On entanglement entropy in ’t Hooft model

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We use the replica trick approach to calculate, in the large-$N$ limit, the entanglement $c$-function in two-dimensional quantum chromodynamics with the gauge group $U(N)$ and quark in the fundamental representation (’t Hooft model). We show explicitly that the result is equal to a sum (with certain numerical coefficients) of entanglement $c$-functions of free massive scalars, which represent mesons in spectrum of the ’t Hooft model. Interestingly, each meson contributes as a point-like particle. We provide an explanation for why this is the case.
1. Introduction

The subject of entanglement entropy in quantum field theory received a considerable attention over the past several years. A general physical setup is the following. One takes a system in the vacuum state $|0\rangle$, and specifies a sub-region $\mathcal{A}$ in the coordinate space. This way the whole system is split into sub-region $\mathcal{A}$ and its complement $\overline{\mathcal{A}}$.\footnote{Below in this section we discuss subtlety of such splitting in systems with gauge interaction.} The state of the subsystem $\mathcal{A}$ is described by a density matrix $\rho$, obtained by tracing out the degrees of freedom of the subsystem $\overline{\mathcal{A}}$,

$$\rho = \text{Tr}_{\overline{\mathcal{A}}} |0\rangle\langle 0|.$$ \hspace{1cm} (1.1)

The problem is to find corresponding entanglement entropy

$$S = -\text{Tr} \rho \log \rho,$$ \hspace{1cm} (1.2)

where trace is taken over the states of $\mathcal{A}$.

One way to calculate entanglement entropy (1.2) is a replica trick \cite{1,2,3}. The simplest system, which this method can be applied to, is a single interval in 2d conformal field theory. If $L$ is a length of the interval, and $c$ is a central charge of the CFT, then entanglement entropy is given by \cite{1,2,3}

$$S_{\text{CFT}} = \frac{c}{3} \log \frac{L}{\epsilon},$$ \hspace{1cm} (1.3)

where $\epsilon$ is a UV cutoff. The entanglement $c$-function for single interval in 2d field theory is defined as

$$c_E = L \frac{dS}{dL},$$ \hspace{1cm} (1.4)

and due to (1.3), for CFT it is equal to

$$c_E = \frac{c}{3}.$$ \hspace{1cm} (1.5)

Moving away from the conformal regime quickly complicates calculation of the entanglement entropy. The question which one can ask is what is the entanglement entropy of a single interval in the theory of free 2d scalar, or fermion, with a mass $m$. In both cases entanglement entropy is expressed in terms of solution of the Painlevé V equation. This solution is known numerically for general $mL$, and is known analytically only in asymptotic UV and IR regions \cite{4,5,6}.
It becomes more challenging to find entanglement entropy in interacting quantum field theory. One of the computational tools has been suggested by Ryu and Takayanagi [7,8], and it gives a prescription to find entanglement entropy in strongly interacting quantum field theory, which has a known gravity dual. The other, purely field theoretical, method involves perturbative computation of entanglement entropy, see, e.g., [9,10] and references therein, for recent developments.

In this paper we use the replica trick approach to calculate entanglement $c$-function in two-dimensional quantum chromodynamics with a single-flavored quark in the fundamental representation of the gauge group $U(N)$. This system is known as 't Hooft model [11]. It is solvable in the large-$N$ limit. The spectrum of the 't Hooft model is given by a tower of mesons, with masses of sufficiently heavy mesons approximated by the formula

$$m_k^2 \simeq \pi^2 \lambda k,$$

where $\lambda$ is the 't Hooft coupling [11,12], see [13] for a review. The mesons are non-interacting in the large-$N$ limit.

Entanglement entropy in pure $U(N)$ gauge theory in two dimensions has been calculated and discussed, e.g., in [14,15,16]. There is a subtlety in defining entanglement entropy for systems with gauge interaction. One way to see the problem is to look at gauge-invariant operators of a finite size. The question is, how to treat the gauge-invariant operators which are partly located in $\mathcal{A}$, and partly in $\overline{\mathcal{A}}$. Following [14,15,16] we assume that entanglement entropy is well-defined in 2$d$ field theory with gauge interaction, and replica trick calculation is applicable, and gives a sensible answer. (See also [17] and references therein for discussion of entanglement entropy in gauge theories.)

Each meson in the spectrum of the 't Hooft model can be viewed as a string of a given tension. Length of such a string increases as mass of the meson grows. The subtlety in defining entanglement entropy arises, e.g., when we look at string which has one end inside the interval $\mathcal{A}$, and the other end outside of the interval. At the same time, due to having a finite length, one would expect that mesons contribute to the entanglement entropy as finite-size objects. However, our replica trick calculation shows that each meson contributes to the entanglement $c$-function as a point-like particle. We explain that this is a consequence of a consistent definition of [16] of entanglement entropy in gauge theories.

In two dimensions entanglement $c$-function (1.4) is a monotonically decreasing function of the (dimensionless) length of the interval [18]. Therefore it can be compared with the
Zamolodchikov c-function. The latter is defined via the two-point function of the stress-energy tensor [19,20]. Schematically, it is given by

\[ C(|z|) \simeq z^4 \langle T(z)T(0) \rangle. \quad (1.7) \]

In ’t Hooft model we have qualitative behavior of Zamolodchikov c-function given by \(^2\)

\[ C(\sqrt{\lambda}|z|) = NF(\sqrt{\lambda}|z|), \quad (1.8) \]

where \( F \) is a smooth function of order one, monotonically decreasing between one in the ultra-violet and zero in the infra-red. At a fixed point Zamolodchikov c-function is equal to central charge of the corresponding CFT. Similarly, entanglement c-function of CFT is proportional to the central charge (1.5).

Due to these similarities between entanglement and Zamolodchikov c-functions, we expect the entanglement c-function for the ’t Hooft model to behave in a way, similar to (1.8). However, our calculation demonstrates a different behavior. In agreement with [18] we have obtained a monotonically decreasing entanglement c-function. We have derived that entanglement c-functions is \( c_E(\sqrt{\lambda}L) = O(1) \) for \( \lambda L^2 \gg 1/N \), unlike the Zamolodchikov c-function (1.8). When \( \lambda L^2 = 1/N \), we obtained that the entanglement c-function approaches the value of \( O(N) \). In this regime interaction of quarks is parametrically suppressed, and perturbative calculation of entanglement entropy should be done.\(^3\) We leave this for future work.

As a warm-up, we start by analyzing a simpler theory with four-fermionic interaction, the Thirring model. This model is well known to be equivalent to a free massless scalar. This can be immediately demonstrated by bosonization of fermions. Therefore the single-interval entanglement entropy in Thirring model is given by the CFT formula (1.3) with central charge \( c = 1 \). We choose to postpone bosonization, and apply the replica trick approach to calculation of entanglement entropy in the Thirring model. We explicitly show that CFT result (1.3) for free complex-valued fermion (\( c = 1 \)) is not changed by Thirring four-fermionic interaction.

\(^2\) See App. B of [21] for results on two-point functions of general bi-fermionic operators in the ’t Hooft model.

\(^3\) See, e.g., [22,23] for examples of perturbative replica trick calculation of entanglement entropy in 2d quantum field theory.
Before proceeding to the 't Hooft model, we study a simpler gauge theory in two dimensions, Schwinger model of a fermion interacting with a $U(1)$ gauge field. Gauge field in two dimensions is non-dynamical and can be explicitly integrated over. This generates a non-local four-fermionic interaction. Unlike the case of Thirring model, in this case we have a relevant interaction, and it affects the result for the entanglement entropy.

It is well known that Schwinger model is equivalent to free massive scalar. We assume that single-interval entanglement entropy of Schwinger model maps to single-interval entanglement entropy of massive scalar field theory. It is not obvious that such an assumption is correct, because of non-local nature of duality between Schwinger model and massive scalar field [16]. Similar subtlety in relation between entanglement entropy in bosonic and fermionic pictures can appear even in CFT. It appears already in the case of multiple-interval entanglement entropy, which is not defined by simple expression analogous to (1.3), in CFTs of free boson and free fermion [24]. Careful treatment shows that the results for multiple-interval entanglement entropy do indeed agree for the theories, related by bosonization [24].

We match the expression for the entanglement entropy in Schwinger model with the result of [5] for massive scalar field. This gives an expression, which will be useful in calculation of entanglement entropy in the 't Hooft model. We show that calculation of entanglement entropy in the 't Hooft model and the Schwinger model goes through similar steps. By explicit calculation we show that single-interval entanglement $c$-function in the 't Hooft model (in the large-$N$ limit) is given by a sum (with certain numerical coefficients) of entanglement $c$-functions of free scalars with masses $m_k$ in the spectrum of the 't Hooft model.

The rest of this paper is organized as follows. In Section 2 we summarize the formulae for entanglement entropy, and outline the general calculation procedure which we will follow. We start with the CFT of free fermions, and then proceed to deforming the CFT by an arbitrary operator. We discuss in general how such an operator contributes to the single-interval entanglement entropy. In Section 3 we calculate entanglement entropy in the Thirring model, testing the formulae derived in Section 2. In Section 4 we discuss entanglement entropy in the Schwinger model, and derive expression, which will be useful for calculation in the 't Hooft model. In Section 5 we derive expression for entanglement entropy in the 't Hooft model. We discuss our results in Section 6, where we provide an argument for why each meson, now matter how heavy it is, contributes to the entanglement entropy as a point-like particle. Appendix A is dedicated to derivation of gap equation governing dressed fermionic propagator in the 't Hooft model. In Appendix B we review derivation of quark-antiquark scattering amplitude in the 't Hooft model.
2. Preliminaries

In Subsection 2.1 we collect the defining formulae for entanglement entropy which we will need in this paper. In Subsection 2.2 we discuss free massless fermions on $n$-sheeted Riemann surface. In Subsection 2.3 we perturb the CFT of free fermions by operator $\mathcal{O}$ and discuss how it contributes to the entanglement entropy.

2.1. Entanglement entropy

Consider $2d$ system in a vacuum state. Suppose the subsystem $A$ is a spatial interval of length $L$. Denote $\rho$ to be density matrix obtained by tracing out all degrees of freedom outside of the $A$. One can compute the trace $\text{Tr} \rho^n$ for positive integer values of $n$. This quantity enters definition of the Rényi entropy

$$S_n = \frac{1}{1-n} \log \text{Tr} \rho^n. \quad (2.1)$$

Then, following the replica trick, one analytically continues the expression (2.1) to any real value of $n$, and calculates entanglement entropy using the formula

$$S = -\text{Tr} \rho \log \rho = -\frac{\partial}{\partial n} \log \text{Tr} \rho^n \bigg|_{n=1} = \lim_{n \to 1} S_n. \quad (2.2)$$

Evaluation of entanglement entropy therefore gets reformulated as a problem of finding the Rényi entropy (2.1). To calculate it, let us put the system on $n$-sheeted Riemann surface $\mathcal{R}_n$. Different sheets of $\mathcal{R}_n$ are glued at the cut $A$, and boundary conditions across the cut are specified. Then the trace $\text{Tr} \rho^n$ can be determined as [1,2,3]

$$\text{Tr} \rho^n = \frac{Z_n}{Z_1}, \quad (2.3)$$

where $Z_n$ is partition function on $\mathcal{R}_n$.

Due to (2.1), (2.3), the Rényi entropy can be determined as

$$S_n = \frac{1}{1-n} (\log Z_n - n \log Z_1). \quad (2.4)$$

The entanglement entropy is therefore given by

$$S = \log Z_1 - \frac{1}{Z_1} \frac{\partial Z_n}{\partial n} \bigg|_{n=1}. \quad (2.5)$$
For the future purposes it is useful to express the partition function as

\[ Z_n = \frac{1}{L^{2N_\Delta_n}} Z(n), \tag{2.6} \]

where \( \Delta_1 = 0 \) and \( \frac{\partial \Delta_n}{\partial n} \big|_{n=1} = \frac{1}{6} \). Then entanglement entropy is

\[ S = \frac{N}{3} \log L + \log Z_1 - \frac{1}{Z_1} \frac{\partial Z(n)}{\partial n} \bigg|_{n=1}. \tag{2.7} \]

To get rid of the logarithm \( \log Z_1 \) in (2.7) one usually defines entanglement \( c \)-function

\[ c_E = L \frac{d}{dL} S = \frac{N}{3} - L \frac{d}{dL} \frac{1}{Z_1} \frac{\partial Z(n)}{\partial n} \bigg|_{n=1}. \tag{2.8} \]

Suppose we start with conformal field theory with central charge equal to \( N \). Then if we perturb it by some interaction, the entanglement entropy will be given by (2.7), where the \( \frac{N}{3} \log L \) term is a CFT input [3], and \( Z(n) \) term originates from non-trivial interaction.

### 2.2. Free massless fermions

Let us consider a free massless fermionic theory on \( n \)-sheeted Riemann surface \( \mathcal{R}_n \). We have left-moving (holomorphic) fermion \( \Psi = \Psi_1 + i\Psi_2 \) and right-moving (anti-holomorphic) fermion \( \overline{\Psi} = \overline{\Psi}_1 + i\overline{\Psi}_2 \), with the free field action

\[ S = \int_{\mathcal{R}_n} d^2w \left( \overline{\Psi}^* \partial \Psi + \overline{\Psi} \partial \overline{\Psi} \right). \tag{2.9} \]

In our notation bar denotes chirality of the fermion, and star denotes complex conjugation. The integral in (2.9) is taken over \( n \)-sheeted Riemann surface \( \mathcal{R}_n \) with the branch points \( w_{1,2} \), and the sheets are glued along the cut \([w_1, w_2] \). Without loss of generality, we choose \( w_1 = 0 \), \( w_2 = L \), and take \( L \) to be real-valued. Instead of considering fermions \( (\Psi, \overline{\Psi}) \) \( \mathcal{R}_n \), consider \( n \) fermions \( (\Psi_j, \overline{\Psi}_j), j = 1, \ldots, n \), living on a complex \( z \)-plane [3]. The action on the complex plane is

\[ S = \int d^2z \sum_{j=1}^n \left( \Psi_j^* \overline{\psi}_j + \overline{\psi}_j \partial \Psi_j \right), \tag{2.10} \]

and we must insert twist operators \( T(0) \tilde{T}(L) \) into all correlation functions. Before describing the action of the twist operators, let us do a conformal transformation of a complex plane,

\[ u = \frac{z - L}{z}, \tag{2.11} \]
which maps twist operators to the points \( u = 0 \) and \( u = \infty \). These twist operators lead to the boundary conditions on fermions [4]

\[
\begin{align*}
\left( \Psi_j \right) \rightarrow \left( \Psi_{j+1} \right), & \quad \left( \Psi_n \right) \rightarrow (-1)^{n+1} \left( \Psi_1 \right) \\
\left( \Psi_j \right) \rightarrow \left( \Psi_{j-1} \right), & \quad \left( \Psi_1 \right) \rightarrow (-1)^{n+1} \left( \Psi_n \right)
\end{align*}
\]  

(2.12)

when one circles clockwise around \( u = 0 \), and

\[
\begin{align*}
\left( \Psi_j \right) \rightarrow \left( \Psi_{j+1} \right), & \quad \left( \Psi_n \right) \rightarrow (-1)^{n+1} \left( \Psi_1 \right) \\
\left( \Psi_j \right) \rightarrow \left( \Psi_{j-1} \right), & \quad \left( \Psi_1 \right) \rightarrow (-1)^{n+1} \left( \Psi_n \right)
\end{align*}
\]  

(2.13)

when one circles around \( u = \infty \) (i.e., when one circles around \( u = 0 \) counter-clockwise). The complex \( u \)-plane can be mapped onto a cylinder with world-sheet coordinates \((\tau, \sigma)\):

\[
u = e^{\tau + i\sigma}.
\]  

(2.14)

The twist operators are inserted at \( \tau = \pm \infty \), at infinite past and infinite future. The coordinate \( \sigma \in [0, 2\pi] \) parametrizes circumference of the cylinder.

Let us switch to diagonal basis of twist operators, by doing a discrete Fourier transform

\[
\Psi_j = \sum_{s=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{\sqrt{n}} e^{2\pi ijs/n} \psi_s, \quad \Psi_j = \sum_{s=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{\sqrt{n}} e^{2\pi ijs/n} \overline{\psi}_s.
\]  

(2.15)

In the basis \( \psi_s, \overline{\psi}_s, s = -\frac{n-1}{2}, \ldots, \frac{n-1}{2} \) the boundary conditions are

\[
\left( \frac{\psi_s}{\psi_s} \right) \rightarrow e^{2\pi is/n} \left( \frac{\psi_s}{\psi_s} \right)
\]  

(2.16)

when one circles around \( u = 0 \) (goes around circumference of the cylinder), and

\[
\left( \frac{\psi_s}{\psi_s} \right) \rightarrow e^{-2\pi is/n} \left( \frac{\psi_s}{\psi_s} \right)
\]  

(2.17)

when one circles around \( u = \infty \) (goes around circumference of the cylinder in the opposite direction). The action for fermions \((\psi_s, \overline{\psi}_s)\) is

\[
S = \int d^2 z \sum_{s=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left[ \psi^*_s \overline{\psi}_s + \overline{\psi}_s^* \overline{\psi}_s \right].
\]  

(2.18)
Fermions, satisfying the boundary conditions (2.16), (2.17), have the mode expansion

\[
\psi_s^*(u) = \sum_{r \in \mathbb{Z} + 1/2} \frac{\psi_s^{(r)} r + \frac{1}{2}}{u^{r + \frac{1}{2} + \frac{1}{2}}} , \quad \psi_s(u) = \sum_{r \in \mathbb{Z} + 1/2} \frac{\psi_s^{(r)} r - \frac{1}{2}}{u^{r -\frac{1}{2} + \frac{1}{2}}},
\]

\[
\overline{\psi}_s^*(\overline{u}) = \sum_{r \in \mathbb{Z} + 1/2} \frac{\overline{\psi}_s^{(r)} r - \frac{1}{2}}{\overline{u}^{r - \frac{1}{2} + \frac{1}{2}}} , \quad \overline{\psi}_s(\overline{u}) = \sum_{r \in \mathbb{Z} + 1/2} \frac{\overline{\psi}_s^{(r)} r + \frac{1}{2}}{\overline{u}^{r + \frac{1}{2} + \frac{1}{2}}},
\]

(2.19)

where the amplitudes of expansion satisfy anti-commutation relations [25,20]

\[
\{\psi_{s_1}^{(r)} r + s_1/n, \overline{\psi}_{s_2}^{(r)} r - s_2/n\} = \delta_{s_1,s_2} \delta_{r_1,-r_2} , \quad \{\overline{\psi}_{s_1}^{(r)} r - s_1/n, \psi_{s_2}^{(r)} r + s_2/n\} = \delta_{s_1,s_2} \delta_{r_1,-r_2}
\]

(2.20)

and the vacuum is defined as 4

\[
\psi_{s_1}^{(r)} r + s_1/n \langle 0 \rangle = 0 , \quad \psi_{s_1}^{(r)} r + s_1/n \langle 0 \rangle = 0 , \quad r > 0 , \quad \overline{\psi}_{s_2}^{(r)} r + s_2/n \langle 0 \rangle = 0 , \quad \overline{\psi}_{s_2}^{(r)} r + s_2/n \langle 0 \rangle = 0 , \quad r > 0,
\]

(2.21)

for all s. Using equations (2.19), (2.20), (2.21) we compute the correlation functions

\[
\langle \psi_{s_1}^{(r)}(u_1) \psi_{s_2}(u_2) \rangle = \delta_{s_1,s_2} \left( \frac{u_2}{u_1} \right)^{\frac{1}{n}} \frac{1}{u_1 - u_2} .
\]

(2.22)

For the anti-holomorphic sector we obtain

\[
\langle \overline{\psi}_{s_1}^{(r)}(\overline{u}_1) \overline{\psi}_{s_2}(\overline{u}_2) \rangle = \delta_{s_1,s_2} \left( \frac{\overline{u}_1}{\overline{u}_2} \right)^{\frac{1}{n}} \frac{1}{\overline{u}_1 - \overline{u}_2} .
\]

(2.23)

We can also arrive at the correlation functions (2.22), (2.23) by computing the limit of four-point functions of fermions and twist operators

\[
\lim_{R \to \infty} R^{2\Delta_s} \langle \sigma_{s_1}^{(r)}(0) \psi_{s}^{(r)}(u_1) \psi_{s}(u_2) \sigma_{s}(R) \rangle
\]

\[
= \frac{1}{u_1 - u_2} \lim_{R \to \infty} \left( \frac{u_2}{u_1 - R} \right)^{\frac{1}{n}} \frac{1}{u_1 - u_2} \left( \frac{u_2}{u_1} \right)^{\frac{1}{n}} .
\]

(2.24)

In (2.24) the \( \Delta_s = s^2/(2n^2) \) is dimension of twist operator \( \sigma_{s}(u) = e^{i\frac{2}{n} \phi_{s}(u)} \). We bosonized the fermions, \( \psi_s(u) = e^{i\phi_{s}(u)} \), where \( \langle \phi_{s}(u_1) \phi_{s}(u_2) \rangle = - \log (u_1 - u_2) \). Similarly one can derive correlation function for anti-holomorphic fermions (2.23).

\[\footnote{4}{In our case s/n is never equal to 1/2, so we do not have to worry about zero modes: \( \psi_{s}(0) \) never comes up.}\]
For the future purposes we also are going to need correlation function for \( \Psi_j \) fermions. Due to (2.15) and (2.22) we obtain

\[
\langle \Psi_j^* (u_1) \Psi_{j_2} (u_2) \rangle = (-1)^{j_1 + j_2} \frac{1}{n} e^{\frac{\pi i j_1}{n} u_1^{\frac{1}{n}} - \frac{1}{2} e^{\frac{\pi i j_2}{n} u_2^{\frac{1}{n}} - \frac{1}{2}}}. \tag{2.25}
\]

Similarly, for anti-holomorphic fermions we obtain

\[
\langle \overline{\Psi}_j^* (\overline{u}_1) \overline{\Psi}_{j_2} (\overline{u}_2) \rangle = (-1)^{j_1 + j_2} \frac{1}{n} e^{-\frac{\pi i j_1}{n} \overline{u}_1^{\frac{1}{n}} - \frac{1}{2} e^{-\frac{\pi i j_2}{n} \overline{u}_2^{\frac{1}{n}} - \frac{1}{2}}}. \tag{2.26}
\]

### 2.3. Interacting fermions

Suppose CFT of free fermions is deformed by the interaction

\[
S_{int} = g \int d^2 z \mathcal{O}(z, \overline{z}), \tag{2.27}
\]

where \( \mathcal{O} \) is a composite operator of \( \Psi, \overline{\Psi} \), and \( g \) is corresponding coupling constant. Suppose \( \mathcal{O} \) in (2.27) is operator of dimensions \((\Delta, \overline{\Delta})\). Then conformal transformation of \( \mathcal{O} \) on complex plane, \( w = f(z) \), is given by

\[
\left( \frac{\partial f(z)}{\partial z} \right)^\Delta \left( \frac{\overline{\partial f(\overline{z})}}{\overline{\partial \overline{z}}} \right)^{\overline{\Delta}} \mathcal{O}(f(z), \overline{f(\overline{z})}) = \mathcal{O}(z, \overline{z}). \tag{2.28}
\]

We can study the model (2.27) on Riemann surface \( \mathcal{R}_n \) by considering \( n \) copies of the original model on complex \( z \)-plane, and specifying boundary conditions for going around singular points \( z = 0, L \). Let us map the singular points to \( u = 0, \infty \) by transformation (2.11). The corresponding action is then

\[
S_{int} = g \int d^2 u J(u, \overline{u}) \sum_{j=1}^n \mathcal{O}_j(u, \overline{u}), \tag{2.29}
\]

and the boundary conditions are

\[
\mathcal{O}_j(e^{2\pi i u}, e^{-2\pi i \overline{u}}) = \mathcal{O}_{j+1}(u, \overline{u}), \quad \mathcal{O}_n(e^{2\pi i u}, e^{-2\pi i \overline{u}}) = \mathcal{O}_1(u, \overline{u}). \tag{2.30}
\]

In (2.29) we have introduced the \( J(u, \overline{u}) \), which is a product of Jacobian of coordinate change (2.11) and factors coming from conformal transformation (2.28) of operators \( \mathcal{O}_j \),

\[
J(u, \overline{u}) = \frac{L^2 - \Delta - \overline{\Delta}}{|1 - u|^4} (1 - u)^{2\Delta} (1 - \overline{u})^{2\overline{\Delta}}. \tag{2.31}
\]
Notice that boundary conditions (2.30) are just conditions for going between sheets of Riemann surface of the $n$th root function $u^{1/n}$. Therefore the sum over $j$ in (2.29) actually stands for summing over $n$ sheets of $u^{1/n}$. This means that we can do the coordinate transformation $u = v^n$, mapping $n$ sheets of $u^{1/n}$ onto complex plane with $v$ coordinate.

The way this transformation acts on $\mathcal{O}$ is described by equation analogous to (2.28),

$$
(nv^{n-1})^\Delta (nv^{n-1})^{\overline{\Delta}} \sum_j \mathcal{O}_j(v^n, \overline{v}^n) = \mathcal{O}(v, \overline{v}).
$$

(2.32)

Now we can perform $u = v^n$ transformation in the action (2.29). Due to (2.32) we obtain

$$
S_{int} = gn^2-(\Delta-\overline{\Delta}) \int d^2v J(v^n, \overline{v}^n) \left(v^{1-\Delta} \overline{v}^{1-\overline{\Delta}}\right)^{n-1} \mathcal{O}(v, \overline{v}).
$$

(2.33)

Up to this moment the discussion has been general and did not rely on the fact that $\mathcal{O}_j$ operators are composed of $\Psi_j$, $\overline{\Psi}_j$ fermions. After we do the change of variables $u = v^n$ we obtain operator $\mathcal{O}(v, \overline{v})$ in (2.33), composed of single-valued on a plane fermions $\Psi(v)$, $\overline{\Psi}(\overline{v})$. The free-field correlation functions of these fermions are

$$
\langle \Psi^*(v_1)\Psi(v_2) \rangle = \frac{1}{v_1 - v_2}, \quad \langle \overline{\Psi}^*(\overline{v}_1)\overline{\Psi}(\overline{v}_2) \rangle = \frac{1}{\overline{v}_1 - \overline{v}_2}.
$$

(2.34)

This conclusion can also be reached in a more explicit way. First, one writes down perturbative expansion of the partition function

$$
Z = \int [D\Psi] e^{-S_{int}},
$$

(2.35)

with $S_{int}$ given by (2.29). Each term in the perturbative expansion can be evaluated, by using correlation functions of fermions (2.25), (2.26). One notices that it is possible to do the transformation $u = v^n$ for each given order of the perturbative expansion. Indeed, for example in holomorphic sector every $u_m^{1/n}$ is multiplied by phase factor $e^{2\pi i j_m/n}$, and $j_m$ is summed over from 1 to $n$, see (2.25). Then all the terms can be assembled back into partition function of fermions (2.34), deformed by interaction (2.33).

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5 Mapping from $\mathcal{R}_n$ to complex plane is frequently used in calculations of entanglement entropy, see, e.g., [3,26].
3. Thirring model

As a warm-up, in this section we consider Thirring model, described by the action

\[
S_T = \int d^2 z \left( \Psi^* \bar{\partial} \Psi + \bar{\Psi} \partial \bar{\Psi} + \lambda_T \Psi^* \bar{\Psi} \bar{\Psi} \Psi \right),
\]

(3.1)

where \( \lambda_T \) is a dimensionless coupling constant. A well-known fact is that one can immediately bosonize the fermions, \( \Psi(z) = e^{i\varphi(z)} \), \( \bar{\Psi}(z) = e^{i\bar{\varphi}(z)} \), and notice that the action (3.1) describes free massless scalar field \( \varphi \). Therefore the entanglement entropy of a single interval in Thirring model is simply given by (1.3) with \( c = 1 \). We prefer to postpone bosonization and instead treat the four-fermionic term in the action (3.1) as a perturbation \( O \). Then we can test the procedure, described in Subsection 2.3.

Let us put the model (3.1) on \( n \)-sheeted Riemann surface \( \mathcal{R}_n \). This can be accomplished by dealing with \( n \) copies of fermions, \( (\Psi_j, \bar{\Psi}_j) \), \( j = 1, \ldots, n \) on a complex \( z \)-plane, with the action

\[
S_T = \int d^2 z \sum_{j=1}^{n} \left( \Psi_j^* \bar{\partial} \Psi_j + \bar{\Psi}_j \partial \bar{\Psi}_j + \lambda_T \Psi_j^* \bar{\Psi}_j \bar{\Psi}_j \Psi_j \right),
\]

(3.2)

and twist operators inserted at \( z = 0, L \). The interaction in (3.2) is of the form (2.27) with \( (\Delta, \bar{\Delta}) = (1, 1) \). In the \( u \) coordinates (2.11), the fermions \( (\Psi_j, \bar{\Psi}_j) \) satisfy boundary conditions (2.12), (2.13).

Due to (2.31) we obtain \( J(u, \bar{u}) = 1 \). The partition function of Thirring model is therefore given by

\[
Z_n^T = \frac{1}{L^{2\Delta_n}} \int \mathcal{D}[\Psi] e^{-\lambda_T \int d^2 u \sum_{j=1}^{n} \Psi_j^* \bar{\Psi}_j \bar{\Psi}_j \Psi_j},
\]

(3.3)

where

\[
\Delta_n = \frac{n^2 - 1}{12n}.
\]

(3.4)

We can apply conformal perturbation theory for computation of (3.3). In the previous section we computed CFT correlation functions (2.25), (2.26) for the \( (\Psi_j, \bar{\Psi}_j) \) fermions. We can either use these correlation functions for explicit calculation, or use general result derived in Subsection 2.3. Due to (2.33) we obtain simply

\[
Z_n = \frac{1}{L^{2\Delta_n}} \left\langle e^{-\lambda_T \int d^2 v \bar{\varphi} \varphi} \right\rangle,
\]

(3.5)

where we have bosonized the fermions, and

\[
\left\langle \varphi(v_1) \varphi(v_2) \right\rangle = -\log (v_1 - v_2), \quad \left\langle \bar{\varphi}(\bar{v}_1) \bar{\varphi}(\bar{v}_2) \right\rangle = -\log (\bar{v}_1 - \bar{v}_2).
\]

(3.6)

From (3.5) we conclude that entanglement entropy of Thirring model is given by CFT formula (1.3), with \( c = 1 \), as expected.
4. Schwinger model

In this section we are going to derive formula for entanglement $c$-function in the Schwinger model ($2d$ QED with massless fermion). Derivation splits into three parts. First, by using the procedure described in Subsection 2.3, we derive an expression for $Z(n)$, defined in (2.6), which contains all the non-trivial contribution from the gauge interaction. The resulting formula contains fermionic four-point function in Schwinger model, which we compute in Subsection 4.2. We assemble all the results in Subsection 4.3 to write down expression for entanglement $c$-function.

The purpose of this section is twofold. First, it can be viewed as a warm-up for 't Hooft model: in this section we study entanglement entropy in $U(1)$ gauge theory with massless fermion, and in the next section we generalize it to $U(N)$ gauge theory with massless fundamental fermion. Second, Schwinger model is equivalent to a free massive $2d$ scalar, for which entanglement entropy is known (numerically) [5]. It turns out that calculation of entanglement entropy in Schwinger model is very similar to calculation in 't Hooft model, as we will show in the next Section. Therefore the result for 't Hooft model can be expressed using known result for Schwinger model.

4.1. Derivation of partition function $Z_m(n)$

The Schwinger model in the $A = 0$ gauge, with the gauge coupling $m$, is described by the action

$$S = \int d^2 z \left( -\frac{1}{2} (\partial A)^2 + m A \Psi^* \Psi + \Psi^* \overline{\Psi} + \Psi^* \overline{\Psi} \Psi + \overline{\Psi} \partial \Psi \right).$$

The gauge field is non-dynamical in two dimensions, so we can integrate over it. This amounts to substitution of

$$\overline{A}(z_1, \overline{z}_1) = m \int d^2 z_2 G(z_1 - z_2) \Psi^*(z_2) \Psi(z_2),$$

$$G(z_1 - z_2) = \frac{1}{2\pi} \frac{z_1 - z_2}{\overline{z}_1 - \overline{z}_2}.$$
It is well known that the Schwinger model is equivalent to a free scalar with mass $m$. This can easily be seen by bosonization of the fermions. Then (4.4) becomes just a mass term for a free scalar. As in the case of Thirring model, let us postpone bosonization for now.

Let us consider Schwinger model on $n$-sheeted Riemann surface $\mathcal{R}_n$, with the sheets glued along the cut of the length $L$. As in the case of Thirring model, this can be accomplished by considering $n$ copies of fermions $(\Psi_j, \bar{\Psi}_j)$ on a complex plane, with the interaction given by

$$S = m^2 \int d^2 z_1, 2 G(z_1 - z_2) \sum_{j=1}^n \Psi_j^*(z_1) \Psi_j(z_1) \Psi_j^*(z_2) \Psi_j(z_2).$$

By doing change of variables (2.11) we obtain

$$S = m^2 L^2 \int d^2 u_1, 2 F(u_1, u_2) \sum_{j=1}^n \Psi_j^*(u_1) \Psi_j(u_1) \Psi_j^*(u_2) \Psi_j(u_2),$$

where we have denoted

$$F(u_1, u_2) = \frac{1}{|1 - u_1|^2|1 - u_2|^2} G(u_1 - u_2).$$

The action (4.6) can be written in a form, similar to (2.29),

$$S = m^2 L^2 \int d^2 u_1, 2 F(u_1, u_2) \sum_{j=1}^n O_j(u_1) O_j(u_2),$$

where operators $O_j(u) = \Psi_j^*(u) \Psi_j(u)$ of dimension $(\Delta, \bar{\Delta}) = (1, 0)$ satisfy boundary conditions (2.30). Therefore for the model (4.8) to be well-defined, the coordinates $u_{1, 2}$ should go around singular points $u = 0, \infty$ at the same time. This means that the sum over $j$ allows to make change of variables $u_{1, 2} = v_{1, 2}^n$. Similarly to (2.32) we obtain

$$n^2 v_1^{n-1} v_2^{n-1} \sum_j O_j(v_1^n) O_j(v_2^n) = O(v_1) O(v_2),$$

where $O(v) = \Psi^*(v) \Psi(v)$, and correlation function for $\Psi(v)$ is given by (2.34). The action therefore becomes

$$S = n^2 m^2 L^2 \int d^2 v_{1, 2} H(v_1, v_2; n) \Psi^*(v_1) \Psi(v_1) \Psi^*(v_2) \Psi(v_2),$$

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where we denoted
\[ H(v_1, v_2; n) = \frac{v_1^{n-1} \bar{v}_2^{n-1}}{|1 - v_1|^2|1 - v_2|^2} \frac{v_1^n - v_2^n}{v_1 - v_2}. \] (4.11)

Again, one can also arrive at the same result (4.10) by using correlation functions (2.25), (2.26) in explicit perturbative calculation of the partition function.

The partition function of Schwinger model on \( \mathcal{R}_n \) is given by
\[ Z_n = \frac{1}{L^{2\Delta_n}} \int [D\Psi] e^{-S_{\Phi}}. \] (4.12)

In (4.12) we have path integral in CFT of free fermion \( \Psi(v) \) with correlation function (2.34), perturbed by interaction (4.10). Following (2.6), let us denote
\[ Z_m(n) = \int [D\Psi] e^{-S_{\Phi}}. \] (4.13)

Therefore \( Z(1) = Z_1^S \) is just a partition function of free massive 2d scalar on a complex plane. For the purpose of calculation of entanglement entropy (2.7), we are interested in
\[ \left. \frac{\partial Z_m(n)}{\partial n} \right|_{n=1} = -m^2 L^2 \int d^2v_{1,2} H_0(v_1, v_2) Z_1^S \left\langle \partial_1 \Phi(v_1, \bar{v}_1) \partial_2 \Phi(v_2, \bar{v}_2) e^{-S_{\Phi}} \right\rangle, \] (4.14)

where we have denoted
\[ H_0(v_1, v_2) = 2H(v_1, v_2; 1) + \left. \frac{\partial H(v_1, v_2; n)}{\partial n} \right|_{n=1}. \] (4.15)

Now we need to compute fermionic four-point function, which enters (4.14).

### 4.2. Calculation of quark four-point correlation function

To compute correlation function in (4.14), let us bosonize the fermions,
\[ \left. \frac{1}{Z_1^S} \frac{\partial Z_m(n)}{\partial n} \right|_{n=1} = m^2 L^2 \int d^2v_{1,2} H_0(v_1, v_2) \left\langle \partial_1 \Phi(v_1, \bar{v}_1) \partial_2 \Phi(v_2, \bar{v}_2) e^{-S_{\Phi}} \right\rangle, \] (4.16)

where due to (4.10), (4.11) we have
\[ S_{\Phi} = -m^2 L^2 \int d^2v_{1,2} \frac{1}{|1 - v_1|^2|1 - v_2|^2} \frac{v_1 - v_2}{\bar{v}_1 - \bar{v}_2} \partial_1 \Phi(v_1, \bar{v}_1) \partial_2 \Phi(v_2, \bar{v}_2). \] (4.17)
We can easily get rid of the extra factors in the integral (4.17) by doing transformation, inverse to (2.11):

\[
\frac{y_{1,2} - 1}{y_{1,2}} = v_{1,2}, \tag{4.18}
\]

where \(y_{1,2}\) are new dimensionless coordinates on complex plane. This gives

\[
S_\Phi = -m^2 L^2 \int d^2 y_{1,2} \frac{y_1 - y_2}{y_1 - \bar{y}_2} \partial_{y_1} \Phi \partial_{y_2} \Phi = m^2 L^2 \int d^2 y \Phi(y, \bar{y})^2, \tag{4.19}
\]

where we have also used (4.3). Applying the same transformation in (4.16) gives

\[
\left. \frac{1}{Z_1^S} \frac{\partial Z(n)}{\partial n} \right|_{n=1} = m^2 L^2 \int d^2 y_{1,2} H_1(y_1, y_2) \partial_{y_1} \partial_{y_2} \langle \Phi(y_1, \bar{y}_1) \Phi(y_2, \bar{y}_2) e^{-S_\Phi} \rangle, \tag{4.20}
\]

where we have denoted

\[
H_1(y_1, y_2) = H_0 \left( \frac{y_1 - 1}{y_1}, \frac{y_2 - 1}{y_2} \right) \frac{1}{\bar{y}_1 \bar{y}_2}. \tag{4.21}
\]

Let us perform Wick rotation and go to light-cone coordinates, \((\bar{y}, y) = (y^+, y^-), (p, \bar{p}) = (p_+, p_-)\). Then \(p \cdot y \equiv p_+ y^+ + p_- y^- = p \bar{y} + \bar{p} y\). We are working in \(A_- = 0\) gauge. Therefore the residual gauge transformations are parametrized by \(\alpha = \alpha(y^+)\). In the light-cone \((y^+, y^-)\) coordinates it becomes manifest why (4.14) is a gauge-invariant object, since fermions as not separated in the \(y^+\) direction, \(y_1^+ = y_2^+ = y^+\).

Due to (4.19) we have in (4.20) correlation function of free massive scalar, with dimensionless mass \(m^2 L^2\),

\[
\langle \Phi(y^+, y_1^-) \Phi(y^+, y_2^-) e^{-S_\Phi} \rangle = \int d^2 q \frac{e^{i q^- (y_1^- - y_2^-)}}{q^2 - m^2 L^2}. \tag{4.22}
\]

Therefore, from (4.20) we arrive at

\[
\left. \frac{1}{Z_1^S} \frac{\partial Z_m(n)}{\partial n} \right|_{n=1} = m^2 L^2 \int dy_1^- dy_2^+ H_1(y_1, y_2) \int d^2 q \frac{e^{i q^- (y_1^- - y_2^-)} q_-^2}{q^2 - m^2 L^2}. \tag{4.23}
\]

The integral (4.23) will also appear in the calculation of entanglement entropy in \('t\) Hooft model. This integral requires regularization, and as written, should be viewed formally. In the next Subsection we derive a useful expression for this integral.
4.3. Entanglement c-function

Due to (2.7) to find entanglement entropy we need to evaluate (4.23). This is a hard exercise and, as mentioned above, requires regularization. Fortunately we know the answer for entanglement entropy $S_m(mL)$ of free scalar field with mass $m$ [5,6,18]. Therefore we can use the known value of $S_m(mL)$ to find (2.7)

$$\frac{1}{Z_1^S} \frac{\partial Z_m(n)}{\partial n} \bigg|_{n=1} = \frac{1}{3} \log L + \log Z_1 - S_m(mL).$$

(4.24)

Differentiating w.r.t. $L$ we obtain

$$L \frac{\partial}{\partial L} \left[ \frac{1}{Z_1^S} \frac{\partial Z_m(n)}{\partial n} \bigg|_{n=1} \right] = \frac{1}{3} - c_m(mL),$$

(4.25)

where $c_m(mL)$ is entanglement c-function of free scalar with mass $m$. Equations (4.23), (4.25) give rise to

$$L \frac{\partial}{\partial L} \left[ m^2L^2 \int dy_1dy_2 H_1(y_1, y_2) \int d^2q e^{iq \cdot (y_1 - y_2)} \frac{q^2}{q^2 - m^2L^2} \right] = \frac{1}{3} - c_m(mL).$$

(4.26)

Equation (4.26) is the main result of this section, and it will play a key role in derivation of the entanglement entropy in ’t Hooft model. When $m = 0$ we have $c_0 = 1/3$, and therefore regularized value of the expression in the l.h.s. of (4.26) is equal to zero.

5. ’t Hooft model

In this section we derive expression for entanglement c-function in the ’t Hooft model. As in the case of Schwinger model, derivation splits into three parts: calculation of $Z(n)$, defined in (2.6); calculation of fermionic four-point function in the ’t Hooft model; and assembling all the results together, to get the expression for the entanglement c-function. Calculation of $Z(n)$ is similar to analogous calculation, performed for the Schwinger model.

5.1. Derivation of partition function $Z(n)$

The ’t Hooft model in the $A^a = 0$ gauge is described by the action [11]

$$S = \int d^2z \left( -\frac{1}{2} (\partial A^a)^2 + \bar{\Psi}^\alpha \gamma^a \Psi^\alpha + \bar{\Psi}^\alpha \gamma^a \Psi^\alpha + g A^a \Psi^\alpha T_{\alpha\beta} A^\beta \right).$$

(5.1)
Here $T^a$, $a = 1, \ldots, N^2$ are generators of the $u(N)$ algebra, and $g$ is coupling constant, with dimension of mass. Fermions $\Psi^\alpha, \overline{\Psi}^{\alpha}$, $\alpha = 1, \ldots, N$ live in fundamental representation of the $U(N)$. Similarly as in the case of Schwinger model, we can integrate over the gauge field, which amounts to substitution of

$$A^a(z_1) = g \int d^2 z_2 \, G(z_1 - z_2) \Psi^{\alpha*}(z_2) T^a_{\alpha\beta} \Psi^\beta(z_2)$$

into the action (5.1). This gives rise to free fermionic theory, deformed by the interaction

$$S_H = g^2 \int d^2 z_{1,2} \, G(z_1 - z_2) \Psi^{\alpha*}(z_1) \Psi^\beta(z_1) \Psi^{\beta*}(z_2) \Psi^\alpha(z_2).$$

In (5.3) we have used

$$T^a_{\alpha_1\alpha_2} T^a_{\alpha_3\alpha_4} = \frac{1}{2} \delta_{\alpha_1\alpha_4} \delta_{\alpha_2\alpha_3}.$$  

The Green’s function for $\partial^2$ operator is given by (4.3).

Let us consider ’t Hooft model on $n$-sheeted Riemann surface $R_n$ with branch points at $z = 0, L$. Let us make coordinate change (2.11), and consider $n$ copies of fermions $(\Psi_j^{\alpha}, \overline{\Psi}_j^{\alpha})$ on a $u$-plane, satisfying boundary conditions (2.12), (2.13). These fermions are described by CFT, deformed by the interaction

$$S_H = g^2 L^2 \int d^2 u_{1,2} \, F(u_1, u_2) \sum_j \Psi_j^{\alpha*}(u_1) \Psi_j^\beta(u_1) \Psi_j^{\beta*}(u_2) \Psi_j^\alpha(u_2),$$

where $F(u_1, u_2)$ is defined in (4.7).

Let us proceed as in the case of Schwinger model. The action (5.5) can be written as

$$S_H = g^2 L^2 \int d^2 u_{1,2} \, F(u_1, u_2) \sum_j \mathcal{O}_j^{\alpha\beta}(u_1) \mathcal{O}_j^{\beta\alpha}(u_2),$$

where $\mathcal{O}_j^{\alpha\beta}(u) = \Psi_j^{\alpha*}(u_1) \Psi_j^\beta(u_1)$. For this action to be well-defined the $u_{1,2}$ should go around singular points $u_{1,2} = 0, \infty$ at the same time. This means that we are allowed to make a change variables $u_{1,2} = v_{1,2}^n$, mapping $n$ sheets of $u^{1/n}$ function onto complex $v$-plane. The transformation law of operators $\mathcal{O}^{\alpha\beta}$ is given by

$$n^2 v_1^{n-1} v_2^{n-1} \sum_j \mathcal{O}_j^{\alpha\beta}(v_1^n) \mathcal{O}_j^{\beta\alpha}(v_2^n) = \mathcal{O}^{\alpha\beta}(v_1) \mathcal{O}^{\beta\alpha}(v_2),$$

where $\mathcal{O}^{\alpha\beta}(v) = \Psi^{\alpha*}(v) \Psi^\beta(v)$. Correlation function of holomorphic fermions on $v$-plane is

$$\langle \Psi^{\alpha*}(v_1) \Psi^\beta(v_2) \rangle = \delta^{\alpha\beta} \frac{1}{v_1 - v_2}.$$  

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In the $v$ coordinates the action (5.6) becomes
\[ S_H = n^2 g^2 L^2 \int d^2 v_{1,2} H(v_1, v_2; n) \Psi^{\alpha*}(v_1) \Psi^\beta(v_1) \Psi^{\beta*}(v_2) \Psi^\alpha(v_2), \] (5.9)
where $H(v_1, v_2; n)$ is defined in (4.11).

The partition function of 't Hooft model on $\mathcal{R}_n$ is therefore given by
\[ Z_n = \frac{1}{L^{2N\Delta_n}} \int [D\Psi] e^{-S_H}. \] (5.10)
In (5.10) we have path integral in CFT of free fermions with correlation functions (5.8), perturbed by interaction (5.9). As in the case of Schwinger model, and following (2.6), let us denote
\[ Z(n) = \int [D\Psi] e^{-S_H} \] (5.11)
in (5.10). To compute entanglement entropy (2.7), we have to find
\[ \frac{\partial Z(n)}{\partial n} \bigg|_{n=1} = -g^2 L^2 \int d^2 v_{1,2} H_0(v_1, v_2) Z_H^1 \langle \Psi^{\alpha*}(v_1) \Psi^\beta(v_1) \Psi^{\beta*}(v_2) \Psi^\alpha(v_2) e^{-S_H|_{n=1}} \rangle, \] (5.12)
where $H_0(v_1, v_2)$ is defined in (4.15), and $Z_H^1$ is partition function of 't Hooft model on a plane. In (5.12) we have $S_H|_{n=1}$, which is just four-fermionic interaction in 't Hooft model on complex plane. Indeed, doing change of coordinates (4.18), we obtain
\[ \frac{1}{Z_H^1} \frac{\partial Z(n)}{\partial n} \bigg|_{n=1} = -g^2 L^2 \int d^2 y_{1,2} H_1(y_1, y_2) \langle \Psi^{\alpha*}(y_1) \Psi^\beta(y_1) \Psi^{\beta*}(y_2) \Psi^\alpha(y_2) \rangle_H, \] (5.13)
where $H_1(y_1, y_2)$ is defined in (4.21). Subscript $H$ stands for correlation function in the theory with interaction
\[ S_{1H} = g^2 L^2 \int d^2 y_{1,2} \frac{y_1 - y_2}{y_1 - \overline{y}_2} \Psi^{\alpha*}(y_1) \Psi^\beta(y_1) \Psi^{\beta*}(y_2) \Psi^\alpha(y_2). \] (5.14)
We can bring the gauge field back, restoring the original 't Hooft model framework, with fermions interacting locally with the gauge field:
\[ S_{1H}^L = \int d^2 y \left( -\frac{1}{2} (\partial A^a)^2 + \hat{g} A^a \Psi^{\alpha*} T^a_{\alpha\beta} \Psi^\beta \right), \] (5.15)
where
\[ \hat{g} = gL \] (5.16)
is dimensionless gauge coupling.
5.2. Calculation of quark four-point correlation function

In (5.13) we have four-quark amplitude in 't Hooft model on a plane,

$$\mathcal{K}(y_1, y_2) = \langle \Psi^{\alpha*}(y_1) \Psi^\beta(y_1) \Psi^{\beta*}(y_2) \Psi^\alpha(y_2) \rangle_H .$$

(5.17)

Due to translational symmetry, \(\mathcal{K}(y_1, y_2) = \mathcal{K}(y_1 - y_2)\). This is gauge-invariant quantity, which has been demonstrated explicitly in [27]. The situation is analogous to Schwinger model, discussed in the previous Section. Remember that we are working in the gauge \(A^a = 0\), and therefore the residual gauge transformations are parametrized by \(\alpha^a\) such that

$$0 = \delta A^a = \partial \alpha^a + f^{abc} A^b \alpha^c = \partial \alpha^a ,$$

(5.18)

that is, \(\alpha^a = \alpha^a(\vec{r})\). Therefore the object \(\Psi^{\alpha*}(y_1) \Psi^\alpha(y_2)\) is gauge-invariant simply because both fermions are not separated in the \(\vec{y}\) direction. Therefore the four-fermionic amplitude in (5.13) is gauge-invariant. In fact one can consider a simpler gauge-covariant “blob” [11]

$$\delta^{\alpha_1 \alpha_2} \Phi_r(y_1, y_2) = \langle r | \Psi^{\alpha_1*}(y_1) \Psi^{\alpha_2}(y_2) | 0 \rangle ,$$

(5.19)

giving an amplitude of creation of meson \(|r\rangle\) by quark-anti-quark pair \(\Psi^{\alpha*}(y_1) \Psi^\alpha(y_2)\).

The Fourier transform of amplitude (5.17) is given by

$$\mathcal{K}(y_1 - y_2) = \int d^2 p_{1,2,3,4} e^{i(y_1 - (p_1 - p_2) + y_2 - (p_3 - p_4))} \left\langle \hat{\Psi}^{\alpha*}(p_1) \hat{\Psi}^\beta(p_2) \hat{\Psi}^{\beta*}(p_3) \hat{\Psi}^\alpha(p_4) \right\rangle_H .$$

(5.20)

Define vertex \(T(p, p'; q)\) [12] for quark-anti-quark scattering as

$$\left\langle \hat{\Psi}^{\alpha_1*}(p_1) \hat{\Psi}^{\alpha_2}(p_2) \hat{\Psi}^{\alpha_3*}(p_3) \hat{\Psi}^{\alpha_4}(p_4) \right\rangle_H = \delta(p_1 - p_2 + p_3 - p_4) P(p_1)P(p_2)P(p_3)P(p_4)$$

$$\times \left[ \delta^{\alpha_1 \alpha_4} \delta^{\alpha_2 \alpha_3} T(p_1, p_2; p_1 - p_4) - \delta^{\alpha_1 \alpha_2} \delta^{\alpha_3 \alpha_4} T(p_1, p_4; p_1 - p_2) \right] ,$$

(5.21)

where \(P(p)\) is the full quark propagator [11,12]. Let us use (5.21) in (5.20), obtaining

$$\mathcal{K}(y_1 - y_2) = \mathcal{K}_1(y_1 - y_2) + \mathcal{K}_2(y_1 - y_2) ,$$

(5.22)

where

$$\mathcal{K}_1(y_1 - y_2) = N^2 \int d^2 p_{1,2,4} e^{i(p_1 - p_2) - (y_1 - y_2)} T(p_1, p_2; p_1 - p_4)$$

$$\times P(p_1)P(p_2)P(-p_1 + p_2 + p_4)P(p_4) ,$$

(5.23)
\[
\mathcal{K}_2(y_1 - y_2) = N \int d^2p_{1,2,4} e^{i(p_1-p_2) \cdot (y_1-y_2)} T(p_1, p_4; p_1 - p_2) 
\times P(p_1)P(p_2)P(-p_1 + p_2 + p_4)P(p_4),
\]

(5.24)

The \(N^2\) in \(\mathcal{K}_1\) appeared after we summed over \(\alpha_1 = \alpha_4 = \alpha\) and \(\alpha_2 = \alpha_3 = \beta\), and \(N\) in \(\mathcal{K}_2\) appeared after we summed over \(\alpha = \beta\).

Let us denote integrated momenta in (5.23) as \(p_1 = p, p_2 = p', p_4 = p - q\), and define

\[
\tilde{\mathcal{K}}_1(q) = \int d^2p d^2p' e^{i(p-p') \cdot (y_1-y_2)} T(p, p'; q)P(p - q)P(p)P(p' - q)P(p').
\]

Therefore

\[
\mathcal{K}_1(y_1 - y_2) = N^2 \int d^2q \tilde{\mathcal{K}}_1(q).
\]

(5.26)

Let us denote integrated momenta in (5.24) and \(p_1 = p, p_2 = p - q, p_4 = p'\), and define

\[
\tilde{\mathcal{K}}_2(q) = \int d^2p d^2p' T(p, p'; q)P(p - q)P(p)P(p' - q)P(p').
\]

Therefore

\[
\mathcal{K}_2(y_1 - y_2) = N \int d^2q e^{iq \cdot (y_1-y_2)} \tilde{\mathcal{K}}_2(q).
\]

(5.28)

The \(\mathcal{K}_2\) is sub-leading in the large-\(N\) limit, and we will not consider it. In what follows we will only be interested in \(\mathcal{K}_1\).

Let us perform Wick rotation and switch to light-cone coordinates, similarly to what we have done in Subsection 4.2, for Schwinger model. Denote \(\hat{g}\) to be gauge coupling constant. The 't Hooft coupling constant is \(\hat{\lambda} = \hat{g}N^2\). The extra hat over the couplings is introduced because at the end we will substitute \(\hat{g} = gL\), where \(g\) is the original gauge coupling constant, and \(L\) is the length of interval, see (5.16) and preceding discussion.

Let us introduce the IR cutoff \(\lambda_{IR}\). Then full quark propagator, in the large-\(N\) limit, is given by [11]

\[
P(p) = \frac{i}{2p_+ - \frac{\hat{\lambda} \text{sgn}(p_-)}{\lambda_{IR}} - \frac{M^2 - i\epsilon}{p_-}},
\]

(5.29)

where

\[
M^2 = m_q^2 - \hat{\lambda}/\pi,
\]

(5.30)

\textit{Equation for full quark propagator} \(P(p)\) is derived in Appendix A. Quark propagator is two by two matrix. In \(A_- = 0\) gauge we are only considering entry of this matrix which corresponds to two-point function of left-moving quarks, since right-moving quarks are non-dynamical.
and $\hat{m}_q = m_q L$ is dimensionless bare quark mass. At the very end of calculation of gauge-invariant observables one sends $\lambda_{IR} \to 0$.

It is known that $T(p, p'; q)$, in the large-$N$ limit, satisfies equation [12] (see Appendix B, where derivation of this equation is reviewed)

$$T(p, p'; q) = \frac{i\hat{g}^2}{(p_+ - p'_+)^2} + \frac{i\hat{\lambda}}{\pi^2} \int dk_- \frac{1}{(k_- - p_-)^2} \int dk_+ P(k) P(k - q) T(k, p'; q). \quad (5.31)$$

Denote

$$x = \frac{p_-}{q_-}, \quad x' = \frac{p'_-}{q_-}, \quad (5.32)$$

which stand for the fraction of total momentum $q_-$, carried by one of in-coming quarks $p_-$, and one of the out-going quarks $p'_-$. The solution to equation (5.31) is [12]

$$T(p_-, p'_-; q) = \frac{\hat{\lambda}}{N(p_- - p'_-)^2} - \frac{\hat{\lambda}^2}{\pi N q_-^2} \sum_r \frac{1}{q^2 - \hat{m}_r^2} + i\epsilon \int_0^1 dy \int_0^1 dy' \frac{\phi^*_r(y') \phi_r(y)}{(y' - y)(y - x')^2}. \quad (5.33)$$

Integrals over $y$ and $y'$ in (5.33) are assumed to be taken with the cutoff: holes of size $\lambda_{IR}/q_-$ are drilled around $y = x$ and $y' = x'$. In (5.33) we denoted $\phi_r(y)$ to be set of eigenfunctions of 't Hooft equation

$$\frac{\pi M^2}{\hat{\lambda} y(1 - y)} \phi_r(y) - \hat{P} \int_0^1 dy' \frac{\phi_r(y')}{(y' - y)^2} = \frac{\hat{m}_r^2}{\hat{\lambda}} \phi_r(y), \quad (5.34)$$

describing meson of mass $\hat{m}_r = m_r L$. Here $\hat{P}$ stands for principal value.

Eigenfunctions $\phi_r(x)$ of 't Hooft equation (5.34) are non-vanishing when $x \in [0, 1]$ and satisfy completeness and orthonormality conditions [11,12]

$$\sum_r \phi_r(x) \phi_r(x') = \delta(x - x'), \quad (5.35)$$

$$\int_0^1 dx \phi_r(x) \phi^*_r(x) = \delta_{rr}. \quad (5.35)$$

For large $r$ (since $N$ does not appear in (5.34), large $r$ is not $\mathcal{O}(N^p)$) we have [11,12]

$$\phi_r(x) \simeq \sqrt{2} \sin (\pi r x), \quad \hat{m}_r^2 \simeq \pi^2 \hat{\lambda} r. \quad (5.36)$$

Let us re-write (5.33) as [21]

$$T(p_-, p'_-; q) = \frac{\hat{\lambda}}{N(p_- - p'_-)^2} + \frac{4\hat{\lambda}^2}{\pi N \hat{\lambda}_{IR}^2} \sum_r \frac{1}{q^2 - \hat{m}_r^2 + i\epsilon} \chi_r(x, q_-) \chi^*_r(x', q_-), \quad (5.37)$$
where we have defined
\[ \chi_r(x,q) = \frac{\lambda_{IR}}{2q_-} \int_0^1 dy \frac{\phi_r(y)}{(y-x)^2} . \] (5.38)

The reason for such reformulation is that one can see that presence of \( \lambda_{IR} \) in the r.h.s. of (5.38) makes the \( \chi_r \) finite in \( \lambda_{IR} \to 0 \) limit [21]. Indeed, we can split the integral over \( y \) into principal value part, which is finite when \( \lambda_{IR} \to 0 \), and infinite part, coming from divergence near \( y = x \):
\[ \int_0^1 dy \frac{\phi_r(y)}{(y-x)^2} = \hat{P} \int_0^1 dy \frac{\phi_r(y)}{(y-x)^2} + \frac{2}{\lambda_{IR}/q_-} \phi_r(x) . \] (5.39)

We conclude that
\[ \chi_r(x,q) = \phi_r(x) + O(\lambda_{IR}) . \] (5.40)

Now let us discuss \( K_1(y_1 - y_2) \), see (5.25), (5.26). We have
\[ \hat{K}_1(q) = \int dp_- dp'_- e^{i(p_- - p'_-)(y_1^- - y_2^-)} T(p_- , p'_- ; q) I_1 I_2 . \] (5.41)

we have introduced integrals \( I_{1,2} \),
\[ I_1 = \int dp_+ e^{i(p_+ - p'_+)(y_1^+ - y_2^+)} P(p - q) P(p) , \]
\[ I_2 = \int dp'_+ e^{-i(p'_+)(y_1^+ - y_2^+)} P(p' - q) P(p') . \] (5.42)

As discussed above, here we should set \( y_1^+ = y_2^+ \). In \( \lambda_{IR} \to 0 \) limit
\[ I_1 = I_2 = \frac{i\pi^2 \lambda_{IR}}{2\lambda} , \] (5.43)

Using definition (5.32) and equations (5.37), (5.40), (5.41), (5.43), and neglecting overall factors of \( \pi \), etc., we obtain
\[ \hat{K}_1(q) = \frac{1}{N} \int_0^1 dx dx' e^{i(x-x')(q_-(y_1^- - y_2^-))} \sum_r \frac{q^2}{q^2 - \hat{m}_r^2} \phi_r(x) \phi^*_r(x') . \] (5.44)

Now we need to plug (5.44) into (5.26) and integrate over \( q \),
\[ \hat{g}^2 K_1(y_1 - y_2) = \hat{\lambda} \int_0^1 dx dx' \int d^2 q e^{i(x-x')(q_-(y_1^- - y_2^-))} \sum_r \frac{q^2}{q^2 - \hat{m}_r^2} \phi_r(x) \phi^*_r(x') \] (5.45)
We can change variables of integration (assuming some smooth regularization when \(x = x'\), which we discuss below), \(q_- \rightarrow q_-/(x-x')\), \(q_+ \rightarrow (x-x')q_+\). Then (5.45) becomes

\[
\hat{g}^2 K_1(y_1 - y_2) = \hat{\lambda} \int_0^1 dx \, dx' \frac{1}{(x-x')^2} \int d^2 q \, e^{iq \cdot (y_1 - y_2)} \sum_r \frac{q^2}{q^2 - \hat{m}_r^2} \phi_r(x) \phi_r^*(x') .
\]  

(5.46)

Divergence in the integral in (5.46) when \(x\) approaches \(x'\) originates from divergence in the IR limit, when momentum \(p_- - p'_-\), flowing between \(y_1\) and \(y_2\), goes to zero. Therefore we can assume that such integral is regularized as in 't Hooft equation, by taking principal value part:

\[
a_r = -\int_0^1 dx \, dx' \frac{1}{(x-x')^2} \phi_r(x) \phi_r^*(x') = -\int_0^1 dx' \phi_r^*(x') \hat{P} \int_0^1 dx \frac{\phi_r(x)}{(x-x')^2} .
\]  

(5.47)

Using 't Hooft equation (5.34), normalization condition (5.35) of 't Hooft wave-functions \(\phi_r(x)\), and definition (5.30), we can re-write (5.47) as

\[
a_r = \frac{\hat{m}_r^2}{\hat{\lambda}} + \left(1 - \frac{\pi \hat{m}_q^2}{\hat{\lambda}}\right) \int_0^1 dx \left|\phi_r(x)\right|^2 \frac{1}{x(1-x)} .
\]  

(5.48)

The integral in (5.48) is positive-valued, and for large \(r\) we can use (5.36) and substitute

\[
\int_0^1 dx \left|\phi_r(x)\right|^2 \frac{1}{x(1-x)} \simeq \log(2\pi r) .
\]  

(5.49)

In this subsection we have been considering the situation of generally non-vanishing bare quark mass \(m_q\). However, our computation of entanglement entropy is performed in a set-up when we perturb CFT by four-fermionic interaction (5.9). If we had also perturbed CFT by non-vanishing bare mass for quarks, then an extra term, besides (5.9), would have been generated, and it would have also modified the total contribution (5.12) to the entanglement entropy. In particular, it is known that entanglement entropy of free massive fermion is given by a non-trivial expression [4,6]. Therefore let us refrain to vanishing bare quark mass. Then from (5.48) we obtain that \(a_r > 0\) for all \(r\),

\[
a_r = \frac{\hat{m}_r^2}{\hat{\lambda}} + \int_0^1 dx \left|\phi_r(x)\right|^2 \frac{1}{x(1-x)} .
\]  

(5.50)

5.3. Entanglement c-function

We can use the result (5.46) in the expression (5.13),

\[
\left. \frac{1}{Z_1^H} \frac{\partial Z(n)}{\partial n} \right|_{n=1} = \hat{\lambda} \sum_r a_r \int dy_1 \, dy_2 H_1(y_1, y_2) \int d^2 q \, e^{iq \cdot (y_1 - y_2)} \frac{q^2}{q^2 - \hat{m}_r^2} .
\]  

(5.51)
To find entanglement $c$-function in the 't Hooft model we need to substitute (5.51) into (2.8), and compute the integral. This is non-trivial, and requires regularization. In fact, the integral which appears in (5.51) is exactly the same as the integral which appears in analogous expression for the Schwinger model (4.23).

To avoid the difficulty of calculation of integral in (5.51), we use (4.26) to obtain entanglement $c$-function in the 't Hooft model, $c_H(\hat{\lambda})$, in terms of entanglement $c$-function of free massive scalars $c(\hat{m}_r)$

$$c_H(\hat{\lambda}) = \frac{N}{3} + \sum_{r=2}^{\infty} a_r \frac{\hat{\lambda}}{\hat{m}_r^2} \left( c(\hat{m}_r) - \frac{1}{3} \right) ,$$

(5.52)

where $\hat{m}_r = m_r L$. We know that $c_H(\hat{\lambda})$ should be a monotonically decreasing function [18], which should fix unambiguously overall sign of the sum in (5.52) to be plus (at the end of previous Subsection it was shown that $a_r > 0$, see (5.50)). Due to (5.50), (5.36), (5.49), we have $a_r \frac{\hat{\lambda}}{\hat{m}_r^2} = 1 + O(r^{-1} \log r)$ for large $r$.

The expression (5.52) cannot be a final answer, and requires regularization. This can be consequence of the way we handled integrals in (4.26) and (5.51). Besides, as written now, (5.52) has a contribution $N/3$ from twist operators, which diverges in the large-$N$ limit, and the sum over mesonic states, which is finite in the sense of large-$N$ counting. To fix this problem, let us take non-trivial $L$-dependent part of (5.52), and regularize the sum by requiring correct infra-red behavior. Namely, we have $c_H|_{L=\infty} = 1/3$, which is the entanglement $c$-function for one free massless meson ($m_1 = 0$) in the spectrum. This way we arrive at

$$c_H(\lambda L^2) = \frac{1}{3} + \sum_{r=2}^{\infty} a_r \frac{\hat{\lambda}}{\hat{m}_r^2} c(\hat{m}_r) .$$

(5.53)

Recall that [5,6] entanglement $c$-function of free massive scalar falls off exponentially at large distances, $c(\hat{m}) \simeq e^{-\hat{m}}$. The masses of higher mesons are given by (5.36). Therefore for interval of length $L$ the number of mesons which contribute to the sum (5.53) is

$$O \left( \frac{1}{\lambda L^2} \right) .$$

(5.54)

As $L$ is decreased, and we go to the ultra-violet, more mesons start to contribute, and when $\lambda L^2 = O(1/N)$ we get $c_H \simeq N$ due to (5.54), with $O(N)$ light mesons contributing. In this regime of parametrically small 't Hooft coupling constant we can neglect gauge interaction of quarks. Therefore we need to switch to picture of free quarks, which gives $c_H = N/3$. It would be interesting to interpolate more smoothly between the regimes of free quarks and mesons.
6. Discussion

In this paper we used the replica trick approach to calculate entanglement $c$-function in three models of a two-dimensional quantum field theory. We have considered Thirring model, Schwinger model, and ’t Hooft model. Thirring model is the simplest model with four-fermionic interaction, and after bosonization it becomes re-formulated as a theory of free massless scalar. Postponing bosonization, we have demonstrated explicitly that single-interval entanglement entropy in the Thirring model is given by the CFT expression (1.3) with central charge $c = 1$.

We have shown that replica trick calculation of entanglement entropy in the Schwinger and ’t Hooft models follows a similar path. In fact, one can derive entanglement entropy in the ’t Hooft model, provided it is known what is the entanglement entropy in the Schwinger model. The latter is indeed known (numerically in general and analytically in asymptotic regions), because the Schwinger model is equivalent to free massive scalar field.

It is essential for our calculation that ’t Hooft model is solvable in the large-$N$ limit. More precisely, it is important that one can compute quark-anti-quark scattering amplitude, and demonstrate that the only intermediate states in this process are mesons. Since each meson is free in the large-$N$ limit, we can use the known result for entanglement entropy of free massive 2$d$ scalar. We have demonstrated by explicit calculation that entanglement $c$-function in the ’t Hooft model (in the large-$N$ limit) is equal to a sum of entanglement $c$-functions of all the mesons in the spectrum (with certain numerical coefficients). It is interesting to observe that the fact that each meson has a finite size, which moreover grows as the meson gets heavier, is not manifested in the entanglement entropy.

A possible explanation of the fact that each meson contributes to the entanglement entropy as a point-like particle is the following. As it was mentioned in Introduction, defining entanglement entropy in systems with gauge fields is subtle. The problem appears with finite-size gauge-invariant operators (like mesons in the ’t Hooft model) which are partly located in $\mathcal{A}$ and partly in $\overline{\mathcal{A}}$. Such operators are in conflict with the very definition of entanglement entropy, which starts with the assumption of splitting of the Hilbert space into a tensor product

$$\mathcal{H}_{\mathcal{A}} \times \mathcal{H}_{\overline{\mathcal{A}}}.$$ (6.1)

Therefore the question is what exactly the replica trick calculation, which we performed in this paper, has given us.
One answer is that in the entanglement entropy calculation the gauge-invariant operators, which are located partly in $\mathcal{A}$ and partly in $\overline{\mathcal{A}}$, are cut into two parts, by the entangling surface. The free ends of those parts are glued to the extra added gauge non-invariant (charged) edge modes, located on the entangling surface [16,28] (see also [29] for the calculation of the contribution of the edge modes to the entanglement entropy). Therefore each of the resulting parts is completed to a gauge-invariant object. One is located entirely in $\mathcal{A}$, the other is located entirely in $\overline{\mathcal{A}}$. This procedure is incorporated in the replica trick calculation [16]. This means that the whole Hilbert space is enhanced to include such charged boundary degrees of freedom, located on the entangling surface. The consequence of such an enhancement is that there are no more gauge-invariant operators, located partly in $\mathcal{A}$ and partly in $\overline{\mathcal{A}}$. This makes definition of the entanglement entropy consistent. At the same time it explains why finite size of mesons is not manifested in the entanglement $c$-function, no matter how long the meson is. 7

If bare quark mass is zero, then the lowest meson is massless. Therefore in the deep infra-red we have CFT of one free massless scalar, and entanglement $c$-function is equal to 1/3. In the deep ultra-violet regime quarks are free, and entanglement $c$-function is equal to $N/3$, where $N$ is the number of colors. One can compare this crossover from the mesonic phase to free quark phase with the phase transition, observed in [31] (see also [32]), where single-interval (slab) entanglement entropy in confining gauge theory has been calculated holographically. In [31] it was shown that there is a critical thickness of the slab $L_c$ so that when $L < L_c$ entanglement entropy behaves as $N^2$, and when $L > L_c$ entanglement entropy behaves as $N^0$. It would be interesting to understand better how these two observations are related.

It would be interesting to address the question, raised in the introduction, and calculate entanglement entropy in the quark language, perturbatively in the coupling constant. Such calculation will shed light on the crossover region, $\lambda L^2 = 1/N$. 7

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7 In a pure two-dimensional Yang-Mills theory there is an additive contribution to the entanglement entropy, which can be interpreted by counting the gauge non-invariant states, located on the entangling surface [16]. See also [30], where entanglement entropy in the $O(N)$ $\sigma$-model has been calculated, and interpretation in terms of counting the states of the UV degrees of freedom has been provided. It might be that there are such parton-counting terms, entering entanglement entropy in the ’t Hooft model, and these terms are $L$-independent, and hence invisible in the calculation of the entanglement $c$-function. We thank A. Wall for discussion of this point.
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Appendix A. Gap equation for fermionic propagator

In this Appendix we derive gap equation on full quark propagator by calculating path integral in the large-$N$ limit. This is different from the conventional derivation, which involves equation satisfied by planar ladder diagrams [11]. Similar derivation for Gross-Neveu model can be found in [33,34].

For generality, let us give fermions the bare mass $m$. The total action is then

$$ S = S_0 + S_{\text{int}}, \quad \text{(A.1)} $$

where

$$ S_0 = \int d^2 z \, \hat{\Psi}^\dagger(z)(\gamma^\mu \partial_\mu + m)\hat{\Psi}(z) \quad \text{(A.2)} $$

and (let us absorb ’t Hooft coupling $\lambda$ into rescaling of coordinates $z_{1,2}$ and the mass $m$)

$$ S_{\text{int}} = \frac{1}{N} \int d^2 z_{1,2} \left[ G(z_1 - z_2) \Psi^\ast(z_2) \Psi^\ast(z_1) \right]. \quad \text{(A.3)} $$

We have denoted Dirac fermion as $\hat{\Psi}$, and its charge conjugate as $\hat{\Psi}^\dagger$,

$$ \hat{\Psi} = \left( \begin{array}{c} \Psi \\ \Psi^\ast \end{array} \right), \quad \hat{\Psi}^i = \left( \begin{array}{c} \Psi^\ast \\ \Psi \end{array} \right), \quad \text{(A.4)} $$

and the Dirac operator is

$$ \gamma^\mu \partial_\mu = \left( \begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array} \right). \quad \text{(A.5)} $$

Let us introduce auxiliary gauge-singlet field $P(z_1, z_2)$, which for the purpose of finding quark two-point function, due to translational symmetry, is taken to be $P(z_1 - z_2)$. We can re-write the action (A.3) as

$$ S_{\text{int}} = \int d^2 z_{1,2} \left[ G(z_1 - z_2) \left( N P(z_1 - z_2) P(z_2 - z_1) + P(z_1 - z_2) \Psi^\ast(z_1) \Psi(z_2) \right) \right] \quad \text{(A.6)} $$

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Then the total action is the sum of quadratic action for $P(z_1 - z_2)$ and the fermionic action:

$$S = S_\Psi + S_P,$$

$$S_\Psi = \int d^2z_{1,2} \bar{\Psi}^\alpha(z_1) \left[ \delta(z_1 - z_2) \gamma^\mu \partial_\mu + \delta(z_1 - z_2) m + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} P(z_1 - z_2) G(z_1 - z_2) \right] \bar{\Psi}^\alpha(z_2),$$

$$S_P = N \int d^2z_{1,2} G(z_1 - z_2) P(z_1 - z_2) P(z_2 - z_1).$$

(A.7)

In momentum representation

$$S_\Psi = \int d^2q \bar{\Psi}^\alpha(-q, -\bar{q}) K \bar{\Psi}^\alpha(q, \bar{q}),$$

$$K = \begin{pmatrix} m & \bar{q} \\ q + T(q) & m \end{pmatrix}, \quad T(q) = \int d^2k \frac{P(k, \bar{k})}{(q - k)^2},$$

$$S_P = N \int d^2z \int d^2q d^2k \frac{P(k, \bar{k}) P(q, \bar{q})}{(q - k)^2}.$$

(A.8)

Integrating over fermions gives

$$Z_\Psi = \int |d\Psi| e^{-S_\Psi} = e^N \text{Tr} \log K,$$

(A.9)

where

$$\text{Tr} \log K = \int d^2z d^2q \log \det K(q, \bar{q}) = \int d^2z d^2q \log \left( m^2 - |q|^2 - \bar{q} \int d^2k \frac{P(k, \bar{k})}{(q - k)^2} \right).$$

(A.10)

Combining (A.10) with $S_P$ from (A.8) we obtain the total action for $P(q, \bar{q})$:

$$S = N \int d^2z d^2q \left[ \int d^2k \frac{P(k, \bar{k}) P(q, \bar{q})}{(q - k)^2} - \log \left( m^2 - |q|^2 - \bar{q} \int d^2k \frac{P(k, \bar{k})}{(q - k)^2} \right) \right].$$

(A.11)

In the large-$N$ limit solution to the theory is obtained simply by extremization of (A.11) w.r.t. $P(q)$:

$$\delta S = N \int d^2z d^2q d^2k_1 \frac{\delta P(k_1, \bar{k}_1)}{(q - k_1)^2} \left[ P(q, \bar{q}) - \frac{\bar{q}}{|q|^2 - m^2 - \bar{q} \int d^2k \frac{P(k, \bar{k})}{(q - k)^2}} \right].$$

(A.12)

The corresponding equation of motion is

$$P(q, \bar{q}) = \frac{\bar{q}}{|q|^2 - m^2 - \bar{q} \int d^2k \frac{P(k, \bar{k})}{(q - k)^2}}.$$

(A.13)
We can see that the equation (A.13) is just 't Hooft gap equation [11] on full fermionic propagator $P(q, \overline{q})$ in leading order in large-$N$ limit. Indeed, the full fermionic propagator is equal to the sum of geometric series of 1PI propagators. On the other hand, 1PI propagator is equal to [11]

$$\Sigma(q, \overline{q}) = \int d^2k \, \frac{P(k, \overline{k})}{(q - k)^2}$$

(A.14)

and therefore the sum of geometric series is equal to

$$\frac{\overline{q}}{1 - \frac{\overline{q}}{|q|^2 - m^2} \Sigma(q, \overline{q})} = \frac{\overline{q}}{|q|^2 - m^2 - \overline{q} \int d^2k \, \frac{P(k, \overline{k})}{(q - k)^2}}.$$  

(A.15)

This agrees with the equation (A.13).

**Appendix B. Bethe-Salpeter equation for four-quark vertex**

Let us give a short review of how one can arrive at the Bethe-Saltpeter equation (5.31) on four-quark vertex $T$. Consider four quarks scattering process, with two incoming quarks

$$\left( \hat{\psi}^{\alpha_1}(p_1), \hat{\psi}^{\alpha_1}(p_2) \right)$$

where $\alpha_1$ is a fixed color, and two outgoing quarks

$$\left( \hat{\psi}^{\alpha_3}(p_3), \hat{\psi}^{\alpha_3}(p_4) \right).$$  

(B.2)

Denote $q = p_1 - p_2$ to be momentum flowing in this process. Due to (5.21) such diagram is equal to

$$\delta(p_1 - p_2 + p_3 - p_4)P(p_1)P(p_2)P(p_3)P(p_4)T(p_1, p_3; q).$$  

(B.3)

On the other hand [12], this diagram is equal to sum of the diagram

$$\delta(p_1 - p_2 + p_3 - p_4)P(p_1)P(p_2)P(p_3)P(p_4)\frac{ig^2}{(p_1 - p_3)^2},$$

(B.4)

describing scattering process in which incoming quarks (B.1) exchange one gluon and fly away as (B.2); and the diagram

$$\delta(p_1 - p_2 + p_3 - p_4)P(p_1)P(p_2) \sum_{\alpha_1} \int d^2k \, ig^2 P(k)P(k-q) \frac{1}{(k-p_1)^2} T(k, p_3; q)P(p_3)P(p_4),$$

(B.5)

describing process in which two incoming quarks (B.1) exchange one gluon with momentum $k - p_1$, scatter in two outgoing quarks

$$\left( \hat{\psi}^{\alpha_1}(k-q), \hat{\psi}^{\alpha_1}(k) \right),$$

(B.6)

which subsequently scatter on vertex $T(k, p_3; q)$, giving two outgoing quarks (B.2). This way we obtain equation (5.31).
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