Topological Higher Gauge Theory — from BF to BFCG theory

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We study generalizations of 3- and 4-dimensional BF-theory in the context of higher gauge theory. First, we construct topological higher gauge theories as discrete state sum models and explain how they are related to the state sums of Yetter, Mackaay, and Porter. Under certain conditions, we can present their corresponding continuum counterparts in terms of classical Lagrangians. We then explain that two of these models are already familiar from the literature: the ΣΦEA-model of 3-dimensional gravity coupled to topological matter, and also a 4-dimensional model of BF-theory coupled to topological matter.

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I. INTRODUCTION

Given a d-dimensional space-time manifold M, a compact Lie group G with Lie algebra g := Lie G, and a principal G-bundle P → M, the gauge theory associated with the action

\[ S_{BF}(A,B) = \int_M \text{tr}_g(B \wedge F_A) \tag{1} \]

is known as BF-theory [1]. Here F_A is the g-valued curvature 2-form associated with a connection A, B denotes a g-valued (d − 2)-form and the notation \text{tr}_g(\ldots) stands for the Cartan–Killing form on g.

Although the classical field equations \( F_A = 0 \) and \( d_A(B) = 0 \) are not very interesting on their own, stating that the connection A be flat and the (d − 2)-form B be covariantly constant, BF-theory serves as an important toy model in various contexts.

First of all, the action of BF-theory is defined on any smooth manifold equipped with a principal G-bundle, and as such it does not require the existence of any Riemannian background metric on M. This characteristic is shared with certain first order formulations of general relativity, e.g. with the first order formulation of 3-dimensional pure Lorentzian [Riemannian] general relativity which can be shown to be a particular case of BF-theory for \( d = 3 \) and with \( G = SO(1,2) \) or \( Spin(1,2) \) [or with \( SO(3) \) or \( Spin(3) \), respectively].

Second, BF-theory has an enhanced local symmetry, i.e. besides the local gauge transformations,

\[ A \mapsto A + \delta A, \quad \delta A = d_A(\alpha), \tag{2} \]
\[ B \mapsto B + \delta B, \quad \delta B = -[\alpha, B], \tag{3} \]

where \( \alpha \) is locally a g-valued function on M, BF-theory is also invariant under the infinitesimal ‘translations’ of the B field,

\[ B \mapsto B + \delta B, \quad \delta B = d_A(\beta), \tag{4} \]

for any g-valued (d − 3)-form \( \beta \).

Finally, since BF-theory is independent of any background metric, it is particularly suitable for the construction of state sum models. For example, in the case of pure 3-dimensional Euclidean general relativity, i.e. for 3-dimensional BF-theory with \( G = SU(2) \), the corresponding state sum model is the Ponzano–Regge model [2]. More generally,
for an arbitrary compact Lie group or finite group \( G \), one obtains the state sum model by specializing the Turaev–Viro state sum \([3,4]\) to the category of finite-dimensional complex representations of \( G \) and by not worrying about convergence of the partition function\(^1\).

In this article, we study generalizations of \( d \)-dimensional \( BF \)-theory, \( d \in \{3,4\} \), to the context of higher gauge theory \([6,7,8,9]\).

Roughly speaking, in addition to the connection 1-form of conventional gauge theory which equips curves with holonomies in the gauge group \( G \), higher gauge theory introduces a connection 2-form which can be used to equip surfaces with a new kind of surface holonomy, given by elements of another group \( H \). More precisely, the algebraic structure that replaces the gauge group in higher gauge theory is a crossed module \((G,H,\triangleright,t)\), as described in Section II A below.

The purpose of the present article is to connect the following different developments in the literature: (i) higher gauge theory, i.e. the generalization of gauge theory from connection 1-forms to both 1-forms and 2-forms, in the topological case in dimension 3 and 4; (ii) state sum invariants of combinatorial 3- and 4-manifolds, familiar from the literature on combinatorial topology, homotopy theory, and higher category theory; (iii) the \( \Sigma \Phi EA \)-model of 3-dimensional gravity coupled to matter and a related model of 4-dimensional \( BF \)-theory coupled to matter, both of which are familiar from the literature on quantum gravity.

Our approach to the generalization of \( BF \)-theory to the framework of higher gauge theory is therefore twofold. First, we present a combinatorial construction of such a topological higher gauge theory as a state sum model, and we show that the model is well-defined for any finite crossed module, i.e. if the groups \( G \) and \( H \) are finite. In particular, it makes sense for arbitrary finite groups, and the group \( H \) by which the surfaces are labeled, is not required to be abelian. We then explain how our state sum model is related to the state sums of Yetter \([10]\), Porter \([11]\), and Mackaay \([12,13]\). In an appendix, we give a self-contained proof in terms of Pachner moves of the results of Yetter \([10]\) and Porter \([11,14]\) that these state sums are independent of triangulation. In fact, both models are invariants of the homotopy type of \( M \) \([15]\). It is an open question which topological invariant generalizes \([3]\) if one studies Lie groups rather than finite groups.

In addition to the combinatorial state sum construction, we present a continuum counterpart of our models in terms of differential forms and classical Lagrangians for the case in which \( G \) and \( H \) are Lie groups. At present, two restrictions apply: first, we require the ‘fake curvature’ (Section IV A below) to vanish in order to make sure that there are well defined curve and surface holonomies. Second, we require the group \( H \) of the crossed module \((G,H,\triangleright,t)\) to be abelian. This is the only case in which the extended local gauge symmetry is presently fully understood\(^2\) \([8]\). There is an obvious candidate for the corresponding continuum model. We show that one can recover the state sum by the standard heuristic discretization procedure, and we recall that special cases of this model have already appeared in the literature, for example, the \( \Sigma \Phi EA \)-model \([16]\).

We emphasize that our state sum models that are available for an arbitrary finite crossed module \((G,H,\triangleright,t)\) yield well-defined continuum theories, just by considering the continuum limit under arbitrary refinement of the triangulation. It is, however, not known whether all such models can be alternatively defined in terms of a classical Lagrangian.

The present article is structured as follows. In Section II we review the relevant algebraic tools involved in the description of higher gauge theory: 2-groups, crossed modules, Lie 2-algebras and differential crossed modules. In Section III we define the discrete state sum models of topological higher gauge theory in dimensions \( d = 3, 4 \). A self-contained proof that these models are well defined, i.e. independent of the chosen triangulation, is contained in Appendix A. We then explain the relationship to Mackaay’s state sum in Appendix B. In Section IV we present the continuum counterparts of our discrete models for the case of Lie groups and comment on their relationship with models known from the quantum gravity literature. In Section V we finally show how the continuum and discrete models can be related to each other by a generalization of the usual heuristic discretization procedure.

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\(^1\) For the topological interpretation in this case, see, for example \([3]\).

\(^2\) In more general cases, it is not understood whether one can obtain a Lagrangian that is invariant under a continuum analogue of the extended local symmetry of \([3]\) and whose fields are locally functions or differential forms on \( M \).
II. PRELIMINARIES

A. 2-Groups

The local symmetry of higher gauge theory is described by an algebraic structure known as a categorical group or as a 2-group. We first give the relevant definitions and then briefly sketch in which way this structure is used to equip both curves and surfaces with holonomies. For more details on 2-groups and for a comprehensive list of references, we refer the reader to [17].

Definition II.1. A strict 2-group $(G_0, G_1, s, t, \iota, \circ)$ consists of groups $G_0$ (group of objects), $G_1$ (group of morphisms), and homomorphisms of groups $s: G_1 \to G_0$ (source), $t: G_1 \to G_1$ (target), $\iota: G_0 \to G_1$ (identity) and $\circ: G_1 \times \_ G_1 \to G_1$ (vertical composition) such that the following conditions are satisfied,

1. $s(\iota(g)) = g$ and $t(\iota(g)) = g$ for all $g \in G_0$,
2. $s(f \circ f') = s(f')$ and $t(f \circ f') = t(f)$ for all $f, f' \in G_1$ for which $s(f) = t(f')$,
3. $\iota(t(f)) \circ f = f$ and $f \circ \iota(s(f)) = f$ for all $f \in G_1$,
4. $(f \circ f') \circ f'' = f \circ (f' \circ f'')$ for all $f, f', f'' \in G_1$ for which $s(f) = t(f')$ and $s(f') = t(f'')$.

Here $G_1 \times G_1 := \{(f, f') \in G_1 \times G_1 \mid s(f) = t(f')\}$ denotes the set of all pairs of vertically composable morphisms. The multiplication of the groups $G_0$ and $G_1$ is referred to as horizontal composition and is denoted either by $\cdot$ or just by simple juxtaposition.

Definition II.2. A strict Lie 2-group is a strict 2-group in which $G_0$ and $G_1$ are Lie groups and the maps $t, s, \iota$ and $\circ$ are homomorphisms of Lie groups. A strict finite 2-group is a strict 2-group in which both $G_0$ and $G_1$ are finite groups.

While in ordinary gauge theory, curves are labeled by holonomies taking values in the gauge group, in higher gauge theory, both curves and surfaces have holonomies with values in the groups $G_0$ and $G_1$, respectively:

![Diagram](image)

The elements $g_1, g_2 \in G_0$ label the source and target curves of the surface, $f \in G_1$ labels the surface, and they are required to satisfy the conditions,

$$s(f) = g_1 \quad \text{and} \quad t(f) = g_2. \quad (6)$$

The algebraic structure of a strict 2-group guarantees that one can change the base points of closed curves and the decomposition of the boundary of a disc into source and target in a consistent manner and that one can compose surfaces and define surface-ordered products. In particular, there is a local gauge symmetry which makes sure that surface-ordered products are independent of the base point and of the source curve of the surface.

Examples of strict 2-groups can be obtained from Whitehead’s crossed modules of groups as follows.

Definition II.3. A crossed module $(G, H, \triangleright, t)$ consists of two groups $G$ and $H$ and two group homomorphisms $t: H \to G$ and $\alpha: G \to \text{Aut}(H)$, $g \mapsto \alpha(g) := (h \mapsto g \triangleright h)$, i.e. an action of $G$ on $H$ by automorphisms, such that for all $g \in G$ and $h, h' \in H$,

$$t(g \triangleright h) = g t(h)g^{-1}, \quad (7)$$
$$t(h) \triangleright h' = hh'h^{-1}. \quad (8)$$

Definition II.4. A Lie crossed module is a crossed module in which $G$ and $H$ are Lie groups and in which $t$ and $\alpha$ are homomorphisms of Lie groups. A finite crossed module is a crossed module in which both $G$ and $H$ are finite groups.

Proposition II.5. Given a [Lie, finite] crossed module $(G, H, \triangleright, t)$, there exists a strict [Lie, finite] 2-group $(G_0, G_1, s, t, \iota, \circ)$ as follows. The groups of objects and morphisms are $G_0 := G$ and $G_1 := H \rtimes G$ where the semi-direct product uses the multiplication $(h_1, g_1) \cdot (h_2, g_2) := (h_1 g_1 \triangleright h_2), g_1 g_2$. The source and target maps are given by $s: H \rtimes G \to G$, $(h, g) \mapsto g$ and $t: H \rtimes G \to G$, $(h, g) \mapsto t(h)g$, the identity by $\iota: G \to H \rtimes G$, $g \mapsto (e, g)$ and vertical composition by $(h, g) \circ (h', g') = (hh', g)$ whenever $g = t(h')g'$. 


In fact, there is a 2-category of crossed modules and a 2-category of strict 2-groups, and these are equivalent as 2-categories, see, for example [17,18]. Note that in any strict 2-group, the vertical composition is already determined by the remaining structure maps as \( f \circ f' = f \cdot \cdot (s(f))^{-1} \cdot f' \) for all \( f, f' \in G \) for which \( s(f) = t(f') \), and every element \( f \in G \) has got a vertical inverse \( f^\times := s(f) \cdot f^{-1} \cdot t(f) \) such that \( f \circ f^\times = s(t(f)) \) and \( f^\times \circ f = s(f) \).

The map \( G_1 \to G_1, f \mapsto f^\times \) is a homomorphism of groups.

Using the data of the crossed module, the vertical inverse is \( (h, g)^\times = (h^{-1}, t(h)g), (h, g) \in H \times G \), and the labeling of the surface of (5) reads,

\[
\begin{array}{c}
G_1 \\
g_1 \\
\downarrow \\
g_2 \\
H
\end{array}
\]

where \( g_1, g_2 \in G \) and \( h \in H \) are such that \( t(h)g_1 = g_2 \).

### B. Lie 2-algebras

The connection of a conventional gauge theory is often described by using its connection 1-form, i.e. by using a locally defined Lie algebra valued 1-form, subject to a certain transformation law under change of the local trivialization. By analogy, higher gauge theory admits a differential formulation too, with the role of the Lie algebra of the gauge group being played by a Lie 2-algebra. Lie 2-algebras can be constructed from differential crossed modules, and in fact, the 2-category of Lie 2-algebras is equivalent as a 2-category to the 2-category of differential crossed modules [19].

Here we just review the definition and refer to [19] for more details and references.

**Definition II.6.** A differential crossed module \((\mathfrak{g}, \mathfrak{h}, \triangleright, \tau)\) consists of Lie algebras \(\mathfrak{g}\) and \(\mathfrak{h}\) and homomorphisms of Lie algebras \(\tau: \mathfrak{h} \to \mathfrak{g}\) and \(d\alpha: \mathfrak{g} \to \text{Der}(\mathfrak{h}), X \to d\alpha(X) := (Y \mapsto X \triangleright Y)\) such that

\[
\begin{align}
\tau(X \triangleright Y) &= [X, \tau(Y)], \\
\tau(Y) \triangleright Y' &= [Y, Y'],
\end{align}
\]

for all \(X \in \mathfrak{g}\) and \(Y, Y' \in \mathfrak{h}\), and with \(\text{Der}(\mathfrak{h})\) denoting the Lie algebra of derivations of \(\mathfrak{h}\).

Since \(d\alpha(X)\) is a derivation of \(\mathfrak{h}\) for all \(X \in \mathfrak{g}\), it is linear and satisfies the relations

\[
X \triangleright [Y_1, Y_2] = (d\alpha(X))(\cdot) [Y_1, Y_2] = [(d\alpha(X))(\cdot)Y_1, Y_2] + [Y_1, (d\alpha(X))(\cdot)Y_2] = [X \triangleright Y_1, Y_2] + [Y_1, X \triangleright Y_2]
\]

for all \(X \in \mathfrak{g}\) and \(Y_1, Y_2 \in \mathfrak{h}\). The map \(d\alpha\) is a homomorphism of Lie algebras, i.e. it is a linear map that satisfies the relations

\[
d\alpha([X_1, X_2]) = d\alpha(X_1) \circ d\alpha(X_2) - d\alpha(X_2) \circ d\alpha(X_1),
\]

for all \(X_1, X_2 \in \mathfrak{g}\), i.e.

\[
[X_1, X_2] \triangleright Y = X_1 \triangleright (X_2 \triangleright Y) - X_2 \triangleright (X_1 \triangleright Y),
\]

for all \(Y \in \mathfrak{h}\). Thus \(\triangleright\) is an action of \(\mathfrak{g}\) on \(\mathfrak{h}\) by derivations.

**Proposition II.7.** Let \((G, H, \triangleright, t)\) be a Lie crossed module. Then there is a differential crossed module \((\mathfrak{g}, \mathfrak{h}, \triangleright, \tau)\) that can be constructed as follows. The Lie algebras are \(\mathfrak{g} := \text{Lie}G\) and \(\mathfrak{h} := \text{Lie}H\), and the homomorphism of Lie algebras \(\tau := Dt\) is the derivative of the homomorphism of Lie groups \(t: H \to G\). If we write \(\alpha: G \to \text{Aut}(H), g \mapsto \alpha(g) := (h \mapsto g \triangleright h)\), its derivative \(d\alpha: \mathfrak{g} \to \text{Der}(\mathfrak{h})\) defines the action \(\triangleright\) in the differential crossed module by \((d\alpha(X))(Y) := X \triangleright Y\) for all \(X \in \mathfrak{g}\) and \(Y \in \mathfrak{h}\).

### III. COMBINATORIAL CONSTRUCTION OF TOPOLOGICAL HIGHER GAUGE THEORY

For conventional gauge theory, one can choose the action in such a way that the theory depends only on the underlying smooth space-time manifold, but not on any background metric. A very simple example is given by \(BF\)-theory [1] whose classical field equations require the gauge connection to be flat. In terms of the holonomy variables, this condition requires the holonomy of any null-homotopic closed curve to be the identity of the gauge group. We
generalize this idea to the framework of higher gauge theory by imposing the higher flatness condition requiring that the surface holonomy around the boundary 2-sphere of any 3-ball be trivial.

In this section, we present a combinatorial description of such a model for any triangulation of any smooth manifold of dimension \( d \in \{3, 4\} \), using the integral formulation of higher gauge theory \([2]\). For \( d = 3 \), this is precisely the Yetter model \([10]\) whereas for \( d = 4 \) it coincides with the Porter’s TQFT \([11]\) for \( d = 4 \) and \( n = 2 \). It is known that the partition function does not depend on the chosen triangulation. In particular, it is invariant under arbitrary refinement and therefore defines a continuum theory on the smooth manifold. The renormalization of this model is therefore fully under control. In fact, we are sitting right on the renormalization fixed point, and the model is scale invariant. This is no surprise since our background is just a smooth manifold with no background metric.

The combinatorially defined model is available for any strict finite 2-group and even for strict compact Lie 2-groups if one is not worried by divergencies similar in nature to those of the Ponzano–Regge model, i.e. to those of the \( SU(2) \) \( BF \)-theory in \( d = 3 \) \([20, 21]\).

Below, we use the following notation. If \( G \) is a finite group with unit element \( e \in G \), we denote by \( \int_G dg := 1/|G| \sum_{g \in G} \) the normalized sum over all group elements and by \( \delta_G \) the corresponding \( \delta \)-distribution on \( G \), i.e. for \( g \in G \) we have \( \delta_G(g) = |G| \) if \( g = e \) and \( \delta_G(g) = 0 \) if \( g \neq e \). If \( G \) is a compact Lie group, \( \int_G dg \) and \( \delta_G \) denote the Haar measure and the usual \( \delta \)-distribution on \( G \), respectively.

We define our model for any closed and oriented combinatorial manifold \( \Lambda \) of dimension \( d \in \{3, 4\} \). These arise precisely as the triangulations of closed and oriented smooth manifolds of dimension \( d \) \([20, 21]\). We denote the set of all \( k \)-simplices, \( 0 \leq k \leq d \), by \( \Lambda_k \). We equip the set of vertices \( \Lambda_0 \) which can be assumed to be finite, with an arbitrary total order and denote the \( k \)-simplices by \((k + 1)\)-tuples of vertices \((i_0 \ldots i_k)\) where \( i_0 ,\ldots ,i_k \in \Lambda_0 \) such that \( i_0 < \cdots < i_k \).

**Definition III.1.** Let \( \Lambda \) be a compact and oriented combinatorial \( d \)-manifold, \( d \in \{3, 4\} \), and \((G, H, \triangleright, t)\) be a finite crossed module. The partition function of topological higher gauge theory is defined by

\[
Z = |G|^{-|\Lambda_0| + |\Lambda_1| - |\Lambda_2|} |H|^{-|\Lambda_0| + |\Lambda_1| - |\Lambda_3|} \left( \prod_{(jk) \in \Lambda_1} \int_G dg_{jk} \right) \left( \prod_{(jk\ell) \in \Lambda_2} \int_H dh_{j\ell k} \right) \times \left( \prod_{(jk) \in \Lambda_1} \delta_G (t(h_{jk\ell} g_{jk} g_{k\ell}^{-1})) \right) \left( \prod_{(j\ell k m) \in \Lambda_3} \delta_H (h_{j\ell k m} h_{j k \ell m} (g_{jk} \triangleright h_{k \ell m}^{-1}) h_{j k m}^{-1}) \right).
\]

(15)

Here we integrate over \( g_{jk} \in G \) for every edge \((jk) \in \Lambda_1\) and over \( h_{j\ell k} \in H \) for every triangle \((j\ell k) \in \Lambda_2\). The \( \delta \)-distributions under the integral impose the condition that \( t(h_{jk\ell} g_{jk} g_{k\ell}^{-1}) = g_{j\ell} \) for each triangle \((j\ell k) \in \Lambda_2\), i.e. that each surface label \( h_{j\ell k} \) has got the appropriate source and target,

(16)

and the condition that the surface holonomy around every tetrahedron \((j\ell k m) \in \Lambda_3\) be trivial. Recall from \([8]\) that

\( ^{3} \) For the relevance of the latter reference, see \([22]\).
the argument of the $\delta_H$ in (15) is precisely the surface ordered product around the tetrahedron:

\begin{equation}
\text{(17)}
\end{equation}

This expression is independent of the choice of the base edge $(jm)$ because $\delta_H$ is a gauge invariant function \[7\]. Similarly, exploiting the local gauge symmetry of higher gauge theory \[7, 8\], it is not difficult to show that the partition function (15) does not depend on the ordering of the vertices.

Using Alexander moves \[23\], Yetter \[10\] has shown for the case $d = 3$ that the partition function does not depend on the chosen triangulation. It seems to have gone unnoticed that the same result for $d = 4$ is in fact already implied by \[11\] in combination with \[14\], again by using Alexander moves:

**Theorem III.2.** Let $\Lambda$ be a closed and oriented combinatorial $d$-manifold, $d \in \{3, 4\}$, and $(G, H, \triangleright, t)$ be a finite crossed module. The partition function (15) is invariant under Pachner moves and therefore well defined on equivalence classes of combinatorial manifolds.

Since the original references may not be very accessible to readers interested in higher gauge theory, we sketch in Appendix A how one can obtain a self contained proof of triangulation independence using Pachner moves \[24\]. This has the advantage that there are only a finite number of moves to verify – a number that is independent of the chosen triangulation – and that it can be done in a direct calculation without any additional machinery from homotopy theory. In Appendix B, we explain under which conditions the model (15) forms a special case of Mackaay’s state sum \[12, 13\] and in which cases it does not.

In the partition function (15), the labeling of edges by elements $g_{jk} \in G$ and of triangles with elements $h_{jk\ell} \in H$ are called colorings. Those colorings for which the $\delta_G(\ldots)$ and $\delta_H(\ldots)$ are non-zero, are called admissible colorings. The partition function (15) counts the number of admissible colorings and multiplies the result by $|G|^{-|\Lambda_0|} |H|^{||\Lambda_0|-|\Lambda_1||}$. This factor may indicate that one has already integrated out further variables associated with the lower-dimensional simplices, see, for example \[23\]. The partition function (15) is known to be an invariant of the homotopy type of $\Lambda$ \[15\].

We emphasize that because of \[20, 21\], the model (15) which we have here defined in the discrete language of combinatorial manifolds, is in fact a proper continuum theory that is well-defined on any smooth $d$-manifold. This argument can be put in a more physical language by saying that the invariance under the $1 \leftrightarrow (d + 1)$ Pachner move \[24\] allows us to pass to an arbitrary refinement of the triangulation and thereby to the continuum limit of the model.

**IV. LAGRANGIAN FORMULATION**

If the state sum model (Definition III.1) is studied for a Lie crossed module $(G, H, \triangleright, t)$ rather than a finite crossed module, the partition function (15) is in general no longer well defined. Work \[5, 26\] on the Ponzano–Regge model, i.e. the $H = \{e\}$, $d = 3$ special case, nevertheless indicates that there are physical observables that can still be defined. The analogy with the Ponzano–Regge model also suggests that in the Lie group case, there is an alternative, continuous, formulation of the model in terms of a classical Lagrangian and fields given locally by differential forms on $M$. For the Ponzano–Regge model, this turned out to be $SU(2)$ BF-theory in $d = 3$. In the following, we present a similar continuum counterpart of the state sum model (15).
A. Higher gauge theory in the differential formulation

In this section, we recall the differential formulation of higher gauge theory \([8]\) for the case in which the structure 2-group is given by a Lie crossed module \((G, H, \triangleright, t)\) with \(H\) abelian. If we consider the associated differential crossed module \((\mathfrak{g}, \mathfrak{h}, \triangleright, \tau)\) (see Proposition IV.7), the connections of the higher gauge theory formalism will be a \(\mathfrak{g}\)-valued connection 1-form \(A\) and an \(\mathfrak{h}\)-valued connection 2-form \(\Sigma\). Here, we consider only the local description of the connection of higher gauge theory. For the global aspects, we refer to \([9]\). Note that our notation is different from \([8]\).

The curvature of higher gauge theory is then given by two differential forms: the curvature 2-form ('fake curvature')

\[
F_A = R_A + \tau(\Sigma),
\]

where \(R_A = dA + \frac{1}{2}[A, A]\) is the conventional curvature of \(A\), and the curvature 3-form

\[
G_\Sigma = dA(\Sigma) := d\Sigma + A \triangleright \Sigma.
\]

If the connection 1- and 2-forms originate from an integral formulation in terms of holonomies, the fake curvature vanishes: \(F_A = 0\). In the model that we introduce in the following section, this condition is enforced on-shell. Together with the requirement that \(H\) be abelian, it ensures that the model has the following extended local gauge symmetry:

\[
A \mapsto A + \delta A, \quad \text{where} \quad \delta A = d\alpha + \tau(\lambda),
\]

\[
\Sigma \mapsto \Sigma + \delta \Sigma, \quad \text{where} \quad \delta \Sigma = -d\lambda - \alpha \triangleright \Sigma,
\]

where the gauge transformation is generated locally by a \(\mathfrak{g}\)-valued 0-form \(\alpha\) and an \(\mathfrak{h}\)-valued 1-form \(\lambda\). The fake curvature and the curvature 3-form transform as follows,

\[
F_A \mapsto F_A + \delta F_A, \quad \text{where} \quad \delta F_A = [F, \alpha],
\]

\[
G_\Sigma \mapsto G_\Sigma + \delta G_\Sigma, \quad \text{where} \quad \delta G_\Sigma = -\alpha \triangleright G_\Sigma - F_A \triangleright \lambda.
\]

For more details, the reader is referred to \([6, 8, 9]\).

We are in particular interested in the Lie crossed module \((G, H, \triangleright, t)\) associated with the adjoint 2-group of a Lie group \(G\). Here \(\triangleright\) is the adjoint action of \(G\) on its Lie algebra \(H := \mathfrak{g} = \text{Lie} G\), and \(t(h) = e\) for all \(h \in H\). Its group of morphisms \(\mathfrak{g} \times G\) is often called the inhomogeneous group associated with \(G\), see, for example \([27]\).

The corresponding differential crossed module (Proposition IV.7) is given by \((\mathfrak{g}, \mathfrak{h}, \triangleright, \tau)\) where \(\mathfrak{h}\) is the vector space underlying \(\mathfrak{g}\) equipped with the abelian Lie algebra structure, \(\mathfrak{g}\) acts on \(\mathfrak{h}\) by the adjoint action, and \(\tau(Y) = 0\) for all \(Y \in \mathfrak{h}\). The corresponding Lie 2-algebra has the semidirect sum \(\mathfrak{h} \oplus \mathfrak{g}\) as its Lie algebra of morphisms, the inhomogeneous algebra associated with \(\mathfrak{g}\).

With this choice of differential crossed module, the vanishing of the fake curvature \(F_A\) implies the vanishing of the conventional curvature \(R_A = 0\).

B. The BFCG theory

With the above considerations, we can now propose a classical action corresponding to the partition function \([18]\) as follows. Let \((G, H, \triangleright, t)\) be the Lie crossed module associated with the adjoint 2-group of the Lie group \(G\) and \((\mathfrak{g}, \mathfrak{h}, \triangleright, \tau)\) be the associated differential crossed module as explained above.

Besides the curvature 1- and 2-forms of higher gauge theory, we consider two additional fields \(B\) and \(C\) which are a \(\mathfrak{g}\)-valued \((d - 2)\)-form and an \(\mathfrak{h}\)-valued \((d - 3)\)-form, respectively, and which are assumed to transform under the 2-gauge transformations in \([20]\) as:

\[
B \mapsto B + \delta B \quad \text{with} \quad \delta B = [B, \alpha] - [C, \lambda],
\]

\[
C \mapsto C + \delta C \quad \text{with} \quad \delta C = -\alpha \triangleright C.
\]

It should be noted that with the above choice of differential crossed module, the local symmetries of both fields \(B\) and \(C\) in \([21]\) are well defined and make sense.

The reason for introducing these fields is the same as in BF-theory: their associated field equations are the conditions that the curvature 2-form \(F_A\) and the curvature 3-form \(G_\Sigma\) vanish. The action of our BFCG theory therefore reads,

\[
S = \int_M \text{tr}_\mathfrak{g}(B \wedge F_A) + \text{tr}_\mathfrak{h}(C \wedge G_\Sigma).
\]
Note that the transformations (24) and (25) are chosen such as to make the action gauge invariant in view of the transformations (22) and (23).

Similar to the traditional $BF$ action in (11), the $BFCG$ action (20) also exhibits an extended local symmetry, in the sense that it is also invariant under the additional infinitesimal gauge transformations:

\[
\begin{align*}
B &\mapsto B + \delta' B \quad \text{with} \quad \delta' B = d_A(\beta) + [\Sigma, \gamma], \\
C &\mapsto C + \delta' C \quad \text{with} \quad \delta' C = d_A(\gamma),
\end{align*}
\]  

where $\beta$ and $\gamma$ are locally a $g$-valued $(d-3)$-form and an $h$-valued $(d-4)$-form, respectively. It should be emphasized that not all of the above gauge transformations of the fields of the theory are irreducible. Indeed, on shell, the gauge transformations for the connection 2-form $\Sigma$ in (21), and in the case $d = 4$ also the gauge transformation for the field $B$ in (24), are themselves invariant under an infinitesimal translation of the gauge parameters

\[
\begin{align*}
\lambda &\mapsto \lambda + \delta' \lambda \quad \text{with} \quad \delta' \lambda = d_A(\rho), \\
\beta &\mapsto \beta + \delta' \beta \quad \text{with} \quad \delta' \beta = d_A(\eta),
\end{align*}
\]

where $\eta$ and $\rho$ are $h$-valued and $g$-valued 0-forms, respectively. While mathematically obvious, the transformations in (28) also have a rather straightforward physical interpretation. A not so complicated counting argument \cite{16} shows that if the above gauge symmetry reducibility is ignored, in both the $d = 3$ and $d = 4$ cases $BFCG$ theory would exhibit a negative number of (local) physical degrees of freedom. The role of the transformations in (28) is to bring the number of (local) physical degrees of freedom up to zero and hence establish the topological character of the theory.

Upon first order variation, the $BFCG$ action yields the equations of motion:

\[
\begin{align*}
d_A(B) + [\Sigma, C] &= 0 \\
d_A(C) &= 0 \\
F_A &= 0 \\
G_\Sigma &= 0
\end{align*}
\]

so that in particular the higher flatness condition $G_\Sigma = 0$ holds. Note that the vanishing of the fake curvature $F_A = 0$ is automatically satisfied on-shell.

In dimensions $d = 3$ and $d = 4$, the $BFCG$-theory can readily be related with topological models that have already been studied in the literature. For $d = 3$, the $BFCG$ model can be related to Euclidean and Lorentzian gravity. Indeed if one chooses $g$ to be the Lie algebra $so(3)$ or $so(2,1)$ and $h$ to be the abelian Lie algebra of 3-dimensional translations $\mathbb{T}^3$, the 1-form field $B$ can be interpreted as the local triad field of the spacetime manifold $M$, and the first term in (26) becomes the action for pure gravity in the Palatini formalism. Under these circumstances, and upon tracing, the $BFCG$ action becomes functionally identical to the action of a topological matter model that has already been studied in the literature \cite{16}, called the $\Sigma \Phi$EA model. Within this latter context, the second term of the $BFCG$ action containing the 0-form field $C$ and the curvature $G_\Sigma$ of the connection 2-form $\Sigma$ can be interpreted as a coupling of topological matter fields to pure 3-dimensional gravity. Consequently, the $BFCG$ model can be shown to admit topological solutions like point-particle solutions, the BTZ black-hole solution and cosmological solutions of the Robertson-Friedman-Walker type \cite{28}.

For $d = 4$, by choosing $g$ to be the Lie algebra $so(4)$ or $so(3,1)$ and $h$ the 6-dimensional abelian Lie algebra, it can be shown — in a manner similar to the 3-dimensional case \cite{28} — that the $BFCG$ action yields (up to surface terms) yet another topological matter model that has been studied previously in the literature \cite{28}. This topological matter model is also non-trivial \cite{30} and is related to topological gravity in 4-dimensional spacetimes in a similar way as ordinary $BF$ theory.

Also note that dropping the second term of (26) does not specialize $BFCG$-theory to the first example in Section 3.9 of \cite{8}.

V. DISCRETIZATION

In this section, we show how the action (25) is related to the state sum model (15) by the usual heuristic discretization procedure. We therefore consider the partition function

\[
Z = \int [DC][DB][DA][D\Sigma] e^{i \int_M \{Tr_8\{B \wedge F_A\} + Tr_8\{C \wedge G_\Sigma\}\}}.
\]

(30)
The formal integration over $C$ and $B$ leads to

$$Z = \int [DA][D\Sigma] \delta(F_A) \delta(G_\Sigma). \quad (31)$$

Similarly to the treatment of $BF$-theory, this partition function is then regularized on a triangulation $\Lambda$ of $M$. This amounts to a translation from the differential picture [8] to the integral picture [7] of higher gauge theory. Similarly to conventional gauge theory, the connection $A$ is discretized by colouring the edges $e = (jk) \in \Lambda_1$ with group elements $g_e \in G$. The connection 2-form $\Sigma$ is in turn represented by group elements $h_f \in H$ decorating the triangles $f = (jk\ell) \in \Lambda_2$. We then recover the discretization described in [10] by reversing the procedure of [8]. The vanishing fake curvature condition is discretized on each triangle $f$ by replacing $\delta(F_A)$ by

$$\delta_G \left( g_{jk} g_{jk} (t(h_{jk}) g_{jk}^{-1}) \right). \quad (32)$$

The condition $\delta(G_\Sigma)$ on the curvature 3-form for every tetrahedron $T = (jk\ell m) \in \Lambda_3$ is turned into

$$\delta_H (h_{j\ell m} h_{j\ell m} (g_{jk} \triangleright h_{k\ell m}^{-1}) h_{jkm}^{-1}). \quad (33)$$

The path integral measures of (31) are discretized by replacing

$$\int [DA] \mapsto \prod_{(jk) \in \Lambda_1} \int_G dg_{jk}, \quad (34)$$

$$\int [D\Sigma] \mapsto \prod_{(jk\ell) \in \Lambda_2} \int_H dh_{j\ell k}, \quad (35)$$

where $dg_{jk}$ and $dh_{j\ell k}$ denote integration with respect to the Haar measures of $G$ and $H$. By inserting (33), (34) and (35) into (31), we obtain an expression proportional to (15).

It then turns out that this expression can be made independent of the triangulation if one multiplies it by the appropriate ‘anomaly’ factors that render the expression equal to (15).

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APPENDIX A: PACHNER MOVE INVARIANCE

In the appendix, we give a self contained proof in terms of Pachner moves that the partition function (15) is independent of the chosen triangulation and therefore well defined on equivalence classes of combinatorial manifolds. By Whitehead’s theorem [20], it is thus even well defined on diffeomorphism classes of smooth manifolds.

1. Three-dimensional case

We first sketch the proof of Pachner move invariance for the case $d = 3$.

a. The 1 $\leftrightarrow$ 4 move

We use the following notation. For every triangle $(jk\ell) \in \Lambda_2$, we write

$$g_{jk\ell} := t(h_{jk\ell}) g_{jk} g_{\ell k} g_{j\ell}^{-1} , \quad (A1)$$

and for every tetrahedron $(jk\ell m) \in \Lambda_3$,

$$h_{jk\ell m} := h_{j\ell m} h_{jk\ell} (g_{jk} \triangleright h_{k\ell m}^{-1}) h_{jkm}^{-1} . \quad (A2)$$
Since the partition function \( \text{[15]} \) is independent of the total order of vertices, we need to verify the move only in one case. We denote the vertices of the left hand side (one tetrahedron) by 1, 2, 3, 4 and the additional vertex on the right hand side (four tetrahedra) by 5. This determines a total order, restricted to our subset of vertices. The partition function on the two sides of the 1 \( \leftrightarrow \) 4 move then differs by the following factors. On the l.h.s., we have the integrand

\[
\delta_H(h_{1234}),
\]

whereas on the r.h.s., we have integrals

\[
\int_{G^*} dg_{15} dg_{25} dg_{35} dg_{45} \int_{H^6} dh_{125} dh_{135} dh_{145} dh_{235} dh_{245} dh_{345}
\]

and the integrand

\[
\left( \prod_{(jkt) \in M_2} \delta_G(g_{jkt}) \right) \left( \prod_{(jk\ell m) \in M_3} \delta_H(h_{jk\ell m}) \right),
\]

where the products are over the following sets of simplices: \( M_2 := \{(125), (135), (235), (245), (345)\} \) and \( M_3 := \{(125), (1245), (1345), (2345)\} \). The numbers of the \( k \)-simplices on both sides of the 1 \( \leftrightarrow \) 4 move are as follows (not taking into account the remainder of the triangulation):

|   | \( |\Lambda_0| \) | \( |\Lambda_1| \) | \( |\Lambda_2| \) | \( |\Lambda_3| \) |
|---|---|---|---|---|
| l.h.s. | 4 | 6 | 4 | 1 |
| r.h.s. | 5 | 10 | 9 | 4 |

In order to verify the 1 \( \leftrightarrow \) 4 move, we consider the r.h.s and first integrate over \( g_{15} \), exploiting \( \delta_G(g_{125}) \), i.e. the integral over \( g_{15} \) and the integrand \( \delta_G(g_{125}) \) both disappear, and all other occurrences of \( g_{15} \) in the integrand are replaced by \( t(h_{125}) g_{125} g_{25} \). We then integrate over \( g_{25} \), exploiting \( \delta_G(g_{235}) \), and over \( g_{35} \), exploiting \( \delta_G(g_{345}) \). At this stage, the integral over \( g_{15} \) is trivial, i.e. over a constant integrand.

Finally, we integrate over \( h_{135} \), exploiting \( \delta_H(h_{1345}) \), i.e. substituting \( h_{135} = h_{125} h_{134}(g_{13} \triangleright h_{345}^{-1}) \) everywhere else in the integrand, and we integrate over \( h_{125} \), exploiting \( \delta_H(h_{1245}) \), and over \( h_{235} \), exploiting \( \delta_H(h_{2345}) \). One can now show that the remaining integrand of the r.h.s. equals

\[
(\delta_G(e))^3 \delta_H(h_{1234}) = |G|^3 \delta_H(h_{1234}),
\]

and so the remaining three integrals over \( h_{145}, h_{245}, \) and \( h_{345} \) are trivial. In order to show this, we make use the condition

\[
g_{j\ell} = t(h_{j\ell})(g_{jk} g_{k\ell})
\]

for \( (jk\ell) \in \{(123), (124), (234)\} \). This is possible because these triangles are present on both sides of the move, and so the corresponding \( \delta_G(g_{jkt}) \) that enforces the condition \( \text{[A7]} \), are part of the integrand. Finally, the prefactor \( |G|^{-|\Lambda_0| + |\Lambda_1| - |\Lambda_2| + |\Lambda_3|} |H|^{3(|\Lambda_0| - |\Lambda_1| + |\Lambda_2| - |\Lambda_3|)} \) is \( |G|^{-2} |H|^{1} \) on the l.h.s. and \( |G|^{-5} |H|^{1} \) on the r.h.s., compensating for the \( |G|^3 \) from the left over \( \delta_G \) of the integrand.

b. The 2 \( \leftrightarrow \) 3 move

The numbers of \( k \)-simplices on the two sides of the 2 \( \leftrightarrow \) 3 move are as follows:

|   | \( |\Lambda_0| \) | \( |\Lambda_1| \) | \( |\Lambda_2| \) | \( |\Lambda_3| \) |
|---|---|---|---|---|
| l.h.s. | 5 | 9 | 7 | 2 |
| r.h.s. | 5 | 10 | 9 | 3 |

We order the vertices in such a way that the l.h.s. has the tetrahedra (1234) and (2345), sharing the triangle (234), whereas the r.h.s has the tetrahedra (1235), (1245) and (1345), all sharing the edge (15) and each two of them sharing one of the triangles (125), (135) and (145).
On the l.h.s. of the $2 \leftrightarrow 3$ move, we therefore have the integral
\[ \int_{H} dh_{234} \] (A8)
and the integrand
\[ \delta_G(g_{234}) \delta_H(h_{1234}) \delta_H(h_{2345}), \] (A9)
whereas on the r.h.s we have the integrals
\[ \int_{G} dg_{15} \int_{H^5} dh_{125} dh_{135} dh_{145} \] (A10)
and the integrand
\[ \delta_G(g_{125}) \delta_G(g_{145}) \delta_G(g_{135}) \delta_H(h_{1235}) \delta_H(h_{1234}) \delta_H(h_{1345}). \] (A11)

All other integrals and all other factors of the integrand are the same on both sides of the move.

In order to simplify the l.h.s., we integrate over $h_{234}$, exploiting $\delta_H(h_{2345})$. In the remaining integrand, we therefore substitute $h_{234} = h_{235}^1 h_{235} (g_{23} \triangleright h_{1345})$. The integrand of the l.h.s. thus reduces to
\[ \delta_G(e) \delta_H(h_{1234} \triangleright (h_{235}^{-1} h_{235})) h_{123}^{-1} (g_{13} \triangleright h_{345}) h_{1345}^{-1}. \] (A12)

In order to simplify the r.h.s., we integrate over $g_{15}$, exploiting $\delta_G(g_{135})$, over $h_{125}$, exploiting $\delta_H(h_{1235})$, and over $h_{135}$, exploiting $\delta_H(h_{1345})$. The remaining integral over $h_{145}$ turns out to be trivial if one uses (A7) for all $(jk \ell) \in \{(123), (235), (134), (345)\}$. The integrand of the r.h.s reduces to
\[ (\delta_G(e))^2 \delta_H(h_{124} \triangleright (h_{245}^{-1} h_{235})) h_{123}^{-1} (g_{13} \triangleright h_{345}) h_{134}^{-1}. \] (A13)

Again, the different powers of $\delta_G(e) = |G|$ are compensated for by the prefactors. These are $|G|^{-3} |H|^4$ on the l.h.s. and $|G|^{-4} |H|^4$ on the r.h.s.

2. Four-dimensional case

We now sketch the proof of Pachner move invariance for the case $d = 4$.

a. The $1 \leftrightarrow 5$ move

The numbers of the $k$-simplices on both sides of the move are as follows:

|        | $|A_0|$ | $|A_1|$ | $|A_2|$ | $|A_3|$ | $|A_4|$ |
|--------|--------|--------|--------|--------|--------|
| l.h.s. | 5      | 10     | 10     | 5      | 1      |
| r.h.s. | 6      | 15     | 20     | 15     | 5      |

Again, since the partition function $\langle 15 \rangle$ does not depend on the total order of the vertices, we need to show this move only for one case. We order the vertices such that the l.h.s consists of the 4-simplex $(23456)$ whereas the r.h.s. contains the five 4-simplices $(13456)$, $(12456)$, $(12356)$, $(12346)$ and $(12345)$. On the r.h.s., we therefore have the triangles $(jk \ell) \in M_2 := \{(123), (124), (125), (126), (134), (135), (136), (145), (146), (156)\}$ and the edges $(jk) \in M_1 := \{(12), (13), (14), (15), (16)\}$.

In order to compare the l.h.s with the r.h.s. of the $1 \leftrightarrow 5$ move, we have to show that the integrals
\[ \int_{G^5} \prod_{(jk) \in M_1} dg_{jk} \int_{H^{10}} \prod_{(jk \ell) \in M_2} dh_{jk \ell} \] (A14)
and the integrand
\[ \left( \prod_{(jk \ell) \in M_2} \delta_G(g_{jk \ell}) \right) \left( \prod_{(jk \ell m) \in M_3} \delta_H(h_{jk \ell m}) \right) \] (A15)
on the r.h.s. reduce to 1. Here, \( M_3 := \{(1234), (1235), (1236), (1245), (1246), (1256), (1345), (1346), (1356), (1456)\} \).

In order to simplify the r.h.s., we integrate over \( h_{123} \), exploiting \( \delta_H(h_{123}) \), and over \( g_{12} \), exploiting \( \delta_G(g_{12}) \), and make use of (A7) for \( (jk\ell) = (234) \). We then integrate over \( h_{124} \), exploiting \( \delta_H(h_{124}) \), and over \( g_{13} \), exploiting \( \delta_G(g_{13}) \), and make use of (A7) for \( (jk\ell) \in \{(234), (235)\} \). We then integrate over \( h_{125} \), exploiting \( \delta_H(h_{125}) \), and over \( g_{14} \), exploiting \( \delta_G(g_{14}) \), and make use of (A7) for \( (jk\ell) = (236) \).

Finally, we integrate over \( g_{15} \), exploiting \( \delta_G(g_{146}) \), and over \( h_{134} \), exploiting \( \delta_H(h_{134}) \), over \( h_{135} \), exploiting \( \delta_H(h_{1356}) \), and over \( h_{145} \), exploiting \( \delta_H(h_{1456}) \). We make use of (A7) for \( (jk\ell) = \{(234), (346), (356), (456)\} \). We then use the condition that

\[
h_{jk\ell m} = h_{j\ell m} h_{jk\ell} (g_{jk\ell} \triangleright h_{k\ell m}^{-1}) \quad (A16)
\]

for all \( (jk\ell m) \in \{(2345), (2346), (2356), (3456)\} \). The remaining integrals over \( h_{126} \), \( h_{136} \), \( h_{146} \) and \( h_{156} \) are trivial as well as that over \( g_{16} \). The integrand reduces to

\[
(\delta_G(e))^6 (\delta_H(e))^3 = |G|^6 |H|^4. \quad (A17)
\]

The prefactor \(|G|^{-|\Lambda_0| + |\Lambda_1| + |\Lambda_2|} |H|^{|\Lambda_0| - |\Lambda_1| + |\Lambda_2| - |\Lambda_3|}\) equals \(|G|^{-5} |H|^0\) on the l.h.s. and \(|G|^{-11} |H|^{-4}\) on the r.h.s. and therefore compensates for these left-over factors.

\[ \]

b. The \( 2 \leftrightarrow 4 \) move

The numbers of the \( k \)-simplices on both sides of the move are as follows:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
| & |\Lambda_0| & |\Lambda_1| & |\Lambda_2| & |\Lambda_3| & |\Lambda_4| \\
\hline
l.h.s. & 6 & 14 & 16 & 9 & 2 \\
\hline
r.h.s. & 6 & 15 & 20 & 14 & 4 \\
\hline
\end{array}
\]

We order the vertices in such a way that on the l.h.s., we have the 4-simplices (23456) and (13456) whereas on the r.h.s., there are (12456), (12356), (12436) and (1245). On the l.h.s., there is one tetrahedron (3456) whereas on the r.h.s., there are six, namely (1234), (1235), (1236), (1245), (1246) and (1256). All other tetrahedra are part of the common boundary of both sides of the move. On the r.h.s., we also have the triangles (123), (124), (125) and (126) and the edge (12).

The integrals and factors of the integrand that differ on both sides of the \( 2 \leftrightarrow 4 \) move are as follows. On the l.h.s., there is the integrand

\[
\delta_H(h_{23456}) \quad (A18)
\]

whereas on the r.h.s., we have the integrals

\[
\int_G dg_{12} \int_{H^4} dh_{123} dh_{124} dh_{125} dh_{126} \quad (A19)
\]

and the integrand

\[
\left( \prod_{(jk\ell) \in M_2} \delta_G(g_{jk\ell}) \right) \left( \prod_{(jk\ell m) \in M_3} \delta_H(h_{jk\ell m}) \right), \quad (A20)
\]

where \( M_2 := \{(123), (124), (125), (126)\} \) and \( M_3 := \{(1234), (1235), (1236), (1245), (1246), (1256)\} \).

The l.h.s. simplifies to \( \delta_H(e) = |H| \) because of the following general result.

**Lemma A.1.** Given a 4-simplex (\( jk\ell m n \)) with a colouring that satisfies (A16) for four of the tetrahedra (\( k\ell m n \), (\( j\ell m n \)), (\( jk\ell m n \)) and (\( jk\ell n \)) and (A7) for all triangles in their boundary, then (A16) also holds on the fifth tetrahedron (\( jk\ell m \)).

**Proof.** Consider \( h_{jk\ell m} = h_{j\ell m} h_{jk\ell} (g_{jk\ell} \triangleright h_{k\ell m}^{-1}) h_{jk\ell m}^{-1} \) and use the condition (A7) for (\( jk\ell \), (\( jk\ell m \)), (\( jk\ell n \)), (\( jk\ell m \)), (\( jk\ell m \)), (\( jk\ell m \)). This implies \( h_{jk\ell m} = e \). \[ \square \]
In order to simplify the r.h.s., we integrate over $h_{124}$, exploiting $\delta_H(h_{1234})$, over $h_{123}$, exploiting $\delta_H(h_{1235})$, and over $h_{125}$, exploiting $\delta_H(h_{1256})$. Use the condition (A7) for all $(jkl) \in \{(134), (234), (135), (235), (156), (256)\}$. Then we integrate over $g_{12}$, exploiting $\delta_G(g_{126})$ and use (A7) for $(jkl) \in \{(156), (256)\}$ again. The last remaining integral over $h_{126}$ is then trivial, and the integrand reduces to

$$(\delta_G(e))^3(\delta_H(e))^3 = |G|^3|H|^3.$$ (A21)

The difference in powers of $|G|$ and $|H|$ is compensated for by the prefactors which equal $|G|^{-8}|H|^{-1}$ on the l.h.s. and $|G|^{-11}|H|^{-3}$ on the r.h.s.

c. The $3 \leftrightarrow 3$ move

We order the vertices in such a way that one the l.h.s. of the 3 \leftrightarrow 3 move, we have the 4-simplices (23456), (13456) and (12456) whereas on the r.h.s. they are (12356), (12346) and (12345). Six tetrahedra therefore form the common boundary of both sides of the move whereas on each side there are three tetrahedra shared by two 4-simplices. On the l.h.s. these are (1456), (2456) and (3456) and on the r.h.s. (1234), (1235) and (1236). On the l.h.s we therefore have the triangle (456) and on the r.h.s (123). All other triangles appear on both sides of the move.

The integral and integrand for the l.h.s. read

$$\int_H dh_{456} \delta_G(g_{456}) \delta_H(h_{3456}) \delta_H(h_{2456}) \delta_H(h_{1456}),$$ (A22)

whereas for the r.h.s. we have

$$\int_H dh_{123} \delta_G(g_{123}) \delta_H(h_{1234}) \delta_H(h_{1235}) \delta_H(h_{1236}).$$ (A23)

In order to simplify the l.h.s., we make use of the fact that $\delta_H(\_)$ is constant on the orbits of $G$ on $H$ and also on the conjugacy classes, i.e.

$$\delta_H(h_{3456}) = \delta_H((g_{23}^{-1} \triangleright h_{346}^{-1})(g_{23}^{-1} \triangleright h_{356})(g_{23}^{-1} \triangleright h_{345})h_{456}^{-1}).$$ (A24)

We then integrate over $h_{156}$, exploiting $\delta_H(h_{3456})$. The integrand reduces to

$$(\delta_G(e)(\delta_H(e))^2 = |G||H|^2$$ (A25)

if we make use of the condition (A7) for all $(jkl) \in \{(134), (234), (135), (235), (346), (356)\}$ and of (A16) for all $(jklm) \in \{(1345), (1346), (1356), (2345), (2346), (2356)\}$.

In order to simplify the r.h.s., we integrate over $h_{123}$, exploiting $\delta_H(h_{1234})$. The integrand reduces to (A25), too, if we make use of (A7) for $(jkl) \in \{(123), (124), (134), (234)\}$ and of (A16) for $(jklm) \in \{(1245), (1246), (1345), (1346), (2345), (2346)\}$. The numbers of $k$-simplices agree on both sides of the $3 \leftrightarrow 3$ move for all $k$, and the prefactors play no role in this case.

APPENDIX B: STATE SUM MODELS WITH 2-CATEGORIES

Under certain conditions, the partition function (15) for $d = 4$ forms a special case of Mackaay’s state sum (12). Here we explain in detail in which case this happens and which assumptions of (12) are violated in more general situations.

First, we follow (13) in which Mackaay specializes his state sum (12) to the case of finite groups, and describe the common special case with our model (15).

Recall that weak 2-groups (17) are algebraic models for pointed and connected homotopy 2-types as follows. Given any path connected CW-complex $X$ with 1-skeleton $X_1$ and base point $p \in X_1 \subseteq X$, there is a weak 2-group $\Pi_2(X, X_1, p)$ defined as follows. It has only a single object $p$. The 1-morphisms are the continuous closed curves in $X_1$ with base point $p$. The 2-morphisms are bigon-shaped surfaces between two such curves, continuously mapped into $X$, up to homotopy. It can be shown that $\Pi_2(X, X_1, p)$ forms a bicategory with one object. Furthermore, it turns out that each 2-morphism has got a vertical inverse and that each 1-morphism has got an inverse up to 2-isomorphism, and so $\Pi_2(X, X_1, p)$ forms a weak 2-group as defined in (17).
Homotopy equivalent based pairs of spaces \((X, X_1, p) \simeq (Y, Y_1, q)\) yield \(\Pi_2(X, X_1, p)\) and \(\Pi_2(Y, Y_1, q)\) that are equivalent as weak 2-groups. There are two characterizations of \(\Pi_2(X, X_1, p)\) up to equivalence of weak 2-groups that are relevant in the following.

First, by the coherence theorem for bicategories, \(\Pi_2(X, X_1, p)\) is equivalent to a (strict) 2-category. It can be shown that it is even equivalent to a strict 2-group and can thus be characterized by a crossed module \((G, H, \triangleright, t)\) of groups. In this case, we have \(G \cong \pi_1(X_1)\) and \(H \cong \pi_2(X, X_1)\). Note that this is in general a non-abelian group. The action \(\triangleright\) is the action of \(\pi_1(X)\) on \(\pi_2(X, X_1)\) by the change of base point, and \(t: \pi_2(X, X_1) \to \pi_1(X_1)\) is the restriction to the boundary. This way, the crossed module \((G, H, \triangleright, t)\) is determined up to equivalence in the 2-category of crossed modules.

Second, every bicategory with one object forms a weak monoidal category, and any weak monoidal category is equivalent (as a weak monoidal category) to any of its skeleta. This result can be extended to weak 2-groups whose skeleta are precisely the equivalent (as a weak monoidal category) to any of its skelet a. This result can be extended to weak 2-groups whose skeleta are precisely the special 2-groups of \([17]\). Passing to a skeleton in this way amounts to characterizing \(\Pi_2(X, X_1, p)\) in terms of its Postnikov data \((K, A, \triangleright, \alpha)\) where \(K := \pi_2(X)\) (abelian), \(K := \pi_1(X)\), \(\triangleright\) denotes the action of \(\pi_1(X)\) on \(\pi_2(X)\) by the change of base point, \(\alpha\) is the Postnikov \(k\)-invariant which is an \(A\)-valued algebraic 3-cocycle on \(K\). Starting from the crossed module \((G, H, \triangleright, t)\), the groups \(A\) and \(K\) appear when one extends the map \(t\) to a 4-term exact sequence of groups,

\[
\begin{align*}
& 0 \longrightarrow A' \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} K \longrightarrow \{e\} \\
&B1
\end{align*}
\]

i.e. \(A \cong \ker(t)\) and \(K \cong \coker(t) \cong G/t\langle H\rangle\). The action \(\triangleright\) of \(G\) on \(H\) by automorphisms induces an action \(\triangleright\) of \(K\) on \(A\) by automorphisms. The Postnikov \(k\)-invariant can then be constructed from a section of the map \(\pi\) (using, in general, the axiom of choice) and a diagram chase. Mackaay’s \(G\) and \(H\) in \([13]\) are our \(K\) and \(A\), respectively.

A close look at \([12]\) reveals that in the case in which the action \(\triangleright\) of \(K\) on \(A\) is trivial, our partition function \([15]\) agrees with Mackaay’s state sum of \([13]\). The case in which \(K\) acts trivially on \(A\), however, is far from generic. For example, let \((G, H, \triangleright, t)\) be a crossed module of groups in which \(H\) is abelian, \(t(h) = e \in G\) for all \(h \in H\) and in which \(G\) acts non-trivially on \(H\). In this case, \(A \cong H, K \cong G\), and so \(K\) acts non-trivially on \(A\). There exist many examples of this type. As soon as \(K\) acts non-trivially on \(A\), however, Mackaay’s construction \([13]\) is no longer a special case of his state sum \([12]\). For the general case, we cannot use \([13]\), but rather have to go back to the state sum as defined in \([12]\).

Let us explain how to define for every crossed module of groups \((G, H, \triangleright, t)\) a semi-strict monoidal 2-category with duals in such a way that Mackaay’s state sum \([12]\) agrees with our partition function \([15]\). This 2-category is semi-strict and pivotal, but not spherical, and so the proof of Pachner move invariance used in \([12]\) no longer applies. The fact that \([15]\) is nevertheless Pachner move invariant as was known from \([11, 14]\) and as we have confirmed in Appendix \([A]\) above, suggests that one ought to generalize the definition of spherical and modify the proof of Pachner move invariance in \([12]\) accordingly in order to encompass our example \([15]\) as well.

The 2-group associated with the crossed module \((G, H, \triangleright, t)\) forms a small category whose objects are elements of \(G\) and whose 1-morphisms \(f: g_1 \to g_2\) are elements \(f = (h, g) \in H \times G\) that satisfy \(g = g_1\) and \(t(h)g = g_2\). The composition of 1-morphisms is the vertical composition. The 2-category to consider is the discrete 2-category on this small category, i.e. its objects are elements of \(G\), its 1-morphisms \(f: g_1 \to g_2\) are elements \(f = (h, g) \in H \times G\) as above, and for every 1-morphism \(f\), there is only the identity 2-morphism.

This 2-category has a semi-strict monoidal structure as follows. For objects \(g_1, g_2 \in G\), we have \(g_1 \otimes g_2 = g_1g_2\). For 1-morphisms \(f = (h, g_1): g_1 \to g_2\) we have \(\hat{g} \otimes f = (e, \hat{g}) \cdot (h, g_1) = (\hat{g} \triangleright h, g_1g_2)\) and \(f \otimes \hat{g} = (h, g_1) \cdot (e, \hat{g}) = (h, \hat{g}g)\). The discreteness of the 2-category determines the monoidal structure on 2-morphisms. The monoidal unit \(\mathbb{1} = e \in G\) is the unit of \(G\).

For 1-morphisms \((h_1, g_1): g_1 \to g_1'\) and \((h_2, g_2): g_2 \to g_2'\), the tensorator 2-isomorphism is the identity 2-morphism associated with the following equality of 1-morphisms:

\[
((h_1, g_1) \otimes g_2') \circ (g_1 \otimes (h_2, g_2)) = (g_1' \otimes (h_2, g_2')) \circ ((h_1, g_1) \otimes g_2). \tag{B2}\]

Duality is defined as follows. The dual of an object \(g \in G\) is its inverse \(g^\ast = g^{-1}\) with unit and counit \(\iota_g = (e, e): \mathbb{1} \to g \otimes g^\ast\) and \(e_g = (e, e): g^\ast \otimes g \to \mathbb{1}\). The triangulator is the identity 2-morphism associated with the following equality of 1-morphisms:

\[
(e_g \otimes g) \circ (g \otimes \iota_g) = \text{id}_g. \tag{B3}\]

The dual of a 1-morphism \((h, g_1): g_1 \to g_2\) is its vertical inverse \((h, g_1)^\ast = (h^{-1}, g_2)\). The unit and counit for this dual as well as the duals of 2-morphisms are already determined because of discreteness of the 2-category.

For each 1-morphism \((h, g): g_1 \to g_2\), the 1-morphisms \(\sharp f\) and \(\sharpt f\) of \([12]\) turn out to be \(\sharp f = (g_2^{-1} \triangleright h, g_2^{-1})\): \(g_2^{-1} \to g_1\) and \(\sharpt f = (g_1^{-1} \triangleright h, g_2^{-1})\): \(g_2^{-1} \to g_1\). And so the semi-strict monoidal 2-category is pivotal because of the identity 2-morphism associated with the equality of 1-morphisms \(f_2 = \sharpt f\).
As remarked in [31], however, left- and right-traces of a 1-morphism $f = (h, g): g \to t(h)g$ turn out to be $\text{tr}_L(f) = (g^{-1} \triangleright h, e)$ and $\text{tr}_R(f) = (h, e)$, respectively. Unless $G$ acts trivially on $H$, these are in general distinct 1-morphisms, and so in the discrete 2-category, there cannot exist any 2-isomorphism between them. Therefore, the 2-category is not spherical. Nevertheless, the state sum defined in [12] agrees with (15) for $d = 4$ for any finite crossed module $(G, H, \triangleright, t)$, up to an overall prefactor $|G|^{-|A_0|} |H|^{|A_0| - |A_1|}$.

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