Well-posedness via Monotonicity – An Overview.

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Abstract. The idea of monotonicity is shown to be the central theme of the solution theories associated with problems of mathematical physics. A “grand unified” setting is surveyed covering a comprehensive class of such problems. We illustrate on the applicability of this setting with a number examples. A brief discussion of stability and homogenization issues within this framework is also included.

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0. Introduction

In this paper we shall survey a particular class of problems, which we like to refer to as “evolutionary equations” (to distinguish it from the class of explicit first order ordinary differential equations with operator coefficients predominantly considered under the heading of evolution equations). This problem class is spacious enough to include not only classical evolution equations but also partial differential algebraic systems, functional differential equations and integro-differential equations. Indeed, by thinking of elliptic systems as time dependent, for example as constant with respect to time on the connected components of $\mathbb{R} \setminus \{0\}$, they also can be embedded into this class. The setting is – in its present state – largely limited to a Hilbert space framework. As a matter of convenience the discussion will indeed be set in a complex Hilbert space framework. For the concept of monotonicity it is, however, more appropriate to consider complex Hilbert spaces as real Hilbert spaces, which can canonically be achieved by reducing scalar multiplication to real numbers and replacing the inner product by its real part. So, a binary relation $R$ in a complex Hilbert space $H$ with inner product $\langle \cdot | \cdot \rangle_H$ would be called strictly monotone if

$$\Re \langle x - y | u - v \rangle_H \geq \gamma \langle x - y | x - y \rangle_H$$

for all $(x,u),(y,v) \in R$ holds and $\gamma$ is some positive real number. In case $\gamma = 0$ the relation $R$ would be called monotone.

The importance of strict monotonicity, which in the linear operator case reduces to strict positive definiteness, is of course well-known from the elliptic case. By a suitable choice of space-time norm this key to solving elliptic partial differential equation problems also allows to establish well-posedness for dynamic problems in exactly the same fashion.

The crucial point for this extension is the observation that the one dimensional derivative itself, acting as the time derivative $\partial_0$ (on the full time line $\mathbb{R}$), can be realized as a maximal strictly positive definite operator in an appropriately exponentially weighted $L^2$-type Hilbert space over the real time-line $\mathbb{R}$. It is in fact this strict positive definiteness of $\partial_0$ which opens access to the problem class we shall describe later.

Indeed, $\partial_0$ simply turns out to be a normal operator with $\Re \partial_0$ being just multiplication by a positive constant. Moreover, this time-derivative $\partial_0$ is continuously invertible and, as a normal operator, admits a straightforwardly defined functional calculus, which can canonically be extended to operator-valued functions. Indeed, since we have control over the positivity constant via the choice of the weight, the norm of $\partial_0^{-1}$ can be made as small as wanted. This observation is the Hilbert space analogue to the technical usage of the exponentially weighted sup-norm as introduced by D. Morgenstern, [26], and allows for the convenient inclusion of a variety of perturbation terms.

Having established time-differentiation $\partial_0$ as a normal operator, we are led to consider evolutionary problems as operator equations in a space-time setting, rather than an ordinary differential equation in a spatial function space. The space-time operator equation perspective implies to deal with sums of unbounded operators, which, however, in our particular context is – due to the limitation of remaining in a Hilbert space setting and considering only sums, where one of the terms is a function of the normal operator $\partial_0$ – not so deep an issue. For more general operator sums or for a Banach space setting more sophisticated and powerful tools from the abstract theory of operator sums initiated by the influential papers by da Prato and Grisvard, [11], and Brezis and Haraux, [8], may have to be employed. In these papers operator sums $\frac{d}{dt} + A$ typically occurring in the

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1We use the term strict positive definiteness for a linear operator $A$ in a real or complex Hilbert space $X$ in the sense naturally induced by the classification of the corresponding quadratic form $Q_A$ given by $u \mapsto \langle u | Au \rangle_X$ on its domain $D(A)$. So, if $Q_A$ is non-negative (mostly called positive semi-definite), positive definite, strictly positive definite, then the operator $A$ will be called non-negative (usually called positive), positive definite, strictly positive definite, respectively. If $X$ is a complex Hilbert space it follows that $A$ must be Hermitian. Note that we do not restrict the definition of non-negativity, positive definiteness, strict positive definiteness to Hermitian or symmetric linear operators.

2We follow here the time-honored convention that physicists practice by labeling the partial time derivative by index zero.
context of explicit first order differential equations in Banach spaces are considered as applications of the abstract theory, compare also e.g. [21, Chapter 2, Section 7]. The obvious overlap with the framework presented in this paper would be the Hilbert space situation in the case $\mathcal{M} = 1$. We shall, however, not pursue to explore how the strategies developed in this context may be expanded to include more complicated material laws, which indeed has been done extensively in the wake of these ideas, but rather stay with our limited problem class, which covers a variety of diverse problems in a highly unified setting. Naturally the results available for specialized cases are likely to be stronger and more general.

For introductory purposes let us consider the typical linear case of such a space-time operator equation

$$\partial_0 V + AU = f,$$

(0.1)

where $f$ are given data, $A$ is a – usually – purely spatial – prototypically skew-selfadjoint\(^4\) – operator and the quantities $U, V$ are linked by a so-called material law

$$V = \mathcal{M}U.$$

Solving such an equation would involve establishing the bounded invertibility of $\partial_0 \mathcal{M} + A$. As a matter of “philosophy”\(^5\) we shall think of the – here linear – material law operator $\mathcal{M}$ as encoding the complexity of the physical material whereas $A$ is kept simple and usually only contains spatial derivatives. If $\mathcal{M}$ commutes with $\partial_0$ we shall speak of an autonomous system, otherwise we say the system is non-autonomous.

Another – more peripheral – observation with regards to the classical problems of mathematical physics is that they are predominantly of first order not only with respect to the time derivative, which is assumed in the above, but frequently even in both the temporal and spatial derivatives. Indeed, acoustic waves, heat transport, visco-elastic and electro-magnetic waves etc. are governed by first order systems of partial differential operators, i.e. $A$ is a first order differential operator in spatial derivatives, which only after some elimination of unknowns turn into the more common second order equations, i.e. the wave equation for the pressure field, the heat equation for the temperature distribution, the visco-elastic wave equation for the displacement field and the vectorial wave equation for the electric (or magnetic) field. It is, however, only in the direct investigation of the first order system that, as we shall see, the unifying feature of monotonicity becomes easily visible. Moreover, the first order formulation reveals that the spatial derivative operator $A$ is of a Hamiltonian type structure and consequently, by imposing suitable boundary conditions, turn out – in the standard cases – to lead to skew-selfadjoint $A$ in a suitable Hilbert space $\mathcal{H}$. So, from this perspective there is also undoubtedly a flavor of the concept of symmetric hyperbolic systems as introduced by K. O. Friedrichs, [16], and of Petrovskii well-posedness, [29], at the roots of this approach.

For illustrational purposes let us consider from a purely heuristic point of view the $(1+1)$-dimensional system

$$\left(\partial_0 M_0 + M_1 + A\right) \begin{pmatrix} p \\ s \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

(0.2)

where

$$M_0 := \begin{pmatrix} \eta & 0 \\ 0 & \alpha \end{pmatrix}, \quad M_1 := \begin{pmatrix} (1 - \eta) & 0 \\ 0 & (1 - \alpha) \end{pmatrix}, \quad \alpha, \eta \in \{0, 1\}, \quad A := \begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix},$$

and $\partial_1$ is simply the weak $L^2(\mathbb{R})$-derivative, compare Footnote\(^6\). Assuming $\eta = 1, \alpha = 1$, in (0.2) clearly results in a (symmetric) hyperbolic system and eliminating the unknown $s$ yields the wave equation in the form

$$\left(\partial_s^2 - \partial_t^2\right) p = \partial_0 f.$$

\(^4\)Note that in our canonical reference situation $A$ is skew-selfadjoint rather than selfadjoint and so we have $\Re \langle u | Au \rangle_H = 0$ for all $u \in D(A)$ and coercitivity of $A$ is out of the question. To make this concrete: let $\partial_1$ denote the weak $L^2(\mathbb{R})$-derivative. Then our paradigmatic reference example on this elementary level would be the transport operator $\partial_0 + \partial_1$ rather than the heat conduction operator $\partial_0 - \partial_1^2$.\(^5\)
For $\eta = 1, \alpha = 0$ we obtain a differential algebraic system, which represents the parabolic case in the sense that after eliminating $s$ we obtain the heat equation
\[(\partial_0 - \partial^2_t) p = f.\]

Finally, if both parameters vanish, we obtain a 1-dimensional elliptic system and as expected after eliminating the unknown $s$ a 1-dimensional elliptic equation for $p$ results:
\[(1 - \partial^2_t) p = f.\]

Allowing now $\alpha, \eta$ to be $L^\infty(\mathbb{R})$-multiplication operators with values in $\{0, 1\}$, which would allow the resulting equations to jump in space between elliptic, parabolic and hyperbolic “material properties”, could be a possible scenario we envision for our framework. As will become clear, the basic idea of this simple “toy” example can be carried over to general evolutionary equations. Also in this connotation there are stronger and more general results for specialized cases. A problem of this flavor of “degeneracy” has been for example discussed for a non-autonomous, degenerate integro-differential equation of parabolic/elliptic type in [22, 23].

A prominent feature distinguishing general operator equations from those describing dynamic processes is the specific role of time, which is not just another space variable, but characterizes dynamic processes via the property of causality. Requiring causality for the solution operator $(\partial_0 \mathcal{M} + A)^{-1}$ results in very specific types of material law operators $\mathcal{M}$, which are causal and compatible with causality of $(\partial_0 \mathcal{M} + A)^{-1}. This leads to deeper insights into the structural properties of mathematically viable models of physical phenomena.

The solution theory can be extended canonically to temporal distributions with values in a Hilbert space. In this perspective initial value problems, i.e. prescribing $V(0+) in [0,1]$ amounts to allowing a source term $f$ of the form $\delta \otimes V_0$ defined by
\[(\delta \otimes V_0) (\varphi) := \langle V_0|\varphi(0) \rangle_H\]
for $\varphi$ in the space $C_c(\mathbb{R}, H)$ of continuous $H$-valued functions with compact support. This source term encodes the classical initial condition $V(0+) = V_0$. For the constant coefficient case – say – $\mathcal{M} = 1$, it is a standard approach to establish the existence of a fundamental solution (or more generally, e.g. in the non-autonomous case, a Green’s functions) and to represent general solutions as convolution with the fundamental solution. This is of course nothing but a description of the continuous one-parameter semi-group approach. Indeed, such a semi-group $U = \{U(t)\}_{t \in [0,\infty]}$ is, if extended by zero to the whole real time line, nothing but the fundamental solution $G = \{G(t)\}_{t \in \mathbb{R}}$

with $G(t) := \begin{cases} U(t) & \text{for } t \in [0,\infty[, \\ 0 & \text{for } t \in ]-\infty,0[. \end{cases}$

In the non-autonomous case, the role of $U$ is played by a so-called evolution family. The regularity properties of such fundamental solutions results in stronger regularity properties of the corresponding solutions. Since we allow $\mathcal{M}$ to be more general, constructing such fundamental solutions/Green’s functions is not always available or feasible. Indeed, we shall focus for sake of simplicity on the case that the data $f$ do not contain such Dirac type sources, which can be achieved simply by subtracting the initial data or by including distributional objects such as $\delta \otimes V_0$ in the Hilbert space structure via extension to extrapolation spaces, which for sake of simplicity we will not burden this presentation with.

As a trade-off for our constraint, which in the simplest linear case would reduce our discussion to considering $\partial_0 + A$ as a sum of commuting normal operators, which clearly cannot support any claim of novelty, see e.g. [54], we obtain by allowing for a large class of material law operators $\mathcal{M}$ access to a large variety of problems including such diverse topics as partial differential-algebraic systems, integro-differential equations and evolutionary equations of changing type in one unified setting.

Based on the linear theory one has of course a first access to non-linear problems by including Lipschitz continuous perturbations. A different generalization towards a non-linear theory can be

\[\text{Note that this perspective specifically excludes the case of a periodic time interval, where “before” and “after” makes little sense.}\]
done by replacing the (skew-selfadjoint) operator $A$ by a maximal monotone relation or allowing for suitable maximal monotone material law relations (rather than material law operators). In this way the class of evolutionary problems also comprises evolutionary inclusions.

Having established well-posedness, qualitative properties associated with the solution theory come into focus. A first step in this direction is done for the autonomous case by the discussion of the issue of "exponential stability". One can give criteria with regards to the material law $\mathcal{M}$ ensuring exponential stability.

Another aspect in connection with the discussion of partial differential equations of mathematical physics is the problem of continuous dependence of the solution on the coefficients. A main application of results in this direction is the theory of homogenization, i.e., the study of the behavior of solutions of partial differential equations having large oscillatory coefficients. It is natural to discuss the weak operator topology for the coefficients and it turns out that the problem class under consideration is closed under limits in this topology if further suitable structural assumptions are imposed. The closedness of the problem class is a remarkable feature of the problem class, which is spacious enough to also include – hidden in the generality of the material law operator – integro-differential evolutionary problems. In this regard it is worth recalling that there are examples already for ordinary differential equations, for which the resulting limit equations are of integro-differential type, showing that differential equations are in this respect a too small problem class.

Although links to the core concepts which have entered the described approach are too numerous to be recorded here to any appropriate extent, we shall try modestly to put them in a bibliographical context. The concept of the time-derivative considered as a continuously invertible operator in a suitably weighted Hilbert space has its source in [30]. It has been employed in obtaining a solution theory for evolutionary problems in the spirit described above only more recently, compare e.g. [33], Chapter 6]. General perspectives for well-posedness to partial differential equations via strict positive definiteness are of course at the heart of the theory of elliptic partial differential equations.

For the theory of maximal monotone operators/relations, we refer to [7, 18, 19, 27]. For non-autonomous equations, we refer to [40, 41] and – with a focus on maximal regularity – to [6]. Note that due to the generality of our approach, one cannot expect maximal regularity of the solution operator in general. In fact, maximal regularity for the solution operator just means that the operator sum is already closed with its natural domain. This is rarely the case neither in the paradigmatic examples nor in our expanded general setting.

For results regarding exponential stability for a class of hyperbolic integro-differential equations, we refer to [38] and to [13, 17, 15] for the treatment of this issue in the context of one-parameter semi-groups. A detailed introduction to the theory of homogenization can be found in [4] and in [10]. We also refer to [43, 42], where homogenization for ordinary differential equations has been discussed extensively.

The paper itself is structured as follows. We begin our presentation with a description of the underlying prerequisites, even to the extent that we review the celebrated well-posedness requirements due to Hadamard, which we found inspirational for a deeper understanding of the case of differential inclusions. A main point in this first section is to introduce the classical concept of maximal strictly monotone relations and to recall that such relations are inverse relations of Lipschitz continuous mappings (Minty’s Theorem [14]. Specializing to the linear case we recall in particular the Lax-Milgram lemma (Corollary [14]) and as a by-product derive a variant of the classical solution theory for elliptic type equations. Moreover, we comment on a general solution theory for (non-linear) elliptic type equations in divergence form relying only on the validity of a Poincaré type estimate (Theorem [16]. We conclude this section with an example for an elliptic type equation with possible degeneracies in the coefficients as an application of the ideas presented.

Based on the first section’s general findings, Section 2 deals with the solution theory for linear evolutionary equations. After collecting some guiding examples in Subsection 2.1 we rigorously establish in Subsection 2.2 the time-derivative as a strictly monotone, normal operator in a suitably
weighted Hilbert space. Based on this and with resulting structural properties, such as a functional calculus, at hand, in Subsection 2.3 (Theorem 2.5) we formulate a solution theory for autonomous, linear evolutionary equations. The subsequent examples review some of those mentioned in Subsection 2.1 in a rigorous functional analytic setting to illustrate the applicability of the solution theory. As further applications we show that Theorem 2.6 also covers integro-differential equations (Theorem 2.9) and equations containing fractional time-derivatives (Theorem 2.12). We conclude this subsection with a conceptual study of exponential stability (Definition 2.13 and Theorem 2.14) in our theoretical context.

In Subsection 2.4 starting out with a short motivating introductory part concerning homogenization issues, we discuss the closedness of the problem class under the weak operator topology for the coefficients. A first theorem in this direction is then obtained as Theorem 2.26. After presenting some examples, we continue our investigation of homogenization problems first for ordinary differential equations (Theorems 2.32 and 2.34). Then we formulate a general homogenization result (Theorem 2.37), which is afterwards exemplified by considering Maxwell’s equations and in particular the so-called eddy current problem of electro-magnetic theory.

In Subsection 2.5 we extend the solution theory to include the non-autonomous case. A first step in this direction is provided by Theorem 2.42, for which the illustrative Example 2.43 is given as an application. A common generalization of the Theorems 2.5 and 2.42 is given in Theorem 2.40. This is followed by an adapted continuous dependence result Theorem 2.44 which in particular is applicable to homogenization problems. A detailed example of a mixed type problem concludes this section.

Section 3 gives an account for a non-linear extension of the theory. Similarly to the previous section, the results are considered in the autonomous case first (Subsection 3.1) and then generalized to the non-autonomous case (Subsection 3.2). Subsection 3.3 concludes this section and the paper with a discussion of an application to evolutionary problems with non-linear boundary conditions. One of the guiding conceptual ideas here is to avoid regularity assumptions on the boundary of the underlying domain. This entails replacing the classical boundary trace type data spaces, by a suitable generalized analogue of 1-harmonic functions. We exemplify our results with an impedance type problem for the wave equation and with the elastic equations with frictional boundary conditions.

Note that inner products, indeed all sesqui-linear forms, are – following the physicists habits – assumed to be conjugate-linear in the first component and linear in the second component.

1. Well-posedness and Monotonicity

To begin with, let us recall the well-known Hadamard requirements for well-posedness. It is appropriate for our purposes, however, to formulate them for the case of relations rather than – as usually done – for mappings. Hadamard proposed to define what “reasonably solvable” should entail. Solving a problem involves to establish a binary relation \( P \subseteq X \times Y \) between “data” in a topological space \( Y \) and corresponding “solutions” in a topological space \( X \), which is designed to cover a chosen pool of examples to our satisfaction. Finding a solution then means, given \( y \in Y \) find \( x \in X \) such that \( (x, y) \in P \). If we wish to supply a solution for all possible data, there are some natural requirements that the problem class \( P \) should have to ensure that this task is reasonably conceived. To exclude cases of trivial failure to describe a solution theory for \( P \), we assume first that \( P \) is already closed in \( X \times Y \). Then well-posedness in the spirit of Hadamard requires the following three properties.
holds, from which the desired continuity estimate follows.

Theorem 1.1 (Minty, \[25\]). Let \((P-c) \subseteq X \times X\) be a maximal monotone relation\(^6\) for some \(c \in [0, \infty[^\]\). Then the inverse relation \(P^{-1}\) defines a Lipschitz continuous mapping with domain \(D(P^{-1}) = X\) and \(\frac{1}{c}\) as possible Lipschitz constant.

Proof. We first note that the monotonicity of \(P-c\) implies

\[\bigwedge_{(x_0,y_0),(x_1,y_1) \in P} \Re (x_0 - x_1|y_0 - y_1)_X \geq c \langle x_0 - x_1|x_0 - x_1\rangle_X. \tag{1.1}\]

Hence, if \(y_0 = y_1\) then \(x_0\) must equal \(x_1\), i.e. the uniqueness requirement is satisfied, making \(P^{-1}: P[X] \to X\) a well-defined mapping. Moreover, \(P[X]\) is closed, since from (1.1) we get

\[\bigwedge_{(x_0,y_0),(x_1,y_1) \in P} |y_0 - y_1 |_X \geq c |x_0 - x_1 |_X.\]

The actually difficult part of the proof is to establish that \(P[X] = X\). This is the part we will omit and refer to \[25\] instead. To establish Lipschitz continuity of \(P^{-1}: X \to X\) we observe that

\[\bigwedge_{y_0,y_1 \in X} c |P^{-1}(y_0) - P^{-1}(y_1)|_X^2 \leq \Re \langle P^{-1}(y_0) - P^{-1}(y_1)|y_0 - y_1\rangle_X \]

\[\leq |P^{-1}(y_0) - P^{-1}(y_1)|_X |y_0 - y_1|_X,\]

holds, from which the desired continuity estimate follows. \(\square\)

For many problems, the strict monotonicity is easy to obtain. The maximality, however, needs a deeper understanding of the operators involved. In the linear case, writing now \(A\) for \(P\), there is a convenient set-up to establish maximality by noting that

\[((\{0\}) A^*)^\perp = A[X]\]

\(^6\)For subsets \(M \subseteq X, N \subseteq Y\) the post-set of \(M\) under \(P\) and the pre-set of \(N\) under \(P\) is defined as \(P[M] := \{y \in Y \mid \forall x \in M (x,y) \in P\}\) and \([N]P := \{x \in X \mid \forall y \in N (x,y) \in P\}\), respectively. The post-set \(P[X]\) of the whole space \(X\) under \(P\) is then the domain of the mapping \(P^{-1}\).

\(^7\)Note here that maximal monotone relations are automatically closed, see e.g. \[7\] Proposition 2.5.

\(^8\)In this case \(P\) would be called \textit{maximal strictly monotone}. 
according to the projection theorem. Here we denote by $A^*$ the adjoint of $A$, given as the binary relation
\[
A^* := \left\{ (u, v) \in X \times X \mid \bigwedge_{(x, y) \in A} \langle y | u \rangle_X = \langle x | v \rangle_X \right\}.
\]
Thus, maximality for the strictly monotone linear mapping (i.e. strictly accretive) $A$ is characterized by
\[
[0] \subset A^* = \{0\},
\]
i.e. the uniqueness for the adjoint problem. Characterization (1.2) can be established in many ways, a particularly convenient one being to require that $A^*$ is also strictly monotone. With this we arrive at the following result.

**Theorem 1.2.** Let $A$ and $A^*$ be closed linear strictly monotone relations in a Hilbert space $X$. Then for every $f \in X$ there is a unique $u \in X$ such that
\[
(A, f) \in A.
\]
Indeed, the solution depends continuously on the data in the sense that we have a (Lipschitz-) continuous linear operator $A^{-1} : X \to X$ with
\[
u = A^{-1} f.
\]
Of course, the case that $A$ is a closed, densely defined linear operator is a common case in applications.

**Corollary 1.3.** Let $A$ be a closed, densely defined, linear operator and $A$, $A^*$ strictly accretive in a Hilbert space $X$. Then for every $f \in X$ there is a unique $u \in X$ such that
\[
u = A f.
\]
Indeed, solutions depend continuously on the data in the sense that we have a (Lipschitz) continuous linear operator $A^{-1} : X \to X$ with
\[
u = A^{-1} f.
\]
In the case that $A$ and $A^*$ are linear operators with $D(A) = D(A^*)$ the situation simplifies, since then strict accretivity of $A$ implies strict accretivity of $A^*$ due to
\[
\Re \langle x | Ax \rangle_X = \Re \langle A^* x | x \rangle_X = \Re \langle x | A^* x \rangle_X
\]
for all $x \in D(A) = D(A^*)$.

**Corollary 1.4.** Let $A$ be a closed, densely defined, linear strictly accretive operator in a Hilbert space $X$ with $D(A) = D(A^*)$. Then for every $f \in X$ there is a unique $u \in X$ such that
\[
u = A f.
\]
Indeed, the solution depends continuously on the data in the sense that we have a continuous linear operator $A^{-1} : X \to X$ with
\[
u = A^{-1} f.
\]
The domain assumption of the last corollary is obviously satisfied if $A : X \to X$ is a continuous linear operator. This observation leads to the following simple consequence.

**Corollary 1.5.** Let $A : X \to X$ be a strictly accretive, continuous, linear operator in the Hilbert space $X$. Then for every $f \in X$ there is a unique $u \in X$ such that
\[
u = A f.
\]
Indeed, the solution depends continuously on the data in the sense that we have a continuous linear operator $A^{-1} : X \to X$ with
\[
u = A^{-1} f.
\]
\(^8\)Recall that $A$ has closed range.
Note that since continuous linear operators and continuous sesqui-linear forms are equivalent, the last corollary is nothing but the so-called Lax-Milgram theorem. Indeed, if \( A : X \to X \) is in the space \( L(X) \) of a continuous linear operators then
\[
(u, v) \mapsto \langle u | Av \rangle_X
\]
is in turn a continuous sesqui-linear form on \( X \), i.e. an element of the space \( S(X) \) of continuous sesqui-linear forms on \( X \), and conversely if \( \beta (\cdot | \cdot) \in S(X) \) then \( \beta (\cdot | v) \in X^* \) and utilizing the unitary Riesz map \( R_X : X^* \to X \) we get via the Riesz representation theorem \( \beta (u | v) = \langle u | A_\beta v \rangle_X \), where \( A_\beta v := R_X \beta (\cdot | v) \), \( v \in X \), defines indeed a continuous linear operator on \( X \). Moreover,
\[
S(X) \to L(X)
\]
\[
\beta \mapsto A_\beta
\]
is not only a bijection but also an isometry. Indeed,
\[
|\beta|_{S(X)} := \sup_{x,y \in B_X(0,1)} |\beta(x,y)| = \sup_{x,y \in B_X(0,1)} |\langle x | A_\beta y \rangle_X| = \|A_\beta\|_{L(X)}.
\]
Strict accretivity for the corresponding operator \( A_\beta \) results in the so-called coercivity\(^{10} \) of the sesqui-linear form \( \beta \):
\[
\Re \beta (u | u) \geq c (u | u)_X \quad (1.3)
\]
for some \( c \in [0, \infty) \) and all \( u \in X \). Thus, as an equivalent formulation of the previous corollary we get the following.

**Corollary 1.6 (Lax-Milgram theorem).** Let \( \beta (\cdot | \cdot) \) be a continuous, coercive sesqui-linear form on a Hilbert space \( X \). Then for every \( f \in X^* \) there is a unique \( u \in X \) such that
\[
\beta (u | v) = f (v)
\]
for all \( v \in X \).

Keeping in mind the latter approach has been utilized extensively for elliptic type problems it may be interesting to note that its generalization in the form of Corollary\(^{12} \) is perfectly sufficient to solve elliptic, parabolic and hyperbolic systems in a single approach. For further illustrating the Lax-Milgram theorem in its abstract form, we discuss an example, which is related to the sesqui-linear forms method.

**Example 1.7.** Let \( H_0, H_1 \) be Hilbert spaces. Denote by \( H_{-1} \) the dual of \( H_1 \) and let \( R_{H_1} : H_{-1} \to H_1 \) be the corresponding Riesz-isomorphism. Consider a continuous linear bijection \( C : H_1 \to H_0 \) and a continuous linear operator \( A : H_0 \to H_0 \) with
\[
\Re \langle x | Ax \rangle_{H_0} \geq \alpha_0 \langle x | x \rangle_{H_0} \quad (x \in H_0)
\]
for some \( \alpha_0 \in \mathbb{R}_{>0} \). Denoting
\[
C^\circ : H_0 \to H_{-1}, \; y \mapsto C^\circ y := \langle y | \cdot \rangle_{H_0},
\]
we consider
\[
C^\circ AC : H_1 \to H_{-1}.
\]
Now, from
\[
R_{H_1}C^\circ AC = C^* AC,
\]
we read off that \( C^\circ AC \) is an isomorphism. This may also be seen as an application of the Lax-Milgram theorem, since the equation
\[
C^\circ ACw = f,
\]
\(^{10}\) The strict positivity in \( 1.3 \) can be weakened to requiring merely \( \int_{u \in X} |\beta (u | u)| \geq c (u | u)_X \), which yields in an analogous way a corresponding well-posedness result. This option is used in some applications.
for given \( f \in H_{-1} \) amounts to be equivalent to the discussion of the sesqui-linear form

\[
(v, w) \mapsto \beta(v|w) := \langle ACv|Cw \rangle_{H_0} = (C^0 ACv)(w)
\]
similar to the way it was done in the above.

In order to establish a solution theory for elliptic type equations, it is possible to go a step further. For stating an adapted well-posedness theorem we recall the following. Let \( G : D(G) \subseteq H_1 \to H_2 \) be a densely defined closed linear operator with closed range \( R(G) = G[H_1] \). Then, the operator \( B_G : D(G) \cap N(G)^{\perp} \subseteq N(G)^{\perp} \to R(G), x \mapsto Gx \), where \( N(G) = \{0\} \) \( G \) denotes the null-space of \( G \), is continuously invertible as it is one-to-one, onto and closed. Consequently, the modulus \( |B_G| \) of \( B_G \) is continuously invertible on \( N(G)^{\perp} \). We denote by \( H_1(|B_G|) \) the domain of \( |B_G| \) endowed with the norm \( ||B_G|| \cdot |H_1| \), which can be shown to be equivalent to the graph norm of \( |B_G| \). We denote by \( H_{-1}(|B_G|) \) the dual of \( H_1(|B_G|) \) with the pivot space \( H_0(|B_G|) := N(G)^{-1} \).

It is possible to show that the range of \( G^* \) is closed as well. Thus, the above reasoning also applies to \( G^* \) in the place of \( G \).

Moreover, the operators \( B_G \) and \( B_G^* \), defined as in Example IX, are unitary transformations from \( H_1(|B_G|) \) to \( H_0(|B_G^*|) \) and from \( H_0(|B_G^*|) \) to \( H_{-1}(|B_G|) \), respectively. Moreover, note that \( B_G^* \) is the continuous extension of \( B_G \) (= \( B_G^* \)). The abstract result asserting a solution theory for homogeneous elliptic boundary value problems reads as follows.

**Theorem 1.8** (Theorem 3.1.1). Let \( H_1, H_2 \) be Hilbert spaces and let \( G : D(G) \subseteq H_1 \to H_2 \) be a densely defined closed linear operator, such that \( R(G) \subseteq H_2 \) is closed. Let \( a \subseteq R(G) \oplus R(G) \) such that \( a^{-1} : R(G) \to R(G) \) is Lipschitz-continuous. Then for all \( f \in H_{-1}(|B_G|) \) there exists a unique \( u \in H_1(|B_G|) \) such that the following inclusion holds

\[
(u, f) \in G^0 aG \quad \text{such that} \quad \left\{ (x, z) \in H_1(|G| + i) \times H_{-1}(|G| + i) \mid \bigwedge_{y \in H_0(|G| + i)} (Gx, y) \in a \land z = G^0 y \right\}.
\]

Moreover, the solution \( u \) depends Lipschitz-continuously on the right-hand side with Lipschitz constant \( |a^{-1}|_{\text{Lip}} \) denoting the smallest Lipschitz-constant of \( a^{-1} \).

In other words, the relation \( (B_G^0 a B_G)^{-1} \subseteq H_{-1}(|B_G|) \oplus H_1(|B_G|) \) defines a Lipschitz-continuous mapping with \( |(B_G^0 a B_G)^{-1}|_{\text{Lip}} = |a^{-1}|_{\text{Lip}} \).

**Proof.** It is easy to see that \((u, f) \in G^0 aG \) for \( u \in H_1(|B_G|) \) and \( f \in H_{-1}(|B_G|) \) if and only if \((u, f) \in B_G^0 a B_G \). Hence, the assertion follows from \((B_G^0 a B_G)^{-1} = B_G^{-1} a^{-1} (B_G^0)^{-1} \), the unitarity of \( B_G \) and \( B_G^0 \), and the fact that \( a^{-1} \) is Lipschitz-continuous on \( R(G) \). \( \square \)

To illustrate the latter result, we give an example.

**Definition 1.9.** Let \( \Omega \subseteq \mathbb{R}^n \) open. We define

\[
\widehat{\text{div}}_c : C_{\infty, c}(\Omega)^n \subseteq \bigoplus_{k=1}^n L^2(\Omega) \to L^2(\Omega)
\]

\[
\phi = (\phi_1, \ldots, \phi_n) \mapsto \sum_{k=1}^n \partial_k \phi_k,
\]

where \( \partial_k \) denotes the derivative with respect to the \( k \)’th variable \((k \in \{1, \ldots, n\}) \) and \( C_{\infty, c}(\Omega) \) is the space of arbitrarily differentiable functions with compact support in \( \Omega \). Furthermore, define

\[
\widehat{\text{grad}}_c : C_{\infty, c}(\Omega) \subseteq L^2(\Omega) \to \bigoplus_{k=1}^n L^2(\Omega)
\]

\[
\phi \mapsto (\partial_1 \phi, \ldots, \partial_n \phi),
\]

\footnote{We use \( H_1(A) \) also as a notation for the graph space of some continuously invertible operator \( A \) endowed with the norm \(|A|\). Similarly, we write \( H_{-1}(A) \) for the respective dual space with the pivot space \( D(A) \).}
Integration by parts gives $\tilde{\text{div}}_{c} \subseteq - (\text{grad}_{c})^{*}$ and consequently $\tilde{\text{grad}}_{c} \subseteq - (\text{div}_{c})^{*}$. We set $\text{div} := - (\text{grad}_{c})^{*}$, $\text{grad} := - (\text{div}_{c})^{*}$, $\text{div}_{c} := - \text{div}^{*}$ and $\text{grad}_{c} := - \text{div}^{*}$.

In the particular case $n = 1$ we set $\partial_{1,c} := \text{grad}_{c} = \text{div}_{c}$ and $\partial_{1} := \text{grad} = \text{div}$. With the latter operators, in order to apply the solution theory above, one needs to impose certain geometric conditions on the open set $\Omega$. Indeed, the above theorem applies to grad or $\text{grad}_{c}$ in the place of $G$ for the homogeneous Neumann and Dirichlet case, respectively (in this case $G^{e}$ is then the canonical extension of $- \text{div}_{c}$ and $- \text{div}$, respectively). The only thing that has to be guaranteed is the closedness of the range of grad (grad$_c$, resp.). This in turn can be warranted, e.g., if $\Omega$ is bounded, connected and satisfies the segment property for the Neumann case or if $\Omega$ is bounded in one direction for the Dirichlet case. In both cases, one can prove the Poincaré inequality, which especially implies the closedness of the corresponding ranges (see e.g. [68, Satz 7.6, p. 120] and [1, Theorem 3.8, p. 24] for the Poincaré inequality for the Dirichlet case and Rellich’s theorem for the Neumann case, respectively. Note that if the domain of the gradient endowed with the graph norm is compactly embedded into the underlying space, a Poincaré type estimate can be derived by a contradiction argument.)

In the remainder of this section, we discuss an elliptic-type problem in one dimension with indefinite coefficients. A similar result in two dimensions can be found in [6].

**Example 1.10.** Let $\Omega := [-\frac{1}{2}, \frac{1}{2}]$ and set

$$\tilde{a}(x) := \begin{cases} \alpha & x \geq 0, \\ \beta & x < 0 \end{cases} \quad (x \in \left[-\frac{1}{2}, \frac{1}{2}\right])$$

for some $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. We denote the corresponding multiplication-operator on $L^{2}\left([-\frac{1}{2}, \frac{1}{2}]\right)$ by $\tilde{a}(m)$ and consider the following equation in divergence-form

$$- \partial_{1} \tilde{a}(m) \partial_{1,c} u = f \quad (1.4)$$

for some $f \in H^{-1}(\{\partial_{1,c}\})$. Clearly, $R(\partial_{1,c})$ is closed and we denote the canonical embedding from $R(\partial_{1,c})$ into $L^{2}\left([-\frac{1}{2}, \frac{1}{2}]\right)$ by $\iota_{R(\partial_{1,c})}$. Then $\iota_{R(\partial_{1,c})}^{*} : L^{2}\left([-\frac{1}{2}, \frac{1}{2}]\right) \rightarrow R(\partial_{1,c})$ is the orthogonal projection onto $R(\partial_{1,c})$ (see e.g. [35, Lemma 3.2]) and we can rewrite (1.4) as

$$- \partial_{1} \iota_{R(\partial_{1,c})}^{*} \iota_{R(\partial_{1,c})} \tilde{a}(m) \iota_{R(\partial_{1,c})} \iota_{R(\partial_{1,c})}^{*} \partial_{1,c} u = f,$$

where we have used that $\partial_{1}$ vanishes on $R(\partial_{1,c})$. Hence, we are in the setting of Theorem 1.8 where $a = \iota_{R(\partial_{1,c})}^{*} \iota_{R(\partial_{1,c})} \tilde{a}(m) \iota_{R(\partial_{1,c})} : R(\partial_{1,c}) \rightarrow R(\partial_{1,c})$. The only thing we have to show is that $a^{-1}$ defines a Lipschitz-continuous mapping on $R(\partial_{1,c})$. As $R(\partial_{1,c})$ is closed, it follows that

$$R(\partial_{1,c}) = N(\partial_{1})^{\perp} = \left\{ 1 \right\}^{\perp},$$

where 1 denotes the constant function $1(x) = 1$ for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. To show that $a$ is invertible, we have to solve the problem

$$a \varphi = \psi$$

for given $\psi \in \left\{ 1 \right\}^{\perp}$. The latter can be written as

$$\psi = a \varphi = \tilde{a}(m) \varphi - \left\langle 1 | \tilde{a}(m) \varphi \right\rangle 1.$$

As $\tilde{a}(m)$ is continuously invertible (since $\alpha, \beta \neq 0$) we derive

$$\varphi = \tilde{a}(m)^{-1} \psi + \left\langle 1 | \tilde{a}(m) \varphi \right\rangle \tilde{a}(m)^{-1} 1.$$

Since $\varphi \in \left\{ 1 \right\}^{\perp}$ we obtain

$$0 = \left\langle 1 | \tilde{a}(m)^{-1} \psi \right\rangle + \left\langle 1 | \tilde{a}(m) \varphi \right\rangle \left\langle 1 | \tilde{a}(m)^{-1} 1 \right\rangle,$$

yielding

$$\left\langle 1 | \tilde{a}(m) \varphi \right\rangle = - \frac{\left\langle 1 | \tilde{a}(m)^{-1} \psi \right\rangle}{\left\langle 1 | \tilde{a}(m)^{-1} 1 \right\rangle}.$$
providing that $1|\bar{a}(m)^{-1}1| \neq 0$. The latter holds if and only if $\alpha \neq -\beta$. Thus, assuming that $\alpha \neq -\beta$ we get

$$a^{-1}\psi = \bar{a}(m)^{-1}\psi - (1|\bar{a}(m)^{-1}\psi),$$

which clearly defines a Lipschitz-continuous mapping. Summarizing, if $\alpha, \beta \neq 0$ and $\alpha \neq -\beta$, then for each $f \in H^{-1}(\Omega_{1,\varepsilon})$ there exists a unique $u \in H_1(\Omega_{1,\varepsilon})$ satisfying (1.4).

Remark 1.11. (a) If (1.4) is replaced by the problem with homogeneous Neumann-boundary conditions, then the constraint $\alpha \neq -\beta$ can be dropped, since in this case $R(\partial_1) = N(\partial_1,\varepsilon)^\perp = L_2([\frac{1}{2}, \frac{1}{2}])$, and thus, $a$ is invertible if $\bar{a}(m)$ is invertible.

(b) Of course in view of Theorem 1.8, the coefficient $a$ in Example 1.10 may also be induced by a relation such that its inverse relation is a (nonlinear) Lipschitz continuous mapping in $R(\partial_1,\varepsilon)$.

2. Linear Evolutionary Equations and Strict Positivity

In this section we shall discuss equations of the form

$$(\partial_{0,\nu}\mathcal{M} + \mathcal{A})U = F,$$  \hspace{1cm} (2.1)

where $\partial_{0,\nu}$ is the time-derivative operator to be introduced and specified below, $\mathcal{M}$ and $\mathcal{A}$ are linear operators, the former – the material law operator – being bounded, and the latter being possibly unbounded. The task is in finding the unknown $U$ for a given right hand side $F$. This is done by showing that both (the closure of) $(\partial_{0,\nu}\mathcal{M} + \mathcal{A})$ and $(\partial_{0,\nu}\mathcal{M} + \mathcal{A})^*$ are strictly accretive operators in a suitable Hilbert space and then using Corollary 1.3. We will comment on the specific assumptions on $\mathcal{M}$ and $\mathcal{A}$ in the subsequent sections as well as on the rigorous (Hilbert space) framework the equation (2.1) should be considered in. Before we discuss the abstract theory, we give four elementary guiding examples which shall lead us through the development of the abstract theory.

2.1. Guiding examples

Ordinary (integro-)differential equations. We shall consider the following easy form of an ordinary differential equation. For a given right hand side $f \in C_c(\mathbb{R} \times \mathbb{R})$, i.e. $f$ is a continuous function on $\mathbb{R} \times \mathbb{R}$ with compact support, and a coefficient $a \in L^\infty(\mathbb{R})$ we consider the problem of finding $u$ in a suitable Hilbert space such that for (a.e.) $(t, x) \in \mathbb{R} \times \mathbb{R}$ the equation

$$u(t, x) = f(t, x)$$

holds. We will also have the opportunity to consider an integro-differential equation of the form

$$u(t, x) + a(x)u(t, x) + \int_{-\infty}^t k(t - s)u(s, x)ds = f(t, x).$$

for a suitable kernel $k : \mathbb{R} \to \mathbb{R}$.

The heat equation. The heat $\partial$ in a given body $\Omega \subseteq \mathbb{R}^n$ can be described by the conservation law

$$\partial(t, x) + \text{div} q(t, x) = f(t, x) \quad ((t, x) \in \Omega \times \text{a.e.}),$$

where $f$ is a given heat source and $q$ is the heat flux given by Fourier’s law as follows

$$q(t, x) = -k(x) \text{grad} \partial(t, x) \quad ((t, x) \in \mathbb{R} \times \Omega \text{ a.e.}),$$

where $k$ is a certain coefficient matrix describing the specific conductivities of the underlying material varying over $\Omega$. Here div and grad are the canonical extensions of the spatial operators div and grad defined in the previous section (Definition 1.9) to the space $L^2(\mathbb{R} \times \Omega, \mu \otimes \lambda) = \mathbb{R}$.
Well-posedness via Monotonicity – An Overview.

$L^2(\mathbb{R}, \mu) \otimes L^2(\Omega, \lambda)$, where $\mu$ is a Borel-measure on $\mathbb{R}$ and $\lambda$ denotes the $n$-dimensional Lebesgue-measure, i.e.

$$\text{div} \, q(t, x) = (\text{div} \, q(t, \cdot))(x)$$

$$\text{grad} \, \vartheta(t, x) = (\text{grad} \, \vartheta(t, \cdot))(x) \quad ((t, x) \in \mathbb{R} \times \Omega \text{ a.e.}).$$

In a block operator matrix form, recalling that $\partial_0$ denotes the derivative with respect to time, we get

$$\begin{pmatrix} \partial_0 & 0 & 0 \\ 0 & \text{div} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & k^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \text{div} & \text{grad} \end{pmatrix} \begin{pmatrix} \vartheta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

assuming that the coefficient $k$ is invertible. Imposing suitable assumptions on data and coefficients and boundary conditions for the operators div and/or grad will be seen to warrant well-posedness of the resulting system. We will comment on the precise details in our discussion of abstract well-posedness results.

**The elastic equations.** In the theory of elasticity, the open set $\Omega \subseteq \mathbb{R}^n$, being the underlying domain, models a body in its non-deformed state (of course, in applications $n = 3$). The displacement field $u$ assigns to each space-time coordinate $(t, x) \in \mathbb{R} \times \Omega$ direction and size of the displacement at time $t$ of the material point at position $x$. The displacement field $u$ satisfies the balance of momentum equation (again writing $\partial_0$ for the time-derivative)

$$\partial_0^2 u - \text{Div} \, \sigma = f,$$

with $f$ being an external forcing term, $\sigma$ being the (symmetric) stress tensor and Div being the row-wise (distributional) divergence acting on suitable elements in the space $H_{\text{sym}}(\Omega)$ of symmetric $n \times n$ matrices of $L^2(\Omega)$-functions as an operator from $H_{\text{sym}}(\Omega)$ to $L^2(\Omega)^n$ with maximal domain.$^{12}$

Endowing $H_{\text{sym}}(\Omega)$ with the Frobenius inner product, we get that the negative adjoint of Div is the symmetrized gradient or strain tensor given by

$$\varepsilon(u) := \text{Grad} \, u := \frac{1}{2} \left( \partial_i u_j + \partial_j u_i \right)_{i,j}$$

with Dirichlet boundary conditions as induced constraint on the domain. Neumann boundary conditions can be modeled similarly. The stress tensor satisfies the constitutive relation involving the elasticity tensor $C$ in the way that

$$\sigma = C \varepsilon(u).$$

Introducing the displacement velocity $v := \partial_0 u$ as a new unknown, we write the elastic equations formally as the following block operator matrix equation

$$\begin{pmatrix} \partial_0 & 0 & 0 \\ 0 & \text{div} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \text{Grad} \end{pmatrix} \begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where we assume that $C$ is invertible.

**The Maxwell’s equations.** The equations for electro-magnetic theory describe evolution of the electro-magnetic field $(E, H)$ in a 3-dimensional open set $\Omega$. As Gauss’ law can be incorporated by a suitable choice of initial data, we think of Maxwell’s equations as Faraday’s law of induction (the Maxwell-Faraday equation), which reads as

$$\partial_0 B + \text{curl}_c E = 0,$$

where $\text{curl}_c$ denotes the (distributional) curl operator in $L^2(\Omega)^3$ with the electric boundary condition of vanishing tangential components. The magnetic field $B$ satisfies the constitutive equation

$$B = \mu H,$$

where $\mu$ is the magnetic permeability. Faraday’s law is complemented by Ampere’s law

$$\partial_0 D + J_c - \text{curl} H = J_0$$

$^{12}$The precise definition will be given later.
for \( J_0, D, J_c \) being the external currents, the electric displacement and the charge, respectively. The latter two quantities satisfy the two equations
\[
D = \varepsilon E, \quad \text{and} \quad J_c = \sigma E.
\]
The former is a constitutive equation involving the dielectricity \( \varepsilon \) and the latter is Ohm’s law with conductivity \( \sigma \). Plugging the constitutive relations and Ohm’s law into Faraday’s law of induction and Ampere’s law and arranging them in a block operator matrix equation, we arrive at
\[
\begin{pmatrix}
\partial_0 
\begin{pmatrix}
\varepsilon & 0 \\
0 & \mu
\end{pmatrix} 
+ 
\begin{pmatrix}
\sigma & 0 \\
0 & 0
\end{pmatrix} 
+ 
\begin{pmatrix}
0 & \text{curl}_c \\
\text{curl}_c & 0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
E \\
H
\end{pmatrix} = 
\begin{pmatrix}
J_0 \\
0
\end{pmatrix}.
\]

Having these examples in mind, we develop the abstract theory a bit further and discuss the time-derivative operator in the next section. After having done so, we aim at giving a unified solution theory for all of the latter examples. In fact we show that all of these equations are of the general form \((2.1)\).

2.2. The time-derivative

When considering evolutionary equations, we need a distinguished direction of time. Anticipating this fact, we define a time-derivative \( \partial_{0,t} \) as an operator in a weighted Hilbert space.

Beforehand, we recall some well-known facts from the (time-)derivative in the unweighted space \( L^2(\mathbb{R}) \). We denote the Sobolev space of \( L^2(\mathbb{R}) \)-functions \( f \) with distributional derivative \( f' \) representable as a \( L^2(\mathbb{R}) \)-function by \( \mathcal{H}_1(\mathbb{R}) \). Then, the operator
\[
\partial : \mathcal{H}_1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R}), f \mapsto f'
\]
is skew-selfadjoint. Indeed, using that the space \( C_{\infty,c}(\mathbb{R}) \) is a core for \( \partial \), we immediately verify with integration by parts that \( \partial \) is skew-symmetric. With the help of some elementary computations it is possible to show that the range of both the operators \( \partial + 1 \) and \( \partial - 1 \) contains \( C_{\infty,c}(\mathbb{R}) \). The closedness of \( \partial \) thus implies the skew-selfadjointness of \( \partial \) (see also \[20\], Example 3.14). Moreover, it is well-known that \( \partial \) admits an explicit spectral representation given by the Fourier transform \( \mathcal{F} \), being the unitary extension on \( L^2(\mathbb{R}) \) of the mapping given by
\[
\mathcal{F}\phi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \phi(x) dx \quad (\xi \in \mathbb{R})
\]
for \( \phi \in C_{\infty,c}(\mathbb{R}) \). Denoting by \( m : D(m) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) the multiplication-by-argument operator given by
\[
m f := (\xi \mapsto \xi \phi(\xi))
\]
for \( f \) belonging to the maximal domain \( D(m) \) of \( m \), we find the following unitary equivalence of differentiation and multiplication (see \[2\] Volume 1, p. 161-163):
\[
\partial = \mathcal{F}^* \text{im} \mathcal{F},
\]
which, due to the selfadjointness of \( m \), confirms the skew-selfadjointness of \( \partial \).

As mentioned above, in evolutionary processes there is a particular bias for the forward time direction. As \( L^2(\mathbb{R}) \) has no such bias, we choose a suitable weight, which serves to express this bias. For \( \nu \in \mathbb{R} \) we let \( L^2_{\nu}(\mathbb{R}) := \{ f \in L^2_{\nu,\text{loc}}(\mathbb{R}) \mid \int_{\mathbb{R}} |f(t)|^2 \exp(-2\nu t) \, dt < \infty \} \), the Hilbert space of (equivalence classes of) functions with \( t \mapsto \exp(-\nu t) f(t) \in L^2(\mathbb{R}) \) (with the obvious norm). In particular, \( L^2_0(\mathbb{R}) = L^2(\mathbb{R}) \). Moreover, it is easily seen that the mapping
\[
e^{-\nu m} : L^2_{\nu}(\mathbb{R}) \rightarrow L^2(\mathbb{R}), f \mapsto (t \mapsto e^{-\nu t} f(t))
\]
defines a unitary mapping. To carry differentiation over to the exponentially weighted \( L^2 \)-spaces, we observe that for \( \phi \in C_{\infty,c}(\mathbb{R}) \) we have
\[
(e^{-\nu m})^{-1} \partial e^{-\nu m} \phi = (e^{-\nu m})^{-1}(-\nu e^{-\nu m} \phi + e^{-\nu m} \phi')
\]
\[
= -\nu \phi + \phi,'
\]
For an evolution to take place, a physically reasonable property is

\[ (e^{-\nu m})^{-1} (\partial + \nu) e^{-\nu m} \phi = \phi'. \]

Defining \( \partial_{0,\nu} := (e^{-\nu m})^{-1} (\partial + \nu) e^{-\nu m} = (e^{-\nu m})^{-1} \partial e^{-\nu m} + \nu, \) we read off that \( \partial_{0,\nu} \) is a realization of the (distributional) derivative operator in the weighted space \( L^2_{\nu}(\mathbb{R}) \). Moreover, we see that \( \partial_{0,\nu} \) is a normal operator, i.e. \( \partial_{0,\nu} \) and \( \partial_{0,\nu}^* \) commute. In particular, \( \Re \partial_{0,\nu} = \frac{1}{2} (\partial_{0,\nu} + \partial_{0,\nu}^*) = \nu, \) due to the skew-selfadjointness of \( \partial \). This also shows that \( \partial_{0,\nu} \) is continuously invertible if \( \nu \neq 0 \).

Indeed, we find the following explicit formula for the inverse: For \( t \in \mathbb{R}, \nu \in \mathbb{R} \setminus \{0\}, f \in L^2_{\nu}(\mathbb{R}) \) we have

\[
\partial_{0,\nu}^{-1} f(t) = \begin{cases} 
\int_{-\infty}^{t} f(\tau) d\tau, & \nu > 0, \\
-\int_{t}^{\infty} f(\tau) d\tau, & \nu < 0.
\end{cases}
\]

For positive \( \nu \), the latter formula also shows that the values of \( \partial_{0,\nu}^{-1} f \) at time \( t \) only depend on the values of \( f \) up to time \( t \). This is the nucleus of the notion of causality, where the sign of \( \nu \) switches the forward and backward time direction. Later, we will comment on this in more detail.

The spectral representation of \( \partial \) induces a spectral representation for \( \partial_{0,\nu} \). Indeed, introducing the Fourier-Laplace transformation\(^{13}\) \( L^\nu := \mathcal{F} e^{-\nu m} : L^2_{\nu}(\mathbb{R}) \to L^2_{\nu}(\mathbb{R}) \) for \( \nu \in \mathbb{R} \), which itself is a unitary operator as a composition of unitary operators, we get that

\[
\partial_{0,\nu} = L^\nu_{\nu}(\text{im} + \nu) L^\nu.
\]

This shows that for \( \nu \neq 0 \) the normal operator \( \partial_{0,\nu}^{-1} \) is unitarily equivalent to the multiplication operator\(^{14}\) \((\text{im} + \nu)^{-1}\) with spectrum \( \sigma(\partial_{0,\nu}^{-1}) = \partial B(\frac{1}{2\nu}, \frac{1}{2\nu}),\) where \( \nu > 0 \) and \( B(a,r) := \{z \in \mathbb{C}||z-a|<r\}, \) \( a \in \mathbb{C}, \) \( r > 0 \). We will use this fact in the next section by establishing a functional calculus for \( \partial_{0,\nu}^{-1} \). Henceforth, if not otherwise stated, the parameter \( \nu \) will always be a positive real number.

### 2.3. The autonomous case

In order to formulate the Hilbert space framework of (2.1) properly, we need to consider the space of \( H \)-valued \( L^2_{\nu} \)-functions. Consequently, we need to invoke the (canonical) extension of \( \partial_{0,\nu} \) to \( L^2_{\nu}(\mathbb{R},H) \) for some Hilbert space \( H \). For convenience, we re-use the notation for the respective extension since there is no risk of confusion. Moreover, we shall do so for the Fourier-Laplace transform \( L^\nu \) which is then understood as a unitary operator from \( L^2_{\nu}(\mathbb{R},H) \) onto \( L^2(\mathbb{R},H) \).

In this section we treat a particular example for the choice of the operators \( M \) and \( A \), namely that of autonomous operators. For this we need to introduce the operator of time translation: For \( h \in \mathbb{R}, \nu \in \mathbb{R} \) we define the time translation operator \( \tau_h \) on \( L^2_{\nu}(\mathbb{R},H) \) by \( \tau_h f := f(\cdot + h) \). Again, we identify \( \tau_h \) with its extension to the \( H \)-valued case.

**Definition 2.1.** We call an operator \( B \colon D(B) \subseteq L^2_{\nu}(\mathbb{R},H) \to L^2_{\nu}(\mathbb{R},H) \) autonomous or time translation invariant, if it commutes with \( \tau_h \) for all \( h \in \mathbb{R} \), i.e.,

\[
\tau_h B \subseteq B \tau_h \quad (h \in \mathbb{R}).
\]

For an evolution to take place, a physically reasonable property is causality. Denoting by \( \chi_{\frac{\cdot}{\lambda}}\) \((m_0)\) the multiplication operator on \( L^2_{\nu}(\mathbb{R},H) \) given by \( \chi_{\frac{\cdot}{\lambda}}\) \((m_0) u \) \( (t) := \chi_{\frac{\cdot}{\lambda}}\) \((t) u(t) \) for \( u \in L^2_{\nu}(\mathbb{R},H) \) and \( t \in \mathbb{R} \), the definition of causality reads as follows.

\( ^{13}\)For the classical Fourier-Laplace transformation, also known as two-sided Laplace transformation, this unitary character is rarely invoked. In fact, it is mostly considered as an integral expression acting on suitably integrable functions, whereas the unitary Fourier-Laplace transformation is continuously extended (thus, of course, including also some non-integrable functions).

\( ^{14}\)In the sense of the induced functional calculus we have

\[
(\text{im} + \nu)^{-1} = \frac{1}{\text{im} + \nu}.
\]
Definition 2.2. We call a closed mapping \( M : D(M) \subseteq L^2_0(\mathbb{R}, H) \to L^2_0(\mathbb{R}, H) \) causal, if for all \( f, g \in D(M) \) and \( a \in \mathbb{R} \) we have
\[
\chi_{\Re z \leq a}(m_0)f = \chi_{\Re z \leq a}(m_0)M(f) = \chi_{\Re z \leq a}(m_0)M(g),
\]
(2.2)

Remark 2.3.
(a) Property (2.2) reflects the idea that the “future behavior does not influence the past”, which may be taken as the meaning of causality.
(b) If, in addition, \( M \) in the latter definition is linear, then \( M \) is causal if and only if for all \( u \in D(M) \)
\[
\chi_{\Re z \leq a}(m_0)u = 0 \Rightarrow \chi_{\Re z \leq a}(m_0)Mu = 0 \quad (a \in \mathbb{R}).
\]
(c) For continuous mappings \( M \) with full domain \( L^2_0(\mathbb{R}, H) \), causality can also equivalently be expressed by the equation
\[
\chi_{\Re z \leq a}(m_0)M = \chi_{\Re z \leq a}(m_0)M\chi_{\Re z \leq a}(m_0)
\]
holding for all \( a \in \mathbb{R} \). If, in addition, \( M \) is autonomous then this condition is in turn equivalent to
\[
\chi_{\Re z \leq a}(m_0)M = \chi_{\Re z \leq a}(m_0)M\chi_{\Re z \leq a}(m_0)
\]
for some \( a \in \mathbb{R} \), e.g. \( a = 0 \).
(d) The respective concept for causality for closable mappings is a bit more involved, see [57].

In order to motivate the problem class discussed in this section a bit further we state the following well-known representation theorem:

Theorem 2.4 (see e.g. [45] Theorem 2.3, [44] Theorem 9.1 or [45]). Let \( H \) be a Hilbert space, \( M : L^2_0(\mathbb{R}; H) \to L^2_0(\mathbb{R}; H) \) bounded, linear, causal and autonomous. Then there exists a unique \( \tilde{M} : \{ z \in \mathbb{C} \mid \Re z > \nu \} \to L(H) \) bounded and analytic, such that for all \( u \in L^2_0(\mathbb{R}; H) \) we have
\[
Mu = L^2_0(\tilde{M}(im + \nu)\mathcal{L}_\nu u)
\]
where \( \tilde{M}(im + \nu)\phi(\xi) := \tilde{M}(i(\xi + \nu))\phi(\xi) \) for \( \phi \in L^2(\mathbb{R}, H), \xi \in \mathbb{R} \).

This theorem tells us that the class of bounded, linear, causal, autonomous operators is described by bounded and analytic functions of \( \partial_{0,\nu} \) or equivalently of \( \partial_{0,\nu}^{-1} \). Thus, we are led to introduce the Hardy space for some open \( E \subseteq \mathbb{C} \) and Hilbert space \( H \):
\[
\mathcal{H}^\infty(E, L(H)) := \{ M : E \to L(H) \mid M \text{ bounded, analytic} \}.
\]
Clearly, \( \mathcal{H}^\infty(E, L(H)) \) (or briefly \( \mathcal{H}^\infty \) if \( E \) and \( H \) are clear from the context) is a Banach space with norm
\[
\mathcal{H}^\infty \ni M \mapsto \| M \|_\infty := \sup \{ \| M(z) \| \mid z \in E \}.
\]
In the particular case of \( M \in \mathcal{H}^\infty(E, L(H)) \) with \( E = B(0, r) \) for some \( r > 0 \), we define for \( \nu > \frac{1}{2r} \)
\[
M(\partial_{0,\nu}^{-1}) : L^2_\nu(\mathbb{R}; H) \to L^2_\nu(\mathbb{R}; H),
\]
\[
\phi \mapsto \mathcal{L}_\nu^* M \left( \frac{1}{im + \nu} \right) \mathcal{L}_\nu \phi.
\]
Here \( \frac{1}{\tilde{M}(im + \nu)} \in L(L^2(\mathbb{R}, H)) \) is given by
\[
\left( \frac{1}{\tilde{M}(im + \nu)} \right) w(t) := \frac{1}{\tilde{M}(i(t + \nu))} w(t)
\]
for \( w \in L^2(\mathbb{R}, H) \) and \( t \in \mathbb{R} \). Note that \( \sup_{z \in B(0, r)} \| M(z) \| = \| M(\partial_{0,\nu}^{-1}) \|_{L(L(H, \omega, (\mathbb{R}; H))} \) according to [44] Theorem 9.1.

Our first theorem asserting a solution theory for evolutionary equations reads as follows.
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\textbf{Theorem 2.5 ([31] Solution Theory, [33] Chapter 6).} Let \(\nu > 0, r > \frac{1}{2\nu}\) and \(M \in \mathcal{H}^\infty(B(r, r), L(H))\), \(A : D(A) \subseteq H \to H\). Assume that

\[ A \text{ is skew-selfadjoint, and} \]

\[ \bigvee_{c > 0} \bigwedge_{z \in B(r, r)} z^{-1}M(z) - c \text{ is monotone.} \]  

(2.3)

Then \(\partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A\) is continuously invertible in \(L^2_\nu(\mathbb{R}, H)\). The closure of the inverse is causal. Moreover, the solution operator is independent of the choice of \(\nu\) in the sense that for \(\rho > \nu\) and \(f \in L^2_\rho(\mathbb{R}, H) \cap L^2_\rho(H, \mathbb{R})\) we have that

\[ (\partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A)^{-1} f = (\partial_{0,\rho} M(\partial_{0,\rho}^{-1}) + A)^{-1} f. \]

\textit{Sketch of the proof.} First, one proves that the operator \(\partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A\) is closable and its closure is strictly monotone. The same holds for its adjoint, which turns out to have the same domain. Thus, the well-posedness of (2.1), where \(A\) is skew-selfadjoint and \(M = M(\partial_{0,\nu}^{-1})\) follows by Corollary [1.3]. The causality of the solution operator can be shown by a Paley-Wiener type result (see e.g. [39] Theorem 19.2)).

According to the latter theorem, the solution operator associated with an evolutionary problem is independent of the particular choice of \(\nu\). Therefore, we usually will drop the index \(\nu\) and write instead \(\partial_0\) and \(M(\partial_{0}^{-1})\).

Since the positive definiteness condition in (2.3) will occur several times, we define \(\mathcal{H}^{\infty,c}\) to be the set of \(M \in \mathcal{H}^{\infty}\) satisfying condition (2.3) with the constant \(c \in \mathbb{R}_{>0}\).

Note that with \(A = 0\) in Theorem 2.3, ordinary differential equations are covered. We shall further elaborate this fact in Subsection 2.4. Here, let us illustrate the versatility of this well-posedness result, by applying the result to several (partial) differential equations arising in mathematical physics.

\textbf{Example 2.6 (The heat equation).} Recall the definition of the operators \(\grad, \grad, \div\) and \(\div\) from Definition 1.9. The domain of \(\grad\) coincides with the classical Sobolev-space \(H^1_0(\Omega)\), the space of \(L^2(\Omega)\)-functions with distributional gradients lying in \(L^2(\Omega)^n\) and having vanishing trace, while the domain of grad is \(H^1(\Omega)\), the Sobolev-space of weakly differentiable functions in \(L^2(\Omega)\). Analogously, \(v \in D(\div v)\) is a \(L^2(\Omega)\)-vector field, whose distributional divergence is in \(L^2(\Omega)\) and satisfies a generalized Neumann-condition \(v \cdot N = 0\) on \(\partial\Omega\), where \(N\) denotes the outward unit normal vector field on \(\partial\Omega\).\(^{15}\) Recall the conservation of energy equation, given by

\[ \partial_0 \vartheta + \div q = f, \]

(2.4)

where \(\vartheta \in L^2_\rho(\mathbb{R}, L^2(\Omega))\) denotes the (unknown) heat, \(q \in L^2_\rho(\mathbb{R}, L^2(\Omega)^3)\) stands for the heat flux and \(f \in L^2_\rho(\mathbb{R}, L^2(\Omega))\) models an external heat source. This equation is completed by a constitutive relation, called Fourier’s law

\[ q = -k \grad \vartheta, \]

(2.5)

where \(k \in L^\infty(\Omega)^{n \times n}\) denotes the heat conductivity and is assumed to be selfadjoint-matrix-valued and strictly positive, i.e. there is some \(c > 0\) such that \(k(x) \geq c\) for almost every \(x \in \Omega\). Plugging Fourier’s law into the conservation of energy equation, we end up with the familiar form of the heat equation

\[ \partial_0 \vartheta - \div k \grad \vartheta = f. \]

However, it is also possible to write the equations (2.4) and (2.5) as the system

\[ \begin{pmatrix} \partial_0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vartheta \\ 0 \\ q \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \div \end{pmatrix} \begin{pmatrix} \vartheta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \]

\(^{15}\)Note that the definition of \(\div\) still makes sense, even if the boundary of \(\Omega\) is non-smooth and hence, the normal vector field \(N\) does not exist. Thus, for rough domains \(\Omega\) the condition \(v \cdot N = 0\) on \(\partial\Omega\)
Requiring suitable boundary conditions, say Dirichlet boundary conditions for the temperature density \( \vartheta \) (i.e. \( \vartheta \in D(\text{grad}_c) \)) the system becomes an evolutionary equation of the form discussed in Theorem 2.5 with

\[
M(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & k^{-1} \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
and the skew-selfadjoint operator

\[
A = \begin{pmatrix} 0 & \text{div} \\ \text{grad}_c & 0 \end{pmatrix}.
\]

By our assumptions on the coefficient \( k \), the solvability condition (2.3) can easily be verified for our material law \( M \). Indeed, with \( k \) being bounded and strictly positive, the inverse operator \( k^{-1} \) is bounded and strictly positive as well. Now, since for \( z \in B(r, r) \) for some \( r > 0 \) the real part of \( z^{-1} \) is bounded below by \( \frac{1}{2r} \), we deduce that

\[
\Re (z^{-1} M(z)) = \Re \left( z^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix} \right) \geq \begin{pmatrix} \frac{1}{2r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}.
\]

**Example 2.7 (Maxwell’s equations).** To begin with, we formulate the functional analytic setting for the operator curl with and without the electric boundary condition. Let \( \Omega \subseteq \mathbb{R}^3 \) be a non-empty open set and define

\[
\widetilde{\text{curl}}_c : C^\infty_c(\Omega)^3 \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3
\]

\[
\phi \mapsto \begin{pmatrix} \partial_2 \phi_3 - \partial_3 \phi_2 \\ \partial_3 \phi_1 - \partial_1 \phi_3 \\ \partial_1 \phi_2 - \partial_2 \phi_1 \end{pmatrix}.
\]

Analogously to the previous example, we let \( \text{curl} := (\widetilde{\text{curl}}_c)^\ast \) and \( \text{curl}_c := \text{curl}^\ast \). Now, let \( \varepsilon, \mu, \sigma \) be bounded linear operators in \( L^2(\Omega)^3 \). We assume that both \( \varepsilon \) and \( \mu \) are selfadjoint and strictly positive. As a consequence, the operator function

\[
z \mapsto \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

belongs to \( H^{\infty,c}(B(r, r), L(\Omega)_0^6) \) for some \( c \in \mathbb{R}_{>0} \), if \( r \) is chosen small enough. Thus, due to the skew-selfadjointness of the operator \( \begin{pmatrix} 0 & -\text{curl} \\ \text{curl}_c & 0 \end{pmatrix} \) in \( L^2(\Omega)_0^6 \), Theorem 2.5 applies to the operator sum

\[
\partial_0 \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\text{curl} \\ \text{curl}_c & 0 \end{pmatrix},
\]

which yields continuous invertibility of the closure of the latter operator in \( L^2(\Omega)^3 \) for sufficiently large \( \nu \). It is noteworthy that the well-posedness theorem of course also applies to the case, where \( \varepsilon = 0 \) and the real part of \( \sigma \) is strictly positive definite, i.e., to the operator of the form

\[
\partial_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\text{curl} \\ \text{curl}_c & 0 \end{pmatrix}.
\]

In the literature this arises when dealing with the so-called “eddy current problem”.

**Example 2.8 (The equations of elasticity and visco-elasticity).** We begin by introducing the differential operators involved. Let \( \Omega \subseteq \mathbb{R}^3 \) open. We consider the Hilbert space \( H_{sym}(\Omega) \) given as the space of symmetric \( L^2(\Omega)^3 \times 3 \)-matrices equipped with the inner product

\[
\langle \Phi | \Psi \rangle_{H_{sym}(\Omega)} := \int_\Omega \text{trace}(\Phi(x)^\ast \Psi(x)) \, dx,
\]

as the appropriate setting.
where $\Phi(x)^*$ denotes the adjoint matrix of $\Phi(x)$. Using this Hilbert space we define the operator $\text{Grad}_c$ as the closure of

$$\text{Grad}_c : C_{\infty,c}(\Omega)^3 \subseteq L^2(\Omega)^3 \rightarrow H_{\text{sym}}(\Omega)$$

$$(\phi_i)_{i \in \{1,2,3\}} \mapsto \left( \frac{\partial_i \phi_j + \partial_j \phi_i}{2} \right)_{i,j \in \{1,2,3\}}$$

and likewise we define $\text{Div}_c$ as the closure of

$$\text{Div}_c : C_{\infty,c}(\Omega)^3 \times 3 \cap H_{\text{sym}}(\Omega) \subseteq H_{\text{sym}}(\Omega) \rightarrow L^2(\Omega)^3$$

$$(\psi_{ij})_{i,j \in \{1,2,3\}} \mapsto \left( \sum_{j=1}^{3} \partial_j \psi_{ij} \right)_{i \in \{1,2,3\}}$$

Integration by parts yields that $\text{Grad}_c \subseteq - (\text{Div}_c)^*$ as well as $\text{Div}_c \subseteq - (\text{Grad}_c)^*$ and we define

$$\text{Grad} := - (\text{Div}_c)^*,$$

$$\text{Div} := - (\text{Grad}_c)^*.$$ 

Similar to the case of grad and div, elements $u$ in the domain of $\text{Grad}_c$ satisfy an abstract Dirichlet boundary condition of the form $u = 0$ on $\partial \Omega$, while elements $\sigma$ in the domain of $\text{Div}_c$ satisfy an abstract Neumann boundary condition of the form $\sigma N = 0$ on $\partial \Omega$, where $N$ denotes the unit outward normal vector field on $\partial \Omega$.

The equation of linear elasticity is given by

$$\partial_0^2 u - \text{Div} \sigma = f,$$  \hspace{1cm} (2.6)

where $u \in L^2_c(\mathbb{R}, L^2(\Omega)^3)$ denotes the displacement field of the elastic body $\Omega$, $\sigma \in L^2_c(\mathbb{R}, H(\Omega))$ is the stress tensor and $f \in L^2_c(\mathbb{R}, L^2(\Omega)^3)$ is an external force. Equation (2.6) is completed by Hooke’s law

$$\sigma = C \text{Grad} u,$$  \hspace{1cm} (2.7)

where $C \in L(H_{\text{sym}}(\Omega))$ is the so-called elasticity tensor, which is assumed to be selfadjoint and strictly positive definite (which in particular gives the bounded invertibility of $C$). Defining $v := \partial_0 u$ as the displacement velocity, (2.6) and (2.7) can be written as a system of the form

$$\begin{pmatrix} \partial_0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} C^{-1} \\ \text{Grad} & - \text{Div} \end{pmatrix} \begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$ 

Imposing boundary conditions on $v$ or $\sigma$, say for simplicity Neumann boundary conditions for $\sigma$, we end up with an evolutionary equation with

$$M(z) = \begin{pmatrix} 1 & 0 \\ 0 & C^{-1} \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & - \text{Div}_c \\ - \text{Grad} & 0 \end{pmatrix}.$$ 

Since $C^{-1}$ is also selfadjoint and strictly positive definite, we obtain that $M$ satisfies the solvability condition (2.3). 

In order to incorporate viscous materials, i.e. materials showing some memory effects, one modifies Hooke’s law (2.7) for example by

$$\sigma = C \text{Grad} u + D \text{Grad} \partial_0 u,$$  \hspace{1cm} (2.8)

where $D \in L(H_{\text{sym}}(\Omega))$. This modification is known as the Kelvin-Voigt model for visco-elastic materials (see e.g. [13, p. 163], [5, Section 1.3.3]). Using $v$ instead of $u$, the latter equation reads as

$$\sigma = (\partial_0^{-1} C + D) \text{Grad} v = D (\partial_0^{-1} D^{-1} C + 1) \text{Grad} v,$$

where we assume that $D$ is selfadjoint and strictly positive definite, while $C \in L(H_{\text{sym}}(\Omega))$ is arbitrary (the assumption on $D$ can be relaxed, by requiring suitable positivity constraints on $C$, etc.)
see e.g. [35] Theorem 4.1). Since \( \| \partial_{0,v}^{-1} \| = \nu^{-1} \) we may choose \( \nu_0 > 0 \) large enough in order to get that \( \| \partial_{0,v}^{-1}D^{-1}C \| < 1 \) for all \( \nu > \nu_0 \). Then, by the Neumann series, we end up with

\[
(1 + \partial_{0}^{-1}D^{-1}C)^{-1} D^{-1} \sigma = \text{Grad} v.
\]

Hence, our new material law operator becomes

\[
M(z) = \begin{pmatrix}
1 & 0 \\
0 & z(1 + zD^{-1}C)^{-1}D^{-1}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + z \left( \begin{pmatrix}
0 & 0 \\
0 & D^{-1}
\end{pmatrix} + \sum_{k=1}^{\infty} (-1)^{k} z^{k+1} \left( \begin{pmatrix}
0 & 0 \\
0 & (D^{-1}C)^{k}D^{-1}
\end{pmatrix} \right) \right),
\]

which satisfies the condition \( (2.3) \) for \( z \geq \nu > 0 \). Moreover, kernels of Hooke’s law \( (2.7) \), see e.g. [14, Section III.7], [12]. The new constitutive relation then reads as

Another way to model materials with memory is to add a convolution term on the right hand side of Hooke’s law \( \text{see e.g.} \ [14] \), see e.g. [12] Section III.7. The new constitutive relation then reads as

\[
\sigma = C \text{Grad} u - k \ast \text{Grad} u = \partial_{0}^{-1}(C - k) \ast \text{Grad} v,
\]

where \( k : \mathbb{R}_{\geq 0} \rightarrow L(H_{\text{sym}}(\Omega)) \) is a strongly measurable function satisfying \( \int_{0}^{\infty} \| k(t) \| e^{-\mu t} \, dt < \infty \) for some \( \mu \geq 0 \) (note that in [12] \( k \) was assumed to be absolutely continuous). Again, choosing \( \nu > 0 \) large enough, we end up with a material law operator (see [51 Subsection 4.1])

\[
M(z) = \begin{pmatrix}
1 & 0 \\
0 & C^{-\frac{1}{2}}(1 - \frac{\nu}{(2\pi)^{\frac{1}{2}}} \hat{k}(-iz^{-1})C^{-\frac{1}{2}})^{-1}C^{-\frac{1}{2}},
\end{pmatrix}
\tag{2.9}
\]

where \( \hat{k} \) denotes the Fourier transform of \( k \) and where we have used that

\[
\mathcal{L}_{\nu}(k) \mathcal{L}_{\nu} = \sqrt{2\pi} \hat{k}(m - iw)
\]

for \( \nu \geq \mu \) (see e.g. [51 Lemma 3.4]). According to the solution theory presented in Theorem 2.5 we have to find suitable assumptions on the kernel \( k \) in order to obtain the positivity condition \( (2.3) \). This is done in the following theorem.

**Theorem 2.9 (Integro-differential equations, [51]).** Let \( H \) be a separable Hilbert space and \( k : \mathbb{R}_{\geq 0} \rightarrow L(H) \) a strongly measurable function satisfying \( \int_{0}^{\infty} \| k(t) \| e^{-\mu t} \, dt < \infty \) for some \( \mu \in \mathbb{R} \). If

\begin{itemize}
  \item[(a)] \( k(t) \) is selfadjoint for almost every \( t \in \mathbb{R}_{\geq 0} \),
  \item[(b)] there exists \( d \geq 0, \nu_0 \geq \mu \) such that
    \[
    t \Im \hat{k}(t - i\nu_0) \leq d
    \]
    for all \( t \in \mathbb{R} \),
\end{itemize}

then there exists \( r > 0 \) such that the material law \( M(z) := 1 + \sqrt{2\pi} \hat{k}(-iz^{-1}) \) with \( z \in B(r, r) \) satisfies \( (2.3) \). If, in addition, \( k \) satisfies

\begin{itemize}
  \item[(c)] \( k(t)k(s) = k(s)k(t) \) for almost every \( s, t \in \mathbb{R}_{\geq 0} \),
\end{itemize}

then there exists \( r' > 0 \) such that the material law \( \tilde{M}(z) := \left( 1 - \sqrt{2\pi} \hat{k}(-iz^{-1}) \right)^{-1} \) with \( z \in B(r', r') \) satisfies \( (2.3) \).

**Remark 2.10.** For real-valued kernels \( k \), a typical assumption is that \( k \) should be non-negative and non-increasing (see e.g. [38]). This, however implies the assumptions (a)-(c) of Theorem 2.9 for \( k \). Moreover, kernels \( k \) of bounded variation satisfy the assumptions of the latter theorem (see [51 Remark 3.6]).

Using Theorem 2.9 we get that \( M \) given by (2.9) satisfies the solvability condition \( (2.3) \), if \( k \) satisfies the assumptions (a)-(c) of Theorem 2.9 and \( k(t) \) and \( C \) commute for almost every \( t \in \mathbb{R}_{\geq 0} \) (see [51 Subsection 4.1] for a detailed study).

Convolutions as discussed in the previous theorem need to be incorporated due to the fact that, for instance, the elastic behavior of a solid body depends on the stresses the body experienced in the
past. One also speaks of so-called memory effects. From a similar type of nature is the modeling of material behavior with the help of fractional (time-)derivatives. In fact, in recent years, material laws for visco-elastic solids were described by using fractional derivatives (see e.g. [5, 28]) as an ansatz to better approximate a polynomial in $\delta_0^{-1}$ by potentially fewer terms of real powers of $\delta_0^{-1}$. In [61] a model for visco-elasticity with fractional derivatives has been analyzed mainly in the context of homogenization issues, which will be discussed below. For the moment, we stick to the presentation of the model and sketch the idea of the well-posedness result for this type of equation presented in [61, Theorem 2.1 and 2.2].

Example 2.11 (Visco-elasticity with fractional derivatives). In this model the Kelvin-Voigt model is replaced by a fractional analogue of the form

$$\sigma = C \nabla u + D \nabla \partial_0^{-\alpha} u,$$

for some $\alpha \in [0, 1]$. We emphasize here that $\partial_0^{-\alpha}$ has a natural meaning as a function of a normal operator. We refer to [35, Subsection 2.1] for a comparison with classical notions of fractional derivatives (see e.g. [27] for an introduction to fractional derivatives). As in the case $\alpha = 1$ we get that $\partial_0^{-\alpha}$ is boundedly invertible for each $\alpha \in [0, 1]$ and $\nu > 0$ and by [35, Lemma 2.1] we can estimate the norm of its inverse by

$$\|\partial_0^{-\alpha}\| \leq \nu^{-\alpha}.$$

Again we assume for simplicity that $D$ is selfadjoint and strictly positive definite, and thus we may rewrite the constitutive relation above as

$$\sigma = (C + D\partial_0^{-\alpha}) \nabla u = D\partial_0^{-\alpha}(D^{-1}C + 1) \nabla u$$

yielding for large enough $\nu > 0$

$$\partial_0^{-1-\alpha}(\partial_0^{-\alpha}D^{-1}C + 1)^{-1}D^{-1}\sigma = \nabla v.$$

Hence, our material law operator is given by

$$
\begin{pmatrix}
1 & 0 \\
0 & z^\alpha (1 + z^\alpha D^{-1}C)^{-1} D^{-1}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
0 & z^\alpha D^{-1}
\end{pmatrix}
+ \sum_{k=1}^{\infty} (-1)^k z^{(k+1)\alpha} \begin{pmatrix}
0 & 0 \\
0 & (D^{-1}C)^k D^{-1}
\end{pmatrix}.
$$ (2.10)

Note that the visco-elastic model under consideration is slightly different from the one treated in [61] as there is no further restriction on the parameter $\alpha$. In [61], we assumed $\alpha \geq \frac{1}{2}$ and showed the positive definiteness of the sum in (2.10) with the help of a perturbation argument. This argument does not apply to the situation discussed here. However, assuming selfadjointness and non-negativity of the operators $C$ and $D$ well-posedness can be warranted even for $\alpha < \frac{1}{2}$.

More generally, if one considers material law operators containing fractional derivatives of the form

$$M(\partial_0^{-1}) = M_0 + \sum_{\alpha \in \Pi} \partial_0^{-\alpha}M_\alpha + \partial_0^{-1}M_1,$$

where $\Pi \subseteq [0, 1]$ is finite and $M_\alpha \in L(H)$ for some Hilbert space $H$ and each $\alpha \in \{0, 1\} \cup \Pi$, one imposes the following conditions on the operators $M_\alpha$ in order to get an estimate of the form (2.3) for the material law (2.11):

Theorem 2.12 ([35, Theorem 3.5]). Let $(\alpha_0, \ldots, \alpha_k)$ be a monotonically increasing enumeration of $\Pi$. Assume that the operators $M_0$ and $M_\alpha$ are selfadjoint for each $j \in \{0, \ldots, k\}$. Moreover, let $P, Q, F \in L(H)$ be three orthogonal projectors satisfying

$$P + Q + F = 1$$

and assume, that $M_0$ and $M_\alpha$ commute with $P, Q$ and $F$ for every $j \in \{1, \ldots, k\}$. If $PM_\alpha P \geq 0$, $QM_\alpha Q \geq 0$, $M_\alpha \geq 0$ and $M_0, \Re \lambda M_1$ and $M_\alpha$ are strictly positive definite on the ranges of $P, Q$ and $F$ respectively, then the material law (2.11) satisfies the solvability condition (2.3).

\footnote{The infinite series in (2.10) can be treated with the help of a perturbation argument.}
With this theorem we end our tour through different kinds of evolutionary equations, which are all covered by the solution theory stated in Theorem \[2.5\].

Besides the well-posedness of evolutionary equations, it is also possible to derive a criterion for (exponential) stability in the abstract setting of Theorem \[2.5\]. Since the systems under consideration do not have any regularizing property, we are not able to define exponential stability as it is done classically, since our solutions \(u\) do not have to be continuous. So, point-wise evaluation of \(u\) does not have any meaning. Indeed, the problem class discussed in Theorem \[2.5\] covers also purely algebraic systems, where definitely no regularity of the solutions is to be expected unless the given data is regular. Thus, we are led to define a weaker notion of exponential stability as follows.

**Definition 2.13.** Let \(A: D(A) \subseteq H \to H\) be skew-selfadjoint\(^{17}\) and \(M \in \mathcal{H}(B(r, r), L(H))\) satisfying \([60]\) for some \(r > 0\). Let \(\nu > \frac{1}{2r}\). We call the operator \((\partial_0 M(\partial_0^{-1}) + A)^{-1}\) exponentially stable with stability rate \(\nu > 0\) if for each \(0 \leq \nu' < \nu_0\) and \(f \in L^2_{\nu', r} \cap L^2_{\nu}(\mathbb{R}, H)\) we have

\[
\nu \geq \inf_{\chi \geq 0} \frac{\nu_0}{\nu_0 - \nu' + \nu_0 h} = \nu_0 e^{-\nu_0 h} = c.
\]

As it turns out, this notion of exponential stability yields the exponential decay of the solutions, provided the solution \(u\) is regular enough. For instance, this can be achieved by assuming more regularity on the given right-hand side (see \[49\] Remark 3.2 (a)). The result for exponential stability reads as follows.

**Theorem 2.14** (\[50\] Theorem 3.2). Let \(A: D(A) \subseteq H \to H\) be a skew-selfadjoint operator and \(M\) be a mapping satisfying the following assumptions for some \(\nu_0 > 0\):

(a) \(M: \mathbb{C} \setminus \overline{B\left(\frac{1}{2\nu_0}, \frac{1}{2\nu_0}\right)} \to L(H)\) is analytic;

(b) for every \(0 < \nu' < \nu_0\) there exists \(c > 0\) such that for all \(z \in \mathbb{C} \setminus \overline{B\left(\frac{1}{2\nu_0}, \frac{1}{2\nu_0}\right)}\) we have

\[
\Re z^{-1} M(z) \geq c.
\]

Then for each \(\nu > 0\) the solution operator \((\partial_0 M(\partial_0^{-1}) + A)^{-1}\) is exponentially stable with stability rate \(\nu_0\).

**Example 2.15** (A parabolic-hyperbolic system, \[49\] Example 4.2). Let \(\Omega \subseteq \mathbb{R}^n\) be an open subset and \(\Omega_0, \Omega_1 \subseteq \Omega\) measurable, disjoint, non-empty, \(c > 0\). Then the solution operator for the equation

\[
\left(\partial_0 \left(\begin{array}{cc}
\chi_{\Omega_0} + \chi_{\Omega_1} & 0 \\
0 & \chi_{\Omega_0}
\end{array}\right) + c \begin{array}{cc}
0 & \mathrm{div} c \\
0 & 0
\end{array} + \left(\begin{array}{cc}
\mathrm{grad} c & 0 \\
0 & 0
\end{array}\right)\right) \left(\begin{array}{c}
v \\
q
\end{array}\right) = \left(\begin{array}{c}
f \\
0
\end{array}\right)
\]

for suitable \(f\) is exponentially stable with stability rate \(c\).

**Remark 2.16.** As in \[49\] Initial Value Problems, the stability of the corresponding initial value problems can be discussed similarly.

**Example 2.17** (Example 2.15 continued (see also \[49\] Theorem 4.4)). Let \(h < 0\) and assume, in addition, that \(c > 1\). Then the solution operator for the equation

\[
\left(\partial_0 \left(\begin{array}{cc}
\chi_{\Omega_0} + \chi_{\Omega_1} & 0 \\
0 & \chi_{\Omega_0}
\end{array}\right) + \tau h + c \begin{array}{cc}
0 & \mathrm{div} c \\
0 & 0
\end{array} + \left(\begin{array}{cc}
\mathrm{grad} c & 0 \\
0 & 0
\end{array}\right)\right) \left(\begin{array}{c}
v \\
q
\end{array}\right) = \left(\begin{array}{c}
f \\
0
\end{array}\right)
\]

is exponentially stable with stability rate \(\nu_0 > 0\) such that

\[
\nu_0 e^{-\nu_0 h} = c.
\]

\(^{17}\)For sake of presentation, we assume \(A\) to be skew-selfadjoint. However, in \[50\] \(A\) was assumed to be a linear maximal monotone operator. We will give a solution theory for this type of problem later on. One then might replace the condition of skew-selfadjointness in this definition and the subsequent theorem by the condition of being linear and maximal monotone.
Remark 2.18. We note that the exponential stability of integro-differential equations can be treated in the same way, see [19] Section 4.3.

2.4. The closedness of the problem class and homogenization

In this section we discuss the closedness of the problem class under consideration with respect to perturbations in the material law $M$. We will treat perturbations in the weak operator topology, which will also have strong connections to issues stemming from homogenization theory. For illustrational purposes we discuss the one dimensional case of an elliptic type equation first.

Example 2.19 (see e.g. [4] Example 1.1.3)]. Let $A: \mathbb{R} \to \mathbb{R}$ be a bounded, uniformly strictly positive, measurable, 1-periodic function. We denote the multiplication operator on $L^2([0,1])$ associated with $A$ by $A(m_1)$. Denoting the one-dimensional gradient on $[0,1]$ with homogeneous Dirichlet boundary conditions by $\partial_x$ (see also Definition 1.19) and $\partial_x$ for its skew-adjoint, we consider the problem of finding $u_\varepsilon \in D(\partial_x)$ such that for given $f \in L^2([0,1])$ and $\varepsilon > 0$ we have

$$-\partial_x A\left(\frac{1}{\varepsilon} m_1\right) \partial_x u_\varepsilon = f.$$ 

Of course, the solvability of the latter problem is clear due to Corollary 1.6. Now, we address the question whether $(u_\varepsilon)_{\varepsilon>0}$ is convergent in any particular sense and if so, whether the limit satisfies a differential equation of “similar type”. Before, however, doing so, we need the following result.

Proposition 2.20 (see e.g. [10] Theorem 2.6]). Let $A: \mathbb{R}^N \to \mathbb{C}$ be bounded, measurable and $[0,1]^N$-periodic, i.e., for all $k \in \mathbb{Z}^N$ and a.e. $x \in \mathbb{R}^N$ we have $A(x + k) = A(x)$. Then

$$A\left(\frac{\cdot}{\varepsilon}\right) \to \int_{[0,1]^N} A(x) dx \quad (\varepsilon \to 0)$$

in the weak*-topology $\sigma(L^\infty(\mathbb{R}^N), L^1(\mathbb{R}^N))$ of $L^\infty(\mathbb{R}^N)$.

Example 2.21 (Example 2.19 continued). For $\varepsilon > 0$, we define $\xi_\varepsilon := A\left(\frac{\cdot}{\varepsilon} m_1\right) \partial_x u_\varepsilon$. It is easy to see that $(\xi_\varepsilon)$ is bounded in $L^2([0,1])$ and also in $H_1([0,1]) = D(\partial_x)$. The Arzela-Ascoli theorem implies that $(\xi_\varepsilon)$ has a convergent subsequence (again labelled with $\varepsilon$), which converges in $L^2([0,1])$. We denote $\xi := \lim_{\varepsilon \to 0} \xi_\varepsilon$. In consequence, by Proposition 2.20 we deduce that

$$\partial_x u_\varepsilon = \frac{1}{A\left(\frac{\cdot}{\varepsilon} m_1\right)} \xi_\varepsilon \to \left(\int_0^1 \frac{1}{A(x)} dx\right)^{-1} \xi$$

weakly in $L^2([0,1])$ as $\varepsilon \to 0$. Hence, $(\partial_x u_\varepsilon)_{\varepsilon>0}$ weakly converges in $L^2([0,1])$, which, again by compact embedding, implies that $(u_\varepsilon)$ converges in $L^2([0,1])$. Denoting the respective limit by $u$, we infer

$$f = -\partial_x \xi = -\partial_1 \left(\int_0^1 \frac{1}{A(x)} dx\right)^{-1} \left(\int_0^1 \frac{1}{A(x)} dx\right)^{-1} \partial_x u.$$ 

Now, unique solvability of the latter equation together with a subsequence argument imply convergence of $(u_\varepsilon)$ without choosing subsequences.

Remark 2.22. Note that examples in dimension 2 or higher are far more complicated. In particular, the computation of the limit (if there is one) is more involved. To see this, we refer to [10] Sections 5.4 and 6.2, where the case of so-called laminated materials and general periodic materials is discussed. In the former the limit may be expressed as certain integral means, whereas in the latter so-called local problems have to be solved to determine the effective equation. Having these issues in mind, we will only give structural (i.e. compactness) results on homogenization problems of (evolutionary) partial differential equations. In consequence, the compactness properties of the differential operators as well as the ones of the coefficients play a crucial role in homogenization theory.
Regarding Proposition 2.20, the right topology for the operators under consideration is the weak operator topology. Indeed, with the examples given in the previous section in mind and modeling local oscillations as in Example 2.19 we shall consider the weak-$*$-topology of an appropriate $L^\infty$-space. Now, if we identify any $L^\infty$-function with the corresponding multiplication operator on $L^2$, we see that convergence in the weak-$*$-topology of the functions is equivalent to convergence of the associated multiplication operator in the weak operator topology of $L(L^2)$. This general perspective also enables us to treat problems with singular perturbations and problems of mixed type.

Before stating a first theorem concerning the issues mentioned, we need to introduce a topology tailored for the case of autonomous and causal material laws.

Definition 2.23 ([55] Definition 3.1]). For Hilbert spaces $H_1, H_2$ and an open set $E \subseteq \mathbb{C}$, we define $\tau_w$ to be the initial topology on $\mathcal{H}_w(E, L(H_1, H_2))$ induced by the mappings

$$\mathcal{H}_w(E, L(H_1, H_2)) \ni M \mapsto (z \mapsto \langle \phi, M(z) \psi \rangle) \in \mathcal{H}(E)$$

for $\phi \in H_2, \psi \in H_1$, where $\mathcal{H}(E)$ is the set of holomorphic functions endowed with the compact open topology, i.e., the topology of uniform convergence on compact sets. We write $\mathcal{H}_w^\infty := (\mathcal{H}_w^\infty, \tau_w)$ for the topological space and re-use the notation $\mathcal{H}_w^\infty$ for the underlying set.

We note the following remarkable fact.

Theorem 2.24 ([55] Theorem 3.4], [61] Theorem 4.3]). Let $H_1, H_2$ be Hilbert spaces, $E \subseteq \mathbb{C}$ open. Then

$$B_{\mathcal{H}_w} := \{ M \in \mathcal{H}_w^\infty(E, L(H_1, H_2)) \mid \| M \|_{\infty} \leq 1 \} \subseteq \mathcal{H}_w^\infty$$

is compact. If, in addition, $H_1$ and $H_2$ are separable, then $B_{\mathcal{H}_w}$ is metrizable.

Sketch of the proof. For $s \in [0, \infty[$ introduce the set $B_{\mathcal{H}(E)}(s) := \{ f \in \mathcal{H}(E) \mid \forall z \in E : |f(z)| \leq s \}$. The proof is based on the following equality

$$B_{\mathcal{H}_w} = \left( \prod_{\phi \in H_1, \psi \in H_2} B_{\mathcal{H}(E)}(\| \phi \|, \| \psi \|) \right) \cap \{ M : E \to L(H_1, H_2) \mid M(z) \text{ sesquilinear } (z \in E) \},$$

which itself follows from a Dunford type theorem ensuring the holomorphy (with values in the space $L(H_1, H_2)$) of the elements on the right-hand side and the Riesz-Fréchet representation theorem for sesquilinear forms. Now, invoking Montel’s theorem, we deduce that $B_{\mathcal{H}(E)}(s)$ is compact for every $s \in [0, \infty[$. Thus, Tikhonov’s theorem applies to deduce the compactness of $B_{\mathcal{H}_w}$. The proof for metrizability is standard. \hfill $\Box$

Recall for $r, c \in \mathbb{R}_{>0}$, and a Hilbert space $H$, we set

$$\mathcal{H}_{\infty,c}(B(r, r), L(H)) = \left\{ M \in \mathcal{H}_w^\infty(B(r, r), L(H)) \mid \bigwedge_{z \in B(r, r)} \Re z^{-1} M(z) \geq c \right\}.$$ 

In accordance to Definition 2.23 we will also write $\mathcal{H}_w^{\infty,c}$ for the set $\mathcal{H}_{\infty,c}$ endowed with $\tau_w$. The compactness properties of $\mathcal{H}_w^\infty$ are carried over to $\mathcal{H}_w^{\infty,c}$. The latter follows from the following proposition:

Proposition 2.25 ([60] Proposition 1.3]). Let $c \in \mathbb{R}_{>0}$. Then the set $\mathcal{H}_w^{\infty,c} \subseteq \mathcal{H}_w^\infty$ is closed.

We are now ready to discuss a first theorem on the continuous dependence on the coefficients for autonomous and causal material laws, which particularly covers a class of homogenization problems in the sense mentioned above. For a linear operator $A$ in some Hilbert space $H$, we denote $D(A)$ endowed with the graph norm of $A$ by $D_A$. If a Hilbert space $H_1$ is compactly embedded in $H$, we write $H_1 \hookrightarrow H$. A subset $M \subseteq \mathcal{H}^\infty$ is called bounded, if there is $\lambda > 0$ such that $M \subseteq \lambda B_{\mathcal{H}_w^\infty}$. The result reads as follows.
Theorem 2.26 ([10] Theorem 3.5, [61] Theorem 4.1). Let \( \nu, c \in \mathbb{R}_{>0}, r > \frac{1}{2\nu}, (M_n)_n \) be a bounded sequence in \( \mathcal{H}^{\infty,c}(B(r,r), L(H)) \), \( A : D(A) \subseteq H \to H \) skew-selfadjoint. Assume that \( D_A \hookrightarrow \hookrightarrow H \). Then there exists a subsequence of \( (M_n)_n \) such that \( (M_{n_k})_k \) converges in \( \mathcal{H}^{\infty,c} \) and

\[
\left( \partial_0 M_{n_k} \left( \partial_0^{-1} \right) + A \right)^{-1} \to \left( \partial_0 M \left( \partial_0^{-1} \right) + A \right)^{-1}
\]

in the weak operator topology.

We first apply this theorem to an elliptic type equation.

Example 2.27. Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded. Let \( \text{grad}_c \) and \( \text{div} \) be the operators introduced in Definition [19]. Let \( (a_k)_{k \in \mathbb{N}} \) be a sequence of uniformly strictly positive bounded linear operators in \( L^2(\Omega)^n \). For \( f \in L^2(\Omega) \) consider for \( k \in \mathbb{N} \) the problem of finding \( u_k \in L^2(\Omega) \) such that the equation

\[
u_k - \text{div} \, a_k \, \text{grad}_c \, u_k = f
\]

holds. Observe that if \( \nu : R(\text{grad}_c) \to L^2(\Omega)^n \) denotes the canonical embedding, this equation is the same as

\[
u_k - \text{div} \, \nu^* a_k \nu^* \, \text{grad}_c \, u_k = f.
\]

(2.12)

Indeed, by Poincaré’s inequality \( R(\text{grad}_c) \subseteq L^2(\Omega)^n \) is closed, the projection theorem ensures that \( \nu^* \) is the orthogonal projection on \( R(\text{grad}_c) \). Moreover, \( N(\text{div}) = R(\text{grad}_c)^\perp \) yields that \( \text{div} = \text{div}(\nu^* + (1 - \nu^*)) = \text{div} \, \nu^* \). Now, we realize that due to the positive definiteness of \( a_k \), so is \( \nu^* a_k \nu^* \). Consequently, the latter operator is continuously invertible. Introducing \( v_k := \nu^* a_k \nu^* \, \text{grad}_c \, u_k \) for \( k \in \mathbb{N} \), we rewrite the equation (2.12) as follows

\[
\begin{pmatrix}
1 & 0 \\
0 & (\nu^* a_k \nu^*)^{-1}
\end{pmatrix}
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix}
= \begin{pmatrix}
f \\
0
\end{pmatrix}.
\]

Now, let \( \nu > 0 \) and lift the above problem to the space \( L^2_\nu(\mathbb{R}, L^2(\Omega) \oplus R(\text{grad}_c)) \) by interpreting \( \begin{pmatrix} f \\ 0 \end{pmatrix} \) as \( \begin{pmatrix} \chi_{\nu\gg0} (t) \begin{pmatrix} f \\ 0 \end{pmatrix} \end{pmatrix} \in L^2_\nu(\mathbb{R}, L^2(\Omega) \oplus R(\text{grad}_c)) \). Then this equation fits into the solution theory stated in Theorem 2.5 with

\[
M_k(\partial_0^{-1}) := \partial_0^{-1} \begin{pmatrix} 1 & 0 \\
0 & (\nu^* a_k \nu^*)^{-1}
\end{pmatrix}, \quad A := \begin{pmatrix} 0 & \text{div} \nu \\
\nu^* \text{grad}_c & 0
\end{pmatrix}.
\]

Note that the skew-selfadjointness of \( A \) is easily obtained from \( \text{div}^* = -\text{grad}_c \). In order to conclude the applicability of Theorem 2.26 we need the following observation.

Proposition 2.28 ([61] Lemma 4.1). Let \( H_1, H_2 \) be Hilbert spaces, \( C : D(C) \subseteq H_1 \to H_2 \) densely defined, closed, linear. Assume that \( D_C \hookrightarrow \hookrightarrow H_1 \). Then \( D_C \cap N(C^*)^\perp u_2 \hookrightarrow \hookrightarrow H_2 \).

Example 2.29 (Example 2.27 continued). With the help of the theorem of Rellich-Kondrachov and Proposition 2.28 we deduce that \( A \) has compact resolvent. Thus, Theorem 2.26 is applicable. We find a subsequence such that \( u := \tau_w - \lim_{t \to \infty} (\nu^* a_k \nu^*)^{-1} \) exists, where we denoted by \( \tau_w \) the weak operator topology. Therefore, \( (u_k)_k \) weakly converges to some \( u \), which itself is the solution of

\[
u - \text{div} \, \nu^{-1} \, \nu^* \, \text{grad}_c \, u = f.
\]

In fact it is possible to show that \( \nu^{-1} \, \nu^* \) coincides with the usual homogenized matrix (if the possibly additional assumptions on the sequence \( (a_k)_k \) permit the computation of a limit in the sense of \( H \)- or \( G \)-convergence, see e.g. [10] Chapter 13 and the references therein).

As a next example let us consider the heat equation.

Example 2.30. Recall the heat equation introduced in Example 2.6

\[
\begin{pmatrix}
\partial_0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & \text{div}
\\
\text{grad}_c & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\varphi \\
q
\end{pmatrix}
= \begin{pmatrix}
f \\
0
\end{pmatrix}.
\]

To warrant the compactness condition in Theorem 2.26 we again assume that the underlying domain \( \Omega \) is bounded. Similarly to Example 2.27 we assume that we are given \( (k_i)_i \), a bounded
sequence of uniformly strictly monotone linear operators in $L(L^2(\Omega)^n)$. Consider the sequence of equations
\[
\left( \partial_0 \left( \begin{array}{ccc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{ccc} 0 & 0 \\ 0 & k_i^{-1} \end{array} \right) + \left( \begin{array}{ccc} \div & 0 \\ 0 & \text{grad} \end{array} \right) \right) \left( \begin{array}{c} \vartheta_t \\ q_i \end{array} \right) = \left( \begin{array}{c} f \\ 0 \end{array} \right).
\]
Now, focussing only on the behavior of the temperature $(\vartheta_t)_t$, we can proceed as in the previous example.

Assuming more regularity of $\Omega$, e.g., the segment property and finitely many connected components, we can apply Theorem 2.26 also to the corresponding homogeneous Neumann problems of Examples 2.27 and 2.30. Moreover, the aforementioned theorem can also be applied to the homogenization of (visco-)elastic problems (see also Example 2.8). For this we need criteria ensuring Examples 2.27 and 2.30. Moreover, the aforementioned theorem can also be applied to the homogenization of (visco-)elastic problems (see also Example 2.8). For this we need criteria ensuring the compactness condition $D_{\text{Grad}_i} \hookrightarrow L^2(\Omega)^n$ (or $D_{\text{Grad}_i} \hookrightarrow L^2(\Omega)^n$). The latter is warranted for a bounded $\Omega$ for the homogeneous Dirichlet case or an $\Omega$ satisfying suitable geometric requirements (see e.g. [64]). An example of a different type of nature is that of Maxwell’s equations:

**Example 2.31.** Recall Maxwell’s equation as introduced in Example 2.7.
\[
\left( \partial_0 \left( \begin{array}{cc} \varepsilon & 0 \\ 0 & \mu \end{array} \right) + \left( \begin{array}{cc} \sigma & \text{curl} \\ 0 & \text{curl} \end{array} \right) \right) \left( \begin{array}{cc} E \\ H \end{array} \right) = \left( \begin{array}{c} J \\ 0 \end{array} \right).
\]
In this case, we also want to consider sequences $(\varepsilon_n)_n, (\mu_n)_n, (\sigma_n)_n$ and corresponding solutions $(E_n, H_n)_n$. In any case the nullspaces of both curl and curl are infinite-dimensional. Thus, the projection mechanism introduced above for the heat and the elliptic equation cannot apply in the same manner. Moreover, considering the Maxwell’s equations on the nullspace of $\left( \begin{array}{cc} 0 & \text{curl} \\ \text{curl} & 0 \end{array} \right)$, we realize that the equation amounts to be an ordinary differential equation in an infinite-dimensional state space. For the latter we have not stated any homogenization or continuous dependence result yet. Thus, before dealing with Maxwell’s equations in full generality, we focus on ordinary (integro-)differential equations next.

**Theorem 2.32** ([60] Theorem 4.4). Let $\nu, \varepsilon \in \mathbb{R}_{>0}, r > \frac{1}{2\nu}, (M_n)_n$ in $H^{\infty,\nu}(B(0,r), L(H)) \cap H^\infty(B(0,\varepsilon), L(H))$ bounded, $H$ separable Hilbert space. Assume that
\[
M_n(0) \geq c \text{ on } R(M_n(0)) = R(M_1(0))
\]
for all $n \in \mathbb{N}$. Then there exists a subsequence $(n_k)_k$ of $(n)_n$ and some $M \in H^\infty$ such that
\[
\left( \partial_0 M_{n_k}(\partial_0^{-1}) \right)^{-1} \rightarrow \left( \partial_0 M(\partial_0^{-1}) \right)^{-1}
\]
in the weak operator topology.

**Remark 2.33.** Note that in the latter theorem, in general, the sequence $\left( M_{n_k}(\partial_0^{-1}) \right)_k$ does not converge to $M(\partial_0^{-1})$. The reason for that is that the computation of the inverse is not continuous in the weak operator topology. So, even if one chose a further subsequence $(n_{k_l})_l$ of $(n_k)_k$ such that $\left( M_{n_{k_l}}(\partial_0^{-1}) \right)_l$ converges in the weak operator topology, then, in general,
\[
M_{n_{k_l}}(\partial_0^{-1}) \not\rightarrow M(\partial_0^{-1})
\]
in $\tau_w$. Indeed, the latter can be seen by considering the periodic extensions of the mappings $a^1, a^2$ to all of $\mathbb{R}$ with
\[
a^1(x) := \begin{cases} \frac{1}{2}, & 0 \leq x < \frac{1}{2}, \\
1, & \frac{1}{2} \leq x < 1,
\end{cases} \quad a^2(x) := \frac{3}{4} \quad (x \in [0,1]).
\]
We let $a_n := a^1(n \cdot)$ for odd $n \in \mathbb{N}$ and $a_n := a^2(n \cdot)$ if $n \in \mathbb{N}$ is even. Then, by Proposition 2.20, we conclude that $a_n \rightarrow \frac{1}{2}, a_{2n+1} \rightarrow \frac{1}{2}, a_{2n-1} \rightarrow \frac{3}{4}$ as $n \rightarrow \infty$ in $\sigma(L^\infty, L^1)$.

\[\text{Note that } M_n \in H^{\infty,\nu}(B(r,r), L(H)) \cap H^{\infty}(B(0,\varepsilon), L(H)) \text{ implies that } M_n(0) \text{ is selfadjoint.} \]
In a way complementary to the latter theorem is the following. The latter theorem assumes analyticity of the \( M_n \)'s at 0. But the zero'th order term in the power series expansion of the \( M_n \)'s may be non-invertible. In the next theorem, the analyticity at 0 is not assumed any more. The (uniform) positive definiteness condition, however, is more restrictive.

**Theorem 2.34 ([55 Theorem 5.2]).** Let \( \nu, \varepsilon \in \mathbb{R}_{>0} \), \( r > \frac{1}{2\nu} \), \( (M_n)_n \) in \( \mathcal{H}^{\infty,(c)}(B(r,r), L(H)) \) bounded, \( H \) separable Hilbert space. Assume that

\[
\Re M_n(z) \geq c \quad (z \in B(r,r))
\]

for all \( n \in \mathbb{N} \). Then there exists a subsequence \( (M_{n_k})_k \) of \( (M_n)_n \) and some \( M \in \mathcal{H}^{\infty} \) such that

\[
(\partial_0 M_{n_k}(\partial_0^{-1}))^{-1} \to (\partial_0 M(\partial_0^{-1}))^{-1}
\]

in the weak operator topology.

Now, we turn to more concrete examples. With the methods developed, we can characterize the convergence of a particular ordinary equation. In a slightly more restrictive context these types of equations have been discussed by Tartar in 1989 (see [43, 42]) using the notion of Young-measures, see also the discussion in [59, Remark 3.8].

**Proposition 2.35.** Let \( (a_n)_n \) in \( L(H) \) be bounded, \( H \) a separable Hilbert space, \( \nu > 2 \sup_{n \in \mathbb{N}} \|a_n\| + 1 \). Then

\[
(\partial_0 + a_n)^{-1}
\]

converges in the weak operator topology if and only if for all \( \ell \in \mathbb{N} \)

\[
(a_\ell^n)_n
\]

converges in the weak operator topology to some \( b_\ell \in L(H) \). In the latter case \( (\partial_0 + a_n)^{-1} \)

converges to

\[
(\partial_0 + \partial_0 \sum_{j=1}^{\infty} (-\sum_{\ell=1}^{\infty} (-\partial_0^{-1})^\ell b_\ell)^j)^{-1}
\]

in the weak operator topology.

**Proof.** The ’if’-part is a straightforward application of a Neumann series expansion of \( (\partial_0 + a_n)^{-1} \), see e.g. [63 Theorem 2.1]. The ’only-if’-part follows from the representation

\[
(\partial_0 + a_n)^{-1} = \sum_{j=0}^{\infty} (-\partial_0^{-1})^j a_n^j \partial_0^{-1} =: M_n(\partial_0^{-1}) \quad (n \in \mathbb{N}),
\]

the application of the Fourier-Laplace transform and Cauchy’s integral formulas for the derivatives of holomorphic functions. For the latter argument note that \( (M_n)_n \) is a bounded sequence in \( \mathcal{H}^{\infty}(B(r,r), L(H)) \) for \( r > \frac{1}{2\nu} \) and, thus, contains a \( \mathcal{H}^{\infty}_w \)-convergent subsequence, whose limit \( M \) satisfies \( M(\partial_0^{-1}) = \tau_w \lim_{n \to \infty} (\partial_0 + a_n)^{-1} \). \( \Box \)

One might wonder under which circumstances the conditions in the latter theorem happen to be satisfied. We discuss the following example initially studied by Tartar.

**Example 2.36 (Ordinary differential equations).** Let \( a \in L^\infty(\mathbb{R}) \). If \( a \) is 1-periodic then \( a(n\cdot) \)

converges to \( \int_0^1 a \) in the \( \sigma(L^\infty, L^1) \)-topology. Regard \( a \) as a multiplication operator \( a(m_1) \) on \( H^1(\mathbb{R}) \). Now, we have the explicit formula

\[
(\partial_0 + a(nm_1))^{-1} \to_{\mathcal{H}^\infty} (\partial_0 + \partial_0 \sum_{j=1}^{\infty} (-\sum_{\ell=1}^{\infty} (-\partial_0^{-1})^\ell \int_0^1 a^\ell)^j)^{-1}
\]

We should remark here that the classical approach to this problem uses the theory of Young-measures to express the limit equation. This is not needed in our approach.
With the latter example in mind, we now turn to the discussion of a general theorem also working for Maxwell’s equation. As mentioned above, these equations can be reduced to the cases of Theorem 2.26 and 2.32. Consequently, the limit equations become more involved. For sake of this presentation, we do not state the explicit formulae for the limit expressions and instead refer to [56, Corollary 4.7].

**Theorem 2.37 ([56 Corollary 4.7])**. Let \( \nu, \varepsilon \in \mathbb{R}_{>0}, r > \frac{1}{2\nu}, (M_n)_n \) in \( H^{\infty,c}(B(r, r), L(H)) \cap H^{\infty}(B(0, \varepsilon), L(H)) \) bounded, \( A : D(A) \subseteq H \to H \) skew-selfadjoint, \( H \) separable. Assume that \( D_A \cap N(A)^\perp \leftrightarrow H \) and, in addition,

\[
M_n(0) \geq c \text{ on } R(M_n(0)) = R(M_1(0))
\]

\[
\iota_{N(A)}^* M_n'(0) \iota_{N(A)} \left( \iota_{N(A)}^* M_n'(0) \iota_{N(A)} \right)^{-1} = \iota_{N(A)}^* M_n'(0)^* \iota_{N(A)} \left( \iota_{N(A)}^* M_n'(0)^* \iota_{N(A)} \right)^{-1},
\]

for all \( n \in \mathbb{N} \), where \( \iota_{N(A)}^* : N(M_1(0)) \cap N(A)^\perp \to H, \iota_{N(A)} : N(M_1(0)) \cap N(A) \to H \) denote the canonical embeddings. Then there exists a subsequence \( (M_{n_k})_k \) of \( (M_n)_n \) such that

\[
\left( \partial_0 M_{n_k} \left( \partial_0^{-1} + A \right)^{-1} \right)^{-1} \to \left( \partial_0 M \left( \partial_0^{-1} + A \right)^{-1} \right)^{-1}
\]

converges in the weak operator topology.

**Remark 2.38**. It should be noted that, similarly to the case of ordinary differential equations, in general, we do not have \( M_{n_k} \left( \partial_0^{-1} \right) \to M \left( \partial_0^{-1} \right) \) in the weak operator topology.

Before we discuss possible generalizations of the above results to the non-autonomous case, we illustrate the applicability of Theorem 2.37 to Maxwell’s equations:

**Example 2.39 (Example 2.31 continued)**. Consider

\[
\left( \partial_0 \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \mu_n \end{pmatrix} + \begin{pmatrix} \sigma_n & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \text{curl}_e \\ \text{curl} & 0 \end{pmatrix} \right) \begin{pmatrix} \mathbf{E}_n \\ H_n \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ 0 \end{pmatrix}
\]

for bounded sequences of bounded linear operators \((\varepsilon_n)_n, (\mu_n)_n, (\sigma_n)_n\). Assuming suitable geometric requirements on the underlying domain \( \Omega \), see e.g. [34], we realize that the compactness condition is satisfied. Thus, we only need to guarantee the compatibility conditions: Essentially, there are two complementary cases. On the one hand, one assumes uniform strict positive definiteness of the (selfadjoint) operators \( \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \mu_n \end{pmatrix} \). On the other hand, we may also consider the eddy current problem, which results in \( \varepsilon_n = 0 \). Then, in order to apply Theorem 2.37, we have to assume selfadjointness of \( \sigma_n \) and the existence of some \( c > 0 \) such that \( \sigma_n \geq c \) for all \( n \in \mathbb{N} \). In this respect our homogenization theorem only works under additional assumptions on the material laws apart from (uniform) well-posedness conditions. We also remark that the limit equation is of integro-differential type, see [56 Corollary 4.7] or [63].

### 2.5. The non-autonomous case

The non-autonomous case is characterized by the fact that the operators \( \mathcal{M} \) and \( \mathcal{A} \) in (2.41) does not have to commute with the translation operators \( \tau_h \). A rather general abstract result concerning well-posedness reads as follows:

**Theorem 2.40 ([52 Theorem 2.4])**. Let \( \nu > 0 \) and \( \mathcal{M}, \mathcal{N} \in L(L^2_{\nu}(\mathbb{R}, H)) \). Assume that there exists \( M \in L(L^2_{\nu}(\mathbb{R}, H)) \) such that

\[
\mathcal{M} \partial_{0, \nu} \subseteq \partial_{0, \nu} \mathcal{M} + M.
\]

Let \( \mathcal{A} : D(\mathcal{A}) \subseteq L^2_{\nu}(\mathbb{R}, H) \to L^2_{\nu}(\mathbb{R}, H) \) be densely defined, closed, linear and such that \( \partial_{0, \nu}^{-1} \mathcal{A} \subseteq A_0^{-1} \mathcal{A} \mathcal{A}_{0}^{-1} \). Assume there exists \( c > 0 \) such that the positivity conditions

\[
\Re(\langle \partial_{0, \nu} \mathcal{M} + \mathcal{N} + \mathcal{A} \rangle \phi|\chi_{\mathcal{A}_{0}^{-1}}(m_0)\phi) \geq c\langle \phi|\chi_{\mathcal{A}_{0}^{-1}}(m_0)\phi\rangle
\]

and

\[
\Re(\langle (\partial_{0, \nu} \mathcal{M} + \mathcal{N})^* + \mathcal{A}^* \rangle \psi|\psi) \geq c\langle \psi|\psi\rangle
\]

for all \( \phi, \psi \in L^2_{\nu}(\mathbb{R}, H) \) and \( \mathcal{M} \).
hold for all $a \in \mathbb{R}$, $\phi \in D(\partial_{0,\nu}) \cap D(A)$, $\psi \in D(\partial_{0,\nu}) \cap D(A^*)$. Then $B := \partial_{0,\nu} M + N + A$ is continuously invertible, $\|B^{-1}\| \leq \frac{1}{\nu}$, and the operator $B^{-1}$ is causal in $L^2_0(\mathbb{R}, H)$.

In order to capture the main idea of this general abstract result, we consider the following special non-autonomous problem of the form

$$\left(\partial_{0,\nu} M_0(m_0) + M_1(m_0) + A\right) u = f, \tag{2.13}$$

where $\partial_{0,\nu}$ denotes the time-derivative as introduced in Subsection 2.2, and $A$ denotes a skew-selfadjoint operator on some Hilbert space $H$ (and its canonical extension to the space $L^2_0(\mathbb{R}, H)$).

Moreover, $M_0, M_1 : \mathbb{R} \to L(H)$ are assumed to be strongly measurable and bounded (in symbols $M_0, M_1 \in L^\infty_0(\mathbb{R}, L(H))$) and therefore, they give rise to multiplication operators on $L^2_0(\mathbb{R}, H)$ by setting

$$(M_i(m_0)u)(t) := M_i(t)u(t) \quad (\text{a.e. } t \in \mathbb{R})$$

for $u \in L^2_0(\mathbb{R}, H)$, where $\nu \geq 0$ and $i \in \{0, 1\}$. Of course, the so defined multiplication operators are bounded with

$$\|M_i(m_0)\|_{L(L^2_0(\mathbb{R}, H))} \leq |M_i|_\infty = \operatorname{ess-sup}_{t \in \mathbb{R}} \|M_i(t)\|_{L(H)}$$

for $i \in \{0, 1\}$ and $\nu \geq 0$. In order to formulate the theorem in a less cluttered way, we introduce the following hypotheses.

**Hypotheses 2.41.** We say that $T \in L^\infty_0(\mathbb{R}, L(H))$ satisfies the property

(a) if $T(t)$ is self-adjoint ($t \in \mathbb{R}$),
(b) if $T(t)$ is non-negative ($t \in \mathbb{R}$),
(c) if the mapping $T$ is Lipschitz-continuous, where we denote the smallest Lipschitz-constant of $T$ by $|T|_{\text{Lip}}$, and
(d) if there exists a set $N \subseteq \mathbb{R}$ of measure zero such that for each $x \in H$ the function $\mathbb{R} \setminus N \ni t \mapsto T(t)x$

is differentiable.\textsuperscript{19}

If $T \in L^\infty_0(\mathbb{R}, H)$ satisfies the hypotheses above, then for each $t \in \mathbb{R} \setminus N$ the operator

$$\hat{T}(t) : H \to H$$

$$x \mapsto (T(t)x)'(t)$$

becomes a self-adjoint linear operator satisfying $\|\hat{T}(t)\|_{L(H)} \leq |T|_{\text{Lip}}$ for every $t \in \mathbb{R} \setminus N$ and consequently $\hat{T} \in L^\infty_0(\mathbb{R}, L(H))$. We are now able to state the well-posedness result for non-autonomous problems of the form \textsuperscript{2.14}.

**Theorem 2.42 (\textsuperscript{36} Theorem 2.13).** Let $A : D(A) \subseteq H \to H$ be skew-selfadjoint and $M_0, M_1 \in L^\infty_0(\mathbb{R}, L(H))$. Furthermore, assume that $M_0$ satisfies the hypotheses (a)-(d) and that there exists a set $N_1 \subseteq \mathbb{R}$ of measure zero with $N \subseteq N_1$ such that

$$\bigvee_{c_0 > 0, \nu_0 > 0} \bigwedge_{t \in \mathbb{R} \setminus N_1, \nu \geq \nu_0} : \nu M_0(t) + \frac{1}{2} \dot{M}_0(t) + \Re M_1(t) \geq c_0. \tag{2.14}$$

Then the operator $\partial_{0,\nu} M_0(m_0) + M_1(m_0) + A$ is continuously invertible in $L^2_0(\mathbb{R}, H)$ for each $\nu \geq \nu_0$. A norm bound for the inverse is $1/c_0$. Moreover, we get that

$$\left(\partial_{0,\nu} M_0(m_0) + M_1(m_0) + A\right)^* = \left(M_0(m_0) \partial^\ast_{0,\nu} + M_1(m_0)^\ast - A\right). \tag{2.15}$$

**Proof.** The result can easily be established, when observing that $M_0(m_0)\partial_{0,\nu} \subseteq \partial_{0,\nu} M_0(m_0) + \dot{M}_0(m_0)$ and using Theorem 2.40. \hfill \Box

\textsuperscript{19}If $H$ is separable, then the strong differentiability of $T$ on $\mathbb{R} \setminus N$ for some set $N$ of measure zero already follows from the Lipschitz-continuity of $T$ by Rademachers theorem.
Independently of Theorem 2.40 note that condition (2.14) is an appropriate non-autonomous analogue of the positive definiteness constraint (2.3) in the autonomous case. With the help of (2.14) one can prove that the operator \( \partial_{0,\nu} M_0(m_0) + M_1(m_0) + A \) is strictly monotone and after establishing the equality (2.15), the same argumentation works for the adjoint. Hence, the well-posedness result may also be regarded as a consequence of Corollary 4.3.

**Example 2.43.** As an illustrating example for the applicability of Theorem 2.42 we consider a non-autonomous evolutionary problem, which changes its type in time and space. Let \( \nu > 0 \). Consider the \((1 + 1)\)-dimensional wave equation:

\[
\partial^2_{0,\nu} u - \partial^2_t u = f \quad \text{on } \mathbb{R} \times \mathbb{R}.
\]

As usual we rewrite this equation as a first order system of the form

\[
\left( \partial_{0,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial^{-1}_{0,\nu} f \\ 0 \end{pmatrix}.
\]

(2.16)

In this case we can compute the solution by Duhamel’s formula in terms of the unitary group generated by the skew-selfadjoint operator

\[
\begin{pmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{pmatrix}.
\]

Let us now, based on this, consider a slightly more complicated situation, which is, however, still autonomous:

\[
\left( \partial_{0,\nu} \begin{pmatrix} \chi_{\mathbb{R} \setminus \{0\}}(m_1) & 0 \\ 0 & \chi_{\mathbb{R} \setminus \{0\}}(m_1) \end{pmatrix} + \begin{pmatrix} \chi_{[-\nu,\nu]}(m_1) & 0 \\ 0 & \chi_{[-\nu,\nu]}(m_1) \end{pmatrix} + \begin{pmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial^{-1}_{0,\nu} f \\ 0 \end{pmatrix},
\]

(2.17)

where \( \chi_i(m_1) \) denotes the spatial multiplication operator with the cut-off function \( \chi_i \), given by \( (\chi_i(m_1)f)(t,x) = \chi_i(x)f(t,x) \) for almost every \( (t,x) \in \mathbb{R} \times \mathbb{R} \), every \( f \in L^2_0(\mathbb{R}, L^2(\mathbb{R})) \) and \( I \subseteq \mathbb{R} \). Hence, (2.17) is an equation of the form (2.13) with

\[
M_0(m_0) := \begin{pmatrix} \chi_{\mathbb{R} \setminus \{-\nu\}}(m_1) & 0 \\ 0 & \chi_{\mathbb{R} \setminus \{-\nu\}}(m_1) \end{pmatrix}
\]

and

\[
M_1(m_0) := \begin{pmatrix} \chi_{[-\nu,\nu]}(m_1) & 0 \\ 0 & \chi_{[-\nu,\nu]}(m_1) \end{pmatrix}
\]

and both are obviously not time-dependent. Note that our solution condition (2.14) is satisfied and hence, problem (2.17) is well-posed in the sense of Theorem 2.42. By the dependence of the operators \( M_0(m_0) \) and \( M_1(m_0) \) on the spatial parameter, we see that (2.17) changes its type from hyperbolic to elliptic to parabolic and back to hyperbolic and so standard semi-group techniques are not at hand to solve the equation. Indeed, in the subregion \([-\nu,0]\) the problem reads as

\[
\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial^{-1}_{0,\nu} f \\ 0 \end{pmatrix},
\]

which may be rewritten as an elliptic equation for \( u \) of the form

\[
u - \partial^2_t u = \partial^{-1}_{0,\nu} f.
\]

For the region \([0,\nu]\) we get

\[
\begin{pmatrix} \partial_{0,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\partial_1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial^{-1}_{0,\nu} f \\ 0 \end{pmatrix},
\]

Indeed, the well-posedness already follows from Theorem 2.40 since \( M \) is autonomous and satisfies (2.3).
which yields a parabolic equation for $u$ of the form
\[ \partial_{0,t}u - \partial_t^2 u = \partial_{0,t}^1 f. \]
In the remaining sub-domain $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$ the problem is of the original form [24.10], which corresponds to a hyperbolic problem for $u$.
To turn this into a genuinely time-dependent problem we now make a modification to problem [2.17]. We define the function
\[ \varphi(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t > 1 \end{cases} \quad (t \in \mathbb{R}) \]
and consider the material-law operator
\[ M_0(m_0) = \varphi(m_0) \begin{pmatrix} x_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_1) & 0 \\ 0 & x_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_1) \end{pmatrix}, \]
which now also degenerates in time. Moreover we modify $M_1(m_0)$ by adding a time-dependence of the form
\[ M_1(m_0) = \left( x_{[-\infty, \varepsilon]}(m_0) + x_{[0, \infty]}(m_0) x_{[-\varepsilon, \varepsilon]}(m_1) \right) + \left( x_{[-\infty, \varepsilon]}(m_0) + x_{[0, \infty]}(m_0) x_{[-\varepsilon, \varepsilon]}(m_1) \right). \]
We show that this time-dependent material law still satisfies our solvability condition. Note that
\[ \varphi'(t) = \begin{cases} 1 & \text{if } t \in ]0, 1[, \\ 0 & \text{otherwise} \end{cases} \]
and thus, for $t \leq 0$ we have
\[ \nu M_0(t) + \frac{1}{2} M_0(t) + \Re M_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq 1. \]
For $0 < t \leq 1$ we estimate
\[ \nu M_0(t) + \frac{1}{2} M_0(t) + \Re M_1(t) \]
\[ = \left( \frac{1}{2} + \nu t \right) \begin{pmatrix} x_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_1) & 0 \\ 0 & x_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_1) \end{pmatrix} + \begin{pmatrix} x_{[-\varepsilon, \varepsilon]}(m_1) & 0 \\ 0 & x_{[-\varepsilon, \varepsilon]}(m_1) \end{pmatrix} \geq \frac{1}{2} \]
and, finally, for $t > 1$ we obtain that
\[ \nu M_0(t) + \frac{1}{2} M_0(t) + \Re M_1(t) \]
\[ = \nu \begin{pmatrix} x_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_1) & 0 \\ 0 & x_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_1) \end{pmatrix} + \begin{pmatrix} x_{[-\varepsilon, \varepsilon]}(m_1) & 0 \\ 0 & x_{[-\varepsilon, \varepsilon]}(m_1) \end{pmatrix} \geq \min \{ \nu, 1 \}. \]
There is also an adapted result on the closedness of the problem class for the non-autonomous situation. The case $\mathcal{A} = 0$ is thoroughly discussed in [59]. We give the corresponding result for the situation where $\mathcal{A}$ is non-zero and satisfies a certain compactness condition.

**Theorem 2.44 [28 Theorem 3.1].** Let $\nu > 0$. Let $\{\mathcal{M}_n\}_n$ be a bounded sequence in $L(L^2(\mathbb{R}, H))$ such that $\{\mathcal{M}_n(\partial_{0,t})\}_n$ is bounded in $L(L^2(\mathbb{R}, H))$. Moreover, let $\mathcal{A}: D(\mathcal{A}) \subseteq L^2(\mathbb{R}, H) \rightarrow L^2(\mathbb{R}, H)$ be linear and maximal monotone commuting with $\partial_{0,t}$ and assume that $\mathcal{M}_n$ is causal for each $n \in \mathbb{N}$. Moreover, assume the positive definiteness conditions
\[ \Re \left( \partial_{0,t} \mathcal{M}_n u |\chi_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_0) u \right) \geq c \left( u |\chi_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_0) u \right), \quad \left( \mathcal{A} u |\chi_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(m_0) u \right) \geq 0 \quad (2.18) \]
for all $u \in D(\partial_{0,t}) \cap D(\mathcal{A})$, $a \in \mathbb{R}$, $n \in \mathbb{N}$ and some $c > 0$.
Assume that there exists a Hilbert space $K$ such that $K \hookrightarrow \hookrightarrow H$ and $D(\mathcal{A}) \hookrightarrow L^2(\mathbb{R}, K)$ and that $\{\mathcal{M}_n\}_n$ converges in the weak operator topology to some $\mathcal{M}$. 

---
Then \( \partial_{0,\nu} \mathcal{M} + \mathcal{A} \) is continuously invertible in \( L^2_v(\mathbb{R}, H) \) and \( (\partial_{0,\nu} \mathcal{M} + \mathcal{A})^{-1} \) in the weak operator topology of \( L^2_v(\mathbb{R}, H) \) as \( n \to \infty \).

As in [58], we illustrate the latter theorem by the following example, being an adapted version of Example 2.43.

**Example 2.45 ([58] Section 1).** Recalling the definition of \( \partial_1, \partial_{1,c} \) on \( L^2([0,1]) \) from Definition 1.19 we treat the following system written in block operator matrix form:

\[
\begin{pmatrix}
\partial_{0,\nu} & \chi_{[0,\frac{1}{2}]}(m_1) \\
0 & \chi_{[\frac{1}{2},1]}(m_1)
\end{pmatrix}
\begin{pmatrix}
0 \\
\chi_{[0,\frac{1}{2}]}(m_1)
\end{pmatrix}
+ \begin{pmatrix}
\chi_{[\frac{1}{2},1]}(m_1) \\
0 \\
\chi_{[0,\frac{1}{2}]}(m_1)
\end{pmatrix}
\begin{pmatrix}
0 \\
\partial_1
\end{pmatrix}
\begin{pmatrix}
u \\
u
\end{pmatrix}
= \begin{pmatrix} f \\
g \end{pmatrix}
\tag{2.19}
\]

where \( f, g \) are thought of being given. We find that \( \mathcal{M} \) is given by

\[
\mathcal{M} = \begin{pmatrix}
\chi_{[0,\frac{1}{2}]}(m_1) \\
0 \\
\chi_{[0,\frac{1}{2}]}(m_1)
\end{pmatrix}
+ \partial_0^{-1}
\begin{pmatrix}
\chi_{[\frac{1}{2},1]}(m_1) \\
0 \\
\chi_{[0,\frac{1}{2}]}(m_1)
\end{pmatrix}
\begin{pmatrix}
0 \\
\partial_1
\end{pmatrix}
\begin{pmatrix} u \\
v
\end{pmatrix}
\]

We realize that

\[
\mathcal{A} = \begin{pmatrix} 0 & \partial_1 \\
\partial_{1,c} & 0 \end{pmatrix}
\]

is skew-selfadjoint and, thus, maximal monotone. Note that the system describes a mixed type equation. The system varies between hyperbolic, elliptic and parabolic type equations either with homogeneous Dirichlet or Neumann data. Well-posedness of the system \( \tag{2.19} \) can be established in \( L^2_v(\mathbb{R}, L^2(0,1)) \).

Now, instead of \( \tag{2.19} \), we consider the sequence of problems

\[
\begin{pmatrix}
\partial_{0,\nu} & \chi_{[0,\frac{1}{2}]}(n \cdot m_1 \mod 1) \\
0 & \chi_{[\frac{1}{2},1]}(n \cdot m_1 \mod 1)
\end{pmatrix}
\begin{pmatrix}
0 \\
\chi_{[\frac{1}{2},1]}(n \cdot m_1 \mod 1)
\end{pmatrix}
+ \begin{pmatrix}
\chi_{[\frac{1}{2},1]}(n \cdot m_1 \mod 1) \\
0 \\
\chi_{[0,\frac{1}{2}]}(n \cdot m_1 \mod 1)
\end{pmatrix}
\begin{pmatrix}
0 \\
\partial_1
\end{pmatrix}
\begin{pmatrix}
u \\
v_n
\end{pmatrix}
= \begin{pmatrix} f \\
g
\end{pmatrix}
\tag{2.20}
\]

for \( n \in \mathbb{N} \), where \( x \mod 1 := x - \lfloor x \rfloor, x \in \mathbb{R} \). With the same arguments from above well-posedness of the latter equation is warranted in the space \( L^2_v(\mathbb{R}, L^2([0,1])) \). Now,

\[
\begin{pmatrix}
\chi_{[0,\frac{1}{2}]}(n \cdot m_1 \mod 1) \\
0 \\
\chi_{[\frac{1}{2},1]}(n \cdot m_1 \mod 1)
\end{pmatrix}
+ \partial_0^{-1}
\begin{pmatrix}
\chi_{[\frac{1}{2},1]}(n \cdot m_1 \mod 1) \\
0 \\
\chi_{[0,\frac{1}{2}]}(n \cdot m_1 \mod 1)
\end{pmatrix}
\begin{pmatrix}
0 \\
\partial_1
\end{pmatrix}
\begin{pmatrix}
u \\
v_n
\end{pmatrix}
\to \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix} + \partial_0^{-1}
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
\]

in the weak operator topology due to periodicity. Theorem 2.44 asserts that the sequence \( \begin{pmatrix} u_n \\
v_n \end{pmatrix} \) weakly converges to the solution \( \begin{pmatrix} u \\
v \end{pmatrix} \) of the problem

\[
\begin{pmatrix}
\partial_{0,\nu} & \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \\
0 & \begin{pmatrix}
\partial_1 \\
\partial_{1,c}
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \begin{pmatrix} f \\
g
\end{pmatrix}
\]

It is interesting to note that the latter system does not coincide with any of the equations discussed above.
Theorem 2.44 deals with coefficients \( M \) that live in space-time. Going a step further instead of treating (2.20), we let \((\kappa_n)_n\) in \( W^1_1(\mathbb{R}) \) be a \( W^1_1(\mathbb{R})\)-convergent sequence of weakly differentiable \( L^1(\mathbb{R})\)-functions with limit \( \kappa \) and support on the positive reals. Then it is easy to see that the associated convolution operators \((\kappa_n)_n\) converge in \( L(L^2(\mathbb{R}_{\geq 0})) \) to \( \kappa \). Moreover, using Young’s inequality, we deduce that

\[
\|\kappa_n \ast \|_{L(L^2(\mathbb{R}^n)))}, \|\kappa' \ast \|_{L(L^2(\mathbb{R}^n)))} \to 0 \quad (\nu \to \infty)
\]

uniformly in \( n \). Thus, the strict positive definiteness of

\[
\partial_{0,\nu} (1 + \kappa_n) \left( \begin{pmatrix} \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (m_1) & 0 \\ 0 & \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (m_1) \end{pmatrix} \right)
\]

in the truncated form as in (2.18) in Theorem 2.44 above follows from the respective inequality for

\[
\partial_{0,\nu} \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (m_1) \quad 0 \\
0 & \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (m_1)
\]

Now, the product of a sequence converging in the weak operator topology and a sequence converging in the norm topology converges in the weak operator topology. Hence, the solutions of

\[
\left( \partial_{0,\nu} (1 + \kappa_n) \left( \begin{pmatrix} \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) & 0 \\ 0 & \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) \end{pmatrix} \right)
\right)
\]

converge weakly to the solution of

\[
\left( \partial_{0,\nu} (1 + \kappa) \left( \begin{pmatrix} \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) & 0 \\ 0 & \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) \end{pmatrix} \right)
\right)
\]

The latter considerations dealt with time-translation invariant coefficients. We shall also treat another example, where time-translation invariance is not warranted. For this take a sequence of Lipschitz continuous functions \((N_n): \mathbb{R} \to \mathbb{R})_n\) with uniformly bounded Lipschitz semi-norm and such that \((N_n)_n\) converges point-wise almost everywhere to some function \( N: \mathbb{R} \to \mathbb{R}\). Moreover, assume that there exists \( c > 0 \) such that \( \frac{1}{c} \geq N_n \geq c \) for all \( n \in \mathbb{N} \). Then, by Lebesgue’s dominated convergence theorem \( N_n(m_0) \to N(m_0) \) in the strong operator topology, where we anticipated that \( N_n(m_0) \) acts as a multiplication operator with respect to the temporal variable. The strict monotonicity in the above truncated sense of

\[
\partial_{0,\nu} \left( \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) \quad 0 \\
0 & \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) \end{pmatrix} \right)
\]

+ \( \partial_{0,\nu}^{-1} \left( \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) \quad 0 \\
0 & \chi_{[0, \frac{1}{2}[|\frac{1}{2}, \frac{3}{2}[} (n \cdot m_1 \mod 1) \end{pmatrix} \right)
\]
is easily seen using integration by parts, see e.g. [36] Lemma 2.6]. Our main convergence theorem now yields that the solutions of
\[
\begin{pmatrix}
\partial_{0,\nu}(N(m_0)) \\
\partial_{0,\nu}^{-1}(u,v) 
\end{pmatrix} + \begin{pmatrix}
\frac{\chi_{[0,1]}(v)}{2} & \frac{\chi_{[1,2]}(v)}{2} \\
0 & \frac{\chi_{[1,2]}(v)}{2}
\end{pmatrix} \begin{pmatrix}
n \cdot m_1 \\ n \cdot m_1
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
converge weakly to the solution of
\[
\begin{pmatrix}
\partial_{0,\nu}(N(m_0)) \\
\partial_{0,\nu}^{-1}(u,v) 
\end{pmatrix} + \begin{pmatrix}
\frac{\chi_{[0,1]}(v)}{2} & \frac{\chi_{[1,2]}(v)}{2} \\
0 & \frac{\chi_{[1,2]}(v)}{2}
\end{pmatrix} \begin{pmatrix}
n \cdot m_1 \\ n \cdot m_1
\end{pmatrix} + \begin{pmatrix}
\partial_{1,c} & 0 \\
0 & \partial_{1,c}
\end{pmatrix} \begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}f \\ g\end{pmatrix}.
\]

3. Nonlinear Monotone Evolutionary Problems

This last section is devoted to the generalization of the well-posedness results of the previous sections to a particular case of non-linear problems. Instead of considering differential equations we turn our attention to the study of differential inclusions. As in the previous section, we begin to consider the autonomous case and present the well-posedness result.

3.1. The autonomous case

Let \( \nu > 0 \). The problem class under consideration is given as follows
\[
(u,f) \in \partial_{0,\nu}M \left( \partial_{0,\nu}^{-1} \right) + A,
\]
where \( M(\partial_{0,\nu}^{-1}) \) is again a linear material law, arising from an analytic and bounded function \( M : B_{\mathbb{C}}(r,r) \to L(H) \) for some \( r > \frac{1}{2\pi} \), \( f \in L^2_{\nu}(\mathbb{R}, H) \) is a given right-hand side and \( u \in L^2_{\nu}(\mathbb{R}, H) \) is to be determined. In contrast to the above problems, \( A \subseteq L^2_{\nu}(\mathbb{R}, H) \) is now a maximal monotone relation, which in particular need not to be linear. By this lack of linearity we cannot argue as in the previous section, where the maximal monotonicity of the operators were shown by proving the strict monotonicity of their adjoints (in other words, we cannot apply Corollary 1.3). Thus, the maximal monotonicity has to be shown by employing other techniques and the key tools are perturbation results for maximal monotone operators.

In the autonomous case, our hypotheses read as follows:

**Hypotheses 3.1.** We say that \( A \) satisfies the hypotheses (H1) and (H2) respectively, if

(H1) \( A \) is maximal monotone and translation-invariant, i.e. for every \( h \in \mathbb{R} \) and \( (u,v) \in A \) we have \( (u(\cdot + h),v(\cdot + h)) \in A \).

(H2) for all \( (u,v),(x,y) \in A \) the estimate \( \int_{-\infty}^{0} R e \langle u(t) - x(t) \rangle \langle v(t) - y(t) \rangle e^{-\nu t} dt \geq 0 \) holds.

Assuming the standard assumption \([23]\) for the function \( M \), the operator \( \partial_{0,\nu}M \left( \partial_{0,\nu}^{-1} \right) - c \) is maximal monotone on \( L^2_{\nu}(\mathbb{R}, H) \). Thus, the well-posedness of \([34]\) just relies on the maximal monotonicity of the sum of \( \partial_{0,\nu}M \left( \partial_{0,\nu}^{-1} \right) - c \) and \( A \). Since \( A \) is assumed to be maximal monotone, we can apply well-known perturbation results in the theory of maximal monotone operators to prove that \( \partial_{0,\nu}M \left( \partial_{0,\nu}^{-1} \right) + A - c \) is indeed maximal monotone, which in particular yields that
\[
\left( \partial_{0,\nu}M \left( \partial_{0,\nu}^{-1} \right) + A \right)^{-1}
\]
is a Lipschitz-continuous mapping on \( L^2_{\nu}(\mathbb{R}, H) \) (see Theorem 2.1). Moreover, using hypothesis (H2) we can prove the causality of the corresponding solution operator \( \left( \partial_{0,\nu}M \left( \partial_{0,\nu}^{-1} \right) + A \right)^{-1} \).

The well-posedness result reads as follows:
Theorem 3.2 (Well-posedness of autonomous evolutionary inclusions, \cite{35}). Let $H$ be a Hilbert space, $M : B_{c} \left( \frac{1}{2\pi}, \frac{1}{2\pi} \right) \rightarrow L(H)$ a linear material law for some $\nu_{0} > 0$ satisfying (2.3). Let $\nu > \nu_{0}$ and $A \subseteq L_{2}^{2}(\mathbb{R}, H) \oplus L_{2}^{2}(\mathbb{R}, H)$ a relation satisfying (H1). Then for each $f \in L_{2}^{2}(\mathbb{R}, H)$ there exists a unique $u \in L_{2}^{2}(\mathbb{R}, H)$ such that

\begin{equation}
(u, f) \in \partial_{\nu,M} (\partial_{\nu}^{-1} + A). \tag{3.2}
\end{equation}

Moreover, $\left( \partial_{\nu,M} (\partial_{\nu}^{-1} + A) \right)^{-1}$ is Lipschitz-continuous with a Lipschitz constant less than or equal to $\frac{1}{\nu}$. If in addition $A$ satisfies (H2), then the solution operator $\left( \partial_{\nu,M} (\partial_{\nu}^{-1} + A) \right)^{-1}$ is causal.

A typical example for a maximal monotone relation satisfying (H1) and (H2) is an extension of a maximal monotone relation satisfying (H1) and (H2). Moreover, $A_{\nu}$ obviously satisfies (H1) and (H2).

Remark 3.3. It is possible to drop the assumption $(0, 0) \in A$, if one considers the differential inclusion on the half-line $\mathbb{R}_{\geq 0}$ instead of $\mathbb{R}$. In this case, an analogous definition of the time derivative on the space $L_{2}^{2}(\mathbb{R}_{\geq 0}, H)$ can be given and the well-posedness of initial value problems of the form

\begin{equation}
(u, f) \in (\partial_{\nu,M} M_{0} + M_{1} + A_{\nu}), \tag{3.3}
\end{equation}

where $A_{\nu}$ is given as the extension of a maximal monotone relation $A \subseteq H \oplus H$ and $M_{0}, M_{1} \in L(H)$ satisfy a suitable monotonicity constraint, can be shown similarly (see \cite{46}).

The general coupling mechanism as illustrated e.g. in \cite{31} also works for the non-linear situation. This is illustrated in the following example.

Example 3.4 (\cite{46} Section 5.1). We consider the equations of thermo-plasticity in a domain $\Omega \subseteq \mathbb{R}^{3}$, given by

\begin{align}
M \partial_{\nu}^2 u - \text{Div} \sigma &= f, \tag{3.4} \\
\varrho \partial_{\nu} \vartheta - \text{div} \kappa \text{ grad } \vartheta + \tau_{0} \text{ trace } \text{Grad} \partial_{\nu} u &= g. \tag{3.5}
\end{align}

The functions $u \in L_{2}^{2}(\mathbb{R}, L^{2}(\Omega)^3)$ and $\vartheta \in L_{2}^{2}(\mathbb{R}, L^{2}(\Omega))$ are the unknowns, standing for the displacement field of the medium and its temperature, respectively. $f \in L_{2}^{2}(\mathbb{R}, L^{2}(\Omega)^3)$ and $g \in L_{2}^{2}(\mathbb{R}, L^{2}(\Omega))$ are given source terms. The stress tensor $\sigma \in L_{2}^{2}(\mathbb{R}, H_{\text{sym}}(\Omega))$ is related to the strain tensor and the temperature by the following constitutive relation, generalizing Hooke’s law,

\begin{equation}
\sigma = C(\text{Grad } u - \varepsilon_{p}) - c \text{ trace }^* \vartheta, \tag{3.6}
\end{equation}

where $c > 0$ and $C : H_{\text{sym}}(\Omega) \rightarrow H_{\text{sym}}(\Omega)$ is a linear, selfadjoint and strictly positive definite operator (the elasticity tensor). The operator trace : $H_{\text{sym}}(\Omega) \rightarrow L^{2}(\Omega)$ is the usual trace for matrices and its adjoint can be computed by trace$^* f = \begin{pmatrix} f & 0 & 0 \\
0 & f & 0 \\
0 & 0 & f \end{pmatrix}$. The function $\varrho \in L^{\infty}(\Omega)$ describes the mass density and is assumed to be real-valued and uniformly strictly positive, $M, \kappa \in L^{\infty}(\Omega)^{3 \times 3}$ are assumed to be uniformly strictly positive definite and selfadjoint and $\tau_{0} > 0$ is a real numerical parameter. The additional term $\varepsilon_{p}$ models the inelastic strain and is related to $\sigma$ by

\begin{equation}
(\sigma, \partial_{\nu,p} \varepsilon_{p}) \in \mathbb{I} \tag{3.7}
\end{equation}

where $\mathbb{I} \subseteq H_{\text{sym}}(\Omega) \oplus H_{\text{sym}}(\Omega)$ is a maximal monotone relation satisfying trace $[\mathbb{I}[H_{\text{sym}}(\Omega)] = \{0\}$, i.e. each element in the post-set of $\mathbb{I}$ is trace-free. If $\varepsilon_{p} = 0$, then (3.4)-(3.6) are exactly the
equations of thermo-elasticity (see [33] p. 420 ff.). The quasi-static case was studied in [31] for a particular relation \( I \), depending on the temperature \( \vartheta \) under the additional assumption that the material possesses the linear kinematic hardening property. We complete the system (3.4)-(3.7) by suitable boundary conditions for \( u \) and \( \vartheta \), for instance \( u, \vartheta = 0 \) on \( \partial \Omega \). We set \( v := \partial_0,\nu u \) and \( q := \tau_0^{-1} c_\kappa \text{grad}_e \vartheta \).

Following [46], Subsection 5.1, the system (3.4)-(3.7) can be written as

\[
\begin{pmatrix}
\vartheta \\
q \\
u \\
s
\end{pmatrix} =
\begin{pmatrix}
\partial_0,\nu \mathcal{M}(\partial_0,\nu^{-1}) + \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
c_{\tau_0}^{-1} f \\
0 \\
F \\
0
\end{pmatrix},
\]

where

\[
\mathcal{M}(\partial_0,\nu^{-1}) = \begin{pmatrix}
c_{\tau_0}^{-1} w + \text{trace } cC^{-1} c \text{trace}^* & 0 & 0 & \text{trace } cC^{-1} \\
0 & 0 & 0 & 0 \\
0 & 0 & M & 0 \\
C^{-1} c \text{trace}^* & 0 & 0 & C^{-1}
\end{pmatrix} + \partial_0,\nu^{-1}.
\]

Thus, we have that \( \mathcal{M}(\partial_0,\nu^{-1}) = \mathcal{M}_0 + \partial_0,\nu^{-1} \mathcal{M}_1 \) with

\[
\mathcal{M}_0 = \begin{pmatrix}
c_{\tau_0}^{-1} w + \text{trace } cC^{-1} c \text{trace}^* & 0 & 0 & \text{trace } cC^{-1} \\
0 & 0 & 0 & 0 \\
0 & 0 & M & 0 \\
C^{-1} c \text{trace}^* & 0 & 0 & C^{-1}
\end{pmatrix},
\]

\[
\mathcal{M}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \kappa^{-1} c_{\tau_0}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

It can easily be verified, that the material law \( \mathcal{M}(\partial_0,\nu^{-1}) \) satisfies \(2.3\). Thus, we only have to check that

\[
A := \begin{pmatrix}
0 & \text{div} & 0 & 0 \\
-\text{grad}_e & 0 & 0 & 0 \\
0 & 0 & -\text{Div} & 0 \\
0 & 0 & -\text{Grad}_e & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

is maximal monotone (note that the other assumptions on \( A \) are trivially satisfied, since \( A \) is given as in \([43]\)). Since \( A \) is the sum of two maximal monotone operators its maximal monotonicity can be obtained by assuming suitable boundedness constraints on \( I \) and applying classical perturbation results for maximal monotone operators [3].

3.2. The non-autonomous case

We are also able to treat non-autonomous differential inclusions. Consider the following problem

\[
(u, f) \in (\partial_0,\nu \mathcal{M}_0(m_0) + \mathcal{M}_1(m_0) + A_\nu),
\]

where \( \mathcal{M}_0, \mathcal{M}_1 \in L^\infty_{\text{loc}}(\mathbb{R}, L(H)) \) and \( A_\nu \) is the canonical extension of a maximal monotone relation \( A \subset H \oplus H \) with \((0, 0) \in A \) as defined in [33]. As in Subsection 2.4 we assume that \( \mathcal{M}_0 \) satisfies Hypotheses [2.4] [31, p. 33 ff.].

Our well-posedness result reads as follows:

\(^{21}\)The easiest assumption would be the boundedness of \( I \), i.e. for every bounded set \( M \) the post-set \( I[M] \) is bounded. For more advanced perturbation results we refer to [33] p. 331 ff.
Theorem 3.5 (Solution theory for non-autonomous evolutionary inclusions, [53]). Let $M_0, M_1 \in L_1^\infty(\mathbb{R}; L(H))$, where $M_0$ satisfies Hypotheses 2.41 (i)-(iv). Moreover, we assume that $N(M_0(t)) = N(M_0(0))$ for every $t \in \mathbb{R}$ and $\nu > 0$.

\[
\max_{c > 0} \left\{ \iota_{R(M_0(0))} M_0(t) \iota_{R(M_0(0))} \geq c \text{ and } \iota_{N(M_0(0))} M_1(t) \iota_{N(M_0(0))} \geq c. \right\}
\]

Let $A \subseteq H \oplus H$ be a maximal monotone relation with $(0, 0) \in A$. Then there exists $\nu_0 > 0$ such that for every $\nu \geq \nu_0$

\[
(\partial_{0,\nu} M_0(m_0) + M_1(m_0) + A_\nu)^{-1} : L_1^2(\mathbb{R}, H) \to L_1^2(\mathbb{R}, H)
\]
is a Lipschitz-continuous, causal mapping. Moreover, the mapping is independent of $\nu$ in the sense that, for $\nu, \nu' \geq \nu_0$ and $f \in L_1^2(\mathbb{R}, H) \cap L_2^2(\mathbb{R}, H)$ we have that

\[
(\partial_{0,\nu} M_0(m_0) + M_1(m_0) + A_\nu)^{-1} (f) = (\partial_{0,\nu'} M_0(m_0) + M_1(m_0) + A_{\nu'})^{-1} (f).
\]

Note that in Subsection 2.4.1 we do not require that $N(M_0(t))$ is $t$-independent. However, in order to apply perturbation results, which are the key tools for proving the well-posedness of (3.8), we need to impose this additional constraint (compare [36, Theorem 2.19]).

3.3. Problems with non-linear boundary conditions

As we have seen in Subsection 3.1, the maximal monotonicity of the relation $A \subseteq L_1^2(\mathbb{R}, H) \oplus L_2^2(\mathbb{R}, H)$ plays a crucial role for the well-posedness of the corresponding evolutionary problem (3.1). Motivated by several examples from mathematical physics, we might restrict our attention to (possibly non-linear) operators $A : D(A) \subseteq L_1^2(\mathbb{R}, H) \to L_2^2(\mathbb{R}, H)$ of a certain block structure. As a motivating example, we consider the wave equation with impedance-type boundary conditions, which was originally treated in [32].

Example 3.6. Let $\Omega \subseteq \mathbb{R}^n$ be open and consider the following boundary value problem

\[
\begin{align*}
\partial_{0,\nu}^2 u - \text{div grad } u &= f \text{ on } \Omega, \\
(\partial_{0,\nu} a(m) u + \text{grad } u) \cdot N &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where $N$ denotes the outward normal vector field on $\partial \Omega$ and $a \in L_1^\infty(\Omega)^n$ such that $\text{div } a \in L_1^\infty(\Omega)$.

Formulating (3.10) as a first order system we obtain

\[
\begin{pmatrix}
\partial_{0,\nu} (v) \\
q
\end{pmatrix}
= \begin{pmatrix}
0 & \text{div } \\
\text{grad } & 0
\end{pmatrix}
\begin{pmatrix}
v \\
q
\end{pmatrix}
= \begin{pmatrix}
f \\
0
\end{pmatrix},
\]

where $v := \partial_{0,\nu} u$ and $q := -\text{grad } u$. The boundary condition (3.11) then reads as

\[
(\partial_{0,\nu} a(m) v - q) \cdot N = 0 \text{ on } \partial \Omega.
\]

The latter condition can be reformulated as

\[
a(m) v - \partial_{0,\nu}^{-1} q \in D(\text{div}_c),
\]

where $\text{div}_c$ is defined as in Definition 1.29. Thus, we end up with a problem of the form

\[
(\partial_{0,\nu} + A) \begin{pmatrix}
v \\
q
\end{pmatrix}
= \begin{pmatrix}
f \\
0
\end{pmatrix},
\]

where $A \subseteq \begin{pmatrix}
0 & \text{div } \\
\text{grad } & 0
\end{pmatrix}$ with $D(A) := \{(v, q) \in D(\text{grad}) \times D(\text{div}) \mid a(m) v - \partial_{0,\nu}^{-1} q \in D(\text{div}_c)\}$. In order to apply the solution theory, we have to ensure that the operator $A$, defined in that way, is maximal monotone as an operator in $L_1^2(\mathbb{R}, L^2(\Omega)) \oplus L^2(\Omega)^n$.

Remark 3.7. In [32] a more abstract version of Example 3.6 was studied, where the vector field $a$ was replaced by a suitable material law operator $a(\partial_{0,\nu}^{-1})$ as it is defined in Subsection 2.3.

22We denote by $\iota_{R(M_0(0))}$ and $\iota_{N(M_0(0))}$ the canonical embeddings into $H$ of $R(M_0(0))$ and $N(M_0(0))$, respectively.

23Here we mean the divergence in the distributional sense.
Following this guiding example, we are led to consider restrictions \( A \) of block operator matrices
\[
\begin{pmatrix}
\ 0 & D \\
G & 0 \\
\end{pmatrix},
\]
where \( G : D(G) \subseteq H_0 \to H_1 \) and \( D : D(D) \subseteq H_1 \to H_0 \) are densely defined closed linear operators satisfying \( D^* \subseteq -G \) and consequently \( G^* \subseteq -D \). We set \( D_c := -G^* \) and \( G_c := -D^* \) and obtain densely defined closed linear restrictions of \( D \) and \( G \), respectively. Regarding the example above, \( G = \text{grad} \) and \( D = \text{div} \), whereas \( G_c = \text{grad}_c \) and \( D_c = \text{div}_c \). Having this guiding example in mind, we interpret \( G_c \) and \( D_c \) as the operators with vanishing boundary conditions and \( G \) and \( D \) as the operators with maximal domains. This leads to the following definition of so-called abstract boundary data spaces.

**Definition 3.8 ([34, Subsection 5.2]).** Let \( G_c, D_c, G, \) and \( D \) as above. We define
\[
BD(G) := D(G_c)^{\perp} D(G) = N(1 - DG),
\]
\[
BD(D) := D(D_c)^{\perp} D(D) = N(1 - GD),
\]
where \( D(G_c) \) and \( D(D_c) \) are interpreted as closed subspaces of the Hilbert spaces \( D(G) \) and \( D(D) \), respectively, equipped with their corresponding graph norms. Consequently, we have the following orthogonal decompositions
\[
D(G) = D(G_c) \oplus BD(G) \tag{3.12}
\]
\[
D(D) = D(D_c) \oplus BD(D).
\]

**Remark 3.9.** The decomposition (3.12) could be interpreted as follows: Each element \( u \) in the domain of \( G \) can be uniquely decomposed into two elements, one with vanishing boundary values (the component lying in \( D(G_c) \)) and one carrying the information of the boundary value of \( u \) (the component lying in \( BD(G) \)). In the particular case of \( G = \text{grad} \) a comparison of \( BD(G) \) and the classical trace space \( H^1(\partial \Omega) \) can be found in [47, Section 4].

Let \( \iota_{BD(G)} : BD(G) \to D(G) \) and \( \iota_{BD(D)} : BD(D) \to D(D) \) denote the canonical embeddings. An easy computation shows that \( G[BD(G)] \subseteq BD(D) \) and \( D[BD(D)] \subseteq BD(G) \) and thus, we may define
\[
\dot{G} := \iota_{BD(G)}^* G \iota_{BD(G)} : BD(G) \to BD(D)
\]
\[
\dot{D} := \iota_{BD(D)}^* D \iota_{BD(D)} : BD(D) \to BD(G).
\]

These two operators share a surprising property.

**Proposition 3.10 ([34, Theorem 5.2]).** The operators \( \dot{G} \) and \( \dot{D} \) are unitary and
\[
(\dot{G})^* = \dot{D} \quad \text{as well as} \quad (\dot{D})^* = \dot{G}.
\]

Coming back to our original question, when \( A \subseteq \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \) defines a maximal monotone operator, we find the following characterization.

**Theorem 3.11 ([47, Theorem 3.1]).** Let \( G \) and \( D \) be as above. A restriction \( A \subseteq \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \) is maximal monotone, if and only if there exists a maximal monotone relation \( h \subseteq BD(G) \oplus BD(G) \) such that
\[
D(A) = \left\{ (u,v) \in D(G) \times D(D) \mid (\iota_{BD(G)}^* u, \dot{D} \iota_{BD(D)}^* v) \in h \right\}.
\]

**Example 3.12.**
(a) In Example 3.6, the operators $G$ and $D$ are grad and div, respectively and the relation $h \subseteq BD(\text{grad}) \oplus BD(\text{div})$ is given by

$$(x, y) \in h \iff \partial_{0, \nu}^{-1} y = \text{div} \, t^*_B D(\text{div}) a(\nu) t_B D(\text{grad}) x.$$ 

Indeed, by the definition of the operator $A$ in Example 3.6, a pair $(v, q) \in D(\text{grad}) \times D(\text{div})$ belongs to $D(A)$ if and only if

$$a(\nu)v - \partial_{0, \nu}^{-1} q \in D(\text{div}_c) \iff t^*_B D(\text{div}) a(\nu) t_B D(\text{grad}) v = \partial_{0, \nu}^{-1} D(\text{div}_c) q = \text{div} \, t^*_B D(\text{div}) a(\nu) t_B D(\text{grad}) v \iff (t^*_B D(\text{grad}) v, \text{div} \, t^*_B D(\text{div}) q) \in h.$$ 

Thus, if we show that $h$ is maximal monotone, we get the maximal monotonicity of $A$ by Theorem 3.11. For doing so, we have to assume that the vector field $a$ satisfies a positivity condition of the form

$$\Re \int_{-\infty}^0 (\langle \text{grad} u, \partial_{0, \nu} a(\nu) u \rangle(t) + \langle u, \text{div} \partial_{0, \nu} a(\nu) u \rangle(t)) e^{-2\nu t} dt \geq 0$$

(3.13)

for all $u \in D(\partial_{0, \nu}) \cap D(\text{grad})$. In case of a smooth boundary, the latter can be interpreted as a constraint on the angle between the vector field $a$ and the outward normal vector field $N$. Indeed, condition 3.11 implies the monotonicity of $h$ and also of the adjoint of $h$ (note that here, $h$ is a linear relation). Both facts imply the maximal monotonicity of $h$ (the proof can be found in [18, Section 4.2]).

(b) In the theory of contact problems in elasticity we find so-called frictional boundary conditions at the contact surfaces. These conditions can be modeled for instance by sub-gradients of lower semi-continuous convex functions (see e.g. [24, Section 5]), which are the classical examples of maximal monotone relations.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. We recall the equations of elasticity from Example 2.8

$$
\left( \partial_{0, \nu} \begin{pmatrix} 1 & 0 \\ 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{pmatrix} \right) \left( \begin{array}{c} v \\ T \end{array} \right) = \left( \begin{array}{c} f \\ 0 \end{array} \right)
$$

(3.14)

and assume that the following frictional boundary condition should hold on the boundary $\partial \Omega$ (for a treatment of boundary conditions just holding on different parts of the boundary, we refer to [17]):

$$
(v, -T \cdot N) \in g,$$

(3.15)

where $N$ denotes the unit outward normal vector field and $g \subseteq L^2(\partial \Omega)^n \oplus L^2(\partial \Omega)^n$ is a maximal monotone relation, which, for simplicity, we assume to be bounded. We note that in case of a smooth boundary, there exists a continuous injection $\kappa : BD(\text{Grad}) \rightarrow L^2(\partial \Omega)^n$ (see [17]) and we may assume that $\kappa[BD(\text{Grad})] \cap [L^2(\partial \Omega)^n] g \neq \emptyset$. Then, according to [17, Proposition 2.6], the relation

$$
\tilde{g} := \kappa^* g \kappa = \{(x, \kappa^* y) \in BD(\text{Grad}) \times BD(\text{Grad}) \mid (\kappa x, y) \in g\}
$$

is maximal monotone as a relation on $BD(\text{Grad})$ and the boundary condition (3.15) can be written as

$$
(t^*_B D(\text{grad}) v, -\text{div} \, t^*_B D(\text{div}) T) \in \tilde{g}.
$$

\footnote{Note that not every maximal monotone relation can be realized as a sub-gradient of a lower semi-continuous convex function. Indeed, sub-gradients are precisely the cyclic monotone relations, see [17, Theoreme 2.5].}
Thus, by Theorem 3.11 the operator
\[
A \subseteq \begin{pmatrix}
0 & - \text{Div} \\
- \text{Grad} & 0
\end{pmatrix}
\]
\[D(A) := \left\{ (v, T) \in D(\text{Grad}) \times D(\text{Div}) \mid \left( t_{BD(\text{Grad})}^* v, - \text{Div} t_{BD(\text{Div})}^* T \right) \in \tilde{g} \right\}
\]
is maximal monotone and hence, Theorem 3.2 is applicable and yields the well-posedness of (3.14) subject to the boundary condition (3.15).

4. Conclusion

We have illustrated that many (initial, boundary value) problems of mathematical physics fit into the class of so-called evolutionary problems. Having identified the particular role of the time-derivative, we realize that many equations (or inclusions) of mathematical physics share the same type of solution theory in an appropriate Hilbert space setting. The class of problems accessible is widespread and goes from standard initial boundary value problems as for the heat equation, the wave equation or Maxwell’s equations etc. to problems of mixed type and to integro-differential-algebraic equations. We also demonstrated first steps towards a discussion of issues like exponential stability and continuous dependence on the coefficients in this framework. The methods and results presented provide a general, unified approach to numerous problems of mathematical physics.

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