Parameter estimation for the FOU(p) process with the same lambda

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Abstract

The FOU(p) processes can be considered as an alternative to ARMA (or ARFIMA) processes to model time series. Also, there is no substantial loss when we model a time series using FOU(p) processes with the same $\lambda$, than using different $\lambda$'s. In this work we propose a new method to estimate the unique value of $\lambda$ in a FOU(p) process. Under certain conditions, we will prove consistency and asymptotic normality. We will show that this new method is more easy and fast to compute. By simulations, we show that the new procedure work well and is more efficient than the general method. Also, we include an application to real data, and we show that the new method work well too and outperforms the family of ARMA(p, q).

Keywords: fractional Brownian motion, fractional Ornstein-Uhlenbeck process, time series. AMS: 62M10
1 Introduction

In [8] the FOU(\(p\)) processes was introduced. The FOU(\(p\)) processes are a continuous time centered and stationary Gaussian process, and is obtained from the iteration of the \(T_\lambda\) functional \(p\) times (in Section 2 can be viewed the explicit definition of this functional) applied to a fractional Brownian motion (fBm), where \(\lambda\) is a positive number. The FOU(\(p\)) processes contains several parameters, for one side \(\sigma\) and \(H\) are the parameters of the fBm (scale and Hurst parameters respectively) and for other side \(\lambda_1, \lambda_2, \ldots, \lambda_q\) are parameters from the application of the functional \(T_\lambda\) \(p\) times, in such way that applying the \(T_\lambda\) functional \(p_i\) times for \(i = 1, 2, \ldots, q\), where \(p_1 + p_2 + \ldots + p_q = p\) (\(p\) is the total number of iterations). More explicitly, the authors proposed to use the notation FOU\(\left(\lambda^{(p_1)}, \lambda^{(p_2)}, \ldots, \lambda^{(p_q)}, \sigma, H\right)\). In the particular case in which \(q = 1\) we have a FOU(\(\lambda^{(p)}, \sigma, H\)) process. The FOU(\(p\)) process has several interesting theoretical properties. First, the fractional Ornstein–Uhlenbeck process defined by [3] is a particular case of FOU(\(p\)) by taking \(p = 1\). Second, when \(H > 1/2\) any FOU(\(p\)) has short range dependence for \(p \geq 2\) and long range dependence for \(p = 1\). Third, as \(p\) grows, the autocorrelation function of the process goes more quickly to zero. Fourth, any FOU(\(p\)) has an explicit and simple formula for the spectral density. Fifth, using a result given in [4], the parameter \(H\) is the local Hölder index of the process. Sixth, if we have observed the process in an equispaced sample of \([0, T]\) where \(T, n \rightarrow +\infty\), under certain conditions between \(T\) and \(n\), it is possible to estimate consistently all the parameters of any FOU(\(p\)). The parameters \(H\) and \(\sigma\) can be estimated by a procedure proposed in [5], and \(\hat{H}\) and \(\hat{\sigma}\) have an explicit formula from the observed data. As a second step, from the formula for the spectral density, it is possible to estimate the \(\lambda_i\) parameters by using a modified Whittle contrast. To obtain these estimators it is necessary to optimize a function that not have an explicit formula, thus the optimum must be found by a numerical optimization procedure. The proof of the above mentioned properties can be found in [8] and [7]. Also, the family of FOU(\(p\)) can be used to model a wide range of time series, including short memory and long memory time series. In [7] can be found three examples of real data modeled by FOU(\(p\)) and your comparison with ARMA, or ARFIMA models, and can be seen the good performance (including measures proposed in [12]) of this new class of models. In this work, in Section 5 we will add a fourth example of real data modeled better than the family of ARMA models. In Section 2 we give a basis to understand the definition of the FOU(\(p\)) processes. In Section 3 we describe the procedure of the parameter estimation of any FOU(\(p\)) model proposed in [8] and [7] (subsections 3.1 and 3.2) and we propose in Subsection 3.3 the main theoretical result of this work, that is to estimate from an explicit formula the parameter \(\lambda\) when we have a FOU(\(\lambda^{(p)}, \sigma, H\)) process, and to prove the consistency and asymptotic normality of \(\hat{\lambda}\). In Section 4 we corroborate the theoretical results (consistency and asymptotic normality) by simulations and we show a comparison between the formula to estimate \(\lambda\) given in Subsection 3.3 with the proposed in [8] and we show an improvement in terms of slightly diminution of the variance of the estimator. In Section 5 we show an application to real data set of the proposed estimation procedure to fit a FOU(\(p\)) model and we
show an improving results than the ARMA family. Our conclusions are given in Section 6 and the proof of the theoretical results proposed in Subsection 3.3 are given in Section 7.

2 Preliminaries

We start recalling the definition of a fractional Brownian motion.

**Definition 1.** A fractional Brownian motion with Hurst parameter \( H \in (0, 1) \), is an almost surely continuous centered Gaussian process \( \{ B_H(t) \}_{t \in \mathbb{R}} \) with

\[
E(B_H(t)B_H(s)) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad t, s \in \mathbb{R}.
\]

We follow with the definition of the iterated Ornstein-Uhlenbeck processes of order \( p \) defined in [8] (FOU(\( p \))).

**Definition 2.** Suppose that \( \{ \sigma B_H(s) \}_{s \in \mathbb{R}} \) is a fractional Brownian motion with Hurst parameter \( H \), and scale parameter \( \sigma \). Suppose further that \( \lambda_1, \lambda_2, ..., \lambda_q \) are distinct positive numbers and that \( p_1, p_2, ..., p_q \in \mathbb{N} \) are such that \( p_1 + p_2 + ... + p_q = p \). We define the iterated Ornstein-Uhlenbeck process of order \( p \) as \( \{ X_t \}_{t \in \mathbb{R}} \) by

\[
X_t := T_{\lambda_1}^{p_1} \circ T_{\lambda_2}^{p_2} \circ ... \circ T_{\lambda_q}^{p_q}(\sigma B_H)(t) = \sum_{i=1}^{q} K_i(\lambda) \sum_{j=0}^{p_i-1} \binom{p_i-1}{j} T_{\lambda_i}^{(j)}(\sigma B_H)(t),
\]

where the numbers \( K_i(\lambda) \) are defined by

\[
K_i(\lambda) = K_i(\lambda_1, \lambda_2, ..., \lambda_q) := \prod_{j \neq i} \frac{1}{(1 - \lambda_j/\lambda_i)}
\]

and the operators \( T_{\lambda_i}^{(j)} \) follows the next formula

\[
T_{\lambda_i}^{(h)}(y)(t) := \int_{-\infty}^{t} e^{-\lambda(t-s)} \left( \frac{-\lambda(t-s)h}{h!} \right)^h dy(s) \quad \text{for} \quad h = 0, 1, 2, ...
\]

When \( h = 0 \) we simply call \( T_{\lambda_i} \), thus

\[
T_{\lambda_i}(y)(t) := \int_{-\infty}^{t} e^{-\lambda(t-s)} dy(s).
\]

**Notation 1.** \( \{ X_t \}_{t \in \mathbb{R}} \sim \text{FOU} \left( \lambda_1^{(p_1)}, \lambda_2^{(p_2)}, ..., \lambda_q^{(p_q)}, \sigma, H \right) \), where \( 0 < \lambda_1 < \lambda_2 < ... < \lambda_q \) or more simply, \( \{ X_t \}_{t \in \mathbb{R}} \sim \text{FOU}(p) \).

**Remark 1.** In [8] can be found the proof of the equality \( T_{\lambda_1}^{p_1} \circ T_{\lambda_2}^{p_2} \circ ... \circ T_{\lambda_q}^{p_q}(\sigma B_H)(t) = \sum_{i=1}^{q} K_i(\lambda) \sum_{j=0}^{p_i-1} \binom{p_i-1}{j} T_{\lambda_i}^{(j)}(\sigma B_H)(t) \) given in Definition 2.
Remark 2. When \( p = 1 \), we obtain a fractional Ornstein–Uhlenbeck process (FOU(\( \lambda, \sigma, H \))).

Throughout this work we will consider the case \( q = 1 \), this is \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda^{(p)}, \sigma, H) \).

Remark 3. As suggested in [7], to model a time series data set from a FOU(\( p \)) process, in several cases may be convenient to standardize the data and then fit FOU(\( p \)) where \( \sigma = 1 \). In this way we avoid the estimation of \( \sigma \). In this work we will use the notation \( \text{FOU}(\lambda^{(p)}, H) \) for any FOU(\( \lambda^{(p)}, \sigma, H \)) where \( \sigma = 1 \).

Remark 4. It is possible to apply the Definition 2, evaluating the functional \( T_{\lambda} \) at another process instead \( \sigma B_H \). Some general properties of this class of processes can be found in [1].

3 Parameter estimation

In [8] it is proposed a method to estimate all the parameters of any FOU(\( p \)) in a consistent way and the estimators of \( \sigma \) and \( H \) have asymptotic Gaussian distribution if the process is observed in a equispaced sample of \([0, T]\). In addition, if the process is observed throughout the interval \([0, T]\), the \( \lambda \)'s parameters also have asymptotic Gaussian distribution. In [7] is proposed a consistent way to estimate the \( \lambda \)'s parameters when the process is observed in an equispaced sample of \([0, T]\). In the FOU(\( \lambda^{(p)}, \sigma, H \)) case, we propose to estimate \( (\sigma, H) \) in the same way as [8], but in the case in which \( 1/2 < H < 3/4 \), we propose a plug-in formula to estimate \( \lambda \) and we will show that this estimator is consistent and has asymptotic Gaussian distribution. Also, we will show by simulations that this estimator has less variance that the one proposed in [7].

3.1 Estimation of \( H \) and \( \sigma \)

We start defining a filter of length \( k + 1 \) an order \( L \).

**Definition 3.** \( a = (a_0, a_1, ..., a_k) \) is a filter of length \( k + 1 \) and order \( L \geq 1 \) if and only if the following conditions hold:

- \( \sum_{i=0}^{k} a_i^l = 0 \) para todo \( 0 \leq l \leq L - 1 \).
- \( \sum_{i=0}^{k} a_i^L \neq 0 \).

Observe that given \( a \) a filter of order \( L \) and length \( k + 1 \), the new filter \( a^2 = (a_0, 0, a_1, 0, a_2, 0, ..., 0, a_k) \) has order \( L \) and length \( 2k + 1 \). Now, we define the quadratic variation of a sample associated to a filter \( a \) as follows. In this work we will use the filters

\[
a_k = \left(-1, \binom{k}{1}, -\binom{k}{2}, ..., (-1)^{k-1}\binom{k}{2}, (-1)^k\binom{k}{1}, (-1)^{k+1}\right).
\]

It is easy to see that \( a_k \) is a filter of order \( k \) and length \( k + 1 \).
Definition 4. Given a filter $a$ of length $k + 1$ and a sample $X_1, X_2, ..., X_n$, we define the quadratic variations associated with filter $a$ by

$$V_{n,a} := \frac{1}{n} \sum_{i=0}^{n-k} \left( \sum_{j=0}^{k} a_j X_{i+j} \right)^2.$$ 

The following theorem defines $\left( \hat{H}, \hat{\sigma} \right)$ and summarizes their asymptotic properties.

Theorem 2 (Kalemkerian & León).

If $X_{\Delta}, X_{2\Delta}, ..., X_{i\Delta}, ..., X_{n\Delta} = X_T$ is an equispaced sample of the process $\{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(p)$ where $H > 1/2$, the filter $a$ is of order $L \geq 2$ and length $k + 1$, $\Delta_n = n^{-\alpha}$ for some $\alpha$ such that $0 < \alpha < \frac{1}{2(H-1)}$ and $T = n\Delta_n \to +\infty$, as $n \to +\infty$. Define

$$\hat{H} = \frac{1}{2} \log_2 \left( \frac{V_{n,a}^2}{V_{n,a}} \right),$$

$$\hat{\sigma} = \left( -\frac{2V_{n,a}}{\Delta_n^2 \sum_{i=0}^{k} \sum_{j=0}^{k} a_i a_j |i-j|^{2H}} \right)^{1/2}. \quad (5)$$

Then

1. $\left( \hat{H}, \hat{\sigma} \right) \overset{a.s.}{\to} (H, \sigma).$

2. $\sqrt{n} \left( \hat{H} - H \right) \overset{w}{\to} N(0, \Gamma_1 (H, \sigma, a)).$

3. $\frac{\sqrt{n}}{\log n} (\hat{\sigma} - \sigma) \overset{w}{\to} N(0, \Gamma_2 (H, \sigma, a)).$

Remark 5. In [7] it is shown that in the case in which $H < 1/2$, the theorem remains valid taking $\alpha > 1/2$.

3.2 Estimation of $\lambda$

In [8] it is obtained an explicit formula for the spectral density of any FOU($p$) and it is proposed a modified Whittle procedure to estimate $\lambda = (\lambda_1, \lambda_2, ..., \lambda_q)$ when the process is observed in the whole interval $[0, T]$ and $H, \sigma$ are known. More explicitly, $\hat{\lambda}_T = \arg \min_{\lambda \in \Lambda} U_T (\lambda)$ where $U_T (\lambda) = \frac{1}{4T} \int_{-\infty}^{+\infty} \left( \log f^{(X)}(x, \lambda) + \frac{I_T(x)}{f^{(X)}(x, \lambda)} \right) w(x) \, dx$ being $I_T(x)$ the periodogram of the second order $I_T(x) = \frac{1}{2\pi T} \left| \int_0^T X_t e^{-itx} \, dt \right|^2$, $f^{(X)}(x) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{T}} \left( \sum_{i=0}^{n-k} \left( \sum_{j=0}^{k} a_j X_{i+j} \right)^2 \right)^{-1/2} & \text{if } x \in [0, T], \\
0 & \text{otherwise.} \end{array} \right.$
\[ \frac{\sigma^2 \Gamma(2H+1) \sin(H\pi)x^2 - 2H}{2\pi \prod_{i=1}^\infty \lambda_i \Gamma(2H+1)} \] is the spectral density of the process, \( \Lambda \) is a compact set and \( w \) be certain weight function. From this procedure and using a result obtained in [10], in [8] can be see the proof of the consistency and asymptotic normality of \( \hat{\lambda}_T \). In [7] it is show that it is possible to take a discretized version of \( U_T \) and \( I_T \) and using \( (\hat{H}, \hat{\sigma}) \) instead the true value of \( (H, \sigma) \), then under certain condition of the weight function \( w \) and the speed in which \( T/n \to 0 \), we have that \( \hat{\lambda}_T \) is consistent. Of course, this method has the drawback in terms of computational cost, due to the lack of explicit formula for the function \( U_T \) and due to the fact that the algorithms to optimize functions beginning with a initial values, and the final estimation can depend of its.

### 3.3 An alternative procedure to estimate \( \lambda \)

In this subsection, we propose to estimate \( \lambda \) from an explicit formula where \( X_1, X_2, \ldots, X_n \) be an equispaced sample of \([0, T]\) of \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda(p), \sigma, H) \) and we will prove consistency and asymptotic normality where \( 1/2 < H < 3/4 \). Two advantages has this procedure with respect to the procedure to estimate \( \lambda \) given in the previous subsection. First is that this procedure is very fast to calculate, and second we obtain consistency and asymptotic normality of the estimator.

In the following proposition, we will show that when \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda(p), \sigma, H) \) the variance of any observation \( X_t \) verify a simple and explicit formula.

**Proposition 1.**

If \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda(p), \sigma, H) \) then

\[ \mathbb{V}(X_t) = \frac{\sigma^2 \Gamma (2H) \prod_{i=1}^{p-1} (i-H)}{(p-1)! \lambda^{2H}} \]  

(7)

where \( \prod_{i=1}^{p-1} (i-H) \) is defined as 1 when \( p=1 \).

From this simple formula and knowing estimate \( \mathbb{V}(X_t), \sigma \) and \( H \), we can obtain an explicit formula to estimate \( \lambda \).

If we call \( \hat{\mu}_2 = \frac{1}{n} \sum_{i=1}^n X_i^2 \), put \( \hat{\mu}_2 \) instead \( \mathbb{V}(X_t) \) and changing \( (\hat{\sigma}, \hat{H}) \) instead \((\sigma, H)\) we can obtain a natural plug-in estimator of \( \lambda \) from

\[ \hat{\lambda} = \left( \frac{\hat{\sigma}^2 \hat{H} \Gamma (2\hat{H}) \prod_{i=1}^{p-1} (i-\hat{H})}{(p-1)! \hat{\mu}_2} \right)^{\frac{1}{2H}} \]  

(8)

Having good asymptotic properties of \( (\hat{\sigma}, \hat{H}) \), from (8), it is natural to find similar properties for \( \hat{\lambda} \). In the next theorem, we will show that when \( 1/2 < H < 3/4 \) and adding a hypothesis about the speed in which \( T \) goes to infinite, we can obtain consistency and asymptotic normality of \( \hat{\lambda} \). The equality \( \hat{\lambda} \) is a generalization of the formula proposed in [2] to estimate the \( \lambda \) parameter in a fractional Ornstein–Uhlenbeck process.
Theorem 3.

If $X_1, X_2, \ldots, X_n$ be an equispaced sample observed in $[0, T]$ of $\{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda^{(p)}, \sigma, H)$, $1/2 < H < 3/4$, $n \left(\frac{T}{n}\right)^k \to 0$, $\frac{T \log^2 n}{n} \to 0$ as $n \to +\infty$ for some $k > 1$, $T \to +\infty$, then
\[ \hat{\lambda} \xrightarrow{a.s.} \lambda \]
and
\[ \sqrt{T} \left(\hat{\lambda} - \lambda\right) \xrightarrow{w} N \left(0, \sigma^2 \alpha(H, \lambda)\right). \]

Remark 6. Observe that $T_n = \log n$ or $T_n = n^\alpha$ where $0 < \alpha < 1/2$ (by taking $k = 2$) verify the conditions requested by Theorem 3.

4 A comparison

In this section we will show a comparison between the performance of the estimator $\hat{\lambda}$ proposed in this work that we will call $\hat{\lambda}_p$ (plug-in) with the estimator proposed in [8] that we will call $\hat{\lambda}_U$ (minimizing the function $U = U_T$). In addition, from the same simulation study, we corroborate the consistency and asymptotic normality of $\hat{\lambda}_p$. Although we didn’t have a theoretical result, we include in Table 1 and Table 4, the results for $H = 0.3$. Tables 1 to 3, show the mean estimation $\left(\hat{\lambda}\right)$, mean error estimation $\left(\hat{\lambda} - \lambda^0\right)$ being $\lambda^0$ the true value of $\lambda$ and deviation of the estimator $\left(\text{sd} \left(\hat{\lambda}\right)\right)$ for $m = 100$ replications, when the observed process is a FOU($\lambda^{(2)}, H$) where $\lambda = 0.8$ and different values of $H$, viewed in $n$ equispaced points of $[0, T]$, for different values of $T$ and $n$. Tables 1 to 3, shows that (for both estimators) the mean error estimation is not necessarily decreasing as $n$ increasing, showing that it is very important the relation between $T$ and $n$. Anyway, in all the cases considered, the mean error estimation take small values. The same occurs with the deviation of the estimators. Table 3 show that when $H = 0.7$, in all the considered cases, we have that $\text{sd} \left(\hat{\lambda}_p\right) < \text{sd} \left(\hat{\lambda}_U\right)$. The same occurs for $H = 0.5$ and $H = 0.3$ for values of $T = 100$ and $T = 50$. In almost all the cases for $H = 0.5$ and $H = 0.7$, the mean error estimation is less for $\left(\hat{\lambda}_p\right)$ than for $\left(\hat{\lambda}_U\right)$. In general, tables 1 to 3, shows better results for $\hat{\lambda}_p$ than for $\hat{\lambda}_U$. Table 4 show the $p$-value of the Truncated Cramér-von Mises test of normality for $\hat{\lambda}_p$ proposed in [6]. For $H = 0.5$ and $H = 0.7$, we non reject normality for all the values of $T$ and $n$ considered (according with our theoretical results). In the $H = 0.3$ case we reject normality only for $T = 25$, but for $T = 50$ and $T = 100$ the test non reject normality. Tables 1 to 4 suggest that for values of $H < 1/2$ it is possible to have consistency and asymptotic normality when $n, T$ goes to infinite where $T/n \to 0$ with certain velocity, this remains as an open problem under which conditions this assertion hold. To estimate $H$, we have
used the Daubechies’ filter of order 2, \( a = \frac{1}{\sqrt{2}}(0.4829629131445341, -0.8365163037378077, 0.2241438680420134, 0.1294095225512603) \).

The results for other filters were similar.

Table 1: Comparison between \( \hat{\lambda}_U \) and \( \hat{\lambda}_p \) as estimators of \( \lambda \). We report the mean estimation \( \bar{\lambda} \), mean error estimation \( \| \hat{\lambda} - \lambda^0 \| \) and deviation \( sd(\hat{\lambda}) \) for a FOU(\( \lambda^2 \),H) viewed in \( n \) equispaced points of \([0, T]\), where \( \lambda = 0.8 \) and \( H = 0.3 \) for \( m = 100 \) replications.

| \( T \) | \( n \) | \( \hat{\lambda}_U \) | \( \hat{\lambda}_p \) | \( \hat{\lambda}_U - \lambda^0 \) | \( \hat{\lambda}_p - \lambda^0 \) | \( sd(\hat{\lambda}_U) \) | \( sd(\hat{\lambda}_p) \) |
|------|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 100  | 1000 | 0.7536         | 0.7416         | 0.0464         | 0.0584         | 0.2185         | 0.2101         |
|      | 5000 | 0.7955         | 0.7874         | 0.0045         | 0.0126         | 0.1671         | 0.1661         |
|      | 10000| 0.8265         | 0.8089         | 0.0265         | 0.0089         | 0.1462         | 0.1244         |
| 50   | 1000 | 0.7713         | 0.7940         | 0.0287         | 0.0060         | 0.2710         | 0.2595         |
|      | 5000 | 0.8249         | 0.7708         | 0.0249         | 0.0292         | 0.2380         | 0.2281         |
|      | 10000| 0.8199         | 0.8022         | 0.0199         | 0.0022         | 0.1983         | 0.2073         |
| 25   | 1000 | 0.7205         | 0.8258         | 0.0795         | 0.0258         | 0.2550         | 0.3027         |
|      | 5000 | 0.8605         | 0.8604         | 0.0605         | 0.0604         | 0.2842         | 0.3284         |
|      | 10000| 0.8742         | 0.8142         | 0.0742         | 0.0142         | 0.2610         | 0.2536         |

Table 2: Comparison between \( \hat{\lambda}_U \) and \( \hat{\lambda}_p \) as estimators of \( \lambda \). We report the mean estimation \( \bar{\lambda} \), mean error estimation \( \| \hat{\lambda} - \lambda^0 \| \) and deviation \( sd(\hat{\lambda}) \) for a FOU(\( \lambda^2 \),H) viewed in \( n \) equispaced points of \([0, T]\), where \( \lambda = 0.8 \) and \( H = 0.5 \) for \( m = 100 \) replications.

| \( T \) | \( n \) | \( \hat{\lambda}_U \) | \( \hat{\lambda}_p \) | \( \hat{\lambda}_U - \lambda^0 \) | \( \hat{\lambda}_p - \lambda^0 \) | \( sd(\hat{\lambda}_U) \) | \( sd(\hat{\lambda}_p) \) |
|------|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 100  | 1000 | 0.7514         | 0.7536         | 0.0486         | 0.0464         | 0.1980         | 0.1400         |
|      | 5000 | 0.7969         | 0.8021         | 0.0481         | 0.0021         | 0.1840         | 0.1330         |
|      | 10000| 0.8159         | 0.8126         | 0.0159         | 0.0126         | 0.1622         | 0.1228         |
| 50   | 1000 | 0.7673         | 0.7880         | 0.0337         | 0.0120         | 0.2633         | 0.1989         |
|      | 5000 | 0.8358         | 0.8309         | 0.0358         | 0.0309         | 0.2132         | 0.1743         |
|      | 10000| 0.8135         | 0.7879         | 0.0135         | 0.0121         | 0.1968         | 0.1666         |
| 25   | 1000 | 0.7331         | 0.7977         | 0.0669         | 0.0023         | 0.2151         | 0.2282         |
|      | 5000 | 0.8541         | 0.8178         | 0.0541         | 0.0178         | 0.2312         | 0.2235         |
|      | 10000| 0.8153         | 0.7963         | 0.0153         | 0.0037         | 0.2850         | 0.2030         |

5 Application to real data

The oxygen saturation in blood of a newborn child has been monitored during seventeen hours. We have observed 304 measures taken at intervals of 200 seconds (\( X_1, X_2, ..., X_{304} \)).
Table 3: Comparison between $\hat{\lambda}_U$ and $\hat{\lambda}_p$ as estimators of $\lambda$. We report the mean estimation ($\hat{\lambda}$), mean error estimation ($|\hat{\lambda} - \lambda^0|$) and deviation ($sd(\hat{\lambda})$) for a FOU($\lambda^{(2)}, H$) viewed in $n$ equispaced points of $[0, T]$, where $\lambda = 0.8$ and $H = 0.7$ for $m = 100$ replications.

| $T$ | $n$ | $\hat{\lambda}_U$ | $\hat{\lambda}_p$ | $|\hat{\lambda}_U - \lambda^0|$ | $|\hat{\lambda}_p - \lambda^0|$ | $sd(\hat{\lambda}_U)$ | $sd(\hat{\lambda}_p)$ |
|-----|-----|---------------------|---------------------|---------------------------------|---------------------------------|---------------------|---------------------|
| 100 | 1000| 0.7209              | 0.7353              | 0.0791                          | 0.0647                          | 0.1890              | 0.1325              |
|     | 5000| 0.7999              | 0.8104              | 0.0001                          | 0.0136                          | 0.1594              | 0.1114              |
|     | 10000| 0.8067             | 0.8012              | 0.0067                          | 0.0012                          | 0.1411              | 0.0932              |
| 50  | 1000| 0.8042              | 0.7703              | 0.0042                          | 0.0297                          | 0.2245              | 0.1815              |
|     | 5000| 0.8379              | 0.8125              | 0.0379                          | 0.0125                          | 0.1868              | 0.1467              |
|     | 10000| 0.7929             | 0.7931              | 0.0071                          | 0.0069                          | 0.1993              | 0.1314              |
| 25  | 1000| 0.8322              | 0.8106              | 0.0322                          | 0.0106                          | 0.3440              | 0.2168              |
|     | 5000| 0.8490              | 0.8322              | 0.0490                          | 0.0322                          | 0.2432              | 0.2030              |
|     | 10000| 0.8410             | 0.8298              | 0.0410                          | 0.0298                          | 0.2319              | 0.1908              |

We have standardized the data set and fitted $\text{FOU}(\lambda^{(p)}, \sigma, H)$ and $\text{FOU}(\lambda^{(p)}, H)$ for $p = 1, 2, 3, 4$ and compared the performance with $\text{ARMA}(p,q)$ for $p,q \in \{0, 1, 2, 3, 4\}$. We measure the performance by taking the lastest 30 and 60 predictions at one step (10% and 20% aproximately of the data set), and computing the quality of the predictions from the mean absolute error of prediction ($\text{MAE}$) for last $m$ observations and their respective predictions, that is,

$$\text{MAE} = \frac{1}{m} \sum_{i=1}^{m} |X_{n-m+i} - \hat{X}_{n-m+i}|$$

where $X(m) := \frac{1}{m} \sum_{i=1}^{m} X_{n-m+i}$, and $X_1, X_2, ..., X_n$ are the real observations, while $\hat{X}_i$ are the predictions given by the model for the value $X_i$. Using this criterion, we obtain that the $\text{ARMA}(1, 1)$ has the better results under the $\text{ARMA}(p, q)$ models being $\text{MAE} = 0.8683$ and $\text{MAE} = 0.6740$ for $m = 30$ and $m = 60$ predictions respectively. Also, the classical techniques to validation the model, resulting in a well adjusted by the $\text{ARMA}(1, 1)$ model. We will show in this section that we can clearly improve the adjusted $\text{ARMA}(1, 1)$ model by taking a $\text{FOU}(\lambda^{(4)}, \sigma, H)$.

According with [7], to fit a $\text{FOU}(p)$ model, previously it is necessary to select a filter $a_k$ and a suitable value of $T$. In this data set, given a filter $a_k$, the value of $\text{MAE}$ for different $\text{FOU}(p)$ models and different values of $T$ are similar. Nevertheless, the performance were different in function of the filter considered. In Figure 1, we show that using $T = 30$, the minimum $\text{MAE}$ for $m = 30$ predictions was reached for the $a_{26}$ filter. According with the theoretical results, see (7), we need to use $T$ large but $T/n$ small, for this reason we report the results for $T = 30$. Anyway, under other values of $T$, the results are similar. About the selection of $T$, although Figure 1 suggest to take values for $T \leq 10$, we have selected $T = 10$ (to avoid the possibility of take $T$ small). In Table 5...
Table 4: p-value for the Truncated Cramér-von Mises test of normality for $\hat{\lambda}$ for the FOU($\lambda^{(2)}, H$) process viewed in $n$ equispaced points of $[0, T]$, where $\lambda = 0.8$, in cases $H = 0.3$, $H = 0.5$ and $H = 0.7$ for $m = 100$ replications.

| $T$  | $n$ | $H = 0.3$ | $H = 0.5$ | $H = 0.7$ |
|------|-----|-----------|-----------|-----------|
| 25   | 1000| 0.001     | 0.177     | 0.942     |
|      | 5000| 0.022     | 0.502     | 0.289     |
|      | 10000| 0.019 | 0.268     | 0.160     |
| 50   | 1000| 0.928     | 0.508     | 0.239     |
|      | 5000| 0.236     | 0.229     | 0.252     |
|      | 10000| 0.354 | 0.358     | 0.490     |
| 100  | 1000| 0.872     | 0.437     | 0.072     |
|      | 5000| 0.704     | 0.198     | 0.201     |
|      | 10000| 0.091 | 0.848     | 0.491     |

we report the results of $MAE$ for 30 and 60 predictions for the different FOU($p$) models considered, using the filter $a_{26}$ and $T = 10$. Table 5 show that FOU($\lambda^{(p)}, H$) and FOU($\lambda^{(p)}, \sigma, H$) (for any $p$) performs similarly (in several cases the difference is until the fivest decimal), slightly better for FOU($\lambda^{(p)}, H$) than FOU($\lambda^{(p)}, \sigma, H$) and clearly outperforms the family of ARMA($p, q$) models. For other side, Figure 2 show that the observed autocorrelation function, is adjusted bad for ARMA(1, 1) and FOU($\lambda^{(p)}, H$) and well for FOU($\lambda^{(p)}, \sigma, H$) being the cases $p = 3$ and $p = 4$ the best models.

![Figure 1](image1.png)

Figure 1: In the left panel, MAE for $m = 30$ predictions for different filters $a_k$ when the data are adjusted by FOU($\lambda^{(2)}, H$) when $T = 30$. The minimum is reached at $k = 26$. In the right panel, MAE for different values of $T$ when the data are adjusted by FOU($\lambda^{(2)}, H$) using the filter $a_{26}$. 

![Figure 2](image2.png)
Figure 2: Autocorrelation function of observed data (black), adjusted ARMA(1, 1) and adjusted FOU($\lambda(p), H$) in the left panel and FOU($\lambda(p), \sigma, H$) in the right panel for $p = 1$ (blue), $p = 3$ (green) and $p = 4$ (purple).

Table 5: MAE for 30 and 60 predictions ($MAE_{30}$ and $MAE_{60}$ respectively) for different FOU($p$) models considered, using $T = 10$ and $\alpha_{26}$.

| Model                  | $H$   | $\hat{\sigma}$ | $\lambda$ | $MAE_{30}$ | $MAE_{60}$ |
|------------------------|-------|-----------------|-----------|------------|------------|
| FOU($\lambda, H$)      | 0.2468| —               | 0.1974    | 0.7434     | 0.6248     |
| FOU($\lambda, \sigma, H$) | 0.2468| 2.3044          | 1.0710    | 0.7509     | 1.0700     |
| FOU($\lambda(2), H$)   | 0.2468| —               | 0.1112    | 0.7434     | 0.6249     |
| FOU($\lambda(2), \sigma, H$) | 0.2468| 2.3044          | 0.6032    | 0.7537     | 0.6280     |
| FOU($\lambda(3), H$)   | 0.2468| —               | 0.0851    | 0.7434     | 0.6250     |
| FOU($\lambda(3), \sigma, H$) | 0.2468| 2.3044          | 0.4619    | 0.7555     | 0.6280     |
| FOU($\lambda(4), H$)   | 0.2468| —               | 0.0715    | 0.7434     | 0.6250     |
| FOU($\lambda(4), \sigma, H$) | 0.2468| 2.3044          | 0.3882    | 0.7567     | 0.6279     |
| ARMA(1, 1)             | —     | —               | —         | 0.8683     | 0.6740     |

Remark 7. The values of $\hat{H}$ and $\hat{\sigma}$ showed in Table 5, are the same in all the models considered, because the estimation of both of them are independent of $p$.

Remark 8. To model a time series from a FOU($p$) process, we have an apparent disadvantage to take only one $\lambda$ than several. Nevertheless in the three real data set worked in [7] we have observed no substantial difference between the performance of FOU($\lambda_1, \lambda_2, \sigma, H$) or FOU($\lambda_1, \lambda_2, \lambda_3, \sigma, H$) than FOU($\lambda(2), \sigma, H$) or FOU($\lambda(3), \sigma, H$) and in the application of this work either.
To conclude this section, we have observed that the FOU($\lambda^3$, $\sigma, H$) and FOU($\lambda^4$, $\sigma, H$) models outperforms clearly the family of the ARMA($p,q$) models.

6 Conclusions

According with Remark 8, there is no substantial loss when we model a time series using FOU($\lambda^p$, $\sigma, H$) processes instead the more general FOU($\lambda^p$) processes. In this work we have proposed a new method to estimate $\lambda$ in a FOU($\lambda^p$, $\sigma, H$) process. We showed that this new method has several advantages. For the one hand, this new method is more easy and fast to compute because it is provenient by an explicit formula. On the other hand, it only requires to have observed the process in a equispaced sample of $[0, T]$, and we have proved consistency and asymptotic normality (at least for $1/2 < H < 3/4$). In this way, we can estimate the three parameters of the model using explicit formulas, avoiding the possible approximation errors of the numerical approximations and estimating more efficiently. By simulations, we show that the new method to estimate $\lambda$ work well and is more efficient than the proposed in [7]. Lastly, we include an application to real data, and we show that the new method work well too and outperforms the familisy of ARMA($p,q$). To finish, we can say that the FOU($p$) processes can be considered as an alternative to ARMA (or ARFIMA) processes to model time series and in this work, we give a way to estimate their parameters efficiently and with desirable asymptotic properties.

7 Proofs

To prove Proposition 1 we need show the following two lemmas.

Lemma 1.

Let $p \geq 2$. The function $g(H)$ defined as

$$g(H) = \frac{(2H - 1)}{\Gamma(2H)} \sum_{i,j=0}^{p-1} \frac{(p-1)(p-1) (-1)^{i+j}}{i!j!} E \left( \int_0^{+\infty} u^i e^{-u} du \int_0^{+\infty} v^j e^{-v} |u-v|^{2H-2} dv \right)$$

is a polynomial of degree $p - 1$ with zeros in $1, 2, ..., p - 1$.

Lemma 2.

Let $p \geq 2$. Then, the function $g(H)$ defined in Lemma 1 is

$$g(H) = \prod_{i=1}^{p-1} (i-H)$$
Proof of Lemma \[7\].

For every \(i, j = 0, 1, 2, \ldots, p - 1\), define

\[
 g_{ij}(H) = \mathbb{E} \left( \int_0^{+\infty} u^i e^{-u} du \int_0^{+\infty} v^j e^{-v} |u - v|^{2H-2} dv \right). \tag{9}
\]

Then

\[
 g(H) = \frac{(2H - 1)}{\Gamma(2H)} \sum_{i,j=0}^{p-1} \frac{(p-1)_i^j (-1)^{i+j}}{i!j!} g_{ij}(H). \tag{10}
\]

If we make \(x = u + v\), \(v = u - v\), we obtain that \(g_{ij}(H) =\)

\[
 \frac{1}{2^{i+j+1}} \int_0^{+\infty} e^{-x} dx \int_0^x (x + y)^i (x - y)^j y^{2H-2} dy + \frac{1}{2^{i+j+1}} \int_0^{+\infty} e^{-x} dx \int_0^x (x + y)^i (x - y)^j (-y)^{2H-2} dy = A_{ij}(H) + B_{ij}(H).
\]

Then

\[
 A_{ij}(H) = \frac{1}{2^{i+j+1}} \int_0^{+\infty} e^{-x} dx \int_0^x \sum_{h=0}^{i} \binom{i}{h} x^h y^{i-h} \sum_{k=0}^{j} \binom{j}{k} x^k (-y)^{j-k} y^{2H-2} dy =
\]

\[
 \frac{1}{2^{i+j+1}} \sum_{h=0}^{i} \binom{i}{h} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \int_0^{+\infty} x^{h+k} e^{-x} dx \int_0^x y^{i+j-h-k+2H-2} dy =
\]

\[
 \frac{\Gamma(i+j+2H)}{2^{i+j+1}} (-1)^j \sum_{h=0}^{i} \binom{i}{h} \sum_{k=0}^{j} \frac{(-1)^k}{i+j-h-k+2H-1}.
\]

\[
 B_{ij}(H) = \frac{1}{2^{i+j+1}} \int_0^{+\infty} e^{-x} dx \int_0^x (x + y)^i (x - y)^j y^{2H-2} dy =
\]

\[
 \frac{1}{2^{i+j+1}} \int_0^{+\infty} e^{-x} dx \int_0^x (x - y)^i (x + y)^j y^{2H-2} dy = A_{ji}(H).
\]

Then

\[
 g(H) = \frac{(2H - 1)}{\Gamma(2H)} \sum_{i,j=0}^{p-1} \frac{(p-1)_i^j (-1)^{i+j}}{i!j!} (A_{ij}(H) + A_{ji}(H)) =
\]

\[
 \frac{(2H - 1)}{\Gamma(2H)} \sum_{i,j=0}^{p-1} \frac{(p-1)_i^j (-1)^i (i+j+2H)}{i!j!2^{i+j}} \sum_{h=0}^{i} \sum_{k=0}^{j} \frac{(-1)^k}{i+j-h-k+2H-1}. \tag{10}
\]
If we replacing in the last equality the expression \( \Gamma(i + j + 2H) \) for \((i + j + 2H - 1)(i + j + 2H - 2) \ldots (1 + 2H) 2H \Gamma(2H) \) we obtain that

\[
g(H) = (2H - 1) \times \\
\sum_{i,j=0}^{p-1} \frac{(p-1)_i}{i!} \frac{(p-1)_j}{j!} (i + j + 2H - 1) \ldots (1 + 2H) 2H \sum_{h=0}^{i} \sum_{k=0}^{j} \frac{(-1)^k}{i+j-h-k+2H-1} \]

where in the case \( i = j = 0 \), the expression

\((i + j + 2H - 1)(i + j + 2H - 2) \ldots (1 + 2H) 2H \Gamma(2H) \)

it means \( \Gamma(2H) \).

Observing that in the case \( i = j = 0 \) we have \( 2H \) and in the rest of summands (where \( i + j \geq 1 \)) we have powers of \( H \). Because for any \( h, k \) the expression \( i + j - h - k + 2H - 1 \) appears in the expansion \((i + j + 2H - 1) \ldots (1 + 2H) \). This concludes the proof that \( g \) is a polynomial.

To prove that \( g \) has degree \( p - 1 \), observe that in the case \( H > 1/2 \) we can write

\[
\sum_{h=0}^{i} \sum_{k=0}^{j} \frac{(i)_h}{h!} \frac{(j)_k}{k!} (-1)^k (i + j - h - k + 2H - 1) = \\
\sum_{h=0}^{i} \sum_{k=0}^{j} \frac{(i)_h}{h!} \frac{(j)_k}{k!} (-1)^{j-k} \\
= \\
\sum_{h=0}^{i} \frac{(i)_h}{h!} \sum_{k=0}^{j} \frac{j!}{k!} (-1)^{j-k} \int_0^1 x^{h+k+2H-2} dx = \\
\sum_{h=0}^{i} \frac{(i)_h}{h!} \int_0^1 x^{h+2H-2} \sum_{k=0}^{j} \frac{j!}{k!} (-1)^{j-k} x^k dx = \\
(-1)^j \sum_{h=0}^{i} \frac{(i)_h}{h!} \frac{\Gamma(h + 2H - 1) j!}{\Gamma(h + j + 2H)}. \tag{12}
\]

Putting (12) in (11) we obtain that

\[
g(H) = (2H - 1) \times \\
\sum_{i,j=0}^{p-1} \frac{(p-1)_i}{i!} \frac{(p-1)_j}{j!} (i + j + 2H - 1) \ldots (1 + 2H) 2H \sum_{h=0}^{i} \frac{(-1)^i}{h!} \frac{\Gamma(h + 2H - 1)}{\Gamma(h + j + 2H)} = \\
(2H - 1) \times \\
\sum_{i,j=0}^{p-1} \sum_{h=0}^{i} \frac{(p-1)_i}{i!} \frac{(p-1)_j}{j!} \frac{(-1)^i}{h!} (i + j + 2H - 1) \ldots (1 + 2H) 2H \\
\sum_{h=0}^{i} \frac{(h)_j}{j!} (h + j + 2H - 1) \ldots (h + 2H - 1). 
\]
Then, $g$ is a polynomial of $p - 1$ degree.

Observe that in the case in which $H < 1/2$, the integral $\int_0^1 x^{h+k+2H-2} \, dx$ does not exist when $h = k = 0$, but the results remain valid if we separate the case $h = k = 0$ and the case $h + k \geq 1$.

To prove that $g(1) = g(2) = \ldots = g(p - 1) = 0$, for values of $H = 1, 2, \ldots, p - 1$ we can develop the binomial formula for $|u - v|^{2H-2}$ and we obtain that

$$E\left(\int_0^{+\infty} u^i e^{-u} \, du \int_0^{+\infty} v^j e^{-v} \, |u - v|^{2H-2} \, dv\right) =$$

$$\sum_{k=0}^{2H-2} \binom{2H-2}{k} E\left(\int_0^{+\infty} u^{i+k} e^{-u} \, du \int_0^{+\infty} v^j e^{2H-2-k} \, dv\right) =$$

$$\sum_{k=0}^{2H-2} \binom{2H-2}{k} (i + k)! (j + 2H - 2 - k)!.$$

(13)

Putting (13) in (9) we obtain that

$$g(H) = \frac{(2H - 1)}{\Gamma(2H)} \sum_{i,j=0}^{p-1} \frac{(p-1)!}{i! j!} \sum_{k=0}^{2H-2} \binom{2H-2}{k} (i + k)! (j + 2H - 2 - k)! =$$

$$\frac{(2H - 1)}{\Gamma(2H)} \sum_{k=0}^{2H-2} \binom{2H-2}{k} \sum_{i=0}^{p-1} \frac{(p-1)!}{i!} (-1)^i \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} (j + 2H - 2 - k)!.$$

Therefore, it is enough to show that

$$\sum_{i=0}^{p-1} \frac{(p-1)!}{i!} (-1)^i = 0 \text{ for every } H = 1, 2, \ldots, p - 1 \text{ and } k = 0, 1, 2, \ldots, 2H - 2.$$

This result it follows from the binomial formula of $\alpha(x) = (1 - x)^{p-1}$ and using that $\alpha(1) = \alpha'(1) = \ldots = \alpha^{(p-1)}(1) = 0$.

This concludes the proof that $g(1) = g(2) = \ldots = g(p - 1) = 0$.

Proof of Lemma 2.

From (10) we deduce that $g(0) = 1$, therefore the corollary follows immediately from Lemma 1. 

Proof of Lemma 2.

From (10) we deduce that $g(0) = 1$, therefore the corollary follows immediately from Lemma 1.
Proof of Proposition 7

It is enough to consider the case $p \geq 2$, because when $p = 1$ we have that (7) is the well known variance of a fractional Ornstein-Uhlenbeck process. If $\{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda(p), \sigma, H)$ then $X_t = \sigma \sum_{i=0}^{p-1} (p-1)_i T^{(i)}_\lambda (B_H) (t)$ where $\{B_H (t)\}_{t \in \mathbb{R}}$ is a fractional Brownian motion with Hurst parameter $H$ and the operators $T^{(i)}_\lambda$ are defined in (2), thus, it is enough to prove the formula in the case in which $\sigma = 1$. Therefore, if $\{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda(p), 1, H)$, then

$$V(X_t) = \mathbb{E} \left( X_t^2 \right) = \mathbb{E} \left( \sum_{i,j=0}^{p-1} \binom{p-1}{i} \binom{p-1}{j} T^{(i)}_\lambda (B_H) (0) T^{(j)}_\lambda (B_H) (0) \right) =$$

$$\mathbb{E} \left( \sum_{i,j=0}^{p-1} \binom{p-1}{i} \binom{p-1}{j} \int_{-\infty}^{0} \frac{(\lambda w)^i}{i!} e^{\lambda w} dB_H (w) \int_{-\infty}^{0} \frac{(\lambda z)^j}{j!} e^{\lambda z} dB_H (z) \right) =$$

$$\mathbb{E} \left( \left| \frac{\lambda^2}{2} \right| \sum_{i,j=0}^{p-1} \binom{p-1}{i} \binom{p-1}{j} \int_{-\infty}^{0} \frac{(\lambda w)^i}{i!} e^{\lambda w} dB_H (w) \int_{-\infty}^{0} \frac{(\lambda z)^j}{j!} e^{\lambda z} |w - z|^{2H - 2} dz \right).$$

(14)

The last equality in (14) is due to the following formula, whose proof can be seen in [11]: if $H \in (1/2, 1)$ and $f, g \in \left\{ f : \mathbb{R} \to \mathbb{R} : \int \int_{\mathbb{R}^2} |f(u)f(v)| |u - v|^{2H - 2} dudv < +\infty \right\}$,

then

$$\mathbb{E} \left( \int_{-\infty}^{+\infty} f(u) dB_H (u) \int_{-\infty}^{+\infty} g(v) dB_H (v) \right) = \int_{-\infty}^{+\infty} f(u) du \int_{-\infty}^{+\infty} g(v) |u - v|^{2H - 2} dv.$$

(15)

If we change $\lambda w = -u$ and $\lambda z = -v$ we obtain that (14) is equal to

$$\frac{H(2H - 1)}{\lambda^{2H}} \sum_{i,j=0}^{p-1} \binom{p-1}{i} \binom{p-1}{j} (-1)^{i+j} \int_{0}^{+\infty} w^i e^{-u} du \int_{0}^{+\infty} v^j e^{-v} |u - v|^{2H - 2} dv =$$

$$\frac{H \Gamma (2H)}{\lambda^{2H}} g(H)$$

where $g(H)$ is the function defined in Lemma 1. From Lemma 2 we obtain the result. This concludes the proof.
Proof of Theorem 3

Throughout this theorem we will call $\lambda^0, \sigma^0$ and $H^0$ the true value of the parameters, also we will call $\mu_2^0$ the true value of the $V(X_t)$ given in (7).

Observe that $\hat{\lambda} = G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)$ where $G(\sigma, H, \mu_2) = \left(\frac{\sigma^2 H^0 (2H) \prod_{i=1}^{n-1}(4-H)}{(\mu_2 - 1)^2}\right)^{1/2}$.

From the ergodic theorem we know that $\frac{1}{T} \int_0^T X_t^2 dt \xrightarrow{a.s.} \mu_2^0$.

Any FOU (FOU) or Gaussian process with Hölder index $H$, then, the conditions $1/2 < H < 3/4$ and $n \left(\frac{T}{n}\right)^k \to 0$ as $n \to +\infty$ for some $k > 1$ allows to affirm that $\sqrt{T} \left(\frac{1}{T} \int_0^T X_t^2 dt - \tilde{\mu}_2\right) \xrightarrow{p} 0$ (Lemma 8 in [9]), thus $\tilde{\mu}_2 \xrightarrow{p} \mu_2^0$.

From continuity of $G$ we obtain immediately that $\hat{\lambda} = G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2) \xrightarrow{a.s.} G(\sigma^0, H^0, \mu_2^0) = \lambda^0$.

Applying the mean value theorem we have $G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2) - G(\sigma^0, H^0, \mu_2^0) = \nabla G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2) \cdot \left(\hat{\sigma} - \sigma^0, \hat{H} - H, \tilde{\mu}_2 - \mu_2^0\right)$ where $\left(\hat{\sigma}, \hat{H}, \tilde{\mu}_2\right) \in \left[(\hat{\sigma}, \hat{H}, \tilde{\mu}_2), (\sigma^0, H^0, \mu_2^0)\right]$. Then

$$\sqrt{T} \left(\hat{\lambda} - \lambda^0\right) = \sqrt{T} \left(\frac{\partial G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)}{\partial \hat{\sigma}} (\hat{\sigma} - \sigma^0) + \frac{\partial G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)}{\partial \hat{H}} (\hat{H} - H) + \frac{\partial G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)}{\partial \tilde{\mu}_2} (\tilde{\mu}_2 - \mu_2^0)\right).$$

Observe that the derivatives of $G$ with respect to $\sigma, H$ and $\mu_2$ are bounded in a neighbourhood of $(\sigma^0, H^0, \mu_2^0)$.

From Theorem 2 and condition $T \frac{\log^2 n}{n} \to 0$ as $n \to +\infty$, we have that $\sqrt{T} \frac{\partial G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)}{\partial \hat{\sigma}} (\hat{\sigma} - \sigma^0) \xrightarrow{p} 0$ and $\sqrt{T} \frac{\partial G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)}{\partial \hat{H}} (\hat{H} - H) \xrightarrow{p} 0$. Therefore, the asymptotic distribution of $\sqrt{T} \left(\hat{\lambda} - \lambda^0\right)$ is the same as that of $\sqrt{T} \frac{\partial G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)}{\partial \tilde{\mu}_2} (\tilde{\mu}_2 - \mu_2^0) = \sqrt{T} \frac{\partial G(\hat{\sigma}, \hat{H}, \tilde{\mu}_2)}{\partial \tilde{\mu}_2} \left(\tilde{\mu}_2 - \mu_2^0\right)$ and $\frac{1}{T} \int_0^T X_t^2 dt \to \mu_2^0$.

\qed
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