A counterexample to a Penrose inequality conjectured by Gibbons

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Abstract
We show that the Brill–Lindquist initial data provide a counterexample to a Riemannian Penrose inequality with charge conjectured by Gibbons. The observation illustrates a sub-additive characteristic of the area radii for the individual connected components of an outermost horizon as a lower bound of the ADM mass.

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1. Introduction

Let $(M, g_0)$ be an asymptotically flat three-dimensional manifold, with nonnegative scalar curvature $S(g_0)$. Given an asymptotically flat end, let us assume that there exists a set of minimal 2-spheres acting as the outermost horizon. In this situation, there are a series of inequalities which relate the asymptotic data to the Riemannian geometry of the manifolds.

The first such inequality is the positive mass theorem [14, 15, 18]. We rephrase the Riemannian version of this result as the following variational statement: among all time-symmetric asymptotically flat initial data sets for the Einstein vacuum equations, flat Euclidean 3-space is the unique minimizer of the total mass. Thus, the total mass satisfies $m \geq 0$ with equality if and only if the data set is isometric to $\mathbb{R}^3$ with the flat metric.

A stronger result is the Riemannian version of the Penrose inequality [1, 8, 12] (see also the review article [10] and references therein), which can be stated in a similar variational vein: among all time-symmetric asymptotically flat initial data sets for the Einstein vacuum equations with an outermost minimal surface of area $A$, the Schwarzschild slice is the unique minimizer of the total mass. In other words, $m \geq R/2$ where $R = \sqrt{A/4\pi}$ is the area radius.
of the outermost horizon, and equality occurs if and only if the data are isometric to the \textit{Schwarzschild slice}: 

\[ g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}. \]

When these results are phrased in this fashion, a natural question is whether similar variational characterizations of the other known stationary solutions of the Einstein equations hold. In particular, one could ask whether among all time-symmetric asymptotically flat Einstein–Maxwell initial data sets with an horizon of area \( A \) and charge \( Q \), the Reissner–Nordström slice is the unique minimizer of the mass. This is equivalent to asking whether the following inequality holds for any data set:

\[ m \geq \frac{1}{2} \left( R + \frac{Q^2}{R} \right), \quad (1) \]

where \( Q \) is the total charge, with equality if and only if the data are a Reissner–Nordström slice. The charge within a 2-surface \( S \) is defined by

\[ Q(S) = \frac{1}{4\pi} \int_S E_i n^i \, dA. \quad (2) \]

Recall that a time-symmetric initial data set \((M, g, E, B)\) for the Einstein–Maxwell equations consists of a Riemannian manifold \((M, g)\), and two vector fields \(E\) and \(B\) on \(M\) such that

\[
S(g) = 2(|E|^2_g + |B|^2_g), \quad \text{div}_g E = \text{div}_g B = 0,
\]

\[ E \times B = 0, \quad \int_S B_i n^i \, dA = 0, \]

where \(S(g)\) is the scalar curvature of \(g\), and \(n\) denotes the unit normal vectors to \(S\). Note that the charge depends only on the homological type of \(S\).

When the horizon is connected, inequality (1) can be proved by using the inverse mean curvature flow \([8, 9]\). Indeed, the argument in [9] relies simply on Geroch monotonicity of the Hawking mass—which still holds for the weak flow introduced by Huisken and Ilmanen in [8]—while keeping track of the scalar curvature term \(S = 2(|E|^2 + |B|^2)\). However, when the horizon has several components, the same argument yields the following inequality:

\[
m \geq \frac{1}{2} \max_i \left( R_i + \frac{\min \sum_j s_j Q_j^2}{R_i} \right), \quad (3)
\]

where \(R_i\) and \(Q_i\) are respectively the area radii and charges of the components of the horizon \(i = 1, \ldots, N, s_i = 0 \text{ or } 1,\) and the minimum is taken over all possible combinations.

In [17] we pointed out that equation (1) does not hold for the case of an horizon with several disconnected components. Namely, there exists a strongly asymptotically flat time-symmetric initial data set \((M, g, E, 0)\) for the Einstein–Maxwell equations such that

\[
m < \frac{1}{2} \left( R + \frac{Q^2}{R} \right). \quad (4)
\]

In 1984, Gibbons [7] conjectured an inequality similar to (1). However, in his conjecture, the right-hand side of (1) is taken to be additive over connected components of the horizon. Thus, Gibbons’s conjecture states that

\[
m \geq \frac{1}{2} \sum_i \left( R_i + \frac{Q_i^2}{R_i} \right). \quad (5)
\]
There is a physical reason to introduce additive quantities on the right-hand side of the inequality. The quantity

\[ m_i = \frac{1}{2} \left( R_i + \frac{Q_i^2}{R_i} \right) \]  

appears to play the role of the quasi-local mass of the \( i \)th black holes. Then, inequality (5) can be interpreted as saying that the total mass of the spacetime is always bigger than the sum of the quasi-local masses of the individual black holes. There is, however, a Newtonian reasoning to doubt inequality (5) in the case of several black holes. When the black holes are separated by a large distance, it is expected that the interaction energy between them, which is asymptotically Newtonian in this limit, will be negative. The total energy of the spacetime should be the sum of the quasi-local masses of the black holes plus this negative interaction energy. Hence the sum of the quasi-local masses is expected to be bigger than the total mass. The counterexample we present in the next section exhibits precisely this behavior.

It is important to note that there is an inequality analogous to (5) for the Kerr black hole:

\[ m \geq \sum_i m_i, \]

where the quasi-local masses are now defined by

\[ m_i = \sqrt{\frac{R_i^2}{4} + \frac{J_i^2}{R_i^2}}. \]

Here, \( J_i \) denotes the angular momentum of the \( i \)th black hole. Unlike the quasi-local charge (2), it is not straightforward to define the quasi-local angular momentum of each black hole in general; see the review article [16] on the problem of quasi-local mass and angular momentum. However, in the case of axial symmetry, there is a natural definition: the Komar integral. In that case, we can ask whether inequality (7) holds. Finally, we can also combine, using the Kerr–Newman black hole solution, both inequalities to obtain the general inequality with charge and angular momentum. Namely, define the \( m_i \) to be

\[ m_i = \sqrt{\frac{1}{4} \left( R_i + \frac{Q_i^2}{R_i} \right)^2 + \frac{J_i^2}{R_i^2}}. \]

While it is still an open problem, inequality (7) for one black hole in axial symmetry is expected to be hold; see the discussion in [10] and [5]. However, if we assume that there are two or more black holes, our counterexample is relevant since when we set the charges and the angular momentum to zero, all these inequalities imply

\[ m \geq \frac{1}{2} \sum_i R_i, \]

which our example violates. This inequality is stronger than the usual Riemannian Penrose inequality [1]

\[ m \geq \frac{1}{2} \left( \sum_i R_i^2 \right)^{1/2}. \]

As mentioned above, and as pointed out also in [6, 17], the natural candidate to violate inequality (10) is a configuration of two Schwarzschild black holes separated by a large distance. This is precisely the counterexample we present. Although all the ingredients used in our argument have been present in the literature, it has not been pointed out before to the best of our knowledge. We believe that this counterexample is important because it sheds some light on the quasi-local aspect of Riemannian Penrose inequalities.
2. Counterexample

The Riemannian manifold that violates inequality (10) proposed by Gibbons [7] is the well-known Brill–Lindquist data [2], which are a conformally flat time-symmetric vacuum data defined on the differentiable manifold $M := \mathbb{R}^3 \setminus \{x_1, x_2\}$ with the metric $h_{ij} = \phi^2 \delta_{ij}$ with the conformal factor

$$\phi = \left(1 + \frac{\mu_1}{2|x - x_1|} + \frac{\mu_2}{2|x - x_2|}\right).$$

‘Time symmetric’ here means that the second fundamental form of the three-manifold in the spacetime that is a solution to the Einstein equation vanishes. As $\phi$ is harmonic on $\mathbb{R}^3$, the scalar curvature of the metric is zero. For the sake of simplicity, we make the assumption $\mu_1 = \mu_2 =: \mu$, and $x_1 = (0, 0, 1), \ x_2 = (0, 0, -1)$.

The manifold $(M, h)$ has three asymptotically flat ends, namely $E_0$ where $|x - x_1|, |x - x_2| \to \infty$, $E_1$ where $x \to x_1$ and $E_2$ on which $x \to x_2$. The end $E_0$ has mass $m(E_0) = \mu_1 + \mu_2 = 2\mu$. Let $\Omega_r = \{|x - x_1| > 1/r, |x - x_2| > 1/r, |x| < r\}$ and let $S_{i,0} = \{|x - x_i| = 1/r\}$ for $i = 1, 2$, and $S_0 = \{|x| = r\}$; then for $r$ large enough, the boundary $S_{i,0} \cup S_{i,1} \cup S_{i,2}$ of $\Omega_r$ has positive mean curvature with respect to the outer normal. Hence, by minimizing area over all surfaces enclosing $S_i$ and $S_j$ and enclosed in $S_0$, one can show that there is a surface of least area $\Sigma$ enclosing $E_1$ and $E_2$, see [11, theorem 1, p 645]. Suppose that $\Sigma = \Sigma_1 \cup \Sigma_2$ where $\Sigma_i$ is a compact minimal surface which encloses $E_i$, and $A(\Sigma_i) = A_i$. In this situation, following Gibbons [6], we have a lower bound $A_1 > 16\pi \mu^2$, or equivalently in terms of the area radius $R_1 > 2\mu$. Indeed, letting

$$\psi = \left(1 + \frac{\mu}{2|x - x_1|}\right),$$

then $\tilde{h} = \psi^2 \delta_{ij}$ is a Schwarzschild metric, $\Sigma = \{|x - x_1| = \mu/2\}$ minimizes area in $\tilde{h}$ among all surfaces enclosing $E_1$, $\phi > \psi$, and hence

$$4\pi R_1^2 = A_1 = A_{\tilde{h}}(\Sigma_1) > A_\tilde{h}(\Sigma_1) \geq A(\Sigma) = 16\pi \mu^2.$$

Similarly, $R_2 > 2\mu$, and it follows immediately that $\frac{1}{2} (R_1 + R_2) > 2\mu = m(E_0)$ violating (10).

The next proposition shows that for $\mu > 0$ sufficiently small, there is no connected minimal sphere enclosing both $E_1$ and $E_2$, showing that for small values of $\mu > 0$, the Brill–Lindquist data indeed provide a counterexample to (10).

**Proposition 1.** For sufficiently small $\mu > 0$, the Brill–Lindquist initial data described above contain no closed connected minimal surface enclosing both punctures $x_1$ and $x_2$.

We remark that the statement follows from a direct application of theorem 3.2 in [4]. For the sake of completeness, we present the proof below. Our situation at hand is simpler than that of [4], and the proof below illustrates the geometry of the Brill–Lindquist metric concisely; see also [3] for a similar argument.

We also remark that this conclusion has been previously claimed based on numerical evidence presented in [2] and [6], and since then extensively numerically confirmed in the literature. Note that $\mu \to 0$ is equivalent to $L \to \infty$ while $\mu$ is kept constant, where $L$ is the separation distance between the punctures (with respect to the flat background metric). This limit can be interpreted as the Newtonian limit of the initial data. The physical content of this lemma is that in this limit the initial data correspond to two separated black holes.
Proof. To show that the claim indeed holds, suppose $\Sigma$ is such a surface. Without loss of generality, we can assume that $\Sigma$ is the outermost minimal surface for the end $E_0$.

The configuration forces $\Sigma \cap \{z = 0\}$ to intersect nontrivially with the $xy$-plane $\{z = 0\}$. Let $p$ be a point in $\Sigma \cap \{z = 0\}$ and define the surface $\Sigma_1 = \Sigma \cap B_r(p)$, with some fixed $r < 1$. Here $B_r(p)$ denotes the (Euclidean) ball of radius $r$ centered at the point $p$. By construction, $\Sigma_1$ is a minimal surface, disjoint from the punctures, with nonempty boundary as the surface $\Sigma$ has to enclose both $x_1$ and $x_2$. Since, by assumption, $\Sigma$ is outermost, it is a stable minimal surface.

The following two consequences are then clearly in contradiction, implying that there is no such surface $\Sigma$.

(i) By the Penrose inequality, the area of the surface $\Sigma$ (and hence the area of $\Sigma_1$) is bounded above by the total mass of the data (which in our configuration is given by $2\mu$):

$$2\mu \geq \sqrt{\frac{A(\Sigma)}{16\pi}} \geq \sqrt{\frac{A(\Sigma_1)}{16\pi}}.$$  \hspace{1cm} (11)

And hence, we have that $A(\Sigma_1) \to 0$ as $\mu \to 0$.

(ii) We have a lower bound on the area $A(\Sigma_1)$ independent of $\mu$. This follows from an estimate of the sup norm of the second fundamental form as in theorem 2 of [13] for stable minimal surfaces. It gives that the norm of the second fundamental form of $\Sigma_1$ is bounded uniformly (in $\mu$) by some positive constant $C$ (which depends on $0 < r < 1$), provided $\mu$ is sufficiently small. Then, there exists a sufficiently small $\varepsilon > 0$ independent of $\mu$, so that the surface $\Sigma \cap B_r(p)$ can be described as a graph over the tangent plane $T_p\Sigma$, which in turn gives a positive lower bound of the area of $\Sigma \cap B_{\varepsilon}(p)$ independent of $\mu$. \hfill $\Box$

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