Almost Ideal Clocks in Quantum Cosmology: A Brief Derivation of Time

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Abstract

A formalism for quantizing time reparametrization invariant dynamics is considered and applied to systems which contain an ‘almost ideal clock.’ Previously, this formalism was successfully applied to the Bianchi models and, while it contains no fundamental notion of ‘time’ or ‘evolution,’ the approach does contain a notion of correlations. Using correlations with the almost ideal clock to introduce a notion of time, the work below derives the complete formalism of external time quantum mechanics. The limit of an ideal clock is found to be closely associated with the Klein-Gordon inner product and the Newton-Wigner formalism and, in addition, this limit is shown to fail for a clock that measures metric-defined proper time near a singularity in Bianchi models.
I. INTRODUCTION

The goal of this work is to derive the usual framework of external time quantum mechanics from the what we will call the ‘almost local formalism’ of [1] for the quantization of time reparametrization invariant finite dimensional systems. By ‘external time quantum mechanics’ we mean the mathematical structure of the Schrödinger picture consisting of a Hilbert space $\mathcal{H}_{ext}$ of states labeled by a parameter $\tau$ together with an algebra generated by some set $L_i$ of Hermitian operators on $\mathcal{H}_{ext}$. The operators should satisfy

$$[L_i, L_j] = C_{ij}^k L_k$$

(1.1)

for some structure constants $C_{ij}^k$ and evolution in the parameter $\tau$ is given by

$$-i\hbar \frac{\partial}{\partial \tau} |\psi(\tau)\rangle = H(\tau)|\psi(\tau)\rangle$$

(1.2)

for some self-adjoint operator $H(\tau)$. Equivalently, we could derive a Heisenberg picture but we will see that construction of the Schrödinger picture is most direct. For a given system with a classical analogue, the commutation relations (1.1) should agree with the Poisson Bracket relations of the corresponding phase space functions.

A derivation of this structure will be possible despite the fact that the almost local formalism contains no fundamental notion of time or evolution. This approach is an implementation of the ideas expressed by, for example DeWitt [2,3] and Rovelli [4–7] that such features are unnecessary and, in fact, inappropriate for a description of time reparametrization invariant systems. Thus, we address the problem of time in quantum gravity.

The main ideas employed below are not new. Actually, they are nearly as old as (canonical) quantum gravity itself as we follow the basic approach outlined in [3] and extended in e.g. [8–13]. That is, we first use a method based on Dirac’s [14] to quantize the system by imposing a Hamiltonian constraint on physical states and we then use a semiclassical approximation to study the correlations so induced between some degree of freedom (which we choose to call a clock) and the rest of the system. We also follow the suggestion of
that these correlations be captured through the use of time reparametrization invariant quantum operators. The construction of these operators will be based on the classical methods of [2].

The new input here lies in bringing together the above ideas with the technology of [1] and its specific implementation of these suggestions in the quantum context. This provides for a much more complete analysis than was previously possible as we will construct the explicit map between the physical Hilbert space $\mathcal{H}_{phys}$ of the almost local formalism and the usual Hilbert space $\mathcal{H}_{ext}$ of external time quantum mechanics while avoiding the difficulties [16] usually associated with such derivations. The complete formalism also forces us to take into account new subtleties but leads in the end to a physically reasonable conclusion.

While a satisfactory formalism for full quantum gravity remains out of reach, the almost local approach was shown in [1] to provide a complete quantization of Hamiltonian vacuum Bianchi models (see, e.g. [17] [19]) and of similar finite dimensional time reparametrization invariant systems. In particular, it was successfully applied [20] to the complicated mixmas-ter cosmology [21]. The details of this approach will be reviewed in section II but for now we mention that it provides [1,20] a positive definite inner product on a physical Hilbert space $\mathcal{H}_{phys}$ as well as a complete set of well-defined symmetric quantum observables on this space (perennials in the language of [16]) that, in a certain sense, capture the notion of ‘the value of quantity $A$ at the point in time at which quantity $Q$ takes a specified value $\tau$.’ In addition, this approach captures in a quantum sense the classical recollapsing behavior of the appropriate cosmological models.

After reviewing the almost local formalism in section II, we describe in section III the class of systems to be considered and the approximations to be used. In particular, we introduce the notion of an ‘almost ideal clock.’ We then derive the external time formalism in three stages in section IV. We begin by showing in IV A that ‘evolution’ in the clock time $\tau$ is unitary to the extent that the clock is ideal and then show in IV B that our observables have the usual algebraic structure associated with external time quantum mechanics (in particular, that they satisfy the canonical commutation relations). Section IV C derives
the explicit map between \( \mathcal{H}_{\text{phys}} \) and \( \mathcal{H}_{\text{ext}} \) and shows that it is of the form associated with the Klein-Gordon inner product and the Newton-Wigner operators. Section [V] points out certain interesting features of our derivation such as the fact that, for clocks that measure metric-defined proper time, our approximations always fail near a singularity in homogeneous models.

II. PRELIMINARIES

We now embark on a brief review of the almost local formalism of [1] for the reader’s convenience. This formalism is a refinement of the original prescription [14] of Dirac for the canonical quantization of constrained systems, combined with a suggestion of DeWitt [2] for the construction of observables (perennials in the language of [14]). A more detailed description which takes a geometric approach toward the classical phase space and an algebraic approach toward the quantum operators was presented in [1] and the appropriate derivations may be found there.

For convenience, we consider systems for which time reparametrization is the only gauge symmetry. In a classical Hamiltonian formalism, such systems are described by a phase space \( \Gamma \) and a constraint \( h = 0 \) where \( h \) is some function on \( \Gamma \). Also for simplicity, consider the case \( \Gamma = T^n R^n \) with coordinate functions \( p_i, q^j \) for \( i, j \) in some index set \( I \). As usual, we choose these to satisfy the Poisson bracket algebra \( \{ q^i, p_j \} = \delta^j_i \). The quantization procedure of [1] can be summarized by introducing the usual operators \( P_i, Q^j \) on the Hilbert space \( \mathcal{H}_{\text{aux}} = L^2(\mathbb{R}^n, d^n q) \) where the \( Q^j \) act by multiplication and the \( P_i \) by derivation with the commutation relations \( [Q^i, P_j] = i\delta^j_i \). Similarly, a self-adjoint operator \( H \) on \( \mathcal{H}_{\text{aux}} \) is to be constructed from some appropriately factor ordered version of the constraint function \( h(p, q) \). As in [14], we take capital letters to refer to quantum operators and lower case letters to refer to their eigenvalues and classical counterparts.

Next introduce a one parameter family of ‘lapse operators’ \( N(t) \) that are proportional to the identity \( \mathbb{1} \) on \( \mathcal{H}_{\text{aux}} \). That is, \( N(t) = n(t) \mathbb{1} \) for some \( n : \mathbb{R} \to \mathbb{R} \). We require \( n(t) \) to
be continuous and to satisfy the boundary conditions
\[ \int_{0}^{\pm\infty} n(t) dt = \pm\infty \]  
\hspace{1cm} (2.1)

but physical operators will be independent of which \( n(t) \) is chosen. For each classical function \( q^j \) and \( p_i \), we now introduce a one parameter family of operators through

\[
Q^j(t) \equiv \exp\left[ i \int_{0}^{t} dt N(t) H \right] Q^j \exp\left[ -i \int_{0}^{t} dt N(t) H \right]
\]

\[
P_i(t) \equiv \exp\left[ i \int_{0}^{t} dt N(t) H \right] P_i \exp\left[ -i \int_{0}^{t} dt N(t) H \right].
\]  
\hspace{1cm} (2.2)

Following [14], we will construct a physical Hilbert space by ‘imposing the constraint’ \( H|\psi_{phys}\rangle = 0 \) on ‘physical states \( |\psi_{phys}\rangle \).’ Precisely, we assume that the Hilbert space \( \mathcal{H}_{aux} \) has a basis \( \{|n, E\rangle : n \in \mathbb{Z}, E \in [a, b]\} \) for the part of \( \mathcal{H}_{aux} \) corresponding to some spectral interval \([a, b]\) of \( H \) with \( 0 \in [a, b] \). These states are to satisfy

\[
\langle n_1, E_1 | n_2, E_2 \rangle = \delta_{n_1, n_2}\delta(E_1 - E_2), \quad H|n, E\rangle = E|n, E\rangle
\]

\hspace{1cm} (2.3)

and we thus require \( H \) to have continuous spectrum that contains the interval \([a, b]\) while having no point spectrum in this interval. We note that this is the case [1,20] for the proper formulation of interesting cosmological models. The physical Hilbert space \( \mathcal{H}_{phys} \) may be defined by stating that the formal symbols \( |n, 0\rangle \) span a dense subset of \( \mathcal{H}_{phys} \) and have the ‘spectral analysis inner products’ [22]

\[
\langle n_1, 0 | n_2, 0 \rangle = \delta_{n_1, n_2}.
\]

\hspace{1cm} (2.4)

Note that this can be described as an infinite renormalization of the ‘bare’ inner product 2.31

Finally, we need to construct physical operators on \( \mathcal{H}_{phys} \). In general, we expect an operator \( A \) on \( \mathcal{H}_{aux} \) to define such a physical operator if \( [H, A] = 0 \). It was shown in [1]

1This inner product has been independently introduced several times in related contexts. See [23, 24] for the cases known to the author.
that a large class of such physical operators $\Omega$ can be constructed from the ‘time-dependent’ operators $\omega(t) = \omega(Q^j(t), P_i(t))$ through

$$\Omega = \int dt N(t) \omega(t).$$  \hfill (2.5)

Convergence of this integral was discussed in [1,20] and will not be considered here. We simply use the result of [1] that, when (2.5) converges, it defines a physical operator $\Omega_{phys}$ on $\mathcal{H}_{phys}$ through

$$\Omega_{phys}|n,0\rangle \equiv 2\pi \sum_m |m,0\rangle \langle m,0|\omega(t)|n,0\rangle_{aux}.$$  \hfill (2.6)

The matrix elements on the right hand side are in fact independent of $t$ and the subscript $aux$ is a reminder that this matrix element is to be evaluated in the auxiliary Hilbert space $\mathcal{H}_{aux}$, using the corresponding inner product. It is readily shown that the map $\Omega \rightarrow \Omega_{phys}$ is a homomorphism; that is, for $[A,H] = 0 = [B,H]$, we have $(AB)_{phys} = A_{phys}B_{phys}$ and $(A + B)_{phys} = A_{phys} + B_{phys}$. Thus, no harm is done by dropping the subscript $phys$ from the operators on $\mathcal{H}_{phys}$. We do so in order to simplify our notation; whether $\Omega$ represents the operator on $\mathcal{H}_{aux}$ or $\mathcal{H}_{phys}$ should be clear from the context.

The particular operators considered in [1] were of the form (2.5) with $\omega(t) = \{A(t), i\hbar[\theta(B(t) - \tau), H]\}$. The associated physical operators describe a certain sort of average over ‘the values of the quantity $A(t)$ at the times for which $B(t) = \tau$,’ even when the corresponding classical quantity $b(t)$ may take on the given value $\tau$ more than once along a trajectory. However, when $b(t)$ takes this value many times the convergence of (2.5) becomes a subtle issue so that our definition of an almost ideal clock below must pay due attention to the corresponding classical solutions. Finally, note that the choice of $\Gamma = \mathbb{R}^{2n}$ and the use of canonical coordinates and momenta is not necessary, but only a simple and familiar case. As in [27], we may start with any Lie algebra of overcomplete functions on $\Gamma$ and use any unitary Hilbert space representation as the auxiliary space $\mathcal{H}_{aux}$.

\begin{footnote}{Note that the notation of 2.5 differs slightly from that in [1].}\end{footnote}
III. THE APPROXIMATION SCHEME

We now describe the systems to be considered and introduce the approximations relevant to our derivation of external time quantum mechanics. That is, we describe what it means for a system to contain an almost ideal clock and in what sense the ideal limit is to be taken. We also provide a number of useful tools for later calculations.

A. Almost Ideal Clocks

The philosophy of this work is that we should, in principle, be able to choose any degree of freedom to be a clock and thereby define a notion of time. Thus, we would like our definition of an almost ideal clock to be as broad as possible. Nevertheless, we must recognize that some clocks are inherently better than others in the sense that the notion of time they define is most like the Newtonian one. This is true even classically and so should come as no surprise in the quantum context. We thus restrict our attention to a clock that, in some sense, interacts weakly both with itself and with the rest of our system.

In order to define almost ideal clocks, it is useful to first establish the concept of an ideal clock. We consider an ideal clock to be a system with a pointer whose position $Q_p$ increments in direct proportion to the passage of proper time. Furthermore, the corresponding constant of proportionality should be independent of the state of the system. This definition is, however, ambiguous for the systems we consider as there will, in general, be two distinct notions of proper time. One such notion is the proper time $\int_{t_1}^{t_2} N(t) dt$ associated with any system described by a Hamiltonian constraint $H$ and lapse $N$ while the other is whatever concept of proper time may be defined by a metric for the system. The point here is that $N$ may not correspond exactly to the metric-defined lapse but may be rescaled in some nontrivial way. This is exactly the case of interest as the almost local formalism can only be applied (see [1,20]) to homogeneous cosmological models by using a constraint $H$ and lapse $N$ which are related to the ADM constraint and lapse by $H = \sqrt{2/3\pi} (\det g)^{1/2} H_{ADM}$
and \( N = \sqrt{3\pi N_{ADM}/\sqrt{2}(\det g)^{1/2}} \). Here, \( \det g \) is the determinant of the 3-metric on a homogeneous spatial slice.

For the purposes of this paper we define an ideal clock to measure the proper time associated with our choice of lapse and Hamiltonian constraint and not the metric-defined proper time. The rationale for this choice is simply that it is in this case that we may derive the usual external time quantum mechanics. While this state of affairs may seem unsatisfactory at first, section \( \PageIndex{3} \) will i) show that the notions of ‘almost ideal clocks’ associated with the two kinds of proper time in fact agree unless the volume of the spatial slices is very small and ii) point out that a clock which explicitly measures metric-based proper time for arbitrarily small spatial volumes cannot even classically define a satisfactory notion of time.

Thus, a time reparametrization invariant system with an ideal clock is described by a constraint of the form

\[
0 = H = P_p + H_1
\] (3.1)

where \( P_p \) is the momentum conjugate to \( Q_p \) and \( H_1 \) is independent of \( P_p \) and \( Q_p \). For such a system, [1] has already shown that the external time formalism can be derived exactly. Our task is to show that when the dynamics and quantum state conspire to create a regime in which the clock is almost ideal, then evolution in the clock time is described by the usual external time formalism to the same degree.

In general, we expect a generic degree of freedom to approximate an ideal clock when it is weakly interacting, both with other degrees of freedom and with itself. We therefore say that a reparametrization invariant system contains a decoupled clock when the Hamiltonian constraint may be written in the form

\[
0 = H = H_0 + H_1
\] (3.2)

where \( H_0 \) depends only on the coordinates and momentum of the clock and \( H_1 \) is independent of the canonical clock variables. While this situation is unphysical, we assume that it may in turn be approximated by systems which differ from (3.2) only by terms that may be treated
adiabatically. However, we will not consider this approximation here and we take $3.2$ as our starting point.

We would like to treat the case where $H_0$ is of the form $H_0 = H_p = \frac{P^2}{2} + V(Q)$. For such systems, we must work in the approximation that the clock coordinate $q$ is far from the classical turning points so that the classical $q(t)$ is a single-valued function of time. Thus, we will be able to describe our clock using a WKB approximation.

Now, if $V(q)$ confines $q$ to some finite region, what $H_p$ describes in not a clock but a pendulum; that is, a part of a clock, or a clock that measures time only modulo some (possibly state dependent) period. Thus, specifying the coordinate $Q$ determines not a unique moment of time but a discrete set of such moments. Even classically, such a pendulum can only be used as a clock during some agreed upon half of an oscillation (such as the right-moving half of the first oscillation after the explosion of a firecracker).

Following the model of a grandfather clock, we will assume that this ambiguity is removed by coupling the pendulum to some counter which increments with each complete oscillation. In contrast to the grandfather clock, however, the primary part of our system will be the pendulum and not the counter. The counter will be used only to specify the oscillation in which we are interested and, in what follows, we use the right-moving half of the cycle in which the counter reads zero. Alternatively, we could couple the counter so that it increments with each half of the oscillation, but the above choice has the advantage that it simultaneously considers the case where $V(q)$ has only one turning point (or none at all) and no counter is present in the system (the counter is irrelevant in this situation). This is the case when our clock is built from the scale factor of a homogeneous cosmological model. Because the counter will only be used in a secondary manner, we expect the details of its construction and coupling to the pendulum to be irrelevant so long as this coupling is weak.

Nevertheless, for a consistent treatment it is useful to have an explicit model of the counter. We therefore take our clock to be described by the canonical pairs $(Q, P)$ for the pendulum and $(Q_c, P_c)$ for the counter with the Hamiltonian:
\[ H_0 = \frac{P^2}{2} + V(Q) + \epsilon \frac{e^{-(Q-q_0(H_p))^2/4\lambda}}{\sqrt{4\pi\lambda}} \]

where we have assumed that \( V(q) \) has no more than a single critical point and that this point (if it exists) is a global minimum. Here, \( H_p = \frac{P^2}{2} + V(Q), \epsilon \geq 0, \) and \( q_0(E) \) is the position of the left classical turning point of a pendulum with energy \( E \) when not coupled to the counter; that is, the leftmost solution of \( E = V(q) \). Note that (3.3) is constructed so that the counter coordinate \( q_c \) advances a distance \( \epsilon \) with each oscillation of the pendulum. Thus, we will say that our system contains an ‘almost ideal clock’ if it is described by a constraint which differs from (3.2) only by adiabatic factors and if \( H_0 \) describes a pendulum coupled (when the pendulum is bound) weakly to a counter.

We now state precisely what is meant by the ideal clock limit. First, we assume that \( |\epsilon p_c| \) is small in the sense that we consider only states \( |\psi\rangle \in \mathcal{H}_{phys} \) whose spectral support in \( P_c \) is concentrated in the region with \( \epsilon p_c \ll \Delta E \), where \( \Delta E \) is the splitting between the energy levels of \( H_p \) on which the physical state \( |\psi\rangle \in \mathcal{H}_{aux} \) is concentrated. Thus, the pendulum-counter coupling is weak and the spectral projections of \( H_0 \) and \( H_p \) on \( |\psi\rangle \) agree. Choosing some pendulum position \( \tau \), we assume that for each eigenvalue \( E \) of \( H_0 \) either \( |	au - q_t| \gg \hbar/\sqrt{E} \) for any solution of \( V(q_t) = E \) or that the corresponding projection of \( |\psi\rangle \) is negligible. Thus, the pendulum is far from its turning points. We also assume that the width \( \lambda \) in (3.3) is large \( (\lambda \gg \hbar/\sqrt{E}) \) so that the notion of a turning point need not be defined too precisely. Finally, we ask that the projection of \( |\psi\rangle \) to the subspace where \( Q = \tau \) have negligible projections onto the eigenvalues 0 and \( \epsilon \) of \( Q_c \) so that the counter may be read clearly.

The observables associated with an almost ideal clock are more complicated than those of an ideal clock as they depend on the counter reading and the direction of pendulum motion as well as the pendulum position. In analogy with [120], we will capture the notion of ‘the

\[^3\]Here, we abuse notation as \(|\psi\rangle\) is to satisfy \( H|\psi\rangle = 0 \) and so is not a normalizable state in \( \mathcal{H}_{aux} \), but only in \( \mathcal{H}_{phys} \).
value of $A$ when the pendulum is moving to the right through position $\tau$ and the counter reads $n'$ with the observables:

$$A_{\tau,n} = \frac{1}{2} \int_{-\infty}^{+\infty} dt \theta(\pm P(t)) \{A(t), \chi_n(t) \frac{\partial}{\partial t} \theta(Q(t) - \tau)\} \theta(\pm P(t)).$$

(3.4)

where $A(t) = A(Q^i(t), P_j(t))$ may be built from the coordinates and momenta of both the clock and the other degrees of freedom and $\chi_n(t)$ is the projection onto the spectral interval $[n\epsilon, (n+1)\epsilon]$ of $Q_c(t)$. We will not address the convergence of (3.4) here, although we expect that, despite the more complicated form of (3.4) as compared with the observables of [1], the same arguments may be applied and that (3.4) converges to give a densely defined operator on $H_{phys}$ for a reasonable set of operators $A$. Note that the commutator of $Q_c$ with $\frac{\partial}{\partial t} Q(t)$ is of order $\epsilon p_c$ and so negligible. Thus, these operators are symmetric on $H_{aux}$ to lowest order in $\epsilon p_c$. Since we are only interested in the case where the counter takes the value zero and the pendulum moves to the right, we drop the labels $(n, \pm)$ on these operators with the understanding that they always implicitly takes the values $(0, +)$. Analogous results follow when the arguments below are applied to the remaining cases.

**B. The Physical States**

We now present the details of the approximation scheme to be used in the rest of this work. As mentioned above, by assuming that the pendulum is far from its classical turning points, we will be able to employ WKB techniques to great effect. In general, we will keep only terms of order $(\hbar)^0$ in our approximation and discard higher terms. However, we will not discard all terms of order $\hbar$, but only those associated with the WKB expansion. Thus, we expect the discarded terms to be small whenever the clock is far from a turning point regardless of the behavior of the rest of the system. In fact, when the pendulum is confined between two exponential walls, this approximation may be turned into an expansion in $1/(\tau - q_t)$ (where $q_t$ is an associated classical turning point) in the usual way.

As in [1] (and in quantum mechanics generally), we will find that it is essential to keep proper account of the normalization of states in order to arrive at physically sensible answers.
Because of the way that the inner product on $\mathcal{H}_{\text{phys}}$ is defined from that of $\mathcal{H}_{\text{aux}}$, we will see that the presence of the counter is crucial to obtaining properly normalized states. It is this subtlety on which we focus in our description of the physical states.

For the moment, we consider only the clock part of the the system and make use of the factorization $\mathcal{H}_{\text{aux}} = \mathcal{H}_0 \otimes \mathcal{H}_1$ where $\mathcal{H}_0 = \hat{H}_0 \otimes \mathbb{1}$ and $\mathcal{H}_1 = \mathbb{1} \otimes \hat{H}_1$ with respect to this decomposition. Since the physical ‘subspace’ of $\mathcal{H}_{\text{aux}}$ is spanned by the states $|E_0, a_0, E_1, a_1\rangle = |E_0, a_0\rangle \otimes |E_1, a_1\rangle$ with $\hat{H}_0|E_0, a_0\rangle = E_0|E_0, a_0\rangle$, $\hat{H}_1|E_1, a_1\rangle = E_1|E_1, a_1\rangle$, and $E_0 = -E_1$, ($a_0$ and $a_1$ are degeneracy labels), we may concentrate on the states $|E_0, a_0\rangle$. The form of the $|E_1, a_1\rangle$ will be unimportant.

In what follows, we will make use of the Hilbert space $\mathcal{H}_p \cong L^2(\mathbb{R})$ associated with the pendulum and the space $\mathcal{H}_c \cong L^2(\mathbb{R})$ associated with the counter. In general, the decomposition $\mathcal{H}_0 = \mathcal{H}_p \otimes_t \mathcal{H}_c$ will depend on a choice of parameter time $t$. Since §2.6 is independent of the choice of $t$, no harm is done by choosing $t = 0$. We do so here and suppress the label $t$ on the states and operators below.

We make no effort to label states explicitly with the Hilbert space to which they belong nor, hereafter, do we distinguish related operators that act on different spaces. Thus, we drop the hats on $\hat{H}_0$ and $\hat{H}_1$ and also refer to these operators as $H_0$ and $H_1$. In addition, we will make use of several bases for each Hilbert space. Both the space in which a state lives and the basis to which it belongs are determined implicitly by the labels inside the bra or ket. For example, $|p_c\rangle$ denotes an eigenstate of $P_c$ in $\mathcal{H}_c$ with eigenvalue $p_c$ while $|q, p_c\rangle$ is an element of $\mathcal{H}_0$. As mentioned above, the issue of normalizations will be subtle and we will often make use of pairs of bases whose members differ only in their normalizations. In this case, one basis will be denoted in the usual way with angle-bracket bra’s ($\langle |$ and ket’s $| \rangle$) while the other will be denoted by round bracket ($|, |$) bra’s and kets. Thus, $|p_c\rangle$ and $|p_c\rangle$ are both eigenstates of $P_c$ in $\mathcal{H}_c$ with eigenvalue $p_c$, but with differing normalizations. In general, the round-bracket basis will be introduced first, but only the angle-bracket basis will be used in the derivation of the external time formalism in section [V]. The normalizations of bases will always be explicitly stated in the text.
Thus, we begin with a basis \( \{|\tau, p_c\}\} \) of \( \mathcal{H}_0 \), where \( P_c|\tau, p_c\rangle = p_c|\tau, p_c\rangle \) and \( Q|\tau, p_c\rangle = \tau|\tau, p_c\rangle \) with \( (\tau, p_c|\tau', p'_c) = \delta(\tau - \tau')\delta(p_c - p'_c) \). We note that \( [P_c, H_0] = 0 \), so that, since the eigenstates of \( H_0 \) for a given value of \( p_c \) are nondegenerate, we may label the \( n \)th eigenstate of \( H_0 \) at \( p_c \) by \( |n, p_c\rangle = |n; p_c\rangle \otimes |p_c\rangle \), where \( (p'_c|p_c) = \delta(p'_c - p_c) \) and \( |n; p_c\rangle \) is the unit normalized \( n \)-th eigenstate on \( \mathcal{H}_p \) of the operator \( H_0^{p_c} \) obtained from \( H_0 \) by replacing \( P_c \) with the eigenvalue \( p_c \). The reader should beware of the difference between \( |n; p_c\rangle \in \mathcal{H}_p \) and \( |n, p_c\rangle \in \mathcal{H}_0 \) – the former is the eigenstate of the operator \( H_0^{p_c} \) (where \( p_c \) determines just which operator this is!) while the latter is the simultaneous eigenstate of \( H_0 \) and \( P_c \).

Invoking the usual WKB approximation [26] and assuming that \( \epsilon p_c \) is small, such states satisfy

\[
\langle \tau, p'_c|n, p_c\rangle = \delta(p'_c - p_c) \frac{C(n, p_c)}{2\pi[2E(n, p_c) - 2V(\tau)]^{1/2}} \left[ \alpha_1 \exp \left( i \int_0^\tau \sqrt{2E(n, p_c) - 2V(q)} \frac{dq}{\hbar} \right) \right] \left[ 1 + \mathcal{O}(\epsilon) + \mathcal{O}(\hbar) \right] \tag{3.5}
\]

where \( C(n, p_c) \) is real and \( \alpha_1, \alpha_2 \in \mathbb{C} \) have unit norm. We expect this approximation to be valid when \( V(\tau) \ll E(n, p_c) \equiv (n; p_c|H_0^{p_c}|n; p_c) \).

In order to make use of 2.6, we must choose our basis \( |E_0, a_0, E_1, a_1\rangle \) to satisfy 2.3. In particular, it will be simplest to proceed by taking these energy states to be delta-function normalized in the total clock energy. Note that the spectrum of \( H_0 \) is, in fact, entirely continuous since we have

\[
\frac{\partial E(n, p_c)}{\partial p_c} = \langle n; p_c|\epsilon Z|n; p_c\rangle > 0 \tag{3.6}
\]

where \( Z \) is the coefficient of \( \epsilon P_c \) in 3.3 and the inequality follows because \( Z \) is a strictly positive operator. Thus, the coupling to the counter broadens the pendulum energy levels into continuous bands. Note that our requirement that \( |\epsilon p_c| \ll \Delta E \) is just the assumption that these bands remain well separated.

Now, the states \( |n, p_c\rangle \) are normalized according to \( (n, p_c|n', p'_c) = \delta(p_c - p'_c)\delta_{n,n'} \). However, we would like to use the states \( |n, p_c\rangle = \left( \frac{\partial E(n, p_c)}{\partial p_c} \right)^{-1/2} |n, p_c\rangle \) which satisfy

\[
\langle n, p_c|n'p'_c\rangle = \delta(p_c - p'_c)\delta(E(n, p_c) - E(n', p'_c)). \tag{3.7}
\]
To derive the WKB approximation for \((\tau, p'_c|n, p_c)\), we evaluate the above derivative semiclassically using 3.6. We begin by replacing the operator \(|P|\) with the value \(\sqrt{2E(n, p_c) - 2V(\tau)}\). Next, note that \(Z\) is peaked about a classical turning point in a Gaussian manner with a width \(\lambda\). Since \(\lambda \gg 1/\sqrt{E(n, p_c)}\), we may use our WKB approximation to evaluate the contribution to 3.6 from the region between the turning points and we may also approximate \(|\alpha_1 \exp(\text{i}f(\tau)) + \alpha_2 \exp(-\text{i}f(\tau))|^2\) by its average value (which is 2). Here, \(f(\tau) = \int_{\tau_0}^{\tau} \sqrt{2E(n, p_c) - 2V(q)}\).

Because the exact wavefunction will decay exponentially beyond \(q_0(E)\), we multiply 3.5 by \(\theta(q - q_0(E))\). Since \(q_0(E)\) is the center of the Gaussian support of \(Z\), the integral over \(q\) contributes a total value of 1/2, canceling the 2 from the oscillatory terms. It thus follows that

\[
\frac{\partial E(n, p_c)}{\partial p_c} = \frac{\epsilon C^2(n, p_c)}{2\pi} \tag{3.8}
\]

to lowest order in \(\epsilon\) and \(\hbar\). Distributing 3.8 among the pendulum and counter factors and moving an \(\hbar\) between them, the states of interest satisfy \(|n, p_c) = |n; p_c) \otimes |p_c)\) where

\[
\langle q_c|p_c) = \frac{e^{iq_c p_c/\hbar}}{\sqrt{\epsilon}} \tag{3.9}
\]

and

\[
\langle \tau|n; p_c) = \frac{\alpha_1}{\sqrt{2\pi \hbar(2E(n, p_c) - 2V(\tau))^{1/2}}} e^{i \int_{\tau_0}^{\tau} \sqrt{2E(n, p_c) - 2V(\tau)}/\hbar \ dq} + \frac{\alpha_2}{\sqrt{2\pi \hbar(2E(n, p_c) - 2V(\tau))^{1/2}}} e^{-i \int_{\tau_0}^{\tau} \sqrt{2E(n, p_c) - 2V(\tau)}/\hbar \ dq} \tag{3.10}
\]

for \(\langle q_c|q'_c) = \delta(q_c - q'_c)\).

Note that we have taken the normalization of \(|p_c)\) to be \(\epsilon\)-dependent and that, in fact, 3.9 diverges at \(\epsilon = 0\). This is because \(\epsilon\) is just the amount that the counter advances after

---

\(^4\)This assumes that \(V\) increases to the left at \(q_0(E)\). If \(V\) has two turning points at \(E\), this follows from our definitions. If \(V\) has only one turning point at \(E\), we may use \(\theta(q_0(E) - q)\) if \(V\) increases to the right of \(q_0(E)\).
each swing of the pendulum and we will always read our clock when the counter is in some
window of width $\epsilon$ using the operators $3.4$. Thus, we will be interested only in matrix
elements which are proportional to $\langle p_c'|\chi_a|p_c \rangle$. For future reference, we note that this is
$$
\langle p_c'|\chi_a|p_c \rangle = e^{i(a+1/2)\epsilon(p_c-p_c')/\hbar} \left[ \frac{2\hbar \sin[(p_c-p_c')\epsilon/2]}{\epsilon(p_c-p_c')} \right].
$$
(3.11)
which takes the value 1 at $p_c' = p_c$ and decays away from $p_c = p_c'$ with a width of $2/\hbar\epsilon$.

Furthermore, we will typically evaluate such matrix elements for $p_c$ and $p_c'$ for which there
are corresponding $n$ and $n'$ such that $E(n,p_c) = E(n',p_c')$. Thus, if the energy levels of $H_p$
are separated by a spacing $\Delta E$, we have $(p_c - p_c')\epsilon \sim \Delta E/C^2$ from $3.8$. But semiclassically
we expect $C \sim \sqrt{\hbar}/L$, where $L$ is the distance between the turning points of $V$. Thus, the
function in brackets is typically evaluated for $\Delta p_c/\hbar \sim \Delta E L^2/\hbar^2$. For a smooth potential
with $V(q)$ unbounded from above, this expression becomes large as we move to high energy
levels (see appendix A). Thus, for this case we have $\epsilon \Delta p_c \hbar \gg 1$. For our purposes then,
$$
\langle p_c'|\chi_a|p_c \rangle = \delta_{p_c,p_c'},
$$
(3.12)
where $3.12$ contains the Kronecker delta function.

IV. THE PROPERTIES OF TIME

In this section, we derive the external time formalism from the almost local formalism in
the ideal clock limit. We address the external time formalism in three stages, first showing
that the operators follow a unitary law of evolution in $[IV.A]$, then showing in $[IV.B]$ that
the operators have the usual algebraic structure, and finally deducing in $[IV.C]$ the explicit
map between $\mathcal{H}_{phys}$ and the Hilbert space $\mathcal{H}_{ext}$ of external time quantum mechanics for the
non-clock degrees of freedom. Note that, by construction, $\mathcal{H}_1$ is isomorphic to $\mathcal{H}_{ext}$. Along
the way, we will show that ‘conservation of existence,’ and thus conservation of probability
in the external time framework follows from unitarity and the algebraic structure.
A. Unitarity

The first topic we address in our derivation of the external time formalism is the issue of unitarity. We will show below that, in the context of our approximations, we have

\[ A_{\tau+\delta} = e^{i\delta P} A_{\tau} e^{-i\delta P} \]  

(4.1)

when \( A = A(Q^1_1, P^2_j) \) is built from the non-clock degrees of freedom. Since, as was noted in section III A, \( P_\tau \) defines a symmetric operator on \( \mathcal{H}_{\text{phys}} \), it follows that 4.1 describes unitary evolution in the clock time \( \tau \).

In a small digression, we note that there is an interesting case for which the evolution in clock time is exactly unitary but is not exactly of the form 4.1. This is the case in which the clock momentum \( P \) is conserved (\( [P, H] = 0 \)) so that \( P(t) = P \) is independent of \( t \) and defines a physical operator. Using this property, a brief study of 3.4 shows that \( A_{\tau+\delta} = e^{i\delta P} A_{\tau} e^{-i\delta P} \) and we have exact unitary evolution. However, \( P \) and \( P_\tau = P \mathbb{1}_\tau \) are different operators unless we have \( \mathbb{1}_\tau = \mathbb{1} \) which, as appendix B describes, is not the case for interesting such systems. Note also that \( i\hbar \frac{\partial}{\partial \tau} \mathbb{1}_\tau = [\mathbb{1}_\tau, P] \) may not vanish. This highlights the difference between unitarity of evolution and the property \( \frac{\partial}{\partial \tau} \mathbb{1}_\tau = 0 \) which we shall call ‘conservation of existence’ and which is related to the more familiar conservation of probability in external time quantum mechanics (see IV C). Unitarity of evolution is associated with invariance of the system under shifting the zero of clock time. On the other hand, conservation of existence in some state is associated with the fact that, in that state, the pendulum moves through not only the value \( \tau \), but also all nearby values of \( q \). Due to geodesic incompleteness, this will fail to hold, for example, when a clock that measures metric-defined proper time finds itself near a spacetime singularity. However, both unitarity and conservation of existence will follow in our ‘ideal clock limit’.

We now proceed to derive 4.1 using the scheme introduced in III. Our strategy will be to compare matrix elements of \([A_\tau, P_\tau]\) with those of \(-i\hbar \frac{\partial}{\partial \tau} A_\tau\) and to show that they agree to within the desired accuracy.
The first step in this process will be to evaluate matrix elements of $P_\tau$ in a convenient basis. To this end, consider the states $|n, p_c, E_1, a\rangle = |n, p_c\rangle \otimes |E_1, a\rangle$ where $\langle E_1, a|E_1', a'\rangle = \delta_{a,a'}\delta^{(2)}(E_1, E_1')$ and $\delta^{(2)}$ is either a Dirac or a Kronecker delta-function as is appropriate to the spectrum of $H_1$. As before, the physical ‘subspace’ is constructed from the states for which $E_1 = -E(n, p_c)$.

For later convenience, we consider not only $P_\tau$, but also the operators $(P^m)_\tau$ for $m \in \{0\} \cup \mathbb{Z}^+$ defined by $3.4$ with $A = P^m$. Thus, using $2.6$ and the above basis of states we have

$$
(P^m)_\tau | n, p_c(n, -E), E, a\rangle = \pi i \sum_{n', E', a'} |n', p_c(n', -E'), E', a'\rangle 
\times \langle n', p_c(n', -E'), E', a'\rangle_\theta(P) \chi_0 \{P^m, [P^2/2, \partial_\tau (Q - \tau)]\} \theta(P) | n, p_c(n, -E), E, a\rangle \tag{4.2}
$$

up to terms of order $\epsilon p_c$ where $p_c(n, -E)$ is the value of $p_c$ for which $-E = E(n, p_c)$. Note that our choice of normalization $3.7$ was crucial in deriving $4.2$. But, using $3.12$ and the fact that the clock coordinates and momenta commute with the canonical variables of the other degrees of freedom, this becomes

$$
P^m_\tau | n, p_c(n, -E), E, a\rangle = \pi i |n, p_c(n, -E), E, a\rangle
\times \langle n, p_c(n, -E)\rangle_\theta(P) \{P^m, [P^2/2, \theta(Q - \tau)]\} \theta(P) | n, p_c(n, -E)\rangle \tag{4.3}
$$

Let $\Pi_\tau = |\tau\rangle \langle \tau|$ so that the commutator in $4.3$ becomes $[P^2/2, \theta(Q - \tau)] = -\frac{i}{\hbar} \{P, \Pi_\tau\}\hbar$. Thus, we need to evaluate the matrix elements $\langle \tau | P^k \theta(P) | n, p_c\rangle$ for $0 \leq k \leq m$.

To do so, we again consider $3.10$ and note that, to the desired accuracy, the projection $\theta(P)$ acting on $|n, p_c\rangle$ yields just the first term ($\langle \tau | n, p_c \rangle_+$) in $3.10$ and the projection $\theta(-P)$ yields the second ($\langle \tau | n, p_c \rangle_-$). This follows since the negative (and zero) frequency components of $|n, p_c\rangle_+$ are given by Fourier transform of the wavefunction $\langle \tau | n, p_c \rangle_+$ and

\[\text{[Footnote: Recall that $3.12$ is valid only when $V$ is unbounded from above. This assumption is not actually needed to derive that $4.4$ holds in the form $\langle \phi | -i\hbar \frac{\partial}{\partial t} A_\tau | \psi \rangle = \langle \phi | [P_\tau, A_\tau] | \psi \rangle$ for states with $\epsilon p_c \ll 1$ as the cross terms removed by using $3.12$ will be irrelevant in this case. However, in the interests of a simpler presentation, we use $3.12$ here.]}\]
the integral which defines this transform may be divided into two parts. The first part comes from the integral over the region between the turning points and, as it is given by the Fourier transform of 3.10 in this region, vanishes faster than any power of \( E \) by the Riemann-Lebesgue theorem. The other contribution comes from the region within a distance of order \( \lambda_0 = \hbar / \sqrt{E} \) of the turning points (the contribution from outside the turning points is negligible). In this region, the magnitude of the wavefunction may be estimated from the value of 3.10 a distance \( \lambda_0 \) from the turning point and is roughly \( \sqrt{\lambda_0 / \hbar} \). Thus, this term is of order \( \lambda_0^{3/2} / \hbar \). In contrast, the largest positive frequency component is of order \( \sqrt{\lambda_0 L / \hbar} \), where \( L \) is the distance between the turning points. Thus, the negative and zero frequency parts of \(| n; p_c \rangle_+ \) are of order \( \lambda_0 / L \sim 1/n \) which is negligible when \( n \) is large enough that we may take \( \tau \) far from a turning point.

Now, to lowest order in \( \hbar \), we compute \( \langle \tau | P^k \theta(P) | n; p_c \rangle = (2E(n, p_c) - 2V(\tau))^{k/2} \langle \tau | n; p_c \rangle_+ \) from 3.10. Since \( |\langle \tau | n; p_c \rangle_+|^2 = 1/(2\pi \hbar [2E(n, p_c) - 2V(\tau)]^{1/2}) \), it follows that

\[
P^m_\tau | n, p_c, -E(n, p_c), a \rangle = (2E(n, p_c) - 2V(\tau))^{m/2} | n, p_c, -E(n, p_c), a \rangle
\] (4.4)

to the accuracy we consider. Note that this holds even when \( m = 0 \). Thus, the existence operator \( \mathbb{1}_\tau \) is the identity to this approximation and, in particular, \( \frac{\partial}{\partial \tau} \mathbb{1}_\tau = 0 \) up to terms of order \( \hbar \) so that we have derived ‘conservation of existence.’

Such results are no surprise; they are exactly what should be expected from a semiclassical approximation to a system of the form 3.2. In fact, there have been attempts to quantize reparametrization invariant systems beginning with an equation of this form \[27, 29\]. That this equation can be derived from the basic formalism of \[1\] is another verification that this approach is physically reasonable. Also, working within this formalism, we see that the issue of ‘what happens when the argument of the square root becomes negative’ is a non-physical issue. In this regime, the approximations considered are not valid and, from general arguments, we have no reason to expect that our pendulum should function well as a clock.
We may now use the result \[4.4\] to compute matrix elements of \([P_\tau, A_\tau]\):

\[
\langle n', p'c, -E(n', p'_c), a'|[P_\tau, A_\tau]|n, p_c, -E(n, p_c), a\rangle_{\text{phys}} = (\sqrt{2 E(n', p'_c) - 2 V(\tau)} - \sqrt{2 E(n, p_c) - 2 V(\tau)}) 
\times \langle n', p'c, -E(n', p'_c), a'|A_\tau|n, p_c, -E(n, p_c), a\rangle_{\text{phys}}
\]

\text{(4.5)}

to lowest order in \(\hbar\). The subscripts \(\text{phys}\) serve as reminders of the Hilbert space in which the inner products are to be taken. We note that there is some subtlety in concluding the above result as the \textit{absolute} accuracy of \[4.5\] is higher than that of the expression for \(\langle \phi|A_\tau P_\tau|\psi\rangle\) or \(\langle \phi|P_\tau A_\tau|\psi\rangle\). The point is that, because this expression is a commutator, we expect that \[4.5\] will be of the form \(h f(P, Q)\), whose commutator with \(A_\tau\) will be subleading, of order \(h^2\).

For comparison, we now compute matrix elements of the time derivative \(-i\hbar \frac{\partial}{\partial \tau} A_\tau\). Applying the same general formalism and making use of the tensor product nature of our basis states we find

\[
-i\hbar \langle n', p'_c, -E(n', p'_c), a'|\frac{\partial}{\partial \tau} A_\tau|n, p_c, -E(n, p_c), a\rangle_{\text{phys}}
= i\pi \hbar^2 \frac{\partial}{\partial \tau} \langle n'; p'_c|\theta(P)\{P, \Pi_\tau\}|n; p_c\rangle \langle -E(n', p'_c), a'|A| - E(n, p_c), a\rangle.
\]

\text{(4.6)}

Using \[3.10\], this agrees with \[4.3\] to leading order in \(\hbar\). When \(V(Q) = 0\), this result and \[4.4\] are exact since there are no corrections to the WKB approximation and the counter is irrelevant so that we may set \(\epsilon p_c = 0\). Thus, within our approximation scheme we have \([P_\tau, A_\tau] = -i\hbar \frac{\partial}{\partial \tau} A_\tau\) on \(\mathcal{H}_{\text{phys}}\), verifying \[4.4\]. Since \(P_\tau\) is symmetric on \(\mathcal{H}_{\text{aux}}\), we have shown that evolution in the parameter \(\tau\) is unitary, modulo issues of convergence of the integrals defining \(P_\tau\) and \(A_\tau\) and of the domain of definition of \(P_\tau\). This is the desired result.

Note that while the same calculation \[4.4\] which showed the unitarity of evolution allowed us to derive conservation of existence \((\frac{\partial}{\partial \tau} \mathbb{I}_\tau = 0)\), this result does \textit{not} follow directly from unitarity. This is because unitarity by itself does not provide a way to compute the commutator \([P_\tau, \mathbb{I}_\tau]\). We again recall (see appendix \[\text{B}\]) that for a homogeneous comoving fleet of ideal metric clocks in a homogeneous cosmology, unitarity may hold exactly but we will not, in general, have conservation of existence.
B. The Equal Time Algebra

We now turn to the second feature of external time quantum mechanics and show that it, too, follows from our formalism. Here, we consider the algebraic structure of the operators $A_\tau$ and, in particular, show that if $Q_i^\perp$ and $P_j^\perp$ are the canonical coordinates and momenta of the non-clock degrees of freedom, then $(Q_i^\perp)_\tau$ and $(P_j^\perp)_\tau$ satisfy the canonical commutation relations to lowest order in $\hbar$. In general, we verify that $\langle \phi | [A_\tau, B_\tau] | \psi \rangle = \langle \phi | ([A, B])_\tau | \psi \rangle (1 + \mathcal{O}(\hbar) + \mathcal{O}(\epsilon_{pc}))$ for $A = A(Q_i^\perp, (P_j^\perp)_\tau)$ and $B = B(Q_i^\perp, P_j^\perp)$ and any two states $|\phi\rangle$ and $|\psi\rangle$ that satisfy the requirements of our approximation scheme. From 3.4 we also have $(A + B)_\tau = A_\tau + B_\tau$, and we will see that the argument given below is also sufficient to derive $\langle \phi | (AB)_\tau | \psi \rangle = \langle \phi | A_\tau B_\tau | \psi \rangle$ to lowest order in $\hbar$. Thus, to this accuracy the map $\tau : A \mapsto A_\tau$ preserves the entire equal time algebraic structure of external time quantum mechanics. Together with the results of [V.A], this in fact implies preservation to this order of the algebraic structure associated with all Heisenberg operators and the Hamiltonian $\sqrt{-2H_1 - 2V(\tau)}$.

Below, we will once again see that the presence of the ‘counter’ is critical. That this should be so can be seen even in the calculation of the classical Poisson Brackets and, in fact, the calculation below is essentially the same as the classical one. This is because we work in the semiclassical approximation only to lowest order in $\hbar$ – the same order to which the ordering of factors may be ignored. Nonetheless, the calculation below is presented in quantum form to show that all of the $\mathcal{O}(\hbar)$ corrections come from ignoring the ordering of clock variables only. Operators associated with the other degrees of freedom are treated in a fully quantum manner so that we expect the terms ignored to be negligible in the ideal clock limit.

It will be easiest to compute the commutator of $A_\tau$ with $B_\tau$ in the nonphysical space $\mathcal{H}_{aux}$ and then take this result over to the physical space. Thus, we must calculate the integrals

$$[A_\tau, B_\tau] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \left[ \theta(P(t)) A(t) \chi(t) \frac{\partial}{\partial \theta} \theta(Q(t) - \tau) \theta(P(t)) \right],$$
\[ \theta(P(t'))A(t')\chi_0(t')\frac{\partial}{\partial t'}\theta(Q(t') - \tau)\theta(P(t')) \]  

(4.7)

where \( A(t) = A(Q_0^t(t), P_j^t(t)) \) and \( B(t) = B(Q_0^t(t), P_j^t(t)) \) are independent of the clock variables. We will not attempt to explicitly expand the commutator in [4.7] but, instead, we simply note that the derivation law \([AB, C] = A[B, C] + [A, C]B\) can be used to express this commutator as a sum of four kinds of terms, depending on whether the \( \theta(P), \chi_0, \) and \( \theta(Q - \tau) \) factors appear inside or outside the commutator. We then treat terms of each type separately.

We first consider the term in which both \( \theta(P(t))\chi_0(t)\frac{\partial}{\partial t}\theta(Q(t) - \tau) \) and \( \theta(P(t'))\chi_0(t')\frac{\partial}{\partial t'}\theta(Q(t') - \tau) \) appear outside of the commutator. For this term, we use the semiclassical approximation to write \( \frac{\partial}{\partial t}\theta(Q(t) - \tau) = N(t)P(t)\Pi_r(t) + O(h) \) and similarly for \( t' \). Again ignoring the commutator of clock variables, move \( \Pi_r(t)\theta(P(t))\chi_0(t)N(t')P(t')\Pi_r(t')\chi_0(t') \) to the right. These operators will now act directly on a state \( |\psi\rangle \) in which the clock behaves semiclassically. Thus, we consider an expression of the form

\[ \langle \tau, q_c, q_\perp | \theta(P(t))\chi_0(t)N(t')P(t')\Pi_r(t')\chi_0(t')\theta(P(t))|\psi\rangle \]

(4.8)

where \( |\tau, q_c, q_\perp\rangle = |\tau\rangle \otimes |q_c\rangle \otimes |q_\perp\rangle \) and \( |q_\perp\rangle \) denotes an arbitrary basis of \( \mathcal{H}_1 \). We now use the fact that a semiclassical expectation value of the above form may be evaluated by considering the values of the corresponding classical quantities along a classical path for which \( q(t) = \tau \). Note that along any such path for which \( p(t), p(t') > 0 \) and both \( q_c(t) \) and \( q_c(t') \) lie in the interval \([0, \epsilon]\), we have \( N(t')p(t')\delta(q(t') - \tau) = \delta(t - t') \). Thus, we may replace the corresponding set of operators by this expression and perform the integral over \( t' \). The remaining commutator is just \([A(t)\theta(P(t)), B(t)\theta(P(t))] = \theta(P(t))[A, B](t)\) so that this term yields just the desired result: \([A, B]_r \).

\[\text{---}\]

\(^6\)Strictly speaking, this is true only when the lapse is chosen to be everywhere positive. That the actual result is equivalent to this for arbitrary \( n(t) \) follows from the fact that the operator \([A_r, B_r]\) is independent of the choice of \( n(t) \), though it may be derived directly as well.
Clearly, we must show that all other terms generated are negligible. We will now address type II terms, where the commutator is of the form \( [\frac{\partial}{\partial t}\theta(Q(t) - \tau), A(t')] \) for some \( A(t') \) and where \( \chi_0(t') \) appears outside the commutator. Using our semiclassical approximation, we may replace this commutator with

\[
\frac{\partial}{\partial t}(\delta(Q(t) - \tau)[Q(t), A(t')])
\]

and integrate by parts with respect to \( t \). The factor of \( \chi_0(t)\delta(Q(t) - \tau) \) guarantees that the boundary terms vanish. We now reorganize the terms as above, with, perhaps, \( \chi_0(t) \) being replaced by \( \frac{\partial}{\partial t}\chi_0(t) \approx \frac{\partial}{\partial t}Q_c(t)(\delta(Q_c(t)) - \delta(Q_c(t) - \epsilon)) \) or \( \theta(P(t)) \) being replaced by \( \frac{\partial}{\partial t}\theta(P(t)) \approx \frac{\partial}{\partial t}P(t)\delta(P(t)) \) in some terms due to the action of \( \frac{\partial}{\partial t} \) in the integration by parts. Even with such replacements, for \( n(t) > 0 \) we have \( \frac{\partial}{\partial t}\theta(q(t') - \tau) = \delta(t - t') \) on all relevant classical solutions with \( q(t) = \tau \). Thus, we may again replace the corresponding operator with \( \delta(t - t') \) so that the commutator is evaluated at equal times. Note that the only factors which may be present on the right side of the commutator which do not commute with the \( Q(t) \) on the left are the \( \theta(P(t)) \). But \( [Q(t), \theta(P(t))] = i\hbar\delta(P(t)) \) and \( \langle \tau|\delta(P)|\psi \rangle \) is as small as the negative frequency components of our WKB states; that is, it is negligible by the argument of section [IV A].

Terms of the third type are just those terms obtained from type II terms by interchanging \( t \) and \( t' \) (which may be handled in the analogous way) and the term containing \( [\frac{\partial}{\partial t}\theta(Q(t) - \tau), \frac{\partial}{\partial t}\theta(Q(t') - \tau)] \). This term is shown to be negligible by integrating by parts with respect to both \( t \) and \( t' \) and using arguments much like those above to set \( t = t' \), whereupon the commutator vanishes.

To deal with the remaining terms, we note that

\[
[\chi_0(t), \mathcal{B}] = [Q_c(t), \mathcal{B}](\delta(Q_c(t) - \delta(Q_c(t) - \epsilon))(1 + \mathcal{O}(\hbar)) \quad (4.10)
\]

and

\[
[\theta(P(t)), \mathcal{B}] = [P(t), \mathcal{B}](\delta(P(t))(1 + \mathcal{O}(\hbar)). \quad (4.11)
\]
The corresponding terms then vanish by the assumption that $|\psi\rangle$ has negligible component for which both $q = \tau$ and $q_c = 0$ or $q_c = \epsilon$ and by the argument of IV A that the zero and negative frequency components of $|\psi\rangle_+$ are negligible.

We thus find that, when acting on a state satisfying our requirements, we have $[A\tau, B\tau] = ([A, B])\tau$ to leading order in $\hbar$, as expected. In particular, $(Q_{\perp}\tau)$ and $(P_{\perp}\tau)$ satisfy the canonical commutation relations. We emphasize that the terms ignored come from commutators of clock variables so that they may be expected to be small whenever the clock, but not necessarily the entire system, behaves semiclassically. While it does not follow from the above argument, the results of section IVB of [1] show that when $V(Q) = 0$ and $H_1 = \sum g^{ij} P_{\perp i} P_{\perp h}$ for any $g^{ij} \in \mathbb{R}$, the $(Q_{\perp}\tau)$ and $(P_{\perp}\tau)$ are just the usual positive frequency Newton-Wigner operators and so satisfy the canonical commutation relations \emph{exactly}. This is a consequence of the particular operator ordering chosen in 3.4, even though the choice of ordering does not effect the above semiclassical derivation. Finally, we note that the arguments applied to the type I terms above can also be used to show that $A\tau B\tau = (AB)\tau$ to leading order in $\hbar$ so that, to this order, the full algebra associated with external time quantum mechanics has been reproduced.

In passing, we note that had we not explicitly shown ‘conservation of existence’ in the previous subsection, the results of IV A combined with the above \emph{would} have been sufficient to derive this feature. That is, conservation of existence follows from the presence of a reasonable generator of time translations and from the usual algebraic structure of the theory. The derivation proceeds as follows. Note that if the operator $A(t)$ is independent of clock variables and is a constant of motion, then $A\tau = A1\tau$. Thus, $[(\sqrt{-2H_1 - 2V(\tau)})_\tau = (-2H_1 - 2V(\tau))^{1/2}1\tau]$. But this is just $P_{\tau}1\tau = P_{\tau}$ by the above. We may calculate the commutator of the existence operator with the time translation generator as $[P_{\tau}, 1\tau] = [(\sqrt{-2H_1 - 2V(\tau)})_\tau, 1\tau]$ and, since $[\sqrt{-2H_1 - 2V(\tau)}, 1] = 0$, this expression vanishes by the result just derived.
C. The Map on States and the Wheeler-DeWitt Equation

We now complete our derivation of external time quantum mechanics by constructing the explicit map between our physical states and states in the space $\mathcal{H}_{\text{ext}} \cong \mathcal{H}_1$. To begin, we consider the expectation value of an operator $A_\tau$ in some state $|\psi\rangle$ that satisfies the requirements of our approximation scheme.

Rewriting the usual commutator $[H, \theta(Q(t) - \tau)]$ as $-i\hbar \sqrt{P(t)} \Pi_\tau(t) \sqrt{P(t)}$ up to $O(\hbar^2)$ corrections, we have

$$\langle \psi | A_\tau | \psi \rangle_{\text{phys}} = 2\pi \hbar \langle \psi | \theta(P) \sqrt{P} \chi_0 A \sqrt{P} \theta(P) | \psi \rangle_{\text{aux}}$$

where all of the operators on the right hand side are evaluated at the parameter time $t = 0$ and the subscripts $\text{phys}$ and $\text{aux}$ serve as reminders of the appropriate Hilbert space in which each inner product is to be taken.

Expressing our states as $|\psi\rangle = \sum_{n,E,a} \psi(n,E,a) |n,p_c(n,-E),E,a\rangle$, equation 4.12 may be written as $\text{Tr}[A \rho(\tau)]$ where $\rho(\tau)$ is a density matrix associated with the Hilbert space $\mathcal{H}_1$ and is defined by:

$$\langle E',a'|\rho(\tau)|E,a\rangle = \sum_{n,n'} \psi^*(n',E',a') \psi(n,E,a) \langle n';p_c(n',-E')|\theta(P)\sqrt{P}|\tau\rangle \times \langle \tau|\sqrt{P}\theta(P)|n;p_c(n,-E)\rangle \langle p_c|\chi_0|p'_c\rangle. \quad (4.13)$$

Note that the $\tau$ dependence of 4.13 is exactly that of a Schrödinger picture density matrix that evolves in time under the Hamiltonian $\sqrt{-2H_1 - 2V(\tau)}$ since $-i\hbar \frac{\partial \rho}{\partial \tau} = [\rho(\tau), \sqrt{-2H_1 - 2V(\tau)}]$. Thus, 4.13 defines a map from states in $\mathcal{H}_{\text{phys}}$ to density matrices in the Schrödinger picture of external time quantum mechanics. In fact, 4.13 would describe a pure state except for the factor $\langle p_c|\chi_0|p'_c\rangle$.

Recall, however, that we consider the limit where the coupling to the counter is small $(p_c \epsilon/\hbar \ll 1)$ so that the bands produced by coupling the counter to the pendulum are well separated. In particular, we consider only those states $|\psi\rangle$ for which the support of $\psi(n,E,a)$ is concentrated on values of $n$ and $E$ for which $|p_c(n,-E)| \leq \beta$ for some $\beta \epsilon/\hbar \ll 1$. Since
\[ \langle p_c | x_0 | p' \rangle \] is a distribution of unit height and width \(2\hbar/\epsilon\), this factor is essentially one in 4.13 when \(\psi(n, E, a)\) is nonzero and \(\rho(\tau)\) effectively describes a pure state.

For our states then, it is useful to consider the projections

\[ \eta : \mathcal{H}_{aux} \to \mathcal{H}_p \otimes \mathcal{H}_1 \]

\[ |\psi\rangle \mapsto \sum_{n,E,a} \psi(n, E, a) |n, E, a\rangle \tag{4.14} \]

and

\[ \sigma_\tau : \mathcal{H}_p \otimes \mathcal{H}_1 \to \mathcal{H}_1 \]

\[ |\phi\rangle = \int d\tau' \sum_{q_\perp} \phi(\tau', q_\perp) |\tau', q_\perp\rangle \mapsto \sum_{q_\perp} \phi(\tau, q_\perp) |q_\perp\rangle \tag{4.15} \]

where \(|\tau, q_\perp\rangle = |\tau\rangle \otimes |q_\perp\rangle\) and \(|q_\perp\rangle\) is an arbitrary basis of \(\mathcal{H}_1\). We may then write \(\rho(\tau) = |\psi; \tau\rangle \langle \psi; \tau|\) where

\[ |\psi; \tau\rangle = \sigma_\tau \theta(P) \sqrt{P} \eta |\psi\rangle \tag{4.16} \]

so that we have derived \(\langle \psi | A_\tau | \psi \rangle = \langle \psi; \tau | A | \psi; \tau \rangle\). Similarly, for two such states \(|\phi\rangle\) and \(|\psi\rangle\), we have

\[ \langle \phi | A_\tau | \psi \rangle = \langle \phi; \tau | A | \psi; \tau \rangle. \tag{4.17} \]

The state \(|\psi; \tau\rangle\) satisfies

\[ -i\hbar \frac{\partial}{\partial \tau} |\psi; \tau\rangle = \sqrt{-2H_1 - 2V(\tau)} |\psi; \tau\rangle \tag{4.18} \]

so that we may interpret \(|\psi; \tau\rangle\) as a Schrödinger picture state in \(\mathcal{H}_{ext} \cong \mathcal{H}_1\) with Hamiltonian \(\sqrt{-2H_1 - 2V(\tau)}\). Note that we arrived at a Schrödinger picture description due to the implicit mapping of the operator \(A_\tau\) to the operator \(A(t = 0)\) in 4.12 and 4.17. Had we used the Heisenberg picture operator \(A_H(t = \tau)\) associated with \(A\) by the Hamiltonian \(\sqrt{-2H_1 - 2V(\tau)}\), we would have arrived at a Heisenberg picture description.

Note that the map 4.16 takes a familiar form. In particular, for \(A = \mathbb{1}\), 4.17 is just

\[ \langle \phi | \mathbb{1}_\tau | \psi \rangle_{phys} = (\sigma_\tau \theta(P) \eta(P) |\phi\rangle, \sqrt{-2H_1 - 2V(\tau)} \sigma_\tau \theta(P) \eta |\psi\rangle) \]
\[
\frac{i\hbar}{2} \left( \frac{\partial}{\partial \tau} \sigma_\tau \theta(P) \eta |\phi\rangle, \sigma_\tau \theta(P) \eta |\psi\rangle \right)_{H_1} - \frac{i\hbar}{2} \left( \sigma_\tau \theta(P) \eta |\phi\rangle, \frac{\partial}{\partial \tau} \sigma_\tau \theta(P) \eta |\psi\rangle \right)_{H_1} \quad (4.19)
\]

and, since \( \mathbb{I}_\tau = \mathbb{I} \) in our approximation, we see that up to constant factors the inner product on \( H_{phys} \) in the ideal clock limit is the positive frequency part of the familiar Klein-Gordon inner product\(^7\). Note, however, that the vector field \( \frac{\partial}{\partial \tau} \) may be either spacelike or timelike with respect to the metric defined by the kinetic terms in the constraint. Similarly, the operators \( A_\tau \) become the positive frequency Newton-Wigner operators in the ideal clock limit and, in this limit, our approach will agree with that of Wald \[^{30}\]. In particular, in contrast to the suggestions of \[^{31,32}\] it follows that in this scheme the square of the ‘wavefunction’ \( \langle q_\perp, \tau |\eta |\psi\rangle^2 \) may not be interpreted as a probability density for ‘the value of \( q_\perp \) when the clock reads \( \tau \).’ Instead, this interpretation belongs to the quantity \( \langle q_\perp |\sigma_\tau \sqrt{\theta(P) \eta} |\psi\rangle^2 \).

A few comments on exact results are now in order. From section IVB of \[^{1}\], it follows that any corrections to \( (4.19) \) vanish when \( V(Q) = 0 \). This is surprising from our derivation since we have completely ignored the effects of the commutator \([P, \Pi_\tau]\). However, when \( V(Q) = \tau \) the WKB approximation is exact and \( P \theta(P) |\psi\rangle = \sqrt{-H_1} \theta(P) |\psi\rangle \). Thus, when \([A, H_1] = 0 \), we have \( \langle \phi |\theta(P) A[P, \Pi_\tau] \theta(P) |\psi\rangle = 0 \) so that \( (4.17) \) is exact. In particular, this follows for \( A = \mathbb{I} \). When \( H_1 = \sum g_{ij} P_i^\perp P_j^\perp \), direct calculation as in section IVB of \[^{1}\] show that \( (4.17) \) holds exactly for \( A = Q_i^\perp \) as well.

Finally, note that \( (4.19) \) may also be written as

\[
\langle \phi |\Pi_\tau |\psi\rangle_{phys} = \langle \phi; \tau |\psi; \tau \rangle_{H_1}. \quad (4.20)
\]

Differentiating both sides with respect to \( \tau \), we see that conservation of probability (that is, of the right hand side) is an explicit consequence of conservation of existence on the left.

\[^7\]The author would like to thank Jim Hartle for bringing this result to his attention in the case where the potential \( V(\tau) \) has no critical points and is asymptotically constant.
V. DISCUSSION

We have now completed our derivation of the three defining properties of external time quantum mechanics. Recall that the entire formalism followed using only the approximations of III that we have an ideal limit of an almost ideal clock as described in III A. In particular, in contrast to standard derivations [8–12] of evolution through the use of the WKB approximation, our assumptions do not imply that the clock is large or massive so that the non-clock degrees of freedom may, in this approach, represent more than a small perturbation and the clock need not be in a state with nearly zero energy. In addition, the complete formalism of III has allowed a more extensive treatment than was previously possible – we were able to address the issue of unitary evolution in the physical inner product and to derive the complete map from the physical Hilbert space to the corresponding Hilbert space of external time quantum mechanics.

That such a derivation was possible may be considered a surprise in light of the results of [33,34] which demonstrate bounds on the extent to which ‘realistic clocks’ (including the free particle) may approximate the ideal clock of 3.1. Note however that the point made in [33,34] is that dispersion effects limit the extent to which the readings of physical clocks may correspond to proper time. Thus, from the above results and the facts that our approximations are exact for a system of free particles and that IV A and 4.19 are exact whenever the clock is a free particle, we may conclude that the use of a clock which

\[\text{Note that, despite the introduction of an inner product in, for example, } [9,10] \text{ for the external time system, the complete map from, in this case, solutions of the Wheeler-DeWitt equation to } \mathcal{H}_{ext} \text{ was not provided. Instead, this method can only define the required map for a collection of states with a common semiclassical prefactor due to the possible addition of a constant to } S \text{ of [3]} \text{ and the ambiguity in the normalization of the factor } C \text{ of [10] which are natural consequences of the lack of a physical inner product. However, within such a class of states, 4.19 agrees with [3,10] up to an overall normalization.} \]
measures proper time in a state-independent way is not fundamental to a derivation of unitary evolution, the canonical commutation relations, or the well-known Hilbert space structure.

However, despite arriving at the expected formalism, the final result was not quantum mechanics in the usual form. Indeed, where we would expect to find $-H_1$ as the Hamiltonian in (4.18), we instead find $\sqrt{-2H_1 - 2V(\tau)}$. This is a result of the above mentioned dispersion. Note that for $A = A(Q^i, P^j)$, we have $[-H_1, A] = \frac{1}{2}\{\sqrt{-2H_1 - 2V(\tau)}, [\sqrt{-2H_1 - 2V(\tau)}, A]\}$ so that, if the state $|\psi\rangle$ has a reasonably well defined value $p$ of the pendulum momentum $P$, this is just $\langle\psi|[-H_1, A]|\psi\rangle = p\langle\psi|\sqrt{-2H_1 - 2V(\tau)}, A|\psi\rangle$. As described in [33,34], this is just the case in which our clock may be said to measure ($p$ times the) proper time. In practice, real clocks are only used to the extent that they measure proper time and thus agree with each other; that is, to the extent that the dispersion above may be neglected. Since every test of time evolution makes use of some sort of dispersive clock, we see that our results are in fact much stronger than what is required to derive

$$-i\hbar \frac{\partial}{\partial T}|\psi; pT\rangle = -H_1|\psi; pT\rangle,$$

(5.1)

where $T = \tau/p$ is the proper time elapsed since the clock read zero, to the extent that this is supported by observations.

It is worthwhile to point out three further interesting features of our derivation. First, note that our discussion required no concept of decoherence as in [11,12] between states (or parts of states) with different semiclassical prefactors. This was the case because the observables themselves impose the condition that the pendulum is moving to the right through the position $\tau$. Second, the signs of the kinetic terms in $H_1$ relative to $H_0$ have not been specified, so that our clock may represent a mechanical clock as in [8] or the scale factor of a cosmological model as in [10]. Third, despite the presence of $\sqrt{-2H_1 - 2V(\tau)}$ in (4.18) and the fact that $H_1$ may not be negative definite, we have no need to concern ourselves as in [28,29] with the negative eigenvalues of $-2H_1 - 2V(\tau)$. This is because [14] guarantees.
that any state with non-negligible components along the corresponding eigenvectors will not be in the image of the map 4.16.

Finally, we would like to return to the discussion of III A and to comment on the physical situations in which our ideal clock approximation should be expected to hold. Recall that our approximation scheme was designed to guarantee that our clock acts like an ideal one. However, this ideal clock records the passing of lapse-defined proper time, not metric-defined proper time. Had the clock instead measured metric-defined proper time (assuming that it is coupled to the description of the gravitational field used in [1] or [20]), our system would be described (see appendix B) by a constraint of the form

\[ 0 = e^{3\alpha} H_0 + H_1 \]  

(5.2)

instead of by 3.2. Here, \( \alpha = \frac{1}{8} \ln \det g \) and \( g \) is the 3-metric on a homogeneous slice of a homogeneous cosmological model. However, the metric-based and lapse-based notions of almost ideal clock will agree when \( e^{3\alpha} \) may be treated as an adiabatic factor. We estimate when this is the case by comparing the contributions coming from \( e^{3\alpha} \) and \( H_0 \) in a typical term of the parameter time derivative \( \frac{\partial}{\partial t} \) of \( e^{3\alpha} H_0 \). Consider, for example, \( \frac{\partial}{\partial t} \left( \frac{e^{3\alpha} P^2}{2} \right) = P^2 e^{3\alpha} \left( \frac{3\dot{\alpha}}{2} + \frac{\dot{P}}{P} \right) \) where a dot denotes \( \frac{\partial}{\partial t} \). Thus, the adiabatic approximation should hold when \( \frac{\dot{P}}{\dot{\alpha}} \gg P \). Using \( \hbar \dot{A} \propto [A, NH] \) and the fact that \( H \) is typically of a form (see [1,20]) such that \( [\alpha, H] = \hbar P \alpha \), this condition is

\[ \frac{V'(Q)}{P \alpha} e^{3\alpha} \gg P \]  

(5.3)

which we may expect to hold when \( e^{3\alpha} \) is large. Thus, these two notions of almost ideal clock agree for large spatial volumes. Inserting the appropriate dimensional factors, this condition says that the 3-volume \( V \) of a homogeneous slice should satisfy \( V \gg (\frac{3}{2}) \frac{m_p}{\ell_p M} \) where \( m_p \) and \( \ell_p \) are the Plank mass and length respectively while \( M \) is the mass of the clock (that is, we replace \( H_p \) by \( P^2/2M + V(Q) \)).

This is in fact the best that we can expect. Note that, on classical solutions in these models, when the spatial volume is small the clock is near the initial or final singularity.
Thus, even if the clock accurately measures metric-defined proper time all the way up to the singularity, the the position of the pendulum will change very little before the end (or beginning) of the universe is reached and conservation of existence should fail for a metric-based clock. In this way it follows that we should not expect the usual external time formalism to hold for a metric-based clock near a spacetime singularity.

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APPENDIX A: LEVEL SPACING FOR UNBOUNDED POTENTIALS

In this appendix we show that if $V$ is unbounded from above and of the type considered in section [III] then $\Delta E L^2 / \hbar^2$ diverges for large $n$, where $\Delta E$ is the spacing between the $n$-th and $(n+1)$-th energy level of the Hamiltonian $P^2/2 + V(Q)$ and $L$ is the distance between the classical turning points associated with energy $E$. As in the main text, we use the approximation scheme of [III A] to invoke semiclassical methods. We proceed by assuming that $D = \Delta E L^2 / \hbar^2$ is bounded and showing that $V$ must in turn be bounded from above.

First, recall that in the semiclassical limit the energy levels are determined by the Bohr-Sommerfeld quantization condition (see, for example, [35]) which is roughly:

$$L_n \sqrt{2E_n} = n\hbar. \quad (A1)$$

Equivalently, $E_n L_n^2 / \hbar^2 = n^2$. Thus,

$$\sqrt{2L^2} \Delta E / \hbar^2 + 2\sqrt{2E} \Delta L / \hbar^2 \approx 2n \to \infty. \quad (A2)$$

If $D$ is bounded by some constant $D_{\text{max}}$, [A2] implies that $EL \Delta L \sim n\hbar^2 / \sqrt{2} \sim 2L \sqrt{E} \hbar$ and that $\Delta L \sim \hbar / \sqrt{E}$. Since, however, $\Delta E \leq D_{\text{max}} \hbar^2 / L^2$, we have
\[ V' \sim \Delta E/\Delta L \leq D_{\text{max}} \hbar \sqrt{E}/2L^2 \]  

(A3)

so that for large \( L \), \( V \leq -\text{const}_1/L + \text{const}_2 \). But, from [11], \( V \) must increase with \( L \). We conclude that \( V \) is bounded above.

**APPENDIX B: A FLEET OF CLOCKS**

While nowhere have we considered full gravity in this work, the almost local formalism is directly applicable [1,20] to the homogeneous cosmological models of [17–19]. Thus, we now investigate the notion of metric-based clocks in such a model. In particular, we show in this appendix that a homogeneous comoving fleet of noninteracting such clocks in a homogeneous spacetime is described by a constraint of the form \( e^{3\alpha} P + H_1 \) and not of the form 3.1.

Let us consider a 3+1 spacetime containing a fleet of metric-based clocks which admits a foliation by a family of spacelike surfaces \( \Sigma_t \) that are homogeneous in the sense of [17] with regard to the metric variables and such that the clocks are evenly distributed with respect to the metric on each slice. Let us also assume that the clocks are comoving in the sense that the 4-velocity field \( U^\alpha \) of the clock fleet is proportional to the normal field \( n^\alpha \) of the family of slices \( \Sigma_t \).

Now such a fleet of clocks is described (see [36]) by the formalism for pressureless dust presented in [28]. Thus, from equations 3.14 and 3.19 of [28], the contribution of this system to the constraints associated with the ADM lapse and shift are

\[
H^D_a = PU_a \\
H^D_\perp = \sqrt{P^2 + g^{ab}H^D_a H^D_b}
\]  

(B1)

where \( a \) is an abstract index on the slice \( \Sigma \), \( g^{ab} \) is the metric on this slice, and \( P \) is the momentum conjugate to the clock reading \( T \). But, since \( U^\alpha \propto n^\alpha \), we have \( U_a = 0 = H^D_a \) and \( H^D_\perp = P \). Thus, when coupled to gravity the homogeneous system may be described by a single Hamiltonian constraint \( H_{\text{ADM}} = H^D_\perp + H^G_{\text{ADM}} \) associated with the ADM lapse. The form of \( H^G_{\text{ADM}} \) may be found in [17,19] and, in the absence of the clock, is not of a
form that is appropriate to the almost local formalism (see [1,20]). Even when the clock is present, such a constraint is inconvenient for the almost local approach because it does not naturally define a unique self-adjoint operator (see [27]). As discussed in [27], this is not merely a technical difficulty but is a reflection of the fact that the corresponding classical hamiltonian vector field is not complete; that is, that this constraint describes a system which forms singularities in finite proper time as defined by the lapse associated with the constraint. The problem may thus be eliminated by rescaling the lapse and constraint as in [1,20] to use instead

\[ 0 = H = e^{3\alpha}(P + H^G_{ADM}) \]  

(B2)

which does naturally define a self-adjoint operator on \( L^2(Q) \). The associated lapse is then \( N = e^{-3\alpha}N_{ADM} \).

On a typical integral curve of the classical hamiltonian vector field corresponding to (B2), our clock will take on only a limited range of values. Thus, we do not classically have conservation of existence \( (\partial_\tau \mathbb{1} = 0) \) or \( \mathbb{1}_\tau = \mathbb{1} \). In the quantum theory, we expect these to hold only to the extent that \( e^{3\alpha} \) may be treated as an adiabatic factor; nevertheless, the results of [V,A] show that evolution in the clock time is always exactly unitary.
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