Models and parameters of Cosserat hexagonal lattices with chiral microstructure

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Abstract The article develops a structural approach to construction of Cosserat lattice models with chiral microstructure. A variant of the phenomenological potential of connecting cites (particles) of the lattice has been introduced. The parameters of this potential have been found for special cases of complex connections of finite-sized particles and complex beam compounds. Based on the potential, a discrete model and a long-wavelength continuum chiral model have been elaborated. Expression of macroparameters, in particular, the Poisson's ratio, through the parameters of the phenomenological potential has been found. The dependence of macroparameters on the parameters of the microstructure has been obtained for chiral hexagonal lattices composed of finite-sized particles with complex connections and composite beam compounds.

1. Introduction

Elaboration of models and investigation of the special properties of chiral lattices are of prime interest. In particular, phenomenological theories and models of specific structures are developed, auxetic properties of lattices for static problems and the property of a frequency filter in dynamics problems [1-6] are studied.

Two approaches can be distinguished in the development of the theory and models of generalized mechanics: phenomenological and structural. The phenomenological approach is quite general, but it requires a theoretical justification for correctness, a study of the limits of variation of parameters, as well as experimental research for practical application. The structural approach is less general, but historically it played a key role in the assessment, development and application of various models and theories of generalized mechanics. One of the advantages of structural models is their ability to establish a relationship between the generalized structural macroparameters and microstructure parameters. In particular, it can be important for designing materials with desired properties.

This article summarizes and develops some results of the paper [5], where the auxetic properties of a particular case of chiral hexagonal lattices composed of finite-sized particles were studied, as well as the papers [6-8], in which the structural approach was applied for non-chiral lattices.

2. The model of the complex connection of finite-sized particles

We consider a complex connection of particles with spring-type connections. Fragments of particles are shown in Fig. 2a. The centers of particles located at points 1 and 2, as well as the attachment points of the connections have been marked. The figure also shows the rigidities of the connections, $e_k$. The
rigidities of the diagonal particle connections can be the same, \( e_2 = e_3 \). In this case, there are a symmetrical connection and a lattice with non-chiral microstructure. If the rigidities are different, i.e. \( e_2 \neq e_3 \), we have a lattice with chiral microstructure.

The potential energy of the springs is proportional to the square of the relative elongation:
\[
P_i = e_i e_i^2 / 2, \quad e_i = (L_i - L_{0,i}) / L_{0,i},
\]
where \( L_i \) and \( L_{0,i} \) are the lengths of the connections before and after deformation. In order to construct linear models, we linearize elongation \( \Delta L = L_i - L_{0,i} \) with respect to small displacements \( u \) and rotations \( \varphi \). For compact recording, let us introduce notation \( E_i = e_i / L_{0,i}^2 \).

The potential of a complex connection is obtained by summarizing the potentials of five springs that make up the connection. After transformation it takes on the form
\[
P = \frac{1}{2} K_{11} \Delta u^2 + \frac{1}{2} K_{22} \gamma^2 + \frac{1}{2} K_{33} \Delta \varphi^2 + K_{12} \gamma \Delta u.
\]

Here the following designations are used:
\[
\Delta u = u_2 - u_1, \quad \gamma = v_2 - v_1 - h(\varphi_2 + \varphi_1) / 2, \quad \Delta \varphi = \varphi_2 - \varphi_1.
\]

Coefficients \( K_{ij} \) have the form:
\[
K_{11} = E_0 + 2E_1 + \left( \frac{b}{d} \right)^2 (E_2 + E_3), \quad K_{22} = \left( \frac{a}{d} \right)^2 (E_2 + E_3), \quad K_{33} = \frac{1}{2} E_0 a^2, \quad K_{12} = \left( \frac{a}{d} \right) \left( \frac{b}{d} \right) (E_2 - E_3),
\]
where \( a = 2r \sin(\psi) \), \( b = h - 2r \cos(\psi) \), \( d = \sqrt{a^2 + b^2} \).
Potential (1) with coefficients (2) can be used to construct lattice models of two types for hexagonal lattices (Fig. 2). The angles $\psi$ are equal to $\pi/6$ and $\pi/3$ for the cells shown in Fig 2a and Fig 2b, respectively. An example of a cell of a chiral lattice with $E_z \neq E_x$, $E_z = 0$, and $E_1 = 0$ is presented in Fig. 3a.

3. The model of the composite beam compound

Let us consider a beam connection composed of three beams (Fig. 1b). For $a = 0$ there are a simple beam connection and an ordinary beam hexagonal lattice. And in the case, when $a \neq 0$, there is a composite beam compound that is used for construction of lattices with chiral microstructure.

By eliminating the degrees of freedom of the internal cites 3 and 4, one can construct a connection potential. Vectors $\vec{U}_{1,2} = [u_1, v_1, \varphi_1, u_2, v_2, \varphi_2]$ and $\vec{U}_{3,4} = [u_3, v_3, \varphi_3, u_4, v_4, \varphi_4]$ are introduced, which components are displacements $u_i, v_i$ and rotations $\varphi_i$ in cites 1-4. The system of static equations is written as $C_{11} \vec{U}_{1,2} + C_{12} \vec{U}_{3,4} = 0$, $C_{21} \vec{U}_{1,2} + C_{22} \vec{U}_{3,4} = 0$. Components $\vec{U}_{3,4} = -C_{12}^{-1} C_{11} \vec{U}_{1,2}$ are found from the first equations of the system. Substitution into the second half of the system of equations gives the rigidity matrix of the connection $\left[ C_{21} - C_{22} C_{11}^{-1} C_{12} \right] \vec{U}_{1,2} = 0$.

The potential of the complex compound has the form: $P = (1/2) \vec{U}_{1,2}^T \left[ C_{21} - C_{22} C_{11}^{-1} C_{12} \right] \vec{U}_{1,2}$. After transformations, it can be reduced to the form of Eq. (1) with coefficients $K_{ij}$

$$K_{11} = 4 \left[ 48a^2 J + Ah^2 \left( h^2 + 4a^2 \right) \right] k, \quad K_{22} = 4 \left[ 12h^2 J + 4a^2 \left( h^2 + 4a^2 \right) \right] k,$$

$$K_{33} = J \sqrt{h^2 + 4a^2}, \quad K_{12} = -6ahA \left( h^2 + 4a^2 \right) k,$$

$$k = E \left( \frac{48JA \sqrt{h^2 + 4a^2}}{7h^2a^2 \left( h^2 + 4a^2 \right)^2} A^2 + 192 \left( h^4 + 4a^4 \right) \left( h^2 + 4a^2 \right) AJ + 9216a^2 h^2 J^2 \right)$$

and standard notation for beams is used: $A$ is a cross-sectional area, $J$ is the second moment of inertia of the cross-section, $E$ is Young’s modulus.

4. The continuum model and the lattice macroparameters

We construct lattice models on the base of potential (1). The discrete equations are derived for the middle particle (middle cite) of the hexagonal cell, but they are bulky. These equations are not given.

Fig. 3. A chiral hexagonal cell composed of finite-sized particles, (a), and a chiral cell with composite beam compounds (b)
here, since the main task of our work is finding expressions of macroparameters in terms of structural parameters. Using the expansion into Taylor series and taking into account the second-order derivatives, it is possible to obtain the long-wavelength continuum equations:

\[ c_1 u_{xx} + c_2 u_{yy} + c_3 v_{xy} + 2 \phi_x = 0, \]

\[ c_4 v_{xx} + c_5 v_{yy} + c_6 u_{xy} - c_4 \phi_x + c_7 u_{xx} - u_{yy} + 2 \phi_y = 0, \]

\[ c_6 (\phi_{xx} + \phi_{yy}) - c_4 (u_y - v_x + 2 \phi) + 2 c_7 (u_x + v_y) = 0. \quad (3) \]

The parameters \( c_i \) are expressed in terms of the connection potential coefficients by the formulas

\[ c_1 = \frac{\sqrt{3}}{4} (3K_{11} + K_{22}), \quad c_2 = \frac{\sqrt{3}}{4} (K_{11} + 3K_{22}), \quad c_3 = \frac{\sqrt{3}}{2} (K_{11} - K_{22}), \]

\[ c_4 = \sqrt{3} K_{22}, \quad c_5 = \frac{\sqrt{3}}{2} K_{12}, \quad c_6 = \frac{\sqrt{3}}{4} (4K_{33} - h^2 K_{22}). \]

These parameters are averaged over the area of the cells \( S = \frac{\sqrt{3}h^2}{2} \).

Account of higher-order derivatives in the expansions enables one obtaining equations of the higher-order gradient micropolar theory and expressing the parameters of this model in terms of the coefficients of potential (1).

Comparing equations (3) with the micropolar theory equations

\[ (\lambda + 2\mu)(u_{xx} + v_{xy}) + (\mu + \alpha)(u_{yy} - v_{xy}) + 2\alpha \phi_x - A(2u_{xy} - v_{xx} + v_{yy} + 2\phi) = 0, \]

\[ (\lambda + 2\mu)(u_{xy} + v_{yy}) + (\mu + \alpha)(v_{xx} - u_{xy}) - 2\alpha \phi_x + A(u_{xx} - u_{yy} + 2v_{xy} - 2\phi) = 0, \]

\[ 4B(\phi_{xx} + \phi_{yy}) - 2\alpha (u_y - v_x + 2\phi) + 2A (u_x + v_y) = 0, \]

one can find the relationships between the parameters of these. As a result, expressions of the micropolar parameters through the parameters of potential (1) yield:

\[ \lambda = \frac{\sqrt{3}}{4} (K_{11} - K_{22}), \quad \mu = \frac{\sqrt{3}}{4} (K_{11} + K_{22}), \quad \alpha = \frac{\sqrt{3}}{2} K_{12}, \quad B = \frac{\sqrt{3}}{16} (4K_{33} - hK_{22}), \quad A = \frac{\sqrt{3}}{2} K_{12}. \]

Due to these equalities it is possible [5] to express the Poisson’s ratio in terms of the connection potential parameters (1):

\[ \nu = \frac{K_{11}K_{22} - K_{22}^2 - 2K_{12}^2}{3K_{11}K_{22} + K_{22}^2 - 2K_{12}^2}. \]

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