The Dirichlet and Neumann problems in Lipschitz and in $C^{1,1}$ domains. Abstract

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Abstract

The main purpose of this paper is to address some questions concerning boundary value problems related to the Laplacian and bi-Laplacian operators, set in the framework of classical $H^s$ Sobolev spaces on a bounded Lipschitz domain of $\mathbb{R}^N$. These questions are not new and a lot of work has been done in this direction by many authors using various techniques since the 80’s. If for regular domains almost everything is elucidated, it is not the case for Lipschitz ones and for $s$ of the form $s = k + 1/2$, with $k$ integer. It is well known that this framework is delicate. Even in these cases many results are well established but sometimes not satisfactory. Several questions remain posed. Our main goal through this work is on one hand to give some improvements to the theory and on another one by using techniques which do not require too intricate calculations. We also tried to obtain maximal regularity for the solutions and as far as we can optimality of the results.

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In this document, only the main results are announced. The full set of results, with their demonstrations, will be submitted shortly.

The purpose of this work is to study the Dirichlet and Neumann problems:

$$(\mathcal{L}_D) \quad -\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = g \quad \text{on } \Gamma,$$

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and
\[
(L_N) \quad - \Delta v = f \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial n} = h \quad \text{on } \Gamma,
\]
with data in some Sobolev spaces and the domain \( \Omega \) is only Lipschitz or sometimes of class \( C^{1,1} \). When \( g = 0 \) (respectively \( h = 0 \)), we denote this problem by \( (L_D^0) \) (respectively \( (L_N^0) \)) and when \( f = 0 \), we denote this problem by \( (L_D^H) \) (respectively \( (L_N^H) \)). These issues have been widely studied since the 1960’s. In [18], Lions and Magenes made a complete study for smooth domains and \( L^2 \) theory. Grisvard in [12] and Nečas [24] treated the case where \( \Omega \) is of class \( C^{r,1} \), with nonnegative integer \( r \). Grisvard [13], [14] was also interested in the case of polygons or polyhedra. We recall that one consequence of Calderon-Zygmund’s theory of singular integrals and boundary layer potential is that, for every \( f \in W^{m-2,p}(\Omega) \) and \( g \in W^{m-1/p,p}(\Gamma) \), the problem \( (L_D^i) \) has a unique solution \( u \in W^{m,p}(\Omega) \) when \( \Omega \) is of class \( C^{r,1} \) with \( r = \max\{1, m - 1\} \). If \( f \in W^{s-2,p}(\Omega) \) and \( g \in W^{s-1/p,p}(\Gamma) \) with \( s > 1/p \), then \( u \in W^{s,p}(\Omega) \) provided that \( \Omega \) is of class \( C^{r,1} \) with \( r = \max\{1, [s]\} \), where \([s]\) is the integer part of \( s \).

The 1980s saw many authors investing in the study of these problems when the domain \( \Omega \) is only Lipschitz, where the situation is completely different (see for instance [3], [4], [5], [8], [7], [15], [16], [26]). Since then and to this day, these questions are still of great interest and many works are devoted to them (see for example [6], [10], [11], [17], [19], [21], [23], [22], [20]).

Recall that when \( \Omega \) is of class \( C^1 \) and \( 1 < p < \infty \), for any \( f \in W^{1-1/p,p}(\Omega) \) and for any \( g \in W^{1-1/p,p}(\Gamma) \), Problem \( (P_D) \) has a unique solution \( u \in W^{1,p}(\Omega) \). In the 80’s, Nečas posed the question of solving the problem \( (P_D^0) \) with the homogeneous boundary condition \( g = 0 \) on Lipschitz domains, when the RHS \( f \in W^{1,p}(\Omega) \). The answer to this question is given in the paper of Jerison and Kenig [17] (see Theorem A), that we rewrite a little more precisely. If \( N \geq 2 \), then for any \( p > 2N/(N - 1) \), there is a Lipschitz domain \( \Omega \) and \( f \in C^\infty(\Omega) \) such that the solution \( u \) of Problem \( (P_D^0) \) with the homogeneous boundary condition \( g = 0 \) does not belong to \( W^{1,p}(\Omega) \). So, as consequence, we don’t have uniqueness in \( W^{1,p}(\Omega) \) for \( p < 2N/(N + 1) \). However, for any Lipschitz bounded domain \( \Omega \), there exists \( q > 2N/(N - 1) \), depending on \( \Omega \), such that if \( q' < p < q \), then the problem \( (P_D^0) \) has a unique solution \( u \in W^{1,p}_0(\Omega) \) satisfying the estimate

\[
\|u\|_{W^{1,p}(\Omega)} \leq C\|f\|_{W^{1-1/p}(\Omega)},
\]

(when \( \Omega \) is \( C^\infty \), we can take \( q = \infty \) as stated above). Observe that the
exponents \(2N/(N-1)\) and \(2N/(N+1)\), corresponding respectively to the limit cases for the existence and the uniqueness of solutions in \(W_0^{1,p}(\Omega)\), are conjugate. This is naturally due to the fact that the operator \(\Delta : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)\) is self-adjoint.

As an example of non-uniqueness, let us consider in the 2D case the following Lipschitz domain for \(1/2 < \alpha < 1\):

\[
\Omega = \{(r, \theta); 0 < r < 1, \quad 0 < \theta < \pi/\alpha\}.
\]

We can easily verify that the following function

\[
 u(r, \theta) = (r^{-\alpha} - r^\alpha)\sin(\alpha \theta)
\]

is harmonic in \(\Omega\) with \(u = 0\) on \(\Gamma\) and \(u \in W^{1,p}(\Omega)\) for any \(p < 2/(\alpha + 1)\). Remark that when \(\alpha\) is near from 1/2 then \(\Omega\) is close to the unit disc and \(2/(\alpha + 1)\) is close to 4/3. Note that the limit value 1/2 of \(\alpha\) corresponds to the case of the cracked disk, which is not a Lipschitz domain.

In this work, we want to investigate the case where data \(f, g\) and \(h\) are non smooth and the domain is only Lipschitz or of class \(C^{1,1}\). These questions are not new and a lot of work has been done in this direction by many authors using various techniques since the 80’s. If for regular domains almost every thing is elucidated, it is not the case for Lipschitz ones and for \(s\) of the form \(s = k + 1/2\), with \(k\) integer and corresponding to limit cases. It is well known that this framework is delicate. So, we are particularly interested in investigating here the maximal regularity in the limit cases. In this direction, some new information concerning the traces of functions belonging to \(L^2(\Omega)\) or to \(H^{1/2}(\Omega)\) and satisfying an adequate additional property are very useful. One of the ideas is to use the interpolation theory and the duality method. Also, we prove new Nečas’ properties under the assumption of data satisfying less restrictive conditions than usual.

In the rest of this introduction, we give our main results.

The first one concerns the \(H^2\)-regularity for Dirichlet problem for the Laplacian, which is a question that has been addressed by many mathematicians. As above for solutions in \(W_0^{1,p}(\Omega)\), we know that, for any \(s > 3/2\), there exist a Lipschitz domain \(\Omega\) and \(f \in C^{\infty}(\overline{\Omega})\) such that the solution \(u\) to the nonhomogeneous Dirichlet problem \((P_0^D)\) for the Laplacian does not belong to \(H^s(\Omega)\) (see [17] for example). In the case of polygonal or polyedral domain \(\Omega\), Grisvard in [13] and [14] gave for \(s = 2\) a necessary and sufficient condition to obtain the solution in \(H^2(\Omega)\). In the following theorem, we extend this result to the case of general bounded Lipschitz domains.
Theorem 0.1 (H²-Regularity). The operator
\[ \Delta : H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega) \perp H_{L^2}(\Omega) \]
is an isomorphism. In particular we have the following inequality: for any \( v \in H^2(\Omega) \cap H^1_0(\Omega) \),
\[ \|v\|_{H^2(\Omega)} \leq C \|\Delta v\|_{L^2(\Omega)}. \]

Independently of the above results, we can show that the following operators
\[ \Delta : H^2_0(\Omega) \longrightarrow L^2(\Omega) \perp H_{L^2}(\Omega) \quad \text{and} \quad \Delta : L^2(\Omega)_{|H_{L^2}(\Omega)} \longrightarrow H^{-2}(\Omega) \]
are isomorphisms. Here the space \( H_{L^2}(\Omega) \) (resp. \( H_{L^2}(\Omega) \)) denotes the subspace of harmonic functions in \( L^2(\Omega) \) (resp. harmonic functions in \( L^2(\Omega) \) vanishing on \( \Gamma \)). So thanks to an interpolation argument, we deduce the following: for any \( 0 < s < 1/2 \), the operator
\[ \Delta : H^{3/2+s}_0(\Omega) \longrightarrow H^{s-1/2}(\Omega) \perp H^{1/2-s}(\Omega) \]
is an isomorphism. Here the space \( H^{1/2-s}(\Omega) \) denotes the subspace of harmonic functions of \( H^{1/2-s}(\Omega) \).

We give a sense to the trace of \( vn \) for harmonic functions \( v \) (or more generally for functions with Laplacian in \( [H^{3/2}_0(\Omega)]' \)) belonging to \( L^2(\Omega) \). This property is a consequence of the density of \( \mathcal{D}(\Omega) \) in
\[ M(\Omega) = \left\{ v \in L^2(\Omega); \Delta v \in [H^{3/2}_0(\Omega)]' \right\}, \]
and the fact that if \( g_0 \in H^1(\Gamma) \) and \( g_1 \in L^2(\Gamma) \) verify the condition
\[ \nabla \tau g_0 + g_1 n \in H^{1/2}(\Gamma), \]
then there exists a function \( u \in H^2(\Omega) \) satisfying \( u = g_0 \) and \( \frac{\partial u}{\partial n} = g_1 \) on \( \Gamma \) with the estimate
\[ \|u\|_{H^2(\Omega)} \leq C \|\nabla \tau g_0 + g_1 n\|_{H^{1/2}(\Gamma)} \]
(see [11] if \( N = 2 \), [2] if \( N = 3 \) and [19] if \( N \geq 2 \)).

The questions of traces of functions belonging to Sobolev spaces are crucial in the study of boundary value problems. We know that if \( u \in H^s(\Omega) \) with \( s > 1/2 \) then the function \( u \) has a trace which belongs to \( H^{s-1/2}(\Gamma) \). Moreover, if \( u \in H^{1/2}(\Omega) \), in general this function \( u \) does not have a trace. In the following theorem, we see that a subtle additional condition on \( \nabla v \) allows then to obtain a trace for the function \( v \).
Theorem 0.2 (Trace Operator in $H^{1/2}(\Omega)$). i) The linear mapping $\gamma_0 : u \mapsto u|_\Gamma$ defined on $\mathcal{D}(\Omega)$ can be extended by continuity to a linear and continuous mapping, still denoted $\gamma_0$, from $E(\nabla; \Omega)$ into $L^2(\Gamma)$, where

$$E(\nabla; \Omega) = \left\{ v \in H^{1/2}(\Omega); \nabla v \in [H^{1/2}(\Omega)]' \right\}. $$

ii) The Kernel of $\gamma_0$ is equal to $H^{1/2}_0(\Omega)$. In particular, we have the following properties:

$$v \in H^{1/2}_0(\Omega) \implies v = 0 \text{ in } L^2(\Gamma)$$

and

$$v \in H^{3/2}_0(\Omega) \implies v = 0 \text{ in } H^1(\Gamma) \text{ and } \frac{\partial v}{\partial n} = 0 \text{ in } L^2(\Gamma). $$

It is well known that if $f \in L^2(\Omega)$, or even in $H^{-s}(\Omega)$ for any $s < 1/2$, then there exists a unique solution $u \in H^{3/2}_0(\Omega)$ to Problem (L$_0^0$). But these assumptions on $f$ are too strong as we can see below. In addition it would be interesting to characterize the range of the Laplacian operator from $H^{3/2}(\Omega) \cap H^1(\Omega)$ into $[H^{1/2}_0(\Omega)]'$. In [17] (see Theorem 0.4) the authors shown that it is not possible for the operator

$$\Delta : H^{3/2}(\Omega) \cap H^1(\Omega) \rightarrow [H^{1/2}_0(\Omega)]'. \quad (0.1)$$

to be an isomorphism, even if $\Omega$ is of class $C^1$. Their proof is based on the argument, which consists to say that if a harmonic function $v$ belongs to $H^{3/2}(\Omega)$, then its trace satisfies $v|_\Gamma \in H^1(\Gamma)$. However this argument is wrong (see Theorem 0.6). In the following theorem, we claim that the operator (0.1) is really an isomorphism and we give, as for Theorem 0.1 above, some versions for solutions in $H^{3/2}_0(\Omega)$ and in $H^{1/2}_0(\Omega)$.

Theorem 0.3 (Solutions in $H^{3/2}_0(\Omega)$). i) The operators

$$\Delta : H^{3/2}_0(\Omega) \rightarrow [H^{1/2}_0(\Omega)]' \text{ and } \Delta : H^{1/2}_0(\Omega) \rightarrow [H^{3/2}_0(\Omega)]'. \quad (0.2)$$

are isomorphisms.

ii) For any $f \in [H^{1/2}(\Omega)]'$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{H}^{1/2}(\Omega), \quad \langle f, \varphi \rangle = 0,$$

there exists a unique solution $u \in H^{3/2}_0(\Omega)$ such that $\Delta u = f$ in $\Omega$. In addition to the boundary Dirichlet condition $u = 0$, the normal derivative of
This solution satisfies $\frac{\partial u}{\partial n} = 0$.

iii) By duality, for any $f \in \left[H_{00}^{3/2}(\Omega)\right]'$, there exists a unique $u \in H^{1/2}(\Omega)$, unique up to an element of $H^{1/2}(\Omega)$, where

$$H^{1/2}(\Omega) = \left\{ v \in H^{1/2}(\Omega); \Delta v = 0 \text{ in } \Omega \right\}.$$ 

As above, using an interpolate argument, we can prove that for any $0 < s < 1/2$ the following operator

$$\Delta : H^{3/2+s}(\Omega) \cap H^1_0(\Omega) \rightarrow H^{s-1/2}(\Omega) \perp H^{1/2-s}_0(\Omega)$$

is an isomorphism. Here the space $H^{1/2-s}_0(\Omega)$ denotes the subspace of harmonic functions of $H^{1/2-s}(\Omega)$ vanishing on $\Gamma$.

**Remark 1.** As mentioned earlier, we know that the operator

$$\Delta : W^{1,p}_0(\Omega) \rightarrow W^{-1,p}(\Omega)$$

is an isomorphism when $p = 2N/(N - 1)$ for any $N \geq 2$, while this is not the case for this operator which is not onto for some Lipschitz domain when $p > 2N/(N - 1)$. In the case of $H^{s}_0(\Omega)$-Sobolev spaces, the corresponding "optimal" value, that is reached, to get a similar isomorphism, is equal to $3/2$. In view of the following Sobolev embeddings

$$H^{3/2}_0(\Omega) \hookrightarrow W^{1,2N/(N-1)}_0(\Omega) \quad \text{and} \quad \left[H^{1/2}_0(\Omega)\right]' \hookrightarrow W^{-1,2N/(N-1)}(\Omega),$$

the first isomorphism in (0.2) can be considered as a regularity result.

**Remark 2.** We investigate also the case of solutions in weighted Sobolev spaces. In particular, we prove that for any function $f$ satisfying $\sqrt{\rho}f \in L^2(\Omega)$, then the solution $u$ belongs to $H^{3/2}_0(\Omega)$ and given in Point i) of the previous theorem satisfies in addition that $\sqrt{\rho} \nabla^2 u \in L^2(\Omega)$. Observe that such a RHS $f$ satisfies the property: $f \in \left[H^{1/2}_0(\Omega)\right]'$. Here the function $\rho$ is the distance function to the boundary.

We give also existence results in the case of boundary data in $L^2(\Gamma)$ or in $H^1(\Gamma)$. Using harmonic analysis techniques, many authors have established similar results (see [15] and [17]). Our proofs, completely different, are essentially based on the first isomorphism given in Theorem 0.3 and the following variant of Nečas’ property:

$$u \in H_0^1(\Omega) \quad \text{with} \quad \Delta u \in \left[H^{1/2}(\Omega)\right]'$$
then
\[ \frac{\partial u}{\partial n} \in L^2(\Gamma). \]

**Theorem 0.4 (Homogeneous Problem in \( H^{1/2}(\Omega) \) and in \( H^{3/2}(\Omega) \)).**

i) For any \( g \in L^2(\Gamma) \), Problem \((\mathcal{L}_D^H)\) has a unique solution \( u \in H^{1/2}(\Omega) \). Moreover \( \sqrt{\rho} \nabla u \in L^2(\Omega) \) and we have the estimate
\[ ||u||_{H^{1/2}(\Omega)} + ||\sqrt{\rho} \nabla u||_{L^2(\Omega)} \leq C ||g||_{L^2(\Gamma)}. \]
The solution \( u \) verifies also the following property: for any positive integer \( k \)
\[ \rho^{k+1/2} \nabla^{k+1} u \in L^2(\Omega). \]

ii) For any \( g \in H^1(\Gamma) \), the problem \((\mathcal{L}_D^H)\) has a unique solution \( u \in H^{3/2}(\Omega) \). Moreover \( \sqrt{\rho} \nabla^2 u \in L^2(\Omega) \) and we have the estimate
\[ ||u||_{H^{3/2}(\Omega)} + ||\sqrt{\rho} \nabla^2 u||_{L^2(\Omega)} \leq C ||g||_{H^1(\Gamma)}. \]
The solution \( u \) verifies also the following property: for any positive integer \( k \)
\[ \rho^{k+1/2} \nabla^{k+2} u \in L^2(\Omega). \]

**Remark 3.** We prove similar results in the case where the domain is of class \( C^{1,1} \) and the Dirichlet boundary condition \( g \in H^2(\Gamma) \).

We then give extensions of the classical Nečas’ property, that will be very useful for the study of the homogeneous Neumann problem \((\mathcal{L}_N^H)\) and also for the Dirichlet-to-Neumann operator for the Laplacian. Usually, the assumption on \( \Delta u \) is stronger than the one we take here.

**Theorem 0.5 (Nečas Property).** Let
\[ u \in H^1(\Omega) \quad \text{with} \quad \Delta u \in [H^{1/2}(\Omega)]'. \]
i) If \( u \in H^1(\Gamma) \), then \( \frac{\partial u}{\partial n} \in L^2(\Gamma) \) and we have the following estimate
\[ \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} \leq C \left( \inf_{k \in \mathbb{R}} ||u + k||_{H^1(\Gamma)} + ||\Delta u||_{[H^{1/2}(\Omega)]'} \right). \]

ii) If \( \frac{\partial u}{\partial n} \in L^2(\Gamma) \), then \( u \in H^1(\Gamma) \) and we have the following estimate
\[ \inf_{k \in \mathbb{R}} ||u + k||_{H^1(\Gamma)} \leq C \left( \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + ||\Delta u||_{[H^{1/2}(\Omega)']'} \right). \]

iii) If \( u \in H^1(\Gamma) \) or \( \frac{\partial u}{\partial n} \in L^2(\Gamma) \), then \( u \in H^{3/2}(\Omega) \).

**Remark 4.** i) Using the properties of the Dirichlet-to-Neumann operator, we give different regularity results for the Neumann problem (\( \mathcal{L}_N \)).

ii) We give also similar Nečas’ properties for functions

\[ u \in H^2(\Omega) \quad \text{with} \quad \Delta u \in H^1(\Omega), \]

when the domain \( \Omega \) is of class \( C^{1,1} \).

In [5] (see also Corollary, Section 6 in [4]), using some specific properties of the distance to the boundary, the authors proved that if \( u \) is harmonic in \( \Omega \) and vanishes at some point \( x_0 \in \Omega \), then

\[ \int_{\Gamma} |u|^2 \leq C \int_{\Omega} \varrho |\nabla u|^2. \quad (0.3) \]

A priori, if the second integral converges, the function \( u \) does not satisfy the property: \( \nabla u \in [H^{1/2}(\Omega)]' \), but \( u \) only satisfies that \( \nabla u \in [H^{1/2}_{00}(\Omega)]' \). We give a counter example of (0.3).

**Theorem 0.6 (Counter example for Are Integral).** There is a Lipschitz domain \( \Omega \) and a harmonic function \( u \in H^{1/2}(\Omega) \) (resp. \( u \in H^{3/2}(\Omega) \)) satisfying

\[ \int_{\Omega} \varrho |\nabla u|^2 < +\infty \quad \text{and} \quad \int_{\Gamma} |u|^2 = +\infty \]

(resp.)

\[ \int_{\Omega} \varrho |\nabla^2 u|^2 < +\infty \quad \text{and} \quad u \notin H^1(\Gamma) \quad \text{or equivalently} \quad \frac{\partial u}{\partial n} \notin L^2(\Gamma). \]

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