The conflict between Bell-Žukowski inequality and Bell-Mermin inequality

Koji Nagata and Jaewook Ahn
Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea
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We consider a two-particle/two-setting Bell experiment to visualize the conflict between Bell-Žukowski inequality and Bell-Mermin inequality. The experiment is reproducible by local realistic theories which are not rotationally invariant. We found that the average value of the Bell-Žukowski operator can be evaluated only by the two-particle/two-setting Bell experiment in question. The Bell-Žukowski inequality reveals that the constructed local realistic models for the experiment are not rotationally invariant. That is, the two-particle Bell experiment in question reveals the conflict between Bell-Žukowski inequality and Bell-Mermin inequality. Our analysis has found the threshold visibility for the two-particle interference to reveal the conflict noted above. It is found that the threshold visibility agrees with the value to obtain a violation of the Bell-Žukowski inequality.

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I. INTRODUCTION

Local and realistic theories assume that physical properties exist irrespective of whether they are measured and that the result of measurement pertaining to one system is independent of any other measurement simultaneously performed on a different system at a distance. As Bell reported in 1964 [1], certain inequalities that correlation functions of a local realistic theory must obey can be violated by quantum mechanics. Bell used the singlet state to demonstrate this. Likewise, a certain set of correlation functions produced by quantum measurements of a single quantum state can contradict local realistic theories. That is, we can see the conflict between local realism and quantum mechanics. Since Bell work, local realistic theories have been researched extensively [2, 3]. Numerous experiments have shown that Bell inequalities and local realistic theories are violated [4, 5, 6, 7, 8, 9, 10, 11, 12].

In 1982, Fine presented [13, 14] the following example. A set of correlation functions can be described with the property that they are reproducible by local realistic theories for a system in two-partite states if and only if the set of correlation functions satisfies the complete set of (two-setting) Bell inequalities. This is generalized to a system described by multipartite [15, 16, 17] states in the case where two dichotomic observables are measured per site. We have, therefore, obtained the necessary and sufficient condition for a set of correlation functions to be reproducible by local realistic theories in the specific case mentioned above.

A violation of ‘standard’ two-settings Bell inequalities [15, 16, 17, 18, 19, 20] is sufficient for experimentalists to show the conflict between local realism and quantum mechanics. However, it is necessary to create an entangled state with sufficient visibility to violate a Bell inequality. Furthermore, measurement settings should be established such that the Bell inequality is violated. We consider, therefore, the following question: What is a general method for experimentalists to see the conflict between local realism and quantum mechanics only from actually measured data?

In many cases one can build a local realistic model for the observed data. However, many such models are artificial and can be disproved if some principles of physics are taken into account. An example of such a principle is rotational invariance of correlation function — the fact that the value of correlation function does not depend on the orientation of reference frames. Taking this additional requirement into account rules out local realistic models even in situations in which standard Bell inequalities allow for an explicit construction of such models [21].

In this paper, we study the physical phenomenon presented in Ref. [21]. Here, we present a method using two Bell operators [22]. To this end, only a two-setting and two-particle Bell experiment reproducible by local realistic theories is needed. Such a Bell experiment also reveals, despite appearances, the conflict between Bell-Žukowski inequality and Bell-Mermin inequality in the sense that the Bell-Žukowski inequality [23] is violated.

Our thesis is as follows. Consider two-qubit states that, under specific settings, give correlation functions reproducible by specific local realistic theory. Imagine that N copies of the states can be distributed among 2N parties, in such a way that each pair of parties shares one copy of the state. The parties perform a Bell-Greenberger-Horne-Zeilinger (GHZ) 2N-particle experiment [15, 16, 17] on their qubits. Each of the pairs of parties uses the measurement settings noted above. The Bell-Mermin operator, B, for their experiment does not show violation of local realism. Nevertheless, one find another Bell operator, which differs from B by a numerical factor, that does show such a violation.

This phenomenon can occur when the system is in a mixed two-qubit state. We analyze threshold visibility for two-particle interference to reveal the conflict mentioned above. It is found that the threshold visibility agrees with the value to obtain a violation of the Bell-Žukowski inequality.
II. EXPERIMENTAL SITUATION

Consider two-qubit states:
\[ \rho_{a,b} = V|\psi\rangle\langle\psi| + (1 - V)\rho_{\text{noise}} \; (0 \leq V \leq 1), \]
where \(|\psi\rangle\) is Bell state as \(|\psi\rangle = \frac{1}{\sqrt{2}}(|+a; +b\rangle - |a; -b\rangle).\)
\[ \rho_{\text{noise}} = \frac{1}{2} \mathbb{I} \] is the random noise admixture. The value of \(V\) can be interpreted as the reduction factor of the interferometric contrast observed in the two-particle correlation experiment. The states \(|\pm k\rangle\) are eigenstates of \(z\)-component Pauli observable \(\sigma_z^k\) for the \(k\)th observer. Here \(a\) and \(b\) are the label of parties (say Alice and Bob). Then we have \(\text{tr} [\rho_{a,b} \sigma_z^a \sigma_z^b] = 0\), \(\text{tr} [\rho_{a,b} \sigma_y^a \sigma_y^b] = 0\), \(\text{tr} [\rho_{a,b} \sigma_z^a \sigma_y^b] = V\), and \(\text{tr} [\rho_{a,b} \sigma_y^a \sigma_z^b] = V\). Here \(\sigma_x^k\) and \(\sigma_y^k\) are Pauli-spin operators for \(x\)-component and for \(y\)-component, respectively. This set of experimental correlation functions is described with the property that they are reproducible by two-settings local realistic theories. See the following relations along with the arguments in [13, 14].

\[
\begin{align*}
|\text{tr} [\rho_{a,b} \sigma_x^a \sigma_x^b] - \text{tr} [\rho_{a,b} \sigma_x^a \sigma_y^b] + \text{tr} [\rho_{a,b} \sigma_y^a \sigma_x^b] + \text{tr} [\rho_{a,b} \sigma_y^a \sigma_y^b]| & = 2V \leq 2, \\
|\text{tr} [\rho_{a,b} \sigma_x^a \sigma_y^b] + \text{tr} [\rho_{a,b} \sigma_y^a \sigma_x^b] - \text{tr} [\rho_{a,b} \sigma_x^a \sigma_y^b] + \text{tr} [\rho_{a,b} \sigma_y^a \sigma_y^b]| & = 0 \leq 2, \\
|\text{tr} [\rho_{a,b} \sigma_x^a \sigma_x^b] + \text{tr} [\rho_{a,b} \sigma_y^a \sigma_y^b] - \text{tr} [\rho_{a,b} \sigma_x^a \sigma_y^b] - \text{tr} [\rho_{a,b} \sigma_y^a \sigma_y^b]| & = 0 \leq 2, \\
|\text{tr} [\rho_{a,b} \sigma_x^a \sigma_y^b] - \text{tr} [\rho_{a,b} \sigma_y^a \sigma_x^b] - \text{tr} [\rho_{a,b} \sigma_x^a \sigma_y^b] + \text{tr} [\rho_{a,b} \sigma_y^a \sigma_y^b]| & = 2V \leq 2. \\
\end{align*}
\]

Therefore, it should be that given \(2^N\) correlation functions are described with the property that they are reproducible by two-settings local realistic theories.

Bell-Mermin operators \(B_{N_2N}\) and \(B'_{N_2N}\) (defined as follows) do not show any violation of local realism as shown below.

Let \(f(x, y)\) denote the function \(\frac{1}{\sqrt{2}} e^{-ix/4}(x + iy), x, y \in \mathbb{R}\). \(f(x, y)\) is invertible as \(x = \Re f - 3f, y = \Im f + 3f\).
Bell-Mermin operators \(B_{N_2N}\) and \(B'_{N_2N}\) are defined by \(\text{tr}[B_{N_2N}, B'_{N_2N}] = \bigotimes_{k=1}^{2^N} f(A_k, A'_k)\). Bell-Mermin inequality can be expressed as [24]

\[ |\langle B_{N_2N} \rangle| \leq 1, \quad |\langle B'_{N_2N} \rangle| \leq 1, \]

where \(B_{N_2N}\) and \(B'_{N_2N}\) are Bell-Mermin operators defined by

\[ f(B_{N_2N}, B'_{N_2N}) = \bigotimes_{k=1}^{2^N} f(A_k, A'_k). \]

We also define \(B_\alpha\) for any subset \(\alpha \subset N_2N\) by

\[ f(B_\alpha, B'_\alpha) = \bigotimes_{k \in \alpha} f(A_k, A'_k). \]

It is easy to see that, when \(\alpha, \beta (\subset N_2N)\) are disjoint,

\[ f(B_{\alpha \cup \beta}, B'_{\alpha \cup \beta}) = f(B_\alpha, B'_\alpha) \otimes f(B_\beta, B'_\beta), \]

which leads to following equations,

\[ B_{\alpha \cup \beta} = (1/2)B_\alpha \otimes (B_\beta + B'_\beta) + (1/2)B'_\alpha \otimes (B_\beta - B'_\beta), \]
\[ B'_{\alpha \cup \beta} = (1/2)B'_\alpha \otimes (B_\beta + B'_\beta) + (1/2)B_\alpha \otimes (B'_\beta - B_\beta). \]

In specific operators \(A_k, A'_k\) given in Eq. [4], where \(\sigma_x^k = |+k\rangle\langle-k| + |-k\rangle\langle+k|\) and \(\sigma_y^k = -i|+k\rangle\langle-k| + i|-k\rangle\langle+k|\),

III. CONFLICT BETWEEN LOCAL REALISM AND QUANTUM MECHANICS

Let \(N_2N\) be \(\{1, 2, \ldots, 2N\}\). Imagine that \(N\) copies of the states introduced in the preceding section can be distributed among \(2N\) parties, in such a way that each pair of parties shares one copy of the state

\[ \rho^{\otimes N} = \rho_{1,2} \otimes \rho_{3,4} \otimes \cdots \otimes \rho_{N-1,N}. \]

Suppose that spatially separated \(2N\) observers perform measurements on each of \(2N\) particles. The decision processes for choosing measurement observables are space-like separated.

We assume that a two-orthogonal-setting Bell-GHZ \(2N\)-particle correlation experiment is performed. We choose measurement observables such that

\[ A_k = \sigma_x^k, A'_k = \sigma_y^k. \]

Namely, each of the pairs of parties uses measurement settings such that they can check the condition [2].
we have (cf. (23))

\[
\begin{align*}
f(A_k, A'_k) &= (e^{-i\pi/2}/\sqrt{2})(\sigma_x^k + i\sigma_y^k) \\
&= e^{-i\pi/2}\sqrt{2}|+\rangle\langle-k|
\end{align*}
\] (10)

and

\[
\begin{align*}
f(B_{N2N}, B'_{N2N}) &= \otimes_{k=1}^{2N} f(A_k, A'_k) \\
&= e^{-i2\pi/2^{2N}} \otimes_{k=1}^{2N} |+\rangle\langle-k|
\end{align*}
\] (11)

Hence we obtain

\[
B_{N2N} = 2^{(2N-1)/2}(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|),
\] (12)

where \(e^{-i2\pi N/2^{2N}}|+\rangle\otimes|2N\rangle = |1\otimes2N\rangle\). Here the states \(|\Psi_{0,1}^\pm\rangle\) are Greenberger-Horne-Zeilinger (GHZ) states (28), i.e.,

\[
|\Psi_0^+\rangle = \frac{1}{\sqrt{2}}(|0\otimes2N\rangle \pm |1\otimes2N\rangle).
\] (13)

Measurements on each of 2N particles enable them to obtain 22N correlation functions. Thus, they get an average value of specific Bell-Mermin operator given in Eq. (12). According to Eq. (4), we obtain

\[
\langle B_{N2N} \rangle = \langle B'_{N2N} \rangle = \prod_{i=2}^N \langle B_{i-1,i} \rangle = V^N(1 \leq 1).
\] (14)

Clearly, Bell-Mermin operators, \(B_{N2N}\) and \(B'_{N2N}\), for their experiment do not show any violation of local realism as we have mentioned above.

Nevertheless, one can find 2N-partite Bell operator, which one may call Bell-Žukowski operator \(Z_{2N}\), which differs from \(B_{N2N}\) only by a numerical factor, that does show such a violation. Bell-Žukowski operator \(Z_{2N}\) is as (cf. Appendix A, Eq. (A22))

\[
Z_{2N} = \frac{1}{2} \left(\frac{\pi}{2}\right)^{2N} (|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|).\n\] (15)

Clearly, we see that Bell-Mermin operator given in Eq. (12) is connected to Bell-Žukowski operator \(Z_{2N}\) in the following relation

\[
Z_{2N} = \frac{1}{2} \left(\frac{\pi}{2}\right)^{2N} \frac{1}{2^{(2N-1)/2}} B_{N2N}.
\] (16)

One can see that specific two settings Bell 2N-particle experiment in question computes an average value of Bell-Žukowski operator \(Z_{2N}\) via an average value of \(\langle B_{N2N} \rangle\). Therefore, from the Bell-Žukowski inequality \(\langle Z_{2N} \rangle \leq 1\), we have a condition on the average value of Bell-Mermin operator \(\langle B_{N2N} \rangle\) which is written by

\[
\langle B_{N2N} \rangle \leq 2 \left(\frac{2}{\pi}\right)^{2N} 2^{(2N-1)/2}.
\] (17)

Please notice that the Bell-Žukowski inequality \(|\langle Z_{2N} \rangle| \leq 1\) is derived under the assumption that there are predetermined ‘hidden’ results of the measurement for all directions in the rotation plane for the system in a state. On the other hand, Bell-Mermin inequality is derived under the assumption that there are predetermined ‘hidden’ results of the measurement for two directions for the system in a state. Thus, Bell-Žukowski inequality governs rotationally invariant descriptions while Bell-Mermin inequality does not.

When \(N \geq 2\) and \(V\) is given by

\[
\left(2\left(\frac{2}{\pi}\right)^{2N} 2^{(2N-1)/2}\right)^{1/N} < V \leq 1,
\] (18)

one can compute a violation of the Bell-Žukowski inequality \(|\langle Z_{2N} \rangle| \leq 1\), that is, the explicit local realistic models are not rotationally invariant. The condition (13) says that threshold visibility decreases when the number of copies \(N\) increases. In extreme situation, when \(N \rightarrow \infty\), we have desired condition \(V > 2(2/\pi)^2\) to show the conflict in question. It agrees with the value to get a violation of the Bell-Žukowski inequality. (It also agrees with the value to get a violation of the generalized Bell inequality presented in Ref. [21].)

Thus the given example using two-qubit states reveals the violation of the Bell-Žukowski inequality. The interesting point is that all the information to get the violation of the Bell-Žukowski inequality can be obtained only by a two-setting and two-particle Bell experiment reproducible by two-settings local realistic theories.

It presents a quantum-state measurement situation that admits local realist descriptions for the given apparatus settings, but no local realist descriptions which are rotationally invariant, even though the experiment should be ruled by rotationally invariant laws. There is no local realist theory for the experiment as a whole and so such a descriptions is only possible for certain setting.

What the result illustrates is that there is a further division among the measurement settings, those that admit rotational invariant local realist models and those that do not. This is another manifestation of the underlying contextual nature of realist theories of quantum experiments.

IV. SUMMARY

In summary, we have presented a Bell operator method. This approach provides a means to check if the explicit model is rotationally invariant, i.e., if a conflict between Bell-Žukowski inequality and Bell-Mermin inequality occurs. Our argument relies only on a two-setting and two-particle Bell experiment reproducible by a local realistic theory which is not rotationally invariant. Given a two-setting and two-particle Bell experiment reproducible by specific local realistic theory, one can compute a violation of Bell-Žukowski inequality. Measured
data indicates that the explicit local realistic models are not rotationally invariant. Thus, the conflict between Bell-Žukowski inequality and Bell-Mermin inequality is, despite appearances, revealed.

This phenomenon can occur when the system is in a mixed state. We also analyzed the threshold visibility for two-particle interference in order to bring about the phenomenon. The threshold visibility agrees well with the value to obtain a violation of the Bell-Žukowski inequality.

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APPENDIX A: BELL-ŽUKOWSKI INEQUALITY

Let us review the Bell-Žukowski inequality proposed in Ref. [23]. Let $L(H)$ be the space of Hermitian operators acting on a finite-dimensional Hilbert space $H$, and $T(H)$ be the space of density operators acting on the Hilbert space $H$. Namely, $T(H) = \{ \rho \in L(H) \land \rho \geq 0 \land \text{tr} (\rho) = 1 \}$. We also consider a classical probability space $(\Omega, \Sigma, M_\rho)$, where $\Omega$ is a nonempty space, $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$, and $M_\rho$ is a $\sigma$-additive normalized measure on $\Sigma$ such that $M_\rho (\Omega) = 1$. The subscript $\rho$ expresses following meaning. The probability measure $M_\rho$ is determined uniquely when a state $\rho$ is specified.

Consider a quantum state $\rho$ in $T(\otimes_{k=1}^n H_k)$, where $H_k$ represents the Hilbert space with respect to party $k \in \mathbb{N}_n = \{1, 2, \ldots, n\}$. Then we can define measurable functions $f_k : \Omega \times \mathbb{R} \to f_k (\omega, \phi) \in [I (\omega), S (\omega)]$, $\omega \in L (H_k)$, $\phi \in \Omega$. Here $S (\omega)$ and $I (\omega)$ are the supremum and the infimum of the spectrum of $\rho \in L (H_k)$, respectively. Those functions $f_k (\omega, \phi)$ must not depend on the choices of $\phi$'s on the other sites in $\mathbb{N}_n \setminus \{k\}$. Using the functions $f_k$, we define a quantum correlation function which admits a local realistic model [27].

Definition 1. A quantum correlation function $\text{tr} [\rho \otimes_{k=1}^n o_k]$ is said to admit a local realistic model if and only if there exist a classical probability space $(\Omega, \Sigma, M_\rho)$ and a set of functions $f_1, f_2, \ldots, f_n$ such that

$$\int \Omega M_\rho (d\omega) \prod_{k=1}^n f_k (\omega, \phi) = \text{tr} [\rho \otimes_{k=1}^n o_k] \quad (A1)$$

for a Hermitian operator $\otimes_{k=1}^n o_k$, where $o_k \in L (H_k)$. Note that there are several (noncommuting) observables per site, however above definition is available for just one $o_k$ per site.

We consider a situation where each of the $n$ spatially separated observers has infinite number of settings of measurements (in the $xy$ plane) to choose from. The operation of each of the measuring apparatuses is controlled by a knob. The knob sets a parameter $\phi$. An apparatus performs measurements of a Hermitian operator $\sigma_\phi$ on two-dimensional space with two eigenvalues $\pm 1$. The corresponding eigenstates are defined as $| \pm \phi \rangle = (1/\sqrt{2}) (| 1 \rangle \pm e^{i\phi} | 0 \rangle)$. The local phases that they are allowed to set are chosen as $0 \leq \phi^k < \pi$ for the $k$th observer. The Bell-Žukowski inequality can be written as

$$|Z_n| \leq 1, \quad (A2)$$

where the corresponding Bell operator $Z_n$ is

$$Z_n = \left( \frac{1}{2^n} \right) \int_0^\pi \int_0^\pi \cdots \int_0^\pi \left[ \cos \left( \sum_{k=1}^n \phi^k \right) \otimes_{k=1}^n \sigma_\phi^k \right], \quad (A3)$$

where

$$\sigma_\phi^k = e^{-i\phi^k} | 1 \rangle \langle 1 | + e^{i\phi^k} | 0 \rangle \langle 0 |, \quad k \in \mathbb{N}_n. \quad (A4)$$

Bell-Žukowski operator $Z_n$ is a sum of infinite number of Hermitian operators, except for fixed number $1/(2^n)$. We shall mention why $Z_n$ given in Eq. (A3) is a Bell operator when Eq. (A2) is a Bell inequality as follows.

Let us assume that all of quantum correlation functions (every setting lies in $xy$ plane) admit a local realistic model. Here each party $k$ performs locally measurements on an arbitrary single state $\rho$.

Then, according to Definition 1 (Eq. (A1)), there exists a classical probability space $(\Omega, \Sigma, M_\rho)$ related to the state in question $\rho$. And there exists a set of functions $f_1, f_2, \ldots, f_n \in [-1, 1]$ such that

$$\int \Omega M_\rho (d\omega) \prod_{k=1}^n f_k (\sigma_\phi^k, \omega) = \text{tr} [\rho \otimes_{k=1}^n \sigma_\phi^k] \quad (A5)$$

for every $0 \leq \phi^k < \pi$, $k \in \mathbb{N}_n$. Hence an expectation of a sum of infinite number of Hermitian operators (i.e., $2^n Z_n$) is bounded by the possible values of

$$S^\omega_{\infty, n} = \int_0^\pi \int_0^\pi \cdots \int_0^\pi \left[ \cos \left( \sum_{k=1}^n \phi^k \right) \prod_{k=1}^n f_k (\sigma_\phi^k, \omega) \right] \quad (A6)$$

where

$$z_k' = \int_0^\pi \phi^k f_k (\sigma_\phi^k, \omega) \exp (i\phi^k).$$

Let us derive an upper bound of $S^\omega_{\infty, n}$. We may assume $f_k = \pm 1$. Let us analyze the structure of the following integral

$$z_k' = \int_0^\pi \phi^k f_k (\sigma_\phi^k, \omega) \exp (i\phi^k) = \int_0^\pi \phi^k f_k (\sigma_\phi^k, \omega) (\cos \phi^k + i \sin \phi^k). \quad (A7)$$
Notice that Eq. (A7) is a sum of the following integrals:

$$\int_0^\pi d\phi f_k(\phi, \omega) \cos \phi$$

(A8)

and

$$\int_0^\pi d\phi f_k(\phi, \omega) \sin \phi .$$

(A9)

We deal here with integrals, or rather scalar products of $f_k(\phi, \omega)$ with two orthogonal functions. One has

$$\int_0^\pi d\phi \cos \phi \sin \phi = 0 .$$

(A10)

The normalized functions $\frac{1}{\sqrt{\pi/2}} \cos \phi$ and $\frac{1}{\sqrt{\pi/2}} \sin \phi$ form a basis of a real two-dimensional functional space, which we shall call $S^{(2)}$. Note further that any function in $S^{(2)}$ is of the form

$$A \frac{1}{\sqrt{\pi/2}} \cos \phi + B \frac{1}{\sqrt{\pi/2}} \sin \phi ,$$

(A11)

where $A$ and $B$ are constants, and that any normalized function in $S^{(2)}$ is given by

$$\cos \psi \frac{1}{\sqrt{\pi/2}} \cos \phi + \sin \psi \frac{1}{\sqrt{\pi/2}} \sin \phi$$

$$= \frac{1}{\sqrt{\pi/2}} \cos(\phi - \psi) .$$

(A12)

The norm $\|f_k^{(l)}\|$ of the projection of $f_k$ into the space $S^{(2)}$ is given by the maximal possible value of the scalar product $f_k$ with any normalized function belonging to $S^{(2)}$, that is

$$\|f_k^{(l)}\| = \max_\psi \int_0^\pi d\phi f_k(\phi, \omega) \frac{1}{\sqrt{\pi/2}} \cos(\phi - \psi) .$$

(A13)

Because $|f_k(\phi, \omega)| = 1$, one has $\|f_k^{(l)}\| \leq 2/\sqrt{\pi/2}$. Since $\frac{1}{\sqrt{\pi/2}} \cos \phi$ and $\frac{1}{\sqrt{\pi/2}} \sin \phi$ are two orthogonal basis functions in $S^{(2)}$, one has

$$\int_0^\pi d\phi f_k(\phi, \omega) \frac{1}{\sqrt{\pi/2}} \cos \phi = \cos \beta_k \|f_k^{(l)}\|$$

(A14)

and

$$\int_0^\pi d\phi f_k(\phi, \omega) \frac{1}{\sqrt{\pi/2}} \sin \phi = \sin \beta_k \|f_k^{(l)}\| ,$$

(A15)

where $\beta_k$ is some angle. Using this fact, one can put the value of (A7) into the following form

$$z_k = \sqrt{\pi/2} \|f_k^{(l)}\| (\cos \beta_k + i \sin \beta_k)$$

$$= \sqrt{\pi/2} \|f_k^{(l)}\| \exp(i\beta_k) .$$

(A16)

Therefore, since $\|f_k^{(l)}\| \leq 2/\sqrt{\pi/2}$, the maximal value of $|z_k|$ is 2. Hence, we have $\|\prod_{k=1}^n z_k\| \leq 2^n$. Then we get

$$|S^{(\infty,n)}| \leq 2^n .$$

(A17)

Let $E(\cdot)$ represent an expectation on the classical probability space. If we integrate this relation (A17) under normalized measure $\mu_\rho$ over space $\Omega$, we obtain the relation (A2). Here we have used the relation that $E(S^{(\infty,n)}) = 2^n tr[\rho Z_n]$ (see Eq. (A3)). Therefore, we have proven the Bell–Zukowski inequality (A2) from an assumption. The assumption is that all of infinite number of quantum correlation functions (every setting lies in $xy$ plane) admit a local realistic model.

Let us consider matrix elements of Bell–Zukowski operator $Z_n$ as given in Eq. (A3) on using GHZ basis

$$|\Psi_\pm^j\rangle = \frac{1}{\sqrt{2}}(|j\rangle|0\rangle \pm 2^{n-1} - j - 1|1\rangle) ,$$

(A18)

where $j = j_1 j_2 \cdots j_{n-1}$ is understood in binary notation. It is clear that no off-diagonal element appears, because of the form of the operator $\sigma_{a_k}$ as given in Eq. (A4).

Let $\beta$ be a subset $\beta \subset N_n$ and $I(\beta)$ be an integer $l_1 \cdots l_n$ in the binary notation with $l_m = 1$ for $m \in \beta$ and $l_m = 0$ otherwise. And let $j(\beta)$ be an integer binary-represented by $l_1 \cdots l_{n-1}$. Then we define a two-to-one function $g : \beta \mapsto g(\beta) \in \{0\} \cup N_2(n-1) \cup N_2(n-1)$ where $g(\beta)$ takes the values $j(\beta)$ and $2^{n-1} - j(\beta) - 1$, respectively, for even and odd values of $l(\beta)$.

In what follows, we show that $\langle \Phi^+_{g(\alpha)} | Z_n | \Phi^+_{g(\alpha)} \rangle = 0$ for any subset $\alpha \subset N_n$ when $\alpha \neq \emptyset, N_n$. We also show that $\langle \Psi^+_{g(\alpha)} | Z_n | \Psi^+_{g(\alpha)} \rangle = \pm \frac{1}{\sqrt{2}} \left(\frac{\pi}{2}\right)^n$ when $\alpha = \emptyset$ or $\alpha = N_n$.

When $\alpha = \emptyset$ or $\alpha = N_n$, we have

$$2^n \langle \Phi^+_{0\alpha} | Z_n | \Phi^+_{0\alpha} \rangle$$

$$= \pm \int_0^\pi d\phi^1 \cdots \int_0^\pi d\phi^n \cos^2 \left(\sum_{k=1}^n \phi_k\right)$$

$$= \pm \frac{1}{2} \int_0^\pi d\phi^1 \cdots \int_0^\pi d\phi^n \left[1 + \cos \left(2 \sum_{k=1}^n \phi_k\right)\right]$$

$$= \pm \frac{1}{2} R \left\{ \int_0^\pi d\phi^1 \cdots \int_0^\pi d\phi^n \left[1 + \exp \left(2i \sum_{k=1}^n \phi_k\right)\right]\right\}$$

$$= \pm \frac{\pi^n}{2} \pm \frac{1}{2} R \left( \prod_{k=1}^n \int_0^\pi d\phi_k \exp(2i\phi_k) \right) .$$

(A19)

Since $\int_0^\pi d\phi_k \exp(2i\phi_k) = 0, k \in N_n$, the last term vanishes. Hence we get

$$\langle \Phi^+_{0\alpha} | Z_n | \Phi^+_{0\alpha} \rangle = \pm \frac{1}{2} \left(\frac{\pi}{2}\right)^n .$$

(A20)

On the other hand, when $\alpha \neq \emptyset, N_n$, we obtain
\[ 2^n \langle \Phi^\pm_{g(\alpha)} | Z_n | \Phi^\pm_{g(\alpha)} \rangle = \int_0^\pi d\phi^1 \cdots \int_0^\pi d\phi^n \cos \left( \sum_{k \in \alpha} \phi^k + \sum_{k \in \mathbb{N}_n \setminus \alpha} \phi^k \right) \times \cos \left( \sum_{k \in \alpha} \phi^k - \sum_{k \in \mathbb{N}_n \setminus \alpha} \phi^k \right) \]

\[ = \frac{1}{2} \int_0^\pi d\phi^1 \cdots \int_0^\pi d\phi^n \left[ \cos \left( 2 \sum_{k \in \alpha} \phi^k \right) + \cos \left( 2 \sum_{k \in \mathbb{N}_n \setminus \alpha} \phi^k \right) \right] \]

\[ = \frac{\pi |\mathbb{N}_n \setminus \alpha|}{2} \Re \left( \prod_{k \in \alpha} \int_0^\pi d\phi^k \exp(2i\phi^k) \right) + \frac{\pi |\alpha|}{2} \Re \left( \prod_{k \in \mathbb{N}_n \setminus \alpha} \int_0^\pi d\phi^k \exp(2i\phi^k) \right). \quad (A21) \]

Since \( \int_0^\pi d\phi^k \exp(2i\phi^k) = 0, k \in \mathbb{N}_n \), the last two terms vanish.

Hence, Bell operator \( Z_n \) as given in Eq. (A3) can be rewritten as

\[ Z_n = \frac{1}{2} \left( \frac{\pi}{2} \right)^n (|\Psi^+_0 \rangle \langle \Psi^+_0 | - |\Psi^-_0 \rangle \langle \Psi^-_0 |). \quad (A22) \]

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