Finite Volume Element Methods for Two-Dimensional Time Fractional Reaction-Diffusion Equations on Triangular Grids

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Abstract: In this paper, the time fractional reaction-diffusion equations with the Caputo fractional derivative are solved by using the classical $L^1$-formula and the finite volume element (FVE) methods on triangular grids. The existence and uniqueness for the fully discrete FVE scheme are given. The stability result and optimal $a$ priori error estimate in $L^2(\Omega)$-norm are derived, but it is difficult to obtain the corresponding results in $H^1(\Omega)$-norm, so another analysis technique is introduced and used to achieve our goal. Finally, two numerical examples in different spatial dimensions are given to verify the feasibility and effectiveness.

Keywords: time fractional reaction-diffusion equation; $L^1$-formula; finite volume element method; stability analysis; $a$ priori error estimate

1 Introduction

In this article, we consider the following time fractional reaction-diffusion equations with the Caputo fractional derivative

\[
\begin{aligned}
\frac{C_0}{\Gamma(\gamma)} D_t^\gamma u(x,t) &- \text{div}(A(x)\nabla u(x,t)) + q(x)u(x,t) = f(x,t), \quad (x,t) \in \Omega \times J, \\
u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times J, \\
u(x,0) &= u_0(x), \quad x \in \bar{\Omega},
\end{aligned}
\]

(1.1)

where $\Omega \subset R^2$ is a bounded convex polygonal domain, $\partial \Omega$ is the corresponding boundary, $J = (0,T]$ is the time interval with $0 < T < \infty$. The source function $f(x,t)$, reaction term coefficient $q(x)$ and initial data $u_0(x)$ are smooth enough, and $q(x) \geq 0$, $\forall x \in \bar{\Omega}$. Moreover, we assume that the diffusion coefficient $A(x) = \{a_{i,j}(x)\}_{2 \times 2}$ is a sufficiently smooth matrix function, which satisfies symmetric and uniformly positive definite, that is, there exists a constant $\beta_0 > 0$ such that

$$
\xi^T A(x) \xi \geq \beta_0 \xi^T \xi, \quad \forall \xi \in R^2, \quad \forall x \in \bar{\Omega}.
$$

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In (1.1), $C_0^\alpha D_t^\alpha (u(x, t))$ is the Caputo fractional derivative defined by
\[
C_0^\alpha D_t^\alpha (u(x, t)) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} (t-s)^{\alpha-1} ds,
\]
$0 < \alpha < 1$.

Fractional partial differential equations (FPDEs) can describe various natural phenomena in physics, chemistry and biology \cite{1,3} and so on. Especially when describing materials with some properties such as memory, heterogeneity or heredity, they often have very good results. Unfortunately, it is difficult to get the exact solutions for most of the FPDEs via the analytical methods. Thus, in past decades, a lot of numerical methods have been proposed to solve various FPDEs.

For the time FPDEs with the Caputo fractional derivative, the $L_1$-formula which was firstly developed in \cite{4,5} is very useful and popular by many scholars. Li et al. \cite{25} developed finite element methods with $L_1$-formula for solving fractional Maxwell’s models. Li et al. \cite{26} designed a numerical approximation based on $L_1$-formula for nonlinear fractional subdiffusion problem, and also discussed a superdiffusion model. In \cite{27}, Feng et al. studied the finite element method with a unstructured mesh for the 2D fractional diffusion model with a time-space Riesz fractional derivative. In \cite{28}, Feng et al. proposed the finite element method for a novel 2D mixed sub-diffusion and diffusion-wave equation with multi-term time-fractional derivative, in which the fractional derivative was approximated by $L_1$-formula and other numerical formulas. Jiang and Ma \cite{6} considered an $L_1$-formula combined with a two-grid mixed element algorithm for solving a nonlinear fourth-order time fractional reaction-diffusion model. Liu et al. \cite{8} applied finite difference/finite element method with $L_1$-formula to treat a nonlinear time fractional reaction-diffusion equation with a fourth-order derivative term. Jin et al. \cite{9} revisited the error analysis of the $L_1$-formula, especially for the nonsmooth initial data, the authors obtained $O(\tau)$ convergence rate. Li et al. \cite{12} developed a finite difference/finite element method with $L_1$-formula to study nonlocal time fractional reaction-diffusion equations obtained the unconditional stability and optimal error estimates, and gave some numerical examples with higher spatial dimension. Zhao et al. \cite{10} constructed two conforming and nonconforming MFE schemes with $L_1$-formula to solve time fractional diffusion equations, and gave the stability and convergence results. Li et al. \cite{11} provided an FE method based on $L_1$-formula to solve time fractional nonlinear parabolic equations, and obtained optimal error estimates for several fully discrete linearized FE methods for nonlinear equations. From the current literatures, we find that there is no report about the finite volume element (FVE) method based on $L_1$-formula for solving the FPDEs.

The FVE methods \cite{13,18}, also known as box methods \cite{19} or generalized difference methods \cite{20,21}, have been widely used in several fields of science and engineering. This methods can preserve the local conservation laws for some physical quantities, which is very important in scientific computing, and have attracted more and more scholars. Recently, some scholars have done valuable works in solving FPDEs by using FVE methods. Sayevand and Arjang \cite{22} proposed a spatially semi-discrete FVE scheme to solve the time fractional sub-diffusion problem $\frac{\partial^{\alpha}_t u}{\partial t^{\alpha}} = \beta \Delta u + f$ with the Caputo fractional derivative on a rectangular partition, gave some error estimates for the FE and FVE schemes. Karaa et al. \cite{23} constructed an FVE scheme for the fractional sub-diffusion equation $u' + \mathcal{R}^\alpha \mathcal{L} u = f$ in a two-dimensional domain, where $\mathcal{L} u = -\Delta u$ and $\mathcal{R}^\alpha$ is the Riemann-Liouville fractional derivative in time, and the authors applied a piecewise linear discontinuous Galerkin method in time and an FVE method in space to construct a fully discrete scheme, and gave the convergence analysis and numerical experiments. Karaa and Pani \cite{24} proposed an FVE scheme for fractional order evolution equations $u' + \partial_t^{\alpha-\alpha} Au = 0$, where $Au = -\Delta u$ and $\partial_t^{\alpha-\alpha}$ is the Riemann-Liouville fractional derivative. In \cite{24}, the authors gave the error analysis for the semi-discrete scheme with smooth, middly smooth and
nonsmooth data, and constructed and analyzed two fully discrete schemes by introducing convolution quadrature in time for smooth and nonsmooth initial data, which were generated by the backward Euler and the second-order difference methods.

In this article, we will construct a fully discrete FVE scheme to treat the time fractional reaction-diffusion equation on triangular grids by using the L1-formula. In spatial discretization, we first construct the primal and dual partitions, select the trial function space (based on primal partitions) and the test function space (based on dual partitions), then integrate the original equation under the control volumes to construct the FVE scheme by using the projection operator \( I_h \). In our theoretical analysis, we give the existence and uniqueness for the fully discrete solution, derive the stability results in \( L^2(\Omega) \)-norm and \( H^1(\Omega) \)-norm, and obtain the optimal \( a \) priori error estimates for \( u \) in \( L^2(\Omega) \)-norm and \( H^1(\Omega) \)-norm. Moreover, we provide two numerical examples in different spatial dimensions, and give some numerical results to examine the feasibility and effectiveness of the fully discrete FVE scheme. Here, because the bilinear \( a(\cdot, I_h \cdot) \) does not necessarily satisfy symmetry, it is difficult to obtain the stability and the optimal error result in \( H^1(\Omega) \)-norm, so we give another analysis technique to achieve our goal.

The layout of this paper is as follows. By introducing the operator \( I_h \) and the L1-formula of approximating the Caputo fractional derivative, a fully discrete FVE scheme for the time fractional reaction-diffusion equation is proposed in Section 2. In Section 3, we give the truncation errors of \( L1 \)-formula and some properties of the operator \( I_h \), and give some important lemma which will be used in theoretical analysis. In Sections 4 and 5, we give the theoretical analysis for the FVE scheme in detailed, including the existence, uniqueness, stability and error estimates. In Section 6, we give two numerical examples in different spatial dimensional to verify the feasibility and effectiveness.

## 2 Fully Discrete FVE Scheme

We use some general definitions and notations of the Sobolev spaces as in Reference [29]. Let \( W^{m,p}(\Omega) \) \((m \geq 0 \text{ and } 1 \leq p \leq \infty)\) be the usual Sobolev space defined in \( \Omega \) with the norm \( \| \cdot \|_{W^{m,p}} \) (abbreviated as \( \| \cdot \|_{m,p} \)). When \( p = 2 \), we denote \( W^{m,2}(\Omega) \) by \( H^m(\Omega) \) with the norm \( \| \cdot \|_{H^m(\Omega)} \) (abbreviated as \( \| \cdot \|_m \)), and denote \( H^0(\Omega) \) by \( L^2(\Omega) \) with the inner product \((\cdot, \cdot)\) and the norm \( \| \cdot \|_{L^2(\Omega)} \) (abbreviated as \( \| \cdot \| \)). We also use \( H^1_0(\Omega) = \{ w \in H^1(\Omega) : w|_{\partial \Omega} = 0 \} \). Furthermore, throughout the article, we use the mark \( C \) to denote a generic positive constant, which is independent of spatial and temporal mesh.

The variational formulation of the problem (1.1) is to find \( u(t) \in H^1_0(\Omega) \) such that

\[
(C^0 D^\alpha_t u, v) + a(u, v) + (qu, v) = (f, v), \quad v \in H^1_0(\Omega),
\]

(2.1)

where \( a(u, v) = \int_\Omega A \nabla u \cdot \nabla v dx \), \( \forall u, v \in H^1_0(\Omega) \).

Now, let \( \mathcal{T}_h = \{ K \} \) be a set of quasi-uniform triangular mesh of the domain \( \Omega \) with \( h = \max \{ h_K \} \), referring to Figure 1 where \( h_K \) denotes the diameter of the triangle \( K \in \mathcal{T}_h \). Then \( \Omega = \cup_{K \in \mathcal{T}_h} K \) and \( Z_h \) denotes all vertices, that is

\[ Z_h = \{ z : z \text{ is a vertex of the element } K, K \in \mathcal{T}_h \}. \]

And \( Z^0_h \subset Z_h \) denotes the set of all interior vertices in \( \mathcal{T}_h \).

Next, let \( \mathcal{T}_h^* \) be the dual mesh based on the primary mesh \( \mathcal{T}_h \). With \( z_0 \in Z^0_h \) as an interior node, let \( z_i \) \((i = 1, 2, \ldots, m)\) be corresponding adjacent nodes (as shown in Figure 1 \((m = 6)\)). we denote the midpoints of \( \frac{z_i + z_{i+1}}{2} \) by \( M_i \), and denote the barycenters of the triangle \( \Delta z_0 z_{i} z_{i+1} \) by \( Q_i \), where \((i = 1, 2, \ldots, m)\) and \( z_{m+1} = z_1 \). We construct the control volume \( K^*_z \) joining successively
Then, the dual mesh $\mathcal{T}_h^*$ is defined by the union of the control volumes $K^*_z$. We denote $Q_i, i = 1, 2, \ldots, m$ be all nodes of the control volume $K^*_z$, and denote $Z_h^* = \{Q: Q$ is a node of a control volume $K^*_z, K^*_z \in \mathcal{T}_h^*\}$.

Then, we define the following finite element space $V_h$ as the trial function space $V_h = \{v \in H^1_0(\Omega) : v|_K \in P_1(K), \forall K \in \mathcal{T}_h\}$, and define $V_h^*$ as the test function space, that is $V_h^* = \{v \in L^2(\Omega) : v|_{K^*_z} \in P_0(K^*_z), \forall K^*_z \in \mathcal{T}_h^*, v|_{\partial \Omega} = 0\}$. It is obvious that $V_h = \text{span}\{\Phi(z): z \in Z_h\}$ and $V_h^* = \text{span}\{\Psi(z): z \in Z_h^\ast\}$, where $\Phi(z)$ is the standard linear basis function with the node $z$, and $\Psi(z)$ is the characteristic function of the control volume $K^*_z$. Next, we will introduce two important interpolation operators. Let $I_h : C(\Omega) \rightarrow V_h$ be the classical piecewise linear interpolation operator and $I_h^* : C(\Omega) \rightarrow V_h^*$ be the piecewise constant interpolation operator, which are defined by

$$I_h v(x) = \sum_{z \in Z_h^\ast} v(z)\Phi(z(x), \text{ and } I_h^* v(x) = \sum_{z \in Z_h^\ast} v(z)\Psi(z(x)).$$

From the Reference [21], we can see that $I_h$ and $I_h^*$ have the following approximation property

$$\|v - I_h v\|_j \leq C h^{2-j}\|v\|_2, \forall v \in H^2(\Omega), \quad (2.2)$$

$$\|v - I_h^* v\| \leq C h\|v\|_1, \forall v \in H^1(\Omega). \quad (2.3)$$

Now, integrating (1.1) on a control volume $K^*_z$ associated with a vertex $z \in Z_h$, and applying the Green formula, we can get

$$\int_{K^*_z} C D_r^\alpha u dx - \int_{\partial K^*_z} A \nabla u \cdot n ds + \int_{K^*_z} qu dx = \int_{K^*_z} f dx, \quad (2.4)$$

where $n$ means the outer-normal direction on $\partial K^*_z$.

Make use of the operator $I_h^*$ to rewrite (2.4) as the following formulation

$$(C D_r^\alpha u, I_h^* v_h) + a(u, I_h^* v_h) + (qu, I_h^* v_h) = (f, I_h^* v_h), \forall v_h \in V_h, \quad (2.5)$$
where $a(\cdot, \cdot)$ defined in (2.1), following the References [13][21], can be rewritten as follows

$$a(\bar{u}, \bar{v}) = \begin{cases} -\sum_{z \in Z_n} \bar{v}(z) \int_{\partial K_z} A \nabla \bar{u} \cdot n ds, & \forall \bar{u} \in V_h, \bar{v} \in V_h^*, \\ \int_{\Omega} A \nabla \bar{u} \cdot \nabla \bar{v} dx, & \forall \bar{u}, \bar{v} \in H_0^1(\Omega). \end{cases} \quad (2.6)$$

Next, we make use of the $L_1$-formula to approximate the Caputo fractional derivative. First, we give an equidistant partition of the time interval $\mathcal{J} = [0, T]$ by $0 = t_0 < t_1 < \cdots < t_N = T$, where $t_n = n \tau$, $n = 0, 1, \cdots, N$, and $\tau = T/N$ for some positive integer $N$. For a given function $\varphi$ on $[0, T]$, let $\varphi^n = \varphi(t_n)$ and $\partial_t \varphi^n = \frac{\varphi^n - \varphi^{n-1}}{\tau}$. Following References [4][5], we can approximate the Caputo time fractional derivative $\frac{\tau}{\Gamma(2-\alpha)} D_0^\alpha u(x, t)$ at $t = t_n$ as follows

$$\frac{\tau}{\Gamma(2-\alpha)} D_0^\alpha u(x, t_n) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t_n - s)^\alpha}$$

$$= \frac{\tau}{\Gamma(2-\alpha)} \left( \sum_{k=0}^{n-1} b_k \frac{u(x, t_{n-k}) - u(x, t_{n-k-1})}{\tau} + R_1^n(x) + R_2^n(x) \right) \quad (2.7)$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $\tilde{b}_n^0 = 1$, $\tilde{b}_n^k = (n-1)^{1-\alpha} - n^{1-\alpha}$, $\tilde{b}_n^k = b_{n-k} - b_{n-k-1}$ ($0 < k < n$), $R_1^n(x) = R_1^n(x) + R_2^n(x)$, and

$$R_1^n(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t_n - s)^\alpha} - \frac{\tau}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \frac{\partial u(x, t_{n-k-1/2})}{\partial s} \quad (2.8)$$

Denote $D_\tau^{\alpha} \varphi^n = \frac{\tau}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \tilde{b}_k^n \varphi^k$, then we have $\frac{\tau}{\Gamma(2-\alpha)} D_0^\alpha u(x, t_n) = D_\tau^{\alpha} u^n + R^n(x)$.

Let $u^n_h$ be the fully discrete approximate solution of $u$ at $t = t_n$. We give the fully discrete FVE scheme to seek $u^n_h \in V_h$, $(n = 0, 1, \ldots, N)$, such that

$$(D_\tau^{\alpha} u^n_h, I^n_h v_h) + a(u^n_h, I^n_h v_h) + (qu^n_h, I^n_h v_h) = (f^n, I^n_h v_h), \forall v_h \in V_h. \quad (2.9)$$

**Remark 2.1.** Making use of the definition of $D_\tau^{\alpha}$, we can rewrite the fully discrete FVE scheme (2.9) as the following or other practical calculation formulation

$$\frac{1}{\Gamma(2-\alpha)} (u^n_h, I^n_h v_h) + \tau^{\alpha} a(u^n_h, I^n_h v_h) + \tau^{\alpha} (qu^n_h, I^n_h v_h)$$

$$= \tau^{\alpha} (f^n, I^n_h v_h) - \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \tilde{b}_k^n (u^n_h, v_h), \forall v_h \in V_h. \quad (2.10)$$

In Section 4, we will use (2.10) to prove the existence and uniqueness of the discrete solutions.
3 Some Lemmas

First, we give some properties for the bilinear forms \( \langle \cdot, I_h^* \rangle \) and \( a(\cdot, I_h^*) \), which are very important in the later theoretical analysis.

**Lemma 3.1.** \([21]\) The bilinear form \( \langle \cdot, I_h^* \rangle \) satisfies the following properties

\[
(v_h, I_h^* w_h) = (w_h, I_h^* v_h), \quad \forall v_h, w_h \in V_h, \tag{3.1}
\]

and there exist constants \( \mu_1 > 0 \) and \( \mu_2 > 0 \) independent of \( h \) such that

\[
(v_h, I_h^* v_h) \geq \mu_1 \| v_h \|^2, \quad \forall v_h \in V_h, \tag{3.2}
\]

\[
(v_h, I_h^* w_h) \leq \mu_2 \| v_h \| \| w_h \|, \quad \forall v_h, w_h \in V_h. \tag{3.3}
\]

**Lemma 3.2.** \([3, 22]\) The bilinear form \( a(\cdot, I_h^*) \) can be expressed as

\[
a(v_h, I_h^* w_h) = a_h(v_h, I_h^* w_h) + b_h(v_h, I_h^* w_h), \quad \forall v_h, w_h \in V_h, \tag{3.4}
\]

where \( a_h(v_h, I_h^* w_h) = \sum_{KQ \in T_h} \{ A(Q) \nabla v_h(Q) \cdot \nabla w_h(Q) \} S_Q \), \( (S_Q \text{ is the area of } K_Q) \). Then, the bilinear form \( a_h(\cdot, I_h^*) \) has the following properties

\[
a_h(v_h, I_h^* w_h) = a_h(w_h, I_h^* v_h), \quad \forall v_h, w_h \in V_h, \tag{3.5}
\]

and there exists constants \( \mu_3 > 0 \) and \( \mu_4 > 0 \) independent of \( h \) such that

\[
a_h(v_h, I_h^* v_h) \geq \mu_3 \| v_h \|^2, \quad \forall v_h \in V_h, \tag{3.6}
\]

\[
a_h(v_h, I_h^* w_h) \leq \mu_4 \| v_h \|_1 \| w_h \|_1, \quad \forall v_h, w_h \in V_h. \tag{3.7}
\]

We also have that there exist a positive constant \( \mu_5 \) such that

\[
|b_h(v_h, I_h^* w_h)| \leq \mu_5 h \| v_h \|_1 \| w_h \|_1, \quad \forall v_h, w_h \in V_h, \tag{3.8}
\]

\[
|a(v_h, I_h^* w_h) - a(w_h, I_h^* v_h)| \leq \mu_5 h \| v_h \|_1 \| w_h \|_1, \quad \forall v_h, w_h \in V_h. \tag{3.9}
\]

**Lemma 3.3.** \([3, 22]\) There exist positive constants \( h_0, \mu_6 \) and \( \mu_7 \) such that, for \( 0 < h \leq h_0 \)

\[
a(v_h, I_h^* v_h) \geq \mu_6 \| v_h \|^2, \quad \forall v_h \in V_h, \tag{3.10}
\]

\[
a(v_h, I_h^* w_h) \leq \mu_7 \| v_h \|_1 \| w_h \|_1, \quad \forall v_h, w_h \in V_h. \tag{3.11}
\]

Next, following the Reference \([4]\), we give the estimates of the truncation errors \( R_1^n \), \( R_2^n \) and \( R_t^n \) defined by \([2, 8]\), and two important lemmas for stability and error analysis.

**Lemma 3.4.** \([2]\) For the truncation error \( R_1^n(x) \), \( R_2^n(x) \) and \( R_t^n(x) \), there exists a constant \( C > 0 \) independent of \( h \) and \( \tau \) such that

\[
\| R_1^n \| \leq C \tau^{2-\alpha}, \quad \| R_2^n \| \leq C \tau^2, \tag{3.12}
\]

\[
\| R_t^n \| \leq C (\tau^2 + \tau^{2-\alpha}). \tag{3.13}
\]
Lemma 3.5. Let $\varphi^k \geq 0$, $k = 0, 1, \ldots, N$, $\zeta > 0$ be a constant, which satisfy

$$
\varphi^n \leq - \sum_{k=1}^{n-1} \tilde{b}_k^n \varphi^k + \zeta,
$$

then there exists a constant $C > 0$ independent of $\tau$ such that

$$
\varphi^n \leq C \tau^{-\alpha} \zeta, \quad n = 1, 2, \ldots, N.
$$

Lemma 3.6. Let $\varphi^k \geq 0$, $k = 0, 1, \ldots, N$, $\zeta > 0$ and $C_0 \geq 1$ be two constants, which satisfy

$$
\varphi^n \leq - C_0 \sum_{k=0}^{n-1} \tilde{b}_k^n \varphi^k + \zeta, \quad (3.12)
$$

Then, we have

$$
\varphi^n \leq C_0^0 (\varphi^0 + b_{n-1}^{-1} \zeta), \quad n = 1, 2, \ldots, N. \quad (3.13)
$$

Furthermore, there exists a constant $C > 0$ independent of $\tau$ such that

$$
\varphi^n \leq C C_0^n (\varphi^0 + \tau^{-\alpha} \zeta), \quad n = 1, 2, \ldots, N. \quad (3.14)
$$

Proof. First, we use the mathematical induction to prove the following result (3.13). Choosing $n = 1$ in (3.12), we have

$$
\varphi^1 \leq C_0 (1 - b_1) \varphi^0 + \zeta \leq C_0 (\varphi^0 + b_{0}^{-1} \zeta). \quad (3.15)
$$

Therefore, (3.13) is proved for the case $n = 1$.

Suppose (3.13) holds for all $n = 1, 2, \ldots, m$. Next, we will prove that (3.13) also holds for the case $n = m + 1$. Choosing $n = m + 1$ in (3.12), we have

$$
\varphi^{m+1} \leq - \sum_{k=0}^{m} C_0 \tilde{b}_k^{m+1} \varphi^k + \zeta
$$

\[ \leq \sum_{k=0}^{m} C_0 (b_{m-k} - b_{m-k+1}) \varphi^k + \zeta \]

\[ = \sum_{k=0}^{m} C_0 (b_k - b_{k+1}) \varphi^{m-k} + \zeta \]

\[ \leq C_0 (1 - b_1) \varphi^{m} + \sum_{k=1}^{m-1} C_0 (b_k - b_{k+1}) \varphi^{m-k} + C_0 b_m \varphi^0 + \zeta. \quad (3.16) \]

Noting that $1 = b_0 > b_1 > b_2 > \cdots > b_n > 0$, $b_n \to 0$ ($n \to 0$), and $b_j^{-1} < b_{j+1}^{-1}$, making use of the
induction assumption, we have

\[ \varphi^{m+1} \leq C_0^{m+1}(1 - b_1)(\varphi^0 + b_{m-1}^{-1} \zeta) + \sum_{k=1}^{m-1} C_0^{m-k+1}(b_k - b_{k+1})(\varphi^0 + b_{m-k-1}^{-1} \zeta) + C_0 b_m (\varphi^0 + b_m^{-1} \zeta) \]

\[ \leq C_0^{m+1}(\varphi^0 + b_m^{-1} \zeta) \left[(1 - b_1) + \sum_{k=1}^{m-1} (b_k - b_{k+1}) + b_m \right] \]

\[ = C_0^{m+1}(\varphi^0 + b_m^{-1} \zeta). \]  

(3.17)

Thus, (3.13) is proved.

Next, following Reference [4], we see that

\[ n^{-\alpha} b_{n-1}^{-1} \leq \frac{1}{1 - \alpha}, \]

and obtain

\[ \varphi^n \leq C_0^n (\varphi^0 + b_{n-1}^{-1} \zeta) \leq C_0^n (\varphi^0 + \frac{n^\alpha \tau^\alpha}{1 - \alpha} \tau^{-\alpha} \zeta) \leq C_0^n (\varphi^0 + \frac{T^\alpha}{1 - \alpha} \tau^{-\alpha} \zeta). \]  

(3.18)

Then we obtain the desired result.

Next, we will give two identical relations of the bilinear forms \((\cdot, I_h^*)\) and \(a(\cdot, I_h^*)\).

**Lemma 3.7.** Let \(\{\varphi^n\}_{n=0}^\infty\) be a function sequence on \(V_h\), then the following relation holds

\[ a\left(\varphi^n, I_h^* \left(\sum_{k=0}^n \tilde{b}_k^n \varphi^k\right)\right) \]

\[ = \frac{1}{2} \left[a(\varphi^n, I_h^* \varphi^n) + \sum_{k=0}^{n-1} \tilde{b}_k^n \left(a(\varphi^k, I_h^* \varphi^k) - a(\varphi^n - \varphi^k, I_h^* (\varphi^n - \varphi^k))\right)\right] \]

\[ + \frac{1}{2} \sum_{k=0}^{n-1} \tilde{b}_k^n \left[a(\varphi^n, I_h^* \varphi^k) - a(\varphi^k, I_h^* \varphi^n)\right]. \]

(3.19)

**Proof.** Noting the fact that \(\sum_{k=0}^{n-1} \tilde{b}_k^n = 1\) and \(\tilde{b}_n^n = 1\), we have

\[ a\left(\varphi^n, I_h^* \left(\sum_{k=0}^n \tilde{b}_k^n \varphi^k\right)\right) \]

\[ = a(\varphi^n, I_h^* \varphi^n) + \sum_{k=0}^{n-1} \tilde{b}_k^n a(\varphi^n, I_h^* \varphi^k) \]

\[ = \frac{1}{2} \left[a(\varphi^n, I_h^* \varphi^n) + \sum_{k=0}^{n-1} \tilde{b}_k^n \left(2a(\varphi^n, I_h^* \varphi^k) - a(\varphi^n, I_h^* \varphi^n)\right)\right] \]

\[ = \frac{1}{2} \left[a(\varphi^n, I_h^* \varphi^n) + \sum_{k=0}^{n-1} \tilde{b}_k^n \left(a(\varphi^k, I_h^* \varphi^k) - a(\varphi^n - \varphi^k, I_h^* (\varphi^n - \varphi^k))\right)\right] \]

\[ + \frac{1}{2} \sum_{k=0}^{n-1} \tilde{b}_k^n \left[a(\varphi^n, I_h^* \varphi^k) - a(\varphi^k, I_h^* \varphi^n)\right]. \]

(3.20)

Thus, we complete the proof of this lemma.

Applying Lemma 3.1 similar to the proof of Lemma 3.7 we can obtain the following identical relation.
Lemma 3.8. Let \( \{ \varphi^n \}_{n=0}^\infty \) be a function sequence on \( V_h \), then the following relation holds
\[
\left( \sum_{k=0}^n \tilde{b}_k^i \varphi_k^i, I_h^i \varphi^n \right) = \frac{1}{2} \left[ (\varphi^n, I_h^i \varphi^n) + \sum_{k=0}^{n-1} \tilde{b}_k^i (\varphi_k^i, I_h^i \varphi^k) - \sum_{k=0}^{n-1} \tilde{b}_k^i (\varphi^n - \varphi_k^i, I_h^i (\varphi^n - \varphi^k)) \right].
\] (3.21)

4 Existence, Uniqueness and Stability Analysis

In this section, we will give the existence, uniqueness and stability results for the fully discrete FVE scheme (2.9).

Theorem 4.1. There exists a unique solution for the fully discrete FVE scheme (2.9).

Proof. Let \( M_Z^0 \) be the number of the vertices in \( Z_h \), and \( \{ \Phi_i : i = 1, 2, \cdots, M_Z^0 \} \) be the abbreviated basis functions of the space \( V_h \), then \( u_h^n \in V_h \) can be expressed as follows
\[
u_h^n(x) = \sum_{i=1}^{M_Z^0} \Phi_i(x).
\]

Substituting the above expression into the FVE scheme (2.9) (or the equivalent formulation (2.10)), and taking \( v_h = \Phi_j (j = 1, 2, \cdots, M_Z^0) \), then (2.9) (or (2.10)) can be rewritten as the following matrix form: find \( U^n \) such that
\[
\frac{1}{\Gamma(2 - \alpha)} B_1 U^n + \tau^\alpha B_2 U^n + \tau^\alpha B_3 U^n = \tau^\alpha F^n - \frac{1}{\Gamma(2 - \alpha)} \sum_{k=0}^{n-1} \tilde{b}_k^i B_1 U^k,
\] (4.1)

where
\[
U^n = (u_1^n, u_2^n, \cdots, u_{M_Z^0}^n)^T, \quad B_1 = ((\Phi_i, I_h^i \Phi_j))_{i,j=1,\cdots,M_Z^0},
\]
\[
B_2 = (a(\Phi_i, I_h^i \Phi_j))_{i,j=1,\cdots,M_Z^0}, \quad B_3 = ((q(\Phi_i, I_h^i \Phi_j))_{i,j=1,\cdots,M_Z^0},
\]
\[
F^n = ((f(t_n), I_h^i \Phi_j))_{j=1,\cdots,M_Z^0}.
\]

Making use of Lemma 3.1, we can easily obtain that the matrix \( B_1 \) is symmetric positive definite. Let \( G = \frac{1}{\Gamma(2 - \alpha)} B_1 + \tau^\alpha B_2 + \tau^\alpha B_3 \), then (4.1) can be rewritten as follows
\[
GU^n = \tau^\alpha F^n - \frac{1}{\Gamma(2 - \alpha)} \sum_{k=0}^{n-1} \tilde{b}_k^i B_1 U^k.
\] (4.2)

Next, we will prove \( G \) is invertible. Applying Lemma 3.3 for \( \forall Y = (y_1, y_2, \cdots, y_{M_Z^0})^T \in R_{M_Z^0}^0 \setminus \{0\} \), we have
\[
Y^T (B_2 + B_3) Y = a(w_h, I_h^i w_h) + (q w_h, I_h^i w_h) \geq \mu_6 \|w_h\|_1^2 > 0,
\]
where \( w_h = \sum_{i=1}^{M_Z^0} y_i \Phi_i \neq 0 \). This means that \( Y^T (B_2 + B_3) Y \) (for \( Y \in R_{M_Z^0}^0 \)) is a positive definite quadratic form generated by nonsymmetric matrix \( B_2 + B_3 \). Therefore, \( Y^T G Y \) (for \( Y \in R_{M_Z^0}^0 \)) is a positive definite quadratic form generated by nonsymmetric matrix \( G \), then we have that \( G \) is invertible. In fact, if \( G \) is noninvertible, then the homogeneous linear equations \( G Y = 0 \) has nonzero solution \( Y_0 \), thus, we have \( Y_0^T G Y_0 = 0 \) which is in contradiction with the definition of positive definite quadratic form. Hence, \( G \) is invertible, then the linear equations (4.1) have a unique solution. This shows that the fully discrete FVE scheme (2.9) has a unique solution. Then, we complete the proof. \( \square \)

Next, we give the stability analysis for the fully discrete FVE scheme (2.9).
Theorem 4.2. Let \( \{u^n_h\}_{n=1}^N \) be the solution of the FVE system (4.9), then there exists a constant \( C > 0 \) independent of \( h \) and \( \tau \) such that

\[
\|u^n_h\| \leq C(\|u^n_0\| + \sup_{t \in [0,T]} \|f(t)\|).
\]

Moreover, let \( \alpha > 0 \) be a constant, then there exist constants \( C > 0 \) and \( \tau_0 \) \((0 < \tau_0 < 1)\) independent of \( h \) and \( \tau \) such that, when \( h \leq C_0 \tau \leq C_0 \tau_0 \) and \( h \leq h_1 \), where \( h_1 = \min\{\frac{\mu_0}{\mu}, 1\} \), we have

\[
\|u^n_h\|_1 \leq C(\|u^n_0\| + \sup_{t \in [0,T]} \|f(t)\|).
\]

Proof. Taking \( v_h = u^n_h \) in (4.9), we can obtain

\[
(D^n_r u^n_h, I^n_h u^n_h) + a(u^n_h, I^n_h u^n_h) + (qu^n_h, I^n_h u^n_h) = (f^n, I^n_h u^n_h).
\]

Make use of Lemma 3.3 and Lemma 3.3 apply the Young inequality to obtain

\[
(D^n_r u^n_h, I^n_h u^n_h) + \mu_6\|u^n_h\|_1^2 \leq C\|f^n\|^2 + \frac{\mu_6}{2}\|u^n_h\|^2.
\]

Applying Lemma 3.8 to rewrite \((D^n_r u^n_h, I^n_h u^n_h)\), we have

\[
(D^n_r u^n_h, I^n_h u^n_h) = \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}(u^n_h, I^n_h u^n_h) + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \tilde{b}^k(u^n_h, I^n_h u^n_h)
\]

\[
- \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \tilde{b}^k(u^n_h - u^n_h, I^n_h(u^n_h - u^n_h)).
\]

Substituting (4.5) into (4.4), and taking note of \( \tilde{b}^k < 0 \) \((0 \leq k \leq n - 1)\), we obtain

\[
(u^n_h, I^n_h u^n_h) + \tau^\alpha\Gamma(2-\alpha)\mu_6\|u^n_h\|^2
\]

\[
\leq - \sum_{k=0}^{n-1} \tilde{b}^k(u^n_h, I^n_h u^n_h) + C\tau^\alpha\Gamma(2-\alpha) \sup_{t \in [0,T]} \|f(t)\|^2.
\]

Applying Lemma 3.6 in (4.6), we have

\[
(u^n_h, I^n_h u^n_h) \leq C((u^n_0, I^n_h u^n_0) + \sup_{t \in [0,T]} \|f(t)\|^2).
\]

Thus, applying Lemma 3.3 in (4.7), we obtain the following result

\[
\|u^n_h\|^2 \leq C(\|u^n_0\|^2 + \sup_{t \in [0,T]} \|f(t)\|^2).
\]

Next, choosing \( v_h = D^n_r u^n_h \) in (2.9), we get

\[
(D^n_r u^n_h, I^n_h D^n_r u^n_h) + a(u^n_h, I^n_h D^n_r u^n_h) = -(qu^n_h, I^n_h D^n_r u^n_h) + (f^n, I^n_h D^n_r u^n_h).
\]

Noting that \((D^n_r u^n_h, I^n_h D^n_r u^n_h) \geq \mu_1\|D^n_r u^n_h\|^2\), and applying the Young inequality in (4.9), we have

\[
\mu_1\|D^n_r u^n_h\|^2 + a(u^n_h, I^n_h D^n_r u^n_h) \leq C(\|u^n_h\|^2 + \|f^n\|^2) + \frac{\mu_1}{2}\|D^n_r u^n_h\|^2.
\]

(4.10)
Thus, we complete the proof of the stability.

Setting \( h \), then there exists a constant \( \tilde{\mu}_6 \) (4.10) to obtain

\[
\mu_1 \|D^\alpha u^n_k\|^2 + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} a(u^n_k, I^*_h u^n_k) + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \tilde{b}^n_k a(u^n_k, I^*_h u^n_k) \\
- \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \tilde{b}^n_k (a(u^n_k - u^n_h, I^*_h (u^n_k - u^n_h)) \\
\le C(\|u^n_n\|^2 + \|f^n\|^2) + \frac{\mu_1}{2} \|D^\alpha u^n_h\|^2 \\
- \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \tilde{b}^n_k [a(u^n_k, I^*_h u^n_k) - a(u^n_h, I^*_h u^n_h)] \tag{4.11}
\]

Making use of Lemma 3.2 and Lemma 3.3, we have

\[
|a(u^n_k, I^*_h u^n_k) - a(u^n_h, I^*_h u^n_h)| \le \mu_5 h \|u^n_n\|_1 \|u^n_k\|_1 \\
\le \frac{\mu_5}{2\mu_6} [a(u^n_n, I^*_h u^n_h) + a(u^n_h, I^*_h u^n_h)] \tag{4.12}
\]

Substituting (4.8) and (4.12) into (4.11), multiplying (4.11) by \( \tau^\alpha \), and noting that \( \tilde{b}^n_k < 0 \) (0 \( \le k \le n - 1 \), we have

\[
(1 - \frac{\mu_5}{2\mu_6} h)a(u^n_h, I^*_h u^n_n) \le -(1 + \frac{\mu_5}{2\mu_6} h) \sum_{k=0}^{n-1} \tilde{b}^n_k a(u^n_k, I^*_h u^n_k) + C\tau^\alpha(\|u^n_n\|^2 + \sup_{t \in [0,T]} \|f(t)\|^2) \tag{4.13}
\]

Setting \( h_1 = \min\{\frac{\mu_5}{2\mu_6}, 1\} \), when \( h \le h_1 \), we have \( (1 - \frac{\mu_5}{2\mu_6} h) \ge \frac{1}{2} \) and

\[
a(u^n_n, I^*_h u^n_n) \le -\frac{1 + \frac{\mu_5}{2\mu_6} h}{1 - \frac{\mu_5}{2\mu_6} h} \sum_{k=0}^{n-1} \tilde{b}^n_k a(u^n_k, I^*_h u^n_k) + C\tau^\alpha(\|u^n_n\|^2 + \sup_{t \in [0,T]} \|f(t)\|^2) \tag{4.14}
\]

Applying Lemma 3.6 in (4.14), we have

\[
a(u^n_k, I^*_h u^n_k) \le C\left(1 + \frac{\mu_5}{2\mu_6} h \right)^n (a(u^n_n, I^*_h u^n_n) + \|u^n_n\|^2 + \sup_{t \in [0,T]} \|f(t)\|^2) \tag{4.15}
\]

Let \( c_0 > 0 \) be a constant. Selecting \( h \) and \( \tau \) to satisfy \( h \le c_0 \tau \) in (4.15) and applying Lemma 3.3, we have

\[
\|u^n_k\|^2 \le C\left(1 + \frac{c_0 \mu_5}{2\mu_6} \tau \right)^n (\|u^n_n\|^2 + \sup_{t \in [0,T]} \|f(t)\|^2) \tag{4.16}
\]

Note that

\[
\lim_{\tau \to 0} \left(1 + \frac{c_0 \mu_5}{2\mu_6} \tau \right)^n = e^{\frac{c_0 T \mu_5}{\mu_6}} \tag{4.17}
\]

then, there exists a constant \( \tau_0 \) (0 \( < \tau_0 < 1 \), when \( \tau \le \tau_0 \), it follows that

\[
\|u^n_k\|^2 \le C e^{\frac{c_0 T \mu_5}{\mu_6}} (\|u^n_n\|^2 + \sup_{t \in [0,T]} \|f(t)\|^2) \tag{4.18}
\]

Thus, we complete the proof of the stability. \( \square \)
Remark 4.1. From Theorem 4.2, it is easy to see that \( u^0_h \) is unconditionally stable, and \( u^n_h \) is stable when the conditions in the Theorem 4.2 are established, because the bilinear \( a(I^n_h) \) does not necessarily satisfy symmetry. When the coefficient \( A \) is a symmetry and positive definite constant matrix, following Reference [13], we can know that \( a(I^n_h) \) satisfy symmetry. And under this condition, we can obtain that \( u^n_h \) is also unconditionally stable.

5 A Priori Error Estimates

In order to obtain the error estimates for the fully discrete FVE scheme \((2.9)\), we introduce an elliptic projection operator \( P_h : H^1_0(\Omega) \cap H^2(\Omega) \rightarrow V_h \), which is defined by the following

\[
a(u - P_h u, I^n_h v_h) = 0, \quad \forall v_h \in V_h.
\]

Following References [21], the above projection operator \( P_h \) satisfies the following estimates.

Lemma 5.1. There exists a constant \( C > 0 \) independent of \( h \) and \( \tau \) such that

\[
\| u - P_h u \|_1 \leq Ch | u |_2, \quad \forall u \in H^1_0(\Omega) \cap H^2(\Omega),
\]

\[
\| u - P_h u \| \leq Ch^2 \| u \|_{3,p}, \quad \forall u \in H^1_0(\Omega) \cap W^{3,p}(\Omega), \quad p > 1.
\]

Next, we give the main results in this paper about the error estimates.

Theorem 5.1. Let \( u \) and \( u^0_h \) be the solutions of system \((2.1)\) and the FVE scheme \((2.9)\), respectively. Assume that \( u^0_h = P_h u_0 \), then there exists a constant \( C > 0 \) independent of \( h \) and \( \tau \) such that

\[
\max_{1 \leq n \leq N} \| u(t_n) - u^n_h \| \leq C(h^2 + \tau^{2-\alpha}),
\]

\[
\max_{1 \leq n \leq N} \| u(t_n) - u^n_h \|_1 \leq C(h + \tau^{\frac{\alpha}{2}}(h^2 + \tau^{2-\alpha})).
\]

Proof. We write \( u(t_n) - u^n_h = \rho^n + \theta^n \), where \( \rho^n = u(t_n) - P_h u(t_n), \theta^n = P_h u(t_n) - u^n_h \), and \( P_h \) is the elliptic projection operator defined in \((5.1)\). According to Lemma 5.1, we only need to estimate \( \theta^n \).

Making use of our definitions, we can have the error equation for \( \theta^n \) as follows

\[
(D^2_\tau \theta^n, I^n_h v_h) + a(\theta^n, I^n_h v_h) + (p\theta^n, I^n_h v_h)
\]

\[
= -(D^2_\tau \rho^n, I^n_h v_h) - (q\theta^n, I^n_h v_h) - (R^n_\theta(x), I^n_h v_h), \quad \forall v_h \in V_h.
\]

Now, choosing \( v_h = \theta^n \) in \((5.6)\), we obtain

\[
(D^2_\tau \theta^n, I^n_h \theta^n) + a(\theta^n, I^n_h \theta^n) + (q\theta^n, I^n_h \theta^n)
\]

\[
= -(D^2_\tau \rho^n, I^n_h \theta^n) - (q\theta^n, I^n_h \theta^n) - (R^n_\theta(x), I^n_h \theta^n).
\]

By virtue of Lemma 3.1 and the Young inequality in \((5.7)\), we have

\[
(D^2_\tau \theta^n, I^n_h \theta^n) + a(\theta^n, I^n_h \theta^n) \leq C(||D^2_\tau \rho^n||^2 + ||\rho^n||^2 + ||R^n_\theta(x)||^2) + \frac{\mu_0}{2} ||\theta^n||^2.
\]
Applying Lemma 3.3 and Lemma 3.8 in (5.8) to obtain

\[
\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}(\theta^n, I^*_h \theta^n) + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \hat{b}_k^n (\theta^k, I^*_h \theta^k)
\]

\[
- \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \hat{b}_k^n (\theta^n - \theta^k, I^*_h (\theta^n - \theta^k)) + \frac{\mu_0}{2} \|\theta^n\|^2_1
\]

\[
\leq C(\|D^n_\tau \rho^n\|^2 + \|\rho^n\|^2 + \|R^n_i(x)\|^2).
\] (5.9)

Multiplying (5.9) by \(2(2-\alpha)\tau^{-\alpha}\), and noting that \(-\sum_{k=0}^{n-1} \hat{b}_k^n (\theta^n - \theta^k, I^*_h (\theta^n - \theta^k)) \geq 0\), we have

\[
(\theta^n, I^*_h \theta^n) + \tau^\alpha (2-\alpha) \mu_0 \|\theta^n\|_1^2
\]

\[
\leq - \sum_{k=0}^{n-1} \hat{b}_k^n (\theta^k, I^*_h \theta^k) + C\tau^\alpha (2-\alpha) (\|D^n_\tau \rho^n\|^2 + \|\rho^n\|^2 + \|R^n_i(x)\|^2).
\] (5.10)

In order to estimate \(\|D^n_\tau \rho^n\|^2\) and \(\|\rho^n\|^2\) in (5.10), we apply Lemma 5.1 to obtain

\[
\|\rho^n\|^2 = \|u(t_n) - P_h u(t_n)\|^2 \leq C h^4 \|u(t_n)\|_{W^3,p} \leq C h^4 \|u\|_{L^\infty(W^3,p)},
\]

\[
\|\partial_t \rho^{n-k}\| = \|\frac{1}{\tau} \int_{t_{n-k}}^{t_n} \rho_t dt\| \leq h^2 \|u_t\|_{L^\infty(W^3,p)}, \quad p > 1.
\] (5.11)

Noting that \(D^n_\tau \rho^n = \frac{\tau^{1-\alpha}}{2(2-\alpha)} \sum_{k=0}^{n-1} b_k \partial_t \rho^{n-k}\) and \(\sum_{k=0}^{n-1} b_k = n^{1-\alpha}\), we have

\[
\|D^n_\tau \rho^n\|^2 \leq \left( \frac{\tau^{1-\alpha}}{2(2-\alpha)} n^{1-\alpha} h^2 \|u_t\|_{L^\infty(W^3,p)} \right)^2 \leq C T^{2-2\alpha} h^4 \|u_t\|^2_{L^\infty(W^3,p)}, \quad p > 1.
\] (5.12)

Substituting (5.11) and (5.12) into (5.10), and applying Lemma 5.4, we have

\[
(\theta^n, I^*_h \theta^n) + \tau^\alpha (2-\alpha) \mu_0 \|\theta^n\|_1^2
\]

\[
\leq - \sum_{k=0}^{n-1} \hat{b}_k^n (\theta^k, I^*_h \theta^k) + C\tau^\alpha (h^4 + \tau^{2(2-\alpha)} + \tau^4).
\] (5.13)

Note that \(\theta^0 = 0\), apply Lemma 5.5 in (5.13) to obtain

\[
(\theta^n, I^*_h \theta^n) + \tau^\alpha (2-\alpha) \mu_0 \|\theta^n\|_1^2 \leq C(h^4 + \tau^{2(2-\alpha)} + \tau^4).
\] (5.14)

By virtue of Lemma 3.1, we have

\[
\mu_1 \|\theta^n\|^2 + \tau^\alpha (2-\alpha) \mu_0 \|\theta^n\|_1^2 \leq C(h^4 + \tau^{2(2-\alpha)} + \tau^4).
\] (5.15)

Finally, apply Lemma 5.1 with (5.15) to complete the proof.

**Remark 5.1.** From Theorem 5.4, we can see that \(\|u(t_n) - u^n_h\|\) has been given the optimal a priori error estimate based on \(L^1\)-formula, and the estimate in time for \(\|u(t_n) - u^n_h\|_{1}\) reduces \(\frac{2}{3}\) by comparing with the optimal estimate. So we need to find and use other estimate methods.

Next, we try to give a better estimate for \(\|u(t_n) - u^n_h\|_{1}\).

**Theorem 5.2.** Let \(u\) and \(u^n_h\) be the solutions of system (2.1) and the FVE scheme (2.9), respectively. Assume that \(u^n_h = P_h u_0\). Let \(c_0 > 0\) be a constant, then there exist two constants \(C > 0\) and
\( \tau_0 \) (0 < \( \tau_0 < 1 \)) independent of \( h \) and \( \tau \) such that, when \( h \leq c_0 \tau \leq c_0 \tau_0 \) and \( h \leq h_1 \), where \( h_1 = \min\{ \frac{\mu_0}{\mu_6}, 1 \} \), we have

\[
\max_{1 \leq n \leq N} \| u(t_n) - u_h^n \|_1 \leq C(h + e^{c_{0}\tau h} (h^2 + \tau^{2-\alpha})).
\]

Proof. First, we take \( v_h = D^\alpha_\tau \theta^n \) in (5.6) to obtain

\[
(D^\alpha_\tau \theta^n, I^*_h D^\alpha_\tau \theta^n) + a(\theta^n, I^*_h D^\alpha_\tau \theta^n)
= -(D^\alpha_\tau \rho^n, I^*_h D^\alpha_\tau \theta^n) - (q \rho^n, I^*_h D^\alpha_\tau \theta^n) - (q \theta^n, I^*_h D^\alpha_\tau \theta^n) - (R^n_t(x), I^*_h D^\alpha_\tau \theta^n).
\]

(5.16)

Noting that \( (D^\alpha_\tau \theta^n, I^*_h D^\alpha_\tau \theta^n) \geq \mu_1 \| D^\alpha_\tau \theta^n \|^2 \), applying Lemma 3.1 and the Young inequality, we have

\[
\mu_1 \| D^\alpha_\tau \theta^n \|^2 + a(\theta^n, I^*_h D^\alpha_\tau \theta^n)
\leq C(\| D^\alpha_\tau \rho^n \|^2 + \| \rho^n \|^2 + \| \theta^n \|^2 + \| R^n_t(x) \|^2) + \frac{\mu_1}{2} \| D^\alpha_\tau \theta^n \|^2.
\]

(5.17)

Apply Lemma 3.7 in (5.17) to obtain

\[
\frac{\mu_1}{2} \| D^\alpha_\tau \theta^n \|^2 + \frac{\tau^{-\alpha}}{2\Gamma(2 - \alpha)} a(\theta^n, I^*_h \theta^n) + \frac{\tau^{-\alpha}}{2\Gamma(2 - \alpha)} \sum_{k=0}^{n-1} b^n_k a(\theta^k, I^*_h \theta^k)
\leq C(\| D^\alpha_\tau \rho^n \|^2 + \| \rho^n \|^2 + \| \theta^n \|^2 + \| R^n_t(x) \|^2)
- \frac{\tau^{-\alpha}}{2\Gamma(2 - \alpha)} \sum_{k=0}^{n-1} b^n_k [a(\theta^k, I^*_h \theta^k) - a(\theta^k, I^*_h \theta^k)].
\]

(5.18)

Making use of Lemma 3.2 and Lemma 3.3, we have

\[
|a(\theta^n, I^*_h \theta^k) - a(\theta^k, I^*_h \theta^n)| \leq \mu_3 h \| \theta^n \|_1 \| I^*_h \|_1 \leq \frac{\mu_5}{2\mu_6} h[a(\theta^n, I^*_h \theta^n) + a(\theta^k, I^*_h \theta^k)].
\]

(5.19)

Noting that \( -\frac{\tau^{-\alpha}}{2\Gamma(2 - \alpha)} \sum_{k=0}^{n-1} b^n_k a(\theta^k, I^*_h (\theta^n - \theta^k)) \geq 0 \), and substituting (5.19) into (5.18), we obtain

\[
\mu_1 \tau_\alpha \Gamma(2 - \alpha) \| D^\alpha_\tau \theta^n \|^2 + (1 - \frac{\mu_5}{2\mu_6} h)a(\theta^n, I^*_h \theta^n)
\leq -(1 + \frac{\mu_5}{2\mu_6} h) \sum_{k=0}^{n-1} b^n_k a(\theta^k, I^*_h \theta^k)
+ C \tau_\alpha \Gamma(2 - \alpha)(\| D^\alpha_\tau \rho^n \|^2 + \| \rho^n \|^2 + \| \theta^n \|^2 + \| R^n_t(x) \|^2).
\]

(5.20)

Substituting (5.11), (5.12) and (5.15) into (5.20), and applying Lemma 3.4, we have

\[
\mu_1 \tau_\alpha \Gamma(2 - \alpha) \| D^\alpha_\tau \theta^n \|^2 + (1 - \frac{\mu_5}{2\mu_6} h)a(\theta^n, I^*_h \theta^n)
\leq -(1 + \frac{\mu_5}{2\mu_6} h) \sum_{k=0}^{n-1} b^n_k a(\theta^k, I^*_h \theta^k) + C \tau_\alpha (h^4 + \tau^{2(2-\alpha)} + \tau^4).
\]

(5.21)
Similar to the proof process of Theorem 4.2 when \( h \leq h_1 \), where \( h_1 = \min\{\frac{h_0}{\mu}, 1\} \), we have

\[
a(\theta^n, I_h^n \theta^n) \leq \frac{1 + \frac{\mu_0 h}{2 \mu h} }{1 - \frac{\mu_0 h}{2 \mu h} } \sum_{k=0}^{n-1} b_k^h a(\theta^k, I_h^k \theta^k) + C \tau^\alpha (h^4 + \tau^{2(2-\alpha)} + \tau^4). \tag{5.22}
\]

Applying Lemma 3.6 in (5.22), noting that \( \theta^0 = 0 \), we easily get

\[
a(\theta^n, I_h^n \theta^n) \leq C \left( \frac{1 + \frac{\mu_0 h}{2 \mu h} }{1 - \frac{\mu_0 h}{2 \mu h} } \right)^n (h^4 + \tau^{2(2-\alpha)} + \tau^4). \tag{5.23}
\]

Let \( c_0 > 0 \) be a constant. Selecting \( h \) and \( \tau \) to satisfy \( h \leq c_0 \tau \), we can obtain

\[
a(\theta^n, I_h^n \theta^n) \leq C \left( \frac{1 + \frac{c_0 \mu_0 h}{2 \mu h} }{1 - \frac{c_0 \mu_0 h}{2 \mu h} } \right)^{\frac{n}{6}} (h^4 + \tau^{2(2-\alpha)} + \tau^4). \tag{5.24}
\]

By virtue of (4.17), there exists a constant \( \tau_0 \) (\( 0 < \tau_0 < 1 \)), when \( \tau \leq \tau_0 \), it follows that

\[
a(\theta^n, I_h^n \theta^n) \leq C e^{\frac{c_0 \tau_0 h}{\mu h}} (h^4 + \tau^{2(2-\alpha)} + \tau^4). \tag{5.25}
\]

Finally, apply Lemma 3.3 and Lemma 5.1 with (5.25) to complete the proof.

**Remark 5.2.** From Theorem 5.2 we can see that the optimal a priori error estimate for \( \|u(t_n) - u_h^n\|_1 \) is obtained when \( h \) and \( \tau \) satisfy \( h \leq c_0 \tau \leq c_0 \tau_0 \). When \( h \) is sufficiently small, we have \( \|u(t_n) - u_h^n\|_1 \leq C(\tau^{2-\alpha}) \), which is validated in the Example 1 in Section 6.

**Remark 5.3.** Similar to Remark 4.4, when the coefficient \( A(x) \) is a symmetry and positive definite constant matrix, in the analysis and results of Theorem 5.2, we can remove the conditions \( h \leq c_0 \tau \leq c_0 \tau_0 \) and \( h \leq h_1 \).

6 Numerical Examples

In this section, we will give two examples to examine the feasibility and effectiveness of the proposed FVE scheme.

**Example 1.** We consider the equations (1.1) in one-dimensional space regions as follows

\[
\begin{align*}
&\frac{D^\alpha u(x, t)}{D t^\alpha} - \frac{\partial}{\partial x} \left( \hat{a}(x) \frac{\partial u(x, t)}{\partial x} \right) + q(x) u(x, t) = f(x, t), \quad (x, t) \in \Omega \times J, \\
u(a, t) = u(b, t) = 0, & \quad t \in J, \\
u(x, 0) = u_0(x), & \quad x \in \Omega.
\end{align*}
\tag{6.1}
\]

where \( \Omega = (a, b) \subset \mathbb{R}^1 \), \( J = (0, T] \) is the time interval with \( 0 < T < \infty \). The functions \( \hat{a}(x), q(x), f(x, t) \) and \( u_0(x) \) are smooth enough, and \( q(x) \geq 0, \forall x \in \Omega \). Suppose that there exist two constants \( \hat{a}_0 \) and \( \hat{a}_1 \) such that \( 0 < \hat{a}_0 \leq \hat{a}(x) \leq \hat{a}_1 \). Following Reference [21], we construct the primal mesh \( \bar{T}_h \) and dual mesh \( \bar{T}_h^* \) and take the spaces \( \bar{V}_h \) and \( \bar{V}_h^* \) as the **trial** function space and **test** function space, respectively, where

\[
\bar{V}_h = \{ w_h \in H^1_0(\Omega) : w_h \in P_1(A), \forall A \in \bar{T}_h \}, \\
\bar{V}_h^* = \{ v_h \in L^2(\Omega) : v_h|_A^*, \in P_0(A^*), \forall A^* \in \bar{T}_h^* \}, \quad v_h|_{\partial \Omega} = 0.
\]
Table 1: Error results for $\max_n \| u(t_n) - u_h^n \|$ with $h = 2.5E - 04$ in Example 1.

| $\alpha$ | $\tau_1 = 1/10$ | $\tau_2 = 1/20$ | $\tau_3 = 1/40$ | $\tau_4 = 1/80$ |
|----------|-----------------|-----------------|-----------------|-----------------|
| 0.1      | 8.4050 300E-05  | 2.4679 2171E-05 | 7.1512 7494E-06 | 2.0528 9204E-06 |
| Rate     | 1.7679 6716    | 1.7870 2425    | 1.8005 4271    | 1.8000 4271    |
| 0.3      | 3.9761 6219E-04| 1.2815 5606E-04| 4.0865 7882E-05| 1.2943 9996E-05|
| Rate     | 1.6334 8002    | 1.6481 9735    | 1.6593 4386    | 1.6593 4386    |
| 0.5      | 1.0802 2005E-03| 3.8961 5121E-04| 1.3967 3166E-04| 4.9862 8084E-05|
| Rate     | 1.4711 7763    | 1.4799 9479    | 1.4860 3977    | 1.4860 3977    |
| 0.7      | 2.5216 2418E-03| 1.0311 9006E-03| 4.2056 0268E-04| 1.7124 1407E-04|
| Rate     | 1.2900 4214    | 1.2939 2668    | 1.2962 8096    | 1.2962 8096    |
| 0.9      | 5.4643 9531E-03| 2.5525 0980E-03| 1.9159 9799E-03| 5.5610 3741E-04|
| Rate     | 1.0981 4536    | 1.0990 1672    | 1.0994 7383    | 1.0994 7383    |

Table 2: Error results for $\max_n \| u(t_n) - u_h^n \|_1$ with $h = 2.5E - 04$ in Example 1.

| $\alpha$ | $\tau_1 = 1/10$ | $\tau_2 = 1/20$ | $\tau_3 = 1/40$ | $\tau_4 = 1/80$ |
|----------|-----------------|-----------------|-----------------|-----------------|
| 0.1      | 1.8008 0487    | 1.8803 6624    | 1.8968 5451    | 1.8968 5451    |
| 0.3      | 6.5343 3449E-05| 2.0975 1531E-05| 6.6047 9132E-06| 2.0357 3159E-06|
| Rate     | 1.6393 3897    | 1.6670 9645    | 1.6979 6563    | 1.6979 6563    |
| 0.5      | 1.7793 1804E-04| 6.4101 6920E-05| 2.2898 8168E-05| 8.1001 3858E-06|
| Rate     | 1.4728 9006    | 1.4850 8938    | 1.4992 5456    | 1.4992 5456    |
| 0.7      | 4.1610 2111E-04| 1.7009 8563E-04| 6.9300 0954E-05| 2.8141 9417E-05|
| Rate     | 1.2905 6665    | 1.2954 4171    | 1.3001 3547    | 1.3001 3547    |
| 0.9      | 9.0324 4861E-04| 4.2186 9751E-04| 1.9687 8771E-04| 9.1813 3309E-05|
| Rate     | 1.0983 1949    | 1.0994 9009    | 1.1053 188     | 1.1053 188     |

As in Reference [21], we also define operator $\hat{I}_h^* : C(\Omega) \rightarrow \hat{V}_h^*$ by

$$\hat{I}_h^* w_h = \sum_{i=1}^{\hat{M}_T^0} w_h(x_i) \chi_{A_i^*}, \ \forall w_h \in \hat{V}_h^*,$$

where $\hat{M}_T^0$ is the number of the interior nodes, and $\chi_{A_i^*}$ is the characteristic function of a set $A_i^* \in \hat{T}_h^*$. Making use of the operator $\hat{I}_h^*$, we can also present the FVE scheme and obtain the corresponding theoretical results as Theorems 4.1-4.2 and Theorems 5.1-5.2. Here, we will not repeat these processes.

In this example, we take $\Omega = (0, 1), T = 1, \tilde{a}(x) = 1 + 2x^2, q(x) = 1 + x^2, u_0(x) = 0$ and

$$f(x,t) = \left( \frac{2}{\Gamma(3 - \alpha)} t^{2 - \alpha} + t^2 (1 + x^2) + 4\pi^2 t^2 (1 + 2x^2) \right) \sin(2\pi x) - 8\pi t^2 x \cos(2\pi x).$$

Thus we can obtain the analytical solution $u(x,t) = t^2 \sin(2\pi x)$.

We give the numerical results with some different parameters $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$ in Tables 1 and 4. In order to test the time convergence rates, we choose the spatial step length $h = 2.5E - 04$ and the time step length $\tau = 1/10, 1/20, 1/40, 1/80$, and give the error behaviors for $u$ with $L^2(\Omega)$-norm (in Table 1) and $H^1(\Omega)$-norm (in Table 2). We can find that the time convergence rates in these two norms are approximate to $2 - \alpha$, which are consistent with the convergence results in Theorems 5.1-5.2. Moreover, choosing different parameter $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$, fixing the time step length
and find that the spatial convergence rates for $\tau = 1$ and choose $\Omega = (0, 1) \times (0, 1)$.

From these numerical results in this example, we can see that the proposed FVE method for the time step length $\tau$ give error behaviors in Tables 11-12, in which the convergence rates still satisfy the theoretical results.

The error results show that the convergence can still be satisfied in the actual calculation, even if the condition $h \leq c_0\tau$ in Theorem 5.2 is not established.

**Example 2.** In this example, we consider the equations (1.1) in two-dimensional space regions, and choose $\Omega = (0, 1) \times (0, 1)$, $J = (0, 1]$, $q(x) = 1 + x_1^2 + x_2^2$, $\forall x = (x_1, x_2) \in \Omega$, and

$$A(x) = \begin{pmatrix}
2 + x_1^2 + x_2^2 & x_1^2 + x_2^2 \\
 x_1^2 + x_2^2 & 2 + x_1^2 + x_2^2
\end{pmatrix}, \forall x = (x_1, x_2) \in \Omega.
$$

We choose the analytical solution $u(x, t) = t^2 \sin(2\pi x_1) \sin(2\pi x_2)$, then we can get the corresponding initial function $u_0(x)$ and the source function $f(x, t)$.

We select some different parameters $\alpha$ and mesh sizes to carry out numerical simulation, and give some error results with $\alpha = 0.1, 0.5, 0.9$ for $u$ in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm in Tables 5-10, where the mesh sizes are selected as $h = \sqrt{2}\tau, 2\sqrt{2}\tau, 4\sqrt{2}\tau, \frac{\sqrt{2}}{2}\tau, \frac{\sqrt{2}}{4}\tau$. The error results show that the convergence rates for $u$ in $L^2(\Omega)$-norm are approximate to 2, and the convergence rates for $u$ in $H^1(\Omega)$-norm are approximate to 1. Moreover, we also ignore the condition $h \leq c_0\tau$ in Theorem 5.2, fix the time step length $\tau = 1 - 0.3$, select the spatial step length $h = \sqrt{2}/10, \sqrt{2}/20, \sqrt{2}/40, \sqrt{2}/80$, and give error behaviors in Tables 11-12 in which the convergence rates still satisfy the theoretical results. From these numerical results in this example, we can see that the proposed FVE method for the time step length $\tau$ will be satisfied in the actual calculation.
Table 5: Error results for \( \max_n \| u(t_n) - u_h^n \| \) in Example 2 with \( \alpha = 0.1 \).

|                  | \( s_1 = 1/5 \)      | \( s_2 = 1/10 \)    | \( s_3 = 1/20 \)    | \( s_4 = 1/40 \)    |
|------------------|----------------------|---------------------|---------------------|---------------------|
| \( \frac{h}{\sqrt{2}} = \tau = s \) | 1.60067189E-01      | 3.71949729E-02      | 9.19333755E-03      | 2.28450645E-03      |
| Rate             | 2.10549806           | 2.01644703          | 2.00879620          |                     |
| \( \frac{h}{\sqrt{2}} = 2\tau = s \) | 1.60069253E-01      | 3.71962047E-02      | 9.19375623E-03      | 2.28463221E-03      |
| Rate             | 2.10546888           | 2.01642910          | 2.00869248          |                     |
| \( \frac{h}{\sqrt{2}} = 4\tau = s \) | 1.60069873E-01      | 3.71965683E-02      | 9.19387809E-03      | 2.28468403E-03      |
| Rate             | 2.10546037           | 2.01642408          | 2.00868875          |                     |
| \( \tau = \frac{2h}{\sqrt{2}} = s \) | 3.71908773E-02      | 9.19194394E-03      | 2.28407438E-03      | 5.69390117E-04      |
| Rate             | 2.01651072           | 2.00875652          | 2.00412027          |                     |
| \( \tau = \frac{4h}{\sqrt{2}} = s \) | 9.18720496E-03      | 2.28261124E-03      | 5.68955265E-04      | 1.42028688E-04      |
| Rate             | 2.00894085           | 2.00429804          | 2.00213285          |                     |

Table 6: Error results for \( \max_n \| u(t_n) - u_h^n \|_1 \) in Example 2 with \( \alpha = 0.1 \).

|                  | \( s_1 = 1/5 \)      | \( s_2 = 1/10 \)    | \( s_3 = 1/20 \)    | \( s_4 = 1/40 \)    |
|------------------|----------------------|---------------------|---------------------|---------------------|
| \( \frac{h}{\sqrt{2}} = \tau = s \) | 2.47401540E+00       | 1.31270541E+00      | 6.65179344E-01      | 3.39757473E-01      |
| Rate             | 0.91431129           | 0.98072792          | 0.99400121          |                     |
| \( \frac{h}{\sqrt{2}} = 2\tau = s \) | 2.47400989E+00       | 1.31270378E+00      | 6.65179079E-01      | 3.39754333E-01      |
| Rate             | 0.91430986           | 0.98072671          | 0.9940081           |                     |
| \( \frac{h}{\sqrt{2}} = 4\tau = s \) | 2.47400823E+00       | 1.31270331E+00      | 6.65179002E-01      | 3.39754212E-01      |
| Rate             | 0.91430942           | 0.98072635          | 0.9940069           |                     |
| \( \tau = \frac{2h}{\sqrt{2}} = s \) | 1.31271081E+00       | 6.65180241E-01      | 3.3975611E-01       | 1.67229229E-01      |
| Rate             | 0.98073191           | 0.99400256          | 0.99791572          |                     |
| \( \tau = \frac{4h}{\sqrt{2}} = s \) | 6.65183228E-01       | 3.3976078E-01       | 1.67229299E-01      | 8.36617901E-02      |
| Rate             | 0.99400702           | 0.99791714          | 0.99918686          |                     |

Table 7: Error results for \( \max_n \| u(t_n) - u_h^n \| \) in Example 2 with \( \alpha = 0.5 \).

|                  | \( s_1 = 1/5 \)      | \( s_2 = 1/10 \)    | \( s_3 = 1/20 \)    | \( s_4 = 1/40 \)    |
|------------------|----------------------|---------------------|---------------------|---------------------|
| \( \frac{h}{\sqrt{2}} = \tau = s \) | 1.59933131E-01      | 3.71319625E-02      | 9.17277091E-03      | 2.2782823E-03       |
| Rate             | 2.10673536           | 2.01723205          | 2.0094532           |                     |
| \( \frac{h}{\sqrt{2}} = 2\tau = s \) | 1.59952720E-01      | 3.71461868E-02      | 9.17869712E-03      | 2.28042518E-03      |
| Rate             | 2.10635950           | 2.0165283           | 2.00898655          |                     |
| \( \frac{h}{\sqrt{2}} = 4\tau = s \) | 1.59959883E-01      | 3.71513345E-02      | 9.18082686E-03      | 2.28121103E-03      |
| Rate             | 2.10622419           | 2.01671804          | 2.00882418          |                     |
| \( \tau = \frac{2h}{\sqrt{2}} = s \) | 3.70930838E-02      | 9.15640676E-03      | 2.27212125E-03      | 5.65372965E-04      |
| Rate             | 2.01829676           | 2.01074173          | 2.00676504          |                     |
| \( \tau = \frac{4h}{\sqrt{2}} = s \) | 9.11176525E-03      | 2.25530613E-03      | 5.59264054E-04      | 1.38601506E-04      |
| Rate             | 2.01440731           | 2.01172176          | 2.01258668          |                     |
Table 8: Error results for $\max_n \|u(t_n) - u_h^n\|_1$ in Example 2 with $\alpha = 0.5$.

| $n$ | $s_1 = 1/5$ | $s_2 = 1/10$ | $s_3 = 1/20$ | $s_4 = 1/40$ |
|-----|-------------|-------------|-------------|-------------|
| $h = \tau = s$ | 2.47419989E+00 | 1.31277495E+00 | 6.65190985E-01 | 3.33977313E-01 |
| Rate | 0.91434245 | 0.98077909 | 0.99401851 | |
| $h = 2\tau = s$ | 2.47414721E+00 | 1.31276008E+00 | 6.65187210E-01 | 3.33976607E-01 |
| Rate | 0.91433247 | 0.98076654 | 0.99401337 | |
| $h = 4\tau = s$ | 2.47412796E+00 | 1.31274252E+00 | 6.65185856E-01 | 3.33976355E-01 |
| Rate | 0.91432874 | 0.98076198 | 0.99401153 | |

Table 9: Error results for $\max_n \|u(t_n) - u_h^n\|_1$ in Example 2 with $\alpha = 0.9$.

| $n$ | $s_1 = 1/5$ | $s_2 = 1/10$ | $s_3 = 1/20$ | $s_4 = 1/40$ |
|-----|-------------|-------------|-------------|-------------|
| $h = \tau = s$ | 1.59738070E-01 | 3.70003648E-02 | 9.11025687E-03 | 2.24984335E-03 |
| Rate | 2.11009679 | 2.02197586 | 2.01766718 | |
| $h = 2\tau = s$ | 1.59801796E-01 | 3.70596396E-02 | 9.14211637E-03 | 2.26514851E-03 |
| Rate | 2.10836287 | 2.01924876 | 2.01292254 | |
| $h = 4\tau = s$ | 1.59831642E-01 | 3.70873701E-02 | 9.15702243E-03 | 2.2731901E-03 |
| Rate | 2.10755318 | 2.01797751 | 2.01071316 | |

Table 10: Error results for $\max_n \|u(t_n) - u_h^n\|_1$ in Example 2 with $\alpha = 0.9$.

| $n$ | $s_1 = 1/5$ | $s_2 = 1/10$ | $s_3 = 1/20$ | $s_4 = 1/40$ |
|-----|-------------|-------------|-------------|-------------|
| $h = \tau = s$ | 2.47455235E+00 | 1.31293700E+00 | 6.65229920E-01 | 3.3398435E-03 |
| Rate | 0.91436987 | 0.98087273 | 0.99406336 | |
| $h = 2\tau = s$ | 2.47437911E+00 | 1.31287363E+00 | 6.65209212E-01 | 3.33981391E-03 |
| Rate | 0.91435638 | 0.98032476 | 0.99402254 | |
| $h = 4\tau = s$ | 2.47429826E+00 | 1.31282030E+00 | 6.65199619E-01 | 3.33979049E-01 |
| Rate | 0.91434996 | 0.98081021 | 0.99402974 | |
| $\tau = 2h = s$ | 3.68740143E-02 | 9.04237452E-03 | 2.2173715E-03 | 5.3952731E-04 |
| Rate | 2.04987183 | 2.07930388 | 2.13926449 | |
Table 11: Error results for $\max_n \| u(t_n) - u_h^n \|$ with $\tau = 1E - 03$ in Example 2.

| $\alpha$   | $h_1 = \sqrt{2}/10$ | $h_2 = \sqrt{2}/20$ | $h_3 = \sqrt{2}/40$ | $h_4 = \sqrt{2}/80$ |
|------------|---------------------|---------------------|---------------------|---------------------|
| 0.1        | 1.60070131E-01      | 3.71967163E-02      | 9.19392661E-03      | 2.28468228E-03      |
| Rate       | 2.20545696          | 2.01642221          | 2.00868760          |                     |
| 0.3        | 1.60018261E-01      | 3.71759572E-02      | 9.18810334E-03      | 2.28319506E-03      |
| Rate       | 2.10579476          | 2.01653090          | 2.00871296          |                     |
| 0.5        | 1.59963911E-01      | 3.71541871E-02      | 9.18198205E-03      | 2.28161733E-03      |
| Rate       | 2.10614975          | 2.01664729          | 2.00874877          |                     |
| 0.7        | 1.59909339E-01      | 3.71322390E-02      | 9.17573371E-03      | 2.27992781E-03      |
| Rate       | 2.10650999          | 2.01677688          | 2.00883538          |                     |
| 0.9        | 1.59857457E-01      | 3.71109600E-02      | 9.16926891E-03      | 2.27776720E-03      |
| Rate       | 2.10686882          | 2.01696671          | 2.00918640          |                     |

Table 12: Error results for $\max_n \| u(t_n) - u_h^n \|_1$ with $\tau = 1E - 03$ in Example 2.

| $\alpha$   | $h_1 = \sqrt{2}/10$ | $h_2 = \sqrt{2}/20$ | $h_3 = \sqrt{2}/40$ | $h_4 = \sqrt{2}/80$ |
|------------|---------------------|---------------------|---------------------|---------------------|
| 0.1        | 2.47400754E+00      | 1.31270311E+00      | 6.65178971E-01      | 3.33975417E-01      |
| Rate       | 0.91430923          | 0.98072620          | 0.99400065          |                     |
| 0.3        | 2.47406090E+00      | 1.31272376E+00      | 6.65181964E-01      | 3.33975808E-01      |
| Rate       | 0.91431766          | 0.98074240          | 0.99400954          |                     |
| 0.5        | 2.47411714E+00      | 1.31274548E+00      | 6.65185122E-01      | 3.33976225E-01      |
| Rate       | 0.91432659          | 0.98075942          | 0.99401050          |                     |
| 0.7        | 2.47417394E+00      | 1.31276744E+00      | 6.65188363E-01      | 3.33976679E-01      |
| Rate       | 0.91433557          | 0.98077652          | 0.99401557          |                     |
| 0.9        | 2.47422847E+00      | 1.31278888E+00      | 6.65191782E-01      | 3.33977288E-01      |
| Rate       | 0.91434381          | 0.98079267          | 0.99402035          |                     |

fractional reaction-diffusion equations with the Caputo fractional derivative in two-dimensional space regions is feasible and effective.

7 Conclusions

We apply the FVE methods based on the $L1$-formula to solve the time fractional reaction-diffusion equations with the Caputo fractional derivative. We construct the fully discrete FVE scheme, give the existence and uniqueness analysis, and derive the stability results which are only depend on the initial data $u_0(x)$ and the source term function $f(x,t)$, where the stability result with $H^1(\Omega)$-norm need to satisfy the condition $h \leq c_0\tau \leq c_0\tau_0$ and $h \leq h_1$. We also obtain the optimal a priori error estimates in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm by using the properties of the operator $I_h^\alpha$ and some important lemmas. Moreover, we give two numerical examples, and find that the convergence can still be satisfied in the actual calculation, even if the condition $h \leq c_0\tau$ is not established.

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