Swimming of a sphere in a viscous incompressible fluid with inertia

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Abstract

The swimming of a sphere immersed in a viscous incompressible fluid with inertia is studied for surface modulations of small amplitude on the basis of the Navier–Stokes equations. The mean swimming velocity and the mean rate of dissipation are expressed as quadratic forms in terms of the surface displacements. With a choice of a basis set of modes the quadratic forms correspond to two Hermitian matrices. Optimization of the mean swimming velocity for a given rate of dissipation requires the solution of a generalized eigenvalue problem involving the two matrices. It is found for surface modulations of low multipole order that the optimal swimming efficiency depends in intricate fashion on a dimensionless scale number involving the radius of the sphere, the period of the cycle, and the kinematic viscosity of the fluid.

Keywords: swimming, fluid with inertia, sphere geometry

1. Introduction

The swimming of fish and the flying of birds still pose problems to theory (Childress 1981). The analysis can be based on the Navier–Stokes equations for the flow of a viscous fluid with a no-slip boundary condition at the surface of the body with periodically changing shape. For simplicity the fluid may be taken to be incompressible. The fluid is then characterized by its shear viscosity and mass density.

Most of the theoretical work has been concerned with either of two limiting situations. The swimming of microorganisms is well described by the time-independent Stokes equations of low
Reynolds number hydrodynamics (Happel and Brenner 1973, Purcell 1977). The work in this area has been reviewed by Lauga and Powers (2009). In the opposite limit of inviscid flow the analysis is based on the Euler equations with the effect of viscosity relegated to a boundary layer. The flow is predominantly irrotational, apart from the boundary layer and a wake of vorticity. The work in this field was reviewed by Sparenberg (1995, 2002) and by Wu (2001, 2011). The modeling of bird flight was reviewed by Pennycuick (2008) and by Shyy et al (2008). The problem has also been addressed in computer simulation (Deng et al 2013).

It is important to have a model covering the full range of kinematic viscosity. The seminal work of Taylor (1951) on the swimming of a sheet in the Stokes limit was extended to a fluid with inertia by Reynolds (1965) and by Tuck (1968). The swimming of a planar slab has also been studied in the full range (Felderhof 2015a). The disadvantage of these models is the infinite length of the system which precludes study of the finite size effects which are believed to be crucial in the inviscid limit.

In earlier work we have studied small amplitude swimming of a body in a viscous fluid with inertia from a general point of view (Felderhof and Jones 1994a). As an example we studied the swimming of a sphere by means of potential flow (Felderhof and Jones 1994b). Later we showed that in the Stokes limit the addition of viscous modes leads to a significantly enhanced optimal efficiency (Felderhof and Jones 2014). In the following we study the small amplitude swimming of a sphere in the full range of kinematic viscosity. For simplicity we assume axisymmetric flow.

The effect of unsteady flow may be characterized by the dimensionless number s, defined by $s^2 = a^2 \omega \rho / (2 \eta)$, where $a$ is the radius of the sphere, $\omega$ is the frequency, $\rho$ is the mass density of the fluid, and $\eta$ is the shear viscosity. Fluid inertia can have an effect in two ways. Either $s$ does not vanish, or the motion is slow, corresponding to zero frequency and $s = 0$, with a flow which is highly nonlinear. In that case the Reynolds stress, proportional to the fluid mass density $\rho$, must be taken into account.

The amplitude of swimming may be characterized by the ratio $\varepsilon = \xi / a$, where $\xi$ is the amplitude of stroke. We use perturbation theory and calculate the mean swimming velocity $\overline{U}$ and the mean rate of dissipation $\overline{D}$ to order $\varepsilon^2$, but consider the full range of scale numbers $s$, i.e., for fixed $a$ and $\omega$ the full range of kinematic viscosity $\nu = \eta / \rho$. The efficiency is proportional to the ratio $\overline{U} / \overline{D}$.

The Reynolds number is defined as usual by $Re = \overline{U} L / \nu$ with $L = 2a$. To second order in the amplitude the mean swimming velocity $\overline{U}$ is of order $\varepsilon^2 \omega L$. The corresponding streaming Reynolds number (Klotsa et al 2015) is defined by $Re_s = \varepsilon^2 L \omega / \nu = 8 \varepsilon^2 s^2$. The complete dependence of the Reynolds number on the two parameters $\varepsilon$ and $s$ is not known. The Strouhal number (Shyy et al 2008) is defined by $St = \varepsilon \omega a / (\pi \overline{U})$.

Wang and Ardekani (2012a) showed that in the swimming of small organisms history and added mass forces are important as the product $St \ Re = 4 \varepsilon s^2 / \pi$ increases above unity. Later (Wang and Ardekani 2012b) they studied the effect of Reynolds stress on the swimming velocity in the zero frequency limit as a function of Reynolds number for a simple spherical squirmer by a singular perturbation method. The latter work was extended by Khair and Chisholm (2014) and by Chisholm et al (2016). Ishimoto (2013) studied the effect of fluid inertia on the swimming of a spherical body, but with neglect of the Reynolds stress. Spelman and Lauga (2017) used a stream function formulation to study the mean swimming velocity of a sphere to second order in the amplitude $\varepsilon$ in the limit of large $s$. In the following we consider the mean swimming velocity to second order in $\varepsilon$ for arbitrary values of $s$, and study in addition the mean rate of dissipation. Optimization of the efficiency is formulated as an eigenvalue problem. This extends our earlier work on the Stokes limit (Felderhof and Jones 2014). The effect of Reynolds stress turns out to be quite important for small amplitude but nonvanishing $s$. At small values of the kinematic viscosity, corresponding to large $s$, it largely
fluids cancels the effect of viscous stress and pressure in the optimal efficiency. As a consequence the efficiency varies little with kinematic viscosity in a wide range. Similar behavior was found for an assembly of interacting rigid spheres (Felderhof 2015b) and for a planar slab (Felderhof 2015a). We find that in swimming of a single sphere with surface distortions consisting of a running wave of three or five low order multipolar modes the optimal efficiency in the inviscid limit tends to the value for potential swimming. We expect that this feature holds more generally.

2. Flow equations

We consider a flexible sphere of radius $a$ immersed in a viscous incompressible fluid of shear viscosity $\eta$ and mass density $\rho$. In the laboratory frame, where the fluid is at rest at infinity, the flow velocity $\mathbf{v}(r, t)$ and the pressure $p(r, t)$ satisfy the Navier–Stokes equations

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \eta \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0. \quad (1)$$

The fluid is set in motion by time-dependent distortions of the sphere. We shall study axisymmetric periodic distortions which lead to a translational swimming motion of the sphere in the $z$ direction in a Cartesian system of coordinates. The surface displacement $\xi(s, t)$ is defined as the vector distance

$$\xi = s' - s \quad (2)$$

of a point $s'$ on the displaced surface $S(t)$ from the point $s$ on the sphere with surface $S_0$. The fluid velocity $\mathbf{v}(r, t)$ in the rest frame is required to satisfy

$$\mathbf{v}(s + \xi(s, t)) = \frac{\partial \xi(s, t)}{\partial t}. \quad (3)$$

This amounts to a no-slip boundary condition. We assume that the integral of $\xi(s, t)$ and of $s \times \xi(s, t)$ over the surface $S_0$ vanishes, so that the displacement does not involve a net translation or rotation of the sphere.

We perform a perturbation expansion in powers of the displacement $\xi(s, t)$. The first order flow velocity $\mathbf{v}_1$ and pressure $p_1$ satisfy the linearized Navier–Stokes equations. The translational swimming velocity of the sphere, averaged over a period, is denoted by $U$. To first order in displacements the mean swimming velocity $U_1$ vanishes. We have previously (Felderhof and Jones 1994a) that to second order the mean swimming velocity may be calculated as the sum of a surface and a bulk contribution

$$\mathbf{U}_2 = \mathbf{U}_{2S} + \mathbf{U}_{2B}. \quad (4)$$

In spherical coordinates ($r, \theta, \varphi$) the surface contribution $\mathbf{U}_{2S}$ may be expressed as an integral of a mean surface velocity $\mathbf{u}_2(\theta)$, defined by

$$\mathbf{u}_2(\theta) = -\frac{1}{(\xi \cdot \nabla)N_1}_{|r=a}, \quad (5)$$

where the overhead bar indicates the time average over a period. The surface contribution $\mathbf{U}_{2S}$ to the swimming velocity is given by the spherical average (Felderhof and Jones 1994b)

$$\mathbf{U}_{2S} = -\frac{1}{4\pi} \int_{S_0} \mathbf{u}_2(\theta) \, d\Omega. \quad (6)$$

The bulk contribution $\mathbf{U}_{2B}$ corresponds to the term $\rho (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1$ in the Navier–Stokes equations. The time-averaged second order flow velocity $\mathbf{v}_2$ and pressure $p_2$ satisfy the inhomogeneous Stokes equations (Felderhof and Jones 1994a)
\[ \eta \nabla^2 \mathbf{v}_2 - \nabla \mathbf{p}_2 = \rho (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1, \quad \nabla \cdot \mathbf{v}_2 = 0, \]  
(7)

with boundary condition
\[ \mathbf{v}_2 |_{r=a} = \mathbf{u}_S(\theta), \]  
(8)

The right-hand side in equation (7) represents a force density \( \mathbf{F}_v = -\rho (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 \). The bulk part of the second order flow \( \mathbf{v}_2, \mathbf{p}_2 \) satisfies equation (7) with the no-slip boundary condition \( \mathbf{v}_2 |_{r=a} = 0 \). The contribution \( \mathbf{U}_{2B} \) to the mean swimming velocity follows from the flow velocity at infinity in the rest frame and the condition that no net force is exerted on the fluid. The integral of the force density \( \mathbf{F}_v \) is canceled by the surface integral of an induced force density on the sphere at rest. As we shall show later, the bulk contribution \( \mathbf{U}_{2B} \) may be calculated with the aid of an antenna theorem (Schmitz and Felderhof 1982).

In the following we consider in particular the case of harmonic time variation. It is then convenient to introduce complex notation. The surface displacement is written as
\[ \xi(\theta, t) = \text{Re}[\xi_\omega(\theta)e^{-i\omega t}], \]  
(9)

with complex amplitude \( \xi_\omega(\theta) \). The corresponding first order flow velocity and pressure are given by
\[ \mathbf{v}_1(\mathbf{r}, t) = \text{Re}[\mathbf{v}_\omega(\mathbf{r})e^{-i\omega t}], \quad p_1(\mathbf{r}, t) = \text{Re}[p_\omega(\mathbf{r})e^{-i\omega t}]. \]  
(10)

The time-averaged surface velocity \( \mathbf{u}_S(\theta) \) may be expressed as
\[ \mathbf{u}_S(\theta) = -\frac{1}{2} \text{Re}[\xi_\omega^* \cdot (\nabla) \mathbf{v}_\omega] |_{r=a}. \]  
(11)

The time-averaged second order rate of energy dissipation is given by (Felderhof and Jones 1994a)
\[ \overline{\mathcal{D}}_2 = -\frac{1}{2} \text{Re} \left[ \int_{S_0} \mathbf{v}_2^* \cdot \mathbf{\sigma}_x \cdot e_r \, dS \right], \]  
(12)

where \( \mathbf{\sigma}_x \) is the first order stress tensor with Cartesian components
\[ \sigma_{x,\omega,\alpha\beta} = \eta \left( \frac{\partial \mathbf{v}_x}{\partial x_\alpha} + \frac{\partial \mathbf{v}_x}{\partial x_\beta} \right) \delta_{\alpha\beta} - \mathbf{p}_x \delta_{\alpha\beta}. \]  
(13)

The efficiency of swimming is defined as
\[ E_T(\omega) = 4\eta \omega a^2 \left[ \frac{\mathcal{D}_2}{\mathcal{D}_2} \right]. \]  
(14)

This quantity may be expressed as a ratio of two forms which are quadratic in the surface displacement, and involve two Hermitian matrices. Earlier we have studied the zero frequency limit of the problem where inertia plays no role (Felderhof and Jones 2014). It has been shown by Shapere and Wilczek (1989) that in this limit the above definition of efficiency is preferable to that of Lighthill (1952). The efficiency defined in equation (14) is essentially the ratio of speed and power and is relevant in the whole range of scale number.

3. Matrix formulation

The explicit calculation requires the choice of a basis set of solutions of the linearized Navier–Stokes equations. After removal of the exponential time-dependent factor the equations read
\[ \eta [\nabla^2 \psi - \alpha^2 \psi] - \nabla p = 0, \quad \nabla \cdot \psi = 0, \]  
(15)

with the variable

\[ \alpha = (-i \omega \rho / \eta)^{1/2} = (1 - i) (\omega \rho / 2 \eta)^{1/2}. \]  
(16)

In our previous work (Felderhof and Jones 1994b) we have chosen a set of modes identical to those used earlier in a hydrodynamic scattering theory of flow about a sphere (Felderhof and Jones 1986). In the present axisymmetric case we can use a reduced set of solutions. Moreover, it turns out that in the numerical work it is advantageous to use a different normalization. Thus we use the modes

\[ \psi_l(r, \alpha) = \frac{2}{\pi} e^{i \alpha r} [l + 1]k_{l-1}(\alpha r)A_l(\hat{r}) + l k_{l+1}(\alpha r)B_l(\hat{r})], \]
\[ u_l(r) = -a \left( \frac{a}{r} \right)^{l+1} B_l(\hat{r}), \quad p_l(r, \alpha) = \eta \alpha^2 a \left( \frac{a}{r} \right)^{l+1} P_l(\cos \theta), \]  
(17)

with \( \hat{r} = r/r \), modified spherical Bessel functions \( k_l(z) \) (Abramowitz and Stegun 1965) and vector spherical harmonics \( \{ A_l, B_l \} \) defined by

\[ A_l = \hat{A}_{l0} = l P_l(\cos \theta) e_r - P_l^1(\cos \theta) e_{\theta}, \]
\[ B_l = \hat{B}_{l0} = -(l + 1) P_l(\cos \theta) e_r - P_l^1(\cos \theta) e_{\theta}, \]  
(18)

with Legendre polynomials \( P_l \) and associated Legendre functions \( P_l^1 \) in the notation of Edmonds (1974). The notation \( \hat{A}_{l0}, \hat{B}_{l0} \) is identical to that used in Cichocki et al (1988). In particular \( A_l = e_r \) and \( B_l = e_r - 3 \cos \theta e_\theta \). The solutions \( \psi_l(r, \alpha) \) are associated with vanishing pressure variation. The notation for the flow field \( u_l(r) \) is identical to that in our previous work for zero frequency (Felderhof and Jones 2014). At zero frequency there is no pressure variation associated with these irrotational flow fields. We remark here that the above basis set is not suitable at low frequency owing to the singularity of the solutions \( \psi_l(r, \alpha) \) at \( \alpha = 0 \). At low frequency we must use a modified set, as discussed later.

We expand the first order flow velocity and pressure in the modes, given by equation (17), as

\[ \psi(r) = -\omega \sum_{l=1}^{\infty} \left[ k_l \psi_l(r, \alpha) + \mu_l u_l(r) \right], \quad p(r) = -\omega \sum_{l=1}^{\infty} \mu_l p_l(r, \alpha), \]  
(19)

with complex coefficients \( \{ k_l, \mu_l \} \), which can be calculated as moments of the function \( \psi(r) \) on the surface \( r = a \). Correspondingly the displacement function \( \xi_s(\hat{r}) \) is expanded as

\[ \xi_s(\hat{r}) = -ia \sum_{l=1}^{\infty} [k_l \psi_l(s, \alpha) + \mu_l u_l(s)]. \]  
(20)

We define the complex multipole moment vector \( \psi \) as the one-dimensional array

\[ \psi = (\kappa_1, \mu_1, \kappa_2, \mu_2, \ldots). \]  
(21)

Then \( U \) can be expressed as (Felderhof and Jones 1994b)

\[ U = \frac{1}{2} \omega \alpha (\psi | B | \psi), \]  
(22)

with a dimensionless Hermitian matrix \( B \) and the notation

\[ (\psi | B | \psi) = \sum_{l,i,l',i'} \psi^b_{l,i,l',i'} B_{l,i,l',i'} \psi_{l',i'}, \]  
(23)
where the subscript $\sigma$ takes the two values $N$, $P$ with $\psi_N = \kappa$ and $\psi_P = \mu$. The subscripts $N$, $P$ correspond to notation used in earlier work (Felderhof and Jones 1986, 1994b) for modes proportional to those in equation (17). We impose the constraint that the force exerted on the fluid vanishes at any time. This requires $\kappa = 0$. We implement the constraint by dropping the first element of $\psi$ and erasing the first row and column of the matrix $\mathbf{B}$. We denote the corresponding modified vector as $\hat{\psi}$ and the modified matrix as $\hat{\mathbf{B}}$.

The time-averaged rate of dissipation can be expressed as

$$\mathcal{D}_2 = 8\pi \eta \omega^2 a^4 (\psi^* \mathbf{A} \psi),$$

with a dimensionless Hermitian matrix $\mathbf{A}$. We denote the modified matrix obtained by dropping the first row and column by $\hat{\mathbf{A}}$.

With the constraint $\kappa = 0$ the mean swimming velocity $\mathcal{U}_2$ and the mean rate of dissipation $\mathcal{D}_2$ can be expressed as

$$\mathcal{U}_2 = \frac{1}{2} \omega a (\psi^* \hat{\mathbf{B}} \psi), \quad \mathcal{D}_2 = 8\pi \eta \omega^2 a^4 (\hat{\psi}^* \hat{\mathbf{A}} \hat{\psi}).$$

Optimization of the mean swimming velocity for given mean rate of dissipation, taken into account with a Lagrange multiplier $\lambda$, leads to the eigenvalue problem

$$\hat{\mathbf{B}} \hat{\psi}_\lambda = \lambda \hat{\mathbf{A}} \hat{\psi}_\lambda.$$  

Both matrices $\hat{\mathbf{B}}$ and $\hat{\mathbf{A}}$ are Hermitian, so that the eigenvalues $\lambda$ are real. With truncation at maximum $l$-value $L$ the truncated matrices $\hat{\mathbf{A}}_{ll}$ and $\hat{\mathbf{B}}_{ll}$ are $2L - 1$-dimensional. The maximum positive eigenvalue $\lambda_{\text{max}}$ is of particular interest. Its corresponding eigenvector provides the swimming mode of maximal efficiency. The truncated matrices correspond to swimmers obeying the constraint that all multipole coefficients for $l > L$ vanish.

With use of equation (6) we find the contribution $\mathbf{B}_S$ to the matrix $\mathbf{B} = \mathbf{B}_S + \mathbf{B}_R$. The remainder $\mathbf{B}_R$ follows from the contribution $\mathcal{U}_{2B}$ to the mean swimming velocity. The elements of the matrices $\mathbf{B}_S$ and $\mathbf{A}$ are complex numbers which can be calculated by substitution of the expansions in equation (19) and (20) into the expressions (11) and (12). We have calculated the elements of the matrices $\mathbf{B}_S$ and $\mathbf{A}$ in our earlier work (Felderhof and Jones 1994b). In order to make contact with our subsequent work on the axisymmetric case in the limit of zero frequency (Felderhof and Jones 2014) we have performed an independent calculation for axisymmetry. In the calculation we use angular integrals which are detailed in appendix A.

The matrix $\mathbf{A}$ is diagonal in $l$, $l'$, and the matrices $\mathbf{B}_S$ and $\mathbf{B}_R$ are tridiagonal in $l$, $l'$. The matrices are frequency-dependent via the variable $\alpha a$. We write

$$\alpha a = (1 - i)s, \quad s = a \sqrt{\frac{\omega \rho}{2\eta}} = \frac{1}{\sqrt{2\eta_b}}, \quad \eta_b = \frac{\eta}{\omega \rho a^2},$$

where $\eta_b$ is the dimensionless viscosity (Sangani and Prosperetti 1993). The maximum eigenvalue $\lambda_{\text{max}}$ depends on the variable $s$. We call $s$ the scale number. It is related to the Roshko number $Ro = L^2 f \rho / \eta$, where $f = \omega/(2\pi)$, by $Ro = 4s^2/\pi$, if in $Ro$ we use the sphere diameter $2a$ as the characteristic length $L$.

From equation (27) we see that for given fluid properties the frequency must decrease with increasing radius as $1/a^2$ in order to keep the scale number $s$ constant. For water the kinematic viscosity takes the value $\eta/\rho = 0.01 \text{ cm}^2 \text{ s}^{-1}$, and in air it takes the value $\eta/\rho = 0.15 \text{ cm}^2 \text{ s}^{-1}$. Hence for air we have $s \approx 5a\sqrt{f}$ with $a$ in cm and $f = \omega/(2\pi)$ in Hz.
4. Effect of Reynolds stress

The force density $F_\nu = -\rho (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1$ in equation (7) can be written alternatively as the divergence of a Reynolds stress $F_\nu = -\rho \nabla \cdot (\mathbf{v}_1 \mathbf{v}_1)$. In order to find the contribution to the mean swimming velocity caused by this stress we must solve equation (7) with no-slip boundary condition at $r = a$. We explained the procedure below equation (8). The mean swimming velocity $U_{2B}$ corresponds to the matrix $B_B$ as in equation (22).

According to theory developed earlier (Felderhof and Jones 1994a) the second order swimming velocity corresponding to the mean Reynolds stress can be written as a sum of two contributions,

$$U_{2B} = U_{V} + \tilde{U}_{V},$$

(28)

where $U_{V}$ corresponds to the integral of the force density

$$F_{V} = \int F_v \, dr,$$

(29)

according to Stokes’ law

$$U_{V} = \frac{-1}{6\pi\mu} F_{V},$$

(30)

The remainder $\tilde{U}_{V}$ corresponds to a solution of the Stokes problem equation (7) corresponding to the force density $F_{V}$ and a freely moving sphere of radius $a$.

In order to find the contribution $\tilde{U}_{V}$ we use an antenna theorem derived earlier (Schmitz and Felderhof 1982). The theorem provides the solution to the Stokes equations for a multipolar force density located on a spherical surface of radius $R$ in infinite fluid. The force density $F_{V}$ is decomposed into vector spherical harmonics as

$$F_{V}(r) = f_A(r)A_1 + f_B(r)B_1 + \delta F_{V}(r),$$

(31)

with scalar functions $f_A(r), f_B(r)$ given by the angular integrals

$$f_A(r) = \frac{1}{2} \int_0^\pi \mathbf{e}_z \cdot F_{V}(r) \sin \theta \, d\theta,$$

$$f_B(r) = \frac{1}{4} \int_0^\pi (\mathbf{e}_z - 3 \cos \theta \, \mathbf{e}_r) \cdot F_{V}(r) \sin \theta \, d\theta$$

(32)

and with remainder $\delta F_{V}(r)$ given by a sum of higher order vector spherical harmonics $\{A_1, B_1\}$. According to the antenna theorem (Schmitz and Felderhof 1982) the Green function solution of the Stokes equations corresponds to a flow velocity for $r < a$ given by

$$v_0(r) = c_{100}v_{100}^+(r) + c_{102}v_{102}^+(r) + \delta v_0(r),$$

(33)

with flow fields $v_{100}^+$, $v_{102}^+$ given by (Cichocki et al 1988)

$$v_{100}^+(r) = A_1 = e_z, \quad v_{102}^+(r) = r^2(5A_1 + B_1),$$

(34)

coefficients

$$c_{100}^+ = \frac{1}{3\eta} \int_a^\infty \left[ 2r^2 f_A^+(r') - r^2 f_B^+(r') \right] \, dr', \quad c_{102}^+ = \frac{1}{15\eta} \int_a^\infty \frac{1}{r} f_B^+(r') \, dr'$$

(35)

and remainder $\delta v_0(r)$ given by a sum of higher order vector spherical harmonics $\{A_1, B_1\}$. The Green function tends to zero at infinity.
The velocity $\mathbf{U}_V$ in equation (28) follows from Faxén’s theorem as (Cichocki et al 1988)

$$\mathbf{U}_V = (c_{100}^+ + 5\sigma^2c_{102}^+) \mathbf{e}_z.$$  \hspace{1cm} (36)

By construction the flow pattern $\mathbf{v}_{2B}(r)$ tends to $-\mathbf{U}_{2B}$ plus a flow which decays faster than $1/r$ at large distance from the origin.

5. Expansion at low frequency

As we have noted in section 3 the singular behavior of the flow fields $v_l(r, \alpha)$ at $\alpha = 0$ leads to numerical difficulties at low frequency (Felderhof and Jones 1986). Thus instead of $v_l(r, \alpha)$ we use

$$v_l^0(r, \alpha) = X_l(\alpha) v_l(r, \alpha) + \frac{2(2l - 1)}{\alpha^2 a^2} u_l(r),$$  \hspace{1cm} (37)

with coefficient

$$X_l(\alpha) = \frac{2\alpha l a^l}{l(2l + 1)(2l - 3)!!} e^{-\alpha a}.$$  \hspace{1cm} (38)

It may be checked that at zero frequency

$$v_l^0(r, 0) = v_l(r),$$  \hspace{1cm} (39)

where $v_l(r)$ is the viscous mode function

$$v_l(r) = \left(\frac{\alpha}{r^l}\right) \left[(l + 1)P_l(\cos \theta) \mathbf{e}_r + \frac{l - 2}{l} P_{l+1}^0(\cos \theta) \mathbf{e}_\theta\right]$$  \hspace{1cm} (40)

used in the zero frequency theory (Felderhof and Jones 2014).

The pressure corresponding to the velocity mode in equation (37) is given by

$$p_l^0(r) = \frac{2(2l - 1)}{\alpha^2 a^2} p_l(r, \alpha).$$  \hspace{1cm} (41)

Correspondingly the expansion in equation (19) must be replaced by

$$v_l(r) = -\omega a \sum_{l=1}^{\infty} [n_l^0 v_l^0(r, \alpha) + \mu_l^0 u_l(r)],$$

$$p_l(r) = -\omega a \sum_{l=1}^{\infty} [n_l^0 p_l^0(r) + \mu_l^0 p_l(r, \alpha)].$$  \hspace{1cm} (42)

With multipole vector $\psi^0$ defined by

$$\psi^0 = (n_1^0, \mu_1^0, n_2^0, \mu_2^0, \ldots)$$  \hspace{1cm} (43)

the mean swimming velocity and mean rate of dissipation can be expressed as

$$\mathbf{U}_2 = \frac{1}{2} \omega a (\psi^0 | \mathbf{B}^0 | \psi^0),$$

$$D_2 = 8\pi a^3 \omega a (\psi^0 | \mathbf{A}^0 | \psi^0),$$  \hspace{1cm} (44)

with matrices $\mathbf{B}^0$ and $\mathbf{A}^0$.

The above expansion is not suitable at high frequency owing to numerical difficulties in the eigenvalue problem analogous to equation (27). Thus we must use two different expansions in the two regimes of low and high frequency. The eigenvalues for the two eigenvalue problems are of course the same, and the eigenvectors are related by the matrix transforming one basis set into the other. At zero frequency the two matrices in the representation of this section are identical to those derived earlier (Felderhof and Jones 2014).
6. Example

With a small number of modes at low multipole order the calculations can be performed analytically. With higher multipoles included the generalized eigenvalue problem can be solved numerically by use of the Eigensystem command of Mathematica. In our earlier work (Felderhof and Jones 1994b) we have performed calculations involving just potential modes. In that case the Reynolds stress vanishes. As we have shown at zero frequency (Felderhof and Jones 2014) the inclusion of viscous modes leads to a significantly higher maximum efficiency.

The qualitative behavior can be seen from a simple model with only modes of orders $l = 1$ and $l = 2$ included. In this case there are just three modes, the dipolar potential mode at $l = 1$, the viscous mode at $l = 2$, and the quadrupolar potential mode at $l = 2$. The modes are given by equation (17) at high frequency, and by equations (37) and (41) for the viscous modes at low frequency. It turns out that in this case the high frequency expansion works well numerically over the whole range of interest. It suffices to compare with the zero frequency results (Felderhof and Jones 2014).

From equations (7.11) and (7.17) of Felderhof and Jones (2014) the matrix $\hat{B}_{12}^0$ at zero frequency is given by

$$
\hat{B}_{12}^0 = \begin{pmatrix}
0 & -\frac{3}{5} & -3 \\
\frac{3}{5} & 0 & 0 \\
\frac{3}{5} & 0 & 0 
\end{pmatrix}
$$

(45)

and the matrix $\hat{A}_{12}^0$ is given by

$$
\hat{A}_{12}^0 = \begin{pmatrix}
3 & 0 & 0 \\
0 & \frac{27}{10} & \frac{18}{5} \\
0 & \frac{18}{5} & 6 
\end{pmatrix}
$$

(46)

We denote the corresponding maximum eigenvalue by $\lambda_{12}^0$. The eigenvalue problem yields (Felderhof 2015c)

$$
\lambda_{12}^0 = \frac{5}{3\sqrt{2}} \approx 1.17851.
$$

(47)

The corresponding eigenvector $\xi_0^0$ has components $(1, -4i\sqrt{2}/3, 11i/(5\sqrt{2}) \approx (1, -1.886i, 1.556i)$.

With fluid inertia included the maximum eigenvalue as a function of the scale number is given by the expression

$$
\lambda_{12}(s) = \left(\frac{N(s)}{D(s)}\right)^{1/2},
$$

(48)

with numerator

$$
N(s) = 225 + 450s + 450s^2 + 282s^3 - 12s^4 - 24s^5 + 104s^6 + 16s^7 + 4s^8 - 8s^9
+ 8s^{10} + 16Re\left((6i - (6 - 6i)s - 3s^2 + (1 + i)s^3 - is^4 - (1 - i)s^5)
\times s^{6}e^{-is}E_1(s - is) + 16s^{12}e^{is}|E_1(s - is)|^2
\right)
$$

(49)

and denominator

$$
D(s) = 18(9 + 18s + 18s^2 + 10s^3).
$$

(50)
The expression is derived by use of the characteristic equation from the matrices \( A \) and \( B \) given explicitly in appendix B. The denominator is related to the determinant of \( A \) by

\[
D(s) = \frac{8s^4}{3} \det[\hat{A}(s)].
\]

In figure 1 we plot the function \( \lambda_{12}(s) \) as a function of \( \log_{10}(s) \). The low frequency expansion is given by

\[
\lambda_{12}(s) = \frac{5}{3 \sqrt{2}} + \frac{8 \sqrt{2}}{135} s^3 - \frac{19 \sqrt{2}}{135} s^4 + \frac{16 \sqrt{2}}{135} s^5 + O(s^6),
\]

in accordance with equation (47). The function shows a weak maximum 1.183 at \( s_0 = 0.865 \). We compare with the maximum value \( 1/\sqrt{2} \) for potential swimming with two modes for \( \lambda_1 \) (dashed line).

The maximum eigenvalue tends to the limiting value

\[
\lambda_{12}^{\text{pot}} = \frac{1}{\sqrt{2}} \approx 0.70711,
\]

the optimal value for potential swimming with just the dipolar and quadrupolar modes. The eigenvalue \( \lambda_{12}^{\text{pot}} \) and the corresponding eigenvector \( \xi_0 \) are found from equations (45) and (46) with the second row and column of the matrices deleted.

The maximum eigenvalue at given \( s \) increases as more modes are included. In figure 2 we show the maximum eigenvalue \( \lambda_{13}(s) \) as a function of \( \log_{10}(s) \) for \( s < 2 \) and the modes of equation (17) for \( s > 2 \). The maximum eigenvalue at zero frequency is \( \lambda_{13}^0 = 1.514 \) and the corresponding eigenvector has components \( \xi_0^0 = (1, -1.553i, 1.824i, 1.373, -1.440) \). There is a weak maximum \( \lambda_{13}(s_0) = 1.516 \) at \( s_0 = 0.962 \) (not visible on the scale of figure 2). The corresponding eigenvector has components \( \xi_0 = (1, -0.715 + 1.592i, 0.262 - 1.861i, 1.291 + 0.198i, -1.385 - 0.060i) \) in terms of the modes of section 5. As \( s \) increases beyond \( s_0 \)
The function \( \lambda_{13}(s) \) decays to \( \lambda_{13}^{\text{opt}} = \sqrt{11/10} \approx 1.049 \), the optimal value for potential swimming with modes \( l = 1, 2, 3 \) corresponding eigenvector \( \xi_0 = (1, 0, \sqrt{11/10}i, 0, -3/5) \).

7. Asymptotics

The behavior shown in figures 1 and 2 requires further discussion of the asymptotic properties for large scale number. The function \( \lambda_{13}(s) \) shown in figure 2 is calculated from a complicated analytic expression involving exponential integrals, like the expression equation (48) for \( \lambda_{12}(s) \). The expression is derived from the explicit expressions for the matrices \( \hat{A}_{13} \), \( \hat{B}_{131} \) and \( \hat{B}_{013} \) given in appendix B, by use of the characteristic equation.

The exponential integrals appear multiplied by exponentials. It is therefore useful to define

\[
F(z) = e^z E_1(z) = \int_0^\infty \frac{e^{-u}}{z + u} \, du.
\]

In the expression for \( \lambda_{13}(s) \) we need the value of \( F(z) \) at \( z = s \pm is \) and at \( z = 2s \) for large positive \( s \). Clearly the function \( F(z) \) is analytic in the complex \( z \) plane apart from a branch cut along the negative real axis. The values of \( F(z) \) which are needed in figure 2 can be found accurately by numerical integration in equation (53). The expression for \( \lambda_{13}(s) \) involves powers of \( z \) up to \( z^{12} \) and down to \( z^{-10} \), so that in numerical calculations for large \( s \) we need integer programming.

To scrutinize the behavior for large values of \( s \) it is useful to derive series expansions. By integration by parts we derive

\[
F(z) = F_n(z) + R_n(z),
\]

where \( F_n(z) \) is given by a sum of \( n + 1 \) terms,

\[
F_n(z) = \sum_{j=0}^{n} (-1)^j \frac{j^n}{z^{j+1}}.
\]
and the remainder $R_n(z)$ is given by

$$R_n(z) = -(n + 1)! \int_0^\infty \frac{e^{-u}}{(z + u)^{n+2}} \, du. \tag{56}$$

The sum $F_n(z)$ corresponds to the sum of the first $n + 1$ terms in the asymptotic expansion of the exponential integral (Abramowitz and Stegun 1965).

By replacing $F_n(z)$ by the sum $F_n(z)$ for low values of $n$ in the expressions for $\lambda_{12}(s)$ and $\lambda_{13}(s)$ we can derive series expansions in inverse powers of $s$. For $\lambda_{12}(s)$ we find

$$\lambda_{12}(s) = \frac{1}{\sqrt{2}} + \frac{16\sqrt{2}}{5s} - \frac{112\sqrt{2}}{5s^2} + \frac{20736\sqrt{2}}{125s^3} + O(s^{-4}). \tag{57}$$

At $s = 100$ the sum $\lambda_{12}^{(4)}(s)$ of the first four terms shown agrees with the exact value to four decimal places. For $\lambda_{13}(s)$ we find by the same method

$$\lambda_{13}(s) = \frac{\sqrt{10}}{10} + \frac{128349}{2695}\frac{\sqrt{2}}{55}\frac{1}{s} + O(s^{-2}). \tag{58}$$

At $s = 100$ the first two terms of the expansion in equation (58) yield a value accurate to 1% and at $s = 10^4$ the value is accurate to five decimal places.

The eigenvector corresponding to the maximum eigenvalue $\lambda_{13}(s)$ can also be found by the same numerical method. For example, at $s = 10^4$ we find the eigenvector $\xi_0 = (1, -3.219 + 3.221i, -1.048i, 1.311 + 1.312i, -0.600)$ for the modes of section 3. This shows that for this scale number all five modes contribute significantly to the eigenvector. The contributions from $\tilde{B}_{133}$ and $\tilde{B}_{113}$ cancel to a large extent. We find the ratios $|\tilde{U}_{123}|/|\tilde{U}_{121}| = 1.472$ and $|\tilde{U}_{221}|/|\tilde{U}_{222} + \tilde{U}_{223}| = 3.112$. This shows again the importance of the Reynolds stress.

We note that for these modes of low order the whole calculation can be performed analytically, though at the expense of quite complicated expressions. It follows from the expressions in appendix B that the elements of the matrix $\tilde{B}_{113}(s)$ remain finite in the limit $s \to \infty$ and apparently there is a cancellation from the matrix $\tilde{B}_{113}(s)$ leading to the limiting behavior shown in figures 1 and 2.

### 8. Mode coefficients

It is worthwhile to discuss the properties of the simple model introduced at the beginning of section 6 in some more detail. We denote the coefficients of the zero frequency modes of Felderhof and Jones (2014) as $(\mu_1^1, \kappa_1^1, \mu_2^1)$. These are the amplitudes of the dipolar mode, the stresslet, and the quadrupolar mode, respectively. By comparison of the amplitudes of vector spherical harmonics on the spherical surface $r = a$ we find that the coefficients are related to the coefficients $(\mu_1, \kappa_2, \mu_2)$ of the modes of section 3 by the relations

$$\begin{align*}
\mu_1 &= \mu_1^1, \\
\kappa_2 &= \frac{1}{5}\frac{z^2}{1 + z} \kappa_2^1, \\
\mu_2 &= \frac{6 + 6z + 3z^2 + z^3}{z^2 + z^3} \mu_2^1, \\
z &= (1 - i)s. \tag{59}
\end{align*}$$

These coefficients give the same surface displacement as the zero frequency modes for the set $(\mu_1^1, \kappa_2^1, \mu_2^1)$. It follows from equation (59) that if one considers surface displacements characterized by a chosen set of coefficients $(\mu_1^1, \kappa_2^1, \mu_2^1)$, then the coefficient $\kappa_2$ of the boundary layer mode grows in absolute magnitude beyond all bounds as the scale number $s$ increases, whereas the amplitudes of the potential modes remain bounded. This causes an
increase of dissipation as $s$ increases, owing to steep gradients in the boundary layer. Hence the efficiency decreases with increasing $s$. As an example we show in figure 3 the ratio

$$\rho_{12} = \frac{\langle \xi(s) | \hat{B}_{12}(s) | \xi(s) \rangle}{\langle \xi(s) | \hat{A}_{12}(s) | \xi(s) \rangle},$$

(60)

where $\xi(s)$ is calculated from the vector $\xi_0$ given below equation (47) by use of equation (59). The ratio starts at the optimal value $\lambda_{12}(0)$ in the limit $s \to 0$, but eventually decreases to zero at large $s$. In figure 4 we show the surface deformation at four chosen times.

The simplest way of constructing a good mode of swimming for large $s$ is to avoid the boundary layer altogether and use surface displacements corresponding to the potential mode $\xi_0 = (1, 0, i/\sqrt{2})$ given below equation (52). In this case $\mu_1 = 1$, $\kappa_2 = 0$, $\mu_2 = i/\sqrt{2}$, independent of $s$. This is a mode of high efficiency, only slightly less efficient than the optimal mode given by the maximum eigenvector of equation (26).

9. Discussion

The analysis shows that for given surface distortion the mean swimming velocity and mean power of a sphere depend in an intricate way on the dimensionless scale number $s$, defined in
equation (27) and related to the Roshko number by $Ro = 4s^2/\pi$. Optimization of the mean swimming velocity for given power at fixed $s$ leads to a generalized eigenvalue problem. The eigenvector with largest eigenvalue within a class of strokes characterizes the optimal stroke in that class. Explicit expressions for the two Hermitian matrices characterizing the eigenvalue problem are given in appendix B. The expressions are the pinnacle of the present analysis.

The results are of particular interest for large values of the scale number. It turns out that in this range it is crucial to take the Reynolds stress into account. Apparently on time average over a period the effect of pressure gradients is largely canceled by the effect of Reynolds stress. Quantitatively the effect is measured by a comparison of the contributions $U_{2B}$ and $U_{2S}$ to the mean swimming velocity. A numerical example is given at the end of section 7.

In the asymptotic range of very large scale number $s$ we find that the optimal efficiency tends to the value for potential flow for the class of axisymmetric strokes involving the three modes of order $l = 1, 2, 3$. The same behavior was found for the swimming of a deformable slab (Felderhof 2015a). It may be conjectured that for a sphere this behavior occurs also for more complicated strokes. It is shown in figures 1 and 2 that in the asymptotic range the swimming is somewhat less effective than in the Stokes limit. There is an appreciable change in the mode of optimal swimming as is seen by a comparison of the eigenvectors of displacements at the end of sections 6 and 7.

For the strokes considered the optimal efficiency shows a maximum in the intermediate range of scale number. In this regime the optimal efficiency varies little with the scale number. For a swimmer the actual value of the scale number is determined by its geometrical dimension and the kinematic viscosity of the fluid. For that particular value of $s$ the swimmer can optimize its stroke to second order in the amplitude from the solution of the eigenvalue problem. It will be of interest to determine how the efficiency varies as the amplitude of stroke is increased. This requires numerical solution of the Navier–Stokes equations with no-slip boundary condition for the moving surface and is beyond the scope of the present investigation.

Appendix A

In this appendix we provide expressions for angular integrals which occur in the calculation. We consider vector-functions of the form

$$v_i(r) = f(r)A_l + g(r)B_l \quad (A1)$$

and scalar functions of the form

$$p_l(r) = h(r)P_l(\cos \theta) \quad (A2)$$

In the calculation of the matrix $A$ we need the angular integral

$$S_l(r) = \int_0^\pi v_i^*(r) \cdot (\nabla v_i(r) + \nabla v_i^*(r)) \cdot e_\theta \sin \theta \, d\theta$$

$$= \frac{2}{2l+1} \left[ (l-1)l(l+1) \left( \frac{f^* + g^*}{r} \right) - l(l+1)(l+2) \left( \frac{f^* + g^*}{r} \right) 
- l(l+1)(f^{*'} + g^{*'}) + l(3l+1)f^*f' + (l+1)(3l+2)g^*g' \right], \quad (A3)$$
where in the last equation we have omitted the dependence on $r$ for brevity. Similarly
\[ Q_j(r) = \int_0^\pi v_\ell^*(r) p_j(r) \cdot \mathbf{e}_z \sin \theta \, d\theta = \frac{2}{2l+1} [f^* h - (l+1) g^* h]. \] (A4)

In the calculation of the matrix $B_\ell$ we need the angular integral
\[ T_j^{(1)}(r) = \int_0^\pi v_\ell^*(r) \cdot \nabla v_{\ell+1}(r) \cdot \mathbf{e}_z \sin \theta \, d\theta \]
\[ = \frac{2l+2}{2l+1} \left[ l(l+1) (f^* + g^*) h - l f^* h' - (l+1) g^* h' \right], \] (A5)
as well as the integral
\[ T_j^{(2)}(r) = \int_0^\pi v_{\ell+1}^*(r) \cdot \nabla v_\ell(r) \cdot \mathbf{e}_z \sin \theta \, d\theta \]
\[ = \frac{2l+2}{2l+3} \left[ -(l+1)(l+2) (f^* + g^*) h - (l+1) f^* h' - (l+2) g^* h' \right]. \] (A6)

In the calculation of the matrix $B_{\ell+1}$ we need the angular integral
\[ V_j^{(1)}(r) = \int_0^\pi v_\ell^*(r) \cdot (\nabla v_{\ell+1}(r)) \cdot B_1 \sin \theta \, d\theta \]
\[ = \frac{2l+2}{2l+1}(2l+3) \left[ -l(l^2+l-3) (f^* + g^*) h + 3l(l+1)(l+3) (f^* + g^*) g \right. \]
\[ - l f^* h' + (l+1) g^* h' - 3l(l+2) h^* h' - 3l(l+1)(l+2) g^* h' \right], \] (A7)
as well as the integral
\[ V_j^{(2)}(r) = \int_0^\pi v_{\ell+1}^*(r) \cdot (\nabla v_\ell(r)) \cdot B_1 \sin \theta \, d\theta \]
\[ = \frac{2l+2}{2l+1}(2l+3) \left[ -3l(l-1)(l+2) (f^* + g^*) h \right. \]
\[ + (l+2)(l^2+3l-1) (f^* + g^*) g \]
\[ - 3l(l+1) f^* h' + 3l(l+2) g^* h' + (l+1)(l+2) h^* h' - (l+2)^2 g^* h' \right]. \] (A8)

The above expressions can be derived by use of Legendre function identities.

**Appendix B**

In this appendix we list the expressions for the matrix elements as functions of $s$ for low multipole orders. We consider the $5 \times 5$ Hermitian matrices $A_{\ell+3}$, $B_{\ell+3}$ and $\hat{B}_{\ell+3}$ corresponding to multipole coefficients $\mu_1, \kappa_2, \mu_2, \kappa_3, \mu_3$ in the representation of section 3. For brevity we denote the matrix elements as $A_{\alpha \beta}$, $B_{\alpha \beta}$, $\hat{B}_{\alpha \beta}$ with $(\alpha, \beta) = 1, \ldots, 5$. We list only
nonvanishing elements. The elements of the matrices $\hat{A}_{12}$, $\hat{B}_{12}$ and $\hat{B}_{12}$ are found by deleting the fourth and fifth rows and columns.

The nonvanishing elements $A_{a,b}$ are given by

\[
A_{11} = 3,
\]
\[
A_{22} = \frac{15}{8s^8}[180 + 360(s + s^2) + 240s^3 + 117s^4 + 42s^5 + 10s^6 + 2s^7],
\]
\[
A_{23} = A_{32}^\alpha = \frac{3}{s^2}[15 + 15(1 + i)s + 12is^2 - 2(1 - i)s^3],
\]
\[
A_{33} = 6,
\]
\[
A_{44} = \frac{21}{16s^{10}}[23625 + 47250(s + s^2) + 31500s^3 + 15606s^4
\]
\[+ 6012s^5 + 1812s^6 + 408s^7 + 64s^8 + 8s^9],
\]
\[
A_{45} = A_{54}^\alpha = \frac{15 - 15i}{4s^5}[105 + 105(1 + i)s + 90is^2 - 20(1 - i)s^3 - 4s^4],
\]
\[
A_{55} = 10. \quad (B1)
\]

The nonvanishing elements $B_{a,b}$ are given by

\[
B_{32} = B_{23}^\alpha = \frac{1}{2s^3}[1 - 45i - 45(1 + i)s - 42s^2 - 12(1 - i)s^3 + 4is^4],
\]
\[
B_{33} = B_{33}^\alpha = -3i,
\]
\[
B_{54} = B_{45}^\alpha = \frac{9 - 9i}{8s^9}[1575 + 3150s + (3150 - 90i)s^2
\]
\[+ (2100 - 180i)s^3 + (1032 - 180i)s^4
\]
\[+ (384 - 120i)s^5 + (104 - 56i)s^6 + 16(1 - i)s^7],
\]
\[
B_{55} = B_{55}^\alpha = \frac{3}{7s^4}[-105i + 105(1 - i)s + 90s^2 + 20(1 + i)s^3 + 4is^4],
\]
\[
B_{34} = B_{43}^\alpha = \frac{9 - 9i}{20s^5}[525 + 525(1 - i)s - 480is^2
\]
\[- 130(1 + i)s^3 - 44s^4 - 4(1 - i)s^5],
\]
\[
B_{35} = B_{53}^\alpha = -6i. \quad (B2)
\]

The nonvanishing elements $B_{a,0}$ are given by

\[
B_{012} = B_{21}^\alpha = \frac{1}{2}[-i - (1 + i)s + s^2 - (1 - i)s^3 - 2is^4F_2],
\]
\[
B_{024} = B_{42}^\alpha = \frac{1 - i}{4s^5}[45 + 90s + (90 - 6i)s^2 + (60 - 12i)s^3 - (6 + 12i)s^4
\]
\[+ (4 + 24i)s^5 - (4 + 48iF_2)s^6 + 8s^7 - 16s^8F_2],
\]
\[
B_{025} = B_{52}^\alpha = \frac{1}{168}[-18i + (18 - 18i)s + 6s^2 - (6 + 6i)s^3 + 9is^4
\]
\[+ (11 - 11i)s^5 + (1 - 24F_3)s^6 - (1 + i)s^7 + 21is^8F_3],
\]
\[
B_{034} = B_{43}^\alpha = \frac{1 - i}{480s}[450 + 450(1 - i)s - 222is^2 + 78(1 + i)s^3
\]
\[- 81s^4 + 83(1 - i)s^5 - is^6 + (1 + i)s^7 + 168is^8F_3 - 2s^8F_3], \quad (B3)
\]
with the abbreviations

\[ F_i = F(s + is), \quad F_- = F(s - is), \quad F_2 = F(2s) \]  \hspace{1cm} (B4)

for values of the function \( F(z) \) defined in equation (53). The vanishing of the elements \( B_{013} \) and \( B_{033} \) follows from a general theorem on the potential flow contribution which we derived earlier (Felderhof and Jones 1994a).

The matrix elements have a quite complicated dependence on the scale number \( s \). The limiting behavior of the maximum eigenvalues \( \lambda_{12}(s) \) and \( \lambda_{13}(s) \) for large \( s \), shown in figures 1 and 2, is due to delicate cancellations.

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