Two new optimal and uniform third-order schemes for singular perturbation problems with initial layers

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Abstract
This article presents two numerical methods of the order of three for singular perturbation problems, with a small positive parameter using finite differences. It is a problem with an initial layer in the neighborhood of the initial nodal point whose width is of the order of the small parameter \( \epsilon \). Explicit order three classical methods are modified, and a new scheme is designed for singular perturbation problems. It is a fitted operator method and it is explicit with a variable fitting factor (VFF) evaluated at all nodal points. To reduce the calculation time of the scheme with VFF, the VFF is replaced by a fixed constant fitting factor (CFF). It is implicit with a CFF which is evaluated only one time at the initial nodal point. These two schemes are both optimal concerning the small parameter \( \epsilon \) and uniform of order three. The three order methods presented in this article are superior to the three order methods available in the literature. To view the initial layer when the mesh size is larger than the parameter in the problem, these two fitted operator methods are extended to fitted mesh methods since fitted mesh methods are layer rescaling. The construction of the fitted mesh method is also provided. That is, the uniform mesh is extended to non-uniform mesh. Experimental results are presented to show the optimal and higher-order performance of the two numerical methods using three test problems.

Keywords
Initial value problem · Singular perturbation problem · Initial layer · Finite difference schemes · fitted operator method · Optimal and uniform scheme · Fitted mesh method · Order · Convergence

1 Introduction
The mathematical model for a problem with initial layer arises in control system considered in this article on the domain \( D = [0, \infty) \) is

\[
Ly(t) \equiv \epsilon \frac{dy(t)}{dt} + a(t)y(t) = f(t), \quad t \in D = (0, \infty) \quad (1a)
\]

\[
y(0) = \eta \quad (1b)
\]

where \( \eta > 0 \), and \( \epsilon \gg 0 \) is a small positive parameter and continuously differentiable functions \( a(t) \) and \( f(t) \) subject to the condition \( a(t) \geq \alpha > 0, \quad t \in D \). The solution \( y(t) \) is unique and uniformly stable, because operator \( L \) in \((1a,b)\) has a maximum principle.

In particular, it is a singular perturbation problem with a layer at the initial point and the layer width is of order \( \epsilon \) (O’Malley 1974, 1991; Smith 1985) at the initial point of the domain \( D \). And so the problem \((1a,b)\) reduces to

\[
a(t)y_0(t) = f(t), \quad t \in D = (0, \infty) \quad (2)
\]

is obtained by reducing the problem \((1a,b)\) by taking \( \epsilon = 0 \) in the original problem \((1a,b)\).

Parameter-dependent problems are the singular perturbation problems which are the mathematical models for problems with initial layers, boundary layers, and interior layers. The survey of numerical methods for singular perturbation problems starting from Prandtl’s work has been done and given in 1980 by Doolan et al. (1980). In the Ph.D. Thesis Selvakumar (1992) a survey up to 1992 is done. Again, a survey from 1905 to 1984 has been done by Kadalbajoo and Reddy (1989) in 1989, and a survey from 1984 to 2000 is done by Kadalbajoo and Patidar Kadalbajoo and Patidar (2002) in 2002. The survey of computational techniques during the period 2000–2005 is done by Kumar et al. (2007) in 2007. The survey on singular perturbation problems with the
interior and turning points are done during the period 1970–2011 by Sharmaa et al. (2013).

Valarmathi and Ramanujam (2002a) in 2002 applied an exponentially fitted finite difference method of order one of Doolan et al. (1980) to the singularly perturbed two-point boundary value problems for third-order ordinary differential equations. The boundary value problem is reduced to a weakly coupled system of one first-order ordinary differential equation with a suitable initial condition and one second-order singularly perturbed subject to boundary conditions. To solve this system of ordinary differential equation, a computational method is suggested. This method combines an exponentially fitted finite difference scheme of Doolan et al. (1980) and a classical finite difference scheme. In general, the boundary value problem possesses two boundary layer regions of different widths at t = 0 and t = 1, which are less severe. Valarmathi and Ramanujam (2002b) in 2002 applied an exponentially fitted finite difference method of order one of Doolan et al. (1980) to the singularly perturbed two-point boundary value problems for third-order ordinary differential equations with a small parameter multiplying the highest derivative. In this method, the given boundary value problem is transformed into a system of two ordinary differential equations subject to suitable initial and boundary conditions. The problem has a boundary layer at t = 0 which is less severe because the boundary conditions are of the Neumann type. Valanarasu and Ramanujam (2003) in 2003 applied an exponentially fitted finite difference method of order one of Doolan et al. (1980) to the singularly perturbed two-point boundary value problems for second-order ordinary differential equations with two small parameters multiplying the derivatives. In this method, using the initial value technique, the boundary value problem is converted into an initial value problem by defining the terminal condition from asymptotic expansion. In general, the BVP possesses two boundary layer regions of different widths at t = 0 and t = 1. Valarmathi and Ramanujam (2002c) in 2003 applied an exponentially fitted finite difference method of order one of Doolan et al. (1980) to the singularly perturbed two-point boundary value problems for third-order ordinary differential equations with convection–diffusion type. In this method, the given boundary value problem is transformed into a system of two ordinary differential equations subject to suitable initial and boundary conditions. The problem has a boundary layer at t = 0 which is less severe.

Ramos (2005) in 2005 designed exponentially fitted finite difference methods of order one for the initial value problems in ordinary differential equations based on Taylor series expansions and piecewise analytical solutions using linearization methods. And, then applied to the singularly perturbed initial value problems in ordinary differential equations. They are the partial linearization method and quadratic partial linearization method. It is a linearization concerning the solution u(t), but not with the independent variable t. On applying the quadratic partial linearization method to the linear problem (1a,b), it reduces to the linear partial linearization method. And hence, on applying these methods of Ramos to the problem (1a,b) which exhibits an initial layer of thickness O(ε) at t = 0, they reduce to the schemes of Doolan et al. (1980) which are of order one explicit and implicit methods (Refer equations (20) and (22) of this article).

Cakir et al. (2016) in 2016 applied an exponentially fitted finite difference method of order one of Doolan et al. (1980) to the quasi-linear second-order initial value problem with an initial condition independent of the parameter ε and initial derivative condition depending on the parameter ε. As ε goes to zero the solution u may exhibit an exponential initial layer near t = 0. Chakravarthy and Kumar (2017) in 2017 applied an exponentially fitted finite difference method of order one of Doolan et al. (1980) to singularly perturbed boundary value problem for a linear second-order delay differential equation. For small values of ε, the boundary value problem exhibits a strong boundary layer at t = 2. Cimen and Cakir (2018) in 2018 applied an exponentially fitted finite difference method of order one of Doolan et al. (1980) to the singularly perturbed non-local boundary value problem for a second-order delay differential equation. The solution of a singularly perturbed problem normally has a boundary layer at t = 0.

It is observed from the literature (Cakir et al. 2016; Chakravarthy and Kumar 2017; Cimen and Cakir 2018; Doolan et al. 1980; Kadlabjoo and Reddy 1989; Kadlabjoo and Patidar 2002; Kumar et al. 2007; Selvakumar 1992; Sharmaa et al. 2013; Valanarasu and Ramanujam 2003; Valarmathi and Ramanujam 2002a, b, c), the fitted operator methods are used in many articles because many of the singular perturbation problems have exponential solutions. And the fitted operator methods yield better results than the generation of grids. This motivates the design of a fitted operator method. On applying the explicit classical order three methods the solution will not give a satisfactory result. And so, the explicit classical order three methods are modified and a new method is presented which provides a satisfactory result.

We first solve the problem (1a,b) by introducing a uniform mesh on D with nodal points ti = ih, and the factor \( \rho = \frac{h}{\epsilon} \). And, we propose the finite difference schemes with VFF for the factor \( \rho = \frac{h}{\epsilon} \) of the form

\[
L^h y_i = \epsilon \sigma (-\rho \alpha_i^h) D_+ y_i + a_i^h y_i = f_i^h \\
y_0 = \eta
\]

(3a)

where the functions \( a_i^h, f_i^h \) and the VFF \( \sigma (-\rho \alpha_i^h) \) are specified in the coming sections. The discrete operator \( D_+ \) is defined as \( D_+ y_i = \frac{y_{i+1} - y_i}{h}, \ h > 0 \). The advantages of the schemes are the VFF model the initial layer and it is uniform.
concerning the small parameter \( \epsilon \). And, the scheme models the reduced problem of (1a, b) and it will work for large time \( t \). An optimal and uniform of order three error estimate,

\[
|y(t_i) - y_i| < C \min(h^3, \epsilon)
\]

will be established to show the convergence. To reduce the computation time a scheme with CFF is also provided. Both schemes satisfy the inequality (4). In Carroll (1982, 1983, 1984, 1986); Doolan et al. (1980); Farrell (1987); Kolmogorov and Shishkin (1997); Selvakumar (1992, 1994a), the authors have designed uniformly convergent and optimal schemes with VFF and CFF for the problem (1a, b). For nonlinear problems, schemes with VFF have been designed for the numerical solution in Byrne and Hindmarsh (1987); Carroll (1982, 1983); O’Reilly (1983, 1987); Ramos (2005). To reduce the computational time, schemes with CFF have been designed in Carroll (1982, 1983, 1984, 1986); Doolan et al. (1980); Farrell (1987); Kolmogorov and Shishkin (1997); O’Reilly (1987); Selvakumar (1992, 1994a).

In Doolan et al. (1980) a scheme of order three for the numerical solution of (1a,b) is presented with a VFF. The scheme with VFF for the factor \( \rho = \frac{h}{\epsilon} \) is of the form

\[
\begin{align*}
L^h y_i & \equiv \epsilon \sigma (-\rho a_i^h D_+ y_i + a_i^h y_i) = Q_i(f), \quad i \geq 0 \\
y_0 & = \eta, \quad \sigma (-\rho a_i^h) = \rho a_i^h [1 - \exp(-\rho a_i^h)]^{-1} \\
Q_i(f) & = \alpha_i f(t_{i-1}) + \beta_i f(t_i) + \gamma_i f(t_{i+1})
\end{align*}
\]

where

\[
\begin{align*}
\alpha_i & = \frac{a_i^h}{a(t_{i-1})} \left[ \frac{1}{2\rho a_i^h} - \frac{a(t_{i-1})}{\rho^2 a_i^h a_{i-1}^h} \right] \\
\gamma_i & = \frac{a_i^h}{a(t_{i+1})} \left[ \left( \frac{(\sigma (-\rho a_i^h) - 1)}{\rho a_i^h} \right) + \left( \frac{\alpha_i a(t_{i-1}) a_i^h}{(a_i^h)^2} \right) \right] \\
\beta_i & = \frac{a_i^h}{a(t_i)} \left[ 1 - \left( \frac{\gamma_i a(t_{i+1})}{\rho a_i^h} \right) - \left( \frac{\alpha_i a(t_{i-1})}{\rho a_i^h} \right) \right]
\end{align*}
\]

and

\[
\alpha_i^h = \frac{1}{12} [5a(t_{i+1}) + 8a(t_i) - a(t_{i-1})]
\]

In Doolan et al. (1980), the error estimate (4) is not obtained for the scheme (5a–g). The numerical result shows from Table 1, that the order three scheme (5a–g) is not better than the order two scheme given in Doolan et al. (1980), for large values of the mesh size. This also motivates to propose a third-order method. And, motivated by the works in Carroll (1982, 1983, 1984, 1986); Doolan et al. (1980); Farrell (1987); Kolmogorov and Shishkin (1997); O’Reilly (1987);

Selvakumar (1992, 1994a), we propose two new finite difference schemes with VFF and CFF of order three.

It is observed from the literature Doolan et al. (1980); Kadgalalpoo and Reddy (1989); Kadgalalpoo and Patidar (2002); Kumar et al. (2007); Selvakumar (1992); Sharmaa et al. (2013); Valanarasu and Ramanujam (2003); Valarmathi and Ramanujam (2002a, b, c), for a larger value of step size than the parameter that we cannot apply fitted operators directly. Boundary value techniques are applied to view and solve the layers for a larger value of step size than the parameter, but it is computationally costlier. It needs iteration to reach the terminal point of the layer. That is, from the literature, to view the initial layer when the step size is larger than the value of the parameter in the problem (1a,b) boundary value techniques can be used. Roberts (1982, 1988) designed a boundary value technique to solve the singular perturbation problems with layers by discretizing the problem with classical finite difference methods. It is applicable also for the larger value of step size than the parameter. Taking subdomains concerning the layer as inner and outer of the layer regions Vrcelj et al. (1991) solved boundary layer problems. Kadgalalpoo (1987a, b, 1987c) also solved using classical methods via boundary value technique. Selvakumar (1994b) applied a boundary value technique using a fitted operator method to solve boundary layer problems. In the literature Doolan et al. (1980); Kadgalalpoo and Reddy (1989); Kadgalalpoo and Patidar (2002); Kumar et al. (2007); Selvakumar (1992); Sharmaa et al. (2013), many numerical scientists have applied boundary value techniques to solve singular perturbation problems. All techniques need iteration to reach the terminal point of the initial layers and there will be a gap in the domain in between the initial layer region and outer region of the layer. To overcome this situation in the boundary value technique, Shishkin (1997), Miller (1994, 2019) introduced a fitted mesh method to the implicit classical numerical methods by locating the terminal point. Motivated by this in this paper, the fitted operator methods are extended to the fitted mesh method to view the layer and to solve the initial layer when the step size is larger than the value of the parameter in the given problem (1a,b).

It is also observed from the literature Kadgalalpoo and Reddy (1989); Kadgalalpoo and Patidar (2002); Kumar et al. (2007); Selvakumar (1992); Sharmaa et al. (2013); Valanarasu and Ramanujam (2003); Valarmathi and Ramanujam (2002a, b, c) that the fitted mesh methods are especially the Shishkin mesh is used to solve singular perturbation problems. The terminal point of the layer is fixed and so there is no iteration procedure as in boundary value technique to reach the terminal point of the layer. But for a larger value of step size than the parameter, we can easily apply. This method can be implemented to linear singular perturbation problems. For nonlinear problems, this method is computationally costlier.
and not suitable for multidimensional problems to extend from the one-dimensional case.

In Sect. 2, it is shown that operator L has a maximum continuous principle and the solution is bounded and stable. And an asymptotic expansion for the solution of (1a,b) is also presented in this section. The explicit classical methods of order three are modified and a scheme with VFF (scheme (9a–m)) is designed in Sect. 3. The VFF is evaluated at all nodal points. With a single evaluation at the initial point only, an implicit scheme with a CFF is given in Sect. 4. This method reduces the calculation time. The construction of the fitted mesh method Kadalbajoo and Reddy (1987a, c) is provided in Sect. 5 to both the fitted operator methods of Sects. 3 and 4. To show the performance and applicability of the schemes proposed in this paper, experimental results are provided with the help of three test problems in Sect. 6.

Throughout this article, C is a generic constant independent of i, the step size ϵ, and the small positive parameter ϵ ≫ 0. Classical methods refer to the numerical methods got from Taylor series expansion for the solution of (1a,b). Classical methods will not fit the factor ρ = h/ε, and hence, there will be a restriction on h and the parameter ϵ for the numerical stability of the numerical solution. Fitted methods or fitted operator method refer to the numerical methods got from Taylor series expansion and asymptotic expansion for the solution of (1a,b). Fitted methods will fit the factor ρ = h/ε, and hence, there will be no restriction on h and the parameter ϵ for the numerical stability of the numerical solution.

2 Results on singular perturbed continuous problem

In this section, it is shown that the operator L has a maximum continuous principle and the solution is bounded and stable in Theorem 1. And an asymptotic is also presented in this section.

**Theorem 1** Let u(t) be a continuously differentiable function defined in domain D.

1. If the function u(t) satisfies u(0) ≥ 0 and L u(t) ≥ 0 for t ∈ D are true, then u(t) ≥ 0 for all t ∈ D.
2. If y(t) is the solution of the stiff problem (la, b), then

\[ |y(t)| ≤ |y(0)| + \frac{1}{α} \max_{t ∈ D} |f(t)|, \quad t ∈ D. \]

**Proof** Doolan et al. (1980). □

Hence, the problem (1a,b) has a unique and bounded solution by Theorem 1. The approximate solution y(t) of the problem (1a,b) can be written as Doolan et al. (1980); Smith (1985)

\[ y(t) = y_H(t) + y_P(t) \] (6)

where \( y_H(t) \) is the solution of problem (1a,b) with f(t) = 0 and \( y_P(t) \) is the particular solution. The term \( y_P(t) \) can be expressed as

\[ y_P(t) = \sum_{j=0}^{∞} y_j(t) \] (7)

where the terms in (7) are \( y_j(t), j ≥ 0 \) are defined as

\[ a(t) y_0(t) = f(t) \] (8a)

\[ a(t) y_{j+1}(t) = -\frac{d y_j(t)}{d t}, j > 0 \] (8b)

This series will become finite if \( y_j(t) \) become constant for some \( j ≥ 0 \).

3 Discrete problem with VFF

The explicit classical method of order three is modified, and a scheme with VFF is proposed in this section. The consistency, stability and uniform of order three and optimal convergence are also discussed. The fitted mesh method is also constructed in this section as an extension.

The explicit scheme with VFF for all 0 ≤ i ≤ N − 1, with the uniform mesh \( h = t_i+1 - t_i \) and the factor \( \rho = h/ε \), is defined as

\[ L^h y_i = ε \sigma(-ρ a_i^h) D_{+} y_i + a_i^h y_i = f_i^h, i ≥ 0 \] (9a)

\[ y_0 = η \] (9b)

where

\[ \sigma(-ρ a_i^h) = \rho a_i^h [1 - \exp(-ρ a_i^h)]^{-1} \] (9c)

\[ \sigma(ρ a_i^h) = \exp(-ρ a_i^h) \sigma(-ρ a_i^h) \] (9d)

\[ a_i^h = \frac{1}{12} [5 a(t_{i+1}) + 8 a(t_i) - a(t_{i-1})] \] (9e)

\[ f_i^h = K_1 f(t_{i-1}) + K_2 f(t_i) + K_3 f(t_{i+1}) \] (9f)

where

\[ K_1 = \frac{1}{ρ a(t_{i-1})} \left[ U_i a_i^h \frac{1}{2} - \frac{S_i}{ρ a(t_i)} \right] \] (9g)

\[ K_2 = \frac{1}{ρ a(t_i)} \left[ \frac{S_i R_i}{ρ} - a_i^h U_i - T_i \right] \] (9h)

\[ K_3 = \frac{1}{ρ a(t_{i+1})} \left[ a_i^h \left( 1 + \frac{U_i}{2} \right) + T_i - \frac{S_i}{ρ a(t_{i+1})} \right] \] (9i)

\[ S_i = \frac{\sigma(-ρ a_i^h) + \sigma(ρ a_i^h) + a_i^h R_i}{2} \] (9j)
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\[ \begin{align*}
R_i &= \frac{a(t_i) + a(t_{i+1})}{a(t_i) a(t_{i+1})} \\
U_i &= \frac{a(t_i) - a(t_{i+1})}{a(t_i) a(t_{i+1})}
\end{align*} \] (9k)

and

\[ T_i = \sigma \left( \rho a_i^h \right) - \frac{a_i^h}{2} R_i \] (9m)

The finite difference scheme (9a–m) with VFF is consistent and satisfying the conditions necessary for uniform convergence Doolan et al. (1980); Farrell (1987)

\[ \lim_{h \to 0} \sigma (-\rho a_i^h) = \sigma (-\rho a(0)) \] (10)

And hence, the scheme (9a–m) is fitted exponentially and models the initial layer accurately. The coefficient \( a_i^h \) defined in the scheme with VFF satisfies the condition

\[ \frac{1}{h} \int_{t_i}^{t_{i+1}} a(t) dt - a_i^h \leq C h^3 \] (11)

The scheme (9a–m) reduces to the reduced problem as the small parameter \( \epsilon \) is very small

\[ y_{i+1} = f(t_{i+1}) \] (12)

And so, the scheme (9a–m) meets the requirement for optimal convergence,

\[ \lim_{\epsilon \to 0} \left| f_i^h - a_i^h f(t_{i+1}) \right| = 0. \] (13)

Due to this condition, the scheme (9a–m) works well for a large time.

The operator \( L^h \) in the discrete problem (9a–m) has a maximum discrete principle which follows from the following lemma.

**Lemma 1** If a mesh function \( u_i \) with \( u_0 \geq 0 \) and \( L^h u_i \geq 0 \) for all \( t_i \) in \( D \), then \( u_i \geq 0 \) for all \( t_i \) in \( D \).

**Proof** Take \( u_i \) such that \( u_0 \geq 0 \) and \( L^h u_i \geq 0 \) and assume that \( L^h \) does not admit the maximum discrete principle. Choose the smallest integer \( k \) for which \( u_k > 0 \) and \( u_{k+1} < 0 \). Then

\[ L^h u_i = \left[ \frac{\sigma (\rho a_i^h)}{\rho} + a_i^h \right] u_{k+1} + a_i^h u_k < 0 \]

is a contradiction. \( \square \)

The operator \( L^h \) in the discrete problem (9a–m) is unconditionally stable which is given in the following.

**Lemma 2** If a mesh function \( y_i \geq 0 \) satisfies (9a–m), then

\[ |y_i| \leq |y_0| + \frac{1}{\alpha} \max_{j \geq 0} \left| L^h y_j \right|, \quad i \geq 0. \]

**Proof** Take two functions

\[ \Psi_i = |y_0| + \frac{1}{\alpha} \max_{j \geq 0} \left| L^h y_j \right| \pm y_i, \quad i \geq 0 \]

Clearly

\[ \Psi_0 = |y_0| + \frac{1}{\alpha} \max_{j \geq 0} \left| L^h y_j \right| \pm y_0, \quad i = 0 \]

and

\[ L^h \Psi_i = a_i^h \left( |y_0| + \frac{1}{\alpha} \max_{j \geq 0} \left| L^h y_j \right| \right) \pm L^h y_i, \quad i \geq 0 \]

\[ > \alpha \left( |y_0| + \frac{1}{\alpha} \max_{j \geq 0} \left| L^h y_j \right| \right) \pm L^h y_i > 0 \]

From the maximum discrete principle for \( L^h \), the function \( \Psi_i > 0 \). \( \square \)

Hence, the scheme (9a–m) has a bounded and unique solution by Lemmas 1 to 2.

**Lemma 3** If the function \( a(t) \) satisfies the inequality,

\[ \left| \frac{1}{h} \int_{t_i}^{t_{i+1}} a(t) dt - a_i^h \right| \leq C h^n \]

then

\[ \left| \frac{\exp \left( -\frac{1}{\epsilon} \int_{t_i}^{t_{i+1}} a(t) dt \right) - \exp (-\rho a_i^h)}{1 - \exp (-\rho a_i^h)} \right| \leq C \min(h^n, \epsilon) \]

**Proof** Doolan et al. (1980). \( \square \)

**Theorem 2** If \( y(t) \) is the solution of the problem (Ia,b) and \( y_i \) is the solution of the method (9a–m) then, for all nodal point \( t_i, \ i \geq 0 \), the following inequality holds:

\[ |y(t_i) - y_i| < C \min(h^n, \epsilon). \]

**Proof** From the stability of the operator \( L^h \), it is sufficient to prove that,

\[ |\tau_i| = |L^h [y(t_i) - y_i]| < C \min(h^n, \epsilon), \quad i \geq 0. \]

Here \( \tau_i, \ i \geq 0 \), is the truncation error.

Take \( i = 0, \tau_0 = \eta - \eta = 0 \).

And, again, for \( i \geq 1, \tau_i = L^h [y(t_i) - y_i] = L^h y(t_i) - L^h y_i = L^h y(t_i) - f_i^h \).
From (6), we can write
\[
\tau_i = L^h y_H(t_i) + L^h y_P(t_i) - f_i^h
\]  
(14)

But
\[
L^h y_P(t_i) = f_i^h + \frac{\varepsilon h^2}{2} a_i^h \left[ u_0^{(3)}(\theta_1) - u_0^{(3)}(\theta_2) \right]
\]

where \( t_i < \theta_j < t_{i+1}, j = 1, 2, \)

\[
L^h y_H(t_i) = \frac{\sigma(\rho a_i^h)}{\rho} \left[ \exp \left( \frac{-1}{\varepsilon} \int_{t_i}^{t_{i+1}} a(t) dt \right) - \exp(-\rho a_i^h) \right] y_H(t_i)
\]

Using (15) and (16), we can write (14) as
\[
|\tau_i| \leq C_{\min}(h^3, \varepsilon)
\]
and hence,
\[
|\tau_i| < C_{\min}(h^3, \varepsilon) + \frac{\varepsilon h^2}{2} |a_i^h| \left| \frac{u_0^{(3)}(\theta_1)}{6a(t_i)} \right| + \frac{\varepsilon h^2}{2} \left| a_i^h \right| \left| \frac{u_0^{(3)}(\theta_2)}{3a(t_i+1)} \right| < C_{\min}(h^3, \varepsilon).
\]

Therefore,
\[
|\tau_i| < C_{\min}(h^3, \varepsilon), \text{ for all } i \geq 0.
\]

On applying Lemma 2,
\[
|y(t_i) - y_i| < C_{\min}(h^3, \varepsilon), \text{ for all } i \geq 0.
\]

Hence, the discrete problem (9a–m) is proved as optimal and uniform of order three.

From (6), we can write
\[
\tau_i = L^h y_H(t_i) + L^h y_P(t_i) - f_i^h
\]

But
\[
L^h y_P(t_i) = f_i^h + \frac{\varepsilon h^2}{2} a_i^h \left[ u_0^{(3)}(\theta_1) - u_0^{(3)}(\theta_2) \right]
\]

where \( t_i < \theta_j < t_{i+1}, j = 1, 2, \)

\[
L^h y_H(t_i) = \frac{\sigma(\rho a_i^h)}{\rho} \left[ \exp \left( \frac{-1}{\varepsilon} \int_{t_i}^{t_{i+1}} a(t) dt \right) - \exp(-\rho a_i^h) \right] y_H(t_i)
\]

Using (15) and (16), we can write (14) as
\[
|\tau_i| < C_{\min}(h^3, \varepsilon) + \frac{\varepsilon h^2}{2} |a_i^h| \left| \frac{u_0^{(3)}(\theta_1)}{6a(t_i)} \right| + \frac{\varepsilon h^2}{2} \left| a_i^h \right| \left| \frac{u_0^{(3)}(\theta_2)}{3a(t_i+1)} \right| < C_{\min}(h^3, \varepsilon).
\]

Therefore,
\[
|\tau_i| < C_{\min}(h^3, \varepsilon), \text{ for all } i \geq 0.
\]

On applying Lemma 2,
\[
|y(t_i) - y_i| < C_{\min}(h^3, \varepsilon), \text{ for all } i \geq 0.
\]

Hence, the discrete problem (9a–m) is proved as optimal and uniform of order three.

On applying classical implicit methods for the given problem, it is computationally costlier. And so the main aim of designing the method (9a–m) is to have a computationally cheaper method that is explicit without any constraint on the step size or the coefficients of the given problem. In this section, a numerical method with uniform mesh is designed which converges nodally at the rate \( O(\min(h^3, \varepsilon)) \).

\section{4 Discrete problem with CFF}

The VFF in the discrete problem (9a–m) is evaluated at all nodal points. With a single evaluation at the initial point only, an implicit scheme with a CFF is given in this section. CFF is evaluated just only once rather than at each step. This scheme reduces the calculation time in real-time situations which is important. The fitted mesh method Kolmogorov and Shishkin (1997); Miller (1994) is also constructed.

The discrete problem CFF for all \( 0 \leq i \leq N - 1 \), with the uniform mesh \( h = t_{i+1} - t_i \) and the factor \( \rho = \frac{h}{\varepsilon} \), is defined as follows:
\[
L^h y_i = \varepsilon \sigma(\rho a_i^h) D_+ y_i + a_i^h y_{i+1} = f_i^h, \quad i \geq 0
\]
(17a)
\[
y_0 = \eta,
\]
(17b)

where
\[
\sigma(-\rho a(0)) = \rho a(0)[1 - \exp(-\rho a(0))]^{-1}
\]
(17c)
\[
\sigma(\rho a(0)) = \exp(-\rho a(0))\sigma(-\rho a(0))
\]
(17d)
\[
a_i^h = \frac{1}{12} \left[ 5a(t_{i+1}) + 8a(t_i) - a(t_{i-1}) \right]
\]
(17e)
\[
f_i^h = K_1 f(t_{i-1}) + K_2 f(t_i) + K_3 f(t_{i+1})
\]
(17f)
\[
A = \frac{1}{2} \left[ \frac{1}{a(t_i)} + \frac{1}{a(t_{i+1})} \right]
\]
(17g)
\[
B = \frac{1}{2} \left[ \frac{1}{a(t_i)} - \frac{1}{a(t_{i+1})} \right]
\]
(17h)
\[
C = \frac{\sigma(-\rho a(0)) + \sigma(\rho a(0))}{2} - a_i^h A
\]
(17i)
\[
D = \sigma(\rho a(0)) - a_i^h A
\]
(17j)
\[
K_1 = \frac{1}{\rho^2 a(t_{i+1}) C}
\]
(17k)
\[
K_2 = \frac{2}{\rho^2 C A - \frac{1}{\rho a(t_i)}} \left[ D + B a_i^h \right]
\]
(17l)

and
\[
K_3 = \frac{a_i^h}{a(t_{i+1})} + \frac{1}{\rho a(t_{i+1})} D + \frac{a_i^h}{2 \rho a(t_{i+1}) B}
\]
(17m)

The coefficient \( a_i^h \) defined in both the scheme (17a–m) and the scheme (9a–m) is the same and it satisfies the condition (11). The finite difference scheme (17a–m) with CFF is consistent, satisfying the conditions necessary for uniform and optimal convergence (10) and (13), respectively. And so this scheme works better for large time. This scheme models both initial and outer layers. The maximum discrete principle for the discrete operator follows from Lemma 1 and
the uniform stability follows from Lemma 2. The error estimate was also obtained by the same procedure adopted in Theorem 2. Tables 2, 3, 4, 5, 6 show that for $\epsilon = 10^{-7}$ the absolute error is approximately $O(10^{-9})$, and for $\epsilon = 10^{-2}$, the error is $O(h^3)$ even for $h = 2^{-3}$. And so, scheme (17a–m) is approximately $O(min(h^3, \epsilon))$.

In this section, a numerical method with uniform mesh is designed which converges at all nodal points at the rate $O(min(h^3, \epsilon))$.

5 Construction of fitted mesh method

In the previous sections, numerical methods with uniform mesh are designed which converges at all nodal points at the rate $O(min(h^3, \epsilon))$. These two methods are applicable when the step size is smaller than the value of the parameter $\epsilon$. To view the initial layer when the mesh size is larger than the parameter, these two fitted operator methods are extended to fitted mesh methods since fitted mesh methods are layer rescaling. The construction of the fitted mesh method is provided in this section. That is, the uniform mesh is extended to non-uniform mesh.

First, we shall apply the fitted mesh method (Kolmogorov and Shishkin 1997; Miller 1994, 2019) to the method (9a–m) since fitted mesh methods are layer rescaling.

Now the uniform mesh can be extended to non-uniform mesh $\{t_k\}_{k=0}^N$ with $N$ subintervals of width $h_{i+1} = t_{i+1} - t_i$ for $0 \leq i \leq N - 1$. The first-order operators are defined as $D^+y_i = (y_{i+1} - y_i)/h_{i+1}$.

A piecewise uniform mesh is sufficient for the construction of a $\epsilon$-uniform method. We construct a piecewise uniform mesh on the interval $D = (0, 1)$. Take $N = 2^r$, $r \geq 2$ and choose a transition point $\phi$ subject to the condition $0 \leq \phi \leq \frac{1}{2}$. The transition point divides the interval $(0, 1)$ into two subintervals $(0, \phi)$ and $(\phi, 1)$. Each of these two subintervals is again subdivided into $\frac{N}{2}$ subintervals, and the corresponding piecewise uniform meshes are constructed. Denote $D_N^+$ to be the piecewise uniform mesh with $N$ subintervals and a transition point $\phi$. Locate $\phi$ at

$$\phi = \min \left\{ \frac{1}{2}, 2\log(N) \right\}.$$ 

where $\phi$ depends on $\epsilon$ and $N$. The location of the transition point $\phi$ changes as $\epsilon$ or $N$ changes its values. On taking larger values for $\phi$, the transition point $\phi$ will be located at $\frac{1}{2}$, and so, the mesh $D_N^+$ becomes the uniform mesh with $N$ subintervals. This situation will happen only when $N = e^{\frac{\phi}{2}}$.

And, for all other values of $\phi$ such that $0 < \phi < \frac{1}{2}$, the subinterval $(0, \phi)$ is smaller than $(\phi, 1)$. In this situation, the width of each of the $\frac{N}{2}$ subintervals of $(0, \phi)$ is of width $\frac{2\phi}{N}$ and the width of each of the $\frac{N}{2}$ subintervals of $(\phi, 1)$ is of width $\frac{1-\phi}{N}$. Hence, the global mesh is piecewise uniform because when $\phi$ is very close to the initial point $t = 0$, the mesh will condense in the neighborhood of $t = 0$. The transition point $\phi$ coincides with the mesh point $t = 0$.

Finally, irrespective of the values of $\phi$, the mesh points of $N$ subintervals will be of the mesh points on $D_N^\phi = \{t_k\}_{k=0}^N$, where $t_k$ are the endpoints of each of the subintervals.

5.1 Fitted mesh method for the scheme (9a–m)

The fitted mesh can be applied to the explicit method (9a–m) since it is layer rescaling. The finite difference operator to the explicit method (9a–m) is applied to the piecewise uniform mesh $D_N^\phi = \{t_k\}_{k=0}^N$. This leads to the fitted mesh method of the form,

Find $\{y_k\}_{k=0}^N \in \mathbb{R}^{N+1}$ such that $y_0 = \eta$ and for all $0 \leq i \leq N - 1$, with the non-uniform mesh $h_{i+1} = t_{i+1} - t_i$ and the factor $\rho = \frac{h_{i+1}}{\epsilon}$,

$$L^{h_{i+1}}y_i \equiv \epsilon \sigma(-\rho a_i^{h_{i+1}})D^+_yi + a_i^{h_{i+1}}y_i = f_i^{h_{i+1}}, i \geq 0$$

(18a)

$$y_0 = \eta.$$  

(18b)

where

$$\sigma(-\rho a_i^{h_{i+1}}) = \rho a_i^{h_{i+1}} \left[ 1 - \exp(-\rho a_i^{h_{i+1}}) \right]^{-1}$$

(18c)

$$\sigma(\rho a_i^{h_{i+1}}) = \exp(-\rho a_i^{h_{i+1}})\sigma(-\rho a_i^{h_{i+1}})$$

(18d)

$$a_i^{h_{i+1}} = \frac{1}{12} \left[ 5a(t_{i+1}) + 8a(t_i) - a(t_{i-1}) \right]$$

(18e)

$$f_i^{h_{i+1}} = K_1 f(t_{i-1}) + K_2 f(t_i) + K_3 f(t_{i+1})$$

(18f)

where

$$K_1 = \frac{1}{\rho a(t_{i-1})} \left[ U_i a_i^{h_{i+1}} - \frac{S_i}{\rho a(t_{i-1})} \right]$$

(18g)

$$K_2 = \frac{1}{\rho a(t_i)} \left[ S_i R_i - a_i^{h_{i+1}}U_i - T_i \right]$$

(18h)

$$K_3 = \frac{1}{\rho a(t_{i+1})} \left[ a_i^{h_{i+1}} \left( 1 + \frac{U_i}{2} \right) + T_i - \frac{S_i}{\rho a(t_{i+1})} \right]$$

(18i)

$$S_i = \frac{\sigma(-\rho a_i^{h_{i+1}}) + \sigma(\rho a_i^{h_{i+1}}) + a_i^{h_{i+1}} R_i}{2}$$

(18j)

$$R_i = \frac{a(t_i) + a(t_{i+1})}{a(t_i) a(t_{i+1})}$$

(18k)

$$U_i = \frac{a(t_i) - a(t_{i+1})}{a(t_i) a(t_{i+1})}$$

(18l)
and

\[ T_i = \sigma \left( \rho a_i h_{i+1} \right) \frac{h_{i+1}}{2} - R_i \]  

\[(18m)\]

On using a step size \( h \) in the fitted mesh method (18a-m) we get \( N \) intervals in the domain \( D = [0, 1] \). Again, on taking \( \frac{N}{2} \) intervals in each subdomain \([0, \phi]\) and \([\phi, 1]\), the step size in the domain \([0, \phi]\) will be \( h_1 = \frac{2\phi}{N} \) and the step size in the domain \([\phi, 1]\) will be \( h_2 = \frac{2(1-\phi)}{N} \). And hence, the relation between \( h, h_1 \) and \( h_2 \) are \( h_1 = 2\phi h \) and \( h_2 = 2(1-\phi)h \) where \( h_1 < h_2 < h \). Applying the fitted mesh method (18a-m) to the problem (1a,b), the error estimate will be of the form from Theorem 2,

\[ |y(t_i) - y_i| < C_{\min}(h^3, \epsilon) \text{ for all } i \geq 0 \text{ in the subdomain } [0, \phi] \]

and

\[ |y(t_i) - y_i| < C_{\max}(\min(h^3, \epsilon), \min(h^3, \epsilon)) \text{ for all } i \geq 0 \text{ in the domain } [0, 1]. \]

Thus, \( h_1 = 2\phi h \) and \( h_2 = 2(1-\phi)h \) where \( h_1 < h_2 < h \).

The error estimate will be of the form

\[ |y(t_i) - y_i| < C_{\min}(h^3, \epsilon) \text{ for all } i \geq 0 \text{ in the domain } [0, 1]. \]

### 5.2 Fitted mesh method for the scheme (17a–m)

The fitted mesh can be applied to the implicit method (17a–m) since it is layer rescaling. The finite difference operator to the implicit method (17a–m) is applied to the piecewise uniform mesh \( D_N^\phi = \{t_k^N\}_{k=0}^N \). This leads to the fitted mesh method of the form:

Find \( \{y_k^N\}_{k=0}^N \in \mathbb{R}^{N+1} \) such that \( y_0 = \eta \) and for all \( 0 \leq i \leq N - 1 \), with the non-uniform mesh \( h_{i+1} = t_{i+1} - t_i \) and the factor \( \rho = \frac{h_{i+1}}{t_{i+1}} \),

\[ L_i^h y_i \equiv \epsilon \sigma(\rho a_i(0))D_+y_i + a_i^h y_{i+1} = f_i^h, i \geq 0 \]

\[(19a)\]

\[ y_0 = \eta, \]

\[(19b)\]

where

\[ \sigma(-\rho a(0)) = \rho a(0) \left[ 1 - \exp(-\rho a(0)) \right]^{-1} \]

\[(19c)\]

\[ \sigma(\rho a(0)) = \exp(-\rho a(0))\sigma(-\rho a(0)) \]

\[(19d)\]

\[ d_i^h = \frac{1}{12} \left[ 5a(t_{i+1}) + 8a(t_i) - a(t_{i-1}) \right] \]

\[(19e)\]

\[ f_i^h = K_1 f(t_{i+1}) + K_2 f(t_i) + K_3 f(t_{i-1}) \]

\[(19f)\]

\[ A = \frac{1}{2} \left[ \frac{1}{a(t_i)} + \frac{1}{a(t_{i+1})} \right] \]

\[(19g)\]

\[ B = \frac{1}{2} \left[ \frac{1}{a(t_i)} - \frac{1}{a(t_{i+1})} \right] \]

\[(19h)\]

\[ C = \frac{\sigma(-\rho a(0)) + \sigma(\rho a(0))}{2} - d_i^h A \]

\[(19i)\]

\[ D = \sigma(\rho a(0)) - d_i^h A \]

\[(19j)\]

\[ K_1 = \frac{1}{\rho^2 a(t_{i+1})} \]

\[(19k)\]

\[ K_2 = \frac{\sigma(\rho a(t_{i+1}))C}{\rho^2} \]

\[(19l)\]

\[ K_3 = \frac{d_i^h}{a(t_{i+1})} + \frac{1}{\rho a(t_{i+1})} D + \frac{d_i^h}{2\rho a(t_{i+1})} B \]

\[(19m)\]

Applying the fitted mesh method (19a–m) to the problem (1a,b), the error estimate will be of the form from Theorem 2,

\[ |y(t_i) - y_i| < C_{\min}(h^3, \epsilon) \text{ for all } i \geq 0 \text{ in the subdomain } [0, \phi] \]

and

\[ |y(t_i) - y_i| < C_{\min}(h^3, \epsilon) \text{ for all } i \geq 0 \text{ in the subdomain } [\phi, 1] \]

Combining the subdomains, we have

\[ |y(t_i) - y_i| < C_{\max}(\min(h^3, \epsilon), \min(h^3, \epsilon)) \text{ for all } i \geq 0 \text{ in the domain } [0, 1]. \]

Thus, \( h_1 = 2\phi h \) and \( h_2 = 2(1-\phi)h \) where \( h_1 < h_2 < h \).

The error estimate will be of the form

\[ |y(t_i) - y_i| < C_{\min}(h^3, \epsilon) \text{ for all } i \geq 0 \text{ in the domain } [0, 1]. \]

The main advantage of having fitted mesh methods (18a–m and 19a–m) is these two methods are applicable when the step size is larger than the parameter \( \epsilon \).
6 Numerical experiment

In this section, first we provide numerical results for fitted operator methods of Sects. 3 and 4. Secondly we provide numerical results for fitted mesh methods of Sect. 5. Finally, graphical results are provided for both fitted operator and fitted mesh methods. To show the performance and applicability of the methods proposed in this paper, experimental results are provided with the help of three problems for large time. Three problems are considered for the numerical experiment. They are

Problem 1 Selvakumar (1994a)

\[
\frac{dy(t)}{dt} = -\mu(1 + t)[y(t) - 1] + 1, \quad 0 < t < 10, \quad y(0) = 1.5, \quad \mu = \frac{1}{\epsilon}
\]

Problem 2 Byrne and Hindmarsh (1987)

\[
\frac{dy(t)}{dt} = -\mu[y(t) - e^{-t}] - e^{-t}, \quad 0 < t < 10, \quad y(0) = 0, \quad \mu = \frac{1}{\epsilon}
\]

Problem 3 Doolan et al. (1980)

\[
\frac{dy(t)}{dt} = -\mu[y(t) - p] + p'(t), \quad 0 < t < 10, \quad \mu = \frac{1}{\epsilon}, \quad y(0) = 10, \quad \mu = 10 - (10 + t)e^{-t}.
\]

Here, define

Absolute error \(= \max |y(t_i) - y_i|\)

and

Relative error \(= \max \left| 1 - \frac{y_i}{y(t_i)} \right|\)

where \(y(t_i)\) and \(y_i\) are solutions to (1a, b) and discrete problems.

The order one scheme Farrell (1987) used to compare are

\[
e\sigma(-\rho a(t))D_y y_i + a(t) y_i = f(t_i), \quad y_0 = \eta, \quad \text{(20)}
\]

\[
e\sigma(\rho a(0))D_y y_i + a(t) y_i + \rho a(t) y_{i+1} = f(t_i), \quad y_0 = \eta, \quad \text{(21)}
\]

\[
e\sigma(\rho a(t))D_y y_i + a(t) y_i + \rho a(t) y_{i+1} = f(t_i), \quad y_0 = \eta, \quad \text{(22)}
\]

\[
e \left[ k\sigma(-\rho a(t)) + (1 - k)\sigma(\rho a(t)) \right] D_y y_i + a(t) \left[ k y_i + (1 - k) y_{i+1} \right]
\]

\[
e kf(t_i) + (1 - k) f(t_{i+1}), \quad y_0 = \eta. \quad \text{(24)}
\]

and order one schemes of Ramos (2005) which are same as the schemes (20) and (22) and order two scheme from Doolan et al. (1980)

\[
\mathbf{L}^h y_i \equiv \epsilon \sigma(-\rho a(t)) D_y y_i + a(t) y_i = f(t), \quad y_0 = \eta, \quad \text{(25a)}
\]

where

\[
a(t) = \frac{1}{2} \left[ a(t_{i+1}) + a(t_i) \right] \quad \text{(25b)}
\]

\[
f(t_i) = \frac{1 - \sigma(\rho a(t))}{\rho a(t_i)} f(t_i) + \frac{\sigma(\rho a(t)) - 1}{\rho a(t_{i+1})} f(t_{i+1}) \quad \text{(25c)}
\]

and order two schemes from O’Reilly (1987)

\[
\mathbf{L}^h y_i \equiv \epsilon \sigma(\rho a(0)) D_y y_i + a(t) y_i = f(t), \quad y_0 = \eta, \quad \text{(26a)}
\]

where

\[
\sigma(\rho a(0)) = \rho a(0) \exp(\rho a(0) - 1) \quad \text{(26b)}
\]

\[
\sigma(\rho a(t)) = \rho a(t) \exp(\rho a(t) - 1) \quad \text{(26c)}
\]

\[
a(t) = \frac{1}{2} \left[ a(t_{i+1}) + a(t_i) \right] \quad \text{(26d)}
\]

\[
f(t_i) = \frac{a(t_i) - \sigma(\rho a(t))}{\rho a(t_i)} f(t_i) + \frac{\sigma(\rho a(t)) - a(t_i)}{\rho a(t_{i+1})} f(t_{i+1}) \quad \text{(26e)}
\]

and

\[
\mathbf{L}^h y_i \equiv \epsilon \sigma(\rho a(0)) D_y y_i + a(t) y_i = f(t), \quad y_0 = \eta, \quad \text{(27a)}
\]

where

\[
a(t) = \frac{1}{2} \left[ a(t_{i+1}) + a(t_i) \right] \quad \text{(27b)}
\]

\[
\sigma(\rho a(0)) = \rho a(0) \exp(\rho a(0) - 1) \quad \text{(27c)}
\]

\[
f(t_i) = \frac{a(t_i) - \sigma(\rho a(0))}{\rho a(t_i)} f(t_i) + \frac{\sigma(\rho a(0)) - a(t_i)}{\rho a(t_{i+1})} f(t_{i+1}) \quad \text{(27d)}
\]

The function value of \(a(t)\) in Problem 1 is not a constant function, and so, the numerical result due to the scheme (9a–m) with VFF and scheme (17 a–m) with CFF is different. In the other two problems, function \(a(t)\) is constant and so the numerical result using the scheme (9a–m) and the scheme (17a–m) are the same.
6.1 Experimental results for the schemes (9a–m) and (17a–m)

The observations are

1. The data in Tables 1 and 2 are absolute and relative errors concerning Problem 1. From Table 1, for $\epsilon = 10^{-7}$ the absolute error is approximately $O(10^{-9})$, at the same time for $\epsilon = 10^{-2}$ the absolute error is approximately $O(h^3)$ on applying the fitted operator method, schemes (9a–m). Similar observation from Table 2, for the fitted operator method, scheme (17a–m). And so, for fitted operator methods, schemes (9a–m) and (17a–m) are approximately $O(min(h^3, \epsilon))$.

2. The data in Tables 3 and 4 are absolute and relative errors, respectively, concerning Problem 1. From Tables 3 and 4, for the fitted operator methods, scheme (9a–m) are better than scheme (17a–m). For the fitted mesh methods, scheme (18a–m) is better than scheme (19a–m). But, the Scheme (17a–m) and Scheme (19a–m) are computationally cheaper.

3. The data in Table 5 are absolute and relative errors. The scheme (9a–m) with VFF and scheme (17a–m) with CFF reduce to the scheme (5a–g) since the coefficient $a(t) = 1$ in the Problem 2 and the numerical result is approximate $O(min(h^3, \epsilon))$.

4. The data in Table 6 are absolute and relative errors. The scheme (9a–m) with VFF and scheme (17a–m) with CFF reduce to the scheme (5a–g) since the coefficient $a(t) = 1$ in the Problem 3 and the numerical result is approximate $O(min(h^3, \epsilon))$.

Hence, for the fitted operator methods, scheme (9a–m) with VFF and scheme (17a–m) with CFF are approximately $O(min(h^3, \epsilon))$.

6.2 Experimental results for the schemes (18a–m) and (19a–m)

On taking large step size $h = \frac{1}{16}$ than the parameter $\epsilon = 0.0001$, one cannot view the initial layer using the fitted operator methods. Fitted mesh methods overcome this issue. The observations are

1. The data in Table 7 give the numerical result to the Test Problem 1 using the fitted operator method, Scheme (9a–m) for $h = \frac{1}{16}$, $\epsilon = 0.0001$. Here $\epsilon < h$, the initial layer is hidden in the subdomain $[0, 0.0625]$ and cannot view the initial layer directly. The data in Table 8 give the numerical result to the Test Problem 1 using the fitted mesh method, Scheme (18a–m) for $h = \frac{1}{16}$, $\epsilon = 0.0001$. Here $\epsilon < h$, the initial layer is not hidden in the subdomain $[0, 0.0625]$ and the initial layer is visible directly. Terminal point of the initial layer is $5.54518E-04$ and the step size $h = \frac{1}{16}$ in the domain $[0,1]$ will automatically becomes a variable mesh, $h_1 = 6.93147E-05$ in the initial layer region $[0, 5.54518E-04]$ and $h_2 = 1.24931E-01$ outside the initial layer region $[5.54518E-04, 1]$.

2. Similar observations are made in Tables 9 and 10 concerning Test Problem 2 with $a(t) = 1$, using the fitted operator method Scheme (9a–m) and fitted, mesh method (18a–m), respectively. Same result on using the fitted operator method Scheme (17a–m) and fitted, mesh method (19a–m) since $a(t) = 1$ in Test Problem 2.

3. Similar observations are made in Tables 11 and 12 concerning Test Problem 3 with $a(t) = 1$, using the fitted operator method Scheme (9a–m) and fitted, mesh method (18a–m), respectively. Same result on using the fitted operator method Scheme (17a–m) and fitted, mesh method (19a–m) since $a(t) = 1$ in Test Problem 2.

### Table 1 Problem 1, Scheme (9a–m)

| $\epsilon$ | $h = 1/8$ | $1/16$ | $1/32$ | $1/64$ |
|------------|-----------|--------|--------|--------|
| (i) maximum absolute error at all nodal points | | | | |
| 0.01       | 2.42583E-04 | 1.04632E-04 | 3.42333E-05 | 9.59652E-05 |
| 0.001      | 2.74378E-05 | 1.42911E-05 | 7.1932E-06 | 3.6764E-06 |
| 0.0001     | 2.77439E-06 | 1.46645E-06 | 7.5302E-07 | 3.7994E-07 |
| 0.00001    | 2.77744E-07 | 1.47017E-07 | 7.3712E-08 | 3.8431E-08 |
| 0.000001   | 2.77774E-08 | 1.47055E-08 | 7.5753E-09 | 3.8456E-09 |
| 0.0000001  | 2.77777E-09 | 1.47058E-09 | 7.5757E-10 | 3.8461E-10 |
| (ii) maximum relative error at all nodal points | | | | |
| 0.01       | 2.15629E-04 | 9.84040E-05 | 3.24438E-05 | 9.1308E-06 |
| 0.001      | 2.43892E-05 | 1.34504E-05 | 6.90358E-06 | 3.3158E-06 |
| 0.0001     | 2.46612E-06 | 1.38019E-06 | 7.3020E-07 | 3.7400E-07 |
| 0.00001    | 2.46883E-07 | 1.38369E-07 | 7.3417E-08 | 3.8113E-07 |
| 0.000001   | 2.46911E-08 | 1.38404E-08 | 7.3457E-09 | 3.7865E-09 |
| 0.0000001  | 2.46913E-09 | 1.38408E-09 | 7.3461E-10 | 3.7869E-10 |
Table 2 Problem 1, Scheme (17a–m)

| $\epsilon$  | $h = 1/8$ | $1/16$ | $1/32$ | $1/64$ |
|-------------|----------|--------|--------|--------|
| ($i$) maximum absolute error at all nodal points | | | | |
| 0.01        | 2.49528E-04 | 9.68433E-05 | 6.89616E-04 | 6.20552E-04 |
| 0.001       | 2.89394E-05 | 1.46680E-05 | 7.20365E-06 | 3.37460E-05 |
| 0.0001      | 2.93670E-06 | 1.51006E-06 | 7.64199E-07 | 3.82699E-07 |
| 0.00001     | 2.94063E-07 | 1.51462E-07 | 7.69713E-08 | 3.87089E-08 |
| 0.000001    | 2.94112E-07 | 1.51510E-07 | 7.69177E-09 | 3.87545E-09 |
| ($ii$) maximum relative error at all nodal points | | | | |
| 0.01        | 2.21812E-04 | 8.60828E-05 | 6.55421E-04 | 6.54898E-04 |
| 0.001       | 2.57239E-05 | 1.38052E-05 | 6.98536E-06 | 3.32263E-06 |
| 0.0001      | 2.60984E-06 | 1.42123E-06 | 7.45148E-07 | 3.76713E-07 |
| 0.00001     | 2.57238E-07 | 1.38052E-07 | 7.45418E-08 | 3.81134E-08 |
| 0.000001    | 2.61433E-08 | 1.42597E-08 | 7.45869E-09 | 3.81583E-09 |

Table 3 Problem 1, maximum absolute error at the nodal points are compared

| Schemes       | $h = 1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---------------|----------|--------|--------|--------|
| $\epsilon = 0.01$ | | | | |
| Backward Euler | 3.31942E-02 | 6.46458E-02 | 9.74873E-02 | 8.97464E-02 |
| Trapezoidal   | 1.19160E+01 | 1.28761E+01 | 1.49662E+01 | 1.99072E+01 |
| Scheme (18)   | 1.24081E-01 | 6.15854E-02 | 3.03383E-02 | 1.47152E-02 |
| Scheme (19)   | 1.24080E-01 | 6.15969E-02 | 3.04699E-02 | 1.50909E-02 |
| Scheme (20)   | 8.88363E-03 | 9.17709E-03 | 8.36967E-03 | 5.72014E-03 |
| Ramos-explicit| 8.88363E-03 | 9.17709E-03 | 8.36967E-03 | 5.72014E-03 |
| Scheme (21)   | 8.88932E-03 | 9.39572E-03 | 8.35967E-03 | 5.72014E-03 |
| Scheme (22)   | 1.78130E-02 | 1.98070E-02 | 9.70137E-03 | 6.21510E-03 |
| Ramos-implicit| 1.78130E-02 | 1.98070E-02 | 9.70137E-03 | 6.21510E-03 |
| Scheme (23a-c)| 4.39405E-04 | 1.94550E-04 | 6.44922E-05 | 1.81198E-05 |
| Scheme (25a-d)| 7.97510E-05 | 2.93595E-04 | 8.00371E-04 | 6.73056E-04 |
| Scheme (24a-e)| 7.89165E-05 | 8.76188E-05 | 7.73668E-05 | 1.81198E-05 |
| Scheme (5a-q) | 1.36971E-04 | 5.41210E-05 | 2.44379E-04 | 1.16825E-05 |
| Scheme (17a-m)| 2.49528E-04 | 9.68433E-05 | 6.89616E-04 | 6.20552E-04 |
| Scheme (19a-m)| 4.72458E-04 | 2.19576E-04 | 2.49859E-04 | 1.59343E-04 |
| Scheme (9a-m) | 2.42583E-04 | 1.04632E-04 | 3.42333E-05 | 9.59652E-05 |
| Scheme (18a-m)| 4.27338E-04 | 2.07218E-04 | 8.52123E-04 | 2.67431E-05 |

$\epsilon = 0.001$

| Schemes       | $h = 1/8$ | $1/16$ | $1/32$ | $1/64$ |
|---------------|----------|--------|--------|--------|
| Backward Euler | 3.53062E-02 | 7.41780E-03 | 1.50483E-02 | 2.96398E-02 |
| Trapezoidal   | 1.10150E+01 | 1.11662E+01 | 1.13571E+01 | 1.10837E+01 |
| Scheme(18)    | 1.24080E-01 | 6.24084E-02 | 3.11584E-02 | 1.55344E-02 |
| Scheme (19)   | 1.24080E-01 | 6.24084E-02 | 3.11584E-02 | 1.55344E-02 |
| Scheme (20)   | 8.88944E-04 | 9.41277E-04 | 9.69768E-04 | 9.84669E-04 |
| Ramos-explicit| 8.88944E-04 | 9.41277E-04 | 9.69768E-04 | 9.84669E-04 |
| Scheme (21)   | 8.88944E-04 | 9.41277E-04 | 9.69768E-04 | 9.84669E-04 |
| Scheme (22)   | 8.81255E-03 | 2.95317E-03 | 1.48833E-03 | 1.12200E-03 |
| Ramos-implicit| 8.81255E-03 | 2.95317E-03 | 1.48833E-03 | 1.12200E-03 |
| Scheme (23a-c)| 5.13792E-05 | 2.75373E-05 | 1.39475E-05 | 6.55651E-06 |
| Scheme (25a-d)| 8.34465E-07 | 9.53674E-07 | 9.53674E-07 | 1.07288E-06 |
### Table 3

Table 3 continued

| Schemes                  | $h = 1/8$       | 1/16       | 1/32       | 1/64       |
|--------------------------|-----------------|------------|------------|------------|
| Scheme (24a–e)           | 8.34465E-07     | 9.53674E-07| 9.53674E-07| 6.55651E-06|
| Scheme (5a–q)            | 1.50204E-05     | 5.84126E-06| 2.74181E-06| 1.43051E-06|
| Scheme (17a–m)           | 2.89394E-05     | 1.46680E-05| 7.20365E-05| 3.37460E-05|
| Scheme (19a–m)           | 5.10189E-05     | 5.10189E-05| 5.73077E-04| 1.61174E-04|
| Scheme (9a–m)            | 2.74378E-05     | 1.42911E-05| 7.11932E-06| 3.36764E-06|
| Scheme (18a–m)           | 4.92486E-05     | 2.76176E-05| 1.40079E-05| 6.94709E-06|
| $\epsilon = 0.00001$    |                 |            |            |            |
| Backward Euler           | 3.55244E-05     | 7.53404E-05| 1.55091E-04| 3.14832E-04|
| Trapezoidal              | 1.10130E+01     | 1.10731E+01| 1.10963E+01| 1.17216E+01|
| Scheme (18)              | 7.82251E-03     | 1.96314E-03| 4.98295E-04| 1.32084E-04|
| Scheme (19)              | 7.82251E-03     | 1.96314E-03| 4.98295E-04| 1.32084E-04|
| Scheme (20)              | 7.90071E-03     | 8.63075E-03| 7.67410E-03| 5.36454E-03|
| Ramos-explicit           | 7.90071E-03     | 8.63075E-03| 7.67410E-03| 5.36454E-03|
| Scheme (21)              | 7.90167E-03     | 8.83639E-03| 7.94518E-03| 5.44262E-03|
| Scheme (22)              | 1.58337E-02     | 1.12675E-02| 9.22036E-03| 5.90694E-03|
| Scheme (5a–q)            | 2.38419E-07     | 1.19209E-07| 1.19209E-07| 1.19209E-07|
| Scheme (17a–m)           | 2.94063E-07     | 1.51462E-07| 7.69713E-08| 3.87098E-08|
| Scheme (19a–m)           | 5.61412E-07     | 3.76530E-07| 2.53635E-07| 1.61847E-07|
| Scheme (9a–m)            | 2.77744E-07     | 1.47017E-02| 7.37121E-08| 3.84138E-08|
| Scheme (18a–m)           | 4.99250E-07     | 2.77701E-07| 1.46988E-07| 7.56936E-08|

### Table 4

Problem 1, maximum relative error at the nodal points are compared

| Schemes                  | $h = 1/8$       | 1/16       | 1/32       | 1/64       |
|--------------------------|-----------------|------------|------------|------------|
| $\epsilon = 0.001$       |                 |            |            |            |
| Backward Euler           | 2.95060E-02     | 6.07977E-02| 9.26534E-02| 8.01909E-02|
| Trapezoidal              | 2.06509E+00     | 2.16940E+00| 3.06724E+00| 3.53353E+00|
| Scheme (18)              | 1.02221E-01     | 4.92319E-02| 2.13697E-02| 9.20743E-02|
| Scheme (19)              | 1.02221E-01     | 4.92319E-02| 2.13946E-02| 9.32878E-03|
| Scheme (20)              | 7.90071E-03     | 8.63075E-03| 7.67410E-03| 5.36454E-03|
| Ramos-explicit           | 7.90071E-03     | 8.63075E-03| 7.67410E-03| 5.36454E-03|
| Scheme (21)              | 7.90167E-03     | 8.83639E-03| 7.94518E-03| 5.44262E-03|
| Scheme (22)              | 1.58337E-02     | 1.12675E-02| 9.22036E-03| 5.90694E-03|
| Scheme (5a–q)            | 1.58337E-02     | 1.12675E-02| 9.22036E-03| 5.90694E-03|
| Scheme (23a–c)           | 3.90589E-04     | 1.82986E-04| 6.09756E-05| 1.72257E-05|
| Scheme (25a–d)           | 7.09295E-05     | 2.15650E-04| 7.66674E-04| 6.01411E-04|
| Scheme (24a–e)           | 7.00951E-05     | 8.23736E-05| 7.35521E-05| 1.72257E-05|
| Scheme (5a–q)            | 1.06813E-04     | 3.96967E-05| 1.74046E-05| 8.34656E-06|
| Scheme (17a–m)           | 2.21812E-04     | 8.60828E-05| 6.55421E-04| 6.54489E-04|
| Scheme (19a–m)           | 4.63821E-04     | 3.07930E-04| 2.31866E-04| 1.47273E-04|
| Scheme (9a–m)            | 2.15629E-04     | 9.84040E-05| 3.24438E-05| 9.13084E-06|
| Scheme (18a–m)           | 3.33548E-04     | 1.76578E-04| 7.55774E-05| 2.38218E-05|
Table 4 continued

| Schemes          | $h = 1/8$ | $1/16$ | $1/32$ | $1/64$ |
|------------------|-----------|--------|--------|--------|
| Backward Euler   | 3.13830E-03 | 6.98149E-03 | 1.45923E-02 | 2.91839E-02 |
| Trapezoidal      | 2.16626E+00 | 2.28175E+00 | 2.31153E+00 | 2.27309E+00 |
| Scheme (18)      | 1.02221E-01 | 5.78823E-02 | 2.93333E-02 | 1.44001E-02 |
| Scheme (19)      | 1.10222E-01 | 5.78823E-02 | 2.93333E-02 | 1.44001E-02 |
| Scheme (20)      | 7.90119E-04 | 8.85963E-04 | 9.40323E-04 | 9.69529E-04 |
| Ramos-explicit   | 7.90119E-04 | 8.85963E-04 | 9.40323E-04 | 9.69529E-04 |
| Scheme (21)      | 7.90119E-04 | 8.85963E-04 | 9.40323E-04 | 9.69529E-04 |
| Scheme (22)      | 7.83336E-03 | 2.77948E-03 | 1.44327E-03 | 1.04718E-03 |
| Ramos-implicit   | 7.83336E-03 | 2.77948E-03 | 1.44327E-03 | 1.04718E-03 |
| Scheme (23a–c)   | 4.56572E-05 | 2.59280E-05 | 1.35303E-05 | 6.43730E-06 |
| Scheme (25a–d)   | 7.15256E-07 | 9.53674E-07 | 9.53674E-07 | 6.43730E-06 |
| Scheme (24a–e)   | 7.15256E-07 | 9.53674E-07 | 9.53674E-07 | 6.43730E-06 |
| Scheme (5a–q)    | 1.23978E-05 | 4.52990E-05 | 2.14577E-05 | 1.07288E-05 |
| Scheme (17a–m)   | 2.57239E-05 | 1.38052E-05 | 6.98536E-06 | 3.32263E-06 |
| Scheme (19a–m)   | 4.79301E-05 | 3.53226E-05 | 3.88854E-05 | 3.31583E-06 |
| Scheme (9a–m)    | 2.43892E-05 | 1.34504E-05 | 6.90358E-06 | 3.31583E-06 |
| Scheme (18a–m)   | 3.93008E-05 | 2.39125E-05 | 1.31025E-05 | 6.68435E-06 |

Table 5  Problem 2., Schemes (9a–m) and (17a–m)

| $\epsilon$ | $h = 1/8$ | $1/16$ | $1/32$ | $1/64$ |
|-------------|-----------|--------|--------|--------|
|             |           |        |        |        |
| (i) maximum absolute error at all nodal points |
| 0.01        | 2.21064E-04 | 9.80022E-05 | 3.27046E-05 | 9.18118E-06 |
| 0.001       | 2.58280E-05 | 1.38937E-05 | 7.01294E-06 | 3.33645E-06 |
| 0.00001     | 2.62331E-05 | 1.43433E-07 | 7.48536E-08 | 3.82038E-08 |
| (ii) maximum relative error at all nodal points |
| 0.01        | 2.60499E-04 | 1.04532E-04 | 3.48854E-05 | 9.71556E-06 |
| 0.001       | 2.92646E-05 | 1.47895E-05 | 7.23556E-06 | 3.38900E-06 |
| 0.00001     | 2.97260E-07 | 1.52684E-07 | 7.72297E-08 | 3.82038E-08 |
### Table 6

Problem 3, Schemes (9a–m) and (17a–m)

| $\epsilon$ | $h = 1/8$ | $1/16$ | $1/32$ | $1/64$ |
|------------|-----------|--------|--------|--------|
| (i) maximum absolute error at all nodal points |
| 0.01       | 2.95100E-03 | 3.84038E-03 | 1.96914E-03 | 1.15187E-03 |
| 0.001      | 3.46145E-04 | 4.02858E-04 | 4.26456E-04 | 4.16105E-04 |
| 0.00001    | 3.51795E-06 | 4.16078E-06 | 4.55340E-06 | 4.76580E-06 |
| (ii) maximum relative error at all nodal points |
| 0.01       | 2.77153E-03 | 5.01426E-03 | 3.52032E-03 | 2.27957E-03 |
| 0.001      | 3.25105E-04 | 7.36276E-04 | 1.53945E-05 | 2.97153E-03 |
| 0.00001    | 3.30411E-06 | 7.64383E-06 | 6.41585E-06 | 3.41261E-05 |

### Table 7

Problem 1, Scheme (9a–m), $h = 1/16$, $\epsilon = 0.0001$

| Time        | Numerical solution | Exact solution | Absolute error |
|-------------|--------------------|----------------|----------------|
| 0.00000E+000 | 1.50000E+000       | 1.50000E+000   | 0.00000E+000   |
| 6.25000E-002 | 1.06250E+000       | 1.06250E+000   | 1.46645E-006   |
| 1.25000E-001 | 1.12500E+000       | 1.12500E+000   | 1.30370E-006   |
| 1.87500E-001 | 1.18750E+000       | 1.18750E+000   | 1.16662E-006   |
| 2.50000E-001 | 1.25000E+000       | 1.25000E+000   | 1.05099E-006   |
| 3.12500E-001 | 1.31250E+000       | 1.31250E+000   | 9.50181E-007   |
| 3.75000E-001 | 1.37500E+000       | 1.37500E+000   | 8.63886E-007   |
| 4.37500E-001 | 1.43750E+000       | 1.43750E+000   | 7.88838E-007   |
| 5.00000E-001 | 1.50000E+000       | 1.50000E+000   | 7.23162E-007   |
| 5.62500E-001 | 1.56250E+000       | 1.56250E+000   | 6.65361E-007   |
| 6.25000E-001 | 1.62500E+000       | 1.62500E+000   | 6.14223E-007   |
| 6.87500E-001 | 1.68750E+000       | 1.68750E+000   | 5.68763E-007   |
| 7.50000E-001 | 1.75000E+000       | 1.75000E+000   | 5.28170E-007   |
| 8.12500E-001 | 1.81250E+000       | 1.81250E+000   | 4.91773E-007   |
| 8.75000E-001 | 1.87500E+000       | 1.87500E+000   | 4.59013E-007   |
| 9.37500E-001 | 1.93750E+000       | 1.93750E+000   | 4.29422E-007   |
| 1.00000E+000 | 2.00000E+000       | 2.00000E+000   | 4.02602E-007   |

### Table 8

Problem 1, Scheme (18a–m), $h = 1/16$, $\epsilon = 0.0001$, Terminal point of the initial layer=5.54518E-004, $h_1 = 6.93147E-005$, $h_2 = 1.24931-001$

| Time        | Numerical solution | Exact solution | Absolute error |
|-------------|--------------------|----------------|----------------|
| 0.00000E+000 | 1.50000E+000       | 1.50000E+000   | 0.00000E+000   |
| 6.93147E-005 | 1.25006E+000       | 1.25006E+000   | 9.93303E-011   |
| 1.38629E-004 | 1.12513E+000       | 1.12513E+000   | 1.49809E-010   |
| 2.07944E-004 | 1.06269E+000       | 1.06269E+000   | 1.73812E-010   |
| 2.77259E-004 | 1.03152E+000       | 1.03152E+000   | 1.86215E-010   |
| 3.46574E-004 | 1.01596E+000       | 1.01596E+000   | 1.92409E-010   |
| 4.15888E-004 | 1.00822E+000       | 1.00822E+000   | 1.95499E-010   |
| 4.85203E-004 | 1.00439E+000       | 1.00439E+000   | 1.97036E-010   |
| 5.54518E-004 | 1.00250E+000       | 1.00250E+000   | 0.00000E+000   |
| 6.25087E-001 | 1.12548E+000       | 1.12548E+000   | 2.77012E-006   |
| 7.50027E-001 | 1.25041E+000       | 1.25041E+000   | 2.21681E-006   |
| 8.75347E-001 | 1.37534E+000       | 1.37534E+000   | 1.81423E-006   |
| 5.00273E-001 | 1.50028E+000       | 1.50028E+000   | 1.51220E-006   |
| 6.25088E-001 | 1.62521E+000       | 1.62521E+000   | 1.27979E-006   |
Table 8 continued

| Time     | Numerical solution | Exact solution | Absolute error |
|----------|--------------------|----------------|----------------|
| 7.50139E-001 | 1.75014E+000 | 1.75014E+000 | 1.09714E-006 |
| 8.75069E-001 | 1.87507E+000 | 1.87507E+000 | 9.50992E-007 |
| 1.00000E+000 | 2.00000E+000 | 2.00000E+000 | 8.32221E-007 |

operator method Scheme (9a–m) and fitted, mesh method (18a–m) respectively. Same result on using the fitted operator method Scheme (17a–m) and fitted, mesh method (19a–m) since \(a(t)=1\) in Test Problem 3.

4. The data in Table 13 give the numerical result to the Test Problem 1 using the fitted operator method with CFF, Scheme (17a–m) for \(h = \frac{1}{16}, \epsilon = 0.0001\). Here \(\epsilon < h\), the initial layer is hidden in the subdomain \([0, 0.0625]\) and cannot view the initial layer directly. The data in Table 14 give the numerical result to the Test Problem 1 using the fitted mesh method, Scheme (19a–m) for \(h = \frac{1}{16}, \epsilon = 0.0001\). Here \(\epsilon < h\), the initial layer is not hidden in the subdomain \([0, 0.0625]\) and the initial layer is visible directly. Terminal point of the initial layer is 5.54518E-04 and the step size \(h = \frac{1}{16}\) will be automatically becomes a variable mesh, \(h_1 = 6.93147E-05\) in the initial layer [0, 5.54518E-04] and \(h_2 = 1.24931E-01\) outside the initial layer [5.54518E-04, 1].

5. Table 15 gives an absolute error and relative error concerning Problem 2 with \(a(t) = 1\). Since \(a(t) = 1\), for fitted mesh methods, Schemes (18a–m) and Scheme (19a–m) are the same, and for small values of the parameter \(\epsilon\), the absolute and relative errors approach \(O(\epsilon)\) even for large values of the step size. The fitted mesh methods are of \(O(\min(h^3, \epsilon))\).

6. Table 16 gives an absolute error and relative error concerning Problem 3 with \(a(t) = 1\). Since \(a(t) = 1\), for the fitted mesh methods, Schemes (18a–m) and Scheme (19a–m) are the same, and for small values of the parameter \(\epsilon\), the absolute and relative errors approach \(O(\epsilon)\) even for large values of the step size. The fitted mesh methods are of \(O(\min(h^3, \epsilon))\).

6.3 Graphical results

The graph of the numerical solution, exact solution and absolute error are plotted in Figs. 1, 2, 3, 4, 5, 6, 7, 8. Tables 7, 8, 9, 10, 11, 12, 13, 14 are used to plot the figures. The absolute error is very closer to the horizontal axis.

In Problem 1, both \(a(t)\) and \(f(t)\) are variable coefficients. And so the scheme with VFF and CFF will differ. On solving Problem 1 using the fitted operator method (9a–m) with VFF, \(\epsilon = 0.0001\), and \(h = 1/16\) in the interval [0, 1], the graph of the numerical solution is shown in Fig. 1. The numerical solution fit with the exact solution. In the interval [0, 0.0001], the correct graph of the numerical solution is not visible since \(h\) is larger than \(\epsilon\). In the neighborhood of \(t = 0\), the solution graph

Table 9 Problem 2, Scheme (9a–m) Scheme (17a–m), \(h = 1/16, \epsilon = 0.0001\)

| Time     | Numerical solution | Exact solution | Absolute error |
|----------|--------------------|----------------|----------------|
| 0.00000E+000 | 0.00000E+000 | 0.00000E+000 | 0.00000E+000 |
| 6.25000E-002 | 9.39414E-001 | 9.39413E-001 | 1.43025E-006 |
| 1.25000E-001 | 8.82498E-001 | 8.82497E-001 | 1.34359E-006 |
| 1.87500E-001 | 8.29030E-001 | 8.29029E-001 | 1.26219E-006 |
| 2.50000E-001 | 7.78802E-001 | 7.78801E-001 | 1.18572E-006 |
| 3.12500E-001 | 7.31617E-001 | 7.31616E-001 | 1.11388E-006 |
| 3.75000E-001 | 6.87290E-001 | 6.87289E-001 | 1.04639E-006 |
| 4.37500E-001 | 6.45650E-001 | 6.45649E-001 | 9.82994E-007 |
| 5.00000E-001 | 6.06532E-001 | 6.06531E-001 | 9.23438E-007 |
| 5.62500E-001 | 5.69784E-001 | 5.69783E-001 | 8.67489E-007 |
| 6.25000E-001 | 5.35262E-001 | 5.35261E-001 | 8.14931E-007 |
| 6.87500E-001 | 5.02832E-001 | 5.02832E-001 | 7.65557E-007 |
| 7.50000E-001 | 4.72367E-001 | 4.72367E-001 | 7.19174E-007 |
| 8.12500E-001 | 4.43748E-001 | 4.43747E-001 | 6.75601E-007 |
| 8.75000E-001 | 4.16863E-001 | 4.16862E-001 | 6.34699E-007 |
| 9.37500E-001 | 3.91606E-001 | 3.91606E-001 | 5.96216E-007 |
| 1.00000E+000 | 3.67880E-001 | 3.67879E-001 | 5.6093E-007 |
### Table 10
Problem 2, Scheme (18a–m), $h = 1/16$, $\epsilon = 0.0001$, Terminal point of the initial layer = 5.54518E-004, $h_1 = 6.93147E-005$, $h_2 = 1.24931-001$

| Time        | Numerical solution | Exact solution | Absolute error |
|-------------|--------------------|----------------|----------------|
| 0.00000E+000 | 0.00000E+000      | 0.00000E+000  | 0.00000E+000   |
| 6.93147E-005 | 4.99931E-001      | 4.99931E-001  | 9.92849E-011   |
| 1.38629E-004 | 7.49861E-001      | 7.49861E-001  | 1.48921E-010   |
| 2.07944E-004 | 8.74792E-001      | 8.74792E-001  | 1.73731E-010   |
| 2.77259E-004 | 9.37223E-001      | 9.37223E-001  | 1.86130E-010   |
| 3.46574E-004 | 9.68403E-001      | 9.68403E-001  | 1.92322E-010   |
| 4.15888E-004 | 9.91702E-001      | 9.91702E-001  | 1.96950E-010   |
| 4.85203E-004 | 9.95539E-001      | 9.95539E-001  | 0.00000E+000   |
| 5.54518E-004 | 9.99931E-001      | 9.99931E-001  | 2.61700E-006   |
| 1.25485E-001 | 8.82071E-001      | 8.82071E-001  | 2.03840E-006   |
| 2.50416E-001 | 7.78479E-001      | 7.78479E-001  | 2.30965E-006   |
| 3.75347E-001 | 6.87053E-001      | 6.87053E-001  | 2.58773E-006   |
| 5.00277E-001 | 6.06364E-001      | 6.06364E-001  | 2.81670E-006   |
| 6.25208E-001 | 5.35152E-001      | 5.35152E-001  | 3.09516E-006   |
| 7.50139E-001 | 4.72302E-001      | 4.72302E-001  | 3.42160E-006   |
| 8.75069E-001 | 4.16834E-001      | 4.16834E-001  | 3.86700E-006   |
| 1.00000E+000 | 3.67881E-001      | 3.67881E-001  | 4.32240E-006   |

### Table 11
Problem 3, Scheme (9a–m), $h = 1/16$, $\epsilon = 0.0001$

| Time        | Numerical solution | Exact solution | Absolute error |
|-------------|--------------------|----------------|----------------|
| 0.00000E+000 | 1.00000E+001      | 1.00000E+001  | 0.00000E+000   |
| 6.25000E-002 | 5.47198E-001      | 5.47198E-001  | 4.14876E-005   |
| 1.25000E-001 | 1.06476E+000      | 1.06476E+000  | 3.88900E-005   |
| 1.87500E-001 | 1.55430E+000      | 1.55430E+000  | 3.64549E-005   |
| 2.50000E-001 | 2.01733E+000      | 2.01733E+000  | 3.41721E-005   |
| 3.12500E-001 | 2.45525E+000      | 2.45525E+000  | 3.20321E-005   |
| 3.75000E-001 | 2.86940E+000      | 2.86940E+000  | 3.00260E-005   |
| 4.37500E-001 | 3.26107E+000      | 3.26107E+000  | 2.81454E-005   |
| 5.00000E-001 | 3.63145E+000      | 3.63145E+000  | 2.63824E-005   |
| 5.62500E-001 | 3.98169E+000      | 3.98169E+000  | 2.47298E-005   |
| 6.25000E-001 | 4.31287E+000      | 4.31287E+000  | 2.31805E-005   |
| 6.87500E-001 | 4.62601E+000      | 4.62601E+000  | 2.17282E-005   |
| 7.50000E-001 | 4.92208E+000      | 4.92208E+000  | 2.03668E-005   |
| 8.12500E-001 | 5.20200E+000      | 5.20200E+000  | 1.90907E-005   |
| 8.75000E-001 | 5.46664E+000      | 5.46664E+000  | 1.78943E-005   |
| 9.37500E-001 | 5.71681E+000      | 5.71681E+000  | 1.67729E-005   |
| 1.00000E+000 | 5.95334E+000      | 5.95334E+000  | 1.57217E-005   |

### Table 12
Problem 3, Scheme (18a–m), $h = 1/16$, $\epsilon = 0.0001$, Terminal point of the initial layer = 5.54518E-004, $h_1 = 6.93147E-005$, $h_2 = 1.24931-001$

| Time        | Numerical solution | Exact solution | Absolute error |
|-------------|--------------------|----------------|----------------|
| 0.00000E+000 | 1.00000E+001      | 1.00000E+001  | 0.00000E+000   |
| 6.93147E-005 | 5.00062E+000      | 5.00062E+000  | 7.94297E-010   |
| 1.38629E-004 | 2.50125E+000      | 2.50125E+000  | 1.19140E-009   |
| 2.07944E-004 | 1.25187E+000      | 1.25187E+000  | 1.38990E-009   |
| 2.77259E-004 | 6.27495E-001      | 6.27495E-001  | 1.48910E-009   |
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Table 12  continued
| Time | Numerical solution | Exact solution | Absolute error |
|------|-------------------|----------------|----------------|
| 3.46574E-004 | 3.15619E-004 | 3.15619E-004 | 1.53865E-009 |
| 4.15888E-004 | 1.59992E-004 | 1.59992E-004 | 1.56338E-009 |
| 4.85203E-004 | 8.24909E-004 | 8.24909E-004 | 1.57570E-009 |
| 5.54518E-004 | 4.40519E-004 | 4.40519E-004 | 0.00000E+000 |
| 1.25485E-001 | 1.06860E+000 | 1.06863E+000 | 2.14182E-005 |
| 2.50416E-001 | 2.02027E+000 | 2.02029E+000 | 1.91914E-005 |
| 3.75347E-001 | 2.87159E+000 | 2.87161E+000 | 1.71922E-005 |
| 5.00277E-001 | 4.40519E-004 | 4.40519E-004 | 0.00000E+000 |
| 1.25485E-001 | 1.06860E+000 | 1.06863E+000 | 2.14182E-005 |
| 2.50416E-001 | 2.02027E+000 | 2.02029E+000 | 1.91914E-005 |
| 3.75347E-001 | 2.87159E+000 | 2.87161E+000 | 1.71922E-005 |
| 5.00277E-001 | 4.40519E-004 | 4.40519E-004 | 0.00000E+000 |

Table 13  Problem 1, Scheme (17a–m), \( h = 1/16, \epsilon = 0.0001 \)
| Time       | Numerical solution | Exact solution | Absolute error |
|------------|--------------------|----------------|----------------|
| 0.00000E+000 | 1.50000E+000 | 1.50000E+000 | 0.00000E+000 |
| 6.25000E-002 | 1.06250E+000 | 1.06250E+000 | 1.51006E-006 |
| 1.25000E-001 | 1.12500E+000 | 1.12500E+000 | 1.41380E-006 |
| 1.87500E-001 | 1.18750E+000 | 1.18750E+000 | 1.32242E-006 |
| 2.50000E-001 | 1.25000E+000 | 1.25000E+000 | 1.23677E-006 |
| 3.12500E-001 | 1.31250E+000 | 1.31250E+000 | 1.15712E-006 |
| 3.75000E-001 | 1.37500E+000 | 1.37500E+000 | 1.08344E-006 |
| 4.37500E-001 | 1.43750E+000 | 1.43750E+000 | 1.01547E-006 |
| 5.00000E-001 | 1.50000E+000 | 1.50000E+000 | 9.52868E-007 |
| 5.62500E-001 | 1.56250E+000 | 1.56250E+000 | 8.95258E-007 |
| 6.25000E-001 | 1.62500E+000 | 1.62500E+000 | 8.42237E-007 |
| 6.87500E-001 | 1.68750E+000 | 1.68750E+000 | 7.53416E-007 |
| 7.50000E-001 | 1.75000E+000 | 1.75000E+000 | 7.48427E-007 |
| 8.12500E-001 | 1.81250E+000 | 1.81250E+000 | 7.06927E-007 |
| 8.75000E-001 | 1.87500E+000 | 1.87500E+000 | 6.86601E-007 |
| 9.37500E-001 | 1.93750E+000 | 1.93750E+000 | 6.33160E-007 |
| 1.00000E+000 | 2.00000E+000 | 2.00000E+000 | 6.00344E-007 |

Table 14  Problem 6.1, Scheme (19a–m), \( h = 1/16, \epsilon = 0.0001 \), Terminal point of the initial layer = 5.54518E-004, \( h_1 = 6.93147E-005, h_2 = 1.24931-001 \)
| Time       | Numerical solution | Exact solution | Absolute error |
|------------|--------------------|----------------|----------------|
| 0.00000E+000 | 1.50000E+000 | 1.50000E+000 | 0.00000E+000 |
| 6.93147E-005 | 1.25000E+000 | 1.25000E+000 | 1.67340E-006 |
| 1.38629E-004 | 1.12513E+000 | 1.12513E+000 | 3.34675E-006 |
| 2.07944E-004 | 1.06270E+000 | 1.06270E+000 | 3.76481E-006 |
| 2.77259E-004 | 1.03152E+000 | 1.03152E+000 | 3.34609E-006 |
| 3.46574E-004 | 1.01596E+000 | 1.01596E+000 | 2.61368E-006 |
| 4.15888E-004 | 1.00822E+000 | 1.00822E+000 | 1.88142E-006 |
| 4.85203E-004 | 1.00439E+000 | 1.00439E+000 | 1.28004E-006 |
| 5.54518E-004 | 1.00251E+000 | 1.00251E+000 | 0.00000E+000 |
Table 14 continued

| Time            | Numerical solution | Exact solution | Absolute error |
|-----------------|--------------------|----------------|----------------|
| 1.25485E-001    | 1.12548E+000       | 1.12549E+000   | 2.93282E-006   |
| 2.50416E-001    | 1.25041E+000       | 1.25042E+000   | 2.56551E-006   |
| 3.75347E-001    | 1.37534E+000       | 1.37535E+000   | 2.24492E-006   |
| 5.00277E-001    | 1.50028E+000       | 1.50028E+000   | 1.97139E-006   |
| 6.25208E-001    | 1.62521E+000       | 1.62521E+000   | 1.73970E-006   |
| 7.50139E-001    | 1.75014E+000       | 1.75014E+000   | 1.54348E-006   |
| 8.75069E-001    | 1.87507E+000       | 1.87507E+000   | 1.37678E-006   |
| 1.00000E+000    | 2.00000E+000       | 2.00000E+000   | 1.23449E-006   |

Table 15 Problem 2. Fitted Mesh Methods: Schemes (18a–m) and (19a–m)

| $\varepsilon$ | $h = 1/8$   | $1/16$   | $1/32$   | $1/64$   |
|---------------|-------------|----------|----------|----------|
| (i) maximum absolute error at all nodal points |          |          |          |          |
| 0.01          | 3.71938E-04 | 1.97298E-04 | 8.27867E-05 | 2.59336E-05 |
| 0.001         | 4.25662E-05 | 2.55607E-05 | 1.37073E-05 | 6.89549E-06 |
| 0.00001       | 4.31638E-07 | 2.62304E-07 | 1.43414E-07 | 7.48414E-08 |
| (ii) maximum relative error at all nodal points |          |          |          |          |
| 0.01          | 4.92735E-04 | 2.34684E-04 | 9.43388E-05 | 2.04464E-05 |
| 0.001         | 5.48275E-05 | 2.01122E-05 | 1.46854E-05 | 7.17193E-06 |
| 0.00001       | 5.54252E-07 | 2.97244E-07 | 1.52672E-07 | 7.72233E-08 |

Table 16 Problem 3. Fitted Mesh Methods: Schemes (18a–m) and (19a–m)

| $\varepsilon$ | $h = 1/8$   | $1/16$   | $1/32$   | $1/64$   |
|---------------|-------------|----------|----------|----------|
| (i) maximum absolute error at all nodal points |          |          |          |          |
| 0.01          | 3.13458E-03 | 1.62262E-03 | 6.74673E-04 | 2.11410E-04 |
| 0.001         | 3.58271E-04 | 2.09337E-04 | 1.10935E-04 | 5.55104E-05 |
| 0.00001       | 3.63254E-06 | 2.14668E-06 | 1.15984E-06 | 6.01927E-07 |
| (ii) maximum relative error at all nodal points |          |          |          |          |
| 0.01          | 1.40006E-03 | 1.12728E-03 | 6.21980E-04 | 2.13882E-04 |
| 0.001         | 1.75646E-04 | 1.8967E-04  | 1.84181E-04 | 1.59668E-04 |
| 0.00001       | 1.88090E-06 | 2.01546E-06 | 2.11762E-06 | 2.16455E-06 |

Fig. 1 Graphical result for the solution of Test Problem 1, $h = 1/16$, $\varepsilon = 0.0001$, fitted method (9a–m)

Fig. 2 Graphical result for the solution of Test Problem 1, $h = 1/16$, $\varepsilon = 0.0001$, $\tau = 5.54518E-004$, fitted mesh method (18a–m)
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Fig. 3 Graphical result for the solution of Test Problem 2, $h = 1/16, \varepsilon = 0.0001$, fitted method (9a–m)

Fig. 4 Graphical result for the solution of Test Problem 2, $h = 1/16, \varepsilon = 0.0001, \tau = 5.54518E-004$, Fitted Mesh Method (18a–m)

Fig. 5 Graphical result for the solution of Test Problem 3, $h = 1/16, \varepsilon = 0.0001$, Fitted Method (9a–m)

Fig. 6 Graphical result for the solution of Test Problem 3, $h = 1/16, \varepsilon = 0.0001, \tau = 5.54518E-004$, Fitted Mesh Method (18a–m)

Fig. 7 Graphical result for the solution of Test Problem 1, $h = 1/16, \varepsilon = 0.0001$, Fitted Method (17a–m)

Fig. 8 Graphical result for the solution of Test Problem 1, $h = 1/16, \varepsilon = 0.001, \tau = 5.54518E-004$, Fitted Mesh Method (19a–m)
is not visible. To view the numerical solution for the same values of $\epsilon = 0.0001$, $h = 1/16$, the Fitted Mesh method (18a–m) with VFF is applied and the graph numerical solution is plotted in Fig. 2. The numerical solution fit with the exact solution. In the neighborhood of $t = 0$, the solution graph is visible.

In Problem 2, $a(t)$ is a constant function and $f(t)$ is a variable function. And so the scheme with VFF and CFF will not differ. On solving Problem 2 using the Fitted Operator method (9a–m) with VFF, $\epsilon = 0.0001$, and $h = 1/16$ in the interval $[0,1]$, the graph of the numerical solution is shown in Fig. 3. The numerical solution fit with the exact solution. In the interval $[0,0.0001]$ the correct graph of the numerical solution is not visible since $h$ is larger than $\epsilon$. In the neighborhood of $t = 0$, the solution graph is not visible. To view the numerical solution for the same values of $\epsilon = 0.0001$, $h = 1/16$, the Fitted Mesh method (18a–m) with VFF is applied and the graph numerical solution is plotted in Fig. 4. The numerical solution fit with the exact solution. In the neighborhood of $t = 0$, the solution graph is visible. Similar results can be seen on applying the scheme (17a–m) with CFF and the scheme (19a–m) with CFF.

In Problem 3, $a(t)$ is a constant function and $f(t)$ is a variable function. And so the scheme with VFF and CFF will not differ. On solving Problem 3 using the Fitted Operator method (9a–m) with VFF, $\epsilon = 0.0001$, and $h = 1/16$ in the interval $[0,1]$, the graph of the numerical solution is shown in Fig 5. The numerical solution fit with the exact solution. In the interval $[0,0.0001]$ the correct graph of the numerical solution is not visible since $h$ is larger than $\epsilon$. In the neighborhood of $t = 0$, the solution graph is not visible. To view the numerical solution for the same values of $\epsilon = 0.0001$, $h = 1/16$, the Fitted Mesh method (18a–m) with CFF is applied and the graph numerical solution is plotted in Fig. 6. The numerical solution fit with the exact solution. In the neighborhood of $t = 0$, the solution graph is visible. Similar results can be seen on applying the scheme (17a–m) with CFF and the scheme (19a–m) with CFF.

Finally, on solving Problem 1 using the Fitted Operator method (17a–m) with CFF, $\epsilon = 0.0001$, and $h = 1/16$ in the interval $[0,1]$, the graph of the numerical solution is shown in Fig. PIC7. The numerical solution fit with the exact solution. In the interval $[0,0.0001]$ the correct graph of the numerical solution is not visible since $h$ is larger than $\epsilon$. In the neighborhood of $t = 0$, the solution graph is not visible. To view the numerical solution for the same values of $\epsilon = 0.0001$, $h = 1/16$, the Fitted Mesh method (19a–m) with CFF is applied and the graph numerical solution is plotted in Fig. PIC7. The numerical solution fit with the exact solution. In the neighborhood of $t=0$, the solution graph is visible.

Graphical results show the Fitted Operator methods (9a–m) with VFF and (17a–m) with CFF fit well with an exact solution which is of order $\min(h^3, \epsilon)$. And, Fitted Mesh methods (18a–m) with VFF and (19a–m) with CFF fit well with an exact solution which is of order $\min(h^3, \epsilon)$. It is observed from the figures that the Fitted Mesh method (18a–m) is an extension of the Fitted Operator methods (9a–m) with VFF. Similarly, the Fitted Mesh method (19a–m) is an extension of the Fitted Operator methods (17a–m) with CFF.

7 Conclusions

This article presents two numerical methods of the order of three for singular perturbation problems with a small parameter using finite differences. It is a problem with an initial layer in the neighborhood of the initial nodal point whose width is of the order of the small parameter $\epsilon$. The main aim of designing the method (9a–m) is to have a computationally cheaper method that is explicit without any constraint on the step size or the coefficients of the given problem. To reduce the computation time again by modifying it an implicit method is introduced. These methods are fitted operator methods. To view the initial layer when the size of the step size is larger than the parameter in the problem fitted operator methods are extended to fitted mesh methods.

On applying the explicit classical order three methods the solution will not give a satisfactory result. In this article, an explicit order of three classical methods is modified and a new scheme (9a–m) is designed for singular perturbation problems. It is explicit with a variable fitting factor (VFF) evaluated at all nodal points. To reduce the calculation time of the scheme with VFF, the VFF is replaced by a fixed Constant Fitting Factor (CFF). It is implicit with a CFF which is evaluated only one time at the initial nodal point. These two schemes are both optimal concerning the small parameter $\epsilon$ and uniform of order three. The three order methods presented in this document are superior to the three order methods available in the literature. These two schemes yield good approximations for large mesh sizes and very small values to the parameter $\epsilon$. These schemes solve the reduced problem exactly. It is observed from experimental results that the finite difference scheme (9a–m) with VFF and scheme (17a–m) with CFF are approximate $O(\min(h^3, \epsilon))$.

These two methods are applicable when the step size is smaller than the value of the parameter. To view the initial layer when the mesh size is larger than the parameter, these two fitted operator methods are extended to fitted mesh methods since fitted mesh methods are layer rescaling. The construction of the fitted mesh method is also provided in this article. That is, the uniform mesh is extended to non-uniform mesh. The Fitted Mesh method (18a–m) is an extension of the Fitted Operator methods (9a–m) with VFF and scheme (17a–m) with CFF are explicit in nature. And, the Fitted Mesh method (19a–m) is an extension of the Fitted Operator methods (17a–m) with CFF are implicit in nature.
One can apply the schemes of this article to higher order and the system of initial value problems. Boundary value problems can be solved by transforming them into a system of initial value problems using the shooting method.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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