CANDY-PASSING GAMES ON GENERAL GRAPHS, II

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ABSTRACT. We give a new proof that any candy-passing game on a graph $G$ with at least $4|E(G)| - |V(G)|$ candies stabilizes. Unlike the prior literature on candy-passing games, we use methods from the general theory of chip-firing games which allow us to obtain a polynomial bound on the number of rounds before stabilization.

1. INTRODUCTION

We let $G$ be an undirected graph and respectively denote the vertex and edge sets of $G$ by $V(G)$ and $E(G)$. The candy-passing game on $G$ is defined by the following rules:

- At the beginning of the game, $c > 0$ candies are distributed among $|V(G)|$ students, each of whom is seated at some distinct vertex $v \in V(G)$.
- A whistle is sounded at a regular interval.
- Each time the whistle is sounded, every student who is able to do so passes one candy to each of his neighbors. (If at the beginning of this step a student holds fewer candies than he has neighbors, he does nothing.)

Tanton [6] introduced this game for cyclic $G$. The authors [4] extended the game to general graphs $G$.

The candy-passing game on $G$ is a special case of the well-known chip-firing game on $G$ introduced by Björner, Lovász, and Shor [2]. Furthermore, terminating candy-passing games on $G$ are actually equivalent to terminating chip-firing games on $G$, by the following key theorem:

Theorem 1 ([2]). The initial configuration of a chip-firing game on $G$ determines whether the game will terminate. If the game does terminate, then both the final configuration and length of the game are dependent only on the initial configuration.

Terminating chip-firing games have been studied extensively and are surprisingly well-behaved. In addition to Theorem 1 it is known that terminating chip-firing processes finish in polynomial time (see [7]). Chip-firing games also have important applications; notably, they are related to Tutte polynomials (see [5]) and the critical groups of graphs (see [1]).

Infinite chip-firing games have received less attention, as the notion of an “end state” of such a game is ambiguous. By contrast, an infinite candy-passing game admits a clear stabilization condition: the game is said to have stabilized if the configuration of candy will never again change.

2000 Mathematics Subject Classification. 05C35, 05C85, 68Q25 (Primary); 37B15, 68R10, 68Q80 (Secondary).

Key words and phrases. candy-passing, chip-firing, graph game, stabilization, polynomial time.

The second author gratefully acknowledges the support of a Harvard Mathematics Department Highbridge Fellowship.
The first author [3] studied the end behavior of candy-passing games on \( n \)-cycles, proving the eventual stabilization of any candy-passing game on an \( n \)-cycle with at least \( 3n - 2 \) candies. The authors [4] extended this analysis to arbitrary connected graphs \( G \), showing that any candy-passing game on such \( G \) with at least \( 4|E(G)| - |V(G)| \) candies will stabilize.

Here, we give a new proof of the stabilization result for general connected graphs, using methods which allow us to obtain a polynomial bound on the stabilization time. Our approach draws from the literature on chip-firing, using in particular a key result from Tardos’s [7] proof that terminating chip-firing games conclude in polynomial time.

2. The Setting

As in the earlier work on candy-passing games, we refer to the interval between soundings of the whistle as a round of candy-passing. We denote by \( \varphi_t(v) \) the total number times a vertex \( v \in V(G) \) has passed candy by the end of round \( t \).

Since infinite candy-passing games differ from infinite chip-firing games, we will continue to distinguish between “candies” and “chips.” However, we drop the student metaphor, treating the candy piles as belonging to the vertices of the graph \( G \). For consistency, we denote the total number of candies in a candy-passing game by \( c \) throughout.

Abusing terminology slightly, we say that a vertex has stabilized in some round if, after that round, the amount of candy held by that vertex will not change during the remainder of the game.

For a vertex \( v \in V(G) \), we denote the degree of \( v \) by \( \deg(v) \). We say that a vertex \( v \in V(G) \) is abundant if it holds at least \( 2 \deg(v) \) pieces of candy.

Any vertex \( v \in V(G) \) with \( k \geq \deg(v) \) candies at the beginning of a round passes \( \deg(v) \) pieces of candy to its neighbors and can, at most, receive one piece of candy from each of its \( \deg(v) \) neighbors. Thus, such a vertex cannot end the round with more than \( k \) candies. In particular, then, the set of abundant vertices of \( G \) can only shrink over the course of a candy-passing game on \( G \).

3. Main Theorem

We will prove the following stabilization theorem:

**Theorem 2.** Let \( G \) be a connected graph with diameter \( d \). In any candy-passing game on \( G \) with

\[
c \geq 4|E(G)| - |V(G)|
\]

candies, every vertex \( v \in V(G) \) will stabilize within \( |V(G)| \cdot d \cdot c \) rounds.

The stabilization component of Theorem 2 was obtained in [4, Theorem 2]. Our methods are inspired by those of Tardos [7]; they are essentially independent of the arguments used in [3] and [4].

We use the following lemma, which is a special case of Tardos’s [7] Lemma 5:

**Lemma 3.** Let \( v, v' \in V(G) \) be adjacent vertices of \( G \). Then, \( |\varphi_t(v) - \varphi_t(v')| \leq c \) for all \( t \).

Additionally, we need an observation about the condition \( c \geq 4|E(G)| - |V(G)| \).
Lemma 4. For $G$ a graph and $c \geq 4|E(G)| - |V(G)|$, in any chip-firing game on $G$ with $c$ candies there is at least one vertex $v_\ast \in V(G)$ which passes candy every round.

Proof. It suffices to find a vertex $v_\ast \in V(G)$ which passes candy every round $t$ during which some vertex $v \in V(G)$ holds fewer than $2\deg(v) - 1$ candies.

As observed above, it is not possible for a vertex $v \in V(G)$ which is not abundant at the beginning of round $t$ to become abundant after round $t$. However, the condition
\[ c \geq 4|E(G)| - |V(G)| \]
guarantees that whenever some $v \in V(G)$ holds fewer than $2\deg(v) - 1$ candies there is also at least one abundant vertex $v' \in V(G)$. The existence of some vertex $v_\ast \in V(G)$ which is abundant in every round when some vertex $v \in V(G)$ has fewer than $2\deg(v) - 1$ candies then follows immediately. □

Remark. Lemma 4 is, in some sense, dual to Tardos’s [7] Lemma 4 which shows that for any terminating chip-firing game on $G$ there is a distinguished vertex $v_\ast \in V(G)$ which never fires.

We may now proceed with the proof of our main result:

Proof of Theorem 2. By Lemma 4 there is some vertex $v_\ast \in V(G)$ which passes candy every round. Denoting the rounds by $t = 1, 2, \ldots$, we then have $\varphi_t(v_\ast) = t$ for all rounds $t$. By Lemma 3 we then know that
\[ |\varphi_t(v_\ast) - \varphi_t(v)| \leq d \cdot c \]
for all $t$ and $v \in V(G)$. Since $\varphi_t(v_\ast)$ is strictly increasing in $t$, no $v \in V(G)$ may fail to pass candy for more than $d \cdot c$ rounds. In the worst case, all but one vertex pass candy in each round when some vertex does not pass candy; hence after $|V(G)| \cdot d \cdot c$ rounds all the vertices of $G$ pass candy every round. □

References

[1] N. L. Biggs, Chip-firing and the critical group of a graph, Journal of Algebraic Combinatorics 9(1), 1999, pp. 25–45.
[2] A. Björner, L. Lovász, and P. Shor, Chip-firing games on graphs, European Journal of Combinatorics 12(4), 1991, pp. 283–291.
[3] P. M. Kominers, The candy-passing game for $c \geq 3n - 2$, Pi Mu Epsilon Journal 12(8), 2008, pp. 459–460.
[4] P. M. Kominers and S. D. Kominers, Candy-passing games on general graphs, I, arXiv:0807.4450.
[5] C. M. López, Chip firing and the Tutte polynomial, Annals of Combinatorics 1(1), 1997, pp. 253–259.
[6] J. Tanton, Today’s puzzler, The St. Mark’s Institute of Mathematics Newsletter, November 2006.
[7] G. Tardos, Polynomial bound for a chip firing game on graphs, SIAM Journal on Discrete Mathematics 1(3), 1998, pp. 397–398.

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