A redesign methodology generating predefined-time differentiators with bounded time-varying gains

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Abstract
There is an increasing interest in designing differentiators, which converge exactly before a prespecified time regardless of the initial conditions, that is, which are fixed-time convergent with a predefined upper bound of their settling time (UBST), due to their ability to solve estimation and control problems with time constraints. However, for the class of signals with a known bound of their $(n+1)^{\text{th}}$ time derivative, the existing design methodologies yield a very conservative UBST, or result in gains that tend to infinity at the convergence time. Here, we introduce a new methodology based on time-varying gains (TVG) to design arbitrary-order exact differentiators with a predefined UBST. This UBST is a priori set as one parameter of the algorithm. Our approach guarantees that the UBST can be set arbitrarily tight, and we also provide sufficient conditions to obtain exact convergence while maintaining bounded TVG. Additionally, we provide necessary and sufficient conditions such that our approach yields error dynamics with a uniformly Lyapunov stable equilibrium. Our results show how TVG offer a general and flexible methodology to design algorithms with a predefined UBST.

KEYWORDS
exact differentiators, finite-time stability, fixed-time stability, prescribed-time, unknown input observers
1 | INTRODUCTION

Real-time exact differentiators are instrumental algorithms for solving various estimation and control problems.\textsuperscript{1-4} Recently, there is interest on extending their applicability to problems with time constraints.\textsuperscript{5,6} In this article, the goal is to design exact differentiators with uniform convergence despite the magnitude of the initial differentiation error, that is, with fixed-time convergence, where the upper bound for their settling time (UBST) can be set a priori by the user. Although some methodologies have been proposed to construct exact differentiators with a predefined UBST, several fundamental gaps remain.

On the one hand, state observer methodologies have been proposed in References 7 and 8, which can be applied to the differentiation problem of signals whose \((n + 1)\)th time derivative is precisely zero. The approach in Reference 7 is based on time-varying gains (TVG) and converges precisely at the prescribed time (algorithms with such a feature are referred in the literature as prescribed-time algorithms). This class of TVG allows a prescribed settling-time even under sensor delay.\textsuperscript{9} However, the TVG diverges to infinity as the time approaches such a prescribed settling-time. Such an unbounded gain is problematic under measurement noise or limited numerical precision.\textsuperscript{10} The approach in Reference 8 is an autonomous differentiator based on homogeneity.\textsuperscript{11,12} However, its UBST is greatly overestimated.\textsuperscript{8} Recent works have been proposed to maintain the TVG finite at the convergence time.\textsuperscript{13,14} However, the magnitude of the TVG is still an unbounded function of the initial condition. An important property to be studied when working with time-varying systems is the uniform Lyapunov stability,\textsuperscript{15} as the absence of uniform stability has an inherent lack of robustness implications. However, such analysis is missing in the literature of prescribed-time observers and differentiators based on TVG.

On the other hand, for the broader class of signals whose \((n + 1)\)th time derivative is bounded by a known constant, several differentiators have been proposed incorporating discontinuities to guarantee their exactness.\textsuperscript{16-18} However, the results in References 16 and 17 are limited to first-order derivatives. Additionally, the Lyapunov techniques used for their design in References 16 and 18 often lead to very conservative estimates of the UBST (e.g., a 130-fold overestimate in Reference 16). A conservative estimation of the UBST results in an over-engineered predefined-time differentiator with a significant slack between the desired UBST and the least one, which typically leads to larger than necessary differentiation errors. Currently, there is no methodology to reduce such over-engineering in high-order differentiators.

In this article, we fill the gaps above by introducing a novel methodology based on a class of TVG to design arbitrary-order exact differentiators with fixed-time convergence and a predefined UBST for the class of signals with a known bound on their \((n + 1)\)th derivative. Specifically, our methodology is based on a class of TVG that subsumes the one used in Reference 7. However, we derive sufficient conditions to guarantee that the TVG of the differentiator remain bounded. This is in contrast to workarounds suggested in Reference 7, where the TVG is maintained bounded at the cost of losing the exactness of the differentiator, producing errors at the prescribed time that grow linearly with the initial condition. Furthermore, we prove that our methodology enables us to set the actual worst-case convergence time of the differentiator arbitrarily close to the desired UBST. Since the resulting differentiator is time-varying, we provide necessary and sufficient conditions for our methodology to yield an error dynamic with an uniformly Lyapunov stable equilibrium.

Methodologies for a predefined UBST have also been proposed for the stabilization problem. Although a least UBST can be obtained based on autonomous controllers and Lyapunov analysis, see, for example, References 19-21, this approach has several technical challenges for higher-order systems, resulting in conservative estimations of an UBST.\textsuperscript{22,23} Methodologies based on TVG were proposed in References 24-28. This class of TVG, has been shown to be effective even in the prescribed-time control of systems described by partial differential equations.\textsuperscript{29} However, such TVG tends to infinity at the prescribed settling-time. Stabilizing controllers with a predefined UBST, based on uniformly bounded TVG have recently been proposed in References 30 and 31 using time-scale transformations.\textsuperscript{26,32} Given that these algorithms are time-varying, the analysis of the uniform Lyapunov stability (with respect to time) is relevant; however, such analysis is missing in such works.

The rest of this article is organized as follows. In Section 2, we introduce the predefined-time differentiation problem and present our main result. Section 3 presents numerical examples illustrating our contributions, in particular, comparing our differentiators with state-of-the-art algorithms. Finally, Section 4 presents some concluding remarks. The proofs are collected in the Appendices.

Notations. Let \(\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}\) and \(\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}\). For \(x \in \mathbb{R}\), \(|x|^\alpha = |x|^\alpha \text{sign}(x)\), if \(\alpha \neq 0\) and \(|x|^{-} = \text{sign}(x)\) if \(\alpha = 0\). For a function \(\phi : I \rightarrow J\), its reciprocal \(\phi^{-1}\), \(\tau \in I\), is such that \(\phi(\tau)^{-1} = 1\) and its inverse function \(\phi^{-1}(t)\), \(t \in J\), is such that \(\phi(\phi^{-1}(t)) = t\). For functions \(\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}\), \(\phi \circ \psi(t)\) denotes the composition \(\phi(\psi(t))\). We use boldface lower case letter for vectors and boldface capital letters for matrices. Given a matrix \(A \in \mathbb{R}^{n \times m}\), \(A^T\) represents its transpose.
Given a vector \( \mathbf{v} \in \mathbb{R}^n \), \( ||\mathbf{v}|| = \sqrt{\mathbf{v}^T \mathbf{v}} \). We use the notation \( \mathbf{b}_i \in \mathbb{R}^{(n+1) \times 1} \), \( i \in \{0, \ldots , n\} \), to denote a vector filled with zeros, except for the \((i+1)\)th component which is 1. For a signal \( y : \mathbb{R}_+ \rightarrow \mathbb{R} \), \( y^{(i)}(t) \) represents its \( i \)th derivative with respect to time at time \( t \). To denote a first-order derivative of \( y(t) \), we simple use the notation \( \dot{y}(t) \). The notation \( \text{Re}(\lambda) \) denotes the real part of the complex number \( \lambda \).

## 2 PROBLEM STATEMENT AND MAIN RESULT

### 2.1 Problem statement and preliminaries

We consider time-varying differentiation algorithms written as the dynamical system

\[
\begin{align*}
\dot{z}_i &= -h_i(e_i(t), p) + z_{i+1}, \quad i = 0, \ldots , n - 1, \\
\dot{z}_n &= -h_n(e_n(t), p),
\end{align*}
\]

(1)

where \( n > 0 \), \( e_0(t) = z_0(t) - y(t) \), for some scalar signal \( y(t) \). Above, \( p \) is used to highlight some parameters of interest and \( \{h_i\}_{i=0}^n \) are the correction functions of the algorithm, which could be discontinuous in the first argument.*

Defining \( e_i(t) = z_i(t) - y^{(i)}(t) \), \( i = 0, \ldots , n \), the differentiation error dynamics is

\[
\begin{align*}
\dot{e}_i &= -h_i(e_i(t), p) + e_{i+1}, \quad i = 0, \ldots , n - 1, \\
\dot{e}_n &= -h_n(e_n(t), p) + d(t),
\end{align*}
\]

(2)

where \( d(t) = -y^{(n+1)}(t) \). We let \( \mathbf{e}(t) := [e_0(t), \ldots , e_n(t)]^T \).

We denote as \( \mathcal{Y}_{(n+1)}^{(n+1)} \) the class of all scalar signals \( y(t) \) defined for \( t \geq 0 \) such that \( |y^{(n+1)}(t)| \leq \mathcal{L}(t) \) for all \( t \geq 0 \), for some known function \( \mathcal{L}(t) \geq 0 \). When \( \mathcal{L}(t) = L \) is constant, we simply write \( \mathcal{Y}_{(n+1)}^{(n+1)} \). When the correction functions \( \{h_i\}_{i=0}^n \) are such that the origin of Equation (2) is globally asymptotically stable\(^{15} \) for scalar signals \( y \) of class \( \mathcal{Y}_{(n+1)}^{(n+1)} \) and some \( \mathcal{L}(t) \), then its settling time function is

\[
T(\mathbf{e}(0)) = \inf \left\{ \xi \in \mathbb{R}_+ : \limsup_{t \to \xi^-} \| \mathbf{e}(t; \mathbf{e}(0), y) \| = 0 \quad \forall y \in \mathcal{Y}_{(n+1)}^{(n+1)} \right\},
\]

where \( \mathbf{e}(0) \) is the initial differentiation error. Here, \( \mathbf{e}(t; \mathbf{e}(0), y) \) is the solution of Equation (2) starting at \( \mathbf{e}(0) \) with signal \( y \in \mathcal{Y}_{(n+1)}^{(n+1)} \). With some abuse of notation, we write \( \mathbf{e}(t) = \mathbf{e}(t; \mathbf{e}(0), y) \) when there is no ambiguity. Then, the origin of system (2) is globally finite-time stable if it is globally asymptotically stable and \( T(\mathbf{e}(0)) < \infty \). The origin of system (2) is globally fixed-time stable if it is globally finite-time stable and there exists \( T_{\max} < \infty \) such that \( T(\mathbf{e}(0)) \leq T_{\max} \) for all \( \mathbf{e}(0) \in \mathbb{R}^{n+1} \). Here, \( T_{\max} \) is an UBST\(^{21,33} \). We say that \( T_{\max}^* \) is the least UBST\(^{19} \) of system (2) if

\[
T_{\max}^* := \sup_{\mathbf{e}(0) \in \mathbb{R}^{n+1}} T(\mathbf{e}(0)).
\]

Finally, the origin of system (2) is uniformly Lyapunov stable if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( s \geq 0 \), \( ||\mathbf{e}(s)|| \leq \delta \) implies \( ||\mathbf{e}(t)|| \leq \epsilon \) for all \( t \geq s \).

With the above definitions, a differentiator is said to be asymptotic, exact, or fixed-time if the origin of its error dynamic is globally asymptotically stable, globally finite-time, or globally fixed-time stable, respectively. Moreover, a differentiator is said to be predefined-time if it is fixed-time with a desired (predefined) UBST; a differentiator is said to be prescribed-time if it is fixed-time with prescribed settling time for every nonzero initial condition (i.e., every nonzero trajectory converges at the same prescribed-time). Additionally, we say that the differentiator is time-invariant if the correction functions \( \{h_i\}_{i=0}^n \) are independent of \( t \).

**Problem 1** (Predefined-time \( n \)th order exact differentiation). Given a desired convergence time \( T_c > 0 \) and any signal \( y \in \mathcal{Y}_{L}^{(n+1)} \), \( L \geq 0 \), obtain estimates \( z_i(t) \) of the time derivatives \( y^{(i)}(t) \), \( i = 0, \ldots , n \), such that the identities \( z_i(t) = y^{(i)}(t) \) hold for all \( t \geq T_c \).

Our problem is:
To solve Problem 1, we provide a methodology to design the correction functions \( \{h_i\}_{i=0}^{n}\) for the differentiator algorithm in Equation (1), to obtain a predefined-time differentiator. To highlight that the differentiator is designed using the desired UBST \( T_c\) as a parameter, we write \( h_i(e_0, t; T_c), i = 0, \ldots, n.\)

### 2.2 Main results

Let the class of functions \( \mathcal{H}^{(n+1)} \), \( L \geq 0 \) and the desired convergence time \( T_c > 0 \) be given. Our main result is a method to “redesign” an existing time-invariant asymptotic differentiator to obtain a time-varying predefined-time differentiator with an UBST given by \( T_c\). We start with an asymptotic differentiator having the form:

\[
\begin{align*}
\dot{z}_i &= -\phi_i(e_0) + z_{i+1}, \quad i = 0, \ldots, n - 1, \\
\dot{z}_n &= -\phi_n(e_0).
\end{align*}
\]

(3)

Here, \( \{\phi_i\}_{i=0}^{n}\) are its specific correction functions. We call algorithm (3) a base differentiator and \( \{\phi_i\}_{i=0}^{n}\) admissible correction functions if \( \{\phi_i\}_{i=0}^{n}\) are continuous, except perhaps at the origin, and there exist an interval \( I_\phi \subseteq \mathbb{R}_+ \) such that for any \( \alpha \in I_\phi\):

\[
\text{A1)}\quad \text{The algorithm (3) is an asymptotic differentiator for the class } \mathcal{H}^{(n+1)} \text{, where}
\]

\[
L(t) = L \exp(-\alpha(n+1)t),
\]

and there exists a constant \( \gamma > 0 \) which may depend on \( e(0) \) and \( \alpha \) such that differentiation error vector \( e(t) \) in algorithm (3) for any \( y \in \mathcal{H}^{(n+1)} \) has an exponential convergence of the form:

\[
\|e(t)\| < \gamma \exp(-\alpha(n+1)t).
\]

Next, let \( T_f \in \mathbb{R}_+ \) denote a bound for the settling time function of the base differentiator for the class of signals \( \mathcal{H}^{(n+1)} \). If such bound does not exist or is unknown, we simply set \( T_f = \infty \).

The next lemma shows that the correction functions as given in Table 1 are admissible correction functions of asymptotic differentiators, and provides bounds for their convergence time. See also References 12, 16, and 34-36 for additional examples.

**Lemma 1.** The correction functions \( \{\phi_i\}_{i=0}^{n}\) as given in Table 1 are admissible correction functions. Namely, there exists an interval \( I_\phi \subseteq \mathbb{R}_+ \) such that, for any \( \alpha \in I_\phi \) the base differentiator is an asymptotic differentiator for the class \( \mathcal{H}^{(n+1)} \), where \( L(t) = L \exp(-\alpha(n+1)t) \) and (A1) holds. Moreover, \( T_f \) as given in Table 1 is an UBST for the base differentiator.

| Correction functions | Design conditions |
|----------------------|-------------------|
| (i) \( \phi_i(w) = r^{i+1}1_i w \) | \( I_\phi = [0, r), T_f = \infty \) |
| (ii) \( \phi_i = 1_i L \) \( \frac{1}{\|w\|^2} \) | \( I_\phi = \mathbb{R}_+, T_f = \infty \) |
| (iii) \( \phi_i(w) = 4\sqrt{L} \left| \frac{1}{w} \right|^2 + k \left| \frac{1}{w} \right|^2 \) | \( I_\phi = \mathbb{R}_+, T_f = T_{\text{max}}^\phi \) |
| (iv) \( \phi_i(w) = \theta^{i+1}k_i \left( \left| w \right|^{i+1}\epsilon^{-1} + \left| w \right|^{i+1}b^{-1} \right) \) | \( I_\phi = \mathbb{R}_+, T_f = \frac{2}{2((1-c)^{-1} + (b-1)^{-1})} \) |

Only if \( \theta \geq 1, \epsilon \in (1-c, 1), b \in (1, 1+c), \epsilon > 0 \) sufficiently small and \( \{k_i\}_{i=0}^{n}\) chosen as in Reference 8.
To introduce our main result, we make the following definitions based on the correction functions \( \{ \phi_i \}_{i=0}^n \) of the base differentiator. First, let \( \alpha \in I_\phi \) and define

\[
Q := \left[ (U - aD)^{\alpha}b_n; \ldots; (U - aD)b_n; b_n \right],
\]

where \( b_n := [0, \ldots, 0, 1]^T \in \mathbb{R}^{(n+1)\times 1} \), \( D := \text{diag}(0, \ldots, n) \) and \( U := [u_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)} \), with \( u_{ij} = 1 \) if \( j = i + 1 \) and \( u_{ij} = 0 \) otherwise, that is,

\[
b_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.
\]

Second, let \( \Phi(e_0) := [\phi_0(e_0), \ldots, \phi_n(e_0)]^T \). Third, define the function \( f : \mathbb{R} \to \mathbb{R}^{n+1} \), as

\[
f(e_0) := \beta \Phi(\beta^{-1}e_0) + (U - aD)^{\alpha+1}b_n e_0.
\]

Here, \( \beta \geq (aT_c/\eta)^{\alpha+1} \), with \( \eta := 1 - \exp(-aT_f) \). Finally, for \( i = 0, \ldots, n \), define \( g_i(e_0; L) := l_i \max(L, \mu) \leq [e_0] \frac{\alpha+1}{\alpha} \) where \( l_i \in \mathbb{R}_+ \) is chosen as in Levant’s arbitrary-order exact differentiator and \( \mu > 0 \) is an arbitrary small positive constant.

Using the above notation, our main result is the following redesigned differentiator:

**Theorem 1.** Given the class of functions \( X^{(n+1)}_L \), \( n > 0 \), \( L \geq 0 \), and admissible correction functions \( \{ \phi_i \}_{i=0}^n \). For a given non-negative \( T_c \), a positive design parameter \( \alpha \in I_\phi \), and any \( \mu > 0 \), let functions \( f(e_0) \) and \( \{ g_i \}_{i=0}^n \) be defined as above. Consider the following “redesigned” correction functions:

\[
h_i(e_0, t; T_c) := \begin{cases} \kappa(t)^{i+1} f_i(e_0) & \text{for } t \in [0, T_c), \\ g_i(e_0; L) & \text{otherwise}, \end{cases}
\]

where \( f_i(e_0), i = 0, \ldots, n \), is the \( (i+1) \)th row of \( f(e_0) \) and \( \kappa(t) \) is a TVG given by

\[
\kappa(t) := \begin{cases} \frac{n}{\alpha(T_c - t)} & \text{for } t \in [0, T_c), \\ 1 & \text{otherwise}. \end{cases}
\]

Then, the differentiator of Equation (1) is exact in \( X^{(n+1)}_L \) and it converges before \( T_c \). That is, with the above correction functions, the origin of system (2) is fixed-time stable and \( T_c \) is an UBST.

**Proof.** See Appendix C.

Given that the redesigned differentiator is time-varying, we next provide necessary and sufficient conditions such that the resulting differentiator is uniformly Lyapunov stable.

**Proposition 1.** Let \( n \geq 1 \). Under the construction in Theorem 1, the differentiation error in (2) is uniformly Lyapunov stable if and only if \( \kappa(t) \) is uniformly bounded.

**Proof.** See Appendix C.1.

**Remark 1.** Note that for \( n = 0 \), uniform Lyapunov stability is obtained also for unbounded \( \kappa(t) \), because the error dynamics is a scalar system. In the context of differentiation, this special case is less important, since no derivatives are computed in this case.

Notice that the error dynamics of the differentiator is time-varying only due to the TVG in the correction functions of our algorithm. Thus, the initialization time when the differentiator is turned on is considered to be \( t_0 = 0 \) without
restriction of generality. In contrast, uniform Lyapunov stability refers to a stability property uniform over the initial time of system trajectories, which is denoted by \( s \) in its definition in Section 2 to avoid confusion, as is usual in the literature.\(^{15}\)

**Remark 2.** Absence of uniform Lyapunov stability is a significant problem in practice. Suppose that briefly before \( T_c \), the differentiator error is accidentally perturbed (by measurement noise, round-off errors, etc.). Proposition 1 and its proof show that, depending on how close to \( T_c \) this happens, an arbitrarily large peak on the differentiator error may occur.

Note that existence of Filippov solutions of the differentiator presented in Theorem 1 is ensured through the time-scaling argument when the gain is not uniformly bounded, that is, when \( \eta = 1 \). However, solutions may cease to exist even with arbitrarily small noise as shown in Reference 10 unless a uniformly bounded gain is guaranteed. Below, we provide the conditions on the base differentiator to yield a uniformly bounded gain.

Compared to the base differentiator of Equation (3), the redesigned differentiator of Equation (1) contains two additional parameters \( \alpha, \beta \) to be tuned. These parameters allow to tune the transient response, but for any admissible value of these parameters, the predefined-time convergence is maintained. The parameter selection of Equation (1) can be mostly performed using existing criteria for the base differentiator (see, e.g., Table 1). When our approach is applied to the case where \( T_f < \infty \), our redesign methodology subsumes the one recently suggested in Reference 18, and it provides an additional degree of freedom \( \alpha \) to tune the transient behavior.

Our approach is based on the time-scale transformation \( \tau = \varphi(t) \) (see Lemma 2 in Appendix B for details), where \( \varphi(t) := -\alpha^{-1} \ln(1 - \eta t / T_c) \). Moreover, the TVG in (4) is obtained by \( \kappa(t) = \frac{d\varphi(t)}{d\tau} \). Figure 1 illustrates such time-scale transformation and its associated TVG. Notice that with a larger value of \( \alpha \), a value \( T_f^* < T_f \) in the \( \tau \)-time (where \( T_f \) is the UBST of the base differentiator) maps closer to \( T_c \) in the \( t \)-time than with a smaller value \( \alpha \). Thus the \( \alpha \) parameter allows to tune the differentiator to converge arbitrarily close to the desired UBST given by \( T_c \), as shown in Proposition 2. Moreover, notice that the TVG becomes singular at \( t = T_c \eta^{-1} \). However, if \( T_f \) is finite, then \( \eta < 1 \), and therefore \( T_f \) in the \( \tau \)-time maps to \( T_c < T_c \eta^{-1} \) in the \( t \)-time. Thus, \( \kappa(t) \) is also finite at \( t = T_c \). This feature is used in Proposition 3 to produce a differentiator with a uniformly bounded TVG.

Moreover, as we show below, this parameter can be used to make the differentiator converge arbitrarily close to the desired UBST \( T_c \), as shown in the following result:

**Proposition 2.** Assume that the base differentiator is such that \( T_f < \infty \) and let \( T_c^* \) denote the least upper bound for the settling time of Equation (1) with the parameter \( \alpha \in I_\phi \) with unbounded \( I_\phi \). Then, for any \( \varepsilon > 0 \), there exists \( \alpha \in \mathbb{R}_+ \) such that \( T_c - T_c^* \leq \varepsilon \).

**Proof.** See Appendix C.1. \(\blacksquare\)

In the simulation results of the following section, we show that \( \alpha, \beta \) can also be used to improve the transient performance and reduce the over-engineering of the differentiator (i.e., an overly conservative bound \( T_c \)).

Note that the prescribed-time observer proposed by Holloway and Krstic,\(^7\) which can be seen as an exact prescribed-time differentiator for signals of class \( Y_0^{n+1} \), has the interesting property that the settling time function (C4) is such that \( T(e(0)) = T_c \) for every nonzero initial condition (i.e., instead of predefining an UBST, the user prescribes the settling-time, which is the same for every nonzero initial condition). However, the time-varying gain will be such
that \( \lim_{t \to T_c} \kappa(t) = \infty \). Actually, an analogous situation happens with controllers based on TVG.\(^{25,31}\) Notice that our approach also yields a prescribed-time differentiator, with similar features, when the base differentiator is not an exact differentiator (e.g., if the base differentiator is linear as in Table 1(i)).

To maintain a bounded gain, Holloway et al. suggest to “turn off” the error correction terms at some time \( t_{\text{stop}} < T_c \). However, turning-off the error correction functions in such a way will destroy the exactness of the differentiator in Reference 7 (the differentiation error will no longer be zero at time \( T_c \)). In fact, since the error dynamics becomes a linear time-varying system with bounded system matrix, it is easy to see that the norm of the error \( \| \varepsilon(t_{\text{stop}}; \varepsilon(0), y) \| \) grows linearly with the initial condition. Our methodology circumvents this limitation in two ways. First, we provide sufficient conditions such that the gain is finite at the convergence time. Nevertheless, the TVG may grow as the magnitude of the initial error grows. Second, we provide sufficient conditions to guarantee that the gain remains uniformly bounded regardless of the initial condition. The following proposition formalizes these two points:

**Proposition 3.** Consider the base differentiator (3) and the redesigned differentiator as in Theorem 1.

(i) If the origin of the error dynamics of the base differentiator of Equation (3) is globally finite-time stable, then in the redesigned differentiator \( \kappa(T(\varepsilon(0))) < \infty \) for all \( \varepsilon(0) \in \mathbb{R}^{n+1} \). But \( \kappa(T(\varepsilon(0))) \) is an unbounded function of the initial condition.

(ii) If the origin of the error dynamics of the base differentiator of Equation (3) is globally fixed-time stable, then, there exists \( \kappa_{\max} < \infty \) such that in the redesigned differentiator \( \kappa(T(\varepsilon(0))) \leq \kappa_{\max} \) for all \( \varepsilon(0) \in \mathbb{R}^{n+1} \). In particular, if \( T_f < \infty \) is a known UBST of the base differentiator (3), then \( \kappa(t) \) is uniformly bounded by

\[
\kappa(t) \leq \kappa_{\max} := \frac{\exp(aT_f) - 1}{aT_c} \quad \text{for all } t \geq 0.
\]

**Proof.** See Appendix C.1.

However, we also note that our approach yields a tradeoff for \( \kappa(t) \): if one tries to make the convergence “tighter” by adjusting \( \alpha \), then this necessarily yields a bigger bound \( \kappa_{\max} \). Conversely, a smaller \( \kappa_{\max} \) will result in a larger “slack” between \( T_c \) and the least UBST of (2).

**Remark 3.** It is straightforward to extend our methodology to filtering differentiators.\(^{35,40}\) Specifically, let \( \{h_i\}_{i=0}^n \) selected as in Theorem 1 for signals in \( \mathcal{Y}(n+1)_L \). Then, the algorithm

\[
\dot{w}_i = -h_{i-1}(w_1, t; T_c) + w_{i+1},
\]

for \( i = 1, \ldots, n_f - 1 \),

\[
\dot{w}_i = -h_{i-1}(w_1, t; T_c) + (z_0 - y),
\]

for \( i = n_f \),

\[
\dot{z}_{i-1} = -h_{i-1}(w_1, t; T_c) + z_{i-1},
\]

for \( i = n_f + 1, \ldots, n_d + n_f \),

\[
\dot{z}_{n_f} = -h_n(w_1, t; T_c),
\]

with \( n = n_d + n_f \), is a predefined-time exact differentiator but now for signals in \( \mathcal{Y}(n+1)_L \). However, in this case, we obtain that, for all \( t \geq T_c \) and every initial condition \( w_0(0), l = 1, \ldots, n_f, z_0(t), i = 0, \ldots, n_d \); \( w_i(t) = 0, l = 1, \ldots, n_f \), and \( z_i(t) = y^{(i)}(t) \) for all \( i = 0, \ldots, n_d \). For \( n_f = 0 \), \( w_1(t) \) is defined as \( w_1(t) = z_0(t) - y(t) \). This observation follows by noticing that by setting the differentiation errors \( e_i(t) = w_{i+1}(t) \), for \( i = 0, \ldots, n_f - 1 \) and \( e_i(t) = z_{i-n_f}(t) - y^{(i-n_f)}(t) \) for \( i = n_f, \ldots, n_d + n_f \) we obtain the error dynamics (2) where \( n = n_d + n_f \) and \( d(t) = -y^{(n+1)}(t) \) with \( |d(t)| \leq L \). The filtering properties of this algorithm can be very useful in the presence of noise,\(^{35}\) as we numerically illustrate in the following section.
Remark 4. The proof of Theorem 1 given in Appendix C is obtained by relating, by the coordinate change (C3) and the time-scale transformation given in Lemma 2, the differentiator’s error dynamics (2) with an asymptotically stable “auxiliary system” (C1) that is built with the selection of correction functions \{\phi_i\}_{i=0}^n. Following the same ideas, different redesigned correction functions can also be obtained. For instance, the proof may straightforwardly be applied to a differentiator with correction functions

\[
 h_i(e_0, t; T_c) := \begin{cases} 
 \kappa(t)^{n+i-\rho} f_{\rho,i}(e_0, t) & \text{for } t \in [0, T_c), \\
 g_i(e_0; L) & \text{otherwise},
\end{cases}
\]

parameterized using \( \rho \in [0, n+1] \), where \( f_{\rho,i}(e_0, t) \) is the \((i+1)\)th component of

\[
 f_\rho(e_0, t) := \beta Q_\rho \Phi(\beta^{-1} \kappa(t)\rho e_0) + \kappa(t)\rho (U - aD_\rho)^{n+1}b_n e_0
\]

with \( D_\rho := \text{diag}(-\rho, 1 - \rho, \ldots, n - \rho) \), \( Q_\rho := [(U - aD_\rho)b_n; \ldots; (U - aD_\rho)b_1; b_0] \), and \( \beta \geq (\alpha T_c/\eta)\rho^{n+1} - \rho \). The corresponding error dynamics are still related with the “auxiliary system” (C1), but with \( \pi(\tau) = \beta^{-1} (\alpha T_c/\eta)\rho^{n+1} - \rho \exp(-a(n+1-\rho)\tau)d(\varphi^{-1}(\tau)) \), where \( \varphi^{-1}(\tau) = t = \eta^{-1} T_c (1 - \exp(-a\tau)) \) is the parameter transformation from Lemma 2 in Appendix B, by the coordinate transformation

\[
 e(t) = \beta K_\rho(t) Q_\rho \chi(\varphi(t)),
\]

where \( K_\rho(t) := \text{diag}(\kappa(t)^{-\rho}, \kappa(t)^{-1-\rho}, \ldots, \kappa(t)^{-n-\rho}) \), and the time-scale transformation given in Lemma 2.

Remark 5. The first-order differentiator in Reference 14 can be derived from Remark 4 with \( \rho = n \), by using as admissible correction functions, the one of Levant’s super-twisting algorithm,41 as in Table 1 with \( n = 2 \). Since Levant’s super-twisting algorithm is finite-time stable with a settling-time function satisfying \( \sup_{t \geq 0} T(e(0)))) = T_f = \infty \), then \( \kappa(T(e(0))) \) is finite, but \( \sup_{t \geq 0} \kappa(T(e(0))) = \infty \). Although the uniform Lyapunov stability result in Proposition 1 is presented for \( \rho = 0 \), it can be shown that the algorithm in Reference 14 is nonuniform Lyapunov stable. To prevent the TVG to grow to infinity, the authors in Reference 14 proposed to detect the settling-time, as the time when the sliding mode starts. However, such approach can only be used in the absence of noise. Moreover, for large initial conditions sliding mode starts arbitrarily close to \( T_c \) with arbitrarily large TVG. On the contrary, as stated in Proposition 3(ii), our approach allows to obtain a predefined-time differentiator based on an uniformly bounded TVG, as illustrated in Section 3. In practice, even if there is a bound \( \kappa_{\max} \), to avoid large values of \( \kappa(t) \), one may wish to detect the convergence of the differentiator by monitoring \( |e_0(t)| \) as in Reference 42 and make the switching in the \( \{h_i\}_{i=0}^n \) functions when the convergence is detected.

3 | NUMERICAL ANALYSIS AND COMPARISONS

Here we present case studies to analyze the performance of the redesigned differentiators, showing in particular the slack of the UBST given by \( T_c \), boundedness of the TVG, and robustness to noise. Since, according to the previous exposition, the main features are obtained when \( T_f \) is finite, we focus in the simulation section in such case.

The simulations below were created in OpenModelica using the Euler integration method with a step size of \( 1 \times 10^{-7} \).

Example 1. Here, we design a first-order exact fixed-time differentiator whose time-varying gain remains bounded. We also systematically study its performance against noises of different magnitudes. Consider the base differentiator with correction functions \( \phi_i(w, \tau) \) given in Table 1(iii) with \( T_f = T_{max} = 1 \). We set \( \alpha = 3 \) and \( \beta = 2(\alpha T_c/\eta)^{n+1} \), and for \( g_i(e_0, t) \), we choose \( l_0 = 1.5 \) and \( l_1 = 1.1 \). Thus, \( Q = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} \) and \( (U - aD)^{n+1}b_n = \begin{bmatrix} -\alpha \\ \alpha^2 \end{bmatrix} \).

In Figure 2A–C, we show the simulation of algorithm (1), with a desired UBST given by \( T_c = 1 \), to differentiate signals with different bounds of the second derivative. That is, different values of \( L \) and different noise magnitude are considered. In Figure 2D, we show the simulation for the algorithm as a filtering differentiator (5) with \( n_f = 1 \) and \( n_d = 0 \). Recall
FIGURE 2 Simulation results for Example 1. For (A) and (B), we apply the algorithm (1) with \( n = 1 \) and \( L = 1 \), to the signal \( y(t) = 0.75 \cos(t) + 0.0025 \sin(10t) + t \) for (A) the noiseless case, (B) with white noise \( \nu(t) \) with standard deviation of 0.1. For (C), we apply the algorithm (1) with \( n = 1 \) and \( L = 10 \), to the signal \( y(t) = 0.75 \cos(t) + 0.0025 \sin(10t) + t \) with white noise \( \nu(t) \) with standard deviation of 0.1. For (D), we apply the algorithm (5) with \( n_f = 1, n_d = 0, \) and \( L = 1 \), to the signal \( y(t) = 0.75 \cos(t) + 0.0025 \sin(10t) + t \) with white noise \( \nu(t) \) with standard deviation of 0.5. For all simulations, \( \kappa(t) \) is bounded by \( \kappa_{\text{max}} = 6.362 \).

FIGURE 3 Simulation results for Example 2. Comparing our approach with \( \alpha = 5 \) and \( \beta = 1.5(\alpha T_c / \eta)^{n+1} \) with the autonomous predefined-time differentiator in Reference 39. Here \( L = 1, y(t) = 0.75 \cos(t) + 0.025 \sin(10t) + t \), and \( z_0(0) = z_1(0) = 10 \). For all simulations, \( \kappa(t) \) is bounded by \( \kappa_{\text{max}} = 29.49 \).

that, in the absence of noise \( \nu(t) = 0 \) and \( z_0(t) = y(t) \) for all \( t \geq T_c \). Figure 2D shows the behavior of this algorithm under noise. In all examples, the initial conditions were chosen as \( z_0(0) = z_1(0) = 10 \).

Recall that in Reference 39, Section 5.1, the convergence is obtained at 0.25 while the UBST is 1. Example 2 illustrates how our redesign significantly reduces the overestimation and allows to tune the transient behavior. It also illustrates the case where predefined-time convergence is obtained with bounded gains.

Example 2. In this example, we compare the performance of our differentiator with the predefined-time differentiator introduced in Reference 39, Section 5.1. To this end, consider the same situation as in Example 1 using the predefined-time differentiator (1) with \( n = 1 \) and differentiating the signal \( y(t) = 0.75 \cos(t) + 0.025 \sin(10t) + t \), and notice that the differentiator in Example 1 can be seen as a redesigned differentiator obtained using the algorithm in Reference 39, Section 5.1 as a base differentiator. In Figure 3, we compare the performance of both algorithms for different initial conditions.

Since the estimate of the UBST in Reference 39 is conservative, the resulting predefined-time exact differentiator is over-engineered and converges sooner than required, producing large estimation errors. Using the algorithm from Reference 39 as a base differentiator, in the redesigned algorithm we can tune \( \alpha \) and \( \beta \) to reduce the slack of the UBST and to obtain a lower maximum error than in Reference 39. As it can be observed from Figure 3, the maximum error is significantly reduced. Figure 4 shows a logarithmic plot of the norm of the error for different initial conditions. In Figure 4A, one can see the performance of the algorithm from Reference 39 designed for \( T_c = 1 \), which is used as base differentiator in our redesign. Figure 4B shows the performance of our algorithm with \( \alpha = 5, \beta = 1.5(\alpha T_c / \eta)^{n+1} \) and \( T_c = 1 \). It can be observed that the slack of the UBST is significantly reduced using our approach. However, as previously
discussed, there exists a trade-off between reducing the slack and the magnitude of the TVG. As it is well known, a large value of the gain is related to a greater sensitivity to numerical errors and chattering. The growing numerical chattering observed in Figure 4B is due to the increasing TVG. However, notice that, in our approach the TVG remains bounded and as a result the numerical chattering remains bounded as well. In contrast, in prescribed-time methods based on unbounded TVG, for example, References 7 and 14, the error due to such numerical chattering would be unbounded.10 Nevertheless, the use of the design in Remark 4 with $\rho = n + 1$ allows reducing such numerical chattering, because in such case, the discontinuous function does not appear multiplied by the TVG. Figure 4C shows the performance of our algorithm, based on Remark 4, with $\rho = n + 1$, $\alpha = 5$, and $\beta = 1.5(\alpha T_c/\eta)^{n+1-r}$ and admissible correction functions from Table 1(iii). For all simulations, $\kappa(t)$ is bounded by $\kappa_{\max} = 29.49$.

As it can be observed the numerical chattering is significantly reducing, having an error norm in the same order of magnitude than with the autonomous approach in Reference 39, but with an improved transient performance and a reduced slack between the desired UBST and the least UBST.

4 | CONCLUSION

We have introduced a methodology to design predefined-time arbitrary-order exact differentiators. Our approach uses a class of TVG that in previous approaches diverged to infinity. In our methodology, we provided sufficient conditions to obtain a predefined-time differentiator with bounded TVG. Such boundedness of the TVG is shown to be necessary and sufficient for uniform Lyapunov stability of the origin of the differentiator error dynamics. Furthermore, we show that the desired upper bound for the convergence can be set arbitrarily tight, contrary to existing fixed-time differentiators which yield very conservative upper bounds for the convergence time or where such upper bound is unknown.

Overall, our results demonstrate how TVG provide a very flexible methodology to design arbitrary-order differentiation algorithms with fixed-time convergence, opening the door to apply them to solve control and estimation problems with time-constraints.

As future work, we consider the discretization of our algorithm. Discretization of differentiators is an active area of research.40,43 It is particularly challenging to provide consistent discretization methods maintaining the predefined-time convergence property.44,45

CONFLICT OF INTEREST

The authors declare no potential conflict of interest.
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Data Availability Statement
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Endnotes
*In the usual Filippov’s interpretation of the solutions of \( \dot{x} = f(x, t), \ x \in \mathbb{R}^n \) it is assumed that \( ||f(x, t)|| \) has an integrable majorant function of time for any \( x \), ensuring existence and uniqueness of solutions in forward time. However, in this work we deal with \( f(x, t) \) for which no majorant function exist, but existence and uniqueness of solutions is still guaranteed by argument similar to Reference 20. In particular, existence of solutions follows directly from the equivalence of solutions to a well-posed Filippov system via the time-scale transformation.
†The correction functions in Theorem 1 correspond to the case \( \rho = 0 \).

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APPENDIX A. ON ADMISSIBLE CORRECTION FUNCTIONS

Proof of Lemma 1. To show that Table 1(i) are admissible correction functions satisfying (A1) let \( I_\phi = [0, r) \), any \( a \in I_\phi \) and consider the error dynamics of the differentiator under such linear correction functions given by \( \dot{e} = \tilde{A} e - b_e d(t) \) where \( |d(t)| \leq \mathcal{L}(t) \), for all \( t \geq 0 \), \( \tilde{A} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)} \) defined by \( a_{i,1} = -r_l l_i, a_{i,j+1} = 1 \) and \( a_{ij} = 0 \) elsewhere, that is,

\[
\tilde{A} = \begin{bmatrix}
-r_l & 1 & 0 & \ldots & 0 \\
-r_l^2 l_1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-r_l^n l_{i-1} & 0 & 0 & \ldots & 1 \\
-r_l^{n+1} l_n & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
\]

and \( b_n = [0, \ldots, 0, 1]^T \in \mathbb{R}^{(n+1) \times 1} \). Notice that the characteristic polynomial of \( \tilde{A} \) is \( s^{n+1} + r_l s^{n} + \ldots + r_l^n l_n \) with eigenvalue set \( \{ r\lambda_0, \ldots, r\lambda_n \} \) and \( \max(\text{Re}(\lambda_0), \ldots, \text{Re}(\lambda_n)) = -(n + 1) \). Moreover, \( \|e(t)\| = \|\exp(\tilde{A}t)e(0) + \int_0^t \exp(\tilde{A}(t-s)b_e d(s)ds)\| \). Additionally, there exists a constant \( c \in \mathbb{R} \) such that \( \|\exp(\tilde{A}t)e(0)\| \leq c \exp(-r(n+1)t)\|e(0)\| \). Consequently,

\[
\|e(t)\| \leq c \exp(-r(n+1)t)\|e(0)\| + \frac{cL}{(r - a)(n + 1)}(\exp(-a(n+1)t) - \exp(-r(n+1)t)).
\]

Furthermore, let \( \gamma(e(0), a) = c\|e(0)\| + cL(r - a)^{-1}(n + 1)^{-1} \). Then, \( \gamma(e(t), a) \exp(-a(n+1)t) \) since \( \exp(-r(n+1)t) < \exp(-a(n+1)t) \). Hence, the correction functions \( \{\phi_i\}_{i=0}^n \) as given in Table 1(i) are admissible.

It follows from References 34 and 39 that with correction functions \( \{\phi_i\}_{i=0}^n \) in Table 1(ii,iii) the base differentiator is an exact differentiator for a class \( \mathcal{Y}_L^{(n+1)} \) with \( L \neq 0 \), while it follows from Reference 8 that with correction functions in Table 1(iv) the base differentiator is an exact differentiator for a class \( \mathcal{Y}_L^{(n+1)} \) with \( L = 0 \). Since, \( \mathcal{L}(t) = L(\exp(-a(n+1)t) \leq L, \) for all \( t \geq 0 \), and any \( a \in I_\phi = [0, r) \), with the correction functions in Table 1(ii,iv) the base differentiator is an exact differentiator for the class \( \mathcal{Y}_L^{(n+1)} \). Moreover, pick \( \gamma(e(0), a) = S(e(0)) \exp(a(n+1)t)T(e(0)) \) where \( S(e(0)) = \sup\{\|e(t)\| : 0 \leq t \leq T(e(0))\} \) and \( T(e(0)) \) is the settling time of \( e(t) \). Therefore, for \( 0 \leq t \leq T(e(0)) \),

\[
\gamma(e(0), a) \exp(-a(n+1)t) = S(e(0)) \exp(a(n+1)(T(e(0)) - t)) \geq S(e(0)) \geq \|e(t)\|.
\]

And \( \gamma(e(0), a) \exp(-a(n+1)t) \geq 0 = \|e(t)\| \) for \( t \geq T(e(0)) \). Hence, the correction functions \( \{\phi_i\}_{i=0}^n \) as given in Table 1(ii,iv) are admissible. The UBST for the convergence time of the error dynamics of the base differentiator with correction functions given in Table 1(iii,iv) is given in References 8 and 39, respectively.

APPENDIX B. TIME-SCALE TRANSFORMATIONS

The trajectories corresponding to the system solutions of (2) are interpreted, in the sense of differential geometry\(^{46}\) as regular parameterized curves. Since we apply regular parameter transformations over the time variable, this reparameterization is referred to as time-scale transformation.

**Definition 1** (46, Definition 2.1). A regular parameterized curve, with parameter \( t \), is a \( C^1(I) \) immersion \( c : I \to \mathbb{R} \), defined on a real interval \( I \subseteq \mathbb{R} \). This means that \( \frac{dc}{dt} \neq 0 \) holds everywhere.

**Definition 2** (46, p. 8). A regular curve is an equivalence class of regular parameterized curves, where the equivalence relation is given by regular (orientation preserving) parameter transformations \( \varphi \), where \( \varphi : I \to I' \) is \( C^1(I) \), bijective and \( \frac{d\varphi}{dt} > 0 \). Therefore, if \( c : I \to \mathbb{R}^n \) is a regular parameterized curve and \( \varphi : I \to I' \) is a regular parameter transformation, then \( c \) and \( c \circ \varphi : I' \to \mathbb{R}^n \) are considered to be equivalent.

**Lemma 2** (20). The bijective function \( \varphi(t) = t = -a^{-1}\ln(1 - \eta t/T_c) \), defines a parameter transformation with \( \varphi^{-1}(\tau) = t = \eta^{-1}T_c(1 - \exp(-a\tau)) \) as its inverse mapping. Moreover, \( \frac{dt}{d\tau} |_{t=\varphi(t)} = \kappa(t)^{-1} \), for \( t \in [0, T_c] \) and \( \kappa \) given in (4).
APPENDIX C. PROOF OF THE MAIN RESULT

Our strategy to prove the main result is as follows. First, we build an “auxiliary system” with the selection of correction functions \{\phi_i\}_{i=0}^n. Then, we show that the auxiliary system and the differentiation error dynamics (2) are related by a coordinate change and the time-scale transformation given in Lemma 2. Thus, the settling-time for (2) can be obtained in the basis of the settling-time of the auxiliary system and the time-scaling.

**Lemma 3.** Suppose that the conditions of Theorem 1 are fulfilled. Then, the origin of the auxiliary system

\[
\frac{d\chi_i}{dt} = -\phi_i(\chi_0) + \chi_{i+1}, \text{ for } i = 0, \ldots, n - 1, \\
\frac{d\chi_n}{dt} = -\phi_n(\chi_0) + \pi(t),
\]

(C1)

is globally asymptotically stable for \(\pi(t) = \beta^{-1}(\alpha T_c/\eta)^{n+1} \exp(-\alpha(n+1)t) d(\phi^{-1}(t))\) where \(\phi(t) = -\alpha^{-1} \ln(1 - \eta t/T_c)\) is the parameter transformation from Lemma 2 in Appendix B. Moreover, let \(\chi = [\chi_0, \ldots, \chi_n]^T\), then for every solution \(e(t)\) of (2) there is a solution \(\chi(t)\) of (C1) such that

\[
e(t) = \beta K(t)Q\chi(\phi(t)),
\]

(C2)

holds for all \(t \in [0, T_c)\) where \(K(t) := \text{diag}(1/k(t), \ldots, k(t)^n)\). That is, the curves \(\beta^{-1}Q^{-1}K(t)^{-1}e(t)\) and \(\chi(t)\) with \(e(0) = \beta K(0)Q\chi(0)\) are equivalent curves under the time-scale transformation \(t = \phi(t)\). Thus, the redesigned differentiator’s error dynamics (1), is asymptotically stable.

**Proof.** Denote \(\hat{\chi}(t) := \beta^{-1}Q^{-1}K(t)^{-1}e(t)\) and define \(A := -Q^{-1}(U - \alpha D)^{n+1}b_n b_n^T\), where \(b_1 = [1, 0, \ldots, 0]^T\). Due to the definition of \(k(t)\), we have \(\kappa(t) - k(t) = \alpha k(t)\) and

\[
\frac{d}{dt} K(t)^{-1} = -k(t)^{-1}k(t)DK(t)^{-1} = -\alpha k(t)DK(t)^{-1}.
\]

We may then write (2) as

\[
\dot{e} = -\kappa(t)K(t)Q(\beta\Phi(\beta^{-1}e_0) - Ae) + Ue + b_n d,
\]

(C3)

in the interval \(t \in [0, T_c)\). Note that \(Q\), by construction, is a lower triangular matrix with ones in the main diagonal. Hence, \(e_0 = \beta \hat{\chi}_0\) and \(Q^{-1}K(t)^{-1}b_n = \kappa(t)^{-n}b_n\). Furthermore, note that \(QU = (U - \alpha D)Q + QA\), and hence \(Q^{-1}(U - \alpha D)Q = U - A\). Using these relations, as well as \(K(t)^{-1}UK(t) = \kappa(t)U\), we obtain the dynamics of \(\hat{\chi}\) in \([0, T_c)\) as

\[
\dot{\hat{\chi}} = \beta^{-1}Q^{-1} \left( \frac{d}{dt} K(t)^{-1} \right) e + \beta^{-1}Q^{-1}K(t)^{-1}\dot{e} \\
= \kappa(t) \left( -\Phi(\hat{\chi}_0) + U\hat{\chi} + \beta^{-1}k(t)^{-n}b_n d \right).
\]

Now, consider the time-scale transformation \(t = \phi(t)\) to obtain an expression for \(\frac{d}{dr} \hat{\chi} = \frac{d}{dt} \hat{\chi} \left( \phi^{-1}(t) \right)\) which yields

\[
\frac{d}{d\tau} \hat{\chi} = -\Phi(\hat{\chi}_0) + U\hat{\chi} + b_n \pi(t),
\]

with \(\pi(t) = \beta^{-1}k(\phi^{-1}(t))^{-n}d(\phi^{-1}(t)) = \beta^{-1}(\alpha T_c/\eta)^{n+1} \exp(-\alpha(n+1)\tau)d(\phi^{-1}(t))\). Comparing this to (C1) and by using \(\hat{\chi}(0) = \chi(0)\), one can see that \(\chi(t) = \hat{\chi} (\phi^{-1}(t))\) is a solution of (C1). Now, let \(\bar{e}(t) := e(\phi^{-1}(t)) = \beta K(\phi^{-1}(t))Q\chi(t)\) and notice that \(\kappa(\phi^{-1}(t)) = (aT_c/\eta)^{-1} \exp(at)\). Thus,

\[
K(\phi^{-1}(t)) = \text{diag}(1,(aT_c/\eta)^{-1} \exp(at), \ldots, (aT_c/\eta)^{-n} \exp(anat)).
\]
Since, \((C1)\) is asymptotically stable and by property \((A1)\), \(\chi(r)\) satisfies
\[
\|\chi(r)\| \leq \gamma \exp(-a(n+1)r)
\]
for some \(\gamma(\chi(0), a) > 0\). Moreover, we have \(\|\hat{e}(r)\| \leq \beta(aT_c/\eta)^{-n}\) \(\exp(anr)\|\chi(r)\|\) by the Rayleigh inequality\(^{47}\)(Theorem 4.2.2) where \(\hat{\sigma}(Q)\) and \((aT_c/\eta)^{-n}\) \(\exp(anr)\) are the largest singular values of \(Q\) and \(K(\varphi^{-1}(r))\) respectively, for sufficiently large \(r\). Then,
\[
\lim_{r \to \infty} \|\hat{e}(r)\| \leq \lim_{r \to \infty} \frac{\beta \gamma \hat{\sigma}(Q)}{(aT_c/\eta)^n} \exp(-ar) = 0.
\]
Thus, the redesigned differentiator’s error dynamics \((2)\) is asymptotically stable.

**Lemma 4.** Suppose that functions \(\{\phi_i\}_{i=0}^n\) are admissible and that the conditions of Theorem 1 are fulfilled. Let \(T(\chi(0))\) be the settling time function of system \((C1)\). Then, \((2)\) is fixed-time stable with settling time function given by
\[
T(e(0)) = \eta^{-1}T_c(1 - \exp(-aT(\chi(0)))),
\]
where \(\chi(0) = \beta^{-1}Q^{-1}K(0)^{-1}e(0)\).

**Proof.** It follows from the equivalence of curves, given in Lemma 3, under the time-scale transformation \(\tau = \varphi(t)\), that since \(\chi(r)\) reaches the origin as \(r \to T(\chi(0))\), where \(\chi(0) = \beta^{-1}Q^{-1}K(0)^{-1}e(0)\), then, \(e(t)\) reaches the origin as \(t \to \varphi^{-1}(T(\chi(0)))\). Thus, \(T(e(0))\) satisfies \((C4)\).

Using these results, we are now ready to show Theorem 1.

**Proof of Theorem 1.** Due to Lemma 4, the settling time function of \((C4)\) satisfies \(T(e(0)) \leq T_c\). Therefore, \((2)\) is fixed time stable with \(T_c\) as an UBST.

**C.1 Proof of the propositions**

Before showing Proposition 1, we first provide an auxiliary lemma.

**Lemma 5.** Consider admissible correction functions \(\{\phi_i\}_{i=0}^n\). Then, for all \(\delta_0 > 0\) there exists a nonzero \(w \in [-\delta_0, \delta_0]\) such that no \(\gamma\) exists satisfying \(\Phi(w) = wUQ^{-1}b_0 - \gamma Q^{-1}b_0\), where \(b_0\) is defined in the notation section.

**Proof.** Assume the opposite, that is, that there exists \(\delta_0\) such that \(\gamma(w)\) as in the lemma exists for all nonzero \(w \in [-\delta_0, \delta_0]\). We will show that this implies existence of trajectories of \((C1)\) with \(\chi(0) = 0\) that satisfy \(\frac{d}{dr}b_n^TQ\chi(r) = -anb_n^TQ\chi(r)\), contradicting the convergence bound in the admissibility requirement \((A1)\). To see this, note that for trajectories satisfying \(|\chi(0)| \leq \delta_0\) one has with \(A\) as in the proof of Lemma 3:
\[
\frac{d}{dr}b_n^TQ\chi = b_n^TQ(-\Phi(\chi(0)) + U\chi)
\]
\[
= b_n^TQ(\gamma Q^{-1}b_0 - \chi_0UQ^{-1}b_0 + U\chi)
\]
\[
= b_n^TQUQ^{-1}(Q\chi - \chi_0b_0)
\]
\[
= b_n^T(U - aD + QAQ^{-1})(Q\chi - \chi_0b_0)
\]
\[
= -anb_n^TDQ\chi,
\]
because \(b_n^TU = 0\) and \(A(\chi - \chi_0Q^{-1}b_0) = 0\). Finally, use \(b_n^TD = nb_n^T\).

**Proof of Proposition 1.** In the following, we consider \(n > 0\) and start by showing that a bounded \(\kappa(t), \forall t \geq 0\) implies uniform Lyapunov stability of \((2)\). First, note that \((2)\) with \([h_i]_{i=0}^n\) as in Theorem 1 is time invariant on the interval \((T_c, \infty)\). Hence, it is sufficient to show uniform Lyapunov stability on the interval \([0, T_c]\). Thus, let \(0 \leq s \leq t < T_c\) and recall the
relation between \( e(t) \) and \( \chi(\varphi(t)) \) in (C2). Note that by the Rayleigh inequality\(^4\)(Theorem 4.2.2) we have that

\[
\beta \sigma(Q)e(K(t))\|\chi(\varphi(t))\| \leq \|e(t)\| \leq \beta \sigma(Q)e(K(t))\|\chi(\varphi(t))\|. \tag{C5}
\]

where \( \sigma(\bullet), \sigma(\bullet) \) denote minimum and maximum singular values respectively. In addition, note that \( 0 < \sigma(K(t)) = \min\{1, \kappa(t), \ldots, \kappa(t)^n\} \) and \( 0 < \sigma(K(t)) = \max\{1, \kappa(t), \ldots, \kappa(t)^n\} < +\infty \) are nondecreasing functions since \( \kappa(t) \) is increasing for \( t \in [0, T_c] \). Choose any \( \epsilon > 0 \) and let \( \epsilon_x = \epsilon \left( \sigma(Q) \sigma(K(T_c)) \right)^{-1} \). Note that \( \epsilon_x > 0 \) since \( \sigma(K(T_c)) < +\infty \). For such \( \epsilon_x > 0 \), there exists \( \delta_x > 0 \) such that \( \|\chi(\varphi(s))\| \leq \delta_x \) implies \( \|\chi(\varphi(s))\| \leq \epsilon_x \) for \( \|\varphi(s)\| \leq \epsilon_x \). Using the second inequality in (C5) it is obtained that \( \|e(s)\| \leq \delta \leq \delta_x \beta \sigma(Q)\sigma(K(s)) \), \( \forall s \in [0, T_c] \). Hence, using the first inequality in (C5) it is obtained that \( \|e(s)\| \leq \delta \leq \delta_x \beta \sigma(Q)\sigma(K(s)) \). This in turn implies \( \|\chi(\varphi(t))\| \leq \epsilon_x \). Using the second inequality in (C5) and the fact that \( \sigma(K(t)) \leq \sigma(K(T_c)) \) for all \( t \in [0, T_c] \), we obtain \( \|e(t)\| \leq \beta \sigma(Q)e(K(T_c))\epsilon_x = \epsilon \), proving Lyapunov stability of (2) on the time interval \( [0, T_c] \).

Now, we show that if \( \kappa(t) \) is not bounded, then (2) is not uniformly Lyapunov stable. In particular, we will show that for any \( \delta, \epsilon > 0 \), there exist \( s, t \) with \( 0 \leq s < t \leq T_c \) and a trajectory \( \varphi(t) \) of (2) which satisfies both \( \|e(s)\| \leq \delta \) and \( \|e(t)\| > \epsilon \). Consider, for fixed \( \delta \), arbitrary \( t_0 \geq 0 \) and \( \pi(t) = 0 \), any trajectory \( \chi(t) \) of (C1) with \( \chi(t_0) = \omega Q^{-1}b_0 \) with nonzero constant \( w \) as in Lemma 5 for \( \delta_0 = \beta^{-1}\delta_1 \) and \( b_i \) as given in the notation section. Now, we show that there is no real number \( \gamma \) such that \( \frac{d}{dr}Q\chi(t) = \gamma b_0 \) for this trajectory. Assume the opposite, which implies that \( \frac{d}{dr}Q\chi(t) = \gamma b_0 \) for any real \( \gamma \). Therefore, there is at least one \( i \in \{1, \ldots, n\} \) such that \( \frac{d}{dr}b_i^TQ\chi(t) = \gamma b_0 \) is nonzero. The previous argument, in addition to the fact that (C1) is time-invariant and \( \chi(t_0) = \omega Q^{-1}b_0 \) are continuous at \( \chi(t_0) \), implies that there exist positive constants \( \bar{\tau}, \bar{\varepsilon} \), which only depend on \( \delta \), such that \( b_i^TQ\chi(t_0 + \bar{\tau}) > \bar{\varepsilon} \). Select now \( s \geq 0 \) such that \( \beta \kappa(s)^{-1}\bar{\varepsilon} > \epsilon \) which is possible since \( \kappa(\bullet) \) is unbounded and set \( t_0 = \varphi(s) \). From (C2), one then obtains

\[
e(s) = \beta K(s)Q\chi(\varphi(s)) = \beta K(s)Q\chi(t_0)
= \beta wK(s)b_0 = \beta w b_0
\]

and consequently \( \|e(s)\| = \beta \|w\| \leq \beta \delta_0 = \delta. \) Moreover, one has for \( t = \varphi^{-1}(\varphi(s) + \bar{\tau}) < T_c \)

\[
e(t) = b_i^Te(t) = \beta \kappa(t)^{-1}b_i^TQ\chi(\varphi(t))
= \beta \kappa(t)^{-1}b_i^TQ\chi(t_0 + \bar{\tau}),
\]

where \( \delta \in \{1, \ldots, n\} \), and hence \( \|e(t)\| \geq |e(t)| \geq \beta \kappa(t)^{-1}\bar{\varepsilon} \geq \beta \kappa(s)^{-1}\bar{\varepsilon} > \epsilon \), since \( \kappa(t) \) is nondecreasing.

**Proof of Proposition 2.** Due to Lemma 4, the settling time function of the error system (2) is given by (C4).

Let \( T^* := \sup_{T \geq 0} T(\chi(0)) \leq T_f \). Hence, the least UBST of (2) is \( T_a = \sup_{e(0) \in \mathbb{R}^{n+1}: T(\chi(0)) < T_f} T(e(0)) = \eta^{-1}T_c(1 - \exp(-aT^*)) \) and thus, the slack \( s_a := T_c - T_a = T_c(1 - \eta^{-1}(1 - \exp(-aT^*))) = \sigma(a)T_c \) where \( \sigma(a) = 1 - (1 - \exp(-aT^*))(1 - \exp(-aT^*))^{-1} \). Moreover, \( \sigma(a) \geq 0 \) for all \( a \) and \( \lim_{a \to \infty} \sigma(a) = 0. \) Hence, for every \( \epsilon \) there exists an \( a \) such that \( \sigma(a) = T_c^{-1}\epsilon \) and consequently \( s_a \leq \epsilon. \)

**Proof of Proposition 3.** Following the settling time function of the error system (2) is given by (C4). For item (i), since (C1) is finite-time stable thus, we set \( T_f = +\infty \) and \( \eta = 1. \) Hence, \( T(e(0)) < T_c \) for any finite \( \|\chi(0)\|, \) leading to \( \kappa(T(e(0))) < +\infty. \) For item (ii), since \( T(\chi(0)) \leq T \) for some \( T < \infty \) then, \( T(e(0)) \leq T_{\text{max}} := \eta^{-1}T_c(1 - \exp(-aT)) \) regardless of \( \epsilon(0) \). Henceforth, \( \sup_{e(0) \in \mathbb{R}^{n+1}} T(e(0)) < +\infty. \) Finally, if we have \( T(\chi(0)) \leq T_f < +\infty \) then, \( T(e(0)) \leq \eta^{-1}T_c(1 - \exp(-aT_f)) = T_c \) independently of \( \chi(0. \) Hence, it can be obtained that \( \kappa_{\text{max}} = \kappa(T_c) \) and \( \kappa(t) \leq \kappa_{\text{max}} < +\infty \) for \( t \in [0, T_c). \)