Regularity Criteria for the 3D Magneto-Hydrodynamics Equations in Anisotropic Lorentz Spaces

Maria Alessandra Ragusa 1,2,* and Fan Wu 3

1 Department of Mathematics, University of Catania, Viale Andrea Doria No. 6, 95128 Catania, Italy
2 RUDN University, Miklukho-Maklay St, 117198 Moscow, Russia
3 Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, China; wufan0319@mail.hunnu.edu.cn
* Correspondence: mariaalessandra.ragusa@unict.it

Abstract: In this paper, we investigate the regularity of weak solutions to the 3D incompressible Magneto-Hydrodynamic (MHD) equations in anisotropic Lorentz space. We provide a regularity criterion for weak solutions involving any two groups functions \((\partial_1 u_1, \partial_1 b_1), (\partial_2 u_2, \partial_2 b_2)\) and \((\partial_3 u_3, \partial_3 b_3)\) in anisotropic Lorentz space.

Keywords: MHD equations; weak solution; regularity criteria; anisotropic Lorentz space

MSC: 76W05; 35Q30; 35B65

1. Introduction

In this paper, we are concerned with regularity criteria for the weak solutions to the incompressible magneto-hydrodynamic (MHD) equations in \(\mathbb{R}^3\) [1,2]:

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p &= (b \cdot \nabla)b, \\
\partial_t b + (u \cdot \nabla) b - \Delta b &= (b \cdot \nabla)u, \\
\nabla \cdot u &= \nabla \cdot b = 0, \\
u(x,0) &= u_0(x), b(x,0) = b_0(x),
\end{aligned}
\]

(1)

where \(u = (u_1, u_2, u_3)\) is the fluid velocity field, \(b = (b_1, b_2, b_3)\) is the magnetic field, \(p\) is a scalar pressure, and \(u_0, b_0\) is the prescribed initial data satisfying the compatibility condition \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\) in the distributional sense. Physically, Equation (1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals, and salt water.

Besides its physical applications, the MHD equations (1) have also mathematically significant. Duvaut and Lions [1] developed a global weak solution to (1) for initial data with finite energy, that is,

\[ u, b \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \quad \text{for any} \quad T > 0. \]

It is well known that the issue of regularity for weak solutions to the 3D incompressible Navier-Stokes equations has been one of the most challenging open problem in mathematical fluid mechanics [3], as well as that for the 3D incompressible magneto-hydrodynamics (MHD) equations (see Sermange and Temam [2]). Many sufficient conditions (see e.g., [4–14] and the references therein) were derived to guarantee the regularity of the weak solution. He and Xin [15] first extended the classical Prodi-Serrin conditions of Navier-Stokes equations to the MHD equations, they obtained regularity criteria involving only on velocity \(u\), i.e.,

\[ u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1 \quad \text{and} \quad 3 < p \leq \infty \]

(2)
\[ \nabla u \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty. \quad (3) \]

Later, He and Wang [16] showed that a weak solution \((u, b)\) is regular, provided only \(\nabla \omega^+ = (u + b)\) or \(\nabla \omega^- = (u - b)\) belongs to Beirao da Veiga’s class, that is,

\[ \nabla \omega^+ \quad \text{or} \quad \nabla \omega^- \in L^{q}(0, T; L^{p,\infty}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 2 \quad \text{and} \quad 3 \leq p \leq \infty. \quad (4) \]

Ni et al. [17] showed that one of the following conditions hold

\[
\begin{align*}
\nabla_h u & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \\
\partial_3 b & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty.
\end{align*}
\]

\[
\begin{align*}
\mu_3 & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1 \quad \text{and} \quad 3 < p \leq \infty, \\
\partial_3 b & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty,
\end{align*}
\]

\[
\begin{align*}
b_3 & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1 \quad \text{and} \quad 3 < p \leq \infty, \\
\partial_3 b & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty,
\end{align*}
\]

\[
\begin{align*}
\nabla_h u & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \\
\nabla_h b & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty,
\end{align*}
\]

then the weak solution \((u, b)\) is regular on \((0, T]\), where \(\nabla_h = (\partial_1, \partial_2)\). Recently, Jia [18] showed that condition (7) can be replaced by

\[
\begin{align*}
\nabla_h \bar{u} & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \\
\nabla_h \bar{b} & \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty,
\end{align*}
\]

where \(\bar{f} = (f_1, f_2)\). Regularity condition (8) was further improved by Xu et al. [19], more precisely, they proved that if any two quantities of

\[
\begin{align*}
A_i^{q,p}(T) & := \partial_i u_i \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \\
B_i^{q,p}(T) & := \partial_i b_i \in L^{q}(0, T; L^{p}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty,
\end{align*}
\]

where \(i = 1, 2, 3\), then the solution is smooth on interval \((0, T]\). For readers interested in this topic for partial components, please refer to [20–26] for recent progresses.

Motivated by papers cited above, the aim of this article is to study the regularity of weak solutions for the 3D MHD equations (1) in term of the two partial derivative of the velocity components and magnetic components on framework of the anisotropic Lorentz space. Before stating our main Theorem, we shall first recall the definitions of some function spaces [27].

**Lorentz Spaces**

Given a measurable function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) define the distribution function of \(f\) by

\[ d_f(a) = \mu(\{x : |f(x)| > a\}), \]

where \(\mu(A)\) (or \(|A|\)) denotes the Lebesgue measure of a set \(A\). We now define its decreasing rearrangement \(f^* : [0, \infty) \rightarrow [0, \infty]\) as

\[ f^*(t) = \inf\{a : d_f(a) \leq t\}, \]
with the convention that \( \inf \emptyset = \infty \). The point of this definition is that \( f \) and \( f^* \) have the same distribution function,
\[
d_{f^*}(x) = d_f(x),
\]
but \( f^* \) is a positive non-increasing scalar function.

**Definition 1.** Let \((p,q) \in [1, \infty]^2\), the Lorentz space \( L_p^q(\mathbb{R}^3) \) consists of all measurable functions \( f \) for which the quantity
\[
\| f \|_{L_p^q} := \begin{cases} 
\left( \int_0^{\infty} \left[ f^{*1}(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} & q < \infty, \\
\sup_{0 < t < \infty} t^q f^*(t) & q = \infty,
\end{cases}
\]
is finite.

In order to give the following definition involving anisotropic Lorentz space, we denote \( f = f(x_1, x_2, x_3) \) be a measurable function defined on \( \mathbb{R}^3 \), \( f^*(t) = f^{*1,*,*,*}(t_1, t_2, t_3) \). Here \( f^{*,*,*,*}(t_1, t_2, t_3) \) is the multivariate decreasing rearrangement of \( f(x_1, x_2, x_3) \) obtained by applying decreasing rearrangement \( f^{*,*}(t_1, x_2, x_3) \) of \( f(x_1, x_2, x_3) \) relating to the first variable \( x_1 \), under fixed the second, the third variables \( x_2, x_3 \), and then applying decreasing rearrangement \( f^{*,*,*}(t_1, x_2, x_3) \) of \( f^{*,*}(t_1, x_2, x_3) \) with respect to the second variable \( x_2 \) under fixed the first variable \( t_1 \) of \( f^{*,*,*}(t_1, x_2, x_3) \) and variable \( x_3 \), finally for variable \( x_3 \), by the same trick, we obtain the multivariate decreasing rearrangement \( f^{*,*,*,*}(t_1, t_2, t_3) \).

Recently, many works have been done for mixed-norm spaces. Stefanov-Torres [28] obtained the boundedness of Calderón-Zygmund operators on mixed-norm Lebesgue spaces. Georgiadis et al. [29] obtained various properties of anisotropic Triebel-Lizorkin spaces with mixed norms. In [30], Chen-Sun introduced the iterated weak and weak mixed-norm spaces and given some applications to geometric inequalities.

**Definition 2.** Let multi indexes \( p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \) be such that if \( 0 < p_i < \infty \), then \( 0 < q_i \leq \infty \), and if \( p_i = \infty \), then \( q_i = \infty \) for every \( i = 1, 2, 3 \) [31]. An anisotropic Lorentz space \( L_{p1}^{q1} \cap L_{p2}^{q2} \cap L_{p3}^{q3}(\mathbb{R}^3) \) is the set of functions for which the following norm is finite:
\[
\| f \|_{L_{p1}^{q1} \cap L_{p2}^{q2} \cap L_{p3}^{q3}} := \left( \int_0^{\infty} \left( \int_0^{\infty} \left( \int_0^{\infty} f^{*1,*,*,*}(t_1, t_2, t_3) \frac{dt_1}{t_1} \right)^{\frac{q_1}{p_1}} \frac{dt_2}{t_2} \right)^{\frac{q_2}{p_2}} \frac{dt_3}{t_3} \right)^{\frac{1}{q_3}}.
\]

Now, our main result reads:

**Theorem 1.** Suppose that \((u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \) and \( \nabla \cdot u_0 = \nabla \cdot b_0 = 0 \) in distributional sense. Let \((u, b)\) be the Leray-Hopf weak solution of (1) on \((0, T)\). If any two quantities
\[
\begin{align*}
A_i(T) := \int_0^T \left\| \partial_t u_i(t) \right\|_{L_{p1}^{q1}} \left\| \frac{1}{t} \left( \frac{2}{p} + \frac{2}{q} + \frac{1}{r} \right) \right\|_{L_{p2}^{q2}} dt, \\
B_i(T) := \int_0^T \left\| \partial_t b_i(t) \right\|_{L_{p1}^{q1}} \left\| \frac{1}{t} \left( \frac{2}{p} + \frac{2}{q} + \frac{1}{r} \right) \right\|_{L_{p2}^{q2}} dt,
\end{align*}
\]
are finite, where \( i = 1, 2, 3 \) with \( 2 < p, q, r \leq \infty \) and \( 1 - \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \geq 0 \), then the weak solution \((u, b)\) is actually smooth on interval \((0, T)\).
Remark 1. While $L^p(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3)$, clearly $L^{p,\infty}$ is a larger space than $L^p$. Therefore, from this point of view, condition (10) can be regarded as an extension of (7)–(9). In addition, our regularity criteria only depends on any two groups of functions $(\partial_1 u_1, \partial_1 b_1)$, $(\partial_2 u_2, \partial_2 b_2)$, and $(\partial_3 u_3, \partial_3 b_3)$). Hence, (10) can be as a significant improvement of condition (7) and (8). In addition, when $b = 0$, it is not to incompressible Navier-Stokes equations.

Remark 2. According to embedding relation $L^p(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3)$, we can obtain the following regularity criteria on framework of anisotropic Lebesgue space,

$$
\begin{align*}
A_i(T) := & \int_0^T \left\| \| \partial_i u_i(t) \|_{L^p} \right\|_{L^{p_2}}^{\frac{2}{2-(\frac{3}{r}+\frac{1}{q}+\frac{1}{p})}} dt < \infty, \\
B_i(T) := & \int_0^T \left\| \| \partial_i b_i(t) \|_{L^p} \right\|_{L^{p_2}}^{\frac{2}{2-(\frac{3}{r}+\frac{1}{q}+\frac{1}{p})}} dt < \infty,
\end{align*}
$$

(11)

where we should point out that for Equation (1), the regularity criterion (11) still new.

Remark 3. Notice that when fix $p = q = r$ in condition (11), the conditions (9) naturally turn out as stated in [19]. Furthermore, if let $p = q = r$ in condition (10), it is not difficult to find that our result improves the condition (4) significantly. Hence, regularity criteria (10) or (11) is much better. In other words, Theorem 1 can be regarded as a generalization of [16,18,19,23].

Before ending this section, we state the following lemmas, which will be used in the proof of our main result.

Lemma 1. (Young’s Inequality for Lorentz Spaces [32,33]) Let $1 < p < \infty, 1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1, \frac{1}{p} + \frac{1}{q'} = 1$. Suppose as well that $1 \leq p_1 < p'$ and $q' \leq q \leq \infty$. If $\frac{1}{p_2} + 1 = \frac{1}{p} + \frac{1}{p_1}$ and $\frac{1}{q_2} = \frac{1}{q} + \frac{1}{q_1}$, then the convolution operator,

$$
* : L^{p,q}(\mathbb{R}^n) \times L^{p_1,q_1}(\mathbb{R}^n) \rightarrow L^{p_2,q_2}(\mathbb{R}^n)
$$

is a bounded bilinear operator.

Lemma 2. (Hölder’s inequality in Lorentz spaces [33]) If $1 \leq p_1 < p_2, q_1 < q_2 \leq \infty$, then for any $f \in L^{p_1,q_1}(\mathbb{R}^n), g \in L^{p_2,q_2}(\mathbb{R}^n)$,

$$
\|fg\|_{L^{p,q}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1,q_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^n)},
$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

For any $s \geq 0$, even if $s$ not an integer, we can define the homogeneous Sobolev space $H^s(\mathbb{R}^n)$:

$$
H^s(\mathbb{R}^n) = \{ f \in S' : \hat{f} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi < \infty \}
$$

with the natural norm

$$
\|f\|_{H^s} = \left( \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},
$$

where $S'$ denotes the space of the tempered distributions on $\mathbb{R}^n$.

Lemma 3. For $2 < p < \infty$, there exists a constant $C = C(p)$ such that $f \in H^\frac{1}{p}(\mathbb{R})$, then $f \in L^{\frac{2p}{p-2}}(\mathbb{R})$ and

$$
\|f\|_{L^{\frac{2p}{p-2}}} \leq C\|f\|_{H^\frac{1}{p}}.
$$

(12)
**Proof.** We first make the pointwise definition, \( \gamma(\xi) = |\xi|^\frac{1}{p} \hat{f}(\xi) \); since \( f \in \dot{H}^\frac{p}{2}(\mathbb{R}) \), \( \gamma \in L^2(\mathbb{R}) \). If we set \( g = \mathcal{F}^{-1}\gamma \), then \( g \in L^2(\mathbb{R}) \) and \( \|g\|_{L^2} = \|\gamma\|_{L^2} = \|f\|_{\dot{H}^\frac{p}{2}} \). Now,

\[
\hat{f}(\xi) = \frac{|\xi|^\frac{1}{p} \hat{f}(\xi)}{|\xi|^\frac{1}{p}} = \hat{g}(\xi)|\xi|^{-\frac{1}{p}}.
\]

Combining the fact that if \( P_\alpha(x) = |x|^{-\alpha} \), then \( \hat{P}_\alpha(\xi) = C_\alpha \frac{1}{|\xi|^{\frac{N-\alpha}{p}}} \). Thus we obtain \( f = g \ast C_\frac{1}{p}^{-1} P_\frac{1}{p} \). The function \( P_\frac{1}{p} \) is \( L^{\frac{p}{p-\alpha}}(\mathbb{R}) \) but not in \( L^{\frac{p}{p-\alpha}}(\mathbb{R}) \). Applying Lemma 1, we find that

\[
\|f\|_{L^{\frac{2p}{2p-\alpha}}} = \left\| g \ast C_\frac{1}{p}^{-1} P_\frac{1}{p} \right\|_{L^{\frac{2p}{2p-\alpha}}} \\
\leq C \|g\|_{L^2} \left\| |x|^{-\frac{1}{p}} \right\|_{L^{\frac{p}{p-\alpha}}} \leq C \|f\|_{\dot{H}^\frac{p}{2}},
\]

\( \square \)

**Lemma 4.** There exists a positive constant \( C \) such that

\[
\left\| f \right\|_{L^{\frac{2p}{2p-\alpha}}} \leq C \|\partial_1 f\|_{L^2} \left\| \frac{1}{p} \right\|_{L^2} \left\| \frac{1}{q} \right\|_{L^2} \left\| \frac{1}{r} \right\|_{L^2} \left\| \frac{1}{\alpha} \right\|_{L^2} \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{\alpha} \right),
\]

for every \( f \in C^\infty_c(\mathbb{R}^3) \) where \( 2 < p, q, r \leq \infty, 1 - \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \geq 0 \).

**Proof.** Let \( \Lambda^p \) be the Fourier multiplier defined as

\[
\mathcal{F}_1(\Lambda^p f)(\xi_1, x_2, x_3) = |\xi_1|^p \mathcal{F}_1 f(\xi_1, x_2, x_3)
\]

with

\[
\mathcal{F}_1 f(\xi_1, x_2, x_3) = \int_{\mathbb{R}} e^{-i\xi_1 x_1} f(x_1, x_2, x_3)dx_1,
\]

\( \Lambda^p \) and \( \Lambda^q \) can be defined analogously. Then by Lemma 3, Minkowski’s inequality and Hölder’s inequality to obtain

\[
\left\| f \right\|_{L^{\frac{2p}{2p-\alpha}}} \leq C \left\| \Lambda^p f \right\|_{L^2} \left\| \Lambda^q f \right\|_{L^2} \left\| \Lambda^r f \right\|_{L^2} \left\| \Lambda^\alpha f \right\|_{L^2} \left\| \frac{1}{p} \right\|_{L^2} \left\| \frac{1}{q} \right\|_{L^2} \left\| \frac{1}{r} \right\|_{L^2} \left\| \frac{1}{\alpha} \right\|_{L^2} \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{\alpha} \right),
\]

\( \leq C \left\| \Lambda^q \Lambda^p f \right\|_{L^2} \left\| \frac{1}{q} \right\|_{L^2} \left\| \frac{1}{r} \right\|_{L^2} \left\| \frac{1}{\alpha} \right\|_{L^2} \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{\alpha} \right),
\]

Combining the Fourier-Plancherel formula and the Hölder’s inequality, we have
\[ C \left\| \Lambda^1 \Lambda^2 \Lambda^3 f \right\|_{L^2} \leq C \left( \int_{\mathbb{R}^3} |\xi_1|^\frac{2}{3} |\xi_2|^\frac{2}{3} |\xi_3|^\frac{2}{3} |Ff(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \right)^\frac{1}{2} \]

\[ \equiv C \left( \int_{\mathbb{R}^3} |\xi_1|^\frac{2}{3} |Ff(\xi)|^\frac{2}{3} |\xi_2|^\frac{2}{3} |Ff(\xi)|^\frac{2}{3} |Ff(\xi)|^2 \left( \frac{2}{3} + \frac{2}{3} \right) d\xi_1 d\xi_2 d\xi_3 \right)^\frac{1}{2} \]

\[ \leq C \|Ff\|^\frac{1}{2} \left( \int_{\mathbb{R}^3} |\xi_1|^2 |Ff|^2 d\xi \right)^\frac{1}{2} \left( \int_{\mathbb{R}^3} |\xi_2|^2 |Ff|^2 d\xi \right)^\frac{1}{2} \left( \int_{\mathbb{R}^3} |\xi_3|^2 |Ff|^2 d\xi \right)^\frac{1}{2} \]

(16)

Remark 4. In fact, since \( L^{\frac{2p}{p-2}, 2} \hookrightarrow L^{\frac{2p}{p-2}, \frac{2}{p}} \) for \( 2 < p < \infty \), we have similar result for estimate (14) in anisotropic Lebesgue space (for more details refer to [34]). However, we should point out that Lemma 4 holds in Lorentz space mainly depends on the Sobolev's embedding in Lemma 3.

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. The proof is based on the establishment of a priori estimates under condition (10).

Firstly, we note that, by the energy inequality, for weak solution \((u, b)\), we have

\[ \|u\|^2_{L^2} + \|b\|^2_{L^2} + 2 \int_0^T \|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2} dt \leq \|u_0\|^2_{L^2} + \|b_0\|^2_{L^2}. \]

(17)

Next, let us convert (1) into a symmetric form. Writing

\[ \omega^\pm = u \pm b, \]

we find by adding and subtracting \((1)_1\) with \((1)_2\),

\[ \begin{cases} \partial_t \omega^+ + (\omega^- \cdot \nabla) \omega^+ - \Delta \omega^+ + \nabla p = 0, \\ \partial_t \omega^- + (\omega^+ \cdot \nabla) \omega^- - \Delta \omega^- + \nabla p = 0, \\ \nabla \cdot \omega^+ = \nabla \cdot \omega^-, \\ \omega^+ (0) = \omega^+_0 \equiv u_0 + b_0, \\ \omega^- (0) = \omega^-_0 \equiv u_0 - b_0. \end{cases} \]

(18)

Taking the inner product of the \(i\)-th equation of (18) with \(|\omega_i^+|^2 \omega_i\) and \((18)_2\) with \(|\omega_i^-|^2 \omega_i\) (for \(i = 1, 2, 3\)) and integrating by parts in \(\mathbb{R}^3\) to get

\[ \frac{1}{4} \frac{d}{dt} \left( \|\omega_i^+\|^4_{L^4} + \|\omega_i^-\|^4_{L^4} \right) + \frac{1}{2} \left( \|\nabla |\omega_i^+|^2\|^2_{L^2} + \|\nabla |\omega_i^-|^2\|^2_{L^2} \right) \]

\[ + \|\omega_i^+ \cdot |\nabla \omega_i^+|\|^2_{L^2} + \|\omega_i^- \cdot |\nabla \omega_i^-|\|^2_{L^2} \]

\[ = - \int_{\mathbb{R}^3} \partial_i p |\omega_i^+|^2 \omega_i^+ dx - \int_{\mathbb{R}^3} \partial_i p |\omega_i^-|^2 \omega_i^- dx \equiv I + J, \]

(19)

we consider the \((u, b)\) satisfying condition (10) with any two quantities of \(A_i(T)\) and \(B_i(T)\) for \((i = 1, 2, 3)\):

\[ \begin{cases} A_i(T) := \int_0^T \left\| \partial_i u_i(t) \right\|_{L^\infty_{t^0} L^1_{\mathbb{R}^3}} \left\| \frac{2}{3} \left( \frac{2}{3} \right) \right\|_{L^\infty_{t^0} L^1_{\mathbb{R}^3}} dt < \infty, \\ B_i(T) := \int_0^T \left\| \partial_i b_i(t) \right\|_{L^\infty_{t^0} L^1_{\mathbb{R}^3}} \left\| \frac{2}{3} \left( \frac{2}{3} \right) \right\|_{L^\infty_{t^0} L^1_{\mathbb{R}^3}} dt < \infty. \end{cases} \]
In order to estimate the term $I$ and $J$ of (19), let us first establish an estimate between the $p$ and the $\omega$. Taking the divergence operator $\nabla \cdot$ on both sides of the first equations of (18), it follows that

$$-\Delta p = \text{div} (w^- \cdot \nabla w^+) = \text{div} (w^- \otimes w^+).$$

Similarly, taking $\nabla \text{div}$ operator on both sides of the first equation of (18) to obtain

$$-\Delta (\nabla p) = \nabla \text{div} (w^- \cdot \nabla w^+) = \nabla \text{div} (w^+ \cdot \nabla w^-).$$

By using the boundedness of Riesz transformations in $L^p$ $(1 < p < \infty)$ space, so we have

\[
\begin{aligned}
\|p\|_{L^p} &\leq C \|w^+\|_{L^{2p}} \|w^-\|_{L^{2p}}, \\
\|\nabla p\|_{L^{2p}} &\leq C \|w^+ \cdot \nabla w^-\|_{L^p}, \\
\|\nabla p\|_{L^p} &\leq C \|w^- \cdot \nabla w^+\|_{L^p}.
\end{aligned}
\]

(20)

Using the Hölder’s inequality, Young’s inequality, Lemma 4 and (20), we can deduce that

\[
I = -\int_{\mathbb{R}^3} \partial_i p |\omega_i^+|^2 \omega_i^+ \ dx \leq C \int_{\mathbb{R}^3} p |\omega_i^+|^2 \partial_i \omega_i^+ \ dx
\]

\[
\leq \left( \int_{\mathbb{R}^3} \|\partial_i \omega_i^+\|_{L^p_{\omega_i^+}} \right) \left( \int_{\mathbb{R}^3} \|p\|_{L^p_{\partial_i \omega_i^+}} \right)
\]

\[
\leq C \|\partial_i \omega_i^+\|_{L^p_{\omega_i^+}} \|\partial_1 p\|_{L^p_{\partial_1 \omega_i^+}} \|\partial_2 p\|_{L^p_{\partial_2 \omega_i^+}} \|\partial_3 p\|_{L^p_{\partial_3 \omega_i^+}} \|p\|_{L^p_{\partial_i \omega_i^+}} \left( \frac{1}{2} + \frac{1}{4} \right) \|\omega_i^+\|^2 \|\omega_i^+\|^2
\]

\[
\leq e \left( \|w^+ \cdot \nabla w^-\|_{L^2}^2 + \|w^- \cdot \nabla w^+\|_{L^2}^2 \right) + C \left( \int_{\mathbb{R}^3} \|\partial_i \omega_i^+\|_{L^p_{\omega_i^+}} \right)
\]

\[
\leq e \left( \|w^+ \cdot \nabla w^-\|_{L^2}^2 + \|w^- \cdot \nabla w^+\|_{L^2}^2 \right) + C \left( \int_{\mathbb{R}^3} \|\partial_i \omega_i^+\|_{L^p_{\omega_i^+}} \right)
\]

(21)

Similarly, for $J$, we have

\[
J = -\int_{\mathbb{R}^3} \partial_i p |\omega_i^-|^2 \omega_i^- \ dx \leq C \int_{\mathbb{R}^3} p |\omega_i^-|^2 \partial_i \omega_i^- \ dx
\]

\[
\leq e \left( \|w^+ \cdot \nabla w^-\|_{L^2}^2 + \|w^- \cdot \nabla w^+\|_{L^2}^2 \right) + C \left( \int_{\mathbb{R}^3} \|\partial_i \omega_i^-\|_{L^p_{\omega_i^-}} \right)
\]

\[
\leq e \left( \|w^+ \cdot \nabla w^-\|_{L^2}^2 + \|w^- \cdot \nabla w^+\|_{L^2}^2 \right) + C \left( \int_{\mathbb{R}^3} \|\partial_i \omega_i^-\|_{L^p_{\omega_i^-}} \right)
\]

(22)

Inserting (21) and (22) into (19) and summing up with respect to the index $i$ from 1 to 3, we get
\[
\frac{1}{4} \left( \| \omega^+ \|_{L^4}^4 + \| \omega^- \|_{L^4}^4 \right) + \frac{1}{2} \int_0^t \left( \| \nabla |\omega^+|^2 \|_{L^2}^2 + \| \nabla |\omega^-|^2 \|_{L^2}^2 \right) dt \\
+ \int_0^t \left( \| |\omega^+| \cdot |\nabla \omega^+| \|_{L^2}^2 + \| |\omega^-| \cdot |\nabla \omega^-| \|_{L^2}^2 \right) dt \\
\leq C \int_0^t \sum_{i=1}^3 \left( \left\| \partial_i \omega_i^+ \right\|_{L^p_{t,1}} \left\| \frac{3}{2} - \left( \frac{2}{p} + \frac{4}{3} + \frac{4}{3} \right) \right\|_{L^q_{t,3}} + \left\| \partial_i \omega_i^- \right\|_{L^p_{t,1}} \left\| \frac{3}{2} - \left( \frac{2}{p} + \frac{4}{3} + \frac{4}{3} \right) \right\|_{L^q_{t,3}} \right) \\
\cdot \left( \| \omega^+ \|_{L^4}^4 + \| \omega^- \|_{L^4}^4 \right) dt + C \int_0^t \left( \| u \cdot \nabla u \|_{L^2}^2 + \| b \cdot \nabla b \|_{L^2}^2 + \| u \cdot \nabla b \|_{L^2}^2 + \| b \cdot \nabla b \|_{L^2}^2 \right) dt \\
+ C \left( \| \omega^+_0 \|_{L^4}^4 + \| \omega^-_0 \|_{L^4}^4 \right),
\]
\[ \sup_{0 \leq t \leq T} \left( \| u(t) \|_{L^4}^4 + \| b(t) \|_{L^4}^4 \right) \]
\[ \leq C \exp C \int_0^T \sum_{i=1}^3 \left( \left\| \| \partial_i \omega_i^+ \|_{L^p_{t,x}} \|_{L^4_{t,x}} \right\|^{\frac{2}{3}} + \left\| \| \partial_i \omega_i^- \|_{L^p_{t,x}} \|_{L^4_{t,x}} \right\|^{\frac{2}{3}} \right) dt \]
\[ \leq C \exp C \int_0^T \sum_{i=1}^3 \left( \left\| \| \partial_i u_i \|_{L^p_{t,x}} \|_{L^4_{t,x}} \right\|^{\frac{2}{3}} + \left\| \| \partial_i b_i \|_{L^p_{t,x}} \|_{L^4_{t,x}} \right\|^{\frac{2}{3}} \right) dt \]
\[ < \infty. \]

Since \( u, b \in L^\infty \left( 0, T; L^4 \left( \mathbb{R}^3 \right) \right) \subset L^8 \left( 0, T; L^4 \left( \mathbb{R}^3 \right) \right) \), combining the classical Serrin-type regularity criterion (2), as in [15], then we complete the proof of Theorem 1.

3. Conclusions

This paper studies the MHD equations, and obtains a regularity criterion only involving the partial components of the \( \nabla u \) and \( \nabla b \). In addition, the anisotropic Lorentz space used in this article is broader than the general Lebesgue and Lorentz spaces. It seems that a slightly modified the technique in Theorem 1 can be applied to other incompressible fluid equations such as micropolar equations and the magneto-micropolar equations.

Author Contributions: Both authors contributed equally to this work. Both authors have read and agreed to the published version of the manuscript.

Data Availability Statement: All data generated or analysed during this study are included in this published article.

Acknowledgments: The first author is partially supported by I.N.D.A.M-G.N.A.M.P.A. 2019 and the “RUDN University Program 5-100”.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Duvaut, G.; Lions, J.L. Inéquations en thermoélasticité et magnétohydrodynamique. Arch. Ration. Mech. Anal. 1972, 46, 241–279. [CrossRef]
2. Sermange, M.; Temam, R. Some mathematical questions related to the MHD equations. Commun. Pure Appl. Math. 1983, 36, 635–664. [CrossRef]
3. Berselli, L.C.; Spirito, S. On the Existence of Leray-Hopf Weak Solutions to the Navier-Stokes Equations. Fluids 2021, 6, 42. [CrossRef]
4. Cao, C.; Wu, J. Two regularity criteria for the 3D MHD equations. J. Differ. Equ. 2010, 248, 2263–2274. [CrossRef]
5. Da Veiga, H.B. A new regularity class for the Navier-Stokes equations in \( \mathbb{R}^n \). Chin. Ann. Math. Ser. B 1995, 16, 407–412.
6. Gala, S.; Ragusa, M.A. A logarithmic regularity criterion for the two-dimensional MHD equations. J. Math. Anal. Appl. 2016, 444, 1752–1758. [CrossRef]
7. Alghamdi, A.M.; Gala, S.; Ragusa, M.A. A regularity criterion of smooth solution for the 3D viscous Hall-MHD equations. Aims Math. 2018, 3, 565–574. [CrossRef]
8. Gala, S.; Ragusa, M.A. A new regularity criterion for the 3D incompressible MHD equations via partial derivatives. J. Math. Anal. Appl. 2020, 481, 123497. [CrossRef]
9. Gala, S. Extension criterion on regularity for weak solutions to the 3D MHD equations. Math. Methods Appl. Sci. 2010, 33, 1496–1503. [CrossRef]
10. Jia, X.; Zhou, Y. Regularity criteria for the 3D MHD equations involving partial components. Nonlinear Anal. Real World Appl. 2012, 13, 410–418. [CrossRef]
11. Prodi, G. Un teorema di unicità per le equazioni di Navier-Stokes. Ann. Mat. Pura Appl. 1959, 48, 173–182. [CrossRef]
12. Serrin, J. On the interior regularity of weak solutions of the Navier-Stokes equations. Arch. Ration. Mech. Anal. 1962, 9, 187–195. [CrossRef]
13. Xu, F. A regularity criterion for the 3D incompressible magneto-hydrodynamics equations. *J. Math. Anal. Appl.* 2018, 460, 634–644. [CrossRef]

14. Zhou, Y.; Gala, S. Regularity criteria for the solutions to the 3D MHD equations in the multiplier space. *Z. Angew. Math. Und Phys.* 2010, 61, 193–199. [CrossRef]

15. He, C.; Xin, Z. On the regularity of weak solutions to the magnetohydrodynamic equations. *J. Differ. Equ.* 2005, 213, 235–254. [CrossRef]

16. He, C.; Wang, Y. Remark on the regularity for weak solutions to the magnetohydrodynamic equations. *Math. Methods Appl. Sci.* 2008, 31, 1667–1684. [CrossRef]

17. Ni, L.; Guo, Z.; Zhou, Y. Some new regularity criteria for the 3D MHD equations. *J. Math. Anal. Appl.* 2012, 396, 108–118. [CrossRef]

18. Jia, X. A new scaling invariant regularity criterion for the 3D MHD equations in terms of horizontal gradient of horizontal components. *Appl. Math. Lett.* 2015, 50, 1–4. [CrossRef]

19. Xu, F.; Li, X.; Cui, Y.; Wu, Y. A scaling invariant regularity criterion for the 3D incompressible magneto-hydrodynamics equations. *Z. Fur Angew. Math. Und Phys.* 2017, 68, 125. [CrossRef]

20. Wu, F. Blow-up criterion of strong solutions to the three-dimensional double-diffusive convection system. *Bull. Malays. Math. Sci. Soc.* 2019, 43, 2673–2686. [CrossRef]

21. Wu, F. A refined regularity criteria of weak solutions to the magneto-micropolar fluid equations. *J. Evol. Equ.* 2020, 1–10. [CrossRef]

22. Wu, F. Navier-Stokes Regularity Criteria in Viskik Spaces. *Appl. Math. Optim.* 2021, 1–15. [CrossRef]

23. Zhang, X. A regularity criterion for the solutions of 3D Navier-Stokes equations. *J. Math. Anal. Appl.* 2008, 34, 336–339. [CrossRef]

24. Zhang, Z. A Serrin-type regularity criterion for the Navier-Stokes equations via one velocity component. *Commun. Pure Appl. Anal.* 2013, 12, 117–124. [CrossRef]

25. Zhang, Z.; Yang, X. Remarks on the blow-up criterion for the MHD system involving horizontal components or their horizontal gradients. *Ann. Pol. Math.* 2016, 116, 87–99. [CrossRef]

26. Zhang, Z.; Chen, Q. Regularity criterion via two components of vorticity on weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. *J. Differ. Equ.* 2005, 216, 470–481. [CrossRef]

27. Lorentz, G. Some new functional spaces. *Ann. Math.* 1950, 51, 37–55. [CrossRef]

28. Stefanov, A.; Torres, R.H. Calderón-Zygmund operators on mixed Lebesgue spaces and applications to null forms. *J. Lond. Math. Soc.* 2004, 70, 447–462. [CrossRef]

29. Georgiadis, A.G.; Johnsen, J.; Nielsen, M. Wavelet transforms for homogeneous mixed-norm Triebel-Lizorkin spaces. *Monatshefte für Math.* 2017, 183, 587–624. [CrossRef]

30. Chen, T.; Sun, W. Iterated weak and weak mixed-norm spaces with applications to geometric inequalities. *J. Geom. Anal.* 2020, 30, 4268–4323. [CrossRef]

31. Bekmaganbetov, K.A.; Toleugazy, Y. On the Order of the trigonometric diameter of the anisotropic Nikol’skii-Besov class in the metric of anisotropic Lorentz spaces. *Anal. Math.* 2019, 45, 237–247.

32. Lemaire-Rieusset, P.G. *Recent Developments in the Navier-Stokes Problem*; CRC Press: Boca Raton FL, USA, 2002. [CrossRef]

33. O’Neil, R. Convolution operators and $L(p,q)$ spaces. *Duke Math. J.* 1963, 30, 129–142. [CrossRef]

34. Liu, Q.; Zhao, J. Blow-up criteria in terms of pressure for the 3D nonlinear dissipative system modeling electro-diffusion. *J. Evol. Equ.* 2018, 18, 1675–1696. [CrossRef]