E-MOTIVES AND MOTIVIC STABLE HOMOTOPY

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Abstract. We introduce in this work the notion of the category of pure $E$-Motives, where $E$ is a motivic strict ring spectrum and construct twisted $E$-cohomology by using six functors formalism of J. Ayoub. In particular, we construct the category of pure Chow-Witt motives $CHW(k)_\mathbb{Q}$ over a field $k$ and show that this category admits a fully faithful embedding into the geometric stable $A^1$-derived category $D_{A^1, gm}(k)_\mathbb{Q}$.

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1. Introduction

One of the main motivations for this work is the embedding theorem of Voevodsky [Voe00], which asserts that there is a fully faithful embedding of the category of Grothendieck-Chow
pure motives \( \text{Chow}(k) \) into the category of geometric motives \( \text{DM}_{gm}(k) \), hence also into the category of motives \( \text{DM}^{-}_{\text{Nis}}(k) \)
\[
\text{Chow}(k)^{\text{op}} \to \text{DM}_{gm}(k),
\]
if \( k \) is a perfect field, which admits resolution of singularities (see e.g. MVW06, Prop. 20.1 and Rem. 20.2], the assumption on resolution of singularities can be removed by using Poincaré duality). In this note, we construct a category \( \text{CHW}(k) \), which we call the category of pure Chow-Witt motives over a field \( k \) and show that \( \text{CHW}(k) \) admits a fully faithful embedding into the geometric \( \mathbb{P}^1 \)-stable \( \mathbb{A}^1 \)-derived category \( D_{\mathbb{A}^1, \text{gm}}(k) \) rationally.

Our work can be viewed as an \( \mathbb{A}^1 \)-version for Voevodsky’s embedding theorem. The advantage here is that by using duality formalism for \( \mathbb{P}^1 \)-stable \( \mathbb{A}^1 \)-derived category \( D_{\mathbb{A}^1}(k) \) (see Hu05, App. A) for stable \( \mathbb{A}^1 \)-motopy categories and the six operations formalism of J. Ayoub Ay08, we do not have to assume the resolution of singularities. However, unlike in motivic setting, one of the main problems here is that we don’t have cancellation theorem for the effective \( \mathbb{A}^1 \)-derived category in general, see [AH11, Rem. 3.2.4], that is the reason why we can prove the embedding result only for \( \mathbb{Q} \)-coefficient. F. Morel conjectured in general that (see [Mor04]):

**Conjecture 1.1.** [Mor04] Let \( S \) be a regular Noetherian scheme of finite Krull dimension. One has a direct decomposition in the rationally motivic stable homotopy category \( \text{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(S) \):

\[
[S^i, G_m^j]_{\mathbb{P}^1} \otimes \mathbb{Q} = H^j_{\mathbb{B}}(S, \mathbb{Q}(j)) \oplus H^{-i}_{\text{Nis}}(S, \mathbb{W} \otimes \mathbb{Q}),
\]

where \( H^j_{\mathbb{B}}(\mathbb{Q}(*) \oplus H^{-i}_{\text{Nis}}(S, \mathbb{W} \otimes \mathbb{Q}) \text{, where } H^j_{\mathbb{B}}(\mathbb{Q}(S))^\text{is the Beilinson motivic cohomology, } \mathbb{W} \text{ is the unramified Witt sheaf and } [-, -]_{\mathbb{P}^1} \otimes \mathbb{Q} \text{ denotes } \text{Hom}_{\text{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(S)}(-, -)_{\mathbb{Q}}.

Over a general base scheme \( S \) one can split the rational motivic sphere spectrum \( 1_{\mathbb{Q}} = 1_{\mathbb{Q}^+} \lor 1_{\mathbb{Q}^+} \). The identification of the plus part \( 1_{\mathbb{Q}^+} = H^j_{\mathbb{B}} \) has been done in [CD10, Thm 16.2.13] over any Noetherian scheme of finite Krull dimension \( S \). The minus part \( 1_{\mathbb{Q}^+} = H^j_{\mathbb{W}^{\mathbb{Q}}}, \text{ where } H^j_{\mathbb{W}^{\mathbb{Q}}} \text{ denotes the Eilenberg-Maclane spectrum associated to the rational Witt homotopy module } \mathbb{W}^{\mathbb{Q}} \text{, is given in the work of A. Ananyeskiy, M. Levine and I. Panin } ([\text{ALP15}, \S 3, \text{Thm. 5}]) \text{ over fields } S = \text{Spec } k. \text{ In general, the conjecture L11 over a regular Noetherian scheme } S \text{ of finite Krull dimension is still widely open, as far as I know. On the other hand, our interest started originally from the study of the existence of 0-cycles of degree one on algebraic varieties. More precisely, Hélène Esnault asked (cf. [Lev10]): Given a smooth projective variety } X \text{ over a field } k, \text{ such that } X \text{ has a zero cycle of degree one. Are there "motivic" explanations which give the (non)-existence of a } k \text{-rational point? In [AH11], A. Asok and C. Haesemeyer show that the existence of zero cycles of degree one over an infinite perfect field of } char(k) \neq 2 \text{ is equivalent to the assertion that the structure map } H^0_{\mathbb{A}^1}(X) \to H^0_{\mathbb{P}^1}(\text{Spec } k) \text{ is a split epimorphism, where } H^0_{\mathbb{P}^1}(X) \text{ denotes the } \mathbb{P}^1 \text{-stable } \mathbb{A}^1 \text{-homology sheaves, while in an earlier work AH11a they also showed that the existence of a } k \text{-rational point over an arbitrary field } k \text{ is equivalent to the condition that the structure map } H^0_{\mathbb{A}^1}(X) \to H^0_{\mathbb{P}^1}(\text{Spec } k) \text{ is split surjective. So roughly speaking, the obstruction to the lifting of a zero cycle of degree one to a rational point arises by passing from } S^1 \text{-spectra to } \mathbb{P}^1 \text{-spectra. As remarked by M. Levine, it is not to expect that the category of Chow-Witt motives } \text{CHW}(k) \text{ contains any information about the existence of rational points. Now we state our main theorem in this work:} \)
Theorem 1.2. Let $k$ be a field. There exists a category of pure Chow-Witt motives $\text{CHW}(k)_{\mathbb{Q}}$, which admits a fully faithful embedding

$$\text{CHW}(k)_{\mathbb{Q}} \rightarrow D_{\mathbb{A}^1,\text{gm}}(k)_{\mathbb{Q}}.$$ 

In fact, one of the main steps in the work of [AH11] is to exhibit a natural isomorphism $H^{\text{st}\mathbb{A}^1}(X)(L) \rightarrow \tilde{\text{CH}}_0(X_L)$ for any separable, finitely generated field extension $L/k$. So one may relate this step to our work as evaluating at a generic point, but much weaker than expected, since we can only prove the result for $\mathbb{Q}$-coefficient. Now our paper is organized as follows: we will review shortly $\mathbb{A}^1$-homotopy theory in section §2. Section §3 is devoted for $\mathbb{A}^1$-derived categories, in fact we will define the geometric $\mathbb{P}^1$-stable $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1,\text{gm}}(k)$ over a field $k$ in §3.9 at the end of §3. In fact, this is the subcategory of compact objects $D_{\mathbb{A}^1,c}(k)$ of $D_{\mathbb{A}^1}(k)$ (see [CD10, Ex. 5.3.43]). In these §2 and §3 we simply steal everything which is needed from the presentation of [AH11]. For a complete treatment we strongly recommend the reader to [Ay08], [CD10] and [Mor12]. In section §4 we introduce the notion of pure $E$-motives, where $E$ is a motivic strict ring spectrum and relate several categories of $E$-correspondences with each other via the twisted $E$-cohomology. The twisted $E$-cohomology appears since we will not assume the motivic ring spectrum $E$ to be orientable.

In topology, if $E$ is a multiplicative cohomology theory and $V$ is an $E$-orientable vector bundle of rank $r$, then one has a Thom-Dold isomorphism

$$E^*(X) \xrightarrow{\cong} \tilde{E}^{*+r}(Th(V)), $$

where $Th(V)$ is the Thom space of $V$ and the right hand side is the reduced cohomology. If $E$ is a ring spectrum, then one can interpret this isomorphism as following: Via the Thom diagonal

$$Th(V) \rightarrow Th(V) \wedge X_+, $$

which is induced by the diagonal

$$X_+ \rightarrow X_+ \wedge X_+ $$

one can express $Th(V)$ as a comodule over $X_+$ and the comodule map is the natural map

$$X_+ \rightarrow Th(V). $$

The geometric Thom isomorphism is the homotopy equivalence

$$E \wedge Th(V) \rightarrow E \wedge Th(V) \wedge X_+ \rightarrow E \wedge \Sigma^n E \wedge X_+ \xrightarrow{\mu_E \wedge \text{id}} E \wedge \Sigma^n X_+. $$

The composition is an $E$-module map, hence one may take function spectrum

$$F_E(E \wedge \Sigma^n X_+, E) \simeq F(\Sigma^n X_+, E) \xrightarrow{\cong} F_E(E \wedge Th(V), E) \simeq F(Th(V), E), $$

which induces the Thom isomorphism on $E$-cohomology. In algebraic geometry one has a similar result. For an oriented motivic ring spectrum $E \in SH(S)$, where $S$ is a regular base, one has ([NSO09, Thm. 2.12])

$$E^{*,*}(X) \xrightarrow{\cong} E^{*+2r,*,*}(Th(V)), $$

where $V$ is a vector bundle of rank $r$ on a smooth $S$-scheme $X$. The key point is that since $E$ is oriented one can define the first Chern class and then prove the projective bundle theorem [NSO09, Thm. 2.11]. The situation becomes much more difficult, even in topology, if $E$ is
not necessary oriented. One has to introduce twisted cohomology. Again in topology, by Atiyah duality one has a commutative diagram in the $(\infty, 1)$-category $\mathbb{S}\text{-Mod}$:

$$
\begin{array}{ccc}
\mathbb{S} & \xrightarrow{PT} & X^\vee \\
\downarrow & & \downarrow \\
Th(-T_X) & \xrightarrow{\simeq} & Th(-T_X)
\end{array}
$$

where $PT : \mathbb{S} \to Th(-T_X)$ is the Pontryagin-Thom collapse map. Let $E$ be an $E_{\infty}$-ring spectrum. By taking $- \wedge_{\mathbb{S}} E$ one obtains a map in the $(\infty, 1)$-category $E\text{-Mod}$

$$
E \to X^\vee \wedge_{\mathbb{S}} E.
$$

Taking function spectrum we have the (twisted) Umkehr map

$$
F_{E}(Th(-T_X) \wedge_{\mathbb{S}} E, E) \simeq F(Th(-T_X), E) \to E.
$$

If $E$ is non-oriented, there is no geometric Thom isomorphism. However, $F_{E}(Th(-T_X) \wedge_{\mathbb{S}} E, E)$ will give the twisted cohomology. This is the motivation from topology for us, since in algebraic geometry we also have the Atiyah-Spanier-Whitehead duality, but I do not know any $\infty$-categorical approach to twisted cohomology like the one in topology [ABGHR14]. So I introduce in section §4 the twisted $E$-cohomology rather through the guide of the six functors formalism of J. Ayoub. The reader may recognize that the notion of $E$-correspondences is similar to the construction of Jack Morava in topology. While it is very simple to define the category of $E$-correspondences $\text{Corr}_E(k)$, it is quite difficult to construct the category $\widetilde{\text{Corr}}_E(k)$ via twisted $E$-cohomology. This category exists only up to a number of natural 2-isomorphisms. This phenomenon reflexes the fact that we rely on six functors formalism, where Thom transformations are only 2-isomorphic to each other. The composition in $\widetilde{\text{Corr}}_E(k)$ is associative only up to a natural isomorphism induced by a natural 2-isomorphism. In §5 we give the proof of the main theorem. In the appendix we give a minimal list of well-known facts and definitions of model categories. We fix now some notations throughout this work. For a pair of adjoint functors $F : A \to B$ and $G : B \to A$, we will adopt the notation in [CD10]

$$
F : A \rightleftarrows B : G,
$$

where $F$ is left adjoint to $G$ and $G$ is right adjoint to $F$. Sometime we will write

$$
\varepsilon_{(F,G)} : FG \to \text{id}, \quad \eta_{(F,G)} : \text{id} \to GF
$$

for the counit and unit of the adjunction respectively. For every morphism $f : FY \to X$ in $\text{Mor}(B)$, there is a unique morphism $g : Y \to GX$ in $\text{Mor}(A)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
FY & \xrightarrow{F(g)} & FG(X) \\
\downarrow & & \downarrow \varepsilon_{(F,G)(X)} \\
X & \xrightarrow{g} & \text{id}
\end{array}
$$
For every morphism \( g : Y \to GX \) in \( \text{Mor}(A) \), there is a unique morphism \( f : FY \to X \) in \( \text{Mor}(B) \), such that the following diagram commutes:

\[
\begin{array}{c}
Y & \xrightarrow{\eta_{F,G}(Y)} & GF(Y) \\
\downarrow{g} & & \downarrow{G(f)} \\
GX & & GX
\end{array}
\]

In a symmetric monoidal category \( (\mathcal{C}, \wedge, 1) \), an object \( A \) is called strongly dualizable if there exists an object \( A^\vee \) and morphisms

\[
\text{coev}_A : 1 \to A \wedge A^\vee, \quad \text{ev}_A : A^\vee \wedge A \to 1,
\]

such that the following compositions

\[
A \cong 1 \wedge A \xrightarrow{\text{coev}_A \wedge \text{id}} A \wedge A^\vee \wedge A \xrightarrow{\text{id} \wedge \text{ev}_A} A \wedge 1 \cong A
\]

and

\[
A^\vee \cong A^\vee \wedge 1 \xrightarrow{\text{id} \wedge \text{coev}_A} A^\vee \wedge A \wedge A^\vee \xrightarrow{\text{ev}_A \wedge \text{id}} 1 \wedge A^\vee \cong A^\vee
\]

are the identities \( \text{id}_A \) and \( \text{id}_{A^\vee} \). The natural isomorphism

\[
\alpha : \text{Hom}_C(-, A) \xrightarrow{\cong} \text{Hom}_C(A^\vee \wedge -, 1)
\]

is given by

\[
\alpha(\phi) = \text{ev}_{A^\vee} \circ (\text{id}_{A^\vee} \wedge \phi),
\]

and its inverse \( \alpha^{-1} \) is given by

\[
\alpha^{-1}(\varphi) = (\text{id}_{A} \wedge \varphi) \circ (\text{coev}_{A^\vee} \wedge \text{id}_-).
\]

Given two smooth \( k \)-schemes \( X, Y \in Sm/k \) and two vector bundles \( \mathcal{E}, \mathcal{E}' \) over \( X \) resp. \( Y \), we write \( \mathcal{E} \times \mathcal{E}'/X \times Y \) for the external sum over \( X \times_k Y \). The \( \mathbb{P}^1 \)-stable homotopy category over a base scheme \( S \) will be denoted by \( \text{StHo}_{A^1, \mathbb{P}^1}(S) \) and we write \( \text{StHo}_{A^1, \mathbb{S}^1}(S) \) for the \( \mathbb{S}^1 \)-stable homotopy category. Sometime when it is clear which category we are talking about, we just abbreviate our \( A^1 \)-stable homotopy category by \( \text{SH}(S) \).

2. \( A^1 \)-HOMOTOPY CATEGORY

2.1. Unstable \( A^1 \)-homotopy category. Let \( Sm/k \) denote the category of separated smooth schemes of finite type over a field \( k \). We write \( \text{Spc}/k \) for the category \( \Delta^{\text{op}} \text{Sh}_{\text{Nis}}(Sm/k) \) consisting of simplicial Nisnevich sheaves of sets on \( Sm/k \). An object in \( \text{Spc}/k \) is simply called a \( k \)-space, which is usually denoted by calligraphic letter \( \mathcal{X} \). The Yoneda embedding \( Sm/k \to \text{Spc}/k \) is given by sending a smooth scheme \( X \in Sm/k \) to the corresponding representable sheaf \( \text{Hom}_{Sm/k}(-, X) \) then by taking the associated constant simplicial object, where all face and degeneracy maps are the identity. We will identify \( Sm/k \) with its essential image in \( \text{Spc}/k \). Denote by \( \text{Spc}_{+}/k \) the category of pointed \( k \)-space, whose objects are \( (\mathcal{X}, x) \), where \( \mathcal{X} \) is a \( k \)-space and \( x : \text{Spec} k \to \mathcal{X} \) is a distinguished point. One has an adjoint pair

\[
\text{Spc}/k \rightleftarrows \text{Spc}_{+}/k,
\]

which means that the functor \( \text{Spc}/k \to \text{Spc}_{+}/k \) sending \( \mathcal{X} \to \mathcal{X}_+ = \mathcal{X} \coprod \text{Spec} k \) is left-adjoint to the forgetful functor \( \text{Spc}_{+}/k \to \text{Spc}/k \). The category \( \text{Spc}/k \) can be equipped with the injective local model structure \( (C_s, W_s, F_s) \), where cofibrations are monomorphisms, weak
equivalences are stalkwise weak equivalences of simplicial sets and fibrations are morphisms with right lifting property wrt. morphisms in $C_s \cap W_s$. Denote by $\text{Ho}_s^{Nis}(k)$ the resulting unpointed homotopy category as constructed by Joyal-Jardine (cf. [MV01, §2 Thm. 1.4]). We will write $\text{Ho}_{s,+}^{Nis}(k)$ for the pointed homotopy category.

**Definition 2.1.** [MV01]

1. A $k$-space $Z \in \text{Spc}/k$ is called $\mathbb{A}^1$-local if and only for any object $X \in \text{Spc}/k$, the projection $X \times \mathbb{A}^1 \to X$ induces a bijection

$$\text{Hom}_{\text{Ho}_s^{Nis}(k)}(X \times \mathbb{A}^1, Z) \xrightarrow{\sim} \text{Hom}_{\text{Ho}_s^{Nis}(k)}(X, Z).$$

2. Let $X \to Y \in \text{Mor}(\text{Spc}/k)$ be a morphism of $k$-spaces. It is an $\mathbb{A}^1$-weak equivalence if and only for any $\mathbb{A}^1$-local object $Z$, the induced map

$$\text{Hom}_{\text{Ho}_s^{Nis}(k)}(Y, Z) \to \text{Hom}_{\text{Ho}_s^{Nis}(k)}(X, Z)$$

is bijective.

In [MV01 §2 Thm. 3.2], F. Morel and V. Voevodsky proved that $\text{Spc}/k$ can be endowed with the $\mathbb{A}^1$-local injective model structure $(C, W_{\mathbb{A}^1}, F_{\mathbb{A}^1})$, where cofibrations are monomorphisms, weak equivalences are $\mathbb{A}^1$-weak equivalences. The associated homotopy category obtained from $\text{Spc}/k$ by inverting $\mathbb{A}^1$-weak equivalences is denoted by $\text{Ho}_{\mathbb{A}^1}(k) \overset{\text{def}}{=} \text{Spc}/k[W_{\mathbb{A}^1}^{-1}]$. This category is called the unstable $\mathbb{A}^1$-homotopy category of smooth $k$-schemes. Let $\text{Ho}_{\mathbb{A}^1-loc}^{Nis}(k) \subset \text{Ho}_s^{Nis}(k)$ be the full subcategory consisting of $\mathbb{A}^1$-local objects. In fact, one has an adjoint pair (cf. [MV01])

$$L_{\mathbb{A}^1} : \text{Ho}_s^{Nis}(k) \rightleftarrows \text{Ho}_{\mathbb{A}^1-loc}^{Nis}(k) : i,$$

where $L_{\mathbb{A}^1}$ is the $\mathbb{A}^1$-localization functor sending $\mathbb{A}^1$-weak equivalences to isomorphisms. $L_{\mathbb{A}^1}$ induces thus an equivalence of categories $\text{Ho}_{\mathbb{A}^1}(k) \to \text{Ho}_{\mathbb{A}^1-loc}^{Nis}(k)$. This will imply that if $X \in \text{Spc}/k$ is any object and $Y$ is an $\mathbb{A}^1$-local object, then one has a canonical bijection

$$\text{Hom}_{\text{Ho}_s^{Nis}(k)}(X, Y) \xrightarrow{\sim} \text{Hom}_{\text{Ho}_{\mathbb{A}^1}(k)}(X, Y).$$

We will write $\text{Ho}_{\mathbb{A}^1,+}(k)$ for the unstable pointed $\mathbb{A}^1$-homotopy category of smooth $k$-schemes.

**Definition 2.2.** Let $X \in \text{Sm}/k$ and $E$ be a vector bundle over $X$. The Thom space of $E$ is the pointed sheaf

$$\text{Th}(E/X) = E/E - s_0(X),$$

where $s_0 : X \to E$ is the zero section of $E$.

Let $T \in \text{Spc}_{+/k}$ be the quotient sheaf $\mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ pointed by the image of $\mathbb{A}^1 - \{0\}$. Then $T \cong S^1_\ast \wedge S^1_s$ in $\text{Ho}_{\mathbb{A}^1,+}(k)$ ([MV01 Lem. 2. 15]). For a pointed space $X \in \text{Spc}_{+/k}$, we denote by $\Sigma_T(X, x) = T \wedge (X, x)$. Remark that $\mathbb{P}^n/\mathbb{P}^{n-1} \cong T^n \overset{\text{def}}{=} T \wedge n$ is an $\mathbb{A}^1$-equivalence. In particular, we have $(\mathbb{P}^1, \ast) \cong T$ ([MV01 Cor. 2.18]). Recall

**Proposition 2.3.** [MV01 §3 Prop. 2. 17] Let $X, Y \in \text{Sm}/k$ and $E, E'$ be vector bundles on $X$ and $Y$ respectively. One has

1. There is a canonical isomorphism of pointed sheaves

$$\text{Th}(E \times E'/X \times Y) = \text{Th}(E/X) \wedge \text{Th}(E'/Y).$$
(2) There is a canonical isomorphism of pointed sheaves
\[ Th(\mathcal{O}_X^n) = \Sigma^n T X_+ \]

(3) The canonical morphism of pointed sheaves
\[ \mathbb{P}(E \oplus \mathcal{O}_X)/\mathbb{P}(E) \rightarrow Th(E) \]
is an \( \mathbb{A}^1 \)-weak equivalence.

The following theorem due to Voevodsky will play an essential role for our purpose. However, as pointed out by M. Levine, the identities in \( K_0(\mathcal{X}) \) are not enough for us to construct maps between twisted E-cohomology. Following a suggestion by M. Levine, we will refine this result of Voevodsky later (see §2.5).

**Theorem 2.4.** \( \text{[Voe03 Thm. 2.11]} \) Let \( X \in SmProj/k \) a smooth projective variety of pure dimension \( d_X \) over a field \( k \). There exists an integer \( n_X \) and a vector bundle \( V_X \) over \( X \) of rank \( n_X \), such that
\[ V_X \oplus T_X = \mathcal{O}_X^{n_X+d_X} \in K_0(X), \]
where \( T_X \) denotes the tangent bundle of \( X \). Moreover, there exists a morphism \( T^{n_X+d_X} \rightarrow Th(V_X) \) in \( \text{Ho}_{\mathbb{A}^1,*}(k) \), such that the induced map \( H^{d_X}(\mathcal{X}, \mathbb{Z}(d_X)) \rightarrow \mathbb{Z} \) coincides with the degree map \( \deg : \text{CH}_0(X) \rightarrow \mathbb{Z} \), where \( T = S^1 \wedge \mathbb{G}_m \).

**Remark 2.5.** One can always add a trivial bundle to \( V_X \) in Voevodsky’s theorem 2.4 to increase \( n_X \) appropriately.

### 2.2. Stable \( \mathbb{A}^1 \)-homotopy category

Let \( \text{Spect}^\Sigma(k) \) be the category of symmetric spectra in \( k \)-spaces, which can be viewed as category of Nisnevich sheaves of symmetric spectra. By applying the construction in \( \text{[Ay08, Def. 4.4.40, Cor. 4.4.42, Prop. 4.4.62]} \), \( \text{Spect}^\Sigma(k) \) has the structure of a monoidal model category. Let \( \text{StHo}_{\Sigma}(k) \) be the resulting homotopy category. The stable \( \mathbb{A}^1 \)-homotopy category of \( S^1 \)-spectra \( \text{StHo}_{\mathbb{A}^1,*}(k) \) is obtained from \( \text{StHo}_{\Sigma}(k) \) by Bousfield localization. Equivalently, the category \( \text{Spect}^\Sigma(k) \) can be equipped with an \( \mathbb{A}^1 \)-local model structure (cf. \( \text{[Ay08, Def. 4.5.12]} \)). The homotopy category of this \( \mathbb{A}^1 \)-local model structure is \( \text{StHo}_{\mathbb{A}^1,*}(k) \), which is also known to be equivalent to the category \( \text{StHo}_{\mathbb{A}^1,*}(Sm/k) \) constructed by F. Morel in \( \text{[Mor05, Def. 4.1.1]} \). The \( \mathbb{A}^1 \)-local symmetric sphere spectrum is defined by taking the functor
\[ n \mapsto L_{\mathbb{A}^1} (S_\omega^{1,n}) \]
with an action of symmetric groups, where \( L_{\mathbb{A}^1} \) denotes the \( \mathbb{A}^1 \)-localization functor. For a pointed space \( (X, x) \), its \( \mathbb{A}^1 \)-local symmetric suspension spectrum is defined as the symmetric sequence
\[ n \mapsto L_{\mathbb{A}^1} (S_\omega^{1,n} \wedge X) \]

**Remark 2.6.** The \( \mathbb{A}^1 \)-local symmetric suspension spectrum is pointed with \( \mathbb{A}^1 \)-localization for \( \mathbb{A}^1 \)-localization of \( Sm/k \) associated to the presheaf
\[ U \mapsto \text{Hom}_{\text{StHo}_{\mathbb{A}^1,*}(k)} (S_\omega^{1,n} \wedge \Sigma^\infty U_+, X). \]

Now we consider the symmetric \( T \)-spectra or \( \mathbb{P}^1 \)-spectra (\( \text{[Jar00]} \)). \( \mathbb{P}^1 \) is pointed with \( \Sigma^\infty \) and \( \mathbb{P}^{1,n} \) has a natural action of \( \Sigma_n \) by permutation of the factors, so the association \( n \mapsto \mathbb{P}^{1,n} \) is a symmetric sequence. A symmetric \( \mathbb{P}^1 \)-spectrum is a symmetric sequence with a module
structure over the sphere spectrum $S^0$. Denote by $\text{Spect}_{\Sigma}^{\mathbb{P}^1}(k)$ the full subcategory of the category of symmetric sequence in $k$-spaces $\text{Fun}(\text{Sym}, \text{Spc}_k/k)$ consisting of symmetric $\mathbb{P}^1$-spectra, which also has a model structure $\text{Ay08}$ Def. 4.5.21. Here we denote by $\text{Sym}$ the groupoid, whose objects are $n$ and morphisms are given by bijections. Let $\text{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)$ be the resulting homotopy category, which is called $\mathbb{P}^1$-stable $\mathbb{A}^1$-homotopy category. For a pointed space $(\mathcal{X}, x)$, we will write $\Sigma_{\mathbb{P}^1}^\infty(\mathcal{X}, x)$ for the suspension symmetric $\mathbb{P}^1$-spectrum, i.e., it is given by the functor $n \mapsto \mathbb{P}^1 \wedge \mathcal{X}$ equipped with an action of symmetric group by permuting the first $n$-factors. Let $S^i$ be the suspension symmetric $\mathbb{P}^1$-spectrum of $S^i_s$. If $\mathcal{E}$ is a symmetric $\mathbb{P}^1$-spectrum, then the $i$-th $\mathbb{P}^1$-stable $\mathbb{A}^1$-homotopy sheaf $\pi^i_{\mathbb{A}^1, \mathbb{P}^1}(\mathcal{E})$ is defined as the Nisnevich sheaf on $Sm/k$ associated to the presheaf (cf. $\text{AH11}$ Def. 2.1.14) 
\[ U \mapsto \text{Hom}_{\text{StHo}_{\mathbb{A}^1, \mathbb{P}^1}(k)}(S^i \wedge \Sigma_{\mathbb{P}^1}^\infty U_+, \mathcal{E}). \]

**Theorem 2.6.** $\text{Mor05}$ Thm. 6.1.8 and Cor. 6.2.9] Let $\mathcal{E}$ be an $\mathbb{A}^1$-local symmetric $S^1$-spectrum. The homotopy sheaves $\pi^i_{\mathbb{A}^1, S^1}(\mathcal{E})$ are strictly $\mathbb{A}^1$-invariant.

One has a canonical isomorphism $\text{AH11}$ Prop. 2.1.16]
\[ \text{colim}_n \text{Hom}_{\text{StHo}_{\mathbb{A}^1, S^1}(k)}(\Sigma_{\mathbb{A}^1}^\infty \mathcal{G}_m^m \wedge \Sigma_{\mathbb{A}^1}^\infty (U_+), \Sigma_{\mathbb{A}^1}^\infty \mathcal{G}_m^m \wedge \Sigma_{\mathbb{A}^1}^\infty (\mathcal{X}, x)) \cong \text{Hom}_{\text{StHo}_{\mathbb{A}^1, S^1}(k)}(\Sigma_{\mathbb{P}^1}^\infty U_+, \Sigma_{\mathbb{P}^1}^\infty (\mathcal{X}, x)). \]

So one may view that $\text{StHo}_{\mathbb{A}^1, S^1}(k)$ is obtained from $\text{StHo}_{\mathbb{A}^1, S^1}(k)$ by formally inverting the $\mathbb{A}^1$-localized suspension spectrum of $\mathcal{G}_m$. So from $\text{Mor04}$ we see that for a pointed $k$-space $(\mathcal{X}, x)$, the homotopy sheaves $\pi^i_{\mathbb{A}^1, S^1}(\mathcal{X})$ are also strictly $\mathbb{A}^1$-invariant. By the computation of F. Morel ($\text{Mor04}$, $\text{Mor12}$), one can identify the Milnor-Witt $K$-theory sheaves with stable homotopy sheaves of spheres
\[ K^M_n \cong \pi^0_{\mathbb{A}^1, S^1}(\Sigma_{\mathbb{A}^1}^\infty (\mathcal{G}_m^m)). \]

This identification allows us to conclude that $K^M_n$ are strictly $\mathbb{A}^1$-invariant sheaves.

### 3. $\mathbb{A}^1$-HOMOLOGICAL ALGEBRA

#### 3.1. Effective $\mathbb{A}^1$-derived category

Let $Ch_-(\mathbb{A}b_k)$ be the category of chain complexes over the category $\mathbb{A}b_k$ of abelian Nisnevich sheaves. Denote by $Ch_{\geq 0}(\mathbb{A}b_k)$ the category of chain complexes of abelian Nisnevich sheaves on $Sm/k$, whose homoglocial degree $\geq 0$. The sheaf-theoretical Dold-Kan correspondence
\[ N : \Delta^{op} \mathbb{A}b_k \rightleftarrows Ch_{\geq 0}(\mathbb{A}b_k) : K, \]
where $\Delta^{op} \mathbb{A}b_k$ is the category of simplicial abelian Nisnevich sheaves, gives us via the inclusion functor $Ch_{\geq 0}(\mathbb{A}b_k) \hookrightarrow Ch_-(\mathbb{A}b_k)$, a functor
\[ \Delta^{op}(\mathbb{A}b_k) \rightarrow Ch_-(\mathbb{A}b_k). \]

By applying this functor on the Eilenberg-Maclane spectrum $HZ$, we obtain a ring spectrum $HZ$ in $\text{Fun}(\text{Sym}, Ch_-(\mathbb{A}b_k))$. Let $\text{Spect}(Ch_-(\mathbb{A}b_k))$ be the full subcategory of the category $\text{Fun}(\text{Sym}, Ch_-(\mathbb{A}b_k))$ consisting of modules over $HZ$. On the other hand, by composing with the free abelian group functor
\[ \mathbb{Z}(\cdot) : \text{Spc}_k \rightarrow \Delta^{op}(\mathbb{A}b_k), \]
one obtains a functor
\[ \text{Fun}(\text{Sym}, \text{Spc}_+/k) \to \text{Fun}(\text{Sym}, \text{Ch}_-(\text{Ab}_k)), \]
which sends the sphere symmetric sequence to \( \widetilde{HZ} \). This induces then a functor between categories of symmetric spectra
\[ \text{Spect}^\Sigma(\text{Spc}/k) \to \text{Spect}^\Sigma(\text{Ch}_-(\text{Ab}_k)). \]
In fact, by [Hov01, Thm. 9.3], this induces a Quillen functor, which one refers as Hurewicz functor
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\[ \widetilde{\mathcal{F}} \]
Now the effective \( \mathbb{A}^1 \)-derived category \( D^\text{eff}_{\mathbb{A}^1}(k) \) is constructed by applying \( \mathbb{A}^1 \)-localization on the category \( \text{Spect}^\Sigma(\text{Ch}_-(\text{Ab}_k)). \). By the work of Cisinski and Déglise (cf. [CD10, §5]), this category is equivalent to the \( \mathbb{A}^1 \)-derived category constructed by F. Morel in [Mor12]. Let \((\mathcal{X}, x) \in \text{Spc}_+/k\) be a pointed space, and \( \Sigma^\infty_*(\mathcal{X}, x) \) its suspension symmetric spectrum. We apply the Hurewicz functor on \( \Sigma^\infty_*(\mathcal{X}, x) \) and then \( L^\text{ab}_{\mathbb{A}^1}(-) \), so we may define a functor
\[ \widetilde{C}^\mathbb{A}^1_* : \text{StHo}^{\mathbb{A}^1}(\mathcal{X}) \to D^\text{eff}_{\mathbb{A}^1}(k), \quad \Sigma^\infty_*(\mathcal{X}, x) \mapsto L^\text{ab}_{\mathbb{A}^1}(\mathcal{S}^\text{eff}_*(\Sigma^\infty_*(\mathcal{X}, x))). \]
Here we write \( L^\text{ab}_{\mathbb{A}^1} \) for the \( \mathbb{A}^1 \)-localization functor on chain complexes to distinguish from the \( \mathbb{A}^1 \)-localization \( L_{\mathbb{A}^1} \) on spaces. If \( \mathcal{X} \in \text{Spc}/k \) is not pointed, then we write \( C^\mathbb{A}^1_* (\mathcal{X}) \) \( \equiv \widetilde{C}^\mathbb{A}^1_* (\mathcal{X}_+). \) Define \( Z[n] = \mathcal{S}^\text{eff}_*(\Sigma^\infty \mathbb{G}_m^n) \).

**Definition 3.1.** Let \( \mathcal{X} \in \text{Spc}/k \) be a \( k \)-space. Its \( i \)-th \( \mathbb{A}^1 \)-homology sheaf is the Nisnevich sheaf \( H^\mathbb{A}^1_i(\mathcal{X}) \) associated to the presheaf
\[ U \mapsto \text{Hom}_{D^\text{eff}_{\mathbb{A}^1}(k)}(C^\mathbb{A}^1_* (U)[i], C^\mathbb{A}^1_* (\mathcal{X})) \equiv \text{Hom}_{D^\text{eff}_{\mathbb{A}^1}(k)}(C^\mathbb{A}^1_* (U) \otimes Z[i], C^\mathbb{A}^1_* (\mathcal{X})). \]

Consider \( (\mathbb{P}^1, \infty) \) pointed by \( \infty \). According to [MV01] Cor. 2.18, we have \( \mathbb{P}^1 = S^1 \vee \mathbb{G}_m \), so we have an identification \( \widetilde{C}^\mathbb{A}^1_* (\mathbb{P}^1) = \widetilde{C}^\mathbb{A}^1_* (S^1 \vee \mathbb{G}_m) \). We define the \( \mathbb{A}^1 \)-Tate complex (called enhanced Tate (motivic) complex) by A. Asok and C. Haesemeyer [AH11] Def. 2.1.25 and Def. 3.2.1 and Lem. 3.2.2] as
\[ Z^\mathbb{A}^1_i(n) \equiv \widetilde{C}^\mathbb{A}^1_* ([\mathbb{P}^1 \wedge n][-2n]) = Z^\mathbb{A}^1_i(1)^{\otimes n}. \]

**Definition 3.2.** Let \( \mathcal{X} \in \text{Spc}/k \) be a \( k \)-space. The bigraded unstable \( \mathbb{A}^1 \)-cohomology group \( H^{p,q}_{\mathbb{A}^1}(\mathcal{X}, \mathbb{Z}) \) is defined as
\[ H^{p,q}_{\mathbb{A}^1}(\mathcal{X}, \mathbb{Z}) = \text{Hom}_{D^\text{eff}_{\mathbb{A}^1}(k)}(C^\mathbb{A}^1_* (\mathcal{X}), Z^\mathbb{A}^1_i(q)[p])). \]

The relationship between unstable \( \mathbb{A}^1 \)-cohomology and Nisnevich hypercohomology with coefficient \( Z^\mathbb{A}^1_i(n) \) is given by the following

**Proposition 3.3.** [AH11] Prop. 3.2.5 Let \( k \) be a field and \( \mathcal{X} \in \text{Spc}/k \) be a \( k \)-space. One has

1. For any \( p, q \), there is a canonical isomorphism
\[ \mathbb{H}^{p}_{\text{Vis}}(\mathcal{X}, Z^\mathbb{A}^1_i(q)) \rightarrow H^{p,q}_{\mathbb{A}^1}(\mathcal{X}, \mathbb{Z}). \]
2. The cohomology sheaves \( H^{p}_{\text{Vis}}(Z^\mathbb{A}^1_i(q)) = 0 \), if \( p > q \).
3. There is a canonical isomorphism \( H^{p}_{\text{Vis}}(Z^\mathbb{A}^1_i(p)) \cong K^M_{p} \), for all \( p > 0 \).
Remark 3.4. By construction the complex $\mathbb{Z}_{A^1}(n)$ is $A^1$-local, hence by definition (cf. [Mor12, Def. 5.17]) one has immediately that the sheaves $H^p(\mathbb{Z}_{A^1}(q))$ are strictly $A^1$-invariant.

3.2. $\mathbb{P}^1$-stable $A^1$-derived category. Having defined an $A^1$-Tate complex, the way that we stabilize the category $D^{eff}_{A^1}(k)$ is to invert formally the $A^1$-Tate complex to obtain the $\mathbb{P}^1$-stable $A^1$-derived category $D_{A^1}(k)$. This can be done by following the construction detailed in [CD10, §5]. As before, we take $D_{A^1}(k)$ as the resulting homotopy category of the model category $\text{Spect}_{\mathbb{P}^1}(\text{Ch}_-(\text{Ab}_k))$ consisting of modules over the $A^1$-localization of the normalized chain complex of the free abelian group on the sphere symmetric $\mathbb{P}^1$-spectrum. For a pointed space $(\mathcal{X}, x) \in \text{Spc}_+/k$, the stable $A^1$-complex $C^{stA^1}(\mathcal{X})$ of $(\mathcal{X}, x)$ is defined as $L^\text{th}_{A^1}(N\mathbb{Z}(\Sigma_{\mathbb{P}^1}^\infty(\mathcal{X}, x)))$ and if $\mathcal{X} \in \text{Spc}/k$ is an unpointed $k$-space, then we write $C^{stA^1}_{+}(\mathcal{X})$ for $C^{stA^1}_{+}(\mathcal{X})$. The category $D_{A^1}(k)$ has an unit object, denoted by $1_k$, which is the complex $C^{stA^1}_{+}(S^0)$. Define $1_k[n] = 1_k \otimes C^{stA^1}_{+}(S^n)$ and $C^{stA^1}_{+}(\mathcal{X})[n] = C^{stA^1}_{+}(\mathcal{X}) \otimes 1_k[n]$ for a $k$-space $(\mathcal{X}, x) \in \text{Spc}_+/k$.

Definition 3.5. Let $\mathcal{X} \in \text{Spc}/k$ be a $k$-space. The $i$-th $\mathbb{P}^1$-stable $A^1$-homology sheaf $H_{i}^{stA^1}(\mathcal{X})$ is the Nisnevich sheaf associated to the presheaf

$$U \mapsto \text{Hom}_{D_{A^1}(k)}(C^{stA^1}_{+}(U)[i], C^{stA^1}_{+}(\mathcal{X})).$$

Just like in case of stable $A^1$-homotopy categories, one has the following result

Proposition 3.6. [AH11] Prop. 2.1.29] Let $U \in \text{Sm}/k$ and $(\mathcal{X}, x) \in \text{Spc}_+/k$. One has a canonical isomorphism

$$(3.1) \quad \text{colim}_n \text{Hom}_{D^{eff}_{A^1}(k)}(C^{A^1}_{*}(U) \otimes \mathbb{Z}_{A^1}(n)[i], C^{stA^1}_{+}(\mathcal{X}) \otimes \mathbb{Z}_{A^1}(n)[i]) \xrightarrow{\simeq} \text{Hom}_{D_{A^1}(k)}(C^{stA^1}_{+}(U), C^{stA^1}_{+}(\mathcal{X})).$$

The Hurewicz formalism induces the following functors, which one still calls Hurewicz functors (or abelianization functors)

$$\text{StHo}_{A^1, \mathbb{P}^1}(k) \to D^{eff}_{A^1}(k),$$
$$\text{StHo}_{A^1, \mathbb{P}^1}(k) \to D_{A^1}(k),$$

which give rise to morphisms of sheaves

$$\pi_i^{stA^1, \mathbb{P}^1}(\Sigma_{\mathbb{P}^1}^\infty(\mathcal{X}_+)) \to H_i^{A^1}(\mathcal{X}),$$
$$\pi_i^{stA^1, \mathbb{P}^1}(\Sigma_{\mathbb{P}^1}^\infty(\mathcal{X}_+)) \to H_i^{stA^1}(\mathcal{X}).$$

Definition 3.7. Let $\mathcal{X} \in \text{Spc}/k$ be a $k$-space. The bigraded $\mathbb{P}^1$-stable $A^1$-cohomology group $H^{p,q}_{stA^1}(\mathcal{X}, \mathbb{Z})$ is defined as

$$H^{p,q}_{stA^1}(\mathcal{X}, \mathbb{Z}) = \text{Hom}_{D_{A^1}(k)}(C^{stA^1}_{+}(\mathcal{X}), \mathbb{Z}_{A^1}(q)[p]).$$

The advantage of $\mathbb{P}^1$-stable $A^1$-derived category $D_{A^1}(k)$ is that one has duality formalism. In the context of stable $A^1$-homotopy theory, it was done in [Hu05, App. A] and also [Ri05]. Firstly we recall that Deligne introduced in [Del87, §4] virtual categories. If $f : X \to \text{Spec } k$ is a smooth $k$-scheme, the category $\mathcal{V}(X)$ of virtual bundles on $X$ is identified to the fundamental groupoid of $\mathcal{K}(X)$ where $\mathcal{K}$ is some $A^1$-fibrant genuine model of algebraic
\( K \)-theory. An actual vector bundle \( \xi \) defines an object \( \xi \in \mathcal{V}(X) \) whose isomorphism class corresponds to \([\xi] \in K_0(X)\). A short exact sequence of vector bundles

\[ 0 \to \xi' \to \xi \to \xi'' \to 0 \]

gives not just an equality \([\xi] = [\xi'] + [\xi'']\) in \( K_0(X) \) but also a specific isomorphism \( \xi \cong \xi' \oplus \xi'' \) of objects in \( \mathcal{V}(X) \). By using universal property of \( \mathcal{V}(X) \) as a Picard category, one can define an isomorphism (see [Rio10 §4])

\[ \text{Th}(\xi/X) \cong \text{Th}(\xi'/X) \wedge \text{Th}(\xi''/X). \]

We haven’t yet introduced in this section the 6 operations formalism of J. Ayoub, however we should mention that the construction of Thom spectrum extends to a functor (cf. [Rio10 Prop. 4.1.1, Def. 4.2.1] and [Ay08, Thm. 1.5.18])

\[ \text{Th}_X : \mathcal{V}(X) \to \text{StHo}_{A^1, \mathbb{P}^1}(X) \xrightarrow{f_\#} \text{StHo}_{A^1, \mathbb{P}^1}(k). \]

We discuss a little bit more about the Thom spectrum of virtual bundles. If \( \xi \) is a virtual vector bundle on an affine variety \( U \), then there exist an actual vector bundle \( \xi' \) on \( U \) and an integer \( n \geq 0 \), such that

\[ \xi \oplus \mathcal{O}_X^\oplus_n = \xi'. \]

So one may define

\[ \Sigma_{p=1}^\infty \text{Th}(\xi/U) = \Sigma_{p=1}^\infty \text{Th}(\xi'/U) \wedge S^{-2n,-n}. \]

If \( X \) is a projective variety, one can define an affine variety, which is \( A^1 \)-weak equivalent to \( X \) (see [Hu05 p. 10]): Consider first of all the projective space \( \mathbb{P}^N \). One defines

\[ U = \mathbb{P}^N \times \mathbb{P}^N \setminus \text{Proj} \ k[x_0, \ldots, x_N, y_0, \ldots, y_N]/(\sum_{i=0}^N x_i y_i = 0), \]

which is an \( A^N \)-bundle \( pr_1 : U \to \mathbb{P}^N \). If \( X \) is a projective variety, one has \( i : X \hookrightarrow \mathbb{P}^N \) and the affine variety \( \pi : i^* U \to X \) is an \( A^1 \)-weak equivalence, where \( \pi \) is the pullback of \( pr_1 \) along the closed immersion \( i \):

\[
\begin{array}{ccc}
\text{U} & \xrightarrow{\pi} & \text{X} \\
i & \downarrow & \downarrow \text{i} \\
\text{U} & \xrightarrow{\text{pr}_1} & \mathbb{P}^N
\end{array}
\]

If \( -T_X \) is the virtual normal bundle on \( X \) of the diagonal embedding \( \Delta_X : X \hookrightarrow X \times_k X \), which is the virtual tangent bundle, then its Thom spectrum is defined to be the Thom spectrum \( \text{Th}(\mu/i^* U) \), where \( \mu \) is the complement of the pullback of the tangent bundle of \( X \) along \( \pi \). We state the following result in \( D_{A^1}(k) \), although the proof in case of \( \text{StHo}_{A^1, \mathbb{P}^1}(k) \) is given in [Hu05 Thm. A1] or see [Rio03 Thm. 2.2]

**Proposition 3.8.** [AH11 Prop. 3.5.2 and Lem. 3.5.3] Let \( X \in \text{SmProj}_k \), then \( C_{\text{st}A^1}(X) \) is a strong dualizable object in \( D_{A^1}(k) \) and its dual is \( C_{\text{st}A^1}(X)^\vee = \text{C}_{\text{st}A^1}(\text{Th}(-T_X)) \). Consequently, one has a canonical isomorphism

\[
\text{Hom}_{D_{A^1}(k)}(1_k, C_{\text{st}A^1}(X)) \cong \text{Hom}_{D_{A^1}(k)}(C_{\text{st}A^1}(X)^\vee, 1_k).
\]

Fortunately, we will use later duality via 6 operations formalism of J. Ayoub, which is good enough for our main purpose. We end up this section by a definition:
**Definition 3.9.** Let $k$ be a field. One defines the geometric stable $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1,\text{gm}}(k)$ over $k$ as the thick subcategory of $D_{\mathbb{A}^1}(k)$ generated by $C^*_*(X)$, where $X \in \text{Sm}/k$.

4. E-Motives

4.1. E-Correspondences. Let $k$ be a field and we denote by $SH(k)$ the motivic stable homotopy category. Throughout this section we fix a motivic spectrum $E \in SH(k)$ together with a multiplication map

$$\mu_E : E \wedge_{S}^L E \to E$$

and a unit map

$$\varphi_E : S \to E,$$

such that the following diagrams commute

$$
\begin{array}{ccc}
E & \xrightarrow{id \wedge \varphi_E} & E \wedge_{S}^L E \\
\downarrow{\mu_E} & & \downarrow{\mu_E} \\
E & & E \\
\end{array}
\quad
\begin{array}{ccc}
E \wedge_{S}^L E \wedge_{S}^L E & \xrightarrow{id \wedge \mu_E} & E \wedge_{S}^L E \\
\downarrow{\mu_E \wedge id} & & \downarrow{\mu_E} \\
E \wedge_{S}^L E & \xrightarrow{\mu_E} & E
\end{array}
$$

Such a triple $(E, \mu_E, \varphi_E)$ is called a motivic ring spectrum.

**Proposition 4.1.** Let $X, Y, Z, W \in \text{SmProj}(k)$. Let $\alpha \in SH(k)[\Sigma_{T,+}^\infty X, \Sigma_{T,+}^\infty Y \wedge_{S}^L E]$, $\beta \in SH(k)[\Sigma_{T,+}^\infty Y, \Sigma_{T,+}^\infty Z \wedge_{S}^L E]$ and $\gamma \in SH(k)[\Sigma_{T,+}^\infty Z, \Sigma_{T,+}^\infty W \wedge_{S}^L E]$. Let’s denote

$$\beta \circ_M \alpha : \Sigma_{T,+}^\infty X \xrightarrow{\alpha} \Sigma_{T,+}^\infty Y \wedge_{S}^L E \xrightarrow{\beta \wedge id_{E}} \Sigma_{T,+}^\infty Z \wedge_{S}^L E \xrightarrow{id_{Z} \wedge \mu_{E}} \Sigma_{T,+}^\infty W \wedge_{S}^L E$$

and similarly for $\gamma \circ_M \beta$. Then $\circ_M$ is associative and unital.

**Proof.** Both $\gamma \circ_M (\beta \circ_M \alpha)$ and $(\gamma \circ_M \beta) \circ_M \alpha$ are equal to the following composition

$$\Sigma_{T,+}^\infty X \xrightarrow{\alpha} \Sigma_{T,+}^\infty Y \wedge_{S}^L E \xrightarrow{\beta \wedge id_{E}} \Sigma_{T,+}^\infty Z \wedge_{S}^L E \xrightarrow{id_{Z} \wedge \mu_{E}} \Sigma_{T,+}^\infty W \wedge_{S}^L E \xrightarrow{id_{W} \wedge \mu_{E}} \Sigma_{T,+}^\infty W \wedge_{S}^L E.$$



**Definition 4.2.** The category of $E$-correspondences $\text{Corr}_E(k)$ is defined as:

$$\text{Obj}(\text{Corr}_E(k)) = \text{Obj}(\text{SmProj}(k))$$

and

$$\text{Corr}_E(k)(X, Y) = SH(k)[\Sigma_{T,+}^\infty X, \Sigma_{T,+}^\infty Y \wedge_{S}^L E],$$

where the composition

$$\circ_M : \text{Corr}_E(k)(X, Y) \otimes \text{Corr}_E(k)(Y, Z) \to \text{Corr}_E(k)(X, Z), \quad (\alpha, \beta) \mapsto \beta \circ_M \alpha$$

is defined as

$$\beta \circ_M \alpha : \Sigma_{T,+}^\infty X \xrightarrow{\alpha} \Sigma_{T,+}^\infty Y \wedge_{S}^L E \xrightarrow{\beta \wedge id_{E}} \Sigma_{T,+}^\infty Z \wedge_{S}^L E \xrightarrow{id_{Z} \wedge \mu_{E}} \Sigma_{T,+}^\infty Z \wedge_{S}^L E.$$
Proposition 4.3. There is a functor
\[ h : \text{SmProj}(k) \to \text{Corr}_E(k), \quad X \mapsto X, \]
which sends a morphism \( f : X \to Y \) of \( k \)-schemes to
\[ \Sigma_{T^+}^\infty(f) \wedge \varphi : \Sigma_{T^+}^\infty X = \Sigma_{T^+}^\infty X \wedge_S L \to \Sigma_{T^+}^\infty Y \wedge_S L E. \]

Proof. The identity morphism in \( \text{Corr}_E(k)(X, X) \) is given by
\[ id_X \wedge \varphi : \Sigma_{T^+}^\infty X \wedge_S L \to \Sigma_{T^+}^\infty X \wedge_S L E. \]
Let \( \alpha \in \text{Corr}_E(k)(X, X) \) be an arbitrary \( E \)-correspondence. By definition we have
\[ \alpha \circ (id_X \wedge \varphi) = (id_X \wedge id) \circ (\alpha \wedge id) \circ (id_X \wedge \varphi). \]
Since \( E \) is a ring spectrum, we must have \( \alpha \circ (id_X \wedge \varphi) = \alpha \). Similarly, \( (id_X \wedge \varphi) \circ \alpha = \alpha \).

We check the compatibility of the composition laws. Let \( X \stackrel{f}{\to} Y \stackrel{g}{\to} Z \) be morphisms of \( k \)-schemes. By definition we have
\[ h(g \circ f) = \Sigma_{T^+}^\infty(g \circ f) \wedge \varphi \]
and
\[ h(g) \circ h(f) = (id_Z \wedge id) \circ (\Sigma_{T^+}^\infty(g) \wedge \varphi \wedge id) \circ (\Sigma_{T^+}^\infty(f) \wedge \varphi). \]
The equality \( h(g \circ f) = h(g) \circ h(f) \) follows from the fact that \( E \) is a ring spectrum. \( \square \)

Let \( \text{Spect}_T^\Sigma(k) \) be the model category of symmetric motivic \( T \)-spectra \([\text{Jar00}]\). Following \([\text{CD10}, \text{Deg13, §2.2}]\) we call \( E \in \text{Spect}_T^\Sigma(k) \) a strict motivic ring spectrum, if \( E \) is a commutative monoid object in \( \text{Spect}_T^\Sigma(k) \). An \( E \)-module spectrum is a pair \((M, \gamma_M)\), where \( M \in \text{Spect}_T^\Sigma(k) \) and \( \gamma_M : M \wedge E \to M \), such that the following diagrams commute:

\[ S \wedge M \xrightarrow{\varphi \wedge id} E \wedge M \xrightarrow{\gamma_M} M \]
\[ \begin{array}{ccc}
E \wedge E \wedge M & \xrightarrow{\mu \wedge id} & E \wedge M \\
\downarrow{\text{id} \wedge \gamma_M} & & \downarrow{\gamma_M} \\
E \wedge M & \xrightarrow{\gamma_M} & M
\end{array} \]

Given two \( E \)-modules \((M, \gamma)\) and \((N, \gamma_N)\), an \( E \)-module map is a map \( f : M \to N \), such that the following diagram commutes:

\[ \begin{array}{ccc}
E \wedge M & \xrightarrow{id \wedge f} & E \wedge N \\
\downarrow{\gamma_M} & & \downarrow{\gamma_N} \\
M & \xrightarrow{f} & N
\end{array} \]

Given a strict motivic ring spectrum \( E \) one can form the model category \( E - \text{Mod}_T^\Sigma \) of \( E \)-modules with respect to the symmetric monoidal model category \( \text{Spect}_T^\Sigma(k) \) (see e.g
and there is a Quillen adjunction of model categories (we will return to this point in the last discussion in the Appendix):

\[ \land_E : \text{Spect}^\Sigma_T(k) \rightleftharpoons \text{E} - \text{Mod}^E : U, \]

where \( U \) denotes the forgetful functor. This Quillen adjunction induces an adjunction between homotopy categories:

\[ (4.1) \quad \land^E_S : SH(k) \rightleftharpoons Ho_k(\text{E} - \text{Mod}) : RU, \]

where we denote by \( Ho_k(\text{E} - \text{Mod}) \) the homotopy category associated to the category of strict \( \text{E} \)-modules.

**Theorem 4.4.** Let \( k \) be a field and \( \text{E} \in \text{Spect}^\Sigma_T(k) \) be a strict motivic ring spectrum. There is a functor

\[ \text{Corr}_E(k) \to Ho(\text{E} - \text{Mod}), \quad X \mapsto \Sigma^\infty_{T,+} X \land_S^E. \]

**Proof.** Recall that we may regard \( \Sigma^\infty_{T,+} X \land_S^E \) as an \( \text{E} \)-modules via the map

\[ \Sigma^\infty_{T,+} X \land_S^E \cong \text{E} \land_S^E \]

Let us denote the association above by

\[ F : \text{Corr}_E(k) \to Ho(\text{E} - \text{Mod}), \quad X \mapsto \Sigma^\infty_{T,+} X \land_S^E. \]

\( F \) maps on morphisms as following: Given \( \alpha : \Sigma^\infty_{T,+} X \to \Sigma^\infty_{T,+} Y \land_S^E \), we associate

\[ \Sigma^\infty_{T,+} X \land_S^E \cong \text{E} \land_S^E \alpha \land S \to \Sigma^\infty_{T,+} Y \land_S^E \land_S^E \to \Sigma^\infty_{T,+} Y \land_S^E \]

We have to check firstly, that \( (id_Y \land \mu_E) \circ (\alpha \land id_E) \) is a morphism of \( \text{E} \)-modules. Since \( \text{E} \) is a ring spectrum, there is a commutative diagram

Now we have to check the compatibility of the composition laws. Given \( \alpha : \Sigma^\infty_{T,+} X \to \Sigma^\infty_{T,+} Y \land_S^E \) and \( \beta : \Sigma^\infty_{T,+} Y \to \Sigma^\infty_{T,+} Z \land_S^E \). Then \( F(\beta) \circ F(\alpha) \) is the following composition

\[ \Sigma^\infty_{T,+} X \land_S^E \alpha \land S \to \Sigma^\infty_{T,+} Y \land_S^E \land_S^E \to \Sigma^\infty_{T,+} Y \land_S^E \beta \land S \to \Sigma^\infty_{T,+} Z \land_S^E \land_S^E \to \Sigma^\infty_{T,+} Z \land_S^E. \]

The composition \( F(\beta \circ_M \alpha) \) is

\[ \Sigma^\infty_{T,+} X \land_S^E \beta \land S \circ_M \alpha \land S \to \Sigma^\infty_{T,+} Z \land_S^E \land_S^E \to \Sigma^\infty_{T,+} Z \land_S^E, \]

where

\[ \beta \circ_M \alpha : \Sigma^\infty_{T,+} X \to \Sigma^\infty_{T,+} Z \land_S^E \land_S^E. \]
Hence, $F(\beta \circ_M \alpha)$ is the following composition

$$
\Sigma_{T,+}^\infty X \wedge_S^L E \xrightarrow{\alpha \wedge \text{id}_E} \Sigma_{T,+}^\infty Y \wedge_S^L E \wedge_S^L E \xrightarrow{\beta \wedge \text{id}_\wedge \wedge \text{id}_E} \Sigma_{T,+}^\infty Z \wedge_S^L E \wedge_S^L E \xrightarrow{\text{id}_Z \wedge \mu_E \wedge \text{id}_E} \Sigma_{T,+}^\infty Z \wedge_S^L E \wedge_S^L E \xrightarrow{\beta \wedge \text{id}_E} \Sigma_{T,+}^\infty Y \wedge_S^L E.
$$

Since $E$ is a ring spectrum, we have a commutative diagram

$$
\begin{array}{ccc}
\Sigma_{T,+}^\infty Y \wedge_S^L E \wedge_S^L E & \xrightarrow{\beta \wedge \text{id}_\wedge \wedge \text{id}_E} & \Sigma_{T,+}^\infty Z \wedge_S^L E \wedge_S^L E \wedge_S^L E \\
\text{id}_Y \wedge \mu_E & & \text{id}_Z \wedge \mu_E \wedge \text{id}_E \\
\Sigma_{T,+}^\infty Y \wedge_S^L E & \xrightarrow{\beta \wedge \text{id}_E} & \Sigma_{T,+}^\infty Y \wedge_S^L E
\end{array}
$$

This implies that $F(\beta \circ_M \alpha) = F(\beta) \circ F(\alpha)$. 

**Definition 4.5.** Let $k$ be a field and $E \in \text{Spect}_{T}^\Sigma(k)$ be a strict motivic ring spectrum. We define the category $\text{Mot}_E(k)$ of pure $E$-motives over $k$ to be the smallest pseudo-abelian subcategory of $H_0(k(E - \text{Mod})$ generated as an additive category by $\{\Sigma_{T,+}^\infty X \wedge_S^L E | X \in \text{SmProj}(k)\}$.

**Remark 4.6.** We know that if $\text{char}(k) = 0$ then there is an equivalence of categories

$$
H_0(k(HZ - \text{Mod}) \cong \text{DM}(k),
$$

where $\text{DM}(k)$ denotes the category of big Voevodsky’s motives (cf. [RO08]). As the category of pure Grothendieck-Chow motives $\text{Chow}(k) \hookrightarrow \text{DM}(k)$ is embedded fully faithful into $DM(k)$, we raise a question: is $\text{Mot}_{HZ}(k)$ equivalent to $\text{Chow}(k)$ via the equivalence above? We only know that

$$
\text{Mot}_{HZ}(k)(X, Y) \cong SH(k)[\Sigma_{T,+}^\infty X, \Sigma_{T,+}^\infty Y \wedge_S^L HZ] \cong H^2_{M}((ny + dy), (ny + dy))(X_+ \wedge Th(V_Y), Z) \\
\cong H^{2dy, dy}(X \times Y, Z) \cong CH^{dy}(X \times Y),
$$

where the first isomorphism comes from the adjunction

$$
- \wedge_S^L HZ : SH(k) \cong H_0(k(HZ - \text{Mod}).
$$

The second isomorphism comes from duality, the third isomorphism is the Thom isomorphism for motivic cohomology and the last isomorphism is the comparison isomorphism of Voevodsky ([MVW06 Cor. 19.2]). The question is, if these isomorphisms are compatible with the equivalence $H_0(k(HZ - \text{Mod}) \cong \text{DM}(k)$? It seems the problem with the first three isomorphisms is not difficult, however it seems that the problem with the last isomorphism is hard.

**Corollary 4.7.** Let $k$ be a field and $E$ be a strict motivic ring spectrum. There is a functor $\text{Mot}_E(k) \rightarrow SH(k)$.

**Proof.** This follows from the adjunction. 

\[\square\]

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4.2. **Functoriality in motivic stable homotopy.** Following [Ay08], we recall that the stable homotopy category of schemes defines a 2-functor from the category of quasi-projective smooth schemes over a field $QSProjSm/k$ to the category of symmetric monoidal closed triangulated categories. Remark that the six operations formalism works much more general than what we here require. However we restrict ourselves only to $QSProjSm/k$, since it is already enough for our aim. We will list now a minimal list of properties of the six operations formalism: for any morphism of schemes $f : T \to S$, there is a pullback functor

$$f^* : SH(S) \to SH(T),$$

such that $(f \circ g)^* = g^* \circ f^*$. Moreover,

1. One has an adjunction for any morphism of schemes $f : T \to S$

$$f^* : SH(S) \rightleftarrows SH(T) : f_*.$$

If $f$ is smooth, then one has an adjunction

$$f^* : SH(T) \rightleftarrows SH(S) : f^*.$$

2. Given a cartesian square

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow g & & \downarrow f \\
T & \xrightarrow{p} & S
\end{array}
\]

and assume $f$ is smooth, then

$$f^#p^* \xrightarrow{\cong} g^#q^*.$$

3. Let $f : Y \to X$ be a smooth morphism, $\mathcal{E} \in SH(Y)$ and $\mathcal{F} \in SH(X)$, the natural transformation

$$f^#(\mathcal{E} \otimes f^*\mathcal{F}) \xrightarrow{\cong} f^#\mathcal{E} \otimes \mathcal{F}$$

is an isomorphism.

4. Let $i : Z \hookrightarrow X$ be a closed immersion with complement $j : U \hookrightarrow X$, then there is a distinguished triangle

$$j^#j^* \to \text{Id} \to i_*i^* \xrightarrow{\pm 1}$$

5. For any closed immersion $i : Z \hookrightarrow X$, one has an adjunction

$$i_* : SH(Z) \rightleftarrows SH(X) : i^!$$

6. Given a cartesian square

\[
\begin{array}{ccc}
T & \xrightarrow{k} & Y \\
\downarrow g & & \downarrow f \\
Z & \xrightarrow{i} & X
\end{array}
\]

where $i : Z \hookrightarrow X$ is a closed immersion, then one has an isomorphism

$$f^*i_* \xrightarrow{\cong} k_*g^*$$
(7) Let \( i : Z \hookrightarrow X \) be a closed immersion, \( \mathcal{E} \in SH(Z) \) and \( \mathcal{F} \in SH(X) \), the natural transformation
\[
i_*(\mathcal{E} \wedge i^* \mathcal{F}) \xrightarrow{\cong} i_* \mathcal{E} \wedge \mathcal{F}
\]
is an isomorphism.

(8) For any separated morphism of finite type \( f : Y \to X \), there is an adjunction
\[
f_* : SH(Y) \rightleftarrows SH(X) : f^!
\]

(9) For a smooth separated morphism of finite type \( f : Y \to X \) with the relative tangent bundle \( T_f \) there are canonical natural isomorphisms, which are dual to each other
\[
f_* \circ f_! \cong f_!(Th_Y(T_f) \wedge_Y -), \quad f^* \cong Th_Y(-T_f) \wedge_Y f^!
\]
Moreover, for any separated morphism of finite type \( f : Y \to X \), there exist natural isomorphisms
\[
Ex(f_!^* \wedge) : (f_! K) \wedge_X L \xrightarrow{\cong} f_!(K \wedge_Y f^* L),
\]
\[
\text{Hom}_X(f_! L, K) \xrightarrow{\cong} f_! \text{Hom}_Y(L, f^! K),
\]
\[
f^! \text{Hom}_X(L, M) \xrightarrow{\cong} \text{Hom}_Y(f^* L, f^! M).
\]

(10) If \( f : Y \to X \) is a smooth projective morphism then \( f_!(1_Y) \) is strongly dualizable in \( SH(X) \) with the dual
\[
D_X(f_!(1_Y)) = f_! Th_Y(-T_f).
\]
Furthermore, one has \( D_X(f_! K) \cong f_! D_Y(K \wedge_Y Th_Y(T_f)) \), \( \forall K \in SH(Y) \).

We will need some facts about cohomology with supports in the next subsection.

4.3. **Cohomology with supports.** Let \( S = \text{Spec } k \). We consider the category \( SH(k) \). For a ring spectrum \( \mathcal{E} \in SH(k) \) and a closed pair \((X,Z)\), where \( \pi_X : X \to S \) is a smooth quasi-projective \( k \)-scheme and \( i : Z \hookrightarrow X \) a smooth closed subscheme, one defines the cohomology with support as
\[
\mathcal{E}^{p,q}_Z(X) = SH(S)[X/X - Z, \mathcal{E} \wedge S^{p,q}] \cong SH(X)[i_* (1_Z), \mathcal{E}_X \wedge S^{p,q}] \cong SH(Z)[1_Z, \pi_X^* \mathcal{E}_X \wedge S^{p,q}],
\]
where we write \( \mathcal{E}_X = \pi_X^* \mathcal{E} \). As \( \Sigma_{T_X X/X - Z} = \pi_X^* i_* (1_Z) \) in \( SH(k) \), so the first isomorphism follows from the adjunction
\[
\pi_X^* : SH(X) \rightleftarrows SH(S) : \pi_X^*
\]
and the last isomorphism comes from the adjunction
\[
i_* : SH(Z) \rightleftarrows SH(X) : i^*.
\]
If \( f : Y \to X \) is a smooth morphism of smooth quasi-projective \( S \)-schemes we have a canonical homomorphism
\[
f^* : \mathcal{E}^{p,q}_Z(X) \to \mathcal{E}^{p,q}_T(Y),
\]
where \( T = Y \times_X Z \) defined as following: Consider the commutative diagram
\[
\begin{array}{ccc}
T & \xrightarrow{f} & Y \\
g \downarrow & & \downarrow f \\
Z & \xrightarrow{f} & X
\end{array}
\]
For a morphism
\[ \alpha : i_* 1_Z \to E_X \land S^{p,q} \]
we can associate to a morphism
\[ f^* \alpha : j_* 1_T \cong j_* g^* 1_Z \overset{(E^*_Z)^{-1}}{\cong} f^* i_* 1_Z \to f^* E_X \land S^{p,q} \overset{\cong}{\to} E_Y \land S^{p,q}. \]

If \( T \overset{j}{\hookrightarrow} Z \overset{i}{\hookrightarrow} X \) are closed immersions, we can define a pushforward on cohomology with supports
\[ j_i : E_T^{p,q}(X) \to E_Z^{p,q}(X) \]
as following: Given a morphism \( \alpha : X/X - T \to E_X \land S^{p,q} \) we associate \( j_i(\alpha) = \alpha \circ j \), where \( j : X/X - Z \to X/X - T \) is the canonical morphism in \( SH(X) \) induced by the immersion \( X - Z \hookrightarrow X - T \). If \( \alpha \in E_T^{p,q}(X) \) and \( \beta \in E_Z^{m,n}(X) \) we define their product in \( E_Z^{p+m,q+n}(X) \) as a morphism
\[ \alpha \cup \beta : X/X - Z \overset{j}{\to} X/(X - Z) \land X/(X - Z) \overset{\alpha \land \beta}{\to} E \land S^{p+m,q+n} \overset{\beta \cdot p}{\to} E \land S^{p+m,q+n}. \]

If \( \xi/X \) is a vector bundle over a smooth \( k \)-scheme \( X \) with the zero section \( s_0 : X \to \xi \), then the \( E \)-cohomology of the Thom spectrum \( Th(\xi) \) is
\[ E^{p,q}(Th(\xi)) = E_X^{p,q}(\xi). \]
The pushforward defined as above works only for closed immersions. We will define later pushforward on \( E \)-cohomology of Thom spectrum for projective smooth morphism using duality.

4.4. Relation to the category of twisted \( E \)-correspondences.

**Notation 4.8.** For a quasi-projective smooth \( k \)-scheme \( \pi_X : X \to \text{Spec} \, k \) and a vector bundle \( p_\xi : \xi \to X \) with 0-section \( s_X : X \to \xi \) we will write \( Th_X(\xi) = p_\xi # s_X!(1_X) \) for the Thom transformation \( Th(s_X, p_\xi) = p_\xi # s_X! \) applying on \( 1_X \). \( Th_X(\xi) \) is an object in \( SH(X) \) and \( Th_X(-\xi) = i_X^! p_\xi^!(1_X) \) for its inverse as the inverse Thom transformation \( Th^{-1}(s_X, p_\xi) \) applying on \( 1_X \). The Thom spectrum will be denoted by \( Th(\xi/X) \), which means
\[ Th(\xi/X) = \pi_X # Th_X(\xi) = \pi_X # p_\xi # s_X!(1_X) \cong \pi_X # p_\xi # s_X!; i_X^! 1_k. \]

Sometime we only write \( Th(\xi) \) for the Thom spectrum, if it is clear which scheme \( X \) we talk about. One can see easily that this definition coincides with the traditional definition of Thom spectrum as follow: Let \( j : \xi - s_X(X) \hookrightarrow \xi \) be the open immersion with the complement \( s_X : X \to \xi \). One has a localization sequence
\[ j_# j^!(1_\xi) \to 1_\xi \to s_X^* s_X^!(1_\xi). \]
Applying \( \pi_# p_\xi # \) and as \( s_X \ast \cong s_X! \) is a natural 2-isomorphism one has a natural isomorphism in \( SH(k) \):
\[ Th_X(\xi) \cong \sum_{\Xi} \xi/\xi - s_X(X). \]

Let \( E \in SH(k) \) be a ring spectrum and \( X/k \) a quasi-projective smooth \( k \)-scheme. Let \( p_\xi : \xi \to X \) be a vector bundle of rank \( r \) with the zero section \( s : X \to \xi \). We define \( E \)-cohomology of \( X \) twisted by a vector bundle as
\[ E^{p,q}(X, \xi) = SH(X)[1_X, s^! p_\xi^* E^{2r,r}_X \land S^{p,q}] = SH(X)[1_X, Th_X(-\xi) \land X E^{2r,r}_X \land S^{p,q}]. \]
where we write $E^{2r,r}_X = E_X \wedge S^{2r,r}$. We denote by $E^{*,*}(X, \xi)$ the bigraded ring

$$E^{*,*}(X, \xi) = \oplus_{p,q} E^{p,q}_X(X, \xi).$$

Remark that $E^{*,*}(X, \xi)$ is bigraded ring. Even if $E$ is a commutative ring spectrum, $E^{*,*}(X, \xi)$ is never bigraded commutative. If $\xi \in V(X)$ is a virtual vector bundle of rank $r < 0$ then $p_\xi : -\xi \to X$ is an actual vector bundle, so we define

$$E^{p,q}(X, \xi) = SH(X)[1_X, p_\xi^* S^{2r,r}_X \wedge S^{p,q}].$$

This group has the following interpretation by Jouanolou trick: As $X$ is quasi-projective, so we have an immersion $i : X \hookrightarrow \mathbb{P}^N$. Via the Segre embedding $\mathbb{P}^N \times \mathbb{P}^N \hookrightarrow \mathbb{P}^{N^2 + 2N}$, $U$ is an affine variety. Let

$$U = \mathbb{P}^N \times \mathbb{P}^N - \text{Proj} \ k[x_0, \cdots, x_N, y_0, \cdots, y_N] / \sum_{i=0}^N x_i y_i.$$

$pr_1 : U \to \mathbb{P}^N$ is an $\mathbb{A}^N$-bundle. Consider the pullback diagram

$$\begin{array}{ccc}
U & \xrightarrow{i^*} & U \\
\downarrow \pi & & \downarrow \text{pr}_1 \\
X & \xrightarrow{i} & \mathbb{P}^N
\end{array}$$

Then

$$E^{p,q}(X, \xi) \cong SH(k)[Th(\xi/U), E^{2(r+n),(r+n)} \wedge S^{p,q}],$$

where $\xi$ is an actual vector bundle on $U$, such that $\pi^* \xi \oplus O^n = \zeta$.

**Proposition 4.9.** Let $f : \xi \xrightarrow{\cong} \xi'$ be an isomorphism of vector bundles on $X$

$$\begin{array}{ccc}
\xi & \xrightarrow{f} & \xi' \\
\downarrow \cong \cong & & \downarrow \cong \\
X & \xrightarrow{p_\xi} & \xi' \xleftarrow{p_{\xi'}} & X
\end{array}$$

There is a natural isomorphism

$$E^{p,q}(X, \xi) \cong E^{p,q}(X, \xi').$$

**Proof.** Consider the Cartesian squares

$$\begin{array}{ccc}
X \xrightarrow{s} \xi & \xrightarrow{p_\xi} & X \\
\downarrow \cong & & \downarrow \cong \\
X \xleftarrow{p_{\xi'}} & \xi' \xrightarrow{f_{\cong}} & X
\end{array}$$

One has two 2-isomorphisms ([Ay08, §1.5.5])

$$Th_X(s, p_\xi) \xrightarrow{\cong} Th_X(s', p_{\xi'}'),$$

and

$$Th_X^{-1}(s', p_{\xi'}) \xrightarrow{\cong} Th_X^{-1}(s, p_\xi),$$

which prove the Proposition. \qed
Proposition 4.10. If $E$ is orientable in sense of [CD10] Def. 12.2.2, then there is a natural isomorphism

$$E^{p,q}(X, \xi) \overset{\cong}{\longrightarrow} E^{p,q}(X).$$

Proof. Since $E$ is orientable, one has by [CD10] Thm. 2.4.50 (3) a canonical natural isomorphism

$$p_\xi^*E_X \overset{\cong}{\longrightarrow} p_\xi^!E_X \wedge S^{-2r,-r}.$$

This induces a natural isomorphism

$$E^{p,q}(X, \xi) = SH(X)[1_X, s^!p_\xi^*E^{2r,r}_X \wedge S^{p,q}] \overset{\cong}{\longrightarrow} SH(X)[1_X, s^!p_\xi^*E_X \wedge S^{p,q}] = E^{p,q}(X).$$

□

Proposition 4.11. (twisted Thom isomorphism) Let $X/k$ be a quasi-projective smooth $k$-scheme and $p_\xi: \xi \to X$ be a vector bundle of rank $r$ with the zero section $s: X \to \xi$. One has a natural isomorphism

$$th_X^E(\xi): E^{p,q}(X, \xi) \cong E^{p+2r,q+r}(Th(\xi)),$$

which we call the twisted Thom isomorphism.

Proof. We have two adjunctions

$$s_!: SH(X) \Rightarrow SH(\xi) : s^!, \quad p_\xi^!: SH(\xi) \Rightarrow SH(X) : p_\xi^*.$$

Hence, we have

$$E^{p,q}(X, \xi) = SH(X)[1_X, s^!p_\xi^*E^{2r,r}_X \wedge S^{p,q}] \cong$$

$$\cong SH(X)[p_\xi^!s_!(1_X), E^{2r,r}_X \wedge S^{p,q}] \cong E^{p+2r,q+r}(Th(\xi)),$$

where the last natural isomorphism is induced by the adjunction $(\pi_X^!, \pi_X^*)$, where $\pi_X: X \to \text{Spec } k$ is the structure morphism. So for a morphism

$$\alpha: 1_X \to s^!p_\xi^*E^{2r,r}_X \wedge S^{p,q}$$

the twisted Thom isomorphism is explicitly given by

$$th_X^E(\xi)(\alpha) = \varepsilon(\pi_X^!, \pi_X^*) \circ \varepsilon(p_\xi^!s_!) \circ p_\xi^* \circ \varepsilon(s_!, s^!) \circ s_!(\alpha).$$

□

Example 4.12. The twisted Chow-Witt group $\widetilde{CH}^p(X, \det \xi)$ defined by J. Fasel (cf. [Fas07] and [Fas08]) and also by F. Morel ([Mor12]) is an example of twisted cohomology. One has a natural isomorphism

$$\widetilde{CH}^p(X, \det \xi) \overset{\text{def}}{=} H^p_{Nis}(X, K_p^{MW}(\det \xi)) \cong H(K_p^{MW})^{2p,p}(X, \xi),$$

where $H(K_p^{MW})$ denotes the Eilenberg-Maclane spectrum associated to the homotopy module $K_p^{MW}$. We will discuss later about $H(K_p^{MW})$ after introducing the homotopy t-structure.

Before going further, we want to give a list of properties of Thom transformations that we will need for our constructions.
Proposition 4.13. [Ay08] Prop. 2.3.19 Let $X$ be a quasi-projective $k$-scheme and $\xi/X$ be a vector bundle. Let $f : Y \to X$ be a morphism. Then one has two natural 2-isomorphisms

$$f^* Th_X(\xi) \cong Th_Y(f^* \xi) f^*, \quad f^* Th_X(-\xi) \cong Th_Y(-f^* \xi) f^*,$$

which satisfy: For all $(K, L) \in \text{Obj}(SH(X)^2)$, there are two commutative diagrams

$$f^* K \wedge_Y (f^* Th_X(\xi)L) \cong f^*(K \wedge_X Th_X(\xi)L) \cong f^* Th_X(\xi)(K \wedge_X L)$$

and

$$f^* K \wedge_Y Th_Y(f^* \xi)f^*L \cong Th_Y(f^* \xi)(f^* K \wedge_Y f^* L) \cong Th_Y(f^* \xi)f^*(K \wedge_X L)$$

Proposition 4.14. [Ay08] Prop. 2.3.20 Let $f : Y \to X$ be a $k$-morphism of quasi-projective schemes and $\xi/X$ be a vector bundle. There are two natural 2-isomorphisms

$$Th_X(\xi) f^* \cong f_* Th_X(f^* \xi), \quad Th_X(-\xi) f^* \cong f_* Th_Y(-f^* \xi),$$

such that the following diagrams commute for all $(K, L) \in SH(X) \times SH(Y)$:

$$K \wedge_X Th_X(\xi) f_* L \cong K \wedge_X f_* Th_Y(f^* \xi)L \cong f_*(f^* K \wedge_Y Th_Y(f^* \xi)L)$$

and

$$K \wedge_X Th_X(-\xi) f_* L \cong K \wedge_X f_* Th_Y(-f^* \xi)L \cong f_*(f^* K \wedge_Y Th_Y(-f^* \xi)L)$$

Let $f : Y \to X$ be any morphism of finite type and separated of quasi-projective smooth $k$-schemes. In the following we define a pullback map on twisted $E$-cohomology

$$E^{p,q}(X, \xi) \to E^{p,q}(Y, f^* \xi).$$

Consider the functor $f^* : SH(X) \to SH(Y)$. $f^*$ induces a map

$$E^{p,q}(X, \xi) = SH(X)[1_x, s^! \pi^*_X E^{2r,r}_{X} \wedge S^{p,q}] \to SH(Y)[f^* 1_x, f^* s^! \pi^*_Y E^{2r,r}_{X} \wedge S^{p,q}] =$$

$$= SH(Y)[1_Y, f^* s^! \pi^*_Y E^{2r,r}_{X} \wedge S^{p,q}].$$

Let $s_Y$ be the 0-section of the vector bundle $p_{f^* \xi} : f^* \xi \to Y$ and we write $f_{\xi} : f^* \xi \to \xi$. One has an exchange transformation (see [Ay08] Prop. 1.4.15])

$$Ex^{s^!} : f^* s^! \to s^! f^*_\xi,$$
which is the following composition \((s_* \cong s_1, s_Y \cong s_{Y*})\) since \(s\) and \(s_Y\) are closed immersion:

\[
\begin{array}{ccc}
  f^* s^! & \cong & f^* p_{\xi#sX^*} \\
  \downarrow f^* \eta_{(sY#sY^*)} & & \downarrow \eta_{(sY#sY^*)} \\
  s_Y^! s_Y s^* s_Y & \overset{f^* \gamma}{\longrightarrow} & s_Y^! s^* s_Y f^*
\end{array}
\]

where \(f_\xi : f^* \xi \to \xi\) is the induced map on vector bundles. Note that the exchange transformation \(Ex^*\) is an isomorphism, when \(f\) is smooth (\cite[Cor. 1.4.17]{Ay08}). At this point we also notice that for an actual bundle \(\xi\), the Thom transformation \(Th_X(\xi)\) behaves well under pullback of a general morphism, since

\[
f^* Th_X(\xi) = f^* p_{\xi#sX^*} \cong p_{f^* \xi#sY^*} s_Y^! \cong p_{f^* \xi#sY} s_Y^! f^* = Th_Y(f^* \xi),
\]

and we have a natural transformation

\[
Ex: f^* Th_X(-\xi) = f^* s_Y^! \cong p_{f^* \xi#sY} s_Y^! \cong s^!_Y p_{f^* \xi} f^* = Th_Y(-f^* \xi),
\]

which is an isomorphism, if \(f\) is smooth (see \cite[Lem. 1.5.4]{Ay08}). However, he showed that \(Th_X(\xi)\) and \(Th_X(-\xi)\) are inverse to each other \cite[Thm. 1.5.7]{Ay08}, hence \(Th_Y(-f^* \xi) \overset{\cong}{\to} f^* Th_X(-\xi)\) (cf. \cite[Rem. 1.5.10]{Ay08}) for all morphism not necessary smooth \(f\). That is a very crucial point. Now consider the pullback diagram

\[
\begin{array}{ccc}
f^* \xi & \overset{f_\xi}{\longrightarrow} & \xi \\
p_{f^* \xi} & \downarrow & \downarrow p_{\xi} \\
Y & \overset{f}{\longrightarrow} & X
\end{array}
\]

We have a natural isomorphism \(f_\xi^* p_{\xi}^* \cong p_{f^* \xi} f^*\). Hence, we obtain a map

\[
E^{p,q}(X, \xi) \to E^{p,q}(Y, f^* \xi),
\]

which we define as pullback of twisted \(E\)-cohomology.

**Remark 4.15.** The composition of pullback on twisted \(E\)-cohomology \(g^* \circ f^*\) is only defined up to the natural isomorphism \((f \circ g)^* \overset{\cong}{\to} g^* f^*\).

**Remark 4.16.** Let \(a : \xi \overset{\cong}{\to} \xi\) be an automorphism of a vector bundle \(\xi\) of rank \(r\) on \(X\). Then one has the cartesian squares

\[
\begin{array}{ccc}
X & \overset{s^*_X}{\longrightarrow} & \xi & \overset{p_\xi^*}{\longrightarrow} & X \\
\downarrow \text{id} & & \cong & & \downarrow \text{id} \\
X & \overset{s_X}{\longrightarrow} & \xi & \overset{p_\xi}{\longrightarrow} & X
\end{array}
\]

As in \cite[§1.5.5 p. 84]{Ay08} \(a\) induces two 2-isomorphisms between the Thom transformations

\[
\omega(a) : Th(s^*_X, p_\xi^*) = p_{s^*_X#s_X^*} \cong Th(s_X, p_\xi) = p_{s^*_X#s_X^*}
\]

and

\[
\omega^{-1}(a) : Th^{-1}(s_X, p_\xi) = s_X^! p_\xi^* \overset{\cong}{\longrightarrow} Th^{-1}(s^*_X, p_\xi^*) = s^!_X p_\xi^*.
\]

This induces an isomorphism, which is not necessary identity

\[
\bar{\omega}(a) : SH(X)[1_X, s^*_X p_\xi^* E^r_{2r} X, S^{p,q}] \overset{\cong}{\longrightarrow} SH(X)[1_X, s^!_X p_\xi^* E^r_{2r} X, S^{p,q}].
\]
However, the two pullbacks induced on twisted $E$-cohomology along a morphism $f : Y \to X$ must not be on the same target $E^{p,q}(Y, f^* \xi)$, as there are two different pullback diagrams

\[ \begin{array}{ccc}
   f^* \xi & \to & \xi \\
   p_{f^* \xi} \downarrow & & \downarrow p_{\xi} \\
   Y & \to & X \\
\end{array} \quad \begin{array}{ccc}
   f^* \xi & \to & \xi \\
   p'_{f^* \xi} \downarrow & & \downarrow p'_{\xi} \\
   Y & \to & X \\
\end{array} \]

Consequently, there is no problem with maps between $E$-cohomology created by automorphisms of $\xi$.

**Remark 4.17.** Thanks to the Proposition 4.13, the pullback of cohomology of virtual vector bundles is defined in the same way.

**Proposition 4.18.** Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of quasi-projective smooth $k$-schemes. Let $\xi/X$ be a vector bundle. Then one has up to a natural isomorphism induced by a natural 2-isomorphism

\[(f \circ g)^* = g^* \circ f^* : E^{p,q}(X, \xi) \to E^{p,q}(Z, g^* f^* \xi).\]

**Proof.** Consider the chain of pullback bundles

\[ \begin{array}{ccc}
   g^* f^* \xi & \xrightarrow{g_\xi} & f^* \xi \\
   p_{g^* f^* \xi} \downarrow & & \downarrow p_{f^* \xi} \\
   Z & \xrightarrow{g} & Y \\
   \downarrow f & & \downarrow f \\
   X \\
\end{array} \]

Let $s_X, s_Y$ and $s_Z$ be the 0-sections of $\xi, f^* \xi$ and $g^* f^* \xi$ respectively. The functoriality up to a natural isomorphism follows easily from the natural 2-isomorphism

\[(f \circ g)^* s_X^! \cong g^* f^* s_X^! \xrightarrow{Ex^!} g^* s_Y^! f^*_\xi \xrightarrow{Ex^!} s_Z^! g^*_\xi f^*_\xi \cong s_Z^!(f_\xi \circ g_\xi)^*).\]

This motivates us to give the following definition:

**Definition 4.19.** A twisted $E$-cohomology pre-theory is an association, which is contravariant in both variables:

\[ E^{*,*}(-, -) : QSP rojSm(k) \times \mathcal{V} \supset \mathcal{A} \to \text{Ring}^*, \]

where $\text{Ring}^*$ denotes the category of bigraded rings and $\mathcal{V}$ is the 2-category, where objects are categories of virtual vector bundles $\mathcal{V}(X)$ for $X \in QSP rojSm(k)$ and

\[ 1 - \text{Mor}_\mathcal{V}(\mathcal{V}(X), \mathcal{V}(Y)) = \text{Fun}(\mathcal{V}(X), \mathcal{V}(Y)) \]

\[ 2 - \text{Mor}_\mathcal{V}(F, G) = \text{Nat}(F, G). \]

$\mathcal{A}$ is the full subcategory of $QSP rojSm(k) \times \mathcal{V}$ consisting of those pairs $(X, \xi)$, where $X \in QSP rojSm(k)$ and $\xi \in \mathcal{V}(X)$. $\text{Mor}_\mathcal{A}((X, \xi), Y(\eta))$ consists of pairs $(f, g)$, where
$f : X \to Y$ is a morphism of quasi-projective smooth $k$-schemes and $g : \xi \to \eta$ is a bundle map

$$
\begin{array}{c}
\xi \\
\downarrow \\
X
\end{array}
\quad
\begin{array}{c}
\eta \\
\downarrow \\
Y
\end{array}
$$

such that $\xi \to f^*\eta$ is a monomorphism in $\mathcal{V}(X)$. $E^{*,*}(-,-)$ sends such a pair $(X, \xi)$ to $E^{*,*}(X, \xi)$. Given an $\mathcal{A}$-morphism $(f, g) : (X, \xi) \to (Y, \eta)$, $E^{*,*}(-,-)$ sends $(f, g)$ to the following composition

$$E^{*,*}(Y, \eta) \to E^{*,*}(X, f^*\eta) \to E^{*,*}(X, \xi),$$

where $f^*$ is the pullback map on twisted $E$-cohomology constructed as above and the last map is induced by

$$\text{Th}_X(\xi) \to \text{Th}_X(f^*\eta),$$

as $\xi \to f^*\eta$ is a monomorphism in $\mathcal{V}(X)$.

**Proposition 4.20.** Let $f : Y \to X$ be a $k$-morphism of quasi-projective smooth $k$-schemes and $p_\xi : \xi \to X$ be a vector bundle of rank $r$ on $X$. There is a commutative diagram up to a natural isomorphism induced by a natural 2-isomorphism

$$
\begin{array}{c}
E^{p,q}(X, \xi) \\
\downarrow \text{th}_E(\xi) \cong \\
E^{p+2r, q+r}(\text{Th}(\xi)) \\
\downarrow f^* \\
E^{p+2r, q+r}(\text{Th}(f^*\xi))
\end{array}
$$

where $f^* : E^{p+2r, q+r}(\text{Th}(\xi)) \to E^{p+2r, q+r}(\text{Th}(f^*\xi))$ is the pullback given by

$$SH(X)[\text{Th}_X(\xi), E_X^{2r,r} \otimes S^{p,q}] \xrightarrow{f^*} SH(Y)[f^*\text{Th}_X(\xi), f^*E_X^{2r,r} \otimes S^{p,q}] \xrightarrow{E_{\xi}^*} \text{Th}_Y(f^*\xi), E_Y^{2r,r} \otimes S^{p,q}].$$

**Proof.** It is obvious by construction. Remark that for general morphism $f$ we always have the pullback

$$SH(X)[\text{Th}_X(\xi), E_X^{2r,r} \otimes S^{p,q}] \to SH(Y)[\text{Th}_Y(f^*\xi), E_Y^{2r,r} \otimes S^{p,q}].$$

Since $\pi_X : X \to \text{Spec} k$ and $\pi_Y : Y \to \text{Spec} k$ are smooth, then one has the natural isomorphisms via the adjunctions $(\pi_X^\#, \pi_X)$ and $(\pi_Y^\#, \pi_Y)$:

$$SH(X)[\text{Th}_X(\xi), E_X^{2r,r} \otimes S^{p,q}] \cong SH(k)[\pi_X^\#\text{Th}_X(\xi), E_X^{2r,r} \otimes S^{p,q}] = E^{p,q}(\text{Th}(\xi)),$$

and

$$SH(Y)[\text{Th}_Y(f^*\xi), E_Y^{2r,r} \otimes S^{p,q}] \cong SH(k)[\pi_Y^\#\text{Th}_Y(f^*\xi), E_Y^{2r,r} \otimes S^{p,q}] = E^{p,q}(\text{Th}(f^*\xi)).$$

Explicitly, given a morphism

$$\alpha : X \to s_X^1 \rho_\xi^* E_X^{2r,r} \otimes S^{p,q}$$

we have

$$f^*\text{th}_E^X(\alpha) = \varepsilon(s_X, s_X^1) \circ E_{\xi}^*(\alpha)^{-1} \circ \eta(s_Y, s_Y^1) \circ f^* \circ \varepsilon(p_\xi, p_\xi^2) \circ \rho_\xi \circ \varepsilon(s_X, s_X^1) \circ s_X(\alpha)$$

$$\text{p}$$
and
\[
    \text{th}^Y_E(f^*\xi)(f^*\alpha) = \varepsilon_{(p^*f^*\xi, p^*f^*\xi)} \circ p^*f^*\xi \circ \varepsilon_{(\text{sh}_{Y\times X}^{1}, \text{sh}_{Y\times X}^{1})} \circ f_*\alpha^{-1} \circ \eta_{(\text{sh}_{Y\times X}^{1}, \text{sh}_{Y\times X}^{1})} \circ f^*(\alpha).
\]

The two composition are natural isomorphism to each other, as we have the natural 2-isomorphisms:
\[
f^*s_X \xrightarrow{\cong} s_Yf^*_{\xi}, \quad p^*\xi \xrightarrow{\cong} f^*p^*_{\xi}.
\]

Let \( f : Y \to X \) be a smooth projective morphism of projective smooth \( k \)-schemes of relative dimension \( d = \dim(Y) - \dim(X) \) and \( p_{\xi} : \xi \to X \) be a vector bundle of rank \( r \) with the zero section \( s : X \to \xi \). We define in the following a pushforward on twisted \( E \)-cohomology: Consider
\[
    \text{SH}(Y)[1_Y, \text{th}_Y(T_f) \wedge Y s_Y^1 p^*_{f^*}\xi E^2_{Y}(r-d, r-d) \wedge S^{p,q}],
\]
where \( T_f \) is the normal bundle of the diagonal immersion \( \delta : Y \to Y \times_X Y \).

\[
    \text{SH}(Y)[1_Y, \text{th}_Y(T_f) \wedge Y s_Y^1 p^*_{f^*}\xi E^2_{Y}(r-d, r-d) \wedge S^{p,q}] \cong \text{SH}(Y)[\text{th}_Y(f^*\xi) \wedge Y \text{th}_Y(-T_f), E^2_{Y}(r-d, r-d) \wedge S^{p,q}],
\]
where \( \text{th}_Y(-T_f) \in \text{SH}(Y) \) is the inverse of \( \text{th}_Y(T_f) \in \text{SH}(Y) \). Since \( E_Y = f^*E_X \), the adjunction
\[
f^* : \text{SH}(Y) \cong \text{SH}(X) : f^*
\]
gives us a natural isomorphism
\[
    \text{SH}(Y)[\text{th}_Y(f^*\xi) \wedge Y \text{th}_Y(-T_f), E^2_{Y}(r-d, r-d) \wedge S^{p,q}] \cong \text{SH}(X)[f^*\text{th}_Y(f^*\xi) \wedge Y \text{th}_Y(-T_f), E^2_{X}(r-d, r-d) \wedge S^{p,q}]
\]
By the projection formula \( \text{Pr}^*_{\#} \) and since \( \text{th}_Y(f^*\xi) \cong f^*\text{th}_X(\xi) \) as \( \xi \) is an actual bundle, we have then a natural isomorphism
\[
f^*\text{th}_Y(f^*\xi) \wedge Y \text{th}_Y(-T_f)) \cong \text{th}_X(\xi) \wedge_X f^*\text{th}_Y(-T_f).
\]
So we have then a natural isomorphism
\[
    \text{SH}(X)[f^*\text{th}_Y(f^*\xi) \wedge Y \text{th}_Y(-T_f), E^2_{X}(r-d, r-d) \wedge S^{p,q}] \cong \text{SH}(X)[\text{th}_X(\xi) \wedge_X f^*\text{th}_Y(-T_f), E^2_{X}(r-d, r-d) \wedge S^{p,q}].
\]
By \cite{CD10} Prop. 2.4.31 we have
\[
f^*\text{th}_Y(-T_f) \cong D_X(f^*1_Y),
\]
where \( D_X(f^*1_Y) \) means the dual of \( f^*1_Y \) in \( \text{SH}(X) \). Hence there is a natural isomorphism
\[
    \text{SH}(X)[\text{th}_X(\xi) \wedge_X f^*\text{th}_Y(-T_f), E^2_{X}(r-d, r-d) \wedge S^{p,q}] \cong \text{SH}(X)[\text{th}_X(\xi), f^*1_Y \wedge_X E^2_{X}(r-d, r-d) \wedge S^{p,q}].
\]
From the counit of the adjunction \((f^* , f^*)\)
\[
f^*1_Y \cong f^*f^*1_X \to 1_X
\]
we have an induced map
\[ SH(X)[Th_X(\xi), f_#1_Y \wedge_X E^2_{X,(r-d),(r-d)} \wedge S^{p,q}] \rightarrow SH(X)[Th_X(\xi), E^2_{X,(r-d),(r-d)} \wedge S^{p,q}] . \]

By the twisted Thom isomorphism, the later group is
\[ SH(X)[Th_X(\xi), E^2_{X,(r-d),(r-d)} \wedge S^{p,q}] \cong E^{p-2d,q-d}(X, \xi) . \]

Now we define formally:

**Definition 4.21.** Let \( f : Y \rightarrow X \) be a smooth projective morphism of projective smooth \( k \)-schemes of relative dimension \( d = \dim(Y) - \dim(X) \) and \( p_\xi : \xi \rightarrow X \) be a vector bundle. We define
\[ E^{p,q}(Y, f^*\xi - T_f) \overset{def}{=} SH(Y)[1_Y, Th_Y(\xi) \wedge_Y s_Y^{p,q}, E^2_{Y,(r-d),(r-d)} \wedge S^{p,q}] , \]

where \( \pi_{f^*\xi} : f^*\xi \rightarrow Y \) is the pullback bundle and \( s_Y \) is its 0-section. The pushforward map is the induced map constructed as above
\[ E^{p,q}(Y, f^*\xi - T_f) \overset{f_*}{\rightarrow} E^{p-2d,q-d}(X, \xi) . \]

Remark that our definition of projective pushforward \( f_* \) doesn’t require \( E \) to be an oriented cohomology theory, however we need the assumption on smoothness of \( f \). The reason that we choose the notation \( E^{p,q}(Y, f^*\xi - T_f) \) is that this group should behave like the so-called cohomology twisted by formal difference of vector bundles. The shifting in the definition \((-2d,-d)\) reminds us that the inverse Thom transformation \( Th_Y(-T_f) \) should behave like the Thom spectrum of the virtual bundle \(-T_f\) after taking \( \pi_Y \), where \( \pi_Y : Y \rightarrow \Spec k \) is the structure morphism of \( Y \), as the rank of the virtual bundle \(-T_f\) is \(-d\). As already mentioned in §3 we refer the reader to [Rio10, §4] and [Del87] for the discussion on Picard category of virtual bundles. But we remind the reader again that we always work with an actual bundle \( \xi \).

**Remark 4.22.** Let \( Z \overset{g}{\rightarrow} Y \overset{f}{\rightarrow} X \) be a sequence of composable morphisms. Then \( f_* \circ g_* \) is not defined for a trivial reason: One has only a natural 2-isomorphism
\[ e^*_g : Th_Z^1(s_Z,p_{T_g}) \overset{\cong}{\rightarrow} Th_Z^1(s_Z,p_T) \circ g^*Th_Y^1(s_Y,p_T) , \]

which comes from the exact sequence
\[ 0 \rightarrow g^*T_f \rightarrow T_{fg} \rightarrow T_g \rightarrow 0 . \]

The 2-isomorphism \( e^*_g \) is not an identity. Consequently \( f_* \circ g_* \) is only defined up to this specific natural 2-isomorphism.

**Remark 4.23.** Another variant to construct pushforward can be obtained as follows: One has a natural isomorphism via Thom transformation adjunctions Apply the functor \( f_* : SH(Y) \rightarrow SH(X) \) we obtain a map
\[ SH(Y)[1_Y, Th_Y(T_f) \wedge_Y Th(-f^*\xi) \wedge_Y E^2_{Y,(r-d),(r-d)} \wedge S^{p,q}] \overset{f_*}{\rightarrow} SH(X)[f_*1_Y, f_*(Th_Y(T_f) \wedge_Y Th_Y(-f^*\xi) \wedge E^2_{Y,(r-d),(r-d)} \wedge S^{p,q})] . \]
By projection formula $Pr_*^*(f)$ we have a canonical isomorphism

$$Ex^q(f, s_X) \circ Ex_*^q(f, \wedge)^{-1} : f_*(Th_Y(T_f)) \wedge_Y Th_Y(-f^*\xi) \wedge_Y E_Y \xrightarrow{\cong} f_*(Th_Y(T_f)) \wedge_X Th_X(-\xi) \wedge_X E_X,$$

which induces a canonical isomorphism

$$Ex^q(f, s_X) \circ Ex_*^q(f, \wedge)^{-1} : SH(X)[f_*1_Y, f_*(Th_Y(T_f)) \wedge_Y Th_Y(-f^*\xi) \wedge E_Y^{2(d-r), (r-d)} \wedge S^{p,q}] \xrightarrow{\cong} SH(X)[f_*1_Y, f_*(Th_Y(T_f)) \wedge_X Th_X(-\xi) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}].$$

From the unit $\eta(f^*, f_*)(X) : 1_X \to f_*f^*1_X \cong f_*1_Y$ of the adjunction $(f^*, f_*)$ one obtains a map

$$- \circ \eta(f^*, f_*)(X) : SH(X)[f_*1_Y, f_*(Th_Y(T_f)) \wedge_X Th_X(-\xi) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}] \rightarrow SH(X)[1_X, f_*(Th_Y(T_f)) \wedge_X Th_X(-\xi) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}].$$

As $f$ is projective we have $f_* \cong f_!$, and since $f$ is smooth we have the canonical purity isomorphism

$$p_f : f_!(-) \xrightarrow{\cong} f_*(Th_Y(T_f)) \wedge_Y -,$$

which induces a canonical isomorphism

$$- \circ p_f^{-1} : SH(X)[1_X, f_*(Th_Y(T_f)) \wedge_X Th_X(-\xi) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}] \xrightarrow{\cong} SH(X)[1_X, f_!1_Y \wedge_X Th_X(-\xi) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}].$$

The counit $\varepsilon(f_!, f^*)(X) : f_!f^*1_Y \cong f_!f^*1_X \to 1_X$ of the adjunction $(f_!, f^*)$ induces then a map

$$\varepsilon(f_!, f^*)(X) \circ : SH(X)[1_X, f_!1_Y \wedge_X Th_X(-\xi) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}] \rightarrow SH(X)[1_X, Th_X(\xi) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}] = E^{p-2d, q-d}(X, \xi).$$

So we obtain a map defined as the composition of the maps above

$$SH(Y)[1_Y, Th_Y(T_f) \wedge_Y s_Y^1 p_*^r E_Y^{2(r-d), (r-d)} \wedge S^{p,q}] \rightarrow E^{p-2d, q-d}(X, \xi).$$

One can check the two constructions are equivalent. And again the Proposition [4.14] tells us that the composition of pushforward maps $f_* \circ g_*$ is only defined up to a specific natural 2-isomorphism.

**Proposition 4.24.** Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be smooth projective morphisms of projective smooth $k$-schemes of relative dimension $e$ resp. $d$. Let $\xi/X$ be a vector bundle of rank $r$. Then one has up to a natural 2-isomorphism

$$(f \circ g)_* = f_* \circ g_* : E^{p,q}(Z, g^*f^*\xi - T_fg) \to E^{p-2(d+e), q-(d+e)}(X, \xi).$$

**Proof.** We have an exact sequence of vector bundles on $Z$ [EGA4 17.2.3]

$$0 \to g^*T_f \to T_{fg} \to T_g \to 0.$$

So we have an isomorphism (cf. [CD10] Rem. 2.4.52)

$$e_\sigma : Th_Z(T_{fg}) \cong Th_Z(T_g) \wedge_Z Th_Z(g^*T_f) \cong Th_Z(T_f) \wedge_Z g^*Th_Y(T_f),$$

where

$$\sigma : Z \to Y \times_k X,$$
where \( \wedge_Z \) means relative wedge product over \( Z \). Since \( g^* \) is strong monoidal and since all \( f \) and \( g \) are smooth, which means that the \( \wedge_Z \)-inverse object of \( Th_Z(T_{f g}) \) is \( Th_Z(-T_{f g}) \) and \( Th_Z(T_g)^{-1} = Th_Z(-T_g) \) and \( (g^*Th_Y(T_f))^{-1} = g^*(Th_Y(-T_f)) \) (cf. [CD10] Thm. 2.4.50 (3)), hence we have

\[
e_{\gamma} : Th_Z(-T_{f g}) \cong Th_Z(-T_g) \wedge_Z g^*Th_Y(-T_f).
\]

Functoriality of pushforward follows from this isomorphism as follow: We write \( h = f \circ g \). Let \( s_Z : Z \to g^*f^*\xi \) be the 0-section of the vector bundle \( p_{g^*f^*\xi} : g^*f^*\xi \to Z \). Let us recall the notation now: For an adjunction between categories

\[
L : A \Rightarrow B : R,
\]

we denote

\[
\varepsilon_{(L,R)} : LR \to id, \quad \eta_{(L,R)} : id \to RL
\]

the counit and unit of the adjunction \( (L, R) \) respectively. The composition \( f_* \circ g_* \) is by construction the following composition:

\[
E^{p,q}(Z, g^*f^*\xi - T_{f g}) \overset{\text{def}}{=} SH(Z)[1_Z, Th_Z(T_{f g}) \wedge_Z s_Z^*p_{g^*f^*\xi}E_{Z}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(1)} \to
\]

\[
SH(Z)[Th_Z(g^*f^*\xi) \wedge_Z Th_Z(-T_{f g}), E_{Z}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(2)} \to
\]

\[
SH(Y)[g_#(Th_Z(g^*f^*\xi) \wedge_Z Th_Z(-T_{f g})), E_{Y}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(3)} \to
\]

\[
SH(Y)[Th_Y(f^*\xi) \wedge_Y g_#(Th_Z(-T_g)) \wedge_Z g^*Th_Y(-T_f)), E_{Y}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(5)} \to
\]

\[
SH(Y)[Th_Y(f^*\xi) \wedge_Y Th_Y(-T_f) \wedge_Y g_#Th_Z(-T_g), E_{Y}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(6)} \to
\]

\[
SH(Y)[Th_Y(f^*\xi) \wedge_Y Th_Y(-T_f) \wedge D_Y(g_#1_Z), E_{Y}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(7)} \to
\]

\[
SH(Y)[Th_Y(f^*\xi) \wedge_Y Th_Y(-T_f), g_#(1_Z) \wedge_Y E_{Y}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(8)} \to
\]

\[
SH(Y)[Th_Y(f^*\xi) \wedge_Y Th_Y(-T_f), E_{Y}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(9)} \to
\]

\[
SH(Y)[f_#(Th_Y(f^*\xi) \wedge_Y Th_Y(-T_f)), E_{X}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(10)} \to
\]

\[
SH(X)[Th_X(\xi) \wedge_X f_#Th_Y(-T_f), E_{X}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(11)} \to
\]

\[
SH(X)[Th_X(\xi) \wedge_X D_X(f_#1_Y), E_{X}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(12)} \to
\]

\[
SH(X)[Th_X(\xi), f_#(1_Y) \wedge_X E_{X}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}]^{(13)} \to
\]

\[
SH(X)[Th_X(\xi), E_{X}^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] \cong E^{p-2(d+e),q-(d+e)}(X, \xi),
\]

where (1) is the natural isomorphism given by the adjunction of the Thom transformations \( Th(s_Z, p_{g^*f^*\xi}) \) and \( Th_Z(T_{f g}) \), (2) is the natural isomorphism given by the adjunction \( (g_#, g^*) \)

\[
(2)(-) = \varepsilon_{(g_#, g^*)}(-) \circ g_#(-),
\]

(3) is the natural isomorphism given by the projection formula \( Pr^*_#(g) \)

\[
(3)(-) = \varepsilon_{(g_#, g^*)}(-) \circ g_#(\eta_{(g_#, g^*)}(-) \wedge_Z id)(-),
\]
(4) is the natural isomorphism given by
\[ e^\gamma_\#: Th_Z(-T_{fg}) \cong Th_Z(-T_g) \wedge_Z g^*Th_Y(-T_f), \]
(5) is the natural isomorphism given by the projection formula \( Pr^*(g) \), (6) is the natural isomorphism given by duality in \( SH(Y) \):
\[ g_#Th_Z(-T_g) \cong D_Y(g_#1_Z), \]
(7) is the natural isomorphism given by adjunction of duality in \( SH(Y) \)
\[ (7)(-) = (id_{g_#1_Z} \wedge -) \circ (coev_{D_Y(g_#1_Z)} \wedge id_-), \]
(8) is the pushforward induced by the counit \( g_#1_Z \cong g_#g^*1_Y \to 1_Y \)
\[ (8) = \varepsilon_{(g_#,g^*)}(-) \wedge_Y -, \]
(9) is the natural isomorphism given by the adjunction \((f_#, f^*)\)
\[ (9) = \varepsilon_{(f_#, f^*)}(-) \circ f_#(-), \]
(10) is the natural isomorphism given by the projection formula \( Pr^*(f) \)
\[ (10)(-) = \varepsilon_{(f_#, f^*)}(-) \circ f_#(\eta_{(f_#, f^*)} \wedge_Y id)(-), \]
(11) is the natural isomorphism given by duality in \( SH(X) \):
\[ f_#Th_Y(-T_f) \cong D_X(f_#1_Y), \]
(12) is the natural isomorphism given by the adjunction of duality in \( SH(X) \)
\[ (12)(-) = (id_{f_#1_Y} \wedge -) \circ (coev_{D_X(f_#1_Y)} \wedge id_-), \]
and finally (13) is the pushforward induced by the counit \( f_#1_Y \cong f_#f^*1_X \to 1_X \)
\[ (13)(-) = \varepsilon_{(f_#, f^*)}(-) \wedge_X -. \]
\( h_* = (f \circ g)_* \) is the following composition:
\[ E^{p,q}(Z, g^*f^*\xi) \overset{def}{=} SH(Z)[1_Z, Th_Z(T_{fg}) \wedge_Z s_{Z}^{l} p_#g^*f^*\xi E^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] (1') \]
\[ \overset{\text{adj}}{\longrightarrow} \]
\[ SH(Z)|Th_Z(g^*f^*\xi) \wedge Z Th_Z(-T_{fg}), E^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] (2') \]
\[ \overset{\text{adj}}{\longrightarrow} \]
\[ SH(X)|h_#(Th_Z(g^*f^*\xi) \wedge Z Th_Z(-T_{fg})), E^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] (3') \]
\[ \overset{\text{adj}}{\longrightarrow} \]
\[ SH(X)|Th_X(\xi) \wedge_X h_#1_Z, E^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] (4') \]
\[ \overset{\text{adj}}{\longrightarrow} \]
\[ SH(X)|Th_X(\xi), h_#1_Z \wedge_X E^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] (5') \]
\[ \overset{\text{adj}}{\longrightarrow} \]
\[ SH(X)|Th_X(\xi), h_#(1_Z) \wedge X E^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] (6') \]
\[ \overset{\text{adj}}{\longrightarrow} \]
\[ SH(X)|Th_X(\xi), E^{2r-2(d+e),r-(d+e)} \wedge S^{p,q}] \cong E^{p-2(d+e),q-(d+e)}(X, \xi), \]
where (1’) is the natural isomorphism given by the adjunction of the Thom transformations \( Th(s_Z, p_{g^*f^*\xi}) \) and \( Th_Z(T_{fg}) \), (2’) is the natural isomorphism given by the adjunction \((h_#, h^*)\)
\[ (2')(-) = \varepsilon_{(h_#, h^*)}(-) \circ h_#(-), \]
(3’) is the natural isomorphism given by the projection formula \( Pr^*(h) \)
\[ (3')(-) = \varepsilon_{(h_#, h^*)}(-) \circ h_#(\eta_{(h_#, h^*)}(-)) \wedge_Z id)(-), \]
(4') is the natural isomorphism by duality in \(SH(X)\):
\[
h_\# T h_Z(-T f_g) \cong D_X(h_\# 1_Z),
\]
(5') is the natural isomorphism given by the adjunction of duality in \(SH(X)\):
\[
(5')(-) = (\text{id}_{h_\# 1_Z} \land_X -) \circ (\text{coev}_{D_X(h_\# 1_Z)} \land \text{id}_-)
\]
and finally (6') is the pushforward induced by the counit \(h_\# 1_Z \cong h_\# h^* 1_X \to 1_X\):
\[
(6')(\cdot) = \varepsilon_{(h_\#, h^*)}(\cdot) \land_X -.
\]
The maps (1) and (1') are identical. The diagram
\[
\begin{array}{ccc}
(2) & \rightarrow & (2') \\
\downarrow & & \downarrow \\
(\alpha_1) & \rightarrow & \\
\end{array}
\]
commutes, because \(h_\# \cong f_\# \circ g_\#\), where
\[
(a_1) : SH(Y)[g_\#(T h_Z(h^* \xi) \land Z T h_Z(-T f_g)), E_{Y}^{2r-2(d+e),r-(d+e)} \land S^{p,q}] \xrightarrow{\cong}
SH(X)[h_\#(T h_Z(h^* \xi) \land Z T h_Z(-T f_g)), E_{X}^{2r-2(d+e),r-(d+e)} \land S^{p,q}]
\]
is the natural isomorphism induced from the adjunction \((f_\#, f^*)\). Indeed, let
\[
\alpha_1 : T h_Z(h^* \xi) \land Z T h(-T f_g) \rightarrow E_{Z}^{2r-2(d+e),r-(d+e)} \land S^{p,q}
\]
be a morphism. Then one has
\[
(2')(\alpha_1) = \varepsilon_{(h_\#, h^*)}(\cdot) \circ h_\#(\alpha_1),
\]
and
\[
(a_1) \circ (2)(\alpha_1) = (\varepsilon_{(f_\#, f^*)} \circ f_\#(\cdot)) \circ (\varepsilon_{(g_\#, g^*)} \circ g_\#(\cdot))(\alpha_1).
\]
So we have \((2') = (a_1) \circ (2)\). Consider the pentagon

\[
\begin{array}{ccc}
(3) & \rightarrow & (3') \\
\downarrow & & \downarrow \\
(\alpha_2) & \rightarrow & \\
\end{array}
\]
where
\[
(a_2) : SH(Y)[T h_Y(f^* \xi) \land Y g_\# T h_Z(-T f_g)), E_{Y}^{2r-2(d+e),r-(d+e)} \land S^{p,q}] \xrightarrow{\cong}
SH(X)[f_\#(T h_Y(f^* \xi) \land Y g_\# T h_Z(-T f_g)), E_{X}^{2r-2(d+e),r-(d+e)} \land S^{p,q}]
\]
is the natural isomorphism induced by the adjunction \((f_\#, f^*)\) and
\[
(a_3) : SH(X)[f_\#(T h_Y(f^* \xi) \land Y g_\# T h_Z(-T f_g)), E_{X}^{2r-2(d+e),r-(d+e)} \land S^{p,q}] \xrightarrow{\cong}
SH(X)[T h_X(\xi) \land X h_\# T h(-T f_g), E_{X}^{2r-2(d+e),r-(d+e)} \land S^{p,q}]
\]
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is the natural isomorphism given by the projection formula \( Pr_*^r(f) \). We remind the reader that the isomorphism in the projection formula \( Pr_*^r(f) \) is given by the composition:
\[
f_#(M \wedge_Y f^*N) \to f_#(f^*f_#(M) \wedge_Y f^*N) \simeq f_#f^*(f_#M \wedge_X N) \to f_# \wedge_X N.
\]
The pentagon commutes since isomorphism induced by the projection formula \( Pr_*^r(h) \) is the composing of isomorphisms coming from projection formulas \( Pr_*^r(g) \) and \( Pr_*^r(f) \). Indeed, let
\[
\alpha_2 : g_#(Th_Z(h^*\xi) \wedge_Z Th_Z(-T_{f^!})) \to E^2_{Y} \wedge S_{p,q}
\]
be a morphism. We have
\[
(3')(−) \circ (a_1)(\alpha_2) = (3')(−)(\varepsilon(f_#, f^*) \circ f_#(\alpha_2)) = Pr_*^r(h)(−)(\varepsilon(f_#, f^*) \circ f_#(\alpha_2))
\]
and
\[
(a3)(−) \circ (a2)(−) \circ (3)(\alpha_2) = (a3)(−) \circ (a2)(−) \circ Pr_*^r(g)(\alpha_2) =
\]
\[
(a3)(−) \circ (a2)(−) \circ \varepsilon(g_#, g^*) \circ f_#(-) \circ g_#(\eta_{g_#, g^*})(−) \wedge \text{id}(\alpha_2)
\]
\[
= (a3)(−) \circ \varepsilon(f_#, f^*) \circ f_#(-) \circ g_#(\eta_{g_#, g^*})(−) \wedge \text{id}(\alpha_2)
\]
\[
\varepsilon^f_#(-) \circ \varepsilon(g_#, g^*) \circ f_#(-) \circ \varepsilon(f_#, f^*) \circ g_#(\eta_{g_#, g^*})(−) \wedge \text{id}(\alpha_2).
\]
So we have
\[
(a3)(−) \circ (a2)(−) \circ (3)(\alpha_2) = (3')(−) \circ (a_1)(\alpha_2).
\]
For any \( K \in SH(Z) \) one has commutative diagram (see [Ay08 §1.4.2, §1.5] and [CD10 Rem. 2.4.52])
\[
\begin{array}{ccc}
h_#K & \xrightarrow{f_#g_#K} & f_#g_#K \\
\downarrow & & \downarrow \\
\simeq & & \simeq \\
\end{array}
\]
\[
\begin{array}{ccc}
f_1(Th_Y(T_f)) \wedge_Y g_1(Th_Z(T_g)) \wedge_Z K & \xrightarrow{f_1g_1} & f_1g_1(Th_Z(T_g)) \wedge_Z Th_Z(T_g) \wedge_Z K \\
\downarrow & & \downarrow \\
\simeq & & \simeq \\
\end{array}
\]
Now we take \( K = 1_Z \) and dualize \( D_X(−) \) the commutative diagram above. One has
\[
D_X(h_1(Th_Z(T_{f^!}))) \cong D_X(h_#1_Z) \cong h_#Th_Z(-T_{f^!}),
\]
\[
D_X(f_1Th_Y(T_f)) \cong D_X(f_#1_Y) \cong f_#Th_Y(-T_f),
\]
and
\[
D_Y(g_1Th_Z(T_g)) = D_Y(g_#1_Z) \cong g_#Th_Z(-T_g).
\]
So we can conclude that
\[
(6') \circ (5') \circ (4') \circ (a_3) \circ (a_2) = (13) \circ (12) \circ \cdots \circ (5) \circ (4).
\]
Putting all together we have
\[
(6') \circ \cdots \circ (1') = (13) \circ \cdots \circ (1),
\]
which means the pushforward on twisted $E$-cohomology satisfies $(f \circ g)_* = f_* \circ g_*$ up to a natural isomorphism induced by the natural 2-isomorphism

$e_\sigma^Y : Th^{-1}_Z(s_Z, p_{Tfg}) \xrightarrow{\cong} Th^{-1}_Z(s_Z, p_{Tg}) \circ g^* Th^{-1}_Y(s_Y, p_{Tg})$.

\[\square\]

Now we follow a suggestion by M. Levine to make a refinement to the result of Voevodsky in [2.4] since as pointed out by M. Levine it is not enough to use the identities in $K_0(-)$ to constructs maps between twisted $E$-cohomology groups.

**Proposition 4.25.** (A refinement of Voevodsky’s theorem) Let $X \in \text{SmProj}(k)$ of dimension $d_X$, where $k$ is a field. After fixing an embedding $X \hookrightarrow \mathbb{P}^d$ there exists a vector bundle $V_X$ on $X$ of rank $d^2 + 2d - d_X$, such that one has a specific isomorphism between objects in the Picard category of virtual bundles $\mathcal{V}(X)$ on $X$:

$$V_X \oplus T_X \cong \mathcal{O}_{X}^{d^2 + 2d}.$$

**Proof.** Case 1: $X = \mathbb{P}^d$. One has an exact sequence

$$0 \to \mathcal{O}_{pd} \to \mathcal{O}_{pd}(1)^{\oplus(d+1)} \to T_{pd} \to 0.$$

By taking dual one also has

$$0 \to \Omega_{pd} \to \mathcal{O}_{pd}(-1)^{\oplus(d+1)} \to \mathcal{O}_{pd} \to 0.$$

There are two isomorphisms between objects in $\mathcal{V}(X)$:

$$\mathcal{O}_{pd} \oplus T_{pd} \cong \mathcal{O}_{pd}(1)^{\oplus(d+1)}$$

and

$$\Omega_{pd} \oplus \mathcal{O}_{pd} \cong \mathcal{O}_{pd}(-1)^{\oplus(d+1)}.$$

Define

$$V_{pd} \overset{def}{=} \Omega_{pd} \oplus (\Omega_{pd} \otimes T_{pd}).$$

As the Picard category $\mathcal{V}(X) = V(\text{Vect}(X))$ (the category of virtual objects associated to the exact category of vector bundles on $X$) has not just $\oplus$, but also a biexact functor $- \otimes - : \mathcal{V}(X) \times \mathcal{V}(X) \to \mathcal{V}(X)$, which is distributive ([Del87]), one has an isomorphism in $\mathcal{V}(X)$:

$$(\Omega_{pd} \oplus \mathcal{O}_{pd}) \otimes (\mathcal{O}_{pd} \oplus T_{pd}) \cong \Omega_{pd} \oplus \mathcal{O}_{pd} \oplus (\Omega_{pd} \otimes T_{pd}) \oplus T_{pd} \cong \mathcal{O}_{pd}^{\oplus(d^2 + 2d + 1)}.$$

This implies that we have an isomorphism in $\mathcal{V}(X)$:

$$V_{pd} \oplus T_{pd} \cong \mathcal{O}_{pd}^{d^2 + 2d}.$$

Case 2: $X$ is smooth projective. Let $i : X \hookrightarrow \mathbb{P}^d$ be a closed embedding. One define

$$V_X \overset{def}{=} N_{X/pd} \oplus i^*(V_{pd}),$$

where $N_{X/pd}$ denotes the normal bundle of $X$ in $\mathbb{P}^d$. In $\mathcal{V}(X)$ one has an isomorphism between objects

$$N_{X/pd} \oplus i^*(V_{pd} \oplus T_{pd}) \cong N_{X/pd} \oplus \mathcal{O}_{X}^{d^2 + 2d}.$$
one has an isomorphism in $\mathcal{V}(X)$:

$$T_X \oplus N_{X/p^d} \cong i^*T_{p^d}.$$  

This implies that we have a isomorphism in $\mathcal{V}(X)$:

$$N_{X/p^d} \oplus i^*V_{p^d} \oplus T_X \oplus N_{X/p^d} \cong N_{X/p^d} \oplus \mathcal{O}_X^{d^2+2d}.$$  

This implies that we have a specific isomorphism in $\mathcal{V}(X)$:

$$V_X \oplus T_X \cong \mathcal{O}_X^{d^2+2d}.$$  

□

P. Hu in [Hu05] didn’t check if her construction is the same as the construction of Voevodsky. We notice that the map constructed by Voevodsky [Voe03 Thm. 2.11]

$$T \wedge n_{X+} \to Th(V_X)$$

is first of all only in $\text{Ho}_{A^1,+} (k)$ and secondly very difficult to follow. We will take the refinement $V_X \oplus T_X \to \mathcal{O}_X^{d^2+2d}$ in $\mathcal{V}(X)$ and construct the Pontryagin-Thom collapse map

$$PTV : \mathbb{S}^0 \to \Sigma_{T_{+},+}^\infty Th(V_X) \wedge S^{-2(d^2+2d),-(d^2+2d)}$$

by unpacking Hu’s construction. Firstly, for a projective smooth $k$-variety $i : X \hookrightarrow \mathbb{P}^d$, we have by definition

$$V_X = N_{X/p^d} \oplus i^*V_{p^d} \cong N_{X/V_{p^d}}.$$  

If $PTV$ is already for $\mathbb{P}^d$ constructed, then $PTV$ for $X$ is defined by the composition

$$\mathbb{S}^0 \longrightarrow \Sigma_{T_{+},+}^\infty Th(V_{p^d}) \wedge S^{-2(d^2+2d),-(d^2+2d)} \longrightarrow \Sigma_{T_{+},+}^\infty Th(V_X) \wedge S^{-2(d^2+2d),-(d^2+2d)},$$

where $q$ is the quotient map

$$Th(V_{p^d}) \to V_{p^d} / (V_{p^d} - X) \cong Th(N_{X/V_{p^d}}) = Th(V_X).$$

The isomorphism $V_{p^d} / (V_{p^d} - X) \cong Th(N_{X/V_{p^d}})$ is the homotopy purity isomorphism ([MV01 §3 Thm.2.23]). For $X = \mathbb{P}^1$ one has a commutative diagram in $\text{Ho}_{A^1,+} (k)$ ([Hu05 pp. 9])

$$((X \times X) - \Delta_X)_+ \xrightarrow{pr_1} (X \times X)_+ \xrightarrow{g} Th(T_X)$$

because $pr_1 : (X \times X) - \Delta_X \to X$ is an affine bundle. So one has a cofiber sequence in $\text{Ho}_{A^1,+} (k)$

$$X_+ \xrightarrow{f} (X \times X)_+ \xrightarrow{g} Th(T_X).$$

For a vector bundle $\xi$ on $X$ one has

$$Th(pr_1^*\xi / X \times X) = Th(\xi / X) \wedge X_+.$$
One the other hand one has commutative diagram ([Hu05, (3.13)])

\[
\begin{array}{ccc}
Th(pr_1^*\xi/X \times X - \Delta_X) & \longrightarrow & Th(pr_1^*\xi/X \times X) \\
\downarrow_{pr_1} \cong & & \downarrow_{pr_1} \\
Th(\xi/X) & \longrightarrow & Th(\xi/X)
\end{array}
\]

So one obtains a cofiber sequence in \(\text{Ho}_{k^1,+}(k)\)

\[
Th(\xi/X) \xrightarrow{f^X} Th(\xi/X) \wedge X_+ \xrightarrow{g^X} Th(T_X \oplus \xi).
\]

Now we take \(\xi = V_X\) and by the refinement \(V_X \oplus T_X \xrightarrow{\cong} \mathcal{O}_X^{d^2+2d}\) in \(\mathcal{V}(X)\) we have then a cofiber sequence

\[
Th(V_X) \xrightarrow{f^V_X} Th(V_X) \wedge X_+ \xrightarrow{g^V_X} Th(V_X \oplus T_X) \cong S^{2(d^2+2d),(d^2+2d)} \wedge X_+.
\]

This gives rise to a map in \(SH(k)\):

\[
\varepsilon : \Sigma^\infty_{T,+} Th(V_X) \wedge X_+ \wedge S^{-2(d^2+2d),-(d^2+2d)} \xrightarrow{g^V_X} \Sigma^\infty_{T,+} X \rightarrow S^0.
\]

To construct the Pontryagin-Thom collapse map

\[
PTV : S^0 \rightarrow \Sigma^\infty_{T,+} Th(V_X) \wedge S^{-2(d^2+2d),-(d^2+2d)},
\]

such that the composition

\[
g^V_X \circ (PTV \wedge \text{id}) : S^0 \wedge \Sigma^\infty_{T,+} X \xrightarrow{PTV \wedge \text{id}} \Sigma^\infty_{T,+} Th(V_X) \wedge X_+ \wedge S^{-2(d^2+2d),-(d^2+2d)} \xrightarrow{g^V_X} \Sigma^\infty_{T,+} X
\]

is the identity \(\text{id}_{\Sigma^\infty_{T,+} X}\) in \(SH(k)\), it is enough to construct a map

\[
PTV : S^0 \rightarrow \Sigma^\infty_{T,+} Th(V_X) \wedge S^{-2(d^2+2d),-(d^2+2d)},
\]

such that the composition \(\varepsilon \circ (PTV \wedge \text{id})\) is the collapse map \(\Sigma^\infty_{T,+} X \rightarrow S^0\), because \(g^V_X\) is the composition

\[
\Sigma^\infty_{T,+} Th(V_X) \wedge X_+ \wedge S^{-2(d^2+2d),-(d^2+2d)} \xrightarrow{\text{id} \wedge \Delta} \Sigma^\infty_{T,+} Th(V_X) \wedge X_+ \wedge X_+ \wedge S^{-2(d^2+2d),-(d^2+2d)} \xrightarrow{\varepsilon \wedge \text{id}} \Sigma^\infty_{T,+} X
\]

By adjunction \(\varepsilon\) gives us a map

\[
\lambda_X : \Sigma^\infty_{T,+} Th(V_X) \wedge S^{-2(d^2+2d),-(d^2+2d)} \rightarrow D_k(\Sigma^\infty_{T,+}(X)) = \text{Hom}(\Sigma^\infty_{T,+} X, S^0).
\]

We remind the reader that P. Hu started with \(X = \mathbb{P}^1\) before [Hu05, Lem. 3.8], since she wanted to prove some particular results for projective quadric. For general \(X\) and a vector bundle \(\xi\) on \(X\) one still has the map

\[
g : (X \times X) \rightarrow Th(T_X) = Th(N_{X/X \times X})
\]

and hence a map

\[
g^\xi : Th(pr_1^*\xi/X \times X) = Th(\xi/X) \wedge X_+ \rightarrow Th(T_X \oplus \xi)
\]
and hence by applying $\xi = V_X$ one has a map

$$\lambda_X : \Sigma^\infty_+ Th(V_X) \wedge S^{-2(d^2+2d), -(d^2+2d)} \to \text{Hom}(\Sigma^\infty_+ X, S^0).$$

Now we consider the linear embedding $i : \mathbb{P}^d \to \mathbb{P}^{d+1}$. By construction

$$V_{\mathbb{P}^{d+1}} = N_{\mathbb{P}^d/\mathbb{P}^{d+1}} \oplus i^*(V_{\mathbb{P}^d})$$

and so the diagram

$$(4.2) \quad \Sigma^\infty_+ Th(V_{\mathbb{P}^{d+1}}) \wedge S^{-2((d+1)^2+2(d+1)), -(d^2+2d)} \xrightarrow{\lambda_{\mathbb{P}^{d+1}}} D(\mathbb{P}^{d+1})$$

commutes, since it is adjoint to the commutativity of the diagram

$$Th(V_{\mathbb{P}^{d+1}}) \wedge \mathbb{P}^d \wedge S^{*,*} \xrightarrow{id \wedge i^*} Th(V_{\mathbb{P}^{d+1}}) \wedge \mathbb{P}^{d+1} \wedge S^{*,*}$$

where we write $q_{\mathbb{P}^{d+1}}$ for the quotient map and the last commutative diagram is obtained by $V_{\mathbb{P}^{d+1}}$-Thomification (i.e. we apply $g_{V_{\mathbb{P}^{d+1}}}$ on $\mathbb{P}^{d+1} \times \mathbb{P}^{d+1} \to Th(T_{\mathbb{P}^{d+1}})$) of the commutative diagram

$$(\mathbb{P}^{d+1} \times \mathbb{P}^d)_+ \xrightarrow{i} (\mathbb{P}^{d+1} \times \mathbb{P}^{d+1})_+$$

Consider the composition

$$(\mathbb{P}^{d+1} - \mathbb{P}^d) \times \mathbb{P}^{d+1} \to \mathbb{P}^{d+1} \times \mathbb{P}^{d+1} \to (\mathbb{P}^{d+1} \times \mathbb{P}^{d+1})/(\mathbb{P}^{d+1} \times \mathbb{P}^{d+1} - \Delta_{\mathbb{P}^{d+1}}).$$

$(\mathbb{P}^{d+1} - \mathbb{P}^d) \times \mathbb{P}^d$ is mapped to $(\mathbb{P}^{d+1} \times \mathbb{P}^{d+1}) - \Delta_{\mathbb{P}^{d+1}}$. So the composition above induces a map

$$Th(j^*V_{\mathbb{P}^{d+1}}) \wedge (\mathbb{P}^{d+1}/\mathbb{P}^d)_+ \wedge S^{*,*} \to \mathbb{P}^{d+1},$$

where $j : (\mathbb{P}^{d+1} - \mathbb{P}^d) \to \mathbb{P}^{d+1}$ denotes the open immersion. After composing with the collapse map $\mathbb{P}^{d+1} \to S^0$ and taking adjoint one obtains a map

$$\lambda_{\mathbb{P}^{d+1}/\mathbb{P}^d} : Th(j^*V_{\mathbb{P}^{d+1}}) \wedge S^{*,*} \to D((\mathbb{P}^{d+1}/\mathbb{P}^d)_+).$$

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By construction there is a commutative diagram in $SH(k)$:

\[\begin{array}{ccc}
S^{−2d,−d} & \xrightarrow{=} & S^{−2d,−d} \\
\downarrow & & \downarrow \\
Th(V_{pd+1}) \wedge S^{**,*} & \xrightarrow{\lambda_{pd+1}} & D(\mathbb{P}^{d+1}_+) \\
\downarrow & & \downarrow \\
Th(V_{pd+1}) \wedge S^{**,*} & \xrightarrow{\lambda_{pd+1}} & D(\mathbb{P}^{d+1}_+) \\
\end{array}\] (4.3)

Now the Claim 1 in the proof of [Hu05, Lem. 3.8] implies that there is a morphism of distinguished triangles given by the commutative diagrams 4.2 and 4.3.

So by induction on $d$ one can conclude that

\[\lambda_{pd} : Th(V_{pd}) \wedge S^{−2(d^2+2d),−(d^2+2d)} \rightarrow D(\mathbb{P}^d_+)\]

is an isomorphism in $SH(k)$ for all $d \geq 0$. Now we can construct the Pontryagin-Thom collapse map

\[PTV : S^0 \rightarrow D(\mathbb{P}^d_+) \xrightarrow{\lambda_{pd}^{-1}} Th(V_{pd}) \wedge S^{−2(d^2+2d),−(d^2+2d)}\].

If $X \hookrightarrow \mathbb{P}^d$ is a smooth projective $k$-variety, we define $PTV$ for $X$ as the composition of $PTV$ for $\mathbb{P}^d$ with the quotient map

\[Th(V_d) \wedge S^{−2(d^2+2d),−(d^2+2d)} \rightarrow Th(V_X) \wedge S^{−2(d^2+2d),−(d^2+2d)}\].

So by construction we have:

**Proposition 4.26.** Let $X$ be a smooth projective $k$-scheme. After fixing an embedding $i : X \hookrightarrow \mathbb{P}^d$, there is a commutative diagram in $SH(k)$

\[\begin{array}{ccc}
S^0 & \xrightarrow{PTV} & \Sigma_{T,+}^\infty Th(V_X) \wedge S^{−2(d^2+2d),−(d^2+2d)} \\
\downarrow & & \downarrow \cong \\
\Sigma_{T,+}^\infty Th(−T_X) & & \end{array}\]

where $PTH : S^0 \rightarrow \Sigma_{T,+}^\infty Th(−T_X)$ is the map constructed in [Hu05, Lem. 3.18].

**Proposition 4.27.** Let $f : Y \rightarrow X$ be a projective smooth morphism of projective smooth $k$-schemes of relative dimension $d = d_Y − d_X$. After fixing an embedding $X \hookrightarrow \mathbb{P}^N$, there is an isomorphism

\[tt^Y_E : E_{p,q}(Y, f^*V_X − T_f) \cong E_{p+2N,q+n_Y}(Th(V_Y)),\]

where $V_X$ and $V_Y$ are vector bundles on $X$ and $Y$ of rank $n_X$ and $n_Y$ as in theorem 2.4 respectively with the refinement in the Proposition 4.25. Moreover, the isomorphism $tt^Y_E$ is
independent from the choice of the projective embeddings up to a unique canonical isomorphism.

Proof. Let us denote by $s_Y$ the 0-section of the vector bundle $p_{f^*V_X}: f^*V_X \to Y$ and by $s'_Y : Y \to T_f$ the 0-section of the relative tangent bundle. Let $\mathcal{V}(X)$ and $\mathcal{V}(Y)$ be the categories of virtual bundles on $X$ and $Y$, respectively. We have

$$E^{p,q}(Y, f^*V_X - T_f) \overset{def}{=} SH(Y)[1_Y, Th_Y(T_f) \wedge_Y s_Y p_{f^*V_X} E^{n_x+d_x-d_y,n_x+d_x-d_y}_Y \wedge S^{p,q}] \cong \begin{array}{c}
SH(Y)[p_{f^*V_X}\#s_Y Th_Y(-T_f), E^{n_x+d_x-d_y}_Y \wedge S^{p,q}] \cong \\
SH(Y)[Th_Y(f^*V_X) \wedge_Y Th_Y(-T_f), E^{n_x+d_x-d_y}_Y \wedge S^{p,q}] \cong SH(Y)[Th_Y(f^*V_X - T_f), E^{n_x+d_x-d_y}_Y \wedge S^{p,q}],
\end{array}$$

where we write $Th_Y(f^*V_X - T_f)$ for the Thom transformation $Th_Y(f^*V_X - T_f) = Th(s_Y, p_{f^*V_X}) \circ Th^{-1}(s'_Y, p_{T_f})(1_Y)$.

Let $\pi_Y : Y \to Spec k$ be the structure morphism of $Y$. By the adjunction $(\pi_Y\#, \pi_Y^*)$ and since $E_Y = \pi_Y^*E$, we have then

$$SH(Y)[Th_Y(f^*V_X - T_f), E^{n_x+d_x-d_y}_Y \wedge S^{p,q}] \cong SH(k)[Th(f^*V_X - T_f), E^{n_x+d_x-d_y}_Y \wedge S^{p,q}],$$

which comes from the fact that (cf. [Ay08] Thm. 1.5.9 and [Ay08] Rem. 1.5.10]):

$$\pi_Y\#(Th(s_Y, p_{f^*V_X}) \circ Th^{-1}(s'_Y, T_f))(1_Y) \cong Th(f^*V_X - T_f).$$

Now we apply the Voevodsky’s theorem 2.4 with a refinement as in Proposition 4.25. After fixing an embedding $X \hookrightarrow \mathbb{P}^N$ we have in $\mathcal{V}(X)$:

$$V_X \oplus T_X \cong \mathcal{O}_X^{N^2+2N}.$$
which gives rise to an isomorphism in $\mathcal{V}(Y)$

\[ f^*T_X \oplus T_f \cong T_Y. \]

So we have then in $\mathcal{V}(Y)$ an isomorphism

\[ f^*V_X - T_f \cong V_Y + \mathcal{O}_Y^{(n_X + d_X) - (n_Y + d_Y)}, \]

where $-$ means $+$ the opposite object as explained in [Del87] and

\[ n_X + d_X = N^2 + 2N \quad \text{and} \quad n_Y + d_Y = ((N + 1)(M + 1) - 1)^2 + 2((N + 1)(M + 1) - 1). \]

Now since $Th$ defines a functor (cf. [Rio10, Def. 4.1.2])

\[ Th : \mathcal{V}(Y) \to SH(k), \]

where $\mathcal{V}(Y)$ is the category of virtual bundles on $Y$, we can conclude that there is canonical isomorphism

\[ Th(f^*V_X - T_f) \cong Th(V_Y) \wedge S^{2(n_X + d_X) - 2(n_Y + d_Y), (n_X + d_X) - (n_Y + d_Y)}, \]

where the right hand side is by [4.26] canonical isomorphic to $D(Y_+) \wedge S^{2(n_X + d_X), (n_X + d_X)}$. So we can conclude that there is an isomorphism

\[ t^Y_E : E^{p,q}(Y, f^*V_X - T_f) \cong SH(k)[Th(V_Y), E^{2n_Y,n_Y} \wedge S^{p,q}] = E^{p+2n_Y,q+n_Y}(Th(V_Y)). \]

Now we have to show that this isomorphism is independent from the projective embeddings up to a unique canonical isomorphism. Let $Y \hookrightarrow \mathbb{P}^N$ be any closed embedding. Then we still have a canonical isomorphism

\[ Th(f^*V'_X - T_f) \cong D(Y_+) \wedge S^{-2*,-*} \]

As $D(Y_+)$ is unique up to a canonical isomorphism we can conclude that $t^Y_E$ is independent from the choice of the embeddings up to a unique canonical isomorphism. \(\square\)

**Remark 4.28.** As pointed out by M. Levine, one can simplify the arguments in the Proposition above by using the maps

\[ S^0 \to S^{-d^2-2d} \wedge \Sigma\Sigma T_+, Th(V_X) \wedge \Sigma\Sigma T_+ X \]

and

\[ S^{-d^2-2d} \wedge \Sigma\Sigma T_+ Th(V_X) \wedge \Sigma\Sigma T_+ X \to S^0, \]

which rigidify the situation considerably.

**Remark 4.29.** We will show later that with the refinement of $Th(V_X)$ as in [4.26] the isomorphism $t^Y_E$ is natural in sense that it is compatible with duality.

**Remark 4.30.** The isomorphism $t^Y_E(V_Y)$ in [4.27] is a natural candidate for a replacement of the twisted Thom isomorphism $t^E_E$ in [4.11] in case of $E$-cohomology twisted by formal difference of vector bundles. But we should remind the reader that we can only compute for a very particular case, namely $\xi = V_X$, where $V_X$ is the vector bundle as in Voevodsky’s theorem [2.4]

Now we can compare:
Corollary 4.31. Let \( f : Y \to X \) be a smooth projective morphism of projective smooth \( k \)-schemes. There is an isomorphism up to a natural isomorphisms induced by the natural canonical isomorphism between duals

\[
E^{p,q}(Y, f^*V_X - T_f) \xrightarrow{tt^Y_E} E^{p+2n_Y,q+n_Y}(Th(V_Y)) \xrightarrow{th^Y_E(V_Y)^{-1}} E^{p,q}(Y, V_Y).
\]

Proof. This is a consequence of \([4.11]\) and \([4.27]\).

Remark 4.32. The Corollary \([4.31]\) is a surprising fact to us. At a first glance we have the impression that \( E^{p,q}(Y, f^*V_X - T_f) \) should depend relatively wrt. \( f \) and \( X \). At the end it turns out that \( E^{p,q}(Y, f^*V_X - T_f) \) is isomorphic to \( E^{p,q}(Y, V_Y) \), which depends absolutely only on \( Y \). But it is clear that this is not the case for a general vector bundle \( \xi \).

Let \( f : Y \to X \) be a smooth projective morphism of smooth projective \( k \)-schemes of dimension \( d_Y \) and \( d_X \) respectively. By Atiyah-Spanier-Whitehead duality and by the Proposition \([4.26]\) we obtain its dual morphism in \( SH(k) \):

\[
f^\vee : X^\vee = \sum_{T,+}Th(V_X) \wedge S^{-2(n_X + d_X), -(n_X + d_X)} \to Y^\vee = \sum_{T,+}Th(V_Y) \wedge S^{-2(n_Y + d_Y), -(n_Y + d_Y)},
\]

where \( V_X \) and \( V_Y \) are vector bundles on \( X \) and \( Y \) of rank \( n_X \) and \( n_Y \) as in theorem \([2.4]\) with a refinement in \([4.25]\) respectively. By taking pullback of this map on \( E \)-cohomology and applying Thom isomorphism we obtain a pushforward

\[
E^{p+2d_Y,q+d_Y}(Y, V_Y) \xrightarrow{th_E} E^{p+2(n_Y + d_Y),q+(n_Y + d_Y)}(Th(V_Y)) \xrightarrow{(f^\vee)^*} E^{p+2(n_X + d_X),q+(n_X + d_X)}(Th(V_X)) \xrightarrow{th_E} E^{p+2d_X,q+d_X}(X, V_X).
\]

We show that the two pushforwards are the same and the isomorphism \( tt^Y_E \) is natural in the sense that it is compatible with the duality in the following:

Proposition 4.33. Let \( f : Y \to X \) be a smooth projective \( k \)-morphism between smooth projective \( k \)-schemes. One has a commutative diagram up to natural isomorphisms induced by natural 2-isomorphisms and the natural canonical isomorphism between duals

\[
\begin{array}{ccc}
E^{p+2d_Y,q+d_Y}(Y, f^*V_X - T_f) & \xrightarrow{f_*} & E^{p+2d_X,q+d_X}(X, V_X) \\
\xrightarrow{tt^Y_E} & & \downarrow \xrightarrow{th^X_E(V_X)} \\
E^{p+2(n_Y + d_Y),q+(n_Y + d_Y)}(Th(V_Y)) & \xrightarrow{(f^\vee)^*} & E^{p+2(n_X + d_X),q+(n_X + d_X)}(Th(V_X))
\end{array}
\]

Proof. Let us denote by \( p_{V_X} : V_X \to X \) the duality vector bundle on \( X \) (cf. \([2.4]\)) with the zero-section \( s_X : X \to V_X \) and \( s_Y : Y \to f^*V_X \) the 0-section of the pullback bundle
\[ p_{f^*V_X} : f^*V_X \to Y. \] By construction, the first pushforward map is the following composition:

\[
    f_* : E^{p+2d_Y,q+d_Y} (Y, f^*V_X - T_f) \xrightarrow{de^f_n} \text{SH}(Y)[1_Y, \text{Th}_Y(T_f) \wedge Y \text{S}_{Y} p_{f^*V_X} E_Y^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}]
\]

\[
    \cong \text{SH}(Y)[p_{f^*V_X} \#s_Y \text{Th}_Y(-T_f), E_Y^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}] \quad (1)
\]

\[
    \text{SH}(Y)[\text{Th}_Y(f^*V_Y) \wedge Y \text{Th}_Y(-T_f), E_Y^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}] \quad (2)
\]

\[
    \text{SH}(X)[f_\#(\text{Th}_Y(f^*V_Y) \wedge Y \text{Th}_Y(-T_f)), E_X^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}] \quad (3)
\]

\[
    \text{SH}(X)[\text{Th}_X(V_X) \wedge X f_\# \text{Th}_Y(-T_f), E_X^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}] \quad (4)
\]

\[
    \text{SH}(X)[\text{Th}_X(V_X) \wedge X D_X(f_\#1_Y), E_X^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}] \quad (5)
\]

\[
    \text{SH}(X)[\text{Th}_X(V_X), f_\#(1_Y) \wedge X E_X^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}] \quad (6)
\]

\[
    \text{SH}(X)[\text{Th}_X(V_X), E_X^{2(n_X+d_X),(n_X+d_X) \wedge S^{p,q}}] \xrightarrow{\text{th}_E^X(V_X)^{-1}} E^{p+2d_X,q+d_X}(X, V_X),
\]

where (1) is the natural isomorphism induced by adjunction of Thom transformations, (2) is the natural isomorphism given by composing Thom transformations, (3) is the natural isomorphism given by the adjunction \((f_\#, f^*)\)

\[
(3)(-) = \varepsilon(f_\#, f^*)(-) \circ f_\#(-),
\]

(4) is the natural isomorphism given by projection formula \(Pr_\#^*(f)\)

\[
(4)(-) = \varepsilon(f_\#, f^*)(-) \circ f_\#(\eta(f_\#, f^*)(-) \wedge_Y \text{id})(-),
\]

(5) is the natural isomorphism induced by \(f_\# \text{Th}_Y(-T_f) \cong D_X(f_\#1_Y)\), (6) is the natural isomorphism induced by adjunction of duality in \(\text{SH}(X)\):

\[
(6)(-) = (\text{id}_{D_X(f_\#1_Y)} \wedge_X -) \circ (\text{coev}_{f_\#1_Y} \wedge \text{id}_-),
\]

and finally (7) is the pushforward induced by the counit \(\eta(f_\#, f^*) : f_\#(1_Y) \cong f_\#f^*(1_X) \to 1_X\):

\[
(7)(-) = \varepsilon(f_\#, f^*)(-) \wedge_X -.
\]

The last isomorphism is the inverse of the twisted Thom isomorphism. So we have

\[ \text{th}_E^X(V_X) \circ f_* = (7) \circ \cdots \circ (1). \]
The map $tt_Y^*: E^{p+2d_Y,q+d_Y}(Y, f^*T_X - T_f) \overset{\text{defn}}{=} SH(Y)[1_Y, Th_Y(T_f) \wedge Y s^!_Y p^!_{f^*V_X} E^2_Y(n_X + d_X) \wedge S^{p,q}] \overset{(1')}{=} SH(Y)[p^!_fV_X \cup s_Y! Th_Y(-T_f), E^2_Y(n_X + d_X) \wedge S^{p,q}] \overset{(2')}{=} SH(Y)[Th_Y(f^*V_X) \wedge Y Th_Y(-T_f), E^2_Y(n_X + d_X) \wedge S^{p,q}] \overset{(3')}{=} SH(Y)[Th_Y(f^*V_X - T_f), E^2_Y(n_X + d_X) \wedge S^{p,q}] \overset{(4')}{=} SH(Y)[Th_Y(V_Y) \wedge S^2_Y - 2(n_Y + d_Y), E^2_Y(n_X + d_X) \wedge S^{p,q}] \overset{(5')}{=} E^{p+2(n_Y + d_Y), q+(n_Y + d_Y)}(Th(V_Y)),

where (1') = (1), (2') = (2), (3') is the natural isomorphism induced by composing Thom transformations, (4') is induced by the isomorphism in $V(Y)$:

$$f^*V_Y - T_f \cong V_Y + O_Y(n_X + d_X) - (n_Y + d_Y),$$

(5') is the cancellation in $SH(Y)$ and finally (6') is the natural isomorphism induced by the adjunction $(\pi_Y^!, \pi_Y^*)$ with $\pi_Y: Y \to \text{Spec} k$ is the structure morphism of $Y$:

$$(6')(\_\_) = \varepsilon(\pi_Y^!, \pi_Y^*)(\_\_) \circ \pi_Y^!(\_\_).$$

Let us consider the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\pi_Y \searrow & & \nearrow \pi_X \\
& \text{Spec} k &
\end{array}
\]

We have

$$[SH(Y)[Th_Y(f^*V_X - T_f), E^2_Y(n_X + d_X) \wedge S^{p,q}] \overset{\pi_Y^!}{=} SH(X)[\pi_Y^! Th_Y(f^*V_X - T_f), E^2_X(n_X + d_X) \wedge S^{p,q}],$$

where the natural isomorphism is induced by the adjunction $(f^!, f^*)$ as $f$ is smooth:

$$\varepsilon(f^!, f^*) \circ f^!(\_\_).$$

By the projection formula $Pr^*_\#(f)$ we have a natural isomorphism

$$SH(X)[\pi^!_Y Th_Y(f^*V_X - T_f), E^2_X(n_X + d_X) \wedge S^{p,q}] \overset{\pi_X^!}{=} SH(X)[Th_X(V_X) \wedge X f^!_Y Th_Y(-T_f), E^2_X(n_X + d_X) \wedge S^{p,q}],$$

which is explicitly written as

$$\varepsilon(f^!, f^*)(\_\_) \circ f^!(\eta(f^!, f^*)(\_\_) \wedge \text{id})(\_\_).$$

But since $f$ is smooth and projective

$$p_f: f^!_Y Th_Y(-T_f) \overset{\cong}{\to} f_* (1_Y) \cong f^* (1_X).$$
Thanks to Proposition 4.26 the composition \((f^\vee)^* \circ tt_k^f\) on \(E\)-cohomology is nothing but just the composition of natural isomorphisms above with the map induced by the unit \(1_X \to f_* f^! 1_X\) of the adjunction \((f^*, f_*)\):

\[
\circ \eta(f^*, f_*) : SH(X)[ThX(V_X) \land_X f_* f^!(1_X), E_X^{2(n_X+d_X), (n_X+d_X)} \land S^{p,q}] \longrightarrow \quad \text{SH}(X)[ThX(V_X), E_X^{2(n_X+d_X), (n_X+d_X)} \land S^{p,q}],
\]

then followed by the natural isomorphism induced from the adjunction \((\pi_X^#, \pi_X^\vee)\):

\[
\varepsilon(\pi_X^#, \pi_X^\vee) \circ \pi_X^# : SH(X)[ThX(V_X), E_X^{2(n_X+d_X), (n_X+d_X)} \land S^{p,q}] \cong \quad SH(k)[\pi_X^# ThX(V_X), E^{2(n_X+d_X), (n_X+d_X)} \land S^{p,q}] = E^{p+2(n_X+d_X)q+(n_X+d_X)}(Th(V_X)).
\]

Indeed, by the very construction of the 6 operations formalism [Ay08, Thm. 4.5.23], the stabilization functor \(\Sigma_{T,+}^\infty : Sm/k \to SH(k)\) induces a morphism in \(SH(k)\):

\[
\Sigma_{T,+}^\infty(f) : \Sigma_{T,+}^\infty(Y) \to \Sigma_{T,+}^\infty(X),
\]

which can be understood as a morphism \(\pi_Y^#(1_Y) \to \pi_X^#(1_X)\), which in turn is the composition \(\pi_X^# \circ \varepsilon(f^#, f^\vee)(1_X)\). In terms of six operations and by the Proposition 4.26 the dual objects in \(SH(k)\) are:

\[
Th(V_X) \land S^{-2(n_X+d_X), -(n_X+d_X)} \cong X^\vee = D_k(\pi_X^# 1_X),
\]

and

\[
Th(V_Y) \land S^{-2(n_Y+d_Y), -(n_Y+d_Y)} \cong Y^\vee = D_k(\pi_Y^# 1_Y).
\]

Hence the pullback map \((f^\vee)^*\) is just the pullback \(SH(k)[-, E^{*,*} \land S^{*,*}]\) of the map

\[
D_k(\pi_X^# \circ \varepsilon(f^#, f^\vee)).
\]

We have to check that

\[
(7) \circ \cdots \circ (3) = D_k(\pi_X^# \circ \varepsilon(f^#, f^\vee)) \circ (6') \circ \cdots \circ (3'),
\]

which means that we have to check

\[
D_k(\pi_X^# \circ \varepsilon(f^#, f^\vee)(-)) \circ \varepsilon(\pi_Y^#, \pi_Y^\vee) \circ \pi_Y^#(-) = (\varepsilon(\pi_Y^#, \pi_Y^\vee)(-) \land_X -) \circ (\text{id}_{D_X(f^# 1_Y)} \land_X -) \circ (\text{coev}_{f^# 1_Y} \land_X \text{id}_{-}) \circ \varepsilon(\pi_Y^#, \pi_Y^\vee)(-) \circ f^#(\eta(f^#, f^\vee)(-) \land_Y \text{id})(-) \circ \varepsilon(f^#, f^\vee) \circ f^#(-).
\]

But this is clear, since for a smooth projective morphism \(\pi : T \to S\) one has a natural 2-isomorphism

\[
D_S(\pi_*(-)) \cong \pi_* D_T(- \land_T Th_T(T_f)),
\]

which is the composition

\[
D_S(f^*(-)) = \text{Hom}_S(\pi_*(-), 1_S) \cong \text{Hom}_S(\pi_*(-), 1_S) \cong \pi_* \text{Hom}_T(-, \pi^! 1_S) \cong \pi_* \text{Hom}_T(-, \pi^! 1_S) \cong \pi_* \text{Hom}_T(-, \pi^! 1_S) \cong \pi_* \text{Hom}_T(-, \pi^! 1_S) = \pi_* D_T(- \land_T Th_T(T_f)).
\]

The equality, which we need to check above, follows simply from this fact and from the fact that, we have a natural 2-isomorphism

\[
f^# Th_Y(s_Y, p_T^! V_X) \cong Th_X(s_X, p_V^! X)_f^#
\]

as \(f\) is assumed to be smooth (cf. Ay08 Thm. 1.5.9}).
Let us construct the pullback for twisted $E$-cohomology of formal difference of vector bundles along a cartesian square. Let

$$
\begin{array}{c}
Y' \\
g \downarrow \quad \downarrow f \\
X'
\end{array}
\xrightarrow{\ }
\begin{array}{c}
Y \\
\downarrow \ \\
X
\end{array}
$$

be a cartesian square of projective smooth $k$-schemes, where $f$ is smooth projective of relative dimension $d = \dim(Y) - \dim(X)$ and $u$ is any morphism. Let $p_{i, \xi} : \xi \to X$ be a vector bundle of rank $r$ and denote by $s_Y : Y \to f^*\xi$ the $0$-section of the pullback bundle $p_{f, \xi} : f^*\xi \to Y$. Consider

$$
E^{p,q}(Y, f^*\xi - T_f) \overset{\text{defn}}{=} SH(Y)[1_Y, T\!h_Y(T_f) \wedge_Y S_Y p_{f, \xi}^* E_Y^{2(r-d),(r-d)} \wedge S^{p,q}].
$$

By adjunction of Thom transformation we have

$$
SH(Y)[1_Y, T\!h_Y(T_f) \wedge_Y S_Y p_{f, \xi}^* E_Y^{2(r-d),(r-d)}] \overset{\approx}{\longrightarrow} SH(Y)[T\!h_Y(f^*\xi), T\!h_Y(T_f) \wedge_Y E_Y^{2(r-d),(r-d)}],
$$

where the isomorphism is

$$
ev_{T\!h_Y(f^*\xi)} \circ (\text{id}_{T\!h_Y(f^*\xi)} \wedge -)
$$

By applying the functor $v^* : SH(Y) \to SH(Y')$ we have an induced map

$$E^{p,q}(Y, f^*\xi - T_f) \to SH(Y')[v^*T\!h_Y(f^*\xi), v^*(T\!h_Y(T_f) \wedge_Y E_Y^{2(r-d),(r-d)} \wedge S^{p,q})].$$

We have $v^*T\!h_Y(f^*\xi) \cong T\!h_{Y'}(v^*f^*\xi) = T\!h_{Y'}(g^*u^*\xi)$ as $\xi$ is an actual bundle. Since $v^*$ is a monoidal functor, so we have

$$v^*(T\!h_Y(T_f) \wedge_Y E_Y^{2(r-d),(r-d)}) \cong v^*T\!h_Y(T_f) \wedge_{Y'} E_{Y'}^{2(r-d),(r-d)}.
$$

But we have $v^*T\!h_Y(T_f) \cong T\!h_{Y'}(v^*T_f)$, since $T_f$ is an actual bundle. By [EGA4, 16.5.12.2] one has $v^*T_f \cong T_g$, so $v^*T\!h_Y(T_f) \cong T\!h_{Y'}(T_g)$. So we obtain the pullback map for twisted $E$-cohomology of formal difference of vector bundle

$$E^{p,q}(Y, f^*\xi - T_f) \to E^{p,q}(Y', v^*f^*\xi - T_g) = E^{p,q}(Y', g^*u^*\xi - T_g).
$$

**Proposition 4.34.** Consider the composition of cartesian squares of smooth projective $k$-schemes

$$
\begin{array}{c}
Y'' \\
h \downarrow \quad \downarrow g \\
X''
\end{array}
\xrightarrow{\ }
\begin{array}{c}
Y' \\
g \downarrow \quad \downarrow f \\
Y
\end{array}
\xrightarrow{\ }
\begin{array}{c}
Y \\
\downarrow \ \\
X
\end{array}
$$

where $f$ is a smooth projective morphism, $u$ and $u'$ are morphisms. Let $\xi$ be a vector bundle on $X$. Then up to natural isomorphisms induced by the natural 2-isomorphism induced by $(- \circ -)^* \overset{\cong}{\longrightarrow} (-)^* \circ (-)^*$ one has

$$(v \circ u')^* = (v^* \circ v^* : E^{p,q}(Y, f^*\xi - T_f) \to E^{p,q}(Y'', v^*v^*f^*\xi - T_f) = E^{p,q}(Y'', h^*u'^* u^*\xi - T_g).
$$

**Proof.** Obvious. \hfill \Box
Proposition 4.35. (projective smooth base change) Consider a cartesian square of projective smooth $k$-schemes

\[
\begin{array}{ccc}
Y' & \overset{u}{\longrightarrow} & Y \\
\downarrow^{g} & & \downarrow^{f} \\
X' & \overset{a}{\longrightarrow} & X
\end{array}
\]

where $f$ is smooth projective of relative dimension $d = \dim(Y) - \dim(X)$ and $u$ is a morphism. Let $\xi/X$ be a vector bundle of rank $r$. One has a commutative diagram up to natural isomorphisms induced by natural 2-isomorphisms

\[
E^{p,q}(Y, f^*\xi - T_f) \xrightarrow{f^*} E^{p-2d,q-d}(X, \xi) \\
\downarrow^{v^*} \\
E^{p,q}(Y', g^*u^*\xi - T_g) \xrightarrow{g^*} E^{p-2d,q-d}(X', u^*\xi)
\]

\[
\text{Proof.} \text{ It is quite straightforward. We write } s_Y : Y \to f^*\xi \text{ and } s_{Y'} : Y \to v^*f^*\xi \text{ for the 0-sections of the vector bundles } p_{f^*\xi} : f^*\xi \to Y \text{ and } p_{v^*f^*\xi} : v^*f^*\xi \to Y' \text{ respectively. } u^*f_* \text{ is the following composition}
\]

\[
E^{p,q}(Y, f^*\xi - T_f) \overset{\text{defn}}{=} SH(Y)[1_Y, Th_Y(T_f) \wedge_Y s_Y p_{f^*\xi} E_Y^{2(r-d), (r-d)} \wedge S^{p,q}] \overset{\alpha_1}{\to} \\
SH(Y)[Th_Y(f^*\xi) \wedge_Y Th_Y(-T_f), E_Y^{2(r-d), (r-d)} \wedge S^{p,q}] \overset{\xi_{(f^*\xi), f^*} f^*}{\cong} \\
SH(X)[f^*(Th_Y(f^*\xi) \wedge_Y Th_Y(-T_f)), E_X^{2(r-d), (r-d)} \wedge S^{p,q}] \overset{Pr_x^*(f)}{\cong} \\
SH(X)[Th_X(\xi) \wedge_X f^* Th_Y(-T_f), E_X^{2(r-d), (r-d)} \wedge S^{p,q}] \overset{\alpha_2}{\cong} \\
SH(X)[Th_X(\xi), f^*(1_Y) \wedge_X E_X^{2(r-d), (r-d)} \wedge S^{p,q}] \overset{\text{oc}(f^* f^*)}{\to} SH(X)[Th_X(\xi), E_X^{2(r-d), (r-d)} \wedge S^{p,q}] \\
\overset{u^*(-)}{\to} SH(X')[u^* Th_X(\xi), E_{X'}^{2(r-d), (r-d)} \wedge S^{p,q}] \overset{\alpha_3}{\to} SH(X')[u^* Th_X(\xi), E_{X'}^{2(r-d), (r-d)} \wedge S^{p,q}] \\
\overset{th_Y^*(u^* \xi)^{-1}}{\cong} E^{p-2d,q-d}(X', u^*\xi),
\]

where

\[
\alpha_1(-) = ev_{Th_Y(f^*\xi)} \circ (id_{Th_Y(f^*\xi)} \wedge -) \circ ev_{Th_Y(T_f)} \circ (id_{Th_Y(T_f)} \wedge -),
\]

and

\[
\alpha_2(-) = (id_{f^*1_Y} \wedge -) \circ (coev_{f^*Th_Y(-T_f)} \wedge id).
\]

\[
\alpha_3 \text{ is the natural isomorphism}
\]

\[
\alpha_3 = Ex^*_a(\Delta_b) \circ Ex^*_a(\Delta_a)^{-1}
\]

$\Delta_a$ is the Cartesian square

\[
\begin{array}{ccc}
u^*\xi & \overset{u_\xi}{\longrightarrow} & \xi \\
\downarrow^{p_{u^*\xi}} & & \downarrow^{p_\xi} \\
X' & \overset{a}{\longrightarrow} & X
\end{array}
\]
\[ \Delta_b \text{ is the Cartesian square} \]

\[
\begin{array}{ccc}
\Delta & \xrightarrow{s} & X' \\
\downarrow & & \downarrow \scriptstyle u' \\
X & \xrightarrow{\scriptstyle s} & \xi
\end{array}
\]

\[
E_{\#} X^* (\Delta_b)^{-1} : u^* p_{\#} \overset{\cong}{\longrightarrow} p_{u^*\xi} u^*.
\]

\[ g_* v^* \text{ is the following composition} \]

\[
E^{p,q}(Y, f^* \xi - T_f) \overset{\text{def}}{=} SH(Y)[1_Y, Th_Y(T_f) \land Y s_Y^1 p_{f^*}\xi E_Y^{2(r-d),(r-d)} \land Sp,q] \xrightarrow{\beta_1} SH(Y)[Th_Y(f^*\xi), Th_Y(T_f) \land Y E_Y^{2(r-d),(r-d)} \land Sp,q] \xrightarrow{v^*(\cdot)} SH(Y)[v' Th_Y(f^*\xi), v'(Th_Y(T_f) \land Y E_Y^{2(r-d),(r-d)} \land Sp,q)] \xrightarrow{\beta_2} SH(Y')[v'^* Th_Y(f^*\xi), v'^*(Th_Y(T_f) \land Y E_Y^{2(r-d),(r-d)} \land Sp,q)] \xrightarrow{\xi(g_{\#}, g^*) \circ g_{\#}} SH(Y')[v'^* Th_Y(f^*\xi), v'^*(Th_Y(T_f) \land Y E_Y^{2(r-d),(r-d)} \land Sp,q)] \xrightarrow{\beta_2} SH(Y)[g_{\#}(Th_Y'(f^*\xi), Th_Y'(-T_g)), E_Y^{2(r-d),(r-d)} \land Sp,q] \xrightarrow{Pr_{p,q}(g)} SH(Y)[g_{\#}(Th_Y'(f^*\xi), Th_Y'(-T_g)), E_Y^{2(r-d),(r-d)} \land Sp,q] \xrightarrow{Pr_{p,q}(g)} \]

where \( \beta_1 \) is the natural isomorphism

\[ \beta_1(-) = ev_{Th_Y(f^*\xi)} \circ (\text{id}_{Th_Y(f^*\xi)} \land -), \]

and

\[ \beta_2(-) = \beta_2 \circ Ex^*_\#(\Delta_4) \circ Ex^*_\#(\Delta_3)^{-1} \circ Ex^*_\#(\Delta_2) \circ Ex^*_\#(\Delta_1)^{-1}. \]

\( \Delta_1 \) is the Cartesian square

\[
\begin{array}{ccc}
Y' & \xrightarrow{v} & Y \\
\downarrow & & \downarrow \scriptstyle v \\
Y & \xrightarrow{\scriptstyle v} & f^*\xi
\end{array}
\]

\[
E_{\#} X^* (\Delta_1)^{-1} : u^* p_{f^*}\xi \overset{\cong}{\longrightarrow} p_{u^*f^*\xi} u^*.
\]

\( \Delta_2 \) is the Cartesian square

\[
\begin{array}{ccc}
Y' & \xrightarrow{sy'} & \xi \\
\downarrow & & \downarrow \scriptstyle v \xi \\
Y & \xrightarrow{\scriptstyle sy} & f^*\xi
\end{array}
\]

\[
E_{\#} X^* (\Delta_2) : u^* s_{Y!} \overset{\cong}{\longrightarrow} s_{Y!} u^* \overset{45}{\longrightarrow} s_{Y!} u^* \overset{\cong}{\longrightarrow} s_{Y!} u^*.
\]
$\Delta_3$ is the Cartesian square

\[
\begin{array}{ccc}
T_g \cong v^* T_f & \xrightarrow{v_T f} & T_f \\
p_{v^* T_f} = p_{T_g} & \downarrow & p_T f \\
Y' & \xrightarrow{v} & Y
\end{array}
\]

\[Ex_#^*(\Delta_3)^{-1} : v^* p_{T_f} \xrightarrow{\cong} p_{v^* T_f #} v_T f \cong p_{T_g #} v_T f.\]

$\Delta_4$ is the Cartesian square

\[
\begin{array}{ccc}
Y' \xrightarrow{\s_{Y'/T_g}} v^* T_f & \cong & T_g \\
v & \downarrow & \downarrow \\
Y & \xrightarrow{s_{Y/T_g}} & T_f
\end{array}
\]

\[Ex_#^*(\Delta_4) : v_T f \circ s_{Y/T_f} \cong v_T f \circ s_{Y'/T_g} \cong s_{Y'/T_g} v_T f. \]

$\beta'_2$ is the natural isomorphism

\[\beta'_2(-) = ev_{Th_y r}(\cdot - T_g) \circ (id_{Th_y r}(\cdot - T_g) \wedge -).\]

$\beta_3$ is the natural isomorphism

\[\beta_3 = (id_{g^#1_{Y'}} \wedge -) \circ (coev_{g^#1_{Th_y r}(\cdot - T_g)} \wedge id_-).\]

Gathering all together we have to check the following equality up to natural 2-isomorphisms:

\[th_E^X(u^* \xi)^{-1} \circ Ex_#^*(\Delta_b) \circ Ex_#^*(\Delta_a)^{-1} \circ \epsilon_{(f^#_\# f^*)} \circ u^* \circ (id_{g^#1_{Y'}} \wedge -) \circ (coev_{g^#1_{Th_y r}(\cdot - T_g)} \wedge X \circ id_-)\]

\[\circ Pr^#(f) \circ \epsilon(f^#_\# (\cdot -) \circ f^#_\# \circ ev_{Th_y r}(\cdot -) \circ (id_{Th_y r}(\cdot -) \wedge -) \circ ev_{Th_y r}(\cdot -) \circ (id_{Th_y r}(\cdot -) \wedge -) = \]

\[th_E^X(u^* \xi)^{-1} \circ \epsilon(g_{\#}, g^*) \circ (id_{g^#1_{Y'}} \wedge -) \circ (coev_{g_{\#}^# Th_y r(\cdot - T_g)} \wedge X \circ id_-) \circ Pr^#(f) \circ \epsilon(g_{\#}, g^*) \circ g_{\#} \circ \]

\[ev_{Th_y r(\cdot - T_g)} \circ (id_{Th_y r(\cdot - T_g)} \wedge -) \circ Ex_#^*(\Delta_4 \circ Ex_#^*(\Delta_3)^{-1} \circ Ex_#^*(\Delta_2) \circ Ex_#^*(\Delta_1)^{-1} \circ u^* \circ ev_{Th_y r(\cdot - T_g)} \circ (id_{Th_y r(\cdot - T_g)} \wedge -).\]

This equality can be chased step by step by using the natural 2-isomorphism

\[f^#_\# u^* \xrightarrow{\cong} g_{\#}^# u^*,\]

which is the following composition

\[g_{\#}^# v^* \xrightarrow{\eta(f^#_\# f^*)} g_{\#}^# f^# f^#_\# \cong g_{\#}^# (f \circ u)^* f^#_\# = g_{\#}^# (u \circ g)^* f^#_\# \cong g_{\#}^# g_{\#}^# u^* f^#_\# \xrightarrow{\epsilon(g_{\#}^# g_{\#}^*)} u^* f^#_\#\]

and also the coherence of the exchange transformations.

Now we construct the exceptional pullback for twisted $E$-cohomology. We keep the notation as above and let $i : T \hookrightarrow Y$ be a regular embedding, where $T$ is a smooth $k$-scheme. Let $N_{T/Y}$ be the normal bundle of $T$ in $Y$. Let $Bl_T(Y)$ be the blow-up of $X$ with the center $Z$. Similarly, $Bl_{T \times \{0\}}(Y \times \mathbb{A}^1)$ is the blow-up of $Y \times \mathbb{A}^1$ with the center $T \times \{0\}$. The deformation space is the $k$-scheme

\[D_T(Y) \overset{def}{=} Bl_{T \times \{0\}}(Y \times \mathbb{A}^1) - Bl_T(Y).\]
Note that $D_T(T) = T \times \mathbb{A}^1$ is a closed subscheme of $D_T(Y)$. The scheme $D_T(Y)$ is fibred over $\mathbb{A}^1$. The flat morphism

$$\pi : D_T(Y) \to \mathbb{A}^1$$

has $\pi^{-1}(1) = Y$ and $\pi^{-1}(0) = N_{T/Y}$. One has a deformation diagram of closed pairs

$$(Y,T) \xrightarrow{\sigma} (D_T(Y), T \times \mathbb{A}^1) \xrightarrow{\varepsilon_{0\pi}} (N_{T/Y}, T).$$

The homotopy purity theorem of Morel-Voevodsky [MV01, §3 Thm. 2.23] states $Y/Y - T \xrightarrow{\sigma_{1\pi}} D_T(Y)/D_T(Y) - T \times \mathbb{A}^1 \xrightarrow{\varepsilon_{0\pi}} Th(N_{T/Y})$ are isomorphism in $\text{Ho}_{k^1,+}(k)$, which is generalized to motivic categories in [CD10, Thm. 2.4.35]. Consider now the adjunction

$$i_! : SH(T) \rightleftarrows SH(Y) : i^!.$$ 

Let

$$
\begin{array}{ccc}
T & \xrightarrow{i} & Y \\
g & & f \\
S & \xrightarrow{k} & X
\end{array}
$$

be a cartesian square of smooth projective $k$-schemes, where $f$ is smooth projective of relative dimension $d = \dim(Y) - \dim(X)$, $k$ and $i$ are regular embeddings. Let $\xi$ be a vector bundle of rank $r$ on $X$. We define the exceptional pullback of twisted $E$-cohomology along a regular embedding $i : T \hookrightarrow Y$ as the following composition:

$$
\begin{aligned}
i^! : E^{p,q}(Y, f^*\xi - T_f) &\overset{\text{def}}{=} SH(Y)[1_Y, Th_Y(T_f) \wedge Y s_Y^! p_f^* E_Y^{2(r-d),(r-d)} \wedge S^{p,q}] \xrightarrow{i^!(-)} \\
&= SH(T)[i^! 1_Y, i^! Th_Y(T_f) \wedge Y s_Y^! p_f^* E_Y^{2(r-d),(r-d)} \wedge S^{p,q}] \xrightarrow{\varepsilon_{(i^!d)}^{(i^!d)}} \\
&= SH(Y)[i^!1_Y, Th_Y(T_f) \wedge Y s_Y^! p_f^* E_Y^{2(r-d),(r-d)} \wedge S^{p,q}] \xrightarrow{\circ Ex^i!} \\
&= SH(Y)[i^!1_Y, Th_Y(T_f) \wedge Y s_Y^! p_f^* E_Y^{2(r-d),(r-d)} \wedge S^{p,q}] \xrightarrow{i^!(-)} \\
&= SH(T)[i^! 1_Y, i^* (Th_Y(T_f)) \wedge Y s_Y^! p_f^* E_Y^{2(r-d),(r-d)} \wedge S^{p,q}] \xrightarrow{\circ Ex^i*} \\
&= SH(T)[1_T, i^* (Th_Y(T_f)) \wedge Y s_Y^! p_f^* E_Y^{2(r-d),(r-d)} \wedge S^{p,q}] = E^{p,q}(T, i^* f^* \xi - T_g).
\end{aligned}
$$

**Proposition 4.36.** Let

$$
\begin{array}{ccc}
T' & \xrightarrow{i'} & T \\
h & & f \\
S' & \xrightarrow{k'} & S & \xrightarrow{k} & X
\end{array}
$$

be a chain of cartesian squares of smooth projective $k$-schemes, where $f$ is smooth projective, $i, i', k, k'$ are regular embeddings. Then we have $(i \circ i')^! = i' \circ i^!$ up to natural isomorphisms induced by natural 2-isomorphisms.
Proof. Obvious. 

By using deformation to the cone as discussed above, one can prove the following result. However, we will not need this result, so we just omit the proof.

**Proposition 4.37.** Consider a cartesian square of projective smooth $k$-schemes

$$
\begin{array}{ccc}
T & \xrightarrow{i} & Y \\
\downarrow g & & \downarrow f \\
S & \xrightarrow{k} & X
\end{array}
$$

where $f$ is smooth projective of relative dimension $d = \dim(Y) - \dim(X)$ and $k$ and $i$ are regular embeddings. Let $p_\xi : \xi \to X$ be a vector bundle of rank $r$. One has a commutative diagram up to a natural isomorphism

$$
\begin{array}{ccc}
E^{p,q}(Y, f^* \xi - T_f) & \xrightarrow{f^*} & E^{p-2d,q-d}(X, \xi) \\
\downarrow \cong & & \downarrow k^* \\
E^{p,q}(T, g^* k^* \xi - T_g) & \xrightarrow{g^*} & E^{p-2d,q-d}(S, k^* \xi)
\end{array}
$$

Let $p_\xi : \xi \to X$ and $p_{\xi'} : \xi' \to X$ be two vector bundles of rank $r$ and $r'$ resp. on $X$ with the zero sections $s : X \to \xi$ and $s' : X \to \xi'$ respectively. Let $s'' : X \to \xi \oplus \xi'$ to be the zero section of the bundle $\xi \oplus \xi'$. We define the cup product

$$
\cup_E : E^{p,q}(X, \xi) \otimes E^{p',q'}(X, \xi') \to E^{p+p',q+q'}(X, \xi \oplus \xi')
$$

as follow: Given morphisms in $SH(X)$

$$
\alpha : 1_X \to s^! p_\xi^* E_X^{2r,r} \otimes S^{p,q}
$$

and

$$
\beta : 1_X \to s'^! p_{\xi'}^* E_X^{2r',r'} \otimes S^{p',q'}.
$$

Then

$$
\alpha \cup_E \beta = \mu_E \circ (\alpha \cup_X \beta) : 1_X = 1_X \otimes 1_X \xrightarrow{\alpha \cup_X \beta} s^! p_\xi^* E_X^{2r,r} \otimes X \otimes S^{p,q} \otimes s'^! p_{\xi'}^* E_X^{2r',r'} \otimes S^{p',q'} \cong
$$

$$
\cong s'^! p_{\xi \oplus \xi'}^* E_X \otimes X \otimes S^{p+q+r+2r+2r'+q'+q+r+r'} \xrightarrow{\mu_E} s'^! p_{\xi \oplus \xi'}^* E_X^{2(r+r'),r'+r'} \otimes S^{p+p',q'+q'}.
$$

**Remark 4.38.** If $f : T \to S$ is a morphism of finite type between schemes, then we have

$$
f^*(E \wedge^L_S F) = f^* E \wedge^L_T f^* F.
$$

**Proposition 4.39.** (Projection formula) Let $f : Y \to X$ be a smooth projective morphism of smooth projective $k$-schemes of relative dimension $d = \dim(Y) - \dim(X)$. Let $\xi$ and $\xi'$ be two vector bundles on $X$. Let $a \in E^{p,q}(X, \xi)$ and $b \in E^{p',q'}(Y, f^* \xi' - T_f)$. Then one has up to natural isomorphisms

$$
f_*(f^* a \cup_E b) = a \cup_E f_* b
$$

in $E^{p+p'-2d,q+q'-d}(X, \xi \oplus \xi')$. 

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Proof. This follows from the projective smooth base change \[4.35\] by standard argument. Consider the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\Gamma_f} & Y \times_k X \\
\downarrow f & & \downarrow f \times \text{id} \\
X & \xrightarrow{\Delta_X} & X \times_k X
\end{array}
\]

We have
\[
\Delta_X^*(f \times \text{id})_* = f_* \Gamma_f^* = f_* \Delta_X^*(f \times \text{id})^*,
\]
where \(\Gamma_f = (f \times \text{id}) \Delta_X\).

\[\square\]

**Proposition 4.40.** Let \(E \in \text{SH}(k)\) be a motivic ring spectrum. Let \(X, Y, Z \in \text{SmProj}(k)\). Let \(\alpha \in E^{2d_Y,d_Y}(X \times Y, pr^{XY}_{Y} V_Y)\) and \(\beta \in E^{2d_Z,d_Z}(Y \times Z, pr^{YZ}_{Z} V_Z)\), where \(V_Y\) and \(V_Z\) are the vector bundles given in theorem \[2.4\] with a refinement in \[4.25\]. Then we have up to natural isomorphisms
\[
pr^{XY}_{XZ} (pr^{XY}_{XY} \alpha \cup_E pr^{YZ}_{YZ} \beta) \in E^{2d_Z,d_Z}(X \times Z, pr^{Z}_{Z} V_Z).
\]

**Proof.** This follows from our construction of pullback, pushforward and cup product and the projections fit to the following commutative diagram

\[
\begin{array}{ccc}
X \times Z & \xleftarrow{pr^Z_{XZ}} & Z \\
\downarrow pr^Y_{XZ} & & \downarrow pr^Z_{Y} \\
X \times Y \times Z & \xrightarrow{pr^Y_{XY}} & Y \times Z \\
\downarrow pr^{XY}_{XZ} & & \downarrow pr^{YZ}_{Z} \\
X \times Y & \xrightarrow{pr^{XY}_{XY}} & Y \\
\downarrow pr^{XY}_{X} & & \downarrow pr^{YZ}_{Y} \\
X & \xrightarrow{pr^{XY}_{X}} & \text{.}
\end{array}
\]

\[\square\]

**Proposition 4.41.** Let \(E \in \text{SH}(k)\) be a motivic ring spectrum. Let \(X, Y, Z, W \in \text{SmProj}(k)\). Let \(\alpha \in E^{2d_Y,d_Y}(X \times Y, pr^{XY}_{XY} V_Y)\), \(\beta \in E^{2d_Z,d_Z}(Y \times Z, pr^{YZ}_{YZ} V_Z)\) and \(\gamma \in E^{2d_W,d_W}(Z \times W, pr^{ZW}_{ZW} V_W)\). Let’s denote
\[
\beta \circ \alpha = pr^{XY}_{XZ} (pr^{XY}_{XY} \alpha \cup_E pr^{YZ}_{Z} \beta),
\]
and similarly for \(\gamma \circ \beta\). Then \(\circ\) is associative up to natural isomorphisms induced by 2-isomorphisms.

**Proof.** We have
\[
\begin{align*}
\gamma \circ (\beta \circ \alpha) & \overset{(1)}{=} pr^{ZW}_{XZ} (pr^{ZW}_{X} (pr^{XW}_{X} (pr^{XY}_{XZ} \alpha \cup_E pr^{XY}_{XY} \beta) \cup_E pr^{XY}_{YZ} \beta)) \cup_E pr^{XW}_{Z} \gamma) \overset{(2)}{=} \\
pr^{XW}_{XZ} (pr^{XY}_{XZ} (pr^{XY}_{XY} \alpha \cup E pr^{XY}_{YZ} \beta)) \cup_E pr^{XY}_{YZ} \beta \cup E pr^{XW}_{Z} \gamma) \overset{(3)}{=} \\
pr^{XY}_{XZ} (pr^{XY}_{XZ} \alpha \cup E pr^{YZ}_{Z} \beta) \cup E pr^{XW}_{Z} \gamma) \overset{(4)}{=} \\
pr^{XY}_{XZ} \cup E (pr^{XY}_{Z} \beta \cup E pr^{XW}_{Z} \gamma),
\end{align*}
\]

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where (1) is the definition, (2) follows from smooth projective base change:

\[ pr_{X/W*}^{XZW} pr_{XZ}^{XY} = pr_{XZW}^{XY} pr_{X/Z}^{YZ*}, \]

(3) follows from the compatibility of pullback and \( \cup_\mathbb{E} \) (4.38), functoriality of pullback (4.18) and the projection formula (4.39), (4) follows from functoriality of pullback (4.18) and push-forward (4.24), (5) follows from the associativity of \( \cup_\mathbb{E} \), which is a consequence of our requirement that \( \mathbb{E} \) is a motivic ring spectrum (see the beginning of §4.1). Symmetrically, the last expression is exactly \((\gamma \circ \beta) \circ \alpha\).

\[ \square \]

**Definition 4.42.** Let \( \mathbb{E} \in SH(k) \) be a motivic ring spectrum. We define the category of twisted \( \mathbb{E} \)-correspondences \( \text{Corr}_{\mathbb{E}}(k) \) to be the category, whose objects are

\[ \text{Obj}(\text{Corr}_{\mathbb{E}}(k)) = \text{Obj}(\text{SmProj}(k)) \]

and morphisms are given by

\[ \text{Corr}_{\mathbb{E}}(k)(X, Y) = \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(X \times Y, pr_Y^{XY*}V_Y), \]

where \( V_Y/Y \) is the vector bundle given in the theorem \[2.4\]. Given \( \alpha \in \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(X \times Y, pr_Y^{XY*}V_Y) \) and \( \beta \in \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(Y \times Z, pr_Z^{YZ*}V_Z) \) we define their composition to be

\[ \beta \circ \alpha = pr_X^{XYZ}(pr_X^{XY*} \cup \mathbb{E} pr_X^{XY*} \cup \mathbb{E} \beta), \]

which is associative up to natural isomorphisms.

**Proposition 4.43.** Let \( \mathbb{E} \in SH(k) \) be a motivic ring spectrum. Let \( X, Y, Z \in \text{SmProj}(k) \). Let \( \alpha \in \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(X_+ \wedge Th(V_Y)) \) and \( \beta \in \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(Y_+ \wedge Th(V_Z)) \). Then the following composition

\[ \beta \circ \alpha : \Sigma_{T,+}^\infty X \wedge Th(V_Z) \xrightarrow{\text{coev}_Y} \Sigma_{T,+}^\infty X \times Y \wedge Th(V_Y) \wedge S^{-2(\nu_\mathbb{L}),-(\nu_\mathbb{L})} \wedge Th(V_Z) \]

\[ \to \Sigma_{T,+}^\infty X \wedge Th(V_Y) \wedge Y \wedge Th(V_Z) \wedge S^{-2(\nu_\mathbb{L}),-(\nu_\mathbb{L})} \xrightarrow{\alpha \wedge \beta} \mathbb{E} \wedge S^{2(\nu_\mathbb{L}),\nu_\mathbb{L}} \xrightarrow{\mu_{E}} \mathbb{E} \wedge S^{2(\nu_\mathbb{L}),\nu_\mathbb{L}} \]

lies in \( \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(X_+ \wedge Th(V_Z)) \), where

\[ \text{coev}_Y : S^0 \to \Sigma_{T,+}^\infty Y \wedge Th(V_Y) \wedge S^{-2(\nu_\mathbb{L}),-(\nu_\mathbb{L})} \]

is the coevaluation map of the Atiyah-Spanier-Whitehead duality on \( Y \).

**Proof.** Trivial.

\[ \square \]

**Proposition 4.44.** Let \( \mathbb{E} \in SH(k) \) be a motivic ring spectrum. Let \( X, Y, Z, W \in \text{SmProj}(k) \). Let \( \alpha \in \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(X_+ \wedge Th(V_Y)) \), \( \beta \in \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(Y_+ \wedge Th(V_Z)) \) and \( \gamma \in \mathbb{E}^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}(Z_+ \wedge Th(V_W)) \). Let us denote by \( \beta \circ \alpha \) for the composition of the above proposition and similarly for \( \gamma \circ \alpha \). Then \( \circ \) is associative and unital.

**Proof.** We have that \( \gamma \circ \beta \circ \alpha \) is the following composition by definition:

\[ \gamma \circ \beta \circ \alpha : \Sigma_{T,+}^\infty X \wedge Th(V_W) \xrightarrow{\text{coev}_W} \Sigma_{T,+}^\infty X \wedge Z \wedge Th(V_Z) \wedge S^{-2(\nu_\mathbb{L}),-(\nu_\mathbb{L})} \wedge Th(V_W) \]

\[ \to \Sigma_{T,+}^\infty X \wedge Th(V_Z) \wedge Z \wedge Th(V_W) \wedge S^{-2(\nu_\mathbb{L}),-(\nu_\mathbb{L})} \xrightarrow{\beta \circ \alpha \circ \gamma} \mathbb{E} \wedge \mathbb{E} \wedge S^{2(\nu_\mathbb{L}),\nu_\mathbb{L}} \wedge S^{2(\nu_\mathbb{L}),\nu_\mathbb{L}} \wedge \mathbb{E} \wedge S^{2(\nu_\mathbb{L}),\nu_\mathbb{L}} \wedge S^{2(\nu_\mathbb{L}),\nu_\mathbb{L}} \wedge S^{2(\nu_\mathbb{L}),\nu_\mathbb{L}}.
\]
which can be rewritten as

\[
\gamma \circ \alpha (\beta \circ \gamma) : \Sigma_{T,+}^{\infty} X \land Th(V_W) \xrightarrow{\text{covev}} \Sigma_{T,+}^{\infty} X \land Z \land Th(V_Z) \land S^{-2(n_z+d_Z)},-(n_z+d_Z) \land Th(V_W) \\
\xrightarrow{\tau} \Sigma_{T,+}^{\infty} X \land Th(V_Z) \land Z \land Th(V_W) \land S^{-2(n_z+d_Z)},-(n_z+d_Z) \land Th(V_W) \\
\xrightarrow{\gamma} \Sigma_{T,+}^{\infty} X \land Th(V_Z) \land Y \land Th(V_Y) \land E \land S^{2(n_w+d_W)},(n_w+d_W) \land Th(V_W) \\
\xrightarrow{\alpha \land \beta} E \land E \land E \land S^{2(n_w+d_W)},(n_w+d_W) \xrightarrow{\mu_E} E \land S^{2(n_w+d_W)},(n_w+d_W).
\]

The composition (\(\gamma \circ \beta \circ \alpha\)) is by definition:

\[
(\gamma \circ \beta \circ \alpha) : \Sigma_{T,+}^{\infty} X \land Th(V_W) \xrightarrow{\text{covev}} \Sigma_{T,+}^{\infty} X \land Y \land Th(V_Y) \land S^{-2(n_y+d_Y)},-(n_y+d_Y) \land Th(V_W) \\
\xrightarrow{\tau} \Sigma_{T,+}^{\infty} X \land Th(V_Y) \land Y \land Th(V_W) \land S^{-2(n_y+d_Y)},-(n_y+d_Y) \land Th(V_W) \\
\xrightarrow{\alpha \land \beta} E \land E \land E \land S^{2(n_w+d_W)},(n_w+d_W) \xrightarrow{\mu_E} E \land S^{2(n_w+d_W)},(n_w+d_W),
\]

which can be rewritten as

\[
(\gamma \circ \beta \circ \alpha) : \Sigma_{T,+}^{\infty} X \land Th(V_W) \xrightarrow{\text{covev}} \Sigma_{T,+}^{\infty} X \land Y \land Th(V_Y) \land S^{-2(n_y+d_Y)},-(n_y+d_Y) \land Th(V_W) \\
\xrightarrow{\tau} \Sigma_{T,+}^{\infty} X \land Th(V_Y) \land Y \land Th(V_W) \land S^{-2(n_y+d_Y)},-(n_y+d_Y) \land Th(V_W) \\
\xrightarrow{\alpha \land \beta} E \land E \land Z \land Th(V_Z) \land S^{-2(n_z+d_Z)},-(n_z+d_Z) \land Th(V_W) \\
\xrightarrow{\nu} \Sigma_{T,+}^{\infty} \Sigma_{T,+}^{\infty} Y \land Th(V_Z) \land Z \land Th(V_W) \land S^{-2(n_z+d_Z)},(n_z+d_Z) \land E \land E \land E \land S^{2(n_w+d_W)},(n_w+d_W) \\
\xrightarrow{\alpha \land \beta \land \gamma} E \land E \land E \land S^{2(n_w+d_W)},(n_w+d_W) \xrightarrow{\mu_E} E \land S^{2(n_w+d_W)},(n_w+d_W).
\]

Both \(\gamma \circ (\beta \circ \alpha)\) and \((\gamma \circ \beta \circ \alpha)\) are equal to the following composition

\[
\Sigma_{T,+}^{\infty} X \land Th(V_W) \xrightarrow{\text{covev} \land \text{coev}_{\gamma}} \Sigma_{T,+}^{\infty} X \land Y \land Th(V_Y) \land S^{-2(n_y+d_Y)},-(n_y+d_Y) \land Z \land Th(V_Z) \land S^{-2(n_z+d_Z)},-(n_z+d_Z) \land Th(V_W) \\
\xrightarrow{\tau} \Sigma_{T,+}^{\infty} X \land Y \land Th(V_Y) \land S^{-2(n_y+d_Y)},-(n_y+d_Y) \land Z \land Th(V_Z) \land S^{-2(n_z+d_Z)},-(n_z+d_Z) \land Th(V_W) \\
\xrightarrow{\alpha \land \beta \land \gamma} E \land E \land E \land S^{2(n_w+d_W)},(n_w+d_W) \xrightarrow{\mu_E} E \land S^{2(n_w+d_W)},(n_w+d_W).
\]

**Definition 4.45.** Let \(E \in SH(k)\) be a motivic ring spectrum. We define the category of Thom-\(E\)-correspondences \(\text{Corr}_{E}(k)^{\dagger}\) to be the category, whose objects are

\[
\text{Obj}(\text{Corr}_{E}(k)^{\dagger}) = \text{Obj}(\text{SmProj}(k))
\]

and morphisms are given by

\[
\text{Corr}_{E}(k)^{\dagger}(X, Y) = E^{2(n_y+d_Y),n_y+d_Y}(X_+ \land Th(V_Y)),
\]

where \(V_Y/Y\) is the duality vector bundle of rank \(n_y\). Given

\[
\alpha \in E^{2(n_y+d_Y),(n_y+d_y)}(X_+ \land Th(V_Y))
\]

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and
\[ \beta \in E^{2(nz + dz), (nz + dz)}(Y_+ \wedge Th(V_Z)), \]
we define their composition to be
\[ \beta \circ \alpha : \Sigma_{T,+}^\infty X \wedge Th(V_Z) \xrightarrow{coev_Y} \Sigma_{T,+}^\infty X \wedge Y \wedge Th(V_Y) \wedge S^{-2(ny + dy), -(ny + dy)} \wedge Th(V_Z) \]
\[ \mapsto \Sigma_{T,+}^\infty X \wedge Th(V_Y) \wedge Y \wedge Th(V_Z) \wedge S^{-2(ny + dy), -(ny + dy)} \xrightarrow{\alpha \circ \beta} E \wedge S^{2(nz + dz), (nz + dz)} \]
where
\[ coev_Y : S^0 \rightarrow \Sigma_{T,+}^\infty Y \wedge Th(V_Y) \wedge S^{-2(ny + dy), -(ny + dy)} \]
is the coevaluation map of the Atiyah-Spanier-Whitehead duality on \( Y \).

As we may write \( X_+ \wedge Th(V_Y) = Th(pr_X^{XY} V_Y) \), we have then the pullback map
\[ pr_X^{XY} : E^{2(ny + dy), (ny + dy)}(Th(pr_X^{XY} V_Y)) \rightarrow E^{2(ny + dy), (ny + dy)}(Th(pr_X^{XY} V_Y)). \]
Similarly
\[ pr_Y^{YZ} : E^{2(nz + dz), (nz + dz)}(Th(pr_Y^{YZ} V_Z)) \rightarrow E^{2(nz + dz), (nz + dz)}(Th(pr_Y^{YZ} V_Z)). \]
By taking cup product
\[ - \cup_E - : E^{2(ny + dy), (ny + dy)}(Th(pr_X^{XY} V_Y)) \otimes E^{2(nz + dz), (nz + dz)}(Th(pr_Y^{YZ} V_Z)) \rightarrow E^{2(ny + dy + nz + dz), (ny + dy + nz + dz)}(Th(pr_X^{XY} V_Y) \wedge Th(pr_Y^{YZ} V_Z)), \]
and applying the pushforward \( pr_X^Z \beta (pr_Y^Z \alpha) \) we see that
\[ pr_X^{XY} (pr_Y^{YZ} \alpha \cup_E pr_Y^{YZ} \beta) \in E^{2(nz + dz), (nz + dz)}(Th(pr_Z^{YZ} V_Z)) = \]
\[ E^{2(nz + dz), (nz + dz)}(X_+ \wedge Th(V_Z)). \]

**Proposition 4.46.** Let \( E \in SH(k) \) be a motivic ring spectrum. Let \( X, Y, Z \in SmProj(k) \) of dimension \( d_X, d_Y, d_Z \) respectively. Let \( \alpha \in E^{2(ny + dy), (ny + dy)}(X_+ \wedge Th(V_Y)) \) and \( \beta \in E^{2(nz + dz), (nz + dz)}(Y_+ \wedge Th(V_Z)) \). Then the composition \( \beta \circ \alpha \) in \( Corr_E(k)^\dagger \) satisfies
\[ \beta \circ \alpha = pr^{XYZ} (pr_X^{XY} \alpha \cup_E pr_Y^{YZ} \beta), \]
where \( pr_X^{YZ} = coev_Y \).

**Proof.** Trivial. \( \square \)

**Theorem 4.47.** (Comparison) Let \( E \in SH(k) \) be a motivic ring spectrum. There is an equivalence of categories up to a natural 2-isomorphism
\[ Corr_E(k) \xrightarrow{\sim} Corr_E(k)^\dagger \xrightarrow{\sim} Corr_E(k). \]

**Proof.** We have the following association
\[ Corr_E(k) \rightarrow Corr_E(k)^\dagger \rightarrow Corr_E(k), \quad X \mapsto X \mapsto X \]
\[ 52 \]
Similarly, given \( \beta \in E^{2,(d_y+n_y),(d_y+n_y)}(Th_{XY}(pr^Y_{XY}*V_Y)) \) we obtain its pullback by

\[
\begin{align*}
\Sigma^\infty_{T,+}Th_{XY}(pr^Y_{XY}*V_Y) & \xrightarrow{\alpha} E^{2,(d_y+n_y),(d_y+n_y)} \\
\Sigma^\infty_{T,+}Th_{XYZ}(pr^YZ_{XY}*V_Y) & \xrightarrow{pr^Y_{YZ}*\alpha} \\
\end{align*}
\]

So \( pr^Y_{XY}*\alpha \cup_E pr^Y_{YZ}*\beta \) is the following composition

\[
pr^Y_{XY}*\alpha \cup_E pr^Y_{YZ}*\beta : \Sigma^\infty_{T,+}Th_{XYZ}(pr^Y_{XY}*V_Y) \wedge \Sigma^\infty_{T,+}Th_{XYZ}(pr^Y_{YZ}*V_Y) \xrightarrow{\sim^-} E \wedge E \wedge S^{2(d_y+d_y+n_y+n_y), dy+d_y+n_y+n_y, dy+d_y+n_y+n_y} \xrightarrow{\mu^E} E \wedge S^{2(d_y+d_y+n_y+n_y), dy+d_y+n_y+n_y} \wedge S^{2(d_y+d_y+n_y+n_y), dy+n_y+n_y, dy+n_y+n_y},
\]

which corresponds to the morphism

\[
\Sigma^\infty_{T,+}Th_{XYZ}(-pr^Y_{XY}*T_Y) \wedge \Sigma^\infty_{T,+}Th_{XYZ}(-pr^Y_{YZ}*T_Z) \rightarrow E.
\]

By definition the composition \( \beta \circ \alpha \) as composition of \( E \)-correspondences is given by the composition

\[
\Sigma^\infty_{T,+}Th_{XY}(pr^X_{XY}*T_Z) \rightarrow \Sigma^\infty_{T,+}Th_{XYZ}(pr^Y_{XY}*T_Y) \wedge \Sigma^\infty_{T,+}Th_{XYZ}(pr^X_{Y}*T_Z) \rightarrow E,
\]

where the first map by construction is given as

\[
\begin{align*}
\Sigma^\infty_{T,+}Th_{XY}(pr^X_{XY}*T_Z) & \xrightarrow{\sim} \Sigma^\infty_{T,+}Th_{XYZ}(pr^Y_{XY}*T_Y) \wedge \Sigma^\infty_{T,+}Th_{XYZ}(pr^X_{Y}*T_Z) \\
\Sigma^\infty_{T,+}X \wedge (\Sigma^\infty_{T,+}Z) & \xrightarrow{id_X \wedge coev \wedge \id_Z} \Sigma^\infty_{T,+}X \wedge \Sigma^\infty_{T,+}Y \wedge (\Sigma^\infty_{T,+}Y) \wedge (\Sigma^\infty_{T,+}Z)
\end{align*}
\]
This implies that the composition \( \beta \circ \alpha \) as \( E \)-correspondences is the same as

\[
X \wedge Y^\vee \wedge Y \wedge Z \xrightarrow{\text{covev}} E \wedge E \xrightarrow{\mu_E} E
\]

This shows that the composition laws of \( \overline{\text{Corr}}_E(k) \) and \( \text{Corr}_E(k) \) are compatible. \( \square \)

5. Proof of theorem 1.2

5.1. Homotopy \( t \)-structure. We recall in this section the notion homotopy \( t \)-structure in terms of generators (cf. \[Ay08\]). Let \( k \) be a field. The subcategory \( SH(k)_{\geq n} \) is generated under homotopy colimits and extensions by

\[
\{ S^{p,q} \wedge \Sigma_{p_1}^\infty (X_+) | X \in Sm/k, p - q \geq n \},
\]

where \( S^{p,q} = S^p_S - q \wedge S^n_q \) denotes the motivic spheres. We set

\[
SH(k)_{\leq n} = \{ E \in SH(k) | [F, E] = 0, \forall F \in SH(k)_{\geq n+1} \}
\]

The bigraded motivic homotopy sheaves are defined as

\[
\pi^A_1^{st}(E) = a_{Nis}(U \mapsto SH(k)[S^{p,q} \wedge \Sigma_{p_1}^\infty (U_+), E]).
\]

We let

\[
\pi^A_1^{st}(E)_n \overset{\text{def}}{=} a_{Nis}(U \mapsto SH(k)[\Sigma^{\infty}_{p_1} U_+, S^{n-p,n} \wedge E])
\]

For a fix \( p \in \mathbb{Z} \), \( \pi^A_1^{st}(E)_s \) is considered as an abelian \( \mathbb{Z} \)-graded sheaf. An abelian Nisnevich sheaf \( F \in Sh_{Nis}(Sm/k) \) is called strictly \( A_1 \)-invariant, if the map induced by the projection \( U \times A_1 \to U \):

\[
H^i_{Nis}(U, F) \to H^i_{Nis}(U \times A_1, F)
\]

is an isomorphism \( \forall U \in Sm/k \) and \( \forall i \geq 0 \). For an abelian Nisnevich sheaf \( F \in Sh_{Nis}(Sm/k) \) we will denote by

\[
F_{-1}(U) = \text{Ker} (F(U \times_k \mathbb{G}_m) \to F(X)),
\]

where the map is induced by the unit section of \( \mathbb{G}_m \).

Definition 5.1. (Morel). A homotopy module is a pair \( (F_*, \varepsilon_*) \), where \( F \) is a strictly \( A_1 \)-invariant \( \mathbb{Z} \)-graded abelian Nisnevich sheaf with

\[
\varepsilon_n : F_n \xrightarrow{\cong} (F_{n+1})_{-1}.
\]

The following description of the homotopy \( t \)-structure is a consequence of F. Morel’s stable \( A_1 \)-connectivity result (see for instance \[Mor04a\]):

Theorem 5.2 (F. Morel). Let \( k \) be field.

1. The triple \( (SH(k), SH(k)_{\geq 0}, SH(k)_{\leq 0}) \) is a \( t \)-structure on \( SH(k) \).
2. The heart of the homotopy \( t \)-structure \( \pi^A_1(k) = SH(k)_{\geq 0} \cap SH(k)_{\leq 0} \) is identified with the category of homotopy modules.

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(3) The homotopy $t$-structure is non-degenerated in the sense that for any $U \in Sm/k$ and any $E \in SH(k)$, one has the morphism
\[
[\Sigma_{p_1}^\infty(U_+), E_{\geq n}] \to [\Sigma_{p_1}^\infty(U_+), E]
\]
is an isomorphism for $n \leq 0$ and the morphism
\[
[\Sigma_{p_1}^\infty(U_+), E] \to [\Sigma_{p_1}^\infty(U_+), E_{\leq n}]
\]
is an isomorphism for $n > \dim(U)$.

By sending $E \mapsto E_{\geq n}$ and $E \mapsto E_{\leq n-1}$ respectively, one has the following adjunctions respectively:
\[
i_{\geq n} : SH(k)_{\geq n} \rightleftarrows SH(k) : \tau_{\geq n}, \quad \tau_{\leq n-1} : SH(k)_{\leq n-1} \rightleftarrows SH(k)_{\leq n-1}:
\]
where we denote by $i_{\geq n}$ and $\tau_{\leq n-1}$ the inclusion functors. We denote by
\[
H : \pi_{\ast}^{\ast}(k) \to SH(k)
\]
the inclusion functor. For a homotopy module $F_\ast \in \pi_{\ast}^{\ast}(k)$ we will call $H(F_\ast)$ the Eisenberg-MacLane spectrum associated to $F_\ast$. Let $S$ be now a Noetherian scheme of finite Krull dimension. We recall the rationally splitting of $SH(S)_Q$ constructed by F. Morel (see [CD10, §16.2]). The permutation isomorphism
\[
\tau : \Sigma_{p_1}^\infty, +G_{m, Q} \wedge \Sigma_{p_1}^\infty, +G_{m, Q} \to \Sigma_{p_1}^\infty, +G_{m, Q} \wedge \Sigma_{p_1}^\infty, +G_{m, Q}
\]
satisfies $\tau^2 = 1$. This defines an element $e \in End_{SH(S)_Q}(1_Q)$, such that $e^2 = 1$. So we may define
\[
e_+ = \frac{e - 1}{2}, \quad e_- = \frac{e + 1}{2}.
\]
Remark that $e_+$ and $e_-$ are idempotents. Hence we can define $1_Q^+ = \text{im}(e_+)$ and $1_Q^- = \text{im}(e_-)$. For any spectrum $E \in SH(S)_Q$, one defines $E_+ = 1_Q^+ \wedge E$ and $E_- = 1_Q^- \wedge E$. This leads to a splitting of stable homotopy category
\[
SH(S)_Q^+ \times SH(S)_Q^- \xrightarrow{\eta} SH(S)_Q, \quad (E_+, E_-) \mapsto E_+ \wedge E_-.
\]
Let us assume now $S = \text{Spec } k$. The algebraic Hopf fibration is the map
\[
A_k^2 - \{0\} \to \mathbb{P}_k^1, \quad (x, y) \mapsto [x : y].
\]
This gives us the stable Hopf map in $SH(k)$
\[
\eta : \Sigma_{T,+}^\infty G_m \to S_k^0.
\]
Remark that from [Mor04a, 6.2.1] one has a homotopy fiber sequence in $SH(k)$:
\[
\Sigma_{T,+}^\infty (A_k^2 - \{0\}) \xrightarrow{S^2,1\eta} \Sigma_{T,+}^\infty \mathbb{P}_k^1 \xrightarrow{\Sigma_{T,+}^\infty i} \Sigma_{T,+}^\infty \mathbb{P}_k^2,
\]
where $i : \mathbb{P}_k^1 \hookrightarrow \mathbb{P}_k^2$ is the linear embedding. Following [Mor12], we define the Milnor-Witt $K$-theory of a field $F$ without any assumption on $\text{char}(F)$:

**Definition 5.3.** Let $F$ be a field. $K^\text{MW}_*(F)$ is the $\mathbb{Z}$-graded associative unital ring freely generated by the symbols $[u]$, where $u \in F^\times$ is of degree 1 and a symbol $\eta$ of degree $-1$ subject to the relation
\[
\begin{align*}
(1) \quad [u] \cdot [1 - u] &= 0, \forall u \in F^\times - \{1\}, \\
(2) \quad [uv] &= [u] + [v] + \eta \cdot [u] \cdot [v], \forall (u, v) \in (F^\times)^2.
\end{align*}
\]
\[ \eta \cdot [u] = [u] \cdot \eta, \forall u \in F^x. \]

(4) Define \( h \overset{\text{defn}}{=} \eta \cdot [-1] + 2. \) Then \( \eta \cdot h = 0. \)

Let \( GW(F) \) be the Grothendieck-Witt ring of non-degenerate bilinear symmetric forms over \( F \), where addition is given by orthogonal sum \( \oplus \) and multiplication is given by tensor product \( \otimes \). There is a surjective ring homomorphism

\[ rk : GW(F) \twoheadrightarrow \mathbb{Z}, \quad Q \mapsto rk(Q). \]

The fundamental ideal is defined as

\[ I(F) \overset{\text{defn}}{=} \text{Ker} (rk : GW(F) \twoheadrightarrow \mathbb{Z}). \]

Denote by \( I^n(F) \) the \( n \)-th power of \( I(F) \). If \( n \leq 0 \) one sets \( I^n(F) = W(F) \), where \( W(F) \) is the Witt ring over \( F \). Remark that \( W(F) = GW(F)/(h) \), where \( (h) \) is the ideal generated by hyperbolic spaces. By [Mor12, Lem. 3.10] there is a ring isomorphism

\[ GW(F) \overset{\cong}{\twoheadrightarrow} K^0_{MW}(F), \quad \langle u \rangle \mapsto 1 + \eta \cdot [u]. \]

Let \( K^M_* (F) \) be the Milnor \( K \)-theory

\[ K^M_*(F) \overset{\text{defn}}{=} \text{Tens}^*(F^x)/(u \otimes (1 - u)). \]

There is a graded surjective homomorphism

\[ U : K^M_*(F) \twoheadrightarrow K^M_*(F), \quad [u] \mapsto \{u\}, \eta \mapsto 0. \]

In fact, one can show that for each \( n \) there is a pullback diagram

\[ \begin{array}{ccc}
K^M_n(F) & \longrightarrow & I^n(F) \\
U \downarrow & & \downarrow \\
K^M_n(F) & \longrightarrow & I^n(F)/I^{n+1}(F)
\end{array} \]

Following [Mor12, §3.2] we let \( K^M_n \) be the \( n \)-th Milnor-Witt sheaf, which is a strictly \( \mathbb{A}^1 \)-invariant sheaf on \( (Sm/k)_{Nis} \). In [Mor04a, p. 437] Morel showed that one can define a homotopy module \( K^M_* \) associated to the Milnor-Witt \( K \)-theory and in fact one has an isomorphism between homotopy modules

\[ \pi_{0}^{st, \mathbb{A}^1}(S^0)_{*} \cong K^M_* . \]

The homotopy module \( \mathbb{W} \) is defined by setting every terms to be the unramified Witt sheaf \( \mathbb{W} = a_{Nis}(U \mapsto W(U) = W(k(U))) \) and all the maps \( \varepsilon_n \) are identity.

**Lemma 5.4.** Let \( k \) be a field. Let \( H\mathbb{K}^M_\ast \) be the Eilenberg-Maclane spectrum associated to the Milnor-Witt \( K \)-theory homotopy module \( K^M_* \). There exists a strict motivic ring spectrum \( \hat{\mathbb{K}}^M_* \in \text{Spect}^N_T(k)_Q \), which is isomorphic to \( H\mathbb{K}^M_* \) in \( SH(k)_Q \).

**Proof.** We have a splitting

\[ H\mathbb{K}^M_* = H(K^M_* ) \vee H(\mathbb{W} \ast Q). \]

The result of Déglise [Deg13, Cor. 4.1.7] asserts that \( H(K^M_* ) \) is a strict \( H\mathbb{Z} \)-module, where \( H\mathbb{Z} \) denotes the motivic cohomology spectrum. The construction in [ALP15, §4] shows that
the cofibrant replacement $H(\mathbf{W}_T^*)$ is a commutative monoid object in $\text{Spect}^\Sigma_T(k)_\mathbb{Q}$, which is isomorphic to $H(\mathbf{W}_T^*)$ and $S^0[\eta^{-1}]$. These imply that

$$\hat{H}\mathbf{K}^{MW}_{*,\mathbb{Q}} = \hat{H}\mathbf{K}^{M}_{*,\mathbb{Q}} \lor H(\mathbf{W}_T^*)$$

is also a strict motivic ring spectrum in $\text{Spect}^\Sigma_T(k)_\mathbb{Q}$, which is isomorphic to $\hat{H}\mathbf{K}^{MW}_{*,\mathbb{Q}}$ in $SH(k)_\mathbb{Q}$.

**Definition 5.5.** Let $k$ be a field. We define the category of pure Chow-Witt motives to be

$$CHW(k)_\mathbb{Q} = \text{Mot}_{\hat{H}\mathbf{K}^{MW}_{*,\mathbb{Q}}}(k)$$

**Corollary 5.6.** Let $k$ be a field. There is a functor

$$CHW(k)_\mathbb{Q} \rightarrow SH(k)_\mathbb{Q}$$

**Proof.** This is a consequence of the Corollary 4.7 and Lemma 5.4. □

5.2. **Isomorphism between Hom-groups.** Let $k$ be a field. In this section we prove that one has a fully faithful embedding

$$CHW(k)_\mathbb{Q} \rightarrow D_{k^1, gm}(k)_\mathbb{Q}.$$

Remark that one has the equivalences of categories:

$$\text{StHo}_{\mathbb{A}^1, \mathbb{Z}^1}(k)_\mathbb{Q} \cong D_{\mathbb{A}^1}^{gm}(k)_\mathbb{Q}, \quad \text{StHo}_{\mathbb{A}^1, \mathbb{Z}^1}(k)_\mathbb{Q} \cong D_{\mathbb{A}^1}(k)_\mathbb{Q}.$$ For $E \in SH(k)$ we define its stable $\mathbb{A}^1$-cohomology as

$$H^{p, q}_{\text{st} \mathbb{A}^1}(E, \mathbb{Z}) = SH(k)(E, S^p q).$$

We denote by $SH(k)_\mathbb{Q}$ the localization of $SH(k)$. One has an adjunction

$$\mathbb{L}L_Q : SH(k) \rightleftarrows SH(k)_\mathbb{Q} : \mathbb{R}U,$$

which is induced by the Quillen adjunction

$$L_Q : \text{Spect}^\Sigma_T(k) \rightleftarrows \text{Spect}^\Sigma_T(k)_\mathbb{Q} : U,$$

where $U : \text{Spect}^\Sigma_T(k)_\mathbb{Q} \rightarrow \text{Spect}^\Sigma_T(k)$ is the forgetful functor by considering

$$E_Q = E \land 1_Q = E \land \text{hocolim}(S^0 \xrightarrow{2} S^0 \xrightarrow{3} S^0 \xrightarrow{4} \cdots)$$

as a symmetric motivic $T$-spectrum in $\text{Spect}^\Sigma_T(k)$. For a motivic spectrum $E \in SH(k)$ we define its rational stable $\mathbb{A}^1$-cohomology as

$$H^{p, q}_{\text{st} \mathbb{A}^1}(E, \mathbb{Q}) = SH(k)_\mathbb{Q}(E, S^p q) = SH(k)(E, S^p q).$$

We remark that by [Lev13, Lem. B2] if $E$ is a compact object in $SH(k)$ then one has an isomorphism

$$H^{p, q}_{\text{st} \mathbb{A}^1}(E, \mathbb{Q}) = H^{p, q}_{\text{st} \mathbb{A}^1}(E, \mathbb{Z}) \otimes \mathbb{Q}.$$ Similarly, we define the motivic cohomology of $E$ as $H^{p, q}_{\mathbb{A}^1}(E, \mathbb{Z}) = SH(k)(E, H\mathbb{Z} \land S^p q)$. If $F_* \in \pi^H_0(k)_*$ is a homotopy module, then the $HF_*$-cohomology of $E$ is defined as

$$H(F_*)^{p, q}(E) = SH(k)(E, S^p q \land HF_*).$$

and if $E = \Sigma^\infty_{T+} \mathcal{X}$, where $\mathcal{X} \in \text{Sp}(k)_+$ is a $k$-space (e.g. Thom spaces), then the later cohomology is $H^{p, q}_{\text{Nis}}(\mathcal{X}, F_q)$, where this cohomology is defined as

$$H^{p, q}_{\text{Nis}}(\mathcal{X}, F_q) = \text{Ho}_{\mathbb{A}^1, +}(k)[\mathcal{X}, K(F_q)[p - q]],$$
where $K(-)$ denotes the Eilenberg-Maclane functor.

**Theorem 5.7.** Let $k$ be a field and $E = \sum_{T, \pm} Th(V/X)$ be the Thom spectrum of a vector bundle $V$ on a smooth $k$-scheme $X$. Let $S^0$ be the motivic sphere spectrum. There exists a canonical isomorphism

$$\varphi : H^{2p,p}_{Wh}(Th(V/X), \mathbb{Q}) \xrightarrow{\cong} H^p_{Nis}(Th(V/X), K^M_{\mathbb{Q}}),$$

where $\varphi$ is induced by the unit $\varphi_{MW} : S^0 \to H\mathbb{K}^M_{\mathbb{Q}}$.

**Proof.** By stable $A^1$-connectivity theorem of Morel [Mor05] the motivic sphere spectrum $S^0$ is $-1$-connective. So we have a distinguished triangle

$$(S^0)_{\geq 1} \to S^0 \to H\pi_0(S^0)^* \xrightarrow{+1}. $$

By the computation of Morel we have $\pi_0(S^0)^* = K^M_{\mathbb{Q}}$. So after smashing with $S^{2p,p}_\mathbb{Q}$ we obtain a distinguished triangle

$$(S^0)_{\geq 1} \wedge S^{2p,p}_\mathbb{Q} \to S^{2p,p}_\mathbb{Q} \to H\mathbb{K}^M_{\mathbb{Q}} \wedge S^{2p,p}_\mathbb{Q} \xrightarrow{+1}. $$

By taking $[Th(V/X), -]$ we have a long exact sequence

$$\cdots \to [Th(V/X), (S^0)_{\geq 1} \wedge S^{2p,p}_\mathbb{Q}] \to [Th(V/X), S^{2p,p}_\mathbb{Q}] \xrightarrow{\varphi} [Th(V/X), H\mathbb{K}^M_{\mathbb{Q}} \wedge S^{2p,p}_\mathbb{Q}] \to [Th(V/X), (S^0)_{\geq 1} \wedge S^{2p+1,p}_\mathbb{Q}] \to \cdots $$

Now we have $(S^0)_{\geq 1} \wedge S^0_\mathbb{Q} = (S^0)_{\geq 1}$. By the work of C. D. Cisinski, F. Déglise ([CD10]) and the work of A. Ananyevskiy, M. Levine, I. Panin ([ALP15]) we have

$$S^0_\mathbb{Q} = HQ \vee H\mathcal{W}_{*, \mathbb{Q}}.$$ 

This implies $(S^0)_{\geq 1} = (HQ)_{\geq 1}$. The motivic cohomology spectrum $HQ$ is also $-1$-connective, so we have a distinguished triangle

$$(HQ)_{\geq 1} \to HQ \to H\pi_0(HQ)^* \xrightarrow{+1}. $$

The homotopy module $\pi_0(HQ)^*$ is $K^M_{*, \mathbb{Q}}$. We have (by [MVW06, Cor. 19.2] and by purity)

$$HQ^{2p,p}(Th(V/X)) \xrightarrow{\cong} H\mathbb{K}^{M,2p,p}_{\mathbb{Q}}(Th(V/X)).$$

Now from the splitting $S^0_\mathbb{Q} = HQ \vee H\mathcal{W}_{*, \mathbb{Q}}$ the map $\varphi$ take the form:

$$[Th(V/X), S^{2p,p}_\mathbb{Q}] \cong HQ^{2p,p}(Th(V/X)) \oplus H^p_{Nis}(Th(V/X), \mathcal{W}_\mathbb{Q}) \xrightarrow{\varphi} H^p_{Nis}(Th(V/X), K^M_{\mathbb{Q}}) \oplus H^p_{Nis}(Th(V/X), \mathcal{W}_\mathbb{Q}) \cong [Th(V/X), H\mathbb{K}^M_{\mathbb{Q}} \wedge S^{2p,p}_\mathbb{Q}].$$

This implies that $\varphi$ is a canonical isomorphism. \qed

**Corollary 5.8.** Let $k$ be a field. The functor constructed in 5.6

$$CHW(k)_\mathbb{Q} \to SH(k)_\mathbb{Q}$$

is fully faithful.

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Proof. By definition \( CHW(k)_Q \) is the smallest pseudo-abelian full subcategory of the homotopy category \( Ho_k(\hat{H}K^MW, Mod) \) generated as an additive category by
\[
\{ \Sigma_{T,+}^\infty X \wedge_S^L \hat{H}K^MW_{*,Q} | X \in SmProj(k) \}.
\]
The adjunction
\[
- \wedge_S^L \hat{H}K^MW_{*,Q} : SH(k)_Q \leftrightarrows Ho_k(\hat{H}K^MW, Mod) : RU
\]
gives us a natural isomorphism
\[
Ho_k(\hat{H}K^MW, Mod)(\Sigma_{T,+}^\infty T, \Sigma_{T,+}^\infty Y \wedge_S^L \hat{H}K^MW_{*,Q}) \cong SH(k)_Q[\Sigma_{T,+}^\infty X, \Sigma_{T,+}^\infty Y \wedge_S^L \hat{H}K^MW_{*,Q}].
\]
By duality \[\text{[4.26]}\] we have
\[
SH(k)_Q[\Sigma_{T,+}^\infty X, \Sigma_{T,+}^\infty Y \wedge_S^L \hat{H}K^MW_{*,Q}] \cong H_{Nis}^{n_Y+d_Y}(Th(V_Y) \wedge X_+, K^MW_{n_Y+d_Y}, Q_Q),
\]
where \( d_Y = \dim(Y) \), \( V_Y \) is the duality vector bundle given in the theorem \[\text{[2.4]}\] and \( n_Y = \text{rank}(V_Y) \). The corollary follows now from the Theorem \[\text{[5.7]}\].

6. Appendix

In this appendix we simply recollect some facts and definitions in model categories. All the results are well-known and classical (see \[\text{[Q67]}, \text{[Hir03]}, \text{[Hov99]}\]).

6.1. Model Categories.

**Definition 6.1.** A model category \( \mathcal{M} \) is a category with three classes of morphisms
\[
(Fib(\mathcal{M}), Cof(\mathcal{M}), W(\mathcal{M}))
\]
called fibrations, cofibrations and weak equivalences, such that:

1. \( \mathcal{M} \) is closed under small limits and colimits.
2. If \( f, g \in Mor(\mathcal{M}) \) are composable and two out of \( f, g, g \circ f \) are in \( W(\mathcal{M}) \), so is the third one.
3. Given a commutative diagram
\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow^i & & \downarrow^p \\
B & \rightarrow & Y
\end{array}
\]
where \( i \in Cof(\mathcal{M}) \), \( p \in Fib(\mathcal{M}) \) and either \( i \) or \( p \) is in \( W(\mathcal{M}) \), then there exists a morphism \( B \rightarrow X \) making the diagram commutative.
4. \( W(\mathcal{M}), Cof(\mathcal{M}) \) and \( Fib(\mathcal{M}) \) are closed under retracts.
5. Given any morphism \( f : X \rightarrow Y \) in \( Mor(\mathcal{M}) \), there exist two functorial factorizations
\[
\begin{array}{c}
\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow & \nearrow & \downarrow^f \\
Y & \rightarrow & Y
\end{array}
\end{array}
\]

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The first axiom implies that there exist an initial object $\emptyset$ and a final object $\star$. We say $\mathcal{M}$ is pointed if $\emptyset \xrightarrow{\cong} \star$.

**Definition 6.2.** Let $X \in \text{Obj}(\mathcal{M})$ be an object. $X$ is called cofibrant if the natural morphism $\emptyset \to X$ is in $\text{Cof}(\mathcal{M})$. $X$ is called fibrant if the natural morphism $X \to \star$ is in $\text{Fib}(\mathcal{M})$.

Let $i : A \to B$ and $p : X \to Y$ be two morphisms in $\text{Mor}(\mathcal{M})$. We say $i$ has left lifting property wrt. $p$ or $p$ has right lifting property wrt. $i$, if for every solide commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{X} & B \\
\downarrow{i} & & \downarrow{p} \\
\star & \xrightarrow{\cong} & Y
\end{array}
\]

the dotted morphism exists and makes the diagram commutative. Given two morphisms $\text{Mor}(\mathcal{M}) \ni f : A \to B$ and $\text{Mor}(\mathcal{M}) \ni g : C \to D$, we say $f$ is a retract of $g$, if there is a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{C} & A \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{D} & B
\end{array}
\]

where the horizontal composites are identities. Given an object $X \in \text{Obj}(\mathcal{M})$, the factorization axiom tells us that we can factor

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{X^{cof}} & \xrightarrow{\cong} & X,
\end{array}
\]

where $X^{cof}$ is cofibrant. We call $X^{cof}$ a cofibrant replacement of $X$. Similarly, we can factor

\[
\begin{array}{ccc}
X & \xrightarrow{\cong} & X^{fib} \xrightarrow{\to} \star,
\end{array}
\]

where $X^{fib}$ is fibrant. We call $X^{fib}$ a fibrant replacement of $X$.

**Definition 6.3.** Let $\mathcal{M}, \mathcal{N}$ be two model categories. A functor $F : \mathcal{M} \to \mathcal{N}$ is called a left Quillen functor, if it has a right adjoint $G : \mathcal{N} \to \mathcal{M}$ and

1. If $i \in \text{Cof}(\mathcal{M})$, then $F(i) \in \text{Cof}(\mathcal{N})$.
2. If $j \in \text{Cof}(\mathcal{M}) \cap \text{W}(\mathcal{M})$, then $F(j) \in \text{Cof}(\mathcal{N}) \cap \text{W}(\mathcal{M})$.

The right adjoint $G : \mathcal{N} \to \mathcal{M}$ is called a right Quillen functor and the adjunction

\[
F : \mathcal{M} \dashv \mathcal{N} : G
\]

is called a Quillen adjunction.

**Definition 6.4.** Let

\[
F : \mathcal{M} \dashv \mathcal{N} : G
\]

be a Quillen adjunction. $F$ is called a left Quillen equivalence, if for every cofibrant object $X \in \text{Obj}(\mathcal{M})$ and every fibrant object $Y \in \text{Obj}(\mathcal{N})$ one has the following: A morphism $f : X \to GY$ is in $\text{W}(\mathcal{M})$ iff its adjoint $g = \varepsilon_{(F,G)} \circ F(f) : FX \to Y$ is in $\text{W}(\mathcal{N})$. $G$ is called then a right Quillen equivalence. The adjunction

\[
F : \mathcal{M} \dashv \mathcal{N} : G
\]

is called a Quillen equivalence.
Definition 6.5. Let $X \in \text{Obj}(\mathcal{M})$ be an object in a model category $\mathcal{M}$. The cylinder object for $X$ is an object $\text{Cyl}(X)$, such that we have a factorization

$$
\begin{array}{cc}
\begin{array}{c}
\xymatrix{
X \coprod X 
\ar[r]^{
\Delta} 
& 
X \times X 
\ar[d]^p 
\ar[dr]^{\text{RH}} \\
\text{Cyl}(X) 
& 
Y 
\end{array}
\end{array}
\end{array}
$$

where $i \in \text{Cof}(\mathcal{M})$ and $s \in \text{W}(\mathcal{M})$.

Definition 6.6. Let $X \in \text{Obj}(\mathcal{M})$ be an object in a model category $\mathcal{M}$. A path object for $X$ is an object $\mathcal{P}(X)$, such that we have a factorization

$$
\begin{array}{cc}
\begin{array}{c}
\xymatrix{
X 
\ar[r]^\Delta 
& 
X \times X 
\ar[d]^r 
\ar[dr]^{\text{R}} \\
\mathcal{P}(X) 
& 
Y \times Y 
\end{array}
\end{array}
$$

where $r \in \text{W}(\mathcal{M})$ and $p \in \text{Fib}(\mathcal{M})$.

Definition 6.7. Let $f, g : X \Rightarrow Y$ be two morphisms in $\text{Mor}(\mathcal{M})$ of a model category $\mathcal{M}$. $f$ is left homotopic to $g$ if there is a cylinder object $\text{Cyl}(X)$ for $X$, such that we have a factorization

$$
\begin{array}{cc}
\begin{array}{c}
\xymatrix{
X \coprod X 
\ar[r]^{(f,g)} 
& 
Y 
\ar[d]^L 
\ar[dr]^{\text{LH}} \\
\text{Cyl}(X) 
& 
Y 
\end{array}
\end{array}
$$

The map $LH$ is called a left homotopy from $f$ to $g$.

Definition 6.8. Let $f, g : X \Rightarrow Y$ be two morphisms in $\text{Mor}(\mathcal{M})$ of a model category $\mathcal{M}$. $f$ is right homotopic to $g$ if there is a path object $\mathcal{P}(Y)$ for $Y$, such that we have a factorization

$$
\begin{array}{cc}
\begin{array}{c}
\xymatrix{
X 
\ar[r]^\Delta 
& 
X \times Y 
\ar[d]^R 
\ar[dr]^{\text{R}} \\
\mathcal{P}(Y) 
& 
Y \times Y 
\end{array}
\end{array}
$$

The map $RH$ is called a right homotopy from $f$ to $g$.

Definition 6.9. Let $f, g : X \Rightarrow Y$ be two morphisms in $\text{Mor}(\mathcal{M})$ of a model category $\mathcal{M}$. $f$ is homotopic to $g$ if $f$ is left and right homotopic to $g$.

Theorem 6.10. (Quillen [Q67, I.1 Thm. 1]). Let $\mathcal{M}$ be a model category. There exists a category $\text{Ho}(\mathcal{M}) = \mathcal{M}[\text{W}(\mathcal{M})^{-1}]$, which is called the homotopy category of $\mathcal{M}$, where

1. $\text{Obj}(\text{Ho}(\mathcal{M})) = \text{Obj}(\mathcal{M})$.
2. $\text{Ho}(\mathcal{M})(X, Y) = \pi((X^\text{cof})^\text{fib}, (Y^\text{cof})^\text{fib})$, where $\pi$ denotes the set of homotopy classes and the composition law is induced by the composition law of $\mathcal{M}$. 

Theorem 6.11. (Quillen [Q67, I.4 Thm. 3]). Let
\[ F : \mathcal{M} \rightleftarrows \mathcal{N} : G \]
be a Quillen adjunction. Then \((F, G)\) induces an adjunction of homotopy categories
\[ \mathbb{L}F : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{N}) : \mathbb{R}G. \]

Definition 6.12. (1) Let \(\mathcal{M}\) be a model category. \(\mathcal{M}\) is left proper, if in any pushout diagram
\[
\begin{array}{ccc}
A & \xrightarrow{h_1} & X \\
\downarrow i & & \downarrow \\
B & \xrightarrow{h_2} & Y
\end{array}
\]
where \(i \in Cof(\mathcal{M})\) and \(h_1 \in W(\mathcal{M})\), so \(h_2 \in W(\mathcal{M})\).

(2) Let \(\mathcal{M}\) be a model category. \(\mathcal{M}\) is right proper, if in any pullback diagram
\[
\begin{array}{ccc}
A & \xleftarrow{h_1} & X \\
\downarrow & & \downarrow \\
B & \xleftarrow{h_2} & Y
\end{array}
\]
where \(p \in Fib(\mathcal{M})\) and \(h_2 \in W(\mathcal{M})\), so \(h_1 \in W(\mathcal{M})\).

(3) \(\mathcal{M}\) is proper, if it is left and right proper.

Let \(\Delta\) denote the category, whose objects are ordered finite sets
\[ \underline{n} = \{0 < 1 < \cdots < n\}, n \geq 0 \]
and
\[ Mor(\Delta)(m, n) = \{ f : m \to n | i \leq j \implies f(i) \leq f(j) \}. \]
There are cofaces \(\delta^i : \underline{n} \to \underline{n+1}\) and codegeneracies \(\sigma^i : \underline{n+1} \to \underline{n}\) defined by
\[
\delta^i(j) = \begin{cases} j, & \text{if } j < i \\ j + 1, & \text{if } j \geq i \end{cases}
\]
\[
\sigma^i(j) = \begin{cases} j, & \text{if } j \leq i \\ j - 1, & \text{if } j > i \end{cases}
\]
Cofaces and codegeneracies are generators for the maps in \(\Delta\). They satisfy a list of relations (cf. [Weib94, §8]). Now one defines the category of simplicial sets as
\[ \textbf{SSets} \overset{def}{=} \Delta^{op}(\text{Sets}). \]
So simplicial sets are just presheaves of sets on \(\Delta\). For a general category \(\mathcal{A}\) the category of simplicial objects and cosimplicial objects in \(\mathcal{A}\) are defined to be \(\Delta^{op}(\mathcal{A})\) and \(\Delta(\mathcal{A})\) respectively. Let \(\textbf{Top}\) be the category of compactly generated Hausdorff topological spaces. The geometric realization functor is defined by
\[ R : \textbf{SSets} \to \textbf{Top}, \quad X \mapsto R(X) = \int_{\underline{n}} X(\underline{n}) \times \Delta^n, \]
where $\Delta^n$ is the presheaf $\text{Mor}(\Delta)(-, n)$. There is an adjunction

$$R : \text{SSets} \leftrightarrows \text{Top} : S,$$

where $S$ is the singular functor

$$S(T) : \Delta^{op} \to \text{Sets}, \quad n \mapsto \text{Top}(R(\Delta^n), T).$$

Here $R(\Delta^n)$ is

$$R(\Delta^n) = \{ (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | x_i \geq 0, \sum_{i=0}^{n} x_i = 1 \}.$$

**Theorem 6.13.** (Quillen [Q67, II.3 Thm. 3]) The category $\text{SSets}$ has a model category structure.

**Definition 6.14.** Let $\mathcal{M}$ be a category. $\mathcal{M}$ is called simplicial if there is a functor

$$\mathcal{M}^{op} \times \mathcal{M} \to \text{SSets}, \quad (X, Y) \mapsto \text{SSMap}(X, Y),$$

such that

1. $\text{SSMap}(X, Y)_0 = \mathcal{M}(X, Y)$.
2. There exists a composition law

$$\circ : \text{SSMap}(Y, Z) \times \text{SSMap}(X, Y) \to \text{SSMap}(X, Z),$$

which is compatible with the composition law in $\mathcal{M}$.
3. There is a simplicial sets map $i_X : \ast \to \text{SSMap}(X, X), \forall X \in \text{Obj}(\mathcal{M})$, where the associativity of the composition law, right and left unit properties of $i_X$ follows from three commutative diagrams ([Hir03, Def. 9.1.2]).

**Definition 6.15.** Let $\mathcal{M}$ be a model category. $\mathcal{M}$ is called a simplicial model category if

1. $\forall X \in \text{Obj}(\mathcal{M})$ there is an adjunction

$$X \otimes - : \text{SSets} \leftrightarrows \mathcal{M} : \text{SSMap}(X, -),$$

which is compatible with the simplicial structure on $\mathcal{M}$.
2. $\forall Y \in \text{Obj}(\mathcal{M})$ there is an adjunction

$$Y^\ast : \text{SSets} \leftrightarrows \mathcal{M}^{op} : \text{SSMap}(-, Y),$$

which is compatible with the simplicial structure on $\mathcal{M}$.
3. For $\text{Cof}(\mathcal{M}) \ni i : A \to B$ and $\text{Fib}(\mathcal{M}) \ni p : X \to Y$ the map

$$\text{SSMap}(B, X) \xrightarrow{(i^*, p^*)} \text{SSMap}(A, X) \times_{\text{SSMap}(A, Y)} \text{SSMap}(B, Y)$$

is in $\text{Fib}(\text{SSets})$, which is also in $W(\text{SSets})$, if either $i$ or $p$ is in $W(\mathcal{M})$.

**Example 6.16.** $\text{SSets}$ has a canonical simplicial model category structure. $\text{SSMap}(X, Y)$ is the simplicial set with

$$\text{SSMap}(X, Y)_n = \text{SSets}(X \times \Delta^n, Y),$$

with faces and degeneracies induced from the cosimplicial object $\Delta^\ast$. 
Proposition 6.17. Let $\mathcal{M}$ be a simplicial model category. If $X$ is cofibrant and $Y$ is fibrant, then

$$\text{Ho}(\mathcal{M})(X, Y) = \pi_0 \text{SSMap}(X, Y).$$

Consequently, for any objects $A, B \in \text{Obj}(\mathcal{M})$ one has

$$\text{Ho}(\mathcal{M})(A, B) = \pi_0 \text{SSMap}((A^{\text{cof}})^{\text{fib}}, (B^{\text{cof}})^{\text{fib}}).$$

6.2. Localization. All model categories in this subsection are being considered simplicial.

Definition 6.18. Let $\mathcal{M}$ be a model category and $\mathcal{V}$ be a class of morphisms in $\text{Mor}(\mathcal{M})$. A left localization of $\mathcal{M}$ wrt. $\mathcal{V}$ is a model category $L_V \mathcal{M}$ together with a left Quillen functor $F : \mathcal{M} \to L_V \mathcal{M}$, such that:

1. The total left derived functor $\mathbb{L}F : \text{Ho}(\mathcal{M}) \to \text{Ho}(L_V \mathcal{M})$ takes the images in $\text{Ho}(\mathcal{M})$ of elements in $\mathcal{V}$ into isomorphisms in $\text{Ho}(L_V \mathcal{M})$.
2. If $\mathcal{N}$ is a model category and $T : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor such that $\mathbb{L}T : \text{Ho}(\mathcal{M}) \to \text{Ho}(\mathcal{N})$ take the images in $\text{Ho}(\mathcal{M})$ of elements in $\mathcal{V}$ into isomorphisms in $\text{Ho}(\mathcal{N})$, then there is a unique left Quillen functor $L_V \mathcal{M} \to \mathcal{N}$, such that

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{F} & L_V \mathcal{M} \\
T \downarrow & \nearrow & \\
\mathcal{N} & & \\
\end{array}$$

Definition 6.19. Let $\mathcal{M}$ be a model category and $\mathcal{V}$ be a class of morphisms in $\text{Mor}(\mathcal{M})$. A right localization of $\mathcal{M}$ wrt. $\mathcal{V}$ is a model category $R_V \mathcal{M}$ together with a right Quillen functor $G : \mathcal{M} \to R_V \mathcal{M}$, such that:

1. The total right derived functor $\mathbb{R}G : \text{Ho}(\mathcal{M}) \to \text{Ho}(R_V \mathcal{M})$ takes the images in $\text{Ho}(\mathcal{M})$ of elements in $\mathcal{V}$ into isomorphisms in $\text{Ho}(R_V \mathcal{M})$.
2. If $\mathcal{N}$ is a model category and $T : \mathcal{M} \to \mathcal{N}$ is a right Quillen functor such that $\mathbb{R}T : \text{Ho}(\mathcal{M}) \to \text{Ho}(\mathcal{N})$ takes the images in $\text{Ho}(\mathcal{M})$ of elements in $\mathcal{V}$ into isomorphisms in $\text{Ho}(\mathcal{N})$, then there is a unique right Quillen functor $R_V \mathcal{M} \to \mathcal{M}$, such that:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{G} & R_V \mathcal{M} \\
T \downarrow & \nearrow & \\
\mathcal{N} & & \\
\end{array}$$

Definition 6.20. Let $\mathcal{M}$ be a model category and $\mathcal{V}$ a class of morphisms in $\text{Mor}(\mathcal{M})$.

1. An object $X \in \text{Obj}(\mathcal{M})$ is called $\mathcal{V}$-local if $X$ is fibrant and for every $f : A \to B$ in $\mathcal{V}$, $\text{SSMap}(B^{\text{cof}}, X) \xrightarrow{\simeq} \text{SSMap}(A^{\text{cof}}, X)$.
2. A morphism $f : X \to Y$ in $\text{Mor}(\mathcal{M})$ is a $\mathcal{V}$-local equivalence if for every $\mathcal{V}$-local object $T$, $\text{SSMap}(Y^{\text{cof}}, T) \xrightarrow{\simeq} \text{SSMap}(X^{\text{cof}}, T)$.
3. $X \in \text{Obj}(\mathcal{M})$ is called $\mathcal{V}$-colocal if $X$ is cofibrant and for every $f : A \to B$ in $\mathcal{V}$, $\text{SSMap}(X, A^{\text{fib}}) \xrightarrow{\simeq} \text{SSMap}(X, B^{\text{fib}})$.
4. $\text{Mor}(\mathcal{M}) \ni f : X \to Y$ is a $\mathcal{V}$-colocal equivalence if for every $\mathcal{V}$-colocal object $T$, $\text{SSMap}(T, X^{\text{fib}}) \xrightarrow{\simeq} \text{SSMap}(T, Y^{\text{fib}})$. 
Definition 6.21. Let $\mathcal{M}$ be a model category and $\mathcal{V}$ be a class of morphisms in $\text{Mor}(\mathcal{M})$. The left Bousfield localization (if it exists) of $\mathcal{M}$ wrt. $\mathcal{V}$ is a model category structure $L_{\mathcal{V}}\mathcal{M}$ on the underlying category $\mathcal{M}$ with:

1. $W(L_{\mathcal{V}}\mathcal{M})$ is the class of $\mathcal{V}$-local equivalences of $\mathcal{M}$.
2. $\text{Cof}(L_{\mathcal{V}}\mathcal{M}) = \text{Cof}(\mathcal{M})$.
3. $\text{Fib}(L_{\mathcal{V}}\mathcal{M}) = \text{RLP}(\text{Cof}(\mathcal{M}) \cap W(L_{\mathcal{V}}\mathcal{M}))$.

Definition 6.22. Let $\mathcal{M}$ be a model category and $\mathcal{V}$ be a class of morphisms in $\text{Mor}(\mathcal{M})$. The right Bousfield localization (if it exists) of $\mathcal{M}$ wrt. $\mathcal{V}$ is a model category structure $R_{\mathcal{V}}\mathcal{M}$ on the underlying category $\mathcal{M}$ with:

1. $W(R_{\mathcal{V}}\mathcal{M})$ is the class of $\mathcal{V}$-colocal equivalences of $\mathcal{M}$.
2. $\text{Fib}(R_{\mathcal{V}}\mathcal{M}) = \text{Fib}(\mathcal{M})$.
3. $\text{Cof}(R_{\mathcal{V}}\mathcal{M}) = \text{LLP}(\text{Fib}(\mathcal{M}) \cap W(R_{\mathcal{V}}\mathcal{M}))$.

6.3. Symmetric motivic $T$-Spectra. The reference for this subsection is [Jar00]. Let $S$ be a Noetherian scheme of finite Krull dimension. Consider the category $\text{Sm}/S$ of smooth of finite type $S$-schemes. A symmetric $T$-spectrum is a collection $\{X_n\}_{n \geq 0}$, where $X_n \in \Delta^{op}(\text{PrSh}_{\text{Nis}}(\text{Sm}/S))_{+}$, together with the left actions

$$\Sigma_n \times X_n \to X_n,$$

where $\Sigma_n$ is the $n$-th symmetric group. There are the bonding maps

$$\sigma_n : T \wedge X_n \to X_{n+1},$$

such that the iterative composition

$$T \wedge X_n \to X_{n+m}$$

is $\Sigma_m \times \Sigma_n$-equivariant. A morphism between symmetric $T$-spectra is a family $\{f_n : X_n \to Y_n\}_{n \geq 0}$, where the following diagram

$$\begin{array}{ccc}
T \wedge X_n & \xrightarrow{\text{id} \wedge f_n} & T \wedge Y_n \\
\sigma_n \downarrow & & \sigma_n \\
X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1}
\end{array}$$

commutes and $f_n$ is $\Sigma_n$-equivariant $\forall n \geq 0$. The category of symmetric $T$-spectra is denoted by $\text{Spect}^\Sigma_T(S)$. A symmetric sequence $X$ is a family $\{X_n | X_n \in \Delta^{op}(\text{PrSh}_{\text{Nis}}(\text{Sm}/S))_{+}\}_{n \geq 0}$ with left actions

$$\Sigma_n \times X_n \to X_n.$$

A morphism $f : X \to Y$ of symmetric sequences is a family $\{f_n : X_n \to Y_n\}$, where $f_n$ are $\Sigma_n$-equivariant $\forall n \geq 0$. We denote the category of symmetric sequences of pointed simplicial presheaves by $\Delta^{op}(\text{PrSh}_{\text{Nis}}(\text{Sm}/S))_{+}^\Sigma$. Recall that there are families of functors

$$F_n : \Delta^{op}(\text{PrSh}_{\text{Nis}}(\text{Sm}/S))_{+} \to \Delta^{op}(\text{PrSh}_{\text{Nis}}(\text{Sm}/S))_{+}^\Sigma,$$

where

$$(F_n(\mathcal{X}))_m = \begin{cases} 
* & \text{if } m \neq n \\
\bigvee_{\sigma \in \Sigma_n} \mathcal{X} & \text{if } m = n
\end{cases}$$
Now we can define the smash product of symmetric $T$ and $\Sigma_n$ for two symmetric sequences $X$ and $\Sigma_n$.

For two symmetric sequences $X$ and $Y$, their product is defined as

$$(X \otimes Y)_n = \bigvee_{p+q=n} \Sigma_n \otimes_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q.$$ 

The notation $\Sigma_n \otimes_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q$ means there is an action $\gamma$ of $\Sigma_p \times \Sigma_q$ on $X_p \wedge Y_q$ via the canonical embedding $\Sigma_p \times \Sigma_q \subset \Sigma_n$ and also another action $\gamma' : \Sigma_p \times \Sigma_q \times (X_p \wedge Y_q) \rightarrow X_p \wedge Y_q$. We let

$$\Sigma_n \otimes_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q = eq[\gamma_{\sigma} - \gamma_{\sigma}']_{\sigma \in \Sigma_p \times \Sigma_q}.$$ 

Now one can define

$$F_n^\Sigma : \Delta^\op(PrSh_{Nis}(Sm/S))_+ \rightarrow \text{Spect}^\Sigma_T(S), \quad \mathcal{X} \mapsto \mathbb{S}^0 \otimes F_n(\mathcal{X}),$$

where $\mathbb{S}^0$ denotes the motivic sphere spectrum

$$\mathbb{S}^0 = (S_+, T \wedge S_+, T^\wedge 2 \wedge S_+, \cdots)$$

$\Sigma_n$ acts on $\mathbb{S}^0$ by permuting the $T^{\wedge n}$ factors and $S_+$ is pointed by $S \coprod S$. One has an adjunction

$$F_n^\Sigma : \Delta^\op(PrSh_{Nis}(Sm/S))_+ \rightleftarrows \text{Spect}^\Sigma_T(S) : E_{n}.$$

In fact, one has $F_0^\Sigma(S_+) = \mathbb{S}^0$. A symmetric $T$-spectrum $X$ can be understood as a symmetric sequence with a module structure $\sigma_X : \mathbb{S}^0 \otimes X \rightarrow X$ over the motivic sphere spectrum $\mathbb{S}^0$.

Now we can define the smash product of symmetric $T$-spectra as

$$X \wedge Y \overset{\text{df}}{=} \text{coeq}(\mathbb{S}^0 \otimes X \otimes Y \xrightarrow{\tau} X \otimes \mathbb{S}^0 \otimes Y \xrightarrow{id_Y \otimes \gamma_Y} X \otimes Y),$$

where the top map is $\sigma_X \otimes id_Y$ and the bottom map is

$$\mathbb{S}^0 \otimes X \otimes Y \xrightarrow{\tau} X \otimes \mathbb{S}^0 \otimes Y \xrightarrow{id_Y \otimes \gamma_Y} X \otimes Y.$$

We just mention the following results of Jardine.

**Theorem 6.23.** (Jardine [Jar00], Thm. 4.2] The category $\text{Spect}^\Sigma_T(S)$ has a model category structure, which is proper and simplicial.

**Theorem 6.24.** (Jardine [Jar00], Prop. 4.19)]). $(\text{Spect}^\Sigma_T(S), \mathbb{S}^0, \wedge)$ is a symmetric monoidal model category.

Now we discuss a little bit about the Quillen adjunction

$$- \wedge E : \text{Spect}^\Sigma_T(S) \rightleftarrows E - \text{Mod}^\Sigma : U,$$

where $E$ is a motivic strict ring spectrum (we always consider only commutative ring spectrum). On the level of the underlying categories the unit and counit of the adjunction are defined by

$$\eta_X : X \cong \mathbb{S}^0 \wedge X \xrightarrow{\varphi_E \wedge id_X} E \wedge X = U(E \wedge X),$$

and

$$\varepsilon_M : E \wedge U(M) = E \wedge M \xrightarrow{\gamma_M} M.$$
By [Jar00, Prop. 4.19] the category \( \text{Spect}_T^{\Sigma}(S) \) satisfies the axiom in [SS00, Def. 3.3]. By [SS00, Thm. 4.1] one can conclude that the adjunction

\[- \wedge : \text{Spect}_T^{\Sigma}(S) \rightleftarrows \text{E} - \text{Mod}^{\Sigma} : U\]

induces a model category structure on \( \text{E} - \text{Mod}^{\Sigma} \). It is clear that the forgetful functor \( U : \text{E} - \text{Mod}^{\Sigma} \to \text{Spect}_T^{\Sigma}(S) \) is a right Quillen functor, because \( \text{Fib}(\text{E} - \text{Mod}^{\Sigma}) \) and \( \text{Fib}(\text{E} - \text{Mod}^{\Sigma}) \cap W(\text{E} - \text{Mod}^{\Sigma}) \) are detected in \( \text{Spect}_T^{\Sigma}(S) \). So we can claim that the adjunction above is a Quillen adjunction. Since \( \text{E} \) is a commutative ring spectrum, \( \text{E} - \text{Mod}^{\Sigma} \) has the closed symmetric monoidal category structure induced by the one on \( \text{Spect}_T^{\Sigma}(S) \) by declaring:

\[- \wedge : \text{E} - \text{Mod}^{\Sigma} \times \text{E} - \text{Mod}^{\Sigma} \to \text{E} - \text{Mod}^{\Sigma}, \quad (M, N) \mapsto M \wedge_E N\]

and

\[\text{Hom}_{\text{E} - \text{Mod}^{\Sigma}} : \text{E} - \text{Mod}^{\Sigma} \times \text{E} - \text{Mod}^{\Sigma} \to \text{E} - \text{Mod}^{\Sigma}, \quad \text{Hom}_{\text{E} - \text{Mod}^{\Sigma}}(M, N),\]

where

\[M \wedge_E N \overset{\text{defn}}{=} \text{coeq}(E \wedge M \wedge N \xrightarrow{i} M \wedge N).\]

The top map is \( \gamma_M \wedge \text{id} \) and the bottom map is the composition

\[E \wedge M \wedge N \xrightarrow{\tau \wedge \text{id}} M \wedge E \wedge N \xrightarrow{\text{id} \wedge \gamma_N} M \wedge N.\]

The internal Hom is defined as

\[\text{Hom}_{\text{E} - \text{Mod}^{\Sigma}}(M, N) \overset{\text{defn}}{=} \text{eq}(\text{Hom}_{\text{Spect}_T^{\Sigma}(S)}(M, N) \xrightarrow{\text{id}} \text{Hom}_{\text{Spect}_T^{\Sigma}(S)}(E \wedge M, N),\]

where the top map is \( \gamma_M^* = \circ \gamma_M \) and the bottom map is

\[\gamma_N^* : \text{Hom}_{\text{Spect}_T^{\Sigma}(S)}(M, N) \xrightarrow{E \wedge} \text{Hom}_{\text{Spect}_T^{\Sigma}(S)}(E \wedge M, E \wedge N) \xrightarrow{\gamma_N^0} \text{Hom}_{\text{Spect}_T^{\Sigma}(S)}(E \wedge M, N).\]

We should also mention the following theorem of Jardine:

**Theorem 6.25.** [Jar00, Thm. 4.31] There is a Quillen equivalence

\[V : \text{Spect}_T(S) \rightleftarrows \text{Spect}_T^{\Sigma}(S) : U,\]

where \( \text{Spect}_T(S) \) is the category of motivic \( T \)-spectra, \( V \) is the symmetrization functor and \( U \) is the forgetful functor.

We remind the reader that throughout this work we take the motivic stable homotopy category as

\[SH(k) = \text{Ho}(\text{Spect}_T^{\Sigma}(k)).\]

The theorem of Jardine allows us to identify \( SH(k) \) equivalently to the \( \mathbb{A}^1 \)-stable homotopy category \( SH^{\mathbb{A}^1}(k) \cong SH^T(k) \) of Morel constructed in [Mor04a, Defn. 5.1, Rem. 5.1.10 and pp. 420], which is defined as the homotopy category of the motivic \( T \)-spectra \( \text{Spect}_T(k) \). Hence, we can use Morel computation of \( \pi_0^{\mathbb{A}^1}(S^0)_* \) and his stable \( \mathbb{A}^1 \)-connectivity result.
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