Our World as an Intersection of Walls and a String

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Abstract

A solution of Einstein equations is obtained for our four-dimensional world as an intersection of a wall and a string-like defect in seven-dimensional spacetime with a negative cosmological constant. A matter energy-momentum tensor localized on the wall and on the string is needed. A single massless graviton is found and is localized around the intersection. The leading correction to the gravitational Newton potential from massive spin 2 graviton is found to be almost identical to that of a wall in five dimensions, contrary to the case of a string in six dimensions. The generalization to the intersection of a string and n orthogonally intersecting walls is also obtained and a similar result is found for the gravitational potential.

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1 Introduction

In recent years, a lot of attention has been paid to topological objects such as domain walls [1], [2], junctions [3]–[5] in field theories, and branes in string theories [6]. An interesting idea has been advocated that our four-dimensional world may be realized on these topological objects embedded in higher dimensional spacetime [7]. In this “brane world” scenario, most of the particles in the standard model should be obtained as modes localized on the topological defects. These models are based on the fact that the Newton’s law for the gravitational force is tested only at short distances of the order of a hundred microns [8]. In view of the possibility of a fundamental theory in higher dimensions such as the ten-dimensional superstring, it is desirable to have a model with more extra dimensions and with supersymmetry. In fact, a model for supersymmetry breaking has been proposed based on the coexistence of branes [9].

Another fascinating possibility has also been proposed to consider walls in five-dimensional spacetime with a negative cosmological constant [10]. It has been recognized that the graviton in our four-dimensional world can be obtained as a zero mode localized on the wall embedded in the five-dimensional curved spacetime [11]. The model has been extended to intersections of walls in higher dimensional AdS spacetime [12]. Topological defects of codimension two [13], [14] and more [16] have also been studied. Following the Kaluza-Klein idea, the graviton field in the bulk is decomposed into spin 2 graviton, spin 1 graviphoton, and spin 0 graviscalar fields in four-dimensional spacetime. It has been shown that the spin 0 field becomes unphysical for an isolated wall, but becomes a physical scalar field called radion if there are two or more walls [17]. The graviphoton has also been studied in the case of walls in five dimensions [18]. However, these scalar and vector fields are not well studied in the case of topological defects in higher dimensional spacetime. The contributions from the spin 2 gravitons to the Newton potential for the gravitational force have been obtained, but the contributions from spin 1 and spin 0 fields are still an open question in these higher dimensional models. A number of works also explored the possible extensions of the model to higher extra dimensions [20]. It is important to study the coexistence of these topological defects in higher dimensional spacetime to build a model from a fundamental theory with higher spacetime dimensions and/or higher supersymmetry [1].

The purpose of this paper is to construct a model for our four-dimensional world as an intersection of a wall (codimension one) and a string-like defect (codimension two) and to obtain corrections to the gravitational Newton potential by working out the spin 2 modes on the background. We find a configuration of a four-dimensional intersection of a wall and a string-like defect in seven-dimensional spacetime. We obtain a four-dimensional graviton as a zero mode localized on the intersection. It turns out necessary to have a matter energy-momentum tensor
besides the cosmological constant on the topological defects. The correction to the Newton’s law is found to be similar to that of the wall in five dimensions contrary to the string-like defect in six dimensions. We also work out generalizations to intersections of a string-like defect and \( n \) orthogonally intersecting walls in \( (6+n) \) dimensional spacetime. A single massless graviton is found to be localized at the intersection and the leading correction to the Newton potential is found to be similar to the single wall case. We leave the study of the spin 1 and spin 0 fields for a future work.

In sect. 2, we will study the solution of the Einstein equation representing the intersection of walls and a string-like defect embedded in seven-dimensional spacetime. The modes on the background of intersection of walls and a string-like defect are obtained, and the correction for the gravitational Newton potential is worked out in sect. 3. In sect. 4 the extension of the solution is obtained for the intersection of a string-like defect and \( n \) orthogonally intersecting walls in higher dimensional spacetime.

2 Intersection of a Wall and a String-like Defect

In order to consider an intersection of parallel walls and a string-like defect, we will work on a seven-dimensional spacetime. The indices for the bulk spacetime coordinates are denoted by capital alphabets \( A, B, \cdots \) throughout this paper. Following Ref.[10], we will compactify one of the extra dimensions and make an orbifold \( S^1/Z_2 \) by identifying points under the reflection : \( y \to -y \). Two parallel five-branes (walls) with vanishing widths are placed at the fixed points of the orbifold: \( y = 0 \) and \( y = y_c \). The polar coordinates \( (r, \theta) \) for the two more extra dimensions perpendicular to the \( y \) direction take values \( 0 \leq r < \infty \), and \( 0 \leq \theta < 2\pi \). Perpendicular to the walls lies a four-brane (string-like defect) at \( r = 0 \). We will consider an infinitesimal thickness of radial size \( \varepsilon \) for the string like defect following Ref.[14]. Denoting the metric and scalar curvature in seven dimensions by \( \hat{g} \) and \( \hat{R} \) respectively, the action is given by

\[
S = \int d^7x \sqrt{\hat{g}} \left( -\frac{1}{2} M_7^5 \hat{R} - \Lambda_7 \right) + \int_{y=0} d^6x \sqrt{-\hat{g}^{(1)}} \left( \mathcal{L}_{\text{wall}}^{(1)} - \Lambda_6^{(1)} \right) \\
+ \int_{y=y_c} d^6x \sqrt{-\hat{g}^{(2)}} \left( \mathcal{L}_{\text{wall}}^{(2)} - \Lambda_6^{(2)} \right) + \int d^7x \sqrt{\hat{g}} \mathcal{L}_{\text{string}}.
\]

where \( M_7 \) is the seven-dimensional Planck scale, \( \Lambda_7 \) is a bulk cosmological constant and \( \Lambda_6^{(1)}; \Lambda_6^{(2)} \) are tensions of five-branes (walls) which lie at \( y = 0, y_c \) respectively. The matter part of the Lagrangian for the walls are denoted as \( \mathcal{L}_{\text{wall}}^{(1)}; \mathcal{L}_{\text{wall}}^{(2)} \) and that for the string-like defect as \( \mathcal{L}_{\text{string}} \). Here we explicitly separated out the wall tensions \( \Lambda_6^{(1)}, \Lambda_6^{(2)} \) from the matter Lagrangians on the walls, but we included the string tension in the Lagrangian \( \mathcal{L}_{\text{string}} \) of the string-like defect. The
induced metrics on the five-branes are denoted by $\bar{g}$

$$
\bar{g}^{(1)}_{AB} \equiv \hat{g}_{AB}|_{y=0} , \quad \bar{g}^{(2)}_{AB} \equiv \hat{g}_{AB}|_{y=y_c} ,
$$

(2.2)

for indices $A, B = 0, 1, 2, 3, r, \theta$. Indices for the coordinates of our four-dimensional spacetime are denoted by Greek alphabets as $x^\mu = 0, \cdots, 3$.

We will look for a solution that respects four-dimensional Poincaré invariance in the intersection. Let us assume the following ansatz for the seven-dimensional metric

$$
d s^2 = \hat{g}_{AB} dx^A dx^B = \sigma(y, r) \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 - dr^2 - \gamma(y, r) d\theta^2 .
$$

(2.3)

This metric has two kinds of warp factors $\sigma(y, r)$ and $\gamma(y, r)$. The seven-dimensional Einstein equations for the action (2.1) is given by

$$
\hat{R}_{AB} - \frac{1}{2} \hat{g}_{AB} \hat{R} = \frac{1}{M_7^5} \left[ \Lambda_7 \hat{g}_{AB} + \hat{T}_{(\text{string})}^{AB} + \left( \hat{T}_{(\text{wall1})}^{AB} + \Lambda_6^{(1)} \right) \sqrt{\bar{g}} \frac{\bar{g}}{\sqrt{\hat{g}}} \bar{g}^{(1)}_{AB} \delta(y) 
+ \left( \hat{T}_{(\text{wall2})}^{AB} + \Lambda_6^{(2)} \right) \sqrt{\bar{g}} \frac{\bar{g}}{\sqrt{\hat{g}}} \bar{g}^{(2)}_{AB} \delta(y - y_c) \right] , \quad A, B = 0, \cdots, 3, r, \theta ,
$$

(2.4)

if the indices $A, B$ do not contain $y$ component. If at least one of the indices $A, B$ corresponds to the $y$ component, the Einstein equation becomes

$$
\hat{R}_{AB} - \frac{1}{2} \hat{g}_{AB} \hat{R} = \frac{1}{M_7^5} \left[ \Lambda_7 \hat{g}_{AB} + \hat{T}_{(\text{string})}^{AB} \right] , \quad A \text{ or } B = y .
$$

(2.5)

The energy-momentum tensors for the string-like defect and the walls are defined as

$$
\hat{T}_{(\text{string})}^{AB} \equiv \frac{2}{\sqrt{\hat{g}}} \frac{\partial \left( \sqrt{\hat{g}} L_{\text{string}}^{(1)} \right)}{\partial \hat{g}^{AB}} ,
$$

$$
\hat{T}_{(\text{wall1})}^{AB} \equiv \frac{2}{\sqrt{-\bar{g}^{(1)}}} \frac{\partial \left( \sqrt{-\bar{g}^{(1)}} L_{\text{wall}}^{(1)} \right)}{\partial \bar{g}^{(1)AB}} , \quad \hat{T}_{(\text{wall2})}^{AB} \equiv \frac{2}{\sqrt{-\bar{g}^{(2)}}} \frac{\partial \left( \sqrt{-\bar{g}^{(2)}} L_{\text{wall}}^{(2)} \right)}{\partial \bar{g}^{(2)AB}} .
$$

(2.6)

Let us note that the ratios $\sqrt{-\bar{g}^{(1)}}/\sqrt{\hat{g}}$ and $\sqrt{-\bar{g}^{(2)}}/\sqrt{\hat{g}}$ at the five-branes in Eq.(2.4) reduce to 1 for the metric (2.3).

The non-zero components of the energy-momentum tensor from the matter sources on the wall at $y = 0$ are assumed as

$$
\hat{T}_{(\text{wall1})}^{\mu\nu} = t_{\nu}^{(1)}(r) \delta^\mu_\nu , \quad \hat{T}_{(\text{wall1})}^{r} = t_{r}^{(1)}(r) , \quad \hat{T}_{(\text{wall1})}^\theta = t_{\theta}^{(1)}(r) ,
$$

(2.7)
where the \( r \)-dependence of \( t_0^{(1)}, t_r^{(1)}, t_\theta^{(1)} \) will be given later. A similar form is assumed for the energy-momentum tensor \( T^{(\text{wall2})\mu}_\nu \) on the other wall at \( y = y_c \). On the other hand, we assume that the matter source for the string-like defect is distributed continuously within the core of radius \( \varepsilon \) and vanishes outside of it. The non-zero components of the energy-momentum tensor are assumed as

\[
\hat{T}^{(\text{string})\mu}_\nu = f_0(y, r) \delta^\mu_\nu , \quad \hat{T}^{(\text{string})y}_y = f_y(y, r) , \quad \hat{T}^{(\text{string})y}_r = f_y(y, r) ,
\]

where the \((y, r)\)-dependence of \( f_0, f_y, f_y r, f_r, f_\theta \) will be given later.

Using the ansatz of the metric (2.3) and the energy-momentum tensors (2.7) (2.8), the Einstein equations (2.4), (2.7) now become

\[
\frac{3 \sigma''}{2 \sigma} + \frac{3 \sigma' \gamma'}{4 \sigma \gamma} + \frac{1 \gamma''}{2 \gamma} - \frac{1 \gamma'^2}{4 \gamma^2} + \frac{3 \delta}{2 \sigma} + \frac{3 \dot{\sigma} \dot{\gamma}}{4 \sigma \gamma} + \frac{1 \dot{\gamma}}{2 \gamma} - \frac{1 \dot{\gamma}^2}{4 \gamma^2} = - \frac{1}{M_7^5} \left[ \Lambda_7 + f_0(y, r) \right] ,
\]

\[
\frac{3 \sigma''}{2 \sigma^2} + \frac{\sigma' \gamma'}{\sigma \gamma} + \frac{2 \ddot{\sigma}}{2 \sigma} + \frac{\dot{\sigma} \ddot{\gamma}}{\sigma \gamma} + \frac{1 \ddot{\gamma}}{2 \gamma} - \frac{1 \dot{\gamma}^2}{4 \gamma^2} = - \frac{1}{M_7^5} \left[ \Lambda_7 + f_y(y, r) \right] ,
\]

\[
\frac{\sigma' \ddot{\sigma}}{\sigma} - \frac{2 \ddot{\sigma} \gamma'}{\sigma} + \frac{1 \gamma' \ddot{\gamma}}{\gamma} - \frac{1 \gamma'}{2 \gamma} = - \frac{1}{M_7^5} f_y r(y, r) ,
\]

\[
\frac{2 \sigma''}{\sigma} + \frac{1 \sigma'^2}{2 \sigma^2} + \frac{\sigma' \gamma'}{\sigma \gamma} + \frac{1 \gamma''}{2 \gamma} - \frac{1 \gamma'^2}{4 \gamma^2} + \frac{\dot{\sigma} \dot{\gamma}}{\sigma \gamma} + \frac{3 \dot{\gamma}^2}{2 \sigma^2} = - \frac{1}{M_7^5} \left[ \Lambda_7 + f_r(y, r) + (t_r^{(1)}(r) + \Lambda_0^{(1)}) \delta(y) + (t_r^{(2)}(r) + \Lambda_0^{(2)}) \delta(y - y_c) \right] ,
\]

\[
\frac{2 \sigma''}{\sigma} + \frac{1 \sigma'^2}{2 \sigma^2} + \frac{2 \ddot{\sigma}}{2 \sigma^2} + \frac{1 \dot{\sigma}^2}{2 \sigma^2} = - \frac{1}{M_7^5} \left[ \Lambda_7 + f_\theta(y, r) + (t_\theta^{(1)}(r) + \Lambda_0^{(1)}) \delta(y) + (t_\theta^{(2)}(r) + \Lambda_0^{(2)}) \delta(y - y_c) \right] ,
\]

where the dash (\(^\prime\)) and the dot (\(^\cdot\)) denote partial differentiations \( \partial_y \) and \( \partial_r \) respectively.

The boundary conditions at \( r = 0 \) are assumed as

\[
\partial_r \sigma |_{r = 0} = 0 , \quad \partial_r \sqrt{\gamma} |_{r = 0} = 1 , \quad \gamma |_{r = 0} = 0 .
\]

These conditions are consistent with the usual regular solution in flat space so that \( r = 0 \) can be regarded as a regular origin of the polar coordinates \((r, \theta)\).
We assume that the solutions outside the core of the string-like defect \((r > \varepsilon)\) take the form

\[
\sigma(y, r) = e^{-a|y| - br}, \quad \gamma(y, r) = R_0^2 e^{-c|y| - dr},
\] (2.15)

which is consistent with the orbifold symmetry \(y \leftrightarrow -y\). The coefficient \(R_0\) is a constant for the length scale and \(a, b, c, d\) are constants with the dimension of mass. We assume that the warp factors do not diverge as \(r\) goes to infinity

\[
b \geq 0, \quad d \geq 0.
\] (2.16)

Substituting Eq. (2.15) into the Einstein equations (2.9)–(2.13), we obtain solutions

\[
c = \pm 2b, \quad d = \mp 2a,
\] (2.17)

where the upper and lower signs are for \(a \leq 0\) and \(a \geq 0\) respectively. The constants \(a\) and \(b\) are related by

\[
a^2 + b^2 = \frac{-2\Lambda_7}{5M_7^5},
\] (2.18)

which means that this solution requires \(\Lambda_7 < 0\). Here we have one parameter family of solutions that are locally related by an \(SO(2)\) rotation of \((y, r)\)-space, apart from the range of \((y, r)\) that is covered.

From the boundary conditions at \(y = 0, y_c\), the components of the energy-momentum tensor from the walls for \(r > \varepsilon\) must be

\[
t_0^{(1)}(r) = M_7^5(3a \pm 2b) - \Lambda_6^{(1)},
\]
\[
t_r^{(1)}(r) = M_7^5(4a \pm 2b) - \Lambda_6^{(1)},
\]
\[
t_\theta^{(1)}(r) = 4M_7^5a - \Lambda_6^{(1)},
\] (2.19)

\[
t_0^{(2)}(r) = M_7^5(-3a \mp 2b) - \Lambda_6^{(2)},
\]
\[
t_r^{(2)}(r) = M_7^5(-4a \mp 2b) - \Lambda_6^{(2)},
\]
\[
t_\theta^{(2)}(r) = -4M_7^5a - \Lambda_6^{(2)}.
\] (2.20)

Therefore we can find a solution for \(a\) and \(b\) only if the energy-momentum tensors \(t^{(1)}, t^{(2)}\)'s from matter sources are present on the walls. It is also interesting to observe that the matter energy-momentum tensors \(t^{(1)}, t^{(2)}\)'s on the walls are independent of the position \((r, \theta)\) on the wall outside the intersection region \(r > \varepsilon\).
Let us examine the Einstein equations near the core of the string-like defect \(0 \leq r < \varepsilon\). The components of the string tension are obtained by integrating over the disk of small radius \(\varepsilon\) containing the string core as

\[
\mu_I = \int_0^\varepsilon dr \sigma^2 \sqrt{\gamma} f_I(y, r) ,
\]

where \(I = 0, y, yr, r, \theta\). Even without specifying the warp factors inside the string core explicitly, the Einstein equations (2.9)-(2.13) and the boundary conditions (2.14) require the components of the string tension to satisfy the following relations

\[
\left[ \sigma (\partial_r \sigma) \sqrt{\gamma} \right]_0^\varepsilon = -\frac{2}{M_7^5} (\mu_r + \mu_\theta) ,
\]

\[
\left[ \sigma^2 \partial_r \sqrt{\gamma} \right]_0^\varepsilon = -\frac{1}{4M_7^5} (4\mu_0 + \mu_r - 3\mu_\theta) ,
\]

\[
4\mu_0 - 4\mu_y + \mu_r + \mu_\theta = 0 ,
\]

neglecting the order \(O(\varepsilon)\) contributions. These quantities can be obtained solely in terms of the values of warp factors at the boundaries of the core \(r = 0\) and \(\varepsilon\), without specifying the warp factors explicitly inside the string core. Note that \(\mu_I\) are functions of \(y\) as there are warp factors which change the scale of the theory along the \(y\)-direction. Since our solutions (2.15) relate the left-hand sides of Eqs.(2.22) and (2.23), we find the following relation between the components of the string tension,

\[
\pm a (\mu_y - \mu_\theta) + 2b M_7^5 (\mu_r + \mu_\theta) = b\sigma^2(y, 0) .
\]

Since more explicit form of the warp factors inside the core \(r < \varepsilon\) of the string-like defect is needed to obtain each component of the energy-momentum tensor \(\tilde{T}^{(\text{string})}\) separately, we will take the following warp factors inside the core as an illustrative example

\[
\sigma(y, r) = e^{-a|y|-br_2} \Theta(\varepsilon_2 - r) + e^{-a|y|-br} \Theta(r - \varepsilon_2) ,
\]

\[
\gamma(y, r) = r^2 \Theta(\varepsilon_1 - r) + \left\{ A(y) (r - \varepsilon_1) + \varepsilon_1^2 \right\} \left\{ \Theta(r - \varepsilon_1) - \Theta(r - \varepsilon_2) \right\} + R_0^2 e^{\mp 2b|y|\pm 2ar} \Theta(r - \varepsilon_2) ,
\]

where \(\Theta\) is a step function, \(0 < \varepsilon_1 < \varepsilon_2 = \varepsilon - 0\) and

\[
A(y) \equiv \frac{R_0^2 e^{\mp 2b|y|\pm 2ar} - \varepsilon_1^2}{\varepsilon_2 - \varepsilon_1} .
\]

The behavior of the warp factors are shown in Fig.4. The components of the string tension needed
We observe that \( \mu_{yr}(y) \) has a discontinuity at the wall \( y = 0 \). Since \( \mu_{yr}(y) \) is an odd function of \( y \), the nonvanishing value of \( \mu_{yr}(y \to +0) \) implies the discontinuity. We believe that the discontinuity comes about because we assumed a vanishing width for the wall, and that it will be smoothed out if the four-brane and five-branes are intersecting smoothly. Therefore we should understand the solution (2.3), (2.15), (2.26) as a limit of the vanishing width for the walls including the intersection region.

When we assume the above warp factors (2.26) even at the intersection region, we also obtain the energy-momentum tensor for the walls even inside the string core \( 0 \leq r < \varepsilon \). We find, for instance, \( t^{(1)} \)'s are modified from Eq.(2.19) by multiplying the constant \( b \) with a function \( B(r) \)

\[
b \to bB(r), \quad B(r) \equiv \left[ 1 + \frac{\varepsilon_2 - r}{r - \varepsilon_1} \left( \frac{\varepsilon_1}{R_0} \right)^2 e^{\mp 2a\varepsilon_2} \right]^{-1}.
\]

Therefore the components of the energy-momentum tensor \( t^{(1)} \)'s on the wall change from the constant value (2.19) outside the string core \( r > \varepsilon \) to another constant value at the center \((0 \leq r < \varepsilon_1)\) smoothly through the string core region \( 0 \leq r < \varepsilon \) where the wall intersects with the string-like defect. The components of the energy-momentum tensor on the other wall \( t^{(2)} \)'s also exhibit a similar behavior.
The four-dimensional reduced Planck scale $M_4$ is related to the seven-dimensional Planck scale $M_7$ as

$$M_4^2 = M_7^5 \int_0^{y_c} dy \int_0^\infty dr \int_0^{2\pi} d\theta \sigma \sqrt{\gamma},$$

which gives after integration

$$M_4^2 = \pm \frac{2\pi R_0}{b^2 - a^2} \left( 1 - e^{-(a \pm b)y_c} \right) M_7^5$$

for $a \pm b \neq 0$, and

$$M_4^2 = \frac{\pi R_0 y_c}{b} M_7^5$$

for $a \pm b = 0$. If we want to consider a single wall configuration which is obtained by taking $y_c \to \infty$ and by removing the regulator brane at $y = y_c$, we must choose a solution which satisfies $a \pm b > 0$.

The choice of $b = 0$ reduces to a rather trivial solution giving a tensor product of the model of parallel walls in the five-dimensional spacetime [10][11] and a two-dimensional model with a string-like defect similar to Ref.[14], because then the $(r, \theta)$-directions become completely separated from the other directions. Instead, in the rest of this paper, we will concentrate on the $a = 0$ case solution and choose the upper sign\footnote{The change of sign merely exchanges the role of two walls in this case. The choice of the upper sign corresponds to choosing the wall at $y = 0$ as the brane with positive cosmological constant.} in (2.17) in order to obtain a finite four-dimensional reduced Planck scale even in the limit $y_c \to \infty$. We will eventually take $y_c \to \infty$ to consider the configuration of a single wall intersecting with a string-like defect at $y = 0$ and $r < \varepsilon$. The metric solution for $r > \varepsilon$ (outside of the string-like defect core) now becomes

$$ds^2 = e^{-br} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 - dr^2 - R_0^2 e^{-2b|y|} d\theta^2,$$

where

$$b = \sqrt{-\frac{2\Lambda}{5M_7^5}}.$$

Taking the upper sign in Eq.(2.13), excluding the regulator wall at $y = y_c$, and setting $a = 0$ in Eq.(2.19), we find the energy-momentum tensor of matter fields on the wall at $y = 0$ satisfying the Einstein equations as

$$t_0^{(1)}(r) = 2bM_7^5 - \Lambda_6^{(1)}, \quad t_r^{(1)}(r) = 2bM_7^5 - \Lambda_6^{(1)}, \quad t_\theta^{(1)}(r) = -\Lambda_6^{(1)}.$$

In this case, we can explicitly construct the energy-momentum tensor on the wall at $y = 0$ by means of matter fields. Let us consider two scalar fields $\phi^a, a = 1, 2$ on the wall with a Lagrangian...
density

\[ \mathcal{L}_{\text{wall}}^{(1)} = \frac{1}{2} \bar{g}^{AB} \partial_A \varphi^a \partial_B \varphi^a - V(\varphi), \]

\[ V(\varphi) = \lambda (\varphi^a \varphi^a - v^2)^2, \]  

(2.40)

where the indices \( A, B \) span only over coordinates on the wall \( A, B = 0, \ldots, 3, r, \theta \). The energy-momentum tensor is then given by

\[ \bar{T}_{(\text{wall})}^{AB} = \bar{g}^{AC} \partial_C \varphi^a \partial_B \varphi^a - \delta^A_B \left( \frac{1}{2} \bar{g}^{CD} \partial_C \varphi^a \partial_D \varphi^a - V(\varphi) \right). \]

The minimum of the potential \( V \) in Eq.(2.40) is at \( \varphi^a \varphi^a = v^2 \). Therefore we can have a winding configuration of the scalar fields

\[ \varphi^1 = v \cos \theta, \quad \varphi^2 = v \sin \theta, \]  

(2.41)

which becomes stable for large values of \( \lambda \). Then the scalar field gives an energy-momentum tensor on the wall with components

\[ t_{(1)}^{(1)} = t_{r}^{(1)} = -t_{\theta}^{(1)} = \frac{v^2}{2 R_0^2}. \]  

(2.42)

Therefore the energy-momentum tensor (2.39) satisfying the Einstein equation is obtained if the vacuum expectation value of the scalar fields is related to the brane tension \( \Lambda_6^{(1)} \), the Planck scale \( M_7 \) in the bulk and the warp factor slope \( b \) in Eq.(2.38) as

\[ \Lambda_6^{(1)} = M_7^5 b = \frac{v^2}{2 R_0^2}. \]  

(2.43)

3 Graviton on the Background of a Wall and a String

3.1 Massless and Massive Spin-2 Fields

Having obtained a solution for the system, we wish to find the modes on the background of the solution. We first examine whether spin 2 graviton is localized on the four-dimensional intersection region, and then check whether the resulting gravitational force is consistent with the four-dimensional experimental gravity. For these purposes, let us consider linearized equations for gravitons. We will only consider the spin 2 fields and neglect the spin 1 and spin 0 fields, which must be included for the complete analysis eventually. Let us consider the following fluctuations of metric,

\[ \hat{g}_{\mu \nu} = \sigma(r) \left[ \eta_{\mu \nu} + h_{\mu \nu}(x^A) \right]. \]  

(3.1)
We will choose a gauge \[1\]
\[\partial_\mu h^{\mu \nu}(x^A) = h_\mu^\nu(x^A) = 0 . \tag{3.2}\]

Then the linearized equations of motion take the form \[2\]
\[\frac{1}{\sqrt{g}} \partial_A \left( \sqrt{g} g^{AB} \partial_B h_{\mu \nu}(x^A) \right) = 0 . \tag{3.3}\]

Since we take \(a = 0\) case \[2.37\], we can separate the variables as
\[h_{\mu \nu}(x^A) = \sum_{m,k,\ell} h^{(mk\ell)}_{\mu \nu}(x^{\alpha}) \phi_{(mk)}(r) \chi_{(k\ell)}(y) e^{i\ell \theta} , \tag{3.4}\]
where \(\ell\) is an integer angular momentum. Substituting the decomposition (3.4) into the linearized equations of motion (3.3), we obtain the following separated equations
\[-\eta^{\lambda \rho} \partial_\lambda \partial_\rho h^{(mk\ell)}_{\mu \nu} = m^2 h^{(mk\ell)}_{\mu \nu} , \tag{3.5}\]
\[-\partial_r \left( \sigma^2 \partial_r \phi_{(mk)} \right) + k^2 \sigma^2 \phi_{(mk)} = m^2 \sigma \phi_{(mk)} , \tag{3.6}\]
\[-\partial_y \left( \gamma^2 \partial_y \chi_{(k\ell)} \right) + \ell^2 \gamma^{-\frac{1}{2}} \chi_{(k\ell)} = k^2 \gamma^\frac{1}{2} \chi_{(k\ell)} . \tag{3.7}\]

Eq.(3.5) shows that the eigenvalue \(m \geq 0\) corresponds to the mass of the four-dimensional field \(h^{(mk\ell)}_{\mu \nu}(x^{\alpha})\). For the differential operator in the left-hand side of Eq.(3.7) to be self-adjoint we impose the boundary conditions\[8\]
\[\chi'_{(k\ell)}(0) = \chi'_{(k\ell)}(y_c) = 0 . \tag{3.8}\]

Then, we can demand the modes \(\chi_{(k\ell)}\) to satisfy the orthonormal condition
\[\int_0^{y_c} dy \gamma^\frac{1}{2} \chi_{(k\ell)}^* \chi_{(k'\ell')} = \delta_{kk'} . \tag{3.9}\]

Multiplying \(\chi_{(k\ell)}\) to Eq.(3.7) and integrating the first term of the left-hand side by parts, we obtain
\[k^2 = \int_0^{y_c} dy \gamma^\frac{1}{2} \left| \chi'_{(k\ell)} \right|^2 + \ell^2 \int_0^{y_c} dy \gamma^{-\frac{1}{2}} \left| \chi_{(k\ell)} \right|^2 \geq 0 . \tag{3.10}\]

This equation together with Eq.(3.9) shows that \(k^2\) is non-negative, more precisely
\[\ell \neq 0 \implies k^2 > \frac{1}{R_0^2} , \tag{3.11}\]
\[\ell = 0 \implies k^2 \geq 0 . \tag{3.12}\]

\[8\] Another possible boundary condition is the Dirichlet boundary condition \(\chi_{(k\ell)}(0) = \chi_{(k\ell)}(y_c) = 0\). However, modes satisfying the Dirichlet boundary condition do not contain the zero mode that we are most interested in. Moreover \(Z_2\) projection allows only the Neumann boundary condition \(3.8\). To obtain the corrections to the Newton potential \(3.36\), it is sufficient to consider \(3.8\).
where the equality \( k^2 = 0 \) holds if and only if \( \chi_{(00)}(y) \) is independent of \( y \). Our boundary condition (3.8) is the most natural one as long as the gravity can be treated semiclassically. We will examine the question of the validity of semiclassical gravity at the end of this subsection.

A similar analysis applies to Eq.(3.6). We can impose the boundary conditions

\[
\dot{\phi}_{(mk)}(0) = \phi_{(mk)}(\infty) = 0,
\]

(3.13)

for the differential operator in Eq.(3.6) to be self-adjoint. Then the modes \( \phi_{(mk)} \) can be made to satisfy the orthonormal condition

\[
\int_0^\infty dr \sigma \phi_{(mk)}^* \phi_{(m'k)} = \delta_{mm'}.
\]

(3.14)

where quantum numbers \( k \) are not summed. Multiplying \( \phi_{(mk)}^* \) to Eq.(3.6) and integrating the first term of the left-hand side by parts, we obtain

\[
m^2 = \int_0^\infty dr \sigma^2 \left| \dot{\phi}_{(mk)} \right|^2 + k^2 \int_0^\infty dr \sigma^2 \left| \phi_{(mk)} \right|^2 \geq 0,
\]

(3.15)

where the equality \( m^2 = 0 \) holds if and only if \( k = 0 \) and \( \phi_{(00)}(r) \) is independent of \( r \). We now obtain from Eqs.(3.10) and (3.15) that \( m = 0 \) requires \( k = \ell = 0 \). Since \( m \) is the mass of the four-dimensional field \( h^{(m\ell)}_{\mu\nu}(x^\alpha) \), we find that there is only one massless four-dimensional mode for spin 2 graviton as required by experiment.

It is easy to see from the wave equations (3.4), (3.7) and the orthonormal conditions (3.9), (3.14) that the unique massless mode is the zero-mode solution with \( m = k = \ell = 0 \),

\[
\chi_{(00)}(y) = \sqrt{\frac{b}{R_0(1 - e^{-by})}} , \quad \phi_{(00)}(r) = \sqrt{b}.
\]

(3.16)

These constant wave functions are normalizable. The definition (2.34) of the four-dimensional reduced Planck scale implies that the probability density of the wave functions \( \phi_{(mk)}(r) \chi_{(k\ell)}(y) \) in the extra dimensions is given by \( \sigma(r) \gamma^2(y) |\phi_{(mk)}(r) \chi_{(k\ell)}(y)|^2 \). The probability density of the zero mode \( m = k = \ell = 0 \) wave function is sharply peaked around the intersection \( y = 0, r = 0 \). Therefore the spin 2 graviton represented by the zero mode is localized around the intersection region.

In order to evaluate the density of states of the non-zero modes precisely, we reintroduce a regulator brane at a finite radial distance cutoff \( r_{\text{max}} \). Accordingly, the boundary condition (3.13) at \( r = \infty \) is modified to a similar boundary condition imposed at \( r = r_{\text{max}} \).

Now consider the non-zero modes. First, let us solve the wave equation (3.7) for \( \chi_{(k\ell)}(y) \).

\footnote{\quad We do not choose the Dirichlet boundary condition by the same reason as that for \( \chi_{(k\ell)}(y) \) except that we do not have \( Z_2 \) projection in \( r \).}
(i) \( \ell = 0 \) modes

The \( k = 0 \) mode solution is a constant \( \chi_{(00)} \) given in Eq. (3.16). The \( 0 < k \leq \frac{b}{2} \) mode solutions turn out to be excluded from the spectrum by the boundary conditions (3.13). The \( k > \frac{b}{2} \) mode solutions are

\[
\chi_{(k0)} = N_{k0} e^{\frac{b}{2}|y|} \left( e^{-i\frac{1}{2}\sqrt{4k^2-b^2}|y|} + \alpha_{k0} e^{i\frac{1}{2}\sqrt{4k^2-b^2}|y|} \right),
\]

where \( N_{k0} \) are normalization constants and \( \alpha_{k0} \) are constants which are determined by the boundary condition at \( y = 0 \) as

\[
\alpha_{k0} = -\frac{b - i\sqrt{4k^2-b^2}}{b + i\sqrt{4k^2-b^2}}.
\]

In addition, there is a boundary condition at \( y = y_c \) which leads to the quantization of \( k \) as

\[
k^2 = \frac{b^2}{4} + \frac{\pi^2}{y_c^2} n_k^2,
\]

where \( n_k = 1, 2, \ldots \). Thus we find that there is a gap in the spectrum of \( k \) between \( k = 0 \) and \( k > b/2 \) even in the limit of \( y_c \to \infty \).

(ii) \( \ell \neq 0 \) modes

Changing the variable from \( y \) to \( u \equiv \ell b_0 e^{b|y|} \) and redefining the wave equation by \( \tilde{\chi}_{(k\ell)} \equiv u^{-\frac{\ell}{2}} \chi_{(k\ell)} \), Eq. (3.7) can be rewritten in the form

\[
\left[ \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} - \left( 1 + \frac{1}{u^2} \left( \frac{1}{4} - \frac{k^2}{b^2} \right) \right) \right] \tilde{\chi}_{(k\ell)} = 0.
\]

The solution to this equation is a linear combination of modified Bessel functions. Therefore, the solution is of the form

\[
\chi_{(k\ell)} = N_{k\ell} u^{\frac{\ell}{2}} \left[ K_{\nu_k}(u) + \alpha_{k\ell} I_{\nu_k}(u) \right],
\]

where

\[
\nu_k = i\sqrt{\frac{k^2}{b^2} - \frac{1}{4}},
\]

\( N_{k\ell} \) are normalization constants and \( \alpha_{k\ell} \) are constant coefficients. Irrespective of the detailed spectrum for the \( \ell \neq 0 \) modes, Eq. (3.11) shows that there is a gap in the spectrum of \( k \) as \( k > 1/R_0 \).

Next, let us turn to Eq. (3.6) to obtain the wave function \( \phi_{(nk)}(r) \).
(i') $m = 0$ mode

As stated before, there is only one $m = 0$ mode corresponding to $m = k = \ell = 0$. The $m = 0$ mode solution is a constant which reduces to the $\phi_{(00)}$ in Eq. (3.16) in the limit $r_{\text{max}} \to \infty$.

(ii') $m > 0$ modes

Changing the variables to $v \equiv \frac{2m}{b}e^{\frac{b}{2}}r$ and redefining the wave equations by $\tilde{\phi}_{(mk)} \equiv v^{-2}\phi_{(mk)}$, Eq. (3.6) can be rewritten in the form

$$\left[\frac{d}{dv} + \frac{1}{v} \frac{d}{dv} \left\{1 - \frac{4}{u_m^2} \left(1 + \frac{k^2}{b^2}\right)\right\}\right] \tilde{\phi}_{(mk)} = 0 . \quad (3.23)$$

The solutions to this equation are linear combinations of Bessel functions. Therefore the solution takes the form

$$\phi_{(mk)} = \tilde{N}_{mk}e^{br} \left[J_{\tilde{v}_k}(v) + \beta_{mk}Y_{\tilde{v}_k}(v)\right] , \quad (3.24)$$

where $\tilde{N}_{mk}$ are the normalization constants, $\beta_{mk}$ are the constant coefficients and

$$\tilde{v}_k = 2\sqrt{1 + \frac{k^2}{b^2}} . \quad (3.25)$$

From the boundary condition (3.13) at $r = 0$, we obtain

$$\beta_{mk} = -\frac{(2 - \tilde{v}_k)J_{\tilde{v}_k}(v_0) + v_0 J_{\tilde{v}_k-1}(v_0)}{(2 - \tilde{v}_k)Y_{\tilde{v}_k}(v_0) + v_0 Y_{\tilde{v}_k-1}(v_0)} , \quad (3.26)$$

where $v_0 \equiv \frac{2m}{b}$ and $\tilde{v}_k = 2\sqrt{1 + \frac{k^2}{b^2}}$. Since the boundary condition at $r = r_{\text{max}}$ gives $\dot{\phi}_{(mk)}(r_{\text{max}}) = 0$, we find

$$\beta_{mk} = -\frac{(2 - \tilde{v}_k)J_{\tilde{v}_k}(v_{\text{max}}) + v_{\text{max}} J_{\tilde{v}_k-1}(v_{\text{max}})}{(2 - \tilde{v}_k)Y_{\tilde{v}_k}(v_{\text{max}}) + v_{\text{max}} Y_{\tilde{v}_k-1}(v_{\text{max}})} \approx -\frac{J_{\tilde{v}_k}(v_{\text{max}})}{Y_{\tilde{v}_k-1}(v_{\text{max}})} , \quad (3.27)$$

where $v_{\text{max}} \equiv \frac{2m}{b}e^{\frac{b}{2}r_{\text{max}}}$. In the limit $\frac{m}{b} \ll 1$, Eq. (3.26) for the $k = 0$ mode is approximated by

$$\beta_{m0} = -\frac{J_1(v_0)}{Y_1(v_0)} \approx \frac{\pi}{4}v_0^2 \ll 1 . \quad (3.28)$$

Eqs. (3.27) and (3.28) imply that the mass spectrum is quantized as

$$m \equiv m_n \simeq \left(n + \frac{1}{4}\right)\frac{\pi b}{2}e^{-\frac{b}{2}r_{\text{max}}} , \quad (3.29)$$
where \( n \) is an integer. The mode function \( \phi_{(m0)} \) at \( r = 0 \) becomes
\[
|\phi_{(m0)}(0)|^2 \sim |\tilde{N}_{m0}|^2 \simeq \pi m e^{-\frac{b}{2} r_{\text{max}}}.
\] (3.30)

On the other hand, Eq. (3.26) for the \( k > 0 \) modes in the limit \( \frac{n}{b} \ll 1 \) is approximated by
\[
\beta_{mk} \sim v_0^{2\tilde{\nu}_k} \ll 1.
\] (3.31)

So the mass spectrum is quantized as
\[
m \equiv m_n \simeq \left( n + \frac{\bar{\nu}_k}{2} - \frac{3}{4} \right) \frac{\pi b}{2} e^{-\frac{b}{2} r_{\text{max}}} ,
\] (3.32)

and the mode function \( \phi_{(mk)} \) at \( r = 0 \) becomes
\[
|\phi_{(mk)}(0)|^2 \sim |\tilde{N}_{mk}|^2 v_0^{2\tilde{\nu}_k} .
\] (3.33)

Let us recall that there is a gap in the spectrum of \( k \) between \( k = 0 \) and \( k > b/2 \) for the \( \ell = 0 \) modes and \( k > 1/R_0 \) for the \( \ell \neq 0 \) modes. On the other hand, the normalization factor \( \tilde{N}_{mk} \) is insensitive to the values of \( k \). Therefore we find that
\[
|\phi_{(m0)}(0)|^2 \gg |\phi_{(mk)}(0)|^2 ,
\] (3.34)

and that the wave function at the intersection \( r = 0 \) is negligible for the modes with nonvanishing values of \( k \).

Let us discuss the question of the validity of semiclassical gravity. The metric (2.37) is locally a product of \( \text{AdS}_6 \) (for \( r, x^\mu \)) and Euclidean \( \text{AdS}_2^E \) (for \( y, \theta \)) and exhibits a conical singularity near \( y \to \infty \), since the effective radius for the angular variable \( \theta \) shrinks to zero. Because of this conical singularity, semiclassical gravity may not be reliable as the regulator brane goes to infinity \( y_c \to \infty \). Therefore our boundary condition (3.8) at \( y = y_c \) is, strictly speaking, reliable only for the regulator brane not too much far away from our brane and may be modified as \( y_c \to \infty \).

Recently a nice resolution of a conical singularity has been proposed by Ponton and Poppitz [23] in a simpler case of string-like defect in six-dimensional gravity. They observed that the string theory as the fundamental theory of gravity offers a dual description of the strong gravity in terms of the conformal field theory. Starting from a solution of supergravity corresponding to the type \( I' \) D4-D8 brane system, they obtained the \( \text{AdS}_6 \times S^4 \) gravity background. From the AdS/CFT correspondence and a chain of duality arguments, they concluded that one way of resolving the singularity leads to a semiclassical gravity description for a long-distance behavior of the gravitational field which is the same as the Randall-Sundrum model in five dimensions. Another
resolution provides a mass gap for all the Kaluza-Klein modes. Consequently the gravitational
Newton potential for the first case receives the same power corrections as those of the model with
one extra dimension (i.e. the Randall-Sundrum model \[11\]), whereas the gravitational Newton
potential for the second case receives only the exponentially suppressed corrections corresponding
to no extra dimensions effectively.

In order to address the issue of strong gravity and the resulting possible modification of the
boundary condition \[3.8\] at \(y = y_c\) for our model, we need to find a system of branes in string
theories which provides a solution of supergravity corresponding locally to our metric \[2.37\] in
the Maldacena limit. We expect that the solution corresponds to a number of coincident branes
localized on other branes, in order to provide two radial coordinates \(r\) for \(AdS_6\) and \(y\) for \(AdS_2^E\).
Semi-localized solutions have been obtained for cases which reduce to a product of an \(AdS\) space
and compact internal spaces such as spheres \[24\]. It is an extremely interesting problem to work
out those supergravity solutions which give a product of noncompact “internal space” and the
\(AdS\) space and to investigate the corresponding system of branes in string theories. We hope
to address these problems in future. The result of ref.\[23\] suggests that there may be choices of
boundary conditions also in our case. On the other hand, our analysis using the semiclassical
gravity gives a mass gap which makes two of the extra dimensions ineffective at low energies. This
will lead to the correction of the Newton potential with the same power as the model with only
one extra dimension (i.e. the Randall-Sundrum model \[11\]), as we will see in the next subsection.
It is interesting to note that this result resembles the result of ref.\[23\] which takes full account
of the strong gravity effects.

### 3.2 Gravitational potential

If we place point sources \(\bar{m}_1\) and \(\bar{m}_2\) on the four-dimensional intersection \((r = y = 0)\) with a large
distance \(R \equiv \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\) apart, we obtain a gravitational potential coming from the
exchanges of zero mode and nonzero modes. The zero mode for spin 2 graviton gives the usual
Newton’s law

\[
V(R) = -G_N \frac{\bar{m}_1 \bar{m}_2}{R}, \quad G_N \equiv \frac{1}{8\pi M_4^2},
\]

where \(G_N\) is the Newton’s constant and \(M_4\) is the reduced Planck scale defined in Eq.\[2.34\]. The
nonzero modes contribute corrections to the Newton potential as \[22\]

\[
\Delta V(R) = -G_N \frac{\bar{m}_1 \bar{m}_2}{R} \sum_n \sum_k \sum_l \frac{|\phi_{(mk)}(0)\chi_{(k\ell)}(0)|^2}{\phi_{(00)}(0)\chi_{(00)}(0)} e^{-m_n R}.
\]

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Because of the gap in the spectrum of \( k \) and \( m \), contributions from the \( k > 0 \) modes are suppressed exponentially. The leading contribution to the gravitational potential comes from the exchange of the nonzero modes of \( m \geq 0, k = \ell = 0 \). In the limit \( r_{\text{max}} \to \infty \), the spectrum (3.29) becomes continuous and the summation of \( n \) is converted into an integral

\[
\sum_n \approx \frac{2}{\pi b} e^{\frac{4\pi}{b} r_{\text{max}}} \int_0^\infty dm .
\] (3.37)

Consequently, Eq. (3.36) becomes

\[
\Delta V(R) \approx -G_N \frac{\bar{m}_1 \bar{m}_2}{R} \frac{2}{b^2} \int_0^\infty dm m e^{-mR}
\]

\[
= -\frac{2G_N \bar{m}_1 \bar{m}_2}{b^2} \frac{1}{R^3} .
\] (3.38)

One should note that the correction term behaves \( \Delta V(R) \sim 1/R^3 \) in the case of the wall in five dimensions [11], and \( \Delta V(R) \sim 1/R^4 \) in the case of the string-like defect in six dimensions [14]. This is natural since the gravitational flux tends to spread out if there are more extra dimensions. The correction term (3.38) to the Newton potential \( \Delta V(R) \sim 1/R^3 \) falls off precisely like the case of a wall in five dimensions even though we have seven dimensional bulk spacetime. Our counter-intuitive result can be understood as follows. The gap for \( k > 0 \) modes implies that only the \( k = 0 \) mode contributes without being suppressed exponentially. Both the gapless continuum massive graviton modes \( (k = \ell = 0) \) as well as the zero mode are localized on the wall at \( y = 0 \), and moreover the zero-mode \( (m = k = \ell = 0) \) is localized at the intersection of the wall and the string. Since the \( k = 0 \) mode is localized at \( y = 0 \), the gravitational flux does not spread out into the direction \( y \). This effectively reduces one dimension. Since the \( \ell \neq 0 \) modes are also exponentially suppressed because of the mass gap, only the \( \ell = 0 \) mode contributes to the leading correction. Therefore the angular direction is not effective either. On the other hand, the remaining direction \( r \) acts precisely like the coordinate of the single extra dimension \( y \) in the original wall model in five-dimensional spacetime. Therefore the same mechanism applies to our case as the wall model in five dimensions, leading to the same correction (3.38) as the wall model.

We have to examine the contributions from spin 1 and spin 0 fields coming from the seven-dimensional graviton in order to obtain the full gravitational potential and also to ascertain the stability of the background metric [17], [18]. It has also been noted that the contributions from the bending of the brane due to the backreaction of the matter source will modify the coefficient of the \( 1/R \) term in Eq. (3.35) and consequently the relative normalization of the correction term

\[1\text{ This } 1/R^4 \text{ behavior of the correction term is modified to } 1/R^3 \text{ behavior or to no power corrections by taking }
\]

\[\text{account of the strong gravity effects in ref. [23]}.\]
Finally we should have in mind that the strong gravity effects near the conical singularity as the regulator brane goes to infinity $y_c \to \infty$ may modify the boundary condition at $y = y_c$ and the corrections to the gravitational Newton potential.

4 Intersection of $n$ Walls and a String-like Defect

In the following, we extend the idea of the previous section to a configuration of mutually intersecting $n$ walls (namely $(5+n)$-branes) and a string-like defect (namely $(4+n)$-brane) in $(6+n)$ spacetime dimensions. Our four-dimensional world will come out as an intersection of these walls and the string-like defect.

We can perform the extension along the lines of [2], where they worked $n$ orthogonally intersecting walls in a locally $AdS_{4+n}$ bulk geometry. Since the bulk geometry of the metric (2.37) is locally $AdS_5 \times AdS_E^2$ where $AdS_E^2$ denotes two-dimensional Euclidean anti-de Sitter space in $(y, \theta)$ directions, we are prompted to extend the $AdS_E^2$ geometry to $AdS_{n+1}^E$. We will work on a $D \equiv 6+n$ dimensional spacetime with the coordinates $x^A \equiv (x^\mu, z_j, r, \theta)$, $j = 1, 2, \cdots, n$. As in the previous case, $(r, \theta)$ are polar coordinates of extra dimensions transverse to the string-like defect, which is placed at $r = 0$. The coordinate for the direction normal to the $j$-th wall is denoted as $z_j$ ($j = 1, 2, \cdots, n$). The action of the extended system is

$$S = S_{\text{gravity}} + S_{\text{string}} + \sum_{j=1}^n S_{\text{wall}}^{(j)},$$

$$S_{\text{gravity}} = \int d^D x \sqrt{\hat{g}} \left( -\frac{1}{2} M_D^{D-2} \hat{R} - \Lambda_D \right),$$

$$S_{\text{string}} = \int_{r<\varepsilon} d^D x \sqrt{\hat{g}} L_{\text{string}}^{(j)},$$

$$S_{\text{wall}}^{(j)} = \int_{z_j=0} d^{D-1} x \sqrt{\hat{g}^{(j)}} L_{\text{wall}}^{(j)},$$

where $M_D$ is the fundamental $D$-dimensional Planck scale and $\Lambda_D$ is a bulk cosmological constant. We included the cosmological constants on branes in the Lagrangians $L_{\text{string}}$ and $L_{\text{wall}}^{(j)}$’s. The $j$-th wall is placed at $z_j = 0$, on which the induced metric is

$$\hat{g}^{(j)}_{A_j B_j} \equiv \hat{g}_{A_j B_j} |_{z_j=0},$$

where $A_j, B_j$ are the $(D-1)$-dimensional indices along the $j$-th wall. Therefore the subscript $j$ on $A, B$ denotes an exclusion of the coordinate $z_j$. 

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Einstein equations for this action are

\[ \hat{R}_{AB} - \frac{1}{2} \hat{g}_{AB} \hat{R} = \frac{1}{M_D^{D-2}} \left[ \Lambda_D \hat{g}_{AB} + \hat{T}^{(\text{string})}_{AB} + \sum_{j=1}^{n} \sqrt{\hat{g}^{(j)}} \hat{T}^{(j)}_{A_{ij} B_{jk}} \delta_{A_{ij}} \delta_{B_{jk}} \delta(z_j) \right], \tag{4.6} \]

where \( \hat{T}^{(\text{string})}_{AB} \) and \( \hat{T}^{(j)}_{A_{ij} B_{jk}} \)'s are the energy-momentum tensor obtained from \( \mathcal{L}_{\text{string}} \) and \( \mathcal{L}^{(j)}_{\text{wall}} \)'s as defined in Eq.(2.10). The tilde (') on \( \hat{T}^{(j)}_{A_{ij} B_{jk}} \) is to denote that the wall tensions are included into the energy-momentum tensor.

We assume an ansatz for the metric of a diagonalized form with warp factors\(^*\) \( \sigma(r, z) \) and \( \Omega(r, z) \)

\[
ds^2 = \sigma(r, z) \eta_{\mu \nu} dx^\mu dx^\nu - dr^2 - \Omega^2(r, z) \left( \sum_{j=1}^{n} dz_j^2 + R_0^2 d\theta^2 \right). \tag{4.7} \]

The ratios of the determinant of metrics reduce as follows,

\[ \frac{\sqrt{\hat{g}^{(j)}}}{\sqrt{\hat{g}}} \delta(z_j) = \sqrt{-\hat{g}^{ij}} \delta(z_j) = \frac{1}{\Omega} \delta(z_j). \tag{4.8} \]

As a natural extension of the solution considered in the last section, we look for a solution of the form

\[ \sigma(r, z) \equiv e^{-br}, \quad \Omega(r, z) \equiv \frac{1}{K \sum_{j=1}^{n} |z_j| + 1}, \tag{4.9} \]

for outside of the string-like defect core \( (r > \varepsilon) \). We assume that the non-zero components of the energy-momentum tensor from the matter sources on the \( j \)-th wall for \( r > \varepsilon \) are

\[
\hat{T}^{(j)}_{\mu \nu} = \tilde{t}^{(j)}_{0 \nu} \delta_{\mu}^r, \quad \hat{T}^{(j)}_{rr} = \tilde{t}^{(j)}_{r r}, \quad \hat{T}^{(j)}_{ik} = \tilde{t}^{(j)}_{z k} \delta^i_z \delta_k, \quad \hat{T}^{(j)}_{d \theta} = \tilde{t}^{(j)}_{d \theta}, \tag{4.10} \]

where \( i, k \neq j \). We assume that the string-like defect is a strictly local defect, \( i.e., \hat{T}^{(\text{string})}_{AB} \neq 0 \) for \( r < \varepsilon \) and \( \hat{T}^{(\text{string})}_{AB} = 0 \) for \( r > \varepsilon \). From the metric ansatz (4.7) and the energy-momentum tensor ansatz (4.10), Einstein equations (4.9) for \( r > \varepsilon \) become

\[
\frac{3 \dot{\sigma}}{2 \sigma} + \frac{n^2(n + 1)}{2} K^2 - 2 n \frac{K}{\Omega} \sum_{i=1}^{n} \delta(z_i) = - \frac{1}{M_D^{D-2}} \left[ \Lambda_D + \frac{1}{\Omega} \sum_{i=1}^{n} \tilde{t}^{(i)}_0 \delta(z_i) \right], \tag{4.11} \]

\[
\frac{3 \dot{\sigma}^2}{2 \sigma^2} + \frac{n^2(n + 1)}{2} K^2 - 2 n \frac{K}{\Omega} \sum_{i=1}^{n} \delta(z_i) = - \frac{1}{M_D^{D-2}} \left[ \Lambda_D + \frac{1}{\Omega} \sum_{i=1}^{n} \tilde{t}^{(i)}_r \delta(z_i) \right], \tag{4.12} \]

** The warp factor \( \Omega \) for the angular coordinate \( \theta \) has to vanish at the origin of the two-dimensional polar coordinates \( r = 0 \) like Eq.\( (2.14) \) in order to have no singularity. Since there is no such requirement for the other coordinates \( z_j \)'s, the warp factors \( \Omega \) for \( \theta \) and \( z_j \)'s can differ inside the string core \( 0 \leq r < \varepsilon \).
where \( \ell \) is an integer angular momentum. Then, the wave equation is separated into modes \( h_{\mu\nu}(x^A) \). As in the last section, we will concentrate only on the linearized fluctuations of spin 2 components to satisfy the boundary conditions imposed at \( r = 0 \) as in Eq. (3.14). A concrete example may be worked out following the illustrative example in Eq. (2.21). The solution inside the string-like defect core \( r < \varepsilon \) can be obtained by adjusting appropriate string tension components to satisfy the boundary conditions imposed at \( r = 0 \) as in Eq. (2.14).

The solution to the above Einstein equations is found to be

\[
b = nK, \quad K = \sqrt{\frac{-2\Lambda_D}{n^2(n + 4)M_D^{D-2}}}.
\]

Therefore the negative cosmological constant \( \Lambda_D < 0 \) in the bulk is required. The \( \delta \)-functions at \( z_j = 0 \) (\( j = 1, 2, \ldots, n \)) imply that the energy-momentum tensor on the walls must satisfy

\[
\tilde{t}_0^{(j)} = \tilde{t}_r^{(j)} = 2nKM_D^{D-2}, \quad \tilde{t}_z^{(j)} = \tilde{t}_\theta^{(j)} = 2(n - 1)K\tilde{M}_D^{D-2}.
\]

The solution inside the string-like defect core \( r < \varepsilon \) can be obtained by adjusting appropriate string tension components to satisfy the boundary conditions imposed at \( r = 0 \) as in Eq. (2.14). A concrete example may be worked out following the illustrative example in Eq. (2.21).

The four-dimensional reduced Planck scale for this solution is given by

\[
M_4^2 = M_D^{D-2} \int_0^{\infty} dr \int_0^{\infty} d^n z \int_0^{2\pi} d\theta \frac{1}{\sigma} \sqrt{g} = \frac{2\pi R_0}{n \cdot n! K^{n+1}} M_D^{D-2}.
\]

Let us now consider metric fluctuations around the background of the solution (4.7), (4.9), (4.13). As in the last section, we will concentrate only on the linearized fluctuations of spin 2 modes \( h_{\mu\nu}(x^A) \) as in Eq. (3.1). The linearized Einstein equations are also given by Eq. (3.3). In order to solve the equations, we separate the variables as

\[
h_{\mu\nu}(x^A) = \sum_{m,k,\ell} h^{(m\ell)}_{\mu\nu}(x^A)^{m,\ell}_\mu(x^\alpha)\phi_{\ell\rho}(r)\chi_{\ell\chi}(z)e^{im\theta},
\]

where \( \ell \) is an integer angular momentum. Then, the wave equation is separated into

\[
-\eta^\alpha^\beta \partial_\alpha \partial_\beta h^{(m\ell)}_{\mu\nu}(x^A) = m^2 h^{(m\ell)}_{\mu\nu}(x^A),
\]

\[
-\partial_r \left( \sigma^2 \partial_r \phi_{nk}(r) \right) + k^2 \sigma^2 \phi_{nk}(r) = m^2 \sigma \phi_{nk}(r),
\]

\[
-\partial_j \left( \Omega^{n-1} \partial_j \chi_{nk}(z) \right) + \frac{\ell^2}{\Omega^{2n} R_0^2} \Omega^{n-1} \chi_{nk}(z) = k^2 \Omega^{n+1} \chi_{nk}(z),
\]

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where $\partial_j$ denotes partial differentiation with respect to $z_j$, and two eigenvalues $k^2$ and $m^2$ are introduced. It is worth emphasizing that the mode equation $\text{(4.20)}$ for $\phi_{(mk)}(r)$ is identical to the previous case $\text{(3.6)}$. The differential operator in Eqs.$\text{(4.21)}$ is self-adjoint if we impose the following boundary conditions††,

\[ \partial_j \chi_{(k\ell)}(z) \big|_{z_j=0} = \partial_j \chi_{(k\ell)}(z) \big|_{z_j=\infty} = 0 . \]

With these conditions we can require the mode $\chi_{(k\ell)}(z)$ to satisfy the orthonormal condition

\[ \int_0^\infty d^n z \Omega^{n+1}(z) \chi^*_\ell(z) \chi_{k\ell}(z) = \delta_{kk'} . \]

Then we obtain from Eq.$\text{(4.21)}$

\[ k^2 = \int_0^\infty d^n z \Omega^{n-1} \left| \partial_j \chi_{(k\ell)} \right|^2 + \frac{\ell^2}{R_0^2} \int_0^\infty d^n z \Omega^{n-1} \left| \chi_{(k\ell)} \right|^2 \geq 0 . \]

The equality holds if and only if $\ell = 0$ and $\chi_{(00)}(z)$ is the zero mode (independent of $z_j$). For $\phi_{(mk)}(r)$, the boundary conditions $\text{(3.13)}$, orthonormal condition $\text{(3.14)}$, and positivity condition $\text{(3.15)}$ still apply as before. Therefore $m^2 \geq 0$ and the equality holds if and only if $k = 0$ and $\phi_{(00)}(r)$ is a zero mode (independent of $r$). These results tell us that there is only one massless spin 2 mode $m = k = \ell = 0$ of graviton, $h_{\mu\nu}^{(00)}(x^\alpha)$. The zero mode wave function in the extra dimensions is

\[ \chi_{(00)} = \text{const.} , \quad \phi_{(00)} = \text{const.} , \]

The zero mode is localized on the four-dimensional intersection region, since the probability distribution in the extra dimensions is strongly peaked around the intersection $r = 0, z_j = 0$ similarly to the case in the previous section. Thus the zero mode corresponds to the massless graviton localized at the four-dimensional intersection. Moreover, we can at least work out all the massive graviton fields $h_{\mu\nu}^{(m00)}(x^\alpha)$ with $k = \ell = 0$, since we know the zero mode $\chi_{(00)}(z)$ for $k = \ell = 0$ and we have already solved the mode equation for $\phi_{(m0)}(r)$ in the previous section. Therefore we find that there is a continuum of massive four-dimensional fields $h_{\mu\nu}^{(m00)}(x^\alpha)$ starting from $m = 0$ similarly to the previous case (and to the original model in Ref.$\text{(11)}$), even though we have not worked out the full spectrum of massive modes corresponding to nonvanishing values of $k$.

The exchange of the massless four-dimensional graviton between two sources $\bar m_1, \bar m_2$ on the four-dimensional intersection with a large distance $R$ gives the usual Newton potential $\text{(3.35)}$

†† We choose the Neumann boundary condition by the same reason as the $n = 1$ case in the previous section. We should have in mind that this boundary condition at $z_j = \infty$ may be modified when the strong gravity effects are taken into account.
also in the present case of $n$ walls. Although we have not worked out the massive modes with nonvanishing $k$ in detail, we have at least corrections coming from the continuum of massive fields $h_{\mu\nu}^{(m00)}(x^a)$ starting from $m = 0$. Since these wave functions are identical to the previous case of the seven-dimensional spacetime with $n = 1$, we find that there is at least the same correction term $\Delta V(R)$ for the Newton potential as the previous $n = 1$ case (3.38). If it turns out that there is a gap in the spectrum of $k$ as in the $n = 1$ case, all the additional contributions are exponentially suppressed. Then the above correction term is the leading contribution to the correction of the Newton’s law of gravitational forces.

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