Discerning Noncommutative Extra Dimensions

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Abstract

Experimental limits on the violation of four-dimensional Lorentz invariance imply that noncommutativity among ordinary spacetime dimensions must be small. Noncommutativity among extra, compactified spatial dimensions, however, is far less constrained and may have discernible collider signatures. Here we study the experimental consequences of noncommutative QED in six dimensions, with noncommutativity restricted to a TeV-scale bulk. Assuming the orbifold $T^2/Z_2$, we construct the effective four-dimensional theory and study interactions unique to the noncommutative case. New vertices involving the Kaluza-Klein (KK) excitations of the photon yield order 100% corrections to the pair production and to the decays of some of the lighter modes. We show that these effects are difficult to resolve at the LHC, but are likely within the reach of a future Very Large Hadron Collider (VLHC).

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I. INTRODUCTION

Interest in the possibility of extra, compactified spatial dimensions at the TeV scale \cite{1} has led to serious consideration of other modifications of spacetime structure that may be manifest at collider energies. One such possibility is that ordinary four-dimensional spacetime may become noncommutative \cite{2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14} at some (not so) high scale $\Lambda_{\text{NC}}$: promoting the position four-vector $x^\mu$ to an operator $\hat{x}^\mu$, one assumes that

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu}$$ (1.1)

where $\theta$ is a real, constant matrix with elements of order $(\Lambda_{\text{NC}})^{-2}$. Field theories defined on such a noncommutative spacetime involve field operators that are functions of the noncommuting coordinates $\hat{x}^\mu$. However, it is possible to map such a theory to a physically equivalent one involving fields that are functions of the classical coordinates $x^\mu$. Given a classical function $f(x)$ with Fourier transform

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^nxe^{ik\cdot x}f(x),$$ (1.2)

one may associate the operator

$$W(f) = \frac{1}{(2\pi)^{n/2}} \int d^nke^{-ik\cdot \hat{x}} \tilde{f}(k)$$ (1.3)

in the noncommuting theory \cite{13}. Requiring that this correspondence holds for the product of functions,

$$W(f)W(g) = W(f \star g)$$ (1.4)

one finds that

$$f \star g = \lim_{y \to x} e^{\frac{i}{2} \int d^dx \theta^{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial y^\mu} f(x)g(y)}.$$ (1.5)

This is the Moyal-Weyl $\star$-product \cite{13}. The starting point for most of the phenomenological study of noncommutative field theories is the construction of gauge-invariant Lagrangians in which ordinary multiplication has been replaced by the star product. The resulting classical field theory is amenable to quantization and study in the usual ways.

Noncommutative field theories (NCFTs) have been shown to arise in the low-energy limit of some string theories \cite{17}, and this has partly motivated a number of the recent
phenomenological studies. On the other hand, NCFTs have been studied frequently without reference to higher-energy embeddings, and are interesting as effective field theories by themselves [13, 18, 19, 20, 21, 22, 23]. We will view the noncommutative theory that we study as a low-energy effective theory, valid up to the Planck scale, which may be relatively low.

Prior phenomenological work on noncommutative extensions of the standard model have focused on direct collider searches [2, 3, 4, 5, 6, 7], astrophysical effects [8], and indirect low-energy bounds [8, 11, 12, 13]. The papers on collider signatures have dealt exclusively with noncommutative QED (NCQED), primarily for two reasons. First, up until recently [21, 22, 23], it was only known how to formulate noncommutative U(1) [18] and U(N) [19] gauge theories consistently, with the matter content of the former restricted to charges of 0 or ±1. Secondly, NCQED has interaction vertices that are strikingly different from those in ordinary QED. In particular, the Abelian field strength tensor in NCQED has the form

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\mu} A_{\nu} - i \omega_{\mu} \times A_{\nu} + i A_{\nu} \times A_{\mu} . \]  

Notice that the last two terms cancel in the \( \theta \to 0 \) limit, but otherwise lead to exotic three- and four-point photon vertices. Provided that the scale \( \Lambda_{NC} \) is sufficiently low, such interactions could be discerned, for example, at the NLC [2, 4, 5, 6, 7].

The prior work on low-energy tests of noncommutativity [9, 10, 11, 12, 13] has focused mainly on searches for Lorentz violation [21], arising from the constant parameter \( \theta^{\mu\nu} \) appearing in Eq. (1.1). Noting that \( \theta^{\mu\nu} \) is antisymmetric, one can regard \( \theta_{0i}^{ij} \) and \( \theta_{ij}^{ij} \) as constant three-vectors indicating preferred directions in a given Lorentz frame. Low-energy tests of Lorentz invariance place bounds on \( \Lambda_{NC} \) of order 10 TeV [10], if one considers NCQED processes at tree-level. As was pointed out by Mocioiu et al. [9], the generation of an effective operator of the form \( \bar{N} \theta^{\mu\nu} \sigma_{\mu\nu} N \) leads to a shift in nuclear magnetic moments and an observable sidereal variation in the magnitude of hyperfine splitting in atoms. The resulting bound on \( \theta^{-1/2} \) is of the same order as the conventional (nonsupersymmetric) grand unification scale. That similar operators are generated radiatively at the quark level was demonstrated in a number of toy models by Anisimov et al. [11], and in a consistent formulation of noncommutative QCD by Carlson et al. [13]. In the latter work, for example, the authors obtained the restriction \( \theta \Lambda^2 < 10^{-29} \), where \( \theta \) is a typical entry in \( \theta^{\mu\nu} \), and \( \Lambda \) is an
ultraviolet regularization scale. For $\Lambda \sim M_{\text{Planck}} \sim 1 \text{ TeV}$, the Lorentz-violation from such radiatively-generated, lower-dimension operators seems to imply that the size of $\theta$ must be much smaller than one might estimate based on any naturalness arguments.

If one accepts the generic conclusions of Refs. [3, 10, 11, 13], then colliders have little chance of probing noncommutative phenomenology. In this paper, we point out that there is a way around this conclusion in the case where noncommutativity is restricted to extra spatial dimensions. For concreteness, we will consider NCQED in six dimensions, where two extra spatial dimensions are compactified on the toroidal orbifold $T^2/\mathbb{Z}_2$. The compact, extra-dimensional space $T^2/\mathbb{Z}_2$ is the minimal choice that can be noncommutative and yield a phenomenologically viable low-energy theory. Orbifolding the torus allows one to project out unwanted scalar photon zero modes in the effective 4D theory and $\mathbb{Z}_2$ is the smallest discrete symmetry that accomplishes the task. (For other scenarios see, for example, Refs. [25, 26].) Since ordinary four-dimensional spacetime is commutative, with itself and with the extra dimensions, there is no violation of four-dimensional Lorentz invariance, and the most stringent bounds described above are evaded. Higher-dimensional Lorentz invariance is broken through compactification in any case, so that the phenomenological constraints should be no stronger than in the case of commutative extra dimensions. We will show that new, $\theta$-dependent interactions are present in our theory, but involve exclusively the Kaluza-Klein (KK) excitation of the photon. We consider how these interactions may be discerned at hadron colliders through the decays and through pair production of some of the lighter modes.

II. FORMALISM

We consider six-dimensional (6D) quantum electrodynamics (QED) with the gauge fields defined on the full space and the fermion fields restricted to a 4D subspace. The Lagrangian is

$$
\mathcal{L}_6 = -\frac{1}{4} \mathcal{F}_{MN} \ast \mathcal{F}^{MN} + \mathcal{L}_{\text{gauge fixing}} + \delta^{(2)}(\vec{y}) \left\{ \bar{\psi} (i \not\partial - m) \psi + \hat{e} \bar{\psi} \ast \mathcal{A} \ast \psi \right\},
$$

(2.1)
where

\[ F_{MN} = \partial_M A_N - \partial_N A_M - i \hat{e} [A_M, A_N] \tag{2.2} \]

and where \( \hat{e} \) is the 6D gauge coupling. The star (*) indicates the Moyal product, Eq. (1.5), and the star commutator \([., .]\) indicates a commutator with Moyal multiplication. Our notation for the position six-vector is

\[ X^M = (x^0, x^1, x^2, x^3, y^5, y^6) \tag{2.3} \]

with \( \vec{y} \equiv (y^5, y^6) \).

We compactify the extra dimensions on the orbifold \( T^2/\mathbb{Z}_2 \), where \( T^2 \) is a general 2-torus. We take into account the possibility of two different radii \( R_5 \) and \( R_6 \) and a relative angle \( \phi \) between the two directions of compactification \([27, 28]\), as illustrated in Fig. 1. The coordinates along the torus are \( \zeta^i \), related to \( y^i \) by

\[ y^5 = \zeta^5 + \zeta^6 \cos \phi \]
\[ y^6 = \zeta^6 \sin \phi \tag{2.4} \]

The periodicity requirements on a function of orbifold coordinates \( f(\zeta^5, \zeta^6) \) are

\[ f(\zeta^5, \zeta^6) = f(\zeta^5 + 2\pi R_5, \zeta^6) = f(\zeta^5, \zeta^6 + 2\pi R_6) \tag{2.5} \]

**FIG. 1:** The two dimensional torus with differing radii and shift angle \( \phi \). Orthogonal axes are \( y_i \) and skewed axes are \( \zeta_i \).
Without orbifolding, Eq. (2.5) implies that bulk fields have 6D wave functions proportional to
\[ \exp \left\{ i \frac{n^5 \zeta^5}{R_5} + i \frac{n^6 \zeta^6}{R_6} \right\}, \]
\[ = \exp \left\{ i \frac{n^5 y^5}{R_5} + i \frac{y^6}{\sin \phi} \left[ \frac{n^6}{R_6} - \frac{n^5}{R_5} \cos \phi \right] \right\}, \tag{2.6} \]
where \( n^5 \) and \( n^6 \) are integers. The masses of the KK modes are eigenvalues of the mass operator \(- \partial^2 - \partial^2_{y^5} \) and are given by
\[ m_{\vec{n}}^2 = \frac{1}{\sin^2 \phi} \left( \frac{n^2_5}{R_5^2} + \frac{n^2_6}{R_6^2} - \frac{2n_5 n_6}{R_5 R_6} \cos \phi \right), \tag{2.7} \]
where \( \vec{n} \equiv (n^5, n^6) \).

The \( \mathbb{Z}_2 \) orbifolding consists of identifying points connected by \( \vec{y} \rightarrow -\vec{y} \) \[27\]. Different components of the gauge field may be Fourier expanded with different \( \mathbb{Z}_2 \)-parities so that zero modes are only present for the first four components,
\[ \mathcal{A}_M(X) = \sum_{\{\vec{n}_+\}} \mathcal{A}_{\vec{n}}^{(M)}(x) \begin{cases} \cos \left( \frac{n^5 \zeta^5 + \xi n^6 \zeta^6}{R} \right), & M = \mu \\ \sin \left( \frac{n^5 \zeta^5 + \xi n^6 \zeta^6}{R} \right), & M = 5, 6 \end{cases}, \tag{2.8} \]
with \( \mu = 0, 1, 2, 3 \). Here, \( \xi \) is the ratio of the radii, with
\[ R_5 \equiv R = \xi R_6. \tag{2.9} \]
Since orbifolding has provided wave functions with distinct parities, the value of \( \vec{n} \) is now restricted to a half plane including the origin,
\[ \{\vec{n}_+\} = \begin{cases} \vec{n} = \vec{0}; & \text{or} \\ n^5 = 0, n^6 > 0; & \text{or} \\ n^5 > 0, n^6 = \text{any integer} \end{cases}, \tag{2.10} \]
also shown in Fig. 2. Within the set \( \{\vec{n}_+\} \), masses are unique for most values of \( \xi \) and \( \phi \).

One obtains the 4D Lagrangian by integrating over the extra dimensions,
\[ \mathcal{L}_4 = \int d^2 y \mathcal{L}_6 = \frac{1}{\xi_0} \int_0^{2\pi R} d\zeta^5 \int_0^{2\pi R} d\tilde{\zeta}^6 \mathcal{L}_6, \tag{2.11} \]
where \( \xi_0 \equiv \xi / \sin \phi \) and \( \tilde{\zeta}^6 \equiv \zeta^6 \).
The set $\{\vec{n}_+\}$ consists of the points to the right of the line with the offset.

The gauge fixing Lagrangian is chosen as

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2} \eta \left( \partial_\mu A^\mu + \frac{1}{\eta} \partial_k A^k \right)^2$$

(2.12)

with $k = 5,6$. Terms in the Lagrangian quadratic in the gauge field become,

$$\left( -\frac{1}{4} F^{MN} F_{MN} + \mathcal{L}_{\text{gauge fixing}} \right)_{\text{free, 4d}}$$

$$= -\frac{1}{4} F^{\mu\nu}_\vec{0} F^{\mu\nu}(-\vec{0}) - \frac{1}{2} \eta \left( \partial_\mu A^{\vec{0}}_\mu \right)^2$$

$$+ \sum' \left\{ -\frac{1}{4} F^{\mu\nu}_\vec{n} F^{\mu\nu}\vec{n} - \frac{1}{2} \eta \left( \partial_\mu A^{\vec{n}}_\mu \right)^2 + \frac{1}{2} \partial_\mu A^{\vec{n}}_L \partial_\mu A^{\vec{n}}_L + \frac{1}{2} \partial_\mu A^{\vec{n}}_H \partial_\mu A^{\vec{n}}_H$$

$$+ \frac{1}{2} m^2_\vec{n} A^{\vec{n}}_\mu A^{\vec{n}}_\mu - \frac{1}{2} m^2_\vec{n} A^{\vec{n}}_L A^{\vec{n}}_L - \frac{1}{2} m^2_\vec{n} A^{\vec{n}}_H A^{\vec{n}}_H \right\}.$$ 

(2.13)

The primed sum is over the Kaluza-Klein (KK) modes, i.e., over $\{\vec{n}_+\}$ excluding $\vec{0}$. The 6D fields $A$ have been rescaled,

$$A^{\vec{0}}_M = \frac{\sqrt{\xi_0}}{2\pi R} A^{\vec{0}}_M,$$

$$A^{\vec{n}}_M = \frac{\sqrt{2\xi_0}}{2\pi R} A^{\vec{n}}_M \quad [\vec{n} \neq \vec{0}],$$

(2.14)

where the fields $A$ have their canonical 4D mass dimensions. The fifth and sixth components have been combined into

$$A^{\vec{n}}_L = \frac{1}{|\vec{n}|} \left( \vec{n}_5 A^6(\vec{n}) - \vec{n}_6 A^5(\vec{n}) \right),$$

$$A^{\vec{n}}_H = \frac{1}{|\vec{n}|} \left( \vec{n}_5 A^5(\vec{n}) + \vec{n}_6 A^6(\vec{n}) \right).$$

(2.15)
where \( \tilde{n} = (\tilde{n}^5, \tilde{n}^6) \) with

\[
\begin{align*}
\tilde{n}^5 &= n^5, \\
\tilde{n}^6 &= \frac{1}{\sin \phi} \left( \xi n^6 - n^5 \cos \phi \right),
\end{align*}
\] (2.16)

and \( m_{\overline{n}} = |\tilde{n}|/R \). The fields \( A_L \) and \( A_H \) are physical and unphysical scalars in the 4D theory, respectively. As \( \eta \to 0 \), the field \( A_H \) is removed from the theory, the extra-dimensional generalization of unitary gauge. We work in the \( \eta \to 0 \) limit henceforth. Thus, from the free gauge Lagrangian the physical states are the ordinary massless photon, the vector KK modes, and the scalar KK modes \( A^{(i)}_L \).

The fermion fields \( \psi \) are defined only at the \( \tilde{y} = \tilde{0} \) orbifold fixed point, and involve no rescaling. Since \( A^{(i)}_L \) is odd under the \( \mathbb{Z}_2 \) parity it vanishes at \( \tilde{y} = \tilde{0} \). Hence the fermions interact only with the photon and its vector KK excitations. The fermion Lagrangian is

\[
L_{f,4d} = \bar{\psi}(i\not\partial - m)\psi + e\bar{\psi} \star A^{(0)} \star \psi \\
+ e\sqrt{2} \sum' \bar{\psi} \star A^{(i)} \star \psi. 
\] (2.17)

The 4D gauge coupling has been identified through the rescaling

\[
\hat{e} = \frac{2\pi R}{\sqrt{\xi_0}} e. 
\] (2.18)

The pure gauge field interactions come from the terms

\[
L_{g \text{ int},6d} = i\hat{e} \partial M A_N \left[ A^M \star A^N \right] \\
+ \frac{1}{4} \hat{e}^2 \left[ A_M \star A_N \right] \star \left[ A^M \star A^N \right], 
\] (2.19)

in which the Moyal commutator may be written as

\[
[A_M \star A_N] = 2i \lim_{X \to Y} \sin \left( \frac{1}{2} \frac{\partial}{\partial X^i} \theta_{ij} \frac{\partial}{\partial Y^j} \right) A_M(X) A_N(Y). 
\] (2.20)

One may now extract the three-photon coupling in the 4D Lagrangian,

\[
L_{3\gamma,4d} = -e\sqrt{2} \sum' \left( \delta_{\tilde{n}_a,\tilde{n}_b+\tilde{n}_c} + \delta_{\tilde{n}_b,\tilde{n}_c+\tilde{n}_a} - \delta_{\tilde{n}_c,\tilde{n}_a+\tilde{n}_b} \right) \\
\times \partial_{\alpha} A^{(\tilde{n}_c)}_\beta A^{(\tilde{n}_a)}_\alpha A^{(\tilde{n}_b)}_\beta \sin \left( \frac{\vec{n}_a \theta_{ij} \vec{n}_b}{2R^2} \right), 
\] (2.21)
where $\tilde{n}$ is defined in Eq. (2.16). The triple-photon couplings involve only the KK modes, and never any ordinary massless photons. The Feynman rule that corresponds to the $3\gamma$ term in the Lagrangian, for the momenta, Lorentz indices, and KK modes labeled in Fig. 3, is given by

$$V_{3\gamma} = -e\sqrt{2} \left( \delta_{i_a}^{i} \theta_{ij} \tilde{n}_j^b + \delta_{i_b}^{i} \theta_{ij} \tilde{n}_j^c - \delta_{i_c}^{i} \theta_{ij} \tilde{n}_j^a \right)$$

$$\times \sin \left( \frac{\tilde{n}_i^j \theta_{ij} \tilde{n}_j^b}{2R^2} \right)$$

$$\times \left[ g_{\mu\nu}(p - q)_\rho + g_{\nu\rho}(q - r)_\mu + g_{\rho\mu}(r - p)_\nu \right].$$

(2.22)

![FIG. 3: The triple KK photon vertex.](image)

When the noncommutativity is only in the extra dimensions, the only independent non-zero component of the noncommutativity tensor is $\theta_{56} \equiv \theta$. (Theories with space-like noncommutativity are known to preserve perturbative unitarity [29].) The argument of the sine simplifies using

$$\tilde{n}_i^a \theta_{ij} \tilde{n}_j^b = \xi_0 \theta \left( n^5_a n^6_b - n^6_a n^5_b \right).$$

(2.23)

A four-photon vertex may be computed in a similar way, but will not be relevant to the physical processes studied in the sections that follow.

III. DECAYS

We first investigate the possibility of detecting noncommutativity in extra dimensions via corrections to the decays of the lighter KK modes. We will discuss production rates of single KK modes later in this section. The two-body decay $\gamma^{(\tilde{n})} \rightarrow f \bar{f}$ is unaffected by
noncommutativity, while $\gamma_n \to \gamma_n \gamma_n$ is kinematically inaccessible for decays allowed by Eq. (2.22). However, by taking one of the external lines in Fig. 3 off-shell, the new noncommutative vertex contributes at tree-level to the three-body decay $\gamma_n \to \gamma_n f \bar{f}$, where $f$ is a fermion at the $\vec{y} = 0$ fixed point. As one can see from Fig. 4, the new contribution to the amplitude occurs at order $\alpha$, and is potentially as large as the ‘standard’ diagrams. Henceforth, we use the term ‘standard’ or ‘nonstandard’ to refer to diagrams that are nonvanishing or vanishing in the $\theta \to 0$ limit.

For concreteness, let us choose the initial KK mode to be $\vec{n}_i = (1, 1)$ and the on-shell KK mode in the final state to be $\vec{n}_f = (1, 0)$. We write the decay width

$$\Gamma = \Gamma_S + \Gamma_{NS}$$

(3.1)

to distinguish the standard and nonstandard contributions. There is no interference between the standard and noncommutative diagrams because they are 90 degrees out of phase. We present our results in terms of the differential decay width, written as a function of the energies of the outgoing fermions. For the nonstandard diagram, we find

$$\frac{d\Gamma_{NS}}{dE_+dE_-} = 16 \frac{\alpha^2}{3 \pi} \frac{1}{m_{11}} \sin^2 \left( \frac{\xi_0 \theta}{2R^2} \right) \cdot f(E_+, E_-) \cdot g(E_+, E_-)$$

(3.2)

where

$$f(E_+, E_-) = \frac{(m_{21}^2 - m_{01}^2)^2}{[2m_{11}(E_+ + E_-) + m_{10}^2 - m_{11}^2 - n_{01}^2]^2[2m_{11}(E_+ + E_-) + m_{10}^2 - m_{11}^2 - m_{21}^2]^2}$$

(3.3)

and

$$g(E_+, E_-) = \frac{1}{2m_{10}^2} [8E_+^2 E_-^2 m_{11}^2 + 4(E_- E_+^3 + E_+ E_-^3)m_{11}^2 + 2(E_+^3 + E_-^3)(m_{11}^3 + 2m_{10}^2 m_{11})]$$
\[-2(E_+ E_-^2 + E_+ E_-^2) (m_{11}^3 - 6 m_{10}^2 m_{11}) + (E_+^2 + E_-^2)(-5 m_{11}^4 - 9 m_{10}^2 m_{11}^2 + 2 m_{10}^4) + 2 E_+ E_- (-3 m_{11}^4 - 5 m_{10}^2 m_{11}^2 + 2 m_{10}^4) + 2 (E_+ + E_-)(2 m_{11}^5 - m_{10} m_{11}^3 - 4 m_{10}^4 m_{11}) + (-m_{11}^6 + 3 m_{10}^2 m_{11}^4 - 2 m_{10}^6)\]

(3.4)

For notational convenience, we have labelled the mass of the \(n = (i,j)\) mode as \(m_{ij}\). The function \(f(E_+, E_-)\) originates from propagators of the off-shell KK modes; the dependence on \(m_{01}\) and \(m_{21}\) reflects that only the \((0, 1)\) and \((2, 1)\) modes may contribute to the internal line, given the choice of external states and the delta functions appearing in Eq. (2.22). For the standard contributions, we find

\[
\frac{d \Gamma_s}{dE_+ dE_-} = \frac{8 \alpha^2}{3 \pi m_{11}} \left[ \frac{(m_{11} - 2E_+)(m_{11} - 2E_-) - m_{10}^2}{(m_{11} - 2E_-)^2} \right.
\]
\[
+ \frac{(m_{11} - 2E_+)(m_{11} - 2E_-) - m_{10}^2}{(m_{11} - 2E_+)^2} + \frac{2(m_{10}^2 - m_{11}^2 + 2(E_+ - E_-)m_{11})(m_{11}^2 + m_{10}^2)}{m_{11}(m_{11} - 2E_-)(m_{11} - 2E_+)} \right]
\]

(3.5)

We present two quantities for our numerical results. We evaluate the partial decay width by integrating Eqs (3.2) and (3.5) over the ranges

\[-E_- + \frac{1}{2m_{11}}(m_{11}^2 - m_{10}^2) \leq E_+ \leq \frac{m_{11}^2 - m_{10}^2 - 2m_{11}E_-}{2(m_{11} - 2E_-)} \]
\[0 \leq E_- \leq \frac{1}{2m_{11}}(m_{11}^2 - m_{10}^2)\]

(3.6)

or alternatively

\[-[(m_{11} - E_T)^2 - m_{10}^2]^{1/2} \leq E_\Delta \leq [(m_{11} - E_T)^2 - m_{10}^2]^{1/2}\]
\[\frac{1}{2m_{11}}(m_{11}^2 - m_{10}^2) \leq E_T \leq m_{11} - m_{10}\]

(3.7)

where \(E_T = E_+ + E_-\) and \(E_\Delta = E_+ - E_-\). (For the latter variable set, one must not forget to include a Jacobian factor of 1/2.) By integrating over \(E_\Delta\) alone, we also obtain the sum energy spectrum of the outgoing fermion-antifermion pair.

Before discussing our numerical results, there is an important subtlety in the analysis related to possible degeneracies in the KK spectrum. Notice, for example, that the \((1, 0)\) and \((0, 1)\) modes are exactly degenerate in the limit that \(R_5 = R_6\). Moreover, these modes have identical couplings to fermions. Thus, there is no way experimentally to determine
whether an observed decay of the \((1,1)\) mode to an on-shell KK mode with mass \(1/R\) is the decay considered above, or the analogous decay with a \((0,1)\) mode in the final state. More concretely, the process considered above might be extracted experimentally through the decay \(\gamma^{(1,1)} \to f\bar{f}f\bar{f}\), by searching for an invariant mass peak in one of the \(f\bar{f}\) pairs; in the case of degenerate KK states, there is no way kinematically to isolate the desired mode. Since the experimental final states are identical, the amplitudes for the \((1,0)\) and \((0,1)\) states must be added. It is straightforward to show, however, that the sum of the nonstandard decay amplitudes then cancel exactly in the stated limit, and all hints of noncommutativity vanish!

This conclusion holds in any Feynman diagram in which two lines of the noncommutative vertex in Fig. \(\Box\) are connected to fermions at the \(\vec{y} = \vec{0}\) fixed point.

This observation does not necessarily imply that tree-level noncommutativity is unobservable, but rather that one must be careful not to make too many simplifying assumptions on the parameters of the compactification. If \(R_5\) and \(R_6\) differ by order one factors, as one would expect generically, then the degeneracy between the \((0,1)\) and \((1,0)\) states is lifted. More generally, taking the parameter \(\xi\) to differ from unity splits the degeneracy between the KK modes \((0,n)\) and \((n,0)\), while varying the shift angle \(\phi\) away from \(\pi/2\) eliminates degeneracies between the states \((m,n)\) and \((m,-n)\). We will typically choose the values \(\xi = 0.8\) and \(\phi = 1.5\) radians for our numerical analysis so that the decay of interest proceeds without any ambiguity in defining the asymptotic states. We aim to draw qualitative conclusions rather than to do a complete survey of the available parameter space. Fig. \(\Box\) shows contributions to the partial decay width in units of the compactification scale \(1/R\) as a function of \(\xi\). Since \(\theta/R^2\) in Eq. \((3.2)\) is unknown, we make the optimistic choice \(\sin^2(\xi_0\theta/2R^2) = 1\) in presenting our results. Fig. \(\Box\) assumes there is a single decay fermion with integral charge. For \(\xi \lesssim 0.9\), the nonstandard contribution to the partial width is dominant, and we expect at least an order 100% correction to the event rate. On the other hand, the branching fraction to the decay of interest is quite small. The partial width for the dominant two-body decay to a single fermion of integral charge, \(\gamma^{(\vec{r})} \to f\bar{f}\), is \(2\alpha m_{11}/3\), which is a factor of \(2 \times 10^5\) larger for \(\xi = 0.8\) and \(\phi = 1.5\). It follows that the branching fraction to the desired three-body decay, summed over fermions \(f\), is \(0.5 \times 10^{-5}\). (We can couple to fermions with fractional electric charge \([18]\) since matter is confined to a 4D subspace where the effects of noncommutativity are absent.)
FIG. 5: Contributions to the partial decay width as a function of $\xi$, for $\sin^2(\xi_0 \theta/2R^2) = 1$ and $\phi = 1.5$.

Typical event rates for the s-channel production of a single KK mode at a large hadron collider are given in Table I. To obtain these estimates we work in the narrow width approximation and use CTEQ5L structure functions [30]. For a 2 TeV initial state produced at a stage 2 VLHC with 200 TeV center of mass energy and 100 fb$^{-1}$ of integrated luminosity we would expect to observe 49.5 three-body decays given the production rate shown in Table I, compared to an expectation of 19.8 ± 4.4, a 6.8-sigma effect. More importantly, the shape of the sum-energy ($E_T$) spectrum for the outgoing fermion-antifermion pair is dramatically different in the case where such an excess is not statistical, as we show in Fig. 6. If collider parameters allow for a sufficient event rate, then an event excess particularly at smaller values of $E_T$ would be a telltale indication of new noncommutative interactions.

Whether or not a future high energy collider would provide for a sufficient event rate to detect noncommutativity in the $\gamma^{(g)} f \bar{f}$ decay channel is a difficult question. The previous example suggests an affirmative answer, though the value of $m_{11} = 2$ TeV is slightly low compared to most precision electroweak bounds on the compactification scale [31]. (The mass spectrum is compatible with current direct collider searches.) Indirect bounds rely strongly on the assumption that no other physics contributes to the relevant low-energy (usually Z-pole) observables. It is likely that these bounds could easily be relaxed by $O(1)$
FIG. 6: Contributions to the sum energy spectrum of the outgoing fermion-antifermion pair.

\[
\begin{array}{|c|c|c|c|c|}
\hline
m_{11} \text{ (TeV)} & \int \mathcal{L} \text{ (fb}^{-1}) & \sqrt{s} \text{ (TeV)} & \text{events (p\bar{p})} & \text{events (pp)} \\
\hline
2 & 100 \text{ fb}^{-1} & 14 & 3.8 \times 10^5 & 9.3 \times 10^4 \\
\cdot & \cdot & 80 & 3.2 \times 10^6 & 2.7 \times 10^6 \\
\cdot & \cdot & 140 & 6.5 \times 10^6 & 6.1 \times 10^6 \\
\cdot & \cdot & 200 & 1.0 \times 10^7 & 9.9 \times 10^6 \\
4 & \cdot & 14 & 1.7 \times 10^4 & 1.3 \times 10^3 \\
\cdot & \cdot & 80 & 3.4 \times 10^5 & 2.2 \times 10^5 \\
\cdot & \cdot & 140 & 6.7 \times 10^5 & 5.6 \times 10^5 \\
\cdot & \cdot & 200 & 1.1 \times 10^6 & 9.6 \times 10^5 \\
\hline
\end{array}
\]

TABLE I: Number of KK modes produced at very large hadron colliders.

factors if Planck-suppressed, higher-dimension operators are taken into account, given that
the Planck scale is low. If one is more conservative and considers the safer 4 TeV results
in Table I, then the event rate drops at a 200 TeV VLHC by about an order of magnitude
(4.8 events with an expectation of 1.9 ± 1.4). It is likely in this case that a higher integrated
luminosity would be required before any definite conclusions could be reached.

To summarize, we have found that noncommutativity in the decays \( \gamma^{(\bar{a})} \rightarrow \gamma^{(\bar{a})} f \bar{f} \) is
potentially observable at a very large hadron collider, providing one is lucky with the param-
eters of the theory. In the next section, we will see that an easier signal is an enhancement in the pair production of some of the lighter KK modes.

IV. PAIR PRODUCTION

Pair production of KK photons at colliders can occur through standard diagrams involving no noncommutative interactions. Additionally, pair production can proceed through the triple 6D-photon vertex which does not exist without noncommutativity.

The standard and non-standard pair production processes are shown at the parton level in Fig. 7. Their matrix elements may be obtained from those of the decay process studied in the last section through crossing symmetry.

![Feynman diagrams for production of KK pairs, including the noncommutative triple photon vertex.](image)

We present the cross section again for the case that the Kaluza-Klein states are the $\vec{n} = (1, 0)$ and $(1, 1)$ modes. While noncommutativity also affects the production of $(0, 1)$-$(1, 0)$ pairs, the nonstandard diagram in this case involves contributions from different intermediate states that tend to cancel, suppressing the rate. At the parton level, for the final states we have chosen, the nonstandard cross section is

$$\sigma_{NS}(\hat{s}) = \frac{\pi \alpha^2}{3\hat{s}^2} \lambda \left[ \frac{(m_{21}^2 - m_{01}^2)/(m_{11}m_{10})}{(\hat{s} - m_{21}^2)(\hat{s} - m_{01}^2)} \right]^2 \sin^2 \left( \frac{\xi_0 \theta}{2R^2} \right) \times \left\{ \hat{s}^4 + 8(m_{11}^2 + m_{10}^2)\hat{s}^3 - [18m_{11}^4 + 32m_{11}^2m_{10}^2 + 18m_{10}^4]\hat{s}^2 \\
+ 8(m_{11}^6 - 4m_{11}^4m_{10}^2 - 4m_{11}^2m_{10}^4 + m_{10}^6)\hat{s} \\
+ (m_{11}^2 - m_{10}^2)^2(m_{11}^4 + 10m_{11}^2m_{10}^2 + m_{10}^4) \right\},$$

(4.1)

where $\hat{s}$ is the partonic center-of-mass (CM) energy squared and $\lambda$ is defined in terms of the CM 3-momentum of either final state particle, $|\vec{p}| = \lambda/2\sqrt{\hat{s}}$ with

$$\lambda = \sqrt{\hat{s}^2 - 2\hat{s}(m_{11}^2 + m_{10}^2) + (m_{11}^2 - m_{10}^2)^2}.$$  

(4.2)
The initial partons are treated as massless. The parton level cross section for the standard process is

\[ \sigma_s(\hat{s}) = \frac{16\pi\alpha^2}{\hat{s}^2} \left\{ -\lambda \frac{m_{11} + m_{10}}{m_{11}} \right. \]

\[ + \left. \frac{\hat{s}^2 + (m_{11}^2 + m_{10}^2)^2}{\hat{s} - m_{11}^2 - m_{10}^2} \ln \frac{\hat{s} - m_{11}^2 - m_{10}^2 - \lambda}{\hat{s} - m_{11}^2 - m_{10}^2 - \lambda} \right\} . \] (4.3)

The collider cross section is

\[ \sigma \left( s, AB \rightarrow \gamma_{11}\gamma_{10}X \right) = \int_\tau^1 dx_1 \int_{\tau/x_1}^1 dx_2 \]

\[ \times \frac{1}{3} \sum_q \left[ f_{q/A}(x_1)f_{q/B}(x_2) + f_{\bar{q}/A}(x_1)f_{\bar{q}/B}(x_2) \right] \]

\[ \times \left\{ e_q^4\sigma_s(\hat{s}) + e_q^2\sigma_{NS}(\hat{s}) \right\} , \] (4.4)

where \( \hat{s} = x_1x_2s, \tau = \hat{s}_{\text{min}}/s, \) and \( \hat{s}_{\text{min}} \) is the square of the sum of the KK excitation masses, or

\[ \hat{s}_{\text{min}} = (m_{10} + m_{11})^2 . \] (4.5)

Also, \( f_{q/A}(x) = f_{q/A}(x, \mu) \) are the parton distribution functions for quark \( q \) in hadron \( A \) evaluated at renormalization scale \( \mu \), and the \( 1/3 \) is from color averaging.

We evaluated the cross section for a proton-proton collider again using the CTEQ5L parton distribution functions at a fixed scale \( \mu = 2 \) TeV, with \( \xi = 0.8, \phi = 1.5 \) and \( \sin^2(\xi_0\theta/2R^2) = 1 \). Fig. 8 shows the event rate over a range of center of mass energies for \( 1/R = 4 \) TeV and 100 fb\(^{-1} \) of integrated luminosity. At a 200 TeV VLHC, for example, we find 294 events where there is an expectation without noncommutativity of 165 \( \pm 12.8 \), a 10.1 sigma effect. This is a significant signal for a choice of \( 1/R \) that is consistent with current indirect bounds on the compactification scale. As in the example of the decays, what is more significant is the way in which noncommutativity emerges. For example, production of \((1,0)-(1,0)\) pairs receives \textit{no} noncommutative corrections while production of \((1,0)-(1,1)\) pairs does. Comparison of these channels may help eliminate uncertainty originating, for example, from parton distribution functions.
V. CONCLUSIONS

We have explored the phenomenological consequences of noncommutativity in extra spatial dimensions. By restricting noncommutativity to the bulk, we avoid conflict with the stringent experimental limits on the violation of four-dimensional Lorentz invariance, which otherwise force the magnitude of noncommutativity to be small. We constructed an explicit example, based on the orbifold $T^2/Z_2$, to illustrate the effects of spatial noncommutativity in 6D QED with fermions confined to an orbifold fixed point. Notably, we find new three- and four-point couplings involving KK excitations of the photon, exclusively. Since all fields involved in these interactions have TeV-scale masses, we confirm that collider signals are most promising at a VLHC, rather than at the LHC. In particular, we find order 100% corrections to the three-body decays $\gamma^{(\vec{m})} \rightarrow \gamma^{(\vec{m})} f \bar{f}$ and to the pair production $f \bar{f} \rightarrow \gamma^{(\vec{m})}\gamma^{(\vec{n})}$, with $\vec{m} \neq \vec{n}$. The former might be discernible at a VLHC, if one is lucky with model parameters, and yield a strikingly different $f \bar{f}$ sum energy spectrum then if the noncommutative interactions are absent. Pair production has a better chance of yielding a statistically significant excess for TeV-scale KK masses. Observing order 100% corrections to the production of certain pairs of KK modes at a VLHC while finding no corrections to others would provide a clear signal of noncommutativity in the bulk.
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