NON-SEMISTABLE EXCEPTIONAL OBJECTS IN HEREDITARY CATEGORIES

GEORGE DIMITROV AND LUDMIL KATZARKOV

Abstract. For a given stability condition $\sigma$ on a triangulated category we define a $\sigma$-exceptional collection as an Ext-exceptional collection, whose elements are $\sigma$-semistable with phases contained in an open interval of length one. If there exists a full $\sigma$-exceptional collection, then $\sigma$ is generated by this collection in a procedure described by E. Macrì.

Constructing $\sigma$-exceptional collections of length at least three in $D^b(A)$ from a non-semistable exceptional object, where $A$ is a hereditary hom-finite abelian category, we introduce certain conditions on the Ext-nontrivial couples (couples of exceptional objects $X,Y \in A$ with $\text{Ext}^1(X,Y) \neq 0$, $\text{Ext}^1(Y,X) \neq 0$).

After a detailed study of the exceptional objects of the quivers $Q_1 = \circ \bullet \circ \bullet \circ$, $Q_2 = \circ \circ \circ \bullet \bullet$ we observe that the needed conditions do hold in $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$.

Combining these findings, we prove that for each $\sigma \in \text{Stab}(D^b(Q_1))$ there exists a full $\sigma$-exceptional collection. It follows that $\text{Stab}(D^b(Q_1))$ is connected.

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1. Introduction

T. Bridgeland introduced in his seminal work [1] the definition of a locally finite stability condition on a triangulated category $\mathcal{T}$, motivated by the work of Douglas on II-stability for Dirichlet branes. He proved that the set of these stability conditions is a complex manifold, denoted by $\text{Stab}(\mathcal{T})$.

Bridgeland’s axioms imply that $\text{Stab}((E)) = \mathbb{C}$ for an exceptional object $E$ in $\mathcal{T}$. The guiding motivation of this paper is the study of $\text{Stab}((E_1, E_2, \ldots, E_n))$, where $(E_1, \ldots, E_n)$ is an exceptional collection in $\mathcal{T}$ and $n \geq 2$. This study was initiated by E. Macrì in [12]. Here, we proceed further.

Collins and Polishchuk defined and studied in [4] a gluing procedure for Bridgeland stability conditions in the situation when $\mathcal{T}$ has a semiorthogonal decomposition $\mathcal{T} = \langle A_1, A_2 \rangle$.

1.1. T. Bridgeland constructed a stability condition $\sigma \in \text{Stab}(\mathcal{T})$ from a bounded t-structure $A \subset \mathcal{T}$ and a stability function $Z : K(A) \to \mathbb{C}$ satisfying certain restrictions. Keeping $A$ fixed and varying $Z$ produces a family of stability conditions, which we denote by $\mathbb{H}^A \subset \text{Stab}(\mathcal{T})$. E. Macrì proved in [12] Lemma 3.14, using results of [3], that the extension closure $A_\mathcal{E}$ of a full Ext-exceptional collection $\mathcal{E} = \langle E_0, E_1, \ldots, E_n \rangle$ in $\mathcal{T}$ is a heart of a bounded t-structure, and for each $\sigma \in \mathbb{H}^{A_\mathcal{E}}$ the objects $E_0, E_1, \ldots, E_n$ are $\sigma$-stable with phases in $(0, 1]$. Motivated by this result, for a given $\sigma \in \text{Stab}(\mathcal{T})$ we define a $\sigma$-exceptional collection (Definition 3.19) as an Ext-exceptional collection $\mathcal{E} = \langle E_0, E_1, \ldots, E_n \rangle$, s. t. the objects $\{E_i\}_{i=0}^n$ are $\sigma$-semistable, and $\{\phi(E_i)\}_{i=0}^n \subset (t, t+1)$ for some $t \in \mathbb{R}$. It follows easily from [12] Lemmas 3.14, 3.16 that for any full Ext-exceptional collection $\mathcal{E}$ the set $\{\sigma \in \text{Stab}(\mathcal{T}) : \mathcal{E} \text{ is } \sigma\text{-exceptional}\}$ coincides with $\mathbb{H}^{A_\mathcal{E}} \cdot \text{GL}^+ (2, \mathbb{R})$ (Corollary 3.20).

---

1. For a subset $S \subset \text{Ob}(\mathcal{T})$ we denote by $\langle S \rangle \subset \mathcal{T}$ the triangulated subcategory of $\mathcal{T}$ generated by $S$.

2. I.e. $Z$ is homomorphism $K(A) \xrightarrow{Z} \mathbb{C}$, s. t. $Z(X) \in \mathbb{H} = \{r \exp(i \pi t) : r > 0 \text{ and } 0 < t \leq 1\}$ for $X \in A \setminus \{0\}$.

3. An exceptional collection $\mathcal{E} = \langle E_0, E_1, \ldots, E_n \rangle$ is said to be Ext-exceptional if $\forall i \neq j \text{ Hom}^{\leq 0}(E_i, E_j) = 0$.

4. Recall that $\text{Stab}(\mathcal{T})$ carries a right action by $\text{GL}^+ (2, \mathbb{R})$. 

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Appendix B. The Kronecker quiver

B.1. There are not Ext-nontrivial couples in $\text{Rep}_k(K(l))$

B.2. $\sigma$-exceptional pairs in $D^b(K(l))$

References
E. Macrì, studying \( \text{Stab}(D^b(K(l))) \) in \([12]\), gave an idea for producing a \( \sigma \)-exceptional pair in \( D^b(K(l)) \) from a non-semistable exceptional object, where \( K(l) \) is the \( l \)-Kronecker quiver.

Throughout sections 4, 5, 6, 7, 8, 9 we develop tools for constructing \( \sigma \)-exceptional collections of length at least three in \( D^b(\mathcal{A}) \), where \( \mathcal{A} \) is a hereditary hom-finite abelian category. Combining them with the findings of Section 2 about \( \text{Rep}_k(Q_1) \) we prove in Section 10 the following theorem:

**Theorem 1.1.** Let \( Q_1 \) be the quiver \( \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \). Let \( k \) be an algebraically closed field. For each \( \sigma \in \text{Stab}(D^b(\text{Rep}_k(Q_1))) \) there exists a full \( \sigma \)-exceptional collection.

**Theorem 1.1** is one novelty of this paper. In particular, it implies that \( \text{Stab}(D^b(\text{Rep}_k(Q_1))) \) is connected (Corollary 10.2).

The \( K(l) \)-analogue of Theorem 1.1 (Lemma B.1) is already treated by E. Macrì in \([12]\) Lemma 4.2 on p.10. For the sake of completeness, we add a proof of this analogue in Appendix B.2.

The proof of **Theorem 1.1** is more complicated than of its \( K(l) \)-analogue not only because the full collections are triples instead of pairs, but also due to the presence of Ext-nontrivial couples in \( \text{Rep}_k(Q_1) \). We circumvent this difficulty by observing remarkable patterns, which the Ext-nontrivial couples obey. These patterns and the notion of regularity-preserving hereditary category, which they imply, are other novelties of the paper.

1.2. We explain now the organization of the paper and give details about the intermediate results.

Here, by \( \mathcal{A} \) we denote a \( k \)-linear hom-finite hereditary abelian category, where \( k \) is an algebraically closed field, and we denote \( D^b(\mathcal{A}) \) by \( \mathcal{T} \).

In Section 4, we analyze the following data: an exceptional object \( E \in D^b(\mathcal{A}) \), which is not \( \sigma \)-semistable for a given stability condition \( \sigma \in \text{Stab}(\mathcal{T}) \). Macrì initiated such an analysis in \([12]\) p.10. We end up in Section 4 with a distinguished triangle, denoted by \( \text{alg}(E) \), which satisfies one of five possible lists of properties, named \( C_1, C_2, C_3, B_1, B_2 \). If the resulting list is one of \( C_1, C_2 \) or \( C_3 \), then we say that the object \( E \) is \( \sigma \)-regular, otherwise - \( \sigma \)-irregular. The triangle \( \text{alg}(R) = \begin{array}{c} V \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} U \end{array} \begin{array}{c} R \end{array} \) of a \( \sigma \)-regular \( R \) has the feature that for any indecomposable components \( S \) and \( E \) of \( V \) and \( U \), respectively, the pair \( (S, E) \) is exceptional with semistable first element \( S \). We denote this relation between a \( \sigma \)-regular object \( R \) and the exceptional pair \( (S, E) \) by \( R \longrightarrow (S, E) \), where \( X \) contains further information as explained in Section 5. This feature is not available in the irregular cases \( B_1 \) and \( B_2 \), and the obstruction to obtaining it are the Ext-nontrivial couples. Such couples exist in \( \text{Rep}_k(Q_1) \) and \( \text{Rep}_k(Q_2) \), as shown in Section 2. Essential part of our efforts concerns the Ext-nontrivial couples. It follows from \([12]\) Lemma 4.1 that there are not such couples in \( \text{Rep}_k(K(l)) \) (Appendix B.1).

Thus, in Sections 4, 5 from each \( \sigma \)-regular exceptional object \( R \) we obtain at least one exceptional pair \( (S, E) \) with \( R \longrightarrow (S, E) \). The first component \( S \) in such a pair is always semistable. If the second component \( E \) is not semistable, which is possible iff \( R \) is non-final as defined in Definition 5.3, then it is natural to ask: Is \( E \) a \( \sigma \)-regular exceptional object?

Motivated by this question, we introduce in Section 6 certain conditions on the Ext-nontrivial couples of \( \mathcal{A} \), which we call **RP property 1 and RP property 2** (Subsection 6.2), and using them we

\footnote{These are couples of exceptional objects \( X, Y \) with \( \text{Ext}^1(X, Y) \neq 0 \), \( \text{Ext}^1(Y, X) \neq 0 \) (Definition 6.2).}
give a positive answer. We say that \( \mathcal{A} \) is a \textit{regularity-preserving category} (Definition 6.1), when the answer is positive. RP properties 1, 2 themselves are not important for the rest of the paper, but that \( \mathcal{A} \) is regularity-preserving, which follows from them.

Whence, in regularity-preserving category \( \mathcal{A} \) the relation \( \cdots \) circumvents the irregular objects, and each non-final \( \sigma \)-regular object \( R \) generates a long sequence\(^6\) of the form:

\[
R \xrightarrow{X_1} (S_1, E_1) \xrightarrow{proj_1} E_1 \xrightarrow{proj_2} (S_2, E_2) \xrightarrow{proj_1} E_2 \xrightarrow{proj_2} (S_3, E_3) \xrightarrow{proj_1} \cdots
\]

(1)

In such a sequence, which we call an \( R \)-sequence, the exceptional objects \( S_1, S_2, \ldots \) are all semistable, and furthermore, if \( E_n \) is final for some \( n \), then, by the very definition of a final object (Definition 5.3), the pair \((S_{n+1}, E_{n+1})\) is semistable and exceptional.

In Section 7 we proceed further in direction \( \sigma \)-exceptional collections by refining on the phases and the degrees of \( \{S_i\} \), and showing various situations, in which the vanishings \( \text{Hom}^\bullet(S_i, S_1) = 0 \) hold for \( i > 1 \). However, these vanishings do not hold in each \( R \)-sequence. Nevertheless, we show that starting from any \( \sigma \)-regular \( R \) through any \( R \)-sequence we reach a final \( \sigma \)-regular object \( E_n \) for some \( n \geq 1 \).

After a careful examination of the final \( \sigma \)-regular objects, in Section 8 we find that an exceptional pair \((S, E)\) produced from such an object is not only semistable, but also \((S, E[−i])\) is a \( \sigma \)-exceptional pair for some \( i \geq 0 \) (e.g., a situation as: \( \phi(S) = \phi(E) \), \( \text{Hom}(S, E) \neq 0 \) cannot happen).

The proofs in Sections 7 and 8 are facilitated by the use of a function \( \theta_\sigma : \text{Ob}(\mathcal{T}) \to \mathbb{N}(\sigma^\ast_{\text{ind}}/\cong) \), introduced in subsection 3.2. For an object \( X \in \text{Ob}(\mathcal{T}) \) the function \( \theta_\sigma(X) : \sigma^\ast_{\text{ind}}/\cong \to \mathbb{N} \) indicates (with multiplicities) the indecomposable components of the Harder-Narasimhan factors of \( X \). The \( X \)-relation \( R \xrightarrow{\cdots} (S, E) \) implies \( \theta_\sigma(E) < \theta_\sigma(R) \) and \( \theta_\sigma(R)(S) > 0 \). This feature gives an upper bound of the lengths of all \( R \)-sequences with a fixed \( R \). It also plays a role in avoiding some situations as the mentioned in the end of the previous paragraph.

In Section 2 we classify exceptional objects of the categories \( \text{Rep}_k(Q_1), \text{Rep}_k(Q_2) \). After that we obtain tables with dimensions of \( \text{Hom}(X, Y), \text{Ext}^1(X, Y) \) for any two exceptional objects \( X, Y \), and observe that one of these always vanishes. RP property 1 and RP property 2 follow by a careful analysis of these tables. For the Ext-non-trivial couples of the quiver \( Q_1 \) we observe an additional pattern: Corollary 2.7 which helps us further to avoid the irregular cases. We refer to it as the additional RP property. It does not hold in \( Q_2 \). In the end of Subsection 2.2 we obtain the lists of all exceptional pairs and triples in \( \text{Rep}_k(Q_1) \).

The results before Section 9 contain the implications (the first is due to regularity-preserving):

\( \sigma \)-regular object \( \Rightarrow \) final \( \sigma \)-regular object \( \Rightarrow \) \( \sigma \)-exceptional pair (Corollary 8.3 and Remark 8.4).

In Section 9 we develop various criteria for existence of \( \sigma \)-exceptional triples in \( D^b(\mathcal{A}) \), assuming that the exceptional objects of \( \mathcal{A} \) obey the global properties observed for \( \text{Rep}_k(Q_1) \) in Section 2. It is shown that any non-final \( C_2 \) or \( C_3 \) object induces such a triple. Thus, if \( R \) is a \( C_2 \) or \( C_3 \) object, then any \( R \)-sequence of length two produces a \( \sigma \)-exceptional triple. If \( R \) is a \( C_1 \) object, then our results imply that any \( R \)-sequence of length three is enough, but for length less or equal to two - only under special circumstances (Lemmas 9.8, 9.13, Corollary 9.11).
If \( R \) is a final \( \sigma \)-regular object, then we have no long \( R \)-sequences, they are all of length one and each of them induces a \( \sigma \)-exceptional pair. To obtain a \( \sigma \)-triple in this case we apply two ideas. The first is to combine the pairs coming from different \( R \)-sequences, which leads to the result that a final \( \sigma \)-regular object \( R \) whose Harder-Narasimhan filtration differs from \( \text{alg}(R) \) induces a \( \sigma \)-exceptional triple. The other idea is to utilize the infimum \( \phi_{\text{min}} \) and the supremum \( \phi_{\text{max}} \) of the set of phases of semistable exceptional objects in \( \cal A \). More precisely, we show that a relation \( R \longrightarrow\ (S[1],E) \) with a final \( C_3 \) object \( R \in \cal A \) and \( \phi(S) > \phi_{\text{min}} \) induces a \( \sigma \)-triple (Corollary 9.7). There is an analogous criterion using a final \( C_2 \) object \( R \in \cal A \) and \( \phi_{\text{min}} \), shown in Corollary 9.10, but there is not an analogue for final \( C_1 \) objects (Lemma 9.13 uses a non-final \( C_1 \) object and in different setting). When \( \phi_{\text{max}} - \phi_{\text{min}} > 1 \), we show that, if \( (S_{\text{min}},E,S_{\text{max}}) \) is an exceptional triple in \( \cal A \) with \( S_{\text{min}} \in \cal P(\phi_{\text{min}}) \) and \( S_{\text{max}} \in \cal P(\phi_{\text{max}}) \), then non-semistability of \( E \) (no matter regular or irregular) implies a \( \sigma \)-exceptional triple. The last is widely used in Subsection 10.3.

The criteria obtained in Section 9 combined with the lists of the exceptional pairs and the exceptional triples of \( \text{Rep}_k(Q_1) \) at our disposal (due to Section 2) turn out to be enough for the proof of the main Theorem 1.1, which is demonstrated in Section 10. The locally finiteness of the stability condition \( \sigma \in \text{Stab}(T) \) plays an important role as well. The proof is divided into two steps: \( \phi_{\text{max}} - \phi_{\text{min}} > 1 \) and \( \phi_{\text{max}} - \phi_{\text{min}} \leq 1 \).

1.3. The following three statements are proved in [7]. In the first and the second statement, \( Q \) is an acyclic Euclidean quiver:

(a) [7, Corollary 3.15]: For each \( \sigma \in \text{Stab}(D^b(Q)) \) the set of semistable phases is either finite or has two limit points in \( S^1 \).

(b) [7, Corollary 3.31]: For any exceptional pair \((A,B)\) in \( D^b(Q) \) and any \( i \in \mathbb{Z} \) holds the inequality \( \text{hom}^i(A,B) \leq 2 \).

(c) [7, Proposition 3.32]: Any connected quiver \( Q \), which is neither Euclidean nor Dynkin, has a family of stability conditions with phases which are dense in an arc. The proof of this fact relies on extendability, as defined in [7, Definition 3.25], of certain stability conditions on a subcategory of \( D^b(Q) \) to the entire \( D^b(Q) \) (the precise setting is described right after Theorem 3.27 in [7]).

We construct in Subsection 3.3 stability conditions \( \sigma \in \text{Stab}(D^b(Q_1)) \) with two limit points in \( S^1 \), concerning (a).

In Remark 2.11 we point out exceptional pairs \((A,B)\) in \( D^b(Q_1) \) with \( \text{hom}^i(A,B) = 2 \), concerning (b).

In Subsection 3.4 we comment on the stability conditions constructed by E. Macrì [12] via exceptional collections. By slightly modifying the statement of [12, Proposition 3.17] and refining its proof is obtained Proposition 3.17 which provides the extendability needed in (c).

1.4. It is known [5] that the Braid group acts transitively on the exceptional collections of \( \text{Rep}_k(Q_1) \). The list of these collections shows that this action is not free (Remark 2.12).

1.5. This paper gives a few answers, and poses many questions. We expect that there is a proof of Theorem 1.1 governed by a general principle. The notion of regularity-preserving hereditary category (Definition 6.1) should be related to this principle. RP property 1 and RP property 2 are our method to prove regularity-preserving. The fact that they hold not only in \( \text{Rep}_k(Q_1) \), but also
in \( \text{Rep}_k(Q_2) \) (Corollary 2.6) seems to be a trace of a larger unexplored picture. We expect that there are further non-trivial examples of regularity-preserving categories.

We do not give an answer to the question: is there a \( \sigma \)-exceptional quadruple for each \( \sigma \in \text{Stab}(D^b(Q_2)) \) (the \( Q_2 \)-analogue of Theorem 1.1). We show that \( \text{Rep}_k(Q_2) \) is regularity-preserving, and the results of Sections 7, 8, and Subsection 9.1 hold for \( \text{Rep}_k(Q_2) \) entirely. These are clues for a positive answer (see especially Corollary 8.5). In section 2 we give the dimensions of \( \text{Hom}(X,Y) \), \( \text{Ext}^1(X,Y) \) for any two exceptional objects \( X,Y \) in \( Q_2 \) as well. This lays a ground for working on the \( Q_2 \)-analogue of Theorem 1.1.

We expect that the results in Section 2 and Theorem 1.1 can be used for the study of the topology of \( \text{Stab}(D^b(\text{Rep}_k(Q_1))) \) further (e. g. to check its simply-connectivity).

Some notations. In these notes the letters \( T \) and \( A \) denote always a triangulated category and an abelian category, respectively, linear over a field \( k \), the shift functor in \( T \) is designated by \([1]\). We write \( \text{Hom}^i(X,Y) \) for \( \text{Hom}(X,Y[i]) \) and \( \text{hom}^i(X,Y) \) for \( \dim_k(\text{Hom}(X,Y[i])) \), where \( X,Y \in T \). For \( X,Y \in A \), writing \( \text{Hom}^i(X,Y) \), we consider \( X,Y \) as elements in \( T = D^b(A) \), i.e. \( \text{Hom}^i(X,Y) = \text{Ext}^i(X,Y) \).

We denote by \( A_{\text{exc}} \), resp. \( D^b(A)_{\text{exc}} \), the set of all exceptional objects of \( A \), resp. of \( D^b(A) \).

An abelian category \( A \) is said to be hereditary, if \( \text{Ext}^i(X,Y) = 0 \) for any two \( X,Y \in A \) and each \( i \geq 2 \).

For an object \( X \in D^b(A) \) of the form \( X \cong X'[j] \), where \( X' \in A \) and \( j \in \mathbb{Z} \), we write \( \text{deg}(X) = j \).

For any quiver \( Q \) we write \( D^b(Q) \) for \( D^b(\text{Rep}_k(Q)) \).

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2. On the Ext-nontrivial couples of some hereditary categories

In Sections 6, 7, 8, 9 we treat hereditary abelian categories whose exceptional objects are supposed to obey specific pairwise relations. In this section we give examples of such categories.

2.1. The categories. For any finite quiver \( Q \) and an algebraically closed field \( k \) we denote the category of \( k \)-representations of \( Q \) by \( \text{Rep}_k(Q) \). It is well known that \( \text{Rep}_k(Q) \) is a hom-finite hereditary \( k \)-linear abelian category (see e. g. [6]).

In this section we classify the exceptional objects of the categories of representations of the following quivers:

\[
Q_1 = \begin{array}{c}
0 \\
\text{○} \\
\text{○}
\end{array} \quad Q_2 = \begin{array}{c}
\text{○} \\
\text{○} \\
\text{○}
\end{array}
\]

\[
(2)
\]

\( \text{in some sections algebraically closedness of } k \text{ is not important, but overall this feature is necessary.} \)
After that we compute the dimensions of $\text{Hom}(X,Y)$, $\text{Ext}^1(X,Y)$ for any two exceptional objects $X,Y$. The obtained information reveals some patterns, which are of importance for the rest of the paper.

More precisely, Corollary 2.6(a) claims that $\text{Rep}_k(Q_1)$ and $\text{Rep}_k(Q_2)$ have RP property 1 and RP property 2 (see subsection 5.2 for definition). These properties ensure that $\text{Rep}_k(Q_1)$ and $\text{Rep}_k(Q_2)$ are regularity-preserving (Definition 6.1, Proposition 7.6), which is of primary importance for Sections 7, 9.

In the end of Section 7 and in Section 9 the property that for any two exceptional objects $X,Y$ at most one of the spaces $\text{Hom}(X,Y)$, $\text{Ext}^1(X,Y)$ is nonzero plays an important role. Corollary 2.6(b) asserts that this property holds for both the quivers $Q_1, Q_2$.

For $Q_1$ we observe the additional RP property (see Corollary 2.7), used in Subsection 9.2. In the end we obtain the lists of exceptional pairs and exceptional triples in $\text{Rep}_k(Q_1)$, which are widely used in Section 10.

We give now more details.

2.2. The dimensions $\text{hom}(X,Y), \text{hom}^1(X,Y)$ for $X,Y \in \text{Rep}_k(Q_i)_\text{exc}$ and $i \in \{1,2\}$

For a representation $\rho = \xymatrix{ k^{\alpha+} \ar[r]^{\rho} & k^{\alpha_1} \ar[r]^1 & k^{\alpha_2} \ar[r] & \cdots \ar[r] & k^{\alpha_n} } \in \text{Rep}_k(Q_2)$, where $\alpha_0, \alpha_-, \alpha_+, \alpha_e \in \mathbb{N}$, we denote its dimension vector by $\text{dim}(\rho) = (\alpha_0, \alpha_-, \alpha_+, \alpha_e)$ and for a representation $\rho = \xymatrix{ k^{\alpha_0} \ar[r] & k^{\alpha_1} \ar[r] & \cdots \ar[r] & k^{\alpha_{n-1}} \ar[r] & k^{\alpha_n} } \in \text{Rep}_k(Q_1)$ we denote $\text{dim}(\rho) = (\alpha_0, \alpha_{mid}, \alpha_e)$. The Euler forms of $Q_1, Q_2$ are:

\[
\langle (\alpha_b, \alpha_{mid}, \alpha_e), (\alpha'_b, \alpha'_{mid}, \alpha'_e) \rangle = \alpha_b \alpha'_b + \alpha_{mid} \alpha'_{mid} + \alpha_e \alpha'_e - \alpha_b \alpha'_e - \alpha_b \alpha'_{mid} - \alpha_{mid} \alpha'_e,
\]

\[
\langle (\alpha_b, \alpha_-, \alpha_+, \alpha_e), (\alpha'_b, \alpha'_-, \alpha'_+, \alpha'_e) \rangle = \alpha_+ \alpha'_e + \alpha_- \alpha'_e - \alpha_b \alpha'_{mid} - \alpha_{mid} \alpha'_e - \alpha_b \alpha'_{mid} - \alpha_{mid} \alpha'_e - \alpha_b \alpha'_{mid} - \alpha_{mid} \alpha'_e.
\]

Recall (see page 8 in [6]) that for any $\rho, \rho' \in \text{Rep}_k(Q)$ we have the formula

\[
\text{hom}(\rho, \rho') - \text{hom}^1(\rho, \rho') = \langle \text{dim}(\rho), \text{dim}(\rho') \rangle.
\]

In particular, it follows that if $\rho \in \text{Rep}_k(Q)$ is an exceptional object, then $\langle \text{dim}(\rho), \text{dim}(\rho) \rangle = 1$.

The vectors satisfying this equality are called real roots (see [6], p. 17). For example, one can show that the real roots of $Q_1$ are $(m + 1, m, m), (m, m + 1, m + 1), (m, m, m + 1), (m + 1, m + 1, m), (m + 1, m, m + 1), (m, m, m + 1), m \geq 0$. The imaginary roots\(^{9}\) of $Q_1$, are $(m, m, m), m \geq 1$. Not every real root is a dimension vector of an exceptional representation. More precisely:

**Lemma 2.1.** Let $m \geq 1$. If $(\alpha_0, \alpha_{mid}, \alpha_e) \in \{(m + 1, m, m), (m, m + 1, m)\}_{m \in \mathbb{N}}$, then $(\alpha_0, \alpha_{mid}, \alpha_e)$ is not dimension vector of any exceptional representation in $\text{Rep}_k(Q_1)$. If $(\alpha_0, \alpha_-, \alpha_+, \alpha_e) \in \{(m, m + 1, m, m), (m, m + 1, m), (m + 1, m, m + 1, m + 1), (m + 1, m + 1, m, m + 1)\}_{m \in \mathbb{N}}$, then $(\alpha_0, \alpha_-, \alpha_+, \alpha_e)$ is not dimension vector of any exceptional representation in $\text{Rep}_k(Q_2)$.

**Sketch of proof.** For the proof of this lemma one can use (see [6], Lemma 1 on page 13) that a representation $\rho \in \text{Rep}_k(Q_1)$ is without self-extensions iff $\text{dim}(\rho) = \text{dim}(\text{Rep}_k(Q_1))$, where $\mathcal{O}_\rho$ is the orbit of $\rho$ in $\text{Rep}_k(Q_1)$ as defined in [6] page 11,12. Using this argument, it can be shown that

\(^{9}\)Imaginary root is a vector $\rho$ with $\langle \text{dim}(\rho), \text{dim}(\rho) \rangle \leq 0$. 
any representation without self-extensions with dimension vector among the listed in the lemma is decomposable.

Now we classify the exceptional objects on $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$ (Propositions 2.2 and 2.3). In these propositions we use the following notations for any $m \geq 1$:

$$\pi^m_+ : k^{m+1} \to k^m, \quad \pi^-_m : k^{m+1} \to k^m, \quad j^m_+ : k^m \to k^{m+1}, \quad j^-_m : k^m \to k^{m+1}$$

$$\pi^m_+ (a_1, a_2, \ldots, a_m, a_{m+1}) = (a_1, a_2, \ldots, a_m) \quad \pi^-_m (a_1, a_2, \ldots, a_m, a_{m+1}) = (a_2, \ldots, a_m, a_{m+1})$$

$$j^m_+ (a_1, a_2, \ldots, a_m) = (a_1, a_2, \ldots, a_m, 0) \quad j^-_m (a_1, a_2, \ldots, a_m) = (0, a_1, \ldots, a_m).$$

**Proposition 2.2.** The exceptional objects up to isomorphism in $\text{Rep}_k(Q_1)$ are $(m = 0, 1, 2, \ldots)$

**Sketch of proof.** We showed that the dimension vectors of the exceptional representations are real roots. The list of real roots is given in Lemma 2.1 and some of them are excluded in Lemma 2.1. Moreover, there is at most one representation without self-extensions of a given dimension vector up to isomorphism [6, p. 13]. Taking into account these arguments, the proposition follows by showing that the endomorphism space of each of the listed representations is $k$ (recall also [3]). The computations, which we skip, are reduced to table 1110 in Appendix A. 

**Proposition 2.3.** The exceptional objects up to isomorphism in $\text{Rep}_k(Q_2)$ are $(m = 0, 1, 2, \ldots)$

**Sketch of proof.** The same as Proposition 2.2.

Now we compute $\text{hom}(\rho, \rho')$, $\text{hom}^1(\rho, \rho')$ with $\rho, \rho'$ varying throughout the obtained lists.
Proposition 2.4. The dimensions of the vector spaces \( \text{Hom}(X, Y) \) and \( \text{Hom}^1(X, Y) \) for any pair of exceptional objects \( X, Y \in \text{Rep}_k(Q_1) \) are contained in the following table:

| \( m \leq n \) | \( \text{hom} \) | \( \text{hom}^1 \) | \( \text{hom} \) | \( \text{hom}^1 \) |
|----------------|----------------|----------------|----------------|----------------|
| \( 0 \leq m < n \) | \( (E_{m}^n, E_{m}^n) \) | 0 | \( n - m - 1 \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( n - m \) | 0 |
| \( 0 \leq n < m \) | \( (E_{m}^n, E_{m}^n) \) | 0 | \( m - n - 1 \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( m - n \) | 0 |
| \( m \geq n \) | \( (E_{m}^n, E_{m}^n) \) | 0 | \( n - m - 1 \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( m - n \) | 0 |

Sketch of proof. Via computations, which we do not write out here, we obtain \( \text{hom}(\rho, \rho') \) for any two representations \( \rho, \rho' \) taken from Proposition 2.2. The computations are reduced to determining the dimensions of some vector spaces of matrices. These spaces and their dimensions are listed in Appendix A, Table 110. Having \( \text{hom}(\rho, \rho') \), the dimension \( \text{hom}^1(\rho, \rho') \) is computed by (4).

Proposition 2.5. The dimensions \( \text{hom}(X, Y) \) and \( \text{hom}^1(X, Y) \) for any pair of exceptional objects \( X, Y \in \text{Rep}_k(Q_2) \) are contained in the following table:

| \( m \leq n \) | \( \text{hom} \) | \( \text{hom}^1 \) | \( \text{hom} \) | \( \text{hom}^1 \) |
|----------------|----------------|----------------|----------------|----------------|
| \( 0 \leq n \leq n \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( m - n \) | 0 | \( (E_{m}^n, E_{m}^n) \) | 0 | \( n - m - 1 \) |
| \( 0 \leq m < n \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( m - n \) | 0 | \( (E_{m}^n, E_{m}^n) \) | 0 | \( n - m - 1 \) |
| \( m \leq n \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( m - n \) | 0 | \( (E_{m}^n, E_{m}^n) \) | 0 | \( n - m - 1 \) |
| \( m \leq n \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( m - n \) | 0 | \( (E_{m}^n, E_{m}^n) \) | 0 | \( n - m - 1 \) |
| \( m \leq n \) | \( (E_{m}^n, E_{m}^n) \) | 1 + \( m - n \) | 0 | \( (E_{m}^n, E_{m}^n) \) | 0 | \( n - m - 1 \) |
| $m$ | $n$ | $\text{hom}$ | $\text{hom}^*$ | $\text{hom}$ | $\text{hom}^*$ |
|-----|-----|-------------|-------------|-------------|-------------|
| $0 < m < n$ | $(E_m^k, E_n^k)$ | $m - n + 1$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $m - n + 1$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $n - m + 1$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $m - n$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $m - n$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $1 + m - n$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $1 + m - n$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $n - m$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $n - m$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $m + n$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |
| $0 < m, 0 < n$ | $(E_m^k, E_n^k)$ | $m + n$ | $0$ | $(E_m^k, E_n^k)$ | $0$ |

Sketch of proof. The table for $\text{Rep}_k(Q_2)$ is obtained by the same method as for $\text{Rep}_k(Q_1)$. □
The next subsection contains corollaries of the obtained tables.

2.3. The Ext-nontrivial couples and their properties. From the table in Proposition 2.4 we see that the only couple \( \{X, Y\} \) of exceptional objects in \( \text{Rep}_k(Q_1) \) satisfying \( \hom^1(X, Y) \neq 0 \) and \( \hom^1(Y, X) \neq 0 \) is \( \{M, M'\} \). We call such a couple an Ext-nontrivial couple (see Definition 6.2).

By Proposition 2.5 we see that the Ext-nontrivial couples in \( \text{Rep}_k(Q_2) \) are \( \{F_+, G_-, \} \), \( \{F_-, G_+\} \).

Corollary 2.6 concerns both \( \text{Rep}_k(Q_1) \) and \( \text{Rep}_k(Q_2) \).

**Corollary 2.6.** The categories \( \text{Rep}_k(Q_1) \), \( \text{Rep}_k(Q_2) \) satisfy the following properties:

(a) RP property 1, RP property 2 (see subsection 6.2 for description).

(b) For any two exceptional objects \( X, Y \in \text{Rep}_k(Q_1) \) at most one degree in \( \{\hom^p(X, Y)\}_{p \in \mathbb{Z}} \) is nonzero, where \( i = \{1, 2\} \).

**Proof.** It follows by a careful case by case check, using the tables in Propositions 2.4 2.5 \( \square \)

The following four corollaries concern only \( \text{Rep}_k(Q_1) \) and are contained in table (4).

**Corollary 2.7.** If \( \{\Gamma_1, \Gamma_2\} \) is an Ext-nontrivial couple in \( \text{Rep}_k(Q_1) \) (see Definition 6.2), then for each exceptional object \( X \in \text{Rep}_k(Q_1) \) we have \( \hom^p(\Gamma_i, X) \neq 0 \) for some \( i = \{1, 2\} \), \( p \in \mathbb{Z} \) and \( \hom^q(X, \Gamma_j) \) for some \( j = \{1, 2\} \), \( q \in \mathbb{Z} \).

**Corollary 2.8.** The exceptional pairs \( (X, Y) \) in \( \text{Rep}_k(Q_1) \) are \( (m \in \mathbb{N}) \):

\[
\begin{align*}
(E_1^{m+1}, E_1^m) & (E_2^m, E_2^{m+1}) (E_3^m, E_3^{m+1}) (E_4^m, E_4^m) (E_0^0, E_0^0) (E_1^0, E_3^0) \\
(E_1^0, E_1^m) & (E_1^0, E_1^m) (E_3^m, E_3^{m+1}) (E_2^m, E_2^{m+1}) (E_0^0, E_0^0) (E_1^0, E_3^0) \\
(E_2^m, M) & (M, E_3^m) (M, E_4^m) (M', E_1^m) (M', E_2^m) (E_3^m, M') (E_4^m, M')
\end{align*}
\]

Using this corollary we obtain the list of the exceptional triples of \( \text{Rep}_k(Q_1) \), which by [5] are the full exceptional collections.

**Corollary 2.9.** The full exceptional collections in \( \text{Rep}_k(Q_1) \) up to isomorphism are \( (m \in \mathbb{N}) \):

\[
\begin{align*}
(E_1^{m+1}, E_1^m, M) & (E_1^{m+1}, E_1^m, E_1^1) (E_1^{m+1}, M, E_4^m) \\
(E_1^0, E_1^m, M) & (E_1^0, E_1^m, E_1^1) (E_1^0, M, E_3^0) \\
(E_2^m, E_2^{m+1}, M) & (E_2^m, E_2^{m+1}, E_2^m) (E_2^m, M, E_4^m) \\
(E_3^m, E_3^{m+1}, M) & (E_3^m, E_3^{m+1}, M') (E_3^m, M', E_3^m) \\
(E_4^m, E_4^{m+1}, E_4^m) & (E_4^m, E_4^{m+1}, E_4^m) (E_4^m, M', E_4^m) \\
(E_0^0, E_2^m) & (E_0^0, E_2^m, M') (E_0^0, M', E_3^m) \\
(E_1^0, E_1^m, E_3^m) & (E_1^0, E_1^m, E_3^m) (E_1^0, M', E_3^m) \\
(M, E_3^m, E_3^{m+1}) & (M, E_4^m, E_4^m) (M, E_1^m, E_0^0) \\
(M', E_1^{m+1}, E_1^m) & (M', E_1^{m+1}, E_1^m) (M', E_1^m, E_0^0)
\end{align*}
\]

The following corollary is a special case of a result in [5]. It also follows from Corollary 2.9.

**Corollary 2.10.** Let \( (A_0, A_1, A_2), (A_0', A_1', A_2') \) be two exceptional triples in \( \text{Rep}_k(Q_1) \). If \( A_i \cong A_j' \), \( A_j \cong A_j' \) for two different \( i, j \in \{0, 1, 2\} \), then \( A_k \cong A_k' \) for the third \( k \in \{0, 1, 2\} \).

**Remark 2.11.** In [7] is shown that any exceptional pair \( (A, B) \) in \( D^b(Q) \) for an acyclic Euclidean quiver \( Q \) satisfies \( \hom^i(A, B) \leq 2 \). Among the pairs of \( \text{Rep}_k(Q_1) \) listed in Corollary 2.8 equality is attained in the following cases: \( 2 = \hom(E_1^{m+1}, E_1^m) = \hom(E_2^m, E_2^{m+1}) = \hom(E_3^m, E_3^{m+1}) = \hom(E_4^m, E_4^m) = \hom(E_1^0, E_2^0) = \hom(E_4^0, E_3^0) \).
Remark 2.12. From Corollaries 2.9 and 2.10 we see that the action of the Braid group $B_3$ on the exceptional collections of $\text{Rep}_k(Q_1)$ is not free. We give an example here.

Example of fixed triples by a Braid group element. For any exceptional triple $(A, B, C)$ we denote here the triple $\bullet^\ast (A, L_B(C), B)$ by $L_1(A, B, C)$. We keep in mind also Corollary 2.10 and that each exceptional object in $D^b(Q_1)$ is a shift of an exceptional object in $\text{Rep}_k(Q_1)$.

The first row in the list of Corollary 2.9 shows that, up to shifts, we have the equalities $L_1(E_1^{m+1}, M, E_4^m) = (E_1^{m+1}, E_1^m, M); L_1(E_1^{m+1}, E_1^m, M) = (E_1^{m+1}, E_4^m, E_1^m); L_1(E_1^{m+1}, E_4^m, E_1^m) = (E_1^{m+1}, M, E_4^m)$. Hence, the triple $(E_1^{m+1}, M, E_4^m)$ is fixed by $(L_1)^3$. The element $(L_1)^3$ is not trivial in the braid group $B_3$, since $B_3$ is torsion free.

Acting with $L_1$ on each of the rest rows, except the last two rows, we find the same behavior.

## 3. PRELIMINARIES

Here we comment on Bridgeland’s stability conditions and on Macri’s construction of stability conditions via exceptional collections.

In Subsection 3.2 for a Krull-Schmidt category $\mathcal{T}$, we introduce a function $\text{Ob}(\mathcal{T}) \xrightarrow{\theta} \mathbb{N}(\sigma_{ind}/\cong)$, depending on a stability condition $\sigma \in \text{Stab}(\mathcal{T})$. It helps us later to encode useful features of the relation $R \quad \xrightarrow{\bullet} \quad (S, E)$ in the simple expressions $\theta_{\sigma}(R) > \theta_{\sigma}(E), \theta_{\sigma}(R)(S) > 0$(see Section 5).

Lemma 3.3, based on the locally finiteness of the elements in $\text{Stab}(\mathcal{T})$, has an important role in Section 10. The simple fact observed in Lemma 3.6 used throughout Sections 6...10 is helpful in our study of long $R$-sequences.

Applying some results of Section 2 we obtain in Subsection 3.3 stability conditions on $D^b(Q_1)$ with two limit points in $S^1$, which concerns $\{\}$ table (1)]

After having recalled Macri’s construction in Subsection 3.4, we define in the final Subsection 3.5 the notion of a $\sigma$-exceptional collection.

### 3.1. Krull-Schmidt property

The function $\theta : \text{Ob}(\mathcal{C}) \to \mathbb{N}(\mathcal{C}_{ind}/\cong)$.

Let $\mathcal{C}$ be an additive category. We denote by $\mathcal{C}_{ind}$ the set of all indecomposable objects in $\mathcal{C}$. We discuss here the well known Krull Schmidt property.

**Definition 3.1.** We say that an additive category $\mathcal{C}$ has Krull-Schmidt property if for each $X \in \text{Ob}(\mathcal{C}) \setminus \{\}$ there exists unique up to isomorphism and permutation sequence $\{X_1, X_2, \ldots, X_n\}$ in $\mathcal{C}_{ind}$ with $X \cong \bigoplus_{i=1}^n X_i$.

For $X \in \text{Ob}(\mathcal{C}) \setminus \{\}$ with a decomposition $X \cong \bigoplus_{i=1}^n X_i$ as above we denote by $\text{Ind}(X)$ the set $\{Y \in \text{Ob}(\mathcal{C}) : Y \cong X_i \text{ for some } i = 1, 2, \ldots, n\}$. If $X$ is a zero object, then $\text{Ind}(X) = \emptyset$.

We will use two simple observations related to this property.

**Lemma 3.2.** Let $\mathcal{A}$ be a hereditary abelian category. If $\mathcal{A}$ has Krull-Schmidt property, then $D^b(\mathcal{A})$ has Krull-Schmidt property.

---

10 Recall that for any exceptional pair $(A, B)$ the exceptional objects $L_A(B)$ and $R_B(A)$ are determined by the triangles $L_A(B) \xrightarrow{\bullet} \text{Hom}^\ast(A, B) \otimes A \xrightarrow{\text{ev}_{A, B}} B$; $\xrightarrow{\text{coev}_{A, B}} \text{Hom}^\ast(A, B) \otimes B \xrightarrow{\bullet} R_B(A)$ and that $(L_A(B), A), (B, R_B(A))$ are exceptional pairs.

11 the set $\mathcal{C}_{ind}$ does not contain zero objects.
Proof. Recall that any object $X \in D^b(A)$ decomposes as follows $X \cong \bigoplus_i H^i(X)[-i]$ and if $X \cong \bigoplus_i X_i[-i]$ for some collection $\{X_i\} \subset A$, then $X_i \cong H^i(X)$ for all $i$. In particular $A$ is a thick subcategory of $D^b(A)$. Now the lemma follows. \hfill \Box

Lemma 3.3. Let $\mathcal{E}$ have Krull-Schmidt property. There exists unique function $\text{Ob}(\mathcal{E}) \xrightarrow{\theta} \mathbb{N}^{\mathcal{E}_{\text{ind}}/\cong}$ satisfying\textsuperscript{12}

(a) If $Y \cong \bigoplus_{i=1}^m Y_i$ in $\mathcal{E}$, then $\theta(Y) = \sum_{i=1}^m \theta(Y_i)$.

(b) For any $X \in \mathcal{E}_{\text{ind}}$ the function $\mathcal{E}_{\text{ind}}/\cong \xrightarrow{\theta(X)} \mathbb{N}$ assigns one to the equivalence class containing $X$, and zero elsewhere.

Proof. For an object $X \in \text{Ob}(\mathcal{E})$ with a decomposition $X \cong \bigoplus_{i=1}^n X_i$ as in Definition 3.1 the function $\mathcal{E}_{\text{ind}}/\cong \xrightarrow{\theta(X)} \mathbb{N}$ assigns to each $u \in \mathcal{E}_{\text{ind}}/\cong$ the number $\# \{ i : X_i \in u \}$.

3.2. Comments on stability conditions. The family $\{ \theta_\sigma : \text{Ob}(\mathcal{T}) \to \mathbb{N}^{(\sigma_{\text{ind}}^\ast/\cong)} \}_{\sigma \in \text{Stab}(\mathcal{T})}$.

Recall that if $\sigma = (\mathcal{P}, Z)$ is a locally finite stability condition on a triangulated category $\mathcal{T}$, then for each $t \in \mathbb{R}$ the subcategory $\mathcal{P}(t)$ is an abelian category of finite length (see [2, p. 6]). Furthermore [1], the short exact sequences in $\mathcal{P}(t)$ are exactly these sequences $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ with $A, B, C \in \mathcal{P}(t)$, s. t. for some $\gamma : C \to A[1]$ the sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is a triangle in $\mathcal{T}$. The first lemma in this subsection, used in Section 4.1, follows from locally finiteness.

Lemma 3.4. Let $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$, $t \in \mathbb{R}$, $A \in \mathcal{P}(t)$. For any object $X \in \mathcal{T}$ denote by $[X] \in K(\mathcal{T})$ the corresponding equivalence class in the Grothendieck group $K(\mathcal{T})$. Then the set

$$\{ [X] \in K(\mathcal{T}) : X \in \mathcal{P}(t) \text{ and there exists a monic arrow } X \to A \text{ in } \mathcal{P}(t) \}$$

is finite.

Proof. Since $\mathcal{P}(t)$ is abelian category of finite length, we have a Jordan-Holder filtration for the given $A \in \mathcal{P}(t)$

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \ldots \longrightarrow E_{n-1} \longrightarrow E_n = A$$

where $E_i \to E_{i+1} \to S_{i+1}$ are short exact sequences in $\mathcal{P}(t)$ and $S_1, S_2, \ldots, S_n$ are simple objects in $\mathcal{P}(t)$. We will show that the set \textsuperscript{13} is finite by showing that it is a subset of:

$$\left\{ \sum_{i=1}^m [S_{\xi(i)}] : \{1, 2, \ldots, m\} \xrightarrow{\xi} \{1, 2, \ldots, n\} \text{ is injective} \right\}.$$

For any monic arrow $X \to A$ in $\mathcal{P}(t)$ we have a Jordan-Holder filtration of $X$

$$0 \longrightarrow E'_1 \longrightarrow E'_2 \longrightarrow \ldots \longrightarrow E'_{n-1} \longrightarrow E'_n = X$$

where $S'_1, S'_2, \ldots, S'_m$ are simple objects in $\mathcal{P}(t)$, s. t. $S'_i \cong S_{\xi(i)}, i = 1, \ldots, m$ for some injection $\xi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$. Since $E'_i \to E'_{i+1} \to S'_{i+1}$ is a short exact sequences in $\mathcal{P}(t)$, it is

\textsuperscript{12} By $\mathbb{N}^{(\mathcal{E}_{\text{ind}}/\cong)}$ we denote the set of functions from $\mathcal{E}_{\text{ind}}/\cong$ to $\mathbb{N}$ with finite support.
Proof. Let \( E'_t \to E'_{t+1} \to S'_{t+1} \to E'[1] \) in \( \mathcal{T} \). Hence by (7) it follows \( [X] = \sum_{i=1}^{n} [S'_i] = \sum_{i=1}^{n} [S_{\xi(i)}] \).

Recall that one of Bridgeland’s axioms \[ \mathbb{I} \] is: for any nonzero \( X \in \text{Ob}(\mathcal{T}) \) there exists a diagram of triangles\[ \mathbb{I} \] called **Harder-Narasimhan filtration**:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n = X \\
& \mathrel{\smash{\rotatebox[origin=c]{180}{\makebox[0pt][l]{\scriptsize \( \mathcal{A}_1 \)}}}} & \mathrel{\smash{\rotatebox[origin=c]{180}{\makebox[0pt][l]{\scriptsize \( \mathcal{A}_2 \)}}}} & \mathrel{\smash{\rotatebox[origin=c]{180}{\makebox[0pt][l]{\scriptsize \( \mathcal{A}_n \)}}}} & & & & & & \\
\end{array}
\]

where \( \{A_i \in \mathcal{P}(t_i)\}_{i=1}^{n} \), \( t_1 > t_2 > \cdots > t_n \) and \( A_i \) is non-zero object for any \( i = 1, \ldots, n \) (the non-vanishing condition makes the factors \( \{A_i \in \mathcal{P}(t_i)\}_{i=1}^{n} \) unique up to isomorphism). In \( \mathbb{I} \) is used the notation \( \phi_\sigma(X) := t_n, \phi_\sigma^+(X) := t_1, \) and the phase of a semistable object \( A \in \mathcal{P}(t) \setminus \{0\} \) is denoted by \( \phi^A := t \), we also use these notations. The objects \( \{A_i\}_{i=1}^{n} \) will be called **HN factors of** \( X \) (HN for Harder-Narasimhan). It is useful to give a name of the minimal HN factor \( A_n \).

**Definition 3.5.** For any \( X \in \mathcal{T} \setminus \{0\} \) we choose\[ \mathbb{I} \] a Harder-Narasimhan filtration as in (8). Having this diagram, we denote the semistable HN factor of minimal phase \( A_n \) by \( \sigma_- (X) \), and the last triangle \( E_{n-1} \longrightarrow X \longrightarrow A_n \longrightarrow E_{n-1}[1] \) by \( \text{HN}_- (X) \). In particular, \( \phi(\sigma_- (X)) = \phi_- (X) \).

In the next Lemma \[ \mathbb{3.6} \] we treat \( \sigma_- (X) \). We recall first (another axiom of Bridgeland \[ \mathbb{1} \]) that from \( \phi(A) > \phi(B) \) with semistable \( A, B \) it follows \( \text{hom}(A,B) = 0 \). This axiom implies that from \( \phi_-(X) > \phi_+(Y) \) it follows \( \text{hom}^{\leq 0}(X,Y) = \text{hom}^{\leq 0}(\sigma_-(X),Y) = 0 \). We get \( \text{hom}^{\leq 1}(\sigma_-(X),Y) = 0 \) in the following situation:

**Lemma 3.6.** If \( \phi_-(X) \geq \phi_+(Y) \) and \( \text{hom}^{\leq 1}(X,Y) = 0 \), then \( \text{hom}^{\leq 1}(\sigma_-(X),Y) = 0 \).

**Proof.** Let \( \text{HN}_-(X) = Z \longrightarrow X \longrightarrow \sigma_-(X) \longrightarrow Z[1] \). Then \( \phi_-(Z) > \phi(\sigma_-(X)) = \phi_-(X) \geq \phi_+(Y) \). Hence \( \text{Hom}^{\leq 0}(Z,Y) = 0 \). We apply \( \text{Hom}(Z[Y[i]], \text{Hom}(\sigma_-(X),Y[Y[i]])) \) to \( i \leq 1 \) to this triangle and obtain: \( 0 = \text{Hom}(Z[1],Y[i]) \rightarrow \text{Hom}(\sigma_-(X),Y[Y[i]]) \rightarrow \text{Hom}(X,Y[Y[i]]) = 0 \). The lemma follows. \[ \square \]

In \[ \mathbb{3.6} \] for a slicing \( \mathcal{P} \) of \( \mathcal{T} \) and an interval \( I \subset \mathbb{R} \) by \( \mathcal{P}(I) \) is denoted the extension closure of \( \{\mathcal{P}(t)\}_{t \in I} \), and \( \mathcal{P}((t,t+1)), \mathcal{P}((t,t+1)) \) are shown to be hearts of bounded t-structures for any \( t \in \mathbb{R} \). If \( \mathcal{P} \) is a part of a stability condition \( \mathcal{P}(Z) \in \text{Stab}(\mathcal{T}) \), then \( \mathcal{P}(t) \) is shown to be abelian. The nonzero objects in the subcategory \( \mathcal{P}(I) \) are exactly those \( X \in \mathcal{T} \setminus \{0\} \), which satisfy \( \phi_{\pm}(X) \in I \).

From these facts it follows that \( \mathcal{P}(I) \) is a thick subcategory for any interval \( I \subset \mathbb{R} \):

**Lemma 3.7.** For any slicing \( \mathcal{P} \) of a triangulated category \( \mathcal{T} \) and any interval \( I \subset \mathbb{R} \) the category \( \mathcal{P}(I) \) is a thick subcategory of \( \mathcal{T} \). In particular, if \( \mathcal{T} \) has Krull-Schmidt property, then \( \mathcal{P}(I) \) has it.

**Proof.** In \[ \mathbb{8} \] t-structures are defined as pairs of subcategories. For any slicing \( \mathcal{P} \) and any \( t \in \mathbb{R} \) the hearts \( \mathcal{P}((t,t+1)), \mathcal{P}((t,t+1)) \) come from the pairs \( \mathcal{P}((t,+,\infty)), \mathcal{P}((-\infty,t+1)) \), \( \mathcal{P}((t,+,\infty)), \mathcal{P}((-\infty,t+1)) \), respectively, which are bounded t-structures. Let us consider for example the t-structure \( \mathcal{P}((t,+,\infty)), \mathcal{P}((-\infty,t+1)) \). In terms of the notations used in \[ \mathbb{8} \] we denote \( \mathcal{T}_{\leq 0} = \mathcal{P}((t,+,\infty)) \), \( \mathcal{T}_{\geq 0} = \mathcal{P}((-\infty,t+1)) \). From the properties of t-structures we know that

\[
X \in \mathcal{T}_{\leq 0} \iff \forall Y \in \mathcal{T}_{\geq 1} \text{ hom}(X,Y) = 0; \quad X \in \mathcal{T}_{\geq 0} \iff \forall Y \in \mathcal{T}_{\leq -1} \text{ hom}(Y,X) = 0.
\]

\[ ^{13} \text{Throughout the whole text the word triangle means distinguished triangle.} \]

\[ ^{14} \text{by the axiom of choice.} \]
Hence $\mathcal{T}^{\leq 0} = \mathcal{P}((t, +\infty))$, $\mathcal{T}^{\geq 0} = \mathcal{P}((-\infty, t + 1])$ are thick subcategories. Similarly $\mathcal{P}([t, +\infty))$, $\mathcal{P}((\infty, t + 1))$ are thick. Since for any interval $I \subset \mathbb{R}$ the subcategory $\mathcal{P}(I)$ is an intersection of two subcategories of the considered types, the lemma follows.

**Corollary 3.8.** Let $X, A, B \in \mathcal{T}$ and $X \cong A \oplus B$, then for any slicing $\mathcal{P}$ of $\mathcal{T}$ we have $\phi_-(X) \leq \phi_-(A) \leq \phi_+(A) \leq \phi_+(X)$.

**Proof.** We have $X \in \mathcal{P}([\phi_-(X), \phi_+(X)])$. From the previous lemma, $A, B \in \mathcal{P}([\phi_-(X), \phi_+(X)])$ and the statement follows.

Thus, if $\mathcal{T}$ has Krull-Schmidt property, then all $\{\mathcal{P}(t)\}_{t \in \mathbb{R}}$ have it (Lemma 3.7). From Lemma 3.3, we obtain a family of functions $\{\mathcal{P}(t) \to \mathbb{N}(\mathcal{P}(t)_{\text{ind}}/\sim)\}_{t \in \mathbb{R}}$. In Definition 3.9 below we build a single function on $\text{Ob}(\mathcal{T})$ from this family of functions, using the HN filtrations. We need first some notations.

For $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$ we denote by $\sigma^{ss}$ the set of $\sigma$-semistable objects, i.e.

\[(9) \quad \sigma^{ss} = \bigcup_{t \in \mathbb{R}} \mathcal{P}(t) \setminus \{0\}.\]

By $\sigma^{ss}_{\text{ind}}$ we denote the set of all indecomposable semistable objects, i.e.

\[(10) \quad \sigma^{ss}_{\text{ind}} = \bigcup_{t \in \mathbb{R}} \mathcal{P}(t)_{\text{ind}} = \sigma^{ss} \cap \mathcal{T}_{\text{ind}}.\]

In (a) of Definition 3.9 we consider $\mathbb{N}(\mathcal{P}(t)_{\text{ind}}/\sim)$ as a subset of $\mathbb{N}(\sigma^{ss}_{\text{ind}}/\sim)$, which is reasonable since the family $\{\mathcal{P}(t)_{\text{ind}}\}_{t \in \mathbb{R}}$ is pairwise disjoint.

**Definition 3.9.** Let $\mathcal{T}$ have Krull-Schmidt property. Let $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$.

We define $\theta_\sigma : \text{Ob}(\mathcal{T}) \to \mathbb{N}(\sigma^{ss}_{\text{ind}}/\sim)$ as the unique function satisfying the following:

(a) For each $t \in \mathbb{R}$ the restriction of $\theta_\sigma$ to $\mathcal{P}(t)$ coincides with the function $\mathcal{P}(t) \to \mathbb{N}(\mathcal{P}(t)_{\text{ind}}/\sim)$, given by Lemmas 3.3, 3.7.

(b) For any non-zero $X \in \text{Ob}(\mathcal{T})$ with a HN filtration\[8\] holds the equality $\theta_\sigma(X) = \sum_{i=1}^n \theta_\sigma(A_i)$.\[10\]

We use freely that $X \cong Y$ implies $\theta_\sigma(X) = \theta_\sigma(Y)$, $X \not\cong 0$ implies $\theta_\sigma(X) \neq 0$, and $\theta_\sigma(X) \leq \theta_\sigma(Y)$ implies $\phi_-(Y) \leq \phi_-(X) \leq \phi_+(X) \leq \phi_+(Y)$. Another property of $\theta_\sigma$, to which we refer later, is:

**Lemma 3.10.** Let $\phi_-(X_1) > \phi_+(X_2)$. For any triangle $X_1 \to X \to X_2 \to X_1[1]$ we have $\theta_\sigma(X) = \theta_\sigma(X_1) + \theta_\sigma(X_2)$.

**Proof.** If the HN factors of $X_1$ and $X_2$ are $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_m$, respectively, then, using the octahedral axiom, one can show that the HN factors of $X$ are $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m$. Now the lemma follows from (b) in Definition 3.9.

The property $\theta_\sigma(X \oplus Y) = \theta_\sigma(X) + \theta_\sigma(Y)$ for $X, Y \in \mathcal{P}(t)$ follows from (a) in Lemma 3.3. To show this additive property for any two objects $X, Y \in \mathcal{T}$ we note first:

**Lemma 3.11.** For any diagram of the type (composed of distinguished triangles):

\[
\begin{array}{cccccc}
0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow \cdots \rightarrow B_{n-1} & \rightarrow & B_n = X, \\
& & A_1 & & A_2 & & \ldots & & A_n & & \rightarrow
\end{array}
\]

where $\{A_i \in \mathcal{P}(t_i)\}_{i=1}^n$, $t_1 > t_2 > \cdots > t_n$, without the constraint that $A_1, A_2, \ldots, A_n$ are non-zero objects, we have $\theta_\sigma(X) = \sum_{i=1}^n \theta_\sigma(A_i)$.

---

\[8\]Recall that $\mathcal{P}(t)$ is thick in $\mathcal{T}$ (Lemma 3.7), hence $\mathcal{P}(t)_{\text{ind}} = \mathcal{P}(t) \cap \mathcal{T}_{\text{ind}}$.

\[10\]Recall that the collection $\{A_i\}_{i=1}^n$ of the HN factors is determined by $X$ up to isomorphism.
Proposition 3.13. We start by recalling a result in [1]:

Application: Stability conditions on category

Lemma 3.12. If \( A \) is zero and in the end we obtain the HN filtration of \( X \), then the equality follows from (b) in Definition 3.9 and \( \theta_\sigma(A_i) = 0 \) if \( A_i \) is a zero object. \( \square \)

Given two non-zero objects \( X_1, X_2 \in \text{Ob}(\mathcal{T}) \), then after inserting triangles of the form \( \text{to their HN filtrations we can obtain two}(i = 1, 2) \) equally long diagrams with distinguished triangles

where \( \{A_j^i \in \mathcal{P}(t_j)\}_{j=1}^n \) and \( i = 1, 2 \) and \( t_1 > t_2 > \cdots > t_n \). Hence, we get a diagram of triangles:

\[
\begin{array}{cccccccc}
0 & \rightarrow & B_1^i & \rightarrow & B_2^i & \rightarrow & \cdots & \rightarrow & B_{n-1}^i & \rightarrow & B_n^i = X_i, \\
& & A_1^i & \rightarrow & A_2^i & \rightarrow & & \rightarrow & A_n^i & \rightarrow & \end{array}
\]

\[
\begin{array}{cccccccc}
& & & & & & & & & & \\
& & & & & & & & & & \end{array}
\]

We have \( \{A_1^j \oplus A_2^j \in \mathcal{P}(t_j)\}_{j=1}^n \) by the additivity of \( \mathcal{P}(t_j) \). Using Lemma 3.11 we obtain: \( \theta_\sigma(X_1 \oplus X_2) = \sum_{j=1}^n \theta_\sigma(A_1^j \oplus A_2^j) = \sum_{j=1}^n \theta_\sigma(A_1^j) + \sum_{j=1}^n \theta_\sigma(A_2^j) = \theta_\sigma(X_1) + \theta_\sigma(X_2) \), i.e. we proved:

Lemma 3.12. For any pair of objects \( X_1, X_2 \) in \( \mathcal{T} \) we have: \( \theta_\sigma(X_1 \oplus X_2) = \theta_\sigma(X_1) + \theta_\sigma(X_2) \).

In the end of this subsection we recall the remaining axioms of Bridgeland [1]. A stability condition \( \sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T}) \) has the properties: \( \mathcal{P}(t)[1] = \mathcal{P}(t+1) \) for each \( t \in \mathbb{R} \), and

\[
X \in \sigma^{ss} \Rightarrow Z(X) = r(X) \exp(i\pi \phi(X)), \quad r(X) > 0.
\]

3.3. Application: Stability conditions on \( \mathcal{D}^b(Q_1) \) with two limit points in \( S^1 \).

We start by recalling a result in [1]:

Proposition 3.13 (Proposition 5.3 in [1]). Let \( \mathcal{A} \subset \mathcal{T} \) be a bounded t-structure in a triangulated category \( \mathcal{T} \) and \( K(\mathcal{A}) \xrightarrow{Z} \mathbb{C} \) be a stability function on \( \mathcal{A} \) with HN property.\textsuperscript{13} Then there exists unique stability condition \( \sigma = (\mathcal{P}, Z_\sigma) \) on \( \mathcal{T} \) satisfying:

(a) \( Z_\sigma(X) = Z(X) \) for \( X \in \mathcal{A} \);

(b) For \( t \in (0, 1] \) the objects of \( \mathcal{P}(t) \) are\textsuperscript{19} \( \text{Ob}(\mathcal{P}(t)) = \{X \in \mathcal{A} : \text{for each A-monic } X' \rightarrow X \text{ arg } Z(X') \leq \text{arg } Z(X) = \pi t\} \).

Conversely, for each stability condition \( \sigma = (\mathcal{P}, Z_\sigma) \) on \( \mathcal{T} \) the subcategory \( \mathcal{P}(0, 1] = \mathcal{A} \) is a heart of a bounded t-structure of \( \mathcal{T} \), the restriction \( Z = Z_\sigma \circ (K(\mathcal{A}) \rightarrow K(\mathcal{T})) \) of \( Z_\sigma \) to \( K(\mathcal{A}) \) is a stability function on \( \mathcal{A} \) with HN property and for \( t \in (0, 1] \) the set of objects of \( \mathcal{P}(t) \) is the same as in (b).

Definition 3.14. We denote by \( \mathbb{H}^A \) the family of stability conditions on \( \mathcal{T} \) obtained by (a), (b) above keeping \( \mathcal{A} \) fixed and varying \( Z \) in the set of all stability functions on \( \mathcal{A} \) with HN property.

Let \( \mathcal{A} = \text{Rep}_k(Q_1) \subset \mathcal{D}^b(\text{Rep}_k(Q_1)) \) be the standard bounded t-structure, where \( k \) is an algebraically closed field. A stability function \( K(\mathcal{A}) \xrightarrow{Z} \mathbb{C} \) is uniquely determined by \( Z(E_0), Z(M), Z(E_1) \in \mathbb{H} \). Here we choose \( Z(E_0), Z(M), Z(E_1) \) as follows:

\textsuperscript{13}HN property for \( K(\mathcal{A}) \xrightarrow{Z} \mathbb{C} \) is defined in [1] Definition 2.3]. If \( \mathcal{A} \) is an abelian category of finite length, then any stability function \( Z \) on \( \mathcal{A} \) satisfies the HN property [1 Proposition 2.4].

\textsuperscript{19}If \( \mathcal{A} \) has finite length and finitely many simple objects, then the obtained stability condition \( \sigma \) is locally finite. For \( u \in \mathbb{H} \) we denote by \( \arg(u) \) the number satisfying \( \arg(u) \in (0, 1], \quad u = \exp(i\arg(u)) \). We set \( \arg(0) = -\infty \).
The values of Lemma 3.15. Let \( \sigma = (\mathcal{P}, Z) \in \mathbb{H}^A \subset \text{Stab}(D^b(A)) \) be the stability condition, uniquely determined by the chosen stability function \( K(A) \rightarrow \mathbb{C} \) and Bridgeland’s Proposition 3.13. Then the set of stable phases \( P_\sigma \), defined by \( P_\sigma = \exp \left( i \frac{1}{2} \arg(Z(E^m_i)) \right) \), has two limit points in \( S^1 \).

Proof. The values of \( Z \) on \( \mathcal{A}_{\text{exc}} \) (see Proposition 2.2) are: \( Z(M), Z(M') = Z(E^0_1) + Z(E^0_2), \) and

\[
Z(E^m_j) = m\delta_Z + Z(E^0_j), \quad m \in \mathbb{N}, j = 1, 2, 3, 4.
\]

By Kac’s theorem, \( \sigma \) is either finite or has two limit points. Hence, it remains to show that any monic \( \mathcal{P} \) satisfies \( \arg(Z(E^m_j)) + i \neq 0 \). By (11) we can write \( \pm Z(E^m_1)/Z(E^m_1) \) for each \( m \in \mathbb{N} \subset P_\sigma \). From (12) we have \( Z(E^m_j) = \lim_{m \rightarrow \infty} Z(E^m_j)/m\delta_Z + Z(E^m_1) \) for each \( m \in \mathbb{N} \subset P_\sigma \). Since \( \delta_Z \) is not collinear with \( Z(E^0_1) \), it follows that \( \pm \delta_Z/Z \) are limit points of \( P_\sigma \).

It remains to show that \( E^m_1 \) satisfies the conditions in Corollary 3.13 (b) for each \( m \in \mathbb{N} \). Since \( \mathcal{A} \) has Krull-Schmidt property, it is enough to show that any monic \( X \rightarrow E^m_1 \) with \( X \in \mathcal{A}_{\text{ind}} \) satisfies \( \arg(Z(X)) \leq \arg(Z(E^m_1)) \). In (12), (13) are given all the values \( \{Z(X) : X \in \mathcal{A}_{\text{ind}}\} \). From the picture we see that if \( u \in \{Z(E^0_1), Z(E^0_2), j\delta_Z, j\delta_Z + Z(M)\} \) \( \ni \{Z(E^m_1)\} \), then \( \arg(u) \leq \arg(Z(E^m_1)) \). Hence, it remains to show that any monic \( X \rightarrow E^m_1 \) with \( X \in \mathcal{A}_{\text{ind}} \) and \( \text{dim}(X) \in \{(j+1,j,j), (j+1,j,j+1)\} \) \( \ni \{(j+1,j,j+1)\} \) satisfies \( \arg(X) \leq \arg(Z(E^m_1)) \).

We consider separately two options.

If either \( \text{dim}(X) = (j+1,j,j) \) or \( \text{dim}(X) = (j+1,j,j+1) \), then a morphism \( X \rightarrow E^m_1 \) consists of three vector space morphisms \( (f_b, f_{\text{mid}}, f_e) : (k^{j+1}, k^j, k) \rightarrow (k^{m+1}, k^{m+1}, k^m) \) where \( x \in \{j,j+1\} \).

\(^{20}\)In [7] is shown that if \( Q \) is an Euclidean quiver without oriented cycles, then for any \( \sigma \in D^b(Q) \) the set of phases \( P_\sigma \) is either finite or has two limit points.

\(^{21}\)saying that the dimension vectors of the indecomposables are the same as the roots

\(^{22}\)Recall that \( \mathcal{P}(t)[1] = \mathcal{P}(t+1) \) for each \( t \in \mathbb{R} \).
Proposition 3.17. The difference is that in the statement of [12, Proposition 3.17] is claimed that one must take these open subsets by

3.17]. The statement of Proposition 3.17 is a slight modification of the first part of [12, Proposition 3.17] says that

Ω

The following proposition ensures extendability of certain stability conditions used in [7, Section 3]. The statement of Proposition 3.17 is a slight modification of the first part of [12, Proposition 3.17]. The difference is that in the statement of [12, Proposition 3.17] is claimed that one must take \( \mathcal{E}_{ij} = (E_i, E_j) \), whereas we take \( \mathcal{E}_{ij} = (E_i, E_{i+1}, \ldots, E_j) \). For the sake of clarity, we give a proof of Proposition 3.17 here.

Proposition 3.17. Let \( \mathcal{E} = (E_0, E_1, \ldots, E_n) \) be a full Ext-exceptional collection in \( \mathcal{I} \). Let \( 0 \leq i < j \leq n \) and denote \( \mathcal{E}_{ij} = (E_i, E_{i+1}, \ldots, E_j) \), \( \mathcal{I}_{ij} = \langle \mathcal{E}_{ij} \rangle \subset \mathcal{I} \). Let \( \mathbb{H}^{\mathcal{E}_{ij}} \subset \text{Stab}(\mathcal{I}_{ij}) \), \( \mathbb{H}^{\mathcal{E}} \subset \text{Stab}(\mathcal{I}) \) be the corresponding families as in Definition 3.14.

Then the map \( \pi_{ij} : \mathbb{H}^{\mathcal{E}} \to \mathbb{H}^{\mathcal{E}_{ij}} \), which assigns to \( (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}} \) the unique \( (\mathcal{P}', Z') \in \mathbb{H}^{\mathcal{E}_{ij}} \) with \( \{Z'(E_k) = Z(E_k)\}_{k=i}^j \) is surjective. For any \( (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}} \) and \( (\mathcal{P}', Z') \in \mathbb{H}^{\mathcal{E}_{ij}} \) holds the implication

\[
\pi_{ij}(\mathcal{P}, Z) = (\mathcal{P}', Z') \quad \Rightarrow \quad \{\mathcal{P}'(t) = \mathcal{P}(t) \cap \mathcal{I}_{ij}\}_{t \in \mathbb{R}}.
\]

Proof. Using the definition of \( \mathbb{H}^{\mathcal{E}}, \mathbb{H}^{\mathcal{E}_{ij}} \) (Definitions 3.14, 3.16), one easily reduces the proof of this proposition to the following lemma (compare with the proof of [12, Proposition 3.17, p.7]).

\footnote{Here regular means that for \( 0 \leq i \leq n - 1 \) at most one degree in \( \{\text{Hom}^p(E_i, E_{i+1}) = 0\}_{p \in \mathbb{Z}} \) does not vanish.}
Lemma 3.18. Let $\mathcal{E}, \mathcal{E}_{ij}$ be as in Proposition 3.17. Let us denote by $A, A_{ij}$ the extension closures of $\mathcal{E}$ and $\mathcal{E}_{ij}$ in $\mathcal{T}$. Then $A_{ij}$ is an exact Serre subcategory of $A$. In particular the embedding functor induces an embedding $\mathcal{K}(A_{ij}) \rightarrow \mathcal{K}(A)$.

Proof. Since both $A, A_{ij}$ are abelian categories (Lemma 3.14), if $A_{ij}$ is a Serre subcategory of $A$ it follows that $A_{ij}$ is an exact subcategory. Whence, it is enough to show that $A_{ij}$ is a Serre subcategory. Let $0 \rightarrow B_1 \rightarrow S \rightarrow B_2 \rightarrow 0$ be any short exact sequence in $A$.

Assume that $B_1, B_2 \in A_{ij}$. Since $A$ is a heart of bounded t-structure, the given short exact sequence is part of a triangle in $\mathcal{T}$. Since $A_{ij}$ is extension closed in $\mathcal{T}$, it follows $S \in A_{ij}$.

Next, assume that $S \in A_{ij}$. We have to show that $B_1, B_2 \in A_{ij}$. By $B_1, B_2 \in A$ and the definition of $A$, we have diagrams of short exact sequences in $A$ for $l = 1, 2$ (the superscript is a power of $E_i$):

\begin{equation}
\begin{array}{cccccccc}
0 & \rightarrow & U_{l,n} & \rightarrow & U_{l,n-1} & \rightarrow & \cdots & \rightarrow & U_{l,1} & \rightarrow & U_{l,0} = B_l \\
E_{n}^{p_{l,n}} & \rightarrow & E_{n-1}^{p_{l,n-1}} & \rightarrow & \cdots & \rightarrow & E_{0}^{p_{l,1}} & \rightarrow & E_{0}^{p_{l,0}} & \rightarrow & 0
\end{array}
\end{equation}

From $S \in A_{ij}$ it follows $\text{Hom}^*(S, E_l) = 0$ for $l < i$ and $\text{Hom}^*(E_i, S) = 0$ for $l > j$. Since we have $A$-epic arrows $S \rightarrow B_2, B_2 \rightarrow E_{0}^{p_{l,0}}$ and $A$-monic arrows $E_{0}^{p_{l,1}} \rightarrow B_1, B_1 \rightarrow S$, it follows that $p_{2,0} = 0$, if $0 < i$ and $p_{1,n} = 0$, if $n > j$. Now by induction it follows:

\begin{equation}
p_{2,k} = 0 \text{ for } k < i, \quad p_{1,k} = 0 \text{ for } k > j.
\end{equation}

We show bellow that $\text{Hom}(E_k, B_2) = 0$ for $k > j$ and $\text{Hom}(B_1, E_k) = 0$ for $k < i$. Since there exist $A$-monic $E_{n}^{p_{2,n}} \rightarrow B_2$ and $A$-epic $B_1 \rightarrow E_{0}^{p_{l,0}}$, by the diagrams (16) and induction we obtain $p_{2,k} = 0$ for $k > j$, $p_{1,k} = 0$ for $k < i$. These vanishings together with (17) imply the lemma.

Having (16) and (17) we can write $B_2 \in \langle E_i, E_{i+1}, \ldots, E_n \rangle$ and $B_1 \in \langle E_0, E_1, \ldots, E_j \rangle$, hence

\begin{equation}
\text{Hom}^*(B_2, E_k) = \text{Hom}^*(S, E_k) = 0 \text{ for } k < i, \quad \text{Hom}^*(E_k, B_1) = \text{Hom}^*(E_k, S) = 0 \text{ for } k > j.
\end{equation}

From the short exact sequence $0 \rightarrow B_1 \rightarrow S \rightarrow B_2 \rightarrow 0$ in $A$ we get a distinguished triangle $B_1 \rightarrow S \rightarrow B_2 \rightarrow B_1[1]$ in $\mathcal{T}$. Since we have (18), applying to this triangle $\text{Hom}(E_k, \_)$ and $\text{Hom}(\_, E_k)$ we obtain the desired $\text{Hom}(E_k, B_2) = 0$ for $k > j$, $\text{Hom}(B_1, E_k) = 0$ for $k < j$.

\section{3.5. $\sigma$-exceptional collections.} Motivated by the work of E. Macri, discussed in the introduction and in the previous Subsection 3.3, we define:

\textbf{Definition 3.19.} Let $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$. We call an exceptional collection $\mathcal{E} = (E_0, E_1, \ldots, E_n)$ \underline{$\sigma$}-exceptional collection if the following properties hold:

- $\mathcal{E}$ is semistable w. r. to $\sigma$ (i. e. all $E_i$ are semistable).
- $\forall i \neq j \text{ hom}^{\leq 0}(E_i, E_j) = 0$ (i. e. this is an Ext-exceptional collection).
- There exists $t \in \mathbb{R}$, s. t. $\{\phi(E_i)\}_{i=0}^{n} \subset (t, t+1]$.

The set stability conditions for which $\mathcal{E}$ is $\sigma$-exceptional coincides with $\Theta^\mathcal{E}_{\sigma} = \mathbb{H}^\mathcal{E} \cdot \widetilde{GL}^+(2, \mathbb{R})$ (Definition 3.16). More precisely, we have:

\textbf{Corollary 3.20} (of Lemmas 3.14, 3.16 in [12]). Let $\sigma = (\mathcal{P}, Z) \in \text{Stab}(\mathcal{T})$. Let $\mathcal{E}$ be a full Ext-exceptional collection in $\mathcal{T}$. Then we have the equivalences:

$\sigma \in \Theta^\mathcal{E}_{\sigma} \iff \mathcal{E} \subset \mathcal{P}(t, t+1)$ for some $t \in \mathbb{R} \iff \mathcal{E}$ is a $\sigma$-exceptional collection.

\footnote{Recall that the short exact sequences in a heart of a t-structure $\mathcal{A}$ are exactly those sequences $A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C$ with $A, B, C \in \mathcal{A}$, s. t. for some $\gamma : C \rightarrow A[1]$ the triangle $A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C \overset{\gamma}{\rightarrow} A[1]$ is distinguished in $\mathcal{T}$.}
Proof. First, note [12, Lemma 3.16] that from \( \{E_i\}_{i=0}^n \subset \mathcal{P}(t, t + 1) \) it follows \( \mathcal{A}_\xi = \mathcal{P}((t, t + 1)) \), and then all \( \{E_i\}_{i=0}^n \) are stable in \( \sigma \), because they are simple in \( \mathcal{A}_\xi = \mathcal{P}((t, t + 1)) \). Indeed, \( \mathcal{A}_\xi \) and \( \mathcal{P}(t, t + 1) \) are both bounded t-structures, therefore the inclusion \( \mathcal{A}_\xi \subset \mathcal{P}(t, t + 1) \) implies equality \( \mathcal{A}_\xi = \mathcal{P}((t, t + 1)) \). Whence, if \( \{E_i\}_{i=0}^n \subset \mathcal{P}(t, t + 1) \), then \( \mathcal{E} \) is \( \sigma \)-exceptional (see Definition 3.19).

Now the corollary follows from the last part of Bridgeland’s Proposition 3.13 and the following comments on the action of \( GL^+(2, \mathbb{R}) \). If \( (\tilde{\mathcal{P}}, \tilde{\mathcal{Z}}) \) is obtained by the action with \( GL^+(2, \mathbb{R}) \) on \( (\mathcal{P}, \mathcal{Z}) \), then \( \{\tilde{\mathcal{P}}(\psi(t)) = \mathcal{P}(t)\}_{t \in \mathbb{R}} \) for some strictly increasing smooth function \( \psi : \mathbb{R} \to \mathbb{R} \) with \( \psi(t+1) = \psi(t) + 1 \), and hence \( \mathcal{P}(0, 1] = \tilde{\mathcal{P}}(\psi(0), \psi(0)+1) \). Conversely, for any \( t \in \mathbb{R} \) and any \( (\mathcal{P}, \mathcal{Z}) \) we can act on it with element in \( GL^+(2, \mathbb{R}) \), so that the resulting \( (\tilde{\mathcal{P}}, \tilde{\mathcal{Z}}) \) satisfies \( \mathcal{P}(t, t + 1] = \tilde{\mathcal{P}}(0, 1] \). □

Since the exceptional collection \( \mathcal{E} \) in Definition 3.19 has finite length, we have:

**Remark 3.21.** The third condition in Definition 3.19 is equivalent to each of the following three conditions: \( \{\phi(E_i)\}_{i=0}^n \subset (t, t + 1] \) for some \( t \in \mathbb{R} \); \( \{\phi(E_i)\}_{i=0}^n \subset [t, t + 1) \) for some \( t \in \mathbb{R} \); \( \max\{\{\phi(E_i)\}_{i=0}^n\} - \min\{\{\phi(E_i)\}_{i=0}^n\} < 1 \).

Furthermore, by Corollary 3.20 we have \( \Theta'_\xi = \{\sigma : \max\{\phi^\sigma(E_i)\}_{i=0}^n - \min\{\phi^\sigma(E_i)\}_{i=0}^n < 1\} = \{\sigma : \mathcal{E} \subset \sigma_{ss} \text{ and } |\phi^\sigma(E_i) - \phi^\sigma(E_j)| < 1 \text{ for } i < j\} \), therefore \( \Theta'_\xi \) is an open subset of \( \text{Stab}(\mathcal{J}) \).

One can now easily show that the assignment:

\[
\Theta'_\xi \ni \sigma = (\mathcal{P}, \mathcal{Z}) \mapsto (|\mathcal{Z}(E_0)|, \ldots, |\mathcal{Z}(E_n)|, \phi^\sigma(E_0), \ldots, \phi^\sigma(E_n))
\]

is well defined, and gives a homomorphism between \( \Theta'_\xi \) and the following simply connected set:

\[
\{(x_0, \ldots, x_n, y_0, \ldots, y_n) \in \mathbb{R}^{2(n+1)} : x_i > 0, |y_i - y_j| < 1\}.
\]

From the first part of this remark and Corollary 3.20 we see that for each \( \sigma \in \Theta'_\xi \) we have an open interval, in which \( \mathcal{P}(x) \) is trivial (take \( t \in \mathbb{R} \) and \( \epsilon > 0 \) so that \( \{\phi(E_i)\}_{i=0}^n \subset (t, t + 1] \cap (t + \epsilon, t + \epsilon + 1] \), then \( (t, t + \epsilon) \) is such an interval). In particular (recall also that \( \mathcal{P}(x)[1] = \mathcal{P}(x + 1) \)), we have:

**Remark 3.22.** Let \( \mathcal{E} \) be as in Corollary 3.21. For each \( \sigma \in \Theta'_\xi \) the set \( P_\sigma \) is not dense in \( S^1 \).

4. Non-semistable exceptional objects in hereditary abelian categories

In this section is written an algorithm, denoted by \( \text{alg} \). In subsection 4.1 we define the input data of the algorithm, in subsection 4.2 the data at the output. The rest sections of the text refer mainly to subsections 4.1 and 4.2.

4.1. Presumptions. For the rest of the paper \( \mathcal{A} \) is an abelian hereditary hom-finite category, linear over an algebraically closed field \( k \). It can be shown that such a category has Krull-Schmidt property (Definition 3.3). Hence, by Lemma 3.2, the derived category \( D^b(\mathcal{A}) \) also satisfies the Krull-Schmidt property. For brevity, we set \( \mathcal{J} = D^b(\mathcal{A}) \). Let \( \sigma = (\mathcal{P}, \mathcal{Z}) \in \text{Stab}(\mathcal{J}) \) be a stability condition. In this setting by Definition 3.9 we obtain the function \( \theta_\sigma : \text{Ob}(\mathcal{J}) \to \mathbb{N}(\sigma_{ss}/\mathbb{R}) \).

The input data of the algorithm \( \text{alg} \) is a non-semistable w. r. to \( \sigma \) exceptional object \( E \in \mathcal{J} \). The output data is a triangle, denoted by \( \text{alg}(E) \). We distinguish five cases at the output, depending

---

25For a fixed nonzero object \( X \in \mathcal{J} \) the functions \( \sigma \mapsto \phi^\sigma(X) \) on the manifold \( \mathcal{J} \) are continuous

26see Lemma 3.3 for the notation \( P_\sigma \)

27In all the sections 4.7 5 6 7 8 9 the symbol \( \mathcal{A} \) denotes such a category.

28using some facts for modules over unital associative ring shown around page 302 of [10], see also [11]
on the features of the triangle \( \text{alg}(E) \), and denote them by \( C_1, C_2, C_3, B_1, B_2 \). Only one of the five possible cases can occur at the output, i.e. \( \text{alg}(E) \) has all the features of exactly one case, say \( X \in \{ C_1, C_2, C_3, B_1, B_2 \} \), and then \( \text{alg}(E) \) is said to be of type \( X \).

We note two facts, which we keep in mind further.

**Remark 4.1.** It can be shown\(^{29}\) that, under the given assumptions on \( \mathcal{A} \), if \( X \in \mathcal{A}_{\text{ind}} \) satisfies \( \text{Ext}^1(X, X) = 0 \), then \( \text{Hom}(X, X) = k \), and hence \( X \) is an exceptional object.

**Remark 4.2.** Since \( \mathcal{A} \) is a hereditary category, for any two indecomposable \( A, B \in D^b(\mathcal{A}) \) with \( \deg(A) = \deg(B) \) from \( \phi_-(A) > \phi_+(B) + 1 \) it follows that \( \text{Hom}^*(A, B) = 0 \).

Another simple observation due to hereditariness, which we will apply throughout, is:

**Lemma 4.3.** Let \( \mathcal{A} \) be a hereditary abelian category and let \( 0 \to X \to Y \to Z \to 0 \) be a short exact sequence in \( \mathcal{A} \). For each \( W \in \mathcal{A} \) hold the following implications:

(a) If \( \text{hom}^1(Y, W) = 0 \), then \( \text{hom}^1(X, W) = 0 \)

(b) If \( \text{hom}^1(W, Y) = 0 \), then \( \text{hom}^1(Y, Z) = 0 \).

**Proof.** To prove (a) we apply \( \text{Hom}(\_, W[1]) \) to the triangle \( X \to Y \to Z \to X[1] \), corresponding to the given exact sequence. It follows \( 0 = \text{Hom}(Y, W[1]) \to \text{Hom}(X, W[1]) \to \text{Hom}(Z[-1], W[1]) = 0 \), where the right vanishing is because \( \mathcal{A} \) is hereditary. In (b) we apply \( \text{Hom}(W, \_) \). \( \square \)

We could work here with weaker assumptions on \( \mathcal{A} \). More precisely:

**Remark 4.4.** Given that \( \mathcal{A} \) is a hereditary \( k \)-linear abelian category with Krull Schmidt property as defined in Definition \[6,7\] without assuming hom-finiteness and that \( k \) is algebraically closed, then everything in Sections \[4,5,6,7,9,10\] remains valid, if we replace “exceptional” by “pre-exceptional”\(^{30}\). Under such seemingly weaker assumptions on \( \mathcal{A} \), we do not have the statement in Remark \[4,1\].

4.2. The cases. Here we explain the features of each of the five cases \( C_1, C_2, C_3, B_1, B_2 \) occurring at the output of \( \text{alg} \). The other subsections of 4 contain the algorithm.

Let \( E \in \mathcal{T} \) be a non-semistable w. r. to \( \sigma \) exceptional object. We recall that the meaning of the notation \( \deg(E) \), used here, is explained in the paragraph Some notations after the introduction. The properties (a),(b),(c) below are common features of \( \text{alg}(E) \) for all the cases, property (d) is common for \( C_1, C_2, C_3 \):

\[
\text{alg}(E) \quad \xrightarrow{\mathcal{A}} \quad E \quad U, V, \in \sigma^{ss}, U \neq 0, V \neq 0, \text{ where:}
\]

(a) \( V \) is the degree \( j \) component of \( \sigma^{-}(E) \), where \( j \in \{ \deg(E), \deg(E) + 1 \} \).

(b) \( \theta_\sigma(U) \prec \theta_\sigma(E) \Rightarrow \phi_-(U) \geq \phi(V) = \phi_-(E) \).

(c) Any \( \Gamma \in \text{Ind}(V) \) satisfies \( \text{hom}(E, \Gamma) \neq 0 \) (see Definition \[3,1\] for the notation \( \text{Ind}(V) \)).

(d) In the cases \( C_1, C_2, C_3 \) hold the vanishings \( \text{hom}^0(U, V) = \text{hom}^1(U, U) = \text{hom}^1(V, V) = 0 \), in particular for any \( S \in \text{Ind}(V) \), \( E' \in \text{Ind}(U) \) the pair \( (S, E') \) is exceptional with \( S \in \sigma^{ss} \).

\( ^{29} \)by adapting the proof of this fact for quivers, given on \[ 6, \text{p. 9,10} \], to \( \mathcal{A} \)

\( ^{30} \)By Pre-exceptional object we mean an indecomposable object \( X \in \mathcal{T} \) with \( \text{Hom}^*(X, X) = 0 \) for \( i \neq 0 \). Pre-exceptional collection is a sequence of pre-exceptional objects \( (E_1, E_2, \ldots, E_n) \) with \( \text{Hom}^*(E_i, E_j) = 0 \) for \( i > j \).

\( ^{31} \)\( \sigma_-(E) \) is defined in Definition \[5,5\]

\( ^{32} \)We write \( f < g \) for two functions \( f, g \in \mathbb{N}^{\text{alg}(E)/\text{alg}(E)} \), if \( f(u) < g(u) \) for some \( u \in \sigma^{ss}/\approx \).

\( ^{33} \)Note below that in cases \( C_3, B_2 \) we have proper inequality \( \phi_-(U) > \phi(V) \).
We give now the complete lists of properties. For simplicity we assume that \( E \in \mathcal{A} \), i. e. \( \deg(E) = 0 \), for other degrees everything is shifted with the corresponding number.

**C1.** The triangle is of the form \( \text{alg}(E) = \begin{array}{ccc} A & \\ & E & \\ B & \end{array} \) with the properties:

1. \( \{A, B\} \subset \mathcal{A}, \ A \neq 0, \ B \neq 0, \ \hom^1(A, A) = \hom^1(B, B) = \hom^*(A, B) = 0, \)
2. \( B \) is the zero degree component of \( \sigma_-(E) \), in particular \( B \) is semistable of phase \( \phi_-(E) \),
3. \( \theta_\sigma(A) < \theta_\sigma(E) \Rightarrow \phi_-(A) \geq \phi_-(E) \),
4. any \( \Gamma \in \text{Ind}(A) \) satisfies \( \hom^1(B, \Gamma) \neq 0 \).

**C2.** The triangle is of the form

\[
\text{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \\ & E & \\ B \end{array}
\]

with the properties:

1. \( \{A_1, A_2, B\} \subset \mathcal{A}, \ A_2 \neq 0, \ B \neq 0, \ A_1 \) is a proper subobject in \( \mathcal{A} \) of \( E \), \( \hom^1(A_2, A_2) = \hom^1(A_1, A_1) = \hom^*(A_1, A_1) = \hom^*(A_2, B) = \hom^*(A_1, A_2) = 0, \)
2. \( B \) is the zero degree component of \( \sigma_-(E) \), in particular \( B \) is semistable of phase \( \phi_-(E) \),
3. \( \theta_\sigma(A_1) + \theta_\sigma(A_2[-1]) < \theta_\sigma(E), \) in particular \( \phi_-(A_1) \geq \phi_-(E) \) and \( \phi_-(A_2[-1]) \geq \phi_-(E) \),
4. any \( \Gamma \in \text{Ind}(A_1) \) satisfies \( \hom(B, \Gamma[1]) \neq 0, \) any \( \Gamma \in \text{Ind}(A_2) \) satisfies the three conditions: \( \hom(B, \Gamma) \neq 0, \hom(\Gamma, E[1]) \neq 0, \hom(E, \Gamma[1]) = 0. \)

**C3.** The triangle is of the form \( \text{alg}(E) = \begin{array}{ccc} A & \\ & E & \\ B[1] \end{array} \) with the properties:

1. \( \{A, B\} \subset \mathcal{A}, \ A \neq 0, \ B \neq 0, \ \hom^1(A, A) = \hom^1(B, B) = \hom^*(A, B) = 0, \)
2. \( \text{alg}(E) \cong \text{IN}_-(E), \) hence \( \theta_\sigma(A) < \theta_\sigma(E) \) and \( \phi_-(A) > \phi_-(E) = \phi(B) + 1, \)
3. any \( \Gamma \in \text{Ind}(B) \) satisfies \( \hom^1(E, \Gamma) \neq 0 \) and \( \hom^1(\Gamma, E) = 0, \) any \( \Gamma \in \text{Ind}(A) \) satisfies \( \hom(B, \Gamma) \neq 0 \) and \( \hom(\Gamma, E) \neq 0. \)

**B1.** The triangle is of the form \( \text{alg}(E) = \begin{array}{ccc} A_1 \oplus A_2[-1] & \\ & E & \\ B \end{array} \) with the properties:

1. \( \{A_1, A_2, B\} \subset \mathcal{A}, \ A_2 \neq 0, \ B \neq 0, \ \hom^1(A_2, A_2) = \hom^1(A_1, A_1) = \hom^*(A_1, A_1) = \hom^*(A_2, B) = 0, \ A_1 \) is a proper subobject in \( \mathcal{A} \) of \( E \),
2. \( B \) is the zero degree component of \( \sigma_-(E) \), in particular \( B \) is semistable of phase \( \phi_-(E) \),
3. \( \theta_\sigma(A_1) + \theta_\sigma(A_2[-1]) < \theta_\sigma(E), \) in particular \( \phi_-(A_1) \geq \phi_-(E) \) and \( \phi_-(A_2[-1]) \geq \phi_-(E) \),
4. there exists \( \Gamma \in \text{Ind}(A_2) \) with \( \hom^1(\Gamma, E) \neq 0, \ \hom^1(E, \Gamma) \neq 0 \) \footnote{A comparison with \( \text{C2.4} \) shows that \( \text{B1} \) and \( \text{C2} \) cannot appear together.}

**B2.** The triangle is of the form \( \text{alg}(E) = \begin{array}{ccc} A & \\ & E & \\ B[1] \end{array} \) with the properties:

1. \( \{A, B\} \subset \mathcal{A}, \ A \neq 0, \ B \neq 0, \ \hom^1(B, B) = \hom^*(A, B) = 0, \)
2. \( \text{alg}(E) \cong \text{IN}_-(E), \) hence \( \theta_\sigma(A) < \theta_\sigma(E) \) and \( \phi_-(A) > \phi_-(E) = \phi(B) + 1, \)

\[34\] A comparison with \( \text{C2.4} \) shows that \( \text{B1} \) and \( \text{C2} \) cannot appear together.
**B2.3** there exists \( \Gamma \in B \) with \( \hom^1(\Gamma, E) \neq 0 \), \( \hom^1(E, \Gamma) \neq 0 \)

### 4.3. The last HN triangle

Now we start explaining alg.

Let \( E \in \mathcal{A}_{\text{exc}}, E \notin \sigma^{ss} \). Macrì initiated in [12] p. 10 an analysis of the last HN triangle of \( E \), when \( E \in \text{Rep}_k(K(l)) \). The arguments on [12] p. 10 are used here in formulas (22), (23), and in the derivation of the vanishing C3.1(Subsection 4.4).

Consider the last HN triangle \( \text{HN}_-(E) \)(see Definition 3.3):

\[
\begin{align*}
\text{HN}_-(E) &= X \xrightarrow{f} E \xrightarrow{\sigma_-} X[1], \quad \phi_-(X) > \phi(\sigma_-(E)) = \phi_-(E). 
\end{align*}
\]

**Lemma 4.5.** The triangle \( \text{HN}_-(E) \) is of the form \((E \oplus B_0, B_1, 1)\):

\[
\begin{align*}
(22) & \quad X \xrightarrow{f} E \xrightarrow{\sigma_-} B_0 \oplus B_1[1] \xrightarrow{X[1]}, \quad \phi_-(X) > \phi(B_0) = \phi(B_1) + 1 = \phi_-(E), \\
(23) & \quad \text{hom}^0(X, B_0) = \text{hom}^0(X, B_1[1]) = 0 \\
(24) & \quad \theta_\sigma(E) = \theta_\sigma(X) + \theta_\sigma(B_0) + \theta_\sigma(B_1[1]).
\end{align*}
\]

For any \( i \in \{0, 1\}, \Gamma \in \text{Ind}(B_i) \) the component of \( f \) to \( \Gamma[1] \) is non-zero and \( \text{hom}(E, \Gamma[i]) \neq 0 \). Any \( \Gamma \in \text{Ind}(X) \) satisfies \( \text{hom}(\Gamma, E) \neq 0 \) and \( \text{hom}(B_0 \oplus B_1[1], \Gamma[1]) \neq 0 \).

**Proof.** We show first that for each \( \Gamma \in \text{Ind}(\sigma_-(E)) \) the component of \( f \) from \( E \) to \( \Gamma \) is non-zero.

Indeed, suppose that for some \( \Gamma \in \text{Ind}(\sigma_-(E)) \) this component vanishes, then by the Krull-Schmidt property we can write \( \sigma_-(E) = U \oplus \Gamma \), and \( f \) is of the form: \( f = (f' : E \to U) \oplus (0 \to \Gamma) \).

After summing the triangles \( X' \xrightarrow{f'} U \xrightarrow{X'[1]} \Gamma[1] \) and \( \Gamma[1] \xrightarrow{0} \Gamma \), we get a triangle \( X' \oplus \Gamma[-1] \xrightarrow{f'} U \xrightarrow{X'[1]} \Gamma \).

From (21) it follows that \( X' \oplus \Gamma[-1] \cong X \). From Corollary 3.3 we see that \( \phi_-(X') \geq \phi_- = \phi(U) \).

By this inequality and the uniqueness of the HN filtration of \( E \) we deduce that \( \sigma_-(E) \cong U \), i.e. \( U \oplus \Gamma \cong U \), which contradicts the Krull-Schmidt property.

Thus, for each \( \Gamma \in \text{Ind}(\sigma_-(E)) \) the component of \( f \) to \( \Gamma \) is non-zero and \( \text{hom}(E, \Gamma) \neq 0 \). Now the triangle (21) reduces to (22), since \( \mathcal{A} \) is hereditary. From \( \phi_-(X) > \phi(B[i]) \) \( (i = 0, 1) \) it follows (23). Applying Lemmas 3.10, 3.12 to (22) we obtain (24). It remains to prove the last property.

Suppose that \( \hom(\Gamma, E) = 0 \) for some \( \Gamma \in \text{Ind}(X) \). Then we can represent \( X \to E \) as a direct sum \( (U \to E) \oplus (\Gamma \to 0) \). By the triangle (21) we get \( Y' \oplus \Gamma[1] \cong \sigma_-(E) \), where \( Y' \) is the cone of \( U \to E \).

From Corollary 3.8 we see \( \phi_-(U) \geq \phi_-(X) > \phi_-(E) = \phi(Y') \). Since \( \phi_-(U) > \phi_-(E) \), we have \( U \cong E \) and \( Y' \cong 0 \). It follows that \( \text{HN}_-(E) = U \xrightarrow{E \to Y'} \xrightarrow{Y'[1]} U \).

Therefore \( X \cong U \), i.e. \( U \oplus \Gamma \cong U \), which contradicts the Krull-Schmidt property.

Suppose that for some \( \Gamma \in \text{Ind}(X) \) we have \( \hom(B_0 \oplus B_1[1], \Gamma[1]) = 0 \), then by similar arguments we get \( E \cong E' \oplus \Gamma \), and hence \( \Gamma \cong E \) (since \( E \) is indecomposable), which contradicts \( \phi_-(\Gamma) \geq \phi_-(X) > \phi_-(E) \). The lemma is proved.

By \( f_i \) will be denoted the component of \( f \) to \( B_i[i] \)(see (22)), i.e. we have commutative diagrams (the right arrow is the projection)

\[
\begin{align*}
(25) & \quad E \xrightarrow{f} B_0 \oplus B_1[1] \\
& \quad \downarrow \text{id} \quad \downarrow \text{id} \quad \downarrow \text{id} \\
& \quad E \xrightarrow{f_i} B_i[i]
\end{align*}
\]

\[\text{A comparison with C3.3 shows that B2 and C3 cannot appear together.}\]
The algorithm \texttt{alg} tests now the condition \( B_0 = 0 \).

4.4. If \( B_0 = 0 \).

This condition leads to one of the cases \textbf{C3}, \textbf{B2} depending on the outcome of one test. Since \( B_0 = 0 \), the triangle (22) is reduced to a short exact sequence \( 0 \rightarrow B_1 \rightarrow X \rightarrow E \rightarrow 0 \), and \( X \in \text{Ob}(A) \). Hence (23) is now the same as \( \text{hom}^*(X, B_1) = 0 \), which by Lemma 4.3 (a) and the given exact sequence implies \( \text{hom}^1(B_1, B_1) = 0 \). By Lemma 4.5 any \( \Gamma \in \text{Ind}(B_1) \) satisfies \( \text{hom}(E, \Gamma[1]) \neq 0 \). Therefore, if \( \text{hom}(\Gamma, E[1]) \neq 0 \) for some \( \Gamma \in \text{Ind}(B_1) \), then the triangle:

\[
\text{HN}_-(E) = \begin{array}{ccc} X & \xrightarrow{\sigma} & E \\
\downarrow & & \downarrow \\
B_1[1] & \rightarrow & \end{array}
\]

satisfies \textbf{B2.1}, \textbf{B2.2}, \textbf{B2.3} (with \( A = X, B = B_1 \)). By setting \texttt{alg}(E) to (26) we get \textbf{B2}.

It remains to consider the case when \( \text{hom}(\Gamma, E[1]) = 0 \) for each \( \Gamma \in \text{Ind}(B_1) \), i. e.

\[
(27) \quad \text{hom}(B_1, E[1]) = 0.
\]

Setting again \texttt{alg}(E) to (26) (with \( X \) replaced by \( A \), \( B_1 \) replaced by \( B \)) we obtain the property \textbf{C3.2} immediately. The property \textbf{C3.3} follows from Lemma 4.5. We have already all the features of \textbf{C3.1} except the vanishing \( \text{hom}^1(X, X) = 0 \).

The vanishing \( \text{hom}^1(X, X) = 0 \) follows from (27), since the triangle (26) and \( \text{Hom}(X, \_\_\_\_) \) give an exact sequence \( \text{Hom}^1(X, B_1) \rightarrow \text{Hom}^1(X, X) \rightarrow \text{Hom}^1(X, E) \), where the left and the right terms vanish. The vanishing \( \text{Hom}^1(X, B_1) = 0 \) is already shown (before (26)). The other vanishing \( \text{hom}^1(X, E) = 0 \) follows from (27), \( \text{hom}^1(E, E) = 0 \), and \( \text{Hom}(\_\_\_, E[1]) \) applied to the same triangle.

Thus, \texttt{alg}(E) is of type \textbf{C3}.

4.5. If \( B_0 \neq 0 \).

Under this condition we obtain one of the cases \textbf{C1}, \textbf{C2}, \textbf{B1} at the output depending on the outcomes of additional tests.

By Lemma 4.5 we have \( f_0 \neq 0 \). Let us take kernel and cokernel of \( f_0 \) in \( A \):

\[
(28) \quad A_1 \xrightarrow{\text{ker}(f_0)} E \xrightarrow{f_0} B_0 \xrightarrow{\text{coker}(f_0)} A_2.
\]

Since \( f_0 \neq 0 \), \( \text{ker}(f_0) \) is a proper subobject of \( E \). Let \( E \xrightarrow{e_0} B'_0 \xrightarrow{\text{im}(f_0)} B_0 \) be a decomposition of \( f_0 \) in \( A \), where \( e_0 \) is \( A \)-epic and \( \text{im}(f_0) \) is \( A \)-monic. In particular, we have an exact sequence in \( A \)

\[
(29) \quad 0 \rightarrow A_1 \xrightarrow{\text{ker}(f_0)} E \xrightarrow{e_0} B'_0 \rightarrow 0.
\]

The next step of the algorithm \texttt{alg} is to test the condition \( A_2 = 0 \). We show first some preliminary facts, which do not depend on the vanishing of \( A_2 \).

4.5.1. Preliminary facts. These facts are (30), (31), (32), (33), and Lemma 4.6.

The equalities below will help us later to obtain \textbf{C1.1}, when \( A_2 = 0 \), and \textbf{C2.1}, when \( A_2 \neq 0 \):

\[
(30) \quad \text{hom}^1(A_1, A_1) = \text{hom}^1(A_2, A_2) = 0
\]

\[
(31) \quad \text{hom}(A_1, B_0) = \text{hom}^*(A_2, B_0) = 0
\]

\[
(32) \quad \text{hom}^1(A_1, E) = \text{hom}(A_1, A_2) = 0
\]

The inequality (33) ensures \textbf{C1.3} and \textbf{C2.3}, and Lemma 4.6 ensures \textbf{C1.4} and half of \textbf{C2.4}.

\[
(33) \quad \theta_\sigma(A_1) + \theta_\sigma(A_2[-1]) < \theta_\sigma(E).
\]
To show these facts we start by recalling that the triangle in $\mathcal{T}$ containing $f_0$ is

$$E \xrightarrow{f_0} B_0 \to C(f_0) \to E[1]$$

where $C(f_0)$ is the cochain complex ($B_0$ is in degree 0)

$$\ldots \to 0 \to E \xrightarrow{f_0} B_0 \to 0 \to \ldots$$

and the non-trivial part of the cochain maps $B_0 \to C(f_0) \to E[1]$ is $\xrightarrow{\theta}$.

Since $A$ is hereditary, we have $C(f_0) \cong \bigoplus_i H^i(C(f_0))[-i]$, which we can reduce by (28) and (35) to

$$C(f_0) \cong A_1[1] \oplus A_2.$$

Since we have the commutative diagram (25) with $i = 0$, by the $3 \times 3$ lemma in triangulated categories [3, Proposition 1.1.11] we can put the triangles (22), (34) in a diagram

$$\begin{array}{ccc}
E & \xrightarrow{f} & B_0 \oplus B_1[1] \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
E & \xrightarrow{f_0} & B_0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & B_1[2] \\
\downarrow & & \downarrow \\
E[1] & \xrightarrow{f_0} & B_0[1] \oplus B_1[2] \\
\downarrow & & \downarrow \\
X[1] & \rightarrow & C(f_0) \rightarrow E[1] \\
\downarrow & & \downarrow \\
0 & \rightarrow & Y \\
\downarrow & & \downarrow \\
0 & \rightarrow & X[2] \\
\downarrow & & \downarrow \\
0 & \rightarrow & E[1]
\end{array}$$

where $X[1] \rightarrow C(f_0) \rightarrow Y \rightarrow X[2]$; $0 \rightarrow B_1[2] \rightarrow Y \rightarrow 0$ are distinguished triangles. Hence $Y \cong B_1[2]$ and we obtain a distinguished triangle

$$X \rightarrow C(f_0)[-1] \rightarrow B_1[1] \rightarrow X[1].$$

The vanishing (30), (31), (32) will be obtained from triangles (37), (34), and the exact sequence (29).

We apply Hom($\cdot$, $B_0$) and Hom($\cdot$, $B_0[-1]$) to (37) and by (29) the result is: Hom($C(f_0)$, $B_0[1]$) = Hom($C(f_0)$, $B_0$) = 0. These vanishings and (33) imply (31). The vanishing hom$^1(A_1, E) = 0$ (the first part of (32)) follows from hom$^1(E, E) = 0$, the exact sequence (29), and Lemma 4.3 (a). Now we can write hom($C(f_0), E[2]$) = hom($A_1[1] \oplus A_2, E[2]$) = hom$^1(A_1, E) = 0$. Having $0 = \text{hom}(C(f_0), B_0[1]) = \text{hom}(C(f_0), E[2])$, we apply Hom($C(f_0)$, _) to (31) and obtain

$$0 = \text{Hom}(C(f_0), B_0[1]) \rightarrow \text{Hom}(C(f_0), C(f_0)[1]) \rightarrow \text{Hom}(C(f_0), E[2]) = 0.$$

Hence hom($A_1[1] \oplus A_2, A_1[2] \oplus A_2[1]$) = 0, which contains (30) and the second vanishing in (32).

The next step is to show (33). From Lemma 3.10 and the triangle (37) we get $\theta_\sigma(C(f_0)[-1]) = \theta_\sigma(X) + \theta_\sigma(B_1[1])$. From $B_0 \neq 0$ it follows $\theta_\sigma(B_0) > 0$, and hence:

$$\theta_\sigma(C(f_0)[-1]) = \theta_\sigma(X) + \theta_\sigma(B_1[1]) < \theta_\sigma(X) + \theta_\sigma(B_1[1]) + \theta_\sigma(B_0) = \theta_\sigma(E),$$

where the last equality is taken from (24). Now (33) follows from (36).

Since alg($E$) in both the cases $A_2 = 0$ and $A_2 \neq 0$ will be set to (34), the following corollary ensures C1.4, and part of C2.4.

**Lemma 4.6.** Each $\Gamma \in \text{Ind}(C(f_0)) = \text{Ind}(A_1[1] \oplus A_2)$ satisfies hom($B_0, \Gamma$) \neq 0, and each $\Gamma \in \text{Ind}(A_2)$ satisfies hom($\Gamma, E[1]$) \neq 0.
Proof. Suppose that \( \text{hom}(B_0, \Gamma) = 0 \) for some \( \Gamma \in \text{Ind}(C(f_0)) \) and split \( C(f_0) = U \oplus \Gamma \), then the arrow \( B_0 \to C(f_0) \) in (34) can be represented as \( (B_0 \to U) \oplus (0 \to \Gamma) \). The sum of the triangle \( E' \to B_0 \to U \to E'[1] \) extending \( B_0 \to U \) and the triangle \( \Gamma[-1] \to 0 \to \Gamma \to \Gamma \) is isomorphic to (34), hence \( E \cong E' \oplus \Gamma[-1] \). Since \( E \) is exceptional and \( \Gamma \neq 0 \), it follows \( E' = 0 \) and \( E \cong \Gamma[-1] \), hence \( \theta_\sigma(E) = \theta_\sigma(\Gamma[-1]) \leq \theta_\sigma(C(f_0)[-1]) < \theta_\sigma(E) \), where we used \( C(f_0) = U \oplus \Gamma \) and the inequality derived before this corollary. Thus, we get a contradiction.

If \( \text{hom}(\Gamma, E[1]) = 0 \) for some \( \Gamma \in \text{Ind}(A_2) \), then we can split \( C(f_0) \cong A_1[1] \oplus A_2 \cong V \oplus \Gamma \), and the last arrow in (34) is of the form \( (V \to E[1]) \oplus (\Gamma \to 0) \). It follows by similar arguments as above that \( B_0 \cong U \oplus \Gamma \) for some \( U \). Therefore \( \text{hom}(A_2, B_0) \neq 0 \), which contradicts (31). □

4.5.2. If \( A_2 = 0 \). Under this condition we get here a triangle or type C1.

Now \( f_0 \) is \text{epic}(see (28)) and (34) becomes a short exact sequence

\[
0 \to A_1 \xrightarrow{\text{ker}(f_0)} E \xrightarrow{f_0} B_0 \to 0.
\]

The triangle al\(\text{g}(E)\) is set to (38), so \( A = A_1 \), and \( B = B_0 \). From (33) we get \( \theta_\sigma(A_1) < \theta_\sigma(E) \), which is the same as C1.3. In Lemma 4.6 we have C1.4, and in 4.15 - C1.2. It remains to show C1.1. We have \( A_1 \neq 0 \), for otherwise \( E \) would be semistable. We have also (30) and (31), therefore we have to show only \( \text{hom}^1(A_1, B_0) = 0 = \text{Hom}^1(B_0, B_0) \).

By \( \text{hom}^1(A_1, E) = 0 \) (see (32)), the sequence (33), and Lemma 4.3(b) we obtain \( \text{hom}^1(A_1, B_0) = 0 \). The same lemma and \( \text{hom}^1(E, E) = 0 \) imply \( \text{hom}^1(E, B_0) = 0 \), hence \( \text{Hom}(\_, B_0[1]) \) applied to (33) gives: \( 0 = \text{Hom}(A_1[1], B_0[1]) \to \text{Hom}(B_0, B_0[1]) \to \text{Hom}(E, B_0[1]) = 0 \), i.e. \( \text{hom}^1(B_0, B_0) = 0 \).

4.5.3. If \( A_2 \neq 0 \).

Under this condition we will obtain either the case C2 or the case B1 depending on the outcome of one additional test. The triangle al\(\text{g}(E)\) is set to (34), which by \( C(f_0) \cong A_1[1] \oplus A_2 \) can be rewritten as:

\[
\text{alg}(E) = \begin{array}{ccc}
A_1 \oplus A_2 & \xrightarrow{-1} & E \\
& B_0 & \\
& \end{array}
\]

From Lemma 4.6 we have \( \text{hom}^1(\_ , E[1]) \neq 0 \) for each \( \Gamma \in \text{Ind}(A_2) \). If \( \text{hom}^1(\_ , \Gamma) \neq 0 \) for some \( \Gamma \in \text{Ind}(A_2) \), then the triangle (39) has all the features of the case B1 due to (29), (30), (31), (33).

Thus, it remains to show that if each \( \Gamma \in \text{Ind}(A_2) \) satisfies \( \text{hom}(\_, \Gamma[1]) = 0 \), in particular

\[
\text{Hom}(\_, A_2[1]) = 0 \quad \Rightarrow \quad \text{Hom}(\_, C(f_0)[1]) = \text{Hom}(\_, A_1[2] \oplus A_2[1]) = 0,
\]

then the triangle (39) satisfies C2.1, C2.2, C2.3, C2.4 (with \( B = B_0 \)).

C2.2 is in (22), C2.3 is (33), and C2.4 is contained in (40), Lemma 4.6. It remains to obtain the vanishings in C2.1, that are not claimed in (30), (31), (32). These vanishings are \( \text{hom}^1(A_1, B_0) = \text{hom}^1(A_1, A_2) = \text{hom}^1(B_0, B_0) = 0 \). We obtain them in this order below.

The equality (40) together with \( \text{hom}^1(E, E) = 0 \) and the triangle (34) imply

\[
\text{hom}(\_, B_0[1]) = 0,
\]
hence by the sequence \( (29) \) and Lemma \( (3.3) \) we get \( \text{hom}^1(A_1, B_0) = 0 \). From this vanishing it follows \( \text{hom}(C(f_0), B_0[2]) = 0 \) and applying \( \text{Hom}(C(f_0), \_ ) \) to \( (31) \) we obtain

\[
0 = \text{Hom}(C(f_0), B_0[2]) \rightarrow \text{Hom}(C(f_0), C(f_0)[2]) \rightarrow \text{Hom}(C(f_0), E[3]) = 0
\]

\[
\Rightarrow 0 = \text{hom}(C(f_0), C(f_0)[2]) = \text{hom}(A[1] \oplus A_2, A_1[3] \oplus A_2[2]) \Rightarrow \text{hom}(A_1, A_2[1]) = 0.
\]

Finally, we apply \( \text{Hom}(_\_ , B_0[1]) \) to \( (31) \): \( 0 = \text{Hom}(C(f_0), B_0[1]) \rightarrow \text{Hom}(B_0, B_0[1]) \rightarrow \text{Hom}(E, B_0[1]) = 0 \), where the left vanishing is contained in \( (31) \), and the right vanishing is above.

Now we have already the complete list \( \text{C2} \) for \( \text{alg}(E) \).

5. Some terminology. The relation \( R \longrightarrow (S, E) \)

The terminology introduced here is important for the rest of the paper. All definitions in this section assume a given stability condition on \( D^b(A) \), which we denote by \( \sigma \). We divide the non-semistable exceptional objects into two types: \( \sigma \)-regular and \( \sigma \)-irregular (Definition \( 5.1 \)). In turn the \( \sigma \)-regular objects are divided into final and non-final (Definition \( 5.3 \)).

We refer to \( \text{C1}, \text{C2}, \text{C3} \) as regular cases and to \( \text{B1}, \text{B2} \) as irregular cases. More precisely:

**Definition 5.1.** Let \( E \in D^b(A)_{\text{exc}} \) and \( E \notin \sigma^s \). If the triangle \( \text{alg}(E) \) given by section \( 4 \) is of type \( X \), where \( X \) is one of \( \text{C1}, \text{C2}, \text{C3}, \text{B1}, \text{B2} \), then \( E \) is said to be an \( X \) object w. r. to \( \sigma \).

The \( \text{Ci} \) objects(for \( i = 1, 2, 3 \)) will be called \( \sigma \)-regular exceptional objects and the \( \text{Bi} \) objects( for \( i = 1, 2 \)) will be called \( \sigma \)-irregular exceptional objects.

We introduce now the relation \( R \longrightarrow (S, E) \). It facilitates the next steps of the exposition.

**Definition 5.2.** Let \( R, S, E \in D^b(A) \) and let \( X \) be one of the symbols \( \text{C1}, \text{C2a}, \text{C2b}, \text{C3} \). By the notation \( R \longrightarrow (S, E) \) we mean the following data:

- \( R \) is a \( \sigma \)-regular exceptional object, in particular \( \text{alg}(R) \) is of type \( \text{Ci}(i \in \{1, 2, 3 \}) \),
- \( S \in \text{Ind}(V), \ E \in \text{Ind}(U) \), where \( (V, U) \) are the lower and the left vertices of \( \text{alg}(R) \) in \( (19) \),
- if \( i \in \{1, 3 \} \) and \( R \) is a \( \text{Ci} \) object, then we set \( X = \text{Ci} \),
- if \( R \) is a \( \text{C2} \) object and \( E \) is a component of \( A_2[-1] \) in diagram \( (20) \), then we set \( X = \text{C2a} \),
- if \( R \) is a \( \text{C2} \) object and \( E \) is a component of \( A_1 \) in \( (20) \), then we set \( X = \text{C2b} \).

In the next sections we refer mainly to the following features(explained below) of the pair \( (S, E) \):

\[
\text{R} \longrightarrow (S, E) \quad X \in \{ \text{C1}, \text{C2a, C2b, C3} \}
\]

(41) \( \{S, E\} \subset D^b(A)_{\text{exc}}, \ \hom^*(E, S) = 0, \ \deg(E) + 1 \geq \deg(S) \geq \deg(R) \geq \deg(E) \)

(42) \( \theta_\sigma(E) < \theta_\sigma(R), \ S \in \sigma^s, \ \theta_\sigma(R)(S) > 0, \ \phi_-(E) \geq \phi(S) = \phi_-(R) \).

The first two statements in \( (41) \) amount to saying that \( (S, E) \) is an exceptional pair \( ^{38} \) which is the same as: \( S, E \) are indecomposable and \( \hom^*(E, S) = \hom^1(S, S) = \hom^1(E, E) = 0 \). This follows from \( (d) \) right after \( (19) \) and \( S \in \text{Ind}(V), E \in \text{Ind}(U) \). In \( (a) \) right after \( (19) \) is specified

---

\(^{36}\) Here and in all sections that follow \( A \) is as in subsection \( (1.1) \).

\(^{37}\) In this text the adjectives “\( \sigma \)-regular”, “\( \sigma \)-irregular” regard either exceptional objects or the cases at the output of \( \text{alg} \). We often omit “exceptional object” after these adjectives, when this is by default. We sometimes omit “\( \sigma \)”, which is akin to writing semistable instead of \( \sigma \)-semistable.

\(^{38}\) In general, this pair is not uniquely determined by \( R \), because we make choices among \( \text{Ind}(U) \) and \( \text{Ind}(V) \).
that $V$ is a direct summand of $\sigma_-(R)$, hence by $S \in \text{Ind}(V)$ and the definition of $\theta_\sigma$(Definition 3.9) it follows that $\theta_\sigma(R)(S) > 0$ and $S \in \sigma^{s\#}$. In (b) right after (19) we have specified $\theta_\sigma(U) < \theta_\sigma(R)$, $\phi_-(U) \geq \phi(V) = \phi_-(R)$, which by $E \in \text{Ind}(U)$, $S \in \text{Ind}(V)$ implies $\theta_\sigma(E) < \theta_\sigma(R)$, $\phi_-(E) \geq \phi(S) = \phi_-(R)$. Thus we obtain (42). The degrees of $R, S, E$ are interrelated as shown in the following table which follows from the very definition of $C1,C2a,C2b,C3$ \[X\] \[
|\quad| \quad| \quad|
\begin{array}{ccc}
C1, C2b & 0 & 0 \\
C2a & 0 & +1 \phi_-(E) \geq \phi(S) \\
C3 & +1 & 0 \phi_-(E) > \phi(S)
\end{array}
\]
The inequalities $\deg(E) + 1 \geq \deg(S) \geq \deg(R) \geq \deg(E)$ follow, so (11) is shown completely.

We divide the $\sigma$-regular objects into final and non-final as follows:

**Definition 5.3.** If $R$ is a $\sigma$-regular object and all the indecomposable components of $U$ (in diagram (19)) are semistable, then $R$ is said to be final, otherwise - non-final.

If $R$ is a non-final regular object then some indecomposable component of $U$ is not semistable. By regularity this component is also an exceptional object and then we can apply to it $\text{alg}$. Now we cannot exclude the occurrence of the irregular cases $B1, B2$, i.e. we cannot exclude the occurrence of an irregular component of $U$.

6. Regularity-preserving categories. RP properties 1,2

Recall that $\text{alg}$ can be applied to any non-semistable exceptional object. Using the terminology from Section 5, we can say that if $R$ is $\sigma$-regular and non-final, then from the output data $\text{alg}(R)$ we can extract some number of non-semistable exceptional objects (the non-semistable components of $U$ in diagram (19)). The algorithm $\text{alg}$ can be applied to any of them again. If the category $A$ has the property that the cases $B1, B2$ cannot occur after this second iteration of $\text{alg}$ we say that $A$ is regularity-preserving. More precisely:

**Definition 6.1.** A hereditary abelian category $A$ will be said to be regularity-preserving, if for each $\sigma \in \text{Stab}(D^b(A))$ from the the following data:

- $R \in D^b(A)$ is a $\sigma$-regular object; $R \xrightarrow{\nu} (S,E)$, where $X \in \{C1, C2a, C2b, C3\}; E \not\in \sigma^{s\#}$
- it follows that $E$ is a $\sigma$-regular object as well.

In this section we show two restrictions on the exceptional objects, called RP property 1 and RP property 2, which ensure that $A$ is regularity-preserving.

6.1. Ext-nontrivial couples. Looking at the description of $B1, B2$ (see B1.4, B2.3) we see that in any of these cases occur couples $\{L, \Gamma\} \subset A$ of exceptional objects with $\text{hom}^1(L, \Gamma) \neq 0, \text{hom}^1(\Gamma, L) \neq 0$. It is useful to give a name to such a couple:

**Definition 6.2.** An Ext-nontrivial couple is a couple of exceptional objects $\{L, \Gamma\} \subset A_{exc}$, s. t. $\text{hom}^1(L, \Gamma) \neq 0$ and $\text{hom}^1(\Gamma, L) \neq 0$.

- Trivially coupling object is an exceptional object $E \in A_{exc}$, s. t. for each $\Gamma \in A_{exc}$ we have $\text{hom}^1(E, \Gamma) = 0$ or $\text{hom}^1(\Gamma, E) = 0$, i.e. for each $\Gamma \in A_{exc}$ the couple $\{E, \Gamma\}$ is not Ext-nontrivial.

\[\text{Recall that for } X \in A \text{ and } j \in \mathbb{Z} \text{ we write } \deg(X[j]) = j.\]

\[\text{the description of } C1, C2, C3 \text{ is in subsection 4.3.}\]
Lemma 6.7. Let $E \in \mathcal{A}_{\text{exc}}$ be a trivially coupling object, then for each stability condition $\sigma \in \text{Stab}(D^b(A))$ it is either $\sigma$-semistable or $\sigma$-regular.

Thus, an object can be $\sigma$-irregular only if it is an element of an Ext-nontrivial couple. The following lemma gives some information about the other element of the couple.

Lemma 6.4. Let each $X \in \mathcal{A}_{\text{exc}}$ satisfy the dichotomy that it is either trivially coupling or there exists unique up to isomorphism another object $Y \in \mathcal{A}_{\text{exc}}$ such that $\{X,Y\}$ is an Ext-nontrivial couple. Then for each Ext-nontrivial couple $\{E,\Gamma\} \subset \mathcal{A}_{\text{exc}}$ and each $\sigma \in \text{Stab}(D^b(A))$ we have:

(a) If $E$ is a $B2$ object, then $\Gamma$ is semistable of phase $\phi_-(E) - 1$.
(b) If $E$ is a $B1$ object, then $\phi_-(\Gamma) \geq \phi_-(E) + 1$.
(c) At most one of the objects $\{E,\Gamma\}$ can be $\sigma$-irregular.

Proof. (a) By B2.3 there exists a semistable $X \in \mathcal{A}_{\text{exc}}$ of phase $\phi_-(E) - 1$, s. t. $\{E,X\}$ is an Ext-nontrivial couple. From the assumption of the lemma it follows $X \cong \Gamma$.

(b) By B1.3 and B1.4 there exists $X \in \mathcal{A}_{\text{exc}}$ with $\phi_-(X) \geq \phi_-(E) + 1$, s. t. $\{E,X\}$ is an Ext-nontrivial couple. From the assumption of the lemma we have $X \cong \Gamma$, hence $\phi_-(\Gamma) \geq \phi_-(E) + 1$.

(c) It is enough to prove that if $E$ is $\sigma$-irregular then $\Gamma$ is not $\sigma$-irregular. If $E$ is $B2$, then by (a) $\Gamma$ is semistable, i. e. it is not $\sigma$-irregular. By (a) applied to $\Gamma$ it follows also that if $E$ is $B1$ then $\Gamma$ is not $B2$. Whence, it remains to show that $E$ and $\Gamma$ cannot both be $B1$. By (b) we see that if both are $B1$ then $\phi_-(\Gamma) \geq \phi_-(E) + 1$ and $\phi_-(E) \geq \phi_-(\Gamma) + 1$ which is impossible. □

The next step is to show that even with the presence of Ext-nontrivial couples $\mathcal{A}$ could be regularity-preserving.

6.2. RP property 1 and RP property 2. Our key to regularity-preserving of $\mathcal{A}$ are the following patterns of the Ext-nontrivial couples of $\mathcal{A}$.

Definition 6.5. Let $\mathcal{A}$ be a hereditary category. We say that $\mathcal{A}$ has

RP Property 1: if for each Ext-nontrivial couple $\{\Gamma,\Gamma'\} \subset \mathcal{A}$ and for each $X \in \mathcal{A}_{\text{exc}}$ from $\text{hom}^*(\Gamma,X) = 0$ it follows $\text{hom}^*(X,\Gamma') = 0$;

RP Property 2: if for each Ext-nontrivial couple $\{\Gamma,\Gamma'\} \subset \mathcal{A}$ and for any two $X,Y \in \mathcal{A}_{\text{exc}}$ from $\text{hom}(\Gamma,X) \neq 0, \text{hom}(X,Y) \neq 0, \text{hom}^*(\Gamma,Y) = 0$ it follows $\text{hom}(\Gamma',Y) \neq 0$.

The main result of Section 6 is:

Proposition 6.6. If $\mathcal{A}$ has RP Property 1 and RP Property 2 then $\mathcal{A}$ is regularity-preserving.

6.3. Proof of Proposition 6.6 We can assume that $R \in \mathcal{A}$. We split the proof in two lemmas. The first lemma uses RP property 1, but does not use RP property 2.

Lemma 6.7. Let $R$ be a $C3$ object with $\text{alg}(R) = \begin{array}{ccc} & \rightarrow & \\ \leftarrow & A & \rightarrow \end{array} R$. Then each non-semistable $E \in \text{Ind}(A)$ is $\sigma$-regular.

---

\(^{41}\)note that $\text{hom}(\Gamma,X) \neq 0, \text{hom}(X,Y) \neq 0, \text{hom}^*(\Gamma,Y) = 0$ imply $X \neq \Gamma$, $X \neq Y$

\(^{42}\)A is as in Subsection 4.1
Proof. Recall that in \textbf{C3.1}, \textbf{C3.2} we have $A, B \neq 0$, $\text{hom}^* (A, B) = \text{hom}^1 (A, A) = \text{hom}^1 (B, B) = 0$, and $\phi_- (A) > \phi (B) + 1$. The last inequality, together with Corollary 3.8 and Remark 3.7 implies:

\begin{equation}
\phi_- (E) > \phi (B) + 1 \ \Rightarrow \ \text{hom}^* (E, B) = 0.
\end{equation}

If $E$ is a \textbf{B1} object, then we get $\text{alg} (E) = \begin{array}{c} A_1 \oplus A_2 [-1] \\ B' \end{array} \xrightarrow{\sigma} E$, where $B' \in A$ is a direct summand of $\sigma_- (E)$ (see \textbf{B1.2}). By \textbf{(44)} we can apply Lemma 3.6 to $E, B$ and obtain $\text{hom}^{\le 1} (B', B) = 0$, hence $\text{hom}^* (B', B) = 0$. From the triangle $\text{alg} (E)$ it follows $\text{hom}^* (A_2, B) = 0$. By \textbf{B1.4} there exists $E' \in \text{Ind} (A_2)$ s. t. $\{ E, E' \}$ is an Ext-nontrivial couple. So, we obtained $\text{hom}^* (E', B) = 0$. Since $\text{hom}^1 (B, B) = 0$, RP property 1 in subsection 6.2 implies $\text{hom}^* (B, E) = 0$, which contradicts \textbf{C3.3}.

If $E$ is \textbf{B2} object, then we get $\text{alg} (E) = \begin{array}{c} A' \\ \text{alg} \end{array} \xrightarrow{\sigma} E$, where $B'[1] = \sigma_- (E)$ (see \textbf{B2.2}). By \textbf{(44)} we can apply Lemma 3.6 to $E, B'[1]$ and obtain $\text{hom}^{\le 1} (B'[1], B'[1]) = 0$, hence $\text{hom}^* (B', B) = 0$. By \textbf{B2.3} there exists $E' \in \text{Ind} (B'[1])$, s. t. $\{ E, E' \}$ is an Ext-nontrivial couple. So, we obtained $\text{hom}^* (E', B) = 0$ which by RP property 1 implies $\text{hom}^* (B, E) = 0$. This contradicts \textbf{C3.3}. \hfill $\square$

The second lemma uses both RP property 1 and RP property 2.

\textbf{Lemma 6.8.} Let $R, E \in \mathcal{A}_{\text{exc}}$, $R \not\in \sigma^s$, $E \not\in \sigma^s$. If $R, E$ fit into any of the following two situations:

\begin{enumerate}
\item[(a)] $R$ is a \textbf{C1} object, $\text{alg} (R) = \begin{array}{c} A \\ \text{alg} \end{array} \xrightarrow{\phi} R$, $E \in \text{Ind} (A)$;
\item[(b)] $R$ is a \textbf{C2} object, $\text{alg} (R) = \begin{array}{c} A' \oplus A_2 [-1] \\ \text{alg} \end{array} \xrightarrow{\phi} R$, $E \in \text{Ind} (A_1)$ or $E \in \text{Ind} (A_2)$;
\end{enumerate}

then $E$ is $\sigma$-regular.

Proof. The arguments for $E \in \text{Ind} (A)$, $R$ is \textbf{C1} and $E \in \text{Ind} (A_1)$, $R$ is \textbf{C2} are similar. We give them first. Recall that in \textbf{C1.3} and \textbf{C2.3} we have $\phi_- (A) \geq \phi (B)$ and $\phi_- (A_1) \geq \phi (B)$, respectively.

By Corollary 3.8 in \textbf{C1} case we have $\phi_- (E) \geq \phi_- (A) \geq \phi (B)$, and in \textbf{C2} case we have $\phi_- (E) \geq \phi_- (A_1) \geq \phi (B)$. In both the cases (see \textbf{C2.1}, \textbf{C1.1}) we have $\text{hom}^* (E, B) = 0$. In both the cases we have also $\text{hom} (E, R) \neq 0$ (recall that in \textbf{C2} case $A_2$ is a subobject of $R$), so we can write

\begin{equation}
\phi_- (E) \geq \phi (B), \ \text{hom}^* (E, B) = 0, \ \text{hom} (E, R) \neq 0 \quad E, B \in \mathcal{A}.
\end{equation}

If we take any $X \in \text{Ind} (B)$, then $\text{hom} (R, X) \neq 0$ (this is valid in all the five cases). Since $R$ is $\sigma$-regular, we have $X, E \in \mathcal{A}_{\text{exc}}$ and combining with \textbf{(45)} we can write:

\begin{equation}
\text{hom} (E, R) \neq 0, \text{hom} (R, X) \neq 0, \text{hom}^* (E, X) = 0, \quad X, E, R \in \mathcal{A}_{\text{exc}}.
\end{equation}

If $E$ is a \textbf{B2} object, then $\text{alg} (E)$ is of the form

\begin{equation}
\text{alg} (E) = \begin{array}{c} A' \\ \text{alg} \end{array} \xrightarrow{\phi} E.
\end{equation}

From \textbf{(45)} we see that Lemma 3.6 can be applied, which implies $\text{hom} (B', B) = 0$. By \textbf{B2.3}, there exists $E' \in \text{Ind} (B')$, s. t. $\{ E, E' \}$ is an Ext-nontrivial couple. Then by \textbf{(46)} and RP property 2 we obtain $\text{hom} (E', X) \neq 0$, which contradicts $\text{hom} (B', B) = 0$.

\textbf{43}by the last part of Lemma 3.5 and since $X$ is a direct summand of $\sigma_- (E)$
If $E$ is $B_1$ object, then $\mathfrak{alg}(E)$ is of the form

$$\mathfrak{alg}(E) = \xymatrix{ A'_1 \oplus A'_2[-1] \ar[r] & E \ar@/^/[r]^{B'} }$$

with $B' \in A$ and for some $E' \in \text{Ind}(A'_2)$ the couple $\{E, E'\}$ is Ext-nontrivial. From (45) and Lemma 3.6 it follows $\hom^{-1}(B', B) = 0$, hence $\hom^*(B', B) = 0$, which combined with $\hom^*(E, B) = 0$ and the triangle (45), implies $\hom^*(A'_2, B) = 0$. Whence, we obtain $\hom^*(E', B) = 0$, which by RP property 1 and $\hom^1(B, B) = 0$ implies $\hom^*(B, E) = 0$. The last contradicts C1.4, C2.4.

Suppose now that we are in the situation (b) and $E \in \text{Ind}(A_2)$ is a $B_2$ object. Then we again have (47) and some $E' \in \text{Ind}(B')$, s. t. $\{E, E'\}$ is an Ext-nontrivial couple. However, now in addition to $\hom^*(E, B) = 0$ we have $\phi_-(E) \geq \phi_-(A_2) = \phi_-(A_2[-1]) + 1 \geq \phi(B) + 1 = \phi(B[1])$. Now Lemma 3.6 gives $\hom^{-1}(B'[1], B[1]) = 0$, i. e. $\hom^*(B', B) = 0$. Thus, we obtain $\hom^*(E', B) = 0$, hence $\hom^*(B, E) = 0$ by RP property 1, which contradicts C2.4.

Finally, suppose that $E \in \text{Ind}(A_2)$ is a $B_1$ object. Then we can use again (48) and take some $E' \in \text{Ind}(A'_2)$, s. t. $\{E, E'\}$ is an Ext-nontrivial couple. As in the preceding paragraph, in addition to $\hom^*(E, B) = 0$, we have again $\phi_-(E) \geq \phi_-(B[1])$. Now Lemma 3.6 gives $\hom^{-1}(B'[1], B[1]) = 0$, i. e. $\hom^*(B', B) = 0$. Combining with $\hom^*(E, B) = 0$ and the triangle (48) we obtain $\hom^*(E', B) = 0$. As in the previous paragraph, the last vanishing gives a contradiction. □

7. SEQUENCE OF REGULAR CASES

In this section we assume that $A$ is regularity-preserving. If we are given a non-final $\sigma$-regular object $R$, then we can apply $\mathfrak{alg}$ iteratively (Definition 6.1). As a result we obtain a sequences of exceptional pairs (between the subsequent iterations we make a choice, whence the resulting sequence is not uniquely determined by $R$ in general):

$$R \xymatrix{ & X_1 \ar[r]^{\text{proj}} & (S_1, E_1) \ar[r]^{\text{proj}} & E_1 \ar[r]^{\text{proj}} & X_2 \ar[r]^{\text{proj}} & (S_2, E_2) \ar[r]^{\text{proj}} & E_2 \ar[r]^{\text{proj}} & X_3 \ar[r]^{\text{proj}} & (S_3, E_3) \ar[r]^{\text{proj}} & \ldots }$$

where $X_i \in \{C_1, C_2a, C_2b, C_3\}$. Such a sequence will be called an $R$-sequence. The number of the objects $\{S_i\}$ will be called length of the $R$-sequence.\footnote{If $R$ is the exceptional object, which is the origin of the sequence, so for example if the length is $\geq 2$, then after removing the first step $X_1$ we get an $E_1$-sequence.} We study here $R$-sequences.

The sequence (49) can be extended after $E_i$ iff $E_i \notin \sigma^*$, which is possible only if $E_{i-1}$ is not final (Definition 5.3). From (42) it follows (recall that $\theta_\sigma(R)$ is an $\mathbb{N}$-valued function with finite support)

$$\theta_\sigma(R) > \theta_\sigma(E_1) > \theta_\sigma(E_2) > \ldots .$$

Hence we see that after finitely many steps we reach a final $\sigma$-regular object. More precisely:

**Lemma 7.1.** Let $R$ be $\sigma$-regular. There does not exist an infinite $R$-sequence. The lengths of all $R$-sequence are bounded above by $\sum_{u \in \sigma^*_\text{Ind}} \theta_\sigma(R)(u)$.

Some features of the individual steps in any $R$-sequence, specified in (41), (42), and Lemma 6.6 are readily integrated to the following basic features of the whole $R$-sequence:
Lemma 7.2. Let \( R \) be \( \sigma \)-regular. Let an \( R \)-sequence as \((49)\) have length \( n \). Then \( \{ (S_i, E_i) \}_{i=1}^{n} \) is a sequence of exceptional pairs, which, in addition to \((50)\), satisfies the following monotonicities:\footnote{Recall that the notation \( \deg(X) \) is explained in \textit{Some notations} right after the introduction.}

\[
\begin{align*}
\phi_\ast(R) = \phi(S_1) &\leq \phi_\ast(E_1) = \phi(S_2) &\leq \phi_\ast(E_2) = \phi(S_3) &\leq \ldots \\
\deg(R) &\geq \deg(E_1) &\geq \deg(E_2) &\geq \deg(E_3) &\geq \ldots
\end{align*}
\]

where \( \{ S_i \}_{i=1}^{n} \) are semistable, \( \{ E_i \}_{i=1}^{n-1} \) are \( \sigma \)-regular, and the last object \( E_n \) is either semistable or again \( \sigma \)-regular (and then the sequence can be extended).

In the rest of this section we make various refinements of Lemma 7.2. Whence, in the rest of this section the objects \( R, \{ (S_i, E_i) \}_{i=1}^{n} \), and the integer \( n \in \mathbb{N} \) will be as in Lemma 7.2 in particular these objects fit in an \( R \)-sequence \((49)\), which ends at \( E_n \). Assuming this data, we will show that under additional conditions some of the inequalities in \((51)\) are strict, and vanishings, other than the already known \( \{ \hom^\ast(E_i, S_i) = 0 \}_{i=1}^{n} \), appear. The basic lemma is:

Lemma 7.3. Let \( 1 \leq i < n \). Then the following implications hold:

- (a) If \( \deg(S_i) \geq \deg(S_{i+1}) \), then \( \hom^\ast(S_{i+1}, S_i) = 0 \).
- (b) If \( \deg(S_i) = \deg(S_{i+1}) \), then \( \hom^\ast(S_{i+1}, S_i) = 0 \) and \( \phi(S_{i+1}) > \phi(S_i) \).
- (c) If \( \deg(S_i) + 1 = \deg(S_{i+1}) \), then \( \hom^\ast(S_{i+1}, S_i) = 0 \).

Proof. Since \( E_i \) and \( E_{i-1} \) are regular, all the four features specified right after \((19)\) hold for \( \alg(E_{i-1}) \) and \( \alg(E_i) \). Now we unfold the definitions and use these features to write:

\[
\alg(E_{i-1}) = \begin{cases} 
U 
& E_{i-1} \quad \text{\( \text{Ind}(V) \)} \\
V 
& \phi(S_i) = \phi(V) \quad \text{\( \text{Ind}(\sigma_\ast(E_i)) \cap \text{Ind}(\sigma_\ast(E_{i-1})) \)}
\end{cases}
\]

(53) \( \hom^\ast(E_i, V) = 0 \), \( \phi(V') = \phi_\ast(E_i) \geq \phi(V) \), \( \theta_\ast(E_i) < \theta_\ast(E_{i-1}) \).

The first two expressions in \((53)\) show that we can apply Lemma 3.6 to \( E_i \) and \( V \). Since \( V' \) is a direct summand of \( \sigma_\ast(E_i) \) and \( \deg(S_{i+1}) = \deg(V'), \deg(V) = \deg(S_i) \), this lemma gives us: \( \hom^\ast(V', V) = 0 \), if \( \deg(S_{i+1}) \leq \deg(S_i) \); \( \hom^\ast(V', V[1]) = 0 \), if \( \deg(S_{i+1}) = \deg(S_i) + 1 \).

So far we proved (a), (c). It remains to show that the inequality \( \phi(S_{i+1}) > \phi(S_i) \) given by \((51)\) is strict inequality \( \phi(S_{i+1}) > \phi(S_i) \) in (b). We first observe the following implication:

\[
\phi(S_{i+1}) = \phi(S_i) \quad \Rightarrow \quad S_{i+1} \in \text{Ind}(\sigma_\ast(E_i)) \cap \text{Ind}(\sigma_\ast(E_{i-1}))
\]

Indeed, by \((42)\) we have \( \theta_\ast(E_i)(S_{i+1}) \neq 0 \). From \((53)\) it follows that \( \theta_\ast(E_{i-1})(S_{i+1}) \neq 0 \), hence \( S_{i+1} \) is an indecomposable component of some \( \text{HN} \) factor of \( E_{i-1} \). This must be \( \sigma_\ast(E_{i-1}) \), because the assumption \( \phi(S_{i+1}) = \phi(S_i) \) implies \( \phi_\ast(E_{i-1}) = \phi(S_{i+1}) \), so we obtain \((51)\).

Suppose that \( \phi(S_i) = \phi(S_{i+1}) \) and \( \deg(S_i) = \deg(S_{i+1}) \), then \( \phi(V) = \phi(V') \) and \( \deg(V) = \deg(V') = j \) for some \( j \in \mathbb{Z} \). Hence \( V \) and \( V' \) are the degree \( j \) terms of \( \sigma_\ast(E_{i-1}) \) and \( \sigma_\ast(E_i) \), respectively. Now \((54)\) and Krull-Schmidt property imply \( S_{i+1} \in \text{Ind}(V) \cap \text{Ind}(V') \), which contradicts the already proven \( \hom^\ast(V', V) = 0 \). Hence (b) and the lemma follow.

Corollary 7.4. If for each \( i \in \{ 1, 2, \ldots, n \} \) we have \( \deg(S_i) \geq \deg(S_{i+1}) \), then:

- (a) the vanishings \( \hom^\ast(S_i, S_{i+1}) = 0 \) hold for each integer \( i \) with \( 2 \leq i \leq n \),
- (b) furthermore, if \( \deg(S_i) = \deg(S_{i+1}) \) for some \( i \geq 2 \) then \( \phi(S_i) < \phi(S_i) \).
The inequalities \( \{ \deg(S_1) \geq \deg(S_i) \}_{i=1}^n \) hold in any of the following cases:

- \( X_1 = C2a \)
- \( X_1 = C3 \)
- \( C3 \) does not occur in the sequence \( \{X_1, X_2, X_3, \ldots, X_n\} \).

Proof. From Lemma 7.2 we have \( \{\phi_-(E_i) \geq \phi(S_1), \phi(S_i) \geq \phi(S_1)\}_{i=1}^n \) and \( \hom^*(E_1, S_1) = 0 \).

Suppose that for some \( i \) with \( 1 \leq i < n \) we are given \( \hom^*(E_i, S_1) = 0 \) (here we make an induction assumption). We use the triangle \( \alg(E_i) \) (it must be of type \( C1, C2, C3 \)):

\[
\begin{array}{ccc}
U & \rightarrow & E_i \\
\downarrow & & \downarrow \\
V & \rightarrow & F
\end{array}
\quad U, V \in \mathcal{T}, U \neq 0, V \neq 0, \quad S_{i+1} \in \Ind(V) \quad E_{i+1} \in \Ind(U)
\]

where \( V \) is a direct summand of \( \sigma_-(E_i) \) and \( V \) is of pure degree.

By \( \hom^*(E_i, S_1) = 0 \), \( \phi_-(E_i) \geq \phi(S_1) \) we can apply Lemma 3.6 and we obtain

\[
\hom^{\leq 1}(V, S_1) = 0.
\]

Therefore, if \( \deg(S_{i+1}) \leq \deg(S_1) \), then \( \hom^*(V, S_1) = 0 \), since \( \deg(V) = \deg(S_{i+1}) \). Now \( \hom^*(V, S_1) = 0 \) together with the induction assumption \( \hom^*(E_i, S_1) = 0 \) and the triangle \( \alg(E_i) \) give \( \hom^*(U, S_1) = 0 \). Hence \( \hom^*(E_{i+1}, S_1) = 0 \) and \( \hom^*(S_{i+1}, S_1) = 0 \). Part (a) follows.

We prove part (b) by contradiction. Suppose that \( \deg(S_i) = \deg(S_1) \) and \( \phi(S_i) = \phi(S_1) \). From (50) and (12) it follows \( \theta_\sigma(R)(S_i) > \theta_\sigma(E_{i-1})(S_i) \), therefore \( S_i \) is a direct summand of some HN factor of \( R \). On the other hand by \( \phi(S_i) = \phi_-(R) \), \( \phi(S_i) = \phi(S_1) \), and \( \deg(S_i) = \deg(S_1) \) it follows \( S_1, S_i \in \Ind(V) \), where \( V \) is the degree \( \deg(S_i) = \deg(S_1) \) term of \( \sigma_-(R) \). Therefore (recall also C1.2, C2.2, C3.2), we can write \( \alg(R) = \begin{array}{ccc}
U & \rightarrow & E_i \\
\downarrow & & \downarrow \\
V & \rightarrow & F
\end{array} \) and \( S_i \in \Ind(V) \). The definition of

\[
(\text{Definition } 5.2)
\]

implies that we can replace \( S_1 \) by \( S_i \) in the \( R \)-sequence which we consider. However now part (a) of the corollary says that \( \hom^*(S_i, S_1) = 0 \), which contradicts \( S_i \neq 0 \). Hence \( \phi(S_i) > \phi(S_1) \), if \( \deg(S_i) = \deg(S_1) \) and part (b) is shown.

To prove the rest of the corollary, we use Table 13 for comparing degrees.

If we are given \( X_1 = C2a \) or \( X_1 = C3 \), then \( \deg(E_1) = \deg(S_1) - 1 \). From (52) in Lemma 7.2 we can write that \( \deg(E_i) \leq \deg(S_i) - 1 \) for \( i = 1, 2, \ldots, n-1 \), hence \( \deg(E_i) + 1 \leq \deg(S_i) \). By (14) \( \deg(S_{i+1}, E_{i+1}) \) and the last expression in (11) we have also \( \deg(S_{i+1}) \leq \deg(E_i) + 1 \). Hence, we obtain \( \deg(S_{i+1}) \leq \deg(S_i) \) for \( i = 1, 2, \ldots, n-1 \).

Finally, assume that the sequence \( \{X_1, X_2, X_3, \ldots, X_n\} \) does not contain \( C3 \). By the already proven, we can assume that \( X_1 = C2b \) or \( X_1 = C1 \), which implies \( \deg(E_1) = \deg(S_1) \). Since \( C3 \) is forbidden, it follows \( \{\deg(S_{i+1}) = \deg(E_i)\}_{i=1}^{n-1} \), hence by (52) we obtain \( \{\deg(S_{i+1}) \leq \deg(S_i)\}_{i=1}^{n-1} \). The corollary is completely proved.

Corollary 7.4 does not ensure the vanishings \( \{\hom^*(S_i, S_1) = \hom^*(E_i, S_1) = 0\}_{i \geq 2} \) for \( R \)-sequences with first step \( C1 \) or \( C2b \) and containing a \( C3 \) step. The obstacle to obtain these vanishings for each \( R \)-sequence is that the data \( \hom^*(X, S) = 0, S \in \sigma^*, \phi_-(X) \geq \phi(S) \) gives \( \hom^{\leq 1}(\sigma_-(X), S) = 0 \), but not \( \hom^*(\sigma_-(X), S) = 0 \) (see Lemma 3.4).

For certain \( R \)-sequences starting with a \( C1 \) step and ending with a \( C3 \) step we obtain these vanishings in the next lemma, but here we use the property in Corollary 7.4(b) for the first time.

Lemma 7.5. Assume that, besides being regularity-preserving, the category \( A \) satisfies the following: for any two \( X, Y \in A_{exc} \) at most one degree in \( \{\hom^p(X, Y)\}_{p \in \mathbb{Z}} \) is nonzero.
If an $R$-sequence (as in Lemma 7.2) obeys the following restrictions (all the three):

$X_1 = C1$; in the sequence $\{X_2, X_3, \ldots, X_{n-1}\}$ do not occur C2a and C3; $X_n = C3$, then it satisfies $\text{hom}^*(S_i, S_1) = \text{hom}^*(E_i, S_1) = 0$ for $i = 2, \ldots, n$.

Proof. Applying the previous lemma to the sequence obtained by truncating the last step $X_n$, we obtain the given vanishings for $i < n$. We have to prove only $\text{hom}^*(S_i, S_1) = \text{hom}^*(E_n, S_1) = 0$.

We first observe that from $B \longrightarrow \cdots \longrightarrow (S, E)$, $X \in \{C2b, C1\}$ it follows by Definition 5.2 that $\deg(B) = \deg(E)$ and there exists a monic $E \to B$ in $A[\deg(B)]$. Therefore we can assume that $0 = \deg(R) = \deg(E_1) = \cdots = \deg(E_{n-1})$ and $E_1, E_2, \ldots, E_{n-1}$ are $A$-subobjects of $R$. Since $X_n = C3$, we have, by C3.2, that $\text{alg}(E_{n-1}) \cong \text{HN}_-(E_{n-1})$, and we can write:

$$\text{alg}(E_{n-1}) = \begin{cases} A & A, B \in A \\ B[1] & E_n \in \text{Ind}(A) \end{cases}$$

Let us take now any $\Gamma \in \text{Ind}(A)$. From Lemma 4.3 we have $\text{hom}(\Gamma, E_{n-1}) \neq 0$. Since $E_{n-1}$ is an $A$-subobject of $R$ and $\Gamma \in A$, it follows that $\text{hom}(\Gamma, R) \neq 0$. By the given property of $A$ it follows that $\text{hom}^1(\Gamma, R) = 0$ (any $\Gamma \in \text{Ind}(A)$ is exceptional object). Therefore we obtain $\text{hom}^1(A, R) = 0$.

Since $X_1 = C1$, we have a diagram $\text{alg}(R) = \begin{cases} A' & A' \to R \\ B' & B' \to S_n \in \text{Ind}(B[1]) \end{cases}$ and $S_n \in \text{Ind}(B[1])$. By Lemma 4.3 (b) it follows $\text{hom}^1(A, B') = 0$. We have also $\phi_-(A) > \phi_-(E_{n-1}) \geq \phi_-(R) = \phi(B')$, therefore $\text{hom}(A, B') = 0$. Thus, we obtain $\text{hom}^*(A, B') = 0$, and hence $\text{hom}^*(A, S_1) = 0$. The triangle (56) and $\text{hom}^*(E_{n-1}, S_1) = 0$ imply $\text{hom}^*(B, S_1) = 0$. The lemma follows. □

8. Final regular cases

Let $R$ be a final $\sigma$-regular object and $(S, E)$ be any exceptional pair satisfying $R \longrightarrow \cdots \longrightarrow (S, E)$, $X \in \{C1, C2a, C2b, C3\}$. We have that $E \in \sigma^{ss}$ from the very definition of final (Definition 5.3). We show here that, besides being semistable, the exceptional pair $(S, E)$ satisfies $\phi(S) < \phi(E)$ (Corollary 8.2). Furthermore, if $R$ is the middle term of an exceptional triple $(S_{\text{min}}, R, S_{\text{max}})$ (see Corollary 8.5), then the quadruple $(S_{\text{min}}, S, E, S_{\text{max}})$ is also exceptional.

All results here, except the second part of Corollary 8.3 hold without regularity-preserving.

The first lemma ensures some strict inequalities. In this respect it is similar to Lemma 7.3 (b) and Corollary 7.4 (b). As in their proofs, the function $\theta_\sigma$ will be useful again here.

**Lemma 8.1.** Let $R$ be a $\sigma$-regular object with $\text{alg}(R) = \begin{cases} U & U \to R \\ V & \text{hom}(V, U) \end{cases}$. For each $\Gamma \in \text{Ind}(U)$ from $\Gamma \in \sigma^{ss}$ it follows that $\phi(V) < \phi(\Gamma)$. In particular, if $R$ is a final, then $\phi_-(U) > \phi(V)$.

Proof. For simplicity, let $R \in A$. If $R$ is a C3 object, then the lemma is true by C3.2, so we can assume that $R$ is a C1 or a C2 object. Then the triangle $\text{alg}(R)$ is of the form (if $R$ is C1, then $A_2 = 0$, otherwise $A_2 \neq 0$)

$$\begin{cases} A_1 \oplus A_2[-1] & A_1 \oplus A_2[-1] \\ B & B \to R \end{cases}$$

$$\text{hom}^*(A_1, B) = \text{hom}^*(A_2, B) = 0$$

$$\theta_\sigma(A_1 \oplus A_2[-1]) < \theta_\sigma(R).$$

We consider first the case $\Gamma \in \sigma^{ss} \cap \text{Ind}(A)$. Then $\theta_\sigma(\Gamma) \leq \theta_\sigma(A_1 \oplus A_2[-1]) < \theta_\sigma(R)$. Since $\Gamma$ is semistable, the last inequality implies $\theta_\sigma(R)(\Gamma) \neq 0$, hence $\Gamma$ is an indecomposable component.
of some HN factor of $R$. If $\phi(\Gamma) = \phi(B)$ then this must be the minimal HN factor $\sigma_-(R)$. On the other hand $\deg(\Gamma) = 0$ and $B$ is the zero degree of $\sigma_-(R)$. Therefore, we see that if $\phi(\Gamma) = \phi(B)$, then $\Gamma \in \Ind(B)$, which contradicts $\hom^*(A_1, B) = 0$.

Now let $\Gamma \in \sigma^\ast \cap \Ind(A_2)$. Then $\theta_\sigma(\Gamma[-1]) \leq \theta_\sigma(A_1 \oplus A_2[-1]) < \theta_\sigma(R)$ and as in the previous case we deduce that $\Gamma[-1]$ is an indecomposable component of an HN factor of $R$. If $\phi(\Gamma[-1]) = \phi(B)$ then this must be $\sigma_-(R)$, but $\deg(\Gamma[-1]) = -1$, which contradicts Lemma 4.5 (a).

If $R$ is final, then each $\Gamma \in \Ind(U)$ is semistable and the lemma follows. 

By this lemma and Definition 5.2 we obtain:

**Corollary 8.2.** Let $R$ be final $\sigma$-regular. Let $R \xrightarrow{X} (S, E)$. Then $S,E \in \sigma^\ast$ and $\phi(E) > \phi(S)$.  

Having $\phi(S) < \phi(E)$, it follows that $(S, E[-i])$ is a $\sigma$-pair (Definition 3.19) for some $i \geq 1$. Indeed, we have $\phi(S) - 1 < \phi(E[-i]) \leq \phi(S)$ for some $i \geq 1$. Since $\deg(S) \geq \deg(E)$ (recall 4.1), the pair $(S, E[-i])$ has all the features of a $\sigma$-pair. Thus, we obtain the first part of the following corollary:

**Corollary 8.3.** Each final $\sigma$-regular object implies the existence of a $\sigma$-exceptional pair.

In particular, if $A$ is regularity-preserving, then each $\sigma$-regular object induces such a pair.

**Proof.** If there exists a $\sigma$-regular object, then by preserving of regularity and Lemma 7.1 we get a final $\sigma$-regular object. Hence, by the first part, we obtain a $\sigma$-exceptional pair. 

If $A$ has not Ext-nontrivial couples, then each non-semistable exceptional object is $\sigma$-regular for each stability condition, hence:

**Remark 8.4.** If there are not Ext-nontrivial couples in $A$, as in $A = \Rep_k(K(l))$, then each non-semistable exceptional object induces a $\sigma$-exceptional pair.

The origin of our main $\sigma$-triples criterion (Proposition 9.16) is in the next corollary.

**Corollary 8.5.** If we are given the following data:

- $S_{\min}, S_{\max} \in \sigma^\ast \cap A_{\exc}$ with $\phi(S_{\min}) \leq \phi(A) \leq \phi(S_{\max})$ for each $A \in \sigma^\ast \cap A_{\exc}$
- $(S_{\min}, R, S_{\max})$ is an exceptional triple, s. t. $R \in A_{\exc}$ is final and $\sigma$-regular
- $R \xrightarrow{X} (S, E)$, $X \in C1, C2a, C2b, C3$,

then $(S_{\min}, S, E, S_{\max})$ is a semistable exceptional quadruple (no two of $R, S, E$ are isomorphic).

**Proof.** We have $\hom^*(E, S) = 0$ (in particular $S \not\cong E$) and we must show that $\hom^*(S_{\max}, S) = \hom^*(S_{\max}, E) = \hom^*(S, S_{\min}) = \hom^*(E, S_{\min}) = 0$. By assumption $R$ is final and then both $S,E$ are semistable. Since $R$ is not semistable, it cannot be isomorphic to $S$ or to $E$.

Let us assume first that $R$ is a $C3$ object. Then we have a triangle $\text{alg}(R) = \xymatrix{A \ar[rr]^-R & & B[1]}$ with $\hom^*(A, B) = 0$ and $E \in \Ind(A), S \in \Ind(B[1])$. The assumptions on $S_{\min}, S_{\max}$ and C3.2 imply $\phi(S_{\max}) \geq \phi(E) > \phi(B) + 1 = \phi(S) + 1 \geq \phi(S_{\min}) + 1$.

Hence $\hom^*(S_{\max}, B) = 0$, which, combined with $\hom^*(S_{\max}, R) = 0$ and the triangle $\text{alg}(R)$, implies $\hom^*(S_{\max}, A) = 0$. Thus, we get $\hom^*(S_{\max}, S) = \hom^*(S_{\max}, E) = 0$. Since each $\Gamma \in \Ind(A)$ satisfies $\phi(\Gamma) > \phi(B) + 1 \geq \phi(S_{\min}) + 1$, we have $\hom^*(A, S_{\min}) = 0$. Now $\hom^*(R, S_{\min}) = 0$ and $\text{alg}(R)$ imply $\hom^*(B, S_{\min}) = 0$. Thus, we get $\hom^*(S, S_{\min}) = \hom^*(E, S_{\min}) = 0$ as well.
Let us assume now that $R$ is a C1 or C2 object. Then the triangle $\tilde{\mathfrak{alg}}(R)$ is of the form (if $R$ is C1, then $A_2 = 0$, otherwise $A_2 \neq 0$):

\[
\begin{array}{ccc}
A_1 \oplus A_2[-1] & \rightarrow & R \\
\downarrow & & \downarrow \\
B & \rightarrow & \end{array}
\]

(58) \hspace{1cm} \tilde{\mathfrak{alg}}(R) = \begin{array}{ccc}
A_1, A_2, B \in A \\
hom^*(A_1, B) = hom^*(A_2, B) = 0 \\
E \in Ind(A_1 \oplus A_2[-1]), \ S \in Ind(B).
\end{array}

Since $B \neq 0$ is semistable and $\text{hom}^1(B, B) = 0$, it follows $\phi(S_{\max}) \geq \phi(B) \geq \phi(S_{\min})$. On the other hand, we have $\text{hom}^*(R, S_{\min}) = 0$ and $\phi_\ast(R) = \phi(B)$. From Lemma 3.8 it follows $\text{hom}^*(B, S_{\min}) = 0$, which, combined with $\text{hom}^*(R, S_{\min}) = 0$ and the triangle $\tilde{\mathfrak{alg}}(R)$, implies $\text{hom}^*(A_1 \oplus A_2[-1], S_{\min}) = 0$. So, we obtained $\text{hom}^*(S, S_{\min}) = \text{hom}^*(E, S_{\min}) = 0$ and it remains to show $\text{hom}^*(S_{\max}, S) = \text{hom}^*(S_{\max}, E) = 0$. From Lemma 8.1 it follows that for each indecomposable component $\Gamma$ of $A_1$, resp $A_2$, we have $\phi(\Gamma) > \phi(B)$, resp. $\phi(\Gamma[-1]) > \phi(B)$, and combining with $\phi(S_{\max}) \geq \phi(\Gamma)$ we see that $\phi(S_{\max}) > \phi(E)$, hence $\text{hom}(S_{\max}, B) = 0$.

Furthermore, if $R$ is C2, then $A_2 \neq 0$ and $\phi(S_{\max}) \geq \phi(\Gamma)$, $\phi(\Gamma[-1]) > \phi(B)$ for each $\Gamma \in Ind(A_2)$. Therefore $\phi(S_{\max}) > \phi(B) + 1$ and $\text{hom}^*(S_{\max}, B) = 0$. The latter together with $\text{hom}^*(S_{\max}, R) = 0$ imply $\text{hom}^*(S_{\max}, A_1) = 0$, and the corollary follows.

Finally, if $R$ is C1, then $A_2 = 0$ in the triangle (58) and we have a short exact sequence $0 \rightarrow A_1 \rightarrow R \rightarrow B \rightarrow 0$. Hence, by Lemma 4.3 and $\text{hom}(S_{\max}, R[1]) = 0$ we get $\text{hom}(S_{\max}, B[1]) = 0$. We showed already that $\text{hom}(S_{\max}, B[1]) = 0$, therefore $\text{hom}^*(S_{\max}, B) = 0$. Using again the triangle (58) and $\text{hom}^*(S_{\max}, R) = 0$ we obtain $\text{hom}^*(S_{\max}, A_1) = 0$. The corollary follows.

9. Constructing $\sigma$-exceptional triples

So far, the property of Corollary 2.6(b) was used only in Lemma 7.5. In this section it is used throughout. We start with a simple observation:

**Lemma 9.1.** Let $A$ be as in Subsection 4.1. Let Corollary 2.6(b) hold for $A$. Then for any two non-isomorphic exceptional objects $A, B \in A$ we have $\text{hom}(A, B) = 0$ or $\text{hom}(B, A) = 0$.

In particular, if $C \in A$ satisfies $\text{hom}^1(C, C) = 0$, then for any two non-isomorphic $A, B \in Ind(C)$ one of the pairs $(A, B), (B, A)$ is exceptional.

**Proof.** Let $\text{hom}(A, B) \neq 0$. Take a nonzero $u : A \rightarrow B$. By Corollary 2.6(b) it follows $\text{hom}^1(A, B) = 0$. One can show that Lemma 1, page 9 holds for $A$, so $\text{hom}^1(A, B) = 0$ implies that every nonzero $f \in \text{hom}(B, A)$ is either monic or epic. Suppose that $f \in \text{hom}(B, A)$ is epic, then $u \circ f \in \text{hom}(B, B) = k$ is nonzero, hence $f$ is also monic. Therefore $f$ is invertible, which contradicts the assumptions. If $f$ is monic, then we consider $f \circ u$ and again get a contradiction.

The second part follows from Remark 4.1.

Besides the restrictions of Subsection 4.1, we assume throughout Section 9 that Corollary 2.6(b) holds for $A$ and that $A$ is regularity-preserving. In Subsection 9.2, besides these features, we assume that $A$ has the additional RP property (Corollary 2.7) and that Corollary 2.10 holds for it. In particular, all results hold for $A = \text{Rep}_k(Q_1)$ (the preserving of regularity follows from Corollary 2.6(a) and Proposition 6.3).

We denote $D^b(A)$ by $\mathcal{T}$, and choose any $\sigma \in \text{Stab}(\mathcal{T})$. In Corollary 8.3 it is shown that any $\sigma$-regular object $R$ induces a $\sigma$-pair. If $R$ is final, then this pair is of the form $(S, E[-j])$ with $j \geq 0$, for any $R \rightarrow (S, E)$. Using a $\sigma$-regular object $R$, we will obtain in this section various criteria for existence of $\sigma$-exceptional triples in $\mathcal{T}$. To obtain a $\sigma$-triple we utilize three approaches: using
long \( R \)-sequences (of length greater than one); combining the \( \sigma \)-pairs induced by several single step \( R \)-sequences with a final \( R \); combining a \( \sigma \)-pair induced from \( R \) with a semistable \( S \in \mathcal{A}_{\text{exc}} \cap \sigma^{\text{ss}} \) of phase close to the minimal/maximal phase. The minimal and maximal phases are defined by

\[
\begin{align*}
\phi_{\text{min}} &= \inf(\{\phi(S) : S \in \sigma^{\text{ss}} \cap \mathcal{A}_{\text{exc}}\}) \\
\phi_{\text{max}} &= \sup(\{\phi(S) : S \in \sigma^{\text{ss}} \cap \mathcal{A}_{\text{exc}}\}).
\end{align*}
\]

Note that if Corollary 2.10 holds for \( \mathcal{A} \), which is assumed in Subsection 9.2 then we have \( -\infty < \phi_{\text{min}} \leq \phi_{\text{max}} < \infty \). Indeed, if some of the strict inequalities fail, then we can construct a sequence \( S_1, S_2, S_3, S_4, \ldots, S_n \) (as long as we want) of semistable exceptional objects in \( \mathcal{A} \), s.t. \( \{\phi(S_i) + 1 < \phi(S_{i+1})\}_{i=1}^{n-1} \), which contradicts Corollary 2.10.

We denote by \( S_{\text{min}}/S_{\text{max}} \) objects in \( \mathcal{A}_{\text{exc}} \cap \sigma^{\text{ss}} \) satisfying \( \phi(S_{\text{min}}) = \phi_{\text{min}}/\phi(S_{\text{max}}) = \phi_{\text{max}} \), this can be expressed by writing \( S_{\text{min}}/S_{\text{max}} \in \mathcal{P}(\phi_{\text{min}}/\phi_{\text{max}}) \cap \mathcal{A}_{\text{exc}} \).

We note in advance that by replacing “C3” with “C2” and “\( \phi_{\text{min}} \)” with “\( \phi_{\text{max}} \)” we obtain the criteria using long \( R \)-sequences with a middle term of an exceptional triple \( \langle S_{\text{min}}, E, S_{\text{max}} \rangle \) induces a \( \sigma \)-exceptional triple (the regularity of \( E \) follows).

### 9.1 Constructions without assuming the additional RP property.

Recall (Definition 3.5) that an exceptional triple \( \langle S_0, S_1, S_2 \rangle \) is said to be \( \sigma \)-exceptional under three conditions: it must be semistable, it must satisfy \( \text{hom}^{\leq 0}(S_0, S_1) = \text{hom}^{\leq 0}(S_0, S_2) = \text{hom}^{\leq 0}(S_1, S_2) = 0 \), and the phases of its elements must be in \( (t, t + 1] \) for some \( t \in \mathbb{R} \). If we are given only that \( \langle S_0, S_1, S_2 \rangle \) is semistable, then we can always ensure the second or the third condition by applying the shift functor to \( S_1, S_2 \), but both together - not always. For example if \( \phi(S_i) = \phi(S_{i+1}) \), \( \text{hom}(S_i, S_{i+1}) \neq 0 \) \( (i = 0, 1) \), then this cannot be achieved (similarly, if \( \phi(S_i) = \phi(S_{i+1}) + 1 \), \( \text{hom}^1(S_i, S_{i+1}) \neq 0 \)).

The following lemma are given some cases in which this can be achieved. We give the arguments for one of them. The rest are also easy. Keeping in mind Remark 3.21 is useful, when checking these implications.

**Lemma 9.2.** Let \( \langle S_0, S_1, S_2 \rangle \) be a semistable exceptional triple, where \( S_0, S_1, S_2 \in \mathcal{A} \). If any of the following conditions holds:

\begin{align*}
(a) & \quad \phi(S_0) < \phi(S_1) < \phi(S_2), \quad 1 + \phi(S_0) < \phi(S_2) \\
(b) & \quad \phi(S_0) \leq \phi(S_1) < \phi(S_2), \quad \text{hom}(S_0, S_1) = 0 \\
(c) & \quad \phi(S_0) < \phi(S_1) \leq \phi(S_2), \quad \text{hom}(S_1, S_2) = 0 \\
(d) & \quad \phi(S_0) < \phi(S_2) \leq \phi(S_1) < \phi(S_2) + 1, \quad \text{hom}(S_1, S_2) = 0 \\
(e) & \quad \phi(S_0) < \phi(S_1) + 1, \quad \phi(S_1) < \phi(S_2), \quad \phi(S_0) < \phi(S_2), \quad \text{hom}(S_0, S_1) = 0 \\
(f) & \quad \phi(S_0) < \phi(S_1) + 1, \quad \phi(S_0) < \phi(S_2) + 1, \quad \phi(S_1) < \phi(S_2) + 1, \quad \text{hom}(S_0, S_1) = \text{hom}(S_0, S_2) = \text{hom}(S_1, S_2) = 0 \\
(g) & \quad \phi(S_0) < \phi(S_2), \phi(S_0) + 1 < \phi(S_1), \phi(S_2) \neq \phi(S_1[-1]), \quad \text{hom}(S_0, S_2) = \text{hom}(S_1, S_2) = 0,
\end{align*}

then for some integers \( 0 \leq i, 0 \leq j \) the triple \( \langle S_0, S_1[-i], S_2[-j] \rangle \) is \( \sigma \)-exceptional.

\[^{46}\text{For the notation } \sigma^{\text{ss}} \text{ see } (9) \text{ and recall that by } \mathcal{A}_{\text{exc}} \text{ we denote the set of exceptional objects of } \mathcal{A} \]
In particular, there exists $S$. Proposition 9.4. hom($S$).

Proof. By Definition 5.2 with $\mathcal{R}$. Let $S$, $S'$ satisfy (60). From the data: $\exists$ such a sequence exists. Since the exceptional triple $S_0$, $S_1$, $S_2$ of the form $S_0$, $S_1$, $S_2$, $S_3$. If $S_0$, $S_1$, $S_2$ $\exists$-exceptional triple. In particular each non-final $\exists$-exceptional triple.

The next lemma is a step in the proof of our basic long $R$-sequences criterion Proposition 9.4.

**Lemma 9.3.** Let $R \rightarrow (S, E)$, where $X \in \{C_1, C_2\}$. Then there exists $S'$, such that

$$
\begin{align*}
R \rightarrow (S', E), \quad \text{hom}(S', E) &= 0, \quad \text{hom}(R, S') \neq 0, \quad \text{hom}(E, R) \\
\end{align*}
$$

Proof. By Definition 5.2 with $X \in \{C_1, C_2\}$, there is a triangle of the form $\mathcal{A}_1 \oplus A_2 \to R \rightarrow B \rightarrow A_1[1] \oplus A_2$ and $S \in Ind(B)$, $E \in Ind(A_1)$. Furthermore, any $A' \in Ind(A_1)$, $B' \in Ind(B)$ satisfy $\text{hom}(B, A') \neq 0$, $\text{hom}(R, B') \neq 0$, $\text{hom}(A', R) \neq 0$ (see C1, C2 and Lemma 4.5). In particular, there exists $S' \in Ind(B)$, with $\text{hom}(S', E) \neq 0$. By Corollary 7.4 (b) it follows that $\text{hom}(S', E) = 0$. The lemma follows.

Now we obtain $\sigma$-triples from certain, but not all, long $R$-sequences.

**Proposition 9.4.** If there exists an $R$-sequence

$$
\begin{align*}
R \rightarrow (S_1, E_1) \rightarrow (S_2, E_2) \rightarrow \ldots \rightarrow (S_n, E_n)
\end{align*}
$$

with $n \geq 2$, $E_{n-1}$ is final, and $\{\deg(S_i) \geq \deg(S_i)\}_{i=1}^n$, then there exists a $\sigma$-exceptional triple.

Proof. Assume that such a sequence exists. Since $E_{n-1}$ is final, Corollary 8.2 implies that $S_n$ and $E_n$ are both semistable and $\phi(E_n) > \phi(S_n)$. Since $\deg(S_1) \geq \deg(S_i)$ for each $i = \{1, 2, \ldots, n\}$, by Corollary 7.4 and table (13) we obtain

$$
\text{hom}^*(S_1, S_1) = \text{hom}^*(E_n, S_1) = 0
$$

$$
\deg(S_1) \geq \deg(S_n) \geq \deg(E_n), \quad \phi(S_1) \leq \phi(S_n) < \phi(E_n).
$$

In particular, the exceptional triple $(S_1, S_n, E_n)$ is semistable and after shifting we obtain a triple of the form $(A, B[-i], C[-i-j])$ with $0 \leq i, 0 \leq j, \phi(A) \leq \phi(B[-i]) < \phi(C[-i-j])$. If $i \neq 0$, then Lemma 9.2 (a) can be applied to the triple $(A, B, C)$ and the proposition follows.

If $i = 0$, then $\deg(S_1) = \deg(S_n)$. By Corollary 7.4 (b) it follows $\phi(S_1) < \phi(S_n)$. Whence, we obtain a semistable triple $(A, B[-i-j])$ with $0 \leq j, \phi(A) < \phi(B) < \phi(C[-j])$. If $j \neq 0$, then the triple $(A, B, C)$ satisfies the conditions in Lemma 9.2 (a). If $j = 0$, then $X_n \in \{C_2b, C_1\}$ and due to Lemma 9.3 we can assume that $\text{hom}(S_n, E_n) = \text{hom}(B, C) = 0$. Now the triple $(A, B, C)$ satisfies the conditions in Lemma 9.2 (c). The proposition follows.

It follows that any long $R$-sequence starting with a $C_3$ or a $C_2a$ step induces a $\sigma$-triple:

**Corollary 9.5.** From the data: $R \rightarrow (S, E), X \in \{C_3, C_2a\}, E \notin \sigma^{**}$ it follows that there exists a $\sigma$-exceptional triple. In particular each non-final $C_3$ object implies such a triple.
Proof. Since $E \not\in \sigma^s$, by Lemma 7.1 we obtain an $R$-sequence with maximal length $n \geq 2$ and with first step the given $R \longrightarrow (S, E)$. This sequence is of the form $60$ with $X_1 = X$, $n \geq 2$. As far as the sequence is of maximal length, the object $E_{n-1}$ must be final and $\sigma$-regular. Since $X_1 = X \in \{C_3, C_2a\}$, Corollary 7.1 gives $\{\deg(S_i) \geq \deg(S_i)\}_{i=1}^n$. Thus, we constructed an $R$-sequence $60$ with the three properties used in Proposition 9.4. The corollary follows. 

The next lemma uses a final regular object $R$, so we do not have long $R$-sequences here.

**Lemma 9.6.** Let $R$ be a final $\sigma$-regular object with $\text{alg}(R) = U \xrightarrow{\rho} R \xleftarrow{\nu} V$. Then we have:

(a) If $\text{alg}(R)$ is not the HN filtration of $R$, then $U$ is not semistable.

(b) If $U$ is not semistable, then there exists a $\sigma$-exceptional triple.

Proof. Without loss of generality we can assume that $R \in A$. Since $R$ is a final $\sigma$-regular object, any $\Gamma \in \text{Ind}(U)$ is a semistable exceptional object, and hence by Lemma 8.1 it satisfies $\phi(\Gamma) > \phi(V)$. Now part (a) is clear and it remains to prove (b).

If $U$ is not semistable, then there exists a pair of non-isomorphic $\Gamma_1, \Gamma_2 \in \text{Ind}(U)$ with different phases. We can assume $\phi(\Gamma_2) > \phi(\Gamma_1)$. In particular, for the rest of the proof we can use

$$\text{hom}(\Gamma_2, \Gamma_1) = 0, \quad \phi(\Gamma_2) > \phi(\Gamma_1) > \phi(V).$$

First, assume that $R$ is a $C_1$ object. Then the triangle $\text{alg}(R)$ and some of its properties are

$$\text{alg}(R) = \begin{array}{ccc} A & \xrightarrow{R} & B \\ V & \xrightarrow{\rho} & U \end{array}, \quad A, B \in A, \text{hom}^1(A, A) = \text{hom}^1(B, B) = \text{hom}^*(A, B) = 0.$$

By $\text{hom}^1(A, A) = 0$ we have $\text{hom}^1(\Gamma_2, \Gamma_1) = 0$, which, combined with $\text{hom}(\Gamma_2, \Gamma_1) = 0$, implies $\text{hom}^*(\Gamma_2, \Gamma_1) = 0$. By $\text{hom}^*(A, B) = 0$ it follows that for each $\Gamma \in \text{Ind}(B)$ we have $\text{hom}^*(\Gamma_2, \Gamma) = 0$, $i = 1, 2$. Hence for each $\Gamma \in \text{Ind}(B)$ the triple $(\Gamma, \Gamma_1, \Gamma_2)$ is exceptional and $\phi(V) = \phi(\Gamma) < \phi(\Gamma_1) < \phi(\Gamma_2)$. By $C_1.4$ we have $\text{hom}^1(B, \Gamma_1) \neq 0$, and hence we can choose $\Gamma$ so that $\text{hom}^1(\Gamma, \Gamma_1) \neq 0$, which by Corollary 2.6 (b) implies $\text{hom}(\Gamma, \Gamma_1) = 0$. Thus, we constructed an exceptional triple $(\Gamma, \Gamma_1, \Gamma_2)$ with $\text{hom}(\Gamma, \Gamma_1) = 0$, $\phi(\Gamma) < \phi(\Gamma_1) < \phi(\Gamma_2)$. By Lemma 9.2 (b), after shifting this triple becomes $\sigma$-exceptional.

In $C_3$ case:

$$\text{alg}(R) = \begin{array}{ccc} A & \xrightarrow{R} & B \\ B[1] & \xrightarrow{\rho} & U \end{array}, \quad A, B \in A \setminus \{0\}, \text{hom}^1(A, A) = \text{hom}^1(B, B) = \text{hom}^*(A, B) = 0.$$

As in the previous case we obtain that for each $\Gamma \in \text{Ind}(B)$ the triple $(\Gamma, \Gamma_1, \Gamma_2)$ is exceptional. Now $61$ becomes $\phi(V) = \phi(\Gamma) + 1 < \phi(\Gamma_1) < \phi(\Gamma_2)$ and Lemma 9.2 (a) gives a $\sigma$-triple.

In $C_2$ case the triangle $\text{alg}(R)$ and some of its properties are:

$$\begin{array}{ccc} A_1 \oplus A_2[-1] & \xrightarrow{R} & A_2, B \in A \setminus \{0\} \\ B & \xrightarrow{\rho} & U \end{array}, \quad \text{hom}^1(A_1, A_1) = \text{hom}^1(A_2, A_2) = \text{hom}^1(B, B) = 0, \quad \text{hom}^*(A_1, A_2) = \text{hom}^*(A_1, B) = \text{hom}^*(A_2, B) = 0.$$

If both $\Gamma_1, \Gamma_2 \in \text{Ind}(A_1)$, then the arguments are the same as in $C_1$ case.

If both $\Gamma_1, \Gamma_2 \in \text{Ind}(A_2[-1])$, then $\text{hom}^1(B, B) = \text{hom}^*(A_2, B) = 0$ imply that for each $\Gamma \in \text{Ind}(B)$ the triple $(\Gamma, \Gamma_1, \Gamma_2)$ is exceptional and now $\Gamma_i[1] \in A, \phi(\Gamma_i[1]) > \phi(B) + 1 = \phi(\Gamma) + 1$, i.e. $\phi(\Gamma) + 1 < \phi(\Gamma_1[1]) < \phi(\Gamma_2[1])$. From this data Lemma 9.2 (a) produces a $\sigma$-exceptional triple.
Before we continue with the other possibility, we note that

\[(63) \quad \text{hom}(A_2, A_1) = 0.\]

Indeed, by C2.4 for each $\Gamma \in \text{Ind}(A_2)$ we have $\text{hom}(\Gamma, R[1]) \neq 0$, then by Corollary 2.6 (b) it follows $\text{hom}(\Gamma, R) = 0$, i.e. $\text{hom}(A_2, R) = 0$. Now $\text{hom}(A_2, A_1) = 0$ follows from the fact that $A_1$ is a proper subobject of $R$ in $A$.

If $\Gamma_1 \in \text{Ind}(A_1)$, $\Gamma_2 \in \text{Ind}(A_2[-1])$, then (see (62)) for each $\Gamma \in \text{Ind}(B)$ the triple $(\Gamma, \Gamma_2, \Gamma_1)$ is exceptional. We will show that $\Gamma \in \text{Ind}(B)$ can be chosen so that the conditions of Lemma 9.2 (g) hold with the triple $(\Gamma, \Gamma_2[1], \Gamma_1)$. These conditions are: $\phi(\Gamma) < \phi(\Gamma_1), \phi(\Gamma) + 1 < \phi(\Gamma_2[1]), \phi(\Gamma_2) \neq \phi(\Gamma_1), \text{hom}(\Gamma, \Gamma_1) = \text{hom}(\Gamma_2[1], \Gamma_1) = 0$.

By C2.4 we see that $\Gamma$ can be chosen so that $\text{hom}^1(\Gamma, \Gamma_1) \neq 0$ and then by Corollary 2.6 (b) $\text{hom}(\Gamma, \Gamma_1) = 0$. We have the vanishing $\text{hom}(\Gamma_2[1], \Gamma_1) = 0$ by $\text{hom}(A_2, A_1) = 0$. The inequalities $\phi(\Gamma_1) > \phi(\Gamma), \phi(\Gamma_2[1]) > \phi(\Gamma) + 1$ hold because $\Gamma_1, \Gamma_2$ are components of $U = A_1 \oplus A_2[-1]$. Finally, we have $\phi(\Gamma_2) \neq \phi(\Gamma_1)$ by assumption and the conditions of Lemma 9.2 (g) are verified. The lemma follows. \square

**Corollary 9.7.** Let $R \in A_{\text{exc}}$ be a C3 object with $\text{alg}(R) = A \xrightarrow{A} R \xleftarrow{B[1]} B$. If $\text{alg}(R)$ differs from the HN filtration of $R$ or they coincide and $\phi_{\text{min}} < \phi(B)$, then there exists a $\sigma$-exceptional triple.

**Proof.** By the previous lemma and Corollary 9.5 we can assume that $\text{alg}(R)$ is the HN filtration, hence $A$ is semistable and $\phi(A) > \phi(B) + 1$. If $\phi_{\text{min}} < \phi(B)$, then $\phi(B) > \phi(S)$ for some $S \in A_{\text{exc}} \cap \sigma^{ss}$, and by $\phi(A) > \phi(B) + 1$ we obtain $\text{hom}^*(A, S) = 0$. Since we have $\phi_-(R) = \phi(B) + 1 > \phi(S) + 1$, it follows $\text{hom}^*(R, S) = 0$, which due to $\text{alg}(R)$ gives $\text{hom}^*(B, S) = 0$. Thus, we see that for any $A' \in \text{Ind}(A), B' \in \text{Ind}(B)$ the triple $(S, B', A')$ is semistable and exceptional with $\phi(S) < \phi(B') < \phi(A'), \phi(S) + 1 < \phi(A')$. Now the corollary follows from Lemma 9.2 (a). \square

We obtain now $\sigma$-triples from some $R$-sequences starting with a C1 object $R$.

**Lemma 9.8.** Let $R \in A$ be a C1 object. Let $R \xrightarrow{\cdots} (S_1, E_1) \xrightarrow{\text{proj}} E_1 \xrightarrow{\cdots} (S_2[1], E_2)$ be an $R$-sequence. Then $(S_1, S_2, E_2)$ is an exceptional triple with $\phi(S_2) + 1 < \phi_-(E_2)$ and $\text{hom}(S_1, S_2) = 0$.

Furthermore, any of the three conditions $E_2 \notin \sigma^{ss}$, $\phi(S_2) > \phi_{\text{min}}$; $\phi(S_1) \neq \phi(S_2) + 1$ implies an existence of a $\sigma$-exceptional triple.

**Proof.** By Lemma 7.5 we see that $(S_1, S_2, E_2)$ is an exceptional triple. Since $E_1$ is a C3 object, we can write $\text{alg}(E_1) = A \xrightarrow{A} E_1 \xleftarrow{B'} E_1'$ and (see C3.2) $\phi_-(A') > \phi(B') + 1$. From $E_2 \in \text{Ind}(A'), S_2 \in \text{Ind}(B')$ we obtain the first property $\phi(S_2) + 1 < \phi_-(E_2)$.

Next, we consider the vanishing $\text{hom}(S_1, S_2) = 0$. From C3.3 it follows $\text{hom}(E_2, E_1) \neq 0$. As far as $R$ is a C1 object, we can write $\text{alg}(R) = A \xrightarrow{A} R \xleftarrow{B}$ and $E_1 \in \text{Ind}(A), S_1 \in \text{Ind}(B)$. In particular $E_1$ is a subobject of $R$ in $A$. Now by $E_1, E_2, R \in A$ and $\text{hom}(E_2, E_1) \neq 0$ it follows that $\text{hom}(E_2, R) \neq 0$, and hence Corollary 9.1 implies $\text{hom}(R, E_2) = 0$. These arguments hold for each

---

**Footnote:** We do not need to consider separately the case: $\Gamma_1 \in \text{Ind}(A_2), \Gamma_2 \in \text{Ind}(A_1)$, for the relation $\phi(\Gamma_2) \neq \phi(\Gamma_1)$ is symmetric.
element in \( \text{Ind}(\mathcal{A}') \), hence \( \text{hom}(R, \mathcal{A}') = 0 \). By the exact sequence \( \text{alg}(E_1) \) we get \( \text{hom}(R, B') = 0 \), and by the exact sequence \( \text{alg}(R) \) we get \( \text{hom}(B, B') = 0 \), hence \( \text{hom}(S_1, S_2) = 0 \).

If \( E_2 \notin \sigma^{ss} \), then we get a \( \sigma \)-triple from Corollary 9.5 so let \( E_2 \in \sigma^{ss} \). If \( \phi(S_2) > \phi_{\text{min}} \), then by Corollary 9.7 the lemma follows.

Finally, consider the condition \( \phi(S_1) \neq \phi(S_2) + 1 \). Since we have also \( \phi(B'[1]) = \phi_\sigma(E_1) \geq \phi(S_1) \), we can write \( \phi(S_1) < \phi(S_2) + 1 \). We already obtained \( \phi(S_2) + 1 < \phi(E_2) \) in the beginning of the proof. Thus, the triple \( (S_1, S_2, E_2) \) satisfies \( \phi(S_1) < \phi(S_2) + 1, \phi(S_2) < \phi(E_2), \phi(S_1) < \phi(E_2) \), \( \text{hom}(S_1, S_2) = 0 \) and by Lemma 9.2(e) it produces a \( \sigma \)-exceptional triple. \( \square \)

9.2. Constructions assuming the additional RP property. In this subsection we restrict \( \mathcal{A} \) further by assuming that the properties in Corollaries 2.7, 2.10 hold.

In the previous subsection we obtained a \( \sigma \)-triple (without using the additional RP property) from any long \( R \)-sequence with a \textbf{C3} object \( R \). One difficulty to obtain analogous criterion when \( R \) is a \textbf{C2} or a \textbf{C1} object is mentioned before Lemma 7.5. It makes it difficult to obtain the vanishings \{ \text{hom}^i(S_1, S_i) = \text{hom}^i(S_1, E_1) \}_{i \geq 2} \) and so to obtain an exceptional triple. Nevertheless, when \( R \) is \textbf{C2}, with some extra efforts and utilizing the additional RP property and the property in Corollary 2.10 we obtain exceptional triples in Proposition 9.9. Furthermore, we show that these exceptional triples can be shifted to \( \sigma \)-triples. We have not an analogous criterion with a \textbf{C1} object. \( ^{51} \)

**Proposition 9.9.** Each non-final \textbf{C2} object produces a \( \sigma \)-exceptional triple.

**Proof.** Let \( R \in \mathcal{A} \) be a non-final \textbf{C2} object. Consider the triangle \( \text{alg}(R) \):

\[
\begin{array}{ccc}
A_1 \oplus A_2[-1] & \xrightarrow{B} & R \\
& \xrightarrow{\text{hom}^1(A_1, A_1) = \text{hom}^1(A_2, A_2) = \text{hom}^1(B, B) = 0} & A_2, B \in \mathcal{A} \setminus \{0\}
\end{array}
\]

\*(C2a)*

For any \( \Gamma_0 \in \text{Ind}(B), \Gamma \in \text{Ind}(A_2[-1]) \) we have \( R \rightarrow (\Gamma_0, \Gamma) \), hence by Corollary 9.5 if \( \Gamma \notin \sigma^{ss} \), the proposition follows. Thus, we can assume that all components of \( A_2 \) are semistable and \( A_1 \neq 0 \).

For any \( \Gamma_0 \in \text{Ind}(B), \Gamma_1 \in \text{Ind}(A_2), \Gamma_2 \in \text{Ind}(A_1) \) the triple \( (\Gamma_0, \Gamma_1, \Gamma_2) \) is exceptional, hence by Corollary 2.10 we see that each of \( \text{Ind}(A_1), \text{Ind}(A_2), \text{Ind}(B) \) has up to isomorphism unique element. Whence we can write

\[
A_1 = \Gamma_2^p, \quad A_2 = \Gamma_1^q, \quad B = \Gamma_0^r \quad (\Gamma_0, \Gamma_1, \Gamma_2) \text{ is exceptional triple.}
\]

We explained that \( \Gamma_1 \in \sigma^{ss} \), furthermore by Lemma 8.1 it follows \( \phi(\Gamma_1[-1]) > \phi(\Gamma_0) \):

\[
\Gamma_0, \Gamma_1 \in \sigma^{ss}, \quad \phi(\Gamma_1) > \phi(\Gamma_0) + 1.
\]

By Proposition 6.6 we know that \( \Gamma_2 \) is \( \sigma \)-regular, so \( \text{alg}(\Gamma_2) \) is of type \( X \in \{\textbf{C1}, \textbf{C2}, \textbf{C3}\} \). We will construct a \( \sigma \)-exceptional triple in each case.

If \( \Gamma_2 \) is a \textbf{C3} object, then by Corollary 9.5 we can assume that \( \Gamma_2 \) is final. For the triangle

\[
\begin{array}{cc}
\text{alg}(\Gamma_2) = A' & \xrightarrow{\Gamma_2} A', B' \in \mathcal{A} \setminus \{0\} \\
\xrightarrow{\text{hom}^1(A', A') = \text{hom}^1(B', B') = 0} & \text{hom}^1(A', A') = \text{hom}^1(B', B') = 0
\end{array}
\]

\( ^{50} \) to which we refer as the additional RP property

\( ^{51} \) Lemma 9.8 and Corollary 9.11 cover all \( R \)-sequences with a \textbf{C1} object \( R \) and of length greater than two.
due to Lemma 9.6 (b) and Corollary 9.7, we can assume also that \( A' \) is semistable with \( \phi(A') > \phi(B') + 1 \) and \( \phi(B') = \phi_{\min} \). We have also \( \phi(B') + 1 = \phi -(\Gamma_2) \geq \phi -(A_1) \geq \phi(B) = \phi(G_0) \geq \phi_{\min} = \phi(B') \). Therefore we can write

\[
(68) \quad \phi(A') > \phi(B') + 1 \geq \phi(G_0) = \phi(B) \geq \phi(B').
\]

For any \( A'' \in \text{Ind}(A'), B'' \in \text{Ind}(B') \) we have \( R \xrightarrow{\text{proj}_2} (\Gamma_0, \Gamma_2) \xrightarrow{\text{C}_2} (B''[1], A''), \) hence by \( \text{deg}(\Sigma_0) + 1 = \text{deg}(B''[1]) \) and Lemma 7.3 (e) we get \( \text{hom}(B'', \Sigma_0) = 0 \), hence

\[
(69) \quad \text{hom}(B', B) = 0.
\]

We show now an implication, which will be used twice later:

\[
(70) \quad \text{If } \text{hom}(B', B) = 0 \text{ and } A'' \in \text{Ind}(A'), \text{ then } A'' \not\equiv \Gamma_1.
\]

Indeed, if \( A'' \equiv \Gamma_1 \), then by C2.4 applied to (64) and recalling (65) we obtain \( \text{hom}(B', A') \neq 0 \), and then by the short exact sequence (67) and \( \text{hom}(B', B') = 0 \) we get \( \text{hom}(B, \Gamma_2) = 0 \). Now from Corollary 2.6 (b) it follows \( \text{hom}^1(\Sigma_0, \Gamma_2) = \text{hom}^1(B, A_1) = 0 \), which contradicts C2.4.

Keeping (68) in mind, we consider two options \( \phi(A') > \phi(B) + 1 \) and \( \phi(A') \leq \phi(B) + 1 \).

If \( \phi(A') > \phi(B) + 1 \), then \( \text{hom}^*(A', B) = 0 \), which, together with \( \text{hom}^*(\Sigma_0, B) = 0 \), implies \( \text{hom}^*(B', B) = 0 \). Therefore (see (65)) \( \text{hom}^*(\Sigma_0, \Gamma_0) = \text{hom}^*(B', \Sigma_0) = \text{hom}^*(A', B') = 0 \), which by Corollary 2.10 imply that \( \text{Ind}(A') / \cong, \text{Ind}(B') / \cong \) have unique elements, say \( A'', B'' \), and \( (\Sigma_0, B'', A'') \) is a semistable exceptional triple with \( \phi(B'') = \phi(B'), \phi(A'') = \phi(A') \). Next, we show that the inequality \( \phi(\Sigma_0) \leq \phi(B'') + 1 \) in (68) must be an equality. Indeed, if \( \phi(\Sigma_0) < \phi(B'') + 1 \), then we have \( \phi(\Sigma_0) < \phi(B'') + 1, \phi(B'') < \phi(A''), \phi(\Sigma_0) < \phi(A'') \) and by Lemma 9.2 (e) we can assume \( \text{hom}(\Sigma_0, \Sigma_0) \neq 0 \), so \( \text{hom}(\Sigma_0, B') \neq 0 \). Hence, the triangle \( \text{alg}^1(\Sigma_0) \) implies \( \text{hom}(\Sigma_0, \Sigma_0) \neq 0, \text{hom}(\Sigma_0, \Sigma_0) \neq 0 \). Now Corollary 2.6 (b) implies \( \text{hom}^1(\Sigma_0, \Sigma_0) = \text{hom}^1(\Sigma_0, \Sigma_0) = 0 \). From the exact sequence \( 0 \to B' \to A' \to \Gamma_2 \to 0 \) and Lemma 4.3 it follows \( \text{hom}^1(\Sigma_0, \Sigma_0) = 0 \). The latter is the same as \( \text{hom}^1(B, A_1) = 0 \), which contradicts C2.4. So, we obtained \( \phi(\Sigma_0) = \phi(B'') + 1 \) and (68) becomes:

\[
(71) \quad \phi(B) = \phi(\Sigma_0) = \phi(B'') + 1 = \phi(B') + 1 \Rightarrow \text{hom}(B, B') = 0.
\]

Now we utilize the semistable \( \Gamma_1 \) in (69). If \( \phi(\Sigma_0) > \phi(B'') + 1 \), then \( \text{hom}^*(\Gamma_0, \Sigma_0) = 0 \) as well as \( \text{hom}^*(\Gamma_0, \Sigma_0) = 0 \), hence the triple \( (\Sigma_0, B'', \Gamma_1) \) is exceptional. From Corollary 2.10 and the triple \( (\Sigma_0, B'', \Gamma_1) \) it follows \( \Gamma_1 \cong A'' \), which contradicts (70). Therefore \( \phi(\Sigma_0) \leq \phi(B'') + 1 \).

Now (71) implies \( \phi(B') \leq \phi(B) \). Since we consider the subcase \( \phi(A') > \phi(B) + 1 \), therefore

\[
(72) \quad \phi(B') < \phi(B) \leq \phi(B') + 1 < \phi(A') \leq \phi(B) + 1 < \phi(\Gamma_1).
\]

These inequalities show that, in addition to \( \text{hom}(B'', B) = 0 \) (equality (69)) and \( \text{hom}^*(\Sigma_0, \Sigma_0) = 0 \), we get \( \text{hom}(B, B') = 0 \) and \( \text{hom}^*(\Sigma_0, B') = 0 \). For clarity, we put together these vanishings:

\[
(73) \quad \text{hom}(B', \Sigma_0) = \text{hom}(\Sigma_0, B') = 0, \quad \text{hom}^*(\Sigma_0, \Sigma_0) = \text{hom}^*(\Sigma_0, B') = 0.
\]

The vanishings \( \text{hom}^*(\Sigma_0, \Sigma_0) = \text{hom}^*(\Sigma_0, B') = 0 \) and the additional RP property (Corollary 2.7) show that for each \( B'' \in \text{Ind}(B') \) the couple \( \{B'', B'\} \) is not Ext-nontrivial, i. e. we have
hom¹(Γ₀, B'') = 0 or hom¹(B'', Γ₀) = 0. Therefore, for each B'' ∈ Ind(B') we have hom*(Γ₀, B'') = 0 or hom*(B'', Γ₀) = 0. If hom*(Γ₀, B'') = 0 for some B'' ∈ Ind(B'), then (B'', Γ₀, Γ₁) is a semistable exceptional triple with φ(B'') < φ(Γ₀) < φ(Γ₁), hom(B'', Γ₀) = 0 and we can apply Lemma 9.2 (b). Hence, we can assume that for each B'' ∈ Ind(B') we have hom*(B'', Γ₀) = 0 and (Γ₀, B'', Γ₁) is an exceptional triple. Therefore the set Ind(B')/ ≈ has unique element, say B''. Thus, we arrive at an exceptional triple

\[(Γ₀, B'', Γ₁), \quad \text{hom(Γ₀, B'') = 0,} \quad B' ≅ (B'')^s.\]

On the other hand, the vanishings hom*(A′, Γ₀) = hom*(Γ₀, Γ₀) = 0 and the triangle (67) imply hom*(A′, A′) = 0. The last vanishing and hom*(A′, B′) = 0 give rise to a triple (Γ₀, B'', A'') with (A'')^u ≅ A'. Both the triples (Γ₀, B'', A'), (Γ₀, B'', Γ₁) imply A'' ≅ Γ₁, which contradicts (70). Thus, the proposition follows, when Γ₂ is a C₃ object.

If Γ₂ is a C₂ object, then alg(Γ₂) and some of its features are

\[A'_1 ⊕ A'_2[-1] \rightarrow Γ₂ \quad A'_2, B' ∈ A \setminus \{0\} \quad \text{hom}(A'_1, A'_1) = \text{hom}(A'_2, A'_2) = \text{hom}(B', B') = 0 \quad \text{hom}(A'_1, A'_2) = \text{hom}(A'_1, B') = \text{hom}(A'_2, B') = 0.\]

For any A'' ∈ Ind(A'_1 ⊕ A'_2[-1]), B'' ∈ Ind(B') we have an R-sequence

\[R \rightarrow (Γ₀, Γ₂) \rightarrow (B'', A'') \quad (B'', A'') \quad \rightarrow (B'', A'') \quad \rightarrow (B'', A'') \quad \rightarrow (B'', A'') \quad \rightarrow (B'', A'') \quad \rightarrow (B'', A'').\]

Thus, Γ₂ becomes final. Furthermore, by deg(B) = deg(B') we have deg(Γ₀) = deg(B'') and we see that the R-sequence R → (Γ₀, Γ₂) → (B'', A''[-1]) satisfies the three conditions of Proposition 9.3. This proposition ensures a σ-exceptional triple. It remains to consider:

Γ₂ is a C₁ object. Denote the corresponding triangle as follows:

\[(76) \quad \text{alg}(Γ₂) = \frac{A'}{B'} \rightarrow Γ₂ \quad A', B' ∈ A \setminus \{0\} \quad \text{hom}(A', A') = \text{hom}(B', B') = 0 \quad \text{hom}(A', B') = 0.\]

Now we have again deg(B') = deg(Γ₀). It follows from Corollaries 7.4, 2.10 that

\[(77) \quad A' ≅ (A'')^s, \quad B' ≅ (B'')^t, \quad (Γ₀, B'', A'') \quad \text{is exceptional,} \quad \phi(B'') > \phi(Γ₀).\]

for some A'', B'' ∈ A_{exc}. The arguments which give (77) are as those giving (75), and (78) follows from Corollary 7.4 (b). If A'' ∈ σ^{ss}, then Γ₂ is final, and Proposition 9.3 produces a σ-sequence from the R-sequence R → (Γ₀, Γ₂) → (B'', A''). Therefore, we can assume that A'' ∉ σ^{ss}.
If $A''$ is $C1$ or $C2$, then we get an $R$-sequence, in which a $C3$ step does not appear as follows:

$$
\begin{array}{c}
\xrightarrow{proj_{1}} R \xrightarrow{C2b} (\Gamma, \Gamma) \xrightarrow{proj_{2}} \Gamma \xrightarrow{proj_{3}} \Gamma \xrightarrow{B''} \xrightarrow{proj_{1}} A'' \xrightarrow{X_{3}} (S, E) \xrightarrow{proj_{2}} E \\
\Gamma_{0} \xrightarrow{B''} S
\end{array}
$$

From Corollary 9.10 it follows that the sequence $(\Gamma, B'', S, E)$ is exceptional, which contradicts Corollary 2.10.

Therefore $A''$ must be a $C3$ object, which ensures a $\Gamma_{2}$-sequence of the form

$$
\Gamma \xrightarrow{proj_{1}} C1 \xrightarrow{proj_{2}} B'' \xrightarrow{proj_{3}} C3 \xrightarrow{proj_{4}} (S[1], E).
$$

In Lemma 9.8 is shown that the triple $(B'', S, E)$ is exceptional. The criteria given there show that $E \in \sigma^{ss}$ and reduce the phases of $(B'', S, E)$ to

$$
\begin{align*}
(79) & \phi(B'') = \phi(S) + 1 < \phi(\Gamma_{0}) \quad (B'', S, E) \text{ is semistable and exceptional.}
\end{align*}
$$

From Corollary 2.10 it follows that $\text{alg}(A'') = E \xrightarrow{i} A'' \xrightarrow{j} S[1]$ for some integers $i, j \in \mathbb{N}$.

If $\phi(E) > \phi(\Gamma_{0}) + 1$, then $\text{hom}^{s}(\phi, E, \Gamma_{0}) = 0$, which, combined with $\text{hom}^{s}(A'', \Gamma_{0}) = 0$ (see 77), implies $\text{hom}^{s}(S, \Gamma_{0}) = 0$. These vanishings and the exceptional triples $(\Gamma_{0}, B'', A'')$, $(B'', S, E)$ imply that $(\Gamma_{0}, B'', S, E)$ is an exceptional sequence, which is impossible.

Thus, $\phi(E) \leq \phi(\Gamma_{0}) + 1$ and we can write (see also 78)

$$
\begin{align*}
\phi(S) + 1 < \phi(E) & \leq \phi(\Gamma_{0}) + 1 < \phi(\Gamma_{1}) \quad \Rightarrow \quad \text{hom}^{s}(\Gamma_{1}, S) = 0.
\end{align*}
$$

Since $\text{hom}^{s}(\Gamma_{1}, \Gamma_{0}) = 0$ as well, the additional RP property (Corollary 2.7) ensures that the couple $(S, \Gamma_{0})$ is not Ext-nontrivial, therefore $\text{hom}^{s}(\Gamma_{0}, S) = 0$ or $\text{hom}^{s}(S, \Gamma_{0}) = 0$. We show below that $\text{hom}(\Gamma_{0}, S) = \text{hom}(S, \Gamma_{0}) = 0$, hence $\text{hom}^{s}(\Gamma_{1}, S) = 0$ or $\text{hom}^{s}(S, \Gamma_{0}) = 0$. It follows that some of the triples $(\Gamma_{0}, \Gamma_{1}, S, F)$ is exceptional.

If $(S, \Gamma_{0}, \Gamma_{1})$ is exceptional, then Lemma 9.2 (a) produces $\sigma$-exceptional triple, due to the inequalities $\phi(S) < \phi(\Gamma_{0})$, $\phi(\Gamma_{0}) + 1 < \phi(\Gamma_{1})$ (see 80).

If $(\Gamma_{0}, S, \Gamma_{1})$ is exceptional, then due to the inequalities $\phi(S) < \phi(\Gamma_{1})$, $\phi(\Gamma_{0}) < \phi(\Gamma_{1})$, $\phi(\Gamma_{0}) < \phi(S) + 1$ (the last comes from 73, 79) and $\text{hom}(\Gamma_{0}, S) = 0$ we can apply Lemma 9.2 (e).

The used in advance $\text{hom}(\Gamma_{0}, S) = 0$ follows from $\phi(S) < \phi(\Gamma_{0})$ (see 80). The other vanishing $\text{hom}(S, \Gamma_{0}) = 0$ follows from $\phi_{+}(A'') < \phi(\Gamma_{0})$, $\text{hom}^{s}(A'', \Gamma_{0}) = 0$ (see 77), and Lemma 3.6.

Now the proposition is completely proved.

It follows now the $C2$-analogue of Corollary 9.7. After a proper reformulation, Corollary 9.7 is transformed to Corollary 9.10 by replacing “$C3$” with “$C2$” and “$\phi_{min}$” with “$\phi_{max}$”.

**Corollary 9.10.** Let $R \in A$ be a $C2$ object with $\text{alg}(R) = A_{1} \oplus A_{2}[−1] \xrightarrow{B} R$. If either $\text{alg}(R)$ differs from the HN filtration of $R$ or they coincide and $\phi(A_{2}) < \phi_{max}$, then there exists a $\sigma$-triple.

**Proof.** Due to the criteria given in Proposition 9.9 and Lemma 9.6 we reduce to the case: $R$ is final and $\text{alg}(R)$ is the HN filtration of $R$. In particular $A_{1} \oplus A_{2}[−1] \in \sigma^{ss}$.

If $\phi(A_{2}) < \phi_{max}$, then $\phi(S) > \phi(A_{2})$ for some $S \in A_{exc} \cap \sigma^{ss}$. Since $\text{alg}(R)$ is the HN filtration of $R$, it follows that $\phi_{+}(R) = \phi(A_{2}) − 1$. Therefore $\phi(S) > \phi_{+}(R) + 1 > \phi(B) + 1$, which

---

52-The part of Corollary 9.7 using $\phi_{min}$ can be reformulated as saying that the data: a final $C3$ object $R \in A_{exc}$, $R \xrightarrow{} (S, F)$, $X \in \{S, F\}$, $\deg(X) ≠ 0$, and $\phi(X) - \deg(X) > \phi_{min}$ implies a $\sigma$-triple.
implies \( \hom^*(S, R) = \hom^*(S, B) = 0 \). From the triangle \( \mathfrak{a}g(R) \) we obtain also \( \hom^*(S, A_2) = 0 \). Therefore, for any \( A' \in \text{Ind}(A_2) \), \( B' \in \text{Ind}(B) \) the semistable triple \((B', A', S)\) is exceptional and it satisfies \( \phi(B') < \phi(A') < \phi(S) \), \( \phi(B') + 1 < \phi(S) \). Now Lemma 7.12 (a) produces a \( \sigma \)-triple. \( \square \)

In the next corollary we obtain \( \sigma \)-triples from some, but not all, long \( R \)-sequences with a \( C_1 \) object \( R \).

**Corollary 9.11.** Let \( R \longrightarrow (S_1, E_1) \). If \( E_1 \) is either a \( C_2 \) or a \( C_1 \) object, then there exists a \( \sigma \)-exceptional triple.

**Proof.** If \( E_1 \) is \( C_2 \), then we have an \( R \)-sequence \( R \longrightarrow (S_1, E_2) \longrightarrow (S_2, E_2[1]) \). By Proposition 9.9 we can assume that \( E_1 \) is final, and then Proposition 9.4 ensures a \( \sigma \)-triple.

If \( E_1 \) is \( C_1 \), then we get a second step \( E_1 \longrightarrow (S_2, E_2) \), for some \( (S_2, E_2) \), and then we go on further until a final object occurs, which will certainly happen by Lemma 7.1. We can assume that in this process a \( C_2 \) step does not occur (otherwise the corollary follows by the proven case). By Corollary 9.3 we can assume that all \( C_3 \) objects are final. Hence, if a \( C_3 \) step occurs, then this is the last step. The other possibility is to reach a final \( C_1 \) case and then Proposition 9.4 gives a \( \sigma \)-triple. Whence, we reduce to an \( R \)-sequence with \( n \geq 3 \) of the form:

\[
\begin{array}{cccccccc}
R & \longrightarrow & (S_1, E_1) & \longrightarrow & (S_2, E_2) & \longrightarrow & (S_3, E_3) & \longrightarrow & (S_n, E_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots & \longrightarrow & S_n
\end{array}
\]

We apply Lemma 7.5 to the \( R \)-sequence above and to the \( E_1 \)-sequence in it, and obtain:

\[
\hom^*(S_n, S_1) = \hom^*(E_n, S_1) = \hom^*(S_n, S_2) = \hom^*(E_n, S_2) = 0.
\]

Furthermore, by Lemma 7.3 (b) and deg \( (S_2) = \deg(S_1) = 0 \) (see table (43) it follows \( \hom^*(S_2, S_1) = 0 \). These vanishings imply that \((S_1, S_2, S_n, E_n)\) is a semistable exceptional sequence, which is a contradiction. \( \square \)

We summarize now the results concerning \( R \)-sequences with a \( C_1 \) object \( R \).

**Corollary 9.12.** Let there be no a \( \sigma \)-exceptional triple. If \( R \longrightarrow (S_1, E_1) \), then the object \( E_1 \) is either semistable or a \( C_3 \) object. If \( E_1 \) is a \( C_3 \) object, then for each \( R \)-sequence

\[
\begin{array}{cccccccc}
R & \longrightarrow & (S_1, E_1) & \longrightarrow & (S_2, E_2) & \longrightarrow & (S_3, E_3) & \longrightarrow & (S_n, E_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots & \longrightarrow & S_n
\end{array}
\]

the triple \((S_1, S_2, E_2)\) is exceptional, semistable, and it satisfies: \( \phi(S_2) = \phi_{\min} \), \( \phi(S_1) = \phi(S_2) + 1 < \phi(E_2) \), \( \hom(S_1, S_2) = 0 \), \( \hom^1(S_1, S_2) \neq 0 \).

**Proof.** Follows from Corollary 9.11 and Lemma 9.8. \( \square \)

A next step to the proof of Proposition 9.16 is to show that, given a \( C_1 \)-object \( R \), each long \( R \)-sequence induces a \( \sigma \)-triple, when \( R \) is part of an exceptional pair \((R, S_{\max})\) or \((S_{\min}, R)\).

**Lemma 9.13.** Let \( R \in \mathcal{A} \) be a non-final \( C_1 \) object. If we are given one of the following:

(a) \( S_{\min} \in \mathcal{A}_{\text{exc}} \) with \( \phi(S_{\min}) = \phi_{\min} \) and \( \hom^*(R, S_{\min}) = 0 \),
(b) \( S_{\max} \in \mathcal{A}_{\text{exc}} \) with \( \phi(S_{\max}) = \phi_{\max} \) and \( \hom^*(S_{\max}, R) = 0 \),

then there exists a \( \sigma \)-exceptional triple.
Proof. By the criterion given in Corollary 9.11 we can assume that there exists an $R$-sequence of the form $R \to (S_1, E_1) \to (S_2[1], E_2)$. The triple $(S_1, S_2, E_2)$ is exceptional by Lemma 9.8 and using the criteria given there we can assume that it is semistable and:

$$\phi(S_1) = \phi(S_2) + 1 = \phi_{\text{min}} + 1 < \phi(E_2), \quad \phi(S_2) = \phi_{\text{min}}.$$  

In part (a) we are given that $\hom^*(R, S_{\text{min}}) = 0$. We claim that the triple $(S_{\text{min}}, S_1, E_2)$ is exceptional. Indeed, we have: $\hom^*(E_2, S_{\text{min}}) = 0$ by $\phi(E_2) > \phi(S_{\text{min}}) + 1$, and $\hom^*(E_2, S_1) = 0$ by the exceptional triple $(S_1, S_2, E_2)$. Finally $\hom^*(S_1, S_{\text{min}}) = 0$ by $\phi(R, S_{\text{min}}) = 0$, $\phi(R) = \phi(S_1) \geq \phi(S_{\text{min}})$ and Lemma 3.6. Thus, we constructed a semistable exceptional triple $(S_{\text{min}}, S_1, E_2)$ with $\phi(S_{\text{min}}) < \phi(S_1) = \phi(S_{\text{min}}) + 1 < \phi(E_2)$. Now Lemma 9.2 (a) produces a $\sigma$-triple.

Let $\hom^*(S_{\text{max}}, R) = 0$ for some $S_{\text{max}} \in \mathcal{A}_{\text{exc}}$ with maximal phase. Unfolding the definition of $\mathcal{C}_1$ we get a short exact sequence $0 \to E \to R \to S \to 0$ with $E_1 \in \text{Ind}(E), S_1 \in \text{Ind}(S)$, $\phi(S) = \phi(S_1)$. Since $S_{\text{max}}$ is of maximal phase, we have $\phi(S_{\text{max}}) \geq \phi(E_2) > \phi(S_2) + 1 = \phi(S_1) = \phi(S)$, which implies $\hom^*(S_{\text{max}}, S_2) = 0$, $\hom^*(S_{\text{max}}, S) = 0$. By Lemma 4.3 and $\hom^*(S_{\text{max}}, R) = 0$ we get also $\hom(S_{\text{max}}, S[1]) = 0$, hence $\hom^*(S_{\text{max}}, S) = 0$, which in turn implies $\hom^*(S_{\text{max}}, E) = 0$. So far, using the conditions of (b), we obtained

$$(81) \quad \hom^*(S_{\text{max}}, S_1) = \hom^*(S_{\text{max}}, E_1) = \hom^*(S_{\text{max}}, S_2) = 0.$$  

We show below that $\hom^*(S_{\text{max}}, E_2)$ also vanishes, and then the sequence $(S_1, S_2, E_2, S_{\text{max}})$ becomes exceptional, which is a contradiction. Then the corollary follows.

Since any relation of the form $E_1 \to (X[1], Y)$ gives by Lemma 9.8 an exceptional triple $(S_1, X, Y)$, it follows from Corollary 2.10 that $\text{alg}(E_1) = \begin{array}{c} E_2 \\ S_2[1] \end{array}$. This triangle and the already shown $\hom^*(S_{\text{max}}, E_1) = \hom^*(S_{\text{max}}, S_2) = 0$ give the desired $\hom^*(S_{\text{max}}, E_2) = 0$. \hfill $\Box$

The additional $\text{RP}$ property gives us another situation, where the irregular cases $\mathbf{B}_1$ and $\mathbf{B}_2$ cannot occur. This is shown in Lemmas 9.14, 9.15 below. In this respect these lemmas are similar to Proposition 6.6, but the latter uses $\text{RP}$ properties 1, 2.

**Lemma 9.14.** If $(S_{\text{min}}, E)$ is an exceptional pair in $\mathcal{A}$ with $S_{\text{min}} \in \mathcal{P}(\phi_{\text{min}})$, then $E$ is not $\mathbf{B}_2$.

**Proof.** If $E$ is a $\mathbf{B}_2$ object, then $\text{alg}(E) = \begin{array}{c} A \\ \downarrow B[1] \end{array}$ with $B \in \sigma^{ss}$, $\phi(B) + 1 = \phi_{\text{max}}(E)$, $\phi_{\text{max}}(A) > \phi(B) + 1$, and for some $\Gamma \in \text{Ind}(B)$ the couple $\{E, \Gamma\}$ is Ext-nontivial. From $\Gamma \in \mathcal{A}_{\text{exc}} \cap \sigma^{ss}$ it follows that $\phi(\Gamma) = \phi(B) \geq \phi_{\text{min}}$, therefore $\phi_{\text{max}}(A) > \phi_{\text{min}} + 1$ and $\hom^*(A, S_{\text{min}}) = 0$. The vanishing $\hom^*(A, S_{\text{min}}) = 0$ imply $\hom^*(B, S_{\text{min}}) = 0$. Thus, we obtain an Ext-nontivial couple $\{\Gamma, E\}$ and $S_{\text{min}} \in \mathcal{A}_{\text{exc}}$ with $\hom^*(E, S_{\text{min}}) = \hom^*(\Gamma, S_{\text{min}}) = 0$, which contradicts the additional $\text{RP}$ property (Corollary 2.7). \hfill $\Box$

**Lemma 9.15.** Let $\phi_{\text{max}} > \phi_{\text{min}} + 1$. If $(S_{\text{min}}, E, S_{\text{max}})$ is an exceptional triple in $\mathcal{A}$ with $S_{\text{min}} \in \mathcal{P}(\phi_{\text{min}})$, $S_{\text{max}} \in \mathcal{P}(\phi_{\text{max}})$, then $E$ is not $\sigma$-irregular.

**Proof.** In the previous lemma we showed that $E$ is not a $\mathbf{B}_2$ object. Suppose that $E$ is a $\mathbf{B}_1$ object. Then $\text{alg}(E) = \begin{array}{c} A_1 \oplus A_2[-1] \\ \downarrow B \end{array}$ with $B \in \sigma^{ss}$, $\phi_{\text{max}}(A_1 \oplus A_2[-1]) \geq \phi(B)$, $\phi(B) = \phi_{\text{max}}(E)$, and for some $\Gamma \in \text{Ind}(A_2)$ the couple $\{E, \Gamma\}$ is Ext-nontivial.
If \( \phi(B) > \phi(S_{\text{min}}) \), then we have \( \phi_-(\Gamma[-1]) \geq \phi_-(A_1 \oplus A_2[-1]) \geq \phi(B) > \phi(S_{\text{min}}) \), hence \( \phi_-(\Gamma) > \phi(S_{\text{min}})+1 \). However, this implies \( \text{hom}^*(\Gamma, S_{\text{min}}) = 0 \) and we have also \( \text{hom}^*(E, S_{\text{min}}) = 0 \), which contradicts the additional RP property (Corollary 2.7).

If \( \phi(B) \leq \phi(S_{\text{min}}) \), then by \( \phi_{\text{max}} > \phi_{\text{min}} + 1 \) we have \( \text{hom}^*(S_{\text{max}}, B) = 0 \), which, combined with \( \text{hom}^*(S_{\text{max}}, E) = 0 \) and the triangle \( \text{alg}(E) \), implies \( \text{hom}^*(S_{\text{max}}, A_2) = 0 \). Thus, we have \( \text{hom}^*(S_{\text{max}}, \Gamma) = \text{hom}^*(S_{\text{max}}, E) = 0 \), which contradicts Corollary 2.7. □

We can prove now easily:

**Proposition 9.16.** Let \( \phi_{\text{max}} - \phi_{\text{min}} > 1 \). Let \( (S_{\text{min}}, E, S_{\text{max}}) \) be an exceptional triple in \( A \) with \( S_{\text{min}} \in \mathcal{P}(\phi_{\text{min}}) \), \( S_{\text{max}} \in \mathcal{P}(\phi_{\text{max}}) \). If \( E \notin \sigma^{ss} \), then there exists a \( \sigma \)-exceptional triple.

**Proof.** From Lemma 9.15 and \( E \notin \sigma^{ss} \) it follows that \( E \) is regular. From Corollary 8.5 it follows that \( E \) cannot be final (due to Corollary 2.10 there are not exceptional sequences of length 4). Now the existence of a \( \sigma \)-exceptional triple follows from Corollary 9.5, Proposition 9.9, and Lemma 9.13. □

10. **Application to \( \text{Stab}(D^b(Q_1)) \)**

The criteria of Section 9 hold for \( A = \text{Rep}_k(Q_1) \), due to Section 2. In this section we apply these criteria to \( \text{Rep}_k(Q_1) \). The result is the following theorem:

**Theorem 10.1.** Let \( k \) be an algebraically closed field. For each \( \sigma \in \text{Stab}(D^b(\text{Rep}_k(Q_1))) \) there exists a \( \sigma \)-exceptional triple.

In Remark 4.4 we pointed out a variant of Sections 4, 5, 6, 7, 8, 9 in which \( k \) is any field. We cannot point out a variant of Theorem 10.1 without the restriction that \( k \) is algebraically closed.

**Corollary 10.2.** The manifold \( \text{Stab}(D^b(\text{Rep}_k(Q_1))) \) is connected.

**Proof.** Let \( \mathcal{E} = (E^0_1, M, E^3_1) \). Let \( \Sigma_\mathcal{E} \) be as in (4). From Corollary 2.6 (b) we see that all triples in \( D^b(Q_1) \) are regular. Therefore \( \Sigma_\mathcal{E} \) is connected [12] Corollary 3.20]. From [13] it follows that all exceptional triples in \( D^b(Q_1) \) are obtained by shifts and mutations of \( \mathcal{E} \). Recalling Corollary 3.20 we see that Theorem 1.1 is the same as the equality \( \text{Stab}(D^b(Q_1)) = \Sigma_\mathcal{E} \). The corollary follows. □

Throughout the proof of Theorem 10.1 the entire Section 10 we fix the notations \( A = \text{Rep}_k(Q_1) \) and \( \mathcal{T} = D^b(A) \). We prove the theorem by contradiction.

Let \( \sigma \in \text{Stab}(D^b(A)) \). In all subsections of Section 10 except subsection 10.1, we assume that there does not exist a \( \sigma \)-exceptional triple.

Loosely speaking, this assumption leads to certain “non-generic” situations (see [86]). However, using the locally finiteness of \( \sigma \), we show that these situations cannot occur (Corollaries 10.5, 10.6) and so we get a contradiction.

The notations \( M, M', E^m_1, E^m_2, E^m_3, E^m_4 \) are explained in Proposition 2.2. We will refer often to table 4 and Corollary 2.9. Whenever we claim that a triple \( (A_0, A_1, A_2) \) is an exceptional triple (with \( A_0, A_1, A_2 \) one of the symbols \( M, M', E^m_1, E^m_2, E^m_3, E^m_4 \)), then we refer implicitly to Corollary 2.9 and whenever we discuss \( \text{hom}^*(A, B) \) with \( A, B \) varying in these symbols, we refer to table 4.

**Remark 10.3.** Recall that (see right after Definition 3.3) \( \text{hom}(A, B) \neq 0 \) implies \( \phi_-(A) \leq \phi_+(B) \). Using table 4 we can write for any \( n \in \mathbb{N} \)

- \( \text{hom}(E^{n+1}_1, E^n_1) \neq 0 \) hence \( \phi_-(E^{n+1}_1) \leq \phi_+(E^n_1) \).
• \( \text{hom}(E^n_3, E^{n+1}_3) \neq 0 \) hence \( \phi_-(E^n_3) \leq \phi_+(E^{n+1}_3) \)
• \( \text{hom}(E^n_3, E^{n+1}_3) \neq 0 \) hence \( \phi_-(E^n_3) \leq \phi_+(E^{n+1}_3) \)
• \( \text{hom}(E^{n+1}_4, E^n_4) \neq 0 \) hence \( \phi_-(E^{n+1}_4) \leq \phi_+(E^n_4) \).

10.1. Basic lemmas. The facts explained here are basic tools used in the following subsections. These facts are individual for \( Q_1 \). The reader may skip this subsection on a first reading and return to it only when we refer to these tools.

In this subsection we do not put any restrictions on \( \sigma \in \text{Stab}(T) \). In all the rest subsections \( \sigma \) is assumed not to admit a \( \sigma \)-exceptional triple.

10.1.1. Useful short exact sequences in \( A \) and two corollaries based on locally finiteness. It is easy to check:

**Lemma 10.4.** There exist arrows in \( A \) as shown below, so that the resulting sequences are exact:

\[
\begin{align*}
(82) & \quad 0 \rightarrow E^{m-1}_2 \rightarrow E^m_2 \rightarrow (E^0_1)^2 \rightarrow 0 \\
(83) & \quad 0 \rightarrow E^m_3 \rightarrow E^m_2 \rightarrow M \rightarrow 0 \\
(84) & \quad 0 \rightarrow E^{m-1}_3 \rightarrow E^m_4 \rightarrow (E^0_4)^2 \rightarrow 0 \\
(85) & \quad 0 \rightarrow M \rightarrow E^m_4 \rightarrow E^m_1 \rightarrow 0
\end{align*}
\]

These short exact sequences combined with the locally finiteness of \( \sigma \) result in Corollaries 10.5 and 10.6. These corollaries exclude the following two situations:

\[
\text{(86)} \quad \{E^m_2\}_{m \in \mathbb{N}} \subset \mathcal{P}(t), \{E^m_1\}_{m \in \mathbb{N}} \subset \mathcal{P}(t+1) \quad \text{or} \quad \{E^m_3\}_{m \in \mathbb{N}} \subset \mathcal{P}(t), \{E^m_4\}_{m \in \mathbb{N}} \subset \mathcal{P}(t+1).
\]

We will sometimes refer to these two cases as non-locally finite cases.

**Corollary 10.5.** Assume that \( \{E^m_1, E^m_2\}_{m \in \mathbb{N}} \subset \sigma^{ss} \) and \( \{E^m_3, E^m_4\}_{m \in \mathbb{N}} \subset \mathcal{P}(t) \) for some \( t \in \mathbb{R} \). Then for each \( m \in \mathbb{N} \) we have \( t \leq \phi(E^m_1) \leq t+1 \), and there exists \( n \in \mathbb{N} \) with \( t \leq \phi(E^n_1) < t+1 \).

**Proof.** By table [4] we have \( \text{hom}(E^m_2, E^1_1) \neq 0 \) and \( \text{hom}(E^m_4, E^m_2) \neq 0 \) for \( m \geq 1 \), hence \( t = \phi(E^m_2) \leq \phi(E^m_1) \leq \phi(E^{m+1}_3) + 1 = t+1 \). It remains to show the last claim.

The short exact sequence (82) gives a distinguished triangle \( E^m_1 \rightarrow (E^0_1)^2 \rightarrow E^{m-1}_2 \rightarrow E^m_1 \).

Suppose that \( \phi(E^m_1) = t+1 \) for each \( m \). Then \( \{E^m_1, (E^0_1)^2, E^{m-1}_2\}_{m \in \mathbb{N}} \subset \mathcal{P}(t+1) \). It follows that \( 0 \rightarrow E^m_1 \rightarrow (E^0_1)^2 \rightarrow E^{m-1}_2 \rightarrow 0 \) is a short exact sequence in the abelian category \( \mathcal{P}(t+1) \) for each \( m \in \mathbb{N} \) (see the beginning of subsection 3.2). Hence \( E^m_1 \rightarrow (E^0_1)^2 \) is a monic arrow in \( \mathcal{P}(t+1) \) for each \( m \in \mathbb{N} \). It follows by Lemma 3.4 that the set \( \{E^1_1\}_{m \in \mathbb{N}} \) is a finite subset of \( K(D^b(A)) \).

On the other hand (see Lemma 2.2) we can write \( \{m + 1, m[M], m[E^0_3]\}_{m \in \mathbb{N}} \), which is infinite in \( K(D^b(A)) \). Thus, the assumption that \( \phi(E^m_1) = t+1 \) for each \( n \) leads to a contradiction.

**Corollary 10.6.** Assume that \( \{E^m_3, E^m_4\}_{m \in \mathbb{N}} \subset \sigma^{ss} \) and \( \{E^m_3\}_{m \in \mathbb{N}} \subset \mathcal{P}(t) \) for some \( t \in \mathbb{R} \). Then for each \( m \in \mathbb{N} \) we have \( t \leq \phi(E^m_4) \leq t+1 \), and there exists \( l \in \mathbb{N} \) with \( t \leq \phi(E^l_4) < t+1 \).

**Proof.** By table [4] we have \( \text{hom}(E^m_3, E^m_4) \neq 0 \) and \( \text{hom}(E^m_3, E^m_1)[1] \neq 0 \) for \( m \geq 1 \), hence \( t \leq \phi(E^m_4) \leq t+1 \). The rest of the proof is the same as the proof of Corollary 10.5, but one must use the short exact sequence (84) instead of (82).

The short exact sequences with middle terms \( E^0_2, E^0_4, M' \) are unique:
Lemma 10.7. If $0 \to A \to C \to B \to 0$ is a short exact sequence in $\mathcal{A}$ with $A \neq 0$ and $B \neq 0$, then we have the following implications:

- if $C \cong E_0^{'0}$, then $A \cong E_0^{'0}$ and $B \cong M$;
- if $C \cong E_0^{'1}$, then $A \cong M$ and $B \cong E_0^{'0}$;
- if $C \cong M'$, then $A \cong E_0^{'0}$ and $B \cong E_0^{'1}$.

Proof. See the representations $E_0^{'0}, E_0^{'2}, E_0^{'3}, M, M'$ in Proposition 2.2. □

10.1.2. Comments on $\mathbf{C1}$ objects. Recall (see Lemma 9.3) that for any $\mathbf{C1}$ object $R \in \mathcal{A}$ there exists an exceptional pair $(X,Y)$ in $\mathcal{A}$ satisfying $R \longrightarrow (X,Y)$, $\text{hom}(X,Y) = 0$, $\text{hom}(R,X) \neq 0$, $\text{hom}(Y,R) \neq 0$. A list of the exceptional pairs in $\mathcal{A}$ is given in Lemma 2.8. Using Table 41 we see that the exceptional pairs $(X,Y)$ in $\mathcal{A}$ with $\text{hom}(X,Y) = 0$ are

$$(87) \quad (E_0^{'0}, E_0^{'2}), (E_1^{'0}, E_0^{'0}), (E_2^{'0}, E_0^{'3}), (E_3^{'0}, E_0^{'0}), (M, E_0^{'3}), (M', E_0^{'2}), (E_0^{'m}, M') \quad m \in \mathbb{N}.$$ 

By setting $R$ to specific objects in $\mathcal{A}_{\text{exc}}$ we can shorten this list further as follows:

Lemma 10.8. Let $R \in \{E_0^{'m} : m \in \mathbb{N}, 1 \leq i \leq 4\}$ and let $R$ be a $\mathbf{C1}$ object. Then there exists a pair $(X,Y) \in P_R$ which satisfies $R \longrightarrow (X,Y)$, where $P_R$ is a set of pairs depending on $R$ as shown in the table:

| $R$          | $P_R$                                                                 |
|-------------|-----------------------------------------------------------------------|
| $E_0^{'m}, m \geq 1$ | $\{(E_0^{'0}, E_0^{'0}), (E_1^{'0}, E_0^{'0}), (E_2^{'0}, E_0^{'0})\} \cup \{(E_0^{'m}, M') : n < m\}$                                    |
| $E_0^{'2}, m \geq 0$ | $\{(E_0^{'0}, E_0^{'0}), (E_1^{'0}, E_0^{'0}), (E_2^{'0}, E_0^{'0})\} \cup \{(M, E_0^{'3}) : n < m\}$                                    |
| $E_0^{'3}, m \geq 1$ | $\{(E_0^{'0}, E_0^{'0}), (E_1^{'0}, E_0^{'0}), (E_2^{'0}, E_0^{'0})\} \cup \{(M', E_0^{'2}) : n < m\}$                                    |
| $E_0^{'4}, m \geq 0$ | $\{(E_0^{'0}, E_0^{'0}), (E_1^{'0}, E_0^{'0}), (E_2^{'0}, E_0^{'0})\} \cup \{(E_0^{'m}, M) : n \leq m\}$                                    |

Proof. We shorten the list (87) using $\text{hom}(R,X) \neq 0$, $\text{hom}(Y,R) \neq 0$ and Table 41. □

Recall that for each $\mathbf{C1}$ object $C \in \mathcal{A}$ we have a short exact sequence $0 \to A \to C \to B \to 0$ with $A \neq 0, B \neq 0$. It follows the first part of:

Lemma 10.9. The simple objects $E_0^{'0}, E_0^{'3}, M$ cannot be $\mathbf{C1}$ objects. Furthermore:

If $E_0^{'2} \longrightarrow (X,Y)$, then $(X,Y) \cong (M, E_0^{'3})$. If $E_0^{'4} \longrightarrow (X,Y)$, then $(X,Y) \cong (E_0^{'0}, M)$.

If $M' \longrightarrow (X,Y)$, then $(X,Y) \cong (E_0^{'0}, E_0^{'3})$.

Proof. The rest of the lemma follows from Lemma 10.7. □

10.1.3. $\sigma$-exceptional triples from the low dimensional exceptional objects $\{E_0^{'i}\}_{i=1}^4, M, M'$.

We have the following corollaries of Lemma 9.2.

Corollary 10.10. Let $\{E_0^{'0}, E_0^{'2}, E_0^{'3}, M\} \subset \sigma^{ss}$. If $\phi(E_0^{'0}) > \phi(E_0^{'0})$ or $\phi(E_0^{'0}) > \phi(E_0^{'0})$, then there exists a $\sigma$-exceptional triple.

Proof. If $\phi(E_0^{'0}) > \phi(E_0^{'0})$, then by $\phi(E_0^{'0}) \leq \phi(E_0^{'0})$ (since $\text{hom}(E_0^{'3}, E_0^{'2}) \neq 0$) we have $\phi(E_0^{'2}) > \phi(E_0^{'0})$. Therefore, it is enough to construct a $\sigma$-exceptional triple assuming $\phi(E_0^{'2}) = \phi(E_0^{'0})$.

By $\text{hom}(E_0^{'2}, M) \neq 0$ we have $\phi(E_0^{'2}) \leq \phi(M)$. If $\phi(E_0^{'2}) < \phi(M)$, then we obtain a $\sigma$-exceptional triple from the triple $(E_0^{'1}, E_0^{'2}, M)$ with $\text{hom}(E_0^{'1}, E_0^{'2}) = 0$ and Lemma 9.2(b). Hence, we reduce to the case $\phi(E_0^{'2}) = \phi(M) > \phi(E_0^{'0})$. 

Next, we consider the triple \((E_1^0, M, E_3^0)\) with \(\text{hom}(E_1^0, M) = \text{hom}(E_1^0, E_3^0) = \text{hom}(M, E_3^0) = 0\). By \(\text{hom}^1(M, E_3^0) \neq 0\) it follows \(\phi(M) \leq \phi(E_3^0) + 1\). If \(\phi(M) < \phi(E_3^0) + 1\), then we obtain a \(\sigma\)-triple from Lemma 9.2 (f), due to the inequalities \(\phi(E_1^0) < \phi(M) < \phi(E_3^0) + 1\). Thus, it remains to consider the case \(\phi(E_1^0) < \phi(E_3^0) + 1 = \phi(E_2^0) = \phi(M)\). In this case we apply Lemma 9.2 (e) to the triple \((E_1^0, E_3^0, E_2^0)\) with \(\text{hom}(E_1^0, E_3^0) = 0\) and obtain a \(\sigma\)-triple.

\[\square\]

**Corollary 10.11.** Let \(\{E_1^0, E_1^0, E_2^0, M'\} \subset \sigma^{ss}\). If \(\phi(E_2^0) > \phi(E_1^0)\) or \(\phi(E_3^0) > \phi(E_1^0)\), then there exists a \(\sigma\)-exceptional triple.

**Proof.** By \(\text{hom}(E_1^0, E_1^0) \neq 0\), we see that \(\phi(E_1^0) > \phi(E_1^0)\) implies \(\phi(E_1^0) > \phi(E_1^0)\). Hence, it is enough to show that the inequality \(\phi(E_1^0) > \phi(E_1^0)\) induces a \(\sigma\)-triple.

The triple \((E_1^0, E_3^0, M')\) has \(\text{hom}(E_1^0, E_3^0) = 0\) and \(\text{hom}(E_3^0, M') \neq 0\), therefore \(\phi(E_1^0) < \phi(E_3^0) \leq \phi(M')\). By Lemma 9.2 (b) we reduce to the case \(\phi(E_1^0) = \phi(M') > \phi(E_1^0)\).

Now, the triple \((E_1^0, M', E_1^0)\) has \(\text{hom}(E_1^0, M') = 0\), \(\text{hom}(M', E_1^0) \neq 0\) and \(\phi(E_1^0) < \phi(M') \leq \phi(E_1^0)\). Therefore, by Lemma 9.2 (b) we can reduce the phases to \(\phi(E_1^0) < \phi(E_1^0) = \phi(E_1^0) = \phi(M')\).

Due to the obtained setting of the phases and \(\text{hom}(E_1^0, E_1^0) = 0\), Lemma 9.2 (c) produces a \(\sigma\)-triple from the exceptional triple \((E_1^0, E_1^0, E_3^0)\). The corollary follows.

\[\square\]

10.2. On the existence of \(S_{min}, S_{max}\). For the rest of section 10 we assume that \(\sigma \in \text{Stab}(D^b(Q_1))\) does not admit a \(\sigma\)-exceptional triple. Hence, Corollaries 9.7, 9.10 imply:

**Corollary 10.12.** If \(R\) is a \(C2\) or a \(C3\) object, then the \(HN\) filtration of \(R\) is \(\text{alg}(R)\) and \(R\) is final.

Moreover, by Corollary 9.7, 9.10 any \(C3/C2\) object induces a semistable \(S_{min/max} \in \mathcal{A}_{exc}\) with \(\phi(S_{min/max}) = \phi_{min/max}\), i.e. each \(C3/C2\) object ensures that \(\mathcal{P}(\phi_{min/max}) \cap \mathcal{A}_{exc} \neq \emptyset\). In this subsection we generalize these implications. The main proposition here is in terms of the numbers \(\phi_{min}, \phi_{max}\) defined in (59). The following lemma gives some information about these numbers.

**Lemma 10.13.** If there exists \(R \in \mathcal{A}_{exc}\) which is either \(C2\) or \(C3\) object, then \(\phi_{max} - \phi_{min} > 1\).

**Proof.** We use that \(R\) is final and apply Corollary 8.2. Therefore, we have either \(R \xrightarrow{C2} (S, E[-1])\) with \(\phi(S) < \phi(E[-1])\) or \(R \xrightarrow{C3} (S[1], E)\) with \(\phi(S[1]) < \phi(E)\), where \(S, E \in \sigma^{ss} \cap \mathcal{A}_{exc}\). Hence there exist \(S, E \in \sigma^{ss} \cap \mathcal{A}_{exc}\) with \(\phi(E) > \phi(S) + 1\), therefore \(\phi_{max} - \phi_{min} > 1\).

The main proposition of this subsection is:

**Proposition 10.14.** If \(\phi_{max} - \phi_{min} > 1\), then \(\mathcal{P}(\phi_{min}) \cap \mathcal{A}_{exc} \neq \emptyset\) and \(\mathcal{P}(\phi_{max}) \cap \mathcal{A}_{exc} \neq \emptyset\).

In the proof of Proposition 10.14 we use Corollaries 10.18, 10.20 proved later independently.

**Proof.** of Proposition 10.14 Suppose first that \(\mathcal{P}(\phi_{max}) \cap \mathcal{A}_{exc} = \emptyset\). It follows that there exists a sequence \(\{S_i\}_{i \in \mathbb{N}} \subset \sigma^{ss} \cap \mathcal{A}_{exc}\) such that

\[\phi_{min} + 1 < \phi(S_0) < \phi(S_1) < \cdots < \phi(S_i) < \phi(S_{i+1}) < \cdots < \phi_{max}\]

\[\lim_{i \to \infty} \phi(S_i) = \phi_{max}.\]

The objects \(\{S_i\}_{i \in \mathbb{N}}\) are pairwise non-isomorphic. Since \(\phi(S_0) - \phi_{min} > 1\), there exists \(S \in \sigma^{ss} \cap \mathcal{A}_{exc}\) with \(\phi(S_0) - \phi(S) \geq \phi_{min}\). In particular, for each \(i \in \mathbb{N}\) holds \(\text{hom}^*(S_i, S) = 0\). From table (4) it follows that either \(S = M\) or \(S = M'\), i.e. there can be at most two elements in \(\sigma^{ss} \cap \mathcal{A}_{exc}\) with phase.
strictly smaller than \( \phi(S_0) - 1 \) and such an element exists. Whence, there exists \( S_{\text{min}} \in \sigma^{ss} \cap \mathcal{A}_{\text{exc}} \) of minimal phase, i.e., \( \phi(S_{\text{min}}) = \phi_{\text{min}} \). Furthermore \( S_{\text{min}} \in \{ M, M' \} \).

If \( S_{\text{min}} = M \). Now, due to \( \text{hom}^*(S_1, M) = 0 \), table (4) shows that \( \{ S_i \}_{i \in \mathbb{N}} \subset \{ E_3^m, E_4^m \}_{m \in \mathbb{N}} \). From Remark 10.3 and the monotone behavior (89) it follows that \( S_i = E_3^m \) and \( m_i < m_{i+1} \) for big enough \( i \in \mathbb{N} \). Later in Corollary 10.18 (a) we show that such a sequence \( \{ S_i \}_{i \in \mathbb{N}} \) with (90) and the equality \( \phi(M) = \phi_{\text{min}} \) imply that all elements of \( \{ E_3^j \}_{j \in \mathbb{N}} \) are semistable. Therefore, from

\[
\phi(M) + 1 < \phi(E_3^m) \leq \phi(E_3^{m+1}) \leq \phi(E_3^{m+2}) \leq \ldots \leq \phi(E_3^{m_{i+1}} - 1) \leq \phi(E_3^{m_{i+1}});
\]

it follows that for some \( j \in \{ m_i, m_i + 1, \ldots, m_{i+1} \} \) we have \( \phi(M) + 1 < \phi(E_3^j) < \phi(E_3^{j+1}) \), hence we can apply Lemma 9.2 (a) to the triple \( (M, E_3^j, E_3^{j+1}) \), which contradicts our assumption on \( \sigma \).

If \( S_{\text{min}} = M' \). Now table (4) shows that \( \{ S_i \}_{i \in \mathbb{N}} \subset \{ E_3^m, E_4^m \}_{m \in \mathbb{N}} \) and Remark 10.3 shows that for big enough \( i \in \mathbb{N} \) we have \( S_i = E_2^m \), \( m_i < m_{i+1} \). By Corollary 10.20 (a) we obtain \( \{ E_2^j \}_{j \in \mathbb{N}} \subset \sigma^{ss} \). Now similar arguments as in the previous case (with an exceptional triple \( (M', E_2^j, E_2^{j+1}) \) for some \( j \in \mathbb{N} \)) lead us to a contradiction.

So far, we derived that there exists \( S_{\text{max}} \in \mathcal{P}(\phi_{\text{min}}) \cap \mathcal{A}_{\text{exc}} \). Next, suppose that \( \mathcal{P}(\phi_{\text{min}}) \cap \mathcal{A}_{\text{exc}} = \emptyset \). Then we have a sequence \( \{ S_i \}_{i \in \mathbb{N}} \subset \sigma^{ss} \cap \mathcal{A}_{\text{exc}} \) with

\[
(91) \quad \phi_{\text{max}} - 1 < \phi(S_i) > \phi(S_{i+1}) > \phi_{\text{min}} \quad \lim_{i \to \infty} \phi(S_i) = \phi_{\text{min}}.
\]

It is clear that \( \text{hom}^*(S_{\text{max}}, S_i) = 0 \) for each \( i \in \mathbb{N} \), hence (by table (4)) we see that \( S_{\text{max}} \in \{ M, M' \} \).

If \( S_{\text{max}} = M \). In this case from table (4) it follows that \( \{ S_i \}_{i \in \mathbb{N}} \subset \{ E_3^m, E_4^m \}_{m \in \mathbb{N}} \). By Remark 10.3 and the monotone behavior (91) we can construct the sequence so that \( S_i = E_4^m \), \( m_i < m_{i+1} \) for \( i \in \mathbb{N} \). Now Corollary 10.18 (b) shows that \( \{ E_4^j \}_{j \in \mathbb{N}} \subset \sigma^{ss} \). Hence, for some \( j \in \mathbb{N} \) we can apply Lemma 9.2 (a) to the triple \( (E_4^{j+1}, E_4^j, M') \), which is a contradiction.

If \( S_{\text{max}} = M' \). Since we have \( \{ \text{hom}^*(M, S_i) = 0 \}_{i \in \mathbb{N}} \), table (4) shows that \( \{ S_i \}_{i \in \mathbb{N}} \subset \{ E_1^m, E_2^m \}_{m \in \mathbb{N}} \). From Remark 10.3 we get \( S_i = E_1^m \), \( m_i < m_{i+1} \) for \( i \in \mathbb{N} \). Corollary 10.20 (b) shows that \( \{ E_1^j \}_{j \in \mathbb{N}} \subset \sigma^{ss} \), hence for some \( j \in \mathbb{N} \) we can use Lemma 9.2 (a) with the triple \( (E_1^{j+1}, E_1^j, M) \), which gives us a contradiction. The proposition is proved.

We divide the proof of Corollaries 10.18 10.20 in several lemmas.

**Lemma 10.15.** Let \( S_{\text{min}} \in \mathcal{P}(\phi_{\text{min}}) \cap \mathcal{A}_{\text{exc}} \). Let \( R \in \mathcal{A}_{\text{exc}} \) be either a C2 object or a C3 object. If \( \text{hom}^*(R, S_{\text{min}}) = 0 \), then there exists \( S \in \sigma^{ss} \cap \mathcal{A}_{\text{exc}} \) with \( \phi^*(S, S_{\text{min}}) = 0 \) and \( \phi(S) + 1 > \phi_{\text{max}} \).

**Proof.** Presenting the arguments below we keep in mind Corollary 10.12.

If \( R \) is C2, then we have \( \text{alg}(R) = \begin{array}{c} A_1 \oplus A_2[-1] \\ \end{array} \begin{array}{c} R \\ \end{array} \), \( A_2, B \in \mathcal{A} \setminus \{ 0 \} \), \( \phi_{\text{max}} \geq \phi(A_2) > \phi(B) + 1 \). From \( \phi_{\text{min}}(R) = \phi(B) \geq \phi(S_{\text{min}}) \), \( \text{hom}^*(R, S_{\text{min}}) = 0 \), and Lemma 3.3 it follows, that \( \text{hom}^*(B, S_{\text{min}}) \). Any \( S \in \text{Ind}(B) \) satisfies the desired properties and the lemma follows.

If \( R \) is C3, then \( \text{alg}(R) = \begin{array}{c} A \\ \end{array} \begin{array}{c} R \\ \end{array} \), \( A, B \in \mathcal{A} \setminus \{ 0 \} \), \( \phi_{\text{max}} \geq \phi(A) > \phi(B) + 1 \geq \phi(S_{\text{min}}) + 1 \), hence \( \text{hom}^*(A, S_{\text{min}}) = 0 \), which, together with \( \text{hom}^*(R, S_{\text{min}}) = 0 \), implies \( \text{hom}^*(B, S_{\text{min}}) = 0 \). Now the lemma follows with any \( S \in \text{Ind}(B) \).
Lemma 10.16. Let $S_{max} \in A_{exc}$ satisfy $\phi(S_{max}) = \phi_{\text{max}}$, and let $R \in A_{exc}$ be either a $C2$ or a $C3$ object. If $\text{hom}^*(S_{max}, R) = 0$, then there exists $S \in \sigma^{ss} \cap A_{exc}$ with $\text{hom}^*(S_{max}, S) = 0$ and $\phi(S) > \phi_{\text{min}} + 1$.

Proof. If $R$ is $C2$, then we can write $\text{alg}(R) = A_1 \oplus A_2[-1] \to R$, $A_2, B \in A \setminus \{0\}$, $\phi_{\text{max}} = \phi(S_{max}) \geq \phi(A_2) \geq \phi(B) + 1 \geq \phi_{\text{min}} + 1$. Hence $\text{hom}^*(S_{max}, B) = 0$, which, together with $\text{hom}^*(S_{max}, R) = 0$, implies $\phi^*(S_{max}, A_2) = 0$. Now the lemma follows with $S \in \text{Ind}(A_2)$.

If $R$ is $C3$, then $\text{alg}(R) = A \to R$, $A, B \in A \setminus \{0\}$, $\phi(S_{max}) \geq \phi(A) > \phi(B) + 1 \geq \phi_{\text{min}} + 1$, hence $\text{hom}^*(S_{max}, B) = \text{hom}^*(S_{max}, A) = 0$. Now any $S \in \text{Ind}(A)$ has the desired properties. \hfill $\Box$

Lemma 10.17. Let $M \in \mathcal{P}(\phi_{\text{min}})$ or $M' \in \mathcal{P}(\phi_{\text{max}})$. If for some $m > 0$ we have $E^m_3 \in \sigma^{ss}$ or $E^m_4 \in \sigma^{ss}$, then there is not a $C1$ object in the set $\{E^j_3, E^j_4\}_{j \in \mathbb{N}}$.

Proof. Suppose that some $R \in \{E^j_3, E^j_4\}_{j \in \mathbb{N}}$ is a $C1$ object. From Lemma 9.13 and $\text{hom}^*(E^{j/4}_3, M) = \text{hom}^*(M', E^{j/4}_3) = 0$ for each $j \in \mathbb{N}$ we see that $R$ must be final, hence $\text{alg}(R)$ is the HN filtration of $R$. In particular, from $R \to \cdots \to (X, Y)$ it follows that $X, Y$ are semistable and $\phi(Y) > \phi(X)$. Now Lemma 10.8 (look at the last two rows in the table) contradicts the following negations $53$:

- $(E^0_3, E^0_1 \in \sigma^{ss} \text{ and } \phi(E^0_3) = \phi(E^0_1))$. Proof: If $E^0_3, E^0_1 \in \sigma^{ss}$, then $E^m_3 \in \sigma^{ss}$ or $E^m_4 \in \sigma^{ss}$, $m > 0$ and $\text{hom}(E^0_3, E^m_3, E^m_4) \neq 0$, $\text{hom}(E^m_3, E^0_1, E^m_4) \neq 0$ if follows $\phi(E^0_3) \leq \phi(E^0_1)$.

- $(E^0_3, E^0_4 \in \sigma^{ss} \text{ and } \phi(E^0_3) > \phi(E^0_4))$. Proof: We are given $m > 0$ with $E^m_3 \in \sigma^{ss}$ or $E^m_4 \in \sigma^{ss}$, hence $\text{hom}(E^0_3, E^m_3, E^m_4) \neq 0$, $\text{hom}(E^m_3, E^0_4, E^m_4) \neq 0$ imply $\phi(E^0_3) \leq \phi(E^0_4)$.

- $(E^0_2, E^0_1 \in \sigma^{ss} \text{ and } \phi(E^0_2) > \phi(E^0_1))$. Proof: If $\phi(M) = \phi_{\text{min}}$, then from $\text{hom}(E^0_2, M) \neq 0$ it follows $\phi(E^0_2) = \phi(M) = \phi_{\text{min}} \leq \phi(E^0_1)$. If $\phi(M') = \phi_{\text{max}}$, then $\text{hom}(M', E^0_1) \neq 0$ implies $\phi(E^0_2) \leq \phi(M') \leq \phi_{\text{max}} = \phi(E^0_1)$. If $M' \in \sigma^{ss}$ and $\phi(E^0_2) > \phi(M')$. Proof: If $\phi(M) = \phi_{\text{min}}$, then by $\text{hom}(E^0_2, M) \neq 0$ we get $\phi(E^0_2) = \phi_{\text{min}}$. If $\phi(M') = \phi_{\text{max}}$, then $\text{hom}(E^0_2, M') \neq 0$.

The lemma follows. \hfill $\Box$

Corollary 10.18. Let $\{S_i\}_{i \in \mathbb{N}}$ be a sequence of pairwise non-isomorphic, semistable objects with $\{S_i\}_{i \in \mathbb{N}} \subset \{E^j_3, E^j_4\}_{j \in \mathbb{N}}$. If any of the two conditions below is satisfied

(a) $M \in \mathcal{P}(\phi_{\text{min}})$, $\lim_{i \to \infty} \phi(S_i) = \phi_{\text{max}}$.

(b) $M' \in \mathcal{P}(\phi_{\text{max}})$, $\lim_{i \to \infty} \phi(S_i) = \phi_{\text{min}}$.

then all the exceptional objects in the set $\{E^j_3, E^j_4\}_{j \in \mathbb{N}}$ are semistable.

Proof. Since each $E \in \{E^j_3, E^j_4\}_{j \in \mathbb{N}}$ is a trivially coupling object, it is neither $B1$ nor $B2$ (Corollary 6.3). From Lemma 10.17 we know that any $E$ is either semistable or $Ci(i=2,3)$. However, if it is $Ci(i=2,3)$, then:

53Recall that we have Corollary 10.12 at our disposal, due to our assumption on $\sigma$.

54For a statement $p$, when we write $\neg p$ we mean: "$p$ is not true".
it follows $\phi$ hom($S, M$) = $\phi_{\min}$, and Lemma 10.15 there exists $S \in \sigma^{ss} \cap A_{exc}$ with hom($S, M$) = 0 and $\phi(M) + 1 \leq \phi(S) + 1 < \phi_{\max}$, which by $\lim_{i \to \infty} \phi(S_i) = \phi_{\max}$ implies that $(M, S, S_i)$ is an exceptional triple for big enough $i$. By Corollary 2.10 this cannot happen, since $\{S_i\}_{i \in \mathbb{N}}$ are pairwise non-isomorphic.

(b) By hom($M', E$) = 0(see table 4), $\phi(M') = \phi_{\max}$, and Lemma 10.16 there exists $S \in \sigma^{ss} \cap A_{exc}$ with hom($M', S$) = 0 and $\phi(M') \geq \phi(S) > \phi_{\min} + 1$, which by $\lim_{i \to \infty} \phi(S_i) = \phi_{\min}$ implies that $(S_i, S, M')$ is an exceptional triple for big enough $i$. This contradicts Corollary 2.10.

The arguments for the proof of Corollary 10.20 are the same, but the role of Lemma 10.17 is played by the following Lemma 10.19.

**Lemma 10.19.** Let $M' \in \mathcal{P}(\phi_{\min})$ or $M \in \mathcal{P}(\phi_{\max})$. If for some $m > 0$ we have $E^m_1 \in \sigma^{ss}$ or $E^m_2 \in \sigma^{ss}$, then there is not a $C1$ object in the set $\{E^j_1, E^j_2\}_{j \in \mathbb{N}}$.

**Proof.** Using that for each $j \in \mathbb{N}$ we have hom($E^j_{1/2}, M'$) = 0, hom($M, E^j_{1/2}$) = 0 and Lemma 10.8 (this time the first two rows in the table) by the same arguments as in Lemma 10.17 we reduce the proof to the negations:

- $(E^0_2, E^0_1 \in \sigma^{ss}$ and $\phi(E^0_0) > \phi(E^0_1))$. Proof: If $E^0_2, E^0_1 \in \sigma^{ss}$, then by hom($E^0_2, E^0_{1/2}) \neq 0$, hom($E^0_{1/2}, E^0_1) \neq 0$, $m > 0$ it follows that $\phi(E^0_2) \leq \phi(E^0_1)$.
- $(E^0_3, E^0_1 \in \sigma^{ss}$ and $\phi(E^0_3) > \phi(E^0_1))$. Proof: Follows from hom($E^0_3, E^0_{1/2} = 0$ and hom($E^0_{1/2}, E^0_1) \neq 0$.
- $(E^0_2, E^0_4 \in \sigma^{ss}$ and $\phi(E^0_3) > \phi(E^0_4))$. Proof: If $\phi(M') = \phi_{\min}$, then from hom($E^0_2, M') \neq 0$ it follows $\phi_{\min} = \phi(E^0_3) \leq \phi(E^0_4)$. If $\phi(M) = \phi_{\max}$, then hom($M, E^0_4) \neq 0$ implies $\phi_{\max} = \phi(E^0_4) \geq \phi(E^0_3)$.
- $(M, E^0_3 \in \sigma^{ss}$ and $\phi(E^0_3) > \phi(M))$. Proof: If $\phi(M') = \phi_{\min}$, then we use hom($E^0_3, M') \neq 0$. If $\phi(M) = \phi_{\max}$, then $E^0_3 \in \sigma^{ss}$ implies $\phi_3 = \phi(M)$.
- $(M, E^0_4 \in \sigma^{ss}$ and $\phi(M') > \phi(E^0_4))$. Proof: If $\phi(M') = \phi_{\min}$, then $E^0_4 \in \sigma^{ss}$ implies $\phi(M') \leq \phi(E^0_4)$. If $\phi(M) = \phi_{\max}$, then the negation follows from hom($M, E^0_4) \neq 0$.

The lemma follows.

**Corollary 10.20.** Let $\{S_i\}_{i \in \mathbb{N}} \subset \{E^j_1, E^j_2\}_{j \in \mathbb{N}}$ be a sequence of pairwise non-isomorphic, semistable objects. Any of the following two settings:

(a) $M' \in \mathcal{P}(\phi_{\min})$, $lim_{i \to \infty} \phi(S_i) = \phi_{\min}$,
(b) $M \in \mathcal{P}(\phi_{\max})$, $lim_{i \to \infty} \phi(S_i) = \phi_{\min}$,

implies that $\{E^j_1, E^j_2\}_{j \in \mathbb{N}} \subset \sigma^{ss}$.

**Proof.** The arguments are the same as those used in the proof of Corollary 10.18 but we use Lemma 10.19 instead of Lemma 10.17.

Note that the conclusions of Corollaries 10.20 and 10.18 namely that $\{E^m_2, E^m_1\} \subset \sigma^{ss}$ and $\{E^m_3, E^m_1\} \subset \sigma^{ss}$, are components of the data in the two non-locally finite cases (80). In the next subsection we derive (80) from the assumption $\phi_{\max} - \phi_{\min} > 1$, and Corollaries 10.20 10.18 will be helpful at some points.

The implications given below are further minor steps towards derivation of the non-locally finite cases (80). These implications will be used in both Subsection 10.3 and Subsection 10.4.

(a) By hom($E, M) = 0$ (see table 4), $\phi(M) = \phi_{\min}$, and Lemma 10.15 there exists $S \in \sigma^{ss} \cap A_{exc}$ with hom($S, M$) = 0 and $\phi(M) + 1 \leq \phi(S) + 1 < \phi_{\max}$, which by $\lim_{i \to \infty} \phi(S_i) = \phi_{\max}$ implies that $(M, S, S_i)$ is an exceptional triple for big enough $i$. By Corollary 2.10 this cannot happen, since $\{S_i\}_{i \in \mathbb{N}}$ are pairwise non-isomorphic.

(b) By hom($M', E) = 0$(see table 4), $\phi(M') = \phi_{\max}$, and Lemma 10.16 there exists $S \in \sigma^{ss} \cap A_{exc}$ with hom($M', S) = 0$ and $\phi(M') \geq \phi(S) > \phi_{\min} + 1$, which by $\lim_{i \to \infty} \phi(S_i) = \phi_{\min}$ implies that $(S_i, S, M')$ is an exceptional triple for big enough $i$. This contradicts Corollary 2.10.
Lemma 10.21.

(a) If $\phi_{\text{max}} = \phi(M')$ and $\{E^m_1 : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E^m_4 : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{\text{max}}$.

(b) If $\phi_{\text{max}} = \phi(M)$ and $\{E^m_1 : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E^m_4 : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{\text{max}}$.

(c) If $\phi_{\text{min}} = \phi(M')$ and $\{E^m_2 : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E^m_3 : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{\text{max}}$.

(d) If $\phi_{\text{min}} = \phi(M)$ and $\{E^m_3 : m \in \mathbb{N}\} \subset \sigma^{ss}$, then $\{E^m_3 : m \in \mathbb{N}\} \subset \mathcal{P}(t)$ for some $t \leq \phi_{\text{max}}$.

Proof. Presenting the proof we keep in mind Remark 10.3.

(a) For any $m \in \mathbb{N}$ we have $\phi(E^{m+1}_4) \leq \phi(E^m_4) \leq \phi(M')$. The triple $(E^{m+1}_4, E^m_4, M')$ has hom$(E^m_4, M') = 0$. Hence from Lemma 9.3 (c) it follows $\phi(E^{m+1}_4) = \phi(E^m_4)$ for each $m \in \mathbb{N}$.

(b) We apply the same arguments as in (a) to the triple $(E^{m+1}_1, E^m_1, M)$ with hom$(E^m_1, M) = 0$.

(c) Now $\phi(M') \leq \phi(E^m_2) \leq \phi(E^{m+1}_2)$, hom$(M', E^m_2) = 0$ and we can apply Lemma 9.2 (b) to the triple $(M', E^m_2, E^{m+1}_2)$, which implies $\phi(E^m_2) = \phi(E^{m+1}_2)$ for each $n \geq 0$.

(d) We apply the same arguments as in (c) to the triple $(M, E^m_3, E^{m+1}_3)$ with hom$(M, E^m_3) = 0$. □

10.3. The case $\phi_{\text{max}} - \phi_{\text{min}} > 1$. In this subsection we show that the inequality $\phi_{\text{max}} - \phi_{\text{min}} > 1$ is inconsistent with the assumption that there is not a $\sigma$-exceptional triple. The inequality $\phi_{\text{max}} - \phi_{\text{min}} > 1$ implies by Proposition 10.14 that (for brevity we denote this product by $\Phi$):

$$\Phi = (\mathcal{P}(\phi_{\text{min}}) \cap A_{\text{exc}}) \times (\mathcal{P}(\phi_{\text{max}}) \cap A_{\text{exc}}) \neq \emptyset.$$  

If $(S_{\text{min}}, S_{\text{max}}) \in \Phi$, then $(S_{\text{min}}, S_{\text{max}})$ is an exceptional pair, since $\phi_{\text{max}} - \phi_{\text{min}} > 1$. Hence there exists unique $E \in A_{\text{exc}}$, s. t. $(S_{\text{min}}, E, S_{\text{max}})$ is an exceptional triple. It is very important for us that $E$ must be necessarily semistable, which follows from 9.16.

For the rest of this subsection we assume that $\phi_{\text{max}} - \phi_{\text{min}} > 1$. In the end we conclude that $\Phi \neq \emptyset$ contradicts the non-existence of a $\sigma$-exceptional triple.

Since any $(S_{\text{min}}, S_{\text{max}}) \in \Phi$ is an exceptional pair in $A$, it must be some of the pairs listed in Corollary 2.8. We show case-by-case (in a properly chosen order) that for each pair $(A, B)$ in this list the incidence $(A, B) \in \Phi$ leads to a contradiction. We show first that $(E^0_1, E^0_3) \notin \Phi$.

Lemma 10.22. $(E^0_1, E^0_3) \notin \Phi$.

Proof. Suppose that $(E^0_1, E^0_3) \in \Phi$. We consider the triple $(E^0_1, M, E^0_3)$. From Proposition 9.16 it follows that $M \in \sigma^{ss}$, hence $\phi_{\text{min}} = \phi(E^0_1) \leq \phi(M) \leq \phi(E^0_3) = \phi_{\text{max}}$. One of these inequalities must be proper. However, by hom$(E^0_1, M) = hom(M, E^0_3) = 0$ and Lemma 9.2 (b), (c) we obtain a $\sigma$-exceptional triple, which is a contradiction. □

We introduce the following formal rules, which facilitate the exposition:

$$\Phi \xrightarrow{(A, B, C)} \Phi \xrightarrow{(A, B, C)} \Phi \xrightarrow{(A, B, C)} \Phi \xrightarrow{(A, B, C)} \Phi \xrightarrow{(A, B, C)} \Phi \xrightarrow{(A, B, C)} \Phi \xrightarrow{(A, B, C)} \Phi$$

In (93), (94), and (95) the triple $(A, B, C)$ is the unique exceptional triple (taken from Lemma 2.9) with first element $A$ and last element $C$. In all the three rules we implicitly use Proposition 9.16 from which it follows $B \in \sigma^{ss}$, and hence $\phi_{\text{min}} = \phi(A) \leq \phi(B) \leq \phi(C) = \phi_{\text{max}}$. The specific arguments assigned to each individual rule are:

- (93) from Lemma 9.2 (a) and $\phi_{\text{max}} - \phi_{\text{min}} > 1$ it follows that either $\phi(A) = \phi(B) = \phi_{\text{min}}$ or $\phi(B) = \phi(C) = \phi_{\text{max}}$, whence we reduce to either $(B, C) \in \Phi$ or $(A, B) \in \Phi$;
by Lemma [9.2] (b) and \( \operatorname{hom}(A,B) = 0 \) we get \( \phi(B) = \phi(C) = \phi_{\text{max}} \), whence \( (A,B) \in \Phi \); 
by Lemma [9.2] (c) and \( \operatorname{hom}(B,C) = 0 \) we get \( \phi(A) = \phi(B) = \phi_{\text{min}} \), whence \( (B,C) \in \Phi \).

Now we eliminate some pairs \((X,Y)\) by showing that \((X,Y) \in \Phi\) implies \((E_1^4, E_3^n) \in \Phi\).

**Corollary 10.23.** For each \( n \in \mathbb{N} \) any of the pairs \((E_1^0, E_3^n), (E_1^0, E_2^n), (M, E_3^n), (E_1^0, M'), (E_1^n, M), (M', E_2^n), (E_1^{n+1}, E_3^n), (E_1^n, E_2^n), (E_1^{n+1}, E_3^{n+1}), (E_1^n, E_2^{n+1})\) is not in \( \Phi \).

**Proof.** We keep in mind the formal rules \((\text{[53]}), (\text{[54]}), (\text{[55]}\)) \textcolor{red}{\text{[52]}}. The following expressions and Lemma [10.22] show that each of the listed pairs is not in \( \Phi \).

\[
\begin{align*}
(E_1^0, E_3^0) & \in \Phi \quad (E_1^n, E_3^n), \operatorname{hom}(E_1^n, E_3^n) = 0 \quad (E_1^0, E_3^n) \notin \Phi. \\
(E_1^0, E_2^0) & \in \Phi \quad (E_1^n, E_2^n), \operatorname{hom}(E_1^n, E_2^n) = 0 \quad (E_1^0, E_2^n) \notin \Phi. \\
(M, E_3^n) & \in \Phi, n \geq 1 \quad (M, E_3^n), \operatorname{hom}(M, E_3^n) = 0 \quad (M, E_3^n) \in \Phi \quad \text{induction} \quad (M, E_3^n) \notin \Phi. \\
(E_1^0, M') & \in \Phi \quad (E_1^n, M'), \operatorname{hom}(E_1^n, M') = 0 \quad (E_1^0, M') \notin \Phi. \\
(E_1^n, M) & \in \Phi, n \geq 1 \quad (E_1^n, M), \operatorname{hom}(E_1^n, M) = 0 \quad (E_1^n, M) \in \Phi \quad \text{induction} \quad (E_1^n, M) \notin \Phi. \\
(M', E_2^0) & \in \Phi \quad (M', E_2^0), \operatorname{hom}(M', E_2^0) = 0 \quad (M', E_2^0) \notin \Phi. \\
(M', E_2^n) & \in \Phi, n \geq 1 \quad (M', E_2^n), \operatorname{hom}(M', E_2^n) = 0 \quad (M', E_2^n) \in \Phi \quad \text{induction} \quad (M', E_2^n) \notin \Phi. \\
(E_1^{n+1}, E_4^n) & \in \Phi, n \geq 0 \quad (E_1^{n+1}, E_4^n), \operatorname{hom}(E_1^{n+1}, E_4^n) = 0 \quad (E_1^{n+1}, E_4^n) \notin \Phi. \\
(E_1^n, E_4^n) & \in \Phi, n \geq 0 \quad (E_1^n, M'), \operatorname{hom}(E_1^n, M') = 0 \quad (E_1^n, M') \notin \Phi. \\
(E_1^{n+1}, E_4^n) & \in \Phi, n \geq 0 \quad (E_1^{n+1}, E_4^n), \operatorname{hom}(E_1^{n+1}, E_4^n) = 0 \quad (E_1^{n+1}, E_4^n) \notin \Phi. \\
(E_1^n, E_4^n) & \in \Phi, n \geq 0 \quad (E_1^n, E_4^n), \operatorname{hom}(E_1^n, E_4^n) = 0 \quad (E_1^n, E_4^n) \notin \Phi. \\
(E_2^n, E_3^{n+1}) & \in \Phi, n \geq 0 \quad (E_2^n, E_3^{n+1}), \operatorname{hom}(M, E_3^{n+1}) = 0 \quad (E_2^n, E_3^{n+1}) \notin \Phi. \\
(E_3^n, E_2^{n+1}) & \in \Phi, n \geq 0 \quad (E_3^n, E_2^{n+1}), \operatorname{hom}(M', E_2^{n+1}) = 0 \quad (E_3^n, E_2^{n+1}) \notin \Phi. \\
(E_3^n, E_2^n) & \in \Phi, n \geq 0 \quad (E_3^n, E_2^n), \operatorname{hom}(M', E_2^n) = 0 \quad (E_3^n, E_2^n) \notin \Phi. \\
(E_3^n, E_2^n) & \in \Phi, n \geq 0 \quad (E_3^n, E_2^n), \operatorname{hom}(M', E_2^n) = 0 \quad (E_3^n, E_2^n) \notin \Phi. \quad \Box
\end{align*}
\]

We eliminated many pairs by using only Section 12 Proposition 9.16 and Lemma [9.2]. It remains to consider the incidences: \((M, E_1^n), (E_2^n, M'), (M', E_2^n), (E_2^n, M) \in \Phi\) for \( n \geq 0 \). From any of these incidences, with the help of Corollaries 10.18, 10.20 and Lemma 10.21, we will derive some of the non-locally finite cases \([80]\), which is excluded by Corollaries 10.6, 10.5. We start with \((M, E_1^n)\).

**Lemma 10.24.** For each \( n \geq 0 \) we have \((M, E_1^n) \notin \Phi\).

**Proof.** Suppose that \((M, E_1^n) \in \Phi\). In the previous corollary we showed that \((E_1^{n+1}, E_3^n) \notin \Phi\). Now from the implication \((M, E_1^n) \in \Phi, n \geq 0 \quad (M, E_1^n, E_4^n) \notin \Phi\) we get \((E_4^{n+1}, E_4^n) \notin \Phi\).
we deduce that \((M, E_4^{n+1}) \in \Phi\), and by induction we obtain \(\phi(E_1^n) = \phi_{\text{max}}\) for \(i \geq n\). We are given also \(\phi(M) = \phi_{\text{min}}\), therefore we can use Corollary 10.18 (a) to obtain \(\{E_3^j, E_4^j\}_{j \in \mathbb{N}} \subset \sigma^{ss}\). By Remark 10.3 we see
\[
\forall i \geq 0 \quad \phi(E_4^i) = \phi_{\text{max}}.
\]
The next step is to show that
\[
\forall i \geq 0 \quad \phi(E_3^i) = \phi_{\text{max}} - 1.
\]
Since \(\text{hom}^1(E_4^0, E_3^0) \neq 0\), we have \(\phi(M) = \phi_{\text{min}} < \phi_{\text{max}} - 1 = \phi(E_4^0) - 1 \leq \phi(E_3^0) \leq \phi_{\text{max}}\). Whence:
\[
\phi(M) < \phi(E_3^0) \leq \phi(E_3^1) \leq \phi(E_3^0) + 1.
\]
If \(\phi(E_4^i) < \phi(E_3^i) + 1\), then we have \(\phi(M) < \phi(E_4^0) \leq \phi(E_4^i) < \phi(E_3^i) + 1\) and Lemma 9.2 (d) applied to the triple \((M, E_4^0, E_3^0)\) gives us a \(\sigma\)-exceptional triple. Therefore \(\phi(E_3^i) = \phi(E_3^0) + 1\). We showed above that \(E_3^i\) is semistable for each \(j \in \mathbb{N}\). From Lemma 10.21 (d) we get \(\phi(E_3^i) = \phi(E_3^0)\) for any \(i \geq 0\), thus we get (97). However (96) and (97) contradict Corollary 10.6.

Lemma 10.25. For each \(n \geq 0\) we have \((E_3^n, M') \notin \Phi\).

Proof. Suppose that \((E_3^n, M') \in \Phi\). We obtain a contradiction of Corollary 10.6 as follows:
\[
(E_3^n, M') \in \Phi \quad \Rightarrow \quad \text{either} \quad (E_3^{n+1}, M') \in \Phi \quad \text{or} \quad (E_3^n, M^n+1) \in \Phi
\]
\[
(E_3^{n+1}, M') \in \Phi \quad \Rightarrow \quad \forall i \geq n \quad \phi(E_3^i) = \phi_{\text{min}}
\]
\[
\{E_3^j, E_4^j\}_{j \in \mathbb{N}} \subset \sigma^{ss}.
\]
By Remark 10.3 we see that \(\phi(E_3^i) = \phi_{\text{min}}\) for \(i \geq 0\). We show below that \(\phi(E_3^i) = \phi_{\text{min}} + 1\) for each \(i \geq 0\), which contradicts Corollary 10.6.

Indeed, by \(\text{hom}^1(E_4^0, E_3^0) \neq 0\) we can write \(\phi_{\text{min}} = \phi(E_3^0) \leq \phi(E_4^0) \leq \phi(E_3^0) + 1 < \phi_{\text{max}} = \phi(M')\). The triple \((E_3^0, E_4^0, M')\) has \(\text{hom}(E_3^0, E_4^0) = 0\), therefore from \(\phi(E_3^0) < \phi(E_4^0) + 1\) it follows that for some \(j \geq 1\) the triple \((E_3^0, E_4^0, M'[-j])\) is \(\sigma\)-exceptional. Therefore \(\phi(E_3^0) = \phi(E_3^0) + 1\). We showed above that \(\{E_3^j\} \subset \sigma^{ss}\). By Lemma 10.21 (a) we conclude that \(\phi(E_3^j) = \phi_{\text{min}} + 1\) for each \(n \geq 0\).

Lemma 10.26. For each \(n \geq 0\) we have \((M', E_1^n) \notin \Phi\).

Proof. Suppose that \((M', E_1^n) \in \Phi\). We show that this contradicts Corollary 10.5 as follows:
\[
(M', E_1^n) \in \Phi \quad \Rightarrow \quad \text{either} \quad (E_1^{n+1}, E_1^n) \in \Phi \quad \text{or} \quad (M', E_1^{n+1}) \in \Phi
\]
\[
(M', E_1^{n+1}) \in \Phi \quad \Rightarrow \quad \forall i \geq n \quad \phi(E_1^i) = \phi_{\text{max}}
\]
\[
\{E_1^j, E_2^j\}_{j \in \mathbb{N}} \subset \sigma^{ss}.
\]
By Remark 10.3 we see that \(\phi(E_1^i) = \phi_{\text{max}}\) for each \(i \geq 0\). Furthermore, using \((M', E_1^n) \in \Phi\) we show below that \(\phi(E_1^i) = \phi_{\text{max}} - 1\) must hold for \(i \geq 0\), which contradicts Corollary 10.5.

Indeed, it follows from \(\text{hom}^1(E_1^n, E_1^n) \neq 0\) that \(\phi_{\text{min}} = \phi(M') < \phi(E_1^n) - 1 \leq \phi(E_1^n) \leq \phi_{\text{max}} = \phi(E_1^n)\). If \(\phi(E_1^n) < \phi(E_2^n) + 1\), then \(\phi(M') < \phi(E_2^n) < \phi(E_1^n) < \phi(E_2^n) + 1\) and the triple \((M', E_1^n, E_2^n)\) with \(\text{hom}(E_1^n, E_2^n) = 0\) gives rise to a \(\sigma\)-triple by Lemma 9.2 (d). Therefore \(\phi(E_1^n) = \phi(E_2^n) + 1\). Since \(\{E_2^n\} \subset \sigma^{ss}\), Lemma 10.21 (c) implies that \(\phi(E_2^n) = \phi(E_2^n)\) for \(n \geq 0\). The lemma is proved.

Lemma 10.27. For each \(n \geq 0\) we have \((E_2^n, M) \notin \Phi\).

Proof. Suppose that \((E_2^n, M) \in \Phi\). We will obtain a contradiction of Corollary 10.3 as follows:
\[
(E_2^n, M) \in \Phi \quad \Rightarrow \quad \text{either} \quad (E_2^{n+1}, M) \in \Phi \quad \text{or} \quad (E_2^n, E_2^{n+1}) \in \Phi
\]
Lemma 10.31. By Remark 10.3 we conclude that \( \phi(E_2^i) = \phi_{\min} \) for \( i \geq 0 \). We show below that \((E_2^i, M) \in \Phi \) implies that \( \phi(E_2^i) = \phi_{\min} + 1 \) for each \( i \geq 0 \), which contradicts Corollary 10.5.

Indeed, it follows from \( \text{hom}^1(E_0^1, E_0^2) \neq 0 \) and \((E_0^1, M) \notin \Phi \) (see Corollary 10.23) that \( \phi_{\min} = \phi(E_0^1) < \phi(E_0^1) \leq \phi(E_0^1) + 1 < \phi_{\max} = \phi(M) \). If \( \phi(E_0^1) < \phi(E_0^1) + 1 \), then for some \( j \geq 1 \) the triple \((E_0^1, E_0^2, M[-j])\) is \( \sigma \)-exceptional, since \((E_0^1, E_0^2, M)\) is exceptional and \( \text{hom}(E_0^1, E_0^2) = 0 \). Therefore \( \phi(E_0^1) = \phi(E_0^1) + 1 < \phi(M) \). Lemma 10.21 (b) gives us \( \phi(E_0^n) = \phi(E_0^n) = \phi_{\min} + 1 \) for \( n \geq 0 \).

Therefore, we reduce to \( \phi_{\max} - \phi_{\min} \leq 1 \), which will be assumed until the end of the proof.

10.4. **The case** \( \phi_{\max} - \phi_{\min} \leq 1 \). From this inequality we obtain a contradiction here again, by deriving the non-locally finite cases [6]. We show first in a series of lemmas that \( \mathcal{A}_{\text{exc}} \subset \mathcal{P}(\phi_{\min}) \cup \mathcal{P}(\phi_{\max}) \), \( \phi_{\max} - \phi_{\min} = 1 \). Lemma 10.13 and Corollary 9.12 imply immediately

**Lemma 10.28.** Any \( E \in \mathcal{A}_{\text{exc}} \) is either semistable or irregular or a final \( \textbf{C}1 \) object.

Any \( X \in \{E_i^j : j \in \mathbb{N}, 1 \leq i \leq 4\} \) is a trivially coupling object, hence by Lemma 6.3 we have only two possibilities: \( X \) is semistable or \( X \) is a final \( \textbf{C}1 \) object (cannot be irregular).

**Corollary 10.29.** The objects \( E_0^1, E_0^3 \) are semistable, and \( M \) is either irregular or semistable.

**Proof.** The objects \( E_0^1, E_0^3, M \) cannot be \( \textbf{C}1 \) by Lemma 10.9.

**Lemma 10.30.** The object \( E_2^2 \) is semistable.

**Proof.** Suppose that \( E_2^0 \) is not semistable.

Therefore \( E_2^0 \) must be \( \textbf{C}1 \), and we have \( E_2^0 \rightarrow \rightarrow (X, Y) \) for some exceptional pair \((X, Y)\). By Lemma 10.9, we see that \((X, Y) = (M, E_3^2)\). Since \( E_2^0 \) is final, we can write

\[
M, E_3^2 \in \sigma^s \quad \phi(M) < \phi(E_3^2).
\]

From the triple \((E_0^1, M, E_0^3)\), which satisfies \( \text{hom}(E_0^1, M) = \text{hom}(E_0^1, E_0^3) = \text{hom}(M, E_0^3) = 0 \), and Lemma 9.2 (f) we see that \( \phi(E_0^3) = \phi(M) + 1 \) (recall that \( E_0^3 \) is semistable). From \( \phi_{\max} - \phi_{\min} \leq 1 \) it is clear that

\[
\phi(M) = \phi_{\min}, \quad \phi(E_0^3) = \phi_{\max} = \phi(M) + 1.
\]

The obtained relations imply that \( E_0^4 \) is semistable. Indeed, if \( E_0^4 \) is not semistable, then it must be final \( \textbf{C}1 \), hence by Lemma 10.9 we have \( E_0^4 \rightarrow \rightarrow (E_0^4, M) \), which in turn implies \( \phi(M) > \phi(E_0^4) \) contradicting \( \phi(E_0^4) = \phi(M) + 1 \). Therefore \( E_0^4 \) is semistable. Now consider the triple \((M, E_0^4, E_3^2)\) with \( \text{hom}(E_0^4, E_3^2) = 0 \). We have \( \phi(M) \leq \phi(E_0^4) \leq \phi(M) + 1, \phi(M) < \phi(E_3^2) \leq \phi(M) + 1 \). If \( \phi(M) < \phi(E_0^4) \), then \( \phi(M) < \phi(E_0^4) \leq \phi(M) \), \( \phi(M) - 1 < \phi(E_0^4) \leq \phi(M) \) and \((M, E_0^4[-1], E_3^2[-1])\) is a \( \sigma \)-exceptional triple. So far, assuming that \( E_2^0 \) is not semistable we get:

\[
\phi_{\min} = \phi(M) = \phi(E_0^4) < \phi(E_0^4) \leq \phi(E_0^4) = \phi(E_0^4) + 1.
\]

Therefore \( \phi(E_0^4) - 1 < \phi(E_3^2[-1]) \leq \phi(E_3^2[-1]) = \phi(E_3^2) \) and then the triple \((E_0^4, E_0^4[-1], E_0^3[-1])\) is a \( \sigma \)-exceptional triple (since \( \text{hom}(E_0^4, E_3^2) = 0 \)). This triple contradicts our assumption on \( \sigma \). 

**Lemma 10.31.** The object \( E_4^1 \) is semistable.
Lemma 10.33. \( \phi \) contradicts the inequality \( \phi(E_3^0) \). From the simple objects triple \( (E_1^0, M, E_3^0) \) and Lemma 9.2 (f) it follows that \( \phi(M) = \phi(E_3^0) + 1 \) (recall that \( E_3^0 \) is semistable), hence by \( \phi_{\text{max}} - \phi_{\text{min}} \leq 1 \):

\[
\phi(E_3^0) = \phi_{\text{min}}, \quad \phi(M) = \phi_{\text{max}} = \phi_{\text{min}} + 1.
\]

Now we have \( \{E_1^0, E_2^0, E_3^0, M\} \subset \sigma^{ss} \). From Corollary 10.10 it follows \( \phi(E_2^0) \leq \phi(E_1^0) \). Whence, we have \( \phi_{\text{min}} = \phi(M) - 1 \leq \phi(E_2^0) \leq \phi(E_1^0) \). The triple \( (E_1^0, E_2^0, M) \) has hom\((E_1^0, E_2^0) = 0 \). We rewrite the last inequalities as follows \( \phi(M[-1]) \leq \phi(E_2^0) \leq \phi(E_1^0) \). Take a \( \sigma \)-exceptional triple \( (E_1^0, E_2^0, M[-1]) \), which is a contradiction. The lemma follows. \( \square \)

Now, using that \( \{E_1^0\}^4 \) \( \subset \sigma^{ss} \), we show that \( M, M' \) cannot be irregular.

Corollary 10.32. There does not exist a \( B2 \) object.

Proof. Suppose that \( E \in A \) is a \( B2 \) object. Since the only Ext-nontrivial couple is \( \{M, M'\} \), we have \( E \in \{M, M'\} \) and we can write

\[
\text{alg}(E) = \begin{array}{c}
A \\
\circ \\
\circ
\end{array} \begin{array}{c}
E \\
\circ \\
\circ
\end{array} \begin{array}{c}
B[1]
\end{array}
\]

\[
\{E, \Gamma\} = \{M, M'\} \text{ for some } \Gamma \in \text{Ind}(B),
\]

\[
\phi_-(A) > \phi(B) + 1 = \phi(\Gamma) + 1.
\]

From hom\((M, E_1^0) \neq 0, \text{hom}(M', E_1^0) \neq 0, \) and \( \{E_1^0, E_2^0\} \subset \sigma^{ss} \) (shown in the preceding lemmas) it follows that there exists \( X \in \sigma^{ss} \cap \text{exc} \) with hom\((E, X) \neq 0 \). Hence hom\((A, X) \neq 0 \) and \( \phi_-(A) \leq \phi(X) \). Whence, we obtain \( \phi(X) \geq \phi_-(A) + 1 = \phi(\Gamma) + 1 \) with \( X, \Gamma \in \sigma^{ss} \cap \text{exc} \), which contradicts the inequality \( \phi_{\text{max}} - \phi_{\text{min}} \leq 1 \). \( \square \)

Lemma 10.33. There does not exist a \( \sigma \)-irregular object.

Proof. By Corollary 10.32 we have to show that neither \( M \) nor \( M' \) can be \( B1 \).

Suppose that \( E \in \{M, M'\} \) is \( B1 \), then we can write:

\[
\text{alg}(E) = \begin{array}{c}
A_1 \oplus A_2[-1]
\end{array} \begin{array}{c}
E \\
\circ \\
\circ
\end{array} \begin{array}{c}
B
\end{array}
\]

\[
\{E, \Gamma\} = \{M, M'\} \text{ for some } \Gamma \in \text{Ind}(A_2),
\]

\[
\text{hom}^1(A_1, A_1) = \text{hom}^1(A_2, A_2) = 0
\]

\[
\phi_-(A_1 \oplus A_2[-1]) \geq \phi(B) = \phi_-(E).
\]

We show first that each \( Y \in \text{Ind}(A_2) \) must be semistable with \( \phi(Y) = \phi(B) + 1 \), which implies

\[
A_2 \in \sigma^{ss}, \quad \phi(A_2[-1]) = \phi(B).
\]

To that end we observe that there exists \( X \in \sigma^{ss} \cap \text{exc} \) with hom\((X, B) \neq 0 \), and hence

\[
\phi(X) \leq \phi(B) \quad X \in \sigma^{ss} \cap \text{exc}.
\]

Indeed, if we find \( X \in \sigma^{ss} \cap \text{exc} \) with hom\((X, E) = 0 \) and hom\((X, \Gamma) \neq 0 \), then from the triangle alg\((E) \) it follows that hom\((X, B) \equiv \text{hom}(X, A_1[1] \oplus A_2) \neq 0 \) (the latter does not vanish by \( \Gamma \in \text{Ind}(A_2) \) and hom\((X, \Gamma) \neq 0 \)). Looking at table 11 we see that hom\((E_2^0, M') = 0, \text{hom}(E_2^0, M) \neq 0, \text{hom}(E_3^0, M') \neq 0 \), therefore

\[
\begin{align*}
X &= E_2^0 & \text{if } E &= M' \\
X &= E_3^0 & \text{if } E &= M.
\end{align*}
\]
Let us take any \( Y \in \text{Ind}(A_2) \). From Lemma [5.4] (c) it follows that \( Y \) cannot be \( \sigma \)-irregular. Hence it is either semistable or a final \( \text{C1} \) object. If \( Y \) is \( \text{C1} \), then \( Y \rightarrow (Z, W) \) for some \( Z, W \in \sigma^{ss} \cap A_{exc} \), and we can write \( \phi(W) > \phi(Z) = \phi_-(Y) \geq \phi_-(A_2) \geq \phi(B) + 1 \geq \phi(X) + 1 \), which contradicts \( \phi_{max} - \phi_{min} \leq 1 \). If \( Y \) is semistable, then by \( \phi_{max} - \phi_{min} \leq 1 \) it follows \( \phi(Y) \leq \phi(X) + 1 \), which, together with \( \phi(Y) > \phi(B) + 1 \geq \phi(X) + 1 \), implies \( \phi(Y) = \phi(B) + 1 = \phi(X) + 1 \). Whence, we proved \([99]\). Furthermore, we see that \([100]\) must be equality.

Being a \( \text{B1} \) object, \( E \) is not semistable. From the triangle \( \text{alg}(E) \), the equality \( \phi(A_2[-1]) = \phi(B) \) and the fact that \( \mathcal{P}(t) \) is an extension closed subcategory of \( \mathcal{T} \) it follows that \( A_1 \neq 0 \) and \( \phi_+(A_1) > \phi(B) \). From [B1.1] we know that \( A_1 \) is a proper \( \mathcal{A} \)-subobject of \( E \). Since \( M \) is simple in \( \mathcal{A} \), it follows that \( E \) cannot be \( M \). Whence, \( E \) must be \( M' \) and then \( X = E_0^0 \) (see \([101]\)). The only proper subobject of \( M' \) in \( \mathcal{A} \) up to isomorphism is \( E_3^0 \) and we know that it is semistable. Whence, we arrive at \( \phi(E_3^0) > \phi(B) = \phi(X) = \phi(E_2^0) \). It follows that \( \text{hom}(E_3^0, E_2^0) = 0 \), which contradicts table \([1]\).

\textbf{Corollary 10.34.} The objects \( M, M' \) are semistable and

\[ \phi(E_3^0) \leq \phi(E_1^0) \quad \phi(E_3^0) \leq \phi(E_1^0) \quad \phi(E_3^0) \leq \phi(E_1^0). \]

\textbf{Proof.} The semistability of \( M \) follows from Corollary [10.29] and Lemma [10.33]. Then from Corollary [10.10] we get \( \phi(E_3^0) \leq \phi(E_1^0) \) and \( \phi(E_3^0) \leq \phi(E_1^0) \).

Using \( \phi_{max} - \phi_{min} \leq 1 \) we showed so far that the cases \( \text{C2}, \text{C3}, \text{B1}, \text{B2} \) cannot appear. Therefore we have only two options for \( M' \): either semistable or final \( \text{C1} \).

Suppose that \( M' \) is final \( \text{C1} \). Lemma [10.9] implies that \( M' \rightarrow (E_0^0, E_3^0) \). Therefore \( \phi(E_3^0) > \phi(E_1^0) = \phi_-(M') \). However, we showed already that \( \phi(E_3^0) \leq \phi(E_1^0) \). Hence, \( M' \) must be also semistable. Now Corollary [10.11] implies \( \phi(E_3^0) \leq \phi(E_1^0) \).

So far, we showed that the low dimensional exceptional objects \( \{E_i^0\}_{i=1}^4, M, M' \) are semistable. The following implications, due to table \([88]\) in Lemma [10.8], will help us to show that \( A_{exc} \subset \sigma^{ss} \).

\textbf{Corollary 10.35.} Let \( R \in \{E_i^m : m \geq 1, 1 \leq i \leq 4 \} \) and let \( R \) be non-semistable.

\begin{itemize}
  \item[(a)] If \( R = E_i^n \), then \( \phi(E_i^n) < \phi(M') \) for some \( n < m \), and hence \( \phi(M) < \phi(M') \).
  \item[(b)] If \( R = E_i^n \), then \( \phi(M) < \phi(E_i^n) \) for some \( n \leq m \), and hence \( \phi(M) < \phi(M') \).
  \item[(c)] If \( R = E_i^n \), then \( \phi(M') < \phi(E_i^n) \) for some \( n < m \), and hence \( \phi(M') < \phi(M) \).
  \item[(d)] If \( R = E_i^n \), then \( \phi(E_i^n) < \phi(M) \) for some \( n \leq m \), and hence \( \phi(M') < \phi(M) \).
\end{itemize}

\textbf{Proof.} Now we have \( M, M' \in \sigma^{ss} \) and any non-semistable \( R \in A_{exc} \) is a final \( \text{C1} \) object. Note also that for each \( n \in \mathbb{N} \) we have \( \text{hom}(M, E_i^n) \neq 0 \), \( \text{hom}(E_i^n, M') \neq 0 \), \( \text{hom}(E_i^n, M) \neq 0 \), \( \text{hom}(M', E_i^n) \neq 0 \), which implies \( \phi(M) \leq \phi_+(E_i^n), \phi_-(E_i^n) \leq \phi(M'), \phi_-(E_i^n) \leq \phi(M) \). Due to Lemma [8.2] and the inequalities \([102]\) in Corollary [10.34], we can remove the pairs \( (E_1^0, E_3^0), (E_1^0, E_3^0), (E_1^0, E_3^0) \) from table \([88]\) in Lemma [10.8]. If \( E_i^n \notin \sigma^{ss} \) for some \( m \geq 1, 1 \leq i \leq 4 \), then \( E_i^n \) is a final \( \text{C1} \) object and the corollary follows from table \([88]\) in Lemma [10.8].

Knowing that the triple \( (E_1^0, M, E_3^0) \) of the simple objects is semistable, we obtain that one of three equalities below must hold, which implies \( \phi_{max} - \phi_{min} = 1 \).

\textbf{Lemma 10.36.} There is an equality \( \phi_{max} - \phi_{min} = 1 \). One of the following equalities must hold:

\[ \phi(E_0^0) = \phi(M) + 1, \quad \phi(E_1^0) = \phi(E_3^0) + 1, \quad \phi(M) = \phi(E_3^0) + 1. \]
Corollary 10.37. \( \phi(M) \in \{ \phi_{\text{min}}, \phi_{\text{max}} \} \).

**Proof.** Suppose that \( \phi_{\text{min}} < \phi(M) < \phi_{\text{max}} \). By Lemma 10.36 we get \( \phi_{\text{min}} = \phi(E_0^0) \) and \( \phi(E_0^0) + 1 = \phi(E_1^0) = \phi_{\text{max}} \). Therefore, we can write \( \phi(E_0^0) < \phi(M) < \phi(E_0^0) + 1 \) and \( \phi(E_0^0) \leq \phi(E_0^0) \leq \phi(E_0^0) + 1 \). Applying (F) of Lemma 9.2 to the triple \( (E_0^0, M, E_3^0) \), we see that one of the equalities (103) holds. Hence \( \phi_{\text{max}} - \phi_{\text{min}} \geq 1 \) and the lemma follows. \( \square \)

Corollary 10.38. We have \( \{ \phi(M), \phi(M'), \phi(E_j^0) \} \subset \{ \phi_{\text{min}}, \phi_{\text{max}} \} \) for \( j = 1, 2, 3 \).

**Proof.** Now we have \( \{ \phi(M), \phi(M'), \phi(E_j^0) \} \subset \sigma^{ss} \) and \( \phi(M) \in \{ \phi_{\text{min}}, \phi_{\text{max}} \} \). It is enough to show \( \{ \phi(E_0^0), \phi(E_0^0) \} \subset \{ \phi_{\text{min}}, \phi_{\text{max}} \} \), because then by formula (11), the equalities \( Z(M') = Z(E_0^1) + Z(E_0^0), Z(E_0^0) = Z(M) + Z(E_0^0), Z(E_0^0) = Z(M) + Z(E_0^0) \), and the inequalities \( \phi_{\text{min}} \leq \phi(M'), \phi(E_j^0), \phi(E_j^0) \leq \phi_{\text{max}} \) it follows that \( \{ \phi(M'), \phi(E_j^0), \phi(E_j^0) \} \subset \{ \phi_{\text{min}}, \phi_{\text{max}} \} \).

If \( \phi(M) = \phi_{\text{min}} \), then by hom\( (E_0^0, M) \neq 0 \) it follows that \( \phi(E_0^0) = \phi_{\text{min}} \), and by Lemma 10.36 it follows that \( \phi(E_0^0) = \phi_{\text{min}} \). Expanding the equality \( Z(E_0^0) = Z(M) + Z(E_0^0) \) by formula (11), and using \( \phi(M) = \phi(E_0^0) = \phi_{\text{min}} \), then by hom\( (M, E_4^0) \neq 0 \) it follows \( \phi(E_0^0) = \phi_{\text{max}} \), and by Lemma 10.36 it follows \( \phi(E_0^0) = \phi_{\text{min}} \). Finally, \( \phi(E_j^0) \in \{ \phi_{\text{min}}, \phi_{\text{max}} \} \) follows from \( \phi(M) = \phi(E_j^0) = \phi_{\text{max}} \), \( Z(E_0^0) = Z(M) + Z(E_0^0) \), and formula (11). The corollary is proved. \( \square \)

The proofs of semistability for \( E_1^m \) and \( E_2^m \) share some steps because the non-semistability of any of them implies \( \phi(M) < \phi(M') \) (Corollary 10.35(a), (b)). Similarly, the starting argument in the proof of Lemma 10.40 that is the non-semistability of \( E_3^m \) or \( E_4^m \) implies \( \phi(M') < \phi(M) \).

Lemma 10.39. All objects in \( \{ E_1^m, E_2^m \}_{m \in \mathbb{N}} \) are semistable.

**Proof.** Suppose that \( E_1^m \) is not semistable for some \( m \in \mathbb{N} \). Corollary 10.35(a) shows that \( E_4^m \subset \sigma^{ss}, \phi(E_0^4) < \phi(M') \) for some \( n \in \mathbb{N} \), and \( \phi(M) < \phi(M') \). The latter inequality implies, due to Corollary 10.35(c) and (d), that \( \{ E_4^m, E_3^m \}_{m \in \mathbb{N}} \subset \sigma^{ss} \), and, due to Corollary 10.38 it implies

\[
\phi_{\text{min}} = \phi(M), \quad \phi(M') = \phi_{\text{max}} = \phi_{\text{min}} + 1.
\]

By Lemma 10.21(a) we can write \( \phi(E_0^1) = \phi(E_0^1) < \phi(M') \) and combining with Corollary 10.34 we arrive at \( \phi_{\text{min}} = \phi(M') - 1 \leq \phi(E_0^4) \leq \phi(E_0^1) < \phi(M') \), hence the triple \( (E_1^m, E_3^m, M'[1]) \) with hom\( (E_1^m, E_3^m) = 0 \) is a \( \sigma^{ss} \)-exceptional triple. Therefore \( \{ E_1^m \}_{m \in \mathbb{N}} \subset \sigma^{ss} \).

Next, suppose that \( E_2^m \) is not semistable for some \( m \in \mathbb{N} \). Then by Corollary 10.35(b) we have \( E_3^m \in \sigma^{ss}, \phi(M) < \phi(E_3^m) \) for some \( n \in \mathbb{N} \), and \( \phi(M) < \phi(M') \). Now by the same arguments as above

\[55\] of the assumption that there is not a \( \sigma^{ss} \)-exceptional triple
we get \([105]\) and \([E^m_3, E^m_4] \subseteq \sigma^{ss}\). By Lemma \([10.21](d)\) we can write \(\phi(E^0_3) = \phi(E^0_3) > \phi(M)\). Combining with Corollary \([10.34]\) we arrive at \(\phi_{min} = \phi(M) < \phi(E^0_3) \leq \phi(E^0_4) \leq \phi(M) + 1\). These inequalities and the exceptional triple \((M, E^0_4, E^0_3)\) with \(\text{hom}(E^0_4, E^0_3) = 0\) provide a \(\sigma\)-exceptional triple \((M, E^0_4[-1], E^0_3[-1])\). The lemma follows.

**Lemma 10.40.** All objects in \([E^m_3, E^m_4]\) are semistable.

**Proof.** Suppose that \(E^m_3\) or \(E^m_4\) is not semistable for some \(m \in \mathbb{N}\). By Lemma \([10.35]\) we get \(\phi(M') < \phi(M)\). Since \(\{\phi(M), \phi(M')\} \subseteq \{\phi_{min}, \phi_{max}\}\) (Corollary \([10.38]\)), we find that:

\[\phi_{min} = \phi(M'), \quad \phi(M) = \phi_{max} = \phi_{min} + 1.\]

We have also \([E^m_1, E^m_2]\) \(\subseteq \sigma^{ss}\). Thus, (b) and (c) in Lemma \([10.21]\) can be used to obtain:

\[\forall m \in \mathbb{N} \quad \phi(E^m_1) = \phi(E^m_1), \quad \phi(E^m_2) = \phi(E^m_2).\]

From \(\text{hom}(M, E^0_3) \neq 0\), and \(\text{hom}(E^0_4, E^0_3) \neq 0\) (note that \(\text{hom}(E^0_4, E^0_1) = 0\) for \(m \geq 1\)) it follows \(\phi(M) = \phi_{max} = \phi(E^0_1)\). On the other hand, from the triple \((M', E^0_1, E^0_2)\) with \(\text{hom}(E^0_4, E^0_2) = 0\) it follows that \(\phi(E^0_3) = \phi(M') = \phi_{min}\) (otherwise \((M', E^0_1[-1], E^0_2[-1])\) would be a \(\sigma\)-exceptional triple). Using \([106]\) we obtain

\[\forall m \in \mathbb{N} \quad \phi_{max} = \phi(M) = \phi(E^m_1), \quad \phi_{min} = \phi(M') = \phi(E^m_2).\]

However, due to (c) and (d) in Corollary \([10.35]\) these equalities contradict the assumption that \(E^m_3\) or \(E^m_4\) is not semistable for some \(m\). The lemma follows.

**Corollary 10.41.** All exceptional objects are semistable and their phases are in \(\{\phi_{min}, \phi_{max}\}\).

**Proof.** We have already proved that the exceptional objects are semistable. As in subsection \([3.3]\) we denote \(\delta_Z = Z(M) + Z(E^0_1) + Z(E^0_3)\). By Bridgeland’s axiom \([11]\) we can rewrite \([12]\) as follows:

\[r(E^m_j) \exp(i\pi \phi(E^m_j)) = m \delta_Z + r(E^0_j) \exp(i\pi \phi(E^0_j)) \quad m \in \mathbb{N}, j = 1, 2, 3, 4.\]

In Corollary \([10.38]\) we have \(\{\phi(M), \phi(E^0_1), \phi(E^0_3)\} \subseteq \{\phi_{min}, \phi_{max}\}\), therefore we can write \(\delta_Z = \Delta \exp(i\pi \gamma)\) with \(\Delta \geq 0\) and \(\gamma \in \{\phi_{min}, \phi_{max}\}\).

Now \([107]\) restricts all the phases in the set \(\{\phi_{min}, \phi_{max}\}\), since \(\phi_{min} \leq \phi(E^0_j) \leq \phi_{max} = \phi_{min} + 1\) for any \(1 \leq j \leq 4, m \in \mathbb{N}\).

We are already close to \([85]\). To derive completely some of the non-locally finite cases in \([86]\) we consider each of the three equalities \([103]\). We showed that one of them holds.

**10.4.1. If** \(\phi(E^0_3) = \phi(M) + 1\). Then \(\phi_{min} = \phi(M)\) and \(\phi_{max} = \phi(E^0_3)\).

Since \(\text{hom}(E^m_3, M) \neq 0\), we have \(\phi(E^m_3) \leq \phi(M) = \phi_{min}\) for \(m \in \mathbb{N}\). Hence \([E^m_3] \subseteq \mathcal{P}(\phi_{min})\).

We will show below that \([E^m_1] \subseteq \mathcal{P}(\phi_{max})\) and so we obtain the first case in \([85]\).

The sequence \(\{\phi(E^m_1)\}_{m \in \mathbb{N}}\) is non-increasing (see Remark \([10.3]\)) and has at most two values. The first value is \(\phi(E^0_1) = \phi_{max} = \phi(M) + 1\). Suppose that \(\phi(E^0_1) = \phi(M)\) for some \(l > 0\). We can assume that \(l\) is minimal, so \(\phi(E^{l-1}_1) = \phi(M) + 1\). In Table \([11]\) we see that \(\text{hom}(M', E^l_1) \neq 0\), hence \(\phi(M') \leq \phi(M) = \phi_{min}\), i.e. \(\phi(M') = \phi(M) = \phi_{min}\). We have the triple \((E^l_1, M, E^l_1)\) with \(\text{hom}(E^l_1, M) = 0\) and \(\phi(E^l_1) = \phi(M)\). It follows that \(\phi(E^{l-1}_1) = \phi(M)\), otherwise Lemma \([9.2](b)\) produces a \(\sigma\)-triple. However, now the exceptional triple \((E^{l-1}_1, M', E^{l-1}_1)\) with \(\text{hom}(E^{l-1}_1, M') = 0\) satisfies \(\phi(M) = \phi(E^{l-1}_1) = \phi(M') < \phi(E^{l-1}_1) = \phi(M) + 1\) and Lemma \([9.2](b)\) gives a contradiction.
Whence, the equality \( \phi(E_0^0) = \phi(M) + 1 \) implies the first case in \((86)\), which contradicts Corollary \(10.5\). Therefore, for the rest of the proof we can use the strict inequality:

\[
\phi(E_1^m) < \phi(M) + 1.
\]

10.4.2. If \( \phi(E_0^0) = \phi(E_3^0) + 1 \) or \( \phi(M) = \phi(E_0^3) + 1 \). In both cases \( \phi_{\min} = \phi(E_0^0) \), \( \phi_{\max} = \phi(E_3^0) + 1 \).

We note first that \( \text{hom}(M, E_1^m) \neq 0 \) and \( \text{hom}(E_1^m, E_1^m) \neq 0 \) for each integer \( m \), hence

\[
\phi(M) \leq \phi(E_1^m) \leq \phi(E_1^m) \leq \phi(E_3^0) + 1 \quad m \in \mathbb{N}.
\]

Therefore, it is enough to consider the case \( \phi(E_0^0) = \phi(E_3^0) + 1 \). The latter equality and \((108)\) imply \( \phi_{\max} = \phi(E_0^0) \) and \( \phi(E_0^0) = \phi_{\min} < \phi(M) \). It follows that \( \phi(M) = \phi_{\max} \). Now \((109)\) implies \( \{E_3^m\}_{m \in \mathbb{N}} \subset \mathcal{P}(\phi_{\min}) \). We will show that \( \{E_3^m\}_{m \in \mathbb{N}} \subset \mathcal{P}(\phi_{\min}) \) and so we obtain the second case in \((86)\).

Now we have \( \phi(E_0^0) = \phi_{\min} \). Suppose that \( \phi(E_0^0) = \phi_{\max} \) for some \( l > 0 \). Choosing the minimal \( l \) with this property, we have \( \phi(E_0^0) = \phi_{\min} \). By \( \text{hom}(E_0^0, M) \neq 0 \) we get \( \phi(M) = \phi_{\max} \). It follows that \( \phi(E_0^0) = \phi_{\min} \), because otherwise \( (E_0^0, M, E_0^0) \) is a \( \tau \)-triple, due to \( \text{hom}(M, E_0^0, M) = 0 \). However, now \( (E_0^0, M, E_0^0) \) is a \( \tau \)-exceptional triple, due to \( \text{hom}(M, E_0^0, M) = 0 \).

Whence, any of the equalities \( \phi(E_0^0) = \phi(E_0^0) + 1 \) and \( \phi(M) = \phi(E_0^0) + 1 \) implies \((86)\), which is the desired contradiction. Theorem \(10.1\) is proved.

**Appendix A.**

In the table below we present the dimensions of some vector spaces of matrices. We skip the computations. For \( m, n \geq 1 \) we denote by \( \mathbb{M}_k(m, n) \) the vector space of \( m \times n \) matrices over the field \( k \). The notations \( \pi^m_\pm, j^m_\pm \) for \( m \in \mathbb{N} \) are explained before Proposition \(2.2\).

\[
\begin{array}{|c|c|c|}
\hline
& V & \text{dim}_k(V) \\
\hline\hline
1 \leq n < m & \{(X, Y) \in \mathbb{M}_k(n + 1, m + 1) \times \mathbb{M}_k(n, m) : X \circ j^m_\pm = j^m_\pm \circ Y, \ X \circ j^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
1 \leq m \leq n & \{(X, Y) \in \mathbb{M}_k(n, m) \times \mathbb{M}_k(n + 1, m + 1) : X \circ j^m_\pm = j^m_\pm \circ Y, \ X \circ j^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
1 \leq n \leq m & \{(X, Y) \in \mathbb{M}_k(n + 1, m + 1) : X \circ \pi^m_\pm = j^m_\pm \circ Y, \ X \circ \pi^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
1 \leq m, 1 \leq n & \{(X, Y) \in \mathbb{M}_k(n + 1, m + 1) : X \circ \pi^m_\pm = j^m_\pm \circ Y, \ X \circ \pi^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
1 \leq m \leq n & \{X \in \mathbb{M}_k(n, m + 1) + \pi^m_\pm \circ X \circ \pi^m_\pm = j^m_\pm \circ X \circ \pi^m_\pm \} & 0 \\
\hline
0 \leq n < m & \{X \in \mathbb{M}_k(n, m + 1) \times \mathbb{M}_k(n + 1, m + 1) : X \circ j^m_\pm = j^m_\pm \circ X \circ j^m_\pm \} & 0 \\
\hline
1 \leq m \leq n & \{X \in \mathbb{M}_k(n + 1, m + 1) : X \circ \pi^m_\pm = j^m_\pm \circ X \circ \pi^m_\pm \} & 0 \\
\hline
0 \leq m < n & \{X \in \mathbb{M}_k(n + 1, m + 1) : X \circ \pi^m_\pm = j^m_\pm \circ X \circ j^m_\pm \} & 0 \\
\hline
1 \leq m, 1 \leq n & \{(X, Y) \in \mathbb{M}_k(m + 1, n + 1) \times \mathbb{M}_k(m + 1, n) : X \circ j^m_\pm = j^m_\pm \circ Y, \ X \circ j^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
0 \leq n < m & \{(X, Y) \in \mathbb{M}_k(n, m + 1) : X \circ \pi^m_\pm = j^m_\pm \circ Y, \ X \circ \pi^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
0 \leq m < n & \{(X, Y) \in \mathbb{M}_k(n + 1, m + 1) : X \circ \pi^m_\pm = j^m_\pm \circ Y, \ X \circ \pi^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
0 \leq m, 0 \leq n & \{(X, Y) \in \mathbb{M}_k(n + 1, m + 1) : X \circ \pi^m_\pm = j^m_\pm \circ Y, \ X \circ \pi^m_\pm = j^m_\pm \circ Y\} & 0 \\
\hline
0 \leq m, 0 \leq n & \{X \in \mathbb{M}_k(n + 1, m + 1) \times \mathbb{M}_k(n + 1, m + 1) : X \circ j^m_\pm = j^m_\pm \circ X \circ j^m_\pm \} & 0 \\
\hline
\end{array}
\]
APPENDIX B. The Kronecker quiver

B.1. There are not Ext-nontrivial couples in $\text{Rep}_k(K(l))$. The quiver with two vertices and $l \geq 2$ parallel arrows will be denoted by $K(l)$. Here we revisit [12, Lemma 4.1]. This lemma implies the title of this subsection.

Following the notations of [12], let $s_0$ and $s_1$ be the exceptional objects in $D^b(K(l))$, such that $s_0[1]$ is the simple representation with $k$ at the source, and $s_1$ is the simple representation with $k$ at the sink, and then define $s_i$ for each $i \in \mathbb{Z}$ as follows:

\[(111) \quad s_{-i} = L_{s_{-i+1}}(s_{-i+2}), \quad s_{i+1} = R_{s_i}(s_{i-1}) \quad i \geq 1.\]

The Braid group $B_2$ is isomorphic to $\mathbb{Z}$. By the transitivity of the action of $B_2$ on the set of full exceptional collections, shown in [5], it follows that, up to shifts, the complete list of the exceptional pairs in $\text{Rep}_k(K(l))$ is $\{(s_i, s_{i+1})\}_{i \in \mathbb{Z}}$. Lemma 4.1 in [12] says that $s_{\leq 0}[1], s_{\geq 1} \in \text{Rep}_k(K(l))$, and:

\[(112) \quad p \neq 0 \Rightarrow \text{hom}^p(s_i, s_j) = 0; \quad p \neq 1 \Rightarrow \text{hom}^p(s_j, s_i) = 0; \quad i < j.\]

Now $\{s_{-i}[1]\}_{i \geq 0} \cup \{s_i\}_{i \geq 1}$ is the complete list of exceptional objects of $\text{Rep}_k(K(l))$, and from the vanishings (112) it follows that for any couple $\{X, Y\}$ in this list $\text{hom}^1(X, Y) \neq 0$ implies $\text{hom}^1(Y, X) = 0$. Thus, there are not Ext-nontrivial couples in $\text{Rep}_k(K(l))$.

One can show that the following inequalities hold for each $i \in \mathbb{Z}$:

\[(113) \quad l = \text{hom}(s_i, s_{i+1}) < \text{hom}(s_i, s_{i+2}) < \ldots; \quad 0 = \text{hom}^1(s_i, s_{i-1}) < \text{hom}^1(s_i, s_{i-2}) < \ldots,\]

\[(114) \quad \dim_k(s_1) = \dim_k(s_0[1]) < \dim_k(s_2) = \dim_k(s_{-1}[1]) < \ldots.\]

which implies that $\{s_{-i}[1]\}_{i \geq 0} \cup \{s_i\}_{i \geq 1}$ are pairwise non-isomorphic. Whence, in this case the action of the Braid group is free (compare with Remark 2.12).

B.2. $\sigma$-exceptional pairs in $D^b(K(l))$.

The full exceptional collections in $D^b(K(l))$ have length two, so the analogue of Theorem 10.1 is:

**Lemma B.1.** For each $\sigma \in \text{Stab}(D^b(K(l)))$ there exists a $\sigma$-exceptional pair.

The statement of [12, Lemma 4.2] is equivalent to the statement of Lemma B.1. For the sake of completeness we give a proof of Lemma B.1 here.

Denote, for brevity $A = \text{Rep}_k(K(l))$, and take any $\sigma = (\mathcal{P}, Z) \in \text{Stab}(D^b(A))$. There are not Ext-nontrivial couples in $A$ and the exceptional pairs of $D^b(A)$, up to shifts, are a sequence $\{(s_i, s_{i+1})\}_{i \in \mathbb{Z}}$, where $\{s_{-i}[1]\}_{i \geq 0} \cup \{s_i\}_{i \geq 1} \subset A$ (see Appendix B.1). By Remark 8.4 we reduce the proof immediately to the case, where all the exceptional objects are semistable. In (113) we have $\{\text{hom}(s_i, s_{i+1}) \neq 0\}_{i \in \mathbb{Z}}$, hence $\{\phi(s_i) \leq \phi(s_{i+1})\}_{i \in \mathbb{Z}}$. If $\phi(s_i) < \phi(s_{i+1})$ for some $i \in \mathbb{Z}$, then there exists $j \geq 1$ with $\phi(s_{i+1}[-j]) \leq \phi(s_i) < \phi(s_{i+1}[-j]) + 1$, and hence, due to (112), the pair $(s_i, s_{i+1}[-j])$ is $\sigma$-exceptional. Thus, we reduce to the case, where all $\{s_i\}_{i \in \mathbb{Z}}$ have the same phase.

\[(115) \quad \{s_i\}_{i \in \mathbb{Z}} \subset \mathcal{P}(t).\]

We show now that the obtained inclusion contradicts the locally finiteness of $\sigma$, i.e. (115) is a non-locally finite case.

\[\text{In the end of Appendix B.1 we pointed out that } \{s_i\}_{i \geq 1} \text{ are pairwise non-isomorphic.}\]
Since all the exceptional pairs in $\mathcal{A}$ are $\{(s_{i-1},1), s_i[1]\}_{i\leq -1} \cup \{(s_0[1], s_1)\} \cup \{(s_i, s_{i+1})\}_{i\geq 1}$, it follows from (115) that:

(116) \hspace{1cm} \text{For each exceptional pair } (S, E) \text{ with } S, E \in \mathcal{A} \text{ we have } \phi(S) \geq \phi(E).\]

We will obtain a contradiction by constructing an exceptional pair $(S, E)$ in $\mathcal{A}$ with $\phi(S) < \phi(E)$. Recall that $Z$ is the central charge of $\sigma$. By (115) and (11) we have $\{Z(s_1), Z(s_0[1]) = -Z(s_0)\} \subset \mathbb{R} \exp(i\pi t)$. Since $K(D^b(\mathcal{A})) \cong \mathbb{Z}^2$ and the simple objects $s_0[1], s_1$ form a basis of $K(D^b(\mathcal{A}))$, it follows that $\im(Z) \subset \mathbb{R} \exp(i\pi t)$. Now using (11) again, we conclude that $\mathcal{P}(x)$ is trivial for $x \in (t - 1, t)$, therefore $\mathcal{P}(t - 1, t) = \mathcal{P}(t)$. From the very foundation [1] given by T. Bridgeland, we know that $\mathcal{P}(t - 1, t)$ is a heart of a bounded $t$-structure of $D^b(\mathcal{A})$, so $\mathcal{P}(t)$ is a heart as well. Due to this property of $\mathcal{P}(t)$, it is also well known that $K(\mathcal{P}(t)) = K(D^b(\mathcal{A}))$ is an isomorphism, so $K(\mathcal{P}(t)) \cong \mathbb{Z}^2$. The locally finiteness of $\sigma$ implies that $\mathcal{P}(t)$ is an abelian category of finite length, which in turn, combined with $K(\mathcal{P}(t)) \cong \mathbb{Z}^2$, implies that $\mathcal{P}(t)$ has exactly two simple objects, say $X, Y \in \mathcal{P}(t)$. It follows by Lemma 3.7 that $\{X, Y\}$ are indecomposable in $D^b(\mathcal{A})$, therefore $X = X'[i], Y = Y'[j]$ for some $i, j \in \mathbb{Z}$ and $X', Y' \in \mathcal{A}$. Viewing $\mathcal{A}$ as the extension closure of $s_0[1], s_1$, we see that $X', Y' \in \mathcal{A} \subset \mathcal{P}(t, t + 1)$. Now from $\{X'[i], Y'[j]\} \subset \mathcal{P}(t)$ it follows that either $\phi(X') = t, i = 0$ or $\phi(Y') = t + 1, i = -1$, and the same holds for $Y', j$. If either $i = i' = -1$ or $i = i' = 0$, then $\hom(s_1, X) = \hom(s_1, Y) = 0$ or $\hom(X, s_0) = \hom(Y, s_0) = 0$, which contradicts the existence of a Jordan-Hölder filtration of $s_0, s_1 \in \mathcal{P}(t)$ via the simples $X, Y$ of $\mathcal{P}(t)$. Thus, we arrive at:

(117) \hspace{1cm} X = X', \hspace{0.5cm} Y = Y'[-1], \hspace{0.5cm} X', Y' \in \mathcal{A} \hspace{0.5cm} \phi(X') = t, \hspace{0.5cm} \phi(Y') = t + 1.

By $\phi(Y') > \phi(X')$ it follows $\hom(Y'[i], X') = 0$. Since $Y'[-1], X'$ are non-isomorphic simple objects in the abelian category $\mathcal{P}(t)$, it follows that $\hom(Y'[-1], X') = 0$ as well, hence $\hom^{*}(Y', X') = 0$.

The pair $(X', Y')$ in $\mathcal{A}$ has $\phi(X') < \phi(Y')$ and $\hom^{*}(Y', X') = 0$, and it almost contradicts (116), but we have no arguments for the vanishings $\Ext^1(X', Y') = 0$ and $\Ext^1(Y', X') = 0$.

Keeping in mind the comments in the beginning of Subsection 3.2 we can view $\mathcal{P}(t)$ as the extension closure in $D^b(\mathcal{A})$ of the set $\{Y'[-1], X'\}$. Denoting the extension closures of $X'$ and $Y'$ by $\mathcal{X}$ and $\mathcal{Y}$ respectively, it is clear that $\mathcal{P}(t)$ is the extension closure of $\mathcal{Y}[-1] \cup \mathcal{X}$ and

\begin{equation}
(118) \hspace{1cm} \mathcal{X} = \mathbb{N}[X'], \hspace{0.5cm} \mathcal{Y} = \mathbb{N}[Y'], \hspace{0.5cm} \hom^{*}(\mathcal{Y}, \mathcal{X}) = 0, \hspace{0.5cm} \mathcal{X} \subset \mathcal{A} \cap \mathcal{P}(t), \hspace{0.5cm} \mathcal{Y} \subset \mathcal{A} \cap \mathcal{P}(t + 1),
\end{equation}

where the first two equalities are between subsets of $K(D^b(\mathcal{A}))$. Using $\hom^{*}(\mathcal{Y}, \mathcal{X}) = 0$ and that $\mathcal{P}(t)$ is the extension closure of $\mathcal{Y}[-1] \cup \mathcal{X}$, as in the case of semi-orthogonal decompositions, one can show that for each $X \in \mathcal{P}(t)$ there exists a triangle $A[-1] \longrightarrow X \longrightarrow B \longrightarrow A$ with $A \in \mathcal{Y}, B \in \mathcal{X}$ and $\hom^{*}(A, B) = 0$. Since $s_j \in \mathcal{A}_{\text{exc}} \cap \mathcal{P}(t)$ for $j \geq 1$, the corresponding triangle for $s_j$ is:

\begin{equation}
(119) \hspace{1cm} s_j \longrightarrow B \longrightarrow A \longrightarrow s_j[1], \hspace{0.5cm} \hom^{*}(A, B) = 0, \hspace{0.5cm} A \in \mathcal{Y}, B \in \mathcal{X}, s_j \in \mathcal{A}_{\text{exc}}.
\end{equation}

To prove Lemma 3.1, we show first that we can assume $A \neq 0$. After that we recall some of the arguments used in Subsection 3.5 for obtaining the properties C2.1 in the triangle (34). These arguments lead to the vanishings $\hom^{1}(B, B) = \hom^{1}(A, A) = 0$. Taking any $S \in \text{Ind}(B), E \in \text{Ind}(A)$, we obtain an exceptional pair $(S, E)$ in $\mathcal{A}$ with $\phi(S) < \phi(E)$, which contradicts (116).

Suppose that $A = 0$. Then $s_j \cong B \in \mathcal{X}$ and by (113) we have $\dim(s_j) = p \dim(X')$ for some $p \in \mathbb{N}$. Since $s_j$ is exceptional and $X'$ is indecomposable, then $\langle \dim(s_j), \dim(s_j) \rangle = 1$ (see (3)) and

\begin{enumerate}
\item This isomorphism is determined by assigning to $[X] \in K(D^b(\mathcal{A}))$, for $X \in \mathcal{A}$, the dimension vector $\dim(X) \in \mathbb{Z}^2$.
\end{enumerate}
It follows that \( \dim(s_j) = \dim(X') \), \( \langle \dim(X'), \dim(X') \rangle \leq 1 \) (see [9, p. 58]). It follows that \( X' \) is simple in \( \mathcal{P}(t) \), which implies \( \text{hom}(X', X') = 1 \). Now formula (3) shows that \( X' \) is an exceptional object, and hence \( \dim(X') = \dim(s_j) \) implies that \( X' \cong s_j \). Thus, \( A = 0 \) implies \( X' \cong s_j \). It follows, since \( \{s_i\}_{i \geq 1} \) are pairwise non-isomorphic, that in (119) the object \( A \) can vanish for at most one integer \( j \geq 1 \). Hence, we can take \( j \geq 1 \) so that \( A \neq 0 \).

Since \( \text{Hom}^1(A, B) = \text{Hom}^2(A, s_j) = 0 \), by applying \( \text{Hom}(A, \_ ) \) to (119) we obtain \( \text{Hom}^1(A, A) = 0 \). Because we have \( \text{hom}^*(A, B) \), it follows that \( \{\text{hom}^1(\Gamma, s_j) \neq 0\}_{\Gamma \in \text{Ind}(A)} \). Since there are no \( \text{Ext} \)-nontrivial couples in \( A \), we obtain \( \{\text{hom}^1(s_j, \Gamma) = 0\}_{\Gamma \in \text{Ind}(A)} \), hence \( \text{hom}^1(s_j, A) = 0 \). Now the triangle (119) and \( \text{Hom}(s_j, \_) \) imply \( \text{hom}^1(s_j, B) = 0 \). Finally, the same triangle and \( \text{Hom}(\_, B[1]) \) imply \( \text{Hom}(B, B[1]) = 0 \). Lemma [1, 1] is proved.

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(Dimitrov) Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Österreich

E-mail address: gkid@abv.bg

(Katzarkov) Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Österreich

E-mail address: lkatzark@math.uci.edu

\footnote{where \( (\_ , \_ ) \) is the Euler form of \( K(l) \).}

\footnote{There is at most one representation without self-extensions of a given dimension vector (2 p. 13).}

\footnote{see the last paragraph of the proof of Lemma 4.6 with \( E \) replaced by \( s_j \), \( A_2 \) by \( A \), \( B_0 \) by \( B \), and letting \( A_1 = 0 \)}