MAGNETIC VIRIAL IDENTITIES, WEAK DISPERSION AND STRICHARTZ INEQUALITIES

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Abstract. We show a family of virial-type identities for the Schrödinger and wave equations with electromagnetic potentials. As a consequence, some weak dispersive inequalities in space dimension \( n \geq 3 \), involving Morawetz and smoothing estimates, are proved; finally, we apply them to prove Strichartz inequalities for the wave equation with a non-trapping electromagnetic potential with almost Coulomb decay.

1. Introduction

In this paper, we consider electromagnetic Hamiltonians in the standard covariant form

\[ H = -\nabla_A^2 + V(x), \]

where

\[ \nabla_A = \nabla - iA, \quad A = (A_1, \ldots, A^n) : \mathbb{R}^n \to \mathbb{R}^n \]

and \( V : \mathbb{R}^n \to \mathbb{R} \); the magnetic potential \( A \) satisfies the Coulomb gauge condition

\[ \text{div} A = 0. \]

Related to these Hamiltonians, we study the magnetic Schrödinger equation

\[
\begin{cases}
iu_t(t,x) - Hu(t,x) = 0 \\
u(0,x) = f(x),
\end{cases}
\]

and the magnetic wave equation

\[
\begin{cases}
u_{tt}(t,x) + Hu(t,x) = 0 \\
u(0,x) = f(x) \\
u_t(0,x) = g(x),
\end{cases}
\]

where in both cases the unknown is a function \( u : \mathbb{R}^{1+n} \to \mathbb{C} \). We are interested in weak dispersive phenomena for equations (1.4) and (1.5); in particular, we will point our attention on the structures of \( A \) and \( V \) which allow the weak dispersion.

In the family of weak dispersive estimates we find, among the others, Morawetz and smoothing estimates.

For the free case \( A \equiv 0 \equiv V \), the Morawetz estimates are, for \( n \geq 3 \),

\[ \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\partial_t e^{it\Delta} f|}{|x|} \leq \|f\|_{H^{1/2}} \]
\[ \int_0^{+\infty} \int_{\mathbb{R}^n} \left| \partial_r e^{it\sqrt{-\Delta}}f \right| \leq \|f\|_{\dot{H}^{\frac{1}{2}}}, \]

where \( \partial_r \) is the tangential derivative. Inequality (1.7) was proved in [15] for the Klein-Gordon equation first, and then extended to the Schrödinger equation. We also recall the smoothing estimates for the free equations

\[ \sup_{R>0} \frac{1}{R} \int_0^{+\infty} \int_{|x|\leq R} |\nabla e^{it\Delta}f|^2 \leq \|f\|_{\dot{H}^{\frac{1}{2}}} \]

(1.8)

\[ \sup_{R>0} \frac{1}{R} \int_0^{+\infty} \int_{|x|\leq R} \left( |\partial_t e^{it\sqrt{-\Delta}}f|^2 + |\nabla e^{it\sqrt{-\Delta}}f|^2 \right) \leq \|f\|_{\dot{H}^{1}}. \]

(1.9)

Inequality (1.8) was proved independently in [6], [17] and [19], and the proof can be easily generalized to obtain (1.9). We also mention the paper [16], in which a proof of (1.9) for the nonlinear wave equation based on a modification of Morawetz ideas [15] is given. Extensions to the Schrödinger equation were considered in [2]. However not so many results are available for the magnetic case \( A \neq 0 \); we mention [1] [7], [10], [8] and [18] where Strichartz and smoothing estimates for the magnetic Schrödinger and wave equations are proved.

The aim of this paper is to show, for non-trapping magnetic Hamiltonians, the relation between virial identities and weak dispersion, in analogy with the results in [16], [2], and [3].

All through the paper we will assume some regularity assumptions on the Hamiltonian \( H \).

(H1) The Hamiltonian \( H_A = -\nabla_A^2 \) is essentially self-adjoint on \( L^2(\mathbb{R}^n) \), with form domain

\[ D(H_A) = \left\{ f : f \in L^2, \int |\nabla_A f|^2 < \infty \right\}. \]

(H2) The potential \( V \) is a perturbation of \( H_A \) in the Kato-Rellich sense, i.e. there exists a small \( \epsilon > 0 \) such that

\[ \|Vf\|_{L^2} \leq (1 - \epsilon)\|H_Af\|_{L^2} + C\|f\|_{L^2}, \]

(1.10)

for all \( f \in D(H_A) \).

Assumptions (H1), (H2) have several consequences about the existence theory for equations (1.4), (1.5). First of all, they imply the self-adjointness of \( H \), by standard perturbation techniques (see e.g. [4]); hence by the spectral theorem we can define the Schrödinger and wave propagators \( S(t) = e^{itH}, W(t) = H^{-\frac{1}{2}} e^{it\sqrt{-\mathcal{N}}} \). Moreover we can define for any \( s \) the distorted norms

\[ \|f\|_{\dot{H}^s} = \|H^s f\|_{L^2}. \]

Since \( H \) and \( H^s \) commute with each other, for any \( s \geq 0 \), the Schrödinger propagator \( S(t) \) satisfy the family of conservation laws

\[ \|e^{itH} f\|_{\dot{H}^s} = \|f\|_{\dot{H}^s}, \]

for all \( t \in \mathbb{R} \). Similarly, the distorted wave energy

\[ E(t) = \frac{1}{2}\|u_t\|_{L^2}^2 + \frac{1}{2}\|u\|_{\dot{H}^1} \]

is conserved on solutions of (1.5).
For the validity of (H1) and (H2) see the standard reference \[4\].

In space dimension \(n = 3\) the magnetic field \(B = \text{curl}A\) is a physically relevant quantity for equations (1.4) and (1.5). In order to continue, we need to define the analogous of curl\(A\) in any space dimension; we give the following definition.

**Definition 1.1.** For any \(n \geq 2\) the matrix-valued field \(B : \mathbb{R}^n \to \mathcal{M}_{n \times n}(\mathbb{R})\) is defined by

\[
B := DA - DA^t, \quad B_{ij} = \frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i}.
\]

We also define the vector field \(B_\tau : \mathbb{R}^n \to \mathbb{R}^n\) as follows:

\[
B_\tau = \frac{x}{|x|} \times B.
\]

Hence \(B\) is defined in terms of the anti-symmetric gradient of \(A\). In dimension \(n = 3\), the previous definition identifies \(B = \text{curl}A\), namely

\[
Bv = \text{curl}A \wedge v, \quad \forall v \in \mathbb{R}^3.
\]

In particular, we have

\[
B_\tau = \frac{x}{|x|} \times \text{curl}A, \quad n = 3.
\] (1.11)

Hence \(B_\tau(x)\) is the projection of \(B = \text{curl}A\) on the tangential space in \(x\) to the sphere of radius \(|x|\), for \(n = 3\). Observe also that \(B_\tau \cdot x = 0\) for any \(n \geq 2\), hence \(B_\tau\) is a tangential vector field in any dimension. Also notice that \(A\) and \(A + \nabla \psi\) produce the same \(B\), for any \(n \geq 2\); actually, in order to preserve the gauge invariance of our results, it is our interest to give always assumptions in terms of \(B\).

We can now state our main results.

### 1.1. Magnetic virial identities.** They are convexity (in time) properties for certain relevant quantities related to the solutions of these equations. A 3D-version of the virial identity for the magnetic Schrödinger equation appears in \[11\], \[12\]; in Theorem 1.2 we present the generalization to any space dimension, while in Theorem 1.3 we give the analogous identity for the magnetic wave equation (1.5).

We start with the Schrödinger equation.

**Theorem 1.2 (Virial for magnetic Schrödinger).** Let \(\phi : \mathbb{R}^n \to \mathbb{R}\) be a radial, real-valued multiplier, \(\phi = \phi(|x|)\), and let

\[
\Theta_S(t) = \int_{\mathbb{R}^n} \phi|u|^2 \, dx.
\] (1.12)

Then, for any solution \(u\) of the magnetic Schrödinger equation (1.4) with initial datum \(f \in L^2\), \(H_{A}f \in L^2\), the following virial-type identity holds:

\[
\dot{\Theta}_S(t) = 4 \int_{\mathbb{R}^n} \nabla A u D^2 \phi \nabla A u \, dx - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx
\]

\[
- 2 \int_{\mathbb{R}^n} \phi' \nabla_x |u|^2 \, dx + 4 \int_{\mathbb{R}^n} u \phi' B_\tau \cdot \nabla A u \, dx,
\] (1.13)

where

\[
(D^2 \phi)_{jk} = \frac{\partial^2}{\partial x^j \partial x^k} \phi, \quad \Delta^2 \phi = \Delta(\Delta \phi),
\]
for \( j, k = 1, \ldots, n \), are respectively the Hessian matrix and the bi-Laplacian of \( \phi \).

The analogous result for the wave equation \((1.5)\) is the following.

**Theorem 1.3** (Virial for magnetic wave). Let \( \phi, \Psi : \mathbb{R}^n \to \mathbb{R} \), be two radial, real-valued multipliers, and let

\[
\Theta_W(t) = \int_{\mathbb{R}^n} \left( \phi|u_t|^2 + \phi|\nabla_A u|^2 - \frac{1}{2} \langle \Delta \phi \rangle |u|^2 \right) dx + \int |u|^2 \phi V dx + \int |u|^2 \Psi dx.
\]

Then, for any solution \( u \) of the magnetic wave equation \((1.5)\) with initial data \( f, g \in L^2 \), \( H_A f, H_A g \in L^2 \), the following virial-type identity holds:

\[
\Theta_W(t) = 2 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \nabla_A u dx - \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi dx + 2 \int |u|^2 \Psi dx - 2 \int |\nabla_A u|^2 \Psi dx + \int |u|^2 \Delta \Psi dx
\]

\[ - \int \phi' \nabla_{\tau} |u|^2 dx + 2 \int \nabla_{\tau} \phi B_{\tau} \cdot \nabla_A u dx. \quad (1.15)\]

We give two immediate corollaries of the previous theorems.

**Corollary 1.4.** Let \( u \) be a solution of the magnetic Schrödinger equation \((1.4)\) with \( f \in L^2 \), \( H_A f \in L^2 \). Then the variance

\[
Q(t) = \int_{\mathbb{R}^n} |x|^2 |u|^2 dx
\]

satisfies the identity

\[
\dot{Q}(t) = 8 \int_{\mathbb{R}^n} |\nabla_A u|^2 dx - 4 \int_{\mathbb{R}^n} |x| |V_{\tau} |u|^2 dx + 8 \int_{\mathbb{R}^n} |x| u B_{\tau} \cdot \nabla_A u dx. \quad (1.16)
\]

For the magnetic wave equation we have the following analogous result:

**Corollary 1.5.** Let \( u \) be a solution of the magnetic wave equation \((1.5)\) with \( f, g \in L^2 \), \( H_A f, H_A g \in L^2 \). Then the quantity

\[
Q(t) = \int_{\mathbb{R}^n} \left\{ |x|^2 \left( |u|^2 + |\nabla_A u|^2 + |u|^2 V \right) - (n - 1)|u|^2 \right\} dx
\]

satisfies the identity

\[
\dot{Q}(t) = 2 \int_{\mathbb{R}^n} |u|^2 + |\nabla_A u|^2 dx - 2 \int V_{\tau} |u|^2 dx + 4 \int u B_{\tau} \cdot \nabla_A u dx. \quad (1.17)
\]

The proofs of the corollaries are immediate applications of identities \((1.13)\) and \((1.15)\) with the choice \( \phi(x) = |x|^2, \Psi \equiv 1 \).

The previous identities suggest that it is relevant to show examples of potentials \( A \) for which \( B_{\tau} \equiv 0 \); we will focus our attention on the 3D case.

**Example 1.6.** First we consider some singular potentials. Take

\[
A = \frac{1}{x^2 + y^2 + z^2} (-y, x, 0) = \frac{1}{x^2 + y^2 + z^2} (x, y, z) \wedge (0, 0, 1). \quad (1.18)
\]

We can check that

\[
\nabla \cdot A = 0, \quad B = -2 \frac{z}{(x^2 + y^2 + z^2)^2} (x, y, z), \quad B_{\tau} = 0.
\]
Another (more singular) example is the following:

\[ A = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right) = \frac{1}{x^2 + y^2} (x, y, z) \wedge (0, 0, 1). \tag{1.19} \]

Here we have \( B = (0, 0, \delta) \), with \( \delta \) denoting Dirac’s delta function. Again we have \( B_\tau = 0 \).

**Example 1.7.** Now we show a natural generalization of the previous examples. Assume that \( B = \text{curl} A : \mathbb{R}^3 \to \mathbb{R}^3 \) is known; since \( \text{div} A = 0 \), we can reconstruct the potential \( A \) using the Biot-Savart formula

\[ A(x) = \frac{1}{4\pi} \int \frac{x - y}{|x - y|^3} \wedge B(y) \, dy. \tag{1.20} \]

Assume now that \( B_\tau = 0 \), namely \( x \wedge B(x) = 0 \); by (1.20) we have

\[ A(x) = \frac{x}{4\pi} \wedge \int \frac{B(y)}{|x - y|^3} \, dy. \tag{1.21} \]

To have \( B_\tau = 0 \) it is necessary \( B(y) = g(y) \frac{y}{|y|} \), for some scalar function \( g : \mathbb{R}^3 \to \mathbb{R} \). Since we want \( A \neq 0 \), \( g \) has not to be radial. As an example we consider

\[ g(y) = h \left( \frac{y}{|y|} \cdot \omega \right) |y|^{-\alpha}, \]

for some fixed \( \omega \in S^2 \), where \( h \) is homogeneous of degree \( 0 \) and \( \alpha \in \mathbb{R} \); consequently, the vector field \( B \) is homogeneous of degree \( -\alpha \). By (1.21) we have

\[ A(x) = \frac{x}{4\pi} \wedge \int \frac{h \left( \frac{y}{|y|} \cdot \omega \right)}{|x - y|^3 |y|^{\alpha}} y \, dy. \tag{1.22} \]

The potential \( A \) is homogenous of degree \( 1 - \alpha \), and by symmetry we have that \( A(\omega) = 0 \). These examples can be easily extended to higher dimensions.

### 1.2. Applications to dispersive estimates

We pass to some applications of Theorems 1.2 and 1.3 to weak dispersive estimates for (1.4) and (1.5). All the following results hold in dimension \( n \geq 3 \).

We need to introduce the following family of norms:

**Definition 1.8.** For any \( f : \mathbb{R}^3 \to \mathbb{R} \) and \( \alpha \in \mathbb{R} \) we define

\[ |||f|||_\alpha := \int_0^{+\infty} \rho^\alpha \sup_{|x| = \rho} |f| \, d\rho. \tag{1.23} \]

We state the following theorems.

**Theorem 1.9** (Weak dispersion for 3D Schrödinger). Let \( n = 3 \); assume that

\[ |||B_\tau^2|||_3 + |||V_\tau^+|||_2 \leq \frac{1}{2}. \tag{1.24} \]

Then, for any solution \( u \) of (1.4) with \( f \in L^2 \), \( H_A f \in L^2 \), the following estimate holds:

\[ \sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} |\nabla_A u|^2 \, dx \, dt \leq C \| f \|_{H^2}^2 \tag{1.25} \]
for some $C > 0$. Moreover, if the strict inequality holds in (1.24), we also have
\[
\sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} |\nabla_A u|^2 \, dx \, dt + \epsilon \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\nabla_A u|^2}{|x|} \, dx \, dt \leq C \|f\|_{L^2}^2,
\]
(1.26)
for some $\epsilon > 0$.

In higher dimension we prove the following Theorem.

**Theorem 1.10** (Weak dispersion for higher dimensional Schrödinger). Let $n \geq 4$; assume that
\[
|B_r(x)| \leq \frac{C_1}{|x|^2}, \quad |V^+_r(x)| \leq \frac{C_2}{|x|^2}, \quad C_1^2 + 2C_2 \leq \frac{2}{3} (n-1)(n-3), \quad (1.27)
\]
for all $x \in \mathbb{R}^n$. Then, for any solution of (1.4) with $f \in L^2$, $H_A f \in L^2$, the following estimate holds:
\[
\sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} |\nabla_A u|^2 \, dx \, dt \leq C \|f\|_{L^2}^2,
\]
for some $C > 0$. Moreover, if the strict inequality holds in (1.27), we also have
\[
\sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} |\nabla_A u|^2 \, dx \, dt + \epsilon \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\nabla_A u|^2}{|x|} \, dx \, dt
+ \epsilon \frac{(n-1)(n-3)}{2} \int_0^{+\infty} \int \frac{|u|^2}{|x|^3} \, dx \, dt \leq C \|f\|_{L^2}^2,
\]
(1.28)
for some $\epsilon > 0$.

For the 3D magnetic wave equation we have the following result.

**Theorem 1.11** (Weak dispersion for 3D wave). Let $n = 3$, and assume that
\[
|B_r(x)| \leq \frac{C_1}{|x|^3}, \quad |V^+_r(x)| \leq \frac{C_2}{|x|^3}, \quad C_1^2 + 2C_2 \leq \frac{1}{2}, \quad (1.29)
\]
Then, for any solution $u$ of (1.5) with $f, g \in L^2$, $H_A f, H_A g \in L^2$, the following estimate holds:
\[
\sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} \left( |u_t|^2 + |\nabla_A u|^2 \right) \, dx \, dt \leq CE(0),
\]
(1.30)
where the energy $E$ is defined by
\[
E(t) = \frac{1}{2} \|u_t\|_{L^2}^2 + \frac{1}{2} \|u\|_{H^1}^2.
\]
Moreover, if the strict inequality holds in (1.24), we also have
\[
\sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} \left( |u_t|^2 + |\nabla_A u|^2 \right) \, dx \, dt + \epsilon \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\nabla_A u|^2}{|x|} \, dx \, dt
+ \epsilon \sup_{R > 0} \frac{1}{R^2} \int_0^{+\infty} \int_{|x| = R} |u|^2 \, d\sigma \, dt \leq CE(0),
\]
(1.31)
for some $C > 0$ and $\epsilon > 0$ small.

The analogous in higher dimension is the following.

**Theorem 1.12** (Weak dispersion for higher dimensional wave). Let $n \geq 4$, and assume that
\[
|B(x)| \leq \frac{C_1}{|x|^2}, \quad |V_+(x)| \leq \frac{C_2}{|x|^3}, \quad C_1^2 + 2C_2 \leq \frac{2}{3}(n-1)(n-3),
\]
for all $x \in \mathbb{R}^n$. Then, for any solution on (1.4) with $f, g \in L^2$, $H_A f, H_A g \in L^2$, the following estimate holds:
\[
\sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} \left( |u_t|^2 + |\nabla A u|^2 \right) \, dx \, dt \leq CE(0),
\]
for some $C > 0$. Moreover, if the strict inequality holds in (1.32), we also have
\[
\sup_{R > 0} \frac{1}{R} \int_0^{+\infty} \int_{|x| \leq R} \left( |u_t|^2 + |\nabla A u|^2 \right) \, dx \, dt + \epsilon \int_{R^n} \frac{|\nabla^2 A u|^2}{|x|^2} \, dx \, dt
\]
\[
+ \frac{\epsilon(n-1)(n-3)}{2} \int_0^{+\infty} \int_{|x| \geq R} \frac{|u|^2}{|x|^3} \, dx \, dt \leq CE(0),
\]
for some $\epsilon > 0$.

### 1.3. Strichartz estimates for the magnetic wave equation.

It is more or less standard to prove Strichartz estimates as applications of Theorems 1.11, 1.12; we do it in Theorem 1.13. The key points are the estimates obtained in the previous section and to write (1.5) as
\[
\begin{aligned}
&u_{tt} - \Delta u = F(t, x) \\
u(0) = f \\
&u_t(0) = g,
\end{aligned}
\]
where
\[
F = -2iA \cdot \nabla A u - A^2 u - Vu.
\]

We recall that a couple $(p, q)$ is said to be wave admissible if
\[
\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \quad 2 \leq p \leq \infty, \quad \frac{2(n-1)}{n-3} \geq q \geq 2, \quad q \neq \infty.
\]

If $(p, q)$ is a wave admissible couple, we say that it is an endpoint couple if $p = 2$. We can state the following theorem.

**Theorem 1.13** (Strichartz for wave). Let $n \geq 3$; assume $(H1)$, $(H2)$ and either (1.29) or (1.32). Moreover assume that
\[
|B(x)| \leq \frac{C}{(1 + |x|)^{2+\delta}}, \quad |V(x)| \leq \frac{C}{(1 + |x|)^{2+\delta}},
\]
for some $C > 0$ and some $\delta > 0$. If $u$ is a solution of (1.5) and $(p, q)$ is any non endpoint wave admissible couple, then the following Strichartz estimate holds:
\[
\|u\|_{L^p_t H^s_x} \lesssim \|f\|_{\dot{H}^1} + \|g\|_{L^2},
\]
(1.39)
where the gap of derivatives is \( \sigma = \frac{1}{q} - \frac{1}{p} + \frac{1}{2} \).

Remark 1.1. It is interesting to compare this result with the ones in [7], [10], [8] and [18]. Assumption (1.38) is made in terms of \( B \); since it is the anti-symmetric gradient of \( A \) and \( \text{div} \ A = 0 \), (1.38) implies that

\[
A \leq \frac{C}{(1 + |x|)^{1+\epsilon}}.
\]

(1.40)

It is relevant to notice that (1.38) is gauge invariant, while it is not the same for (1.40). Also the non-trapping and repulsivity conditions given by (1.29) and (1.32) imply the non-existence either of 0-resonances or eigenvalues.

Remark 1.2. Also notice that (1.38) requires \( \alpha > 2 \) in the homogeneous example [7]. For these examples \( A^2 \) is homogeneous of degree strictly bigger than two. From the counterexamples given in [13] it is natural to expect that Strichartz estimates will fail if \( \alpha < 2 \). Notice that in [17] \( A^2(\omega) = 0 = \min_{|x| = 1} A^2(x) \), and this is a necessary condition for the results in [13] to hold.

Remark 1.3. It would be interesting to extend this result to the magnetic Schrödinger equation (1.4). In order to do that, we should prove the versions of Theorems 1.9, 1.10 with \( L^2 \)-initial data. This will be done elsewhere.

2. Virial identities: proofs of Theorems 1.2 and 1.3

This section is devoted to the proofs of the virial identities for the magnetic equations, Theorems 1.2 and 1.3. Let us start with the magnetic Schrödinger equation.

Proof of Theorem 1.2. Let us start by considering a solution \( u \in H^3 \) of (1.4). Using equation (1.4) in the form

\[
 u_t = -iHu,
\]

we can easily compute

\[
\dot{\Theta}_S(t) = -i \langle u, [H, \phi]u \rangle
\]

(2.2)

\[
\ddot{\Theta}_S(t) = -\langle u, [H, [H, \phi]]u \rangle,
\]

(2.3)

where the brackets [ , ] are the commutator and the brackets \( \langle , \rangle \) are the hermitian product in \( L^2 \). In order to simplify the notations, let us denote by

\[
 T = -[H, \phi],
\]

(2.4)

By the Leibnitz formula

\[
 \nabla_A(fg) = g\nabla_Af + f\nabla g,
\]

(2.5)

which implies that

\[
 H(fg) = (Hf)g + 2\nabla_Af \cdot \nabla g + f(\Delta g),
\]

(2.6)

we can write explicitly

\[
 T = 2\nabla \phi \cdot \nabla_A + \Delta \phi.
\]

(2.7)

Observe that \( T \) is anti-symmetric, namely

\[
 \langle f, Tg \rangle = -\langle Tf, g \rangle;
\]
moreover, in the case $A ≡ 0$ the operator $T$ coincides with the usual one introduce by Morawetz in \[15\], which is $2\nabla \phi \cdot \nabla + \Delta \phi$.

Hence we can rewrite (2.3) in the following form

$$\ddot{\Theta}_S(t) = \langle u, [H, T]u \rangle,$$

where $T$ is given by (2.7).

We can compute explicitly the commutator $[H, T]$; by (2.7) we have

$$[H, T] = -[\nabla_A^2, 2\nabla \phi \cdot \nabla_A] - [\nabla_A^2, \Delta \phi] + [V, T] =: I + II + III.$$

Let us introduce the following notations: for $f : \mathbb{R}^n \to \mathbb{C}$,

$$f_j = \frac{\partial f}{\partial x^j};$$

$$f_j^\pm = f_j - iA_j^f;$$

$$f_j^\ast = f_j + iA_j^f.$$

With these notations we have

$$(fg)_j^\pm = f_j g + fg_j^\pm;$$

moreover, the formula of integrations by parts is

$$\int_{\mathbb{R}^n} f_j(x)g(x) \, dx = -\int_{\mathbb{R}^n} f(x)g_j^\pm(x) \, dx.$$

Now we compute the terms $I$, $II$ and $III$ in (2.9).

The term $III$ in (2.9) is easily computed:

$$III = [V, T] = 2[V, \nabla_A \cdot \nabla] = -2\nabla \phi \cdot \nabla V = -2\phi' V_r.$$

For $I$, we have

$$-I = 2 \sum_{j,k=1}^n \left( \partial_j \partial_k \phi_k \partial_j^\ast - \phi_k \partial_k \partial_j \partial_j^\ast \right)$$

$$= \sum_{j,k=1}^n \left( 2\phi_{kjj} \partial_k^\ast + 4\phi_{jk} \partial_j \partial_k^\ast + 2\phi_k \left( \partial_j^\ast \partial_j^\ast \partial_k - \partial_k \partial_k \partial_j^\ast \right) \right).$$

Notice that

$$\partial_j \partial_k^\ast - \partial_k \partial_j^\ast = i \left( A_j^k - A_k^j \right),$$

$$\partial_j^\ast \partial_k - \partial_k \partial_j^\ast = i \left( A_j^k - A_k^j \right) + 2i \left( A_j^k - A_k^j \right) \partial_j^\ast;$$

hence, by (2.11) we obtain

$$- I = \sum_{j,k=1}^n \left( 2\phi_{kjj} \partial_k^\ast + 4\phi_{jk} \partial_j \partial_k^\ast + 2i\phi_j \left( A_j^k - A_k^j \right) \partial_k^\ast + 4i\phi_j \left( A_k^j - A_j^k \right) \partial_k^\ast \right)$$

(2.12)
For the term $II$ in (2.9) we compute

$$-II = \sum_{j,k=1}^{n} \left( \partial_k \partial_j \phi_{jj} - \phi_{jj} \partial_k \partial_j \right) = \sum_{j,k=1}^{n} \left( \phi_{jkk} + 2\phi_{jk} \partial_k \right).$$ \hspace{1cm} (2.13)

By (2.12) and (2.13) we can write

$$\langle u, [\nabla^2 T, u] \rangle = \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2u \phi_{kjj} \overline{u}_k + 4u \phi_{jk} \partial_j \overline{u}_k + 2u \phi_{kjj} \overline{u}_k \right) dx$$

$$+ \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2i \phi_j \left( A^j_k - A^k_j \right) \overline{u}_k \right) dx$$

$$+ \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx.$$ \hspace{1cm} (2.14)

Observe that

$$\overline{\partial_j \partial_k u} = \partial_j \partial_k \overline{u},$$

as a consequence, integrating by parts the first three terms of (2.14) we have

$$\sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2u \phi_{kjj} \overline{u}_k + 4u \phi_{jk} \partial_j \overline{u}_k + 2u \phi_{kjj} \overline{u}_k \right) dx$$

$$= \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} -4u \phi_{jk} \overline{u}_k \, dx = -4 \int_{\mathbb{R}^n} \nabla_A u \Delta^2 \phi \overline{u} \, dx.$$ \hspace{1cm} (2.15)

For the 4th and 5th term in (2.14) we notice that

$$\sum_{j,k=1}^{n} \phi_{jk} \left( A^j_k - A^k_j \right) = 0,$$

and integrating by parts we obtain

$$\sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2i \phi_j \left( A^j_k - A^k_j \right) \overline{u}_k \right) dx$$

$$= 4 \Im \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} u \phi_j \left( A^j_k - A^k_j \right) \overline{u}_k \, dx$$

$$= 4 \Im \int_{\mathbb{R}^n} u \phi \cdot \overline{B} \cdot \nabla_A u \, dx.$$ \hspace{1cm} (2.16)

whit $B_\tau u$ as in Definition 1.1.

By (2.10), (2.14), (2.15) and (2.16) we conclude that

$$\langle u, [H, T] u \rangle = 4 \int_{\mathbb{R}^n} \nabla_A u \Delta^2 \phi \overline{\nabla_A u} - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi$$

$$- 2 \int_{\mathbb{R}^n} \phi \overline{V} |u|^2 + 4 \Im \int_{\mathbb{R}^n} u \phi \cdot \overline{B} \cdot \nabla_A u.$$ \hspace{1cm} (2.18)
Identities (2.8) and (2.18) prove (1.13).

Remark 2.1. In the above arguments the highest order term in \( u \) that appears is of the form
\[
\int \nabla^2 A u \nabla \phi \cdot \nabla A u;
\]

it makes sense thanks to assumption (H2) and the condition \( f \in L^2, H_A f \in L^2 \), which implies \( H_A e^{itH} f \in L^2 \), and by interpolation \( \nabla A e^{itH} f \in L^2 \). Consequently, all the performed integration by parts are permitted.

□

Proof of Theorem 1.3. The proof of Theorem 1.3 is analogous to the previous one. Let us write
\[
\Theta_W(t) = \langle u_t, \phi u_t \rangle + \langle \phi \nabla A u, \nabla A u \rangle - \frac{1}{2} \langle u \Delta \phi, u \rangle + \langle u, \phi V u \rangle + \langle u, \Psi u \rangle. \tag{2.19}
\]

Differentiating in (2.19) with respect to time and using equation (1.5) we obtain
\[
\begin{align*}
\frac{d}{dt} \langle u_t, \phi u_t \rangle &= -2 \Re \langle u_t, \phi H u \rangle = 2 \Re \langle u_t, \phi \nabla^2 A u \rangle - 2 \Re \langle u_t, \phi V u \rangle, \\
\frac{d}{dt} \langle \phi \nabla A u, \nabla A u \rangle &= -2 \Re \langle u_t, \phi \nabla^2 A u \rangle - 2 \Re \langle u_t, \nabla \phi \cdot \nabla A u \rangle, \\
- \frac{d}{dt} \frac{1}{2} \langle u \Delta \phi, u \rangle &= - \Re \langle u_t, (\Delta \phi) u \rangle, \\
\frac{d}{dt} \langle u, \phi V u \rangle &= 2 \Re \langle u_t, \phi V u \rangle \\
\frac{d}{dt} \langle u, \Psi u \rangle &= 2 \Re \langle u_t, \Psi u \rangle.
\end{align*}
\]

Hence, recalling the operator \( T = -[H, \phi] = 2 \nabla \phi \cdot \nabla A + \Delta \phi \), we have by (2.19)
\[
\dot{\Theta}_W(t) = - \Re \langle u_t, Tu \rangle + 2 \Re \langle u_t, \Psi u \rangle. \tag{2.20}
\]

Consider the first term on the RHS of (2.20), differentiating and using the equation we see that
\[
- \frac{d}{dt} \langle u_t, Tu \rangle = \langle u, HTu \rangle - \langle u_t, Tu_t \rangle. \tag{2.21}
\]

Since \( T \) is anti-symmetric, we have
\[
\Re \langle u_t, Tu_t \rangle = 0; \tag{2.22}
\]

moreover
\[
\langle u, HTu \rangle = \langle u, THu \rangle + \langle u, [H, T]u \rangle = - \langle HTu, u \rangle + \langle u, [H, T]u \rangle,
\]

and then
\[
\Re \langle u, HTu \rangle = \frac{1}{2} \langle u, [H, T]u \rangle. \tag{2.23}
\]

Recollecting (2.21), (2.22) and (2.23) we arrive at
\[
\frac{d}{dt} \Re \langle u_t, Tu \rangle = \frac{1}{2} \langle u, [H, T]u \rangle. \tag{2.24}
\]
For the second term on the RHS of \((2.20)\), we observe that
\[
\frac{d}{dt} 2\Re \langle u_t, \Psi u \rangle = 2\Re \langle u_t, \Psi u_t \rangle + 2\Re \langle Hu, \Psi u \rangle. \tag{2.25}
\]
By integration by parts we see that
\[
\Re \langle Hu, \Psi u \rangle = -\int |\nabla_A u|^2 \Psi - \Re \int \nabla_A u \cdot \nabla \Psi u. \tag{2.26}
\]
Moreover,
\[
\int \nabla_A u \cdot \nabla \Psi u = -\int |u|^2 \Delta \Psi - \int \nabla_A u \cdot \nabla u,
\]
and consequently
\[
\Re \int \nabla_A u \cdot \nabla \Psi u = -\frac{1}{2} \int |u|^2 \Delta \Psi. \tag{2.27}
\]
In conclusion, by \((2.25)\), \((2.26)\) and \((2.27)\) we obtain
\[
\frac{d}{dt} 2\Re \langle u_t, \Psi u \rangle = 2 \int |u_t|^2 \Psi - 2 \int |\nabla_A u|^2 \Psi + \int |u|^2 \Delta \Psi. \tag{2.28}
\]
Finally, by \((2.20)\), \((2.24)\) and \((2.28)\) we conclude that
\[
\ddot{\Theta}_W(t) = \frac{1}{2} \Re \langle u, [H, T] u \rangle + 2 \int |u_t|^2 \Psi - 2 \int |\nabla_A u|^2 \Psi + \int |u|^2 \Delta \Psi. \tag{2.29}
\]
The term \([H, T]\) on the RHS of \((2.29)\) has been already computed in the previous section, modulo the constant \(1/2\). The analogous to Remark 2.1 concludes the proof of \((1.15)\).

\[\square\]

3. Proofs of the smoothing estimates

We devote this section to the proofs of Theorems 1.9, 1.10, 1.11, 1.12. The proofs are based on suitable choices of the multiplier \(\phi\) in the virial identities \((1.13)\) and \((1.15)\). As we see in the following, the choice of the multipliers is different in the cases \(n = 3\) and \(n \geq 4\), and it follows the ideas of the paper [2]. We start with the Schrödinger equation in space dimension \(n = 3\).

3.1. Proof of Theorem 1.9. Recalling \((2.3)\) and \((2.4)\), let us rewrite identity \((1.13)\) as follows
\[
\frac{d}{dt} \dot{\Theta}_S(t) = -\Re \langle u, [H, T] u \rangle.
\]
By \((2.2)\) and \((2.7)\), integrating by parts we see that
\[
\dot{\Theta}_S(t) = 2\Im \int_{\mathbb{R}^n} \overline{u}(x, t) \nabla_A u(x, t) \cdot \nabla \phi(x) \, dx.
\]
Hence we can rewrite \((1.13)\) as follows
\[
2 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \nabla_A u - \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi - \int_{\mathbb{R}^n} \phi' V_r |u|^2 + 2\Im \int_{\mathbb{R}^n} u \phi' B_r \cdot \nabla_A u = \mathcal{R}(t), \tag{3.1}
\]
where
\[
\mathcal{R}(t) = \int_{\mathbb{R}^n} u^2 \phi' B_r \cdot \nabla_A u.
\]
for all $t > 0$, where
\[
\mathcal{R}(t) = \frac{d}{dt} \left( 3 \int_{\mathbb{R}^n} \nabla u(x, t) \cdot \nabla \phi(x) \, dx \right),
\]

We start with an interpolation Lemma that will be used for the estimate of the right hand side of (3.1).

**Lemma 3.1.** Let $\phi = \phi(|x|) : \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, be a radial function such that $\phi'(r)$ and $r\phi''(r)$ are bounded; then the following estimate holds:
\[
\left| \int_{\mathbb{R}^n} \nabla u(x, t) \cdot \nabla \phi(x) \, dx \right| \leq C \|f\|_{\dot{H}^{1/2}}. \tag{3.2}
\]

**Proof.** Let us consider the quadratic form
\[
T(f, g) = \int_{\mathbb{R}^n} \nabla A f(x) \cdot \nabla \phi(x) \, dx.
\]
Since $\phi'$ is bounded, we have the inequality
\[
|T(f, g)| \leq C_1 \|f\|_{L^2} \|\nabla A g\|_{L^2}. \tag{3.3}
\]
By integration by parts, we see that
\[
T(f, g) = -\int_{\mathbb{R}^n} g(x) \nabla A \bar{f}(x) \cdot \nabla \phi(x) \, dx - \int_{\mathbb{R}^n} \bar{f} g \Delta \phi \, dx; \tag{3.4}
\]
under the assumptions on $\phi$ we have that $\Delta \phi(x) \leq C/|x|$, hence using the magnetic Hardy’s inequality (A.1) we have by (3.4) that
\[
|T(f, g)| \leq C_2 \|g\|_{L^2} \|\nabla A f\|_{L^2}. \tag{3.5}
\]
By interpolation between (3.3) and (3.5) we get
\[
|T(f, g)| \leq C \|f\|_{\dot{H}^{1/2}} \|g\|_{\dot{H}^{1/2}}.
\]

Now we choose an explicit multiplier $\phi$. For some $M > 0$, let us consider
\[
\phi_0(x) = \int_0^x \phi'(s) \, ds,
\]
where
\[
\phi_0 = \phi_0(r) = \begin{cases} 
M + \frac{1}{3}r, & r \leq 1 \\
M + \frac{1}{2} - \frac{1}{6r^2}, & r > 1
\end{cases}
\]
A direct computation shows that
\[
\phi_0''(r) = \begin{cases} 
\frac{1}{3}, & r \leq 1 \\
\frac{1}{3r^3}, & r > 1
\end{cases}
\]
and the bilaplacian is given by
\[
\Delta^2 \phi_0(r) = -4\pi \delta_{x=0} - \delta_{|x|=1},
\]
where on the right hand side we have the Dirac masses concentrated at zero and on the unit sphere, respectively. By scaling, for any $R > 0$ we define
\[
\phi(r) = R \phi_0 \left( \frac{r}{R} \right),
\]
hence
\[ \phi'(r) = \begin{cases} M + \frac{r}{3R}, & r \leq R \\ M + \frac{1}{2} - \frac{R^2}{6r^2}, & r > R \end{cases} \] (3.6)

\[ \phi''(r) = \begin{cases} \frac{1}{3R}, & r \leq R \\ \frac{R}{2} + \frac{R^3}{3r^3}, & r > R \end{cases} \] (3.7)

\[ \Delta^2 \phi(r) = -4\pi \delta_{x=0} - \frac{1}{R^2} \delta_{|x|=R}. \] (3.8)

Observe that the assumptions of Lemma 3.1 are satisfied by \( \phi \).

We can pass to the estimate on the left hand side of (3.1). Let us introduce the following formula, which holds in any dimension:

\[ \nabla_A u D^2 \phi \nabla_A u = \phi'' |\nabla_A u|^2 + \frac{\phi'}{r} |\nabla_A^r u|^2. \] (3.9)

Here \( \nabla_A^r \) is the projection of \( \nabla_A \) on the tangent plane to the sphere, such that

\[ |\nabla_A^r u(x)| + |\nabla_A^r u(x)| = |\nabla_A u(x)|, \]

\[ \nabla_A^r u(x) = \left( \nabla_A u(x) \cdot \frac{x}{|x|} \right) \frac{x}{|x|}, \quad \nabla_A^r u \cdot \nabla_A^r u = 0. \]

Now we insert (3.6), (3.7) and (3.8) in (3.1); neglecting the negative part \( V_r^- \) of the electric potential, we have

\[ \frac{2}{3R} \int_{|x| \leq R} |\nabla_A u|^2 \, dx + 2M \int_{\mathbb{R}^n} \frac{|\nabla_A u|^2}{|x|} \, dx + \frac{1}{2R^2} \int_{|x|=R} |u|^2 \, d\sigma \] (3.10)

\[ - \int_{\mathbb{R}^n} \phi' V_r^+ |u|^2 \, dx + 23 \int_{\mathbb{R}^n} u \phi' B_r \cdot \nabla_A u \, dx \]

\[ \leq \frac{d}{dt} \left( \int_{\mathbb{R}^n} \overline{u(x,t)} \nabla_A u(x,t) \cdot \nabla \phi(x) \, dx \right), \]

for all \( R > 0 \). Let us now consider the term involving \( B_r \) on the left hand side of (3.10); since the vector field \( B_r \) is tangential, by (3.6) we can estimate

\[ 23 \int_{\mathbb{R}^n} u \phi' B_r \cdot \nabla_A u \, dx \geq -2 \left| \int_{\mathbb{R}^n} u \phi' B_r \cdot \nabla_A u \, dx \right| \] (3.11)

\[ \geq -2 \left( M + \frac{1}{2} \right) \int_{\mathbb{R}^n} |u| \cdot |B_r| \cdot |\nabla_A^r u| \, dx \]

\[ \geq -2 \left( M + \frac{1}{2} \right) \left( \int_{\mathbb{R}^n} \frac{|\nabla_A^r u|^2}{|x|} \, dx \right)^{\frac{1}{2}} \left( \int_{0}^{+\infty} \frac{d\rho}{\rho} \int_{|x|=\rho} |x| \cdot |u|^2 \cdot |B_r|^2 \, d\sigma \right)^{\frac{1}{2}} \]

\[ \geq -2 \left( M + \frac{1}{2} \right) K_1 \left( \int_{0}^{+\infty} \frac{d\rho}{\rho^2} \left( \sup_{|x|=\rho} |B_r|^2 |x|^2 \right) \right)^{\frac{1}{2}} \left( \int_{0}^{+\infty} \sup_{|x|=\rho} |B_r|^2 |x|^3 \, d\rho \right)^{\frac{1}{2}} \]

\[ \geq -2 \left( M + \frac{1}{2} \right) K_1 \left( \frac{1}{R^2} \int_{|x|=R} |u|^2 \, d\sigma \right)^{\frac{1}{2}} \left( \int_{0}^{+\infty} \sup_{|x|=\rho} |B_r|^2 |x|^3 \, d\rho \right)^{\frac{1}{2}} \]

\[ = -2 \left( M + \frac{1}{2} \right) K_1 K_2 \|B_r^2\|_3^{1/2}, \]
where
\[
K_1 = \left( \int_{\mathbb{R}^n} \frac{\nabla^2 u^2}{|x|} \, dx \right)^{\frac{1}{2}},
\]
\[
K_2 = \left( \sup_{R > 0} \frac{1}{R^2} \int_{|x|=R} |u|^2 \, d\sigma \right)^{\frac{1}{2}},
\]
and we recall the definition
\[
|||B^2_\tau|||_3 = \int_0^{+\infty} \rho^3 \sup_{|x|=\rho} |B_\tau|^2 \, d\rho.
\]

In an analogous way we treat the term involving \(V^+_r\) in (3.10):
\[
- \int \phi' V^+_r |u|^2 \, dx \geq - \left| \int \phi' V^+_r |u|^2 \, dx \right| \geq - \left( M + \frac{1}{2} \right) \int_0^{+\infty} d\rho \int_{|x|=\rho} |V^+_r| \cdot |u|^2 \, d\sigma
\]
\[
\geq - \left( M + \frac{1}{2} \right) \int_0^{+\infty} d\rho \left( \sup_{|x|=\rho} |V^+_r| \cdot |x|^2 \right)^{\frac{1}{2}} \rho^2 \int_{|x|=\rho} |u|^2 \, d\sigma
\]
\[
\geq - \left( M + \frac{1}{2} \right) \left( \int_0^{+\infty} \sup_{|x|=\rho} |V^+_r| \cdot |x|^2 \, d\rho \right) \cdot \left( \sup_{R > 0} \frac{1}{R^2} \int_{|x|=\rho} |u|^2 \, d\sigma \right)
\]
\[
= - \left( M + \frac{1}{2} \right) |||V^+_r|||_2 K_2^2
\]
where \(K_2\) is as before and
\[
|||V^+_r|||_2 = \int_0^{+\infty} \rho^2 \sup_{|x|=\rho} |V^+_r| \, d\rho.
\]

Using (3.11), (3.12) and taking the supremum over \(R > 0\) in (3.10), we obtain
\[
\sup_{R > 0} \frac{2}{3R} \int_{|x| \leq R} |\nabla A u|^2 + 2MK_1^2 + \frac{1}{2}K_2^2
\]
\[
\leq \mathcal{R}(t) + \left( M + \frac{1}{2} \right) \cdot \left( 2K_1 K_2 |||B^2_\tau|||_3^{1/2} + K_2^2 |||V^+_r|||_2 \right),
\]
or equivalently
\[
\sup_{R > 0} \frac{2}{3R} \int_{|x| \leq R} |\nabla A u|^2 \, dx + C(M, K_1, K_2, B) \leq \mathcal{R}(t), \quad (3.13)
\]
where
\[
C(M, K_1, K_2, B) = 2MK_1^2 + \left[ \frac{1}{2} - \left( M + \frac{1}{2} \right) |||V^+_r|||_2 \right] K_2^2
\]
\[
- 2 \left( M + \frac{1}{2} \right) |||B^2_\tau|||_3^{1/2} K_1 K_2.
\]

Notice that all the quantities that appear in the above calculation are finite thanks to the assumptions (H1) and (H2) and the diamagnetic inequality.
In order to conclude the proof, it is sufficient now to optimize the small-
ness condition on \( \| B^2_\tau \|_3 \) and \( \| V^+ \|_2 \) under which we can ensure that 
\[ C(M, K_1, K_2, B) \geq 0. \]

Due to the homogeneity of \( C \), it is not restrictive to fix \( K_1 = 1 \) and impose that 
\[ \left[ \frac{1}{2} - \left( M + \frac{1}{2} \right) \| V^+ \|_2 \right] K_2^2 - 2 \left( M + \frac{1}{2} \right) \| B^2_\tau \|_3^{1/3} K_2 + 2M > 0, \]
for all \( K_2 > 0 \). This gives the following condition:
\[ \frac{(M + \frac{1}{2})^2}{M} \| B^2_\tau \|_3 + 2 \left( M + \frac{1}{2} \right) \| V^+ \|_2 \leq 1. \quad (3.14) \]

In order to optimize \( (3.14) \) in terms of the size of \( B \), we choose \( M = 1/2 \), such that the coefficient of \( \| B^2_\tau \|_3 \) is the minimum possible. Hence we obtain
\[ \| B^2_\tau \|_3 + \| V^+ \|_2 \leq \frac{1}{2} \implies C(M, K_1, K_2, B) \geq 0. \quad (3.15) \]

As a consequence, if \( (1.24) \) is satisfied we have by \( (3.13) \) and \( (3.15) \) that
\[ \sup_{R>0} \frac{1}{R} \int_{|x|\leq R} |\nabla_A u|^2 \, dx \leq C R(t), \]
for some \( C > 0 \). Moreover, if the strict inequality holds in \( (1.24) \), we also have
\[ \sup_{R>0} \frac{1}{R} \int_{|x|\leq R} |\nabla_A u|^2 \, dx + \epsilon \int_{\mathbb{R}^n} \frac{|\nabla_A u|^2}{|x|} \, dx \]
\[ + \epsilon \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 \, d\sigma \leq C R(t), \]
for some \( \epsilon > 0 \).

At this point, the thesis immediately follows by integrating in time the two last inequalities and applying Lemma 3.1 and the conservation of the \( \mathcal{H}^{\frac{1}{2}} \)-norm to the right hand side.

3.2. **Proof of Theorem 1.10** For the proof of the higher dimensional Theorem 1.10 we use the same techniques of the previous one, but with different multipliers. First of all, let us start again from \( (3.1) \). We divide the estimate of the left hand side into two steps, choosing two different multipliers.

**Step 1.** Let us consider the multiplier \( \phi(x) = |x| \), for which
\[ \phi'(r) = 1, \quad \phi''(r) = 0, \quad \Delta^2 \phi(r) = -\frac{(n-1)(n-3)}{r^3}. \]

With this choice, by \( (3.9) \) we can rewrite \( (3.1) \) as follows:
\[ 2 \int_{\mathbb{R}^n} \frac{|\nabla_A u|^2}{|x|} \, dx + \frac{(n-1)(n-3)}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} \, dx \]
\[ - \int_{\mathbb{R}^n} V_r |u|^2 \, dx + 2\Im \int_{\mathbb{R}^n} u B_r \cdot \nabla_A u \, dx = \mathcal{R}(t). \quad (3.16) \]
Here we used again the same notations of the previous Section. As in the
previous case, the main goal is to prove the positivity of the left hand side.
Let us assume that
\[
|B_\tau(x)| \leq \frac{C_1}{|x|^2}
\]
and estimate
\[
-23 \int_{\mathbb{R}^n} uB_\tau \cdot \nabla A u \, dx \geq -2 \left( \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} \, dx \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^n} |x|^3 |B_\tau|^2 |\nabla A u|^2 \, dx \right)^{\frac{1}{2}}
\geq -2C_1K_1K_2,
\]
where
\[
K_1 = \left( \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} \, dx \right)^{\frac{3}{2}}
K_2 = \left( \int_{\mathbb{R}^n} \frac{|\nabla A u|^2}{|x|} \, dx \right)^{\frac{1}{2}}.
\]
Analogously, assume that
\[
|V^+_\tau(x)| \leq \frac{C_2}{|x|^3}
\]
and estimate
\[
- \int_{\mathbb{R}^n} V_\tau |u|^2 \, dx \geq -C_2K_1^2,
\]
where \(K_1\) is as before.
Consequently, for the left hand side of (3.16) we have
\[
2 \int_{\mathbb{R}^n} |\nabla A u|^2 \, dx + \frac{(n-1)(n-3)}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} \, dx
\geq 2K_2^2 - 2C_1K_1K_2 - C_2K_1^2 + \frac{(n-1)(n-3)}{2} K_1^2 =: C(C_1, C_2, K_1, K_2).
\]
Once again, we want to optimize the condition on \(C_1\) and \(C_2\) under which
the right hand side of (3.18) is positive for all \(K_1, K_2\).
Also here it is not restricting to fix \(K_1 = 1\) and requiring that
\[
\left[ \frac{(n-1)(n-3)}{2} - C_2 \right] K_1^2 - 2C_1K_1 + 2 \geq 0,
\]
which gives the following condition:
\[
C_1^2 + 2C_2 \leq (n-1)(n-3).
\]
As a consequence, if (3.19) is satisfied, then
\[
2 \int_{\mathbb{R}^n} |\nabla A u|^2 \, dx + \frac{(n-1)(n-3)}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} \, dx
\geq \int_{\mathbb{R}^n} V^+_\tau |u|^2 \, dx - 23 \int_{\mathbb{R}^n} uB_\tau \cdot \nabla A u \, dx \geq 0.
\]
Moreover, if the strict inequality holds in (3.19), we have
\[
2 \int_{\mathbb{R}^n} \frac{|\nabla \tau A u|^2}{|x|} + \frac{(n-1)(n-3)}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} - \int_{\mathbb{R}^n} V_r |u|^2 - 23 \int_{\mathbb{R}^n} u B_r \cdot \nabla \tau A u
\geq \epsilon \left( \int_{\mathbb{R}^n} \frac{|\nabla \tau A u|^2}{|x|} + \frac{(n-1)(n-3)}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} \right),
\]
for some \( \epsilon > 0 \).

**Step 2.** Now we perturb the multiplier \( \phi \) to complete the proof. Let us consider
\[
\tilde{\phi} = \phi + \varphi,
\]
where \( \phi(r) = r \) and \( \varphi(r) = \int_0^r \varphi'(s) \, ds \), with
\[
\varphi'(r) = \begin{cases} \frac{n-1}{2n} r, & r \leq 1 \\ \frac{1}{2} - \frac{1}{2n} r, & r > 1. \end{cases}
\]
An explicit computation shows that
\[
\varphi''(r) = \begin{cases} \frac{n-1}{2n} r, & r \leq 1 \\ \frac{n-1}{2n} r - \frac{1}{2} + \frac{1}{2n} r, & r > 1. \end{cases}
\]
\[\Delta^2 \varphi = -\frac{n-1}{2} \delta_{|x|=1} - \frac{(n-1)(n-3)}{2r^3} \chi_{[1, +\infty)}.\]
Finally, for any \( R > 0 \) we define the scaled multiplier
\[
\tilde{\phi}_R(r) = R \tilde{\phi} \left( \frac{r}{R} \right);
\]
we have explicitly that
\[
\tilde{\phi}_R(r) = r + R \varphi \left( \frac{r}{R} \right),
\]
where
\[
\varphi'_R(r) = \begin{cases} \frac{n-1}{2nR} r, & r \leq R \\ \frac{1}{2} - \frac{R}{2n} r, & r > R. \end{cases}
\]
\[
\varphi''_R(r) = \begin{cases} \frac{1}{R} - \frac{n-1}{2nR^2} r, & r \leq R \\ \frac{n-1}{2nR^2} r - \frac{1}{R}, & r > R. \end{cases}
\]
\[\Delta^2 \varphi_R = -\frac{n-1}{2R^2} \delta_{|x|=R} - \frac{(n-1)(n-3)}{2r^3} \chi_{[R, +\infty)}.\]
At this point, we put the multiplier \( \tilde{\phi}_R \) in (3.1). Observe that, in this case,
\[
\sup_{r \geq 0} \tilde{\phi}'(r) = \frac{3}{2};
\]
hence for the terms involving \( V_r \) and \( B_r \) we can repeat the same computation in (3.17), with the scaled constants \( \tilde{C}_1 = \frac{3}{2} C_1, \tilde{C}_2 = \frac{3}{2} C_2 \); in this way, the final condition on \( B_r \) and \( V_r \) turns out to be (1.27). Under this assumption, when we put \( \tilde{\phi}_R \) into (3.1), we can use the term \( |x| \) in (3.23) to control
the perturbative terms with the potentials. Finally, by (3.20), (3.21), (3.24), (3.25), (3.26), we have proved that

$$\sup_{R>0} \frac{1}{R} \int_{|x| \leq R} |\nabla_A u|^2 \, dx \leq CR(t),$$

(3.27)

for some $C > 0$, if (1.27) is satisfied. Moreover, if the strict inequality holds in (1.27), we also have

$$\epsilon \left( \int_{\mathbb{R}^n} \frac{|\nabla_A u|^2}{|x|} \, dx + \frac{(n-1)(n-3)}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} \, dx \right) \leq CR(t),$$

(3.28)

analogously to (3.21). Now the proof continues exactly the one of the 3D Theorem 1.9, by integration in time and application of Lemma 3.1.

3.3. Proofs of Theorems 1.11 and 1.12. The proofs of the smoothing Theorems for the magnetic equation are identical to the ones for the Schrödinger equations. The starting point is now the virial identity (1.15), which can be written as follows:

$$2 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \overline{\nabla_A u} \, dx - \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx$$

(3.29)

$$+ 2 \int_{\mathbb{R}^n} |u|^2 \Psi \, dx - 2 \int_{\mathbb{R}^n} |\nabla_A u|^2 \Psi \, dx + \int_{\mathbb{R}^n} |u|^2 \Delta \Psi \, dx$$

$$- \int_{\mathbb{R}^n} \phi' V_r |u|^2 \, dx + 2 \int_{\mathbb{R}^n} \omega B_r \cdot \nabla_A u \, dx = \mathcal{R}(t),$$

where

$$\mathcal{R}(t) = - \frac{d}{dt} \int_{\mathbb{R}^n} u_t (2 \nabla \phi \cdot \nabla_A u + \bar{\mu} \Delta \phi + \overline{\nabla_A u}) \, dx.$$  (3.30)

For the LHS of (3.29), we use the same multiplier $\phi$ of the Schrödinger theorems, while the choice of $\Psi$ is the following:

$$\Psi(x) = \begin{cases} \frac{1}{2R}, & |x| \leq R \\ 0, & |x| > R. \end{cases}$$

By direct computation we see that

$$\Delta \Psi = \begin{cases} \frac{a}{R^2}, & |x| \leq R \\ \frac{a-1}{|x|}, & |x| > R, \end{cases}$$

hence $\Delta \Psi \geq 0$ and the term involving it in (3.29) can be neglected. The only thing to control is the positivity of the term

$$\int_{|x| \leq R} (\nabla_A u D^2 \phi \overline{\nabla_A u} - |\nabla_A u|^2 \Psi) \, dx,$$

which is ensured by the choice of the constant $1/2$ in the definition of $\Psi$ and the explicit formulas for $\phi'$ and $\phi''$ introduced in the previous sections.

Finally, after integration in time, with the same techniques involving Cauchy-Schwartz, magnetic Hardy’s inequality and energy conservation, the RHS of (3.29) turns out to be controlled by $CE(0)$, and this concludes the proof.
4. Strichartz estimates for the magnetic wave equation

The final section of this paper is devoted to the proof of Theorem 1.13 as application of the smoothing estimates, Theorems 1.11 and 1.12. We start with a preliminary Lemma.

Lemma 4.1. Let \((p, q)\) be a non endpoint wave admissible couple; then, for all \(T \in \mathbb{R}\), the following estimate holds

\[
\left\| \int_0^T \sin \left( (T - \tau) \sqrt{-\Delta} \right) F(\tau, \cdot) \, d\tau \right\|_{L^p H^q} \lesssim \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| F_j \|_{L^2 L^2},
\]

(4.1)

where \(\sigma = \frac{1}{q} - \frac{1}{p} + \frac{1}{2}\) and

\[F_j(t, x) = \begin{cases} F(t, x), & |x| \in [2^j, 2^{j+1}] \\ 0, & |x| \in [0, \infty) \setminus [2^j, 2^{j+1}] \end{cases}.
\]

Proof. We recall the usual Strichartz estimate for the free wave equation (see [9], [14])

\[
\left\| (\sqrt{-\Delta})^{-1} e^{it\sqrt{-\Delta}} f \right\|_{L^p H^q} \lesssim \| f \|_{L^2},
\]

(4.2)

with \(\sigma = \frac{1}{q} - \frac{1}{p} + \frac{1}{2}\) and the inequality

\[
\sup_{R > 0} \int_0^\infty \int_{|x| \leq R} \left| e^{it\sqrt{-\Delta}} f \right|^2 \, dx \, dt \lesssim \| f \|_{L^2},
\]

(4.3)

analogous to (1.9). By (4.2) we get

\[
\left\| \int_0^T \frac{\sin \left( (T - \tau) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} F(\tau, \cdot) \, d\tau \right\|_{L^p H^q} \leq \left\| \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} \int_0^T e^{-i\tau\sqrt{-\Delta}} F(\tau, \cdot) \, d\tau \right\|_{L^p H^q}
\]

(4.4)

\[
\leq \left\| \int_0^T e^{-i\tau\sqrt{-\Delta}} F(\tau, \cdot) \, d\tau \right\|_{L^2}.
\]

The dual of estimate (1.3) is

\[
\left\| \int e^{-i\tau\sqrt{-\Delta}} F(\tau, \cdot) \, d\tau \right\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| F_j \|_{L^2 L^2}
\]

(see e.g. [16]); as a consequence and by a standard application of the Christ-Kiselev Lemma ([5]), we have (4.1). □

Now we pass to the proof of Theorem 1.13. As observed in Section 1.3, we can rewrite (1.5) in the form (1.35), with \(F\) given by (1.36). The solution of (1.35) is represented by

\[
u(t, \cdot) = \cos(t\sqrt{-\Delta}) f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \int_0^t \frac{\sin \left( (t - \tau) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} F(\tau, \cdot) \, d\tau.
\]

(4.5)
For the first two terms we use (4.2). For the last term, recalling (1.36), by (4.1) we estimate:

$$
\int_0^t \sin \left((t-\tau)\sqrt{\Delta}\right) (A \cdot \nabla A u + A^2 u + V u) \, d\tau \lesssim (4.6)
$$

$$
\lesssim \sum_{j \in \mathbb{Z}} 2^j \left\{ \left( \int_0^t \int_{|x| \in [2^j, 2^{j+1}]} A \cdot \nabla A u \right)^{\frac{1}{2}} + \left( \int_0^t \int_{|x| \in [2^j, 2^{j+1}]} (A^2 + V u) \right)^{\frac{1}{2}} \right\}.
$$

As observed in Remark 1.1, assumption (1.38) implies

$$
|A(x)| \leq C \left( 1 + |x| \right)^{1+\epsilon},
$$

moreover, (1.38) is compatible with (1.29) and (1.32). Consequently, by Holder inequality and the smoothing estimates (1.31) and (1.33) we obtain

$$
\sum_{j \in \mathbb{Z}} 2^j \| (A \cdot \nabla A u)_j \|_{L^2 L^2} \leq (4.7)
$$

$$
\leq \left( \sum_{j<0} 2^j + \sum_{j \geq 0} 2^j 2^{j(-\frac{1}{2}-\epsilon)} \left( \sup_{j \in \mathbb{Z}} \frac{1}{2^j} \int_0^t \int_{2^j |x| < 2^{j+1}} |\nabla A u|^2 \right)^{\frac{1}{2}} \right) \leq C \sqrt{E(0)};
$$

$$
\sum_{j \in \mathbb{Z}} 2^j \| (A^2 u + V u)_j \|_{L^2 L^2} \leq (4.8)
$$

$$
\leq \left( \sum_{j<0} 2^j + \sum_{j \geq 0} 2^{-2j} \right) \left( \sup_{j \in \mathbb{Z}} \frac{1}{2^{2j}} \int_0^t \int_{2^j |x| < 2^{j+1}} |u|^2 \right)^{\frac{1}{2}}
$$

$$
\sim C \left( \sup_{j \in \mathbb{Z}} \frac{1}{2^{2j}} \int_0^t \int_{|x| = 2^j} |u|^2 \, d\sigma \right)^{\frac{1}{2}} \leq C \sqrt{E(0)}.
$$

By (4.5), (4.6), (4.7) and (4.8) the proof is complete. □

**Appendix A. Magnetic Hardy’s inequality**

We devote an Appendix to the magnetic version of Hardy’s Inequality.

**Theorem A.1.** Let $n \geq 3$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$, $\nabla A = \nabla - iA$. Then, for any $f \in D(H_A)$ the following inequality holds:

$$
\int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} \, dx \leq \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |\nabla A f|^2 \, dx. \quad \text{(A.1)}
$$

**Proof.** We only need to prove (A.1) for $f \in C_0^\infty$, then we conclude by density. Let us observe that, for all $\alpha \in \mathbb{R}$,

$$
0 \leq \int_{\mathbb{R}^n} |\nabla A f + \alpha \frac{x}{|x|^2} f|^2 \, dx \quad \text{(A.2)}
$$

$$
= \int_{\mathbb{R}^n} |\nabla A f|^2 \, dx + \alpha^2 \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} \, dx + 2\alpha \Re \int_{\mathbb{R}^n} \frac{x}{|x|^2} \cdot \nabla A f \, dx.
$$

By integration by parts, using the Leibnitz formula

$$
\nabla A (fg) = g\nabla A f + f\nabla g,
$$
we see that
\[
2\alpha \Re \int_{\mathbb{R}^n} \frac{\nabla A f}{|x|^2} \cdot \nabla f \, dx = -\alpha \int_{\mathbb{R}^n} |f|^2 \text{div} \frac{x}{|x|^2} \, dx = -(n - 2)\alpha \int_{\mathbb{R}^n} |f|^2 \, dx.
\]
Using this in (A.2) we have
\[
\left\{ -\alpha^2 + (n - 2)\alpha \right\} \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^n} |\nabla A f|^2 \, dx,
\]
for all $\alpha \in \mathbb{R}$. Now we observe that
\[
\max_{\alpha \in \mathbb{R}} \left\{ -\alpha^2 + (n - 2)\alpha \right\} = \frac{(n - 2)^2}{4},
\]
and this completes the proof. □

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