On the set of divisors with zero geometric defect

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Abstract

Let $f : \mathbb{C} \to X$ be a transcendental holomorphic curve into a complex projective manifold $X$. Let $L$ be a very ample line bundle on $X$. Let $s$ be a very generic holomorphic section of $L$ and $D$ the zero divisor given by $s$. We prove that the geometric defect of $D$ (defect of truncation 1) with respect to $f$ is zero. We also prove that $f$ almost misses general enough analytic subsets on $X$ of codimension 2.

Keywords: Nevanlinna theory, tangent current, density current, holomorphic curve

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1 Introduction

Let $X$ be a compact Kähler manifold and $D$ an effective divisor in $X$. Let $f : \mathbb{C} \to X$ be a holomorphic curve such that $f(\mathbb{C}) \not\subset \text{Supp} D$. In Nevanlinna’s theory, we are interested in understanding how often $f(\mathbb{C})$ intersects $D$.

Denote by $L$ the line bundle generated by $D$. Let $D_r$ be the disk of radius $r$ centered at the origin in $\mathbb{C}$. Based on the exhaustion $\mathbb{C} = \bigcup_{r > 0} \mathbb{D}_r$, we will count the number of intersection points between $f(\mathbb{D}_r)$ and $D$, which is finite. Precisely, taking the $k$-truncated degrees of the divisor $f^*D$ on disks by

$$n_f^{[k]}(t, D) := \sum_{z \in D_t} \min\{k, \text{ord}_zf^*D\} \quad (t > 0),$$

the truncated counting function of $f$ at level $k$ with respect to $D$ is then defined by taking the logarithmic average

$$N_f^{[k]}(r, D) := \int_1^r \frac{n_f^{[k]}(t, D)}{t} \, dt \quad (r > 1).$$

When $k = \infty$, we write $n_f(r, D)$, $N_f(r, D)$ instead of $n_f^{[\infty]}(r, D)$, $N_f^{[\infty]}(r, D)$. These functions count the number of points in $f(\mathbb{D}_r) \cap D$, taking into account the multiplicity. On the other hand, when $k = 1$, the function $n_f^{[1]}(r, D)$ gives us the number of points in $f(\mathbb{D}_r) \cap D$ as a set. We then call $N_f^{[1]}(r, D)$ the geometric counting function of $f$ with respect to $D$.

For every smooth $(1, 1)$-form $\eta$ on $X$, we put

$$T_f(r, \eta) := \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} f^*\eta \quad (r > 1).$$

Observe that if $\eta, \eta'$ are two smooth closed $(1, 1)$-forms in the same cohomology class, then $T_f(r, \eta) = T_f(r, \eta') + O(1)$ as $r \to \infty$ by the Lelong-Jensen formula. It follows that given a smooth Chern form $\omega_L$ of $L$, the characteristic function of $f$ with respect to $L$ given by

$$T_f(r, L) := T_f(r, \omega_L)$$
is well-defined up to a bounded term as \( r \to \infty \). Let \( \omega \) be a Kähler form on \( X \). The function \( f \) is said to be transcendental, if
\[
\liminf_{r \to \infty} \frac{T_f(r, \omega)}{\log r} = \infty.
\]
Note that in the case where \( X \) is projective, the last condition agrees with the usual definition of transcendental functions (see [15, Th. 2.5.27]).

A major problem in Nevanlinna’s theory is to compare \( N_f^{[1]}(r, D) \) with \( T_f(r, L) \). By the First Main Theorem [15, Th. 2.3.31], there holds
\[
N_f^{[1]}(r, D) \leq N_f(r, D) \leq T_f(r, L) + O(1).
\]

The defect of \( D \) with respect to \( f \) is defined as
\[
\delta_f(D) := \liminf_{r \to \infty} \left( 1 - \frac{N_f(r, D)}{T_f(r, L)} \right).
\]

The geometric defect \( \delta_f^{[1]}(D) \) of \( D \) is then defined in a similar way with \( N_f^{[1]}(r, D) \) in place of \( N_f(r, D) \). It is trivial that
\[
0 \leq \delta_f(D) \leq \delta_f^{[1]}(D) \leq 1.
\]

A divisor with zero geometric defect roughly signifies that the logarithmic average growth of the cardinality of the set \( f(D_r) \cap D \) is the same as that of the area of \( f(D_r) \) as \( r \to \infty \). In the other extreme case where \( \delta_f(D) = 1 \), the logarithmic average of the cardinality of the set \( f(D_r) \cap D \) counted with multiplicity is negligible with respect to the area of \( f(D_r) \). Let us recall the following conjecture ([15, p. 412]).

**Conjecture 1.1.** (Griffiths-Noguchi-Winkelmann) Let \( K_X \) be the canonical line bundle of \( X \). Assume that \( D \) is an ample divisor of simple normal crossing type and \( f \) is algebraically non-degenerate, i.e., its image is Zariski dense in \( X \). Then for any given constant \( \epsilon > 0 \), we have
\[
T_f(r, L) + T_f(r, K_X) \leq N_f^{[1]}(r, D) + \epsilon T_f(r, L),
\]
for \( r \) outside a set of finite Lebesgue measure, possibly depending on \( \epsilon \).

Note that if \( K_X > 0 \), then \( T_f(r, K_X) \geq c T_f(r, L) \) for some constant \( c > 0 \), hence, \( 1.1 \) can not hold. So Conjecture \( 1.1 \) implies a weak version of the Green-Griffiths-Lang conjecture stipulating that if \( K_X > 0 \), then there is no algebraically non-degenerate holomorphic curve in \( X \). On the other hand, for some class of \( X \), there exist many algebraically non-degenerate curves in \( X \), e.g. \( X \) is an abelian variety or a complex projective space. More generally, a recent result of Campana-Winkelmann [4] implies that this is also the case if \( X \) is rationally connected. In this context, Conjecture \( 1.1 \) is fully confirmed only in the case where \( X \) is an abelian variety or semi-abelian variety by the celebrated Second Main Theorem of Noguchi-Winkelmann-Yamanoi [17, 16, 19]. Recall that \( K_X = 0 \) in the last situation. Here is our first main result giving a weak version of Conjecture \( 1.1 \) when \( K_X \leq 0 \).

**Theorem 1.2.** Let \( X \) be a complex projective manifold and \( L \) a very ample line bundle on \( X \). Let \( E \) be the space of holomorphic sections of \( L \). Let \( f : \mathbb{C} \to X \) be a transcendental holomorphic curve. Then for every effective divisor \( D \) of \( L \) outside a countable union of proper algebraic subsets of \( \mathbb{P}(E) \), we have \( \delta_f^{[1]}(D) = 0 \).
Note that we don’t require that \( f \) is algebraically non-degenerate in the above theorem. The above result is clear if \( f \) is a non-constant rational curve because the image of \( f \) is an algebraic curve in \( X \). By using a basis of \( E \), we can embed \( X \) into a projective space and the problem can be reduced to the case where \( X = \mathbb{P}^n \) and \( L \) is the hyperplane bundle. But this reduction doesn’t make the problem easier. We would like to note that it is known that in the setting of the above theorem, \( \delta_f(D) = 0 \) for \( D \) outside a pluripolar subset of \( \mathbb{P}(E) \), see [3, 14] and [15, Th. 2.3.39].

Consider now an analytic subset \( V \) of \( X \). By using local basis of the sheaf of ideals of holomorphic functions defining \( V \), we can also define the counting function \( N_f(r, V) \) even if \( \text{codim} \ V \geq 2 \), see [15, Sec. 2.4.1] for details. Here is our second main result.

**Theorem 1.3.** Let \( Z \) be a complex manifold. Let \( X \) be a compact Kähler manifold and \( \omega \) a Kähler form on \( X \). Let \( f : \mathbb{C} \to X \) a transcendental holomorphic curve. Let \( V \) be a complex smooth submanifold of \( X \times Z \) of codimension \( s \geq 2 \). Denote by \( p_1, p_2 \) the natural projections from \( X \times Z \) to \( X, Z \) respectively. Assume that the restriction \( p_1 \vert_V \) of \( p_1 \) to \( V \) is a submersion and the restriction \( p_2 \vert_V \) of \( p_2 \) to \( V \) is a surjection. Then for any \( \alpha \) outside a countable union of proper analytic subsets of \( Z \), we have

\[
\liminf_{r \to \infty} \frac{N_f(r, V_\alpha)}{T_f(r, \omega)} = 0,
\]

where \( V_\alpha := V \cap (X \times \{\alpha\}) \).

Observe that the fiber of \( p_2 \vert_V \) at a point \( \alpha \in Z \) is \( V_\alpha \). Since \( p_2 \) is surjective and proper, the set of critical values of \( p_2 \) is a proper analytic subset of \( Z \) and for every \( \alpha \) outside this set, we see that \( V_\alpha \) is of codimension \( s \) in \( X \) and \( V \) is transverse to \( X \times \{\alpha\} \). Hence \( V \) essentially is a family of analytic subsets of codimension \( s \geq 2 \) in \( X \). The above result roughly says that for an analytic subset \( V \) of codimension 2 general enough, then \( \liminf_{r \to \infty} \frac{N_f(r, V)}{T_f(r, \omega)} = 0 \), i.e., \( f \) almost misses such \( V \).

A simple example to which we can apply Theorem 1.3 is when \( X = \mathbb{P}^n \), \( Z \) is the space of projective subspaces of codimension \( s \geq 2 \) of \( X \) and \( V \) is the family of subspaces of codimension \( s \) of \( \mathbb{P}^n \) which is viewed as a submanifold of \( \mathbb{P}^n \times Z \). Even in this situation, it seems that the conclusion of the above theorem is still new. Note that when \( X \) is an abelian variety, by Noguchi-Winkelmann-Yamanoi [17, 16, 19], one has \( \liminf_{r \to \infty} \frac{N_f(r, V)}{T_f(r, \omega)} = 0 \) for every analytic set \( V \) of codimension \( \geq 2 \) in \( X \).

To prove Theorem 1.3 we look at a Nevanlinna’s current \( S \) of bidimension \((1,1)\) associated to \( f \). We study the intersection of \( S \) with \( V_\alpha \) in terms of the theory of density currents coined recently by Dinh-Sibony in [10] and developed in [7, 12, 18]. Using the fact that the codimension of \( V_\alpha \) is at least 2, we can show that the density current associated to \( S, V_\alpha \) is zero for \( \alpha \) outside a countable union of proper analytic subsets of \( Z \). The subtleties in our use of density currents is that we have to deal with the associativity of density currents in some setting which is necessary for our proof and work with non-closed (or even non-ddc-closed) currents.

The proof of Theorem 1.2 goes first by lifting \( f \), \( D \) to a curve \( \hat{f} \) and an analytic subset \( \hat{D} \) of codimension 2 in the projectivisation \( \mathbb{P}(TX) \) of the tangent bundle \( TX \) of \( X \). The intersection of \( \hat{f} \) and \( \hat{D} \) encodes the points where \( f \) is tangent to \( D \). This is an idea inspired by [2, 6]. To estimate the mass of \( \hat{f} (\mathbb{D}_r) \cap \hat{D} \), we look at a Nevanlinna’s current of \( \hat{f} \) and by the proof of Theorem 1.3, the intersection of that current with \( \hat{D} \) is zero for almost \( \hat{D} \). This is a key of the proof. Another important detail is that we can show a continuity property of total tangent classes for non-ddc currents arising in our setting, see Lemma 3.4 below.

The paper is organized as follows. In Section 2 we present preparatory results about density currents which will be needed for our proofs of main theorems. We prove Theorems 1.3 and 1.2.
in Sections 3 and 4 respectively.

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2 Density currents

In this section, we will prove several results concerning density currents which are crucial for our proofs of main results. Firstly, we need to recall some known facts about the density currents proved in [10], see also [13, 12, 7] for recent developments.

Let $Y$ be a compact Kähler manifold. Let $V$ be a smooth submanifold of dimension $\ell$ of $Y$. Let $[V]$ be the current of integration along $V$. Let $T$ be a positive closed current of bi-degree $(p, p)$ in $Y$. Assume that $T$ has no mass on $V$. We denote by $\{T\}, \{V\}$ the cohomology class of $T, [V]$ respectively. Denote by $\pi : E \to V$ the normal bundle of $V$ in $Y$ and $E := \mathbb{P}(E \oplus \mathbb{C})$ the projective compactification of $E$. The hypersurface at infinity $H_\infty := E \setminus \overline{E}$ of $E$ is naturally isomorphic to $\mathbb{P}(E)$ as fiber bundles over $V$. We also have a canonical projection $\pi_\infty : \overline{E} \setminus V \to H_\infty$.

A smooth diffeomorphism $\tau$ from an open subset $U$ of $Y$ to an open neighborhood of $V$ in $E$ is called admissible if $\tau$ is the identity map on $V \cap U$ and the restriction of its differential $d\tau$ to $E|_{V \cap U}$ is the identity map. By [10, Le. 4.2], there exists an admissible map $\tau : U \to E$ such that $U$ is a tubular neighborhood of $V$. In general, such a $\tau$ is not holomorphic.

For $\lambda \in \mathbb{C}^*$, let $A_{\lambda} : E \to E$ be the multiplication by $\lambda$ on fibers of $E$. Let $\tau$ be an admissible map defining on a tubular neighborhood of $V$. By [10], the family of closed currents $(A_{\lambda})_{*}\tau_{*}T$ has a uniformly bounded mass on compact sets of $E$. Any limit current of this family in $E$ is called a tangent current to $T$ along $V$. Such a current is a positive closed current invariant by $A_{\lambda}$ and can be extended to a current in $E$. Tangent currents are independent of $\tau$, this means that if $R := \lim_{n \to \infty}(A_{\lambda_n})_{*}\tau_{*}T$ is a tangent current, then for every admissible map $\tau' : U' \to E$, we also have $R = \lim_{n \to \infty}(A_{\lambda_n})_{*}\tau'_{*}T$ on $\pi_\infty^{-1}(U' \cap V)$. This property allows us crucial flexibility in choosing admissible maps when we need to estimate tangent currents in practice. Thus if we work locally, then after trivializing $E$, the admissible map in the definition of tangent currents can be chosen to be the identity.

In general, the tangent currents to $T$ along $V$ are not unique but their cohomology classes in $E$ are unique. That unique class is denote by $\kappa^V(T)$ and called the total tangent class to $T$ along $V$. Let $h^E_T$ be the Chern class of the dual of the tautological line bundle of $E$ respectively. We can write $\kappa^V(T)$ uniquely as

$$\kappa^V(T) = \min\{\ell, p - 1\} \sum_{j = \max\{0, \ell - p\}} \pi^* \kappa_j \wedge h^E_T^{p - (\ell - j)},$$

where $\kappa_j$ is a cohomology class in $H^{\ell - j, \ell - j}(V)$, which is called the $j^{th}$ component of $\kappa^V(T)$.

Let $T_1, \ldots, T_m$ be positive closed currents on $Y$. Let $T_{1} \otimes \cdots \otimes T_{m}$ be the tensor current of $T_1, \ldots, T_m$ on $Y^m$. A density current associated to $T_1, \ldots, T_m$ is a tangent current of $T_1 \otimes \cdots \otimes T_m$ along the diagonal $\Delta_m := \{(y, \ldots, y) : y \in Y\}$ of $Y^m$. If there is a unique density current $T_\infty$ to $T_1, \ldots, T_m$ and $T_\infty = \pi_m^* S$ for some positive closed current $S$ on $\Delta_m$, where $\pi_m : E_{\Delta_m} \to \Delta_m$ is

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the projection from the normal bundle $E_\Delta$, then we say that the Dinh-Sibony product $T_1 \wedge \cdots \wedge T_m$ is well-defined and put $T_1 \wedge \cdots \wedge T_m := S$.

Consider a particular case where $T_1 := T$ and $T_2 := [V]$. Observing that we have natural identifications $T(Y^2) \approx TY \times TY$ between vector bundles, where $TY$ is the tangent bundle of $Y$ and $\Delta \approx Y$. Since $V \subset Y \approx \Delta$, there is a canonical inclusion $i$ from $TV$ to $(TY \times \{0\})|_{\Delta}$ which is a subbundle of $T(Y^2)|_{\Delta}$. Let $F$ be the image of $i(TV)$ in the normal bundle $E_\Delta = T(Y^2)/T\Delta$. Put $\Delta_V := \{(y, y) \in Y^2 : y \in V\}$. Let $E_{\Delta,V}$ be the restriction of $E_\Delta$ to $\Delta_V$. Observing that $F$ is a subbundle of $E_{\Delta,V}$ of rank $\ell$ and the natural map

$$\Psi : E_{\Delta,V}/F \to E = TY/TV$$

is an isomorphism. Let $p_V : E_{\Delta,V} \to E_{\Delta,V}/F$ be the natural projection. The following result tells us that a density current associated to $T, [V]$ corresponds naturally to a tangent current of $T$ along $V$.

**Lemma 2.1.** ([10] Le. 5.4) or ([18] Le. 2.3]) If $T_\infty$ is a tangent current of $T$ along $V$, then the current $p_V^*\Psi^*T_\infty$ is a tangent current of $T \otimes [V]$ along $\Delta$. Conversely, every tangent current of $T \otimes [V]$ along $\Delta$ can be written as $p_V^*\Psi^*T_\infty$ for some tangent current $T_\infty$ of $T$ along $V$.

We present below some more properties of tangent currents which will be used in the sequel. Let $\sigma : \hat{Y} \to Y$ be the blowup along $V$ of $Y$ and $\hat{V} := \sigma^{-1}(V)$ the exceptional hypersurface. Recall that $\hat{V}$ is naturally biholomorphic to $\mathbb{P}(E)$.

Let $\sigma_E : \hat{E} \to E$ be the blowup along $V$ of $E$. The projection $\pi$ induces naturally a vector bundle projection $\pi_E$ from $\hat{E}$ to $\sigma_E^{-1}(V)$. Let $\bar{E}$ be the projective compactification of the vector bundle $\hat{E}$. The map $\pi_E$ can be extended to a projection $\pi_{\bar{E}} : \bar{E} \to \sigma_{\bar{E}}^{-1}(V)$. We also denote by $\sigma_E$ its natural extension from $\bar{E}$ to $\bar{E}$. The vector bundle $\pi_{\bar{E}} : \bar{E} \to \sigma_{\bar{E}}^{-1}(V)$ is naturally identified with the normal bundle of $\hat{V}$ in $\hat{Y}$. Hence we can identify $\sigma_{\bar{E}}^{-1}(V)$ with $\hat{V}$ and use $\bar{E}$ as the normal bundle of $\hat{V}$ in $\hat{Y}$.

Recall that since $Y$ is Kähler, so is $\hat{Y}$. If $\text{codim} V \geq 2$, let $\tilde{\omega}_h$ be a smooth Chern form of the line bundle $O(-\hat{V})$ whose restriction to each fiber of $\hat{V} \approx \mathbb{P}(E)$ is strictly positive, otherwise we simply put $\tilde{\omega}_h := 0$. By rescaling $\omega$ if necessary, we can assume that $\tilde{\omega} := \sigma^*\omega + \tilde{\omega}_h > 0$.

Observe that the hypersurface at infinity $\hat{H}_\infty$ of $\bar{E}$ is biholomorphic to that of $\bar{E}$ via $\sigma_E$. We use $\hat{\pi}_{\infty}$ to denote the natural projection from $\bar{E} \setminus \hat{V}$ to $\hat{H}_\infty$. Since the rank of $\bar{E}$ over $\hat{V}$ is 1, we can extend $\hat{\pi}_{\infty}$ to a projection from $\bar{E}$ to $\hat{H}_\infty$. Thus, $\hat{V}$ is naturally identified with $\hat{H}_\infty$ which is in turn naturally identified with $H_\infty$.

Let $\hat{T}$ be the pull-back of $T$ on $\hat{Y} \setminus \hat{V}$ by $\sigma|_{\hat{Y} \setminus \hat{V}}$. The mass of $\hat{T}$ is finite by [8]. We thus can extend $\hat{T}$ trivially through $\hat{V}$ to a current on $\hat{Y}$.

For any positive current $S$ on $H_\infty$, the positive current $\pi_{\infty}^*S$ has a finite mass on $\bar{E} \setminus V$. Hence we can extend it trivially through $V$. Denote also by $\pi_{\infty}^*S$ this extension. Since

$$\pi_{\infty} = \hat{\pi}_{\infty} \circ (\sigma_E|_{\bar{E}})^{-1},$$

we can check that

$$\pi_{\infty}^*S = (\sigma_E)_* \circ (\hat{\pi}_{\infty})^*S.$$

By the last formula, the map $\pi_{\infty}^*$ induces natural maps on the cohomology groups and $\pi_{\infty}^*$ is continuous. For a cohomology class $\alpha$ in $X$, we denote by $\alpha|_V$ the restriction of $\alpha$ to $V$. 

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Let $T_\infty$ be a tangent current to $T$ along $V$. We can check directly that there exists a tangent current $\hat{T}_\infty$ to $\hat{T}$ along $\hat{V}$ satisfying

$$T_\infty = (\sigma_E)_* \hat{T}_\infty. \quad (2.2)$$

We have the following important property.

**Proposition 2.2.** (10) The current $T_\infty$ is $V$-conic, i.e., $A_\lambda^* T_\infty = T_\infty$ for every $\lambda \in \mathbb{C}^*$. Equivalently, there exists a positive closed current $S_\infty$ on $H_\infty$ such that

$$T_\infty = \pi^* S_\infty. \quad (2.3)$$

Moreover, we have

$$\kappa^V(T) = \pi^* \{S_\infty\} = \pi^* \{\{\hat{T}_\infty\} - \{\hat{V}\}\}, \quad (2.4)$$

where recall that we identified $\hat{V}$ with $\hat{H}_\infty$ and $\hat{H}_\infty$ with $H_\infty$.

We are going to present the first main result of this section concerning the density currents associated to a current and slices of a given submersion. Recall that the mass of a current $S$ of order $0$ on a manifold $M$ is given by

$$\|S\|_M := \sup_{\|\Phi\|_{C^0} \leq 1} \langle S, \Phi \rangle,$$

where the supremum is taken over all smooth differential forms $\Phi$ on $M$. From now on, the notations $\preceq, \succeq$ are used to indicate $\leq, \geq$ modulo a multiplicative constant, respectively.

Let $W$ be a compact Kähler manifold and $\pi_W : Y \to W$ a holomorphic submersion. Put $Y_\theta := \pi_W^{-1}(\theta)$ for every $\theta \in W$. Let $E_\theta$ be the normal bundle of $Y_\theta$ in $Y$ and $\pi(E_\theta) : \mathbb{P}(E_\theta) \to Y_\theta$ be the natural projection. Let $m := \dim W$ and $\ell := \dim Y_\theta$, which is independent of $\theta$.

**Theorem 2.3.** There exists a subset $A$ of $W$ which is a countable union of proper analytic subsets of $W$ such that for any $\theta \notin A$, the component $\kappa^T_\theta (T)$ of the total density tangent class to $T$ along $Y_\theta$ is zero for any $j > \ell - p$. In particular, if $p > \ell$, then the tangent current to $T$ along $Y_\theta$ is zero, or equivalently $T \perp \{Y_\theta\} = 0$.

**Proof.** Let $\omega$ be a Kähler form on $Y$. For $j \geq \max\{\ell - p, 0\}$, the current

$$T_j := (\pi_W)_* (T \wedge \omega^j)$$

is positive closed of bidegree $(s, s)$ in $W$, where $s := p + j - \ell$. If $j > \ell - p$, then $s \geq 1$. Let $A_j$ be the set of $\theta \in W$ such that $\nu(T_j, \theta) > 0$. Observing that $A_j$ is a countable union of proper analytic subsets of $W$ by Siu’s semi-continuity theorem if $j > \ell - p$. Set

$$A := \bigcup_{j \geq \max\{\ell - p + 1, 0\}} A_j.$$

Let $\theta_0 \notin A$. Thus for any $j \geq \ell - p + 1$, one has $\nu(T_j, \theta_0) = 0$.

Consider a tangent current $R$ to $T$ along $Y_{\theta_0}$, i.e., $R = \lim_{n \to \infty} (A_{\lambda_n})_* \tau_n T$, where $\tau$ is an admissible map from a tubular neighborhood of $Y_{\theta_0}$ to $E_{\theta_0}$. Let $j_0$ be the maximal $j$ such that $\kappa^T_j (T) \neq 0$. Note that by a bi-degree reason, we have $j_0 \geq \max\{0, \ell - p\}$.

Suppose that $j_0 > \ell - p$ because otherwise we have nothing to prove. Recall that $\kappa^T_{j_0}(T)$ is a class in $H^{\ell - j, \ell - j}(Y_{\theta_0})$. It follows that the mass of the positive closed current $R_{j_0} := R \wedge \pi^* (\mathbb{P}(E_\theta))_{\theta_0}^{\omega_{j_0}}$
Let \( \omega_\theta \) is the restriction of \( \omega \) to \( Y_\theta \). Moreover since \( R \) is \( Y_\theta \)-conic, so is \( R_{j_0} \). Thus, the mass of \( R_{j_0} \) on \( \overline{E_\theta} \) is bounded by a constant times the mass of \( R_{j_0} \) on every open tubular neighborhood of \( Y_{\theta_0} \), see [10, Le. 3.16]. Hence there exists a local chart \( U \) on \( Y \) such that

\[
\| R_{j_0} \|_U > 0, \tag{2.5}
\]

where we identified \( U \) with a local chart of \( E_{\theta_0} \).

From now on we work locally on \( U \) and without loss of generality, we can assume that \( U = U_1 \times U_2 \) of \( Y \) and \( (x, \theta) \) the coordinate system on \( U \) such that \( Y_{\theta_0} \cap U = \{ \theta = 0 \} \). Hence \( U_2 \) is a local chart of \( W \) centered at \( \theta_0 \) and \( x \) is a local coordinate system on \( Y_\theta \). We can identify \( E_\theta \) with \( U_1 \times \mathbb{C}^m \) and \( A_\lambda \) is given by the multiplication \( (x, \theta) \mapsto (x, \lambda \theta) \). Denote by \( A^*_\lambda \) the multiplication \( \theta \mapsto \lambda \theta \) in \( \mathbb{C}^m \), one sees that \( A^*_\lambda \circ \pi_W = \pi_W \circ A_\lambda \).

The natural projection \( \pi_{U_1} : U_1 \times U_2 \to U_1 \) is the restriction of \( \pi_{\mathbb{P}(E_\theta)} \) to \( \mathbb{P}^{-1}(U_1) \). Denote by \( id_U \) the identity map on \( U \). Observe that \( id_U \) is an admissible map by our identification of \( E_\theta \) with \( U_1 \times \mathbb{C}^m \). On \( U_1 \times \mathbb{C}^m \), we get

\[
R = \lim_{n \to \infty} (A_{\lambda_n})*(id_U)_* T = \lim_{n \to \infty} (A_{\lambda_n})* T. \tag{2.6}
\]

For each \( j > \ell - p \), recall that \( T_j \) is a current on \( W \) of bi-degree \((s, s)\) with \( s \geq 1 \). We then have

\[
\nu(T_j, \theta_0) \geq \limsup_{\lambda \to \infty} \| (A^*_\lambda) T_j \|_{U_2} = \limsup_{\lambda \to \infty} \| (A^*_\lambda) \pi_W(T \wedge \omega^j) \|_{U_2} = \limsup_{\lambda \to \infty} \| (\pi_W)_* ((A^*_\lambda) T \wedge (U_1)^* \omega^j) \|_{U_2}. \tag{2.7}
\]

Let \( \omega_W \) be a \( \mathbb{K} \)ahler form on \( W \). Observe that \( \omega \lesssim \omega_{\theta_0} + \omega_W \). It follows that for \( r := \dim Y - (p + j_0) \), one has

\[
\limsup_{\lambda \to \infty} \| (A^*_\lambda) T \wedge (U_1)^* \omega^{j_0 + r'} \| \lesssim \limsup_{\lambda \to \infty} \sum_{r' = 0}^r \|| (\pi_W)_* ((A^*_\lambda) T \wedge (U_1)^* \omega^{j_0 + r'}) \| \lesssim \sum_{r' = 0}^r \nu(T_{j_0 + r'}, \theta_0) = 0
\]

by (2.7) and our choice of \( \theta_0 \). This together with (2.6) and (2.5) gives a contradiction. Hence \( j_0 \leq \ell - p \). It follows that \( j_0 = \ell - p \). We also deduce that if \( p > \ell \), then the tangent current \( R \) is zero. This finishes the proof. \( \square \)

The next two results concerns the associativity of density currents: given currents \( T, R, S \) such that \( T \wedge R \) is well-defined and \((T \wedge R) \wedge S \) is well-defined, is \( T \wedge R \wedge S \) well-defined and equal to \((T \wedge R) \wedge S \)? We are not able to answer this question in general but we can show that this is the case in some situations which are enough for our applications later.

**Theorem 2.4.** Let \( R_1, R_2 \) be positive closed currents of bi-degree \((p_1, p_1), (p_2, p_2)\) respectively on \( Y \) such that the Dinh-Sibony product \( T := R_1 \wedge R_2 \) is well-defined. Assume that \( p := p_1 + p_2 > \ell \). Then there exists a subset \( A \) of \( W \) which is a countable union of proper analytic subsets of \( W \) such that for any \( \theta \notin A \), we have \( R_1 \wedge R_2 \wedge [Y_\theta] = 0 \).

**Proof.** Note that we proved in Theorem 2.3 that \((R_1 \wedge R_2) \wedge [Y_\theta] = T \wedge [Y_\theta] = 0 \) for \( \theta \) outside a countable union of proper analytic subsets of \( Z \). Our desired assertion doesn’t follow directly from the last property because in general we don’t know whether \( R_1 \wedge R_2 \wedge [Y_\theta] \) is well-defined and equal to \((R_1 \wedge R_2) \wedge [Y_\theta] \).

Let \( \Delta_3 \) be the diagonal of \( Y^3 \) which is the set of \( (y, y, y) \) for \( y \in Y \). Let \( E_3 \) be the normal bundle of \( \Delta_3 \) in \( Y^3 \). Let \( A_{\lambda, 3} \) be the multiplication by \( \lambda \) along fibers of \( E_3 \). Let \( \Delta \) be the diagonal
of \( Y^2 \) and \( E \) the normal bundle of \( \Delta \) in \( Y^2 \). Let \( A_\lambda \) be the multiplication by \( \lambda \) along fibers of \( E \). Let \( \pi_E \) be the projection from \( E \) to \( \Delta \).

Let \( T_j, A_j \) be the currents and the set given in the proof of Theorem 2.3 for \( j \geq 0 \). Since \( p > \ell \), the set \( A_j \) is a countable union of proper analytic subsets of \( W \). The desired assertion is a direct consequence of the following inequality:

\[
\limsup_{\lambda \to \infty} \| (A_{\lambda,3})_*(R_1 \otimes R_2 \otimes [Y_\theta]) \| \lesssim \sum_{j \geq 0} \nu(T_j, \theta) \tag{2.8}
\]

for \( \theta \in W \). Indeed, if \( \theta \notin A := \bigcup_{j \geq 0} A_j \), then the right-hand side of the above inequality is zero, which implies that the density current associated to \( R_1, R_2, Y_\theta \) is zero.

Let us now prove the inequality (2.8). Let \( \theta_0 \in W \). From now on, we work locally near \( Y_{\theta_0} \).

Let \( U = U_1 \times U_2 \) be a local chart near \( Y_{\theta_0} \) with the coordinates \( y = (x, \theta) \) such that \( \theta_0 = 0 \) and \( \pi_W(x, \theta) = \theta \). We obtain induced coordinates \( (y', y'') \) on \( U_3 \subset Y^3 \), where \( y' = (x', \theta') \), \( y'' = (x'', \theta'') \). Put \( \tilde{y}' = (\tilde{x}', \tilde{\theta}') := y' - y \), \( \tilde{y}'' = (\tilde{x}'', \tilde{\theta}'') := y'' - y \).

Then \( (y, \tilde{y}', \tilde{y}'') \) are new local coordinates on \( U^3 \) and

\[ \Delta_3 = \{ \tilde{y}' = \tilde{y}'' = 0 \}, \quad \Delta = \{ \tilde{y}' = 0 \}. \]

Identify \( E_3 \) over \( \Delta_3 \cap U^3 \) with \( U \times \mathbb{C}^{\dim Y} \) and \( E \) with \( U \times \mathbb{C}^{\dim Y} \). The multiplication \( A_{\lambda,3} \) is given by \( (y, \tilde{y}', \tilde{y}'') \mapsto (y, \lambda \tilde{y}', \lambda \tilde{y}'') \) and \( A_\lambda \) is given by \( (y, \tilde{y}') \mapsto (y, \lambda \tilde{y}') \). Let \( \Phi(y, \tilde{y}', \tilde{y}'') \) be a positive test form with compact support. Let \( \tau_3 \) be the change of coordinates from \( (y, y', y'') \) to \( (y, \tilde{y}', \tilde{y}'') \) and \( \tau \) the change of coordinates from \( (y, y') \) to \( (y, \tilde{y}') \).

Put

\[ Q_\lambda := \langle (A_{\lambda,3})_*(\tau_3)_*(R_1 \otimes R_2 \otimes [Y_{\theta_0}]), \Phi(y, \tilde{y}', \tilde{y}'') \rangle, \]

which is equal to

\[ \langle \tau_*(R_1 \otimes R_2), \int_{\{x'' \in Y_{\theta_0} \}} \Phi(y, \lambda \tilde{y}', \lambda (x'' - x), -\lambda \theta) \rangle = \langle \tau_*(R_1 \otimes R_2), \int_{\{\tilde{x}'' \in \mathbb{C}^\ell \}} \Phi(y, \lambda \tilde{y}', \tilde{x}'', -\lambda \theta) \rangle. \]

It follows that

\[ Q_\lambda = \langle R_\lambda, \Phi_\lambda \rangle, \]

where

\[ R_\lambda := (A_\lambda)_* \tau_*(R_1 \otimes R_2), \quad \Phi_\lambda(y, \tilde{y}', \theta) := \int_{\{\tilde{x}'' \in \mathbb{C}^\ell \}} \Phi(y, \tilde{y}', \tilde{x}'', -\lambda \theta) \].

Observe that

\[ \Phi_\lambda(y, \tilde{y}', \theta) \lesssim \sum_{j=0}^q \Omega_j(x, \tilde{y}') \wedge \omega_W^j(\lambda \theta), \]

for some positive integer \( q \), where \( \omega_W \) is a Kähler form on \( W \) and \( \Omega_j \) are positive test forms with compact supports. Put \( R_{\lambda,j} := \langle (\pi_W)_*(\pi_E)_*(R_\lambda \wedge \Omega_j(x, \tilde{y}')), \Phi_\lambda \rangle \). This implies that

\[ Q_\lambda \lesssim \langle R_{\lambda,j}, \Omega_j(x, \tilde{y}') \wedge \omega_W^j(\lambda \theta) \rangle = \langle R_{\lambda,j}, \omega_W^j(\lambda \theta) \rangle, \tag{2.9} \]

recall here that we identified \( \Delta \) with \( Y \) and the bracket is computed over \( U_2 \). By hypothesis that

\[ \lim_{\lambda \to \infty} R_\lambda = \pi_E^* (R_1 \wedge R_2) = \pi_E^* T, \]
we get
\[ \lim_{\lambda \to \infty} R_{\lambda,j} = (\pi_W)_* (T \wedge (\pi_E)_* \Omega_j). \quad (2.10) \]

For any positive closed current \( S \) of bi-dimension \((j, j)\) on \( U_2 \) and every constant \( \epsilon > 0 \), put
\[ \nu(S, \theta_0, \epsilon) := \epsilon^{-2j} \langle S, 1_{|\theta| \leq \epsilon} \omega_W^j \rangle, \]
where \( 1_B \) denotes the characteristic function of a set \( B \). Since we are working on \( U_2 \), we can take \( \omega_W \) to be the standard Kähler form on \( \mathbb{C}^m \). The function \( \nu(S, \theta_0, \epsilon) \) decreases to the Lelong number \( \nu(S, \theta_0) \) as \( \epsilon \to 0 \). Let \( \epsilon_0 \) be a strictly positive constant. Without loss of generality, we can assume that \( U_2 \) is contained in the unit ball in \( \mathbb{C}^m \). A direct computation shows that
\[ \langle R_{\lambda,j}, \omega_W^j (\lambda \theta) \rangle \leq c \nu(R_{\lambda,j}, \theta_0, c|\lambda|^{-1}) \leq c \nu(R_{\lambda,j}, \theta_0, \epsilon_0) \]
for \(|\lambda|\) big enough and some constant \( c > 0 \) independent of \( \lambda \). This combined with (2.10) and (2.9) yields that
\[ \limsup_{\lambda \to \infty} Q_\lambda \leq \sum_{j=0}^m \limsup_{\lambda \to \infty} \nu(R_{\lambda,j}, \theta_0, \epsilon_0) \leq \sum_{j=0}^m \nu((\pi_W)_* (T \wedge (\pi_E)_* \Omega_j), \theta_0, 2\epsilon_0) \]
for every \( \epsilon_0 > 0 \). Letting \( \epsilon_0 \to 0 \) in the last inequality gives
\[ \limsup_{\lambda \to \infty} Q_\lambda \leq \sum_{j \geq 0} \nu(T_j, \theta_0). \]
So (2.8) follows. This finishes the proof. \( \square \)

**Lemma 2.5.** Let \( Y \) be a compact Kähler manifold and \( T \) a positive closed current on \( Y \). Let \( V_1, V_2 \) be two smooth complex submanifolds of \( Y \). Assume that \( V_1 \) is transverse to \( V_2 \). If \( T \wedge [V_1] \wedge [V_2] = 0 \), then \( T \wedge [V_1 \cap V_2] = 0 \).

**Proof.** Let \( \Delta_3 = \{(x, x, x) : x \in Y\} \) be the diagonal of \( Y^3 \). Let \( E_3 \) be the normal bundle of \( \Delta_3 \) in \( Y^3 \). We work locally. Let \( x = (x_1, x_2, x') \) be a local coordinate system on a local chart \( U = U_1 \times U_2 \times U' \) of \( Y \) such that \( V_j \cap U = \{x_j = 0\} \) for \( j = 1, 2 \). We obtain induced coordinates \((x, y, z)\) on \( U^3 \subset Y^3 \), where \( y = (y_1, y_2, y'), z = (z_1, z_2, z') \).

Let \( \ell_j = \dim V_j \) for \( j = 1, 2 \) and \( k := \dim Y \). Put
\[ \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}') := x - z, \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}') := y - z. \]
Observe that \((\tilde{x}, \tilde{y}, z)\) are new local coordinates on \( U^3 \) and \( \Delta_3 \) is given by the equations \( \tilde{x} = \tilde{y} = 0 \). Identify \( E_3 \) over \( \Delta_3 \cap U^3 \) with \( U \times \mathbb{C}^{2k} \). Let \( \tau \) be the identity map from \( U^3 \to U \times \mathbb{C}^{2k} \). The fiberwise multiplication \( A_\lambda \) by \( \lambda \) on the normal bundle of \( \Delta_3 \) in \( Y^3 \) is given by \((\tilde{x}, \tilde{y}, z) \mapsto (\lambda \tilde{x}, \lambda \tilde{y}, z)\). On the other hand, the normal bundle of \( V_1 \cap V_2 \) in \( Y \) can be identified with \( U' \times \mathbb{C}^{\ell_1 + \ell_2 - k} \) and the fiberwise multiplication \( A_\lambda \) by \( \lambda \) on the normal bundle of \( V_1 \cap V_2 \) in \( Y \) is given by \((x_1, x_2, x') \mapsto (\lambda x_1, \lambda x_2, x')\).

Let \( \Phi(\tilde{x}, \tilde{y}, z) \) be a smooth form with compact support on \( U \times \mathbb{C}^{2k} \). We have
\[ \langle (A_\lambda)_* \tau_* ([V_1] \otimes [V_2] \otimes T), \Phi \rangle = \langle [V_1] \otimes [V_2] \otimes T, \Phi (\lambda \tilde{x}, \lambda \tilde{y}, z) \rangle \]
\[ = \langle T(z), \int_{(x_2, x') \in V_1} \int_{(y_1, y') \in V_2} \Phi (-\lambda x_1, \lambda x_2, \lambda x', \lambda y_1, -\lambda x_2, \lambda y', z) \rangle \]
\[ = \langle T(z), \int_{\tilde{x}_2, \tilde{x}', \tilde{y}_1, \tilde{y}'} \Phi (-\lambda z_1, \tilde{x}_2, \tilde{x}', \tilde{y}_1, -\lambda z_2, \tilde{y}', z) \rangle \]
\[ = \langle T(z), \int_{\tilde{x}_2, \tilde{x}', \tilde{y}_1, \tilde{y}'} \Phi (-\lambda z_1, \tilde{x}_2, \tilde{x}', \tilde{y}_1, -\lambda z_2, \tilde{y}', 0, 0, z') \rangle + O(|\lambda|^{-1}), \]
where \(O(|\lambda|^{-1})\) is a current of mass \(\lesssim |\lambda|^{-1}\) as \(\lambda \to \infty\) because \((\tilde{A}_\lambda)_* T\) is of uniformly bounded mass on compact subsets of \(U' \times \mathbb{C}_1^{1+\ell_2-k}\).

Letting \(\lambda \to \infty\) in the last equality, we get a density current associated to \([V_1],[V_2], T\) in the left-hand side and a tangent current to \(T\) along \(V_1 \cap V_2\) in the right-hand side. The desired assertion thus follows. \(\square\)

In the above proof, we actually proved that there is a natural 1-1 correspondence between the set of tangent currents to \(T\) along \(V_1 \cap V_2\) and the set of density currents associated to \(T,[V_1],[V_2].\) But the conclusion of Lemma 2.5 is enough for our purpose later.

### 3 Proof of Theorem 1.3

For every submanifold \(Z\) of a manifold \(Y\) with smooth boundary, denote by \([Z]\) the current of integration along \(Z\). Let \(\mu_r\) be the Haar measure on the boundary \(\partial D_r\) of \(D_r\). Direct computations show that

\[
\text{dd}^c \int_1^r \frac{d}{t} [D_t] = \mu_r - \mu_1, \tag{3.1}
\]

where \(\text{dd}^c := \frac{i}{2\pi} (\bar{\partial} - \partial)\) and \(\text{dd}^c = \frac{i}{2\pi} \bar{\partial} \partial\).

Let \(X, Z, V, p_{1,V}, \omega, f\) be as in the statement of Theorem 1.3. Put

\[S_r := c_r^{-1} \int_1^r \frac{d}{t} f_* [D_t],\]

where \(c_r := T_f(r, \omega)\). We have the following equality

\[
\frac{N_f(r, D)}{T_f(r, \omega)} = 1 + \langle \varphi_D, \text{dd}^c S_r \rangle, \tag{3.2}
\]

where \([D] := \text{dd}^c \varphi_D + \omega\), which is known as the First Main Theorem in Nevanlinna’s theory. Using (3.1), we get

\[
\text{dd}^c S_r = c_r^{-1} f_* \mu_r - c_r^{-1} f_* \mu_1
\]

which implies that

\[
||\text{dd}^c S_r|| \lesssim c_r^{-1}. \tag{3.3}
\]

It follows that every limit current of the family \((S_r)_{r \in \mathbb{R}^+}\) as \(r \to \infty\) is \(\text{dd}^c\)-closed. It is a well-known fact that at least such a limit current is \(\text{d}\)-closed. Let \((r_k)_{k \in \mathbb{N}}\) be a sequence of positive real numbers converging to \(\infty\) such that \(S_{r_k}\) converges to a positive closed current \(S\) as \(k \to \infty\).

Since \(p_{1,V}\) is a submersion, the pull-back \(p_{1,V}^* S\) is a well-defined current on \(V\). Here is an interpretation of the last current in terms of density currents.

**Lemma 3.1.** The Dinh-Sibony product \((p_{1,V}^* S) \wedge [V]\) of \(p_{1,V}^* S\) and \([V]\) is well-defined and equal to \(p_{1,V}^* S\).

**Proof.** We only need to work locally near \(V\). Let \(U = U_1 \times U_2 \times U_3\) be a local chart of \(X \times Z\) and \((x_1, x_2, x_3)\) be its coordinate system such that \(V = \{x_3 = 0\}, p_1(x_1, x_2, x_3) = x_1\) and \(p_{1,V}(x_1, x_2) = x_1\). Identify the normal bundle of \(V\) over \(U\) with \(U_1 \times U_2 \times \mathbb{C}^s\). We have \(A_\lambda(x_1, x_2, x_3) = (x_1, x_2, \lambda x_3)\). Let \(\Phi\) be a test function with compact support in \(U\). Observe that

\[
\langle (A_\lambda)_* (p_{1,V}^* S), \Phi \rangle = \langle S(x_1), \int_{x_2, x_3} \Phi(x_1, x_2, \lambda x_3) \rangle = \langle S(x_1), \int_{x_2, x_3} \Phi(x_1, x_2, \lambda x_3) \rangle
\]
which is equal to
\[
\langle S(x_1), \int_{x_2,x_3} \Phi(x_1,x_2,x_3) \rangle = \langle \pi^* p_1^* \nu S, \Phi \rangle,
\]
where \( \pi \) is the natural projection from the normal bundle of \( \mathcal{V} \) to \( \mathcal{V} \). This finishes the proof. \( \blacksquare \)

Let \( X_a := X \times \{a\} \) for \( a \in Z \).

**Proposition 3.2.** There exists a countable union \( \mathcal{A} \) of proper analytic subsets of \( Z \) such that for \( a \in Z \setminus \mathcal{A} \), we have that \( \mathcal{V}_a \) is smooth and
\[
S \ast [\mathcal{V}_a] = 0.
\]

**Proof.** By Theorem 2.4 and Lemma 3.1, for \( a \) outside a countable union \( \mathcal{A} \) of proper analytic sets of \( Z \), the Dinh-Sibony product \( S \ast [\mathcal{V}] \ast [X_a] \) is zero because the bi-degree of \( S \ast [\mathcal{V}] \) is \( \dim X - 1 + s > \dim X = \dim X_a \). Using the comment right after Theorem 1.3 by enlarging \( \mathcal{A} \) if necessary, one can assume that \( \mathcal{V} \) is transverse to \( X_a \) for \( a \notin \mathcal{A} \). This together with Lemma 2.5 yields
\[
S \ast [\mathcal{V}_a] = S \ast [\mathcal{V} \cap X_a] = 0
\]
for such \( a \). This finishes the proof. \( \blacksquare \)

From now on, fix an \( a \in Z \setminus \mathcal{A} \). For simplicity, we write \( \mathcal{V} \) for \( \mathcal{V}_a \). Let \( \rho : \tilde{X} \to X \) be the blowup of \( X \) along \( \mathcal{V} \). Denote by \( \tilde{\mathcal{V}} \) the exceptional hypersurface of \( \mathcal{V} \). Let \( \tilde{f} \) be the lift of \( f \) to \( \tilde{X} \), we then have \( \rho \circ \tilde{f} = f \).

**Lemma 3.3.** There exists a Kähler form \( \tilde{\omega} \) on \( \tilde{X} \) such that
\[
T_f(r, \tilde{\omega}) \leq T_f(r, \omega) + O(1).
\]

**Proof.** See [11] 2.5.1 for a proof. \( \blacksquare \)

Let \( \tilde{S}_r \) be the pull-back of \( S_r \) by \( \rho \). Let \( \tilde{S} \) be the strict transform of \( S \) by \( \rho \). We have
\[
\tilde{S}_r = c_{r}^{-1} \int_{1}^{r} \frac{dt}{t} \tilde{f}_s[D_t].
\]
Using this and Lemma 3.3 yields that any limit current \( \tilde{S}' \) of the sequence \( \{ \tilde{S}_r \} \) is a dd\( \bar{c} \)-closed positive current on \( \tilde{X} \). This combined with the fact that \( \tilde{S}_r \to \tilde{S} \) outside \( \tilde{\mathcal{V}} \) gives
\[
\tilde{S}' = \tilde{S} + \tilde{S}'',
\]
where \( \tilde{S}'' \) is a dd\( \bar{c} \)-closed positive current of bi-dimension \((1,1)\) supported on \( \tilde{\mathcal{V}} \). Hence, by a support theorem of Bassanelli [11], \( \tilde{S}'' \) is a dd\( \bar{c} \)-closed current on \( \tilde{\mathcal{V}} \).

Note that since \( X \) is Kähler, so is \( \tilde{X} \). By dd\( \bar{c} \)-Lemma, for every closed smooth \((n-1, n-1)\)-form \( \xi \) on \( \tilde{V} \), where \( n := \dim X \), the quantity \( \langle \tilde{S}'' \rangle \langle \xi \rangle \) depends only on the cohomology class of \( \xi \). This combined with Serre’s duality shows that the cohomology class \( \{ \tilde{S}'' \} \) of \( \tilde{S}'' \) in \( H^{1,1}(\tilde{V}) \) is well-defined.

Let \( \eta \) be a closed form in the cohomology class of \([\tilde{\mathcal{V}}]\). Recall that \( \tilde{\mathcal{V}} \) is naturally isomorphic to the fiber bundle \( \mathbb{P}(F) \), where \( F \) is the normal bundle of \( \mathcal{V} \) in \( X \). Denote by \( \pi_{\mathcal{F}} : \mathbb{P}(F) \to \mathcal{V} \) the natural projection. The restriction of the cohomology class of \([\tilde{\mathcal{V}}]\) to \( \tilde{\mathcal{V}} \) is the opposite of the Chern class \( \omega_{\mathcal{O}_{\mathcal{P}(F)}(1)} \) of the line bundle \( \mathcal{O}_{\mathcal{P}(F)}(1) \) which is the dual of the tautological line bundle of \( \mathbb{P}(F) \).
Lemma 3.4.

\[ \lim_{k \to \infty} \langle \tilde{S}_{r_k}, \eta \rangle = 0. \]

We emphasize that the above lemma is not a direct consequence of the (semi-)continuity of total tangent classes given in [10, Th. 4.11] and [18, Pro. 2.10] because \( \tilde{S}_r \) is not closed.

**Proof.** By the First Main Theorem, we have

\[ \frac{N_f(r, \tilde{V})}{T_f(r, \omega)} \leq \langle \tilde{S}_r, \eta \rangle, \tag{3.4} \]

which yields

\[ \lim_{k \to \infty} \inf \langle \tilde{S}_{r_k}, \eta \rangle \geq 0. \tag{3.5} \]

By Proposition [3.2] and (2.4), we obtain that \( \{ \tilde{S} \} \sim \eta = 0 \). To simplify the notation, we assume \( \lim_{k \to \infty} \tilde{S}_{r_k} = \tilde{S}' \). Thus,

\[ \lim_{k \to \infty} \langle \tilde{S}_{r_k}, \eta \rangle = \langle \tilde{S}, \eta \rangle + \langle \tilde{S}'', \eta \rangle = \langle \tilde{S}'', \eta \rangle = \int_{\tilde{V}} \langle \tilde{S}'', \eta \rangle = \int_{\tilde{V}} \langle \tilde{S}'', \eta \rangle = \int_{\tilde{V}} \omega_{\mathcal{O}(F)}(1). \tag{3.6} \]

On the other hand, since

\[ S + \rho_\ast \tilde{S}'' = \rho_\ast \tilde{S} + \rho_\ast \tilde{S}'' = \lim_{k \to \infty} \rho_\ast \tilde{S}_{r_k} = \lim_{k \to \infty} S_{r_k} = S, \]

we get \( \rho_\ast \tilde{S}'' = 0 \). Since \( \tilde{S}'' \) is supported on \( \tilde{V} \approx \mathbb{P}(F) \), we obtain

\[ (\pi_{\mathbb{P}(F)\ast} S)'' = 0. \tag{3.7} \]

By Leray’s decomposition (see [2]), we can write

\[ \{ \tilde{S}' \} = \pi_{\mathbb{P}(F)\ast} \kappa_0 + \pi_{\mathbb{P}(F)\ast} \kappa_1 \sim \omega_{\mathcal{O}(F)}(1), \]

where \( \kappa_j \) is a cohomology class of bidimension \((j, j)\) on \( V \) for \( j = 0, 1 \). By (3.7), we obtain

\[ \kappa_1 = (\pi_{\mathbb{P}(F)\ast}) \{ \tilde{S}' \} =\{ (\pi_{\mathbb{P}(F)\ast}) \tilde{S}'' \} = 0. \]

This implies that \( \{ \tilde{S}'' \} = \pi_{\mathbb{P}(F)\ast} \kappa_0 \). It follows particularly that \( \kappa_0 \geq 0 \). Combining this with (3.6) gives

\[ \lim_{k \to \infty} \langle \tilde{S}_{r_k}, \eta \rangle = -\int_{\mathbb{P}(F)} \pi_{\mathbb{P}(F)\ast} \kappa_0 \wedge \omega_{\mathcal{O}(F)}(1) = -\kappa_0 \leq 0. \]

This together with (3.5) implies the desired equality. \( \square \)

Recall \( n = \dim X \). Write

\[ \tilde{f}^\ast [\tilde{V}] = \sum_z \nu_{z, f, \tilde{V}} \delta_z, \]

where \( \delta_z \) is the Dirac mass at \( z \). Recall that \( V \) is smooth. We define \( \nu_{z, f, V} \) as follows. Put \( \nu_{z, f, V} := 0 \) if \( z \notin f^{-1}(V) \). Consider now \( z \in f^{-1}(V) \). Let \( U \) be a local chart around \( f(z) \) on \( X \) and \( x = (x_1, \ldots, x_n) \) a coordinate system on \( U \) such that \( V = \{ x_j : 1 \leq j \leq s \} \). Write \( f = (f_1, \ldots, f_n) \) in these local coordinates. Let \( \nu_{z, f, V} \) be the smallest number among the multiplicities of \( z \) in the zero divisors of \( f_j \) for \( 1 \leq j \leq s \) on \( f^{-1}(U) \). This definition is independent of the choice of local coordinates. Recall that \( N_f(r, V) \) is the counting function of \( f \) with respect to the divisor \( \sum_z \nu_{z, f, V} \delta_z \) on \( C \).
Lemma 3.5. We have

\[ \nu_{z,f,V} = \nu_{z,f,\tilde{V}} \]  \hspace{2cm} (3.8) 

and

\[ N_f(r, \tilde{V}) = N_f(r, V). \]

Proof. The second desired equality is a direct consequence of the first one. We prove now the first one. Observe that \( f^{-1}(V) = \tilde{f}^{-1}(\tilde{V}) \) because \( \rho \circ \tilde{f} = f \) and \( \rho^{-1}(V) = \tilde{V} \). Hence it is enough to prove (3.8) for \( z \in f^{-1}(V) \). Consider \( z_0 \in f^{-1}(V) \).

Let \((U,x)\) be a local chart around \( f(z_0) \) such that \( V = \{x_j = 0 : 1 \leq j \leq s\} \). Write \( f = (f_1, \ldots, f_s) \) as above. Denote by \( \nu_{z,f_j} \) the multiplicity of \( z \) in the zero divisor of \( f_j \) on \( f^{-1}(U) \) for \( 1 \leq j \leq s \).

We have

\[ \nu_{z_0,f,V} = \min_{1 \leq j \leq s} \{ \nu_{z_0,f_j} \}. \]  \hspace{2cm} (3.9) 

Let us now recall how to construct the blowup \( \tilde{X} \) along \( V \) on \( U \). Let \( \tilde{U} := \rho^{-1}(U) \). Let \( w := [w_1, \ldots, w_s] \in \mathbb{P}^{s-1} \). The set \( \tilde{U} \) is the submanifold of \( U \times \mathbb{P}^{s-1} \) given by the equations

\[ x_j w_\ell = x_\ell w_j \]

for \( 1 \leq j, \ell \leq s \). Observe that \( \tilde{f}(z) = (f(z), [f_1(z), \ldots, f_s(z)]) \in U \times \mathbb{P}^{s-1} \) for \( z \) such that \( f(z) \in U \).

The set \( \tilde{U} \) can be covered by \( s \) standard local charts which we will describe as follows. Let \( \tilde{U}_j \) be the subset of \( \tilde{U} \) consisting of \((x, [w])\) with \( w_j = 1 \) and \( |w_\ell| \leq c \) for \( 1 \leq \ell \neq j \leq s \), where \( c \) is a constant big enough. For \( c \) big enough, the local charts \( \tilde{U}_j \) cover \( \tilde{U} \). Since the role of \( \tilde{U}_j \) is the same, we now consider only \( \tilde{U}_s \). The natural induced coordinates on \( \tilde{U}_s \) are \((x, w_1, \ldots, w_{s-1})\) and \( \tilde{f} = (f, f_1/f_s, \ldots, f_{s-1}/f_s) \) on \( \tilde{f}^{-1}(\tilde{U}_s) \). Hence, we have \( |f_j/f_s| \leq c \) on \( \tilde{f}^{-1}(\tilde{U}_s) \) for \( 1 \leq j \leq s-1 \).

The hypersurface \( \tilde{V} \) is given by \( x_s = 0 \) on \( \tilde{U}_s \). We deduce that if \( \tilde{f}(z_0) \in \tilde{U}_s \), then we must have

\[ \nu_{z_0,f,V} = \nu_{z_0,f_s} = \nu_{z_0,f,\tilde{V}}. \]

This finishes the proof.

End of Proof of Theorem 1.3

Let \( a \in Z \setminus A \) as above. By Lemmas 3.4, 3.5 and (3.4), we get

\[ 0 \leq \lim \inf_{r \to \infty} \frac{N_f(r, V_a)}{T_f(r, \omega)} \leq \lim \inf_{k \to \infty} \frac{N_f(r_k, V_a)}{T_f(r_k, \omega)} = \lim \inf_{k \to \infty} \frac{N_f(r_k, \tilde{V}_a)}{T_f(r_k, \omega)} = 0. \]

This finishes the proof.

4 Proof of Theorem 1.2

Let \( X, L, f, E \) be as in the statement of Theorem 1.2. By using a basis of \( E \), we obtain an embedding from \( X \) to a complex projective space. So from now on, we can assume \( X = \mathbb{P}^n \) and \( L \) is the hyperplane line bundle of \( X \). This reduction is not essential but it simplifies some computations.

Consider first the case where \( n \geq 2 \). Let \( TX \) be the tangent bundle of \( X \). Let \( \tilde{X} := \mathbb{P}(TX) \) be the projectivisation of \( TX \) and \( \pi : \tilde{X} \to X \) the natural projection. Let \( \tilde{f} \) be the lift of \( f \) to \( \tilde{X} \) defined by \( \tilde{f}(z) := (f(z), [f'(z)]) \), where \( f' \) is the derivative of \( f \) and \( z \in \mathbb{C} \). Hence \( \tilde{f} \) is an entire curve in \( \tilde{X} \). We also have \( \hat{S}_r, \tilde{c}_r \) for \( \hat{f} \) as \( S_r, c_r \) for \( f \).

Let \( \mathcal{O}_{\tilde{X}}(1) \) be the dual of the tautological line bundle of \( \tilde{X} \). Let \( \tilde{\omega} \) be a Kähler form on \( \tilde{X} \) such that \( \tilde{\omega} = \omega + c \omega \mathcal{O}_{\tilde{X}}(1) \), where \( c \) is a strictly positive constant and \( \omega \mathcal{O}_{\tilde{X}}(1) \) is a smooth Chern form of \( \mathcal{O}_{\tilde{X}}(1) \) whose restriction to each fiber of \( \pi \) is strictly positive. Recall the following equality.
Lemma 4.1. We have
\[ T_f(r, \bar{\omega}) = T_f(r, \omega) + o(T_f(r, \omega)) \]
(4.1)
as \( r \to \infty \) outside a set of finite Lebesgue measure of \( \mathbb{R} \).

Proof. The desired assertion is equivalent to the equality
\[ T_f(r, \mathcal{O}_X(1)) = o(T_f(r, \omega)). \]
This is the tautological inequality of McQuillan [13] which is in fact a consequence of the Lemma
on logarithmic derivative. For the readers’ convenience, we briefly recall how to prove it.

Let \((x_j)\) be a partition of unity of \( X \) subordinated to a finite covering \((U_j)\) of \( X \) where \( U_j \) are
local charts on \( X \). Trivialize \( \hat{X} \approx U_j \times \mathbb{P}^{n-1} \) on \( U_j \). Let \( x_j = (x_{j1}, \ldots, x_{jn}) \) be the coordinates on
\( U_j \). Write \( f(z) = (f_{j1}, \ldots, f_{jn}) \) accordingly for \( z \in f^{-1}(U_j) \). Put
\[ h_j(x_j, v) := \chi_j(x_j) \sum_{l=1}^n |v_l|^2, \]
where \( [v] \in \mathbb{P}^{n-1} \). Thus \( h := \sum_j h_j \) is a Hermitian metric on \( \mathcal{O}_X(1) \). This combined with the
Lelong-Jensen formula gives
\[ T_f(r, \mathcal{O}_{\text{TX}(1)}) = \int_{\partial \hat{D}_r} \log \sum_j \chi_j(f) \sum_{\ell=1}^n |f'_{j\ell}|^2 d\mu_r + O(1). \]
Standard estimates in proofs of Lemma on logarithmic derivatives (see [15] Le. 4.7.1]) show that
the last integral is \( o(T_f(r, \omega)) \) as \( r \to \infty \) outside a set of finite Lebesgue measure. This finishes the proof. \( \square \)

We fix a sequence \( r := (r_k) \in \mathbb{R}^+ \) converging to \( \infty \) such that \( \hat{S}_{r_k} \) converges to a closed
positive current \( \hat{S} \) as \( k \to \infty \) and (4.1) holds for \( r = r_k \). For \( a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1}\setminus\{0\} \), put
\[ \sigma_a(x) := \sum_{j=0}^n a_j x_j, \text{ where } x = [x_0 : \ldots : x_n] \in \hat{X} = \mathbb{P}^n. \]
Identify \( \mathbb{P}(E) \) with \( \mathbb{P}^n \) via \( a \leftrightarrow \sigma_a \). Denote by \( D_a \) the hyperplane generated by \( \sigma_a \). Let \( \pi_j : \hat{X} \times \mathbb{P}(E) \) be the natural projections to its
components for \( j = 1, 2 \).

Let \( \hat{D}_a \) be the set of \((x, [v]) \in \hat{X} \) for \( x \in D_a \) and \( v \in T_x X \setminus \{0\} \) tangent to \( D_a \). Let \( U \) be a local chart of \( X \) over which \( L \) is trivial. We identify \( \sigma_j \) with functions on \( U \) and \( d\sigma_j \) with 1-form on \( U \)
hence a function on \( TU \). Put
\[ H_1 := \{(x, [v], [a]) \in \hat{X} \times \mathbb{P}^n : \sigma_a(x) = 0\} \]
and
\[ H_{2, U} := \{(x, [v], [a]) \in (\hat{X}|_U) \times \mathbb{P}^n : (d \sigma_a(x), v) = 0\}. \]
Let \( U' \) be another local chart similar to \( U \). Trivialize \( L \) on \( U' \). Let \( \sigma'_a \) be the trivialisation of \( \sigma_a \)
on \( U' \). We have \( \sigma'_a = \gamma \sigma_a \) for some nowhere vanishing holomorphic function \( \gamma \) on \( U \cap U' \) which is
independent of \( a \). Thus \( d \sigma'_a = d(\gamma \sigma_a) = d \gamma \sigma_a + \gamma d \sigma_a \). We deduce that \( H_1 \cap H_{2, U'} = H_1 \cap H_{2, U} \)
on \( \pi_1^{-1}(T(U \cap U')) \). Gluing \( H_1 \cap H_{2, U} \) together, we obtain a well-defined analytic subset \( \mathcal{V} \) of
\( \hat{X} \times \mathbb{P}(E) \).

Let \( p_j|_\mathcal{V} \) be the restriction of \( p_j \) to \( \mathcal{V} \) for \( j = 1, 2 \). We have

Lemma 4.2. The set \( \mathcal{V} \) is a smooth submanifold of codimension 2 of \( X \times \mathbb{P}(E) \), the map \( p_{1, \mathcal{V}} \) is a
submersion and the fiber of \( p_{2, \mathcal{V}} \) at \( a \in \mathbb{P}(E) \) is \( \hat{D}_a \).
Proof. The fact that the fiber of \( p_{2, V} \) at \( a \in \mathbb{P}(E) \) is \( \hat{D}_a \) is clear from the construction. It is sufficient to check the remaining desired assertions for \( x = [x_0 : \ldots : x_n] \) in a local chart of \( X = \mathbb{P}^n \). Hence, consider the local chart \( U := \{ x \in \mathbb{P}^n : x_0 = 1 \} \). We see that

\[
V = \{ a_0 + \sum_{j=1}^n a_j x_j = 0, \sum_{j=1}^n a_j v_j = 0 \},
\]

For \( x, [v] \) fixed, these two defining equations of \( V \) give two hyperplanes in \( \mathbb{P}(E) \) which are transverse to each other because \( n \geq 2 \). So \( V \) is smooth. Let \( W_{j_0} \) be the local chart of \( \mathbb{P}(E) \) containing \([a]\) with \( a_{j_0} = 1 \). Consider first the case where \( j_0 \neq 0 \). Observing that \( V \) is the set of \((x, [v], [a])\) such that \( x_{j_0} = -a_0 - \sum_{j \neq j_0} a_j x_j \) and \( v_{j_0} = -\sum_{j \neq j_0} a_j v_j \). So the map \( p_{1, V} \) can be identified with the map

\[
\begin{align*}
(a_0, \ldots, a_{j_0-1}, a_{j_0+1}, \ldots, a_n, [v_1, \ldots, v_{j_0-1}, v_{j_0+1}, \ldots, v_n], x_1, \ldots, x_{j_0-1}, x_{j_0+1}, \ldots, x_n) & \\
\rightarrow (x_1, \ldots, x_{j_0-1}, -a_0 - \sum_{j \neq j_0} a_j x_j, x_{j_0+1}, \ldots, x_n),
\end{align*}
\]

which is of maximal rank. The case where \( j_0 = 1 \) is treated similarly by observing that \((a_1, \ldots, a_n) \neq 0 \) if \((x, [v], [a]) \in V \) and \( a_0 = 1 \). This finishes the proof. \( \square \)

Lemma 4.2 combined with Proposition 3.2 applied to \( \hat{X} \) in place of \( X \) and \( Z := \mathbb{P}(E) \) gives

**Corollary 4.3.** There exists a countable union \( A \) of proper analytic subsets of \( \mathbb{P}(E) \) such that for \( a \in \mathbb{P}(E) \backslash A \), we have that

\[
\hat{S} \cap [\hat{D}_a] = 0.
\]

From now on fix \( a \in \mathbb{P}(E) \backslash A \) and write \( D, \hat{D} \) for \( D_a, \hat{D}_a \) to simplify the notation. Let \( \rho : \hat{X} \to \tilde{X} \) be the blowup of \( \hat{X} \) along \( \hat{D} \). Let \( D \) be the exceptional divisor of that blowup. Lift \( \hat{f} \) to a curve \( \tilde{f} \) in \( \tilde{X} \). Let \( \tilde{\omega} \) be a Kähler form on \( \tilde{X} \) such that

\[
T_{\tilde{f}}(r, \tilde{\omega}) \leq T_{\hat{f}}(r, \hat{\omega}) + O(1),
\]

see Lemma 3.3. Let \( \hat{S}_r \) be the pull-back of \( \hat{S}_r \) by \( \rho \). Let \( \tilde{S} \) be the strict transform of \( \hat{S} \) by \( \rho \). Let \( \eta \) be a closed form in the cohomology class of \( \hat{D} \). By Lemma 3.4 applied to \( \hat{S}, \hat{D} \), we get

\[
\lim_{k \to \infty} \langle \tilde{S}_{r_k}, \eta \rangle = 0. \tag{4.2}
\]

**Lemma 4.4.** We have

\[
\langle \varphi_{\hat{D}}, dd^c \hat{S}_r \rangle = \langle \varphi_{\hat{D}}, dd^c S_r \rangle + c_r^{-1} O(1)
\]

as \( r \to \infty \).

**Proof.** Since \( \varphi_{\hat{D}} \) is a potential of \( \hat{D} \), we get

\[
|\varphi_{\hat{D}}(\bar{x}) - \log \text{dist}(\bar{x}, \hat{D})| \lesssim 1.
\]

Thus

\[
\lim_{k \to \infty} \langle \log \text{dist}(\bar{x}, \hat{D}), dd^c \tilde{S}_{r_k} \rangle = 0.
\]

Using the fact that \((\pi \circ \rho)(\hat{D}) = D \) gives

\[
| \log \text{dist}(\bar{x}, \hat{D}) - \log \text{dist}(\pi \circ \rho(\bar{x}), D) | \lesssim 1.
\]

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We also have $(\pi \circ \rho)_* \mathcal{S}_r = S_r$ because $\hat{f}$ is a lift of $f$ to $\hat{X}$. Recall that the mass of $dd^c \mathcal{S}_r$ is $O(1)c_r^{-1}$ as $r \to \infty$. It follows that
\[
\langle \log \text{dist}(\hat{x}, \hat{D}), dd^c \mathcal{S}_{r_k} \rangle = \langle \log \text{dist}(x, D), (\pi \circ \rho)_* dd^c \mathcal{S}_{r_k} \rangle + O(1)c_r^{-1},
\]
which is equal to $\langle \log \text{dist}(x, D), dd^c S_{r_k} \rangle + O(1)c_r^{-1}$ as $r \to \infty$. The proof is finished. □

Write
\[
f^* [D] = \sum_z \nu_{z,f,D} \delta_z,
\]
\[
\hat{f}^* [\hat{D}] = \sum_z \nu_{z,\hat{f},\hat{D}} \delta_z.
\]
For $z \in \mathbb{C}$, let $\nu_{z,\hat{f},\hat{D}}$ be the multiplicity of $z$ with respect to $\hat{f}, \hat{D}$ as in the setting of Lemma 3.5.

Applying (3.8) to $f, \hat{D}, \hat{D}$, we obtain $\nu_{z,\hat{f},\hat{D}} = \nu_{z,f,D}$. In a local coordinates $(U, x)$ of $X$, write $f(z) = (f_1(z), \ldots, f_n(z))$. Let $f'_j$ be the derivative of $f_j$ for $1 \leq j \leq n$. For $z_0 \in \mathbb{C}$, let $\nu_{z_0,f'}$ be the smallest non-negative integer such that $(z - z_0)^{-k}f'_j(z)$ for $1 \leq j \leq n$ are holomorphic functions which are not simultaneously zero at $z_0$. This definition is independent of the choice of local charts. Thus the current $R_f := \sum_{z_0 \in \mathbb{C}} \nu_{z_0,f'} \delta_{z_0}$ is well-defined. Observe that the support of $f, R_f$ consists of at most a countable number of points.

**Lemma 4.5.** For $z \in \text{Supp} f^* D$, we have
\[
\nu_{z,f,D} - 1 = \nu_{z,\hat{f},\hat{D}} + \nu_{z,f'}.
\]
Consequently, for $D$ with $D \cap \text{Supp} f R_f = \emptyset$, there holds
\[
N_f(r, \hat{D}) = N_{\hat{f}}(r, \hat{D}) = N_f(r, D) - N_{f}^{[1]}(r, D).
\]
Note that since $\text{Supp} f R_f$ is an at most countable set, the set of $D$ for which $D \cap \text{Supp} f R_f \neq \emptyset$ is a countable union of hyperplanes in $\mathbb{P}(E)$.

**Proof.** We already proved the second inequality of (4.3). It remains to prove the first one. Consider now $z_0 \in \text{Supp} f^* [D]$. Thus $\nu_{z_0,f,D} \geq 1$.

Let $(U, x)$ be a local chart around $f(z_0)$ such that $x = (x_1, \ldots, x_n)$ and $D = \{x_1 = 0\}$. We then obtain an induced coordinate system $(x, v)$ on $TX|_U = U \times \mathbb{C}^n$ and $\hat{D}$ is given by $x_1 = v_1 = 0$ there. Write $f = (f_1, \ldots, f_n)$ in these coordinates. Hence $f^* [D]$ is the divisor generated by $f_1$ on $f^{-1}(U)$. In these coordinates, we have $\mathbb{P}(TU) = U \times \mathbb{P}^{n-1}$.

Put $f'(\nu) := (f'_1, \ldots, f'_n)$. We have $\hat{f} = (f, [f']) \in U \times \mathbb{P}^{n-1}$ which is a holomorphic curve. Without loss of generality, since $(z - z_0)^{-\nu_{z_0,f'}} f'_j(z)$ isn’t zero at the same time at $z_0$ for $1 \leq j \leq n$, we can assume that $(z - z_0)^{-\nu_{z_0,f'}} f'_n(z)$ is not zero at $z = z_0$.

Using standard local charts on $\mathbb{P}^{n-1}$, we can cover $U \times \mathbb{P}^{n-1}$ by a finite number of local charts. Let $\hat{U}$ be the local chart with $v_n = 1$. So the coordinates on $\hat{U}$ are $(x, v_1, \ldots, v_{n-1})$ and $\hat{D}$ is still given by $x_1 = v_1 = 0$. In these coordinates, $\hat{f} = (f_1, \ldots, g_{n-1}, \ldots, g_n)$, where $g_j := f'_j/f'_n$ for $1 \leq j \leq n - 1$. Observe that the order of $g_1$ at $z_0$ is $(\nu_{z_0,f,D} - 1 - \nu_{z_0,f'}) < \nu_{z_0,f,D}$. Thus,
\[
\nu_{z_0,\hat{f},\hat{D}} = \nu_{z_0,R_f,D} - 1 - \nu_{z_0,f'}.
\]
So (4.3) follows. This finishes the proof. □

**Proposition 4.6.** For $D$ with $D \cap \text{Supp} f R_f = \emptyset$, we have
\[
1 - \frac{N_{f}^{[1]}(r, D)}{T_{f}(r, \omega)} = \langle \mathcal{S}_r, \eta \rangle + \frac{O(1)}{T_f(r, \omega)} (r \to \infty).
\]
Proof. Starting from First Main Theorem and using Lemma 4.1, we get
\[
\frac{N_f(r, D)}{T_f(r, \omega)} = \langle \tilde{S}_r, \eta \rangle + \langle \varphi_D, \mathcal{d}^e \tilde{S}_r \rangle
\]
[Use Lemma 4.4]
\[
= \langle \tilde{S}_r, \eta \rangle + \langle \varphi_D, \mathcal{d}^e S_r \rangle + O(1)[T_f(r, \omega)]^{-1}
\]
[Use First Main Theorem]
\[
= \langle \tilde{S}_r, \eta \rangle + \frac{N_f(r, D)}{T_f(r, \omega)} - 1 + \frac{O(1)}{T_f(r, \omega)}
\]
[Use Lemma 4.5]
\[
= \langle \tilde{S}_r, \eta \rangle + \frac{N_f(r, \tilde{D}) + N_f^{[1]}(r, D)}{T_f(r, \omega)} - 1 + \frac{O(1)}{T_f(r, \omega)},
\]
which yields the desired equality.

End of the proof of Theorem 1.2 when \( n \geq 2 \). Recall that we reduced the problem to the case where \( X = \mathbb{P}^n \) and \( L \) is the hyperplane line bundle. We also have fixed a hyperplane \( D \) on \( X \) such that the tangent current of \( \tilde{S} \) along \( \tilde{D} \) is zero and \( \text{Supp} D \cap \text{Supp} f \cdot R_f = \emptyset \). We had also shown that the set of such \( D \) contains the complement of a countable union of proper analytic subsets of \( \mathbb{P}(E) \). By Proposition 4.6 and (4.2), one gets
\[
T_f(r_k, \omega) = o(T_f(r_k, \omega))
\]
as \( k \to \infty \). Hence \( \delta_f^{[1]}(D) = 0 \). This finishes the proof.

Finally, to finish the proof of Theorem 1.2, we treat the remaining case where \( n = 1 \), i.e., \( X = \mathbb{P}^1 \). In this case, Theorem 1.2 is a direct consequence of the following general result.

Lemma 4.7. Let \( f : \mathbb{C} \to \mathbb{P}^n \) be a non-constant holomorphic curve. There exists a countable union \( A \) of proper linear subspaces of the space of hyperplanes of \( \mathbb{P}^n \) such that for any hyperplane \( D \notin A \), we have
\[
\delta_f^{[n]}(D) = 0.
\]

Proof. Without loss of generality, we can assume that \( f \) is linearly non-degenerate because otherwise we can consider the smallest linear subspace of \( \mathbb{P}^n \) containing \( f(\mathbb{C}) \). Let \( (D_j)_{j=1}^q \) be a family of hyperplanes in general position. By Cartan’s Second Main Theorem [5], we get the following defect relation
\[
\sum_{j=1}^q \delta_f^{[n]}(D_j) \leq n + 1. \tag{4.4}
\]
For any positive number \( k \), set \( A_k := \{ D : \delta_f^{[n]}(D) \geq 1/k \} \). It is clear that the desired equality holds true for all \( D \notin \bigcup_{k=1}^\infty A_k \). Thus the problem reduces to proving that for each integer number \( k \), the set \( A_k \) is contained in a countable union of proper linear subspaces of \( \mathbb{P}^n \).

Suppose on the contrary that this is not the case for some \( k \). For any positive integer \( q \), we now construct a family of hyperplanes \( \{D_i\}_{i=1}^q \subset A_k \) in general position in \( \mathbb{P}^n \).

We first taking a divisor \( D_0 \in A_k \setminus \{0\} \) and consider the linear subspace \( E_0 \) of \( E \) generated by \( D_0 \). Since \( E_0 \) is a proper linear subspace of \( \mathbb{P}^n \), \( A_k \setminus E_0 \) is nonempty. Taking \( D_1 \in A_k \setminus E_0 \) and let \( E_1 \) be the vector space generated by \( D_0, D_1 \). Again since \( E_1 \) is a linear space, \( A_k \setminus E_1 \) is not empty and we can pick in this set a hyperplane \( D_2 \). Iterating this process, we obtain at the \( (q+1) \)th step a family \( \{D_i\}_{0 \leq i \leq q} \subset A_k \) of \( q+1 \) divisors in general position in \( \mathbb{P}^n \). Applying (4.4) to this family gives
\[
n + 1 \geq q/k.
\]
Letting \( q \to \infty \) yields a contradiction. This finishes the proof.
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