Mass generation mechanism for spin 1/2 fermions in Dirac–Yang–Mills model equations with a symplectic gauge symmetry

Nikolay Marchuk

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Abstract

In the Standard Model of electroweak interactions the fundamental fermions acquire masses by the Yukawa interaction with the (spin 0) Higgs field. In our model spin 1/2 fermions acquire masses by an interaction with (spin 1) gauge field with symplectic symmetry.

In [2, 3, 4, 5, 8] we develop a new approach to field theory, which based on so called model equations of field theory. In this paper we introduce Dirac–Yang–Mills model equations for spin 1/2 fermions interacting with two gauge fields simultaneously. One field has a unitary gauge symmetry and another has a symplectic gauge symmetry. There is no fermion’s mass ($m$) term in the model Dirac equation. But there is the term $3m^3/16$ in the right hand part of the Yang–Mills equations with symplectic gauge symmetry. Hence, the constant $3m^3/16$ can be considered as a constant (charge) describing the interaction of a fermion with the symplectic gauge field.

Clifford algebra. Let $\mathcal{C}(1, 3)$ be the complex Clifford algebra [1] with the unity element $e$ and with generators $e^a$, $a = 0, 1, 2, 3$, which satisfy the relations

$$e^a e^b + e^b e^a = 2\eta^{ab} e, \quad a, b = 0, 1, 2, 3,$$

where $\eta = ||\eta^{ab}||$ = diag$(1, -1, -1, -1)$ is the diagonal matrix.

Let $\mathcal{C}_k(1, 3)$, $(k = 0, 1, 2, 3, 4)$ be subspaces of rank $k$ Clifford algebra elements and $\mathcal{C}_{\text{Even}}(1, 3), \mathcal{C}_{\text{Odd}}(1, 3)$ be the subspaces of even and odd Clifford algebra elements respectively. By $\mathcal{C}^{\mathbb{R}}(1, 3)$ denote the real Clifford algebra.
Denote \( \beta = e^0 \in \mathcal{C}(1, 3) \). Consider an operation of pseudo-Hermitian conjugation \( * : \mathcal{C}(1, 3) \to \mathcal{C}(1, 3) \) such that \( (e^a)^* = e^a, a = 0, 1, 2, 3 \) and
\[
(\lambda U)^* = \bar{\lambda} U^*, \quad (UV)^* = V^* U^*, \quad (U + V)^* = U^* + V^*,
\]
where \( U, V \) are arbitrary elements of \( \mathcal{C}(1, 3) \) and \( \lambda \in \mathbb{C} \). Now we can define an operation of Hermitian conjugation of Clifford algebra elements by the formula
\[
U^\dagger = \beta U^* \beta.
\]

\section*{Symplectic Lie group and its real Lie algebra}

Consider the real symplectic Lie group of matrices of even order \( n = 2m \) and its Lie algebra
\[
\begin{align*}
\operatorname{Sp}(m, \mathbb{R}) &= \{ U \in \operatorname{Mat}(n, \mathbb{R}) : U^T S U = S \}, \\
\operatorname{sp}(m, \mathbb{R}) &= \{ u \in \operatorname{Mat}(n, \mathbb{R}) : u^T S = -S u \},
\end{align*}
\]
where \( U^T \) is the transposed matrix, \( S \) is the block matrix
\[
S = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix},
\]
and \( I_m \) is the identity matrix of order \( m \). Note that \( S^2 = -1 \) (1 is the identity matrix of order \( 2m \)).

\section*{Symplectic Lie group of the Clifford algebra and its Lie algebra}

Let us define two sets of Clifford algebra elements
\[
\begin{align*}
\operatorname{Sp}(\mathcal{C}(1, 3)) &= \{ V \in \mathcal{C}^\mathbb{R}_{\text{Even}}(1, 3) \oplus i \mathcal{C}^\mathbb{R}_{\text{Odd}}(1, 3) : V^* V = e \}, \\
\operatorname{sp}(\mathcal{C}(1, 3)) &= \{ v \in i \mathcal{C}^\mathbb{R}_1(1, 3) \oplus \mathcal{C}^\mathbb{R}_2(1, 3) \}.
\end{align*}
\]
The set \( \operatorname{Sp}(\mathcal{C}(1, 3)) \) is closed with respect to the multiplication of Clifford algebra elements and forms a (Lie) group. This group is called the symplectic group of Clifford algebra \( \mathcal{C}(1, 3) \). The set \( \operatorname{sp}(\mathcal{C}(1, 3)) \) is closed w.r.t. the commutator \( [u, v] = uv - vu \) and forms the Lie algebra.

The following proposition is proved in \cite{7}: \textit{The group \( \operatorname{Sp}(\mathcal{C}(1, 3)) \) is isomorphic to the group \( \operatorname{Sp}(2, \mathbb{R}) \) and the Lie algebra \( \operatorname{sp}(\mathcal{C}(1, 3)) \) is isomorphic to the Lie algebra \( \operatorname{sp}(2, \mathbb{R}) \), i.e.}
\[
\begin{align*}
\operatorname{Sp}(\mathcal{C}(1, 3)) &\cong \operatorname{Sp}(2, \mathbb{R}), \\
\operatorname{sp}(\mathcal{C}(1, 3)) &\cong \operatorname{sp}(2, \mathbb{R}).
\end{align*}
\]
Hermitian idempotents. Let $t \in \mathcal{O}(1, 3)$ be a nonzero element such that
\[ t^2 = t, \quad t^\dagger = t, \quad \bar{t}J = Jt, \] (2)
where $J = -e^1 e^3$. Such an element is called a Hermitian idempotent. In particular, we may take the Hermitian idempotents
\[ t^{(1)} = \frac{1}{4}(e + e^0)(e + i e^{12}), \]
\[ t^{(2)} = \frac{1}{2}(e + e^0), \]
\[ t^{(3)} = \frac{1}{4}(3e + e^0 + i e^{12} - i e^{012}), \]
\[ t^{(4)} = e. \]

A Hermitian idempotent $t$ generates the left ideal $I(t)$, the two sided ideal $K(t)$, the Lie algebra $L(t)$, and the Lie group $G(t)$
\[
I(t) = \{ U \in \mathcal{O}(1, 3) : U = Ut \}, \\
K(t) = \{ U \in I(t) : U = tU \}, \\
L(t) = \{ U \in K(t) : U^\dagger = -U \}, \\
G(t) = \{ U \in \mathcal{O}(1, 3) : U^\dagger U = e, U - e \in K(t) \}.
\] (3)

The Minkowski space. Let $\mathbb{R}^{1,3}$ be the Minkowski space with cartesian coordinates $x^\mu$, where $\mu = 0, 1, 2, 3$ and $\partial_\mu = \partial / \partial x^\mu$ are partial derivatives. We use Greek indices $\mu, \nu, \alpha, \beta, \ldots$ (run from 0 to 3) as tensor indices relative to coordinates $x^\mu$. The Minkowski metric is given by the diagonal matrix $\eta$. By $T^r_s$ denote the set of tensor fields of type $(r, s)$ (of rank $r+s$) in Minkowski space. By $T_{[s]}$ denote the set of rank $s$ antisymmetric covariant tensor fields. In the sequel we consider tensors with values in Lie algebras. For example, if $u_\mu$ is a covector with values in a Lie algebra $L(t)$, then we write $u_\mu \in L(t)T_1$.

In what follows the generators $e^0, e^1, e^2, e^3$ of Clifford algebra $\mathcal{O}(1, 3)$ and the fixed Hermitian idempotent $t$ do not depend on $x$ and they are scalars of the Minkowski space, i.e. they do not transform under Lorentzian changes of coordinates.

Model Dirac–Yang–Mills equations. Consider the model Dirac–Yang–
Mills equations

\[ i h^\mu (\partial_\mu \phi + \phi A_\mu - C_\mu \phi) - m \phi = 0, \]
\[ \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu \nu}, \]
\[ \partial_\mu F^{\mu \nu} - [A_\mu, F_{\mu \nu}] = \phi^\dagger \beta i h^\nu \phi, \]
\[ \partial_\mu h^\nu - [C_\mu, h^\nu] = 0, \]

where

1. The vector \( i h^\mu = i h^\mu (x) \in \text{sp}(\mathcal{C}(1, 3)) T^1 \) is such that
   \[ h^\mu h^\nu + h^\nu h^\mu = 2 \eta^{\mu \nu} e, \quad \mu, \nu = 0, 1, 2, 3. \]

2. The element \( \phi = \phi(x) \in I(t) \) is a scalar of Minkowski space (it does not transform under Lorentzian changes of coordinates) \( (\phi(x) \rightarrow \phi(x(\dot{x}))) \).

3. \( A_\mu = A_\mu(x) \in L(t) T_1. \)

4. \( F_{\mu \nu} = F_{\mu \nu}(x) \in L(t) T_2. \)

5. The mass \( m \) is a real constant.

6. \( C_\mu = C_\mu(x) \in \text{sp}(\mathcal{C}(1, 3)) T_1. \)

We suppose that the idempotent \( t \), the constant \( m \), and the generators of Clifford algebra \( e^a \) are known and the variables \( h^\mu, \phi, A_\mu, F_{\mu \nu}, C_\mu \) are unknown. In this case equations (4) are called model Dirac–Yang–Mills equations (with local symplectic symmetry).

From the first equation in (4), using the identity \( \frac{1}{4} h^\mu h_\mu = e \), we get the equation
\[ i h^\mu (\partial_\mu \phi + \phi A_\mu - B_\mu \phi) = 0, \]
where
\[ B_\mu = C_\mu - \frac{m}{4} i h_\mu \in \text{sp}(\mathcal{C}(1, 3)) T_1. \]

If we substitute the expression
\[ C_\mu = B_\mu + \frac{m}{4} i h_\mu \in \text{sp}(\mathcal{C}(1, 3)) T_1 \]
into the equalities (the second equality is a consequence of the first one)
\[ \partial_\mu h^\nu - [C_\mu, h^\nu] = 0, \]
\[ \partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = 0, \]
then we get
\[ \partial_{\mu} i h^\nu - [B_{\mu}, i h^\nu] = \frac{m}{4} [i h_{\mu}, i h^\nu], \]  
(6)
\[ \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} - [B_{\mu}, B_{\nu}] = -\left( \frac{m}{4} \right)^2 [i h_{\mu}, i h_{\nu}]. \]

From the first equality it follows that
\[ \partial_{\mu} h^\mu - [B_{\mu}, h^\mu] = 0. \]

We denote
\[ G_{\mu\nu} = -\left( \frac{m}{4} \right)^2 [i h_{\mu}, i h_{\nu}]. \]

Using the relations (6) and the relations
\[ \frac{1}{4} h^\mu h_{\mu} = e, \quad h^\mu h^\nu h_{\mu} = h_{\mu} h^\nu h^\mu = -2 h^\nu, \]
we see that
\[ \partial_{\mu} G^{\mu\nu} - [B_{\mu}, G^{\mu\nu}] = \frac{3}{16} m^3 i h^\nu. \]

Therefore we have proved that if the variables \( \phi, h^\mu, A_{\mu}, C_{\mu}, F_{\mu\nu} \) satisfy conditions (5) and equations (4), then the variables
\[ \phi, h^\mu, A_{\mu}, B_{\mu} = C_{\mu} - \frac{m}{4} i h_{\mu}, F_{\mu\nu}, G_{\mu\nu} = -\left( \frac{m}{4} \right)^2 [i h_{\mu}, i h_{\nu}] \]
satisfy the equations
\[ i h^\mu (\partial_{\mu} \phi + \phi A_{\mu} - B_{\mu} \phi) = 0, \]
\[ \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - [A_{\mu}, A_{\nu}] = F_{\mu\nu}, \]
\[ \partial_{\mu} F^{\mu\nu} - [A_{\mu}, F^{\mu\nu}] = \phi^\dagger \beta i h^\nu \phi, \]
\[ \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} - [B_{\mu}, B_{\nu}] = G_{\mu\nu}, \]
\[ \partial_{\mu} G^{\mu\nu} - [B_{\mu}, G^{\mu\nu}] = \frac{3}{16} m^3 i h^\nu. \]

This system of equations contains two pairs of Yang–Mills equations for the fields \( (A_{\mu}, F_{\mu\nu}) \) and \( (B_{\mu}, G_{\mu\nu}) \) respectively.

Consider the system of equations (7), where the idempotent \( t \), the real constant \( m \), and the generators of Clifford algebra \( e^a \) are known and the variables \( h^\mu, \phi, A_{\mu}, F_{\mu\nu}, B_{\mu}, G_{\mu\nu} \) are unknown and such that
1a. The vector $ih^\mu = ih^\mu(x) \in \text{sp}(\mathcal{O}(1,3))T^1$ satisfies conditions (5).

2a. The element $\phi = \phi(x) \in I(t)$ is a scalar of the Minkowski space.

3a. $A_\mu = A_\mu(x) \in L(t)T_1$.

4a. $F_{\mu\nu} = F_{\mu\nu}(x) \in L(t)T^2$.

5a. $B_\mu = B_\mu(x) \in \text{sp}(\mathcal{O}(1,3))T_1$.

6a. $G_{\mu\nu} = G_{\mu\nu}(x) \in \text{sp}(\mathcal{O}(1,3))T^2$.

This system of equations is called the model Dirac–Yang–Mills system of equations with two Yang–Mills fields.

Suppose that the variables

\[ \phi, h^\mu, A_\mu, B_\mu, F_{\mu\nu}, G_{\mu\nu} \]

satisfy the conditions 1a-6a and satisfy equations (7). We see the vector $h^\mu$ at the right hand part of the Yang–Mills equations

\[
\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = G_{\mu\nu},
\]

\[
\partial_\mu G^{\mu\nu} - [B_\mu, G_{\mu\nu}] = \frac{3}{16}m^3ih^\nu.
\]

Therefore the vector field $h^\mu$ satisfies the non-abelian conservation law

\[
\partial_\mu h^\mu - [B_\mu, h^\mu] = 0.
\] (8)

However the identities (6) can’t be fulfilled. Hence we may consider the system of equations (7) as a generalization of the system of equations (4).

**Properties of the model Dirac–Yang–Mills equations.** A transformation of variables in the system of equation (7) is called equivalent transformation if this system of equation written for transformed variables has the same form as the system of equation in initial variables. In this case we say that system (7) is covariant w.r.t. this transformation of variables.

An equivalent transformation of variables in the system of equation (7) is called symmetry if the generators $e^a$ and the Hermitian idempotent $t$ do not transform (see [4, 8] for details).

Let us discuss the properties of the model equations (7) that related to equivalent transformations and symmetries. Let $\Theta = \{h^\mu, \phi, B_\mu, G_{\mu\nu}, A_\mu, F_{\mu\nu}\}$ satisfy the equations (7).
1. (Symmetry). All the variables in the system of equation (7) are tensors (scalars are rank 0 tensors). Therefore this system of equations is covariant under Lorentzian changes of coordinates.

2. Consider bilinear forms of the model Dirac–Yang–Mills equations (7)

\[ iJ^{\mu_1\ldots\mu_k} = i^{\frac{k(k-1)}{2}} + 1 \phi^{\dagger}\beta h^{[\mu_1} \ldots h^{\mu_k]} \phi \in L(t)T^k. \]

Bilinear forms \( J^{\mu_1\ldots\mu_k} \) are the components of contravariant antisymmetric tensors of rank \( k \) with values in Hermitian elements of the Clifford Algebra \( \mathcal{C}(1,3) \). Eigenvalues of these bilinear forms are real.

3. The vector

\[ iJ^\mu = \phi^{\dagger}\beta ih^\mu \phi \in L(t)T^1 \]

satisfy non-abelian conservative law

\[ \partial_\mu J^\mu - [A_\mu, J^\mu] = 0. \]

4. The equations (7) are covariant under the following global transformation defined by a unitary element \( U \in \mathcal{C}(1,3) \), \( U^\dagger = U^{-1} \), \( \partial_\mu U = 0 \),

\[ \phi \rightarrow \phi U, \quad A_\mu \rightarrow U^{-1}A_\mu U, \quad F_{\mu\nu} \rightarrow U^{-1}F_{\mu\nu}U, \quad t \rightarrow U^{-1}tU. \quad (9) \]

5. (Symmetry). The equations (7) are covariant under the local (gauge) transformation with \( U = U(x) \in G(t) \)

\[ \phi \rightarrow \phi U, \quad A_\mu \rightarrow U^{-1}A_\mu U - U^{-1}\partial_\mu U, \quad F_{\mu\nu} \rightarrow U^{-1}F_{\mu\nu}U. \quad (10) \]

Note that for \( U \in G(t) \) we have \([U, t] = 0 \) and under the considered transformation the Hermitian idempotent \( t \) does not transform.

6. (Symmetry). System of equation (7) is covariant w.r.t. the local (gauge) transformation of variables \( \Theta \rightarrow \hat{\Theta} \) induced by an element \( W = W(x) \in \text{Sp}(\mathcal{C}(1,3)) \):

\[ \hat{\phi} = W^{-1}\phi, \quad \hat{h}^\mu = W^{-1}h^\mu W, \quad \hat{B}_\mu = W^{-1}B_\mu W - W^{-1}\partial_\mu W. \]

7. System of equation (7) is covariant w.r.t. the discreet transformation (complex conjugation) of variables

\[ ih^\mu \rightarrow \overline{i}h^\mu, \quad (h^\mu \rightarrow \overline{h}^\mu), \quad t \rightarrow \overline{t}, \]
\[ \phi \rightarrow \bar{\phi}, \quad A_\mu \rightarrow \bar{A}_\mu, \quad F_{\mu\nu} \rightarrow \bar{F}_{\mu\nu}, \quad B_\mu \rightarrow \bar{B}_\mu, \]

here we suppose that \( \bar{e}^{a_1...a_k} = e^{a_1...a_k} \).

8. (Symmetry). System of equation (7) is covariant w.r.t. the discreet transformation of variables

\[ i\hbar^\mu \rightarrow \overline{i\hbar^\mu}, \quad (\hbar^\mu \rightarrow -\overline{\hbar^\mu}), \quad \phi \rightarrow \overline{\phi}J, \quad B_\mu \rightarrow \bar{B}_\mu, \quad A_\mu \rightarrow J^{-1} \bar{A}_\mu J, \quad F_{\mu\nu} \rightarrow J^{-1} \bar{F}_{\mu\nu} J, \]

where \( J = -e^1 e^3 \).

**Discussion of the model.** We have introduced system of equation (7), which consists of three parts

- The model Dirac equation

\[ i\hbar^\mu (\partial_\mu \phi + \phi A_\mu - B_\mu \phi) = 0 \]

for the wave function \((i\hbar^\mu, \phi)\) of spin 1/2 particle.

- The first pair of Yang–Mills equations

\[ \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu}, \]

\[ \partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = \phi^\dagger \beta i\hbar^\nu \phi, \]

(11)

describes the Yang–Mills field \((A_\mu, F_{\mu\nu})\) with the gauge group \( G(t) \) that is isomorphic to a subgroup of the unitary group \( U(4) \). According to the Standard Model, if the gauge group \( G(t) \) is isomorphic to one of three groups – \( U(1), U(1) \times SU(2), SU(3) \), then system of equation (11) can be used for the description of the electromagnetic (QED) interaction, the electroweak (EW) interaction, and the strong (QCD) interaction respectively.

- The second pair of Yang–Mills equations

\[ \partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = G_{\mu\nu}, \]

\[ \partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] = \frac{3}{16} m^3 i\hbar^\nu \]

(12)

describes the Yang–Mills field \((B_\mu, G_{\mu\nu})\) with the symplectic group \( Sp(O(1,3)) \) of gauge symmetry. The dimension of the Lie algebra
sp(\mathcal{O}(1, 3)) is equal to 10. Hence the Yang–Mills field \((B_\mu, G_{\mu\nu})\) describes 10 types of spin 1 elementary particles (mediators), which interact with the initial spin 1/2 particle (wave function \((ih^\mu, \phi)\)). The model Dirac equation does not contain the mass \(m\) of spin 1/2 particle. We see mass \(m\) only at the right hand part of Yang–Mills equations \((12)\) in the term \(3m^3/16\). Therefore the constant \(3m^3/16\) can be considered as a charge of spin 1/2 particle relevant to the gauge field \((B_\mu, G_{\mu\nu})\).

**Conclusion.** In considered model, which based on equations \((7)\), spin 1/2 particles acquire masses by interaction between these fermions and the (spin 1) gauge field with symplectic symmetry.

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**References**

[1] Lounesto P., *Clifford Algebras and Spinors*, Cambridge Univ. Press (1997, 2001)

[2] Marchuk N.G., Technique of Tensors with Values in the Atiyah-Kähler Algebra in the New Representation of Dirac-Yang-Mills Equations, Russian Journ. of Math. Phys, Vol.13, No.3, (2006), pp.299-314.

[3] Marchuk N.G., New Representation of Dirac-Yang-Mills Equations, Russian Journ. of Math. Phys, Vol.13, No.4, (2006), pp.397-413.

[4] Marchuk N.G., Model Dirac-Maxwell equations with pseudounitary symmetry, Theoretical and Mathematical Physics, 157(3): pp.1723-1732 (2008),

[5] Marchuk N.G., A concept of Dirac-type tensor equations, Nuovo Cimento, 117B, 12, (2002), pp.1357-1388.

[6] Marchuk N., Shirokov D.S., Unitary Spaces on Clifford Algebras, Advances in Applied Clifford Algebras, v.18, N.2, (2008),
[7] Marchuk N., Dyabirov R., A Symplectic Subgroup of a Pseudounitary Group as a Subset of Clifford Algebra, Advances in Applied Clifford Algebras, 20, pp.343-350, (2010),

[8] Marchuk N.G., Field theory equations and Clifford algebras, (in Russian), R&C Dynamics, (2009).