Two Numerical ways RO (SuMM) and RO (MSuSu)) to Evaluate triple integrals

Safaa M. Aljassas, Wafaa H. Hanoon and Showq Mohammed E.

Email: safaa.musa@uokufa.edu.iq

Abstract. The main goal of this research is to calculate the triple integrals which has continuous functions numerically by two methods are RO(SuMM) and RO(MSuSu) that obtained by Romberg acceleration with two composite rules. The first rule is the Mid-point method on the first dimension X and the second dimension Y with a suggested method on the third dimension Z and, which is denoted by SuMM. The second rule is a suggested method on the first dimension X and the second dimension Y with the Mid-point method on the third dimension Z and, which is denoted by MSuSu, where the number of divisions on these three dimensions is exactly the same. We represented two theorems with their proofs to find such rules and the correction terms (the error terms) for each of rule. Therefore, we obtained high accuracy results in a relatively few sub intervals and short time.

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1. Introduction

There are many authors was interested in evaluating the triple integrals such as Dheyaa [4], in 2009, she introduced numerical composite method (RMRM(RS), RMRM(RM), RMRS(RM) and RMRS(RS)). These methods have obtained from Romberg acceleration method with Midpoint method (RM) on the third dimension (Z), RS(RS), RS(RM), RM(RM), RM(RS) on the second dimension (Y) and first dimension (X) she got good results.

In 2010, Eghaar [7], introduced numerical method to calculate the value of triple integrals by Romberg acceleration method on the resulting values from applying Midpoint method on three dimensions X, Y and Z. She got good results in terms of accuracy and a relatively few sub intervals.
In 2017, Muosa and others [14] presented a numerical method to calculate triple integrals using Aitken’s Acceleration on the conclusion values that obtained from applied Mid-point rule on the dimension z and Simpson’s rule on the two dimensions x and y and symbolized by AMSS. And they got good results.

Mohammed et al. [2] presented in 2013 numerical method to evaluate the value of triple integrals with continuous functions by RSSS method that obtained from Romberg acceleration with Simpson’s rule on three dimensions X, Y and Z as the same approach of Eghaar [7].

In 2015, Aljassas [13] introduced a numerical method \( RM (RMM) \) to calculating triple integrals with continuous functions by using Romberg acceleration with Mid-point rule on the three dimensions when the number of divisions on the first dimension is equal to the number of divisions on the second dimension, but both of them are different from the number of divisions on the third dimension and she got a high accuracy in the results in a little sub-intervals relatively and a short time.

There are many researchers who introduced numerical methods that mixed between the Mid-point and, Simpson, … etc. and used Romberg acceleration or Aitken’s Acceleration to get good results in terms of accuracy and speed depending on the number of sub-intervals and times, see [5,15].

Also in 2015, Sarada et al. [11] use the generalized Gaussian Quadrature to evaluate triple integral and got a good results.

Many researchers presented a new methods to increase the accuracy of numerical integrals results, see [1,8,10,12].

In our research, we introduced two theorems with their proofs to find two new numerical rules to evaluate approximate values for triple integrals with continuous functions and their correction terms. These two methods are derived from the application of Romberg acceleration besides the two composite rules, (first from the Mid-point method on the first and second dimensions with a suggested method on the third dimension) and (second from the suggested method on the first and second dimensions with the Mid-point method on the third dimension) where the number of divisions on these three dimensions is exactly the same. We recalled these rules by \( RO(SuMM) \) and \( RO(MSuSu) \) respectively.

### 2. Mid-Point Rule

Suppose that the integral \( H \) defined as the following

\[
H = \int_{r}^{s} k(\chi) d\chi = \bar{M}(p) + \sigma_M(p) + R_M \quad \ldots(1)
\]

such that \( \bar{M}(p) \) is Mid-Point rule to evaluate the value of integral \( H \) and \( \sigma_M(p) \) is correction terms for \( M(p) \) and \( R_M \) is the reminder that related by truncation from \( \sigma_M(p) \) after using specified terms from \( \sigma_M(p) \) and \( p = \frac{s-r}{n} \), \( n \) is the number of partial intervals of \( [r,s] \).

The general formula for the Mid-Point rule \( \bar{M}(p) \) is:

\[
\bar{M}(p) = p \sum_{r=1}^{n} k(r + \frac{2(2e-1)}{2} p) \quad \ldots(2) \quad [9].
\]
And the correction terms of Mid-Point rule for continuous integrations by using the mean value theorem for the formulas (2) we get:-
\[
\sigma_{M}(p) = \frac{(s-r)}{6}p^2k^{(2)}(\sigma_1) - \frac{7(s-r)}{360}p^4k^{(4)}(\sigma_2) + \frac{31(s-r)}{15120}p^6k^{(6)}(\sigma_3) - \cdots \quad (3)
\]
Such that \( i = 1, 2, 3, \ldots \), \( \sigma_i \in (r, s) \), [7].

3. Suggested method (Su)
In [3] were introduced a suggested method, where is considers one of the methods of computing singular integrals. The general form is
\[
Su(p) = \frac{p}{4} \left[ k(r) + k(s) + 2k(r + (n - 0.5)p) + 2\sum_{i=1}^{n-1} k(r + (i - 0.5)p) + k(r + ep) \right] \quad (4)
\]
And the correction terms of this rule by using the mean value theorem are:-
\[
\sigma_{Su}(p) = \frac{(s-r)}{24}p^2k^{(2)}(\tau_1) - \frac{5(s-r)}{1440}p^4k^{(4)}(\tau_2) + \frac{61(s-r)}{60480}p^6k^{(6)}(\tau_3) - \cdots \quad (5)
\]
Such that \( i = 1, 2, 3, \ldots \), \( \tau_i \in (r, s) \).

4. Romberg Accelerating
Suppose that \( \beta_1 \) and \( \beta_2 \) are two approximations values of the integral
\[
H = \int_{r}^{s} k(\chi)d\chi
\]
for two different values of \( p \) say \( p_1 \) and \( p_2 \) by using any rule of Newton-Cotes formulas then:
\[
H = \int_{r}^{s} k(\chi)d\chi = \beta_1 + \sum_{f=1}^{\infty} A_f p_1^f \quad \cdots (6)
\]
\[
H = \int_{r}^{s} k(\chi)d\chi = \beta_2 + \sum_{f=1}^{\infty} A_f p_2^f \quad \cdots (7)
\]
Where \( \beta_1 \) and \( \beta_2 \) are approximately values for the integral by the Mid-point rule, \( H \) is exactly integral value and \( \sum_{f=1}^{\infty} A_f p_1^f \) is corrections bounds. By solve the equations (6) and (7) we get:-
\[
H = \int_{r}^{s} k(\chi)d\chi = \frac{p_1^2 \beta_2 - \beta_1}{p_1^2 - p_2^2} + \sum_{f=1}^{\infty} A_f p_1^f \left( \frac{p_2^f - p_1^f}{p_1^2 - p_2^2} \right) \quad \cdots (8)
\]
If suppose that \( p_2 = 2 \) \( p_1 \) in equation (8), then we get :-
\[
H = \frac{2^c \beta_2 - \beta_1}{2^c - 1} + \sum_{f=1}^{\infty} \frac{2^c - 2^f}{2^c - 1} A_f p^f \quad \cdots (9)
\]
Where \( p = \frac{s-r}{n} \), \( n \) is the number of partial intervals of \([r, s]\).
And \( \frac{2^c - 2^f}{2^c - 1} A_f \) are constants do not depend on \( p \) and \( c \) is the power of corrections terms,

So \( H \approx \frac{2^c \beta_2 - \beta_1}{2^c - 1} \) \hspace{1cm} \text{(10)}

The above form is Romberg accelerating to improve integral values numerically, [6].

5. The Numerical Rule SuMM and Its Corrections Terms

**Theorem 1.**

Let \( k(\chi, \gamma, \zeta) \) is continuous function and derivable at each point of the region \([r,s] \times [t,u] \times [v,w] \), then the approximate value of \( H = \int_{r}^{s} \int_{t}^{u} \int_{v}^{w} k(\chi, \gamma, \zeta) d\chi d\gamma d\zeta \) can be evaluated by the following rule:

\[
\text{SuMM} = \frac{1}{4} \sum_{j=1}^{n} \sum_{v=1}^{n} \left[ k(\chi_v, \gamma_j, \nu) + k(\chi_v, \gamma_j, \nu) + 2k(\chi_v, \gamma_j, \nu) + (n - 0.5)p \right] \\
+ 2 \sum_{\ell=1}^{n-1} \left[ k(\chi_v, \gamma_j, \nu + (\ell - 0.5)p) + k(\chi_v, \gamma_j, \nu + tp) \right]
\]

Where \( \chi_v = r + \frac{(2e - 1)}{2} p \), \( e = 1, 2, \ldots, n \), \( \gamma_j = t + \frac{(2j - 1)}{2} p \), \( j = 1, 2, \ldots, n \)

And the correction form (error formula) is: \( H - \text{SuMM}(p) = \eta_1 p^2 + \eta_2 p^4 + \eta_3 p^6 + \ldots \)

Where \( \eta_1, \eta_2, \eta_3, \ldots \in R \)

**Proof.** The integral \( H \) can be written as:

\[
H = \int_{r}^{s} \int_{t}^{u} \int_{v}^{w} k(\chi, \gamma, \zeta) d\chi d\gamma d\zeta = \text{SuMM}(p) + \sigma_{\text{SuMM}}(p) \hspace{1cm} \text{(11)}
\]

Where \( \text{SuMM}(p) \) represents the approximation value of integral using a Mid-point method on the first and second dimensions and suggested method on the third dimension, \( \sigma_{\text{SuMM}}(p) \) are the correction terms which could be added to the values of \( \text{SuMM}(p) \), and \( p = \frac{s-r}{n} = \frac{u-t}{n_1} = \frac{v-w}{n_2}, (n = n_1 = n_2) \)

The single integral \( \int_{r}^{s} k(\chi, \gamma, \zeta) d\chi \) can be calculated numerically by Mid-point method on the dimension \( \chi \) (dealing with \( \gamma \) and \( \zeta \) as constants), so:

\[
\int_{r}^{s} k(\chi, \gamma, \zeta) d\chi = \frac{1}{2} \int_{r}^{s} \int_{\gamma}^{u} k(\chi, \gamma, \zeta) d\gamma d\zeta = \frac{1}{2} \int_{r}^{s} \int_{\gamma}^{u} k(\chi, \gamma, \zeta) d\gamma d\zeta \\
= \frac{31(s-r)}{15120} p^2 k_3(\alpha, \gamma, \zeta) - \ldots \hspace{1cm} \text{(12)}
\]
By integrate the form (12) numerically on the interval \([r, u]\) by using the Mid-point method on \(\gamma\), then we get:

\[
\int_{r}^{u} \int_{r}^{u} k(\chi, \gamma, \zeta) d\chi d\gamma = p^4 \sum_{j=0}^{\infty} \left[ k(r + \frac{(2j-1)}{2}p, \gamma, \zeta) + \frac{31(s-r)}{15120} p^4 k_j(\alpha_j, \gamma, \zeta) \right] + \frac{s-r}{6} p^4 k_j(\alpha_j, \gamma, \zeta) - \frac{7(u-r)}{360} p^4 k_j(\alpha_j, \gamma, \zeta)
\]

And by integrate the above formulas numerically on the interval \([v, w]\) by using suggested method on \(\zeta\), we will get:

\[
\int_{v}^{w} \int_{v}^{w} k(\chi, \gamma, \zeta) d\chi d\gamma = \frac{p^4}{4} \sum_{j=0}^{\infty} \left[ k(\chi, \gamma, \zeta) + k(\chi, \gamma, (n-0.5)p) + \frac{31(s-r)}{15120} p^4 k_j(\alpha_j, \gamma, \zeta) \right] + \frac{s-r}{6} p^4 k_j(\alpha_j, \gamma, \zeta) - \frac{7(u-r)}{360} p^4 k_j(\alpha_j, \gamma, \zeta) + \frac{31(s-r)}{15120} p^4 k_j(\alpha_j, \gamma, \zeta)
\]

And since \(k_{\chi}, k_{\gamma}, k_{\zeta}, \ldots, k_{\chi}, k_{\gamma}, k_{\zeta}, \ldots\) are continuous functions at \([r, s] \times [t, u] \times [v, w]\), so the formula of the correction term for the triple integral \(H\) using SumM becomes:

\[
\begin{align*}
\sigma_{\text{sumM}}(p) &= (s-r)(u-t)(w-v) \left( \frac{p^4}{6} \frac{\partial^2 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \chi^2} - \frac{7p^4}{360} \frac{\partial^4 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \chi^4} + \frac{31p^4}{15120} \frac{\partial^6 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \chi^6} \right) \\
+ (s-r)(u-t)(w-v) \left( \frac{p^4}{6} \frac{\partial^2 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \gamma^2} - \frac{7p^4}{360} \frac{\partial^4 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \gamma^4} + \frac{31p^4}{15120} \frac{\partial^6 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \gamma^6} \right) \\
+ (s-r)(u-t)(w-v) \left( \frac{p^4}{24} \frac{\partial^2 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \zeta^2} - \frac{5p^4}{1440} \frac{\partial^4 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \zeta^4} + \frac{61p^4}{60480} \frac{\partial^6 k(\bar{\chi}_{\gamma}, \bar{\mu}_{\gamma}, \bar{\psi}_{\gamma})}{\partial \zeta^6} \right)
\end{align*}
\]
evaluated by the following rule:

\[ w_{\text{MSuS}}(p) = (s-r)(u-t)(w-v)p^\ell \left\{ \begin{array}{c} \frac{1}{6} \frac{\partial^3 k}{\partial \chi^3} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] \\ \frac{1}{6} \frac{\partial^3 k}{\partial \gamma^3} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] \\ \frac{24}{24} \frac{\partial^3 k}{\partial \zeta^3} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] + \ldots \end{array} \right\} \\
+ (s-r)(u-t)(w-v)p^\ell \left\{ \begin{array}{c} \frac{7}{360} \frac{\partial^4 k}{\partial \chi^4} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] \\ \frac{360}{360} \frac{\partial^4 k}{\partial \gamma^4} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] \\ \frac{1440}{5} \frac{\partial^4 k}{\partial \zeta^4} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] + \ldots \end{array} \right\} \\
+ (s-r)(u-t)(w-v)p^\ell \left\{ \begin{array}{c} \frac{31}{15120} \frac{\partial^5 k}{\partial \chi^5} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] \\ \frac{15120}{15120} \frac{\partial^5 k}{\partial \gamma^5} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] \\ \frac{60480}{60480} \frac{\partial^5 k}{\partial \zeta^5} \left[ \varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}} \right] + \ldots \end{array} \right\} \right. \\
\right. \\
\where \left( \frac{1}{\varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}}}, \frac{1}{\varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}}} \right) \left( \frac{1}{\varphi_{\text{m}}, \mu_{\text{w}}, \psi_{\text{w}}} \right) \in [s,r] \times [u,t] \times [w,v] , m = 1, 2, 3, \ldots. 

If the integrand is continuous function and the partial derivatives is existing at each point of the integral region \([s,r] \times [u,t] \times [w,v] \), then the error form can be written as following

\[ H - \text{SuMM}(p) = \eta_1 p^2 + \eta_2 p^4 + \eta_3 p^6 + \ldots \quad (15) \]

Where \( \eta_1, \eta_2, \eta_3, \ldots \) are constants and they depend on partial derivatives for function at the integral region.

6. The Numerical Rule MSuS And Its Corrections Terms

Theorem 2.

Let \( k(\chi, \gamma, \zeta) \) is continuous function and derivable at each point of the region \([r,s] \times [t,u] \times [v,w] \), then the approximate value of \( \int \int \int k(\chi, \gamma, \zeta) d\chi d\gamma d\zeta \) can be evaluated by the following rule:

\[ \text{MSuS} = \frac{1}{2\pi} \sum_{j=1}^{\infty} k(r,t,\zeta) + k(r,u,\zeta) + k(s,t,\zeta) + k(s,u,\zeta) + 2k(r,t+(n-0.5)p,\zeta) + k(s,t+(n-0.5)p,\zeta) \]

\[ + k(r+(n-0.5)p,t,\zeta) + k(r+(n-0.5)p,u,\zeta) + 2k(r+(n-0.5)p,t+(n-0.5)p,\zeta) + \ldots \]

\[ + k(s+(n-0.5)p,t,\zeta) + k(s+(n-0.5)p,u,\zeta) + 2k(s+(n-0.5)p,t+(n-0.5)p,\zeta) + \ldots \]

\[ + 2k(s+(n-0.5)p,t,\zeta) + k(s+(n-0.5)p,u,\zeta) + k(r+i+p,t,\zeta) + k(r+i+p,u,\zeta) + 2k(r+i+p,t+(n-0.5)p,\zeta) + \ldots \]

\[ + 2k(r+i+p,t+(n-0.5)p,\zeta) + \ldots \]

\[ \zeta = v + \frac{(2\ell - 1)}{2} p \quad , \quad \ell = 1, 2, \ldots, n \]
And the correction form (error formula) is: \( H - MSuSu(p) = \zeta_1 p^2 + \zeta_2 p^4 + \zeta_3 p^6 + \cdots \)

Where \( \zeta_1, \zeta_2, \zeta_3, \ldots \in \mathbb{R} \)

**Proof.** The integral \( H \) can be written as:

\[
H = \iiint_{V} k(\chi, \gamma, \zeta) \, \chi \, d\chi \, d\gamma \, d\zeta = MSuSu(p) + \sigma_{MSuSu}(p) 
\]

(16)

Where \( MSuSu(p) \) represents the approximation value of integral using a suggested method on the first and second dimensions and the Mid-point rule on the third dimension, \( \sigma_{MSuSu}(p) \) are the correction terms which could be added to the values of \( MSuSu(p) \), and

\[ p = (s - r)/n = (u - t)/n_1 = (v - v)/n_2. \]

By following the same procedure in the previous proof we get:

\[
MSuSu = \frac{1}{16} \sum_{i=1}^{n} \left[ k(r,t,\zeta) + k(r,u,\zeta) + k(s,t,\zeta) + k(s,u,\zeta) + 2k(r,t+(n-0.5)p,\zeta) + k(s,t+(n-0.5)p,\zeta) \right. \\
+ k(r+(n-0.5)p,t,\zeta) + k(r+(n-0.5)p,u,\zeta) + 2k(r+(n-0.5)p,t+(n-0.5)p,\zeta) + k(r+(n-0.5)p,u+(n-0.5)p,\zeta) \left. \right] \\
+ 2k(r+(n-0.5)p,t+(n-0.5)p,u,\zeta) + k(r+i,p,t,\zeta) + k(r+i,p,u,\zeta) + 2k(r+(i-0.5)p,t+(n-0.5)p,\zeta) + k(r+i,p,t+(j-0.5)p,\zeta) \\
+ 2k(r+i,p,t+(n-0.5)p,\zeta) + \sum_{j=1}^{n_2} \left[ k(r+(i-0.5)p,t+j-0.5)p,\zeta) + k(r+i,p,t+(j-0.5)p,\zeta) \right] \\
+ k(r+0.5)p,t+j,p,\zeta) + (r+i,p,t+j,p,\zeta) \right].
\]

And the error formula is:

\[
H - MSuSu(p) = \zeta_1 p^2 + \zeta_2 p^4 + \zeta_3 p^6 + \cdots \quad (17)
\]

Where \( \zeta_1, \zeta_2, \zeta_3, \ldots \) are constants and they depend on partial derivatives for function at the integral region. To evaluate the triple integral numerically by \( RO(StMM) \) and \( RO(MSuSu) \), start by putting \( n = 1 \) and calculating the approximate value of triple integral by \( StMM \) and fix it in our tables, then when \( n = 2, n = 4 \) and so on. After that we will improve the results by using Romberg accelerating.

**Note that**: We used the Matlab language to write programs for this search to obtain the blog results in tables for the triple integrations below.

7. **Examples and Results**:

**Example(1)**: The integral \( \iiint_{V} (1.2 \chi + 0.4 \gamma) e^{\chi \gamma} \, d\chi \, d\gamma \, d\zeta \) which its analytical value is 0.11826551199798 (Rounded to 14 decimal places) with integrand is defined for all \((\chi, \gamma, \zeta) \in [0, 0.5] \times [0, 0.5] \times [0, 0.5] \), so the correction terms formula for this integral will...
be similar to the two formulas (15) and (17). The two tables (1) and (2) show the above integral value numerically by, \( RO(SuMM), RO(MSuSu) \) respectively.

1) We deduce from table (1) where \( n = 16 \) the value is correct for four decimal places using \( SuMM \) Then applying \( RO(SuMM) \) method, we obtained similar value to the analytical value (Rounded to 14 decimal places)

2) We can note from table (2) where \( n = 16 \) the value is correct for five decimal places using \( MSuSu \) , Then applying \( RO(MSuSu) \) method, we obtained similar value to the analytical value (Rounded to 14 decimal places).

Thus MSuSu method is the best from SuMM in accuracy.

Example(2): The integral \( I = \iiint_{2,2,2} \sqrt[3]{3} \log(\chi + \gamma + \zeta) d\chi d\gamma d\zeta \) which its analytical value is 3.18050262724506 (Rounded to 14 decimal places) with function is defined for all \((\chi, \gamma, \zeta) \in [2,3] \times [2,3] \times [2,3]\), so the correction terms formula for this integral will be similar to the two formulas (15) and (17). The two tables (5), (6) show the above integral value numerically by, \( RO(SuMM), RO(MSuSu) \) respectively. We deduce from two tables (5) and (6) where \( n = 32 \) the value is correct for five decimal places using \( SuMM \) and \( MSuSu \) , Then applying \( RO(SuMM) \) and \( RO(MSuSu) \) methods, we obtained similar value to the analytical value.

Example(3): The integral \( I = \iiint_{3,3,3} \frac{1}{\chi \gamma \zeta} d\chi d\gamma d\zeta \) which analytical value is unknown, and it have a continuous function for each point \((\chi, \gamma, \zeta) \in [3,4] \times [3,4] \times [3,4]\). we used \( RO(SuMM) \) and \( RO(MSuSu) \) are respectively, we obtained the results that are listed in tables (9) and (10) some details of these two tables are:-

The value is constant (for 14 decimal times) which are 1.36264629334865 when \( n = 16 \) and \( n = 32 \) for the two methods that are mentioned above.

8. Discussion and Conclusion

The results in this research show that the calculation of the approximate values for the triple integrals with continuous functions using two composite methods \( RO(SuMM) \) and \( RO(MSuSu) \) where the number of divisions on these three dimensions are equal give exact values (for several decimal times) with Romberg acceleration compared with exact values of integral using number from sub-intervals. In addition we obtained accuracy which is equal to twelve decimal times with sub-intervals between (16-32) for both previous methods.
| n  | SuMM       | RO(SuMM)       |
|----|------------|----------------|
| 1  | 0.11085491589780 | 0.11823435894437 |
| 2  | 0.11638949818273  | 0.11826551199493 |
| 4  | 0.11779503311790  | 0.11826549048420 |
| 8  | 0.11814779982535  | 0.11826551199798 |
| 16 | 0.11823607817280  | 0.11826551199798 |

| n  | MSuSu       | RO(MSuSu)       |
|----|-------------|-----------------|
| 1  | 0.11938581207088 | 0.11826981353063 |
| 2  | 0.11854881318243 | 0.11826981353063 |
| 4  | 0.11833654000712 | 0.11826951353063 |
| 8  | 0.11828328168673 | 0.11826951353063 |
| 16 | 0.1182695521334  | 0.11826951353063 |

| n  | MSSu       | RO(MSSu)       |
|----|------------|----------------|
| 1  | 3.18132843369472 | 3.18050262724518 |
| 2  | 3.18071102114072 | 3.18050262724518 |
| 4  | 3.180548812781  | 3.18050262724518 |
| 8  | 3.1805169013226  | 3.18050262724518 |
| 16 | 3.1805089344628  | 3.18050262724507 |
| 32 | 3.1805034482534  | 3.18050262724506 |

| n  | MSuSu       | RO(MSuSu)       |
|----|-------------|-----------------|
| 1  | 1.36150035928791 | 1.36230804336586 |
| 2  | 1.3623515252165040 | 1.36230804336586 |
| 4  | 1.36257205819242  | 1.36230804336586 |
| 8  | 1.36262796960522  | 1.36230804336586 |
| 16 | 1.36264164271866  | 1.36230804336586 |
| 32 | 1.36264513055387  | 1.36230804336586 |

| n  | MSSu       | RO(MSSu)       |
|----|------------|----------------|
| 1  | 3.18011085491589780 | 3.18011085491589780 |
| 2  | 3.18096530977597  | 3.18011085491589780 |
| 4  | 3.18050262689022  | 3.18011085491589780 |
| 8  | 3.18050262724518  | 3.18011085491589780 |
| 16 | 3.18050262724507  | 3.18011085491589780 |
| 32 | 3.18050262724506  | 3.18011085491589780 |

| n  | MSuSu       | RO(MSuSu)       |
|----|-------------|-----------------|
| 1  | 1.36203197842682 | 1.36203197842682 |
| 2  | 1.36248582326336 | 1.36203197842682 |
| 4  | 1.36260571503297 | 1.36203197842682 |
| 8  | 1.362661944387  | 1.36203197842682 |
| 16 | 1.36264374830110 | 1.36203197842682 |
| 32 | 1.36264565904040 | 1.36203197842682 |
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