Time regularity of stochastic convolutions and stochastic evolution equations in duals of nuclear spaces

Christian A. Fonseca-Mora
Escuela de Matemática, Universidad de Costa Rica, San José, Costa Rica

ABSTRACT
Let $U$ be a locally convex space and let $W$ be a quasi-complete bornological nuclear space (like spaces of smooth functions and distributions) with dual spaces $U'$ and $W'$. In this work we introduce sufficient conditions for time regularity properties of the $W_0$-valued stochastic convolution $\int_0^t \int_0^\infty (t-r) R(r,u) M(dr,du), t \in [0,T]$, where $(S(t)' : t \geq 0)$ is the dual semigroup to a $C_0$-semigroup $(S(t) : t \geq 0)$ on $\Psi$, $R(r,\omega,u)$ is a suitable operator form $\Phi'$ into $\Psi'$, and $M$ is a cylindrical-martingale valued measure on $\Phi'$. Our result is latter applied to study time regularity of solutions to $W_0$-valued stochastic evolutions equations.

ARTICLE HISTORY
Received 11 March 2022
Accepted 28 October 2022

KEYWORDS
Cylindrical martingale-valued measures; dual of a nuclear space; stochastic convolution; stochastic evolution equations

2020 MATHEMATICS SUBJECT CLASSIFICATION
60G17; 60H05; 60H15; 60G20

1. Introduction
Let $\Phi$ be a locally convex space with strong dual $\Phi'$. Let $M = (M(t,A) : t \geq 0, A \in \mathcal{R})$ be a cylindrical martingale-valued measure on $\Phi'$, i.e. $(M(t,A) : t \geq 0)$ is a cylindrical martingale in $\Phi'$ for each $A \in \mathcal{R}$ and $M(t,\cdot)$ is finitely additive on $\mathcal{R}$ for each $t \geq 0$. Here $\mathcal{R} \subseteq B(U)$ is a ring and $U$ a topological space. Moreover, let $\Psi$ be a quasi-complete bornological nuclear space, $Z_0$ a $\Psi'$-valued random variable, and let $A$ be the generator of a $S_0$-semigroup $(S(t) : t \geq 0)$ of continuous linear operators on $\Psi$. In this paper we study time regularity of solutions to the following class of stochastic evolution equations

$$dX_t = (A'X_t + B(t,X_t)) dt + \int_U F(t,u,X_t) M(dt,du),$$

with initial value $X_0 = Z_0$, where $A'$ is the dual operator to $A$ and the coefficients $B$ and $F$ satisfy appropriate measurability conditions.

In [1], sufficient growth and Lipschitz conditions on the coefficients has been introduced to show existence and uniqueness of a weak solution to (1.1). In particular, this weak solution is given by the mild or evolution solution: for every $t \geq 0$, $\mathbb{P}$-a.e.
\[ X_t = S(t)Z_0 + \int_0^t S(t-r)B(r,X_r)dr + \int_0^t \int_U S(t-r)F(r,u,X_r)M(dr,du). \] (1.2)

The second integral at the right-hand side of (1.2) is the stochastic convolution of the dual semigroup \( (S(t)^\prime : t \geq 0) \) and the coefficient \( F \). This stochastic integral is defined by means of the theory of (strong) stochastic integration with respect to the cylindrical martingale-valued measure \( M \) as defined in [1].

In order to show that the mild solution (1.2) has a continuous or a càdlàg version, we will require to show first that for \( R \) integrable with respect to \( M \) (see Sec. 2.3), the stochastic convolution \( \int_0^t \int_U S(t-r)R(r,u)M(dr,du) \) possesses a continuous or a càdlàg version under mild assumptions on the \( C_0 \)-semigroup.

To be more specific, let us assume \( (S(t) : t \geq 0) \) is a \( (C_0,1) \)-semigroup of continuous linear operators on \( \Psi \) (see Sec. 2.1). According to Theorem 2.8 in [2] there exists a family \( \Pi \) of seminorms generating the topology on \( \Psi \) such that for each \( p \in \Pi \) there exist \( \theta_p \geq 0 \) such that

\[
p(S(t)p) \leq e^{\theta_p t}p(\psi), \text{ for all } t \geq 0, \psi \in \Psi. \] (1.3)

In this work we will say that \( (S(t) : t \geq 0) \) is a Hilbertian \( (C_0,1) \)-semigroup if such a family \( \Pi \) can be chosen such that each \( p \in \Pi \) is a Hilbertian seminorm.

In Sec. 3 we will show (see Theorem 3.2) that if \( (S(t) : t \geq 0) \) is a Hilbertian \( (C_0,1) \)-semigroup then the stochastic convolution \( \int_0^t \int_U S(t-r)R(r,u)M(dr,du) \) possesses a continuous or a càdlàg version if either \( M \) has respectively (as a cylindrical process) continuous or càdlàg version. We will indeed show that this version takes values in a Hilbert space \( \Psi_p' \) continuously embedded in \( \Psi' \).

In the context of a Hilbert space with norm \( p \) a \( C_0 \)-semigroup satisfying (1.3) is often referred to as exponentially bounded and it is known from the work of Kotelenez [3] that the stochastic convolution for such a semigroup and with respect to a Hilbert-space valued square integrable martingale possesses a continuous or a càdlàg version. The work we carried out in Sec. 3 can be thought a natural extension of the work of Kotelenez to the context of the stochastic convolution in the dual of a nuclear space.

To the extent of our knowledge only the works [4, 5] studied time regularity of stochastic convolutions with values in the dual of a nuclear space. In each of these works the integrand \( R \) is trivial (i.e. the identity operator) and the convolution is defined with respect to a \( \Phi' \)-valued processes for a reflexive nuclear space \( \Phi \). Below we briefly describe the results in [4, 5].

Time regularity of stochastic convolutions of the form \( \int_0^t S(t-r)\prime dM_r \) has been studied in ([4], Theorem 4.1) under the assumption that \( \Psi = \Phi \) is a Fréchet nuclear space and \( M \) is a \( \Psi' \)-valued square integrable martingale. In [4] it is not explicitly assumed that \( (S(t) : t \geq 0) \) is a Hilbertian \( (C_0,1) \)-semigroup but this property is used in their proofs. Indeed, it is a common practice in the literature of stochastic analysis in duals of nuclear spaces that a \( (C_0,1) \)-semigroup is assumed to be Hilbertian (see for example [6–8]).

In ([5], Theorem 5.7) and under the assumption that both \( \Phi \) and \( \Psi \) are quasi-complete bornological nuclear spaces and \( M \) is a \( \Phi' \)-valued Lévy process, it is shown that if the generator \( A \) of \( (S(t) : t \geq 0) \) is a continuous operator, then \( \int_0^t S(t-r)\prime dM_r \) has a
cadlág version. It is worth to mention that in [5] it is not assumed that the \((C_0,1)\)-semigroup \((S(t) : t \geq 0)\) is Hilbertian.

Now in Sec. 4 we return to our study of time regularity of solutions to stochastic evolution equations. Indeed, by applying our aforementioned result on time regularity of stochastic convolutions we show (see Theorem 4.2) that under some growth and Lipschitz conditions introduced in [1] the solution (1.2) to (1.1) has a unique continuous or cadlág version taking values and having square moments in a Hilbert space \(\Psi_p\) continuously embedded in \(\Psi\) for each bounded interval of time.

Finally, in Sec. 5 we provide some examples and applications of Hilbertian \((C_0,1)\)-semigroups and of stochastic evolution equations for which our results can be applied to show the existence of continuous or cadlág solutions. In particular, equations driven by Lévy noise will be considered.

2. Preliminaries

2.1. Locally convex spaces and linear operator

Let \(\Phi\) be a locally convex space (we will only consider vector spaces over \(\mathbb{R}\)). \(\Phi\) is quasi-complete if each bounded and closed subset of it is complete. \(\Phi\) is called bornological (respectively ultrabornological) if it is the inductive limit of a family of normed (respectively Banach) spaces. A barreled space is a locally convex space such that every convex, balanced, absorbing and closed subset is a neighborhood of zero. For further details see [9–11].

If \(p\) is a continuous semi-norm on \(\Phi\) and \(r > 0\), the closed ball of radius \(r\) of \(p\) given by \(B_p(r) = \{ \phi \in \Phi : p(\phi) \leq r \}\) is a closed, convex, balanced neighborhood of zero in \(\Phi\). A continuous seminorm (respectively norm) \(p\) on \(\Phi\) is called Hilbertian if \(p(\phi)^2 = Q(\phi, \phi)\), for all \(\phi \in \Phi\), where \(Q\) is a symmetric, non-negative bilinear form (respectively inner product) on \(\Phi \times \Phi\). For any given continuous seminorm \(p\) on \(\Phi\) let \(\Phi_p\) be the Banach space that corresponds to the completion of the normed space \((\Phi/\ker(p), \hat{p})\), where \(\hat{p}(\phi + \ker(p)) = p(\phi)\) for each \(\phi \in \Phi\). If \(\Phi_p\) is separable then we say that \(p\) is separable. We denote by \(\Phi_p^\prime\) the Banach space dual to \(\Phi_p\) and by \(p^\prime\) the corresponding dual norm. Observe that if \(p\) is Hilbertian then \(\Phi_p\) and \(\Phi_p^\prime\) are Hilbert spaces. If \(q\) is another continuous seminorm on \(\Phi\) for which \(p \leq q\), we have that \(\ker(q) \subseteq \ker(p)\) and the inclusion map from \(\Phi/\ker(q)\) into \(\Phi/\ker(p)\) has a unique continuous and linear extension that we denote by \(i_{p,q} : \Phi_q \to \Phi_p\). Furthermore, we have the following relation: \(i_p = i_{p,q} \circ i_q\).

Let \(p\) and \(q\) be continuous Hilbertian semi-norms on \(\Phi\) such that \(p \leq q\). The space of continuous linear operators (respectively Hilbert-Schmidt operators) from \(\Phi_q\) into \(\Phi_p\) is denoted by \(\mathcal{L}(\Phi_q, \Phi_p)\) (respectively \(\mathcal{L}_2(\Phi_q, \Phi_p)\)) and the operator norm (respectively Hilbert-Schmidt norm) is denote by \(\| \cdot \|_{\mathcal{L}(\Phi_q, \Phi_p)}\) (respectively \(\| \cdot \|_{\mathcal{L}_2(\Phi_q, \Phi_p)}\)).

We denote by \(\Phi^\prime\) the topological dual of \(\Phi\) and by \(\langle f, \phi \rangle\) the canonical pairing of elements \(f \in \Phi^\prime\), \(\phi \in \Phi\). Unless otherwise specified, \(\Phi^\prime\) will always be consider equipped with its strong topology, i.e. the topology on \(\Phi^\prime\) generated by the family of semi-norms \((\eta_B)\), where for each \(B \subseteq \Phi\) bounded we have \(\eta_B(f) = \sup\{ |\langle f, \phi \rangle| : \phi \in B \}\) for all \(f \in \Phi\).
Φ′. Recall that Φ is called semi-reflexive if the canonical (algebraic) embedding of Φ into Φ′ is onto, and is called reflexive if the canonical embedding is indeed an isomorphism (of topological vector spaces).

Let us recall that a (Hausdorff) locally convex space (Φ, T) is called nuclear if its topology T is generated by a family Π of Hilbertian semi-norms such that for each p ∈ Π there exists q ∈ Π, satisfying p ≤ q and the canonical inclusion i_{p,q} : Φ_q → Φ_p is Hilbert-Schmidt. Other equivalent definitions of nuclear spaces can be found in [11, 12]. Recall that any quasi-complete bornological nuclear space is barreled and semi-reflexive, therefore is reflexive (see Theorem IV.5.6 in [10], p. 145).

The following are all examples of complete ultrabornological (hence barreled) nuclear spaces: the spaces of functions $\mathcal{E}_K := C^\infty(K)$ (K: compact subset of $\mathbb{R}^d$) and $\mathcal{E} := C^\infty(\mathbb{R}^d)$, the rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$, and the space of test functions $\mathcal{D}(U) := C^\infty(U)$ (U: open subset of $\mathbb{R}^d$), as well as the spaces of distributions $\mathcal{E}'_K$, $\mathcal{E}'$, $\mathcal{S}'(\mathbb{R}^d)$, and $\mathcal{D}'(U)$. Other examples are the space of harmonic functions $\mathcal{H}(U)$ (U: open subset of $\mathbb{R}^d$), the space of polynomials $\mathcal{P}_n$ in n-variables and the space of real-valued sequences $\mathbb{R}^N$ (with direct product topology). For references see [10–12].

Let $\Psi$ a locally convex space. A family $(S(t) : t \geq 0) \subseteq \mathcal{L}(\Psi, \Psi)$ is a $C_0$-semigroup on $\Psi$ if

i. $S(0) = I$, $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$, and

ii. $\lim_{t \to 0} S(t)\phi = S(s)\psi$, for all $s \geq 0$ and any $\psi \in \Psi$.

The infinitesimal generator $A$ of $(S(t) : t \geq 0)$ is the linear operator on $\Psi$ defined by $A\psi = \lim_{h \to 0} \frac{S(h)\psi - \psi}{h}$ (limit in $\Psi$), whenever the limit exists; the domain of $A$ being the set Dom$(A) \subseteq \Psi$ for which the above limit exists. If the space $\Psi$ is reflexive, then the family $(S(t) : t \geq 0)$ of dual operators is a $C_0$-semigroup on $\Psi'$ with generator $A'$, that we call the dual semigroup and the dual generator respectively.

Recall (see [2]) that a $C_0$-semigroup $(S(t) : t \geq 0)$ is a $(C_0, 1)$-semigroup if for each continuous seminorm $p$ on $\Psi$ there exists some $\vartheta_p \geq 0$ and a continuous seminorm $q$ on $\Psi$ such that $p(S(t)\psi) \leq e^{\vartheta_p t} q(\psi)$, for all $t \geq 0$ and $\psi \in \Psi$. If in the above inequality $\vartheta_p = \omega$ with $\omega$ a positive constant (independent of $p$) $(S(t) : t \geq 0)$ is called quasiequicontinuous, and if $\omega = 0$ $(S(t) : t \geq 0)$ is called equicontinuous. It is worth to mention that even if $\Psi$ is reflexive, the dual semigroup $(S(t)' : t \geq 0)$ to a $(C_0, 1)$-semigroup $(S(t) : t \geq 0)$ is not in general a $(C_0, 1)$-semigroup on $\Psi'$ (see [2], Sec. 6). However, if $(S(t) : t \geq 0)$ is equicontinuous and $\Psi$ is reflexive we have that $(S(t)' : t \geq 0)$ is equicontinuous (see [13], Theorem 1 and its Corollary).

### 2.2. Cylindrical and stochastic processes

Throughout this work we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and consider a filtration $(\mathcal{F}_t : t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies the usual conditions, i.e. it is right continuous and $\mathcal{F}_0$ contains all subsets of sets of $\mathcal{F}$ of $\mathbb{P}$-measure zero. We denote by $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the space of equivalence classes of real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We always consider the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the topology of
convergence in probability and in this case it is a complete metrizable topological vector space. We denote by \( \mathcal{P}_\infty \) the predictable \( \sigma \)-algebra on \([0, \infty) \times \Omega\) and for any \( T > 0 \), we denote by \( \mathcal{P}_T \) the restriction of \( \mathcal{P}_\infty \) to \([0, T] \times \Omega\).

Let \( \Phi \) be a locally convex space. A \textit{cylindrical random variable} in \( \Phi' \) is a linear map \( X : \Phi \to L^0(\Omega, \mathcal{F}, \mathbb{P}) \) (see [14]). If \( X \) is a cylindrical random variable in \( \Phi' \), we say that \( X \) is \textit{n-integrable} \( (n \in \mathbb{N}) \) if \( \mathbb{E}(|X(\phi)|^n) < \infty, \forall \phi \in \Phi \), and has \textit{zero-mean} if \( \mathbb{E}(X(\phi)) = 0, \forall \phi \in \Phi \).

Let \( X \) be a \( \Phi' \)-valued random variable, i.e. \( X : \Omega \to \Phi' \) is a \( \mathcal{F}/\mathcal{B}(\Phi') \)-measurable map. For each \( \phi \in \Phi \) we denote by \( \langle X, \phi \rangle : \Omega \to \mathbb{R} \) the real-valued random variable defined by \( \langle X, \phi \rangle(\omega) := \langle X(\omega), \phi \rangle, \forall \omega \in \Omega \). The linear mapping \( \phi \mapsto \langle X, \phi \rangle \) is called the \textit{cylindrical random variable induced/defined} by \( X \). We will say that a \( \Phi' \)-valued random variable \( X \) is \textit{n-integrable} if the cylindrical random variable induced by \( X \) is \( n \)-integrable.

Let \( J = \mathbb{R}_+ := [0, \infty) \) or \( J = [0, T] \) for \( T > 0 \). We say that \( X = (X_t : t \in J) \) is a \textit{cylindrical process} in \( \Phi' \) if \( X_t \) is a cylindrical random variable for each \( t \in J \). Clearly, any \( \Phi' \)-valued stochastic processes \( X = (X_t : t \in J) \) \textit{induces/denotes} a cylindrical process under the prescription: \( \langle X, \phi \rangle = \langle X_t, \phi \rangle : t \in J \), for each \( \phi \in \Phi \).

If \( X \) is a cylindrical random variable in \( \Phi' \), a \( \Phi' \)-valued random variable \( Y \) is called a \textit{version} of \( X \) if for every \( \phi \in \Phi, X(\phi) = \langle Y, \phi \rangle \) \( \mathbb{P} \)-a.e. A \( \Phi' \)-valued process \( Y = (Y_t : t \in J) \) is said to be a \( \Phi' \)-valued \textit{version} of the cylindrical process \( X = (X_t : t \in J) \) on \( \Phi' \) if for each \( t \in J \), \( Y_t \) is a \( \Phi' \)-valued version of \( X_t \).

For a \( \Phi' \)-valued process \( X = (X_t : t \in J) \) terms like continuous, càdlàg, purely discontinuous, adapted, predictable, etc., have the usual (obvious) meaning.

A \( \Phi' \)-valued random variable \( X \) is called \textit{regular} if there exists a weaker countably Hilbertian topology \( \theta \) on \( \Phi \) (see Sec. 2 in [14]) such that \( \mathbb{P}(\omega : X(\omega) \in (\Phi_\theta)'') = 1 \); here \( \Phi_\theta \) denotes the space \( (\Phi, \theta) \) and \( \Phi_\theta \) denotes its completion. If \( \Phi \) is barreled, the property of being regular is equivalent to the property that the law of \( Y \) is a Radon measure on \( \Phi' \) (see Theorem 2.10 in [14]). A \( \Phi' \)-valued process \( Y = (Y_t : t \in J) \) is said to be regular if \( Y_t \) is a regular random variable for each \( t \in J \).

A cylindrical process \( Y = (Y_t : t \in J) \) in \( \Phi' \) is a cylindrical martingale if \( Y(\phi) = (Y_t(\phi) : t \in J) \) is a real-valued martingale for each \( \phi \in \Phi \). A \( \Phi' \)-valued process is a martingale if the induced cylindrical process is a cylindrical martingale in \( \Phi' \).

2.3. Stochastic integration in duals of nuclear spaces

**Assumption 2.1.** From now on \( \Phi \) denotes a locally convex space and \( \Psi' \) denotes a quasi-complete bornological nuclear space.

In this section we briefly recall the theory of stochastic integration in duals of nuclear spaces in introduced in [1]. We start with the definition of cylindrical martingale-valued measure.

Let \( U \) be a topological space and consider a ring \( \mathcal{R} \subseteq \mathcal{B}(U) \) that generates \( \mathcal{B}(U) \). A \textit{cylindrical martingale-valued measure} on \( \mathbb{R}_+ \times \mathcal{R} \) is a collection \( M = (M(t, A) : t \geq 0, A \in \mathcal{R}) \) of cylindrical random variables in \( \Phi' \) such that:

\[
(1) \quad \forall \ A \in \mathcal{R}, M(0, A)(\phi) = 0 \ \mathbb{P}\text{-a.s., } \forall \phi \in \Phi.
\]
(2) \( \forall t \geq 0, M(t, \emptyset)(\phi) = 0 \) \( \mathbb{P} \)-a.s. \( \forall \phi \in \Phi \) and if \( A, B \in \mathcal{R} \) are disjoint then
\[
M(t, A \cup B)(\phi) = M(t, A)(\phi) + M(t, B)(\phi) \quad \mathbb{P} - \text{a.s.,} \quad \forall \phi \in \Phi.
\]

(3) \( \forall A \in \mathcal{R}, (M(t, A) : t \geq 0) \) is a cylindrical zero-mean square integrable càdlàg martingale.

(4) For disjoint \( A, B \in \mathcal{R} \), \( \mathbb{E}(M(t, A)(\phi)M(s, B)(\psi)) = 0 \), for each \( t, s \geq 0, \phi, \psi \in \Phi \).

We will further assume that the following properties are satisfied:

(5) For \( 0 \leq s < t, M((s, t], A)(\phi) := (M(t, A) - M(s, A))(\phi) \) is independent of \( \mathcal{F}_s \), for all \( A \in \mathcal{R}, \phi \in \Phi \).

(6) For each \( A \in \mathcal{R} \) and \( 0 \leq s < t \),
\[
\mathbb{E}(|M((s, t], A)(\phi)|^2) = \int_s^t \int_A q_{r,u}(\phi)^2 \mu(du)\lambda(dr), \quad \forall \phi \in \Phi,
\]
where

(a) \( \mu \) is a \( \sigma \)-finite measure on \((U, B(U))\) satisfying \( \mu(A) < \infty, \forall A \in \mathcal{R} \),

(b) \( \lambda \) is a \( \sigma \)-finite measure on \((\mathbb{R}_+, B(\mathbb{R}_+))\), finite on bounded intervals,

(c) \( \{q_{r,u} : r \in \mathbb{R}_+, u \in U\} \) is a family of continuous Hilbertian semi-norms on \( \Phi \),

such that for each \( \phi, \psi \in \Phi \), the map \( (r, u) \mapsto q_{r,u}(\phi, \psi) \) is \( B(\mathbb{R}_+) \otimes B(U)/B(\mathbb{R}_+) \)-measurable and bounded on \([0, T] \times U\) for all \( T > 0 \).

Here, \( q_{r,u}(\cdot, \cdot) \) denotes the positive, symmetric, bilinear form associated to the Hilbertian semi-norm \( q_{r,u} \).

The stochastic integral with respect to the above class of cylindrical martingale-valued measures is introduced in [1] in a two step construction. First, a real-valued stochastic integral, known as the \textit{weak stochastic integral}, is constructed via a Itô isometry. Later, by using a procedure of regularization a vector-valued stochastic integral, known as the \textit{strong stochastic integral}, is constructed using the weak stochastic integral as a building block.

We start by recalling the class of \textit{weak stochastic integrands}: Given \( T > 0 \), we denote by \( \Lambda_w^2(T) \) the collection of families \( X = \{X(r, \omega, u) : r \in [0, T], \omega \in \Omega, u \in U\} \) of Hilbert space-valued maps satisfying the following:

(1) \( X(r, \omega, u) \in \Phi_{q_{r,u}}, \) for all \( r \in [0, T], \omega \in \Omega, u \in U \),

(2) \( X \) is \( q_{r,u} \)-predictable, i.e. for each \( \phi \in \Phi \), the mapping \([0, T] \times \Omega \times U \rightarrow \mathbb{R}_+\) given by \( (r, \omega, u) \mapsto q_{r,u}(X(r, \omega, u), \phi) \) is \( \mathcal{P}_r \otimes B(U)/B(\mathbb{R}_+) \)-measurable.

(3) \( ||X||_{w,T}^2 := \mathbb{E} \int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr) < \infty. \) (2.2)

It is not difficult to check that \( \Lambda_w^2(T) \) is a Hilbert space when equipped with the inner product \( \langle \cdot, \cdot \rangle_{w,T} \) corresponding to the Hilbertian norm \( || \cdot ||_{w,T} \).

The stochastic integral is first defined for the following class of \textit{simple weak integrands}: Let \( S_w(T) \) be the collection of all the families \( X = \{X(r, \omega, u) : r \in [0, T], \omega \in \Omega, u \in U\} \) of Hilbert space valued maps of the form:
\[ X(r, \omega, u) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{I}_{[s_j, t_j]}(r) \mathbb{I}_{F_i}(\omega) \mathbb{I}_{A_i}(u) i_{r, u} \phi_{i,j}. \] (2.3)

for all \( r \in [0, T], \omega \in \Omega, u \in U, \) where \( m, n \in \mathbb{N}, \) and for \( i = 1, \ldots, n, j = 1, \ldots, m, 0 \leq s_j < t_j \leq T, F_i \in \mathcal{F}_{s_j}, A_i \in \mathcal{R} \) and \( \phi_{i,j} \in \Phi. \)

Let \( X \in \mathcal{S}_w(T) \) be of the form (2.3). The weak stochastic integral is defined as follows:

\[
\int_{0}^{t} X(r, u) M(dr, du) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{I}_{F_i}(\omega) \mathbb{I}_{A_i}(u) \phi_{i,j}.
\]

It is shown in Theorems 4.6 and 4.7 in [1] that the stochastic integral defined above is a real-valued zero-mean, square integrable, càdlàg martingale with second moments for every \( t \in [0, T] \) by

\[
\mathbb{E} \left( \left| \int_{0}^{t} X(r, u) M(dr, du) \right|^2 \right) = \mathbb{E} \left( \int_{0}^{t} q_{r, u}(X(r, u))^2 \mu(du) \lambda(dr) \right). \quad (2.4)
\]

By (2.4) we have an Itô isometry from \( (\mathcal{S}_w(T), \| \cdot \|_{w,T}) \) into the space \( \mathcal{M}_F^2(\mathbb{R}) \) of real-valued zero-mean, square integrable, càdlàg martingales on \([0, T], \). Since \( \mathcal{S}_w(T) \) is dense in \( \Lambda_2^w(T) \) (Proposition 4.5 in [1]), the above mapping extends to an isometry \( I^w : \Lambda_2^w(T) \to \mathcal{M}_F^2(\mathbb{R}) \) called the weak integral mapping.

Now we recall the class of strong stochastic integrands: Let \( \Lambda^2_2(T) \) denote the collection of families \( R = \{ R(r, \omega, u) : r \in [0, T], \omega \in \Omega, u \in U \} \) of operator-valued maps satisfying the following conditions:

1. \( R(r, \omega, u) \in \mathcal{L}(\Psi, \Phi) \) for all \( r \in [0, T], \omega \in \Omega, u \in U, \)

2. \( R \) is \( q_{r, u} \)-predictable, i.e. for each \( \phi \in \Phi, \psi \in \Psi, \) the mapping \([0, T] \times \Omega \times U \to \mathbb{R}_+ \) given by \((r, \omega, u) \mapsto q_{r, u}(R(r, \omega, u)^\prime \psi, \phi) \) is \( \mathcal{P}_T \otimes \mathcal{B}(U) \)-measurable.

3. \( \mathbb{E} \int_{0}^{T} q_{r, u}(R(r, u)^\prime \psi)^2 \mu(du) \lambda(dr) < \infty, \quad \forall \psi \in \Psi. \) (2.5)

The space \( \Lambda_2^2(T) \) is clearly a vector space but the condition (2.5) is too weak to introduce a norm structure and hence the strong stochastic integral cannot be introduced by following an Itô isometry as done for the weak integrands. A different construction of the strong stochastic integral is carried out using the tools of regularization of cylindrical stochastic processes developed in [14]. We sumarize this construction below.

Let \( R \in \Lambda_2^2(T). \) By Theorem 5.9 in [1] we can identify \( R \) with a unique element \( \Delta(R) \) in \( \mathcal{L}(\Psi, \Lambda_2^w(T)) \) by means of the isomorphism:

\[
R \mapsto (\psi \mapsto R\psi := \{ R(r, \omega, u)^\prime \psi : r \in [0, T], \omega \in \Omega, u \in U \}).
\]

Then we use the continuity and linearity of the weak integral map to show that

\[
I^w : I^w \circ \Delta(R) : \Psi \to \mathcal{M}_F^2(\mathbb{R}), \quad \psi \mapsto I^w_\psi = \int_{0}^{t} R(r, u)^\prime \psi M(dr, du),
\]
defines a cylindrical martingale in \( \Psi' \) such that for each \( 0 \leq t \leq T \) the linear mapping \( I_t^\Psi : (L^0(W, \mathcal{F}, \mathbb{P})) \to (L^0(\Omega, \mathcal{G}, \mathbb{P})) \) is continuous. Then by Theorems 5.11 in [14] there exists a \( \Phi'_p \)-valued zero-mean, square integrable, càdlàg martingale

\[
\int_0^t \int_\Omega R(r, u) M(dr, du), \quad 0 \leq t \leq T,
\]

satisfying for each \( 0 \leq t \leq T \), for which the stochastic convolution has a \( \Phi'_p \)-valued version with range on a Hilbert space embedded in \( \Psi' \).

Let \( p \) be a continuous Hilbertian semi-norm on \( \Psi \). Let \( \Lambda^2_p(T, \mathbb{P}) \) denote the collection of families \( \tilde{R} = \{\tilde{R}(r, \omega, u)\} \) of linear operators \( \tilde{R}(r, \omega, u) \in \mathcal{L}_2(\Phi'_p, \Psi'_p), r \in [0, T], \omega \in \Omega, u \in U \), which are \( u^p \)-predictable, and for which

\[
\mathbb{E} \left( \int_0^T \left( \int_0^T \tilde{R}(r, u) M(dr, du) \right)^2 \right) < \infty.
\]

For \( \tilde{R} \in \Lambda^2_p(T, \mathbb{P}) \), it follows by Theorem 3.3.16 in [15] that the stochastic integral

\[
\int_0^T \tilde{R}(r, \omega, u) M(dr, du)
\]

is a \( \Phi'_p \)-valued zero-mean, square integrable, càdlàg martingale satisfying for each \( 0 \leq t \leq T \),

\[
\mathbb{E} \left( p' \left( \int_0^T \tilde{R}(r, u) M(dr, du) \right)^2 \right) = \mathbb{E} \left( \int_0^T \left\| \tilde{R}(r, u) \right\|_{\mathcal{L}_2(\Phi'_p, \Psi'_p)}^2 \mu(dr) \lambda(dr) \right)
\]

Moreover, by Theorem 5.11 in [1] for every \( R \in \Lambda^2(T) \) there exists a continuous Hilbertian seminorm \( p \) on \( \Phi \) and some \( \tilde{R} \in \Lambda^2_p(T, \mathbb{P}) \) such that \( R(r, \omega, u) = \tilde{R}(r, \omega, u) \) for \( \lambda \otimes \mathbb{P} \otimes \mu \)-a.e. \( (r, \omega, u) \) and the stochastic integral

\[
\int_0^T \tilde{R}(r, \omega, u) M(dr, du)
\]

is a \( \Phi'_p \)-valued version of \( \int_0^T R(r, \omega, u) M(dr, du) \).

### 3. Regularity of paths of the stochastic convolution

**Assumption 3.1.** Al through this section \( M = (M(t, A) : t \geq 0, A \in \mathcal{R}) \) denotes a cylindrical martingale-valued measure as in Sec. 2.3 with \( \lambda \) being the Lebesgue measure on \( (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \), and let \( R \in \Lambda^2(T) \).

Let \( (S(t) : t \geq 0) \) be a \( (C_0, 1) \)-semigroup on \( \Psi \). In Proposition 6.8 in [1] it is shown that the stochastic convolution

\[
X_t = \int_0^t \int_\Omega S(t - r) R(r, u) M(dr, du), \quad \forall \ t \in [0, T],
\]

is a \( \Psi' \)-valued regular, adapted, square integrable process. Moreover, Theorem 6.14 in [1] shows that there exists a continuous Hilbertian seminorm \( q \) on \( \Phi \) such that the stochastic convolution has a \( \Phi'_q \)-valued, mean-square continuous, predictable version with second moments. In the following result we show that a càdlàg version exists if \( (S(t) :
$t \geq 0$) is Hilbertian, i.e. if the family $\Pi$ of seminorms generating the topology on $\Psi$ and satisfying (1.3) can be chosen such that every $p \in \Pi$ is a Hilbertian seminorm.

**Theorem 3.2.** Assume that $(S(t) : t \geq 0)$ is a Hilbertian $(C_0, 1)$-semigroup on $\Psi$. Then there exists a continuous Hilbertian seminorm $p$ on $\Phi$ such that the stochastic convolution $(X_t : t \in [0, T])$ has a $\Psi'_p$-valued, square integrable, adapted, càdlàg version $(Y_t : t \in [0, T])$ satisfying $\sup_{t \in [0, T]} \mathbb{E}(p'(Y_t)^2) < \infty$. In particular, the stochastic convolution has a $\Psi'_p$-valued càdlàg version.

Moreover if for each $A \in \mathcal{R}$ and $\phi \in \Phi$, the real-valued process $(M(t, A)(\phi) : t \geq 0)$ is continuous, then the results above remain valid replacing the property càdlàg by continuous.

For our proof of Theorem 3.2 we will require the following terminology. For $(S(t) : t \geq 0)$ let $\Pi$ be the corresponding family of Hilbertian seminorms satisfying (1.3). For each $p \in \Pi$ it is shown in ([12], Theorems 2.3 and 2.6) that there exists a $C_0$-semigroup $(S_p(t) : t \geq 0)$ on the Banach space (Hilbert space since $p$ is Hilbertian) $\Psi_p$ into itself such that

$$S_p(t)i_p\psi = i_pS(t)\psi, \quad \forall \psi \in \Psi, \quad t \geq 0. \quad (3.2)$$

Our first step is to prove the following Kotelenez type inequality.

**Lemma 3.3.** Let $(S(t) : t \geq 0)$ be a Hilbertian $(C_0, 1)$-semigroup with corresponding family $\Pi$ of Hilbertian seminorms satisfying (1.3). For any continuous Hilbertian seminorm $p \in \Pi$, $F \in \Lambda_2^2(p, T)$, $C > 0$ and countable $D \subseteq [0, T]$,

$$\mathbb{P} \left( \sup_{r \in D} p': \left( \int_0^t \int_U S_p(t - r)'F(r, u) \, M(dr, du) \right) > C \right) \leq \frac{e^{2\theta_1T}}{C^2} \mathbb{E} \int_0^T \|F(r, u)\|_{\Psi'_p}^2 \mu(du) dr,$$

where $(S_p(t) : t \geq 0)$ is the corresponding $C_0$-semigroup on the Hilbert space $\Psi_p$ satisfying (1.3) (replacing $S(t)$ with $S_p(t)$) and (3.2).

**Proof.** We modify to our context the arguments used in the proof of Proposition 9.18 in [16].

For fixed $t \in [0, T]$, one can easily check that the family $\{S_p(t - r)'F(r, \omega, u) : r \in [0, t], \omega \in \Omega, u \in U\}$ belongs to $\Lambda_2^2(p, t)$. In particular,

$$\mathbb{E} \int_0^t \int_U \|S_p(t - r)'F(r, u)\|_{\Psi'_p}^2 \mu(du) dr \leq e^{2\theta_1T} \mathbb{E} \int_0^T \|F(r, u)\|_{\Psi'_{p}}^2 \mu(du) dr < \infty.$$

Hence, from Theorem 3.3.16 in [15] the stochastic convolution

$$Y(t) := \int_0^t \int_U S_p(t - r)'F(r, u) \, M(dr, du), \quad t \in [0, T],$$
is a $\Phi_p'$-valued adapted process such that for each $t \in [0, T],
\begin{align*}
\mathbb{E} \left[ p' \left( \int_0^t \int_U S_p(t - r)'F(r, u) M(dr, du) \right)^2 \right] \\
= \int_0^t \int_U \left\| S_p(t - r)'F(r, u) \right\|_{\mathcal{L}_2(\Phi_{p,u}', \Phi_p')}^2 \mu(du) dr < \infty.
\end{align*}

(3.3)

Let $0 = t_0 < t_1 < \ldots < t_n = T$ and $C > 0$. Then,
\begin{align*}
\mathbb{P} \left( \max_{1 \leq k \leq n} p'(Y(t_k)) > C \right) &= \sum_{k=1}^n \mathbb{P} \left( \cap_{j=1}^{k-1} \{ p'(Y(t_j)) \leq C \} \cap \{ p'(Y(t_k)) > C \} \right) \\
&\leq \frac{1}{C^2} \sum_{k=1}^n \mathbb{E} (p'(Y(t_k))^2 \mathbb{1}_{\{ p'(Y(t_k)) > C \}} \mathbb{1}_k),
\end{align*}

(3.4)

where $\mathbb{1}_k = \prod_{j=1}^{k-1} \mathbb{1}_{\{ p'(Y(t_j)) \leq C \}}$. Now, observe that for each $1 \leq k \leq n$, the semigroup property of $(S_p(t) : t \geq 0)$ and the action of the continuous linear operators on the stochastic integral (as for example in Proposition 5.18 in [1]) show that we have

$$Y(t_k) = S_p(t_k - t_{k-1})' Y(t_{k-1}) + \int_{t_{k-1}}^{t_k} \int_U S_p(t_k - r)'F(r, u)M(dr, du).$$

Then, by the martingale property of the stochastic integral, the Itô isometry (3.3) and (1.3) (replacing $S(t)$ with $S_p(t)$), we get

$$\mathbb{E} (p'(Y(t_k))^2 \mathbb{1}_k) = \mathbb{E} (p'(S_p(t_k - t_{k-1})' Y(t_{k-1}))^2 \mathbb{1}_k) + \mathbb{E} \left( p' \left( \int_{t_{k-1}}^{t_k} \int_U S_p(t_k - r)'F(r, u)M(dr, du) \right)^2 \right) \mathbb{1}_k \\
\leq e^{2\theta_p(t_k - t_{k-1})} \left( \mathbb{E} (p'(Y(t_{k-1}))^2 \mathbb{1}_{k-1}) + \mathbb{E} \left( \int_{t_{k-1}}^{t_k} \int_U \left\| F(r, u) \right\|_{\mathcal{L}_2(\Phi_{p,u}', \Phi_p')}^2 \mu(du) dr \right) \right).$$

Then, by iteration we have

$$\sum_{k=1}^n \mathbb{E} (p'(Y(t_k))^2 \mathbb{1}_k) \leq e^{2\theta_p T} \sum_{k=1}^n \mathbb{E} \left( \int_{t_{k-1}}^{t_k} \int_U \left\| F(r, u) \right\|_{\mathcal{L}_2(\Phi_{p,u}', \Phi_p')}^2 \mu(du) dr \right) \\
= e^{2\theta_p T} \mathbb{E} \left( \int_0^T \int_U \left\| F(r, u) \right\|_{\mathcal{L}_2(\Phi_{p,u}', \Phi_p')}^2 \mu(du) dr \right).$$

Combining the above inequality with (3.4) we conclude the estimate in Lemma 3.3. ✓

**Proof of Theorem 3.2.** Let $p$ be a continuous Hilbertian seminorm on $\Phi$ and $\tilde{R} \in \mathcal{A}_\chi^2(\rho, T)$ such that $R(r, \omega, u) = \frac{\partial}{\partial r} \tilde{R}(r, \omega, u) \lambda \otimes \mathbb{P} \otimes \mu$-a.e. Since the family of Hilbertian seminorms $\Pi$ generates the topology on $\Psi$, then we can assume $p \in \Pi$. The corresponding $C_0$-semigroup $(S_p(t) : t \geq 0)$ on the Hilbert space $\Psi_p$ satisfies (1.3) (replacing $S(t)$ with $S_p(t)$) and (3.2).
Then, for fixed \( t \in [0, T] \) it follows from the above properties that for Leb \( \otimes \mu - \text{a.e.} (r, u) \in [0, T] \times U \),
\[ \mathbb{I}_{[0, t]}(r)S(t - r)'R(r, \omega, u) = i_p S_p(t - r)'\tilde{R}(r, \omega, u). \] (3.5)
Moreover, the family \( \{S_p(t - r)'\tilde{R}(r, \omega, u) : r \in [0, t], \omega \in \Omega, u \in U \} \) belongs to \( \Lambda^2_{X}(p, t) \) and the stochastic convolution
\[ \int_0^t \int_U S_p(t - r)'\tilde{R}(r, u) M(dr, du), \quad t \in [0, T], \]
is a \( \Psi'_p \)-valued adapted process satisfying the Itô isometry (3.3) (with \( F \) being replaced by \( \tilde{R} \)). Furthermore, it follows from (3.5), Theorem 3.3.17 in [15], and Theorem 5.11 in [1], that for each \( t \in [0, T], \) \( \mathbb{P} \)-a.e.
\[ \int_0^t \int_U S(t - r)'R(r, u) M(dr, du) = i_p \int_0^t \int_U S_p(t - r)'\tilde{R}(r, u) M(dr, du). \] (3.6)
Therefore, \( \int_0^t \int_U S_p(t - r)'\tilde{R}(r, u) M(dr, du) \) is a \( \Psi'_p \)-valued square integrable version for the stochastic convolution \( \int_0^t \int_U S(t - r)'R(r, u) M(dr, du) \).
Hence in order to prove Theorem 3.2 it suffices to show that the stochastic convolution \( \int_0^t \int_U S_p(t - r)'\tilde{R}(r, u) M(dr, du) \) has a càdlàg version.

Given \( k \in \mathbb{N}, \) let \( r(k) = iT/2^k \) if \( r \in (iT/2^k, (i + 1)T/2^k] \) for \( i = 0, 1, ..., 2^{k-1} \) and consider the \( \Psi'_p \)-valued process
\[ Y^k(t) = \int_0^t \int_U S_p(t - r(k))'\tilde{R}(r, u) M(dr, du), \quad \forall \ t \in [0, T]. \]
Observe that because for each \( t \in [0, T], \)
\[ \mathbb{E} \int_0^t \int_U \left| \left| (S_p(t - r(k))' - S_p(t - r)'\right|\left|\tilde{R}(r, u)\right| \right|^2 \mathcal{L}_{X(\Psi'_r, \Psi'_p)} \mu(du) dr \]
\[ \leq 2e^{2\theta_t} \mathbb{E} \int_0^T \int_U \left| \left| \tilde{R}(r, u) \right| \right|^2 \mathcal{L}_{X(\Psi'_r, \Psi'_p)} \mu(du) \lambda(dr) < \infty, \]
then by the strong continuity of the \( C_0 \)-semigroup \( S_p(t) \) and dominated convergence we have that
\[ \lim_{k \to \infty} \mathbb{E} \int_0^t \int_U \left| \left| (S_p(t - r(k))' - S_p(t - r)'\right|\left|\tilde{R}(r, u)\right| \right|^2 \mathcal{L}_{X(\Psi'_r, \Psi'_p)} \mu(du) dr = 0. \]
Therefore, by the Itô isometry (3.3) (with \( F \) being replaced by \( \tilde{R} \)) we conclude that
\[ \lim_{k \to \infty} \mathbb{E} \left[ p'(\int_0^t \int_U (S_p(t - r(k))' - S_p(t - r)'\tilde{R}(r, u) M(dr, du))^2 \right] = 0. \]
So for each \( t \in [0, T], Y^k(t) \) converges in \( L^2(\Omega, \mathcal{F}, \mathbb{P}; \Psi'_p) \) (the space of square integrable random variables in \( \Psi'_p \)) to \( \int_0^t \int_U S_p(t - r)'\tilde{R}(r, u) M(dr, du). \)
Now, observe that for each \( k \in \mathbb{N}, \) the process \( Y^k \) is càdlàg. To see why this is true, note that if \( t \in (iT/2^k, (i + 1)T/2^k], \) then
\[ Y^k(t) = S_p \left( t - \frac{i T}{2^k} \right) \int_U^t \tilde{R}(r, u) \, M(dr, du) \]
\[ + \sum_{j=1}^{T} S_p \left( t - \frac{(j-1) T}{2^k} \right) \int_U^t \tilde{R}(r, u) \, M(dr, du). \]

Thus, the strong continuity of the semigroup \( S_p \) and because the stochastic integral \( \int_0^T \tilde{R}(r, u) \, M(dr, du) \) have càdlàg paths we conclude that each \( Y^k \) is càdlàg.

Our next objective is to show that the sequence \( (Y^k : k \in \mathbb{N}) \) has a subsequence that converges in probability uniformly on \([0, T]\). In that case, it will follows that

\[ \int_0^T S_p(t - r) \tilde{R}(r, u) \, M(dr, du), \quad \forall \ t \in [0, T], \]

has a càdlàg version.

Assume \( m \geq k \). Then, because \( r(m) \geq r(k) \), we have for every \( t \in [0, T] \) that

\[ Y^m(t) - Y^k(t) \]
\[ = \int_U^t (S_p(t - r(m)) - S_p(t - r(k))) \tilde{R}(r, u) \, M(dr, du) \]
\[ = \int_U^t S_p(t - r(m)) \tilde{R}(r, u) \, M(dr, du), \]

where \( I_p \) is the identity in \( \Psi_p \). Let \( F^{m,k}(r, \omega, u) = (I'_p - S_p(r(m) - r(k))) \tilde{R}(r, \omega, u) \) for \((r, \omega, u) \in [0, T] \times \Omega \times U\). By (1.3) (replacing \( S(t) \) with \( S_p(t) \)) we have

\[ \mathbb{E} \int_0^T \| F^{m,k}(r, u) \|_{\mathcal{L}^2(\Psi_p')}^2 \mu(du)dr \]
\[ \leq (1 + e^{2\theta T}) \mathbb{E} \int_0^T \| \tilde{R}(r, u) \|_{\mathcal{L}^2(\Psi_p')}^2 \mu(du)dr \]< \infty.

Hence \( F^{m,k} \in \mathcal{A}^2(p, T) \). Then, from Lemma 3.3, the fact that \( Y^m \) and \( Y^k \) are càdlàg and from (3.7) it follows that

\[ \mathbb{P} \left( \sup_{t \in [0, T]} p' \left( Y^m(t) - Y^k(t) \right)^2 > C \right) \]
\[ \leq \frac{e^{2\theta T}}{C^2} \mathbb{E} \int_0^T \| F^{m,k}(r, u) \|_{\mathcal{L}^2(\Psi_p')}^2 \mu(du)dr. \]

Now, the strong continuity of the \( C_0 \)-semigroup \( S_p(t) \) shows that for each \((r, \omega, u) \in [0, T] \times \Omega \times U\) we have

\[ \lim_{m, k \to \infty} \| F^{m,k}(r, \omega, u) \|_{\mathcal{L}^2(\Psi_p')} = 0. \]

Therefore, from (3.8), dominated convergence and (3.9) it follows that

\[ \lim_{m, k \to \infty} \mathbb{P} \left( \sup_{t \in [0, T]} p' \left( Y^m(t) - Y^k(t) \right)^2 > C \right) = 0. \]
Then, by standard arguments we can show the existence of a subsequence of \((Y^k : k \in \mathbb{N})\) that converges \(\mathbb{P}\)-a.e. uniformly on \([0, T]\) to a càdlàg version \((Y_t : t \in [0, T])\) of the process \(\int_0^t \int_U S_p(t - r)\tilde{R}(r, u) M(dr, du), t \in [0, T]\). Moreover, observe that by (3.3) we have

\[
\sup_{t \in [0, T]} \mathbb{E}(\rho'(Y_t)^2) \leq e^{2\theta_T} \mathbb{E} \int_0^T \left\| \tilde{R}(r, u) \right\|^2 \mathcal{L}_2(\Phi_{\psi, u}, \Phi_{\psi, u}) \mu(du) dr < \infty.
\]

Finally, if each \((M(t, A)(\phi) : t \geq 0)\) is continuous then the stochastic integrals defined with respect to \(M\) are continuous processes (see Proposition 5.12 in [1]). In such a case each member of the approximation sequence \((Y^k : k \in \mathbb{N})\) is continuous and since there exist a subsequence that converges \(\mathbb{P}\)-a.e. uniformly on \([0, T]\) we conclude that \(\int_0^t \int_U S_p(t - r)\tilde{R}(r, u) M(dr, du)\) has a continuous version. This completes the proof of Theorem 3.2.

4. Time regularity for solutions to stochastic evolution equations

In this section we apply our result in Theorem 3.2 to show the existence of càdlàg solutions to the following class of stochastic evolution equations

\[
dX_t = (A'X_t + B(t, X_t)) dt + \int_U F(t, u, X_t) M(dt, du), \quad \text{for } t \geq 0,
\]

with initial condition \(X_0 = Z_0\), where we will assume the following:

**Assumption 4.1.**

(A1) Every continuous seminorm on \(\Psi'\) is separable (e.g. if either \(\Psi'\) is nuclear or separable).

(A2) \(Z_0\) is a \(\Psi'\)-valued, regular, \(\mathcal{F}_0\)-measurable, square-integrable random variable.

(A3) \(A\) is the infinitesimal generator of a \((C_0, 1)\)-semigroup \((S(t) : t \geq 0)\) on \(\Psi\) and the dual semigroup \((S(t)' : t \geq 0)\) is a \((C_0, 1)\)-semigroup.

(A4) \(M = (M(t, A) : t \geq 0, A \in \mathcal{R})\) is a cylindrical martingale-valued measure as in Sec. 2.3 with \(\lambda\) being the Lebesgue measure on \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\).

(A5) \(B : \mathbb{R}_+ \times \Psi' \rightarrow \Psi'\) is such that the map \((r, g) \mapsto \langle B(r, g), \psi \rangle\) is \(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\Psi')\)-measurable, for every \(\psi \in \Psi\).

(A6) \(F = \{F(r, u, g) : r \in \mathbb{R}_+, u \in U, g \in \Psi'\}\) is such that

(a) \(F(r, u, g) \in \mathcal{L}(\Phi_{\psi, u}, \Psi')\), \(\forall r \in \mathbb{R}_+, u \in U, g \in \Psi'\).

(b) The mapping \((r, u, g) \mapsto q_{r, u, F(r, u, g)}(\psi, \phi)\) is \(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U) \otimes \mathcal{B}(\Psi')\)-measurable, for every \(\phi \in \Phi, \psi \in \Psi\).

(A7) There exist two functions \(a, b : \Psi \times \mathbb{R} \rightarrow \mathbb{R}\) satisfying:

(1) For each \(T > 0\) and \(K \subseteq \Psi\) bounded,

\[
\int_0^T \sup_{\psi \in K} (a(\psi, r)^2 + b(\psi, r)^2) dr < \infty.
\]
(2) (Growth conditions) For all \( r \in \mathbb{R}_+ \), \( g \in \Psi' \),
\[
|\langle B(r, g), \psi \rangle| \leq a(\psi, r)(1 + |\langle g, \psi \rangle|),
\]
\[
\int_U q_{r,u}(F(r,u,g)'\psi)^2 \mu(du) \leq b(\psi, r)^2(1 + |\langle g, \psi \rangle|)^2.
\]

(3) (Lipschitz conditions) For all \( r \in \mathbb{R}_+ \), \( g_1, g_2 \in \Psi' \),
\[
|\langle B(r, g_1), \psi \rangle - \langle B(r, g_2), \psi \rangle| \leq a(\psi, r)|\langle g_1, \psi \rangle - \langle g_2, \psi \rangle|,
\]
\[
\int_U q_{r,u}(F(r,u,g_1)'\psi - F(r,u,g_2)'\psi)^2 \mu(du) \leq b(\psi, r)|\langle g_1, \psi \rangle - \langle g_2, \psi \rangle|^2.
\]

It is shown in Theorem 6.23 in [1] that (4.1) has a unique weak solution with initial condition \( X_0 = Z_0 \), that is a \( \Psi' \)-valued regular and predictable process \( X = (X_t : t \geq 0) \) such that for every \( \psi \in \text{Dom}(A) \) and \( t \geq 0 \), \( \mathbb{P} \)-a.e.
\[
\langle X_t, \psi \rangle = \langle X_0, \psi \rangle + \int_0^t \langle X_r, A\psi \rangle + \langle B(r, X_r), \psi \rangle dr + \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du),
\]
where the first integral is a Lebesgue integral for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and the second integral is a weak stochastic integral.

This solution is also a mild solution, i.e. for every \( t \geq 0 \), \( \mathbb{P} \)-a.e.
\[
X_t = S(t)\gamma Z_0 + \int_0^t S(t-r)'B(r,X_r)dr + \int_0^t \int_U S(t-r)'F(r,u,X_r)M(dr,du). \tag{4.2}
\]

The first integral at the right-hand side of (4.2) is a \( \Psi' \)-valued regular, adapted process such that for all \( t \geq 0 \) and \( \psi \in \Psi \), for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \),
\[
\left\langle \int_0^t S(t-r)'B(r,X_r(\omega))dr, \psi \right\rangle = \int_0^t \langle S(t-r)'B(r,X_r(\omega)), \psi \rangle dr, \tag{4.3}
\]
where for each \( t \geq 0 \), \( \psi \in \Psi \), the integral on the right-hand side of (4.3) is defined for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) as a Lebesgue integral. The second integral at the right-hand side of (4.2) is the stochastic convolution.

For every \( T > 0 \), it is proved in Theorem 6.23 in [1] that there exists a continuous Hilbertian seminorm \( \rho = \rho(T) \) on \( \Psi \) such that \( X = (X_t : t \in [0, T]) \) has a \( \Psi'_\rho \)-valued predictable version \( \tilde{X} = (\tilde{X}_t : t \in [0, T]) \) satisfying \( \sup_{t \in [0,T]} \mathbb{E}(\rho'(\tilde{X}_t)^2) < \infty \).

**Theorem 4.2.** Apart from Assumption 4.1, suppose we have that \( (S(t) : t \geq 0) \) is a Hilbertian \( (C_0,1) \)-semigroup on \( \Psi \). Then (4.1) has a unique càdlàg weak solution with initial condition \( X_0 = Z_0 \). Furthermore, for every \( T > 0 \) there exists a continuous Hilbertian seminorm \( \rho = \rho(T) \) on \( \Psi \) such that \( X = (X_t : t \in [0, T]) \) is a \( \Psi'_{\rho} \)-valued adapted càdlàg process satisfying \( \sup_{t \in [0,T]} \mathbb{E}(\rho'(X_t)^2) < \infty \).

Moreover if for each \( A \in \mathcal{R} \) and \( \phi \in \Phi \), the real-valued process \( (M(t,A)(\phi) : t \geq 0) \) is continuous, then the results above remain valid replacing the property càdlàg by continuous.
Proof. We already know that (4.1) has a unique weak solution with initial condition \(X_0 = Z_0\), hence to prove the result it suffices to show that for every \(T > 0\) each term in (4.2) has a Hilbert space-valued version satisfying the conditions in Theorem 4.2.

In effect, in the proof of Lemma 6.24 in [1] (Step 1) it is shown the existence of a continuous Hilbertian seminorm \(q_0\) on \(\Psi\) such that \((Y^0_t : t \in [0, T]) := (S(t)Z_0 : t \in [0, T])\) has a \(\Psi''_0\)-valued continuous adapted version \((\tilde{Y}^0_t : t \in [0, T])\) satisfying 
\[
\sup_{t \in [0, T]} \mathbb{E}(q'_0(\tilde{Y}^0_t)^2) < \infty.
\]
Likewise, for \((Y^1_t : t \in [0, T]) := (\int^t_0 S(t - r)B(r, X_t)dr : t \in [0, T])\) it is shown in the proof of Lemma 6.24 in [1] (Step 2) that there exists a continuous Hilbertian seminorm \(q_1\) on \(\Psi\) such that \((Y^1_t : t \in [0, T])\) has a \(\Psi''_1\)-valued continuous adapted version \((\tilde{Y}^1_t : t \in [0, T])\) satisfying 
\[
\sup_{t \in [0, T]} \mathbb{E}(q'_1(\tilde{Y}^1_t)^2) < \infty.
\]
Finally, for \((Y^2_t : t \in [0, T]) := (\int^T_0 \int^r_0 S(t - r)F(r, u, X_t)M(dr, du) : t \in [0, T])\) it is shown in Lemma 6.24 in [1] (Step 3) that 
\[
F_X = \{F(r, u, X_t(\omega)) : r \in [0, T], \omega \in \Omega, u \in U\} \in \Lambda^2_x(T).
\]
Then, Theorem 3.2 shows that there exists a continuous Hilbertian seminorm \(q_2\) on \(\Psi\) such that \((Y^2_t : t \in [0, T])\) has a \(\Psi''_2\)-valued càdlàg adapted version \((\tilde{Y}^2_t : t \in [0, T])\) satisfying 
\[
\sup_{t \in [0, T]} \mathbb{E}(q'_2(\tilde{Y}^2_t)^2) < \infty.
\]
Let \(\rho\) be a continuous Hilbertian semi-norm on \(\Psi\) such that \(q_i \leq \rho\), for \(i = 0, 1, 2\). Then, the inclusions \(i_{q_i, \rho} : \Psi_{\rho} \to \Psi_{q_i}, i = 0, 1, 2\) are linear and continuous. Hence, if we take
\[
Y_t = i_{q_0, \rho}\tilde{Y}^0_t + i_{q_1, \rho}\tilde{Y}^1_t + i_{q_2, \rho}\tilde{Y}^2_t,
\]
we have that \((Y_t : t \in [0, T])\) is a \(\Psi''_\rho\)-valued adapted càdlàg version for \(X = (X_t : t \in [0, T])\) for which it is true that 
\[
\sup_{t \in [0, T]} \mathbb{E}(\rho'(Y_t)^2) < \infty.
\]
Finally, if for each \(A \in \mathcal{R}\) and \(\phi \in \Phi\), the real-valued process \((M(t, A)(\phi) : t \geq 0)\) is continuous, then Theorem 3.2 shows that \((\tilde{Y}^2_t : t \in [0, T])\) has continuous paths, hence the same is true for \((Y_t : t \in [0, T]). \)

\[\square\]

5. Examples and applications

Example 5.1. The following is an example of the construction of a Fréchet nuclear space and a Hilbertian \((C_0, 1)\)-semigroup on it which is commonly used on the literature of stochastic analysis in duals of nuclear spaces. See [17], Example 1.3.2, for full details.

Let \((H, \langle \cdot, \cdot \rangle_H)\) be a separable Hilbert space and—\(L\) be a closed densely defined self-adjoint operator on \(H\) such that \(\langle -L\phi, \phi \rangle_H \leq 0\) for each \(\phi \in \text{Dom}(L)\). Let \((S(t) : t \geq 0)\) be the \(C_0\)-contraction semigroup on \(H\) generated by—\(L\). Assume moreover that there exists some \(r_1\) such that \((\lambda I + L)^{-r_1}\) is Hilbert-Schmidt. Given these conditions, there exist a complete orthonormal system \((\phi_i : i \in \mathbb{N})\) in \(H\) and \(0 \leq \lambda_1 \leq \lambda_2 \leq \cdots\) such that 
\[
L\phi_i = \lambda_i\phi_i \text{ for } i = 1, 2, \ldots.
\]
Define
\[ \Psi = \left\{ \psi \in H : \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \psi, \phi_j \rangle_H^2 < \infty, \ \forall r \in \mathbb{R} \right\}, \]
and for every \( \psi \in \Psi \) and \( r \in \mathbb{R} \), define the Hilbertian norm
\[ |\psi|_r^2 = \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \psi, \phi_j \rangle_H^2. \]

It can be shown that \( \Psi \) is a nuclear Fréchet space when equipped with the topology generated by the family \( \{ | \cdot |_n : n \geq 0 \} \). Moreover, \( (S(t) : t \geq 0) \) restricts to an equicontinuous \( C_0 \)-semigroup \( (S(t) : t \geq 0) \) on \( \Psi \), i.e. \( |S(t)|_n \leq |\psi|_n, \ n \geq 0 \). Hence, \( (S(t) : t \geq 0) \) is a Hilbertian \( (C_0,1) \)-semigroup on \( \Psi \). Furthermore, the restriction \( A \) of \(-L\) to \( \Psi \) is the infinitesimal generator of \( (S(t) : t \geq 0) \) on \( \Psi \) and \( A \in \mathcal{L}(\Psi, \Psi) \). In particular, we have for all \( \psi \in \Psi, \ t \geq 0, \)
\[ A\psi = -\sum_{j=1}^{\infty} \lambda_j \langle \psi, \phi_j \rangle_H \phi_j, \quad S(t)\psi = \sum_{j=1}^{\infty} e^{-t\lambda_j} \langle \psi, \phi_j \rangle_H \phi_j. \tag{5.1} \]

As an example, let \( H = L^2(\mathbb{R}) \) and \(-L = \frac{d^2}{dx^2} - \frac{x^2}{4} \). Consider the sequence of Hermite functions \( (\phi_n : n \in \mathbb{N}) \) defined as:
\[ \phi_{n+1}(x) = \sqrt{g(x)} h_n(x), \quad n = 0, 1, 2, \ldots, \]
for \( g(x) = (\sqrt{2\pi})^{-1} \exp(-x^2/2) \) and where \( (h_n : n = 0, 1, 2, \ldots) \) is the sequence of Hermite polynomials:
\[ h_n(x) = \frac{(-1)^n}{\sqrt{n!}} g(x)^{-1} \frac{d^n}{dx^n} g(x), \quad n = 0, 1, 2, \ldots. \]

We have \( \sum_{n=1}^{\infty} \| (I + L)^{-r} \phi_n \|^2_H = \sum_{n=1}^{\infty} (n + \frac{1}{2})^{-2r} < \infty \) for \( r > \frac{1}{2} \). Hence the operator \( (\lambda I + L)^{-r} \) is Hilbert-Schmidt for \( r > \frac{1}{2} \). Then from the above construction one can check (see Theorem 1.3.2 in [17]) that \( \Psi = \mathcal{S}(\mathbb{R}) \) and the generator \( A \) and equicontinuous semigroup \( S(t) \) can be described by (5.1) for \( \lambda_n = n - \frac{1}{2} \) and the Hermite functions \( \phi_n \). See [17–19] for other examples using the above construction.

**Example 5.2.** Let \( \Phi \) be a barreled nuclear space. Recall from [20] that a \( \Phi' \)-valued adapted continuous zero-mean Gaussian process \( W = (W_t : t \geq 0) \) is called a generalized Wiener process if \( W_t - W_s \) is independent of \( \mathcal{F}_s \) for \( 0 \leq s < t \), and
\[ \mathbb{E}(\langle W_t, \phi \rangle \langle W_s, \varphi \rangle) = \int_0^{t \wedge s} q_r(\phi, \varphi) dr, \quad \forall \ t, s \in R_+, \ \phi, \varphi \in \Phi. \tag{5.2} \]
where \( \{q_r : r \in \mathbb{R}_+\} \) is a family of continuous Hilbertian semi-norms on \( \Phi \), such that the map \( r \mapsto q_r(\phi, \varphi) \) is Borel measurable and bounded on finite intervals, for each \( \phi, \varphi \in \Phi \).

Consider the stochastic evolution equation
\[ dX_t = (A'X_t + B(t, X_t))dt + F(t, 0, X_t)dW_t, \quad \text{for } t \geq 0, \tag{5.3} \]
with initial condition \(X_0 = Z_0\). The process \(W\) induces the cylindrical martingale valued measure \(M(t, A) = W_t \delta_0(A)\) where

i. \(U = \{0\}, \mathcal{R} = \mathcal{B}\{0\}\) and \(\mu = \delta_0\), and

ii. \(q_{r,0} = q_r\), where \(q_r: r \in \mathbb{R}_+\) are as in (5.2).

Hence, in comparison with (4.1) we have \(\int_0^t \int_U F(r, u, X_r)M(dr, du) = \int_0^t F(r, 0, X_r)dW_r\).

Suppose that \(\Psi', Z_0, A, (S(t): t \geq 0), (S(t)': t \geq 0)\), \(B\) and \(F\) satisfy Assumption 4.1 and that \((S(t): t \geq 0)\) is a Hilbertian \((C_0, 1)\)-semigroup on \(\Psi\). By Theorem 4.2 we have that (5.3) has a unique weak solution with continuous paths and initial condition \(X_0 = Z_0\). Moreover, for every \(T > 0\) there exists a continuous Hilbertian seminorm \(\rho = \rho(T)\) on \(\Psi\) such that \(X = (X_t: t \in [0, T])\) is a \(\Psi_\rho\)-valued adapted continuous process satisfying \(\sup_{t \in [0, T]} \mathbb{E}(\rho'(X_t)^2) < \infty\).

**Example 5.3.** Suppose \(\Phi\) is a barreled nuclear space and consider a \(\Phi'\)-valued Lévy process \(L = (L_t: t \geq 0)\) (see Sec. 3.2 in [21]). It is shown in ([21], Theorem 4.17) that the Lévy process \(L\) admits a Lévy-Itô decomposition, i.e. for each \(t \geq 0\),

\[
L_t = mt + W_t + \int_{\mathcal{B}_\rho'(1)^c} f\tilde{N}(t, df) + \int_{\mathcal{B}_\rho'(1)^c} f\tilde{N}(t, df).
\]

In (5.4), we have that \(m \in \Phi'\), \((W_t: t \geq 0)\) is a \(\Phi'\)-valued Wiener process with zero-mean and covariance functional \(Q\) (see Sec. 3.4 in [21]). Moreover, \(N(t, A) = \sum_{0 \leq s \leq t} 1_A(\Delta L_s), \forall t \geq 0, A \in \mathcal{B}(\Phi' \setminus \{0\})\), is the Poisson random measure associated to \(L\) with respect to the ring \(\mathcal{A}\) of all the subsets of \(\Phi' \setminus \{0\}\) that are bounded below (i.e. \(A \in \mathcal{A}\) if \(0 \notin \overline{A}\)), \(\tilde{N}(dt, df) = N(dt, df) - dt \, \nu(df)\) where \(\nu\) is a Lévy measure on \(\Phi'\) (see Sec. 4.3 in [21]) with a continuous Hilbertian semi-norm \(\rho\) on \(\Phi\) such that \(\int_{\mathcal{B}_\rho'(1)^c} \rho(f)^2 \nu(df) < \infty\). The process \(\int_{\mathcal{B}_\rho'(1)} f\tilde{N}(t, df), t \geq 0\), is a \(\Phi'\)-valued zero-mean, square integrable, càdlàg Lévy process, and the process \(\int_{\mathcal{B}_\rho'(1)^c} f\tilde{N}(t, df) \forall t \geq 0\) is a \(\Phi'\)-valued càdlàg Lévy process defined by means of a Poisson integral (see Sec. 4.1 in [21]) with respect to the Poisson random measure \(N\) of \(L\) on the set \(\mathcal{B}_\rho'(1)^c\). All the random components of the representation (5.4) are independent.

Consider the following Lévy-driven stochastic evolution equation:

\[
dX_t = (A'X_t + B(t, X_t))dt + F(t, 0, X_t)dW_t + \int_{\mathcal{B}_\rho'(1)^c} F(t, u, X_t)\tilde{N}(dt, du) + \int_{\mathcal{B}_\rho'(1)^c} F(t, u, X_t)\tilde{N}(dt, df).
\]

Suppose that \(\Psi', Z_0, A, (S(t): t \geq 0), (S(t)': t \geq 0)\), \(B\) and \(F\) satisfy Assumption 4.1 for \(U = \Phi'\), \(\mu = \nu\), and with the family of continuous Hilbertian semi-norms \(\{q_{r,u}: r \in \mathbb{R}_+, u \in \Phi'\}\) given by \(q_{r,0}(\phi) = Q(\phi, \phi)^{1/2}\) and \(q_{r,u}(\phi) = |\langle u, \phi \rangle|\) if \(u \neq 0\).

We know by Theorem 7.1 in [1] that (5.5) has a weak and mild solution. If we further assume that \((S(t): t \geq 0)\) is a Hilbertian \((C_0, 1)\)-semigroup on \(\Psi\), then if in the
proof of Theorem 7.1 in [1] we use Theorem 4.2 instead of Theorem 6.23 in [1], we can show that (5.5) has a unique càdlàg weak solution with initial condition $X_0 = Z_0$.

Acknowledgements

The author thank two anonymous reviewers whose comments helped improve the manuscript.

Funding

The author was financially supported by The University of Costa Rica through the grant “821-C2-132—Procesos cilíndricos y ecuaciones diferenciales estocásticas.”

ORCID

Christian A. Fonseca-Mora http://orcid.org/0000-0002-9280-8212

References

[1] Fonseca-Mora, C. A. (2018). Stochastic integration and stochastic PDEs driven by jumps on the dual of a nuclear space. Stoch. PDE: Anal. Comp. 6(4):618–689. DOI: 10.1007/s40072-018-0117-x.
[2] Babalola, V. A. (1974). Semigroups of operators on locally convex spaces. Trans. Am. Math. Soc. 199:163–179. DOI: 10.1090/S0002-9947-1974-0383142-8.
[3] Kotelenez, P. (1982). A submartingale type inequality with applications to stochastic evolution equations. Stochastics. 8(2):139–151. DOI: 10.1080/17442508208833233.
[4] Pérez-Abreu, V., Tudor, C. (1992). Regularity and convergence of stochastic convolutions in duals of nuclear Fréchet spaces. J. Multivariate Anal. 43(2):185–199. DOI: 10.1016/0047-259X(92)90033-C.
[5] Fonseca-Mora, C. A. (2021). Stochastic evolution equations with Lévy noise in duals of nuclear spaces. Preprint. ArXiv:2105.12812.
[6] Ding, D. (1999). Stochastic evolution equations in duals of nuclear Fréchet spaces. In: Nonlinear Evolution Equations and Their Applications (Macau, 1998). River Edge, NJ: World Sci. Publ., pp. 57–71.
[7] Wu, J.-L. (1994). On the regularity of stochastic difference equations in hyperfinite-dimensional vector spaces and applications to $D$-valued stochastic differential equations. Proc. R. Soc. Edinburgh Sect. A. 124(6):1089–1117.
[8] Wu, J.-L. (1995). On the existence of solutions to $D$-valued stochastic differential equations involving evolution drift. Acta Math. Sci. (English Ed.). 15(suppl):91–102.
[9] Jarchow, H. (1981). Locally Convex Spaces. Mathematische Leitfäden. Stuttgart: B. G. Teubner.
[10] Schaefer, H. H., Wolff, M. P. (1999). Topological Vector Spaces, Vol. 3: Graduate Texts in Mathematics. 2nd ed. New York: Springer-Verlag.
[11] Trèves, F. (2006). Topological Vector Spaces, Distributions and Kernels. Mineola, NY: Dover Publications, Inc., Unabridged Republication of the 1967 Original.
[12] Pietsch, A. (1972). Nuclear locally convex spaces. Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, Band 66. New York and Heidelberg: Springer-Verlag.
[13] Komura, T. (1968). Semigroups of operators in locally convex spaces. J. Funct. Anal. 2: 258–296.
[14] Fonseca-Mora, C. A. (2018). Existence of continuous and Càdlàg versions for cylindrical processes in the dual of a nuclear space. J. Theor. Probab. 31(2):867–894. DOI: 10.1007/s10959-016-0726-0.
[15] Fonseca-Mora, C. A. (2015). Stochastic analysis with Lévy Noise in the dual of a nuclear space. PhD thesis. Sheffield, UK.

[16] Peszat, S., Zabczyk, J. (2007). Stochastic Partial Differential Equations with Lévy Noise. Volume 113 of Encyclopedia of Mathematics and Its Applications. Cambridge: Cambridge University Press.

[17] Kallianpur, G., Xiong, J. (1995). Stochastic Differential Equations in Infinite-Dimensional Spaces. Volume 26 of Institute of Mathematical Statistics Lecture Notes—Monograph Series. Hayward, CA: Institute of Mathematical Statistics.

[18] Kallianpur, G., Pérez-Abreu, V. (1988). Stochastic evolution equations driven by nuclear-space-valued martingales. Appl. Math. Optim. 17(1):237–272. DOI: 10.1007/BF01448369.

[19] Kallianpur, G., Wolpert, R. (1984). Infinite-dimensional stochastic differential equation models for spatially distributed neurons. Appl. Math. Optim. 12(1):125–172. DOI: 10.1007/BF01449039.

[20] Bojdecki, T., Jakubowski, J. (1990). Stochastic integration for inhomogeneous Wiener process in the dual of a nuclear space. J. Multivariate Anal. 34(2):185–210. DOI: 10.1016/0047-259X(90)90035-G.

[21] Fonseca-Mora, C. A. (2020). Lévy processes and infinitely divisible measures in the dual of a nuclear space. J. Theor. Probab. 33(2):649–691. DOI: 10.1007/s10959-019-00972-3.