Symplecton for $\mathcal{U}_h(sl(2))$ and Representations of $SL_h(2)$

N. Aizawa

Department of Applied Mathematics
Osaka Women’s University
Sakai, Osaka 590-0035, JAPAN

Abstract

Polynomials of boson creation and annihilation operators which form irreducible tensor operators for Jordanian quantum algebra $\mathcal{U}_h(sl(2))$, called $h$-symplecton, are introduced and their properties are investigated. It is shown that many properties of symplecton for Lie algebra $sl(2)$ are extended to $h$-symplecton. The $h$-symplecton is also a basis of irreducible representation of $SL_h(2)$ dual to $\mathcal{U}_h(sl(2))$. As an application of the procedure used to construct $h$-symplecton, we construct the representation bases of $SL_h(2)$ on the quantum $h$-plane.
I Introduction

It is no doubt that well-developed representation theories are necessary when we apply algebraic objects to physics. The simplest examples in quantum physics are angular momentum algebra $su(2)$ and rotation matrices in 3 dimensional space $SO(3)$. Their algebraic structure is simple but contents of representation theories are quite rich [1]. To investigate these algebraic objects or their complexification could be a foundation for further investigation of higher dimensional objects.

As for deformation of Lie groups and Lie algebras, $q$-deformation of Lie algebra $sl(2)$ and Lie group $SL(2)$ (and their real form) is studied quite well. Their representation theories have attracted much interest in both physics and mathematics and give a way to higher dimensional cases [2]. There exists, however, some other deformation of Lie groups and algebras and these are generally called nonstandard deformation. The most studied one may be the so-called Jordanian deformation obtained by Drinfeld twist from a Lie algebra or a known quantum algebra. The simplest examples are, of course, the Jordanian deformation of Lie algebra $sl(2)$ and its dual. The Jordanian deformation of Lie group $SL(2)$, denoted by $SL_h(2)$, is studied in [3, 4, 5] and then Ohn introduced its dual algebra, namely, Jordanian deformation of $sl(2)$ denoted by $U_h(sl(2))$ [6]. The Jordanian quantum algebra $U_h(sl(2))$ is more natural than the $q$-deformed $sl(2)$ in the sense that it is regarded as the angular momentum algebra with nonstandard coproduct (see §3) and we can use ordinary boson operators to represent $U_h(sl(2))$, while it is hard to regard the $q$-deformed $sl(2)$ as angular momentum and $q$-deformed boson algebras are used for representations [1]. However, the representation theories of $U_h(sl(2))$ and $SL_h(2)$ have not been developed yet. We do not know, for example, the Racha coefficients and matrix elements of the universal $R$-matrix for $U_h(sl(2))$. As for $SL_h(2)$, even its representation matrices are not obtained.

In this article, in order to develop representation theories for Jordanian deformed algebras, we study symplecton for $U_h(sl(2))$ and apply it to investigate representation matrices of $SL_h(2)$. The use of symplecton could be legitimated by recalling the properties of symplecton and $q$-deformed case. The symplecton, introduced by Biedenharn and Louck [7, 8], is a polynomial of boson creation and annihilation operators which form an irreducible tensor operator of $sl(2)$, that is, symplecton is a basis of irreducible representation (irrep.) for both $sl(2)$ and $SL(2)$. It is known that the symplecton is written in terms of Gauss hypergeometric function and product of two symplecton is reduced to a series of symplecton with Racha coefficients. In Ref. [7], application of symplecton to the Elliot model for nuclei is discussed, then it is found that Weyl-ordered polynomials for position and momentum operators are equivalent to symplecton [9]. Many properties of symplecton are inherited from $sl(2)$ to $q$-deformed case [2, 10, 11]. The $q$-deformed symplecton, called $q$-symplecton, is a irreducible tensor operator so that it is a irrep. basis for $q$-deformed $sl(2)$ and $SL(2)$. The $q$-symplecton is written in terms of $q$-hypergeometric function and product of two $q$-symplecton is reduced to a series of $q$-symplecton with $q$-Racha coefficients. $q$-Deformation of Weyl-ordered polynomial [12] is formulated with $q$-symplecton. These facts show that symplecton is a powerful tool to investigate representation.

†There exist mappings from the ordinary boson algebra to $q$-deformed ones [27, 13, 28]. It is, however, simpler to use the $q$-deformed boson algebras for representation theories.
The plan of this article is as follows. Next three sections are mainly preparation for symplecton of \( U_h(sl(2)) \). We often call the symplecton for \( U_h(sl(2)) \) \( h \)-symplecton. The next section is a review of symplecton for \( sl(2) \). Some of the properties of symplecton listed in §2 will be extended to \( h \)-symplecton. §3 is devoted to the Jordanian quantum algebra \( U_h(sl(2)) \) and Jordanian quantum group \( SL_h(2) \). We give new results on the twist element and Racha coefficients for \( U_h(sl(2)) \). In §4, tensor operators for a Hopf algebra is introduced according to Ref. [13] and the relation between tensor operators for a Lie algebra and a Hopf algebra obtained by Drinfeld twist is discussed. Applying the result in §4, the \( h \)-symplecton is constructed from the \( sl(2) \) symplecton in §5. The properties of \( h \)-symplecton are studied in §5 and §6. We shall consider another irreducible tensor operators obtained from the quantum \( h \)-plane for \( U_h(sl(2)) \) in §7 and using these tensor operators, as well as \( h \)-symplecton, irreps. of \( SL_h(2) \) are considered. §8 is concluding remarks.

II Symplecton for \( sl(2) \)

The symplecton realization of \( sl(2) \) is said to be "minimal", since only one kind of boson operator is used. It is in marked contrast to the well-known Jordan-Schwinger realization where two kinds of bosons are necessary. Let us first review the definition and important properties of the \( sl(2) \) symplecton [4, 5].

Let \( \bar{a}, a \) be boson operators satisfying \( [\bar{a}, a] = 1 \), and define

\[
J_+ = -\frac{1}{2} a^2, \quad J_- = \frac{1}{2} \bar{a}^2, \quad J_0 = \frac{1}{2} (a\bar{a} + \bar{a}a).
\] (II.1)

It is easy to verify that (II.1) satisfies the \( sl(2) \) commutation relations

\[
[J_0, J_{\pm}] = \pm 2 J_{\pm}, \quad [J_+, J_-] = J_0.
\] (II.2)

The symplecton is a polynomial in \( \bar{a} \) and \( a \) and form a irreducible tensor operator of \( sl(2) \) belonging to the spin \( j \) representation \((j = \frac{1}{2}, 1, \frac{3}{2}, \ldots)\). Namely the symplecton, denoted by \( P_j^m(a, \bar{a}) \), is defined by

\[
[J_{\pm}, P_j^m] = \sqrt{(j \mp m)(j \pm m + 1)} P_j^{m \pm 1},
\]

\[
[J_0, P_j^m] = 2m P_j^m.
\] (II.3)

The basic idea of symplecton is to treat \( \bar{a} \) and \( a \) in a symmetric way. To this end, the usual "boson calculus" is replaced with the so-called "symplecton calculus", that is, instead of the boson vacuum \( |0\rangle \) satisfying \( \bar{a} |0\rangle = 0 \), the formal ket \( |\rangle \) which is not annihilated by both \( \bar{a} \) and \( a \) is introduced. The representation bases in the realization (II.1) are formed by letting \( P_j^m \) act on \( |\rangle \), and the action of generators on the bases is defined by \( J_a |jm\rangle = [J_a, P_j^m] |\rangle \). There exists an appropriate definition of an inner product for these \( |jm\rangle \), so that we obtain the usual unitary representations of \( sl(2) \) with spin \( j \).
The explicit form of the polynomials $P^m_j(a, \bar{a})$ is found by solving $[J_+, P^j] = 0$ to obtain $P^j_j = a^{2j}$, and then using the action of $J_-$ to calculate $P^m_j$.

\[
P^m_j(a, \bar{a}) = \frac{1}{2^{j-m}} \left[ \frac{(2j)!}{(j+m)!} \right]^{1/2} \sum_{s=0}^{j-m} \frac{\bar{a}^{j-m-s}a^{j+m-s}}{s!(j-m-s)!}.
\]

An alternative form for $P^m_j$ is obtained by starting with $P^{-j} = \bar{a}^{2j}$ and then using the action of $J_+$.

\[
P^m_j(a, \bar{a}) = \frac{1}{2^{j+m}} \left[ \frac{(2j)!}{(j-m)!} \right]^{1/2} \sum_{s=0}^{j+m} \frac{a^sa^{j-m-s}}{s!(j+m-s)!}.
\]

We would like to list some properties of $sl(2)$ symplecton. For their proof or detail, we refer the reader to Refs. [7, 8].

1. A set of polynomials \( \{P^m_j(a, \bar{a}) \mid m = -j, -j + 1, \ldots, j\} \) forms representation bases for the Lie group $SL(2)$ as well as the Lie algebra $sl(2)$. The boson commutation relation is covariant under the action of $SL(2)$ defined by

\[
(a', \bar{a}') = (a, \bar{a}) \begin{pmatrix} x & u \\ v & y \end{pmatrix},
\]

where the $2 \times 2$ matrix is an element of $SL(2)$. The transformed polynomial $P^m_j(a', \bar{a}')$ is decomposed into $P^m_j(a, \bar{a})$ multiplied by polynomials in the entries of $SL(2)$ matrix.

\[
P^m_j(a', \bar{a}') = \sum_n P^n_j(a, \bar{a}) d^1_{nm}(g), \quad g \in SL(2)
\]

The $(2j+1) \times (2j+1)$ matrix $d^1_{nm}(g)$ gives an irrep. of $SL(2)$ and is called Wigner's $d$-function in terminology of physics.

2. The polynomials $P^m_j(a, \bar{a})$ have a generating function. Let $\xi, \eta$ be ordinary c-numbers commuting with $a, \bar{a}$. Then

\[
(\xi a + \eta \bar{a})^{2j} = \sqrt{(2j)!} \sum_{m=-j}^j \Phi^j m(\xi, \eta) P^m_j(a, \bar{a}),
\]

where $\Phi^j m$ are well-known representation bases of both $sl(2)$ and $SL(2)$,

\[
\Phi^j m(\xi, \eta) = \frac{\xi^{j+m} \eta^{j-m}}{\sqrt{(j+m)!(j-m)!}}.
\]

Irreps. of $sl(2)$ are constructed on (II.9) by the realization

\[
J_+ = \xi \frac{d}{d\eta}, \quad J_- = \eta \frac{d}{d\xi}, \quad J_0 = \xi \frac{d}{d\xi} - \eta \frac{d}{d\eta},
\]

while irreps. of $SL(2)$ are obtained by the following transformation

\[
(\xi', \eta') = (\xi, \eta) \begin{pmatrix} x & u \\ v & y \end{pmatrix},
\]

\[\text{(II.11)}\]
it follows that
\[ \Phi_{\eta'}^\eta(\xi', \eta) = \sum_n \Phi_n^\eta(\xi, \eta) d_{nm}(g), \] (II.12)
where we have obtained the same d-function as (II.7).

(3) The symplecton polynomials can be expressed in terms of Gauss hypergeometric function \( _2F_1(a, b; c; z) \). The polynomial \( _2F_1(a, b; c; z) \) is defined by
\[ _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n, \] (II.13)
where \((a)_n\) stands for the sifted factorial
\[ (a)_n = \begin{cases} 1 & n = 0 \\ a(a + 1) \cdots (a + n - 1) & n = 1, 2, \cdots \end{cases} \] (II.14)
Now define the operator \( N = a\bar{a} \), then the symplecton \( P^m_j(a, \bar{a}) \) is written in terms of \( _2F_1(a, b; c; z) \) with \( z = -1 \) and the parameters \( a, c \) become functions of operator \( N \). The expression (II.4) becomes
\[ P^m_j = \frac{1}{2^{j+m}} \left[ \frac{(2j)!}{(j+m)! (j-m)!} \right]^{1/2} \frac{(N+j-m)!}{(N-2m)!} \times _2F_1(-N+2m, -j+m; -N-j+m; -1)(\bar{a})^{-2m}. \] (II.15)
In this way, properties of \( P^m_j \) are reduced to properties of the hypergeometric function. Especially, the equivalence of two form (II.4) and (II.5) is explained by the formula
\[ _2F_1(a, b; c; z) = (1-z)^{a-b} _2F_1(c-a, c-b; c; z). \] (II.16)

(4) The polynomials \( P^m_j(a, \bar{a}) \) are transformed under the action \( a \to \bar{a}, \bar{a} \to -a \)
\[ P^m_j(a, -a) = (-1)^{j-m} P^m_j(a, \bar{a}). \] (II.17)
To define an inner product for the bases \( \langle jm \rangle = P^m_j \) ), the property (II.17) and the product formula discussed below play a crucial role.

(5) Let \( P^m_j \) and \( P^{m'}_{j'} \) be the symplecton polynomials, then they obey the product law
\[ P^m_j P^{m'}_{j'} = \sum_{k=|j-j'|}^{j+j'} \langle k | j \rangle \langle j' | k \rangle C^m_{m', m, m+m'} P_k^{m+m'}, \] (II.18)
where
\[ \langle k | j \rangle \langle j' | k' \rangle = 2^{k-j-j'} (2k+1)^{-1/2} \nabla(k,j,j'), \]
\[ \nabla(abc) = \left[ \frac{(a+b+c+1)!}{(a+b-c)! (a-b+c)! (-a+b+c)!} \right]^{1/2}, \] (II.19)
and \( C^m_{m', m, m+m'} \) is the Clebsch-Gordan coefficient (CGC) for \( sl(2) \). The associativity of the products \( (P^\alpha_a P^\beta_b P^\gamma_c) = P^\alpha_a (P^\beta_b P^\gamma_c) \) gives a relation between "triangle functions"
\[ \nabla(acf) \nabla(bdf) = (2f+1) \sum_e W(abcd; ef) \nabla(abc) \nabla(cde), \] (II.20)
where $W(\abcd;ef)$ is the Racha coefficient.

The inner product for $|jm\rangle$ is defined by

$$\langle jm|jm'\rangle = \langle (-1)^{j-m}P_j^{-m} \cdot P_j^{m'} | \rangle,$$  

and the operation $\langle \cdots | \rangle$ means to take only the $j=0$ part of the expression $\cdots$.

Applying the product law (II.18) to the RHS of (II.21), we see that that the $j=0$ part is given by the CGC $C^{i,j,0}_{m,m',0}$, so that the bases $|jm\rangle$ are orthonormal.

### III Jordanian Deformation of $sl(2)$ and $SL(2)$

The Jordanian quantum algebras $\mathcal{U}_h(\mathfrak{g})$ are obtained from the (universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of ) Lie algebras $\mathfrak{g}$ from Drinfeld twist \cite{14}. We denote the coproduct, counit and antipode for $\mathcal{U}(\mathfrak{g})$, when it is regarded as a Hopf algebra, by $\Delta, \epsilon, S$, respectively. With the invertible element $\mathcal{F} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ satisfying

$$\epsilon \otimes \mathcal{I} \mathcal{F} = \mathcal{I} \epsilon \mathcal{F} = 1,$$  

\begin{equation}
\mathcal{F}_{12} \Delta \otimes \mathcal{I} \mathcal{F} = \mathcal{F}_{23} \mathcal{I} \otimes \Delta \mathcal{F},
\end{equation}

the algebra $\mathcal{U}_h(\mathfrak{g})$ is defined by the same commutation relations as $\mathfrak{g}$ and the following Hopf algebra mappings

$$\tilde{\Delta} = \mathcal{F} \Delta \mathcal{F}^{-1}, \quad \tilde{\epsilon} = \epsilon, \quad \tilde{S} = uSu^{-1},$$  

where $u = m(id \otimes S)(\mathcal{F})$, $u^{-1} = m(S \otimes id)(\mathcal{F}^{-1})$, $m$ denotes the usual product in $\mathfrak{g}$. This is a triangular Hopf algebra whose universal $R$-matrix is given by $R = \mathcal{F}_{21} \mathcal{F}^{-1}$.

For the case of $\mathfrak{g} = sl(2)$, $\mathcal{F}$ is given by \cite{13}

$$\mathcal{F} = \exp \left( -\frac{1}{2} J_0 \otimes \sigma \right), \quad \sigma = -\ln(1 - 2hJ_+),$$  

The twist element $\mathcal{F}$ used here gives different form of $\mathcal{U}_h(sl(2))$ from the one in Ref. \cite{13}. The relationship between these two form is given in Appendix A. The explicit form of Hopf algebra mappings for $\mathcal{U}_h(sl(2))$ is summarized in Appendix B (some of them will be used in the later computation). An application of the $\mathcal{U}_h(sl(2))$ to the Heisenberg spin chain is found in Ref. \cite{13}. The finite dimensional highest weight irreps. for $\mathcal{U}_h(sl(2))$ are same as $sl(2)$, because of the same commutation relations. We shall use the following lemmas on tensor product representations in subsequent sections.

**Lemma III.1** \cite{13} Let $V^{j_1}, V^{j_2}$ be the representation space with the highest weight $j_1, j_2$. Then the tensor product of them is completely reducible, i.e.

$$V^{j_1} \otimes V^{j_2} = \bigoplus_{j=|j_1-j_2|} V^j,$$

and the bases of $V^j$ are given by

$$e_m^{(j_1j_2)j} = \sum C_{m_1,m_2,m}^{j_1,j_2,j} F_{k_1,k_2,m_1,m_2} e_{k_1}^{j_1} \otimes e_{k_2}^{j_2},$$  

where $C_{m_1,m_2,m}^{j_1,j_2,j}$ is the CGC of $sl(2)$ and $F_{k_1,k_2,m_1,m_2}^{j_1,j_2}$ is the matrix element of $\mathcal{F}$ on $V^{j_1} \otimes V^{j_2}$.
The explicit form of matrix elements $F_{j_1,k_1, m_1, j_2,k_2, m_2}$ is given in Appendix C. It seems to be the first time to show the explicit form of $F_{j_1,k_1, m_1, j_2,k_2, m_2}$ in the literature, and this also gives the explicit form of the $R$-matrix for $U_h(sl(2))$.

**Lemma III.2** The Racha coefficients for $sl(2)$ and $U_h(sl(2))$ coincide.

Lemma III.2 is proved in Appendix D.

The matrix quantum group dual to $U_h(sl(2))$ is called the Jordanian quantum group $SL_h(2)$. It is generated by four elements $x,y,u$ and $v$ subject to the relations

\[ [v, x] = hv^2, \quad [u, x] = h(1 - x^2), \]
\[ [v, y] = hv^2, \quad [u, y] = h(1 - y^2), \]
\[ [x, y] = h(xv - yv), \quad [v, u] = h(xv + vy). \]  \hspace{1cm} (III.6)

It follows that the central element of $SL_h(2)$ which gives the determinant of the quantum matrix

\[ T = \begin{pmatrix} x & u \\ v & y \end{pmatrix}, \]  \hspace{1cm} (III.7)

is defined by

\[ detT = xy - uv - hv = 1. \]  \hspace{1cm} (III.8)

The $SL_h(2)$ has a Hopf algebra structure. The relations (III.6) and Hopf algebra mappings are summarized in the FRT-formalism with the $R$-matrix

\[ R = \begin{pmatrix} 1 & h & -h & h^2 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  \hspace{1cm} (III.9)

The coproduct, the counit and the antipode are given by

\[ \Delta(T) = T \otimes T, \]
\[ \epsilon(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
\[ S(T) = T^{-1} = \begin{pmatrix} y - hv & -u - h(y - x) + h^2v \\ -v & x + hv \end{pmatrix}. \]

Let us define the $d$-function for $SL_h(2)$ using the notion of comodule. A vector space $M$ is called right $SL_h(2)$ comodule if there is a map $\rho : M \rightarrow M \otimes SL_h(2)$ such that the following relations are satisfied

\[ (\rho \otimes id) \circ \rho = (id_M \otimes \rho) \circ \rho, \quad (id_M \otimes \epsilon) \circ \rho = id_M, \]  \hspace{1cm} (III.10)

where $id_M$ stands for the identity map in $M$. Using bases $e_i$ of $M$, the map $\rho$ is written as

\[ \rho(e_i) = \sum_j e_j \otimes \tilde{d}_{ji}, \]  \hspace{1cm} (III.11)
it follows that the relations (III.10) are rewritten as

\[ \Delta(\tilde{d}ij) = \sum_k \tilde{d}ik \otimes \tilde{d}kj, \quad \epsilon(\tilde{d}ij) = \delta_{ij}. \quad (III.12) \]

We call the \( \tilde{d}ij \) satisfying (III.11) and (III.12) the \( d \)-function for \( SL_h(2) \). In the following sections, we deal with the case in which the vector space \( M \) has an algebraic structure. It is natural, in this case, to require that the map \( \rho \) should respect the extra structure on \( M \).

### IV Tensor Operators and Twist

To define the symplecton for \( U_h(sl(2)) \), it is necessary to extend the notion of tensor operators to Hopf algebra. This has been carried out by Rittenberg and Scheunert [13]. Tensor operators are defined for each realization of the Hopf algebra \( \mathcal{H} \) under consideration. Assuming that we have a realization of \( \mathcal{H} \), we first define the adjoint action.

**Definition IV.1** Let \( W, W' \) be representation space of \( \mathcal{H} \), and let \( t \) be an operator which carries \( W \) into \( W' \). Then the adjoint action of \( X \in \mathcal{H} \) on \( t \) is defined by

\[ adX(t) = m(id \otimes S)(\Delta(X)(t \otimes 1)). \quad (IV.1) \]

The adjoint action has two important properties

\[ adXX'(t) = adX \circ adX'(t), \quad adX(t \otimes s) = \sum_i adX_i(t) \otimes adX'_i(s), \quad (IV.2) \]

where the coproduct for \( X \) is written as \( \Delta(X) = \sum_i X_i \otimes X'_i \). From these properties, we see that the adjoint action gives a representation of \( \mathcal{H} \)

\[ ad[X, X'](t) = [adX, adX'](t). \quad (IV.3) \]

Tensor operators for \( \mathcal{H} \) are defined as operators which form representation bases of \( \mathcal{H} \) under the adjoint action.

**Definition IV.2** Let \( D(X) \) be a representation matrix of \( X \in \mathcal{H} \). The operators \( t_\alpha \) are called the tensor operator, if they satisfy the relation

\[ adX(t_\alpha) = \sum_\beta D(X)_{\beta\alpha}t_\beta. \quad (IV.4) \]

If the representation is irreducible, the tensor operators are called irreducible tensor operators.

The explicit form of the adjoint action for \( U_h(sl(2)) \) reads

\[ adJ_0(t) = [J_0, t]e^{-\sigma}, \]

\[ adJ_+(t) = e^{-\sigma}[J_+e^\sigma, t], \quad (IV.5) \]

\[ adJ_-(t) = [J_- + hJ_0 + \frac{h}{2}J_0^2, t]e^{-\sigma} - h[J_0, t]e^{-2\sigma} - \frac{h}{2}[J_0, [J_0, t]]e^{-2\sigma}. \]
Some examples of the $U_h(sl(2))$ tensor operators are considered in Ref.\cite{17} and they are applied to construct boson algebra which is covariant under the action of Jordanian matrix quantum groups \cite{18}.

Since the coproduct for Lie algebra and Jordanian quantum algebra is related via twist element (III.3), tensor operators for these algebras are also related by twisting via $\mathcal{F}$ \cite{13}.

**Lemma IV.1** Let $t_\alpha$ be tensor operators for Lie algebra $\mathfrak{g}$ and $\tilde{t}_\alpha$ be corresponding ones for Jordanian quantum algebra $U_h(\mathfrak{g})$. Then these tensor operators are related via the twist element $\mathcal{F}$

$$
\tilde{t}_\alpha = m(id \otimes \tilde{S})(\mathcal{F}(t_\alpha \otimes 1)\mathcal{F}^{-1}), \quad (IV.6)
$$

$$
t_\alpha = m(id \otimes S)(\mathcal{F}^{-1}(\tilde{t}_\alpha \otimes 1)\mathcal{F}). \quad (IV.7)
$$

**Proof** : The first relation (IV.6) is derived in Ref.\cite{19} (Proposition 3). The second one (IV.7) is its inverse. The expression used in Lemma [IV.1] is different form Ref.\cite{19}, it may be good to show the second relation as an example of the proof. It is proved by showing the substitution of (IV.7) into (IV.6) gives the identity map.

Let us write the twist element and its inverse as

$$
\mathcal{F} = \sum f^a \otimes f_a, \quad \mathcal{F}^{-1} = \sum g^a \otimes g_a,
$$

then

$$
u = \sum f^a S(f_a), \quad \nu^{-1} = \sum S(g^a)g_a,
$$

and the relation (IV.6) becomes

$$
\tilde{t}_\alpha = \sum f^a t_\alpha g^b \tilde{S}(f_ag_b) = \sum f^a t_\alpha g^b uS(f_ag_b)u^{-1} = \sum f^a t_\alpha S(f_a)u^{-1}, \quad (IV.8)
$$

where we used

$$
\sum g^b uS(g_b) = \sum g^b f^a S(g_b f_a) = m(id \otimes S)(\mathcal{F}^{-1}\mathcal{F}) = 1.
$$

On the other hand, the relation (IV.7) is rewritten

$$
t_\alpha = \sum g^a \tilde{t}_\alpha f^b S(g_a f_b) = \sum g^a \tilde{t}_\alpha uS(g_a). \quad (IV.9)
$$

Substituting (IV.9) into (IV.8)

$$
\tilde{t}_\alpha = \sum f^a g^b \tilde{t}_\alpha uS(f_ag_b)u^{-1} = \sum f^a g^b \tilde{t}_\alpha \tilde{S}(f_ag_b) = m(id \otimes \tilde{S})(\mathcal{F}\mathcal{F}^{-1}(\tilde{t}_\alpha \otimes 1)) = \tilde{t}_\alpha.
$$

This proves the second relation in Lemma [IV.1]. \qed
V  Symplecton Polynomials for $\mathcal{U}_h(sl(2))$

In this section, we derive the explicit form of the symplecton for $\mathcal{U}_h(sl(2))$ and investigate its properties. Since $\mathcal{U}_h(sl(2))$ has the same commutation relations as $sl(2)$, $\mathcal{U}_h(sl(2))$ and $sl(2)$ have the same realizations. Therefore the symplecton realization for $\mathcal{U}_h(sl(2))$, which is identical to the one for $sl(2)$, is the realization in terms of the usual boson operators. This is a contrast to the $q$-symplecton where the $q$-deformed boson operators are used.

Let $\bar{a}$ and $a$ be boson operators satisfying $[\bar{a}, a] = 1$, then the generators of $\mathcal{U}_h(sl(2))$ are realized by

$$J_+ = -\frac{1}{2}a^2, \quad J_- = \frac{1}{2}\bar{a}^2, \quad J_0 = \frac{1}{2}(a\bar{a} + \bar{a}a), \quad (V.1)$$

The $h$-symplecton, denoted by $\tilde{P}_j^m(a, \bar{a})$, is defined as a polynomial in $\bar{a}, a$ satisfying

$$adJ_\pm(\tilde{P}_j^m) = \sqrt{(j \mp m)(j \pm m + 1)}\tilde{P}_j^{m\pm 1},$$
$$adJ_0(\tilde{P}_j^m) = 2m\tilde{P}_j^m, \quad (V.2)$$

where the adjoint action on the LHS is given by (IV.5). Using Lemma IV.1, the explicit form of $h$-symplecton is obtained from the corresponding one for $sl(2)$.

**Proposition V.1** The explicit form of the $h$-symplecton defined by (V.2) is given by

$$\tilde{P}_j^m(a, \bar{a}) = P_j^m(a, \bar{a})e^{m\sigma}, \quad (V.3)$$

where $\sigma$ is given in (III.4) and $P_j^m(a, \bar{a})$ denotes $sl(2)$ symplecton.

**Proof**: By definition of $sl(2)$ symplecton, it holds that

$$(J_0 - 2m)P_j^m = P_j^m J_0.$$ 

Using this and the RHS of (IV.8),

$$\tilde{P}_j^m = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)^n P_j^m (J_0 + 2m)^n S(\sigma)^n u^{-1} = P_j^m \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^n (2m)^s}{2^s (n-s)!} J_0^{n-s} S(\sigma)^n u^{-1}. \quad (V.4)$$

Changing the order of sum, then replacing $n-s$ with $n$, we obtain

$$\tilde{P}_j^m = P_j^m \sum_{s,n=0}^{\infty} \frac{(-1)^{n+s} (2m)^s}{2^{n+s} n!} J_0^n S(\sigma)^{n+s} u^{-1}. \quad (V.4)$$

Note that

$$u = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \frac{1}{n!} J_0^n S(\sigma)^n,$$

and (III.3), it follows that (V.4) is rewritten as

$$\tilde{P}_j^m = P_j^m \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s \frac{(2m)^s}{s!} S(\sigma)^s = P_j^m e^{m\sigma},$$

where (B.4) is used in the last equality. \qed
We would like to show some explicit form of $h$-symplecton. For $j = 1/2$

$$\hat{P}^{-1/2}_{1/2} = \bar{a}e^{-\sigma/2} \equiv \bar{a}_h, \quad \hat{P}^{1/2}_{1/2} = \bar{a}e^{\sigma/2} \equiv a_h,$$

(V.5)

and for $j = 1$

$$\hat{P}^{-1}_{1} = \bar{a}^2e^{-\sigma} = \bar{a}_h^2 + h\bar{a}_ha_h,$$

$$\hat{P}^{0}_{1} = (\bar{a}a + a\bar{a})/\sqrt{2} = (\bar{a}_ha_h + a_h\bar{a}_h - ha_h^2)/\sqrt{2},$$

$$\hat{P}^{1}_{1} = a^2e^{\sigma} = a_h^2.$$ (V.6)

The $j = 1/2$ $h$-symplecton forms covariant $h$-deformed oscillator algebra

$$[\bar{a}_h, a_h] = 1 - ha_h^2,$$ (V.7)

i.e., the commutation relation (V.7) is preserved under the action of $SL_h(2)$

$$(a'_h, \bar{a}'_h) = (a_h, \bar{a}_h) \left( \begin{array}{cc} x & u \\ v & y \end{array} \right).$$ (V.8)

This shows that it is possible to construct representations of $\text{SL}_h(2)$ on $h$-symplecton. We shall discuss it later. It may be worth noting that the action (V.8) is different from the ones in [18, 19] where $a$ and $\bar{a}$ are not mixed by the action of quantum groups.

The $j = 1$ $h$-symplecton forms an algebra isomorphic to $sl(2)$. Its commutation relations are

$$[P^0_1, P^1_1] = 2\sqrt{2}P^1_1(1 - hP^1_1),$$

$$[P^0_1, P^{-1}_1] = -2\sqrt{2}P^{-1}_1(1 - hP^1_1),$$

$$[P^1_1, P^{-1}_1] = -2\sqrt{2}(1 - hP^1_1)P^0_1.$$ (V.9)

The generators of $sl(2)$ are written in terms of $P^m_1$

$$J_+ = -\frac{1}{2}P^1_1(1 - hP^1_1), \quad J_0 = \frac{1}{\sqrt{2}}P^0_1, \quad J_- = \frac{1}{2}P^{-1}_1(1 - hP^1_1).$$ (V.10)

We see, from the explicit form of $h$-symplecton (V.3), that the $h$ dependence of polynomial $\hat{P}^m_j(a, \bar{a})$ is absorbed in $\sigma$ which is a infinite polynomial in $a^2$. Recall that the relationship between $sl(2)$ symplecton and Gauss hypergeometric function $\text{2F}_1$ is given in terms of the operator $N = a\bar{a}$, then we see that the factor $e^{m\sigma}$ in (V.3) does not affect this relationship. Therefore the specific hypergeometric function for $h$-symplecton may be again $\text{2F}_1$.

The fact that the $j = 1/2$ $h$-symplecton forms covariant $h$-oscillator algebra may suggest that it is useful to write $h$-symplecton in terms of covariant $h$-oscillators (V.3).

**Proposition V.2** The $h$-symplecton is written in terms of covariant $h$-oscillators as follows. The corresponding expression for (V.4) is

$$\hat{P}^m_j(a_h, \bar{a}_h) = \frac{1}{2j - m} \frac{1}{[2j!(j - m)!]}^{1/2} \sum_{s=0}^{j-m} \frac{1}{s!(j - m - s)!} \bar{a}_h(\bar{a}_h + ha_h) \cdots (\bar{a}_h + (j - m - s - 1)ha_h)a_h^{j+m}$$

$$\times \bar{a}_h - (2m + s)ha_h \cdots (\bar{a}_h - (2m + s - 1)ha_h) \cdots (\bar{a}_h - (2m + 1)ha_h),$$ (V.11)
and for (II.4) is
\[ \tilde{P}_j^m(a_h, \bar{a}_h) = \frac{1}{2j+m} \left[ \frac{(2j)!(j+m)!}{(j-m)!} \right]^{1/2} \frac{1}{j+m} \sum_{s=0}^{j+m} \frac{1}{s!(j+m-s)!} \times a_h^s (\bar{a}_h - hsa_h) \{ \bar{a}_h + h(1-s)a_h \} \cdots \{ \bar{a}_h + h(j-m-1-s)a_h \} a_h^{j+m-s}. \] (V.12)

**Proof:** From (V.3)
\[ \tilde{a} = \tilde{a}_he^{\sigma/2}, \quad a = a_he^{-\sigma/2}. \]
Substituting these into (II.4) and (II.5), then straightforward calculation proves the proposition. \( \square \)

In order to discuss generating functions for h-symplecton, it is possible to apply Lemma IV.1 to the generating function (II.8) for sl(2) symplecton, since the RHS of (II.8) is a sum of tensor operators of sl(2). It follows that the RHS of (II.8) becomes the sum of h-symplecton; \( \sqrt{(2j)!} \sum_{m=-j}^{j} \Phi_{jm} \tilde{P}_j^m. \) However the LHS may be quite complicated and may not be in closed form. Another way to obtain generating functions for h-symplecton is to substitute (V.3) and (V.5) into (II.8)
\[ (\xi a_h e^{-\sigma/2} + \eta \bar{a}_h e^{\sigma/2})^{2j} = \sqrt{(2j)!} \sum_{m=-j}^{j} \Phi_{jm}(\xi, \eta) \tilde{P}_j^m(a_h, \bar{a}_h) e^{-m\sigma}. \] (V.13)

It is possible to remove \( \sigma \) from (V.13) by using the relation \( e^\sigma + ha_h^2 = 1 \), however the obtained relation is quite complicated. Therefore the simplest generating function for h-symplecton may be (V.13) where the \( \sigma \) is regarded as an independent quantity subject to the relations
\[ [\sigma, a_h] = 0, \quad [\sigma, \bar{a}_h] = 2ha_h, \] (V.14)
and \( \lim_{h \to 0} \sigma = 0. \)

**VI Product Law for h-Symplecton**

It is possible to extend the product law (II.18) for sl(2) symplecton to h-symplecton. The product law plays a crucial role when the symplecton calculus is considered. In this section, we first prove the product law for h-symplecton by using the one for sl(2) symplecton, then consider the symplecton calculus for \( U_h(sl(2)) \).

**Theorem VI.1** Let \( \tilde{P}_j^m \) and \( \tilde{P}_{j'}^{m'} \) be h-symplecton, then these obey the product law
\[ \tilde{P}_j^m \tilde{P}_{j'}^{m'} = \sum_{k=|j-j'|}^{j+j'} \sum_{n,n'} \langle k | j | j' \rangle (F^{-1})_{n,n',m,m'} C_{n',j',j,m,n,m',m} \tilde{P}_k^{n+m}, \] (VI.1)
where $C^{j_1 j_2 j}_m$ is CGC for $sl(2)$ and

$$
\langle k | j | j' \rangle = 2^{k-j-j'}(2k+1)^{-1/2} \nabla(kjj'),
$$

$$
\nabla(abc) = \left[ \frac{(a + b + c + 1)!}{(a + b - c)!(a - b + c)!(a + c)!!} \right]^{1/2}.
$$

Proof: From Proposition VI.1.

Using the Hopf algebra mappings for $\sigma$ given in (B.4), we see that the adjoint action of $\sigma^{e,\sigma}$ is given by

$$
adj_{\sigma^{e,\sigma}}(t) = \sigma^{e,\sigma}te^{-\sigma^{e,\sigma}}.
$$

h-Symplecton is a irreducible tensor operator of $U_h(sl(2))$, it follows that

$$
e^{\sigma_{\alpha}} \tilde{P}_{j'}^{m'} e^{-\sigma_{\alpha}} = adj_{\sigma^{e,\sigma}}(\tilde{P}_{j'}^{m'}) = \sum_{n',m'} (\sigma^{e,\sigma})_{n',m'} \tilde{P}_{j'}^{m'}
$$

$$
= \sum_{n',m'} \delta_{n,m} (\sigma^{e,\sigma})_{n',m'} \tilde{P}_{j'}^{m'}
$$

$$
= \sum_{n',m'} (F^{-1})_{n',m',m} \tilde{P}_{j'}^{m'},
$$

where the matrix elements of $F$ (C.4) is used in the last equality. Therefore, we have

$$
\tilde{P}_{j}^{m} \tilde{P}_{j'}^{m'} = \sum_{n',m'} (F^{-1})_{n',m',m} \tilde{P}_{j}^{m} \tilde{P}_{j'}^{m'} e^{\sigma_{\alpha}} = \sum_{n',m'} (F^{-1})_{n',m',m} \tilde{P}_{j}^{m} \tilde{P}_{j'}^{m'} e^{\sigma_{\alpha}(n'+m)}
$$

Applying the product law (II.18) for $sl(2)$ symplecton, Theorem VI.1 is proved. \hfill \Box

Corollary VI.1 The associativity of the products $(\tilde{P}_{a}^{\alpha} \tilde{P}_{b}^{\beta} \tilde{P}_{c}^{\gamma}) = \tilde{P}_{a}^{\alpha} (\tilde{P}_{b}^{\beta} \tilde{P}_{c}^{\gamma})$ gives the same relation as (II.20) for the triangle function $\nabla(abc)$ appeared in Theorem VI.2.

Proof: The associativity gives the same relation as (II.20), but the Racha coefficients are replaced with the ones for $U_h(sl(2))$. From Lemma III.2, these two kinds of Racha coefficients coincide. \hfill \Box

Let us now consider the $h$-symplecton calculus. We assume the formal ket $| \rangle$ and that both $a| \rangle$ and $a| \rangle$ are nonvanishing vectors. Then the vectors defined by $|jm\rangle = \tilde{P}_{j}^{m} | \rangle$ are irreps. bases of $U_h(sl(2))$ provided that the action of $X \in U_h(sl(2))$ is defined by $X|jm \rangle = adj X(\tilde{P}_{j}^{m}) | \rangle$. The dual bases are defined by $\langle jm | = \langle \tilde{P}_{j}^{-m}(-1)^{j-m} \rangle$ in order to keep the correspondence with the $h = 0$ case. The action of $X \in U_h(sl(2))$ is, of course, given by $\langle jm | = \langle adj X(\tilde{P}_{j}^{-m})(-1)^{j-m}$ The inner product is defined in the same manner as $h = 0$ case, namely,

$$
\langle jm | j' m' \rangle = \langle \tilde{P}_{j}^{-m}(-1)^{j-m} \rangle,
$$

12
the operation \( \langle j | \cdots | j' \rangle \) means to take only the \( j = 0 \) part of the expression \( \cdots \). Applying the product law (VI.1) for \( h \)-symplecton, we obtain
\[
\langle jm | j'm' \rangle = \delta_{j,j'} 2^{-2j} F_{-m,m}^{j,j'}.
\] (VI.5)

Therefore the vectors \(|jm\rangle\) and \(|j'm'\rangle\) are orthonormal if they belong to different irreps. but not orthonormal if they belong to a same irrep. The nonvanising part on the RHS of (VI.5) depends on only the twist element \( F \).

From the product law, we can show the following relations for \( h \)-symplecton

**Proposition VI.1** The following relations hold for \( h \)-symplecton.
\[
\sum_{m,m'} \tilde{P}^m_j \hat{P}^{m'}_{j'} F_{m,m'}^{j,j'} = \sum_k \langle k | j | j' \rangle C^{j,j'}_{j',j,k} \hat{P}^k_{k+k'},
\] (VI.6)
\[
\hat{P}^{m'}_{j'}(a_h, \bar{a}_h + 2hma_h) = \sum_{\ell=0} \langle F^{-1} \rangle_{n,n'}^{j,j'} m,m' \tilde{P}^{m'}_{j'}(a_h, \bar{a}_h).
\] (VI.7)

**Proof:** The relation (VI.6) is easily proved by multiplying the product law (VI.1) by \( F_{n,n'}^{j,j'} m,m' \) and summing over \( m,m' \). The relation (VI.7) is derived by moving \( e^{m\sigma} \) to the right of \( P_{j'}^{m'} \) in (VI.4). One can do that by using the relations
\[
e^{m\sigma} a_h = a_h e^{m\sigma}, \quad e^{m\sigma} \bar{a}_h = (\bar{a}_h + 2hma_h)e^{m\sigma}.
\]

\[\blacksquare\]

**VII Quantum \( h \)-Plane and Representations of \( SL_h(2) \)**

The Jordanian quantum algebra \( \mathcal{U}_h(sl(2)) \) and Jordanian quantum group \( SL_h(2) \) are dual each other. It follows that any representation basis of \( \mathcal{U}_h(sl(2)) \) is also representation basis for \( SL_h(2) \) belonging to the same representation. Since \( h \)-symplecton is a irrep. basis of \( \mathcal{U}_h(sl(2)) \), it is also a irrep. basis of \( SL_h(2) \). We have seen this for \( j = 1/2 \) in §5. The relation (V.8) can be generalized to arbitrary \( j \)
\[
\hat{P}^m_j(a'_h, \bar{a}'_h) = \sum_n \hat{P}^n_j(a_h, \bar{a}_h) \hat{d}^n_{nm}(g) \quad g \in SL_h(2).
\] (VII.1)

We can obtain \( d \)-functions for \( \mathcal{U}_h(sl(2)) \) by substituting (V.8) into the explicit form of \( h \)-symplecton given in Proposition VI.2. However, as is seen from the explicit form, the actual computation seems to be complicated.

The use of quantum \( h \)-plane [20] provides us a procedure which is a little bit simpler in computation. In this section, we shall find irrep. bases for \( SL_h(2) \) in terms of quantum \( h \)-plane which give the same irreps. as \( h \)-symplecton by using the tensor operator approach. Recall that the functions \( \Phi_{jm}(\xi, \eta) \) defined by (II.9) are irrep. bases of \( sl(2) \) in the realization (II.10) and irrep. bases of \( SL(2) \) under (II.11) as well. We can regard \( \Phi_{jm}(\xi, \eta) \) as a irreducible tensor operator of \( sl(2) \), since it is easy to verify that
\[
[J_\pm, \Phi_{jm}] = \sqrt{(j \mp m)(j \pm m + 1)} \Phi_{jm \pm 1},
\]
\[
[J_0, \Phi_{jm}] = 2m \Phi_{jm}.
\] (VII.2)
From Lemma [IV.1] it is easy to find the corresponding irreducible tensor operators for $U_h(sl(2))$.

**Proposition VII.1** Let $\xi, \eta$ be commutative numbers, then the followings are irreducible tensor operators for $U_h(sl(2))$

$$\tilde{\Phi}_jm(\xi, \eta) = \Phi jm(\xi, \eta)e^{m\sigma},$$  \hspace{1cm} (VII.3)

where $\sigma = -\ln(1 - 2h\xi \frac{d}{dy}).$

For $j = 1/2$, we have

$$\Phi \frac{1}{2} \frac{1}{2} = \xi e^{\sigma/2} \equiv \xi_h, \quad \tilde{\Phi} \frac{1}{2} - \frac{1}{2} = \eta e^{-\sigma/2} \equiv \eta_h,$$  \hspace{1cm} (VII.4)

and they satisfy the commutation relation

$$[\xi_h, \eta_h] = h\xi_h^2,$$  \hspace{1cm} (VII.5)

this corresponds to the commutation relation of quantum $h$-plane in Ref.[20]. It is easily verified that the commutation relation (VII.5) is preserved under the action of $SL_h(2)$

$$(\xi'_h, \eta'_h) = (\xi_h, \eta_h) \begin{pmatrix} x & u \\ v & y \end{pmatrix}.$$  \hspace{1cm} (VII.6)

It is an easy exercise to write $\tilde{\Phi}_jm$ in terms of $\xi_h$ and $\eta_h$. Then $\tilde{\Phi}_jm(\xi_h, \eta_h)$ forms irrep. bases of $SL(2)$, that is, the $d$-functions for $SL_h(2)$ are obtained by substituting (VII.6) into $\tilde{\Phi}_jm(\xi_h, \eta_h)$.

**Proposition VII.2** Irreps. of $SL_h(2)$ on the quantum $h$-plane are obtained by

$$\tilde{\Phi}_jm(\xi'_h, \eta'_h) = \sum_k \tilde{\Phi}_jk(\xi_h, \eta_h) d^j_{km},$$  \hspace{1cm} (VII.7)

where the irrep. bases are given by

$$\tilde{\Phi}_jm = c_{jm} \xi_h^{j+m}(\eta_h - h(j + m)\xi_h)(\eta_h - h(j + m - 1)\xi_h)\cdots(\eta_h - h(2m + 1)\xi_h),$$

$$= c_{jm} \eta_h(\eta_h + h\xi_h)\cdots(\eta_h + (j - m - 1)h\xi_h) \xi_h^{-j-m},$$  \hspace{1cm} (VII.8)

with

$$c_{jm} = \frac{1}{\sqrt{(j + m)!(j - m)!}}.$$

Since $\tilde{P}_jm$ and $\tilde{\Phi}_jm$ give the same irreps. of $U_h(sl(2))$, they also give the same $d$-functions of $SL_h(2)$. Indeed, the explicit computation shows that we obtain the same $d$-functions for $j = 1/2$ and $j = 1$. The $j = 1/2$ case gives the $2 \times 2$ quantum matrix $T$ (III.7) itself, while $j = 1$ $d$-function reads

$$d^1 = \begin{pmatrix} x^2 + hxv & \sqrt{2}(ux + hv) & u^2 + hu(x + y + hv) \\ \sqrt{2}xv & 1 + 2uv & \sqrt{2}(uy + hv) \\ uv^2 & \sqrt{2}yv & y^2 + hyv \end{pmatrix}.$$  \hspace{1cm} (VII.9)

The $d$-functions for $SL_h(2)$ are also discussed in Ref.[21] where the authors assert that the $d$-functions can be obtained from the q-deformed ones via a contraction method and show some explicit examples. Another way to obtain the $d$-functions is to use the recurrence relations for $d$-functions. This will be discussed in a separate publication.
VIII Concluding Remarks

We have constructed \( h \)-symplecton in this article and investigated some of its properties. It has been seen that many properties of \( sl(2) \) symplecton are inherited to \( h \)-symplecton. Unfortunately, \( h \)-dependence of \( h \)-symplecton is absorbed in \( \sigma \), namely, twist element \( \mathcal{F} \), so that we can not see specific hypergeometric function for \( h \)-deformation. It will become clear what kind of hypergeometric functions are specific to \( h \)-deformed quantities if we obtain explicit form of \( d \)-function for \( SL_h(2) \) as in the case of \( q \)-deformed \( SU(2) \)[22]. The \( sl(2) \) symplecton has a simple generating function. We presented (V.1 3) as a generating function for \( h \)-symplecton. However, this may be one of possible choices, we might find simpler generating function. The use of quantum \( h \)-plane \( \xi_h, \eta_h \) instead of \( \xi, \eta \) is one of the possibilities. We have done some calculation to find simpler form of generating function in terms of \( \xi_h \) and \( \eta_h \), however, all what we obtained have more complicated form.

We would like to emphasize the usefulness of Lemma [IV.1]. This provides us a much simpler procedure to obtain \( h \)-symplecton than starting with the definition (V.2) and using the lemma, we could easily find another irrep. bases (VII.8) for \( SL_h(2) \). This lemma is, of course, applicable to any Jordanian quantum algebra, since we usually know the explicit form of twist element. Furthermore, the lemma is extended to quasitriangular Hopf algebras [23]. For quasitriangular Hopf algebras, the twist elements are usually not known, they are known up to certain order of the deformation parameters. It is expected that many properties of tensor operators for quasitriangular Hopf algebras are studied based on the present knowledge of the tensor operators for Lie algebras via Lemma [V.1], even if the explicit form of tensor operators is not obtained. It may also be possible to apply Lemma [V.1] to the investigation of \( q \)-symplecton.

Appendices

A Relation to Ohn’s \( U_h(sl(2)) \)

Ohn defined in Ref.[6] \( U_h(sl(2)) \) as an algebra generated by \( H, X \) and \( Y \) subject to

\[
[X, Y] = H, \quad [H, X] = 2 \frac{\sinh hX}{h}, \quad [H, Y] = -Y(\cosh hX) - (\cosh hX)Y. \tag{A.1}
\]

Meanwhile, the commutation relations of \( J_\pm, J_0 \), which are generators of \( U_h(sl(2)) \) in this article, are same as \( sl(2) \). These two kinds of generators are related by

\[
H = e^{-\sigma/2}J_0, \quad X = \frac{\sigma}{2h}, \quad Y = e^{-\sigma/2}(J_- + \frac{h}{2}J_0^2) - \frac{h}{8}e^{\sigma/2}(e^{-\sigma} - 1). \tag{A.2}
\]

By this relation, not only the commutation relations but also the Hopf algebra mappings are transformed each other. The relation (A.2) corresponds to the one parameter case discussed in Ref.[24] where two parameter Jordanian deformation of \( gl(2) \) is considered.
B Hopf Algebra Structure of $\mathcal{U}_h(sl(2))$

We here give explicit formulae for the coproduct, counit and antipode of $\mathcal{U}_h(sl(2))$ calculated from (III.3).

(i) coproduct
\[
\tilde{\Delta}(J_0) = J_0 \otimes e^\sigma + 1 \otimes J_0 \\
\tilde{\Delta}(J_+) = J_+ \otimes 1 + e^{-\sigma} \otimes J_+ \\
\tilde{\Delta}(J_-) = J_- \otimes e^\sigma + 1 \otimes J_- - hJ_0 \otimes e^\sigma J_0 - \frac{h}{2} J_0 (J_0 + 2) \otimes e^\sigma (e^\sigma - 1).
\]

(ii) counit
\[
\tilde{\epsilon}(X) = 0, \quad X = J_+, J_0.
\]

(iii) antipode
\[
\tilde{S}(J_0) = -J_0 e^{-\sigma}, \\
\tilde{S}(J_+) = -J_+ e^\sigma, \\
\tilde{S}(J_-) = -J_- e^\sigma - \frac{h}{2} J_0^2 (e^\sigma + 1) e^{-\sigma} + h J_0 (e^{-\sigma} - 1) e^\sigma.
\]

All of these are reduced to the ones for $sl(2)$ in the limit of $h = 0$. The Hopf algebra mappings for $\sigma$ have simple form
\[
\tilde{\Delta}(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma, \quad \tilde{\epsilon}(\sigma) = 0, \quad \tilde{S}(\sigma) = -\sigma.
\]

C Matrix Elements of $\mathcal{F}$

In this appendix, we show the explicit formula of matrix elements of the twist element $\mathcal{F}$ (III.4) and some of their properties. We denote a irrep. basis of $\mathcal{U}_h(sl(2))$ by the bracket notation $|jm\rangle$ for the sake of simplicity.

It is easily verified the following relations from (III.3)
\[
\tilde{\Delta}(J_\pm)\mathcal{F}|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\mathcal{F}|j_1 m_1 \pm 1\rangle \otimes |j_2 m_2\rangle \\
+ \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\mathcal{F}|j_1 m_1\rangle \otimes |j_2 m_2 \pm 1\rangle,
\]
\[
\tilde{\Delta}(J_0)\mathcal{F}|j_1 m_1\rangle \otimes |j_2 m_2\rangle = 2 (m_1 + m_2)\mathcal{F}|j_1 m_1\rangle \otimes |j_2 m_2\rangle.
\]

It shows that the vectors $\mathcal{F}|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ for $\mathcal{U}_h(sl(2))$ play the same role as $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ for $sl(2)$. Eq. (III.3) is readily obtained from this. Another proof of Lemma III.1 with the bases of $\mathcal{U}_h(sl(2))$ in Ref. [1] is found in Refs. [25, 26].

In the bracket notation, matrix elements of $\mathcal{F}$ are defined by
\[
F_{j_1,j_2, m_1, m_2}^{k_1,k_2} = \langle j_1 k_1 | \otimes \langle j_2 k_2 | \mathcal{F}|j_1 m_1\rangle \otimes |j_2 m_2\rangle.
\]

We first show a relationship between the matrix elements of $\mathcal{F}$ and its inverse
\[
F_{-n_1,-n_2, -m_1,-m_2}^{j_1,j_2} = (F^{-1})_{m_1,m_2}^{j_1,j_2}.
\]
The LHS of (C.2) is calculated as
\[
\text{LHS} = \langle j_1 - n_1 | \otimes | j_2 - n_2 | \sum_{\ell=0}^{\infty} \frac{(-J_0)^\ell \otimes \sigma^\ell}{2^\ell \ell!} | j_1 - m_1 \rangle \otimes | j_2 - m_2 \rangle,
\]
where \( \langle j_1 - n_1 | J_0 = -2n_1 \langle j_1 - n_1 | \) is used. While the RHS is
\[
\text{RHS} = \langle j_1 m_1 | \otimes \langle j_2 m_2 | \sum_{\ell=0}^{\infty} \frac{J_0^\ell \otimes \sigma^\ell}{2^\ell \ell!} | j_1 n_1 \rangle \otimes | j_2 n_2 \rangle = \delta_{m_1, n_1} \langle j_2 m_2 | \exp(n_1 \sigma) | j_2 n_2 \rangle.
\]

Note that
\[
J_+ | jm \rangle = \sqrt{(j - m)(j + m + 1)} | jm + 1 \rangle,
\]
\[
\langle j - m | J_+ = \sqrt{(j - m)(j + m + 1)} | j - m - 1 \rangle.
\]

It follows that any polynomials in \( J_+ \), denoted by \( f(J_+) \), satisfies
\[
\langle jm | f(J_+) | jn \rangle = \langle j - n | f(J_+) | j - m \rangle.
\]

Since \( \sigma \) is a polynomial in \( J_+ \), we see that the (C.3) equals to (C.4). Thus (C.2) has been proved.

We next show that the matrix elements of \( \mathcal{F} \) are given by
\[
F_{j_1,j_2, m_1,m_2}^{k_1,k_2} = \delta_{k_1,m_1} \theta(m_2 \leq k_2 \leq j_2) S_{k_2,m_2}^{j_2} \times \begin{cases}
\frac{(2k_2 - 2m_1 - 2m_2 - 2)!}{(k_2 - m_2)!(-2m_1 - 2)!!} k_2^{-m_2}, & \text{for } m_1 \leq 0 \\
(2\ell + 2m_1 - 2)!! \frac{2^\ell \ell!}{2^\ell \ell!(-2m_1 - 2)!!} \sum_{\ell=0}^{m_2} (-1)^\ell \left( \frac{2m_1}{k_2 - m_2 - \ell} \right) & \text{for } m_1 > 0
\end{cases}
\]
where \( n!! = 1 \) for \( n \leq 0 \) and \( \theta(m_2 \leq k_2 \leq j_2) = 1 \) if and only if the inequality in the parenthesis holds, otherwise \( \theta \) vanishes. \( S_{k_2,m_2}^{j_2} \) is defined by
\[
S_{k_2,m_2}^{j_2} = \left\{ \frac{(j_2 - m_2)!(j_2 + k_2)!}{(j_2 + m_2)!(j_2 - k_2)!} \right\}^{1/2}.
\]
To prove (C.6), note that similar to (C.4) we have
\[
F_{j_1,j_2, m_1,m_2}^{k_1,k_2} = \delta_{k_1,m_1} \langle j_2 k_2 | e^{-m_1 \sigma} | j_2 m_2 \rangle.
\]

One can use the power series expansion in order to compute the RHS of (C.7)
\[
(1 - X)^{-\ell/2} = \sum_{n=0}^{\infty} \frac{(2n + \ell - 2)!!}{2^n n!(\ell - 2)!!} X^n, \quad \ell \in \mathbb{Z}_+.
\]

(i) For \( m_1 \leq 0 \)
Let \( m_1 = -\ell/2 \ (\ell \in \mathbb{Z}_+) \) and using (C.8)
\[
e^{-m_1\sigma} |j_2 m_2\rangle = (1 - 2hJ_+)^{-\ell/2} |j_2 m_2\rangle
\]
\[
= \sum_{n=0}^{j_2 - m_2} \frac{(2n + \ell - 2)!!}{2^{n!(\ell - 2)!!}} (2h)^n \left\{ \frac{(j_2 - m_2)!}{(j_2 + m_2)!} \frac{(j_2 + m_2 + n)!}{(j_2 - m_2 - n)!} \right\}^{1/2} |j_2 m_2 + n\rangle.
\]
Therefore \( \langle j_2 k_2 | e^{-m_1\sigma} |j_2 m_2\rangle \) takes values if and only if \( k_2 = m_2 + n \). This proves the first part of (C.6).
(ii) For \( m_1 > 0 \)
Let \( m_1 = \ell/2 \ (\ell \in \mathbb{Z}_+) \). Since
\[
e^{-m_1\sigma} |j_2 m_2\rangle = e^{\ell\sigma/2} e^{\ell\sigma/2} |j_2 m_2\rangle,
\]
we can apply the previous result to compute \( e^{\ell\sigma/2} |j_2 m_2\rangle \) and then applying the binomial expansion to \( e^{\ell\sigma} = (1 - 2hJ_+)^\ell \)
\[
e^{-m_1\sigma} |j_2 m_2\rangle
\]
\[
= \sum_{t=0}^{j_2 - m_2} \frac{(2t + \ell - 2)!!}{2^{t!(\ell - 2)!!}} (2h)^t \left\{ \frac{(j_2 - m_2)!}{(j_2 + m_2)!} \frac{(j_2 + m_2 + t)!}{(j_2 - m_2 - t)!} \right\}^{1/2} |j_2 m_2 + t\rangle
\]
\[
= \sum_{n=0}^{\ell} \sum_{t=0}^{j_2 - m_2} \left( \frac{\ell}{n} \right) \frac{(2t + \ell - 2)!!}{2^{t!(\ell - 2)!!}} (-1)^n (2h)^{t+n} \left\{ \frac{(j_2 - m_2)!}{(j_2 + m_2)!} \frac{(j_2 + m_2 + t + n)!}{(j_2 - m_2 - t - n)!} \right\}^{1/2} |j_2 m_2 + t + n\rangle.
\]
Replacing \( t + n \) with \( n \), we obtain
\[
e^{-m_1\sigma} |j_2 m_2\rangle = \sum_{t,n} (-1)^t (-2h)^n \left( \frac{\ell}{n - t} \right)
\]
\[
\times \frac{(2t + \ell - 2)!!}{2^{t!(\ell - 2)!!}} \left\{ \frac{(j_2 - m_2)!}{(j_2 + m_2)!} \frac{(j_2 + m_2 + n)!}{(j_2 - m_2 - n)!} \right\}^{1/2} |j_2 m_2 + n\rangle.
\]
Again \( \langle j_2 k_2 | e^{-m_1\sigma} |j_2 m_2\rangle \) takes values if and only if \( k_2 = m_2 + n \). This completes the proof of (C.6).
We can obtain the explicit formula for the universal \( R \)-matrix in the irreps. with highest weight \( j_1 \) and \( j_2 \) by combining the (C.6) and relation (C.2), since universal \( R \)-matrix for \( \mathcal{U}_h(sl(2)) \) is given by \( R = \mathcal{F}_{21} \mathcal{F}^{-1} \).

D Proof of Lemma III.2

Let \( V^a, V^b \) and \( V^c \) be representation spaces of \( \mathcal{U}_h(sl(2)) \) with highest weight \( a, b \) and \( c \), respectively. Bases of each space are denoted as \( e^a_{\alpha}, \ -a \leq \alpha \leq a \). We would like to construct irrep. bases in the space \( V^a \otimes V^b \otimes V^c \) in two ways, namely, \( (V^a \otimes V^b) \otimes V^c \) and \( V^a \otimes (V^b \otimes V^c) \). According to the discussion in Appendix B, irrep. bases in the space \( V^a \otimes V^b \) are given by
\[
e^{(ab)d}_\delta = \sum C^{a,b,d}_{\alpha,\beta,\delta} \mathcal{F} e^a_\alpha \otimes e^b_\beta.
\]
Then we couple these with the bases in $V^c$ to obtain
\[
\psi_e^{\epsilon} = \sum C^{d,c,e}_{\delta,\gamma,\epsilon} (\tilde{\Delta} \otimes \text{id})(\mathcal{F}) e^{ab\delta}_{\epsilon} \otimes e^{\epsilon}_{\gamma} = \sum C^{d,c,e}_{\delta,\gamma,\epsilon} C^{a,b,d}_{\alpha,\beta,\delta} (\Delta \otimes \text{id})(\mathcal{F}) F_{12} e^{a}_{\alpha} \otimes e^{b}_{\beta} \otimes e^{\epsilon}_{\gamma}.
\]
Similarly we obtain the following bases when we couple $V^b$ and $V^c$ first
\[
\psi_e^{\epsilon} = \sum C^{a,f,e}_{\alpha,\rho,\epsilon} C^{b,c,f}_{\beta,\gamma,\rho} (\text{id} \otimes \tilde{\Delta})(\mathcal{F}) F_{23} e^{a}_{\alpha} \otimes e^{b}_{\beta} \otimes e^{\epsilon}_{\gamma}.
\]
From (B.4)
\[
(id \otimes \tilde{\Delta})(\mathcal{F}) = \exp(-\frac{1}{2} J_0 \otimes \tilde{\Delta}(\sigma)) = F_{12} F_{13}.
\]
Using the relations (III.3), (III.2) and above
\[
(\tilde{\Delta} \otimes \text{id})(\mathcal{F}) = F_{23}(id \otimes \Delta)(\mathcal{F}) F_{12}^{-1}
\]
and
\[
F_{12} F_{13} F_{23} F_{12}^{-1}.
\]
It follows that $\psi_e^{\epsilon}$ and $\psi_e^{\epsilon}$ are rewritten as
\[
\psi_e^{\epsilon} = \sum e^{a}_{\alpha} \otimes e^{b}_{\beta} \otimes e^{\epsilon}_{\gamma},
\]
\[
\psi_e^{\epsilon} = \sum C^{a,f,e}_{\alpha,\rho,\epsilon} C^{b,c,f}_{\beta,\gamma,\rho} F_{12} F_{13} F_{23} e^{a}_{\alpha} \otimes e^{b}_{\beta} \otimes e^{\epsilon}_{\gamma}.
\]
The Racha coefficients $W_h(abce; df)$ for $U_h(sl(2))$ is defined by
\[
\psi_e^{\epsilon} = \sum f \sqrt{(2d + 1)(2f + 1)} W_h(abce; df) \psi_e^{\epsilon}.
\]
It it now obvious that the Racha coefficients for $U_h(sl(2))$ satisfy the relation
\[
\sum_{\delta} C^{d,c,e}_{\delta,\gamma,\epsilon} C^{a,b,d}_{\alpha,\beta,\delta} = \sum_{f,\rho} C^{a,f,e}_{\alpha,\rho,\epsilon} C^{b,c,f}_{\beta,\gamma,\rho} \sqrt{(2d + 1)(2f + 1)} W_h(abce; df).
\]
This is the same relation for the Racha coefficients for $sl(2)$. This proves Lemma III.2.

References

[1] See for example, L. C. Biedenharn and J. D. Louck, Angular Momentum in Quantum Physics: Theory and Application, Encyclopedia of Mathematics and Its Applications 8, Addison-Wesley, Reading, Massachusetts (1981).

[2] See for example, L. C. Biedenharn and M. A. Lohe, Quantum Group Symmetry and q-Tensor Algebras, World Scientific (1995).

[3] E. E. Demidov et al, Prog. Theor. Phys. Suppl. 102, 203 (1990).

[4] S. Zakrewski, Lett. Math. Phys. 22, 287 (1991).

[5] H. Ewen, O. Ogievetsky and J. Wess, Lett. Math. Phys. 22, 297 (1991).
[6] Ch. Ohn, Lett. Math. Phys. 25, 85 (1992).

[7] L. C. Biedenharn and J. D. Louck, Ann. Phys. (N.Y.) 63, 459 (1971).

[8] L. C. Biedenharn and J. D. Louck, The Racah-Wigner Algebra in Quantum Theory, Encyclopedia of Mathematics and Its Applications 9, Addison-Wesley, Reading, Massachusetts (1981).

[9] M. A. Lohe, L. C. Biedenharn and J. D. Louck, Phys. Rev. D43, 617 (1991).

[10] L. C. Biedenharn and M. A. Lohe, in Quantum Groups, Proceedings of the Argonne Workshop, ed. T. Curtright, D. Fairlie and C. Zachos, World Scientific (1991).

[11] M. Nomura, J. Math. Phys. 33, 3636 (1992).

[12] I. M. Gelfand and D. B. Fairlie, Comm. Math. Phys. 136, 487 (1991).

[13] V. Rittenberg and M. Scheunert, J. Math. Phys. 33, 436 (1992).

[14] V. G. Drinfeld, Leningrad Math. J. 1, 1419 (1990). See also, N. Yu. Reshetikhin, Lett. Math. Phys. 20, 331 (1990).

[15] P. P. Kulish and A. A. Stolin, Czech. J. Phys. 47, 123 (1997).

[16] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, Leningrad Math. J. 1, 193 (1990).

[17] N. Aizawa, J. Phys. A:Math. Gen. 31, 5467 (1998).

[18] C. Quesne, Czech. J. Phys. 48, 1471 (1998).

[19] G. Fiore, J. Math. Phys. 39, 3437 (1998).

[20] V. Karimipour, Lett. Math. Phys. 30, 87 (1994).

[21] R. Chakrabarti and C. Quesne, preprint, math.QA/9811064.

[22] T. Masuda et al., J. Funct. Anal. 99, 357 (1991), and references therein.

[23] G. Fiore, preprint, q-alg/9708017.

[24] N. Aizawa, Czech. J. Phys. 48, 1273 (1998).

[25] N. Aizawa, J. Phys. A:Math. Gen. 30, 5981 (1997).

[26] J. Van der Jeugt, J. Phys. A:Math. Gen 31, 1495 (1998), Czech. J. Phys. 47, 1283 (1997).

[27] P. P. Kulish and E. V. Damaskinsky, J. Phys. A:Math. Gen. 23, L415 (1990). M. Chaichian, P. P. Kulish and J. Lukierski, Phys. Lett. B262, 43 (1991).