A NOTE ON THE SMOOTHNESS OF FLOW MAPS FOR THE YANG-MILLS SYSTEM IN THE LORENZ GAUGE

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Abstract. We study the failure of the smoothness of flow maps for the (1 + 3) dimensional Yang-Mills system in the Lorenz gauge by Knapp type counterexamples. This shows a gap between the scaling critical regularity exponents and the best attainable regularity via Picard’s iteration in the Yang-Mills system under the Lorenz gauge condition.

1. Introduction

In this note, we prove the failure of the $C^2$ smoothness of flow maps for the Yang-Mills system in the Lorenz gauge, which is given by the following nonlinear wave system [3]:

\begin{align}
\square A_\nu &= -2[A^\mu, \partial_\mu A_\nu] + [A^\mu, \partial_\nu A_\mu] - [A^\mu, [A_\mu, A_\nu]], \\
\square F_{\mu\nu} &= -2[A^\lambda, \partial_\lambda F_{\mu\nu}] + 2[\partial_\nu A^\lambda, \partial_\lambda A_\mu] - 2[\partial_\mu A^\lambda, \partial_\lambda A_\nu] \\
&\quad + 2[\partial_\lambda A_\mu, \partial_\nu A_\nu] + 2[\partial_\mu A_\lambda, \partial_\nu A_\lambda] - [A^\lambda, [A_\lambda, F_{\mu\nu}]] \\
&\quad + 2[F_{\lambda\mu}, [A^\lambda, A_\nu]] - 2[F_{\lambda\nu}, [A^\lambda, A_\mu]] - 2[[A^\lambda, A_\mu], [A_\lambda, A_\nu]],
\end{align}

(1.1)

Let $G$ be a compact Lie group and $\mathfrak{g}$ its Lie algebra. The gauge potential $A$ is $\mathfrak{g}$-valued 1-form and the curvature 2-form $F$ is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

Here $[\cdot, \cdot]$ is the matrix commutator. From now on, we assume $G = SU(n, \mathbb{C})$ (the group of unitary matrices of determinant one) and $\mathfrak{g} = \mathfrak{su}(n, \mathbb{C})$ (the algebra of trace-free skew-Hermitian matrices). We denote the initial data for $(A, F)$ at $t = 0$ by

\begin{align}
A(0) &= a, \quad \partial_t A(0) = \dot{a}, \quad F(0) = f, \quad \partial_t F(0) = \dot{f}.
\end{align}

(1.2)

We note that the initial data $f$ for $F$ is completely determined by $(a, \dot{a})$:

\begin{align}
\begin{aligned}
f_{ij} &= \partial_i a_j - \partial_j a_i + [a_i, a_j], \\
of_{0i} &= \dot{a}_i - \partial_i a_0 + [a_0, a_i], \\
\dot{f}_{ij} &= \partial_i \dot{a}_j - \partial_j \dot{a}_i + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\
\dot{f}_{0i} &= \partial^i f_{ji} + [a^i, f_{ai}].
\end{aligned}
\end{align}

(1.3)

and we also have

\begin{align}
\begin{aligned}
\dot{a}_0 &= \partial^i a_i, \\
\partial^i f_{0i} &= [a^i, \dot{a}_i].
\end{aligned}
\end{align}

(1.4)

In [3], the present author studied the local well-posedness of (1.1).

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Theorem 1.1 (Theorem 1.1 of [3]). Let \( \sigma \geq 1 \). Suppose that given initial data \((a, \dot{a}, f, \dot{f})\) which satisfy

\[
\langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}} \times \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}} \times \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}} \times \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{3}{2}},
\]

where \( \langle \Omega \rangle^{\sigma} \) is the \( \sigma \)-angular derivative. Then there exists local existence time \( T > 0 \) depending on the initial data norm such that there exist solutions

\[
\begin{align*}
A &\in C([-T, T]; \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}}) \cap C^{1}((-T, T); \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}}), \\
F &\in C([-T, T]; \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}}) \cap C^{1}((-T, T); \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{3}{2}}),
\end{align*}
\]

of (1.1), respectively.

In [3], we revealed the null structure in \([A^\mu, \partial_\mu A_\nu]\) and \([A^\mu, \partial_\nu A_\mu]\). The first bilinear term turns out to be the \( Q \)-type null form and hence the treatment is not difficult. On the other hand, the second bilinear term is problematic.

Lemma 1.2 (Lemma 3.5 of [3]). In the Lorenz gauge \( \partial^\mu A_\mu = 0 \), we have the following identity:

\[
[A^\lambda, \partial_\mu A_\lambda] = \sum_{j=1}^{3} \Gamma_j^\mu(A, \partial A, F, \partial F),
\]

where

\[
\begin{align*}
\Gamma_1^\mu(A, \partial A, F, \partial F) &= -[A_0, \partial_\mu A_0] + [D^{-1} R_j(\partial_\mu A_0), D^{-1} R^j \partial_l (\partial_\mu A_0)], \\
\Gamma_2^\mu(A, \partial A, F, \partial F) &= \frac{1}{2} \epsilon^{ijk} \epsilon_{klm} (Q_{ij} [D^{-1} R^n A_n, D^{-1} R^l \partial_\mu A_m] \\
&\quad + Q_{ij} [D^{-1} R^n \partial_\mu A_n, D^{-1} R^l A_m]), \\
\Gamma_3^\mu(A, \partial A, F, \partial F) &= [D^{-2} \nabla \times F, D^{-2} \nabla \times \partial_\mu F] \\
&\quad - [D^{-2} \nabla \times F, D^{-2} \partial_\mu \nabla \times (A \times A)] \\
&\quad - [D^{-2} \nabla \times (A \times A), D^{-2} \nabla \times \partial_\mu F] \\
&\quad + [D^{-2} \nabla \times (A \times A), D^{-2} \partial_\mu \nabla \times (A \times A)].
\end{align*}
\]

Here \( F = (F_{23}, F_{31}, F_{12}) \).

The \( \Gamma^1, \Gamma^2 \) are essentially \( Q \)-type null forms. From now on, we focus on the \( \Gamma^3 \). In [3], we observe that for initial data \((a, f) \in H^s \times H^r\), the regularity we can attain is at most \( r = -\frac{1}{4} \). More explicitly, if we only use bilinear estimates, then we have the relation between \( s \) and \( r \): \( s - 1 \leq 2r \), which is significantly far from \( s - r = 1 \). (See [3] Remark 6.1 for details.) From this observation, we expect the Picard’s iteration to fail if \( s - 1 > 2r \). Indeed, we have the following theorem:

Theorem 1.3. For \( s, r \in \mathbb{R} \), suppose that \( s - 1 > 2r \). Let \( T > 0 \) be given. Then the flow map of \((a, f) \mapsto (A(t), F(t))\) from \( H^s \times H^r \) to \( C([-T, T]; H^s \times H^r) \) cannot be \( C^2 \) at the origin.

This shows the threshold for the contraction mapping principle and reinforce our previous study [3]. To prove Theorem 1.3, we give explicit counterexample which is Knapp type and follow the argument by [2, 4]. We also refer the readers to [1, 5] for study on regularity of flow maps for other dispersive equations. In the rest of the paper, we shall introduce the decomposition of d’Alembertian and then give the proof of Theorem 1.3.
Notations. As usual different positive constants dependent on only dimension is denoted by the same letter $C$, if not specified. $A \lesssim B$ and $A \gtrsim B$ means that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

The spatial Fourier transform on $\mathbb{R}^3$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx,$$

where $\xi \in \mathbb{R}^3$. Also we denote $\mathcal{F}_x(u) = \hat{u}$.

We put $D := |\nabla|$ whose symbol is $|\xi|$.

2. Preliminaries

2.1. Decomposition of d’Alembertian. Given nonlinear wave equation $\Box u = G(u, \partial u)$ with initial data $(u(0), \partial_t u(0))$, we consider the following transform: $(u, \partial_t u) \rightarrow (u_+, u_-)$, where

$$u_\pm(t) = \frac{1}{2} \left( u \pm \frac{1}{iD} \partial_t u \right).$$

Then given nonlinear wave equation is rewritten as

$$(io \partial_t \pm D)u_\pm = \mp \frac{1}{2D} \check{G}(u_+, u_-),$$

where $\check{G}(u_+, u_-) = G(u, \partial u)$. The Duhamel integral is given by

$$(2.1) \quad u_\pm(t) = u_\pm^\text{hom} \pm \int_0^t e^{\mp i(t-t')D} \frac{1}{2iD} \check{G}(u_+, u_-)(t') \, dt',$$

where

$$(2.2) \quad u_\pm^\text{hom}(t) = \frac{1}{2} e^{\mp iuD} \left( u(0) \pm \frac{1}{iD} \partial_t u(0) \right).$$

3. The failure of smoothness: Proof of Theorem 1.3

We prove Theorem 1.3 by contradiction. We assume $A_0(t = 0) = 0$, $A_j(t = 0) = \delta a_j$, and $\partial_t A_j(t = 0) = 0$, where $0 < \delta \ll 1$. We shall denote the local solutions to $\Box \partial_\mu$ by $(A_\mu(\delta, t), F_{\mu\nu}(\delta, t))$.

3.1. Set up. By (2.2) and (1.3), we have

$$\sum_{\pm} F_{\mu, 0}^\text{hom} = \frac{1}{2} \sum_\pm e^{\mp iuD} \left( f_{ij} \pm \frac{1}{iD} f_{ij} \right) = \frac{1}{2} \sum_\pm e^{\mp iuD} \left( \delta \partial_i a_j - \delta \partial_j a_i + \delta^2 [a_i, a_j] \right),$$

Now we write

$$\partial_\mu^3 \Gamma_\mu^3 (A, \partial A, F, \partial F)|_{\delta = 0} = 2[D^{-2} \nabla \times \partial_\delta F|_{\delta = 0}, D^{-2} \nabla \times \partial_\mu \partial_\delta F|_{\delta = 0}]$$

since the derivative of cubic and quartic term in $\Gamma_\mu^3$ vanish at $\delta = 0$. Thus we recall (2.1) to get (we ignore the first bilinear term $[A^\mu, \mu A_\nu]$)

$$\partial_\mu^3 A_\nu|_{\delta = 0} = 2 \sum_{\pm 1, \pm 2, \pm 3} \int_0^t e^{\mp i(t-t')D} (2iD)^{-1} [D^{-2} \nabla \times \partial_\delta F_{\pm 2}(t')|_{\delta = 0}, D^{-2} \nabla \times \partial_\nu \partial_\delta F_{\pm 3}(t')|_{\delta = 0}] \, dt'.$$

Hence if the flow is $C^2$ in $H^s \times H^r$ within a local existence time interval $[0, T]$, we have the following estimates:

$$\sup_{0 \leq t \leq T} \| \partial_\mu^3 A_\nu(t)|_{\delta = 0} \|_{H^s} \lesssim \| F \|^2_{H^r}.$$
We note that
\[ \partial_b F_{ij}(t)|_{\delta=0} = \frac{1}{2} \sum e^{\pm i t D}(\partial_i a_j - \partial_j a_i). \]
We further assume that \( a_3 = 0 \) and \( \partial_3 a_2 = 0 \) for simplicity. That is, \( \tilde{a}_2(\xi) = 0 \) for all \( \xi_3 \neq 0 \), where \( \xi = (\xi_1, \xi_2, \xi_3) \). Then
\[ \nabla \times \partial_b F(t)|_{\delta=0} = \frac{1}{2} \sum e^{\pm i t D}(\partial_1 \partial_2 a_2 - \partial_2^2 a_1, -\partial_1^2 a_2 + \partial_1 \partial_2 a_1, 0) + \frac{1}{2} \sum e^{\pm i t D}(-\partial_3^2 a_1, 0, \partial_3 \partial_3 a_1). \]
We also assume \( a_1 = a_1^1 T^1, a_2 = a_2^2 T^2 \), where \( T^a \) are infinitesimal generators of \( \mathfrak{su}(n, \mathbb{C}) \), \( a = 1, 2, \ldots, n^2 - 1 \). From now on, we simply put \( a_1^1 = a_1 \) and \( a_2^2 = a_2 \). We let \( \lambda \gg 1 \). Define \( W_\lambda = \{ \xi = (\xi_1, \xi_2, \xi_3) : |\xi_1 - \lambda| \leq 10^{-6}, |\xi_2 - \xi_3| \leq 10^{-6} \lambda^{\frac{2}{3}} \} \) and \( c W_\lambda = \{ c \xi : \xi \in W_\lambda \} \) for \( c \in \mathbb{R} \). We also define \( W_\lambda' = \{ \xi = (\xi_1, \xi_2, 0) : |\xi_1 - \lambda| \leq 10^{-6}, |\xi_2| \leq 10^{-6} \lambda^{\frac{2}{3}} \} \). Now we take
\[ (3.4) \quad \tilde{a}_1(\xi) = \chi_{2 W_\lambda}(\xi), \quad \tilde{a}_2(\xi) = \chi_{- W_\lambda'}(\xi). \]
We note that the integration of \( \tilde{a}_2 \) is 0, whereas the integration of \( \tilde{a}_1 \tilde{a}_2 \) is nonzero.

3.2. Failure of \( \hat{\mathcal{R}} \) for \( s - 1 > 2r \). Since \( \nabla \times \mathbf{F}^\text{hom} \) consists of so many integrations, at first glance it seems tremendous. However, by the support condition of \( \mathbf{F}^\text{hom} \) it is no harm to ignore the most of integrations. In fact, for \( \xi \in W_\lambda \), we have \( \xi_1 \approx \lambda \) whereas \( \xi_2, \xi_3 \approx \lambda^{\frac{2}{3}} \). Then we can write \( \partial_b^2 A_\nu = \Lambda_1^2 + \Lambda_2^2 + \cdots \), where the leading terms \( \Lambda_1^2, \Lambda_2^2 \) are given by
\[ (3.5) \quad \Lambda_1^2 = \frac{1}{2} \sum_{\pm_1, \pm_2, \pm_3} \int_0^t e^{\pm i t L(t') D} (2i D)^{-1} (D^{-2} e^{\pm i t' D} \partial_1^2 a_2 D^{-2} \partial_2 e^{\pm i t' D} \partial_1 \partial_2 a_2) \partial_{t'} [T^1, T^2], \]
\[ (3.6) \quad \Lambda_2^2 = \frac{1}{2} \sum_{\pm_1, \pm_2, \pm_3} \int_0^t e^{\pm i t L(t') D} (2i D)^{-1} (D^{-2} e^{\pm i t' D} \partial_1^2 a_2 D^{-2} \partial_2 e^{\pm i t' D} \partial_1 \partial_3 a_2) \partial_{t'} [T^1, T^2]. \]
Thus it suffices to consider the lower bound of \( \mathcal{F}_\nu[\Lambda_1^2, \Lambda_2^2] \). We consider \( \nu = 1 \). First,
\[ \mathcal{F}_\nu[\Lambda_1^2](t, \xi) = \frac{1}{4i} \sum_{\pm_1, \pm_2, \pm_3} e^{\pm i t L[\xi]} \int_{m_{123}} \frac{1}{|\xi||\xi - \eta|^2|\eta|^2} \left[ (\xi_1 - \eta_1)^2 \eta_1^2 \eta_2 \chi_w^\lambda (\xi - \eta) \chi_{2 W_\lambda}(\eta) - (\xi_1 - \eta_1)(\xi_2 - \eta_2) \eta_1^2 \chi_{2 W_\lambda}(\xi - \eta) \chi_{- W_\lambda'}(\eta) \right] d\eta, \]
where
\[ m_{123}(t, \xi, \eta) = \frac{e^{\pm i t \omega_{123}} - 1}{i \omega_{123}}, \quad \omega_{123} = \pm_1 |\xi| - \pm_2 |\xi - \eta| - \pm_3 |\eta|. \]
The resonance case \( \mathcal{R} \) occurs only when \( \pm_1 = \pm_2 = \pm_3 \) for which \( |\omega_{123}| \lesssim \lambda^{\frac{2}{3}} \). For other cases we have \( |\omega_{123}| \approx \lambda \). We put \( t = \epsilon \lambda^{-\frac{2}{3}} \), for some \( 0 < \epsilon \ll 1 \). We also have
\[ \mathcal{F}_\nu[\Lambda_1^2](t, \xi) = \frac{1}{4i} \sum_{\pm_1, \pm_2, \pm_3} e^{\pm i t L[\xi]} \int_{m_{123}} \frac{1}{|\xi||\xi - \eta|^2|\eta|^2} \left[ (\xi_1 - \eta_1)^2 \eta_1^2 \eta_3 \chi_{- W_\lambda'}(\xi - \eta) \chi_{2 W_\lambda}(\eta) - (\xi_1 - \eta_1)(\xi_3 - \eta_3) \eta_1^2 \chi_{2 W_\lambda}(\xi - \eta) \chi_{- W_\lambda'}(\eta) \right] d\eta. \]
For $\eta \in -W'$, $\eta_1$ is negative and hence $[\cdots]$ in the integrand is nonnegative. Now we write
\[
\mathcal{F}_\tau[A_1^1](t, \xi) \geq \frac{1}{4} \left| \sum_{\pm, j \in \mathbb{R}} e^{\mp i \tau |\xi|} \int \lambda^{-1} \lambda^{\frac{1}{2}} (1 + O_{123}(\epsilon)) \chi_{W_\lambda}(\xi - \eta) \chi_{-W_\lambda}(\eta) \, d\eta \right| 
\]
\[
- \frac{1}{4} \left| \sum_{\pm, j \in \mathbb{R}} \int \lambda^{-1} \lambda^{\frac{1}{2}} \chi_{W_\lambda}(\xi - \eta) \chi_{-W_\lambda}(\eta) \, d\eta \right| 
\]
\[
:= I + II.
\]
Here, for $\xi - \eta \in 2W_\lambda$, $\eta \in W_\lambda$, and then for some $0 < \rho < 1$ we have $(1 - \rho)\lambda < |\xi| < (1 + \rho)\lambda$. Clearly, we have $|II| \lesssim \lambda^2$. For $I$, we see that
\[
I \geq \frac{1}{4} |\cos(t|\xi|)| \int t^{1 - \frac{1}{2}} \chi_{W_\lambda}(\xi - \eta) \chi_{-W_\lambda}(\eta) \, d\eta - \frac{1}{2} \int t O_{123}(\epsilon) \lambda^{-\frac{1}{2}} \chi_{W_\lambda}(\xi - \eta) \chi_{-W_\lambda}(\eta) \, d\eta
\]
\[
:= I_1 + I_2.
\]
Then we have $|I_2| \lesssim \epsilon^2 \lambda$. Take $\lambda = \lambda(\epsilon, \rho, k)$ such that for each integer $k$,
\[
\frac{2k\pi - \epsilon}{\epsilon(1 - \rho)} < \lambda^2 < \frac{2k\pi + \epsilon}{\epsilon(1 + \rho)}.
\]
Then we have $|\cos(t|\xi|)| \approx 1$ and hence $|I_1| \gtrsim \epsilon \lambda$. Thus we get $|\mathcal{F}_\tau[A_1^1](t, \xi)| \gtrsim \epsilon \lambda$ for $\xi \in W_\lambda$. The treatment of $A_2^1$ is very similar. Since the integrand of $A_1^2$ and $A_1^1$ is both nonnegative, we obtain
\[
|\mathcal{F}_\tau[A_1^1 + A_1^2](t, \xi)| \gtrsim \epsilon \lambda.
\]
On the other hand,
\[
\|F\|_{H^r} \lesssim \|\partial_1 a_2\|_{H^r} + \|\partial_2 a_1\|_{H^r} + \|a_1 a_2\|_{H^r} \lesssim 0 + \lambda^{r + \frac{3}{2}} + \lambda^{r + 1}.
\]
Thus for $C^2$ smoothness to hold, we must have $\epsilon \lambda^{r + 2} \lesssim \lambda^{2r + 2} + \lambda^{2r + 3}$, and hence $s - 1 \leq 2r$.

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