A Giraud-type characterization of the simplicial categories associated to closed model categories as ∞-pretopoi

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In SGA 4 [1], one of the principal building blocks of the theory of topoi is Giraud’s theorem, which says that the condition of a 1-category $A$ being the category of sheaves on a site, may be characterized by intrinsic, internal conditions in $A$. The intrinsic conditions are basically existence of certain limits and colimits, plus a condition about generation by a small set of objects.

In this paper, we will present a generalization of this theorem to the situation of simplicial categories (by which we mean simplicially enriched categories) or equivalently Segal categories ([11] [28], [32]), or complete Segal spaces (Rezk [25]). One can easily imagine generalizing the internal conditions of existence of limits or colimits (these become conditions of existence of homotopy limits or colimits). On the other hand, the condition which we take as a generalization of the condition of being the category of sheaves on a site, is the condition of coming from a closed model category [22]. Recall that Dwyer and Kan associate to any closed model category $M$ its simplicial localization $L(M)$ which is a simplicial category [3]. If $M$ is a simplicial closed model category in the sense of Quillen, then $L(M)$ is equivalent to the simplicial category of fibrant and cofibrant objects of $M$. It is this simplicial category $L(M)$ which represents the homotopy theory (including information about all higher-order homotopies) which comes out of $M$.

We attack the very natural question of characterizing which simplicial categories $A$ are equivalent to ones of the form $L(M)$ for closed model categories $M$. This formulation of the question is closely related to some of the entries in the “Model Category” section of M. Hovey’s recent “problem list” [18].

The first, easy but fundamental observation is that if $M$ is a closed model category (admitting all small limits and colimits as it is now customary to assume), then $L(M)$ admits small homotopy limits and colimits. In particular, not every simplicial category will be equivalent to one of the form $L(M)$. Our characterization is that this necessary condition is basically suffi-
cient; however, one has to add in an additional set-theoretic hypothesis about small generation which in practice will always hold. This first easy observation came from thinking about C. Rezk’s terminology of calling his version of the closed model category of Segal categories, the “homotopy theory of homotopy theories”.

Our answer is, as stated above, analogous to Giraud’s theorem. To be quite precise, the analogy is not complete. In effect, the internal conditions on \( A \) which come out are existence of homotopy colimits, and small generation. These turn out to imply existence of homotopy limits; however one does not get any sort of exactness properties allowing one to commute limits and colimits, and indeed one can find examples of closed model categories \( M \) such that \( L(M) \) does not have these exactness properties. Thus, in the statement of our theorem, we refer to our equivalent conditions as defining a notion of \( \infty \)-pretopos, and reserve the name \( \infty \)-topos for an \( \infty \)-pretopos satisfying additional exactness properties.

Another remark is that we are not able to treat all closed model categories, nor does this seem natural in the context of a Giraud-type theorem. Rather we speak only of cofibrantly generated closed model categories see [8] [17] [15]. Almost all known closed model categories (here as usual we only consider ones in which all small limits and colimits exist) are cofibrantly generated.

Hovey also states in [18] that D. Dugger has shown that any cofibrantly generated closed model category is Quillen-equivalent to a simplicial one; thus the reader of the present introduction who is unfamiliar with Dwyer-Kan may assume that we are speaking of simplicial model categories and may replace \( L(M) \) by the simplicial category \( M_{cf} \) of cofibrant and fibrant objects.

Here is a shortened version of the statement. As a matter of notation, we speak in the introduction of “simplicial categories”; the notion of equivalence is that which was explored by Dwyer and Kan [9]. This is just the obvious notion of “fully faithful and essentially surjective” where “fully faithful” means inducing weak equivalences of simplicial \( \text{Hom} \) sets, and “essentially surjective” means essential surjectivity of the truncated morphism on homotopy 1-categories. However, with this definition an equivalence between two simplicial categories means a string of functors which are equivalences, possibly going in different directions. See below for a bit more explanation. We also refer to the body of the paper for the definitions of homotopy colimit, generation by homotopy colimits, and smallness.

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Theorem 1  (cf Theorem 14 p. 42) Suppose $A$ is a simplicial category. The following conditions are equivalent:

(i) There is a cofibrantly generated closed model category $M$ such that $A$ is equivalent to the Dwyer-Kan simplicial localization $L(M)$;
(ii) $A$ admits all small homotopy colimits, and there is a small subset of objects of $A$ which are $A$-small, and which generate $A$ by homotopy colimits.

We call a simplicial category satisfying the conditions of the theorem, an $\infty$-pretopos. If in addition a certain exactness condition is satisfied (see the statement of Theorem 14 for details) then we say that $A$ is an $\infty$-topos.

The possibility of having a reasonable notion of $n$-topos was predicted in [29]. This prediction came about due to the influence of correspondence with C. Teleman who at the time was telling me about pullbacks of simplicial presheaves under morphisms of sites. Of course, like most of what we do here, this idea is very present in spirit throughout [13].

A word about rigour and level of detail in this version of the present paper. At several places in the argument, we skip verification of some details. These are mostly details concerning “homotopy-coherent category theory” as done with Segal categories. They are all generalizations to the “weak-enriched” setting of classical statements in category theory, so it seems completely clear that the statements in question are true. It also seems clear that in the relatively near future, techniques will have sufficiently advanced in order to cover these questions. Finally, it seems likely that using some of the other approaches (such as Cordier-Porter [7] or the model category of Dwyer-Hirschhorn-Kan [8]), a significant number of these details could be verified relatively easily—the reason I haven’t taken that route is lack of familiarity with those approaches. However, at the time of writing of the present version, I have not verified the details any further than what is written down below. One could say that the present paper is premature in this sense, but the result seemed interesting enough to justify writing it up quickly. In order to clarify matters, the places where this problem occurs are marked with the symbol ($\otimes$).

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Segal categories
A simplicial category means a category $C$ enriched over simplicial sets. In other words, for every pair of objects $x, y \in \text{ob}(C)$, we have a simplicial set $\text{Hom}_C(x, y)$. In order to conform with our notations for Segal categories, we shall denote this simplicial set by $C_1(x, y)$. In the case of a simplicial category, composition of morphisms is a map of simplicial sets

$$C_2(x, y, z) := C_1(x, y) \times C_1(y, z) \to C_1(x, z),$$

and this operation is strictly associative. In view of the strict associativity, we obtain a bisimplicial set (i.e. simplicial simplicial set) by setting

$$C_p := \prod_{x_0, \ldots, x_p} C_{p/}(x_0, \ldots, x_p),$$

with

$$C_{p/}(x_0, \ldots, x_p) := C_1(x_0, x_1) \times \ldots \times C_1(x_{p-1}, x_p).$$

Here we set $C_0 := C_0/ := \text{ob}(C)$. This bisimplicial set has the property that for any $m$, the “Segal map”

$$C_m/ \to C_1/ \times_{C_0/} \ldots \times_{C_0/} C_1/$$

is an isomorphism; and conversely any bisimplicial set such that the simplicial set $C_0/ is a discrete set which we denote $C_0 or $\text{ob}(C)$, and such that the above Segal maps are isomorphisms, corresponds to a simplicial category. The composition is obtained by using the third face map $C_2/ \to C_1/.

The above presentation of the notion of “simplicial category” motivates the definition of “Segal category”—a Segal category is just a bisimplicial set such that the simplicial set $C_0/ is a discrete set which we denote $C_0 or $\text{ob}(C)$, and such that the Segal maps are weak equivalences of simplicial sets. A simplicial category thus gives rise to a Segal category, and we shall sometimes call the Segal categories which arise in this way “strict”.

Suppose $C$ is a Segal category. As suggested by the previous notation, for any sequence of objects $x_0, \ldots, x_p \in \text{ob}(C)$ we obtain a simplicial set $C_{p/}(x_0, \ldots, x_p)$ defined as the inverse image of $(x_0, \ldots, x_p)$ under the map (given by the $p + 1$ “vertex” maps)

$$C_{p/} \to C_0 \times \ldots \times C_0.$$
We think of the simplicial set $C_1/(x, y)$ as being the space of maps from $x$ to $y$ in $C$. The Segal condition can be rewritten as saying that the morphism (given by the $p$ “principal edge” maps)

$$C_p/(x_0, \ldots, x_p) \to C_1/(x_0, x_1) \times \cdots \times C_1/(x_{p-1}, x_p)$$

is a weak equivalence. In particular, the “composition of morphisms in $C$” is given by the diagram

$$C_1/(x, y) \times C_1/(y, z) \xrightarrow{\cong} C_2/(x, y, z) \to C_1/(x, z).$$

The notion of Segal category is based in an obvious way on Segal’s weakened notion of “topological monoid” [27] (which is the case where $\text{ob}(C)$ contains only one element), although Segal himself never seems to have written anything suggesting to look at this notion for several objects. This notion per se first appears in Dwyer-Kan-Smith [11] where they also show the equivalence between Segal categories-up-to-equivalence and simplicial categories-up-to-equivalence (see below).

This notion later appeared in an ad hoc way in my preprint [28] (I was unaware of [11] at the time and until fairly recently); and it appears as the basic idea which is iterated in Tamsamani’s definition of weak $n$-category [34]. Further occurrences are in my preprint [32] and the joint paper [16].

A couple of closely related notions are used by Rezk in [25]. He defines a notion of Segal space which is a simplicial set satisfying the condition that the Segal maps are equivalences but not necessarily the condition that $C_0$ be a discrete set; consequently he includes a “Reedy fibrant” condition in the definition in order to make sure that the fiber products involved in the definition of the Segal maps are homotopically correct ones. He also defines a notion of complete Segal space which basically says that the simplicial set $C_0$ should itself correspond to the space which is the realization of the subcategory obtained by only looking at invertible (up-to-homotopy) morphisms in $C$. We will state without proof below what should be the relation between Rezk’s notions and our own.

There are also other related notions such as various notions of $A_{\infty}$-category see for example Batanin [4]; and more generally there are several definitions of $n$-category alternative to Tamsamani’s definition and which should also have variants for weak simplicial categories, see Baez-Dolan [3].
for example. These other notions should be directly related to our own but we don’t go into that here.

Finally, we note that the above notions should be viewed as substitutes for the notion of “1-groupic ∞-category” i.e. an ∞-category in which the i-morphisms are invertible (up to equivalence) for i ≥ 2. We leave it to the reader to make this notational translation.

We shall use the framework of “Segal categories” throughout the rest of the paper, although we sometimes speak of the relationship with the classical notion of simplicial category. The reader is referred to [31] and [16] for any further details and introductory material that we may leave out in our brief discussion which follows.

By abuse of notation, we may sometimes forget to put in the qualifier “Segal” and just use the word “category” for “Segal category”. In order to avoid confusion, we will try to systematically use the terminology 1-category for classical (non-simplicial) categories.

A morphism of Segal categories C → D is said to be fully faithful if for every x, y ∈ ob(C), the morphism C_1/(x, y) → D_1/(x, y) is a weak equivalence of simplicial sets. This is the natural generalization of the corresponding notion in category theory; however one should be careful that the separate notions of “full” and “faithful” don’t have reasonable generalizations to the present theory, because there is no way of decomposing the condition of being a weak equivalence of simplicial sets, into “injectivity plus surjectivity”. For this reason, huge swaths of the argumentation which is employed in SGA 4 [1] are no longer available and we are forced to look for more intrinsic reasoning.

If C is a Segal category, define a 1-category denoted ho(C) with the same objects as C, by setting

\[ \text{ho}(C)_{1/}(x, y) := \pi_0(C_{1/}(x, y)). \]

We say that a morphism of Segal categories C → D is essentially surjective if the resulting morphism of 1-categories

\[ \text{ho}(C) \rightarrow \text{ho}(D) \]

is essentially surjective. We say that a morphism of Segal categories is an equivalence if it is fully faithful and essentially surjective.

In the context of simplicial categories this notion of equivalence was introduced by Dwyer and Kan in [1]. (A morphism of simplicial categories is
an equivalence if and only if the corresponding morphism of Segal categories is an equivalence.) In the context of \(n\)-categories this notion was called “external equivalence” by Tamsamani in \([34]\). In his situation of Segal spaces, this notion was called “Dwyer-Kan equivalence” by Rezk in \([25]\).

We say that a morphism in \(C\) (i.e. a vertex of \(C_1/(x, y)\)) is an equivalence if its image in \(ho(C)_{1/}(x, y)\) is an isomorphism in \(ho(C)\). This corresponds to what Tamsamani called “internal equivalence” in \([34]\). The essential surjectivity condition can be expressed as saying that every object of \(D\) is equivalent (in this “internal” sense) to an object coming from \(C\).

We often use the terminology full subcategory for a fully faithful functor of Segal categories \(C \to D\) which is injective on objects. In this case, up to equivalence in the variable \(C\), we may assume that the morphism is actually an isomorphism on all of the \(C_{p/}(x_0, \ldots, x_p)\). With this convention, the intersection of full subcategories is again a full subcategory. Furthermore, we say that a full subcategory \(C \subset D\) is saturated if it satisfies the “saturation condition” that whenever \(x \in ob(C)\) and \(y\) is (internally) equivalent to \(x\), then \(y \in ob(C)\) too. Again, the intersection of saturated full subcategories is again a saturated full subcategory.

**Strictification**

We can now explain the comparison result of Dwyer, Kan, Smith which was alluded to above. Let \(spl\text{-Cat}\) denote the 1-category of simplicial categories, and let \(Seg\text{-Cat}\) denote the 1-category of Segal categories. Let

\[
Ho(spl\text{-Cat})(\text{resp. } Ho(spl\text{-Cat}))
\]

 denote the Gabriel-Zisman localizations of these categories by inverting the equivalences. We say that two simplicial categories (or two Segal categories) are equivalent if they project to isomorphic objects in these homotopy categories. Dwyer, Kan and Smith in the last few pages of \([11]\) show that the morphism

\[
Ho(spl\text{-Cat}) \to Ho(Seg\text{-Cat})
\]

is an equivalence of categories. Among other things, this says that any Segal category can be “strictified”, i.e. made equivalent (in the above sense) to a simplicial category. We should take this occasion to stress that, as \(Ho(spl\text{-Cat})\) and \(Ho(spl\text{-Cat})\) are Gabriel-Zisman localizations, one can have
two objects (simplicial categories or Segal categories) which are equivalent but without there being any actual morphism between the two; the “equivalence” in question might be realizable only as a chain of morphisms which are equivalences, going in different directions. This situation is improved by the introduction of closed model structures as we shall explain below (and in particular if ever it is necessary to go through a chain of equivalences, at least one can restrict to looking at chains of length 2). In the statement of Theorem 1, it is the present notion of equivalence which is used.

In view of the strictification result of Dwyer-Kan-Smith, we may at many places in the present paper assume that the Segal categories we are dealing with are actually simplicial categories. This can simplify the problem of composing morphisms and the like.

**Closed model structures**

There are several possible closed model structures which can be used to attack the homotopy category $\text{Ho}(\text{splCat}) \cong \text{Ho}(\text{splCat})$. What seems to be historically the first is that of Dwyer-Hirschhorn-Kan \[8\].

To introduce the structure of \[8\], we first point out that Dwyer and Kan obtained (essentially trivially) a closed model structure on splCat in \[9\] where the weak equivalences were the equivalences which induce isomorphisms on objects. In this structure, the fibrations are the morphisms of simplicial categories which induce fibrations of the individual simplicial $\text{Hom}$ sets. The cofibrations are closely related to the free resolutions which are used throughout \[9\], and the cofibrant objects are just the simplicial categories which are free at each stage. This closed model structure is not the one which we are actually interested in (although it can be useful in a preliminary way), because we are interested in understanding the equivalences which are essentially surjective but not isomorphisms on objects. This problem was rectified in \[8\] where a closed model structure on splCat is given, with the following properties. The cofibrations are the same as in the previous structure; and the weak equivalences are the “Dwyer-Kan equivalences” as described above. This leads to a more restrictive notion of fibration than that which occurs in their first structure. However, the fibrant objects are the same as in the previous structure, namely the simplicial categories $C$ with $C_1/(x, y)$ being

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1 The draft of \[8\] that I have is dated at approximately the same time as \[30\] but earlier versions of \[8\] had apparently been in limited circulation for some time.
fibrant simplicial sets. To sum up what is going on here, we can say that in order to correctly calculate the morphisms between two simplicial categories \( C \) and \( D \), one must make a replacement \( D \rightarrow D' \) by an equivalent one in which the simplicial \( Hom \)-sets are fibrant, and one must make a replacement \( C^c \rightarrow C \) with \( C^c \) cofibrant (which essentially means taking a free resolution). Now \( Hom(C^c, D') \) contains representatives for all of the homotopy classes of morphisms from \( C \) to \( D \) in \( Ho(\text{splCat}) \).

The main drawback of the Dwyer-Hirschhorn-Kan closed model structure is that the cofibrant replacement \( C^c \rightarrow C \) is not compatible with direct product. Thus one does not obtain (in any direct way) an internal \( Hom(C, D) \). This internal \( Hom \) will be crucial for the arguments in the present paper.

In [30] is given a closed model structure for \( n \)-categories. This is essentially the same problem as for Segal categories, and in [32], the closed model structure for Segal categories was announced with the statement that the proof is the same as in [30]. A complete proof was written up in [16]. This closed model structure yields as underlying homotopy category \( Ho(\text{SegCat}) \), and it is “internal” i.e. admits a homotopically correct internal \( Hom \). We shall use this structure in the present paper.

Before getting to a more detailed description, we note that Rezk constructs a closed model structure for what he calls complete Segal spaces in [25]. Rezk’s closed model structure again yields as underlying homotopy category a category which is equivalent to \( Ho(\text{SegCat}) \cong Ho(\text{SegCat}) \) (this fact follows immediately from the statements in [25] plus the strictification result of Dwyer-Kan-Smith). And Rezk’s closed model structure is “internal”, in other words it can be used to calculate \( Hom(C, D) \). Thus it should be possible to write the present paper using Rezk’s structure rather than my own. The obvious conjecture is that Rezk's structure and my own are Quillen-equivalent. We don’t prove this here, but it is probably an easy consequence of everything that is said in Rezk’s preprints and my own—the only problem being to digest all of that!

We should also at this point mention another approach, which is the “homotopy-coherent” approach of Cordier and Porter [7]. They define a simplicial category \( Coh(C, D) \) for any two simplicial categories \( C \) and \( D \). This should be equivalent to the \( Hom \) constructed in either my model category or Rezk’s model category, and again it should be possible to write the present paper using Cordier-Porter’s theory (and indeed this might be advantageous in many places).
Invoking the principle that the author of a paper is allowed to choose which approach he wants to use, we will use the closed model structure of \cite{30}, \cite{32} and \cite{16}, which we now describe. The first step is to define a 1-category of \textit{Segal precats} denoted \textit{SePC}. The objects are the bisimplicial sets (denoted as above \( p \mapsto X_p \)) with \( X_p \) denoting a simplicial set such that \( X_0 \) is a discrete set which we denote \( C_0 \) or \( \text{ob}(X) \). The morphisms in \textit{SePC} are just morphisms of bisimplicial sets. This category admits all small limits and colimits. We define the \textit{cofibrations} to be the monomorphisms in this category, in other words the injections of bisimplicial sets. It remains to be seen how to define the weak equivalences. For this, note that the category \textit{SegCat} is a subcategory of \textit{SePC}. The main step (we refer to \cite{30}, \cite{32} and \cite{16} for the details of which) is an essentially unique “projection functor”

\[ \textit{SeCat} : \textit{SePC} \to \textit{SegCat} \subset \textit{SePC}, \]

together with a natural transformation \( \eta_X : X \to \textit{SeCat}(X) \), such that \( \eta_X \) is an equivalence if \( X \) is already a Segal category. This is a variant of the well-known notion of “monad” in category theory, a variant which uses the notion of equivalence (rather than isomorphism) in the target subcategory \textit{SegCat}. We think of \( \textit{SeCat}(X) \) as being the Segal category generated by the “generators and relations” \( X \). In \cite{32} the operation \( X \mapsto \textit{SeCat}(X) \) is analyzed explicitly and shown to have good effectivity properties.

Now we say that a morphism \( X \to Y \) is a \textit{weak equivalence} if the resulting morphism of Segal categories \( \textit{SeCat}(X) \to \textit{SeCat}(Y) \) is an equivalence in the sense explained above. This gives rise to the notion of \textit{trivial cofibration} (a cofibration which is a weak equivalence) and hence to the notion of \textit{fibration} (a morphism which satisfies lifting for all trivial cofibrations). It is shown in \cite{30} and \cite{16} that \textit{SePC} with these three classes of morphisms is a cofibrantly generated closed model category. One thing to note is that the fibrant objects of \textit{SePC} are themselves Segal categories, i.e.

\[ \textit{SePC}_f \subset \textit{SegCat}. \]

It follows that

\[ \text{Ho}(\textit{SePC}) \cong \text{Ho}(\textit{SePC}_f) \cong \text{Ho}(\textit{SegCat}). \]

The closed model category \textit{SePC} is “internal”, see \cite{30} and \cite{16}. This basically means that the cartesian product is a monoidal structure in the
sense of Hovey et al. The effect of this property is that we have a notion of internal $\text{Hom}$ in $\text{SePC}$. This is defined by the adjunction property that for any Segal precat $E$, a morphism

$$E \to \text{Hom}(A, B)$$

is the same thing as a morphism (in $\text{SePC}$)

$$A \times E \to B.$$

Now if $B$ is a fibrant Segal category (i.e. a fibrant object in $\text{SePC}$) then for $A$ any Segal precat, $\text{Hom}(A, B)$ is again a fibrant Segal category. In the case where the second variable is fibrant, formation of the internal $\text{Hom}$ is compatible with weak equivalences in both variables. We will make heavy use of this internal $\text{Hom}$, bearing in mind that whenever it is used, the second variable has to be made fibrant.

The above discussion leads to the notion of natural transformation between two functors of Segal categories. If $A$ and $B$ are Segal categories (with $B$ assumed to be fibrant) and if $f, g : A \to B$ are morphisms, a natural transformation from $f$ to $g$ is a vertex of the simplicial set

$$\eta \in \text{Hom}(A, B)_{1}/(f, g).$$

In general for a Segal category $C$, a vertex of $C_{1}/(x, y)$ is the same thing as a morphism $I \to C$ (where $I$ is the 1-category with two objects 0, 1 and one arrow $0 \to 1$) such that 0 goes to $x$ and 1 goes to $y$. Apply this with $C = \text{Hom}(A, B)$. We get that a natural transformation from $f$ to $g$ is the same thing as a morphism

$$\eta : A \times I \to B$$

such that $\eta|_{A \times 0} = f$ and $\eta|_{A \times 1} = g$.

The internal $\text{Hom}$ is used in [16] (following the same idea in the case of $n$-categories in [30]) to define the Segal 2-category $1\text{SeCAT}$ of all Segal categories. This has for objects the fibrant Segal categories, and between two objects $A, B$ one takes as Segal category of morphisms the internal $\text{Hom}(A, B)$. We get a strict category enriched over fibrant Segal categories, which yields a Segal 2-category. We refer to [30] and [16] for more details; this will not be used in the remainder of the present paper.
We now indicate a sketch of how one should obtain the relationship bet-

between the above closed model category and Rezk’s closed model category \[25\] of complete Segal spaces which we shall denote \( RC \) for the present discus-
sion. If \( A \) is a Segal category, let \( rf(A) \) be a Reedy-fibrant replacement of \( A \) as bisimplicial set. Then \( rf(A) \) is a Segal space in Rezk’s terminology. Now Rezk has a construction which replaces a Segal space by a complete Segal space, which we will denote by \( crf(A) \). This gives a functor going from the category of Segal categories to the category of complete Segal spaces. It descends to the Gabriel-Zisman (or even Dwyer-Kan) localizations where we divide out by equivalences (Rezk states that his construction takes Dwyer-
Kan equivalences of Segal spaces, to equivalences of complete Segal spaces). In the other direction, given a complete Segal space \( X \), we can discretize the space of objects and chop up the other spaces accordingly (in the minimal way so that the transition morphisms remain continuous). This yields a Segal category. Again, this construction takes equivalences to equivalences. Thus we obtain an equivalence of 1-categories between the homotopy category of Segal categories, and the homotopy category of complete Segal spaces:

\[
\text{Ho}(\text{SegCat}) \cong \text{Ho}(\text{SePC}) \cong \text{Ho}(RC).
\]

Furthermore, on the level of Dwyer-Kan localizations we obtain an equivalence of simplicial categories

\[
L(\text{SePC}) \cong L(RC).
\]

Technically speaking, there is probably some remaining verification to be done here, for example verifying that the two constructions are really inverses. It would also be nice to set up a Quillen equivalence between the two model categories, and to verify that the equivalences are compatible with internal \( Hom \).

This last compatibility is already obtained on a homotopy-theoretic level in the following way: it was observed (e.g. in \[16\]) that if a closed model category \( M \) is “internal”, then its Dwyer-Kan localization \( L(M) \) is a simplicial category admitting internal \( Hom \) as defined in an appropriate way. In this case, the internal \( Hom(X,Y) \) (for \( X,Y \in L(M) \)) may be characterized in a way which is internal to \( L(M) \). This applies both to \( \text{SePC} \) and to Rezk’s closed model category \( RC \). Since the two localizations \( L(\text{SePC}) \) and \( L(RC) \) are equivalent (by the argument sketched above), this shows that the
internal $\text{Hom}(X, Y)$ are equivalent in $L(SePC)$ and $L(RC)$. Another way
to recast this remark is to point out that, applying the result of Dwyer-Kan-
Smith [11] we obtain an equivalence with the Dwyer-Kan localization of the
Dwyer-Hirschhorn-Kan model category (we denote the latter by $DHK$)

\[ L(DHK) \cong L(\text{splCat}) \cong L(\text{SegCat}) \cong L(SePC) \cong L(RC), \]

and existence of the internal closed model categories $SePC$ and Rezk's $RC$
can be viewed as ways of proving that the simplicial category $L(DHK)$
adopts an internal $\text{Hom}$. 

We close this subsection on a slightly more technical note. In many pla-
ces, the notation $\Upsilon$ introduced in [31] is crucial for correctly manipu-
lating Segal categories in our method. We refer to there (or to any of my
more recent preprints where this notation is used) for details and ex-
amples. A rapid overview would say that if $E$ is a simplicial set then we obtain a Segal
precat $\Upsilon(E)$ having two objects denoted 0, 1, and having $E$ as simplicial set
of morphisms from 0 to 1. In the case $E = *$ we recover $\Upsilon(*) = I$, the 1-
category with objects 0 and 1 and a single morphism $0 \to 1$. This has a sort
of universal property: for any Segal precat $A$, a morphism $E \to A_{1/(x, y)}$ is
the same thing as a morphism $\Upsilon(E) \to A$
sending 0 to $x$ and 1 to $y$.

More generally if $E, F$ are simplicial sets then we obtain $\Upsilon^2(E, F)$ which
has objects 0, 1, 2 and $E$ as morphisms from 0 to 1; $F$ as morphisms from 1
to 2; and $E \times F$ as morphisms from 0 to 2. This latter is useful for dividing
up a square into two triangles: one has the pushout formula

\[ \Upsilon(E) \times \Upsilon(F) \cong \Upsilon^2(E, F) \cup^{(E \times F)} \Upsilon^2(F, E). \]

Finally, the existence of weak compositions is manifested in the statement
that the inclusion

$\Upsilon(E) \cup (1) \Upsilon(F) \to \Upsilon^2(E, F)$

is a trivial cofibration.

**Simplicial sets and cartesian families**
Let $S$ denote the simplicial category of all fibrant simplicial sets. It has for objects the fibrant simplicial sets $K$, and for simplicial $Hom$ sets the internal $Hom(K, L)$ of simplicial sets.

Unfortunately, $S$ is not fibrant as a Segal category. Thus we must fix a fibrant replacement $S \to S'$ (i.e. an equivalence of Segal categories with $S'$ fibrant). Note here that $S'$ cannot be a strict simplicial category. This fibrant replacement is a source of most of the technical difficulties which were encountered in [31] and [16]. The best way to get around these problems, at least in the context of the theory we are exposing here, is the canonical fibrant replacement defined using the notion of “cartesian family” in [33]. This was constructed in the context of $n$-categories, giving a fibrant replacement for the $n+1$-category $nCAT$ of all $n$-categories. We describe here the variant for obtaining a fibrant replacement for $S$ (note that in the notation of [16], a simplicial set is a Segal 0-category and $S = 0SeCAT$; the variant we are about to describe is obtained from the discussion in [33] by substituting “0Se” for “$n$”).

For ease of use in the rest of the paper, we consider “contravariant” cartesian families; these will correspond to functors $A' \to S'$, and this constitutes a change with respect to [33] where “covariant” cartesian families were considered.

Suppose $A$ is a Segal category, considered as a bisimplicial set. A (contravariant) precartesian family (of simplicial sets) over $A$ is a morphism of bisimplicial sets

$$\mathcal{F} \to A$$

satisfying the “cartesian property” which we now explain. We first establish some notations: $\mathcal{F}_{p/}$ is the simplicial set obtained by putting $p$ in the first bisimplicial variable; thus $\mathcal{F}_{p/} \to A_{p/}$. For objects $x_0, \ldots, x_p \in ob(A)$, we denote by

$$\mathcal{F}_{p/}(x_0, \ldots, x_p)$$

the inverse image of $A_{p/}(x_0, \ldots, x_p)$. It is also the inverse image of $(x_0, \ldots, x_p)$ under the map $\mathcal{F}_{p/} \to A_0 \times \ldots \times A_0$. We do not make the assumption that $\mathcal{F}_{0/}$ is a discrete set, and indeed for $x \in ob(A)$ the simplicial set $\mathcal{F}_{0/}(x)$ is exactly the one which is considered to be parametrized by the object $x$. We have a map of simplicial sets

$$\mathcal{F}_{p/}(x_0, \ldots, x_p) \to A_{p/}(x_0, \ldots, x_p) \times \mathcal{F}_{0/}(x_p)$$
given by the projection $F \to A$ and the structural map for $F$ with respect to
the arrow $0 \to p$ in $\Delta$ corresponding to the last vertex. The “(contravariant)
cartesian condition” is that the above map should be a weak equivalence of
simplicial sets. Note that the “covariant cartesian condition” would be the
same but using the structural map to $F_0/(x_0)$ rather than to $F_0/(x_p)$.

A cartesian family corresponds to a weak functor $A^o \to S$ in much the
same way as the Segal condition encodes the notion of weak category: the
action of the space of morphisms $A_1/(x,y)$ is given by the diagram

$$F_0/(y) \times A_1/(x,y) \xrightarrow{\cong} F_1/(x,y) \to F_0/(x),$$

the second morphism being the structural morphism for the map $0 \to 1$ in $\Delta$
corresponding to the first vertex. The higher $F_p/\cdot$ encode homotopy-coherent
associativity of this action.

In [33] the notion of cartesian family is defined by saying that it is a
precartesian family which satisfies a certain quasi-fibrant condition. This
quasi-fibrant condition (which is analogous to the classical notion of quasi-
fibration and is somewhat similar to Rezk’s notion of “sharp map” [20]) is
designed to guarantee that cartesian families over Segal precats can be glued
together. This glueing property ensures representability of the associated
functor of Segal precats, and allows us to define a Segal category $S'$ with
the property that a morphism $A^o \to S'$ is exactly the same thing as a con-
travariant cartesian family over $A$. In [33] it is shown that there is a natural
morphism $S \to S'$, that this is an equivalence of Segal categories, and that $S'$
is fibrant; thus $S'$ is a canonical fibrant replacement for $S$. This fact means
that “weak families” of simplicial sets parametrized by a Segal category $A$,
i.e. weak functors $A^o \to S$, may be viewed as cartesian families. The proofs
in [33] are given in the context of $n$-categories but the same work in the Segal
category context (or more generally for Segal $n$-categories [10]).

In practice, there is no essential difference between the notion of pre-
cartesian family and the notion of cartesian family. Generally speaking, the
natural constructions that one can make are precartesian but not cartesian;
than one should make a fibrant replacement (which is consequently quasi-
fibrant) to get a cartesian family. We will systematically ignore this point
in the remainder of the paper, and speak only of precartesian families but
use the terminology “cartesian family”. The reader should note that in order
to be precise, one must make fibrant replacements sometimes. Since these
are essentially unique (i.e. unique up to coherent homotopy) this doesn’t pose any homotopy-coherence problems. Of course one should check that the previous phrase is true (⊗).

**Segal categories of presheaves**

The fundamental construction underlying SGA 4 [1] is the Yoneda embedding of a category into the category of presheaves over itself. We have the same thing for Segal categories. For this section I should acknowledge the suggestion of A. Hirschowitz who pointed out that it would be interesting to look at the notion of representable functor in the context of n-categories. And J. Tapia who pointed out to me that this was the fundamental thing in SGA 4; he is working on an altogether different generalization of it.

Let $S$ be the simplicial category of fibrant simplicial sets, and let $S'$ be its replacement by an equivalent fibrant Segal category. If $A$ is any Segal category, put

$$\hat{A} := \text{Hom}(A^o, S').$$

Recall that $A^o$ is the “opposite” Segal category, with the same objects as $A$ and obtained by putting

$$A^o_p(x_0, \ldots, x_p) := A_p(x_p, \ldots, x_0).$$

The first step is that we would like to construct a natural morphism

$$h_A : A \to \hat{A}.$$

In view of the definition of the internal $\text{Hom}(A^o, S')$ (see above), constructing the morphism $h_A$ is equivalent to constructing the “arrow family”

$$\text{Arr}_A : A^o \times A \to S'.$$

We give two discussions of the construction of $\text{Arr}_A$. Both of these constructions were done for $n$-categories in [3]. We should also note that in the simplicial case, the “arrow family” is certainly very classical; among other things it occurs in Cordier-Porter [7].

The easy case is when $A$ is a strict simplicial category with fibrant simplicial $\text{Hom}$ sets. In this case, the formula

$$\text{Arr}_A(x, y) := A_1(x, y)$$
defines in an obvious way a morphism of strict simplicial categories

\[ A^o \times A \to S. \]

There is a canonical fibrant replacement within the category of simplicial sets, compatible with direct product (namely taking the singular complex of the topological realization of a simplicial set), so we obtain a way of replacing any simplicial category by one whose simplicial Hom sets are fibrant. This can be composed with the Dwyer-Kan strictification described above, so if \( A \) is any Segal category then we can replace \( A \) by an equivalent strict simplicial category with fibrant Hom spaces and then apply the construction of \( Arr_A \) given in the present paragraph. Thus this construction technically speaking suffices in order to define the morphism \( h_A \) and the reader wishing to avoid technicalities may skip the subsequent paragraph.

The more complicated case is to treat directly the case where \( A \) is a Segal category. This has the advantage of avoiding a number of equivalences used in the previous paragraph; however it makes use of the notion of “cartesian family” described above (and for which the reader must refer to [3]). We choose for fibrant replacement that \( S' \) which was obtained using the notion of cartesian family. Thus, in order to define the morphism

\[ Arr_A : A^o \times A \to S', \]

we have to define a contravariant cartesian family over \( A \times A^o \). We do this by first defining a natural precartesian family \( \mathcal{F} \), then replacing by a fibrant replacement \( \mathcal{F}' \). The precartesian family \( \mathcal{F} \) has the very simple formula

\[ \mathcal{F}_p/((x_0, y_0), \ldots, (x_p, y_p)) := A_{2p+1}/(x_0, \ldots, x_p, y_p), \ldots, y_0). \]

Note that

\[ (A \times A^o)_p/((x_0, y_0), \ldots, (x_p, y_p)) = A_p/(x_0, \ldots, x_p) \times A_p/(y_p), \ldots, y_0). \]

The Segal condition for \( A \) implies that the map

\[ \mathcal{F}_p/((x_0, y_0), \ldots, (x_p, y_p)) \to A_p/(x_0, \ldots, x_p) \times A_p/(y_p), \ldots, y_0) \times A_1/(x_p, y_p) \]

is an equivalence. This is the cartesian condition for \( \mathcal{F} \), so \( \mathcal{F} \) is a precartesian family. The morphism \( Arr_A \) is defined by choosing a fibrant replacement \( \mathcal{F}' \) for \( \mathcal{F} \).
In the above discussion, the Segal category $A$ must be small. For a “big” Segal category (by which we always mean one in which the objects can form a class, but in which the $A_{p/}(x_0,\ldots,x_p)$ are still sets), it doesn’t seem to be reasonable to define $\hat{A}$. However, we will run across the following intermediate situation: suppose

$$C \to A$$

is a morphism from a small Segal category $C$ to a “big” Segal category $A$. Then we still obtain a morphism

$$i : A \to \hat{C}.$$

Define this by exhausting $A$ by small Segal categories $A_\beta$, and on each of these define $i$ as the composition

$$A_\beta \to \hat{A}_\beta \to \hat{C}.$$

Here is the statement of our main “Yoneda-type” theorem.

**Theorem 2** If $A$ is any small Segal category then the morphism

$$h_A : A \to \hat{A}$$

is fully faithful.

**Proof:** We prove the following more general statement: if $G \in \hat{A}$ and if $x \in A$ then there is a natural equivalence

$$\hat{A}_{1/}(h_A(x), G) \cong G(x)$$

(which is required to be compatible with $h_A$, see below).

We first point out how to go from here to the statement of the theorem: for $x,y \in \text{ob}(A)$, apply the above to $G := h_A(y)$. We get

$$\hat{A}_{1/}(h_A(x), h_A(y)) \cong h_A(y),$$

but $h_A(y) \cong A_{1/}(x,y)$ by construction (recall that $h_A$ comes from the arrow family). Thus

$$\hat{A}_{1/}(h_A(x), h_A(y)) \cong A_{1/}(x,y).$$
This equivalence will be compatible with the morphism $h_A : A \to \hat{A}$, so it shows that $h_A$ is fully faithful.

Now we show how to prove the more general statement. We can view $G$ as being a cartesian family over $A$. In order to define a morphism

$$G(x) \to \hat{A}_{1/}(h_A(x), G)$$

we need to define a morphism

$$\Upsilon(G(x)) \to \hat{A}$$

or equivalently a morphism

$$[\Upsilon(G(x)) \times A]^p \to S'$$

restricting over $0 \times A^p$ to $G$, and restricting over $0 \times A^p$ to $G$. This latter morphism corresponds to a contravariant cartesian family

$$\mathcal{F} \to \Upsilon(G(x)) \times A,$$

with $\mathcal{F}$ restricting as above to $Arr_A(-, x)$ and $G$ on the endpoints. In order to define the family $\mathcal{F}$, given that we already know its restrictions to $0 \times A$ and $1 \times A$, it suffices to define

$$\mathcal{F}_{p/}(u_0, \ldots, u_a; v_0, \ldots, v_b) := G_{p+1/}(u_0, \ldots, u_a, v_0, \ldots, v_b, x).$$

for $a, b \geq 0$ and $a + b + 1 = p$. Here $u_i, v_j \in ob(A)$ and the variables $u_i$ indicate objects considered in $0 \times A$; the variables $v_j$ indicate objects considered in $1 \times A$. The simplicial restriction maps are obtained by those of $G$ whenever the sequence of objects still contains an object of $0 \times A$, otherwise it is obtained by composing with the morphism $G \to A$. The structural morphism to $\Upsilon(G(x)) \times A$ will be seen in the upcoming verification. We check the cartesian condition:

$$G_{p+1/}(u_0, \ldots, u_a, v_0, \ldots, v_b, x) \cong A_{p+1/}(u_0, \ldots, u_a, v_0, \ldots, v_b, x) \times G(x)$$

$$\cong A_{p/}(u_0, \ldots, u_a, v_0, \ldots, v_b) \times h_A(x)(v_b) \times G(x)$$

$$\cong [\Upsilon(G(x)) \times A]_{p/}(u_0, \ldots, u_a, v_0, \ldots, v_b) \times h_A(x)(v_b).$$
Thus $\mathcal{F}$ is a precartesian family. As said previously, we are ignoring the difference between cartesian and precartesian families. Thus we have defined our morphism

$$G(x) \rightarrow \hat{A}_{1/}(h_A(x), G).$$

The next step is to define a morphism in the other direction:

$$\hat{A}_{1/}(h_A(x), G) \rightarrow G(x).$$

For this, note that the restriction along $\{x\} \rightarrow A$ gives a morphism

$$\hat{A} \rightarrow S'.$$

We obtain a morphism

$$\hat{A}_{1/}(h_A(x), G) \rightarrow S'_{1/}(h_A(x)(x), G(x)).$$

On the other hand, the identity element gives a morphism $\ast \rightarrow h_A(x)(x) = A_{1/}(x, x)$, and “composing” with this gives

$$S'_{1/}(h_A(x)(x), G(x)) \rightarrow S'_{1/}(\ast, G(x)) \cong G(x).$$

As usual this “composition” requires inverting some equivalences which come up in the notion of Segal category. We don’t write out the details of that here (although this neglect doesn’t actually merit a $\otimes$). We get our morphism

$$\hat{A}_{1/}(h_A(x), G) \rightarrow G(x).$$

To complete the proof, we have to say that these two morphisms are inverses up to homotopy. In one direction it is basically easy (modulo struggling with the details of the weak compositions everywhere) that the composition

$$G(x) \rightarrow \hat{A}_{1/}(h_A(x), G) \rightarrow G(x)$$

is homotopic to the identity of $G(x)$. For this direction, one way to proceed would be to note that, for an appropriate Dwyer-Kan-Smith strictification and then Dwyer-Hirschhorn-Kan cofibrant replacement, $A$ can be assumed to be a strict simplicial category and $G$ a strict diagram $A \rightarrow S$. In this setup we obtain (by just simplicially-enriching the easy discussion for 1-categories) a sequence

$$G(x) \rightarrow \text{Hom}(A, S)_{1/}(h_A(x), G) \rightarrow G(x)$$

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whose composition is the identity of $G(x)$ on-the-nose. In this formula, the simplicial category $\text{Hom}(A, S)$ is not necessarily the “right” one but it maps into $\hat{A}$, and this is sufficient to check that the above composition that we are interested in, is homotopic to the identity. Note that the morphism in the strictified setup is homotopic to the morphism we have constructed in the original weak situation.

It is somewhat more problematic to see why the composition

$$\hat{A}_1/(h_A(x), G) \to G(x) \to \hat{A}_1/(h_A(x), G)$$

is the identity. This is because it is not clear (to me at least) whether all of $\hat{A}_1/(h_A(x), G)$ can in some way—and after appropriate replacements of $A$ and $G$—be supposed to consist entirely of strict natural transformations between strict diagrams.

Instead, we again make a more general statement, namely the naturality of the morphism

$$G(x) \to \hat{A}_1/(h_A(x), G)$$

in the variable $G$. This says that if $F$ and $G$ are objects in $\hat{A}$ then the diagram

$$
\begin{array}{ccc}
F(x) \times \hat{A}_1/(F, G) & \to & G(x) \\
\downarrow & & \downarrow \\
\hat{A}_1/(h_A(x), F) \times \hat{A}_1/(F, G) & \to & \hat{A}_1/(h_A(x), G)
\end{array}
$$

commutes up to homotopy.

For this statement and its proof, we first take note of the following remark: if $U$ and $V$ are diagrams in $\hat{A}$ then a morphism $E \to \hat{A}_1/(U, V)$ is by definition a morphism

$$\Upsilon(E) \to \text{Hom}(A^o, S')$$

or equivalently a contravariant cartesian family over

$$A \times \Upsilon(E)$$

restricting to $V$ on $A \times 0$ and to $U$ on $A \times 1$ (in this last reduction we use the natural isomorphism $\Upsilon(E)^o \cong \Upsilon(E)$ which interchanges 0 and 1). It is easy to see that a precartesian family over $A \times \Upsilon(E)$, with restrictions $U$ and $V$, is exactly the same thing as a precartesian family over $A \times I$ with restrictions $V$ on $A \times 0$, and $U \times E$ on $A \times 1$. 

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With this remark in mind, we can return to the above diagram and (ap-
plying the remark to the vertical arrows) note that it is the same thing as
giving a diagram in \( \widehat{A} \) of the form

\[
\begin{align*}
h_A(x) \times F(x) \times \widehat{A}_{1/}(F, G) & \rightarrow h_A(x) \times G(x) \\
\downarrow & \quad \downarrow \\
F \times \widehat{A}_{1/}(F, G) & \rightarrow \quad \widehat{A}_{1/}(F, G) \rightarrow G.
\end{align*}
\]

Again applying the remark of the previous paragraph (but to the horizontal
arrows this time) with \( E = \widehat{A}_{1/}(F, G) \) we get that the above diagram is the
same thing as a precartesian family over

\[ A \times I \times \Upsilon(E) \]

whose restrictions to the corners are respectively:

\[
\begin{align*}
A \times 0 \times 0 & : \quad G \\
A \times 0 \times 1 & : \quad F \\
A \times 1 \times 0 & : \quad h_A(x) \times G(x) \\
A \times 1 \times 1 & : \quad h_A(x) \times F(x).
\end{align*}
\]

The restrictions to the edges \( A \times I \times 0 \) and \( A \times I \times 1 \) should be the cartesian
families constructed above for \( G \) and \( F \) respectively; the restrictions to \( A \times 0 \times \Upsilon(E) \) and \( A \times 1 \times \Upsilon(E) \) should be the tautological families. The construction
of this cartesian family is done in the same way as the previous construction
for the morphism \( h_A(x) \times G(x) \rightarrow G \), but starting with the tautological
cartesian family over \( A \times \Upsilon(E) \) corresponding to the morphism \( F \times E \rightarrow G \).
We leave it to the reader to write down the details (\( \otimes \)). This gives the
homotopy-commutative diagram of naturality.

Let’s now look at how to go from the above naturality statement to the
fact that our composition of morphisms is homotopic to the identity. For
this, apply the naturality statement with \( F = h_A(x) \) and \( G \) as given. Then
the naturality statement is a diagram

\[
\begin{align*}
A_{1/}(x, x) \times \widehat{A}_{1/}(h_A(x), G) & \rightarrow G(x) \\
\downarrow & \quad \downarrow \\
\widehat{A}_{1/}(h_A(x), h_A(x)) \times \widehat{A}_{1/}(h_A(x), G) & \rightarrow \quad \widehat{A}_{1/}(h_A(x), G).
\end{align*}
\]
Plugging in the identity from $x$ to $x$ we get a diagram

$$
\begin{array}{ccc}
\hat{A}_1/(h_A(x), G) & \rightarrow & G(x) \\
\downarrow & & \downarrow \\
\{1_{h_A(x)}\} \times \hat{A}_1/(h_A(x), G) & \rightarrow & \hat{A}_1/(h_A(x), G).
\end{array}
$$

The composition along the top followed by the right is the morphism we are interested in; the other composition is the identity. Therefore, homotopy-commutativity of the square shows that the composition in question

$$
\hat{A}_1/(h_A(x), G) \rightarrow G(x) \rightarrow \hat{A}_1/(h_A(x), G)
$$
is the identity. This completes the proof of the theorem.

A lemma which will be used in several places below (and indeed, which is at the origin of the statement of Theorem 14) is the following calculation of $\hat{C}$. Recall that the Heller model category of simplicial presheaves [14] was the precursor to the now standard Joyal-Jardine model category [20] [19]; Heller’s result was the special case of a category with trivial Grothendieck topology (which is the case we need here).

**Lemma 3** Suppose $C$ is a small 1-category. Then $\hat{C}$ is equivalent to $L(M)$ where $M = S^C$ is the Heller model category of simplicial presheaves over $C$.

**Proof:** This is a special case of Théorème 12.1 of [14]. To obtain the special case, replace $n$ by 0 in the statement of that theorem, and note that (in the notation of [14]) a “Segal 0-category” is the same thing as a simplicial set. One should also refer to Théorème 11.11 of the same reference.

This statement is also given by Rezk in [23], and the proof Rezk gives uses some results of Dwyer-Kan. (The results of Dwyer-Kan were of course much prior to [14]).

**Adjoint functors**

There is a notion of adjunction between functors of simplicial categories or Segal categories, which is a direct generalization of the classical notion of adjunction of functors. In making this generalization, it is best to specify only one of the adjunction transformations and impose the condition that
it induces an equivalence between the appropriate simplicial $Hom$ sets. If one tried to specify both of the classical adjunction transformations, this would run into the homotopy-coherence problem that it would be necessary (in order to obtain a well-behaved notion) to specify higher order homotopy coherencies.

The basic historical reference for this section is Cordier and Porter [7], who treat the case of adjunctions of homotopy-coherent functors between simplicial categories. This should be completely equivalent to what we say here. Furthermore, referring to their approach might allow easy removal of the many $\otimes$ which appear in the following discussion.

Suppose $A, B$ are Segal categories (which we suppose fibrant) and suppose $F : A \to B$ and $G : B \to A$ are functors. Suppose $\eta : 1_B \to FG$ is a natural transformation; technically speaking, this means

$$\eta \in Hom(B, B)_{1/(1_B, FG)},$$

which in turn means that $\eta$ is a morphism of Segal categories

$$B \times I \to B$$

restricting to $1_B$ on $B \times 0$ and to $FG$ on $B \times 1$. Here as throughout, $I$ denotes the category with two objects 0, 1 and a single (non-identity) morphism $0 \to 1$. Generally we consider $I$ as a Segal category.

We obtain the following morphisms:

$$\overline{Hom}(A^o \times A, S')^{(G^o \times 1)^*} \to \overline{Hom}(B^o \times A, S'),$$

and

$$\overline{Hom}(B^o \times B, S')^{(1 \times F)^*} \to \overline{Hom}(B^o \times A, S').$$

In particular, we have two elements

$$(G^o \times 1)^*(Arr_A), \ (1 \times F)^*(Arr_B) \in \overline{Hom}(B^o \times A, S').$$

These represent respectively

$$(x, y) \mapsto A_{1/(Gx, y)}$$

and

$$(x, y) \mapsto B_{1/(x, Fy)}.$$
In the same way as for the classical 1-category case, the natural transformation $\eta$ gives rise to a morphism $\text{adj}(\eta)$ in the Segal category $\text{Hom}(B^o \times A, S')$ relating the above two elements; this morphism arises as a morphism $I \times B^o \times A \to S'$ restricting to $(G^o \times 1)^*(\text{Arr}_A)$ over $0 \in I$ and to $(1 \times F)^*(\text{Arr}_B)$ over $1 \in I$.

In the case where $F$ and $G$ are strict morphisms of strict simplicial categories and $\eta$ is a strict natural transformation between them, the adjunction morphism $\text{adj}(\eta)$ is easy to describe; it is just given by exactly the same formula as in the classical case.

The paragraph which follows contains a more technical description of how to construct the morphism refered to above, in our framework of Segal categories. This construction in turn relies on the explicit construction of a certain cartesian family $\Phi_F$ which is left to the intrepid reader. The less intrepid who are willing to accept that everything works as usual, may skip the following paragraph.

Note that $(B \times I)^o \cong B^o \times I$ using $I^o \cong I$ (an involution which switches 0 and 1). Look in $\text{Hom}(I \times B^o \times A, S')$ at $(\eta \times F)^*(\text{Arr}_B)$.

Over $0 \in I$ this restricts to $(1_B \times F)^*(\text{Arr}_B)$. Over 1 this restricts to $((FG)^o \times F)^*(\text{Arr}_B)$.

Essentially speaking, this means that we have a natural transformation

$$B_1/(FGx, Fy) \to B_1/(x, Fy).$$

Note that $(FG)^o \times F$ is the composition

$$B^o \times A \xrightarrow{G^o \times 1} A^o \times A \xrightarrow{F^o \times F} B^o \times B.$$ 

Thus

$$((FG)^o \times F)^*(\text{Arr}_B) = (G^o \times 1)^*((F^o \times F)^*(\text{Arr}_B)).$$

The morphism of functoriality for $F$ is a natural transformation

$$A_1/(x, z) \to B_1/(Fx, Fz),$$
which translates in our language to a morphism

\[ \Phi_F : I \times A^o \times A \to S', \]

restricting over 0 to \( Arr_A \), and over 1 to \((F^o \times F)^* Arr_B \). Technically speaking, \( \Phi_F \) needs to be constructed as a cartesian family (recall that \( Arr_A \) and \( Arr_B \) are themselves cartesian families). We leave this construction to the reader (\( \otimes \)). Now look at

\[(1 \times G^o \times 1)^*(\Phi_F) : I \times B^o \times A \to S'.\]

Heuristically it is the natural transformation

\[ A_{1/}(Gx, y) \to B_{1/}(FGx, Fy). \]

We can “compose” this with the previous transformation to obtain a natural transformation

\[ A_{1/}(Gx, y) \to B_{1/}(FGx, Fy) \to B_{1/}(x, Fy). \]

Technically speaking, this means using the above two morphisms to give the 01 and 12 edges which can be filled in to a morphism

\[ \Upsilon^2(\ast, \ast) \times B^o \times A \to S', \]

the third (02) edge of which is a morphism

\[ \text{adj}(\eta) : I \times B^o \times A \to S' \]

restricting on the endpoints to \((1 \times G)^*(Arr_A)\) and \((F^o \times 1)^*(Arr_B)\) respectively. This is the technical description of how we get from the natural transformation \( \eta : 1_B \to FG \) to a natural transformation

\[ \text{adj}(\eta)(x, y) : A_{1/}(Gx, y) \to B_{1/}(x, Fy). \]

Now getting back to our discussion of adjoint functors, we say that \( \eta \) is an adjunction between \( F \) and \( G \) if the natural transformation \( \text{adj}(\eta) \) is an equivalence between \((G^o \times 1)^*(Arr_A)\) and \((1 \times F)^*(Arr_B)\) (by “equivalence” here we mean internal equivalence in the Segal category \( \text{Hom}(B^o \times A, S') \)).
Remark: In order to check the adjunction condition, it suffices to check that for every pair of objects $x \in \text{ob}(B)$ and $y \in \text{ob}(A)$, the morphism

$$\text{adj}(\eta)(x, y) : A_1/(Gx, y) \to B_1/(x, Fy)$$

is a weak equivalence of simplicial sets. This is a general fact about natural transformations between functors of Segal categories: being a levelwise equivalence implies being an equivalence. It is Corollary 2.5.8 of [31] (which was stated for $n$-categories but which works the same way for Segal categories); a similar early result was shown in [28].

Lemma 4 Suppose $F : A \to B$, $G : B \to A$ are functors of fibrant Segal categories, and $\eta : B \times I \to B$ is a natural transformation $1_B \to FG$ which is an adjunction. Suppose that $C$ is another Segal category. Let $F_C, G_C$ be the induced functors between $\text{Hom}(C, A)$ and $\text{Hom}(C, B)$, and let $\eta_C$ denote the functor

$$\text{Hom}(C, B) \times I \to \text{Hom}(C, B)$$

defined by the composed morphism

$$\text{Hom}(C, B) \times I \times C = C \times \text{Hom}(C, B) \times I \to B \times I \xrightarrow{\eta} B.$$ 

Then $\eta_C$ is a natural transformation

$$1_{\text{Hom}(C, B)} \to F_C G_C,$$

which is an adjunction between $F_C$ and $G_C$.

Proof: After the details of how to define everything, we will end up with a natural transformation

$$\text{adj}(\eta_C)(u, v) : \text{Hom}(C, A)_{1/}(Gu, v) \to \text{Hom}(C, B)_{1/}(u, Fv).$$

According to the previous remark, we have to show that this is an equivalence for every $u : C \to B$ and $v : C \to A$. To check this, note that

$$\text{Hom}(C, A)_{1/}(Gu, v)$$

is calculated by a homotopy-coherence calculation using the

$$A_1/(Gu(c), v(c')).$$
for $c, c' \in C$ (something like a coend, see Cordier-Porter [7]). Similarly,

$$\text{Hom}(C, B)_{1/}(u, Fv)$$

is calculated by the same homotopy-coherence calculation using

$$B_{1/}(u(c), Fv(c')).$$

The fact that the adjunction induces an equivalence

$$A_{1/}(Gu(c), v(c')) \cong B_{1/}(u(c), Fv(c'))$$

for any $c, c' \in \text{ob}(C)$, implies that the two calculations give the same answer; thus $\text{adj}(\eta_C)(u, v)$ is an equivalence. This completes the proof, but a number of details need to be followed through ($\otimes$).

Construction: We can apply this to the case where $C = A$, where $u = F$ and where $v = 1_A$. We obtain an equivalence

$$\text{adj}(\eta_A)(F, 1_A) : \text{Hom}(A, A)_{1/}(GF, 1_A) \xrightarrow{\cong} \text{Hom}(A, B)_{1/}(F, F).$$

In particular, there is an essentially unique element

$$\zeta \in \text{Hom}(A, A)_{1/}(GF, 1_A)$$

which goes to $1_F$ under the above equivalence. (To be more precise, what is essentially unique—i.e. parametrized by a contractible space—is the pair consisting of $\zeta$ plus a path between the image of $\zeta$ and $1_F$).

We leave it to the reader ($\otimes$) to check that $\zeta$ is an adjunction morphism going in the other direction between $F$ and $G$ (reversing the appropriate things in the above discussion/definition). We will use this construction of the other adjunction morphism, at some point in the argument below.

Lemma 5 With the above notations, the composed morphisms

$$F \xrightarrow{\eta_F} FGF \xrightarrow{F(\zeta)} F$$

and

$$G \xrightarrow{G(\eta)} GFG \xrightarrow{CG(\zeta)} G$$

are homotopic to the identity natural transformations of $F$ and $G$ respectively.
We don’t give a proof of this here (⊗).

For the above places where details are left out in our discussion of adjunc-
tion, the necessary arguments can probably be obtained from Cordier-Porter [7].

**Homotopy colimits**

It would be impossible to give a complete list of references to everything
pertaining to homotopy colimits (and limits). A non-exhaustive list includes
[6] [37] [38] [12] [15] [8] . . . .

Recall the notion of *homotopy colimit* in a simplicial category or Segal
category. If $A$ is a Segal category (which we suppose fibrant) and if $J$ is a
small Segal category, then we can form the *category of diagrams* $\text{Hom}(J, A)$. This is the “homotopically correct” one if $A$ is fibrant. There is a morphism $c_J : A \to \text{Hom}(J, A)$ induced by the projection $J \to *$; thus $c_J(x)$ is the constant diagram with values $x$. Suppose $F : J \to A$ is a diagram. If $x$ is
an object of $A$ and $f : F \to c_J(x)$ is a morphism, then we say that $x$ is the *homotopy colimit* of the diagram $F$ and write

$$(x, f) = \text{colim}_J(F)$$

(or just $x = \text{colim}_J(F)$ if there is no confusion about $f$), if for any object
$y$ of $A$, the morphism of “composition with $f$”, which can be seen as the
composition

$$A_1/F(x, y) \to \text{Hom}(J, A)_1/F(c_J(x), c_J(y)) \to \text{Hom}(J, A)_1/F(c_J(y)), $$

is an equivalence of simplicial sets. Here the second morphism is essentially
well-defined as “composition” in the Segal category $\text{Hom}(J, A)$, see above.

Note that we never speak of actual limits or colimits in a simplicial cate-
gory, so the notation $\text{colim}$ means homotopy colimit. If we forget to include
the qualiﬁer “homotopy” in front of the word “colimit” in the text below, the
reader will insert it. However, for homotopy limits or colimits of simplicial
sets, we keep the classical notation $\text{holim}$ or $\text{hocolim}$ so as not to confuse
these with 1-limits or 1-colimits in the 1-category of simplicial sets.

Note that

$$\text{Hom}(J, A)_1/F(c_J(y)) \cong \text{holim}_{j \in J} A_1/F(j), y$$

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where the holim on the right is the homotopy limit of simplicial sets. Thus, we can rewrite the condition for being a homotopy colimit as saying that for any object \( y \), the composition morphism with \( f \) gives an equivalence of simplicial sets

\[
A_1/(x, y) \xrightarrow{\approx} \text{holim}_{j \in J} A_1/(F(j), y).
\]

In this sense, the homotopy colimit is in a certain sense dual to the homotopy limit on the level of the simplicial \( \text{Hom} \)-sets of \( A \) (i.e. the \( A_1/(-, -) \)). In particular, we can verify certain formulae for homotopy colimits by verifying the dual formulae for homotopy limits of simplicial sets. For example it follows from \([38]\) that homotopy colimits commute with other homotopy colimits.

There is an analogous definition of homotopy limit which we leave to the reader to write down in our current language.

We now remark that colimits over a Segal category \( J \) can be transformed into colimits over a 1-category \( J' \); thus, in the above discussion, there would be no loss of generality in considering the indexing category \( J \) to be a 1-category. This remark follows from the following statement, which we isolate as a lemma because it will also be used in the proof of the main theorem.

**Lemma 6** If \( C \) is a Segal category (which we assume fibrant), then there is a 1-category \( D \) and a morphism \( D \to C \) such that for any fibrant Segal category \( A \), the induced morphism

\[
\text{Hom}(C, A) \to \text{Hom}(D, A)
\]

is fully faithful. Furthermore, we can assume that \( D \) is a “Reedy poset”, i.e. a poset with a Reedy structure such that the Reedy function is compatible with the ordering.

**Proof:** It suffices to construct a 1-category \( D \) and a subcategory \( W \subset D \) with a morphism \( D \to C \) sending the arrows of \( W \) to equivalences in \( C \), such that this morphism induces an equivalence

\[
L(D, W) \xrightarrow{\approx} C.
\]

To see that this suffices, recall from ([10] Proposition 8.6–Corollaire 8.9) which in turn comes from ([31] Theorem 2.5.1), that

\[
\text{Hom}(L(D, W), A) \subset \text{Hom}(D, A)
\]
is the saturated full Segal subcategory consisting of the morphisms $D \to A$ which send the morphisms of $W$ to equivalences in $A$. (In the case $A = S'$ this result is basically the same as the result of Dwyer-Kan in [10]).

Now for the construction of $D$ and $W$, we refer to [10] Lemmas 16.1, 16.2. These basically say that one can construct $D$ and $W$ using barycentric subdivision and the Grothendieck construction in the style of Thomason. ///

**Caution:** One must be careful in combining this lemma with the Yoneda result of Theorem 2. In effect, one obtains (in the situation of the lemma with $A = S'$) a sequence of three morphisms

$$D \to C \to \hat{C} \to \hat{D}.$$  

The last two morphisms are fully faithful. The Yoneda morphism $D \to \hat{D}$ is also fully faithful. However, the composition of these three morphisms is not in general the Yoneda morphism of $D$, so one cannot conclude that $D \to C$ must be fully faithful (which visibly it isn’t, in general). In fact, the composition of the above three morphisms is homotopic to the Yoneda morphism for $D$ if and only if the original morphism $D \to C$ is fully faithful.

**Corollary 7** If $J$ is a Segal category (which we may assume fibrant) and if $F : J \to A$ is a morphism to another Segal category, then there is a strict 1-category (which we may assume to be a Reedy poset) $J'$ and a morphism $g : J' \to F$ such that if $\text{colim}_J^A F \circ g$ exists then $\text{colim}_J^A F$ exists and the two colimits are equivalent.

**Proof:** Choose $g : J' \to J$ (with $J'$ a Reedy poset) so that

$$\text{Hom}(J, S') \to \text{Hom}(J', S')$$

is fully faithful. Now note that if $G : J \to S'$ is a simplicial set diagram over $J$, we have

$$\text{holim}_J^{S'} G \cong \text{Hom}(J, S')_{1/J}(\ast, G).$$

The same holds for $J'$. Therefore the fully faithful property implies that

$$\text{holim}_J^{S'} G \cong \text{holim}_J^{S'} G \circ g.$$

Now the fact that homotopy colimits in $A$ are dual to homotopy limits of the simplicial $\text{Hom}$ sets, implies that for any diagram $F : J \to A$,

$$\text{colim}_J^A F \circ g \to \text{colim}_J^A F$$
is an equivalence, and in fact existence of the first colimit implies existence of the second one. For the statement about existence we use the fully faithful property of the lemma (for target \( A \) this time) to say that

\[
\text{Hom}(J, A)_{1/}(F, c_J(\text{colim}^A_J F \circ g)) \rightarrow \text{Hom}(J', A)_{1/}(F \circ g, c_{J'}(\text{colim}^A_J F \circ g))
\]

is an equivalence, so there exists a morphism of \( J \)-diagrams from \( F \) to \( \text{colim}^A_J F \circ g \) restricting to the colimit morphism over \( J' \); now we can apply the previous discussion about \( \text{holim}^S \) to get that this morphism is a \( J \)-colimit.

We say that \( A \) admits all small homotopy colimits if for any small Segal category \( J \) and for any diagram \( J \rightarrow F \), the homotopy colimit exists. From the previous corollary, it suffices to check the existence of colimits over 1-categories \( J \) which we can furthermore assume are Reedy posets.

**Lemma 8** Suppose \( A \rightarrow B \) is a fully faithful morphism of Segal categories. Suppose \( F: J \rightarrow A \) is a diagram. If

\[
\text{colim}^B_J (F)
\]

is in \( A \), then the natural morphism

\[
\text{colim}^B_J (F) \rightarrow \text{colim}^A_J (F)
\]

is an equivalence.

**Proof:** The facts that \( \text{colim}^B_J (F) \) is in \( A \) and that the inclusion of \( A \) in \( B \) is fully faithful imply that we have a morphism of \( J \)-diagrams in \( A \)

\[
F \rightarrow c_J[\text{colim}^B_J (F)].
\]

Therefore there is up to homotopy a unique morphism

\[
\text{colim}^A_J (F) \rightarrow \text{colim}^B_J (F)
\]

whose composition with the canonical morphism of diagrams for the \( \text{colim}^A_J \), is the above morphism. In the other direction, we have a morphism of \( B \)-diagrams

\[
F \rightarrow c_J[\text{colim}^A_J (F)].
\]
Again we get an essentially unique morphism

\[ \text{colim}^B_j(F) \to \text{colim}^A_j(F) \]

(which is the morphism in the statement of the lemma). Essential uniqueness implies that the compositions in both directions are homotopic to the identity, thus our morphism is an equivalence.

Lemma 9 If \( C \) is a small Segal category, then homotopy colimits in \( \hat{C} \) exist and are calculated object-by-object.

Proof: According to Lemma 3 there is a small 1-category \( D \) and a morphism \( D \to C \) such that this induces a fully faithful morphism

\[ \hat{C} \to \hat{D}. \]

Furthermore, from the proof of Lemma 3 we may assume that there is a subcategory \( W \subset D \) such that \( C \) is equivalent to the localization \( L(D, W) \). This implies that \( \hat{C} \) is the full subcategory of \( \hat{D} \) consisting of diagrams \( X : D^o \to S' \) such that for any arrow \( w \in W \), \( X(w) \) is an equivalence. This situation is identical to that of Dwyer-Kan in [10], and all of the elements going into here are due to [10] in this case.

Next we recall that \( \hat{D} \) is equivalent to \( L(M) \) where \( M \) is the Heller closed model category \( S^D \) of simplicial presheaves over \( D \) (Lemma 3). Now, suppose we have a diagram \( F : J \to \hat{C} \), which we may also consider as a diagram in \( \hat{D} \). We may assume that \( J \) is a Reedy poset. The argument of [10] (see chapter 18 for example) allows us to “strictify” and assume that \( F \) is the projection of a diagram \( F' : J \to M \). Furthermore we may assume that \( F' \) is Reedy-cofibrant in the variable \( J \). Then

\[ \text{colim}^{\hat{D}}_j(F) = \text{colim}^{L(M)}_j(F) \]

exists and is calculated by taking the 1-colimit of \( F' \) in \( M \) (see the discussion at the proof of \( (i) \Rightarrow (ii) \) in Theorem L4 below). This 1-colimit is calculated object-by-object over \( D \) (recall that \( M \) is the category of simplicial presheaves on \( D \)). On the other hand, the Reedy cofibrant condition for \( F' \) also holds object-by-object. Therefore for any \( d \in \text{ob}(D) \), the 1-colimit of \( F'(j)(d) \) over
$j \in J$, is also the homotopy colimit. This shows that the homotopy colimit is calculated object-by-object, i.e.

$$
colim_{j \in J}^\hat{D}(F)(d) = \colim_{j \in J}^{S'} F(j)(d) = hocolim_{j \in J} F(j)(d)
$$

for $d \in \text{ob}(D)$. On the other hand, the fact that $F$ is a diagram in $\hat{C}$ means that for arrows $w$ in $W$, and for any $j \in J$, we have that $F'(j)(w)$ is an equivalence of simplicial sets. Homotopy-invariance of the 1-colimit of a Reedy cofibrant diagram \cite{15} implies that the arrow

$$
colim_{j}^\hat{D}(F)(w)
$$

is an equivalence for any arrow $w$ in $W$. It follows that

$$
colim_{j}^\hat{D}(F) \in \text{ob}(\hat{C}).
$$

Now by Lemma \cite{8}

$$
colim_{j}^\hat{C}(F) \cong \colim_{j}^\hat{D}(F)
$$

including the statement that the homotopy colimit in $\hat{C}$ exists. Finally, we have

$$
colim_{j}^\hat{C}(F)(d) = \colim_{j}^\hat{D}(F)(d) = \colim_{j \in J}^{S'} F(j)(d).
$$

This completes the proof. ///

**Smallness and rearrangement of colimits**

Recall from \cite{15} the notion of *sequential colimit*. This is a colimit indexed by an ordinal $\beta$ (where the ordered set $\beta$ is considered as a category with morphisms going in the increasing direction) with the additional property that if $i \in \beta$ is a limit element then the $i$-th object $X_i$ is equivalent to the colimit of the $X_j$ for $j < i$. A diagram giving rise to a sequential colimit will be called a *sequential diagram*. In giving these definitions for a Segal category $A$, the notion of colimit which occurs is the notion of homotopy colimit as defined above.

A diagram or colimit is *essentially sequential* if it satisfies the sequential condition at sufficiently large points. In what follows we shall make no
distinction between sequential and essentially sequential (an essentially sequential diagram can be replaced by a sequential one which gives the same colimit, by starting out with a constant diagram in low degrees).

Suppose $A$ is a Segal category admitting all small colimits. An object $z \in \text{ob}(A)$ is said to be $\beta$-small in $A$ if for any ordinal $\delta$ of size $\geq \beta$ and any sequential diagram $X: \delta \to A$, the natural morphism

$$\text{hocolim}_{i \in \delta} A_1(z, X_i) \to A_1(z, \text{colim}_\delta A X)$$

is an equivalence. We say that $z$ is small in $A$ if there is a cardinal $\beta$ such that $z$ is $\beta$-small in $A$.

Example: Lemma 9 shows that the objects of $C$ are small in $\hat{C}$. On the other hand, every object of $\hat{C}$ can be expressed as a small homotopy colimit of objects of $C$ (see Lemma 11 below). From this it follows easily that every object in $\hat{C}$ is small in $\hat{C}$ (although of course there is no bound uniform over the class $\text{ob}(C)$).

We will now treat some aspects of colimits which are useful in connection with the notion of smallness.

Suppose $J$ is a small 1-category, and $F : J \to A$ is a diagram. Choose a well-ordering of the objects of $J$, in other words choose an ordinal $\beta$ and an isomorphism $\text{ob}(\beta) \cong \text{ob}(J)$. We assume that $\beta$ is the first ordinal of its cardinality. For each $i \in \beta$ let $J_i$ denote the full subcategory of objects $< i$. Put

$$X_i := \text{colim}_{J_i} (F|_{J_i}).$$

Then the $X_i$ form a sequential diagram, and we have

$$\text{colim}_J (F) = \text{colim}_\beta (X_i).$$

This can be proved by noting the dual property for homotopy limits of simplicial sets.

We call the above expression a normalized reindexing of the colimit. The word “normalized” refers to the condition that $\beta$ be the first ordinal of its cardinality. With this condition, we get that each $X_i$ is a colimit of size $< |\beta|$ (this latter notation is the cardinality of $\beta$).

Now we discuss another aspect of rearranging colimits. Let $A$ be a Segal category admitting small colimits and let $C \subset A$ be a small full subcategory. For any ordinal $\beta$ we will define a full subcategory

$$A(\beta C) \subset A,$$
by the following prescription: it is the smallest saturated full subcategory of $A$ containing $C$ and closed under colimits of size $< |\beta|$.

It is clear that if there is an ordinal $\beta' < \beta$ of the same cardinality, then $A(<\beta'C) = A(<\beta C)$.

The following lemma is our main statement giving a normal form for successive colimits.

**Lemma 10** Suppose $\beta$ is the first ordinal of its cardinality. There are two cases.

1. If $\beta$ is a limit of ordinals of strictly increasing cardinality, then
   
   $$A(<\beta C) = \bigcup_{\gamma < \beta} A(<\gamma C).$$

2. On the other hand, if $\beta$ is the limit of ordinals $\gamma$ all having the same cardinality, then (letting $\gamma$ denote the first of these) $A(<\beta C)$ is the saturated full subcategory of $A$ consisting of objects which are $\gamma$-sequential colimits of objects of $A(<\gamma C)$.

**Proof:** In the first case (1), take the union in question, and note that it is indeed closed under colimits of size $< \beta$, because any such colimit over $J$ has the property that there exists $\gamma < \beta$ with $|J| < \gamma$ so the colimit exists in $A(<\gamma C)$.

The main problem is to treat the second case. Let $A' \subset A$ be the saturated full subcategory of $A$ consisting of objects which are $\gamma$-sequential colimits of objects of $A(<\gamma C)$. Suppose $F : J \to A'$ is a diagram of size $< \beta$. Note that this implies that $|J| \leq |\gamma|$. Again we isolate two cases:

- (a) where $|J| < |\gamma|;
- (b) where $F$ is a sequential diagram taken over $J = \gamma$.

In these two cases, we will show that $\text{colim}_J F \in A'$. This suffices, in view of the reorganization of the colimit over a $J$ of cardinality $|\gamma|$.

In the first case, note that by Corollary 4, we can assume that $J$ is a 1-category which is a Reedy poset (note that the operation of Corollary 4 doesn’t increase the size of $J$ beyond the countable cardinal).

Now by doing an induction on the Reedy function of the poset $J$ (and using the assumption $|J| < |\gamma|$) we can rearrange the individual limits expressing our objects $F(j)$ as objects in $A'$, so that we have a doubly indexed system $F_i(J)$ for $i \in \gamma$ and $j \in J$, such that

$$F(j) = \text{colim}_{i \in \gamma} F_i(J).$$
Here the systems in the variable \( i \) are sequential. Now we may set \( G_i := \text{colim}_J F_i(j) \). This colimit lies in \( A(<\gamma C) \), and we have

\[
\text{colim}_J F = \text{colim}_\gamma G.
\]

The diagram \( G \) may then be replaced by a sequential diagram giving the same colimit. This proves that

\[
\text{colim}_J F \in A',
\]

so we have finished treating case (a).

For case (b), we can again (by induction on the ordered set \( \gamma \)) suppose that our colimit comes from a doubly-indexed family \( F_i(j) \in A(<\gamma C) \) this time indexed by \( \gamma \times \gamma \). By a diagonal reindexing of this family we can express \( \text{colim}_J F \) as a \( \gamma \)-sequential colimit of objects of \( A(<\gamma C) \), so again

\[
\text{colim}_J F \in A',
\]

and we have finished case (b). As remarked above, this suffices to obtain case (2) of the lemma.

///

**Generating subcategories**

We have the following notions of generation: these are for a saturated full sub-Segal category \( C \subset A \), and we suppose that \( A \) is closed under colimits.

We say that \( C \) **strongly generates** \( A \) if the morphism \( A \to \hat{C} \) is fully faithful.

We say that \( C \) **generates** \( A \) by colimits if the smallest saturated full sub-Segal category of \( A \) which contains \( C \) and is closed under \( A \)-colimits, is \( A \) itself.

As a preliminary for the subsequent proposition we have the following lemma. The main problem in giving the proof is to be careful to avoid an error related to the “caution” after Lemma 6. It should be possible to give a more conceptual proof of this lemma using the “coend” construction of Cordier-Porter [7].

**Lemma 11** If \( D \) is a small Segal category, then every object of \( \hat{D} \) can be expressed as a small homotopy colimit of objects of \( D \) (in the Yoneda embedding).

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Proof: First suppose $D$ is a 1-category. Then $\hat{D} \cong L(M)$ where $M$ is the Heller model category $S^D$ of simplicial presheaves over $D$ (Lemma 3). Any object of $M$ is equivalent to a simplicial object in the category of formal disjoint unions of objects of $D$ (C. Teleman pointed this out to me). From there we can go to an expression for any object as a homotopy colimit of objects of $D$. This treats the case of a 1-category.

Suppose now that $D$ is a Segal category, and choose a 1-category $C$ with a morphism $f : C \to D$ which induces a fully faithful morphism

$$i : \hat{D} \to \hat{C}.$$ 

We know the present lemma for $\hat{C}$. Plugging in $A := \hat{D}$ in the argument which will be given below in the proof of Theorem 4 (the part $(ii) \Rightarrow (iii)$) we obtain existence of an adjoint $\psi : \hat{C} \to \hat{D}$. Note that the present lemma is used in that argument, for $C$; but this we know from the previous paragraph. Full-faithfulness of $i$ implies that $\psi \circ i \cong 1_{\hat{D}}$. Suppose now that $U \in \hat{D}$. Express

$$iU \cong \text{colim}_J \hat{C} \circ h_C \circ F$$

for a small diagram $F : J \to C$ composed with the Yoneda $h_C : C \to \hat{C}$. Now

$$U \cong \psi iU \cong \psi (\text{colim}_J \hat{C} \circ h_C \circ F).$$

The fact that $\psi$ is an adjoint to $i$ implies that $\psi$ preserves colimits. Therefore we get

$$U \cong \text{colim}_J \hat{D} \psi \circ h_C \circ F.$$ 

However, we have the formula

$$\psi \circ h_C \cong h_D \circ f.$$ 

(this follows from the adjunction property of $\psi$). Therefore we get

$$U \cong \text{colim}_J \hat{D} h_D \circ f \circ F.$$ 

In particular $U$ is a colimit of $h_D$ composed with the diagram $f \circ F : D \to \hat{D}$. \\\n
The following proposition gives an equivalence between our two notions of generation.

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Proposition 12  Suppose $A$ is a Segal category closed under small homotopy colimits. The following conditions are equivalent.

(a) there exists a small saturated full subcategory $C \subset A$ consisting of objects which are small in $A$ and which strongly generates $A$;

(b) there exists a small saturated full subcategory $C$ consisting of objects which are small in $A$, such that $C$ generates $A$ by colimits.

Proof: For (a) $\Rightarrow$ (b), suppose $X \in \text{ob}(A)$ and write (by Lemma [11])

$$X = \text{colim}^\hat{C}_J F$$

for a diagram $F : J \to \hat{C}$. The strong generation hypothesis that $A \to \hat{C}$ is fully faithful implies (cf Lemma [8] above) that $X$ is also the colimit $\text{colim}^A_J (F)$. Thus $C$ generates $A$ by colimits (and in fact we obtain that every element of $A$ is expressed as a single colimit of a diagram in $C$). This shows that (a) $\Rightarrow$ (b).

We now show that (b) $\Rightarrow$ (a). Let $C$ be a saturated full subcategory which generates $A$ by colimits, and suppose that there is a cardinal $\gamma$ such that the objects of $C$ are $\gamma$-small in $A$. We may also assume that the objects of $C$ are $\gamma$-small in $\hat{C}$. Recall that we have defined a saturated full subcategory $A(< \beta C)$, which is the smallest one containing $C$ and closed under colimits of size $< \beta$.

We claim that for $\beta$ big enough (say bigger than $\gamma$), the elements of $A(< \beta C)$ are $\beta$-small in $A$. Indeed, the objects of $C$ will be $\beta$-small, and it is easy to see that a colimit of size $< \beta$ of objects which are $\beta$-small, is again $\beta$-small.

Now choose $\beta$ big enough, and set

$$D := A(< \beta C).$$

From the previous paragraph, the elements of $D$ are $\beta$-small in $A$. We claim that the morphism

$$A \xrightarrow{i_D} \hat{D}$$

is fully faithful. Suppose $Y$ is an object of $A$. Look at the saturated full subcategory $A' \subset A$ of objects $X$ such that

$$A_{1/}(X, Y) \to \hat{D}_{1/}(i_{D}X, i_{D}Y)$$

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is an equivalence. We note first of all that $A'$ contains $D$. To see this, note that if $X \in \text{ob}(D)$ then by definition

$$i_D(Y)(X) = A_1/(X, Y).$$

On the other hand, from Theorem 3

$$\hat{D}_1/(i_D X, i_D Y) \cong i_D(Y)(X).$$

Therefore we obtain that $X \in A'$.

Next we claim that $A'$ is closed under sequential colimits of size $\geq \beta$. Suppose $X$ is a $\delta$-sequential $A$-colimit of $X_i$ with the $X_i$ in $A'$, and with $|\delta| \geq \beta$. We have

$$A_1/(X, Y) = \text{holim}_\delta A_1/(X_i, Y).$$

This maps by an equivalence to

$$\lim_\delta \hat{D}_1/(X_i, Y).$$

In turn, this is equivalent to

$$\hat{D}_1/(\text{colim}_\delta \hat{D} X_i, Y).$$

Now we have a morphism in $\hat{D}$

$$\text{colim}_\delta \hat{D} X_i \rightarrow X.$$

We claim that this morphism is an equivalence. Indeed, suppose $Z$ is in $D$. Then

$$\hat{D}_1/(Z, \text{colim}_\delta \hat{D} X_i) \cong \text{colim}_{i \in \delta} \hat{D}_1/(Z, X_i)$$

(filtered colimits of simplicial-set diagrams over $D$ are calculated object-by-object). On the other hand,

$$\hat{D}_1/(Z, X) \cong X(Z) \cong A_1/(Z, X)$$

and finally

$$A_1/(Z, X) = A_1/(Z, \text{colim}^A_{i \in \delta} X_i) \cong \text{hocolim}_{i \in \delta} A_1/(Z, X_i)$$

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the latter because $|\delta| \geq \beta$ and as remarked above, the elements of $D$ are $\beta$-small in $A$. Finally note that

$$A_1/(Z, X_i) = X_i(Z) \cong \tilde{D}_1/(Z, X_i).$$

We obtain, after all of this, that the morphism

$$\text{colim}_\delta \tilde{D} X_i \to X.$$ 

induces an equivalence on morphism spaces from any object of $D$. This implies that it is an equivalence, as claimed.

Recall from above that we have equivalences

$$A_1/(X, Y) \cong \lim_\delta A_1/(X_i, Y) \cong \lim_\delta \tilde{D}_1/(X_i, Y). \cong \tilde{D}_1/(\text{colim}_\delta \tilde{D} X_i, Y).$$

On the other hand the composed morphism factors as

$$A_1/(X, Y) \to \tilde{D}_1/(X, Y) \to \tilde{D}_1/(\text{colim}_\delta \tilde{D} X_i, Y).$$

From the claim of the previous paragraph, the second morphism is an equivalence; therefore the first morphism is an equivalence, which shows that $X \in A'$.

We have now shown that $A'$ is closed under sequential colimits. The result of Lemma 10 implies that $A' = A$. Therefore the morphism $A \to \tilde{D}$ is fully faithful, and $D$ strongly generates $A$ giving condition (a) of the proposition. ///

We now turn to generation for classes of morphisms. Suppose $\mathcal{F}$ is a set of homotopy classes of morphisms in $A$. An $\mathcal{F}$-fibration is a morphism $f : X \to Y$ which satisfies the weak lifting property for elements of $\mathcal{F}$ (i.e. whose image satisfies the lifting property in $\text{ho}(A)$). Here the morphism in $\mathcal{F}$ goes on the left in the square diagram, and the morphism $f$ goes on the right. Suppose $\mathcal{F}$ is a subset of homotopy classes of morphisms. The class of morphisms generated (in terms of lifting) by $\mathcal{F}$ is the largest subclass of (homotopy classes of) morphisms $\mathcal{F}$ such that the $\mathcal{F}$-fibrations are the same as the $\overline{\mathcal{F}}$-fibrations.

**Lemma 13** Suppose $\mathcal{F}$ is a set of homotopy classes of morphisms. Then any morphism in $A$ which is a sequential limit of pushouts by morphisms in $\mathcal{F}$, is in $\overline{\mathcal{F}}$. 

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Proof: Lifting for morphisms in \( F \) implies lifting for morphisms which are pushouts by morphisms in \( F \), and also for sequential colimits of such.  

We will apply the above in the following case: where we are given a morphism of Segal categories \( \psi : B \to A \). We will say that an arrow \( f \) in \( B \) (i.e. a vertex of some \( B_1(x,y) \)) is \( \psi \)-trivial if \( \psi(f) \) is an internal equivalence in \( A \). We say that an arrow in \( B \) is a \( \psi \)-fibration if it is a fibration in the sense of the paragraph before the previous lemma, for the class \( F \) of \( \psi \)-trivial morphisms. In the main theorem below, we will be interested in when the class \( F \) of \( \psi \)-trivial morphisms is generated (in terms of lifting as per the above definition) by a small subset of \( \psi \)-trivial morphisms.

**The main theorem**

The following theorem is a generalization of Giraud’s theorem characterizing Grothendieck topoi.

**Theorem 14** Suppose \( A \) is a Segal category (which we may assume fibrant). The following conditions are equivalent.

(i) There exists a cofibrantly generated closed model category \( M \) such that \( A \) is equivalent to \( L(M) \);  
(ii) All small homotopy colimits exist in \( A \), and there exists a cardinal \( \beta \) and a small subset of objects \( \mathcal{G} \subset A^0 \) such that the objects of \( \mathcal{G} \) are \( \beta \)-small in \( A \), and such that \( \mathcal{G} \) generates \( A \) by colimits;  
(iii) There exists a small 1-category \( C \) and a morphism \( g : C \to A \) sending objects of \( C \) to objects which are small in \( A \), which induces a fully faithful inclusion

\[
i : A \to \hat{C};
\]

and there is a morphism \( \psi : \hat{C} \to A \) together with a natural transformation

\[
\eta_X : X \to i\psi(X)
\]

such that \( \eta \) induces an adjunction between \( i \) and \( \psi \).  
(iv) The category \( A \) admits all small homotopy colimits, and there exists a small 1-category \( C \) and a functor \( \psi : \hat{C} \to A \) commuting with colimits, such that \( A \) is the localization of \( \hat{C} \) by inverting the morphisms which \( \psi \) takes to equivalences, and such that the \( \psi \)-trivial morphisms of \( \hat{C} \) are generated (in terms of lifting) by a small subset of \( \psi \)-trivial morphisms of \( \hat{C} \).
Note that (i) implies that \( A \) admits all small homotopy limits too. In (iii), the fully faithful condition implies that the adjunction morphism going in the other direction is an equivalence between \( \psi \circ i \) and \( 1_A \).

We call a Segal category \( A \) which satisfies these equivalent conditions, an \( \infty \)-pretopos. If furthermore there exists \( C \rightarrow A \) as in condition (iii) such that the adjoint \( \psi \) preserves finite homotopy limits, then we say that \( A \) is an \( \infty \)-topos.

The terminology “\( \infty \)-topos” naturally gives rise to a number of conjectures, definitions, generalizations, predictions etc., which would be too numerous to list here. As an example, we mention that the applications of the theory of topoi to mathematical logic (cf e.g. Moerdijk-MacLane [21]), should give rise to generalizations in the case of \( \infty \)-topoi—which if they exist could be called “\( \infty \)-categorical logic” or “higher-dimensional logic”.

Remark: It would be interesting to know what conditions on the closed model category \( M \) correspond to the \( \infty \)-topos condition. (In a similar vein, it would be good to know that the \( \infty \)-topos condition is independent of the choice of \( C \) in condition (iii).) Charles Rezk ([24] and later [26]) points out that in the case of presheaves on categories with Grothendieck pretopologies, the exactness of the associated sheaf functor corresponds exactly to the condition that the pretopology be a topology (this letter from Rezk to Hirschhorn was one of the main elements motivating the present paper). We might expect a similar sort of behavior here; one might even go so far as to conjecture that the \( \infty \)-topoi are exactly the Segal categories of simplicial presheaves on Grothendieck sites. Another conjecture (more reasonable) would be that the \( \infty \)-topoi correspond exactly to the right proper closed model categories.

In keeping with the above remark, one should note that our theorem is not strictly speaking an exact generalization of Giraud’s theorem, because we treat the case where \( \psi \) may not be exact, and we obtain a weaker result (existence of a closed model structure, rather than existence of a site).

P. Hirschhorn points out that if \( M \) is a cofibrantly generated closed model category, then the opposite category \( M^\circ \) (which is again a closed model category admitting all small limits and colimits) will not in general be cofibrantly generated. Similarly, the opposite of an \( \infty \)-pretopos will generally not be an \( \infty \)-pretopos. The point is that the generation condition is asymmetric. For example (as Hirschhorn pointed out in an email) the closed model category \( Sets^\circ \) is not cofibrantly generated; indeed the only small objects are those
which correspond to the sets $\emptyset$ and $\ast$ (as these are the only sets which are
cosmall in $Sets$). These sets don’t generate $Sets$ by inverse limits, so they
don’t generate $Sets^o$ by colimits.

Our strategy for the proof of Theorem [14] is similar to the strategy of the
proof of Giraud’s theorem [1]: we prove

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).$$

For various reasons, our statements $(i)$–$(iv)$ are not precise generalizations
of the statements which are numbered in the same way in Giraud’s theorem.
Among other things, there is too much “element-wise” reasoning with respect
to the $Hom$ sets in SGA 4 [1] to make a direct generalization possible. One
major difference in our proof is that at the last step, we replace the notion
of Grothendieck topology by the notion of localization of a closed model
category. This explains why the theorem concerns pretopoi rather than topoi.

**The proof of $(i) \Rightarrow (ii)$**

It is not difficult to see that if $M$ is a closed model category (admitting
small limits and colimits), then $L(M)$ admits all small homotopy limits and
homotopy colimits. A version of this statement was known to Edwards and
Hastings [12]. Our technique comes out of the Dwyer-Hirschhorn-Kan meth-
ods for calculating homotopy (co)limits and their methods for calculating the
function spaces in $L(M)$. The argument we are about to present was done
for products and coproducts in Lemme 8.4 of [16], but it works the same way
for arbitrary colimits (resp. limits). We briefly review this for the reader’s
convenience.

Suppose we want to calculate the homotopy colimit of a diagram

$$F : J \to L(M)'.$$

It is shown in [16] that we may assume that this diagram comes from a strict
diagram $F : J \to M$. As in Corollary [4], we may also assume that $J$ is a Reedy
category and a poset with the ordering compatible with the Reedy structure.
We may replace $F$ by a levelwise equivalent Reedy-cofibrant diagram, so we
may assume that $F$ is Reedy-cofibrant. With these hypotheses we will show
that the strict 1-colimit of $F$ in $M$,

$$\text{colim}^M_J (F)$$
is a representative for the homotopy colimit \( \text{colim}^{L(M)}_J F \) which implies that the latter exists. To do this, consider an object \( Y \in M \). Choose a Reedy-fibrant simplicial resolution (see [8] or [15])

\[ Y \to Z. \]

Recall from [8] and [15] that if \( X \) is any object of \( M \), we have

\[ L(M)_{1/}(X, Y) \cong M_{1/}(X, Z), \]

where the latter is a simplicial set using the simplicial variable of the resolution.

We claim that

\[ j \mapsto M_{1/}(F(j), Z), \]

is a Reedy-fibrant diagram of simplicial sets over \( J \). To prove this, note that the Reedy-fibrant condition is a condition involving strict limits and colimits, and it is dual to the Reedy-cofibrant condition. Thus taking \( \text{Hom}_M = M_{1/} \) of a Reedy-cofibrant diagram \( F \), into anything, gives a Reedy-fibrant diagram as claimed.

Using this claim, and the fact that homotopy limits of simplicial sets may be calculated using Reedy-fibrant diagrams, we get that

\[ \text{lim}^{\text{str}}_j M_{1/}(F(j), Z) \cong \text{holim}_{j \in J} M_{1/}(F(j), Z). \]

Here the notation \( \text{lim}^{\text{str}}_j \) means the strict limit taken in the 1-category of simplicial sets. On the other hand,

\[ \text{lim}^{\text{str}}_{j \in J} M_{1/}(F(j), Z) = M_{1/}(\text{colim}^M_J F, Z). \]

Putting these all together we get that

\[ L(M)_{1/}(\text{colim}^M_J F, Y) \cong \text{holim}_{j \in J} L(M)_{1/}(F(j), Y). \]

This exactly says that

\[ \text{colim}^M_J F = \text{colim}^{L(M)}_J F, \]

meaning in particular that the latter exists.

The case of homotopy limits in \( L(M) \) is dual and identical to the above. Thus we get existence of homotopy limits and colimits in \( L(M) \).
Similarly, the cofibrant generation condition implies the second condition in (ii). Let $G$ be a small set of objects containing those which occur in a generating set of cofibrations \[15\]. Any object of $M$ is weak equivalent to an object which is obtained as a sequential limit of pushouts by objects in $G$ (this is the small object argument \[22\]). Therefore, $G$ generates $A$ by colimits. Furthermore, the objects of $G$ are $\beta$-small in $M$ for some $\beta$ (this is in the definition of “cofibrantly generated”), and since these objects are cofibrant, $\beta$-smallness in $M$ implies that their images are $\beta$-small in $L(M)$. Thus we obtain condition (ii).

The proof of (ii) $\Rightarrow$ (iii)

Starting with (ii), we can replace our subcategory $C$ which generates $A$ by colimits, with a subcategory $C$ which strongly generates $A$ by Proposition \[14\]. Therefore we may now assume that the morphism $i : A \to \hat{C}$ is fully faithful.

The next step is to use colimits in $A$ to construct the adjoint functor $\psi$. Do this as follows. Define the Segal category of arrows $V := Hom(I, \hat{C}) \times \hat{C} A$ where the first structural morphism in the fiber product is evaluation at $1 \in I$ denoted $ev(1)$. Thus $V$ is the Segal category of arrows $x \to y$ with $x \in \hat{C}$ and $y \in A$. Say that such an arrow is universal if it is an initial object in the fiber over $x$ for the evaluation map at $0$

$$ev(0) : V \to \hat{C}.$$  

This condition means that for any object $z \in A$ the morphism of composition with our arrow,

$$A_1/(y, z) \to \hat{C}_1/(x, z)$$

is an equivalence of categories.

Let $U \subset V$ be the full sub-Segal category consisting of universal arrows.

The same definitions can be made with respect to any morphism of Segal categories $A \to B$ (in the above we have written $B = \hat{C}$).

**Lemma 15** For any functor $i : A \to B$, if we construct the Segal category $U$ of universal arrows from objects of $B$ to objects of $A$, then the evaluation at $0$ is a fully faithful morphism $ev(0) : U \to B$. 

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Proof: This statement was shown (in a particular example, but with a technique which works in general) as a part of the “second construction” in the proof of Lemma 6.4.3 in [33]. It was this proof which was added in the revised v2 of [33]. We rewrite the proof here.

We may suppose that $A$ and $B$ are fibrant Segal categories. We have

$$V := \text{Hom}(I, B) \times_B A.$$  

This maps by a fibration $ev(0)$ to $B$. Suppose that $u, v \in U \subset V$ are universal arrows; they are maps $u, v : I \to B$ together with objects $u_1, v_1$ in $A$ with $u(1) = i(u_1)$ and similarly for $v$. Then

$$U_1/(u, v) = \text{Hom}(I, B)_{1/(u, v)} \times_{B_{1/(u(1), v(1))}} A_{1/(u_1, v_1)}.$$  

This maps by a fibration to $B_{1/(u(0), v(0))}$. We will show that this latter map is an equivalence by showing that it is a trivial fibration, namely showing that it satisfies lifting for any cofibration $E \hookrightarrow E'$. Thus we suppose that we have a diagram

$$
\begin{array}{ccc}
E & \to & U_1/(u, v) \\
\downarrow & & \downarrow \\
E' & \to & B_{1/(u(0), v(0))}.
\end{array}
$$

The top map amounts to a diagram

$$\Upsilon(E) \times I \to B$$

coupled with a lifting

$$\Upsilon(E) \times 1 \to A,$$

such that over 0 (resp. 1) in $\Upsilon(E)$ these restrict to $u$ (resp. $v$). The bottom map amounts to a diagram

$$\Upsilon(E') \times 0 \to B.$$  

We would like to extend the above to a diagram

$$\Upsilon(E') \times I \to B$$

plus lifting

$$\Upsilon(E') \times 1 \to A.$$
Divide the square $\Upsilon(E') \times I$ into two triangles,

$$\Upsilon(E') \times I = \Upsilon^2(\ast, E') \cup^{T(E')} \Upsilon^2(E', \ast).$$

The first triangle is the one containing the edge corresponding to $u$ as its first edge; the second is the one containing the edge corresponding to $v$ as its second edge.

We need to define a morphism from these triangles into $B$ (plus a lifting on the second edge of the first triangle, into $A$). These are already defined over the subobjects where one puts $E$ instead of $E'$. Furthermore, the first edge of the second triangle is already defined.

We treat first the second triangle: the inclusion corresponding to the $01$ and $12$ edges

$$[\Upsilon(E') \cup^* \Upsilon(\ast)] \cup^* \Upsilon^2(E, \ast)
\hookrightarrow \Upsilon^2(E', \ast)$$

is a trivial cofibration (cf [31]) so the extension in question exists. In particular we obtain an extension along the diagonal.

We now turn to the first triangle $\Upsilon^2(\ast, E')$. We have an extension which is already specified along the diagonal (i.e. the $02$ edge) and we have the specification $u$ which is a universal morphism, along the $01$ edge. We claim that we can choose the required extension plus lifting along the $12$ edge into $A$. For this, note (by going back from the notations $\Upsilon$ to the usual notation) that we are looking at the following problem. We are given

$$E' \rightarrow B_{1j}/(u(0), v(1)),$$

and

$$E \rightarrow \{u\} \times_{B_{1j}/(u(0), u(1))} B_{2j}/(u(0), u(1), v(1)) \times_{B_{1j}/(u(1), v(1))} A_{1j}/(u_1, v_1),$$

such that the restriction of this map to the $02$ edge is the same as the restriction of the first map to $E$.

We look for an extension of the above to a map

$$E \rightarrow \{u\} \times_{B_{1j}/(u(0), u(1))} B_{2j}/(u(0), u(1), v(1)) \times_{B_{1j}/(u(1), v(1))} A_{1j}/(u_1, v_1).$$

Recall that

$$\{u\} \times_{B_{1j}/(u(0), u(1))} B_{2j}/(u(0), u(1), v(1)) \cong B_{1j}/(u(1), v(1)).$$
In particular,\[
\{u\} \times_{B_1/(u(0),u(1))} B_2/(u(0),u(1),v(1)) \times_{B_1/(u(1),v(1))} A_1/(u_1,v_1) \cong A_1/(u_1,v_1).
\]

The condition that \(u\) is a universal map means that the 02-restriction morphism
\[
\{u\} \times_{B_1/(u(0),u(1))} B_2/(u(0),u(1),v(1)) \times_{B_1/(u(1),v(1))} A_1/(u_1,v_1)
\to B_1/(u(0),v(1))
\]
is an equivalence. (In other words the “composition with \(u\)” from \(A_1/(u_1,v_1)\) to \(B_1/(u(0),v(1))\) is an equivalence.) This restriction map is also fibrant, so it is a trivial fibration and satisfies lifting for all cofibrations. The lifting condition is exactly the condition that we need to show.

This completes treatment of the first triangle, and finishes the proof of the lifting property which shows that the map
\[
U_1/(u,v) \to B_1/(u(0),v(0))
\]
is an equivalence.

///

**Lemma 16** Suppose in the situation of the previous lemma that \(B = \hat{C}\). If the functor \(i : A \to \hat{C}\) comes from a morphism \(a : C \to A\), and if \(A\) admits arbitrary small homotopy colimits, then the morphism \(U \to \hat{C}\) is essentially surjective, so it is an equivalence of Segal categories.

**Proof:** It suffices to show that if \(G \in \hat{C}\) is any object, then there exists a universal morphism \(f : G \to iX\) to an object \(X \in ob(A)\). To construct \(f\), we first note that any such \(G\) can be expressed as a colimit of objects of \(C\) (Lemma \((\square)\)): there is a small category \(J\) and a morphism \(F : J \to C\) such that
\[
G = colim_J^\hat{C}(i \circ a \circ F).
\]
Now we set
\[
X := colim_J^A(a \circ F).
\]
We have a morphism of \(J\)-diagrams in \(A\)
\[
a \circ F \to c_f(X),
\]
which gives a morphism of $J$-diagrams in $\hat{C}$

$$i \circ a \circ F \to c_J(iX).$$

In turn this can be factored through an essentially unique morphism $f : G \to iX$ because $G$ is the colimit of $i \circ a \circ F$. We claim that $f$ is universal. To see this, suppose $Y \in A$. In what follows we pretend that weak compositions are actually compositions (this avoids tedious references to things like $A_2$). We get the following diagram (well-defined and commuting, up to homotopy):

$$\begin{array}{ccc}
A_1/(X,Y) & \to & \text{Hom}(J,A)_1/(a \circ F, c_J(Y)) \\
\downarrow & & \downarrow \\
\hat{C}_1/(G,iY) & \to & \text{Hom}(J,\hat{C})_1/(i \circ a \circ F, c_J(iY)).
\end{array}$$

The top arrow is an equivalence because $X$ is the colimit of $a \circ F$ in $A$. The bottom arrow is an equivalence because $G$ is the colimit of $i \circ a \circ F$ in $\hat{C}$. The right vertical arrow is an equivalence because of the hypothesis that $i$ is fully faithful (it is the morphism of functoriality of $i$). Therefore the left vertical arrow is an equivalence, which exactly says that $f : G \to iX$ is a universal arrow.

Go back to our previous situation (which is the situation of the second half of the lemma). We have an equivalence $U \to \hat{C}$ and the evaluation at 1 provides a morphism $U \to A$; this gives an essentially well-defined morphism $\psi : \hat{C} \to A$.

There is a tautological morphism

$$U \times I \to \hat{C}.$$ 

This corresponds to a natural transformation of functors, which (when we compose with the inverse of the equivalence $U \cong \hat{C}$) gives a natural transformation of functors $\hat{C} \to \hat{C}$,

$$\eta_X : X \to i\psi(X).$$

On each object $X$, $\eta_X$ is a universal map.

Now for $Y$ an object of $A$, we look at the morphism induced by $\eta_X$,

$$A_1/(\psi(X),Y) \to \hat{C}_1/(X,i(Y)).$$

The universality condition for $\eta_X$ implies that this map is an equivalence (of simplicial sets). This means that $(\psi, \eta)$ is an adjoint functor to $i$. This proves condition (iii).
Proof of the additional statement in (iii)

In the last paragraph of Theorem 14 was the additional statement that in (iii), the adjunction morphism going in the other direction is an equivalence between \( \psi \circ i \) and \( 1_A \). In other words, that \( \psi \) is a retract of the inclusion \( i \).

In our discussion of adjunctions, we explained how to obtain the adjunction morphism in the other direction

\[
\zeta_Y : \psi(iY) \to Y.
\]

(Recall, however, that it was left to the reader to show that the morphism thus constructed was an adjunction.)

We claim that \( \zeta_Y \) is an equivalence. To prove this, we will show that for any \( Z \in \text{ob}(A) \), the morphism of composition with \( \zeta_Y \)

\[
A_1/(Y, Z) \to A_1/(\psi(iY), Z)
\]

is an equivalence. Follow this with the two morphisms

\[
A_1/(\psi(iY), Z) \to \tilde{C}_1/(i\psi(iY), iZ) \to \tilde{C}_1/(iY, iZ),
\]

the first of which is functoriality for \( i \) and the second of which comes from the first adjunction morphism \( \eta_{iY} : iY \to i\psi(iY) \). The composition of these last two morphisms is an equivalence (that is the condition that \( \eta \) be an adjunction). We have to verify that the composed morphism

\[
A_1/(Y, Z) \to \tilde{C}_1/(iY, iZ)
\]

is homotopic to the morphism of functoriality for \( i \). For this, note that the diagram

\[
\begin{array}{ccc}
A_1/(Y, Z) & \to & A_1/(\psi(iY), Z) \\
\downarrow & & \downarrow \\
\tilde{C}_1/(iY, iZ) & \to & \tilde{C}_1/(i\psi(iY), iZ) \to \tilde{C}_1/(iY, iZ)
\end{array}
\]

commutes up to homotopy. The bottom row comes from composition with the sequence

\[
iY \xrightarrow{\eta_Y} i\psi(iY) \xrightarrow{\zeta_Y} iY.
\]

This composition is homotopic to the identity by Lemma 5.

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Given the above verification, we get that the composed morphism

\[ A_1/(Y, Z) \rightarrow A_1/(\psi(iY), Z) \rightarrow \hat{C}_1/(iY, iZ) \]

is an equivalence (since \( i \) is by hypothesis fully faithful), therefore the first morphism of composition with \( \zeta_Y \) is an equivalence

\[ A_1/(Y, Z) \xrightarrow{\sim} A_1/(\psi(iY), Z). \]

Since this works for all \( Z \), it follows that \( \zeta_Y \) is an internal equivalence in \( A \). This proves the claimed statement.

**The proof of \((iii) \Rightarrow (iv)\)**

This part of the proof is mainly concerned with establishing the Segal-category version of the small generation condition for trivial cofibrations, which is the main part of the “cofibrantly generated” condition for a closed model category. This benefits from P. Hirschhorn’s book \([15]\) where this type of condition is widely discussed; and from discussions and correspondence with A. Hirschowitz about how to put the cardinality arguments characteristic of Jardine’s paper \([19]\), into a general framework.

Recall from Proposition \([12]\) that strong generation implies generation by colimits. Also, the existence of the adjoint \( \psi \) implies that \( A \) admits all small homotopy colimits (and that \( \psi \) preserves homotopy colimits). In particular, \((iii) \Rightarrow (ii)\) and for the present step we may use the hypotheses of \((ii)\) and \((iii)\) together. Recall also that in the previous section of the proof we showed that in the situation of hypothesis \((iii)\), the adjunction morphism \( \zeta \) is an equivalence between \( \psi \circ i \) and \( 1_A \).

Let \( W \subset \hat{C} \) be the subcategory of morphisms which go to equivalences under \( \psi \). We obtain morphisms

\[ L(\hat{C}, W) \xrightarrow{\psi} A, \]

and

\[ A \xrightarrow{i} L(\hat{C}, W). \]

The composition \( \psi \circ i \) is already the identity before localization. On the other hand, the composition \( i \circ \psi \) is related to the identity by a natural transformation which lies in \( W \) (we give the argument for this in the paragraph...
which follows); thus on the level of the localization $i \circ \psi$ is also homotopic to the identity. This proves that the above morphisms between $A$ and the localization $L(\widehat{C}, W)$ are equivalences, and gives the statement in (iv) about localization.

In the previous paragraph we left open the detail of verifying that for any $X \in ob(\widehat{C})$, the adjunction morphism $\eta_X : X \to i\psi(X)$ is $\psi$-trivial. Thus we have to look at
\[
\psi(\eta_X) : \psi(X) \to \psi(i\psi(X)).
\]
This fits into a sequence
\[
\psi(X) \xrightarrow{\psi(\eta_X)} \psi(i\psi(X)) \xrightarrow{\zeta_X} \psi(X).
\]
By Lemma 5, the composition is homotopic to the identity of $\psi(X)$. Also the second morphism is an equivalence as we have shown above. Therefore $\psi(\eta_X)$ is an equivalence, as claimed.

Turn now to our situation
\[
C \to A \to \widehat{C}
\]
where $C$ is a small 1-category, consists of objects which are small in $A$, and where $A$ admits small colimits and the morphism $i$ is fully faithful. The fact that $C$ is small means that the bound for smallness in $A$ of objects of $C$ can be assumed uniform. Thus there is a cardinal $\delta$ such that every object of $C$ is $\delta$-small in $A$.

One has to be careful (cf the “caution” after Lemma 6) that the composition $C \to A \to \widehat{C}$ is not the Yoneda morphism for $C$.

By Lemma 9, homotopy colimits in $\widehat{C}$ are calculated object-by-object. Thus we can write
\[
\text{hocolim}_\beta^S \widehat{C}_1/(Z, X_i) \xrightarrow{\cong} \widehat{C}_1/(Z, \text{colim}_\beta^C X_i).
\]
We claim that for any ordinal $\beta$ of size $\geq \delta$, sequential $\beta$-colimits in $A$ agree with those in $\widehat{C}$. Using the fact that the objects of $C$ are $\delta$-small in $A$, we get that for ordinals $\beta \geq \delta$ and sequential $\beta$-diagrams $\{X_i\}$, we have
\[
\text{hocolim}_\beta^S \widehat{C}_1/(Z, X_i) \xrightarrow{\cong} \widehat{C}_1/(Z, \text{colim}_\beta^A X_i).
\]
Comparing with the previous result we get that for any $Z \in C$, the morphism
\[
\widehat{C}_1/(Z, \text{colim}_\beta^C X_i) \to \widehat{C}_1/(Z, \text{colim}_\beta^A X_i)
\]
is an equivalence. This implies that the morphism
\[ \text{colim} \beta X_i \to \text{colim} \beta X_i \]
is an equivalence in \( \hat{C} \). Thus the two colimits agree.

Now go directly to the case of looking at the functor \( \psi : \hat{C} \to A \). Recall that an arrow \( U \to V \) in \( \hat{C} \) is \( \psi \)-trivial if \( \psi U \to \psi V \) is an equivalence in \( A \). An arrow \( F \to G \) in \( \hat{C} \) is \( \psi \)-fibrant if for every \( \psi \)-trivial morphism \( U \to V \), the morphism
\[ \text{Hom}(V, F) \to \text{Hom}(V, G) \times_{\text{Hom}(U, G)} \text{Hom}(U, F) \]
is an equivalence (where the fiber product is a homotopy fiber product of simplicial sets). We call this condition the lifting condition. A subset \( G \) of \( \psi \)-trivial morphisms is a generating subset if a morphism \( A \to B \) which satisfies the above lifting condition with respect to morphisms in \( G \), is necessarily \( \psi \)-fibrant.

We would like to show that the class of \( \psi \)-trivial morphisms admits a small generating subset. For this, we adopt the strategy used by Jardine in [19]. We will take an arbitrary \( \psi \)-trivial morphism \( U \to V \) and express it as a sequential colimit of pushouts by \( \psi \)-trivial morphisms between smaller objects (until getting back to objects of a fixed size \( \delta \), where we stop and say that we have a generating subset). To start, we need to be able to talk about the “size” of an object. We say that an object \( U \in \hat{C} \) has size \( \leq \beta \) if \( U \) can be expressed (as in Lemma [11]) as a colimit of objects of \( h_C(C) \), over a 1-category \( J \) with \( |J| = \beta \). If this is the case, then rearrangement of the colimit allows one to express \( U \) as a \( \beta \)-sequential colimit of objects \( U_i \) such that the \( U_i \) are of size \( \leq \gamma_i < \beta \).

Fix an ordinal \( \delta \) (at least as big as the \( \delta \) above, but also at least uncountable, at least as big as \( |C| \), etc.). Let \( \mathcal{F} \) be the small set of \( \psi \)-trivial morphisms between objects of size \( \leq \delta \) in \( \hat{C} \) (technically speaking, this would still be a class but can be replaced by a small subset containing morphisms equivalent to all those in the class). We claim that the class of morphisms \( \mathcal{F} \) generated by \( \mathcal{F} \) in the sense of lifting, is equal to the full class of \( \psi \)-trivial morphisms. (This will complete the proof of \( (iii) \Rightarrow (iv) \).)

To prove this, we proceed by transcendental induction: suppose it isn’t the case, and let \( \beta \) be the smallest ordinal such that there exists a morphism \( U \to V \) between objects of size \( \leq \beta \), which is \( \psi \)-trivial but not in \( \mathcal{F} \). We will
show that the morphism may be expressed as a sequential colimit of pushouts by smaller \(\psi\)-trivial morphisms. By the induction hypothesis and since they are smaller, these \(\psi\)-trivial morphisms are in \(\mathcal{F}\); but then this implies that our original morphism is in \(\mathcal{F}\), a contradiction showing the claim.

Note that \(\beta\) is the first ordinal of its cardinality; also \(\beta > \delta\). Express \(U\) and \(V\) as sequential colimits
\[
U = \text{colim}^\beta_{\hat{C}} U_i
\]
and
\[
V = \text{colim}^\beta_{\hat{C}} V_i,
\]
with the \(U_i\) and \(V_i\) of size \(\leq |i| < \beta\). The \(U_i\) are \(\beta\)-small in \(\hat{C}\) which implies that, after possibly reindexing the second colimit, we can assume that the map \(U \to V\) comes from a collection of maps \(U_i \to V_i\).

By assumption, \(\psi U \to \psi V\) is an equivalence in \(\mathcal{A}\). The fact that \(\psi\) preserves colimits means that the morphisms
\[
\text{colim}^\beta_{\hat{C}} \psi U_i \to \psi U
\]
and
\[
\text{colim}^\beta_{\hat{C}} \psi V_i \to \psi V
\]
are equivalences. The fact that \(\beta\)-colimits agree in \(\mathcal{A}\) and \(\hat{C}\) means that the morphisms
\[
\text{colim}^\beta_{\hat{C}} \psi U_i \to \psi U.
\]
and
\[
\text{colim}^\beta_{\hat{C}} \psi V_i \to \psi V.
\]
are equivalences. A similar argument shows that these colimits are essentially sequential, so (by restricting our attention to big enough indices \(i\)) we may assume that they are sequential.

Furthermore by Lemma \(\Box\) the above colimits in \(\hat{C}\) are calculated object-by-object. Thus, for every \(z \in C\) the morphism
\[
\text{hocolim}^\beta_{\hat{C}}(\psi U_i)(z) \to \text{hocolim}^\beta_{\hat{C}}(\psi V_i)(z)
\]
is a weak equivalence of simplicial sets. This implies that there are subsequences \(i_k\) and \(j_k\) in \(\beta\) (which are again indexed by an ordinal which we denote \(\kappa\) even though it is isomorphic to \(\beta\)), such that
\[
U_{j_k}(z) \to (\psi V_{i_k})(z) \cup (\psi U_{j_k})(z) (\psi U_{i_k})(z)
\]
are equivalences. This fact about simplicial sets comes from Jardine’s argument [19]. Furthermore, since $\beta$ is big with respect to the cardinality of $C$, we can assume that these same subsequences work for all $z \in C$. Therefore the morphism

$$U_{jk} \to (\psi V_{ik}) \cup^{\psi U_{jk}} (\psi U_{jk})$$

is an equivalence in $\hat{C}$.

Now apply Lemma 8 which says that since the $\hat{C}$-coproduct

$$(\psi V_{ik}) \cup^{\psi U_{jk}} (\psi U_{jk})$$

is in $A$ (because it is equivalent by the previous paragraph to $\psi U_{jk}$) then this is also the coproduct in $A$. Now the fact that $\psi$ commutes with colimits means that

$$(\psi V_{ik}) \cup^{\psi U_{jk}} (\psi U_{jk}) \simeq \psi(V_{ik} \cup^{U_{jk}} U_{jk}).$$

Therefore we finally get that the morphism

$$\psi U_{jk} \to \psi(V_{ik} \cup^{U_{jk}} U_{jk})$$

is an equivalence. Thus, the morphism

$$U_{jk} \to V_{ik} \cup^{U_{jk}} U_{jk}$$

is $\psi$-trivial. Defining

$$U_k' := U_{jk}, \quad V_k' := V_{ik} \cup^{U_{jk}} U_{jk},$$

we still have $U = \text{colim}_k U_k'$ and $V = \text{colim}_k V_k'$, but now the morphisms $U_k' \to V_k'$ are $\psi$-trivial. The $U_k'$ and $V_k'$ have size $\leq |j_k| < \beta$, so these $\psi$-trivial morphisms are in $\mathcal{F}$. It follows that the morphism $U \to V$ is in $\mathcal{F}$.

The proof of $(iv) \Rightarrow (i)$

Let $N$ be the Heller model category of simplicial presheaves on $C$ [14]. Thus $L(N) \cong \hat{C}$ (Lemma 3).

Let $M$ be the model category with the same underlying category as $N$, and the same class of cofibrations, but where a morphism is said to be a weak equivalence if its image in $\hat{C}$ is $\psi$-trivial. As generating set of trivial cofibrations, we can take a generating set for $N$ plus a set of cofibrant
representatives for our small generating set given in the hypothesis of (iv). It is easy to see that a morphism is a fibration in \( M \) if and only if it is a fibration in \( N \), and if its image in \( \hat{C} \) is a \( \psi \)-fibration. This implies that the given generating set indeed generates the trivial cofibrations (that is, lifting for the given generating set is equivalent to being a fibration i.e. to lifting for all trivial cofibrations).

We use the criterion of [16] Lemma 2.5 to obtain a closed model structure for \( M \) (the numbers in the present paragraph refer to the conditions in that lemma). As a historical point, note that this lemma is just a synopsis of the techniques of Dwyer, Kan and Hirschhorn [8] [15]. Start by noting that \( M = N \) admits small limits and colimits (0). Since \( M \) is a category of simplicial presheaves, any small subset is adapted to the small object argument so conditions (4) and (5) are automatic. The three for two condition (2) is automatic in view of the definition of weak equivalence. A morphism which satisfies lifting for all cofibrations is an equivalence in \( N \) already so it is an equivalence in \( M \); this gives (3). The cofibrations are the same as for \( N \) so condition (6) comes from that of \( N \). Condition (7), that the trivial cofibrations are stable under coproduct and sequential colimit, comes from the same property for \( \psi \)-trivial morphisms in \( \hat{C} \). In effect, a coproduct or sequential colimit of cofibrations, calculated in \( M \), is a homotopy colimit (cf [8] [13] [16]), in other words it is a colimit in \( \hat{C} \); and the \( \psi \)-trivial morphisms in \( C \) are stable under coproduct and sequential colimit because \( \psi \) preserves colimits by hypothesis. Finally, for condition (1) note that cofibrations are stable under retracts because they are the same as for \( N \). As for weak equivalences, a morphism is by definition a weak equivalence if and only if it is an equivalence in \( A \), and this condition is equivalent to saying that it projects to an isomorphism in \( \text{ho}(A) \). The class of isomorphisms in \( \text{ho}(A) \) is closed under retracts, so this implies that the class of weak equivalences in \( M \) is closed under retracts. This gives (1).

Therefore by Lemma 2.5 of [16], we obtain a cofibrantly generated closed model structure \( M \).

To complete the proof of (i) we just have to show that \( L(M) \cong A \). For this, note that \( L(M) \) is obtained from \( L(N) \) by inverting the images of \( M \)-weak equivalences (since \( L(N) \) was obtained from \( N = M \) by inverting a subset of the weak equivalences). We have that \( L(N) \cong \hat{C} \), and the images of the \( M \)-weak equivalences are exactly the morphisms of \( \hat{C} \) whose image by \( \psi \) is an equivalence in \( A \). The hypothesis of (iv) says that this localization
gives exactly $A$. This completes the proof of $(iv) \Rightarrow (i)$. We have now finished the proof of Theorem 14.

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