New Quadratic Mass Relations for Heavy Mesons

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Abstract

By assuming the existence of (quasi)-linear Regge trajectories for heavy mesons, we derive new quadratic mass relations of non-Gell-Mann–Okubo type,

\[6M^2(q\bar{q}) + 3M^2(c\bar{c}) = 8M^2(c\bar{q}), \quad 20M^2(q\bar{q}) + 5M^2(b\bar{b}) = 16M^2(b\bar{q}), \quad q = n(=u, d), s\]

which show excellent agreement with experiment. We also establish the sum rule \[M^2(i\bar{i}) + M^2(j\bar{j}) - 2M^2(j\bar{i}) \approx \text{const}\] for any pair of flavors, \((i, j)\).

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1 Introduction

The generalization of the standard $SU(3)$ Gell-Mann–Okubo mass formula \[ \text{A} \] to higher symmetry groups, e.g., $SU(4)$ and $SU(5)$, became a natural subject of investigation after the discovery of the fourth and fifth quark flavors in the mid-70’s \[ \text{B} \]. Attempts have been made in the literature to derive such a formula, either quadratic or linear in mass, by a) using group theoretical methods \[ \text{C} \], \[ \text{D} \], \[ \text{E} \], b) generalizing the perturbative treatment of $U(3) \times U(3)$ chiral symmetry breaking and the corresponding Gell-Mann-Oakes-Renner relation \[ \text{F} \] to $U(4) \times U(4)$ \[ \text{G} \], \[ \text{H} \], c) calculating the corrections to the Gell-Mann–Okubo formula in the charmed-quark sector due to second order $SU(4)$ breaking effects, using current algebra techniques \[ \text{I} \], d) assuming the asymptotic realization of $SU(4)$ symmetry in the algebra $[A_\alpha, A_\beta] = i f_{\alpha\beta\gamma} V_\gamma$ (where $V_\alpha, A_\beta$ are vector and axial-vector charges, respectively) \[ \text{J} \], e) extending the Weinberg spectral function sum rules \[ \text{K} \] to accommodate the higher symmetry breaking effects \[ \text{L} \], and f) applying alternative methods, such as Regge phenomenology \[ \text{M} \], \[ \text{N} \], and the linear mass spectrum for meson multiplets \[ \text{O} \], \[ \text{P} \].

In the following, $\eta_n, \eta_s, \eta_c, \eta_b, K, D_n, D_s, B_n, B_s, B_c$ stand for the masses of the $n\bar{n}$ (where $n \equiv u, d$), $s\bar{s}, c\bar{c}, b\bar{b}, s\bar{n}, c\bar{n}, c\bar{s}, b\bar{n}, b\bar{s}, b\bar{c}$ mesons, respectively \[ \text{Q} \], unless otherwise specified. The linear mass relations (here the symbol for the meson represents its mass value)

$$D_n = \frac{\eta_n + \eta_c}{2}, \quad D_s = \frac{\eta_s + \eta_c}{2},$$

$$B_n = \frac{\eta_n + \eta_b}{2}, \quad B_s = \frac{\eta_s + \eta_b}{2}, \quad B_c = \frac{\eta_c + \eta_b}{2}$$

found in \[ \text{R} \], \[ \text{S} \], although perhaps justified for vector mesons, since a vector meson mass is given approximately by a sum of the corresponding constituent quark masses,

$$m(ij) \approx m(i) + m(j)$$

(in fact, for vector mesons, the relations (1),(2) hold with an accuracy of up to $\sim 4\%$), are expected to fail for other meson multiplets, as confirmed by direct comparison with experiment. Similarly, the quadratic mass relation

$$D_s^2 - D^2 = K^2 - \pi^2$$

\[ \text{3} \]Here, we speak of linear spectrum over the additive, $I_3$ and $Y$, multiplet quantum numbers, taking proper account of degeneracy, but do not (directly) make use of linear Regge trajectories.

\[ \text{2} \]Here $\eta_n$ stands for the masses of both isovector and isoscalar $n\bar{n}$ states which coincide on a naive quark model level.

\[ \text{3} \]Since these designations apply to all spin states, vector mesons will be confusingly labelled as $\eta$’s. We ask the reader to bear with us in this in the interest of minimizing notation.
obtained in ref. [7] by generalizing the $SU(3)$ Gell-Mann-Oakes-Renner relation [3] to include the $D$ and $D_s$ mesons,

$$\frac{\pi^2}{2n} = \frac{K^2}{n+s} = \frac{D^2}{n+c} = \frac{D_s^2}{s+c};$$  

(4)

(and therefore $D_s^2 - D^2 = K^2 - \pi^2 \propto (s - n)$, also found in refs. [4, 10, 12]), does not agree with experiment. For pseudoscalar mesons, for example, one has (in GeV$^2$) 0.388 for the l.h.s. of (3) vs. 0.226 for the r.h.s.. For vector mesons, the corresponding quantities are 0.424 vs. 0.199, with about 100% discrepancy. The reason that the relation (3) does not hold is apparently due to the impossibility of perturbative treatment of $U(4) \times U(4)$ symmetry breaking, as a generalization of that of $U(3) \times U(3)$, due to very large bare mass of the $c$-quark as compared to those of the $u$-, $d$- and $s$-quarks.

It was concluded in ref. [9] that second order $SU(4)$ breaking effects shift the masses of the charmed vector and pseudoscalar mesons, respectively, upwards and downwards from the values predicted by the quadratic Gell-Mann–Okubo type formula ($I$ stands for isospin)

$$\frac{1}{2} \left( M^2(n\bar{n}, I = 1) + M^2(n\bar{n}, I = 0) \right) + M^2(c\bar{c}) = 2M^2(c\bar{n})$$  

(5)

obtained by generalizing the standard $SU(3)$ relation [17]

$$\frac{1}{2} \left( M^2(n\bar{n}, I = 1) + M^2(n\bar{n}, I = 0) \right) + M^2(s\bar{s}) = 2M^2(s\bar{n}),$$  

(6)

and hence [4, 9]

$$\frac{1}{2} \left( \rho^2 + \omega^2 \right) + (J/\psi)^2 - 2D^2 \simeq -0.60 \text{ GeV}^2,$$  

(7)

$$\frac{\pi^2}{2} + \frac{\eta^2}{3} + \frac{\eta^2}{6} + \eta_c^2 - 2D^2 \simeq 0.80 \text{ GeV}^2.$$  

(8)

In fact, charmed meson masses of all (four) well-established multiplets are shifted downwards, and the magnitudes of these shifts seem not to depend on the quantum numbers of the corresponding multiplets; indeed, using the measured meson masses, one obtains

$$\frac{1}{2} \left( \rho^2 + \omega^2 \right) + (J/\psi)^2 - 2D^2 = 2.12 \pm 0.02 \text{ GeV}^2,$$  

(9)

The reason for the appearance of a peculiar combination of the pseudoscalar meson squared masses in Eq. (8) is the following assumed flavor content of the $\eta$ and $\eta'$ [8],

$$\eta \simeq 0.58 (u\bar{u} + d\bar{d}) - 0.57 s\bar{s} \simeq \frac{u\bar{u} + d\bar{d} - s\bar{s}}{\sqrt{3}},$$

$$\eta' \simeq 0.40 (u\bar{u} + d\bar{d}) + 0.82 s\bar{s} \simeq \frac{u\bar{u} + d\bar{d} + 2s\bar{s}}{\sqrt{6}},$$

as explained below in the text.
\[ \frac{\pi^2}{2} + \frac{\eta^2}{3} + \frac{\eta^\prime_1}{6} + \eta^2_c - 2D^2 = 2.17 \pm 0.02 \text{ GeV}^2, \]  

(10)

and for pseudovector and tensor mesons, respectively,

\[ \frac{1}{2} \left( b_1^2 + h_1^2 \right) + h_2^2(1P) - 2D_2^2 = 2.14 \pm 0.09 \text{ GeV}^2, \]  

(11)

\[ \frac{1}{2} \left( a_2^2 + f_2^2 \right) + \chi_{2s}(1P) - 2D_2^2 = 2.23 \pm 0.04 \text{ GeV}^2, \]  

(12)

so that Eqs. (9)-(12) agree with each other with an accuracy of up to \( \sim 5\% \). Similar conclusion seems to hold in the bottom sector where for two established vector and tensor multiplets the corresponding shifts

\[ \frac{1}{2} \left( \rho^2 + \omega^2 \right) + \Upsilon^2 - 2B^s = 33.38 \pm 0.04 \text{ GeV}^2, \]  

(13)

\[ \frac{1}{2} \left( a_2^2 + f_2^2 \right) + \chi_{2b}(1P) - 2B_2^s = 35.01 \pm 0.28 \text{ GeV}^2 \]  

(14)

agree with each other to \( \sim 4.5\% \).

With this background of the apparent failure of the standard methods to reproduce proper mass relations for higher symmetry group, alternative methods turn out to be quite successful. Regge phenomenology explored by the present authors in refs. \([13, 14] \) leads to sixth and fourteenth power mass relations (for \( SU(4) \) and \( SU(5) \) meson multiplets, respectively), out of which only the former may be tested so far giving an accuracy of up to \( \sim 5\% \) for all (four) well-established multiplets. The relation

\[ 12\bar{D}^2 = 7\eta_0^2 + 5\eta_c^2, \]  

(15)

obtained by two of the present authors in ref. \([16] \) by the application of the linear spectrum to \( SU(4) \) meson hexadecuplet (here \( \bar{D} \) is the average mass of the \( D_n \) and \( D_s \) states which are mass degenerate when flavor \( SU(4) \) symmetry is broken down to \( SU(3) \) by \( m(c) \neq m(s) = m(n) \), and \( \eta_0 \) is the mass average of the corresponding \( SU(3) \) nonet which is also mass degenerate in this case), holds with a similar accuracy of up to \( 5\% \).

The purpose of the present paper is to show that, in addition to the quite successful higher power relations obtained in \([13, 14] \), quadratic mass formulas may be also derived. We recall that such formulas have already been presented in refs. \([13, 14] \) by fitting the values of the Regge slopes of the corresponding meson trajectories; e.g.

\[ 8.13 K^2 + 4.75 \eta_c^2 = 6 \left( D_n^2 + D_s^2 \right), \]  

(16)

with an accuracy of \( \sim 1\% \). Non-integer coefficients in Eq. (16) reflect the uncertainty in fitting the values of the Regge slopes. We shall show that no such fitting is required to obtain quadratic meson mass relations which, in contrast to (16), contain integer coefficients (similar to quadratic baryon mass relations obtained by the present authors in ref. \([19] \), and are therefore more suitable for practical use, e.g., to make predictions for the masses of the states yet to be discovered in experiment.
2 Regge phenomenology

It is well known that the hadrons composed of light (u, d, s) quarks populate linear Regge trajectories; i.e., the square of the mass of a state with orbital momentum ℓ is proportional to ℓ : $M^2(ℓ) = ℓ/α' + \text{const}$, where the slope $α'$ depends weakly on the flavor content of the states lying on the corresponding trajectory,

\[ α'_{nn} \simeq 0.88 \text{ GeV}^{-2}, \quad α'_{ss} \simeq 0.84 \text{ GeV}^{-2}, \quad α'_{s\bar{s}} \simeq 0.80 \text{ GeV}^{-2}. \quad (17) \]

In contrast, the data on the properties of Regge trajectories of hadrons containing heavy quarks are almost nonexistent at the present time, although it is established [20] that the slope of the trajectories decreases with increasing quark mass (as seen in (17)) in the mass region of the lowest excitations. This is due to an increasing (with mass) contribution of the color Coulomb interaction, leading to a curvature of the trajectory near the ground state. However, as the analyses show [20, 21, 22], in the asymptotic regime of the highest excitations, the trajectories of both light and heavy quarkonia are linear and have the same slope $α' \simeq 0.9 \text{ GeV}^{-2}$, in agreement with natural expectations from the string model.

If one assumes the (quasi)-linear form of Regge trajectories for hadrons with identical $J^{PC}$ quantum numbers (i.e., belonging to a common multiplet), then one has for the states with orbital momentum ℓ

\[
\begin{align*}
\ell &= α'_i m_i^2 + a_i(0), \\
\ell &= α'_{ji} m_{ji}^2 + a_{ji}(0), \\
\ell &= α'_{jj} m_{jj}^2 + a_{jj}(0).
\end{align*}
\]

Using now the relation among the intercepts [25, 26],

\[ a_{ii}(0) + a_{jj}(0) = 2a_{ji}(0), \quad (18) \]

one obtains from the above relations

\[ α'_i m_i^2 + α'_{jj} m_{jj}^2 = 2α'_{ji} m_{ji}^2. \quad (19) \]

In order to eliminate the Regge slopes from this formula, we need a relation among the slopes. Two such relations have been proposed in the literature,

\[ α'_i \cdot α'_{jj} = (α'_{ji})^2, \quad (20) \]

which follows from the factorization of residues of the $t$-channel poles [27, 28], and

\[ \frac{1}{α'_i} + \frac{1}{α'_{jj}} = \frac{2}{α'_{ji}}, \quad (21) \]

based on topological expansion and the $q\bar{q}$-string picture of hadrons [26].

For light quarkonia (and small differences in the $α'$ values), there is no essential difference between these two relations; viz., for $α'_{ji} = α'_i/(1 + x)$, $x \ll 1$, Eq. (21)
gives $\alpha^{'}_{jj} = \alpha^{'}_{ii}/(1 + 2x)$, whereas Eq. (20) gives $\alpha^{'}_{jj} = \alpha^{'}_{ii}/(1 + x)^2 \approx \alpha^{'}/(1 + 2x)$, i.e., essentially the same result to order $x^2$. However, for heavy quarkonia (and expected large differences from the $\alpha'$ values for the light quarkonia) these relations are incompatible; e.g., for $\alpha^{'}_{ji} = \alpha^{'}_{ii}/2$, Eq. (20) will give $\alpha^{'}_{jj} = \alpha^{'}_{ii}/4$, whereas Eq. (21) $\alpha^{'}_{jj} = \alpha^{'}_{ii}/3$. One has therefore to choose between these relations in order to proceed further. Here we use Eq. (21), since it is much more consistent with (19) than is Eq. (20), which we tested by using measured quarkonia masses in Eq. (19). We shall justify this choice in more detail in a separate publication [29]. Here we only wish to show an explicit relation of Eq. (21) to the shifts of the masses of the charmed and beauty mesons downwards from their Gell-Mann–Okubo values which are independent of the multiplet quantum numbers, as we have seen above in Eqs. (9)-(12) and (13),(14).

Assume, as usually, that both $J^{PC}$ and $(J+1)^{-P,-C}$ states belong to common Regge trajectory, and two trajectories on which $(\bar{n}n, I = 0)$ and $(\bar{n}n, I = 1)$ states lie have approximately equal slopes. Then

$$\alpha^{'}_{nn} = \frac{1}{M^2((J+1)^{-P,-C}, \bar{n}n, I = 1) - M^2(J^{PC}, \bar{n}n, I = 1)}$$

$$\approx \frac{1}{M^2((J+1)^{-P,-C}, \bar{n}n, I = 0) - M^2(J^{PC}, \bar{n}n, I = 0)}$$

$$\approx \frac{2}{M^2((J+1)^{-P,-C}, \bar{n}n, I = 1) - M^2(J^{PC}, \bar{n}n, I = 1)}$$

$$+ \frac{2}{M^2((J+1)^{-P,-C}, \bar{n}n, I = 0) - M^2(J^{PC}, \bar{n}n, I = 0)},$$

$$\alpha^{'}_{cc} = \frac{1}{M^2((J+1)^{-P,-C}, \bar{c}\bar{c}) - M^2(J^{PC}, \bar{c}\bar{c})},$$

$$\alpha^{'}_{cn} = \frac{1}{M^2((J+1)^{-P,-C}, \bar{c}n) - M^2(J^{PC}, \bar{c}n)}.$$
which is the sum rule discussed above. Since all trajectories in a given \((ij)\) sector have approximately equal slopes, the numerical value of the difference of the squared masses in Eq. (22) does not depend on the quantum numbers \(J^{PC}\), as we have seen in Eqs. (9)-(12). By replacing the \(c\)-quark by the \(b\)-quark, a similar sum rule may be derived for the bottom sector (Eqs. (13),(14)). Moreover, by repeating the above analysis for arbitrary pair of flavors, \((i, j)\), one may easily establish the general sum rule

\[
M^2(\bar{i}i) + M^2(\bar{j}j) - 2M^2(\bar{j}i) \approx \text{const},
\]

where \(M^2(n\bar{n})\) is given in Eq. (24) below.

We note that the sum rule (22) cannot be obtained by starting from Eq. (20). As easily seen, Eq. (20) would lead to, e.g.,

\[
\frac{1}{2} \left( a_2^2 - \rho^2 + f_2^2 - \omega^2 \right) \left( \chi_{2a}(1P) - (J/\psi)^2 \right) \approx \left( D_2^2 - D^*2 \right)^2,
\]

which gives 3.3 GeV\(^2\) on the l.h.s. vs. 4.0 GeV\(^2\) on the r.h.s., with an accuracy of \(\sim 20\%\) which is much worse than the accuracy of Eqs. (9)-(12) and (13),(14). This fact should be considered as the best experimental evidence for additivity of the inverse Regge slopes, Eq. (21), and against factorization of the slopes, Eq. (20).

No standard Gell-Mann–Okubo type quadratic mass formula is compatible with Eqs. (19),(21), except for the standard \(SU(3)\) Gell-Mann–Okubo formula itself. By standard Gell-Mann–Okubo type one we mean a quadratic mass relation the sums of coefficients on both sides of which coincide (e.g., Eq. (15) where \(12 = 7 + 5\)). Indeed, as follows from (19),(21),

\[
\alpha'_i m^2_{ni} + \alpha'_j m^2_{nj} = \frac{4\alpha'_i \alpha'_j}{\alpha'_i + \alpha'_j} m^2_{ji}.
\]

Equating the sums of coefficients on both sides of this relations gives

\[
\left( \alpha'_i - \alpha'_j \right)^2 = 0, \quad \text{i.e.,} \quad \alpha'_i = \alpha'_j,
\]

and this holds (approximately) only in the \(SU(3)\) sector (for \(i = n, j = s\)). Thus, proper generalization of the standard quadratic Gell-Mann–Okubo formula to higher symmetry groups must be of non-Gell-Mann–Okubo type.

It is also clear from the comparison of the relations (6), and (19) with \(i = n, j = s\):

\[
\eta_n^2 + \eta_s^2 = 2K^2,
\]

that

\[
\eta_n^2 = \frac{1}{2} \left( M^2(n\bar{n}, I = 1) + M^2(n\bar{n}, I = 0) \right).
\]

This relation explains the appearance of a peculiar combination of the pseudoscalar meson masses in Eq. (8). Indeed, the pure “non-strange” and “strange” pseudoscalar...
isoscalar states which may be constructed from the physical $\eta$ and $\eta'$ states should have the masses

$$
\eta^2(\bar{n}n) = \frac{2}{3} \eta^2 + \frac{1}{3} \eta'^2,
$$
$$
\eta^2(s\bar{s}) = \frac{1}{3} \eta^2 + \frac{2}{3} \eta'^2,
$$
respectively, according to their flavor content, as described in Footnote 4. Therefore, in view of (24),

$$
\eta_n^2 = \frac{1}{2} \left( \pi^2 + \eta^2(\bar{n}n) \right) = \frac{\pi^2}{2} + \frac{\eta^2}{3} + \frac{\eta'^2}{6},
$$
$$
\eta_s^2 = \eta^2(s\bar{s}) = \frac{\eta^2}{3} + \frac{2\eta'^2}{3}.
$$

Theoretical basis for such quadratic meson mass relations of non-Gell-Mann–Okubo type with integer coefficients has been established by Balázs in ref. [30] where, by using the dual topological unitarization approach to confinement region of QCD which is based on analyticity and generalized ladder-graph dynamics and takes into account the effect of planar sea-quark loops, it was shown that

$$
\alpha'_{\bar{c}c} = \frac{\alpha'_{\bar{n}n}}{N_1}, \quad \alpha'_{\bar{b}b} = \frac{\alpha'_{\bar{n}n}}{N_2}, \quad N_1, N_2 \text{ are integer},
$$

and $N_2 > N_1 > 1$ (i.e., $\alpha'_{\bar{b}b} < \alpha'_{\bar{c}c} < \alpha'_{\bar{n}n}$). The approach of ref. [30] does not however specify the numerical values of $N_1$ and $N_2$. It was suggested by Balázs that

$$
N_1 = 3, \quad N_2 = 9.
$$

Also, in ref. [31], Balázs has suggested

$$
\alpha'_{\bar{c}c} = \frac{\alpha'_{\bar{n}n}}{3}, \quad \alpha'_{\bar{c}n} = \frac{\alpha'_{\bar{n}n}}{2},
$$
$$
\alpha'_{\bar{b}b} = \frac{\alpha'_{\bar{n}n}}{9}, \quad \alpha'_{\bar{b}n} = \frac{\alpha'_{\bar{n}n}}{5},
$$
so that

$$
\frac{2}{\alpha'_{\bar{c}n}} = \frac{1}{\alpha'_{\bar{c}c}} + \frac{1}{\alpha'_{\bar{n}n}} = \frac{4}{\alpha'_{\bar{n}n}},
$$
and

$$
\frac{2}{\alpha'_{\bar{b}n}} = \frac{1}{\alpha'_{\bar{b}b}} + \frac{1}{\alpha'_{\bar{n}n}} = \frac{10}{\alpha'_{\bar{n}n}},
$$
confirming (21).

We now show that these values are motivated by the ratio of the constituent quark masses, and therefore may be perhaps justified for vector mesons. Indeed, by
viewing a vector meson as \( m(ij) = m(i) + m(j) \), and solving Eq. (21) by introducing \( x \equiv \alpha'_{\bar{a} i}/\alpha'_{\bar{a} j} \), as follows:

\[
\alpha'_{\bar{a} i} = \frac{\alpha'_{\bar{a} i}}{(1 + x)/2}, \quad \alpha'_{\bar{a} j} = \frac{\alpha'_{\bar{a} j}}{x}.
\]

Eq. (19) may be rewritten as

\[
4m^2(i) + \frac{4m^2(j)}{x} = \frac{4(m(i) + m(j))^2}{1 + x},
\]

leading to

\[
x = \frac{m(j)}{m(i)}.
\]

For \( i = s, j = c (b) \), \( m(i) \approx 0.5 \text{ GeV} \), \( m(j) \approx 1.5 (4.5) \text{ GeV} \), and \( x \approx 3 (9) \), which are the values suggested by Balázs which enter Eq. (23) (in the approximation \( \alpha'_{s\bar{s}} \approx \alpha'_{n\bar{n}} \)). However, even for vector mesons, the formula obtained from (19), (21), (26),

\[
3\tilde{\rho}^2 + (J/\psi)^2 = 3D^2, \quad \tilde{\rho} \equiv \frac{\rho^2 + \omega^2}{2},
\]

gives \( 11.4 \text{ GeV}^2 \) on the l.h.s. vs. 12.1 GeV\(^2 \) on the r.h.s., with an accuracy of \( \sim 6\% \). This accuracy is not bad by itself but much worse than that of a new mass relation for vector mesons we suggest in the next Section. The reason for this is the physically incorrect assumption of a meson mass being a sum of the corresponding constituent quark masses. It is well known that in order to reproduce meson spectroscopy correctly, one has to include hyperfine (spin-spin, spin-orbit and tensor) interaction, in addition to a sum of the constituent quark masses, even for vector mesons \[32\].

In this paper we suggest the values for \( N_1 \) and \( N_2 \) which are different from those of Eq. (26); namely,

\[
N_1 = 2, \quad N_2 = 4.
\]

The value \( N_1 = 2 \), although seems to be a guess consistent with (25), is quite natural (in contrast to \( N_1 = 3 \) of Balázs), since it leads to

\[
\alpha'_{c\bar{c}} = \frac{\alpha'_{n\bar{m}}}{2} \approx 0.45 \text{ GeV}^{-2},
\]

in agreement with \( \alpha'_{c\bar{c}} \approx 0.5 \text{ GeV}^{-2} \) which has long been discussed in the literature \[20, 33, 34, 35\]. As we show below, mass relation obtained on the basis of \( N_1 = 2 \) is in excellent agreement with experiment. The value \( N_2 = 4 \) cannot be predicted by any known (at least, to the authors) theoretical approach, but may be anticipated on the basis of simple phenomenological arguments which we present below.

\[5\]

A corresponding formula in the bottom sector,

\[
45\tilde{\rho}^2 + 5\Upsilon^2 = 18B^2,
\]

holds with an accuracy of \( \sim 7.5\% \).
3 New quadratic mass relations

Let us start with quadratic mass relation for the $SU(4)$ multiplet built of the $u$, $d$, $s$, and $c$-quarks. This relation follows from (19), (21) with $i = n$, $j = c$, and (21) with $N_1 = 2$:

$$6 \eta_n^2 + 3 \eta_c^2 = 8 D_n^2. \tag{32}$$

In order to test this relation, we calculate the value of $D_n$, as given by (32), using the measured values of $\eta_n$ and $\eta_c$, and compare it with experiment. Our results are shown in Table I. The combination $(\pi^2/2 + \eta^2/3 + \eta'^2/6)$ has been used for $\eta_n^2$ in the case of pseudoscalar mesons, as explained above. One sees excellent agreement with experiment.

In the approximation $\alpha_{s\bar{s}}' \approx \alpha_{n\bar{n}}'$, another mass relation may be written down (which is obtained by replacing the $s$-quark for the non-strange quarks in Eq. (32)),

$$6 \eta_s^2 + 3 \eta_c^2 = 8 D_s^2. \tag{33}$$

This relation is tested in Table II, again by comparing predictions for $D_s$ given by (33) with experiment. Now $(\eta^2/3 + 2\eta'^2/3)$ has been used for $\eta_s^2$ in the case of pseudoscalar mesons. Again one sees very good agreement with experiment. The reason for poorer agreement with experiment in the case of the $1^{+-}$ multiplet may be that the $D_{s1}$ is an axial-vector meson, not a pseudovector one, or the mixture of both states. If the $D_{s1}$ does belong to the $1^{++}$ multiplet, calculation using Eq. (33) will give $D_{s1} = 2525 \pm 2$ MeV, with the accuracy of 0.4 %. Let us note that in this case the accuracy of Eq. (32), as applied to predict the mass of the $D_1$, will not change since both the $a_1$ and $b_1$ mesons have approximately equal mass of $\sim 1230$ MeV \[36\].

In order to determine the value of $N_2$, we use the following phenomenological arguments. First, as we have already remarked in ref. [14], the difference in the constituent quark masses is the only reason for the different slopes of the corresponding trajectories. Indeed, if one considers sub-$SU(3)$ symmetry which incorporates the $u$, $d$- and $c$-quarks (with the $c$-quark playing the same role as $s$-quark in a real world), one has to obtain the same quadratic mass relation in this sub-$SU(3)$ sector as the standard Gell-Mann–Okubo one in the sub-$SU(3)$ sector which incorporates the $u$, $d$- and $s$-quarks,

$$m^2 = a + bZ + c \left[ \frac{Z^2}{4} - I(I+1) \right], \tag{34}$$

since charm ($C$) is now playing the role of strangeness, and the “supercharge” $Z = B + C$ is playing the role of hypercharge. However, the actual mass relation in the $(u, d, c)$ sector, Eq. (32), differs considerably from the standard Gell-Mann–Okubo formula, as applied to this sector,

$$\eta_n^2 + \eta_c^2 = 2D_n^2.$$
corresponding trajectories, and leads to non-Gell-Mann–Okubo type mass relation (32). This difference is also responsible for peculiar numerical coefficients of Eq. (32), viz., 6, 3 and 8.

It is quite natural to assume that the ratio of the slopes in the $\bar{\eta}i$ and $\bar{j}j$ sectors is solely determined by the ratio of the corresponding constituent quark masses, i.e.,

$$\frac{\alpha'_{\bar{i}i}}{\alpha'_{\bar{j}j}} = F\left(\frac{m(j)}{m(i)}\right),$$

(35)

where $F(x) \equiv m(j)/m(i)$, is some unknown function. Then, in particular,

$$\frac{\alpha'_{\bar{s}s}}{\alpha'_{\bar{c}c}} = F\left(\frac{m(c)}{m(s)}\right) \simeq F\left(\frac{1.5 \text{ GeV}}{0.5 \text{ GeV}}\right) = F(3).$$

Since

$$\frac{\alpha'_{\bar{c}c}}{\alpha'_{\bar{b}b}} = F\left(\frac{m(b)}{m(c)}\right) \simeq F\left(\frac{4.5 \text{ GeV}}{1.5 \text{ GeV}}\right) = F(3) \simeq \frac{\alpha'_{\bar{s}s}}{\alpha'_{\bar{c}c}},$$

and the coefficients of a mass relation are solely determined by the ratio of the constituent quark masses, as discussed above, we conclude that the mass relation for mesons composed of the $c$- and $b$-quarks should have the same coefficients as the mass relation for mesons composed of the $s$- and $c$-quarks, respectively (Eq. (33)), i.e.,

$$6 \eta_c^2 + 3 \eta_b^2 = 8 B_c^2.$$  

(36)

This relation will now enable one to obtain the value of $N_2$. It follows from (19), (21) with $i = c$, $j = b$, and (25) with $N_1 = 2$ that

$$\frac{\eta_c^2}{2} + \frac{\eta_b^2}{N_2} = \frac{2 B_c^2}{1 + N_2/2}.$$  

(37)

By comparing Eqs. (36) and (37), one finds

$$N_2 = 4.$$  

(38)

Once $N_2$ is known, one can easily derive mass relations for the $SU(4)$ multiplet built of the $u$-, $d$-, $s$-, and $b$-quarks, in a way which is completely analogous to that for the derivation of Eqs. (32) and (33) above. These are

$$20 \eta_u^2 + 5 \eta_b^2 = 16 B_u^2,$$  

(39)

$$20 \eta_d^2 + 5 \eta_b^2 = 16 B_d^2,$$  

(40)

and tested in Tables III and IV, respectively, again by comparing their predictions for $B_u$ and $B_s$ with experiment. So far, firm comparison is only possible for vector mesons. As for the tensor multiplet, we consider the states $B_u^*(5732)$ and $B_s^*(5850)$.

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6 An example of such a function, $F(x) = x$, is given by Eqs. (27), (28), where $\frac{\alpha'_{\bar{j}j}}{\alpha'_{\bar{i}i}} = \frac{m(j)}{m(i)}$. 

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discovered recently, whose quantum numbers are uncertain, as tensor mesons, since the former has the dominant decay modes $B^*\pi$ and $B\pi^{[36]}$, and the latter lies in the proper mass interval.

Finally, we can use Eq. (36) to predict the masses of the $(b\bar{c})$ states not measured so far. In Table V these predictions are compared with rough quark model-motivated estimate $B_c \simeq (\eta_b + \eta_c)/2$, for vector and tensor mesons.

| $J^PC$ | Eq. (28), MeV | Ref. [34], MeV | Accuracy, % |
|--------|---------------|----------------|-------------|
| 0$^+$  | 1877 ± 1      | 1867 ± 3       | 0.5         |
| 1$^-$  | 2012          | 2008 ± 2       | 0.2         |
| 1$^{++}$ | 2397 ± 6     | 2422 ± 2       | 1.0         |
| 2$^{++}$ | 2450 ± 1      | 2459 ± 3       | 0.4         |

Table I. Comparison of predictions for the masses of the $(c\bar{n})$-mesons given by Eq. (32) with the measured masses provided by the Particle Data Group [36], for four well-established meson multiplets. Electromagnetic corrections are included as uncertainties in the mass values.

| $J^PC$ | Eq. (29), MeV | Ref. [34], MeV | Accuracy, % |
|--------|---------------|----------------|-------------|
| 0$^+$  | 1965.5 ± 1    | 1968.5 ± 0.5   | 0.1         |
| 1$^-$  | 2092          | 2112 ± 0.5     | 0.9         |
| 1$^{++}$ | 2468 ± 8     | 2535           | 2.6         |
| 2$^{++}$ | 2547 ± 2      | 2573.5 ± 1.5   | 1.0         |

Table II. Comparison of predictions for the masses of the $(c\bar{s})$-mesons given by Eq. (33) with the measured masses provided by the Particle Data Group [36], for four well-established meson multiplets.

| $J^PC$ | Eq. (35), MeV | Ref. [34], MeV | Accuracy, % |
|--------|---------------|----------------|-------------|
| 0$^+$  | 5359          | 5325 ± 2       | 0.6         |
| 1$^{++}$ | 5728 ± 1      | 5698 ± 12      | 0.5         |

Table III. Comparison of predictions for the masses of the $(b\bar{n})$-mesons given by Eq. (39) with the measured masses provided by the Particle Data Group [36], for vector and tensor mesons.

| $J^PC$ | Eq. (36), MeV | Ref. [34], MeV | Accuracy, % |
|--------|---------------|----------------|-------------|
| 1$^-$  | 5410          | 5416 ± 3       | 0.1         |
| 2$^{++}$ | 5798 ± 2      | 5853 ± 15      | 0.9         |

Table IV. Comparison of predictions for the masses of the $(b\bar{s})$-mesons given by Eq. (40) with the measured masses provided by the Particle Data Group [36], for vector and tensor mesons.

| $J^PC$ | Eq. (32), MeV | $(\eta_b + \eta_c)/2$, MeV | Discrepancy, % |
|--------|---------------|-----------------------------|---------------|
| 1$^-$  | 6384          | 6278.5                      | 1.6           |
| 2$^{++}$ | 6807          | 6734.5                      | 1.1           |

Table V. Comparison of predictions for the masses of the $(b\bar{c})$-mesons given by Eq. (36) with rough estimate $(\eta_b + \eta_c)/2$, for vector and tensor mesons.
4 Concluding remarks

In this paper we have further explored Regge phenomenology for heavy hadrons initiated in our previous publications \[13, 14\]. We have obtained new non-Gell-Mann–Okubo quadratic meson mass relations which show excellent agreement with experiment (as seen in Tables I-IV, the accuracy of these relations for pseudoscalar, vector and tensor mesons does not exceed 1%). They are

\[
\begin{align*}
6 \eta_n^2 + 3 \eta_c^2 &= 8 D_n^2, \\
6 \eta_s^2 + 3 \eta_c^2 &= 8 D_s^2, \\
6 \eta_c^2 + 3 \eta_b^2 &= 8 B_c^2, \\
20 \eta_n^2 + 5 \eta_b^2 &= 16 B_n^2, \\
20 \eta_s^2 + 5 \eta_b^2 &= 16 B_s^2.
\end{align*}
\]

We have shown that the sum rules (9)-(12), and (13),(14) are easily explained in the framework discussed in this paper.

The values \(N_1 = 2\) and \(N_2 = 4\) that we have suggested predict the following values for the Regge slopes:

\[
\begin{align*}
\alpha'_{c\bar{n}} &\simeq \frac{\alpha'_{n\bar{n}}}{1.5} \simeq 0.60 \text{ GeV}^{-2}, \\
\alpha'_{c\bar{c}} &= \frac{\alpha'_{n\bar{n}}}{2} \simeq 0.45 \text{ GeV}^{-2}, \\
\alpha'_{b\bar{n}} &\simeq \frac{\alpha'_{n\bar{n}}}{2.5} \simeq 0.36 \text{ GeV}^{-2}, \\
\alpha'_{b\bar{c}} &= \frac{\alpha'_{n\bar{n}}}{3} \simeq 0.30 \text{ GeV}^{-2}, \\
\alpha'_{b\bar{b}} &= \frac{\alpha'_{n\bar{n}}}{4} \simeq 0.225 \text{ GeV}^{-2}.
\end{align*}
\]

It is interesting to compare these values with experiment. Since no more than one state is known to lie on each of the heavy meson trajectories, we again, as in Section 2, use the assumption that both \(J^{PC}\) and \((J + 1)^{-P,-C}\) states belong to common trajectory, and calculate the slope by the formula

\[
\alpha'_{ii} = \frac{1}{M^2((J + 1)^{-P,-C}, ii) - M^2(J^{PC}, ii)}.
\]

Our results are shown in Table VI (we assume that radially excited states \(2^3S_1\) and \(2^3P_2\) lie on daughter trajectory which is parallel to the leading one to which \(1^3S_1\) and \(1^3P_2\) belong).

It is seen that the slopes increase as the mass of the state is increased. For \(c\bar{c}\) and \(b\bar{b}\) states, the highest calculated values are not far from the values predicted in the present paper. We may conclude, therefore, that the trajectories are not linear but rather have curvatures in the region of lower spin, and the values for the slopes
predicted in this paper are achieved in the region of higher spin. Similar curvature is also a feature of the pion trajectory. Since quadratic relations that we have obtained in the paper are very accurate even for lower spin states, we conclude that the additivity of the inverse slopes (which is the basis of these relations) is a universal feature of Regge trajectories which does not depend on spin and holds in the curvature region as well.

It is very interesting to determine the actual form of the function \( F(x) \) discussed in this paper. Since

\[
N_1 = \frac{\alpha_{\bar{s}s}}{\alpha_{\bar{c}c}} \equiv F \left( \frac{m(c)}{m(s)} \right) \approx F(3) = 2,
\]

\[
N_2 = \frac{\alpha_{\bar{s}s}}{\alpha_{\bar{b}b}} \equiv F \left( \frac{m(b)}{m(s)} \right) \approx F(9) = 4,
\]

one may fit \( F \) as

\[ F(x) = \frac{x}{3} + 1, \]

i.e.,

\[ \frac{\alpha_{\bar{j}j}}{\alpha_{\bar{j}j}} \approx \frac{1}{3} \frac{m(j)}{m(i)} + 1. \]

Theoretical (or phenomenological) models for the form of this ratio of the slopes \( F(x) \) are called for.

| \( jj \) | \( 1^1S_0 - 1^1P_1 \) | \( 1^3S_1 - 1^3P_2 \) | \( 1^3P_0 - 1^3D_1 \) | \( 2^3S_1 - 2^3P_2 \) | Present paper |
|-------|----------------|----------------|----------------|----------------|----------------|
| \( c\bar{n} \) | 0.418 | 0.496 | | | | 0.60 |
| \( c\bar{s} \) | 0.392 | 0.462 | | | | 0.60 |
| \( c\bar{c} \) | 0.281 | 0.327 | 0.392 | | | 0.45 |
| \( b\bar{n} \) | 0.243 | | | | | 0.36 |
| \( b\bar{s} \) | 0.203 | | | | | 0.36 |
| \( b\bar{b} \) | 0.114 | 0.201 | | | | 0.225 |

**Table VI.** The slopes of the heavy meson trajectories given by Eq. (37), in which pairs of states shown in the Table are used (in GeV\(^{-2}\)), vs. predictions of the present paper.

\(^7\)If one tries, apart from its Goldstone nature, to fit the pion to the linear trajectory on which the \( b_1(1231) \) and \( \pi_2(1670) \) lie [\( \ell = 0.80 \) \( M^2(\ell) - 0.20 \)], extrapolation down to \( \ell = 0 \) gives \( m(\pi) \approx 0.5 \) GeV, much higher than the physical value \( m(\pi) = 0.138 \) GeV, which means that the pion trajectory has curvature near \( \ell = 0 \).
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