Non-linear $\sigma$-model for long range disorder and quantum chaos.

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We suggest a new scheme of derivation of a non-linear ballistic $\sigma$-model for a long range disorder and quantum billiards. The derivation is based on writing equations for quasiclassical Green functions for a fixed long range potential and exact representation of their solutions in terms of functional integrals over supermatrices $Q$ with the constraint $Q^2 = 1$. Averaging over the long range disorder or energy we are able to write a ballistic $\sigma$-model for all distances exceeding the electron wavelength (Eq. 3.13). Neither singling out slow modes nor a saddle-point approximation are used in the derivation. Carrying out a coarse graining procedure that allows us to get rid off scales in the Lapunov region we come to a reduced $\sigma$-model containing a conventional collision term. For quantum billiards, we demonstrate that, at not very low frequencies, one can reduce the $\sigma$-model to a one-dimensional $\sigma$-model on periodic orbits. Solving the latter model, first approximately and then exactly, we resolve the problem of repetitions.

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1. INTRODUCTION

The $\sigma$-model approach to disordered systems, first written within the replica trick, proved to be a powerful method of calculations when formulated in the supersymmetric form. This method allows to describe the electron motion at large distances assuming that at shorter distances the motion is diffusive. For study of such phenomena as localization, level statistics in a system of a size $L$ much exceeding the elastic mean free path, etc., the information obtained from the $\sigma$-model is sufficient. Although the $\sigma$-model is not valid at distances smaller than the mean free path $l$, the information about the motion at distances below $l$ is not very interesting for these phenomena.

Success in nano-fabrication made possible producing and studying clean systems with the size smaller than the elastic mean free path $l$ (for a review, see, e.g. [4]). In such small systems called now quantum dots electrons move ballistically being scattered mainly the walls. There are many interesting questions about transport in the quantum dots and related systems that cannot be answered using the picture of the diffusive motion. In order to describe the ballistic motion one has to go beyond the diagrammatic and field theoretical methods developed for disordered systems.

Another motivation to study the ballistic motion originates from the field called now quantum chaos. The subject of research in the quantum chaos is to understand the quantum behavior within models that are chaotic in the classical limit. There are many books and reviews related to this field (see, e.g. [5]). The most popular analytical tool for studying the quantum chaos is the Gutzwiller trace formula, which reduces calculation of the density of states to a sum over periodic orbits.

This method (complemented by different approximation schemes) allows one to study very well the limit of not very long times $t$ when the motion along a periodic orbit is well defined. At the same time, calculations with the trace formulae become very difficult in the limit $t \to \infty$ when one expects an universal behavior described by the Wigner-Dyson statistics.

Both the experimental and theoretical interest to investigations of the ballistic motion resulted in several attempts to construct a generalization of the supersymmetric $\sigma$-model to distances smaller than the mean free path $l$ due to scattering on impurities or the walls in the system. Muzykantskii and Khmelnitskii (MK) decoupled as usual the $\psi^4$ interaction in the effective Lagrangian by gaussian integration over a supermatrix $Q$ but did not use after that a saddle-point approximation. Instead, they derived a quasi-classical equation for effective Green functions $g$ analogous to the Wess-Zumino-Novikov-Witten (WZNW) term. So, they replaced the solution of the semiclassical equations by a functional integral containing the functional $\Phi$, which allowed to average over the supermatrix $Q$. This could be done provided the equation of the Green functions corresponded to a deep minimum of the functional $\Phi$, such that fluctuations near the minimum could be neglected. Although the authors of Ref. [6] conjectured that their field theory could be applicable even in the limit of a vanishing disorder, they did not confirm this point of view by any calculations.

A more traditional way of derivation was used by Andreev et al. [7], who tried to derive the $\sigma$-model for a bal-
listic quantum billiard. Instead of averaging over disorder they averaged over the energy. After decoupling the $\psi^4$ term by integration over the supermatrix $Q$ they used the saddle-point approximation, which is equivalent to the self-consistent Born approximation (SCBA). This had to fix the eigenvalues of $Q$ such that one could put $Q^2 = 1$ and what remained to do was to expand the action in gradients of $Q$ and in the frequency $\omega$. Shortly after it became clear that the saddle-point did not fix the eigenvalues of the supermatrix $Q$ and modes that were usually massive for disordered system became massless in the ballistic limit. This problem was discussed in the publications. It is relevant to notice that the existence of the additional massless modes is not just a consequence of the bad saddle-point approximation. As discussed in Ref. [1], the same problem is encountered when deriving the $\sigma$-model with the help of the so-called “color-flavor” transformation. Within this approach, although one does not need to use any saddle-point approximation, the expansion in gradients still remains to be performed. However, there is no parameter that would allow one to take into account the lowest gradients only. Zirnbauer suggested [1] to perform an additional averaging over ensembles in order to suppress short range fluctuations.

The saddle-point approximation can actually be useful if one considers a system with a long range disorder. In this case, the single-particle mean free path $l$ can be much smaller than the transport mean free path $l_{tr}$. At distances exceeding the length $l$ the saddle-point approximation and the expansion in gradients of $Q$ can be used and one comes to a ballistic $\sigma$-model that reduces to the diffusion one only at distances exceeding $l_{tr}$. Thus, in the interval between $l$ and $l_{tr}$ one can obtain the ballistic $\sigma$-model in a reliable way (see also a subsequent discussion in Ref. [1]). However, this does not solve the problem completely because a reasonable semi-classics should be applicable at all distances exceeding the wavelength $\lambda_F$.

A ballistic $\sigma$-model should describe low lying excitations that exist for any long range disorder. At the same time, the conventional saddle-point approximation, being equivalent to the SCBA, can be good for a short range disorder only and, hence, may not be used for derivation of a $\sigma$-model for a long range disorder and quantum chaos at arbitrary distances. The same is true for the decoupling of the $\psi^4$ term. The integration over the supermatrix $Q$ is usually used after singling out slowly varying pairs $\psi \bar{\psi}$. However, if the random potential is very long ranged or one averages over the energy, one has slowly varying pairs from the beginning and there is no necessity of integration over the supermatrix $Q$ instead of integration over the initial random potential $U(r)$. The same is true when applying the color-flavor transformation [1]. The replacement of an integration over $u(r)$ by an integration over a supermatrix $Z(r)$ does not seem to correspond to physical processes and is a purely (although exact) mathematical transformation. Thus, the correct scheme of the derivation of a field theory describing the low lying excitations should not be based on the Hubbard-Stratonovich or color-flavor decoupling and the saddle-point approximation.

In this paper, we present a derivation of a ballistic $\sigma$-model using neither the Hubbard-Stratonovich decoupling with a supermatrix $Q$ nor the saddle-point approximation determining the eigenvalues of $Q$. As in Ref. [1], we derive quasiclassical equations for Green functions but we write them without making the Hubbard-Stratonovich transformation. This is justified because we use a long range potential. Only if a short range potential is added we must single out slow pairs in corresponding term in the Lagrangian and decouple it by the integration over supermatrices. The crucial step of the derivation is an exact representation of the solution of the quasiclassical equations in terms of a functional integral over $8 \times 8$ supermatrices $Q_n(r)$ with the constraint $Q_n^2(r) = 1$, where $r$ is the coordinate and $n = p_F/|p_F|$ is the normalized vector on the Fermi-surface. An effective action $\Phi_n[g_n]$ entering the functional integral is similar the one written in Ref. [1]. We show that supersymmetric properties of the matrix $Q_n(r)$ make the representation exact, which was not noticed in the MK variational approach. Moreover, the solution written for an arbitrary long range potential $u(r)$ is applicable even for non-averaged quantities.

Averaging over the random potential leads to an effective action $\Phi[g_n]$ that has a form different from those discussed previously. Analyzing properties of the new ballistic non-linear $\sigma$-model with the action $\Phi[g_n]$ we demonstrate that a new length $l = v_F \tau_L$ introduced by Aleiner and Larkin [1], where $v_F$ is the Fermi velocity and $\tau_L$ is the inverse Lapunov exponent, determines different regimes. The importance of this length was also discussed recently in Ref. [1]. Integrating over variations of the supermatrix $Q_n(r)$ at distances smaller than $l$ we come to another form of the ballistic $\sigma$-model containing the conventional collision term.

We show that without an internal disorder the calculation of the functional integral can be reduced to study of the $\sigma$-model for periodic orbits. Only the presence of a regularizer analogous to the one introduced in Ref. [1] may mix the periodic orbits. The problem of repetitions is discussed and we are able to demonstrate that the contradiction between the references [1] and [2] is rather a consequence of an unjustified approximation used in Ref. [2] than a deficiency of the $\sigma$-model.

The paper is organized as follows: In Chapter II, we express correlation functions of interest in terms of functional integrals over supervectors and write equations for generalized Green functions. In Chapter III, we represent the solution of the quasiclassical equations in terms of functional integrals over supermatrices and average over disorder, thus obtaining a ballistic $\sigma$-model applicable at all distances exceeding the wavelength. In Chapter IV, we integrate over a Lapunov region and derive a reduced ballistic $\sigma$-model containing a collision term. In Chapter V, we show how one can derive equations for correlation functions. In Chapter VI, we show how calculations
within the ballistic $\sigma$-model can be reduced to calculations for periodic orbits. We explain how the so called “repetition problem” can be resolved. Chapter VII is devoted to a discussion of the results obtained. The Appendix contains a derivation of the boundary conditions.

II. FORMULATION OF THE PROBLEM. QUASICLASSICAL APPROXIMATION.

The aim of the present paper is to find a convenient representation that would allow us to consider an electron motion in a smooth potential at large times or low frequencies. Of course, with the formalism presented one can consider wave scattering in microwave cavities and other interesting problems but, to simplify notations, we will use the condensed matter language.

We want to extend the supersymmetry method developed for disordered systems to distances smaller than the mean free path. Actually, the only assumption we will use in the derivation is that all physical quantities vary at distances exceeding the Fermi wavelength $\lambda_F = 2\pi p_F^{-1}$. Although this assumption is much less restrictive than those used in Ref. it allows to simplify essentially the consideration. Averaging over the energy, which is the standard procedure for quantum chaos, can be considered as the limiting case for an infinite range random potential.

The method developed in this paper is applicable for calculation of gauge invariant quantities like density- or level-level correlation functions. Such quantities as average one-particle Green functions at different points will not be considered here. We choose the Hamiltonian $\hat{H}$ of the system in the standard form

$$\hat{H} = \hat{H}_0 + u(\mathbf{r}) + u_\sigma(\mathbf{r}), \quad (2.1)$$

$$\hat{H}_0 = -\nabla^2/2m - \varepsilon_F$$

where $u(\mathbf{r})$ is a long range potential, which is of the main interest now, and $u_\sigma(\mathbf{r})$ is a short range impurity potential. The latter is added in order to make the models somewhat more general. The presence of the short range potential will help to understand better the procedure we will use. However, nothing is assumed about the strength of $u_\sigma(\mathbf{r})$ and it can be safely put to zero in all formulae written below.

As usual\textsuperscript{9,10}, one can express correlation functions of interest in terms of a functional integral over 8-component supervectors $\psi(\mathbf{r})$ with an effective Lagrangian $L$

$$L[\psi] = \int \left[ -i \bar{\psi}(\mathbf{r}) \left( \hat{H}_0 + u(\mathbf{r}) \right) \psi(\mathbf{r}) \right. - \frac{i}{2}(\omega + i\delta) \bar{\psi}(\mathbf{r}) \Lambda \psi(\mathbf{r}) + \left. \frac{1}{4\pi\nu\tau_s} (\bar{\psi}(\mathbf{r}) \psi(\mathbf{r}))^2 \right] \, d\mathbf{r}, \quad (2.2)$$

where $\varepsilon$ is the energy at which the physical quantities are considered and $\omega$ is the frequency.

The Lagrangian $L$, Eq. (2.2), is written after the averaging over the short range potential $u_\sigma(\mathbf{r})$ implying the standard gaussian correlations of the type

$$\langle u_\sigma(\mathbf{r}) u_\sigma(\mathbf{r}') \rangle = \frac{1}{2\pi\nu\tau_s} \delta (\mathbf{r} - \mathbf{r}') \quad (2.3)$$

It is relevant to emphasize that averaging over the long range potential $u(\mathbf{r})$ has not been performed.

The correlation functions we are interested in can be obtained adding proper source terms in the Lagrangian $L[\psi]$. An important class of the correlation functions can be obtained writing the Lagrangian $L_{\alpha}[\psi]$ including the sources in the form

$$L_{\alpha}[\psi] = L[\psi] + i \int \bar{\psi}(\mathbf{r}) \hat{a}(\mathbf{r}) \psi(\mathbf{r}) \, d\mathbf{r} \quad (2.4)$$

where $\hat{a}(\mathbf{r})$ is a matrix depending on coordinates. Its explicit form depends on what type of the correlation function is calculated.

The level-level correlation function $R(\omega)$

$$R(\omega) = \frac{1}{2\pi^2 \nu^2 V^2} \left\langle \text{Re} \int \left( n(\varepsilon - \omega) - n(\varepsilon) \right) \times G_{\varepsilon-\omega}(\mathbf{r}, \mathbf{r}') \left( G^{R}_{\varepsilon}(\mathbf{r}', \mathbf{r}') - G^{A}_{\varepsilon}(\mathbf{r}', \mathbf{r}') \right) \, d\mathbf{r} d\mathbf{r}' \right\rangle \quad (2.5)$$

where $n(\varepsilon)$ is the Fermi distribution, contains the product $G^{A}_{\varepsilon}G^{R}_{\varepsilon}$. For calculation of this product one should choose $\hat{a}$ in the form

$$\hat{a} = \begin{pmatrix} \hat{a}_1 & 0 \\ 0 & -\hat{a}_2 \end{pmatrix}, \quad \hat{a}_1, 2 = \frac{a_{1,2}}{2} (1 - k)$$

where both $k$ and $\tau_3$ denoting different blocks have the form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Below we use the same notations as in the book\textsuperscript{11,12}.

The product $\langle G^{A}_{\varepsilon-\omega}(\mathbf{r}, \mathbf{r}) G^{A}_{\varepsilon}(\mathbf{r}', \mathbf{r}') \rangle$ is trivial, whereas the first product in Eq. (2.5) can be written as

$$\langle G^{A}_{\varepsilon-\omega}(\mathbf{r}, \mathbf{r}) G^{R}_{\varepsilon}(\mathbf{r}', \mathbf{r}') \rangle \quad (2.6)$$

and we reduce the function $R(\omega)$ to the form

$$R(\omega) = \frac{1}{2}$$

(2.7)
For computation of the density-density correlation function one has to calculate the averages of the type

\[
Y^{00}(r_1, r_2; \omega) = 2 \langle G^A_{\omega} (r_2, r_1) G^R (r_1, r_2) \rangle
\]  

(2.8)

This product can be obtained from the following source term

\[
\hat{a} = \begin{pmatrix}
0 & \hat{a}_0 (r) \\
-\hat{a}_0 (r) & 0
\end{pmatrix}
\]

(2.9)

where

\[
\hat{a}_0 (r) = \begin{pmatrix}
\hat{a}_1 \delta (r - r_1) & 0 \\
0 & \hat{a}_2 \delta (r - r_2)
\end{pmatrix}
\]

(2.10)

with the same expression for \( \hat{a}_{1,2} \) as in Eq. (2.6). Then, we have for \( Y^{00} (r_1, r_2, \omega) \)

\[
Y^{00} (r_1, r_2; \omega) = 2 \int \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \exp (-L_\alpha [\psi]) \, D\psi \bigg|_{\alpha_1 = \alpha_2 = 0}
\]

(2.10)

In principle we can proceed calculating in Eqs. (2.7, 2.10) in the standard way by singling out slowly varying pairs \( \psi \bar{\psi} \) in the term \( \psi^4 \) in the Lagrangian \( L \), Eqs. (2.2, 2.4), and decoupling the products of these terms by Gaussian integration over \( 8 \times 8 \) supermatrices \( M \). A word of caution should be said at this point. The separation into the products of slowly varying pairs \( \psi \bar{\psi} \) makes a sense only for distances exceeding the range of the random potential. This means that one may not this approximation for distances smaller than the potential range. Of course, the same is true when averaging over the spectrum. In the latter case the range of the random potential is just the system size. Therefore, the previous derivations where this separation was used can hardly be justified. The same problem apparently arises when doing the color-flavor transformation. Although the transformation is formally exact, there is no reason for neglecting higher gradients when making the expansions in gradients.

In order to avoid the problem we keep the long range potential \( u (r) \) in Eq. (2.2, 2.4) as it stands and do not average over it. This will be done later. At the same time, the decoupling of the term corresponding to the short range potential by integration over the supermatrix \( M \) can be safely done (We repeat again that the presence of the short range potential is not crucial for our derivation and its strength can be put zero).

After the decoupling for the short range potential the calculation of the correlation functions is reduced to the computation of an effective partition function \( Z_1 [J] \)

\[
Z_1 [J] = \int \exp (-L_J [\psi]) \, D\psi
\]

(2.11)

where

\[
J (r) = i \hat{a} (r) + \frac{M (r)}{2r_s}
\]

(2.12)

and \( L_J [\psi] \) is obtained from \( L_\alpha [\psi] \), Eqs. (2.2, 2.4), after the replacement of the parameter \( \hat{a} \) by \( J \) according to Eq. (2.12) and neglecting the quartic term. The function \( J \) satisfies the standard symmetry relation \( J = \bar{J} \), where \( J = CJ^T G^T \). The bar means the usual “charge conjugation” of Refs. \( J \), the matrix \( C \) is defined as follows

\[
C = \Lambda \otimes \left( \begin{array}{cc}
c_1 & 0 \\
0 & c_2
\end{array} \right), \quad c_1 = \left( \begin{array}{cc}
0 & -1 \\
1 & 0
\end{array} \right), \quad c_2 = \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right)
\]

The correlation functions for a given long range potential \( u (r) \) can be calculated by differentiating in \( \alpha_{1,2} \) the following integral

\[
Z_u = \int Z_1 [J] \exp \left( -\frac{\pi \nu}{8r_s} \int \text{Str} M^2 (r) \, dr \right) \, DM
\]

(2.13)

Averaged correlation functions can be obtained from the quantity \( Z = \langle Z \rangle_u \), where \( \langle \ldots \rangle_u \) means averaging over \( u (r) \). Of course, we could immediately average over \( u (r) \) in Eq. (2.13) but we want to avoid the standard scheme. The approximations that worked so well for short range potential are not applicable to the long range one. In particular, not only the separation of slow modes is not justified but also the saddle-point approximation for the integral over supermatrices is no longer good. It is clear that the existence of low lying excitations like diffusons and cooperons is more general than the SCBA and one should try to avoid the latter.

In this paper we follow the method of quasiclassical Green functions first introduced for study of superconductivity. This method was used recently by Muzykantski and Khmelnitski for study of the ballistic transport. Our calculations are partially equivalent to those by MK but there are essential differences. First, we keep the long range potential \( u (r) \) fixed and average over it at the later stage of the derivation. Second, which is the most crucial step, we show how to write the solution of the quasiclassical equations exactly. The possibility of writing an exact solution of the equations for quasiclassical Green functions in terms of functional integrals is a consequence of the supersymmetry and has not been realized before.

As in MK, we consider the logarithmic derivative of the partition function \( Z_1 [J (r)] \)

\[
\frac{\delta \ln Z_1 [J (r)]}{\delta J_{\alpha \beta} (r)} = K_{\beta \gamma} \langle \psi_\alpha (r) \bar{\psi}_\beta (r') \rangle \psi
\]

(2.14)

where \( \langle \ldots \rangle \) is the average with the functional \( L_J [\psi] \), Eqs. (2.2, 2.4, 2.12) and
\[ K = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \]

Introducing the matrix function \( G(r, r') \) as
\[
G(r, r') = 2(\psi(r)\bar{\psi}(r'))\psi
\]
we can write for this function the following equation
\[
\left[ \hat{H}_{0r} + u(r) + \frac{\Lambda}{2} \omega + i\delta \right] G(r, r') = i\delta(r - r')
\]
(2.16)

where the subscript \( r \) of \( \hat{H}_{0r} \) means that the operator acts on \( r \).

Conjugating Eq. (2.14) and using the property
\[
\hat{G}(r, r') = C\hat{G}^T(r', r)C^T = G(r, r')
\]
(2.17)
we obtain another equation for the matrix \( G(r, r') \) with the operator \( \hat{H}_{0r} \) acting on its second variable
\[
\dot{G}(r, r') \left[ \hat{H}_{0r} + u(r') + \frac{\Lambda}{2} \omega + i\delta \right] = i\delta(r - r')
\]
(2.18)

Until now no approximations have been done and Eqs. (2.16, 2.18) are exact. Now we can use the assumption that the potential \( u(r) \) changes slowly on the wavelength \( \lambda_F \). If the mean free path for the scattering on the random potential exceeds \( \lambda_F \) the Green function varies as a function of \( r - r' \) at distances of the order of \( \lambda_F \) but, at the same time, is a slow function of \( (r + r')/2 \). The Fourier transform \( G_p((r + r')/2) \) of \( G(r, r') \) respective to \( r - r' \) has a sharp maximum near the Fermi surface. In order to cancel large terms we subtract Eq. (2.18) from Eq. (2.16). Using the assumption that the potential \( u(r) \) is smooth and expanding it in gradients we obtain in the lowest order
\[
\left[ -i\frac{\nabla}{m} + \frac{i}{2} \left\{ \nabla u(R), \frac{\partial}{\partial p} \right\} \right] G_p(R) = \frac{i\Lambda}{2} G_p(R) + i[J(R), G_p(R)] = 0
\]
(2.19)

where \( R = (r + r')/2 \) and \( \left\{ \right\} \) stands for the commutator.

When deriving Eq. (2.19), not only the potential \( u(r) \) but also the function \( J(r) \) was assumed to be smooth. The supermatrix \( M(r) \) is smooth by the construction. As concerns \( \bar{a} \), this is not always so, as is seen from Eq. (2.14). However, we can slightly smear the coordinates in the definition of the correlation functions and obtain after this procedure a smooth function \( \bar{a} \).

The dependence of the Green function \( G_p(R) \) on \( |p| \) is more sharp than on other variables. In order to avoid this sharp dependence we integrate Eq. (2.19) over \( |p| \). Of course, this procedure makes a sense for very large samples when the level discreteness can be neglected. However, this procedure can also be performed in finite samples provided it is complemented by an averaging over the energy.

The most interesting contribution in the integral over \( |p| \) comes from the vicinity of the Fermi-surface. A contribution given by momenta considerably different from \( p_F \) is proportional to the unity matrix and drops out from Eq. (2.19).

Introducing the function \( g_n(r) \)
\[
g_n(r) = \frac{1}{\pi} \int G_{pn}(\xi) d\xi, \quad \xi = \frac{p^2 - p_F^2}{2m}
\]
(2.20)

where \( n \) is a unit vector pointing a direction on the Fermi surface, we obtain the final quasiclassical equation
\[
\left( v_F n \nabla - p_F^2 \nabla u(r) \partial_n \right) g_n(r) + i \frac{\delta}{2} \left\{ \Lambda, g_n(r) \right\} - [J, g_n] = 0
\]
(2.21)

where
\[
\partial_n = \nabla_n = -[n \times [n \times \frac{\partial}{\partial n}]]
\]

The function \( g_n(r) \) is self-conjugate
\[
g_n(r) = C g_{-n}(r) C^T = g_n(r),
\]
(2.22)

which follows from Eqs. (2.17), (2.20).

Eq. (2.21) should be complemented by a boundary condition at the surface of the sample. This boundary condition is derived in Appendix A. Considering a closed sample we assume that the current across the border is equal to zero. This leads the boundary condition at the surface
\[
g_n(\mathbf{r}) \big|_{surface} = g_{-n}(\mathbf{r}) \big|_{surface}
\]
(2.23)

where \( n \perp \) is the component of the vector \( n \) perpendicular to the surface.

The correlation functions considered here can easily be expressed in terms of the quasiclassical Green functions \( g_n(r) \). The functional derivative of the partition functions \( Z_1 \), Eq. (2.13), can be written through the function \( g_n(r) \) as follows
\[
\langle \psi_\alpha(r) \bar{\psi}_\beta(r) \rangle = \frac{1}{2} G_{\alpha\beta}(r, r)
\]
\[
= \frac{\nu}{2} \int d\mathbf{n} \int d\mathbf{r} G_{pn,\alpha\beta}(\mathbf{r}, \mathbf{r}) \approx \frac{\pi \nu}{2} \int d\mathbf{n} g_{n,\alpha\beta}(\mathbf{r})
\]
normalized to its surface area \( S_d \). With this definition we have e.g.

\[
\int d n = 1 \quad (2.25)
\]

As in the theory of superconductivity, the solution for the Eq. (2.21) satisfies the condition \( g^2_n(r) = 1 \).

Eq. (2.21) is written for a non-averaged potential \( u(r) \). It is valid also in the absence of the long range potential. If the system is finite, Eq. (2.21) can still be obtained provided averaging over the spectrum is performed.

In principle, one could average over the potential \( u(r) \) by expanding in this potential and averaging terms of supermatrices provided averaging over the spectrum is performed.

If the system is finite, Eq. (2.21) can still be obtained like the quasiclassical Eq. (2.21), there is an important exception when the functional integral becomes an exact one, namely spoken of as a variational or phenomenological one. Unfortunately, conditions providing such a description have not been found as yet and the MK theory is usually considered here when the equation is linear in supermatrices \( g^0(r) \) are supermatrices. Surprisingly, this rather simple fact has not been realized before.

However, they did not try to solve this equation but noticed in analogy with an equation of motion for a ferromagnet that Eq. (2.21) is an extremum of a functional containing a Wess-Zumino-Novikov-Witten term. As in the theory of superconductivity, the solution for Eq. (2.20), we notice that the integration in Eq. (3.3) is performed over all supermatrices \( Q_n(r) \) with the same structure as the supermatrices \( g_n(r) \).

In the next chapter we write this solution and average over the long range potential \( u(r) \).

### III. SOLUTION OF THE QUASICLASSICAL EQUATION. THE BALLISTIC \( \sigma \)-MODEL.

The quasiclassical equation, Eq. (2.21), has been written previously by MK, Ref. [3] (the long range random potential \( u(r) \) and the source term were not included). However, they did not try to solve this equation but noticed in analogy with an equation of motion for a ferromagnet that Eq. (2.21) is an extremum of a functional containing a Wess-Zumino-Novikov-Witten term. Assuming that this had to be a deep minimum MK have written the solution of the quasiclassical equation in terms of a functional integral with this functional. Unfortunately, conditions providing such a description have not been found as yet and the MK theory is usually spoken of as a variational or phenomenological one. Although this is generally so for “equations of motion” like the quasiclassical Eq. (2.21), there is an important exception when the functional integral becomes an exact solution of the problem. This exception is just the case considered here when the equation is linear in \( g_n(r) \) and the functions \( g_n(r) \) are supermatrices. Surprisingly, this rather simple fact has not been realized before.

It is clear that a solution for Eq. (2.21) is not unique because the equation is homogenous. To determine the solution uniquely one should put \( J = 0, u(r) = 0 \) and calculate the corresponding \( g^{(0)}_n(r) \) directly without using the quasiclassical equation for the Green function. It is easy to see that in this case

\[
g^{(0)}_n(r) = \Lambda \quad (3.1)
\]

Eq. (3.1) plays the role of a boundary condition. Let us show that the exact solution for Eq. (2.21) satisfying the boundary condition, Eq. (3.1), can be written as

\[
g_n(r) = Z_2^{-1}[J] \int_{Q^2_n=1} Q_n(r) \exp \left( -\frac{\pi \nu}{2} \Phi_J[Q_n(r)] \right) DQ_n,
\]

\[
\Phi_J[Q_n(r)] = Str \int d r d n [\Lambda T_n(r)(v_F n\nabla_r
\]

\[
-p_F^{-1} \nabla r u(r) \nabla_n) T_n(r) + \left( i(\omega + i\delta) / 2 \Lambda - J(r) \right) Q_n(r)],
\]

In Eq. (3.2), the partition function \( Z_2[J(r)] \) is

\[
Z_2[J] = \int_{Q^2_n=1} \exp \left( -\frac{\pi \nu}{2} \Phi_J[Q_n(r)] \right) DQ_n \quad (3.3)
\]

and the integration is performed over the self-conjugate supermatrices \( Q_n = \bar{Q}_n(r) \) satisfying the following relations

\[
Q^2_n(r) = 1 \quad (3.4)
\]

everywhere in the bulk and, in addition,

\[
Q_n(r)|_{surface} = Q_{n|surface} = 0 \quad (3.5)
\]

at the surface of the sample.

In order to prove the statement that the integral, Eq. (3.2), is equal to the solution \( g_n(r) \) for Eq. (2.20) we notice that the integration in Eq. (3.3) is performed over all supermatrices \( Q_n(r) \) with the constraints, Eqs. (3.4, 3.5), and therefore cannot change under the following replacement of the variable of the integration

\[
Q_n(r) \to \bar{Q}_n(r) = U_n(r)Q_n(r)\bar{U}_n(r), \quad U_n(r)\bar{U}_n(r) = 1 \quad (3.6)
\]

On the other hand, one can formally consider the integral, Eq. (3.3), with the transformed matrix \( \bar{Q}_n(r) \) as a functional of the transformation matrix \( U_n(r) \). The fact that the integral \( Z_2[J(r)] \), Eq. (3.3), does not change under the transformation means, in particular, that the first variation of the considered functional must be zero for any unitary matrix \( U_n(r) \).

In order to find the first variation, we make a small rotation, resulting in the replacement \( U_n(r) \to U_n(r) + \delta U_n(r)\bar{U}_n(r) \) and compute in the linear approximation in the matrix \( \delta U_n(r) \) the difference between the functionals with the changed and initial matrices \( U_n(r) \).

For the supermatrix \( \bar{Q}_n(r) \), this means the replacement \( \bar{Q}_n(r) \to \bar{Q}_n(r) + [\delta U_n(r), \bar{Q}_n(r)] \), which follows from the relation \( \delta U_n(r) = -\delta \bar{U}_n(r) \). Then, the first variation \( \delta Z_2 \) of the partition function \( Z_2[J(r)] \), Eq. (3.3), takes the form:

\[
\delta Z_2[J] = -\frac{\pi \nu}{2} \int_{Q^2_n=1} DQ_n \delta \Phi[Q_n] \exp \left( -\frac{\pi \nu}{2} \Phi[Q_n] \right)
\]

(3.7)
The variation $\delta \Phi[Q_n]$ can easily be calculated from Eq. (3.2) and we write it as

$$
\delta \Phi[Q_n] = - Str \int d\mathbf{r} d\mathbf{n} \delta U_n(\mathbf{r}) \{ (v_F \mathbf{n} \nabla \mathbf{r}) 
-p_F^{-1} \nabla \mathbf{r} u(\mathbf{r}) \partial_n Q_n(\mathbf{r}) + \left[ i \frac{\omega + i \delta}{2} \Lambda - J(\mathbf{r}) \right] Q_n(\mathbf{r}) \}$$

$$
+ v_F Str \int_S d\mathbf{n} (n dS) \delta U_n(\mathbf{r}) Q_n(\mathbf{r}) \tag{3.8}
$$

The integration in the last term in Eq. (3.8) is performed over the surface of the sample. At the surface, not only $Q_n(\mathbf{r})$ but also $\delta U_n(\mathbf{r})$ must be invariant under the replacement $\mathbf{n} \rightarrow - \mathbf{n}$ (see Eq. (2.3)). It follows from this property that the surface term in Eq. (3.8) is equal to zero. Then, substituting Eq. (3.8) into Eq. (3.7) and taking into account that $\delta Z_2[J]$ must be zero for any $\delta U_n(\mathbf{r})$ we come immediately to Eq. (2.21).

Eq. (2.21) is a differential one and a class of different solutions may exist. The fact that the integral, Eq. (2.2), satisfies Eq. (2.21), does not guarantee that it is equal to $g_n(\mathbf{r})$, Eq. (2.22), and this should be checked separately. It is the supersymmetric structure of the supermatrices $Q_n(\mathbf{r})$ that allows the integral, Eq. (2.2), to satisfy both the equations.

Putting $J = 0$ and $u(\mathbf{r}) = 0$ one can calculate $g_n(0)(\mathbf{r})$ directly from Eq. (2.22) and come to Eq. (3.1). The same can be done using the functional integral in Eq. (2.2). As the functional $\Phi_J[Q_n(\mathbf{r})], \text{Eq.} \ (3.3)$, contains at $J = 0$, $u(\mathbf{r}) = 0$ only the matrices $Q_n(\mathbf{r})$ and $\Lambda$, averaging with such a functional gives according to general rules

$$
\langle Q_n(\mathbf{r}) \rangle_a = \Lambda \tag{3.9}
$$

Comparing Eqs. (3.9) with Eq. (3.1) we conclude that the solution, Eq. (3.2), is compatible with Eq. (2.20). Although Eq. (3.9) is trivially fulfilled for the supermatrices, it would not be necessarily correct if the matrices $Q$ did not have the supersymmetric structure. In the latter case one could obtain e.g. a non-trivial function of $\omega$ in the r.h.s. of it. This would invalidate the present approach for such symmetries. Actually, Eq. (3.9) tells us that the average density of states is a constant, which is the case for weakly disordered systems.

The partition functions $Z_1$, Eqs. (2.2, 2.3, 2.11, 2.12), and $Z_2$, Eq. (3.3), are equal to unity at $J = 0$. As their logarithmic derivatives coincide for all $J$, we come to the conclusion that

$$
Z_1[J] = Z_2[J] \tag{3.10}
$$

Thus, we replaced the integration over electron modes, Eqs. (2.2, 2.3, 2.11, 2.12), by the integration over low-lying excitations that were called diffusion modes in the diffusive limit. Apparently, the name kinetic modes would be a more proper one for the limit under consideration.

We checked Eq. (3.10) additionally by a direct expansion of the both sides in $J$ up to terms $J^2$. Eq. (3.10) was written in Ref. [4] at $u(\mathbf{r}) = 0$ and $\partial_n = 0$ but its accuracy remained unclear.

The form of the partition function $Z_2[J]$, Eqs. (3.2, 3.3) allows us to average it immediately over both the long range potential $u(\mathbf{r})$ and the supermatrix $M(\mathbf{r})$. Assuming that fluctuations of the random potential $u(\mathbf{r})$ are gaussian with the correlation

$$
\langle u(\mathbf{r}) u(\mathbf{r}') \rangle = W(\mathbf{r} - \mathbf{r}') \tag{3.11}
$$

where $W(\mathbf{r})$ is a function decaying at distances $b$ much exceeding the wavelength $\Lambda_F$, $b \gg \Lambda_F$, we average the partition function $Z_2(J)$ over $u(\mathbf{r})$ and $M(\mathbf{r})$ with the help of Eqs. (3.13), (3.11) and find for the final averaged partition function $Z(\hat{a})$

$$
Z(\hat{a}) = \int Z_{\hat{a}}(\lambda ) = \exp \left( - \frac{\lambda}{\mu} \right) DQ_n(\mathbf{r}) \tag{3.12}
$$

The “free energy” functional $F$ takes the form

$$
F[Q_n(\mathbf{r})] = F_{\text{kin}} + F_{\text{imp}} + F^{(s)}_{\text{imp}}, \tag{3.13}
$$

$$
F_{\text{kin}}[Q_n(\mathbf{r})] = \frac{\pi \mu}{4} Str \int d\mathbf{r} d\mathbf{n} [v_F \hat{\Lambda} (\mathbf{r}) \mathbf{n} \nabla T_n(\mathbf{r}) + 2i(\omega + i \delta)(\Lambda - \hat{a}) Q_n(\mathbf{r})]
$$

$$
F_{\text{imp}}[Q_n(\mathbf{r})] = - \frac{1}{8} \left( \frac{\pi \mu}{\lambda_F} \right)^2 \int d\mathbf{r} d\mathbf{r}' d\mathbf{n} \nabla^\perp_r W(\mathbf{r} - \mathbf{r}') \times Str[\hat{\Lambda} T_n(\mathbf{r}) \nabla^\perp_n T_n(\mathbf{r})] Str[\hat{\Lambda} T_{n'}(\mathbf{r}') \nabla^\perp_n T_{n'}(\mathbf{r}')]
$$

$$
F^{(s)}_{\text{imp}}[Q_n(\mathbf{r})] = - \frac{\pi \mu}{8 \tau_s} Str \left( \int Q_n(\mathbf{r}) d\mathbf{n} \right)^2
$$

and $Q_n(\mathbf{r}) = T_n(\mathbf{r}) \hat{\Lambda} T_n(\mathbf{r})$.

As we see from Eqs. (3.12,3.13), the free energy functional $F[Q_n(\mathbf{r})]$ consists of three parts.

The first part $F_{\text{kin}}[Q_n(\mathbf{r})]$ describes the kinetic modes in the absence of any impurities. Correlation functions of interest can be obtained differentiating $Z(\hat{a})$ in $\hat{a}$ as written in Eqs. (2.7, 2.10). The first term in $F_{\text{kin}}[Q_n(\mathbf{r})]$ can be written also in the form of the Wess-Zumino-Novikov-Witten integral with an additional variable of integration. The second part $F[Q_n(\mathbf{r})]$ is responsible for scattering on the long range potential, whereas the third term is due to scattering on the short range impurities.

Eqs. (3.12, 3.13) are valid even in the absence of any impurities as long as one may use the quasiclassical approximation. This description fails near boundaries of
the sample because “turning points” where the quasiclassical description fails are inevitable in those regions. Apparently, this can lead to an additional term in the free energy functional analogous to the regularizer introduced in Ref.\[12\]. In all other parts of the sample, Eqs. (3.12, 3.13) are valid at all distances exceeding the wavelength \(\lambda_L\).

It is important to emphasize that the problem of the “mode locking”\[12\] does not exist in the present approach. As \(n^2 = 1\), there are no fluctuations transverse to the constant energy shell and no additional averaging is necessary. Moreover, there are no fluctuations of the eigenvalues of the supermatrix \(Q_n(r)\) because by construction \(Q_n^2 = 1\). The limit of the vanishing short range potential \(s_\tau \to \infty\) can be taken if some long range potential is present. The purely ballistic case without any randomness is more sophisticated because one needs a regularizer. However, this is a more delicate effect than the problem of the mode locking.

The term \(F_{imp} [Q_n(r)]\), Eq. (3.13), does not have a form of a collision integral of the Boltzmann equation. It differs from what one obtains using the saddle-point approximation and expanding in gradients. If one expands the functional \(F [Q_n(r)]\) in small deviations \(Q\) from \(\Lambda\) using e.g. a parametrization like

\[
T = iP + (1 - P^2)^{1/2}, \quad PA + AP = 0 \quad (3.14)
\]

the first non-vanishing contribution is of the order of \(P^4\), which means that \(F_{imp} [Q_n(r)]\) does not contribute to the bare propagator at all.

At the same time, it was demonstrated\[13\] that the ballistic \(\sigma\)-model with a term in the form of the collision integral could be obtained at large distances exceeding a single particle mean free path \(l\). In the next chapter we will clarify this question by demonstrating that the standard form of the collision integral can really be obtained at large distances as a result of a course-graining procedure.

**IV. REDUCED BALLISTIC \(\sigma\)-MODEL**

The free energy functional \(F [Q_n(r)]\), Eqs. (3.12, 3.13) is most general and applicable at all distances exceeding the wavelength \(\lambda_L\). We neglect now the part \(F_s [Q_n(r)]\) originating from the short range impurities and concentrate on studying properties of the long range scattering.

At very large distances exceeding the mean free path \(l_{tr}\) (we evaluate this length later, see Eq. (3.39)) the \(\sigma\)-model must acquire the standard diffusive form. However, if the long range potential \(u(r)\) is weak such that \(l_{tr} \gg b\), one more intermediate scale is important for describing the behavior of the system, namely, the one related to the Lapunov exponent \(\lambda_L = \tau L^{-1}\). The corresponding time \(\tau L\) is a time required for two particles moving initially parallel to each other to increase the distance between them by a factor of order unity (This definition gives, of course, the order of magnitude of \(\tau L\) only). The importance of this time was first pointed out in Ref.\[12\] where a weak localization correction was calculated for scattering on a long range potential. This time is also relevant for correlations of wave functions discussed recently\[14\] for a model of weak scatterers where it was demonstrated that only at distances \(R > l_L\) the notion of separate diffusions makes a sense.

The length \(l_L\) was estimated for the model of long range weak scatterers corresponding to the case considered here as

\[
l_L \sim l_{tr} (b/l_{tr})^{2/3} \quad (4.1)
\]

which shows that \(l_L\) is between \(b\) and \(l_{tr}\), \(b \ll l_L \ll l_{tr}\).

At distances \(R\) smaller than \(l_L\), two particles that started their motion along parallel trajectories still move parallel to each other and, following Ref.\[12\], we call the region inside \(l_L\) the Lapunov region.

In this chapter, we want to show that the ballistic non-linear \(\sigma\)-model, Eqs. (3.12, 3.13), acquires a more familiar form\[14\] provided an integration over \(Q_n(r)\) within the Lapunov region is performed. This integration is some kind of the course-graining procedure used very often in statistical physics. As a result of such an integration, we obtain another field theory that can be called a reduced ballistic \(\sigma\)-model.

Proceeding in the standard way we separate fluctuations of the matrix \(Q_n(r)\) into fast and slow parts. By fast variations of the supermatrix \(Q_n(r)\) we mean fluctuations changing fast within the Lapunov region and the rest is classified as slow ones. This procedure is very similar to the renormalization group (RG) scheme applied in 2D in the diffusive region (see, e.g.\[15\]). Carrying out the course-graining procedure when deriving the RG equations is simplified by the fact that all arising integrals are logarithmic and one needs to know only the order of magnitude of the cutoffs at each step of the shell integration. In the course-graining procedure used here, resulting integrals are not logarithmic. At first glance, this would make the entire procedure rather tricky because a renormalized free energy functional would depend on the length \(l_L\) estimated by the order of magnitude only.

Fortunately, there exists a method of integration over the fast modes resulting in a renormalized free energy functional \(F[\tilde{Q}_n]\) Eq. (4.22) that does not contain \(l_L\) as a parameter. This length will be implied for \(F[\tilde{Q}_n]\) as an ultraviolet cutoff only.

The separation into slow and fast fluctuating parts is performed as follows

\[
T_n(r) = \tilde{T}_n(r)V_n(r) \quad (4.2)
\]

where both \(\tilde{T}_n(r)\) and \(V_n(r)\) are unitary supermatrices. The supermatrix \(\tilde{T}_n(r)\) is supposed to describe slow modes, whereas \(V_n(r)\) is responsible for fast ones. Substitution of Eq. (4.1) into Eq. (3.13) results in a new action.
in which the two fields interact in complicated way. We write this action as follows

\[
\mathcal{F}_{\text{kin}}[T] = \mathcal{F}_{\text{kin}}[V] + \mathcal{F}'_{\text{kin}}[\tilde{T}, V] \\
\mathcal{F}_{\text{imp}}[T] = \mathcal{F}_{\text{imp}}[V] + \mathcal{F}'_{\text{imp}}[\tilde{T}, V]
\]

where

\[
\mathcal{F}_{\text{kin}}[V] = \frac{\pi \nu}{4} \text{Str} \int d\mathbf{r} d\mathbf{n} [2v_F A \tilde{V}_n(r) \nabla V_n(r) + i(\omega - 2i\dot{\alpha}) A Q_n^{(0)}(r)]
\]

\[
\mathcal{F}'_{\text{kin}}[\tilde{T}, V] = \frac{\pi \nu}{4} \text{Str} \int d\mathbf{r} d\mathbf{n} \nabla_i \nabla_j \tilde{V}_n(r) \nabla^i \nabla^j V_n(r)
\]

\[
\mathcal{F}_{\text{imp}}[V] = -\frac{1}{8} \left( \frac{\pi \nu}{p_F} \right)^2 \int d\mathbf{r} d\mathbf{n} \nabla_i \nabla_j \Phi_i^{(0)}(r) \Phi_j^{(0)}(r)
\]

\[
\times \text{Str}[A \tilde{V}_n(r) \nabla^i V_n(r)] \text{Str}[A \tilde{V}_n(r') \nabla^i V_n(r')]
\]

\[
\mathcal{F}'_{\text{imp}}[\tilde{T}, V] = -\frac{1}{8} \left( \frac{\pi \nu}{p_F} \right)^2 \int d\mathbf{r} d\mathbf{n} \nabla_i \nabla_j \tilde{V}_n(r) \nabla^i \nabla^j V_n(r)
\]

\[
\times [\text{Str}(Q_n^{(0)}(r) \Phi_i^{(0)}(r)) \text{Str}(Q_n^{(0)}(r') \Phi_j^{(0)}(r'))]
\]

\[
+ 2 \text{Str}(A \tilde{V}_n(r) \nabla^i V_n(r)) \text{Str}(Q_n^{(0)}(r) \Phi_i^{(0)}(r'))
\]

\[
Q_n^{(0)} = V_n(r) A \tilde{V}_n(r), \quad \Phi_i^{(0)}(r) = \tilde{T}_n(r) \nabla^i \tilde{T}_n(r).
\]

The separation into the fast and slow parts, Eq. \((4.4)\), is not simple for the problem involved. This is because, for a given \(\mathbf{n}\), a characteristic dependence of \(Q_n(r)\) on the coordinate \(r\) is extremely anisotropic and one cannot introduce an isotropic momentum shell for integration over the fast modes. Besides, we should not violate the rotational invariance when integrating over the fast modes. This goal is achieved by writing the term with the Lapunov exponent \(\lambda_L\) in \(\mathcal{F}_{\text{kin}}[V]\), Eq. \((4.4)\). This is similar to an invariant integration over a momentum shell for a diffusive \(\sigma\)-model.

Introducing a notation

\[
\langle \ldots \rangle_0 \equiv \int \langle \ldots \rangle \exp(-\mathcal{F}_{\text{kin}}[V]) DQ_n^{(0)}
\]

we write the reduced free energy \(F\) as

\[
F = -\ln(\exp(-\mathcal{F}_{\text{kin}}[\tilde{T}, V] - \mathcal{F}_{\text{imp}}[V] - \mathcal{F}'_{\text{imp}}[\tilde{T}, V]))_0
\]

(due to the supersymmetry \((1) = 0\)).

We calculate the average in Eq. \((4.10)\) by expansion of the exponential and computing averages of all terms obtained in this way. In the first order, using the relation

\[
\langle Q_n^{(0)}(r) \rangle_0 = \Lambda
\]

we obtain

\[
\langle \mathcal{F}_{\text{kin}}[\tilde{T}, V] \rangle_0 = \mathcal{F}_{\text{kin}}[\tilde{Q}_n(r)]
\]

with \(\mathcal{F}_{\text{kin}}[\tilde{Q}_n(r)]\) from Eq. \((3.13)\).

\[
\langle \mathcal{F}_{\text{imp}}[V] \rangle_0 = 0,
\]

which is due to the supersymmetry, as well as the second term in \(\mathcal{F}_{\text{imp}}[T, V]\), Eq. \((4.7)\).

\[
\langle \mathcal{F}'_{\text{imp}}[\tilde{T}, V] \rangle_0 = -\frac{1}{8} \left( \frac{\pi \nu}{p_F} \right)^2 \int d\mathbf{r} d\mathbf{n} \nabla_i \nabla_j \tilde{V}_n(r) \nabla^i \nabla^j V_n(r - r')
\]

\[
\times \langle \text{Str}(Q_n^{(0)}(r) \Phi_i^{(0)}(r)) \text{Str}(Q_n^{(0)}(r') \Phi_j^{(0)}(r')) \rangle_0
\]

If we replaced the average of the product of \(Q_n^{(0)}\) in Eq. \((4.13)\) by the product of the averages we would simply come back using Eq. \((4.10)\) to \(\mathcal{F}_{\text{imp}}[\tilde{Q}_n(r)]\) in Eq. \((3.13)\). However, now the supermatrices \(Q_n(r)\) vary at much longer distances than the radius \(b\) of the correlation function \(W(r - r')\). As the integrand in Eq. \((4.13)\) contains derivatives \(\nabla_r W(r - r')\), the integral is small and can be neglected.

The main contribution comes from the irreducible part of the correlation function which we denote as

\[
\langle \langle Q_n^{(0)}(r) \rangle^{\alpha \beta} \rangle_0
\]

(at the moment we do not distinguish between the advanced/retarded blocks and the others and write all indices standing for matrix elements as superscripts). We calculate this average in Chapter V in a general case including the impurity potential (see Eq. \((5.9)\)). Now we neglect the impurity potential and make the replacement \(\delta \to \lambda_L\) when determining the function \(G_{\mathbf{nn}}^{(0)}(\rho)\) from Eq.\((5.8)\). Then, we substitute Eq.\((5.9)\) into Eq.\((4.13)\) and transform the result of the substitution by introducing new integration variables \(\mathbf{R} = \frac{r + r'}{2}, \rho = r - r'\). Then, we note that the integrand contains the function \(G_{\mathbf{nn}}^{(0)}(\rho)\) that changes on the distance \(l_\omega\) (the limit \(\omega \ll \lambda_L\) is implied), while the supermatrix \(\Phi_i^{(0)}(r)\) changes slower. This allows us to rewrite Eq. \((4.13)\) as follows

\[
\langle \mathcal{F}'_{\text{imp}}[\tilde{T}, V] \rangle_0 = -\frac{\pi \nu}{2p_F} \text{Str} \int d\mathbf{r} d\mathbf{n} \nabla_i \tilde{V}_n(r) \nabla^i \tilde{V}_n(r') \langle \Phi_i^{(0)}(r) \rangle_0
\]

\[
\times \langle \Phi_j^{(0)}(r') \rangle_0 \int d\rho \nabla^i \nabla^j W(\rho) \Lambda G_{\mathbf{nn}}^{(0)}(\rho)
\]

(4.15)
The notation $(\ldots)^\dagger$ means here the part of the supermatrices anticommuting with $\Lambda$. Eq. (4.15) can be reduced to a more simple form. In the Fourier transformed representation, the integral $I_{nn'}$ over $\rho$ in Eq. (4.15) takes the form

$$I_{nn'} = -\int \frac{dq}{(2\pi)^d} q q^i W(q) \mathcal{G}_{nn'}(-q)$$

(4.16)

Solving Eq. (5.5) we find for the Green function in the limit $\omega \ll \lambda_L$

$$\mathcal{G}_{nn'}(-q) = -\frac{\delta_{nn'}}{iv_F nq + \lambda_L \Lambda}$$

(4.17)

The function $W(q)$ decays at momenta much smaller than $p_F$ and only such momenta give the main contribution in Eq. (4.16). Therefore, we can approximately write the function $\mathcal{G}_{nn'}(-q)$ in the integral, Eq. (4.16), as

$$I_{nn'} = i\delta_{nn'} \int \frac{dq}{(2\pi)^d} q^i q^j W(q) \left( \frac{(p-q)^2}{2m} - \varepsilon_F + i\lambda_L \Lambda \right)^{-1}$$

(4.18)

where $p = p_F n$.

Using the fact that

$$\langle \Phi_n(r) \rangle^2 = \left( \Phi_n(r) \right)^\dagger = -\frac{1}{4} \nabla_i \tilde{Q}(r) \nabla^i \tilde{Q}(r)$$

we reduce Eq. (4.15) to the form

$$\left\langle F_{\text{imp}}' \tilde{T}, V \right\rangle = \frac{i\nu}{8p_F^2} \mathrm{Str} \int dndr \nabla_i \tilde{Q}(r) \nabla^i \tilde{Q}(r) \left( \frac{(p-q)^2}{2m} - \varepsilon_F + i\lambda_L \Lambda \right)^{-1}$$

(4.19)

Changing the variables of the integration in the integral Eq. (4.19) to $p' = p - q$ and integrating separately over $\xi = p'/2m - \varepsilon_F$ and $n = p'/p'$ we obtain finally

$$\left\langle F_{\text{imp}}' \tilde{T}, V \right\rangle = \frac{\langle \nu \rangle^2}{8} \int dr \int dndn' (n-n')^i (n-n')^j \times W_{nn'} \mathrm{Str} \left[ \nabla_i \tilde{Q}(r) \nabla^i \tilde{Q}(r) \right]$$

(4.20)

where $W_{nn'} = W(p_F (n-n'))$.

This result corresponds to the collision integral in the Boltzmann kinetic equation in the limit of small angles of the scattering.

Proceeding further we can calculate, in principle, not only $\left\langle F_{\text{imp}}' \tilde{T}, V \right\rangle$ but also its higher cummulants like

$$\left\langle \left\langle F_{\text{imp}}' \tilde{T}, V \right\rangle \left\langle F_{\text{imp}}' \tilde{T}, V \right\rangle \right\rangle$$

To estimate these cummulants we should consider an average of the type

$$\left\langle (\nabla_n \nabla_n' Q_n(0)(r) Q_{n'}(0)(r')) \nabla_n \nabla_n' Q_{n}(0)(r_1) Q_{n'}(0)(r_1') \right\rangle$$

(4.21)

where $|r - r'| \sim |r_1 - r_1'| \lesssim b$ and we do not write explicitly indices.

The correlation function for the product of four $Q^{(0)}$, Eq. (4.21), resembles a correlation function calculated in Ref. where it was demonstrated that, being a complicated function of small distances, the reducible correlation function decoupled into 2 diffusion as soon as the distance between the points $r, r'$ and $r_1, r_1'$ exceeded the Lapunov length $l_L$. This means that the irreducible correlation function of the type as in Eq. (4.21) decays at distances of the order of the length $l_L$. As the second cummulant contains a product of four slow functions $\Phi_n(r)$, Eq. (4.13), we conclude that the cummulant expansion is effectively a series in $l_L \nabla r$. Therefore, beyond the Lapunov region we may keep the average $F_{\text{imp}}'[\tilde{T}, V]$ only.

Using Eqs. (4.9, 4.12, 4.20) we write the reduced ballistic $\sigma$-model applicable beyond the Lapunov region in the form

$$F[\mathcal{Q}_n(r)] = \frac{\pi \nu}{4} \mathrm{Str} \int dr \left[ \int (2v_F \Lambda \mathcal{T}_n(r) \nabla r \mathcal{T}_n(r)) + i(\omega - 2\delta \Lambda) \Lambda \mathcal{Q}_n(r) \right] dn$$

$$+ \frac{\pi \nu}{2} \int W_{nn'}(n-n')^i (n-n')^j \times \nabla^i \mathcal{Q}_n(r) \nabla^j \mathcal{Q}_n(r) dn$$

(4.22)

The reduced ballistic $\sigma$-model, Eq. (4.22), has the same form as the corresponding $\sigma$-model derived for a long range potential [4] provided the limit of small angle scattering is considered. The non-linear $\sigma$-model was obtained in Ref. under the assumption that all distances exceeded the single particle mean free path $l$ using, as the first step, the saddle point approximation. However, the saddle point approximation is not good for a long range potential and, hence, the length $l$ cannot be a good quantity. Now we see that one should use the length $l_L$. The reduced $\sigma$-model is applicable at distances larger than $l_L$ and this agrees with the suggestion of Ref. It is remarkable, that the reduced $\sigma$-model does not contain explicitly the length $l_L$ as a parameter. It can enter as an ultraviolet cutoff only. Therefore, it is sufficient to know $l_L$ by order of magnitude.

One can further simplify the reduced ballistic $\sigma$-model, Eq. (4.22), at distances exceeding the transport mean free path $l_r \gg l_L$. This route is well developed [4,4,14,14]. Separating again fast modes (strongly varying at distances smaller than $l_r$) from slow ones with the zero angular harmonics (fluctuating at distances exceeding $l_r$) we write the supermatrix $T_n(r)$ in Eq. (4.22) as

$$T_n(r) = U(r) T_n^{(0)}(r)$$

(4.23)

Then, the free energy functional $F[\mathcal{Q}_n(r)]$, Eq. (4.22), takes the form
Writing the reduced $\sigma$-model, Eq. (4.22), in a limited volume one can come rather easily to the Wigner-Dyson level-level correlation functions. This can be done separating fluctuations of $T_n$ with the zero space and angle harmonics from other degrees of freedom. We write this separation in a form similar to Eqs. (4.2, 4.23)

$$T_n (r) = U T_n'(r)$$

(4.31)

where $U$ depends neither on the vector $n$ nor on the coordinate $r$, while $T_n'(r)$ contains only non-zero harmonics in the both coordinates. Integration over $T_n'(r)$ can be performed using a parametrization analogous to Eq. (4.28). Excitations corresponding to the matrix $T_n'(r)$ have a gap and their contribution can be neglected in the collision region in the main approximation in the parameter $\omega_{min}(\tau_{tr}, \tau_L)$. Then, we come to the zero-dimensional version $F_0 [Q]$ of the $\sigma$-model

$$F_0 [Q] = \frac{\pi i (\omega + i \delta)}{4 \Delta} \text{Str} (\Lambda Q)$$

(4.32)

where $\Delta = (\nu V)^{-1}$ ($V$ is the volume) is the mean level spacing. This leads directly to the Wigner-Dyson statistics.

The above discussion describes completely the method of derivation of the $\sigma$-model based on quasiclassical Green functions in all regions. However, explicit calculations with the ballistic $\sigma$-model are not simple and calculational schemes are not necessarily the same as those used in the diffusive limit. The most unusual is the Lapunov region. In the next chapters we consider calculational schemes that can be useful for clean systems.

V. CORRELATION FUNCTIONS FOR A LONG RANGE DISORDER

The ballistic $\sigma$-model derived in the previous chapters has a form of a functional integral over the supermatrix $Q_n(r)$ with the constraint $Q_n^2(r) = 1$. Using parametrizations of the type of Eqs. (3.14, 4.28) and expanding in $P_n$ is a potentially dangerous procedure because the rotational invariance may be lost. From study of disordered metals we know that such a perturbation theory works well in the diffusive region but fails in the regime of strong localization. Interestingly enough, the perturbation theory in terms of the ballistic excitations does not work also in the Lapunov region. Using the expansion in $P_n$ one comes to propagators

$$(v_F n q + \omega \Lambda)^{-1}$$

(5.1)

and integrals over both $n$ and vectors $q$.

These integrals are not generally convergent and the result depends strongly on a regularization. In Ref. the regularization was carried out by writing an additional term in the $\sigma$-model that would arise if a small amount of short range impurities was added. It was
suggested that, at the end of calculations, the regularizer could be put to zero. In contrast, the authors of Ref. suggested that a regularizer had to remain finite and demonstrated in a subsequent publication that its presence in the \( \sigma \)-model leads to an anomalous contribution. Problems with diagrammatic expansions in the ballistic excitations were also discussed in Ref. Here.

At the same time, the integration over \( Q_n \) in the ballistic \( \sigma \)-model, Eq. (3.12, 3.13) is well defined and the problems arise only after using a parametrization like those in Eqs. (2.10, 3.10) and a subsequent expansion in \( P_n \). Therefore, we should understand what one can do if one may not use such a parametrization. It turns out that the most proper way of computations of correlation functions is deriving equations for them. This is analogous to what one does in models for turbulence. Unfortunately, this method is not as general as the perturbation theory but for some correlation functions closed equations can be derived without difficulties.

As an example, we consider the function \( Y^{00}(r_1, r_2; \omega) \), Eq. (2.8) (without averaging over the long range potential). Using Eqs. (2.10, 5.10) we reduce this correlation function to the form

\[
Y^{00}(r_1, r_2; \omega) = -2(\pi \nu)^2 \int d\mathbf{n}_1 d\mathbf{n}_2
\]

\[
\times \int Q_{n_1}(r_1)Q_{n_2}(r_2) \exp \left( -\frac{\pi \nu}{2} \Phi_0[Q_n(r)] \right) DQ_n
\]

with the functional \( \Phi_0[Q_n(r)] \) from Eq. (3.4) in which we neglect the short range impurities and put \( J = 0 \).

In order to find the average \( \langle Q_{n_1}(r_1)Q_{n_2}(r_2) \rangle \) (here and below the symbol \( \langle \ldots \rangle_Q \) is used for averaging with the functional \( \Phi_0[Q_n(r)] \)) we introduce the function \( g_n(r; \tilde{a}_n) \) (superscripts are omitted):

\[
g_n(r; \tilde{a}_n) = Z_0^{-1} \langle \tilde{a}_n \rangle \int Q_n(r) \exp \left( -\frac{\pi \nu}{2} \Phi_0[Q_n(r)] \right) DQ_n
\]

(5.3)

\[
Z_0[\tilde{a}_n] = \int Q_{n_1}(r_1) \exp \left( -\frac{\pi \nu}{2} \Phi_0[Q_n(r)] \right) DQ_n
\]

\[
\Phi_0[Q_n] = \Phi_0[Q_n] - i Str \int d\mathbf{r} d\mathbf{n} \hat{a}_n(\mathbf{r}) Q_n(\mathbf{r})
\]

(5.4)

Below, the source \( \hat{a}_n \) is assumed to be a matrix anti-commuting with \( \Lambda \). Besides, in contrast to Eq. (3.2), it includes now the dependence on the direction \( \mathbf{n} \) and is no longer self-conjugate: \( \hat{a}_n(\mathbf{r}) \neq \tilde{a}_n(\mathbf{r}) \). Then, the first order \( g_n^{(1)}(r; \hat{a}_n) \) of the expansion of the function \( g_n(\mathbf{r}; \tilde{a}_n) \) in the source gives the irreducible correlation function

\[
g_n^{(1)}(r; \hat{a}_n) = i \frac{\pi \nu}{2} \int d\mathbf{r}' d\mathbf{n}' \langle (Q_n(r)) \times Str (Q_n'(\mathbf{r}') \hat{a}_n(\mathbf{r}')) \rangle_Q
\]

(5.5)

At the same time, the function \( g_n(r; \tilde{a}_n) \), Eq. (5.4), is a solution for the equation (c.f. with Eq. (2.21))

\[
\langle (Q_n(r)) \rangle = i \frac{\pi \nu}{2} \left[ \hat{a}_n(\mathbf{r}) + \tilde{a}_n(\mathbf{r}) , g_n(r; \tilde{a}_n) \right]
\]

(5.6)

\[
\langle (Q_n(r)) \rangle = 0.
\]

As the zero order in \( \tilde{a}_n(\mathbf{r}) \) we may take \( g_n^{(0)}(r) = \Lambda \) using again the supersymmetry of the integral Eq. (5.3) when the source is disregarded. Then, we obtain in the first order

\[
g_n^{(1)}(r; \tilde{a}_n) = i \frac{\pi \nu}{2} \int d\mathbf{r}' d\mathbf{n}' \langle (Q_n(\mathbf{r})), (Q_n(\mathbf{r}')) \times Str (Q_n'(\mathbf{r}'), \Lambda) \rangle = 0
\]

(5.7)

The kernel \( G_{nn}(r, r') \) in this expression is a Green function for the equation:

\[
\langle (Q_n(r)) \rangle = i \frac{\pi \nu}{2} \left[ \hat{a}_n(\mathbf{r}) + \tilde{a}_n(\mathbf{r}) , g_n(r; \tilde{a}_n) \right]
\]

(5.8)

\[
\langle (Q_n(r)) \rangle = 0.
\]

As the zero order in \( \tilde{a}_n(\mathbf{r}) \) we may take \( g_n^{(0)}(r) = \Lambda \) using again the supersymmetry of the integral Eq. (5.3) when the source is disregarded. Then, we obtain in the first order

\[
g_n^{(1)}(r; \tilde{a}_n) = i \frac{\pi \nu}{2} \int d\mathbf{r}' d\mathbf{n}' \langle (Q_n(\mathbf{r})), (Q_n(\mathbf{r}')) \times Str (Q_n'(\mathbf{r}'), \Lambda) \rangle = 0
\]

(5.7)

The kernel \( G_{nn}(r, r') \) in this expression is a Green function for the equation:

\[
\langle (Q_n(r)) \rangle = i \frac{\pi \nu}{2} \left[ \hat{a}_n(\mathbf{r}) + \tilde{a}_n(\mathbf{r}) , g_n(r; \tilde{a}_n) \right]
\]

(5.8)

\[
\langle (Q_n(r)) \rangle = 0.
\]

As the zero order in \( \tilde{a}_n(\mathbf{r}) \) we may take \( g_n^{(0)}(r) = \Lambda \) using again the supersymmetry of the integral Eq. (5.3) when the source is disregarded. Then, we obtain in the first order

\[
g_n^{(1)}(r; \tilde{a}_n) = i \frac{\pi \nu}{2} \int d\mathbf{r}' d\mathbf{n}' \langle (Q_n(\mathbf{r})), (Q_n(\mathbf{r}')) \times Str (Q_n'(\mathbf{r}'), \Lambda) \rangle = 0
\]

(5.7)

The kernel \( G_{nn}(r, r') \) in this expression is a Green function for the equation:

\[
\langle (Q_n(r)) \rangle = i \frac{\pi \nu}{2} \left[ \hat{a}_n(\mathbf{r}) + \tilde{a}_n(\mathbf{r}) , g_n(r; \tilde{a}_n) \right]
\]

(5.8)

\[
\langle (Q_n(r)) \rangle = 0.
\]
We use here the notations \( j \equiv (n_j, r_j) \), \( \tilde{\jmath} \equiv (-n_j, r_j) \) of Ref. [4].

The quantity \( Y^{00}(r_1, r_2, \omega) \), Eq. (5.11), is an extension of the non-averaged density-density correlation function to arbitrary scales. We emphasize that the result is exact within the quasiclassical approximation.

A similar computation can be carried out for a higher order correlation function \( Y^{00}_2(r_1, r_2, r_3, r_4; \omega) \),

\[
Y^{00}_2(r_1, r_2, r_3, r_4; \omega) = G_{\ell}^{\ell}(r_1, r_2)G_{\ell'-\omega}^{\ell}(r_2, r_1)G_{\ell'}^{\ell}(r_3, r_4)G_{\ell'-\omega}^{\ell}(r_4, r_3) \tag{5.14}
\]

This function can be written as an integral over the supervectors \( \psi(n) \) as

\[
Y^{00}_2(r_1, r_2, r_3, r_4; \omega) = \int d\psi(r_1, r_2) \int d\psi(r_3, r_4) \psi^* r_1, r_2) \psi r_3, r_4) e^{-\frac{\pi \nu}{2}(\phi_{26}^{\eta}(r_1, r_2) - \phi_{26}^{\eta}(r_3, r_4))} \tag{5.15}
\]

In order to calculate the correlation function \( \mathcal{M} \), Eq. (5.11), one should expand Eqs. (5.3), (5.6) up to the third orders in the supermatrix elements. We emphasize that the result is exact within the quasiclassical approximation.

We used here the notations \( j \equiv (n_j, r_j) \), \( \tilde{\jmath} \equiv (-n_j, r_j) \) of Ref. [4].

This function can be written as an integral over the supervectors \( \psi(n) \) as

\[
Y^{00}_2(r_1, r_2, r_3, r_4; \omega) = \int d\psi(r_1, r_2) \int d\psi(r_3, r_4) \psi^* r_1, r_2) \psi r_3, r_4) e^{-\frac{\pi \nu}{2}(\phi_{26}^{\eta}(r_1, r_2) - \phi_{26}^{\eta}(r_3, r_4))} \tag{5.15}
\]

In order to calculate the correlation function \( \mathcal{M} \), Eq. (5.11), one should expand Eqs. (5.3), (5.6) up to the third order in the source and then compare them with each other. Writing the integral Eq. (5.15) for \( \alpha = 8, \beta = 4 \) and assuming that only the elements \( \phi_{26}^{\eta}(r_1, r_2) \), \( \phi_{26}^{\eta}(r_3, r_4) \) of the supermatrix are not equal to zero we expand it in the element \( \phi_{26}^{\eta}(r_1, r_2) \). Keeping only the first order in the expansion and putting \( \phi_{26}^{\eta}(r) = \delta_{n_1 n_2} \delta(r_1 - r_2) \) we obtain the following integral generalizing Eq. (5.10)

\[
\mathcal{D}_{n_1 n_2}(r_1, r_2; \phi_{26}^{\eta}) = -\frac{\pi \nu}{2} \int Q_{n_2}^{n_1}(r_1, r_2) \phi_{26}^{\eta}(r_2) - e^{-\frac{\pi \nu}{2} (Q_{n_2}^{n_1}(r_1, r_2) - \phi_{26}^{\eta}(r_2))} DQ_{n_2} \tag{5.19}
\]

The new source \( \phi_{26}^{\eta} \) in Eq. (5.15) differs from the previous one \( \phi_{26}^{\eta} \) by the substitution \( \phi_{26}^{\eta}(r) = \phi_{26}^{\eta}(r_1, r_2) \). At the same time, the source \( \phi_{26}^{\eta} \) in the source \( \phi_{26}^{\eta} \) bilinear in the elements \( \phi_{26}^{\eta}(r_1, r_2) \).

As concerns the elements \( \phi_{26}^{\eta}(r_1, r_2; \phi_{26}^{\eta}) \), \( \phi_{26}^{\eta}(r_1, r_2; \phi_{26}^{\eta}) \), they satisfy Eq. (5.6) implying the replacement \( \phi_{26}^{\eta}(r_1, r_2) \) by \( \phi_{26}^{\eta}(r_1, r_2) \).

Due to the structure of the source each of the equations is closed and contains no terms with the source. Their solutions are therefore \( \pm 1 \), respectively. Expanding further \( Z_2[\phi_{26}^{\eta}] \) in the elements \( \phi_{26}^{\eta}(r_1, r_2) \) and substituting the result in Eq. (5.21) we obtain

\[
[v_F \mathbf{n}_2 \nabla_2 - p_F^{-1} \nabla_2 u(r_2)] \partial_{n_2} - i(\omega + i\delta) \mathcal{M}(1, 4; 2, 3) = \delta_{12} D(3; 4) \tag{5.21}
\]

The operator in the l.h.s. of Eq. (5.21) acts on the variables of a one of the “diffusons”. In order to get also an equation with respect to the variables of the other diffusion we consider the integral, Eq. (5.3), with the source having now only the elements \( \phi_{26}^{\eta}(r_1, r_2) \), \( \phi_{26}^{\eta}(r_1, r_2) \), \( \phi_{26}^{\eta}(r_3, r_4) \). Putting there \( \alpha = 6, \beta = 2 \) and repeating successively all the steps that lead us to Eqs. (5.21) we come to an equation like Eq. (5.21) in which \( n_2, \mathbf{r}_2 \) in the left side are replaced by \( n_3, \mathbf{r}_3 \) and the variables 1, 2 in the right side by 3, 4 respectively. Adding this equation with Eq. (5.21) we find

\[
[v_F \mathbf{n}_2 \nabla_2 - p_F^{-1} \nabla_2 u(r_2)] \partial_{n_2} - i(\omega + i\delta) \mathcal{M}(1, 4; 2, 3) = \delta_{12} D(3; 4) + \delta_{34} D(1; 2) \tag{5.22}
\]

Eq. (5.22) agrees with the corresponding equation of Ref. [4]. This equation is written before averaging over the long range potential and is again exact within the quasiclassical approximation. However, the averaging over the long range potential is not trivial. Analyzing this equation Aleiner and Larkin demonstrated that the averaged function \( \mathcal{M}(1, 4; 2, 3) \) decouples into 2 diffusons (one can connect the Green functions \( G^{R, A} \) in 2 different ways forming 2 diffusons) at lengths exceeding the Lapunov length \( L_L \). We call here this region collisional. A similar analysis starting from a different formulation was carried out in Ref. [4] where the authors came to the same conclusion. This corresponds to the possibility of expanding in small fluctuations of the supermatrix \( Q_n \) using a parametrization like the one given by Eqs. (5.10) or (7.28).

In the Lapunov region, such an expansion is impossible and the only way to analyze correlation functions is to write equations for them for a fixed potential and then average the solution over the potential. Unfortunately, we do not know how to derive exact equations for averaged correlation functions. The form of the ballistic \( \sigma \)-model, Eq. (3.12), (3.13), does not allow us to derive such equations because of the complicated form of the term \( F_{imp} \).

Apparently, using the ballistic \( \sigma \)-model for calculations in the Lapunov region does not bring considerable advantages. At the same time, we see that the \( \sigma \)-model is applicable also in this region, although one cannot use the perturbation theory. Therefore, the conclusion of Ref. [4] that a completely different theory should be constructed for the Lapunov region is too pessimistic.
VI. PERIODIC ORBITS

In this chapter, we consider a clean finite system (quantum billiard). In principle, there can be holes (antidots) in the system but this can be discussed in terms of a more complicated surface.

A standard tool for computation of the one-particle density of states is the Gutzwiller trace formula (for a review, see, e.g., Ref. 19). This formula allows one to express the non-averaged density of states in terms of a sum over periodic orbits. This sum is actually divergent but there are methods to obtain reasonable results from it. As long as energies involved are of the order of the inverse period of short orbits one can extract a detailed information and compute also level-level correlation functions. Very often a statistical information is of the main interest and one calculates quantities averaged over spectrum.

The situation becomes considerably more difficult if one studies behavior at small energies of the order of the mean level spacing \( \Delta \). Even calculation of the first non-trivial term of the sum in Eq. (6.1) is not simple and difficulties grow when calculating next orders. On the other hand, the Wigner-Dyson statistics can easily be obtained from the zero-dimensional \( \sigma \)-model and the only question is when this \( 0D \) \( \sigma \)-model description is valid.

In Chap. IV we came to the conclusion that the \( 0D \) \( \sigma \)-model can be used for the model of weak long range scatterers if the frequency \( \omega \) is smaller than \( \min (\tau_1^{-1}, \tau_2^{-1}) \). For the level-level correlation function \( R(\omega) \), Eq. (6.2), the parameter \( \omega \) is the energy difference between two levels but what are the parameters \( \tau_1 \) and \( \tau_2 \) or, respectively, \( \tau_1 = v_F t_1 \) and \( \tau_2 = v_F t_2 \)?

For a clean quantum billiard the transport mean free path should be of order of the system size \( L \). At the same time, an important role should be played by the Ehrenfest time

\[
t_E = \lambda^{-1} \ln (L/\lambda_F)
\]  

(6.1)

where \( \lambda \) is the Lapunov exponent for scattering on the boundaries. The corresponding length \( l_E = v_F t_E \) is much larger or of the order of the system size \( L \). This time determines the crossover from the classical to the quantum regime. \( 1/\tau_1 \) is the signature of the classical trajectory to quantum trajectory. In the present work, we are not able to obtain this time within the approximations used. Averaging over the spectrum is equivalent to averaging over infinite range impurities and we see from Eqs. (6.12, 6.13) that only the term \( F_{kin} [Q_n(r)] \) is present in this case.

At first glance, it was not necessary to average over the energy when deriving the quasiclassical equations and the ballistic \( \sigma \)-model in Chap. II. However, this depends on what limit is taken first: the infinite size of the systems at finite disorder or vanishing disorder in a finite system. In the former case, an additional averaging over the energy is really not necessary. However, energy levels of a finite system are quantized and we cannot directly follow the arguments of Chap. II for the latter case. For example, the function \( g_n(r) \) introduced in Eq. (2.20) is not a smooth function because one should sum over \( \xi \) instead of integrating over it. However, the averaging over the energy improves the situation and makes possible using the quasiclassical equations. Since we should average the partition function with the sources, Eq. (2.1), and, hence, the Green functions, one can simply add averaging over the energy to the summation over \( \xi \) in the definition of \( g_n(r) \), Eq. (2.20). Then, the function \( g_n(r) \) is a smooth function of \( r \) and we can repeat all the subsequent arguments leading to the \( \sigma \)-model. So, although there is no disorder in the free energy functional \( F_{kin} [Q_n(r)] \), the energy averaging is implied for the clean quantum billiards.

As we have mentioned, our quasiclassical approach should be valid everywhere except in the vicinity of the boundaries, where the approximation fails near the turning points. Therefore, a more accurate computation might produce an additional term in the ballistic \( \sigma \)-model near the boundary. Aleiner and Larkin \( 17 \) introduced such a term modelling the quantum diffraction by fictitious short range impurities.

We think that an additional term in the \( \sigma \)-model due to the quantum diffraction is really necessary but leave its derivation for a future work. Instead, we will try now to derive the function \( R(\omega) \) neglecting this term. In other words, we consider energies \( \omega \) exceeding \( t_E^{-1} \) whatever it is.

In Ref. 12 an additional term was added as a regularizer, which had to be put to zero at the end of the calculations. However, proceeding in this way the authors of Ref. 12 got a result that did not agree with the one obtained from the Gutzwiller trace formula. The discrepancy has been called “repetition problem” and was discussed in a number of works. \( 14, 15 \)

We want to show now that the result obtained with the ballistic \( \sigma \)-model containing only the term \( F_{kin} [Q_n(r)] \), Eq. (6.13), agrees with what one can expect from the trace formulae. In contrast, the perturbative approach of Ref. 12 is not accurate in this limit.

We start our discussion with Eqs. (5.12, 5.13), where only the term \( F_{kin} [Q_n(r)] \) is left. The level-level correlation function \( R(\omega) \), Eq. (5.7), can be written as

\[
R(\omega) = 1 - \text{Re} I(\omega),
\]

(6.2)

\[
I(\omega) = \frac{1}{2\sqrt{\pi}} \int \left( (Q_{n}^{44}(r) - 1) \times (Q_{n}^{68}(r') + 1) \right)_{kin} dr dr' d\mathbf{d} d\mathbf{d}'
\]

(6.3)

where we use Eqs. (5.7), (5.8) and introduce a notation

\[
<...>_kin = \int (...) \exp \left( -F_{kin}(Q_n(r)) \right) DQ_n
\]
The free energy functional $\mathcal{F}^{(0)}_{kin}$ is obtained from $\mathcal{F}_{kin}$, Eq. (5.13) by putting $\tilde{a} = 0$.

Due to the absence of any regularizer a perturbation expansion in ballistic excitations (diffusion modes) cannot be good because one obtains diverging integrals with propagators like the one in Eq. (6.1) in the integrand. Therefore, this method should not be applied and we should try something different. At the same time, the specific form of the functional $\mathcal{F}^{(0)}_{kin}[Q_n(r)]$ that does not contain second space derivatives allows us to simplify the functional integral by reducing it to functional integrals on periodic orbits.

In order to proceed in this way we discretize the phase space writing the functional integral as a definite integral over $Q$ at all sites of a lattice in the phase space. In this way we write the function $I(\omega)$, Eq. (6.3), as

$$I(\omega) = \frac{[\Delta \Omega]^2 f^2}{2V^2} \sum_{r_i, r_j, n_i, n_j} ((Q_{44}^{n_i}(r_i) - 1) \times (Q_{44}^{n_j}(r_j) + 1))_{kin} \tag{6.4}$$

The free energy functional $\mathcal{F}^{(0)}_{kin}$ on this lattice takes the form

$$\mathcal{F}^{(0)}_{kin}[Q_n(r)] = \sum_{n_i} F_{n_i}[Q] \tag{6.5}$$

$$F_{n_i}[Q] = \frac{\pi \nu [\Delta \Omega] f}{4} \text{Str}\left[ \sum_{\{r_i, r'_i\}} \frac{2v_P}{f} \Lambda T_n(r_i) T_n(r'_i) \right]$$

$$+ \sum_{\{r_i, r'_i\}} i(\omega + i\delta) \Lambda Q_n(r_i)$$

In Eqs. (6.4, 6.5), $f$ is the elementary length in the coordinate space along a trajectory and $[\Delta \Omega]$ is the elementary phase volume in the phase space perpendicular to it (i.e. to the unit vector $n_i$). The summation in the first term in $F_{n_i}[Q]$ is performed over nearest neighbors on the trajectory and in a certain order. The elementary volume $[\Delta \Omega]$ in Eqs. (6.4, 6.5) can be written as

$$[\Delta \Omega] = S_d^{-1} \prod_{i=1}^{d-1} (\Delta n_i^t \Delta r_i^t) \tag{6.6}$$

where $S_d$ is the surface of the unit sphere in the $d$-dimensional space. In Eq. (6.4), $\Delta r_i^t$ and $\Delta n_i^t$ are the elementary length and momentum in a direction perpendicular to the path.

In principle, the length $f$ and the space volume $[\Delta \Omega]$ may be arbitrary. At the same time, we should remember that we have used the quasiclassical approximation and the length $f$ may not be smaller than the wavelength $\lambda_P$. However, only for specific choice of $[\Delta \Omega]$, the functional $F_{n_i}[Q]$, Eq. (6.5), remains single valued. As we will see, this choice corresponds to the Bohr-Sommerfeld quantization rules.

In order to reduce the multiple integral over all $Q$ on the lattice sites in the phase space to a simpler form we use the following equalities that can be proven using the methods of integration over supermatrices:

$$\int \exp(-F_n[Q]) DQ = 1 \tag{6.7}$$

$$\int (Q_{ss}^r(r) - 1) \exp(-F_n[Q]) DQ \tag{6.8}$$

$$= \int (Q_{ss}^r(r) + 1) \exp(-F_n[Q]) DQ = 0$$

for any $r$ and $n$.

Let us understand first how to simplify the function $I(\omega)$, Eq. (6.3), for an infinite sample. Using Eqs. (6.7, 6.8) we conclude immediately that only the terms with $n_i = n_j$ contribute in the sum in Eq. (6.4). As concerns the free energy functional, one can integrate over all $Q_n$ with $n_i \neq n_j$ using Eq. (6.7), which leaves only one term $F_{n_i}[Q]$ in the exponential. Only the term with $n_i$ parallel to the line connecting the points $r$ and $r'$ gives a non-zero contribution and, moreover, integration over $Q$ on sites outside the line gives 1 because the free energy functional $\mathcal{F}^{(0)}_{kin}[Q_n(r)]$, Eq. (6.3), does not contain couplings of $Q$ on these sites with $Q$ on sites on the line between $r$ and $r'$.

Thus, we come to an integral over all $Q$ on sites along the line connecting the points $r$ and $r'$

$$I(\omega) = I_+ (\omega) + I_- (\omega), \tag{6.9}$$

$$I_+ (\omega) = \frac{[\Delta \Omega]^2 f^2}{2V^2} \sum_{r_i, r_j} \int (Q_{44}^{r_i}(r_i) - 1) (Q_{44}^{r_j}(r_j) + 1) \times \exp(-F_{+}[Q] - F_{-}[Q]) DQ,$$

$$F_{\pm}[Q_{\pm}] = \frac{\pi \nu [\Delta \Omega] f}{4} \text{Str}\left[ \sum_{\{r_i, r'_i\}} \frac{2v_P}{f} \Lambda T_{\pm}(r_i) T_{\pm}(r'_i) \right]$$

$$+ \sum_{\{r_i, r'_i\}} i(\omega + i\delta) \Lambda Q_{\pm}(r_i)$$

The signs + and − correspond to different directions of the trajectory; the summation over the pairs $\{r_i, r'_i\}$ in the free energies $F_{\pm}[Q_{\pm}]$ should be fulfilled in the order conforming with its direction. One can see that the free energies and, hence, both the terms $I_+ (\omega), I_- (\omega)$ are equal to each other. This relation follows formally from the definition of the conjugation and the equality $Q_n(r) = \overline{Q_n}(r)$. It is important that the time-reversal symmetry is not violated.

We see from Eqs. (6.9) that the functional integral, Eq. (6.3), over $Q_n(r)$ on all sites in the phase space of an infinite sample has been reduced to a functional integral along a line and averaging over all directions of this
line. This is due to a specific form of the free energy functional containing only first space derivatives. Any regularizer containing second derivatives would make such a reduction impossible.

What happens if the sample is finite? We can reduce as before the functional integral, Eq. (6.3), to an integral over the line. However, we can follow this line until we reach the boundary, where we have a degeneracy that follows from the boundary condition, Eq. (3.3). Namely, the supermatrix $Q_n$ belongs to 2 different lines. This means that, having reached the surface, we can follow the line obtained from the first one by a specular reflection. We can keep going along the second line until we reach another boundary, etc. In principle, we have two possibilities:

1. After several reflections from the boundaries we come to the same point in the phase space or, in other words we get a periodic orbit.

2. The broken line obtained after the reflections on the boundaries does not close in the phase space, which can be considered as a periodic orbit with an infinite period. Of course, we should speak rather of tubes than of lines. However, this is not important because we are interested now in comparatively high frequencies $\nu$ of the order of a typical period of the orbit. Very long orbits would contribute to the function $I(\omega)$, Eq. (6.3), at much smaller frequencies. The question about the finite thickness of the lines can arise for times larger than the Ehrenfest time $t_F$, Eq. (1.1), which we do not consider here.

So, let us assume that we have got a periodic orbit. All supermatrices $T$ and $Q$ in Eqs. (6.3) are assumed to be on this orbit and we can write these equations in a more convenient form choosing the thickness of the paths and changing to the continuous limit along them. Taking the continuous limit along the paths the elementary length $f$ will disappear. The only quantity to be chosen is the elementary phase volume $[\Delta\Omega]$ in Eq. (6.6).

In the $d$-dimensional space the density of states $\nu$ can be written as

$$\nu = \frac{dn_{\Omega}}{dF} = \frac{\Omega_d d}{(2\pi\hbar)^d} v_F^{d-1}$$  \hspace{1cm} (6.10)

where $\Omega_d$ is the volume of the unit $d$-dimensional sphere.

Then, using the relation between the surface of the unit sphere and its volume $S_d = \Omega_d d$ we reduce the coefficient $\nu[\Omega]$ entering the free energy functional $F_{\pm}[Q]$ in Eqs. (6.3) to the form

$$\nu[\Delta\Omega] = \frac{1}{(2\pi\hbar)^d} \prod_{i=1}^{d-1} (\Delta p_i^\perp \Delta r_i^\perp)$$  \hspace{1cm} (6.11)

Now we have to choose the product $\prod_{i=1}^{d-1} (\Delta p_i^\perp \Delta r_i^\perp)$ and we do this using a standard quasiclassical rule according to which we write

$$\prod_{i=1}^{d-1} (\Delta p_i^\perp \Delta r_i^\perp) = (2\pi\hbar)^{d-1}$$  \hspace{1cm} (6.12)

With Eq. (6.12) we obtain

$$\nu[\Omega] = (2\pi\hbar \nu_F)^{-1}$$  \hspace{1cm} (6.13)

Putting as everywhere before $\hbar = 1$ we can rewrite Eqs. (6.9) as a sum over periodic orbits

$$I(\omega) = \frac{\Delta^2}{(2\pi v_F)^2} \int \left( (Q^{44}(x) - 1) \times (Q^{88}(x') + 1) \right) dx dx'$$  \hspace{1cm} (6.14)

where $< ... >_1$ stands for the functional integral

$$\langle ... \rangle_1 = \int (...) \exp(-F_1[Q]) DQ$$  \hspace{1cm} (6.15)

and $\Delta = (\nu V)^{-1}$ is the mean level spacing for the billiard under consideration. The one-dimensional free energy functional $F_1[Q]$ for an orbit takes the form

$$F_1[Q] = \frac{1}{2} Str \int \left( \Lambda T(x) \frac{dT(x)}{dx} + i(\omega + i\delta) \frac{\Lambda Q}{2\nu_F} \right) dx$$  \hspace{1cm} (6.16)

and is the result of the adding of the free energies $F_{\pm}[Q_{\pm}]$ Eq.(6.9). The sum over $p$ in Eq. (6.14) means the sum over all periodic orbits and, in principle, $Q(x)$ should depend on $p$. In order to simplify notations we omit writing this dependence explicitly. The integrals over $x$ and $x'$ are taken along the orbits (a certain direction is implied to be already chosen).

The overall coefficient in the functional $F_1[Q]$ is determined by the quasiclassical rule, Eq. (6.12). It is not difficult to understand that Eq. (6.12) is the only reasonable choice for the “thickness” of the classical paths. The functional $F_1[Q]$ is multivalved because any replacement of the type $T(x) \rightarrow T(x)h(x)$, where $h(x)h(x) = 1$ and $[h(x), A] = 0$, does not change the supermatrix $Q(x)$ but changes the functional $F_1[Q]$ as

$$F_1[Q] \rightarrow F_1[Q] + \frac{1}{2} \int Str \left( \tilde{h}_1(x) \frac{dh_1(x)}{dx} - \tilde{h}_2(x) \frac{dh_2(x)}{dx} \right) dx$$  \hspace{1cm} (6.17)

where $h_1$ and $h_2$ are the upper and lower diagonal blocks, respectively. Writing $h_m$, $m = 1, 2$ as

$$h_m(x) = \begin{pmatrix} \exp(i\hat{\phi}_m(x)) & 0 \\ 0 & \exp(i\hat{\chi}_m(x)) \end{pmatrix}$$  \hspace{1cm} (6.18)

$$\hat{\phi}_m(x) = \begin{pmatrix} \phi^+_m(x) & 0 \\ 0 & -\phi^-_m(x) \end{pmatrix}$$  \hspace{1cm} (6.19)

$$\hat{\chi}_m(x) = \begin{pmatrix} \chi^+_m(x) & 0 \\ 0 & -\chi^-_m(x) \end{pmatrix}$$  \hspace{1cm} (6.20)
we write the change $\Delta F_1 [Q]$ of the functional $F_1 [Q]$ as

$$\Delta F_1 [Q] = \frac{i}{2} \sum_{m=1,2} (-1)^{m-1} \left( \Delta \phi_m^+ - \Delta \phi_m^- + \Delta \chi_m^+ - \Delta \chi_m^- \right)$$

(6.19)

where $\Delta \phi_m^+$ and $\Delta \chi_m^+$ are changes of the phases when going around the periodic orbit. These changes must be integer multiple of $2\pi$. The phases $\phi_m^+(x)$ and $\phi_m^-(x)$ (as well as $\chi_m^+(x)$ and $\chi_m^-(x)$) are not independent of each other. If $\phi_m^+(x) = (2\pi k_m / L_p) x$, then $\phi_m^-(x) = - (2\pi k_m / L_p) x$ ($k_m$ is an integer and $L_p$ is the length of a $p$-orbit). In a general case we can write

$$\phi_m^+(x) = \pm \frac{2\pi k_m}{L_p} x \tilde{\phi}_m^+(x) \quad \tilde{\phi}_m^+(0) = \tilde{\phi}_m^+(L_p)$$

(6.20)

and the same for $\chi_m^+$.

Then, we obtain for the changes of the phases $\Delta \phi_m^+ = - \Delta \phi_m^-$, $\Delta \chi_m^+ = - \Delta \chi_m^-$, which reduces Eq. (6.13) to the form

$$\Delta F_1 [Q] = 2\pi i M$$

(6.21)

where $M$ is integer.

With Eq. (6.21) we come to the result that, although the functional $F_1 [Q]$ is multivalued, the partition function $\exp (- F_1 [Q])$ is not and one can integrate over $Q$ in a standard way. In principle, we could proceed in the opposite way and determine the thickness of the classical paths by demanding the partition function be single valued. Then, we would obtain the quasiclassical quantization rule, Eq. (6.14), automatically.

Now, let us calculate the functional integral, Eq. (6.14), using two different approaches. First, we compute the functional integral following the scheme of Ref. [12]. We can use different parametrizations like those specified by Eqs. (6.14) or (4.28). The parametrization, Eq. (4.28), is very convenient because the Jacobian is equal to unity. At the same time, the ballistic $\sigma$-model contains also non-harmonic in $P$ terms. In the quadratic approximation, one writes the free energy as

$$F^{(2)}_1 [Q] = \frac{1}{2} \text{Str} \int \left( \Delta P \frac{dP}{dx} \right)(x) dx - i \frac{\omega}{v_F} \int \left( P^2 \right)(x) dx$$

(6.22)

The supermatrices $Q(x)$ entering Eq. (6.14) should also be expanded up to quadratic terms. Calculating gaussian integrals we obtain for $I(\omega)$

$$I(\omega) = \frac{\Delta^2}{\pi^2 v_F} \sum_p \sum_{m=-\infty}^{\infty} \left( q_m^{(p)} + \frac{\omega}{v_F} \right)^{-2}$$

(6.23)

where

$$q_m^{(p)} = \frac{2\pi m}{L_p}$$

are momenta corresponding to a $p$-periodic orbit, $L_p$ is the length of the orbit.

Eq. (6.23) corresponds to the “perturbative part” of the level-level correlation function of Ref. [12], although here we sum over momenta on periodic orbits instead of summation over eigenvalues of the Perron-Frobenius operator. However, it is clear that Eq. (6.23) cannot correspond to a good perturbation theory because this expression contains resonances at arbitrarily high frequencies.

One might guess that next orders of the expansion in $P$ had to be taken into account and the expansion would not be good at the resonances. Curiously enough, it is not so. One can see immediately that the quadratic form, Eq. (6.22) is exact in the parametrization, Eq. (6.14) (the overall coefficient is 2 times larger). At the same time, the part $Q^0$ of the supermatrix $Q$ commuting with $\Lambda$ and entering Eq. (6.14) is exactly $Q^0 = 1 + 2P^2$. Therefore, we do not obtain any perturbative corrections to Eq. (6.23) (at the same time, the contribution of the Jacobian in the parametrization, Eq. (6.14), is not as clear).

Nevertheless, Eq. (6.23) is not exact. The matrices $P$ in Eq. (6.14) vary on non-trivial manifolds and extending the integration over these matrices from $-\infty$ to $+\infty$ as it is implied in any gaussian integration is unjustified. To make the discussion simpler, let us rewrite Eqs. (6.23) using the Poisson summation formula as

$$R(\omega) = 1 + \sum_p \left( \frac{T_p \Delta}{\pi} \right)^2 \text{Re} \sum_{n=1}^{\infty} n \exp (i (\omega + i\delta) T_p n)$$

(6.24)

where $T_p = L_p / v_F$ is the period of the motion on the $p$-orbit. Eq. (6.24) corresponds to an expansion in periodic orbits of a classical flow. Strictly speaking, Eq. (6.24) is different from what one writes for classical flows by absence of a factor containing the monodromy matrix. It is clear that in our simple consideration the stability of the periodic orbit is not taken into consideration.

As any periodic orbit we consider has a finite thickness, the monodromy matrix would appear in a more accurate calculation. However, the aim of this chapter is only to clarify the origin of the repetition problem and therefore we use the simplest approximation.

The factor $n$ in front of the exponential is a characteristic feature of expansions for classical flows (see e.g. Eq. (52) of Ref. [10] which leads to this dependence after taking the logarithm of both parts and taking second derivative in $s$). In other words, we have now an expansion in periodic orbits of the Perron-Frobenius operator and this corresponds to the result of Ref. [12] in the limit of the vanishing regularizer.

However, although the perturbative approximation of Ref. [12] works very well in the diffusion limit, we do not see any justification for it in the ballistic limit. Therefore, we should try to calculate the integral in Eqs. (6.14)

(6.15) without using this approximation.

Fortunately, the functional integral in Eq. (6.14)
can be calculated exactly even easier than approximately. The free energy functional \( F_1 (Q) \) entering these equations corresponds to a one-dimensional ring without any impurities, provided the averaging over the spectrum has been performed. This energy averaging is necessary to get a smooth quasiclassical function \( g_m (r) \), Eq. (2.20). Without averaging this function would not be smooth due to quantization of the energy levels in the ring. So, we conclude that the calculation of functional integral over \( Q \) with the free energy functional \( F_1 [Q] \) is equivalent to calculation of the averaged level-level correlation function \( R_p (\omega) \) for a clean electron system on a ring.

The level-level correlation function \( R_{1p} (\omega) \) for such a ring can be written as

\[
R_{1p} (\omega) = (\nu_1 L_p)^2 \sum_{m,m'= - \infty} \infty (\delta (\varepsilon - \omega - \varepsilon (q_m))) \times \delta (\varepsilon - \varepsilon (q_m')) \varepsilon (q_m) = \frac{g_m^2 - p_F^2}{2m} \approx v_F \left( |q_m| - p_F^{(p)} \right)
\]  

(6.26)

Using the Poisson formula and Eq. (6.26) we write

\[
\sum_{m=0} \infty (\delta (\varepsilon - \varepsilon (q_m))) \approx \frac{L_p}{\pi v_F} \sum_{n= - \infty} \infty \exp \left( i \left( 2\pi n_F^{(p)} + \varepsilon T_p \right) n \right)
\]  

(6.27)

After averaging over \( \varepsilon \), only the term with \( n = n' \) gives the contribution in Eq. (6.28). On the other hand, repeating all the steps of the derivation of the \( \sigma \)-model, we come to Eqs. (6.14) with the only difference that we should replace the mean level spacing \( \Delta \) of the entire system by the level spacing of the one-dimensional ring \( \Delta_p = \pi / T_p \). This allows us to write the level-level correlation function \( R (\omega) \) of the quantum billiard under consideration as

\[
R (\omega) = 1 + \sum_p \left( \frac{T_p \Delta_p}{\pi} \right)^2 \Re \sum_{n=1} \infty \exp (i (\omega + i\delta) T_p n)
\]  

(6.29)

Comparing Eqs. (6.24) and (6.29) with each other we see that the only difference between them is the presence of the prefactor \( n \) in Eq. (6.24). So, we conclude that the factor \( n \) in the expansion in periodic orbits (a consequence of a unjustified approximation of Ref. 12) and this solves the problem of repetitions [13]. The assumption of Ref. 13 that the regularizer can be put to zero at the end of calculations does not seem to be correct. We believe (following Ref. 14) that the presence of a finite regularizer is inevitable in a quantum system and a very important problem is to calculate it.

The problem of repetitions was discussed recently for weak scatterers in Ref. [15] where the problem was related to the question of a possibility of separating 4 point correlation functions into two diffusions. From the above discussion, we see that the problem is even more delicate because Eq. (6.23) is perturbatively exact and the difference comes from oscillating exponentials.

In order to understand better what has been neglected in our derivation, we compare Eq. (6.29) with a corresponding diagonal contribution obtained from the Gutzwiller trace formula, (see, e.g. [1])

\[
R^{(d)} (x) = 1 + \frac{2}{T_H} \sum_{p \sigma} \delta \sum_{n=1} \infty g_p |M_p^\sigma - I| \cos \left( \frac{2\pi n T_p}{T_H} \right)
\]  

(6.30)

where \( g_p \) is the action-multiplicity of the \( p^{th} \) primitive orbit (for the orthogonal ensemble \( g_p = 2 \)), and \( M_p^\sigma \) is the monodromy matrix that describes the flow linearized in its vicinity. In Eq. (6.30), \( T_H = 2\pi / \Delta \) is the Heisenberg time, \( T_p \) is the orbit period, and \( x = \omega / \Delta \). Except for the factor \( |M_p^\sigma - I|^{-1} \) Eqs. (6.29) and (6.30) agree. As we carried out computation without any regularizer like the one of Ref. 15, we conclude again that its presence is absolutely necessary and it must be related to the monodromy matrix \( M_p \). We see that without this term correlations between orbits do not exist and one cannot pass to the universal limit when lowering the frequency \( \omega \). In the language of the field theoretical approach, one cannot reduce the ballistic \( \sigma \)-model to the zero-dimensional one without this regularizing term that must describe quantum diffraction on irregularities of the boundary.

Adding a term like

\[
F_{reg} [Q] = \text{Str} \int \beta_n (r) \left( \frac{\partial Q (r)}{\partial n} \right)^2 d r d n
\]  

(6.31)

where \( \beta_n (r) \) is a function in the phase space, we may obtain effectively a coupling between the periodic orbits of the type

\[
- \sum_{p,p'} \beta_{pp'} \text{Str} \int Q(p) (q) Q(p') (q') dx_p d x_{p'}
\]  

(6.32)

which resembles coupling between grains in a granular system. For small \( \beta_{pp'} \) the orbits are not coupled and we have separate periodic orbits. As the coupling \( \beta_{pp'} \) grows, the relative fluctuations of \( Q(p) \) with respect to
each other get suppressed and one needs to consider rotations of the system as the whole. Then, one obtains the zero-dimensional $\sigma$-model and, hence, the Wigner-Dyson statistics. This is a scenario for a granular metal and we believe that it is relevant for the quantum billiard, the role of the grains being played by separate periodic orbits.

VII. DISCUSSION.

In the present work, we made an attempt to put the field theoretical approach to systems with a long range disorder on a solid basis. The conventional method of derivation of the supermatrix $\sigma$-model is based on singling out slow modes, performing Hubbard-Stratonovich transformation and using a saddle-point approximation. Although this approach worked well for a short range disorder, its validity is not justified for a long range disorder and quantum billiards. We suggested a scheme that allows us to overcome these difficulties and derive a modified non-linear ballistic $\sigma$-model (see Eq. (5.13)).

The method resembles the approach of Ref. [12] and is based on writing quasiclassical equations for generalized Green functions. At the same time, the quasiclassical equations are written for non-averaged over the long range potential quantities and singling out slow modes is performed only for a part originating from a short range disorder. In addition, the short range disorder does not play an important role and can be put to zero. The crucial step of the derivation is that the solution of the quasiclassical equations can be found exactly, which is a consequence of the supersymmetric structure of the Green functions. This possibility was overlooked in the previous study [17].

The scheme developed now leads to a considerable progress in describing disordered systems with long range disorder because the derivation is applicable for all lengths exceeding the wavelength $\lambda_F$. Representing the solution of the quasiclassical equations and also the partition function for the electron Lagrangian with sources in terms of a functional integral over supermatrices $Q_n (r)$, $n^2 = 1$, with the constraint $Q_n^2 (r) = 1$ we were able to average over the disorder exactly and obtain a ballistic $\sigma$-model in a new form that has not been written before. The so-called “mode locking” problem does not arise here because the eigenvalues of the supermatrix $Q_n (r)$ are fixed by the construction and do not fluctuate. The method suggested resembles the method of bosonization, well known in field theory, see e.g. a book [16] where a fermionic system is replaced by a bosonic one. In our approach, we also replace the electron system by a system of ballistic excitations that can be considered as quasiparticles. At large scales, these quasiparticles are well known diffusons and cooperons. In analogy, our scheme can be called superbosonization.

For weak scatterers, there should exist 2 more scales: the Lapunov length $l_L$ and the transport mean free path $l_t$. The single particle mean free path $l$ does not appear in our consideration (actually, we do not consider one-particle Green functions at different points restricting our study to gauge invariant quantities). Integrating out degrees of freedom related to distances smaller than the Lapunov length $l_L$ we obtained a reduced ballistic $\sigma$-model. A propagator describing small fluctuations within this reduced $\sigma$-model corresponds to the kinetic Boltzmann equation with a collision term. Integrating further on scales up to the transport mean free path $l_t$, we obtain the standard diffusive $\sigma$-model.

Trying different calculational schemes we conclude that one can do perturbative calculations with the ballistic $\sigma$-model only at scales exceeding the Lapunov length (we call this range “collision region”). At smaller lengths (following Ref. [17] we call this range Lapunov region) no perturbation expansions in diffusons and cooperons are possible. In this region one can carry out calculations deriving equations for correlation functions and investigating them in different approximations.

It seems that an infinite system with a weak long range disorder is adequately described by the ballistic $\sigma$-model we have derived. However, when describing quantum billiards, our approach is not accurate near the boundaries, where the quasiclassical approximation may not be used (turning points). We believe that a more accurate derivation may result in a new term in the $\sigma$-model. This term was suggested phenomenologically in Refs. [17, 18] but has not been derived yet microscopically. Its presence seems to be absolutely necessary because it must introduce a new scale: the Ehrenfest time $t_E$. One may not put the regularizer to zero at the end of calculations. We have demonstrated that neglecting such a term we reduced the ballistic $\sigma$-model for the billiard to ballistic $\sigma$-models for periodic orbits. Proceeding in this way we demonstrated explicitly where the contradiction between the work [14] and Ref. [17] (repetition problem) comes from.

Our conclusion is that the representation of the level-level correlation function in terms of eigenvalues of the Perron-Frobenius operator suggested in Ref. [17] is not justified in the ballistic case. This approach is valid only if there are no repetitions but this would rather correspond the diffusive case. In the opposite limit, one comes to a description in terms of periodic orbits without correlations between actions of different orbits. This is the region where the description of Ref. [17] may be applicable. At the same time, the Ehrenfest time can hardly be identified on the basis of the trace formula and therefore the limits of applicability of the result of Ref. [17] have not been specified. The hypothetical regularizer seems to be related to the monodromy matrix entering the Gutzwiller trace formula.

We believe that the field theoretical approach presented here and the formalism based on the Gutzwiller trace formula can be complementary to each other describing quantum systems in different regions of parameters. At times smaller than the Ehrenfest time, the trace
formula can be more convenient. However, at a larger
time, trying to extract physical quantities from the trace
formula does not make much sense because the $\sigma$-model is
a much more convenient tool for such calculations. This
concerns especially the universal limit where the
$\sigma$-model approach leads for most correlation functions to definite
integrals that can be computed rather easily.

It is important to notice, that diagrammatic expan-
sions like those attempted in Ref. can hardly be suc-
scessful. The authors of Ref.\(^\dagger\) found that the results de-
pended crucially on the way how the ultraviolet cutoff
was introduced. Now we understand that the ultraviolet
cutoff must be imposed by the requirement that the in-
tegration is performed over the manifold $Q^2_n(r) = 1$ in
an invariant way. Any artificial ultraviolet cutoffs in the
perturbation theory would correspond to a violation the
rotational invariance in the space of the supermatrices
$Q_n(r)$ and lead to wrong results. At the same time, it
is not clear how to develop a perturbation theory in an
invariant way.

An important question of an averaging procedure was
discussed in several works.\(^\dagger\dagger\)\(^\dagger\dagger\)\(^\ddagger\) In all these publica-
tions an opinion was expressed that an averaging over
energy was not sufficient for study of quantum chaos in
quantum systems and different types of an additional
averaging were suggested. We do not agree with this
point of view because, in our derivation of the quasiclassical
equations, averaging over the energy allowed us to
smooth generalized Green functions and this was all we
needed. The only condition is that the averaging should
be performed in an interval of energies much exceeding
the mean level spacing.

The source of the discrepancy is simple: the authors of
the works\(^\dagger\dagger\)\(^\dagger\dagger\)\(^\ddagger\) used the saddle point approxima-
tion and the expansion in gradients. Although the saddle
point approximation was not necessary in Ref.\(^\ddagger\), the gra-
dient expansion still had to be carried out. Therefore, an
additional averaging was necessary to justify these ap-
proximation. Since we do not do such approximations, no
additional averaging is needed in our scheme.

In conclusion, our approach enables us to carry out
calculations for long range disorder and chaos in a reli-
able way. Still, a derivation of a new term \((3.13)\) describing quantum diffraction on bound-
aries has to be done to make the theory complete but we
believe that this is not impossible.

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APPENDIX A: BOUNDARY CONDITIONS

In this Appendix, we derive boundary conditions for the
boundary of the sample. We describe the boundary by an external field $u_\beta(r)$ which is negligible inside the
sample and grows sharply at the surface of the sample.
In such a situation, the electron wave function decays
fast outside the sample and, in the limit of infinite poten-
tial walls, one can just put the wave function equal to
zero at the boundary. Unfortunately, such a boundary
condition is not very helpful because in the quasiclassical
approximation we use it looses its validity at a distance
of several wave lengths from the boundary and matching
the wave functions in the bulk and at the boundary is
necessary.

In order to find effective boundary conditions for qua-
siclassical Green functions we follow methods well devel-
oped in superconductivity theory.\(^\dagger\dagger\)\(^\dagger\dagger\)\(^\ddagger\) First, we write the
supermatrix $G^{\alpha\beta}(r; r')$ in a form of a sum over eigensu-
per vectors $\psi_k(r)$

$$G^{\alpha\beta}(r; r') = \sum_k \frac{\psi^\alpha_k(r)\bar{\psi}^\beta_k(r')}{\varepsilon_k - \varepsilon} \tag{A1}$$

satisfying the Schrödinger equation

$$\left[ H_{0r} + u(r) + \Lambda \frac{\omega + i\delta}{2} + iJ(r) \right] \psi_k(r) = \varepsilon_k \psi_k(r) \tag{A2}$$

We assume that the potential $u(r)$ in Eq. \((A2)\) contains
not only the impurity field but also the potential $u_B(r)$
describing the boundary. The conjugated equation can be
written as

$$\bar{\psi}_k(r) \left[ H_{0r} + u(r) + \Lambda \frac{\omega + i\delta}{2} + iJ(r) \right] = \varepsilon_k \bar{\psi}_k(r) \tag{A3}$$

The summation in Eq. \((A1)\) should be performed over the complete set of eigenfunctions, so that

$$\sum_k \psi^\alpha_k(r)\bar{\psi}^\beta_k(r') = \delta^{\alpha\beta}\delta(r-r') \tag{A4}$$

It means that the choice of the set of eigenfunctions $\psi^\alpha_k(r)$ and the operation of the
conjugation must conform with each other. For ex-
ample, at distances from the boundary much larger than
the wavelength $\lambda_F$ but much smaller than the radius $b$
of the random potential, we choose the eigenfunctions
$\psi^\alpha_k(r)$ in a form of plane waves

$$\psi_{p}(r) = e^{ipr}\bar{\psi}_{p}, \tag{A4}$$

where $\bar{\psi}_{p}$ is a normalized vector from the superspace.
Then, we have to adjust the definition of the conjugation
written in the book and add to it the momentum inver-
sion $p \to -p$. If the motion cannot be treated as free
changing the sign of the momentum has to be generalized by replacing this operation by the time reversal. Since the energy $\varepsilon_k$ in Eq. (A2) remains the same after the time reversal, the spectral expansion written in Eq. (A1) is in agreement with Eq. (2.17). Below, we will use this relation between supervectors $\psi^\alpha(r)$ and those conjugated to them.

Now we introduce local coordinates $(z, r_i)$ in the vicinity of the boundary. The coordinate $r_i$ is a coordinate along the boundary surface and $z$ is a distance between a given point and the surface. Points on the surface have the coordinates $(0, r_i)$. If the boundary is rough, the coordinate system $(z, r_i)$ is not very useful. However, if the boundary is smooth, which means that the derivative of the field $u_B(r)$ along it is small in comparison to that in the perpendicular direction, the coordinates $(z, r_i)$ are very convenient for the quasiclassical approximation.

If the radius of the curvature of the boundary is large the electron wave function in the vicinity of the boundary can be represented as a sum of one-dimensional solutions with respect to the $z$-direction with amplitudes slowly dependent on the coordinates $r_i$. Since we need to know the wave function in the domain in which the potential $V_B(r)$ vanishes, we can write the asymptotic form of the wave functions as

$$
\psi_k(r) = e^{ipz} \psi_k(r_0) \left( e^{ipz} \varphi_{k,c}(r_0) + e^{-ipz} \varphi_{k,r}(r_0) \right) \tag{A5}
$$

where $\varphi_{k,c}(r_0)$ and $\varphi_{k,r}(r_0)$ are some slowly varying supervector functions depending on the potential $u_B(r)$ that play the role of amplitudes of the coming and reflecting waves respectively. They comprise the minimal knowledge about the potential $u_B(r)$ that is needed to find the required boundary conditions.

If the boundary is impenetrable and the potential reflects all waves, then the amplitudes $\varphi_{k,c}(r_0)$, $\varphi_{k,r}(r_0)$ can be determined from the condition that the component of the current perpendicular to it is equal to zero. This condition is valid not only in the region in which the potential $u_B(r)$ is relevant but also in the quasiclassical region because the current cannot considerably change at distances of the order of the wavelength.

An expression for the current can be obtained in a standard way from the particle conservation law that follows from Eqs. (A2), (A3). Its $z$-component perpendicular to the surface is proportional to the difference $\partial_z \psi \dot{\psi} - \psi \partial_z \dot{\psi}$. Substituting Eq. (A3) and its conjugate into the difference and putting the result to zero we find $\dot{\varphi}_{k,c}(r_0) \varphi_{k,c}(r_0) = \dot{\varphi}_{k,r}(r_0) \varphi_{k,r}(r_0)$. A relation between the amplitudes of the coming and reflected waves in the case of the impenetrable boundary can also be established by the demand that they should transform one into the other by the time reversal. A general expression that satisfies it can be chosen in the form:

$$
\psi_k(r) = e^{ipz} \psi_k(r_0) \left( e^{ipz} \varphi_{k,c}(r_0) - e^{-ipz} \varphi_{k,r}(r_0) \right) \tag{A6}
$$

where $z_0$ determines an unknown phase that can be found only by matching the function $\psi_k(r)$, Eq. (A6), with the corresponding decaying asymptotics at the opposite side of the turning point (its value is of the order $\lambda_F$). This is generally not an easy task but, fortunately, the parameter $z_0$ is not important for finding the boundary conditions for the Green functions.

Substituting Eq. (A6) into the spectral expansion, Eq. (A1), we find an expression for the matrix $G^{\alpha\beta}(r;r')$ at the boundary. To determine the matrix $g_{n}(r)$, Eq. (2.20), it is necessary to carry out both the summation in $\xi$ (the system may be finite and we have to sum instead of integrating over $\xi$) and averaging over the energy. The latter is absolutely necessary because only this averaging guarantees vanishing of all terms containing the products $ip_F n(r + r')$ in the exponents ($n$ is a unit vector parallel to $p$). As soon as the terms containing $ip_F n(r + r')$ vanish, the parameter $z_0$ drops out. Then, the Green function $G^{\alpha\beta}(r;r')$ can be written in the vicinity of the boundary as

$$
G^{\alpha\beta}(r;r') = i \sum_k (\varepsilon_k - \varepsilon)^{-1} (A7)
$$

$$
\times e^{ipz} (r_0 - r_0') (e^{-ipz} (z' - z) + e^{ipz} (z - z')) f_k(r_0', r_0') \varepsilon
$$

where $f_k(r_0', r_0') = \varphi_k(r_0) \varphi_k(r_0')$ and $\langle \ldots \rangle_\varepsilon$ stands for averaging over the energy.

Carrying out in Eq. (A7) summation over $\xi$ and averaging over the energy, which is equivalent to integration over $\xi$, and Fourier transforming with respect to $r - r'$ we obtain the quasiclassical function $g_n(r)$. The function obtained from Eq. (A7) is a slow function of the coordinate $r + r'$ and does not change at the boundary under the replacement $n_\perp \to -n_\perp$. Hence, we come to the boundary condition for the quasiclassical Green functions $g_n(r)$

$$
g_{n_\perp}(r) = g_r - n_\perp(r), \tag{A8}
$$

$$
r = (z = 0, r_0)
$$

where $n_\perp$ is the component of the vector $n$ perpendicular to the surface. Although Eq. (A8) is rather simple, it has not been written in previous works on the ballistic $\sigma$-models.

1. F. Wegner, Z. Phys. B 35, 207 (1979)
2. K.B. Efetov, A.I. Larkin, and D.E. Khmelnitskii, Zh. Eksp. Teor. Fiz. 79, 1120 (1980) (Sov. Phys. JETP 52, 568); L. Schäfer and F. Wegner, Z. Phys. B38, 113 (1980)
3. K. B. Efetov, Adv. Phys. 32, 53 (1983); Supersymmetry in Disorder and Chaos (Cambridge Univ. Press, 1997).
4. L.P. Kouwenhoven, C.M. Marcus, P.L. McEuen, S. Tarucha, R.M. Westervelt, and N.C. Wingreen, in Mesoscopic Electron Transport, ed. by L.L. Sohn, L.P. Kouwenhoven, and G. Schön (Kluwer, 1997)
M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, (Springer, New York, 1990)

F. Haake, *Quantum Signatures of Chaos*, (Springer, Berlin, 1991)

*Chaos in Quantum Physics*, eds. M.-J. Gianonni, A. Voros and J. Zinn-Justin, Les Houches, Session LI 1989 (North-holland, Amsterdam, 1991)

M.C. Gutzwiller, J. Math. Phys., 12, 343 (1971)

M.L. Mehta, *Random Matrices*, (Academic Press, New York, 1991).

B. A. Muzykantskii and D. E. Khmelnitskii, Pis'ma Zh. Eksp. Theo. Fiz. 62, 68 (1995) [JETP Lett. 62, 76 (1995)]; also see D.E. Khmelnitskii and B.A. Muzykantskii, in *Supersymmetry and Trace formulae*, edited by I.V. Lerner, J.P. Keating, D.E. Khmelnitskii, NATO ASI Series B (Kluwer Academic, Dordrecht, 1999), Vol. 370, p. 327.

G. Eilenberger, Z. Phys. 214, 195 (1968)

M.R. Zirnbauer in in *Supersymmetry and Trace formulae*, edited by I.V. Lerner, J.P. Keating, D.E. Khmelnitskii, NATO ASI Series B (Kluwer Academic, Dordrecht, 1999), Vol. 370, p. 153.

P. Cvitanović, in *Supersymmetry and Trace formulae*, edited by I.V. Lerner, J.P. Keating, D.E. Khmelnitskii, NATO ASI Series B (Kluwer Academic, Dordrecht, 1999), Vol. 370, p. 17.

I.P. Keating, in *Supersymmetry and Trace formulae*, edited by I.V. Lerner, J.P. Keating, D.E. Khmelnitskii, NATO ASI Series B (Kluwer Academic, Dordrecht, 1999), Vol. 370, p. 1.

J.P. Keating, *Bosonization and Strongly Correlated Systems*, Cambridge University Press, Cambridge GB, (1998)

A.V. Zaitsev, Sov. Phys. JETP 59, 1015 (1984)