x^*\| \leq r_t\). Then for every \(\alpha \in (0, 1)\) it holds that,

\[
f(x_{t+1}) - f(x^*) \leq (1 - \alpha) \left( f(x_t) - f(x^*) \right) + \beta \alpha^2 \min \{ \rho^2 r_t^2, D^2 \}
\]

**Proof.** By the \(\beta\)-smoothness of \(f(x)\) and the update step of algorithm \[2\] we have,

\[
f(x_{t+1}) = f(x_t + \alpha (p_t - x_t)) \leq f(x_t) + \alpha \nabla f(x_t)^\top (p_t - x_t) + \beta \alpha^2 \|p_t - x_t\|^2
\]

Since \(\|x_t - x^*\| \leq r_t\), by the definition of the oracle \(\mathcal{A}\) it holds that i) \(p_t^\top \nabla f(x_t) \leq x^* \nabla F_t(x_t)\) and ii) \(\|x_t - p_t\| \leq \min\{\rho r_t, D\}\). Thus we have that,

\[
f(x_{t+1}) \leq f(x_t) + \alpha \nabla f(x_t)^\top (x^* - x_t) + \beta \alpha^2 \min \{ \rho^2 r_t^2, D^2 \}
\]

Using the convexity of \(f(x)\) and subtracting \(f(x^*)\) from both sides we have,

\[
f(x_{t+1}) - f(x^*) \leq (1 - \alpha) \left( f(x_t) - f(x^*) \right) + \beta \alpha^2 \min \{ \rho^2 r_t^2, D^2 \}
\]

\[\square\]

**Lemma 2.** Let \(\alpha\) be as in algorithm \[2\]. Denote \(h_t = f(x^*) - f(x_t)\) and let \(C \geq h_1\). Then

\[
h_t \leq C e^{-\frac{\sigma}{4\rho^2} (t-1)} \quad \forall t \geq 1
\]

**Proof.** The proof is by a simple induction. For \(t = 1\) we have that \(h_1 = f(x^*) - f(x_1) \leq C\).

Now assume that the theorem holds for \(t \geq 1\). This implies via the strong convexity of \(f(x)\) that

\[
\|x_t - x^*\|^2 \leq \frac{1}{\sigma} (f(x^*) - f(x_t)) = \frac{1}{\sigma} h_t \leq C e^{-\frac{\sigma}{4\rho^2} (t-1)}
\]

Setting \(r_t\) such that \(r_t^2 = \frac{C}{\sigma} e^{-\frac{\sigma}{4\rho^2} (t-1)}\), we have that \(x^* \in P \cap \mathbb{B}_{r_t}(x_t)\). Applying lemma [1] with respect to \(x_t\) and by induction we get,

\[
h_{t+1} \leq \left( 1 - \alpha \right) Ce^{-\frac{\sigma}{4\rho^2} (t-1)} + \frac{\alpha^2 \beta \rho^2}{\sigma} \cdot C e^{-\frac{\sigma}{4\rho^2} (t-1)}
\]

\[
\leq C e^{-\frac{\sigma}{4\rho^2} (t-1)} \left( 1 - \alpha + \frac{\alpha^2 \beta \rho^2}{\sigma} \right)
\]

By plugging the value of \(\alpha\) and using \((1 - x) \leq e^{-x}\) we have

\[
h_{t+1} \leq C e^{-\frac{\sigma}{4\rho^2} t}
\]

\[\square\]

We can now prove theorem [1].

**Proof.** In lemma [2] we prove that a LLO oracle with parameter \(\rho = \sqrt{n} \mu\) and in lemma [2] we shows that this construction requires a single call to the oracle \(\mathcal{O}_P\). The convergence results thus follows from [2].

\[\square\]
5 Constructing a Local Linear Oracle

In this subsection we show how to construct an algorithm for the procedure $A(x, r, c)$ given only an oracle that minimizes a linear objective over the polytope $P$.

As an exposition for our construction of a local linear oracle for general polyhedral sets, we first consider the specific case of constructing a local linear oracle for the $n$-dimensional simplex, that is the set $S_n = \{x \in \mathbb{R}^n | \forall i: x_i \geq 0, \sum_{i=1}^n x_i = 1\}$. Given a point $x \in S_n$, a radius $r$ and a linear objective $c \in \mathbb{R}^n$, consider the optimization problem,

$$\min_{y \in S_n} y^\top c$$

s.t. $\|x - y\|_1 \leq \sqrt{nr}$ \hspace{1cm} (4)

where $\|x - y\|_1 = \sum_{i=1}^n |x(i) - y(i)|$.

Observe that a solution $p$ to problem (4) satisfies:

1. $\forall y \in S_n \cap \mathbb{B}_r(x) : p^\top c \leq y^\top c$
2. $\|x - p\| \leq \sqrt{nr}$

Thus a procedure that returns solutions to problem (4) is a local linear oracle for the simplex with parameter $\rho = \sqrt{n}$.

Assume for now that $\frac{\sqrt{nr}}{2} \leq 1$. Problem (4) is solved optimally by the following very simple algorithm:

1. $i^* \leftarrow \arg\min_{i \in [n]} c(i)$
2. $p \leftarrow x$
3. let $i_1, ..., i_n$ be a permutation over $[n]$ such that $c(i_1) \geq c(i_2) \geq ... \geq c(i_n)$
4. let $k \in [n]$ be the smallest integer such that $\sum_{j=1}^k x(i_j) \geq \frac{\sqrt{nr}}{2}$
5. $\delta \leftarrow \sum_{j=1}^k x(i_j) - \frac{\sqrt{nr}}{2}$
6. $\forall j \in [k-1] : p(i_j) \leftarrow 0$
7. $p(i_k) \leftarrow \delta$
8. $p(i^*) \leftarrow p(i^*) + \frac{\sqrt{nr}}{2}$

Observe that step 1 of the above algorithm is equivalent to a single linear optimization step over the simplex.

We now turn to generalize the above simple construction for the simplex to arbitrary polyhedral sets. The algorithm for the local linear oracle is given below. Note that the algorithm assumes that the input point $x$ is given in the form of a convex combination of vertices of the polytope. Later on we show that maintaining such a decomposition of the input point $x$ is easy.
Algorithm 3

1: Input: a point $x \in \mathcal{P}$ such that $x = \sum_{i=1}^{k} \lambda_i v_i$ \hspace{1mm} $\lambda_i > 0$, $\sum_{i=1}^{k} \lambda_i = 1$, $v_i \in \mathcal{V}$, \hspace{1mm} radius $r > 0$, linear objective $c \in \mathbb{R}^n$.
2: $\Delta \leftarrow \min \{ \frac{N_0}{c^\top r}, 1 \}$.
3: $\forall i \in [k]$: $l_i \leftarrow c^\top v_i$.
4: Let $i_1, ... , i_k$ be a permutation over $[k]$ such that $l_{i_1} \geq l_{i_2} \geq ... l_{i_k}$.
5: for $j = 1 ... k$ do
6: $\lambda_j' \leftarrow \max\{0, \lambda_{i_j} - \Delta\}$.
7: $\Delta \leftarrow \Delta - (\lambda_{i_j} - \lambda_j')$.
8: end for
9: $v \leftarrow \mathcal{O}_p(c)$.
10: return $p \leftarrow \sum_{i=1}^{k} \lambda_i' v_i + \left(1 - \sum_{i=1}^{k} \lambda_i'\right) v$.

Lemma 3. Let $x \in \mathcal{P}$ the input to algorithm 3 and let $y \in \mathcal{P}$. Write $y = \sum_{i=1}^{k} (\lambda_i - \Delta_i) v_i + (\sum_{i=1}^{k} \Delta_i) z$ for some $\Delta_i \in [0, \lambda_i]$ and $z \in \mathcal{P}$ such that the sum $\sum_{i=1}^{k} \Delta_i$ is minimized. Then $\forall i \in [k]$ there exists an index $j \in [m]$ such that $A_2(j) v_i < b_2(j)$ and $A_2(j) z = b_2(j)$.

Proof. Denote $\Delta = \sum_{i=1}^{k} \Delta_i$. Assume the lemma is false and let $i' \in [k]$ such that $\forall j \in [m]$ it holds that if $A_2(j) v_{i'} < b_2(j)$ then $A_2(j) z < b_2(j)$. Given $j \in [m]$ we consider two cases. If $A_2(j) v_{i'} = b_2(j)$ then for any $\gamma \in (0, 1]$ it holds that $A_2(j)(z - \gamma v_{i'}) \leq b_2(j) - \gamma b_2(j) = (1 - \gamma)b_2(j)$

If $A_2(j) v_{i'} < b_2(j)$ then by the assumption $A_2(j) z < b_2(j)$ and there exists scalars $\epsilon_j, \delta_j > 0$ such that $A_2(j) v_{i'} = b_2(j) - \delta_j$ and $A_2(j) z = b_2(j) - \epsilon_j$. Now given a scalar $\gamma > 0$ it holds that $A_2(j)(z - \gamma v_{i'}) = b_2(j) - \epsilon_j - \gamma(b_2(j) - \delta_j) = (1 - \gamma)b_2(j) - (\epsilon_j - \gamma \delta_j)$. Choosing $\gamma \leq \frac{\epsilon_j}{\delta_j}$ we get that $A_2(j)(z - \gamma v_{i'}) \leq (1 - \gamma)b_2(j)$.

Combining the two cases above we conclude that for all $\gamma \leq \min\{1, \min_{j \in [m]} (\frac{1}{\lambda_j})\}$ it holds that $A_2(z - \gamma v_{i'}) \leq (1 - \gamma)b_2$. Since it also holds that $A_1(z - \gamma v_{i'}) = (1 - \gamma)b_1$ we have that $z - \gamma v_{i'} \in (1 - \gamma)\mathcal{P}$.

Thus in particular by choosing $\gamma$ such that $\gamma \leq \frac{\Delta_i}{\lambda_i}$, there exists $w \in \mathcal{P}$ such that $z = (1 - \gamma)w + \gamma v_{i'}$ and,

\[
y = \sum_{i=1}^{k} (\lambda_i - \Delta_i) v_i + \Delta z
= \sum_{i=1}^{k} (\lambda_i - \Delta_i) v_i + \Delta((1 - \gamma)w + \gamma v_{i'})
= \left(\sum_{\substack{i=1, i \neq i' \atop \lambda_i - \Delta_i}} (\lambda_i - \Delta_i) v_i + (\lambda_{i'} - (\Delta_{i'} - \gamma \Delta) - \gamma \Delta) v_{i'}\right) + \Delta(1 - \gamma)w + \gamma \Delta v_{i'}
= \left(\sum_{\substack{i=1, i \neq i' \atop \lambda_i - \Delta_i}} (\lambda_i - \Delta_i) v_i + (\lambda_{i'} - (\Delta_{i'} - \gamma \Delta) v_{i'})\right) + \Delta(1 - \gamma)w
\]

Thus by defining $\forall i \in [k], i \neq i' \Delta_i = \Delta_i$ and $\Delta_{i'} = \Delta_{i'} - \gamma \Delta$ we have that $y = \ldots$
Proof. Denote \( \sum_{i=1}^{k} \lambda_i v_i + (\sum_{i=1}^{k} \Delta_i) w \) with \( \sum_{i=1}^{k} \Delta_i = (1 - \gamma) \sum_{i=1}^{k} \Delta_i \), which contradicts the minimality of \( \sum_{i=1}^{k} \Delta_i \).

Lemma 4. Let \( z \in P \) and denote \( C(z) = \{ i \in [m] : A_2(i)z = b_2(i) \} \) and let \( C_0(z) \subseteq C(z) \) be such that the set \( \{ A_2(i) \}_{i \in C_0(z)} \) is a basis for the set \( \{ A_2(i) \}_{i \in C(z)} \). Then given \( y \in P \), if there exists \( i \in C(z) \) such that \( A_2(i)y < b_2(i) \) then there exists \( i_0 \in C_0(z) \) such that \( A_2(i_0)y < b_2(i_0) \).

Proof. Assume by way of contradiction that there exists \( y \in P \) and \( i \in C(z) \) such that \( A_2(i)y < b_2(i) \) and for any \( j \in C_0(z) \) it holds that \( A_2(j)y = b_2(j) \). Since \( A_2(i) \) is a linear combination of vectors from \( \{ A_2(i) \}_{i \in C_0(z)} \) it follows that the linear system \( A_2(j)x = b_2(j), j \in C_0(z) \cup \{ i \} \) has no solution which is a contradiction to the assumption that \( A_2(i)z = b_2(i) \forall i \in C(z) \).

Lemma 5. Let \( x \in P \) the input to algorithm \( 3 \) and let \( y \in P \) such that \( \| x - y \| \leq r \). Write \( y = \sum_{i=1}^{k} (\lambda_i - \Delta_i) v_i + (\sum_{i=1}^{k} \Delta_i) z \) for some \( \Delta_i \in [0, \lambda_i] \) and \( z \in P \) such that the sum \( \sum_{i=1}^{k} \Delta_i \) is minimized. Then \( \sum_{i=1}^{k} \Delta_i \leq n \psi \frac{\psi}{\xi} r \).

Proof. Denote \( C(z) = \{ j \in [m] : A_2(j)z = b_2(j) \} \) and let \( C_0(z) \subseteq C(z) \) such that the set of vectors \( \{ A_2(i) \}_{i \in C_0(z)} \) is a basis for the set \( \{ A_2(i) \}_{i \in C(z)} \). Denote \( A_{2,z} = \mathbb{R}^{[C(z)] \times n} \) the matrix \( A_2 \) after deleting every row \( i \notin C_0(z) \) and recall that by definition \( \| A_{2,z} \| \leq \psi \). Then it holds that,

\[
\| x - y \|^2 = \| \sum_{i=1}^{k} \Delta_i (v_i - z) \|^2 \geq \frac{1}{\| A_{2,z} \|^2} \| A_{2,z} \left( \sum_{i=1}^{k} \Delta_i (v_i - z) \right) \|^2 \\
\geq \frac{1}{\psi^2} \| \sum_{i=1}^{k} \Delta_i A_{2,z} (v_i - z) \|^2 = \frac{1}{\psi^2} \sum_{j \in C_0(z)} \left( \sum_{i=1}^{k} \Delta_i (A_2(j) v_i - b_2(j)) \right)^2
\]

Note that \( |C_0(z)| \leq n \) and that for any vector \( x \in \mathbb{R}^{[C_0(z)]} \) it holds that \( \| x \| \geq \frac{1}{\sqrt{|C_0(z)|}} \| x \|_1 \). Thus we have that,

\[
\| x - y \|^2 \geq \frac{1}{n \psi^2} \left( \sum_{j \in C_0(z)} \left| \sum_{i=1}^{k} \Delta_i (A_2(j) v_i - b_2(j)) \right| \right)^2 = \frac{1}{n \psi^2} \left( \sum_{j \in C_0(z)} \sum_{i=1}^{k} \Delta_i (b_2(j) - A_2(j) v_i) \right)^2
\]

Combining lemma \( 3 \) and lemma \( 4 \) we have that for all \( i \in [k] \) such that \( \Delta_i > 0 \) there exists \( j \in C_0(z) \) such that \( A_2(j) v_i \leq b_2(j) - \xi \). Hence,

\[
\| x - y \|^2 \geq \frac{1}{n \psi^2} \left( \sum_{i=1}^{k} \Delta_i \xi \right)^2 = \frac{\xi^2}{n \psi^2} \left( \sum_{i=1}^{k} \Delta_i \right)^2
\]

Since \( \| x - y \|^2 \leq r^2 \) we conclude that \( \sum_{i=1}^{k} \Delta_i \leq \frac{\sqrt{n \psi} r}{\xi} \).
The following lemma establishes that algorithm $\mathcal{O}$ is a local linear oracle for $\mathcal{P}$ with parameter $\rho = \sqrt{n\mu}$.

**Lemma 6.** Assume that the input to algorithm $\mathcal{O}$ is $x = \sum_{i=1}^{k} \lambda_i v_i$ such that $\forall i \in [k]$, $\lambda_i > 0$, $v_i \in \mathcal{V}$ and $\sum_{i=1}^{k} \lambda_i = 1$. Let $p$ be the point returned by algorithm $\mathcal{O}$. Then the following conditions hold:

1. $p \in \mathcal{P}$.
2. $\|x - p\| \leq \sqrt{n\mu r}$.
3. $\forall y \in B_r(x) \cap \mathcal{P}$ it holds that $c^T y \geq c^T p$.

**Proof.** Condition 1. holds since $p$ is given as convex combination of points in $\mathcal{V}$. For condition 2. note that,

\[
\|x - p\| = \left\| \sum_{i=1}^{k} (\lambda_i - \lambda'_i) v_i - \left(1 - \sum_{i=1}^{k} \lambda'_i\right) v \right\|
\]

\[
= \left\| \sum_{i=1}^{k} (\lambda_i - \lambda'_i) v_i - \sum_{i=1}^{k} (\lambda_i - \lambda'_i) v \right\|
\]

\[
= \left\| \sum_{i=1}^{k} (\lambda_i - \lambda'_i) (v_i - v) \right\| \leq \sum_{i=1}^{k} (\lambda_i - \lambda'_i) \|v_i - v\|
\]

where the last inequality is due to the triangle inequality.

According to algorithm $\mathcal{O}$ it holds that $\sum_{i=1}^{k} (\lambda_i - \lambda'_i) \leq \sqrt{n\mu} \xi r$ and thus $\|x - p\| \leq \sqrt{n\mu} \xi r = \sqrt{n\mu r}$.

Finally, for condition 3, let $y \in B_r(x) \cap \mathcal{P}$. From lemma $\mathcal{O}$ we can write $y = \sum_{i=1}^{k} (\lambda_i - \Delta_i) v_i + \left(\sum_{i=1}^{k} \Delta_i\right) z$ such that $\Delta_i \in [0, \lambda_i]$, $z \in \mathcal{P}$ and $\sum_{i=1}^{k} \Delta_i = \min\{\sqrt{n\mu} \xi r, 1\}$. Thus,

\[
c^T y = \sum_{i=1}^{k} (\lambda_i - \Delta_i) c^T v_i + \left(\sum_{i=1}^{k} \Delta_i\right) c^T z
\]

\[
= \sum_{i=1}^{k} (\lambda_i - \Delta_i) l_i + \left(\sum_{i=1}^{k} \Delta_i\right) c^T z
\]

\[
\geq \sum_{i=1}^{k} (\lambda_i - \Delta_i) l_i + \left(\sum_{i=1}^{k} \Delta_i\right) c^T v
\]

\[
= \sum_{j=1}^{k} (\lambda_{ij} - \Delta_{ij}) l_{ij} + \left(\sum_{i=1}^{k} \Delta_i\right) c^T v
\]

Since algorithm $\mathcal{O}$ reduces the weights of the vertices $v_i$ according to a decreasing order of $l_i$, we have that $\sum_{j=1}^{k} (\lambda_{ij} - \Delta_{ij}) l_{ij} \geq \sum_{j=1}^{k} \lambda'_{ij} l_{ij}$. Thus we conclude that $c^T y \geq \sum_{j=1}^{k} \lambda'_{ij} l_{ij} + \left(\sum_{i=1}^{k} \Delta_i\right) c^T v = c^T p$. \qed
Note that algorithm 3 assumes that the input point \( x \) is given by its convex decomposition into vertices. All optimization algorithms in this work use 3 in the following way: they give as input to 3 the current iterate \( x_t \) and then given the output of algorithm 3 \( p_t \) they produce the next iterate \( x_{t+1} \) by a taking a convex combination \( x_{t+1} \leftarrow (1 - \alpha) x_t + \alpha p_t \) for some parameter \( \alpha \in (0, 1) \). Thus if the convex decomposition of \( x_t \) is given, updating it to the convex decomposition of \( x_{t+1} \) is straightforward. Moreover, denoting \( V_t \subseteq V \) the set of vertices that form the convex decomposition of \( x_t \), it is clear from algorithm 3 that \( |V_{t+1} \setminus V_t| \leq 1 \), since at most a single vertex \((v)\) is added to the decomposition.

Lemma 7. Algorithm 3 has an implementation such that each invocation of the algorithm requires a single call to the oracle \( \mathcal{O}_P \) and additional \( O(T(n + \log T)) \) time where \( T \) is the total number of calls to algorithm 3.

Proof. Clearly algorithm 3 calls \( \mathcal{O}_P \) only once. The complexity of all other operations depends on \( k \) - the number of vertices in the convex decomposition of the input point \( x \). As we discussed, if we denote by \( x_t, x_{t+1} \) the inputs to the algorithm on calls number \( t, t+1 \) to the algorithm and by \( k_t, k_{t+1} \) the number of vertices in the convex decompositon of \( x_t, x_{t+1} \) respectively then \( k_{t+1} \leq k_t + 1 \). Thus if the algorithm is called a total number of \( T \) times and the initial point \((x_1)\) is a vertex, then at all times \( k \leq T \). Since all other operations except for calling \( \mathcal{O}_P \) consist of computing \( k \) inner products between vectors in \( \mathbb{R}^n \) and sorting \( k \) scalars, the lemma follows.

Note that we can get rid of the linear dependence on \( T \) in the bound in lemma 7 by decomposing the iterate \( x_t \) into a convex sum of fewer vertices in case the number of vertices in the current decomposition, \( k \) becomes too large. From Caratheodory’s theorem we know that there exists a decomposition with at most \( n+1 \) vertices and for many polytopes of interest there is an efficient algorithm for computing such a decomposition. From previous discussions, we will need to invoke this decomposition algorithm only every \( O(n^\mu) \) iterations which will keep the amortized iteration complexity low.

Another approach for the above problem that relies only on the use of the oracle \( \mathcal{O}_P \) is the following. If \( k \) is too large we can compute a new decomposition of the input point \( x \) by finding an approximated solution to the optimization problem \( \min_{y \in \mathcal{P}} \|x - y\|^2 \) using theorem 1 up to precision \( r^2 \). The result will be a point \( x' \) given by a decomposition into \( O(n\mu^2 \log(1/r)) \) vertices such that \( \|x' - x\| \leq r \). Thus this gives us a construction of a local linear oracle with parameter \( \rho = 3\sqrt{n\mu} \). We will need to invoke this decomposition procedure only every \( O(n\mu^2 \log(1/r)) \) iterations which leads to the following lemma.

Lemma 8. Assume that on all iterations the input \( r \) to the LLO algorithm is lower-bounded by \( r_1 > 0 \). Then there exists an implementation for a local linear oracle with parameter \( \rho = 3\sqrt{n\mu} \) such that the amortized linear oracle complexity per iteration is 2 and the additional amortized complexity per iteration is \( O(n\mu^2 \log(1/r_1)) (n + \log(n\mu^2 \log(1/r_1))) \).
In our online algorithm the bound $r_1$ will always satisfy $\frac{1}{r_1} = O(T)$ where $T$ is the length of the game, and thus the running time per iteration depends only logarithmically on $T$.

6 Online and Stochastic Convex Optimization

In this section we present algorithms for the general setting of online convex optimization that are suitable when the decision set is a polytope. We present regret bounds for both general convex losses and for strongly convex losses. These regret bounds imply convergence rates for general convex stochastic optimization and non-smooth convex optimization over polyhedral sets as described in subsections 2.2.1 and 2.2.2. We also present an algorithm for the bandit setting.

Our algorithm for online convex optimization in the full information setting is given below. The functions $F_t(x)$, used by the algorithm (in line 6), will be specified precisely in the analysis. Informally, $F_t(x)$ aggregates information on the loss functions on all times $1\ldots t$ plus some regularization term.

**Algorithm 4**

1: Input: horizon $T$, set of radii $\{r_t\}_{t=1}^T \in [T]$, optimization parameter $\alpha \in (0, 1)$, LLO $\mathcal{A}$ with parameter $\rho$.
2: Let $x_1$ be an arbitrary vertex in $\mathcal{V}$.
3: for $t = 1\ldots T$ do
4: play $x_t$.
5: receive $f_t$.
6: $p_t \leftarrow \mathcal{A}(x_t, r_t, \nabla F_t(x_t))$.
7: $x_{t+1} \leftarrow x_t + \alpha(p_t - x_t)$.
8: end for

We have the following two main results.

Denote $G = \sup_{x \in \mathcal{P}, t \in [T]} \|\nabla f_t(x)\|$ and recall that we have a construction for a local linear oracle with parameter $\rho = O(\sqrt{n\mu})$.

**Theorem 2.** Given a LLO with parameter $\rho = O(\sqrt{n\mu})$, there exists a choice for the parameters $\eta, \alpha, \{r_t\}_{t=1}^T$ such that for general convex losses the regret of algorithm 4 is $O(GD\mu\sqrt{nT})$.

**Theorem 3.** Given a LLO with parameter $\rho = O(\sqrt{n\mu})$, there exists a choice for the parameters $\eta, \alpha, \{r_t\}_{t=1}^T$ such that if all loss functions $f_t(x)$ are $H$-strongly convex then the regret of algorithm 4 is $O((G + HD)^2n\mu^2/H) \log T)$.

Applying the above two theorems with the reduction of stochastic optimization to online optimization, described in subsection 2.2.1 yields the following two corollaries.

**Corollary 1.** Given a stochastic objective $F(x) = \mathbb{E}_{f \sim \mathcal{D}}[f(x)]$ where $\mathcal{D}$ is a distribution over arbitrary convex functions, there exists an iterative algorithm that after
Corollary 2. Given a stochastic objective $F(x) = \mathbb{E}_{f \sim D}[f(x)]$ where $D$ is a distribution over $H$-strongly convex functions, there exists an iterative algorithm that after $t$ linear optimization steps over the domain and after viewing $t$ i.i.d samples from $D$ outputs a point $x_t \in \mathcal{P}$ such that

$$F(x_t) - \min_{x^* \in \mathcal{P}} F(x^*) = O\left(\frac{(G + HD)\eta \mu^2 \log(t)}{Ht}\right)$$

Lemma 9. There is a choice for the parameters $\eta, \alpha, r_t$ such that for any $\epsilon > 0$ it holds that for all $t \in [T]$: $\|x_t - x_t^*\| \leq \sqrt{\epsilon}$.

Proof. We prove by induction that for all $t \in [T]$ it holds that $F_{t-1}(x_t) - F_{t-1}(x_t^*) \leq \epsilon$. By the strong-convexity of $F_{t-1}$ this yields that $\|x_t - x_t^*\| \leq \sqrt{\epsilon}$.

The proof is by induction on $t$. For $t = 1$ it holds that $x_1 = x_1^*$ and thus the lemma holds. Thus assume that for time $t \geq 1$ it holds that $F_{t-1}(x_t) - F_{t-1}(x_t^*) \leq \epsilon$. By the strong-convexity of $F_{t-1}(x)$ and the assumption that the lemma holds for time $t$ we have that,

$$\|x_t - x_t^*\| \leq \sqrt{\epsilon}$$

By the definition of $F_t(x)$ and $x_t^*$ we have that $F_t(x_t^*) - F_t(x_{t+1}^*) = F_{t-1}(x_t^*) - F_{t-1}(x_{t+1}^*) + \eta \nabla f_t(x_t)\top(x_t^* - x_{t+1}^*) \leq \eta G \|x_t^* - x_{t+1}^*\|$ and thus again by the strong convexity of $F_t(x)$ we have that

$$\|x_{t+1}^* - x_t^*\| \leq \eta G$$

Combining (5), (6) we have,

$$\|x_t - x_{t+1}^*\| \leq \sqrt{\epsilon} + \eta G$$

By induction,

$$F_t(x_t) - F_t(x_{t+1}^*) = F_{t-1}(x_t) - F_{t-1}(x_{t+1}^*) + \eta \nabla f_t(x_t)\top(x_t - x_{t+1}^*) \leq \epsilon + \eta G \|x_t - x_{t+1}^*\| \leq \epsilon + \eta G \sqrt{\epsilon_t} + \eta^2 G^2$$

6.1 Analysis for general convex losses

For time $t \in [T]$ we define the function $F_t(x) = \eta \left(\sum_{t=1}^T \nabla f_t(x_t)\top x_t + \|x_t - x_t^*\|^2\right)$ where $\eta$ is a parameter that will be determined in the analysis.

Denote $x_t^* = x_1$ and for all $t \in [T]$ $x_t^* = \arg \min_{x \in \mathcal{P}} F_t(x)$. Denote also $x^* = \arg \min_{x \in \mathcal{P}} \sum_{t=1}^T f_t(x)$. Observe that $F_t(x)$ is 1-smooth and 1-strongly convex.

In the following two subsections we prove theorems 2, 3.
Setting \( r_t = \sqrt{\epsilon} + \eta G \) we can apply lemma 1 with respect to \( F_t(x) \) and get,

\[
F_t(x_{t+1}^*) - F_t(x_{t+1}) \leq (1 - \alpha)(F_t(x_t) - F_t(x_{t+1}^*)) + \alpha^2 \rho^2 \left( \sqrt{\epsilon} + \eta G \right)^2
\]

Plugging (7),

\[
F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq (1 - \alpha) \left( \epsilon + \eta G \sqrt{\epsilon} + \eta^2 G^2 \right) + 2\alpha^2 \rho^2 \left( \epsilon + \eta G \sqrt{\epsilon} + \eta^2 G^2 \right)
\]

Setting \( \alpha = \frac{1}{3\rho^2} \) we get,

\[
F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq \left( \epsilon + \eta G \sqrt{\epsilon} + \eta^2 G^2 \right) \left( 1 - \frac{1}{9\rho^2} \right)
\]

Plugging \( \eta = \frac{\sqrt{\epsilon}}{18G\rho^2} \) gives

\[
F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq \epsilon \left( 1 + \frac{1}{9\rho^2} \right) \left( 1 - \frac{1}{9\rho^2} \right) < \epsilon
\]

we are now ready to prove theorem 2.

**Proof.** Observe that playing the point \( x_{t+1}^* = \arg \min_{x \in P} F_t(x) \) on each time is equivalent to playing the leader on each time with respect to the loss functions \( f'_1(x) = \nabla f_1(x_1)^\top x + \frac{1}{\eta} \| x - x_1 \|^2 \) and \( f'_t(x) = \nabla f_t(x)^\top x \) for every \( t > 1 \). This strategy of playing on each time according to the leader is known to achieve overall zero regret, see [20]. Thus,

\[
\sum_{t=1}^{T} \nabla f_t(x_t)^\top (x_{t+1}^* - x_t^*) \leq \frac{1}{\eta} (\| x_{t+1}^* - x_t^* \|^2 - \| x_{t+1}^* - x_1 \|^2) \leq \frac{D^2}{\eta}
\]

By the definition of \( F_t(x) \), \( x_t^* \) and the use of strong-convexity we have that,

\[
\| x_{t+1}^* - x_{t+1} \|^2 \leq F_t(x_{t+1}^*) - F_t(x_{t+1}^*) \leq \eta G \| x_{t+1} - x_{t+1} \|
\]

Which implies by the triangle inequality that,

\[
\sum_{t=1}^{T} \nabla f_t(x_t)^\top (x_{t}^* - x_t^*) \leq \frac{D^2}{\eta} + T\eta G^2
\]

Setting \( \eta \) to the value determined in lemma 1, plugging lemma 1 and by the convexity of \( f_t() \) we get that for all \( \epsilon > 0 \),

\[
\sum_{t=1}^{T} f_t(x_t) - f_t(x^*) \leq \sum_{t=1}^{T} \nabla f_t(x_t)^\top (x_t - x^*) \leq \frac{18GD^2\rho^2}{\sqrt{\epsilon}} + \frac{T\sqrt{\epsilon}}{18\rho^2} + TG\sqrt{\epsilon}
\]

The theorem now follows from plugging \( \epsilon = \frac{(D\rho)^2}{\sqrt{T}} \).

\( \square \)
6.2 Analysis for strongly convex losses

Assume all loss functions are $H$-strongly-convex. For time $t \in [T]$ define $\tilde{f}_t(x) = \nabla f_t(x) + H\|x - x_t\|^2$ and $F_t(x) = \left(\sum_{\tau=1}^t \tilde{f}_\tau(x)\right) + HT_0\|x - x_1\|^2$ for some constant $T_0$ that will be determined later. Observe that $F_t(x)$ is $H(t + T_0)$-smooth and $H(t + T_0)$-strongly convex.

Denote $L = G + 2HD$.

**Lemma 10.** For all $t \in [T]$, $\tilde{f}_t(x)$ is $L$-Lipschitz over $\mathcal{P}$.

**Proof.** Given two points $x, y \in \mathcal{P}$ it holds that

$$\tilde{f}_t(x) - \tilde{f}_t(y) = \nabla f_t(x)^\top (x - y) + H\|x - x_t\|^2 - H\|y - x_t\|^2 \leq (1 + L)\|x - y\|$$

Using the convexity of the function $g(x) = \|x - x_t\|^2$ we have,

$$f_t(x) - f_t(y) \leq G\|x - y\| + 2H\|x - x_t\|^T(x - y) \leq G\|x - y\| + 2H D\|x - y\|$$

The same argument clearly holds for $\tilde{f}_t(y) - \tilde{f}_t(x)$ and thus the lemma follows.

**Lemma 11.** There is a choice for the parameters $\alpha, r_t, T_0$ such that for any $t \in [T]$ it holds that $\|x_t - x_t^*\| \leq \frac{100\alpha^2 L}{Ht}$.

**Proof.** The proof is similar to that of lemma 3. We prove that for any time $t \in [T]$, $F_{t-1}(x_t) - F_{t-1}(x_t^*) \leq \epsilon_t$ for some $\epsilon_t > 0$ which by strong convexity implies that $\|x_t - x_t^*\| \leq \sqrt{\frac{\epsilon_t}{H(t - 1 + T_0)}}$.

Clearly for time $t = 1$ the lemma holds since $x_1 = x_1^*$. Assume that on time $t$ it holds that $F_{t-1}(x_t^*) - F_{t-1}(x_t) \leq \epsilon_t$. Thus as we have shown,

$$\|x_t^* - x_t\| \leq \sqrt{\frac{\epsilon_t}{H(t - 1 + T_0)}} \quad (8)$$

It also holds that $F_t(x_t^*) - F_t(x_{t+1}^*) = F_{t-1}(x_t^*) - F_{t-1}(x_{t+1}^*) + \tilde{f}_t(x_t^*) - \tilde{f}_t(x_{t+1}^*) \leq \tilde{f}_t(x_t^*) - \tilde{f}_t(x_{t+1}^*)$. By lemma 10 we thus have that $F_t(x_t^*) - F_t(x_{t+1}) \leq L\|x_t^* - x_{t+1}^*\|$. By strong-convexity of $F_t(x)$ we have,

$$\|x_{t+1}^* - x_t^*\| \leq \frac{L}{H(t + T_0)} \quad (9)$$

Combining 8, 9 we have,

$$\|x_t - x_{t+1}\| \leq \sqrt{\frac{\epsilon_t}{H(t - 1 + T_0)}} + \frac{L}{H(t + T_0)} \quad (10)$$

By induction and lemma 3,

$$F_t(x_t) - F_t(x_{t+1}^*) = F_{t-1}(x_t) - F_{t-1}(x_{t+1}^*) + \tilde{f}_t(x_t) - \tilde{f}_t(x_{t+1}^*) \leq \epsilon_t + \frac{L\sqrt{\epsilon_t}}{\sqrt{H(t - 1 + T_0)}} + \frac{L^2}{H(t + T_0)}$$

21
Setting \( r_t \) to the bound in (10), applying lemma 1 with respect to \( F_t(x) \) we have,

\[
F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq (1 - \alpha)(F_t(x_t) - F_t(x_{t+1}^*)) + H(t + T_0)\alpha^2 \rho^2 r_t^2
\]

\[
\leq (1 - \alpha) \left( \epsilon_t + \frac{L \sqrt{\epsilon_t}}{\sqrt{H(t - 1 + T_0)}} + \frac{L^2}{H(t + T_0)} \right) + 2H(t + T_0)\alpha^2 \rho^2 \left( \frac{\epsilon_t}{H(t - 1 + T_0)} + \frac{L^2}{H^2(t + T_0)^2} \right)
\]

\[
\leq (1 - \alpha) \left( \epsilon_t + \frac{L}{\sqrt{H(t - 1 + T_0)}} \sqrt{\epsilon_t} + \frac{L^2}{H(t + T_0)} \right) (1 - \alpha + 4\alpha^2 \rho^2)
\]

Setting \( \alpha = \frac{1}{5\rho^2} \) we get,

\[
F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq \left( \epsilon_t + \frac{L \sqrt{\epsilon_t}}{\sqrt{H(t - 1 + T_0)}} + \frac{L^2}{H(t + T_0)} \right) \left( 1 - \frac{1}{25\rho^2} \right)
\]

Assume now that \( \epsilon_t \leq \frac{(100\rho^2L)^2}{H(t + T_0)} \). Then we have,

\[
F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq \frac{(100\rho^2L)^2}{H(t + T_0)} \left( 1 + \frac{1}{50\rho^2} + \frac{1}{(50\rho^2)^2} \right) \left( 1 - \frac{1}{25\rho^2} \right)
\]

\[
\leq \frac{(100\rho^2L)^2}{H(t + T_0)} \left( 1 + \frac{1}{25\rho^2} \right) \left( 1 - \frac{1}{25\rho^2} \right)
\]

\[
= \frac{(100\rho^2L)^2}{H(t + T_0)} \left( 1 - \frac{1}{(25\rho^2)^2} \right)
\]

Finally, setting \( T_0 = (25\rho^2)^2 \) we have that,

\[
F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq \frac{(100\rho^2L)^2}{H(t + T_0)} \left( 1 - \frac{1}{T_0} \right) \leq \frac{(100\rho^2L)^2}{H(t + T_0)} \left( 1 - \frac{1}{t + 1 + T_0} \right)
\]

\[
= \frac{(100\rho^2L)^2}{H(t + T_0)} \cdot \frac{t + T_0}{t + 1 + T_0} = \frac{(100\rho^2L)^2}{H(t + T_0)}
\]

Thus for all \( t \), \( F_t(x_{t+1}) - F_t(x_{t+1}^*) \leq \frac{(100\rho^2L)^2}{H(t + (25\rho^2)^2)} \), and the lemma follows. \( \square \)

We are now ready to prove theorem 3.

**Proof.** Following the lines of theorem 2 and noticing that on time \( t \), \( x_{t+1}^* \) is the leader with respect to the loss functions \( f_t^*(x) = \tilde{f}_t(x) + HT_0\|x - x_t\|^2 \) and \( f_t^*(x) = \tilde{f}_t(x) \) for all \( t > 1 \) we have that,

\[
\sum_{t=1}^{T} \tilde{f}_t(x_{t+1}^*) - \tilde{f}_t(x^*) \leq HT_0(\|x^* - x_1\|^2 - \|x_{t+1}^* - x_1\|^2) \leq HT_0D^2
\]
By the definition of $F_t(x)$, $x_t^*$, the use of strong-convexity and lemma 10 we have that,

$$Ht\|x_t^* - x_{t+1}^*\|^2 \leq H(t + T_0)\|x_t^* - x_{t+1}^*\|^2 \leq F_t(x_t^*) - F_t(x_{t+1}^*) = \tilde{f}_t(x_t^*) - \tilde{f}_t(x_{t+1}^*) \leq L\|x_t^* - x_{t+1}^*\|$$

Which implies that,

$$\sum_{t=1}^{T} \tilde{f}_t(x_t) - \tilde{f}_t(x^*) = \sum_{t=1}^{T} \tilde{f}_t(x_t) - \tilde{f}_t(x^*) + \tilde{f}_t(x_{t+1}^*) - \tilde{f}_t(x_{t+1}^*)$$

$$\leq \sum_{t=1}^{T} \tilde{f}_t(x_{t+1}^*) - \tilde{f}_t(x_t^*) + \sum_{t=1}^{T} \tilde{f}_t(x_t) - \tilde{f}_t(x_{t+1}^*)$$

$$\leq HT_0D^2 + \sum_{t=1}^{T} \tilde{f}_t(x_t) - \tilde{f}_t(x_t^*) + \sum_{t=1}^{T} \tilde{f}_t(x_t^*) - \tilde{f}_t(x_{t+1}^*)$$

$$\leq HT_0D^2 + \sum_{t=1}^{T} L \left(\|x_t - x_t^*\| + \|x_t^* - x_{t+1}^*\|\right)$$

$$\leq HT_0D^2 + \sum_{t=1}^{T} L \left(\|x_t - x_t^*\| + \frac{L}{Ht}\right)$$

Plugging the value of $T_0$ chosen in lemma 11 and the result of lemma 11 we have,

$$\sum_{t=1}^{T} \tilde{f}_t(x_t) - \tilde{f}_t(x^*) \leq O(HD^2\rho^4) + 2 \sum_{t=1}^{T} \frac{100\rho^2L^2}{Ht} = O \left(\frac{\rho^3L^2}{H} \log T\right)$$

The theorem follows from the observation that for all $t$, since $f_t(x)$ is $H$-strongly convex it holds that,

$$f_t(x_t) - f_t(x^*) \leq \nabla f_t(x_t)\top (x_t - x^*) - H\|x_t - x^*\|^2$$

$$= \nabla f_t(x_t)\top (x_t - x^*) - H(\|x_t - x^*\|^2 - \|x_t - x_t\|^2)$$

$$= \tilde{f}_t(x_t) - \tilde{f}_t(x^*)$$

\[\square\]

### 6.3 Bandit Algorithm

In this section we give an online algorithm for the partial information setting (bandits). The derivation is basically straightforward using our algorithm for the full information setting (algorithm 11) and the technique of [8].

For this section assume that without loss of generality the polytope $P$ contains the origin and it holds that $rB \subseteq P \subseteq R^2$ for some positive scalars $r, R$. We assume that the loss function $f_t(x)$ chosen by the adversary on time $t$, is chosen with knowledge of the history of the game but without any knowledge of possible randomization used by the decision maker on time $t$. We also assume that $f_t(x)$ is $L$-Lipschitz for some positive scalar $L$ and $|f_t(x)| \leq C$ for some positive scalar $C$.  

23
Our algorithm is given below.

Algorithm 5
\begin{algorithm}
1: Input: horizon $T$, parameters $\alpha, \delta$.
2: let $x_1$ be an arbitrary vertex in $V$.
3: for $t = 1 \ldots T$ do
4: let $u_t \in S(x_t)$ chosen uniformly at random.
5: play $y_t = (1 - \alpha)x_t + \delta u_t$.
6: receive $f_t(y_t)$.
7: let $g_t = \frac{\delta}{\eta} f_t(y_t)u_t$.
8: let $F_t(x) = \frac{\eta}{2} \sum_{\tau=1}^{t} g_t^\top x + \|x - x_1\|^2$.
9: update $x_{t+1}$ according to update step in algorithm 4.
10: end for
\end{algorithm}

The regret analysis of algorithm 5 closely follows the analysis [9] but instead of using Zinkevich’s algorithm [33] for the reduction from bandit feedback to full feedback, we use our algorithm 4.

We make the following definitions:
\begin{enumerate}
\item $\hat{f}_t(x) = \mathbb{E}_{v \in B_1(x)}[f_t(x + \delta v)]$
\item $z_t = (1 - \alpha)x_t$
\item $h_t(x) = \hat{f}_t(x) + x^\top (g_t - \nabla \hat{f}_t(z_t))$
\item $G$ satisfies: $\|g_t\| \leq G$.
\end{enumerate}

An important observation is that $\nabla h_t(z_t) = g_t$.

The following lemma states a few conditions, all were proved in [9].

Lemma 12. The following conditions hold.
\begin{enumerate}
\item For $\delta \leq \alpha \cdot r$: $y_t \in \mathcal{P}$
\item $\mathbb{E}[h_t(z_t)] = \mathbb{E}[\hat{f}_t(z_t)] = \mathbb{E}[\hat{f}_t((1 - \alpha)x_t)]$.
\item For any fixed $x \in \mathcal{P}$: $\mathbb{E}[h_t(x)] = \hat{f}_t(x)$.
\item $\mathbb{E} \left[ \min_{x \in \mathcal{P}} \sum_{t=1}^{T} h_t(x) \right] \leq \mathbb{E} \left[ \min_{x \in \mathcal{P}} \sum_{t=1}^{T} \hat{f}_t(x) \right] + O(G\sqrt{T})$
\end{enumerate}

Theorem 4. Algorithm 5 achieves the following regret bound:
\[ \mathbb{E} \left[ \sum_{t=1}^{T} f_t(y_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x) \right] = O \left( \frac{\sqrt{n^{3/2}CL\mu R T^{3/4}}}{\sqrt{r}} \right) \]

Proof. Applying theorem 2 with respect to the losses $\{x^\top g_t\}_{t=1}^{T}$ we have that,
\[ \sum_{t=1}^{T} x_t^\top g_t - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} x^\top g_t = O(G\sqrt{T}) \]
where $\rho = \sqrt{n}\mu$ is the LLO parameter.

Using $z_t = (1 - \alpha)x_t$ and the fact that $0 \in \mathcal{P}$ we have,

$$
\frac{1}{1 - \alpha} \sum_{t=1}^{T} z_t^\top g_t - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} x^\top g_t = \sum_{t=1}^{T} z_t^\top g_t - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} x^\top g_t + \frac{\alpha}{1 - \alpha} \sum_{t=1}^{T} (z_t - 0)^\top g_t
$$

$$= O(GR\rho\sqrt{T})$$

Using $g_t = \nabla h_t(z_t)$ and the convexity of $h_t(x)$ we have,

$$
\sum_{t=1}^{T} h_t(z_t) - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} h_t(x) + \frac{\alpha}{1 - \alpha} \sum_{t=1}^{T} (h_t(z_t) - h_t(0)) = O(GR\rho\sqrt{T})
$$

Taking expectation, plugging conditions 2,3,4 in lemma 12 and assuming $\alpha \leq 1/2$ we have,

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \hat{f}_t(z_t) \right] - \mathbb{E} \left[ \min_{x \in \mathcal{P}} \sum_{t=1}^{T} \hat{f}_t(x) \right] = O(GR\rho\sqrt{T}) + 2\alpha \sum_{t=1}^{T} \mathbb{E}[|\hat{f}_t(z_t) - \hat{f}_t(0)|] \quad (11)
$$

Since $f_t$ is $L$-Lipschitz it holds that for all $x \in \mathcal{P}$, $|f_t(x) - \hat{f}_t(x)| \leq L\delta$. Thus for all $t \in [T]$ the following three conditions hold:

1. $|\hat{f}_t(z_t) - f_t(z_t)| \leq L\delta$.

2. $\left| \min_{x \in \mathcal{P}} \sum_{t=1}^{T} \hat{f}_t(x) - \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x) \right| \leq TL\delta$.

3. $|\hat{f}_t(z_t) - \hat{f}_t(0)| \leq |f_t(z_t) - f_t(0)| + 2L\delta \leq 2L(R + \delta)$.

Plugging the above three conditions into (11) gives,

$$
\mathbb{E} \left[ \sum_{t=1}^{T} f_t(z_t) \right] - \mathbb{E} \left[ \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x) \right] = O(GR\rho\sqrt{T} + L\delta T + \alpha LRT)
$$

We are interested in the regret with respect to the points that we actually played $y_1, \ldots, y_T$. Using $\|y_t - z_t\| = \delta$ and again the Lipschitz property of $f_t$ we have,

$$
\mathbb{E} \left[ \sum_{t=1}^{T} f_t(y_t) \right] - \mathbb{E} \left[ \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x) \right] = O(GR\rho\sqrt{T} + L\delta T + \alpha LRT)
$$

By definition it holds that $G = \max_{t \in [T]} \|g_t\| \leq \frac{\mu}{\delta} C$. Hence,

$$
\mathbb{E} \left[ \sum_{t=1}^{T} f_t(y_t) \right] - \mathbb{E} \left[ \min_{x \in \mathcal{P}} \sum_{t=1}^{T} f_t(x) \right] = O(\frac{n}{\delta} C R \rho \sqrt{T} + L\delta T + \alpha LRT)
$$
Setting $\alpha = \frac{4}{r}$ have,

$$
E \left[ \sum_{t=1}^{T} f_t(y_t) \right] - E \left[ \min_{x \in P} \sum_{t=1}^{T} f_t(x) \right] = O\left( \frac{n CRp \sqrt{T}}{\delta} + \frac{\delta LR}{r} T \right)
$$

Finally setting $\delta = \frac{\sqrt{nCp}}{\sqrt{LT}}$, we get,

$$
E \left[ \sum_{t=1}^{T} f_t(y_t) \right] - E \left[ \min_{x \in P} \sum_{t=1}^{T} f_t(x) \right] = O\left( \frac{\sqrt{nCLpR}}{\sqrt{r}} T^{3/4} \right)
$$

\[\Box\]

### 7. Lower bounds

In this section we show that for some cases, our algorithm and analysis are tight and cannot be improved within the conditional-gradient-type family of methods.

We have considered the optimization of smooth and strongly convex functions with access to a linear optimization oracle. Our algorithm, the first linearly converging conditional gradient algorithm, also produces sparse solutions in the following sense: after $t$ iterations the solution is a convex combination of at most $t$ vertices.

This sparsity property is tight in the following sense: as long as $t = o(n)$, any $\Theta(\frac{1}{t})$-approximate solution must be $t$-dense. This fact is formally given in the following folklore lemma (see Clarkson [5]):

**Lemma 13.** There exists a 1-smooth and 1-strongly convex $f$ over the $n$-dimensional simplex for which if $x_t \in \mathbb{R}^n$ is such that $f(x_t) \leq \min_{x^* \in P} f(x^*) + \frac{1}{t} - \frac{1}{n}$, then $\|x_t\|_0 \geq t$.

**Proof.** We consider an optimization problem over the $n$-dimensional simplex $\Delta_n$. Let $f(x) = \|x - \bar{1}\|^2_2 - n + 2$, where $\bar{1} \in \mathbb{R}^n$ is the all-ones vector, be the 1-smooth and 1-strongly convex objective function.

Sparsity over the simplex is simply measured by the number of non-zero coordinates, or the $\ell_0$ norm. Consider any $t$-sparse solution, $x_t$. The solution to the elementary optimization problem:

$$
\min_{\|x\|_0 \leq t, x \in \Delta_n} \sum_{i=1}^{n} (x_i - 1)^2
$$

is given by

$$
\hat{x}_i = \begin{cases} 
\frac{1}{t} & i \leq t \\
0 & \text{o/w}
\end{cases}
$$

This solution has value $\sum_{i=1}^{n} (\hat{x}_i - 1)^2 = t\left(1 - \frac{1}{t}\right)^2 + (n-t) = n - 2 + \frac{1}{t}$. Hence,

$$
f(x_t) \geq f(\hat{x}) = (n - 2 + \frac{1}{t}) - n + 2 = \frac{1}{t}
$$

On the other hand, the optimal solution $x^* = \frac{1}{n} \bar{1}$ satisfies $f(x^*) = \frac{1}{n}$. \[\Box\]
This lemma implies that any conditional-gradient-type algorithm, which inherently produces sparse solutions, must run at least $n$ iterations before entering the linearly-converging phase. Indeed - our analysis satisfies this condition. Furthermore, for certain cases our analysis is tight and cannot be improved up to constant and poly-logarithmic terms.

Consider, for example, the setting of 1-smooth and 1-strongly convex optimization over the simplex, as in the lower bound above. The linear inequalities defining the simplex can be written in vector form as $-I \mathbf{x} \leq \mathbf{0}$ where $I$ is the $n$-dimensional identity matrix and $\mathbf{0}$ is the all zeros $n$-dimensional vector. Thus it follows that $\psi(\Delta_n) = 1$. The vertices of $\Delta_n$ are the standard basis vectors $e_i \in \mathbb{R}^n$, $i \in [n]$ and thus it is easily verified that $\xi(\Delta_n) = 1$. Finally since $D(\Delta_n) = \sqrt{2}$ we have that $\mu(\Delta_n) = \sqrt{2}$ and thus by theorem 1 applied to this setting guarantees that Algorithm 2 returns a solution $x_t$ for which

$$f(x_{t+1}) - f(x^*) \leq (f(x_1) - f(x^*)) \exp\left(-\frac{1}{4n\mu^2}t\right) \leq 2e^{-\frac{t}{8n}}$$

Thus, over the $n$-dimensional simplex our algorithm returns an $\epsilon$-approximate solution after $O(n \log \frac{1}{\epsilon})$ iterations, which is nearly tight by the aforementioned lower bound.

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