The long-time behaviour of a stochastic SIR epidemic model with distributed delay and multidimensional Lévy jumps

Driss Kiouach* and Yassine Sabbar

LPAIS Laboratory, Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco.

Abstract

In this article, taking into account distributed delay and multidimensional Lévy disturbances, we present an exhaustive study on the dynamic of the Susceptible-Infected-Removed (SIR) model. We aim to ameliorate the mathematical tools to obtain the long-run characteristics of the perturbed delayed model. Within this scope, we give sufficient conditions for two interesting asymptotic properties: extinction and the mean persistence of the epidemic. Numerical simulations on different Lévy disturbances are carried out to verify the obtained theoretical results.

Keywords: Distributed delay; epidemic model; Lévy jumps; extinction; persistence in the mean.

Mathematics Subject Classification: 92B05; 93E03; 93E15.

1. Introduction

Recently, considerable attention has been paid to the analysis of susceptible-infectious-recovered (SIR) type models, which have been constructed to describe the dissemination and the spread of some known infectious diseases [1]. In this epidemic model, it is often assumed that recovered individuals can get continuous immunity [2]. Many works have paid close attention to the characteristics of the long-term epidemic immunity [3, 4, 5]. To confer the realistic aspect of the epidemic model and make it biologically reasonable, numerous scholars considered the SIR epidemic model with time delay because an individual may not be infectious until some time after becoming infected [6, 7]. In the mentioned works, the time delay is assumed to be single-valued. The constant delay may be considered if the variation of the time is known exactly, which is not real for biological cases [8]. Considering the variable infectivity in the time interval yields a model with a distributed delay [9]. Therefore, it is more realistic to introduce a continuously distributed delay in biological modeling [10, 11]. Analyzing the characteristics of the SIR model with a distributed time delay still a rich subject that may deliver new comprehension on the propagation of epidemics which motivates our work. According to the approach of Muroya et al. [12], we introduce a delay kernel \(K(s)\) into the classical SIR epidemic model. We consider the delay Kernel \(K : [0, \infty) \to [0, \infty)\) as a normalized \(L^1\)-function where \(\int_0^{\infty} K(s)ds = 1\).

The average delay for the kernel \(K\) can be presented by the following quantity \(\int_0^{\infty} sK(s)ds\). Hence, the infection force at time \(t\) can be presented as the following form: \(\beta S(t) \int_{-\infty}^{0} K(t - s)I(s)ds\), where \(\beta\) denotes the transmission rate, \(S(t)\) and \(I(t)\) represent the fractions of susceptible and infective individuals at time \(t\). The SIR epidemic model with distributed delay can be expressed as follows:

\[
\begin{aligned}
\frac{dS(t)}{dt} &= A - \mu_1 S(t) - \beta S(t) \int_{-\infty}^{0} K(t - s)I(s)ds, \\
\frac{dI(t)}{dt} &= \beta S(t) \int_{-\infty}^{0} K(t - s)I(s)ds - (\mu_2 + \gamma)I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu_3 R(t),
\end{aligned}
\]

(1)

where \(R(t)\) is the fraction number of recovered populations at time \(t\). The parameter \(A\) is the recruitment rate of susceptible individuals corresponding to births and immigration. \(\mu_1, \mu_2\) and \(\mu_3\) are the natural death rates. \(\gamma\) is the rate of individuals leaving the infected compartment \(I\) to the recovered compartment \(R\). The threshold number of the deterministic system (1) is \(\mathcal{R}_0 = \frac{A\beta}{\mu_1(\mu_2 + \gamma)}\) which determines the persistence or the extinction of the epidemic. Many studies showed that the deterministic epidemic model (1) is suitable to describe the transmission process of some

* Corresponding author.

E-mail addresses: d.kiouach@uiz.ac.ma (D. Kiouach), yassine.sabbar@edu.uiz.ac.ma (Y. Sabbar)

Preprint submitted to March 19, 2020
known epidemics such as Rubella, Whooping cough, Measles, and Smallpox. Despite their advantages, those studies ignore the random fluctuations and perturbations which can affect the dissemination of an epidemic [13]. Therefore, the stochastic delayed SIR epidemic can be an accurate tool to predict the long-run dynamics of infectious epidemics [14, 15, 16, 17, 18, 19, 20]. In [21], the authors inserted the stochastic perturbation in the model (1). They studied the following stochastic system

\[
\begin{align*}
\frac{dS(t)}{dt} &= \left( A - \mu_1 S(t) - \beta S(t) \int_{-\infty}^{0} K(t-s)I(s)ds \right) dt + \sigma S(t) dW(t), \\
\frac{dI(t)}{dt} &= \left( \beta S(t) \int_{-\infty}^{0} K(t-s)I(s)ds - (\mu_2 + \gamma) I(t) \right) dt, \\
\frac{dR(t)}{dt} &= \left( \gamma I(t) - \mu_3 R(t) \right) dt,
\end{align*}
\]

where \( W(t) \) is the standard Brownian motion with intensity \( \sigma > 0 \). Specifically, they proved the existence and uniqueness of a stable stationary distribution to the model (2). Then, they established sufficient conditions for the extinction of a disease. In this paper, we will develop and generalize the stochastic model proposed in [21]. So, we aim to describe the strong fluctuations by introducing a Lévy jump process into the dynamical model (1). As far as we know, the jump-diffusion decomposition can describe the phenomena that cause a big jump to occur occasionally [22, 23]. As a result, the model (1) becomes:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \left( A - \mu_1 S(t) - \beta S(t) \int_{-\infty}^{0} K(t-s)I(s)ds \right) dt + \sigma_1 S(t) dW_1(t) + \int_{\mathcal{Z}} \lambda_1(u) S(t^-) \tilde{N}(dt, du), \\
\frac{dI(t)}{dt} &= \left( \beta S(t) \int_{-\infty}^{0} K(t-s)I(s)ds - (\mu_2 + \gamma) I(t) \right) dt + \sigma_2 I(t) dW_2(t) + \int_{\mathcal{Z}} \lambda_2(u) I(t^-) \tilde{N}(dt, du), \\
\frac{dR(t)}{dt} &= \left( \gamma I(t) - \mu_3 R(t) \right) dt + \sigma_3 R(t) dW_3(t) + \int_{\mathcal{Z}} \lambda_3(u) R(t^-) \tilde{N}(dt, du),
\end{align*}
\]

where \( S(t^-), I(t^-) \) and \( R(t^-) \) are the left limits of \( S(t), I(t) \) and \( R(t) \), respectively. \( W_i(t) \) \((i = 1, 2, 3)\) are independent Brownian motions and \( \sigma_i > 0 \) \((i = 1, 2, 3)\) are their intensities. \( N \) is a Poisson counting measure with compensating martingale \( \tilde{N} \) and characteristic measure \( \nu \) on a measurable subset \( \mathcal{Z} \) of \((0, \infty)\) satisfying \( \nu(\mathcal{Z}) < \infty \). \( W_i(t) \) \((i = 1, 2, 3)\) are independent of \( N \). We assumed that \( \nu \) is a Lévy measure such that \( \tilde{N}(dt, du) = N(dt, du) - \nu(du)dt \). The bounded function \( \lambda : \mathcal{Z} \times \Omega \to \mathbb{R} \) is \( \mathcal{F}_t \)-measurable and continuous with respect to \( \nu \).

Since the random variable \( R(t) \) does not appear in the equations of \( S(t) \) and \( R(t) \), it is sufficient to analyze the dynamic behavior of the following stochastic model:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \left( A - \mu_1 S(t) - \beta S(t) \int_{-\infty}^{0} K(t-s)I(s)ds \right) dt + \sigma_1 S(t) dW_1(t) + \int_{\mathcal{Z}} \lambda_1(u) S(t^-) \tilde{N}(dt, du), \\
\frac{dI(t)}{dt} &= \left( \beta S(t) \int_{-\infty}^{0} K(t-s)I(s)ds - (\mu_2 + \gamma) I(t) \right) dt + \sigma_2 I(t) dW_2(t) + \int_{\mathcal{Z}} \lambda_2(u) I(t^-) \tilde{N}(dt, du),
\end{align*}
\]

To establish the stochastic characteristics of the model (4) and due to biological considerations [21], we consider the following delay kernel with Gamma distribution:

\[
K(s) = \frac{s^n \eta^{n+1} e^{-\eta s}}{n!}, \quad s \in (0, \infty)
\]

where the constant \( \eta > 0 \) is the rate of exponential fading memory, which means the retrogradation of the effect of past memories. In this paper, we consider the low kernel function \( K \) with \( n = 0 \). By using the linear chain approach [21], we obtain the following equation

\[
dD(t) = \eta (I(t) - D(t)) dt,
\]

where \( D(t) = \int_{-\infty}^{t} \eta e^{-\eta(t-s)} I(s) ds \). In order to address the effects of delay in epidemic spreading, we consider the perturbed component \( D(t) \) in the model (4). Hence, the system (4) can be transformed into the following equivalent system:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \left( A - \mu_1 S(t) - \beta S(t) D(t) \right) dt + \sigma_1 S(t) dW_1(t) + \int_{\mathcal{Z}} \lambda_1(u) S(t^-) \tilde{N}(dt, du), \\
\frac{dI(t)}{dt} &= \left( \beta S(t) D(t) - (\mu_2 + \gamma) I(t) \right) dt + \sigma_2 I(t) dW_2(t) + \int_{\mathcal{Z}} \lambda_2(u) I(t^-) \tilde{N}(dt, du), \\
\frac{dD(t)}{dt} &= \eta (I(t) - D(t)) dt + \sigma_4 D(t) dW_4(t) + \int_{\mathcal{Z}} \lambda_4(u) D(t^-) \tilde{N}(dt, du),
\end{align*}
\]

where \( W_4(t) \) is an independent Brownian motion with the intensity \( \sigma_4 > 0 \), and \( \lambda_4 \) denotes the Lévy jump intensity. After presenting the model of our study and its parameters, we shall clarify the main contributions of this paper in the following items:
1. In order to investigate the sufficient condition of the disease extinction in the model (5), we applied a new approach based on the stochastic comparison theorem and the average property of the positive solution to the following subsystem:

\[
\begin{align*}
\begin{cases}
    d\psi(t) &= (A - \mu_1\psi(t))dt + \sigma_1\psi(t)dW_1(t) + \int_Z \eta_1(u)\psi(t^-)d\tilde{N}(dt, du) \quad \forall t > 0 \\
    \psi(0) &= S(0) > 0.
\end{cases}
\end{align*}
\] (6)

Without using the stationary distribution of the auxiliary process (6), we will directly compute the values of the averages \( \frac{1}{t} \int_0^t \psi(s)ds \) and \( \frac{1}{t} \int_0^t \psi(s)ds \) which can close the gap left by using the classical method presented for example in [24].

2. For the purpose of well analyzing the dynamics of the delayed model (5), we give a sufficient condition of the disease persistence.

Throughout this paper, we let \((\Omega, \mathcal{F}, \mathbb{P})\) denotes a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying these conditions: right continuous and \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets. We assume that \(W_i(t)\) be defined on this probability space.

This work is organized as follows. In section 2, we verify the well-posedness of the stochastic model (5). In section 3, we give a sufficient condition for the extinction of the disease. In section 4, we give a sufficient condition for the persistence in the mean of the disease. Finally, in section 5, numerical simulations are carried out to confirm the theoretical study.

2. Well-posedness of the model and useful lemmas

For the purpose of well analyzing our model (5), it necessary that we make the following standard assumptions:

- (A1) We assume that for a given \(K > 0\), there exists a constant \(L_K > 0\) such that

\[
\int_Z |F_i(x, u) - F_i(y, u)|^2\nu(du) < L_K|x - y|^2, \quad \forall |x| \vee |y| \leq K,
\]

where \(F_i(x, u) = x\eta_i(u)\) (i = 1, 2, 4).

- (A2) \(\forall u \in Z\), we assume that \(1 + \eta_i(u) > 0\), (i = 1, 2, 4) and \(\int_Z (\lambda_i(u) - \ln(1 + \lambda_i(u)))\nu(du) < \infty\).

- (A3) We suppose that exists a constant \(\kappa > 0\), such that \(\int_Z (\ln(1 + \lambda_i(u)))^2\nu(du) \leq \kappa < \infty\).

- (A4) We suppose that exists a constant \(\kappa_2 > 0\), such that \(\int_Z ((1 + \lambda_i(u))^2 - \lambda_i(u))^2\nu(du) \leq \kappa_2 < \infty\).

- (A5) We assume that for some \(p > 2\), \(\chi_1 = \min\{\mu_1, \mu_2\} - \frac{1}{2} \max\{\sigma_1^2, \sigma_2^2, \sigma_4^2\} - \frac{1}{p} \ell > 0\), where

\[
\ell = \int_Z (1 + \eta_1(u) \vee \eta_2(u) \vee \eta_4(u))^2p - 1 - \eta_1(u) \wedge \eta_2(u) \wedge \eta_4(u)\nu(du) < \infty.
\]

By the assumption (A1), the coefficients of the system (5) are locally Lipschitz continuous, then for any initial value \((S(0), I(0), D(0)) \in \mathbb{R}_+^3\) there is a unique local solution \((S(t), I(t), D(t))\) on \([0, \tau_c)\), where \(\tau_c\) is the explosion time. In the following theorem, our goal is to show that the solution is positive and global.

**Theorem 2.1.** For any initial value \((S(0), I(0), D(0)) \in \mathbb{R}_+^3\), there exists a unique positive solution \((S(t), I(t), D(t))\) of the system (5) on \(t \geq 0\), and the solution will stay in \(\mathbb{R}_+^3\) almost surely.

**Proof.** We prove that \(\tau_c = \infty\) a.s. Let \(\epsilon_0 > 0\) be sufficiently large, such that \(S(0), I(0), R(0)\) lie within the interval \([\frac{1}{\epsilon_0}, \epsilon_0]\). For each integer \(\epsilon \geq \epsilon_0\), we define the following stopping time:

\[
\tau_c = \inf \left\{ t \in [0, \tau_c) : \min\{S(t), I(t), R(t)\} \leq \frac{1}{\epsilon} \text{ or } \max\{S(t), I(t), R(t)\} \geq \epsilon \right\}.
\]

Evidently, \(\tau_c\) is increasing as \(\epsilon \rightarrow \infty\). Set \(\tau_\infty = \lim_{\epsilon \rightarrow \infty} \tau_c\), whence \(\tau_\infty \leq \tau_c\). If we can prove that \(\tau_\infty = \infty\) a.s., then \(\tau_c = \infty\) and the solution \((S(t), I(t), R(t)) \in \mathbb{R}_+^3\) for all \(t \geq 0\) almost surely. Specifically, we need to prove that \(\tau_\infty = \infty\) a.s. Suppose the opposite, then there is a pair of positive constants \(T > 0\) and \(k \in (0, 1)\) such that \(\mathbb{P}\{\tau_\infty \leq T\} > k\). Hence, there is an integer \(\epsilon_1 \geq \epsilon_0\) such that

\[
\mathbb{P}\{\tau \leq T\} \geq k \quad \text{for all} \quad \epsilon \geq \epsilon_1.
\] (7)
Define a $C^2$-function $V : \mathbb{R}_+^3 \to \mathbb{R}_+$ by

$$V(S, I, R) = \left( S - m - m \ln \frac{S}{m} \right) + (I - 1 - \ln I) + \frac{\mu_2 + \gamma}{\eta} (D - 1 - \ln D),$$

where $m > 0$ is a positive constant to be determined later. Obviously, this function is nonnegative which can be seen from $x - \ln x > 0$ for $x > 0$.

For $0 \leq t \leq \tau_\epsilon + T$, using Itô’s formula, we obtain that

$$dV(S, I, R) = LV(S, I, R)dt + \left(1 - \frac{m}{S}\right) \sigma_1 SdW_1(t) + \left(1 - \frac{1}{I}\right) \sigma_2 IdW_2(t)$$

$$+ \frac{\mu_2 + \gamma}{\eta} (1 - \frac{1}{R}) \sigma_4 DdW_4(t) + \int_{\mathbb{R}} \left\{ \frac{\lambda_1(u)S(t^-) - m \ln(1 + \lambda_1(u))}{\lambda_2(u)I(t^-) - \ln(1 + \lambda_2(u)) + \frac{\mu_2 + \gamma}{\eta} (\lambda_4(u)D(t^-) - \ln(1 + \lambda_4(u)))} \right\} \tilde{N}(dt, du),$$

where,

$$LV(S, I, R) = A - \mu_1 S - \frac{m \lambda_1(u) - m \lambda_2(u) - \ln(1 + \lambda_1(u))}{\lambda_2(u) - \ln(1 + \lambda_2(u))}$$

$$+ \frac{\mu_2 + \gamma}{\eta} \sigma_4 D + m\beta \frac{S}{I} + m \beta D + m \mu_1 - (\mu_2 + \gamma) I - \frac{\beta SD}{I}$$

$$+ (\mu_2 + \gamma) + (\mu_2 + \gamma) I - (\mu_2 + \gamma) D - \frac{\mu_2 + \gamma}{D} + (\mu_2 + \gamma)$$

$$+ m \sigma_1^2 + m \sigma_2^2 + \frac{\mu_2 + \gamma}{\eta} \sigma_4^2 + \int_{\mathbb{R}} \left\{ \frac{m \lambda_1(u) - m \lambda_2(u) - \ln(1 + \lambda_2(u))}{\lambda_4(u) - \ln(1 + \lambda_4(u))} + \frac{\mu_2 + \gamma}{\eta} \right\} \mu(du).$$

Then

$$LV(S, I, R) \leq A + 2(\mu_2 + \gamma) + m \mu_1 + (m \beta - (\mu_2 + \gamma)) D$$

$$+ \frac{m \sigma_1^2}{2} + \frac{m \sigma_2^2}{2} + \frac{\mu_2 + \gamma}{\eta} \sigma_4^2 + \int_{\mathbb{R}} \left\{ \frac{m \lambda_1(u) - m \lambda_2(u) - \ln(1 + \lambda_2(u))}{\lambda_4(u) - \ln(1 + \lambda_4(u))} + \frac{\mu_2 + \gamma}{\eta} \right\} \mu(du).$$

Given the fact that $x - \ln(1 + x) \geq 0$ for all $x > 1$ and the hypothesis $(A_2)$, we define

$$J_1 = \int_{\mathbb{R}} \left\{ \frac{m \lambda_1(u) - m \lambda_2(u) - \ln(1 + \lambda_2(u))}{\lambda_4(u) - \ln(1 + \lambda_4(u))} + \frac{\mu_2 + \gamma}{\eta} \right\} \mu(du).$$

To simplify, we choose $m = \frac{\mu_2 + \gamma}{\beta}$, then we obtain

$$LV(S, I, R) \leq A + 2(\mu_2 + \gamma) + m \mu_1 + \frac{m \sigma_1^2}{2} + \frac{m \sigma_2^2}{2} + \frac{\mu_2 + \gamma}{\eta} \sigma_4^2 + J_1 \equiv J_2.$$

The proof of the remainder is similar to the proof [25, 26], so we omitted it.

For convenience, we introduce the following lemmas which will be used later.

**Lemma 2.2.** [26] Let $(S(t), I(t), D(t))$ be the positive solution of the system (5) with any given initial condition $(S(0), I(0), D(0)) \in \mathbb{R}_+^3$. Let also $(\psi(t)) \in \mathbb{R}_+$ be the solution of the equation (6) with any given initial value $\psi(0) = S(0) \in \mathbb{R}_+$. Then

1. $\lim_{t \to \infty} \frac{\psi(t)}{t} = \lim_{t \to \infty} \frac{S(t)}{t} = \lim_{t \to \infty} \frac{I(t)}{t} = \lim_{t \to \infty} \frac{D(t)}{t} = 0 \text{ a.s.}$

2. $\lim_{t \to \infty} \int_{0}^{t} \frac{\psi(s)}{t} dW_1(s) = \lim_{t \to \infty} \int_{0}^{t} \frac{S(s)}{t} dW_1(s) = 0,$

$$\lim_{t \to \infty} \int_{0}^{t} \frac{I(s)}{t} dW_2(s) = 0, \quad \lim_{t \to \infty} \int_{0}^{t} \frac{D(s)}{t} dW_4(s) = 0 \text{ a.s.}$$
\[ \lim_{t \to \infty} \int_0^t \int_Z \lambda_1(u) \psi(s^-) \tilde{N}(ds, du) = 0, \quad \lim_{t \to \infty} \int_0^t \int_Z \lambda_1(u) S(s^-) \tilde{N}(ds, du) = 0, \]
\[ \lim_{t \to \infty} \int_0^t \int_Z \lambda_2(u) I(s^-) \tilde{N}(ds, du) = 0, \quad \lim_{t \to \infty} \int_0^t \int_Z \lambda_1(u) D(s^-) \tilde{N}(ds, du) = 0 \text{ a.s.} \]

Lemma 2.3. Let \( \psi(t) \) be solution of the system (6) with an initial value \( \psi(0) \in \mathbb{R}_+ \). Then,
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \psi(s) ds = \frac{A}{\mu_1} \text{ a.s.} \]
and
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \psi^2(s) ds = \frac{2A^2}{\mu_1 \chi_2} \text{ a.s.} \]
where \( \chi_2 = 2\mu_1 - \sigma_1^2 - \int_Z (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \nu(du) > 0 \).

Proof. Integrating from 0 to \( t \) on both sides of (6) yields
\[ \frac{\psi(t) - \psi(0)}{t} = A - \frac{\mu_1}{t} \int_0^t \psi(s) ds + \frac{\sigma_1}{t} \int_0^t \psi(s) dW_1(s) + \frac{1}{t} \int_0^t \int_Z \lambda_1(u) \psi(t^-) \tilde{N}(ds, du). \]

Clearly, we can derive that
\[ \frac{1}{t} \int_0^t \psi(s) ds = \frac{A}{\mu_1} + \frac{\sigma_1}{\mu_1 t} \int_0^t \psi(s) dW_1(s) + \frac{1}{\mu_1 t} \int_0^t \int_Z \lambda_1(u) \psi(t^-) \tilde{N}(dt, du). \]

Hence
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \psi(s) ds = \frac{A}{\mu_1} \text{ a.s.} \]

Applying the generalized Itô’s formula to model (6) leads to
\[ d\psi^2(t) = \left( 2\psi(t) \left( A - \mu_1 \psi(t) \right) + \sigma_1^2 \psi^2(t) + \int_Z \psi^2(t) \left( (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \right) \nu(du) \right) dt \]
\[ + 2\sigma_1 \psi(t) dW_1(t) + \int_Z \psi^2(t^-) \left( (1 + \lambda_1(u))^2 - \lambda_1(u) \right) \tilde{N}(dt, du). \]

Integrating both sides from 0 to \( t \), yields
\[ \psi^2(t) - \psi^2(0) = 2A \int_0^t \psi(s) ds - \left( 2\mu_1 - \sigma_1^2 - \int_Z \left( (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \right) \nu(du) \right) \int_0^t \psi^2(s) ds \]
\[ + 2\sigma_2 \int_0^t \psi^2(s) dW_1(s) + \int_0^t \int_Z \psi^2(s) \left( (1 + \lambda_1(u))^2 - \lambda_1(u) \right) \tilde{N}(ds, du). \]

Let \( \chi_2 = 2\mu_1 - \sigma_1^2 - \int_Z \left( (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \right) \nu(du) > 0 \). Therefore
\[ \frac{1}{t} \int_0^t \psi^2(s) ds = \frac{2A^2}{\mu_1 \chi_2} + \frac{2\sigma_1 (\psi^2(0) - \psi^2(t))}{\chi_2 t} + \frac{2\sigma_1}{\chi_2 t} \int_0^t \psi^2(s) W_1(s) \]
\[ + \frac{2\sigma_1}{\chi_2} \int_0^t \int_Z \psi^2(s) \left( (1 + \lambda_1(u))^2 - \lambda_1(u) \right) \tilde{N}(ds, du). \]

By using the same method as that in [26], assumption \( (A_4) \) and the large number theorem for martingales, we can easily verify that
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t X^2(s) ds = \frac{2A^2}{\mu_1 \chi_2} \text{ a.s.} \]
3. Extinction of the disease

In this section, we establish a sufficient condition for the extinction of the disease in our system (5).

**Theorem 3.1.** Let \((S(t), I(t), D(t))\) be a solution of the stochastic system (5) with any initial data \((S(0), I(0), D(0))\) \(\in \mathbb{R}_+^3\). Then

\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left( \frac{1}{\mu_2 + \gamma} I(t) + \frac{\sqrt{\psi_0}}{\eta} D(t) \right) \leq \vartheta \quad a.s.,
\]

where

\[
\vartheta = \min \{ \mu_2 + \gamma, \eta \} \sqrt{\psi_0} \mathbf{1}_{\{ \psi_0 \leq 1 \}} + \max \{ \mu_2 + \gamma, \eta \} \sqrt{\psi_0} \mathbf{1}_{\{ \psi_0 > 1 \}} + \eta \left( \frac{T_0 \zeta}{\lambda^2} \right)^{\frac{1}{2}},
\]

\[
\zeta = \sigma_1^2 + \int_Z \left( (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \right) \nu(du) > 0.
\]

*Notably, if \( \vartheta < 0 \), then \( I(t) \) will go to zero exponentially with probability one.*

**Proof.** At first, we use theorem 1.4 in [27] to prove that there is a left eigenvector of the matrix

\[
\mathcal{M}_0 = \begin{pmatrix}
0 & \frac{\beta A}{\mu_1 (\mu_2 + \gamma)} \\
1 & \frac{\mu_1 (\mu_2 + \gamma)}{\beta A}
\end{pmatrix}
\]
corresponding to \( \sqrt{\psi_0} \). This vector will be denoted by \((e_1, e_2) = (1, \sqrt{\psi_0})\). Then

\[
\sqrt{\psi_0} (e_1, e_2) = (e_1, e_2) \mathcal{M}_0.
\]

On the other hand, we define a \( C^2 \)-function \( M : \mathbb{R}_+^2 \to \mathbb{R}_+ \) by

\[
M(I(t), D(t)) = \omega_1 I(t) + \omega_2 D(t),
\]

where \( \omega_1 = \frac{\mu_1}{\mu_2 + \gamma} \) and \( \omega_2 = \frac{\mu_2}{\eta} \). By applying the generalized Itô’s formula with Lévy process we obtain

\[
\begin{align*}
\frac{d \ln M(I(t), D(t))}{dt} &= \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left\{ \omega_1 \left( \beta S(t) D(t) - (\mu_2 + \gamma) I(t) \right) + \omega_2 \eta \left( I(t) - D(t) \right) \right\} \\
&\quad + \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left\{ \omega_1 \sigma_2 I(t) dW_2(t) + \omega_2 \sigma_4 D(t) dW_4(t) \right\} \\
&\quad - \frac{1}{2(\omega_1 I(t) + \omega_2 D(t))^2} \left\{ \omega_1^2 \sigma_2^2 I^2(t) + \omega_2^2 \sigma_4^2 D^2(t) \right\} dt \\
&\quad + \int_Z \left[ \ln \left( 1 + \frac{\omega_1 \lambda_2(u) I(t) + \omega_2 \lambda_4(u) D(t)}{\omega_1 I(t) + \omega_2 D(t)} \right) - \frac{\omega_1 \lambda_2(u) I(t) + \omega_2 \lambda_4(u) D(t)}{\omega_1 I(t) + \omega_2 D(t)} \right] \nu(du) dt \\
&\quad + \int_Z \ln \left( 1 + \frac{\omega_1 \lambda_2(u) I(t) + \omega_2 \lambda_4(u) D(t)}{\omega_1 I(t) + \omega_2 D(t)} \right) \tilde{N}(dt, du) \\
&\quad + \int_Z \ln (1 + \lambda(u)) \tilde{N}(dt, du),
\end{align*}
\]

where \( \lambda(u) = \lambda_2(u) \lor \lambda_4(u) \) and

\[
\mathcal{L} \ln M(I(t), D(t)) = \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left\{ \omega_1 \left( \beta S(t) D(t) - (\mu_2 + \gamma) I(t) \right) + \omega_2 \eta \left( I(t) - D(t) \right) \right\} \\
&\quad - \frac{1}{2(\omega_1 I(t) + \omega_2 D(t))^2} \left\{ \omega_1^2 \sigma_2^2 I^2(t) + \omega_2^2 \sigma_4^2 D^2(t) \right\} dt \\
&\quad + \int_Z \left[ \ln \left( 1 + \frac{\omega_1 \lambda_2(u) I(t) + \omega_2 \lambda_4(u) D(t)}{\omega_1 I(t) + \omega_2 D(t)} \right) - \frac{\omega_1 \lambda_2(u) I(t) + \omega_2 \lambda_4(u) D(t)}{\omega_1 I(t) + \omega_2 D(t)} \right] \nu(du) dt.
\]
Notice that $\ln(1 + x) \leq x$, for all $x > -1$, result in

$$
\mathcal{L} \ln M(I(t), D(t)) \leq \frac{\omega_1 \beta D(t)}{\omega_1 I(t) + \omega_2 D(t)} \left( S(t) - \frac{A}{\mu_1} \right)
+ \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left\{ \omega_1 \left( \frac{\beta A}{\mu_1} D(t) - (\mu_2 + \gamma) I(t) \right) + \omega_2 \eta \left( I(t) - D(t) \right) \right\}
- \frac{1}{2(\omega_1 I(t) + \omega_2 D(t))^2} \left\{ \omega_1^2 \sigma_1^2 I^2(t) + \omega_2^2 \sigma_2^2 D^2(t) \right\} dt.
$$

In accordance with the positivity of the solution, we can derive that

$$
\mathcal{L} \ln M(I(t), D(t)) \leq \frac{\omega_1 \beta D(t)}{\omega_1 I(t) + \omega_2 D(t)} \left( S(t) - \frac{A}{\mu_1} \right)
+ \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left\{ - \frac{e_1}{\mu_2 + \gamma} \left( \frac{\beta A}{\mu_1} D(t) - (\mu_2 + \gamma) I(t) \right) + \frac{\omega_2}{\eta} \left( I(t) - D(t) \right) \right\}.
$$

By using the stochastic comparison theorem, we get

$$
\mathcal{L} \ln M(I(t), D(t)) \leq \frac{\omega_1 \beta D(t)}{\omega_1 I(t) + \omega_2 D(t)} \left( \psi(t) - \frac{A}{\mu_1} \right)
+ \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left\{ - \frac{e_1}{\mu_2 + \gamma} \left( \frac{\beta A}{\mu_1} D(t) - (\mu_2 + \gamma) I(t) \right) + \frac{\omega_2}{\eta} \left( I(t) - D(t) \right) \right\}.
$$

We then find that

$$
\mathcal{L} \ln M(I(t), D(t)) \leq \frac{\omega_1 \beta}{\omega_2} \left| \psi(t) - \frac{A}{\mu_1} \right| + \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left( e_1, e_2 \right) \left( \mathcal{M}_0(I(t), D(t))^T - (I(t), D(t))^T \right)
= \frac{\omega_1 \beta}{\omega_2} \left| \psi(t) - \frac{A}{\mu_1} \right| + \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left( \sqrt{T_0 - 1} \right) \left( e_1 I(t) + e_2 D(t) \right)
= \frac{\omega_1 \beta}{\omega_2} \left| \psi(t) - \frac{A}{\mu_1} \right| + \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \left( \sqrt{T_0 - 1} \right) \left( \omega_1 (\mu_2 + \gamma) I(t) + \omega_2 D(t) \right)
\leq \min \left\{ \mu_2 + \gamma, \eta \right\} \left( \sqrt{T_0 - 1} \right) \left\{ \sqrt{T_0} \leq 1 \right\} + \max \left\{ \mu_2 + \gamma, \eta \right\} \left( \sqrt{T_0 - 1} \right) \left\{ \sqrt{T_0} > 1 \right\}
+ \frac{\omega_1 \beta}{\omega_2} \left| \psi(t) - \frac{A}{\mu_1} \right|.
$$

Hence we deduce that

$$
d\ln M(I(t), D(t)) \leq \ln M(I(0), D(0)) + \frac{\ln \left( \frac{\omega_1 \beta}{\omega_2} \right)}{t} + \min \left\{ \mu_2 + \gamma, \eta \right\} \left( \sqrt{T_0 - 1} \right) \left\{ \sqrt{T_0} \leq 1 \right\}
+ \max \left\{ \mu_2 + \gamma, \eta \right\} \left( \sqrt{T_0 - 1} \right) \left\{ \sqrt{T_0} > 1 \right\}
+ \frac{\omega_1 \beta}{\omega_2} \int_0^t \left| \psi(s) - \frac{A}{\mu_1} \right| ds + \frac{U_1(t)}{t} + \frac{U_2(t)}{t},
$$

where

$$
U_1(t) = \frac{1}{\omega_1 I(t) + \omega_2 D(t)} \int_0^t \left\{ \omega_1 \sigma_2 I(s) dW_2(s) + \omega_2 \sigma_4 D(s) dW_4(s) \right\},
U_2(t) = \int_0^t \int_Z \ln(1 + \lambda(u)) \tilde{N} \left( ds, du \right).
$$
By the strong law of large numbers for local martingales, we get \( \lim_{t \to \infty} \frac{U_i(t)}{t} = 0 \) (i = 1, 2) a.s. Now by using the Hölder’s inequality, we deduce that
\[
\int_0^t |\psi(s) - \frac{A}{\mu_1}\| ds \leq \left( \int_0^t \left( \psi(s) - \frac{A}{\mu_1} \right)^2 ds \right)^{\frac{1}{2}} = \left( \int_0^t \left( \psi^2(s) - \frac{2A}{\mu_1} \psi(s) + \left( \frac{A}{\mu_1} \right)^2 \right) ds \right)^{\frac{1}{2}}.
\]
It follows from lemma 2.3 that
\[
\lim_{t \to \infty} \int_0^t |\psi(s) - \frac{A}{\mu_1}\| ds \leq \left( \frac{2A^2}{\mu_1 \chi_2} - \frac{2A^2}{\mu_1^2} + \frac{A^2}{\mu_1^2} \right)^{\frac{1}{2}} = \left( \frac{A^2 \left( \sigma_1^2 + \int_Z \left( (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \right) \nu(du) \right)}{\mu_1 \chi_2} \right)^{\frac{1}{2}}.
\]
Taking the superior limit on both sides, we obtain
\[
\limsup_{t \to \infty} \frac{\ln M(I(t), D(t))}{t} \leq \frac{\ln M(I(0), D(0))}{t} + \min\{\mu_2 + \gamma, \eta\} (\sqrt{\tau_0} - 1) 1_{\{\sqrt{\tau_0} \leq 1\}} + \max\{\mu_2 + \gamma, \eta\} (\sqrt{\tau_0} - 1) 1_{\{\sqrt{\tau_0} > 1\}} + \frac{\omega_1 \beta}{\omega_2} \left( \frac{A^2 \left( \sigma_1^2 + \int_Z \left( (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \right) \nu(du) \right)}{\mu_1 \chi_2} \right)^{\frac{1}{2}}.
\]
Which implies,
\[
\limsup_{t \to \infty} \frac{\ln M(I(t), D(t))}{t} \leq \min\{\mu_2 + \gamma, \eta\} (\sqrt{\tau_0} - 1) 1_{\{\sqrt{\tau_0} \leq 1\}} + \max\{\mu_2 + \gamma, \eta\} (\sqrt{\tau_0} - 1) 1_{\{\sqrt{\tau_0} > 1\}} + \eta \left( \frac{T_0 \zeta}{\chi_2} \right)^{\frac{1}{2}} =: \vartheta \text{ a.s.}
\]
where \( \zeta = \sigma_1^2 + \int_Z \left( (1 + \lambda_1(u))^2 - 1 - \lambda_1(u) \right) \nu(du) \). That is to say, if \( \vartheta < 0 \), then
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} < 0, \text{ and } \limsup_{t \to \infty} \frac{\ln D(t)}{t} < 0 \text{ a.s.}
\]
This implies that the disease will die out with probability one. This completes the proof.

4. Persistence in mean of the disease
The study of the persistence in the mean is a significant characteristic to know more about epidemic dynamics. For this reason, in this section, we will give the condition for the disease persistence. For simplify, we define the following quantity
\[
T_0 = \beta \left( \frac{A}{\mu_1 + \sigma_1} \right) \left( (\mu_2 + \gamma + \sigma_2) + \beta \left( \frac{A}{\mu_1 + \sigma_1} \right) \frac{\sigma_1}{\eta} \right)^{-1},
\]
where \( \sigma_1 = 0.5 \sigma_1^2 + \int_Z (\lambda_i(u) - \ln(1 + \lambda_i(u))) \nu(du) dt, i = 1, 2, 4. \)

**Theorem 4.1.** Let \((S(t), I(t), D(t))\) be a solution of the stochastic system (5) with any initial data \((S(0), I(0), D(0))\) \(\in \mathbb{R}^3_+\). The stochastic model (5) has the property, if \( T_0 > 1 \) holds, then the disease persists in the mean almost surely.

**Proof.** We begin by considering the following function
\[
N(S(t), T(t), D(t)) = -c_1 \ln S(t) - \ln I(t) - c_2 \ln D(t) + c_3 D(t).
\]
where \( c_i \), \( i = 1, 2, 3 \) are positive constants to be determined in the following. Then

\[
dN(S(t), T(t), D(t)) = \mathcal{L}N(S(t), T(t), D(t))dt - c_1\sigma_1dW_1(t) - \sigma_2dW_2(t) - c_2\sigma_4dW_4(t)
+ c_3\sigma_4D(t)dW_4(t) - \int_{\mathbb{Z}} \left\{ \begin{array}{l}
c_1 \ln(1 + \lambda_1(u)) \\
+ \ln(1 + \lambda_2(u)) \\
+ c_2 \ln(1 + \lambda_4(u)) \\
- c_3\lambda_4(u)D(t)
\end{array} \right\} \tilde{N}(dt, du),
\]

where

\[
\mathcal{L}N(S(t), T(t), D(t)) = -\frac{c_1}{S(t)}(A - \mu_1 - \beta S(t)D(t)) + \frac{c_1\sigma_1^2}{2} - \frac{1}{I(t)}(\beta S(t)D(t) - (\mu_2 + \gamma)I(t))
+ \frac{\sigma_2^2}{2} - \frac{c_2\eta}{D(t)}(I(t) - D(t)) + \frac{c_2\sigma_4^2}{2} + c_3\eta(I(t) - D(t))
+ \int_{\mathbb{Z}} \left\{ \begin{array}{l}
c_1(\lambda_1(u) - \ln(1 + \lambda_1(u)) ) \\
+ (\lambda_2(u) - \ln(1 + \lambda_2(u))) \\
+ c_2(\lambda_4(u) - \ln(1 + \lambda_4(u))) \\
\end{array} \right\} \nu(du)dt.
\]

Note that \( \tilde{\sigma}_i = 0.5\sigma^2_i + \int_{\mathbb{Z}}(\lambda_i(u) - \ln(1 + \lambda_i(u)))\nu(du)dt, \ i = 1, 2, 4 \). We then find that

\[
\mathcal{L}N(S(t), T(t), D(t)) = -\frac{\beta S(t)D(t)}{I(t)} - \frac{c_1A}{S(t)} - \frac{c_2\eta I(t)}{D(t)} + (c_1\beta - c_3\eta)D(t)
\]

\[
+ c_1(\mu_1 + \sigma_1) + c_2(\eta + \tilde{\sigma}_4) + (\mu_2 + \gamma + \tilde{\sigma}_2) + c_3\eta I(t)
\]

\[
\leq -3(\beta\eta c_1c_2) + (c_1\beta - c_3\eta)D(t) + c_1(\mu_1 + \tilde{\sigma}_1)
\]

\[
+ c_2(\eta + \tilde{\sigma}_4) + (\mu_2 + \gamma + \tilde{\sigma}_2) + c_3\eta I(t).
\]

Now let

\[
c_1 = \beta \left( \frac{A}{\mu_1 + \tilde{\sigma}_1} \right)^2 (\eta + \tilde{\sigma}_4)/\eta,
\]

\[
c_2 = \beta \left( \frac{A}{\mu_1 + \tilde{\sigma}_1} \right) / (\eta + \tilde{\sigma}_4),
\]

\[
c_3 = c_1\beta / \eta.
\]

So that we clearly have

\[
\mathcal{L}N(S(t), T(t), D(t)) \leq -\beta \left( \frac{A}{\mu_1 + \tilde{\sigma}_1} \right) + (\mu_2 + \gamma + \tilde{\sigma}_2) + \beta \left( \frac{A}{\mu_1 + \tilde{\sigma}_1} \right) \frac{\sigma_2^2}{\eta} + c_1\beta D(t)
\]

\[
= -\beta \left( \frac{A}{\mu_1 + \tilde{\sigma}_1} \right) \left( 1 - \frac{1}{I(t)} \right) + c_1\beta I(t)
\]

Hence we obtain

\[
dN(S(t), T(t), D(t)) \leq \left( -\beta \left( \frac{A}{\mu_1 + \tilde{\sigma}_1} \right) \left( 1 - \frac{1}{I(t)} \right) + c_1\beta I(t) \right) dt - c_1\sigma_1dW_1(t) - \sigma_2dW_2(t)
\]

\[
- c_2\sigma_4dW_4(t) + c_3\sigma_4D(t)dW_4(t) - \int_{\mathbb{Z}} \left\{ \begin{array}{l}
c_1 \ln(1 + \lambda_1(u)) \\
+ \ln(1 + \lambda_2(u)) \\
+ c_2 \ln(1 + \lambda_4(u)) \\
- c_3\lambda_4(u)D(t)
\end{array} \right\} \tilde{N}(dt, du).
\]

Integrating from 0 to \( t \) on both sides, yields

\[
N(S(t), T(t), D(t)) = \frac{N(S(0), T(0), D(0))}{t} - \beta \left( \frac{A}{\mu_1 + \tilde{\sigma}_1} \right) \left( 1 - \frac{1}{I(t)} \right)
\]

\[
+ c_1\beta \int_0^t I(s)ds + \frac{U_3(t)}{t} + \frac{U_4(t)}{t},
\]

9
where
\[ U_3(t) = - \left( c_1 \sigma_1 W_1(t) + \sigma_2 W_2(t) + c_2 \sigma_4 W_4(t) \right) + c_3 \sigma_4 \int_0^t D(s)dW_4(s), \]
\[ U_4(t) = - \int_0^t \int_Z \begin{pmatrix} c_1 \ln(1 + \lambda_1(u)) \\
+ \ln(1 + \lambda_2(u)) \\
+ c_2 \ln(1 + \lambda_4(u)) \\
- c_3 \lambda_4(u) D(s) \end{pmatrix} \tilde{N}(ds, du). \]

Then by the large numbers theorem for martingales, we can obtain \( \lim_{t \to \infty} \frac{U_i(t)}{t} = 0 \) \((i = 3, 4)\) a.s. Therefore
\[ \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t I(s)ds \geq \frac{1}{c_1} \left( \frac{A}{\mu_1 + \sigma_1} \right) \left( 1 - \frac{1}{T_0} \right) > 0 \text{ a.s.} \]
This shows that the disease persistence in the mean. The proof is completed. \( \square \)

5. Example

In this section, we will validate our theoretical result with the help of numerical simulations taking parameters from the theoretical data mentioned in 1. We numerically simulate the solution of the system (5) with initial value \((S(0), I(0), D(0)) = (0.9, 0.5, 0.1)\) and use Matlab software to perform numerical examples.

| Parameters | Description | Value |
|------------|-------------|-------|
| \(A\)      | The recruitment rate | 0.8   |
| \(\mu_1\)  | The natural mortality rate of \(S\) | 0.9   |
| \(\beta\)  | The transmission rate | 0.1   |
| \(\gamma\) | The recovered rate | 0.2   |
| \(\mu_2\)  | The natural mortality of \(I\) | 0.2   |
| \(\eta\)   | The exponentially fading memory rate | 0.2   |
| \(\sigma_1\) | The intensity of \(W_1(t)\) | 0.2   |
| \(\sigma_2\) | The intensity of \(W_2(t)\) | 0.25  |
| \(\sigma_4\) | The intensity of \(W_4(t)\) | 0.15  |
| \(\lambda_1\) | The jump intensity of \(S\) | 0.3   |
| \(\lambda_2\) | The jump intensity of \(I\) | 0.2   |
| \(\lambda_4\) | The jump intensity of \(D\) | 0.1   |

Table 1: Theoretical parameter values of the model (5) used in figure 1.

By using the parameters listed in table 1, we get the following results: \(T_0 = 0.222 < 1, \zeta = 0.043 > 0, \chi_2 = 1.370 > 0, \) and \(\vartheta = -0.052 < 0.\) The result in theorem 3.1 gives the condition under which the disease dies out exponentially almost surely. This is illustrated in figure 1.

When the parameter \(T_0^*\) exceeds one, the epidemic persists in the mean with probability one. This confirms the behavior of the epidemic in figure 2 with parameters given in table 2 and a value of \(T_0^* = 1.2684 > 1.\)

| Parameters | Description | Value |
|------------|-------------|-------|
| \(A\)      | The recruitment rate | 0.65  |
| \(\mu_1\)  | The natural mortality rate of \(S\) | 0.5   |
| \(\beta\)  | The transmission rate | 0.8   |
| \(\gamma\) | The recovered rate | 0.8   |
| \(\mu_2\)  | The natural mortality of \(I\) | 0.7   |
| \(\eta\)   | The exponentially fading memory rate | 0.6   |
| \(\sigma_1\) | The intensity of \(W_1(t)\) | 0.12  |
| \(\sigma_2\) | The intensity of \(W_2(t)\) | 0.17  |
| \(\sigma_4\) | The intensity of \(W_4(t)\) | 0.15  |
| \(\lambda_1\) | The jump intensity of \(S\) | 0.1   |
| \(\lambda_2\) | The jump intensity of \(I\) | 0.2   |
| \(\lambda_4\) | The jump intensity of \(D\) | 0.1   |

Table 2: Theoretical parameter values of the model (5) used in figure 2.
6. Conclusion

In this paper, by considering distributed delay and a multidimensional Lévy process, we studied the dynamic behaviors of the stochastic SIR model. Firstly, we employed the linear chain approach to transforming the model with a low kernel case into an equivalent system (5). Then, we analyzed the long term behavior of the stochastic model (5). Our analysis based on new calculus to obtain sufficient conditions for two interesting asymptotic proprieties: extinction and persistence in the mean. The novelty of this study can summarise into the following points:

1. The extension of the study presented in [21] to the case of the multidimensional Lévy process.
2. The calculation of the temporary averages of the solution to (5) instead of the classic method based on the explicit form of the stationary distribution of the model (5).

Our theoretical studies deliver some new insights for understanding the propagation of diseases with distributed delay.

References

[1] W. O. Kermack and A. G. McKendrick, “A contribution to the mathematical theory of epidemics,” Proceedings of The Royal Society A Mathematical Physical and Engineering Sciences, vol. 115, no. 772, pp. 700–721, 1927.

[2] E. Beretta, T. Hara, and W. Ma, “Global asymptotic stability of an SIR epidemic model with distributed time delay,” Nonlinear Analysis, vol. 47, pp. 4107–4115, 2001.
Figure 2: The numerical simulation of the solution \((S(t), I(t), D(t))\) to the system (5).

[3] H. Guo, M. Li, and Z. S. Shuai, “Global stability of the endemic equilibrium of multigroup SIR epidemic models,” *Canadian Applied Mathematics Quarterly*, vol. 14, pp. 259–284, 2006.

[4] X. Z. Meng and L. S. Chen, “The dynamics of a new SIR epidemic model concerning pulse vaccination strategy,” *Applied Mathematics and Computation*, vol. 197, pp. 528–597, 2008.

[5] M. Roy and R. D. Holt, “Effects of predation on host-pathogen dynamics in SIR models,” *Theoretical Population Biology*, vol. 73, pp. 319–331, 2008.

[6] Y. Kyrychko and K. Blyuss, “Global properties of a delayed SIR model with temporary immunity and nonlinear incidence rate,” *Nonlinear Analysis*, vol. 6, pp. 495–507, 2005.

[7] E. Beretta, T. Hara, and W. Ma, “Global asymptotic stability of an SIR epidemic model with distributed time delay,” *Nonlinear Analysis*, vol. 47, pp. 4107–4115, 2001.

[8] W. Ma, Y. Takeuchi, T. Hara, and E. Beretta, “Epermanence of an SIR epidemic model with distributed time delays,” *Tohoku Mathematical Journal*, vol. 54, pp. 581–591, 2002.

[9] E. Beretta and Y. Takeuchi, “Global stability of an SIR epidemic model with time delay,” *Journal of Mathematical Biology*, vol. 33, pp. 250–260, 1995.

[10] Y. Takeuchi, W. Ma, and E. Beretta, “Global asymptotic properties of a delay SIR epidemic model with finite incubation times,” *Nonlinear Analysis*, vol. 42, pp. 931–947, 2000.
[11] H. Shu, D. Fan, and J. Wei, “Global stability of multi-group seir epidemic models with distributed delays and nonlinear transmission,” *Nonlinear Analysis: Real World Applications*, vol. 13, pp. 1581–1591, 2012.

[12] Y. Muroya, T. Kuniya, and J. Wang, “Stability analysis of a delayed multi-group SIS epidemic model with nonlinear incidence rates and patch structure,” *Journal of Mathematical Analysis and Applications*, vol. 425, pp. 415–439, 2015.

[13] G. Chen and T. Li, “Stability of stochastic delayed SIR model,” *Stochastics and Dynamics*, vol. 9, no. 2, pp. 231–252, 2009.

[14] Q. Liu, Q. Chen, and D. Jiang, “The threshold of a stochastic delayed SIR epidemic model with temporary immunity,” *Physica A*, vol. 450, pp. 115–125, 2016.

[15] Q. Liu, D. Jiang, N. Shi, T. Hayat, and A. Alsaedi, “Asymptotic behaviors of a stochastic delayed sir epidemic model with nonlinear incidence,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 40, pp. 89–99, 2016.

[16] E. tornatore, S. Buccellato, and P. Vetro, “Stability of a stochastic SIR system,” *Physica A*, vol. 354, pp. 111–126, 2005.

[17] C. Ji, D. Jiang, and N. Shi, “The behavior of an SIR epidemic model with stochastic perturbation,” *Stochastic analysis and applications*, vol. 30, pp. 755–773, 2012.

[18] C. Ji and D. Jiang, “Threshold behaviour of a stochastic SIR model,” *Applied Mathematical Modelling*, vol. 38, pp. 5067–5079, 2014.

[19] C. Ji, D. Jiang, and N. Shi, “Asymptotic behavior of global positive solution to a stochastic SIR model,” *Applied Mathematical Modelling*, vol. 45, pp. 221–232, 2011.

[20] Y. Lin, D. Jiang, and P. Xia, “Long-time behavior of a stochastic SIR model,” *Applied Mathematics and Computation*, vol. 236, pp. 1–9, 2014.

[21] Q. Liu, D. Jiang, T. Hayat, and A. Alsaedi, “Dynamics of a stochastic SIR epidemic model with distributed delay and degenerate diffusion,” *Journal of the Franklin Institute*, vol. 356, pp. 7347–7370, 2019.

[22] X. Zhang and K. Wang, “Stochastic SIR model with jumps,” *Applied Mathematics letters*, vol. 826, pp. 867–874, 2013.

[23] Y. Zhou and W. Zhang, “Threshold of a stochastic SIR epidemic model with Levy jumps,” *Physica A*, vol. 446, pp. 204–2016, 2016.

[24] D. Zhao, S. Yuan, and H. Liu, “Stochastic dynamics of the delayed chemostat with Levy noises,” *International Journal of Biomathematics*, vol. 12, no. 5, 2019.

[25] D. Kiouach and Y. Sabbar, “Stability and threshold of a stochastic SIRS epidemic model with vertical transmission and transfer from infectious to susceptible individuals,” *Discrete Dynamics in Nature and Society*, no. 7570296, 2018.

[26] Y. Zhou, S. Yuan, and D. Zhao, “Threshold behavior of a stochastic SIS model with Levy jumps,” *Discrete Dynamics in Nature and Society*, vol. 275, pp. 255–267, 2016.

[27] A. Berman and R. Plemmons, “Nonnegative matrices in the mathematical sciences,” *Academic Press, New York*, 1979.