Abstract

The Cayley-Hamilton-Newton theorem - which underlies the Newton identities and the Cayley-Hamilton identity - is reviewed, first, for the classical matrices with commuting entries, second, for two $q$-matrix algebras, the RTT-algebra and the RLRL-algebra. The Cayley-Hamilton-Newton identities for these $q$-algebras are related by the factorization map. A class of algebras $\mathcal{M}(\hat{R}, \hat{F})$ is presented. The algebras $\mathcal{M}(\hat{R}, \hat{F})$ include the RTT-algebra and the RLRL-algebra as particular cases. The algebra $\mathcal{M}(\hat{R}, \hat{F})$ is defined by a pair of compatible matrices $\hat{R}$ and $\hat{F}$. The Cayley-Hamilton-Newton theorem for the algebras $\mathcal{M}(\hat{R}, \hat{F})$ is stated. A nontrivial example of a compatible pair is given.
1 Introduction

In this lecture we discuss several results from the classical multilinear algebra and their \( q \)-generalizations. The following simple observation lies in the origin of our discussion: the trace of the Cayley-Hamilton identity reproduces one of the Newton identities. Thus, it is natural to ask if other Newton identities can be “detraced”; in other words, can one find matrix identities whose trace reproduces the Newton identities. Such matrix identities do exist and we call them the Cayley-Hamilton-Newton (CHN) identities.

In fact, we have discovered the classical CHN identities in attempts to \( q \)-deform the Cayley-Hamilton theorem. We formulate the CHN identities (which imply the Cayley-Hamilton theorem) for two standard \( q \)-matrix algebras and explain the relationship of the CHN identities for the RTT-algebra and the RLRL-algebra using the factorization map. The formulation requires an understanding of some basic notions, like powers of \( q \)-matrices and \( q \)-matrices acting in different copies of a vector space. An analysis of these notions manifests different roles of the permutation matrix and suggests a definition of a class of \( q \)-matrix algebras \( \mathcal{M}(\hat{R}, \hat{F}) \) which includes the standard \( q \)-matrix algebras. The different roles of the permutation matrix are played by different Yang-Baxter matrices \( \hat{R} \) and \( \hat{F} \) for the algebras \( \mathcal{M}(\hat{R}, \hat{F}) \).

We formulate the CHN identities for the algebras \( \mathcal{M}(\hat{R}, \hat{F}) \) and give a simple example.

The lecture is mainly based on papers [1], [2], [3].

Section 2 contains an elementary introduction to the classical case. The CHN theorem is formulated.

In Section 3 two commonly used quantum matrix algebras, the RTT-algebra and the RLRL-algebra, are presented. A \( q \)-generalization of the CHN theorem for these algebras is given.

In Section 4 we explain that in the factorizable case the CHN theorem for the RLRL-algebra follows from the CHN theorem for the RTT-algebra.

In Section 5 we discuss more general \( q \)-matrix algebras for which the \( q \)-CHN theorem holds. These algebras are defined by a “compatible” pair of Yang-Baxter matrices \( \hat{R} \) and \( \hat{F} \). The section contains also a simplest nontrivial example of a compatible pair.

2 Classical Case

In this lecture we shall be talking about \( q \)-generalizations of the following two facts from the classical multilinear algebra:

2.1 Fact 1. Consider a set of \( n \) variables \( x_1, \ldots, x_n \). The ring of symmetric functions in \( x_1, \ldots, x_n \) has several useful sets of generators (see, e.g. [4]):
Power sums:

\[ s_k := \sum x_i^k, \quad k = 1, \ldots, n, \quad (1) \]

elementary symmetric functions

\[ \sigma_k := \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}, \quad k = 1, \ldots, n \quad (2) \]

and complete symmetric functions

\[ \tau_k := \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}, \quad k = 1, \ldots, n. \quad (3) \]

The classical Newton identities relate these generating sets:

\[
(-1)^{k+1} k \sigma_k = \sum_{j=1}^{k-1} (-1)^j s_{k-j} \sigma_j, \quad (4)
\]

\[
k \tau_k = \sum_{j=1}^{k-1} s_{k-j} \tau_j. \quad (5)
\]

Here one additionally defines \( \sigma_0 := 1 \) and \( \tau_0 := 1 \).

2.2 Fact 2. Let \( X \) be an operator in a vector space \( V \) of dimension \( n \). Let \( \chi(t) \) be the characteristic polynomial of \( X \), \( \chi(t) := \det(tI - X) \) where \( I \) is the identity operator in \( V \).

The Cayley-Hamilton theorem says that \( \chi(X) = 0 \).

2.3 In attempts to \( q \)-generalize these two facts we have found that there is a common predecessor of both of them. To present it, we first reformulate the Newton identities and the Cayley-Hamilton theorem.

Let \( \Lambda^k V \) and \( S^k V \) be the \( k \)-th wedge power and the \( k \)-th symmetric power of the vector space \( V \) respectively. The operator \( X \) induces operators \( \Lambda^k X \) and \( S^k X \) acting in \( \Lambda^k V \) and \( S^k V \): \( \Lambda^k X \) and \( S^k X \) act on polyvectors as follows:

\[
\Lambda^k X(v_1 \wedge v_2 \wedge \ldots \wedge v_k) := Xv_1 \wedge Xv_2 \wedge \ldots \wedge Xv_k \quad (6)
\]

\[
S^k X(Symm(v_1 \otimes v_2 \otimes \ldots \otimes v_k)) := Symm(Xv_1 \otimes Xv_2 \otimes \ldots \otimes Xv_k). \quad (7)
\]

Define three sets of invariants,

\[
s_k(X) := \text{Tr}_{V} X^k, \quad \sigma_k(X) := \text{Tr}_{\Lambda^k V} (\Lambda^k X), \quad \tau_k(X) := \text{Tr}_{S^k V} (S^k X). \quad (8)
\]

If an operator \( X \) is diagonalizable with eigenvalues \( x_1, \ldots, x_n \) one finds (using a basis in which \( X \) is diagonal) that \( s_k(X) = s_k, \quad \sigma_k(X) = \sigma_k \) and \( \tau_k(X) = \tau_k \) where \( s_k, \sigma_k \) and \( \tau_k \) are symmetric functions in \( x_1, \ldots, x_n \) given by (1), (2) and (3).
Thus, Newton identities relate traces of powers of an operator with traces of wedge or symmetric powers of an operator.

The elementary symmetric functions $\sigma_k(X)$ of the operator $X$ are sums of principal $k$-minors of a matrix (in an arbitrary basis) of the operator $X$: for a subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, \nu\}$ compute the determinant of a submatrix with rows and columns $i_1, \ldots, i_k$. The sum over all $k$-subsets equals to $\sigma_k(X)$.

On the other hand the sum of principal $k$-minors is exactly what enters the characteristic polynomial: the coefficient in $t^k$ in $\chi(t)$ gets contributions from principal $(n-k)$-minors - we have to select $t$ on the diagonal of $tI - X$ on $k$ places, then the corresponding term in $\chi(t) \equiv \det(tI - X)$ is $(-1)^{n-k}$ (the minor in complementary rows and columns). In other words the characteristic polynomial equals to

$$\sum_{j=0}^{n} t^j (-1)^{n-j} \sigma_{n-j}(X) \equiv (-1)^n \sum_{j=0}^{n} (-t)^{n-j} \sigma_j(X) \, .$$

(9)

Let us compare the Cayley-Hamilton theorem

$$0 = \sum_{j=0}^{n} (-X)^{n-j} \sigma_j(X)$$

(10)

with the Newton identities (4)

$$k\sigma_k(X) = -\sum_{j=0}^{k-1} (-1)^{k-j} s_{k-j}(X) \sigma_j(X) \equiv -\sum_{j=0}^{k-1} \text{Tr}(-X)^{k-j} \sigma_j(X) \, .$$

(11)

The comparison makes clear that the trace of the Cayley-Hamilton identity (10) reproduces the Newton identity with $k = n+1$ ($\sigma_{n+1} = 0$, the $(n+1)$-st wedge power of an $n$-dimensional space vanishes).

Therefore we conclude that there is a natural way to “detrace” the $(n+1)$-st Newton identity. A question arises: can we detrace other Newton identities?

2.4 It turns out that the answer is positive.

**Theorem.** Let $X^{[j]} := \text{Tr}_{(1, \ldots, j-1)} \Lambda^j X$ (the operator $\Lambda^j X$ acts on $j$-polyvectors; we take the trace in all indices but the last one). Then

$$kX^{[k]} = -\sum_{j=0}^{k-1} (-X)^{k-j} \sigma_j(X) \quad \forall \, k \, .$$

(12)

This theorem underlies both Newton and Cayley-Hamilton identities: the trace of eqs. (12) gives the Newton identities; eq. (12) for $k = n+1$ is the Cayley-Hamilton identity. We couldn’t find this theorem in the literature and we gave it a name: the Cayley-Hamilton-Newton (CHN) theorem.
2.5 Remarks. 1. There is the Cayley-Hamilton-Newton theorem for the symmetric powers. Let $X^{(j)} := \text{Tr}_{(1...j-1)} S^j X$. Then

$$kX^{(k)} = \sum_{j=0}^{k-1} X^{k-j} \tau_j(X) \quad \forall k .$$

(13)

Now the sequence $X^{(j)}$ does not terminate; there is no analogue of the Cayley-Hamilton theorem. However, taking the trace of (13) one finds another sequence (5) of the Newton identities.

2. A matrix $X$ generates an algebra $k[X]$ of polynomials in $X$. As a vector space, $k[X]$ has a filtration $F^j k[X] = <1, X, \ldots, X^j>$ which stabilizes after at most $n$ steps. The CHN theorem implies that $X^{[j]}$ (and $X^{(j)}$) belongs to $k[X]$ and moreover to $F^j k[X]$.

3. The following qualitative argument shows that an identity of the type (12) should exist, or, in other words, that the operator $X^{[k]}$ should be expressible in terms of usual powers $X^j$.

The space $\Lambda^k V$ can be embedded into $V^{\otimes k}$ as an image of a projector (antisymmetrizer) $A_k$ acting in $V^{\otimes k}$,

$$(A_k)^{i_1 \ldots i_k}_{j_1 \ldots j_k} = \frac{1}{k!} \delta^{i_1}_{[j_1} \ldots \delta^{i_k}_{j_k]} .$$

(14)

Here $[\ldots]$ means antisymmetrization.

The space $V^{\otimes k}$ decomposes into a direct sum of subspaces corresponding to the Young diagrams. Therefore, the subspace $\Lambda^k V$ has a well defined complement. One can interpret the operator $\Lambda^k X$ as an operator acting in $V^{\otimes k}$ as $\Lambda^k X = A_k X \otimes \ldots \otimes X$ ($k$ times): its restriction to $\Lambda^k V$ coincides with (3) and it acts as zero on the complement to $\Lambda^k V$.

Eq. (14) shows that $\Lambda^k X$ equals

$$(\Lambda^k X)^{i_1 \ldots i_k}_{j_1 \ldots j_k} = \frac{1}{k!} X^{i_1}_{[j_1} \ldots X^{i_k}_{j_k]} .$$

(15)

This is a sum of $k!$ terms. Tracing each term in the spaces $1, \ldots, k - 1$ one obtains some power of $X$ times some products of traces of powers of $X$. Therefore we have $X^{[k]} = \sum X^{k-j} \mu_j(X)$. Eq. (12) gives an exact expression for the scalar coefficients $\mu_j(X)$.

Similar arguments hold for the CHN identities (13).

3 Two quantum matrix algebras

3.1 The standard Drinfeld-Jimbo $\hat{R}$-matrix

$$\hat{R}^{ij}_{kl} = q^{\delta^{ij}_{kl}} \delta^{i}_{l} \delta^{j}_{k} + (q - q^{-1}) \theta(l - k) \delta^{i}_{k} \delta^{j}_{l}$$

(16)
has two eigenvalues, $q$ and $-q^{-1}$. In other words, its projector decomposition has two terms, $\hat{R} = qS - q^{-1}A$. Conventionally, the projector $S$ is called $q$-symmetrizer and the projector $A - q$-antisymmetrizer.

One says that Yang-Baxter matrices having two projectors are of Hecke type. The $q$-CHN theorem which we shall formulate is valid for arbitrary Yang-Baxter matrices, without any conditions on ranks of projectors $S$ and $A$.

### 3.2 Two types of $q$-matrix algebras are commonly used in the literature.

The first one is the RTT-algebra $[5]$. It is generated by a matrix $T_{ij}$ and relations

$$\hat{R}T_1T_2 = T_1T_2\hat{R}. \quad (17)$$

Here $T_1$ is the matrix $T$ in the first copy of the space $V$, $T_2$ is the matrix $T$ in the second copy of the space $V$.

This algebra is the algebra of functions on the $q$-group.

The second algebra is the RLRL-algebra (see $[6]$ and references therein). It is generated by a matrix $L_{ij}$ and relations

$$\hat{RL}_1\hat{RL}_1 = L_1\hat{RL}_1\hat{R}. \quad (18)$$

For a deformation-type Hecke $\hat{R}$ (like $\hat{R}$ in $(10)$) this algebra can be interpreted as the $q$-universal enveloping algebra: in the first orders, $\hat{R} = P(1 + \alpha r) + O(\alpha^2)$ and $L = 1 + \alpha l + O(\alpha^2)$ ($P$ is the permutation matrix and $\alpha$ is the deformation parameter); the eq. $(18)$ implies, in the second order in $\alpha$, that $[l_1, l_2] = [r + r_{21}, l_1]$; the sum $r + r_{21}$ is proportional to the permutation $P$; the commutation relations $[l_1, l_2] = [P, l_1]$ is just a compact way to write the commutation relations for the Lie algebra $gl(n)$ (or, with a condition of zero traces, for $sl(n)$).

### 3.3 CHN theorem for the RTT-algebra.

The result will look almost the same as in the classical case, we only have to explain meanings of $T^{[k]}$, $T^k$, $\sigma_k(T)$ (in the present talk we shall give a $q$-version of the eq. $(12)$ only; for a $q$-version of $(13)$ see $[1]$).

In the Hecke situation there is a well defined sequence of projectors (we assume that $q$ is not a root of unity), antisymmetrizers, defined inductively $[7]$ by

$$A_1 := I, \ A_k := \frac{1}{k_q} A_{k-1} \left( q^{k-1} - (k-1)q^{-1} \hat{R}_{k-1} \right) A_{k-1}. \quad (19)$$

Here $\hat{R}_{k-1}$ is the operator $\hat{R}$ acting in the $(k - 1)$-st and $k$-th copies of the space $V$ and $k_q$ is the $q$-number, $k_q := (q^k - q^{-k})/(q - q^{-1})$. The second antisymmetrizer, $A_2$, coincides with the projector $A$ entering the spectral decomposition of $\hat{R}$.

To define $X^{[k]}$ classically, it does not matter if we take traces in the spaces $1, \ldots, k - 1$ or $2, \ldots, k$. For a general $\hat{R}$ these two possibilities differ and we define two versions of the $k$-th wedge power of the matrix $T$:

$$T^{[k]} := \text{Tr}_{(1 \ldots k-1)} (A_k T_1 \ldots T_k) \quad (20)$$
and
\[ T^{[k]} := \text{Tr}_{(2\ldots k)}(A_k T_1 \ldots T_k) . \]  

The elementary symmetric functions in the “spectrum of \( T \)” are defined by
\[ \sigma_k(T) := q^k \text{Tr}_{(1\ldots k)}(A_k T_1 \ldots T_k) . \]  

The definition of powers of the matrix \( T \) is more interesting. To \( q \)-deform a classical object or notion whose definition involves the permutation matrix, one always has to analyse whether the permutation stays a permutation on the \( q \)-level or it becomes the \( \hat{R} \)-matrix. There is a way to define the square of a classical matrix \( X \) using the permutation matrix \( P \):
\[ X^2 = \text{tr}_1(PX_1X_2) . \]  
It turns out that the right choice is to replace \( P \) by \( \hat{R} \). Again, there are two versions:
\[ T^k := \text{Tr}_{(1\ldots k-1)}\left( \hat{R}_1 \hat{R}_2 \ldots \hat{R}_{k-1} T_1 T_2 \ldots T_k \right) \]  
and
\[ T^{\overline{k}} := \text{Tr}_{(2\ldots k)}\left( \hat{R}_1 \hat{R}_2 \ldots \hat{R}_{k-1} T_1 T_2 \ldots T_k \right) . \]  

With these preliminaries we can formulate the \( q \)-CHN theorem for the RTT-algebra:

**Theorem.** The following identities hold:
\[ k_q T^{[k]} = - \sum_{j=0}^{k-1} (-1)^{k-j} \sigma_j(T) T^{\overline{k-j}} \quad \forall \ k \]  
and
\[ k_q T^{\overline{k}} = - \sum_{j=0}^{k-1} (-1)^{k-j} T^{\overline{k-j}} \sigma_j(T) \quad \forall \ k . \]  

**Remarks.** 1. The elementary symmetric functions \( \sigma_j(T) \) form a commutative set but they are not central; the order of terms in the right hand sides of (25) and (26) is essential.

2. Taking \( k = n + 1 \) one obtains two \( q \)-versions of the characteristic identity. However, there is only one version of the Newton identities: taking trace of either (23) or (26) one finds that \( s_k(T) \) can be expressed in terms of \( \sigma_j(T) \)'s. Therefore \( s_k(T) \) commute with \( \sigma_j(T) \) and the tracing of the identities (25) and (26) produce the same result.

**3.4 CHN theorem for the RLRL-algebra.** Again, the result looks as the classical one after we give the right meaning to the notation.

It turns out that for the RLRL-algebra the subtle point is the definition of the \( L \)-matrix “acting in a \( k \)-th copy” of the space \( V \). It is not at all \( L_k \) and moreover,
it acts in all spaces from 1 till \( k \). We denote it by \( L_\mathcal{T} \) (to distinguish from \( L_k \)). The definition is inductive:

\[
L_\mathcal{T} := L_1 , \quad L_\mathcal{R} := R_{k-1}^{-1}L_\mathcal{T}R_k^{-1} .
\]

The definitions of \( L^{[k]} \) and \( \sigma_j(L) \) are as follows:

\[
L^{[k]} := \text{Tr}_q(2\ldots k)(A_k L_\mathcal{T} \ldots L_\mathcal{T})
\]

and

\[
\sigma_j(L) := \text{Tr}_q(1\ldots j)(A_j L_\mathcal{T} \ldots L_\mathcal{T}) .
\]

Here \( \text{Tr}_q \) is the \( q \)-trace, \( \text{Tr}_q(Z) := \text{Tr}(DZ) \) for an arbitrary matrix \( Z \); the matrix \( D \) is defined by

\[
\text{Tr}_2(\hat{R}D) = I .
\]

Finally, the \( q \)-power of the matrix \( L \) is just the usual power.

We are ready to state the \( q \)-CHN theorem for the RLRL-algebra.

**Theorem.** The following identities hold:

\[
k_q L^{[k]} = -\sum_{j=0}^{k-1} \sigma_j(L) (-L)^{k-j} \quad \forall \, k .
\]

**Remark.** The elements \( \sigma_j(L) \) are central; there is only one version of the CHN identities.

## 4 Factorization

Denote by \( \mathcal{U} \) the algebra dual to the RTT-algebra and by \( \mathcal{U}^\ast \) the RTT-algebra itself. Assume that \( \mathcal{U} \) is quasitriangular with the universal \( R \)-matrix \( R \in \mathcal{U} \otimes \mathcal{U} \). Assume also that the numerical matrix \( P\hat{R} \) is the image of \( R \) in the representation in the vector space \( V \).

Contracting the second argument of the element \( \mathcal{R}_{21} \mathcal{R} \in \mathcal{U} \otimes \mathcal{U} \) with an arbitrary element \( x \in \mathcal{U}^\ast \) we obtain an element from \( \mathcal{U} \). This defines a mapping \( \phi : \mathcal{U}^\ast \to \mathcal{U} \) which is called the factorization map \([8],[9]\),

\[
\phi(x) := \langle \mathcal{R}_{21} \mathcal{R}, x \rangle_2 , \quad x \in \mathcal{U}^\ast .
\]

Here \( \langle, \rangle \) is the pairing between \( \mathcal{U} \) and \( \mathcal{U}^\ast \), the index 2 in \( \langle, \rangle_2 \) means the second copy of \( \mathcal{U} \) of the first argument.

The map \( \phi \) is not a homomorphism, the matrix \( L = \phi(T) \) satisfies the RLRL-algebra \([18]\). However, as a linear map, \( \phi \) transforms any identity to an identity.
The matrix $D$ defined by (30) satisfies the equality $\hat{R}D_1D_2 = D_1D_2\hat{R}$. It follows then that the matrix $\tilde{T} := DT$ satisfies (17) or, in other words, $T \mapsto \tilde{T}$ is an automorphism of the RTT-algebra. Since $\tilde{T}$ generates the same RTT-algebra, the matrix $\tilde{T}$ satisfies the CHN identities (26).

**Theorem.** The map $\phi$ transforms the CHN identities for $\tilde{T}$ (26) to the CHN identities (31) for $L = \phi(T)$.

The algebra $\mathcal{U}$ is called factorizable if the map $\phi$ is an isomorphism of the underlying vector spaces.

For a factorizable $\mathcal{U}$ the theorem above gives another way to prove the CHN identities (31) for the RLRL-algebra using the CHN identities (26) for the RTT-algebra.

5 **Further $q$-generalizations**

5.1 The statement of the CHN theorem for the RTT-algebra and the RLRL-algebra looks the same but the meaning of objects (like powers of matrices, wedge powers of matrices, matrices acting in a $j$-th copy of the space) is different. It seems natural to try to construct a wider class of algebras among which the RTT-algebra and the RLRL-algebra are just particular cases.

Such a class of algebras indeed exists. The idea behind the construction appears already at the classical level and again shows up in the analysis of the roles of the permutation matrix. There are two quite different uses of the permutation matrix $P$.

Let $X$ be a matrix acting in the vector space $V$. Consider the tensor square $V \otimes V$ of the space $V$ and let $X_1$ be the operator $X$ acting in the first copy of $V$, $X_1 = X \otimes I$. The operator $X_2$ acting in the second copy of $V$ can be obtained from $X_1$ with the help of the permutation matrix: $X_2 = PX_1P$. This is the first use of $P$.

The second use of $P$: to say that the matrix elements of $X$ commute, one writes $PX_1X_2 = X_1X_2P$.

At the $q$-level we can play with different possibilities of $q$-deforming the uses of $P$. In particular, there is a room for two different Yang-Baxter matrices to appear and one can expect an existence of $q$-algebras defined by a pair of Yang-Baxter matrices $\hat{R}$ and $\hat{F}$, the matrix $\hat{R}$ governs the commutation relations between the matrix elements of a quantum matrix, the matrix $\hat{F}$ is responsible for shifting from a copy of the space $V$ to the next copy.

A detailed analysis shows that the Yang-Baxter matrices $\hat{R}$ and $\hat{F}$ should be “compatible” in the following sense:

$$\hat{R}_1\hat{F}_2\hat{F}_1 = \hat{F}_2\hat{F}_1\hat{R}_2 \quad \text{and} \quad \hat{R}_2\hat{F}_1\hat{F}_2 = \hat{F}_1\hat{F}_2\hat{R}_1. \tag{33}$$

Such compatible pairs appear in a particular kind [11] of the Drinfeld twist [12], the matrix $\hat{R}^F = \hat{F}\hat{R}\hat{F}^{-1}$ is again a Yang-Baxter matrix.
One easily checks that the pair $\hat{R}^F$ and $\hat{F}$ is again compatible and one can twist the second time to obtain a Yang-Baxter matrix $\hat{R}^F F$.

The generalized algebra $\mathcal{M}(\hat{R}, \hat{F})$ is the algebra generated by a matrix $M^j$ and relations
\[ \hat{R} M^i M^j = M^i M^j \hat{R}^F F . \] (34)

Here $M^i := M_1$ and $M^j := \hat{F} M^i \hat{F}^{-1}$.

For $\hat{F} = P$ the algebra $\mathcal{M}(\hat{R}, \hat{F})$ becomes the RTT-algebra; for $\hat{F} = \hat{R}$ the algebra $\mathcal{M}(\hat{R}, \hat{F})$ becomes the RLRL-algebra.

5.2 To conclude, we shall formulate the $q$-CHN theorem for the algebra $\mathcal{M}(\hat{R}, \hat{F})$. To this end, define the $k$-th power of $M$ by
\[ M^k := \text{Tr}_{F(2...k)} \left( \hat{R}_1 \hat{R}_2 ... \hat{R}_{k-1} M^i M^j ... M^k \right) , \] (35)
the $k$-th wedge power of $M$ by
\[ M^{[k]} := \text{Tr}_{F(2...k)} \left( A_k M^i M^j ... M^k \right) \] (36)
and the elementary symmetric functions in the spectrum of $M$ by
\[ \sigma_k(M) := \text{Tr}_F(M^{[k]}) . \] (37)

Here the matrices $M^i$ are defined inductively, $M^{i+1} := \hat{F}_k M^i \hat{F}^{-1}_k$; the antisymmetrizer $A_k$ is built with the help of the Yang-Baxter matrix $\hat{R}$ by eqs. (19); $\text{Tr}_F$ is the quantum trace defined by the Yang-Baxter matrix $F$, that is, for an arbitrary matrix $Z$, $\text{Tr}_F(Z) := \text{Tr}(D(F)Z)$ where the matrix $D(F)$ is defined by $\text{Tr}_2(F D(F)) = I$.

**Theorem.** The following identities hold
\[ (-1)^{k-1} q^j M^{[k]} = \sum_{j=0}^{k-1} (-q)^j M^{k-j} \sigma_j(M) \quad \forall \ k . \] (38)

There are versions of the $q$-CHN theorem for $M^{[k]}$ and for the $q$-symmetric powers of $M$ as well [3].

5.3 For the standard Yang-Baxter matrix [10] the multiparametric deformation is given by a twist with the Yang-Baxter matrix $\hat{F} = \Delta P$, where $\Delta$ is a diagonal matrix. In this case one has $\hat{R}^F F = \hat{R}$.

We shall give a simplest example of a compatible pair of Yang-Baxter matrices $\hat{R}$ and $\hat{F}$, for which $\hat{R}^F F \neq \hat{R}$. 


Consider the Cremmer-Gervais $R$-matrix [10],

$$
\hat{R}(b, y) = \begin{pmatrix}
q & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & b & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & b^2/q & \ldots & \ldots \\
\ldots & 1/b & \lambda & \ldots & \ldots & \ldots \\
\ldots & y & q & -b^2 y/q^2 & \ldots & \ldots \\
\ldots & \ldots & \ldots & b & \ldots & \ldots \\
\ldots & q/b^2 & \ldots & \lambda & \ldots & \ldots \\
\ldots & \ldots & \ldots & 1/b & \lambda & \ldots \\
\end{pmatrix}.
$$

(39)

Here $\lambda = q - 1/q$; zero entries of the matrix are denoted by dots.

The parameter $y$ is irrelevant, it can be set to 1 by a rescaling of the coordinates of the space $V$. The parameter $y$ is introduced for convenience, with its help we will write the matrix $\hat{R}^{FF}$ in a compact way.

The matrix $\hat{F}$ has the form

$$
\hat{F} = \begin{pmatrix}
1 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \beta & \ldots & \ldots \\
\ldots & \lambda & \ldots & \ldots & -1 \\
\ldots & \alpha & \ldots & \ldots & \ldots \\
\ldots & \gamma & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \alpha \\
\ldots & \ldots & \ldots & -1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \beta \\
\ldots & \ldots & \ldots & \ldots & 1 \\
\end{pmatrix},
$$

(40)

where $\alpha^2 = \beta^2 = \gamma^2 = -1$.

The Yang-Baxter matrices (39) and (40) form a compatible pair.

In [3] it is shown that for a compatible pair $\hat{R}$ and $\hat{F}$ one has

$$
\hat{R}^{FF} = D(F)_1 D(F)_2 \hat{R} (D(F)_1 D(F)_2)^{-1}.
$$

(41)

The matrix $D(F)$ for the Yang-Baxter matrix $\hat{F}$ is given by

$$
D(F) = \text{diag}\{1, \gamma^{-1}, 1\}.
$$

(42)

Finally, for the compatible pair (39) and (40) one finds

$$
D(F)_1 D(F)_2 R(b, y) (D(F)_1 D(F)_2)^{-1} = R(b, -y).
$$

(43)

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