Upper bounds for $B_h[g]$-sets with small $h$

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Abstract

For $g \geq 2$ and $h \geq 3$, we give small improvements on the maximum size of a $B_h[g]$-set contained in the interval $\{1, 2, \ldots, N\}$. In particular, we show that a $B_3[g]$-set in $\{1, 2, \ldots, N\}$ has at most $(14.3gN)^{1/3}$ elements. The previously best known bound was $(16gN)^{1/3}$ proved by Cilleruelo, Ruzsa, and Trujillo. We also introduce a related optimization problem that may be of independent interest.

1 Introduction

Let $A \subseteq [N] := \{1, 2, \ldots, N\}$ and let $h$ and $g$ be positive integers. We say that $A$ is a $B_h[g]$-set if for any integer $n$, there are at most $g$ distinct multi-sets $\{a_1, a_2, \ldots, a_h\} \subseteq A$ such that

$$a_1 + a_2 + \cdots + a_h = n.$$  

Determining the maximum size of a $B_h[g]$-set in $A \subseteq [N]$ is a well-studied problem in number theory. Initial bounds on $B_h[g]$-sets were obtained combinatorially. Indeed, if $A$ is a $B_h[g]$-set, then consider the $\binom{|A| + h - 1}{h}$ multi-sets of size $h$ in $A$. The sum of the elements in each of the multi-sets represents each integer in $\{1, 2, \ldots, hN\}$ at most $g$ times. Therefore,

$$\binom{|A| + h - 1}{h} \leq ghN \quad (1)$$

which implies $|A| \leq (h!ghN)^{1/h}$. The breakthrough papers of Cilleruelo, Ruzsa, Trujillo [3], Cilleruelo, Jiménez-Urroz [2], and Green [4] introduced methods from analysis and probability to obtain significant improvements on (1). Several of the results in these papers have yet to be improved upon. For more on $B_h[g]$-sets, we recommend the survey papers of O’Bryant [5] and Plagne [6]. We will be concerned with $B_h[g]$-sets where $g \geq 2$ and $h \geq 3$. For $3 \leq h \leq 6$ and $g \geq 2$, the best known upper bound on the size of a $B_h[g]$-set $A \subseteq [N]$ is

$$|A| \leq \left( \frac{h!hgN}{1 + \cos^h(\pi/h)} \right)^{1/h} \quad (2)$$

due to Cilleruelo, Ruzsa, and Trujillo [3]. For $h \geq 7$, the best known bound is

$$|A| \leq \left( \sqrt{3}hh!gN \right)^{1/h} \quad (3)$$

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which was proved by Cilleruelo and Jiménez-Urroz [2] using an idea of Alon. For \( g = 1 \), the best bounds can be found in [1] and [4]. In the case that \( h = 2 \) and \( g \geq 2 \), Yu [7] was able to make some improvements to the results of Green [4]. In this note we improve (2) and make a small improvement upon (3).

**Theorem 1.1** (i) Let \( g \geq 2 \) and \( h \geq 4 \) be integers. If \( A \subseteq [N] \) is a \( B_h[g] \)-set, then

\[
|A| \leq (1 + o_N(1)) \left( \frac{x_h h! h g N}{\pi} \right)^{1/h}
\]

where \( x_h \) is the unique real number in \((0, \pi)\) that satisfies

\[
\sin \left( \frac{\pi}{x_h} \right) = \left( \frac{4}{3 - \cos(\pi/h)} - 1 \right)^h.
\]

(ii) If \( A \subseteq [N] \) is a \( B_3[g] \)-set, then for large enough \( N \),

\[
|A| < (14.3 g N)^{1/3}.
\]

Our improvements for small \( h \) are contained in the following table.

| \( h \) | upper bound of [3], [2] | new upper bound |
|---|---|---|
| 3 | \((16gN)^{1/3}\) | \((14.3gN)^{1/3}\) |
| 4 | \((76.8gN)^{1/4}\) | \((71.49gN)^{1/4}\) |
| 5 | \((445.577gN)^{1/5}\) | \((413.07gN)^{1/5}\) |
| 6 | \((3054.7gN)^{1/6}\) | \((2774.16gN)^{1/6}\) |
| 7 | \((23096.19gN)^{1/7}\) | \((21294.74gN)^{1/7}\) |

Table 1: Upper bounds on \( B_h[g] \)-sets in \([1, 2, \ldots, N] \) for sufficiently large \( N \).

By looking at Table 1, it is clear that Theorem 1.1 improves (2) for \( 3 \leq h \leq 6 \). The inequality

\[
\frac{\sin(\pi \sqrt{3/h})}{\pi \sqrt{3/h}} < \left( \frac{4}{3 - \cos(\pi/h)} - 1 \right)^h
\]

holds for all \( h \geq 3 \); a fact that can be verified using Taylor series. Since \( \frac{\sin x}{x} \) is decreasing on \([0, \pi]\), we must have \( x_h < \pi \sqrt{3/h} \) for all \( h \geq 3 \) which shows that Theorem 1.1 improves (3). The improvement, however, is \((1 - o_h(1))\) since \( \frac{\pi \sqrt{3}}{x_h \pi} \to 1 \) as \( h \to \infty \).

In the next section we prove Theorem 1.1. Our arguments rely heavily on [3] and [4]. In Section 3 we introduce an optimization problem that is motivated by our work in Section 2.

**2 Proof of Theorem 1.1**

First we show how to improve (2) using the arguments of [3] and [4]. Let \( A \subseteq [N] \) be a \( B_h[g] \)-set where \( h \geq 2 \). Define \( f(t) = \sum_{a \in A} e^{iat} \), \( t_h = \frac{2\pi}{hN} \), and

\[
r_h(n) = |\{ (a_1, \ldots, a_h) \in A^h : a_1 + \cdots + a_h = n \}|.
\]

The first lemma is a variation of inequality (40) from [4].
Lemma 2.1 (Green [4]) For any $j \in \{1, 2, \ldots, hN - 1\}$,
\[
|f(t_{hj})| \leq (1 + o_N(1))|A| \left( \frac{\sin(\pi Q_h)}{\pi Q_h} \right)^{1/h}
\]
where $Q_h = \frac{|A|^h}{h! h g N}$.

**Proof.** Let $j \in \{1, 2, \ldots, hN - 1\}$. Define $g : \mathbb{Z}_{hN} \to \{0, 1, \ldots\}$ by $g(n) = h! g - r_h(n)$.

Following [3], we observe that
\[
f(t_{hj})^h = \sum_{n=1}^{hN} r_h(n)e^{\frac{2\pi inj}{hN}} = -\sum_{n=1}^{hN} (h! g - r_h(n))e^{\frac{2\pi inj}{hN}}. \tag{4}
\]

Let $\hat{g}$ be the Fourier transform of $g$ so $\hat{g}(j) = \sum_{n=1}^{hN} g(n)e^{\frac{2\pi inj}{hN}}$ for $j \in \mathbb{Z}_{hN}$. From (4) and the definition of $g$,
\[
|f(t_{hj})|^h = |\hat{g}(j)|. \tag{5}
\]

Since $A$ is a $B_h[g]$-set, the inequality $0 \leq g(n) \leq h! g$ holds for all $n$. Furthermore, $\sum_{n=1}^{hN} g(n) = h! ghN - |A|^h$. Lemma 26 of [4] gives
\[
|\hat{g}(j)| \leq h! g \left| \frac{\sin(\frac{\pi h g N - |A|^h}{h! g h N})}{\sin(\frac{\pi}{hN})} \right| = h! g \left| \frac{\sin(\pi Q_h - \frac{\pi}{hN})}{\sin(\frac{\pi}{hN})} \right|. \tag{6}
\]

By (2), the value $Q_h$ satisfies $0 \leq Q_h \leq 1$ for all $N$. Therefore,
\[
|\hat{g}(j)| \leq h! g(1 + o_N(1)) \frac{\sin(\pi Q_h)}{\pi h N} = (1 + o_N(1))|A|^h \frac{\sin(\pi Q_h)}{\pi Q_h}.
\]

Combining this inequality with (5), we get
\[
|f(t_{hj})| \leq (1 + o_N(1))|A| \left( \frac{\sin(\pi Q_h)}{\pi Q_h} \right)^{1/h}
\]
which completes the proof of the lemma.

Again following [3], we need to choose a function $F(x) = \sum_{j=1}^{hN} b_j \cos(jx)$ such that
\[
\sum_{a \in A} F \left( \left( a - \frac{N + 1}{2} \right) t_h \right)
\]
is large and $\sum_{j=1}^{hN} |b_j|$ is small. For $h \geq 3$, the function $F(x) = \frac{1}{\cos(\pi/h)} \cos x$ gives
\[
\sum_{a \in A} F \left( \left( a - \frac{N + 1}{2} \right) t_h \right) \geq |A|
\]
and \( \sum_{j=1}^{hN} |b_j| = \frac{1}{\cos(\pi/h)} \). This is the function that is used in \([3]\). We will choose a different function \( G \) that does better than \( F \) and still has a simple form. Let

\[
G(x) = \left( \frac{2}{3 - \cos(\pi/h)} \right) \frac{1}{\cos(\pi/h)} \cos(x) - \left( 1 - \frac{2}{3 - \cos(\pi/h)} \right) \frac{1}{\cos(\pi/h)} \cos(hx). \tag{7}
\]

The minimum value of \( G(x) \) on the interval \([-\frac{\pi}{h}, \frac{\pi}{h}]\) is \( \frac{1}{\cos(\pi/h)} \left( \frac{4}{3 - \cos(\pi/h)} - 1 \right) \) and so

\[
\sum_{a \in A} G \left( \left( a - \frac{N+1}{2} \right) t_h \right) \geq \frac{1}{\cos(\pi/h)} \left( \frac{4}{3 - \cos(\pi/h)} - 1 \right) |A|. \tag{8}
\]

Here we are using the fact that \( |(a - (N+1)/2)t_h| < \frac{\pi}{h} \) for any \( a \in A \). If the constants \( c_j \) are defined by \( G(x) = \sum_{j=1}^{hN} c_j \cos(jx) \), then \( \sum_{j=1}^{hN} |c_j| = \frac{1}{\cos(\pi/h)} \). Using (8), we have

\[
\frac{1}{\cos(\pi/h)} \left( \frac{4}{3 - \cos(\pi/h)} - 1 \right) |A| \leq \sum_{a \in A} G \left( \left( a - \frac{N+1}{2} \right) t_h \right) = \text{Re} \left( \sum_{j=1}^{hN} c_j \sum_{a \in A} e^{(a-(N+1)/2) \frac{2\pi ij}{hN}} \right) \leq \sum_{j=1}^{hN} |c_j||f(t_hj)| \leq \frac{1}{\cos(\pi/h)}(1 + o_N(1))|A| \left( \frac{\sin(\pi Q_h)}{\pi Q_h} \right)^{1/h}
\]

where in the last line we have used Lemma \([2, 3]\) and \( \sum_{j=1}^{hN} |c_j| = \frac{1}{\cos(\pi/h)} \). Some rearranging gives

\[
\left( \frac{4}{3 - \cos(\pi/h)} - 1 \right)^{h} \leq (1 + o_N(1)) \frac{\sin(\pi Q_h)}{\pi Q_h}. \tag{9}
\]

We remark that \( \frac{4}{3 - \cos(\pi/h)} - 1 > \cos(\pi/h) \) is equivalent to \((1 - \cos(\pi/h))^2 > 0 \). The point of this is that using \( G \) defined by (7) instead of \( F(x) = \frac{1}{\cos(\pi/h)} \cos x \) (which would give the value 1 on the left hand side of (9)) does lead to a better upper bound.

Recalling that \( 0 \leq Q_h \leq 1 \), lower bounds on \( \frac{\sin(\pi Q_h)}{\pi Q_h} \) translate to upper bounds on \( \pi Q_h \). Let \( x_h \) be the unique real number in the interval \((0, \pi)\) that satisfies

\[
\left( \frac{4}{3 - \cos(\pi/h)} - 1 \right)^{h} = \frac{\sin(x_h)}{x_h}.
\]

Then by (9), \( \pi Q_h \leq (1 + o_N(1))x_h \) since the function \( \frac{\sin x}{x} \) is decreasing on \([0, \pi]\). We can rewrite \( \pi Q_h \leq (1 + o_N(1))x_h \) as

\[
|A| \leq (1 + o_N(1)) \left( \frac{x_h h! h g N}{\pi} \right)^{1/h}. \tag{10}
\]
The upper bounds obtained from (10) for \( h \in \{4, 5, 6, 7\} \) are given in Table 1. We have chosen to round the values so that all of the bounds in Table 1 hold for sufficiently large \( N \). In particular, (10) implies that a \( B_3[g] \)-set \( A \subseteq [N] \) has at most \((14.65gN)^{1/3}\) elements. We can improve this bound by considering the distribution of \( A \) in the interval \([N]\).

Assume now that \( A \) is a \( B_3[g] \)-set. Let \( \delta \) be a real number with \( 0 < \delta < \frac{1}{4} \) and set \( l = \lfloor \frac{1}{2 \delta} \rfloor \). For \( 1 \leq k \leq l \), let

\[
C_k = (A \cap ((k - 1)\delta N, k\delta N]) \cup (A \cap [(1 - k\delta)N, (1 - (k - 1)\delta)N]).
\]

The definition of \( l \) ensures that the sets \( C_1, \ldots, C_l \) together with \( A \cap (l\delta N, (1 - l\delta)N) \) form a partition of \( A \). Using the same counting argument that is used to obtain (1), we show that if some \( C_k \) contains a large proportion of \( A \), then \( |A| \leq (14.295gN)^{1/3} \). To this end, define real numbers \( \alpha_1(\delta), \ldots, \alpha_l(\delta) \) by

\[
\alpha_k(\delta)|A| = |C_k|
\]

for \( 1 \leq k \leq l \). The value \( \alpha_k(\delta) \) represents the proportion of \( A \) that is contained in the union \((C_k) = (k - 1)\delta N, k\delta N] \cup [(1 - k\delta)N, (1 - (k - 1)\delta)N] \).

**Lemma 2.2** If \( 0 < \delta < \frac{1}{4} \), \( l = \lfloor \frac{1}{2 \delta} \rfloor \), and \( \alpha_1(\delta), \ldots, \alpha_l(\delta) \) are defined by (11), then for any \( N > \frac{2}{\delta} \) and \( 1 \leq k \leq l \),

\[
|A| \leq \left( \frac{72g\delta N}{\alpha_k(\delta)^3} \right)^{1/3}.
\]

**Proof.** Let \( 1 \leq k \leq l \) and consider \( C_k \). Since \( C_k \) is a \( B_3[g] \)-set,

\[
\left( \frac{|C_k| + 3 - 1}{3} \right) \leq g|C_k + C_k + C_k|
\]

where \( C_k + C_k + C_k = \{a + b + c : a, b, c \in C_k\} \). The set \( |C_k + C_k + C_k| \) is contained in the union of the intervals

\[
[3(k - 1)\delta N, 3k\delta N], [(1 + (k - 2)\delta)N, (1 + (k + 1)\delta)N],
\]

\[
[(2 - (k + 1)\delta)N, (2 - (k - 2)\delta)N], \text{ and } [(3 - 3k\delta)N, (3 - 3(k - 1)\delta)N].
\]

Each of these four intervals has length \( 3\delta N \) so \( |C_k + C_k + C_k| \leq 12\delta N \). Combining this inequality with (12) we have \( \left( \frac{C_k| + 2}{3} \right) \leq 12g\delta N \) which implies \( \alpha_k(\delta)|A| = |C_k| \leq (312g\delta N)^{1/3} \).

Now we consider two cases.

**Case 1:** For some \( 0 < \delta < \frac{1}{4} \) and \( 1 \leq k \leq l = \lfloor \frac{1}{2 \delta} \rfloor \), we have

\[
\left( \frac{72\delta}{14.295} \right)^{1/3} < \alpha_k(\delta).
\]

In this case, we apply Lemma 2.2 to get \( |A| \leq (14.295gN)^{1/3} \) and we are done.
Case 2: For all $0 < \delta < \frac{1}{4}$ and $1 \leq k \leq l = \lfloor \frac{1}{128} \rfloor$, we have

$$\alpha_k(\delta) \leq \left( \frac{72\delta}{14.295} \right)^{1/3}. \tag{13}$$

Let $H(x) = 1.6 \cos x - 0.3 \cos 3x + 0.1 \cos 6x$. Partition the interval $[-\pi/3, \pi/3]$ into 128 subintervals $I_1, \ldots, I_{128}$ of equal width so

$$I_j = \left[ -\frac{\pi}{3} + \frac{2\pi(j - 1)}{3 \cdot 128}, -\frac{\pi}{3} + \frac{2\pi j}{3 \cdot 128} \right]$$

for $1 \leq j \leq 128$. Let $v_j = \min_{x \in I_j} H(x)$ for $1 \leq j \leq 128$. Since $H$ is an even function, $v_j = v_{128-j+1}$ for $1 \leq j \leq 64$. The values $v_j$ can be approximated numerically. They satisfy

$$v_1 < v_2 < v_3 < v_4 < v_5 < v_{35} \leq v_j$$

for all $6 \leq j \leq 64$. The sum

$$\sum_{a \in A} H \left( \left( a - \frac{N + 1}{2} \right) t_3 \right) \tag{15}$$

is minimized when $J = \{ (a - \frac{N + 1}{2}) t_3 : a \in A \}$ contains as many elements as possible in $I_1 \cup I_2 \cup \cdots \cup I_5$ and the remaining elements of $J$ are contained in $I_{35}$. This follows from (14). Furthermore, in order to minimize (15), $J$ must intersect $I_1$ in as many elements as possible, and the remaining elements in $J$ intersect $I_2$ in as many elements as possible, and so on. By (13) with $\delta = 1/128$,

$$\alpha_k(1/128) \leq \left( \frac{72(1/128)}{14.295} \right)^{1/3}$$

thus,

$$|J \cap I_1| \leq \left( \frac{72(1/128)}{14.295} \right)^{1/3} |A|.$$  

Similarly, by (13) with $\delta = j/128$ for $j \in \{2, 3, 4, 5\}$,

$$\alpha_k(j/128) \leq \left( \frac{72(j/128)}{14.295} \right)^{1/3}.$$  

We conclude that

$$|J \cap (I_1 \cup I_2 \cup \cdots \cup I_j)| \leq \left( \frac{72(j/128)}{14.295} \right)^{1/3} |A|$$

for $1 \leq j \leq 5$. From this inequality and (14), we deduce that

$$\sum_{a \in A} H \left( \left( a - \frac{N + 1}{2} \right) t_3 \right) \geq \sum_{j=1}^{5} v_j \left( \left( \frac{72(j/128)}{14.295} \right)^{1/3} - \left( \frac{72((j-1)/128)}{14.295} \right)^{1/3} \right) |A|$$

$$+ v_{35} \left( 1 - \left( \frac{72(5/128)}{14.295} \right)^{1/3} \right) |A| > 1.2455 |A|.$$
Using 1.2455 in the derivation of (9) instead of \( \frac{1}{\cos(\pi/3)} \left( \frac{4}{3 - \cos(\pi/3)} - 1 \right) \) gives

\[
1.2455 |A| \leq \frac{1}{\cos(\pi/3)} (1 + o_N(1)) |A| \left( \frac{\sin(\pi Q_3)}{\pi Q_3} \right)^{1/3}.
\]

This inequality can be rewritten as

\[
\left( \frac{1.2455}{2} \right)^3 \leq (1 + o_N(1)) \left( \frac{\sin(\pi Q_3)}{\pi Q_3} \right)
\]

which leads to the bound \(|A| < (14.296gN)^{1/3}\) for large enough \(N\).

### 3 An optimization problem

In this section we introduce an optimization problem that is motivated by (8) from the previous section.

Given integers \(K\) and \(h \geq 2\), define

\[
\mathcal{F}_{K,h} = \left\{ \sum_{j=1}^{K} b_j \cos(jx) : \sum_{j=1}^{K} |b_j| = \frac{1}{\cos(\pi/h)} \right\}.
\]

For \(A \subseteq [N]\) and \(F \in \mathcal{F}_{K,h}\), define

\[
w_F(A) = \sum_{a \in A} F \left( \left( a - \frac{N + 1}{2} \right) \frac{2\pi}{hN} \right)
\]

and

\[
\psi(N, K, h) = \min_{A \subseteq [N], A \neq \emptyset} \sup \left\{ \frac{w_F(A)}{|A|} : F \in \mathcal{F}_{K,h} \right\}.
\]

Our interest in \(\psi(N, K, h)\) is due to the following proposition.

**Proposition 3.1** If \(A \subseteq [N]\) is a \(B_h[1]\)-set and \(K \leq hN\), then

\[
|A| \leq (1 + o_N(1)) \left( \frac{y_h h! h^N}{\pi} \right)^{1/h}
\]

where \(y_h\) is the unique real number in \([0, \pi]\) with \(\frac{\sin y_h}{y_h} = (\cos(\pi/h) \psi(N, K, h))^h\).

The function \(G\) defined by (7) shows that

\[
\psi(N, h, h) \geq \frac{1}{\cos(\pi/h)} \left( \frac{4}{3 - \cos(\pi/h)} - 1 \right).
\]

When \(h = 3\), this gives \(\psi(N, 3, 3) \geq 1.2\) which implies \(\psi(N, 6, 3) \geq 1.2\). This is because the collection of functions \(\mathcal{F}_{3,3}\) is a subset of \(\mathcal{F}_{6,3}\). By considering more than one
function, we can improve the bound \( \psi(N, 6, 3) \geq 1.2 \). The method by which we achieve
this can be stated just as easily for general \( K \) and \( h \) so we do so.

To estimate \( \psi(N, K, h) \), we will consider finite subsets of \( \mathcal{F}_{K,h} \). Given a subset \( \mathcal{F}'_{K,h} \subseteq \mathcal{F}_{K,h} \), we obviously have

\[
\sup \left\{ \frac{w_F(A)}{|A|} : F \in \mathcal{F}'_{K,h} \right\} \leq \sup \left\{ \frac{w_F(A)}{|A|} : F \in \mathcal{F}_{K,h} \right\}
\]

for every \( A \subseteq [N] \) with \( A \neq \emptyset \). When \( \mathcal{F}_{K,h} \) is finite, then the supremum on the left hand side of (16) can be replaced with the minimum. Let \( m \) be a positive integer and partition the interval \( [-\pi/h, \pi/h] \) into \( m \) subintervals \( I^m_1, \ldots, I^m_m \) where

\[
I^m_j = \left[ -\frac{\pi}{h} + \frac{2\pi(j-1)}{hm}, -\frac{\pi}{h} + \frac{2\pi j}{hm} \right]
\]

for \( 1 \leq j \leq m \). Any \( F \in \mathcal{F}_{K,h} \) is continuous and thus obtains its minimum value on \( I^m_j \). Given \( F \in \mathcal{F}_{K,h} \), define

\[
v_{m,j}(F) = \min_{x \in I^m_j} F(x).
\]

Given \( A \subseteq [N] \), define

\[
\alpha_{m,j}(A) = \frac{1}{|A|} \left| \left\{ a - \frac{N+1}{2} \cdot \frac{2\pi}{hN} : a \in A \right\} \cap I^m_j \right|.
\]

With this notation, we have that for any \( A \subseteq [N] \) and \( F \in \mathcal{F}_{K,h} \),

\[
w_F(A) \geq \sum_{j=1}^{m} \alpha_{m,j}(A)|A|v_{m,j}(F).
\]

Therefore, given a finite set \( \{F_1, \ldots, F_n\} \subseteq \mathcal{F}_{K,h} \),

\[
\psi(N, K, h) \geq \min_{A \subseteq [N], A \neq \emptyset} \max \left\{ \sum_{j=1}^{m} \alpha_{m,j}(A)v_{m,j}(F_k) : 1 \leq k \leq n \right\}.
\]

We now put the above discussion to use by proving the following result.

**Theorem 3.2** For sufficiently large \( N \), the function \( \psi(N, 6, 3) \) satisfies the estimate

\[
\psi(N, 6, 3) \geq 1.2228.
\]

**Proof.** Let

\[
F_1(x) = 1.7 \cos x - 0.3 \cos 3x, \quad F_2(x) = 1.6 \cos x - 0.3 \cos 3x + 0.1 \cos 6x,
\]

\[
F_3(x) = 1.5 \cos x - 0.4 \cos 3x + 0.1 \cos 6x, \quad F_4(x) = 1.2 \cos x - 0.6 \cos 3x + 0.2 \cos 6x,
\]

\[
F_5(x) = -2 \cos 3x,
\]

and \( \mathcal{F} = \{ F_1, F_2, F_3, F_4, F_5 \} \). Observe that \( \mathcal{F} \subseteq \mathcal{F}_{5,3} \). We take \( m = 12 \) and we must compute the numbers \( v_{12,j}(F_k) \) for \( 1 \leq j \leq 12 \) and \( 1 \leq k \leq 5 \). Since each \( F_k \) is an even
function, \( v_{12,j}(F_k) = v_{12,12-j+1}(F_k) \) for \( 1 \leq j \leq 6 \). To prove Theorem 3.2, we will only need to estimate these values from below.

Let \( A \subseteq [N] \) with \( A \neq \emptyset \). We assume that no element of the form \((a - \frac{N+1}{2})N\sqrt{A}\) is contained in two of the intervals \( I_1^{(2)}, \ldots, I_6^{(2)} \). For large \( A \), this will not affect \(|A|\), at least in an asymptotic sense. Under this assumption, the non-negative real numbers \( \alpha_{12,1}(A), \ldots, \alpha_{12,12}(A) \) satisfy

\[
\alpha_{12,1}(A) + \cdots + \alpha_{12,12}(A) = 1.
\]

We will consider several cases which depend on the distribution of \( A \). For notational convenience, we write \( \alpha_j \) for \( \alpha_{12,j}(A) \).

**Case 1:** \( \alpha_1 + \alpha_{12} \leq 0.6 \).

Here we will use the function \( F_1(x) \). Lower estimates on the \( v_{12,j}(F_1) \) are

\[
\begin{align*}
v_{12,1}(F_1) &\geq 1.15, \quad v_{12,2}(F_1) \geq 1.3525, \quad v_{12,3}(F_1) \geq 1.4522, \\
v_{12,4}(F_1) &\geq 1.4474, \quad v_{12,5}(F_1) \geq 1.4143, \quad \text{and} \quad v_{12,6}(F_1) \geq 1.4.
\end{align*}
\]

In fact, these values satisfy

\[
v_{12,1}(F_1) \leq v_{12,2}(F_1) \leq v_{12,6}(F_1) \leq v_{12,5}(F_1) \leq v_{12,4}(F_1) \leq v_{12,3}(F_1).
\]

Since \( \alpha_1 + \alpha_{12} \leq 0.6 \), we must have

\[
w_{F_1}(A) \geq (0.6v_{12,1}(F_1) + 0.4v_{12,2}(F_1))|A| \geq (0.6(1.15) + 0.4(1.3525))|A| > 1.23|A|.
\]

**Case 2:** \( 0.6 \leq \alpha_1 + \alpha_{12} \leq 0.7 \).

Here we use the function \( F_2(x) \). A close look at Case 1 shows that if \( v_{12,1}(F_2) \) is one of the two smallest values in the set \( \{v_{12,j}(F_2) : 1 \leq j \leq 6\} \), then essentially the same estimate applies. The two smallest values are \( v_{12,1}(F_2) \geq 1.2 \) and \( v_{12,4}(F_2) \geq 1.2834 \). Since \( 0.6 \leq \alpha_1 + \alpha_{12} \leq 0.7 \),

\[
w_{F_2}(A) \geq (0.7(1.2) + 0.3(1.2834))|A| > 1.225|A|.
\]

**Case 3:** \( 0.7 \leq \alpha_1 + \alpha_{12} \leq 0.8 \).

Here we use the function \( F_3(x) \). In this range of \( \alpha_1 + \alpha_{12} \), our estimate behaves a bit differently. Lower estimates on the \( v_{12,j}(F_3) \) are

\[
\begin{align*}
v_{12,1}(F_3) &\geq 1.25, \quad v_{12,2}(F_3) \geq 1.299, \quad v_{12,3}(F_3) \geq 1.199, \\
v_{12,4}(F_3) &\geq 1.1595, \quad v_{12,5}(F_3) \geq 1.1595, \quad \text{and} \quad v_{12,6}(F_3) \geq 1.18.
\end{align*}
\]

In this case, \( w_{F_3}(A) \) will be minimized when \( \alpha_1 + \alpha_{12} \) is as small as possible. In the previous two cases, \( w_{F_1}(A) \) was minimized when \( \alpha_1 + \alpha_1 \) was as large as possible. We conclude that

\[
w_{F_3}(A) \geq (0.7(1.25) + 0.3(1.1595))|A| > 1.2228|A|.
\]

**Case 4:** \( 0.8 \leq \alpha_1 + \alpha_{12} \leq 0.9 \).

In this case we use the function \( F_4(x) \). Lower estimates on the \( v_{12,j}(F_4) \) are
\[ v_{12,1}(F_4) \geq 1.3909, \quad v_{12,2}(F_4) \geq 1.1192, \quad v_{12,3}(F_4) \geq 0.8392, \]
\[ v_{12,4}(F_4) \geq 0.7276, \quad v_{12,5}(F_4) \geq 0.7264, \quad \text{and} \quad v_{12,6}(F_4) \geq 0.7621. \]

We have
\[ w_{F_i}(A) \geq (0.8(1.3909) + 0.2(0.7264))|A| > 1.25|A|. \]

**Case 5:** \( 0.9 \leq \alpha_1 + \alpha_{12} \leq 1. \\)

Lower estimates on the \( v_{12,j}(F_5) \) are
\[ v_{12,1}(F_5) \geq 1.73, \quad v_{12,2}(F_5) \geq 1, \quad v_{12,3}(F_5) \geq -0.01, \]
\[ v_{12,4}(F_5) \geq -1, \quad v_{12,5}(F_5) \geq -1.8, \quad \text{and} \quad v_{12,6}(F_5) \geq -2. \]

As in Cases 3 and 4, \( w_{F_5}(A) \) is minimized when \( \alpha_1 + \alpha_{12} \) is as small as possible. Hence,
\[ w_{F_5}(A) \geq (0.9(1.73) + 0.1(-2))|A| > 1.35|A|. \]

In all five cases, we can find a function \( F_i \in \mathcal{F} \) such that \( w_{F_i}(A) > 1.2228|A| \). This completes the proof of Theorem 3.2.

\[ \square \]

### 4 Concluding Remarks

Although it is an improvement of \( \psi(N, 6, 3) \geq 1.2 \), Theorem 3.2 is not enough to prove part (ii) of Theorem 1.1. The improvement on \( B_3[g] \)-sets uses the \( B_3[g] \) property to increase the 1.2 to 1.2455 which exceeds the 1.2228 provided by Theorem 3.2. Similar arguments can be done for \( B_h[g] \)-sets with \( h > 3 \), but the improvements in the results of Table 1 are minimal. Aside from \( B_3[g] \)-sets, the bounds in Table 1 come from lower bounds on \( \psi(N, h, h) \) together with Lemma 2.1.

The function \( \psi(N, K, h) \) is relevant to an inequality of Cilleruelo. Let \( A \) be a finite set of positive integers. For an integer \( h \geq 2 \), let
\[ r_h(n) = |\{(a_1, \ldots, a_h) \in A^h : a_1 + \cdots + a_h = n\}| \text{ and } R_h(m) = \sum_{n=1}^{m} r_h(m). \]

Generalizing the argument of [3], Cilleruelo proved the following result.

**Theorem 4.1 (Cilleruelo [1])** Let \( A \subseteq [N], \ h \geq 2 \) be an integer, and \( \mu \) be any real number. For any positive integer \( H = o(N) \),
\[ \sum_{n=h}^{hN+H} |R_h(n) - R_h(n-H) - \mu| \geq (L_h + o(1))H|A|^h \]

where \( L_2 = \frac{4}{(\pi+2)^2} \) and \( L_h = \cos^h(\pi/h) \) for \( h > 2 \).

By slightly modifying the argument in [1] that is used to prove Theorem 4.1, it is easy to prove the next proposition.
Proposition 4.2 Let \( A \subseteq [N] \), \( h \geq 2 \) be an integer, and \( \mu \) be a real number. For any positive integers \( H = o(N) \) and \( K \leq \frac{N}{H} \),

\[
\sum_{n=h}^{hN+H} |R_h(n) - R_h(n - H) - \mu| \geq (\psi(N, K, h)^h L_h + o(1)) H|A|^h
\]

where \( L_2 = \frac{4}{(\pi+2)^2} \) and \( L_h = \cos^h(\pi/h) \) for \( h > 2 \).

For instance, Theorem 3.2 gives

\[
\sum_{n=3}^{3N+H} |R_3(n) - R_3(n - H) - \mu| \geq (1.2228^3 L_3 + o(1)) H|A|^3.
\]

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