From Peierls brackets to a generalized Moyal bracket for type-I gauge theories

Giampiero Esposito,1,2 Cosimo Stornaiolo1,2

1Istituto Nazionale di Fisica Nucleare, Sezione di Napoli,
Complesso Universitario di Monte S. Angelo,
Via Cintia, Edificio N’, 80126 Napoli, Italy

2Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S. Angelo,
Via Cintia, Edificio N’, 80126 Napoli, Italy

(Dated: March 27, 2022)

Abstract

In the space-of-histories approach to gauge fields and their quantization, the Maxwell, Yang–Mills and gravitational field are well known to share the property of being type-I theories, i.e. Lie brackets of the vector fields which leave the action functional invariant are linear combinations of such vector fields, with coefficients of linear combination given by structure constants. The corresponding gauge-field operator in the functional integral for the in-out amplitude is an invertible second-order differential operator. For such an operator, we consider advanced and retarded Green functions giving rise to a Peierls bracket among group-invariant functionals. Our Peierls bracket is a Poisson bracket on the space of all group-invariant functionals in two cases only: either the gauge-fixing is arbitrary but the gauge fields lie on the dynamical sub-space; or the gauge-fixing is a linear functional of gauge fields, which are generic points of the space of histories. In both cases, the resulting Peierls bracket is proved to be gauge-invariant by exploiting the manifestly covariant formalism. Moreover, on quantization, a gauge-invariant Moyal bracket is defined that reduces to $i\hbar$ times the Peierls bracket to lowest order in $\hbar$. 

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I. INTRODUCTION

The modern formulations of quantum field theory and quantum gravity still reflect two basic attitudes: either one follows a Lagrangian path, starting from an action functional with the associated functional integral formulation \[1\], or the Hamiltonian road to quantization is chosen, with the associated constraint analysis \[2\] and functional differential equations (the latter being extremely difficult, especially for gravitation). The two approaches are not obviously equivalent in all cases \[3\], and one can indeed build a sum-over-histories which does not solve the constraint equations of the quantum theory via Hamiltonian methods \[4\]. A further relevant example is provided by quantum supergravity: the supersymmetry constraints lead to equations solved exactly by a wave function which is finite \[5\], whereas the analysis of counterterms within the framework of covariant perturbation methods shows no hope for finiteness of quantum supergravity \[6, 7\].

It is therefore important to re-assess the foundations of covariant methods on the one hand, and their relation with ‘covariant’ formulations of Hamiltonian quantization on the other hand \[8\]. In particular, a cornerstone of the space-time approach to quantum field theory \[9\] is the Peierls bracket \[10\], that makes it possible to have a Poisson bracket which is completely invariant under the (proper) gauge group \[1\] (this being an infinite-dimensional Lie group). For our purposes, the framework we are interested in can be described as follows \[1, 11\].

To begin, an action functional \(S\) is given, which is a real-valued functional defined on the space \(\Phi\) of field histories. On \(\Phi\), a set of vector fields \(Q^\alpha\) exist which leave the action invariant, i.e.

\[ Q^\alpha S = 0, \]  

and having components \(Q^i_\alpha\). Lower case Latin indices are used for components of fields \(\varphi^i\) (e.g. \(\varphi^i = A_\mu(x)\), or \(A^\alpha_\mu(x)\), or \(g_{\mu\nu}(x)\)), while Greek indices from the beginning of the alphabet are Lie-algebra indices. Whenever Latin or Greek indices are summed over, this means contraction jointly with integration, e.g.

\[ S_{ij} w^j = \int \frac{\delta^2 S}{\delta \varphi^i(x) \delta \varphi^j(x')} w^i(x') dx', \]

while infinitesimal gauge transformations read

\[ \delta \varphi^i = \int Q^i_\alpha(x, x') \delta \xi^\alpha dx' = Q^i_\alpha \delta \xi^\alpha. \]  

(1.2)
The vector fields $Q_\alpha$ are linearly independent and have Lie brackets satisfying
\[
\left[ Q_\alpha, Q_\beta \right] = C^\gamma_{\alpha\beta} Q_\gamma + S_i T^i_{\alpha\beta}.
\] (1.3)

For type-I theories, that we consider hereafter and include Maxwell, Yang–Mills and general relativity, the $C^\gamma_{\alpha\beta}$ are structure constants in that
\[
\frac{\delta}{\delta \varphi^i} C^\gamma_{\alpha\beta} = C^\gamma_{\alpha\beta,i} = 0,
\] (1.4)
and $T^i_{\alpha\beta}$ vanishes as well. The components $Q^i_\alpha$ can be taken to depend linearly on the fields $\varphi^i$, i.e.
\[
Q^i_{\alpha,jk} = 0,
\] (1.5)
since, on acting with $Q_\rho$ on both sides of the Lie bracket
\[
\left[ Q_\alpha, Q_\beta \right] = C^\gamma_{\alpha\beta} Q_\gamma,
\] (1.6)
the resulting second functional derivatives $Q^i_{\alpha,jk}$ multiply vanishing terms like $Q^j_{\beta} Q^k_{\gamma}$ weighted with opposite signs. Moreover, the $Q^i_\alpha$ are a sum of Dirac delta $\delta(x, x')$ and/or their first derivatives, multiplied by local functions of $\varphi^i$ and their first derivatives. For example, in the infinitesimal gauge transformation for Maxwell theory:
\[
\delta A_\mu(x) = \partial_\mu \delta \xi(x) = \int -\delta_\mu(x, x') \delta \xi(x') dx',
\] (1.7)
$Q^i_\alpha$ reduces to $Q_\mu(x, x') = -\delta_\mu(x, x')$.

By virtue of Eq. (1.6) the proper gauge group $G$ obtained by exponentiating the transformations (1.2) decomposes $\Phi$ into sub-spaces, called orbits, to which the vector fields $Q_\alpha$ are tangent. The space $\Phi$ is a principal fibre bundle over $\Phi/G$, and the base space $\Phi/G$ is the space of orbits. Fibre-adapted coordinates consist of abstract coordinates $I^A$ which label fibres of $\Phi \to \Phi/G$, jointly with $P^\alpha$ coordinates which label points within each fibre. The $P^\alpha$ correspond to a choice of gauge-fixing functional $P^\alpha$ in the functional integral for the $\langle \text{out} | \text{in} \rangle$ amplitude. On going from $(I^A, P^\alpha)$ coordinates to field variables $\varphi^i$, the loop expansion of the $\langle \text{out} | \text{in} \rangle$ amplitude involves eventually two invertible operators, i.e. the gauge-field operator
\[
F_{ij} = S_{ij} + P^\alpha_{\alpha} \omega_{\alpha\beta} P^\beta_{ji},
\] (1.8)
$\omega_{\alpha\beta}$ being taken to be, for the sake of manifest covariance, a $\varphi$-dependent local distribution obeying the gauge transformation law (see, however, the end of section 3 for an alternative scheme)

$$
\delta \omega_{\alpha\beta} = \omega_{\alpha\beta,i}Q^i_\gamma\delta \xi^\gamma = -\left(\omega_{\delta\beta}C^\delta_{\gamma\alpha} + \omega_{\alpha\delta}C^\delta_{\beta\gamma}\right)\delta \xi^\gamma,
$$

(1.9)
as well as the ghost operator

\[ \hat{F}_\beta^\alpha = Q_\beta^\alpha P^\alpha = P^\alpha_i Q^i_\beta. \] (1.10)

Since $P^\alpha$ is chosen in such a way that $F_{ij}$ and $\hat{F}_\beta^\alpha$ are invertible, one can consider their Green functions $G^{ij}$ and $G^{\alpha\beta}$, for which

\[ F_{ij}G^{jk} = -\delta^k_i, \] (1.11)

\[ \hat{F}_{\alpha\beta}G^{\beta\gamma} = -\delta^\gamma_\alpha. \] (1.12)

From the advanced and retarded Green functions of $F_{ij}$, hereafter denoted as $G^{+ij}$ and $G^{-ij}$, respectively, one can build the super-commutator function

\[ \tilde{G}^{ij} \equiv G^{+ij} - G^{-ij}, \] (1.13)

and hence the Peierls bracket

\[ (A, B) \equiv A_i \tilde{G}^{ij} B_j = \int d^4x \int d^4y \frac{\delta A}{\delta \varphi^i(x)} \tilde{G}^{ij}(x, y) \frac{\delta B}{\delta \varphi^j(y)}, \] (1.14)

where $A$ and $B$ are any pair of gauge-invariant functionals of the fields, i.e.

\[ Q_\alpha A = 0 \implies A_i Q^i_\alpha = 0, \quad Q_\alpha B = 0 \implies B_i Q^i_\alpha = 0. \] (1.15)

Since the gauge-field operator $F_{ij}$ is the naturally occurring invertible operator in the quantum theory of gauge fields from the point of view of functional-integral approach, we have been led to define the Peierls bracket as in Eq. (1.14). The same definition has been proposed in Ref. [12].

Section 2 proves under which conditions Eq. (1.14) defines indeed a Poisson bracket on the space of all gauge-invariant functionals obeying Eq. (1.15). Section 3 proves gauge invariance of the Peierls bracket (1.14). A covariant Moyal bracket on the space of histories is proposed in section 4, while concluding remarks and open problems are presented in section 5.
II. JACOBI IDENTITY FOR THE PEIERLS BRACKET

Since advanced and retarded Green functions for type-I theories are related by

\[ G^{+ij} = G^{-ji}, \] (2.1)

which implies \( \tilde{G}^{ij} = -\tilde{G}^{ji} \), the antisymmetry of (1.14), i.e.

\[ (A, B) = -(B, A), \] (2.2)

follows immediately from the definition. Bilinearity is also obtained at once from (1.14):

\[ (A, B + C) = A_i \tilde{G}^{ij} (B_j + C_j) = (A, B) + (A, C). \] (2.3)

The only non-trivial task is the verification of the Jacobi identity. Indeed, one finds

\[ P(A, B, C) \equiv (A, (B, C)) + (B, (C, A)) + (C, (A, B)) = A_{il} B_{jl} C_{lk} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{il} \tilde{G}^{ki} \right) \]

\[ + A_i B_{jl} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{il} \tilde{G}^{ki} \right) + A_i B_{jl} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{il} \tilde{G}^{ki} \right) \]

\[ + A_i B_{jl} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{il} \tilde{G}^{ki} \right). \] (2.4)

The antisymmetry of the supercommutator function \( \tilde{G}^{ij} \), jointly with commutation of functional derivatives: \( T_{il} = T_{li} \) for all \( T = A, B, C \), implies that the first three terms on the last equality in (2.4) vanish. For example, one finds

\[ A_{il} B_{jl} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{il} \tilde{G}^{ki} \right) = A_{il} B_{jl} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{il} \tilde{G}^{ki} \right) \]

\[ = -A_{il} B_{jl} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{il} \tilde{G}^{ki} \right) = 0, \] (2.5)

and an entirely analogous procedure can be applied to the terms containing the second functional derivatives \( B_{jl} \) and \( C_{kl} \).

The last term in (2.4) requires more labour because it contains functional derivatives of \( \tilde{G}^{ij} \). To begin note that, from infinitesimal variations of Eq. (1.11), one finds

\[ \delta G^\pm = G^\pm (\delta F) G^\pm, \] (2.6)

and hence, for any Green function of \( F_{ij} \),

\[ G^{lm}_{\ldots} = G^{di} F_{ij,k} G^{jm} = G^{di} S_{ijk} G^{jm} + G^{di} P_{\alpha,i}^{\omega_{\alpha\beta,k}} P_{\beta,j} G^{jm} \]

\[ + G^{di} P_{\alpha,i}^{\omega_{\alpha\beta,k}} P_{\beta,j} G^{jm} + G^{di} P_{\alpha,i}^{\omega_{\alpha\beta,k}} P_{\beta,j} G^{jm}. \] (2.7)
In Eq. (2.7), the terms involving third functional derivatives of the action give vanishing contribution \( U(A, B, C) \) to the Jacobi identity (2.4), because

\[
U(A, B, C) = A_i B_j C_k \left[ (G^{+ia} - G^{-ia})(G^{+jb}G^{-kc} - G^{-jb}G^{+kc}) 
+ (G^{+ib} - G^{-ib})(G^{+kc}G^{-ia} - G^{-kc}G^{+ia}) 
+ (G^{+kc} - G^{-kc})(G^{+ia}G^{+jb} - G^{-ia}G^{+jb}) \right] S_{abc}.
\] (2.8)

This sum vanishes since it involves six pairs of triple products of Green functions with opposite signs, i.e.

\[
G^{+ia}G^{+jb}G^{-kc}, G^{-ia}G^{-jb}G^{+kc}, G^{+ib}G^{+kc}G^{-ia}, G^{-jb}G^{-kc}G^{+ia},
\]

\[
G^{+kc}G^{+ia}G^{-jb}, G^{-kc}G^{-ia}G^{+jb}.
\]

In the evaluation of the Jacobi identity we therefore deal eventually, from Eqs. (2.4) and (2.7), with three terms like

\[
V(A, B, C) = A_i B_j C_k \left[ G^{+ir} P_{\alpha}^{\beta}, \omega_{\alpha \beta}, P_{\alpha}^{\beta} G^{+sj} 
+ G^{+ir} P_{\alpha}^{\beta}, \omega_{\alpha \beta}, P_{\alpha}^{\beta} G^{+sj} 
+ G^{+ir} P_{\alpha}^{\beta}, \omega_{\alpha \beta}, P_{\alpha}^{\beta} G^{+sj} 
+ G^{+ir} P_{\alpha}^{\beta}, \omega_{\alpha \beta}, P_{\alpha}^{\beta} G^{+sj} \right].
\] (2.9)

If \( \omega_{\alpha \beta} \) were taken to be just a non-singular, symmetric, continuous matrix, independent of field variables, \( V(A, B, C) \) would vanish, and hence the Jacobi identity would be fulfilled, provided that either

\[
Z_i G^{ir} P_{\alpha}^{\beta} = 0, \text{ with } Z = A, B, C,
\] (2.10)

or

\[
P^{\alpha}_{\cdot i j} = 0,
\] (2.11)

where (2.10) and (2.11) are sufficient conditions for Jacobi to hold. If instead \( \omega_{\alpha \beta} \) is taken to be a \( \varphi \)-dependent local distribution, as in Sec. 1 following Ref. [14], the desired sufficient condition is expressed by Eq. (2.10) only, so that, in both cases, we are led to consider functional identities involving the left-hand side of Eq. (2.10). For this purpose, we first note that

\[
F_{ik} Q_{\alpha}^k = S_{ik} Q_{\alpha}^k + P_{\cdot i j}^{\beta} \omega_{\beta \gamma} P_{\cdot j}^{\gamma} Q_{\alpha}^k = -S_{ik} Q_{\alpha}^k + P_{\cdot i j}^{\beta} \omega_{\beta \gamma} \Gamma_{\alpha}^{\gamma}.
\] (2.12)
from the gauge invariance of the action (Eq. (1.1)) and from the definition of ghost operator (Eq. (1.10)). By acting on both sides of Eq. (2.12) with any Green function of the gauge-field operator one finds, from Eq. (1.11),

$$Q_i^j = G_{ij} S_{jk} Q^{k} - G_{ij} P_{ir} \omega_{r \gamma} \hat{F}_\alpha^\gamma. \quad (2.13)$$

At this stage, we can exploit Eq. (1.15) for any gauge-invariant functional $Z[\varphi]$ to find, from the identity (2.13),

$$0 = Z_i Q_i^j = Z_i G_{ij} S_{jk} Q^{k} - Z_i G_{ij} P_{ir} \omega_{r \gamma} \hat{F}_\alpha^\gamma, \quad (2.14)$$

and hence, from Eq. (1.12),

$$0 = Z_i Q_i^j G^{\alpha \beta} = Z_i G_{ij} S_{jk} Q^{k} G^{\alpha \beta} + Z_i G_{ij} P_{ir} \omega_{r \beta}^\alpha. \quad (2.15)$$

Eventually, the definition of inverse matrix

$$\omega_{\rho \beta} \omega^{\beta \alpha} = \omega_{\rho}^\beta \omega_\beta^\alpha = \delta_\rho^\alpha \quad (2.16)$$

yields, from Eq. (2.15), the desired identity in the form

$$Z_i G_{ij} P_{ir} = -Z_i G_{ij} S_{jk} Q^{k} G^{\gamma \beta} \omega_\beta^\alpha. \quad (2.17)$$

Thus, the sufficient condition (2.10) holds if and only if the gauge fields lie on the dynamical subspace where the action is stationary, i.e.

$$S_{jk} = 0. \quad (2.18)$$

This is a very restrictive condition for us to be able to prove the Jacobi identity with a field-dependent matrix $\omega_{\alpha \beta}$ in the gauge-field operator (1.8). If we relax this assumption and just work with a non-singular, symmetric $\omega_{\alpha \beta}$, we obtain instead, from Eq. (2.9), the sufficient condition (2.11). In other words, linear covariant gauges are naturally picked out if Eq. (1.14) is required to define a good Peierls bracket which obeys the Jacobi identity.

Last, the Leibniz rule

$$(A, BC) = (A, B)C + B(A, C) \quad (2.19)$$

is immediately obtained from (1.14) and from the Leibniz rule for functional derivatives in type-I theories, i.e.

$$(BC)_j = B_j C + BC_j. \quad (2.20)$$
III. GAUGE INVARIANCE OF THE PEIERLS BRACKET

When the gauge fields are subject to infinitesimal gauge transformations according to Eq. (1.2), the Peierls bracket (1.14) follows the gauge transformation law

$$\delta(A, B) = (\delta A, B) + A(i \delta \tilde{G}^{ij})B + A(i \tilde{G}^{ij})(\delta B),$$

(3.1)

where

$$\delta A = A, k Q^k \delta \xi = - A, k Q^k \delta \xi$$

(3.2)

from the gauge-invariance condition in Eq. (1.15), and the same holds for $\delta B$. As far as the gauge-transformation law of the supercommutator function $\tilde{G}^{ij}$ is concerned, this is obtained by imposing Eq. (1.9) jointly with

$$\delta S_{ij} = - (S_{ik} Q^k + S_{ik} Q^i(\delta \tilde{G}^{ij})) \delta \xi,$$

(3.3)

$$\delta P^\alpha_i = P^\alpha_j Q^i \delta \xi = \left( C^\alpha \gamma, P^\gamma_i - P^\alpha_j Q^j \right) \delta \xi,$$

(3.4)

which imply

$$\delta F_{ij} = - (F_{kj} Q^k + F_{kj} Q^j) \delta \xi,$$

(3.5)

and hence, bearing in mind Eq. (2.6),

$$\delta G^{ij} = \left( Q^i_{a, k} G^{kj} + Q^j_{a, k} G^{ik} \right) \delta \xi,$$

(3.6)

$$\delta \tilde{G}^{ij} = \delta G^{+ij} - \delta G^{-ij} = \left( Q^i_{a, k} \tilde{G}^{kj} + Q^j_{a, k} \tilde{G}^{ik} \right) \delta \xi.$$

(3.7)

Equation (3.3), in particular, is obtained from the second Ward identity [1, 9, 11] for functional derivatives of the gauge-invariant action $S$:

$$S_{ijk} Q^i + S_{ij} Q^i + S_{,ik} Q^i + S_{,jk} Q^i = 0,$$

(3.8)

jointly with the linearity of $Q^i_{a, k}$ expressed by Eq. (1.5). In the course of deriving the law (3.5), the four terms involving structure constants from the use of gauge transformation laws (1.9) and (3.4) cancel each other exactly.
By virtue of Eqs. (3.1), (3.2) and (3.7) we prove immediately gauge invariance of the
Peierls bracket (1.14), because
\[ \delta(A, B) = \left[ -A_kQ^k_{\alpha,i}G_{ij}B_j + A_iQ^i_{\alpha,k}G_{kj}B_j \right. \]
\[ \left. + A_iQ^i_{\alpha,k}G_{ik}B_j - A_iG_{ij}Q^k_{\alpha,j}B_k \right] \delta\xi^\alpha = 0. \] (3.9)

Since the sufficient condition (2.18) is very restrictive, it would be desirable to use the
sufficient condition (2.11) only while still being able to prove gauge invariance of the Peierls
bracket (1.14). This is indeed possible because, in its final form (3.5), the gauge transfor-
mation law for the gauge-field operator \( F_{ij} \) is independent of the functional derivatives \( \omega_{\alpha\beta,i} \)
and \( P^\alpha_{\cdot ij} \). We can therefore first assume that Eq. (3.5) holds and later take \( \omega_{\alpha\beta} \) to be a
non-singular, symmetric continuous matrix independent of field variables.

IV. TOWARDS A MOYAL BRACKET ON THE SPACE OF HISTORIES

Since the Peierls bracket (1.14) is a Poisson bracket, and bearing in mind that the Poisson
bracket of two functions on a manifold is the coefficient of the term linear in \( i\hbar \) in the
corresponding Moyal bracket \([15]\), we are now led to study how a Moyal bracket among
gauge-invariant functionals can be defined. Our starting point is a careful consideration of a
formula defining the star-product of phase-space functions. Following the appendix of Ref.
\([15]\), we recall that such a product may be expressed in the form
\[ f \star g \equiv fg + \frac{i\hbar}{2} \{ f, g \} + \sum_{k=2}^{\infty} \left( \frac{i\hbar}{2} \right)^k \frac{1}{k!} D_k(f, g), \] (4.1)
where \( D_k \) is a bidifferential operator defined by
\[ D_k(f, g)(q, p) \equiv \left[ \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial q_2} \frac{\partial}{\partial p_1} \right]^k f(q_1, p_1)g(q_2, p_2) \] \[ q_1 = q_2, p_1 = p_2 = p. \] (4.2)
The star-product (4.1) may be recovered from the (asymptotic) expansion of
\[ f \text{ exp} \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial \xi^i} \omega^i_{\cdot jk} \frac{\partial}{\partial \xi^j} \right) \right] g, \]
where \( \xi^i \) takes the \( 2N \) values \( q^1, ..., q^N, p_1, ..., p_N \). The Moyal bracket is eventually obtained
from the definition
\[ [f, g]_M \equiv f \star g - g \star f = i\hbar \{ f, g \} + \sum_{k=2}^{\infty} \left( \frac{i\hbar}{2} \right)^k \frac{1}{k!} \left[ D_k(f, g) - D_k(g, f) \right], \] (4.3)
where even values of $k$ give vanishing contribution to the Moyal bracket.

In our field-theoretical framework, the inverse $\omega^{jl}$ of the symplectic form (also denoted by $\Lambda^{jl}$) is replaced by the supercommutator $\tilde{G}^{jl}$, and we are led to the following heuristic definition of the star-product of gauge-invariant functionals:

$$A \star B \equiv A \exp \left[ \frac{i\hbar}{2} \left( \frac{\delta}{\delta \phi^j} \tilde{G}^{jk} \frac{\delta}{\delta \phi^k} \right) \right] B.$$  (4.4)

Upon expansion of the exponential, Eq. (4.4) yields, bearing in mind the definition (1.14),

$$A \star B = AB + \frac{i\hbar}{2} (A, B) + \frac{(i\hbar/2)^2}{2!} A_{jl} \tilde{G}^{jk} \tilde{G}^{lm} B_{km}
+ \frac{(i\hbar/2)^2}{2!} \left[ A_{jl} \tilde{G}^{jk} \tilde{G}^{lm} B_{km} + A_{jl} \tilde{G}^{jk} \tilde{G}^{lm} B_{km} + A_{jl} \tilde{G}^{jk} \tilde{G}^{lm} B_{km} \right]
+ O(\hbar^3).$$  (4.5)

For our star-product to be associative, we have to assume that the associative law of multiplication holds also for our contractions involving both summation over repeated indices and integration over a space-time region. Ultimately, this amounts to requiring a suitable rate of fall-off at infinity or a suitable choice of boundary conditions (Section 4.1 of Ref. [9]).

Interestingly, the functional derivatives of $\tilde{G}^{lm}$ provide additional terms with respect to the formulae of ordinary quantum mechanics. However, the gauge-invariant Moyal bracket

$$[A, B]_M \equiv A \star B - B \star A$$  (4.6)

retains the same functional form as in ordinary quantum mechanics. Indeed, on using the super-condensed DeWitt notation \[9\],

$$A_{i} \equiv A_1, \ A_{ij} \equiv A_2, ..., A_{i_1...i_l} \equiv A_l,$$  (4.7)

one finds, from Eqs. (4.4)–(4.6),

$$[A, B]_M = i\hbar (A, B) + \frac{(i\hbar/2)^2}{2!} \left[ A_{2j} \tilde{G} \tilde{G} B_2 - B_2 \tilde{G} \tilde{G} A_2 
+ A_2 \tilde{G} \tilde{G}_1 B_1 + A_1 \tilde{G}_1 \tilde{G}_1 B_1 + A_1 \tilde{G}_1 \tilde{G} B_2 
- B_2 \tilde{G} \tilde{G}_1 A_1 - B_1 \tilde{G}_1 \tilde{G}_1 A_1 - B_1 \tilde{G}_1 \tilde{G} A_2 \right] + O(\hbar^3),$$  (4.8)

where exact cancellations occur among the various terms in square brackets in Eq. (4.8), because

$$A_{2j} \tilde{G} \tilde{G} B_2 - B_2 \tilde{G} \tilde{G} A_2 = A_{jl} \tilde{G}^{jk} \tilde{G}^{lm} B_{km} - B_{jl} \tilde{G}^{jk} \tilde{G}^{lm} A_{km}
= A_{km} \tilde{G}^{kj} \tilde{G}^{ml} B_{jl} - B_{jl} \tilde{G}^{jk} \tilde{G}^{lm} A_{km} = (-1)^2 A_{km} \tilde{G}^{jk} \tilde{G}^{lm} B_{jl} - B_{jl} \tilde{G}^{jk} \tilde{G}^{lm} A_{km}
= B_{jl} \tilde{G}^{jk} \tilde{G}^{lm} A_{km} - B_{jl} \tilde{G}^{jk} \tilde{G}^{lm} A_{km} = 0,$$  (4.9)
\[ A_2 \tilde{G}_1 B_1 - B_1 \tilde{G}_1 A_2 = A_{jk} \tilde{G}^{jm} \tilde{G}^{lm} A_{km} - B_j \tilde{G}^{jk} \tilde{G}^{lm} A_{km} \]

\[ = A_{jm} \tilde{G}^{kj} \tilde{G}^{lm} A_{km} - B_j \tilde{G}^{jk} \tilde{G}^{lm} A_{km} = A_{km} \tilde{G}^{kl} \tilde{G}^{mj} B_j - B_j \tilde{G}^{kj} \tilde{G}^{ml} A_{km}( -1)^2, \]

\[ = B_j \tilde{G}^{mj} \tilde{G}^{kl} A_{km} - B_j \tilde{G}^{kj} \tilde{G}^{ml} A_{km} = B_j \tilde{G}^{mj} \tilde{G}^{kl} A_{km} - B_j \tilde{G}^{kj} \tilde{G}^{ml} A_{km} \]

\[ = B_j \tilde{G}^{jk} \tilde{G}^{ml} A_{km} = 0, \tag{4.10} \]

\[ A_1 \tilde{G}_1 \tilde{G}_2 B_1 - B_2 \tilde{G}_1 \tilde{G}_1 B_1 = A_{jm} \tilde{G}^{ml} \tilde{G}^{jk} B_{jj} - B_{jj} \tilde{G}^{jm} \tilde{G}^{ml} A_{jm} = ( -1)^2 B_{jj} \tilde{G}^{jm} \tilde{G}^{ml} A_{jm} - B_{jj} \tilde{G}^{jm} \tilde{G}^{ml} A_{jm} \]

\[ = 0, \tag{4.11} \]

\[ A_1 \tilde{G}_1 \tilde{G}_1 B_1 - B_1 \tilde{G}_1 \tilde{G}_1 A_1 = A_{jm} \tilde{G}^{ml} \tilde{G}^{jk} B_{jm} - B_{jm} \tilde{G}^{jm} \tilde{G}^{ml} A_{jm} \]

\[ = B_{jm} \tilde{G}^{jm} \tilde{G}^{ml} A_{jm} - B_{jm} \tilde{G}^{jm} \tilde{G}^{ml} A_{jm} = 0. \tag{4.12} \]

In the course of deriving Eqs. (4.9)–(4.12), besides relabelling dummy indices and exploiting anti-symmetry of \( \tilde{G}^{jk} \) and commutation of second functional derivatives for type-I theories, we have assumed that the associative law of multiplication holds in agreement with the assumption made following Eq. (4.5).

We have therefore proved explicitly that the definition (4.6) of gauge-invariant Moyal bracket may engender the asymptotic expansion

\[ [A, B]_M = i\hbar (A, B) + O(\hbar^3). \tag{4.13} \]

It remains to be seen whether, order by order in \( \hbar \), the functional derivatives of the super-commutator function \( \tilde{G}^{jk} \) give always vanishing contribution to \( [A, B]_M \). This can be done by hand with finitely many terms in the expansion of Eq. (4.4), but more powerful methods are necessary to obtain a proof to all orders.

V. CONCLUDING REMARKS AND OPEN PROBLEMS

We have studied structural issues which are relevant for the manifestly covariant approach to quantization of gauge theories. In Secs. II and III we have put on firm ground the choice of linear covariant gauges from the point of view of Peierls-bracket formalism: the
Jacobi identity and the choice of generic gauge fields on the space of histories enforce the choice of linear covariant gauges. In Sec. IV we have defined, by close inspection of the quantum mechanical formalism, a Moyal bracket on the space of histories (which is new to our knowledge), proving that it reduces to $i\hbar$ times the Peierls bracket to lowest order in $\hbar$. Several important problems deserve now investigation, i.e.

(i) How to replace the definition (4.4) by exploiting a suitable integral kernel, along the lines of Appendix A of Ref. [15], in such a way that Moyal brackets for type-I gauge theories are put on firm ground, without relying upon formal series.

(ii) How to apply such a formalism to quantized general relativity [16].

(iii) How to define Peierls brackets for noncommutative extensions of general relativity, bearing in mind the work in Refs. [17, 18].

(iv) What is the relation, if any, with modern non-perturbative approaches to quantum gravity [19].

Hopefully, the years to come will tell us whether a renaissance of Peierls-bracket formalism may lead to a better understanding of the difficulties faced by any attempt of quantizing the gravitational field [1]. Note also that, if the Moyal bracket is viewed as being more fundamental, Eq. (4.13) suggests defining the Peierls bracket from the relation

$$\left( A, B \right) \equiv \lim_{\hbar \to 0} \frac{1}{i\hbar} [A, B]_M, \quad (5.1)$$

which provides, to our knowledge, a novel way of looking at the Peierls bracket.

Acknowledgments

Previous collaboration with Giuseppe Bimonte and Giuseppe Marmo on related topics has taught us a lot and has provided the appropriate motivation for the present research. The authors are also grateful to Fedele Lizzi and Patrizia Vitale for clarifying the work in Ref. [15], and to the INFN for financial support. Their work has been partially supported
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