Optimal Strategies for Static Black-Peg AB Game With Two and Three Pegs

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Abstract

The AB Game is a game similar to the popular game Mastermind. We study a version of this game called Static Black-Peg AB Game. It is played by two players, the codemaker and the codebreaker. The codemaker creates a so-called secret by placing a color from a set of $c$ colors on each of $p \leq c$ pegs, subject to the condition that every color is used at most once. The codebreaker tries to determine the secret by asking questions, where all questions are given at once and each question is a possible secret. As an answer the codemaker reveals the number of correctly placed colors for each of the questions. After that, the codebreaker only has one more try to determine the secret and thus to win the game.

For given $p$ and $c$, our goal is to find the smallest number $k$ of questions the codebreaker needs to win, regardless of the secret, and the corresponding list of questions, called a $(k + 1)$-strategy. We present a $\lceil 4c/3 \rceil - 1$-strategy for $p = 2$ for all $c \geq 2$, and a $\lfloor (3c - 1)/2 \rfloor$-strategy for $p = 3$ for all $c \geq 4$ and show the optimality of both strategies, i.e., we prove that no $(k + 1)$-strategy for a smaller $k$ exists.

Keywords: game theory, Mastermind, AB Game, optimal strategy

1. Introduction

The AB Game (also known as “bulls and cows game”) is a game similar to the popular game Mastermind. Whereas the first one dates back more than a century, the latter was invented by Meirowitz in 1970. Mastermind has since turned out to have interesting applications in fields such as cryptography, bioinformatics and privacy protection. In both games a codemaker and

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a codebreaker play against each other. The codemaker chooses a secret code by placing colors from a set of \( c \) available colors on \( p \) pegs. In the original version of the AB Game and Mastermind, respectively, \( p = 4 \), \( c = 6 \) and \( p = 4 \), \( c = 10 \), respectively, are fixed, but to make the computational and mathematical properties of the game interesting, at least one of these parameters must be made variable. The goal of the codebreaker is to discover the secret code by making a sequence of guesses until the secret has been found. Each guess is a possible secret. The corresponding answers of the codemaker consist of black and white pegs, a black one for each peg of the question which is correct in both position and color, and a white one for each peg which is correct in color but not in position. The goal of the codebreaker is to minimize the number of questions needed to find the secret, i.e., to receive \( p \) black pegs as the last answer. We call games of this kind codebreaking games.

Mastermind can be turned into a decision problem, Mastermind Satisfiability, as follows: given a list of questions and corresponding answers, are these answers compatible with at least one possible secret? Interestingly, in [23] this problem was shown to be \( \mathcal{NP} \)-complete. For further results on (non-static) Mastermind see for example [3, 16, 20].

The difference between Mastermind and the AB Game is that a possible secret or question of the latter may contain repeated colors whereas the AB Game requires the secret as well as each question to consist of \( p \) distinct colors. Thus, the distinction between the two games mirrors the well-known distinction made in combinatorics between an ordered choice of \( p \) items from a set of \( c \) elements either with or without replacement.

Both Mastermind and the AB Game also have a black-peg variant, where the answers of the codebreaker contain only black pegs (i.e., in addition to how many pegs are correct in both position and color, no further information about correct colors placed on the wrong pegs is provided).

While strategies for the Black-Peg AB Game were previously studied in [4, 17, 18], we continue our study of optimal strategies for the static black-peg variant of codebreaking games which was started in [10, 11, 13, 14, 15]. The static black-peg variant differs from the standard version described as follows.

The codebreaker is required to present all questions except the last one (which reveals the secret) right at the beginning of the game. Thus, the game is static, meaning that the codebreaker cannot adapt later questions to previous answers.

In [3] it was shown that the original and the static version of Mastermind need \( \mathcal{O}(n \log \log n) \) questions for the most prominent case \( n = c = p \). For the Static Black-Peg AB Game, for this case \( n = p = c \), a lower bound of \( \Omega(n \log n) \) and an upper bound of \( \mathcal{O}(n^{1.525}) \) were presented in [11], and this upper bound has recently been improved to \( \mathcal{O}(n \log n) \) [19].

A \((k + 1)\)-strategy is a sequence of \( k \) questions such that the answers to these questions uniquely determine the secret (i.e., the codebreaker can win the game with the \((k + 1)\)-th question). We are interested in optimal strategies – \((k + 1)\)-strategies where \( k + 1 \) is as small as possible. Let us denote this optimal \( k + 1 \) (depending on \( p \) and \( c \)) by \( s(p, c) \). Erdős and Rényi [5] and Söderberg and Shapiro [22] showed independently that \( s(p, c) \in \mathcal{O}(p/\log p) \) for \( c = 2 \), a result
that was later generalized to $c \leq p^{1-\epsilon}$ for $\epsilon > 0$ by Chvátal [2]. Goddard [12] developed a $([2c/3] + 1)$-strategy for two pegs and $c$-strategies for three and four pegs. For sufficiently large $c$, these strategies are optimal.

In [13], we presented an optimal $[(4c - 1)/3]$-strategy for Static Black-Peg Mastermind in the case of $p = 2$ pegs and an arbitrary number $c \geq 1$ of colors, and in [15] an optimal $([3c/2] + 1)$-strategy for Static Black-Peg Mastermind in the case of $p = 3$ pegs and an arbitrary number $c \geq 2$ of colors. We now continue this line of research by considering the corresponding cases of the Static Black-Peg AB Game with $p = 2$ and $p = 3$, respectively. We start with an investigation of the simpler $p = 2$ case and an arbitrary number $c$ of colors (where by definition of the AB Game, $c \geq 2$ must hold) resulting in an optimal $([4c/3] - 1)$-strategy. This strategy improves an earlier optimal strategy that was presented in [10] without explicit proof of feasibility and optimality. While our new strategy obviously has the same number of questions as the earlier one, the improvement lies in its structural simplicity.

Our main result of this work is an optimal $\lceil(3c - 1)/2\rceil$-strategy for the Static Black-Peg AB Game with $p = 3$ and an arbitrary number $c \geq 4$ of colors. The strategies for both cases for the Static Black-Peg AB Game with $p = 2$ and $p = 3$ have a similar structure. This structure can also be applied to and simplify our earlier strategies for Static Black-Peg Mastermind with $p = 2$ and $p = 3$ [13, 15]. Our expectation is that extending and comparing the strategies for Static Black-Peg Mastermind of [13, 15] and of the AB Game in this work will eventually make it possible to develop generic optimal strategies for arbitrary $p$ for both of these codebreaking games.

Interestingly, there is also a graph-theoretic interpretation of strategies for Static Black-Peg Mastermind to the so-called metric dimension. For an undirected graph $G$, the metric dimension of $G$ is defined as the minimum number of vertices in a subset $S$ of $G$ such that all other vertices are uniquely determined by their distances to the vertices in $S$, see for example [6, 21, 24] for results on special graph classes. Furthermore, the decision variant of this problem is $\mathcal{NP}$-complete [9].

Concretely, Static Black-Peg Mastermind with $p$ pegs and $c$ colors needing exactly $k$ questions is equivalent to the metric dimension of the graph $\mathbb{Z}_p^c$ being $k$ (see [15] for more details). Thus, the results of [12, 15] show that the metric dimension of $\mathbb{Z}_n \times \mathbb{Z}_n$ is $[(4c - 1)/3] - 1$ and that of $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ is $[3c/2]$. An open question is whether the results of the present paper can also be connected to the metric dimension of other types of graphs. In any case, our new way to assemble optimal strategies by iterating color-shifted copies of a fixed block can similarly be used to compose simpler optimal strategies for Static Black-Peg Mastermind than those in [13, 15]. In turn, this results in simplified proofs of the fact that the metric dimensions of $\mathbb{Z}_n \times \mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ are $[(4c - 1)/3] - 1$ and $[3c/2]$, respectively.

This work is set up as follows. After giving some preliminaries in Section 2, we present an optimal $([4c/3] - 1)$-strategy for $p = 2$ in Section 3 and an optimal $[(3c - 1)/2]$-strategy for $p = 3$ in Section 4. Finally, we conclude and suggest possible future work in Section 5.
2. Preliminaries

Let $p \geq 1$ denote the number of pegs and $c \geq p$ the number of colors. W.l.o.g., throughout the remainder of this paper, let the pegs be numbered by $1, 2, \ldots, p$, and the colors by $1, 2, \ldots, c$. We write the questions and secrets in the form $Q = (q_1 | q_2 | \ldots | q_p)$. The possible answers are written as $0B, 1B, 2B, \ldots, pB$.

For $k \in \mathbb{N}$, a $(k + 1)$-strategy for Static Black-Peg AB Game consists of $k$ questions which the codebreaker has to ask altogether at the beginning of the game. These are the so-called main questions. Such a strategy is feasible if every secret $S$ is uniquely determined by the $k$ answers. Having received these answers, the codebreaker can ask the final question $S$ to win the game.

We use the letter $k$ to denote the number of main questions, excluding the final question. Since we will only be concerned with the main questions, in the following we will generally omit the term “main”, referring to the main questions simply as questions. A $(k + 1)$-strategy is called optimal if there is no feasible $k$-strategy. Clearly, all questions of an optimal strategy must be distinct. Therefore, we shall in the following only consider strategies in which all questions are distinct, without explicitly mentioning this fact.

**Remark 1.** Determining an optimal strategy for the Static Black-Peg AB Game is trivial in the case $p = 1$ because the questions $(1), (2), \ldots, (c−1)$ reveal the secret and, obviously, no set consisting of fewer than $c−1$ questions does. Thus, in the case $p = 1$ this strategy is an optimal strategy for all $c$ (and so is any other strategy consisting of $c−1$ questions). In other words, there is an optimal $c$-strategy for $p = 1$ and every $c \geq 1$.

We need the following definition.

**Definition 1.** Let $Q = (q_1 | q_2 | \ldots | q_p)$ and $Q’ = (q’_1 | q’_2 | \ldots | q’_p)$ be questions.

(a) $Q$ and $Q’$ are called neighboring, if $q_i = q’_i$ for some $i \in \{1, 2, \ldots, p\}$. We say that they overlap in peg $i$ and call peg $i$ an overlapping peg (of $Q$ and $Q’$).

(b) $Q$ and $Q’$ are are double neighboring if they overlap in at least two distinct pegs.

(c) $Q$ and $Q’$ are disjoint, if $q_i \neq q’_j$ for all $i, j \in \{1, 2, \ldots, p\}$. More generally, we say that $Q$ and $Q’$ are disjoint in pegs $i_1, i_2, \ldots, i_k$ if $\{q_{i_1}, q_{i_2}, \ldots, q_{i_k}\} \cap \{q’_{i_1}, q’_{i_2}, \ldots, q’_{i_k}\} = \emptyset$.

(d) For $a_1, a_2, \ldots, a_p \in \mathbb{N}$, $Q$ is a $(a_1, a_2, \ldots, a_p)$-question of a given strategy if the $i$-th color of $Q$ occurs exactly $a_i$ times on the $i$-th peg of the

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Note the different notation in comparison to the question itself, where we separate the numbers by the symbol “|”. 

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In the remainder of the paper, we investigate the cases \( p = 2, 3 \). For both cases, the feasible strategy starts with so-called base questions for a small number of colors and then repeats a copy of a fixed block of questions, shifting the colors in each copy appropriately. This structure makes the proof of feasibility relatively easy. However, for the proof of optimality all possible feasible strategies have to be considered, without making specific assumptions regarding their structure, and it has to be excluded that they may require fewer questions.

### 3. Two Pegs

In this section let \( p = 2 \).

#### 3.1. A \( ([4c/3] - 1) \)-Strategy

We introduce a \( ([4c/3] - 1) \)-strategy for each \( c \geq 2 \) which we will later show to be feasible and optimal. We start with presenting such a strategy explicitly for \( c = 2, 3, 4 \) in Table 1. These three strategies have been found by brute-force computer search.

As mentioned, we create \( ([4c/3] - 1) \)-strategies for all \( c \geq 2 \) by starting with a block of so-called base questions and then repeatedly adding copies of a fixed block of questions, shifting the colors in each copy appropriately. As for the base questions, we start with one of the three strategies of Table 1c, which we repeat a suitable number of times (with shifted colors).

Concretely, let \( c \geq p = 2 \) and \( c = 3s + t \), where \( s \in \mathbb{N}_0 \), \( t \in \{2, 3, 4\} \) are uniquely determined. For \( t = 2 \) we start with the \( k = 1 \) question of the strategy of Table 1a for \( t = 3 \) with the \( k = 2 \) questions of the strategy of Table 1b and for \( t = 4 \) with the \( k = 4 \) questions of the strategy of Table 1c. As mentioned, we call this first group of questions the base questions.

In all three cases the base questions are followed \( s \) times by the 4 questions of Table 1c. For \( l = 1, 2, \ldots, s \), the number \( t + 3(l - 1) \) is added to the colors.
of the questions of Table 1. We call this second group of questions the *iterated* questions.

As examples, we present the corresponding \((\lceil 4c/3 \rceil - 1)\)-strategies for \(5 \leq c \leq 10\) explicitly in Table 2.

In Subsections 3.2 and 3.3 we prove the following theorem.

**Theorem 1.** The presented strategy is a feasible and optimal \((\lceil 4c/3 \rceil - 1)\)-strategy for \(p = 2\) and for the corresponding \(c \geq 2\).

In the following let \(h \equiv c \mod 3\).

**Remark 2.** (a) For \(h \in \{0, 2\}\), our presented strategy for the Static Black-Peg AB Game needs one question less than the strategy of \([13]\) for Static Black-Peg Mastermind (see Strategies 1 and 3, respectively, in \([13]\)).
(b) For \( h = 1 \), our presented strategy for the Static Black-Peg AB Game needs the same number of questions as the strategy of [13] for Static Black-Peg Mastermind (see Strategy 2 in [13]).

3.2. Feasibility of the \( \lceil 4c/3 \rceil - 1 \)-Strategy

Analyzing the \( \lceil 4c/3 \rceil - 1 \)-strategy, we observe the following:

**Observation 1.** (a) For \( h = 0 \), the following holds:

The \( \lceil 4c/3 \rceil - 1 \)-strategy contains only \((1, 2)\)-questions and \((2, 1)\)-questions, except the first two questions \( Q_1 = (1 | 2) \) and \( Q_2 = (3 | 1) \) which are both \((1, 1)\)-questions.

On the first peg only the color 2 is missing (throughout the entire strategy), and on the second peg only the color 3 is missing.

(b) For \( h = 1 \), the following holds:

The \( \lceil 4c/3 \rceil - 1 \)-strategy contains only \((1, 2)\)-questions and \((2, 1)\)-questions.

Both on the first and the second peg only the color 4 is missing.

(c) For \( h = 2 \), the following holds:

The \( \lceil 4c/3 \rceil - 1 \)-strategy contains only \((1, 2)\)-questions and \((2, 1)\)-questions, except the first question \( Q_1 = (1 | 2) \) which is a \((1, 1)\)-question.

On the first peg only the color 2 is missing, and on the second peg only the color 1 is missing.

Consider one fixed \((1, 2)\)-question or \((2, 1)\)-question, and assume that it receives a non-empty answer 2B or 1B, respectively. Then the following conclusions can be drawn.

(I) The answer is 1B.

Here we have to find out which one of the two pegs is correct. This is determined by the answer to the neighboring question: if that answer is also non-empty, then the color of the overlapping peg is correct, otherwise the color of the other peg is correct.

As an example, consider the strategy for \( c = 9 \) in Table 2e.

For the secret \((4 | 9)\), the answer to the question \( Q_3 = (4 | 6) \) is 1B. The neighboring question to \( Q_3 \) is \( Q_5 = (5 | 6) \). As \( Q_5 \) gives the answer 0B, we know that color 4 is correct on the first peg, but we do not know yet the color of the second peg.

On the other hand, for the secret \((3, 6)\), the answer to the question \( Q_3 \) is 1B and to \( Q_5 \) also 1B. Then we know that color 6 is correct on the second peg.

(II) The answer is 2B.

The secret is found.
So as soon we have a non-empty answer to a $(1,2)$-question or to a $(2,1)$-question, we know the color of one peg. However, having this information about one peg, also the other peg can be determined from the answers to all questions of the strategy, as by Observation 1 on each peg only one color is missing.

So we have shown the feasibility for the case that at least one $(1,2)$-question or at least one $(2,1)$-question receives a non-empty answer. To show the feasibility also for the remaining case that all $(1,2)$-questions and all $(2,1)$-questions receive an empty answer, we distinguish between $h = 0, 1, 2$.

(I) $h = 0$:
Six possible secrets are consistent with answers $0B$ to all $(1,2)$-questions and $(2,1)$-questions. We list them and the combination of answers to the $(1,1)$-questions $Q_1 = (1 \, | \, 2)\,$ and $Q_2 = (3 \, | \, 1)$:

(i) Secret $(1 \, | \, 2)$: Combination of answers $(2B, 0B)$.
(ii) Secret $(1 \, | \, 3)$: Combination of answers $(1B, 0B)$.
(iii) Secret $(2 \, | \, 1)$: Combination of answers $(0B, 1B)$.
(iv) Secret $(2 \, | \, 3)$: Combination of answers $(0B, 0B)$.
(v) Secret $(3 \, | \, 1)$: Combination of answers $(0B, 2B)$.
(vi) Secret $(3 \, | \, 2)$: Combination of answers $(1B, 1B)$.

(II) $h = 1$:
Only one possible secret is consistent with the answer $0B$ to all $(1,2)$-questions and $(2,1)$-questions, namely $(4 \, | \, 4)$.

(III) $h = 2$:
Two possible secrets are consistent with the answer $0B$ to all $(1,2)$- and $(2,1)$-questions, namely $(1 \, | \, 2)$ and $(2 \, | \, 1)$. However, these are distinguished by the only $(1,1)$-question $Q_1 = (1 \, | \, 2)$.

Thus, each possible secret is uniquely determined in all three cases, which finishes the proof of feasibility.

3.3. Optimality of the $(\lceil 4c/3 \rceil - 1)$-Strategy

We need the following lemma.

**Lemma 1.** For the Static Black-Peg AB Game with $p = 2$ the following statements hold:

(a) For each feasible strategy and for each peg there exists at most one color which does not occur on this peg.

(b) A feasible strategy cannot contain two disjoint $(1,1)$-questions.

(c) If a feasible strategy contains three $(1,1)$-questions, by possibly permuting the colors, these questions can be written in the form $(1 \, | \, 2)$, $(2 \, | \, 3)$, $(3 \, | \, 1)$. 
(d) If a feasible strategy contains three \((1,1)\)-questions, then on at least one of the two pegs of the questions, all \(c\) colors must occur.

(e) A feasible strategy cannot contain four \((1,1)\)-questions.

**Proof.**

1. The assertion is clear for \(c = 2\). Thus, let \(c \geq 3\). Assume without loss of generality that there are two colors \(a\) and \(b\) which do not occur on the first peg. Choose an arbitrary color \(x\) which is neither \(a\) nor \(b\) (\(x\) exists because \(c \geq 3\)). Then the possible secrets \((a|x)\) and \((b|x)\) receive the same combination of answers. Thus, the two possible secrets are indistinguishable, contradicting the feasibility of the strategy.

By symmetry, this also proves the corresponding statement for the second peg.

(b) Suppose that the strategy contains two disjoint \((1,1)\)-questions. So their four colors are distinct, say \((1|2)\) and \((3|4)\), and thus \(c \geq 4\). Then the possible secrets \((1|4)\) and \((3|2)\) receive the same answer \((1B)\) for both questions \((1|2)\) and \((3|4)\), and also the same answer \((0B)\) for all other questions. Thus, the two possible secrets are indistinguishable, leading to a contradiction.

(c) Let a feasible strategy contain three \((1,1)\)-questions, say \((q_1|q_1'), (q_2|q_2')\) and \((q_3|q_3')\). Without loss of generality, we may permute colors so that \(q_1 = 1, q_2 = 2, q_3 = 3\). (Note that \(q_1, q_2, q_3\) are distinct by the definition of \((1,1)\)-questions, and so are \(q_1', q_2', q_3'\).)

If some \(q_i\), say \(q_1\), differs from \(1, 2\) and \(3\), then not both \(q_2'\) and \(q_3'\) can be equal to \(1\). Thus, in this case \((1|q_1')\) and \((2|q_2')\), or \((1|q_1')\) and \((3|q_3')\) would be disjoint. By (b), this contradicts the feasibility of the strategy, thus showing that \(\{q_1', q_2', q_3'\} = \{1, 2, 3\}\).

As \(q_i = i\), the definition of the AB Game yields \(q_i' \neq i\), which means that

\[
q_1' \in \{2, 3\}, \quad q_2' \in \{1, 3\}, \quad q_3' \in \{1, 2\},
\]

which leaves only two possibilities, namely the claimed

\[
(q_1|q_1') = (1|2), \quad (q_2|q_2') = (2|3), \quad (q_3|q_3') = (3|1)
\]

or

\[
(q_1|q_1') = (1|3), \quad (q_2|q_2') = (2|1), \quad (q_3|q_3') = (3|2).
\]

In the second case, the claimed situation is obtained by switching colors 2 and 3 and switching the second and the third question.

(d) By (c), a feasible strategy may be assumed to contain three \((1,1)\)-questions \((1|2), (2|3), \) and \((3|1)\). Suppose that there is a color \(a\) which does not occur on the first peg of the questions and a color \(b\) which does not occur on the second peg of the questions. Then the possible secrets \((1|b)\) and \((a|2)\) receive the answer \(1B\) for the question \((1|2)\), and \(0B\) for all other questions, i.e., the same combination of answers. Thus, these possible secrets \((1|b)\) and \((a|2)\) are indistinguishable – a contradiction.
If a feasible strategy contains four \((1,1)\)-questions, by (c) the first three of them can be written in the form \((1\mid 2), (2\mid 3), \text{and} (3\mid 1)\). Adding a fourth \((1,1)\)-question would automatically lead to two disjoint \((1,1)\)-questions. This contradicts (b).

Now we come to the final part of the optimality proof. Note that the presented strategy uses 
\[ k = \left\lceil \frac{4c}{3} \right\rceil - 2 \] questions. In the following we prove that every feasible strategy consists of at least \(k\) questions. Thus, consider any feasible strategy. Let \(l_i\) be the number of colors which occur exactly once on the \(i\)-th peg of this strategy for \(i = 1,2\). Furthermore, let \(m\) be the number of \((1,1)\)-questions of the strategy. Observe that the total number of questions is at least \(l_1 + l_2 - m\). Again we distinguish between \(h = 0,1,2\).

(I) \(h = 0\).

We have
\[ k = \frac{4c}{3} - 2. \]

On the one hand, if \(l_i \leq 2c/3\) for some \(i \in \{1,2\}\), since at least \(c - 1\) colors occur on peg \(i\) (see Lemma 1(a)), we have at least
\[ \frac{2c}{3} + 2 \cdot \left( \frac{c}{3} - 1 \right) \geq \frac{4c}{3} - 2 = k \]
questions.

If, on the other hand, \(l_i \geq \frac{2c}{3} + 1\) for \(i = 1,2\) it follows that there are at least
\[ l_1 + l_2 - m \geq 2 \cdot \left( \frac{2c}{3} + 1 \right) - 3 \geq \frac{4c}{3} - 1 = k + 1 \]
questions, because we know from Lemma 1(e) that \(m \leq 3\).

(II) \(h = 1\).

We have
\[ k = \frac{4c - 1}{3} - 1. \]

We distinguish between two sub-cases.

(i) On both pegs only \(c - 1\) colors occur.

On the one hand, analogously to the case \(h = 0\), if \(l_i \leq \frac{2c-2}{3}\) for some \(i \in \{1,2\}\), we have at least
\[ \frac{2c - 2}{3} + 2 \cdot \left( \frac{c - 1}{3} \right) \geq \frac{4c - 1}{3} - 1 = k \]
(1)

\[ ^2 \text{Recall that } k \text{ does not include the final question.} \]
questions.
If instead \( l_i \geq \frac{2c+1}{3} \) for \( i = 1, 2 \), we have

\[
l_1 + l_2 - m \geq 2 \cdot \left( \frac{2c+1}{3} \right) - 2 \geq \frac{4c}{3} + \frac{2}{3} - \frac{6}{3} = k,
\]

because we know from Lemma 1(d) that \( m \leq 2 \).

(ii) On at least one peg, say peg 1, all \( c \) colors occur.
If \( l_1 \leq \frac{2c+1}{3} \), then there are at least

\[
\frac{2c+1}{3} + 2 \cdot \left( \frac{c-1}{3} \right) \geq \frac{4c-1}{3} = k + 1
\]

questions.
Recall also from Eq. (1) that we have at least \( k \) questions if \( l_2 \leq \frac{2c-2}{3} \), and assume thus that \( l_1 \geq \frac{2c+4}{3} \) and \( l_2 \geq \frac{2c+2}{3} \). Since \( m \leq 3 \) by Lemma 1(e), it follows that

\[
l_1 + l_2 - m \geq \frac{2c+4}{3} + \frac{2c+1}{3} - 3 = \frac{4c-1}{3} = k.
\]

(III) \( h = 2 \).
We have

\[
k = \frac{4c-2}{3} - 1.
\]

Analogously to the previous cases, if \( l_i \leq \frac{2c-1}{3} \) for some \( i \in \{1, 2\} \), then we have at least

\[
\frac{2c-1}{3} + 2 \cdot \left( \frac{c-2}{3} \right) \geq \frac{4c-2}{3} - 1 = k
\]

questions.
It remains to consider the case where \( l_i \geq \frac{2c+2}{3} \) for \( i = 1, 2 \). Again using the fact that \( m \leq 3 \) by Lemma 1(e) it follows that

\[
l_1 + l_2 - m \geq 2 \cdot \left( \frac{2c+2}{3} \right) - 3 = \frac{4c+4}{3} - 3 = \frac{4c-2}{3} - 1 = k,
\]

which completes the last case and thus the proof. \( \square \)

4. Three Pegs

In this section let \( p = 3 \).
We introduce a $\lfloor (3c-1)/2 \rfloor$-strategy for each $c \geq 4$ which we will later show to be feasible and optimal. Interestingly, such a feasible strategy does not exist for $c = 3$, i.e., no feasible 4-strategy exists although there are only $3! = 6$ possible secrets. One feasible and optimal 5-strategy for $c = 3$ is shown in Table 3.

We start by explicitly presenting $\lfloor (3c-1)/2 \rfloor$-strategies for $c = 4, 5, 6, 7, 8, 9$ in Table 4. These six strategies have been found by brute-force computer search.

Then we create $\lfloor (3c-1)/2 \rfloor$-strategies for all $c \geq 4$ in a way similar to the case $p = 2$. We start with one of the six strategies of Table 4 and append the strategy of Table 5 a suitable number of times, shifting the colors of each such block of questions appropriately.

Concretely, let $c = 6s + t$, where $s \in \mathbb{N}_0$, $t \in \{4, 5, 6, 7, 8, 9\}$ are uniquely determined. For $t = 4, 5, \ldots, 9$ we start with the questions of the strategies in Tables 4a–4f, respectively. Again we call this first group of questions the base questions.

In all six cases the base questions are followed $s$ times by the 9 questions of Table 5. For $l = 1, 2, \ldots, s$, the $l$-th of these $s$ blocks is obtained by adding the number $t + 6(l - 1)$ to the colors in Table 5. Again we call this second group of questions the iterated questions.

As examples, we present the corresponding $\lfloor (3c-1)/2 \rfloor$-strategies for $10 \leq c \leq 15$ explicitly in Table 6.

In Subsections 4.2 and 4.3 we prove the following theorem.

**Theorem 2.** The presented strategy is a feasible and optimal $\lfloor (3c-1)/2 \rfloor$-strategy for $p = 3$ and for the corresponding $c \geq 4$.

**Remark 3.**

(a) For even $c$, our presented strategy for the Static Black-Peg AB Game needs two questions less than the strategy from [13] for Static Black-Peg Mastermind (see Strategies 1 and 3 in [13]).

(b) For odd $c$, our presented strategy for the Static Black-Peg AB Game needs one question less than the strategy from [13] for Static Black-Peg Mastermind (see Strategies 2 and 4 in [13]).

---

Observe that $\lfloor (3 \cdot 3 - 1)/2 \rfloor = 4$, but this includes the final question.

---

Table 3: Feasible and optimal strategy for $p = 3$, $c = 3$, $k = 4$.

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| $Q_1$ | 1 | 2 | 3 |
| $Q_2$ | 1 | 3 | 2 |
| $Q_3$ | 2 | 1 | 3 |
| $Q_4$ | 2 | 3 | 1 |
Table 4: Feasible and optimal \( ([3c - 1]/2)] \)-strategies for \( p = 3 \) and \( 4 \leq c \leq 9 \).

Table 5: Feasible (but not optimal) strategy for \( p = 3, c = 6, k = 9 \).
Table 6: Feasible and optimal \([[\lfloor (3c-1)/2 \rfloor]])-strategies for \(p = 3\) and \(10 \leq c \leq 15\).

(a) \(c = 10, k = 13\).

(b) \(c = 11, k = 15\).

(c) \(c = 12, k = 16\).

(d) \(c = 13, k = 18\).

(e) \(c = 14, k = 19\).

(f) \(c = 15, k = 21\).

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \(Q_1\) | 1 | 2 | 3 |
| \(Q_2\) | 1 | 3 | 4 |
| \(Q_3\) | 2 | 3 | 4 |
| \(Q_4\) | 3 | 1 | 5 |
| \(Q_5\) | 4 | 2 | 5 |
| \(Q_6\) | 5 | 3 | 1 |
| \(Q_7\) | 6 | 3 | 2 |
| \(Q_8\) | 7 | 10 | 11 |
| \(Q_9\) | 8 | 9 | 10 |
| \(Q_{10}\) | 9 | 6 | 8 |
| \(Q_{11}\) | 10 | 7 | 9 |
| \(Q_{12}\) | 10 | 6 | 7 |
| \(Q_{13}\) | 10 | 8 | 7 |

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \(Q_1\) | 1 | 2 | 3 |
| \(Q_2\) | 1 | 3 | 4 |
| \(Q_3\) | 2 | 3 | 4 |
| \(Q_4\) | 3 | 1 | 5 |
| \(Q_5\) | 4 | 2 | 5 |
| \(Q_6\) | 5 | 3 | 1 |
| \(Q_7\) | 6 | 3 | 2 |
| \(Q_8\) | 7 | 10 | 11 |
| \(Q_9\) | 8 | 9 | 10 |
| \(Q_{10}\) | 9 | 6 | 8 |
| \(Q_{11}\) | 10 | 7 | 9 |
| \(Q_{12}\) | 10 | 6 | 7 |
| \(Q_{13}\) | 10 | 8 | 7 |
| \(Q_{14}\) | 11 | 9 | 8 |

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \(Q_1\) | 1 | 2 | 3 |
| \(Q_2\) | 1 | 3 | 4 |
| \(Q_3\) | 2 | 3 | 4 |
| \(Q_4\) | 3 | 1 | 5 |
| \(Q_5\) | 4 | 2 | 5 |
| \(Q_6\) | 5 | 3 | 1 |
| \(Q_7\) | 6 | 3 | 2 |
| \(Q_8\) | 7 | 10 | 11 |
| \(Q_9\) | 8 | 9 | 10 |
| \(Q_{10}\) | 9 | 6 | 8 |
| \(Q_{11}\) | 10 | 7 | 9 |
| \(Q_{12}\) | 10 | 6 | 7 |
| \(Q_{13}\) | 10 | 8 | 7 |
| \(Q_{14}\) | 11 | 9 | 8 |

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \(Q_1\) | 1 | 2 | 3 |
| \(Q_2\) | 1 | 3 | 4 |
| \(Q_3\) | 2 | 3 | 4 |
| \(Q_4\) | 3 | 1 | 5 |
| \(Q_5\) | 4 | 2 | 5 |
| \(Q_6\) | 5 | 3 | 1 |
| \(Q_7\) | 6 | 3 | 2 |
| \(Q_8\) | 7 | 10 | 11 |
| \(Q_9\) | 8 | 9 | 10 |
| \(Q_{10}\) | 9 | 6 | 8 |
| \(Q_{11}\) | 10 | 7 | 9 |
| \(Q_{12}\) | 10 | 6 | 7 |
| \(Q_{13}\) | 10 | 8 | 7 |
| \(Q_{14}\) | 11 | 9 | 8 |
| \(Q_{15}\) | 12 | 10 | 9 |

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \(Q_1\) | 1 | 2 | 3 |
| \(Q_2\) | 1 | 3 | 4 |
| \(Q_3\) | 2 | 3 | 4 |
| \(Q_4\) | 3 | 1 | 5 |
| \(Q_5\) | 4 | 2 | 5 |
| \(Q_6\) | 5 | 3 | 1 |
| \(Q_7\) | 6 | 3 | 2 |
| \(Q_8\) | 7 | 10 | 11 |
| \(Q_9\) | 8 | 9 | 10 |
| \(Q_{10}\) | 9 | 6 | 8 |
| \(Q_{11}\) | 10 | 7 | 9 |
| \(Q_{12}\) | 10 | 6 | 7 |
| \(Q_{13}\) | 10 | 8 | 7 |
| \(Q_{14}\) | 11 | 9 | 8 |
| \(Q_{15}\) | 12 | 10 | 9 |
4.2. Feasibility of the \( \lceil (3c - 1)/2 \rceil \)-Strategy

We start with the following observations.

**Observation 2.** For the 10-strategy for \( c = 6 \) of Table 5, the following statements hold:

(a) It consists entirely of \((1, 2, 2)\)-questions, \((2, 1, 2)\)-questions and \((2, 2, 1)\)-questions.

(b) It contains neighboring pairs of questions, but no double neighboring pairs.

(c) No color is missing on any of the pegs.

(d) It consists of three blocks of three questions each, all having the same structure, namely one \((1, 2, 2)\)-question, one \((2, 1, 2)\)-question and one \((2, 2, 1)\)-question. Each two of them are neighboring, but not double neighboring, and they are not neighboring to any other question.

**Observation 3.** On each peg of the six \( \lceil (3c - 1)/2 \rceil \)-strategies of Table 4, at most one color is missing.

Now we start with the proof of feasibility. Consider one fixed \((1, 2, 2)\)-question, \((2, 1, 2)\)-question or \((2, 2, 1)\)-question which lies in one of the copies of the 9-strategy of Table 5. We begin our reasoning by discussing which conclusions can be drawn if the question receives a non-empty answer 1B, 2B or 3B.

(I) The answer is 1B.

Here it is not clear which peg is correct. Again, this can be decided by the answers to the two neighboring questions. If both answers to the neighboring questions are empty, then the color of the non-overlapping peg is the correct one. Otherwise the color of the peg which overlaps with the neighboring question whose answer contains the larger number of blacks is the correct one.

As an example, consider the strategy for \( c = 12 \) in Table 6c. For the secret \((7 \mid 9 \mid 2)\), the answer to the question \( Q_8 = (7 \mid 11 \mid 12) \) is 1B. The two neighboring questions to \( Q_8 \) are \( Q_9 = (10 \mid 7 \mid 12) \) and \( Q_{10} = (10 \mid 11 \mid 7) \). As the answers to \( Q_9 \) and \( Q_{10} \) are both empty, we know that color 7 is correct on the first peg.

On the other hand, for the secret \((5 \mid 11 \mid 9)\), the answer to both the questions \( Q_8 \) and \( Q_{10} \) is 1B and the answer to \( Q_9 \) is empty. Consequently, color 11 must be correct on the second peg.

(II) The answer is 2B.

Here it is not clear which two pegs are correct. However, this can be decided by the answers to the two neighboring questions (which lie in the same block) because at least one of the answers to these neighboring questions is also non-empty. If both answers are non-empty, then the colors
of both overlapping pegs are correct. Otherwise the color of the peg which overlaps with the neighboring question that received the empty answer is the incorrect one.

As an example, consider again the strategy for \( c = 12 \) in Table 6c. For the secret (7|11|3), the answer to the question \( Q_8 \) is 2B. As \( Q_9 \) gives the empty answer and \( Q_{10} \) gives the answer 1B, we know that color 12 is not correct on the third peg, so that color 7 is correct on the first peg and color 11 is correct on the second peg.

On the other hand, for the secret (8|11|12), the answer to the question \( Q_8 \) is 2B, and the answers to \( Q_9, Q_{10} \) are 1B. Then we know that color 11 is correct on the second peg and that color 12 is correct on the third peg.

(III) The answer is 3B.

The secret is found.

So as soon we have a non-empty answer \( iB \) to a \( (1, 2, 2) \)-question, a \( (2, 1, 2) \)-question or a \( (2, 2, 1) \)-question from one of the copies of the 9-strategy of Table 5 we can determine the \( i \) pegs which gave rise to the non-empty answers, and thus we know their colors.

So assume that we start with the iterated questions before we ask the base questions. Then we have four cases after having seen the resulting answers:

(I) No colors of any of the three pegs have been determined.

By Observation 4[6], this means that the correct colors are those occurring in the base questions. Thus, they are uniquely determined by the answers to those questions, thanks to the feasibility of the strategies for \( c = 4, 5, 6, 7, 8, 9 \) in Table 4. Their feasibility has been checked by a computer program using brute-force search [25].

(II) One peg has been determined.

Then the colors of two pegs are unknown. To prove that they can be determined from the answers to the base questions, it has to be shown that the \( ⌊(3c−1)/2⌋ \)-strategies for \( c = 4, 5, 6, 7, 8, 9 \) in Table 4 remain feasible if we remove one arbitrary column (corresponding to the already found peg). The feasibility of these three sub-strategies was again checked by the computer program [25].

For motivation, in Table 7 we present an example of a feasible \( ⌊(3c−1)/2⌋ \)-strategy for \( c = 4 \), where adding one copy of the iterated questions leads to an infeasible \( ⌊(3c−1)/2⌋ \)-strategy for \( c = 10 \). For the latter, the two possible secrets (1|4|5) and (2|3|5) lead to the same combination of answers, namely 1B, 1B, 1B, 0B, 0B, 0B, 1B, 0B, ..., 0B which shows that the strategy is not feasible. The reason is that the sub-strategy consisting of the first two columns for \( c = 4 \) is not feasible, as the secrets (3|1) and (4|2) are indistinguishable.
(III) Two pegs have been determined.

Only the color of one peg is still unknown. This peg is determined by the answers to the base questions as by Observation 3) on each peg only one color is missing.

(IV) Three pegs have been determined.

The secret is found without making use of the base questions.

So we have shown the feasibility of the \(\left\lfloor \frac{3c-1}{2} \right\rfloor\)-strategy in all cases.

4.3. Optimality of the \(\left\lfloor \frac{3c-1}{2} \right\rfloor\)-Strategy

We use the following lemma, which in parts (a)-(e) is a generalization of Lemma \(\textbf{1}\) and which holds only for \(c \geq 5\).

**Lemma 2.** For Static Black-Peg AB Game with \(p = 3\) and \(c \geq 5\) the following statements hold:

(a) For each feasible strategy and for each peg there exists at most one color which does not occur on this peg.

(b) A feasible strategy cannot contain two \((1, 1, \ast)\)-questions which are disjoint in the first two pegs.\(^4\)

---

\(^4\)Recall that \((q_1 | q_2 | q_3)\) and \((q_1' | q_2' | q_3')\) are said to be disjoint in the first two pegs if \(\{q_1, q_2\} \cap \{q_1', q_2'\} = \emptyset\).
(c) If a feasible strategy contains three (1, 1, *)-questions, by possibly permuting the colors, these questions can be written in the form $(1 \mid 2 \mid *)_1$, $(2 \mid 3 \mid *)_2$, $(3 \mid 1 \mid *)_3$, for suitable colors $*_1$, $*_2$, $*_3$.

(d) If a feasible strategy contains three (1, 1, *)-questions, then on at least one of the first two pegs of the questions, all $c$ colors must occur.

(e) A feasible strategy cannot contain four (1, 1, *)-questions.

(f) Let a feasible strategy have a (1, 1, *)-question $(q_1 \mid q_2 \mid *)_1$, for a suitable color $*_1$, and let $q_3$ not occur on the first peg and $q_4$ not occur on the second peg. Then $q_3 \neq q_4$ holds and, furthermore, $q_1 = q_4$ or $q_2 = q_3$.

(g) Let a feasible strategy have two (1, 1, *)-questions, and let $q_1$ not occur on the first peg and $q_2$ not occur on the second peg. Then, by permuting colors and reordering questions, we can assume that $q_1 = 3$ and $q_2 = 1$, and the two (1, 1, *)-questions can be written in the form $(1 \mid 2 \mid *)_1$, $(2 \mid 3 \mid *)_2$, for suitable colors $*_1$, $*_2$.

By symmetry, statements analogous to (b)-(g) hold for the first and the third peg, and for the second and the third peg.

Proof. (a) Suppose that there are two colors $a$ and $b$ which do not occur on the first peg. Choose arbitrary colors $x$, $y$ with $x \neq y$ which are neither $a$ nor $b$, where $x$, $y$ exist because of $c \geq 5$. Then the possible secrets $(a \mid x \mid y)$ and $(b \mid x \mid y)$ are indistinguishable, contradicting the feasibility of the strategy.

(b) Suppose that the strategy contains two (1, 1, *)-questions. By permuting colors and using the definition of a (1, 1, *)-question, we can assume that these questions are $(1 \mid 2 \mid *)_1$ and $(3 \mid 4 \mid *)_2$, for suitable colors $*_1$, $*_2$. Let $x$ be a color which is different from 1, 2, 3, 4, where $x$ exists because of $c \geq 5$. Then the possible secrets $(1 \mid 4 \mid x)$ and $(3 \mid 2 \mid x)$ receive the same answers (1B or 2B) for the first two questions and the same answer (0B or 1B) for all other questions, i.e., they are indistinguishable.

(c)-(e) The proofs work analogously to the proofs of Lemma (c)-(e) respectively, if we add arbitrary entries on the third peg.

(f) Let $x$ be an arbitrary color not in $\{q_1, q_2, q_3, q_4\}$. Again, $x$ exists because of $c \geq 5$.

First, suppose that $q_3 = q_4$. By assumption, $q_1 \neq q_3 = q_4 \neq q_2$ holds, and thus $(q_1 \mid q_3 \mid x)$ and $(q_3 \mid q_2 \mid x)$ are different and well-defined secrets that lead to the same combination of answers. Thus, $q_3 \neq q_4$ holds.

Second, suppose that $q_1 \neq q_4$ and $q_2 \neq q_3$. Then the possible secrets $(q_1 \mid q_4 \mid x)$ and $(q_3 \mid q_2 \mid x)$ lead to the same combination of answers. Thus, $q_1 = q_4$ or $q_2 = q_3$.
Note that for \( c = 4 \) this statement does not hold. (See the strategy of Table 7a where the question \( Q_4 \) is a \((1, 1, \ast)\)-question and where the four numbers \( q_1 = 4, q_2 = 1, q_3 = 3, q_4 = 2 \) are disjoint. Nevertheless the strategy is feasible.)

Using (b), permuting colors, and possibly switching the two \((1, 1, \ast)\)-questions, we can assume that these questions are \((1 \mid 2 \mid \ast_1)\) and \((2 \mid a \mid \ast_2)\), for suitable colors \( \ast_1, \ast_2 \). The color \( a \) cannot be equal to \( 1 \) because then \( q_1, q_2 \notin \{1, 2\} \) would hold and that would make the possible secrets \((1 \mid q_2 \mid q_4)\) and \((q_1 \mid 2 \mid q_4)\) indistinguishable (where \( q_4 \) is an arbitrary color with \( q_4 \notin \{1, 2, q_1, q_2\} \)). Hence, we can assume that \( a = 3 \), as claimed. Applying (f) to the questions \((1 \mid \ast_1 \mid q_4)\) and \((q_1 \mid 2 \mid \ast_2)\), it follows that \( q_1 = 3 \vee q_2 = 1 \). As not both \( q_1 \) and \( q_2 \) can be equal to 2, it follows that \( q_1 = 3, q_2 = 1 \).

**Lemma 3.** For Static Black-Peg AB Game with \( p = 3 \) and \( c \geq 5 \), the following statements hold:

(a) Let a feasible strategy contain two \((1, 1, 1)\)-questions. Then it contains no additional \((\ast, 1, 1)\)-question, \((1, \ast, 1)\)-question or \((1, 1, \ast)\)-question.

(b) A feasible strategy contains at most two \((1, 1, 1)\)-questions.

(c) Let a feasible strategy contain one \((1, 1, 1)\)-question and let there be one color for each peg which does not occur on this peg. Then the strategy contains no further \((\ast, 1, 1)\)-questions, \((1, \ast, 1)\)-questions or \((1, 1, \ast)\)-questions.

(d) A feasible strategy, where for each peg there exists one color which does not occur on this peg, contains at most one \((1, 1, 1)\)-question.

**Proof.** (a) Let \( Q_1, Q_2 \) be the two \((1, 1, 1)\)-questions.

Suppose that \( Q_3 \) is a \((1, 1, \ast)\)-question in the strategy. By Lemma 2(c) we can assume that the three questions have the form

\[
\begin{array}{c|c|c|c}
\text{Peg} & 1 & 2 & 3 \\
Q_1 & 1 & 2 & \ast_1 \\
Q_2 & 2 & 3 & \ast_2 \\
Q_3 & 3 & 1 & \ast_3 \\
\end{array}
\]  

(2)

where \( \ast_1, \ast_2, \ast_3 \) are suitable colors and \( \ast_1 \neq \ast_2 \) (since \( Q_1 \) and \( Q_2 \) are \((1,1,1)\)-questions).

By applying Lemma 2(b) to \( Q_1 \) and \( Q_2 \) twice, namely to the first and third peg, and to the second and third peg, the form can be assumed to be

\[
\begin{array}{c|c|c|c}
\text{Peg} & 1 & 2 & 3 \\
Q_1 & 1 & 2 & 3 \\
Q_2 & 2 & 3 & 1 \\
Q_3 & 3 & 1 & \ast_3 \\
\end{array}
\]  

(3)

□
Then the possible secrets \( (2 \mid 1 \mid 3) \) and \( (3 \mid 2 \mid 1) \) receive the same combination of answers, namely 1B for \( Q_1, Q_2, Q_3 \) and 0B for the remaining questions. This is a contradiction, and so the strategy contains no additional \((1, 1, *)\)-question.

By symmetry, the strategy contains no additional \((1, *, 1)\)-question or \((*, 1, 1)\)-question either.

\( \textbf{(b)} \) This follows directly from \( \text{(a)} \)

\( \textbf{(c)} \) Let \( Q'_1 \) be the \((1, 1, 1)\)-question and \( Q'_2 \) be the question consisting of the colors which do not occur on the first, second and third peg, respectively.

Suppose that \( Q'_3 \) is a \((1, 1, *)\)-question in the strategy. By Lemma 2(g), we can assume that the three questions have the form \( (2) \). By Lemma 2(f), we get the form \( (3) \).

Then the possible secrets \( (2 \mid 1 \mid 3) \) and \( (3 \mid 2 \mid 1) \) receive the same combination of answers, namely 1B for \( Q'_1 \) and \( Q'_3 \), and 0B for the remaining questions. This is a contradiction, and so the strategy contains no additional \((1, 1, *)\)-question.

Symmetrically, it follows that the strategy contains no \((1, *, 1)\)-question and no \((*, 1, 1)\)-question either.

\( \textbf{(d)} \) This follows directly from \( \text{(c)} \) \( \square \)

**Lemma 4.** For Static Black-Peg AB Game with \( p = 3 \) and \( c \geq 5 \), the following statements hold:

\( \textbf{(a)} \) If a feasible strategy contains a \((1, 1, 1)\)-question, then the strategy contains in total at most three questions which are \((\geq 2, 1, 1)\)-questions, \((1, \geq 2, 1)\)-questions or \((1, 1, \geq 2)\)-questions.

\( \textbf{(b)} \) If a feasible strategy is such that, for each peg, there exists one color which does not occur on this peg, then the strategy contains in total at most three questions which are \((\geq 2, 1, 1)\)-questions, \((1, \geq 2, 1)\)-questions or \((1, 1, \geq 2)\)-questions.

**Proof.** \( \textbf{(a)} \) Let \( Q_1 \) be a \((1, 1, 1)\)-question of the strategy. We prove two assertions, which together prove part \( \text{(a)} \)

(I) The strategy cannot have two \((1, 1, \geq 2)\)-questions and two \((1, \geq 2, 1)\)-questions at the same time.

Suppose that \( Q_2, Q_3 \) are two \((1, 1, \geq 2)\)-questions and \( Q_4, Q_5 \) are two \((1, \geq 2, 1)\)-questions in the strategy. We present two indistinguishable possible secrets, thus contradicting the assumed feasibility of the strategy.
By the definition of the AB Game, Definition 1(d) and Lemma 2(c) applied to the first and the second peg and applied to the first and the third peg, we can assume that the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| Q₁  | 1 | 2 | 4 |
| Q₂  | 2 | 3 | *₁ |
| Q₃  | 3 | 1 | *₂ |
| Q₄  | 4 | *₃ | 5 |
| Q₅  | *₄ | 1 |   |

where *₁, *₂ ∉ \{1, 4, 5\} and *₃, *₄ ∉ \{1, 2, 3\} are colors which occur at least twice on their peg.

Then the possible secrets (3 | 2 | 1) and (5 | 1 | 4) receive the same combination of answers, namely 1B for Q₁, Q₃, Q₅ and 0B for the remaining questions.

Symmetrically, it follows that the strategy cannot have two (1, 1, ≥ 2)-question and two (≥ 2, 1, 1)-questions at the same time, or (1, ≥ 2, 1)-question and two (≥ 2, 1, 1)-questions at the same time.

(II) The strategy cannot have one (1, ≥ 2, 1)-question, one (≥ 2, 1, 1)-question, and two (1, 1, ≥ 2)-questions at the same time.

Suppose that Q₂, Q₃ are the two (1, 1, ≥ 2)-questions, Q₄ is the (1, ≥ 2, 1)-question and Q₅ is the (≥ 2, 1, 1)-question of the strategy.

In each case or sub-case we present two indistinguishable possible secrets, thus contradicting the assumed feasibility of the strategy.

By the definition of the AB Game, Definition 1(d) and Lemma 2(c) applied to the first and the second peg, we can assume that the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| Q₁  | 1 | 2 | b |
| Q₂  | 2 | 3 | *₃ |
| Q₃  | 3 | 1 | *₄ |
| Q₄  | 4 | *₂ | d |
| Q₅  | *₁ | a | e |

where *₁, *₂, *₃, *₄ are colors which occur at least twice on their peg, and b, d are colors which occur only once on their peg.

By Lemma 2(b) applied to the first and third peg of questions Q₁ and Q₄, and to the second and third peg of questions Q₁ and Q₅, we have

\[ b = 4 \quad \lor \quad d = 1, \]

Note that the parameter c is reserved for the number of colors. So we use the parameters a, b, d, e, ... here.
and

\[ e = 2 \quad \lor \quad a = b. \]

This leads to the following four cases:

(i) \( a = b = 4. \)

Here the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \( Q_1 \) | 1 | 2 | 4 |
| \( Q_2 \) | 2 | 3 | *3 |
| \( Q_3 \) | 3 | 1 | *4 |
| \( Q_4 \) | 4 | *2 | d |
| \( Q_5 \) | *1 | 4 | e |

We have three sub-cases:

(1) All combinations of \( (d, e) \) except \( d = 3 \) and except \( e = 1. \)

Here the possible secrets \( (3 \mid 4 \mid d) \) and \( (4 \mid 1 \mid e) \) receive the same combination of answers, namely \( 1B \) for \( Q_3, Q_4, Q_5 \) and \( 0B \) for the remaining questions. (Note that \( d \neq 4 \) and \( e \neq 4 \) as \( Q_4 \) and \( Q_5 \) are questions in the AB Game.)

(2) \( d = 3. \)

Here the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \( Q_1 \) | 1 | 2 | 4 |
| \( Q_2 \) | 2 | 3 | *3 |
| \( Q_3 \) | 3 | 1 | *4 |
| \( Q_4 \) | 4 | *2 | 3 |
| \( Q_5 \) | *1 | 4 | e |

Then the possible secrets \( (2 \mid 4 \mid 3) \) and \( (4 \mid 3 \mid e) \) receive the same combination of answers, namely \( 1B \) for \( Q_2, Q_4, Q_5 \) and \( 0B \) for the remaining questions. (Note that \( e \neq 3, 4 \), as \( Q_5 \) is a \((\geq 2, 1, 1)-question\) having the color \( e \) on the last peg.)
(3) $e = 1$.

Here the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| $Q_1$ | 1 | 2 | 4 |
| $Q_2$ | 2 | 3 | $\star_1$ |
| $Q_3$ | 3 | 1 | $\star_4$ |
| $Q_4$ | 4 | $\star_2$ | $d$ |
| $Q_5$ | $\star_1$ | 4 | 1 |

Then the possible secrets $(1 \mid 4 \mid d)$ and $(4 \mid 2 \mid 1)$ receive the same combination of answers, namely 1B for $Q_1$, $Q_4$, $Q_5$ and 0B for the remaining questions. (Note that $d \neq 1, 4$, analogously to the case $d = 3$.)

(ii) $a \neq b = 4 \land e = 2$.

Here the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| $Q_1$ | 1 | 2 | 4 |
| $Q_2$ | 2 | 3 | $\star_3$ |
| $Q_3$ | 3 | 1 | $\star_4$ |
| $Q_4$ | 4 | $\star_2$ | $d$ |
| $Q_5$ | $\star_1$ | $a$ | 2 |

Then the possible secrets $(1 \mid 3 \mid 2)$ and $(2 \mid a \mid 4)$ receive the same combination of answers, namely 1B for $Q_3$, $Q_4$, $Q_5$ and 0B for the remaining questions. (Note that $a \neq 2, 4$.)

(iii) $d = 1 \land e = 2$.

Here the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| $Q_1$ | 1 | 2 | $b$ |
| $Q_2$ | 2 | 3 | $\star_1$ |
| $Q_3$ | 3 | 1 | $\star_2$ |
| $Q_4$ | 4 | $\star_3$ | 1 |
| $Q_5$ | $\star_4$ | $a$ | 2 |

Then the possible secrets $(3 \mid a \mid 1)$ and $(4 \mid 1 \mid 2)$ receive the same combination of answers, namely 1B for $Q_3$, $Q_4$, $Q_5$ and 0B for the remaining questions. (Note that $a \neq 1, 3$.)
(iv) $a = b \neq 4 \land d = 1$.

Here the five questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| $Q_1$ | 1 | 2 | $a$ |
| $Q_2$ | 2 | 3 | *3 |
| $Q_3$ | 3 | 1 | *4 |
| $Q_4$ | 4 | *2 | 1 |
| $Q_5$ | *1 | $a$ | * | e |

Then the possible secrets $(3 \mid 2 \mid 1)$ and $(4 \mid 1 \mid a)$ receive the same combination of answers, namely 1B for $Q_1$, $Q_3$, $Q_4$ and 0B for the remaining questions. (Note that $a \neq 1, 4$.)

Symmetrically, it follows that the strategy cannot have one $(1, 1, \geq 2)$-question, one $(\geq 2, 1, 1)$-question and two $(1, \geq 2, 1)$-questions, or one $(1, 1, \geq 2)$-question, one $(1, \geq 2, 1)$-question and two $(\geq 2, 1, 1)$-questions at the same time.

(b) Consider a feasible strategy such that there are colors $q_1$, $q_2$, and $q_3$ which do not occur on pegs 1, 2, and 3, respectively. Let $Q_1 = (q_1 \mid q_2 \mid q_3)$. We aim to apply Lemma 4(a) by adding the question $Q_1$ to the given strategy. To be able to do so, we have to verify that it is a valid question, i.e., the three colors are distinct. For doing so, we can assume that the strategy contains four questions which are $(\geq 2, 1, 1)$-questions, $(1, \geq 2, 1)$-questions or $(1, 1, \geq 2)$-questions (because otherwise the conclusion of Lemma 4(a) is fulfilled).

By Lemma 4(d) and possibly switching the questions, we can assume that there are two $(1, 1, \geq 2)$-questions $Q_2$ and $Q_3$, a $(1, \geq 2, 1)$-question $Q_4$, and a question $Q_5$ which is either also a $(1, \geq 2, 1)$-question or a $(\geq 2, 1, 1)$-question. This yields two cases.

Consider first the case where both $Q_4$ and $Q_5$ are $(1, \geq 2, 1)$-questions. Applying Lemma 4(g) applied once to $Q_2$, $Q_3$ and once to $Q_4$, $Q_5$, yields the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| $Q_2$ | $q_2$ | *1 | *2 |
| $Q_3$ | *1 | $q_1$ | *3 |
| $Q_4$ | $q_3$ | *5 | *4 |
| $Q_5$ | *4 | *6 | $q_1$ |

where $q_1 \neq q_2$ and $q_1 \neq q_3$, and *2, *3, *5, *6 are colors which occur at least twice on their peg, and *1, *4 are colors occurring only once on their peg. However, we also know that $Q_2$ is a $(1, 1, \geq 2)$-question, which implies that $q_2 \neq q_3$. Hence, all three colors are distinct.
If \( Q_5 \) is a \((\geq 2, 1, 1)\)-question, the reasoning is easier: applying Lemma 2(f) thrice, namely to \( Q_2, Q_4, \) and \( Q_5 \) proves immediately that \( q_1 \neq q_2, q_1 \neq q_3, \) and \( q_2 \neq q_4, \) respectively. Hence, again, all three colors \( q_1, q_2, \) and \( q_3 \) are distinct.

Thus, in either case \( Q_1 \) is a valid question. It follows that the strategy obtained by adding question \( Q_1 \) to the already feasible strategy is feasible. In this extended strategy, \( Q_1 \) is a \((1, 1, 1)\)-question. Hence, Lemma 4(a) applies, the conclusion being that the modified strategy (and thus the original one) contains in total at most three questions which are \((\geq 2, 1, 1)\)-questions, \((1, \geq 2, 1)\)-questions or \((1, 1, \geq 2)\)-questions. This completes the proof.

\[ \square \]

**Lemma 5.** For Static Black-Peg AB Game with \( p = 3 \) and \( c \geq 5 \), the following statements hold:

(a) If a feasible strategy contains no \((1, 1, 1)\)-question, then it contains in total at most six questions that are \((1, 1, \geq 2)\)-questions, \((1, \geq 2, 1)\)-questions or \((\geq 2, 1, 1)\)-questions.

(b) If a feasible strategy contains no \((1, 1, 1)\)-question, and for each of two pegs there is one color which does not occur on this peg, then the strategy contains in total at most five questions that are \((1, 1, \geq 2)\)-questions, \((1, \geq 2, 1)\)-questions or \((\geq 2, 1, 1)\)-questions.

**Proof.** (a) We prove two assertions, which together prove part (a).

(I) The strategy cannot have three \((1, 1, \geq 2)\)-questions, three \((1, \geq 2, 1)\)-questions and one \((\geq 2, 1, 1)\)-question at the same time.

Suppose that \( Q_1, Q_2, Q_3 \) are \((1, 1, \geq 2)\)-questions, \( Q_4, Q_5, Q_6 \) are \((1, \geq 2, 1)\)-questions and \( Q_7 \) is a \((\geq 2, 1, 1)\)-question of the strategy.

By Lemma 4(c) applied to the first and the second peg and applied to the first and the third peg, we can assume that the seven questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \( Q_1 \) | 1 | 2 | *5 |
| \( Q_2 \) | 2 | 3 | *6 |
| \( Q_3 \) | 3 | 1 | *7 |
| \( Q_4 \) | 4 | *2 | 5 |
| \( Q_5 \) | 5 | *3 | 6 |
| \( Q_6 \) | 6 | *4 | 4 |
| \( Q_7 \) | *1 | a | b |

where \( *_1, *_2, *_3, *_4, *_5, *_6, *_7 \) are colors which occur at least twice on their peg, and \( a, b \) are colors occurring only once on their peg. In particular, \( a \notin \{1, 2, 3\} \) and \( b \notin \{4, 5, 6\} \). Note that the appearance

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of a 7-th color \( \star_1 \) in the questions already proves the assertion for 
\( c = 5, 6 \) (or, expressed differently, that the assumptions imply \( c \geq 7 \).

Choose a question \( Q = (q_1 \mid q_2 \mid q_3) \) from the set 
\( S_1 := \{Q_1, Q_2, Q_3\} \) so that \( b \neq q_2 \). (Note that at least two questions in \( S_1 \) fulfill this condition.)

Similarly, choose a question \( Q' = (q_4 \mid q_5 \mid q_6) \) from the set 
\( S_2 := \{Q_4, Q_5, Q_6\} \) so that \( a \neq q_6 \).

By construction, the possible secrets 
\( (q_1 \mid a \mid q_6) \) and \( (q_4 \mid q_2 \mid b) \) receive the same combination of answers, namely \( 1B \) for \( Q, Q', Q_7 \) and \( 0B \) for the remaining questions, contradicting the feasibility of the strategy.

As an example for \( a = 4, b = 3 \) choose \( Q = Q_3 \) and \( Q' = Q_5 \). Then the possible secrets \( (3 \mid 4 \mid 6) \) and \( (5 \mid 1 \mid 3) \) receive the same combination of answers, namely \( 1B \) for \( Q_3, Q_5, Q_7 \) and \( 0B \) for the remaining questions.

Symmetrically, it follows that the strategy cannot have three \((1, 1, \geq 2)\)-questions, three \((\geq 2, 1, 1)\)-questions and one \((1, \geq 2, 1)\)-question, or three \((1, \geq 2, 1)\)-questions, three \((\geq 2, 1, 1)\)-questions and one \((1, 1, \geq 2)\)-question at the same time.

\((\text{II})\) The strategy cannot have three \((1, 1, \geq 2)\)-questions, two \((1, \geq 2, 1)\)-questions and two \((\geq 2, 1, 1)\)-questions at the same time.

Suppose that \( Q_1, Q_2 \) and \( Q_3 \) are \((1, 1, \geq 2)\)-questions, \( Q_4 \) and \( Q_5 \) are \((1, \geq 2, 1)\)-questions and \( Q_6 \) and \( Q_7 \) are \((\geq 2, 1, 1)\)-questions of the strategy. Below, we consider an exhaustive set of cases, presenting in each of them two indistinguishable possible secrets, thus contradicting the feasibility of the strategy.

By Lemma 2(b) applied to the first and the third peg and applied to the second and the third peg, by possibly switching questions \( Q_4 \) and \( Q_5 \), and questions \( Q_6 \) and \( Q_7 \), and by applying Lemma 2(c) to the first and the second peg, we can assume that the seven questions have the form

\[
\begin{array}{c|ccc}
\text{Peg} & 1 & 2 & 3 \\
Q_1 & 1 & 2 & \star_5 \\
Q_2 & 2 & 3 & \star_6 \\
Q_3 & 3 & 1 & \star_7 \\
Q_4 & 4 & \star_3 & d \\
Q_5 & 5 & \star_4 & 4 \\
Q_6 & \star_1 & a & e \\
Q_7 & \star_2 & b & a \\
\end{array}
\]

where \( \star_1, \star_2, \star_3, \star_4, \star_5, \star_6, \star_7 \) are colors which occur at least twice on their peg. and \( a, b, d, e \) are colors occurring only once on their peg.
Here the appearance of a 6-th color $\star_1$ in the questions already proves the assertion for $c = 5$ (or, expressed differently, that the assumptions imply $c \geq 6$).

We consider the following four cases:

(i) $a \neq 5 \land b \neq 4$.

Then the possible secrets $(1 \mid b \mid 4)$ and $(5 \mid 2 \mid a)$ receive the same combination of answers, namely 1B for $Q_1, Q_5, Q_7$ and 0B for the remaining questions. (Note that $a \neq 2, 5$ and $b \neq 1, 4$.)

(ii) $a = 5 \land e \neq 3$.

Here the seven questions have the form

\[
\begin{array}{c|c|c|c}
\text{Peg} & 1 & 2 & 3 \\
\hline
Q_1 & 1 & 2 & \star_5 \\
Q_2 & 2 & 3 & \star_6 \\
Q_3 & 3 & 1 & \star_7 \\
Q_4 & 4 & \star_3 & d \\
Q_5 & 5 & \star_4 & 4 \\
Q_6 & \star_1 & 5 & e \\
Q_7 & \star_2 & b & 5 \\
\end{array}
\]

Then the possible secrets $(2 \mid 5 \mid 4)$ and $(5 \mid 3 \mid e)$ receive the same combination of answers, namely 1B for $Q_2, Q_5, Q_6$ and 0B for the remaining questions. (Note that $e \neq 3, 5$.)

(iii) $a = 5 \land e = 3$.

Here the seven questions have the form

\[
\begin{array}{c|c|c|c}
\text{Peg} & 1 & 2 & 3 \\
\hline
Q_1 & 1 & 2 & \star_5 \\
Q_2 & 2 & 3 & \star_6 \\
Q_3 & 3 & 1 & \star_7 \\
Q_4 & 4 & \star_3 & d \\
Q_5 & 5 & \star_4 & 4 \\
Q_6 & \star_1 & 5 & 3 \\
Q_7 & \star_2 & b & 5 \\
\end{array}
\]

Then the possible secrets $(1 \mid 5 \mid 4)$ and $(5 \mid 2 \mid 3)$ receive the same combination of answers, namely 1B for $Q_1, Q_5, Q_6$ and 0B for the remaining questions.
(iv) \( a \neq 5 \land b = 4 \land d \neq 1 \).
Here the seven questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \( Q_1 \) | 1 | 2 | *5 |
| \( Q_2 \) | 2 | 3 | *6 |
| \( Q_3 \) | 3 | 1 | *7 |
| \( Q_4 \) | 4 | *3 | d |
| \( Q_5 \) | 5 | *4 | 4 |
| \( Q_6 \) | *1 | a | e |
| \( Q_7 \) | *2 | 4 | a |

Then the possible secrets \((1 \mid 4 \mid d)\) and \((4 \mid 2 \mid a)\) receive the same combination of answers, namely \(1\mathrm{B}\) for \(Q_1, Q_4, Q_7\) and \(0\mathrm{B}\) for the remaining questions. (Note that \(a \neq 2, 4\) and \(d \neq 1, 4\).)

(v) \( a \neq 5 \land b = 4 \land d = 1 \).
Here the seven questions have the form

| Peg | 1 | 2 | 3 |
|-----|---|---|---|
| \( Q_1 \) | 1 | 2 | *5 |
| \( Q_2 \) | 2 | 3 | *6 |
| \( Q_3 \) | 3 | 1 | *7 |
| \( Q_4 \) | 4 | *3 | 1 |
| \( Q_5 \) | 5 | *4 | 4 |
| \( Q_6 \) | *1 | a | e |
| \( Q_7 \) | *2 | 4 | a |

Then the possible secrets \((3 \mid 4 \mid 1)\) and \((4 \mid 1 \mid a)\) receive the same combination of answers, namely \(1\mathrm{B}\) for \(Q_3, Q_4, Q_7\) and \(0\mathrm{B}\) for the remaining questions. (Note that \(a \neq 1, 4\).)

Again, symmetric arguments prove that the strategy cannot simultaneously have three \((1, \geq 2, 1)\)-questions, two \((\geq 2, 1, 1)\)-questions and two \((1, 1, \geq 2)\)-questions, or three \((\geq 2, 1, 1)\)-questions, two \((1, \geq 2, 1)\)-questions and two \((1, 1, \geq 2)\)-questions.
This completes the proof of (a).

(b) Similarly to the proof of Lemma 4(b), this case can be proved by modifying the strategy in order to be able to apply Lemma 4(a). Let \(q_1, q_2\) be the colors which do not occur on the first and second peg, respectively. By Lemma 4(f) \(q_1 \neq q_2\) holds. Now, let \(Q_1 = (q_1 \mid q_2 \mid q_3)\), where the third color \(q_3\) is chosen in such a way that it appears at least once on the third peg of another question but is different from \(q_1\) and \(q_2\). Such a color \(q_3\) exists because by Lemma 4(a) there are at least \(c - 1 \geq 4\) distinct colors on the third peg, so we have at least two to choose from (as we have to exclude \(q_1\) and \(q_2\)).
We have thus shown that we can add $Q_1$ to the given strategy, where it becomes a $(1, 1, \geq 2)$-question. The extended strategy fulfills the assumptions of Lemma[2(a)] which completes the proof. □

Now we come to the actual proof of optimality. For $c = 4$ it can be easily checked by brute-force search that there is no $\left(\lfloor (3c - 1)/2 \rfloor - 1\right)$-strategy, i.e., no strategy with $k = \lfloor (3 \cdot 4 - 1)/2 \rfloor - 2 = 3$ questions. Thus, assume in the following that $c \geq 5$. The presented $\lfloor (3c-1)/2 \rfloor$-strategy has $k = \lfloor (3c-1)/2 \rfloor - 1$ questions. We show that this is optimal by considering any feasible strategy and proving that it must contain at least $k$ questions.

In the following let $e$ be the number of $(1, 1, 1)$-questions and $f$ be the total number of $(1, 1, \geq 2)$-questions, $(1, \geq 2, 1)$-questions and $(\geq 2, 1, 1)$-questions of this assumed strategy. Further, for $i = 1, 2, 3$ let $l_i$ be the number of colors which occur exactly once on the $i$-th peg of the strategy.

We will need the following observation, which is easy to see.

**Observation 4.** Every feasible strategy contains at least

$$l_1 + l_2 + l_3 - 2e - f$$

questions. Hence, we can establish optimality of our strategy by showing that

$$l_1 + l_2 + l_3 - 2e - f \geq k.$$ 

We distinguish between even and odd numbers of colors.

(I) $c$ even.

We have

$$k = \frac{3c}{2} - 2.$$

By Lemma[2(a)] for each peg there exists at most one color which does not occur throughout the entire strategy. We start by proving two claims which verify optimality in two restricted cases.

**Claim 1.** If $l_i \leq \frac{c}{2}$ for some peg $i \in \{1, 2, 3\}$ on which not all colors occur, then the strategy contains at least $k$ questions.

This is clear because $l_i \leq \frac{c}{2}$ means that $\frac{c}{2} - 1$ colors appear at least twice on peg $i$, yielding at least

$$\frac{c}{2} + 2 \cdot \left(\frac{c}{2} - 1\right) = \frac{3c}{2} - 2 = k$$

questions.
Claim 2. If \( l_i \leq \frac{c}{2} + 2 \) for some peg \( i \in \{1, 2, 3\} \) on which all colors occur, then the strategy contains at least \( k \) questions.

Here, the assumption \( l_i \leq \frac{c}{2} + 2 \) implies that we have at least

\[
\frac{c}{2} + 2 + 2 \cdot \left( \frac{c}{2} - 2 \right) = \frac{3c}{2} - 2 = k
\]

questions.

Now we finish the case (I) of the proof by considering two sub-cases.

(i) For each of the three pegs there exists one color which does not occur throughout the entire strategy.

By Claim 1, the situation which remains to be considered is when \( l_i \geq \frac{c}{2} + 1 \) for every \( i \in \{1, 2, 3\} \). In this case,

\[
l_1 + l_2 + l_3 \geq 3 \cdot \left( \frac{c}{2} + 1 \right) = \frac{3c}{2} + 3 = k + 5.
\]

By Observation 3 it thus remains to be shown that

\[
2e + f \leq 5.
\]

By Lemma 3(b) at most two \((1, 1, 1)\)-questions exist.

(1) If we have two \((1, 1, 1)\)-questions, then Lemma 3(a) yields

\[
2e + f = 2 \cdot 2 + 0 = 4.
\]

(2) If we have one \((1, 1, 1)\)-question, then Lemma 3(a) yields

\[
2e + f \leq 2 \cdot 1 + 3 = 5.
\]

(3) If we have no \((1, 1, 1)\)-question, then Lemma 2(b) yields

\[
2e + f \leq 2 \cdot 0 + 3 = 3.
\]

So we have finished the optimality proof for case (I).

(ii) For at most two of the three pegs there exists one color which does not occur throughout the entire strategy.

By Claims 1 and 2 it remains to consider the situation where \( l_i \geq \frac{c}{2} + 1 \) for at most two \( i \in \{1, 2, 3\} \) and \( l_i \geq \frac{c}{2} + 3 \) for the remaining ones, i.e.,

\[
l_1 + l_2 + l_3 \geq \left( \frac{c}{2} + 3 \right) + 2 \cdot \left( \frac{c}{2} + 1 \right) = \frac{3c}{2} + 5 = k + 7.
\]
Again using Observation
 we thus have established that there are at least \( k \) questions if we can show that

\[
2e + f \leq 7.
\]

Again we have at most two \((1, 1, 1)\)-questions.

1. If we have two \((1, 1, 1)\)-questions, as in case

\[
2e + f = 2 \cdot 2 + 0 = 4.
\]

2. If we have one \((1, 1, 1)\)-question, as in case

\[
2e + f \leq 2 \cdot 1 + 3 = 5.
\]

3. If we have no \((1, 1, 1)\)-question, then by Lemma

\[
2e + f \leq 2 \cdot 0 + 6 = 6.
\]

So we have finished the optimality proof for case \((\text{II})\) and thus for case \((\text{I})\).

(II) \( c \) odd.

We have

\[
k = \frac{3c - 3}{2}.
\]

We proceed similarly to case \((\text{I})\) starting by proving two claims.

Claim 3. If \( l_i \leq \frac{c - 1}{2} \) for some peg \( i \in \{1, 2, 3\} \) on which not all colors occur, then the strategy contains at least \( k \) questions.

Clearly, if \( l_i \leq \frac{c - 1}{2} \), then we have at least

\[
\frac{c - 1}{2} + 2 \cdot \left( \frac{c - 1}{2} \right) = \frac{3c - 3}{2} = k
\]

questions, which verifies the claim.

Claim 4. If \( l_i \leq \frac{c + 1}{2} + 1 \) for some peg \( i \in \{1, 2, 3\} \) on which all colors occur, then the strategy contains at least \( k \) questions.

This is clear as well, because the assumption \( l_i \leq \frac{c + 1}{2} + 1 \), together with the fact that all colors occur on peg \( i \), implies that we have at least

\[
\frac{c + 1}{2} + 1 + 2 \cdot \left( \frac{c - 1}{2} - 1 \right) = \frac{3c - 3}{2} = k
\]

questions.

Now we finish the part \((\text{III})\) of the proof by considering three sub-cases.
(i) For each of the three pegs there exists one color which does not occur throughout the entire strategy.

By Claim 3, we only need to consider the situation where \( l_i \geq \frac{c-1}{2} + 1 \) for every \( i \in \{1, 2, 3\} \), which means that

\[
l_1 + l_2 + l_3 \geq 3 \cdot \left( \frac{c + 1}{2} \right) = \frac{3c}{2} + \frac{3}{2} = k + 3.
\]

By Observation 4 we thus have to show that

\[2e + f \leq 3.\]

By Lemma 3(d) at most one \((1, 1, 1)\)-questions exists, which yields two cases:

1. If we have one \((1, 1, 1)\)-question, then by Lemma 3(c)
   \[2e + f = 2 \cdot 1 + 0 = 2.\]
2. If we have no \((1, 1, 1)\)-question, then by Lemma 4(b)
   \[2e + f \leq 2 \cdot 0 + 3 = 3.\]

So we have completed case (i).

(ii) For exactly two of the three pegs there exists one color which does not occur throughout the entire strategy.

By Claims 3 and 4 we need to consider the case where \( l_i \geq \frac{c+1}{2} + 1 \) for exactly two pegs \( i \in \{1, 2, 3\} \) and \( l_i \geq \frac{c+1}{2} + 2 \) for the remaining peg, which yields

\[
l_1 + l_2 + l_3 \geq \left( \frac{c + 1}{2} + 2 \right) + 2 \cdot \left( \frac{c + 1}{2} \right) = \frac{3c}{2} + \frac{7}{2} = k + 5.
\]

Again using Observation 4 this means that we have to show that

\[2e + f \leq 5.\]

By Lemma 3(b) at most two \((1, 1, 1)\)-questions exist, and so we have the following cases:

1. If there are two \((1, 1, 1)\)-questions, then by Lemma 3(a)
   \[2e + f = 2 \cdot 2 + 0 = 4.\]
2. If there is one \((1, 1, 1)\)-question, then by Lemma 4(a)
   \[2e + f \leq 2 \cdot 1 + 3 = 5.\]
(3) If there is no \((1, 1, 1)\)-question, then by Lemma \[ b \]
\[ 2e + f \leq 2 \cdot 0 + 5 = 5. \]

So we have finished the proof for case \[ ii \].

(iii) For at most one of the three pegs there exists a color which does not occur throughout the entire strategy.

Again using Claims \[ 3 \] and \[ 4 \] the case to be studied is the one where

\[ l_1 + l_2 + l_3 \geq 2 \cdot \left( \frac{c + 1}{2} + 2 \right) + \left( \frac{c + 1}{2} \right) \]

\[ = \frac{3c}{2} + \frac{11}{2} \]

\[ = k + 7. \]

By Observation \[ 4 \] we thus need to show that

\[ 2e + f \leq 7. \]

Again we have at most two \((1, 1, 1)\)-questions, and thus the following cases:

(1) If there are two \((1, 1, 1)\)-questions, then by Lemma \[ 3(a) \]
\[ 2e + f = 2 \cdot 2 + 0 = 4. \]

(2) If there is one \((1, 1, 1)\)-question, then by Lemma \[ 4(a) \]
\[ 2e + f \leq 2 \cdot 1 + 3 = 5. \]

(3) If there is no \((1, 1, 1)\)-question, then by Lemma \[ 5(a) \]
\[ 2e + f \leq 2 \cdot 0 + 6 = 6. \]

So the proof for case \[ iii \] is complete, and so is the optimality proof in its entirety. \[ \square \]

5. Conclusions and Future Work

We have introduced a new game, called Static Black-Peg AB Game, a variant of the AB Game for which, to the best of our knowledge, only the case of equal number of pegs and colors (called “Static Permutation Mastermind”) has been studied before \[ 11, 19 \]. We have developed optimal strategies for this game for two and three pegs, extending similar results for Static Black-Peg Mastermind with two and three pegs \[ 13, 14, 15 \].

Some of the methods and strategies used in this work would also simplify the earlier strategies \[ 12, 15 \] of the related Static Black-Peg Mastermind, which
makes them easier to read and reason about. In particular, the iterated questions of Table 1c and 5 can be used in optimal strategies for Static Black-Peg Mastermind with two and three pegs, respectively.

It would be an interesting task to extend the strategies to more than three pegs. However, even though such strategies obviously exist, the optimality proofs will likely be quite challenging. An attempt to do so should probably aim at developing a schema for constructing optimal strategies for an arbitrary numbers of pegs \( p \). The fact that our strategies for \( p = 2 \) and \( p = 3 \) pegs clearly share some structural characteristics may provide clues. A definition of optimal strategies parameterized with \( p \) may also facilitate an optimality proof that works for all of them.

Further, solving Static Black-Peg Mastermind is equivalent to the well-studied problem of determining the metric dimension of undirected graphs, in our case the graph \( Z^p_c \). Thus, the fact that the strategy construction principle of iterated blocks used in this paper applies to Static Black-Peg Mastermind as well yields simplified proofs of the metric dimension of \( Z^2_c \) and \( Z^3_c \). We believe that our methods can furthermore be applied to Static Black-Peg Mastermind for constant \( p > 3 \), and also to the situation where the pegs have independent numbers of colors. This would lead to new discoveries regarding the metric dimension of \( Z^p_c \) for \( p > 3 \) and of \( Z_{c_1} \times Z_{c_2} \times \ldots \times Z_{c_3} \), where \( c_1, c_2, c_3 \) may differ.

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