Local centrally essential subalgebras of triangular algebras

O. V. Lyubimtseva and A. A. Tuganbaev

Nizhny Novgorod State University, Nizhny Novgorod, Russian Federation; National Research University 'MPEI', Lomonosov Moscow State University, Moscow, Russian Federation

ABSTRACT

We study local centrally essential subalgebras in the algebra of all upper triangular matrices over a field of characteristic $\neq 2$. It is proved that the algebras of upper triangular $3 \times 3$ or $4 \times 4$ matrices have only commutative local centrally essential subalgebras. Every algebra of upper triangular matrices of order exceeding 6 contains a non-commutative local centrally essential subalgebra.

1. Introduction

We only consider associative rings which are not necessarily unital.

1.1. Centrally essential rings

A ring $R$ is said to be centrally essential if either it is commutative or for any non-central element $a \in R$, there exist two non-zero central elements $x, y$ with $ax = y$. A ring $R$ with non-zero identity element is centrally essential if and only if the following condition (*) holds:

for any non-zero element $a \in R$, there exist two non-zero central elements $x, y \in R$ with $ax = y$. Any non-zero ring with zero multiplication is centrally essential but does not satisfy (*).\(^1\)

Centrally essential rings with non-zero identity element are studied in papers [1–7]. Every centrally essential semiprime ring with $1 \neq 0$ is commutative; see [1, Proposition 3.3]. In [1], examples of non-commutative group algebras over fields are given. For example, if $Q_8$ is the quaternion group of order 8, then its group algebra over the field of order 2 is a finite local non-commutative centrally essential ring of order 256. In addition, in [2] it is proved that the external algebra of a three-dimensional linear space over the field of order 3 is a finite non-commutative centrally essential ring, as well. In [4], there is an example of a centrally essential ring $R$ with $1 \neq 0$ such that the factor ring of $R$ with respect
to the prime radical is not a PI ring. Abelian groups with centrally essential endomorphism rings are considered in [8].

The main result of this paper is Theorem 1.1.

**Theorem 1.1:** For any field $F$ of characteristic $\neq 2$ and an arbitrary positive integer $n \geq 7$, there exists a local non-commutative centrally essential subalgebra of the algebra $T_n(F)$ of upper triangular $n \times n$ matrices.

**Remark 1.2:** For $n \geq 2$ and a field $F$, the complete matrix algebra $M_n(F)$ over $F$ and the algebra $T_n(F)$ of upper triangular matrices over $F$ are not centrally essential, since all idempotents of any centrally essential ring with $1 \neq 0$ are central by Markov and Tuganbaev [1, Lemma 2.3].

An algebra $A$ is said to be centrally essential if $A$ is a centrally essential ring. In this paper, we consider local centrally essential subalgebras of the algebra $T_n(F)$ of all upper triangular matrices, where $F$ is a field of characteristic $\neq 2$. In particular, such subalgebras are of interest, since, for $F = \mathbb{Q}$, they are quasi-endomorphism algebras of strongly indecomposable torsion-free Abelian groups of finite rank $n$. Quasi-endomorphism algebras of all such groups are local matrix subalgebras in algebra $M_n(\mathbb{Q})$ of all matrices of order $n$ over the field $\mathbb{Q}$; e.g. see [9, Chapter I, § 5]. We remark that the algebra $\mathbb{Q}E$ is the quasi-endomorphism algebra of a strongly indecomposable torsion-free Abelian group of prime rank $p$ if and only if $\mathbb{Q}E$ is isomorphic to a local subalgebra of $T_p(\mathbb{Q})$. Indeed, $\mathbb{Q}E/J(\mathbb{Q}E) \cong \mathbb{Q}$ in this case; see [10, Theorem 4.4.12], where $J(\mathbb{Q}E)$ is the Jacobson radical which is nilpotent, since $\mathbb{Q}E$ is Artinian. It follows from the Weddenburn-Malcev theorem that $\mathbb{Q}E \cong \mathbb{Q}E_p \oplus J(\mathbb{Q}E)$, where $E_p$ is the identity matrix. It is known that every nilpotent subalgebra of a full matrix algebra $M_n(F)$ over an arbitrary field $F$ is transformed by conjugation to a nil-triangular subalgebra; see [11, Chapter 2, Theorem 6]. Since diagonal matrices of a local matrix algebra have equal elements on the main diagonal, they are transformed to itself under conjugation. Consequently, the quasi-endomorphism algebras of such Abelian groups can be realized as matrix subalgebras if and only if these subalgebras are conjugated to some local subalgebra of $T_p(\mathbb{Q})$. The necessary information on Abelian groups can be found in [9,12].

Let $F$ be a field and $A$ a finite-dimensional algebra over $F$. An element $a$ of the algebra $A$ is said to be nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$. The minimal value of $n$ with this property is called the nilpotence index of the element $a$. An algebra $A$ is called a nil-algebra if every element is nilpotent. For a nil-algebra $A$, the maximal nilpotence index $\nu(A)$ of its elements is called the nil-index. A positive integer $k$ such that $A^k = (0)$ and $A^{k-1} \neq (0)$ is called the nilpotence index of the algebra $A$. If such an integer $k$ exists, then algebra is said to be of nilpotent index $k$.

For a field $F$ and an associative algebra $A$ over $F$, we denote by $Z(A)$ and $J(A)$ the center and the Jacobson radical of the algebra $A$, respectively. An algebra $A$ with 1 is said to be local if the factor algebra $A/J(A)$ is a division algebra. Further, $A$ denotes a local subalgebra in the algebra $T_n(F)$ and $N_n(F)$ denotes the subalgebra of nilpotent matrices in $A$ (i.e. the algebra of strictly upper triangular matrices). We note that any matrix $A \in A$ is
We denote by $E_{ij}$ the matrix unit, i.e. the matrix with 1 on the position $(i, j)$ and zeros on the remaining positions; $E_k$ denotes the identity $k \times k$ matrix. We denote by $\langle S \rangle$ the linear hull of a subset $S$ of some linear space.

2. Some general results

We recall that a ring $R$ is said to be semiprime if $R$ does not contain two non-zero ideals with zero multiplication. A ring is said to be reduced if it does not contain zero-square elements. The center of a semiprime ring is a reduced ring.

**Proposition 2.1:** If $R$ is a centrally essential ring and its center is a semiprime ring, then the ring $R$ is commutative.

**Proof:** This assertion follows from [13, Theorem 1.3(a)] in the case, where $R$ satisfies condition 1.1(*).

We consider the general case. Suppose that the ring $R$ is not commutative, i.e. there exist $x, y \in R$ such that $xy - yx \neq 0$. Since $R$ is a centrally essential ring and $Z(R)$ is a reduced ring, there exist $c, d \in Z(R)$ such that $d = (xy - yx)c \in Z(R) \setminus \{0\}$. We note that $xd \neq 0$; otherwise $d^2 = (xy - yx)cd = ((xd)y - y(xd))c = 0$, which is impossible. If $xd \notin Z(R)$, then we can repeat the proof of [13, Theorem 2.1(c)]. Namely, there exists an element $z \in Z(R)$ such that $xdz \in Z(R) \setminus \{0\}$. We consider the set $I = \{i \in Z(R) \mid ix \in Z(R)\}$. It is clear that $dz \in I$. Now assume that $di = 0$. Then $d(dz) = 0$, $(dz)^2 = 0$ and $dz = 0$; this is a contradiction. Therefore, $di \neq 0$ for some $i \in I$. However,

$$di = (xy - yx)ci = ((ix)y - y(ix))c = 0,$$

and we obtain a contradiction, as well.

We assume that $xd \in Z(R)$. Then $xdy \neq 0$, since otherwise $d^2 = 0$. In addition, $(xy)d = (yx)\overline{d}$. Therefore, $(xy - yx)d = 0$. However, then we have $d^2 = (xy - yx)cd = 0$; this is a contradiction. Thus, the ring $R$ is commutative. 

The following assertion expands Lemma 2.3 of [1] to the case of rings which do not necessarily have $1 \neq 0$; in addition, $1$ is replaced by an arbitrary idempotent $e$.

**Proposition 2.2:** In any centrally essential ring, the following condition holds:

$$\forall \ n \in \mathbb{N}, \ x_1, \ldots, x_n, y_1, \ldots, y_n, r, \quad e = e^2 \in R,$$

$$\begin{cases} x_1y_1 + \cdots + x_ny_n = e \\ x_1rey_1 + \cdots + x_nrey_n = 0 \end{cases} \Rightarrow re = 0. \quad (1)$$

In particular, all idempotents of a centrally essential ring are central.
Proof: We assume that $R$ is a centrally essential ring which satisfies condition (1), but $re \neq 0$. If $re \in Z(R)$, then

$$re = re^2 = re(x_1y_1 + \cdots + x_ny_n) = x_1rey_1 + \cdots + x_nrey_n = 0;$$

this is a contradiction.

Let $re \notin Z(R)$. Then there exist two elements $c, d \in Z(R)$ such that $cre = d \neq 0$. We note that $d = cre = (cre)e = de$. Therefore,

$$d = de = d(x_1y_1 + \cdots + x_ny_n) = c(x_1rey_1 + \cdots + x_nrey_n) = 0.$$ 

Now let $e = e^2 \notin Z(R)$. We have the relations $e \cdot e = e$ and $c(re)e = 0$, where $r = x - ex$ for any $x \in R$. Consequently, $re = 0$ and $xe = exe$. We note that condition (1) remains true if we replace $re$ by $er$. In this case, $ex = exe$. Therefore, all idempotents of the ring $R$ are central. 

Proposition 2.3: Let $A$ be a local subalgebra of $T_n(F)$ with Jacobson radical $J(A)$. The algebra $A$ is centrally essential if and only if $J(A)$ is a centrally essential algebra. 

Proof: Let us have a matrix $A \in J(A)$ with $A \notin Z(J(A))$. Since $A$ is a centrally essential algebra, there exists a matrix $B \in Z(A)$ such that $0 \neq AB = C \in Z(A)$. Since $J(A)$ is an ideal, $C \in Z(J(A))$. If $B \notin J(A)$, then $A = CB^{-1} \in Z(J(A)))$; this contradicts to the choice of the matrix $A$.

Conversely, we have a decomposition $A = FE_n \oplus J(A)$. Since $FE_n \subset Z(A)$,

$$Z(J(A)) \subset Z(A). \hspace{1cm} (2)$$ 

If $0 \neq A \in A$ and $A \in Z(A)$, then $0 \neq AE_n \in Z(A)$. Let $A \notin Z(A)$ and $A \in J(A)$. Then there exists a matrix $B \in Z(J(A))$ such that $0 \neq AB = C \in Z(J(A))$. It follows from relation (2) that $B \in Z(A)$ and $C \in Z(J(A))$.

Let $A \notin J(A)$. Then $A = A' + A''$, where $0 \neq A' \in FE_n, A'' \in J(A)$. If $A'' = 0$, then $A \in Z(A)$. Otherwise, $0 \neq A''B \in Z(J(A))$ for some matrix $B \in Z(J(A))$. Then

$$AB = A'B + A''B = BA' + BA'' = BA.$$ 

Since $A'B, A''B \in Z(J(A))$, we have $AB \in Z(J(A)) \subset Z(A)$. We also note that $AB \neq 0$, since the matrix $A$ is invertible. 

By considering Remark 1.2, we obtain the following corollary.

Corollary 2.4: The algebra $N_n(F)$ for $n \geq 3$ is not a centrally essential algebra.
Let $\mathcal{A}$ be a subalgebra of the algebra $N_n(\mathbb{F})$ of nilpotence index $n$. We assume that $\nu(A) = n$. Then there exists a matrix $A \in \mathcal{A}$ such that $A^{n-1} \neq 0$. We transform $A$ to the Jordan form,

$$A = E_{12} + E_{23} + \cdots + E_{(n-1)n},$$

and pass to the corresponding conjugated subalgebra $\mathcal{A}_c$. We denote by $C(A)$ the centralizer of the matrix $A$ in $\mathcal{A}_c$. Since the minimal polynomial of the matrix $A$ is equal to its characteristic polynomial, $C(A) = \mathbb{F}[A]$, where $\mathbb{F}[A]$ is the ring of all matrices which can be represented in the form $f(A)$, $f(x) \in \mathbb{F}[x]$; see [11, Chapter 1, Theorem 5]. For $B \in C(A)$, we have

$$B = f(A) = \alpha_0 E_n + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}.$$

In addition, $\alpha_0 = 0$, since the matrix $B$ is nilpotent.

**Lemma 2.5:** If $Z(\mathcal{A}_c) = C(A)$, then the algebra $\mathcal{A}_c$ is commutative.

**Proof:** Indeed, if $A' \notin C(A)$, then $AA' \neq A'A$. However, $A \in C(A) = Z(\mathcal{A}_c)$. This is a contradiction. $\blacksquare$

**Lemma 2.6:** Let $\mathcal{A}_c$ be a centrally essential algebra and $Z(\mathcal{A}_c) = \langle A^{n-1} \rangle$. Then the algebra $\mathcal{A}_c$ is commutative.

**Proof:** Indeed, if $\mathcal{A}_c$ is not commutative, then for the matrix $A' \notin Z(\mathcal{A}_c)$ we have $BA' = 0$ for any matrix $B \in Z(\mathcal{A}_c)$. $\blacksquare$

### 3. Nilpotent centrally essential subalgebras of algebras $N_3(\mathbb{F})$ and $N_4(\mathbb{F})$

In what follows, we assume that the ground field $\mathbb{F}$ is of characteristic $\neq 2$.

**Proposition 3.1:** Any centrally essential subalgebra of the algebra $N_3(\mathbb{F})$ is commutative.

**Proof:** Every matrix $A \in N_3(\mathbb{F})$ is of the form

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{A}$ be a non-commutative centrally essential subalgebra of the algebra $N_3(\mathbb{F})$ of nilpotence index 3. Then $\nu(\mathcal{A}) = 3$. Let a matrix $A \in \mathcal{A}$ be of nilpotence index 3. We transform $A$ to the Jordan form: $A = E_{12} + E_{23}$. Now, if $B \in C(A)$, then $B = \alpha_1 A + \alpha_2 A^2$. We note that $Z(\mathcal{A}_c) \subseteq C(A)$; in addition, $\nu(Z(\mathcal{A}_c)) = 3$ by Lemma 2.6. However, then $Z(\mathcal{A}_c) = C(A)$ and the algebra $\mathcal{A}_c$ is commutative by Lemma 2.5. This is a contradiction. $\blacksquare$

In particular, it follows from Proposition 3.1 that all centrally essential endomorphism rings of strongly indecomposable Abelian torsion-free groups of rank 3 are commutative; cf. [8, Example 3.4].
Proposition 3.2: Any centrally essential subalgebra \( A \) of the algebra \( N_4(\mathbb{F}) \) is commutative.

Proof: If the algebra \( A \) is of nilpotence index 2, then it is commutative. Let the nilpotence index of \( A \) be equal to 4. There exists \( A \in A \) such that \( A^3 \neq 0 \). Indeed, the algebra \( A \) contains three matrices \( S = (s_{ij}), T = (t_{ij}), P = (p_{ij}) \) with \( s_{12} \neq 0, t_{23} \neq 0, p_{34} \neq 0 \). Otherwise, the nilpotence index \( A \) is lower than 4. As the required matrix, we can take a matrix \( A = (a_{ij}) \) such that \( a_{i(i+1)} \neq 0, i = 1, 2, 3 \). We transform \( A \) to the Jordan form,

\[
A = E_{12} + E_{23} + E_{34},
\]

and pass to the corresponding conjugated subalgebra \( A_c \). For a matrix \( B \in C(A) \), we have

\[
B = \alpha_1 A + \alpha_2 A^2 + \alpha_3 A^3,
\]

where \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F} \). It follows from Lemmas 2.5 and 2.6, then \( Z(A_c) \neq C(A) \) and \( Z(A_c) \neq (A^3) \) provided the algebra \( A_c \) is not commutative. Then any matrix \( C \in Z(A_c) \) is of the form

\[
C = \begin{pmatrix}
0 & 0 & c_{13} & c_{14} \\
0 & 0 & 0 & c_{13} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since \( A_c \) is a centrally essential algebra, we have that for the non-zero matrix \( D \neq Z(A_c) \), there exists a matrix \( C \in Z(A_c) \) such that \( 0 \neq DC \in Z(A_c) \). Since the matrix \( D \) is nilpotent, \( \text{tr}D = 0 \). In addition, \( A_c \) is local; therefore all elements on the main diagonal of the matrix \( D \) are equal to zero. In this case, it is directly calculated that

\[
D = \begin{pmatrix}
0 & d_{12} & d_{13} & d_{14} \\
0 & 0 & d_{23} & d_{24} \\
0 & 0 & 0 & d_{12} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

If \( d_{12} = 0 \) and \( D \neq Z(A_c) \), then \( DC = 0 \) for any matrix \( C \in Z(A_c) \); this is a contradiction. Let \( d_{12} \neq 0 \) and \( DF \neq FD \) for some matrix \( F = (f_{ij}) \in A_c \). We find an element \( \lambda \in \mathbb{F} \) such that \( f_{12} = \lambda d_{12} \). We set \( G = \lambda D - F, G = (g_{ij}) \). Then

\[
FG = F(\lambda D - F) = \lambda FD - F^2,
\]

\[
GF = (\lambda D - F)F = \lambda DF - F^2.
\]

Therefore, \( G \notin Z(A_c) \) and \( g_{12} = 0 \). It follows from the obtained contradiction that the algebra \( A_c \) is commutative.

Let the nilpotence index of the algebra \( A \) be equal to 3. Then \( \nu(A) = 3 \), i.e. \( A \) contains a matrix \( A \) such that \( A^2 \neq 0 \). Indeed, let us assume the contrary, \( A^2 = 0 \) for all \( A \in A \). If \( A \notin Z(A) \), then \( 0 \neq AB \in Z(A) \) for some matrix \( B \in Z(A) \). Then

\[
(A + B)^2 = A^2 + 2AB + B^2 = 2AB = 0.
\]

Hence \( AB = 0 \). This is a contradiction.
We transform the matrix $A$ to the Jordan form,

$$A = E_{12} + E_{23}.$$ 

In the corresponding conjugated subalgebra $A_c$, the centralizer $C(A)$ consists of matrices $B$ of the form

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{12} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{43} & 0 \end{pmatrix};$$ (3)

see [11, Chapter 3, §1]. In addition, if $C \in Z(C(A))$, then we have

$$C = \begin{pmatrix} 0 & c_{12} & c_{13} & 0 \\ 0 & 0 & c_{12} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

Let $Z(A_c)$ be of nilpotence index 3. Then we can take a matrix from $Z(A_c)$ as the matrix $A$; see [11, Chapter 1, Proposition 5, Corollary]. In this case, all matrices from $A_c$ are contained in $C(A)$. Then $A_c$ consists of the matrices of the form (3) and the matrices from $Z(A_c)$ is form (4). If $B = (b_{ij}) \notin Z(A_c)$ and $b_{12} = 0$, then $BC = 0$ for all $C \in Z(A_c)$. Then $A_c$ is not a centrally essential algebra. Let $b_{12} \neq 0$ and $BD \neq DB$ for some matrix $D = (d_{ij}) \in A_c$. Let $d_{12} = \lambda b_{12}$ and $F = \lambda B - D, F = (f_{ij})$. Then $f_{12} = 0$ and $F \notin Z(A_c)$. This is a contradiction.

Let $Z(A_c)$ be of nilpotence index 2. Then for $C \in Z(A_c)$, we obtain

$$C = \begin{pmatrix} 0 & 0 & c_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

It follows from relation $AC = CA$ for $A \in A_c$ that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix}.$$ 

However, then $AC = 0$ for any matrix $C \in Z(A_c)$. Consequently, if $A_c$ is a centrally essential algebra, then $A_c$ is commutative.

4. The proof of Theorem 1.1

In [2, Proposition 2.5], it is proved that the external algebra $\Lambda(V)$ over a field $\mathbb{F}$ of characteristic $\neq 2$ is a centrally essential algebra if and only if the dimension of the space $V$ is odd. By considering the regular matrix representation of the algebra $\Lambda(V)$, we obtain that for an odd positive integer $n > 1$, there exists a non-commutative centrally essential subalgebra of the algebra $N_{2n}(\mathbb{F})$; also see [8, Example 3.5]. Therefore, the minimal order of matrices of a
non-commutative the external centrally essential algebra is equal to 8. In the next example, we construct a non-commutative centrally essential algebra of $7 \times 7$ matrices.

**Example 4.1:** We consider a subalgebra $\mathcal{A}$ of $N_7(F)$ consisting of matrices $A$ of the form

$$A = \begin{pmatrix}
0 & a & b & c & d & e & f \\
0 & 0 & 0 & b & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Let $A' \in \mathcal{A}$, $a' = a + 1$ and the remaining components of the matrix $A'$ coincide with the corresponding components of the matrix $A$. Then $AA' \neq A'A$ if $a \neq 0$ and $b \neq 0$. Therefore, the algebra $\mathcal{A}$ is not commutative. It is easy to see that $Z(\mathcal{A})$ contain matrices $B$ of the form

$$B = \begin{pmatrix}
0 & 0 & 0 & c & d & e & f \\
0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

If $0 \neq A \notin Z(\mathcal{A})$, then $0 \neq AB \in Z(\mathcal{A})$ for some matrix $B \in Z(\mathcal{A})$. Consequently, $\mathcal{A}$ is a centrally essential algebra.

**4.1. The completion of the proof of Theorem 1.1**

In $N_n(F)$, we consider the subalgebra $\mathcal{A}$ matrices $A$ of the form

$$A = \begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} & \ldots & a_{1n-2} & a_{1n-1} & a_{1n} \\
0 & 0 & 0 & a_{13} & 0 & \ldots & 0 & 0 & a_{1n-2} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & a_{1n-1} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & a_{12} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & a_{13} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}.$$

We remark that the algebra $\mathcal{A}$ is not commutative; also see Example 4.1. If $B \in Z(\mathcal{A})$, then

$$B = \begin{pmatrix}
0 & 0 & 0 & b_{14} & b_{15} & \ldots & b_{1n-2} & b_{1n-1} & b_{1n} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & b_{1n-2} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & b_{1n-1} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}.$$
For $A = (a_{ij}) \notin Z(A)$, we have $a_{12} \neq 0$, $a_{13} \neq 0$. Let $B = (b_{ij}) \in Z(A)$ and $b_{1n-2} = a_{12}$, $b_{1n-1} = a_{13}$. Then $0 \neq AB \in Z(A)$. Indeed, let $AB = C = (c_{ij})$, $BA = D = (d_{ij})$. Then $c_{ij} = d_{ij} = 0$ for all $i \neq 1, j \neq n$. In addition, $c_{1n} = d_{1n} = a_{12}^2 + a_{13}^2$. Therefore, $A$ is a centrally essential algebra.

It is known (e.g. see [14]) that every rational algebra of dimension $n$ can be realized as the quasi-endomorphism ring of torsion-free Abelian group of rank $n$. By considering [8, Proposition 3.1], we obtain the following corollary.

**Corollary 4.2:** For every positive integer $n > 6$, there exists an Abelian torsion-free group $A(n)$ of rank $n$ such that its endomorphism ring is a non-commutative centrally essential ring.

### 5. Remarks and open questions

It follows from Proposition 2.3 that for a centrally essential local ring $R$, the ring $J(R)$ is centrally essential. The converse is not always true. Indeed, let $R$ be the local ring of upper triangular matrices $2 \times 2$ over a division ring $D$ which is not a field. Then $J(R)$ is a commutative ring with zero multiplication. However, the ring $R$ is not centrally essential, since the ring $R/J(R)$ is not commutative; see [2, Proposition 3.3].

**Open question 5.1:** Is it true that for any centrally essential ring $R$, the Jacobson radical $J(R)$ is a centrally essential ring? $\blacksquare$

**Open question 5.2:** In Theorem 1.1, it is proved that for any positive integer $n > 6$, there exists a local non-commutative centrally essential subalgebra of the algebra $T_n(F)$ of nilpotence index 3. Is it true that there exists a positive integer $n$ such that $T_n(F)$ contains a local non-commutative centrally essential subalgebra of any nilpotence index $2 < k \leq n$? $\blacksquare$

**Open question 5.3:** Is it true that there exist local non-commutative centrally essential subalgebras of the algebra $T_n(F)$ for $n = 5$ and $n = 6$? $\blacksquare$

**Open question 5.4:** Is it true that for every positive integer $k > 2$, there exists a positive integer $n = n(k)$ such that the algebra $T_n(F)$ contains a local non-commutative centrally essential subalgebra of nilpotence index $k$? $\blacksquare$

### Notes

1. Not necessarily unital rings with $(\ast)$ are considered in [13].
2. See, for example, [15, Theorem 6.2.1].

### Disclosure statement

No potential conflict of interest was reported by the author(s).

### Funding

The work of O.V. Lyubimtsev is done under the State assignment No 0729-2020-0055. A.A. Tuganbaev is supported by Russian Scientific Foundation, project 16-11-10013P.
References

[1] Markov VT, Tuganbaev AA. Centrally essential group algebras. J Algebra. 2018;512(15):109–118.
[2] Markov VT, Tuganbaev AA. Centrally essential rings. Discrete Math Appl. 2019;29(3):189–194.
[3] Markov VT, Tuganbaev AA. Rings essential over their centers. Comm Algebra. 2019;47(4):1642–1649.
[4] Markov VT, Tuganbaev AA. Rings with polynomial identity and centrally essential rings. Beitr Algebra Geometrie/Contributions to Algebra and Geometry. 2019. doi:10.1007/s13366-019-00447-w
[5] Markov VT, Tuganbaev AA. Uniserial artinian centrally essential rings. Beitr Algebra Geometrie/Contributions to Algebra and Geometry. 2020;61(1):23–33.
[6] Markov VT, Tuganbaev AA. Uniserial noetherian centrally essential rings. Comm Algebra. 2020;48(1):149–153.
[7] Markov VT, Tuganbaev AA. Constructions of centrally essential rings. Comm Algebra. 2019. doi:10.1080/00927872.2019.1677698
[8] Lyubimtsev OV, Tuganbaev AA. Centrally essential endomorphism rings of abelian groups. Comm Algebra. Also see Archive: http://arxiv.org/abs/1910.01222. doi:10.1080/00927872.2019.1635611
[9] Krylov PA, Mikhalev AV, Tuganbaev AA. Endomorphism rings of Abelian groups. Dordrecht/Boston/London: Springer Netherlands (Kluwer); 2003.
[10] Faticoni T. Direct sum decompositions of Torsion-free finite rank groups. Boca Raton/London/New York: Taylor&Francis Group; 2007.
[11] Suprunenko DA. Commutative matrices. New York: Academic Press; 1968.
[12] Fuchs L. Abelian groups. 2015. Heidelberg/New York/Dordrecht/London: Springer monographs in mathematics.
[13] Markov VT, Tuganbaev AA. Centrally essential rings which are not necessarily unital or associative. Discrete Math Appl. 2019;29:215–218.
[14] Pierce RS, Vinsonhaler C. Realizing central division algebras. Pacific J Math. 1983;109(1):165–177.
[15] Drozd YA, Kirichenko VV. Finite dimensional algebras. Berlin Heidelberg: Springer-Verlag; 1994.