THE GENERIC ISOMETRY AND MEASURE PRESERVING HOMEOMORPHISM ARE CONJUGATE TO THEIR POWERS

CHRISTIAN ROSENDAL

Abstract. It is known that there is a comeagre set of mutually conjugate measure preserving homeomorphisms of Cantor space equipped with the coin-flipping probability measure, i.e., Haar measure.

We show that the generic measure preserving homeomorphism is moreover conjugate to all of its powers. It follows that the generic measure preserving homeomorphism extends to an action of \( (\mathbb{Q},+) \) by measure preserving homeomorphisms.

Similarly, S. Solecki has proved that there is a comeagre set of mutually conjugate isometries of the rational Urysohn metric space. We prove that these are all conjugate with their powers and hence also embed into \( \mathbb{Q} \)-actions. By consequence, the generic isometry of the full Urysohn metric space has roots of all orders.

We also consider a notion of topological similarity in Polish groups and use this to give simplified proofs of the meagreness of conjugacy classes in the automorphism group of the standard probability space and in the isometry group of the Urysohn metric space.

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1. Introduction

Suppose \( M \) is a compact metric space and let \( \text{Homeo}(M) \) be its group of homeomorphisms. We equip \( \text{Homeo}(M) \) with the topology of uniform convergence or...
what is equivalent, since $M$ is compact metric, the compact-open topology. Thus, in this way, a neighbourhood basis at the identity $1$ consists of the sets

$$\{ h \in \text{Homeo}(M) \mid h(C_1) \subseteq V_1 \& \ldots \& h(C_n) \subseteq V_n \},$$

where $V_i \subseteq M$ is open and $C_i \subseteq V_i$ compact. Under this topology the group operations are continuous and thus $\text{Homeo}(M)$ is a topological group. Moreover, the topology is Polish, that is, $\text{Homeo}(M)$ is separable and its topology can be induced by a complete metric.

Now consider the case when $M$ is Cantor space $2^N$. Then, as any two disjoint closed sets in $2^N$ can be separated by a clopen set, we get a neighbourhood basis at the identity consisting of sets of the form

$$\{ h \in \text{Homeo}(2^N) \mid h(C_1) = C_1 \& \ldots \& h(C_n) = C_n \},$$

where $C_1, \ldots, C_n \subseteq 2^N$ is a partition of $2^N$ into clopen sets.

By Stone duality, the homeomorphisms of Cantor space are just the automorphisms of the boolean algebra of clopen subsets of $2^N$, which we denote by $B_\infty$. Thus, viewed in this way, the neighbourhood basis at the identity has the form

$$\{ h \in \text{Homeo}(2^N) \mid h|_C = \text{id} \},$$

where $C$ is a finite subalgebra of $B_\infty$.

Cantor space $2^N$ is of course naturally homeomorphic to the Cantor group $(\mathbb{Z}_2)^N$ and therefore comes equipped with Haar measure $\mu$. Up to a homeomorphism of Cantor space $\mu$ is the unique atomless Borel probability measure on $2^N$ such that

- if $C \in B_\infty$, then $\mu(C)$ is a dyadic rational, i.e., on the form $\frac{n}{2^k}$.
- if $C \in B_\infty$ and $\mu(C) = \frac{n}{2^k}$, then for all $l \geq k$, there is some clopen $B \subseteq C$ such that $\mu(B) = \frac{1}{2^l}$.
- if $\emptyset \neq C \in B_\infty$, then $\mu(C) > 0$.

The measure $\mu$ is of course the product probability measure of the coinflipping measure on each factor $2 = \{0, 1\}$. For simplicity, we call $\mu$ Haar measure on $2^N$.

One easily sees that the group of Haar measure preserving homeomorphisms $\text{Homeo}(2^N, \mu)$ of $2^N$ is a closed subgroup of $\text{Homeo}(2^N)$ and therefore a Polish group in its own right. It was proved by Kechris and Rosendal in [8] that there are comeagre conjugacy classes in both $\text{Homeo}(2^N)$ and $\text{Homeo}(2^N, \mu)$. In fact, the result for $\text{Homeo}(2^N, \mu)$ is rather simple and also holds for many other sufficiently homogeneous measures on $2^N$ (see Akin [1]). This result allows us to refer to the generic measure preserving homeomorphism of Cantor space (with Haar measure) knowing that they are all mutually conjugate. One of the aims of this paper is to show that they are all conjugate to their non-zero powers, which will in turn show that they all are part of an action of the additive group $\mathbb{Q}$ by measure preserving homeomorphisms of $2^N$. Notice that this is to some extent an optimal result, for as $\text{Homeo}(2^N, \mu)$ is totally disconnected there are no non-trivial continuous homomorphism (or even measurable homomorphisms) from $\mathbb{R}$ into $\text{Homeo}(2^N, \mu)$ and thus $\mathbb{R}$ cannot act non-trivially by measure preserving homeomorphisms on $2^N$.

The Urysohn metric space $U$ is a universal separable metric space first constructed by Urysohn in the posthumously published [13]. It soon went out of fashion following the discovery that many separable Banach spaces are already universal separable metric spaces, but has come to the forefront over the last twenty years as an analogue of Fraïssé theory in the case of metric spaces.
The Urysohn space $U$ is characterised up to isometry by being separable and complete together with the following extension property.

If $\phi: A \to U$ is an isometric embedding of a finite metric space $A$ into $U$ and $B = A \cap \{y\}$ is a one point metric extension of $A$, then $\phi$ extends to an isometric embedding of $B$ into $U$.

There is also a rational variant of $U$ called the rational Urysohn metric space, which we denote by $QU$. This is up to isometry the unique countable metric space with only rational distances such that the following variant of the above extension property holds.

If $\phi: A \to QU$ is an isometric embedding of a finite metric space $A$ into $QU$ and $B = A \cap \{y\}$ is a one point metric extension of $A$ whose metric only takes rational distances, then $\phi$ extends to an isometric embedding of $B$ into $QU$.

We denote by $\text{Iso}(QU)$ and $\text{Iso}(U)$ the isometry groups of $QU$ and $U$ respectively. These are Polish groups when equipped with the topology of pointwise convergence on $QU$ seen as a discrete set and $U$ seen as a metric space respectively. Thus, the basic neighbourhoods of the identity in $\text{Iso}(QU)$ are of the form

$$\{h \in \text{Iso}(QU) \mid h|_A = \text{id}_A\},$$

where $A$ is a finite subset of $QU$. On the other hand, the basic open neighbourhoods of the identity in $\text{Iso}(U)$ are of the form

$$\{h \in \text{Iso}(U) \mid \forall x \in A \ d(hx,x) < \epsilon\},$$

where $A$ is a finite subset of $U$ and $\epsilon > 0$.

In [12] S. Solecki proved, building on work of Herwig and Lascar [6], the following result.

**Theorem 1.** Let $A$ be a finite rational metric space. Then there is a finite rational metric space $B$ containing $A$ and such that any partial isometry of $A$ extends to a full isometry of $B$.

This is turn has the consequence that $\text{Iso}(QU)$ has a comeagre conjugacy class and we can therefore refer to its elements as *generic* isometries of $QU$. The second aim of our paper is to prove that these are all conjugate to their non-zero powers, which again suffices to show that they all are part of an action of the additive group $\mathbb{Q}$ by isometries of $QU$.

In the last section we briefly consider a coarse notion of conjugacy in Polish groups. We say that $f$ and $g$ belonging to a Polish group $G$ are *topologically similar* if for all increasing sequences $(s_n)$ we have $f^{s_n} \xrightarrow{n \to \infty} 1$ if and only if $g^{s_n} \xrightarrow{n \to \infty} 1$. As opposed to automorphism groups of countable structures there tend not to be comeagre conjugacy classes in large connected Polish groups and we shall provide new simple proofs of this for $\text{Aut}([0,1],\lambda)$ and $\text{Iso}(U)$ by showing that in fact their topological similarity classes are meagre.

2. **Powers of generic measure preserving homeomorphisms**

2.1. **Free amalgams of measured boolean algebras.** Suppose $B_1, B_2, \ldots, B_n$ are finite boolean algebras containing a common subalgebra $A$. We define the *free*
amalgam

\[ \otimes^i_A B_i = B_1 \otimes_A B_2 \otimes_A \ldots \otimes_A B_n \]

of \( B_1, \ldots, B_n \) over \( A \) as follows.

By renaming, we can suppose that \( B_i \cap B_j = A \) for all \( i \neq j \). We then take as our atoms the set of formal products

\[ b_1 \otimes \ldots \otimes b_n, \]

where each \( b_i \) is an atom in \( B_i \) and such that for some atom \( a \) of \( A \) we have \( b_i \leq a \) for all \( i \). Also, for simplicity, if \( c_i \in B_i \) is not necessarily an atom, but nevertheless we have some atom \( a \) of \( A \) such that \( c_i \leq a \) for all \( i \), we write

\[ c_1 \otimes \ldots \otimes c_n = \bigvee \{ b_1 \otimes \ldots \otimes b_n \mid b_i \text{ is an atom in } B_i \text{ and } b_i \leq c_i \}. \]

We can now embed each \( B_i \) into \( \otimes^i_A B_i \) by defining for each \( b \in B_i \) minorising an atom \( a \in A \)

\[ \pi_i(b) = a \otimes \ldots \otimes a \otimes b \otimes a \otimes \ldots \otimes a, \]

where the \( b \) appears in the \( i \)’th position. In particular,

\[ \pi_i(a) = a \otimes \ldots \otimes a \]

for all atoms \( a \) of \( A \). Thus, for each \( i \), \( \pi_i : B_i \hookrightarrow \otimes^i_A B_i \) is an embedding of boolean algebras and if \( \iota_i : A \hookrightarrow B_i \) denotes the inclusion mapping, then the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_i} & B_i \\
\downarrow{\iota_j} & & \downarrow{\pi_i} \\
B_j & \xrightarrow{\pi_j} & \otimes^i_A B_i
\end{array}
\]

Now, if \( \mu_i \) are measures on \( B_i \) agreeing on \( A \), then we can define a new measure \( \mu \) on \( \otimes^i_A B_i \) by setting for all \( b_i \in B_i \) minorising the same atom \( a \in A \)

\[ \mu(b_1 \otimes \ldots \otimes b_n) = \frac{\mu_1(b_1) \cdots \mu_n(b_n)}{\mu_1(a)^{n-1}}. \]

Thus,

\[ \mu(\pi_i(b)) = \mu(a \otimes \ldots \otimes a \otimes b \otimes a \otimes \ldots \otimes a) \]

\[ = \frac{\mu_1(a) \cdots \mu_{i-1}(a) \mu_i(b) \mu_{i+1}(a) \cdots \mu_n(a)}{\mu_1(a)^{n-1}} \]

\[ = \frac{\mu_1(a) \cdots \mu_1(a) \mu_i(b) \mu_1(a) \cdots \mu_1(a)}{\mu_1(a)^{n-1}} \]

\[ = \mu_i(b). \]

So \( \pi_i : (B_i, \mu_i) \hookrightarrow (\otimes^i_A B_i, \mu) \) is an embedding of measured boolean algebras.

A special case is when \( A \) and each \( B_i \) are equidistributed dyadic algebras, i.e., have \( 2^k \) atoms each of measure \( 2^{-k} \) for some \( k \geq 0 \). Then this implies that for each \( i \), all atoms of \( A \) are the join of the same number of atoms of \( B_i \), namely, \( 2^{k_i-m} \), where \( A \) has \( 2^m \) atoms and \( B_i \) has \( 2^{k_i} \) atoms. In this case, one can verify that \( \otimes^i_A B_i \) has \( 2^{k_1+\ldots+k_n-(n-1)m} \) atoms each of measure \( 2^{(n-1)m-k_1-\ldots-k_n} \). So again this is an equidistributed dyadic algebra.
A similar construction works for *equidistributed* algebras, i.e., those having a finite number of atoms of the same (necessarily rational) measure. In this case, the amalgam is also equidistributed.

In general, an automorphism of a finite boolean algebra arises from a permutation of the atoms, but in the case of equidistributed (dyadic) algebras, any permutation of the atoms conversely gives rise to a measure preserving automorphism. Thus, for equidistributed algebras an automorphism is necessarily a measure preserving automorphism and we can therefore be a bit forgetful about the measure.

**Lemma 2.** Let $A$ be an equidistributed (dyadic) finite boolean algebra. Then any partial measure preserving automorphism of $A$ extends to an automorphism of $A$.

*Proof.* Suppose that $B$ and $C$ are subalgebras of $A$ and $g : B \rightarrow C$ a measure preserving isomorphism. Then if $b$ is an atom of $B$, we have, as $g$ is measure preserving, that $b$ and $g(b)$ are composed of the same number of atoms of $A$. Therefore, we can extend $g$ to an automorphism of $A$ by choosing a bijection between the constituents of $b$ and $g(b)$ for each atom $b$ of $B$. \qed

### 2.2. Roots of measure preserving homeomorphisms.

**Proposition 3.** Suppose $A \subseteq B$ are equidistributed (dyadic) boolean algebras, $g$ an automorphism of $A$ and $f$ an automorphism of $B$ such that $f|A = g^n$. Then there is an equidistributed (dyadic) algebra $C \supseteq B$ and an automorphism $h$ of $C$ extending $g$ and such that $h^n|B = f$.

*Proof.* Enumerate the atoms of $A$ as $a_1, \ldots, a_m$ and the atoms of $B$ as

$$b^1_1, b^1_2, \ldots, b^k_1, b^1_2, b^2_2, \ldots, b^k_2, \ldots, b^1_m, b^2_m, \ldots, b^k_m,$$

where

$$a_i = b^1_i \lor b^2_i \lor \ldots \lor b^k_i.$$

Since $g$ is an automorphism of $A$ we can find a permutation $\phi$ of $\{1, \ldots, m\}$ such that

$$g(a_i) = a_{\phi(i)}$$

for all $i$. Similarly, we can find a function $\psi : \{1, \ldots, m\} \times \{1, \ldots, k\} \rightarrow \{1, \ldots, k\}$ such that for all $i$ and $j$

$$f(b^j_i) = b^{\psi(i,j)}_{\phi^n(i)}.$$

For $f(a_i) = g^n(a_i) = a_{\phi^n(i)}$ and thus $f(b^j_i) \leq f(a_i) = a_{\phi^n(i)}$, whence $f(b^j_i) = b^{\psi(i,j)}_{\phi^n(i)}$ for some $\psi(i,j) \in \{1, \ldots, k\}$. Also, since

$$b^{\psi(i,1)}_{\phi^n(i)} \lor b^{\psi(i,2)}_{\phi^n(i)} \lor \ldots \lor b^{\psi(i,k)}_{\phi^n(i)} = f(b^1_i \lor b^2_i \lor \ldots \lor b^k_i) = f(a_i) = a_{\phi^n(i)} = b^{3}_{\phi^n(i)} \lor b^{2}_{\phi^n(i)} \lor \ldots \lor b^{k}_{\phi^n(i)},$$

we see that $\psi(i, \cdot) : \{1, \ldots, k\} \rightarrow \{1, \ldots, k\}$ is a bijection for each $i$.

Let $B_1 = B_2 = \ldots = B_n = B$ and consider the free amalgam $\bigotimes_{A} B_1$. We can now define the automorphism $h$ of $\bigotimes_{A} B_1$ as follows.

$$h(b^{\psi(i,j_n)}_{\phi(i)} \otimes b^{\psi(i,j_1)}_{\phi(i)} \otimes \ldots \otimes b^{\psi(i,j_{n-1})}_{\phi(i)}) = b^{\psi(i,j_n)}_{\phi(i)} \otimes b^{\psi(i,j_1)}_{\phi(i)} \otimes \ldots \otimes b^{\psi(i,j_{n-1})}_{\phi(i)}.$$
It follows from the fact that $\psi(i, \cdot)$ is a bijection that $h$ also is a bijection of the atoms of $\otimes A B_i$ and thus defines an automorphism of $\otimes A B_i$. Consider now

\[
h^n(b_i^{j_1} \otimes b_i^{j_2} \otimes \ldots \otimes b_i^{j_n}) = h^{n-1}(b_{\phi(i)}^{j_1} \otimes b_{\phi(i)}^{j_2} \otimes \ldots \otimes b_{\phi(i)}^{j_{n-1}})
\]
\[
= h^{n-2}(b_{\phi^2(i)}^{j_1} \otimes b_{\phi^2(i)}^{j_2} \otimes b_{\phi^2(i)}^{j_{n-2}}) \ldots
\]
\[
= b_{\phi^{n-1}(i), j_1} \otimes b_{\phi^{n-2}(i), j_2} \otimes \ldots \otimes b_{\phi^n(i), j_n}.
\]

Thus,

\[
h^n(a_i \otimes a_i \otimes \ldots \otimes a_i \otimes b_i^{j_n})
\]
\[
= h^n\left(\bigvee_{j_1=1}^{k} \bigvee_{j_2=1}^{k} \ldots \bigvee_{j_{n-1}=1}^{k} b_i^{j_1} \otimes b_i^{j_2} \otimes \ldots \otimes b_i^{j_n}\right)
\]
\[
= \bigvee_{j_1=1}^{k} \bigvee_{j_2=1}^{k} \ldots \bigvee_{j_{n-1}=1}^{k} h^n(b_i^{j_1} \otimes b_i^{j_2} \otimes \ldots \otimes b_i^{j_n})
\]
\[
= \bigvee_{j_1=1}^{k} \bigvee_{j_2=1}^{k} \ldots \bigvee_{j_{n-1}=1}^{k} b_{\phi^{n-1}(i), j_1} \otimes b_{\phi^{n-2}(i), j_2} \otimes \ldots \otimes b_{\phi^n(i), j_n}
\]
\[
= a_{\phi^n(i)} \otimes a_{\phi^n(i)} \otimes \ldots \otimes a_{\phi^n(i)} \otimes b_{\phi^n(i)}^{j_n}.
\]

Similarly,

\[
h(a_i \otimes a_i \otimes \ldots \otimes a_i) = h\left(\bigvee_{j_1=1}^{k} \bigvee_{j_2=1}^{k} \ldots \bigvee_{j_{n-1}=1}^{k} b_i^{j_1} \otimes b_i^{j_2} \otimes \ldots \otimes b_i^{j_n}\right)
\]
\[
= \bigvee_{j_1=1}^{k} \bigvee_{j_2=1}^{k} \ldots \bigvee_{j_{n-1}=1}^{k} h^{j_1} \otimes h^{j_2} \otimes \ldots \otimes h^{j_n}
\]
\[
= \bigvee_{j_1=1}^{k} \bigvee_{j_2=1}^{k} \ldots \bigvee_{j_{n-1}=1}^{k} b_{\phi(i)}^{j_1} \otimes b_{\phi(i)}^{j_2} \otimes \ldots \otimes b_{\phi(i)}^{j_{n-1}}
\]
\[
= a_{\phi(i)} \otimes a_{\phi(i)} \otimes \ldots \otimes a_{\phi(i)}.
\]

We now identify $B$ with the image of $B_n$ by the embedding $\pi_n$ of $B_n$ into $\otimes A B_i$. Thus, the atoms of $B$ are of the form

\[a_i \otimes a_i \otimes \ldots \otimes a_i \otimes b_i^j\]

and the atoms of $A$ are

\[a_i \otimes a_i \otimes \ldots \otimes a_i.
\]

Moreover, $g$ acts by

\[g(a_i \otimes a_i \otimes \ldots \otimes a_i) = g(a_i) \otimes g(a_i) \otimes \ldots \otimes g(a_i)
\]
\[
= a_{\phi(i)} \otimes a_{\phi(i)} \otimes \ldots \otimes a_{\phi(i)},
\]

while $f$ acts by

\[f(a_i \otimes a_i \otimes \ldots \otimes a_i \otimes b_i^j) = a_{\phi^n(i)} \otimes a_{\phi^n(i)} \otimes \ldots \otimes a_{\phi^n(i)} \otimes b_{\phi^n(i)}^{\psi(i,j)}.
\]

Therefore, $h$ extends $g$, while $h^n$ extends $f$, which was what we wanted.
Proposition 4. Let \( n \geq 1 \). Then the generic measure preserving homeomorphism of Cantor space is conjugate with its \( n \)’th power.

Proof. Notice first that a basic open set in \( \text{Homeo}(2^N, \mu) \) is of the form

\[
U(h, A) = \{ g \in \text{Homeo}(2^N, \mu) \mid g|_A = h|_A \},
\]

where \( A \) is a finite equidistributed subalgebra of \( B_\infty \) and \( h \in \text{Homeo}(2^N, \mu) \). We claim that for any \( U(h, A) \) there is some finite equidistributed \( B \subset B_\infty \) containing \( A \) and some measure preserving homeomorphism \( k \) leaving \( B \) invariant, such that \( U(k, B) \subset U(h, A) \). To see this, suppose \( h \) and \( A \) are given. Choose an equidistributed \( B \) containing both \( A \) and \( h(A) \) and notice that the partial automorphism \( h: A \to h(A) \) of \( B \) extends to an automorphism \( \hat{h} \) of \( B \). So let \( k \) be any measure preserving homeomorphism of \( 2^N \) that extends \( \hat{h} \). Then \( B \) is \( k \)-invariant while \( U(k, B) \subset U(h, A) \).

For simplicity, if \( k \) is an automorphism of a finite equidistributed algebra \( B \), we also write \( U(k, B) \) to denote the set \( \{ g \in \text{Homeo}(2^N, \mu) \mid g|_B = k \} \).

Let now \( C \) be the comeagre conjugacy class of \( \text{Homeo}(2^N, \mu) \) and find dense open sets \( V_i \subset \text{Homeo}(2^N, \mu) \) such that \( C = \bigcap_i V_i \). Enumerate the clopen subsets of \( 2^N \) as \( a_0, a_1, a_2, \ldots \). We shall define a sequence of finite equidistributed algebras \( A_0 \subset A_1 \subset A_2 \subset \ldots \) of clopen sets and automorphisms \( g_i \) and \( f_i \) of \( A_i \) such that

1. \( a_i \in A_{i+1} \),
2. \( g_{i+1} \) extends \( g_i \),
3. \( f_{i+1} \) extends \( f_i \),
4. \( g_i^n = f_i \),
5. \( U(g_{i+1}, A_{i+1}) \subset V_i \),
6. \( U(f_{i+1}, A_{i+1}) \subset V_i \).

To begin, let \( A_0 \) be the trivial algebra with automorphism \( g_0 = f_0 \). So suppose \( A_i, g_i \), and \( f_i \) are defined. We let \( B \) be an equidistributed algebra containing both \( a_i \) and \( A_i \) and let \( h \) be any automorphism of \( B \) extending \( g_i \). As \( V_i \) is dense open we can find some \( U(k, C) \subset V_i \), where \( C \) is a \( k \)-invariant equidistributed algebra containing \( B \) and \( k \) extends \( h \). Again, as \( V_i \) is dense open, we can find some \( U(p, D) \subset V_i \), where \( D \) is a equidistributed algebra containing \( C \), \( p \) a measure preserving homeomorphism leaving \( D \) invariant and extending \( k^n|_C \).

Now, by Proposition 3 we can find an equidistributed algebra \( E \) containing \( D \) and an automorphism \( q \) of \( E \) extending \( k|_C \) such that \( q^n \) extends \( p|_D \). Finally, set \( A_{i+1} = E \),

\[
g_{i+1} = q|_C \supset h \supset g_i,
\]

and

\[
f_{i+1} = q^n \supset p|_D \supset k^n|_C \supset g_i^n = f_i.
\]

Then \( U(g_{i+1}, A_{i+1}) \subset U(k, C) \subset V_n \) and \( U(f_{i+1}, A_{i+1}) \subset U(p, D) \subset V_n \).

Set now \( g = \bigcup_i g_i \) and \( f = \bigcup_i f_i \). By (1),(2), and (3), \( f \) and \( g \) are measure preserving automorphisms of \( B_\infty \) and, thus by Stone duality, measure preserving homeomorphisms of \( 2^N \). And by (4), \( g^n = f \), while by (5) and (6), \( f, g \in \bigcap_i V_i = C \). Thus, \( f \) and \( g \) belong to the comeagre conjugacy class and are therefore mutually conjugate. \( \square \)

Proposition 5. Let \( G \) be a Polish group with a comeagre conjugacy class. Then the generic element of \( G \) is conjugate to its inverse.
Proof. Let \( C \) be the comeagre conjugacy class of \( G \). Then also \( C^{-1} \) is comeagre, so must intersect \( C \) in some point \( g \). Thus both \( g \) and \( g^{-1} \) are generic and hence conjugate. Now, being conjugate with your inverse is a conjugacy invariant property and thus holds generically in \( G \). \( \square \)

**Theorem 6.** Let \( n \neq 0 \). Then the generic measure preserving homeomorphism of Cantor space is conjugate with its \( n \)'th power and hence has roots of all orders.

Thus, for the generic measure preserving homeomorphism \( g \), there is an action of \( (\mathbb{Q}, +) \) by measure preserving homeomorphisms of \( 2^N \) such that \( g \) is the action by \( 1 \in \mathbb{Q} \).

**Proof.** We know that the generic \( g \) is conjugate to all its positive powers and to \( g^{-1} \). But then \( g^{-1} \) is generic and thus conjugate to \( (g^{-1})^n = g^{-n} \), whence \( g \) is conjugate with \( g^{-n} \), \( n \geq 1 \).

So suppose \( g \) is generic and \( n \geq 1 \). Then, there is some \( f \) such that \( (fgf^{-1})^n = fg^n f^{-1} = g \), and hence \( g \) has a generic \( n \)'th root, namely, \( fgf^{-1} \). This means that we can define a sequence \( g = g_1, g_2, \ldots \) of generic elements such that \( g_{n+1} \) is an \( n + 1 \)'th root of \( g_n \). The following therefore defines an embedding of \((\mathbb{Q}, +)\) into \( \text{Homeo}(2^N, \mu) \) with \( 1 = \frac{1}{n!} \mapsto g_1, \frac{k}{n!} \mapsto g_n^k \), \( k \in \mathbb{Z}, n \geq 1 \). \( \square \)

### 3. Powers of generic isometries

#### 3.1. Free amalgams of metric spaces.** Suppose \( A \) and \( B_1, \ldots, B_n \) are non-empty finite metric spaces and \( \iota_i : A \hookrightarrow B_i \) is an isometric embedding for each \( i \). We define the **free amalgam** \( \bigcup A B_i \) of \( B_1, \ldots, B_n \) over \( A \) and the embeddings \( \iota_1, \ldots, \iota_n \) as follows.

Denote by \( d_i \) the metric on \( B_i \) for each \( i \) and let \( C_i = B_i \setminus \iota_i[A] \). By renaming elements, we can suppose that \( C_1, \ldots, C_n \) and \( A \) are pairwise disjoint.

We then let the universe of \( \bigcup A B_i \) be \( A \cup \bigcup_{i=1}^n C_i \) and define the metric \( \partial \) by the following conditions

\[
\begin{align*}
(1) \quad \partial(x, y) &= d_i(\iota_i x, \iota_i y) \quad \text{for } x, y \in A, \\
(2) \quad \partial(x, y) &= d_i(\iota_i x, y) \quad \text{for } x \in A \text{ and } y \in C_i, \\
(3) \quad \partial(x, y) &= d_i(x, y) \quad \text{for } x, y \in C_i, \\
(4) \quad \partial(x, y) &= \min_{z \in A} d_i(x, \iota_i z) + d_j(\iota_j z, y) \quad \text{for } x \in C_i \text{ and } y \in C_j, i \neq j.
\end{align*}
\]

We notice first that in (1) the definition is independent of \( i \) since each \( \iota_i \) is an isometry. Also, a careful checking of the triangle inequality shows that this indeed defines a metric \( \partial \) on \( A \cup \bigcup_{i=1}^n C_i \).

We define for each \( i \) an isometric embedding \( \pi_i : B_i \hookrightarrow \bigcup A B_i \) by

\[
\begin{align*}
\pi_i(x) &= x \quad \text{for } x \in C_i, \\
\pi_i(\iota_i x) &= x \quad \text{for } x \in A.
\end{align*}
\]

Notice that in this way the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_i} & B_i \\
\downarrow & & \downarrow \pi_i \\
B_j & \xrightarrow{\iota_j} & \bigcup A B_i
\end{array}
\]
3.2. Roots of isometries.

**Proposition 7.** Let \( A \subseteq B \) be finite rational metric spaces, \( f \) and isometry of \( A \) and \( g \) an isometry of \( B \) leaving \( A \) invariant and such that \( f^n = g|_A \) for some \( n \geq 1 \). Then there is a finite rational metric space \( D \supseteq B \) and an isometry \( h \) of \( D \) such that \( h^n \) leaves \( B \) invariant and \( h^n|_B = g \).

**Proof.** Let \( B_1 = \ldots = B_n = B \) and define isometric embeddings \( \iota_i : A \hookrightarrow B_i \) by

\[
\iota_i(x) = f^{-i}(x).
\]

To distinguish between the different copies of \( B \), we let for \( x \in B \setminus A \), \( x^i \) denote the copy of \( x \) in \( C_i = B_i \setminus \iota_i[A] = B_i \setminus A \). Note also that \( B = B_1 = \ldots = B_n \) all have the same metric, which we denote by \( d \). We now define \( h \) on \( \bigsqcup \bigcup A B_i \) as follows.

- \( h(x) = f(x) \) for \( x \in A \),
- \( h(x^i) = x^{i+1} \) for \( x \in B \setminus A \) and \( 1 \leq i < n \),
- \( h(x^n) = (gx)^i \) for \( x \in B \setminus A \).

Now, obviously, \( h \) is a permutation of \( A \) and for \( 1 \leq i < n \), \( h \) is a bijection between \( C_i \) and \( C_{i+1} \). Moreover, \( h \) is a bijection between \( C_n \) and \( C_1 \). Therefore, \( h \) is a permutation of \( \bigsqcup A B_i \). We check that \( h \) is 1-Lipschitz.

Suppose first that \( x, y \in A \). Then

\[
\partial(hx, hy) = \partial(fx, fy) = d(\iota_i fx, \iota_i fy) = d(f^{-i} fx, f^{-i} fy) = d(f^{1-i} x, f^{1-i} y) = d(f^{-i} x, f^{-i} y) = d(\iota_i x, \iota_i y) = \partial(x, y).
\]

Also, \( h \) is clearly an isometry between \( C_i \) and \( C_{i+1} \) for \( 1 \leq i < n \). So consider the case \( C_n \). Fix \( x, y \in B \setminus A \). Then

\[
\partial(h(x^n), h(y^n)) = \partial((gx)^i, (gy)^i) = d(gx, gy) = d(x, y) = \partial(x^n, y^n).
\]

Now, if \( x \in A \), \( y \in B \setminus A \), and \( 1 \leq i < n \), then

\[
\partial(h(x), h(y^i)) = \partial(fx, y^{i+1}) = d(\iota_{i+1} fx, y) = d(f^{-i} x, y) = d(\iota_i x, y) = \partial(x, y^i).
\]
Also, if $x \in A$, $y \in B \setminus A$, then
\[
\partial(h(x), h(y^n)) = \partial(fx, (gy)^1)
= d(ι_1 fx, gy)
= d(f^{-1} fx, gy)
= d(x, gy)
= d(g^{-1} x, y)
= d(f^{-n} x, y)
= d(ι_n x, y)
= \partial(x, y^n).
\]

And finally, if $x, y \in B \setminus A$ and $1 \leq i < j \leq n$, we pick $z \in A$ such that the distance $\partial(x^i, y^j)$ is witnessed by $z$, i.e.,
\[
\partial(x^i, y^j) = d(x, ι_iz) + d(ι_jz, y) = d(x, f^{-i}z) + d(f^{-j}z, y).
\]

Assume first that $j < n$. Then
\[
\partial(h(x^i), h(y^j)) = \partial(x^{i+1}, y^{j+1})
\leq d(x, ι_{i+1}fz) + d(ι_{j+1}fz, y)
= d(x, f^{-i}z) + d(f^{-j}z, y)
= \partial(x^i, y^j).
\]

And if $j = n$, we have
\[
\partial(h(x^i), h(y^n)) = \partial(x^{i+1}, (gy)^1)
\leq d(x, ι_{i+1}fz) + d(ι_1 fz, gy)
= d(x, f^{-i}z) + d(z, f^ny)
= d(x, f^{-i}z) + d(f^{-n}z, y)
= \partial(x^i, y^n).
\]

Thus, $h$ is an isometry of $\bigsqcup_A B_l$.

Now see $g$ and $f$ as isometries of the first copy $B_1$ of $B$, i.e., $g(x^1) = (gx)^1$ for $x^1 \in C_1$. Let $π_1: B_1 \hookrightarrow \bigsqcup_A B_l$ be the canonical isometric embedding defined by
\begin{itemize}
  \item $π_1(x^1) = x^1$ for $x^1 \in C_1$,
  \item $π_1(ι_1 x) = x$ for $x \in A$.
\end{itemize}

To finish the proof, we need to show that the following diagram commutes
\[
\begin{array}{ccc}
B_1 & \xrightarrow{g} & B_1 \\
\downarrow{π_1} & & \downarrow{π_1} \\
\bigsqcup_A B_l & \xrightarrow{h^n} & \bigsqcup_A B_l
\end{array}
\]

First, suppose $y = ι_1 x \in A$. Then
\[
h^nπ_1y = h^nπ_1ι_1x = h^nx = f^nx
= π_1ι_1f^nx = π_1f^{-1}f^nx = π_1f f^{-1}x
= π_1f^nι_1x = π_1f^n y = π_1gy.
\]
Now suppose that $x \in B \setminus A$. Then
\[ h^n\pi_1(x') = h^n(x^1) = h(x^n) = (gx)^1 = \pi_1(gx)^1 = \pi_1(g(x^1)). \]
\[
\]

**Proposition 8.** Let $n \geq 1$. Then the generic isometry of the rational Urysohn metric space is conjugate with its $n$’th power.

**Proof.** A basic open set in $\text{Iso}(QU)$ is on the form

\[ U(h,A) = \{g \in \text{Iso}(QU) \mid g|_A = h|_A\}, \]

where $A$ is a finite subspace of $QU$ and $h \in \text{Iso}(QU)$. We claim that for any $U(h,A)$ there is some finite $B \subseteq QU$ containing $A$ and some isometry $k$ leaving $B$ invariant, such that $U(k,B) \subseteq U(h,A)$. For if $h$ and $A$ are given, choose by Theorem $[\mathcal{I}]$ some finite $B \subseteq QU$ containing both $A$ and $h(A)$ such that the partial isometry $h: A \to h(A)$ of $A \cup h(A)$ extends to an isometry $h$ of $B$. Let $k$ be any isometry of $QU$ that extends $h$. Then $B$ is $k$-invariant while $U(k,B) \subseteq U(h,A)$.

Again, if $k$ is an isometry of some finite $B \subseteq QU$, we let $U(k,B) = \{g \in \text{Iso}(QU) \mid g|_B = k\}$. Let now $C$ be the comeagre conjugacy class of $\text{Iso}(QU)$ and find dense open sets $V_i \subseteq \text{Iso}(QU)$ such that $C = \bigcap_i V_i$. Enumerate the points of $QU$ as $a_0, a_1, a_2, \ldots$. We shall define a sequence of finite subsets $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq QU$ and isometries $g_i$ and $f_i$ of $A_i$ such that

1. $a_i \in A_{i+1}$,
2. $g_{i+1}$ extends $g_i$,
3. $f_{i+1}$ extends $f_i$,
4. $g_i^n = f_i$,
5. $U(g_{i+1}, A_{i+1}) \subseteq V_i$,
6. $U(f_{i+1}, A_{i+1}) \subseteq V_i$.

To begin, let $A_0 = \emptyset$ with trivial isometries $g_0 = f_0$. So suppose $A_i$, $g_i$, and $f_i$ are defined. We let $B \subseteq QU$ be a finite subset containing both $a_i$ and $A_i$ and such that there is some isometry $h$ of $B$ extending $g_i$. As $V_i$ is dense open we can find some $U(k,C) \subseteq V_i$, where $C \subseteq QU$ is a $k$-invariant finite set containing $B$ and $k$ extends $h$. Again, as $V_i$ is dense open, we can find some $U(p,D) \subseteq V_i$, where $D \subseteq QU$ is a finite set containing $C$, $p$ an isometry of $QU$ leaving $D$ invariant and extending $k^n|C$.

Now, by Proposition $[\mathcal{I}]$ we can find a finite subset $E \subseteq QU$ containing $D$ and an isometry $q$ of $E$ extending $k|C$ such that $q^n$ extends $p|D$. Finally, set $A_{i+1} = E$,

\[ g_{i+1} = q \supseteq k|C \supseteq h \supseteq g_i, \]

and

\[ f_{i+1} = q^n \supseteq p|D \supseteq k^n|C \supseteq g_i^n = f_i. \]

Then $U(g_{i+1}, A_{i+1}) \subseteq U(k,C) \subseteq V_n$ and $U(f_{i+1}, A_{i+1}) \subseteq U(p,D) \subseteq V_n$.

Set now $g = \bigcup_i g_i$ and $f = \bigcup_i f_i$. By (1),(2), and (3), $f$ and $g$ are isometries of $QU$. And by (4), $g^n = f$, while by (5) and (6), $f, g \in \bigcap_i V_i = C$. Thus, $f$ and $g$ belong to the comeagre conjugacy class and are therefore mutually conjugate. \[ \square \]

Now in exactly the same way as for measure preserving homeomorphisms, we can prove
Theorem 9. Let $n \neq 0$. Then the generic isometry of the rational Urysohn metric space is conjugate with its $n$'th power and hence has roots of all orders.

Thus, for the generic isometry $g$, there is an action of $(\mathbb{Q}, +)$ by isometries of $\mathbb{Q}U$ such that $g$ is the action by $1 \in \mathbb{Q}$.

4. Comeagre conjugacy classes and Baire category

We now set up the framework allowing us to pass between a Polish group and a dense subgroup with a comeagre conjugacy class.

Definition 10. Let $\pi : X \to Y$ be a Borel map between Polish spaces. We say that $\pi$ is categorical if the following conditions hold.

1. If $A \subseteq X$ is a nonmeagre analytic set, then $\pi(A) \subseteq Y$ is nonmeagre,
2. if $A \subseteq X$ is a comeagre analytic set, then $\pi(A) \subseteq Y$ is comeagre,
3. if $B \subseteq Y$ is a meagre analytic set, then $\pi^{-1}(B) \subseteq X$ is meagre,
4. if $B \subseteq Y$ is a nonmeagre analytic set, then $\pi^{-1}(B) \subseteq X$ is nonmeagre,
5. and if $B \subseteq Y$ is a comeagre analytic set, then $\pi^{-1}(B) \subseteq X$ is comeagre.

We say that $\pi$ is strongly categorical if, moreover,

6. if $A \subseteq X$ is a meagre analytic set, then $\pi(A) \subseteq Y$ is meagre.

We notice that (1) ⇔ (3) ⇔ (5) and (2) ⇔ (4). But (2) $\nRightarrow$ (1) $\nRightarrow$ (2). However, if $\pi(X)$ is comeagre in $Y$, then (6) $\Rightarrow$ (2).

Lemma 11. Let $\pi : X \to Y$ be a surjective, continuous, and open map between Polish spaces. Then $\pi$ is categorical.

Proof. (5) follows from exercise (8.45) in Kechris [7]. So to see (2), assume towards a contradiction that $A \subseteq X$ is comeagre, but $\pi(A)$ is not comeagre. Then, pick some non-empty open $U \subseteq Y$ in which $\pi(A)$ is meagre and let $X_0 = \pi^{-1}(U)$, which is non-empty and open. The map $\pi : X_0 \to U$ is then surjective, continuous, and open, whereby (5) holds for this map. But then $\pi(A \cap X_0)$ is nonmeagre in $U$ being the image of the nonmeagre set $A \cap X_0$ and using the equivalence of (1) and (5).

This contradicts that $\pi(A)$ is meagre in $U$. $\square$

Before we state the next result, recall that, by a theorem of Marker and Sami [2], if a Polish group $G$ has a comeagre conjugacy class $C$, then $C$ is $G_\delta$ in $G$ and thus a Polish space in its induced topology. Moreover, $G$ acts continuously and transitively on $C$ by conjugation.

Proposition 12. Suppose $G$ is a Polish group with a comeagre conjugacy class $C$. Suppose also that an element of $C$ is conjugate with all its non-zero powers. Then, for every $n \neq 0$, the map $\pi : C \to C$ defined by

$$\pi(g) = g^n$$

is a surjective, continuous, and open $G$-map. In particular, $\pi$ is categorical.

Proof. First notice that $\pi$ is a continuous $G$-map, where $G$ acts on $C$ by conjugation,

$$\pi(ghg^{-1}) = (ghg^{-1})^n = gh^ng^{-1} = g\pi(h)g^{-1}. $$

Also, $\pi$ is surjective. For if $f \in C$, we know that $f$ and $f^n$ are conjugate and thus for some $g \in G$, $gfg^{-1} = f$, whence $\pi(gfg^{-1}) = gfg^{-1} = f$. Finally, to see that $\pi$ is open, suppose that $U \subseteq C$ is open and let $g \in U$. As $G$ acts continuously on $C$ by the conjugacy action, which we write as $h.g = hgh^{-1}$, we
can find some open neighbourhood $V$ of 1 in $G$ such that $Vg \subseteq U$. But then by Effros’ Theorem (see Becker and Kechris [2]) the set $V, \pi(g)$ is open in $C$ and $\pi(g) \in V, \pi(g) = \pi(Vg) \subseteq \pi(U)$. Thus $\pi(U)$ is open. 

**Proposition 13.** Suppose $H$ is a Polish group with a dense subgroup $G$, which is Polish in a finer topology. Assume that $G$ has a comeagre conjugacy class $C$ whose elements are conjugate with their non-zero powers. Assume $n \neq 0$ and let $\pi : H \to H$ be the continuous $H$-map $\pi(g) = g^n$.

Then $\pi$ is categorical.

**Proof.** Suppose towards a contradiction that $A \subseteq H$ is nonmeagre, while $\pi(A)$ is meagre. Find closed nowhere dense sets $F_n \subseteq H$ such that $\pi(A) \subseteq \bigcup F_n$. Then $\pi^{-1}(F_n)$ is closed and $A \subseteq \bigcap \pi^{-1}(F_n)$ is nonmeagre, whereby some $\pi^{-1}(F_n)$ must be nonmeagre and thus have non-empty interior $U$. Now, $C$ is dense in $H$ and thus $U \cap C$ is open in the topology of $C$, whereby $\pi(U \cap C)$ is nonmeagre and hence somewhere dense in $C$. Therefore, $\pi(U \cap C) \subseteq F_n$ is also somewhere dense in $H$, which is a contradiction. Thus, images of nonmeagre sets are nonmeagre.

We now need to show that also images of comeagre sets are comeagre. So suppose $A \subseteq H$ is comeagre and let $H_0 \leq H$ be a countable dense subgroup. Set $B = \bigcap_{g \in H_0} gAg^{-1}$, which is comeagre and $H_0$-invariant. But then $\pi(B)$ is nonmeagre and $H_0$-invariant, whence, as the action of $H_0$ on $H$ is topologically transitive, $\pi(B)$ must be comeagre. Thus also $\pi(A)$ is comeagre. 

**5. Measure preserving automorphisms and isometries**

We can now apply our Theorems 6 and 9 in combination with Proposition 13 to deduce results, the first of which is due to J.L.F. King, about respectively the automorphism group of a standard probability space and the isometry group of the Urysohn metric space.

Before stating the next result we recall the so called weak topology on the group $\text{Aut}([0, 1], \lambda)$ of Lebesgue measure preserving automorphisms of the unit interval: It is the weakest topology such that for all Borel sets $A, B \subseteq [0, 1]$ the map $g \mapsto \lambda(gA \triangle B)$ is continuous.

**Theorem 14** (J.L.F. King [9]). The generic measure preserving automorphism of the unit interval has roots of all orders.

**Proof.** It is well-known that the group of Haar measure preserving homeomorphisms embeds continuously as a dense subgroup of the group $H$ of measure preserving automorphisms of the unit interval (see, e.g., Halmos [5]). Thus, by Theorem 9 and Proposition 13 we know that for each $n \neq 0$, the map $\pi : H \to H$ given by $\pi(g) = g^n$ is categorical. In particular, $\pi(H)$ is comeagre in $H$ and thus the generic element of $H$ has an $n$’th root. As $n \geq 1$ is arbitrary, this thus holds for all $n \geq 1$ simultaneously. 

We should mention that there are now significantly better results known for the generic automorphism. Thus T. de la Rue and J. de Sam Lazaro [11] prove, in
response to a question of J.L.F. King [9], that the generic automorphism is the
time 1 map of a measure preserving R-flow and hence of course has roots of all
orders.

**Theorem 15.** The generic isometry of the Urysohn metric space has roots of all
orders.

*Proof.* We repeat the proof of Theorem 14 using that the isometry group of the
rational Urysohn metric space embeds continuously and densely into the isometry
group of the Urysohn metric space.

It is important to notice that there are in fact isometries of the Urysohn space
without square roots. This is proved by J. Melleray in his thesis [10].

6. **Topological similarity and Rohlin’s Lemma for isometries**

Suppose $G$ is a Polish group and $f, g \in G$. We say that $f$ and $g$ are topologically
similar if the topological groups $\langle f \rangle \leq G$ and $\langle g \rangle \leq G$ are isomorphic.

Notice first that any $f$ is topologically similar to $f^{-1}$. If $\psi(f^n) = f^{-n}$, then
$\psi$ is an involution homeomorphism, since inversion is continuous in $G$. Of course,
if $f$ and $g$ have infinite order, then any isomorphic homeomorphism $\phi$ between
$\langle f \rangle$ and $\langle g \rangle$ must send the generators to the generators and so either $\phi(f) = g$ or
$\phi(f) = g^{-1}$. But then composing with $\psi$ we can always suppose that $\phi(f) = g$.

Notice also that as each $\langle f \rangle$ is metrisable, $f$ and $g$ are topologically similar if and
only if for all increasing sequences $(s_n) \subseteq \mathbb{N}$, $f^{s_n} \rightarrow 1$ if and only if $g^{s_n} \rightarrow 1$.

Thus, in particular, topological similarity is a coanalytic equivalence relation. We
notice also that topological similarity is really independent of the ambient group
$G$. For example, if $G$ is topologically embedded into another Polish group $H$, then
$f$ and $g$ are topologically similar in $G$ if and only if they are topologically similar
in $H$.

Topological similarity is an obvious invariant for conjugacy, that is, if there is
any way to make $f$ and $g$ conjugate in some Polish group, then they have to be
topologically similar.

Of particular interest are the cases $G = \text{Aut}([0, 1], \lambda)$, $G = U(\ell_2)$, and $G =
\text{Iso(U)}$. Here of course $\text{Aut}([0, 1], \lambda)$ sits inside of $U(\ell_2)$ via the Koopman
representation and two measure preserving transformations $f$ and $g$ are said to be spectrally
equivalent if they are conjugate in $U(\ell_2)$. By the spectral theorem, spectral equivalence is Borel.
Also, topological similarity is coarser than spectral equivalence. To see this, we notice that mixing is not a topological similarity invariant, whereas
it is a spectral invariant. For if $f$ is mixing, then the automorphism $f \oplus \text{id}$ is
a non-mixing transformation of $[0, 1] \oplus [0, 1]$ but generates a discrete subgroup of
$\text{Aut}([0, 1] \oplus [0, 1], \lambda \oplus \lambda)$. So taking a transformation $h \in \text{Aut}([0, 1], \lambda)$ conjugate
with $f \oplus \text{id}$, we see that $f$ and $h$ are topologically similar, since they both generate
discrete groups. A survey of the closely related topic of topological torsion elements
in topological groups is given by Dikranjan in [3].

**Proposition 16.** Let $G$ be a non-trivial Polish group such that for all infinite
$S \subseteq \mathbb{N}$ the set $\mathcal{A}(S) = \{g \in G \mid \exists s \in S \ g^s = 1\}$ is dense. Then every topological
similarity class of $G$ is meagre.

Moreover, for every infinite $S \subseteq \mathbb{N}$ the set
\[ \mathcal{C}(S) = \{g \in G \mid \exists (s_n) \subseteq S \ g^{s_n} \rightarrow 1\}. \]


is dense $G_δ$ and invariant under topological similarity.

Proof. Let $V_0 ⊆ V_1 ⊆ \ldots$ be a basis of open neighbourhoods of the identity and set

$$\mathcal{B}(S, k) = \{g ∈ G \mid ∃n ∈ [1, k] \ g^n ∈ V_k\},$$

which is open and dense since it contains $\mathcal{A}(S \setminus [1, k])$. Now set

$$\mathcal{C}(S) = \{g ∈ G \mid ∃s_n \subseteq S \ g^{s_n} \xrightarrow{n} 1\}$$

$$= \{g ∈ G \mid ∀ k \ ∃ n ∈ [1, k] \ g^n ∈ V_k\}$$

$$= \bigcap_k \mathcal{B}(S, k).$$

Then $\mathcal{C}(S)$ is invariant under topological similarity and dense $G_δ$.

Now if some topological similarity class $C$ was nonmeagre, then

$$C \subseteq \bigcap_{S ∈ \mathbb{N} \text{ infinite}} \mathcal{C}(S)$$

and hence for all $g ∈ C$, $g^n \xrightarrow{n} 1$, implying that $g = 1$, which is impossible. $\square$

Since Rohlin’s Lemma the sets $\{g ∈ \text{Aut}(\bar{[0,1]}), \ λ) \mid ∃ s ∈ \mathcal{N} g^s = 1\}$ are dense in $\text{Aut}(\bar{[0,1]}), \ λ)$ for all infinite $S ⊆ \mathbb{N}$, we have

**Corollary 17.** Every topological similarity class is meagre in $\text{Aut}(\bar{[0,1]}), \ λ)$.

This improves a result of Rohlin saying that all conjugacy classes are meagre in $\text{Aut}(\bar{[0,1]}), \ λ)$. We clearly see the importance of Rohlin’s Lemma in these matters. However, interestingly, Rohlin’s Lemma can also be used to prove the existence of dense conjugacy classes in $\text{Aut}(\bar{[0,1]}), \ λ)$.

We now have the following analogue of Rohlin’s Lemma for isometries of the Urysohn metric space.

**Proposition 18** (Rohlin’s Lemma for isometries). Suppose $S ⊆ \mathbb{N}$ is infinite. Then the set

$$\{g ∈ \text{Iso}(\mathbb{U}) \mid ∃ n ∈ S \ g^n = 1\}$$

is dense in $\text{Iso}(\mathbb{U})$.

A finite circular order is a finite subset $\mathbb{F}$ of the unit circle $S^1$. If $x ∈ \mathbb{F}$, we denote by $x^+$ the first $y ∈ \mathbb{F}$ encountered by moving counterclockwise around $S^1$ beginning at $x$. We then denote $x$ by $y^-$, i.e., $x^+ = y$ if and only if $y^- = x$.

**Lemma 19.** Suppose $h$ is an isometry of $\mathbb{U}$ and $\delta > 0$. Then for all finite $A ⊆ \mathbb{U}$ there is an isometry $f$ of $\mathbb{U}$ such that $d(f(a), h(a)) ≤ \delta$ for all $a ∈ A$ while $d(a, f(b)) ≥ \delta$ for all $a, b ∈ A$.

Proof. Let $B = A ∪ h[A]$ and let $C = B × \{0, \delta\}$ be equipped with the $\ell_1$-metric $d_1((b, x), (b', y)) = d(b, b') + |x − y|$. Clearly, $B$ is isometric with $B × \{0\}$ and $B × \{\delta\}$, so we can assume that $B$ is actually $B × \{0\} ⊆ C ⊆ \mathbb{U}$. Now, let $f$ be any isometry of $\mathbb{U}$ such that $f(a, 0) = (h(a), \delta)$ for $a ∈ A$. $\square$

Now for the proof of Proposition 18.
Proof. Suppose $A \subseteq U$ is finite, $h$ an isometry of $U$, and $\epsilon > 0$. We wish to find some isometry $g$ such that $d(g(a), h(a)) < \epsilon$ for all $a \in A$ and such that for some $s \in S$, $g^s = 1$. Find first some $f$ such that $d(f(a), h(a)) < \epsilon$ for all $a \in A$ while $d(a, f(b)) > \epsilon/2$ for all $a, b \in A$. It is therefore enough to find some $g$ that agrees with $f$ on $A$ while $g^s = 1$ for some $s \in S$.

We let $\Delta = \text{diam}(A \cup f[A])$ and $\delta = \min(d(x, f(y)) \mid x, y \in A)$. Fix a number $s \in S$ such that $\delta \cdot (s - 2) \geq \Delta$ and take a finite circular order $F$ of cardinality $s$. Now let

$$B = \{a \cdot x \mid a \in A \& x \in F\},$$

where $a \cdot x$ are formally new points.

A path in $B$ is a sequence $p = (a_0 \cdot x_0, a_1 \cdot x_1, \ldots, a_n \cdot x_n)$ where $n \geq 1$ and such that for each $i$, $x_{i+1}$ is either $x_i^-$, $x_i^+$, or $x_i^-$. We define the length of a path by

$$\ell(p) = \sum_{i=0}^{n-1} \rho(a_i \cdot x_i, a_{i+1} \cdot x_{i+1}),$$

where

$$\rho(a \cdot x, b \cdot y) = \begin{cases} d(a, b), & \text{if } y = x; \\ d(a, f(b)), & \text{if } y = x^+; \\ d(f(a), b), & \text{if } y = x^-, \end{cases}$$

and put $|p| = n + 1$.

Therefore, if $\tilde{p}$ denotes the reverse path of $p$ and $p \cdot q$ the concatenation of two paths (whenever it is defined), then $\ell(\tilde{p}) = \ell(p)$ and $\ell(p \cdot q) = \ell(p) + \ell(q)$. Thus, $\ell$ is the distance function in a finite graph with weighted edges and hence the following defines a a metric on $B$

$$D(a \cdot x, b \cdot y) = \inf \{\ell(p) \mid p \text{ is a path with initial point } a \cdot x \text{ and end point } b \cdot y\}.$$ 

We say that two paths are equivalent if they have the same initial point and the same end point. We also say that a path $p$ is positive if either $p = (a \cdot x, b \cdot x)$ for some $x \in F$ or $p = (a_0 \cdot x_0, a_1 \cdot x_1, \ldots, a_n \cdot x_n)$, where $x_{i+1} = x_i^+$ for all $i$. Similarly, $p$ is negative if either $p = (a \cdot x, b \cdot x)$ for some $x \in F$ or $p = (a_0 \cdot x_0, a_1 \cdot x_1, \ldots, a_n \cdot x_n)$, where $x_{i+1} = x_i^-$ for all $i$. So $p$ is positive if and only if $\tilde{p}$ is negative. Notice also that if $p$ is positive, then $\ell(p) \geq \delta \cdot (|p| - 2)$.

Lemma 20. For every path $p$ there is an equivalent path $q$, with $\ell(q) \leq \ell(p)$, which is either positive or negative.

Proof. If $p$ is not either positive or negative, then there is a segment of $p$ of one of the following forms

$$(1) \quad (a \cdot x, b \cdot x, c \cdot x), \quad (5) \quad (a \cdot x, b \cdot x^-, c \cdot x),$$

$$(2) \quad (a \cdot x^+, b \cdot x, c \cdot x), \quad (6) \quad (a \cdot x, b \cdot x, c \cdot x^+),$$

$$(3) \quad (a \cdot x^-, b \cdot x, c \cdot x), \quad (7) \quad (a \cdot x, b \cdot x, c \cdot x^-),$$

$$(4) \quad (a \cdot x, b \cdot x^+, c \cdot x).$$

We replace these by respectively

$$(1') \quad (a \cdot x, c \cdot x), \quad (5') \quad (a \cdot x, c \cdot x),$$

$$(2') \quad (a \cdot x^+, c \cdot x), \quad (6') \quad (a \cdot x, c \cdot x^+),$$

$$(3') \quad (a \cdot x^-, c \cdot x), \quad (7') \quad (a \cdot x, c \cdot x^-),$$

$$(4') \quad (a \cdot x, c \cdot x).$$
and see that by the triangle inequality for \( d \) we can only decrease the value of \( \ell \).

For example, in case (3), we see that
\[
\rho(a \cdot x^-, b \cdot x) + \rho(b \cdot x, c \cdot x) = d(a, f(b)) + d(b, c) = d(a, f(b)) + d(f(b), f(c)) \geq d(a, f(c)) = \rho(a \cdot x^-, c \cdot x).
\]

We can then finish the proof by induction on \(|p|\).

We now claim that \( D(a \cdot x, b \cdot x) = d(a, b) \). To see this, notice first that \( D(a \cdot x, b \cdot x) \leq d(a, b) \). For the other inequality, let \( p \) be an either positive or negative path from \( a \cdot x \) to \( b \cdot x \). By symmetry, we can suppose \( p \) is positive. But then, unless \( p = (a \cdot x, b \cdot x) \), we must have \(|p| \geq s + 1\), whence also \( \ell(p) \geq \delta \cdot (|p| - 2) \geq \delta \cdot (s - 1) \geq \Delta \geq d(a, b) \). A similar argument shows that \( D(a \cdot x, b \cdot x^+) = d(a, f(b)) \).

This shows that for any \( x_0 \in F \), \( A \cup f[A] \) is isometric with \( A \times \{x_0, x_0^+\} \) by the function \( a \mapsto a \cdot x_0 \) and \( f(a) \mapsto a \cdot x_0^+ \). So we can just identify \( A \cup f[A] \) with \( A \times \{x_0, x_0^+\} \). Notice also that the following mapping \( g \) is an isometry of \( B \):
\[
a \mapsto \mapsto a \cdot x^+.
\]

Moreover, it agrees with \( f \) on their common domain \( A \times \{x_0\} \). Realising \( B \) as a subset of \( U \) containing \( A \), we see that \( g \) acts by isometries on \( B \) with \( g^s = 1 \). It then follows that \( g \) extends to a full isometry of \( U \) still satisfying \( g^s = 1 \).

\[\square\]

Corollary 21. Every topological similarity class is meagre in Iso(U).

Again this strengthens a result of Kechris \[11\] saying that all conjugacy classes are meagre.

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Address of C. Rosendal:
Department of mathematics,
University of Illinois at Urbana-Champaign,
273 Altgeld Hall, MC 382,
1409 W. Green Street,
Urbana, IL 61801,
USA.
rosendal@math.uiuc.edu