Increased critical temperature $T_c$ in the ideal anisotropic boson gas

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Abstract

Thermal anisotropy has a positive effect on Bose-Einstein condensation. Any large temperature anisotropy $A = T_\perp/T_\parallel - 1 > 0$ shifts the critical temperature $T_c = T_\perp c$ into the higher temperature range while increasing the condensate density.

Key words: Bose gases, thermal anisotropy, Bose-Einstein condensation

There is a vital long-time interest in the low-temperature behaviour of massive boson gases which was stimulated by the discovery of superfluidity in the He II phase (Kapitza, 1938; Allen and Misener, 1938) and the development of its theory by Landau (1941, 1949) and his followers (Penrose and Onsager, 1956; Ginzburg and Pitaevskii, 1958; Pitaevskii, 1961; Gross, 1961). These theories culminated in the BCS theory of superconductivity (Bardeen et al., 1957) where at low temperatures the bosonic electron-pair fluid close above the Fermi boundary escapes resistive friction by becoming superfluid. This theory applies to very low temperatures. The discovery (Bednorz and Müller, 1986) of high-temperature superconductivity, for instance, has opened up the question for any possibility of raising the critical temperature (Leggett, 2006; Kordyuk, 2012; Drozdov et al., 2015) of boson
gases. Non-ideal boson gases also exhibit condensation but their repulsive potentials lower the critical temperature. Here we check for the effect of a finite temperature anisotropy $\tau = T_\parallel/T_\perp < 1$ on the state of an ideal massive boson gas.

The bosonic temperature anisotropy is arbitrarily assumed in the third direction of a local cartesian coordinate system $\{1, 2, 3\}$ with $T_3 \equiv T_\parallel < T_\perp$ the parallel temperature, and $T_\perp$ the isotropic reference temperature in the two perpendicular directions. Defining $A = T_\perp/T_\parallel - 1 = (1 - \tau)/\tau$, the distribution of states $n_p$ with respect to the boson momentum $p$ becomes

$$n_p(A) - n_0 = \left\{ \exp \frac{1}{T_\perp} \left[ (\epsilon_p - \mu)(1 + A \cos^2 \theta) \right] - 1 \right\}^{-1} \quad (1)$$

where $\epsilon_p = (p_\perp^2 + p_\parallel^2)/2m$, $\tan \theta = p_\perp/p_\parallel$, and the zero momentum (ground) state occupation $n_0 = n_{p=0}$ has been separated out. This ground state still depends on the chemical potential $\mu$, $\tau$ (through $A$), and angle $\theta = \tan^{-1}(\mu_\perp/\mu_\parallel)$. It is clear that the angle is the same as that for the momenta. It is also clear that, like for all ideal boson gases, independent of the presence of an anisotropy, the chemical potential $\mu \lesssim 0$ is negative with modulus approaching $|\mu| \to 0$ at some low critical temperature $T_{\perp,c}(\mu = 0)$ which for massless bosons vanishes but is finite for massive bosons. This anisotropy-caused angular dependence complicates the problem. One may note that for $x = 0$ the ordinary Bose distribution is recovered, independent of any anisotropy. Hence for all anisotropies the ground state occupation $n_0$ corresponds to the perpendicular direction at zero energy, thereby in retrospect justifying the choice of $T_\perp$ as reference temperature.

Putting $\mu = 0$ and as usually integrating over energy [Kittel and Kroemer 1980; Huang 1987; Lifshitz and Pitaevskii 1998] yields an estimate for the
critical temperature under the assumption of constant total particle number \( N \), constant volume \( \mathcal{V} \), and density \( N \equiv N/\mathcal{V} \) from

\[
N - N_0 = \frac{2g}{\sqrt{\pi \lambda_\perp^2}} \int_0^1 \frac{dx}{(1 + A x^2)^3} \int_0^{\infty} \frac{y^{\frac{3}{2}} dy}{e^y - 1}
\]

(2)

with \( \lambda_\perp^2 = 2\pi \hbar^2 / m T_\perp \) the (perpendicular) squared thermal wavelength, \( y \) the dimensionless energy variable, and \( x = \cos \theta \). The average density \( N_0 \equiv N_{\epsilon=0} \) of particles in the zero energy ground state \( n_0 \) refers to the Bose-Einstein condensate (BEC). No distinction has been made in the degeneracies \( g \) of the parallel and perpendicular energy levels. Integration yields

\[
N - N_0 = \frac{2g\sqrt{\tau}}{\sqrt{\pi \lambda_\perp^3}} \zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right) = \frac{g\sqrt{\tau}}{\lambda_\perp^3} \zeta\left(\frac{3}{2}\right)
\]

(3)

with \( \tau = T_\parallel / T_\perp \). This expression behaves regularly at \( \tau = 1 \) which corresponds to the ordinary isotropic case.

At critical temperature \( T_\perp = T_{\perp c} \) one has \( N_0 = 0 \). Resolving the thermal wavelength for \( T_{\perp c} \), the critical temperature is defined by the right-hand side of the last expression through the constant density \( N \) and temperature anisotropy \( \tau_c = T_\parallel / T_{\perp c} \) at \( T_{\perp c} \):

\[
T_{\perp c} = \frac{2\pi \hbar^2}{m \tau_c^{\frac{3}{4}}} \left[ \frac{N}{g \zeta\left(\frac{3}{2}\right)} \right]^{\frac{2}{3}} = \frac{T_c}{\tau_c^{\frac{1}{3}}}
\]

(4)

where \( T_c \) is the critical temperature of the isotropic boson gas with \( \tau = 1 \). Remembering that the effect of anisotropy in the Bose distribution disappeared in the perpendicular direction \( x = 0 \), the temperature \( T_c \) corresponds to the invariant critical temperature in the perpendicular direction. The inverse proportionality to the anisotropy \( \tau_c \) indicates an increase of the critical temperature with increasing anisotropy \( A \) respectively decreasing \( \tau \), i.e. decreasing parallel temperature \( T_\parallel \). This is written explicitly for the critical
perpendicular temperature as function of the parallel temperature

\[ T_{\perp c}(T_{\parallel}) = \left( \frac{2\pi \hbar^2}{m} \right)^{\frac{3}{2}} \left[ \frac{N}{g \zeta^{\left(\frac{3}{2}\right)}} \right] T_{\parallel}^{-\frac{1}{2}} \equiv \left( \frac{T_{\perp c}}{T_{\parallel}} \right)^{\frac{1}{2}} \quad (5) \]

The effect of the anisotropy on the critical temperature is obvious from the last expression. The shift \( \Delta T_c = T_{\perp c} - T_c \) in the critical temperature from the isotropic state becomes

\[ \frac{\Delta T_c}{T_c} = \left( \frac{T_c}{T_{\parallel}} \right)^{\frac{1}{2}} - 1, \quad T_{\parallel} \leq T_{\perp} \quad (6) \]

This is always positive and increases with decreasing parallel temperature. In an anisotropic boson gas the critical temperature for condensate formation thus shifts towards higher temperatures for any \( \tau < 1 \). Though this shift may not be large it may become of interest in superfluidity, BEC and possibly in high-temperature superconductivity.

At temperatures \( T_{\perp} < T_{\perp c} \) the chemical potential remains negative and about zero. The density of particles above ground state is thus given by Eq. (3), and one obtains an estimate for the density fraction in the ground-state \( \epsilon_p(0) \) as function of perpendicular temperature

\[ \frac{N_0}{N} \approx \left( 1 - \frac{T_\perp}{T_{\perp c}} \right), \quad T_\perp < T_{\perp c} \quad (7) \]

The explicit dependence on the parallel temperature has dropped out of this expression but is implicit to \( T_{\perp c} \). Similar to the isotropic case, the fraction of density in the condensate ground state depends only on the ratio of (perpendicular) temperature to critical perpendicular temperature. However, the power of the ratio is different. With decreasing perpendicular temperature the fraction of density in the condensate increases faster. Comparing with
Figure 1: The normalised critical temperature $T_{\perp c}$ as function of the normalised parallel temperature $T_{\parallel}$. Isotropic critical temperatures are around $T_c \sim 1$ K. Increasing the anisotropy $A$ by reducing the parallel temperature below values of $T_{\parallel} \sim \text{mK}$ causes substantial increases of the effective critical temperature.

the isotropic case and using Eq. (5) this can be written

$$\frac{N_0}{N} = 1 - \left(\frac{T_{\perp}}{T_c}\right)^{\frac{3}{2}} \frac{1}{\tau^2} = 1 - \frac{N_c}{N}$$

with $N_c$ the critical density. Now the temperature ratio has same power as in the isotropic case, it is, however, multiplied with the anisotropy factor $\tau$ which takes care of its reduction. Clearly, increasing the critical temperature allows more particles in the energy distribution to enter the condensate. We consider these findings as sufficiently important for communication. Superfluid condensates are at the heart of the microscopic theory of superconductivity. The inclusion of temperature anisotropies perhaps helps in the race for higher temperature superconductors.

From a thermodynamic point of view it will be interesting to develop
the full thermodynamics of an anisotropic boson gas and to also make the transition to non-ideal gases. The related questions we delegate to a separate investigation which requires returning to the grand thermodynamic potential in its integral version taking into account that the temperature is anisotropic and therefore the angular integration is implicit. This gives

$$\Omega = -PV = P_0 V + \frac{4gVT_\perp}{\sqrt{\pi} \lambda_\perp^3} \int_0^1 dx \int_0^\infty dy \times$$

$$\times \ y^{\frac{3}{2}} \log \left( 1 - e^{(\mu/T_\perp - y)(1+Ax^2)} \right)$$

where $P_0$ is the pressure in the zero-energy ground state $\epsilon_p = 0$. Carrying out the partial integrations should yield the equation of state, mean energy and the dependence on anisotropy of the heat capacity at constant volume. In the anisotropic case this becomes quite involved as one should keep in mind that the pressure becomes anisotropic. In order to hint on a possible tackling of the problem one returns to the definition of the average density and includes the finite chemical potential $\mu \neq 0$

$$N - N_0 = \frac{4g}{\sqrt{\pi} \lambda_\perp^3} \int_0^1 \frac{dx}{(1 + Ax^2)^{\frac{3}{2}}} \int_0^\infty \frac{y^{\frac{3}{2}} dy}{z^{1-e^y} - 1}$$

where $z = z_0 \exp[-|\mu|Ax^2/T_\perp]$ with $z_0 = \exp(-|\mu|/T_\perp)$ the fugacity, which is generalised to the finite anisotropy and with $\mu < 0$. Here $z_0 = \exp \mu/T_\perp$ is the ordinary fugacity referring to the perpendicular reference temperature $T_\perp$. The $y$-integral is recognised as the common function (cf., e.g., Huang, 1987)

$$g_2(z) \equiv \sum_{l=1}^\infty z_l(x)/l^{\frac{3}{2}}$$

where now $z(x)$ is a function of $x$. The integration with respect to $x$, though quite involved, can be performed stepwise by parts. This yields for the density

$$N_0 = \frac{4g}{\sqrt{\pi} \lambda_\perp^3} \int_0^1 \frac{dx}{(1 + Ax^2)^{\frac{3}{2}}} \int_0^\infty \frac{y^{\frac{3}{2}} dy}{z^{1-e^y} - 1}$$
at finite chemical potential \( \mu \) the expression

\[
N - N_0 \equiv \frac{g}{\lambda^3_{\perp}} g_{\frac{3}{2}}(z_0, \tau) = \frac{g}{\lambda^3_{\perp}} \sum_{l=1}^{\infty} \frac{z_0^l}{l^{3/2}} \left[ e^{-la} \sqrt{\tau} + \frac{2}{5} la \Phi_1 \left( \frac{3}{5}, \frac{1}{2}, \frac{5}{2}; \frac{1 - \tau}{\tau}; -la \right) \right]
\]

(12)

where \( a = |\mu|(1 - \tau)/T_{\perp} \tau \), and \( \Phi_1(\alpha, \beta, \gamma; x, y) \) is the degenerate hypergeometric series. The sum expression can be taken as a redefinition of \( g_{\frac{3}{2}}(z) \) in the presence of a temperature anisotropy. This has been written in terms of the anisotropy ratio \( \tau = T_{\parallel}/T_{\perp} \). The generating function \( g_{\frac{3}{2}}(z) \) can in principle be obtained from the former function by integration with respect to \( z \) after having performed the \( x \) integral. This is not necessary since its expansion is also known. In exactly the same way as done above one obtains the equation of state for the ideal anisotropic boson gas with the replacement \( l^{3/2} \rightarrow l^{5/2} \) in the denominator. Similarly we have

\[
g_{\frac{5}{2}}(z_0) = \sum_{l=1}^{\infty} \frac{z_0^l}{l^{5/2}} \left[ e^{-la} \sqrt{\tau} + \frac{2}{5} la \Phi_1 \left( \frac{3}{5}, \frac{1}{2}, \frac{5}{2}; \frac{1 - \tau}{\tau}; -la \right) \right]
\]

(13)

One recognises that in this case

\[
g_{\frac{3}{2}}(z_0, \tau) \equiv z_0 \frac{\partial}{\partial z_0} g_{\frac{5}{2}}(z_0, \tau)
\]

(14)

For \( \tau = 1 \) both functions \( g_{\frac{3}{2}}(z_0, \tau) \) and \( g_{\frac{5}{2}}(z_0, \tau) \) reproduce their isotropic versions. On the other hand, for \( \mu = 0 \), only the exponential survives, which is the critical anisotropic state. This allows to write for the density

\[
\frac{N}{N_c} = \frac{g_{\frac{3}{2}}(z_0, \tau)}{g_{\frac{3}{2}}(1, 0)}, \quad \text{with} \quad \lambda^3_{\perp} N_c = gg_{\frac{3}{2}}(1, 0)
\]

(15)

which also implies that \( N_0 \), the ground state density \([8]\), is finite for \( N\lambda^3_{\perp}/g \leq g_{\frac{3}{2}}(1, 0) \) and vanishes for \( N\lambda^3_{\perp}/g \geq g_{\frac{3}{2}}(1, 0) \).
With these expressions at hand one can find the average energy \( U(z_0, V, T_\perp, \tau) \) and specific heats in the ground state and for any finite temperature and negative chemical potential. In particular \( \mu = 0 \) yields \( z_0 = 1 \) and \( a = 0 \) such that the last terms in the above expressions cancel, and for the density one obtains the ground state density as given above. One can also ask for the excitation levels above ground state and find the energy gap as function of \( \tau \). This requires considering the excitation of a spectrum of phonons in the anisotropic boson gas at low temperatures.

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Competing Interests

The authors declare that they have no competing interests.

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