Unit system independent formulation of electrodynamics

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Abstract

Nowadays Maxwell’s equations are formulated merely in the SI system. Physicists should also be familiar with other systems, like the systems of Gauss and Heaviside, but a comparison of equations in different systems is sometimes annoying. In a unified notation it is easier to switch from one system to another and to recognize the consequences of the chosen system. In covariant electrodynamics the comparison of the different systems is based on a frequently used representation of the pseudo-Euclidean metric with four-dimensional electromagnetic quantities. The key role between different systems have the mechanical forces between charges, currents and magnetic poles. They build the basis of the systems and their units.

Keywords: Maxwell’s equations, covariant electrodynamics, units in electrodynamics

1. Introduction

The history of electromagnetism is connected with different attempts to establish an appropriate system of units. Our starting point is a treatise on electricity and magnetism of Maxwell [1873]. There the important unit systems, the electrostatic units (esu) and the electromagnetic units (emu) are derived in detail. Electric and magnetic quantities are accessible via mechanical forces. In this context three equations are of interest: the Coulomb force between electric charges, the magnetostatic force between magnetic poles, and Ampère’s force law between electric currents (Maxwell 1873, §41, §374 and §687). Although the last law is the most important one, because it defines the ampere, applies our interest to the Coulomb-forces between electric respectively magnetic quantities.

Maxwell (1873) (§623) determined the dimensions of the electromagnetic variables using length, mass, time and the electric charge (L, M, T, q) or the pole strength (L, M, T, ρb). Instead
of formulating a system with four fundamental units, the electric charge has been expressed in the basic dimensions (L, M, T) using Coulomb’s law and the pole strength with the corresponding magnetic Coulomb law. Thus one obtained the esu and the emu, where the second is considered as the more important one. A disadvantage of these systems is that the dimensions of the electromagnetic quantities differ in powers of a velocity \( c \). Weber and Kohlrausch (1856) have shown, that \( c \) is (nearly) the speed of light. This difference in the dimensions has its origin in the definition of the electromotive force by Maxwell (1873) (§599) \( E_L = v \times B \) that a particle feels when moving with \( v \) in a magnetic field \( B \). By replacing \( v \) through \( v/c \) according to the Lorentz transformation, the differences in the dimensions disappear.

Especially from the technical side the introduction of a fourth, electric base unit was desired, and Giorgi had already laid the basis for SI with the MKS\( \Omega \) system in 1901. Then the differences of \( E \) and \( D \) respectively \( B \) and \( H \) appear also in dimensions and units. At this point one may remember that even in mechanics different physical phenomena like energy (joule) and torque (newton metre) have the same dimension in the MKS-system. In the following decades a few proposals were made for different unit systems, whereby the international electric unit system from 1908 is worth to be picked out. It is a non-rational system with the base units international ohm, international ampere, centimetre and second (Dellinger 1917, p 606).

In textbooks, electrodynamics is usually presented in the SI or Gauss system, and formulas and tables for the conversion to other unit systems are preferably given in an appendix. In some articles the whole background concerning equations, dimensions and units is given in detail (Venkates 1963, Gelman 1966) or more or less restricted to SI and Gauss (Ivanov 2015, Garg 2018). This is not equivalent to an approach of formulating electrodynamics in a form independent of specific systems, as presented by Carron (2015), and in a more restricted form by Vrejoiu (2008), Heras and Báez (2009).

It is intended to go beyond the previously mentioned articles. Above all one has to hold the number of required parameters as small as possible. In vacuum two parameters are used, the first one is necessary to define the electric charge density \( \rho \), the current density \( j \), the electric field \( E \), and the magnetic field \( B \), and the second one serve as an additional scaling factor for \( B \). These two parameters suffice to include ‘all’ unit systems of relevance, i.e. emu, esu, Gaussian units, the rational system of Heaviside–Lorentz and SI. Two auxiliary fields, \( D \) and \( H \) are used in matter, and as a consequence at least a third parameter is required, which can be the electric constant (permittivity of the vacuum). In the Gaussian system all three parameters are equal to one, and the equations have therefore a shape known from the Gaussian system.

One may believe that it is straightforward to transfer the system independent notation to the covariant formulation of electrodynamics. However, difficulties arise from different representations of the pseudo-Euclidean metric, and different definitions existing for nearly all four-dimensional quantities. Therefore the unified formulation of covariant electrodynamics is based on a commonly used representation of the pseudo-Euclidean metric together with four-dimensional vectors and tensors associated to this representation (Jackson 1999, chapter 11).

### 2. Maxwell equations in vacuum

Electromagnetic quantities are accessible via mechanical forces acting on electric charges, that are \( \text{et al} \) the Coulomb force \( F_c \) and the magnetic part \( F_L \) of the Lorentz force. If a charge \( q' \) is located at the origin, then the charge \( q \) experiences at the position \( x \) the force

\[
F_c = k_c q q' \frac{x}{r^3} = q E.
\]
$k_c$ is a freely choosable parameter which defines the dimensions of the electrical charge $q$ (charge density $\rho$), the current density $\mathbf{j}$ and the electric field $\mathbf{E}$. This requires that $\mathbf{E}$ via (1) and $\mathbf{j}$ are equally defined in all systems. Then the equation of continuity

$$\dot{\rho} + \nabla \cdot \mathbf{j} = 0 \quad (2)$$

is independent of the system$^1$. The force acting on a charged particle moving with $\mathbf{v}$ in the magnetic field $\mathbf{B}$ is

$$\mathbf{F}_l = q \frac{\mathbf{v}}{k_c} \times \mathbf{B} = q \mathbf{E}_l. \quad (3)$$

c is the speed of light and $k_l$ the second parameter which defines $\mathbf{B}$. The parameter values for the most important systems are listed in table 1. $\epsilon_0$ is the electric constant, which is connected to the magnetic constant $\mu_0$ by means of

$$\mu_0 = 1/k_c^2 \epsilon_0. \quad (4)$$

$k_c$, $k_l$ and the assumption that electromagnetic waves propagate with $c$ determine Maxwell’s equations:

$$\nabla \cdot \mathbf{E} = 4\pi k_c \rho, \quad \nabla \times \mathbf{E} = -(k_c/c) \mathbf{B},$$
$$\nabla \times \mathbf{B} = \left(4\pi k_c \mathbf{j} + \dot{\mathbf{E}}\right)/ck_l, \quad \nabla \cdot \mathbf{B} = 0. \quad (5)$$

If the magnetic field is redefined by $\mathbf{B}_l = k_l \mathbf{B}$, Maxwell’s equations are independent of $k_l$. Only two choices for $k_l$ are in use: For $k_l = c$ Maxwell’s equations, especially for SI, have a simple shape as shown on (11), while for $k_l = 1$ $\mathbf{E}$ and $\mathbf{B}$ have the same dimension$^2$, and velocities appear only in the ratio $v/c$ and time derivatives in the form $\partial / \partial \tau c$ as suggested by the Lorentz transformation. In Lorenz gauge

$$(1/c k_l) \dot{\phi} + \nabla \cdot \mathbf{A} = 0 \quad (6)$$

$$\phi(x, t) = k_c \int \frac{d^3 x'}{\left|x-x'\right|} \rho(x', t'), \quad \mathbf{A}(x, t) = \frac{k_c}{ck_l} \int \frac{d^3 x'}{\left|x-x'\right|} \mathbf{j}(x', t'), \quad (7)$$

where $t_r = t - \frac{|x-x'|}{c}$ is the retarded time. The electromagnetic fields are given by

$$\mathbf{E} = -\nabla \phi - (k_c/c) \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$  

From (7) it follows that $\mathbf{E}$ and $\mathbf{B}$ are linear functions of $k_c$ and $k_c/k_l$. Thus, the electromagnetic force $\mathbf{F}$ is independent of $k_l$ and linear in $k_c$ like the Coulomb force (1):

$$\mathbf{F} = q \left(\mathbf{E} + k_c \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \quad (8)$$

Therefore the electric charge $q$ (and the current density $\mathbf{j}$) can be fixed with a proper choice of $k_c$ and $F$ or better $F_c$. $k_c$ determines the relation between electric and magnetic quantities, but has no influence on $\mathbf{F}_l$, $\mathbf{E}$, $\rho$ or $\mathbf{j}$.

$^1$The so-called ‘variant Gauss system’ (Carron 2015. (72)), in which $\mathbf{j}$ is defined with the factor $1/c$, is thus excluded.

$^2$Systems with $k_l = 1$ are called symmetric. The SI system can also be symmetrized: In this new system remains $\epsilon_0$ unchanged, but according to (4) is $\mu_0^{sym} = 1/\epsilon_0$, from which $\mathbf{H} = \epsilon_0 \mathbf{B}$ follows.
Table 1. Parameter values for some systems with its prefactors appearing in Maxwell equations (5) in vacuum (left side). In matter (right side) one has to consider the additional parameter $\varepsilon_0$, esu is the abbreviation for electrostatic units and emu for electromagnetic units.

| System | $k_c$ | $k_L$ | $4\pi k_c$ | $4\pi\varepsilon_0 k_c$ | $\varepsilon_0$ | $\mu = \frac{1}{\varepsilon_0} k_L$ | $4\pi k_c \varepsilon_0$ |
|--------|-------|-------|-------------|-----------------|----------------|-----------------------------|------------------|
| SI     | $\frac{1}{4\pi \mu_0}$ | $c$   | $\frac{1}{\varepsilon_0}$ | $\mu_0$ | $\varepsilon_0$ | $\frac{1}{4\pi \varepsilon_0}$ | 1 |
| Heaviside | $\frac{1}{c}$ | 1     | 1           | $\frac{1}{\mu_0}$ | 1   | 1                         | 1 |
| Gauss  | 1  | 1     | $4\pi$ | $\frac{1}{\mu_0}$ | 1   | $4\pi$                     | 1 |
| esu    | $\frac{c^2}{2}$ | $c$   | $4\pi c^2$ | $\frac{1}{\mu_0}$ | 1   | $4\pi$                     | 1 |
| emu    | $\frac{c^2}{2}$ | $c$   | $4\pi c^2$ | $\frac{1}{\mu_0}$ | 1   | $4\pi$                     | 1 |

3. Maxwell’s equations in media

In media, the charge density is split into the density of free charges $\rho_f$ and bound charges, i.e. the polarization charges $\rho_p = -\nabla \times \mathbf{P}$, where the polarization is denoted by $\mathbf{P}$. In the analogous division of the current density one obtains (again for ‘all’ unit systems) the bound current densities of the polarization $j_p = \mathbf{P}$ and the magnetization $j_m = (c/k_c) \nabla \times \mathbf{M}$, where $\mathbf{M}$ is the magnetization of the medium:

$$\rho = \rho_f - \nabla \cdot \mathbf{P}, \quad j = j_f + (\varepsilon_0/c) \nabla \times \mathbf{M}.$$ 

$\rho$ and $j$ are inserted in Maxwell’s equations (5). From Gauss’s law one may verify the definition of the dielectric displacement field $\mathbf{D}$ and from Ampère–Maxwell’s equation the auxiliary magnetic field $\mathbf{H}$:

$$\mathbf{D} = \varepsilon_0 (\mathbf{E} + 4\pi k_c \mathbf{P}), \quad \mathbf{B} = \mu_0 (\mathbf{H} + 4\pi k_c \varepsilon_0 \mathbf{M}).$$

For their definition it was necessary to introduce the additional parameter $\varepsilon_0$. $\varepsilon_0$ fulfils (4), i.e. $\mu_0 = 1/k_c^2 \varepsilon_0$. This is certainly not the most general representation, but sufficient for all systems considered. Alternatively $\varepsilon_0$ may be considered as the fundamental parameter. Then $k_c = 1/4\pi \varepsilon_0$ for rational systems and $k_c = 1/\varepsilon_0$ for the non-rational ones. The parameter values for all considered systems are listed in the table 1. One obtains in the formulation applicable to ‘all’ systems

\[(a) \nabla \cdot \mathbf{D} = 4\pi k_c \varepsilon_0 \rho_f, \quad \text{(B) } \nabla \times \mathbf{E} + \frac{k_L}{c} \mathbf{B} = 0,\]

\[(c) \nabla \times \mathbf{H} = \frac{k_L}{c} \left(4\pi k_c \varepsilon_0 j_f + \dot{\mathbf{D}}\right), \quad \text{(d) } \nabla \cdot \mathbf{B} = 0.\]

For the SI-system, a rational system with $k_c = c$, one gets

\[(a) \nabla \cdot \mathbf{D} = \rho_f, \quad \text{(B) } \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0,\]

\[(c) \nabla \times \mathbf{H} = j_f + \dot{\mathbf{D}}, \quad \text{(d) } \nabla \cdot \mathbf{B} = 0.\]

Starting from the SI-system, in non-rational systems $\rho$ and $j$ must be multiplied by $4\pi$ (Gauss, esu and emu) and in symmetric systems $\mathbf{j}$, $\mathbf{B}$ and $\mathbf{D}$ have to be multiplied by $1/c$ (Gauss, Heaviside).

3 The original definition of the dielectric displacement $\mathbf{D} = \varepsilon_0 \mathbf{E}/4\pi$ (Gauss, esu, emu) is not taken into account.
4. Covariant formulation of electrodynamics

As already mentioned in the abstract, a comparison of nearly all existing variations of covariant electrodynamics will not be given here. In the first papers on special relativity (SRT) it was attempted to maintain the Euclidean metric using an imaginary time axis with $x_4 = ict$. This notation has been used by Minkowski (1908), Stratton (2007), Sommerfeld (1952), among others, and is treated separately in the appendix. The definitions, concerning the metric and four-dimensional tensors are in accordance with Jackson (1999). The formulas are emphasized for different unit systems as given in Table 1. Within our notation the time axis has the index 0, greek indices run from 0 to 3 and latin indices from 1 to 3, but it should be easy to rewrite the equations, if the indices run from 1 to 4. Two different metric tensors are used, which differ only in sign:

$$\left(g^{\mu\nu}\right) = \left(g_{\mu\nu}\right) = k_s \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\delta_{01} & -\delta_{02} & -\delta_{03} \\ 0 & -\delta_{11} & -\delta_{12} & -\delta_{13} \\ 0 & -\delta_{21} & -\delta_{22} & -\delta_{23} \\ 0 & -\delta_{31} & -\delta_{32} & -\delta_{33} \end{pmatrix},$$

with $k_s = +1 \Leftrightarrow$ time-like convention: signature $(1, 3)$ and $k_s = -1 \Leftrightarrow$ space-like convention: signature $(3, 1)$.

The two signatures\(^4\) are taken into account in the following by the factor $k_s = \pm 1$. The value $k_s = 1$ is assigned to the time-like convention, which is e.g. used by Einstein (1916), Landau and Lifshitz (1975), Jackson (1999). The co- and contra-variant spacetime vectors are given by $(x_\mu) = k_s(ct, -\mathbf{x})$ and $(x^\mu) = (ct, \mathbf{x})$ and the four-vectors of the gradient, potential and current are defined by

$$\nabla A_\mu = \frac{4\pi k_c}{c k_s} \rho, \quad \Box = \partial^\alpha \partial_\alpha = k_s \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right).$$

From Maxwell’s equations (5) one deduces in Lorenz gauge $A^\mu = 0$ the inhomogeneous wave equations

$$\Box A_\mu = \frac{4\pi k_c^2}{c k_s} \rho, \quad \Box = \partial^\alpha \partial_\alpha = k_s \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right).$$

$k_s A^\mu$ is independent of $k_s$. The definition of the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

differs from its originally definition of Einstein (1916) (1) by a factor of $-1$. Although a few authors (Bjorken and Drell 1964, Becker and Sauter 1973, Schwabl 2008) kept the old definition, we continue with (13) as defined by Jackson (1999) and Landau and Lifshitz (1975). Then one gets

$$F^{\mu\nu} = k_s \begin{pmatrix} 0 & -E^i/k_s \\ E^i/k_s & -(-\epsilon_{mnk} B_k) \end{pmatrix}.\quad (14)$$

\(\epsilon_{mnk}\) is an element of the total antisymmetric pseudo tensor with $\epsilon_{123} = 1$. Note that latin indices go from 1 to 3. The dual field tensor depends on the definition of the totally antisymmetric tensor, which can be defined either by $\epsilon_{0123} = 1$ (Sexl and Urbanke 2001, (5.5.8)) or by $\epsilon_{0123} = 1$ (Jackson 1999, (11.139)), where $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$. Both definitions are equally common. According to Jackson, one obtains for the dual field strength tensor ($\epsilon_{0123} = 1$)

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (\tilde{F}^{\mu\nu}) = k_s \begin{pmatrix} 0 & -\mathbf{B} \\ \mathbf{B} & (-\epsilon_{mnk} E_k/k_s) \end{pmatrix}.\quad (15)$$

\(^4(n_+, n_-): n_\pm = \text{number of positive/negative diagonal elements.}\)
One gets $\tilde{F}^{\mu\nu}$ from $F^{\mu\nu}$ (14) by the following substitutions:

$$ F^{\mu\nu} \rightarrow \tilde{F}^{\mu\nu}: \ E/k \rightarrow B, \ B \rightarrow -E/k. $$

Now the inhomogenous Maxwell’s equations can be represented in a covariant form with the field strength tensor and the homogenous equations with the dual field strength tensor:

$$ F^{\mu\nu} = \frac{4\pi k_c}{ck_l} j^\mu, \quad \tilde{F}^{\mu\nu} = 0. $$

The invariants of the field strength tensors are

$$ \frac{1}{2} F^{\mu\nu} F_{\mu\nu} = B^2 - \frac{1}{k^2} E^2, \quad \frac{1}{4} \tilde{F}^{\mu\nu} F_{\mu\nu} = -\frac{1}{k^2} E \cdot B $$

The pointing vector is given by

$$ S(x, t) = \frac{ck_l}{4\pi k_c} E \times B. $$

The symmetric energy–momentum tensor, also called symmetric stress tensor (Jackson 1999, (12.113)) is defined as

$$ T^{\mu\nu} = \left( \frac{1}{8\pi k_c} \begin{pmatrix} E^2 + k^2 B^2 & S^2/c \\ S/c & -(\sigma_{\mu\nu}) \end{pmatrix} \right), $$

whereby $\sigma_{ij}$ denotes the Maxwell stress tensor

$$ \sigma_{ij} = \frac{1}{4\pi k_c} \left[ E_i E_j + k^2 B_i B_j - \frac{1}{2} (E^2 + k^2 B^2) \delta_{ij} \right] $$

$$ T^{\mu\nu} = k_s \frac{k^2}{8\pi k_c} \left( g_{\rho\sigma} F^{\mu\rho} F_{\sigma\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right). $$

One may note that the factor $k_s g_{\rho\sigma}$ is independent of the signature as well as (19). The Lagrangian density $L$ of the electromagnetic field consists of a free Lagrangian density $L_f$ and a source term $L_s$. The gauge invariant, free Lagrangian density of the electromagnetic field is then

$$ L_f = -\frac{k^2}{16\pi k_c} F^{\mu\nu} F_{\mu\nu} = \frac{1}{8\pi k_c} \left( E^2 - k^2 B^2 \right). $$

In the presence of sources one obtains the Lagrangian density (Jackson 1999, (12.85)):

$$ L = L_f + L_s, \quad L_s = -k_s \frac{k_l}{c} j^\mu A_\mu. $$

The Euler–Lagrange equations are now

$$ \partial_\mu \left[ \frac{\partial L}{\partial (\partial_\mu A_\nu)} \right] = \frac{\partial L}{\partial A_\nu} = -k_s \frac{k_l}{c} j^\nu, $$
where the field terms may be evaluated as
\[
\frac{\partial}{\partial (\partial_\mu A_\nu)} F_{\rho\sigma} F^{\rho\sigma} = 2 \frac{\partial}{\partial (\partial_\rho A_\sigma)} (\partial_\rho A_\sigma) F^{\rho\sigma} = 4 F^{\mu\nu}.
\]

Inserted in the Euler–Lagrange equations one obtains, as expected, the inhomogeneous Maxwell equations (16). The above description of covariant electrodynamics is merely a collection of formulas than an explanation of electrodynamics in the special theory of relativity. The differences between the electromagnetic quantities in various unit systems are simple factors, but this is not the whole truth. In particular exist a few representations of the metric. It is easy to change from (ct, x) to (x, ct), except for ($\tilde{F}^{\mu\nu}$), where one has to be careful because of the $\epsilon$-tensor. It becomes more complicated if the pseudo-Euclidean metric is introduced via an imaginary time axis. Then some elements of the tensors are imaginary and it is difficult to compare these tensors with our notation. The signature has for imaginary time axis always the opposite sign, i.e. (3, 1).

5. Mechanical forces between electric charges, magnetic poles and electric currents

As already mentioned in the introduction Maxwell (1873) (§623) compared the dimensions of the electromagnetic quantities in an ‘electricsystem’ (L, M, T, q) with those in a ‘magnetic’ one (L, M, T, $p_b$) using $k_i = c$ in both systems. $q$ is the electric charge (quantity of electricity) and $p_b$ the pole strength (quantity of free magnetism). In these systems with four base variables $q$ and $p_b$ has been replaced by its definitions within the cgss system using Coulomb’s law for charges and pole strengths. The resulting cgss systems are the electrostatic and the electromagnetic system.

5.1. Coulomb’s law and the unit of charge

Maxwell (1873) (§41). ‘The electrostatic unit of electricity is that a quantity of positive electricity which, when placed at unit of distance from an equal quantity, repels it with the unit of force’

\[
F_c = qq'/r^2.
\]

(25)

The electromagnetic unit of electricity has been calculated from Ampère’s force law (34) and compared with the electrostatic unit (Maxwell 1873, §675). Different force laws can be avoided using the more general form (1) of the Coulomb force. The unit of the electric charge is defined by setting $q = q'$, $r = 1$ cm, and $F_c = 1$ dyn, where $c = c_0$ cm s$^{-1}$:

\[
q = r \frac{F_c}{k_c} = \begin{cases} 1 \text{ cm} \sqrt{\text{dyn}} \sqrt{4\pi \varepsilon_0} = 10 \text{C}/c_0 & \text{SI} \\ 1 \text{ cm} \sqrt{\text{dyn}} = 1 \text{ statC} & \text{Gauss, esu} \\ 1 \text{ cm} \sqrt{\text{dyn}/c_0} \text{ cm s}^{-1} = 1 \text{ abC}/c_0 & \text{emu}. \end{cases}
\]

(26)

The charge can be expressed by the electric flux, when the entire charge distribution is integrated over a sphere $V$

\[
\int_V d^3x \nabla \cdot \mathbf{D} = 4\pi k_c \varepsilon_0 q = \int_{\partial V} df \cdot \mathbf{D} = \Phi_0.
\]
If $\Phi_d$ denotes the electric flux through a spherical surface that encloses the charge $q$, then

$$\text{SI: } \Phi_d = q, \quad \text{Gauss, esu, emu: } \Phi_d = 4\pi q.$$  

So the electric flux in the above systems has the same dimension as the charge, but is different, following from Gaussian law, in the transformation by a factor of $4\pi$, which means that $q$ and $\Phi_d$ are measured in the same units, but with different conversion factors (Jackson 1999, table A.4).

$$q: 1 \text{ statC} \equiv 10C/c_0, \quad \Phi_d: 1 \text{ statC} \equiv 10C/4\pi c_0.$$  

5.2. Magnetic charge and pole strength

The magnetic Coulomb law formulated by Maxwell (1873) (§374) is

$$F_M = \frac{p_B}{r^2}, \text{ cgs: } \|p_B\| = \text{cm} \sqrt{\text{dyne}} = \sqrt{\text{g cm}^3 \text{s}^{-2}}.$$  

(27)

Following Maxwell (1873) (§623), the pole strength is determined by magnetic flux $\Phi_B$ outgoing from the pole of a long and thin bar magnet or solenoid. Here we divide the magnetic field into the part of the coil $B_s$ and that of a point source $B_p$ (Petrascheck and Schwabl 2019, (4.2.47)). Integrating over a volume $V$, which includes the pole, one obtains

$$\int_V d^3 x \cdot B_p = 4\pi k_c \epsilon_0 p_B = \Phi_B.$$  

The magnetic Coulomb law reads in its general form

$$F_m = \frac{k_c \epsilon_0}{\mu_0} p_B \frac{x}{r^3} = p_B H.$$  

(28)

If the force $F_m = 1 \text{ dyn}$, is acting on the magnetic poles at a distance of 1 cm, the (cgs) unit pole strength

$$p_B = r \sqrt{\frac{F_m \mu_0}{k_c \epsilon_0}} = \begin{cases} r\sqrt{\frac{4\pi \mu_0 F_m}{k_c \epsilon_0}} = 4\pi \times 10^{-8} \text{ Wb} & \text{SI} \\ r\sqrt{\frac{F_m}{k_c \epsilon_0}} = 1 \text{ cm}^3 \text{ g}^{-1} \text{s}^{-2} = 1 \text{ emu, Gauss} \\ r\sqrt{\frac{F_m}{c}} = 1 \text{ cm} \sqrt{\text{g}/c_0} = (1/c_0) \text{ esu, esu} \end{cases}.$$  

(29)

The unit emu is a placeholder for the unit pole strength (Semat and Katz 1958, (29–2)), which equals to $4\pi \times 10^{-8}\text{ Wb}$. Analogous to the (electric) Coulomb force $F_c$, one obtains for the magnetic flux emanating from the unit pole

$$\Phi_B = \frac{4\pi r}{k_c} \sqrt{F_m k_c} = 4\pi \begin{cases} 10^{-8} \text{ Wb} & \text{SI} \\ Mx \end{cases}.$$  

(30)

In contrast to $F_c$, where the charge $q$ was used for the conversion $1 \text{ statC} \equiv 10C/c_0$, it is here the flux $\Phi_B$, which determines the conversion $1 \text{ Mx} \equiv 10^{-8}\text{ Wb}$.

In the next step the current $I$ is calculated, which is necessary to generate in a unit coil the magnetic flux required for the unit pole. In a solenoid the magnetic flux is given by

$$\Phi_B = B a^2 \pi = \frac{4\pi k_c I_n}{k_c c} a^2 \pi.$$  

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$I_n$ is the current per unit length, i.e. $I_n = I/l$. Then it follows for the current $I$, if $F_M = 1$ dyn, $l = r = 1$ cm and $\alpha^2 \pi = 1$ cm$^2$:

$$I = \frac{clr \sqrt{F_M}}{\sqrt{k_c} \alpha^2 \pi} = \frac{c}{\sqrt{k_c}} \sqrt{\text{dyn}} = \left\{ \begin{array}{ll} 4\pi/\mu_0 \sqrt{10^{-2} \text{N}} = 10 \text{ A SI} \\
1 \sqrt{\text{cm} \text{gs}^{-2}} = 1 \text{abA emu} \\
10 \sqrt{\text{cm}^2 \text{gs}^{-4}} = c_0 \text{statA esu} \\
\end{array} \right.$$ \hspace{1cm} (31)

This result is consistent with the force exerted by two parallel wires on each other, although it can only be proven experimentally to a limited extent.

The ambiguity of (27) should be pointed out, since the quantity of magnetism (pole strength) can also be understood as the effective magnetic charge (Jackson 1999, (5.96))

$$\rho_m = -\nabla \cdot M = \frac{\nabla \cdot H}{4\pi k_c \epsilon_0}, q_M = \int_V d^3\!x \rho_m(x).$$

Sommerfeld (1952) (6) defines the pole strength according to

$$F_M = \frac{k_c^2}{k_c} q_M \left(\frac{\mathbf{x}}{r^3}\right) = q_M B.$$ \hspace{1cm} (32)

The unit of the magnetic charge (pole strength) is now

$$q_m = r \sqrt{\frac{F_M k_c^2}{k_c}} = \left\{ \begin{array}{ll} 1 \sqrt{\text{cm} \text{dyn} 4\pi/\mu_0} = 0.1 \text{ m A SI} \\
1 \sqrt{\text{cm}^2 \text{gs}^{-2}} = 1 \text{ emu emu, Gauss} \\
c_0 \sqrt{\text{cm}^2 \text{gs}^{-4}} = c_0 \text{ statA esu} \\
\end{array} \right.$$ \hspace{1cm} (33)

The magnetization $\mathbf{M}$ has in emu the unit Oersted (Cardarelli 2003, table 3–1), although sometimes Gauss is used (Jackson 1999, table A.4). Therefore one possible choice for the placeholder emu is Oe cm$^2$. Sommerfeld’s definition of the pole strength (32) is in accordance with the electrodynamics in vacuum using only $\mathbf{E}$ and $\mathbf{B}$.

5.3. Amperes force law

The magnetic Coulomb law was sufficient for the definition of the electromagnetic system, but for the determination of its units the force $F_i$ between parallel currents has been used until 2019 (Maxwell 1873, §687). If the force between two parallel current lines at a distance $d$ is $F_i = 2 \times 10^{-7} \text{N} = 0.02 \text{ dyn}$ for a length $l = d$, then the current intensity, measured in the SI system, is equal to 1 A:

$$F_i = \frac{k_c}{c^2} \frac{2l}{d} I_i^2 = \left\{ \begin{array}{ll} 4\pi/\mu_0 \sqrt{10^{-7} \text{N}} = 1 \text{ A SI} \\
0.1 \sqrt{\text{dyn}} = 0.1 \text{ abA emu Gauss} \\
0.1 c \sqrt{\text{dyn}} = 0.1 c_0 \text{ statA esu} \\
\end{array} \right.$$ \hspace{1cm} (34)

The electromagnetic charge unit 1 abC = 1 abA corresponds exactly to $c_0$ electrostatic charge units statC = statAs, regardless if the current was determined using $F_i$ or the charge using $F_c$.

Summary

It has been shown that the two parameters $k_c$ and $k_l$ are sufficient to write Maxwell’s equations in vacuum in a notation valid in all systems if $\mathbf{F}_c = q\mathbf{E}$ and $\mathbf{\dot{\rho}} = -\nabla \cdot \mathbf{j}$. $k_c$ is used to give the
magnetic quantities their own dimensions at the cost of physical symmetry; \( k_c \) determines the electric charge, current and the electromagnetic fields. In matter at least one additional parameter, \( \tilde{\epsilon}_0 \) has to be added for \( D \) and via \( \tilde{\mu}_0 = 1/k_c^2 \tilde{\epsilon}_0 \) simultaneously for \( H \). As already mentioned in the introduction, our parameters are chosen so that they all have the value 1 in the Gaussian system. Heras and Báez (2009) used parameters which are more orientated on rationalized systems. If \( k_c \) is replaced by \( k_h = 4\pi k_c \), all parameters are equal to one in the Heaviside–Lorentz system, which has been used by Minkowski (1908) and Einstein (1916) in their work to the theory of relativity. Since the author is not aware of any textbook of classical electrodynamics written in the Heaviside–Lorentz system, this idea has been discarded.

In the covariant formulation of the electrodynamics one has to consider the pseudo-Euclidean geometry in all formulas. This can be done in different ways, and our notation is therefore not as general as in the ‘classical’ electrodynamics.

The units may be fixed by mechanical forces between charges, currents or magnetic quantities. These forces are linear functions in \( k_c \), independent of \( k_l \). Historically, the electrostatic units were defined by \( F_c \), the electromagnetic units by \( F_m \) and \( F \), which has defined the Ampere until 2019. It has been shown that these force laws in the system independent notation are valid for all systems, and therefore the choice of one of these laws (electric or magnetic Coulomb force, Lorentz force or Ampère’s force law) is sufficient to fix the units for all systems (SI, Gauss, emu, esu) inclusive their conversion factors. Changing from one system to another it is not enough to know the conversion factor between the units, because this factor may differ between different quantities of the same dimension like the electric charge \( q \) and the electric flux \( \Phi_d \).

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Appendix A. The four-dimensional metric with imaginary time

In the representation of an event by a four-dimensional vector has Poincaré (19051906) (p. 168) with the choice of imaginary components on the time axis the shape of the Euclidean metric retained:

\[
(x_\mu) = (x_1, x_2, x_3, ic t), \quad \Rightarrow \quad x_\mu x_\mu = x^2 - c^2 t^2.
\]

(A.1)

This corresponds to the signature (3,1) with the parameter \( k_s = -1 \) (space-like convention). Minkowski (1908) elaborated within this metric the covariant formalism for the SRT. From the imaginary time follows that some components of four-vectors and tensors become imaginary. The four-vector of the gradient potential and current are

\[
(\partial_\mu) = \left( \nabla, -i\frac{\partial}{c\partial t} \right), \quad (A_\mu) = (A, i\phi/k_s), \quad (j_\mu) = (j, ic\rho).
\]

(A.2)

It is an advantage that the four-dimensional quantities are uniformly defined\(^5\). The Lorenz gauge (6) can in turn be expressed by the divergence \( A_{\mu\rho} = 0 \), and the wave equations for the

\(^5\) Exceptions are the definitions of the current \( (j_\mu) \) by Minkowski (1908), Pauli (19211958) and the dual field strength tensor of Møller (1955).
four-potential are exactly the same as before (12):

\[ \Box A_\mu = \partial_\nu \partial_\nu A_\mu = k_s \frac{4\pi k_c}{ck_L} j_\mu, \quad k_s = -1. \]

The field strength tensor (13)

\[ (F_{\mu\nu}) = \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) = k_s \begin{pmatrix} -\epsilon_{\mu
uho\sigma} B_\rho & iE_\nu/k_L \ & -iE_\mu/k_L \end{pmatrix}, \quad k_s = -1. \quad (A.3) \]

is compared with its contravariant counterpart (14) by replacing \( E \) with \( iE \). The dual field strength tensor is generally defined by

\[ \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \quad \text{with} \quad \epsilon_{1234} = 1. \quad (A.4) \]

Once more it should be mentioned that all \( \tilde{F}_{\mu\nu} \) which are constructed by summing up from \( \mu = 1–4 \) have the opposite sign to those summed up from 0–3. According to our notation one gets

\[ \langle \tilde{F}_{\mu\nu} \rangle = k_s \begin{pmatrix} (i\epsilon_{\mu
u\rho\sigma} E_\rho/k_L) & -B \ & 0 \end{pmatrix}, \quad F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} : \begin{cases} iE/k_L \rightarrow -B \\ B \rightarrow -iE/k_L. \end{cases} \quad (A.5) \]

From the field strength tensors one obtains its invariants (17) and Maxwell’s equations (16):

\[ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} = B^2 - \frac{E^2}{k_L^2}, \quad \frac{1}{4} \tilde{F}_{\mu\nu} F_{\mu\nu} = -i\frac{1}{k_L} E \cdot B. \]

\[ F_{\nu\mu,\nu} = k_s \frac{4\pi k_c}{ck_L} j_\mu, \quad \tilde{F}_{\nu\mu,\nu} = 0. \]

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