On transverse hyperplanes to self-similar Jordan arcs.

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Abstract

We consider self-similar Jordan arcs $\gamma$ in $\mathbb{R}^d$, different from a line segment and show that they cannot be projected to a line bijectively. Moreover, we show that the set of points $x \in \gamma$, for which there is a hyperplane, intersecting $\gamma$ at the point $x$ only, is nowhere dense in $\gamma$.

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1 Introduction.

The first examples of self-similar fractals which appeared in the beginning of XX century were the constructions of self-similar curves with predefined geometrical properties [10] [15]. Though the study of geometrical properties of self-similar curves is so close to historical origins of fractal geometry, some of their elementary geometric properties were established only in recent times.

For example, it was a common opinion that self-similar curves have no tangent at any of their points. But in 2005 A.Kravchenko [11] found that there are self-affine curves which are differentiable everywhere and therefore

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have a tangent at any of their points. In 2011 the problem of differentiability for self-affine curves with 2 generators and the problem of existence of tangent subspaces for self-similar sets found their exhaustive solution in the paper of Ch. Bandt and A. Kravchenko [3].

In this note we study the projections of self-similar Jordan arcs in $\mathbb{R}^d$ to the real line along families of parallel hyperplanes. Analysing the case when there is a bijective projection of a self-similar Jordan arc $\gamma$ to a straight line segment, we show that this is possible only when the arc is a straight line segment itself.

**Theorem 1** Let $\gamma$ be a self-similar Jordan arc in $\mathbb{R}^d$. Suppose there is such hyperplane $\sigma$, that for any $x \in \gamma$ the parallel copy of $\sigma$ passing through the point $x$ intersects $\gamma$ only once, then $\gamma$ is a straight line segment.

Really, we prove a much more general statement, in which transverse hyperplanes $\sigma(x)$ at different points $x$ of $\gamma$ need not be parallel to each other, and transversality is understood in the sense of Definition 6:

**Theorem 2** Let $\gamma$ be a self-similar Jordan arc in $\mathbb{R}^d$. Suppose there is a dense subset $D \subset \gamma$ such that for any $x \in D$ there is a hyperplane $\sigma$, which is weakly transverse to $\gamma$ at the point $x$, then $\gamma$ is a straight line segment.

The proof is based on a simple and almost obvious observation (Theorem 5), that the invariant set of a multizipper of similarity dimension 1 is always a collection of straight line segments. We prove it in Section 3.

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## 2 Preliminaries.

We give some definitions needed in current paper. Some of them are slightly different from generally accepted ones, but they are best fit for our further argument.

**Self-similar arcs.** A contraction similarity $S$ in $\mathbb{R}^d$ is a map of the form $S(x) = q \cdot O(x - x_0) + x_0$, where $x_0$ is the fixed point of $S$, $q \in (0, 1)$ is the contraction ratio, and $O$ is the orthogonal transformation called the **orthogonal part** of $S$.

Let $\mathcal{S} = \{S_1, ..., S_m\}$ be a system of contraction similarities in $\mathbb{R}^d$. A compact set $K$ is called the **invariant set** or the **attractor** of the system $\mathcal{S}$,
if \( K = \bigcup_{i=1}^{m} S_i(K) \). If this invariant set is an arc \( \gamma \) we call \( \gamma \) a **self-similar arc** defined by the system \( S \).

We denote the semigroup generated by \( S_1, \ldots, S_m \) by \( G(S) \).

**Directed multigraphs.** A directed multigraph (or digraph) \( \Gamma \) is defined by a set of vertices \( V(\Gamma) \), a set of edges \( E(\Gamma) \) and maps \( \alpha, \omega : E(\Gamma) \to V(\Gamma) \). Here \( \alpha(e) \) is the **beginning** of the edge \( e \) and \( \omega(e) \) is its **end**.

By \( E_{uv} \) we denote the set of all edges \( e \in E \) for which \( \alpha(e) = u, \omega(e) = v \), and by \( E_u = \bigcup_{v \in V} E_{uv} \) — the set of all edges with the starting point at \( u \).

To make the further argument more convenient, the set \( V \) will be supposed to be equal to \( \{1, 2, \ldots, n\} \), where \( n = \#V \). In this case \( u \in V \) means the same as \( 1 \leq u \leq n \). We also denote the numbers \( \#E_{uv} \) by \( m_{uv} \) and \( \#E_u \) by \( m_u \).

A **path** \( \sigma \) from a vertex \( \alpha(e_1) = u \) to \( \omega(e_n) = v \) in a digraph \( \Gamma \) is a sequence of edges \( \sigma = e_1e_2 \ldots e_n \), with \( \omega(e_i) = \alpha(e_{i+1}) \) for every \( 1 \leq i \leq n-1 \). The set of all paths \( \sigma \) of the length \( n \) with the beginning \( u \) and the end \( v \) is denoted by \( E(n)_{uv} \) and \( E^*(uv) = \bigcup_{n=1}^{\infty} E(n)_{uv} \) is the set of all paths from \( u \) to \( v \).

A digraph \( \Gamma \) is **strongly connected** if for every two vertices \( u \) and \( v \) it has a path from \( u \) to \( v \).

**Graph-directed systems of contraction similarities.**

A **graph-directed system of contraction similarities** \( \mathcal{S} \) with **structural graph** \( \Gamma = \langle V, E, \alpha, \omega \rangle \) is a finite collection of metric spaces \( \{X_v\}_{v \in V} \), together with a collection of contraction similarities \( \{S_e : X_{\omega(e)} \to X_{\alpha(e)}\}_{e \in E} \).

We denote the contraction ratios of the similarities by \( q_e = \text{Lip}(S_e) \).

Throughout this paper all the spaces \( X_u \) will be different copies of the same space \( \mathbb{R}^d \) for certain \( d \).

A graph-directed system of similarities \( \mathcal{S} \) is called **regular**, if its structural graph \( \Gamma \) is strongly connected.

A finite collection of compact subsets \( \{K_v\}_{v \in V} \), is called the **invariant set**, or the **attractor** of the system \( \mathcal{S} \), if for every \( v \in V \)

\[
K_u = \bigcup_{\alpha(e) = u} S_e(K_{\omega(e)}). \tag{1}
\]

The sets \( \{K_u\}_{u \in V} \) are called the **components of the attractor** of the system \( \mathcal{S} \).

We use the following definition of a similarity dimension of graph-directed system of similarities \([5],[14]\):
Definition 3 Let $S$ be a regular graph-directed system of similarities with a structure graph $\Gamma = (V, E, \alpha, \omega)$. For each positive real number $s$, let $B(s)$ be the matrix (with rows and columns indexed by $V$ ) with entry $B_{uv}(s) = \sum_{e \in E_{uv}} q^s_e$ in row $u$ column $v$. Let $\Phi(s) = r(B(s))$ be the spectral radius of $B(s)$. The unique solution $s_1 \geq 0$ of $\Phi(s) = 1$ is the similarity dimension of the system $S$.

3 Multizippers of similarity dimension 1.

A method of construction of self-similar curves, used by many authors [15, 13, 9] was studied in 2002 by V.V.Aseev [1] as a zipper construction. This construction proved to be an efficient tool in the investigation of geometrical properties of self-similar curves and continua [2]. Its graph-directed version was introduced by the author in 2006 and was called a multizipper construction; it gives a complete description of self-similar Jordan arcs in $\mathbb{R}^d$ [16, Theorem 4.1]:

Theorem 4 Let $S$ be a regular graph directed system of similarities in $\mathbb{R}^d$ with Jordan attractor $\vec{\gamma}$. If one of the components $\gamma_u$ of the attractor $\vec{\gamma}$ is different from a straight line segment, then there is a multizipper $Z$ such that the set of the components of the attractor of $Z$ contains each of the arcs $\gamma_u$.

Definition of a multizipper. Consider a graph-directed system $Z$ of similarities with structural graph $\Gamma$, which satisfies the following conditions:

MZ1. In each of the spaces $X_u, u \in V$, a chain of points $\{z^{(u)}_0, \ldots, z^{(u)}_{m_u}\}$, is specified. These chains are defined in such a way that

$$\|z^{(u)}_i - z^{(u)}_{i-1}\| < \|z^{(v)}_{m_v} - z^{(v)}_0\|$$

for any $u, v \in V, i = 1, \ldots, m_u$.

MZ2. There is a bijection $\epsilon$ from the set of all pairs $\{(u, i), u \in V, 1 \leq i \leq m_u\}$ to the set $E$.

MZ3. For any pair $(u, i)$, the map $S_e$, corresponding to the edge $e = \epsilon(u, i)$ with $v = \omega(e)$, sends two-point set $\{z^{(v)}_{m_v}, z^{(v)}_0\}$ to the set $\{z^{(u)}_{i-1}, z^{(u)}_i\}$.

The graph-directed system $Z$, satisfying the conditions MZ1—MZ3 is called a multizipper with structural graph $\Gamma$ and node points $z^{(u)}_i$.

Let $L^{(u)}$ be the polygonal line specified by the sequence $\{z^{(u)}_0, z^{(u)}_1, \ldots, z^{(u)}_{m_u}\}$ of the nodes of the multizipper $Z$. Denote the distance $\|z^{(u)}_{m_u} - z^{(u)}_0\|$ by $l_u$. 

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Observe that if \( S_e(\{ z_0^{(u)}, z_m^{(u)} \}) = \{ z_{i-1}^{(u)}, z_i^{(u)} \} \), then \( ||z_i^{(u)} - z_{i-1}^{(u)}|| = q_e l_v \). So, the length of the polygonal line \( L^{(u)} \) is equal to \( \sum_{v=1}^{n} \sum_{e \in E_{uv}} q_e l_v \).

**Theorem 5** Let \( Z \) be a regular self-similar multizipper whose similarity dimension is 1. Then all the components \( \gamma^{(u)} \) of its invariant set are line segments.

**Proof.** Suppose there is a component \( \gamma^{(u)} \) of the attractor of \( Z \), which is not a line segment. Since \( Z \) is regular, for any \( v \in V \) there is a path \( \sigma = e_1 \ldots e_k \in E^{(v)}_{vu} \), so the similarity \( S_\sigma = S_{e_1} \cdots S_{e_k} \) maps the arc \( \gamma^{(u)} \) to a subarc of \( \gamma^{(v)} \). Therefore, each \( \gamma^{(v)} \) is also different from a straight line.

Then, choosing appropriate refinement of the multizipper \( Z \), we may suppose that all the polygonal lines \( L^{(u)} \) are different from a straight line. For each component \( \gamma^{(u)} \) we have:

\[
\gamma^{(u)} = \bigcup_{v=1}^{n} \bigcup_{e \in E_{uv}} S_e(\gamma^{(v)}).
\]

The similarity dimension of the multizipper \( Z \) is equal to such value of a parameter \( s \), that the spectral radius of the matrix \( B(s) \) whose entries are \( B_{uv}(s) = \sum_{e \in E_{uv}} q_e^s \), is equal to 1.

So, the spectral radius of the matrix \( B(1) \) with entries \( B_{uv}(1) = \sum_{e \in E_{uv}} q_e \) is equal to 1.

Since all the polygonal lines \( L^{(u)} \) are not straight lines, they obey the inequality

\[
l_u < \sum_{v=1}^{n} \sum_{e \in E_{uv}} q_e l_v = (B(1)\vec{l})_u.
\]

Therefore, for a vector \( \vec{l} = (l_1, \ldots, l_n) \) and for the matrix \( B(1) \) we have the inequality

\[
\min_{1 \leq u \leq n} \frac{(B\vec{l})_u}{l_u} > 1.
\]

The structural graph of the system \( Z \) is strongly connected. Then the matrix \( B(1) \) is a positive irreducible matrix. According to [8, Remark 4, §2, Ch.XIII] its spectral radius is equal to

\[
r = \max_{\vec{l} \neq 0} \min_{1 \leq u \leq n} \frac{(B\vec{l})_u}{l_u}.
\]
So, if \( r = 1 \), then for any \( l \), there is such \( u \), that \( \frac{(B\bar{l})_{u}}{l_{u}} \leq 1 \).

The contradiction shows that all \( L^{(u)} \) are straight line segments, so all \( \gamma^{(u)} \) are straight line segments too. \( \blacksquare \)

4 Theorem on transverse hyperplanes.

Jordan arcs and transverse hyperplanes. Let \( \gamma : [0, 1] \to \mathbb{R}^{d} \) be a Jordan arc in \( \mathbb{R}^{d} \). For any point \( x = \gamma(t) \) we define the half-open subarcs \( \gamma_{x}^{+} = \gamma((t, 1]) \) and \( \gamma_{x}^{-} = \gamma([0, t)) \).

Let \( x, y \in \gamma \), and \( y \in \gamma_{x}^{+} \). We denote the open subarc \( \gamma_{x}^{+} \cap \gamma_{y}^{-} \) by \( (x, y) \) and \( \gamma_{x}^{+} \cap \gamma_{y}^{-} \) by \([x, y]\).

A hyperplane containing the origin 0 is denoted by \( \sigma \), while \( V^{+}(\sigma) \) and \( V^{-}(\sigma) \) are open half-spaces, defined by \( \sigma \). A hyperplane parallel to \( \sigma \) and containing \( x \) is denoted by \( \sigma(x) \) or \( \sigma + x \). The open half-spaces defined by \( \sigma(x) \) are denoted by \( V^{+}(\sigma, x) \) and \( V^{-}(\sigma, x) \) or \( V^{+}(\sigma) + x \) and \( V^{-}(\sigma) + x \).

Definition 6 We say a hyperplane \( \sigma \) is weakly transverse to the arc \( \gamma \) at the point \( x \), if \( \gamma_{x}^{+} \subset \bar{V}^{-}(\sigma, x) \), \( \gamma_{x}^{-} \subset \bar{V}^{+}(\sigma, x) \).

We say a hyperplane \( \sigma \) is transverse to the arc \( \gamma \) at the point \( x \), if \( \gamma_{x}^{+} \subset V^{+}(\sigma, x) \), \( \gamma_{x}^{-} \subset V^{-}(\sigma, x) \).

The cones \( Q^{+} \) and \( Q^{-} \). By \( Q^{+}(x, y) \) (respectively, \( Q^{-}(x, y) \)) we denote the intersection of all closed half-spaces \( V^{+}(\sigma, z) \) (resp. \( V^{-}(\sigma, z) \)) corresponding to the hyperplanes \( \sigma(z) \), weakly transverse to \( \gamma \) at the points \( z \in [x, y] \). These sets are convex and closed and they satisfy the relations

\[ \gamma^{+}(y) \subset Q^{+}(x, y) \quad \text{and} \quad \gamma^{-}(x) \subset Q^{-}(x, y). \]

Taking \( x = y \) we come to the sets \( Q^{+}(x) \) (\( Q^{-}(x) \)) which are the intersections of all closed half-spaces \( V^{+}(\sigma, x) \) (\( V^{-}(\sigma, x) \)) corresponding to hyperplanes \( \sigma(x) \), weakly transverse to \( \gamma \) at the point \( x \). We can also consider the set \( Q^{+}(x) \cup Q^{-}(x) \) as the intersection of all unions \( Q^{+}_{i} \cup Q^{-}_{i} \) of pairs of convex closed cones symmetric with respect to \( x \) which satisfy relations \( \gamma^{+}(x) \subset Q^{+}_{i} \) and \( \gamma^{-}(x) \subset Q^{-}_{i} \).

Lemma 7 Let \( \gamma \) be a Jordan arc in \( \mathbb{R}^{n} \). Suppose a sequence of points \( x_{n} \in \gamma \) converges to a point \( x_{0} \), while a sequence of hyperplanes \( \sigma_{n} \), weakly transverse to \( \gamma \) at points \( x_{n} \), converges to a hyperplane \( \sigma_{0} \). Then \( \sigma_{0} \) is weakly transverse to \( \gamma \) at the point \( x_{0} \).
For any \( n \), \( \bar{\gamma}^+ + x_n \subset V^+ (\sigma_n, x_n) \). Since \( x_n \to x_0 \), \( \sigma_n \) converge to \( \sigma_0 \) if and only if \( \sigma_n (x_n) \) converge to \( \sigma_0 (x_0) \). Taking the closed half-spaces, corresponding to \( \sigma_n (x_n) \), we get \( \lim_{n \to \infty} V^+ (\sigma_n, x_n) = V^+ (\sigma_0, x_0) \). At the same time, \( \lim_{n \to \infty} \bar{\gamma}^+_{x_n} = \bar{\gamma}^+_{x_0} \). Therefore, \( \bar{\gamma}^+_{x_0} \subset V^+ (\sigma_0, x_0) \). The same way we get \( \bar{\gamma}^-_{x_0} \subset V^- (\sigma_0, x_0) \). ■

Denote by \( \Sigma (x) \) the set of all hyperplanes, weakly transverse to the arc \( \gamma \) at the point \( x \in \gamma \). This set is a compact subset of \( \mathbb{RP}^d \). It follows from the Lemma 7, that \( \Sigma (x) \supset \limsup_{y \to x, y \in \gamma} \Sigma (y) \).

This inclusion implies that the cones \( Q^+ (x) \) and \( Q^- (x) \) satisfy the following semicontinuity condition:

**Lemma 8** Let \( \gamma \) be a Jordan arc in \( \mathbb{R}^d \) and \( x \in \gamma \). Then,

\[
Q^+ (x) \subset \liminf_{y \to x, y \in \gamma} Q^+ (y).
\]

**Proof.** Since

\[
Q^+ (x) = \bigcap_{\sigma \in \Sigma (x)} \bar{V}^+ (\sigma, x),
\]

using basic properties of upper and lower limits[12, §29], we can write

\[
Q^+ (x) = ( \bigcup_{\sigma \in \Sigma (x)} V^- (\sigma, x))^c \subset (\limsup_{y \to x, y \in \gamma} \bigcup_{\sigma \in \Sigma (y)} V^- (\sigma, y))^c = \\
= \liminf_{y \to x, y \in \gamma} ( \bigcup_{\sigma \in \Sigma (y)} V^- (\sigma, y))^c = \liminf_{y \to x, y \in \gamma} Q^+ (y). ■
\]

**Lemma 9** Let \( \gamma \) be a self-similar Jordan arc. If for any \( x \in \gamma \) there is a hyperplane, weakly transverse to \( \gamma \) at the point \( x \), then there is a hyperplane \( \sigma \), which is transverse to \( \gamma \) at any point \( x \in \gamma \).

**Proof.** Suppose the affine hull of \( \gamma \) is \( \mathbb{R}^d \) so it is not contained in a hyperplane.

Take some \( \delta > 0 \).

Consider the family of all the cones \( A = \{ Q^+ (x), x \in \gamma \} \). Taking the parallel copy of each cone \( Q^+ (x) \) having the vertex at the center 0 of the unit ball \( B \subset \mathbb{R}^d \), we denote its intersection with the ball \( B \) by \( Q(x) \). This turns the family \( A \) to a subset of the hyperspace \( Conv (B) \) of compact convex
subsets of the unit ball $B$. Observe that the inclusion $Q(x) \subset \liminf_{y \to x, y \in \gamma} Q(y)$ in the statement of Lemma 8 holds for the cones $Q(x)$ as well.

Let $S$ be a contraction similarity, for which $S(\gamma) \subset \gamma$. Let $x_0$ be its fixed point. Let $O$ be the orthogonal part of the similarity $S$.

Since $Q(x_0) \subset \liminf_{x \to x_0} Q(x)$, there is an open subarc $(y, z) \ni x_0$ such that for any $x \in (y, z)$, the cone $Q(x_0)$ is contained in $\delta$–neighborhood $N_\delta(Q(x))$ of a cone $Q(x)$.

For some sufficiently large $k$, the subarc $S^k(\gamma)$ is contained in $(y, z)$. Then for any $\xi \in \gamma$, the point $x = S^k(\xi)$ lies in $(y, z)$ and $N_\delta(Q(x)) \supset Q(x_0)$. Since $Q(x) = O^k_s(Q(\xi))$, and $Q(x_0) = O^k_s(Q(x_0))$ and $O^k_s$ is an isometry, $N_\delta(Q(\xi))$ must also contain $Q(x_0)$.

Thus, if $S : \gamma \to \gamma$ is a similarity and $\text{fix}(S) = x$, then for any $\delta > 0$ and any $\xi \in \gamma$, $N_\delta(Q(\xi)) \supset Q(x)$. Therefore, $Q(\xi) \supset Q(x)$ for all $\xi \in \gamma$. If we take for $\xi$ a fixed point of some other similarity $S' : \gamma \to \gamma$, we get that $Q(\xi) = Q(x)$. Thus, the minimal cone $Q(x)$ is the same, no matter which fixed point we choose, and we denote it by $Q$. If $x$ is not a fixed point of any $S \in G(S)$, then $Q(x) \subset Q$. If $\sigma(x)$ is a support hyperplane to the cone $Q^+(x)$ at some fixed point $x$, then for any $\xi \in \gamma$ parallel hyperplane $\sigma(\xi)$ is a support hyperplane for $Q^+(\xi)$ and is thus weakly transverse to $\gamma$ at the point $\xi$.

Suppose for some $x$ and $w$ in $\gamma$, $w \in \gamma^+(x)$ and $w \in \sigma(x)$. Then $\sigma(x) = \sigma(w)$ and $V^+ (\sigma, x) = V^+ (\sigma, w)$. By weak transversality of $\sigma$ at the points $x$ and $w$, the subarc $[x, w]$ lies in $V^+ (\sigma, x) \cap V^-(\sigma, w) = \sigma$. Then the whole arc $\gamma$ lies in a hyperplane. The contradiction shows that the hyperplanes parallel to $\sigma(x)$ are transverse to $\gamma$ at any point.$\blacksquare$

**Lemma 10** Let $\gamma$ be a self-similar Jordan arc, which has a hyperplane transverse to $\gamma$ at any of its points. Then there is such transverse hyperplane $\sigma$, that for any similarity $S_i \in S$, $O_i(\sigma) = \sigma$.

**Proof.** Let $G_O$ be a group generated by orthogonal parts $O_i$ of the similarities $S_i \in S$. For any $O \in G_O$, the image $O(Q)$ is either $Q$ or $-Q$. The space $\mathbb{R}^d$ is a direct sum of two orthogonal subspaces $X_0 \oplus X_1$, where $X_0$ is the space of all such $x$ that for any $O \in G_O$, $O(\{x, -x\}) = \{x, -x\}$ and $X_1 = X_0^\perp$.

Consider the intersection $X_0 \cap Q$. This intersection is a convex cone $Q'$ in $X_0$. Take a support hyperplane $Y$ to the cone $Q'$ at the point $0$ in the space $X_0$. Then $Y + X_1$ is a support hyperplane for $Q$ in $\mathbb{R}^d$.

Suppose contrary. Then there is some $z \in (Y + X_1) \cap \bar{Q}$. The point $z$ has unique representation in the form $z = x + y$, where $x \in X_1$, $x \neq 0$ and
\( y \in Y \). Consider the convex hull \( W \) of the orbit \( G_O(x) \). It’s barycenter is fixed by the group \( G_O \), therefore it is 0. Then the barycenter of the convex hull of the orbit \( G_O(z) \) is \( y \).

Take a ball \( B(z, \varepsilon) \subset Q \). The convex hull of \( \bigcup_{O \in G_O} O(B(z, \varepsilon)) \) contains the ball \( B(y, \varepsilon) \), therefore \( y \in \hat{Q} \), which is impossible. So \( \hat{Q} \cap (Y + X_1) = \emptyset \).

At the same time, for any \( O \in G_O \) the transformation \( O \) sends the hyperplane \( Y + X_1 \) to itself. 

\[ \blacksquare \]

The proof of Theorem 2

Let \( \gamma \) be a self-similar Jordan arc, which is not a line segment. By Theorem 4.1 in [16], the arc \( \gamma \) may be represented as a component \( \gamma^{(u)} \) of the invariant set of some multizipper \( Z \), for which the maps \( S_e, e \in E \) are the elements of the semigroup \( G(S) \). Let \( z_i^{(u)} \) be the node points and \( \Gamma = \langle V, E, \alpha, \omega \rangle \) be the structural graph of \( Z \). Passing, if necessary, to a subarc of \( \gamma \), we may suppose that the graph \( \Gamma \) is strongly connected and the multizipper \( Z \) is regular.

If \( \gamma \) contains such dense subset \( D \subset \gamma \), that for any \( x \in D \) there is a hyperplane \( \sigma(x) \), weakly transverse to \( \gamma \), then by Lemma [7] such hyperplane \( \sigma(x) \) exists for any \( x \in \gamma \). By Lemma [9] there is a hyperplane \( \sigma \), transverse to \( \gamma \) at any \( x \in \gamma \).

By Lemma [10] there is a hyperplane \( \sigma \), transverse to \( \gamma \) at any of its points, which is preserved by any of \( O_i \in G_O \). Then the duplicates of \( \sigma \) are transverse to the components \( \gamma^{(u)}, u \in V \) of the attractor of the multizipper \( Z \) at any of their points and are preserved by the orthogonal parts \( O_e \) of the similarities \( S_e \).

Let \( \Lambda^{(u)} \) be a line, orthogonal to \( \sigma \) in the copy \( X^{(u)} \) of the space \( \mathbb{R}^d \). Let \( \gamma^{(u)} \) be the component of the invariant set of \( Z \) lying in \( X^{(u)} \). Consider the orthogonal projection \( \pi \) of each arc \( \gamma^{(u)} \) to the \( \Lambda^{(u)} \).

Since the similarities \( S_e \) send the hyperplanes, parallel to \( \sigma \), to the hyperplanes, parallel to \( \sigma \), for each similarity \( S_e \in Z, S_e : \gamma^{(u)} \rightarrow \gamma^{(u)} \) there is a similarity \( \hat{S}_e : \Lambda^{(u)} \rightarrow \Lambda^{(u)} \), satisfying the condition

\[ \pi \circ S_e = \hat{S}_e \circ \pi. \]

Due to this condition each map \( \hat{S}_e \) sends the set \( \{\pi(z_i^{(u)}), \pi(z_{i+m}^{(u)})\} \) to the set \( \{\pi(z_i^{(u)}), \pi(z_{i+m}^{(u)})\} \).

The system \( \hat{Z} \) is a linear multizipper with node points \( z_i^{(u)} = \pi(z_i^{(u)}) \).
Since for any $S_e$, $\text{Lip}(\hat{S}_e) = \text{Lip}(S_e)$ the similarity dimension of the multizipper $\mathcal{Z}$ is equal to the similarity dimension of $\hat{\mathcal{Z}}$ and therefore it is equal to 1. By Theorem 5 its invariant set is a collection of straight line segments. ■

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