Cohomology of weighted Rota-Baxter Lie algebras and Rota-Baxter paired operators

Apurba Das

Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur 208016, Uttar Pradesh, India. Email: apurbadas348@gmail.com

Abstract

In this paper, we define representations and cohomology of weighted Rota-Baxter Lie algebras. As applications of cohomology, we study abelian extensions and formal 1-parameter deformations weighted Rota-Baxter Lie algebras. Finally, we consider weighted Rota-Baxter paired operators that induces a weighted Rota-Baxter Lie algebra and a representation of it. We define suitable cohomology for such paired operators that govern deformation.

2020 MSC classification: 17B38, 17B56, 16S80.

Keywords: Weighted Rota-Baxter Lie algebras, Representations, Cohomology, Deformations, Rota-Baxter paired operators.

Contents

1 Introduction 1
2 Representations of weighted Rota-Baxter Lie algebras 3
3 Cohomology of weighted Rota-Baxter Lie algebras 7
4 Applications of cohomology 12
5 Rota-Baxter paired operators 16

1 Introduction

Rota-Baxter operators were first appeared in the work of Baxter in his study of the fluctuation theory [6] and further developed by Rota in combinatorics [25]. Subsequently, Cartier [7] and Atkinson [2] studied further properties of Rota-Baxter operators. In last twenty years, Rota-Baxter operators on associative algebras pay very much attention due to its connection with combinatorics of shuffle algebras [15], Yang-Baxter equations [1], dendriform algebras [1,13], renormalizations in quantum field theory [8], multiple zeta values in number theory [16] and splitting of algebraic operads [3]. See [17] for more details about Rota-Baxter operators on associative algebras. On the other hand, Rota-Baxter operators on Lie algebras was first considered by Kuperscmidt in the study of classical $r$-matrices [19]. They are also connected with pre-Lie algebras [3], integrable systems [26] and combinatorics of Lyndon-Shirshov words [23].
Deformation theory of some algebraic structure goes back to Gerstenhaber [14] for associative algebras and Nijenhuis-Richardson [24] for Lie algebras. Recently, deformation theory has been adapted to Rota-Baxter operators on Lie algebras [27] and subsequently developed in [9] for associative algebras. In both these papers, the authors only considered deformations of Rota-Baxter operators by keeping the underlying algebras intact. In [11,21] the authors dealt with deformations of Rota-Baxter algebras in which they simultaneously deform Rota-Baxter operators and the underlying algebras. Note that all these works are concern about Rota-Baxter operators of weight zero.

Rota-Baxter operators with arbitrary weight (also called weighted Rota-Baxter operators) was considered in [4,5]. They are related with triendriform algebras [12], post-Lie algebras and modified Yang-Baxter equations [4], weighted infinitesimal bialgebras and weighted Yang-Baxter equations [32], combinatorics of rooted forests [31], among others. Recently, the authors in [18] defined the cohomology of Rota-Baxter operators of weight 1 on Lie algebras and Lie groups. This motivates the present author to study cohomology and deformations of weighted Rota-Baxter operators on both associative and Lie algebras [10]. However, the simultaneous deformations of (associative) algebras and weighted Rota-Baxter operators are considered in a recent preprint of Wang and Zhou [30]. More precisely, they considered weighted Rota-Baxter associative algebras and define cohomology of them with coefficients in a suitable Rota-Baxter bimodule. When considering the cohomology with coefficients in itself, it governs the simultaneous deformations of algebras and weighted Rota-Baxter operators.

Our aim in this paper is to apply the approach of [30] to weighted Rota-Baxter Lie algebras. More precisely, we first consider representations of weighted Rota-Baxter Lie algebras and provide various constructions of representations. Then we define the cohomology of a weighted Rota-Baxter Lie algebra with coefficients in a representation. This cohomology is obtained as a byproduct of the standard Chevalley-Eilenberg cohomology of the underlying Lie algebra and the cohomology of the underlying weighted Rota-Baxter operator. When the weight is zero, our cohomology coincides with the one introduced in [21]. We interpret our second cohomology group as the isomorphism classes of abelian extensions of weighted Rota-Baxter Lie algebras. Then we consider deformations of weighted Rota-Baxter Lie algebras in which we simultaneously deform underlying Lie algebras and weighted Rota-Baxter operators. The infinitesimals of such deformations are 2-cocycles in the cohomology and equivalent deformations produce cohomologous 2-cocycles. Hence they correspond to the same element in the second cohomology group. We also find a sufficient condition for the rigidity of a weighted Rota-Baxter Lie algebra.

Finally, given a Lie algebra and a representation, we introduce Rota-Baxter paired operators that induces a weighted Rota-Baxter Lie algebra and representation of it. The terminology of Rota-Baxter paired operators is motivated by a paper by Zheng, Guo and Zhang [33] where the authors introduced Rota-Baxter paired modules in the associative context. We define a suitable differential graded Lie algebra that characterize Rota-Baxter paired operators as its Maurer-Cartan elements. This suggests us to define the cohomology of Rota-Baxter paired operators that control formal deformations.

The paper is organized as follows. In Section 2 we consider weighted Rota-Baxter Lie algebras and their representations. We also give several new constructions of representations. The cohomology of a weighted Rota-Baxter Lie algebra with coefficients in a representation is defined in Section 3. Applications of such cohomology to abelian extensions and formal 1-parameter deformations are given in Section 4. Finally, in Section 5, we define Rota-Baxter paired operators. We provide the cohomology of such paired operators and consider their deformations.

All vector spaces, (multi)linear maps, wedge products are over a field $k$ of characteristic 0.
2 Representations of weighted Rota-Baxter Lie algebras

In this section, we consider weighted Rota-Baxter Lie algebras [28] and introduce their representations. We also provide various examples and new constructions. Let $\lambda \in \mathbb{k}$ be a fixed scalar unless specified otherwise.

2.1 Definition. (i) Let $\mathfrak{g}$ be a Lie algebra. A linear map $\mathfrak{T} : \mathfrak{g} \to \mathfrak{g}$ is said to be a $\lambda$-weighted Rota-Baxter operator if $\mathfrak{T}$ satisfies

\[ [\mathfrak{T}(x), \mathfrak{T}(y)] = \mathfrak{T}([\mathfrak{T}(x), y] + [x, \mathfrak{T}(y)] + \lambda [x, y]), \quad \text{for } x, y \in \mathfrak{g}. \]  

(ii) A $\lambda$-weighted Rota-Baxter Lie algebra is a pair $(\mathfrak{g}, \mathfrak{T})$ consisting of a Lie algebra $\mathfrak{g}$ together with a $\lambda$-weighted Rota-Baxter operator on it.

2.2 Example. (i) For any Lie algebra $\mathfrak{g}$, the pair $(\mathfrak{g}, \text{id}_\mathfrak{g})$ is a $(-1)$-weighted Rota-Baxter Lie algebra.

(ii) Let $(\mathfrak{g}, \mathfrak{T})$ be a $\lambda$-weighted Rota-Baxter Lie algebra. Then for any $\mu \in \mathbb{k}$, the pair $(\mathfrak{g}, \mu\mathfrak{T})$ is a $(\mu \lambda)$-weighted Rota-Baxter Lie algebra.

(iii) Let $(\mathfrak{g}, \mathfrak{T})$ be a $\lambda$-weighted Rota-Baxter Lie algebra. Then $(\mathfrak{g}, -\lambda \text{id}_\mathfrak{g} - \mathfrak{T})$ is so.

(iv) Given a $\lambda$-weighted Rota-Baxter Lie algebra $(\mathfrak{g}, \mathfrak{T})$ and an automorphism $\psi \in \text{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, the pair $(\mathfrak{g}, \psi^{-1} \circ \mathfrak{T} \circ \psi)$ is a $\lambda$-weighted Rota-Baxter Lie algebra.

(v) Let $\mathfrak{g}$ be a Lie algebra which splits as the direct sum of two subalgebras $\mathfrak{g}_-$ and $\mathfrak{g}_+$. Then $(\mathfrak{g}, \mathfrak{T})$ is a $\lambda$-weighted Rota-Baxter Lie algebra, where $\mathfrak{T} : \mathfrak{g} \to \mathfrak{g}$ is given by

\[ \mathfrak{T}(x_-, x_+) = (0, -\lambda x_+), \quad \text{for } (x_-, x_+) \in \mathfrak{g}_- \oplus \mathfrak{g}_+ = \mathfrak{g}. \]

(vi) This example generalizes the previous one. Let $\mathfrak{g}$ be a Lie algebra which splits as the direct sum of three subalgebras $\mathfrak{g}_-$, $\mathfrak{g}_0$ and $\mathfrak{g}_+$ in which $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are both $\mathfrak{g}_0$-modules (i.e., representations of $\mathfrak{g}_0$). If $(\mathfrak{g}_0, \mathfrak{T}_0)$ is a $\lambda$-weighted Rota-Baxter Lie algebra then $(\mathfrak{g}, \mathfrak{T})$ is so, where $\mathfrak{T} : \mathfrak{g} \to \mathfrak{g}$ is given by

\[ \mathfrak{T}(x_-, x_0, x_+) = (0, \mathfrak{T}_0(x_0), -\lambda x_+), \quad \text{for } (x_-, x_0, x_+) \in \mathfrak{g}. \]

2.3 Definition. Let $(\mathfrak{g}, \mathfrak{T})$ and $(\mathfrak{g}', \mathfrak{T}')$ be two $\lambda$-weighted Rota-Baxter Lie algebras. A morphism from $(\mathfrak{g}, \mathfrak{T})$ to $(\mathfrak{g}', \mathfrak{T}')$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ satisfying additionally $\phi \circ \mathfrak{T} = \mathfrak{T}' \circ \phi$. It is called an isomorphism if $\phi$ is so.

Let $\mathfrak{g}$ be a Lie algebra. Recall that a representation of $\mathfrak{g}$ is a vector space $\mathcal{V}$ with a linear map (called the action map) $\rho : \mathfrak{g} \to \text{End}(\mathcal{V})$ satisfying

\[ \rho[x, y] = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x), \quad \text{for } x, y \in \mathfrak{g}. \]

We denote a representation of $\mathfrak{g}$ as above by $(\mathcal{V}, \rho)$ or simply by $\mathcal{V}$ when no confusion arises. Note that any Lie algebra $\mathfrak{g}$ is a representation of itself with the action map $\rho : \mathfrak{g} \to \text{End}(\mathfrak{g})$ given by $\rho(x)(y) = [x, y]$, for $x, y \in \mathfrak{g}$. This is called the adjoint representation.

2.4 Definition. Let $(\mathfrak{g}, \mathfrak{T})$ be a $\lambda$-weighted Rota-Baxter Lie algebra. A representation of it is a pair $(\mathcal{V}, \mathcal{T})$ in which $\mathcal{V} = (\mathcal{V}, \rho)$ is a representation of the Lie algebra $\mathfrak{g}$ and $\mathcal{T} : \mathcal{V} \to \mathcal{V}$ is a linear map satisfying

\[ \rho(Tx)(Tu) = \mathcal{T}(\rho(T)x)u + \rho(T)(x)(Tu) + \lambda \rho(x)u, \quad \text{for } x \in \mathfrak{g}, u \in \mathcal{V}. \]  

2.5 Example. Any $\lambda$-weighted Rota-Baxter Lie algebra $(\mathfrak{g}, \mathfrak{T})$ is a representation of itself. We call this the adjoint representation.
2.6 Example. Let \( \mathfrak{g} \) be a Lie algebra and \( \mathcal{V} \) be a representation of it. Then the pair \((\mathcal{V}, \text{id}_\mathcal{V})\) is a representation of the \((-1)\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \text{id}_\mathfrak{g})\).

2.7 Example. Let \((\mathfrak{g}, \Xi)\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \((\mathcal{V}, \mathcal{T})\) be a representation of it. Then for any scalar \(\mu \in \mathbf{k}\), the pair \((\mathcal{V}, \mu\mathcal{T})\) is a representation of the \((\mu\lambda)\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \mu\Xi)\).

2.8 Proposition. Let \((\mathfrak{g}, \Xi)\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \((\mathcal{V}, \mathcal{T})\) be a representation of it. Then \((\mathcal{V}, -\lambda\text{id}_\mathcal{V} - \mathcal{T})\) is a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, -\lambda\text{id}_\mathfrak{g} - \Xi)\).

Proof. For any \(x \in \mathfrak{g}\) and \(u \in \mathcal{V}\), we observe that
\[
\rho((-\lambda\text{id}_\mathfrak{g} - \Xi)x)(-\lambda\text{id}_\mathcal{V} - \mathcal{T})(u) = \lambda^2 \rho(x)u + \lambda\rho(\Xi)xu + \lambda\rho(\mathcal{T}x)u + \rho(\mathcal{T}x)u
\]
\[
= \lambda^2 \rho(x)u + \lambda\mathcal{T}(\rho(x)u) + \rho(\mathcal{T}(\rho(x)u) + \lambda \rho(x)\mathcal{T}u + \mathcal{T}(\rho(x)\mathcal{T}u).
\]

On the other hand,
\[
(-\lambda\text{id}_\mathcal{V} - \mathcal{T})\left((\rho(-\lambda\text{id}_\mathfrak{g} - \Xi(x))u + \rho(x)(-\lambda\text{id}_\mathcal{V} - \mathcal{T})(u) + \rho(x)u\right)
\]
\[
= \lambda^2 \rho(x)u + \lambda\mathcal{T}(\rho(x)u) + \rho(\mathcal{T}(\rho(x)u) + \lambda \rho(x)\mathcal{T}u + \mathcal{T}(\rho(x)\mathcal{T}u))
\]
\[
- \lambda^2 \rho(x)u - \lambda\mathcal{T}(\rho(x)u).
\]

The expressions in (3) and (4) are same. Hence \((\mathcal{V}, -\lambda\text{id}_\mathcal{V} - \mathcal{T})\) is a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, -\lambda\text{id}_\mathfrak{g} - \Xi)\).

2.9 Proposition. Let \((\mathfrak{g}, \Xi)\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \(\{(\mathcal{V}_i, \mathcal{T}_i)\}_{i \in I}\) be a family of representations of it. Then \((\bigoplus_{i \in I} \mathcal{V}_i, \bigoplus_{i \in I} \mathcal{T}_i)\) is also a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \Xi)\).

Proof. Let \(\rho_i : \mathfrak{g} \to \text{End}(\mathcal{V}_i)\) denote the action of the Lie algebra \(\mathfrak{g}\) on the representation \(\mathcal{V}_i\). Then it follows that \(\rho : \mathfrak{g} \to \text{End}(\bigoplus_{i \in I} \mathcal{V}_i), \rho(x)(u_i)_{i \in I} = (\rho_i(x)u_i)_{i \in I}\) is a representation of \(\mathfrak{g}\) on \(\bigoplus_{i \in I} \mathcal{V}_i\). Moreover, for any \(x \in \mathfrak{g}\) and \((u_i)_{i \in I} \in \bigoplus_{i \in I} \mathcal{V}_i\), we have
\[
\rho(\Xi x)(\bigoplus_{i \in I} \mathcal{T}_i)(u_i)_{i \in I} = \left(\rho_i(\Xi x)\mathcal{T}_i(u_i)\right)_{i \in I}
\]
\[
= \left(\mathcal{T}_i(\rho_i(\Xi x)u_i + \rho_i(x)\mathcal{T}_i(u_i) + \lambda \rho_i(x)u_i)\right)_{i \in I}
\]
\[
= \left(\bigoplus_{i \in I} \mathcal{T}_i\left(\rho(\Xi x)(u_i)_{i \in I} + \rho(x)\left(\bigoplus_{i \in I} \mathcal{T}_i(u_i)_{i \in I} + \lambda \rho(x)(u_i)_{i \in I}\right)\right)\right).
\]

Hence the result follows.

Let \(\mathfrak{g}\) be a Lie algebra and \((\mathcal{V}, \rho)\) be a representation. Then there is a representation of the Lie algebra \(\mathfrak{g}\) on the space \(\text{End}(\mathcal{V})\) with the action given by
\[
\bar{\rho} : \mathfrak{g} \to \text{End}(\text{End}(\mathcal{V})), \ (\bar{\rho}(x)f)u = -f(\rho(x)u), \ \text{for} \ x \in \mathfrak{g}, \ f \in \text{End}(\mathcal{V}) \text{ and } u \in \mathcal{V}.
\]

With this representation, we have the following.

2.10 Proposition. Let \((\mathfrak{g}, \Xi)\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \((\mathcal{V}, \mathcal{T})\) be a representation of it. Then \((\text{End}(\mathcal{V}), \bar{\mathcal{T}})\) is also a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \Xi)\), where
\[
\bar{\mathcal{T}} : \text{End}(\mathcal{V}) \to \text{End}(\mathcal{V}), \ \bar{\mathcal{T}}(f)(u) = -\lambda f(u) - f(\mathcal{T}(u)), \ \text{for} \ u \in \mathcal{V}.
\]
Proof. For any \( x,y \in \mathfrak{g}, \ f \in \text{End}(\mathcal{V}) \) and \( u \in \mathcal{V} \),

\[
\left( \tilde{T}(\mathfrak{T}x)\tilde{T}(f) \right)u = -\tilde{T}(f)(\rho(\mathfrak{T}x)u) = \lambda f(\rho(\mathfrak{T}x)u) + f(\mathcal{T}(\rho(\mathfrak{T}x)u)).
\]  

(5)

On the other hand,

\[
\left( \tilde{T}(\rho(\mathfrak{T}x)f + \rho(x)\tilde{T}(f) + \lambda \tilde{T}(f)) \right)u
\]

\[
= -\lambda(\rho(\mathfrak{T}x)f + \rho(x)\tilde{T}(f) + \lambda \tilde{T}(f))u - (\rho(\mathfrak{T}x)f + \rho(x)\tilde{T}(f) + \lambda \tilde{T}(f))\mathcal{T}(u)
\]

\[
= \lambda\left( f(\rho(\mathfrak{T}x)u) + \tilde{T}(f)(\rho(x)u) + \lambda f(\rho(x)u) + \tilde{T}(f)(\mathcal{T}(u)) + \lambda f(\mathcal{T}(u)) \right)
\]

\[
= \lambda\left( f(\rho(\mathfrak{T}x)u) - \lambda f(\rho(x)u) - \tilde{T}(f)(\mathcal{T}(u)) + \lambda f(\mathcal{T}(u)) \right)
\]

\[
+ f\left( \mathcal{T}(\rho(\mathfrak{T}x)u) + \tilde{T}(\rho(x)(\mathcal{T}u)) + \mathcal{T}(\rho(x)u) - \lambda \rho(x)(\mathcal{T}u) - \mathcal{T}(\rho(x)(\mathcal{T}u)) + \lambda \rho(x)(\mathcal{T}u) \right)
\]

\[
= \lambda f(\rho(\mathfrak{T}x)u) + f(\mathcal{T}(\rho(\mathfrak{T}x)u)).
\]

(6)

It follows from (5) and (6) that \((\text{End}(\mathcal{V}), \tilde{T})\) is a representation of \((\mathfrak{g}, \mathfrak{T})\). \(\square\)

In the following, we construct the semidirect product in the context of \(\lambda\)-weighted Rota-Baxter Lie algebras.

2.11 Proposition. Let \((\mathfrak{g}, \mathfrak{T})\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \((\mathcal{V}, \mathcal{T})\) be a representation of it. Then \((\mathfrak{g} \oplus \mathcal{V}, \mathfrak{T} \oplus \mathcal{T})\) is a \(\lambda\)-weighted Rota-Baxter Lie algebra, where the Lie bracket on \(\mathfrak{g} \oplus \mathcal{V}\) is given by the semidirect product

\[
[(x, u), (y, v)]_\lambda := ([x, y], \rho(x)v - \rho(y)u), \text{ for } x,y \in \mathfrak{g}, u,v \in \mathcal{V}.
\]

(7)

Proof. We have

\[
[(\mathfrak{T} \oplus \mathcal{T})(x,u), (\mathfrak{T} \oplus \mathcal{T})(y,v)]_\lambda
\]

\[
= ([\mathfrak{T}x, \mathfrak{T}y], \mathfrak{T}(\rho(\mathfrak{T}x)v - \rho(\mathfrak{T}y)(\mathcal{T}u))
\]

\[
= ([\mathfrak{T}x, \mathfrak{T}y], \mathcal{T}(\rho(\mathfrak{T}x)v - \rho(\mathfrak{T}y)u)) + ([x, y], \mathcal{T}(\rho(x)(\mathcal{T}v) - \rho(y)(\mathcal{T}u)) + \lambda([x, y], \mathcal{T}(\rho(x)v - \rho(y)u))
\]

\[
= ([\mathfrak{T} \oplus \mathcal{T}][x, y], \mathcal{T}(\rho(x)v - \rho(y)u)) + \lambda([x, y], \mathcal{T}(\mathfrak{T}v) - \mathfrak{T}(\mathcal{T}u)) + \lambda([x, y], \mathcal{T}(\rho(x)v - \rho(y)u))
\]

This shows that \((\mathfrak{T} \oplus \mathcal{T})\) is a \(\lambda\)-weighted Rota-Baxter operator on the semidirect product Lie algebra. Hence the result follows. \(\square\)

2.12 Remark. The converse of the above proposition is also true. More precisely, let \((\mathfrak{g}, \mathfrak{T})\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra. Let \(\mathcal{V}\) be a vector space and \(\rho: \mathfrak{g} \to \text{End}(\mathcal{V})\), \(\mathcal{T}: \mathcal{V} \to \mathcal{V}\) be two linear maps. Then \((\mathcal{V} = (\mathcal{V}, \rho), \mathcal{T})\) is a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \mathfrak{T})\) if and only if \((\mathfrak{g} \oplus \mathcal{V}, \mathfrak{T} \oplus \mathcal{T})\) is a \(\lambda\)-weighted Rota-Baxter Lie algebra, where \(\mathfrak{g} \oplus \mathcal{V}\) is equipped with the bracket (7).

2.13 Remark. Let \(\mathfrak{g}\) be a Lie algebra and \(\mathcal{V}\) be a representation of it. Let \(\mathfrak{T}: \mathfrak{g} \to \mathfrak{g}\) be a \(\lambda\)-weighted Rota-Baxter operator on \(\mathfrak{g}\) which makes \((\mathfrak{g}, \mathfrak{T})\) a \(\lambda\)-weighted Rota-Baxter Lie algebra. Intuitively, it follows from the above proposition that \(\mathcal{T}: \mathcal{V} \to \mathcal{V}\) can be considered as a representation of \(\mathfrak{T}\) with respect to the \(\mathfrak{g}\)-representation \(\mathcal{V}\). In Section 5, we call the tuple \((\mathfrak{T}, \mathcal{T})\) as \(\lambda\)-weighted Rota-Baxter paired operators.

2.14 Proposition. Let \((\mathfrak{g}, \mathfrak{T})\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra. Then we have the followings.
(i) The pair \((\mathfrak{g}, \cdot, \cdot)\) is a Lie algebra, where
\[
[x, y]_\mathfrak{g} := [\Xi(x), y] + [x, \Xi(y)] + \lambda [x, y], \quad \text{for } x, y \in \mathfrak{g}.
\]
We denote this Lie algebra by \(\mathfrak{g}_\Xi\).

(ii) The pair \((\mathfrak{g}_\Xi, \Xi)\) is a \(\lambda\)-weighted Rota-Baxter Lie algebra and the map \(\Xi : \mathfrak{g}_\Xi \rightarrow \mathfrak{g}\) is a morphism of \(\lambda\)-weighted Rota-Baxter Lie algebras.

**Proof.** (i) This is a standard result that \(\mathfrak{g}_\Xi = (\mathfrak{g}, \cdot, \cdot, \cdot)_\Xi\) is a Lie algebra. See for instance [10, Proposition 5.5]. We also observe that
\[
[\Xi(x), \Xi(y)]_\mathfrak{g} = [\Xi^2(x), \Xi(y)] + [\Xi(x), \Xi^2(y)] + \lambda [\Xi(x), \Xi(y)]
= \Xi([\Xi(x), y]_\mathfrak{g} + [x, \Xi(y)]_\mathfrak{g} + \lambda [x, y]_\mathfrak{g})
\]
which shows that \(\Xi\) is a \(\lambda\)-weighted Rota-Baxter operator on the Lie algebra \(\mathfrak{g}_\Xi\).

(ii) Since \(\Xi\) is a \(\lambda\)-weighted Rota-Baxter operator on \(\mathfrak{g}\), it follows from (1) that
\[
\Xi([x, y]_\mathfrak{g}) = [\Xi(x), \Xi(y)], \quad \text{for } x, y \in \mathfrak{g}_\Xi.
\]
This implies that \(\Xi : \mathfrak{g}_\Xi \rightarrow \mathfrak{g}\) is a morphism of \(\lambda\)-weighted Rota-Baxter Lie algebras from \((\mathfrak{g}_\Xi, \Xi)\) to \((\mathfrak{g}, \cdot, \cdot)\).

**2.15 Theorem.** Let \((\mathfrak{g}, \Xi)\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \((\mathcal{V}, T)\) be a representation of it. Define a map \(\overline{\mathcal{V}} : \mathfrak{g} \rightarrow \text{End}(\mathcal{V})\) by
\[
\overline{\mathcal{V}}(x)u := \rho(\Xi(x))u + \rho(x)(Tu) + \lambda \rho(x)u, \quad \text{for } x \in \mathfrak{g}, u \in \mathcal{V}.
\]
Then
(i) \(\overline{\mathcal{V}}\) satisfies \(T(\overline{\mathcal{V}}(x)u) = \rho(\Xi(x))Tu\),
(ii) \((\overline{\mathcal{V}} = (\mathcal{V}, \overline{\mathcal{V}}), T)\) is a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}_\Xi, \Xi)\).

**Proof.** The part (i) follows from (2). To prove the part (ii) we first observe that
\[
\overline{\mathcal{V}}(x)\overline{\mathcal{V}}(y)u - \overline{\mathcal{V}}(y)\overline{\mathcal{V}}(x)u
= \rho(\Xi x)\rho(\Xi y)u + \rho(x)\rho(y)\overline{\mathcal{V}}(x)u + \rho(y)\rho(x)\overline{\mathcal{V}}(y)u + \lambda \rho(y)\rho(x)u
= \rho(\Xi x)\rho(\Xi y)u + \rho(x)\rho(y)\rho(x)u + \rho(y)\rho(x)\rho(y)u
+ \lambda \rho(y)\rho(x)u + \lambda \rho(x)\rho(y)u + \lambda \rho(x)\rho(y)u
= \rho([\Xi x, y])u + \rho([\Xi y, x])u + \lambda \rho([\Xi y, x])u + \lambda \rho([\Xi x, y])u
= \rho([\Xi x, y])u + \rho([\Xi y, x])u + \lambda \rho([\Xi y, x])u + \lambda \rho([\Xi x, y])u
= \rho([\Xi x, y])u + \rho([\Xi y, x])u + \lambda \rho([\Xi y, x])u + \lambda \rho([\Xi x, y])u
\]
This shows that \((\overline{\mathcal{V}}, T)\) is a representation of \((\mathfrak{g}_\Xi, \Xi)\).
2.16 Remark. When considering the adjoint representation \((g, \mathfrak{g})\) of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((g, \mathfrak{g})\), the representation of the above theorem is the adjoint representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((g, \mathfrak{g})\).

In the following, we will prove another relevant result that will be useful in the next section to construct the cohomology of \(\lambda\)-weighted Rota-Baxter Lie algebras.

2.17 Theorem. Let \((g, \mathfrak{g})\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \((\mathcal{V}, \mathcal{T})\) be a representation of it. Define a map \(\tilde{\rho} : g \to \text{End}(\mathcal{V})\) by

\[
\tilde{\rho}(x)u = \rho(\mathfrak{g}(x))u - \mathcal{T}(\rho(x)u), \quad \text{for } x \in g, u \in \mathcal{V}.
\]

Then \(\tilde{\rho}\) defines a representation of the Lie algebra \(g\) on \(\mathcal{V}\). Moreover, \((\mathcal{V}, \tilde{\rho}, \mathcal{T})\) is a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((g, \mathfrak{g})\).

Proof. For any \(x, y \in g\) and \(u \in \mathcal{V}\), we have

\[
\begin{align*}
\tilde{\rho}(x)\tilde{\rho}(y)u & - \tilde{\rho}(y)\tilde{\rho}(x)u \\
& = \tilde{\rho}(x)(\rho(\mathfrak{g}(y))u - \mathcal{T}(\rho(y)u)) - \tilde{\rho}(y)(\rho(\mathfrak{g}(x))u - \mathcal{T}(\rho(x)u)) \\
& = \rho(\mathfrak{g}(x))\rho(\mathfrak{g}(y))u - \rho(\mathfrak{g}(x))\mathcal{T}(\rho(y)u) - \rho(\mathfrak{g}(y))\rho(\mathfrak{g}(x))u + \rho(\mathfrak{g}(x))\mathcal{T}(\rho(y)u) \\
& \quad - \rho(\mathfrak{g}(x))\rho(\mathfrak{g}(y))u + \rho(\mathfrak{g}(y))\mathcal{T}(\rho(x)u) + \mathcal{T}(\rho(y)\rho(\mathfrak{g}(x))u - \rho(y)\mathcal{T}(\rho(x)u)) \\
& = \rho(\mathfrak{g}(x))\rho(\mathfrak{g}(y))u - \mathcal{T}(\rho(\mathfrak{g}(x))\rho(\mathfrak{g}(y))u + \rho(\mathfrak{g}(x))\rho(\mathfrak{g}(y))u - \rho(\mathfrak{g}(x))\mathcal{T}(\rho(y)u)) \\
& \quad - \rho(\mathfrak{g}(y))\rho(\mathfrak{g}(x))u + \mathcal{T}(\rho(\mathfrak{g}(y))\rho(\mathfrak{g}(x))u + \rho(\mathfrak{g}(y))\rho(\mathfrak{g}(x))u - \rho(\mathfrak{g}(y))\mathcal{T}(\rho(x)u)) \\
& = \rho([\mathfrak{g}(x), \mathfrak{g}(y)]u - \mathcal{T}([\mathfrak{g}(x), \mathfrak{g}(y)]u + \mathcal{T}(\rho([\mathfrak{g}(x), \mathfrak{g}(y)]u))) \\
& = \tilde{\rho}([x, y]\mathfrak{g}u).
\end{align*}
\]

This shows that \(\tilde{\mathcal{V}} = (\mathcal{V}, \tilde{\rho})\) defines a representation of the Lie algebra \(g\). Moreover, we have

\[
\begin{align*}
\tilde{\rho}(\mathfrak{g}(x))(\mathcal{T}u) & = \rho(\mathfrak{g}(x))(\mathcal{T}u) - \mathcal{T}(\rho(\mathfrak{g}(x))(\mathcal{T}u)) \\
& = \mathcal{T}((\rho(\mathfrak{g}(x))u + \rho(\mathfrak{g}(x))(\mathcal{T}u)) + \rho(\mathfrak{g}(x))(\mathcal{T}u)) - \mathcal{T}((\rho(\mathfrak{g}(x))u + \rho(\mathfrak{g}(x))(\mathcal{T}u)) + \rho(\mathfrak{g}(x))(\mathcal{T}u)) \\
& = \mathcal{T}(\rho(\mathfrak{g}(x))u - \mathcal{T}(\rho(\mathfrak{g}(x))u)) + \mathcal{T}(\rho(\mathfrak{g}(x))(\mathcal{T}u) - \mathcal{T}(\rho(\mathfrak{g}(x))(\mathcal{T}u))) + \mathcal{T}(\rho(\mathfrak{g}(x))u - \mathcal{T}(\rho(\mathfrak{g}(x))u)) \\
& = \mathcal{T}(\rho(\mathfrak{g}(x))u + \rho(\mathfrak{g}(x))(\mathcal{T}u)) + \mathcal{T}(\rho(\mathfrak{g}(x))u).
\end{align*}
\]

which shows that \((\tilde{\mathcal{V}}, \mathcal{T})\) is a representation of \((g, \mathfrak{g})\).

3 Cohomology of weighted Rota-Baxter Lie algebras

In this section, we first recall the Chevalley-Eilenberg cohomology of a Lie algebra with coefficients in a representation. Then we define the cohomology of a weighted Rota-Baxter Lie algebra with coefficients in a representation. This cohomology is obtained as a byproduct of the Chevalley-Eilenberg cohomology of the underlying Lie algebra with the cohomology of the weighted Rota-Baxter operator. Applications of cohomology to abelian extensions and formal 1-parameter deformations are given in the next section.

Let \(g\) be a Lie algebra and \(\mathcal{V} = (\mathcal{V}, \rho)\) be a representation of it. The Chevalley-Eilenberg cohomology of \(g\) with coefficients in \(\mathcal{V}\) is given by the cohomology of the cochain complex \(\{C^*_\mathcal{CE}(g, \mathcal{V}), \delta_\mathcal{CE}\}\), where
$C_{\text{CE}}^n(g, V) = \text{Hom}(\wedge^n g, V)$ for $n \geq 0$ and the coboundary map $\delta_{\text{CE}} : C_{\text{CE}}^n(g, V) \to C_{\text{CE}}^{n+1}(g, V)$ given by

$$(\delta_{\text{CE}} f)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+n} \rho(x_i) f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+n+1} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}),$$

for $f \in C_{\text{CE}}^n(g, V)$ and $x_1, \ldots, x_{n+1} \in g$.

Let $(g, \Xi)$ be a $\lambda$-weighted Rota-Baxter Lie algebra and $(V, T)$ be a representation of it. Then we have seen in Theorem 2.17 that $\tilde{V} = (V, \tilde{\rho})$ is a representation of the Lie algebra $g_\Xi$. Therefore, one can define the corresponding Chevalley-Eilenberg cohomology. More precisely, for each $n \geq 0$, we define

$$C_{\text{CE}}^n(g_\Xi, \tilde{V}) = \text{Hom}(\wedge^n g_\Xi, V)$$

and a coboundary map $\partial_{\text{CE}} : C_{\text{CE}}^n(g_\Xi, \tilde{V}) \to C_{\text{CE}}^{n+1}(g_\Xi, \tilde{V})$ given by

$$(\partial_{\text{CE}} f)(x_1, \ldots, x_{n+1})$$

$$= \sum_{i=1}^{n+1} (-1)^{i+n} \tilde{\rho}(x_i) f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+n+1} f([\Xi(x_i), x_j]_\Xi, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1})$$

$$= \sum_{i=1}^{n+1} (-1)^{i+n} \rho(\Xi(x_i)) f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) - \sum_{i=1}^{n+1} (-1)^{i+n} T(\rho(x_i) f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}))$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+n+1} f([\Xi(x_i), x_j] + [x_i, \Xi(x_j)] + \lambda[x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

Then $\{C_{\text{CE}}^n(g_\Xi, \tilde{V}), \partial_{\text{CE}}\}$ is a cochain complex. The corresponding cohomology groups are called the cohomology of $\Xi$ with coefficients in the representation $T$.

### 3.1 Remark.
When $(V, T) = (g, \Xi)$ is the adjoint representation of the $\lambda$-weighted Rota-Baxter Lie algebra $(g, \Xi)$, one may consider the cohomology of $\Xi$ with coefficients in the representation $\Xi$ itself. In [10] the author defines the cohomology of a $\lambda$-weighted Rota-Baxter operators motivated from their Maurer-Cartan characterizations. It follows that the cohomology of $\Xi$ in the sense of [10] is isomorphic to our cohomology of $\Xi$ with coefficients in the representation $\Xi$ itself.

We will now in a position to define the cohomology of the $\lambda$-weighted Rota-Baxter Lie algebra $(g, \Xi)$ with coefficients in the representation $(V, T)$. We first consider the two cochain complexes, namely,

- the Chevalley-Eilenberg cochain complex $\{C_{\text{CE}}^n(g, V), \delta_{\text{CE}}\}$ defining the cohomology of the Lie algebra $g$ with coefficients in the representation $V$,

- the complex $\{C_{\text{CE}}^n(g_\Xi, \tilde{V}), \partial_{\text{CE}}\}$ defining the cohomology of $\Xi$ with coefficients in the representation $T$.

The following result is similar to [30, Proposition 5.1].

### 3.2 Proposition.
The collection of maps $\{\Phi^n : C_{\text{CE}}^n(g, V) \to C_{\text{CE}}^n(g_\Xi, \tilde{V})\}_{n \geq 0}$ defined by

$$\Phi^0 = \text{id}_V,$$

$$\Phi^n(f)(x_1, \ldots, x_n) = f(\Xi(x_1), \ldots, \Xi(x_n)) - \sum_{k=0}^{n-1} \lambda^{n-k-1} \sum_{1 \leq i_1 < \cdots < i_k} T \circ f(x_1, \ldots, \Xi(x_{i_1}), \ldots, \Xi(x_{i_k}), \ldots, x_n)$$
is a morphism of cochain complexes from \( \{ C^n_{CE}(\mathfrak{g}, V), \delta_{CE} \} \) to \( \{ C^n_{CE}(\mathfrak{g}, \tilde{V}), \partial_{CE} \} \), i.e., \( \partial_{CE} \circ \Phi^n = \Phi^{n+1} \circ \delta_{CE} \), for \( n \geq 0 \).

Let \((\mathfrak{g}, \Sigma)\) be a \( \lambda \)-weighted Rota-Baxter Lie algebra and \((V, \mathcal{T})\) be a representation of it. For each \( n \geq 0 \), we define an abelian group \( C^n_{RB}(\mathfrak{g}, V) \) by

\[
C^n_{RB}(\mathfrak{g}, V) = \begin{cases} 
C^n_{CE}(\mathfrak{g}, V) & \text{if } n = 0 \\
C^n_{CE}(\mathfrak{g}, V) \oplus C^{n-1}_{CE}(\mathfrak{g}, \tilde{V}) & \text{if } n \geq 1
\end{cases}
\]

and a map \( \delta_{RB} : C^n_{RB}(\mathfrak{g}, V) \to C^{n+1}_{RB}(\mathfrak{g}, V) \) by

\[
\delta_{RB}(v) = (\delta_{CE}(v), -v), \quad \text{for } v \in C^n_{RB}(\mathfrak{g}, V) = V,
\]

\[
\delta_{RB}(f, g) = (\delta_{CE}(f), -\partial_{CE}(g) - \Phi^n(f)), \quad \text{for } (f, g) \in C^n_{RB}(\mathfrak{g}, V).
\]

Note that

\[
(\delta_{RB}^2)(v) = \delta_{RB}(\delta_{CE}(v), -v) = ((\delta_{CE})^2(v), \delta_{CE}(v) - \Phi^2 \circ \delta_{CE}(v)) = 0 \quad (\because \delta_{CE} \circ \Phi^0 = \Phi^1 \circ \delta_{CE})
\]

and

\[
(\delta_{RB}^2)(f, g) = \delta_{RB}(\delta_{CE}(f), -\partial_{CE}(g) - \Phi^n(f)) = ((\delta_{CE})^2(f), (\delta_{CE})^2(g) + \delta_{CE} \circ \Phi^n(f) - \Phi^{n+1} \circ \delta_{CE}(f)) = 0 \quad (\because \delta_{CE} \circ \Phi^n = \Phi^{n+1} \circ \delta_{CE}).
\]

This shows that \( \{ C^n_{RB}(\mathfrak{g}, V), \delta_{RB} \} \) is a cochain complex. Let \( Z^n_{RB}(\mathfrak{g}, V) \) and \( B^n_{RB}(\mathfrak{g}, V) \) denote the space of \( n \)-cocycles and \( n \)-coboundaries, respectively. Then we have \( B^n_{RB}(\mathfrak{g}, V) \subset Z^n_{RB}(\mathfrak{g}, V) \), for \( n \geq 0 \). The corresponding quotients

\[
H^n_{RB}(\mathfrak{g}, V) := \frac{Z^n_{RB}(\mathfrak{g}, V)}{B^n_{RB}(\mathfrak{g}, V)}, \quad \text{for } n \geq 0
\]

are called the cohomology of the \( \lambda \)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \Sigma)\) with coefficients in the representation \((V, \mathcal{T})\).

Observe that there is a short exact sequence of cochain complexes

\[
0 \to C^n_{CE}(\mathfrak{g}, \tilde{V}) \to C^n_{RB}(\mathfrak{g}, V) \to C^n_{CE}(\mathfrak{g}, V) \to 0
\]

given by \( i(g) = (0, (-1)^n g) \) and \( p(f, g) = f \), for \( (f, g) \in C^n_{RB}(\mathfrak{g}, V) \). This short exact sequence induces the following long exact sequence on cohomology groups

\[
\cdots \to H^n_{CE}(\mathfrak{g}, \tilde{V}) \to H^n_{CE}(\mathfrak{g}, V) \to H^n_{CE}(\mathfrak{g}, \tilde{V}) \to H^n_{RB}(\mathfrak{g}, V) \to H^n_{CE}(\mathfrak{g}, V) \to H^n_{CE}(\mathfrak{g}, \tilde{V}) \to \cdots
\]

3.3 Proposition. Let \((\mathfrak{g}, \Sigma)\) be a \( \lambda \)-weighted Rota-Baxter Lie algebra and \((V, \mathcal{T})\) be a representation of it. Then the cohomology of \((\mathfrak{g}, \Sigma)\) with coefficients in the representation \((V, \mathcal{T})\) is isomorphic to the cohomology of \((\mathfrak{g}, -\lambda \text{id}_\mathfrak{g} - \Sigma)\) with coefficients in the representation \((V, -\lambda \text{id}_V)\).

Proof. Let \( \Sigma' = -\lambda \text{id}_\mathfrak{g} - \Sigma \) and \( \mathcal{T}' = -\lambda \text{id}_V - \mathcal{T} \). Then it is easy to see that the Lie algebra \( \mathfrak{g}_{\Sigma'} \) of Proposition 2.14 (i) is negative to the Lie algebra \( \mathfrak{g}_\Sigma \). Moreover, the representation \( \tilde{\rho}' \) of the Lie algebra \( \mathfrak{g}_{\Sigma'} \) on \( V \) (we denote this representation by \( \tilde{\rho}' \)) as of Theorem 2.17 is given by the negative of the representation \( \tilde{\rho} \) of the Lie algebra \( \mathfrak{g}_\Sigma \) on \( V \). Hence the corresponding Chevalley-Eilenberg differentials

\[
\partial_{CE} : C^n_{CE}(\mathfrak{g}, \tilde{V}) \to C^n_{CE}(\mathfrak{g}, \tilde{V}) \quad \text{and} \quad \partial_{CE}' : C^n_{CE}(\mathfrak{g}_{\Sigma'}, \tilde{V}') \to C^n_{CE}(\mathfrak{g}_{\Sigma'}, \tilde{V}')
\]
are related by $\partial^\gamma_{CE}f = -\partial_{CE}f$. Finally, if $(\Phi')^n : C^n_{CE}(g, V) \to C^n_{CE}(g_{\mathbb{T}}, \hat{V}')$ be the map as of Proposition 3.2 replacing $\mathbb{T}$ by $\mathbb{T}'$ and replacing $\mathbb{T}$ by $\mathbb{T}'$, then we have $(\Phi')^n(f) = (-1)^n \Phi^n(f)$.

For each $n \geq 0$, define an isomorphism of vector spaces

$$
\Xi^n : C^n_{CE}(g, V) \oplus C^{n-1}_{CE}(g_{\mathbb{T}}, \hat{V}) \to C^n_{CE}(g, V) \oplus C^{n-1}_{CE}(g_{\mathbb{T}}, \hat{V}') \quad \text{by} \quad \Xi^n(f, g) = (f, (-1)^{n-1}g).
$$

Moreover,

$$(\partial'_{RB} \circ \Xi^n)(f, g) = \delta'_{RB}(f, (-1)^n-1g) = \left(\delta_{CE}f, (-1)^n\delta_{CE}g - (\Phi')^n(f)\right)$$

$$= \left(\delta_{CE}f, (-1)^{n+1}\delta_{CE}g - (-1)^n\Phi^n(f)\right)$$

$$= (\Xi^{n+1} \circ \delta_{RB})(f, g).$$

This shows that the collection of maps $\{\Xi^n\}_{n \geq 0}$ commute with the respective coboundary maps. Hence they induce isomorphism on cohomology.

3.1 $H^0$ and $H^1$

Let $(g, \mathbb{T})$ be a $\lambda$-weighted Rota-Baxter Lie algebra and $(V, \mathbb{T})$ be a representation of it. An element $v \in V$ is in $Z^0_{RB}(g, V)$ if and only if $(\delta_{CE}(v), -v) = 0$. This holds only when $v = 0$. Therefore, it follows from the definition that $H^0_{RB}(g, V) = 0$.

A pair $(\gamma, v) \in \text{Hom}(g, V) \oplus V$ is said to be a derivation on the weighted Rota-Baxter Lie algebra $(g, \mathbb{T})$ with coefficients in the representation $(V, \mathbb{T})$ if they satisfies

$$\gamma([x, y]) = \rho(x)(\gamma(y)) - \rho(y)(\gamma(x)),$$

$$\gamma(\mathbb{T}(x)) - \mathbb{T}(\gamma(x)) = \mathbb{T}(\rho(x)v) - \rho(\mathbb{T}(x)v), \quad \text{for} \quad x, y \in g.$$

It follows from the first condition that $\gamma$ is a derivation on the Lie algebra $g$ with coefficients in $V$. The second condition says that the obstruction of vanishing $\gamma \circ \mathbb{T} - \mathbb{T} \circ \gamma$ is measured by the presence of $v$. We denote the set of all derivations by $\text{Der}(g, V)$.

A derivation is said to be inner if it is of the form $(-\delta_{CE}(v), v)$, for some $v \in V$. The set of all inner derivations are denoted by $\text{InnDer}(g, V)$.

It follows from the definition that $Z^1_{RB}(g, V) = \text{Der}(g, V)$ and $B^1_{RB}(g, V) = \text{InnDer}(g, V)$. Hence we have $H^1_{RB}(g, V) = \frac{\text{Der}(g, V)}{\text{InnDer}(g, V)}$, the space of outer derivations.

3.2 Relation with the cohomology of weighted Rota-Baxter associative algebras

In [30] Wang and Zhou defined the cohomology of a weighted Rota-Baxter associative algebra with coefficients in a Rota-Baxter bimodule. In this subsection, we show that our cohomology is related to the cohomology of [30] by suitable skew-symmetrization.

3.4 Definition. (i) A $\lambda$-weighted Rota-Baxter associative algebra is a pair $(\mathfrak{A}, \mathcal{R})$ in which $\mathfrak{A}$ is an associative algebra and $\mathcal{R} : \mathfrak{A} \to \mathfrak{A}$ is a linear map satisfying

$$\mathcal{R}(a) \cdot \mathcal{R}(b) = \mathcal{R}(\mathcal{R}(a) \cdot b + a \cdot \mathcal{R}(b) + \lambda a \cdot b), \quad \text{for} \quad a, b \in \mathfrak{A}.$$

(ii) Let $(\mathfrak{A}, \mathcal{R})$ be a $\lambda$-weighted Rota-Baxter associative algebra. A Rota-Baxter bimodule over it consists of a pair $(\mathcal{M}, \mathcal{R})$ in which $\mathcal{M}$ is an $\mathfrak{A}$-bimodule (denote both left and right actions by cdot) and
\[ R : M \rightarrow M \] is a linear map satisfying for \( a \in A, m \in M, \)
\[
R(a) \cdot R(m) = R(R(a) \cdot m + a \cdot R(m) + \lambda \ a \cdot m), \\
R(m) \cdot R(a) = R(R(m) \cdot a + m \cdot R(a) + \lambda \ m \cdot a).
\]

### 3.5 Remark
It follows from the above definition that any \( \lambda \)-weighted Rota-Baxter associative algebra \((A, R)\) is a Rota-Baxter bimodule over itself. This is called the adjoint Rota-Baxter bimodule.

The following result is straightforward.

### 3.6 Proposition
Let \((A, R)\) be a \( \lambda \)-weighted Rota-Baxter associative algebra. Then \((A, c)\) is a \( \lambda \)-weighted Rota-Baxter Lie algebra, where \( A_c \) is the vector space \( A \) with the commutator Lie bracket
\[
[a, b]_c = a \cdot b - b \cdot a, \text{ for } a, b \in A_c.
\]
(\text{This is called the skew-symmetrization}). Moreover, if \((M, R)\) is a Rota-Baxter bimodule over \((A, R)\), then \((M, R)\) is a representation of the \( \lambda \)-weighted Rota-Baxter Lie algebra \((A, R)\), where \( M \rightarrow M \) is a Rota-Baxter bimodule over itself. This is called the adjoint Rota-Baxter bimodule.

Let \((A, R)\) be a \( \lambda \)-weighted Rota-Baxter associative algebra. Then it is known that \((A, \ast_{RB})\) is an associative algebra \([12]\), where
\[
a \ast_{RB} b = R(a) \cdot b + a \cdot R(b) + \lambda \ a \cdot b, \text{ for } a, b \in A.
\]
We denote this associative algebra by \( A_{RB} \). Moreover, if \((M, R)\) is a Rota-Baxter bimodule over the \( \lambda \)-weighted Rota-Baxter associative algebra \((A, R)\), then it has been observed in \([30]\) that \( M \) carries a bimodule structure over the associative algebra \( A_{RB} \) with left and right actions
\[
a \ast_m = R(a) \cdot m - R(a \cdot m) \text{ and } m \ast a = m \cdot R(a) - R(m \cdot a), \text{ for } a \in A_{RB} \text{ and } m \in M.
\]
Denote this \( A_{RB} \)-bimodule by \( \bar{M} \). To define the cohomology of the \( \lambda \)-weighted Rota-Baxter associative algebra \((A, R)\) with coefficients in the Rota-Baxter bimodule \((M, R)\), the authors in \([30]\) considered two Hochschild cochain complexes, namely \( \{C^n_H(A, M), \delta_H\} \) and \( \{C^n_{ass}(A_{RB}, \bar{M}), \delta_{ass}\} \). The first one is the Hochschild complex of the given algebra \( A \) with coefficients in the bimodule \( M \), whereas the second one is the Hochschild complex of the algebra \( A_{RB} \) with coefficients in the bimodule \( \bar{M} \). They proved that the collection \( \{\Psi^n : C^n_H(A, M) \rightarrow C^n_{ass}(A_{RB}, \bar{M})\}_{n \geq 0} \) of maps given by
\[
\Psi^0 = \text{id}_M, \text{ and } \\
\Psi^n(f)(a_1, \ldots, a_n) = f(R(a_1), \ldots, R(a_n)) - \sum_{k=0}^{n-1} \sum_{i_1 < \cdots < i_k} R \circ f(a_{i_1}, \ldots, R(a_{i_k}), \ldots, a_n)
\]
defines a morphism of cochain complexes. This allows the authors to consider the cochain complex \( \{C^0_{RB}(A, M), \delta_{ass}^{RB}\} \), where
\[
C^0_{RB}(A, M) = C^0_H(A, M) = M \text{ and } C^n_{RB}(A, M) = C^n_{ass}(A_{RB}, \bar{M}) \oplus C^{n-1}_H(A_{RB}, \bar{M}), \text{ for } n \geq 1
\]
and \( \delta_{ass}^{RB} : C^n_{RB}(A, M) \rightarrow C^{n+1}_{RB}(A, M) \) given by
\[
\delta_{ass}^{RB} = (\delta_{H}, - \delta_{ass}(g) - \Psi^n(f)), \text{ for } (f, g) \in C^n_{ass}(A_{RB}, \bar{M}).
\]
The corresponding cohomology groups are called the cohomology of the \( \lambda \)-weighted Rota-Baxter associative algebra \((A, R)\) with coefficients in the Rota-Baxter bimodule \((M, R)\).
To find the connection between the cohomology of a weighted Rota-Baxter associative algebra and the cohomology of the corresponding skew-symmetrized weighted Rota-Baxter Lie algebra, we need the following result.

3.7 Proposition. Let \((\mathfrak{A}, \mathcal{R})\) be a \(\lambda\)-weighted Rota-Baxter associative algebra and \((\mathcal{M}, \mathcal{R})\) be a Rota-Baxter bimodule. Then \((\mathfrak{A}, \mathcal{R})\) is a Lie algebra structure on \(\mathfrak{A}\) induced by the \(\lambda\)-weighted Rota-Baxter operator \(\mathcal{R}\) on the Lie algebra \(\mathfrak{A}_c\) as in Proposition 2.14 (i). Moreover, the representation of the Lie algebra \((\mathfrak{A}, \mathcal{R})\) on \(\mathcal{M}_c\) and the representation of the Lie algebra \((\mathfrak{A}_c, \mathcal{R}_c)\) on \(\mathcal{M}_c\) coincide.

It is known that the standard skew-symmetrization gives rise to a morphism from the Hochschild cochain complex of an associative algebra to the Chevalley-Eilenberg cochain complex of the corresponding skew-symmetrized Lie algebra [20]. Hence the following diagrams commute:

\[
\begin{array}{ccc}
C^n_H(\mathfrak{A}, \mathcal{M}) & \overset{\delta h}{\longrightarrow} & C^{n+1}_H(\mathfrak{A}, \mathcal{M}) \\
\delta_{CE} & \downarrow & \downarrow \delta_{CE} \\
C^n_{CE}(\mathfrak{A}_c, \mathcal{M}_c) & \overset{\partial c_e}{\longrightarrow} & C^{n+1}_{CE}(\mathfrak{A}_c, \mathcal{M}_c)
\end{array}
\quad \begin{array}{ccc}
C^n_H(\mathfrak{A}_c, \mathcal{M}_c) & \overset{\partial c_e}{\longrightarrow} & C^{n+1}_H(\mathfrak{A}_c, \mathcal{M}_c) \\
\delta h & \downarrow & \downarrow \delta h \\
C^n_{CE}(\mathfrak{A}_c, \mathcal{M}_c) & \overset{\partial c_e}{\longrightarrow} & C^{n+1}_{CE}(\mathfrak{A}_c, \mathcal{M}_c)
\end{array}
\]

Here \(S_*\) are the skew-symmetrization maps. As a consequence, we get the following.

3.8 Theorem. Let \((\mathfrak{A}, \mathcal{R})\) be a \(\lambda\)-weighted Rota-Baxter associative algebra and \((\mathcal{M}, \mathcal{R})\) be a Rota-Baxter bimodule. Then the collection of maps

\[
S_n : C^n_{RB}(\mathfrak{A}, \mathcal{M}) \rightarrow C^n_{RB}(\mathfrak{A}_c, \mathcal{M}_c), \quad \text{for} \ n \geq 0
\]

defined by \(S_0 = \text{id}_\mathcal{M}\) and \(S_n = (S_n, S_{n-1})\) for \(n \geq 1\), induces a morphism from the cohomology of \((\mathfrak{A}, \mathcal{R})\) with coefficients in the Rota-Baxter bimodule \((\mathcal{M}, \mathcal{R})\) to the cohomology of \((\mathfrak{A}_c, \mathcal{R}_c)\) with coefficients in the representation \((\mathcal{M}_c, \mathcal{R}_c)\).

Proof. We only need to check that the maps \(\{S_n\}_{n \geq 0}\) commute with corresponding coboundary maps. For \((f, g) \in C^n_{RB}(\mathfrak{A}, \mathcal{M})\),

\[
(\delta_{RB} \circ S_n)(f, g) = \delta_{RB}(S_n f, S_{n-1} g) = (\delta_{CE} \circ S_n)(f) - \Phi^n \circ (S_n \circ \Psi^n)(g) = (S_{n+1} \circ \delta_H)(f) - S_n \circ \delta_H(g) - \Phi^n \circ (S_n \circ \Psi^n) = S_{n+1} \circ \delta_{RB}(f, g).
\]

Hence the result follows.

4 Applications of cohomology

In this section, we study abelian extensions and formal 1-parameter deformations of weighted Rota-Baxter Lie algebras in terms of cohomology.

4.1 Abelian extensions and \(H^2\)

Let \((\mathfrak{g}, \mathfrak{T})\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra. Let \((\mathcal{V}, \mathcal{T})\) be a pair of a vector space \(\mathcal{V}\) and a linear map \(\mathcal{T} : \mathcal{V} \rightarrow \mathcal{V}\). Note that \((\mathcal{V}, \mathcal{T})\) can be considered as a \(\lambda\)-weighted Rota-Baxter Lie algebra where the Lie bracket on \(\mathcal{V}\) is assumed to be trivial.
4.1 Definition. An abelian extension of \((\mathfrak{g}, \Xi)\) by \((\mathcal{V}, \mathcal{T})\) is a short exact sequence of morphisms of \(\lambda\)-weighted Rota-Baxter Lie algebras

\[
0 \longrightarrow (\mathcal{V}, \mathcal{T}) \xrightarrow{i} (\hat{\mathfrak{g}}, \hat{\Xi}) \xrightarrow{p} (\mathfrak{g}, \Xi) \longrightarrow 0. \tag{8}
\]

In this case, we say that \((\hat{\mathfrak{g}}, \hat{\Xi})\) is an abelian extension of \((\mathfrak{g}, \Xi)\) by \((\mathcal{V}, \mathcal{T})\).

4.2 Definition. Let \((\hat{\mathfrak{g}}, \hat{\Xi})\) and \((\hat{\mathfrak{g}}', \hat{\Xi}')\) be two abelian extensions of \((\mathfrak{g}, \Xi)\) by \((\mathcal{V}, \mathcal{T})\). They are said to be isomorphic if there exists an isomorphism \(\phi : (\hat{\mathfrak{g}}, \hat{\Xi}) \to (\hat{\mathfrak{g}}', \hat{\Xi}')\) of \(\lambda\)-weighted Rota-Baxter Lie algebras which makes the following diagram commutative

\[
\begin{array}{ccc}
0 & \longrightarrow & (\mathcal{V}, \mathcal{T}) \xrightarrow{i} (\hat{\mathfrak{g}}, \hat{\Xi}) \xrightarrow{p} (\mathfrak{g}, \Xi) \longrightarrow 0 \\
0 & \longrightarrow & (\mathcal{V}, \mathcal{T}) \xrightarrow{i'} (\hat{\mathfrak{g}}', \hat{\Xi}') \xrightarrow{p'} (\mathfrak{g}, \Xi) \longrightarrow 0 \\
\phi & & \downarrow \\
\end{array}
\]

Let \((\hat{\mathfrak{g}}, \hat{\Xi})\) be an abelian extension of \((\mathfrak{g}, \Xi)\) by \((\mathcal{V}, \mathcal{T})\) as of (8). A section of the map \(p\) is a linear map \(s : \mathfrak{g} \to \hat{\mathfrak{g}}\) satisfying \(p \circ s = \text{id}_{\mathfrak{g}}\). Note that a section of \(p\) always exists.

Let \(s : \mathfrak{g} \to \hat{\mathfrak{g}}\) be a section of the map \(p\). We define a linear map \(\rho : \mathfrak{g} \to \text{End}(\mathcal{V})\) by \(\rho(x)(u) := [s(x), i(u)]_{\hat{\mathfrak{g}}}, \) for \(x \in \mathfrak{g}\) and \(u \in \mathcal{V}\). Then it can be easily check that \(\mathcal{V} = (\mathcal{V}, \rho)\) is a representation of the Lie algebra \(\mathfrak{g}\). More generally, \((\mathcal{V}, \mathcal{T})\) is a representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \Xi)\). One can easily check that this representation does only depend on the choice of the section \(s\). We call this as the induced representation on \((\mathcal{V}, \mathcal{T})\).

Suppose \((\mathcal{V}, \mathcal{T})\) is a given representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \Xi)\). We denote by \(\text{Ext}(\mathfrak{g}, \mathcal{V})\) the isomorphism classes of abelian extensions of \((\mathfrak{g}, \Xi)\) by \((\mathcal{V}, \mathcal{T})\) for which the induced representation on \((\mathcal{V}, \mathcal{T})\) is the prescribed one.

4.3 Theorem. Let \((\hat{\mathfrak{g}}, \hat{\Xi})\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra and \((\mathcal{V}, \mathcal{T})\) be a representation of it. Then there is a one-to-one correspondence between \(\text{Ext}(\mathfrak{g}, \mathcal{V})\) and the second cohomology group \(H^2_{\text{RB}}(\mathfrak{g}, \mathcal{V})\).

Proof. Let \((\psi, \chi) \in Z^2_{\text{RB}}(\mathfrak{g}, \mathcal{V})\) be a 2-cocycle, i.e., we have \(\delta_{\text{CE}}\psi = 0\) and \(-\partial_{\text{CE}}(\chi) - \Phi^2(\psi) = 0\). Consider the space \(\mathfrak{g} \oplus \mathcal{V}\) with the bracket

\[
[(x, u), (y, v)] = ([x, y], \rho(x)v - \rho(y)u + \psi(x, y)).
\]

Since \(\delta_{\text{CE}}\psi = 0\), it follows that the above bracket makes \(\mathfrak{g} \oplus \mathcal{V}\) into a Lie algebra. We denote this Lie algebra by \(\hat{\mathfrak{g}}\). We also define a map \(\hat{\Xi} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}\) by \(\hat{\Xi}(x, u) = (\Xi(x), \mathcal{T}(u) + \chi(x)), \) for \((x, u) \in \hat{\mathfrak{g}}\). Since \(-\partial_{\text{CE}}(\chi) - \Phi^2(\psi) = 0\), it follows that \(\hat{\Xi}\) is a \(\lambda\)-weighted Rota-Baxter operator on the Lie algebra \(\hat{\mathfrak{g}}\). In other words, \((\hat{\mathfrak{g}}, \hat{\Xi})\) is a \(\lambda\)-weighted Rota-Baxter Lie algebra. Moreover, the short exact sequence

\[
0 \longrightarrow (\mathcal{V}, \mathcal{T}) \xrightarrow{i} (\hat{\mathfrak{g}}, \hat{\Xi}) \xrightarrow{p} (\mathfrak{g}, \Xi) \longrightarrow 0
\]

defines an abelian extension of \((\mathfrak{g}, \Xi)\) by \((\mathcal{V}, \mathcal{T})\), where \(i(u) = (0, u)\) and \(p(x, u) = x\), for \(x \in \mathfrak{g}\) and \(u \in \mathcal{V}\). Note that the canonical section \(s : \mathfrak{g} \to \hat{\mathfrak{g}}\), \(s(x) = (x, 0)\) induces \((\mathcal{V}, \mathcal{T})\) with the original representation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((\mathfrak{g}, \Xi)\).

Let \((\psi', \chi') \in Z^2_{\text{RB}}(\mathfrak{g}, \mathcal{V})\) be another 2-cocycle cohomologous to \((\psi, \chi)\). Then there exists a pair \((\gamma, v) \in \text{Hom}(\mathfrak{g}, \mathcal{V}) \oplus \mathcal{V}\) such that

\[
(\psi, \chi) - (\psi', \chi') = (\delta_{\text{CE}}(\gamma), -\partial_{\text{CE}}v - \Phi^1(\gamma)).
\]
Consider the map $\phi: g \oplus V \rightarrow g \oplus V$ given by

$$\phi(x, u) = (x, u - \gamma(x) - \partial_{CE}(v)(x)).$$

Then it can be easily checked that $\phi$ defines an isomorphism of abelian extensions from $(\hat{g}, \hat{\Xi})$ to $(\tilde{g}^', \tilde{\Xi}^')$. Therefore, there is a well-defined map $\Upsilon : H^2_{RB}(g, V) \rightarrow \text{Hom}(g, V)$.

Conversely, let $(\hat{g}, \hat{\Xi})$ be an abelian extension given by (8). Let $s: g \rightarrow \hat{g}$ be a section of the map $p$. We define elements $\psi \in \text{Hom}(\wedge^2 g, V)$ and $\chi \in \text{Hom}(g, V)$ by

$$\psi(x, y) = [s(x), s(y)]_{\hat{g}} - s[x, y] \quad \text{and} \quad \chi(x) = \hat{\Xi}(s(x)) - s(\hat{\Xi}(x)), \quad \text{for } x, y \in g.$$ 

Then it follows from a straightforward computation that the pair $(\psi, \chi)$ defines a 2-cocycle in $Z^2_{RB}(g, V)$. Moreover, the corresponding cohomology class in $H^2_{RB}(g, V)$ doesn’t depend on the choice of the section $s$.

Let $(\hat{g}, \hat{\Xi})$ and $(\tilde{g}', \tilde{\Xi}')$ be two isomorphic abelian extensions as of (9). Let $s: g \rightarrow \hat{g}$ be a section of the map $p$. Then we have

$$p' \circ (\phi \circ s) = p \circ s = \text{id}_g.$$

This shows that $s' = \phi \circ s : g \rightarrow \hat{g}'$ is a section of $p'$. If $(\psi', \chi')$ denotes the 2-cocycle in $Z^2_{RB}(g, V)$ corresponding to the abelian extension $(\hat{g}', \hat{\Xi}')$ and section $s': g \rightarrow \hat{g}'$ of the map $p'$, then

$$\psi'(x, y) = [s'(x), s'(y)]_{\hat{g}'} - s'[x, y]$$

$$= [\phi \circ s(x), \phi \circ s(y)]_{\hat{g}} - \phi \circ s[x, y]$$

$$= \phi([s(x), s(y)]_{\hat{g}} - s[x, y])$$

$$= \phi(\psi(x, y)) = \phi(x, y) \quad (\because \phi|_V = \text{id}_V)$$

and

$$\chi'(x) = \hat{\Xi}'(s'(x)) - s'(\hat{\Xi}(x))$$

$$= \hat{\Xi}'(\phi \circ s(x)) - \phi \circ s(\hat{\Xi}(x))$$

$$= \phi(\hat{\Xi}(s(x)) - s(\hat{\Xi}(x))) = \phi(\chi(x)) = \chi(x) \quad (\because \phi|_V = \text{id}_V).$$

This show that $(\psi, \chi)$ and $(\psi', \chi')$ correspond to the same element in $H^2_{RB}(g, V)$. Therefore, we have a well-defined map $\Omega : \text{Ext}(g, V) \rightarrow H^2_{RB}(g, V)$. Finally, the map $\Upsilon$ and $\Omega$ are inverses to each other. This completes the proof.

4.2 Formal deformations

In this subsection, we will consider formal deformations of a $\lambda$-weighted Rota-Baxter Lie algebra $(g, \Xi)$. Here we will simultaneously deform the Lie bracket on $g$ and the $\lambda$-weighted Rota-Baxter operator $\Xi$. We find the relation between such deformations and cohomology of $(g, \Xi)$ with coefficients in the adjoint representation.

Let $(g, \Xi)$ be a $\lambda$-weighted Rota-Baxter Lie algebra. Let $\mu \in C^2_{CE}(g, g) = \text{Hom}(\wedge^2 g, g)$ be the element that corresponds to the Lie bracket on $g$, i.e., $\mu(x, y) = [x, y]$, for $x, y \in g$. Consider the space $g[[t]]$ of formal power series in $t$ with coefficients from $g$. Then $g[[t]]$ is a $k[[t]]$-module.

4.4 Definition. A formal 1-parameter deformation of $(g, \Xi)$ consists of a pair $(\mu_t, \Xi_t)$ of two formal power
Proof. Since \( \mu_t \) defines a Lie algebra structure on \( g[[t]] \) and the \( k[[t]] \)-linear map \( \xi_t : g[[t]] \to g[[t]] \) is a \( \lambda \)-weighted Rota-Baxter operator. In other words, \( (g[[t]], \mu_t, \xi_t) \) is a \( \lambda \)-weighted Rota-Baxter algebra over \( k[[t]] \).

Thus, \( (\mu_t, \xi_t) \) is a formal 1-parameter deformation of \( (g, \xi) \) if and only if

\[
\begin{align*}
\mu_t(x, \mu_t(y, z)) + \mu_t(y, \mu_t(z, x)) + \mu_t(z, \mu_t(x, y)) &= 0, \\
\mu_t(\xi_t(x), \xi_t(y)) &= \xi_t(\mu_t(\xi_t(x), y) + \mu_t(x, \xi_t(y)) + \lambda \mu_t(x, y)),
\end{align*}
\]

for \( x, y, z \in g \). They are equivalent to the following two systems of identities: for \( n \geq 0 \) and \( x, y, z \in g \),

\[
\begin{align*}
\sum_{i+j=n} \mu_i(x, \mu_j(y, z)) + \mu_i(y, \mu_j(z, x)) + \mu_i(z, \mu_j(x, y)) &= 0, \quad (10) \\
\sum_{i+j+k=n} \mu_i(\xi_j(x), \xi_k(y)) &= \sum_{i+j+k=n} \xi_i(\mu_j(\xi_k(x), y) + \mu_j(x, \xi_k(y)) + \lambda \xi_i(\mu_j(x, y))), \quad (11)
\end{align*}
\]

4.5 Definition. Two formal deformations \( (\mu_t, \xi_t) \) and \( (\mu_t', \xi_t') \) of a \( \lambda \)-weighted Rota-Baxter Lie algebra \( (g, \xi) \) are said to be equivalent if there is a formal isomorphism

\[
\phi_t = \sum_{i \geq 0} \phi_i t^i : g[[t]] \to g[[t]], \quad \text{where } \phi_i \in \text{Hom}(g, g) \text{ with } \phi_0 = \text{id}_g
\]

such that the \( k[[t]] \)-linear map \( \phi_t \) is a morphism of \( \lambda \)-weighted Rota-Baxter Lie algebras from \( (g[[t]]') = (g[[t]], \mu_t', \xi_t') \) to \( (g[[t]], \mu_t, \xi_t) \).

Therefore, \( (\mu_t, \xi_t) \) and \( (\mu_t', \xi_t') \) are equivalent if the followings are hold:

\[
\phi_t(\mu_t'(x, y)) = \mu_t(\phi_t(x), \phi_t(y)) \quad \text{and} \quad \phi_t \circ \xi_t' = \xi_t \circ \phi_t.
\]

They can be equivalently expressed by the following system of equations: for \( n \geq 0 \),

\[
\begin{align*}
\sum_{i+j=n} \phi_i(\mu_j'(x, y)) &= \sum_{i+j+k=n} \mu_i(\phi_j(x), \phi_k(x)), \quad (12) \\
\sum_{i+j=n} \phi_i \circ \xi_j'(x) &= \sum_{i+j=n} \xi_i \circ \phi_j(x). \quad (13)
\end{align*}
\]

4.6 Theorem. Let \( (\mu_t, \xi_t) \) be a formal 1-parameter deformation of the \( \lambda \)-weighted Rota-Baxter Lie algebra \( (g, \xi) \). Then \( (\mu_1, \xi_1) \in C^2_{RB}(g, g) \) is a 2-cocycle in the cohomology of \( (g, \xi) \) with coefficients in the adjoint representation. Moreover, the corresponding cohomology class in \( H^2_{RB}(g, g) \) depends only on the equivalence class of the deformation \( (\mu_t, \xi_t) \).

Proof. Since \( (\mu_t, \xi_t) \) is a formal 1-parameter deformation, we get from (10) and (11) for \( n = 1 \) that

\[
[x, \mu_1(y, z)] + \mu_1(x, [y, z]) + [y, \mu_1(z, x)] + \mu_1(y, [z, x]) + [z, \mu_1(x, y)] + \mu_1(z, [x, y]) = 0
\]
and

\[
\mu_1(x, y) = [\xi_1(x), y] + [x, \xi(y)] + [\xi(x), \xi(y)]
\]

\[
= \xi_1([\xi(x), y] + [x, \xi(y)]) + \xi([\xi_1(x), y] + \mu_1(x, y) + \mu_1(x, \xi(y)) + \lambda \xi(\mu_1(x, y)) + \lambda \xi_1([x, y]).
\]

The first identity is equivalent to \(\delta_{\text{CE}} \mu_1 = 0\) while the second identity is equivalent to \(-\partial_{\text{CE}}(\xi_1) - \Phi^2(\mu_1) = 0\). This implies that

\[
\delta_{\text{RB}}(\mu_1, \xi_1) = (\delta_{\text{CE}} \mu_1, -\partial_{\text{CE}}(\xi_1) - \Phi^2(\mu_1)) = 0.
\]

This proves the first part. For the second part, we let \((\mu_1, \xi_1)\) and \((\mu_1', \xi_1')\) be two equivalent deformations. For \(n = 1\), it follows from (12) and (13) that

\[
(\mu_1' - \mu_1)(x, y) = [\phi_1(x), y] + [x, \phi_1(y)] - \phi_1(x, y),
\]

\[
(\xi_1' - \xi_1)(x) = \xi(\phi_1(x)) - \phi_1(\xi(x)).
\]

Hence \((\mu_1', \xi_1') - (\mu_1, \xi_1) = (\delta_{\text{CE}} \phi_1, -\Phi^1(\phi_1)) = \delta_{\text{RB}}(\phi_1, 0).\) This shows that \((\mu_1, \xi_1)\) and \((\mu_1', \xi_1')\) correspond to the same cohomology class in \(H_{\text{RB}}^1(g, \xi)\). This completes the proof.

We will now prove the following interesting result about formal 1-parameter deformations.

4.7 Theorem. Let \((g, \xi)\) be a \(\lambda\)-weighted Rota-Baxter Lie algebra. If \(H^2_{\text{RB}}(g, \xi) = 0\) then any formal 1-parameter deformation of \((g, \xi)\) is equivalent to the trivial one \((\mu_1 = \mu, \xi_1 = \xi)\).

Proof. Let \((\mu_1, \xi_1)\) be a formal 1-parameter deformation of the \(\lambda\)-weighted Rota-Baxter Lie algebra \((g, \xi)\). It follows from Theorem 4.6 that \((\mu_1, \xi_1)\) is a 2-cocycle. Thus, from the hypothesis, there exists \((\psi_1, v) \in \text{Hom}(g, V) \oplus V\) such that

\[
(\mu_1, \xi_1) = \delta_{\text{RB}}(\psi_1, v) = (\delta_{\text{CE}} \psi_1, -\partial_{\text{CE}} \xi - \Phi^1(\psi_1)).
\]  

(14)

Let \(\phi_1 : g \to g\) be the map \(\phi_1 = \text{id}_g + \phi_1 t\). Then

\[
(\mu_1, \xi_1) = \Phi^{-1}_t \circ \mu_1 \circ (\phi_t \otimes \phi_t), \quad (\xi_1) = \Phi^{-1}_t \circ \xi_t \circ \phi_t)
\]

is a deformation of \((g, \xi)\) equivalent to \((\mu_1, \xi_1)\). Using (14), it can be easily check that the linear terms of \(\mu_t\) and \(\xi_t\) are vanish. In other words, \(\mu_t\) and \(\xi_t\) are of the form

\[
\mu_t = \mu + \mu_2 t^2 + \cdots \quad \text{and} \quad \xi_t = \xi + \xi_2 t^2 + \cdots.
\]

Hence by repeating this argument, we conclude that \((\mu_t, \xi_t)\) is equivalent to \((\mu_1 = \mu, \xi_1 = \xi)\). Hence the proof.

4.8 Remark. A \(\lambda\)-weighted Rota-Baxter Lie algebra \((g, \xi)\) is said to be rigid if any deformation of it is equivalent to the trivial one. It follows from the above theorem that the vanishing of the second cohomology group \(H^2_{\text{RB}}(g, \xi)\) is a sufficient condition for the rigidity of \((g, \xi)\).

5 Rota-Baxter paired operators

Let \(g\) be a Lie algebra and \(V = (V, \rho)\) be a representation of it. In this section, we introduce weighted Rota-Baxter paired operators that are related to weighted Rota-Baxter Lie algebras together with their representations. We introduce a differential graded Lie algebra whose Maurer-Cartan elements are weighted
Rota-Baxter paired operators. Using such characterization, we define the cohomology of Rota-Baxter paired operators. Finally, we discuss deformations of weighted Rota-Baxter paired operators in terms of cohomology.

5.1 Definition. Let \( g \) be a Lie algebra and \( V = (V, \rho) \) be a representation of it. A pair \( (\mathfrak{z}, \mathcal{T}) \) of linear maps \( \mathfrak{z} : g \to g \) and \( \mathcal{T} : V \to V \) is said to be a \( \lambda \)-weighted Rota-Baxter paired operators if they satisfy the followings: for \( x, y \in g \) and \( u \in V \),

\[
\mathcal{T}(x, \mathfrak{z}(y)) = \mathfrak{z}(\mathcal{T}(x), y) + [x, \mathfrak{z}(y)] + \lambda[x, y],
\]

\[
\rho(Tx)(Tu) = (\mathcal{T}(\rho(x)u) + \rho(x)(Tu) + \lambda \rho(x)u).
\]

5.2 Remark. Observe that the first condition is equivalent to say that \((g, \mathfrak{z})\) is a \( \lambda \)-weighted Rota-Baxter Lie algebra, while the second condition is equivalent to say that \((V, \mathcal{T})\) is a representation of \((g, \mathfrak{z})\). Therefore, \( \lambda \)-weighted Rota-Baxter paired operators are closely related with weighted Rota-Baxter Lie algebras and their representations. Hence examples of Rota-Baxter paired operators follow from the examples of Section 2.

Let \( g \) be a Lie algebra. Then the direct sum \( g \oplus g \) carries a Lie algebra structure with bracket given by

\[
[[x, x'], (y, y')]] = ([x, y], [x, y'] + [x', y] + \lambda[x', y']).
\]

We denote this Lie algebra by \((g \oplus g)_\lambda\). Moreover, if \( V = (V, \rho) \) is a representation of the Lie algebra \( g \), then \( V \oplus V \) can be equipped with a representation of the Lie algebra \((g \oplus g)_\lambda\) with the action given by

\[
\rho_\lambda((x, x'))(u, u') = (\rho(x)u, \rho(x)u' + \rho(x')u + \lambda \rho(x')u').
\]

Hence the direct sum \( g \oplus g \oplus V \oplus V \) carries the semidirect product Lie algebra structure whose bracket is explicitly given by

\[
[[[x, x', u, u'], (y, y', v, v')]] = ([x, y], [x, y'] + [x', y] + \lambda[x', y'], \rho(x)v - \rho(y)u, \rho(x)v' + \rho(x')v + \lambda(x')v' - \rho(y)u' - \rho(y')u - \rho(y')u').
\]

We denote this Lie algebra by \((g \oplus g \oplus V \oplus V)_\lambda\). The following proposition is straightforward, hence we omit the details.

5.3 Proposition. Let \( g \) be a Lie algebra and \( V \) be a representation of it. A pair \( (\mathfrak{z}, \mathcal{T}) \) of linear maps \( \mathfrak{z} : g \to g \) and \( \mathcal{T} : V \to V \) forms a \( \lambda \)-weighted Rota-Baxter paired operators if and only if

\[
\text{Gr}_{\mathfrak{z}, \mathcal{T}} := \{(\mathfrak{z}x, x, Tu, u) \mid x \in g, u \in V \} \subset g \oplus g \oplus V \oplus V
\]

is a Lie subalgebra of \((g \oplus g \oplus V \oplus V)_\lambda\).

Let \( g \) be a Lie algebra and \( V \) be a representation of it. We denote \( g' \) and \( V' \) by another copies of \( g \) and \( V \), respectively. Consider the vector space \( l = g' \oplus g' \oplus V' \oplus V' \) and the Nijenhuis-Richardson bracket on the space \( \oplus_{n \geq 1} \text{Hom}(\wedge^n l, l) \) of skew-symmetric multilinear maps on \( l \), given by

\[
[f, g]_{\text{NR}} := f \circ g - (-1)^{(m-1)(n-1)} g \circ f,
\]

where

\[
(f \circ g)(l_1, \ldots, l_{m+n-1}) = \sum_{\sigma \in S_{m+n-1}} (-1)^{\sigma} f(g(l_{\sigma(1)}, \ldots, l_{\sigma(n)}), l_{\sigma(n+1)}, \ldots, l_{\sigma(m+n-1)}),
\]

17
for \( f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \) and \( g \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \). The Nijenhuis-Richardson bracket \([\ , \ ]_{\text{NR}}\) makes the shifted graded vector space \( \oplus_{n \geq 0} \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \) into a graded Lie algebra. Moreover, it can be easily checked that the graded subspace \( \mathfrak{a} = \oplus_{n \geq 0} (\text{Hom}(\wedge^{n+1} \mathfrak{g}', \mathfrak{g}) \oplus \text{Hom}(\wedge^n \mathfrak{g} \otimes \mathcal{V}', \mathcal{V})) \) is an abelian subalgebra.

In the subsequent, we will use the following notations. Let

\[ \diamond \mu \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}) \text{ and } \mu' \in \text{Hom}(\wedge^2 \mathfrak{g}', \mathfrak{g}') \text{ denote the multiplications of } \mathfrak{g} \text{ and } \mathfrak{g}', \text{ respectively; } \]
\[ \diamond \text{ ad}' : \mathfrak{g} \times \mathfrak{g}' \to \mathfrak{g}' \text{ be the adjoint representation of } \mathfrak{g} \text{ on } \mathfrak{g}'; \]
\[ \diamond \rho : \mathfrak{g} \times \mathcal{V} \to \mathcal{V} \text{ and } \rho' : \mathfrak{g} \times \mathcal{V}' \to \mathcal{V}' \text{ denote the representations of } \mathfrak{g} \text{ on } \mathcal{V} \text{ and } \mathcal{V}', \text{ respectively; } \]
\[ \diamond g : \mathfrak{g}' \times \mathcal{V} \to \mathcal{V} \text{ and } g' : \mathfrak{g}' \times \mathcal{V}' \to \mathcal{V}' \text{ denote the representations of } \mathfrak{g}' \text{ on } \mathcal{V} \text{ and } \mathcal{V}', \text{ respectively.} \]

We consider the element

\[ \theta = \mu + \text{ad}' + \rho + \rho' + g \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}). \]

It is easy to see that \( \theta \) is a Maurer-Cartan element in the graded Lie algebra \( (\oplus_{n \geq 0} \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}), [\ , \ ]_{\text{NR}}) \).

Therefore, it induces a differential \( d_\theta := [\theta, -]_{\text{NR}} \) on the graded vector space \( \oplus_{n \geq 0} \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \). Hence by the derived bracket construction of Voronov [29], the shifted graded space

\[ \mathfrak{a}[-1] = \oplus_{n \geq 1} (\text{Hom}(\wedge^n \mathfrak{g}', \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g}' \otimes \mathcal{V}', \mathcal{V})) \]

inherits a graded Lie algebra structure with bracket given by

\[ [P, Q] = (-1)^m [d_\theta(P), Q]_{\text{NR}}, \quad (15) \]

for \( P = (P_1, P_2) \in \text{Hom}(\wedge^n \mathfrak{g}', \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g}' \otimes \mathcal{V}', \mathcal{V}) \) and \( Q = (Q_1, Q_2) \in \text{Hom}(\wedge^n \mathfrak{g}', \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g}' \otimes \mathcal{V}', \mathcal{V}). \)

On the other hand, the element \( \theta' = - (\mu' + \rho') \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}) \) is also a Maurer-cartan element in the graded Lie algebra \( (\oplus_{n \geq 0} \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}), [\ , \ ]_{\text{NR}}) \). Hence it induces a differential \( d_{\theta'} = [\theta', -]_{\text{NR}} \) on \( \oplus_{n \geq 1} \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \). The differential \( d_{\theta'} \) restricts to a differential \( d \) on the space \( \mathfrak{a}[-1] \). Finally, the elements \( \theta \) and \( \theta' \) additionally satisfies \([\theta, \theta'] = 0\) which in turn implies that \( d \) is a derivation for the bracket \([\ , \ ]\) on \( \mathfrak{a}[-1] \). In other words, the triple \((\mathfrak{a}[-1], [\ , \ ], d)\) is a differential graded Lie algebra.

By replacing \( \mathfrak{g}' \) by \( \mathfrak{g} \) and replacing \( \mathcal{V}' \) by \( \mathcal{V} \), we get the following.

5.4 Theorem. Let \( \mathfrak{g} \) be a Lie algebra and \( \mathcal{V} \) be a representation of it. Then the triple

\[ (\oplus_{n \geq 1} (\text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes \mathcal{V}, \mathcal{V})), [\ , \ ], d) \]

is a differential graded Lie algebra. A pair \( \mathbf{T} = (\Sigma, \mathcal{T}) \) is a Rota-Baxter paired operators if and only if \( \mathbf{T} \) is a Maurer-Cartan element in the above differential graded Lie algebra.

Proof. The first part is clear from previous discussions. For any \( \mathbf{T} = (\Sigma, \mathcal{T}) \in \text{Hom}(\mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\mathcal{V}, \mathcal{V}) \), by expanding (15) similar to [27], we get that

\[ [\mathbf{T}, \mathbf{T}]_1(x, y) = 2(\Sigma([\Sigma(x), y] + [x, \Sigma(y)]) - [\Sigma(x), \Sigma(y)]), \]
\[ [\mathbf{T}, \mathbf{T}]_2(x, u) = 2(\mathcal{T}(\rho(\Sigma x)u + \rho(x)(\mathcal{T} u)) - \rho(\Sigma x)(\mathcal{T} u)), \]

for \( x \in \mathfrak{g} \) and \( u \in \mathcal{V} \). On the other hand, we can also check (similar to [10]) that

\[ (dT)_1(x, y) = \lambda \Sigma [x, y] \quad \text{and} \quad (dT)_2(x, u) = \lambda \mathcal{T}(\rho(x)u). \]

Hence it follows that \( d\mathbf{T} + \frac{1}{2}[\mathbf{T}, \mathbf{T}] = 0 \) if and only if \( \mathbf{T} = (\Sigma, \mathcal{T}) \) is a Rota-Baxter paired operators. \( \square \)
It follows from the above theorem that a Rota-Baxter paired operators \( T = (\mathcal{T}, \mathcal{T}) \) induces a differential

\[
d_T : \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes \mathcal{V}, \mathcal{V}) \to \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^n \mathfrak{g} \otimes \mathcal{V}, \mathcal{V}), \quad \text{for } n \geq 1,
\]

\[
d_T = d + [T, -].
\]

We define \( C^n_{\text{RBp}}(\mathfrak{g}, \mathcal{V}) = \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes \mathcal{V}, \mathcal{V}), \) for \( n \geq 1. \) Then \( \{C^n_{\text{RBp}}(\mathfrak{g}, \mathcal{V}), d_T\} \) is a cochain complex. The corresponding cohomology groups are called the cohomology of the Rota-Baxter paired operators \( (\mathcal{T}, \mathcal{T}) \).

Note that, for a Rota-Baxter paired operators \( T = (\mathcal{T}, \mathcal{T}) \), the differential \( d_T \) makes the tuple

\[
(\bigoplus_{n \geq 1} C^n_{\text{RBp}}(\mathfrak{g}, \mathcal{V}), [\cdot, \cdot], d_T)
\]

into a differential graded Lie algebra. In the following, we will show that this differential graded Lie algebra governs the Maurer-Cartan deformations of \( T \).

5.5 Proposition. Let \( \mathfrak{g} \) be a Lie algebra and \( \mathcal{V} \) be a representation of it. Let \( T = (\mathcal{T}, \mathcal{T}) \) be a Rota-Baxter paired operators. Then for a pair \( T' = (\mathcal{T}', \mathcal{T}') \) of maps \( \mathcal{T}' : \mathfrak{g} \to \mathfrak{g} \) and \( \mathcal{T}' : \mathcal{V} \to \mathcal{V}, \) the sum \( T + T' = (\mathcal{T} + \mathcal{T}', \mathcal{T} + \mathcal{T}') \) is a Rota-Baxter paired operators if and only if \( T' \) is a Maurer-Cartan element in the differential graded Lie algebra \( (\bigoplus_{n \geq 1} C^n_{\text{RBp}}(\mathfrak{g}, \mathcal{V}), [\cdot, \cdot], d_T). \)

Proof. We observe that

\[
d(T + T') + \frac{1}{2} [T + T', T + T']
\]

\[
= dT + dT' + \frac{1}{2}([T, T'] + 2[T, T'] + [T', T'])
\]

\[
= dT' + [T, T'] + \frac{1}{2} [T', T'] = d_T(T') + \frac{1}{2} [T', T'].
\]

Hence the result follows from the Maurer-Cartan characterization of Rota-Baxter paired operators. \( \square \)

5.6 Remark. In \([10]\) the present author studied formal 1-parameter deformations of weighted Rota-Baxter operators by keeping the underlying algebra intact. Given a Lie algebra \( \mathfrak{g} \) and a representation \( \mathcal{V}, \) one could also study formal 1-parameter deformations of Rota-Baxter paired operators by keeping the algebra \( \mathfrak{g} \) and the representation \( \mathcal{V} \) intact. Then it is easy to see that the above-defined cohomology governs such deformations. More precisely, the infinitesimals lie in the first cohomology group, and the obstruction to extending a finite order deformation lies in the second cohomology group.

Acknowledgements. The research is supported by the fellowship of Indian Institute of Technology (IIT) Kanpur.

References

[1] M. Aguiar, Pre-Poisson algebras, *Lett. Math. Phys.* 54 (2000), 263-277.

[2] F. V. Atkinson, Some aspects of Baxter’s functional equation, *J. Math. Anal. Appl.* 7 (1963), 1-30.

[3] C. Bai, O. Bellier, L. Guo and X. Ni, Splitting of operations, Manin products and Rota-Baxter operators, *Int. Math. Res. Not. IMRN* 2013 (2013), 485-524.
[4] C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, Comm. Math. Phys. 297 (2010), 553-596.

[5] C. Bai, L. Guo and X. Ni, Relative Rota-Baxter algebras and tridendriform algebras, J. Algebra Appl. 12 (2013) 1350027.

[6] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, Pacific J. Math. 10 (1960), 731-742.

[7] P. Cartier, On the structure of free Baxter algebras, Adv. Math. 9 (1972) 253-265.

[8] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, Comm. Math. Phys. 210 (2000), 249-273.

[9] A. Das, Deformations of associative Rota-Baxter operators, J. Algebra 560 (2020) 144-180.

[10] A. Das, Cohomology and deformations of weighted Rota-Baxter operators, arXiv:2108.05411

[11] A. Das and S. K. Mishra, The $L_\infty$-deformations of associative Rota-Baxter algebras and homotopy Rota-Baxter operators, arXiv:2008.11076

[12] K. Ebrahimi-Fard, Loday-type algebras and the Rota-Baxter relation, Lett. Math. Phys. 61 (2002) 139-147.

[13] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras and dendriform algebras, J. Pure Appl. Algebra 212 (2008) 320-339.

[14] M. Gerstenhaber, On the deformation of rings and algebras, Ann. of Math. (2) 79 (1964), 59-103.

[15] L. Guo and W. Keigher, Baxter algebras and shuffle products, Adv. Math. 150 (2000) 117-149.

[16] L. Guo and B. Zhang, Renormalization of multiple zeta values, J. Algebra 319 (2008) 3770-3809.

[17] L. Guo, An introduction to Rota-Baxter algebra, Surveys of Modern Mathematics, 4. International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

[18] J. Jiang, Y. Sheng and C. Zhu, Cohomologies of relative Rota-Baxter operators on Lie groups and Lie algebras, arXiv:2108.02627v1

[19] B. A. Kupershmidt, What a classical r-matrix really is, J. Nonlinear Math. Phys. 6 (1999), no. 4, 448-488.

[20] J.-L. Loday, Cyclic homology, Grundlehren der Mathematischen Wissenschaften 301, Springer-Verlag, Berlin, 1998. xx+513 pp.

[21] A. Lazarev, Y. Sheng and R. Tang, Deformations and homotopy theory of relative Rota-Baxter Lie algebras, Comm. Math. Phys. 383 (2021) 595-631.

[22] J. B. Miller, Some properties of Baxter operators, Acta Math. Acad. Sci. Hungar. 17 (1966), 387-400.

[23] J. Qiu and Y. Chen, Gröbner-Shirshov bases for Lie $\Omega$-algebras and free Rota-Baxter Lie algebras, J. Algebra Appl. Vol. 16, No. 10 (2017) 1750190.

[24] A. Nijenhuis and R. Richardson, Deformation of Lie algebra structures, J. Math. Mech. 17 (1967), 89-105.
[25] G.-C. Rota, Baxter algebras and combinatorial identities, I, II, Bull. Amer. Math. Soc. 75 (1969), 325-329; ibid. 75 1969 330-334.

[26] M. A. Semenov-Tyan Shanskii, What is a classical r-matrix? Funct. Anal. Appl. 17 (1893) 259-272.

[27] R. Tang, C. Bai, L. Guo and Y. Sheng, Deformations and their controlling cohomologies of \(O\)-operators, Comm. Math. Phys. 368 (2019), no. 2, 665-700.

[28] R. Tang, C. Bai, L. Guo and Y. Sheng, Homotopy Rota-Baxter operators, homotopy \(O\)-operators and homotopy post-Lie algebras, arXiv:1907.13504

[29] Th. Voronov, Higher derived brackets and homotopy algebras, J. Pure Appl. Algebra 202 (2005), 133-153.

[30] K. Wang and G. Zhou, Deformations and homotopy theory of Rota-Baxter algebras of any weight, arXiv:2108.06744

[31] Y. Zhang, D. Chen, X. Gao and Y. Luo, Weighted infinitesimal unitary bialgebras on rooted forests and weighted cocycles, Pacific J. Math. 302 (2019), 741-766.

[32] Y. Zhang, X. Gao and J. Zheng, Weighted infinitesimal unitary bialgebras on matrix algebras and weighted associated Yang-Baxter equations, arXiv:1811.00842

[33] H. Zheng, L. Guo and L. Zhang, Rota-Baxter paired modules and their constructions from Hopf algebras, J. Algebra 559 (2020) 601-624.