Spin-Flavor Oscillations of Dirac Neutrinos Described by Relativistic Quantum Mechanics*

M. S. Dvornikov**

Pushkov Institute of Terrestrial Magnetism, Ionosphere, and Radiowave Propagation, Russian Academy of Sciences, Troitsk, Moscow oblast, 142190 Russia; Inst. de Fisica, Univ. de Sao Paulo (IFUSP), BR-05389-970 Sao Paulo, Brasil

Received March 16, 2011; in final form, June 24, 2011

Abstract—Spin–flavor oscillations of Dirac neutrinos in matter and a magnetic field are studied using the method of relativistic quantum mechanics. Using the exact solution of the wave equation for a massive neutrino, taking into account external fields, the effective Hamiltonian governing neutrino spin–flavor oscillations is derived. Then the The consistency of our approach with the commonly used quantum mechanical method is demonstrated. The obtained correction to the usual effective Hamiltonian results in the appearance of the new resonance in neutrino oscillations. Applications to spin–flavor neutrino oscillations in an expanding envelope of a supernova are discussed. In particular, transitions between right-polarized electron neutrinos and additional sterile neutrinos are studied for realistic background matter and magnetic field distributions. The influence of other factors such as the longitudinal magnetic field, the matter polarization, and the non-standard contributions to the neutrino effective potential, is also analyzed.

DOI: 10.1134/S1063778812020068

1. INTRODUCTION

It was confirmed by numerous experiments that neutrinos are massive particles and there is a mixing between different neutrino generations. Besides these non-standard model neutrino properties it is believed that neutrinos can also have non-zero diagonal and transition magnetic moments. The latter mix both eigenstates with opposite helicities as well as different neutrino flavors. Thus the existence of transition magnetic moments implies that in presence of an external electromagnetic field conversions of left-polarized neutrinos into their right-polarized counterparts of another flavor are possible. This process is known as neutrino spin–flavor oscillations [1, 2]: \( \nu_\alpha^- \leftrightarrow \nu_\beta^+ \), where \( \alpha \neq \beta \) and the index \( \pm \) corresponds to the different projections of the neutrino spin to the particle momentum.

In the present work we will consider spin–flavor oscillations of Dirac neutrinos in dense matter under the influence of a strong external magnetic field. Although the existence of Majorana neutrinos is favored in various neutrino mass generation models, like seesaw mechanism [3], the question whether neutrinos are Dirac or Majorana particles is still open [4].

Note that Dirac and Majorana neutrinos have completely different structure of the magnetic moments matrix. Dirac neutrinos can have both diagonal and transition magnetic moments whereas Majorana neutrinos only transition ones (see, e.g., Ref. [5]). Moreover the magnetic moments matrix is symmetric in the Dirac case and anti-symmetric for Majorana neutrinos [5].

The strongest laboratory constraint on the neutrino effective magnetic moment, obtained by the GEMMA collaboration [6], is \( 3.2 \times 10^{-11} \mu_B \), where \( \mu_B \) is the Bohr magneton. Slightly weaker upper bound on the magnetic moments of neutrinos was obtained by the BOREXINO collaboration [7]. Note that the astrophysical limits on the magnetic moments of Dirac and Majorana neutrinos extracted from the white dwarfs cooling rate [8] can be even stronger than the laboratory ones. Therefore in most realistic situations, where the magnetic field is not too strong, neutrino spin and spin–flavor oscillations are likely to play a sub-leading role [9]. Nevertheless in some astrophysical media the dynamics of neutrino oscillations can be significantly affected by a magnetic field because of its interaction with neutrino magnetic moments. For example, during a supernova explosion or in the vicinity of a neutron star magnetic fields can reach values up to \( 10^{16} \) G [10], which is enough to influence the neutrino oscillations process in a significant way.
Neutrino spin and spin-flavor oscillations in a supernova were discussed in Refs. [2, 11]. The impact of neutrino magnetic moments and spin-flavor oscillations on $r$-process nucleosynthesis during a supernova explosion was studied in Ref. [12]. The neutrino spin flip, i.e. the transformation like $\nu_{\alpha} \leftrightarrow \nu_{\beta}$ within the same flavor, can happen not only because of the magnetic moment interaction with an external magnetic field, but also in collisions with background particles during a supernova explosion, that was predicted in Ref. [13].

In the present work we study neutrino spin-flavor oscillations in frames of the relativistic quantum mechanics [14–18]. Within the developed formalism we find the wave functions of neutrino mass eigenstates, which are the superposition of flavor neutrinos, for a given initial condition (Sec. 2). Using this method for the description of the neutrino evolution one can exactly take into account neutrino masses and the influence of external fields since we use exact solutions of the Dirac equation for massive neutrinos. Note that neutrino oscillations in dense matter and strong magnetic fields can also be described with help of the methods of finite temperature field theory [19].

Then, in Sec. 3, we analyze the quantum mechanical approach to the description of neutrino spin-flavor oscillations. In particular, we examine neutrino oscillations in various bases of wave functions and demonstrate the equivalence of the method, based on relativistic quantum mechanics, to the standard approach for the treatment of neutrino oscillations. Moreover we obtain the correction to the standard effective Hamiltonian and show that it results in the appearance of the new resonance in neutrino oscillations.

In Sec. 4, we discuss a possible application of our results to the conversion of right-polarized electron neutrinos, which can be created in a supernova explosion [13], into a quasi-degenerate in mass sterile neutrino when particles propagate in an expanding envelope of a supernova and interact with its magnetic field. The importance of other factors, which can also influence the neutrino oscillations process, is considered in Sec. 5. Finally, in Sec. 6, we summarize our results.

2. RELATIVISTIC QUANTUM MECHANICS DESCRIPTION

Let us study the time evolution of the system of two mixed flavor neutrinos ($\nu_\alpha, \nu_\beta$) in matter and in an external electromagnetic field $F_{\mu \nu} = (E, B)$. The indexes $\alpha$ and $\beta$ can stand for any of the neutrino flavors: $e, \mu$ or $\tau$. The interaction with background matter can be represented in terms of the external axial-vector fields $f^{\mu}_{\lambda \gamma}$. The Lagrangian for our system has the form,

$$L = \sum_{\lambda = \alpha, \beta} \bar{\nu}_\lambda i \gamma^\mu \partial_\mu \nu_\lambda$$

$$- \sum_{\lambda \lambda' = \alpha, \beta} \bar{\nu}_\lambda \left( m_{\lambda \lambda'} + \gamma^\mu f^{\mu}_{\lambda \lambda'} + \frac{1}{2} M_{\lambda \lambda'} \sigma_{\mu \nu} F^{\mu \nu} \right) \nu_{\lambda'},$$

where $(m_{\lambda \lambda'})$ is the non-diagonal mass matrix and $(M_{\lambda \lambda'})$ is the matrix of the neutrino magnetic moments. Note that in general case the matrices $(m_{\lambda \lambda'})$ and $(M_{\lambda \lambda'})$ are independent, i.e. the diagonal form of the matrix $(m_{\lambda \lambda'})$ in a certain basis does not imply that of the matrix $(M_{\lambda \lambda'})$.

In the case of the standard model neutrino interactions the matrix $(f^{\mu}_{\lambda \lambda'})$ is diagonal: $f^{\mu}_{\lambda \lambda'} = f^{\mu}_{\lambda \lambda} \delta_{\lambda \lambda'}$, where $f^{\mu}_{\lambda \lambda} = (f_0^\lambda, f_\lambda)$. The possible nondiagonal elements of the matrix $f^{\mu}_{\lambda \lambda}$, with $\lambda \neq \lambda'$, correspond to the non-standard neutrino interactions with matter [16]. In the following we will study neutrino oscillations in non-moving and unpolarized matter with $f_\lambda = 0$. The explicit form of the zero-th component $f_0^\lambda \equiv f_\lambda$ of the four vector $f_\lambda$ for the electoneutral medium consisting of electrons, protons and neutrons can be found in Ref. [20]. In the case of sterile neutrinos $f_{\nu_\alpha} = 0$.

To study the time evolution of the system (1) it is necessary to formulate the initial condition for the flavor neutrinos $\nu_\lambda$. We suppose that only one neutrino flavor, e.g., $\beta$, is present initially (see Refs. [14–18]), i.e. $\nu_\alpha(r, 0) = 0$, and $\nu_\beta(r, 0) = \nu_\beta^{(0)}(r)$, where $\nu_\beta^{(0)}(r)$ is the given function. Then we will look for the wave function $\nu_\beta$ at $t > 0$. If we choose $\alpha = \mu$ or $\tau$ and $\beta = e$, it will correspond to the typical situation of neutrinos emitted in the Sun: for the given initial flux of electron neutrinos we study the presence of other neutrino flavors at subsequent moments of time. Note that the analogous initial condition problem for neutrino wave packets was studied in Ref. [21].

To analytically study the evolution of the system (1) in presence of external fields we will consider the case of the coordinate independent functions $f_\lambda$ and $F_{\mu \nu}$. The analysis of the validity of this approximation will be made in Sec. 5. For the coordinate independent external fields the momentum of the particles $p$ is conserved.

Moreover, the additional constraint can be imposed on the wave function $\nu_\beta^{(0)}(r)$,

$$P_{\pm} \nu_{\beta}^{(0)}(r) = \nu_{\beta}^{(0)}(r), \quad P_{\pm} = \left( 1 \pm \frac{\Sigma \cdot p}{|p|} \right),$$

where $\Sigma = -\gamma^0 \gamma^5 \gamma$ are the Dirac matrices. Eq. (2) implies that initially neutrinos of the flavor $\beta$ have a
certain helicity. If we act with the operator \( P_\pm \) on the final state \( \nu_\alpha(r, t) \), we can study the appearance of the opposite helicity eigenstates among neutrinos of the flavor \( \alpha \), i.e. this situation corresponds to the neutrino spin-flavor oscillations \( \nu_\beta \Leftrightarrow \nu_{\beta'} \).

Note that, unlike the helicity states defined in Eq. (2), the chirality \( \kappa \) of a particle is the eigenvalue of the matrix \( \gamma^5 \): \( \gamma^5 \psi = \kappa \psi \). In case of massive Dirac particles the helicity and the chirality do not coincide [22]. The helicity is a conserving number whereas the chirality—not, since the operator \( \gamma^5 \) does not commute with a Hamiltonian.

Let us introduce the neutrino mass eigenstates \( \psi_a \), \( a = 1, 2 \), as

\[
\nu_\lambda = \sum_{a=1,2} U_{\lambda a} \psi_a, \quad (U_{\lambda a}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},
\]

(3)

to diagonalize the mass matrix \((m_{\lambda\lambda'})\). In Eq. (3) we take into account that for the two neutrinos system the mixing matrix \((U_{\lambda a})\) can be parameterized with help of the one vacuum mixing angle \( \theta \). We suppose that the mass eigenstates \( \psi_a \) are Dirac particles with the masses \( m_a \).

The Lagrangian (1) expressed in terms of the fields \( \psi_a \) takes the form,

\[
\mathcal{L} = \sum_{a=1,2} \bar{\psi}_a (i \gamma^\mu \partial_\mu - m_a) \psi_a - \sum_{a,b=1,2} \bar{\psi}_a \left( \gamma^L g_{ab}^\mu + \frac{1}{2} \mu_{ab} \sigma_{\mu\nu} F^{\mu\nu} \right) \psi_b,
\]

(4)

where

\[
(g_{ab}) = U^\dagger (f_{\lambda\lambda'}) U = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}, \quad \mu_{ab} = U^\dagger (M_{\lambda\lambda'}) U = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_1 \end{pmatrix},
\]

(5)

are the nondiagonal matrices of neutrino magnetic moments and the neutrino interaction with matter in the mass eigenstates basis. In Eq. (5) we take into account that background matter is non-moving and unpolarized, i.e. only the zero-th component \( g_{ab} = g_{ab}^0 \) of the four vector \( g_{ab}^0 \) of the four vector \( g_{ab}^0 \) survives.

To discuss the time evolution of the system (4) we write down the wave equations which result from Eq. (4),

\[
i \dot{\psi}_a = \mathcal{H}_a \psi_a + V \psi_b, \quad a = 1, 2, \quad a \neq b,
\]

(6)

\[
\mathcal{H}_a = (\mathbf{I} \mathbf{p}) + \beta m_a - \mu_a \beta \Sigma B + g_a (1 - \gamma^5)/2,
\]

\[
V = -\mu \beta \Sigma B + g (1 - \gamma^5)/2,
\]

where \( \alpha = \gamma^0 \gamma \) and \( \beta = \gamma^0 \) are the Dirac matrices. Here we study the neutrino motion along the \( x \)-axis in only transversal magnetic field: \( B = (0, 0, B) \) and \( E = 0 \).

Note that we cannot directly solve the wave equations (6) because of the nondiagonal interaction \( V \) which mixes different mass eigenstates. In vacuum, i.e. when \((\mu_{ab}') = 0\) and \( F_{\mu\nu} = 0\), the mass eigenstates \( \psi_{1,2} \) decouple and the system (6) can be easily solved. Nevertheless we can point out an exact solution of the wave equation \( i \dot{\psi}_a = \mathcal{H}_a \psi_a \), for a single mass eigenstate \( \psi_a \), that exactly accounts for the influence of the external fields.

We look for the solution of Eq. (6) in the following form [14–18]:

\[
\psi_a(r, t) = e^{-i E_a t/2} \int \frac{d^3 p}{(2\pi)^{3/2}} e^{i p r} \sum_{\zeta = \pm 1} \left[ a_\zeta^{(\mathcal{C})}(t) u_\zeta^{(\mathcal{C})} \exp (-i E_\zeta^{(\mathcal{C})} t) + \right. \\
\left. + b_\zeta^{(\mathcal{C})}(t) v_\zeta^{(\mathcal{C})} \exp (i E_\zeta^{(\mathcal{C})} t) \right],
\]

(7)

where the energy levels, which were found in Ref. [17], have the form,

\[
E_\zeta^{(\mathcal{C})} = \sqrt{M_a^2 + \mu_3^2 + \mu_1^2 + \mu_2^2 + 2 \zeta R_2^2},
\]

(8)

where \( R_2^2 = \sqrt{\mu_3^2 + (\mu_1 B)_3^2 + (\mu_2 B)_3^2} \) and \( M_a = \sqrt{(\mu_1 B)_3^2 + (\mu_2 B)_3^2} \). The basis spinors can be found in the limit of a small neutrino mass [17],

\[
\begin{pmatrix} u_\zeta^{(\mathcal{C})} \\ v_\zeta^{(\mathcal{C})} \end{pmatrix} = \frac{1}{2 \sqrt{2 M_a (M_a - \zeta g_0/2)}} \begin{pmatrix} \mu_1 B + \zeta M_a - g_0/2 \\ \mu_2 B - \zeta M_a + g_0/2 \end{pmatrix},
\]

(9)

\[
\begin{pmatrix} u_\zeta^{(\mathcal{C})} \\ v_\zeta^{(\mathcal{C})} \end{pmatrix} = \frac{1}{2 \sqrt{2 M_a (M_a + \zeta g_0/2)}} \begin{pmatrix} M_a - \zeta [\mu_1 B - g_0/2] \\ M_a + \zeta [\mu_1 B + g_0/2] \end{pmatrix}.
\]

It should be noted that the discrete quantum number \( \zeta = \pm 1 \) in Eqs. (7)–(9) does not coincide with the helicity.
Now our goal is to find the time dependent coefficients $a_{a}^{(C)}(t)$ and $b_{b}^{(C)}(t)$ since the mixing potential $V$ is present in Eq. (6). In the case of neutrino propagation in vacuum these coefficients do not depend on time and their values are completely defined by the initial condition. If we put the ansatz (7) in the wave equations (6), we get the following ordinary differential equation for the function $a_{a}^{(C)}(t)$:

$$i\dot{a}_{a}^{(C)} = e^{i(g_{a} - g_{b})t/2} \exp(iE_{a}^{(C)}t)u_{a}^{(C)}t V \times \sum_{\zeta = \pm 1} \left[ a_{b}^{(C)}(t) u_{b}^{(C)} \exp(-iE_{b}^{(C)}t) + \bar{a}_{b}^{(C)}(t) u_{b}^{(C)} \exp(iE_{b}^{(C)}t) \right].$$

To obtain Eq. (10) we use the orthonormality of the basis spinors (9). Note that the differential equation for the function $b_{b}^{(C)}$ is analogous to Eq. (10) and thus omitted. Moreover, taking into account the fact that $\langle a_{a}^{(C)} | V | b_{b}^{(C)} \rangle = 0$, we get that the equations for $a_{a}^{(C)}(t)$ and $b_{b}^{(C)}(t)$ decouple, i.e. the interaction $V$ does not mix positive and negative frequency eigenstates.

Let us rewrite Eq. (10) in a more conventional effective Hamiltonian form. For this purpose we introduce the “wave function” $Ψ^{T} = (a_{1}^{-}, a_{2}^{-}, a_{1}^{+}, a_{2}^{+})$. Directly from Eq. (10) we derive the equation for $Ψ'$,

$$i\frac{dΨ'}{dt} = H'Ψ',$$

$$H' = \begin{pmatrix}
0 & h_{-}e^{i\omega_{-}t} & 0 & H_{-}e^{i\Omega_{-}t} \\
0 & h_{+}e^{-i\omega_{+}t} & 0 & 0 \\
0 & H_{-}e^{-i\Omega_{-}t} & 0 & h_{+}e^{i\omega_{+}t} \\
H_{+}e^{-i\Omega_{+}t} & 0 & 0 & H_{-}e^{i\omega_{-}t}
\end{pmatrix},$$

where $h_{\pm} = \langle u_{a}^{\pm} | V | u_{b}^{\pm} \rangle$, $H_{\pm} = \langle u_{b}^{\pm} | V | u_{b}^{\pm} \rangle$, $\omega_{\pm} = E_{1}^{\pm} - E_{2}^{\pm} + (g_{1} - g_{2})/2$, and $\Omega_{\pm} = E_{1}^{\pm} - E_{2}^{\pm} + (g_{1} - g_{2})/2$. Note that we do not give here the explicit form of the scalar products $h_{\pm}$ and $H_{\pm}$ in order not to encumber the text.

Instead of $Ψ'$ it is more convenient to use the transformed “wave function” $Ψ$ defined by $Ψ = UΨ'$, where

$$U = \text{diag}\{e^{i(\Omega_{-} - \Omega_{+})t/2}, e^{i(\Omega_{-} + \Omega_{+})t/2}, e^{-i(\Omega_{-} - \Omega_{+})t/2}, e^{-i(\Omega_{-} + \Omega_{+})t/2}\},$$

and $Ω = (Ω_{-} - Ω_{+})/2$. Using the property $ω_{+} + ω_{-} = Ω_{+} + Ω_{-}$, we arrive to the new Schrödinger equation for the “wave function” $Ψ$,

$$i\frac{dΨ}{dt} = HΨ,$

$$H = UH'U^{-1} - iU\dot{U} = \begin{pmatrix}
(\Omega + ω_{-})/2 & h_{-} & 0 & H_{-} \\
h_{-} & (\Omega - ω_{-})/2 & H_{+} & 0 \\
0 & H_{+} & -(Ω - ω_{+})/2 & h_{+} \\
H_{-} & 0 & h_{+} & -(Ω + ω_{+})/2
\end{pmatrix}. \quad (12)$$

$$× \left[ \frac{ψ_{1}(t)}{\sqrt{M_{1} + g_{1}/2}} + \frac{ψ_{3}(t)}{\sqrt{M_{1} - g_{1}/2}} \right]^{2}. \quad (13)$$

To obtain Eq. (13) we suppose that initially we have rather broad spatial wave packet, corresponding to the initial condition $ν_{β}^{(0)}(r) \sim e^{iπr}$. Eqs. (12) and (13) are completely new. However, in Sec. 3, we will show that the relativistic quantum mechanics approach to the description of neutrino spin-flavor oscillations is consistent with the conventional quantum mechanical approach for the case of ultrarelativistic neutrinos.
3. QUANTUM MECHANICAL DESCRIPTION

In this section we analyze spin-flavor oscillations of Dirac neutrinos in frames of the standard quantum mechanical approach. The main concept of this approach is the construction of an effective Hamiltonian acting in the space of quantum mechanical neutrino “wave functions”. Thus the proper choice of the basis of wave functions is as important as the correct form of the effective Hamiltonian. In the majority of works devoted to neutrino spin-flavor oscillations the basis of helicity eigenstates was adopted (see, e.g., Ref. [2]). As we will see below, this choice is justified only in the case of a relatively weak external magnetic field and a dense matter. We also consider the quantum mechanical derivation of the effective Hamiltonian in the opposite situation of a strong magnetic field and a low density matter. Finally we demonstrate the consistency of the relativistic quantum mechanics approach, developed in Sec. (2), to the standard quantum mechanical treatment.

As a rule the Schrödinger equation which describes the dynamics of the Dirac neutrinos system was formulated in the flavor eigenstates basis [1, 2]. Definitely it is more convenient to use the flavor eigenstates basis since one gets the transition probability directly from the solution of the Schrödinger equation without any additional matrix transformation like (3). Nevertheless we will formulate the dynamics of the neutrinos system in the mass eigenstates basis since the energies are well defined only for the neutrino mass eigenstates and one can distinguish the nature of neutrinos, i.e. say whether neutrinos are Dirac or Majorana particles, only in this basis.

First let us discuss the situation when a neutrino moves in a sufficiently dense matter and interacts with a weak magnetic field. The opposite case of a strong magnetic field and a low density medium will be considered later. If the matter density is great, then in Eq. (6) the averaged interaction of the diagonal magnetic moment with an external magnetic field is less than the averaged diagonal interaction with background matter: \( \langle \mu_a \beta \Sigma_3 B \rangle \ll (g_a (1 - \gamma^2)/2). \) In this approximation it is convenient to rewrite the diagonal part of the Hamiltonian in Eq. (6) as

\[
\mathcal{H}_a \rightarrow \mathcal{H}_a + \mathcal{V}_a, \tag{14}
\]

\[
\mathcal{H}_a = (\mathbf{\alpha p} + \beta m_a + g_a (1 - \gamma^2)/2, \hspace{1cm} \mathcal{V}_a = -\mu_a \beta \Sigma_3 B, \hspace{1cm}
\]

to extract the small interaction \( \mathcal{V}_a. \) We also rewrite the nondiagonal interaction \( V \) in Eq. (6) in the following form: \( V = V_B + V_m, \) where \( V_B = -\mu \beta \Sigma_3 B \) and \( V_m = g(1 - \gamma^2)/2, \) to separate the nondiagonal magnetic and matter interactions.

One can notice that now the helicity operator defined in Eq. (2) commutes with the modified Hamiltonian \( \mathcal{H}_a. \) Therefore one can choose the helicity eigenstates basis for the quantum mechanical “wave functions” instead of the more complete set of basis functions (9). If we study neutrinos propagating along the \( x \)-axis, these basis functions should satisfy the condition, \( (1/2)(1 \pm \Sigma_1) u^\pm_a = \pm u^\pm_a. \) The explicit form of these spinors can be found in Refs. [16, 23]. Note that we do not take into account the negative energy spinors \( u^{(c)}_a \) since in Sec. 2 we demonstrated that the evolution equations (10) for \( a^{(c)}_a \) and \( b^{(c)}_a \) decouple in matter and a magnetic field.

To construct the effective Hamiltonian \( H_1, \) which governs the dynamics of the “wave function” \( \Psi^f = (\psi^f_1, \psi^f_2, \psi^f_3, \psi^f_4), \) where \( \psi^f_\alpha \) are the time dependent \( e \)-number wave functions corresponding to a definite helicity, we notice that the diagonal elements of \( H_1 \) can be calculated as the mean values of \( \mathcal{H}_a \) over the states \( u^{(c)}_a, \) i.e. they are equal to the energies of a neutrino moving only in background matter [16, 23],

\[
E_a^- = p + g_a + \frac{m^2_a}{2p} + \ldots, \tag{15}
\]

\[
E_a^+ = p + \frac{m^2_a}{2p} + \ldots,
\]

where we use the limit of ultrarelativistic particles. The nondiagonal elements of the matrix \( H_1 \) are the mean values of the operators \( V_a, V_B \) and \( V_m \) over the same helicity eigenstates \( u^{(c)}_a. \) Finally we arrive to the effective Hamiltonian in the ultrarelativistic limit,

\[
H_1 = \begin{pmatrix} \Phi + g_1 & g & -\mu_1 B & -\mu B \\ g & -\Phi + g_2 & -\mu B & -\mu_2 B \\ -\mu_1 B & -\mu B & \Phi & 0 \\ -\mu B & -\mu_2 B & 0 & -\Phi \end{pmatrix}, \tag{16}
\]

where \( \Phi = \delta m^2/4p \) is the phase of vacuum oscillations and \( \delta m^2 = m^2_2 - m^2_1. \)

One can see that in Eq. (16) we have reproduced the standard effective Hamiltonian proposed in Ref. [2] to study spin-flavor oscillations of Dirac neutrinos. In our analysis it is important that the helicity eigenstates are used as the basis functions. It is correct only if the diagonal magnetic interaction \( \langle \mu_a \beta \Sigma_3 B \rangle \) is small. Despite this basis is widely used in the studies of neutrino spin-flavor oscillations (see, e.g., Ref. [24]), the correct form of the effective Hamiltonian (16) is obtained for ultrarelativistic neutrinos in case of the small diagonal magnetic interaction.

Now we discuss the opposite situation when a neutrino interacts with a very strong magnetic field...
and moves in a low density matter. Hence in Eq. (6) the diagonal magnetic interaction is much bigger than the diagonal interaction with background matter: \( \langle \mu_a \beta \Sigma_3 B \rangle \gg \langle g_a (1 - \gamma^5)/2 \rangle \). In this case it is also convenient to redefine the diagonal Hamiltonian in Eq. (6) in the following way:

\[
\mathcal{H}_a \to \mathcal{H}_a + \mathcal{V}_a, \\
\mathcal{H}_a = (\alpha p) + \beta m_a - \mu_a \beta \Sigma_3 B, \\
\mathcal{V}_a = g_a (1 - \gamma^5)/2.
\]

Note that now the Hamiltonian \( \mathcal{H}_a \) does not conserve the helicity of a particle. Thus as the basis wave functions we may choose the eigenvectors of the operator \( \Pi_a = m_a \Sigma_3 + i \gamma^0 \gamma^5 (\Sigma \times p)_3 - \mu_a B \), which characterizes the spin direction with respect to the magnetic field. The explicit form of these spinors is given in Ref. [18].

The effective Hamiltonian \( H_2 \) acting in the space of the quantum mechanical neutrino “wave functions” \( \Psi^T_2 = (\psi^+_1, \psi^+_2, \psi^-_1, \psi^-_2) \) can be constructed in a straightforward way as in the previous case. Here \( \psi^\pm_a \) are the time dependent \( \epsilon \)-number functions representing neutrino states with a definite spin projection on the magnetic field direction. However we can notice that the Hamiltonian \( H_2 \) can be obtained by the similarity transformation, \( H_2 = \mathcal{V}_{21} \mathcal{H}_1 \mathcal{V}_{21}^T \), with the orthogonal matrix \( \mathcal{V}_{21} = (\gamma^0 - \gamma^3)/\sqrt{2} \), with the Dirac matrices \( \gamma^0 \) and \( \gamma^3 \) taken in the standard representation [25]. It should be noted that the direct calculation of the Hamiltonian \( H_2 \) gives the same result.

Despite we demonstrated the consistency of the descriptions of neutrino oscillations in different bases, there is a conceptual discrepancy between these cases. The quantum mechanical treatment of neutrino evolution requires the correct choice of both the effective Hamiltonian and the basis of “wave functions” where this Hamiltonian acts. The quantum mechanical “wave functions” \( \Psi_{1,2} \) are applicable to the description of neutrino evolution for the cases of weak and strong diagonal magnetic interaction respectively since they correspond to different conserved quantum numbers in each case. Although for ultrarelativistic neutrinos the description of neutrino oscillations is identical in these bases, there can be a difference if we calculate the corrections to the leading term.

Now let us check the consistency between the relativistic quantum mechanics method, developed in Sec. 2, and the standard quantum mechanical description of neutrino oscillations. In the approximation of ultrarelativistic neutrinos the decomposition of the energy levels (8) takes the form,

\[
E_a^{(\zeta)} = p + \frac{g_a}{2} - \zeta M_a + \frac{m_a^2}{2p} + \zeta \frac{m_2^2 g_a^2}{8p^2 M_a} + \ldots,
\]

where we keep the term \( \sim m_2^2/p^2 \) to examine the corrections to the leading term.

Using the orthogonal matrix \( \mathcal{V} \) of the following form:

\[
\mathcal{V} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-\sqrt{1 + g_1/2M_1} & 0 & \mu_1 B/\sqrt{M_1(M_1 + g_1/2)} & 0 \\
0 & -\sqrt{1 + g_2/2M_2} & 0 & \mu_2 B/\sqrt{M_2(M_2 + g_2/2)} \\
\sqrt{1 - g_1/2M_1} & 0 & \mu_1 B/\sqrt{M_1(M_1 - g_1/2)} & 0 \\
0 & \sqrt{1 - g_2/2M_2} & 0 & \mu_2 B/\sqrt{M_2(M_2 - g_2/2)}
\end{pmatrix},
\]

we can transform the effective Hamiltonian \( H \) in Eq. (12) to \( \mathcal{V}^T H \mathcal{V} \approx H_0 + \delta H \), where \( H_0 = H_1 - I/i \text{Tr}(H_1)/4 \), \( I \) is the \( 4 \times 4 \) unit matrix, and

\[
\delta H = \frac{1}{16p^2} \text{diag} \left( -m_1^2 \frac{g_1^3}{M_1^3}, -m_2^2 \frac{g_2^3}{M_2^3}, m_1^2 \frac{g_1^3}{M_1^3}, m_2^2 \frac{g_2^3}{M_2^3} \right),
\]

is the correction to the standard effective Hamiltonian. It should be noted that the transformation matrix \( \mathcal{V} \) in Eq. (19) depends on the magnetic field strength and the matter density, whereas the matrix \( \mathcal{V}_{21} \) is external fields independent.

The effective Hamiltonian \( H_0 \) is equivalent to \( H_1 \) in Eq. (16) since the unit matrix does not change the particles dynamics. If we omit the correction (20), that is valid for ultrarelativistic neutrinos, we can see that the relativistic quantum mechanics approach is equivalent to the standard quantum mechanical

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method. The correction $\delta H$ results from the fact that in Sec. 2 we use the correct energy levels (8) and the wave functions (9) for a neutrino moving in dense matter and a strong magnetic field.

Note that in Eq. (20) we keep only the diagonal terms since the appearance of resonances in neutrino oscillations is sensitive to the diagonal elements of a Hamiltonian. We should remind that the expressions for the basis spinors (9) were obtained in the approximation of small masses of neutrinos, whereas in the expansion of the energy levels (18) we keep the terms up to $\sim m^2_\nu/p^2$. However, using Eqs. (12) and (19), we get that the additional small contributions $\sim m^2_\nu/p^2$ from the basis spinors are washed out in diagonal entries in Eq. (20).

4. APPLICATIONS OF THE OBTAINED RESULTS

Now let us consider the application of the obtained Hamiltonians (16) and (20) to the description of oscillations between electron and hypothetical sterile neutrinos in an expanding envelope of a supernova. The existence of sterile neutrinos closely degenerate in mass with electron, muon- or tau-neutrinos was recently discussed in Ref. [26] in connection to solar and supernova neutrinos. The mass squared differences considered in these publications were in the following range: $10^{-19} < \delta m^2 < 10^{-12}$ eV$^2$.

On the contrary, the mixing angles of these additional neutrinos cannot be well constraint. Therefore can assume that the vacuum mixing angle is small, $\theta \ll 1$. In this case the mixing matrix between mass, $\psi_a$, $a = 1, \ldots, 4$, and flavor, $\nu_\lambda$, $\lambda = e, \mu, \tau, s$, neutrino eigenstates has the form $U_3 \approx \text{diag}(U_3, 1)$, where $U_3$ is the mixing matrix of the three Dirac neutrinos system (see Eq. (3) and Ref. [27]). It should be noted that it is very difficult to detect additional neutrino flavor if it is weakly mixed with active neutrinos and $\delta m^2$ is small. Such a neutrino can be revealed through spin-flavor oscillations only if it has a transition magnetic moment.

Besides the huge amount of left-polarized neutrinos from a supernova, a smaller flux of right-polarized particles is predicted [13]. These right-polarized neutrinos can be created in the following reaction: $\nu_e^- (p, N) \rightarrow \nu_e^- (p, N)$, with electrons $e^-$, protons $p$, and nuclei $N$ in the dense matter of a proto-neutron star. For the neutrino spin-flip in matter to happen within one generation $\nu_e^- \rightarrow \nu_e^+$, a neutrino should be a Dirac particle with a nonzero diagonal magnetic moment. Left-polarized supernova neutrinos are strongly degenerate and can occupy energy levels above the Fermi surface. Right-polarized neutrinos created in the spin-flip reactions have the energy of the order of the left-polarized neutrinos. The detailed analysis of Ref. [13] shows that the energy of these particles is in the $E_\nu = 100$–200 MeV range.

The generation of electron neutrinos with right-handed polarization can provide additional supernova cooling since they do not interact with background matter and thus freely carry away the supernova energy. Moreover these particles can be potentially detected in a terrestrial detector owing to their spin precession, back to left-polarized states, in the galactic magnetic field. However, if we point out an additional neutrino oscillations channel, which contributes to the right-polarized neutrinos dynamics, these particles are unlikely to be observed.

Thus let us study the appearance of a resonance in $\nu_e^- \leftrightarrow \nu_s^-$ oscillations channel. It is known that a resonance in a certain channel of neutrino oscillations can appear if the difference between two diagonal elements in the effective Hamiltonian is small [28]. Using Eqs. (16) and (20) in the approximation of $\theta \ll 1$, as well as the results of Ref. [20] we obtain the resonance condition for these oscillations as,

$$\delta m^2 \approx 5.0 \times 10^{-17} \text{eV}^2 \times (3Y_e - 1) \quad (21)$$

where $Y_e = n_e/(n_e + n_n)$ is the electrons fraction, $\rho$ is the mass density of background matter, and $m_\nu$ is the absolute mass of a neutrino. Since $\delta m^2$ is supposed to be small, we can set $m_\nu$ to be equal to either $m_1$ or the mass the additional neutrino mass eigenstate appearing due to the presence of a sterile neutrino. In Eq. (21) we suppose that matter is electroneutral and the diagonal magnetic moment of the active neutrino is small: $\mu_1 B \ll f_e$.

Suppose that the flux of right-polarized electron neutrinos is crossing an expanding envelope of a supernova, where a shock wave is usually formed. Approximately 1 s after the core collapse, the matter density in the shock wave region $L \approx 10^8$ cm can be up to $10^6$ g/cm$^3$ (see Ref. [29] and references therein). We can also suppose that the matter density is approximately constant inside the shock wave. From Eq. (21) we can see that a resonance in neutrino oscillations happens if $\rho \approx 10^6$ g/cm$^3$, $Y_e > 1/3$, $E_\nu \approx 100$ MeV, $m_\nu \approx 0.2$ eV, and $\delta m^2 \approx 2 \times 10^{-18}$ eV$^2$, that is consistent with the modern cosmological limits on the absolute neutrino mass [30], the mentioned above estimates of the mass squared differences with a sterile neutrino as well as to the parameters of a shock wave.

Besides the fulfillment of the resonance condition (21), to have the significant $\nu_e^+ \leftrightarrow \nu_s^-$ transitions rate the oscillations length should be comparable to
the the shock wave size $L$. We can express this condition in the following form:

$$B \approx 5.3 \times 10^7 \text{ G} \quad \text{(22)}$$

$$\times \left(\frac{\mu}{10^{-12} \mu_B}\right)^{-1} \left(\frac{L}{10^3 \text{ km}}\right)^{-1}.$$  

For the transition magnetic moment $\mu = 3 \times 10^{-12} \mu_B$ [31] and $L \sim 10^8 \text{ cm}$ (see above), we get that $B \sim 10^7 \text{ G}$. Supposing that the magnetic field of a protoneutron star depends on the distance as $B_{\text{dip}}(r) = B_0 (R/r)^3$, where $R = 10 \text{ km}$ is the typical protoneutron star radius and $B_0 = 10^{13} \text{ G}$ is the magnetic field on the surface of a protoneutron star, we get that at the distance $r = 10^8 \text{ cm}$ the magnetic field reaches $10^7 \text{ G}$, which is consistent with the estimates of Eq. (22). Note that magnetic fields in a supernova explosion can be even stronger than $10^{13} \text{ G}$ and reach the values up to $\sim 10^{16} \text{ G}$ [10].

In Fig. 1a we present the numerical solution of the Schrödinger equation with the Hamiltonians (16) and (20) taking into account the diagonal magnetic moment of an electron neutrino which was neglected while obtaining the resonance conditions. It is necessary to remind that in Ref. [13] it was found that to get a significant $\nu^+_{e}$ luminosity $\sim 10^{50} \text{ erg/s}$ the diagonal magnetic moment should be $\mu_1 = 10^{-13} \mu_B$. We suppose that both Eqs. (21) and (22) are satisfied. One can see in Fig. 1a that diagonal magnetic moments $\mu_1$ do not significantly influence the dynamics of spin-flavor oscillations since the numerical transition probability practically coincide with the approximate analytical expression $P(x) = \sin^2(\mu B x)$ obtained from Eqs. (16) and (20)–(22) in the limit of small $\mu_1$.

We plot Fig. 1a in the approximation of constant external fields. Despite the matter density decreases as $1/r^3$ inside the envelope [32], we can suppose that it is approximately constant in the shock wave region [29]. However the approximation of the constant magnetic field is quite rough. In Fig. 1b we present the solution of the exact Schrödinger equation for the coordinate dependent magnetic field, $B_{\text{dip}}(r)$, for various values of the magnetic fields $B_0$ at the neutrinosphere. We again suppose that the resonance condition (21) is fulfilled.

Note that in Sec. 2 we assumed that the magnetic field is coordinate independent. However, if we suggest that the length of the spatial variation of a magnetic field is much bigger than the typical size of a neutrino wave packet (see also Sec. 5), we can neglect the coordinate dependence of a magnetic field in the analysis of the neutrino wave equation (6). Thus, if this condition is satisfied, the magnetic field is supposed to be spatially constant in deriving of main Eqs. (12), (16), and (20) in Secs. 2 and 3. Nevertheless we take into account the magnetic field variation while solving the effective Schrödinger equation with the Hamiltonians (16) and (20). This requirement for spatial variations of magnetic fields can be characterized as the microscopic adiabaticity condition in contrast to the macroscopic adiabaticity typically used in the analysis of neutrino oscillations (see, e.g., Ref. [33]).

One can see in Fig. 1b that at relatively weak magnetic fields $B_0 \sim 10^{13} \text{ G}$ the transition probability is a strictly increasing function reaching the asymptotic value $\sim 0.5$ and never is equal to one. This magnetic field corresponds to $B_{\text{dip}}(r = R_{\text{int}}) = 10^7 \text{ G}$, where $R_{\text{int}} = 10^8 \text{ cm}$ is the internal radius of an expanding envelope. We should remind that oscillations in a constant magnetic field of such a strength are at resonance, cf. Eq. (22), and hence the transition probability can reach a unit value.

We also notice that at the big distances traveled by neutrinos the transition probability becomes constant. Indeed, using Eqs. (16) and (20) we get that in the limit of the small diagonal magnetic moment of an electron neutrino and if the resonance condition (21) is fulfilled, at big distances $x \gg R_{\text{int}}$ the transition probability approaches to $\sin^2(\mu B_0 R^3/2R_{\text{int}}^2)$. The values of the asymptotic transition probability shown in Fig. 1 are in agreement with this expression. At strong magnetic fields ($B_0 = 10^{14} \text{ G}$ or $B_{\text{dip}}(r = R_{\text{int}}) = 10^8 \text{ G}$) the transition probability can reach big values at the outer edge of a broad envelope, $r \sim 10^9 \text{ cm}$. Note that analogous behaviour of transition probability was found in Ref. [34] while studying spin-flavor oscillations of Majorana neutrinos in a supernova.

5. ANALYSIS OF APPROXIMATIONS

First we should remind that we use the relativistic quantum mechanics approach, with the external fields being independent of spatial coordinates. If external fields depend on the spatial coordinates, the Dirac wave packets theory reveals various additional phenomena such as pairs creation by the external field inhomogeneity [35]. For the approximation of the spatially constant external fields to be valid, the typical length scale of the external field variation $L_{\text{ext}}$ should be much greater than the Compton length of a neutrino [35]: $L_{\text{ext}} \gg \lambda_c = h/m_\nu c$ [20]. For a particle with $m_\nu \sim 0.2 \text{ eV}$ this condition reads $L_{\text{ext}} \gg 10^{-4} \text{ cm}$, that is fulfilled for almost all realistic external fields.

We should also make a remark on the accounting for the mixing potential $V$ in Eq. (10). Eq. (6)
contains the nondiagonal term $\sim V$. The analysis of analogous equations is typically made in frames of the perturbation theory [36] using the expansion on powers of the mixing potential, i.e. on the coupling constants which are proportional to $g$ and $\mu B$. As it was mentioned in Sec. 2, the wave equations for massive neutrinos in vacuum decouple and the evolution of these states depends on the initial condition only. While solving Eq. (10) we could have taken into account the terms linear in $g$ and $\mu B$ as it was made in Refs. [15, 18]. However in subsequent calculations, which lead to Eqs. (11) and (12), these coefficients were accounted for exactly.

Now let us discuss other factors which can also give the contributions, comparable with Eq. (20), to the effective Hamiltonian. While deriving the effective Hamiltonian (12) in Sec. 2 we supposed that the magnetic field is transverse with respect to the neutrino motion. The effect of the longitudinal magnetic field $B_\parallel$ on neutrino oscillations was studied in Ref. [37]. Using the results of that work we get that to neglect the longitudinal magnetic field contribution to the effective Hamiltonian, the following condition should be satisfied:

$$\frac{B_\parallel}{B_\perp} \ll \frac{1}{16kB_\perp |\mu_a m_a - \mu_b m_b|}$$

$$\times \left[ \frac{m_a^2 g_a^3}{M_a^2} - \frac{m_b^2 g_b^3}{M_b^2} \right],$$

where $B_\perp$ is the transverse component of the magnetic field. The condition (23) is satisfied for neutrinos emitted inside the solid angle near the equatorial plane with the spread $2\theta \sim 2B_\parallel/B_\perp$. Assuming the radially symmetric neutrino emission we find that about 2% of the total neutrino flux are affected by the new resonance (21), i.e. the influence of the longitudinal magnetic field is negligible for oscillations of such particles.

The next important approximation made in the deviation of Eq. (12) was the assumption of negligible polarization of matter which can be not true if we study rather strong magnetic fields. The effect of matter polarization on neutrino oscillations was previously discussed in Refs. [20, 38, 39]. It was found that in the leading order in $m_a/k$ matter polarization produces the following contribution to the diagonal entry of the effective Hamiltonian, corresponding to right-polarized neutrinos: $g_a(\lambda_f \beta_\nu)(m_a/k)$, where $\beta_\nu$ is the neutrino velocity and $\lambda_f$ is the mean polarization vector of background fermions. Note that in polarized matter the effective energy of left-polarized neutrinos also changes. However this process does not contribute to the considered oscillations channel $\nu^e_\nu \leftrightarrow \nu^e_\nu$.

It is clear that one should take into account only the polarization of electrons since nucleons are much heavier. Using the results of Refs. [38, 40] we obtain that in the shock wave region electrons are relativistic and weakly degenerate. Finally we get that the new correction to the effective Hamiltonian (20) becomes bigger than the contribution of matter polarization to the effective potential of right-polarized neutrinos if the electron temperature exceeds $10 \text{ MeV}$ which is comparable with the mean temperature in an expanding envelope [40].
The presence of relatively big Dirac neutrino magnetic moments implies the existence of new interactions, beyond the standard model, which electromagnetically couple left- and right-handed neutrinos. It is possible that these new interactions also contribute to the effective potential of the right-polarized neutrino interaction with background matter. Thus one should compare this effective potential with the contribution of Eq. (20).

The most generic SU(2)_L × U(1)_Y gauge invariant and renormalizable interaction which produces Dirac neutrino magnetic moment was discussed in Ref. [41]. The effective Lagrangian of this interaction involves the dimension n = 6 operators \( \mathcal{O}_j \),

\[
\mathcal{L}_{\text{eff}} = (1/\Lambda^2) \sum_j C_j \mathcal{O}_j + \text{h.c.},
\]

where \( \Lambda \sim 1 \text{ TeV} \) is the energy scale of the new physics, \( C_j \) are the effective operator coupling constants and sum spans all the operators of the given dimension.

One of the operators \( \mathcal{O}_j \) also contributes to the effective potential of a right-polarized neutrino in matter, \( \mathcal{O} = \kappa \bar{\nu}_{aL} \sigma^{\mu\nu} \nu_{R \mu} W^a_{\mu\nu} \), where \( \kappa \) is the coupling constant, \( \tau_a \) are Pauli matrices, \( L^T = (\nu_L, e_L) \) is the SU(2)_L isodoublet, \( \bar{\phi} = i\tau_2 \phi^* \), with \( \phi \) being a Higgs field, and \( W^a_{\mu\nu} = \partial_{\mu} W^a_{\nu} - \partial_{\nu} W^a_{\mu} - \kappa \epsilon_{abc} W^b_{\mu} W^c_{\nu} \) is the SU(2)_L field strength tensor. Assuming the spontaneous symmetry breaking at the electroweak scale, \( \phi^T \to (0, v/\sqrt{2}) \), we can rewrite the contribution of the operator \( \mathcal{O} \) to the effective Lagrangian in the form

\[
\mathcal{L}_{\text{eff}} = \frac{C_{\nu e}v}{\sqrt{2}} e_L^a \sigma^{\mu\nu} \nu_{R} (W^a_{\mu\nu} - iW^a_{\nu\mu}) + \text{h.c.}, \quad (24)
\]

which implies that a process \( e^- + \nu_+ \to e^- + \nu_+ \) should happen in a background matter.

Using the results of Ref. [41] we can evaluate the contribution of the new interactions to the effective Hamiltonian (16) as

\[
\delta V_R \approx V_{\text{sm}} \frac{|\kappa|^2}{G_F M_W^2} \frac{\left( \frac{\mu_\nu}{\mu_B} \right)}{\left( \frac{E_\nu}{m_e} \right)^2}, \quad (25)
\]

where \( V_{\text{sm}} \sim G_F n_e \) is the standard model effective potential, \( G_F \) is the Fermi constant, and \( M_W \) is the W-boson mass. Taking the following neutrino properties (see Sec. 4): \( \mu_\nu \sim 10^{-12} \mu_B \), \( E_\nu \sim 100 \text{ MeV} \), and \( m_\nu \sim 0.2 \text{ eV} \), we can get that the ratio of the correction to the effective potential \( \delta V_R \) and new correction (20) is \( \sim 10^{-2} \). It means that the influence of new interactions, which generate neutrino magnetic moments, are not important for neutrino spin-flavor oscillations.

The constraint on the Dirac neutrino magnetic moment obtained in Ref. [41] is \( 10^{-14} \mu_B \). Nevertheless in Sec. 4 we used the magnetic moments in the \( (10^{-13} - 10^{-12})\mu_B \) range since analogous constrains on the Dirac neutrino magnetic moments were obtained in Refs. [13, 31] on the basis of the analysis of astrophysical data.

6. CONCLUSION

In this paper we have described neutrino spin-flavor oscillations in dense matter and a strong magnetic field in the frame of relativistic quantum mechanics. The advantage of this formalism, compared to the commonly used quantum mechanical approach, is that one can exactly take into account the neutrino properties like the initial momentum, masses, mixing angles, and magnetic moments, as well as the matter density and the strength of the magnetic field since we use the exact solutions of the Dirac equation for massive neutrinos in presence of external fields.

In Sec. 2 it was demonstrated that the initial condition problem for the system of two mixed flavor neutrinos, each of them represented as four-component Dirac spinors, can be reduced to a Schrödinger like equation (12). In Sec. 3 it was shown that the Hamiltonian of this evolution equation formally coincides with the effective Hamiltonian (16) previously proposed in Ref. [2].

It should be, however, noted that the dynamics of neutrino spin-flavor oscillations in matter and a magnetic field in frames of the quantum mechanical description is defined by both the effective Hamiltonian and the correct basis of the neutrinos “wave functions”. In Sec. 3, we showed that the choice of the helicity eigenstates basis, used in the previous works on neutrino spin-flavor oscillations, is justified only in the case of the weak diagonal magnetic interaction. The situation when the diagonal magnetic interaction is strong was also discussed in Sec. 3. In this limit the correct basis consists of the eigenvectors of the operator \( \Pi_\alpha \).

Besides the demonstration of the consistency of our results with the standard approach for the description of neutrino spin-flavor oscillations, we found the correction (20) to the commonly used effective Hamiltonian (16) which is usually omitted [2]. It was possible to obtain this correction in the explicit form since we used the energy levels (8) and the basis spinors (9) which exactly account for external fields.

In Sec. 4 we discussed the realistic situation when the correction (20) is important. We considered \( \nu_\mu^+ \leftrightarrow \nu_s \) oscillations channel, where \( \nu_s \) is the additional sterile neutrino almost degenerate in mass with other
neutrino states \[26\]. The right-polarized electron neutrinos were supposed to be produced during the supernova explosion due to the scattering of left-polarized neutrinos with the non-zero diagonal magnetic moment off background fermions \[13\]. The flux of these right-polarized neutrinos was taken to propagate through the expanding envelope of a supernova and interact with a strong magnetic field.

We found that the new resonance in neutrino spin-flavor oscillations can appear if the strength of the magnetic field and the matter density are close to the values recently discussed in Refs. \[10, 29\]. Note that new resonance condition \(21\) depends on the absolute value of the neutrino mass. Thus the observation of this new resonance would cast light upon the absolute scale of the neutrino masses. Although \(\nu_e^+ \leftrightarrow \nu_e^-\) oscillations do not change the dynamics of a supernova explosion, the appearance of this additional resonance channel of neutrino spin-flavor oscillations makes impossible a terrestrial observation of the right-polarized supernova neutrinos proposed in Ref. \[13\].

To analyze the dynamics of the Schrödinger equation with the exact Hamiltonians \(16\) and \(20\), in Sec. 4 we presented the numerical transition probability which accounts for all magnetic moments and is built for various magnetic fields configurations. In particular we analyze the constant magnetic field and the more realistic coordinate dependent magnetic field of a magnetic dipole \(B_{\text{dip}}(r)\). It was revealed that the diagonal magnetic moment of an electron neutrino does not significantly influence the transition probability. Then we found that at a rather strong magnetic field strength the transition probability can reach big values when a neutrino leaves an expanding envelope.

In Sec. 5 we considered other factors which can be comparable with the correction obtained \(20\). We analyzed the contributions of the longitudinal magnetic field and the matter polarization which were omitted in the derivation of the effective Hamiltonian \(12\). In particular we have found that a longitudinal magnetic field is not important for neutrinos emitted near the equator of a star. The matter polarization does not influence the dynamics of neutrino oscillations if the temperature of background electrons is higher than a few MeV. We also evaluated the possible contributions of the new interactions, which generate neutrino magnetic moments \[41\], to the effective potentials of right-polarized neutrinos. It was found that these contributions are negligible compared to the correction \(20\).

ACKNOWLEDGMENTS

This work has been supported by CONICYT (Chile) through Programa Bicentenario PSD-91-2006 and by Deutscher Akademischer Austausch Dienst and FAPESP. The author is thankful to G.G. Raffelt and V.B. Semikoz for fruitful discussions as well as to the referee, whose comments helped to improve this paper.

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Translated by the author