Dual Solutions for Yang-Mills Field Theory in Minkowski Space

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Abstract
In this paper, we use the spherical symmetrical ansatz to construct a dual equation for the non-abelian gauge theory in Minkowski space. The symmetry in the solution function space is examined. The analytically continued instanton satisfies the dual equation but it assumes a more ansatz. However, it is not a solution of the reduced theory under spherical ansatz which has the MIT new solution. This suggests that both instanton and MIT solutions may not be the lowest energy solution. The relevant physics is reviewed.

1 Introduction

It is well known that the classical solutions of non-abelian gauge field theory have many important physics effects[1],[2]. Especially, the instanton solution is related to the tunnelling[3] between different topological vacuums. This tunneling can cause explicit violations of conservation laws by the anomalies[1]. As results of this, fermion number is non-conserved in the electroweak theory[2], the $\theta$-vacuum arises a strong-CP problem, and the chirality in QCD is violated[2]. However, all of these effects are understood by using the instanton as a tunnelling path in Euclidean space[3]. The main reason for this is that the instanton is only well-behaved and have a well defined topological identity in Euclidean space. The analytically continuation of instanton into Minkowski space makes the solution very singular. The first semiclassical calculation[2] shows that the tunnelling probability is exponentially suppressed and really small. However, recently some group find that when the temperature[5] or energy[6] becomes very high, the field configuration will pass over the barrier instead of tunnelling. This would make the probability very large. But as you know the instanton is a solution in the Euclidean space which always corresponds to the tunneling process. To know what is going on in real physics space time when the energy becomes bigger than the barrier between the the vacuums, we need to figure out the
actual path in Minkowski space. The classical solution in Minkowski space will take the role as the instanton in the tunnelling process.

Since BPST[4] found the instanton solution, there are several new solutions discovered [7][8][9][10][12]. Recently, MIT group find [11] a new numerical solution to the equations of motion in Minkowski space by taking the spherical ansatz and further restrictions. They found that their solution can have non-integer topological number which leads to the collapse of the periodic picture of vacuum structure. Moreover, V.V.Khoze[13] has studied the effect of non-integer topological gauge field configurations.

In this paper, we assume the spherical ansatz[11][12] and calculate the dual equations in Minkowski space. We find that there is a local rotation and translation symmetry between the parameter function which keep the dual solutions. This symmetry is shown to the residual gauge transformations which keep the spherical ansatz. By using the rotational symmetry, one can construct new solutions. The instanton is analytically continued to Minkowski space. It is shown that the Minkowski space version of instanton is satisfying the dual equations. To our surprise is that this solution is not satisfying the MIT equations which is the equations of motion in the spherical ansatz. However, by directly plugging the solution into the original field equations, we find it is a solution of the field equations. In section 2, we introduce the spherical ansatz and derived the dual equations. The symmetry of the dual solutions is investigated in section 3. In section 4, we study the solutions of the dual equations. Conclusions and discussions are in section 5.

2 The spherical ansatz and dual equations in the Minkowski space

In this section, we study the spherical ansatz[11,12] for $SU(2)$ gauge group. We follow the notations of MIT’s paper. The action for the pure $SU(2)$ Yang-Mills theory is:

$$S = -\frac{1}{2g^2}\int d^4x \text{tr}(F_{\mu\nu}F^{\mu\nu})$$

where

$$F_{\mu\nu} = F^a_{\mu\nu} \tau^a = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

is the field strength and $A_\mu = A^a_{\mu} \tau^a$. The space-time metric is given as $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. $\tau^a$, $a = 1, 2, 3$ are the Pauli matrices. The spherical ansatz is given in terms of the four functions $a_0, a_1, a_2, a_3$ by:

$$A_0(\vec{x}, t) = \frac{1}{2}a_0(r, t)\vec{r} \cdot \hat{x}$$
\[ A_i(\vec{x}, t) = \frac{1}{2} [a_1(r, t) e_i^3 + \alpha(r, t) e_i^1 + \frac{1 + \beta(r, t)}{r} e_i^2] \] (2)

where the matrix-valued functions \( \{e_i^k\} \) are defined as:

\[
\begin{align*}
  e_i^1 &= \tau_i - (\tau \cdot \hat{x}) \hat{x}_i \\
  e_i^2 &= \epsilon_{ijk} \hat{x}_j \tau_k \\
  e_i^3 &= \tau \cdot \hat{x} \hat{x}_i
\end{align*}
\] (3)

and where \( \hat{x} \) is a unit three-vector in the radial direction. Now in spherical ansatz, the field strength takes the following form:

\[
\begin{align*}
  F_{ij} &= \frac{1}{2} \left\{ -\left(\alpha' + a_1 \beta\right) (\tau_i \hat{x}_j - \tau_j \hat{x}_i) + \left[ \frac{\left(\beta' - a_1 \alpha\right)}{r} + \frac{\alpha^2 - 2(1 + \beta)}{r^2} \right] \epsilon_{ijk} \tau_k \\
  &\quad + \frac{(1 + \beta)^2}{r^2} \epsilon_{ijk} \tau_k \right\} \\
  F_{0i} &= \frac{1}{2} \left\{ \dot{\alpha} + a_0 \beta \right\} e_i^1 + \frac{\dot{\beta} - \alpha a_0}{r} e_i^2 + (\dot{a}_1 - a_0') e_i^3 \right\} \\
\end{align*}
\] (4)

\[
\begin{align*}
  F_{ij} &= \frac{1}{2} \left\{ -\left(\alpha' + a_1 \beta\right) (\tau_i \hat{x}_j - \tau_j \hat{x}_i) + \left[ \frac{\left(\beta' - a_1 \alpha\right)}{r} + \frac{\alpha^2 - 2(1 + \beta)}{r^2} \right] \epsilon_{ijk} \tau_k \\
  &\quad + \frac{(1 + \beta)^2}{r^2} \epsilon_{ijk} \tau_k \right\} \\
\end{align*}
\] (5)

where \( \alpha' = \frac{\partial \alpha}{\partial r} \), and \( \dot{\alpha} = \frac{\partial \alpha}{\partial t} \), etc. It is well known that

\[ D_\mu \tilde{F}^{\mu \nu} = 0 \] (6)

This is the Bianchi identity in gauge theory. So if

\[ F^{\mu \nu} = \pm \tilde{F}^{\mu \nu} \] (7)

one can immediately get

\[ D_\mu F^{\mu \nu} = \pm D_\mu \tilde{F}^{\mu \nu} \equiv 0 \] (8)

which is the equation of motion. So, \( F^{\mu \nu} \) is the solution of the field equation. By using this fact, we require:

\[ F^{\mu \nu} = \pm \tilde{F}^{\mu \nu} \] (9)

to get the first order differential equations. Now in Minkowski space, the dual of \( F^{\mu \nu} \) is given by:

\[ \tilde{F}_{\mu \nu} = \frac{i}{2} \varepsilon_{\mu \nu}^{\rho \sigma} F_{\rho \sigma} \] (10)

(10) becomes

\[ F_{\mu \nu} = \pm \frac{i}{2} \varepsilon_{\mu \nu}^{\rho \sigma} F_{\rho \sigma} \]
This is equivalent to the following equations

\[ F_{0i} = \pm \frac{i}{2} \epsilon_{ijk} F_{jk} \]

In the parameter of \((\alpha, \beta, a_0, a_1)\), the dual equations is given as:

\[
\begin{align*}
\dot{\alpha} + a_0 \beta &= \mp i(\beta' - a_1 \alpha) \\
\dot{\beta} - \alpha a_0 &= \pm i(\alpha' + a_1 \beta) \\
\dot{a}_1 - a_0' &= \pm i(\frac{\alpha^2 + \beta^2 - 1}{r^2})
\end{align*}
\]

(11)

This is from \(F_{0i} = \pm \frac{i}{2} \epsilon_{ijk} F_{jk}\). The Euclidean version of (11) was constructed by E. Witten[12].

3 symmetries of the dual equations

To keep the spherical symmetry, the only residual gauge transformation is as follows[11]

\[ U(\vec{x}, t) = \exp(i\theta(r, t) \frac{\tau \cdot \hat{x}}{2}) \]

(12)

which is actually a abelian \(U(1)\) local transformation. Under this transformation, the parameters are transformed as

\[ A'_\mu = U A_\mu U^\dagger + iU \partial_\mu U^\dagger \]

(13)

which is

\[
\begin{align*}
a'_\mu &= a_\mu + \partial_\mu \theta(r, t) \\
\chi' &= \exp(i\theta(r, t)) \chi
\end{align*}
\]

(14)

where \(\chi = \alpha + i\beta\). Under this gauge transformations, the dual equation (11) is transformed as:

\[ F'_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \]

(15)

with \(F'_{\mu\nu} = UF_{\mu\nu} U^\dagger\). So both spherical ansatz and the dual equations are preserved under (13).

Now suppose we have a group of functions \((\alpha, \beta, a_0, a_1)\) which satisfy equations (11), from (14) or by directly plugging into (11), one can prove

\[
\begin{align*}
\alpha' &= \cos \theta \alpha - \sin \theta \beta \\
\beta' &= \sin \theta \alpha + \cos \theta \beta \\
a'_0 &= a_0 + \dot{\theta}(r, t) \\
a'_1 &= a_1 + \theta'(r, t)
\end{align*}
\]

(16)
is also a solution of (11), here $\theta$ is an arbitrary local angle function. By using this symmetry, one can simplify the dual equation. For instance, we can work in the $A_0 = 0$ gauge, and we have a residual rotation with the $\theta$ function only depending on space coordinates. By using this rotation, one can further simplify the equation. If we are only interested in the static solution, we can rotate the $(\alpha, \beta)$ into $(\alpha, 0)$ or $(0, \beta)$. Furthermore, if $(\alpha, \beta, a_0, a_1)$ is the solution of the dual equations, then $(\alpha, \beta, -a_0, -a_1)$ is the, anti-dual solutions. Through (13), one also can construct new solutions in a certain gauge. Also, we know the conformal transformation keeps the dual structure[10,11], one can also use those transformations to make the new solution[11].

4 The solutions of the spherical dual equations

Now, by looking at the instanton solution in Euclidean space

$$A_\mu = \frac{-ix^2}{x^2 + \lambda^2} U \partial_\mu U^\dagger$$  \hspace{1cm} (17)

where $x^2 = \hat{x}^2 + x_0^2$. With the analytical continuation $x_0 \rightarrow it, A_0 \rightarrow iA_0$. Compared with the ansatz form, one can get

$$a_0 = \frac{-2ir}{r^2 - t^2 + \lambda^2}$$

$$a_1 = \frac{2it}{r^2 - t^2 + \lambda^2}$$

$$\alpha = \frac{2irt}{r^2 - t^2 + \lambda^2}$$

$$\beta = \frac{r^2 + t^2 - \lambda^2}{r^2 - t^2 + \lambda^2}$$  \hspace{1cm} (18)

By simplifying substituting this into the equation (11), one can see that it is the solution of (11) with the lower sign in the left of (11). We also directly plug (18) into the field equation to make sure it is a solution of the field equation:

$$D^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu} - i[A^\mu,F_{\mu\nu}] = 0$$  \hspace{1cm} (19)

However, to make us suprise, it is not the solution of MIT reduced second order equation which is supposed to be the field equation in the spherical ansatz. To look at this, we introduce the MIT methods as follows[11]:

By studying moving spherical shells of energy, MIT group find a new numerical solution of their reduced field equation. The action in the spherical
ansatz take the form:

\[
S = \frac{4\pi}{g^2} \int dt \int_0^\infty dr \left[ -\frac{1}{4} r^2 f_{\mu\nu} f^{\mu\nu} - (D_\mu \chi)^* (D^\mu \chi) \right. \\
\left. - \frac{1}{2r^2} (|\chi|^2 - 1)^2 \right] \tag{20}
\]

where \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \) \( \mu, \nu = (t, r), \chi = \alpha + i\beta, D_\mu \chi = (\partial_\mu - i a_\mu) \chi. \) The metric in the (1 + 1) dimension space is \( \bar{\eta}_{\mu\nu} = \text{diag}(-1, 1). \) From (a), one can immediately construct the reduced (1 + 1) dimensional field equations by \( \delta S = 0 \) which are

\[
\begin{align*}
\partial_\mu \partial (\partial_\mu a_\nu) &= \frac{\partial L}{\partial a_\nu} \\
\partial_\mu \frac{\partial L}{\partial (\partial_\mu \chi)} &= \frac{\partial L}{\partial \chi}
\end{align*}
\tag{21}
\]

where \( S = \int d^4x L. \) And the results are the following second order differential equations[12]:

\[
-\partial^\mu (r^2 f_{\mu\nu}) = i [(D_\nu \chi)^* \chi - \chi^* D_\nu \chi] \\
[-D^2 + \frac{1}{r^2} (|\chi|^2 - 1)] \chi = 0 \tag{22}
\]

By solving the (22) numerically, MIT found a new solution which can have noninteger topological numbers.

Here, we substitute the solution (18) into (22) and find the solution does not hold the equations. This means the equation (22) is not equivalent to the original field equation. Mathematically, this is understandable: when you take some certain ansatz of the solution, it means you put more restrictions on the equation, so the solution for original equations is not necessarily the solution of the restricted equations. In the Euclidean space, the more restriction means the higher energy of the solution. If that is the case in the Minkowski space, the solutions for the equation (22) will have higher energy than that corresponding solutions of the equations (19). Here, due to the singular of the solution (18), we have not prove this yet.

To look at this carefully, we write out (19) explicitly with the functions \( \alpha, \beta, a_1, a_2. \) First, \( \nu = 0: \)

\[
\frac{(\dot{a}_1 - a'_1)}{2} + \frac{a_1 - a_0'}{r} + \frac{\beta(\dot{a} + a_0 \beta)}{r^2} - \frac{\alpha (\dot{\beta} - a_0 \alpha)}{r^2} = 0 \tag{23}
\]

and \( \nu = i \)

\[
(\dot{\alpha} + a_1 \beta)' + a_1 (\beta' - a_1 \alpha) - \frac{\alpha (a^2 + \beta^2 - 1)}{r^2} - \frac{(\dot{a} + a_0 \beta)}{\dot{t}} - a_0 (\dot{\beta} - a_0 \alpha) = 0
\]
\[
(\beta' - a_1 \alpha)' - \frac{\beta(\beta^2 + \alpha^2 - 1)}{r^2} - \frac{\partial(\beta^2 - \alpha a_0)}{\partial t} + a_0(\dot{\alpha} + a_0 \beta) - a_1(\alpha' + a_1 \beta) = 0
\]

\[
- \beta(\alpha' + a_1 \beta) + \alpha(\beta' - a_1 \alpha) - \frac{\partial(\dot{a}_1 - a_0')}{2\partial t} = 0
\]

where we let the respective coefficients of \( \epsilon_i^j \) equal zero to get the above equations. Now it is an exercise to check the (11) satisfying the above equation. Also, we note that when the functions \( \alpha, \beta, a_0, a_1 \) are real, (23) and the last equation of (24) together can be written as the first equation of (22), and the first and the second of (24) can be written as the second equation. However, the solution (18) is not real. But the equation is appropriate for any case. Nonetheless, the equation (11) means the dual solutions are intrinsically not real but complex. So the solutions of (11) are not the solutions of (22), however they are always the solutions of (23) and (24).

Furthermore, the solution (18) make the second equation of (11) decoupled. It is:(18) is satisfying the following equations:

\[
\dot{\alpha} + a_0 \beta = i(\beta' - a_1 \alpha)
\]

\[
\dot{\beta} - a_0 \alpha = 0
\]

\[
\alpha' + a_1 \beta = 0
\]

\[
\dot{a}_1 - a_0' = -i\frac{(\alpha^2 + \beta^2 - 1)}{r^2}
\]

instead of (11). Obviously, (18) is not the most general solutions of (11). (18) assumes more restrictions than the spherical ansatz.

5 Conclusions and Discussions

By taking the spherical ansatz and assuming the dual relation between fields, we construct the dual equations in Minkowski space which claims the analytically continued instanton as its solutions. The residual symmetry which is a rotation and translation in the parameter function space can keep the solutions. The instanton is a restricted spherical dual solutions which assume more ansatz. This suggest the instanton may be not the lowest energy solutions. On the other hand, the solution to equation (11) is not necessarily the solutions of equation (22) which means the solution of (22) is not the lowest energy solutions either. If this is the case, the vacuum structure will have a fruitful picture which will have a great effect on the anomalies physics. We will try to find the general solution to equation (11), and investigate the Minkowski field configurations’ effect on the tunnelling process.
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