Synchronization of Chaotic Maps by Symmetric Common Noise

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Abstract
Synchronization of identical chaotic systems subjected to common noise has been the subject of recent research. Studies on several chaotic systems have shown that, the synchronization is actually induced by the non-zero mean of the noise, and symmetric noise with zero-mean cannot lead to synchronization. Here it is presented that synchronization can be achieved by \textit{zero-mean} noise in some chaotic maps with large convergence regions.

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Running title: Synchronization by Noise
The effect of common noise on the synchronization of identical chaotic systems has attracted much attention since the work by Maritan and Banavar[1], which claimed that two identical chaotic systems subjected to the same strong enough noise can be synchronized, with the logistic map and the Lorenz system as examples. Some authors have reconsidered their conclusion. Pikovsky[2] pointed out that the largest Lyapunov exponent of the noisy logistic map is positive which is in contradiction to the criterion of negative largest Lyapunov exponent for synchronization[3]. It was also pointed out[2] that the synchronization observed in [1] is an outcome of finite precision in numerical simulations, which was further confirmed by a detailed study by Longa et al[4].

Several other authors, on the other hand, reconsidered the problem by examining the properties of the noise added to the systems. For the case of noisy logistic map

$$x_{n+1} = 4x_n(1 - x_n) + \xi,$$

(1)

where the random number $\xi$ is chosen from the interval $[-W, W]$ with the constraint $x_{n+1} \in (0, 1)$; otherwise, a new random number is chosen. Such a state-dependent noise is no longer symmetric[5], but has a negative mean[6], and it is this nonzero mean that plays an important role in the coalescence of trajectories. A zero-mean noise, although is still state-dependent, cannot lead to synchronization[6].

For the case of the Lorenz system, synchronization was observed by Maritan and Banavar for uniform noise in $[0, W]$, but not for symmetric noise. In [5], it was shown that the largest Lyapunov exponent of the noisy Lorenz system is the same as that of the system driven constantly by the mean value of the noise, indicating that the bias of the noise plays the central role in synchronization. It was pointed out that the origin of nonchaotic behavior is that the Lorenz system driven by large enough constant perturbations is actually stable at the fixed points[5,7]. Very recently, Sanchez et al[8] analyzed the synchronization of chaotic systems by noise in an experiment with the Chua circuit, again drawing the conclusion that synchronization may be achieved only by biased noise, and not symmetric noise.

So it seems that symmetric noise cannot convert a chaotic system into a nonchaotic one, so that synchronization will occur for systems in common noise. In this letter, we are going to present an example that synchronization can actually be achieved by symmetric, zero-mean common noise. In order to avoid the effect of the boundary of a system, such as the logistic map, on the realization of noise, we choose a system that can be driven by noise of any level. The chaotic systems driven by noise are written as

$$x_{n+1} = f(x_n) + \xi,$$

(2)

$$y_{n+1} = f(y_n) + \xi,$$

(3)

with

$$f(x) = \tanh(A_1 x) - B \tanh(A_2 x),$$

(4)

where $A_1$, $A_2$ and $B$ are parameters of the map. Chaos can easily occur in such a nonlinear system. With $A_1 = 20$ and $A_2 = 2$, the Lyapunov exponent of the noise-free system

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \ln |f'(x_n)|,$$

(5)
is calculated as a function of $B$, as shown in Fig. 1(a). Chaos occurs in large regions of $B$ with $\lambda > 0$. In the following simulations, $B$ is fixed at 1.5, and the corresponding chaotic map is shown in Fig. 1(b).

For the noisy chaotic system, the Lyapunov exponent can be estimated by exactly the same formula as Eq. (5), but $\{x_n\}$ is now a noise trajectory. For synchronization to occur, $\lambda$ should be negative. $\lambda < 0$ is possible for an appropriate level of noise so that the state of the system has higher probability to reach $|f'(x_n)| < 1$. Such a region

$$ C = \{x \mid |f'(x)| < 1\} $$

is called the convergence region of the system, and is employed to realize entrainment and migration control of chaotic systems[9].

Our first simulation is to evaluated the Lyapunov exponent of the noisy system. The noise is simulated with uniform random number having a zero-mean and a variance $\sigma^2$. With 100 random initial conditions for the same realization of noise, the averaged Lyapunov exponent is estimated as a function of $\sigma^2$, as shown in Fig. 2. Synchronization with $\lambda < 0$ occurs for $\sigma^2 > 0.24$. So, in contrast to the examples in [1-8], the sensitivity of this chaotic map can be suppressed by zero-mean noise, so that systems starting from different initial conditions will finally collapse into the same final orbit.

The synchronization can be understood from the view point of the convergence region of the map. We perform two calculations: one is the distribution of the state of the system; and the other the distribution of the finite-time Lyapunov exponents defined as [5]

$$ \lambda^{(m)} = \frac{1}{m} \sum_{n=1}^{m} \ln |f'(x_n)|, $$

which measures the average expansion or contraction rate in $m$ steps. The results for noise-free system ($\sigma^2 = 0$), chaotic noisy system ($\sigma^2 = 0.1$) and non-chaotic noisy system ($\sigma^2 = 0.3$) are shown in Fig. 3 (a) for the distribution of the state, and Fig. 3(b) for the distribution of $\lambda^{(10)}$. In Fig. 3(a), the plot of $|f'| < 1$ (solid line) is superimposed onto the normalized distribution of $x_n$. It is seen that for the noise-free case, the system spends most of its time out of the convergence region, and two slightly different orbits will diverge after almost any 10 iterations, because it is very rare that $\lambda^{(10)} < 0$. When the system is subjected to noise, it is driven to spend more time in the convergence region. However, if the noise level is not high enough so that temporal convergence is overcome by divergence in the system dynamics, the system will remain chaotic, and synchronization will not occur. When the noise level is higher then some threshold, the sensitivity is suppressed, and synchronization is achieved.

Our next simulation is to examine the biased noise on synchronization. A biased noise can be denoted by its mean value and a symmetric noise as

$$ \xi_{asym} = a_v + \xi_{sym}, $$

where $\xi_{sym}$ is a uniform random number with mean value 0 and variance $\sigma^2$. For the Lorenz system studied in [1], it is shown in [5,6] that, the largest Lyapunov of the noise system is the same when the fluctuations $\xi_{asym}$ are replaced by the mean value $a_v$ of the noise, showing that it is the non-zero mean which plays the central role in the synchronization of the system. So in our simulations, the Lyapunov exponent is estimated as a function of $a_v$ for the noise system

$$ x_{n+1} = f(x_n) + \xi_{asym}, $$

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and the constantly driven system

\[ x_{n+1} = f(x_n) + a_v. \]  \hspace{1cm} (10)

The results for Eq. (9) with \( \sigma^2 = 0.3 \) and Eq. (10) are shown in Fig. 4. The system of Eq. (10) is still chaotic for most of \( a_v \) in the region \( a_v \in (0, 0.5) \). The noise system, however, is nonchaotic for all the values of \( a_v \). In fact, the constantly driven system of Eq. (10) can be viewed as a new chaotic system \( f_1(x_n) = f(x_n) + a_v \), and the system subjected to biased noise can be regarded as a new chaotic system driven by symmetric noise.

\[ x_{n+1} = f_1(x_n) + \xi_{sym}. \]  \hspace{1cm} (11)

Again, synchronization is achieved by zero-mean noise.

Similar behavior is observed for other parameters of the map. We have also studied other maps having similar large convergence regions, such as the Gaussian map \( f(x) = r\alpha x \exp(-2x^2 + ax) \), where \( \alpha = \sqrt{e}/2 \). For example, with \( a = 0 \) and \( r = 7 \), synchronization can be shown to occur for \( \sigma^2 > 0.099 \).

The counterintuitive effect of noise on chaotic map was reported in some earlier studies[10-12] on the BZ map which is directly connected to the real chemical reaction, the Belousov-Zhabotinshy reaction, and has a similar structure as the maps studied in this letter, i.e., having steep and flat regions. There, a small noise may change the chaotic orbit of the system into a state similar to a periodic orbit with noise[10], which leads to a negative Lyapunov exponent[10], a slower decay of correlations[11] and an improvement of state predictability[12]. This noise-induce order was attributed to the steepness of the BZ map. However, in our examples, the flat regions outside the original chaotic attractors of the map play an important role in synchronization for large enough noise. Unlike the BZ map, the resulted synchronized state is not similar to some periodic orbits.

Synchronization of chaotic systems has potential applications in secure communication[13]. However, information masked by low-dimensional chaos may be attacked using some prediction-based methods[14-16]. Since multiplicative noise may impose great difficulties in dynamical analysis, communication using synchronized noisy maps may provide additional security. We are now investigating an appropriate realization of this idea.

In summary, we have shown that synchronization of identical chaotic systems subjected to the same zero-mean noise can be achieved for large enough noise levels if the systems have large convergence regions. Synchronization is achieved not because of the constant bias of the noise, but because of the fact that noise of a sufficiently high level drives the system deep into the convergence region, where the difference between nearby trajectories shrinks quickly. The general claim that synchronization of identical chaotic system subjected to the same noise is induced by the nonzero bias is not valid. This mechanism of synchronization is quite different from other synchronization approaches where synchronization is achieved when the invariant synchronization manifold is stable[17], thus providing a new understanding of the behavior of driven nonlinear systems. For future research, it is worthwhile to study the synchronization of identical continuous chaotic systems subject to common noise from the viewpoint of convergence regions, and application of the system in secure communication.

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Figure Captions

Fig. 1. (a) Lyapunov exponent of the map via the parameter $B$ at $A_1 = 20, A_2 = 2$. (b) The shape of the map for $A_1 = 20, A_2 = 2$ and $B = 1.5$.

Fig. 2. Lyapunov of the noisy system as a function of noise level $\sigma^2$.

Fig. 3. Normalized histograms of the state $x_n$ (a), and finite-time Lyapunov exponent (b) for $\sigma^2 = 0$ (dots), $\sigma^2 = 0.1$ (pluses) and $\sigma^2 = 0.3$ (circles) from $5 \times 10^6$ iterations. The plot of $|f'| < 1$ (solid line) is superposed onto the figure (a) to indicate the convergence regions.

Fig. 4. Lyapunov exponents as a function of $a_v$. Circles, noise system with $\sigma^2 = 0.3$; Stars, noise replaced by its mean value.
Fig. 1

(a) (b)
Fig. 3

(a)  (b)

![Graphs showing frequency distributions.](image-url)
Fig. 4