On Non-central Stirling Numbers of the First Kind

Milan Janjić

Abstract

It is shown in this note that non-central Stirling numbers $s(n, k, \alpha)$ of the first kind naturally appear in the expansion of derivatives of the function $x^{-\alpha} \ln^\beta x$, where $\alpha$ and $\beta$ are arbitrary real numbers. We first obtain a recurrence relation for these numbers, and then, using Leibnitz rule we obtain an explicit formula for them. We also obtain a formula for $s(n, 1, \alpha)$ and then derive several combinatorial identities related to these numbers.

1 Introduction

We are dealing here with a special kind of numbers introduced by D. S. Mitricnović in his note [4]. In the paper [5] tables are given for the numbers which we called non-central Stirling numbers of the first kind. Following [3], they will be denoted by $s(n, k, \alpha)$. Several other names are in use for these numbers. One of them is $r$-Stirling numbers, as in [1]. The definition in this paper is restricted to the case when $\alpha$ is a nonnegative integer, and $\alpha \leq n$. L. Carlitz [2] used the name: weighted Stirling numbers. In the well known encyclopedia [6] they are called the generalized Stirling numbers. Here we use the name and the notation from the book [3]. For instance, $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$ are falling factorials, and $s(n, k)$ are Stirling numbers of the first kind.

2 Derivatives of $x^{-\alpha} \ln^\beta x$.

We shall investigate derivatives of the function

$$f(x) = x^{-\alpha} \ln^\beta x, \; (\alpha, \beta \in \mathbb{R}),$$

obtaining them in two different way.

**Theorem 1.** Let $\alpha$ be real, and $n$ nonnegative integer. Then

$$f^{(n)}(x) = x^{-\alpha - n} \sum_{i=0}^{n} s(n, i, \alpha)(\beta)_i \ln^{\beta - i} x. \; (1)$$

where $s(n, i, \alpha)$, $0 \leq i \leq n$ are polynomials of $\alpha$ with integer coefficients.
Proof. The assertion is true for \( n = 0 \) if we take \( s(0, 0, \alpha) = 1 \).

Taking
\[
s(1, 0, \alpha) = -\alpha, \ s(1, 1, \alpha) = 1,
\]
we see that the assertion is true for \( n = 1 \).

Suppose that the assertion is true for \( n \geq 1 \).

Taking derivative in \( 1 \) we obtain
\[
f^{(n+1)}(x) = (-\alpha - n)x^{-\alpha - n-1}\sum_{i=0}^{n} s(n, i, \alpha)(\beta)_i \ln^{\beta-i} x +
\]
\[
+x^{-\alpha - n-1}\sum_{i=0}^{n-1} s(n, i, \alpha)(\beta)_{i+1} \ln^{\beta-i-1} x =
\]
\[
x^{-\alpha - n-1}\left[(-\alpha - n)\sum_{i=0}^{n} s(n, i, \alpha)(\beta)_i \ln^{\beta-i} x + \sum_{i=0}^{n} s(n, i, \alpha)(\beta)_{i+1} \ln^{\beta-i-1} x \right].
\]

Replacing \( i + 1 \) by \( i \) in the second sum we obtain
\[
f^{(n+1)}(x) = x^{-\alpha - n-1}\left[\sum_{i=0}^{n} (-\alpha - n)s(n, i, \alpha)(\beta)_i + \sum_{i=1}^{n+1} s(n, i - 1, \alpha)(\beta)_i \ln^{\beta-i} x \right] =
\]
\[
x^{\alpha - n-1}(\alpha - n)s(n, 0, \alpha) +
\]
\[
+x^{\alpha - n-1}\sum_{i=1}^{n} \left[(-\alpha - n)s(n, i, \alpha) + s(n, i - 1, \alpha) \ln^{\beta-i} x \right](\beta)_i +
\]
\[
+s(n, n, \alpha)(\beta)_{n+1} \ln^{\beta-n-1} x.
\]

It follows that the assertion is true if we take
\[
s(n + 1, 0, \alpha) = (-\alpha - n)s(n, 0, \alpha),
\]
\[
s(n + 1, i, \alpha) = (-\alpha - n)s(n, i, \alpha) + s(n, i - 1, \alpha), \ (i = 1, \ldots, n), \quad (2)
\]
\[
s(n + 1, n + 1, \alpha) = s(n, n, \alpha).
\]

The preceding equation are well-known recurrence relations for non-central Stirling numbers of the first kind [3, pp.316].

**Note 1.** It is obvious that \( s(n, i, 0) = s(n, i) \) are Stirling numbers of the first kind.

Since \( p(0, 0, \alpha) = 1 \), for from \( n = 1, 2, \ldots \) from the first equation in \( 2 \) we obtain:
\[
p(n, 0, \alpha) = (-\alpha)_n,
\]
and, since \( p(1, 1, \alpha) = 1 \), from the last equation in \( 2 \) follows:
\[
p(n, n, \alpha) = 1, \ (n = 0, 1, 2, \ldots).
\]
By the use of Leibnitz formula we shall obtain an explicit expression for $s(n, k, \alpha)$. The following equation holds:

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (x^{\alpha})^{(k)} (\ln^{\beta} x)^{(n-k)}.$$  \hspace{1cm} (3)

First we have

$$(x^{-\alpha})^{(k)} = (-\alpha)_k x^{\alpha-k}.$$  

Using induction it is easy to prove that:

$$[f(\ln x)]^{(n)} = x^{-n} \sum_{k=1}^{n} s(n, k) f^{(k)}(t), \text{ (} t = \ln x \text{).}$$

Taking particular $f(t) = t^\beta$ we obtain:

$$(\ln^\beta x)^{(n-k)} = x^{-n+k} \sum_{i=1}^{n-k} s(n-k, i) (\beta)_i \ln^{\beta-i} x.$$  

Replacing these in (3) we have the following.

**Theorem 2.** Let $\alpha \neq 0$ be real, and $n, i, (i \leq n)$ be nonnegative integers. Then

$$s(n, i, \alpha) = \sum_{k=0}^{n-i} \binom{n}{k} (-\alpha)_k s(n-k, i).$$  \hspace{1cm} (4)

**Note 2.** Theorem is true even in the case $\alpha = 0$ if we additionally define $(0)_0 = 1$.

### 3 Some combinatorial identities

Taking $i = 1$ in (4) we obtain the following:

**Corollary 1.** Let $\alpha$ be a real number, and $n$ be a positive integer. Then

$$s(n, 1, \alpha) = n! \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{(-\alpha)^k}{n-k}.$$  

For $s(n, 1, \alpha)$ we have the following recurrence relation:

$$s(1, 1, \alpha) = 1, \; s(n, 1, \alpha) = (-\alpha - n + 1) s(n-1, 1, \alpha) + (-\alpha)_{n-1}, \text{ (} n \geq 2 \text{).}$$  \hspace{1cm} (5)

Particularly, we have $s(2, 1, \alpha) = -2\alpha - 1$.

We shall now prove that polynomials $r(n, 1, \alpha), \text{ (} n = 1, 2, \ldots \text{)}$ defined by:

$$r(n, 1, \alpha) = \sum_{k=0}^{n-1} (k+1) s(n, k+1)(-\alpha)^k,$$
satisfy the above recurrence relation. For \( n = 1 \) it is obviously true.

Using two terms recurrence relations for Stirling numbers of the first kind we obtain:

\[
\begin{align*}
  r(n, 1, \alpha) &= \sum_{k=0}^{n-1} (k+1)[s(n-1, k) - (n-1)s(n-1, k+1)](-\alpha)^k = \\
  &= \sum_{k=0}^{n-1} (k+1)s(n-1, k)(-\alpha)^k - (n-1)\sum_{k=0}^{n-2} (k+1)s(n-1, k+1)(-\alpha)^k.
\end{align*}
\]

Since \( s(n-1, 0) = 0 \), by replacing \( k+1 \) instead of \( k \) in the first sum on the right we have:

\[
\begin{align*}
  r(n, -\alpha) &= \sum_{k=0}^{n-2} (k+2)s(n-1, k+1)(-\alpha)^{k+1} - (n-1)\sum_{k=0}^{n-2} (k+1)s(n-1, k+1)(-\alpha)^k = \\
  &= (-\alpha - n + 1)r(n-1, \alpha) + \sum_{k=0}^{n-2} s(n-1, k+1)(-\alpha)^{k+1}.
\end{align*}
\]

Furthermore, a well known property of Stirling numbers implies:

\[
\sum_{k=0}^{n-2} s(n-1, k+1)(-\alpha)^{k+1} = \sum_{k=1}^{n-1} s(n-1, k)(-\alpha)^k = (-\alpha)_{n-1}.
\]

We thus obtain that \( r(n, 1, \alpha) \) satisfies (5). In this way we have proved the following identity:

**Theorem 3.** Let \( \alpha \) be a real number, and \( n \geq 1 \) be an integer. Then:

\[
\begin{align*}
  n! \sum_{k=0}^{n-1} (-1)^{k} \frac{(-\alpha)^{k}}{n - k} &= \sum_{k=0}^{n-1} (k+1)s(n, k+1)\alpha^{k},
\end{align*}
\]

where \( s(n, k+1) \) are unsigned Stirling numbers of the first kind.

**Note 3.** Theorem is true for \( \alpha = 0 \) with the convention that \( 0^0 = 1 \).

Some particular values for \( \alpha \) in (6) gives several interesting combinatorial identities. For \( \alpha = -1 \) we obtain an identity expressing factorials in terms of Stirling numbers of the first kind.

**Corollary 2.** The following formula is true:

\[
(-1)^n(n-2)! = \sum_{k=0}^{n-1} (k+1)s(n, k+1), \ (n \geq 2).
\]

For \( \alpha = 1 \) we obtain a formula for the sum of reciprocals of natural numbers.
Corollary 3. For each \( n = 1, 2, \ldots \) we have:

\[
n! \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) = \sum_{k=0}^{n-1} (k+1)s(n, k+1).
\]

Consider now the case that \( \alpha < 0 \) is an integer and \( n \geq 1 - \alpha \). In this case the function\( \]

\[
q(n, 1, \alpha) = (-1)^{n+\alpha-1}(-\alpha)!(n + \alpha - 1)!, \ (n \geq 1 - \alpha)
\]
satisfies the recurrence relation \( (6) \). In fact, in this case we have \( (-\alpha)_{n-1} = 0 \), since a factor in this product must be zero. Thus:

\[
(-\alpha - n + 1)q(n-1, 1, \alpha) = -(n + \alpha - 1)(-1)^{n-\alpha-2}(-\alpha)!(n + \alpha - 2)! =
\]

\[
= (-1)^{n-\alpha-1}(-\alpha)!(n + \alpha - 1)! = q(n, 1, \alpha).
\]

We have thus obtained the following:

Corollary 4. Let \( n, \alpha \) be positive integers and \( n \geq \alpha + 1 \). Then:

\[
n! \sum_{k=0}^{\alpha} \frac{(-1)^{\alpha-k}{\alpha \choose k}}{n-k} = \alpha!(n - \alpha - 1)!
\]

In other words, the following equation holds:

\[
(\alpha + 1) \sum_{k=0}^{\alpha} \frac{(-1)^{\alpha-k}{\alpha \choose k}}{n-k} = \frac{1}{(\frac{n}{\alpha+1})}.
\]

In the case that \( \alpha \) is a negative integer and \( n \leq -\alpha \) the identity \( (6) \) is closely related to the harmonic numbers. Namely, denote \( H_0 = 0 \), \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) (\( n = 1, 2, \ldots \)), and define:

\[
h(n, 1, \alpha) = (H_{-\alpha} - H_{-\alpha-n})\frac{(-\alpha)!}{(-\alpha - n)!}, \ (\alpha = -1, -2, \ldots; n = 1, 2, \ldots, -\alpha).
\]

It will be shown that \( h(n, 1, \alpha) \) satisfies \( (6) \). We use the induction on \( n \). For \( n = 1 \) we have:

\[
h(1, 1, \alpha) = \frac{(-\alpha)!}{(-\alpha - 1)!}(H_{-\alpha} - H_{-\alpha-1}) = 1.
\]

Furthermore we have:

\[
(-\alpha-n+1)h(n-1, 1, \alpha)+(-\alpha)_{n-1} = \frac{(-\alpha)!}{(-\alpha - n)!}(H_{-\alpha} - H_{-\alpha-n+1})+(-\alpha)_{n-1} =
\]

\[
= \frac{(-\alpha)!}{(-\alpha - n)!}(H_{-\alpha} - H_{-\alpha-n}) - \frac{(-\alpha)!}{(-\alpha - n)!(\alpha - n + 1)} + (-\alpha)_{n-1} =
\]
= \frac{(-\alpha)!}{(-\alpha-n)!} (H_{-\alpha} - H_{-\alpha-n}),

since \frac{(-\alpha)!}{(-\alpha-n)!(-\alpha-n+1)} = (-\alpha)_{n-1}. We thus obtain:

h(n, 1, \alpha) = (-\alpha - n + 1) b(n - 1, 1, \alpha) + (-\alpha)_{n-1},

that is, h(n, 1, \alpha) satisfies (5). We have thus proved the following.

**Corollary 5.** Let \( \alpha \) be a positive integer and \( 1 \leq n \leq \alpha \) be integers. Then:

\[
H_\alpha - H_{\alpha-n} = \frac{(-1)^{n+1}}{(n)} \sum_{k=0}^{n-1} \left(\frac{(-1)^k (n^k)}{n-k}\right).
\]

Equivalently:

\[
H_\alpha - H_{\alpha-n} = \frac{\sum_{k=0}^{n-1} (k+1) s(n, k+1) \alpha^k}{\sum_{k=0}^{n} s(n, k) \alpha^k}.
\]

In the case \( \alpha = n \) we have the following expressions for harmonic numbers.

**Corollary 6.** Let \( n \) be a positive integer. Then:

\[
H_n = (-1)^{n+1} \sum_{k=0}^{n-1} \left(\frac{(-1)^k (n^k)}{n-k}\right),
\]

and

\[
H_n = \frac{1}{n!} \sum_{k=0}^{n-1} (k+1) s(n, k+1) n^k.
\]

**References**

[1] A. Z. Broder, The r-Stirling numbers, IT Department of Computer Science, Stanford University, Stanford, CA 94305, 1982.

[2] L. Carlitz, Weighted Stirling Numbers of the First and Second kind. IT. *The Fibonacci Quarterly* 18(1980):242-257.

[3] Ch. A. Charalambides, Enumerative Combinatorics, Chapman & Hall/CRC, Boca Raton, Florida, 2002.

[4] D. S. Mitrinović, Sur une classe de nombres reîles aux nombres de Stirling, *Les Comptes rendus de l’Academie des Sciences de Paris, 1.* 252, 1961, p. 2354-2356.

[5] D. S. Mitrinović et R. S. Mitrinović, Tableaux d’une classe de nombres reîles aux nombres de Stirling, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No. 77 1962, 77 pp.

[6] N. J. Sloane, The Encyclopedia of Integer Sequences. Electronically published at www.research.att.com