On the containment hierarchy for simplicial ideals

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Abstract

The purpose of this note is to study containment relations and asymptotic invariants for ideals of fixed codimension skeletons (simplicial ideals) determined by arrangements of $n + 1$ general hyperplanes in the $n$–dimensional projective space over an arbitrary field.

Keywords symbolic powers, simplicial ideals, resurgence, containment relations

1 Introduction

The last few years have seen a number of exciting developments at the intersection of commutative algebra, algebraic geometry and combinatorics. A number of new methods and intriguing asymptotic invariants have evolved out of ground breaking papers by Ein, Lazarsfeld and Smith [7] in characteristic zero and Hochster and Huneke [11] in positive characteristic. In particular, star configurations have emerged as a natural testing ground for numerous conjectures related to the containment relations between symbolic and ordinary powers of ideals, see [6] for a very nice introduction to this circle of ideas. In the present note we study a special case of star configurations, namely simplices $\Delta(n)$ cut out by $n + 1$ coordinate hyperplanes $H_0, \ldots, H_n$ in $\mathbb{P}^n$, or more precisely simplicial complexes arising by intersecting all possible tuples of these hypersurfaces. Thus a codimension $c$ face $F(i_1, \ldots, i_c)$ is the intersection $H_{i_1} \cap \ldots \cap H_{i_c}$ for $i_1, \ldots, i_c \in \{1, \ldots, n\}$. Let $I(n, c)$ denote the ideal of the union of all codimension $c$ faces of $\Delta(n)$. We call such ideals simplicial. There
are obvious containments

\[
I(n,1) \subset I(n,2) \subset \ldots \subset I(n,n-1) \subset I(n,n).
\] (1)

The containment problem for symbolic and usual powers of ideals has been intensively studied in recent years, see e.g. [2], [3], [9]. The first counterexample to the \(I(3) \subset I^2\) containment for an ideal of points in \(\mathbb{P}^2\) announced in [5] has prompted another series of papers [1], [10], [12]. In all these works the authors study containment relations of the type \(I^{(m)} \subset I^r\) for a fixed homogeneous ideal \(I\). Along these lines we obtain the following result for simplicial ideals.

**Theorem A.** For \(n \geq 1\) and \(c \in \{1, \ldots, n\}\), there is the containment

\[
I^{(m)}(n,c) \subset I^r(n,c)
\]

if and only if \(m\) and \(r\) satisfy the inequality

\[
(n - c + 2) \cdot r \leq \left\lfloor \frac{n + 1}{c} \cdot m \right\rfloor.
\]

However our approach is a little bit more general. It is motivated by the hierarchy established in (1). Thus we extend the containment problem to inclusion relations between symbolic powers of various simplicial ideals. Our main result in this direction is the following.

**Theorem B.** Let \(n\) be a positive integer and let \(c, d \in \{1, \ldots, n\}\) be integers such that \(c \leq d\). Then there is the containment

\[
I^{(m)}(n,c) \subset I^s(n,d)
\]

if and only if \(m \cdot c \leq s \cdot d\).

Bocci and Harbourne introduced in [3] an interesting invariant, the resurgence \(\rho(I)\) measuring in effect the asymptotic discrepancy between symbolic and ordinary powers of a given ideal. This is a delicate invariant and the family of ideals for which it is known is growing slowly, see e.g. [4]. Here we expand this knowledge a little bit.

**Theorem C.** For a positive integer \(n\) and \(c \in \{1, \ldots, n\}\) there is

\[
\rho(I(n,c)) = \frac{c(n + 2 - c)}{n + 1}.
\]

Note that the \(\geq\) inequality was established in [2] Theorem 2.4.3 b] and the case \(I(n,n)\) was computed in [2] Theorem 2.4.3 a].
2 Preliminaries

In this section we recall basic definitions and introduce some notation. We work over an arbitrary field $\mathbb{K}$. Let $S(n) = \mathbb{K}[x_0, \ldots, x_n]$ be the ring of polynomials over $\mathbb{K}$.

**Definition 2.1.** Let $I \subseteq S(n)$ be a homogenous ideal and let $m \geq 1$ be a positive integer. The $m$-th symbolic power of $I$ is

$$I^{(m)} = S(n) \cap \left( \bigcap_{Q \in \text{Ass}(I)} I^{m}_Q \right),$$

where the intersection takes place in the field of fractions of $S(n)$.

Although symbolic powers are defined algebraically, they have a nice geometrical interpretation due to the following result of Zariski and Nagata (see [8], Theorem 3.14 and [13], Corollary 2.9):

**Theorem 2.2.** (Nagata-Zariski) Let $I \subseteq S(n)$ be a radical ideal and let $V$ be the set of zeroes of $I$. Then $I^{(m)}$ consists of all polynomials vanishing to order at least $m$ along $V$.

It follows immediately from the above theorem that there are inclusions

$$I = I^{(1)} \supseteq I^{(2)} \supseteq I^{(3)} \supseteq \ldots$$

Of course the same is true for usual powers

$$I = I^1 \supseteq I^2 \supseteq I^3 \supseteq \ldots$$

It is natural to wonder for what $m$ and $r$ there are inclusions

$$a) \ I^r \subseteq I^{(m)} \quad \text{and} \quad b) \ I^{(m)} \subseteq I^r \quad (2)$$

It is easy to see that there is the inclusion in a) if and only if $m \leq r$. More generally it follows from Theorem 2.2 that there is always the inclusion

$$\left(I^{(a)}\right)^b \subseteq I^{(ab)} \quad (3)$$

As for b) Ein, Lazarsfeld and Smith in characteristic zero and Hochster and Huneke in positive characteristic showed that there is always containment for $m \geq n \cdot r$. Of
course, in certain cases this bound is not optimal and the problem has to be studied individually in any given case.

Here we study ideals $I(n, c)$ of codimension $c$ skeletons of the simplex spanned by all coordinate points in $\mathbb{P}^n$. More exactly, if $H_i$ is the hyperplane $\{x_i = 0\}$ for $i = 0, \ldots, n$, then the set of zeroes of $I(n, c)$ is the union of all $c$-fold intersections $H_{i_1} \cap \ldots \cap H_{i_c}$ for mutually distinct indices $i_1, \ldots, i_c \in \{0, \ldots, n\}$.

In connection with containment b) in [2] Bocci and Harbourne introduced in [2] the following quantity.

**Definition 2.3.** Let $I \subseteq S(n)$ be a homogenous ideal. The resurgence of $I$ is the real number

$$\rho(I) := \sup \left\{ \frac{m}{r} : I^m \not\subseteq I^r \right\}.$$  

This invariant is of interests as it guarantees the containment

$$I^{(m)} \subseteq I^r$$

for $\frac{m}{r} > \rho(I)$.

**2.1 Monomial ideals**

We identify monomials $x^a = x^{a_0} \cdot \ldots \cdot x^{a_n} \in S(n)$ with vectors $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$ in the usual way. For a monomial ideal $I$ generated by $x^{a(1)}, \ldots, x^{a(r)}$, we define its associated shape $\text{Sh}(I)$ as

$$\text{Sh}(I) = \bigcup_{i=1}^r \left( a(i) + \mathbb{R}_{\geq 0}^{n+1} \right).$$

**Example 2.4.** Let $I = \langle x_0^3, x_0 x_1^2, x_1^4 \rangle$ be a monomial ideal in $S(1)$. Then its associated shape $\text{Sh}(I)$ is the shaded area in Figure 1.
Fact 2.5. Let $I \subseteq S(n)$ be a monomial ideal. Then

$$x^a \in I \text{ if and only if } a \in \text{Sh}(I).$$

This criterion can be easily extended to containments of monomial ideals.

Lemma 2.6. Let $I, J \subseteq S(n)$ be monomial ideals. Then

$$I \subseteq J \text{ if and only if } \text{Sh}(I) \subseteq \text{Sh}(J).$$

Proof. It is enough to check the containment for generators of $I$. The claim follows then from the definition of $\text{Sh}(I)$ and Fact 2.5.

Our approach to Theorems A, B and C relies heavily on the fact that the ideals $I(n,c)$ are Stanley–Reisner ideals, i.e. ideals generated by square free monomials. More exactly we have the following fact.

Lemma 2.7. The ideal $I(n,c)$ is generated by all monomials of the form

$$x_{i_1} \cdot \ldots \cdot x_{i_{n+2-c}}$$

for mutually distinct $i_1, \ldots, i_{n+2-c} \in \{0, \ldots, n\}$.

The symbolic powers of simplicial ideals are also monomial ideals.
Proposition 2.8. Let \( m \) be a positive integer. For a monomial \( x^a = \prod_{i=0}^{n} x_i^{a_i} \) the following conditions are equivalent:

\( a) \) \( x^a \in I^{(m)}(n, c) \);

\( b) \) for all \( c \)-tuples of mutually distinct indices \( i_1, \ldots, i_c \in \{0, \ldots, n\} \)

\[ a_{i_1} + \ldots + a_{i_c} \geq m. \tag{4} \]

Proof. Based on (\cite{H}, Theorem 3.1.) we claim that \( I^{(m)}(n, c) \) is the monomial ideal. Assume that \( x^a \in I^{(m)}(n, c) \). This means that on all faces of codimension \( c \) any derivative of \( x^a \) of order \( m - 1 \) is equal to zero which enforces for all \( c \)-tuples of exponents the condition (4).

On the other hand suppose that \( x^a \) is a monomial such that (4) holds. Since (4) is invariant under permutation of variables, it is enough to check that \( x^a \) vanishes to order \( \geq m \) along one of the codimension \( c \) faces. To this end let \( F \) be defined by equations \( x_n = x_{n-1} = \ldots = x_{n-c+1} = 0 \). For \( j_1 = n, j_2 = n-1, \ldots, j_c = n-c+1 \) the monomial \( x^a \) is divisible by a monomial of degree greater or equal to \( m \) in variables \( x_n, \ldots, x_{n-c+1} \), namely by \( \prod_{i=n}^{n-c+1} x_i^{a_i} \), hence it is in the \( m \)-th symbolic power of \( I(F) \).

We can similarly characterize usual powers of simplicial ideals.

Proposition 2.9. Let \( r \) be a positive integer. Let \( x^a = \prod_{i=0}^{n} x_i^{a_i} \) be a monomial in \( S(n) \). The following conditions are equivalent.

\( a) \) \( x^a \in I^r(n, c) \);

\( b) \) i) \( \sum_{i=0}^{n} a_i = (n - c + 2)r \);

\( ii) \) \( a_i \leq r \) for all \( 0 \leq i \leq n \);

\( iii) \) \( a_{i_1} + \ldots + a_{i_c} \geq r \) for all \( c \)-tuples of mutually distinct indices \( i_1, \ldots, i_c \in \{0, \ldots, n\} \).

Proof. The implication from a) to b) is obvious in the view of Lemma \ref{2.7}

For the reverse implication, we proceed by induction. The case \( r = 1 \) follows again from Lemma \ref{2.7}. Suppose thus that the statement is proved for all integers up to \( r - 1 \). Let \( x^a \) be a monomial satisfying b).
Renumbering the variables if necessary, we may assume that the powers \(a_0, \ldots, a_n\) are ordered
\[
a_0 \geq a_1 \geq \ldots \geq a_{n+1-c} \geq \ldots \geq a_n.
\] (5)

The conditions b.i) and b.ii) imply that \(a_{n+1-c} \geq 1\). Hence \(x^a\) is divisible by the generator \(g = x_0 \cdot \ldots \cdot x_{n+1-c} \in I(n,c)\). Let
\[
a'_i := \begin{cases} 
a_i - 1 & \text{for } 0 \leq i \leq n + 1 - c \\
a_i & \text{for } i \geq n + 2 - c
\end{cases}
\]
and let \(x^{a'} = x_0^{a'_0} \cdot \ldots \cdot a_n^{a'_n}\). We need to show that the numbers \(a'_0, \ldots, a'_n\) satisfy conditions b) with \(r-1\) in place of \(r\). Indeed, if they do, then by the induction assumption \(x^{a'}\) is an element of \(I^{r-1}(n,c)\) and thus \(x^a = g \cdot x^{a'} \in I^r(n,c)\).

Turning to that claim, note that the first two conditions are satisfied by construction. So it sufficed to check the condition b.iii). To this end let \(s\) be the maximal integer (possible equal 0) such that
\[
a_{n+1-c-s} = a_{n+1-c-(s-1)} = \ldots = a_{n+1-c} = a_{n+2-c} = \ldots = a_{n+1+s-c} =: \alpha.
\]
Passing to the primed powers, note that the first \(s+1\) numbers will be decreased by 1, whereas the last \(s\) will remain unchanged. Thus, ordering the primed powers we get as the \(c\)-tuple with the least sum
\[
(s + 1)(\alpha - 1) + a_{n+1-c+s+1} + \ldots + a_n.
\] (6)

We need to show that this number is greater or equal \(r-1\). It is convenient to abbreviate
\[
L := a_0 + \ldots + a_{n+1-c-s-1} \quad \text{and} \quad R := a_{n+1-c+s+1} + \ldots + a_n.
\]

Thus
\[
L + (2s+1)\alpha + R = (n-c+2)r
\] (7)

By assumption we have also
\[
(s + 1)\alpha + R \geq r
\] (8)

and
\[
L \leq r(n-c-s+1).
\] (9)

Assume that (10) fails, i.e.
\[
(s + 1)(\alpha - 1) + R \leq r - 2.
\] (10)
Using (10), (7), (8) and (9) we obtain
\[(n + 2 - c)r \leq (n - c - s + 1)r + r - 2 + (\alpha + 1)s + 1,\]
which gives
\[sr \leq (\alpha + 1)s - 1.\]
This in turn implies \(r \leq \alpha\), which finally contradicts (10).

3 Triangle

Before we pass to proving the general statements, we want to examine the cases \(n = 2\) and \(n = 3\) in more detail. In these cases we obtain more containment relations, which suggest that there might be even more regularity also in the general case. We hope to come back to this problem in the next future.

In this section we consider the ideal \(E = I(2, 1) = \langle x_0x_1x_2 \rangle\) and \(V = I(2, 2) = \langle x_0x_1, x_0x_2, x_1x_2 \rangle\) in \(\mathbb{P}^2\). We introduce the following notation
\[P_0 = [1 : 0 : 0], \quad P_1 = [0 : 1 : 0], \quad P_2 = [0 : 0 : 1].\]

During the investigation of ordinary and symbolic powers we observed the following behavior of these ideals.

**Lemma 3.1.** The ideal \(E\) is a complete intersection ideal.

Since the powers of a complete intersection ideal ([N], p.466) are arithmetically Cohen-Macaulay this implies that for all positive integers \(k\) we have \(E^k = E^{(k)}\).

**Proposition 3.2.** The ideal \(V^{(m)}\) is generated by monomials of the form \(x_{\sigma(0)}^{m-k}x_{\sigma(1)}^{m-k}x_{\sigma(2)}^{k}\) for any permutation \(\sigma \in \Sigma\) and \(k \in \{0, 1, \ldots, \lfloor \frac{m}{2} \rfloor \}\).

**Proof.** Based on Proposition 2.8 we take any \(x^a = x_{\sigma(0)}^i x_{\sigma(1)}^j x_{\sigma(2)}^k \in V^{(m)}\), where \(i, j, k\) fulfill (11) with \(c = 2\). We are looking for a monomial \(x^a\) of the smallest degree. Without lose of generality we may assume that \(0 \leq k \leq j \leq i \leq m\). The monomial \(x_{\sigma(0)}^{i-k}x_{\sigma(1)}^{j-k}x_{\sigma(2)}^k\) divides \(x^a\). We are asking about the maximal possible value of \(k\) for which the condition (11) is fulfilled. We obtain
\[m \leq i - k + j - k \leq 2m - 2k,\]
or equivalently \(k \leq \lfloor \frac{m}{2} \rfloor\). \(\square\)
Example 3.3. $V^{(2)} = \langle x_0^2 x_1^2, x_0^2 x_2^2, x_1^2 x_2^2, x_0 x_1 x_2 \rangle$.

The following result is a special case of Theorem A.

**Proposition 3.4.** For all positive integers $r$ and $q$, $V^{(m)} \subset V^r$ if and only if $2r \leq \lceil \frac{3m}{2} \rceil$.

**Proposition 3.5.** For all positive integers $m$ we have

1. $E^{(m)} \subset V^{(2m)}$,
2. $E^{(m+1)} \subset V^{(2)} \cdot E^{(m)} \subset V \cdot E^{(m)}$.

**Proof.**

a) The ideal $E^{(m)}$ consists of all forms vanishing along edges up to order at least $m$, and this set is the subset of the ideal consisting of all forms vanishing along the vertices up to order $2m$, i.e. $V^{(2m)}$.

b) We have $V^{(2)} = \langle x_0^2 x_1^2, x_0^2 x_2^2, x_1^2 x_2^2, x_0 x_1 x_2 \rangle$ and $E = \langle x_0 x_1 x_2 \rangle$. It is easy to see that then $E \subset V^{(2)}$. That implies $E^{m+1} \subset V^{(2)} \cdot E^{m}$, which by Lemma 3.1 is equivalent to $E^{(m+1)} \subset V^{(2)} \cdot E^{(m)}$. The inclusion $V^{(2)} \cdot E^{(m)} \subset V \cdot E^{(m)}$ is obvious.

From Propositions 3.2, 3.4 and 3.5 we obtain a more interesting observation about containment relations. The direction of arrows symbolizes inclusion between the ideals.

**Proposition 3.6.** For all positive integers $m$ it holds $V^{(2m)} = (V^{(2)})^m$ and $V^{(2m+1)} = V^{(2m)} V$.
Proof. By \((3)\) we have \((V^{(2)})^m \subseteq V^{(2m)}\).

On the other hand by Proposition \((3,2)\) any generator of \(V^{(2m)}\) has the form
\[ x^{2m-j} x^{2m-j} x^j \]
for some \(j \in \{0, 1, \ldots, m\} \). Then
\[ x^{2m-j} x^{2m-j} x^j = (x^{2m-j} x^{2m-j} x^j)^m \in (V^{(2)})^m. \]

This ends the first part of the proof of the Proposition. In the second part, using Proposition \((3,2)\) again, we obtain
\[ V^{(2m)} V = V^{(2m)} \cdot (x_0 x_1, x_0 x_2, x_1 x_2) = (x_0^{2m+1}, x_1^{2m+1}, \ldots, x_1^{m+1}, x_0^m) = V^{(2m+1)}. \]

We conclude this section with the following corollary from Proposition \((3,2)\)

**Corollary 3.7.** There is the following relation
\[ V^{(2)} = E + V^2. \]

**Proof.** It is a simple observation. Directly from the definition of \(V^2\) and from Proposition \((3,2)\) we get
\[ E + V^2 = (x_0^{2m+1}, x_1^{2m+1}, \ldots, x_1^{m+1}, x_0^m) = V^{(2)}. \]

\[ \square \]

4 Tetrahedron

As a natural generalization of previous results we consider the tetrahedron in \(\mathbb{P}^3\).

We write
\[ P_0 = [1 : 0 : 0 : 0], P_1 = [0 : 1 : 0 : 0], P_2 = [0 : 0 : 1 : 0], P_3 = [0 : 0 : 0 : 1] \]
for the vertices of the tetrahedron.

As before, we denote by \(V = I(3,3) = \langle x_0 x_1, x_0 x_2, x_1 x_2, x_2 x_3, x_1 x_3, x_0 x_3 \rangle\) the ideal of vertices, \(E = I(3,2) = \langle x_0 x_1 x_2, x_1 x_2 x_3, x_0 x_2 x_3, x_0 x_1 x_3 \rangle\) the ideal of edges and by \(F = I(3,1) = \langle x_0 x_1 x_2 x_3 \rangle\) the ideal of faces.

Once again we take a closer look at the symbolic and ordinary powers of these ideals. We obtain following results.
Lemma 4.1. The ideal $F$ is a complete intersection ideal and this implies the equality $F^{(k)} = F^k$ for all positive integers $k$.

Proposition 4.2. For any permutation $\sigma \in \Sigma$ the generators of

1) $E^{(m)}$ are of the form $x_{\sigma(0)}^{m-j} x_{\sigma(1)}^{m-j} x_{\sigma(2)}^{m-j} x_{\sigma(3)}$ for $j \in \{0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \}$,

2) $V^{(m)}$ are of the form $x_{\sigma(0)}^{m-i-j} x_{\sigma(1)}^{m-i-j} x_{\sigma(2)}^{i} x_{\sigma(3)}^{j}$ for $0 \leq 2i + j \leq m$ and $0 \leq i + 2j \leq m$.

Proof. Proof of the case 1) is similar to the proof of Theorem 3.2 with one additional variable.

In the proof of 2) there is only one thing demanding explanation. After taking a partial derivative of order $m - 1$ of $x_{\sigma(0)}^{i} x_{\sigma(1)}^{j} x_{\sigma(2)}^{k} x_{\sigma(3)}^{l}$ it has to contain at least two variables $x_t, x_s$. Using the method described in Theorem 3.2 we conclude that $m - i - j + m - i - j + i \geq m$ and $m - i - j + m - i - j + j \geq m$ and from these inequalities we obtain

$$2i + j \leq m \quad \text{and} \quad i + 2j \leq m.$$ 

\[ \square \]

Proposition 4.3. For all positive integers $m$ it holds $E^{(2m)} = (E^{(2)})^{m}$ and $E^{(2m+1)} = E^{(2m)} E$.

Proof. This repeats the reasoning in the proof of Proposition 3.6 and is left to a motivated reader. \[ \square \]

Immediately from Proposition 4.2 Theorem A and Theorem B basing on generators shape we can derive many more containment relations between the ideals $V, E, F$ and their ordinary and symbolic powers. We present them on the following diagram. The direction of arrows symbolizes inclusion between the ideals.
As an example we prove here the inclusion $E^{(3)} \subset E^2$.

**Example 4.4.** Proposition 4.2 shows that generators of $E^{(3)}$ are in two forms

$$x^{3}_{\sigma(0)}x^{3}_{\sigma(1)}x^{3}_{\sigma(2)} = e \cdot x^{2}_{\sigma(0)}x^{2}_{\sigma(1)}x^{2}_{\sigma(2)},$$

and

$$x^{2}_{\sigma(0)}x^{2}_{\sigma(1)}x^{2}_{\sigma(2)}x^{2}_{\sigma(3)} = e \cdot x_{\sigma(3)},$$

where $e = x^{2}_{\sigma(0)}x^{2}_{\sigma(1)}x^{2}_{\sigma(2)}$. The definitions of $E^2$ gives us that $e \in E^2$ which finishes the proof.

We finish this section with the following corollary from Proposition 4.2.

**Corollary 4.5.** $E^{(2)} = F + E^2$.

**Remark 4.6.** This generalizes easily to $I^{(2)}(n, 2) = I(n, 1) + I^2(n, 2)$ in $\mathbb{P}^n$ case.

## 5 General case

We begin by proving Theorem B.

**Proof of Theorem B.** Let $x^a \in I^{(m)}(n, c)$. By Proposition 2.8 we have $a_{j_1} + \ldots + a_{j_c} \geq m$ for any $c$-tuple $j_1, \ldots, j_c \in \{0, 1, \ldots, n\}$. Recall that we assume that $d \geq c$. 

Consider the following system of inequalities
\[
\begin{align*}
  a_{j_1} + \ldots + a_{j_c} & \geq m \\
  a_{j_2} + \ldots + a_{j_{c+1}} & \geq m \\
  a_{j_3} + \ldots + a_{j_{c+2}} & \geq m \\
  \vdots \\
  a_{j_i} + \ldots + a_{j_{n-1}} + a_{j_d} & \geq m
\end{align*}
\] (11)

Taking into account that every element \(a_{j_i}\) appears in \(c\) inequalities and taking the sum of all of them we get
\[
c(a_{j_1} + \ldots + a_{j_d}) \geq dm.
\]
This means that \(a_{j_1} + \ldots + a_{j_d} \geq \frac{md}{c}\) and as a consequence \(x^a \in I^{(m)}(n,d)\) again by Proposition 2.8. \(\square\)

**Proof of Theorem A.** By the definition of the ordinary power, the associated shape of \(I'(n,c)\) is the union of all translates of the positive octant by integral lattice points \(a = (a_0, \ldots, a_n)\) in \(\mathbb{Z}_{\geq 0}^{n+1} \subseteq \mathbb{R}^{n+1}\) satisfying condition b) in Proposition 2.9.

On the other hand Proposition 2.8 tells us that \(I^{(m)}\) is generated by elements \(x_0^{k_0} \cdot \ldots \cdot x_n^{k_n}\) subject to conditions:
\[
k_{s_1} + \ldots + k_{s_c} \geq m,
\] (12)
for every \(c\)-tuple \(s_1, \ldots, s_c \in \{0, \ldots, n\}\). It follows that the associated shape of \(I^{(m)}(n,c)\) is contained in the convex set in \(\mathbb{R}^{n+1}\), consisting of solutions to inequalities in (12). In fact we can make this set a little bit smaller.

**Claim.** \(\text{Sh}(I^{(m)}(n,c))\) is contained in the halfspace
\[
a_0 + \ldots + a_n \geq (n+1)\left\lceil \frac{m}{c} \right\rceil.
\]

Turning to the proof of the claim, it suffices to note that the strip
\[
\left\{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \mid (n+1)\frac{m}{c} < a_0 + \ldots + a_n < (n+1)\left\lceil \frac{m}{c} \right\rceil \right\}
\]
contains no integral points. Since the point \(\left\lceil \frac{m}{c} \right\rceil, \ldots, \left\lceil \frac{m}{c} \right\rceil\) \(\in \mathbb{R}^{n+1}\) satisfies (12) the monomial
\[
x_0^{\left\lceil \frac{m}{c} \right\rceil} \cdot \ldots \cdot x_n^{\left\lceil \frac{m}{c} \right\rceil} \in I^{(m)}(n,c).
\]
On the other hand by Proposition 2.9 b), the condition
\[
(n-c+2)r \leq \left\lceil \frac{(n+1)m}{c} \right\rceil,
\]
must be satisfied for the inclusion \(I^{(m)}(n,c) \subseteq I'(n,c)\). \(\square\)
This result lets to find the resurgences of ideals $I(n,c)$.

**Proof of Theorem C.** Let $r_0(q) = \frac{1}{n-c+2} \cdot \left\lceil \frac{(n+1)q}{c} \right\rceil$. From Theorem A we know that for all positive integers $r \leq r_0(q)$ there is the containment $I^{(q)} \subset I^{(r)}$. We conclude that

$$\sup\left\{ \frac{q}{r} : I^{(q)}(n,c) \nsubseteq I^{(r)}(n,c) \right\} = \sup_{q \in \mathbb{Z}_+} \frac{q}{[r_0(q)]+1}.$$ 

Let $\rho(q) = \frac{q}{[r_0(q)]+1}$. We have the following obvious sequence of inequalities

$$\frac{q}{[r_0(q)]+1} \leq \frac{q}{r_0(q)} \leq \frac{c \cdot (n-c+2)}{n+1}. \quad (13)$$

That gives us $\rho(q) \leq \frac{c(n-c+2)}{n+1}$ for every $q$. On the other hand we have

$$\frac{q}{[r_0(q)]+1} \geq \frac{q}{r_0(q)} = \frac{q \cdot (n-c+2)}{\left\lceil \frac{q(n+1)}{c} \right\rceil + n-c+2} \geq \frac{q \cdot (n-c+2)}{\frac{q(n+1)}{c} + n-c+3}. \quad (14)$$

Finally from (13) and (14) we get

$$\frac{c \cdot (n-c+2)}{n+1} \geq \frac{q}{[r_0(q)]+1} \geq \frac{q \cdot (n-c+2)}{\frac{q(n+1)}{c} + n-c+3}.$$ 

what in a natural way implies that $\lim_{q \to \infty} \rho(q) = \frac{c(n-c+2)}{n+1}$. This limit and the fact that the inequality $\rho(q) \leq \frac{c(n-c+2)}{n+1}$ holds for every $q$ gives us

$$\rho(I(n,c)) = \sup_{q \in \mathbb{Z}_+} \frac{q}{[r_0(q)]+1} = \frac{c \cdot (n-c+2)}{n+1}.$$ 

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