Abstract—A trading system is said to be robust if it generates a positive return regardless of market direction. To this end, a consistently positive expected trading gain is often used as a robustness metric for a trading system. In this letter, we propose a new class of trading policies called the double linear policy in an asset trading scenario when transaction costs are involved. Unlike many existing papers, we first show that the desired robust positive expected gain may disappear when transaction costs are involved. Then we quantify under what conditions the desired positivity can still be preserved. In addition, we conduct heavy Monte-Carlo simulations for an underlying asset whose prices are governed by a geometric Brownian motion with jumps to validate our theory. A more realistic backtesting example involving historical data for cryptocurrency Bitcoin-USD is also studied.

Index Terms—Finance, stochastic systems, robustness, algorithmic trading.

I. INTRODUCTION

A TRADING system is said to be robust if it generates a positive return regardless of market direction. In particular, a consistently positive expected trading gain-loss is often used as a robustness metric. The earliest contribution to robust return can be found in papers such as [1], [2]. It is known that the so-called Simultaneous Long-Short (SLS) control scheme, see [3], [4], [5], [6], [7], [8], guarantees the satisfaction of the so-called Robust Positive Expectation (RPE); i.e., the cumulative trading gain-loss function is guaranteed to have positive expected value for a broad class of stock price processes. This fact attracts many new extensions and modifications to the SLS theory.

For example, the SLS theory involves RPE results with respect to stock prices having time-varying drift and volatility; see [9], prices generated from Merton’s diffusion model and more general models [8], [10], generalization from a static linear policy to the case of Proportional–Integral (PI) controllers [11], and discrete-time systems with delays [12].

More recently, in [13], an SLS control with cross-coupling is proposed to trade two stocks. In [14], a generalized RPE Theorem to the case of an SLS controller, which can have different parameters for the long and short sides of the trade, is studied. In [15], an $H_\infty$ approach for selecting the SLS controller parameters is proposed.

To close this brief introduction, we mention some related studies using the control-theoretic approach in stock trading scenarios; e.g., see [16] for studying optimal selling rule, [8] for studying RPE problems in discrete-time setting when there are no transaction costs involved, [17], [18] for studying optimal pair trading, and [19] for studying a new feedforward control in transaction level price model. While various extensions and ramifications are proposed to improve the SLS theory, the effect of transaction costs, which is believed to erode the desired positivity, is typically studied with empirical backtests; see [20]. To this end, we propose a new SLS-based trading structure, which we call the double linear policy, in a discrete-time setting to study theoretical positive expectation property when the transaction costs are involved.

The contributions of this letter are threefold: First, we propose a new double linear policy scheme for a class of asset price processes involving independent returns; see Section II. In the financial market with nonzero transaction costs, we derive explicit expressions for the expected value and variance of the cumulative gain-loss function using the proposed policy scheme; see Section III. Second, we go beyond the existing literature by quantifying under what condition the desired expected positivity can still be preserved when transaction costs are involved. A novel asymptote approach is used to obtain a relatively conservative positivity; see Theorem 1 in Section II. Third, we validate the theory via heavy Monte-Carlo simulations. An empirical backtesting example using the historical data for cryptocurrency Bitcoin USD is also provided; see Section IV.

II. PRELIMINARIES

In this section, we provide some necessary preliminaries that are useful for the analysis to follow.

A. Asset Prices and Returns

For $k = 0, 1, 2, \ldots$, let $S(k) > 0$ be the price of an underlying risky asset, often to be stock, at stage $k$. Then the associated per-period return, call it $X(k)$, is given by
X(k) := \frac{S(k+1)-S(k)}{S(k)}. Assuming that X_{min} \leq X(k) \leq X_{max} for all k almost surely with known bounds -1 < X_{min} < 0 < X_{max} < \infty and X_{min} and X_{max} are in the support of X(k). Unlike many existing papers in finance that models the asset price dynamics, we assume that X(k) are independent in k and have arbitrary distribution with common but unknown mean \mathbb{E}[X(k)] := \mu and unknown variance \text{var}(X(k)) = \mathbb{E}[\{(X(k) - \mu)^2\}] := \sigma^2 \geq 0 for all k.

Remark 1: In practice, since the true mean \mu and variance \sigma are unknown, one often works with the estimated surrogates such as sample mean and sample standard deviation; see [21, Ch. 9] and Section IV-B for an illustration. While this approach may incur some estimation errors, as seen in the sections to follow, we shall quantify a range for \mu such that the desired positivity of expected return can be preserved.

B. Double Linear Policy and Transaction Costs

With initial account value V(0) := V_{0} > 0, we split it into two parts as follows: Taking an allocation constant \alpha \in [0, 1], we define V_L(k) := \alpha V_{0} as the initial account value for long position and V_S(k) := (1 - \alpha) V_{0} for short position. Then V(0) = V_L(0) + V_S(0). The trading policy or controller \pi(\cdot) is given by \pi(k) := \pi_L(k) + \pi_S(k) where \pi_L and \pi_S are of double linear forms:

\[
\pi_L(k) = w V_L(k);
\pi_S(k) = -w V_S(k)
\] (1)

where the weight w satisfies w \in \mathcal{W} for some admissible set \mathcal{W}. Specifically, when there are transaction costs per trade with percentage rate \varepsilon \in [0, 1], we take \varepsilon \pi_L(k) for long trade or \varepsilon |\pi_S(k)| for short trade. In the sequel, we take \mathcal{W} := [0, w_{\text{max}}] with

w_{\text{max}} := \min \left\{ \frac{1}{1 + \varepsilon}, \frac{1}{X_{\text{max}} + \varepsilon} \right\}.

We refer to the control scheme above as the double linear policy with pair (\alpha, w).

Remark 2: (i) The double linear form can be viewed as a natural extension of the pure long linear strategy by taking \alpha = 1; e.g., see [22], [23], [24]. (ii) The choice of the upper bound for \mathcal{W} above corresponds to assuring survival trades, see Lemma 1 in the next Section III-A to follow, and the trades are cash-financed; i.e., |\pi(k)| \leq V(k) for all k with probability one; see Lemma 4. Additionally, as seen later in Section III to follow, the double linear policy also enjoys some additional properties such as convexity and positive expectation.

C. Account Value Dynamics

The associated account values under \pi_L and \pi_S, denoted by V_L(k) and V_S(k) respectively, are described by the following stochastic recursive equations:

\[
\begin{align*}
V_L(k+1) &= V_L(k) + X(k)\pi_L(k) - \varepsilon \pi_L(k); \\
V_S(k+1) &= V_S(k) + X(k)\pi_S(k) - \varepsilon |\pi_S(k)|.
\end{align*}
\] (2)

By induction, one can readily verify that account values V_L(k) and V_S(k) are nonnegative for all k. Moreover, for stage k = 0, 1, ..., the account values for long and short trades, respectively, are as follows: V_L(k) = \prod_{j=0}^{k-1} (1 + w X_L(j)) V_L(0) and V_S(k) = \prod_{j=0}^{k-1} (1 - w X_S(j)) V_S(0) where X_L(k) := X(k) - \varepsilon and X_S(k) := X(k) + \varepsilon. Therefore, the overall account value at stage k is given by

\[
V(k) = V_L(k) + V_S(k) = V_0 \left( \alpha \prod_{j=0}^{k-1} (1 + w X_L(j)) + (1 - \alpha) \prod_{j=0}^{k-1} (1 - w X_S(j)) \right).
\]

Note that the account value V(k) depends on the allocation constant \alpha, the decision weight w, transaction costs rate \varepsilon and return sequence X = \{X(i)\}_{i=0}^{\infty}.

Remark 3: (i) According to classical SLS theory; e.g., see [5], [6], the associated control law can be written as

\[
\begin{align*}
\pi_L(k) &= \pi_0 + w (V_L(k) - V_L(0)); \\
\pi_S(k) &= -\pi_0 - w (V_S(k) - V_S(0))
\end{align*}
\]

for some initial investment \pi_0 > 0. It is readily verified that the classical SLS controller is equivalent to the double linear policy with \alpha = 1/2 if one takes \pi_0 := w V_0/2 and V_L(0) = V_S(0) := V_0/2. (ii) Our framework can easily adapt to involve the risk-free asset, such as a bond or bank account, with per-period interest rate r(k) \geq 0 for all k with probability one; see [25]. Then the account value dynamics for the long trade can be modeled as

\[
V_L(k+1) = (1 + r(k)) V_L(k) + (X(k) - \varepsilon - r(k)) \pi_L(k)
\]

and the account value dynamics V_S(k) for short trade stays the same. By taking the risk-free asset as a cash which yields r(k) := 0 for all k, we replicate Equation (2).

III. Analysis of Trading Performance

In this section, we study the trading performance of the proposed double linear policy. Specifically, we provide several results such as survivability, convexity, positive expectation, and cash-financing lemma.

A. Survivability Considerations

As mentioned in the previous subsection, every w \in \mathcal{W} assures survivability, i.e., bankruptcy avoidance. That is, for all k, the w-value that can potentially lead to V(k) < 0 is disallowed.

Lemma 1 (Survivability): Let \varepsilon \in [0, 1]. The double linear policy with a pair (\alpha, w) \in (0, 1] \times \mathcal{W} assures survival trades; i.e., for k = 0, 1, ..., V_L(k) > 0, V_S(k) \geq 0, and V(k) > 0 with probability one.

Proof: Note that for k = 0, we have V(0) = V_0 > 0. Hence V_L(0) = \alpha V_0 > 0 and V_S(0) = (1 - \alpha) V_0 \geq 0 for \alpha \in (0, 1]. Fix an integer k > 0. Observe that the account value for long trade is given by

\[
V_L(k) = \alpha \prod_{j=0}^{k-1} (1 + w X_L(j)) V_0 \\
\geq \alpha (1 + w (X_{\text{min}} - \varepsilon))^k V_0.
\]
To see $V_L(k) > 0$, it suffices to show that the lower bound is positive. Since $w \in \mathcal{W}$ and $X_{\min} > -1$, it implies that

$$1 + w(X_{\min} - \varepsilon) > 1 - w_{\max}(1 + \varepsilon) \geq 0$$

where the last inequality holds since $w_{\max} \leq 1/(1 + \varepsilon)$. Hence $(1 + w(X_{\min} - \varepsilon))^k > 0$ for all $k$. With the aid that $\alpha \in (0, 1]$ and $V_0 > 0$, it follows that $V_L(k) > 0$ for all $k$ with probability one. On the other hand, the account value for short trade is given by

$$V_S(k) = (1 - \alpha) \prod_{j=0}^{k-1} (1 - w\tilde{X}_S(j))V_0 \geq (1 - \alpha)(1 - w(X_{\max} + \varepsilon))^kV_0.$$

Since $0 \leq w \leq 1/(X_{\max} + \varepsilon)$, it follows that $1 - w(X_{\max} + \varepsilon) \geq 0$. Hence, $(1 - w(X_{\max} + \varepsilon))^k \geq 0$. With the fact that $\alpha \in (0, 1]$, we conclude that $V_S(k) \geq 0$ for all $k$ with probability one. Finally, the overall account value satisfies

$$V(k) = V_L(k) + V_S(k) \geq (\alpha(1 + w(X_{\min} - \varepsilon))^k + (1 - \alpha)(1 - w(X_{\max} + \varepsilon))^k)V_0 \equiv V_{\min}(k).$$

In combination with the results above, we conclude that $V_{\min}(k) > 0$ for all $k$ with probability one.

### B. Expected Trading Gain-Loss and Variance

We take $\overline{G}_k(\alpha, w, X, \varepsilon) := V(k) - V_0$ to be the cumulative trading gain-loss function up to stage $k$ where $X := \{X(i)\}_{i=0}^{k-1}$ is the sequence of returns $X(0), \ldots, X(k-1)$. Then the expected cumulative gain-loss is given by

$$\overline{G}_k(\alpha, w, \mu, \varepsilon) := \mathbb{E}[\overline{G}_k(\alpha, w, X, \varepsilon)].$$

The following lemma states the closed-form expression of the expected value and variance of the cumulative gain-loss function when there are transaction costs.

**Lemma 2 (Expected Trading Gain-Loss and Variance):** Fix $\varepsilon \in [0, 1]$. For stage $k = 0, 1, \ldots, \alpha(w, \varepsilon) \in [0, 1] \times \mathcal{W}$ yields expected cumulative gain-loss function

$$\overline{G}_k(\alpha, w, \mu, \varepsilon) := V_0\left(\alpha(1 + w(\mu - \varepsilon))^k + (1 - \alpha)(1 - w(\mu + \varepsilon))^k - 1\right).$$

Moreover, the corresponding variance is given by

$$\text{var}(\overline{G}_k(\alpha, w, X, \varepsilon)) = \alpha^2V_0^2\left(1 + w(\mu - \varepsilon))^2 + w^2\sigma^2\right)^k + (1 - \alpha)^2V_0^2\left(1 + w(\mu + \varepsilon))^2 + w^2\sigma^2\right)^k + 2\alpha(1 - \alpha)V_0^2\left(1 + 2w\varepsilon - w^2(\mu^2 + \sigma^2) - \varepsilon^2\right)^k - 2\alpha(1 - \alpha)V_0^2\left(1 + w(\mu - \varepsilon))^k - 2(1 - \alpha)V_0^2(1 - w(\mu + \varepsilon))^k + V_0^2 \left(\alpha(1 + w(\mu - \varepsilon))^k + (1 - \alpha)(1 - w(\mu + \varepsilon))^k - 1\right)^2.$$

Of course, the standard deviation of cumulative trading gain or loss is $\text{std}(\overline{G}_k(\alpha, w, X, \varepsilon)) = \sqrt{\text{var}(\overline{G}_k(\alpha, w, X, \varepsilon))}$.

**Proof:** Let $k$ be fixed. Set $R_L(k) := \prod_{i=0}^{k-1}(1 + w\tilde{X}_L(i))$ and $R_S(k) := \prod_{j=0}^{k-1}(1 - w\tilde{X}_S(j))$. Using the facts that the returns $X(k)$ are independent in $k$ and $\mathbb{E}[X(k)] = \mu$ for all $k$, we have

$$\overline{G}_k(\alpha, w, \mu, \varepsilon) = V_0(\alpha\mathbb{E}[R_L(k)] + (1 - \alpha)\mathbb{E}[R_S(k)] - 1) = V_0\left(\alpha(1 + w(\mu - \varepsilon))^k + (1 - \alpha)(1 - w(\mu + \varepsilon))^k - 1\right).$$

(3)

Subsequently, to obtain the variance, we use the fact $\text{var}(\overline{G}_k(\alpha, w, X, \varepsilon)) = \mathbb{E}[\overline{G}_k^2(\alpha, w, X, \varepsilon)] - \overline{G}_k^2(\alpha, w, \mu, \varepsilon)$. Since the second term can be obtained from Equation (3), it remains to calculate the second moment $\mathbb{E}[\overline{G}_k^2(\alpha, w, X, \varepsilon)]$. Without loss of generality, we set $V_0 := 1$. Using the facts that $X(k)$ are independent, $\mathbb{E}[X(k)] = \mu, \mathbb{E}[X^2(k)] = \sigma^2 + \mu^2$ for all $k$, and the linearity of expected value, a straightforward calculation leads to

$$\mathbb{E}[\overline{G}_k^2(\alpha, w, X, \varepsilon)] = \mathbb{E}[(\alpha R_L(k) + (1 - \alpha)R_S(k) - 1)^2] = \alpha^2\mathbb{E}[R_L^2(k)] + (1 - \alpha)^2\mathbb{E}[R_S^2(k)] + 2(1 - \alpha)\alpha\mathbb{E}[R_L(k)R_S(k)] - 2\alpha\mathbb{E}[R_L(k)] - 2(1 - \alpha)\mathbb{E}[R_S(k)] + 1$$

where the three terms $\mathbb{E}[R_L^2(k)], \mathbb{E}[R_S^2(k)],$ and $\mathbb{E}[R_L(k)R_S(k)]$ are obtained via lengthy but straightforward calculations. That is, we have

$$\mathbb{E}[R_L^2(k)] = \left(1 + w(\mu - \varepsilon))^2 + w^2\sigma^2\right)^k,$nand $\mathbb{E}[R_S^2(k)] = \left(1 + w(\mu + \varepsilon))^2 + w^2\sigma^2\right)^k$, and the cross term $\mathbb{E}[R_L(k)R_S(k)] = (1 + 2w\varepsilon - w^2(\mu^2 + \sigma^2) - \varepsilon^2)^k$. Therefore, in combination with the results above, the desired expression for the variance is obtained. To complete the proof, we note that the standard deviation $\text{std}(\overline{G}_k(\alpha, w, X, \varepsilon)) = \sqrt{\text{var}(\overline{G}_k(\alpha, w, X, \varepsilon))}$, which is desired.

**Remark 4:** (i) As a sanity test, if one decides not to trade; i.e., $w = 0$, then $\overline{G}_k(\alpha, w, \mu, \varepsilon) = 0 = \text{var}(\overline{G}_k(\alpha, w, X, \varepsilon))$. (ii) If there are no transaction costs and no uncertainty on the return; i.e., $\varepsilon := 0 = \sigma$, then a straightforward calculation leads to $\text{var}(\overline{G}_k(\alpha, w, X, \varepsilon)) = 0$. (iii) If the returns have no trend; i.e., $\mu = 0$, then it follows that $\overline{G}_k(\alpha, w, \mu, \varepsilon) = V_0(1 - w\varepsilon)^k < 0$. Thus, with nonzero transaction costs $\varepsilon > 0$, one should invest with zero weight $w = 0$ to avoid loss in expected gain-loss. (iii) Lemma 2 above can be viewed as a generalization of the result for the classical discrete-time SLS controller considered in [18].

### C. Convexity and Positive Expectation

To study the expected positivity, the following convexity result of the expected gain-loss function is useful.

**Lemma 3 (Convexity):** Fix $\varepsilon \in [0, 1]$. For $k > 1$, consider the double linear policy with pair $(\alpha, w) \in (0, 1) \times \mathcal{W} \setminus \{0\}$, then the expected cumulative gain-loss function $\overline{G}_k(\alpha, w, \mu, \varepsilon)$ is strictly convex in $w$ and in $w \in (-1, X_{\max}]$.

**Proof:** According to Lemma 2, we have $\overline{G}_k(\alpha, w, \mu, \varepsilon) = V_0\overline{G}_k(\alpha, w, \mu, \varepsilon)$ where

$$f_k(\alpha, w, \mu, \varepsilon) := \alpha(1 + w(\mu - \varepsilon))^k + (1 - \alpha)(1 - w(\mu + \varepsilon))^k - 1.$$

(4)
Note that any strict convex function multiplied by a strictly positive scalar is still strict convex. Fix $k \geq 2$. Since $V_0 > 0$, it suffices to show that $f_k(\alpha, w, \mu, \varepsilon)$ is strictly convex in $w \in \mathcal{W}$. Taking the derivative with respect to $w$ twice yields the second derivative
\[
\frac{\partial^2 f_k(\alpha, w, \mu, \varepsilon)}{\partial w^2} = ak(k-1)^2(1+w(\mu-\varepsilon))^{k-2}(\mu-\varepsilon)^2 + (1-\alpha)k(k-1)^2(1-w(\mu+\varepsilon))^{k-2}(\mu+\varepsilon)^2.
\]
To establish the desired convexity, it suffices to verify positivity of $(1+w(\mu-\varepsilon))^{k-2}$ and $(1-w(\mu+\varepsilon))^{k-2}$. Since $w \in \mathcal{W}\setminus\{0\}$ and $\mu > -1$, we have $1+w(\mu-\varepsilon) > 1-w(1+\varepsilon) > 0$. Therefore, $(1+w(\mu-\varepsilon))^{k-2} > 0$. On the other hand, note that $1-w(\mu+\varepsilon) \geq 1-w(X_{\text{max}}+\varepsilon) \geq 0$, which implies that $(1-w(\mu+\varepsilon))^{k-2} > 0$. Therefore, in combination with the fact that $\alpha \in (0, 1)$, it implies that $\frac{\partial^2 f_k(\alpha, w, \mu, \varepsilon)}{\partial w^2} > 0$ for all $w \in \mathcal{W}$. Hence, the strict convexity for $\overline{G}_k(\alpha, w, \mu, \varepsilon)$ is established.

To establish the convexity in $\mu$, a similar idea used above would work. That is, we take the second derivative with respect to $\mu$ and verify the strict positivity.

\[
\frac{\partial^2 f_k(\alpha, w, \mu, \varepsilon)}{\partial \mu^2} = a(k(k-1)kw^2(1+w(\mu-\varepsilon))^{k-2} + (1-\alpha)k(k-1)kw^2(1-w(\mu+\varepsilon))^{k-2} > 0
\]
where the last inequality holds since $w(\mu-\varepsilon) + 1 > 0$ and $1-w(\mu+\varepsilon) \geq 0$ as seen in the earlier derivations. Hence, the proof is complete.

**Remark 5:** For $\varepsilon > 0$ and $\alpha = 1/2$, the expected gain-loss function has the minimum at $\mu_0 = 0$. For $\alpha \neq 1/2$, then the minimum, call it $\mu_0$, requires to solve a non-linear equation $e^{\frac{1}{w} \log \frac{\mu}{\pi}} = \frac{1+w(\mu-\varepsilon)}{1+w(\mu+\varepsilon)}$. In either case, the associated gain-loss function $\overline{G}_k(\alpha, w, \mu_0, \varepsilon) < 0$; see also example in Section IV-A. Using the fact that $\overline{G}_k(\cdot)$ is continuous and strictly convex in $\mu$, Intermediate Theorem; e.g., see [26], indicates that there exist two critical points $\mu_{\pm}$ such that $\overline{G}_k(\alpha, w, \mu_{\pm}, \varepsilon) = 0$. Therefore, strict convexity assures that for $\mu > \mu_{+}$ or $\mu < \mu_{-}$, $\overline{G}_k(\alpha, w, \mu, \mu_{\pm}, \varepsilon) > 0$. However, it is still necessary to check that $\mu_+$ and $\mu_-$, the values at which $\overline{G}_k(\alpha, w, \mu, \mu_{\pm}, \varepsilon) = 0$ both belong to this $(-1, X_{\text{max}}]$ to establish convexity. Also, obtaining the two critical points $\mu_{\pm}$ analytically require solving high-order nonlinear functions, which is intractable in general. In the following theorem, we propose an asymptote approach for $\overline{G}_k$ to determine a moderately conservative positive expectation property.

**Theorem 1 (Positive Expectation):** Consider the double linear policy with pair $(\alpha, w)$. For all $k > 1$, the following statements hold true:

(i) If $\varepsilon = 0$, then any pair $(\alpha, w) \in [1/2] \times \mathcal{W}\setminus\{0\}$ guarantees the positive expected cumulative trading gain-loss; i.e., $\overline{G}_k(1/2, w, \mu, \varepsilon) > 0$ for all $\mu \neq 0$. Moreover, if $\mu = 0$, we have $\overline{G}_k(1/2, w, \mu, \varepsilon) = 0$.

(ii) If $\varepsilon > 0$, then any pair $(\alpha, w) \in (0, 1) \times \mathcal{W}\setminus\{0\}$ guarantees a positive expected cumulative trading gain-loss; i.e., $\overline{G}_k(\alpha, w, \mu, \varepsilon) > 0$ for all $\mu > v_+$ or $\mu < v_-$ where $v_+ := \frac{1}{w}(e^{\frac{1}{w} \log \frac{\mu}{\pi}} - 1) + \varepsilon$ and $v_- := \frac{1}{w}(1-e^{\frac{1}{w} \log \frac{\mu}{\pi}}) - \varepsilon$ with $v_- < 0 < v_+$.

**Proof:** To prove part (i), take $\varepsilon = 0$ and fix $k > 1$. Take the pair $(\alpha, w) \in [1/2] \times \mathcal{W}\setminus\{0\}$. Lemma 2 tells us that $\overline{G}_k(1/2, w, \mu, 0) = \frac{V_0}{2}(1+w(\mu)^k + (1-w(\mu)^k - 2)$. If $\mu = 0$, it is trivial to see that $\overline{G}_k(1/2, w, 0, 0) = 0$. On the other hand, if $\mu \neq 0$, then $w(\mu)^k \neq 0$ for all $w \in \mathcal{W}\setminus\{0\}$. By virtue of the fact that $(1+x)^k + (1-x)^k > 2$ for all $x \neq 0$ and $k > 1$, the desired positivity is guaranteed by $x := w(\mu)$.

To prove part (ii), we take $\varepsilon > 0$, fix $k > 1$. Take a pair $(\alpha, w) \in (0, 1) \times \mathcal{W}\setminus\{0\}$. Lemma 2 tells us that $\overline{G}_k(\alpha, w, \mu, \varepsilon) = V_0(\alpha, w, \mu, \varepsilon)$ where

\[
f_k(\alpha, w, \mu, \varepsilon) := \alpha(1+w(\mu-\varepsilon))^k + (1-\alpha)(1-w(\mu+\varepsilon)) - 1.
\]
Since $V_0 > 0$, without loss of generality, we set $V_0 := 1$. Note that $f_k(\alpha, w, \mu, \varepsilon) \geq \alpha(1+w(\mu-\varepsilon))^k - 1$ and $f_k(\alpha, w, \mu, \varepsilon)$ is continuous in $\mu$. The asymptote $\alpha(1+w(\mu-\varepsilon))^k - 1$ is continuous and monotonically increasing in $\mu$. By solving the zero crossing root for the asymptote; i.e., solving $\alpha(1+w(\mu-\varepsilon))^k - 1 := 0$ for $\mu$ yields

\[
\mu = \frac{1}{w}(e^{\frac{1}{w} \log \frac{\mu}{\pi}} - 1) + \varepsilon := v_+
\]
and $v_- < 0$. Hence, for $\mu > v_+$, we have $f_k(\alpha, w, \mu, \varepsilon) > 0$. Similarly, it is also readily verified that

\[
f_k(\alpha, w, \mu, \varepsilon) \geq (1-\alpha)(1-w(\mu+\varepsilon)) - 1
\]
and the asymptote $(1-\alpha)(1-w(\mu+\varepsilon)) - 1$ is again continuous and monotonically decreasing in $\mu$. Hence, solving $(1-\alpha)(1-w(\mu+\varepsilon)) - 1 = 0$ yields

\[
\mu = \frac{1}{w}(1-e^{\frac{1}{w} \log \frac{\pi}{\mu}}) - \varepsilon := v_-.
\]
and $v_- < 0$. Therefore, for $\mu < v_-$, $f_k(\alpha, w, \mu, \varepsilon) > 0$. It is also readily verified that $v_- < 0 < v_+$. Hence, in combination with the results above, we conclude that the desired positivity $f_k(\alpha, w, \mu, \varepsilon) > 0$ holds for $\mu > v_+$ or $\mu < v_-$. ■

**Remark 6:** The two critical points $v_{\pm}$ obtained in Theorem 1 relies on zero crossing roots of the asymptote of $\overline{G}_k$. Hence, it gives a conservative criterion for assuring the positive expectation. In addition, if $k \to \infty$, then $v_+, v_- \to \varepsilon, -\varepsilon$, respectively. When $k \to \infty$ and $\varepsilon = 0$, part (ii) in Theorem 1 reduces to part (i) in the same theorem.

**D. Cash-Financing Lemma**

As mentioned in Remark 2, the double linear policy with pair $(\alpha, w) \in [0, 1] \times \mathcal{W}$ assures a cash-financed trade.

**Lemma 4 (Cash-Financing):** The double linear policy with a pair $(\alpha, w) \in [0, 1] \times \mathcal{W}$ satisfies the cash-financing condition. That is, $|\pi(k)| \leq V(k)$ for all $k \geq 0$ with probability one.

**Proof:** With $V_0 > 0$, fix $(\alpha, w) \in [0, 1] \times \mathcal{W}$. For $k = 0$, we begin by noting that $\pi_L(0) = wV_L(0) = wV_0$ and $\pi_S(0) = -wV_S(0) = -(1-\alpha)wV_0$. Thus, observe that

\[
|\pi(0)| = |\pi_L(0) + \pi_S(0)| \leq |\pi_L(0)| + |\pi_S(0)| = wV_0 \leq V_0.
\]
where the first inequality follows from the triangle inequality. Fix \( k \geq 1 \). We have \( \pi(k) = \pi_L(k) + \pi_S(k) \) where \( \pi_L(k) = w V_L(k) \) and \( \pi_S(k) = -w V_S(k) \). Since \( w \in \mathcal{W} \), applying Lemma 1, it is readily verified that \( V_L(k), V_S(k) \geq 0 \) for all \( k \geq 1 \) with probability one.

\[
\begin{align*}
|\pi(k)| &= |\pi_L(k) + \pi_S(k)| \\
&\leq w|V_L(k) + V_S(k)| \\
&= wV(k) \leq V(k)
\end{align*}
\]

where the last inequality holds since \( w \in \mathcal{W} = [0, w_{\text{max}}] \) and \( w_{\text{max}} \leq \frac{1}{\pi + \epsilon} \leq 1 \).

IV. ILLUSTRATIVE EXAMPLES

This section provides two examples to illustrate our double linear policy scheme. The first example examines the positive expectation by carrying large numbers of Monte-Carlo simulations for prices modeled by geometric Brownian motion with jumps. The second example is for illustrating the trading performance using cryptocurrency historical data.

A. Monte-Carlo Simulations: GBM With Jumps

For \( t \in [0, T] \), consider a stock whose price \( S(t) \) follows a geometric Brownian motion (GBM) with jump model; see [27]; i.e.,

\[
S(t) = S_0 \exp \left( \left( \mu^* - \frac{1}{2} \sigma^* \right) t + \sigma^* W(t) \right) (1 - \delta)^N(t)
\]

where \( \{W(t)\}_{t \geq 0} \) is a standard Wiener process and \( \{N(t)\}_{t \geq 0} \) is a standard Poisson process with \( P(N(t) = k) = \frac{e^{-\lambda t} \lambda^k}{k!} \) that is independent of \( W(t) \). \( \mu^* \) is the drift constant, \( \sigma^* > 0 \) is the volatility constant, \( \lambda \) is the average rate of the jump occurs for the process, and \( \delta \in (0, 1) \) denotes that for each jump reduces the stock price by a fraction \( \delta \).

Taking a basic period length \( \Delta t := 1/252 \) and \( T := 1 \), we simulate the stock behavior for one year with an annualized\(^2\) drift rate \( \mu^* \in [-0.9, 0.9] \) and annualized volatility \( \sigma^* := 2|\mu^*/Z| \) with \( Z \) being a uniformly distributed random variable on interval \([0, 1]\), jump intensity \( \lambda = 0.1 \) with jump size \( \delta = 0.05 \). For each \( \mu^* \in [-0.9:0.01:0.9] \) with incremental values 0.01 and \( \sigma^* \), we generate 10,000 GBM with jump sample paths, hence, a total of 1,810,000 paths; see also [21] for details on this topic.

With \( \epsilon = 0.01\% \) and \( V_0 = 51 \), we consider a double linear policy with pair \( (\alpha, w) \in [0, 1] \times [1/(1+\epsilon)] \). Then we evaluate the associated expected trading gain-loss functions versus various \( \mu^* \); see Figure 1, where the red-colored dotted lines are obtained by Lemma 2, and the gray area indicates the negative expected gain-loss. From the figure, for each choice of \( \alpha \), we see that the Monte-Carlo results are consistent with the theory. In addition, it indicates that the desired expected positivity occurs after \( \mu > v_+^* \) and \( \mu < v_-^* \) for some \( v_+^* \) and \( v_-^* \). Similar patterns are also found for different choices of \( \epsilon, \lambda, \delta, \) and \( \sigma \). While not studied here, it would be of interests as a future work to test a more general Merton’s jump diffusion model; see [8].

B. Backtesting With Cryptocurrency

In this example, we apply the double linear policy to trade Bitcoin USD (BTC-USD) within the period from January 02, 2020 to August 01, 2022 with a total of 952 days; see Figure 2 for the corresponding prices trajectory. Compared to conventional stocks, Bitcoin prices are much more volatile and sensitive to regulatory and market events. In this case, the maximum and minimum daily returns are \( X_{\text{max}} \approx 0.1875 \) and \( X_{\text{min}} \approx -0.3717 \), respectively. With an initial account value of \( V_0 = 500,000 \), transaction costs\(^3\) \( \epsilon = 0.1\% \) and weight \( w = 1/4 \), the trading performances in terms of gain-loss functions are shown in Figure 3. For higher transaction costs, say \( \epsilon = 0.1\% \) with the same weight \( w = 1/4 \), the trading performances are shown in Figure 4. From the figures, we see the effect of transaction costs, and one should choose a proper allocation constant \( \alpha \) of the fund when \( \epsilon > 0 \).

\[^2\]For a one-year 252 trading days, annualized drift rate and annualized volatility can be approximated by the daily drift rate and daily volatility; i.e., \( \mu^* = 252\mu \) and \( \sigma^* = \sqrt{252}\sigma \).

\[^3\]Generally, most cryptocurrency exchanges charge between 0% and 1.5% per trade, depending on the trading volume and whether it is buying or selling; e.g., see [28]. Regular transaction costs charged by common cryptocurrency exchanges such as Binance is about \( \epsilon = 0.1\% \); see also https://www.binance.com/en/fee/schedule.
Another possible direction for future research is the use of a model-free feedback controller. A model for different fee structures, different values for the feedback gains in the long and short positions can still be preserved. We also validate our theory via heavy Monte-Carlo simulations using GBM with jumps and provide a backtesting example using Bitcoin-USD historical data.

Regarding future research, it would be of interest to determine an “optimal” pair \((a, w)\) that also preserves desired positivity; see [29] for studies on determining the pair by solving a robust optimal gain selection problem without transaction costs. See also [15] for some initial studies along this line. Another possible direction for future research is the use of different values for the feedback gains in the long and short branches; e.g., see [14]. A model for different fee structures, including maker fee or taker fee, may be interesting to pursue. The other interesting direction is to consider a multi-asset portfolio case; see [13], [30] for a preliminary extension to pair tradings.

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