Lifts of Poisson structures to Weil bundles

VADIM V. SHURYGIN, JR

ABSTRACT. In the present paper, we study complete and vertical lifts of tensor fields from a smooth manifold $M$ to its Weil bundle $T^A M$ defined by a Frobenius Weil algebra $A$. For a Poisson manifold $(M, \omega)$, we show that the complete lift $\omega^C$ and the vertical lift $\omega^V$ of the Poisson tensor $\omega$ are Poisson tensors on $T^A M$ and establish their properties. We prove that the complete and the vertical lifts induce homomorphisms of the Poisson cohomology spaces. We compute the modular classes of the lifts of Poisson structures.

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1 Introduction

Complete and vertical lifts of Poisson structures from a smooth manifold $M$ to its tangent bundle $TM$ were studied in several papers, see, e.g., J. Grabowski and P. Urbański [8, 9], G. Mitric and I. Vaisman [20, 45]. In the present paper we discuss the generalization of these lifts to the case of a Weil bundle $T^A M$ for a Frobenius Weil algebra $A$.

Differential-geometrical properties of lifts of tensor fields and connections to tangent bundles were studied by K. Yano and S. Ishihara [51]. Various aspects of geometry of
Weil bundles and, in particular, lifts of geometric structures were studied by I. Kolář [13], A. Morimoto [22], E. Okassa [25], L.-N. Patterson [26], P.C. Yuen [53] and other researchers (see for references the book of I. Kolář, P.W. Michor and J. Slovák [14]). Weil bundles were also studied in connection with product preserving functors [14].

A.P. Shirokov [35] discovered that the Weil bundle $T^A M$ carries a structure of a smooth manifold over $A$, which made it possible to apply to the study of Weil bundles the theory of smooth manifolds over algebras (see, e.g., [35, 39]). In particular, the theory of realization of tensor fields and tensor operations over finite-dimensional modules over Frobenius algebras developed in the papers of V.V. Vishnevskii [46] and G.I. Kruchkovich [17] allowed ones to simplify the theory of lifts of tensor fields and linear connections from a manifold $M$ to its Weil bundle $T^A M$.

This paper is structured as follows.

In Section 2 we recall basic facts and definitions concerning Poisson manifolds and Poisson cohomology.

In Section 3 we recall the notion of a Weil algebra (a local algebra in the sense of A. Weil), basic concepts of the theory of smooth mappings over Weil algebras and describe the structure of a smooth manifold over algebra $A$ on the Weil bundle $T^A M$. In this section, we also analyze the structure of a Frobenius Weil algebra and prove some auxiliary statements which we use later.

In Section 4, we develop the theory of lifts of skew-symmetric covariant and contravariant tensor fields from a smooth manifold $M$ to its Weil bundle $T^A M$ on the base of the theory of realization of tensor operations. We show that the complete lift of exterior forms induces a homomorphism of the de Rham cohomology spaces $H^*_dR(M) \to H^*_dR(T^A M)$ and prove that this homomorphism is either a zero map or an isomorphism depending on the choice of a Frobenius covector (Theorem 4.1). We also show that the complete lift preserves the Schouten-Nijenhuis bracket of multivector fields (Proposition 4.5). We prove that for a Frobenius Weil algebra $A$ it is possible to introduce a uniquely defined vertical lift of tensor fields. We study the Schouten-Nijenhuis brackets of lifts of multivector fields (Proposition 4.9). In this section formulas for the complete lift of the tensor product and relations between the lifts and the Lie derivative are derived.

In Section 5 we study the complete and the vertical lifts of the Poisson structure $w$ from a Poisson manifold $(M, w)$ to its Weil bundle $T^A M$. We prove that the complete and the vertical lifts induce homomorphisms of the Poisson cohomology spaces $H^*_P(M, w) \to H^*_P(T^A M, w^C)$ and $H^*_P(M, w) \to H^*_P(T^A M, w^V)$ and establish the structure of these homomorphisms for some Poisson manifolds, in particular, for symplectic manifolds. Finally, we compute the modular classes of the lifts of Poisson structures. Namely, we prove that the modular class of Poisson manifold $(T^A M, w^C)$ is represented by the vector field $\dim A \cdot \Delta^V$ for every modular vector field $\Delta$ of the base manifold $(M, w)$ and that the modular class of Poisson manifold $(T^A M, w^V)$ is zero (Theorem 5.2).

The research of this paper was motivated by the works of G. Mitric and I. Vaisman [20] and J. Grabowski and P. Urbański [8, 9, 10].
2 Poisson manifolds

Let $M$ be a smooth manifold, $\dim M = m$. We will denote by $C^\infty(M)$ the algebra of smooth functions on $M$ and by $T^{r,s}(M)$ the space of tensor fields of type $(r, s)$ on $M$. The algebra of smooth exterior forms on $M$ will be denoted by $\Omega^*(M) = \bigoplus_{k=0}^{m} \Omega^k(M)$, and the exterior algebra of skew-symmetric contravariant tensor fields (multivector fields) on $M$ by $\mathcal{V}^*(M) = \bigoplus_{k=0}^{m} \mathcal{V}^k(M)$. We assume all manifolds and maps between manifolds under consideration to be of class $C^\infty$.

Throughout the paper we use the Einstein summation convention.

For $u \in \mathcal{V}^k(M)$ we denote by $i(u) : \Omega^p(M) \to \Omega^{p-k}(M)$ the interior product of a $p$-form with $u$. In local coordinates

$$(i(u)\alpha)_{i_1 \ldots i_{p-k}} = u^{i_1 \ldots i_k} \alpha_{j_1 \ldots j_k i_1 \ldots i_{p-k}}.$$  

We will denote by $d$ the exterior differential on $\Omega^*(M)$ and by $H^*_d(M) = H^*_{dR}(M, \mathbb{R})$ the de Rham cohomology of $M$.

By $L_X$ we will denote the Lie derivative along the vector field $X$.

The Lie bracket of vector fields on $M$ can be uniquely extended to an $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]$ on $\mathcal{V}^*(M)$, called Schouten-Nijenhuis bracket \cite{21,32}, such that $(\mathcal{V}^*(M), [\cdot, \cdot])$ is a graded superalgebra.

The Schouten-Nijenhuis bracket is an $\mathbb{R}$-bilinear map $[\cdot, \cdot] : \mathcal{V}^p(M) \times \mathcal{V}^q(M) \to \mathcal{V}^{p+q-1}(M)$ defined as follows. Let $X_1, \ldots, X_p, Y_1, \ldots, Y_q$ be vector fields on $M$. Then

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q] = \sum (-1)^{i+j} X_1 \wedge \ldots \wedge \widehat{X_i} \ldots \wedge X_p \wedge [X_i, Y_j] \wedge Y_1 \wedge \ldots \wedge \widehat{Y_j} \ldots \wedge Y_q,$$

where $[X_i, Y_j]$ in the right-hand side is the Lie bracket of vector fields and $\widehat{X}$ means the omission of $X$.

Let $u \in \mathcal{V}^p(M)$ and $v \in \mathcal{V}^q(M)$ be given in terms of a local coordinates as $u = u^{i_1 \ldots i_p} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_p}}$, $v = v^{j_1 \ldots j_q} \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_q}}$. Then for their Schouten-Nijenhuis bracket we have

$$[u, v]^{k_2 \ldots k_{p+q}} = \varepsilon^{k_2 \ldots k_{p+q}}_{i_2 \ldots i_p j_2 \ldots j_q} u^{r_{i_2} \ldots i_p} \frac{\partial}{\partial x^{r}} v^{j_1 \ldots j_q} + (-1)^{p} \varepsilon^{k_2 \ldots k_{p+q}}_{i_2 \ldots i_p j_2 \ldots j_q} v^{r_{i_2} \ldots i_p} \frac{\partial}{\partial x^{r}} u^{j_1 \ldots j_q}. \quad (2)$$

where $\varepsilon^{k_1 \ldots k_s}_{i_1 \ldots i_s}$ is the antisymmetric Kronecker symbol.

The Schouten-Nijenhuis bracket is supercommutative

$$[u, v] = (-1)^{|u|\cdot |v|} [v, u],$$

it satisfies the super-Jacobi identity

$$(-1)^{|u|\cdot |v|} [[v, y], u] + (-1)^{|v|\cdot |y|} [[y, u], v] + (-1)^{|y|\cdot |u|} [[u, v], y] = 0 \quad (3)$$
and the super-Leibniz identity
\[ [u, v \wedge y] = [u, v] \wedge y + (-1)^{|u|-1} |v| v \wedge [u, y], \] (4)
where \(|u|\) denotes the degree of \(u\). For more detail see, e.g., [11, 19, 24, 32, 41].

A Poisson bracket on a smooth manifold \(M\) is a bilinear skew-symmetric mapping \(\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)\), satisfying the Leibniz rule
\[ \{f, gh\} = \{f, g\} h + g \{f, h\} \]
and the Jacobi identity
\[ \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0. \]

A Poisson manifold is a smooth manifold \(M\) endowed with a Poisson bracket.

A Poisson bracket on \(M\) uniquely defines a bivector field \(w \in \mathcal{V}^2(M)\), called a Poisson bivector, such that
\[ \{f, g\} = i(w)(df \wedge dg) \] (5)
for any \(f, g \in C^\infty(M)\). It is known (see, e.g., [16, 18, 43]) that a bracket on \(C^\infty(M)\) defined by (5) for a bivector field \(w\) satisfies the Jacobi identity if and only if
\[ [w, w] = 0. \] (6)

In terms of local coordinates, (6) takes the form
\[ w^{js} \frac{\partial w^{k\ell}}{\partial x^s} + w^{ks} \frac{\partial w^{tj}}{\partial x^s} + w^{ts} \frac{\partial w^{jk}}{\partial x^s} = 0. \]

In what follows we will denote a Poisson manifold by \((M, w)\).

A Poisson bivector determines a bundle map
\[ \tilde{w} : T^* M \to TM, \] (7)
defined by
\[ (\tilde{w} \alpha)(\beta) := w(\alpha, \beta), \quad \alpha, \beta \in T^* M. \]
A Poisson bracket induces a bracket of 1-forms on \(M\) by
\[ \{\alpha, \beta\} = \mathcal{L}_{\tilde{w} \alpha} \beta - \mathcal{L}_{\tilde{w} \beta} \alpha - d(w(\alpha, \beta)). \] (8)
This bracket naturally extends the bracket \(\{df, dg\} := d\{f, g\}\) from \(B^1(M) := \{df | f \in C^\infty(M)\}\) to \(\Omega^1(M)\), and \((\Omega^1(M), \{\cdot, \cdot\})\) is a Lie algebra [12, 42].

To each function \(f \in C^\infty(M)\) there is associated a vector field \(X_f = X_f^w \in \mathcal{V}^1(M)\) called Hamiltonian vector field of \(f\) defined by \(X_f(g) := \{f, g\}\). Locally, Hamiltonian vector fields on \((M, w)\) are of the form [20, 41, 43]
\[ X_f^w = \{f, \cdot\}_w = w^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, \quad f \in C^\infty(M). \] (9)
A smooth function \( f \in C^\infty(M) \) is called a Casimir function if \( \{f, g\} = 0 \) for every \( g \in C^\infty(M) \), which is equivalent to the fact that the Hamiltonian vector field \( X^w_f \) of \( f \) is trivial. In terms of local coordinates, Casimir functions satisfy

\[
 w^{ij} \frac{\partial f}{\partial x^i} = 0.
\]

The rank of a Poisson bracket (Poisson structure) at a point \( x \in M \) is defined to be the rank of \( w(x) \). The rank of a Poisson bracket on \( M \) is the number

\[
 \max_{x \in M} \text{rank } w(x).
\]

A Poisson manifold \((M, w)\) is said to be regular, if the rank of \( w \) is constant on \( M \). A smooth map \( \varphi : (M, w) \to (M', w') \) between two Poisson manifolds is called a Poisson map if

\[
 \varphi^*\{f, g\}_{M'} = \{\varphi^*f, \varphi^*g\}_M.
\]

There is also an alternative characterization of Poisson maps [41]. Let \( X \in \mathcal{V}^k(M) \) and \( Y \in \mathcal{V}^k(M') \) be two multivector fields. We say that \( X \) is \( \varphi \)-related to \( Y \), writing \( Y = \varphi_*X \), if

\[
 (\wedge^k T_x \varphi) X(x) = Y(\varphi(x)) \quad \text{for all } x \in M.
\]

Then \( \varphi \) is a Poisson map if and only if

\[
 w' = \varphi_*w. \tag{10}
\]

For a Poisson manifold \((M, w)\) A. Lichnerowicz [18] introduced an operator

\[
 \sigma = \sigma_w : \mathcal{V}^k(M) \to \mathcal{V}^{k+1}(M)
\]

defined by \( \sigma u := [w, u] \). From the super-Jacobi identity (3) and the super-Leibniz rule (4) it follows that

\[
 \sigma(u \wedge v) = \sigma u \wedge v + (-1)^{|u|} u \wedge \sigma v
\]

and

\[
 \sigma \circ \sigma = 0.
\]

Therefore the Poisson cohomology spaces

\[
 H^k_P(M, w) := \frac{\ker \sigma : \mathcal{V}^k(M) \to \mathcal{V}^{k+1}(M)}{\text{im } \sigma : \mathcal{V}^{k-1}(M) \to \mathcal{V}^k(M)}
\]

are defined. In the general situation this cohomology is very difficult to compute (see, e.g., [6, 7, 21, 23, 27, 30, 42, 50]).

A map \( \bar{w} \) can be extended to a map \( \Omega^k(M) \to \mathcal{V}^k(M) \) defined by the formula

\[
 \bar{w} \theta(\alpha_1, \ldots, \alpha_k) = (-1)^k \theta(\bar{w}\alpha_1, \ldots, \bar{w}\alpha_k), \tag{11}
\]
where $\alpha_i \in \Omega^1(M)$. In terms of local coordinates,

$$(\tilde{w}\theta)^{j_1 \cdots j_k} = (-1)^k w^{i_1 \cdots i_k} \cdots w^{i_k j_1 \cdots j_k} \theta_{i_1 \cdots i_k}.$$  \hspace{1cm} (12)

Clearly, for a symplectic manifold map (11) is an isomorphism. It can be shown that [42]

$$\sigma \circ \tilde{w} = (-1)^k \tilde{w} \circ d.$$ \hspace{1cm} (13)

It follows that there arise natural homomorphisms

$$\rho^k : H^k_{dR}(M) \to H^k_P(M, w).$$

In the case of a symplectic manifold, these homomorphisms are isomorphisms, and the Poisson cohomology is isomorphic to the de Rham cohomology. See [16, 18, 42] for details.

**Example 2.1.** Let $M$ be a smooth manifold with zero Poisson structure ($w = 0$). Then

$$H^k_P(M, w) \cong \mathcal{V}^k(M).$$

**Example 2.2.** Let $S$ be a symplectic manifold and $N$ an arbitrary smooth manifold. Let $M = S \times N$ be the regular Poisson manifold whose Poisson structure $w$ is induced from $S$. Suppose that $\dim H^\cdot_{dR}(S) < \infty$. Then [12, 43]

$$H^r_P(M, w) \cong \bigoplus_{0 \leq k \leq r} H^k_{dR}(S) \otimes \mathcal{V}^{r-k}(N).$$ \hspace{1cm} (14)

For a Poisson manifold $(M, w)$, the canonical cohomology class $[w] \in H^2_P(M, w)$ is defined. This class is zero if and only if there exists $X \in \mathcal{V}^1(M)$ such that $\mathcal{L}_X w = w$.

A vector field $X$ such that $\mathcal{L}_X w = w$ is called a Liouville vector field for $w$ [11]. A Poisson manifold $(M, w)$ admitting a Liouville vector field is said to be exact (or homogeneous) [43].

Assume that $M$ is oriented, and let $\mu$ be a volume form on $M$. The divergence $\text{div}_\mu X$ of a vector field $X \in \mathcal{V}^1(M)$ is the smooth function defined by

$$\mathcal{L}_X \mu = (\text{div}_\mu X) \mu.$$  

For a Poisson manifold $(M, w)$ with volume form $\mu$, the operator

$$\Delta_\mu = \Delta_{\mu, w} : f \in C^\infty(M) \mapsto \text{div}_\mu X_f \in C^\infty(M)$$

is a derivation on $C^\infty(M)$, hence a vector field [45]. This vector field is called the modular vector field of $(M, w, \mu)$.

The modular vector field satisfies $\sigma \Delta_\mu = 0$ [15]. For another volume form $a \mu$, where $a \in C^\infty(M)$ is a non-vanishing function, the modular vector field changes to $\Delta_{a\mu} = \Delta_\mu + X_{-\log a}$ [48]. Since Hamiltonian vector fields form the space of 1-coboundaries of $\sigma$ [11, 18], it follows that the set of modular vector fields for all volume forms on $M$ is a cohomology
class from $H^1_P(M, w)$. This cohomology class is called the modular class of $(M, w)$. We denote it by $\text{mod}(M, w)$.

From (9) it follows that in terms of local coordinates on $M$ the modular vector field is given by

$$\Delta_\mu = \sum_{j=1}^m \left( \frac{\partial w^{ij}}{\partial x^j} + w^{ij} \frac{\partial \log \rho}{\partial x^j} \right) \frac{\partial}{\partial x^i},$$

where $\mu = \rho \, dx^1 \wedge \ldots \wedge dx^m$.

In the case when $M$ is non-orientable, one defines the modular class in a similar way using a smooth density instead of a volume form.

## 3 Weil algebras and Weil bundles

### 3.1 Weil bundles as smooth manifolds over Weil algebras

A Weil algebra $[14, 38]$ is a finite-dimensional associative commutative $\mathbb{R}$-algebra $\mathbb{A}$ with unit $1_\mathbb{A}$ whose nilpotent elements form a unique maximal ideal $\mathring{\mathbb{A}}$. The linear span of $1_\mathbb{A}$ form a subalgebra isomorphic to $\mathbb{R}$. We will identify it with $\mathbb{R}$. As a vector space, $\mathbb{A}$ is the direct sum $\mathbb{R} \oplus \mathring{\mathbb{A}}$. In what follows $n = \dim_{\mathbb{R}} \mathring{\mathbb{A}}$ and so $\dim \mathbb{A} = n + 1$.

By $\mathring{\mathbb{A}}^a$ we denote the $r$th power of $\mathring{\mathbb{A}}$. The positive integer $h$ defined by the relations $\mathring{\mathbb{A}}^h \neq 0$, $\mathring{\mathbb{A}}^{h+1} = 0$ is called the height of $\mathbb{A}$. Let $d_k(\mathbb{A}) = \dim_{\mathbb{R}} \mathring{\mathbb{A}}^k/\mathring{\mathbb{A}}^{k+1}$ for $k = 1, \ldots, h$ and $d_0(\mathbb{A}) = \dim_{\mathbb{R}} \mathbb{A}/\mathring{\mathbb{A}} = 1$. The number $d_1(\mathbb{A})$ is usually called the width of $\mathbb{A}$.

The chain of embedded ideals

$$\mathbb{A} \supset \mathring{\mathbb{A}} \supset \mathring{\mathbb{A}}^2 \supset \ldots \supset \mathring{\mathbb{A}}^h \supset 0$$

can be extended to the chain of ideals called the Jordan-Hölder composition series $[38]$

$$\mathbb{A} \supset \mathring{\mathbb{A}} = \mathfrak{I}_1 \supset \mathfrak{I}_2 \supset \ldots \supset \mathfrak{I}_n \supset 0,$$

where $\mathfrak{I}_a/\mathfrak{I}_{a+1}$ is a 1-dimensional algebra with zero multiplication. Here

$$\mathring{\mathbb{A}}^k = \mathfrak{I}_{1+d_1(\mathbb{A})+\ldots+d_{k-1}(\mathbb{A})} \quad \text{for} \quad 2 \leq k \leq h.$$ 

This is a particular case of the general construction for rings, see $[28]$.

Using the Jordan-Hölder composition series one can choose a basis ($a$ Jordan-Hölder basis)

$$\{e_a\} = \{e_0, e_\hat{a}\}, \quad a = 0, 1, \ldots, n = \dim \mathring{\mathbb{A}}, \quad \hat{a} = 1, \ldots, n,$$

in $\mathbb{A}$ such that $e_0 = 1_\mathbb{A} \in \mathbb{R}$, $e_\hat{a} \in \mathfrak{I}_\hat{a}$, $e_\hat{a} \not\in \mathfrak{I}_{\hat{a}+1}$. For $X = x^a e_a = x^0 + x^\hat{a} e_\hat{a} \in \mathbb{A}$ we set $\hat{X} = x^\hat{a} e_\hat{a}$, then $X = x^0 + \hat{X}$. Let $\delta^a$ be the coordinates of unit of $\mathbb{A}$, i.e., $1_\mathbb{A} = \delta^a e_a$.

Sometimes we denote the multiplication in $\mathbb{A}$ by a dot in order to avoid confusion.
We denote by \((\gamma^c_{ab})\) the structure tensor of \(A\) with respect to a Jordan-Hölder basis (16). We have \(e_a e_b = \gamma^c_{ab} e_c, \gamma^b_{0a} = \delta^b_a\) (Kronecker’s deltas), and \(\gamma^c_{ab} = 0\) for \(a \geq c\). The conditions of commutativity and associativity are \(\gamma^c_{ab} = \gamma^c_{ba}\) and

\[
\gamma^c_{ab} \gamma^d_{ef} = \gamma^d_{ae} \gamma^c_{bf},
\]

respectively. The relations \(e_a = e_a \cdot 1_A = e_a \delta_b^a e_b = \delta_b^a \gamma^c_{ab} e_c\) imply that

\[
\delta_b^a \gamma^c_{ab} = \delta_c^a.
\]

A smooth function \(f : U \subset A \rightarrow A\) is said to be \(A\)-differentiable (\(A\)-smooth) if its differential \(df\) is an \(A\)-linear map. The conditions of \(A\)-differentiability of \(f\), usually called Scheffers’ equations, are (see [31, 33, 47]):

\[
\frac{\partial f^b}{\partial x^c} \gamma^c_{ad} = \gamma^b_{ac} \frac{\partial f^c}{\partial x^d},
\]

Scheffers’ equations are equivalent to

\[
\frac{\partial f^b}{\partial x^a} = \gamma^b_{ac} \delta^d \frac{\partial f^c}{\partial x^d}.
\]

Let \(A^m = A \times \cdots \times A\) be the \(A\)-module of \(m\)-tuples of elements of \(A\). We will enumerate the real coordinates in \(A^m\) corresponding to a basis (16) by the double indices \(ia\). For a smooth function of several variables \(f : U \subset A^m \rightarrow A\), \(f : \{X^i = x^i a\} \mapsto f(X^i) = f^b(x^i a)e_b\), Scheffers’ conditions of \(A\)-differentiability are of the form [38, 47]:

\[
\frac{\partial f^b}{\partial x^a} = \gamma^b_{ac} \delta^d \frac{\partial f^c}{\partial x^d}.
\]

If \(f\) satisfies (21), its differential can be represented in the form \(df = f_i dX^i\), where \(f_i = \delta^a_\alpha \frac{\partial f}{\partial x^a} \) is the partial derivative with respect to the variable \(X^i \in A\). We will denote the latter by \(\frac{df}{dX^i}\). Thus,

\[
f_i = \frac{\partial f}{\partial X^i} = \delta^\alpha_\alpha \frac{\partial f}{\partial x^\alpha}.
\]

The functions \(f_i(X^j), i = 1, \ldots, m\), are also \(A\)-differentiable.

Recall that a smooth map \(f : M \rightarrow N\) of a foliated manifold \((M, F)\) is called projectable (or basic) if \(f\) is constant along the leaves of \(F\). The natural epimorphism \(\pi^m : A^m \rightarrow \mathbb{R}^m\) determines the canonical \(A^m\)-foliation on \(A^m\). The following theorem (see [38]) describes the local structure of an \(A\)-differentiable map of the form \(F : U \subset A^m \rightarrow A^k\) for a Weil algebra \(A\).

Theorem 3.1. [38] 1) Let \(U \subset A^m\) be an open set and \(\varphi : U \rightarrow A^k\) a projectable map with respect to the canonical \(A^m\)-foliation. Then the formula

\[
X^{i'} = \varphi^{i'} + \sum_{|p|=1}^{h} \frac{1}{p!} \frac{D^p\varphi^{i'}}{Dx^p} X^p,
\]

(23)
where $i = 1, \ldots, m$, $i' = 1, \ldots, k$, $p = (p_1, \ldots, p_m)$ is a multiindex of length $m$ and $p! = p_1! \ldots p_m!$, $X^i = x^i + \hat{X}^i$ is the decomposition with respect to (16), $\hat{X}^p = (\hat{X}^1)^{p_1} \ldots (\hat{X}^m)^{p_m}$, determines an $\mathbb{A}$-smooth map $\Phi : U \to \mathbb{A}^k$.

2) Any projectable $\mathbb{A}$-differentiable map $\Phi : U \to \mathbb{A}^k$ is of the form (23) for some basic functions $\varphi^i : U \to \mathbb{A}$.

**Definition.** Let $\varphi : U \to \mathbb{A}^k$ be a projectable map. Then the map $\Phi : U \to \mathbb{A}^k$ given by (23) is called the analytic prolongation ($\mathbb{A}$-prolongation) of $\varphi$.

The analytic prolongation of a map $\varphi$ will be denoted by $\varphi^\mathbb{A}$.

**Proposition 3.1.** Analytic prolongations satisfy the following relations:

1°. $(\varphi + \psi)^\mathbb{A} = \varphi^\mathbb{A} + \psi^\mathbb{A}$.
2°. $(\varphi \cdot \psi)^\mathbb{A} = \varphi^\mathbb{A} \cdot \psi^\mathbb{A}$.
3°. $(\varphi^\mathbb{A} \circ \psi^\mathbb{A}) = \varphi^\mathbb{A} \circ \psi^\mathbb{A}$.
4°. $(D^p \varphi / D x^p)^\mathbb{A} = D^p \varphi^\mathbb{A} / D X^p$ for $\varphi : U \subset \mathbb{A}^m \to \mathbb{A}$.

Let now $\mathbb{L}$ be an arbitrary $\mathbb{A}$-module, $\dim_{\mathbb{R}} \mathbb{L} < \infty$, and let $M$ be a smooth manifold.

An $\mathbb{L}$-chart on $M$ is a pair $(U, h)$ consisting of an open set $U \subset M$ and a diffeomorphism $h : U \to U' \subset \mathbb{L}$. An $\mathbb{L}$-atlas on $M$ is a collection of $\mathbb{L}$-charts $\{(U_\kappa, h_\kappa)\}_{\kappa \in K}$ such that $\{U_\kappa\}_{\kappa \in K}$ is a covering of $M$ and the tangent map

$$T_{h_\lambda(x)} (h_\kappa \circ h_\lambda^{-1}) : T_{h_\lambda(x)} \mathbb{L} \equiv \mathbb{L} \to \mathbb{L} \equiv T_{h_\kappa(x)} \mathbb{L}$$

is an isomorphism of $\mathbb{A}$-modules for all $x \in M$ and $\kappa, \lambda \in K$. This condition is equivalent to the $\mathbb{A}$-differentiability of all transition functions $h_{\kappa \lambda} := h_\kappa \circ h_\lambda^{-1}$.

**Definition.** A quadruple $(\mathbb{A}, \mathbb{L}, M, \mathcal{A})$ consisting of a Weil algebra $\mathbb{A}$, an $\mathbb{A}$-module $\mathbb{L}$, a smooth manifold $M$, and a maximal $\mathbb{L}$-atlas $\mathcal{A}$ on $M$ is called an $\mathbb{L}$-manifold or an $\mathbb{A}$-smooth manifold, modeled on $\mathbb{L}$.

The isomorphism (24) allows ones to transfer by means of $T_x h_\kappa^{-1}$ the structure of an $\mathbb{A}$-module from $T_x U'_\kappa \cong \mathbb{L}$ to the tangent space $T_x M$ at $x \in M$.

**Definition.** $\mathbb{A}^n$-manifold is called an $n$-dimensional $\mathbb{A}$-smooth manifold.

Let $M$ and $M'$ be two $\mathbb{A}$-smooth manifolds modeled, respectively, by $\mathbb{A}$-modules $\mathbb{L}$ and $\mathbb{L}'$. A smooth map $f : M \to M'$ is said to be $\mathbb{A}$-smooth if $T_x f$ is an $\mathbb{A}$-linear map for all $x \in M$. This is equivalent to the $\mathbb{A}$-differentiability of all chart representatives $h_{\kappa'} \circ f \circ h_\kappa^{-1}$ of $f$.

For any Weil algebra $\mathbb{A}$ and smooth manifold $M$, the Weil bundle $\pi_{\mathbb{A}} : T^\mathbb{A} M \to M$ of $\mathbb{A}$-points is defined as follows. An $\mathbb{A}$-point near to $x \in M$ is a homomorphism $X : C^\infty(M) \to \mathbb{A}$ such that the real part of $X(f) \in \mathbb{A}$ coincides with $f(x)$. The set $T^\mathbb{A} M$ of all $\mathbb{A}$-points near to points of $M$ can be endowed with a structure of smooth manifold. Bundle projection $\pi_{\mathbb{A}}$ sends $X$ to $x$. Let $x^i$ be local coordinates on a neighborhood $U$ of $M$. These coordinates induce local coordinates $x'^a$ on $\pi_{\mathbb{A}}^{-1}(U) \subset T^\mathbb{A} M$ defined by $X(x^i) = x'^a(X)e_a$, $a = 0, 1, \ldots n = \dim \mathbb{A}$, where $x^0 = x^i$. The functions $X^i = x'^a e_a$ are
A-valued coordinates on $\pi^{-1}(U)$. A.P. Shirokov proved [35] that these coordinates define the structure of a smooth manifold over $\mathbb{A}$ on $T^A M$.

The correspondence which assigns to a manifold $M$ the Weil bundle $T^A M$ and to a smooth map $\varphi : M \to N$ the map $\varphi^A : T^A M \ni X \mapsto X \circ \varphi^* \in T^A N$, where $\varphi^* : C^\infty(N) \ni g \mapsto g \circ \varphi \in C^\infty(M)$, is a functor called the Weil functor (see [14, 38, 49]). It is well-known that Weil functors preserve products, i.e., $T^A(M \times N) \cong T^A M \times T^A N$.

### 3.2 The structure of a Frobenius Weil algebra

**Definition.** [4] A Frobenius Weil algebra is a pair $(\mathbb{A}, q)$ where $\mathbb{A}$ is a Weil algebra and $q : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$ is a nondegenerate bilinear form which satisfies the following associativity condition:

$$q(XY, Z) = q(X, YZ) \text{ for any } X, Y, Z \in \mathbb{A}. \tag{25}$$

In terms of basis (16) the condition (25) is written as

$$q_{ac} \gamma^c_{bd} = \gamma^c_{ab} q_{cd}. \tag{26}$$

The form $q$ is called the Frobenius form. Frobenius algebras play an important role in the theory of smooth manifolds over algebras in constructing realizations of tensor operations [17, 17].

For a Frobenius algebra $\mathbb{A}$ the linear form $p : \mathbb{A} \to \mathbb{R}$ defined by

$$p(X) = q(X, 1_{\mathbb{A}}) \tag{27}$$

is called the Frobenius covector. Its coordinates satisfy

$$p_a \gamma^a_{bc} = q_{bc}. \tag{28}$$

Contracting (28) with $\delta^c$ gives

$$p_b = q_{bc} \delta^c. \tag{29}$$

From (25) and (28) it follows that

$$q(X, Y) = p(XY) \text{ for any } X, Y \in \mathbb{A}. \tag{30}$$

Let $\mathbb{A}^*$ be the dual space of $\mathbb{A}$, i.e., the space of $\mathbb{R}$-linear functions $\xi : \mathbb{A} \to \mathbb{R}$. The Frobenius form $q$ induces the isomorphism

$$\varphi : \mathbb{A} \to \mathbb{A}^*, \quad \varphi(X)Y := q(X, Y) = p(XY). \tag{31}$$

From the definition of $p$ it follows that $\varphi(1_{\mathbb{A}}) = p$, and (25) implies that

$$\varphi(XY)(Z) = \varphi(X)(YZ).$$
The isomorphism \( \varphi \) allows ones to transfer the multiplication operation from \( \mathbb{A} \) to \( \mathbb{A}^\ast \):

\[
\xi \ast \eta := \varphi(\varphi^{-1}(\xi) \cdot \varphi^{-1}(\eta))
\]

(the dot means the multiplication in \( \mathbb{A} \)). The multiplication \( \ast \) turns \( \mathbb{A}^\ast \) into a Weil algebra isomorphic to \( \mathbb{A} \). The bilinear form \( q \) induces the form \( \tilde{q} : \mathbb{A}^\ast \times \mathbb{A}^\ast \to \mathbb{R} \),

\[
\tilde{q}(\xi, \eta) := \xi(\varphi^{-1}\eta).
\]

It is obvious that \( \tilde{q} \) is symmetric.

Let \( \{ \tilde{e}^a \} \) be the basis in \( \mathbb{A}^\ast \) dual to a basis \( \{ e_a \} \) of \( \mathbb{A} \). Then \( \varphi(e_a) = q_{ab} \tilde{e}^b \), \( \varphi^{-1}(\tilde{e}^a) = \tilde{q}^{ab} e_b \) and \( \| \tilde{q}^{ab} \| = \| q_{ab} \|^{-1} \). In what follows we will omit the tilde in \( \tilde{q}^{ab} \) and write \( q^{ab} \) instead of \( \tilde{q}^{ab} \).

**Proposition 3.2.** For any \( \xi, \eta \in \mathbb{A}^\ast \)

\[
\tilde{q}(\xi, \eta) = (\xi \ast \eta)(1_\mathbb{A}). \tag{32}
\]

**Proof.** Let \( \xi = \varphi(X) \), \( \eta = \varphi(Y) \) for some \( X, Y \in \mathbb{A} \). Then \( \tilde{q}(\xi, \eta) = \xi(\varphi^{-1}\eta) = \varphi(X)Y = p(XY) = p(\varphi^{-1}(\xi) \cdot \varphi^{-1}(\eta)) = p(\varphi^{-1}(\xi \ast \eta)) \). Let us show that \( p(\varphi^{-1}\zeta) = \zeta(1_\mathbb{A}) \) for any \( \zeta \in \mathbb{A}^\ast \). In fact, if \( \zeta = \varphi(Z) \), then \( p(\varphi^{-1}\zeta) = p(Z) = \varphi(Z)(1_\mathbb{A}) = \zeta(1_\mathbb{A}) \). \( \square \)

Every function \( F : \mathbb{A} \to \mathbb{A} \) obviously determines the function

\[
\hat{F} = \varphi \circ F \circ \varphi^{-1} : \mathbb{A}^\ast \to \mathbb{A}^\ast.
\]

The proof of the following proposition is immediate.

**Proposition 3.3.** If \( F \) is \( \mathbb{A} \)-differentiable, then \( \hat{F} \) is \( \mathbb{A}^\ast \)-differentiable.

**Example 3.1.** Let \( \mathbb{A} \) be the Weil algebra of plural numbers

\[
\mathbb{D}^n = \mathbb{R}(\varepsilon^n) = \{ x_0 + x_1 \varepsilon + \cdots + x_n \varepsilon^n \mid x_i \in \mathbb{R}, \varepsilon^{n+1} = 0 \}
\]

(the algebra of truncated polynomials in one variable \( \varepsilon \) of degree not greater than \( n \)). For \( n = 1 \) we get the algebra of dual numbers (also known as Study numbers)

\[
\mathbb{D} = \mathbb{R}(\varepsilon) = \{ x_0 + x_1 \varepsilon \mid x_0, x_1 \in \mathbb{R}, \varepsilon^2 = 0 \}.
\]

There is a natural Jordan-Hölder basis in \( \mathbb{R}(\varepsilon^n) \), namely, \( e_0 = 1, e_a = \varepsilon^a, a = 1, \ldots, n \). Let \( p = (p_0, \ldots, p_n) \) be an arbitrary covector on \( \mathbb{R}(\varepsilon^n) \). Then the matrix \( \| p_c \gamma^{ac}_{ab} \| \) is

\[
\begin{pmatrix}
  p_0 & p_1 & \cdots & p_{n-1} & p_n \\
  p_1 & p_2 & \cdots & p_n & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{n-1} & p_n & \cdots & 0 & 0 \\
  p_n & 0 & \cdots & 0 & 0
\end{pmatrix}. \tag{33}
\]
Since $\det \| p \gamma^c_ab \| = p_n^{n+1}$, it follows that $p$ is a Frobenius covector if and only if $p_n = p(e_n) \neq 0$ (cf. Proposition 3.4).

In what follows we assume all Weil algebras under consideration to be Frobenius algebras.

Let $(A, q)$ be a Frobenius Weil algebra of height $h$ and let $p$ be its Frobenius covector. We denote $n = \dim A$. Let us fix a Jordan-Hölder basis (16) in $A$.

**Proposition 3.4.** For a Frobenius Weil algebra $(A, q)$ the following conditions hold:

1) $\dim A^h = 1$, that is, $A^h = I$;
2) $p|_{A^h} \neq 0$.

**Proof.** Denote $\text{Ann} A := \{ X \in A \mid X \cdot A = 0 \}$. Let $0 \neq X \in \text{Ann} A$. Then for any $Y = y^0 + \hat{Y} \in A$ we have $XY = Xy^0$ and $q(X, Y) = p(XY) = y^0 p(X)$. The nondegeneracy of $q$ implies that $p(X) \neq 0$. Hence $\text{Ann} A \cap \ker p = 0$, which means that $\dim \text{Ann} A \leq 1$. However, it is obvious that $0 \neq A^h \subset \text{Ann} A$. Consequently, $\dim A^h = \dim \text{Ann} A = 1$. □

The second statement of Proposition 3.4 is equivalent to the inequality $p_n \neq 0$. In what follows we will always assume that a Jordan-Hölder basis (16) is chosen in such a way that $p_n = p(e_n) = 1$. Then the matrix of $q$ is of the form

$$\| q_{ab} \| = \begin{pmatrix}
* & * & \ldots & * & 1 \\
* & * & \ldots & * & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & \ldots & * & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix},$$

and the inverse matrix $\| q^{ab} \|$ is of the form

$$\| q^{ab} \| = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & * & \ldots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & * & \ldots & * & * \\
1 & * & \ldots & * & *
\end{pmatrix}.$$  (34)

**Remark 3.1.** Let $e^0 \in A^*$ be defined by $\bar{e}^0(X) := x^0$ (i.e., the projection onto $\mathbb{R}$ along $\hat{A}$).

We show that if $p \in A^*$ is a Frobenius covector then $\bar{p} := p - p(1)e^0$ is also a Frobenius covector. Suppose the contrary. Then there exists $X \in A$ such that $\bar{p}(XY) = 0$ for any $Y \in A$. This means that $x^0 p(Y) + y^0 p(X) + p(XY) = 0$. Let $Z \in A^h$, $p(Z) = 1$ (in terms
of a Jordan-Hölder basis under consideration, \( Z = e_n \) and let \( \tilde{X} = X - x^0 Z \). Then for any \( Y \in \mathbb{A} \) we have \( \tilde{X}Y = XY - x^0 y^0 Z = x^0 y^0 + x^0 \tilde{Y} + y^0 \tilde{X} + \tilde{X} \tilde{Y} - x^0 y^0 Z \). Obviously, \( p(\tilde{X}Y) = 0 \), which contradicts to the fact that \( p \) is a Frobenius covector. This means that having an arbitrary Frobenius covector \( p \in \mathbb{A}^* \) we can obtain another Frobenius covector \( \tilde{p} \) satisfying \( \mathbb{R} \subset \ker \tilde{p} \).

Let \( \{e^a\} \) be the basis in \( \mathbb{A} \), corresponding to the dual basis \( \{\hat{e}^a\} \) in \( \mathbb{A}^* \) with respect to the isomorphism \( \varphi \). Then

\[
e^a = q^{ab} e_b, \quad e_a = q_{ab} e^b.
\]

Denote by \( \gamma_{ab}^c \) the components of the structure tensor of \( \mathbb{A} \) with respect to the basis \( \{e^a\} \). Then

\[
p(e^a e^b) = q^{ab}.
\]

The components \( \gamma_{ab}^c \) and \( \gamma_{bc}^a \) are related as follows \( \gamma_{ab}^c = q^{ad} q^{bf} q_{eh} \gamma_{df}^c \). Using (29), we have

\[
\gamma_{ab}^c \delta^d = q^{ab}.
\]

From (18) it follows that

\[
\gamma_{bc}^a \delta^d = q^{ab}.
\]

From (37) and (17), we have

\[
\gamma_{bc}^a \gamma_{df}^c = \gamma_{d}^f \gamma_{bc}^a.
\]

Using (37), one can obtain the following formulas for the products \( e_a e^c \):

\[
e_a e^c = \gamma_{ab}^c e^b = \gamma_{bc}^a e^b.
\]

In fact, we have \( e_a e^c = q_{ad} e^d e^c = q_{ad} \gamma_{dc}^b e^b = \gamma_{bc}^a e^b \). The second relation is derived in the similar way. From (40) it follows that

\[
p(e_a e^c) = \delta^c_a
\]

and

\[
p(e_a e^b e^c) = \gamma_{ab}^c, \quad p(e_a e^b e^c) = \gamma_{ac}^b.
\]

Contracting (29) with \( q^{ab} \) we obtain

\[
q^{ab} p_b = \delta^a.
\]

The following two formulas (44) and (45) for an \( \mathbb{A} \)-smooth function \( F : \mathbb{A} \to \mathbb{A} \), \( F = F^a(x^b)e_a = F_a(x^b)e^a \) will be used in the sequel.

\[
F^a p_a = F_b \delta^b.
\]

\[
\frac{\partial(\delta^a F_a)}{\partial x^b} = \delta^c \frac{\partial F_b}{\partial x^c}.
\]

The proof of (44) is obvious. Formula (45) follows from (20) and (18): \( \frac{\partial(\delta^a F_a)}{\partial x^b} = \delta^c \frac{\partial F_b}{\partial x^c} = \delta^c \frac{\partial(\delta^c F_a)}{\partial x^c} = \delta^c \frac{\partial F_b}{\partial x^c} \).
4 Lifts of tensor fields to Weil bundles

Let \((\mathbb{A}, q)\) be a Frobenius Weil algebra of height \(h\), and let \(p\) be the corresponding Frobenius covector on \(\mathbb{A}\). We will use the indices \(i, j, \ldots\), and \(\alpha, \beta, \ldots\) to enumerate coordinates on manifolds and the indices \(a, b, c, d, \ldots\) to enumerate coordinates in \(\mathbb{A}\).

The Weil bundle \(T^h M\) of an \(m\)-dimensional smooth manifold \(M\) is an \(m\)-dimensional \(\mathbb{A}\)-smooth manifold. For each \(X \in T^h M\) the tangent space \(T_X T^h M\) is an \(m\)-dimensional \(\mathbb{A}\)-module. This allows us to consider \(\mathbb{A}\)-linear \(\mathbb{A}\)-valued tensors at any point \(X \in T^h M\) and \(\mathbb{A}\)-smooth tensor fields on \(T^h M\) (see, e.g., [36]).

For an arbitrary \(\mathbb{A}\)-smooth manifold \(M_\mathbb{A}\) the following notation is used:

\[
T^{r,s}_\mathbb{A} M_\mathbb{A} \text{ is the bundle of } \mathbb{A}\text{-valued tensors of type } (r, s),
\]

\[
\mathbb{A} \otimes T^{r,s}_\mathbb{A} M_\mathbb{A} \text{ is the bundle of } \mathbb{A}\text{-valued tensors of type } (r, s),
\]

\[
T^{r,s}_\mathbb{A}\text{-lin} M_\mathbb{A} \text{ is the subbundle in } \mathbb{A} \otimes T^{r,s}_\mathbb{A} M_\mathbb{A} \text{ consisting of } \mathbb{A}\text{-linear tensors.}
\]

Local sections of these bundles (local \((r, s)\)-tensor fields on \(M_\mathbb{A}\)) are expressed in terms of local coordinates \(X^i = x^i_a c_a\) on \(M_\mathbb{A}\) as follows

\[
t^{i_1 \ldots i_r}_{a_1 \ldots a_r} dX^{i_1} \otimes \cdots \otimes dX^{i_r} \otimes \mathbb{A} \frac{\partial}{\partial X^{b_1}} \otimes \cdots \otimes \mathbb{A} \frac{\partial}{\partial X^{b_s}},
\]

where in the first case \(t^{i_1 \ldots i_r}_{a_1 \ldots a_r}\) are smooth real-valued functions of \(x^{kc}\), and in the second and the third cases they are smooth \(\mathbb{A}\)-valued functions \(t^{i_1 \ldots i_r}_{a_1 \ldots a_r} = \epsilon^{i_1 \ldots i_r}_{1 \ldots r} c_c\). Local sections of the third bundle can also be expressed in the form

\[
t^{i_1 \ldots i_r}_{a_1 \ldots a_r} = \omega^{i_1 \ldots i_r}_{i_1 \ldots i_r} \frac{\partial}{\partial X^{j_i}} \otimes \cdots \otimes \frac{\partial}{\partial X^{j_s}}.
\]

The total space of the bundle \(T^{r,s}_\mathbb{A}\text{-lin} M_\mathbb{A}\) carries the natural structure of an \(\mathbb{A}\)-smooth manifold.

**Definition.** \(\mathbb{A}\)-smooth tensor field of type \((r, s)\) on \(M_\mathbb{A}\) is an \(\mathbb{A}\)-smooth section of the bundle \(T^{r,s}_\mathbb{A}\text{-lin} M_\mathbb{A}\).

For an \(\mathbb{A}\)-smooth tensor field functions \(t^{i_1 \ldots i_r}_{a_1 \ldots a_r}\) are \(\mathbb{A}\)-smooth functions of \(X^k\).

We denote the space of \(\mathbb{A}\)-smooth tensor fields of type \((r, s)\) on \(M_\mathbb{A}\) by \(T^{r,s}_\mathbb{A}\text{-diff}(M_\mathbb{A})\), the space of \(\mathbb{A}\)-smooth exterior forms on \(M_\mathbb{A}\) by \(\Omega^{r,s}_\mathbb{A}\text{-diff}(M_\mathbb{A})\) and the space of \(\mathbb{A}\)-smooth multivector fields by \(\mathcal{V}^r_s(M_\mathbb{A})\).

For an \(\mathbb{A}\)-smooth exterior form \(\Theta = \Theta_{i_1 \ldots i_k} dX^{i_1} \wedge \ldots \wedge dX^{i_k} \in \Omega^{r,s}_\mathbb{A}\text{-diff}(M_\mathbb{A})\) and \(\mathbb{A}\)-smooth multivector fields \(U \in \mathcal{V}^{r}_\mathbb{A}\text{-diff}(M_\mathbb{A}), V \in \mathcal{V}^{s}_\mathbb{A}\text{-diff}(M_\mathbb{A})\) the exterior differential \(d\Theta\) and the Schouten-Nijenhuis bracket \([U, V]\) can be represented in terms of coordinates \(X^i\), respectively, as follows.

\[
d\Xi = \frac{\partial \Xi_{i_1 \ldots i_k}}{\partial X^j} dX^j \wedge \cdots \wedge dX^{i_k};
\]

\[
[U, V]^{k_2 \ldots k_{g+\ell}} = \epsilon^{k_2 \ldots k_{g+\ell}}_{i_2 \ldots i_{g+1} \ldots i_{g+\ell}} \frac{\partial}{\partial X^{i_{g+1}}} V^{i_{g+2} \ldots i_{g+\ell}}
\]

\[
+ (-1)^g \epsilon^{k_2 \ldots k_{g+\ell}}_{i_1 \ldots i_{g+2} \ldots i_{g+\ell}} U^{i_{g+2} \ldots i_{g+\ell}} \frac{\partial}{\partial X^{i_{g+1}}}.
\]
4.1 Realizations of tensor operations

Let $\mathbb{L}$ be a finite dimensional free $\mathbb{A}$-module. Choose a basis $\{f_i\}$ in $\mathbb{L}$ (over $\mathbb{A}$) and let $\{f^i\}$ be the dual basis in $\mathbb{L}^*$. For any $X \in \mathbb{L}$ we have $X = X^i f_i = x^i a f_i e_a$ where $x^i a \in \mathbb{R}$. Thus, the elements $f_i a := f_i e_a$ form a basis of $\mathbb{L}$ considered as an $\mathbb{R}$-module. Let $f^i a = p \circ (f^i e^a) : \mathbb{L} \to \mathbb{R}$. Then $\{f^i a\}$ form a basis of $\mathbb{L}^*$ considered as an $\mathbb{R}$-module, dual to $\{f_i a\}$. In fact, $f^i a (f_j b) = p(f^i e^a (f_j b)) = p(e^a b) \delta^i_j = \delta^i_j a$ by (41) and $f^i a (X) = x^i a$.

Let $t : \mathbb{L} \times \cdots \times \mathbb{L} \to \mathbb{A}$ be an $\mathbb{A}$-linear covariant tensor.

**Definition.** The realization of $t$ is the $\mathbb{R}$-linear tensor

$$R(t) := p \circ t : \mathbb{L} \times \cdots \times \mathbb{L} \to \mathbb{R}.$$  

Let $t = t_{i_1 \ldots i_k} f^{i_1} \otimes \cdots \otimes f^{i_k}$. Then $t(X_1, \ldots, X_k) = t_{i_1 \ldots i_k} X_1^{i_1} \cdots X_k^{i_k}$, hence

$$R(t)(X_1, \ldots, X_k) = p(t_{i_1 \ldots i_k} X_1^{i_1} \cdots X_k^{i_k}) = p(t_{i_1 \ldots i_k} e_a_1 \cdots e_{a_k} x_1^{i_1 a_1} \cdots x_k^{i_k a_k}) = t_{i_1 a_1 \ldots i_k a_k} x_1^{i_1 a_1} \cdots x_k^{i_k a_k},$$

where

$$t_{i_1 a_1 \ldots i_k a_k} := p(t_{i_1 \ldots i_k} e_a_1 \cdots e_{a_k}). \tag{47}$$

Thus

$$R(t) = t_{i_1 a_1 \ldots i_k a_k} f^{i_1 a_1} \otimes \cdots \otimes f^{i_k a_k},$$

and its components $t_{i_1 a_1 \ldots i_k a_k}$ can be calculated by (47).

Let $\mathbb{L}'$ be another finite-dimensional $\mathbb{A}$-module and $\Psi : \mathbb{L}' \to \mathbb{L}$ be an $\mathbb{A}$-linear map. Let us show that

$$\Psi^* R(t) = R(\Psi^* t). \tag{48}$$

In fact, let $\{g_a\}$ be a basis in $\mathbb{L}'$. Denote the components of $\Psi$ by $\Psi^i_a = \Psi^i_a c e_c$. Then the components $\Psi^i_a a_b$ of $\Psi$ considered as an $\mathbb{R}$-linear map $\mathbb{L}' \to \mathbb{L}$ are $\Psi^i_a a_b = \Psi^i_a c^a e_c$. We have

$$(\Psi^* t)_{a_1 \ldots a_k} = \Psi^i_a \cdots \Psi^i_k t_{i_1 \ldots i_k},$$

whence

$$(R(\Psi^* t))_{a_1 b_1 \ldots a_k b_k} = p(\Psi^i_{a_1} \cdots \Psi^i_k t_{i_1 \ldots i_k} e_{b_1} \cdots e_{b_k})$$

$$= p(\Psi^i_{a_1} c^1 e_{c_1} e_{b_1} \cdots \Psi^i_k c_k e_{c_k} e_{b_k} t_{i_1 \ldots i_k})$$

$$= \Psi^i_{a_1} c^a_{c_1} b_1 \cdots \Psi^i_k c_k a_k \ t_{i_1 \ldots i_k} e_{a_1} \cdots e_{a_k}$$

$$= \Psi^i a_{a_1 b_1} \cdots \Psi^i_k a_k b_k R(t)_{i_1 a_1 \ldots i_k a_k} = (\Psi^* R(t))_{a_1 b_1 \ldots a_k b_k}.$$
space over \( \mathbb{R} \)). The isomorphism \( R \) transfers the structure of an \( \mathbb{A} \)-module from \( \mathbb{L}^* \) to \( \mathbb{L}^*_c \); if \( \xi = R(\omega), \alpha \in \mathbb{A} \), then \( \alpha \xi = R(\alpha \omega) \). The structure of an \( \mathbb{A} \)-module on \( \mathbb{L}^*_c \) can be described as follows: if \( v \in \mathbb{L} \), then \( (\alpha \xi)(v) = \xi(\alpha v) \). In fact, \( (\alpha \xi)(v) = R(\alpha \omega)(v) = p \circ \alpha \omega(v) = p \circ (\alpha \omega)(v) = R(\omega)(\alpha v) = \xi(\alpha v) \). It will be convenient in the sequel to identify the modules \( \mathbb{L}^* \) and \( \mathbb{L}^*_c \) and consider \( \mathbb{L}^* \) as a dual \( \mathbb{A} \)-module to \( \mathbb{L} \) with contraction \( \mathbb{L} \times \mathbb{L}^* \ni (\omega, v) \mapsto \langle \omega, v \rangle_{\mathbb{A}} \in \mathbb{A} \) and as a dual vector space to \( \mathbb{L} \) with contraction \( \mathbb{L} \times \mathbb{L}^* \ni (\omega, v) \mapsto p \circ \langle \omega, v \rangle_{\mathbb{A}} \in \mathbb{R} \).

Now we describe the realization of contravariant tensors.

Let \( \mathbb{L} \) be a finite-dimensional \( \mathbb{A} \)-module and \( u : \mathbb{L}^* \times \cdots \times \mathbb{L}^* \to \mathbb{A} \) be an \( \mathbb{A} \)-linear contravariant tensor.

**Definition.** The realization of \( u \) is the \( \mathbb{R} \)-linear tensor

\[
R(u) := p \circ u : \mathbb{L}^* \times \cdots \times \mathbb{L}^* \to \mathbb{R}.
\]

Making use of the diagram

\[
\begin{array}{ccc}
\mathbb{L}^* \times \cdots \times \mathbb{L}^* & \xrightarrow{u} & \mathbb{A} \\
p & & \varphi \\
& & \downarrow 1_{\mathbb{A}} \\
& & \mathbb{R}
\end{array}
\]

we may represent the realization \( R(u) \) as \( 1_{\mathbb{A}} \circ \varphi \circ u \), where \( \varphi : \mathbb{A} \to \mathbb{A}^* \) is isomorphism \([31]\) induced by the Frobenius form \( q \).

Let \( u = u^{i_1 \cdots i_k} f_{i_1} \otimes \cdots \otimes f_{i_k} \) be the representation of \( u \) in terms of a basis \( \{f_i\} \) in \( \mathbb{L} \), and let \( u^{i_1 \cdots i_k} e^{a_1} \cdots e^{a_k} = u^{i_1 a_1 \cdots i_k a_k} e^b \) be the expansion in terms of the basis of \( \mathbb{A} \). From \([14]\) we have \( p(u^{i_1 a_1} e^{a_1} \cdots e^{a_k}) = u^{i_1 a_1 \cdots i_k a_k} \delta^b \). Denote

\[
u^{i_1 a_1 \cdots i_k a_k} := p(u^{i_1 a_1} e^{a_1} \cdots e^{a_k}).\]

Then

\[
R(u) = u^{i_1 a_1 \cdots i_k a_k} f_{i_1 a_1} \otimes \cdots \otimes f_{i_k a_k}.
\]

Let now \( \mathbb{L}' \) be another finite-dimensional \( \mathbb{A} \)-module, and let \( \Psi : \mathbb{L} \to \mathbb{L}' \) be an \( \mathbb{A} \)-linear map. Suppose that a contravariant \( \mathbb{A} \)-linear tensor \( u \) on \( \mathbb{L} \) is \( \Psi \)-related to a contravariant \( \mathbb{A} \)-linear tensor \( v \) on \( \mathbb{L}' \).

**Proposition 4.1.** \( R(u) \) is \( \Psi \)-related to \( R(v) \).

**Proof.** The condition that \( u \) is \( \Psi \)-related to \( v \) is as follows \([14, 41]\)

\[
u^{a_1 \cdots a_k} = \Psi^{a_1}_{i_1} \cdots \Psi^{a_k}_{i_k} u^{i_1 \cdots i_k}.
\]

We have

\[
(R(v))^{a_1 b_1 \cdots a_k b_k} = p(\Psi^{a_1}_{i_1} \cdots \Psi^{a_k}_{i_k} u^{i_1 \cdots i_k} e^{b_1} \cdots e^{b_k}) = p(\Psi^{a_1 c_1}_{i_1} e_{c_1} e^{b_1} \cdots \Psi^{a_k c_k}_{i_k} e_{c_k} e^{b_k} u^{i_1 \cdots i_k}).
\]
Note that $e^b c_1 = \delta^b_{ac} a^d c_a$ by (40). Thus
\[(R(v))^{\alpha b_1 \ldots \alpha b_k} = \Psi_{t_1}^{\alpha_{i_1} c_1} \gamma_{i_1 c_1}^{b_1} \ldots \Psi_{i_k}^{\alpha_{i_k} c_k} \gamma_{i_k c_k}^{b_k} p(u^{i_1 \ldots i_k} e_a^{i_1} \ldots e_a^{i_k}) = \Psi_{i_1 a_1}^{\alpha_{i_1} b_1} \ldots \Psi_{i_k a_k}^{\alpha_{i_k} b_k} (R(u))^{i_1 a_1 \ldots i_k a_k}.
\]

□

Let us find the expression for $R(v)$ where $v \in \mathbb{L}$. For a basis \{fi\} in $\mathbb{L}$ we have $v = v^j f_j$. Let $v^j = v^b c_a = v^0 e^b$. Then $v^j e^a = v^b c_a e = v^j g^a b$. Since $p(e_a e^b) = \delta^b_{a}$ and $p(e^a e^b) = q^{ab}$ by (4) and (30), respectively, we get
\[(R(v))^{j a} = v^j a = v^j g^{ab}.
\]

**Proposition 4.2.** Let $\mathbb{L}$ be a finite-dimensional $\mathbb{A}$-module, $v \in \mathbb{L}$, and let $t \in T^k, \theta$ be a covariant $\mathbb{A}$-tensor. Then
\[R(i(v)t) = i(R(v))R(t),
\]
where $i(v)$, $i(R(v))$ are defined by (4).

**Proof.** Let \{fi\} be a basis in $\mathbb{L}$, $v = v^j f_j$, and $t = t^{i_1 \ldots i_k} f^{i_1} \otimes \ldots \otimes f^{i_k}$. Denote $\theta = i(v)t$. We compute the components of $R(v)$, $R(t)$ and $R(\theta)$.

We have $(R(v))^{j a} = v^j g^{ab}$ by (50).
We define $\gamma^{b}_{i_1 \ldots i_k} \in \mathbb{R}$ by
\[e_{a_1} \ldots e_{a_k} = \gamma^{b}_{i_1 \ldots i_k} e_{b}.
\]

Clearly, $\gamma^{b}_{i_1 \ldots i_k} = \gamma^{c_1}_{i_1 a_2} \gamma^{c_2}_{i_2 a_3} \ldots \gamma^{c_k}_{i_k a_k}$ (see \[17, 47\]).

Let $t_{i_1 \ldots i_k} = t_{i_1 \ldots i_k}^{s} e_{s}$ be the expansion in terms of the basis in $\mathbb{A}$. Then
\[t_{i_1 \ldots i_k} e_{a_1} \ldots e_{a_k} = t_{i_1 \ldots i_k}^{s} e_{a_1} \ldots e_{a_k} = t_{i_1 \ldots i_k}^{s} \gamma^{c}_{a_1 \ldots a_k} e_{c}.
\]

Contracting with $p$ we obtain
\[(R(t))^{ja_1 \ldots a_k} = t_{ja_1 \ldots a_k} = t_{i_1 \ldots i_k}^{s} \gamma^{c}_{a_1 \ldots a_k} e_{c}.
\]

We also have
\[\theta_{i_1 \ldots i_k} = v^{j} t_{i_1 \ldots i_k}^{s} = v^{j} t_{i_1 \ldots i_k}^{s} e_{a_1} \ldots e_{a_k} = v^{j} g^{b} t_{i_1 \ldots i_k}^{s} e_{a_1} \ldots e_{a_k} = v^{j} g^{b} t_{i_1 \ldots i_k}^{s} \gamma^{c}_{a_1 \ldots a_k} e_{c}
\]
and
\[\theta_{i_1 \ldots i_k} e_{a_1} \ldots e_{a_k} = v^{j} t_{i_1 \ldots i_k}^{s} q^{b} \gamma^{c}_{a_1 \ldots a_k} e_{c} = v^{j} t_{i_1 \ldots i_k}^{s} q^{b} \gamma^{c}_{a_1 \ldots a_k} e_{c}.
\]

Hence
\[(R(\theta))^{ja_1 \ldots a_k} = v^{j} t_{i_1 \ldots i_k}^{s} q^{b} \gamma^{c}_{a_1 \ldots a_k} e_{c}.
\]
which means that
\[(i(R(v))R(t))_{i_1 a_1 \ldots i_k a_k} = v^{j a} t_{j a i_1 a_1 \ldots i_k a_k} - v^j p_{s a_1 a_k} t^c_{s a a_1 a_k} = (R(\theta))_{i_1 a_1 \ldots i_k a_k}.
\]

\[\square\]

**Remark 4.1.** Note that for a Frobenius Weil algebra \(\mathbb{A}\) the realization of tensors is an injective operation. Indeed, if \(t : \mathbb{L} \times \ldots \times \mathbb{L} \to \mathbb{A}\) is a covariant \(\mathbb{A}\)-tensor then \(R(t) = p \circ t = q(t, 1_\mathbb{A})\). Since \(q\) is nondegenerate, it follows that \(t\) and \(R(t)\) vanish or do not vanish simultaneously. For the contravariant \(\mathbb{A}\)-tensors proof is similar.

**Remark 4.2.** The realization of a tensor \(t\) of type \((k, \ell)\) for \(k, \ell \geq 1\) can be constructed in the same way. Namely, if
\[
t = t^{i_1 \ldots i_k} f_{i_1} \otimes \mathbb{A} \otimes \ldots \otimes \mathbb{A} f_{i_\ell} \otimes \mathbb{A}.
\]

then
\[
R(u) = t^{i_1 j_1 a_1 \ldots i_k j_k a_k} f_{i_1 a_1} \otimes \mathbb{R} \otimes \ldots \otimes \mathbb{R} f_{i_\ell j_\ell a_\ell} = p(t^{i_1 \ldots i_k e_{a_1} \ldots e_{a_k}} e_{b_1} \ldots e_{b_\ell}).
\]

### 4.2 The complete lift of a covariant tensor field

Let \(M\) be a smooth manifold of dimension \(m\) and \(\pi_A : T^A M \to M\) its Weil bundle. For a local chart \((U, x^1, \ldots, x^m)\) on \(M\) the functions \(X^i = (x^i)^A = x^i a_a\) form a system of \(\mathbb{A}\)-valued local coordinates on \(T^A U \subset T^A M\), and \((x^a)\) are real local coordinates on \(T^A U\), \(x^{i_0} = x^i \circ \pi_\mathbb{A}\).

Let \(\xi \in T^{k,0}(M)\) be a tensor field of type \((k, 0)\) on \(M\). In local coordinates
\[
\xi = \xi_{i_1 \ldots i_k} dx^{i_1} \otimes \ldots \otimes dx^{i_k}.
\]

Consider the analytic prolongations \(\Xi_{i_1 \ldots i_k} = (\xi_{i_1 \ldots i_k})^A\) of the functions \(\xi_{i_1 \ldots i_k}\). The analytic prolongation \(\xi^A \in T_{\mathbb{A} - \text{diff}}^{k,0}(T^A M)\) of \(\xi\) locally is of the form \(\xi^A = \Xi_{i_1 \ldots i_k} dX^{i_1} \otimes \ldots \otimes dX^{i_k}\). Denote
\[
\xi_{i_1 a_1 \ldots i_k a_k} := p(\Xi_{i_1 \ldots i_k e_{a_1} \ldots e_{a_k}}).
\]

We have
\[
R(\xi^A) = \xi_{i_1 a_1 \ldots i_k a_k} dx^{i_1 a_1} \otimes \ldots \otimes dx^{i_k a_k}.
\]

**Definition.** [5, 35] The complete lift of \(\xi\) is the tensor field
\[
\xi^C = \xi^C_A := R(\xi^A)
\]

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From (48) it follows that for every smooth map \( \varphi : N \to M \)
\[
(T^A \varphi)^* (\xi^C) = (\varphi^* \xi)^C.
\]

**Remark 4.3.** It follows from the Remark 4.1 that for \( k \geq 1 \) the complete lift induces the injective map \( T^{k,0}(M) \to T^{k,0}(T^A M) \), that is, \( \xi^C = 0 \) if and only if \( \xi = 0 \).

**Proposition 4.3.** The complete lift is an injective map \( C^\infty(M) \to C^\infty(T^A M) \) if and only if \( p(1_A) = p_0 \neq 0 \). When \( p(1_A) = 0 \) its kernel is the space of locally constant functions.

**Proof.** In fact, let \( f \in C^\infty(M) \) be a non-zero function and \( f^h = f^a e_a = f^0 e_0 + f^\hat{a} e_{\hat{a}} \). Then \( f^C = p_0 f^0 \) (recall that the indices \( \hat{a}, \hat{b} \) run through the set of values \( 1, \ldots, n \)). It follows from (23) that the functions \( f^\hat{a} \) are locally of the form
\[
f^\hat{a} = \frac{\partial f}{\partial x^i} x^{ia} + \text{summands of degree } \geq 2 \text{ in } x^{j\hat{b}}.
\]
Thus \( f^\hat{a} \equiv 0 \) if and only if all partial derivatives \( \frac{\partial f}{\partial x^i} \) vanish, that is, if \( f \equiv \text{const} \). This means that the condition \( f^C = p_0 f^0 + p_\hat{a} f^\hat{a} = 0 \) is equivalent to \( p_0 f^0 = p_\hat{a} f^\hat{a} = 0 \). When \( p(1_A) \neq 0 \) the first equation gives \( f = 0 \). In case \( p(1_A) = 0 \) the second equation, by virtue of the fact that \( p_n \neq 0 \), means that \( f^n = 0 \). Consequently, \( f = \text{const} \). \( \Box \)

**Proposition 4.4.** Let \( M_A \) be an \( \mathbb{A} \)-smooth manifold and let \( \Xi \in \Omega^k_{\mathbb{A}^- \text{diff}}(M_A) \) be an \( \mathbb{A} \)-smooth exterior form. Then
\[
R(d \Xi) = d (R(\Xi)),
\]

where \( d \) is the exterior differential.

**Proof.** If
\[
\Xi = \Xi_{i_1...i_k} dX^{i_1} \wedge \cdots \wedge dX^{i_k}
\]
in terms of local coordinates, then
\[
d \Xi = \frac{\partial \Xi_{i_1...i_k}}{\partial X^j} dX^j \wedge dX^{i_1} \wedge \cdots \wedge dX^{i_k}.
\]
Let \( \Xi_{i_1...i_k} e_{a_1} \ldots e_{a_k} = \Xi^c_{i_1a_1...i_k a_k} e_c \) be expansions in terms of the basis of \( \mathbb{A} \). Then
\[
(R(\Xi))_{i_1a_1...i_k a_k} = \Xi^c_{i_1a_1...i_k a_k} p_c.
\]
From (21) we obtain
\[
(d R(\Xi))_{jbi_1a_1...i_k a_k} = \frac{\partial}{\partial X^j} (R(\Xi))_{i_1a_1...i_k a_k} = \frac{\partial}{\partial X^j} \Xi^c_{i_1a_1...i_k a_k} p_c
\]
\[
= \gamma^c_{bd} \delta^b_{i_1} \Xi^d_{i_1a_2...i_k a_k} p_c = q_{bd} \delta^b_{i_1} \Xi^d_{i_1a_2...i_k a_k}.
\]
On the other hand, by (22),
\[
(d\Xi)_{j_1\ldots j_k} e_{a_1} \ldots e_{a_k} = \frac{\partial}{\partial x^j} \Xi_{i_1 a_1 \ldots i_k a_k} e_{c} e_{b} = \delta^a \frac{\partial}{\partial x^j} \Xi_{j_1 a_1 \ldots j_k a_k} \gamma^d_{bc} e_d.
\]
Therefore
\[
R(d\Xi)_{j b i_1 \ldots i_k a_k} = \delta^a \frac{\partial}{\partial x^j} \Xi_{j_1 a_1 \ldots j_k a_k} \gamma^d_{bc} p_d = \delta^a \frac{\partial}{\partial x^j} \Xi_{j_1 a_1 \ldots j_k a_k} q_{bc},
\]
which coincides with (55). □

**Corollary 4.1.** Let $M$ be a smooth manifold and $T^A M$ be its Weil bundle. The complete lift commutes with the exterior differential, i.e.
\[
(d\xi)^C = d(\xi^C)
\]
for every $\xi \in \Omega^*(M)$.

This means that the complete lift of exterior forms induces a homomorphism of de Rham cohomology spaces
\[
H^*_dR(M) \rightarrow H^*_dR(T^A M), \quad [\xi] \mapsto [\xi^C].
\]

**Theorem 4.1.** Let $(\mathbb{A}, q)$ be a Frobenius Weil algebra and $p$ its Frobenius covector. If $p(1_A) \neq 0$, then the homomorphism (58) is an isomorphism. If $p(1_A) = 0$, then the homomorphism (58) is zero.

**Proof.** The manifold $M$ may be embedded into $T^A M$ by means of the zero section
\[
s_A : M \rightarrow T^A M.
\]
It is shown in [37] that the complexes $(\Omega^*_{\mathbb{A}} - \text{diff}(T^A M), d)$ and $\mathbb{A} \otimes (\Omega^*(M), d)$ are isomorphic. In fact, to any $\mathbb{A}$-valued exterior form $\xi$ on $M$ there corresponds its $\mathbb{A}$-prolongation $\xi^A$ which is an $\mathbb{A}$-smooth form on $T^A M$. Moreover, each $\mathbb{A}$-smooth form $\theta$ on $T^A M$ coincides with the $\mathbb{A}$-prolongation of $\theta|s_A(M) = s_A^*(\theta)$. Denote the cohomology of $(\Omega^*_{\mathbb{A}} - \text{diff}(T^A M), d)$ by $H^*_{\mathbb{A}} - \text{diff}(T^A M)$. Thus,
\[
H^*_{\mathbb{A}} - \text{diff}(T^A M) \cong \mathbb{A} \otimes H^*_dR(M).
\]
It is also clear that the maps $\pi_A^* : H^*_{dR}(M) \rightarrow H^*_{dR}(T^A M)$ and $s_A^* : H^*_{dR}(T^A M) \rightarrow H^*_dR(M)$ are mutually inverse isomorphisms of de Rham cohomology (we use here the symbols $\pi_A^*$ and $s_A^*$ simultaneously for the maps of exterior forms and for the corresponding maps of the de Rham cohomology). Therefore we get the isomorphism
\[
\begin{align*}
H^*_{\mathbb{A}} - \text{diff}(T^A M) & \xrightarrow{s_A^*} \mathbb{A} \otimes H^*_{dR}(M) \xrightarrow{\mathbb{A} \otimes \pi_A^*} \mathbb{A} \otimes H^*_dR(T^A M).
\end{align*}
\]
Let \( \xi \in \Omega^*(M) = \mathbb{R} \otimes \Omega^*(M) \subset A \otimes \Omega^*(M) \) be a closed form and \( \xi^A = \xi^0 e_0 + \ldots + \xi^n e_n \in \Omega^*_A - \text{diff}(T^A M) \) its analytic prolongation. It is easily seen that

\[
s^\ast_A : \Omega^*_A - \text{diff}(T^A M) \to \mathbb{A} \otimes \Omega^*(M)
\]

maps \( \xi^0 \) to \( \xi \) and all the forms \( \xi^1, \ldots, \xi^n \) to zero. Consequently,

\[
\pi^\ast_A \circ s^\ast_A : H^*_A - \text{diff}(T^A M) \to \mathbb{A} \otimes H^*_dR(T^A M)
\]

maps the cohomology class \( \{ \xi^0 \} \) to \( 1_A \otimes \{ \xi^0 \} \) and the classes \( \{ \xi^1 \}, \ldots, \{ \xi^n \} \) to zero. It follows from (53) that \( \xi^C = p_0 \xi^a = p_0 \xi^0 + p_1 \xi^1 + \ldots + p_n \xi^n \). Hence \( \{ \xi^C \} = p_0 \{ \xi^0 \} = p_0 \{ \pi^\ast_A \xi \} \in H^*_dR(T^A M) \). □

Thus, each Frobenius Weil algebra \((\mathbb{A}, q)\) determines the endomorphism of the cohomology spaces

\[
\Delta_{\mathbb{A}, q} : H^*_dR(M) \xrightarrow{C} H^*_dR(T^A M) \xrightarrow{(\pi^\ast_A)^{-1}} H^*_dR(M),
\]

where the first arrow means the complete lift. It follows from the Theorem 4.1 that this endomorphism is the multiplication by \( p(1_A) \).

**Example 4.1.** Let \( \tau = \pi_{\mathbb{R}(\varepsilon)} : TM \to M \) be the tangent bundle. We denote the local coordinates on \( M \) by \( (x^i) \) and the corresponding local coordinates on \( TM \) by \( (x^i, y^i) \).

Take the basis \( \{ e_0 = 1, e_1 = \varepsilon \} \) in \( \mathbb{R}(\varepsilon) \), and introduce two Frobenius covectors \( p_{(0)} \) and \( p_{(1)} \) on \( \mathbb{R}(\varepsilon) \) defined by \( p_{(0)}(1) = 0, p_{(0)}(\varepsilon) = 1 \) and \( p_{(1)}(1) = 1, p_{(1)}(\varepsilon) = 1 \), respectively.

Let \( \xi = \xi_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Omega^k(M) \) be a closed form. For \( p_{(0)} \) the corresponding complete lift \( \xi^C_{(0)} \) is

\[
\xi^C_{(0)} = y^j \frac{\partial \xi_{i_1 \ldots i_k}}{\partial x^j} dx^{i_1} \wedge \ldots \wedge dx^{i_k} + \xi_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} \wedge dy^{i_k}.
\]

It is easily seen that

\[
\eta := y^j \xi_{j_{i_1 \ldots i_{k-1}}} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}
\]

is a well-defined form on \( TM \) and that \( \xi \) is closed if and only if \( \xi^C_{(0)} = d\eta \). Thus, \( \{ \xi^C_{(0)} \} = 0 \in H^*_dR(TM) \).

For \( p_{(1)} \) the complete lift \( \xi^C_{(1)} \) is

\[
\xi^C_{(1)} = \xi_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} + y^j \frac{\partial \xi_{i_1 \ldots i_k}}{\partial x^j} dx^{i_1} \wedge \ldots \wedge dx^{i_k} + \xi_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} \wedge dy^{i_k} =
\]

\[
= \tau^\ast \xi + \xi^C_{(0)}.
\]

Therefore \( \{ \xi^C_{(1)} \} = [\tau^\ast \xi] \in H^*_dR(TM) \).

**Remark 4.4.** Let \( \xi \in T^{k, 0}(M) \) be a tensor field of type \((k, 0)\) on \( M \) and \( \xi^A \in T^{k, 0}_A - \text{diff}(T^A M) \) its analytic prolongation to \( T^A M \). Then \( e_n \xi^A \) is also an \( A \)-smooth tensor field. If \( \xi = \xi_{i_1 \ldots i_k} dx^{i_1} \otimes \ldots \otimes dx^{i_k} \), then \( e_n \xi^A = e_n \xi_{i_1 \ldots i_k} dX^{i_1} \otimes \ldots \otimes dX^{i_k} = e_n \xi_{i_1 \ldots i_k} dx^{i_10} \otimes \ldots \otimes dx^{i_k0} \). It follows that \( R(e_n \xi^A) \in T^{k, 0}(T^A M) \) is of the form

\[
R(e_n \xi^A) = \xi_{i_1 \ldots i_k} dx^{i_10} \otimes \ldots \otimes dx^{i_k0},
\]

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that is, it coincides with $\pi^*_A \xi$. Thus,
\[
\pi^*_A \xi = R(e_n e^A_k). \tag{59}
\]

### 4.3 The complete lift of a contravariant tensor field

Let $M$ be a real smooth manifold, $T^k M$ its Weil bundle, and let $u \in T^{0,k}(M)$ be a contravariant tensor field on $M$. In terms of local coordinates,
\[
u = u^{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}}.
\]
Consider the analytic prolongations $U^{i_1 \ldots i_k} = (u^{i_1 \ldots i_k})^A$ of the functions $u^{i_1 \ldots i_k}$. The analytic prolongation of $u$ is
\[
u^A = U^{i_1 \ldots i_k} \frac{\partial}{\partial X^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial X^{i_k}}.
\]

Let $U^{i_1 \ldots i_k} e^{a_1} \ldots e^{a_k} = U^{i_1 \ldots i_k} e_b$ be the expansions in terms of the basis in $\mathbb{A}$.

Denote
\[
u^{i_1 a_1 \ldots i_k a_k} := p(U^{i_1 \ldots i_k} e^{a_1} \ldots e^{a_k}) = U^{i_1 a_1 \ldots i_k a_k} \delta^b.
\]
We have
\[
R(u^A) = \nu^{i_1 a_1 \ldots i_k a_k} \frac{\partial}{\partial x^{i_1 a_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k a_k}}.
\]

**Definition.** The complete lift $u^C$ of $u$ is the tensor field
\[
u^C = u^C_A := R(u^A)
\]
on $T^k M$.

It follows immediately from the Proposition 4.1 that if $\varphi : M \rightarrow N$ is a smooth map and a tensor field $u$ is $\varphi$-related with a tensor field $v$ on $N$, then $u^C$ is $T^k \varphi$-related to $v^C$.

**Remark 4.5.** It follows from the Remark 4.1 that for $k \geq 1$ the complete lift is an injective map $T^{0,k}(M) \rightarrow T^{0,k}(T^k M)$.

**Proposition 4.5.** Let $M_A$ be an $\mathbb{A}$-smooth manifold and let $U, V \in \mathcal{V}_{A-diff}(M_A)$ be two $\mathbb{A}$-smooth multivector fields. Then
\[
[R(U), R(V)] = R([U, V]). \tag{61}
\]

**Proof.** Let $U \in \mathcal{V}_{A-diff}(M_A)$, $V \in \mathcal{V}_{A-diff}(M_A)$. According to (2), in terms of local coordinates, the Schouten-Nijenhuis bracket $[U, V]$ is of the form
\[
[U, V]^{k_2 \ldots k_g + \ell} = \varepsilon_{i_2 \ldots i_g i_{g+1} \ldots i_g + \ell} U^{r i_2 \ldots i_g} \frac{\partial}{\partial X^r} V^{i_{g+1} \ldots i_g + \ell} + (-1)^g \varepsilon_{i_2 \ldots i_g i_{g+1} i_{g+2} \ldots i_g + \ell} \frac{\partial}{\partial X^r} U^{r i_2 \ldots i_g + 1}. \tag{62}
\]

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Let us multiply each side of (62) by \( e^{a_1} \cdots e^{a_g} \) and then contract with \( \delta^s \). In the left-hand side we get \((R([U,V]))^{k_2 \cdots k_g+\ell} \). Using (22) and (15), we transform the first summand in the right-hand side of (62), omitting the coefficient \( \varepsilon_{i_2 \cdots i_g+1 \cdots i_g+\ell} \) as follows.

\[
U^{r_1 \cdots r_g} e^{a_1} \cdots e^{a_g} \frac{\partial}{\partial x^{r}} V^{i_g+1 \cdots i_g+\ell} e^{a_{g+1}} \cdots e^{a_{g+\ell}} \\
= U^{r_1 \cdots r_g} e^{a_1} \cdots e^{a_g} b \frac{\partial}{\partial x^{r}} V^{i_g+1 a_{g+1} \cdots i_g+\ell a_{g+\ell}} \\
= U^{r_1 \cdots r_g} e^{a_1} \cdots e^{a_g} \frac{\partial}{\partial x^{r}} (\delta^b V^{i_g+1 a_{g+1} \cdots i_g+\ell a_{g+\ell}}) \\
= U^{r_1 \cdots r_g} e^{a_1} \cdots e^{a_g} \frac{\partial}{\partial x^{r}} ((R(V))^{i_g+1 a_{g+1} \cdots i_g+\ell a_{g+\ell}}).
\]

Contracting the result with \( \delta^s \) we get

\[
(R(U))^{r_1 \cdots r_g} e^{a_1} \cdots e^{a_g} \frac{\partial}{\partial x^{r}} ((R(V))^{i_g+1 a_{g+1} \cdots i_g+\ell a_{g+\ell}}).
\]

The commutativity of multiplication in \( A \) yields that \( \varepsilon_{i_2 \cdots i_g+\ell} = \varepsilon_{i_2 \cdots i_g+\ell} \). In the same manner we transform the second summand in (62). As a result, we have

\[
(R([U,V]))^{k_2 \cdots k_g+\ell} \\
= \varepsilon_{i_2 \cdots i_g+\ell} (R(U))^{r_1 \cdots r_g} e^{a_1} \cdots e^{a_g} \frac{\partial}{\partial x^{r}} ((R(V))^{i_g+1 a_{g+1} \cdots i_g+\ell a_{g+\ell}}) \\
+ (-1)^g \varepsilon_{i_2 \cdots i_g+\ell} (R(V))^{r_1 \cdots r_g} e^{a_1} \cdots e^{a_g} \frac{\partial}{\partial x^{r}} ((R(U))^{i_1 a_1 \cdots i_g a_g}) ,
\]

which coincides with (61).  \( \square \)

**Corollary 4.2.** Let \( M \) be a smooth manifold and \( T^h M \) its Weil bundle. The complete lift commutes with the Schouten-Nijenhuis bracket, i.e.,

\[
[u, v]^C = [u^C, v^C].
\]

for every \( u, v \in \mathcal{V}^*(M) \).

**Proposition 4.6.** Let \( M \) be a smooth manifold, and let \( \xi \in \Omega^*(M), \ v \in \mathcal{V}^1(M) \).

Then

\[
i(v^C)\xi^C = (i(v))\xi^C.
\]

(64)
Proof. Follows from the Proposition 4.2. □

Remark 4.6. For \( v \in \mathcal{V}^\ell(M), \ell \geq 2 \), relation (64), in general, does not remain valid. Consider, for example, \( \xi = \xi_{ij} dx^i \wedge dx^j \in \Omega^2(M) \), \( v = v^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \in \mathcal{V}^2(M) \). Their complete lifts to the tangent bundle \( TM \) corresponding to the Frobenius covector \( p(0) \) (see Example 4.1) in terms of standard coordinates \((x^i, y^i)\) are of the form

\[
\xi^C = y^k \frac{\partial \xi_{ij}}{\partial x^k} dx^i \wedge dx^j + \xi_{ij} dx^i \wedge dy^i, \quad v^C = v^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + y^k \frac{\partial v^{ij}}{\partial x^k} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}.
\]

Consequently, the inner product of the lifts is

\[
i(v^C)\xi^C = v^{ij} \xi_{ij}.
\]

But the lift of the inner product is

\[
(i(v))^C = y^k \frac{\partial}{\partial x^k} (v^{ij} \xi_{ij}).
\]

This may be explained by the fact that the complete lift of the tensor product is not equal to the tensor product of complete lifts: \( (u \otimes v)^C \neq u^C \otimes v^C \) (cf. Remark 4.8 below).

4.4 The vertical lift of a tensor field

The vertical lift of a tensor field

\[
u = u_{i_1 \ldots i_k}^{j_1 \ldots j_\ell} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}}, \tag{65}\]

in terms of the local coordinates \((x^i, y^i)\) on \( TM \) is of the form

\[
u^V = u_{i_1 \ldots i_k}^{j_1 \ldots j_\ell} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial y^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{j_\ell}}.
\]

We generalize the notion of the vertical lift to the case of an arbitrary Frobenius Weil algebra \( A \) in the following way. Consider a Jordan-Hölder basis \( \{e_0, \ldots, e_n\} \) in \( A \). Let \( u \in T^{k,\ell}(M) \) be a tensor field \( (65) \) on \( M \) and \( u^A \) its analytic prolongation to \( T^A M \). We define the vertical lift \( u^V \in T^{k,\ell}(T^A M) \) of \( u \) by

\[
u^V = u^V := R(e_n u^A).
\]

The above definition does not depend on the choice of a Jordan-Hölder basis because \( e_n \) is a basis of the one-dimensional ideal \( \mathfrak{A}^n \) (see Proposition 3.4) and is fixed by the condition \( p(e_n) = 1 \).

Proposition 4.7. In terms of local coordinates \((x^{i_a})\) on \( T^A M \),

\[
u^V = u_{i_1 \ldots i_k}^{j_1 \ldots j_\ell} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}}.
\]

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Proof. It follows from (33) (see also Remark 4.5) that $(e_n u^A)_{i_1 \ldots i_k}^{j_1 \ldots j_k} e_{a_1} \ldots e_{a_k} e_{b_1} \ldots e_{b_\ell}$ vanishes for all values of indices except for the case $a_1 = \ldots = a_k = 0$ and $b_1 = \ldots = b_\ell = n$, when it is equal to $e_n (u_{i_1 \ldots i_k}^A) = e_n u_{i_1 \ldots i_k}^A$. Contracting with $p$, we obtain
\[
(u^V)^b_{i_1 a_1 \ldots i_k a_k} = \begin{cases} 
  u_{i_1 a_1 \ldots i_k a_k}^A, & \text{if } a_1 = \ldots = a_k = 0, \ b_1 = \ldots = b_\ell = n, \\
  0, & \text{otherwise}.
\end{cases}
\]
\]
\]
\]
\]
\]

The following proposition follows immediately from (67).

**Proposition 4.8.** For every $u, v \in T^{\ast,*}(M)$ we have
\[
(u \otimes v)^V = u^V \otimes v^V. \tag{68}
\]

It should be noted that, in accordance with (59), the vertical lift of a smooth function $f$ is equal to $f \circ \pi_h$ [35].

**Remark 4.7.** For the discussion in Section 3 it is convenient to present here a direct coordinate proof of the fact that (67) defines a tensor field on $T^hM$ (cf. [40]). Let $x' = x'(x)$ be a coordinate change on $M$ and
\[
\frac{\partial}{\partial x'^i} = \frac{\partial x'^i}{\partial x^j}. \tag{69}
\]

Let us find the change of coordinates $x'^a = x'^a(x'^b)$ on $T^hM$ corresponding to a change $x' = x'(x)$. By (23), we have
\[
X'^i = x'^0 + x'^a e_a = x'^0 + \sum_{|p|=1}^h \frac{1}{p!} D^p x'^i \frac{\partial}{\partial x^p}. \tag{70}
\]

Then, for $T^hM$ we have
\[
x'^0 = x'^0(x'^0), \quad x'^a = x'^a(x'^b), \quad \hat{a} = 1, \ldots, n.
\]

Let us show that
\[
\frac{\partial x'^a}{\partial x'^b} = 0 \quad \text{for } b > \hat{a}.
\]

In fact, the coefficients $\frac{1}{p!} D^p x'^i$ in (70) depend only on $x^i = x'^0$, while $x'^b$ occurs in an expression $(\tilde{X}^1)^{p_1} \ldots (\tilde{X}^m)^{p_m}$ only as a coefficient of $e_b$ in $\tilde{X}^i$. Since $\gamma^s_{cd} = 0$ for $c > s$, the coefficient of $e_{\hat{a}}$ in the expansion of $(\tilde{X}^1)^{p_1} \ldots (\tilde{X}^m)^{p_m}$ does not depend on $x'^b$. Moreover,
it can depend on \( x^i^\hat{a} \) only when \( |p| = 1 \). The part of the sum (70) corresponding to \( p = 1 \) is of the form \( \frac{\partial x'}{\partial x^j} x^j e^{i} \), and the variable \( x^i^\hat{a} \) appears in this expression only when \( \hat{c} = \hat{a} \). Therefore,

\[
\frac{\partial x'^i}{\partial x^b} = \begin{cases} 0, & \text{if } \hat{b} > \hat{a}, \\ \frac{\partial x'}{\partial x^i}, & \text{if } \hat{b} = \hat{a}, \end{cases}
\]

and the Jacobi matrix \[ \begin{vmatrix} \frac{\partial x'^i}{\partial x^b} \end{vmatrix} \] of the coordinate change \( x''^i = x'^i(a^b) \) on \( T^a M \) has the following block structure:

\[
\begin{array}{cccccc}
\frac{\partial x'}{\partial x^i} & \ast & \ast & \cdots & \ast & \ast \\
0 & \frac{\partial x'}{\partial x^i} & \ast & \cdots & \ast & \ast \\
0 & 0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{\partial x'}{\partial x^i} & \ast \\
0 & 0 & \cdots & 0 & 0 & \frac{\partial x'}{\partial x^i} \\
\end{array}
\]

(71)

where \( \ast \) denotes the blocks which are unessential for our consideration. Now (69) is obvious.

**Proposition 4.9.** For \( u, v \in \mathcal{V}^*(M) \) we have

\[
a) \quad [u, v]^V = [u^V, v^C] = [u^C, v^V]; \\
b) \quad [u^V, v^V] = 0.
\]

(72)

**Proof.** Let \( u^A \) and \( v^A \) be the \( A \)-prolongations of \( u \) and \( v \), respectively. Then

\[
[u, v]^V = R(e_n [u^A, v^A]) = R([e_n u^A, v^A]) = [R(e_n u^A), R(v^A)] = [u^V, v^C].
\]

Similarly, \( [u, v]^V = [u^C, v^V] \).

The second equality is proved in the same way:

\[
[u^V, v^V] = [R(e_n u^A), R(e_n v^A)] = R(e_n e_n [u^A, v^A]) = 0.
\]

For the case of the tangent bundle \( TM \) the relations (72) were proved by J. Grabowski and P. Urbański [9, 10].

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Remark 4.8. J. Grabowski and P. Urbański have also proved \cite{9} that for the tangent bundle one has

\[(u \otimes v)^C = u^C \otimes v^V + u^V \otimes v^C\]  

(73)

for every \(u, v \in T^{*,*}(M)\) (see also the monograph of K. Yano and S. Ishihara \cite{52}). This formula can be generalized for the case of Frobenius Weil algebras of height \(h > 1\) in the following way. Let \(u \in T^{k,\ell}(M)\) be a smooth tensor field and \(u^A \in T^{k,\ell}_{\text{diff}}(T^A M)\) its analytic prolongation. Define the \(a\)-lift \(u^{(a)} \in T^{k,\ell}(T^A M)\) of \(u\) for a fixed basis \(\{e_a\}\) by

\[u^{(a)} := R(e_a u^A)\]  

(74)

In particular, when \(\{e_a\}\) is the Jordan-Hölder basis, \(u^{(0)} = u^C, u^{(n)} = u^V\).

For \(u, v \in T^{*,*}(M)\) the following generalization of the relation (73) holds:

\[(u \otimes v)^C = \sum_{a,b} q^{ab} u^{(a)} \otimes v^{(b)}.\]  

(75)

In fact, let \(u^A, v^A \in T^{*,*}_{\text{diff}}(T^A M)\) be the analytic prolongations of \(u\) and \(v\), respectively. To prove (75) it suffices to verify that

\[p(XY) = q^{ab} p(e_a X)p(e_b Y).\]

We have

\[p(XY) = X^c Y^d p(e_c e_d) = X^c Y^d q_{cd},\]

\[q^{ab} p(e_a X)p(e_b Y) = q^{ab} X^c p(e_a e_c) Y^d p(e_b e_d) = q^{ab} X^c Y^d q_{ac} q_{bd} = X^c Y^d q^{cd} = X^c Y^d q_{cd}.\]

In the similar manner one can prove that

\[(u \otimes v)^{(a)} = \sum_{b,d} \gamma^{bd}_{la} u^{(b)} \otimes v^{(d)}.\]  

(76)

The tangent bundle \(T^n(M) = T^{R(\varepsilon^n)} M\) of order \(n\) is equivalent to the Weil bundle corresponding to the algebra of plural numbers \(R(\varepsilon^n)\). In this case the so called \(\lambda\)-lifts \(u^{(\lambda)} = R(\varepsilon^\lambda u^{R(\varepsilon^n)})\) of a tensor field \(u\) on \(M\) \((\lambda = 0, \ldots, n)\) are defined. These lifts were considered, for example, in the papers of Ch.-S. Houh and S. Ishihara \cite{11} and A.P. Shirokov \cite{34}. Relation (76), in this case takes the form \cite{11,34}

\[(u \otimes v)^{(\lambda)} = \sum_{\kappa=0}^\lambda u^{(\kappa)} \otimes v^{(\lambda-\kappa)}.\]

Proposition 4.10. Let \(v \in \mathcal{V}^1(M)\) be a vector field on \(M\). The Lie derivative with respect to \(v\) has the following properties:

\[a) \quad (\mathcal{L}_v t)^C = \mathcal{L}_{v^C} t^C;\]

\[b) \quad (\mathcal{L}_v t)^V = \mathcal{L}_{v^V} t^V = \mathcal{L}_{v^V} t^C;\]

\[c) \quad \mathcal{L}_{v^V} t^V = 0\]  

(77)
for every $t \in T^*\ast(M)$.

**Proof.** a) Recall that the components of the Lie derivative of a tensor field

$$t = t^i_{j_1 \ldots j_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_k}}$$

with respect to a vector field $v$ are of the form $[29, 51]$

$$(L_v t)^{j_1 \ldots j_k}_{i_1 \ldots i_k} = \frac{\partial t^i_{j_1 \ldots j_k}}{\partial x^m} v^m + t^i_{m j_1 \ldots j_k} \frac{\partial v^m}{\partial x^{i_1}} + \cdots + t^i_{1 j_1 \ldots j_k} \frac{\partial v^m}{\partial x^{i_k}}$$

$$- t^i_{m j_2 \ldots j_k} \frac{\partial v^1}{\partial x^m} - \cdots - t^i_{1 j_1 \ldots j_{k-1}} \frac{\partial v^{j_k}}{\partial x^m}. \quad (78)$$

For simplicity, we prove (77) in the case $k = \ell = 1$. In the general case this formula is proved in the same way. Let $t = t^i dx^i \otimes \frac{\partial}{\partial x^j}$ be a $(1, 1)$-tensor field on $M$ and let $T$ and $V$ be the analytic prolongations of $t$ and $v$, respectively. Then the analytic prolongations satisfy the relation

$$(L_V T)^i_j = \frac{\partial T^i_j}{\partial x^m} V^m + T^i_m \frac{\partial V^m}{\partial x^j} - T^i_m \frac{\partial V^j}{\partial x^m}. \quad (79)$$

Recall that if $V^i = v^i_e a^e = v^i_b^a$, then $(v^C)^ia = v^ia$ and then contract the result with $p$. On the left-hand side we have $((L_a t)^C)^{ja}_i$, and it remains to prove that the right-hand is of the form

$$\frac{\partial t_{ia}}{\partial x^{mc}} v^{mc} + t_{mb} \frac{\partial v^{mc}}{\partial x^{ia}} - t_{ia} \frac{\partial v^{jb}}{\partial x^{mc}}$$

where $t^b_{ia} = (t^C)^{jb}_i$ and $v^ia = (v^C)^{ia}$.

First, let $T^i_j e^a = (T^j_{ia} c^e c^c).$ Then, by virtue of $[22], [15$ and $[36]$,

$$p\left(\frac{\partial T^i_j}{\partial x^m} V^m e^a\right) = p\left(\frac{\partial (T^j_{ia} c^d e^c v^m s^e)}{\partial x^{md}}\right) = \frac{\partial (T^j_{ia} d^c e^c v^m s^e)}{\partial x^{mc}} = \frac{\partial t_{ia}}{\partial x^{mc}} v^{mc}.$$ 

Second, by (24) we have

$$T^j_m \frac{\partial V^m}{\partial x^i} e^a e^b = T^j_m \delta^d \frac{\partial v^m}{\partial x^{id}} e^d e^a e^b = T^j_m \delta^d \frac{\partial v^m}{\partial x^{id}} \gamma^{ag} e^c e^b = T^j_m e^c e^b \frac{\partial v^m}{\partial x^{ia}} = (T^j_{mc} d^c) e^d \frac{\partial v^m}{\partial x^{ia}}.$$ 

Whence

$$p\left(T^j_i \frac{\partial V^m}{\partial x^i} e^a e^b\right) = t_{mc} \frac{\partial v^m}{\partial x^{ia}}.$$ 

Finally, let $V^i e^b = V^j_c e^c.$ Then

$$T^m_i \frac{\partial V^j}{\partial x^m} e^a e^b = T^m_i e^a \delta^d \frac{\partial V^j}{\partial x^{md}} e^c = T^m_i e^a \delta^d \frac{\partial V^j}{\partial x^{md}} = T^m_i e^a e^c \frac{\partial v^m}{\partial x^{ic}} = T^m_i e^a e^c \frac{\partial v^m}{\partial x^{ic}}.$$ 

Therefore,

$$p\left(T^m_i \frac{\partial V^j}{\partial x^m} e^a e^b\right) = t_{mc} \frac{\partial v^m}{\partial x^{ic}}.$$ 

Formulas (77) b) and c) are proved in the same way. □

For tangent bundles relations (77) has been proved in $[52]$ and $[15]$.
5 Weil bundles of Poisson manifolds

In this section we consider the complete and the vertical lifts of a Poisson tensor $w$ and establish some properties of the Poisson structures arised. In particular, we compute the modular classes of these structures.

5.1 The complete lift of a Poisson tensor

Let $(M, w)$ be a Poisson manifold, and let $w^C$ be the complete lift of $w$ to $T^hM$. By virtue of (6) and (63),

$$[w^C, w^C] = 0.$$ 

Hence $w^C$ is a Poisson tensor on $T^hM$.

Let $w^A$ be the analytic prolongation of $w$ and let $(w^{ij})^A = (w^{ij})^s e_s$ be the expansions in terms of a basis in $A$. Then $(w^{ij})^A e^a e^b = (w^{ij})^s e_s e^a e^b$, therefore $(w^C)^{iajb} = (w^{ij})^s \gamma_{ab}^s$ by (42). Thus, the components of $w^C$ in terms of the local coordinates on $T^hM$ are as follows

$$w^C_{iajb} = (w^{ij})^s \gamma_{ab}^s.$$ 

(80)

Proposition 5.1. If $(M, w)$ is a symplectic manifold, then $(T^hM, w^C)$ also is a symplectic manifold.

Proof. By the Darboux theorem [41], we can choose a local coordinate system $(x^i)$ on $M$, in terms of which the components $w^{ij}$ are constant. Then, by (23), the analytic prolongations $(w^{ij})^A$ coincide with $w^{ij}$ and so $(w^{ij})^0 = w^{ij}$ and $(w^{ij})^k = 0$ for $k \geq 1$. Therefore $w^{iajb} = w^{ij} \gamma_0^{ab} = w^{ij} q^{ab}$. Thus,

$$w^C = w^{ij} q^{ab} \frac{\partial}{\partial x^{ia}} \wedge \frac{\partial}{\partial x^{jb}}$$

where the matrix $\|(w^C)^{iajb}\| = \|w^{ij} q^{ab}\|$ is nondegenerate as the tensor (Kronecker) product of nondegenerate matrices $\|w^{ij}\|$ and $\|q^{ab}\|$. □

One can easily see that if $\omega = \omega_{ij} \, dx^i \wedge dx^j$ is the symplectic form on $M$ corresponding to $w$, then the complete lift

$$\omega^C = \omega_{ij} q_{ab} \, dx^{ia} \wedge dx^{jb}$$

is the symplectic form on $T^hM$ corresponding to $w^C$. In the same way one can prove that if $(M, w)$ is a regular Poisson manifold, then $(T^hM, w^C)$ is also a regular Poisson manifold.

Remark 5.1. The fact that for a Frobenius Weil algebra $(A, q)$ the total space $T^hM$ of a symplectic manifold $(M, w)$ carries a natural symplectic structure, was pointed out by A.V. Brailov [2].
Example 5.1. Let \((M, w)\) be a Poisson manifold and \(TM\) its tangent bundle. The complete lift of \(w\) corresponding to \(p(0)\) (Example 4.1) is of the form
\[
w^C = w_{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + y^k \frac{\partial w_{ij}}{\partial x^k} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j},
\]
and the complete lift \(w^C\), corresponding to \(p(1)\), is
\[
w^C = w_{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \left( y^k \frac{\partial w_{ij}}{\partial x^k} - w_{ij} \right) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}.
\]
Poisson bivector (81) was studied by several authors, e.g., T.J. Courant [3], J. Grabowski and P. Urbański [8, 9, 10], G. Mitric and I. Vaisman [20, 45].

Proposition 5.2. Let \(w\) be a bivector field on \(M\). The complete lift \(w^C\) is a Poisson bivector on \(T^A M\) if and only if \(w\) is a Poisson bivector on \(M\).

Proof. Follows from (63) and Remark 4.5. □

Proposition 5.3. Let \((M, w)\) and \((M', w')\) be Poisson manifolds and let \(\varphi : (M, w) \to (M', w')\) be a Poisson map. Then \(T^A \varphi : (T^A M, w^C) \to (T^A M', (w')^C)\) is also a Poisson map.

Proof. Follows from (10) and Proposition 4.1. □

Proposition 5.4. The complete lift of multivector fields induces the homomorphism of Poisson cohomology spaces
\[
H^*_P(M, w) \longrightarrow H^*_P(T^A M, w^C), \quad [u] \longmapsto [u^C].
\]

Proof. From (63) it follows that
\[
(\sigma_{w} u)^C = \sigma_{w^C} u^C,
\]
which implies that the map (82) is well-defined. □

It follows from Proposition 4.5 that if \(f \in C^\infty(M)\) is a Casimir function of \(w\) then \(f^C = R(f^A)\) and \(f^V = f \circ \pi_A\) are Casimir functions of \(w^C\). For an arbitrary smooth function \(f\) on \(M\) one has
\[
(X^w_f)^C = X^{w^C}_f.
\]

Proposition 5.5. If \(p(1_A) \neq 0\), then the homomorphism (82) in the dimension 0
\[
H^0_P(M, w) \longrightarrow H^0_P(T^A M, w^C)
\]
is a monomorphism. If \(p(1_A) = 0\) then the kernel of (84) is the space of constant functions.
Proof. Follows from Remark 4.3 □

Proposition 5.6. Let $(M, w)$ be a Poisson manifold. For every Frobenius Weil algebra $(A, q)$ the following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{V}^*(M) & \xrightarrow{C} & \mathcal{V}^*(T^A M) \\
\hat{w} & & \hat{w}^C \\
\Omega^*(M) & \xrightarrow{C} & \Omega^*(T^A M)
\end{array}
$$

where the horizontal arrows mean the complete lift.

Proof. Define $\gamma^{a_1\ldots a_k}_b$ by

$$e^{a_1}\ldots e^{a_k} = \gamma^{a_1\ldots a_k}_b e^b.$$

Consider an exterior form $\xi \in \Omega^k(M)$ and a multivector field $v \in \mathcal{V}^k(M)$. In terms of local coordinates,

$$\xi = \xi_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \quad \text{and} \quad v = v^{j_1\ldots j_k} \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_k}}.$$

Let $(\xi_{i_1\ldots i_k})^A = \xi_{i_1\ldots i_k}^f e^f$ and $(v^{j_1\ldots j_k})^A = (v^{j_1\ldots j_k})^s e^s$ be the analytic prolongations of $\xi_{i_1\ldots i_k}$ and $v^{j_1\ldots j_k}$, respectively. Then

$$(\xi^C)_{i_1 a_1\ldots i_k a_k} = p(\xi_{i_1\ldots i_k}^f e^f e_{a_1} \ldots e_{a_k}) = \xi_{i_1\ldots i_k}^f \gamma^{d}_{a_1\ldots a_k} p,$$

and

$$(v^C)^{j_1 b_1\ldots j_k b_k} = p((v^{j_1\ldots j_k})^s e^s e^{b_1} \ldots e^{b_k}) = (v^{j_1\ldots j_k})^s q_{sf} e^f e^{b_1} \ldots e^{b_k}) = (v^{j_1\ldots j_k})^s q_{sf} \gamma^{f}_{d b_1\ldots b_k} \delta^d,$$

where $\gamma^{b}_{a_1\ldots a_k}$ are defined by (52).

In our case $v = \hat{w}\xi$. According to (12), $\nu^{j_1\ldots j_k} = (-1)^k w^{i_1 j_1} \ldots w^{i_k j_k} \theta_{i_1\ldots i_k}$. Thus, we need to show that

$$(w^C)^{i_1 a_1 j_1 b_1} \ldots (w^C)^{i_k a_k j_k b_k} (\xi^C)_{i_1 a_1\ldots i_k a_k} = (v^C)^{j_1 b_1\ldots j_k b_k}.$$

We have

$$(w^C)^{i_1 a_1 j_1 b_1} \ldots (w^C)^{i_k a_k j_k b_k} (\xi^C)_{i_1 a_1\ldots i_k a_k} = (w^{i_1 j_1})^{c_1} \gamma^{a_1 b_1}_{c_1} \ldots (w^{i_k j_k})^{c_k} \gamma^{a_k b_k}_{c_k} \xi^{f}_{i_1 a_1\ldots i_k a_k} p.$$

On the other hand, it follows from Proposition 3.1 that

$$(v^{j_1\ldots j_k})^A = (w^{i_1 j_1})^{c_1} \ldots (w^{i_k j_k})^{c_k} \xi^{f}_{i_1 a_1\ldots i_k a_k} e_{c_1} \ldots e_{c_k}.$$
Therefore
\[
(v^C)_{j_1 \cdots j_k} = (w^{i_1 j_1}) e_1 \cdots (w^{i_k j_k}) e_k \mathcal{C}_{s_1 \cdots s_k} \mathcal{P}(e_f c_1 \cdots e_c e_{b_1} \cdots e_{b_k}).
\]

Thus, it suffices to prove that
\[
\gamma_{c_1}^{a_1 b_1} \cdots \gamma_{c_k}^{a_k b_k} \mathcal{P}_{a_1 \cdots a_k} \mathcal{P}_d = \mathcal{P}(e_f c_1 \cdots e_c e_{b_1} \cdots e_{b_k}).
\]

We have
\[
p(e_f c_1 \cdots e_c e_{b_1} \cdots e_{b_k})
\]
\[
= q^{a_1 b_1} \cdots q^{a_k b_k} p(e_a \cdots e_e e_f c_1 \cdots e_c) = q^{a_1 b_1} \cdots q^{a_k b_k} \mathcal{P}_{a_1 \cdots a_k} \mathcal{P}_d.
\]

According to (82), \(q^{a_1 b_1} = \gamma_{c_1}^{a_1 b_1} \mathcal{D}_c, \ldots, q^{a_k b_k} = \gamma_{c_k}^{a_k b_k} \mathcal{D}_c\). Since for a Jordan-Hölder basis \(\mathcal{D}_0 = 1\) and \(\mathcal{D}_0 = 0\), it follows that \(\mathcal{P}_{a_1 \cdots a_k} \mathcal{P}_d \mathcal{D}_c = \mathcal{P}_d \mathcal{P}_s = p_d\). Hence the result.

**Theorem 5.1.** The following diagram of morphisms of complexes is commutative
\[
\begin{array}{ccc}
(V^*(M), \sigma_w) & \xrightarrow{C} & (V^*(T^h M), \sigma_w C) \\
\hat{w} & & \hat{w} C \\
(\Omega^*(M), d) & \xrightarrow{C} & (\Omega^*(T^h M), d)
\end{array}
\]

**Proof.** The commutativity of the three-dimensional diagram
\[
\begin{array}{ccc}
V^k(M) & \xrightarrow{C} & V^k(T^h M) \\
\sigma_w & & \sigma_w C \\
V^{k-1}(M) & \xrightarrow{C} & V^{k-1}(T^h M) \\
\hat{w} & & \hat{w} C \\
\Omega^k(M) & \xrightarrow{C} & \Omega^k(T^h M) \\
d & & d \\
\Omega^{k-1}(M) & \xrightarrow{C} & \Omega^{k-1}(T^h M)
\end{array}
\]

follows from (57), (83) and (13). □

**Corollary 5.1.** Let \((A, q)\) be a Frobenius Weil algebra and let \(p\) be the corresponding Frobenius covector. Let \((M, w)\) be a symplectic manifold. If \(p(1_h) \neq 0\), then (82) is an isomorphism. If \(p(1_h) = 0\), then (82) is the zero map.
Proof. For a symplectic manifold the maps $\tilde{w}$ and $\tilde{w}^C$ are isomorphisms. Thus, the vertical arrows in (86) are isomorphisms. The rest of the proof follows from Theorem 4.1. □

Remark 5.2. It has been shown in [20] and [44] that for the tangent bundle $TM$ of a Poisson manifold $(M, w)$ the complete lift $w^C$ (81) is an exact Poisson structure. In fact, for $E = y^i \frac{\partial}{\partial y^i}$ we have $w^C = \sigma_{w,C}E = [w^C, E]$. One can easily verify that the vertical lift $w^V = w^i_j \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$ of $w$ is also exact: $w^V = [w^V, E]$.

In the case of an arbitrary Frobenius Weil algebra, Corollary 5.1 implies the following proposition.

Proposition 5.7. Let $(A, q)$ be a Frobenius Weil algebra, and let $p$ be the corresponding Frobenius covector such that $p(1_A) = 0$. If $(M, w)$ is a symplectic manifold then $(T^k M, w^C)$ is an exact symplectic manifold.

The following example shows that homomorphism (82) may be a monomorphism or have a nonzero kernel depending on the dimension of the cohomology space.

Example 5.2. Let $T^2 = S^1 \times S^1$ be the two-dimensional torus and let $M = T^2 \times \mathbb{R}^k$. We denote the standard angle coordinates on the torus by $(x^1, x^2)$ and the standard coordinates on $\mathbb{R}^k$ by $(t^1, \ldots, t^k)$. Consider the case of the algebra $\mathbb{R}(\varepsilon)$ with Frobenius covector $p(0)$ (see Example 4.1). The corresponding Weil bundle of $M$ coincides with its tangent bundle: $T^{\mathbb{R}(\varepsilon)} M = TM$. The tangent bundle of $M = T^2 \times \mathbb{R}^k$ is trivial: $TM \cong M \times (\mathbb{R}^2 \times \mathbb{R}^k)$. Denote by $(y^1, y^2)$ and $(s^1, \ldots, s^k)$ the standard coordinates in $\mathbb{R}^2$ and $\mathbb{R}^k$, respectively.

The bivector field $w = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ defines a regular Poisson structure on $M$. The complete lift of $w$ is of the form $w^C = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^2} + \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial x^2}$.

In the dimension 0, by (14) we have

\[
H^0_P(M, w) \cong C^\infty(\mathbb{R}^k) \cong \mathbb{R} \oplus C_0^\infty(\mathbb{R}^k),
\]
\[
H^0_P(TM, w^C) \cong C^\infty(\mathbb{R}^{2k}),
\]

where $C_0^\infty(\mathbb{R}^k)$ is the subring of $C^\infty(\mathbb{R}^k)$, consisting of smooth functions vanishing at 0 $\in \mathbb{R}^k$. The complete lift of a function $f \in C^\infty(M)$ is $f^C = y^i \frac{\partial f}{\partial y^i} + s^a \frac{\partial f}{\partial s^a}$. The kernel of homomorphism (82) consists of constant functions (see Remark 4.3). Hence, its image is isomorphic to $C_0^\infty(\mathbb{R}^k)$.

In the dimension 1, we have

\[
H^1_P(M, w) \cong \mathcal{V}(\mathbb{R}^k) \oplus C^\infty(\mathbb{R}^k) \oplus C^\infty(\mathbb{R}^k),
\]
\[
H^1_P(TM, w^C) \cong \mathcal{V}(\mathbb{R}^{2k}) \oplus C^\infty(\mathbb{R}^{2k}) \oplus C^\infty(\mathbb{R}^{2k}).
\]

The cohomology classes of the following vector fields form a complete system of generators of $H^1_P(M, w)$:

\[
f^i \frac{\partial}{\partial x^i}, \quad f^a \frac{\partial}{\partial s^a}.
\]
where \( f^i, f^a \in C^\infty(\mathbb{R}^k) \). The complete lifts of the above indicated vector fields, respectively, are

\[
\left( f^i \frac{\partial}{\partial x^i} \right)^C = f^i \frac{\partial}{\partial x^i} + s^b \frac{\partial f^i}{\partial y^b} \frac{\partial}{\partial y^i}, \quad \left( f^a \frac{\partial}{\partial t^a} \right)^C = f^a \frac{\partial}{\partial t^a} + s^b \frac{\partial f^a}{\partial t^b} \frac{\partial}{\partial s^b}.
\]

One can easily see that

\[
f_1 \frac{\partial}{\partial x^1} = -\sigma_{w^C}(y^2 f^1), \quad f_2 \frac{\partial}{\partial x^2} = \sigma_{w^C}(y^1 f^2),
\]

and that the classes \([s^b \frac{\partial f^i}{\partial y^b} \frac{\partial}{\partial y^i}]\) (where at least one of \( f^i \) is not a constant function) and \(([f^a \frac{\partial}{\partial t^a}]^C)\) are nonzero classes in \( H^1_P(TM, w^C) \). Therefore, in the dimension 1, the kernel of homomorphism (82) is isomorphic to \( \mathbb{R} \oplus \mathbb{R} \), and the image of (82) is isomorphic to \( V_1^1(\mathbb{R}^k) \oplus C^\infty_0(\mathbb{R}^k) \).

In the dimension 2, we have

\[
H^2_P(M, w) \cong V_2^2(\mathbb{R}^k) \oplus V_1^1(\mathbb{R}^k) \oplus V_1^1(\mathbb{R}^k) \oplus C^\infty_0(\mathbb{R}^k),
\]

\[
H^2_P(TM, w^C) \cong V_2^2(\mathbb{R}^2k) \oplus V_1^1(\mathbb{R}^2k) \oplus V_1^1(\mathbb{R}^2k) \oplus C^\infty(\mathbb{R}^2k).
\]

The cohomology classes of the following bivector fields are generators of \( H^2_P(M, w) \):

\[
f \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}, \quad f^{ia} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial t^a} \quad \text{and} \quad f^{ab} \frac{\partial}{\partial t^a} \wedge \frac{\partial}{\partial t^b},
\]

where \( f, f^{ia}, f^{ab} \in C^\infty(\mathbb{R}^k) \). Among the cohomology classes defined by these bivector fields, only the classes \([([f \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}]^C)\) with \( f = \text{const} \) are zero. Thus, in the dimension 2, the kernel of (82) is isomorphic to \( \mathbb{R} \), and the image of (82) is isomorphic to

\[
V_2^2(\mathbb{R}^k) \oplus V_1^1(\mathbb{R}^k) \oplus V_1^1(\mathbb{R}^k) \oplus C^\infty_0(\mathbb{R}^k).
\]

In the dimensions \( s \geq 3 \), for every generator of \( H^s_P(M, w) \) one can find a representative which has the form \( u = u^a \wedge \frac{\partial}{\partial x^a}, u^a \in V^{s-1}(M) \). Therefore, the cohomology class of \( u^C \) is nonzero. Thus, in each dimension \( s = 3, \ldots, k + 2 \), the complete lift induces a monomorphism of Poisson cohomology spaces.

### 5.2 The vertical lift of a Poisson tensor

It follows from (72) that the vertical lift \( w^V \) of a Poisson tensor (as well as of any bivector) \( w \) is a Poisson tensor on \( T^k M \).

The following example shows that the cohomology of \( \sigma_{w^V} \) depends on the choice of \( w \in V^2(M) \).

**Example 5.3.** Let \( M = \mathbb{R}^{2m} \times \mathbb{R}^{2k} \) and let \( w_1 \) be the regular structure induced by the standard symplectic structure on \( \mathbb{R}^{2m} \), \( w_2 \) the regular Poisson structure induced by the standard symplectic structure on \( \mathbb{R}^{2k} \). The sum \( w_1 + w_2 \) is the standard symplectic structure on \( M \).
In the case of the algebra \( \mathbb{R}(\varepsilon) \), \( TM \cong \mathbb{R}^{4(m+k)} = \mathbb{R}^{2m+2k} \times \mathbb{R}^{2m} \times \mathbb{R}^{2k} \). By virtue of (14), one can obtain

\[
\begin{align*}
H^r_P(TM, w^1) &\cong \mathcal{V}^r(\mathbb{R}^{2m+2k} \times \mathbb{R}^{2k}), \\
H^r_P(TM, w^2) &\cong \mathcal{V}^r(\mathbb{R}^{2m+2k} \times \mathbb{R}^{2m}), \\
H^r_P(TM, (w_1 + w_2)^V) &\cong \mathcal{V}^r(\mathbb{R}^{2m+2k}).
\end{align*}
\]

If \( k > m \), then \( H^r_P(TM, (w_1 + w_2)^V) = 0 \) for \( r > 2m+2k \), \( H^r_P(TM, w^2) = 0 \) for \( r > 4m+2k \), and \( H^r_P(TM, w^1) = 0 \) for \( r > 4k + 2m \).

In the case of an arbitrary Frobenius Weil algebra \( A \) of dimension \( n + 1 \),

\[
\begin{align*}
H^r_P(T^A M, w^1) &\cong \mathcal{V}^r(\mathbb{R}^{2(m+k)n} \times \mathbb{R}^{2k}), \\
H^r_P(T^A M, w^2) &\cong \mathcal{V}^r(\mathbb{R}^{2(m+k)n} \times \mathbb{R}^{2m}), \\
H^r_P(T^A M, (w_1 + w_2)^V) &\cong \mathcal{V}^r(\mathbb{R}^{2(m+k)n}).
\end{align*}
\]

**Proposition 5.8.** Let \( (M, w) \) and \( (M', w') \) be Poisson manifolds and let \( \varphi : (M, w) \to (M', w') \) be a Poisson map. Then \( T^A \varphi : (T^A M, w^V) \to (T^A M', (w')^V) \) is also a Poisson map.

**Proof.** Similar to that of Proposition 5.3. \( \square \)

Obviously, if \( (M, w) \) is a regular Poisson manifold, then \( (T^A M, w^V) \) is also a regular Poisson manifold.

**Proposition 5.9.** The vertical lift of multivector fields induces the homomorphism of Poisson cohomology spaces

\[
H^r_P(M, w) \longrightarrow H^r_P(T^A M, w^V), \quad [u] \longmapsto [u^V]. \tag{87}
\]

**Proof.** From (72) it follows that if \( \sigma_w u = [u, w] = 0 \), then \( \sigma_w w^V = [w^V, u^V] = 0 \), and if \( u = \sigma_w v = [w, v] \), then \( u^V = \sigma_w v^C = [w^V, v^C] \). \( \square \)

If \( f \in C^\infty(M) \) is a Casimir function of \( w \), then \( f^C \) and \( f^V \) are Casimir functions of \( w^V \). For an arbitrary smooth function \( f \) on \( M \), it follows from Proposition 4.9 that

\[
(X^w)^V = X^{w^V} = X^{w^C}. \tag{87}
\]

**Proposition 5.10.** In the dimension 0, homomorphism (87)

\[
H^0_P(M, w) \longrightarrow H^0_P(T^A M, w^V)
\]

is a monomorphism.

**Proof.** If \([f^V] = [f \circ \pi_A] = 0\), then \( f = 0 \). \( \square \)

The next example shows that the homomorphism (87) may be the zero map in every dimension except for zero dimension.

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Example 5.4. Let \((M, w)\) be a symplectic manifold and let \((x^i)\) be the local coordinate system on \(M\) with respect to which the components \(w^{ij}\) are constant. Let \((x^i, y^i)\) be the induced local coordinate system on \(TM\).

The space \(H^0_p(M, w)\) coincides with the set of constant functions on \(M\). The space \(H^0_p(TM, w^V)\) coincides with the set of constant functions on \(TM\) having constant values along fibers. Therefore, it is isomorphic to \(C^\infty(M)\). By Proposition 5.10, the vertical lift induces a monomorphism \(H^0_p(M, w) \cong \mathbb{R} \to H^0_p(TM, w^V) \cong C^\infty(M)\).

For \(k \geq 1\), homomorphism \([87]\) is the zero map. In fact, if \(v = v^{i_1 \ldots i_k} \frac{\partial}{\partial y^1} \wedge \ldots \wedge \frac{\partial}{\partial y^k} \in \mathcal{V}^k(M)\), then \(v^V = v^{i_1 \ldots i_k} \frac{\partial}{\partial y^1} \wedge \ldots \wedge \frac{\partial}{\partial y^k}\). Denote \(u^{i_1 \ldots i_{k-1}} = y^j \omega_{ji} v^{i_1 \ldots i_{k-1}}\), where \(\omega = \omega_{ij} dx^i \wedge dx^j\) is the symplectic form corresponding to \(w\). Then \(u = u^{i_1 \ldots i_{k-1}} \frac{\partial}{\partial y^1} \wedge \ldots \wedge \frac{\partial}{\partial y^{k-1}}\) is a well-defined multivector field on \(TM\). One can easily verify that \(v^V = [u^V, u] = \sigma_w v u\).

The following example shows that the homomorphism \([87]\) may have a non-zero kernel in each dimension except for zero dimension.

Example 5.5. Let, as in Example 5.2 \(M = T^2 \times \mathbb{R}^k\) and \(w = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\). Then \(w^V = \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2}\).

In the dimension 1,
\[
H^1_p(M, w^V) \cong \mathcal{V}^1(\mathbb{R}^k) \oplus C^\infty(\mathbb{R}^k) \oplus C^\infty(\mathbb{R}^k),
\]
\[
H^1_p(TM, w^V) \cong \mathcal{V}^1(T^2 \times \mathbb{R}^{2k}).
\]

The cohomology classes of the vertical lifts \((f^i \frac{\partial}{\partial x^i})^V = f^i \frac{\partial}{\partial y^i}\) vanish and the cohomology classes of the vertical lifts \((f^a \frac{\partial}{\partial x^a})^V = f^a \frac{\partial}{\partial y^a}\) are linearly independent in \(H^1_p(TM, w^V)\). Therefore, in the dimension 1, the kernel of \([87]\) is isomorphic to \(C^\infty(\mathbb{R}^k) \oplus C^\infty(\mathbb{R}^k)\), and the image of \([87]\) is isomorphic to \(\mathcal{V}^1(\mathbb{R}^k)\).

In the dimensions \(\ell = 2, \ldots, k+2\),
\[
H^\ell_p(M, w) \cong \mathcal{V}^\ell(\mathbb{R}^k) \oplus \mathcal{V}^{\ell-1}(\mathbb{R}^k) \oplus \mathcal{V}^{\ell-2}(\mathbb{R}^k) \oplus \mathcal{V}^{\ell-1}(\mathbb{R}^k) \oplus \mathcal{V}^{\ell-2}(\mathbb{R}^k),
\]
\[
H^\ell_p(TM, w^V) \cong \mathcal{V}^\ell(T^2 \times \mathbb{R}^{2k}).
\]

The cohomology classes \([f^{a_1 \ldots a_{\ell-1}} \frac{\partial}{\partial x^{a_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{a_{\ell-1}}}]^V\) are linearly independent and generate the image of \([87]\). Hence, the kernel of \([87]\) is isomorphic to \(\mathcal{V}^{\ell-1}(\mathbb{R}^k) \oplus \mathcal{V}^{\ell-1}(\mathbb{R}^k) \oplus \mathcal{V}^{\ell-2}(\mathbb{R}^k)\), and the image of \([87]\) is isomorphic to \(\mathcal{V}^\ell(\mathbb{R}^k)\).

Remark 5.3. As it seems, there are no natural nonzero homomorphisms between the cohomology spaces \(H^p_p(T^k M, w)\) and \(H^p_p(T^k M, w^V)\). Simple examples show that the identity map \(\text{id} : \mathcal{V}^*(T^k M) \to \mathcal{V}^*(T^k M)\) in general is not a cochain map of the complexes
\[
(\mathcal{V}^*(T^k M), \sigma_w) \quad \text{and} \quad (\mathcal{V}^*(T^k M), \sigma_{w^V}).
\]

Proposition 5.11. Let \((M, w)\) be a Poisson manifold. For the complete lift \(w^C\), the vertical lift \(w^V\) of \(w\) to the Weil bundle \(T^k M\), and for any exterior form \(\xi \in \Omega^*(M)\) we have
1) $\widetilde{w^C}(\pi^*_M \xi) = (\tilde{w} \xi)^V$,
2) $\tilde{w^V}(\pi^*_M \xi) = 0$,
3) $\tilde{w^V}(\xi^C) = \begin{cases} (\tilde{w} \xi)^V, & \text{if } |\xi| = 1, \\ 0, & \text{if } |\xi| \geq 2. \end{cases}$

**Proof.** 1) Let in terms of a local coordinate system $\xi = \xi_1 \ldots \xi_m dx^1 \wedge \ldots \wedge dx^m \in \Omega^1(M)$. Then $\pi^*_M \xi = \xi_1 \ldots \xi_m dx^1 \wedge \ldots \wedge dx^m$. It follows from Remark 4.3 that $(w^C)^{ij} = 0$ for $b < n$ and $(w^C)^{ij_0} = w^{ij}$. The rest of the proof is obvious from (67).

2) The proof follows from (67).

3) For $|\xi| = 1$ we have $(\xi^C)_{in} = p((\xi^i)^A e_n) = \xi^i$ and $\tilde{w^V}(\xi^C) = w^{ij} \xi^i \frac{\partial}{\partial x^j} = (\tilde{w} \xi)^V$. For $|\xi| = k \geq 2$ we have $(\xi^C)_{i_1 i_2 n} = p((\xi^i)^A e_n \ldots e_n) = 0$. □

**Remark 5.4.** Let $\{e_a\}$ be a Jordan-Hölder basis in $A$. From Proposition 4.3 it follows that for every $a = 0, 1, \ldots, n$ the $a$-lift

$$w_a := w^{(a)} = R(e_a w^A)$$

of $w$ is a Poisson tensor on $T^A M$ and that these Poisson tensors are pairwise compatible, that is,

$$[w_a, w_b] = 0.$$

In addition, for every $\varepsilon = \varepsilon^a e_a \in A$ the bivector field

$$w_\varepsilon := R(\varepsilon w^A)$$

on $T^A M$ also is the Poisson tensor on $T^A M$ and $w_\varepsilon = \varepsilon^a w_a$.

### 5.3 Modular classes of lifts of Poisson structures

Let $(M, w)$ be an orientable Poisson manifold, and let $A = \{(U_\kappa, h_\kappa)\}_{\kappa \in K}$ be the maximal oriented atlas on $M$ [29]. The atlas $A$ induces the oriented atlas $\overline{A} = \{(\overline{U}_\kappa, \overline{h}_\kappa)\}_{\kappa \in K}$, $\overline{U}_\kappa = \pi_\kappa^{-1}(U_\kappa)$, $\overline{h}_\kappa = h^\kappa$, on $T^A M$. It follows from (71) that the Jacobian $\det \left| \frac{\partial x'^i}{\partial x^j} \right|$ of a coordinate change on $M$ and the Jacobian $\det \left| \frac{\partial x'^m}{\partial x^m} \right|$ of the corresponding coordinate change on $T^A M$ satisfy the following relation

$$\det \left| \frac{\partial x'^i}{\partial x^j} \right| = \left( \det \left| \frac{\partial x'^m}{\partial x^m} \right| \right)^{n+1}, \quad n + 1 = \dim A. \quad (89)$$

Let $\mu$ be a volume form on $M$ and let

$$\mu(U_\kappa, h_\kappa) = \rho(U_\kappa, h_\kappa) dx^1 \wedge \ldots \wedge dx^m, \quad m = \dim M,$$

be the coordinate representation of $\mu$. The family $\rho = \{\rho(U_\kappa, h_\kappa)\}_{\kappa \in K}$ defines a smooth density on $M$ [29]. We let

$$\overline{\rho}(U_\kappa, \overline{h}_\kappa) = (\rho(U_\kappa, h_\kappa))^{\dim A}. \quad (90)$$
From (89) it follows that the family $\rho = \{\rho(\mathcal{T}_\kappa, \overline{\mathcal{T}_\kappa})\}_{\kappa \in K}$ defines a smooth density on $T^hM$. Then the exterior form $\overline{\mu}$ with the coordinate representation

$$\overline{\mu} = \rho(\mathcal{T}_\kappa, \overline{\mathcal{T}_\kappa}) \, dx^1 \wedge \ldots \wedge dx^m \wedge \ldots \wedge dx^{1n} \wedge \ldots \wedge dx^{mn},$$

in every local chart $(\overline{U}_\kappa, \overline{h}_\kappa)$ is a volume form on $T^hM$.

Let $\Delta_\mu$ be the modular vector field of an oriented Poisson manifold $(M, w, \mu)$.

In this subsection, we compute the modular class of a Poisson structure $w_\varepsilon$ defined by an arbitrary $\varepsilon \in \mathbb{A}$. Let $\{e_a\}$ be a Jordan-Hölder basis in $\mathbb{A}$, $\varepsilon = \varepsilon^a e_a$.

We will consider the two cases: 1) $\varepsilon^0 \neq 0$, that is, $\varepsilon \not\in \mathbb{A}$, 2) $\varepsilon^0 = 0$, that is, $\varepsilon \in \mathbb{A}$.

**Theorem 5.2.** Let $(M, w, \mu)$ be an oriented Poisson manifold, $(\mathbb{A}, q)$ the $(n + 1)$-dimensional Frobenius Weil algebra and $\varepsilon \in \mathbb{A}$.

i) If $\varepsilon \not\in \mathbb{A}$, then the modular vector field of $(T^hM, w_\varepsilon, \overline{\mu})$ is

$$\Delta_{\overline{\mu}, w_\varepsilon} = \varepsilon^0 (n + 1) \Delta_\mu.$$

In particular, the modular vector field of $(T^hM, w^C, \overline{\mu})$ is

$$\Delta_{\overline{\mu}, w^C} = (n + 1) \Delta_\mu.$$

ii) If $\varepsilon \in \mathbb{A}$, then the modular vector field of $(T^hM, w_\varepsilon, \overline{\mu})$ is zero. In particular, the modular vector field of $(T^hM, w^V, \overline{\mu})$ is zero.

**Proof.** It suffices: to verify relation (92) and to show that the modular vector field of a Poisson structure $w_\varepsilon = R(e_\varepsilon w^h)$ is zero when $c \geq 1$. By (15), in terms of a local chart $(U_\kappa, h_\kappa)$ on $M$, the modular vector field $\Delta_\mu$ is of the form

$$\Delta_\mu = \sum_{j=1}^m \left( \frac{\partial w^{ij}}{\partial x^j} + w^{ij} \frac{\partial \ln \rho(U_\kappa, h_\kappa)}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$ 

From (90) it follows that, in terms of the local chart $(U_\kappa, h_\kappa)$ on $T^hM$, we have

$$\frac{\partial \ln \rho(U_\kappa, h_\kappa)}{\partial x^j} = \begin{cases} (n + 1) \frac{\partial \ln \rho(U_\kappa, h_\kappa)}{\partial x^j}, & b = 0, \\ 0, & b = 1, 2, \ldots, n. \end{cases}$$

(93)

1) First, we need to show that

$$\frac{\partial (w^C)^{iaj}}{\partial x^b} = \begin{cases} \frac{\partial w^{ij}}{\partial x^j}, & a = n, \\ 0, & a = 0, 1, \ldots, n - 1. \end{cases}$$

(94)

By (80), we have $(w^C)^{iaj} = (w^{ij})^s \gamma^{ab}_s = (w^{ij})^s q^{ac} \gamma^{bc}_s$. 

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Arguing as in Remark 4.7, we find that
\[
\frac{\partial (w^{ij})^s}{\partial x^j} = 0 \quad \text{for } s < b, \quad \text{and } \quad \frac{\partial (w^{ij})^b}{\partial x^j} = \frac{\partial w^{ij}}{\partial x^j},
\]
where there is no summation over \( j \) or \( b \). Since \( \gamma^b_{cs} = 0 \) when \( s > b \), it follows that the only nonzero summand in
\[
\frac{\partial (w^C)^{iajb}}{\partial x^j} = \frac{\partial (w^{ij})^s q^{ac} \gamma^b_{sc}}{\partial x^j} \quad \text{(no summation over } j \text{ or } b)\]
corresponds to \( s = b \). Since \( p_a = p(e_n) = 1 \), it follows that \( \gamma^b_{ab} = q^{ad} \gamma^b_{bd} = 1 \) when \( a = n \) and \( \gamma^b_{ab} = 0 \) when \( a \neq n \). Thus,
\[
\frac{\partial (w^C)^{iajb}}{\partial x^j} = 0, \quad a \neq n, \quad \text{and } \quad \frac{\partial (w^C)^{iajb}}{\partial x^j} = \frac{\partial (w^{ij})^b}{\partial x^j} = \frac{\partial w^{ij}}{\partial x^j},
\]
where, as above, there is no summation over \( j \) or \( b \).

Now we show that
\[
w^{iaj0} = \begin{cases} 
w^{ij}, & a = n, \\ 0, & a = 0, 1, \ldots, n - 1. \end{cases} \tag{95}
\]
In fact, \((w^C)^{iaj0} = (w^{ij})^s \gamma^a_0\) where \( \gamma^a_0 = q^{ac} \gamma^0_{sc} \). The only nonzero summand in \( q^{ac} \gamma^0_{sc} \) corresponds to \( c = s = 0 \). In addition, \( q^{a0} = 0 \) when \( a \neq n \) and \( q^{a0} = 1 \). We also have \((w^{ij})^0 = w^{ij}\), which implies (95).

The modular vector field of the complete lift \( w^C \) on \( T^hM \) is
\[
\Delta_{\mu, w^C} = \sum_{jb} \left( \frac{\partial (w^C)^{iajb}}{\partial x^j} + (w^C)^{iajb} \frac{\partial \ln \rho^{(U_k,h_k)}}{\partial x^j} \right) \frac{\partial}{\partial x^{ia}}.
\]
Since the index \( b \) runs through \( n + 1 \) different values, (94) implies that
\[
\sum_{jb} \frac{\partial (w^C)^{iajb}}{\partial x^j} \frac{\partial}{\partial x^{ia}} = (n + 1) \frac{\partial w^{ij}}{\partial x^j} \frac{\partial}{\partial x^{in}}.
\]
By (93), all summands with \( b \) different from 0 in the sum
\[
\sum_{jb} (w^C)^{iajb} \frac{\partial \ln \rho^{(U_k,h_k)}}{\partial x^j} \frac{\partial}{\partial x^{ia}}
\]
are zero. Then, using (95), we obtain
\[
\sum_{jb} (w^C)^{iajb} \frac{\partial \ln \rho^{(U_k,h_k)}}{\partial x^j} \frac{\partial}{\partial x^{ia}} = (n + 1) \sum_j w^{ij} \frac{\partial \ln \rho^{(U_k,h_k)}}{\partial x^j} \frac{\partial}{\partial x^{in}},
\]
which proves (92).
2) Consider now a Poisson structure \( w_c = R(e_c w^k), \ c \geq 1 \). By virtue of (44), we have 
\[(w_c)^{iaj b} = p((w^{ij})^s e_s e_a e^b) = (w^{ij})^s \gamma^{s f}_c \gamma^{a b}_f p(e_f e^g) = (w^{ij})^s \gamma^{s f}_c \gamma^{a b}_f \]. The modular vector field of \( w_c \) is 
\[\Delta_{\mathfrak{m}, w_c} = \sum_j (w_c)^{iaj b} \left( \frac{\partial (w_c)^{iaj b}}{\partial x^j b} + (w_c)^{iaj b} \frac{\partial \ln \rho \nu}{\partial x^j b} \right) \frac{\partial}{\partial x^{ia}}. \tag{96}\]

Let us show that the first summand in the brackets in (96) is zero for all values of indices. By (39), we have 
\[\frac{\partial (w_c)^{iaj b}}{\partial x^j b} = \gamma^{s f}_c \gamma^{a b}_f \frac{\partial (w^{ij})^s}{\partial x^j b} = \gamma^{b f}_a \gamma^{a f}_s \frac{\partial (w^{ij})^s}{\partial x^j b}, \tag{97}\]

Now, arguing as in the case of the structure \( w^C \), we conclude that the only nonzero summand in the right-hand side of (97) corresponds to \( b = s \). But for the Jordan-Hölder basis \( \gamma^{b f}_a \neq 0 \) (no summation over \( b \)) only when \( f = 0 \). On the other hand, \( \gamma^{0a}_c \) does not vanish only when \( a = n \) and \( c = 0 \). Thus, the first summand in the brackets in (96) vanishes identically.

Let us consider the second summand in the brackets in (96). Since \( \mathfrak{m}^{(U, \overline{\mathfrak{m}})} \) does not depend on \((x^j)\) for \( b > 0 \), it remains to consider only the case when \( b = 0 \). The coordinates \((w_c)^{iaj 0} = (w^{ij})^s \gamma^{s f}_c \gamma^{a f}_j \) are nonzero only when \( f = 0, a = n \). But \( \gamma^{0a}_c = 0 \) if \( c \geq 1 \), which completes the proof. \( \square \)

**Corollary 5.2.** In the hypotheses of Theorem 5.2, the modular class of the Poisson manifold \((T^h M, w^C)\) is represented by the vector field \((n + 1)\Delta^V_\mu\) for any modular vector field \(\Delta_\mu\) of the base manifold \((M, w)\). The modular class of the Poisson manifold \((T^h M, w^V)\) is zero.

**Theorem 5.3.** Let \((M, w, \mu)\) be an oriented Poisson manifold and \((\mathfrak{A}, q)\) the \((n + 1)\)-dimensional Frobenius Weil algebra. The modular class of \((T^h M, w^C, \overline{\mu})\) vanishes if and only if the modular class of \((M, w, \mu)\) vanishes.

**Proof.** Let \(\Delta_\mu\) be the modular vector field of \((M, w)\). Suppose that \([\Delta_\mu] = 0\), that is, \(\Delta_\mu = X^w_g = [w, g]\) for some \(g \in C^\infty(M)\). Then, by Theorem 5.2, the modular vector field of \((T^h M, w^C)\) is \(\Delta_{\mathfrak{m}} = (n + 1)\Delta^V_\mu = (n + 1)[w, g]^V = (n + 1)[w^C, g^V] = (n + 1)[w^C, g \circ \pi_{\mathfrak{A}}]\). Therefore \([\Delta_{\mathfrak{m}}] = 0\).

Conversely, let \([\Delta_{\mathfrak{m}}] = 0\), then \(\Delta_{\mathfrak{m}} = [w^C, f]\) for some \(f \in C^\infty(T^h M)\). Let \(s_{\mathfrak{A}} : M \to T^h M\) denote the zero section, and let \(g = \frac{1}{n + 1}(f \circ s_{\mathfrak{A}}) \in C^\infty(M)\). We claim that \(\Delta_\mu = X^w_g\).

By virtue of (92) and (80), in terms of local coordinates,
\[\Delta_{\mathfrak{m}} = (n + 1) \sum_k \left( \frac{\partial w^{jk}}{\partial x^k} + w^{jk} \frac{\partial \ln \rho}{\partial x^k} \right) \frac{\partial}{\partial x^{jn}}.\]
and
\[
X^w_f = (w^C)^{iajb} \frac{\partial f}{\partial x^ia} \frac{\partial}{\partial x^jb} = (w^{ij})^{s \alpha \beta}_{\gamma \delta} \frac{\partial f}{\partial x^ia} \frac{\partial}{\partial x^jb},
\]
where \( \mu = \rho dx^1 \wedge \ldots \wedge dx^m \). From the condition \( \Delta_\mu = X^w_f \) we obtain
\[
(w^{ij})^{s \alpha \beta}_{\gamma \delta} \frac{\partial f}{\partial x^ia} = (n+1) \sum_k \left( \frac{\partial w^{jk}}{\partial x^k} + w^{jk} \frac{\partial \ln \rho}{\partial x^k} \right),
\]
\[
\tag{98}
\]
\[
(w^{ij})^{s \alpha \beta}_{\gamma \delta} \frac{\partial f}{\partial x^ia} = 0 \quad \text{for} \quad b = 0, \ldots, n-1.
\]
Note that
\[
\frac{\partial f}{\partial x^ia} \circ s_h = \frac{\partial (f \circ s_h)}{\partial x^ia}.
\]
The restriction of \((w^C)^{iajb}\) to the zero section is \(w^{ij0}_{\alpha \beta} = w^{ij} q^{\alpha \beta}\). Thus, restricting \tag{98} to
the zero section, we obtain
\[
(w^{ij})^{q \alpha \beta}_{\gamma \delta} \frac{\partial f}{\partial x^ia} = (n+1) \sum_k \left( \frac{\partial w^{jk}}{\partial x^k} + w^{jk} \frac{\partial \ln \rho}{\partial x^k} \right),
\]
\[
\tag{99}
\]
\[
(w^{ij})^{q \alpha \beta}_{\gamma \delta} \frac{\partial f}{\partial x^ia} = 0 \quad \text{for} \quad b = 0, \ldots, n-1.
\]
Therefore, contracting the left-hand side of \tag{99} with \(p_b\) by virtue of \tag{43}, we obtain
\[
(w^{ij})_{\beta}^{q \alpha \delta} p_b \frac{\partial (f \circ s_h)}{\partial x^ia} = w^{ij} \frac{\partial f \circ s_h}{\partial x^ia} q^{\alpha \beta} p_b = w^{ij} \frac{\partial f \circ s_h}{\partial x^ia} \delta^0_{\alpha \beta} = w^{ij} \frac{\partial (f \circ s_h)}{\partial x^ia}.
\]
Since \(p_n = 1\), the contraction of the right-hand side of \tag{99} with \(p_b\) gives
\[
(n+1) \sum_k \left( \frac{\partial w^{jk}}{\partial x^k} + w^{jk} \frac{\partial \ln \rho}{\partial x^k} \right).
\]
Conversely
\[
\sum_k \left( \frac{\partial w^{jk}}{\partial x^k} + w^{jk} \frac{\partial \ln \rho}{\partial x^k} \right) = w^{ij} \frac{\partial f \circ s_h}{\partial x^ia} q^{\alpha \beta} p_b = w^{ij} \frac{\partial (f \circ s_h)}{\partial x^ia} = w^{ij} \frac{\partial g}{\partial x^i}.
\]
\[
\square
\]
\textbf{Remark 5.5.} In the case when a Poisson manifold \((M, w)\) is non-orientable, all the results of this subsection remain valid. One only should consider smooth densities instead of volume forms.

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**Geometry Department**

**Branch of Mathematics**

**Chebotarev Research Institute of Mathematics and Mechanics**

**Kazan State University**

**Universitetskaya, 17, Kazan, 420008**

**Russia**

*E-mail: vadimjr@ksu.ru, vshjr@yandex.ru*