Edge Excitations of the $\nu = \frac{2}{3}$ Spin-Singlet Quantum Hall State

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The spectrum of edge excitations is derived for the spin-unpolarized $\nu = 2$ and $\nu = 2/3$ FQHE. Numerical diagonalization of a system of six electrons on a disc confirms that the edge $\nu = 2/3$ spin-singlet FQHE state consists of oppositely directed spin and charge branches on the same physical edge. The highly correlated $\nu = 2/3$ singlet edge is shown to have the same spin branch as the $\nu = 2$ singlet edge, providing evidence that the same $SU(2)_{k=1}$ Kac-Moody algebra describes all unmixed spin branches. The spin and charge branches of the singlet state at $\nu = 2/3$ are less coupled than the two branches of the spin-polarized state at the same filling factor, suggesting that the conductance along an edge may increase sharply across the polarized-unpolarized transition.

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I. INTRODUCTION

The possibility of spin-singlet ground states in the fractional quantum Hall effect (FQHE) has received considerable experimental and theoretical attention. The ground state at filling factor $\nu = 2/3$ is found experimentally and numerically to be spin-unpolarized for low magnetic field but spin-polarized for high field. Halperin first suggested this possibility on the grounds that the ratio of Zeeman energy to cyclotron energy is $\sim 1$ for GaAs in a weak magnetic field, so that electron-electron interactions may override the tendency of electron spins to align with the magnetic field while still being too weak to push electrons into higher Landau levels. The spin-polarized state at $\nu = 2/3$ is now well understood as the particle-hole transform of the Laughlin $\nu = 1/3$ state, and its edge has been shown numerically to be a combination of two spatially separated sub-edges, an inner edge where the density of holes drops from 1/3 to 0, and an outer edge where the density of electrons drops from 1 to 0.

This paper studies singlet FQHE edges and in particular the edge of the $\nu = 2/3$ unpolarized state, which also contains two oppositely directed branches but does not break down into separate edges. The $\nu = 2/3$ unpolarized state carries oppositely chiral spin and charge branches of excitations on the same physical edge. Its spin excitations are described by the $SU(2)_{k=1}$ Kac-Moody algebra and thus have a different structure from previously studied $U(1)_{k=1}$ charge excitations. The $\nu = 2/3$ edge inherits its structure from the properties of the bulk FQHE liquid, and part of this structure is shown to arise as a nontrivial prediction of the Chern-Simons effective theory of the FQHE.

The excitation spectra of FQHE edges are of interest for a number of reasons. Edge excitations remain gapless in the thermodynamic limit, unlike excitations in the bulk fluid, and thus may determine the specific heat. They can provide a nonzero conductivity between two contacts connected by an edge. Finally, FQHE edges are more accessible experimentally than the bulk liquids and may provide the only practical probe of the rich structure of FQHE liquids.

The same $SU(2)$ algebra is expected to apply equally well to all spin branches which are not mixed with other modes. The spin and charge branches of the $\nu = 2/3$ edge are found numerically to be independent to 1 part in $10^4$. Hence the spectrum of the spin branch should be given by the same algebra describing spin excitations at $\nu = 2$. We can derive the pattern of excitations for a simple spin-singlet system, the $\nu = 2$ state, and expect the same pattern to describe the spin branch of the $\nu = 2/3$ highly correlated system. (How the $SU(2)$ algebra arises from the Chern-Simons bulk theory of the $\nu = 2/3$ state is discussed briefly below and in more detail in McDonald and Haldane.) The stability of the $SU(2)$ Kac-Moody algebra, not present in the $U(1)$ case, is mathematically a consequence of the additional non-Abelian structure of the $SU(2)$ algebra. In physical terms, strong electron-electron interactions might be expected to modify the effective charge of an electron but not its spin.

The $\nu = 2/3$ unpolarized edge has previously attracted theoretical interest as the simplest spin-singlet FQHE edge, but often at a rather high level of abstraction. We carried out a numerical diagonalization of a system of six interacting electrons at $\nu = 2/3$ and observed the predicted pattern of excitations. The disc geometry was used as the simplest geometry possessing an edge. The numerical results indirectly confirm the topological properties ($K$-matrix and shift) of the wavefunction proposed by Wu, Dev, and Jain. The $\nu = 2/3$ edge is seen to consist of two oppositely directed branches, verifying a nontrivial prediction of the chiral Luttinger liquid theory of FQHE edges. The same prediction is shown to arise from the quantization of total angular momentum of electrons on a disc. The differences at the edge between polarized and unpolarized states at $\nu = 2/3$ may yield a useful experimental probe of the polarized-unpolarized transition.
II. THEORY OF THE SINGLET EDGE

The edge excitations of FQHE liquids have a rich topological structure which is closely related to the Chern-Simons effective theory of the FQHE. The Hilbert space of excitations of the \( \nu = 2/3 \) unpolarized state is a direct product of oppositely directed spin and charge sectors.\(^4\) The \( U(1) \) Kac-Moody algebra of the charge sector and its associated spectrum are now well understood in the FQHE context, but the \( SU(2) \) algebra of the spin sector gives a more complicated excitation spectrum related to a different conformal field theory. For concreteness, we examine the \( \nu = 2 \) and \( \nu = 2/3 \) singlet edges and obtain testable predictions about their excitation spectra.

The spectrum for a single branch of edge excitations on a droplet of \( N \) electrons has been derived from quantization of hydrodynamic perturbations on the edge of a charged fluid\(^2\) and from the polynomial representation of lowest-Landau-level wavefunctions.\(^3\) This is the spectrum of excitations for the Laughlin-type states at \( \nu = 1/m, m \) odd. The result is

\[
E = \frac{Mv}{L} = \frac{\hbar v}{L} \sum_{m=0}^{\infty} m e^{\frac{i}{4} c_m c_m^\dagger} \tag{1}
\]

where \( v \) is the branch velocity, \( L = 2\pi R \) the droplet circumference, \( M \) total angular momentum, and the \( c_m \) are independent bosonic modes of angular momentum \( m\hbar \).

The simplest singlet state is the \( \nu = 2 \) state consisting of filled spin-up and spin-down lowest Landau levels. The excitations of this state can be explicitly calculated within the lowest Landau level using the rotationally symmetric single-electron wavefunctions

\[
\psi_m = \frac{e^{i\hbar m \cdot p e^{-r^2/4a^2}}}{\ell^{m+1} 2^{m+1} \pi^m} = \frac{e^{-|z|^2/4a^2}}{\ell^{m+1} 2^{m+1} \pi^m} \tag{2}
\]

The ground state at \( \nu = 2 \) of \( N = 2k \) electrons confined by some radial potential has all orbitals \( m = 0, \ldots, k-1 \) filled with both up and down electrons. This state has total angular momentum \( M_0 = k(k-1) \) in units with \( \hbar = 1 \). Its excitations within the lowest Landau level consist of (i) some combination of excitations of the up spins, with corresponding creation and annihilation operators \( b_i, b_i^\dagger \), (ii) some combination of excitations of the down spins, denoted by \( c_i, c_i^\dagger \), and (iii) some (positive or negative) number \( S_z \) of up spins being converted to down spins. Clearly \( S_z \) is the z-component of spin of the resulting state, and all excited states will be eigenstates of \( S_z \) because the Hamiltonian is spin-symmetric. The total angular momentum of an excited state is, using \( \hbar \),

\[
M = M_0 + S_z^2 + \sum_i lb_i^\dagger b_i + \sum_m mc_m^\dagger c_m \tag{3}
\]

The excitations must still be classified into spin and charge branches, which may have different velocities. Some excitations are purely spin excitations, some are purely charge excitations, and some are mixed. The pure charge excitations are the hydrodynamic modes of a charged droplet and thus have the spectrum \( [0] \). Hence one of the two sums over bosonic modes in \( [0] \) corresponds to the charge branch. The other sum must correspond to the spin branch, so that the total edge Hamiltonian is

\[
H = \frac{v_s}{L} (S_z^2 + \sum_i lb_i^\dagger b_i) + \frac{v_e}{L} \sum_m mc_m^\dagger c_m \tag{4}
\]

Here \( v_s \) and \( v_e \) are the (positive) velocities of the spin and charge branches. Pure spin (charge) excitations are given by states which are annihilated by all \( c_m \) (\( b_m \)). The resulting structure of pure spin excitations is given in Table\(^3\). If the system is exactly spin-symmetric then the excited states are unified into exact \( SU(2) \) multiplets.

The spin branch in \( [0] \) is closely related to the Virasoro operator \( L_0 \) of a conformal field theory (CFT) of free bosons in \( 1+1 \) dimensions. The spin branch corresponds to a different CFT than the charged excitations of a \( \nu = 1/m \) state. The charged excitations of a single charge branch with \( \Delta N \geq 0 \) have angular momentum

\[
L = \frac{(\Delta N)^2}{2} + \sum_{m=0}^{\infty} c_m^\dagger c_m \tag{5}
\]

Free bosonic CFT’s in \( 1+1 \) dimensions contain in the generator \( L_0 \) a term proportional to \( N^2 \) with coefficient depending on the boson radius. The single-branch spectrum \( [0] \) is given by the CFT with boson radius of compactification \( 1 \), while the boson radius is \( 1/2 \) for the spin branch if \( S_z \) is taken to correspond to winding number \( N \).

The spectrum \( [0] \) describes the edge excitations of the \( \nu = 2 \) polarized state, which has two branches moving in the same direction. The charge and spin branches have oppositely directed velocities for \( \nu = 2/3 \), however. This is a nontrivial prediction of the Chern-Simons effective theory of the FQHE\(^2\), which yields a “chiral Luttinger liquid” theory on the edge.\(^4\) In this theory the \( \nu = 2/3 \) unpolarized state is associated with the K-matrix \( (1/2) \) (up to \( SL(2,\mathbb{Z}) \) equivalence). The edge consists of two oppositely directed branches because this matrix has one positive and one negative eigenvalue. The \( \nu = 2/3 \) unpolarized edge has the same charge and spin branches as derived above for independent electrons; the sole difference is that the branches are oppositely directed. The charge branch, corresponding to hydrodynamic perturbations, has excitations of positive angular momentum, and the spin branch has excitations of negative angular momentum.

The fact that the edge branches propagate in opposite directions is not clearly evident in the explicit wavefunction of the ground state\(^3\), nor can it be derived from a hydrodynamic approach because the \( \nu = 2 \) and \( \nu = 2/3 \) state would seem to have the same hydrodynamics. The
reversed direction of the spin branch is observed in numerical studies, confirming the prediction of the effective theory of the FQHE and reflecting the deep connection between the bulk FQHE liquid and edge excitations. The Halperin-Laughlin multicomponent wavefunctions all have positive definite $K$-matrices and hence cannot describe the observed $\nu = 2/3$ state. The $\nu = 2/3$ FQHE edge is particularly significant for theory because it is described by a non-Fermi-liquid theory at constant $N$. The $U(1)$ Kac-Moody algebra which describes a single edge branch exhibits a non-Fermi-liquid structure only with variable $N$. Fig. 1 shows ideal spectra for the $\nu = 2/3$ (opposite velocities) and $\nu = 2$ (parallel velocities) edges, where the velocities have been drawn with magnitudes as is generically the case. The states have been combined into exact $SU(2)$ multiplets.

Some insight into the nature of the spin branch in the $\nu = 2$ electrons can be gained by considering an ideal system of electrons which interact only through Pauli exclusion. Suppose that there is one orbital at each value of the angular momentum $m$, which can be filled by at most one spin-up electron and one spin-down electron, and that these orbitals have energies $E(m)$. If the ground state is a singlet, its energy must be increased by the operation of transferring one spin-up electron to spin-down (and vice versa). Since energy is a smooth function of occupancies in the thermodynamic limit, there will be a term in the Hamiltonian containing $(N_\uparrow - N_\downarrow)^2 = S_z^2$. Excitations which do not change $S_z$ will have energies (and hence velocity) proportional to $dE/dm$. The prediction derived above for the spectrum amounts to a fixed ratio between the $S_z$ term and the constant-spin excitation term. The ratio is the same value as obtained for an exactly linear dispersion relation $E(m)$, i.e., the same as the requirement used above that the energies of edge excitations be linear in the angular momentum.

III. CONNECTION TO THE CHIRAL LUTTINGER LIQUID THEORY

These results on the structure of singlet edges can be placed in the context of field theories of the FQHE edge. The simplest edge is the $\nu = 1/m$ edge, which possesses only a single branch of excitations. This edge is described as a “chiral Luttinger liquid” with $1 + 1$-dimensional action

$$S = -\frac{\hbar}{4\pi} \int \partial_x \phi \partial_t \phi + v(\partial_x \phi)^2 dx. \quad (6)$$

The edge state theory predicts that the charge-neutral edge excitations of an abelian quantum Hall state containing $N$ species of quasiparticles form a representation of $N U(1)$ Kac-Moody algebras. This result can be obtained from quantization of the above edge action for $\nu = 1/m$ states or from the multispecies chiral Luttinger liquid picture of more complicated states. The condition for a general multispecies edge to admit an $SU(2)$ symmetry can also be derived from the theory. For a two-quasiparticle state such as the $\nu = 2$ or $\nu = 2/3$ unpolarized states, there are two families of edge operators satisfying the $U(1)$ Kac-Moody algebra commutation relations:

$$[\rho_{k'}, \rho_k] = v_1 k \delta_{k+k'} \quad (7)$$
$$[\tilde{\rho}_{k'}, \tilde{\rho}_k] = v_2 k \delta_{k+k'}. \quad (8)$$

These can be combined to give raising and lowering operators for two families of harmonic oscillators. The angular momentum and Hamiltonian of the system are

$$L = L_0 + \sum_{k=1}^{\infty} \frac{v_1}{|v_1|} \rho_{k} \rho_{-k} + \frac{v_2}{|v_2|} \tilde{\rho}_{k} \tilde{\rho}_{-k} \quad (9)$$
$$H = \sum_{k=1}^{\infty} \frac{|v_1|}{L} \rho_{k} \rho_{-k} + \frac{|v_2|}{L} \tilde{\rho}_{k} \tilde{\rho}_{-k}. \quad (10)$$

At this point we must add operators to the system reflecting its $SU(2)$ symmetry. These operators cannot be implicit in the $K$-matrix describing the bulk state because the long-distance behavior reflected in the effective theory does not determine the symmetries. As a concrete example, the $\nu = 2/3$ polarized and unpolarized states have the same $K$-matrix but different symmetry properties.

The spin operators commute with the $U(1)$ excitation operators and appear in the angular momentum and Hamiltonian only as terms proportional to $S_z^2$. For a discrete example, the $\nu = 2/3$ polarized and unpolarized states have the same $S_z$ term.

$$[S_z, \rho_k] = [S_z, \tilde{\rho}_k] = [S_z, \tilde{\rho}_k] = 0 \quad (11)$$
$$[S_z, \rho_k] = [S_y, \rho_k] = [S_z, \rho_k] = 0 \quad (12)$$

$$L = L_0 + \sum_{k=1}^{\infty} \frac{v_1}{|v_1|} (S_z^2 + \rho_k \rho_{-k}) + \frac{v_2}{|v_2|} \tilde{\rho}_{k} \tilde{\rho}_{-k} \quad (13)$$
$$H = \sum_{k=1}^{\infty} \frac{|v_1|}{L} (S_z^2 + \rho_k \rho_{-k}) + \frac{|v_2|}{L} \tilde{\rho}_{k} \tilde{\rho}_{-k}. \quad (14)$$

The edge operators in the chiral Luttinger liquid picture of a singlet state form a representation of the $SU(2)$ Kac-Moody algebra. At constant $N$ but variable spin, the excitation spectrum differs from the ordinary $U(1)$ spectrum because of the $S_z^2$ term (Table I). The states in Table I must be reunified into $SU(2)$ multiplets because of the $SU(2)$ spin symmetry of the system. Thus one of the $\Delta M = 2 \bar{S}_z = 0$ states is exactly degenerate with the $S_z = \pm 1$ states, for example.
Now we examine the numerical spectrum obtained by diagonalizing the Hamiltonian for a disc of six interacting electrons in the lowest Landau level:

$$H = \frac{1}{2} \sum_{i, j, k, l} V_{ij} a_i^\dagger a_j a_k a_l + \sum_i U_i a_i a_i^\dagger.$$  \hspace{1cm} (15)

The electrons in our simulation are confined by a harmonic potential $U(r) = \alpha r^2$ where $\alpha$ is chosen to give central density $\nu/2\pi\ell^2$, where $\ell = \sqrt{\hbar c/eB}$ is the magnetic length. Any radial potential retains the conservation of total angular momentum, so that each angular momentum subspace can be diagonalized independently.

The qualitative nature of the edge excitations is independent of the choice of $U(r)$ as long as the edge separates $\nu = 2/3$ from $\nu = 0$ with no intermediate Hall liquid.

We study the $\nu = 2/3$ state rather than the $\nu = 2$ unpolarized state because all excitations of the $\nu = 2$ state are edge excitations, because all orbitals inside the edge are fully occupied for states not too excited from the ground state. In the same way all low-lying polarized excitations of the $\nu = 1$ state are edge excitations. The number of excitations at given angular momentum can be exactly calculated for the $\nu = 2$ unpolarized state (as done above) and the $\nu = 1$ polarized state. Numerical calculation is not especially illuminating for these states because the numbers of excitations of given angular momentum and spin are fixed simply by state counting; the only information gained by a numerical calculation is the velocity of the branches, which is not expected to be universal anyway. Calculations of $\nu = 1/3, 2/3, 2/5$ etc. are useful precisely because not every excited state is to be interpreted as an edge state. If the low-lying excitations, which are much smaller in number than the set of all excitations, follow the theoretically predicted pattern for one of these states, it is strong confirmation of the theory.

The effect of the interaction potential $V$ in the lowest Landau level is completely determined by the pseudopotentials:

$$V_m = \int_0^\infty dr r^{2m+3} \frac{\exp(-r^2/4\ell^2)}{\int_0^\infty dr r^{2m+1} \exp(-r^2/4\ell^2)}.$$  \hspace{1cm} (16)

The highly correlated structure of FQHE wavefunctions is typically destroyed by strong long-range pseudopotentials. This is manifested numerically by the disappearance of the bulk excitation gap characteristic of the FQHE. Such an excitation gap is useful for edge studies because (gapless) edge excitations are clearly distinguishable in the spectrum from bulk excitations. The low-lying excited states can be verified to be edge excitations by checking that they differ from the ground state only near the edge.

The rotational symmetry of the disc geometry makes convenient the single-electron basis \[.\] The basis states of the diagonalization are Slater determinants formed from these orbitals. The Hilbert space is made finite by neglecting those $\psi_m$ which are negligible in the vicinity of the droplet of electrons.

The observed pattern of the spectrum near the ground state is remarkably similar for a variety of short-range interactions. Long-range interactions such as the unscreened Coulomb interaction $V_m = \Gamma((n + 1)/2)/2\Gamma(n/2 + 1)$ destroy the Kac-Moody $SU(2)_k = 1$ pattern of edge excitations but also destroy the FQHE. None of the interactions tested destroys the edge excitations without eliminating the excitation gap characteristic of the FQHE. Fig. 2 shows the lowest-energy states for the screened Coulomb potential

$$V(r) = \frac{e^2 \exp(-r/\ell)}{r^2}.$$  \hspace{1cm} (17)

The pattern begins to disappear when the screening length in \[.] is increased to $2\ell$.

Fig. 3 shows the excitations of negative angular momentum relative to the ground state. Single-particle density plots confirm that the low-lying states differ from the ground state only in the vicinity of the edge. Each angular momentum subspace contains of order $10^3$ states, of which only the lowest few are shown. There is a well-separated ground state at $M = 24 = \frac{3}{2}N\phi$. This is the ground state angular momentum for the wavefunction of Wu et al., which in the disc geometry is

$$\Psi = e^{-\sum_i \frac{r^2}{4\ell^2}} (\prod_{i<j} (\partial z_i - \partial z_j)^{\delta \sigma_i \sigma_j} e^{\frac{i}{\sqrt{2}} \text{sgn}(\sigma_i - \sigma_j)}) \times (\prod_{i<j} (z_i - z_j)^2).$$  \hspace{1cm} (18)

Hence $N\phi = \frac{3N}{2} - 1$ is the number of flux quanta, confirming that 1 is the shift of the singlet state on the sphere.

The Hamiltonian in the absence of a Zeeman term contains no spin terms, so that the calculated spectrum contains exactly degenerate $SU(2)$ multiplets. For example, each $S^2 = 1$ energy in Fig. 3 represents three states with $S_z = -1, 0, 1$. The basis for the diagonalization consists of all six-particle Slater determinants formed from spin-up and spin-down versions of the $\psi_m$ in [3]. The basis states are eigenstates of $S_z$ but not $S^2$, but $S^2$ is easily calculated for the energy eigenstates using $S^2 = S_+ S_- + S_- S_+ + S_z^2$.

The calculated spectrum (Fig. 3) shows some splitting in the degeneracies of the pattern [3], presumably caused by finite-size effects. But the features of the numerical spectrum are well described by the pure spin branch spectrum given in Table [3]. The pure spin excitations up to $M = 20$ are clearly separated from the clutter of bulk excitations. $M = 20 + \Delta M = 4$ is the first subspace found to have a low-lying state with $S^2 = 2$, matching the theoretical prediction. This pattern is strong confirmation for the theory derived above partly because the naive addition of excitations gives a different pattern: for example, the sum of two $\Delta M = 1$ excitations with
obtained above for the results from this calculation match the numerical results of the topological shift of the bulk wavefunction. The relation between the direction of the spin branch at the edge and the numerical result applied to singlet states yields a relation between the spin branch on the edge and the spin branch in the bulk due to the quantum Hall effect. Since the perturbation $V$ from finite-size effects is essentially arbitrary, we can expect the unpolarized edge to have reduced backscattering (relative to the polarized edge) for a generic perturbation. This should be manifested in real systems as a decrease in the finite two-point resistance along an edge as the polarized-unpolarized transition occurs. The reduced backscattering can be understood simply: almost all excitations in the spin sector have non-zero $S^2$, so that spin-independent perturbations will cause no backscattering.

V. IMPLICATIONS OF QUANTIZATION OF TOTAL ANGULAR MOMENTUM

The total angular momentum of a system of $N$ electrons is of course quantized in units of $\hbar$. This simple statement applied to singlet states yields a relation between the direction of the spin branch at the edge and the topological shift of the bulk wavefunction. The results from this calculation match the numerical results obtained above for the $\nu = 2/3$ state and also predict an excess energy for systems containing an odd number of electrons.

The total angular momentum of the ground state of $N$ electrons at filling factor $\nu$ is $M = \frac{1}{2}N\nu \phi$, where

$$N\phi = \frac{N}{\nu} - S$$

is the number of flux quanta in the state and $S$ is the “shift” of the wavefunction on the sphere. The angular momentum of the ground state is thus

$$L_0 = \frac{N^2 - \nu NS}{2\nu} \pm S_z^2$$

with the positive sign corresponding to a spin branch of positive velocity, i.e., of the same sign as the charge branch, and the negative sign to an oppositely directed branch.

The states with $\nu = 2/m$ are the simplest singlet states. For $N$ even, $S_z = 0$ in the singlet ground state and $L_0$ is integral since $(N^2m/4) - (NS/2)$ is integral for even $N$ for any integers $m$ and $S$. For $N$ odd the constraint that $L$ be integral requires that $(m/4) - (S/2) \pm S_z^2$ be integral. The lowest-energy state for odd $N$ has $S_z^2 = 1/4$, so there are four possibilities:

I. $S$ odd, $m = 4k + 1$, spin velocity positive
II. $S$ odd, $m = 4k + 3$, spin velocity negative
III. $S$ even, $m = 4k + 1$, spin velocity positive
IV. $S$ even, $m = 4k + 3$, spin velocity negative.

The singlet $\nu = 2$ state belongs to category I and the singlet $\nu = 2/3$ state to category II, since both of these states have shift $S = 1$. If the $\nu = 2/5$ unpolarized state has $K$ and $S$ equal to their values in the hierarchy $\nu = 2/5$ polarized state, as happens at $\nu = 2/3$, then a positively directed spin branch exists at $\nu = 2/5$.

Another consequence is that states with an odd number of electrons should be displaced in energy from states with an even number of electrons. The term $\hbar v_s S_z^2/L$ in the excitation Hamiltonian (4) should contribute an energy $\hbar v_s/4L$ to the doubly degenerate $S_z = \pm 1/2$ ground multiplet of odd numbers of electrons. Thus the ground state energy $E(N)$ of $N$ electrons should satisfy

$$E(2k - 1) + E(2k + 1) \frac{\hbar v_s}{2L} - E(2k) = \frac{\hbar v_s}{4L}$$

while the right side of this equation would be 0 for an ordinary system with a well-defined chemical potential, i.e., $\mu = E'(N) \gg E^{(k)}(N), k > 1$. This excess energy may also receive a contribution from the $\Delta N = 1$ charged mode at the edge, since the ground state of an odd number of particles is a $\Delta N = 1$ excitation of an even-number ground state. The energy of this charge excitation can be estimated by writing $N = N_0 + \Delta N$ and interpreting the resulting term

$$L_{\text{charge}} = \frac{(\Delta N)^2}{2\nu}$$

as resulting from the charged mode. The interpretation is correct in the $\nu = 2$ case for a linearized Fermi surface, but it is unclear to what degree the energy of the charged mode will be renormalized by interactions. This predicts a total energy excess of

$$\Delta E = \frac{\hbar v_c}{2\nu} + \frac{\hbar v_s}{4L}.$$ 

We have attempted to see this effect numerically, but for small systems nonlinearities in $E(N)$ from finite size swamp the desired effect.

VI. CONCLUSIONS

The spin branch of a singlet FQHE edge has a complicated structure which is difficult to explain from a purely hydrodynamical point of view. The $SU(2)$ symmetry of the singlet FQHE is likely to be the only exact symmetry of accessible FQHE states, so the spin-branch spectrum
described here and the previously derived charge-branch spectrum may be the only excitation patterns observable in experiments.

The numerical results obtained are strong evidence for the \( SU(2) \otimes U(1) \) description of the \( \nu = 2/3 \) singlet edge. The validity of the spectrum obtained from free fermions for the highly correlated \( \nu = 2/3 \) state suggests that the same pattern of singlet spin excitations describes all unmixed spin branches, including the \( \nu = 2 \) spin branch. The \( \nu = 2/3 \) FQHE is experimentally quite robust, and theoretical predictions about the edge excitations may be directly observable.

Edges of FQHE states inherit their structure from the bulk electron liquid. Their transformation properties under gauge transformations of the Chern-Simons effective theory of the bulk give rise to quantitative connections between the edge spectra and bulk properties. The \( \nu = 2/3 \) unpolarized state has an indefinite \( K \)-matrix in the effective theory and hence has edge modes of both chiralities, as seen numerically. One consequence of this is that the \( \nu = 2/3 \) singlet state shows non-Fermi-liquid behavior even at constant \( N \), unlike \( \nu = 1/m \) edges.

Some predictions of the effective theory arise simply from the requirement of integral angular momentum, which gives a relation between filling factor, shift, and edge chirality satisfied by the \( \nu = 2 \) and \( \nu = 2/3 \) states and possibly by all singlet states. Another result is a prediction that the ground state energy of an odd number of electrons is raised by an energy \( \hbar v_s/4 \) relative to the energy of an even number of electrons, but we have not yet been able to verify this conjecture numerically.

The description of the excitation spectrum as a direct product is more accurate in the unpolarized edge than in the polarized edge, suggesting that the unpolarized edge may have decreased backscattering and increased two-probe conductance along an edge. The measurement of edge conductance may reveal interesting properties of the \( \nu = 2/3 \) polarized-unpolarized transition.

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TABLE I. Number of edge excitations for given values of \( S \) and \( \Delta_M \). Excitations with the same \( \Delta_M \) belong to the same Kac-Moody multiplet and are expected to be approximately degenerate. The states will form exactly degenerate ordinary \( SU(2) \) multiplets because of the unbroken spin symmetry of the Hamiltonian.

| \( S \) | \( \Delta_M \) | \( \nu \) |
| --- | --- | --- |
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 3 | 3 |
| 4 | 4 | 4 |
| 5 | 5 | 5 |
| 6 | 6 | 6 |
| 7 | 7 | 7 |
| 8 | 8 | 8 |

---

FIG. 1. Predicted spectrum of edge excitations for two-component edge states possessing an \( SU(2) \) symmetry. The states have been combined into \( SU(2) \) multiplets, so that \( (0,1) \) denotes a fourfold degeneracy of one \( S^z = 0 \) and three \( S^z = 2 \hbar \) states.

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FIG. 2. Energy levels from numerical diagonalization of six interacting electrons in two dimensions. The electrons interact through a screened Coulomb potential and are confined by a (radially symmetric) harmonic potential.
| Number of modes for given $S_z, \Delta M$ | | | | |
|---|---|---|---|---|
| $9\hbar$ | 26 | 20 | 7 | 1 |
| $8\hbar$ | 20 | 14 | 5 | |
| $7\hbar$ | 14 | 11 | 3 | |
| $6\hbar$ | 11 | 7 | 2 | |
| $5\hbar$ | 7 | 5 | 1 | |
| $4\hbar$ | 5 | 3 | 1 | |
| $3\hbar$ | 3 | 2 | | |
| $2\hbar$ | 2 | 1 | | |
| $\hbar$ | 1 | 1 | | |
| 0 (ground) | | | | 1 |

| $\Delta M$ | $S_z = 0$ | $S_z = \pm 1$ | $S_z = \pm 2$ | $S_z = \pm 3$ | |

Table I

Table II

| $M$ | $S^2$ |
|---|---|
| 19 | 0,0,1,1,2,2 |
| 20 | 0,0,1,1,2 |
| 21 | 0,1,1 |
| 22 | 0,1 |
| 23 | 1 |
| 24 | 0 |

$M \equiv \hbar$
Figure 1

\[
\begin{array}{cccc}
-3 & -2 & -1 & 0 \\
\hline
0,0,1,1 & 0,0,0,1,1 & 0,0,1,1,1 & 0,0,0,1,1,1 \\
0,0,1,1 & 0,1 & 0 & 0 \\
0 & 0,1,1 & 1,1 & 0,0,0 \\
0 & 0,1 & 1 & 0,0 \\
0,0 & 0,0 & 0,0 & 0 \\
M
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\hline
-0,0,0 & -0,0,1,1 & -0,0,1,1,1 & -0,0,1,1,1,1 \\
-0,0,1,1,1 & -0,0,1,1,1,1 & -0,0,1,1,1,1,1 & -0,0,1,1,1,1,1,1 \\
-0,0,1,1,1,1 & -0,0,1,1,1,1,1 & -0,0,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1 \\
-0,0,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1,1 \\
-0,0,1,1,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
-0,0,1,1,1,1,1,1,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1 & -0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
\end{array}
\]

\[v_s < 0, v_c > 0\]  \[v_s > 0, v_c > 0\]
