Heisenberg versus standard scaling in quantum metrology with Markov generated states and monitored environment

Catalin Catana$^1$ and Madalin Guta$^1$

$^1$School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, UK

Finding optimal and noise-robust probe states, is a key problem in quantum metrology. In this paper we propose Markov dynamics as a possible mechanism for generating such states. Additionally, we model noisy channels by coupling the Markov output to ‘environment’ ancillas, and consider the scenario where the ancillas are monitored to increase the quantum Fisher information of the output. We focus on dynamics where the memory system has multiple ‘phases’ (stationary states) which can exhibit Heisenberg scaling in the case of unitary channels. For noisy channels we find that the survival of the Heisenberg limit depends on whether the environment receives ‘which phase’ information about the memory. If ‘phase purification’ occurs, the quantum Fisher information of the conditional output state has an initial quadratic scaling, but in the long run the coherent superposition of the ‘output phases’ is destroyed leading to the standard scaling. However, if the environment cannot distinguish between distinct dynamical phases, the Heisenberg scaling is preserved.

The accurate estimation of unknown parameters is a fundamental task in quantum technologies, with applications ranging from spectroscopy [1] and interferometry [2,3], to atomic clocks [4–6] and gravitational wave detectors [7,8]. The typical metrological protocol [9] can be described as follows. A quantum transformation $T_\theta$, depending on an unknown parameter $\theta$, is applied (in parallel) to each component of a ‘probe’ ensemble of $n$ quantum systems which are initially prepared in a state $|\psi^n\rangle$. The ensemble is subsequently measured and an estimator $\theta_n$ of $\theta$ is computed. For independently prepared systems, the estimation precision scales as $1/n$, known as the ‘standard quantum limit’. In contrast, if quantum resources such as entanglement or squeezing are used in the preparation stage, and if $T_\theta$ is a unitary, the precision can be enhanced to the ‘Heisenberg limit’ $1/n^2$ [10,12].

However, when noise and decoherence are taken into account, they typically lead to a ‘downgrading’ of the Heisenberg scaling to the standard one, but a ‘quantum enhancement’ is still achievable in the form of a constant factor that increases with decreasing noise levels [13–15]. In this setup, new tools for deriving upper bounds on the quantum Fisher information of the final state have been developed in [15–18]. This being said, we note that the Heisenberg limit can be preserved for certain noise models [19–21] or by using quantum error correction techniques [22–24] in modified metrological setups.

The aim of this paper is to explore quantum metrology in a setup similar to that of [25], which is characterised by two key features. Firstly, the probe state is generated as output of a quantum Markov chain, which allows us to use previous results on system identification for quantum Markov systems [20,27]. Secondly, we model the noisy channel $T_\theta$ as coupling with an ancilla (environment) and we assume that the latter can be monitored by means of measurements. If the Markov transition operator is primitive [28] (memory converges to unique stationary state), then the quantum Fisher information (QFI) scales linearly even with full access to the environment, and the standard limit holds [27]. We therefore consider Markov systems with multiple ‘phases’ (invariant spaces), and investigate the evolution of the QFI of the probe state, conditional on the measurement outcomes. We will show that one of the following two scenarios can occur. If the measurement does not distinguish between the dynamical phases, and the memory is started in a superposition of different phases, then the QFI of the conditional output state scales as $n^2$, and the Heisenberg scaling holds. If the environment receives ‘which phase’ information about the memory, the QFI may have an initial quadratic scaling but becomes linear as the memory system is ‘localised’ to one of the phases. Next we describe the details of our metrology setup and discuss an example exhibiting the two behaviours.

The noise is $\mathcal{N}(|\psi^n\rangle,\mathcal{E})$. The noise is monitored by the ancilla $|\psi^n\rangle$. Final state conditional on ancilla measurements has quantum Fisher information $F(|\Psi^n(\theta|\mathcal{E})\rangle)$.

FIG. 1. (Color online) Quantum metrology with $n$ noisy channels acting on a pure state input $|\Psi^n\rangle$. The noise is monitored by the ancilla $|\chi\rangle$. Final state conditional on ancilla measurements has quantum Fisher information $F(|\Psi^n(\theta|\mathcal{E})\rangle)$. 
Metrology with monitored environment. We assume that each of the noisy channels $T^i_\theta$ is modelled by coupling the $i$th probe system unitarily to a corresponding ancilla representing the environment, as illustrated in Figure [1] the ancilla is then monitored by performing a measurement with outcome $c_i$, and the joint conditional state of the probe is $|\Psi^\theta_\theta^c\rangle$, where $c = (c_1, \ldots, c_n)$. The data $c$ is fed into the design of the final measurement which aims to extract the maximum amount of information about $\theta$ from the state. As customary in the quantum metrology literature, we will use the quantum Fisher information as a figure of merit for estimation. This is justified by the quantum Cramér-Rao bound [29] which states that for any smooth family of states ($\rho_\theta: \theta \in \mathbb{R}$), and any unbiased measurement with outcome $\hat{\theta}$ the following lower bound holds

$$E[(\hat{\theta} - \theta)^2] \geq F(\rho_0)^{-1},$$

where $F(\rho_0) = \text{Tr}(\rho_0 L_\theta^2)$ is the QFI and $L_\theta$ is the symmetric logarithmic derivative defined by $2 \frac{d\rho_\theta}{d\theta} = \{L_\theta, \rho_\theta\}$. In particular, if $\rho_0 = |\psi_0\rangle\langle \psi_0|$ is a pure state rotation family with $|\psi_0\rangle = \exp(i\theta G)|\psi\rangle$ then

$$F(\rho_0) = 4\text{Var}(G) := 4(\langle G^2 \rangle - \langle G \rangle^2) \quad (1)$$

where $\langle \cdot \rangle$ denotes the expectation with respect to $|\psi\rangle$.

By the convexity of the quantum Fisher information, the following inequality holds

$$F(\rho^n_0) \leq \sum_c p(c) F(|\Psi^n\theta\langle c|\rangle),$$

where $\rho^n_0 = T^n_\theta(\langle\psi^n\rangle\langle\psi^n|)$ is the state obtained by averaging the conditional state over all $c$. The interpretation is that monitoring the environment provides additional information leading to an increase of the QFI.

We illustrate our ‘monitored environment’ set-up with the following toy example. The noisy channel $T^n_\theta$ is the convex combination of two unitary rotations on a qubit

$$T^n_\theta(\rho) = \sum_{j \in \{x, z\}} \lambda_j e^{i\sigma_j |\rho|} e^{-i\sigma_j |\rho|}.$$

Since $T^n_\theta$ is not extremal, the estimation rate in the standard metrology setup is $n^{-1}$ [13]. Consider now that by monitoring the environment, we know which of the two unitaries has been applied. If the probe is prepared in the state $|\Psi^n\rangle = (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$, then it can be shown that the QFI of the conditional output state depends only the number of outcomes $(n_x, n_z)$ of each type and equals $F(|\Psi^n_\theta\rangle|n_x, n_z\rangle) = 2n^2_x + 2n^2_z$. Since $n_j \sim \lambda_j n$ for large $n$, the QFI scales as $n^2$.

Markov generated probe states. We now introduce the second main ingredient of our analysis: a Markovian mechanism for generating the initial probe state $|\Psi^n\rangle$.

This ansatz is motivated by the fact that Markovian output states are closely related to finitely correlated states [30] and matrix product states (MPS) [31], which have often been used as efficient and tractable approximations of complex many-body states [32]. The dynamics of the preparation stage and subsequent metrology protocol is illustrate in Figure [2]. The top row represents a ‘memory system’ $A$ which interacts sequentially (moving from right to left) with a chain of $n$ identically prepared probe systems (row B), by applying the same unitary $U^{AB}$.

![Figure 2](image_url)

FIG. 2. (Color online) Model of discrete dynamics with Markov generated probe state. The memory system $A$ interacts successively (from right to left) with the probe systems $B$ via the unitary $U^{AB}$. The channel $T^n_\theta B$ is implemented by unitary coupling of the probe systems with ancillas $C$.

After the interaction the chain $B$ together with the memory are in the state

$$|\Psi^n_{AB}\rangle = \sum_{f, b} |f|K^b_k|f\rangle \otimes |b\rangle,$$

where $b = (b_1, \ldots, b_n)$ is the index of the product basis for the B row, $K^b_k := K^{bn_1}_b \cdots K^{bn_k}_b$, and $K^b_k := |b\rangle U^{AB}|\xi\rangle$ are the Kraus operators associated to the unitary $U^{AB}$ and the initial state $|\xi\rangle$ of the B systems. After this probe preparation stage, each system undergoes a separate unitary interaction $W^{BC}_{\theta}$ with an ancilla (environment) in row C, prepared initially in state $|\chi\rangle$. The channel $T^n_\theta$ is recovered by tracing out the ancilla

$$T^n_\theta(\rho_B) = \text{Tr}_C(W^{BC}_{\theta}(\rho_B \otimes |\chi\rangle\langle \chi|) W^{BC}_{\theta}^\dagger)$$

where $A^n_{\theta} := \langle c|W^{BC}_{\theta}|\chi\rangle$ are Kraus operators. By commutativity, the final ABC state is the same irrespective of whether the unitaries $W^{BC}_{\theta}$ are applied at the end of the preparation stage or each of them is applied immediately after the corresponding $U^{AB}$. With similar notations as
above, the joint ABC final state is
\[ |\Psi_{ABC}(\theta)\rangle = \sum_f \langle f| K_{fi}^\theta |i\rangle \otimes |f\rangle \otimes |\xi\rangle, \] (2)
\[ = \sum_c \left( I_A \otimes A_c^\theta |\Psi_{AB}^c\rangle \right) \otimes |\xi\rangle = \sum_c |\tilde{\Psi}^n_{AB}(\theta|c\rangle) \otimes |\xi\rangle \]
where \(|\tilde{\Psi}^n_{AB}(\theta|c\rangle)\) is the unnormalised conditional state of \(AB\) given outcome \(c\) and
\[ K_{bc}^\theta = |b\rangle \otimes \langle c| \left( I^4 \otimes W_{\theta}^{BC} \right) \left( U^{AB} \otimes I^C \right) |\xi\rangle \otimes |\chi\rangle. \]
are ‘extended’ Kraus operators.

The reduced evolution of \(A\) is obtained either by tracing out the \(B\) systems in \(|\Psi_{AB}^c\rangle\) or the \(B\) and \(C\) systems in \(|\Psi_{ABC}\rangle\), so that the one step transition operator is
\[ Z_0(\rho) = \sum_b K_b^\rho \rho K_b^\dagger = \sum_{b,c} K_{b,c}^\rho \rho K_{b,c}^\dagger. \]

Before further delving into the analysis of the Markov model let summarise how the set-up of Figure 2 fits into the metrology scheme. The metrological probe is prepared via the subsequent interaction \(W^{BC}\) between the \(B\) and their corresponding \(C\) systems. As explained in the introduction, we distinguish two scenarios: the experimentalist has access only to the \(B\) systems (which typically leads to standard rates for non-unitary channels \(T_b\), or the environment \(C\) can be monitored and the collected data can be used to improve the estimation rates.

Seen from a system identification perspective, this set-up has been investigated in [26, 27], which show that if the transition operator \(Z_0\) is primitive [28] (system \(A\) converges to a unique stationary state) then the quantum Fisher information of the state \(|\Psi_{ABC}^n(\theta)\rangle\) increases linearly with \(n\), so \(\theta\) can only be estimated with rate \(n^{-1}\). Therefore, from the metrology viewpoint it is interesting to consider models in which the memory system \(A\) has several invariant subspaces (or ‘phases’). In this case, the quantum Fisher information of the full state \(|\psi_{ABC}^\theta\rangle\) may increase as \(n^2\) [26], a similar divergence from the linear regime being observed in classical models [33].

In a two phases system for instance, the memory space \(\mathcal{H}_A\) decomposes into orthogonal subspaces (phases) \(\mathcal{H}_A = \mathcal{H}_A^0 \oplus \mathcal{H}_A^1\), such that the Kraus operators \(K_b\) (and similarly for \(K_{b,c}\)) are block-diagonal with respect to this decomposition
\[ K_b = (i|U^{AB}|\xi\rangle = \begin{pmatrix} K_{b}^0 & 0 \\ 0 & K_{b}^1 \end{pmatrix}, \]
and the restricted evolutions are primitive, with unique stationary states \(\rho_{ss}, \rho_{ss}^1\). Assuming that the initial state of \(A\) is a coherent superposition of states from the two phases, e.g. \(|i\rangle = (|i,0\rangle + |i,1\rangle)/\sqrt{2}\), the joint states \(|\Psi_{AB}^n\rangle\) and \(|\Psi_{ABC}^n\rangle\) have a similar decomposition
\[ |\Psi_{AB}^n\rangle = \frac{1}{\sqrt{2}} \left(|\Psi_{AB}^{n,0}\rangle + |\Psi_{AB}^{n,1}\rangle\right) \in \mathcal{H}_A^0 \otimes \mathcal{H}_B^0 \otimes \mathcal{H}_A^1 \otimes \mathcal{H}_B^1. \]
For concreteness let us consider that the unitary \(W_{\theta}^{BC}\) on \(BC\) is of the form
\[ W_{\theta}^B = (U_{\theta}^B \otimes I^C)V^{BC} \]
where \(U_{\theta}^B = \exp(-i\theta G)\) is a phase rotation on the probe system \(B\) with generator \(G\) and \(V^{BC}\) is a fixed unitary describing the interaction with the environment. Since \(|\Psi_{ABC}^n\rangle\) is a unitary rotation family,
\[ |\psi_{ABC}^\theta\rangle = \exp(-i\theta G)|\psi_{ABC}^0\rangle, \quad G = \sum_{i=1}^n G^{(i)} \]
its quantum Fisher information is proportional to the variance of the ‘total generator’ \(G\). The distribution of \(G\) is the mixture of \(\mathbb{P}_n = \mathbb{P}^n/2\) of the distributions corresponding to the two phases, which can be calculated from the states \(|\Psi_{ABC}^0\rangle\) and \(|\Psi_{ABC}^1\rangle\). Under each \(\mathbb{P}_n\) and \(\mathbb{P}_1\), the following convergence to the normal distribution (Central Limit Theorem) holds [26]
\[ \frac{1}{\sqrt{n}}(G - ng_\theta) \xrightarrow{\mathcal{L}} N(0, V^n), \quad a = 0, 1. \] (3)
Therefore, if \(g_\theta \neq g_1\) the distribution of \(G\) with respect to the output state \(|\Psi_{ABC}^n(\theta)\rangle\) has variance of the order \(n^2\), as illustrated in Figure 3 and by [4] the Heisenberg scaling holds.

We now investigate what happens when the ancillas in row \(C\) are measured, as described in our environment
monitoring scheme. The conditional state $|\Psi_{AB}(\theta;\xi)\rangle$ has a similar phase decomposition

$$|\Psi_{AB}(\theta;\xi)\rangle = p_0(\xi)|\Psi_{AB}^{(0)}(\theta;\xi)\rangle + p_1(\xi)|\Psi_{AB}^{(1)}(\theta;\xi)\rangle$$

where $p_0(\xi)$ is the probability that the system $A$ is in phase $a$ given the outcome $\xi$. The normalised state $|\Psi_{AB}^{(a)}(\theta;\xi)\rangle$ is the posterior state corresponding to the system $A$ being initially in state $|a\rangle_A$ belonging to phase $a$. Using large deviation theory $34$, one can show that for each phase $a$, the conditional distribution of $\tilde{G}/n$ concentrates around the mean $g^n$ with high probability with respect to the outcome $\xi$. By the same argument as above, the variance of $\tilde{G}$ with respect to the conditional state $|\Psi_{AB}^{(a)}(\theta;\xi)\rangle$ increases as $n^2$ provided that the ‘weights’ $p_0(\xi)$ and $p_1(\xi)$ of the two components of the mixture stay away from the extreme values 0, 1. Unfortunately however, this can happen only in special situations as the following argument shows. Let $Q^0(\xi)$ and $Q^1(\xi)$ be the probability distributions of measurements on the environment corresponding to the two phases. If these distributions are different in the stationary regime, it means that the observer can distinguish between them at a certain exponential rate, similarly to the case of discrimination between the two.

The normalised state $|\Psi_{AB}^{(a)}(\theta;\xi)\rangle$ has the following expression

$$|\Psi_{AB}^{(a)}(\theta;\xi)\rangle = (|0\rangle_A \otimes U_B^0 A_{\alpha}^n |\psi_0\rangle \otimes \cdots \otimes U_B^0 A_{\alpha}^n |\psi_0\rangle + |1\rangle_A \otimes U_B^0 A_{\alpha}^n |\psi_1\rangle \otimes \cdots \otimes U_B^0 A_{\alpha}^n |\psi_1\rangle)$$

The probability distribution $Q(\xi) = ||\Psi_{AB}^{(a)}(\theta;\xi)||^2$ is the mixture $(Q_0 + Q_1)/2$ where both components are product measures (independent samples from $\{0,1\}$) with probabilities

$$q_0^a = p_0, \quad q_1^a = p_1 = (1 - p)\alpha_\pm, \quad q_\pm^a = (1 - p)\beta_\pm,$$

where $\alpha_\pm = |\langle v_\pm |\psi_0\rangle|^2, \beta_\pm = |\langle v_\pm |\psi_1\rangle|^2$. In particular, the weights $p_\pm(\xi)$ of the two phases and the quantum Fisher information of the conditional state $|\Psi_{AB}^{(a)}(\theta;\xi)\rangle$ depend only on the total number for each outcome $n_\pm$ and $n_0$, the latter being the number of systems which have not been affected by the noise. The Fisher information has the following expression

$$F(|\Psi_{AB}^{(a)}(\theta;\xi)\rangle) = F(n_\pm, n_0) = 4 \left\{ n_0 \alpha_\pm \alpha_{\mp} \alpha_+ \beta_- + \beta_+ \beta_- \alpha_\mp \alpha_- \varepsilon + \alpha_\mp \alpha_- + \alpha_\pm \beta_\mp \beta_- \right\}$$

where since $n_0 \approx mn$, the state has Heisenberg scaling if and only if the second term in (5) does not converge to zero. As noted before this coefficient is equal to $(g_0 - g_1)^2 p_0(\xi)p_1(\xi)$, so the Heisenberg scaling holds when the means of $G$ in the two phases are different ($\alpha_\mp \neq \beta_\mp$), and phase purification does not occur ($q_0^a = q_1^a$).

Now, it is easy to verify that for any noise direction $v$ different from $z$, there exist Kraus operators $K_v$ such that these conditions are met. In terms of the Bloch sphere, the requirement is that the Bloch vectors of $|\psi_0\rangle$ and $|\psi_1\rangle$ lie symmetrically with respect to $v$ such that the (environment) measurement in this direction cannot distinguish between the two states.

In the special case when $v$ is along the $z$ axis, the means of $G$ in the two phases are equal and therefore

The distribution of the generator $\tilde{G}$ with respect to this state is a mixture of two binomial distributions Bin$(n, g_0 = \alpha_1)$ and Bin$(n, g_1 = \beta_1)$, and the quantum Fisher information is

$$F(|\Psi_{AB}^{(a)}(\theta;\xi)\rangle) = 2n(\alpha_0\alpha_1 + \beta_0\beta_1) + n^2(\alpha_1 - \beta_1)^2.$$
after that, the exponential decay kicks in and \( F \) becomes negligible, and quantum Fisher information per probe system converges to the relative entropy of the conditional output state has for unitary channels, the Heisenberg limit is attainable for small \( n \) exponentially with rate \( S \), to the relative entropy \( S(q^0|q^1) \). This implies that for small \( n \) the weights of the two phases are comparable, and quantum Fisher information per probe system \( f(q) = F(|\Psi_{AB}^q(\theta)|)/n \) increases linearly with \( n \); after that, the exponential decay kicks in and \( f(q) \) converges to the Fisher information of the corresponding limiting phase. In Figure 4 we illustrate this behaviour with two trajectories having different limiting phases, with the phases weights shown in the inset plot.

Conclusions and outlook. We proposed Markov dynamics as a possible mechanism for generating probe states for quantum metrology. We pointed out that even for unitary channels, the Heisenberg limit is attainable only if the memory system has multiple ‘phases’ (stationary states) and is initialised in a coherent superposition of different phases. We then modelled noisy channels by coupling the Markov output to ‘environment’ ancillas, and considered the scenario where the environment is monitored to increase the quantum Fisher information of the output. In this setup we found that the survival of the Heisenberg limit depends on whether the environment receives ‘which phase’ information about the memory system. If ‘phase purification’ occurs, the quantum Fisher information of the conditional output state has an initial quadratic scaling, but in the long run the environment wins, and the coherent superposition of the ‘output phases’ is destroyed leading to the standard scaling. However, in a simple example we showed that the Heisenberg scaling is preserved if the Markov dynamics is chosen such that the environment cannot distinguish between the dynamical phases.

These preliminary results open several lines of investigation in the input-output setting with monitored environment, e.g. finding the ‘optimal’ Markov dynamics and ‘stopping times’ which maximise the constant of the standard scaling, analysing the use of feedback control based on the measurement outcomes. An appropriate framework for answering these questions may be that of ‘thermodynamics of trajectories’ and ‘dynamical phase transitions’ and ‘thermodynamics of trajectories’ [30, 37].

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* pmxcc@nottingham.ac.uk

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