WEAK LAWS OF LARGE NUMBERS FOR SUBLINEAR EXPECTATION

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Abstract. In this paper we study the weak laws of large numbers for sublinear expectation. We prove that, without any moment condition, the weak laws of large numbers hold in the sense of convergence in capacity induced by some general sublinear expectations. For some specific sublinear expectation, for instance, mean deviation functional and one-side moment coherent risk measure, we also give weak laws of large numbers for corresponding capacity.

1. Introduction. The mathematical theory of non-additive set-functions or capacities got its first important contribution with Choquet’s Theory of Capacities in 1953, see [6]. Choquet’s interest was applications to statistical mechanics and potential theory. On the other hand capacities started to attract economists’ attention after the seminal contribution of [15] because of their applications to the study of cooperative games, also attract finance theorists’ attention for the integral with respect to capacities after the seminal work of [1] because of the assessing of the risks of financial positions.

The last decade has witnessed a steady increase in the study of the laws of large numbers for capacities, rather than probabilities. The authors have investigated different kinds of laws of large numbers for capacity or sublinear expectation. The resulting type of law of large numbers for capacities has the form that the averages do not generally converge to a point, but they are asymptotically confined in a limit set, which was already considered by some authors.

In [11], Maccheroni and Marinacci proved this kind of law of large numbers for totally monotone capacities under some other conditions that the space Ω be a complete separable metric space and random variables be continuous. In [12], Peng proved a law of large numbers under sublinear expectation by applying a deep interior estimate of fully nonlinear PDE. In [4], Chen proved a strong law of large numbers under sublinear expectation based on the limit result of [13], and soon
after, [5] improved the proof process. In [16], Terán proved laws of large numbers under some weaker topological condition when capacity is a completely monotone set function. Some other related works have been studied by [2, 3, 7] and therein.

In this paper, we will also prove weak type of laws of large numbers for capacities induced by the sublinear expectations. It is remarked that our assumption on capacity is not completely monotone capacity or 2-alternating capacity but only induced by sublinear expectation. The assumption for random variable

$$\lim_{n \to \infty} nV(|X| \geq n) = 0,$$

is sufficient for our weak laws large numbers for capacities. This is in accord with the classical weak laws large numbers for probability measures.

There are some interesting examples of sublinear expectation about risk measures in mathematical finance, for instance, upper expectation, Choquet integral with 2-alternating capacity, distortion risk measure with concave distortion function, mean deviation functional, one-sided moment coherent risk measure, et al. We also obtain some results for corresponding capacities induced by some specific sublinear expectation.

The paper is organized as follows. Section 2 describes some setting of the sublinear expectations, the corresponding capacities and the independence. Section 3 uses the independence to show weak laws of large numbers for capacities induced by general sublinear expectation under the conditions from strong to weak in the sublinear expectation theory. In Section 4, we give some weak laws of large numbers for corresponding capacities which be induced by mean deviation functional and one-sided moment coherent risk measure.

2. Preliminaries.

2.1. Setting of the sublinear expectation. Let \((\Omega, \mathcal{F})\) be a given measure space, \(\mathcal{H}\) be a linear space of real valued \(\mathcal{F}\)-measurable functions defined on \(\Omega\). Suppose that \(\mathcal{H}\) satisfies \(c \in \mathcal{H}\) for each constant \(c\) and \(|X| \in \mathcal{H}\) if \(X \in \mathcal{H}\). The space \(\mathcal{H}\) can be considered as the space of random variables.

**Definition 2.1.** A sublinear expectation \(E\) on \(\mathcal{H}\) is a functional \(E : \mathcal{H} \rightarrow \mathbb{R}\) satisfying the following properties:

(a): Monotonicity: \(X \geq Y\) implies \(E[X] \geq E[Y]\).

(b): Constant preserving: \(E[c] = c, \forall c \in \mathbb{R}\).

(c): Sub-additivity: \(E[X + Y] \leq E[X] + E[Y]\).

(d): Positive homogeneity: \(E[\lambda X] = \lambda E[X], \forall \lambda \geq 0\).

The triple \((\Omega, \mathcal{H}, E)\) is called a sublinear expectation space.

An equivalence form for property (d) is

\[E[\lambda X] = \lambda^+ E[X] + \lambda^- E[-X], \ \text{for any } \lambda \in \mathbb{R}.\]

Given a sublinear expectation \(E\), let us denote the conjugate expectation \(E^\ast\) of sublinear expectation \(E\) by

\[E^\ast[X] := -E[-X], \ \forall X \in \mathcal{H}.\]

Obviously, for all \(X \in \mathcal{H}\), \(E^\ast[X] \leq E[X]\). Furthermore, denote a pair \((V, v)\) of capacities, capacity is a from \(\mathcal{F}\) to \([0, 1]\) set function which is normalized and monotone in the sense that \(V(\emptyset) = 0, V(\Omega) = 1\) and \(V(A) \leq V(B)\) if \(A \subset B\) by

\[V(A) := E[I_A] \ \text{and} \ v(A) := E^\ast[I_A],\]
here, \( A \) belongs to the Borel \( \sigma \)-algebra \( F \) of \( \Omega \). It is easy to check that for any \( A \in F \),

\[ V(A) + v(A^c) = 1, \]

where \( A^c \) is the complement set of \( A \). From \( E[X + Y] \leq E[X] + E[Y] \) for \( X = I_A, X = I_B \), we can get that

\[ V(A \cup B) \leq V(A) + V(B). \]

In what follows we provide some concrete examples to illustrate the sublinear expectation.

**Example 1.** (Choquet integral) Let \( V : F \to [0, 1] \) be a capacity. The Choquet integral of a measurable function \( X \) with respect to \( V \) is defined as

\[ E_C[X] := \int_0^\infty X dV := \int_0^\infty (V(X > x) - 1)dx + \int_0^\infty V(X > x)dx. \]

It is easy to check that the Choquet integral satisfies the monotonicity, constant preserving and positive homogeneity. If \( V \) is submodular (or 2-alternating), i.e.,

\[ V(A \cap B) + V(A \cup B) \leq V(A) + V(B) \]

for all \( A, B \in F \), then the Choquet integral is sub-additivity, hence is a sublinear expectation.

**Example 2.** (Distortion risk measure) Distortion risk measure can be viewed as a Choquet integral with respect to the set function \( V = \psi(P(\cdot)) \), that is,

\[ E_D[X] := \int_0^\infty (\psi(P(X > x)) - 1)dx + \int_0^\infty \psi(P(X > x))dx. \]

If the distortion function \( \psi \) is concave, then the distortion risk measure is a sublinear expectation.

**Example 3.** (Mean deviation functional) Given a probability measure \( P \) in the measure space \((\Omega, F)\), let \( X \in L^1(\Omega, F, P) \) and consider mean deviation functional

\[ E_M[X] := E[X] + \lambda E[|X - E[X]|], \quad \text{for any } \lambda \in [0, 1]. \]

Here and hereafter, \( E[\cdot] \) refer to the classical expectation with respect to probability measure \( P \). It can be shown that the mean deviation functional is a sublinear expectation. In this case, we have a set function which is induced by the mean deviation functional:

\[ V(A) := \rho(I_A) = P(A)((1 + 2\lambda) - 2\lambda P(A)). \]

**Example 4.** (One-sided moment coherent risk measure) Let \( X \in L^p(\Omega, F, P) \), here given a probability measure \( P \) as in Example 3, and consider the so-called one-sided moment coherent risk measure

\[ E_O[X] := E[X] + \alpha(E[(X - E[X])^+])^{1/p}, \quad \text{for any } \alpha \in [0, 1] \text{ and } p \geq 1. \]

It can be shown that the one-sided moment coherent risk measure is a sublinear expectation. In this case, we have a set function which is induced by the one-sided moment coherent risk measure

\[ V(A) := \rho(I_A) = P(A) + \alpha(1 - P(A))(P(A))^{1/p}. \]
These examples are all viewed as coherent/convex risk measures, see [9, 14] for more information about these risk measures. In this paper, we provide these specific examples to illustrate the sublinear expectation regardless of the interrelation of these examples. In fact, under certain mild condition, there is a unified representation of the form as follows consists of some of these examples above like mean deviation functional and one-sided moment coherent risk measure,

$$\mathbb{E}[X] = \sup \left\{ \int_{(0,1]} AV_{\alpha}(X) d\mu(\alpha) : \mu \in \mathcal{M}_0 \right\},$$

here $\mathcal{M}_0$ is a subset of the set of all probability measures on $(0, 1]$, and

$$AV_{\alpha}(X) := \frac{1}{\alpha} \int_0^\alpha q_X(\alpha) d\alpha,$$

where $q_X$ denotes any quantile function of the distribution $F_X$ of $X$. It is noted that the subset $\mathcal{M}_0$ of the set of all probability measures on $(0, 1]$ may be vary based on the specified law-invariant coherent risk measures.

**Remark 1.** In fact, we can make further assumption: for each sequence $\{X_n \in \mathcal{H}, n = 1, 2, \cdots \}$ such that $X_n(\omega) \downarrow 0$ for any $\omega \in \Omega$, then $\mathbb{E}[X_n] \downarrow 0$. In this case, the sublinear expectation $\mathbb{E}$, also known as upper expectation, has the following representation:

$$\mathbb{E}[X] = \sup_{\theta \in \Theta} E_\theta[X] = \sup_{\theta \in \Theta} \int_\Omega X(\omega) dP_\theta(\omega),$$

for some subset of probability measure $\{P_\theta : \theta \in \Theta \}$, (see also [13]).

### 2.2. Independence.

In classical probability theory, the independence of random variables and the Fubini’s theorem imply that

$$E[\varphi(X,Y)] = E[E[\varphi(x,Y)]_{x=X}] = E[E[\varphi(X,y)]_{y=Y}],$$

where $E$ is the classical linear expectation with respect to $P$.

More precisely, we have the following statement:

**Proposition 1.** Given a probability space $(\Omega, \mathcal{F}, P)$, suppose that two random variables $X$ and $Y$ are independent and that $\varphi(x,y)$ is a Borel measurable function on $\mathbb{R}^2$ such that $E[|\varphi(X,Y)|] < \infty$. Then

$$E[\varphi(X,Y)] = E[E[\varphi(x,Y)]_{x=X}] \quad (1)$$

$$= E[E[\varphi(X,y)]_{y=Y}]. \quad (2)$$

**Remark 2.** There are two main reasons, for the most part, why the equalities in the above proposition hold: (i) we make assumption of the independence between $X$ and $Y$; (ii) The Fubini’s theorem holds in classical probability theory.

In the capacity theory, in general, the Fubini’s theorem does not hold unless some other condition (for instant, slice-comonotonic, see [10]) is assumed. In Appendix, we will give a simple example\(^1\) to illustrate that the following equality, in general, does not hold:

$$E[E[\varphi(x,Y)]_{x=X}] \neq E[E[\varphi(X,y)]_{y=Y}].$$

Motivated by the Proposition 1 and this example in Appendix, in this paper, we adopt the following notion of independence under sublinear expectation, see also [13].

\(^1\)In [13], another example is given to verify that $E[E[\varphi(x,Y)]_{x=X}] \neq E[E[\varphi(X,y)]_{y=Y}].$
Definition 2.2. In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random variable $Y \in \mathcal{H}$ is said to be independent to another random variable $X \in \mathcal{H}$ under $\mathbb{E}$ if for each measurable function $\varphi$ on $\mathbb{R}^2$ such that $\mathbb{E}[|\varphi(X, Y)|] < \infty$, we have
\[
\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)|x=X]].
\]

In fact, in this paper, we only need the following independence under the sublinear expectation:

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random variable $Y \in \mathcal{H}$ is said to be independent to another random variable $X \in \mathcal{H}$ under $\mathbb{E}$, if for any Borel set $D$ of $\mathbb{R}$, we have
\[
V(X + Y \in D) = \mathbb{E}[\mathbb{E}[I_{\{x+y \in D\}}|x=X]]; \quad (3)
\]
or if for each bounded continuous function $\varphi$ on $\mathbb{R}$, we have
\[
\mathbb{E}[\varphi(X + Y)] = \mathbb{E}[\mathbb{E}[\varphi(x + Y)|x=X]]. \quad (4)
\]

A sequence random variables $\{X_n, n = 1, 2, \cdots\}$ is said to be a sequence of independent random variables, if $X_{n+1}$ is independent of $(X_1, \cdots, X_n)$ for each $n$.

Remark 3. In fact, we can just assume that (3) holds for $D = [a, \infty)$ or $D = (-\infty, b]$. So, we can only suppose (3) holds for any interval $D$, since every interval is a Borel set. However, in this paper, we use the assumption (4) in the proof of our theorem since we don’t want to deviate from our main goal, the weak laws of large numbers.

We concluded this section by introducing the notion of identical distribution which has been introduced by Peng in [13]:

Definition 2.4. In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, two random variable, $X, Y \in \mathcal{H}$ are called identically distributed, if for each measurable function $\varphi$ on $\mathbb{R}$, we have
\[
\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)].
\]

3. Weak LLNs for general sublinear expectations. In general many classical results are false for the sublinear expectation. In this section, we try to establish the weak law of large numbers for capacities that induced by the general sublinear expectations in the sublinear expectation theory. The main way to proceed in trying to formulate this results is to restrict the cluster point of empirical averages lies in a interval, but in classical law of large numbers the cluster point of empirical averages is quite equal to the expectation of random variable. This claim is proved by the following weak type of laws of large numbers, which is established by assuming some mild technical conditions while the independence of random variables possesses the form (4).

Theorem 3.1. Given a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Let $\{X, X_i\}_{i=1}^\infty$ be a sequence of independent (4), identical distributed random variables under the sublinear expectation $\mathbb{E}$, such that
\[
\lim_{n \to \infty} nV(|X| \geq n) = 0, \quad \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[|X|^2I_{\{|X| \leq n\}}] = 0. \quad (5)
\]
Then, for any constant $\varepsilon > 0$,
\[
\lim_{n \to \infty} \mathbb{P}\left(-\mathbb{E}[-XI_{\{|X| \leq n\}}] - \varepsilon < \frac{1}{n} \sum_{i=1}^n X_i < \mathbb{E}[XI_{\{|X| \leq n\}}] + \varepsilon\right) = 1.
\]
Proof. To simplify the notation, we denote
\[ \mu_n := -\mathbb{E}[-X I_{\{|X| \leq n\}}], \quad \overline{p}_n := \mathbb{E}[X I_{\{|X| \leq n\}}] \]
and
\[ \varphi_n(x) := I_{\{x \geq \overline{p}_n + \varepsilon\}}. \]
It is easy to check that
\[ \sup_{\mu_n \leq x \leq \overline{p}_n} \varphi_n(x) = 0. \]
Taking \( \delta_n \in (\overline{p}_n, \overline{p}_n + \frac{\varepsilon}{4}) \). For each \( \varepsilon > 0 \), we construct a sequence of increasing functions \( \Phi_n(x) \in C^2_0(\mathbb{R}) \) by \(^2\)
\[ \Phi_n(x) := \begin{cases} 
0, & x \leq \overline{p}_n + \frac{\varepsilon}{4}; \\
(x - \overline{p}_n - \frac{\varepsilon}{4})^3, & \overline{p}_n + \frac{\varepsilon}{4} < x \leq \overline{p}_n + \frac{3\varepsilon}{4}; \\
g(x - \overline{p}_n), & \overline{p}_n + \frac{3\varepsilon}{4} < x \leq \overline{p}_n + \frac{5\varepsilon}{4}; \\
2 + (x - \overline{p}_n - \frac{7\varepsilon}{4})^3, & \overline{p}_n + \frac{5\varepsilon}{4} < x \leq \overline{p}_n + \frac{7\varepsilon}{4}; \\
x \geq \overline{p}_n + \frac{7\varepsilon}{4}.
\end{cases} \]
Obviously, for each \( n \), we have \( \Phi_n(\delta_n) = 0 \),
\[ \|\Phi_n''\| := \sup_x \Phi_n''(x) = C(\varepsilon) < \infty, \]
here \( C(\varepsilon) \) be a constant independent of \( n \), and
\[
\mathbb{E}
\left[
\varphi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)
\right]
- \sup_{\mu_n \leq x \leq \overline{p}_n} \varphi_n(x)
= \mathbb{E}
\left[
\varphi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)
\right]
\leq \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)
\right]
= \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)
\right]
- \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i + \frac{\delta_n}{n}\right)
\right]
+ \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i + \frac{2\delta_n}{n}\right)
\right]
+ \cdots
+ \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i + \frac{(n-2)\delta_n}{n}\right)
\right]
- \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i + \frac{(n-1)\delta_n}{n}\right)
\right]
+ \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} X_1 + \frac{(n-1)\delta_n}{n}\right)
\right]
- \Phi_n(\delta_n)
= \sum_{m=1}^{n}
\left[
\mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{m} X_i + \frac{n-m}{n} \delta_n\right)
\right]
- \mathbb{E}
\left[
\Phi_n\left(\frac{1}{n} \sum_{i=1}^{m-1} X_i + \frac{n-m-1}{n} \delta_n + \frac{1}{n} \delta_n\right)
\right]
\right].
\]

\(^2\)Here we can choose
\[ g(x) = -1440x^3 - 384x^2 + 720x - 1920x^4 - 1412x^3 - 3760x^5 + 1356x^3 - 3600x^2 \\
- 3 \cdot 3401x^3 - 9000x + \frac{1883x^3}{16} - \frac{621}{2}. \]
By (4), we get
\[
\mathbb{E} \left[ \Phi_n \left( \frac{1}{n} \sum_{i=1}^{m} X_i + \frac{n-m}{n} \delta_n \right) \right] 
\]
\[= \mathbb{E} \left[ \Phi_n \left( \frac{1}{n} \sum_{i=1}^{m-1} X_i + \frac{1}{n} X_m + \frac{n-m}{n} \delta_n \right) \right] 
\]
\[= \mathbb{E} \left[ \mathbb{E} \left[ \Phi_n \left( x + \frac{1}{n} X_m \right) \right] \right] 
\]
\[= \mathbb{E} \left[ \mathbb{E} \left[ \Phi_n \left( x + \frac{1}{n} X_m \right) \right] \right] 
\]

So, by the equality above, we have
\[
\mathbb{E} \left[ \Phi_n \left( \frac{1}{n} \sum_{i=1}^{m} X_i + \frac{n-m}{n} \delta_n \right) \right] 
\]
\[\leq \sup_{x \in \mathbb{R}} \left\{ \mathbb{E} \left[ \Phi_n \left( x + \frac{1}{n} X_m \right) \right] \right\} \]
\[= \sum_{x \in \mathbb{R}} \left\{ \mathbb{E} \left[ \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) \right] - \Phi_n(x) \right\} \]
\[= \sup_{x \in \mathbb{R}} \left\{ \mathbb{E} \left[ \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) \right] - \Phi_n(x) \right\} \] (6)

here, we have used the sublinear property of $\mathbb{E}$. For $\mathbb{E} \left[ \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) \right] - \Phi_n(x)$ in above (6), we have
\[
\mathbb{E} \left[ \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) - \Phi_n(x) \right] 
\]
\[\leq \mathbb{E} \left[ \left( \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) - \Phi_n(x) \right) I_{\{|X_m| \geq n\}} \right] 
\]
\[+ \mathbb{E} \left[ \left( \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) - \Phi_n(x) \right) I_{\{|X_m| \leq n\}} \right] 
\]
\[\leq \mathbb{E} \left[ \left( \left| \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) - \Phi_n(x) \right| \right) I_{\{|X_m| \geq n\}} \right] 
\]
\[+ \mathbb{E} \left[ \Phi_n(x) \frac{X_m - \delta_n}{n} I_{\{|X_m| \leq n\}} \right] + \Phi''_n(x) \frac{(X_m - \delta_n)^2}{2n^2} I_{\{|X_m| \leq n\}} \right] 
\]
\[\leq I_1 + I_2 + I_3, \] (7)

here
\[
I_1 := \mathbb{E} \left[ \left( \left| \Phi_n \left( x + \frac{X_m - \delta_n}{n} \right) - \Phi_n(x) \right| \right) I_{\{|X_m| \geq n\}} \right], 
\]
\[
I_2 := \mathbb{E} \left[ \Phi_n(x) \frac{X_m - \delta_n}{n} I_{\{|X_m| \leq n\}} \right] 
\]
\[
I_3 := \mathbb{E} \left[ \Phi''_n(x) \frac{(X_m - \delta_n)^2}{2n^2} I_{\{|X_m| \leq n\}} \right]. 
\]

For the term $I_1$, let $\|\Phi_n\| := \sup_x \Phi_n(x)$, we have $\|\Phi_n\| \leq C$ and
\[
I_1 \leq 2\|\Phi_n\| \mathbb{E} \left[ I_{\{|X_m| \geq n\}} \right] \leq CV(\{|X_m| \geq n\}) = CV(\{|X| \geq n\}). \] (8)

For the term $I_2$, by $\Phi'_n \geq 0$ and the positive homogeneity of sublinear expectation we have
\[
I_2 = \frac{\Phi'_n(x)}{n} \mathbb{E} \left[ (X_m - \delta_n) I_{\{|X_m| \leq n\}} \right]. 
\]
Here, $\|\Phi_n\'\| := \sup_x \Phi_n'(x)$ and $\|\Phi_n\'\| < C$ and by the subadditivity of sublinear expectation, we have

$$I_2 = \frac{\Phi_n'}{n} \mathbb{E}\left[X_m I(\{X_m \leq n\}) - \delta_n + \delta_n I(\{X_m \geq n\})\right]$$

$$\leq \frac{\Phi_n'}{n} \left\{ \mathbb{E}\left[X_m I(\{X_m \leq n\}) - \delta_n\right] + \mathbb{E}\left[\delta_n I(\{X_m \geq n\})\right] \right\}$$

$$\leq \frac{\Phi_n'}{n} \mathbb{E}\left[\delta_n I(\{X_m \geq n\})\right],$$

here, we have used the following fact

$$\mathbb{E}\left[X_m I(\{X_m \leq n\})\right] = \mathbb{E}\left[XI(\{|X| \leq n\})\right] = \overline{\mu} \leq \delta_n.$$

By $\overline{\mu}_n \leq \delta_n \leq \overline{\mu}_n + \frac{\varepsilon}{4}$, we have

$$|\delta_n| \leq |\overline{\mu}_n| + \frac{\varepsilon}{4} = |\mathbb{E}[X I(\{|X| \leq n\})]| + \frac{\varepsilon}{4} \leq |\mathbb{E}[n I(\{|X| \leq n\})]| + \frac{\varepsilon}{4} = nV(|X| \leq n) + \frac{\varepsilon}{4}.$$

Hence, we can choose large enough $n$ such that

$$I_2 \leq \frac{|\delta_n||\Phi_n'|}{n} \mathbb{E}\left[I(\{|X| \geq n\})\right]$$

$$\leq \frac{(n + \frac{\varepsilon}{4})||\Phi_n'||}{n} V(|X_m| \geq n)$$

$$\leq CV(|X_m| \geq n)$$

$$= CV(|X| \geq n).$$

(9)

For the term $I_3$, we have

$$I_3 \leq \frac{||\Phi_n'||}{2n^2} \mathbb{E}\left[(X_m - \delta_n)^2 I(\{|X_m| \leq n\})\right]$$

$$\leq \frac{||\Phi_n'||}{2n^2} \mathbb{E}\left[2(X_m^2 + \delta_n^2) I(\{|X_m| \leq n\})\right]$$

$$\leq \frac{||\Phi_n'||}{n^2} \left( \mathbb{E}\left[X_m^2 I(\{|X_m| \leq n\})\right] + \delta_n^2 \right).$$

Noting that

$$\delta_n^2 \leq 2 \left| \mathbb{E}[XI(\{|X| \leq n\})] \right|^2 + \frac{\varepsilon^2}{8} \leq 2 \mathbb{E}[X^2 I(\{|X| \leq n\})] + \frac{\varepsilon^2}{8},$$

hence, we get

$$I_3 \leq \frac{||\Phi_n'||}{n^2} \left(3 \mathbb{E}\left[X_m I(\{|X_m| \leq n\})\right] + \frac{\varepsilon^2}{8}\right) \leq \frac{||\Phi_n'||}{n^2} \left(3 \mathbb{E}\left[X^2 I(\{|X| \leq n\})\right] + \frac{\varepsilon^2}{8}\right).$$

(10)

Combining three inequalities above (8), (9), (10) with (7), we get

$$\mathbb{E}\left[\Phi_n\left(x + \frac{X_m - \delta_n}{n}\right) - \Phi_n(x)\right] \leq 2CV(|X| \geq n) + \frac{C}{n^2} \left(3 \mathbb{E}\left[X^2 I(\{|X| \leq n\})\right] + \frac{\varepsilon^2}{8}\right)$$

Hence, we get

$$\mathbb{E}\left[\varphi_n\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)\right] \leq \sup_{\mu_n \leq x \leq \overline{\mu}_n} \varphi(x)$$

$$\leq \sum_{m=1}^{n} \left\{2CV(|X| \geq n) + \frac{C}{n^2} \left(3 \mathbb{E}\left[X^2 I(\{|X| \leq n\})\right] + \frac{\varepsilon^2}{8}\right)\right\}$$

$$= 2CnV(|X| \geq n) + \frac{3C}{n} \mathbb{E}\left[X^2 I(\{|X| \leq n\})\right] + \frac{C\varepsilon^2}{8n^2}$$
\[ \rightarrow 0 \]
as \( n \to \infty \), since \( nV(|X| \geq n) \to 0 \) and \( \frac{1}{n}E\left[X^2I_{\{|X| \leq n\}}\right] \to 0 \) by (5).

It follows that
\[
\lim_{n \to \infty} \left\{ E\left[ \varphi_n\left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \right] - \sup_{\mu_n \leq x \leq \overline{\mu}_n} \varphi_n(x) \right\} \leq 0.
\]

Hence, using \( \varphi_n(x) = I_{\{x \geq \mu_n + \epsilon\}} \), we get
\[
\lim_{n \to \infty} V\left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq \mu_n + \epsilon \right) = 0.
\]

Similarly, consider \( \psi_n(x) = I_{\{x \leq \mu_n - \epsilon\}} \), we can obtain
\[
\lim_{n \to \infty} V\left( \frac{1}{n} \sum_{i=1}^{n} X_i \leq \mu_n - \epsilon \right) = 0.
\]

By \( v(A^c) + V(A) = 1 \) and \( V(A \cup B) \leq V(A) + V(B) \), we get
\[
\lim_{n \to \infty} v\left( \mu_n - \epsilon < \frac{1}{n} \sum_{i=1}^{n} X_i < \mu_n + \epsilon \right) = 1,
\]
and this is our desired result. This completes the proof. \( \square \)

**Theorem 3.2.** Given a sublinear expectation space \((\Omega, \mathcal{H}, E)\). Let \( \{X, X_i\}_{i=1}^{\infty} \) be a sequence of independent (4), identical distributed random variables under the sublinear expectation \( E \), such that
\[ E[X] = \overline{\mu} \quad \text{and} \quad \overline{-E[-X]} = \underline{\mu}. \]

Furthermore, we assume
\[ \lim_{n \to \infty} nV(|X| \geq n) = 0, \quad \lim_{n \to \infty} \frac{1}{n}E[|X|^2I_{\{|X| \leq n\}}] = 0, \]
then, for any constant \( \epsilon > 0 \),
\[ \lim_{n \to \infty} v\left( \mu_n - \epsilon < \frac{1}{n} \sum_{i=1}^{n} X_i < \mu_n + \epsilon \right) = 1. \]

**Proof.** The proof of this theorem follows along the same process as the proof of Theorem 3.1 with some minor changes. Therefore we omit the proof of this theorem. \( \square \)

**Remark 4.** (1) The conditions (5) are obviously weaker than the second moment hypothesis under sublinear expectation, that is \( E[|X|^2] < \infty \) implies (5) hold.

(2) The conditions (5) are, furthermore, weaker than the following condition
\[ \lim_{n \to \infty} E[|X|I_{\{|X| \geq n\}}] = 0. \]
This claim holds since
\[ nV(|X| \geq n) = E[nI_{\{|X| \geq n\}}] \leq E[|X|I_{\{|X| \geq n\}}] \]
and for any \( \epsilon > 0 \)
\[
\frac{1}{n}E[|X|^2I_{\{|X| \leq n\}}] \leq \frac{1}{n}E[|X|^2I_{\{\epsilon \sqrt{n} \leq |X| \leq n\}}]\]
In the following theorem, we will continue to weaken the conditions of the theorem 3.1 above.

**Theorem 3.3.** Given a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Let $\{X, X_i\}_{i=1}^{\infty}$ be a sequence of independent (4), identical distributed random variables under the sub-linear expectation $\mathbb{E}$, such that

$$\lim_{n \to \infty} n V(|X| \geq n) = 0.$$  

Then, for any constant $\varepsilon > 0$,

$$\lim_{n \to \infty} n \left( -\mathbb{E}[-X I_{|X| \leq n}] - \varepsilon < \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}[X I_{|X| \leq n}] + \varepsilon \right) = 1.$$  

**Proof.** Compare with the condition of Theorem 3.1, it suffices to show that

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[^{2}X I_{|X| \leq n}] = 0$$

provided $\lim_{n \to \infty} n V(|X| \geq n) = 0$. In fact, we have

$$\frac{1}{n} \mathbb{E}[^{2}X I_{|X| \leq n}] = \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^{n} |X|^{2} I_{|X| \leq i} \right]$$

$$\leq \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^{n} i^{2} I_{|X| \leq i} \right]$$

$$\leq \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^{n} \left( \sum_{j=1}^{i} 2j \right) I_{|X| \leq i} \right]$$

$$= \frac{1}{n} \mathbb{E} \left[ \sum_{j=1}^{n} 2j \sum_{i=j}^{n} I_{|X| \leq i} \right]$$

$$\leq \frac{1}{n} \mathbb{E} \left[ \sum_{j=1}^{n} 2j I_{|X| \geq j-1} \right]$$

$$= \frac{2}{n} \mathbb{E} \left[ \sum_{j=0}^{n-1} (j + 1) I_{|X| \geq j} \right]$$

$$\leq \frac{2}{n} \mathbb{E} \left[ 1 + 2 \sum_{j=1}^{n-1} j I_{|X| \geq j} \right]$$

$$\leq \frac{2}{n} + \frac{4}{n} \sum_{j=1}^{n-1} j \mathbb{E}[I_{|X| \geq j}]$$

$$= \frac{2}{n} + \frac{4}{n} \sum_{j=1}^{n-1} j V(|X| \geq j).$$

From this, we can get our desired results

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[X^2 I_{|X| \leq n}] = 0,$$

here we have used the fact if $\lim_{n \to \infty} x_n = 0$, then $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$.  \( \square \)
Remark 5. Compare with the classical weak law of large numbers in probability measure theory, the condition in Theorem 3.3 can be thought as the optimal in sublinear expectation theory frame, because the condition
\[nP(|X| > n) \to 0 \text{ as } n \to \infty\]
in the classical weak law of large numbers for probability is the same as our condition.

4. Weak LLNs for some sublinear expectations. In this section we turn to the examples in Section 2 and get some particular weak LLNs for the specific sublinear expectations. One result concerns the weak LLN for the mean deviation functional (Example 3) and another result concerns the weak LLN for the one-sided moment coherent risk measure (Example 4). Since both the mean deviation functional and the one-sided moment coherent risk measure are sublinear expectations, we can directly obtain the following results by using the theorem in Section 3.

As a further preparation for the following theorems, we first assume that a probability measure \(P\) is given and \(\{X, X_i\}_{i=1}^{\infty}\) is a sequence of independent, identically distributed random variables under a given probability measure space \((\Omega, \mathcal{F}, P)\).

**Theorem 4.1.** Consider the sublinear expectation
\[E_M[X] = E[X] + \lambda E[|X - E[X]|], \text{ for any } \lambda \in [0, \frac{1}{2}].\]
The capacity induced by this sublinear expectation is
\[V_M(A) = P(A)(1 + 2\lambda - 2\lambda P(A)).\]
If
\[\lim_{n \to \infty} nV_M(|X| \geq n) = 0,\]
then, for any constant \(\varepsilon > 0\) and \(\lambda \in [0, \frac{1}{2}]\),
\[\lim_{n \to \infty} v_M\left(E[X] - \lambda E[|X - E[X]|] - \varepsilon < \frac{1}{n} \sum_{i=1}^{n} X_i < E[X] + \lambda E[|X - E[X]|] + \varepsilon\right) = 1,
here, the set function \(v_M(A) := P(A)((1 - 2\lambda) + 2\lambda P(A)).\)

**Proof.** It is easy to check that if \(\{X, X_i\}_{i=1}^{\infty}\) are independent, identically distributed random variables under a given probability measure space \((\Omega, \mathcal{F}, P)\), then \(\{X, X_i\}_{i=1}^{\infty}\) are independent, identically distributed random variables under the sublinear expectation \(E_M\).

From \(E_M[X] = E[X] + \lambda E[|X - E[X]|]\), we get
\[-E_M[-X] = E[X] - \lambda E[|X - E[X]|],\]
and from \(V_M(A) = P(A)((1 + 2\lambda) - 2\lambda P(A))\), we get
\[v_M(A) = 1 - V_M(A^c) = P(A)((1 - 2\lambda) + 2\lambda P(A)).\]
Hence, we obtain our desired result by previous theorems. 

\(^{3}\) see [8] for details.
Theorem 4.2. Consider the sublinear expectation
\[ E_O[X] = E[X] + \alpha(1 - P(A))P(A)^{\frac{1}{2}}, \]
for any \( \alpha \in [0, 1] \).
The capacity induced by this sublinear expectation is
\[ V_O(A) = P(A) + \alpha(1 - P(A))P(A)^{\frac{1}{2}}. \]
If
\[ \lim_{n \to \infty} nV_O(|X| \geq n) = 0, \]
then, for any constant \( \varepsilon > 0 \) and \( \alpha \in [0, 1] \),
\[ \lim_{n \to \infty} v_O\left( E[X] - \alpha(1 - P(A))P(A)^{\frac{1}{2}} \right) - \varepsilon < \frac{1}{n} \sum_{i=1}^{n} X_i < E[X] + \alpha(1 - P(A))P(A)^{\frac{1}{2}} + \varepsilon \]
\[ = 1, \]
here, the set function \( v_O(A) := P(A)(1 - \alpha(1 - P(A)^{\frac{1}{2}})). \)

Proof. It is easy to check that \( \{X, X_i\}_{i=1}^{\infty} \) are independent, identically distributed random variables under the sublinear expectation \( E_O \).

From \( E_O[X] = E[X] + \alpha(E[(X - E[X])^+]^{\frac{1}{2}}) \), we get
\[ -E_O[-X] = E[X] - \alpha(E[(X - E[X])^-]^{\frac{1}{2}}), \]
and from \( V_O(A) = P(A) + \alpha(1 - P(A))(P(A)^{\frac{1}{2}}) \), we get
\[ V_O(A) = 1 - V_O(A^c) = 1 - P(A^c) - \alpha P(A)(1 - P(A)^{\frac{1}{2}}) = P(A)(1 - \alpha(1 - P(A)^{\frac{1}{2}})). \]
Hence, we obtain our desired result by previous theorems. \( \square \)

Remark 6. Observe that if \( v(A) = P(A) \) for all \( A \in \mathcal{F} \), then the results above also hold for probability measure \( P \), i.e., the classical weak LLN hold, for any \( \varepsilon > 0 \),
\[ \lim_{n \to \infty} P\left( E[X] - \varepsilon < \frac{1}{n} \sum_{i=1}^{n} X_i < E[X] + \varepsilon \right) = 1, \]
due to \( \lambda = 0 \) or \( \alpha = 0 \), while \( v(A) = P(A) \) for all \( A \in \mathcal{F} \).

Remark 7. We can explain the previous theorems as follows. There are many different opinions on the same thing from each people, hence there are many different type LLN along with different coherent risk measure.

Finally, let us consider a situation of model ambiguity where probability measure \( P \) is replaced by a whole class \( \mathcal{P} \) of probability measures on \((\Omega, \mathcal{F})\). For example, we assume that all probability measures \( P \in \mathcal{P} \) are equivalent to some reference probability measure \( P_0 \) on \((\Omega, \mathcal{F})\), and that the family of densities
\[ \left\{ \frac{dP}{dP_0} | P \in \mathcal{P} \right\} \]
is convex and weakly compact in \( L^1(\mathbb{R}) \). However, we can assume that there exists the set \( \mathcal{P} \) of probability measures which are not equivalent but even mutually singular.

For a probability measure \( P \) on \((\Omega, \mathcal{F})\), we can construct many sublinear expectations, for example,
\[ E_M^P[X] = E[X] + \lambda E[|X - E[X]|], \]
and
\[ E^P_O[X] = E[X] + \alpha(E[((X - E[X])^+)^\frac{1}{p})]. \]
Here, the superscript \(P\) indicates the dependence on a specific measure \(P \in \mathcal{P}\). In the face of model ambiguity, we consider the robust versions of \(E^P_M[X]\) and \(E^P_O[X]\) defined by
\[ E^P_M[X] := \sup_{P \in \mathcal{P}} \{E[X] + \lambda E[|X - E[X]|]\}, \quad \text{for any } \lambda \in [0, \frac{1}{2}] \]
and
\[ E^P_O[X] := \sup_{P \in \mathcal{P}} \{E[X] + \alpha(E[((X - E[X])^+)^\frac{1}{p})]\}, \quad \text{for any } \alpha \in [0, 1]. \]

Let \(\{X, X_i\}_{i=1}^\infty\) be a sequence of independent (4), identical distributed random variables under the sublinear expectations \(E^P_M\) and \(E^P_O\), respectively. Applying the same argument with Theorem 4 and Theorem 5, we also obtain the ensuing weak laws of large numbers for capacities \(V^P_M\) and \(V^P_O\) which be induced by the sublinear expectations \(E^P_M\) and \(E^P_O\), respectively.

**Theorem 4.3.** If
\[ \lim_{n \to \infty} nV^P_M(|X| \geq n) = 0, \]
then, for any constant \(\varepsilon > 0\) and \(\lambda \in [0, \frac{1}{2}]\),
\[ \lim_{n \to \infty} v^P_M \left( \inf_{P \in \mathcal{P}} \left\{E[X] - \lambda E[|X - E[X]|]\right\} - \varepsilon < \frac{1}{n} \sum_{i=1}^n X_i \right) < \sup_{P \in \mathcal{P}} \left\{E[X] + \lambda E[|X - E[X]|]\right\} + \varepsilon \right) = 1, \]
here, the set function \(v^P_M(A) := \inf_{P \in \mathcal{P}} \{P(A)\cdot(1 - 2\lambda) + 2\lambda P(A)\}\).

**Theorem 4.4.** If
\[ \lim_{n \to \infty} nV^P_O(|X| \geq n) = 0, \]
then, for any constant \(\varepsilon > 0\) and \(\alpha \in [0, 1]\),
\[ \lim_{n \to \infty} v^P_O \left( \inf_{P \in \mathcal{P}} \left\{E[X] - \alpha(E[|(X - E[X])^-|^\frac{1}{p})]\right\} - \varepsilon < \frac{1}{n} \sum_{i=1}^n X_i \right) < \sup_{P \in \mathcal{P}} \left\{E[X] + \alpha(E[|(X - E[X])^+|^\frac{1}{p})]\right\} + \varepsilon \right) = 1, \]
here, the set function \(v^P_O(A) := \inf_{P \in \mathcal{P}} \{P(A)(1 - \alpha(1 - P(A)\cdot\frac{1}{p})\}\}.

**Appendix.**

**Proof of Proposition 1.** For completion of this paper, we give a proof of Proposition 1:

**Proof.** Denote \(\mu(\cdot) = P(X \in \cdot)\) and \(\nu(\cdot) = P(Y \in \cdot)\). Then use the Fubini’s theorem, we have
\[
E[\varphi(X,Y)] = \int \int_{x \in \mathbb{R}, y \in \mathbb{R}} \varphi(x,y)\mu(dx)\nu(dy)
= \int_y \left( \int_{x \in \mathbb{R}} \varphi(x,y)\mu(dx) \right)\nu(dy)
\]
\[
\int_{y \in \mathbb{R}} \nu(dy) = \int_{y \in \mathbb{R}} \psi(y) \nu(dy) = E[\psi(Y)]
\]

with \( \psi(y) = E[\varphi(X, y)] \), hence the desired first equality (1) follows. Note that, from the Fubini’s theorem,

\[
\int_{y \in \mathbb{R}} \left[ \int_{x \in \mathbb{R}} \varphi(x, y) \mu(dx) \right] \nu(dy) = \int_{y \in \mathbb{R}} \left[ \int_{x \in \mathbb{R}} \varphi(x, y) \nu(dx) \right] \mu(dy),
\]

hence we have the second equality holds.

**An example.** We now give a simple example to illustrate that the following fact:

\[
E\left[E[\varphi(x, Y)|x=X]\right] \neq E[E[\varphi(X, y)|y=Y]].
\]

Here, we choose the function \( \varphi(x, y) = |x + y| \) for clarity. Let \( X \sim U[-1, 1], Y \sim g(U[-1, 1]) \), here the distorted function \( g(x) = \sqrt{x} \). Letting the sublinear expectation \( E \) be a Choquet expectation, be defined for a capacity \( V = g \circ P \), as

\[
E[\xi] = \int_{0}^{\infty} V(\xi \geq t)dt + \int_{-\infty}^{0} (V(\xi \geq t) - 1)dt.
\]

On the one hand, it is obviously to see that the Choquet expectation is quite equal to the classical expectation when the capacity \( V \) is a probability measure, that is the distorted function \( g(x) = x \). So, after some computation, we have

\[
E[|X + Y|] = E[\frac{1}{2}(1 + y^2)_{y=Y}] = \frac{1}{2}(1 + E[Y^2]) = \frac{23}{30},
\]

here, we have used the positive homogeneity and constant preservation of the Choquet expectation, and

\[
E[Y^2] = \int_{0}^{\infty} g \circ P(Y^2 > t)dt = \int_{0}^{1} g \circ P(Y > \sqrt{t} \text{ or } Y < -\sqrt{t})dt = \int_{0}^{1} g(1 - \sqrt{t})dt = \frac{8}{15}.
\]

On the other hand, we have

\[
E[|x + Y|] = \frac{1}{2} \int_{-1}^{1} \int_{0}^{\infty} g \circ P(|x + Y| > t)dt dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} \int_{0}^{2} g \circ P(Y > t - x \text{ or } Y < -t - x)dt dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} \int_{0}^{2} (P(Y > t - x) + P(Y < -t - x))\frac{1}{2} dt dx
\]

\[
= \frac{14}{15}.
\]
From (11) and (12), we easy to see that,

\[ \mathbb{E}[\mathbb{E}[|x + Y|_{x=X}] \neq \mathbb{E}[\mathbb{E}[|X + y|_{y=Y}]]. \]

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REFERENCES

[1] P. Artzner, F. Delbaen, J. M. Eber and D. Heath, Coherent measures of risk, Mathematical Finance, 9 (1999), 203–228.
[2] X. Chen and Z. Chen, Weak and strong limit theorems for stochastic processes under nonadditive probability, Abstract and Applied Analysis, 2014 (2014), Art. ID 645947, 7 pp.
[3] X. Chen, Strong law of large numbers under an upper probability, Applied Mathematics, 3 (2012), 2056.
[4] Z. Chen, Strong laws of large numbers for sub-linear expectations, Science China Mathematics, 59 (2016), 945–954.
[5] Z. Chen, P. Wu and B. Li, A strong law of large numbers for non-additive probabilities, International Journal of Approximate Reasoning, 54 (2013), 365–377.
[6] C. Gustave, Theory of capacities, Annales de l’institut Fourier, 5 (1953), 131–295.
[7] G. De Cooman and E. Miranda, Weak and strong laws of large numbers for coherent lower previsions, Journal of Statistical Planning and Inference, 138 (2008), 2409–2432.
[8] R. Durrett, Probability: Theory and Examples, Cambridge university press, 2010.
[9] H. Föllmer and A. Schied, Stochastic Finance: An Introduction in Discrete Time, Walter de Gruyter, 2011.
[10] P. Ghirardato, On independence for non-additive measures with a Fubini theorem, J. Econom. Theory, 73 (1997), 261–291.
[11] F. Maccheroni and M. Massimo, A strong law of large numbers for capacities, Annals of Probability, 33 (2005), 1171–1178.
[12] S. Peng, Law of large numbers and central limit theorem under nonlinear expectations, Sci. China Ser. A, 52 (2009), 1391–1411, arXiv:math/0702358.
[13] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty, preprint, arXiv:1002.4546.
[14] G. C. Pflug, W. Römisch, Modeling, Measuring and Managing Risk, Singapore: World Scientific, 2007.
[15] L. S. Shapley, A value for n-person games, Contributions to the Theory of Games, 2 (1953), 307–317.
[16] P. Terán, Laws of large numbers without additivity, Transactions of American Mathematical Society, 366 (2014), 5431–5451.

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