New $L^2$-type exponentiality tests

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Abstract

We introduce new consistent and scale-free goodness-of-fit tests for the exponential distribution based on Puri-Rubin characterization. For the construction of test statistics we employ weighted $L^2$ distance between $V$-empirical Laplace transforms of random variables that appear in the characterization. The resulting test statistics are degenerate $V$-statistics with estimated parameters. We compare our tests, in terms of the Bahadur efficiency, to the likelihood ratio test, as well as some recent characterization based goodness-of-fit tests for the exponential distribution. We also compare the powers of our tests to the powers of some recent and classical exponentiality tests. In both criteria, our tests are shown to be strong and outperform most of their competitors.

keywords: goodness-of-fit; exponential distribution; Laplace transform; Bahadur efficiency; V-statistics

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1 Introduction

The exponential distribution is one of most widely studied distributions in theoretical and applied statistics. Many models assume exponentiality of the data. Ensuring that those models can be used is of great importance. For this reason, a great variety of goodness of fit tests for the particular case of the exponential distribution, have been proposed in literature.

Different constructions have been used to build test statistics. They are mainly based on empirical counterparts of some special properties of the exponential distribution. Some of those tests employ properties connected to different integral transforms such as: characteristic functions (see e.g. [9], [10], [12]); Laplace transforms (see e.g. [11], [16], [19]); and other integral transforms (see e.g. [17], [20]). Other properties include maximal correlations (see [7], [8]), entropy (see [4]), etc.

The simple form of the exponential distribution gave rise to many equidistribution type characterizations. The equality in distribution can be expressed in many ways (equality of distribution functions, densities, integral transforms, etc.). This makes them suitable for building different types of test statistics.

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Such tests have become very popular in recent times, as they are proven to be rather efficient. Tests that use U-empirical and V-empirical distribution functions, of integral-type (integrated difference) and supremum-type, can be found in [28], [33], [15], [23], [21], [25]. A class of weighted integral-type tests that uses U-empirical Laplace transforms is presented in [22].

Motivated by the power and efficiency of those tests, here we create a similar test based on an equidistribution characterization. The test statistics measure the distance between two V-empirical Laplace transforms of the random variables that appear in the characterization, but, for the first time, using weighted $L^2$-distance. This guarantees the consistency of the test against all alternatives.

The paper is organized as follows. In Section 2 we introduce the test statistics and derive their asymptotic properties. In Section 3 we calculate the approximate Bahadur slope of our tests, for different close alternatives, and inspect the impact of the tuning parameter to the efficiencies of the test. We also compare the proposed tests to their recent competitors, via approximate local relative Bahadur efficiency. In Section 4 we conduct a power study. We obtain empirical powers of our tests, against different common alternatives, and compare them to some recent and classical exponentiality tests. We also apply an algorithm for data driven selection of tuning parameter and obtain the corresponding powers in small sample case.

## 2 Test statistic

Puri and Rubin [30] proved the following characterization theorem.

**Characterization 2.1.** Let $X_1$ and $X_2$ be two independent copies of a random variable $X$ with pdf $f(x)$. Then $X$ and $|X_1 - X_2|$ have the same distribution, if and only if for some $\lambda > 0$, $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.

Let $X_1, X_2, \ldots, X_n$ be independent copies of a non-negative random variable $X$ with unknown distribution function $F$. We consider the transformed sample $Y_i = \hat{\lambda} X_i$, $i = 1, 2, \ldots, n$, where $\hat{\lambda}$ is the reciprocal sample mean. For testing the null hypothesis $H_0 : F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, in view of the characterization 2.1 we propose the following family of test statistics, depending on the tuning parameter $a > 0$,

$$M_{n,a}(\hat{\lambda}) = \int_0^\infty \left( L_n^{(1)}(t) - L_n^{(2)}(t) \right)^2 e^{-at} dt,$$

where

$$L_n^{(1)}(t) = \frac{1}{n} \sum_{i=1}^n e^{-tY_i}$$

$$L_n^{(2)}(t) = \frac{1}{n^2} \sum_{i_1, i_2=1}^n e^{-t|Y_{i_1} - Y_{i_2}|}$$

are V-empirical Laplace transforms of $Y_1$ and $|Y_1 - Y_2|$ respectively.
In order to explore the asymptotic properties we rewrite (1) as

\[ M_{n,a}(\hat{\lambda}) = \int_0^\infty \left( \frac{1}{n^2} \sum_{i=1}^n e^{-tX_i} \hat{\lambda} - \frac{1}{n^2} \sum_{i_1,i_2=1}^n e^{-t|X_{i_1} - X_{i_2}|\hat{\lambda}} \right)^2 e^{-at} dt \]

\[ = \frac{1}{n^4} \int_0^\infty \sum_{i_1,i_2,i_3,i_4} \left( e^{-tX_{i_1}} - e^{-t|X_{i_1} - X_{i_2}|\hat{\lambda}} \right) \left( e^{-tX_{i_3}} - e^{-t|X_{i_3} - X_{i_4}|\hat{\lambda}} \right) e^{-at} dt \]

\[ = \frac{1}{n^4} \sum_{i_1,i_2,i_3,i_4} \int_0^\infty g(X_{i_1}, X_{i_2}, t; \hat{\lambda}) g(X_{i_3}, X_{i_4}, t; \hat{\lambda}) e^{-at} dt \]

\[ = \frac{1}{n^4} \sum_{i_1,i_2,i_3,i_4} h(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}, a; \hat{\lambda}), \]

where \( \hat{\lambda} = \bar{X}^{-1} \) is a consistent estimator of \( \lambda \).

Let’s focus, for a moment, on \( M_{n,a}(\lambda) \), for a fixed \( \lambda > 0 \). Notice that \( M_{n,a}(\lambda) \) is a \( V \)-statistic with kernel \( h \). Moreover, under the null hypothesis its distribution does not depend on \( \lambda \), so we may assume \( \lambda = 1 \). It is easy to show that its first projection on a basic observation is equal to zero. After some calculations, one can obtain its second projection given by

\[ \tilde{h}_2(x, y, a) = E(h(X_1, X_2, X_3, X_4, a|X_1 = x, X_2 = y) \]

\[ = -\frac{1}{2} + \frac{1}{3}(e^{-x} + e^{-y}) + \frac{1}{6}e^{a-x-y}\text{Ei}(-a)(a(e^x - 2)(e^y - 2) - e^x - e^y + 4) \]

\[ + \frac{1}{6} e^{a-x-y} \left( \text{Ei}(a)(4a + e^x + e^y - 4) - (\text{Ei}(a + x)(4a + x - 1) + e^y) \right) + \frac{1}{6}(a + x + y), \]

where \( \text{Ei}(x) = -\int_x^\infty \frac{e^{-t}}{t} dt \) is the exponential integral. The function \( \tilde{h}_2 \) is non-constant for any \( a > 0 \). Its plot, for \( a = 1 \), is shown in Figure 1. Hence, the kernel \( h \) is degenerate with degree 2.

Figure 1: Second projection \( \tilde{h}_2(x, y, 1) \)

The asymptotic distribution of \( M_{n,a}(\hat{\lambda}) \) is given in the following theorem.
Theorem 2.2. Let $X_1, \ldots, X_n$ be i.i.d. sample with distribution function $F(x) = 1 - e^{\lambda x}$ for some $\lambda > 0$. Then
\[ nM_{n,a}(\hat{\lambda}) \xrightarrow{d} 6 \sum_{k=1}^{\infty} \delta_k W_k, \]
where $\{\delta_k\}$ are the eigenvalues of the integral operator $M_a$ defined by
\[ M_a q(x) = \int_0^{+\infty} h_2(x, y, a)q(y)dF(y) \]
and $\{W_k\}$ is the sequence of i.i.d standard Gaussian random variables.

Proof. Since the kernel $h$ is bounded and degenerate, from the theorem for the asymptotic distribution of $U$-statistics with degenerate kernels [13, Corollary 4.4.2], and the Hoeffding representation of $V$-statistics, we get that, $M_{n,a}(1)$, being a $V$-statistic of degree 2, has the asymptotic distribution from (2). Hence, it suffices to show that $M_{n,a}(\hat{\lambda})$ and $M_{n,a}(1)$ have the same distribution.

Our statistic $M_{n,a}(\hat{\lambda})$ can be rewritten as
\[ M_{n,a}(\hat{\lambda}) = \int_0^{\infty} \left( \frac{1}{n^2} \sum_{i_1, i_2=1}^n g(X_{i_1}, X_{i_2}, t; \hat{\lambda}) \right)^2 e^{-at} dt \]
\[ = \int_0^{\infty} V_n(\hat{\lambda})^2 e^{-at} dt. \]

Here $V_n(\hat{\lambda})$ is a $V$-statistic of order 2 with estimated parameter, and kernel $g(X_{i_1}, X_{i_2}, t; \hat{\lambda})$.

Since the function $g(x_1, x_2, t; \gamma)$ is continuously differentiable with respect to $\gamma$ at the point $\gamma = \lambda$, the mean-value theorem gives us
\[ V_n(\hat{\lambda}) = V_n(\lambda) + (\hat{\lambda} - \lambda) \frac{\partial V_n(\gamma)}{\partial \gamma} |_{\gamma = \lambda^*}, \]
for some $\lambda^*$ is between $\lambda$ and $\hat{\lambda}$.

Using the Law of large numbers for $V$-statistics [32, 6.4.2.], we have that $\frac{\partial V_n(\gamma)}{\partial \gamma}$ converges to
\[ E \left( t|X_1 - X_2| e^{-t|X_1 - X_2|} - tX_1 e^{-tX_1} \right) = 0. \]

Since $\sqrt{n}(\hat{\lambda} - \lambda)$ is stochastically bounded, we conclude that statistics $\sqrt{n}V_n(\hat{\lambda})$ and $\sqrt{n}V_n(1)$ are asymptotically equally distributed. Therefore, $nM_{n,a}(\hat{\lambda})$ and $nM_{n,a}(1)$ will have the same limiting distribution, which completes the proof. \qed
3 Local Approximate Bahadur efficiency

One way to compare tests is to calculate their relative Bahadur efficiency. We briefly present it here. For more details we refer to [4] and [26].

For two tests with the same null and alternative hypotheses, $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$, the asymptotic relative Bahadur efficiency is defined as the ratio of sample sizes needed to reach the same test power, when the level of significance approaches zero. For two sequences of test statistics, it can be expressed as the ratio of Bahadur exact slopes, functions proportional to exponential rates of decrease of their sizes, for the increasing number of observations and a fixed alternative. The calculation of these slopes depends on large deviation functions which are often hard to obtain. For this reason, in many situations, the tests are compared using the approximate Bahadur efficiency, which is shown to be a good approximation in the local case (when $\theta \to \partial \Theta_0$).

Suppose that $T_n = T_n(X_1, ..., X_n)$ is a test statistic with its large values being significant. Let the limiting distribution function of $T_n$, under $H_0$, be $F_{T_n}$, whose tail behavior is given by $\log(1 - F_{T_n}(t)) = -\frac{a_T t^2}{2} (1 + o(1))$, where $a_T$ is positive real number, and $o(1) \to 0$ as $t \to \infty$. Suppose also that the limit in probability $\lim_{n \to \infty} T_n / \sqrt{n} = b_T(\theta) > 0$ exists for $\theta \in \Theta_1$. Then the relative approximate Bahadur efficiency of $T_n$, with respect to another test statistic $V_n$ (whose large values are significant), is

$$e_{V,T}^*(\theta) = \frac{c_T^*(\theta)}{c_V^*(\theta)},$$

where $c_T^*(\theta) = a_T b_T^2(\theta)$ and $c_V^*(\theta) = a_V b_V^2(\theta)$ are approximate Bahadur slopes of $T_n$ and $V_n$, respectively.

We may suppose, without loss of generality, that $\Theta_0 = \{0\}$. Consequently, the approximate local relative Bahadur efficiency is given by

$$e_{V,T}^* = \lim_{\theta \to 0} e_{V,T}^*(\theta).$$

Let $\mathcal{G} = \{G(x, \theta), \theta > 0\}$ be a family of alternative distribution functions such that $G(x, \theta) = 1 - e^{-\lambda x}$, for some $\lambda > 0$, if and only if $\theta = 0$, and the regularity conditions for V-statistics with weakly degenerate kernels from [27] Assumptions WD are satisfied.

The logarithmic tail behaviour of the limiting distribution of $M_{n,a}(\hat{\lambda})$, under the null hypothesis, is derived in the following lemma.

**Lemma 3.1.** For the statistic $M_{n,a}(\lambda)$ and the given alternative density $g(x, \theta)$ from $\mathcal{G}$, the Bahadur approximate slope satisfies the relation $c_{M}(\theta) \sim \frac{b_M(\theta)}{b_M(\theta)}$.

where $b_M(\theta)$ is the limit in $P_\theta$ probability of $M_{n,a}(\lambda)$, and $\delta_1$ is the largest eigenvalue of the sequence $\{\delta_k\}$ from [27].

**Proof.** Using the result of Zolotarev [35], we have that the logarithmic tail behavior of limiting distribution function of $M_{n,a}(\lambda)$ is

$$\log(1 - F_M(t)) = -\frac{\lambda^2}{12\delta_1} + o(t^2), \quad t \to \infty.$$
Therefore, we obtain that \( a_{\lambda_{\theta}} = \frac{1}{\delta} \). The limit in probability \( P_\theta \) of \( M_{n,a}(\lambda)/\sqrt{n} \) is

\[
b_{\lambda_{\theta}} = \sqrt{b_M(\theta)}.
\]

Inserting this into the expression for Bahadur slope, we complete the proof. \( \square \)

The limit in probability of our test statistic, under a close alternative, can be derived using the following Lemma.

**Lemma 3.2.** For a given alternative density \( g(x; \theta) \) whose distribution belongs to \( \mathcal{G} \), we have that the limit in probability of the statistic \( M_{n,a}(\lambda) \) is

\[
b_M(\theta) = 6 \int_{0}^{\infty} \int_{0}^{\infty} \tilde{h}_2(x, y)f(x)f(y)dx dy \cdot \theta^2 + o(\theta^2), \theta \rightarrow 0,
\]

where \( f(x) = \frac{\partial g(x; \theta)}{\partial \theta}\bigg|_{\theta=0} \).

**Proof.** For brevity, let us denote \( x = (x_1, x_2, x_3, x_4) \) and \( G(x; \theta) = \prod_{i=1}^{4} G(x_i; \theta) \). Since \( X \) converges almost surely to its expected value \( \mu(\theta) \), using the Law of large numbers for \( V \)-statistics with estimated parameters (see [13]), we have that \( \lambda_{n,a}(\hat{\lambda}) \) converges to

\[
b_M(\theta) = E_\theta(h(X, a; \mu(\theta)))
\]

\[
= \int_{(R^+)^4} \mu(\theta) \left( \frac{x_1 + x_3 + a \mu(\theta)}{x_3 + |x_1 - x_2| + a \mu(\theta)} - \mu(\theta) \right) dG(x; \theta).
\]

We may assume that \( \mu(0) = 1 \) due to the scale freeness of test statistic under the null hypothesis. After some calculations we get that \( b_M'(0) = 0 \). Next, we obtain that

\[
b''(0) = \int_{(R^+)^4} h(x, a; 1) \frac{\partial^2}{\partial \theta^2} dG(x; 0) = 6 \int_{(R^+)^2} \tilde{h}_2(x, y)f(x)f(y)dx dy.
\]

Expanding \( b_M(\theta) \) into the Maclaurin series we complete the proof. \( \square \)

To calculate the efficiency one needs to find \( \delta_1 \), the largest eigenvalue. Since we cannot obtain it analytically, we use the following approximation, introduced in [6].

It can be shown that \( \delta_1 \) is the limit of the sequence of the largest eigenvalues of linear operators defined by \( (m+1) \times (m+1) \) matrices \( M^{(m)} = |m^{(m)}_{i,j}| \), \( 0 \leq i \leq m, 0 \leq j \leq m \), where

\[
m^{(m)}_{i,j} = \tilde{h}_2 \left( B_{i} \left( \frac{B_{i}}{m} \right) + B_{j} \left( \frac{B_{j}}{m} \right) \right) \sqrt{e^{\frac{m}{2}} - e^{\frac{m+1}{2}}} \cdot \sqrt{e^{\frac{m}{2}} - e^{\frac{m+1}{2}}} \cdot \frac{1}{1 - e^{-B_{i}}}, \quad (4)
\]

when \( m \) tends to infinity and \( F(B) \) approaches 1.

In Table [4] we present the largest eigenvalues for \( a = 0.5, 1, 2 \) and 5, obtained using [4] with \( m = 4500 \) and \( B = 10 \).
### Table 1: Approximate eigenvalues of $M_a$

| $a$  | 0.5 | 1   | 2    | 5    |
|------|-----|-----|------|------|
| $\delta_1$ | $1.32 \cdot 10^{-2}$ | $5.32 \cdot 10^{-3}$ | $1.73 \cdot 10^{-3}$ | $2.80 \cdot 10^{-4}$ |

#### 3.1 Efficiencies with respect to LRT

Lacking a theoretical upper bound, the approximate Bahadur slopes are often compared (see e.g. [19]) to the approximate Bahadur slopes of the likelihood ratio tests (LRT), which are known to be optimal parametric tests in terms of Bahadur efficiency. Hence, we may consider the approximate relative Bahadur efficiencies against the LRT as a sort of "absolute" local approximate Bahadur efficiencies. We calculate it for the following alternatives:

- a Weibull distribution with density
  \[ g(x, \theta) = e^{-x^{\theta+1}}(1 + \theta)x^\theta, \theta > 0, x \geq 0; \]

- a Gamma distribution with density
  \[ g(x, \theta) = \frac{x^\theta e^{-x}}{\Gamma(\theta + 1)}, \theta > 0, x \geq 0; \]

- a Linear failure rate distribution with density
  \[ g(x, \theta) = e^{-x^2(1 + \theta x)}(1 + \theta x), \theta > 0, x \geq 0; \]

- a mixture of exponential distributions with negative weights (EMNW($\beta$)) with density (see [14])
  \[ g(x, \theta) = (1 + \theta)e^{-x} - \theta e^{-\beta x}, \theta \in \left(0, \frac{1}{\beta - 1}\right], x \geq 0; \]

It is easy to show that all densities given above belong to family $\mathcal{G}$.

The efficiencies, as functions of the tuning parameter $a$, are shown on Figures 2–5.

We can notice that the local efficiencies range from reasonable to high, and for some values of $a$ they are very high. Also, their behaviour with respect to the tuning parameter $a$ is very different. In the cases of Weibull and Linear failure rate alternatives (Figures 2 and 4), they are increasing functions of $a$, while in the Gamma case (Figure 3), the function is decreasing. In the case of EMNW(3) (Figure 5), the efficiencies increase up to a certain point and then decrease.

#### 3.2 Comparison of efficiencies

In this section, we calculate the local approximate Bahadur relative efficiency of our tests against some recent, characterization based integral-type tests, for the previously mentioned alternatives.
Figure 2: Local approximate Bahadur efficiencies w.r.t. LRT for a Weibull alternative

Figure 3: Local approximate Bahadur efficiencies w.r.t. LRT for a gamma alternative

Figure 4: Local approximate Bahadur efficiencies w.r.t. LRT for a linear failure rate alternative

The characterizations are of the equidistribution type and take the following form.
Let $X_1, \ldots, X_{\max(m,p)}$ be i.i.d with d.f. $F$, $\omega_1 : R^m \mapsto R^1$ and $\omega_2 : R^p \mapsto R^1$ two sample functions. Then the following relation holds

$$\omega_1(X_1, \ldots, X_m) \overset{d}{=} \omega_2(X_1, \ldots, X_p)$$

if and only if $F(x) = 1 - e^{-\lambda x}$, for some $\lambda > 0$.

Notice that the Puri-Rubin characterization is an example of such characterizations.

The first class of competitor tests consists of the integral-type tests with test statistic

$$I_n = \int_0^\infty \left( G_n^{(1)}(t) - G_n^{(2)}(t) \right) dF_n(t),$$

where $G_n^{(1)}(t)$ and $G_n^{(2)}(t)$ are $V$-empirical distribution functions of $\omega_1$ and $\omega_2$, respectively.

In particular, we consider the following integral-type test statistics:

- $I_{n,k}^{(1)}$, proposed in [15], based on the Arnold and Villasenor characterization, where $\omega_1(X_1, \ldots, X_k) = \max(X_1, \ldots, X_k)$ and $\omega_2(X_1, \ldots, X_k) = X_1 + \frac{X_2}{2} + \cdots + \frac{X_k}{k}$ (see [3], [24]);

- $I_n^{(2)}$, proposed in [23], based on the Milošević-Obradović characterization, where $\omega_1(X_1, X_2) = \max(X_1, X_2)$ and $\omega_2(X_1, X_2, X_3) = \min(X_1, X_2) + X_3$ (see [24]);

- $I_n^{(3)}$, proposed in [21], based on the Obradović characterization, where $\omega_1(X_1, X_2, X_3) = \max(X_1, X_2, X_3)$ and $\omega_2(X_1, X_2, X_3, X_4) = X_1 + \text{med}(X_2, X_3, X_4)$ (see [29]);

- $I_n^{(4)}$, proposed in [33], based on the Yanev-Chakraborty characterization, where $\omega_1(X_1, X_2, X_3) = \max(X_1, X_2, X_3)$ and $\omega_2(X_1, X_2, X_3) = \frac{X_1}{3} + \max(X_2, X_3)$ (see [34]).

We also consider integral-type tests of the form

$$J_{n,a} = \int_0^\infty \left( L_n^{(1)}(t) - L_n^{(2)}(t) \right) X e^{-at} dt,$$

where $L_n^{(1)}(t)$ and $L_n^{(2)}(t)$ are $V$-empirical distribution functions of $\omega_1$ and $\omega_2$, respectively.
where \( L^{(1)}_n(t) \) and \( L^{(2)}_n(t) \) are \( V \)-empirical Laplace transforms of \( \omega_1 \) and \( \omega_2 \), respectively. This approach has been originally proposed in [22]. There, particular cases of Desu characterization, with \( \omega_1(X_1) = X_1 \) and \( \omega_2 = 2 \min(X_1, X_2) \), and Puri-Rubin characterization were examined. We denote the corresponding tests statistics with \( J_{n,a}^P \) and \( J_{n,a}^D \), respectively. The results are presented in Table 2. We can notice that in most cases tests that employ \( V \)-empirical Laplace transforms are more efficient than those based on \( V \)-empirical distribution functions. On the other hand, new tests are comparable with \( J_{n,a}^P \) and more efficient than \( J_{n,a}^D \).

Table 2: Relative Bahadur efficiency of \( M_{n,a} \) with respect to its competitors

| \( a \) | 0.5 | 1   | 2   | 5   |
|-------|-----|-----|-----|-----|
| \( I^{(1)}_{n,2} \) | Weibull | 1.27 | 1.33 | 1.37 | 1.42 |
|       | Gamma  | 1.14 | 1.13 | 1.10 | 1.06 |
|       | LFR    | 2.44 | 3.13 | 3.93 | 5.08 |
|       | EMNW(3) | 1.25 | 1.34 | 1.40 | 1.42 |
| \( I^{(1)}_{n,3} \) | Weibull | 1.19 | 1.24 | 1.28 | 1.32 |
|       | Gamma  | 1.17 | 1.15 | 1.12 | 1.09 |
|       | LFR    | 1.59 | 2.04 | 2.56 | 3.31 |
|       | EMNW(3) | 1.08 | 1.17 | 1.22 | 1.23 |
| \( I^{(2)}_n \) | Weibull | 1.05 | 1.10 | 1.14 | 1.17 |
|       | Gamma  | 1.04 | 1.02 | 1.00 | 0.97 |
|       | LFR    | 1.22 | 1.56 | 1.96 | 2.53 |
|       | EMNW(3) | 1.02 | 1.10 | 1.15 | 1.17 |
| \( I^{(3)}_n \) | Weibull | 1.06 | 1.10 | 1.14 | 1.18 |
|       | Gamma  | 1.18 | 1.16 | 1.14 | 1.10 |
|       | LFR    | 0.82 | 1.05 | 1.32 | 1.71 |
|       | EMNW(3) | 0.94 | 1.02 | 1.06 | 1.08 |
| \( I^{(4)}_n \) | Weibull | 1.21 | 1.27 | 1.31 | 1.35 |
|       | Gamma  | 1.30 | 1.28 | 1.25 | 1.21 |
|       | LFR    | 1.23 | 1.57 | 1.98 | 2.56 |
|       | EMNW(3) | 1.04 | 1.12 | 1.16 | 1.18 |
| \( J_{n,a}^P \) | Weibull | 0.97 | 0.97 | 1.01 | 1.00 |
|       | Gamma  | 0.98 | 0.99 | 1.00 | 1.02 |
|       | LFR    | 0.97 | 0.93 | 0.91 | 0.93 |
|       | EMNW(3) | 0.97 | 0.98 | 0.99 | 1.00 |
| \( J_{n,a}^D \) | Weibull | 1.00 | 0.95 | 0.93 | 0.95 |
|       | Gamma  | 2.16 | 1.64 | 1.33 | 1.13 |
|       | LFR    | 1.17 | 1.07 | 1.01 | 0.99 |
|       | EMNW(3) | 1.42 | 1.18 | 1.06 | 0.99 |
4 Power study

In this section we compare the empirical powers of our tests with those of some common competitors, listed in [12] and [22]. The Monte Carlo study is done for small sample size $n = 20$ and the moderate sample size $n = 50$, with $N = 10000$ replicates, for level of significance $\alpha = 0.05$.

The powers are presented in Tables 3 and 4. The labels used are identical to the ones in [12] and [22].

| Alt. | W(1.4) | T(2) | HN | U | CH(0.5) | CH(1) | CH(1.5) | LF(2) | LF(4) | EW(1.5) |
|------|--------|------|----|---|---------|-------|---------|-------|-------|---------|
| EP   | 36     | 48   | 21 | 66| 63      | 15    | 84      | 28    | 42    | 45      |
| KS   | 35     | 46   | 24 | 72| 47      | 18    | 79      | 32    | 44    | 48      |
| CM   | 32     | 47   | 21 | 66| 61      | 16    | 83      | 30    | 43    | 47      |
| $\omega^2$ | 34   | 47   | 21 | 66| 61      | 14    | 79      | 28    | 41    | 43      |
| KS   | 28     | 40   | 18 | 52| 56      | 13    | 67      | 24    | 34    | 35      |
| KL   | 29     | 44   | 16 | 61| 77      | 11    | 76      | 23    | 34    | 37      |
| S    | 35     | 46   | 21 | 70| 63      | 15    | 84      | 29    | 42    | 46      |
| CO   | 37     | 54   | 19 | 50| 80      | 13    | 81      | 25    | 37    | 37      |
| $J_{n,1}^D$ | 42 | 64   | 20 | 45| 15      | 15    | 15      | 29    | 40    | 36      |
| $J_{n,2}^D$ | 47 | 66   | 25 | 59| 18      | 19    | 18      | 33    | 48    | 46      |
| $J_{n,3}^D$ | 48 | 64   | 28 | 70| 20      | 21    | 21      | 36    | 52    | 53      |
| $J_{n,1}^P$ | 49 | 65   | 29 | 73| 21      | 22    | 21      | 38    | 51    | 54      |
| $J_{n,2}^P$ | 50 | 64   | 31 | 77| 21      | 21    | 23      | 40    | 54    | 57      |
| $J_{n,3}^P$ | 48 | 62   | 32 | 79| 23      | 23    | 23      | 41    | 56    | 58      |
| $M_{n,0,5}$ | 46 | 66   | 25 | 64| 19      | 18    | 19      | 35    | 49    | 46      |
| $M_{n,1}$ | 49 | 66   | 28 | 72| 21      | 21    | 21      | 38    | 52    | 53      |
| $M_{n,2}$ | 50 | 67   | 31 | 75| 22      | 23    | 23      | 40    | 55    | 56      |
| $M_{n,5}$ | 48 | 62   | 32 | 80| 22      | 23    | 24      | 40    | 56    | 58      |

It can be noticed that in the majority of cases the tests based on V-empirical Laplace transforms are most powerful. Among them, those tests that are based on same characterization have more or less the same empirical powers, and the similar sensibility to the change of tuning parameter, for each considered alternative.

4.1 On a data-dependent choice of tuning parameter

The powers of proposed tests depend on the values of tuning parameter $a$, and the well-chosen value of $a$ would help us make the right decision. However, since the "right" value of $a$ is rather different for different alternatives, a general conclusion, which $a$ is most suitable in practice, can not be made. Hence, in what follows, we present an algorithm for data driven selection of tuning parameter, proposed initially by Allison and Santana [2]:

1. fix a grid of positive values of $a$, $(a_1, ..., a_k)$;
2. obtain a bootstrap sample $X_n^*$ from empirical distribution function of $X_n$;
3. determine the value of test statistic $M_{n,a_i}, i = 1, ..., k$, for the obtained sample;
Table 4: Percentage of rejected hypotheses for $n = 50$

| Alt. | $W(1.4)$ | $T(2)$ | $HN$ | $U$ | $CH(0.5)$ | $CH(1)$ | $CH(1.5)$ | $LF(2)$ | $LF(4)$ | $EW(1.5)$ |
|------|----------|--------|------|----|-----------|---------|---------|---------|---------|-----------|
| EP   | 80       | 91     | 54   | 98 | 94        | 38      | 100     | 69      | 87      | 90        |
| KS   | 71       | 86     | 50   | 99 | 90        | 36      | 100     | 65      | 82      | 88        |
| CM   | 77       | 90     | 53   | 99 | 94        | 37      | 100     | 69      | 87      | 90        |
| $\omega^2$ | 75 | 90 | 48   | 95 | 32        | 100     | 64      | 83      | 86      |
| KS   | 64       | 83     | 39   | 93 | 92        | 26      | 98      | 53      | 72      | 75        |
| KL   | 72       | 93     | 37   | 97 | 99        | 23      | 100     | 54      | 75      | 79        |
| S    | 79       | 90     | 54   | 99 | 94        | 38      | 100     | 69      | 87      | 90        |
| CO   | 82       | 96     | 45   | 91 | 99        | 30      | 100     | 60      | 80      | 78        |
| $j_{n,1}^D$ | 78 | 96 | 36   | 76 | 23        | 24      | 23      | 51      | 71      | 64        |
| $j_{n,2}^D$ | 83 | 97 | 46   | 90 | 31        | 30      | 31      | 62      | 83      | 79        |
| $j_{n,5}^D$ | 86 | 97 | 55   | 97 | 41        | 40      | 40      | 72      | 89      | 89        |
| $j_{n,1}^P$ | 85 | 96 | 54   | 97 | 38        | 38      | 38      | 70      | 87      | 87        |
| $j_{n,2}^P$ | 86 | 96 | 59   | 98 | 41        | 42      | 42      | 73      | 89      | 90        |
| $j_{n,5}^P$ | 86 | 96 | 63   | 99 | 46        | 46      | 45      | 77      | 91      | 93        |
| $M_{n,1}$ | 85 | 97 | 54   | 97 | 38        | 38      | 38      | 69      | 87      | 86        |
| $M_{n,2}$ | 86 | 96 | 57   | 98 | 41        | 41      | 41      | 73      | 89      | 90        |
| $M_{n,5}$ | 87 | 96 | 63   | 99 | 45        | 45      | 45      | 76      | 91      | 93        |

4. repeat steps 2 and 3 $B$ times and obtain series of values of test statistics for every $a$, $M_{j,a,i}^*, i = 1, \ldots, k, j = 1, \ldots, B$;

5. determine the empirical power of the test for every $a$, i.e.

$$\hat{P}_a = \frac{1}{B} \sum_{j=1}^{B} I\{M_{j,a_i} \geq \hat{C}_{n,a_i}(\alpha)\}, i = 1, \ldots, k;$$

6. for the next calculation $\hat{a} = \arg\max_{a \in \{a_1, \ldots, a_k\}} \hat{P}_a$ will be used.

The critical value $\hat{C}_{n,\hat{a}}$ is determined using the Monte Carlo procedure with $N_1$ replicates. Then, the empirical power of the test is determined based on the new sample from the alternative distribution

$$p = \frac{1}{N_1} \sum_{i=1}^{N_1} I\{M_{n,\hat{a}} \geq \hat{C}_{n,\hat{a}}(\alpha)\}.$$

The previously described procedure is being repeated $n$ times and the average value is taken as the estimated power:

$$\hat{P} = \frac{1}{N} \sum_{i=1}^{N} p_i.$$ 

The results are presented in Table 5 and 6. The numbers in the parentheses represent the percentage of times that each value of $a$ equaled the estimated optimal one. It is important to note that this bootstrap powers are comparable to the maximum achievable power for the tests calculated over a grid of values of the tuning parameter.
Table 5: Percentage of rejected samples for different value of $a$, $n = 20$, $\alpha = 0.05$

|       | 0.5 | 1   | 2   | 5   | $\hat{a}$ |
|-------|-----|-----|-----|-----|---------|
| $W(1.4)$ | 46 (50) | 49 (12) | 50 (15) | 48 (23) | 48 |
| $\Gamma(2)$ | 66 (63) | 65 (12) | 65 (10) | 63 (15) | 65 |
| $HN$ | 25 (35) | 28 (14) | 30 (17) | 32 (34) | 29 |
| $U$ | 64 (20) | 72 (9) | 75 (21) | 80 (50) | 75 |
| $CH(0.5)$ | 19 (37) | 21 (15) | 22 (17) | 22 (31) | 21 |
| $CH(1)$ | 18 (35) | 21 (15) | 23 (16) | 23 (34) | 21 |
| $CH(1.5)$ | 19 (35) | 20 (11) | 20 (20) | 24 (34) | 21 |
| $LF(2)$ | 35 (33) | 37 (12) | 38 (20) | 41 (35) | 38 |
| $LF(4)$ | 49 (35) | 53 (14) | 54 (16) | 54 (35) | 52 |
| $EW(1.5)$ | 46 (24) | 53 (12) | 56 (20) | 58 (44) | 54 |

Table 6: Percentage of rejected samples for different value of $a$, $n = 50$, $\alpha = 0.05$

|       | 0.5 | 1   | 2   | 5   | $\hat{a}$ |
|-------|-----|-----|-----|-----|---------|
| $W(1.4)$ | 84 (43) | 86(19) | 86(16) | 87(22) | 85 |
| $\Gamma(2)$ | 97 (68) | 97(15) | 96(11) | 95(6) | 97 |
| $HN$ | 48(21) | 53(13) | 57(23) | 62(43) | 57 |
| $U$ | 95(31) | 97(12) | 98(20) | 99(37) | 98 |
| $CH(0.5)$ | 34(19) | 37(11) | 41(20) | 44(50) | 41 |
| $CH(1)$ | 33(18) | 37(13) | 41(18) | 46(51) | 41 |
| $CH(1.5)$ | 33(18) | 37(13) | 42(19) | 44(50) | 41 |
| $LF(2)$ | 65(20) | 69(12) | 74(24) | 76(44) | 72 |
| $LF(4)$ | 83(25) | 86(16) | 89(20) | 91(39) | 88 |
| $EW(1.5)$ | 81(17) | 87(13) | 89(22) | 93(48) | 89 |
5 Real data examples

In this section we apply our tests to two real data examples.

The first data set represents inter-occurrence times of the British scheduled data, measured in number of days and listed in the order of their occurrence in time (see [31]):

\[
\begin{array}{cccccccccccccccccc}
20 & 106 & 14 & 78 & 94 & 20 & 21 & 136 & 56 & 232 & 89 & 33 & 181 & 424 & 14 & 430 & 205 & 117 & 253 & 86 & 260 & 213 & 58 & 276 & 263 & 246 & 341 & 1105 & 50 & 136.
\end{array}
\]

Applying the algorithm for data-driven tuning parameter we get \( \hat{a} = 1 \). The value of the test statistic \( M_{31,1} \) is \( 6.07 \times 10^{-4} \), and the corresponding \( p \)-value is 0.49, so we cannot reject exponentiality in this case.

The second data set represents failure times for right rear breaks on D9G-66A Caterpillar tractors (see [5]):

\[
\begin{array}{ccccccccccccccccccccccc}
56 & 83 & 104 & 116 & 244 & 305 & 429 & 452 & 453 & 503 & 552 & 614 & 661 & 673 & 683 & 685 & 753 & 763 & 806 & 834 & 838 & 862 & 897 & 904 & 981 & 1007 & 1008 & 1049 & 1060 & 1107 & 1125 & 1141 & 1153 & 1154 & 1193 & 1201 & 1253 & 1313 & 1329 & 1347 & 1454 & 1464 & 1490 & 1491 & 1532 & 1549 & 1568 & 1574 & 1586 & 1599 & 1608 & 1723 & 1769 & 1795 & 1927 & 1957 & 1975 & 2005 & 2010 & 2016 & 2022 & 2037 & 2065 & 2096 & 2139 & 2150 & 2156 & 2160 & 2190 & 2210 & 2220 & 2248 & 2285 & 2325 & 2353 & 2351 & 2337 & 2364 & 2546 & 2569 & 2584 & 2624 & 2675 & 2701 & 2755 & 2877 & 2879 & 2922 & 2986 & 3092 & 3160 & 3185 & 3191 & 3439 & 3617 & 3685 & 3756 & 3826 & 3995 & 4007 & 4159 & 4300 & 4487 & 5074 & 5579 & 5623 & 6869 & 7739.
\end{array}
\]

Here we get \( \hat{a} = 0.5 \). The value of the test statistic \( M_{107,0.5} \) is 0.0239, and the corresponding \( p \)-value is less than 0.0001, so our test rejects the null exponentiality hypothesis.

6 Conclusion

In this paper we propose new consistent scale-free exponentiality tests based on Puri-Rubin characterization. The proposed tests are shown to be very efficient in Bahadur sense. Moreover, in small sample case, the tests have reasonable to high empirical powers. They also outperform many recent competitor tests in terms of both efficiency and power, which makes them attractive for use in practice.

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