Numerical Analysis of Thirring Model under White Noise

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Abstract. Today, the effects of noise on dynamical systems are an attractive area of research. The noise acts as a driving term in the equations of motion in nonlinear systems. In this work, we present conformally invariant pure spinor nonlinear Thirring model. Thirring model describes Dirac fermions in (1+1) space-time dimensions with local current-current interaction. This model has rich dynamic of the quantization of relativistic quantum field theories. We investigate the response of Thirring oscillator to white noise by constructing phase space displays.

1. Introduction
It is well known that the Korteweg-de Vries (KdV) equation derived by Korteweg and de Vries in 1895 which describes weakly nonlinear shallow water waves [1]. Zabusky and Kruskal found that the stable pulse-like waves could exist in a system described by the KdV equation. These solitary waves could collide with each other and preserve their shapes and speeds after the collision [2]. They were named as solitons by Zabusky and Kruskal. There are four leading solitonic characters: instanton, monopole, vortex, and kink ones. Instantons are localized, classical topological solutions in quantum field theories. They have finite action with zero energy so they have been considered as configurations of quantum fields that provide a tunnelling effect between the vacuums [2, 3]. Thirring model which describes Dirac fermions in (1+1) space-time dimensions with the non-linear $(\bar{\psi}\psi)^2$ self-interaction term [4].

Thirring model is also conformally invariant and with no mass term. It played a very fruitful role in the progress of quantum field theory [4]. The spinor-type instanton solutions of Thirring model were found via breaking of conformal symmetry and were shown that these solutions were stable [5].

In this paper, we present the conformal invariant pure spinor Thirring model, which admit instanton-type solutions. We focus the effects of white noise in the long time behaviors of excited 2D Thirring instantons. For this purpose, we consider the general dynamics of Thirring nonlinear differential equations system formed by the use of Heisenberg ansatz with the addition of a white noise term.
2. The Model

The wave equation of model as below

\[ i\sigma_\mu \partial_\mu \psi + g(\bar{\psi}\psi) \psi = 0 \]  

follows from the conformal invariant Lagrangian

\[ \mathcal{L} = i\bar{\psi}\partial_t \psi + \frac{g}{2}(\bar{\psi}\psi)^2 \]  

Where \( g \) is the positive coupling constant and the fermion field \( \psi(x) \) has scale dimension \( \frac{1}{2} \) [4]. The complex form of the Euclidian configuration of Heisenberg ansatz [6] is

\[ \psi = [ix_\mu \gamma_\mu \chi(s) + \varphi(s)]c \]  

with an arbitrary spinor constant, \( c \). \( \chi(s) \) and \( \varphi(s) \) are real functions of \( s = x^2 + t^2 \). Inserting ansatz into wave equation of model we get

\[ \chi(s) + s \frac{d\chi(s)}{ds} + \alpha[s\chi(s)^2 + \varphi(s)^2]\varphi(s) = 0 \]  

\[ \frac{d\varphi(s)}{ds} - \alpha[s\chi(s)^2 + \varphi(s)^2]\chi(s) = 0 \]  

where \( \alpha = g(\bar{CC}) \). By writing \( \chi = As^{-\sigma}F(u) \) and \( \varphi = Bs^{-\tau}G(u) \) with \( u \equiv \ln s \) and \( \sigma = \tau + \frac{1}{2}, \tau = \frac{1}{4} \) and \( A^2 = B^2 \) [7], we achieved the dimensionless form of the non-linear ordinary coupled differential equation system (4) as

\[ 2\frac{dF(u)}{du} + \frac{1}{2}F(u) - \alpha AB\left(F(u)^2 + G(u)^2\right)G(u) = 0 \]  

\[ 2\frac{dG(u)}{du} - \frac{1}{2}G(u) + \alpha AB\left(F(u)^2 + G(u)^2\right)F(u) = 0 \]  

Here \( F \) and \( G \) are dimensionless functions of \( u \) and \( A, B \) are constants [4, 7, 5]. In Ref. [5], the two types of classical solution were found in the case of the two-dimensional conformal invariant Lagrangian exploiting the same symmetry arguments as for gauge and scalar fields. Either of them leads to the vanishing of energy-momentum tensor and finite action for \( \alpha(AB) = 1 \). It is called instanton solution (Thirring Instantons) [4, 5]. Recently, the role of the coupling constant in the evolution of Thirring instantons in phase space has been investigated [8]. Also common behaviours of spinor type 2D Thirring instantons and 4D Gursey instantons have been investigated by constructing their vector field [9].

In this paper we focus on effects of white noise on the dynamics of 2D excited Thirring instantons. For this purpose we perturbe this fermionic model by an external force under white noise as

\[ 2\frac{dF(u)}{u} + \frac{1}{2}F(u) - \beta\left(F(u)^2 + G(u)^2\right)G(u) = 0 \]  

\[ 2\frac{dG(u)}{u} - \frac{1}{2}G(u) + \beta\left(F(u)^2 + G(u)^2\right)F(u) - ACos(\omega H(u)) + \eta(u) = 0 \]  

\[ \frac{H(u)}{du} = \Omega \]  

with a constant \( \Omega \) by adding an extra dimension for numerical calculations. Here \( A \) is the amplitude of the external forcing, \( \omega \) is its frequency and \( \eta(u) \) is white noise term.
3. Simulation Results

Today, noise interference is a long-term problem for researchers in detection technology. In this paper we investigate the response of 2D excited Thirring instantons to white noise, a random signal with a constant power spectral density, to provide a better understanding of the spinor type instanton dynamics in phase space. Firstly, we show the phase space dynamics of Thirring instantons for the unperturbed case without and with the addition of noise respectively in Figures 1 (a) and (b). They exhibit stability behaviours in phase space as it is realized in Figure 1.

![Phase space displays of Thirring instantons](image)

**Figure 1.** Phase space displays of Thirring instantons with the initial conditions $F(0) = G(0) = 1.29904$ and $\beta = 1$ (a) without noise (b) with noise.

In Figure 2-3, phase space displays of excited Thirring instantons with the initial conditions $F(0) = 1.1, G(0) = 0.1$, for $\beta = 1, \omega = \pi$ are seen. It is known that the effect of the external force on the spinor-type instantons sensitively depend on the initial conditions [10]. From the obtained time evolution and phase space displays, we can say that the system exhibits similar behaviours to forced Duffing oscillator [11] type characterization without noise in phase space. Also the system show small changes in phase space for not large values of noise (noise ratio $r=0.015$). It is interesting to note that 2D excited Thirring instantons remains forced Duffing-type characterization under white noise. This state can be interpreted as excited Thirring instantons have noise immunity in statistical sense.

![Phase space displays and time evolutions](image)

**Figure 2.** Phase space displays and time evolutions of the excited Thirring instantons without (first row) and with noise (second row) at $A = 0.6$. 
Figure 3. Phase space displays and time evolutions of the excited Thirring instantons without (first row) and with noise (second row) at $A = 0.7$.

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