Dark matter in $f(\mathcal{R})$ gravity

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Abstract

We discuss here the possibility of explaining the rotational velocity curves of the galaxies in $f(\mathcal{R})$ gravity by solving field equations numerically. In a given constant rotational velocity region, we prove that all values of rotational velocities does not lead to an analytic solution of the field equations in vacuum. We then obtain the numerical solutions of the field equations which suggests that a very slightly modification from linear relations of $\mathcal{R}$ can be an alternative to the so called dark matter.

Keywords: Dark matter, $f(\mathcal{R})$ gravity.

1 Introduction

There is no priori reason to consider that the gravity is linear in Ricci scalar $\mathcal{R}$ except it best matches the observation (excluding dark matter and dark energy). Dark energy is related to the mystery of accelerated expanding Universe whereas dark matter is related to the observed rotation velocity curves of the galaxy. Here, we address the problem of the dark matter. The present Newtonian dynamics and Einstein’s gravity can not explain these phenomenas which leads to idea that either one or both sides of the Einstein field equations (EFE) are incomplete. The right hand side of the EFE comprise of the matter part whereas left side is the geometric part. Einstein field equations simply tells that if we put some matter in the space, the space counter acts to tell the matter how to react in a given geometry. There are many proposals that address the dark matter problem. A few of them are weakly interacting massive particles (WIMP) [8, 6], scalar field from extensions of SM [7], modified Newtonian dynamics (MOND) [10], modified gravity [16, 13, 1]. The modified gravity modifies the left hand side (geometric part) of the Einstein field equations.
(field equations then called). It then can be used to explain the observed rotational velocity curves to be a geometric effect only.

The mysterious dark matter is the type of matter supposed to be surrounding galaxies that give the observed rotational velocity curves. Einstein Hilbert action could also be one of the reason that there is some discrepancy between observed and expected tangential velocity of a particle moving around the galaxy.

In this article, we present a simple version of $f(\mathcal{R})$ gravity model by solving field equations numerically that can describe the dark matter as a geometric effect only. The scheme of the article is as follows; using a metric that can describe the constant velocity region of space around a galaxy we try to solve the field equations for the metric in $f(\mathcal{R})$ gravity analytically and numerical results of the field equations suggest that very slight modification from Einstein’s gravity could explain the discrepancy between observed and expected rotational velocity curves. The article is as follows; first a metric that can describe the constant velocity region of space around a galaxy is given. We then move on to solve the field equations for the metric in $f(\mathcal{R})$ gravity analytically. The numerical results of the field equations succeed that.

2 $f(\mathcal{R})$ gravity field equations in vacuum

The metric which describes the static and spherically symmetry in the constant rotational velocity regions is

$$dS^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2.$$  \hspace{1cm} (1)

The metric coefficient $\nu(r)$ can be calculated from the Euler Lagrange equations of the Lagrangian of the metric. From Euler Lagrange equations, we get $\nu(r) = 2m\ln(r/r_0)$ where $m = V_{tg}^2/c^2$; $V_{tg}$ being the rotational velocity of a particle around galaxy, $c$ is the speed of light and $r_0$ is the constant of integration obtained in deriving $\nu(r)$.

The $f(\mathcal{R})$ modified gravity action is written as

$$S = \int \sqrt{-g} \ f(\mathcal{R}) \ d^4x ,$$  \hspace{1cm} (2)

here $f(\mathcal{R})$ is an arbitrary analytic function of Ricci scalar $\mathcal{R}$. The variation of the above action with respect to the metric $g_{\mu\nu}$ gives the following field equations

$$F(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) F(R) = 0 ,$$  \hspace{1cm} (3)

where $F(\mathcal{R}) = df/d\mathcal{R}$. The contraction of the above equation gives

$$F(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) + 3\Box F(R) = 0 .$$  \hspace{1cm} (4)

Upon using eq. (4) in eq. (3) we get modified field equations as

$$F(\mathcal{R})\mathcal{R}_{\mu\nu} - \nabla_\mu \nabla_\nu F = \frac{1}{4}g_{\mu\nu} (F\mathcal{R} - \Box F) ,$$  \hspace{1cm} (5)
also differentiation of eq. (4) with respect to ‘r’ gives

\[ \mathcal{R} F' - \mathcal{R}' F + 3 (\Box F)' = 0 . \]  

(6)

Where ′ represents derivative with respect to ‘r’. If we use modified field equations to find the solution then the solution must also satisfy eq. (4) to be a solution of original field equations given by eq. (3). From eq. (5), we see that

\[ A_\mu \equiv (F R_{\mu\nu} - \nabla_\mu \nabla_\nu) / g_{\mu\nu} \]  

is independent of the index \( \mu \), thus \( A_\mu - A_\nu = 0 \). From the relation \( A_\mu - A_\nu = 0 \) we can write three equations \( (A_0 - A_1 = 0, A_1 - A_2 = 0 \) and \( A_0 - A_2 = 0) \). The equations \( A_0 - A_1 = 0, A_1 - A_2 = 0, \) the rr component of the field equations and eq. (4) can be written as [16, 13],

\[ F'' - \frac{1}{2} (\nu' + \lambda') F' + \frac{1}{r} (\nu' + \lambda') F = 0 , \]  

(7)

\[ \nu'' + \nu^2 - \frac{1}{2} (\nu' + 2/r) (\nu' + \lambda') - \frac{2}{r^2} + 2 e^\lambda / r^2 = 2 F'' - \left( \lambda' + \frac{2}{r} \right) F' / F , \]  

(8)

\[ f = F e^{-\lambda} \left[ \nu'' - \frac{1}{2} \nu' (\nu' + \lambda') - \frac{2}{r} \lambda' \right] + \left( \nu' + \frac{4}{r} \right) F' / F ] , \]  

(9)

\[ \mathcal{R} = 2 \frac{F}{F} - 3 e^{-\lambda} \left[ F'' \frac{1}{F} + \frac{1}{2} (\nu' - \lambda') + \frac{2}{r} \right] F' / F ] , \]  

(10)

respectively.

Introducing \( \eta = \ln(r/r^*) \) with \( r^* = 1 \), eq. (7)-(10) now become

\[ \frac{d^2 F}{d\eta^2} - \left( 1 + \frac{1}{2} \left( \frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) \right) \frac{dF}{d\eta} + \left( \frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) F = 0 , \]  

(11)

\[ \frac{d^2 \nu}{d\eta^2} - \frac{d\nu}{d\eta} + \left( \frac{d\nu}{d\eta} \right)^2 - \frac{1}{2} \left( \frac{d\nu}{d\eta} + 2 \right) \left( \frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) \right. \]  

\[ - \left. 2 (1 - e^\lambda) = 2 \frac{d^2 F}{F d\eta^2} - \left( \frac{d\lambda}{d\eta} + 4 \right) \frac{1}{F} \frac{dF}{d\eta} , \]  

(12)

\[ f = F e^{-\lambda - 2\eta} \left[ \frac{d^2 \nu}{d\eta^2} - \frac{d\nu}{d\eta} - \frac{1}{2} \left( \frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) \frac{d\nu}{d\eta} - \frac{2}{d\eta} d\lambda + \left( \frac{d\nu}{d\eta} + 4 \right) \frac{1}{F} \frac{dF}{d\eta} \right] , \]  

(13)

\[ \mathcal{R} = 2 \frac{f}{F} - 3 e^{-\lambda - 2\eta} \left[ \frac{1}{F} \frac{d^2 F}{d\eta^2} - \frac{1}{F} \frac{dF}{d\eta} + \left( \frac{1}{2} \left( \frac{d\nu}{d\eta} - \frac{d\lambda}{d\eta} \right) + 2 \right) \frac{1}{F} \frac{dF}{d\eta} \right] . \]  

(14)
With the introduction $1/F' (dF/d\eta) = u$, eqs. (11) and (12) simplifies to

$$
\frac{du}{d\eta} + u^2 - \left(1 + \frac{1}{2} \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta}\right)\right) u + \frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} = 0 ,
$$

(15)

even though above equations are second order differential equations in $\nu$ but since $\nu$ has already been given as a function of $r$ earlier, the introduction of $u$ makes eqs. (11) and (12) first order with the increase in number of equations. Using $du/d\eta + u^2$ from eq. (16) into eq. (15) we get

$$
\frac{d^2\nu}{d\eta^2} - \frac{d\nu}{d\eta} + \left(\frac{d\nu}{d\eta}\right)^2 - \frac{1}{2} \left(\frac{d\nu}{d\eta} + 2\right) \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta}\right)
- 2 \left(1 - e^\lambda\right) = 2 \left(\frac{du}{d\eta} + u^2\right) - \left(\frac{d\lambda}{d\eta} + 4\right) u ,
$$

(16)

Eq. (17) and eq. (15) or eq. (16) provide the solution of $\lambda$ and $u$ ($\nu$ is known).

3 Dark matter as a geometric effect

To obtain the geometric interpretation of the constant velocity regions, we solve eq. (15) and eq. (17) for $\lambda$ and $u$ with $\nu = 2m(\eta - \eta_0)$. Eq. (15) and eq. (17) can now be written as

$$
\frac{du}{d\eta} + u^2 - \left(1 + m + \frac{1}{2} \frac{d\lambda}{d\eta}\right) u + 2m + \frac{d\lambda}{d\eta} = 0 ,
$$

(18)

$$
-2m + 4m^2 - (1 + m) \left(2m + \frac{d\lambda}{d\eta}\right) - 2 \left(1 - e^\lambda\right) + 2 (1 - m) u + 2 \left(2m + \frac{d\lambda}{d\eta}\right) = 0 .
$$

(19)

Using $u$ and $du/d\eta$ from eq. (19) into eq. (18) we obtain

$$
\frac{1}{2} \frac{d^2\lambda}{d\eta^2} + \frac{1}{1 - m} \left(\frac{3}{2} e^\lambda - m(1 - m)\right) \frac{d\lambda}{d\eta} - \frac{1}{2} \left(\frac{d\lambda}{d\eta}\right)^2
- \frac{(1 - e^\lambda)^2}{(1 - m)^2} - \frac{e^\lambda(1 + m^2) - 1}{(1 - m)^2} + 2m = 0 ,
$$

(20)

This is a second order differential equation in $\lambda$. After obtaining $\lambda$ from this equation one can use either eq. (18) or eq. (19) to obtain $u$. The obtained $\lambda$ and $u$ must satisfy
the third equation (which has not been used in getting \( \lambda \) and \( u \)). As discussed in [1], the tangential velocity of a test particle in stable circular orbit around the galactic center is about \( 200 - 300 \) Km/s [14, 12, 2] thus \( m \approx \mathcal{O}(10^{-6}) \). Hence, we must get an approximate function of Ricci scalar around \( m \approx \mathcal{O}(10^{-6}) \).

Introducing new variable as \( d\lambda/d\eta = \rho \) then we can write \( d/d\eta = \rho \, d/d\lambda \). This allows us to write eq. (20) as

\[
\frac{\rho}{2} \frac{d\rho}{d\lambda} + \frac{1}{1-m} \left( \frac{3}{2} e^\lambda - m(1-m) \right) \rho - \frac{1}{2} \rho^2 \\
- \frac{(1-e^\lambda)^2}{(1-m)^2} - \frac{e^\lambda(1+m^2) - 1}{(1-m)^2} + 2m = 0 .
\]  

(21)

Now, using the transformation \( \rho = 1/\omega \) and \( e^\lambda = \theta \), eq. (21) after simplification can be written as

\[
\frac{d\omega}{d\theta} = \left[ \frac{m}{\theta} - \frac{2(1-\theta)^2}{\theta(1-m)^2} - \frac{2(\theta(1+m^2) - 1)}{\theta(1-m)^2} \right] \omega^3 + \left[ 3 - \frac{2m(1-m)}{\theta} \right] \omega^2 - \frac{1}{\theta} \omega .
\]  

(22)

This is the nonlinear first order Abel differential of first kind of the form

\[
\frac{d\omega}{d\theta} = p(\theta)\omega^3 + q(\theta)\omega^2 + r(\theta)\omega + s(\theta) ,
\]

with

\[
p(\theta) = \frac{m}{\theta} - \frac{2(1-\theta)^2}{\theta(1-m)^2} - \frac{2(\theta(1+m^2) - 1)}{\theta(1-m)^2} ,
\]

\[
q(\theta) = 3 - \frac{2m(1-m)}{\theta} , \quad r(\theta) = -\frac{1}{\theta} , \quad s(\theta) = 0 .
\]

If we choose that the particular solution \( y_p \) of the above equation satisfies

\[
3 \left[ \frac{m}{\theta} - \frac{2(1-\theta)^2}{\theta(1-m)^2} - \frac{2(\theta(1+m^2) - 1)}{\theta(1-m)^2} \right] y_p^2 + 2 \left[ 3 - \frac{2m(1-m)}{\theta} \right] y_p - \frac{1}{\theta} = 0 ,
\]  

(23)

then eq. (22) becomes Abel differential equation of second kind [9], of the form

\[
\frac{d\omega}{d\theta} = \left[ \frac{m}{\theta} - \frac{2(1-\theta)^2}{\theta(1-m)^2} - \frac{2(\theta(1+m^2) - 1)}{\theta(1-m)^2} \right] \omega^3 \\
+ \left[ 3 \left( \frac{m}{\theta} - \frac{2(1-\theta)^2}{\theta(1-m)^2} - \frac{2(\theta(1+m^2) - 1)}{\theta(1-m)^2} \right) y_p + \left( 3 - \frac{2m(1-m)}{\theta} \right) \right] \omega^2 .
\]  

(24)

In the case when,

\[
\frac{d}{d\theta} \left( \frac{p(\theta)}{\sqrt{q(\theta)^2 - 3p(\theta)r(\theta)}} \right) = K\sqrt{q(\theta)^2 - 3p(\theta)r(\theta)} ,
\]  

(25)
where $K$ is a arbitrary constant then the general solution of eq. (22) is given by [9],

$$\omega(\theta) = \pm \frac{\sqrt{q(\theta)^2 - 3p(\theta)r(\theta)}}{p(\theta)} V + \left(\frac{-q(\theta) \pm \sqrt{q(\theta)^2 - 3p(\theta)r(\theta)}}{3p(\theta)}\right),$$

(26)

where $V$ satisfies the separable differential equation

$$\frac{dV}{d\theta} = \left(V^3 + V^2 + K V\right)\left(\frac{q(\theta)^2 - 3p(\theta)r(\theta)}{p(\theta)}\right).$$

(27)

To get the solution, we must satisfy the condition given by eq. (25). The plot of

$$\alpha = \frac{1}{\sqrt{q(\theta)^2 - 3p(\theta)r(\theta)}} \frac{p(\theta)}{\sqrt{q(\theta)^2 - 3p(\theta)r(\theta)}}\frac{d}{d\theta}\left(\frac{p(\theta)}{\sqrt{q(\theta)^2 - 3p(\theta)r(\theta)}}\right)$$

is given in fig. (1) and fig. (2). The plot allows us to understand when (i.e. for what $m$ and $\theta$) could we get an analytic solution to the eq. (22).

Figure 1: When $\theta = 0$ we have eq. (25) satisfied for all values of $m$. This demands an arbitrary negative large value of $\lambda(r)$. 

6
It has also be proven in [15] that $\lambda = \text{constant}$ is not the solution of eq. (18) and (19) in the limit that $m^2$ and higher order terms in $m$ can be neglected.

4 Numerical solution to the field equations

From the figures (1) and (2), any analytic solution of eq. (22) in the presented scheme around $m = O(10^{-6})$ can not be obtained for all possible values of $\theta$. We then find the numerical solution to the field eqs. (7) and (8). The initial conditions taken were

$$r_{ini} = 10 \text{ Kpc}$$
$$F(r_{ini}) = 1$$
$$F'(r_{ini}) = 0$$
$$\lambda(r_{ini}) = 2 \times 10^{-6}.$$

The initial conditions were chosen such that initially our model of $f(\mathcal{R})$ gravity matches onto Einstein’s gravity (the value of $\lambda$ was chosen by brute force attack).
Figure 3: The radial coefficient of the metric. As we move away from the center of the galaxy $\lambda(r)$ decreases till $r \approx 300$ Kpc then it starts to increase.

From the fig. 3 we see that the metric coefficient $\lambda(r)$ decreases initially till $r \approx 300$ Kpc. For $r > 300$ Kpc we see a monotonically increasing function.

Figure 4: Normalized Ricci scalar as a function of the radial parameter $r$, moving away from the center of the galaxy we have $R \to 0$.

The Ricci scalar $R$ decreases as we move away from galactic center. As we get around $r \approx 100 - 200$ Kpc, Ricci scalar goes to zero showing that we have reached the galactic edge. This should be the case as away from galaxy the volume of a geodesic sphere should match on to that of a ball in Euclidean space.
Figure 5: Normalized $f$ as a function of the radial parameter $r$, moving away from the center of the galaxy we have $f \to 0$.

The $f$ as a function of $r$ is given in fig. (5). In moving away from the galactic center $f$ decreases and goes to zero thus matches with $\mathcal{R} = 0$ case.

Figure 6: The blue thick dashed line is our model whereas the solid black line is $f(\mathcal{R}) = \mathcal{R}$. Moving slightly away from $r_{ini}$ our $f(\mathcal{R})$ model starts to deviate from linear relations very slightly (a clearer version of this statement is given in table (1)). As we reach the boundary of the galaxy we have $\mathcal{R} \approx 0$ and $f(\mathcal{R}) \approx 0$.

The fig. (6) is the parametric graph of $f(\mathcal{R})$ and $\mathcal{R}$. We see from the graph as we move away from the galaxy the line of $f(\mathcal{R})$ is approximately linear. The solid black line corresponds to $f(\mathcal{R}) = \mathcal{R}$, it overlaps our model. The black line has been drawn to compare Einstein’s gravity and our $f(\mathcal{R})$ gravity model. Due to very slight modification from linear relation it is very hard to see any difference between $f(\mathcal{R}) = \mathcal{R}$ and our model. But we give
some numerical values to see that difference.

| r (Kpc) | Normalized Abs[$\mathcal{R}$] | Normalized Abs[$f$] |
|---------|-------------------------------|-------------------|
| 10      | 1                             | 1                 |
| 12      | 0.694395                      | 0.694347          |
| 15      | 0.444373                      | 0.444302          |
| 20      | 0.249931                      | 0.249868          |
| 40      | 0.0624127                     | 0.0623267         |
| 70      | 0.0203219                     | 0.0202357         |
| 100     | 0.00991653                    | 0.00983378        |
| 200     | 0.00241698                    | 0.00233415        |
| 500     | 0.000314893                   | 0.000229789       |

Table 1: Some numerical values of $\mathcal{R}$ and $f(\mathcal{R})$ in our model. The model starts from $f(\mathcal{R}) = \mathcal{R}$, moving away from galactic center it deviates from Einstein’s gravity. ‘Abs’ represents the absolute value. The precision of the numerical values is the machine precision of the MATHEMATICA software set at 15.9546 whereas the accuracy goal is set to 10. Thus, we expect the numerical error in each result to be less than or equal to $10^{-10} + |x|10^{-15.9546/2}$, where $|x|$ here is the $\mathcal{R}$ at different values of $r$.

From the table (1), a very slight deviation from Einstein’s gravity can explain the constant rotational velocity of a particle in moving away from the galaxy. The table (1) suggests that the a very slight negative correction to the $f(\mathcal{R}) = \mathcal{R}$ can explain the constant galactic rotation curves i.e. $f(\mathcal{R}) < \mathcal{R}$.

5 Conclusions and discussion

In this article, we started of with a metric that describes the constant rotational velocity region in spherically symmetric and static space-time. We then obtained the numerical solution to the field equation to get an appropriate estimate of $f(\mathcal{R})$ gravity that could be an alternative to the dark matter. The resultant graphs showed that a very slight modification to the Einstein’s gravity can be an alternative to the dark matter. As said in the last section, the correction up to $r = 500$Kpc we find is negative i.e. $f(\mathcal{R}) < \mathcal{R}$, shown in table (1). Although we are not giving any numerical result beyond $r = 500$Kpc but it is worth mentioning that in our numerical solutions after $r \gtrsim 885$Kpc, the modifications in the $f(\mathcal{R}) = \mathcal{R}$ becomes positive i.e. for $r \gtrsim 885$Kpc, $f(\mathcal{R}) > \mathcal{R}$.

The weak field limit of the $f(\mathcal{R})$ gravity models has been discussed for star like objects in [3, 5, 4]. If we assume that $f(\mathcal{R})$ is an analytical function at the constant curvature $R_0$, that $m_\phi/r \ll 1$, where $m_\phi$ is the effective mass of the scalar degree of freedom of the theory, and that the fluid is pressureless, the post-Newtonian potentials $\Psi(r)$ and $\Phi(r)$ are obtained for a metric

$$dS^2 = - (1 - 2\Psi(r)) \, dt^2 + (1 + 2\Phi(r)) \, dr^2 + r^2 \, d\Omega^2.$$
In that way, the behavior of $\Psi(r)$ and $\Phi(r)$ outside the star can be estimated. This then gives the value of the post-Newtonian parameter $\gamma$ to be $1/2$ \cite{1}. From the solar system observations, we know that $\gamma = 1$ \cite{1}. This inconsistency between expected and measured value of $\gamma$ then is used to rule out most of the modified gravity models.

T. Chiba, T.L. Smith and A.L. Erickcek have done the same analysis for the $f(\mathcal{R})$ gravity function of the form $f(\mathcal{R}) = (\mathcal{R}/\alpha)^{1+\delta}$ in \cite{3}. They concluded that in their section III. Case Studies “...this analysis is incapable of determining whether $f(\mathcal{R}) = \mathcal{R}^{1+\delta}$ gravity with $\delta \neq 1$ conflicts with solar system tests.”. Thus, we can conclude that results presented in fig. (6) and table (1) does not contradict with the solar model tests.

One should keep in mind that the graphs presented above are obtained in approximation that the rotational velocity of a particle around the galaxy in radial direction remains constant however this is not exactly the case observed but much close to the observations, trend of rotational velocity curve for different galaxies is different but slightly. The assumption is valid for at least a significant region of the total tangential velocity profile. If one can establish an exact relation between the rotational velocity and distant from the galactic center then one can obtain more accurate numerical results of the field equations. In that case, we might even also be able to find the exact solution to the field equations.

In conclusion, we found that regions with flat galactic rotations curves do not require any kind of dark matter if we try to explain these in $f(\mathcal{R})$ gravity. The constant rotation curves are then a consequence of the additional geometrical structure provided by the modified gravity function $f(\mathcal{R})$. We found that a very slight modification to the Einstein-Hilbert Lagrangian may account for the existence of “dark matter”. In the $f(\mathcal{R})$ gravity modified theory in principle we can rewrite the field equations in terms of the Einstein tensor $\mathcal{R}$ and interpret the remaining term(s) as a geometrical energy-momentum tensor which then gives us the option not to include the exotic type of matter called dark matter. In this way, the presence of the remaining terms can provide us with an elegant geometric interpretation of the dark matter.

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