Quantum phase transition (QPT) is one of the most interesting problems in these days.\(^1\) It is often argued that the simple Ginzburg-Landau theory does not apply to certain class of the QPT’s. In this paper we shall study \(s = 1/2\) antiferromagnetic (AF) Heisenberg model on 2-dimensional square lattice with nonuniform exchange couplings,

\[
H_{\text{AF}} = \sum_{x,j} \beta_{xj} \vec{S}_x \cdot \vec{S}_{x+j}
\]

where \(x\) denotes site of the spatial lattice, \(j\) is the direction index \((j = 1, 2)\) and \(\vec{S}_x\) is the spin operator at site \(x\). We rename the even lattice sites \(x = (o, i)\) where \(o\) denotes odd site and the index \(i = 1, 2, 3\) and 4 specifies its four nearest-neighbor (NN) even sites (see Fig.1).

The exchange couplings \(J_{xj} = J_{oi}\) are position dependent and we explicitly consider the following case which corresponds to the dimer configuration,

\[
\begin{align*}
J_{oi} &= J + \Delta J_{oi}, \\
\Delta J_{oi} &= \begin{cases} \\
\Delta J_{o1} = \alpha J, & i = 1 \\
\Delta J_{oi} = -\alpha J, & i = 2, 3, 4 \\
\end{cases}
\end{align*}
\]

where \(0 \leq \alpha \leq 1\) is a control parameter which connects the uniform Heisenberg model to the dimer model.

It is not so difficult to derive the \(CP^1\) field-theory model\(^2,3\) from Eq.(1).\(^4\) The spin operator \(\vec{S}_x\) can be expressed in terms of the \(CP^1\) variable \(z_x = (z^1_x, z^2_x)^+\) as

\[
\vec{S}_x = \frac{1}{2} z_x^1 \bar{\sigma} z_x,
\]

where \(\bar{\sigma}\) are the Pauli matrices and the \(CP^1\) constraint \(\sum_{o=1,2} |z^o_x|^2 = 1\) guarantees the magnitude of the localized spin as \(1/2\).

From our assignment of \(J_{oi}\) (2), it is obvious that \(J_{o1}\) is larger than the others. We use the path-integral formalism and parameterize the \(CP^1\) variable \(z_o\) by referring to \(z_{o1} \equiv z_e\),

\[
z_o = p_o z_e + \sqrt{1 - |p_o|^2} e^{i\theta} z_e,
\]

where \(p_o\) is a parameter, \(e^{i\theta}\) is a phase factor and \(z_e = i\sigma_2 z_e^*\). At vanishing temperature \((T)\), spins tend to point antiparallel their NN spins, and then the parameter \(p_o\) can be treated as a small parameter. We expand \(\sqrt{1 - |p_o|^2} \simeq 1 - \frac{1}{2} |p_o|^2 \cdots\) and retain only terms up to quadratic of \(p_o\). Then we perform the Gaussian integration of \(p_o\)’s to obtain an effective model of \(z_e\)’s for which smooth configurations dominate at \(T = 0\).

Calculation is rather long but straightforward\(^4\) and we obtain an effective field theory of the AF Heisenberg model under study,

\[
\mathcal{L}_{\text{CP}} = \sum_{\mu = r, t} |D_\mu z_x|^2 + \sigma_x \left( |z_x|^2 - \frac{1}{\mu_{\text{eff}}} \right) + \mathcal{L}_B,
\]
where \( D_\mu z_x = (\partial_\mu + iA_\mu)z_x \), \( A_\mu = iz^\dagger \partial_\mu z \) is the \( U(1) \) gauge field, \( \sigma_x \) is the Lagrange multiplier for the \( CP^1 \) constraint and \( \mathcal{L}_B \) is the Berry phase term. We have rescaled the imaginary time \( \tau \) and \( x \) as taking the continuum limit (the spatial coordinate \( x \) in Eq. (5) denotes the \textit{even site} of the original lattice). The effective coupling constant \( f_{\text{eff}} \) and the “speed of light” \( c \), which is often set unity, are explicitly given as

\[
\frac{1}{f_{\text{eff}}} = \frac{1}{2\sqrt{2}a} \cdot \frac{1 - \alpha \sqrt{\frac{2(2 + \alpha)}{1 - \alpha}}}{\frac{1}{f_{\text{AFH}}} b(\alpha)}, \quad (6)
\]

\[
c = \sqrt{\frac{2aJ}{\hbar}} \cdot \sqrt{\frac{\frac{2(2 + \alpha)(1 - \alpha)}{2}}{\frac{1}{f_{\text{AFH}}} (\alpha)}}, \quad (7)
\]

where \( a \) is the lattice spacing of the original lattice and \( f_{\text{AFH}} = 2\sqrt{2}a \) and \( c_{\text{AFH}} = \sqrt{\frac{2aJ}{\hbar}} \).

We should comment on the Berry phase terms, \( \mathcal{L}_B \), which appear in the \( CP^1 \) nonlinear-\( \sigma \) model representation of the AF Heisenberg model. First we consider AF spin chains in one dimension in order to obtain important insight for the effect of the Berry phase terms. For the uniform AF chains, the Berry phase reduces to the \( \text{AFH} \) (6) and it gives terms like \( \sigma \theta \) by the time evolution of \( \mathcal{L}_B \) and \( \sigma \theta \) appears as \( \mathcal{L}_B \). We have the effective coupling \( f_{\text{eff}} \) and \( c_{\text{AFH}} \) as

\[
V_{\text{eff}} = \sigma (|z^2|^2 - \frac{1}{f_{\text{eff}}} + 1) \left[ (\sigma + \Lambda^2)\sqrt{\sigma + \Lambda^2 - \Lambda^2 - \sigma \sqrt{\sigma}} \right],
\]

where the cutoff \( \Lambda = \frac{1}{\sqrt{2\pi}} \). The effective potential \( V_{\text{eff}} \) indicates that there exists a critical coupling \( f_C = \frac{4}{\pi} \). The existence of the phase transition has been verified by the numerical calculation of the equivalent \( O(3) \) nonlinear-\( \sigma \) model in \((2 + 1)\) dimensions. In the weak-coupling region \( f_{\text{eff}} < f_\text{C} \), the spontaneous symmetry breaking occurs and \( \langle z^2 \rangle \neq 0 \). As a result, the Higgs phase is realized in the gauge-theory terminology. Low-energy excitations are gapless spin waves which are described by \( z^2 \).

In the strong-coupling phase \( f_{\text{eff}} > f_\text{C} \), \( \langle z^2 \rangle \neq 0 \) whereas \( \langle z^2 \rangle = 0 \). Local Maxwell terms appear in the effective action of the gauge field \( \tilde{A}_\mu \) and the confinement phase is realized. Low-energy excitations are \( s = 1 \) composites of the spinons which correspond to \( \tilde{n}_x \) with (mass) \( 2\pi \alpha \) for \( \sigma \).

The effective coupling \( f_{\text{eff}} \) in (6) first decreases as \( \alpha \) increases but above certain value of \( \alpha \) it starts to increase and goes to infinity at the dimer limit \( \alpha = 1 \). In the uniform case \( \alpha = 0 \), \( f_{\text{eff}}(\alpha = 0) < f_\text{C} \) and this means that the ordered Néel state is realized at the vanishing \( T \) in the AF Heisenberg model in two spatial dimensions as it is now widely believed. The behavior of the effective coupling \( f_{\text{eff}}(\alpha) \) shows the existence of a critical value \( \alpha_C \) at which the phase transition occurs. This result indicates that the strong-coupling phase of the \( CP^1 \) model (5) corresponds to the dimer phase in which the ground state is nothing but spin-singlet pairs formed by the alternative strong exchange couplings and excitations have \( s = 1 \). In fact this Néel-dimer transition was observed by the numerical calculations some years ago.

Hereafter we are interested in the critical point at \( f_{\text{eff}} = f_\text{C} \) which separates the Néel and dimer phases. In order to investigate that “phase”, study on the gauge dynamics is required. At \( f_{\text{C}}, \langle \sigma_x \rangle = \langle z^2 \rangle = 0 \). Effective action of the gauge field and the field \( \sigma_x \) is obtained by integrating out \( z^2 \) (\( \alpha = 1, 2 \)). The resultant effective action becomes nonlocal and therefore it is possible for the gauge dynamics to belong to \textit{different} universality class from that of the usual gauge theory in \( 2 + 1 \) dimensions. By the continuum field-theory calculation, the effective action of the gauge field \( A_\mu \) is obtained as

\[
\mathcal{L}_A \propto \int d^3x \int d^3y \sum_{\mu,\nu} F_{\mu\nu}(x) \frac{1}{|x-y|^2} F_{\mu\nu}(y), \quad (8)
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Similarly the effective action of the field \( \sigma_x \) is obtained as
\[ \mathcal{L}_\sigma \propto \int d^3p \; \hat{\sigma}(-p) \frac{1}{|p|} \hat{\sigma}(p), \] (9)

where \( \hat{\sigma}(p) \) is the Fourier transformed field of \( \sigma_x \). Equation (9) shows that fluctuations of the field \( \sigma_x \) are strongly suppressed at large distances. In the \( CP^1 \) model on the 3-dimensional space-time cubic lattice, a similar expression of the action \( \mathcal{L}_A \) and \( \mathcal{L}_\sigma \) is obtained by the hopping expansion of \( \varepsilon_a^\tau \) \((a = 1, 2)\), in the effective action \( S_A \) of the compact gauge field \( U_\mu = e^{iA_\mu} \), the following nonlocal terms appear

\[ S_A \sim \sum_{\Gamma} \gamma_{|\Gamma|} \prod_{\Gamma} U_\mu(x), \] (10)

where the summation over closed loops \( \Gamma \) includes loops of an arbitrary large size, \(|\Gamma|\) is the length of \( \Gamma \) and the parameter \( \gamma \) is estimated as \( \gamma \sim \frac{1}{2d} \) for massless \( \varepsilon_a^\tau \) with the dimension of the lattice \( d = 3 \). We shall focus our interest on the gauge dynamics of the above nonlocal action which is one of the most important problems in the theoretical studies on the strongly-correlated electron systems in these days and is still controversial. At present it is known that there exists only the confinement phase in the \((2 + 1)\)-dimensional compact \( U(1) \) gauge theory without matter couplings.\(^9\) However phase structure is not clear when the \( U(1) \) gauge field couples to matter fields, particularly gapless matter fields\(^10\)-\(^14\). In particular in Ref.\(^10,11\), it is argued that a deconfinement phase is realized by the gapless fermion couplings. Simple mean-field type argument is not applicable for the nonlocal gauge systems and numerical studies are required.

We shall study lattice gauge model with a nonlocal action which is related with (10) but slightly more tractable. The summation over \( \Gamma \) in \( S_A \) (10) becomes (logarithmically) divergent for the configuration \( U_\mu = 1 \) since the massless relativistic bosons \( \varepsilon_a^\tau \) \((a = 1, 2)\), which appear at the critical point, give divergent hopping expansion for \( U_\mu = 1 \). From the above discussion we shall consider the following \((2 + 1)\) dimensional lattice gauge model with the long-range interaction in the \( \tau \) direction,

\[
S_G = g_1 \sum_{n=1}^{N_\tau - 1} \sum_{x, \mu = 1, 2} \frac{1}{n} U_\mu(x) W_{x+\mu}(n) U_\mu^\dagger(x + n\tau) W_{x}^\dagger(n) + g_2 \prod_{p_1, \ldots, p_l} U,
\] (11)

where \( W_n = U_0(x) U_0(x + \tau) \cdots U_0(x + n\tau) \), \( N_\tau \) is the system size in the \( \tau \) direction and \( g_i \) \((i = 1, 2)\) are coupling constants for the time and spatial directions, respectively. From action (11), it is obvious that the gauge model under study has nonlocal coupling in the \( \tau \) direction whereas it has the usual local Maxwell-type correlation in the spatial directions.\(^15\) Reason why we take the action (11) is that the \( \tau \) direction terms logarithmically divergent for \( U_\mu = 1 \) and also Monte-Carlo simulations are easier for the model than those with full-nonlocal interaction terms. We think that studies of the model (11) give important insight for the full-nonlocal gauge system (10). More comments on this point will be given after showing the results of the Monte-Carlo simulations of the model (11).

![FIG. 2. Fluctuation of energy as a function of \( g_1 \) is plotted for the fixed \( g_2 = 1.0 \) and 3.0. Lattice size is \( 12^2 \times 16 \) and \( 9^2 \times 12 \). The results show the existence of a phase transition.](image)

![FIG. 3. The correlations of Polyakov lines as functions of their distance are plotted. The results for \( g_1 = 0.12 \) and 0.14 are shown in (a), and one for \( g_1 = 0.16 \) is shown in (b). Value of \( g_2 \) is fixed as \( g_2 = 1.0 \). The results show the existence of the confinement-deconfinement phase transition.](image)
the coupling constant $g_1 \sim 1$ corresponds to (10) in which the damping factor $\gamma |r|$ balances the entropy factor of the paths $\Gamma$.

In the deconfinement phase of (11), topologically nontrivial configurations are suppressed and the field-theory result (8) gives a qualitatively correct picture. Charges interacting through $A_\mu$ have the potential $V(r) \propto 1/r$ where $r$ is the spatial distance between the two charges.

Let us comment on the effects of the Berry phase. Since $\mathcal{L}_B$ is neglected in (11), one may doubt the deconfinement phase transition observed above. However, as the Berry phase generates extra phases for topologically nontrivial configurations in the path integral, the Berry phase enhances the deconfinement. In fact without these extra phases, all instanton contributions contribute additively to disorder the gauge system. Thus the existence of the deconfinement phase in the gauge system (11) guarantees its existence even in the presence of the Berry phase. Similar argument was used for the deconfinement transition at finite $T^{18}$, which is established at present.

We summarize the phase structure of the original spin model. In the region $\alpha < \alpha_C$ ($f_{\text{eff}} < f_C$), the low-energy excitations are the massless spin waves whereas in the region $\alpha > \alpha_C$ ($f_{\text{eff}} > f_C$), they are $s = 1$ excitations $\vec{n}_x$. On the critical point $\alpha = \alpha_C$ ($f_{\text{eff}} = f_C$), the gauge dynamics is in the "Coulomb" phase and the low-energy excitations are the $s = 1/2$ bosonic spinons $z^n_x$ ($a = 1, 2$) which are interacting with each other by the potential $1/r$. The spin correlation function decays algebraically both in the Néel state and at the criticality but exponent is different. In the Néel state, the spin operator is given as $\vec{S}_x = 1/2 z^1_x \sigma_x \sim \langle z^1_x \rangle \sigma_x \sim z^1_x$ whereas at the criticality $\vec{n}_x$, the bilinear of $z^n_x$ and $z^{\alpha \dagger}_x$ ($a = 1, 2$). Phase structure of the nonuniform AF Heisenberg model is schematically shown in Fig.4.

![Schematic phase diagram of the Heisenberg model. Phases A, B, and C are the Higgs (Néel), Coulomb (critical) and confinement (dimer) phases, respectively.](image)

Recently a similar phase transition from the Néel to the dimer states was discussed$^{19}$. There they conclude that instanton effects are irrelevant at the critical point. Our numerical investigation is consistent with their result but our study shows the long-range interactions of the gauge field, which appear as a result of the coupling to the massless boson $z^n_x$, play an essentially important role. Results of more detailed studies on the long-range gauge theories will be reported in near future.$^{20}$

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15. In the strong-coupling phase, the field $z_x$ acquires nonvanishing mass by $\langle \sigma_x \rangle \neq 0$. In this case, the summation over $\Gamma$ is convergent even for $U_\tau = 1$.
16. The model (11) can be extended for the strong-coupling phase in which $\langle \sigma_x \rangle \neq 0$. The nonlocal terms become as $\sum n=1 \frac{1}{n} \langle \sigma_{y,\tau} \rangle U_{\tau, n} W_{\tau, n} (x) W_{\tau, n}^\dagger (x + n \tau)$.
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