Detecting deformed commutators with exceptional points in optomechanical sensors

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Abstract

Models of quantum gravity imply a modification of the canonical position-momentum commutation relations. In this paper, working with a binary mechanical system, we examine the effect of quantum gravity on the exceptional points of the system. On the one side, we find that the exceedingly weak effect of quantum gravity can be sensed via pushing the system towards a second-order exceptional point, where the spectra of the non-Hermitian system exhibits non-analytic and even discontinuous behavior. On the other side, the gravity perturbation will affect the sensitivity of the system to deposition mass. In order to further enhance the sensitivity of the system to quantum gravity, we extend the system to the other one which has a third-order exceptional point. Our work provides a feasible way to use exceptional points as a new tool to explore the effect of quantum gravity.

1. Introduction

Cavity optomechanics [1], exploring the interaction between light and mechanical systems, has made a profound impact in recent years due to its wide variety of applications including optomechanical sensors. Optomechanical sensors have achieved ultrasensitive performance in gravitational wave detection [2, 3], high-precision measurements [4], detection for mass [5], acceleration [6], displacement [7], and force [8–11]. For practical applications, the optomechanical system is unavoidably coupled with its surroundings, leading to a non-Hermitian optomechanical system. Several earlier studies have shown that non-Hermitian spectral degeneracies, also known as exceptional points (EPs) [12–16], governs the dynamics of parity-time (PT) symmetric system [17] subject to environment. In contrast to level degeneracy points in Hermitian systems, the EP is associated with level coalescence, in which the eigenenergies and the corresponding eigenvectors simultaneously coalesce [18, 19]. Besides, the intriguing phenomena of EPs in unidirectional invisibility [20], topological chirality [21] and low-threshold lasers [22, 23] have been predicted.

In recent years, sensitivity enhancement of the sensor operating at EPs has been explored both theoretically [13, 24, 25] and experimentally [15, 26–28] in a number of systems including nanoparticle detector [13, 15], mass sensor [24], and gyroscope [25–27, 29]. These studies have shown that if a second-order exceptional point (EP2) where the coalescence of two levels occurs is subjected to a perturbation of strength ε, the frequency splitting (the energy spacing of the two levels) is typically proportional to the square root of the perturbation strength ε. This is the so-called complex square-root topology. Moreover, the splitting is significantly enhanced for sufficiently small perturbation. This suggests that the use of EP can enhance the sensitivity of a quantum sensor.

In standard quantum mechanics, on the basis of Heisenberg uncertainty principle \( \Delta x \Delta p \geq \frac{\hbar}{2} \) [30], the position \( x \) and the momentum \( p \) of an particle cannot be simultaneously measured to arbitrary precision, however, the uncertainty of \( x \) can reach zero in case \( \Delta p \) approaches infinity. This is not the case when the quantum gravity is taken into account. It has been suggested that the uncertainty relation should be...
modified when gravitational effects have been taken into consideration [31]. Such generalized uncertainty principle (GUP) is found in various approaches to quantum gravity, such as the finite bandwidth approach to quantum gravity [32, 33], string theory [34–38], the theory of double special relativity [40, 41], relative locality [42] and black holes [43].

The simplest and most commonly used generalized Heisenberg uncertainty principle that counts the gravitational effects is \( \Delta x \Delta p \geq \frac{\hbar}{2}(1 + \mu \Delta p^2) \) [37–39]. Here \( \mu = \frac{\hbar}{M_p c^3} = \frac{\hbar c^2}{p_0} \), \( \beta_0 \) is a dimensionless parameter, \( M_p \) is Planck mass, \( M_p c^2 \) is Planck energy and \( L_p \) is Plank length. This inequality means that \( \Delta x \geq L_p \sqrt{\frac{\hbar}{\mu}} \). So if \( \beta_1 = 1 \) [44], the minimal length is equal to the Planck length \( (L_p) \) beyond which the concepts of time and space will lose their meaning. The GUP has been extensively explored in various fields, including high energy physics, cosmology and black holes [45]. Due to experiments that can test minimal length scale directly require energies much higher than that currently available, most of the work has been devoted to find indirect evidences of quantum gravity in high energy particle collisions and astronomical observations [46, 47].

In this paper, we theoretically propose the other sensing scheme to explore the effects of quantum gravity via GUP. We will consider a binary and a ternary mechanical system separately within an optomechanical configuration. Controlling the gain and loss of the mechanical oscillators and driving the two cavities with blue and red detuned lasers as well as manipulating the strength of the electromagnetic field \( (\alpha^m) \), we can set the system into a self-sustained regime for the mechanical oscillations. Since the gravity always exists, we can first use a mechanical resonator with a very small mass, such that the effect of gravity can be ignored and we can get an EP. Then we increase the mass such that the effect mentioned above cannot be ignored, while keeping the other parameters unchanged. When the gravitational effect increases by several orders of magnitude, the supermodes of the system are obviously shifted away from the EP2. The frequency splitting induced by gravitational effects can be read out in the mechanical spectrum. This result has been further extended to a third-order exceptional point (EP3) by taking a more complicated ternary mechanical system into account. Compared with the scheme utilizing EP2, optomechanical sensor based on EP3 performs better. The physics of this EP-based sensor is that the eigenvalues of non-Hermitian Hamiltonian may exhibit non-analytic and even discontinuous behavior, which in principle enables an unlimited spectral sensitivity.

The rest of this paper is organized as follows. In section 2, we introduce the physical model and derive a \( \beta_0 \) set of equation for the dynamics of our system. In section 3, we study the sensitivity of the system to gravity perturbation at EP2 and analyze the influence of gravitational effect on mass sensing. In section 4, we extend the study on the sensitivity of the system to the gravity perturbation to EP3. In section 5, we discuss experimental feasibility of the proposed sensing scheme and analyze the limitation of the proposed quantum gravity sensor. In parallel, we derive the bound on the modification parameter \( \beta_0 \). Finally, the conclusions are drawn in section 6. In appendix A, we present details of derivation for the effective Hamiltonian.

2. General framework

We start by briefly recalling the description of harmonic oscillator with mass \( m \) when the effect of gravity is taken into consideration. Afterwards we would apply this result to our model, in which the mechanical resonator is modeled as a harmonic oscillator.

2.1. Deformed harmonic oscillations under quantum gravity

From the aspect of commutation relation, the gravity would modify the relation leading to the GUP given in reference [39],

\[
[x, p] = i\hbar(1 + \mu p^2) .
\]  

(1)

Here, \( \mu = \frac{\hbar}{(M_p c^3)} = \frac{\hbar c^2}{p_0} \). Define [44]

\[
p = \left(1 + \frac{1}{2} \mu \tilde{p}^2\right) \tilde{p},
\]  

(2)

where \( x, \tilde{p} \) satisfying the (non deformed) canonical commutation relations \([x, \tilde{p}] = i\hbar\). It can be seen that equation (2) is written up to the first order in \( \mu \) (the terms of order \( \mu^2 \) and higher are neglected). For a
harmonic oscillator with frequency $\omega$ and mass $m$, we assume that the Hamiltonian takes
$$H = \frac{1}{2} m \omega^2 x^2 + \frac{p^2}{2m} \quad [48].$$
In terms of $\tilde{p}$, the Hamiltonian of harmonic oscillator can be rewritten as
$$H = \frac{1}{2} m \omega^2 x^2 + \frac{\tilde{p}^2}{2m} \quad [49].$$
Introducing canonical creation and annihilation operators,
$$b^\dagger = \sqrt{\frac{m \omega}{2\hbar}} (x - \frac{i \tilde{p}}{m \omega}), \quad b = \sqrt{\frac{m \omega}{2\hbar}} (x + \frac{i \tilde{p}}{m \omega}) \quad [50],$$
and rewriting the Hamiltonian (3) in terms of $b$, we can get
$$H = \hbar \omega \left( b^\dagger b + \frac{1}{2} \right) + \frac{1}{12} \hbar^2 \omega^2 m \mu (b - b^\dagger)^4. \quad [51]$$
The first term represents the free Hamiltonian of the harmonic oscillator. The second term comes from the gravitational effects.

2.2. Modeling and dynamical equations
We consider two coupled identical mechanical resonators with optomechanically induced gain and loss. Each of the resonators is characterized by frequency $\omega_j$, damping rate $\gamma_m$ and coupling strength $J$. The schematic diagram is presented in figure 1. In this configuration, we assume that only the mechanical commutation relation are modified, while the optical commutation relation remains unchanged, i.e. $[a, a^\dagger] = 1$. This restriction can be justified as those are expected to be negligible compared with the deformations of the massive mechanical modes [49]. The total Hamiltonian of the whole system can be written as ($\hbar = 1$)
$$H = H_\Omega + H_i + H_d + H_g, \quad [52]$$
where,
$$H_\Omega = \sum_{j=1,2} \omega_{\phi, j} a^\dagger_j a_j + \omega_j b^\dagger_j b_j,$n$$H_i = \sum_{j=1,2} \left\{ -g a^\dagger_j a_j (b_j^\dagger + b_j) \right\} - J (b_1 b_1^\dagger + b_2 b_2^\dagger), \quad [53]$$
$$H_d = \sum_{j=1,2} i E (a^\dagger_j e^{-i \omega_j t} - a_j e^{i \omega_j t}), \quad [54]$$
$$H_g = \sum_{j=1,2} \frac{1}{12} \omega_j^2 m \mu_j (b_j - b_j^\dagger)^4. \quad [55]$$
In this expression, $H_\Omega$ represents the sum of free Hamiltonian of the optomechanical system, $a^\dagger_j (b_j^\dagger)$ and $a_j (b_j)$ are the creation and annihilation operators of the $j$th cavity (mechanical resonator) ($j = 1, 2$). The frequencies of the cavities and mechanical resonators are $\omega_{\phi, j}$ and $\omega_j$, respectively. $H_i$ describes the interaction Hamiltonian of the configuration. The first term represents the coupling of the cavities to the corresponding mechanical resonators with optomechanical coupling strength $g$. The second term describes the coupling between the two mechanical resonators with coupling strength $J$, where the rotating wave approximation was applied. $H_d$ indicates that the two cavities are driven by external fields with amplitude $E$ and frequency $\omega_p$. $H_g$ describes the gravitational effects in mechanical resonators. The effective mass of the $j$th mechanical mode is $m_j$. In the frame rotating at the input laser frequency $\omega_p$, the Hamiltonian of the system reads,
$$H = \sum_{j=1,2} \left\{ -\Delta_j a^\dagger_j a_j + \omega_j b^\dagger_j b_j - g a^\dagger_j a_j (b_j^\dagger + b_j) + \frac{1}{12} \omega_j^2 m \mu_j (b_j - b_j^\dagger)^4 + i E (a^\dagger_j - a_j) \right\} - J (b_1 b_1^\dagger + b_2 b_2^\dagger), \quad [56]$$
where, $\Delta_j = \omega_p - \omega_{aj}$ represents the detuning of the driving field with respect to the cavity. In the limit of large photon and phonon numbers, we replace the quantum operators with their mean values, i.e. $\alpha_j = \langle a \rangle$ and $\beta_j = \langle b \rangle$. By introducing dissipation terms, the evolution of the system operators is obtained as follows

\begin{equation}
\begin{aligned}
\frac{d\alpha_j}{dt} &= \left[i(\Delta_j + g(\beta_j^2 + \beta_j)) - \frac{\kappa}{2}\right] \alpha_j + \sqrt{\kappa}\alpha_j^{in}, \\
\frac{d\beta_j}{dt} &= -\left(i\omega_j + \frac{\gamma_m}{2}\right) \beta_j + i\beta_{3,j} + ig_\alpha^\tau \alpha_j + \frac{1}{3}i\gamma_m^\mu j((\beta_j - \beta_j^*)^3),
\end{aligned}
\end{equation}

where $\kappa$ and $\gamma_m$ are the intrinsic damping rates of the cavities and mechanical resonators, respectively.

$E = \sqrt{\kappa}\alpha_j^{in}$ is the amplitude of the driving field, where $\alpha_j^{in} = \sqrt{\gamma_m}p_m$ characterizes the input field driving the cavity. For the sake of simplicity, we assume the two mechanical resonators identical, this means $\omega_j = \omega_m$, $n_j = m$, and $\mu_j = \mu$. We apply the input lasers with the same power ($p_m$) to drive the two mechanical resonators, i.e. $\alpha_j^{in} = \alpha^{in}$. Throughout the work, the parameters satisfy the following condition, $\gamma_m^\mu, g \ll \kappa \ll \omega_m$, similar to those chosen in reference [50, 51]. Under this hierarchy, the amplitude and phase of the mechanical resonators slowly evolving on the time scale of the cavity dynamics.

We will pay our attention to the steady state of the mechanical resonators. In this regime, $\beta_j(t) = \beta_j^0 + B_j e^{i\theta} e^{-i\omega_j t}$ [52, 53], where $\beta_j^0$ is the center of the mechanical oscillations and amplitude $B_j$ can be regarded as a slowly evolving function of time, so we neglected the time dependence of $B_j$ (see figure 2(a)). In the limit-cycle states, the amplitude of other frequency components of the mechanical resonator is very weak compared with the term oscillating at $\omega_l$ (see figure 2(b)). Throughout this paper, we set $\theta = 0$. In parallel, we ignored all terms in mechanical dynamics except for the constant one and the term oscillating at $\omega_l$.

Using this analytic approximation, we solve the equation for $\alpha_j$ assuming a fixed mechanical amplitude and then substitute the result into the equation for $\beta_j$, resulting in the following set of equations of motion.
where, \( \Theta_j = \mu_m \omega_j^2 B_j^2 \) \((j = 1, 2)\). \( \omega_{\text{eff}}^{(j)} = \omega_j + \Omega_j \) and \( \gamma_{\text{eff}}^{(j)} = \gamma_m + \Gamma_j \) \((j = 1, 2)\) represent the effective frequency and damping of the \( j \)th mechanical oscillator \((j = 1, 2)\), respectively. The modal field evolution in this configuration obeys \( \frac{d}{dt} \Phi = H_{\text{eff}} \psi \), where \( \Phi = (\beta_1, \beta_2, \beta_1^*, \beta_2^*)^T \) is the state vector and \( t \) represents time. \( H_{\text{eff}} \) is the associated 4 \times 4 non-Hermitian Hamiltonian (see more details in appendix A):

\[
H_{\text{eff}} = \begin{pmatrix}
\omega_{\text{eff}}^{(1)} - \frac{i \gamma_{\text{eff}}^{(1)}}{2} & -J & -\Theta_1 & 0 \\
-J & \omega_{\text{eff}}^{(2)} - \frac{i \gamma_{\text{eff}}^{(2)}}{2} & 0 & -\Theta_2 \\
-\Theta_1 & 0 & -\omega_{\text{eff}}^{(1)} - \frac{i \gamma_{\text{eff}}^{(1)}}{2} & \Theta_1 \\
0 & \Theta_2 & J & -\omega_{\text{eff}}^{(2)} - \frac{i \gamma_{\text{eff}}^{(2)}}{2} - \Theta_2
\end{pmatrix}.
\]

Here, \( \Omega_j(\Gamma_j) \) represents the optical spring effect (the optomechanical damping rate). These quantities are given as (see more details in appendix A)

\[
\Omega_j = -\frac{2\kappa(\alpha_{\text{in}})^2}{\omega_\epsilon} \text{Re} \left( \sum_n I_{n+1}(-\epsilon_j)I_n(-\epsilon_j) \right),
\]

and

\[
\Gamma_j = \frac{2(\kappa \alpha_{\text{in}})^2}{\epsilon_j} \sum_n \left| I_{n+1}(-\epsilon_j)I_n(-\epsilon_j) \right|^2.
\]

Here \( \epsilon_j = 2gB_j/\omega_j \), \( K_n^0 = i(n\omega_j - \Delta_j^0) + \frac{\kappa}{2} \) and \( J_n \) is the Bessel function of the first kind. We can first use a mechanical resonator with a very small mass, then we can ignore gravitational effect. The eigenvalues of the above effective Hamiltonian are given by

\[
\lambda_\pm = \omega_1 \pm \frac{i}{2} \left( \gamma_{\text{eff}}^{(1)} + \gamma_{\text{eff}}^{(2)} \right) \pm \frac{1}{2} \delta,
\]

where,

\[
\delta = \sqrt{16\delta^2 + \left( 2(\omega_{\text{eff}}^{(1)} - \omega_{\text{eff}}^{(2)}) + i(\gamma_{\text{eff}}^{(2)} - \gamma_{\text{eff}}^{(1)}) \right)^2}.
\]

Here, \( \omega_1 = \frac{\omega_{\text{eff}}^{(1)} + \omega_{\text{eff}}^{(2)}}{2} \). Replace the conventional vibrational modes, we now have new mechanical modes, which can be called as the mechanical supermodes. The effective frequencies and spectral linewidths of the system are defined as the real \((\omega)\) and imaginary \((\gamma)\) parts of eigenvalues, respectively. The solid lines in figures 3(a) and (b) show the real and imaginary parts of the eigenvalues vs the driving strength \( \alpha_{\text{in}} \) corresponding to the case where the gravitational effect can be ignored. At the specific point, both these pairs of effective frequencies and effective dampings of the system coalesce. It is evident that for a critical driving strength \( \alpha_{\text{in}} \) the pairs of eigenvalues merge at \( 4J = \gamma_{\text{eff}}^{(2)} - \gamma_{\text{eff}}^{(1)} \).

3. Sensitivity at the second-order exceptional point

3.1. Sensitivity of a system at the second-order exceptional point to the gravitational effect

We now consider the case where the gravitational effect increases by several orders of magnitude. We numerically solve the eigenvalues of this mechanical effective Hamiltonian and show the results in figures 3(c) and (d). The effective Hamiltonian has four eigenvalues forming two pairs, one pair is due to the appearance of \( \beta_j^* \) in the dynamics. Throughout this paper, we only study the eigenvalue corresponding to \( \beta_j^* \).

As shown in figures 3(c) and (d), we see that the splitting of effective frequency (real part of the eigenvalue) and linewidth (imaginary part of the eigenvalue) increases as the mass of the mechanical resonators increases. A typical detection strategy is to observe the associated response of the system to the
Figure 3. (a) The real and (b) the imaginary parts of the eigenvalues vs the driving strength $\alpha$ corresponding to the case where the gravitational effect can be ignored. (c) The real and (d) the imaginary parts of the eigenvalues as a function of $\mu m$ around an EP2. The two eigenvalues of the effective Hamiltonian are marked with red solid and blue dashed lines. The other system parameters are the same as in figure 2. We only study the eigenvalue corresponding to $\beta$.

Figure 4. Sensitivity (red solid line) vs $\mu m$ around a second-order EP. The blue dashed line denotes the fitted curve with $\xi = 86.12\omega_m^{1/2}$. The parameters are the same as that in figure 2.

The perturbation of gravitational effects can shift the EP2, and thereby the degeneracy of the effective frequencies are released and cause the supermodes to split. The frequency splitting caused by gravitational effects can be fitted using

$$\Delta\omega \approx \xi (\mu m)^{1/2}.$$  

Here, $\xi$ is the fitting coefficient. Figure 4 shows $\Delta\omega$ as a function of the $\mu m$ near the EP2. The blue dashed lines represent the fitting result according to equation (17) with $\xi = 86.12\omega_m^{1/2}$. Therefore, it can be inferred that the mechanical frequency splitting in response to the $\mu m$ obeys the square root behavior. Due to the gravitational effect, usually the frequency splitting or the frequency shift. In this paper, in order to quantify the frequency splitting caused by the gravitational effect, we define the sensitivity as follows,

$$\Delta\omega = |\text{Re } \lambda_+ - \text{Re } \lambda_-|.$$  

(16)

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$$\Delta\omega \approx \xi (\mu m)^{1/2}.$$  

(17)
3.2. Sensitivity of the system to deposition mass at the second-order exceptional point with gravitational effect

In order to gain insight into the influence of gravitational effects on mass sensing, we assume that a small mass $\delta m$ has been deposited on the mechanical oscillator driven by the blue-detuned electromagnetic field, which would induce the frequency shift given in equation (14), i.e. replacing $\omega_2$ with $\omega_2 + \delta \omega$. For an ordinary mass sensor, the relation between the deposited mass $\delta m$ and the frequency shift $\delta \omega$ is given by [24, 54–56]

$$\frac{\delta \omega}{\omega_m} = \frac{\delta m}{2m}. \tag{18}$$

We can define the gap as

$$\chi_{\pm} = \lambda_{\pm}(m', \delta \omega) - \lambda_{\pm}(m \ll m', \delta \omega = 0). \tag{19}$$

Figure 5 shows that for mechanical frequency shift $\delta \omega = 10^{-2}\omega_m$, the larger the mass of the mechanical resonators, the larger the gap between effective frequencies. However, the gap between the effective dampings does not change significantly. Therefore, the mass of the mechanical resonators should be small enough to minimize the disturbance caused by gravitational effects.

4. Sensitivity of a system at the third-order exceptional point to the gravitational effect

Inspired by these results, we now extend this scheme to the third-order exceptional point (EP3) [57]. A possible configuration that supports a EP3 would be a system consisting of two cavities and three coupled mechanical oscillators where the two cavities are symmetrically driven by red- and blue-detuned lasers, and the corresponding mechanical resonators are coupled together (see figure 6).
From equation (21), one can write the following nonlinear equations of motion, the time dependence of the amplitudes (see figure 7(a)). Figure 7(b) shows the corresponding Fourier spectra. The inset of figure 7(b) shows limit cycle oscillations at $\beta_i(t)$ is still applicable. By the use of this formal solution, equation (22) can be further reduced.

Proceeding in a similar way, one can write the following the Hamiltonian of the system,

$$H = H_f + H_i + H_d + H_g,$$

with

$$H_f = \sum_{j=1,2} - \Delta a_j^\dagger a_j + \sum_{j=1,2,3} \omega_j b_j^\dagger b_j,$$

$$H_i = -g a_1^\dagger a_1 (b_1^\dagger + b_1) - g a_2^\dagger a_2 (b_2^\dagger + b_2) - J (b_1 b_2 + b_1^\dagger b_2^\dagger),$$

$$H_d = \sum_{j=1,2} i E (a_j^\dagger - a_j),$$

$$H_g = \sum_{j=1,2,3} \frac{1}{12} \omega_j^2 m_j \mu_j (b_j - b_j^\dagger)^4.$$ 

From equation (21), one can write the following nonlinear equations of motion,

$$\frac{d\alpha_j}{dt} = \left[ i (\Delta_j + g (\beta_j^2 + \beta_j^3)) - \frac{\kappa}{2} \right] \alpha_j + \sqrt{\kappa} \alpha_j^m,$$

$$\frac{d\beta_1}{dt} = -\left( \omega_1 + \frac{\gamma_1}{2} \right) \beta_1 + iJ \beta_2 + i \gamma_1 \alpha_1 + \Xi_1 (\beta_1^2 - \beta_1^4)^3,$$

$$\frac{d\beta_2}{dt} = -\left( \omega_2 + \frac{\gamma_2}{2} \right) \beta_2 + iJ \beta_1 + i \gamma_2 \alpha_2 + \Xi_2 (\beta_2 - \beta_2^4)^3,$$

$$\frac{d\beta_3}{dt} = -\left( \omega_3 + \frac{\gamma_3}{2} \right) \beta_3 + iJ \beta_2 + i \gamma_3 \alpha_3 + \Xi_3 (\beta_3 - \beta_3^4)^3.$$ 

Here, $\Xi_j = \frac{1}{4} i \mu_j \omega_j^3$ ($j = 1, 2, 3$), $\alpha_j = \langle a_j \rangle$ ($j = 1, 2$), and $\beta_j = \langle b_j \rangle$ ($j = 1, 2, 3$). For the convenience of discussion, we assume $\omega_j = \omega_m$ ($j = 1, 2, 3$).

In figure 7, we show the overall properties of the steady state solutions of the mechanical resonators. It can be seen that the amplitudes of the mechanical resonators change very slowly over time, so we neglected the time dependence of the amplitudes (see figure 7(a)). Figure 7(b) shows the corresponding Fourier spectra. The inset of figure 7(b) shows limit cycle oscillations at $\alpha^m = 158.1 \sqrt{\omega_m}$. So in this case, the formal solution for $\beta_i(t)$ is still applicable. By the use of this formal solution, equation (22) can be further reduced.
to

\[ \frac{d\beta_1}{dt} = - \left( i\omega_{\text{eff}}^{(1)} + \frac{\gamma_{\text{eff}}^{(1)}}{2} + i\Theta_1 \right) \beta_1 + i\beta_2 + i\Theta_1 \beta_2^*, \]

\[ \frac{d\beta_2}{dt} = - \left( i\omega_2 + \frac{\gamma_m}{2} + i\Theta_2 \right) \beta_2 + i\beta_1 + i\beta_3 + i\Theta_2 \beta_3^*, \]

\[ \frac{d\beta_3}{dt} = - \left( i\omega_{\text{eff}}^{(3)} + \frac{\gamma_{\text{eff}}^{(3)}}{2} + i\Theta_3 \right) \beta_3 + i\beta_2 + i\Theta_3 \beta_2^*. \]

Here \( \Theta_j = \mu m \omega_j^2 R_j^2 \) \((j = 1, 2, 3)\). The modal field evolution in this configuration obeys \( i \frac{d\psi}{dt} = H_{\text{eff}} \psi \), where \( \psi = (\beta_1, \beta_2, \beta_3, \beta_1^*, \beta_2^*, \beta_3^*)^T \) represents the modal state vector and \( t \) represents time. \( H_{\text{eff}} \) is the associated \( 6 \times 6 \) non-Hermitian Hamiltonian,

\[
\begin{pmatrix}
\omega_{\text{eff}}^{(1)} - \frac{\gamma_{\text{eff}}^{(1)}}{2} + \Theta_1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\gamma_{\text{eff}}^{(1)}}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{\gamma_{\text{eff}}^{(2)}}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\gamma_{\text{eff}}^{(3)}}{2} + \Theta_3 & 0 & 0 & -\Theta_3 & 0 \\
0 & 0 & -\omega_{\text{eff}}^{(3)} - \frac{\gamma_{\text{eff}}^{(3)}}{2} - \Theta_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\omega_{\text{eff}}^{(3)} - \frac{\gamma_{\text{eff}}^{(3)}}{2} - \Theta_3
\end{pmatrix}.
\]

(24)

It can be found that the effective Hamiltonian has six eigenvalues to form two groups, one group is due to the appearance of \( \beta_j^* \) in the dynamics.

This characteristic feature of the EP3 has been demonstrated in figures 8(a) and (b), where we show the dependence of the eigenvalues on driving strength \( \alpha_m \).

Now to take this discussion further to show how the system reacts around the EP3. The real (figure 8(c)) and imaginary parts (figure 8(d)) of the eigenvalues are plotted as a function of \( \mu m \) around a third-order exceptional point. The three eigenvalues of the effective Hamiltonian are marked with red solid, orange dashed, and blue dash-dotted lines. The other system parameters are the same as in figure 7. We only study the eigenvalue corresponding to \( \beta_j^* \).

\[ \Delta \omega \approx \varsigma (\mu m)^{1/3}. \]

(25)

Here \( \varsigma \) is the fitting coefficient. The blue dashed line represents the fitting results according to equation (25) with \( \varsigma = 2.933 \omega_m^{4/3} \), confirming thus that the mechanical frequency splitting in response to \( \mu m \) obeys the
Figure 9. Frequency splitting (red solid line) in the mechanical supermodes vs $\mu m$ near EP3. The blue dashed line denotes the fitted curve with $\delta = 2.933\omega_m^{3/2}$. The other parameters are the same as in figure 7.

cube root behavior. This indicates that it is feasible to further enhance the sensitivity by means of third-order exceptional point (EP3).

5. Experimental feasibility and ultimate limits of the sensing scheme

There are many types of optomechanical systems. For concreteness, we choose one of them, where the mechanical degree of freedom is a dielectric membrane placed inside a Fabry–Perot cavity [58]. Here we use two coupled Si beam, which possess the mass of $m = 5.3 \times 10^{-18}$ kg and thickness $t = 80$ nm [54]. In general, various basic physical noise processes will limit the sensitivity of the sensing scheme. For this nanomechanical resonators, the main noise source is the thermomechanical noise [54]. In order to obtain this basic limits imposed upon measurements by thermomechanical fluctuations, we need to consider the minimum detectable frequency shift ($\delta \omega$) that can be resolved in a practical noisy system. An estimate for $\delta \omega$ can be obtained by [54]

$$\delta \omega \approx \left( \frac{K_B T \omega_m \Delta f}{Q} \right)^{1/2}.$$  \hspace{1cm} (26)

Here $Q$ is the mechanical quality factor, $K_B$ is the Boltzmann constant, $T$ is the effective temperature of the mechanical resonator, and $E_c = m \omega_m^2 \langle x_c^2 \rangle$, which describes the maximum drive energy. $\langle x_c \rangle$ can be approximated as [54]

$$\langle x_c \rangle \approx 0.53t.$$ \hspace{1cm} (27)

In order to obtain the ultimate sensitivity limits of the system to the effect of gravity, we assume that the frequency splitting caused by gravitational effects is exactly equal to the minimum measurable frequency shift ($\delta \omega$) determined by the thermomechanical fluctuations, i.e. $\Delta \omega_{\text{min}} = \delta \omega$. We plot $\Delta \omega_{\text{min}}$ as a function of the bandwidth for thermomechanical fluctuations in figure 10. The result shows that small bandwidth $\Delta f$ and high quality factor $Q$ of the mechanical resonator are essential for the superresolution. Assuming that $\Delta f = 10^3$ Hz, we can obtain the quantum-noise-limited sensitivity of the system to gravitational effects with equations (17) and (26), i.e. $\Delta \omega_{\text{min}} = 0.7394$ Hz for $Q = 2 \times 10^3$. Then we find upper bound on the modification parameter $\beta_0$, i.e. $\beta_0 < 5.921 \times 10^5$. We can see that the upper bound on the modification parameter $\beta_0$ is mainly determined by the mass $m$, quality factor $Q$, effective temperature $T$, and bandwidth $\Delta f$ of the mechanical resonator. This bound on $\beta_0$ is much better than the one set by reference [44] (10^21) and [60] (10^6).

Finally, we note that the implications of using composite particles to probe quantum gravity effects are not clear. This is because the deformed commutator like equation (1) have been derived for point particles and not for centre-of-mass (COM) modes of composite systems. In reference [61], the author constructs a deformed commutation relation for a composite system consisting of a number $N$ of elementary constituents, and shows that the deformation parameter should scale as $N^{-2}$. Therefore, even assuming the
constituent particles or degrees of freedom to be affected by Planck scale physics, the COM of a composite macroscopic object would be much more weakly affected. This shows that free elementary degrees of freedom should feel quantum gravity effects in a different way, i.e. spacetime properties should depend on the kind of elementary particles [62, 63]. On the other side, we do not know at which constituent-particle level quantum gravitational effects could intervene [63], and there are no theories even showing what such elementary constituents should be. Other works suggest instead that the quantum gravity corrections should scale as the number of elementary interactions [64].

In reference [65], the authors define a new parameter \( \alpha \) that accounts for the suppression of corrections to the canonical commutation relations with the number of constituent particles, regardless of how these constituent particles are defined. This deformation is given by \[ [x, p] = i\hbar \left( 1 + \frac{\beta_0}{N^\alpha (M_p c^2)^2} p^2 \right), \] (28)

where \( N \) is the number of constituent particles in the quasi-rigid macroscopic bodies [61]. Then, we obtain the bound \( \beta_0 N^{-\alpha} < 5.921 \times 10^5 \). For the determination of \( N \) we assume that the nucleons form the elementary particles which leads to \( N = \frac{m}{m_{\text{nuc}}} = 3.1737 \times 10^9 (m_{\text{nuc}} \simeq 1.67 \times 10^{-27} \text{ kg}) \) [65] and therefore we obtain \( \alpha > -0.60752 \) for \( \beta_0 = 1 \). If the optimistic parameters required can be obtained through experiments promises significant improvement in bound of \( \alpha \). On the other side, \( N^\alpha \) should also be considered for sensitivity analysis. According to equation (17), we show that the introduction of \( N^\alpha \) greatly enhances the gravitational effect for \( m = 5.3 \times 10^{-18} \text{ kg} (\beta_0 = 1 \text{ updated to } \beta_0 N^{-\alpha} = 1 \times 5.921 \times 10^5, \mu = \frac{\beta_0}{(M_p c^2)^2} \).

6. Conclusion

In conclusion, we have presented a scheme for sensing the effect of quantum gravity. Starting with a system consisting of two coupled mechanical resonators with driving and dissipation, we show that the system eigenenergy is sensitive to the effect of quantum gravity when the system is in an second-order exceptional point. The response of the binary mechanical system to the gravity exhibits square root behaviour, and the sensitivity of the system at EPs increases significantly with the decrease of the perturbation. Moreover, we found that small mass of the mechanical resonator benefits the sensitivity of the system to deposition mass. In order to further enhance the sensitivity of the system to the effect of gravity, we extend the sensing scheme to a third-order exceptional point by taking a more complicated ternary mechanical system into account. The response of the ternary mechanical systems to perturbation exhibits cube root behaviour. The quantum-noise-limited sensitivity of the system to gravitational effects due to thermomechanical noise is also discussed. From the quantum-noise-limited sensitivity and eigenfrequency splitting of the system we derive bounds on the modified commutator. Better bounds on \( \beta_0 \) can be found in the future. It is
worthwhile to note that our scheme could, in principle, be extended to various photonic and phononic systems with optomechanically induced gain and loss. Moreover, our study can also be extended to other modified uncertainty relation, and bounds on $\beta_0$ are typically much weaker than in the other scenarios under previous experiments. These findings may pave the ways for utilizing EPs as a novel tool to probe effect of quantum gravity.

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Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

Appendix A. The derivation of mechanical effective Hamiltonian

Based on this formal solution: $\beta(t) = \tilde{\beta}_0 + B_j e^{-i\alpha t} e^{-i\omega_d t} (\tilde{\beta}_0 \ll B_j) [52, 53]$, equation (9) can be further simplified as

$$\frac{d\beta_1}{dt} = -\left(\frac{i\omega_1 + \frac{\gamma_m}{2}}{2}\right)\beta_1 + i\beta_2 + i\alpha \tilde{\beta}_0 (-\beta_1 + \beta_2^*),$$

$$\frac{d\beta_2}{dt} = -\left(\frac{i\omega_2 + \frac{\gamma_m}{2}}{2}\right)\beta_2 + i\beta_1 + i\alpha \tilde{\beta}_0 (-\beta_2 + \beta_1^*),$$

(A1)

where, $\Theta_j = \mu m \omega_j^2 B_j^2 (j = 1, 2)$. We substitute this formal solution into the equation for $\alpha_j$, one then obtain the dynamics of the cavity field in the form,

$$\alpha_j(t) = \exp(-i\varphi_j(t)) \sum_n A_n^j \exp(in\omega_jt),$$

(A2)

with

$$A_n^j = \sqrt{\kappa_n^j \alpha_n^j J_n(-\epsilon_j) K_n^j},$$

(A3)

where $\epsilon_j = 2gB_j/\omega_1$ is normalized amplitude, $\Delta_j' = \Delta_j + 2g \Re(\tilde{\beta}_j)$, $K_n^j = i(n\omega_j - \Delta_j' + \frac{\gamma_j}{2})$, the global phase is $\varphi_j(t) = -\epsilon_j \sin(\omega_j t - \theta)$ and $J_n$ is the Bessel function of the first kind.

As we pay our attention to the limit-cycle states of the mechanical resonators, we neglected all terms in mechanical dynamics except for the constant one and the term oscillating at $\omega_1$. We substitute equation (A2) into equation (A1) which leads to the following equations of motion for the oscillating part of $\beta_j (\tilde{\beta}_0 \ll B_j)$,

$$\frac{d\beta_1}{dt} = -\left(\omega_1^{(1)} + \frac{\gamma_1^{(1)}}{2} + i\Theta_1\right)\beta_1 + i\beta_2 + i\alpha \tilde{\beta}_0 (-\beta_1 + \beta_2^*),$$

$$\frac{d\beta_2}{dt} = -\left(\omega_2^{(1)} + \frac{\gamma_2^{(1)}}{2} + i\Theta_2\right)\beta_2 + i\beta_1 + i\alpha \tilde{\beta}_0 (-\beta_2 + \beta_1^*),$$

(A4)

where, $\omega_j^{(i)} = \omega_j + \Omega_j$ and $\gamma_j^{(i)} = \gamma_m + \Gamma_j$ (j = 1, 2) represent the effective frequency and the effective damping of the jth mechanical oscillator (j = 1, 2), respectively. The optical spring effect ($\Omega_j$) and optomechanical damping rate ($\Gamma_j$) of the mechanical resonator due to the cavity are given by [53]

$$\Omega_j = -\frac{2\kappa (g\alpha_{in})^2}{\omega_1 \epsilon_j} \Re\left(\sum_n \frac{J_{n+1}(-\epsilon_j) J_n(-\epsilon_j)}{K_{n+1}^j K_n^j}\right),$$

(A5)

and

$$\Gamma_j = 2(g\kappa \alpha_{in})^2 \sum_n \frac{J_{n+1}(-\epsilon_j) J_n(-\epsilon_j)}{\epsilon_j K_{n+1}^j K_n^j},$$

(A6)
Here, $\Omega_j(\Gamma_j)$ can be controlled by the external drive signal. If the optomechanical system satisfies the resolved-sideband condition, $\gamma_m g < \kappa < \omega_m$ [50, 51], the optical spring effect can be ignored. Further, the effective Hamiltonian of the mechanical modes can be derived as

$$H_{\text{eff}} = \begin{pmatrix}
\omega_{\text{eff}}^{(1)} - \frac{i\gamma_{\text{eff}}^{(1)}}{2} & -J & -\Theta_1 & 0 \\
-J & \omega_{\text{eff}}^{(2)} - \frac{i\gamma_{\text{eff}}^{(2)}}{2} & 0 & -\Theta_2 \\
\Theta_1 & 0 & -\omega_{\text{eff}}^{(1)} - \frac{i\gamma_{\text{eff}}^{(1)}}{2} & 0 \\
0 & \Theta_2 & 0 & -\omega_{\text{eff}}^{(2)} - \frac{i\gamma_{\text{eff}}^{(2)}}{2} \end{pmatrix}.$$  \hspace{1cm} (A7)

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