A REMARK ON MISHCHENKO-FOMENKO ALGEBRAS AND REGULAR SEQUENCES

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Abstract. In this note, we show that the free generators of the Mishchenko-Fomenko subalgebra of a complex reductive Lie algebra, constructed by the argument shift method at a regular element, form a regular sequence. This result was proven by Serge Ovsienko in the type A at a regular and semisimple element. Our approach is very different, and is strongly based on geometric properties of the nilpotent bicone.

1. Introduction

Let \( q \) be a finite-dimensional Lie algebra over the field of complex numbers \( \mathbb{C} \). The symmetric algebra \( S(q) \cong \mathbb{C}[q^*] \) carries a natural Poisson structure. Denote by \( Z(q) \) the Poisson center of \( S(q) \). Let \( \xi \in q^* \) and consider the Mishchenko-Fomenko subalgebra \( \mathcal{F}_\xi(q) \) of \( S(q) \) constructed by the so-called argument shift method [13]. It is generated by the \( \xi \)-shifts of elements in \( Z(q) \), that is, \( \mathcal{F}_\xi(q) \) is generated by all the derivatives \( D^j_\xi(p) \) for \( p \in Z(q) \) and \( j \in \{0, \ldots, \deg p - 1\} \), where

\[
D^j_\xi(p)(x) = \frac{d^j}{dt} p(x + t\xi)|_{t=0}, \quad x \in q^*.
\]

It is well-known that \( \mathcal{F}_\xi(q) \) is a Poisson-commutative subalgebra of \( S(q) \). Furthermore,

\[
\text{trdeg}(\mathcal{F}_\xi(q)) \leq \frac{\dim q + \text{ind } q}{2} =: b(q)
\]

where \( \text{ind } q \) is the index of \( q \), that is, the minimal dimension of the stabilizers of linear forms on \( q \) for the coadjoint representation [7]. Let \( q^*_\text{reg} \) be the set of regular elements of \( q^* \), that is, those elements whose stabilizer in \( q \) has the minimal dimension \( \text{ind } q \), and \( q^*_\text{sing} := q^* \setminus q^*_\text{reg} \).

Theorem 1.1 (Panyushev-Yakimova [16]). Assume that the following two conditions are satisfied:

1. \( Z(q) \) contains algebraically independent homogeneous elements \( f_1, \ldots, f_\ell \), with \( \ell = \text{ind } q \), such that \( \sum_{i=1}^\ell \deg f_i = b(q) \),

2. the codimension of \( q^*_\text{sing} \) in \( q^* \) is greater than or equal to 3.

Then for any \( \xi \in q^*_\text{reg} \), the Mishchenko-Fomenko algebra \( \mathcal{F}_\xi(q) \) is a polynomial algebra of Krull dimension \( b(q) \), and it is a maximal Poisson-commutative algebra of \( S(q) \).

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Theorem 1.1 generalizes the result of Tarasov [19] for semisimple Lie algebras which are known to satisfy the above conditions (in [19], the maximality is proved for \( \xi \) regular and semisimple).

**Question 1.** In the case that \( q \) satisfies the conditions (1) and (2) of Theorem 1.1, do the free generators \( D_\xi^j(f_i), i = 1, \ldots, \ell, j = 0, \ldots, \deg f_i - 1 \) of \( F_\xi(q) \), for \( \xi \in g_{\text{reg}} \), form a regular sequence?

The above question is discussed for instance in [16, Remark 3.4]. The motivations come from Gelfand-Zetlin modules (cf. [20, 15]), and quantizations of Mishchenko-Fomenko algebras [8, 18].

In more details, if the answer to Question 1 is positive and if \( F_\xi(q) \) admits a quantization, that is, a (maximal) commutative subalgebra \( \tilde{F}_\xi(q) \subset U(q) \) such that \( \text{gr} \tilde{F}_\xi(q) \cong F_\xi(q) \), then \( U(q) \) is free over \( \tilde{F}_\xi(q) \) [9]. This implies for instance that any \( \mu \in \text{Specm} \tilde{F}_\xi(q) \) lifts to a simple \( U(q) \)-module, i.e., there exists a simple \( U(q) \)-module \( M_\mu \) generated by \( m \in M_\mu \), such that for all \( \gamma \in \tilde{F}_\xi(q) \), \( \gamma m = \chi_\mu(\gamma)m \), where \( \chi_\mu: \tilde{F}_\xi(q) \to \mathbb{C} \) is the character corresponding to \( \mu \) [9].

From now on, let \( g \) be a reductive Lie algebra with adjoint group \( G \), and identify \( g \) with \( g^* \) through an invariant inner product \( (\cdot | \cdot) \). According to a result of Chevalley, the algebra \( S(g)^\theta = Z(g) \) is polynomial in \( \ell \) variables, where \( S(g)^\theta = S(g)^G \) is the subalgebra of \( S(g) \) consisting of \( G \)-invariant elements. The nilpotent cone \( \mathcal{N} \) of \( g \) is by definition the subscheme of \( g \) defined by the augmentation ideal of \( S(g)^\theta \). It is well-known since Kostant [11] that \( \mathcal{N} \) is a complete intersection of codimension \( \ell \).

In other words, homogeneous generators \( p_1, \ldots, p_\ell \) of \( S(g)^\theta \) form a regular sequence in \( g \).

Let us fix such generators, and order them so that \( d_1, \ldots, d_\ell \) is an increasing sequence with \( d_i \) the degree of \( p_i \). The Mishchenko-Fomenko algebra \( F_\xi(g) \), for \( \xi \in g \), is then generated by the elements \( D_\xi^j(p_i) \) for \( i = 1, \ldots, \ell \) and \( j = 0, \ldots, d_i - 1 \).

Let \( g_{\text{reg}} \) be the set of regular elements of \( g \) and set \( b := b(g) \).

**Theorem 1.2.** Assume that \( \xi \in g_{\text{reg}} \). Then the free generators of \( F_\xi(g) \) form a regular sequence. Namely, for \( \xi \in g_{\text{reg}} \), the family \( \{ D_\xi^j(p_i) : i = 1, \ldots, \ell, j = 0, \ldots, d_i - 1 \} \) forms a regular sequence in \( g \). Equivalently, the natural morphism

\[
\begin{align*}
\sigma: & \quad g \longrightarrow \text{Spec} F_\xi(g) \cong \mathbb{C}^b, \\
& \quad x \mapsto (D_\xi^j(p_i)(x) : i = 1, \ldots, \ell, j = 0, \ldots, d_i - 1)
\end{align*}
\]

induced by the inclusion of algebras \( F_\xi(g) \subset S(g) \) is faithfully flat, that is, the extension \( S(g) \) of \( F_\xi(g) \) is faithfully flat.

As mentioned in [16, Remark 3.4], the above result was proved by Ovsienko [15] for \( g = \mathfrak{sl}_n(\mathbb{C}) \) and \( \xi \) regular and semisimple.

Our proof is very different. It is based on geometric properties of the nilpotent bicone (cf. Definition 2.1) introduced and studied in [3]. We recall in Section 2 the main results of [3] on the nilpotent bicone. As a consequence we get Theorem 1.2 for \( \xi \) nilpotent and regular. The proof of Theorem 1.2 for an arbitrary regular \( \xi \) is completed in Section 3. In Section 4 we discuss the case where \( q \) is the centralizer a nilpotent element of \( g \), and formulate a conjecture.
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2. Nilpotent bicone

We assume in this section that $g$ is simple, and we identify $g$ with $g^*$ through the Killing form $(\ | \ )$.

For $p$ a homogeneous element of $S(g)^g$, define elements $p^{(j)}$ of $(S(g) \otimes_C S(g))^g \cong C[g \times g]^g$ by

$$p(sx + ty) = \sum_{j=0}^{\deg p} p^{(j)}(x, y)s^{\deg p-j}t^j,$$

for all $(s, t) \in \mathbb{C}^2$ and $(x, y) \in g \times g$. Thus for $j = 0, \ldots, \deg p$ and $(x, \xi) \in g \times g$,

$$p^{(j)}(x, \xi) = \frac{1}{j!} D^j \xi(p)(x).$$

Definition 2.1 ([3]). The nilpotent bicone $\tilde{\mathcal{N}}$ of $g$ is by definition the subscheme of $g \times g$ defined by the ideal generated by the elements $p^{(j)}_i$ for $i = 1, \ldots, \ell$ and $j = 0, \ldots, d_i$,

$$\tilde{\mathcal{N}} = \text{Spec } C[g \times g]/(p^{(j)}_i, i = 1, \ldots, \ell, j = 0, \ldots, d_i).$$

Thus a point $(x, y) \in g \times g$ lies in $\tilde{\mathcal{N}}$ if and only if the vector span generated by $x$ and $y$ is contained in nilpotent cone $\mathcal{N}$.

Set

$$\Omega := \{(x, y) \in g \times g \mid \text{span}_\mathbb{C}(x, y) \setminus (0, 0) \subset g_{\text{reg}} \text{ and dim span}_\mathbb{C}(x, y) = 2\}.$$

Denote by $\varpi_1$ and $\varpi_2$ the first and second projections from $g \times g$ to $g$,

$$\varpi_1: \quad g \times g \longrightarrow g \quad \varpi_2: \quad g \times g \longrightarrow g$$

$$(x, y) \longmapsto x, \quad (x, y) \longmapsto y.$$

Theorem 2.2 ([3]).

(1) The nilpotent bicone is a complete intersection of dimension $3(b - \ell)$.

(2) The images by $\varpi_1$ and $\varpi_2$ of any irreducible component of $\tilde{\mathcal{N}}$ are equal to $\mathcal{N}$.

(3) The intersection $\Omega \cap \tilde{\mathcal{N}}$ is precisely the set of smooth points of $\tilde{\mathcal{N}}$, that is, the set of $(x, y)$ such that the differentials of the $p^{(j)}_i$‘s at $(x, y)$ are linearly independent.

Note that the scheme $\tilde{\mathcal{N}}$ is not reduced [3]. Since the algebra $C[g \times g]$ is Cohen-Macaylay, and since the elements $p^{(j)}_i$ are homogenous, part (1) of Theorem 2.2 implies that any subset of the set $(p^{(j)}_i, i = 1, \ldots, \ell, j = 0, \ldots, d_i)$ forms a regular sequence in $g \times g$. [12].

From Theorem 2.2, (1) and (2), we get the following.
Corollary 2.3. Let $e$ be a regular nilpotent element of $\mathfrak{g}$. Then the fiber $\tilde{\mathcal{N}}_e$ of the restriction to $\mathcal{N}$ of $\varpi_1$ (resp. $\varpi_2$) at $e$ is a complete intersection of dimension $b - \ell$.

3. PROOF OF THEOREM 1.2

For $\xi \in \mathfrak{g}$, denote by $\mathcal{Y}_\xi(\mathfrak{g})$ the subscheme of $\mathfrak{g}$ defined by the elements $D^i_j(\mu_i)$, $i = 1, \ldots, \ell, j = 0, \ldots, d_i - 1$ of $F_\xi(\mathfrak{g}) \subset S(\mathfrak{g})$. Since the algebra $\mathbb{C}[\mathfrak{g}]$ is Cohen-Macaulay and since the elements $\mathcal{D}_j^i(\mu_i)$ are homogeneous, to prove that for $\xi \in \mathfrak{g}_{\text{reg}}$, the elements $D^i_j(\mu_i)$, $i = 1, \ldots, \ell, j = 0, \ldots, d_i - 1$ form a regular sequence, we have to prove that for $\xi \in \mathfrak{g}_{\text{reg}}$, the scheme $\mathcal{Y}_\xi(\mathfrak{g})$ is equidimensional of dimension $b - \ell$.

Note that each irreducible component of $\mathcal{Y}_\xi(\mathfrak{g})$ has at least dimension $b - \ell$.

Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ be the simple factors of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{z} \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s$, with $\mathfrak{z}$ the center of $\mathfrak{g}$, and fix $\xi \in \mathfrak{g}_{\text{reg}}$. From

$$\mathcal{Y}_\xi(\mathfrak{g}) \cong \mathcal{Y}_\xi(\mathfrak{g}_1) \times \cdots \times \mathcal{Y}_\xi(\mathfrak{g}_s),$$

we can assume that $\mathfrak{g}$ is simple.

Corollary 2.3 gives Theorem 1.2 for $\xi$ regular and nilpotent since for such $\xi$, $\mathcal{Y}_\xi(\mathfrak{g}) \cong \tilde{\mathcal{N}}_e$. It remains to generalize the statement for an arbitrary regular $\xi \in \mathfrak{g}$.

Let $(e, h, f)$ be a principal $\mathfrak{sl}_2$-triple, that is, $e$ is regular nilpotent. Then consider the Kostant’s slice

$$\mathcal{S}_e := e + \mathfrak{g}^f,$$

where $\mathfrak{g}^f$ is the centralizer of $f$ in $\mathfrak{g}$. This is an affine subspace of $\mathfrak{g}$ which consists of regular elements. Moreover, for any regular element $\mu \in \mathfrak{g}$, the $G$-orbit of $\mu$ intersects $e + \mathfrak{g}^f$ at one point [11]. Thus

$$\mathfrak{g}_{\text{reg}} = G.\mathcal{S}_e.$$  

Since $\dim \mathcal{Y}_\xi(\mathfrak{g}) = \dim \mathcal{Y}_\xi(e)$ for any $g \in G$, we can assume that $\xi \in \mathcal{S}_e$.

Let $U$ be the set of $\mu \in \mathfrak{g}$ such that $\dim \mathcal{Y}_\mu(\mathfrak{g}) = b - \ell$. It is an open subset of $\mathfrak{g}$ which contains $e$ by Corollary 2.3. So $U \cap \mathcal{S}_e$ is a nonempty subset of $\mathcal{S}_e$ which contains $e$. Hence for any $\mu$ in a nonempty neighborhood $W$ of $e$ in $\mathcal{S}_e$, $\dim \mathcal{Y}_\mu(\mathfrak{g}) = b - \ell$. Consider the one-parameter subgroup $\rho: \mathbb{C}^* \to G$ of $G$ defined by

$$\forall t \in \mathbb{C}^*, \quad \rho(t).x = t^{-2} \bar{\rho}(t).x$$

where $\bar{\rho}: \mathbb{C}^* \to G$ is the one-parameter subgroup of $G$ defined by $h$. Then $\rho$ induces a contracting $\mathbb{C}^*$-action on $\mathcal{S}_e$, meaning that

$$\forall x \in \mathcal{S}_e, \quad \rho(t).x \in \mathcal{S}_e, \quad \rho(t).e = e \quad \text{and} \quad \lim_{t \to 0} \rho(t).x = e.$$  

So for some $t \in \mathbb{C}^*, \rho(t).\xi \in W$. But for any $t \in \mathbb{C}^*$,

$$\dim \mathcal{Y}_\xi(\mathfrak{g}) = \dim \mathcal{Y}_{\rho(t).\xi}(\mathfrak{g}),$$

whence $\dim \mathcal{Y}_\xi(\mathfrak{g}) = b - \ell$, as desired.

Remark 3.1. To generalize the statement to any arbitrary regular $\xi$, we have used Kostant’s slice. This can also be deduced from the construction of Borho-Kraft [2] about deformations of $G$-orbits.
It remains to prove that the morphism $\sigma$ is faithfully flat for $\xi \in g_{\text{reg}}$. As $\mathcal{F}_\xi(g)$ is generated by homogeneous functions, the fiber at 0 of the morphism $\sigma$ has maximal dimension. But by what foregoes, $\sigma^{-1}(0) \cong \mathcal{F}_\xi(g)$ has codimension $b$ in $g$. On the other hand, by [16, Theorem 0.1], $\mathcal{F}_\xi(g)$ is a polynomial algebra in $b$ variables. So $\sigma$ is an equidimensional morphism and by [12, Ch. 8, Theorem 21.3], $\sigma$ is a flat morphism. In particular by [10, Ch. III, Exercise 9.4], it is an open morphism whose image contains 0. So $\sigma$ is surjective. Hence $\sigma$ is faithfully flat, according to [12, Ch. 3, Theorem 7.2].

4. Centralizers of nilpotent elements

Other interesting examples to consider come from the centralizers of nilpotent elements.

Assume that $q$ is the centralizer $g^e$ of a nilpotent element $e$ of $g$. Then the index of $q = g^e$ is equal to $\ell$ by [4], and the algebra $S(g^e)^q$ is known to be polynomial for a large number of element $e$ (cf. e.g. [17, 5]).

According to the main results of [5, 6], we have a characterization of nilpotent elements $e$ for which $S(g^e)^q$ is polynomial, and homogeneous free generators form a regular sequence. They are called good elements in [5]. In more details, for $p \in S(g)$, let $^q p$ be the initial homogeneous component of its restriction to $\mathcal{F}_e = e + g^f$, with $(e, h, f)$ an $\mathfrak{sl}_2$-triple of $g$. By [17], if $p \in S(g)^q$, then $^q p \in S(g^e)^q$. Consider now the following condition:

\[ (*) : \text{ for some homogeneous free generators } q_1, \ldots, q_\ell \text{ of } S(g)^q, \text{ we have } \sum_{i=1}^\ell \deg q_i = b(g^e). \]

By [5, 6], the condition $(*)$ is satisfied if and only if $e$ is good. In addition, we have the following result:

**Theorem 4.1** (Arakawa-Premet [1]). Assume that the condition $(*)$ and the condition (2) of Theorem 1.1 are satisfied. Then $\mathcal{F}_\xi(g^e)$ admits a quantization $\mathcal{F}_\xi(g^e) \subset U(g)$.

The conditions $(*)$ and (2) are satisfied for $e = 0$, and (at least) in the following cases: $g = \mathfrak{sl}_n(\mathbb{C})$ and $e$ arbitrary ([17, 22]), $g$ is simple not of type $E_8$ and $e$ is in the minimal nilpotent orbit of $g$ ((17, 1)).

The fact that homogeneous free generators of $S(g^e)^q$ form a regular sequence when $g = \mathfrak{sl}_n(\mathbb{C})$ was known by [17, Theorem 5.4]. The fact that $\mathcal{F}_\xi(g^e)$ admits a quantization $\mathcal{F}_\xi(g^e) \subset U(g)$ for $e = 0$ comes from [18, 8].

In view of the above remarks, we formulate a conjecture.

**Conjecture 1.** Assume that the condition $(*)$ and the condition (2) of Theorem 1.1 are satisfied. Then the free generators of $\mathcal{F}_\xi(g^e)$ form a regular sequence for any $\xi \in (g^e)^*_\text{reg}$.

Conjecture 1 holds for $e = 0$ (Theorem 1.2), for regular nilpotent $e$ (since $g^e$ is commutative in this case), for subregular nilpotent $e \in \mathfrak{sl}_n(\mathbb{C})$ (easy computations),
was proved by Tomoyuki Arakawa and Vyacheslav Futorny for minimal nilpotent $e \in \mathfrak{sl}_n(\mathbb{C})$ (private communication) and by Wilson Fernando Mutis Cantero for any nilpotent $e \in \mathfrak{gl}_n(\mathbb{C}), n = 3, 4$. Note that $S(\mathfrak{g}e)$ is not always polynomial, cf. [21, 5, 22]. Also, even when $S(\mathfrak{g}e)$ is free, it may happen that the free generators do not form a regular sequence (cf. [5, Examples 7.5 and 7.6]). At last, the codimension of $(\mathfrak{g}e)^{\text{sing}}$ in $(\mathfrak{g}e)^*$ is not always greater than or equal to 2 (cf. [17]), even if $e$ is good [5, Remark 7.7].

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