Embedding dimension of a good semigroup

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Abstract

In this paper, we study good semigroups of \( \mathbb{N}^n \), a class of semigroups that contains the value semigroups of algebroid curves with \( n \) branches. We give the definition of embedding dimension of a good semigroup showing that, in the case of good semigroups of \( \mathbb{N}^2 \), some of its properties agree with the analogue concepts defined for numerical semigroups.

Introduction

The concept of good subsemigroup of \( \mathbb{N}^n \) was formally introduced in [1]. Its definition arises from the properties of the value semigroups of one dimensional analytically unramified rings (for example the local rings of an algebraic curve) that were initially studied in [2, 4, 5, 9, 11, 10, 14]. In [1], the authors proved that the class of good semigroups is actually larger than the one of value semigroups. Thus, such semigroups can be seen as a natural generalization of numerical semigroups and studied without necessarily referring to the ring theory context, using a more combinatorial approach.

Although, as we have already pointed out, good semigroups share traits with the numerical semigroups, there are some important properties of the latter that cannot be generalized to them. For instance, they are not finitely generated as monoids, and they are not closed under finite intersections. This makes the study of good semigroups much more difficult than the numerical ones.

Thus, a relevant part of the literature dedicated to these objects is concerned to find a suitable way to represent them by means of a finite set of data.

For instance, for what concerns good semigroups which are also value semigroups, in [14, 18] singularities with only two branches are studied. In these papers, the finite set considered is the set of maximal elements (in [11], it is possible to find a generalization of this approach to the case of more than two branches). In [7], the authors considered a new approach that is still valid for good semigroups not realizable as value semigroups of curves. They firstly notice that the set of small elements of the semigroup, that is, the finite set of elements between 0 and the conductor of the semigroup with the usual partial order, completely describes it. Then they proved the uniqueness of the minimal subset \( G \subseteq \text{Small}(S) \), called minimal good generating system, from which is possible to recover completely the semigroup \( S \), if also the conductor of \( S \).
is known. Another interesting approach is the one presented in [6], where the authors introduced the semiring of values $\Gamma$ of an algebroid curve $R$ where also the valuation of the zero-divisors elements are considered ($v(0) = (\infty, \ldots, \infty)$). Thus $\Gamma$ contains the value semigroup of $R$ and $(\Gamma, +)$ is a semigroup setting $\gamma + \infty = \infty$ for all $\gamma \in \Gamma$. The key point is that $\Gamma$, equipped with the tropical operations

$$\alpha \oplus \beta = \min \{\alpha, \beta\} := (\min \{\alpha_1, \beta_1\}, \ldots, \min \{\alpha_n, \beta_n\}) \quad \text{and} \quad \alpha \odot \beta = \alpha + \beta,$$

is a finitely generated semiring. This leads the authors to introduce the concept of minimal standard basis.

The aim of this paper is to continue this kind of investigation, in order to find the smallest possible finite set that is able to encode some of the information of a good semigroup with two branches. Specifically, we introduce the concept of minimal set of representatives of a good subsemigroup $S$ of $\mathbb{N}^2$. Although a minimal set of representatives $\eta$ of $S$ does not univocally describe the semigroup (however $S$ is still among the minimal good semigroups containing $\eta$), it is possible to show that it stores relevant data. For instance, in the case of value semigroup, a system of representatives contains all the information regarding the valuation of a minimal system of generators of the corresponding ring. This leads us to generalize in a reasonable way, to the good semigroups of $\mathbb{N}^2$, the concept of embedding dimension that plays an important role in the numerical case.

The structure of the paper is the following.

In Section 1 we give all the basic definitions and we introduce all the main tools of the paper. In particular, in Subsection 1.1 we recall the definition of good semigroup and we explain how to associate to a good semigroup $S$ of $\mathbb{N}^2$ a semiring $\Gamma_S$. Then, in Proposition 1.3, we prove that, in the case of value semigroups, our semiring coincides with the one given in [6]. In subsection 1.2 we define the concept of irreducible and absolute point of $\Gamma_S$, and in Theorem 1.8, we prove that $\Gamma_S$ is generated as a semiring by its set $I_A$ of irreducible absolute points generalizing to all good semigroups a result proved Carvalho E. and Hernandes M.E. [6, Thm 11, Cor 20] for the value semigroups of a ring.

In Section 2, we introduce the notation $S_\eta$ for the set of the minimal good semigroups containing $\eta$. In Proposition 2.2 we give some conditions on $\eta$ in order to have finitely many elements in $S_\eta$. Then, given a good semigroup $S$, a set $\eta$ is called a system of representatives of $S$ if $S \in S_\eta$. This lets us to define the embedding dimension of a good semigroup $S$ as the smallest cardinality of a system of representatives of $S$. Starting from this point we work on good semigroups of $\mathbb{N}^2$ in order to study the property of the embedding dimension. In Subsection 2.1 we introduce the definition of track of a good semigroup $S$ and with Lemma 18 we show how to obtain a good semigroup $S'$ contained in $S$ by removing one of its tracks. Using this lemma we can compute an inferior bound for the embedding dimension. In Subsection 2.2 it is given the definition of reducibility of an element of $I_A(S)$ with respect to a subset $\eta \subseteq I_A(S)$. Then, using Theorem 28 this concept is used to develop a strategy to find a superior bound for the embedding dimension. In Subsection 2.3 we present a series of functions implemented in GAP [13] that, using the computational vantages of calculating the previous bounds, allow
us to describe a fast algorithm to find the embedding dimension. In the examples proposed in this section, for reasons of legibility and space, some verifications are not reported; these were made using functions written in GAP [13]. Finally, Section 3 is dedicated to studying whether the embedding dimension defined in \( \mathbb{N}^2 \) retains some of the features of the numerical case. In particular in Theorem 36 we prove that a good semigroup \( S \), realizable as a value semigroup, has embedding dimension greater or equal than the corresponding ring (as in the numerical case). Then, we give some examples when the previous inequality is strict, where it is possible to observe the limits of the combinatorial structure of a good semigroup that is not always able to store all the information contained in the ring in the same amount of data given by a system of generators. In Subsection 3.2 we give the definition of levels of the Apéry set of a good semigroup as in [8], and we use it to prove that \( edim(S) \leq e_1 + e_2 \) where \( e = (e_1, e_2) \) is the multiplicity vector of \( S \) (extending the relation \( edim(S) \leq e \) of the numerical case and the corresponding relation for one-dimensional rings). This result also lets us to prove Corollary 46, where we show that the Arf good semigroups of \( \mathbb{N}^2 \) have maximal embedding dimension, generalizing another important property valid in the numerical case.

1 Semiring associated to a good semigroup and Irreducible Absolutes

1.1 Semiring \( \Gamma_S \) and basic properties

We start this section recalling the definition of good semigroup introduced in [1].

**Definition 1.** \( S \subseteq \mathbb{N}^n \) will be called good semigroup if it satisfies the following properties:

1. If \( \alpha, \beta \in S \), then \( \min(\alpha; \beta) = (\min\{\alpha_1, \beta_1\} \ldots \min\{\alpha_n, \beta_n\}) \in S \);
2. There exists \( \delta \in \mathbb{N}^n \) such that \( S \supseteq \delta + \mathbb{N}^n \);
3. If \( (\alpha, \beta) \in S; \alpha \neq \beta \) and \( \alpha_i = \beta_i \) for some \( i \in \{1, \ldots, n\} \); then there exists \( \epsilon \in S \) such that \( \epsilon_i > \alpha_i = \beta_i \) and \( \epsilon_j \geq \min\{\alpha_j, \beta_j\} \) for each \( j \neq i \) (and if \( \alpha_j \neq \beta_j \), the equality holds).

Furthermore, we always suppose to work with a *local* good semigroup \( S \), i.e. if \( \alpha = (\alpha_1, \ldots, \alpha_n) \in S \) and \( \alpha_i = 0 \) for some \( i \in \{1, \ldots, n\} \), then \( \alpha = 0 \). As a consequence of property (G2), the element \( c = \min\{\delta|S \supseteq \delta + \mathbb{N}^n\} \) is well defined and it is called conductor of the good semigroup.

Furthermore, we denote by \( e = \min S \setminus \{0\} \) the multiplicity vector of the good semigroup. In order to simplify the notation and some proofs, in this paper, we often work with good semigroups \( S \subseteq \mathbb{N}^2 \) but most of the definitions and proofs remain true also in the general case. According to the work of Carvalho and Hernandes [6], we wish to introduce a semiring \( \Gamma_S \) associated with the semigroup \( S \).
We set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, where, $\infty$ is just a symbol that will correspond to the valuation of the element 0 if the semigroup is the semigroup of valuation of a ring. We extend the natural order and the sum over $\overline{\mathbb{N}}$ to $\overline{\mathbb{N}}$, setting respectively, $a < \infty$ for all $a \in \mathbb{N}$ and $x + \infty = \infty + x = \infty$.

We set:

$$S_1^\infty = \{(a,\infty) | \exists \tilde{y} \in \mathbb{N} : (a, y) \in S \ \forall y \geq \tilde{y}\}$$

$$S_2^\infty = \{(\infty, b) | \exists \tilde{x} \in \mathbb{N} : (x, b) \in S \ \forall x \geq \tilde{x}\}$$

$$S^\infty = S_1^\infty \cup S_2^\infty \cup \{\infty, \infty\}$$

$\Gamma_S = S \cup S^\infty$

If $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \Gamma_S$, we set $\min\{\alpha, \beta\} := (\min\{\alpha_1, \beta_1\}, \min\{\alpha_2, \beta_2\})$.

Now we define over $\Gamma_S$ the following tropical operations:

$$\oplus : \alpha \oplus \beta = \min\{\alpha, \beta\}$$

$$\odot : \alpha \odot \beta = \alpha + \beta$$

It is easy to prove that, with these operations, $(\Gamma_S, \oplus, \odot)$ is a semiring.

Now we recall some facts and fix some notations that will be useful for the following.
Let be $R = \mathbb{K}[x_1, \ldots, x_n] / Q$ a two-branches algebroid curve, where $Q$ is an ideal of $\mathbb{K}[x_1, \ldots, x_n]$ with $Q = P_1 \cap P_2,$ where $P_1, P_2$ are prime ideals.

We can embed $R \hookrightarrow R_1 \times R_2$ where $R_i = \mathbb{K}[x_1, \ldots, x_n] / P_i, i = 1, 2$; Furthermore $R \hookrightarrow \overline{\mathbb{R}} \cong \overline{\mathbb{R}_1} \times \overline{\mathbb{R}_2} \cong \mathbb{K}[t_1] \times \mathbb{K}[t_2]$. Given $r \in R$, $r = (r_1, r_2) \in \mathbb{K}[t_1] \times \mathbb{K}[t_2]$ that is a product of DVRs, so we can associate to each element of $R$ a valuation. If $v_i$ is the valuation function on $\mathbb{K}[t_i]$, we set:

$$v_i(r) = \begin{cases} v_i(r_i) & \text{if } r_i \neq 0 \\ \infty & \text{if } r_i = 0 \end{cases}$$

and $v(r) = (v_1(r), v_2(r))$.

According to the notation of Carvalho and Hernandes [6], we introduce the following sets:

$$\Gamma_S = \{v_i(r) \mid r \in R\} \subseteq \overline{\mathbb{N}};$$

$$S_i = \{v_i(r) \mid r \text{ is not a zerodivisor in } R\} \subseteq \mathbb{N};$$

$$\Gamma_R = \{v(r) := (v_1(r), v_2(r)) \mid r \in R\} \subseteq \overline{\mathbb{N}}^2;$$

$$S = \{v(r) := (v_1(r), v_2(r)) \mid r \text{ is not a zerodivisor in } R\} \subseteq \mathbb{N}^2.$$

$\Gamma_R$ and $S$ will be called respectively semiring of values and semigroup of values associated with $R$. It is easy to observe that $S = \Gamma_R \cap \mathbb{N}^2$. 

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At this point, we wish to prove that, if $R$ is a two-branches algebroid curve, and $S$ is its semi-group of values, then $\Gamma_S = \Gamma_R$.

**Lemma 2.** The following statements hold:

i) $(a, \infty) \in \Gamma_S$ if and only if $(a, y) \in S$ for any $y \geq c_2$.

ii) $(\infty, b) \in \Gamma_S$ if and only if $(x, b) \in S$ for any $x \geq c_1$.

**Proof.** We prove $i)$, the other statement is analogue. If $(a, \infty) \in \Gamma_S$, then there exists $\tilde{y} \in \mathbb{N}$ such that $(a, \tilde{y}), \ldots, (a, \tilde{y} + n) \in S$ for any $n \in \mathbb{N}$. If $\tilde{y} \leq c_2$ the statement is proved, otherwise $\tilde{y} = c_2 + n$, with $n \in \mathbb{N}$. Since $S$ is a good semigroup, for all $i < n$, $a < c_1$, we have that $(a, c_2 + i) = \min\{(a, \tilde{y}), (c_1, c_2 + i)\} \in S$.

**Proposition 3.** If $R$ is a two-branches algebroid curve and $S$ is its semigroup of values, then $\Gamma_S = \Gamma_R$.

**Proof.** We have observed that $S = \Gamma_R \cap \mathbb{N}^2$, so we need to prove that $\Gamma_R \setminus S = S^\infty$. If $\alpha \in \Gamma_R \setminus S$ we can write $\alpha = v(r)$, where $r$ is a zerodivisor in $R$ or $r = 0$; so we have $r \in P_1 \cup P_2$. If $r = 0$, $v(r) = (\infty, \infty)$; if $r \in P_1$, then $r = 0$ in $R_1$, $v_1(r) = \infty$ so $r \in S_2^\infty$; if $r \in P_2$, then $r = 0$ in $R_2$, $v_2(r) = \infty$ so $r \in S_1^\infty$. If $s \in S^\infty$, without loss of generality, we can suppose $s \in S_2^\infty$, we can write $s = (\infty, b)$, and, as a consequence of Lemma 1, $(c_1, b) \in S$. Because $S = v(R)$ and the conductor ideal is $\mathcal{C} = (t^{c_1}, u^{c_2})(\mathbb{K}[t] \times \mathbb{K}[u])$, there exists an element in $R$ of the form $(t^{c_1}, b_y(u))$ with $v(b_y(u)) = y$. As the element $(t^{c_1}, 0) \in R$, we have that the element $(0, -b_y(u)) \in R$, then $(\infty, y) \in \Gamma_R$.

1.2 A system of generators of $\Gamma_S$ as a semiring

**Definition 4.** We will say that an element $\alpha \in \Gamma_S$ is irreducible if, from $\alpha = \beta + \gamma$, it follows $\alpha = \beta$ or $\alpha = \gamma$. An element that is not irreducible will be said reducible.

We denote with $I(S)$ the set of irreducible elements of $\Gamma_S$.

**Remark 5.** We observe that:

1. If $\alpha = (a, b) \in \Gamma_S$ with $a \geq c_1 + e_1$ and $b \geq c_2 + e_2$, then $\alpha$ is reducible.
2. If $\alpha = (a, \infty) \in \Gamma_S$ with $a \geq c_1 + e_1$, then $\alpha$ is reducible.
3. If $\alpha = (\infty, b) \in \Gamma_S$ with $b \geq c_2 + e_2$, then $\alpha$ is reducible.
Given a good semigroup $S \subseteq \mathbb{N}^2$, and an element $\alpha \in \mathbb{N}^2$, following the notation in [1], we set:

$$
\Delta_i(\alpha) := \{ \beta \in \mathbb{Z}^2 | \alpha_i = \beta_i \text{ and } \alpha_j < \beta_j \text{ for } j \neq i \} \\
\Delta(\alpha) := \Delta_1(\alpha) \cup \Delta_2(\alpha) \\
\Delta^S_i(\alpha) := S \cap \Delta_i(\alpha) \\
\Delta^S(\alpha) := S \cap \Delta(\alpha).
$$

Furthermore we define:

$$
i\Delta(\alpha) := \{ \beta \in \mathbb{Z}^2 | \alpha_i = \beta_i \text{ and } \beta_j < \alpha_j \text{ for } j \neq i \} \\
i\Delta^S(\alpha) := S \cap_i \Delta(\alpha).
$$

Extending the previous definitions to infinite elements of $\mathbb{N}^2$, we set:

$$
1\Delta((\alpha_1, \infty)) := \{ \beta \in \mathbb{Z}^2 | \beta_1 = \alpha_1 \} \\
2\Delta((\alpha_1, \infty)) := \emptyset \\
1\Delta((\infty, \alpha_2)) := \emptyset \\
2\Delta((\infty, \alpha_2)) := \{ \beta \in \mathbb{Z}^2 | \beta_2 = \alpha_2 \} \\
i\Delta^S(\alpha) := S \cap_i \Delta(\alpha).
$$

**Definition 6.** An element $\alpha \in \Gamma_S$ will be said absolute in $\Gamma_S$ if $\alpha \in S$ and $\Delta^S(\alpha) = \emptyset$ (finite absolute), or if $\alpha \in S^\infty$ (infinite absolute).

We denote with $A_f(\Gamma_S)$ the set of finite absolutes in $\Gamma_S$, with $A^\infty(\Gamma_S)$ the set of infinite absolutes in $\Gamma_S$ and with $A(\Gamma_S)$ the set of all absolutes in $\Gamma_S$. We call $I_{A_f}(\Gamma_S)$ the set of finite irreducible absolutes in $\Gamma_S$, $I^\infty_{A_f}(\Gamma_S)$ the set of infinite irreducible absolutes in $\Gamma_S$ and $I_A(\Gamma_S)$ the set of all irreducible absolutes in $\Gamma_S$. As a consequence of the Remark 5, the set of irreducible absolutes is finite. Now we introduce other sets that will be considered in the following:

$$
\text{small}(S) = \{(a, b) \in S | a \leq c_1, b \leq c_2 \} \\
\text{small}(\Gamma_S) = \text{small}(S) \cup \{ (\infty, b) \in S^\infty_2, b \leq c_2 \} \cup \{ (a, \infty) \in S^\infty_1, a \leq c_1 \} \\
B^\infty_1(\Gamma_S) = \{ (a, \infty) \in \Gamma_S | c_1 < a \leq c_1 + e_1 \} \subseteq S^\infty_1 \\
B^\infty_2(\Gamma_S) = \{ (\infty, b) \in \Gamma_S | c_2 < b \leq c_2 + e_2 \} \subseteq S^\infty_2 \\
B^\infty(\Gamma_S) = B^\infty_1(\Gamma_S) \cup B^\infty_2(\Gamma_S) \subseteq S^\infty(C)
$$

The sets $\text{small}(S)$, $\text{small}(\Gamma_S)$, $B^\infty(\Gamma_S)$ will be said respectively: small elements of $S$, small elements of $\Gamma_S$ and beyond elements of $\Gamma_S$. 

\[ 6 \]
Remark 7. It easy to observe the following facts:

i) Each element in the semiring can be written as tropical product of irreducible elements, i.e, if \( \alpha \in \Gamma_S \), \( \alpha = \alpha_1 \odot \ldots \odot \alpha_n \) where \( \alpha_i \in I(\Gamma_S) \).

ii) Each element in the semiring can be written as tropical sum of two absolute elements, i.e, if \( \beta \in \Gamma_S \), \( \beta = \beta_1 \oplus \beta_2 \) where \( \beta_1, \beta_2 \in A(\Gamma_S) \).

Now we prove that the set of irreducible absolutes generates \( \Gamma_S \) as semiring.

Theorem 8. \((\Gamma_S, \oplus, \odot)\) is generated as semiring by the irreducible absolutes i.e, if \( \alpha \in \Gamma_S \),

\[
\alpha = \sum_{i=1}^{m} (\odot_{j=1}^{n} \gamma_{ij}), \gamma_{ij} \in I_A(S)
\]

Proof. First of all, we observe that we can reduce to prove the thesis only for the elements \( \alpha \in \text{small}(\Gamma_S) \cup B(\Gamma_S) \). Indeed, if \( \alpha \notin \text{small}(\Gamma_S) \cup B(\Gamma_S) \), then there exists \( k \in \mathbb{N} \) such that \( \beta = \alpha - ke \in \text{small}(\Gamma_S) \cup B(\Gamma_S) \). In this case we would have \( \alpha = \beta \odot ke \), where \( \beta \in \text{small}(\Gamma_S) \cup B_S \) and \( e \) is trivially irreducible.

We can reduce again the proof only for the elements \( \alpha \in I(\Gamma_S) \cap S \) (finite irreducibles). In fact, if \( \alpha \) is reducible, by Remark 7, we can write \( \alpha = \alpha^{(1)} \odot \ldots \odot \alpha^{(n)} \), with \( \alpha^{(i)} \) irreducibles. Furthermore, we observe that if \( \alpha^{(i)} \in S^\infty \), then \( \alpha^{(i)} \in I_A(\Gamma_S) \); so we can write:

\[
\alpha = \alpha^{(1)} \odot \ldots \odot \alpha^{(f)} \odot \left( \odot \gamma \in I_A(S) \gamma \right)
\]

where \( \alpha^{(i)} \in I_A(\Gamma_S) \). Therefore we can suppose \( \alpha \in I(\Gamma_S) \cap S \) and prove the thesis. By Remark 7, we can write \( \alpha = \beta \oplus \gamma \) with \( \beta = (\beta_1, \beta_2) \in A, \gamma = (\gamma_1, \gamma_2) \in A \) and we can
assume: $\beta_1 = \alpha_1 \leq \gamma_1$ and $\gamma_2 = \alpha_2 \leq \beta_2$.

We consider

$$\beta = \beta^{(1)} \odot \ldots \odot \beta^{(n)}$$

$$\gamma = \gamma^{(1)} \odot \ldots \odot \gamma^{(m)}$$

the decompositions in irreducible elements of $\beta$ and $\gamma$. We define $\beta^{(i)} = \beta^{(i)} \oplus \gamma$, for all $i \in \{1, \ldots, n\}$ and $\gamma^{(j)} = \gamma^{(j)} \oplus \beta$ for all $j \in \{1, \ldots, m\}$. If we define $\beta' = \beta^{(1)} \oplus \ldots \oplus \beta^{(n)}$, $\gamma' = \gamma^{(1)} \oplus \ldots \oplus \gamma^{(m)}$, it is easy to observe that $\beta'_1 = \beta_1$ and $\gamma'_2 = \gamma_2$, so we have $\alpha = \beta' \oplus \gamma'$.

We can definitely write:

$$\alpha = (\beta^{(1)} \odot \ldots \odot \beta^{(n)}) \oplus (\gamma^{(1)} \odot \ldots \odot \gamma^{(m)})$$

where each $\beta^{(i)}$ and $\gamma^{(j)}$ is strictly smaller than $\alpha$ (that is $\gamma^{(j)} \leq \alpha$ and $\gamma^{(j)} \neq \alpha$). If we express each of these elements as a tropical product of irreducibles, we can write $\alpha$ as a tropical sum of tropical products, where all the terms are irreducible and strictly smaller than $\alpha$. This means that if we repeat the same argument on each element in this expression, in a finite number of iteration we will surely obtain the required expression.

**Remark 9.** In the case of good semigroups that are value semigroup of a ring, the theorem above follows by [6, Thm 11] and [6, Thm 19].

But we remember that not all good semigroups are value semigroup of a ring (for an example cf.[1, Example 2.16]).

Thus, the previous theorem generalizes this property to all semirings obtained by semigroups of $\mathbb{N}_2^2$, also if they are not value semigroup of a ring.

## 2 Embedding dimension of a good semigroup

It is a note fact that every numerical semigroup $S \subseteq \mathbb{N}$ admits a unique minimal system of generators as a monoid and the embedding dimension of the numerical semigroup is defined as the number of these generators. This name follows from the fact that it is equal to the embedding dimension of the monomial curve associated with the numerical semigroup.

Now we will define a set of vectors that, although it does not uniquely determine a good semigroup, will allow us to give a definition of embedding dimension of a good semigroup. This embedding dimension, in the case of good semigroup of $\mathbb{N}_2^2$, will satisfy some of the properties that are valid in the case of numerical semigroups.

Starting from this point, in order to enlighten the notations, when we consider a good semigroup $S$, we suppose that it coincides with the semiring $\Gamma_S$, i.e. we treat the infinite elements as elements of $S$.

Given a set of vectors $\eta \subseteq \mathbb{N}_2^n$, we denote with $S_\eta$ the family of all the good semigroups containing $S$ and that are minimal with respect the set inclusion. $S_\eta$ can be infinite or equal to the empty set.

Furthermore, given a set of vectors $\eta \subseteq \mathbb{N}_2^n$, we denote by $\langle \eta \rangle_\oplus$ the semiring generated by $\eta$. 


Example 10. Let us consider \( \eta = \{[2,2], [3,3]\} \subseteq \mathbb{N}^2 \), and suppose that there exists a good semigroup \( S \in S_\eta \).

First of all we prove that, for any \( n \in \mathbb{N} \setminus \{1\} \), we have \( (n,n) \in S \). In fact, it is easy to observe that each natural number \( n \neq 1 \) can be written as \( n = 2\alpha + 3\beta \), with \( \alpha, \beta \in \mathbb{N} \). Hence we can write \( (n,n) = (2\alpha + 3\beta, 2\alpha + 3\beta) = \alpha(2,2) + \beta(3,3) \in S \).

We denote with \( c(S) = (c_1,c_2) \) the conductor of \( S \). If \( c_1 = 1 \), we have that \((1,2) = \min \{(1,c_2), (2,2)\} \in S \); hence \( c(S) = (1,2) \) or \( S = \mathbb{N}^2 \). In both cases, if we consider \( S' \) such that \( \text{small}(S') = \{(0,0), (2,2)\} \) we have that \( S' \) is a good semigroup containing \( \eta \) and such that \( S' \subset S \); but this contradicts the minimality of \( S \). So we have obtained \( c_1 \neq 1 \) and, using the same argument, we can suppose \( c_2 \neq 1 \).

If \( c_1 > 1 \) and \( c_2 > 1 \) we prove that \( c(S) = (c,c) \), with \( c \in \mathbb{N} \). Let us assume by contradiction that \( c(S) = (c_1,c_2) \) with \( c_1 < c_2 \); in this case, there exists \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 \geq c_1 \), \( c_1 \leq \alpha_2 < c_2 \) such that \( \alpha \notin S \). If \( \alpha_1 \leq \alpha_2 \) we would have \( \alpha = \min \{(\alpha_1, c_2), (\alpha_2, c_2)\} \in S \), so necessarily we have \( \alpha_1 > \alpha_2 \). Now we observe that \((c_1, \alpha_2) = \min \{c(S), (\alpha_2, c_2)\} \in S \) and for property \((G3)\) applied to \( c(S) \) and \((c_1, \alpha_2) \), there exists \((x_1, \alpha_2) \in S \) with \( x_1 > c_1 \). If \( x_1 \geq \alpha_1 \) we would have \( \alpha = \min \{(x_1, \alpha_2), (\alpha_1, c_2)\} \in S \) that is a contradiction. Thus we necessarily have \( x_1 < \alpha_1 \). Now, if we consider \((x_1, \alpha_2), (x_2, c_2) \in S \) using property \((G3)\) we observe that there exists \((x_2, \alpha_2) \in S \) with \( x_2 > x_1 \). We can repeat this argument until we find an element \((x_1, \alpha_2) \in S \) with \( x_i \geq \alpha_1 \). In this case we obtain \( \alpha = \min \{(x_i, \alpha_2), (\alpha_1, c_2)\} \in S \), that is a contradiction.

Now, using the properties \((G2)\) and \((G3)\) repeatedly, it is easy to observe that, \( \text{small}(S) = \{(0,0), (2,2), (3,3), \ldots, (c-1, c-1), (c,c)\} \). If we define \( S' \) such that \( \text{small}(S') = \{(0,0), (2,2), (3,3), \ldots, (c,c), (c+1, c+1)\} \), we have found a minimal good semigroup containing \((2,2), (3,3)\) and strictly contained in \( S \), in contradiction with the minimality of \( S \).

The following proposition gives a condition that guarantees that \( S_\eta \) is finite.

Proposition 11. Suppose that we have \( \eta = \{\eta^{(1)} = (\eta_1^1, \ldots, \eta_n^1), \ldots, \eta^{(k)} = (\eta_1^k, \ldots, \eta_n^k)\} \subseteq \mathbb{N}^n \).

Then the set \( S_\eta \) is finite if the following conditions hold:

- \( \gcd \{\eta_h^i, h = 1, \ldots, k\} = 1 \) for \( i = 1, \ldots, n \);
- For all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \) there exists a \( l \in \{1, \ldots, k\} \) such that \( \eta^i_l \neq \eta^j_l \).

Proof. We denote by \( \langle \eta \rangle \) the semiring generated by \( \eta \). We claim that for each \( i = 1, \ldots, n \), we can obtain two vectors \( \alpha^{(i)} = (\alpha_1^i, \ldots, \alpha_n^i) \) and \( \beta^{(i)} = (\beta_1^i, \ldots, \beta_n^i) \) in \( \langle \eta \rangle \) such that 
\[
\alpha_i^i = \beta_i^i \quad \text{and} \quad \alpha_i^j < \beta_i^j \quad \text{for all} \quad j \neq i.
\]

We will prove this fact by induction on \( n \).

- **Base case** \( n = 2 \). Suppose that \( i = 1 \). For the second property of the set \( \eta \), there exists a \( \eta^{(i)} \in \eta \) such that \( \eta_1^1 \neq \eta_2^1 \). Then \( \eta \) must contain a vector \( \eta^{(m)} \) such that \( \frac{\eta_2^m}{\eta_1^m} \neq \frac{\eta_2^1}{\eta_1^1} \). In
fact, if we had $\gamma_i^{(2)} = \gamma_j^{(2)} \neq 1$ for all $h = 1, \ldots, k$ then it would follow that $m_2$ divides $\gcd\{h_2, h = 1, \ldots, k\} = 1$ and it is a contradiction. Then we consider $\eta^{(m)}$ such that $\gamma_i^{(m)} \neq \gamma_j^{(m)}$ and the vectors

\begin{align*}
\alpha^{(1)} &= (\eta_1^1 \eta_1^m, \eta_2^2 \eta_2^m), \\
\beta^{(1)} &= (\eta_1^1 \eta_1^m, \eta_1^1 \eta_2^m)
\end{align*}

satisfy our condition because $\eta_2^1 \eta_2^m \neq \eta_1^1 \eta_1^m$ and they belong to $\langle \eta \rangle_{\oplus}$. For $i = 2$ we can use the same strategy.

- **Inductive step:** Let us suppose that the claim is true for $n - 1$ and we prove it for $n$. We suppose that $i = 1$ (the other cases can be treated in the same way). We consider the element $\eta_1 = \{\eta^{(h)} = (\eta_1^h, \ldots, \eta_{n-1}^h), h = 1, \ldots, k\}$ that satisfies the conditions of the theorem. Then for the inductive step it easily follows that in $\langle \eta \rangle_{\oplus}$ there exist two vectors $\gamma^{(1)} = (\gamma_1^1, \ldots, \gamma_n^1)$ and $\delta^{(1)} = (\delta_1^1, \ldots, \delta_n^1)$ such that

$$
\gamma_j^1 = \delta_j^1 \text{ and } \gamma_j^1 < \delta_j^1 \text{ for all } j = 2, \ldots, n - 1.
$$

If $\gamma_j^2 < \delta_j^1$, then the claim is true for $\alpha^{(1)} = \gamma^{(1)}$ and $\beta^{(1)} = \delta^{(1)}$. If $\gamma_j^2 > \delta_j^1$, we consider $\alpha^{(1)} = \min(2\gamma_j^{(1)}, 2\delta^{(1)})$ and $\beta^{(1)} = \gamma^{(1)} + \delta^{(1)}$. In fact we have $\alpha_1^1 = 2\gamma_1^1 = \beta_1^1$. If $j \in \{2, \ldots, n - 1\}$ then $\alpha_j^1 = 2\gamma_j^1 < \gamma_j^1 + \delta_j^1 = \beta_j^1$. Finally we have $\alpha_n^1 = 2\delta_n^1 < \gamma_n^1 + \delta_n^1 = \beta_n^1$. Thus suppose that $\gamma_j^1 = \delta_j^1$. In this case we can consider $\eta_2 = \{\eta^{(h)} = (\eta_1^h, \eta_3^h, \ldots, \eta_{n-1}^h), h = 1, \ldots, k\}$. For the inductive step there exist two vectors $\gamma^{(2)} = (\gamma_1^2, \ldots, \gamma_n^2)$ and $\delta^{(2)} = (\delta_1^2, \ldots, \delta_n^2) \in \langle \eta \rangle_{\oplus}$ such that

$$
\gamma_j^2 = \delta_j^2 \text{ and } \gamma_j^2 < \delta_j^2 \text{ for all } j = 3, \ldots, n.
$$

Then, as we have just seen, if $\gamma_j^2 \neq \delta_j^2$ the claim is true. Therefore we suppose that $\gamma_j^2 = \delta_j^2$. Then it is very easy to check that the claim is true with $\alpha^{(1)} = \gamma^{(1)} + \gamma^{(2)}$ and $\beta^{(1)} = \delta^{(1)} + \delta^{(2)}$.

Now denote by $c^{(i)}$ the conductor of the numerical semigroup generated by $\{\eta^h : h = 1, \ldots, k\}$ and we choose $\alpha^{(i)} = (\alpha_1^i, \ldots, \alpha_n^i)$ and $\beta^{(i)} = (\beta_1^i, \ldots, \beta_n^i)$ in $\langle \eta \rangle_{\oplus}$ as in the previous claim. We will prove that for each $i \in \{1, \ldots, n\}$ there exist $c_{i,j}$ for $j = 1, \ldots, i - 1, i + 1, \ldots, n$ such that the vectors

$$
c^{(i)}(y) = (c_{i,1}, \ldots, c_{i,i-1}, c^{(i)} + \alpha_i^1 + y, c_{i,i+1}, \ldots, c_{i,n}) \in S,
$$

for each $S \in S_\eta$, and $y \in \mathbb{N}$. If this is true then it is clear that

$$
c_\eta = \sum_{i=1}^n c^{(i)}(0) + \mathbb{N}^n \subseteq S,
$$

for all $S \in S_\eta$. 

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Suppose that \( i = 1 \) (the proof is identical in the other cases). Let us consider an arbitrary \( S \in S_\eta \). We obviously have \( \langle \eta \rangle_\oplus \subseteq S \). We will denote by \( m = \alpha_1 = \beta_1 \). Because \( c^{(1)} \) is the conductor of \( \langle \{ \eta^h : h = 1, \ldots, k \} \rangle \), we can find the vectors:

\[
\sigma^{(h)} = (\sigma_1^h, \ldots, \sigma_n^h) \in \langle \eta \rangle_\oplus, \quad \text{for } h = 0, \ldots, m - 1,
\]

such that \( \sigma_i^h = c^{(i)} + h \) for all \( h = 0, \ldots, m - 1 \).

For each \( i = 0, \ldots, m - 1 \) we consider \( \lambda^{(i)} = \min(\sigma^{(i)}, \ldots, \sigma^{(m-1)}) \). Then we have \( \lambda^{(0)} \leq \ldots \leq \lambda^{(m-1)} \) and, if \( \lambda^{(h)} = (\lambda_1^h, \ldots, \lambda_n^h) \), then \( \lambda_1^h = c^{(1)} + h \).

Now we want to show that \( (c^{(1)} + m + y, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S \) for each \( y \in \mathbb{N} \).

We notice that

\[
\lambda^{(0)} + \alpha^{(1)} = (c^{(1)} + m, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S
\]

\[
\lambda^{(0)} + \beta^{(1)} = (c^{(1)} + m, \lambda_2^0 + \beta_2^1, \ldots, \lambda_n^0 + \beta_n^1) \in S,
\]

thus, recalling that \( \alpha_1 < \beta_1^i \) for all \( j = 2, \ldots, n \), it follows by (G3) that there exists \( x > c^{(0)} + m \) such that \( (x, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S \).

Now we consider

\[
\lambda^{(1)} + \beta^{(1)} = (c^{(1)} + m + 1, \lambda_2^1 + \beta_2^1, \ldots, \lambda_n^1 + \beta_n^1) \in S.
\]

Since \( \lambda_h^0 \leq \lambda_h^1 \) for all \( h = 2, \ldots, n \) and \( \alpha_j^1 < \beta_j^1 \) for all \( j = 2, \ldots, n \), we have

\[
(x, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \oplus (\lambda^{(1)} + \beta^{(1)}) = (c^{(1)} + m + 1, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S.
\]

Now as before from

\[
(c^{(1)} + m + 1, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S
\]

\[
(c^{(1)} + m + 1, \lambda_2^1 + \beta_2^1, \ldots, \lambda_n^1 + \beta_n^1) \in S,
\]

we can deduce that there exists \( x > c^{(1)} + m + 1 \) such that \( (x, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S \).

Repeating the previous considerations and using the fact that \( \lambda^{(0)} \leq \lambda^{(i)} \) for each \( i \leq m - 1 \), we can show that

\[
(c^{(1)} + m + y, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S,
\]

for all \( y = 0, \ldots, m - 1 \). Now, we can consider

\[
\lambda^{(0)} + \beta^{(1)} = (c^{(1)} + m, \lambda_2^0 + \beta_2^1, \ldots, \lambda_n^0 + \beta_n^1) \in S
\]

\[
\ldots
\]

\[
\lambda^{(m-1)} + \beta^{(1)} = (c^{(1)} + 2m - 1, \lambda_2^{m-1} + \beta_2^{1}, \ldots, \lambda_n^{m-1} + \beta_n^{1}) \in S
\]

and because \( \lambda_j^h + \beta_j^1 > \lambda_j^0 + \alpha_j^1 \) for all \( j = 0, \ldots, m - 1 \) and \( h = 2, \ldots, n \), we can use the same strategy to show that

\[
(c^{(1)} + m + y, \lambda_2^0 + \alpha_2^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S,
\]
for all \( y = 0, \ldots, 2m - 1 \). Now it is clear that we can endlessly repeat the strategy and we finally proved that

\[
(e^{(1)} + m + y, \lambda_2^0 + \alpha_1^1, \ldots, \lambda_n^0 + \alpha_n^1) \in S,
\]

for all \( y \in \mathbb{N} \) and for all the \( S \in S_\eta \) (\( S \) was arbitrarily chosen). So we proved that if \( S \in S_\eta \) then the conductor of \( S \) is smaller than \( c_\eta \). Now we know that a good semigroup is completely characterized by its small elements. This implies that the set of good semigroups with a conductor smaller than \( c_\eta \) is finite and therefore also \( S_\eta \) must be finite. \( \square \)

**Definition 12.** Given a set of vector \( \eta \subseteq I_A(S) \), we say that \( \eta \) is a system of representatives of \( S \), or more simply sor, if \( S \subseteq S_\eta \).

**Remark 13.** As a consequence of the Theorem 8, \( I_A(S) \) is a sor of \( S \), since every semigroup containing the elements of \( I_A(S) \) must contain \( S \).

**Definition 14.** A system of representatives \( \eta \) of \( S \) is minimal, if given another set of representatives, \( \eta' \subseteq \eta \), it follows \( \eta' = \eta \). We call such a set a msor of \( S \).

It is possible to show that two msor can have different cardinalities (see Example 32).

**Definition 15.** Given a good semigroup \( S \), we define embedding dimension of \( S \):

\[
edim(S) = \min \{|\eta| : S \in S_\eta \text{ and } \eta \subseteq I_A(S)\}
\]

Starting from this point we will start to analyze the properties of the embedding dimension. We will consider only good semigroups \( S \subseteq \mathbb{N}^2 \). To compute all the minimal good semigroups containing a set of vectors is computationally very dispensing, also in the two-branches case. At this point, our first aim is to produce a "fast" algorithm that, in the case of good semigroup \( S \subseteq \mathbb{N}^2 \), returns a msor of \( S \). In order to do this we will introduce two bounds for the embedding dimension.

### 2.1 An inferior bound for the embedding dimension

First of all we want to produce an inferior bound for the embedding dimension. In order to do this we give the following definitions.

**Definition 16.** Given \( \alpha, \beta \in I_A(S) \) we say that \( \alpha \) and \( \beta \) are connected by a piece of track if they are not comparable, i.e. \( \alpha \not\leq \beta \) and \( \beta \not\leq \alpha \), and denoted with \( \gamma = \min\{\alpha, \beta\} \), we have

\[
\Delta^S(\gamma) \cap (S \setminus I(S)) = \emptyset.
\]

**Definition 17.** Given \( \alpha_1, \ldots, \alpha_n \in I_A(S) \), with \( \alpha_{11} < \ldots < \alpha_{n1} \) we say that \( \alpha_1, \ldots, \alpha_n \) are connected by a track if we have:

- \( 2\Delta^S(\alpha_1) \cap (S \setminus I(S)) = \emptyset; \)
- \( 1\Delta^S(\alpha_n) \cap (S \setminus I(S)) = \emptyset; \)
• $\alpha_i$ and $\alpha_{i+1}$ are connected by a piece of track for all $i \in \{1, \ldots, n-1\}$.

In this case, denoted with $\gamma_i = \min\{\alpha_i, \alpha_{i+1}\}$ for $i \in \{1, \ldots, n-1\}$, we denote with:

$$T((\alpha_1, \ldots, \alpha_n)) = 2\Delta^S(\alpha_1) \cup \bigcup_{i=1}^{n-1} \Delta^S(\gamma_i) \cup \Delta^S(\alpha_n)$$

the track connecting $\alpha_1, \ldots, \alpha_n$.

We will simply say that $T \subseteq S$ is a track in $S$ if there exist $\alpha_1, \ldots, \alpha_n \in I_A(S)$ such that $T$ is the track connecting $\alpha_1, \ldots, \alpha_n$.

In the following lemma we will show how these definitions are related to embedding dimension.

**Lemma 18.** Given a good semigroup $S$, if there exists a track $T = T((\alpha_1, \ldots, \alpha_n)) \in S$, then, $S' = S \setminus T$ is a good semigroup strictly contained in $S$.

**Proof.** If $\alpha, \beta \in S'$, because $\alpha, \beta \in S$ and $T \cap (S \setminus I(S)) = \emptyset$, we have $\alpha + \beta \in S'$, so $S'$ is a semigroup. Now, we have to check that $S'$ satisfy the property (G1); so considering $\alpha, \beta \in S'$, we have to prove that $\min\{\alpha, \beta\} \in S'$. If we suppose $\min\{\alpha, \beta\} \in T$ then: there exists a $\gamma_i = \min\{\alpha_i, \alpha_{i+1}\}$ such that $\min\{\alpha, \beta\} \in \Delta^S(\gamma_i)$; or $\min\{\alpha, \beta\} \in \Delta^S(\alpha_1)$; or $\min\{\alpha, \beta\} \in \Delta^S(\alpha_n)$. But in all the previous cases, for the definition of track, these would imply that $\alpha, \beta \in T$. Furthermore, for all $\alpha \in S$ with $\alpha \geq c(S) + e(S)$ we have $\alpha \in S'$, so $S'$ satisfy property (G2). We complete the proof verifying the property (G3). So we take $\alpha, \beta \in S'$ and suppose that $\beta \in \Delta^S(\alpha)$, we need to show that $\Delta^S(\alpha) \neq \emptyset$, where $j \in \{1, 2\} \setminus \{i\}$. Because $\alpha, \beta \in S$, for property (G3), there exists $\gamma \in \Delta^S(\alpha)$. If $\gamma \in \Delta^S(\alpha)$ the thesis is proved so we suppose the contrary, so $\gamma$ necessarily belongs to $T'$. We have two cases. Case 1: there exists $\gamma_k = \min\{\alpha_k, \alpha_{k+1}\}$ such that $\gamma \in \Delta^S(\gamma_k)$, but this implies $\gamma_k \in \Delta^S(\alpha)$. Case 2: there exists $\gamma_k = \min\{\alpha_k, \alpha_{k+1}\}$ such that $\gamma \in \Delta^S(\gamma_k)$. We notice that if $\gamma \in I_A(S)$ we can reduce to the previous case, so we can suppose that there exists $\rho \neq \gamma$ with $\rho \in \Delta^S(\gamma_k) \cap I_A(S)$. But since $\rho, \gamma \in S$, for property (G3) in $S$ and for the definition of track, $\Delta^S(\gamma) \neq \emptyset$, then we have $\Delta^S(\alpha) \neq \emptyset$. Case 3: $\gamma \in \Delta^S(\alpha_1)$ if $i = 2$ or $\gamma \in \Delta^S(\alpha_n)$ if $i = 1$; in this case we can conclude the proof with the same argument of "Case 2". 

**Definition 19.** Given $M \subseteq I_A(S)$ we say that $M$ is an hitting set (HS) of $S$, if for any track $T$ in $S$ there exists an element $\alpha \in M$ such that $\alpha \in T$. We say that $M$ is a minimal hitting set (MHS), if for any hitting set $M$ such that $M' \subseteq M$, we have $M' = M$.

**Remark 20.** In an hypergraph $(V, E)$, with $E = \{E_1, \ldots, E_n\}$, $E_i \subseteq V$, a set of vertices $H \subset V$ such that $H \cap E_i \neq \emptyset$ for all $i = 1, \ldots, n$ is called transversal or hitting set [3].

If we consider the hypergraph with vertices $V = I_A(S) \subset \Gamma_S$ and edges $E = \{T \subset S|T$ is a track$\}$, then the hitting sets of the good semigroup $S$ correspond exactly to the hitting sets of this hypergraph. The problem of finding the minimal hitting set of an hypergraph is an NP-hard problem and there are several algorithms related to computation of them (see for example [12], [16]).
We denote with \( \mathcal{H} = \{ M \mid M \text{ is a HS} \} \).

**Proposition 21.** If \( M \) is a sor then \( M \in \mathcal{H} \).

**Proof.** If we suppose that \( M \) is not a HS, then it would exist a track \( T \) in \( S \) that does not contain elements of \( M \). Using the same construction of Lemma 18 we could build a good semigroup \( S' \) such that \( M \subseteq S' \subseteq S \), but it is a contradiction. \( \Box \)

The converse of the previous theorem it is not true in general as it is shown by the following example.

**Example 22.** Let us consider the good semigroup \( S \) with the following set of irreducible absolute points:

\[
I_A(S) = \{(6, 3), (12, 17), (18, 25), (19, 6), (24, \infty), (25, 28), (27, 9), (31, \infty), (33, 20),
(39, \infty), (41, \infty), (44, \infty), (46, \infty), (\infty, 15), (\infty, 23), (\infty, 31)\}.
\]

From the picture we can easily deduce that \( S \) contains only the following tracks:

- \( T_1 = T((6, 3)) \);  
- \( T_2 = T((12, 17), (19, 6)) \);  

![Figure 2: \( \bigcirc \): Irreducible Absolutes; \( \circ \): Reducible Elements](image-url)
\[ T_3 = T((39, \infty), (\infty, 31)); \]
\[ T_4 = T((41, \infty), (\infty, 23)); \]
\[ T_5 = T((41, \infty), (\infty, 31)); \]
\[ T_6 = T((41, \infty)); \]
\[ T_7 = T((46, \infty), (\infty, 15)); \]
\[ T_8 = T((46, \infty), (\infty, 23)); \]
\[ T_9 = T((46, \infty), (\infty, 31)). \]

Then it is easy to verify that \( M = \{(6, 3), (12, 17), (39, \infty), (41, \infty), (46, \infty)\} \) is a MHS for \( S \). However, \( M \) is not a sor for \( S \), in fact it is possible to check that there exists a good semigroup \( S' \) with

\[ I_A(S') = \{(6, 3), (12, 17), (19, 6), (24, \infty), (39, \infty), (41, \infty), (46, \infty), (50, \infty), (\infty, 18), (\infty, 29), (\infty, 34)\}, \]

such that \( S' \) is strictly contained in \( S \) and we have \( M \subseteq S' \).

Now we define: \( \text{bedim}(S) = \min\{|M|, M \in \mathfrak{S}\} \).

**Corollary 23.** Given a good semigroup \( S \subseteq \mathbb{N}^2 \), \( \text{bedim}(S) \leq \text{edim}(S) \).

**Example 24.** The inequality of Corollary 23 can be strict. In fact, for instance, it is possible to check that each minimal hitting set of the semigroup \( S \) described in Example 22 is not a sor for \( S \), implying that \( \text{bedim}(S) < \text{edim}(S) \).

### 2.2 A superior bound for the embedding dimension

Let be \( S \subseteq \mathbb{N}^2 \) a good semigroup; given \( \eta \subseteq I_A(S) \), and \( \alpha \in I_A(S) \), we want to define the reducibility of \( \alpha \) with respect to \( \eta \). By convention we will say that all the elements \( \alpha \in \eta \) are reducible by \( \eta \). We take \( \alpha \in I_A(S) \setminus \eta \) and we will treat the finite and infinite elements separately.

**Finite case:** We suppose \( \alpha = (\alpha_1, \alpha_2) \in I_A(S) \setminus \eta \).

Let us consider the following sets:

\[ U^\eta(\alpha) = \{(\alpha_1, y) \in \langle \eta \rangle_\oplus | y < \alpha_2\}, \quad R^\eta(\alpha) = \{(x, \alpha_2) \in \langle \eta \rangle_\oplus | x < \alpha_1\}. \]

If \( U^\eta(\alpha) \neq \emptyset \), we introduce:

\[ u^\eta(\alpha) = \max\{y | (a, y) \in U^\eta(\alpha)\}; \]
\[ M_U(\alpha) = \max\{M|(\alpha_1, M) \in S\setminus\{(\alpha_1, \alpha_2)\}\} \]

If \( R^\eta(\alpha) \neq \emptyset \), we introduce:

\[ r^\eta(\alpha) = \max\{x|(x, \alpha_2) \in R^\eta(\alpha)\}; \]
\[ M_R(\alpha) = \max\{M|(M, \alpha_2) \in S\setminus\{(\alpha_1, \alpha_2)\}\} \]

If there is not ambiguity on the set \( \eta \), in order to simplify the notation we write simply \( U(\alpha) \), \( R(\alpha) \), \( u(\alpha) \), \( r(\alpha) \).

**Remark 25.** If \( U(\alpha) \neq \emptyset \) we have \( u(\alpha) \leq M_U(\alpha) \) and, similarly, if \( R(\alpha) \neq \emptyset \) we have \( r(\alpha) \leq M_R(\alpha) \).

If \( U(\alpha) \neq \emptyset \), we define \( Y_U(\alpha) = \{y \in \{u(\alpha), \ldots, M_U(\alpha)\)|(\alpha_1, y) \in S\} \) and if \( R(\alpha) \neq \emptyset \) we define \( X_R(\alpha) = \{x \in \{r(\alpha), \ldots, M_R(\alpha)\)|(x, \alpha_2) \in S\} \).

**Definition 26.** We say that \( \alpha = (\alpha_1, \alpha_2) \in I_A \setminus \eta \) is reducible by \( \eta \) if \( (\alpha_1, \alpha_2) \in U(\alpha) \cup R(\alpha) \neq \emptyset \) and one of the following conditions are satisfied:

- \( U)\) \( U(\alpha) \neq \emptyset \), and for all \( y \in Y_U(\alpha) \), there exists \( (x, y) \in \langle \eta \rangle_{\oplus} \) such that \( x > \alpha_1 \).
- \( R)\) \( R(\alpha) \neq \emptyset \), and for all \( x \in X_R(\alpha) \), there exists \( (x, y) \in \langle \eta \rangle_{\oplus} \) such that \( y > \alpha_2 \).

**Infinite case:** We suppose \( \alpha = (\alpha_1, \infty) \in I_A \setminus \eta \) (if \( \alpha \) is in the form \( (\infty, \alpha_2) \) we can repeat the following definitions moving all on the second component).

Let us consider:

\[ U^\eta(\alpha) = \{(\alpha_1, y) \in \langle \eta \rangle_{\oplus}\} \]

If \( U^\eta(\alpha) \neq \emptyset \), we introduce:

\[ u^\eta(\alpha) = \max\{y|(a, y) \in U^\eta(\alpha)\} \]

Also in this case, if there is not ambiguity, we write simply \( U(\alpha) \) and \( u(\alpha) \). Let be \( \tilde{y} \) such that \((\alpha_1, y) \in S\) for all \( y \geq \tilde{y} \) (it exists for Lemma 2). Let us consider the set:

\[ Y_{U}(\alpha) = \{y \in \{u(\alpha), \ldots, \max\{\tilde{y}, u(\alpha)\} + e_2 - 1\)|(\alpha_1, y) \in S\} \].

**Definition 27.** We say that \( \alpha \) is reducible by \( \eta \) if \( U(\alpha) \neq \emptyset \) and for all \( y \in Y_{U}(\alpha) \), there exists an element \( (x, y) \in \langle \eta \rangle_{\oplus} \) with \( x > \alpha_1 \).

This condition essentially ensures us that removing an infinite element we do not lose elements in the semigroup that cannot be recovered with the third property by elements of \( \eta \). Given \( \eta \subseteq I_A \), we denote with:

\[ \langle \eta \rangle := \{\alpha \in I_A | \alpha \text{ is reducible by } \eta\} \].
Let us consider the following algorithm:

**Algorithm 1:** Algorithm for find \( \text{red}(\eta) \)

We denote with \( \text{red}(\eta) \) the output of the previous algorithm and we introduce the set \( \mathcal{R}(S) = \{ \eta \subseteq I_A(S) \mid \text{red}(\eta) = I_A(S) \} \). We will say that \( \eta \subseteq I_A(S) \) satisfy the reducibility condition if \( \eta \in \mathcal{R}(S) \).

We have the following statement:

**Theorem 28.** If \( \eta \in \mathcal{R}(S) \) then \( \eta \) is a sor.

**Proof.** If \( \eta \in \mathcal{R}(S) \), we prove that \( \eta \) is a sor. From \( \eta \in \mathcal{T} \) follows that, there exists a chain of subset of \( I_A(S) \):

\[
\eta \subset \eta_1 \subset \ldots \subset \eta_{n-1} \subset \eta_n = \text{red}(\eta) = I_A(S)
\]

such that \( \eta_i = \langle \langle \eta_{i-1} \rangle \rangle \) We prove that \( \eta \) is a sor using a decreasing induction on this chain. We have that \( \eta_n = I_A(S) \) is a sor for Remark 13, now we prove that if \( \eta_{n+1} \) is a sor, then \( \eta \) is a sor.

We assume by contradiction that \( S \not\subseteq S_{\eta_n} \); in this case there exists a good semigroup \( S_i \) such that \( \eta_i \subseteq S_i \not\subseteq S \).

If we suppose \( \eta_{i+1} \subseteq I_A(S_i) \), we would have \( \eta_{i+1} \subseteq \langle \langle \eta_{i+1} \rangle \rangle \subseteq \langle I_A(S_i) \rangle = S_i \subseteq S \), against the fact that \( \eta_{i+1} \) is a sor for \( S \). For this reason, we can always suppose that there exists \( \alpha = (\alpha_1, \alpha_2) \in \eta_{i+1} \setminus I_A(S_i) \). Furthermore we observe that \( \alpha \not\subseteq \eta_i \), indeed, assuming the opposite, we should have \( \alpha \in S_i \) and because \( S_i \subseteq S \) and \( \alpha \in \eta_{i+1} \subseteq I_A(S) \), it would imply that \( \alpha \in I_A(S) \). We distinguish two case: \( \alpha \in \eta_{i+1} \cap I_A(S) \) and \( \alpha \in \eta_{i+1} \cap I_A^\infty(S) \).

**Case 1:** \( \alpha \in \eta_{i+1} \cap I_A(S) \).

Because \( \langle \langle \eta_i \rangle \rangle = \eta_{i+1} \), \( \alpha \) is reducible by \( \eta \). Without loss of generality we can assume \( U(\alpha) \neq \emptyset \); in this case there exists \( (\alpha_1, u(\alpha)) \in \langle \langle \eta_i \rangle \rangle \subseteq S_i \). We have \( u(\alpha) \in Y_{U(\alpha)} \) and, from the reducibility of \( \alpha \) by \( \eta_i \), there exists \( (x_{u(\alpha)}, u(\alpha)) \in \langle \langle \eta_i \rangle \rangle \subseteq S_i \). We have obtained two elements \( (\alpha_1, u(\alpha)) \), \( (x_{u(\alpha)}, u(\alpha)) \) in \( S_i \), from property (G3), there exists \( (\alpha_1, y_1) \in S_i \), with \( y_1 > u(\alpha) \). We observe that, from the definition of \( M_U(\alpha), y_1 \leq M_U(\alpha) \). So \( y_1 \in Y_{U(\alpha)} \). We can repeat the same argument until we obtain that \( (\alpha_1, M_U(\alpha)) \in S_i \). Using again the property (G3) we should obtain \( \alpha \in S_i \) (notice that \( \Delta^S(\alpha) = \emptyset \)), but this is an absurd.

**Case 2:** \( \alpha \in \eta_{i+1} \cap I_A^\infty(S) \).

Without loss of generality we can suppose \( \alpha = (\alpha_1, \infty) \). Because \( \alpha \) is reducible by \( \eta \), we have \( U(\alpha) \neq \emptyset \). We denote with \( M(\alpha) := \max\{\tilde{y}, u\} + e_2 - 1 \), where \( \tilde{y} \) is such that \( (\alpha_1, y) \in S \) for any \( y > \tilde{y} \). Now, using the same argument of the finite case, we obtain that \( (\alpha_1, M(\alpha)) \in S_i \),
but for Lemma 2, this implies \((\alpha_1, \infty) \in S_i\) which is a contradiction.

The following example shows that the converse of the previous theorem is not true in general.

\textbf{Example 29.} Let us consider the good semigroup \(S\) with the following set of irreducible absolute points:

\[ I_A(S) = \{(3, 4), (6, \infty), (7, 8), (10, 15), (14, 18), (17, 25), (\infty, 12), (\infty, 19), (\infty, 22), (\infty, 29)\}. \]

Notice that since \(S\) contains only the tracks \(T_1 = T((3, 4)), T_2 = T((6, \infty), (7, 8)), T_3 = T((6, \infty), (10, 15), (\infty, 12))\) and \(T_4 = T((10, 15), (\infty, 12))\), we have that \(\eta = \{(3, 4), (7, 8), (10, 15), (14, 18), (\infty, 12), (\infty, 22)\}\) is a HS for \(S\). Let us show that \(\text{red}(\eta) \neq I_A(S)\). It suffices to show that \(\langle \langle \eta \rangle \rangle = \eta\), i.e. all the elements in \(I_A(S) \setminus \eta\) are not reducible by \(\eta\). We have

- \(\alpha_1 = (6, \infty)\) is not reducible by \(\eta\). Notice that there exists \((6, 8) = 2(3, 4) \in U^\eta(\alpha_1)\), thus we have \(u^\eta(\alpha_1) = 8\). Furthermore, \(\tilde{y} = 22\) and we have:

\[
Y_{U(\alpha_1)} = \{y \in \{u(\alpha_1), \ldots, \max\{\tilde{y}, u(\alpha_1)\}\} + e_2 - 1|6, y \in S\} = \{12, 15, 16, 18, 19, 20, 22, 23, 24, 25\}.
\]

For each element \(y\) in \(Y_{U(\alpha_1)}\) we need to find \((x, y) \in \langle \eta \rangle \) with \(x > 6\). It is not difficult to notice that for \(y = 25 \in Y_{U(\alpha_1)}\), it is not possible to produce such an element in \(\langle \eta \rangle \).

- \(\alpha_2 = (17, 25)\) is not reducible by \(\eta\). Notice that there exists \((17, 23) = (7, 8) \odot (10, 15) \in U^\eta(\alpha_2)\), thus we have \(u^\eta(\alpha_2) = 23\) (while \(R^\eta(\alpha_2) = \emptyset\)). Furthermore, \(M_U(\alpha_2) = 24\) and we have:

\[
Y_{U(\alpha_2)} = \{y \in \{u(\alpha_2), \ldots, M_U(\alpha_2) = 24|17, y \in S\} = \{23, 24\}.
\]
For each element \( y \) in \( Y_{U(\alpha_2)} \) we need to find \((x, y) \in \langle \eta \rangle_\square\) with \( x > 17 \). However for \( y = 23 \in Y_{U(\alpha_2)} \), it is not possible to produce such an element in \( \langle \eta \rangle_\square \)

- \( \alpha_3 = (\infty, 19) \) is not reducible by \( \eta \). Notice that there exists \((13, 19) = (3, 4) \odot (10, 15) \in R^\eta(\alpha_3) \), thus we have \( r^\eta(\alpha_3) = 13 \). Furthermore, \( \bar{x} = 15 \) and we have:

\[
X_{R(\alpha_3)} = \{ x \in \{ r(\alpha_3), \ldots, \max\{ \bar{x}, r(\alpha_3) \} \} + e_1 - 1 | (x, 19) \in S \} = \{13, 15, 16, 17\}.
\]

For each element \( x \) in \( X_{R(\alpha_3)} \) we need to find \((x, y) \in \langle \eta \rangle_\square\) with \( y > 19 \). It is not difficult to notice that for \( x = 13 \in X_{R(\alpha_3)} \), it is not possible to do that.

- \( \alpha_4 = (\infty, 29) \) is not reducible by \( \eta \), since \( R^\eta(\alpha_4) = \emptyset \).

However it is possible to check that there are no good semigroups \( S' \) such that \( \eta \subseteq S' \subset S \). Thus \( \eta \) is actually a sor for \( S \) and it is not difficult to check that the minimal hitting set \( M = \{ (3, 4), (7, 8), (10, 15) \} \) contained in it, is a sor itself, thus a msor for \( S \).

Now we define: \( \text{Bedim}(S) = \min\{|\eta|, \eta \in \mathcal{R}(S)\}, \)

**Corollary 30.** Given a good semigroup \( S \subseteq \mathbb{N}^2 \), \( \text{edim}(S) \leq \text{Bedim}(S) \).

**Example 31.** The inequality in Corollary 30 can be strict. An example of this behaviour is the good semigroup \( S \) with the following set of irreducible absolute points:

\[
I_A(S) = \{(7, 7), (14, 20), (17, 14), (24, \infty), (25, 21), (32, 30), (39, 45), (42, \infty), (43, 35), (44, 37), (46, \infty), (47, 50), (50, \infty), (54, \infty), (\infty, 32), (\infty, 34), (\infty, 42), (\infty, 51), (\infty, 57)\}.
\]

It is possible to prove that for each MHS \( \eta \) of \( S \) we have that \( \eta \) is a sor for \( S \) but \( \text{red}(\eta) \neq I_A(S) \). This easily implies that \( \text{edim}(S) < \text{Bedim}(S) \).

**Example 32.** Let us consider the good semigroup \( S \), with

\[
I_A(S) = \{(4, 3), (6, 7), (8, 8), (9, 6), (11, \infty), (12, \infty), (13, \infty), (14, \infty), (\infty, 9), (\infty, 11), (\infty, 13)\}.
\]

This is an example of good semigroup having msor with distinct cardinalities. In fact, it is possible to prove that the sets \( \eta_1 = \{(4, 3), (6, 7), (8, 8), (11, \infty), (13, \infty)\} \) and \( \eta_2 = \{(4, 3), (6, 7), (8, 8), (11, \infty), (\infty, 9), (\infty, 11)\} \) are both MHS of \( S \) satisfying the reducibility condition. In particular \( \text{edim}(S) = 5 \).

### 2.3 An algorithm for the computation of the embedding dimension of a semigroup \( S \subseteq \mathbb{N}^2 \)

We will conclude this section presenting an algorithm for the computation of the embedding dimension and with some remarks concerning the definition that we have given.

We proved that:

\[
\text{bedim}(S) \leq \text{edim}(S) \leq \text{Bedim}(S)
\]

and both inequalities are sharp as we will see in Example 35.

We implemented in GAP [13] the following functions:
• ComputeMHS(S): it takes in input a good semigroup and returns the set of its MHS.

• VerifyReducibility(list): it takes in input a list of subsets of $I_A(S)$ and returns the first set that satisfy the condition of reducibility if there exists, otherwise it returns "fail".

• IsThereAMGSContainedInAndContaining(S,V): it takes in input a good semigroup $S$ and a subset $V$ of $I_A(S)$ and returns "true" if there exists a good semigroup $S'$ such that $V \subseteq S' \subsetneq S$

Remark 33. Testing in GAP these function on a sample of about 200000 semigroup, we observed empirically that VerifyReducibility is about seventy times faster than IsThereAMGSContainedInAndContaining.

We introduce the following algorithm to compute the embedding dimension and a set of representatives with minimal cardinality.

```
Algorithm 2: Algorithm for find an msor of minimal cardinality
```

- **input**: A good Semigroup $S$
- **output**: A minimal system of representatives of minimal cardinality

```plaintext
M ← ComputeMHS(S)
H ← M
n ← bedim(S)
Stop ← false

while Stop=false do
    H ← \{η ∈ I_A(S) | |η| = n and H ⊆ η for some H ∈ H\} ∪ \{η ∈ M||η| = n\}
    if VerifyReducibility(H)=η then
        Stop=true, return η
    end
    if VerifyReducibility(H)=fail then
        if ForAny η ∈ H, IsThereAMGSContainedInAndContaining(S,η)=false then
            Stop=true, return η
        else
            n ← n + 1
        end
    end
end
```
Remark 34. We tested the algorithm on a sample of 200000 good semigroups and we noticed that, for $n = \text{bedim}(S)$:

- The condition "VerifyReducibility($\mathcal{S}$) = fail" occurred only in 82 cases.
- Both the conditions "VerifyReducibility($\mathcal{S}$) = fail" and "IsThereAMGSContainedInAndContaining($S, \eta$) = true for all $\eta \in \mathcal{S}$" occurred only in 2 cases. So in these cases $\text{bedim}(S) \neq \text{edim}(S)$.
- The situation which all MSH of minimal cardinality are not reducible and at least one of them is a sor occurred only in one case. So only in this case $\text{Bedim}(S) \neq \text{edim}(S)$.

For this reasons and for Remark 33 this algorithm is considerably faster than to computing the embedding dimension using the definition.

Example 35. Let us consider the good semigroup $S$, represented in Figure 35, we want to find a msor for $S$ and the embedding dimension of $S$.

We have that $I_A(S) = \{(4, 3), (7, 13), (11, 17), (14, \infty), (15, \infty), (16, 20), (24, \infty), (\infty, 12), (\infty, 16), (\infty, 26)\}$.

First of all we need to compute the minimal hitting sets of $S$. It contains the following tracks:

- $T_1 = T((4, 3))$;
- $T_2 = T((7, 13))$;
- $T_3 = T((11, 17), (\infty, 16))$;
- $T_4 = T((15, \infty), (16, 20), (\infty, 12))$;
- $T_5 = T((15, \infty), (16, 20), (\infty, 16))$;
- $T_6 = T((24, \infty), (\infty, 26))$.

Thus the following is the complete list of the MHS of $S$.

- $\eta_1 = \{(4, 3), (7, 13), (\infty, 12), (\infty, 16), (\infty, 26)\}$;
- $\eta_2 = \{(4, 3), (7, 13), (11, 17), (15, \infty), (\infty, 26)\}$;
- $\eta_3 = \{(4, 3), (7, 13), (11, 17), (16, 20), (24, \infty)\}$;
- $\eta_4 = \{(4, 3), (7, 13), (11, 17), (16, 20), (\infty, 26)\}$;
- $\eta_5 = \{(4, 3), (7, 13), (15, \infty), (24, \infty), (\infty, 16)\}$;
- $\eta_6 = \{(4, 3), (7, 13), (15, \infty), (\infty, 16), (\infty, 26)\}$;
• $\eta_7 = \{(4,3), (7,13), (16,20), (24,\infty), (\infty,16)\}$;
• $\eta_8 = \{(4,3), (7,13), (16,20), (\infty,16), (\infty,26)\}$;
• $\eta_9 = \{(4,3), (7,13), (24,\infty), (\infty,12), (\infty,16)\}$;
• $\eta_{10} = \{(4,3), (7,13), (11,17), (15,\infty), (24,\infty)\}$.

Figure 4: o: Reducible elements; [ ]: Irreducible Absolutes; $\eta$: Elements of $\langle \text{red}(\eta) \rangle$ ⊕

Thus for this semigroup $\text{bedim}(S) = 5$.

We consider $\eta = \eta_1 = \{(4,3), (7,13), (\infty,12), (\infty,16), (\infty,26)\}$ and we want to show that $\eta \in \mathcal{R}(S)$.

We have

$\eta' = \langle \langle \eta \rangle \rangle = \{(4,3), (7,13), (11,17), (14,\infty), (16,20), (24,\infty), (\infty,12), (\infty,16), (\infty,26)\}$

In fact
• $\alpha_1 = (11,17)$ is reducible by $\eta$ because we have $U^\eta(\alpha_1) \neq \emptyset$ since $(4,3) \odot (7,13) = (11,16) \in \langle \eta \rangle_\oplus$. Furthermore $u^\eta(\alpha_1) = 16$.

Since $M_U(\alpha_1) = 16$ we need only to find an element of the type $(x,16) \in \langle \eta \rangle_\oplus$ with $x > 11$. The element $(\infty,16) \in \eta$ satisfies this property.

• $\alpha_2 = (14,\infty)$ is reducible by $\eta$ because we have $U^\eta(\alpha_2) \neq \emptyset$; in fact we have $2(7,13) = (14,26) \in \langle \eta \rangle_\oplus$. Furthermore $u^\eta(\alpha_2) = 26$.

Since $\tilde{y} = 18$, for all

$y \in Y_U(\alpha_1) = \{y \in \text{u}(\alpha_2) = 26, \ldots, \max\{\tilde{y}, u^\eta(\alpha_2)\} + e_2 - 1 = 28\} | (14, y) \in S\}$

$= \{26,27,28\}$,
we need to find an element of the type \((x, y) \in \langle \eta \rangle_{\oplus} \) with \(x > 14\). The following elements of \(\langle \eta \rangle_{\oplus} \) satisfy this property:

\[(\infty, 26), \ 9(4, 3) = (36, 27), \ 5(4, 3) \odot (7, 13) = (27, 28).\]

- \(\alpha_3 = (16, 20)\) is reducible by \(\eta\). In fact we have \(U^\eta(\alpha_3) \neq \emptyset\); since \(4(4, 3) = (16, 12) \in \langle \eta \rangle_{\oplus}\). Furthermore \(u^\eta(\alpha_3) = 12\).

Since \(M_U(\alpha_3) = 19\), for all \(y \in Y_U(\alpha_3) = \{12, 15, 16, 18, 19\}\) we need to find an element of the type \((x, y) \in \langle \eta \rangle_{\oplus} \) with \(x > 16\). The following elements of \(\langle \eta \rangle_{\oplus} \) satisfy this property:

\[(\infty, 12), \ 5(4, 3) = (20, 15), \ 6(4, 3) = (24, 18), \ 4(3) \odot (\infty, 16) = (\infty, 19).\]

- \(\alpha_4 = (24, \infty)\) is reducible by \(\eta\). In fact \(U^\eta(\alpha_4) \neq \emptyset\) since \(6(4, 3) = (24, 18) \in \langle \eta \rangle_{\oplus}\). Thus \(u^\eta(\alpha_4) = 18\). Since \(\tilde{y} = 24\), for all

\(y \in Y_U(\alpha_4) = \{y \in \{u(\alpha_4) = 18, \ldots, \max\{\tilde{y}, u^\eta(\alpha_4)\}\} + e_2 - 1 = 26|24, y) \in S\} = \{18, 19, 21, 22, 24, 25, 26\},\)

we need to find an element of the type \((x, y) \in \langle \eta \rangle_{\oplus} \) with \(x > 24\). The following elements of \(\langle \eta \rangle_{\oplus} \) satisfy this property:

\[2(4, 3) \odot (\infty, 12) = (\infty, 18), \ 4(3) \odot (\infty, 16) = (\infty, 19), \ 7(4, 3) \odot (\infty, 12) = (\infty, 25), \ (\infty, 26).\]

Notice that \(\alpha_5 = (15, \infty)\) is not reducible by \(\eta\), but it is reducible by \(\eta^1\). In fact \(U^{\eta^1}(\alpha_5) \neq \emptyset\) since \((4, 3) \odot (11, 17) = (15, 20) \in \langle \eta^1 \rangle_{\oplus}\). Thus \(u^{\eta^1}(\alpha_5) = 20\). Since \(\tilde{y} = 18\), for all

\(y \in Y_U(\alpha_5) = \{y \in \{u(\alpha_5) = 20, \ldots, \max\{\tilde{y}, u^{\eta^1}(\alpha_5)\}\} + e_2 - 1 = 22|14, y) \in S\} = \{20, 21, 22\},\)

we need to find an element of the type \((x, y) \in \langle \eta^1 \rangle_{\oplus} \) with \(x > 15\). The following elements of \(\langle \eta^1 \rangle_{\oplus} \) satisfy this property:

\[(16, 20), \ 3(4, 3) \odot (\infty, 12) = (\infty, 21), \ 3(4, 3) \odot (7, 13) = (19, 22).\]

Thus \(\langle \eta^1 \rangle \) is \(I_A(S)\), and this means \(\eta \in \mathcal{R}(S)\) since \(\text{red}(\eta) = I_A(S)\). Hence \(\text{Bedim}(S) \leq 5 = |\eta|\). Since we have

\[5 = \text{bedim}(S) \leq \text{edim}(S) \leq \text{Bedim}(S) \leq 5,\]

we can finally deduce that \(\text{edim}(S) = 5\) and \(\eta\) is an \textit{msor}. It is possible to check that all the minimal hitting sets previously found satisfy the reducibility condition, thus they are all \textit{msor} for \(S\).

All the previous computations were realized implementing all the previous algorithms in GAP [13].
3 Properties of embedding dimension

3.1 Relationship between embedding dimension of a ring and embedding dimension of its value semigroup

**Theorem 36.** Let $S$ be a good semigroup of $\mathbb{N}^2$ such that there exists an algebroid curve $R$ with $v(R) = S$. Then $\text{edim}(S) \geq \text{edim}(R)$.

**Proof.** Let us consider an algebroid curve $R$ such that $v(R) = S$ and denote by $\varepsilon$ the embedding dimension of $S$. Thus there exists $\eta \subset I_A(S)$, $\text{msor}$ of $S$, with $|\eta| = \varepsilon$. We want to prove $\text{edim}(R) \leq \varepsilon$.

We denote by 
$$\eta = \{\alpha_1, \ldots, \alpha_\varepsilon\},$$

and we want to show that it is possible to choose elements $\phi_1, \ldots, \phi_\varepsilon$ in $R$, such that:

- $v(\phi_j) = \alpha_j$ for each $j = 1, \ldots, \varepsilon$;
- $R_1 = \mathbb{K}[\phi_1, \ldots, \phi_\varepsilon]$ is an algebroid curve.

It is clear that $R_1 = \mathbb{K}[\phi_1, \ldots, \phi_\varepsilon]$ is an algebroid curve if and only if $v(R_1)$ is a good semigroup. By construction, the subsemigroup $\nu(R_1)$ always satisfies the properties G1) and G3) of good semigroups, thus we need to guarantee the existence of a conductor. This can be done by forcing in $\nu(R_1)$ the presence of vectors that fulfil the conditions of Proposition 11 (it is not difficult to do that by accordingly adding to the $\phi_i$ elements of $R$ with valuation greater than its conductor).

Now, 
$$\eta \subseteq \nu(R_1) \subseteq \nu(R) = S,$$

and, since $\eta$ is a $\text{msor}$ of $S$ and $\nu(R_1)$ is a good semigroup, we have $\nu(R_1) = S$ that easily implies $R_1 = R$.

Thus $\text{edim}(R) = \text{edim}(R_1) \leq \varepsilon = \text{edim}(S)$. \hfill \square

We want to show that the inequality can be strict and we want to analyze the cases when this happens.

**Example 37.** Let us consider the ring $R \cong \mathbb{K}[(t^4, u^4), (t^6 + t^9, u^6 + u^7), (2t^{15} + t^{18}, 2u^{13} + u^{14})]$, and observe that $R = \mathbb{K}[(t^4, u^4), (t^6 + t^9, u^6 + u^7), (2t^{15} + t^{18}, 2u^{13} + u^{14})] = \mathbb{K}[(t^4, u^4), (t^6 + t^9, u^6 + u^7)]$, in fact:

$$\left(2t^{15} + t^{18}, 2u^{13} + u^{14}\right) = (t^6 + t^9, u^6 + u^7)^2 - (t^4, u^4)^3.$$

So we have that $\text{edim}(R) = 2$, but $\text{edim}(v(R)) = 3$, since $M = \{(4, 4), (6, 6), (15, 13)\}$ is the only one hitting set of the semigroup $v(R)$.

This fact happens because in the ring $R$ the element of valuation $(15, 13)$ is obtained by the
element of evaluation \((4, 4)\) and \((6, 6)\) with a cancellation on both components. This situation cannot be controlled by the third property of good semigroup; so this gap in embedding dimension can be justified by the fact that this piece of information is lost in the passage from the ring to the semigroup. For this value semigroup it is possible to find a ring, namely \(T = \mathbb{K}[[t^4, u^4], (t^6, u^6), (t^{15}, u^{13})]\) with \(v(T) = v(R)\), and such that \(\text{edim}(T) = \text{edim}(v(T))\). This situation is not guaranteed to happen as it is shown in the following example.

**Figure 5:** Semigroup \(v(R)\) of the Example 37

**Example 38.** Let us consider the ring \(R = \mathbb{K}[[t^4, u^3], (t^7, u^{13}), (t^{11}, u^{17}), (t^{16}, u^{20})]\) that has embedding dimension 4. Its value semigroup is the good semigroup that appeared in Example 35, where we proved that its embedding dimension is five. We focus on one of its \(msor\), namely \(\eta = \{(4, 3), (7, 13), (11, 17), (16, 20), (\infty, 26)\}\). If we analyze in detail what happens we observe that \((t^{23}, u^{33}) = (t^7, u^{13}) \cdot (t^{16}, u^{20}) \in R\) and \((t^{23}, u^{26}) = (t^{11}, u^{17}) \cdot (t^{12}, u^{9}) \in R\), so \((0, u^{26} - u^{33}) \in R\). But \(\eta = \{(4, 3), (7, 13), (11, 17), (16, 20)\}\) is not a \(sor\), since we have seen in the Example 35 that all MHS have to contain either \((\infty, 26)\) or \((24, \infty)\). This fact happens because in the ring \(R\) all the elements of valuation \((x, 26)\) with \(x \geq 25\) appear because we have a complete cancellation on the first component (we obtain 0 on the first component).

In the semiring \(\langle \eta \rangle\) the existence of the elements \((23, 33)\) and \((23, 26)\) guarantees, for the third property, only the existence of one element of valuation \((> 23, 26)\), but not the presence of all
elements \((x, 26)\), with \(x \geq 24\).

Also in this case in the semigroup we lose a piece of information present in the ring.

Differently from the previous example, it is not possible to find a ring \(T\) such that \(v(T) = v(R)\) and \(\text{edim}(T) = \text{edim}(v(T)) = 5\). To see it, let us suppose by contradiction that such a ring \(T\) exists. Let us consider \(\psi_1, \ldots, \psi_5 \in T\), such that

- \(v(\psi_1) = (4, 3)\);
- \(v(\psi_2) = (7, 13)\);
- \(v(\psi_3) = (11, 17)\);
- \(v(\psi_4) = (16, 20)\);
- \(v(\psi_5) = (\infty, 26)\).

From the proof of Theorem 36 we have that \(T \cong \mathbb{K}[\psi_1, \psi_2, \psi_3, \psi_4, \psi_5]\). Let us consider the ring \(T' \cong \mathbb{K}[\psi_1, \psi_2, \psi_3, \psi_4]\). We must have \(v(T') \subset v(T)\), because otherwise \(T = T'\) against the fact that \(\text{edim}(T) = 5\). Now we have that \(\{(4, 3), (7, 13), (11, 17), (16, 20)\} \subseteq v(T')\) and it is not difficult to show that there exists only one good semigroup \(D\) containing these vectors and contained in \(v(T)\). The good semigroup \(D\) is the one appeared in \([1, \text{Example 2.16}]\) as the first example of a good semigroup that cannot be a value semigroup of a ring. Thus \(v(T') = v(T)\) and we have a contradiction.

### 3.2 Relationship between embedding dimension and multiplicity

Now we want to prove the following theorem.

**Theorem 39.** Let \(S\) be a good semigroup. Denote by \(e = (e_1, e_2)\) the multiplicity vector of \(S\). Then \(\text{edim}(S) \leq e_1 + e_2\).

We remember that, if \(S\) is a numerical semigroup with multiplicity \(e(S)\), it is possible to prove that \(\text{edim}(S) \leq e(S)\) using the fact that the set \(\text{Ap}(S) \setminus \{0\} \cup \{e(S)\}\) is a system of generators of \(S\) with cardinality \(e(S)\). Using the properties of the Apéry set of a good semigroup, introduced in \([8]\), we wish to prove the same inequality for good semigroups contained in \(\mathbb{N}^2\).

First of all, we recall the notion of Apéry set and levels.

**Definition 40.** The Apéry set of the good semigroup \(S\) (with respect to the multiplicity) is defined as the set:

\[
\text{Ap}(S) = \{\alpha \in S \setminus 0 : \alpha - e \notin S \setminus 0\}.
\]

Notice that we are slightly changing the definition of Apéry set introduced in \([8]\) in order to have \(e \in \text{Ap}(S)\).

We say that \((\alpha_1, \alpha_2) \leq (\beta_1, \beta_2)\) if and only if \((\alpha_1, \alpha_2) = (\beta_1, \beta_2)\) or \((\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)\) and we have \((\alpha_1, \alpha_2) \ll (\beta_1, \beta_2)\) where the last means \(\alpha_1 < \beta_1\) and \(\alpha_2 < \beta_2\).
As described in [8], it is possible to build up a partition of the Apéry set, in the following way. Let us define, \( D_0 = \emptyset \):

\[
B^{(i)} = \{ \alpha \in \text{Ap}(S) \setminus (\bigcup_{j<i} D^{(j)}) : \alpha \text{ is maximal with respect to } \leq \}
\]

\[
C^{(i)} = \{ \alpha \in B^{(i)} : \alpha = \beta_1 \oplus \beta_2 \text{ for some } \beta_1, \beta_2 \in B^{(i)} \}
\]

\[
D^{(i)} = B^{(i)} \setminus C^{(i)}.
\]

For a certain \( N \in \mathbb{N} \), we have \( \text{Ap}(S) = \bigcup_{i=1}^{N} D^{(i)} \) and \( D^{(i)} \cap D^{(j)} = \emptyset \). In accordance to notation of [8], we rename these sets in an increasing order setting \( A_i = D^{(N+1-i)} \). So we have

\[
\text{Ap}(S) = \bigcup_{i=1}^{N} A_i.
\]

It was proved [Thm. 3.4 [8]] that \( N = e_1 + e_2 \), a key result in the proof of our inequality. In order to prove Theorem 39, it is useful to introduce the following new definition of reducibility of an element of \( I_A(S) \) by a subset \( \eta \subseteq I_A(S) \).

**Definition 41.** Let \( \alpha = (\alpha_1, \alpha_2) \in I_A(S) \) and \( \eta \subseteq I_A(S) \).

- **Case** \((\alpha_1, \alpha_2) \in I_A(S)\). Denote, as in the previous definition,

\[
M_R(\alpha) = \max \{ M | (M, \alpha_2) \in S \setminus \{ \alpha \} \}.
\]

Then \( \alpha \) is \( \rho \)-reducible by \( \eta \) if

1. \( \exists h_1, \ldots, h_k \in \eta \text{ such that } h_1 \odot \cdots \odot h_k = (\beta_1, \alpha_2) \) with \( \beta_1 < \alpha_1 \).
2. \( \forall x \in \{ x \in \{ \beta_1, \ldots, M_R(\alpha) \} : (x, \alpha_2) \in S \} \) we can find \( j_1, \ldots, j_l \in \eta \) such that \( j_1 \odot \cdots \odot j_l = (\bar{x}, \beta_2) \) with \( \beta_2 > \alpha_2 \).

- **Case** \( (\infty, \alpha_2) \in I_A(S)^\infty \). Denote, as we did before, by \( \bar{x} \) the minimal element such that \( (x, \alpha_2) \in S \) for all \( x \geq \bar{x} \). Then \( (\infty, \alpha_2) \) is \( \rho \)-reducible by \( \eta \) if

1. \( \exists h_1, \ldots, h_k \in \eta \text{ such that } h_1 \odot \cdots \odot h_k = (\beta_1, \alpha_2) \) with \( \beta_1 < \infty \).
2. \( \forall \bar{x} \in \{ x \in \{ \beta_1, \ldots, \max(\beta_1, \bar{x}) + e_1 - 1 \} : (x, \alpha_2) \in S \} \) we can find \( j_1, \ldots, j_l \in \eta \) such that \( j_1 \odot \cdots \odot j_l = (\bar{x}, \beta_2) \) with \( \beta_2 > \alpha_2 \).

- **Case** \( (\alpha_1, \infty) \in I_A(S)^\infty \). Such an element is never \( \rho \)-reducible by \( \eta \).

**Remark 42.** If an element of \( I_A(S) \) is \( \rho \)-reducible by \( \eta \), it is also reducible by \( \eta \).

**Remark 43.** If an element \((\alpha_1, \alpha_2) \) of \( I_A(S) \) is \( \rho \)-reducible by \( \eta \), then it is also \( \rho \)-reducible by \( \eta_{\alpha_1} = \{ (x, \alpha_2) \in \eta : x < \alpha_1 \} \). In fact, the elements required to satisfy the condition 1) and 2) of Definition 41 cannot be obtained by using irreducible absolute points of \( S \) with first component bigger than \( \alpha_1 \) (because we only allow the \( \odot \) operation to produce them).
Now we write

\[ I_A(S) = \{ \alpha(1) = (\alpha_1^{(1)}, \alpha_2^{(1)}), \ldots, \alpha(n) = (\alpha_1^{(n)}, \alpha_2^{(n)}) \}, \]

where the elements are ordered in decreasing order with respect the first coordinate, i.e. if \( j < l \), then \( \alpha_j^{(j)} > \alpha_l^{(l)} \) or \( \alpha_j^{(j)} = \alpha_l^{(l)} = \infty \) and \( \alpha_j^{(j)} > \alpha_l^{(l)} \). Let us consider the following algorithm to produce, starting from \( I_A(S) \), a set \( \eta \) that is still a sor for \( S \).

```
input : The set of irreducible absolute point \( I_A(S) \)
output: A subset \( \eta \subseteq I_A(S) \)
\eta ← I_A(S)
for \( k ← 1 \) to \( n \) do
  if \( \alpha(k) \) is \( \rho \)-reducible by \( I_A(S) \setminus \{ \alpha(k) \} \) then
    \( \eta ← \eta \setminus \{ \alpha(k) \} \)
end
\eta
```

Let us prove by induction on \( k \) that the subset \( \eta \) produced by the algorithm is a sor for \( S \). By Theorem 28, we can do it by showing that it satisfies the reducibility condition. At the first step \( \eta = I_A(S) \), so we have a sor for \( S \). Suppose that at the \( k \)-th step of the algorithm \( \eta \in \mathcal{R}(S) \) and let us show that it still satisfies the reducibility condition after the \( k + 1 \)-th step. If \( \alpha^{(k+1)} \) is not \( \rho \)-reducible by \( I_A(S) \setminus \{ \alpha^{(k+1)} \} \) then we have nothing to prove because \( \eta \) remains unchanged. So let us suppose that \( \alpha^{(k+1)} \) is \( \rho \)-reducible by \( I_A(S) \setminus \{ \alpha^{(k+1)} \} \). We need to prove that \( \eta \setminus \{ \alpha^{(k+1)} \} = \eta' \in \mathcal{R}(S) \). For Remark 43, \( \alpha^{(k+1)} \) is \( \rho \)-reducible by the set \( W = \{ (\alpha_1, \alpha_2) \in I_A(S) \setminus \{ \alpha^{(k+1)} \} : \alpha_1 < \alpha_1^{(k+1)} \} = \{ \alpha^{(k+2)}, \ldots, \alpha^{(n)} \} \). But at this step of the algorithm \( W \subseteq \eta' \), thus \( \alpha^{(k+1)} \) is \( \rho \)-reducible by \( \eta' \), thus also reducible and this means that \( \eta \subseteq \text{red}(\eta') \). Thus by the inductive step \( I_A(S) = \text{red}(\eta) \subseteq \text{red}(\eta') \), hence \( \eta' \in \mathcal{R}(S) \) and it is still a sor.

**Proposition 44.** If \( \alpha = (\alpha_1, \alpha_2) \in I_A(S) \) is such that \( 2\Delta^S(\alpha) \cap (e + S) \neq \emptyset \), then \( \alpha \) is \( \rho \)-reducible by \( I_A(S) \setminus \{ \alpha \} \).

**Proof.** Let us choose \((\beta_1, \alpha_2) \in e + S\) with the greatest possible \( \beta_1 \). Denote by \( k \) the integer such that \((\tilde{\alpha}_1, \tilde{\alpha}_2) \otimes k(e_1, e_2) = (\beta_1, \alpha_2)\), where \((\tilde{\alpha}_1, \tilde{\alpha}_2) \in \text{Ap}(S) \cup \{0\}\).

If \((\tilde{\alpha}_1, \tilde{\alpha}_2) \neq 0\) we write it as

\[ (\tilde{\alpha}_1, \tilde{\alpha}_2) = h_1 \otimes \cdots \otimes h_l, \]

where the \( h_j \) are irreducible element of \( S \).

Each \( h_j = (\alpha_1^j, \alpha_2^j) \) is an absolute point. In fact, if it were possible to write it as

\[ (x, \alpha_2^j) \oplus (\alpha_1^j, y), \text{ with } x > \alpha_1^j \text{ and } y > \alpha_2^j, \]

and \((x, \alpha_2^j), (\alpha_1^j, y) \in S\), then it would follow that

\[ h_1 \otimes \cdots \otimes (x, \alpha_2^j) \otimes \cdots \otimes h_l \otimes k(e_1, e_2) = (\gamma_1, \alpha_2) \in e + S, \]
and \( \gamma_1 > \beta_1 \), and this is against the maximality of \( \beta_1 \).

Thus \( h_i \in I_A(S) \) for all \( i \) (and they are clearly distinct from \( (\alpha_1, \alpha_2) \)).

Now if \( (e_1, e_2) \in I_A(S) \), then

\[
(\beta_1, \alpha_2) = k(e_1, e_2) \odot h_1 \odot \cdots \odot h_l
\]
is already the element required to fulfill condition 1. in Definition 41.

Thus, let us suppose that \( (e_1, e_2) = (\tilde{e}_1, \tilde{e}_2) \oplus (e_1, e_2) \), where \( \tilde{e}_1 > e_1, \tilde{e}_2 > e_2 \) and \( (\tilde{e}_1, \tilde{e}_2), (e_1, e_2) \in I_A(S) \setminus \{(\alpha_1, \alpha_2)\} \) (obviously \( (\alpha_1, \alpha_2) \) cannot be of the type \( (\tilde{e}_1, \tilde{e}_2) \) or \( (e_1, e_2) \)).

First of all notice that \( \tilde{e}_1 \neq \infty \). In fact, if it were equal to \( \infty \), then there would exist \( \bar{x} \) such that \( (x, e_2) \in S \) for all \( x \geq \bar{x} \). This implies that

\[
k(x, e_2) \odot h_1 \odot \cdots \odot h_l = (kx + q, \alpha_2) \in S
\]
for all \( x \geq \bar{x} \). Thus \( (\alpha_1, \alpha_2) = (\infty, \alpha_2) \) and this is a contradiction because

\[
(\alpha_1, \alpha_2) = (\infty, \alpha_2) = k(\infty, e_2) \odot h_1 \odot \cdots \odot h_l,
\]
is not an element of \( I_A(S) \) being reducible. Thus \( \tilde{e}_1 \neq \infty \), and the element

\[
(\tilde{\alpha}_1, \alpha_2) = k(\tilde{e}_1, e_2) \odot h_1 \odot \cdots \odot h_l,
\]
is the required element that satisfies the condition 1. of Definition 41.

Now we want to show that we can satisfy the condition 2. of \( \rho \)-reducibility. Let us suppose that \( \alpha = (\alpha_1, \alpha_2) \in I_A(S) \) (all the following considerations can be adapted to the case \( (\alpha_1, \alpha_2) = (\infty, \alpha_2) \)).

We have to show that for each \( \bar{x} \in X = \{x \in \{\beta_1, \ldots, M_R(\alpha)\} : (x, \alpha_2) \in S\} \) we can find \( j_1, \ldots, j_l \in \eta \) such that \( \{j_1 \odot \cdots \odot j_l = (\bar{x}, \beta_2) \) with \( \beta_2 > \alpha_2 \).

Thus, let us consider an arbitrary \( \bar{x} \in X \). Since \( (\bar{x}, \alpha_2), (\alpha_1, \alpha_2) \in S \), for the G3) property of Definition 1, there exists \( \beta_2 > \alpha_2 \) such that \( (\bar{x}, \beta_2) \in S \).

Theorem 8 ensures that we can write

\[
(\bar{x}, \beta_2) = \sum_{i=1}^{m} n \gamma_{j_i}, \gamma_{j_i} \in I_A(S).
\]

It must exist an index \( \bar{j} \) such that

\[
\bigotimes_{j=1}^{n} \gamma_{j_{\bar{j}}} = (\bar{x}, \tilde{\beta}_2).
\]

Notice that \( \gamma_{j_{\bar{j}}} \in I_A(S) \setminus \{(\alpha_1, \alpha_2)\} \) for all \( j = 1, \ldots, n \) (they all have first coordinate less than \( \bar{x} \leq \alpha_1 \)). Furthermore \( \tilde{\beta}_2 \geq \beta_2 > \alpha_2 \), thus it is the element which we were looking for in order to satisfy the condition 2. of \( \rho \)-reducibility.
From Proposition 44 and Algorithm 3, we can immediately deduce the following Corollary.

**Corollary 45.** Let $S$ be a good semigroup. Then the set

$$\eta_S = \{ \alpha \in I_A(S) : 2\Delta^S(\alpha) \cap (e + S) = \emptyset \}$$

is a sor for $S$.

Now we are ready to give a proof of Theorem 39.

**Proof of Theorem 39.** Using Corollary 45 and the definition of embedding dimension, it suffices to show that $|\eta_S| \leq e_1 + e_2$.

Let us write $\eta_S = \{ h^{(1)} = (\alpha_1^{(1)}, \alpha_2^{(1)}), \ldots, h^{(k)} = (\alpha_1^{(k)}, \alpha_2^{(k)}) \}$ where if $i < j$ then $\alpha_2^{(i)} < \alpha_2^{(j)}$ or $\alpha_2^{(i)} = \alpha_2^{(j)} = \infty$ with $\alpha_1^{(i)} < \alpha_1^{(j)}$. Furthermore we denote by $c = (c_1, c_2)$ the conductor of $S$. Now to each element $h^{(i)}$ of $\eta_S$ we associate an element $\overline{h}^{(i)}$ in the following way:

- **Case** $h^{(i)} = (\alpha_1, \infty)$. Then we set $\overline{h}^{(i)} = (\alpha_1, c_2 + i)$.

- **Case** $h^{(i)} = (\alpha_1, \alpha_2)$, with $\alpha_2 \neq \infty$. Then we set $\overline{h}^{(i)} = \min(2\Delta^S(h^{(i)}))$.

We consider the set $\eta' = \{ \overline{h}^{(1)}, \ldots, \overline{h}^{(k)} \}$, and we want to show that distinct elements of $\eta'$ belong to distinct levels of the Apéry set of $S$. In order to do that we consider two arbitrary elements $\overline{h}^{(i)}$ and $\overline{h}^{(j)}$ of $\eta'$ and we prove that they cannot be on the same level of the Apéry set. We have four possible configuration:

- **Case** $\overline{h}^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)})$ and $\overline{h}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)})$, with $\alpha_1^{(i)} < \alpha_1^{(j)}$ and $\alpha_2^{(i)} < \alpha_2^{(j)}$.

  In this case $\overline{h}^{(i)} \preceq \overline{h}^{(j)}$ and from definition of Apéry levels it follows that $\overline{h}^{(j)} \in A_n$ and $\overline{h}^{(i)} \in A_m$ with $m < n$.

- **Case** $\overline{h}^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)})$ and $\overline{h}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)})$, with $\alpha_1^{(i)} < \alpha_1^{(j)}$ and $\alpha_2^{(i)} = \alpha_2^{(j)}$.

  This configuration is not possible, because it is against the minimality of the element $\overline{h}^{(j)}$ (it is easy to check that this situation cannot involve elements that come from $h^{(i)}$ of the type $(\alpha_1, \infty)$).

- **Case** $\overline{h}^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)})$ and $\overline{h}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)})$, with $\alpha_1^{(i)} < \alpha_1^{(j)}$ and $\alpha_2^{(i)} > \alpha_2^{(j)}$.

  This configuration is not possible, because the element $\overline{h}^{(i)} \oplus \overline{h}^{(j)} \in S$ is against the minimality of the element $\overline{h}^{(j)}$, (it is also easy to check that this situation cannot involve elements that come from $h^{(i)}$ of the type $(\alpha_1, \infty)$).
Case $\mathbf{h}^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)})$ and $\mathbf{h}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)})$, with $\alpha_1^{(i)} = \alpha_1^{(j)}$ and $\alpha_2^{(i)} > \alpha_2^{(j)}$. Suppose by contradiction that there exists $n \in \mathbb{N}$ such that $\mathbf{h}^{(i)}, \mathbf{h}^{(j)} \in A_n$. From the definition of $\eta_S$, it follows that $\Delta^2 S(\mathbf{h}^{(j)}) \subseteq A_p(S)$. Thus from Lemma 3.3 (3) of [8], the minimal element $\beta$ of $\Delta^2 S(\mathbf{h}^{(j)}) \in A_m$ with $m \leq n$. On the other hand $\mathbf{h}^{(j)} \leq \beta$, thus $\beta \in A_l$ with $l \geq n$. Thus $\beta \in A_n$ and this is a contradiction because we have

$$\mathbf{h}^{(i)} \oplus \beta = \mathbf{h}^{(j)} \in A_n,$$

that is against the definition of Apéry set level. Because Theorem 3.4 of [8], states that the levels of the Apéry Set are exactly $e_1 + e_2$, it follows that

$$\text{edim}(S) \leq |\eta_S| = |\eta'| \leq e_1 + e_2,$$

and the proof of Theorem 39 is complete.

We recall that a good semigroup is said to be Arf if and only if $S(\alpha) = \{ \beta \in S | \beta \geq \alpha \}$ is a semigroup for any $\alpha \in S$. In [1, Proposition 3.19 and Corollary 5.8] the authors proved that an Arf semigroup can be always seen as the value semigroup of an Arf ring. From this result and Theorem 39 we can deduce the following corollary.

**Corollary 46.** Let $S$ be an Arf good subsemigroup of $\mathbb{N}^2$. Then, denoted as usual by $e = (e_1, e_2)$ the multiplicity vector of $S$, we have $\text{edim}(S) = e_1 + e_2$.

**Proof.** By Theorem 39 we have $\text{edim}(S) \leq e_1 + e_2$. Denote by $R$ an Arf ring such that $v(R) = S$. By Theorem 36 we have $\text{edim}(S) \geq \text{edim}(R)$. But $R$ is an Arf ring, thus its embedding dimension is equal to its multiplicity (cf.[15, Theorem 2.2]). Because the multiplicity of $R$ is also equal to $e_1 + e_2$, we have

$$e_1 + e_2 = \text{edim}(R) \leq \text{edim}(S) \leq e_1 + e_2,$$

and the proof of the corollary is complete.

We say that a good semigroup $S \subseteq \mathbb{N}^2$ is maximal embedding dimension if $\text{edim}(S) = e_1 + e_2$. Thus Arf good semigroups constitute a particular class of maximal embedding dimension semigroups. It is known that a numerical semigroup is maximal embedding dimension if and only if $M + M = e + M$ where $M = S \setminus \{0\}$ is its maximal ideal and $e$ is its multiplicity [17]. Thus, we propose the following conjecture.

**Conjecture 47.** Let $S$ be a good subsemigroup of $\mathbb{N}^2$. Then $S$ is maximal embedding dimension if and only if $M + M = e + M$, where $e$ is its multiplicity vector and $M = S \setminus \{0\}$.
At the moment we have tested Conjecture 47 for a large number of good semigroup, and we have a proof of the fact that $M + M = e + M$ implies $\text{edim}(S) = e_1 + e_2$.

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