Heun-type solutions for Schwarzschild metric with electromagnetic fields

T. Birkandan¹ and M. Hortaçsu²

1 Istanbul Technical University - Department of Physics, Istanbul, Turkey
2 Mimar Sinan Fine Arts University - Department of Physics, Istanbul, Turkey

Introduction. – Heun functions [1,2] seem to be still a novelty among theoretical physicists although they were introduced nearly 130 years ago. After the centennial conference which took place in 1989 and the papers presented in this conference were published in a book [2], there was an explosion of papers in this field [3]. Many equations whose exact solutions were not known turned out to have solutions in this set. After referring to people [4–6] who tried to show whether the exact solutions of the celebrated “Teukolsky Master Equation” [7] can be written down in terms of the confluent forms of the Heun equation, Batic and Schmidt [8] showed that the “Teukolsky Master Equation” (and similar equations) could be transformed in any relevant type-D metric into a Heun form. Although the Heun equation and its confluent forms are much better known today in theoretical physics community and included in some mathematical packages, we still find some authors who do not identify the equations they find properly.

Here we give two examples of metrics which yield confluent Heun solutions for the equations describing a test particle whose wave equation is written in the background metric of these metrics in the linear approximation, ignoring the backreaction and nonlinear terms in the Einstein-Maxwell equations. In the first example the massless Dirac equation and in the second, the massless Klein-Gordon equation.

Dirac equation. – In a very interesting paper [9] Al-Badawi and Owaidat study the Dirac equation in the background of the spherically symmetric solution of the Einstein-Maxwell equations, analogous to the Schwarzschild metric, in the presence of spherically symmetric static electromagnetic field.

For the metric they use a solution found by one of these authors with Halilsoy [10]. Actually this metric was previously discovered by Ray and Wei [11]. Metrics, when a Schwarzschild metric is in a homogeneous electromagnetic field, were given by Halilsoy [12,13], as stated in [14], and were included in the book by Griffiths and Podolsky [15]. This solution is a superposition of the Schwarzschild [16] solution with an external, stationary magnetic field, were given by Halilsoy [12,13], as stated in [14], and were included in the book by Griffiths and Podolsky [15]. This solution is a superposition of the Schwarzschild [16] solution with an external, stationary electromagnetic Bertotti-Robinson solution [17,18].

The metric is given as

\[ ds^2 = \frac{r^2 - 2Mr}{r^2 f(r)} \left[ dt - Mq(1 + a^2) \cos \theta d\phi \right]^2 - \frac{r^2 f(r)}{r^2 - 2Mr} dr^2 - r^2 f(r) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad (1) \]

\[ r^2 f(r) = \frac{1}{2} \left( r - 2M \right) \left[ p \left( 1 + a^2 \right) + a^2 - 1 \right]
+ 2Mar + M^2 \left[ p \left( 1 + a^2 \right) - 2a \right]. \quad (2) \]

This is a type-D metric. \( M \) is a constant parameter which mimics the role of the source mass in Newtonian approximations for the geodesics of test particles in the Schwarzschild metric when one considers large values of the luminosity radius \( r \); \( p \) is the twisting parameter of the external electromagnetic field and \( a \) is the interpolation parameter between two metrics used. This metric is in the class named as Plebanski and Demianski solutions [19].

When the parameter \( a \) is zero one gets a metric which can be transformed into the Bertotti-Robinson solution
by setting the twisting parameter to one. As stated in [15], in general this space-time is a direct product of two two-dimensional spaces of constant curvature, namely the 2-sphere and two-dimensional anti-de Sitter space-times. When \( a \) is between zero and one, it contains a family of expanding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.

The authors consider only a spinor test particle, in the linear approximation, ignoring the backreaction and non-linear terms in the Einstein-Maxwell equations. They use the Newman-Penrose formalism [20] to separate the Dirac linear approximation, ignoring the backreaction and non-zero.

metrics. Here, as in [10,11] we take the cosmological constant between zero and one, it contains a family of ex-

panding Schwarzschild-Reissner-Nordström-de Sitter metrics. Here, as in [10,11] we take the cosmological constant equal to zero.
where

\[-1/2M (−4kM + i)ka^2 + M (2kM + i)ka + i/2Mk − \lambda^2/3+ 3/8, 1/2(\frac{r}{M})\].

(7)

In the following, we do not consider it, since it has a square-root irregularity at \(r = 0\), our point of expansion.

The standard form of the confluent Heun equation is given as [29,30]

\[
\frac{d^2 H_C}{dz^2} + \left(\frac{\alpha + \gamma + 1}{z} + \beta + 1 \right) \frac{dH_C}{dz}
+ \left(\frac{\mu}{z} + \frac{\nu}{z-1}\right) H_C = 0,
\]

with solution \(H_C(\alpha, \beta, \gamma, \delta, \eta, z)\), and the parameters have the relations

\[
\delta = \mu + \nu - \alpha \left(\frac{\beta + \gamma + 2}{2}\right),
\]

\[
\eta = \alpha(\beta + 1) - \mu - \beta + \gamma + \beta\gamma.
\]

Since we are interested in the region \(r > 2M\), outside the event horizon, the solution we gave above does not suit our purposes if we want to investigate the behavior of the wave for \(r > 2M\). To find a solution to suit our purpose, we have to transform to the variable \(u = r - 2M\). This will give us one solution which is analytic around \(r = 2M\).

This solution may not be analytic around \(r = 0\). Since we are not interested in the region \(0 < r < 2M\), this will not cause any problems.

From the two first-order differential equations given above, eqs. (3) and (4), we derive a second-order equation for \(F_1\). This equation reads

\[
A \frac{d^2 T_1}{du^2} + B \frac{d}{du} T_1(u) + (C + D + E) T_1(u) = 0,
\]

where

\[
A = (u + 2M)^2 u^2,
\]

\[
B = (M + u)(u + 2M) u,
\]

\[
C = \left((p/2 + 1/2)a^2 + p/2 - 1/2\right)(u + 2M)^2 - \left((p + 1)a^2 - 2a + p - 1\right) M (u + 2M)
+ M^2 (\alpha^2 + p - 2a + p)\right)^2 k^2,
\]

\[
D = \left((i/2 + i/2p)a^2 + i/2p - i/2\right)(u + 2M)^3
- 3/2 iM \left((p + 1)a^2 + p - 1\right)(u + 2M)^2
+ iM (a - 1) M^2 (a + 1)(u + 2M)
+ iM^3 (\alpha^2 + p - 2a + p)k,
\]

\[
E = \lambda^2 (u + 2M) u.
\]

Here we again take the solution which is analytic around \(u = 0 (r = 2M)\), namely

\[
T_1(u) = e^{-i/2u((p+1)a^2+p-1)}(u + 2M)^{i/2k(\alpha^2 - 2a + p)} M
\]

\[
\times u^{-i/2(\alpha^2 + p - 2a + p)k} M
\]

\[
\times H_C(2iM (p + 1) a^2 + p - 1) k, -1/2
\]

\[
- (a^2 + 2a + p) k M,
\]

\[
- 1/2 + i k (a^2 - 2a + p) M,
\]

\[
- M ((p + 1)a^2 + p - 1) (4kMa + i) k,
\]

\[
1/2 M^2 k^2 (a^2 + 1)^2 p^2 + 1/2 M (a^2 + 1)
\]

\[
\times (2Ma^2 k + 4kMa - 2M k + i) k p
\]

\[
+ 2k^2 a^3 M^2 + 1/2M (4M k + i) k^2
\]

\[
+ M k (-2M k + i) a - i/2M k - \lambda^2 + 3/8, -1/2(\frac{u}{2M})\].

(17)

The second solution is given below:

\[
T_{12}(u) = e^{-i/2u((p+1)a^2+p-1)}(u + 2M)^{i/2k(\alpha^2 - 2a + p)} M
\]

\[
\times u^{-i/2(\alpha^2 + p - 2a + p)k} M
\]

\[
\times H_C(2iM (p + 1) a^2 + p - 1) k, 1/2
\]

\[
+ (a^2 + 2a + p) k M,
\]

\[
- 1/2 + i k (a^2 - 2a + p) M,
\]

\[
- M ((p + 1)a^2 + p - 1) (4kMa + i) k,
\]

\[
1/2 M^2 k^2 (a^2 + 1)^2 p^2 + 1/2 M (a^2 + 1)
\]

\[
\times (2Ma^2 k + 4kMa - 2M k + i) k p
\]

\[
+ 2k^2 a^3 M^2 + 1/2M (4M k + i) k^2
\]

\[
+ M k (-2M k + i) a - i/2M k - \lambda^2 + 3/8, -1/2(\frac{u}{2M})\].

(18)

We discard it since it has a second root nonanalyticity at \(u = 0\), our point of expansion.

Using \(p = 10, k = 0.2, \ a = 0.1, \lambda = 0.7\) and \(M = 5\), we give the plots of the first solution for \(0 < r < 2M\) and \(u > 0\) in fig. 1 and fig. 2, respectively.

Polynomial solutions can be given for the confluent Heun equation under some conditions [30,31]. The identity \(\mu + \nu = -N\), \(N\) being the degree of the polynomial solution, should be satisfied along with a vanishing determinant. However, this identity is not useful in this case as \(\mu + \nu = 0\).

**Solution around infinity.** – Solution of the confluent Heun equation around the irregular singularity at infinity can be given by the Thomé solution as [32]

\[
\lim_{z \to \infty} U(z) \sim e^{\pm \omega z^\pm \eta} (B_k/2).
\]

(19)

20002-p3
The confluent Heun equation is written in the form
\[
 z(z-1)\frac{d^2U}{dz^2} + (B_1 + B_2z)\frac{dU}{dz} + \left[ B_3 - 2\eta\omega(z-1) + \omega^2z(z-1) \right] U = 0, \quad (20)
\]
for \( \omega \neq 0 \) and all other parameters are constants. This form is called the generalized spheroidal wave equation and finding the correspondence with the general form given by eq. (8) needs some algebra as the solutions are not in the same form (i.e., we need to define \( V(z) = e^{i\omega z}U(z) \) first and then proceed with the solution). Studying the solution for our case, we find the first term in this asymptotic series as
\[
 \lim_{u \to \infty} T_1(u) \sim e^{(2i[(p+1)a^2+p-1])Mu}, \quad (21)
\]
as the solution around infinity. There is a second solution which behaves as \( u^{-4kMa^{-1}} \), i.e., it vanishes as \( u \) goes to infinity.

As to the physical interpretation of our solutions, we can get information only by plotting our solutions, since the general behavior of Heun functions is not generally known explicitly. We see the approach to singularity at \( r = 2M \) when we plot our function for the range \( 0 < r < 2M \). When we expand around one of the regular singular points, we expect such a behaviour around the second singular point.

To our surprise, the regular solution resembles a plane wave for \( u > 0 \) (fig. 2) with almost constant frequency and constant amplitude. This was a surprise, since this is the behaviour only in the asymptotic region for the quasi-classical solution in ref. [9]. The irregular solution, besides being nonanalytic at \( u = 0 \), goes to zero, oscillating with vanishing amplitude. This behaviour is reflected in the behaviour of the second solution (asymptotic solution) which behaves as a reciprocal power of \( u \).

**Klein-Gordon equation.** – In another recent paper by Al-Badawi, the Dirac equation is studied in a Schwarzschild black hole immersed in an electromagnetic universe with charge coupling [14]. Here the electromagnetic radiation is not attributed to the parameter \( M \). This solution again interpolates the Schwarzschild [16] and Bertotti-Robinson [17,18] solutions [14]. The metric is
\[
 ds^2 = \frac{\Delta}{r^2}dt^2 - \frac{r^2}{\Delta}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (22)
\]
where \( \Delta = r^2 - 2Mr + M^2(1 - a^2) \) [33,34]. Here, \( M \) is the parameter used in the Schwarzschild solution and \( a \) (\( 0 < a \leq 1 \)) is the external parameter. This metric has two horizons, the outer horizon at \( r_1 = M(1 + a) \) and the inner horizon at \( r_2 = M(1 - a) \). The external electromagnetic field shrinks at the outer horizon and expands at the inner horizon [33]. Note that the \( a = 0 \) case can be transformed into the Bertotti-Robinson solution [34].

We study the massless Klein-Gordon equation in the background of this metric, since we cannot identify the solution for the massive case,
\[
 \frac{1}{\sqrt{-g}}\partial_\mu \left( \sqrt{-g}g^{\mu\nu} \partial_\nu \Phi \right) = 0, \quad (23)
\]
in this background, namely,
\[
\Phi \sin^2 \theta r^4 \omega^2 + \Delta^2 \sin^2 \theta \frac{\partial^2 \Phi}{\partial r^2} + \Delta \left( \frac{\partial \Phi}{\partial r} \right) \sin^2 \theta \frac{\partial \Delta}{\partial r} \\
- \Delta^2 \Phi + \Delta \left( \frac{\partial \Phi}{\partial \theta} \right) \sin \theta \cos \theta \\
+ \Delta \sin^2 \theta \frac{\partial^2 \Phi}{\partial \theta^2} = 0.
\] (24)

This equation can be separated into the radial and angular parts with the Ansatz
\[
\Phi = e^{-i\omega t}e^{i\phi}F(r)S(\theta).
\] (25)

After defining the separation constant \( \lambda \), the radial and angular parts are obtained as
\[
\frac{\Delta^2 F(r)}{dr^2} - \frac{F(r) \lambda}{\Delta} + \frac{F(r) r^4 \omega^2 + \Delta \left( \frac{\partial F(r)}{\partial r} \right) \frac{d\Delta}{dr}}{\Delta^2} = 0,
\] (26)

\[
\frac{\Delta^2 S(\theta)}{d\theta^2} + \frac{S(\theta) \lambda}{\sin^2 \theta} \left( \sin \theta \cos \theta - S(\theta) \frac{n^2}{\sin^2 \theta} \right) = 0.
\] (27)

The angular part is in the form of the associated Legendre equation and the radial part can be solved in terms of confluent Heun functions. We change our parameter \( r \) to \( u = r - r_1, r_1 \) being the outer event horizon in order to study the behavior outside the event horizons. We note that the event horizon is located at \( r_1 = M(1 + a) \) and the inner horizon is located at \( r_2 = M(1 - a) \). The radial solution is
\[
F(u) = e^{-i\omega t}u^{\frac{\nu_1 + \Delta}{2}} (u + r_1 - r_2)^{\frac{\nu_2 + \Delta}{2}} \\
\times H_C \left( 2i\omega (r_1 - r_2), \frac{2i\nu_2 + \Delta}{r_1 - r_2}, \frac{2i\nu_2 + \Delta}{r_1 - r_2}, (-2r_1^2 + 2r_2^2) \omega^2, \\
2r_1^4 \omega^2 - 4r_1^3 \omega^2 r_2 - \lambda r_1^2 + 2r_1r_2 \lambda - \lambda r_2^2 \\
(r_1 - r_2)^2 \right),
\] (28)

and the second solution, namely
\[
F_2(u) = e^{-i\omega t}u^{-\frac{\nu_1 + \Delta}{2}} (u + r_1 - r_2)^{-\frac{\nu_2 + \Delta}{2}} \\
\times H_C \left( 2i\omega (r_1 - r_2), \frac{-2i\nu_2 + \Delta}{r_1 - r_2}, \frac{2i\nu_2 + \Delta}{r_1 - r_2}, (-2r_1^2 + 2r_2^2) \omega^2, \\
2r_1^4 \omega^2 - 4r_1^3 \omega^2 r_2 - \lambda r_1^2 + 2r_1r_2 \lambda - \lambda r_2^2 \\
(r_1 - r_2)^2 \right),
\] (29)

These solutions may be interpreted as two waves with different phases, but both moving in the same direction asymptotically, since for large values of \( u \), \( hu \) is much smaller than \( u \).

We just wanted to state that this test particle in this metric, too, has a Heun family solution. We will study other properties of this solution in further papers.

**Conclusion.** – Here we studied two different metrics given by [10] and [33]. In the first case we studied the Dirac equation given in [9] and found that the radial solution can be expressed in terms of confluent Heun functions. We found the same structure in the second metric case [14,33] for the Klein-Gordon equation.

***

We thank the anonymous referee for correcting our “careless” use of the physical and mathematical terminology. MH thanks Prof. IBRAHIM SEMIZ for providing important literature and Prof. NADIR GHANAZAFARI for technical assistance. He also thanks the Science Academy, Turkey for support. This work is supported by TUBITAK, the Scientific and Technological Council of Turkey.

REFERENCES

[1] HEUN K., Math. Ann., 33 (1899) 161.
[2] RONVEAUX A. (Editor), Heun’s Differential Equations (Oxford University Press) 1995.
[3] HORTACSU M., Heun Functions and their uses in Physics, in Proceedings of the 13th Regional Conference on Mathematical Physics, Antalya, Turkey, October 27–31, 2010, edited by CAMCI U. and SEMIZ I. (World Scientific, Singapore) 2013, pp. 23–29 (arXiv:1101.0471).
[4] BLAUDIN J., POONS R. and MARCILHAC Y., Lett. Nuovo Cimento, 38 (1983) 561.
[5] LEAVER E. W., J. Math. Phys., 27 (1986) 1238.
[6] SUZUKI H., TAKASUGI E. and UMETSU H., Prog. Theor. Phys., 100 (1998) 491.
[7] TEUKOLSKY S. A., Phys. Rev. Lett., 29 (1972) 1114.
[8] BATIC D. and SCHMIDT H., J. Math. Phys., 48 (2007) 042502.
[9] AL-BADawi A. and OWAIdat M. Q., Gen. Relativ. Gravit., 49 (2017) 110 (arXiv:1702.00368).
[10] HALISoy M. and AL-BADawi A., Class. Quantum Grav., 12 (1995) 3013.
[11] RAY J. M. and WEI M. S., Nuovo Cimento B, 42 (1977) 151.
[12] HALISoy M., Gen. Relativ. Gravit., 25 (1993) 275.
[13] HALISoy M., Gen. Relativ. Gravit., 25 (1993) 975.
[14] AL-BADawi A., arXiv:1702.01380 (2017).
[15] GRIFFITHS J. B. and PODOLSKY J., Exact Space-Times in Einstein’s General Relativity (Cambridge University Press) 2009, p. 320.
[16] SCHWARZSCHILD K., Sitzungsber. K. Preuss. Akadem. Wiss., 7 (1916) 189.
[17] BERTOTTI B., Phys. Rev., 116 (1959) 1331.
[18] ROBINSON I., Bull. Acad. Pol. Sci., 7 (1959) 351.
[19] PLEBANski J. and DEMANSKI M., Ann. Phys. (N. Y.), 98 (1986) 98.
[20] NEWMAN E. T. and PENROSE R., J. Math. Phys., 3 (1962) 566.
[21] PHILIPP D. and PERLICK V., arXiv:1503.08101 [gr-qc] (2015).
[22] LEAVER E., Proc. R. Soc. Lond. A, 402 (1985) 285.
[23] FIZIEV P., arXiv:gr-qc/0603003 (2006).
[24] FIZIEV P., Class. Quantum Grav., 23 (2006) 2447.
[25] Fiziev P., *J. Phys.: Conf. Ser.*, **66** (2007) 012016.

[26] Eguchi T. and Hanson A. J., *Phys. Lett. B*, **74** (1978) 249.

[27] Birkandan T. and Hortacsu M., *J. Math. Phys.*, **49** (2008) 054101.

[28] Arscott F. M. and Ronveaux A. (Editors), *Heun’s Differential Equations* (Oxford University Press) 1995, p. 31.

[29] Fiziev P., *Class. Quantum Grav.*, **27** (2010) 135001.

[30] Fiziev P., *J. Phys. A: Math. Theor.*, **43** (2010) 035203.

[31] Ciftci H., Hall R. L., Saad N. and Dogu E., *J. Phys. A: Math. Theor.*, **43** (2010) 415206.

[32] El-Jaick L. J. and Figueiredo B. D. B., *J. Math. Phys.*, **49** (2008) 083508.

[33] Halilsoy M. and Al-Badawi A., *Nuovo Cimento B*, **113** (1998) 761.

[34] Ovgun A., *Int. J. Theor. Phys.*, **55** (2016) 2919 (arXiv:1508.04100).