Optimal damping concept: features and applications

E. I. Veremey

Applied Mathematics and Control Processes Faculty, Saint Petersburg University, 7/9 Universitetskaya Emb., Saint Petersburg 199034, Russia
e_veremey@mail.ru

Abstract. This work presents new ideas related to the synthesis of nonlinear stabilizing control laws in the framework of optimization approach. The focus is done on the optimal damping concept, proposed by V.I. Zubov in the early 60s of the last century. This theory is used to reduce significant computational costs in solving optimal stabilization problems. The essential features of the aforementioned concept are taken into account, allowing the construction of new methods for practical synthesis of control systems with desired dynamic properties. Various modern aspects of the optimal damping theory’s practical implementation are discussed. Special attention is paid to the specific choice of the functional to be damped to provide the desirable stability and performance features of the closed-loop connection.

1. Introduction

The zero equilibrium stabilization for nonlinear and non-autonomous plants is one of the most important regimes for the operation of modern automatic control systems. Nonlinear time-dependent control of dynamic plants can now be treated as one of the most practically significant and theoretically considerable problems in the area of automatic control analysis and design. The main requirement is to ensure asymptotic stability and the desired performance of control processes. Many approaches associated with the design of feedback control laws for nonlinear and plants have already been extensively researched and presented. Various methods are widely used ([1, 2]): Bellman’s dynamic programming principle, Pontryagin’s maximum principle, prediction control technique (MPC) and others. However, the corresponding problems are extremely complex owing to the existence of many dynamical requirements, restrictions, and conditions to be satisfied by the use of control. There is a need to develop persistently analytical and numerical methods of the design of nonlinear controllers to improve their quality and reduce computational consumptions.

This work focuses on a different concept that can be applied to design stabilizing controllers using the theory of optimal damping (OD). This theory, which was first proposed and developed by V.I. Zubov in his works [3 – 5], provides effective analytical and numerical methods for control calculations with essentially reduced computational consumptions with respect to classical techniques.

The obvious advantage of OD approach is the ideological simplicity of the process of optimal feedback design. The capabilities of modern computer technologies now make it possible to pose a question about the broad practical application of this approach to improve the efficiency and performance of automatic control systems.

This work is devoted to taking into account the features of the theory of optimal damping in order to develop new methods to ensure its practical applicability for the synthesis of stabilizing controllers with the desired dynamic properties.
The paper is organized as follows. In Section 2, certain issues of optimal stabilization are discussed to formalize the practical requirements for the properties of the nonlinear control system. In Section 3, emphasis is given to the specific features of Zubov’s OD concept and its connection with the traditional optimization approach. Section 4 is devoted to particular cases of OD synthesis to illustrate the technique of its practical implementation. The most important ways of practical use of OD approach are presented in Section 5. Section 6 concludes the paper by discussing the overall results of this study.

2. The essence of the OD concept
We first consider a stabilization problem for a nonlinear dynamic control plant presented by the following system of ordinary differential equations:

\[ \dot{x} = f(x, u), \quad t \in [0, \infty), \]  

where the vectors \( x \in E^n \) and \( u \in E^m \) represent a state and a control action correspondingly. Here, the function \( f: E^{n+m} \to E^n \) is continuous with respect to all its arguments in the space \( E^{n+m} \). Let us suppose that \( f(0,0) = 0 \), i.e., the system (1) has zero equilibrium.

The essence of the aforementioned problem is to synthesize a nonlinear control law of the form

\[ u = u(x) \]

with the piecewise continuous function \( u(x) \) that maintains the zero equilibrium for the closed loop connection (1), (2). The zero equilibrium point must be locally (globally) asymptotically stable (LAS or GAS). The control \( u \) for any time \( t \) must belong to an admissible set \( U \subseteq E^m \), which is a metric compact set in the space \( E^m \).

The practical problem statements are usually formulated as certain additional requirements to be undeviatingly satisfied with the help of the controller (2). In most cases, the aforementioned requirements can be presented as follows:

\[ x(t,x^0,u(\cdot)) \in X, \quad \forall t \geq 0, \quad \forall x^0 \in B_r, \quad \forall u \in U, \]  

where the vector function \( x(t,x^0,u(\cdot)) \) is the motion of plant (1) closed by controller (2) under the initial condition \( x(0) = x^0 \). Herein, an admissible set \( X \) determines the complex of requirements to be satisfied. This set, in particular, can be determined by some constraints of the system's characteristics (transient time, overshoot, etc.). The set \( B_r \subseteq E^n \) is an \( r \)-neighborhood of the origin.

From the formalized point of view, it is not convenient to use condition (3) directly for the control synthesis. Accordingly, various optimization approaches are considered to reflect this condition indirectly. The most popular and widely used of them connects our opinion about the process performance with values of the certain integral functionals of the form

\[ J = J(u(\cdot)) = \int_0^\infty F_0(x,u)\, dt, \]  

where the subintegral function \( F_0 \) is positively definite, i.e. \( F_0 \geq 0 \forall x \in B_r, \forall u \in U \), \( F_0(x,u) = 0 \iff x = 0, u = 0 \).

It is possible to pose the following optimization problem (MIF - minimization of integral functional)

\[ J = J(u(\cdot)) \rightarrow \inf_{u \in U_0}, \quad u_{c0}(x) = \arg \inf_{u \in U_0} J(u(\cdot)), \quad J_0 = J(u_{c0}(\cdot)) \]

(5)

on the set \( U_0 \) of stabilizing controllers. One of the most effective methods of solving this problem is Bellman's dynamic programming technique, which allows obtaining the optimal MIF controller.
Let us notice especially that the computational scheme for the problem (7) solution is significantly simpler than that for the MIF one, and this is the main advantage of the OD approach. Let us notice that we have

\[ W = W(x, u) \rightarrow \min_{u \in U}, \quad u = u_\alpha(x) = \arg \min_{u \in U} W(x, u), \]  

where the function \( W \) determines the rate of change of the functional \( L \) on the motions of plant (1): 

\[ W(x, u) = (dL/dt)|_{x} = (\partial V(x)/\partial x)f(x, u) + F(x, u). \]

In accordance with the optimal damping concept, the more rapidly the functional (6) decreases based on the motions of the closed-loop connection, the more the process is improved. This idea ultimately leads to implementation of the inequality \( dV/dt|_{x} < 0 \), i.e., to ensure an asymptotic stability for the zero equilibrium position. Let us notice that the presence of an integral item in (6) inherently determines the penalty for a closed-loop system with the help of the function \( F \) associated with the performance of the motion. This approach is implemented by using the OD feedback control law \( u = u_\alpha(x) \) (OD controller).

Let us notice especially that the computational scheme for the problem (7) solution is significantly simpler than that for the MIF one, and this is the main advantage of the OD approach. Actually, the values \( u = u_\alpha(x) \) of the control action can be directly calculated not only analytically but also numerically based on the pointwise minimization of the function \( W(x, u) \) using the choice of \( u \) for current values of the variables \( t, x \).

It is very important that OD controller (7) has certain specific features, which should be used as a basis for practical feedback control laws synthesis implementation.

3. Specific features of the OD controller

The following statements are valid with respect to the OD problem.

1. If continuously differentiable function \( V \) in the functional (6) satisfies the condition \( \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \) \( \forall x \in B_r \), and the ratio \( W(x, u_{\alpha}(x)) \leq -\alpha_3(\|x\|) \) is true, where \( \alpha_1, \alpha_2, \alpha_3 \) are comparison Hahn functions from the class \( K \) (6), then function \( V(x) \) is control Lyapunov function (CLF) for plant (1), and the OD controller \( u = u_{\alpha}(x) \) provides asymptotic stability of the zero equilibrium point for the closed-loop system.

To prove this statement, let us note that we have

\[ W(x, u_{\alpha}(x)) = \min_{u \in U} W(x, u) = \min_{u \in U} \left[ \frac{dV}{dt}|_{x}(x, u) + F(x, u) \right] \leq -\alpha_3(\|x\|), \]

that determines the relationship

\[ \min_{u \in U} \frac{dV}{dt}|_{x}(x, u) \leq -\alpha_4(\|x\|) - \min_{u \in U} F(x, u) \leq -\alpha_4(\|x\|), \]
according to the features of subintegral function $F$. This means that $V(x)$ is CLF ([6]), and that the OD controller stabilizes the plant (1).

2. If we have some continuously differentiable function $V$, which is CLF for the plant (1), and satisfies the condition $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ $\forall x \in B_r$, then there exists the function $F(x,u)$ with aforementioned properties such that OD-controller (7) with respect to functional (6) asymptotically stabilizes zero equilibrium of the closed-loop system (1), (7).

For the proof, let us firstly note that for any positive definite function $F$ by virtue of (7), we have

$$W_0(t,x) := \min_{u \in U} W(x,u) \leq \frac{dV}{dt}(x,u) + F(x,u), \quad \forall x \in B_r,$$

where $u = \tilde{u}(x) = \arg \min_{u \in U} \{dV/dt\}(x,u)$. However, it follows from here that

$$\frac{dV}{dt}(x,u_d(x)) \leq \frac{dV}{dt}(x,\tilde{u}(x)) + F(x,\tilde{u}(x)) - F(x,u_d(x)), \quad \forall x \in B_r. \quad (8)$$

For the CLF $V$ we have $(dV/dt)(x,\tilde{u}(x)) \leq -\alpha_3(\|x\|), \forall x \in B_r$, and the positive definiteness of $F$ yields that $-F(x,u_d(x)) \leq 0$. Then, based on (8), we obtain

$$\frac{dV}{dt}(x,u_d(x)) \leq -\alpha_3(\|x\|) + F(x,\tilde{u}(x)) \forall x \in B_r. \quad (9)$$

Now, let us specify any function $\alpha_4 \in K$, such that $\alpha_4(\|x\|) < \alpha_3(\|x\|)$ $\forall x \in B_r$, and construct a function $F$ satisfying the conditions

$$0 \leq F(x,\tilde{u}(x)) \leq \alpha_3(\|x\|), \quad (10)$$

where $\alpha_3(\|x\|) = -\alpha_4(\|x\|) + \alpha_3(\|x\|) \geq 0$, $\alpha_3 \in K$. It is obvious that condition (10) determines a choice of the function $F(x,u)$: if (10) holds, then from (9), we obtain

$$\frac{dV}{dt}(x,u_d(x)) \leq -\alpha_4(\|x\|) \forall t \geq t_0, \forall x \in E^n,$$

for any function $\alpha_4 \in K$ that proves the statement.

3. Let us suppose that the MIF problem (5) has a unique solution, and let the control law $u = u_d(t,x)$ be a solution for the OD problem (7) with respect to functional (6) with the subintegral function $F(x,u) = F_0(x,u)$ and with function $V$, which is the value function for the MIF problem (i.e. $V$ is a solution for the correspondent Hamilton-Jacobi-Bellman equation). Then the controller $u = u_d(t,x)$ is simultaneously a solution for the MIF problem (5), i.e., $u_{\alpha_4}(t,x) \equiv u_d(t,x).

If the mentioned solution is not unique, then any OD controller can be taken as the MIF optimal feedback.

This statement can be proven using the same scheme as for analogous theorems from the works [3 − 5], where the correspondent problems where considered by V.I. Zubov with respect to integral functionals with finite limits.

4. Let the right-hand part of equation (1) admit expansion of the form $f(x,u) \equiv Ax + Bu + G(x,u)$ into the power series in the neighborhood of the zero point $(0,0)$, wherein the equality $\lim_{\|u\| \to 0} \frac{G(x,u)}{\|u\|} = 0$ is valid. Let us suppose that the pair $\{A, B\}$ is controllable. Then the quadratic form $V(x) = x^T S x$ can be used for the preceding statements in solving the corresponding
**OD problems, where \( Q \geq 0 \) and \( R > 0 \) are any symmetric matrices, matrix \( S \) is a solution of the Riccati equation \( A^T S + SA - S B R^{-1} B^T S = -Q \).

Really, it is a matter of simple calculations to prove that the quadratic form \( V(x) = x^T S x \) is CLF for the control plant (1), using a scheme proposed in [6].

### 4. Particular cases of the OD synthesis

To illustrate the technique of applying the OD approach, let us consider two particular cases of optimal damping problems, which are often used in practical applications.

First, we consider a linear-quadratic problem that is extremely significant for various branches of automatic control.

Let the plant be presented as follows:

\[
x = Ax + Bu,
\]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and pair \( \{A, B\} \) is controllable.

As a Lyapunov function candidate, we adopt a positively defined quadratic form \( V(x) = x^T P x \) with a symmetric matrix \( P > 0 \). Next, we address the OD problem (7) with the following functional

\[
L(t, x, u) = V(x) + \int_0^t u^T(\tau) Tu(\tau) d\tau = x^T P x + \int_0^t u^T(\tau) Tu(\tau) d\tau
\]

to be damped, where \( T > 0 \) is given symmetric matrix.

The rate of change of the functional \( L \) on the motions of plant (11) can be presented as follows:

\[
W(x, u) = \frac{dL(t, x, u)}{dt} \Bigg|_{(11)} = \frac{dV(x)}{dx} \dot{x}_{(11)} + u^T Tu = 2x^T P (Ax + Bu) + u^T Tu =
\]

\[
= 2x^T PAx + 2x^T PBu + u^T Tu.
\]

Using the necessary condition of extremum, we obtain

\[
\frac{dW(x, u)}{du} = 2x^T PB + 2u^T T = 0 \Leftrightarrow Tu + B^T P x = 0,
\]

hence we have an expression for the OD controller:

\[
u = u_{od}(x) = -T^{-1} B^T P x.
\]

This result can be discussed in connection with the MIF problem for the same plant (11) with widely used LQR functional

\[
J = \int_0^\infty (x^T Q x + u^T R u) dt, \quad R > 0, \ Q \geq 0.
\]

As is known, the solution of the MIF LQR problem with the functional (13) has the form

\[
u = u_{LQR}(x) = -R^{-1} B^T P x,
\]

where \( S \) is a solution for the algebraic Riccati equation \( A^T S + SA - S B R^{-1} B^T S = -Q \).

One can easily see that the controller (14) completely coincides with (12), if we accept for the aforementioned OD problem \( T = R, P = S \). This is in full compliance with statement 3 of the preceding Section.

Second, let us consider an affine-quadratic OD problem. Let us especially notice that among various dynamic plants of control, a special place is occupied by affine in control plants with nonlinear mathematical models. This primarily includes mobile objects such as robots, marine vehicles, aircrafts, and cars.
A mathematical model of the controlled plant is presented by the following affine-control nonlinear differential equations:

\[ \dot{x} = f(x) + g(x)u, \]  

where \( x \in \mathbb{E}^n, \ u \in \mathbb{E}^m, \ t \in [0, \infty), \ g := (g_1, g_2, \ldots, g_m), \ f, g_i : \mathbb{E}^n \to \mathbb{E}^n, i = 1, m, \) the functions \( f \) and \( g \) are continuously differentiable for any \( x \in B_j \).

As in the previous case, we adopt a positively defined quadratic form \( V(x) = x^T P x \) with a symmetric matrix \( P > 0 \), as a Lyapunov function candidate.

Next, let us consider the OD problem (7) with the functional

\[ L(t, x, u) = V(x) + \int_0^T u^T(\tau) Tu(\tau) d\tau = x^T P x + \int_0^T u^T(\tau) Tu(\tau) d\tau \]

to be damped, where \( T > 0 \) is given symmetric matrix.

The rate of the functional \( L \) change on the motions of plant (15) can be presented as follows:

\[ W = W(x, u) := \frac{dL}{dt}_{(15)} = \frac{dV}{dt}_{(15)} + u^T Tu = f^T P x + x^T Pf + x^T P x + 2x^T Pg u + u^T Tu. \]

Providing the extremum for the function \( W \), we have

\[ \frac{dW(t, x, u)}{du} = 2u^T T + 2x^T P g = 0 \iff Tu + g^T P x = 0, \]

hence we have an expression for the OD controller:

\[ u = u_d(x) = -T^{-1} g^T (x) P x. \]  

(16)

Note that as for the linear plant (11), feedback (16) will not be stabilizing for any \( P \) and \( Q \) matrices: we must verify that the conditions for the statement 1 from the previous Section are met.

5. About practical applications of the OD approach

Aforementioned features of the OD controllers make it possible to determine the following ways of practical use of the optimal damping concept.

a) Within statements 1 and 2, optimal damping feedback \( u = u_d(x) \) allows direct implementation in real time regime. This can be done most easily if the function \( u_d(x) \) is found analytically. Otherwise, we can search for its values for the current state \( x \) numerically: in particular, if the dimension of the vector \( x \) is not large, we can look for an approximation to solving the problem (7) on the finite set \( U_{ns} \subset U \).

b) If the MIF problem (5) has a certain self-sufficient significance with respect to the requirements (3), then the OD problem (7) can be used to find the approximate optimal MIF controller. Such an approach makes sense either if the direct solution of the problem (5) is too difficult, or if the optimal controller \( u = u_0(x) \) is not fully convenient for practical implementation.

c) The OD problem (7), without direct connection to any integral functional (4) and MIF problem (5), can be directly implemented to provide stability of the equilibrium and fulfillment of the requirements (3) for performance of control processes. This implementation is based on a certain choice of the functions \( V \) and \( F \) for the functional (6) to be damped.

One of the specific approaches to implement the presented practical ways is based on the parameterization of admissible sets of the aforementioned functions. Let us consider this approach with respect to the function \( F(x, u) \), supposing that the primary choice of \( V \) as a Lyapunov function candidate is done (preferably, as a CLF). At the same time, the subintegral function \( F \) should be varied to provide certain desirable features of the closed-loop connection. We assume that desired
properties are associated with a certain additional functional \( I = I(u) \), which is determined on the motions of the closed-loop connection.

Note that this idea originates from the following statement proven in [7]: Any CLF \( V(x) \) is a value function for certain performance index, i.e., this function satisfies the HJB equation associated with a corresponding integral functional.

Let us next consider the suggested OD oriented approach in detail. Suppose that function \( V(x) \) is assigned to the functional \( L(x,u) \) and that this function meets the aforementioned condition \( \alpha_1(\|x\|) \leq \alpha_2(\|x\|) \) \( \forall x \in B_r \). Let us introduce a certain class \( \mathcal{F}_k \) of positively definite functions \( F = F(x,u,h) \), which are parameterized by the vector \( h \in E^p \).

In particular, consider the quadratic form \( F = u^TQ(h)u \) with a positive definite symmetric matrix of the form \( Q = \text{diag}(h_1^2, h_2^2, \ldots, h_p^2) \), \( p = m \).

For any fixed vector \( h \in E^p \), one can specify a functional to be damped as follows:

\[
L = L(x,u,h) = V(x) + \int_0^T u^T Q(h)u \, dt,
\]
which determines a solution of OD problem (7) as

\[
u = u_d(x,h) := \arg\min_{u \in U} W(x,u,h).
\]

Suppose that there exists a metric compact set \( \mathcal{H}_c \subset E^p \) such that the condition is met

\[
W_{d,0}(x,h) := W(x,u_d(x,h),h) \leq -\alpha_3(\|x\|) \quad \forall x \in B_r, \quad \forall h \in \mathcal{H}_c,
\]
where \( \alpha_3 \in \mathbb{K} \). If this condition is valid, using statement 1, one can conclude that controller (17) stabilizes plant (1) for any parameter \( h \) from the set \( \mathcal{H}_c \).

Next, the following finite dimensional minimization problem

\[
I_d(h) := I(u_d(x,h)) \rightarrow \inf_{h \in \mathcal{H}_c}
\]
can be posed. If the vector \( h = \tilde{h} \in \mathcal{H}_c \) is obtained in the course of this problem solution, then the corresponding OD controller

\[
u = u_d(x,\tilde{h}) := \arg\min_{u \in U} W(x,u,\tilde{h})
\]
is asymptotically stabilizing one for the plant (1). This controller minimizes the additional functional \( I(u) \) on the set \( \mathcal{H}_c \), providing the desired control quality.

6. Conclusions
The aim of this work is to discuss some vital questions connected to various design applications of the modern optimization theory for the practical synthesis of nonlinear automatic control systems. The advantages of the optimization approach are widely known: in numerous practical applications, mathematical formalization can be carried out, leading to the problem of optimization.

Nevertheless, most such problems involve providing desirable dynamic features, usually presented in the form of (3). This allows one to attract different ideas for their formalization using Bellman’s theory [1] or Zubov’s optimal damping concept [3 – 5]. These approaches are closely connected, but the latter has certain advantages related to the practical requirements for the dynamic features of a closed-loop connection.

First of all, the numerical solution of the OD problem is considerably simpler than that of the MIF problem. This factor facilitates the fair formalization of functional choice considering the optimal
damping concept. This is one of the main issues discussed above, which is based on the fundamental coincidence of the mention problems’ solutions under the execution of certain conditions.

The paper considers the features of feedback synthesized on the basis of the optimal damping concept. The main ways of using these features for solving practical problems with nonlinear plants are proposed. Possible approaches for the implementation of these paths are indicated.

The results of the presented study can be developed and expanded taking into account output-feedback control laws synthesis, taking into account the transport delay for control and output, providing the desired measure of robustness for the closed-loop connection. The adaptation of the proposed approach is of particular importance for the practical control of marine vehicles.

Acknowledgments
This work was supported by the Russian Foundation for Basic Research (RFBR) (research project number 20-07-00531) controlled by the Government of the Russian Federation.

References
[1] Geering H P 2007 Optimal Control with Engineering Applications (Berlin, Heidelberg: Springer-Verlag)
[2] Lewis F L, Vrabie D L and Syrmos V L 2012 Optimal Control (Hoboken, NJ: John Wiley & Sons)
[3] Zubov V I 1962 Oscillations in Nonlinear and Controlled Systems (Leningrad: Sudpromgiz) (in Russian)
[4] Zubov V I 1966 Theory of Optimal Control of Ships and Other Moving Objects (Leningrad: Sudpromgiz) (in Russian)
[5] Zubov V I 1978 Theorie de la Commande (Moscow: Mir)
[6] Khalil H 2002 Nonlinear Systems (Englewood Cliffs, NJ: Prentice Hall)
[7] Freeman R A and Kokotović P V 1996 Inverse optimality in robust stabilization. SIAM J. of Control and Optimization 34 1365.