Vertex-connectivity and $Q$-index of graphs with fixed girth

Huicai Jia, Hong-Jian Lai, Ruifang Liu, Ju Zhou

Abstract

Let $q(G)$ denote the $Q$-index of a graph $G$, which is the largest signless Laplacian eigenvalue of $G$. We prove best possible upper bounds of $q(G)$ and best possible lower bounds of $q(G)$ for a connected graph $G$ to be $k$-connected and maximally connected, respectively. Similar upper bounds of $q(G)$ and lower bounds of $q(G)$ to assure $G$ to be super-connected are also obtained. Upper bounds of $q(G)$ and lower bounds of $q(G)$ to assure a connected triangle-free graph $G$ to be $k$-connected, maximally connected and super-connected are also respectively investigated.

AMS Classification: 05C50, 05C40

Keywords: vertex-connectivity; girth; $Q$-index; triangle-free graphs; maximally connected; super-connected

1 Introduction

We consider simple, undirected and connected graphs. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. Thus $G$ can be viewed as a spanning subgraph of $K_n$. Define the complement of $G$ to be the graph $\overline{G} = K_n - E(G)$. We denote by $d(v)$ and $\delta(G)$ the degree of a vertex $v$ in $G$ and the minimum degree of $G$, respectively. Let $K_n$ and $K_{a,b}$ denote complete graphs and complete bipartite graphs on $n$ vertices, where $a + b = n$. For two disjoint subsets $X$ and $Y$ of $V(G)$, let $E(X,Y)$ be the set of edges with one end in $X$ and the other end in $Y$. The join of $G$ and $H$, denoted by $G \vee H$, is the graph obtained from a disjoint union of $G$ and $H$ by adding all possible edges between them. Let $G \cup H$ and $G[V_0]$ ($V_0 \subseteq V(G)$) denote the disjoint union of $G$ and $H$ and the subgraph of $G$ induced by $V_0$, respectively. Assume $e = uv \in E(G)$, let $G - e$ be the subgraph of $G$ by deleting $e$ from $G$. Let

---

*School of Mathematics, Renmin University of China, Beijing, 100872, China; College of Science, Henan University of Engineering, Zhengzhou, Henan 451191, China. Email: hjc607@163.com
†Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA. Email: hjlai@math.wvu.edu
‡Corresponding author. School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China. Email: rfliu@zzu.edu.cn
§Department of Mathematics, Kutztown University of Pennsylvania, Kutztown, PA 19530, USA. Email: zhou@kutztown.edu
Let $G - V_0$ be the induced subgraph obtained from $G$ by deleting the vertices of $V_0$ together with the edges. The girth of a graph $G$, is defined as

$$g(G) = \begin{cases} \min \{|E(C)| : C \text{ is a cycle of } G\} & \text{if } G \text{ is not acyclic,} \\ \infty & \text{if } G \text{ is acyclic.} \end{cases}$$

A vertex subset $C$ of a connected graph $G$ is called a vertex-cut if $G - C$ is not connected or $G - C = K_1$. The vertex connectivity $\kappa(G)$ of a connected non-complete graph $G$ is the minimum number of vertices whose deletion disconnects $G$. A vertex-cut $C$ is minimum if $|C| = \kappa(G)$.

A well-known result of Whitney [13] states that $\kappa(G) \leq \delta(G)$ for any graph $G$. A graph $G$ is $k$-connected if $\kappa(G) \geq k$, maximally connected if $\kappa(G) = \delta(G)$, and super-$\kappa$ (or super-connected) if each minimum vertex-cut isolates a vertex of minimum degree. Hence every super-$\kappa$ graph must be maximally connected. A triangle-free graph is an undirected graph with no induced 3-cycle. We follow Bondy and Murty [2] for notation and terminologies not defined here.

The adjacency matrix of $G$ is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and otherwise $a_{ij} = 0$. Let $D(G)$ be the diagonal matrix of the vertex degrees of $G$. The matrix $Q(G) = D(G) + A(G)$ is known as the $Q$-matrix or the signless Laplacian matrix of $G$. We denote the largest eigenvalue of $Q(G)$ by $q(G)$, which is called $Q$-index or the signless Laplacian spectral radius of $G$.

There have been quite a few recent studies on the relationship between vertex-connectivity and eigenvalues of graphs. O [10] presented the relation between vertex-connectivity and the second largest eigenvalue of regular multigraphs. Abiad et al. [1] proved upper bounds for the second largest eigenvalues of regular graphs and multigraphs which guarantee a desired vertex-connectivity. Recently, Liu et al. [9] investigated functions $f(\delta, \Delta, g, k)$ with $\Delta \geq \delta \geq k \geq 2$ and girth $g \geq 3$ such that any graph $G$ satisfying $\lambda_2(G) < f(\delta, \Delta, g, k)$ has connectivity $\kappa(G) \geq k$. On the other hand, Li [8] presented sufficient conditions for a graph to be $k$-connected in terms of the spectral radius and $Q$-index. Hong et al. [7] found sufficient conditions for a connected graph and a connected triangle-free graph with given minimum degree to be $k$-connected, maximally connected and super-connected in terms of the spectral radius of the graph and of its complement, respectively. Zhang et al. [6] proved a sufficient condition for a connected graph with fixed minimum degree to be $k$-connected based on $Q$-index for sufficiently large order $n$.

Motivated by these results, the purpose of the current research focuses on the following general problem.

**Problem 1.1** For a connected graph $G$ with fixed girth $g \geq 3$ and minimum degree $\delta \geq k \geq 2$, find optimal sufficient conditions in terms of $Q$-index of the graph and of its complement to describe the properties of being $k$-connected, maximally connected and super-connected.

In particular, we in this paper investigate the problem above for the two special cases: connected graphs ($g \geq 3$) and connected triangle-free graphs ($g \geq 4$). In the next section, we display some useful tools to be deployed in our arguments. In the subsequent sections, our main results for the generic study and for the special cases are presented and justified.
2 Preliminaries

We in this section will present some former results that will be utilized in our arguments. The following bounds of the $Q$-index of a graph $G$, stated in Lemmas 2.1 and 2.2, are applied frequently.

**Lemma 2.1** (Cvetković, Rowlinson and Simić [3]) Let $G$ be a graph with order $n$ and size $m$. Then

$$q(G) \geq \frac{4m}{n}.$$ 

If $G$ is connected, then the equality holds if and only if $G$ is a regular graph.

**Lemma 2.2** (Feng and Yu [5]) Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$q(G) \leq \frac{2m}{n-1} + n - 2,$$

and the equality holds if and only if $G$ is $K_n$ or $K_{1,n-1}$.

Given positive integers $\delta$, $g$ and $\kappa$, define $t = \left\lfloor \frac{g-1}{2} \right\rfloor$ and

$$\nu(\delta, g, \kappa) = \begin{cases} 1 + (\delta - \kappa) \sum_{i=0}^{t-1}(\delta - 1)^i & \text{if } g = 2t + 1, \\ 2 + (2\delta - 2 - \kappa) \sum_{i=0}^{t-1}(\delta - 1)^i & \text{if } g = 2t + 2 \text{ and } \delta \geq 3, \\ 2t + 1 & \text{if } g = 2t + 2 \text{ and } \delta = 2. \end{cases}$$

Liu et al. [9] proved the following result which is crucial to our main results in Section 3.

**Lemma 2.3** (Liu, Lai, Tian and Wu [9]) Let $G$ be a simple connected graph with $\kappa(G) = \kappa$, minimum degree $\delta \geq k \geq 2$ and girth $g \geq 3$. Let $C$ be a minimum vertex cut of $G$ with $|C| = \kappa$ and $V_0$ be a connected component of $G - C$. If $\kappa \leq k - 1 < \delta$, then

$$|V_0| \geq \nu(\delta, g, \kappa) \geq \nu(\delta, g, k - 1).$$

In [4], Füredi et al. proved the following girth and Turán number result.

**Lemma 2.4** (Füredi and Simonovits [4]) Let $G$ be a simple connected graph with order $n$, size $m$ and girth $g \geq 3$. Then

$$m < \begin{cases} \frac{n+1}{2} + \frac{1}{2}n & \text{if } g = 2t + 1, \\ \frac{1+\kappa}{2+t} n^{1+t} + \frac{1}{2}n & \text{if } g = 2t + 2. \end{cases}$$

**Lemma 2.5** Let $G = K_\kappa \cup (K_a \cup K_b)$, where $\delta - \kappa + 1 \leq a \leq b \leq n - \delta - 1$ and $a + b = n - \kappa$, then $q(G) > n - 2$.

**Proof.** Since $\delta - \kappa + 1 \leq a \leq b \leq n - \delta - 1$ and $a + b = n - \kappa$, we have $a \leq \frac{n-\kappa}{2} \leq b$. By Lemma 2.1, then

$$q(G) \geq \frac{4m}{n} = \frac{4}{n} \left[ \frac{n(n-1)}{2} - ab \right] = 2(n-1) - \frac{4}{n} a(n-\kappa - a)$$

$$= 2(n-1) - \frac{4}{n} (a - \frac{n-\kappa}{2})^2 - \frac{(n-\kappa)^2}{4} \geq 2(n-1) - \frac{4}{n} (\frac{n-\kappa}{2})^2$$

$$\geq 2(n-1) - \frac{4}{n} \left( n - \frac{n-\kappa}{2} \right)^2 > n - 2.$$
The result follows.

The following Turán’s Theorem is well known.

**Theorem 2.6** (Mantel [11] and Turán [12]) For any triangle-free graph $G$ of order $n$ and size $m$, we have

$$m \leq \left\lfloor \frac{1}{4}n^2 \right\rfloor,$$

with equality if and only if $G \cong K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$.

3 Vertex-connectivity and $Q$-index of graphs with fixed girth

Motivated by the methods deployed in [7, 9], in this section, we mainly give sufficient conditions on $q(G)$ and $q(G)$ to predict a connected graph $G$ with fixed girth $g$ to be $k$-connected.

First, we present a crucial and technical lemma.

**Lemma 3.1** Let $G$ be a connected graph of order $n$, size $m$, minimum degree $\delta \geq k \geq 2$ and girth $g \geq 3$. Define $\nu = \nu(\delta, g, k - 1)$. If

$$m(G) \geq \begin{cases} \frac{1}{2}(\nu + k - 1)^{1+\frac{1}{t}} + \frac{1}{2}(n - \nu)^{1+\frac{1}{t}} + \frac{1}{2}(n + k - 1) & \text{if } g = 2t + 1, \\ \frac{1}{2}(\nu + k - 1)^{1+\frac{1}{t}} + \frac{1}{2}(n - \nu)^{1+\frac{1}{t}} + \frac{1}{2}(n + k - 1) & \text{if } g = 2t + 2, \end{cases}$$

then $G$ is $k$-connected.

**Proof.** Assume that $\kappa \leq k - 1$. Let $C$ be a minimum vertex-cut of $G$, then $|C| = \kappa \leq k - 1 < \delta$. Let $V_0, V_1, \ldots, V_t$ ($t \geq 2$) be the vertex sets of connected components of $G - C$ with $|V_0| \leq |V_1| \leq \cdots \leq |V_t|$, and let $U = \bigcup_{i=1}^{t-1} V_i$ (see Figure 1).

![Figure 1](image_url)

**Figure 1.** The partition of $V(G)$ into $V_0$, $C$ and $U$.

By Lemma 2.3, for any $i$ with $0 \leq i \leq t - 1$, we have

$$|V_i| \geq \nu(\delta, g, \kappa).$$
In particular, \( \nu(\delta, g, \kappa) \leq |V_0| \leq |U| \leq n - \kappa - \nu(\delta, g, \kappa) \) and \( |V_0| + |U| = n - \kappa \). In the following, we proceed our proof according to the different parities of the girth \( g \).

**Case 1.** \( g = 2t + 1 \) is odd.

By Lemma 2.4, and since \( \nu(\delta, g, \kappa) \leq |V_0| \leq \frac{n-k}{2} \leq |U| \), we have

\[
m(G) = |E(G[V_0 \cup C])| + |E(G[C \cup U])| - |E(G[C])| \\
\leq |E(G[V_0 \cup C])| + |E(G[C \cup U])| \\
< \frac{1}{2}(|V_0| + |C|)^{1+\frac{1}{2}} + \frac{1}{2}(|V_0| + |C|) + \frac{1}{2}(|C| + |U|)^{1+\frac{1}{2}} + \frac{1}{2}(|C| + |U|) \\
= \frac{1}{2}(|V_0| + \kappa)^{1+\frac{1}{2}} + \frac{1}{2}(n - |V_0|)^{1+\frac{1}{2}} + \frac{1}{2}(n + \kappa) \quad \text{(decreasing on } |V_0|) \\
\leq \frac{1}{2}((\nu(\delta, g, \kappa) + \kappa)^{1+\frac{1}{2}} + \frac{1}{2}(n - \nu(\delta, g, \kappa))^1 + \frac{1}{2}(n + \kappa) \quad \text{(increasing on } \kappa) \\
\leq \frac{1}{2}((\nu + k - 1)^{1+\frac{1}{2}} + \frac{1}{2}(n - \nu)^{1+\frac{1}{2}} + \frac{1}{2}(n + k - 1),
\]

contrary to (1) and so Case 1 is justified.

**Case 2.** \( g = 2t + 2 \) is even.

By Lemma 2.4, with a similar argument as in Case 1, we obtain

\[
m(G) = |E(G[V_0 \cup C])| + |E(G[C \cup U])| - |E(G[C])| \\
< \frac{1}{2^{1+\frac{1}{2}}}(|V_0| + |C|)^{1+\frac{1}{2}} + \frac{1}{2}(|V_0| + |C|) + \frac{1}{2}(|C| + |U|)^{1+\frac{1}{2}} + \frac{1}{2}(|C| + |U|) \\
= \frac{1}{2^{1+\frac{1}{2}}}(|V_0| + \kappa)^{1+\frac{1}{2}} + \frac{1}{2^{1+\frac{1}{2}}}(n - |V_0|)^{1+\frac{1}{2}} + \frac{1}{2}(n + \kappa) \\
\leq \frac{1}{2^{1+\frac{1}{2}}}((\nu(\delta, g, \kappa) + \kappa)^{1+\frac{1}{2}} + \frac{1}{2}(n - \nu(\delta, g, \kappa))^1 + \frac{1}{2}(n + \kappa) \\
\leq \frac{1}{2^{1+\frac{1}{2}}}((\nu + k - 1)^{1+\frac{1}{2}} + \frac{1}{2^{1+\frac{1}{2}}}(n - \nu)^{1+\frac{1}{2}} + \frac{1}{2}(n + k - 1),
\]

contrary to (1) and so Case 2 is justified. This completes the proof of the lemma. \( \square \)

**Theorem 3.2** Let \( G \) be a connected graph of order \( n \), minimum degree \( \delta \geq k \geq 2 \) and girth \( g \geq 3 \), and let \( \nu = \nu(\delta, g, k - 1) \). If

\[
q(G) \geq \begin{cases} 
\frac{1}{2^{1+\frac{1}{2}}}((\nu + k - 1)^{1+\frac{1}{2}} + (n - \nu)^{1+\frac{1}{2}}) + \frac{k}{n-1} + (n - 1) & \text{if } g = 2t + 1, \\
\frac{1}{2^{1+\frac{1}{2}}}((\nu + k - 1)^{1+\frac{1}{2}} + (n - \nu)^{1+\frac{1}{2}}) + \frac{k}{n-1} + (n - 1) & \text{if } g = 2t + 2, 
\end{cases}
\]

then \( G \) is \( k \)-connected.

**Proof.** By Lemma 2.2, we have

\[
\frac{2m}{n-1} + n - 2 \geq q(G) \geq \begin{cases} 
\frac{1}{2^{1+\frac{1}{2}}}((\nu + k - 1)^{1+\frac{1}{2}} + (n - \nu)^{1+\frac{1}{2}}) + \frac{k}{n-1} + (n - 1) & \text{if } g = 2t + 1, \\
\frac{1}{2^{1+\frac{1}{2}}}((\nu + k - 1)^{1+\frac{1}{2}} + (n - \nu)^{1+\frac{1}{2}}) + \frac{k}{n-1} + (n - 1) & \text{if } g = 2t + 2.
\end{cases}
\]

Then

\[
m \geq \begin{cases} 
\frac{1}{2^{1+\frac{1}{2}}}((\nu + k - 1)^{1+\frac{1}{2}} + (n - \nu)^{1+\frac{1}{2}} + \frac{1}{2}(n + k - 1) & \text{if } g = 2t + 1, \\
\frac{1}{2^{1+\frac{1}{2}}}((\nu + k - 1)^{1+\frac{1}{2}} + (n - \nu)^{1+\frac{1}{2}} + \frac{1}{2}(n + k - 1) & \text{if } g = 2t + 2.
\end{cases}
\]

5
By Lemma 3.1, G is k-connected.

**Theorem 3.3** Let G be a connected graph of order n, minimum degree \( \delta \geq k \geq 2 \) and girth \( g \geq 3 \), and let \( \nu = \nu(\delta, g, k - 1) \). If

\[
q(G) \leq \begin{cases} 
2(n - 1) - \frac{2}{n}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{2}{n}(n - \nu)^{1+\frac{1}{n}} - \frac{2}{n}(n + k - 1) & \text{if } g = 2t + 1, \\
2(n - 1) - \frac{2}{n}2^{1+\frac{1}{n}}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{2}{n}2^{1+\frac{1}{n}}(n - \nu)^{1+\frac{1}{n}} - \frac{2}{n}(n + k - 1) & \text{if } g = 2t + 2,
\end{cases}
\]

then G is k-connected.

**Proof.** By contradiction, we assume that \( \kappa(G) \leq k - 1 \). We argue according to the different parities of the girth g.

**Case 1.** \( g = 2t + 1 \) is odd.

As \( \kappa(G) \leq k - 1 \), by Lemma 3.1, we have \( m(G) < \frac{1}{2}(\nu + k - 1)^{1+\frac{1}{n}} + \frac{1}{2}(n - \nu)^{1+\frac{1}{n}} + \frac{1}{2}(n + k - 1) \).

It follows from \( m(G) + m(G) = \frac{n(n - 1)}{2} \) that

\[
m(G) = \frac{n(n - 1)}{2} - m(G) > \frac{n(n - 1)}{2} - \frac{1}{2}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{1}{2}(n - \nu)^{1+\frac{1}{n}} - \frac{1}{2}(n + k - 1).
\]

By Lemma 2.1, a contradiction to (2) is obtained.

\[
q(G) \geq \frac{4m(G)}{n} > \frac{4}{n}\left[ \frac{n(n - 1)}{2} - \frac{1}{2}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{1}{2}(n - \nu)^{1+\frac{1}{n}} - \frac{1}{2}(n + k - 1) \right] = 2(n - 1) - \frac{2}{n}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{2}{n}(n - \nu)^{1+\frac{1}{n}} - \frac{2}{n}(n + k - 1).
\]

**Case 2.** \( g = 2t + 2 \) is even.

As \( \kappa(G) \leq k - 1 \), by Lemma 3.1, we have \( m(G) < \frac{1}{2^{1+\frac{1}{n}}}(\nu + k - 1)^{1+\frac{1}{n}} + \frac{1}{2^{1+\frac{1}{n}}}(n - \nu)^{1+\frac{1}{n}} + \frac{1}{2}(n + k - 1) \). Since \( m(G) + m(G) = \frac{n(n - 1)}{2} \), we have

\[
m(G) = \frac{n(n - 1)}{2} - m(G) > \frac{n(n - 1)}{2} - \frac{1}{2^{1+\frac{1}{n}}}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{1}{2^{1+\frac{1}{n}}}(n - \nu)^{1+\frac{1}{n}} - \frac{1}{2}(n + k - 1).
\]

By Lemma 2.1, we obtain a contradiction to (2) again.

\[
q(G) \geq \frac{4m(G)}{n} > \frac{4}{n}\left[ \frac{n(n - 1)}{2} - \frac{1}{2^{1+\frac{1}{n}}}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{1}{2^{1+\frac{1}{n}}}(n - \nu)^{1+\frac{1}{n}} - \frac{1}{2}(n + k - 1) \right] = 2(n - 1) - \frac{2}{n \cdot 2^{1+\frac{1}{n}}}(\nu + k - 1)^{1+\frac{1}{n}} - \frac{2}{n \cdot 2^{1+\frac{1}{n}}}(n - \nu)^{1+\frac{1}{n}} - \frac{2}{n}(n + k - 1).
\]

These contradictions establish Theorem 3.3. □
Remark 3.4 In fact, by taking \( k = \delta \) in Lemma 3.1 and \( \kappa = \delta \) in the proof of Lemma 3.1, we can prove sufficient conditions on size \( m \) for a connected graph with fixed girth to be maximally connected and super-connected, respectively. Using sufficient conditions on size \( m \), we can also obtain sufficient conditions on \( q(G) \) and \( q(\overline{G}) \) to ensure a connected graph with fixed girth to be maximally connected and super-connected, respectively. In view of complex mathematical expressions, we omit these results here. However, for two special cases: connected graphs \((g \geq 3)\) and connected triangle-free graphs \((g \geq 4)\), we will provide improved and specific theorems in subsequent sections.

4 Vertex-connectivity and \( Q \)-index of connected graphs \((g \geq 3)\)

Throughout this section, we assume that \( k \) and \( \delta \) are positive integers. The goal of this section is to investigate the relationship between the connectivity and the \( Q \)-index of a graph.

4.1 \( k \)-connected graphs \((g \geq 3)\)

We will present a lower bound on \( q(G) \) for a connected graph to be \( k \)-connected. Define \( q_0 = q(K_{k-1} \cup (K_\delta+2 \cup K_{n-\delta-1})) \). Direct computation yields that \( q_0 \) is the largest root of the equation

\[
\lambda^3 - (3n + k - 7) \lambda^2 + (2n^2 - 15n + 3nk - 4k + 16 + 4(\delta - k + 2)(n - \delta - 1)) \lambda - 4(\delta - k + 2) (n - \delta - 1)(n-2) - (n-k+1)(2n-k+1)(k-3) - (3n-2k+2)(k-3)^2 -(k-3)^3 = 0.
\]

Theorem 4.1 Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta \geq 2 \). Suppose that \( q(G) \geq q_0 \). Then \( G \) is \( k \)-connected if and only if \( G \not\cong K_{k-1} \cup (K_\delta+2 \cup K_{n-\delta-1}) \).

Proof. By definition, \( K_{k-1} \cup (K_\delta+2 \cup K_{n-\delta-1}) \) has a \((k-1)\) vertex-cut and so \( \kappa(K_{k-1} \cup (K_\delta+2 \cup K_{n-\delta-1})) = k - 1 \). Therefore, it suffices to prove the sufficiency.

By contradiction, we assume that \( G \not\cong K_{k-1} \cup (K_\delta+2 \cup K_{n-\delta-1}) \) and \( k \leq k - 1 \). Let \( C \) be a minimum vertex-cut of \( G \), then \( |C| = \kappa \leq k - 1 < \delta \). Let \( V_0, V_1, \ldots, V_{t-1} \) \((t \geq 2)\) be the vertex sets of connected components of \( G - C \) with \( |V_0| \leq |V_1| \leq \cdots \leq |V_{t-1}| \).

By Lemma 2.3 with \( g \geq 3 \), for each \( i \) with \( 0 \leq i \leq t - 1 \), we have

\[
|V_i| \geq \nu(\delta, g, \kappa) \geq \nu(\delta, 3, \kappa) = \delta - \kappa + 1.
\]

Let \( U = \bigcup_{i=0}^{t-1} V_i \). Then \( \delta - \kappa + 1 \leq |V_0| \leq |U| \leq n - \delta - 1 \) and \( |V_0| + |U| = n - \kappa \). As \( E(V_0, U) = \emptyset \), \( G \) can be viewed as a subgraph of \( K_\kappa \cup (K_{|V_0|} \cup K_{|U|}) \), and so

\[
q(G) \leq q(K_\kappa \cup (K_{|V_0|} \cup K_{|U|})).
\]

Let \( G(\kappa, a, b) = K_\kappa \cup (K_a \cup K_b) \), where \( \delta - \kappa + 1 \leq a \leq b \leq n - \delta - 1 \) and \( \kappa + a + b = n \). Let \( X = (x_1, x_2, \ldots, x_n)^T \) be the Perron vector of \( G \) corresponding to \( q(G(\kappa, a, b)) \). Without loss of generality, let \( x := x_i, i \in K_a \); \( y := x_j, j \in K_b \); \( z := x_l, l \in K_\kappa \). As \( \lambda X = (D + A)X \), we have

\[
\begin{align*}
\lambda x &= (a - 1 + \kappa)x + (a - 1)x + \kappa y, \\
\lambda y &= ax + (n - 1)y + (\kappa - 1)y + bz, \\
\lambda z &= \kappa y + (b - 1 + \kappa)z + (b - 1)z.
\end{align*}
\]
It follows that \( q(K_n \vee (K_a \cup K_b)) \) is the largest root of the equation
\[
\lambda^3 - (3n + \kappa - 6)\lambda^2 + (2n^2 - 12n + 3n\kappa - 4\kappa + 12 + 4ab)\lambda - 4ab(n - 2) - (n - \kappa)(2n - \kappa)(\kappa - 2) - (3n - 2\kappa)(\kappa - 2)^2 - (\kappa - 2)^3 = 0.
\]

By algebraic manipulation, for \( \lambda \geq n - 2 \), we have
\[
f(\lambda; \kappa, a, b) = f(\lambda; \kappa, a, b) = 4(\lambda - n + 2)(ab - (\delta - \kappa + 1)(n - \delta - 1)] \geq 0.
\]

By Lemma 2.5, \( q(G(\kappa, a, b)) > n-2 \). Substituting \( \lambda \) with \( q(G(\kappa, a, b)) \) in (3), we have \( f(q(G(\kappa, a, b)); \kappa, \delta - \kappa + 1, n - \delta - 1) \leq 0 \), and so
\[
q(G(\kappa, a, b)) \leq q(G(\kappa, \delta - \kappa + 1, n - \delta - 1)).
\]

Therefore,
\[
q(G) \leq q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})) \leq q(K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1})). \tag{4}
\]

Since \( \kappa \leq k - 1 \), we conclude that \( K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1}) \) is a subgraph of \( K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1}) \), and
\[
q(G) \leq q(K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1})) \leq q(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})). \tag{5}
\]

By the hypothesis of Theorem 4.1, \( q(G) \geq q(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})) \), and so we must have
\[
q(G) = q(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})).
\]

It follows that all the inequalities in (4) and (5) must be equalities. Hence we must have \( |V_0| = \delta - k + 2 \), \( |U| = n - \delta - 1 \) and \( \kappa = k - 1 \). Therefore \( G \cong K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1}) \), contrary to our assumption. This completes the proof of the theorem.

Hong et al. obtained a sufficient condition on size \( m \) for \( k \)-connected graphs, in which the lower bound of the size is the special case when \( g \geq 3 \) of Lemma 3.1.

**Theorem 4.2** (Hong, Xia, Chen and Volkmann [7]) Let \( k \geq 2 \) be an integer. Let \( G \) be a connected graph of order \( n \), size \( m \), and minimum degree \( \delta \geq k \). If
\[
m \geq \frac{1}{2}n(n - 1) - (\delta - k + 2)(n - \delta - 1),
\]
then \( G \) is \( k \)-connected unless \( G \cong K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1}) \).

Theorem 4.2 can be applied to show an explicit lower bound of \( q(G) \) to predict \( k \)-connected graphs.

**Corollary 4.3** Let \( G \) be a connected graph with \( n = |V(G)| \), \( m = |E(G)| \) and \( \delta = \delta(G) \geq k \geq 2 \).

Suppose that
\[
q(G) \geq 2(n - \delta + k - 3) + \frac{2\delta(\delta - k + 2)}{n - 1}.
\]

Then \( G \) is \( k \)-connected.
Proof. Suppose that $G$ is not $k$-connected. By assumption and Lemma 2.2, we have
\[ 2(n - \delta + k - 3) + \frac{2\delta(\delta - k + 2)}{n - 1} \leq q(G) \leq \frac{2m}{n - 1} + n - 2. \] (6)
Then $m \geq \frac{1}{2}n(n - 1) - (\delta - k + 2)(n - \delta - 1)$. By Theorem 4.2, $G \cong K_{k-1} \lor (K_{\delta-k+2} \lor K_{n-\delta-1})$. Since
\[ |E(G)| = \frac{1}{2}n(n - 1) - (\delta - k + 2)(n - \delta - 1), \]
the inequalities in (6) must be equalities. By Lemma 2.2, $G \cong K_n$ or $K_{1,n-1}$. As $K_{k-1} \lor (K_{\delta-k+2} \lor K_{n-\delta-1})$ is isomorphic to neither $K_n$ nor $K_{1,n-1}$, a contradiction is obtained. \hfill \square

Finally, we present a sufficient condition for a $k$-connected graph in terms of $q(G)$ to conclude this section.

Theorem 4.4 Let $a, \delta, k, n$ be positive integers satisfying $\delta - k + 2 \leq a \leq n - \delta - 1$, and $G$ be a connected graph of order $n$ and minimum degree $\delta \geq k \geq 2$. Suppose that
\[ q(G) \leq n - k + 1. \]
Then $G$ is $k$-connected if and only if $G \not\cong K_{k-1} \lor (K_n \lor K_{n-k+1-a})$.

Proof. By definition, $K_{k-1} \lor (K_n \lor K_{n-k+1-a})$ has a vertex-cut of cardinality $k - 1$. Thus we only need to prove the sufficiency of the theorem. Suppose that $G \not\cong K_{k-1} \lor (K_n \lor K_{n-k+1-a})$ and $\kappa \leq k - 1$. Let $C$ be a minimum vertex-cut of $G$. Then $|C| = \kappa \leq k - 1 < \delta$, and for some integer $t \geq 2$, $G - C$ has $t$ components. Let $V_0, V_1, \ldots, V_{t-1}$ be the vertex sets of connected components of $G - C$ satisfying $|V_0| \leq |V_1| \leq \cdots \leq |V_{t-1}|$. By Lemma 2.3 with $g \geq 3$, we have, for any $i$ with $0 \leq i \leq t - 1$,
\[ |V_i| \geq \nu(\delta, g, \kappa) \geq \nu(\delta, 3, \kappa) = \delta - \kappa + 1. \]
Let $U = \bigcup_{i=0}^{t-1} V_i$. Then $\delta - \kappa + 1 \leq |V_0| \leq |U| \leq n - \delta - 1$ and $|V_0| + |U| = n - \kappa$. Since $E(V_0, U) = \emptyset$, we conclude that $K_{|V_0|, |U|}$ must be a subgraph of $\overline{G}$, and so
\[ q(\overline{G}) \geq q(K_{|V_0|, |U|}) = n - \kappa \geq n - k + 1. \]
It follows from the hypothesis of Theorem 4.4 that $q(\overline{G}) = n - k + 1$. Hence we have $\kappa = k - 1$ and $\overline{G} = K_{|V_0|, |U|}$. Let $|V_0| = a$. Then $G \cong K_{k-1} \lor (K_n \lor K_{n-k+1-a})$, where $\delta - k + 2 \leq a \leq n - \delta - 1$, contrary to our assumption. \hfill \square

4.2 Maximally connected graphs ($g \geq 3$)

We consider the problem how the $Q$-index of a graph warrants the property that $G$ is maximally connected, that is, the condition $\kappa(G) = \delta(G)$ holds. These can be obtained by taking $k = \delta$ in Theorem 4.1, Corollary 4.3 and Theorem 4.4, and so we have the following corollaries of the main results in Section 4.1. Let $q_1 = q(K_{\delta-1} \lor (K_2 \lor K_{n-\delta-1}))$, the largest root of the equation
\[ \lambda^3 - (3n + \delta - 7)\lambda^2 + (2n^2 - 7n + 3n\delta - 125 + 8)\lambda - 8(n - \delta - 1)(n - 2) - (n - \delta + 1)(\delta - 3)(2n - \delta + 1) - (3n - 2\delta + 2)(\delta - 3)^2 - (\delta - 3)^3 = 0. \]
Corollary 4.5 Let \( G \) be a connected graph with order \( n \) and minimum degree \( \delta \geq 2 \), and let \( q(G) \geq q_1 \). Then \( G \) is maximally connected if and only if \( G \not\cong K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1}) \).

Corollary 4.6 Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta \geq 2 \). If

\[
q(G) \geq 2(n - 3) + \frac{4\delta}{n - 1},
\]

then \( G \) is maximally connected.

Corollary 4.7 Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta \geq 2 \). If

\[
q(G) \leq n - \delta + 1,
\]

then \( G \) is maximally connected if and only if \( G \not\cong K_{\delta-1} \vee (K_a \cup K_{n-\delta-1-a}) \), where \( 2 \leq a \leq n-\delta-1 \).

4.3 Super-connected graphs \((g \geq 3)\)

Let \( q_2 = q(K_4 \vee (K_2 \cup K_{n-\delta-2})) \). By definition, \( q_2 \) is the largest root of the equation

\[
\lambda^3 - (3n + \delta - 6)\lambda^2 + (2n^2 - 4n + 3n\delta - 12\delta - 4)\lambda - 8(n - \delta - 2)(n - 2) - (n - \delta)(2n - \delta)(\delta - 2) - (3n - 2\delta)(\delta - 2)^2 - (\delta - 2)^3 = 0.
\]

Theorem 4.8 Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta \). If \( q(G) \geq q_2 \), then \( G \) is super-\( \kappa \).

Proof. By contradiction, we assume that \( G \) is not super-\( \kappa \). Then \( G \) contains a minimum vertex-cut with \( |C| = \kappa \leq \delta \). Therefore, for some integer \( t \geq 2 \), \( G - C \) has \( t \) components, whose vertex sets are respectively denoted by \( V_0, V_1, \ldots, V_{t-1} \), such that \( 2 \leq |V_0| \leq |V_1| \leq \cdots \leq |V_{t-1}| \). Let \( U = \bigcup_{i=1}^{t-1} V_i \). Then we have \( 2 \leq |V_0| \leq |U| \leq n - \kappa - 2 \) and \( |V_0| + |U| = n - \kappa \). As \( E(V_0, U) = \emptyset \), it follows that \( G \) is a subgraph of \( K_\kappa \vee (K_{|V_0|} \cup K_{|U|}) \), and so

\[
q(G) \leq q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})).
\]

With an argument similar to that of Theorem 4.1, we conclude that \( q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})) \) is the largest root of the equation

\[
\lambda^3 - (3n + \kappa - 6)\lambda^2 + (2n^2 - 12n + 3n\kappa - 4\kappa + 12 + 4|V_0| \cdot |U|)\lambda - 4|V_0| \cdot |U|(n - 2) - (n - \kappa)(2n - \kappa)(\kappa - 2) - (3n - 2\kappa)(\kappa - 2)^2 - (\kappa - 2)^3 = 0.
\]

Direct computation yields that, if \( \lambda \geq n - 2 \), then

\[
f(\lambda; \kappa, |V_0|, |U|) - f(\lambda; \kappa, 2, n - \kappa - 2) = 4(\lambda - n + 2)(|V_0| \cdot |U| - 2(n - \kappa - 2)) \geq 0. \tag{7}
\]

By Lemma 2.5, \( q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})) \leq n - 2 \). Substituting \( \lambda \) with \( q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})) \) in (7), we have \( f(q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})); \kappa, 2, n - \kappa - 2) \leq 0 \). It follows that

\[
q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})) \leq q(K_\kappa \vee (K_2 \cup K_{n-\kappa-2})),
\]

which implies that

\[
q(G) \leq q(K_\kappa \vee (K_{|V_0|} \cup K_{|U|})) \leq q(K_\kappa \vee (K_2 \cup K_{n-\kappa-2})). \tag{8}
\]

10
By the assumption of Theorem 4.11, we have \( q(G) = q(K_\delta \vee (K_2 \cup K_{n-\delta-2})) \). Thus the inequalities in (8) and (9) must be equalities. It follows that \( \kappa = \delta, |V_0| = 2 \) and \(|U| = n - \delta - 2 \), and so \( G \cong K_\delta \vee (K_2 \cup K_{n-\delta-2}) \). However, \( \delta(G) = \delta + 1 > \delta \), contrary to the choice of \( G \). This justifies the theorem.

Hong et al. obtained the following sufficient condition on size \( m \) for super-connected graphs. This again, can be applied to obtain a relationship between the \( Q \)-index and super-\( \kappa \) property of a connected graph \( G \).

**Theorem 4.9** (Hong, Xia, Chen and Volkmann [7]) Let \( G \) be a connected graph of order \( n \), size \( m \) and minimum degree \( \delta \). If

\[
m \geq \frac{1}{2}(n - 2)(n - 3) + 2\delta,
\]

then \( G \) is super-\( \kappa \) unless \( G \cong (K_\delta \vee (K_2 \cup K_{n-\delta-2})) - e \), where \( e = xy \) is an edge of \( K_\delta \vee (K_2 \cup K_{n-\delta-2}) \), and \( d(x) = \delta + 1, d(y) = n - 1 \).

**Corollary 4.10** Let \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta \). If

\[
q(G) \geq 2(n - 3) + \frac{4\delta + 2}{n - 1},
\]

then \( G \) is super-\( \kappa \).

**Proof.** Suppose that \( G \) is not super-\( \kappa \). By assumption and Lemma 2.2,

\[
2(n - 3) + \frac{4\delta + 2}{n - 1} \leq q(G) \leq \frac{2m}{n - 1} + n - 2.
\]

Then we have \( m \geq \frac{1}{2}(n - 2)(n - 3) + 2\delta \). By Theorem 4.9, \( G \cong (K_\delta \vee (K_2 \cup K_{n-\delta-2})) - e \), where \( e = xy \) is an edge of \( K_\delta \vee (K_2 \cup K_{n-\delta-2}) \) with \( d(x) = \delta + 1, d(y) = n - 1 \). Since

\[
|E(G)| = \frac{n(n - 1)}{2} - 2(n - \delta - 2) - 1 = \frac{(n - 2)(n - 3)}{2} + 2\delta,
\]

the inequalities in (10) should be equalities. By Lemma 2.2, \( G \cong K_n \) or \( K_{1,n-1} \). As \( (K_\delta \vee (K_2 \cup K_{n-\delta-2})) - e \) is isomorphic to neither \( K_n \) nor \( K_{1,n-1} \), a contradiction is obtained. Thus \( G \) must be super-\( \kappa \).

Finally, we present a sufficient condition for a super-connected graph in terms of \( q(\overline{G}) \) to conclude the section.

**Theorem 4.11** Let \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta \). If

\[
q(\overline{G}) \leq n - \delta,
\]

then \( G \) is super-\( \kappa \).
Theorem 5.1

Let $G$ be a connected triangle-free graph of order $n$ and minimum degree $\delta \geq k \geq 2$. If

$$m \geq \delta^2 + \frac{1}{4} (n - 2\delta + k - 1)^2,$$

then $G$ is $k$-connected unless $V(G) = X \cup C \cup Y$, and $C$ is a minimum vertex-cut of $G$ with $G[C] \cong K_{k-1}$, $G[X \cup C] \cong K_{\delta,\delta}$ and $G[Y \cup C] \cong K_{\lceil \frac{n-2\delta + k-1}{2} \rceil, \lceil \frac{n-2\delta + k-1}{2} \rceil}$.

Corollary 5.2 Let $G$ be a connected triangle-free graph of order $n$ and minimum degree $\delta \geq k \geq 2$. If

$$q(G) \geq n + k - 2\delta - 2 + \frac{2\delta^2}{n-1} + \left\lceil \frac{1}{2} (n-1 + \frac{(k-2\delta)^2}{n-1}) \right\rceil,$$

then $G$ is $k$-connected.

Proof. Suppose that $G$ is not super-$\kappa$. Then $G$ has a minimum vertex-cut $C$ with $|C| = \kappa \leq \delta$ such that for some integer $t \geq 2$, $G - C$ has $t$ components. Let $V_0, V_1, \ldots, V_{t-1}$ be the vertex sets of connected components of $G - C$ with $|V_0| \leq |V_1| \leq \cdots \leq |V_{t-1}|$. Let $U = \bigcup_{i=1}^{t-1} V_i$. Then $2 \leq |V_0| \leq |U| \leq n - \kappa - 2$ and $|V_0| + |U| = n - \kappa$. As $E(V_0, U) = \emptyset$, we conclude that $K_{|V_0|, |U|}$ is a subgraph of $G$, and so

$$q(G) \geq q(K_{|V_0|, |U|}) = n - \kappa \geq n - \delta.$$

By assumption, $q(G) \leq n - \delta$, and so we have $q(G) = n - \delta$ and $G = K_{|V_0|, |U|}$. Thus we must have $\kappa = \delta$ and $G \cong K_{\delta} \cup (K_{|V_0|} \cup K_{|U|})$. Since $\delta (K_{\delta} \cup (K_{|V_0|} \cup K_{|U|})) \geq \delta + 1 > \delta$, contrary to the assumption on the choice of $G$. \hfill $\Box$

5 Vertex-connectivity and Q-index of triangle-free graphs ($g \geq 4$)

5.1 $k$-connected triangle-free graphs ($g \geq 4$)

Hong et al. obtained a sufficient condition on size $m$ to warrant $k$-connected graphs, in which the lower bound on the graph size is a special case of Lemma 3.1 when $g \geq 4$. This can, once again, be applied to obtain results relating the $Q$-index and the connectivity in a connected triangle-free graph.

Theorem 5.1 (Hong, Xia, Chen and Volkmann [7]) Let $G$ be a connected triangle-free graph of order $n$, size $m$ and minimum degree $\delta \geq k \geq 2$. If

$$m \geq \delta^2 + \frac{1}{4} (n - 2\delta + k - 1)^2,$$

then $G$ is $k$-connected unless $V(G) = X \cup C \cup Y$, and $C$ is a minimum vertex-cut of $G$ with $G[C] \cong K_{k-1}$, $G[X \cup C] \cong K_{\delta,\delta}$ and $G[Y \cup C] \cong K_{\lceil \frac{n-2\delta + k-1}{2} \rceil, \lceil \frac{n-2\delta + k-1}{2} \rceil}$.

Corollary 5.2 Let $G$ be a connected triangle-free graph of order $n$ and minimum degree $\delta \geq k \geq 2$. If

$$q(G) \geq n + k - 2\delta - 2 + \frac{2\delta^2}{n-1} + \left\lceil \frac{1}{2} (n-1 + \frac{(k-2\delta)^2}{n-1}) \right\rceil,$$

then $G$ is $k$-connected.

Proof. Suppose that $G$ is not $k$-connected. By assumption and Lemma 2.2, we have

$$n + k - 2\delta - 2 + \frac{2\delta^2}{n-1} + \left\lceil \frac{1}{2} (n-1 + \frac{(k-2\delta)^2}{n-1}) \right\rceil \leq q(G) \leq \frac{2m}{n-1} + n - 2. \quad (11)$$

Then $m \geq \delta^2 + \frac{1}{4} (n - 2\delta + k - 1)^2$. By Theorem 5.1, $V(G) = X \cup C \cup Y$, and $C$ is a minimum vertex-cut of $G$ with $G[C] \cong K_{k-1}$, $G[X \cup C] \cong K_{\delta,\delta}$ and $G[Y \cup C] \cong K_{\lceil \frac{n-2\delta + k-1}{2} \rceil, \lceil \frac{n-2\delta + k-1}{2} \rceil}$. Since

$$|E(G)| = \delta^2 + \frac{1}{4} (n - 2\delta + k - 1)^2,$$
the inequalities in (11) must be equalities. By Lemma 2.2, \( G \cong K_n \) or \( K_{1,n-1} \). However, \( G \) is isomorphic to neither \( K_n \) nor \( K_{1,n-1} \), a contradiction. \( \square \)

Finally, we present a sufficient condition for a \( k \)-connected triangle-free graph in terms of \( q(G) \) to conclude the section.

**Theorem 5.3** Let \( G \) be a connected triangle-free graph of order \( n \) and minimum degree \( \delta \geq k \geq 2 \), and \( \overline{G} \) be connected. If
\[
q(G) \leq 2(n-1) - \frac{4\delta^2}{n} - \left\lfloor \frac{(n-2\delta+k-1)^2}{n} \right\rfloor,
\]
then \( G \) is \( k \)-connected.

**Proof.** By contradiction, assume that \( G \) has a minimum vertex-cut \( C \) with \( |C| = \kappa(G) \leq k-1 < \delta \). Let \( V_0, V_1, \ldots, V_{t-1} \), for some integer \( t \geq 2 \), be the vertex sets of connected components of \( G - C \) satisfying \( |V_0| \leq |V_i| \leq \cdots \leq |V_{t-1}| \). By Lemma 2.3 with \( g \geq 4 \), we have, for any \( i \) with \( 0 \leq i \leq t-1 \),
\[
|V_i| \geq \nu(\delta, g, \kappa) \geq \nu(\delta, 4, \kappa) = 2\delta - \kappa.
\]

Let \( U = \bigcup_{i=1}^{t-1} V_i \). Then \( 2\delta - \kappa \leq |V_0| \leq |U| \leq n - 2\delta \) and \( |V_0| + |U| = n - \kappa \).

By Theorem 2.6, with similar analysis of Theorem 5.2 in [7], we have
\[
m(G) = |E(G[V_0 \cup C])| + |E(G[U \cup C])| - |E(G[C])| = \frac{1}{4} |(V_0 + |C|)^2| + \frac{1}{4} (|U| + |C|)^2 - |E(G[C])| \leq \frac{n^2 + \kappa^2}{4} - \frac{|V_0| \cdot |U|}{2} = \frac{n^2 + \kappa^2}{4} - \frac{(2\delta - \kappa)(n - 2\delta)}{2} \leq \frac{\delta^2 + \frac{(n-2\delta+k+1)^2}{4}}{4} \leq \frac{(n-2\delta+k+1)^2}{4}.
\]

Since \( m(G) + m(\overline{G}) = \frac{n(n-1)}{2} \), then
\[
m(\overline{G}) = \frac{n(n-1)}{2} - m(G) \geq \frac{n(n-1)}{2} - \delta^2 - \left\lfloor \frac{(n-2\delta+k-1)^2}{4} \right\rfloor.
\]

By Lemma 2.1,
\[
q(G) \geq \frac{4m(\overline{G})}{n} \geq \frac{4}{n} \left[ \frac{n(n-1)}{2} - \delta^2 - \left\lfloor \frac{(n-2\delta+k-1)^2}{4} \right\rfloor \right] = 2(n-1) - \frac{4\delta^2}{n} - \left\lfloor \frac{(n-2\delta+k-1)^2}{n} \right\rfloor.
\]

Combine (12) and (14) to get \( q(G) = 2(n-1) - \frac{4\delta^2}{n} - \left\lfloor \frac{(n-2\delta+k-1)^2}{n} \right\rfloor \). Then all the inequalities in (13) and (14) must be equalities. It follows that \( |C| = \kappa = k-1 \), \( |V_0| = 2\delta - k + 1 \), \( |U| = n - 2\delta \), \( |E(G[C])| = 0 \), \( |E(G[V_0 \cup C])| = \delta^2 \), \( |E(G[U \cup C])| = \left\lfloor \frac{(n-2\delta+k-1)^2}{4} \right\rfloor \), \( G[V_0 \cup C] = K_{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}} \cup K_{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}} \cup K_{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}} \) and \( \overline{G} \) is regular. Therefore, \( G[C] = K_{k-1} \), \( G[V_0 \cup C] = K_{k, \delta} \) and \( G[U \cup C] = K_{k-2, 2\delta-1} \). However, \( G \) is not regular, and so \( G \) cannot be regular, a contradiction. \( \square \)
5.2 Maximally connected triangle-free graphs \((g \geq 4)\)

Naturally, by setting \(k = \delta\) in Corollary 5.2 and Theorem 5.3, we can obtain the following results on maximally connected triangle-free graphs.

**Corollary 5.4** Let \(G\) be a connected triangle-free graph of order \(n\) and minimum degree \(\delta \geq 2\). If
\[
q(G) \geq n - \delta - 2 + \frac{2\delta^2}{n-1} + \left\lfloor \frac{n-1}{2} + \frac{\delta^2}{n-1} \right\rfloor,
\]
then \(G\) is maximally connected.

**Corollary 5.5** Let \(G\) be a connected triangle-free graph of order \(n\) and minimum degree \(\delta \geq 2\), and \(\overline{G}\) be connected. If
\[
q(\overline{G}) \leq 2(n-1) - 4\delta - \left\lfloor \frac{(n-\delta)^2}{n} \right\rfloor,
\]
then \(G\) is maximally connected.

5.3 Super-connected triangle-free graphs \((g \geq 4)\)

We start quoting a theorem by Hong et al. [7] again, to be applied in one of our results.

**Theorem 5.6** (Hong, Xia, Chen and Volkmann [7]) Let \(G\) be a connected triangle-free graph of order \(n\), size \(m\) and minimum degree \(\delta \geq 2\). If
\[
m \geq \delta^2 + \left\lfloor \frac{1}{4}(n - \delta)^2 \right\rfloor,
\]
then \(G\) is super-\(\kappa\).

**Corollary 5.7** Let \(G\) be a connected triangle-free graph of order \(n\) and minimum degree \(\delta \geq 2\). If
\[
q(G) \geq n - \delta - 1 + \frac{2\delta^2}{n-1} + \left\lfloor \frac{1}{2}(n - 1 + \frac{(\delta - 1)^2}{n-1}) \right\rfloor,
\]
then \(G\) is super-\(\kappa\).

**Proof.** By assumption and Lemma 2.2, we have
\[
n - \delta - 1 + \frac{2\delta^2}{n-1} + \left\lfloor \frac{1}{2}(n - 1 + \frac{(\delta - 1)^2}{n-1}) \right\rfloor \leq q(G) \leq \frac{2m}{n-1} + n - 2,
\]
and so \(m \geq \delta^2 + \left\lfloor \frac{1}{4}(n - \delta)^2 \right\rfloor\). By Theorem 5.6, \(G\) is super-\(\kappa\). 

**Theorem 5.8** Let \(G\) be a connected triangle-free graph of order \(n\) and minimum degree \(\delta \geq 2\), and \(\overline{G}\) be connected. If
\[
q(\overline{G}) \leq 2(n-1) - 4\delta - \left\lfloor \frac{(n-\delta)^2}{n} \right\rfloor,
\]
then \(G\) is super-\(\kappa\).
Proof. Suppose that $G$ is not super-$\kappa$. Note that
\[
q(G) \leq 2(n - 1) - \frac{4\delta^2}{n} - \left\lfloor \frac{(n - \delta)^2}{n} \right\rfloor
\]
\[
\leq 2(n - 1) - \frac{4\delta^2}{n} - \left\lfloor \frac{(n - \delta - 1)^2}{n} \right\rfloor,
\]
by Corollary 5.5, we have $\kappa = \delta$.

Let $C$ be the minimum vertex-cut with $|C| = \delta$. Let $V_0, V_1, \ldots, V_{t-1}$ ($t \geq 2$) be the vertex sets of connected components of $G - C$ with $2 \leq |V_0| \leq |V_1| \leq \cdots \leq |V_{t-1}|$. Let $U = \bigcup_{i=1}^{t-1} V_i$. Then $|V_0| + |U| = n - \delta$. By Lemma 2.3 with $g \geq 4$, then $|V_0| \geq \nu(g, \delta, \kappa) \geq \nu(\delta, 4, \kappa) = 2\delta - \kappa$. So we have
\[
2 \leq \delta = 2\delta - \kappa \leq |V_0| \leq |U| \leq n - 2\delta.
\]
By Theorem 2.6, we have
\[
m(G) = |E(G[V_0 \cup C])| + |E(G[U \cup C])| - |E(G[C])| \\
\leq \frac{|(|V_0| + |C|)^2}}{4} + \frac{|(|U| + |C|)^2}}{4} - |E(G[C])| \\
\leq \frac{|(|V_0| + |C|)^2}}{4} + \frac{|(|U| + |C|)^2}}{4} \\
= \frac{|(|V_0| + |U| + |C|)^2}}{4} - |V_0| \cdot |U| \\
= \frac{n^2 + \kappa^2}{4} - \frac{4}{2} \cdot \frac{|V_0| \cdot |U|}{2} \leq \frac{n^2 + \kappa^2}{4} - \frac{\delta \cdot (n - 2\delta)}{2} \\
= \delta^2 + \frac{(n - \delta)^2}{4}.
\]
Since $m(G) + m(\overline{G}) = \frac{n(n-1)}{2}$, then
\[
m(\overline{G}) = \frac{n(n-1)}{2} - m(G) \geq \frac{n(n-1)}{2} - \delta^2 - \left\lfloor \frac{(n - \delta)^2}{4} \right\rfloor. \tag{16}
\]
By Lemma 2.1,
\[
q(\overline{G}) \geq \frac{4m(\overline{G})}{n} \geq \frac{4}{n} \cdot \frac{n(n-1)}{2} - \delta^2 - \left\lfloor \frac{(n - \delta)^2}{4} \right\rfloor = 2(n - 1) - \frac{4\delta^2}{n} - \left\lfloor \frac{(n - \delta)^2}{n} \right\rfloor. \tag{17}
\]
Combine (15) and (17) to get $q(\overline{G}) = 2(n - 1) - \frac{4\delta^2}{n} - \left\lfloor \frac{(n - \delta)^2}{n} \right\rfloor$. Then all the inequalities in (16) and (17) must be equalities. It follows that $|C| = \delta$, $|V_0| = \delta$, $|U| = n - 2\delta$, $|E(G[C])| = 0$, $|E(G[V_0 \cup C])| = \delta^2$, $|E(G[U \cup C])| = \left\lfloor \frac{(n - \delta)^2}{4} \right\rfloor$, $G[V_0 \cup C] = K_{\left\lfloor \frac{\delta}{2} \cdot \frac{(n-\delta)^2}{4} \right\rfloor}$, $G[U \cup C] = K_{\left\lfloor \frac{\delta}{2} \cdot \frac{(n-\delta)^2}{4} \right\rfloor}$ and $\overline{G}$ is regular. Therefore, $G[C] = \overline{K}_\delta$, $G[V_0 \cup C] = K_{\left\lfloor \frac{\delta}{2} \cdot \frac{(n-\delta)^2}{4} \right\rfloor}$ and $G[U \cup C] = K_{\left\lfloor \frac{\delta}{2} \cdot \frac{(n-\delta)^2}{4} \right\rfloor}$. However, $G$ is not regular. So $\overline{G}$ cannot be regular, a contradiction.

\[\Box\]

Acknowledgement. The research of Huicai Jia is supported by NSFC (No. 11701148) and Natural Science Foundation of Education Ministry of Henan Province (18B110005). The research of Hong-Jian Lai is supported by NSFC (Nos. 11771039 and 11771443). The research of Ruifang Liu is supported by Outstanding Young Talent Research Fund of Zhengzhou University (No. 1521315002), China Postdoctoral Science Foundation (No. 2017M612410) and Foundation for University Key Teacher of Henan Province (No. 2016GGJS-007).
References

[1] A. Abiad, B. Brimkov, X. Martínez-Rivera, S. O. J. Zhang, Spectral bounds for the connectivity of regular graphs with given order, Electronic J. Linear Algebra 34 (2018) 428-443.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, New York, 2008.

[3] D. Cvetković, P. Rowlinson, S. K. Simić, Eigenvalue bounds for the signless Laplacian, Publ. Inst. Math. (Beograd) 81 (2007) 11-27.

[4] Z. Füredi, M. Simonovits, The history of degenerate (bipartite) extremal graph problems, In Erdős centennial, volume 25 of Bolyai Soc. Math. Stud., pages 169-264. János Bolyai Math. Soc., Budapest, 2013.

[5] L.H. Feng, G.H. Yu, On three conjectures involving the signless Laplacian spectral radius of graphs, Publ. Inst. Math. (Beograd) 85 (2009) 35-38.

[6] P.L. Zhang, X.D. Zhang, L.H. Feng, W.J. Liu, The Q-index and connectedness of graphs, Submitted for publication, 2018.

[7] Z.M. Hong, Z.J. Xia, F.Y. Chen, L. Volkmann, Sufficient conditions for graphs to be k-connected, maximally connected and super-connected, arXiv:1708.05396v1 [math.CO] 17 Aug 2017.

[8] R. Li, Spectral conditions for a graph to be k-connected, Ann. Pure Appl. Math. 8 (2014) 11-14.

[9] R.F. Liu, H-J Lai, Y.Z. Tian, Y. Wu, Vertex-connectivity and eigenvalues of graphs with fixed girth, Applied Math. Computation 344-345 (2019) 141-149.

[10] S. O, The second largest eigenvalues and vertex-connectivity of regular multigraphs, arXiv:1603.03960v3 [math.CO] 4 Oct 2016.

[11] W. Mantel, Problem 28, Wiskundige Opgaven, 10 (1907) 60-61.

[12] P. Turán, On an extremal problem in graph theory (Hungarian), Mat. Fiz. Lapok 48 (1941) 436-452.

[13] H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math. 54 (1932) 150-168.