The Resource Theory of Quantum States Out of Thermal Equilibrium

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(Dated: November 17, 2011)

The ideas of thermodynamics have proved fruitful in the setting of quantum information theory, in particular the notion that when the allowed transformations of a system are restricted, certain states of the system become useful resources with which one can prepare previously inaccessible states. The theory of entanglement is perhaps the best-known and most well-understood resource theory in this sense. Here we return to the basic questions of thermodynamics using the formalism of resource theories developed in quantum information theory and show that the free energy of thermodynamics emerges naturally from the resource theory of energy-preserving transformations. Specifically, the free energy quantifies the amount of useful work which can be extracted from asymptotically-many copies of a quantum system when using only reversible energy-preserving transformations and a thermal bath at fixed temperature. The free energy also quantifies the rate at which resource states can be reversibly interconverted asymptotically, provided that a sublinear amount of coherent superposition over energy levels is available, a situation analogous to the sublinear amount of classical communication required for entanglement dilution.

Quantum resource theories are specified by a restriction on the quantum operations (state preparations, measurements, and transformations) that can be implemented by one or more parties. This singles out a set of states which can be prepared under the restricted operations. If the parties facing the restriction acquire a quantum state outside the restricted set of states, then they can use this state to implement measurements and transformations that are outside the class of allowed operations, consuming the state in the process. Thus, such states are a resource.

A few prominent examples serve to illustrate the idea: if two or more parties are restricted to communicating classically and implementing local quantum operations, then entangled states become a resource [1]; if a party is restricted to quantum operations that have a particular symmetry, then states that break this symmetry become a resource [2, 4]; if a party is restricted to preparing states that are completely mixed and performing unitary operations, then any state that is not completely mixed, i.e. any state that has some purity, becomes a resource [3].

In this Letter we develop the quantum resource theory of athermal states. This provides a useful new formulation of nonequilibrium thermodynamics for finite-dimensional quantum systems, and allows us to apply new mathematical tools to the subject. The restricted class of operations which defines our resource theory are those that can be achieved through energy-conserving unitaries and the preparation of any ancillary system in a thermal state at temperature \( T \), as first studied by Janzing et al. [6] in the context of Landauer’s principle. Here the ancillary systems can have an arbitrary Hilbert space and an arbitrary Hamiltonian, and may be described as having access to a single heat bath at temperature \( T \). States that are not in thermal equilibrium, that is, which are athermal, are the resource in this approach.

Quantum resource theories provide answers to questions such as: how does one measure the quality of different resource states? Can one particular resource state be converted to another deterministically? If not, can it be done nondeterministically, and if so with what probability? What if one has access to a catalyst? A particularly fundamental problem, addressed in this Letter, is to identify the equivalence classes of states that are reversibly interconvertible in the limit of asymptotically-many copies of the resource and to determine the rates of interconversion. We show that all nonthermal states are reversibly interconvertible asymptotically and that the free energy of a state is the quantity governing the rate of interconversion.

It is worth pausing to consider the merits of the resource theory approach. The examples of entanglement, asymmetry and purity demonstrate that studying a given property of quantum states as a resource is a very efficient way of coming to understand it better. Naïve intuitions and preconceptions about such properties are typically replaced by a more refined view through the simple process of solving the variety of quantitative and operationally well-defined problems presented by the resource paradigm. Another feature of such an approach
is its generality: rather than considering the behavior of the property of interest for some particular system with particular dynamics (as is typical in thermodynamics), one considers instead the fundamental limits that are imposed by the restriction defining the resource and the laws of quantum theory.

On the practical side, a better understanding of a given resource helps to determine how best to implement the tasks that make use of it, and, more fundamentally, such an understanding may serve to clarify what sorts of resources are even relevant for a given task [7]. Finally, a resource theory approach can provide a framework for organizing, synthesizing, and consolidating the results in a given field, and thermodynamics is a field that is particularly in need of such a framework [8]. In particular, studying the interconvertibility of finite resources leads to useful notions of free energy in that case, as shown by two of us in [9]. Similar results were reported by Åberg [10], who investigated the work extractable from finite resources.

Allowed Operations & Resource States.— We now define the restricted class of operations and the resource states more precisely. Given a quantum system $s$ with Hilbert space $H_s$ and Hamiltonian $H_s$, the restricted operations are the completely-positive trace-preserving (CPTP) maps $\mathcal{E} : \mathcal{L}(H_s) \rightarrow \mathcal{L}(H_s)$ of the form

$$\mathcal{E}(\rho_s) = \text{Tr}_a \left( V (\rho_s \otimes \gamma_a) V^\dagger \right).$$

(1)

Here $a$ is an arbitrary ancillary system with Hilbert space $H_a$ and Hamiltonian $H_a$, $\gamma_a$ is the thermal (Gibbs) state at inverse temperature $\beta = 1/k_BT$, $\gamma_a = e^{-\beta H_a}/\text{Tr}_a \left( e^{-\beta H_a} \right)$, and $V$ is an arbitrary unitary operation on $H_s \otimes H_a$ that satisfies $[V,H] = 0$, where $H = H_s \otimes I_a + I_s \otimes H_a$ is the sum of the free Hamiltonians of system and ancilla. Any state $\rho \neq \gamma$ is a resource state in system $s$. While we here consider the case that input and output systems and their Hamiltonians are identical, this framework can be easily extended to the more general case, as done by Janzing et al. [6].

The restricted class of operations is singled out as particularly relevant for thermodynamics because the operations it includes cannot, on their own, be used to do work. Moreover, it is not too difficult to see that various different kinds of nonthermal states can be used, via the restricted class of operations, to do work: for thermal states at a temperature $T'$, distinct from $T$, work can be drawn using a heat engine (such states simulate having a second heat bath at a different temperature); for pure states within a degenerate energy eigenspace, work can be drawn using a Szilard engine [11]; for pure energy eigenstates, work can be drawn directly by an energy-conserving unitary. One is led to expect that one can extract work form any athermal state. We shall show that asymptotically this is indeed the case.

Note that within a resource theory framework, any interaction between the system and ancilla is assumed to be under the control of the experimenter and is therefore included in $V$. However, as described in the Appendix, the operations needed to establish our main result are mappings between macroscopic observables and do not require fine-grained control. The Appendix also shows that a number of different classes of operations for thermodynamics are in fact equivalent.

Resource Interconvertibility & Free Energy.— A central question in any resource theory is that of resource interconversion: Which resources can be transformed into which others, and how easily? Generally there exists a partial order, or quasiorder, of resources: We say $A \geq B$ if resource $A$ can be transformed into $B$ using the allowed operations. Functions which respect this quasiorder are known as resource monotones. For instance, the relative entropy of entanglement is a well-known resource monotone in the resource theory of local operations and classical communication (LOCC) [1].

Here we are interested in determining the optimal rate $R(A \rightarrow B)$ at which resource $A$ can generate resource $B$, in the limit of an infinite supply of $A$. A simple argument, going back to Carnot [12], implies that if the transformation is reversible in the sense that $R(B \rightarrow A) \neq 0$, then the rate at which two resources can be reversibly interconverted must achieve the optimal rate. Otherwise, it would be possible to generate arbitrary amounts of a resource state from a small number via cyclic transformations to and from another resource state.

That reversible interconversion is optimal (when possible) also gives a simple means of characterizing the interconversion rate by using a “standard” reference resource. Consider a transformation from $A$ to $B$ which proceeds via the standard resource $C$: $A \rightarrow C \rightarrow B$. Composing this with $B \rightarrow A$ must give a combined transformation of unit rate, again to avoid the possibility of spontaneously generating resources. Composing the rates, we have $R(A \rightarrow C)R(C \rightarrow B)R(B \rightarrow A) = 1$, or

$$R(A \rightarrow B) = \frac{R(A \rightarrow C)}{R(B \rightarrow C)},$$

(2)

using the fact that $R(A \rightarrow B)R(B \rightarrow A) = 1$. With this framework, we only need to define the relative entropy $D(\rho || \gamma) = \text{Tr}[\rho \log \rho - \log \gamma]$ to state the main result of this Letter.

**Theorem 1** Asymptotic interconversion at nonvanishing rate is possible between all states $\rho$ and $\sigma$ of system with fixed Hamiltonian. Furthermore, the optimal rate is given by

$$R(\rho \rightarrow \sigma) = \frac{D(\rho || \gamma)}{D(\sigma || \gamma)}.$$  

(3)

Simple calculation reveals that $D(\rho || \gamma) = \beta F_\beta(\rho) - \beta F_\beta(\gamma)$, where $F_\beta(\rho) \equiv \langle H \rangle_\rho - TS(\rho)$ is the free energy. Thus, the free energy directly determines the optimal rate of resource interconversion in our resource theory.
The proof of this result proceeds in two steps. First, it is relatively simple to establish that if asymptotic interconversion is possible at nonvanishing rate, then the interconversion rate must be the expression given. The more difficult part is then to establish that interconversion is possible for any pair of states.

The rate expression Eq. (3) follows almost immediately from [13], Theorem 1 (see also [14], Theorem 4) once reversibility of transformation has been shown. This result states that any asymptotically-continuous resource monotone \( f \) determines the interconversion rate via its regularized version \( f^\infty(\rho) = \lim_{n \to \infty} \frac{1}{n} f(\rho^\otimes n) \) as \( R(\rho \to \rho') = f^\infty(\rho)/f^\infty(\rho') \), provided this quantity is nonzero and finite.

To make use of this theorem we need to show that one can define \( f(\rho) = D(\rho||\gamma) \) using the relative entropy and that we can interconvert states reversibly. The relative entropy is a resource monotone because it is contractive under all completely-positive maps, not just the allowed operations. Further, regularization is unnecessary in this case because the thermal state on \( n \) systems is just \( n \) copies of the thermal state on one system, and therefore \( f^\infty(\rho^\otimes n) = D(\rho^\otimes n||\gamma^\otimes n) = n f(\rho) \). Finally, extensivity of energy implies asymptotic continuity of the relative entropy using Proposition 2 of [15], the derivation of which we leave to the Appendix.

Extensivity is more important than it might initially appear. For instance, the relative entropy \( D(\gamma||\rho) \) is also a resource monotone, but does not lead to the interconversion rate as the extensivity argument does not go through and this quantity fails to be asymptotically continuous. Nonetheless, the relative entropy \( D(\gamma||\rho) \) as a monotone plays an important role in the resource requirements of creating low-temperature states [6].

Reversible Interconversion of Arbitrary States.—Now it remains to show that all states are interconvertible at the rate given in Theorem 1. We do this by constructing protocols for both distillation of resource states into a standard state and formation of resource states from standard states. The standard state is chosen to have energy but no entropy so as to represent available work.

For purposes of exposition, we make two simplifications which nevertheless capture the essential aspects of the problem. The more general details are left to the Appendix. First, we consider resources having just two nondegenerate energy levels, i.e. qubits. Second, we assume that the resource states are stationary, meaning \( [\rho, H] = 0 \) and therefore they are diagonal in the energy eigenbasis. Such states are called quasiequilibrium.

Both distillation and formation protocols must satisfy three requirements, up to error terms smaller than \( O(n) \): (1) energy conservation, (2) unitarity, and (3) equality of input and output dimensions. Without loss of generality, we may take the system Hamiltonian to be \( H = E_0 \sum_i |i\rangle \langle i| \) for some energy \( E_0 \), where the sum runs over all involved qubits.

We begin the distillation protocol with \( \ell \) copies of the Gibbs state \( \gamma_n \) and \( n \) copies of the resource \( \rho \), where \( \rho = \text{diag}(1-p, p) \) for arbitrary \( 0 \leq p \leq 1 \) and \( \gamma_n = \text{diag}(1-q, q) \) for \( q = e^{-\beta E_0}/(1 + e^{-\beta E_0}) \). The aim is to effect a transformation of the form \( \gamma_n \otimes \rho^\otimes n \to \sigma(k) \otimes |1\rangle \otimes m \) by an energy-conserving unitary, such that \( m \) is as large as possible. The resulting exhaus state \( \sigma \) of \( k \) systems is arbitrary, though as shown in the Appendix, the optimality of the protocol implies that it has near-Gibbs form. We denote by \( R = \frac{n}{\ell} \) the rate of distillation and \( \epsilon = \frac{q}{p} \) the ratio between initial resource states and Gibbs states. The Gibbs states are free, so we allow for \( \epsilon \to 0 \) as \( n \to \infty \).

We now use the fact that for large \( n \), \( \rho^\otimes n \) consists of mixtures of strings with roughly \( np \) 1s. The number \( t \) of 1s in a string is known as its type, and more concretely we have that, to an error which vanishes as \( n \to \infty \),

\[
\rho^\otimes n \approx \sum_t p_t P_t. \tag{4}
\]

Here the \( t \) summation runs over strongly typical types, the types for which \( t = np \pm O(\sqrt{n}) \) [16], and \( P_t \) denotes projector onto the type \( t \). An entirely similar statement holds for \( \gamma_n \otimes \sigma(k) \). For simplicity we shall first pretend that both \( \gamma_n \otimes \sigma(k) \) consist of a single type and subsequently show how to extend the argument to more types.

Thus, we begin with the tensor product of two types, one coming from the resource state and the other the Gibbs state. This produces an equal mixture of strings of length \( n + \ell \), in which each string consists of two substrings: the first having \( \ell q \) 1s and the second \( np \) 1s. There are roughly \( 2^{\ell q}\times 2^{np} \) such strings, where \( H_\gamma(p) = -p \log_2 p - (1-p) \log_2(1-p) \) is the binary entropy.

Now consider a transformation which maps these strings to new strings having at least \( m \) 1s in the rightmost positions,

\[
\begin{array}{cccccccc}
00&01&1\ldots 1&00&01&1\ldots 1& \ldots & \ldots \\
\ell q&np&\ldots & \ldots & \ldots & \ldots & & \ldots \\
\end{array}
\rightarrow
\begin{array}{cccccccc}
00&01&1\ldots 1&1\ldots 1&1\ldots 1& \ldots & \ldots & \ldots \\
k= m&1\ldots 1&1\ldots 1&1\ldots 1&1\ldots 1& \ldots & \ldots & \ldots \\
\end{array}
\]

where \( k = n + \ell - m \) is conservation of dimension, and \( r \) and \( m \) are to be determined. First, \( r \) is fixed by the conservation of energy, which requires that the number of 1s is conserved. Hence \( \ell q + np = r k + m \). Then unitarity requires that there are at least as many strings of length \( k \) with \( rk \) 1s as the number of initial strings: \( 2^{kH_\gamma(r)} \geq 2^{(\ell+H(p))}\times 2^{np} \). Roughly speaking this is conservation of entropy. Using these three conditions, we find that the transformation is possible if

\[
H(q) + \epsilon H(p) \leq (1+\epsilon - Re)H(q+ep-\epsilon R) \leq \frac{1+\epsilon - \epsilon R}{1+\epsilon - \epsilon R}. \tag{5}
\]

We now expand this with respect to \( \epsilon \) to first order and let \( \epsilon \to 0 \). This means the heat reservoir is much larger
than the resource systems. As a result we obtain that the following rate can be achieved

\[ R \leq \frac{H(q) - H(p) + \beta(p - q)}{H(q) + \beta(1 - q)} = \frac{D(\rho|\gamma)}{D(|1\rangle\langle1||\gamma)}, \]  

(6)

establishing one direction necessary for Theorem 1.

In the above argument we worked with a single type, whereas in actuality the initial state is a mixture of products of types \( Q_1 \otimes P_\ell \). We thus apply the protocol separately to each type, with the same number \( m \) of required output excited states, for all types, \( m \) fitted to the product of less numerous types (the ones with smallest number of 1s, i.e. \( np - O(\sqrt{n}) \) and \( \ell q - O(\sqrt{\ell}) \) 1s, respectively). To carry out our estimates we need to simultaneously fulfill \( \sqrt{\ell} \ll m = Rn \), in order for \( R \) to be achievable, and \( \ell \gg n \), in order for \( \epsilon \to 0 \). Choosing \( \ell = (Rn)^{3/2} \) therefore ensures that our estimate \( \mathbf{[9]} \) will be accurate in the limit \( n \to \infty \). \( \mathbf{[7]} \)

The construction of the formation protocol is similar to that of the distillation protocol and is again based on considering type transformations satisfying the three requirements of energy conservation, unitarity, and dimension conservation. The major difference is that whereas the ideal distillation output is simply the fixed-type state \(|1\rangle^\otimes m \), the ideal formation output must recreate a good approximation to the probabilistic mixture of type classes found in \( \rho^\otimes n \).

As outlined in more detail in the Appendix, the formation protocol is constructed in three stages. The first two are similar to the distillation protocol. In the first, a given type class of the Gibbs state together with the standard resource is transformed into a desired type class of the target resource \( \rho^\otimes n \). In the second, this construction is extended to all the strongly typical types of the Gibbs state. Finally, in the third step an additional number of Gibbs states are used to probabilistically select which type class of the target should be constructed, in order to recreate the appropriate distribution over types of the target state. In principle this step is irreversible, but since the number of type classes grows only polynomially with \( n \), the number of extra resources required for the third step of the formation protocol vanishes in the \( n \to \infty \) limit. The similarity of the first two steps with the distillation protocol then ensures that the formation protocol achieves the inverse rate.

The distillation of arbitrary states is related to that of stationary states, and we can recycle part of the previous distillation protocol. Suppose the resource state has the diagonal form \( \rho = p|\phi_1\rangle\langle\phi_1| + (1 - p)|\phi_2\rangle\langle\phi_2| \), implying an average energy of \( \langle E \rangle = p\langle|\phi_1| \rangle^2 + (1 - p)\langle|\phi_2| \rangle^2 \rangle E_0 \). In \( n \) instances of \( \rho \) the total energy will overwhelmingly likely be \( n\langle E \rangle \pm O(\sqrt{n}) \). Now imagine projecting the resource state onto the various energy subspaces, destroying any coherence between them. Just as in \( \mathbf{[4]} \), \( \rho^\otimes n \) is supported almost entirely on its typical subspace, whose size not larger than \( 2^{nS(\rho) + O(\sqrt{n})} \). Thus, the state support in every energy subspace is at most this large.

Now we may imagine applying the same scheme as in the previous distillation protocol, creating as many copies of \( |1\rangle \) as possible. The three conditions now become \( k = m + n - \ell \), \( \ell q - O(\sqrt{\ell}) \) 1s, respectively). To carry out our estimates we need to simultaneously fulfill \( \sqrt{\ell} \ll m = Rn \), in order for \( R \) to be achievable, and \( \ell \gg n \), in order for \( \epsilon \to 0 \). Choosing \( \ell = (Rn)^{3/2} \) therefore ensures that our estimate \( \mathbf{[9]} \) will be accurate in the limit \( n \to \infty \). \( \mathbf{[7]} \)

The formation of arbitrary resource states is more complicated than their distillation. Strictly speaking, the transformation we are interested in is impossible, since the inputs are states diagonal in the energy basis and the allowed transformations cannot change this fact. However, to create the appropriate coherences between energy subspaces it suffices to use a small additional resource in the form of a superposition over energy eigenstates. This is akin to a similar phenomenon in the resource theory of entanglement where distillation of entangled states does not require any communication, but formation requires an amount sublinear in the number of inputs \( n \). This provides an interesting asymmetry between formation and distillation as explained in more detail in the Appendix. The formation process consumes only a small amount, sublinear in \( n \), of the extra resource, and therefore does not affect the rate calculations.

Conclusions.—In the preceding we have shown that well-known results from thermodynamics can be derived quite naturally within the framework of resource theories. We should emphasize that although the procedures we have described for the conversion of resource states may seem quite unnatural from a physical point of view, their use is to establish the “in principle” interconversion rate given in Theorem \( \mathbf{[4]} \). Any more realistic reversible transformation, for instance the Hamiltonian method of \( \mathbf{[18]} \), will necessarily extract the same amount of work.

We thank Jochen Rau and Dominik Janzing for helpful conversations. JMR acknowledges support from the Center for Advanced Security Research Darmstadt (CASED).

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Often entanglement is asserted to be the necessary resource for a given operational capability, but it is often another resource that is actually relevant. For instance, the resource for quantum metrology is asymmetry.

For instance, the physicist Percy Bridgman commented that there are almost as many formulations of the second law as there have been discussions of it [13].

Additionally, the number of strings of a given type np is given by poly(n)2^{nH(p)}, not simply 2^{nH(p)}. However, in the limit n → ∞, the polynomial factor is again not relevant in our estimates.
APPENDIX

This appendix contains eight sections. The first shows that the relative entropy is asymptotically continuous. The next four sections discuss in detail the state transformation protocols for the case of two level systems. Section II presents a distillation protocol for diagonal states, while section III describes a formation protocol also for diagonal states. The following two sections extend these protocols to the case of arbitrary nonstationary two-level resources. Then in Section VI we outline how the results can be easily generalized to higher dimensions by considering as an example the distillation protocol for diagonal states. Section VII discusses some characteristics of the exhaust states produced in these protocols. Finally, Section VIII discusses the equivalence of our formulation to other models of thermodynamics and the degree of control one needs to implement our thermal operations.

Extensivity and asymptotic continuity

To show that $D(\rho||\gamma)$ is asymptotically-continuous, we make use of the following, from [1]:

**Proposition 2** Suppose a function $f$ satisfies (1) “approximate affinity”

$$|pf(\rho) + (1-p)f(\sigma) - f(p\rho + (1-p)\sigma)| \leq c$$

for some constant $c > 0$ any $0 \leq p \leq 1$, and (2) “subextensivity” $f(\rho) \leq M \log d$, where $M > 0$ is constant and $d = \dim(\mathcal{H})$ for $\mathcal{H}$ the state space on which $\rho$ has support. Then $f$ is asymptotically continuous,

$$|f(\rho_1) - f(\rho_2)| \leq M\|\rho_1 - \rho_2\|_1 \log d + 4c.$$  

The entropy relative to the Gibbs state satisfies both conditions. To see the first, let $\rho = \sum_k p_k \rho_k$ for some arbitrary density operators $\rho_k$ and $\sigma$ be another arbitrary density operator. Then

$$D(\rho||\sigma) = \text{Tr} \left[ \sum_k p_k (\rho_k \log \rho_k - \rho_k \log \sigma + \rho_k \log \rho - \rho_k \log \rho_k) \right]$$

$$= \sum_k p_k D(\rho_k||\sigma) + \sum_k p_k S(\rho_k) - S(\rho).$$

Since $S(\rho) \leq \sum_k p_k S(\rho_k) + H(p_k)$ and the relative entropy is convex, this implies

$$0 \leq \sum_k p_k D(\rho_k||\sigma) - D(\rho||\sigma) \leq H(p_k).$$

In the present setting this means

$$pD(\rho||\gamma) + (1-p)D(\sigma||\gamma) - D(p\rho + (1-p)\sigma||\gamma) \leq H_2(p) \leq 1,$$  

where $H_2$ is the binary entropy function. Furthermore, $D(\rho||\gamma)$ is approximately extensive if the maximum energy of the system is extensive. First, note that $D(\rho||\gamma) = \beta F_\beta(\rho) - \beta F_\beta(\gamma)$, for $F_\beta(\rho) = \langle H \rangle_\rho - H/\beta$ is the free energy. Thus, the maximum of $D(\rho||\gamma)$ occurs for $\rho = \rho_{\text{max}}/\rho_{\text{max}}$ where $\rho_{\text{max}}$ is the eigenstate of maximum energy. Direct calculation shows

$$D(\rho_{\text{max}}||\gamma) = \text{Tr} [E_{\text{max}}] (\log |E_{\text{max}}|/|E_{\text{max}}| - \log |\gamma|) = -|E_{\text{max}}| \log |E_{\text{max}}|$$

$$= \beta E_{\text{max}} + \log Z_\beta = \beta E_{\text{max}} + \log \sum_k e^{-\beta E_k} \leq \beta E_{\text{max}} + \log d.$$  

Here we have assumed that the energy values $E_j > 0$. When the maximum energy is extensive, i.e. $E_{\text{max}} \leq K \log d$ for some constant $K$, we obtain $D(\rho_{\text{max}}||\gamma) \leq M \log d$ for $M = \beta K + 1$. 
Distillation of diagonal states

For simplicity of presentation we consider qubit systems with the Hamiltonian given by $H = \sum_i |1\rangle_i \langle 1|$, where sum runs over all involved qubits. We start with $l$ copies of the Gibbs state $\tau$ and $n$ copies of resource state $\rho$, where

$$\rho = (1 - p)|0\rangle|0\rangle + p|1\rangle|1\rangle; \quad \tau = (1 - q)|0\rangle|0\rangle + q|1\rangle|1\rangle. \quad (15)$$

with $q = e^{-\beta}/(1 + e^{-\beta})$ and $\beta$ the inverse temperature, which we take as a constant parameter. The aim is to obtain the maximal number of copies possible of qubits in the pure excited state $|1\rangle$ by a unitary that commutes with $H$ and partial tracing.

$$\tau^{\otimes l} \otimes \rho^{\otimes n} \rightarrow \sigma^{(k)} \otimes |1\rangle^{\otimes m}. \quad (16)$$

We denote $R = \frac{m}{n}$ (the rate of distillation) and $\epsilon = \frac{n}{l}$ (the ratio between number of used Gibbs states and resource states). The Gibbs states are free, so we accept that $\epsilon$ asymptotically vanishes.

In the protocol we shall use the fact that up to a small error (vanishing for a large number of qubits)

$$\rho^{\otimes n} \approx \sum_t p_t P_t \quad (17)$$

where $t$ run over strongly typical types, i.e. the types containing strings with the number of 1’s within the interval $(n p - O(\sqrt{n}), n p + O(\sqrt{n}))$, and $P_t$ denotes the projector onto type $t$. Similarly

$$\tau^{\otimes l} \approx \sum_t q_t Q_t \quad (18)$$

again with $q_t \approx q n$. The errors in both approximations is smaller than $2^{-s\sqrt{n}}$ when quantified by the trace norm.

For simplicity we shall first pretend that both $\tau^{\otimes l}$ and $\rho^{\otimes n}$ consist of a single type. Then further we will show how to extend the argument to a mixture of types.

So we start with a tensor product of two types (one from Gibbs, the other from $\rho$), i.e. an equal mixture of strings of length $l + n$. The string consist of two substrings: the first has $q_l$ 1’s and the second has $p n$ 1’s:

$$\begin{array}{c}
000 \ldots 0 \\
\underbrace{11 \ldots 1}_{lq} \\
00 \ldots 0 \\
\underbrace{111 \ldots 1}_{np}
\end{array} \quad (19)$$

There are roughly

$$2^{lH(q)} \times 2^{nH(p)} \quad (20)$$

of such strings (with the error being a multiplicative poly($n$) factor).

We now apply unitary transformation to these strings mapping them into strings of the same total length which have $m$ 1’s to the right:

$$\begin{array}{c}
000 \ldots 0 \\
\underbrace{11 \ldots 1}_{lq} \\
00 \ldots 0 \\
\underbrace{111 \ldots 1}_{np} \\
\underbrace{0000 \ldots 0000}_{rk} \\
\underbrace{111 \ldots 111 \ldots 1}_{m}
\end{array} \quad (21)$$

where

$$k = m + n - l \quad (22)$$

(conservation of dimension), and $r$ and $m$ are about to be determined. First, $r$ is fixed by conservation of energy, which requires that the number of 1’s is conserved:

$$lq + np = rk + m \quad (23)$$

with $k = n + l - m$. Then unitarity requires that there are at least as many strings of length $k$ with $rk$ 1’s as the number of initial strings $2^{lH(q) + nH(p)}$.

$$2^{kH(r)} \geq 2^{lH(q) + nH(p)} \quad (24)$$
Roughly speaking this is conservation of entropy. Using [23] and [24] we obtain that our transformation is possible if

$$H(q) + \epsilon H(p) \leq (1 + \epsilon - R\epsilon) H\left(\frac{q + \epsilon p - \epsilon R}{1 + \epsilon - R}\right),$$

where recall that $\epsilon$ is the ratio of Gibbs states used, and $R$ the ratio of pure excited states obtained. We now expand this with respect to $\epsilon$ to first order and let $\epsilon \to 0$. This means that we take much more Gibbs states than the resource states, and it says that we work with an infinite sized heat reservoir (i.e. much larger than our system). As a result (recalling that $q = e^{-\beta}/(1 + e^{-\beta})$) we obtain that the following rate can be achieved

$$R \leq \frac{H(q) - H(p) + \beta(p) - q}{H(q) + \beta(1 - q)} = \frac{S(\rho\|\tau)}{S(1\|1\|\tau)},$$

i.e. the protocol achieves the upper bound obtained by the monotonicity argument in the main text.

So far we worked with a single type. But our initial state is actually a mixture of products of types $Q_i \otimes P_i$. We thus apply the protocol separately to each type, with the same number $m$ of required output excited states, for all types, with this number being fitted to the product of less numerous types (i.e. the one with smallest number of 1’s, i.e. $(np - O(\sqrt{n}))(lq - O(\sqrt{l}))$ 1’s). Note however that the tails will disappear asymptotically, as we divide the equations by $l$. Also the approximation to the number of strings in each type by the the exponential of the entropy is correct up to multiplicative polynomial factor, which is also irrelevant asymptotically.

**Formation of diagonal states**

We are going to construct the formation protocol in three stages. The first is to show that for a particular type $T_q$ of the $l$ copies of the Gibbs state, we can create a particular type $T_p$ of the state we want to form. We then show that we can do this for all types of the Gibbs state. Finally we show how to correctly get the distribution over types of the target state.

We shall need the following useful lemma:

**Lemma 3 (Birkhoff primitive)** The following operation can be done by means of thermodynamical operations with arbitrary accuracy:

$$\rho \to \sum_k p_k U_k \rho U^\dagger_k$$

where $p_k$ is arbitrary probability distribution, and the $U_k$ commute with Hamiltonian. In particular, a random permutation of the systems is a valid operation.

**Remark.** The accuracy depends on the number of Gibbs states we will use, but in our paradigm they are for free.

**Proof.** First let us note that the following unitary transformation preserves energy:

$$\sum_i |i\rangle\langle i| \otimes \hat{U}_i$$

provided that $|i\rangle$ are eigenvectors of the Hamiltonian and the $\hat{U}_i$ commute with the Hamiltonian. To obtain the required transformation we will take $l$ copies of Gibbs states. Let $q_i$ denote the probability distribution of single strings. Clearly $q_i \leq q^l$. We now divide the set of $q_i$’s into groups, such that within each group they sum up to the required $p_k$’s from (27). This can be done with arbitrary accuracy, by taking $l$ large enough. Then within each group we set $\hat{U}_i = U_k$.

We want to form $n$ copies of the state $\rho = (1 - p)|0\rangle\langle 0| + p|1\rangle\langle 1|$ from a pure excited state. Let us first show how we can form any of the typical states of the type.

**Formation of a maximally mixed state over a fixed type.**

Consider a fixed type $T_p$ of the $n$ systems we want to create, and let it be one with $np$ 1’s. For other types, the reasoning is the same and the asymptotic rate will be the same. We start with $l$ copies of Gibbs states $\tau$, and $m$ copies of excited state $|1\rangle$. We consider a final exhaust system of $k$ systems. Consider a particular typical type $T_q$ of the Gibbs state with $lq$ 1’s. In that type, there are $\approx 2^{lH(q)}$ strings up to some $2^{lH(q)}$ factor. We want to map these initial strings onto the $N$ final strings in the type $T_p$ as follows. Take $\{u_i\}$ to be the strings in $T_p$ and for each $u_i$
FIG. 1. Strings mapping in formation protocol

consider some set \( \{ v_i^j \}_{j} \) of strings on the exhaust system. We now map each of the initial strings to some string \( u_i \). This is illustrated on figure.

For each \( i \), we can take the number of strings \( \{ v_i^j \}_{j} \) to be the same or off by 1, simply by assigning the strings in an order determined by fixing \( j = 1 \), then incrementing the \( i \) register until \( i = N \), then incrementing the \( j \) register by 1, resetting \( i \) to 1 and again increment the \( i \) register until \( i = N \) and repeating. This is all done to ensure that when we trace out the exhaust system, we get an even mixture over per mutations within the type class.

We can now use the analogues of equations (22), (23) and (24) to ensure that we can perform the unitary which implements this mapping:

\[
    m + l = n + k \tag{29}
\]
\[
    lq + m = rk + np \tag{30}
\]
\[
    2^lH(q) \leq 2^kH(r) + nH(p) \tag{31}
\]

We now take \( l \propto m^\alpha \) with \( 1 < \alpha < 2 \) and do as before. We take \( \alpha > 1 \) so that we can take \( \epsilon = m/l \) small, and \( b < 2 \) so that \( \sqrt{l} \) is sublinear in \( m \) and we can ignore such terms.

This maps type \( T_q \) of the \( l \) Gibbs states onto the type \( T_p \) of \( \rho^\otimes n \). We can map each of the initial Gibbs types on to \( T_p \) in this manner using a unitary \( U_{pq} \). For each such mapping, the above three equations will change, but only by some \( \sqrt{l} \) factors which we took to be sublinear in \( m \). We can thus choose \( m \) to ensure conservation of energy in the worst case of Eq. (30), and \( kH(r) \) is chosen to ensure the inequality Eq. (31) in the worst case.

We now need to implement \( U_{pq} \) conditioned on the initial type \( T_q \). Since the typical initial types are on orthogonal and diagonal subspaces, we first do a conditional copying of the type class \( q \) onto an initialised register. We then act \( U_{pq} \) conditioned on this register. Since the number of types is polynomial in \( l \), the register only needs to be of size \( \log l \), and thus this resource does not matter as it is sublinear in \( m \). It is an interesting question whether the formation protocol can be made to work without this sublinear supply of pure states.

Let us call this unitary \( U_p \) which creates a particular type \( T_p \). To get the distribution over types, we simply use the Birkoff primitive of Lemma 3 to implement \( \sum_p |p\rangle\langle p| \otimes U_p \). This is irreversible, but since the number of typical types is polynomial in \( n \), the rate of entropy that is created by this procedure is negligible, i.e. logarithmic in \( n \). It is not hard to see that Eqs. (29)–(31) give the required rate (i.e. the inverse of the distillation rate).

**Distillation of arbitrary states**

We now extend distillation to the case where our states are not diagonal in the energy eigenbasis.

Consider a state \( \rho = p|\phi_1\rangle\langle \phi_1| + (1-p)|\phi_2\rangle\langle \phi_2| \). The average energy of the state is \( \langle E \rangle = p|\langle \phi_1|1 \rangle|^2 + (1-p)|\langle \phi_2|1 \rangle|^2 \). As before, we consider \( n \) copies of \( \rho \) and \( l \) copies of systems in a Gibbs state \( \tau \). Regarding \( \rho^\otimes n \), only the blocks with energy \( E \in [n\langle E \rangle - \sqrt{n}, n\langle E \rangle + \sqrt{n}] \) will be relevant, i.e. we have

\[
    \text{Tr} \left( \sum_E P_E \rho^\otimes n \right) \geq 1 - 2^{-O(n)} \tag{32}
\]
where the sum runs over $E \in [n(E) - O(\sqrt{n}), n(E) + O(\sqrt{n})]$, with $P_E$ the projector onto the energy $E$ eigenspace. (This follows from Eq. \ref{eq:energy_bound}, proven in Section \ref{sec:proof}).

Our protocol has two stages:

(i) unitary rotation within energy blocks of a resource system (consisting of $n$ qubits) solely.

(ii) drawing work by string permutations on the total system resource (with $n$ qubits) plus heat bath (with $l$ qubits)

We write down the resource state in the energy eigenbasis. As said above, only blocks with energy $\approx nE$ will appear.

We will use the fact that the state is, up to exponentially vanishing error, equal to its projection onto the typical subspace, having dimension $2^{nS(\rho) + O(\sqrt{n})}$ where $S(\rho)$ is von Neumann entropy of $\rho$. Therefore, within every block the rank of the state is not larger than $\approx 2^n$ (as a projection cannot increase the rank).

Now stage (i) is the following: within each energy block, we apply unitary rotation, which diagonalizes the state restricted to the block in the energy basis. Then there is stage (ii), in which we apply the protocol of distillation of diagonal states as in Section \ref{sec:distillation}, i.e. we permute strings in such a way that the output strings have $m$ 1's to the right. Such protocol produces $m$ systems in a pure excited state. (note that all coherences initially present in the state are now left in the garbage). Using the same notation as in Section \ref{sec:distillation} for a product of two single types, e.g. with $lq$ and $n(E)$ 1's, respectively, the contraints now become

$$k = m + m - l$$
$$lq + n(E) = rk + m$$
$$2^{kH(r)} \geq 2^{H(q) + nS(\rho)}$$

As before, taking the limit $\frac{n}{l} \to 0$ we obtain that any rate $R$ is achievable, provided it satisfies

$$R \leq \frac{H(q) - S(\rho) + \beta((E) - q)}{H(q) + \beta(1 - q)} = \frac{S(\rho||\tau)}{S(1||\tau)}.$$  \hspace{1cm} (33)

Thus also for nondiagonal state, we can reach the upper bound given by relative entropy distance from the Gibbs state.

### Formation of arbitrary states

We now show that we have reversibility even in the off-diagonal case, if we allow a sublinear amount of superposition over energy-eigenstates, which is a resonable assumption since the rate at which this is consumed vanishes in the asymptotic limit. This is very similar to the fact that in entanglement theory, distillation requires no communication but formation requires a sublinear amount of it. The reason is also the same and in entanglement theory is known as entanglement spread \cite{3, 4}. Here, distillation of work requires no superposition over energy eigenstates, but formation does.

Generally, say we want to implement some unitary

$$U = \sum_{ij} u_{ij} |E_i\rangle \langle E_j|$$ \hspace{1cm} (34)

which does not conserve energy. Introduce some reference frame state

$$|H\rangle = \sum f(h)|h\rangle$$ \hspace{1cm} (35)

and instead implement

$$U^{inv} = \sum_{ij} u_{ij} |E_i\rangle \langle E_j| \otimes |h - E_i + E_j\rangle \langle h|.$$ \hspace{1cm} (36)

Then if we act this on a state

$$|\psi\rangle = \sum_{i \in S} c_i |E_i\rangle$$ \hspace{1cm} (37)

with $S$ such that $|u_{ij}|^2$ is small between states with large $|E_i - E_j|$, the reference frame state will not be disturbed too much and we can take $f(h)$ to have small support i.e. we do not need a big superposition. This will be the case
here, since on the typical subspace, the variation in energy is sublinear in the number of copies of the state we want to create.

To see how this works, note by way of example, that if in Equation (35), we take \( f(h) \) to be \( 1/\sqrt{N} \) for energies \( E_j = J, \ j = 1...N \) and \( f(h) = 0 \) otherwise, then removing a unit of energy and adding it to another system, does not change the state of Equation (35) much. i.e. the inner product between \( |H \rangle \) and \( \sum_h |h-1\rangle\langle h|H \rangle \) approaches 1

\[
\sum_{h=0}^{N-1} 1/\sqrt{N} |h\rangle \sum_{h=0}^{N-1} 1/\sqrt{N} |h\rangle = 1 - 1/N
\]

which approaches 1 for large \( N \). State like that of Equation (35) therefore allow us to lift the superselection rule of energy, without being consumed much in the process.

This gives some insight into embezzling states [2], which is a resource often used in entanglement theory in similar situations. E.g. one can use a state similar in form to Equation (35) to create an arbitrary state at a rate given by the relative entropy distance to Gibbs states. Let \( \rho \) be the state of Equation (35)

\[
\rho = \sum_{k,g} \frac{p_k}{N} |\Psi_{k,g}\rangle \langle \Psi_{k,g}|
\]

which is a superposition of a different number of entangled EPR pairs \( |\phi\rangle_{AB} \). As stated previously, these states can be used to implement operations which require entanglement spread. Just as removing one unit of energy, doesn’t change the state of Equation (35) much, likewise, removing one EPR pair from the state of Equation (39) and adding it to another system, doesn’t change that state by much. We can embezzle energy, just as one can embezzle entanglement. We therefore see that a superposition over some resource can create an embezzling state for that resource, and will allow us to lift some superselection rule or restriction.

With this small superposition over energy states, let us now show we can create an arbitrary state at a rate given by the relative entropy distance to Gibbs states. Let \( \rho := p|\phi_1\rangle \langle \phi_1| + (1-p)|\phi_2\rangle \langle \phi_2| \) and

\[
\rho^{\otimes n} = \sum_{k,g} p_k |\Psi_{k,g}\rangle \langle \Psi_{k,g}|
\]

with

\[
|\Psi_{k,g}\rangle := \pi_g |\phi_1\rangle^{\otimes k} \otimes |\phi_2\rangle^{\otimes n-k},
\]

for \( \pi_g \) a permutation.

The idea of the protocol is as follows: we will first create a diagonal state

\[
\varrho_n = \sum_{t_k,s_g} p_k |t_k,s_g\rangle \langle t_k,s_g|
\]

which has the same spectrum as \( \rho^{\otimes n} \) and where each eigenstate has the same average energy as an eigenstate in the typical subspace of \( \rho^{\otimes n} \). From the result of the previous section it is not hard to see that this can be done at a rate given by the relative entropy distance of \( \rho \) to the Gibbs state, since in the limit of many copies the regularised relative entropy distance is the same. We would then like to rotate the diagonal basis to the \( \Psi_{k,g} \)-basis. This cannot be done by unitaries which commute with the Hamiltonian unless we allow for a reference frame \( |H\rangle \) which is a superposition over energy states. We then want to show that the reference frame which allows us to break the energy superselection rule is consumed at a vanishingly small rate. We do so by showing that the reference frame superposition is over a size sublinear in \( n \). This can be understood as coming from the fact that in the typical subspace, the superposition over different types is sublinear.

We consider only typical \( |\Psi_{k,g}\rangle \) with \( k \in \text{Typ}_n := [np - \sqrt{n}, np + \sqrt{n}] \). Then

\[
\left\| \rho^{\otimes n} - \sum_{k \in \text{Typ}_n,g} p_k |\Psi_{k,g}\rangle \langle \Psi_{k,g}| \right\|_1 \leq 2^{-\Omega(\sqrt{n})}
\]

Let also

\[
|\Psi_{k,g}\rangle := \sum_{t',s'} c'_{t's'} |t',s'\rangle
\]

where \( |t',s'\rangle \) is an eigenstate of the Hamiltonian with energy \( t' \) (s' labels the degeneracy). From Eq. (44) it follows that the sum in Eq. (44) will be peaked around only a few energy values \( t' \). Indeed, with

\[
|\phi_1\rangle := a|0\rangle + b|1\rangle,
\]

(45)
and

$$|\phi_2\rangle := b(0) - a|1\rangle,$$

set Typ := \([nE_t - \sqrt{n}, nE_t + \sqrt{n}]\), where \(E_t := ((n - t)|b|^2 + t|a|^2)/n\). Then

$$\left\| \sum_{t' \in \text{Typ}, s'} c_{t's'}^{k} |t', s'\rangle \right\| = 2^{-\Omega(\sqrt{n})}$$

Note that since \(E_t := ((n - t)|b|^2 + t|a|^2)/n\) the degeneracy of each energy state \(|t_g\rangle\) is at least as large as the degeneracy of \(|\Psi_{k}\rangle\).

Now we construct the reference frame. Let \(|w\rangle\) be an energy eigenstate with energy \(n(p|b|^2 + (1 - p)|a|^2) - n^{2/3}\). It is needed to pad the dimension of the reference frame, since although the probability that it happens is vanishingly small, the unitary does connect states with large energy difference. After the protocol, we will see that \(|w\rangle\) will hardly be changed, and thus is only used as a catalyst. We define the reference system as follows

$$|H\rangle := \frac{1}{\sqrt{|H|}} \sum_{h \in H} |h\rangle$$

with \(|h\rangle := |h'\rangle \otimes |w\rangle\), where \(|h'\rangle\) is an energy eigenstate of energy \(h'\) and \(H := \{0, \ldots, 2n^{2/3}\} \cup \{\psi\}\).

Consider the energy preserving unitary

$$U := \sum_{h, t, s, t', s'} c_{t's'}^{k} |t_k, s_g\rangle (t_k, s_g) \otimes |h + t - t'\rangle \langle h|,$$

Then in the sequel we prove that

$$\left\| U \left( \sum_{k \in \text{Typ}, s} p_k |t_k, s_g\rangle (t_k, s_g) \otimes |H\rangle \langle H| \right) U^\dagger - \rho^{\otimes n} \otimes |H\rangle \langle H| \right\|_1 = O(n^{-1/6}).$$

We first analyse the action of \(U\) in \(|s, t\rangle \otimes |H\rangle\):

$$U (|t_k, s_g\rangle \otimes |H\rangle) = \sum_{t', s'} c_{t's'}^{k} |t'_k, s'_g\rangle \otimes \left( \frac{1}{\sqrt{|H|}} \sum_{h \in H} |h + t_k - t'\rangle \right)$$

$$= |\nu_1\rangle + |\nu_2\rangle + |\nu_3\rangle$$

where the non-normalized pure states \(|\nu_k\rangle\) are given by

$$|\nu_1\rangle := \sum_{t' \in \text{Typ}, s'} c_{t's'}^{k} |t'_k, s'_g\rangle \otimes |H\rangle,$$

$$|\nu_2\rangle := \sum_{t' \in \text{Typ}, s'} c_{t's'}^{k} |t'_k, s'_g\rangle \otimes |\text{err}_{t'}\rangle$$

with

$$|\text{err}_{t'}\rangle := \frac{1}{\sqrt{|H|}} \sum_{h \in H} |h + t - t'\rangle - |H\rangle,$$

and

$$|\nu_3\rangle := \sum_{t' \notin \text{Typ}, s'} c_{t's'}^{k} |t'_k, s'_g\rangle \otimes \left( \frac{1}{\sqrt{|H|}} \sum_{h \in H} |h + t - t'\rangle \right).$$
Set $t_k = E_k$ and let us take $s_g = g$. We can do the latter since as we mentioned, the degeneracy of $|t_k, s_g\rangle$ is larger than the degeneracy of $|\Psi_{k,g}\rangle$. Then,

$$\|U(|t_k, s_g\rangle \otimes |H\rangle) - |\Psi_{k,g}\rangle \otimes |H\rangle\| \leq \|\nu_1\| - \|\Psi_{k,g}\rangle \otimes |H\rangle\| + \|\nu_2\| + \|\nu_1\|.$$  \hfill (56)

We now show that the three terms in the R.H.S. are small. For $|\nu_2\rangle$ we first note that for $t' \in \text{Typ}_t$ \n
$$\|\nu_2\| \leq n^{-1/6}.$$  \hfill (57)

by taking the worst case. Then

$$\|\nu_2\|^2 = \sum_{t' \in \text{Typ}_t, s'} |c_{t's'}^{kg}|^2 \|\nu_{t'}\|^2 \leq \max_{t' \in \text{Typ}_t} \|\nu_{t'}\|^2 \leq n^{-1/3}. \hfill (58)$$

For $|\nu_3\rangle$, in turn, we have

$$\|\nu_3\|^2 \leq \sum_{t' \notin \text{Typ}_s, s'} |c_{t's'}^{kg}|^2 \|\nu_{t'}\|^2 \leq 2^{-\Omega(\sqrt{n})}. \hfill (59)$$

Finally, for $|\nu_1\rangle$,

$$\|\nu_1\| = \|\Psi_{k,g}\rangle \otimes |H\rangle\| = \sum_{t' \in \text{Typ}_t, s'} |c_{t's'}^{kg}|^2 \|\nu_{t'}\| \leq 2^{-\Omega(\sqrt{n})}. \hfill (60)$$

From Eq. (56) it thus follows that

$$\|U(|t_k, s_g\rangle \otimes |H\rangle) - |\Psi_{k,g}\rangle \otimes |H\rangle\| \leq O(n^{-1/6}). \hfill (61)$$

Since \(\|\psi\langle\psi - |\phi\rangle\langle\phi|\| \leq \sqrt{2}\|\psi - |\phi\|\) for every two states $|\psi\rangle, |\phi\rangle$, we find

$$\|U(|t_k, s_g\rangle \langle t_k, s_g| \otimes |H\rangle\langle H|) - |\Psi_{k,g}\rangle\langle\Psi_{k,g}| \otimes |H\rangle\langle H|\|_1 \leq O(n^{-1/6}). \hfill (62)$$

Eq. (56) then follows from the triangle inequality for trace-norm and Eq. (61).

**Distillation of quasiclassical states in arbitrary dimensions**

In this section we present the details of the distillation protocol for stationary, or quasiclassical, states for the general case of $d$-dimensional systems. This is presented as an example of how the results can be extended to arbitrary dimensions by very similar arguments to the two-dimensional case.

The input to the protocol consists of $n$ copies of the initial resource $\rho$ and $\ell$ copies of the Gibbs state $\gamma$ of the same Hamiltonian $H$. Since the states are quasiclassical, the overall state of the input is fully described by a collection of strings $s \in \{0, \ldots, d-1\}^{n+\ell}$ listing the energy level occupied by each system, each string weighted by its probability of occurrence. Here $d$ is the total number of energy levels of $H$, and the first $n$ entries of $s$ correspond to the state of $\rho$ and the remainder to $\gamma$. If some of the energy levels are degenerate, we simply work in the eigenbasis of $\rho$ to remove the degeneracy in labeling. If even that proves insufficient, we are free to make any choice we like among the remaining degenerate levels. Because permutations within each of the two substrings do not change the overall probability, we can therefore instead work with the collection of occupation frequencies $f$ of the state, which describe the number of systems in the ground, first excited, second excited states and so on (divided by the total number of systems), again each weighted by an appropriate probability.
We would now like to define a protocol which creates as many standard resources in the form of work as possible. In the qubit case the standard resource had a very simple form, namely the excited state of the Hamiltonian. Here, however, the setup is more cumbersome, as there are $d-1$ excited states and no guarantees that their energy differences are in any way commensurate (so that as, e.g. in a harmonic oscillator, the energy of the state $|1\rangle$ could be transferred to two instances of the state $|1\rangle$). To handle this issue most simply, we imagine that, in addition to the resource and thermal states, we also have a work system at our convenience. The work system is capable of accepting arbitrary amounts of energy, i.e. it has energy transitions which precisely correspond to those of $H$, but it cannot accept any entropy. Now the goal of the protocol is to change the occupation of the energy levels so as to transfer as much energy to the work system as possible.

Let us now restrict attention to a fixed occupation frequency $f_p$ of the resource and $f_\gamma$ for the Gibbs state. We will later design the protocol so that it works for every such frequency pair which has appreciable probability. Suppose that we now change the occupation numbers by an amount described by the vector $-nx$ (whose prefactor is chosen for later convenience). This results in a new occupation vector $\nu$ defined by $(n+\ell)\nu \equiv nf_p + \ell f_\gamma - nx$. For this to be an allowable transformation in our framework, this mapping must satisfy two constraints: energy conservation and unitarity. In contrast to the qubit case, here the input and output dimensions are equal by design.

Energy conservation is simply enforced by requiring the work system to take up the change in energy of the input systems. Using the vector $H$ to describe the energy of each energy level, the initial energy is given by $E_{in} = nH \cdot f_p + \ell H \cdot f_\gamma$, while the final energy is $E_{out} = nH \cdot f_p + \ell H \cdot f_\gamma - nH \cdot x$. Energy conservation is then the statement that the work extracted is given by $W = E_{in} - E_{out} = nH \cdot x$.

Unitarity is enforced by making sure that the total number of configurations (strings) consistent with each occupation vector is conserved by the process. The total number of possible input strings in this case, $N_{in}$, is just the product of the multinomial coefficients using the frequency vectors:

$$N_{in} = M(nf_p) M(\ell f_\gamma) = \frac{n!}{(n(\nu_0))! \cdots (n(\nu_{d-1}))!} \frac{\ell!}{(\ell(\nu_0))! \cdots (\ell(\nu_{d-1}))!}.$$  \hspace{1cm} (63)

The maximum number of strings $N_{out} = M((n+\ell)\nu)$ which can be created in the $n+\ell$ systems given the new occupation frequency $\nu$ is the multinomial coefficient of the new occupation frequency vector,

$$N_{out} = M((n+\ell)\nu) = \frac{(n+\ell)!}{(n+\ell)(\nu_0)! \cdots (n+\ell)(\nu_{d-1})!}. \hspace{1cm} (64)$$

Therefore, a sufficient condition for unitarity is $N_{in} \leq N_{out}$, or $M(nf_p) M(\ell f_\gamma) \leq M((n+\ell)\nu)$.

It can be shown that the multinomial coefficients obey the bounds

$$\frac{e}{(ne)^d} f_1 \cdots f_d 2^{nH(\nu)} \leq \frac{n!}{(n(\nu_0))! \cdots (n(\nu_{d-1}))!} \leq \frac{n!}{e^d - 1} 2^{nH(\nu)},$$

and therefore $M(n\nu) \approx 2^{nH(\nu) + O(\log n)}$. The unitarity condition then becomes

$$nH(f_p) - O(\log n) + \ell H(f_\gamma) - O(\log \ell) \leq (n+\ell)H(\nu) + O(\log(n+\ell)) \hspace{1cm} (66)$$

Defining $\epsilon = \frac{n}{x}$ we may express this as

$$\epsilon H(f_p) + H(f_\gamma) \leq (1 + \epsilon)H(\nu) + O(\frac{\log \ell}{\epsilon}). \hspace{1cm} (67)$$

Using the expression for $\nu$ above and assuming that $\epsilon \ll 1$ gives

$$H(\nu) = H\left(\frac{nf_p + \ell f_\gamma - nx}{n+\ell}\right) \hspace{1cm} (68)$$

$$= H(f_\gamma) + \epsilon |f_\gamma + x - f_p| \cdot 1 - H(f_\gamma) + (x - f_p) \cdot \log f_\gamma + O(\epsilon^2) \hspace{1cm} (69)$$

$$= H(f_\gamma) - \epsilon (f_p - x) \cdot \log f_\gamma - \epsilon H(f_\gamma) + O(\epsilon^2). \hspace{1cm} (70)$$

Here $1$ is the vector of all ones, and we have made use of the fact that $f \cdot 1 = 1$ for any frequency vector $f$, which also implies $x \cdot 1 = 0$. Altogether we obtain the relation

$$-x \cdot \log f_\gamma \leq D(f_p||f_\gamma) + O(\frac{\log \ell}{\epsilon}). \hspace{1cm} (71)$$
The next step is to fix the protocol to the worst case among the likely frequency vectors \( f_\rho \) and \( f_\gamma \). Their probabilities sharply peaked around the individual distributions \( \rho \) and \( \gamma \), respectively. Specifically, fixing an error parameter \( \delta \), the probability that \( ||f_\rho - \rho||_1 \geq \delta \) is less than a quantity of order \( e^{-n\delta^2} \). Thus we can choose \( \epsilon = O(\frac{1}{\sqrt{n}}) \) and still be sure that the particular frequency \( f_\rho \) strays more than a distance \( \epsilon \) from \( \rho \) with only a small probability. Even in the worst, but still probable case we have

\[-x \cdot \log \gamma \leq D(\rho||\gamma) - O(\frac{1}{\sqrt{n}}). \tag{72}\]

Now \( \log \gamma = -\beta H - 1 \log Z \), so this condition becomes

\[\beta H \cdot x \leq D(\rho||\gamma) - O(\frac{1}{\sqrt{n}}). \tag{73}\]

This equation gives the minimum amount of extractable work among all the likely frequencies, which is taken to be the target amount for the process. As the extraction is unitary for each frequency, and these correspond to disjoint quantum states, we can thus find a unitary for the entire input capable of generating \( \frac{1}{\beta} \left[D(\rho||\gamma) - O(\frac{1}{\sqrt{n}})\right] \) units of useful work per input resource state, with probability greater than \( 1 - O(\frac{1}{\sqrt{n}}) \).

**Structure of the exhaust state**

When doing a transformation of \( \rho^\otimes n \) into \( \sigma^\otimes m \), in the end of the protocol we actually obtain \( \sigma^\otimes m \otimes \pi_k \) and we trace out \( \pi_k \), which lives in \( k = \Omega(n) \) copies of system. Although \( \pi_n \) is usually far away from many copies of a Gibbs state, in fidelity, we show its reductions are very close to a Gibbs state. The main observation is that because \( \pi_n \) should be useless for extracting more copies of \( \sigma \) at a non-zero rate, we must have

\[S(\pi_k||\rho^\otimes k_\beta) \leq k^{1-\delta}, \tag{74}\]

for \( \delta > 0 \). But by by subadditivity of the entropy we have

\[S(\pi_k||\rho^\otimes k_\beta) = -S(\pi_k) - \sum_{l=1}^{k} \text{Tr}(\rho_{k,l} \log \rho_\beta) \]

\[\geq - \sum_{l=1}^{k} S(\pi_{k,l}) - \sum_{l=1}^{k} \text{Tr}(\rho_{k,l} \log \rho_\beta) \]

\[= \sum_{l=1}^{k} S(\pi_{k,l}||\rho_\beta), \tag{75}\]

where \( \pi_{k,l} := \text{Tr}_{\setminus l}(\pi_k) \) is reduced state of \( \pi_k \) obtained by partial tracing all the systems except the \( l \)-th one. Let us assume for simplicity that all \( \pi_{k,l} \) are the identical. Then

\[S(\pi_{k,1}||\rho_\beta) \leq k^{-\delta}, \tag{76}\]

which by Pinsker’s inequality implies

\[||\pi_{k,1} - \rho_\beta||_1 \leq \Omega(k^{-2\delta}). \tag{77}\]

More generally, repeating the same argument for larger blocks we get that

\[||\text{Tr}_{L,L+1,...,k}(\pi_k) - \rho^\otimes L_\beta||_1 \leq \Omega(Lk^{-2\delta}). \tag{78}\]

**Equivalence and degree of control for thermal operations**

Here, we address two questions. The first is how our paradigm, where we use unitaries \( V \) which commute with the total Hamiltonian \( H \), relates to other approaches. The second is how much control an experimenter needs over the choice of unitaries \( V \). To answer the first question, consider a common approaches to thermodynamics, which is to manipulate thermodynamical systems using an external apparatus. In this model, the systems are manipulated using
a time-dependent Hamiltonian, $H(t)$. Another approach is to add an interaction term $H_{\text{int}}$ between various systems we are trying to manipulate (e.g. the resource, and the heat bath), and then bring these systems in contact. Let us now see that these are equivalent to considering unitaries $V$ which commute with the original Hamiltonian $H$.

First note, that in the case of a time-dependent Hamiltonian, we can simply include the clock as one of our systems, so that $H(t)$ becomes $H(\tau)+\Pi_\tau$ with $[\tau, \Pi_\tau] = -i\hbar$. Then evolves like a clock, as can be seen by solving the Heisenberg equations of motion to get $\tau(t) = t$. Thus $H(\tau) = H(t)$, however, we conserve energy in this picture. We can go from a picture with a changing Hamiltonian, to one with a fixed one. This model is thus equivalent to the one considered here, with fixed Hamiltonian.

Likewise, in the case where an interaction term is added, we can take the total Hamiltonian to be $H_{\text{tot}} = H + H_{\text{int}}$ and assume that initially, $(H + H_{\text{int}})|\psi\rangle \approx H|\psi\rangle$ i.e. the systems are initially far apart. They can then evolve unitarily, such that the systems interact, and then move far enough apart that the interaction terms are negligible again. In such a picture, an eigenstate of the initial Hamiltonian $H$, will evolve into an eigenstate of $H$ with the same energy (by conservation of energy, and the fact that the interaction is negligible at initial and final times). Thus, all that happens here, is that eigenstates of fixed energy evolve to other eigenstates of the same energy, and this can be accomplished by means of a fixed Hamiltonian $H$ and a unitary $V$ which commutes with it. We thus see that also the picture of adding interaction terms is equivalent to having a fixed Hamiltonian $H$, and operations $V$ which commute with it.

Let us now turn to the second question. It might appear that an experimenter who wished to implement our protocols would need to very carefully manipulate all the many degrees of freedom of the systems and the heat bath. However, this is not the case, as we will now see by taking distillation of work as an example. There, we were mapping eigenstates which had a type $lq$ on the heat bath $\tau$, and $pn$ on the resource $\rho$ to microstates which had type $rk$ on the garbage $\sigma$, and $m$ 1’s on the work system. However, although for any implementation of the protocol, we need a particular mapping of strings (i.e. microstates) of these initial types, to strings of the final type, any mapping will do. The only important thing which is required is just that the unitary operation map the initial types to the final types. Thus an experimenter who wishes to implement the protocol, does not need fine-grained control over the mapping of microstates within one type to microstates within another type. She only needs to know that the unitary maps one type into another. I.e. there are an exponentially large number of possible implementations of our protocols each of which map particular strings within the initial types to particular strings in the final types. However, it doesn’t matter which implementation is chosen, and the experimenter thus does not need the fine degree of control of the particular implementation.

We can think of the type as being like a macroscopic variable such as the total magnetisation of many system, or total energy (indeed it is the latter). In the distillation protocol, we map the macroscopic variables of energy on two large systems ($\tau^\otimes l$ and $\rho^\otimes m$) to the macroscopic variable of energy on the final system. Any unitary which accomplishes this, will successfully implement our protocol. Thus, the experimenter only needs control over the macroscopic variables, not the microscopic ones.

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