Quantum $SU(3)$ Skyrme model with noncanonical embedded $SO(3)$ soliton

D. Jurčiukonis and E. Norvaiša

Vilnius University Institute of Theoretical Physics and Astronomy,
Goštauto 12, Vilnius 01108, Lithuania

Abstract

The new ansatz which is the $SO(3)$ group soliton was defined for the $SU(3)$ Skyrme model. The model is considered in noncanonical bases $SU(3) \supset SO(3)$ for the state vectors. A complete canonical quantization of the model have been investigated in the collective coordinate formalism for the fundamental $SU(3)$ representation of the unitary field. The independent quantum variables manifold cover all the eight dimensions $SU(3)$ group manifold due to the new ansatz. The explicit expressions of the Lagrangian and Hamiltonian densities are derived for this modified quantum skyrmion.
1. INTRODUCTION

The topological soliton models, and the Skyrme model among them, have generated a rising interest. It is related to expansive field of applications of the model. The traditional phenomenological application the model both in nuclear and elementary particle physics as well as the Skyrme model description of the quantum Hall effect, Bose-Einstein condensate and black hole physics, are elaborated.

The first comprehensive phenomenological application of the model to nucleon structure was the semiclassical calculation of the static properties of the nucleon in. The original model was defined for a unitary field $U(x,t)$ that belongs to fundamental representation of the $SU(2)$ group. Semiclassical quantization suggests that the skyrmion rotates as a "rigid body" and the mass of the pion (asymptotic behavior of the mass density) are introduced through an external chiral symmetry breaking term in the Lagrangian density. Constructive realization of ab initio quantization provides in Hamiltonian a term which may be interpreted as an effective pion mass term. The quantum mass corrections stabilize solution of quantum skyrmion. The model was generalized to unitary field in arbitrary irreducible representation (irrep) of the $SU(2)$ group and the $SU(3)$ group. The extension of the Skyrme model to $SU(N)$ group represents the common structure of the Skyrme Lagrangian. The intriguing rich geometrical structure with polyhedral symmetry for winding (baryon) number larger than 1 gives impetus to wide applications of the model.

The aim of this work is to discuss the group-theoretical aspects of the canonical quantization of the $SU(3)$ Skyrme model with new $SO(3)$ ansatz which differs from proposed by A.P. Balachandran et al. The ansatz is defined in noncanonical $SU(3)$ bases. These bases were developed by J. P. Elliott to consider collective motion of nuclei.

The present manuscript is organized as follows. In Section 2 we introduce noncanonical $SU(3)$ bases and $SO(3)$ hedgehog ansatz. In Section 3 the $SO(3)$ soliton is canonically quantized on $SU(3)$ manifold. The explicit expression of the Lagrangian and Hamiltonian densities of the quantum skyrmion are presented in Section 4. The results are discussed in Section 5.

2. DEFINITIONS FOR SOLITON IN NONCANONICAL SU(3) BASES

The unitary field $U(x,t)$ is defined for $SU(3)$ group in the arbitrary irrep $(\lambda, \mu)$. The modified Skyrme model is based on standard Lagrangian density

$$L = -\frac{f_*^2}{4} \text{Tr} \{ R_\mu R^\mu \} + \frac{1}{32\epsilon^2} \text{Tr} \{ [R_\mu, R_\nu] [R_\mu, R_\nu] \},$$

(2.1)

where the "right" and "left" chiral currents are defined as

$$R_\mu = (\partial_\mu U) \hat{U} = \partial_\mu \alpha^i \xi^{(B)}_i(\alpha) \langle J^{(1,1)}_{B} \rangle,$$

(2.2)

$$L_\mu = \hat{U} (\partial_\mu U) = \partial_\mu \alpha^i \xi^{(B)}_i(\alpha) \langle J^{(1,1)}_{B} \rangle,$$

(2.3)

and have the values on the $SU(3)$ algebra. The $f_*$ and $\epsilon$ in (2.1) are the phenomenological parameters of the model. The explicit expressions of functions $\xi^{(B)}_i(\alpha)$ and
classical Lagrangian and winding number have the same factor \( N \). This ensures the winding (baryon) number \( B \) to be \( SU(2) \) free. The relations between canonical bases (reduction chain on subgroup \( SU(3) \)) were the symbol in brackets denotes the \( SU(2) \) Clebsch - Gordan coefficient, \( Y_{l,m}(\theta, \varphi) \) are the spherical harmonics. The boundary conditions \( F(0) = \pi \) and \( F(\infty) = 0 \) ensure the winding (baryon) number \( B = 1 \) for all irreps \( j \) due to the reason that the classical Lagrangian and winding number have the same factor \( N = \frac{\pi}{2} j(j+1)(2j+1) \) which can be reduced \( [7] \). At this work we choose for the ansatz three dimensional \( SU(2) \) group representation which is the fundamental \( SO(3) \) group representation too. The radial dependent functions in (2.5) for such ansatz are as follows:

\[
D_{a,a'}^j(\hat{x}, F(r)) = \frac{2\sqrt{\pi}}{2j+1} w^1_j(F) \left[ \begin{array}{cc} j & l \\ a & m \end{array} \right] Y_{l,m}(\theta, \varphi),
\]

were the symbol in brackets denotes the \( SU(2) \) Clebsch - Gordan coefficient, \( Y_{l,m}(\theta, \varphi) \) are the spherical harmonics. The boundary conditions \( F(0) = \pi \) and \( F(\infty) = 0 \) ensure the winding (baryon) number \( B = 1 \) for all irreps \( j \) due to the reason that the classical Lagrangian and winding number have the same factor \( N = \frac{\pi}{2} j(j+1)(2j+1) \) which can be reduced \( [7] \). At this work we choose for the ansatz three dimensional \( SU(2) \) group representation which is the fundamental \( SO(3) \) group representation too. The radial dependent functions in (2.5) for such ansatz are as follows:

\[
w^0_1(F) = \sqrt{3}(3 - 4\sin^2 F), \\
w^1_1(F) = i2\sqrt{3}\sin 2F, \\
w^2_2(F) = -4\sin^2 F.
\]

In this case it is convenient to define the noncanonical bases of the \( SU(3) \) irrep states vectors by reduction chain on subgroup \( SU(3) \supset SO(3) \). As we shall see later the structure of the quantum skyrmion depends on a choice of bases for ansatz. For general \( SU(3) \) irreps \( (\lambda, \mu) \) the \( SO(3) \) subgroup parameters \( (L, M) \) and its multiplicity can be sorted out by different methods, see \( [14, 15] \). Here considered fundamental \( (1, 0) \) and adjoint \( (1, 1) \) representations of \( SU(3) \) group are multiplicity free. The relations between canonical bases (reduction chain \( SU(3) \supset SU(2) \)) vectors

\[
\left( \begin{array}{c} 1, 0 \\ z, j, m \end{array} \right)
\]

where hypercharge \( y = \frac{2}{3}(\mu - \lambda) - 2z \), and noncanonical bases state vectors

\[
\left( \begin{array}{c} 1, 0 \\ \lambda, \mu, \nu \end{array} \right)
\]

are straightforward:

\[
\left( \begin{array}{c} 1, 0 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{array} \right) = \left( \begin{array}{c} 1, 0 \\ 1, 1 \end{array} \right); \\
\left( \begin{array}{c} 1, 0 \\ 0, 0, 0 \end{array} \right) = \left( \begin{array}{c} 1, 0 \\ 1, 0 \end{array} \right); \\
\left( \begin{array}{c} 1, 0 \\ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \end{array} \right) = \left( \begin{array}{c} 1, 0 \\ 1, -1 \end{array} \right).
\]

The system of noncanonical \( SU(3) \) generator in terms of canonical generators \( J_{(z,l,m)}^{(1,1)} \)
which are defined in \cite{8} can be expressed as follows:

\[
\begin{align*}
J_{(1,1)} &= \sqrt{2} \left( J^{(1,1)}_{\left(\frac{1}{2}, \frac{1}{2} \right)} - J^{(1,1)}_{\left(-\frac{1}{2}, \frac{1}{2} \right)} \right), \\
J_{(1,-1)} &= \sqrt{2} \left( J^{(1,1)}_{\left(-\frac{1}{2}, -\frac{1}{2} \right)} + J^{(1,1)}_{\left(\frac{1}{2}, -\frac{1}{2} \right)} \right), \\
J_{(2,1)} &= -\sqrt{2} \left( J^{(1,1)}_{\left(\frac{1}{2}, \frac{1}{2} \right)} + J^{(1,1)}_{\left(-\frac{1}{2}, \frac{1}{2} \right)} \right), \\
J_{(2,-1)} &= -\sqrt{2} \left( J^{(1,1)}_{\left(-\frac{1}{2}, -\frac{1}{2} \right)} - J^{(1,1)}_{\left(\frac{1}{2}, -\frac{1}{2} \right)} \right),
\end{align*}
\]

They satisfy the commutation relations

\[
[J_{(L',M')}, J_{(L'',M'')} ] = -2\sqrt{3}\left[ \begin{array}{ccc} (1,1) & (1,1) & (1,1) \\ L' & L'' & L \end{array} \right] \left[ \begin{array}{ccc} L'' & L'' & L \\ M' & M' & M \\ M & M & M \end{array} \right] J_{(L,M)}. \tag{2.9}
\]

The coefficients on the rhs of (2.9) are \( SU(3) \) noncanonical isofactor and \( SU(2) \) Clebensch - Gordan coefficient. The relations between the noncanonical base state vectors and canonical state vectors for the adjoint representation (1, 1) are similar to relations of generators (2.5) only with difference in normalization factor \( \frac{1}{2} \) to keep the states vectors normalized.

3. SOLITON QUANTIZATION ON \( SU(3) \) MANIFOLD

We take the \( SO(3) \) skyrmion \cite{2.5} with \( j = 1 \) as the classical ground state \( U_0 \) for ansatz. The quantization of the Skyrme model can be achieved by means of collective coordinates \( q^\alpha(t) \)

\[
U(\hat{x}, F(r), q(t)) = A(q(t)) U_0(\hat{x}, F(r)) A^\dagger(q(t)). \tag{3.10}
\]

We shall consider the Skyrme Lagrangian quantum mechanically \textit{ab initio} \cite{4} and eight unconstraint angles \( q^\alpha(t) \) to be quantum variables. Because the ansatz \( U_0 \) doesn’t commute with all \( SU(3) \) generators the quantization is realized on eight parameter group manifold on the contrary to the usual seven-dimensional homogeneous space \( SU(3)/U(1) \) \cite{12}. The generalized coordinates \( q^\beta(t) \) and velocities \( (d/dt)q^\alpha(t) = \dot{q}^\alpha(t) \) satisfy the commutation relations

\[
[q^\alpha, q^\beta] = -if^{\alpha\beta}(q). \tag{3.11}
\]

Here the symmetric tensor \( f^{\alpha\beta}(q) \) is a function of coordinates \( q \) only. The explicit form of this tensor is determined after the quantization condition has been imposed.

After substitution of the ansatz \cite{2.5} into model Lagrangian density \cite{2.1} it takes a form which is quadratic concerning the velocities \( \dot{q}^\alpha \)

\[
L = \int L d^3x \approx \frac{1}{2} \dot{q}^\alpha \ddot{g}_{\alpha\beta}(q, F) \dot{q}^\beta + \left( \langle \dot{q}^\alpha \rangle^0 - \text{terms} \right), \tag{3.12}
\]

where the metric tensor

\[
g_{\alpha\beta}(q, F) = C_\alpha^{(L,M)}(q)E_{(L,M)}^{(L',M')}(F)C_\beta^{(L',M')}(q), \tag{3.13}
\]

and

\[
E_{(L,M)(L',M')}^{(L,M')} = (-1)^M a_L(F) \delta_{L,L'} \delta_{M,-M'}. \tag{3.14}
\]
The soliton moments of inertia are given as integrals over dimensionless variable \( \tilde{r} = \epsilon \bar{f}_r r \)

\[
a_1(F) = \frac{1}{e^3 \bar{f}_r} \frac{8\pi}{3} \int d\tilde{r} \tilde{r}^2 \sin^2 F \left[ 1 + F'^2 + \frac{1}{F^2} \sin^2 F \right],
\]

\[
a_2(F) = \frac{1}{e^3 \bar{f}_r} \frac{8\pi}{5} \int d\tilde{r} \tilde{r}^2 \left[ \sin^2 F \left( 3 + 2 \cos 2F + (9 + 8 \cos 2F) F'^2 \right) + (9 + 4 \cos 2F) \frac{\sin^2 F}{\tilde{r}^2} \right].
\]

The \( a_1(F) \) coincides to the \( SU(2) \) soliton momenta of inertia, nevertheless the \( a_2(F) \) differs from the second soliton momenta of inertia in the usual \( SU(3) \supset SU(2) \) case. The use of noncanonical \( SU(3) \) bases lead to momenta \( (3.15a, 3.15b) \) which contrast with \( SO(3) \) soliton momenta of inertia defined in [11].

The canonical momenta are defined as

\[
p_\beta = \frac{\partial L}{\partial \dot{q}_\beta} = \frac{1}{2} \left\{ \dot{q}^\alpha, g_{\alpha\beta} \right\}.
\]

They are conjugate to coordinates and satisfy the commutation relations \( [p_\beta, q^\alpha] = -i \delta_{\alpha\beta} \). The curl brackets in \( (3.16) \) denotes the anticommutator. The commutation relations and \( (3.16) \) fixe the explicit expressions of the functions in \( (3.11) \)

\[
f^{\alpha\beta}(q) = (g_{\alpha\beta}(q))^{-1}.
\]

The eight right transformation generators are defined as

\[
\hat{J}_{(L,M)} = \frac{i}{2} \left\{ p_\alpha, C_{(L,M)}^{(L',M')}(q) \right\} = \frac{i}{2} \left\{ \dot{q}_\beta, C_{\beta}^{(L',M')}(q) \right\} E_{(L',M')(L,M)}.
\]

They satisfy the commutation relations \( (2.39) \). Here the functions \( C_{(L,M)}^{(L',M')}(q) \) are dual to function defined in \( (2.39) \)

\[
\sum_{\alpha} C_{(L,M)}^{(L',M')}(q) C_{\alpha}^{(L',M')}(q) = \delta_{(L,M)(L',M')},
\]

\[
\sum_{L,M} C_{(L,M)}^{(L',M')}(q) C_{\alpha'}^{(L',M')}(q) = \delta_{\alpha\alpha'}.
\]

The action of right transformation generators on the Wigner matrix of \( SU(3) \) irrep is well defined

\[
\left[ \hat{J}_{(L,M)}, D_{(\alpha,L',M') \langle (\beta,L'',M'')}(q) \right] = \left\langle (\lambda,\mu)\alpha, L', M' \left| \hat{J}_{(L,M)} \right| (\lambda,\mu)\alpha_0, L_0, M_0 \right\rangle \times D_{(\alpha_0,L_0,M_0) \langle (\beta,L'',M'')}(q).
\]

The indices \( \alpha \) and \( \beta \) label the multiplets of \( (L,M) \). The substitution of the ansatz \( (3.10) \) into model Lagrangian density \( (2.1) \) and integration over spatial coordinates leads to effective Lagrangian in the form

\[
L_{\text{eff}} = \frac{1}{2a_2(F)} (-1)^M \hat{J}_{(L,M)} \hat{J}_{(L,-M)} + \left( \frac{1}{2a_1(F)} - \frac{1}{2a_2(F)} \right) \times (-1)^m \left( \hat{J}_{(1,m)} \cdot \hat{J}_{(1,-m)} \right) - M_0 - \Delta M_1 - \Delta M_2 - \Delta M_3,
\]
were $M_{cl}$ is classical skyrmion mass, $\Delta M_k = \int d^3x \Delta M_k(F)$ are quantum corrections to the semiclassical skyrmion mass:

$$\Delta M_1 = -\frac{2 \sin^2 F}{a_1^2(F)} \left[ f_1^2 (2 - \cos 2F) + \frac{3}{e^2} \left( F'^2 (2 + \cos 2F) + \frac{\sin^2 F}{r^2} \right) \right];$$  

$$\Delta M_2 = -\frac{2 \sin^2 F}{a_2^2(F)} \left[ f_1^2 (14 + 11 \cos 2F) + \frac{3}{e^2} \left( F'^2 (42 + 41 \cos 2F) \right. \right.$$
$$\left. + (25 + 12 \cos 2F) \frac{\sin^2 F}{r^2} \right];$$

$$\Delta M_3 = -\frac{4 \sin^2 F}{a_1(F) \cdot a_2(F)} \left[ f_1^2 (4 + \cos 2F) + \frac{3}{e^2} \left( F'^2 (6 + 5 \cos 2F) \right. \right.$$
$$\left. + (1 - \cos 2F) \frac{\sin^2 F}{r^2} \right].$$

Two operators in (3.21) are quadratic Casimir operators of $SU(3)$ and $SO(3)$ groups.

4. STRUCTURE OF THE HAMILTONIAN DENSITY AND THE SYMMETRY BREAKING TERM

To find explicit expression of Lagrangian and Hamiltonian density of the quantum skyrmion we take into account the explicit commutation relations (3.11) and (3.17). Some lengthy manipulation with ansatz yields the expression of Lagrangian density

$$\mathcal{L}_{SK} = \mathcal{K} - M_{cl} - \Delta M_1 - \Delta M_2 - \Delta M_3,$$

were the kinetic (operator) part of the Lagrangian density is as follows:

$$\mathcal{K} = \frac{4}{a_2^2(F)} (-1)^M \hat{J}_{(L,M),\hat{J}_{(L,M')}} \left\{ \frac{f_1^2}{4} \left( \delta_{-M,-M'} - D_{-M,M'}^L(\hat{x},F(r)) \right) \right. \right.$$
$$\left. \left. + \frac{3}{e^2} \left( \delta_{-M,-M'} - D_{-M,M'}^L(\hat{x},F(r)) \right) \right\} \left\{ \left( F'^2 - \frac{1}{r^2} \sin^2 F \right) \right. \right.$$
$$\left. \times 2 \sqrt{\pi} (2L + 1) \sqrt{\frac{L + 1}{2L + 1}} \times \left[ \frac{L}{M} \frac{L}{M'} \frac{l}{m} \right] Y_{l,m}(\theta,\varphi) + \frac{1}{r^2} \sin^2 F \frac{1}{(5 - 2L) \delta_{-M-MM'}} \right\}. \right.$$  

The coefficient in curl brackets are $6j$ coefficient of the $SU(2)$ group. In (4.24) the Wigner $D_{L,M}^L(\hat{x},F(r))$ matrices are used which in fact are hedgehog anzatz for irrep $L = 1, 2$ in (2.5). For representation $L = 2$ the radial dependent functions are:

$$w_0^2 = (5 - 20 \sin^2 F + 16 \sin^4 F),$$

$$w_1^2 = i \sqrt{2} (\sin 2F + 2 \sin 4F),$$

$$w_2^2 = -\frac{2 \sqrt{2 - 5}}{\sqrt{7}} (7 - 8 \sin^2 F) \sin^2 F,$$

$$w_3^2 = -i 4 \sqrt{2} \sin^2 F \sin 2F,$$

$$w_4^2 = \frac{8 \sqrt{2}}{\sqrt{7}} \sin^4 F.$$
The maximal spherical functions which appear in (4.24) are $Y_{4,m}(\theta, \phi)$.

We define the momentum density as $P_\beta = \frac{\partial L}{\partial \dot{q}_\beta}$. The kinetic energy density is defined as $2K = \frac{1}{2} \{ P_\beta, \dot{q}_\beta \}$. And the Skyrme model Hamiltonian density takes the form

$$H = \frac{1}{2} \{ P_\beta, \dot{q}_\beta \} - \mathcal{L}_{SK} = K + M_{cl} + \Delta M_1 + \Delta M_2 + \Delta M_3. \quad (4.26)$$

The operator (kinetic) part of Lagrangian (3.21) and kinetic part of corresponding Hamiltonian depend on quadratic Casimir operators of $SU(3)$ and $SO(3)$ groups which are constructed using right transformation generators (3.18). The eigenstates of the Hamiltonian $H = \int d^3x H$ are

$$\left| (\Lambda, \Theta)_{\alpha, \beta, S, N} \right> = \sqrt{\dim(\Lambda, \Theta)} D^*(\Lambda, \Theta)(\alpha, S, N)(\beta, S', N') |0\rangle, \quad (4.27)$$

were complex conjugate Wigner matrix elements of the $(\Lambda, \Theta)$ representation depends on eight quantum variables $q^\beta$. The indices $\alpha$ and $\beta$ label the multiplets of $SO(3)$ group. $|0\rangle$ denotes the vacuum state. Due to the structure of the density operator (4.24) the noncanonical soliton mass distribution has a complex but well defined tensorial structure which depends on radial functions $F(r)$ and spherical harmonics $Y_{l,m}(\theta, \phi)$ of order $l = 1, 2, 3, 4$.

The mass or energy functional of (4.27) state is as follows

$$M = \frac{2}{3a_2(F)} (\Lambda^2 + \Theta^2 + \Lambda \Theta + 3\Lambda + 3\Theta) + \left( \frac{1}{2a_1(F)} - \frac{1}{2a_2(F)} \right) S(S + 1) + M_{cl} + \Delta M_1 + \Delta M_2 + \Delta M_3. \quad (4.28)$$

In contrast to the positive impact of Casimir operators (quantum rotation) to the classical mass $M_{cl}$ the quantum corrections $\Delta M$ which appear from commutation relations are negative.

We take account of chiral symmetry breaking effects by introducing the term

$$\mathcal{M}_{SB} = \frac{1}{4N} f_x m_0^2 Tr(U + U^\dagger - 2), \quad (4.29)$$

which takes an explicit form

$$\mathcal{M}_{SB} = \frac{1}{2} f_x m_0^2 \sin^2 F. \quad (4.30)$$

In (4.29) we used the same normalization factor $N = 4$ which is defined for $SO(3)$ classical soliton $j = 1$.

The direct calculation shows that the Wess-Zumino-Witten term is equal to zero $L_{WZ} = 0$ for noncanonical embedded $SO(3)$ soliton.

5. CONCLUSION

In this paper we have considered a new ansatz for Skyrme model which is noncanonical embedded $SU(3) \supset SO(3)$ soliton. The strict canonical quantization of the soliton leads to new expressions of momenta of inertia and negative quantum corrections $\Delta M$. The quantum corrections which appear from commutation relations
compensate the effect of positive $SU(3)$ and $SO(3)$ "rotation" kinetic energy. The variation of quantum energy functional \([4.28]\) allow to find the stable solutions of quantum skyrmion even without symmetry breaking term. The shape of quantum skyrmion are not fixed like in semiclassical "rigid body" case and infinite tower of solutions for the higher representations \((\Lambda, \Theta)\) is absent. It means that the "fast rotation" destroys the quantum skyrmion. For details of canonical $SU(2)$ skyrmion quantization see \([7]\). The unitary field $U(x, t)$ for $SU(3)$ Skyrme model can be define in arbitrary irrep \((\lambda, \mu)\). The $SU(2)$ ($SO(3)$) ansatzes can be constructed as reducible representations of $SU(2)$ embedded into $SU(3)$ irrep \((\lambda, \mu)\) in different ways. It can generates different types of quantum skyrmions.

Acknowledgments

The authors would like to thank S. Ališauskas for discussions on $SU(3)$ non-canonical bases.

[1] T.H.R. Skyrme, Proc. Roy. Soc. A260, 127 (1961).
[2] S.L. Sondhi, A. Karlhede, S.A. Kivelson, and E.H. Rezayi, Phys. Rev. B 47 16419 (1993).
[3] U. A. Khawaja and H. Stoof, Nature (London) 411, 918 (2001).
[4] N. Shiiki and N. Sawado, Phys. Rev. D71, 104031 (2005).
[5] G.S. Adkins, C.R. Nappi, E. Witten, Nucl. Phys. B228, 552 (1983).
[6] K. Fujii, A. Kobushkin, N. Toyota, Phys. Rev. Lett. 58, 651 (1987), Phys. Rev. D 35, 1896 (1987).
[7] A. Acus, E. Norvaišas, D.O. Riska, Phys. Rev. C 57, 2597 (1998).
[8] D. Jurčiukonis, E. Norvaišas, D.O. Riska, J. Math. Phys. 46, 072103 (2005).
[9] H. Walliser, Nucl. Phys. A 548, 649 (1992).
[10] N. Manton and P. Sutcliffe, Topological Solitons (Cambridge University Press, Cambridge, 2004).
[11] A. P. Balachandran, F. Lizzi, V. G. J. Rodgers, Nucl. Phys. B256, 525 (1985).
[12] J. P. Elliott, Proc. Roy. Soc. A245, 128 (1958).
[13] E. Norvaišas, D.O. Riska, Phys. Scr. 50, 634 (1994).
[14] V.N. Tolstoy, Phys. Atom. Nucl. 69, 1058 (2006).
[15] S. Ališauskas, J. Phys. A: Math. Gen. 20, 1045 (1987).
[16] E.Witten, Nucl. Phys. B223, 422 (1983); B223, 433 (1983).