On the metric hypercomplex group alternative-elastic algebras for n mod 8 = 0.

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In this article the fine-tuning of the algorithm 9.1 [2], [3] is considered. In this connection, answers to the following questions are given.

1. How to construct the metric hypercomplex Cayley-Dickson algebra by means of the algorithm 9.1 for \( n = 2^k \)?

2. How to construct the metric hypercomplex orthogonal group alternative-elastic algebra by means of the algorithm 9.1 for \( n \mod 8 = 0 \)?

3. How to decompose the metric hypercomplex orthogonal homogenous group alternative-elastic algebra on an algebraic basis?

4. That is the generator of the metric hypercomplex orthogonal homogenous group alternative-elastic algebra and as it to construct?

5. How technically to realize the algorithm 9.1 and to construct the canonical sedenion algebra for \( n=16 \)?

Let product of elements of the hypercomplex group alternative-elastic algebra \( \mathbb{A} \) [1], [5], [2], [3] over the field \( \mathbb{R} \) and the vector space \( \mathbb{R}^n \) (n mod 8=0) defined as

**Axiom 1.** For any elements \( a \) and \( b \), their product \( c \) is uniquely defined:

\[ c = ab. \]

**Axiom 2.** There exists the unique identity element \( e \). For any element \( a \):

\[ ae = ea = a. \]

**Axiom 3.** For any element \( a \neq 0 \), the inverse element \( a^{-1} \) is uniquely determined:

\[ aa^{-1} = a^{-1}a = e. \]

**Axiom 4.** (The weakly alternative identity.) For any elements \( a \) and \( b \):

\[ (aa)b - a(ab) = b(aa) - (ba)a. \]

**Axiom 5.** (The flexible identity.) For any elements \( a \) and \( b \):

\[ a(ba) = (ab)a. \]

**Axiom 6.** (The distributive identity.) For any elements \( a, b \) and \( c \):

\[ a(b + c) = ab + ac, \quad (b + c)a = ba + ca. \]
Build one of such the algebras. Suppose that the vector space $\mathbb{R}^n$ is equipped with the metric $(i, j, ..., = 1, n)$

$$<a, b > e := \frac{1}{2}(a\bar{b} + b\bar{a}), \quad a = a_0 e + \sum_{r=1}^{n-1} a_r e_r, \quad \bar{a} = a_0 e - \sum_{r=1}^{n-1} a_r e_r,$$

(1)

Let’s consider that the metric $g_{ij}$ is Euclidean metric: $<a, b> = \delta_{ij}a^ib^j$ in the special orthogonal basis.

$$a^{-1} = \frac{\bar{a}}{<a, a>}, \quad \forall a \neq 0,$$

$$a + \bar{a} = 2a_0 e = 2 <a, e > e,$$

$$a(a + \bar{a}) = 2 <a, e > a,$$

$$a^2 = - <a, a > e + 2 <a, e > a,$$

(2)

At the same time, $\eta$ are executed by definition for all $a$ and $b$, where $\eta$ are the structural constants of the unital algebra with the algebra identity $e = \frac{1}{\sqrt{2}}\eta^k$ and the distributive identity. Such the algebra we name the orthogonal hypercomplex algebra $\mathbb{A}$.

**Theorem 1.** The orthogonal hypercomplex algebra will be the metric hypercomplex group alternative-elastic algebra.

**Proof.** The equations (3a) are a consequence of the Clifford equation for $n \mod 8 = 0$ [2], [3] $(A, B, ..., = 1, N, \quad N = 2^{2r-1})$

$$\eta_i^{AB}\eta_{jCB} + \eta_j^{AB}\eta_{iCB} = g_{ij}\delta^{CB}.$$

(4)

At the same time, $\eta_i^{AB} = \sum_{I=0}^{n-1} \frac{1}{2}(\eta_I)_i(\varepsilon_I)^{AB}$. Among the tensor $(\varepsilon_I)^{AB}$ it has only one symmetric tensor $((\varepsilon_0)^ {AB} = : \varepsilon^{AB} -$ the metric spin-tensor), the remaining tensors are antisymmetric tensors. Then the structure constants of the orthogonal hypercomplex algebra $\mathbb{A}$ have the form

$$\eta_{ij}^k := \sqrt{2}\eta_i^{AB}\eta_{j\varepsilon AB}\eta_k^{CD}\theta^{CD}, \quad \bar{X}^{A'} = S_A^{A'}X^A,$$

(5)

where $\theta^{CD}(\varepsilon_{CD}, \varepsilon_{CD} = 2)$ is an arbitrary symmetric tensor, and $S_A^{A'}$ is generated by the real inclusion $H_\varepsilon^A : \mathbb{R}^n \rightarrow \mathbb{C}^n (\Lambda, \Psi, ..., = 1, n)$ [2], [3]. Then $e = \frac{1}{\sqrt{2}}\eta^k (\eta^k := (\eta_0)^k)$. Therefore,

$$\eta_{ij}^k := \sum_{I=0}^{n-1} \frac{1}{\sqrt{2}}(\eta_I)_i(g_I)_j^k$$

(6)
where \( g_{ij} := (g_0)_{ij} \), and the remaining tensors \( (g_1)_{ij} \) are antisymmetric tensors. Indeed,
\[
\eta^{(i)(j)^k} := \sqrt{2}(\eta^{iAB}\eta^{(j)(CA)}\delta^{k}_{DB}\theta^{CD} - \eta^{iAB}\eta^{(j)(AC)}\delta^{k}_{DB}\theta^{CD}) = 2 \left( \frac{1}{\sqrt{2}} \eta^{(i)}(\delta_j)^k - g_{ij} \left( \frac{1}{\sqrt{2}} \eta^k \right) \right)
\]
then the axioms 1-3, identity (3a) are executed. Analogically,
\[
\eta^{(i)(j)(k)} = \sqrt{2} \eta^{iAB}\eta^{(j)(CA)}\eta^{(k)}_{DB}\theta^{CD} = \left( \frac{1}{\sqrt{2}} \eta^{(i)} \right) g_{ik} = \sqrt{2}(\eta^{iAB}\eta^{(j)(CA)}\eta^{(k)}_{DB}\theta^{CD} - \eta^{iAB}\eta^{(j)(AC)}\eta^{(k)}_{DB}\theta^{CD}) = \sqrt{2}\eta^{(j)g_{ik}} + \frac{1}{2}\eta^{g_{jk}} = \left( \frac{1}{\sqrt{2}} \eta^{(j)} \right) g_{jk}
\]
then the identity (3b) is executed,
\[
\eta^{(j)\eta^{(k)}} = \sqrt{2}(\eta^{iAB}\eta^{(j)(CA)}\eta^{(k)}_{DB}\theta^{CD} - \eta^{iAB}\eta^{(j)(AC)}\eta^{(k)}_{DB}\theta^{CD}) = 2 \eta^{iAB}\eta^{(j)(CA)}\theta^{BC} - \eta^{iAB}\eta^{(j)(AC)}\theta^{BC} = 0,
\]
then the identity (3c) is executed as a consequence from (3d). Indeed \( \varepsilon^{[ABCD]} := \eta^{AB}\eta^{CD} \square, \\eta^{XY} = \theta^{XY} \)
\[
\frac{1}{2}\eta^{i(m}\eta^{(j)k]} = \eta^{iAB}\eta^{(j)(CA)}\eta^{(k)_{DB}\theta^{CD} = \eta^{iAB}\eta^{(j)(CA)}\theta^{BC} - \eta^{iAB}\eta^{(j)(AC)}\theta^{BC} = 0,
\]
then the identity (3d) is executed as a consequence from (3e). Indeed \( \varepsilon^{[ABCD]} \)
\[
\frac{1}{2}\eta^{i(m}\eta^{(j)k]} = \eta^{iAB}\eta^{(j)(CA)}\eta^{(k)_{DB}\theta^{CD} = \eta^{iAB}\eta^{(j)(CA)}\theta^{BC} - \eta^{iAB}\eta^{(j)(AC)}\theta^{BC} = 0,
\]
Otherwise, from (3)
\[
\frac{1}{2}((ab)a + a(ab)) = - <a, (ab) > e + < a, e > (ab) + < (ab), e > a = \quad<br, e, < a, a> <a, b, e, < a, a> <a, b, e, < a, a>
\]
\[
= - < a, b + a > e + < a, e > (ab) + < b, e > ( - < a, a + e > 2 < a, e > a) = a (- < a, b > e + < a, e > b + < b, e > a),
\]
\[
(a) = a (ab),
\]
There is the alternative-elastic identity

\[((aa)b) - (a(ab)) - (a(ba)) = ((b(aa)) - ((ba)a) - ((ab)a)).\]  \hspace{1cm} (11)

\[((aa)b) - (a(ab)) - (a(ba)) = - < a, a > b + 2 < a, e > (ab) - \\
- 2(a(- < a, b > e + < a, e > b+ < b, e > a)) = \\
= - < a, a > b + 2 < a, b > a - 2 < b, e > (- < a, a > e + 2 < a, e > a) = \\
= 2 < b, e > < a, a > e + (2 < a, b > - 4 < b, e > < a, e >) - < a, a > b,

\[(b(aa)) - ((ba)a) - ((ab)a) = - < a, a > b + 2 < a, e > (ba) - \\
- 2((- < a, b > e + < a, e > b+ < b, e > a) = \\
= - < a, a > b + 2 < a, b > a - 2 < b, e > (- < a, a > e + 2 < a, e > a) = \\
= 2 < b, e > < a, a > e + (2 < a, b > - 4 < b, e > < a, e >) - < a, a > b,

then the axioms 4-5 are executed too. Note that the equations (3) provide the execution of the axiom 3-5, the equations (5), (4) provide the execution of the axiom 1-2,6 and the equations (3). From (3) the common Jordan identity

\[a^k(ba^l) = (a^kba^l)\]  \hspace{1cm} (13)

follows.

In addition,

\[g_{k\ell}η_{[i]j}^kη_{[m]}^{jr} = g_{kr}η_{j[i]}^rη_{[m]}^{jr}.\]  \hspace{1cm} (14)

This identity follows from (3) and it is called the weakly normalization identity. This identity is the normalization identity for \(n=8\) only \((g_{kr}η_{j[i]}^rη_{[m]}^{jr} = g_{jm}g_{it}\) in this case). And so this algebra is normalized [2], [3].

**Theorem 2.** The metric real numbers, complex numbers, quaternions, octonions, sedenions, hypercomplex Cayley-Dickson numbers possess the identities (3).

**Proof.** Let \(r = \frac{1}{n-1}\) then for the Euclidean metric \(δ_{ij} \forall x\)

\[x = x_0 e + \sum_{r=1}^{n-1} i_r x_r, \quad \bar{x} = x_0 e - \sum_{r=1}^{n-1} i_r x_r,\]  \hspace{1cm} (15)

\[i_r i_s = -i_s i_r, \quad i_r i_s = -e, \quad ei_r = i_r e = i_r, \quad e e = e, \quad < i_r, e > = 0.\]

Let \(x := a + bi, \ y := c + di\) where \(i := i_{n/2} (r = 1, n/2 - 1)\) then for the Euclidean metric \(δ_{ij} \forall a, b\)

\[a = a_0 e + \sum_{r=1}^{n/2-1} a_r i_r, \quad i e = i, \quad i_r i = i_{r+n/2}, \quad < e, ai >= 0.\]  \hspace{1cm} (16)

**Definition 2.** (according to [6, pp.300-303]) 1. Let’s define the multiplication for an inductive step according to the Cayley-Dickson double procedure as

\[a(bi) = (ba)i, \quad (ai)b = (ab)i, \quad (ai)(bi) = -ba.\]  \hspace{1cm} (17)

2. Let’s define the conjugation for an inductive step according to the Cayley-Dickson double procedure as

\[\bar{a} + bi = \bar{a} - bi.\]  \hspace{1cm} (18)
Metric hypercomplex Cayley-Dickson algebra possesses the following identities.

1. Set ∀a, c : \( \frac{1}{2}(ac + ca) = < a, c > e \) by the induction then the following identity is received

\[
\frac{1}{2}(x\bar{y} + y\bar{x}) = < x, y > e,
\]

\[
(x\bar{y} + y\bar{x}) = (a + bi)(\bar{c} - di) + (c + di)(\bar{a} - bi) = a\bar{c} + db + (bc - da)i + c\bar{a} + bd - (bc - da)i = 2(< a, c > e + < d, b > e) = 2 < x, y > e.
\]

2.

\[
\frac{1}{2}(yx + xy) = -\frac{1}{2}(y\bar{x} + x\bar{y}) + < x, e > y + < y, e > x = -< x, y > e + < x, e > y + < y, e > x.
\]

3.

\[
2 < ai, b > = (ai)\bar{b} + b(a\bar{i}) = (ai)\bar{b} - b(ai) = (ab)i - (ab)i = 0.
\]

4. Set ∀a, c : \( \overline{ac} = \bar{c}\bar{a} \) by the induction then the following identity is received

\[
\overline{xy} = \frac{(a + bi)(c + di) - ac - db + (bc + da)i}{(\bar{c}a - bd) - (\bar{b}c + da)i} = \overline{\bar{y}x}.
\]

5. Set ∀a, c : \( < ac - ca, e > = 0 \) by the induction then the following identity is received

\[
< xy - yx, e >= < ((ac - db) + (bc + da)i) - ((ca - \bar{bd}) + (bc + da)i), e >=
\]

\[
= < (ac - ca) + (-b + 2 < b, e >)d - (-d + 2 < d, e >)b, e >= 0.
\]

6. Set ∀a, c : \( < ac, a >= < ca, a >= < ac, a >= < aa, e > < a, a > \) by the induction then the following identity is received

\[
< xy, x >= < y, e > < x, x >,
\]

\[
< xy, x > =< ac - \bar{db} + (\bar{bc} + da)i, a + bi >=
\]

\[
= < ac, a > + < bc, b > + < da, b > + < db, a > - 2 < d, e > < b, a > = -< d(a - b), (a - b) > + < d, a > + < d, e > < b, b > - 2 < d, e > < b, a > = 0
\]

\[
= < y, e > < x, x >.
\]

On the other hand, on the base of the corollary 8.2 \[2, \ 3\] it is executed

\[
\eta_{ij}^k := \sum_{l=0}^{n-1} \left( \frac{1}{\sqrt{2}}(\eta_l)^k \right) \left( -3(h_l)_{ij} + (\eta_{ij}^k)_{ij} \right) =
\]

\[
\sqrt{2} \eta_{AB} \eta_{jCA} \eta_{DB} \frac{2\varepsilon CD}{N} - 3 \sum_{l=1}^{n-1} \left( \frac{1}{\sqrt{2}}(\eta_l)^k \right) (h_l)_{ij} =
\]

\[
= \sqrt{2}(2 - n) \eta_{AB} \eta_{jCA} \eta_{DB} \frac{2\varepsilon CD}{N} + \sum_{l=1}^{n-1} \left( -3 \left( \frac{1}{\sqrt{2}}(\eta_l)^k \right) (h_l)_{ij} + \sqrt{2} \eta_{AB} \eta_{jCA} \eta_{DB} \frac{2\varepsilon CD}{N} \right)
\]

\[
(25)
\]
where \( g_{ij} =: 3(h_0)_{ij} \), and the remaining tensors \((h_I)_{ij}\) are antisymmetric tensors. But \((h_I)_{ij}\) are not arbitrary tensors, there are the compatibility conditions

\[
\eta_{ij}^{(k)} = \left( \frac{1}{\sqrt{2}} \eta_j^k \right) g_{ik}, \quad \eta_{i(j)}^{(k)} = \left( \frac{1}{\sqrt{2}} \eta_i^k \right) g_{jk},
\]

\[
\sum_{i=1}^{n-1} \left( \left( \frac{1}{\sqrt{2}} \eta_i \right) (h_I)_{ij} \right) = 0, \quad \sum_{i=1}^{n-1} \left( \left( \frac{1}{\sqrt{2}} \eta_i \right) (h_I)_{ij} \right) = 0.
\]

(26)

So the equation (25) takes the form

\[
\eta_{ij}^{(k)} := \sum_{i=0}^{n-1} \left( \frac{1}{\sqrt{2}} \eta_i \right) (-3(h_I)_{ij} + \eta_{ij} (\eta_1)_{ij}) = \sqrt{2}(2 - n) \eta_i \eta_j \eta_{ij} \eta_{il} \epsilon_{il}^{CD} + \sum_{i=1}^{n-1} \left( \left( \frac{1}{\sqrt{2}} \eta_i \right) (h_I)_{ij} \right) - \left( \left( \frac{1}{\sqrt{2}} \eta_i \right) (h_I)_{ij} \right) + \sqrt{2} \eta_i \eta_j \eta_{ij} \eta_{il} \epsilon_{il}^{CD}.
\]

(27)

**Note 1.** Since for any special (non-special) orthogonal transformation \( S^i_j \) according to corollary 8.3 [2], [3], the equation

\[
S^i_j \eta_{ij}^{AB} = \eta_{ij}^{CD} S^i_j A B C D (S^i_j \eta_{ij}^{AB} = \eta_{ij}^{CD} S^i_j A B C D)
\]

(28)

is executed then any special (non-special) orthogonal transformation \( S^i_j \) keeping the algebra identity \((S := \tilde{S} = \tilde{S})\) will transform the structural constants as \( S^i_j S^m_i \eta_j^{k} S_{r_k} S^r_i \) that generates the transformation of the controlling spin-tensor \( \theta^{AB} \rightarrow \theta^{CD} S^A C S B D \), \((\theta^{AB} \rightarrow 4 \epsilon^{AB} - \theta^{CD} S^A C S B D)\) keeping without a change \( \eta_i^{CD} \) from [3].

**Definition 3.** Hypercomplex orthogonal algebra \( \mathbb{A} \) is called the homogenous algebra if the orthogonal transformations \( S_I \) exist for all \( I: (h_I)_{ij} = \alpha_i (S_I)_{ij} h_0 (h_{gen})_{ml} (S_I)_{ji} \), \((\eta_I)_i = (S_i)_i m (h_{gen})_m (\alpha_I \in \mathbb{R}, I = 1, n - 1)\).

So, in order to construct an hypercomplex orthogonal homogenous algebra \( \mathbb{A} \), the algebra identity \( \frac{1}{\sqrt{2}} \eta_j^k \) and generator \( \frac{1}{\sqrt{2}} \eta_{ij}^k \) is necessary to know. Then from this generator using orthogonal transformations keeping the algebra identity, \( n - 1 \) basic elements (27) are constructed.

\[
\eta_{ij}^k := \sum_{i=0}^{n-1} \alpha_i \left( \sqrt{2} \eta_i \eta_j \eta_{ij} \eta_{il} \epsilon_{il}^{CD} \right) + \sum_{i=1}^{n-1} \left( \left( \frac{1}{\sqrt{2}} \eta_i \right) (h_I)_{ij} \right) - \left( \left( \frac{1}{\sqrt{2}} \eta_i \right) (h_I)_{ij} \right) + \sqrt{2} \eta_i \eta_j \eta_{ij} \eta_{il} \epsilon_{il}^{CD}.
\]

(29)

Obviously, the equation (29) is a decomposition of the hypercomplex orthogonal homogenous algebra \( \mathbb{A} \) \( \eta_{ij}^k \) on an algebraic basis \( \mathbb{A}_I \) \( \eta_{ij}^k \). In other way, \( \eta_{ij}^k := \sqrt{2} \eta_i \eta_j \eta_{ij} \eta_{il} \epsilon_{il}^{CD} \). Define \( \theta^{CD} := \sum_{i=0}^{n-1} \alpha_i \left( \theta_I \right) \eta_{ij}^k \) then \( \eta_{ij}^k \) determine the hypercomplex orthogonal homogenous algebra \( \mathbb{A} \) according to [5].

Consider the algorithm 9.1 [2], [3] based on the Bott periodicity [4]:
Algorithm 1. 1. Λ,... = 1, n, i,... = 1, n, A,... = 1, 2^{n-1}, α,... = 1, n + 6, a,... = 1, 2^{n-1}. Suppose there is an orthogonal algebra Λ with the structural constants generated from the connection operators η^A with the metric spinor ε^XZ and the inclusion operator H^A. We assume that the metric tensor g_{AB} on the main diagonal contains «+» only. Then we can construct the antisymmetric operators for the space C^{n+6}

$$\eta_{ab} = -\eta_{ba} := \begin{pmatrix} 0 & 0 & 0 & \xi \eta^A Q & 0 & \gamma \eta^A K & -\alpha \eta^{AD} & \eta^A_{AB} \\ 0 & 0 & -\xi \eta^{CR} & 0 & -\gamma \eta^{CM} & 0 & -\alpha \eta_{NM} & (\eta^T)_{ACD} \\ 0 & \xi \eta^{NY} & 0 & 0 & -\alpha \eta_{NM} (\eta^T)_{\Lambda N K} & 0 & \beta \eta_{CB} & \delta \eta_{NB} \\ -\xi \eta^{LZ} & -\gamma \eta^{PY} & \alpha \eta^{PR} & -\xi \eta^{NZ} & 0 & 0 & 0 & \delta \eta_{PB} \\ 0 & -\gamma \eta^{SZ} & 0 & -\eta \Lambda R & -\xi \eta^{QZ} & 0 & 0 & \xi \eta^{SD} \\ \alpha \eta^{XZ} & -\eta \Lambda X Y & 0 & -\gamma \eta^{XQ} & 0 & -\xi \eta^{XK} & 0 & 0 \\ -(\eta^T)_{\Lambda T} Z & -\beta \eta^{TY} & -\delta \eta^{TR} & 0 & \xi \eta^{TM} & 0 & 0 & 0 \end{pmatrix},$$

(30)

$$\eta_{a b} = -\eta_{b a} := \begin{pmatrix} 0 & 0 & 0 & \xi \eta^A Q & 0 & -\delta \eta^A K & -\beta \eta_{AD} & \eta_{AB} \\ 0 & 0 & -\xi \eta^{CR} & 0 & \xi \eta^{CM} & 0 & (\eta^T)_{ACD} & \alpha \eta_{CB} \\ -\xi \eta^{NY} & 0 & 0 & -\beta \eta_{NM} & \eta\Lambda M \Lambda N K & 0 & \gamma \eta_{LD} & \xi \eta^{NB} \\ 0 & -\delta \eta^{PY} & \beta \eta^{PR} & -\xi \eta^{NZ} & 0 & 0 & \gamma \eta_{LD} & 0 \\ -\delta \eta^{SZ} & 0 & -\eta \Lambda R & -\alpha \eta^{SQ} & 0 & 0 & -\xi \eta^{SD} & 0 \\ \beta \eta^{XZ} & -\eta \Lambda X Y & 0 & -\gamma \eta^{XQ} & 0 & \xi \eta^{XK} & 0 & 0 \\ -(\eta^T)_{\Lambda T} Z & -\alpha \eta^{TY} & \xi \eta^{TR} & 0 & -\xi \eta^{TM} & 0 & 0 & 0 \end{pmatrix},$$

(31)

α := \frac{i}{2}(i\eta_{n+1} + i\eta_{n+2}), \quad \beta := \frac{i}{2}(-i\eta_{n+1} + i\eta_{n+2}),

γ := \frac{i}{2}(i\eta_{n+3} + i\eta_{n+4}), \quad δ := \frac{i}{2}(-i\eta_{n+3} + i\eta_{n+4}),

ξ := \frac{i}{2}(i\eta_{n+5} + i\eta_{n+6}), \quad ζ := \frac{i}{2}(-i\eta_{n+5} + i\eta_{n+6}).

2. Λ,... = 1, n + 8, i,... = 1, n + 8, A,... = 1, 2^{n+8-1}, α,... = 1, n + 6, a,... = 1, 2^{n+8-1}. Transition to the connection operators of the space C^{n+8} is carried out as follows:

$$\eta^A_{AB} := \left( \begin{array}{cc} \eta^A_{ab} & \phi \delta^a_d \\ \psi \delta^b_c & - (\eta^T)_{acd} \end{array} \right),$$

(32)

$$\phi := \frac{1}{2}(i\eta_{n+7} + i\eta_{n+8}), \quad \psi := \frac{1}{2}(-i\eta_{n+7} + i\eta_{n+8})$$

(33)

with the metric spinor ε^XZ := \begin{pmatrix} 0 & \delta^a_d \\ \delta^b_c & 0 \end{pmatrix}, \quad \varepsilon_{XZ} := \begin{pmatrix} 0 & \delta^a_d \\ \delta^b_c & 0 \end{pmatrix}.

Then we go to the connection operators of the space \mathbb{R}^{n+8} \subset \mathbb{C}^{n+8} using the corresponding inclusion operator. And such the operators generate the structure constants of the sedenion algebra with dimension equal to n + 8.

Note 2. In the conditions of the algorithm \( \mathbb{O} \) and the examples \( \mathbb{O} \) \( \mathbb{O} \) the algebra identity is \( \frac{1}{\sqrt{2}} \eta_{n+8} \) (or accordingly \( \frac{1}{\sqrt{2}} \eta_n \)). Therefore, for reduction of designations in conformity, it is necessary to make the redesignation: n + 8 \( \mapsto \) 0 (or accordingly n \( \mapsto \) 0).
Example 1. Let $\mathcal{A}_{\text{gen}}$ is defined by the condition on the controlling spinor

\[
X^1 := 1, \quad X^2 := 1. \tag{34}
\]

Let $\|H_{n+8}\| (i_{n+8}, \ldots = 1, n + 8, A_{n+8}, \ldots = 1, 2^{n+8} - 1)$ is identity matrix. Then

\[
(\theta_{\text{gen}})^{C_{n+8}D_{n+8}} = X^{C_{n+8}}X^{D_{n+8}}, \tag{35}
\]

\[
\eta_{i_{n+8}j_{n+8}} k_{n+8} := \sqrt{2}\eta_{i_{n+8}A_{n+8}B_{n+8}} \frac{\eta_{j_{n+8}C_{n+8}A_{n+8}X^{C_{n+8}}X^{D_{n+8}}} \eta^{k_{n+8}D_{n+8}B_{n+8}}}{:=P_{j_{n+8}A_{n+8}} := P^k_{B_{n+8}B_{n+8}}} . \tag{36}
\]

Therefore, $(i_n, \ldots = 1, n, A_n, \ldots = 1, 2^{n+8} - 1)$

\[
X_{n+8} := (X^{B_n}, 0, 0, 0, 0, 0, 0, Y_{C_n}, 0, 0, 0, 0, 0, 0), \quad X^1 = 1, \quad Y_1 = 1, \tag{37}
\]

\[
P_{j_{n+8}A_{n+8}} := H_{j_{n+8}}^{A_{n+8}} (\psi Y_{A_n}, 0, 0, -\xi X_{B_n}, 0, -\delta X_{C_n}, -\beta X_{D_n}, \eta_{A_nK_n} M_n X^K_n, Y_{L_n}, 0, 0, \gamma Y_{P_n}, 0, -\alpha Y_{R_n}, \eta_{A_nZ_n} Z_{T_n} Z_{n+8}). \tag{38}
\]

Define

\[
h_{i_{n+8}j_{n+8}} := H_{i_{n+8}}^{A_{n+8}} H_{j_{n+8}}^{B_{n+8}} (\phi \psi - \xi \xi - \gamma \delta + \alpha \beta + \eta_{A_n} A_{B_n} Y_{A_n} \eta_{C_n} C_{B_n} X^{C_n}). \tag{39}
\]

Then

\[
g_{i_{n+8}j_{n+8}} = 2h_{(i_{n+8}j_{n+8})}, \quad (h_{\text{gen}})_{i_{n+8}j_{n+8}} := 2ih_{[i_{n+8}j_{n+8}]} . \tag{40}
\]

Whence, $(i_n, \ldots = 1, n + 8, A_n, \ldots = 1, 2^{n+8} - 1)$ for all $n$ mod $8 = 0$

Table 1: The matrix table example $(h_{\text{gen}})_{ij}$.

|   | 1 | -1 |
|---|---|----|
| 1 | 1 | -1 |
| -1 | -1 | 1 |

\[
(\eta_{\text{gen}})_{ij} k := \left(\frac{1}{\sqrt{2}} (\eta_{n+8})_i \delta_j^k + \left(\frac{1}{\sqrt{2}} (\eta_{n+8})_j \delta_i^k - g_{ij} \left(\frac{1}{\sqrt{2}} (\eta_{n+8})_i \right) + \left(\frac{1}{\sqrt{2}} (\eta_{n+8})_j \right) (h_{\text{gen}})_i k - \left(\frac{1}{\sqrt{2}} (\eta_{n+8})_j \right) (h_{\text{gen}})_i k - \left(\frac{1}{\sqrt{2}} (\eta_{n+8})_j \right) (h_{\text{gen}})_i k. \tag{41}
\]

Thus, the generating algebra $\mathcal{A}_{\text{gen}}(h_{\text{gen}})$ is constructed. This algebra is unique in that, it is the generator for the metric Cayley-Dickson algebra for $n + 8 = 2^k$. And because the metric Cayley-Dickson algebra with $\eta_{ij}^k$ satisfies the equations and definition then it must have the generating algebra with, for example, $(h_{\text{gen}})_{ij} := \eta_{ij} k (\sqrt{2} (\eta_{n+8})_k), \quad h_{\text{gen}} = \alpha S h_{\text{gen}} S$ for some $\alpha \in \mathbb{R}$ where $S$ is the orthogonal transformation keeping the algebra identity.
Table 2: The multiplication table example of the algebra $A_{gen} ((\eta_{gen})_{ij}^k)$.

| $*$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_{n+1}$ | $e_{n+2}$ | $e_{n+3}$ | $e_{n+4}$ | $e_{n+5}$ | $e_{n+6}$ | $e_{n+7}$ | $e_{n+8}$ |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|------------|------------|------------|------------|------------|------------|------------|------------|
| $e_1$ | $-e_{n+8}$ | $e_{n+7}$ | | | | | | | | | | | | | | |
| $e_2$ | | $-e_{n+7}$ | $e_{n+6}$ | | | | | | | | | | | | | |
| $e_3$ | | | $-e_{n+6}$ | $e_{n+5}$ | | | | | | | | | | | | |
| $e_4$ | | | | $-e_{n+5}$ | $e_{n+4}$ | | | | | | | | | | | |
| $e_5$ | | | | | $-e_{n+4}$ | $e_{n+3}$ | | | | | | | | | | |
| $e_6$ | | | | | | $-e_{n+3}$ | $e_{n+2}$ | | | | | | | | | |
| $e_7$ | | | | | | | $-e_{n+2}$ | $e_{n+1}$ | | | | | | | | |
| $e_8$ | | | | | | | | $-e_{n+1}$ | $e_n$ | | | | | | | |
| $e_9$ | $e_n$ | $e_{n+1}$ | $e_{n+2}$ | $e_{n+3}$ | $e_{n+4}$ | $e_{n+5}$ | $e_{n+6}$ | $e_{n+7}$ | $e_{n+8}$ | | | | | | | |

**Note 3.** The controlling spinor $X^A = (1, 0, 0, 0, 1, 0, 0, 0)$ from (37) generates the oktonion algebra entirely. But tensor

$$h_{ij} := H_i^A H_j^\Psi (\eta_A^B Y_A \eta_C^B X^C), \quad X^C = (1, 0, 0, 0, 0, 0, 0, 0), \quad Y_A = (1, 0, 0, 0, 0, 0, 0, 0)$$

is generated by the equation (37) as usual. Therefore, $n = 8$ is the initial induction step.

The last paragraph of the example 1 should be clarified. Let the hypercomplex orthogonal algebra is given by the structural constant $\eta_{ij}^k$. Then these constants can be expanded according to (3) as

$$\eta_{ij}^k := (\eta_0)_{ij}^k + (\eta_a)_{ij}^k.$$  

(43)

From (29) and corollary 8.2 [2], [3]

$$(\eta_0)_{ij}^k := (\frac{1}{\sqrt{2}} \eta_j)\delta_i^k + (\frac{1}{\sqrt{2}} \eta_i)\delta_j^k - g_{ij}(\frac{1}{\sqrt{2}} \eta^k) = \sqrt{2} \eta^A B \eta^{CA} \eta^k_{DB} \frac{2 \eta^{CD}}{(\theta_0)^{CD}},$$

$$\eta_{(ij)k} = (\eta_0)_{(ij)k}, \quad \eta_{(ij)[k]} = (\eta_0)_{(ij)[k]}, \quad \eta_{(jk)k} = (\eta_0)_{(jk)k}, \quad (\eta_a)_{(ij)k} = 0,$$

(44)

Define

$$(\eta_a)_{ij}^k := \pm \sqrt{2} \eta^{AB} \eta_{CA} \eta^k_{DB} (\theta_a)^{CD} = -\sqrt{2} \eta^A \eta^{AB} \eta_{CA} \eta^k_{DB} (\theta_a)^{CD}.$$  

(45)

This equation always has the particular solution

$$(\theta_a)^{CD} = (\theta_a)^{DC} := \frac{4}{3 \sqrt{2N}} (\eta_a)_{lm} \eta^l_{XY} \eta^m_{XC} \eta^r_{DY} = \frac{4}{3 \sqrt{2N}} (\eta_a)_{lm} \eta^l_{XY} \eta^m_{XC} \eta^r_{DY}.$$  

(46)
This statement follows from the equation executed for all even $n \geq 8$ (note 16.1 \[2, 3\])

\[
\eta_{ij}^{[AB} \eta_{k]}^{AC} \eta_{m}^{[XC] \eta_{l]}^{XY} \eta_{r}^{DY} \eta_{kDB} = \frac{\sqrt{2}}{n} \left( \delta_{r}^{[i] \delta_{j}^{m] \delta_{k}^{i]} - g^{k}[\delta_{j}^{m]} g_{i} r] + \frac{\sqrt{2}}{n} \delta_{r}^{k} \delta_{i}^{[i] \delta_{j}^{m]} \right),
\]

(47)

And this identity is a consequence of the identities (16.28) and (16.31) \[2, 3\]. Thus,

\[
\theta^{CD} := (\theta_{0})^{CD} + (\theta_{a})^{CD}
\]

(48)

that proves the theorem.

**Theorem 3.** Every hypercomplex orthogonal algebra $A$ admits the decomposition \[2\].

**Note 4.** The orthogonal transformations $S_{ij}^{ab}$ from the group $O(n, \mathbb{R})$ ($SO(n, \mathbb{R})$) generate the pinor (spinor) transformations $S_{A}^{B}$ from the group $pin(n, \mathbb{R})$ ($Spin(n, \mathbb{R})$) which are allocated with the real structure by the involution $S_{A}^{B'}$ according to (6.41) from \[2, 3\]. The pinor (spinor) transformations represent the subgroup of the orthogonal group $O_{\mathbb{R}}(2^{\frac{n}{2}} - 1, \mathbb{C})$ ($SO_{\mathbb{R}}(2^{\frac{n}{2}} - 1, \mathbb{C})$) (in the sense $S_{A}^{B}S_{C}^{D} \varepsilon_{BD} = \varepsilon_{AC}$, $\tilde{S}_{A}^{B'} = \tilde{S}_{A}^{C}S_{C}^{D}S_{D}^{B'}$ where $\varepsilon_{BD}$ is the metric spinor). The orthogonal transformations from the group $O(n, \mathbb{R})$ ($SO(n, \mathbb{R})$) keeping the algebra identity cause the transformations of the controlling spinor $\theta^{CD}$ without changing the connection operators $\eta_{ij}^{AB}$. Therefore, the quotient group $O_{\mathbb{R}}(2^{\frac{n}{2}} - 1, \mathbb{C})/pin(n, \mathbb{R})$ ($SO_{\mathbb{R}}(2^{\frac{n}{2}} - 1, \mathbb{C})/Spin(n, \mathbb{R})$) will implement the classification of such the hypercomplex orthogonal homogenous algebras $A$. Besides, the classification is carried out on own values of the controlling spinor $\theta^{CD}$ \[4\] because any symmetric spinor $\theta^{CD}$ \[4\] is led to a diagonal form by the orthogonal transformation from the group $SO_{\mathbb{R}}(2^{\frac{n}{2}} - 1, \mathbb{C})$.

**Theorem 4.** Hypercomplex metric Cayley-Dickson algebra is the hypercomplex special orthogonal homogenous algebra $A$.

**Proof.** Let $x := a + bi$ where $i := i_{n/2}$. Set that any hypercomplex metric Cayley-Dickson algebra $A^{n}$ is the hypercomplex special orthogonal homogenous algebra by the induction. Then it has $(h_{gen})_{ij} := (h_{1})_{ij}$ and $(h_{I})_{ij} = \alpha_{I}(S_{I})_{ij} m(h_{gen})_{ml}(S_{I})_{j}^{l}$ ($\alpha_{I} \in \mathbb{R}$, $I = 1, \frac{n}{2} - 1$, $i, j, m, l = 0, \frac{n}{2} - 1$, $a, b, c = \frac{n}{2}, n - 1$, $\alpha, \beta = 0, n - 1$). Let the $h_{gen}$ has the form $(h_{ab} := \delta_{a}^{i} \delta_{b}^{j} h_{ij}, h_{a} := \delta_{a}^{i} h_{ij}, h_{b} := \delta_{b}^{j} h_{ij}, \delta_{a}^{i} : i_{n/2} \rightarrow i_{j})$

\[
(h_{gen})_{a\beta} = \begin{pmatrix}
(h_{1})_{ij} & 0 \\
0 & -(h_{1})_{ab} + \frac{1}{2}((\eta_{n/2})_{a}(\eta_{1+n/2})_{b} - (\eta_{1+n/2})_{a}(\eta_{n/2})_{b})
\end{pmatrix}.
\]

(49)

Then for the hypercomplex metric Cayley-Dickson algebra $A^{n}$ exists three types of the basic elements only.

1. $(h_{I})_{a\beta} = \begin{pmatrix}
(h_{I})_{ij} & 0 \\
0 & -(h_{I})_{ab} + \frac{1}{2}((\eta_{n/2})_{a}(\eta_{1+n/2})_{b} - (\eta_{1+n/2})_{a}(\eta_{n/2})_{b})
\end{pmatrix}.
\]

(50)

In this case the special orthogonal transformations $(S_{I})_{ij}^{m}$ leave motionless the identity vector $e = \frac{1}{\sqrt{2}} \eta_{0}$. Hence, analogical transformations $(S_{I})_{a}^{b}$ leave motionless
the vector \( i = ei = \frac{1}{\sqrt{2}} \eta_n/2 \). Thus, the special orthogonal transformations have the form

\[
(S_I)_{\alpha}^\beta := \left( \begin{array}{cc}
(S_I)^m_i & 0 \\
0 & (S_I)^a_c
\end{array} \right).
\] (51)

2. \[
\begin{pmatrix}
-(h_I)_{bj} - \frac{i}{2}(\eta_{n/2})_b(\eta_I)_j \\
-(h_I)_{ia} + \frac{i}{2}(\eta_I)_i(\eta_{n/2})_a
\end{pmatrix}.
\] (52)

In this case the special orthogonal transformations leave motionless the identity vector and the vector rail line \( i \) with changing the direction, convert the vector \( i_1 \) to the vector \( i_{I+n/2} \) and vice versa, and have the form \((S_I+\frac{1}{2})_{\alpha}^\beta := (S_I)_{\alpha}^\beta\) where

\[
(S_I)_{\alpha}^\beta := \begin{pmatrix}
-\frac{1}{\sqrt{2}} \delta_i^k + \frac{i}{2 \sqrt{2}}(\eta_I)_i(\eta_I)_k + \frac{i}{2}(1 + \frac{1}{\sqrt{2}})(\eta_0)_i(\eta_0)_k \\
\frac{1}{\sqrt{2}} \delta_b^j + \frac{i}{2}(1 - \frac{1}{\sqrt{2}})(\eta_{I+n/2})_b(\eta_I)_j - \frac{i}{2 \sqrt{2}}(\eta_{n/2})_b(\eta_0)_k \\
\frac{1}{2 \sqrt{2}} \delta_a^j + \frac{i}{2}(1 - \frac{1}{\sqrt{2}})(\eta_I)_i(\eta_{I+n/2})_a - \frac{i}{2 \sqrt{2}}(\eta_0)_i(\eta_{n/2})_a \\
\frac{1}{2 \sqrt{2}} \delta^c_k - \frac{i}{2 \sqrt{2}}(\eta_{I+n/2})_b(\eta_{I+n/2})_a - \frac{i}{2}(1 + \frac{1}{\sqrt{2}})(\eta_{n/2})_b(\eta_{n/2})_a
\end{pmatrix}.
\] (53)

3. In this case the special orthogonal transformation \( S_{n/2} \) leave motionless all the vectors with \( \{ \) even index \( r, \quad r < n/2; \)

odd index \( r, \quad r \geq n/2. \) Then the remaining transformations is such \(< i_1, i_1 >= -1): \)

\[
\begin{cases}
-i_1 & \rightarrow \ i_{n/2}, \quad r = 1; \\
i_{n/2} & \rightarrow \ i_1, \quad r = n/2; \\
i_{r-1} & \rightarrow \ i_r, \quad r = 2s + 1 < n/2, \quad r > 1; \\
i_{r+1} & \rightarrow \ i_{r-2}, \quad r = 2s > n/2.
\end{cases}
\] (54)

Thus, all is made necessary for technical realization of the algorithm □

Example 2. Let \( n=16. \) The algorithm □ is realized in the Appendix.

1. This article contains the file "sedenion.pas" (by the operator \( \{ \text{input} \{ \text{sedenion.pas} \} \) being a programming unit adapted to the LaTex (LaTex version this article on [http://arxiv.org/]) for the Delphi. At the same time, this file is the Appendix to this article. You must create a project with this "unit sedenion" and put on the form Button1: TButton, StringGrid1: TStringGrid (the lines 22-24).

2. At error occurrence "Stack overflow" it is necessary to adjust the line 15.

3. At the lines 146-163 the connecting operators \( \sqrt{2} \eta_i^{AB} \) for \( \mathbb{R}^8 \) \( (i, A, B = 1, 8) \) is constructed.

4. At the lines 164-166 the metric spinor \( \varepsilon_{AB} \) is constructed.

5. At the lines 167-183 the connecting operators \( \sqrt{2} \eta_i^A, \sqrt{2} \eta_i^B, \sqrt{2}(\eta^T)_i^A, \sqrt{2}(\eta^T)_i^B \) is constructed.
6. At the lines 186-251 the connecting operators multiplied by $\sqrt{2}$ for $n=14$ is constructed according to the step 1 of the algorithm $[4]$.

7. At the lines 252-289 the connecting operators for $n=16$ is constructed according to the step 2 of the algorithm $[4]$.

8. At the lines 290-294 the controlling spinors $X^A$ is constructed according to $(34)$. It should be optimized.

9. At the lines 295-303 the inclusion operators $P_j^A := \eta_{ijAB}X^A$ is constructed according to $(38)$.

10. At the lines 304-333 the structural constants $(\eta_{\text{gen}})^k_{ij} := \sqrt{2}\eta^{AB} P_{iA} P_{kB}$ is constructed according to $(5), (35)$.

11. At the lines 334-479 the basic orthogonal transformation $S_i$ is constructed according to $(29)$.

12. At the lines 480-534 the canonical sedenion structural constants is constructed according to $(29)$.

13. At the lines 535-548 the canonical sedenion structural constants is outputted.

14. At the lines 549-584 the canonical sedenion structural constants is outputted into the file.

**Table 3: The canonical sedenion multiplication table.**

| $*$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|
| $e_0$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_1$ | $e_1$ | $-e_0$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $e_7$ | $-e_6$ | $e_9$ | $-e_8$ | $e_{11}$ | $-e_{10}$ | $e_{13}$ | $-e_{12}$ | $e_{15}$ | $-e_{14}$ |
| $e_2$ | $e_2$ | $-e_3$ | $-e_0$ | $e_1$ | $-e_6$ | $e_7$ | $-e_4$ | $e_{10}$ | $-e_9$ | $e_{11}$ | $-e_8$ | $e_{12}$ | $-e_{13}$ | $e_{14}$ | $-e_{15}$ | $e_{16}$ |
| $e_3$ | $e_3$ | $e_2$ | $-e_1$ | $-e_6$ | $e_5$ | $-e_4$ | $e_{11}$ | $-e_{10}$ | $e_9$ | $-e_8$ | $e_{15}$ | $-e_{14}$ | $e_{13}$ | $-e_{12}$ | $e_{16}$ | $-e_{15}$ |
| $e_4$ | $e_4$ | $-e_5$ | $-e_6$ | $e_7$ | $-e_0$ | $e_1$ | $-e_2$ | $e_3$ | $-e_4$ | $e_{10}$ | $-e_9$ | $e_{11}$ | $-e_8$ | $e_{12}$ | $-e_{13}$ | $e_{14}$ |
| $e_5$ | $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | $e_0$ | $-e_3$ | $e_2$ | $-e_1$ | $e_{12}$ | $-e_{13}$ | $e_{14}$ | $-e_{15}$ | $e_{16}$ | $-e_{17}$ | $e_{18}$ |
| $e_6$ | $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $e_2$ | $e_3$ | $-e_0$ | $e_1$ | $-e_4$ | $e_{12}$ | $-e_{13}$ | $e_{14}$ | $-e_{15}$ | $e_{16}$ | $-e_{17}$ | $e_{18}$ |
| $e_7$ | $e_7$ | $-e_6$ | $e_5$ | $-e_3$ | $e_6$ | $-e_0$ | $e_1$ | $-e_5$ | $e_{12}$ | $-e_{13}$ | $e_{14}$ | $-e_{15}$ | $e_{16}$ | $-e_{17}$ | $e_{18}$ | $-e_{19}$ |
| $e_8$ | $e_8$ | $-e_9$ | $-e_{10}$ | $e_{11}$ | $-e_{12}$ | $e_{13}$ | $-e_{14}$ | $-e_{15}$ | $-e_{16}$ | $-e_{17}$ | $-e_{18}$ | $-e_{19}$ | $-e_{20}$ | $-e_{21}$ | $-e_{22}$ | $-e_{23}$ |
| $e_9$ | $e_9$ | $e_8$ | $-e_{11}$ | $e_{10}$ | $-e_{13}$ | $e_{12}$ | $-e_{14}$ | $-e_{15}$ | $-e_{16}$ | $-e_{17}$ | $-e_{18}$ | $-e_{19}$ | $-e_{20}$ | $-e_{21}$ | $-e_{22}$ | $-e_{23}$ |
| $e_{10}$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ | $e_{16}$ | $e_{17}$ | $e_{18}$ | $e_{19}$ | $e_{20}$ | $e_{21}$ | $e_{22}$ | $e_{23}$ | $e_{24}$ | $e_{25}$ |
| $e_{11}$ | $e_{11}$ | $e_{10}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ | $e_{16}$ | $e_{17}$ | $e_{18}$ | $e_{19}$ | $e_{20}$ | $e_{21}$ | $e_{22}$ | $e_{23}$ | $e_{24}$ | $e_{25}$ |
| $e_{12}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ | $e_{16}$ | $e_{17}$ | $e_{18}$ | $e_{19}$ | $e_{20}$ | $e_{21}$ | $e_{22}$ | $e_{23}$ | $e_{24}$ | $e_{25}$ | $e_{26}$ | $e_{27}$ |
| $e_{13}$ | $e_{13}$ | $e_{12}$ | $e_{15}$ | $e_{14}$ | $e_{16}$ | $e_{17}$ | $e_{18}$ | $e_{19}$ | $e_{20}$ | $e_{21}$ | $e_{22}$ | $e_{23}$ | $e_{24}$ | $e_{25}$ | $e_{26}$ | $e_{27}$ |
| $e_{14}$ | $e_{14}$ | $e_{15}$ | $e_{12}$ | $e_{13}$ | $e_{16}$ | $e_{17}$ | $e_{18}$ | $e_{19}$ | $e_{20}$ | $e_{21}$ | $e_{22}$ | $e_{23}$ | $e_{24}$ | $e_{25}$ | $e_{26}$ | $e_{27}$ |
| $e_{15}$ | $e_{15}$ | $e_{14}$ | $e_{13}$ | $e_{12}$ | $e_{11}$ | $e_{10}$ | $e_{9}$ | $e_{8}$ | $e_{7}$ | $e_{6}$ | $e_{5}$ | $e_{4}$ | $e_{3}$ | $e_{2}$ | $e_{1}$ | $e_{0}$ |

15. Algorithm $[4]$ is realized by the scheme "one-to-one" and therefore, it requires more memory allocation (the line 15). In order to apply it to higher dimensions, it should be optimized.
16. System characteristics of the computer on which the program is tested:
   HP Pavilion dv7-6b50er i3-2330M/4096/500/Radeon HD6770 2Gb/Win7 HP64

17. Run time <3c.
1 //The sedenion multiplication table.
2 
3 { Sedenion Unit for Delphi
4 
5 Copyright (c) 2013 Konstantin Andreev All Rights Reserved
6 THIS SOFTWARE IS PROVIDED "AS IS" WITHOUT WARRANTY OF ANY KIND: EITHER EXPRESSED OR IMPLIED,
7 INCLUDING BUT NOT LIMITED TO THE IMPLIED MERCHANTABILITY AND/OR FITNESS FOR A PARTICULAR
8 PURPOSE. KONSTANTIN ANDREEV CANNOT BE HELD RESPONSIBLE FOR ANY LOSSES, EITHER DIRECT OR
9 INDIRECT, OF ANY PARTY MAKING USE OF THIS SOFTWARE. IN MAKING USE OF THIS SOFTWARE, YOU
10 AGREE TO BE BOUND BY THE TERMS AND CONDITIONS FOUND IN THE ACCOMPANYING LICENSE.
11 
12-unit sedenion
13 {SM 16384.9000000}
14 
16 interface
17 uses
18 Windows, Messages, SysUtils, Classes, Graphics, Controls, Forms, Dialogs,
19 StdCtrls, Grids, IniFiles;
20 type
21 TForm1 = class(TForm)
22  Button1: TButton;
23  StringGrid1: TStringGrid;
24  procedure Button1Click(Sender: TObject);
25 end;
26 
27 var
28  Form1: TForm1;
29  implementation
30 
{ $R * TForm1. Button1Click(Sender: TObject)
31 
32 
33 procedure TForm1.Button1Click(Sender: TObject);
34 type
35  complex = record
36     x,y : real;
37  end;
38 
39 var
40  eta_16__ : array [1..16,1..128,1..128] of complex;
41  eta_16__T : array [1..16,1..128,1..128] of complex;
42  eta_8__ : array [1..8,1..8,1..8] of complex;
43  eta_8__T : array [1..8,1..8,1..8] of complex;
44  eta_8 : array [1..8,1..8,1..8] of complex;
45  eta_8_T : array [1..8,1..8,1..8] of complex;
46  eta_16 : array [1..16,1..16,1..16] of complex;
47  eta_16_T : array [1..16,1..16,1..16] of complex;
48  eta : array [1..16,1..128,1..128,1..16] of complex;
49  m_3 : complex;
50  i,j,k,l,m,r : integer;
51  unit_ : array [1..8,1..8,1..8] of complex;
52  x : array [1..128] of complex;
53  g : array [0..15,0..15] of complex;
54  s_orthogonal : array [1..15,0..15,1..16] of complex;
55  i,j,k,l,m,r : integer;
56  mi : complex;
57  m_1 : complex;
58  m_2 : complex;
59  m_3 : complex;
60  IniFile : TStrings;
61  array_str : string;
62  s_orthogonal : array [1..15,0..16,0..16,0..16] of complex;
63  eta_orthogonal : array [1..15,0..16,0..16,0..16,0..16] of complex;
64  eta_orthogonal : array [1..15,0..16,0..16,0..16,0..16] of complex;
65  eta_orthogonal : array [1..15,0..16,0..16,0..16,0..16] of complex;
66  eta_orthogonal : array [1..15,0..16,0..16,0..16,0..16] of complex;
67 
68 
69 //The connection operators
70 // for n=16 $\eta iA B$
71 // The connection operators
72 // for n=16 $\eta iA B$
73 // The complex factor
74 // contracted with the metric
75 // $\eta i A B$
76 // The connection operators
77 // contracted with the metric
78 // for n=8 $\eta i A B$
79 // spinor for n=8 $\eta i A B$
80 // The connection operators
81 // contracted with the metric
82 // for n=8 $\eta i A B$
83 // spinor for n=8 $\eta i A B$
84 // The connection operators
85 // contracted with the metric
86 // for n=8 $\eta i A B$
87 // spinor for n=8 $\eta i A B$
88 // The structural constant of
89 // a generating hypercomplex
90 // algebra for n=16 ($\eta_{sym}$)$A^k$ from 88
91 // The auxiliary variable
92 // for n=16 $\eta i A k$
93 // The metric spinor for n=16
94 // $X^A$
95 // The inclusion operator
96 // for n=16
97 // $P_{AB}$ =$P_{BA}$ $\eta i A X B$
98 // The controlling spinor
99 // for n=16 $X^A$
100 // The algebra identity $e = \sum \eta i$
101 // $\parallel \eta i \parallel ^2 = 1$
102 // The metric of the Euclidean
103 // space $\parallel \parallel$
104 // $g_{ij} = \parallel \parallel g_{ij} \parallel ^2 = \parallel \parallel$
105 // Indices of an array element.
106 // The complex factor $i$
107 // The complex factor $-i$
108 // The complex factor $-1$
109 // The complex factor $-12$
110 // The output file
111 // A line of the output file.
112 // The basic orthogonal transformations
113 // for n=16 ($S_i$)$l$ (l = 1..16) from note 18
114 // The structural constant
115 // of hypercomplex basic
116 // algebras for n=16 ($\eta i_A^k$)
117 // The auxiliary variable.
118 // The structural constant
119 // of the canonical sedenion
120 // algebra $\eta_A^k$.}
Addition of complex numbers.

function add(c11: c12: complex): complex;
begin
  c13.x:=c11.x+c12.x;
  c13.y:=c11.y+c12.y;
  add:=c13;
end;

Multiplication of complex numbers.

function mul(c11, c12: complex): complex;
var
  c13: complex;
begin
  c13.x:=c11.x*c12.x-y*c12.y;
  c13.y:=c11.x*y+c12.x*c12.y;
  mul:=c13;
end;

Initialization of a complex number.

procedure Init(var c13: complex);
begin
  c31.x:=0; c31.y:=0;
end;

//Initialization of the connection operators for n=8.
for i:=1 to 8 do
  for j:=0 to 15 do
  for k:=1 to 8 do
    begin
      Init(eta_8[i,j,k]);
      Init(eta_8[i,j,k]);
    end;

//Initialization the metric spinor for n=8.
for i:=1 to 8 do
  for j:=1 to 16 do
    begin
      Init(e_8[i,j]);
    end;

//Initialization the connection operators for n=16.
for i:=1 to 16 do
  for j:=1 to 16 do
    begin
      Init(eta_16[i,j,k]);
      Init(eta_16[i,j,k]);
    end;

//Initialization the orthogonal transformation for n=16.
for m:=1 to 15 do
  for i:=1 to 16 do
    begin
      Init(s_orthogonal[m,i,j]);
    end;

//Initialization the sedenion algebra identity for n=16.
for i:=0 to 15 do
  begin
    Init(unit[i]);
  end;

//Initialization the metric tensor.
for i:=0 to 15 do
  begin
    for j:=0 to 15 do
      begin
        Init(g[i,j]);
      end;

//Construction the connection operators (generating the octonion algebra) for n=8 (multiplied by \sqrt{2}).
begin
  eta_8[1,2].y:=+1;
  eta_8[1,3,4].y:=+1;
  eta_8[1,5,6].y:=-1;
  eta_8[1,7,8].y:=-1;

  eta_8[2,1,2].x:=+1;
  eta_8[2,3,4].x:=+1;
  eta_8[2,5,6].x:=+1;
  eta_8[2,7,8].x:=+1;

  eta_8[3,1,2].y:=+1;
  eta_8[3,2,4].y:=+1;
  eta_8[3,5,6].y:=+1;
  eta_8[3,7,8].y:=+1;

  eta_8[4,1,2].x:=+1;
  eta_8[4,2,4].x:=+1;
  eta_8[4,5,7].x:=+1;
  eta_8[4,6,8].x:=+1;

  eta_8[5,1,4].y:=+1;
  eta_8[5,2,3].y:=+1;
  eta_8[5,5,8].y:=+1;
  eta_8[5,6,7].y:=+1;

  eta_8[6,1,4].x:=+1;
  eta_8[6,2,3].x:=+1;
  eta_8[6,5,8].x:=+1;
  eta_8[6,6,7].x:=+1;

end;

//Construction the connection operators contracted with the metric spinor for n=8 (multiplied by \sqrt{2}).
begin
  for i:=1 to 8 do
    for j:=1 to 8 do
      for k:=1 to 8 do
        begin
          eta_8[i,j,k] := add(eta_8[i,j,k], mul(eta_8[i,j,m], e_8[m,k]));
          eta_8[i,j,k] := add(eta_8[i,j,k], mul(eta_8[i,j,k], e_8[m,k]));
          eta_8[i,j,k] := add(eta_8[i,j,k], mul(eta_8[i,j,k], e_8[m,k]));
        end;

end;
The constant factors: \( (\mu_i, -i(x_i), -1(m_i) \)

\[ \text{Construction the connection operators for } n=14 \]

for \( i=1 \) to \( 8 \) do
begin
eta\[16\][i, j, k] := \( \mu_i \) \( \mu_i \) ;
end;

3. \( n=15,16 \)

for \( i=1 \) to \( 8 \) do
begin
eta\[16\][i, j, k] := \( \mu_i \) \( \mu_i \) ;
end;

The skew-symmetry.

2. \( n=9,10 \)

for \( i=1 \) to \( 8 \) do
begin
eta\[8\][i, j, k] := \( \mu_i \) \( \mu_i \) ;
end;

// Construction the connection operators for \( n=16 \) from \( 14 \) (multiplied by \( \sqrt{2} \)).
for i=1 to 16 do
for j=1 to 128 do
for k=1 to 128 do
begin
eta_16[i,j,k].x:=eta_16[i,j,k].x/sqrt(2);
eta_16[i,j,k].y:=eta_16[i,j,k].y/sqrt(2);
eta_16[i,j,k].z:=eta_16[i,j,k].z/sqrt(2);
end;
end;
//Initialization the controlling spinors X^A from (6)
for j=1 to 128 do
Init_(x[j]);
x[1..64]:=1;
//Construction of the inclusion operators P_jA from (8)
for i=1 to 4 do
for i=1 to 16 do
for j=1 to 128 do
begin
Init_(P[i,j]);
for k=1 to 128 do
P[i,j]:=add(P[i,j],mul(eta_16[i,k,j],x[k]));
end;
//Construction of the structure constants of the generating algebra for n=16 \eta_{gen}(\eta_{gen})_G from (9) (10).
//1. Multiply the connection operators by P.
begin
for i=1 to 16 do
for j=1 to 16 do
for k=1 to 16 do
begin
eta[i,j,k].x:=0;
eta[i,j,k].y:=0;
eta[i,j,k].z:=0;
end;
end;
//2. Multiply by \sqrt{\tau}
for i=1 to 16 do
for j=1 to 16 do
for k=1 to 16 do
begin
eta[i,j,k].x:=eta[i,j,k].x*sqrt(2);
eta[i,j,k].y:=eta[i,j,k].y*sqrt(2);
end;
//Initialization the basic orthogonal transformations (S_t)_{ij}^k (i=1,16) according to note(2).
end;
end;
end;
//3.
s_orthogonal[1,3,1]:=1;
s_orthogonal[1,2,2]:=1;
s_orthogonal[1,5,3]:=1;
s_orthogonal[1,4,4]:=1;
s_orthogonal[1,7,5]:=1;
s_orthogonal[1,6,6]:=1;
s_orthogonal[1,9,7]:=1;
s_orthogonal[1,8,8]:=1;
s_orthogonal[1,11,9]:=1;
s_orthogonal[1,10,10]:=1;
s_orthogonal[1,13,11]:=1;
s_orthogonal[1,12,12]:=1;
s_orthogonal[1,15,13]:=1;
s_orthogonal[1,14,14]:=1;
s_orthogonal[1,17,15]:=1;
s_orthogonal[1,16,16]:=1;
end;
//4.
for i=1 to 16 do
for j=1 to 16 do
begin
eta[i,j].x:=eta[i,j].x/sqrt(2);
eta[i,j].y:=eta[i,j].y/sqrt(2);
eta[i,j].z:=eta[i,j].z/sqrt(2);
end;
end;
end;
end;
end;
end;
end;
end;
end;
end;
end;
1. Constructing the structural constants of the basic algebras

Constructing the structural constant of the canonical sedenion algebra for n=16.

```
begin
eta_orthogonal[m, i, j, k] := 0;
for l=1 to 16 do
  eta_orthogonal[m, i, j, k] := add(eta_orthogonal[m, i, j, k], mul(s_orthogonal[m, j, l], eta_orthogonal[m, l, j, k]));
for k=0 to 15 do
  begin
    for j=0 to 15 do
      begin
        for i=1 to 16 do
          for k=0 to 15 do
            begin
              m_3.y := 0;
              begin
                for i=1 to 16 do
                  add(eta_constant[i, j, k], eta_orthogonal[m, i, j, k]);
                for k=0 to 15 do
                  begin
                    for j=0 to 15 do
                      begin
                        for i=1 to 16 do
                          begin
                            m_12.y := 0;
                            begin
                              for j=0 to 15 do
                                begin
                                  for i=1 to 16 do
                                    begin
                                      m_12.x := 1;
                                      begin
                                        for j=0 to 15 do
                                          begin
                                            for i=1 to 16 do
                                              begin
                                                m_3.x := 1/3;
                                                begin
                                                  for k=0 to 15 do
                                                    begin
                                                      for j=0 to 15 do
                                                        begin
                                                          for i=1 to 16 do
                                                            begin
                                                              eta_constant[i, j, k] := add(eta_constant[i, j, k], eta_orthogonal[m, i, j, k]);
                                                            end;
                                                          end;
                                                        end;
                                                      end;
                                                    end;
                                                  end;
                                                end;
                                              end;
                                            end;
                                          end;
                                        end;
                                    end;
                                  end;
                                end;
                            end;
                          end;
                      end;
                    end;
                  end;
```
IniFile.WriteString('Вариант_0000000*
array_str="
|________________________|
|                       |
|________________________|
for i:=0 to 15 do
begin
array_str="
for j:=0 to 15 do
begin
for k:=0 to 15 do
if Round(eta_constant[i,j,k].x)>0 then
array_str:=array_str+Format('%12s',[IntToStr(Round(eta_constant[i,j,k].x)]
'x'+chr(36)+'e_{'+IntToStr(k)+'}'+chr(36)]);
if j<15 then array_str:=array_str+'&'
end;
if i<10 then
IniFile.WriteString('Вариант_0'+chr(36)+'e_{'+IntToStr(i)+'}'+chr(36),array_str+'\newline')
else
IniFile.WriteString('Вариант_0'+chr(36)+'e_{'+IntToStr(i)+'}'+chr(36),array_str+'\newline');
end;
end;
end.
//
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