TARSKI’S RELEVANCE LOGIC; VERSION 2

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Abstract. Tarski’s relevance logic is defined and shown to contain many formulas and derived rules of inference. The definition arises from Tarski’s work on first-order logic restricted to finitely many variables. It is a relevance logic because it contains the Basic Logic of Routley-Plumwood-Meyer-Brady, has Belnap’s variable-sharing property, and avoids the paradoxes of implication. It does not include several formulas used as axioms in the Anderson-Belnap system $R$. For example, the Axiom of Contraposition is not in Tarski’s relevance logic. On the other hand, the Rules of Contraposition and Disjunctive Syllogism are derived rules of inference in Tarski’s relevance logic. It also contains a formula (not previously known or considered as an axiom for any relevance logic) that provides a counterexample to a completeness theorem of T. Kowalski (that the system $R$ is complete with respect to the class of dense commutative relation algebras).

1. Introduction

In 1975, Alfred Tarski delivered a pair of lectures on relation algebras at the University of Campinas. The lectures were videotaped and transcriptions of them appeared in the book Alfred Tarski: Lectures at UniCamp in 1975 published in 2016. At the end of his second lecture, Tarski said (p. 154),

“And finally, the last question, if it is so, you could ask me a question whether this definition of relation algebra which I have suggested and which I have founded — I suggested it many years ago — is justified in any intrinsic sense. If we know that these are not all equations which are needed to obtain representation theorems, this means, to obtain the algebraic expression of first-order logic with two-place predicate, if we know that this is not an adequate expression of this logic, then why restrict oneself to these equations? Why not to add strictly some other equations which hold in representable relation algebras or maybe all?”

Tarski’s question arises from the fact that the equations he chose for his axiomatization of relation algebras are all simple and natural and occur throughout the nineteenth century literature on the algebra of logic, such as the works of Peirce [39, 40, 41, 42, 43, 44, 45, 46, 47, 48] and Schröder [51, 52, 53, 54, 55] and yet the choice is clearly arbitrary. Furthermore, the axioms were shown to be incomplete, hence insufficient for proving representability, by Lyndon [23] in 1950. Back in 1941 Tarski [50] pp. 87–88] wrote,
"Is it the case that every sentence of the calculus of relations which is true in every domain of individuals is derivable from the axioms adopted under the second method? This problem presents some difficulties and remains open. I can only say that I am practically sure that I can prove with the help of the second method all of the hundreds of theorems to be found in Schröder’s *Algebra und Logik der Relative.*"  

The “second method” is Tarski’s equational axiomatization. The problem Tarski posed was to find a true equation that his axioms can’t prove. Lyndon solved Tarski’s open problem in his 1950 paper by showing the answer is “no”. This left only Tarski’s rather practical reason for adopting his axioms: they are good enough to prove a lot.  

Besides what could be proved from his axioms, Tarski was also concerned from the outset with what could be expressed with equations. This topic had been considered already by Schröder and Löwenheim [16, 17, 18, 19, 20, 21, 22]. By the early 1940s Tarski had proved that the equations of relation algebras have the same expressive power as first-order logic restricted to three variables. Tarski took a first-order language with an equality symbol and other binary relation symbols (but no function symbols or constants), reduced the usual stock of countably many variables to just three, added a binary operator | on relation symbols, and included a definition asserting that the operator produces the relative product of the relations denoted by the inputs:

\[(A|B)(x, y) \iff \exists z(A(x, z) \land B(z, y)).\]

He included other operators on relation symbols, for union, complementation, and converse, along with their definitions

\[(A \cup B)(x, y) \iff A(x, y) \lor B(x, y),\]
\[\overline{A}(x, y) \iff \neg A(x, y),\]
\[A^{-1}(x, y) \iff A(y, x).\]

Finally, Tarski introduced a new form of sentence called an equation, written \(A = B,\) made out of two relation symbols \(A\) and \(B\) and a new equality symbol, with this definition

\[A = B \iff \forall x\forall y(A(x, y) \iff B(x, y)).\]

In Tarski’s definition for |, \(z\) is the first variable distinct from \(x\) and \(y\). Such a variable always exists because Tarski’s language has three variables. To illustrate, the associative law for relative multiplication is

\[(A|B)|C = A|(B|C),\]

and its expansion according to the definition of | is

\[\forall x\forall y\left(\exists z(\exists y(A(x, y) \land B(y, z)) \land C(z, y)) \iff \exists z(A(x, z) \land \exists y(B(z, x) \land C(y, x)))\right).\]

The burden of parentheses can be reduced by resorting to subscripts.

\[\forall x\forall y\left(\exists z(\exists y(A_{x,y} \land B_{y,z}) \land C_{z,y}) \iff \exists z(A_{x,z} \land \exists y(B_{z,x} \land C_{x,y}))\right).\]

Tarski observed that every relation-algebraic equation expands to a formula in first-order logic of binary relations restricted to three variables, as was just done for the associative law, and then he proved that every formula is equivalent to such an
expansion, i.e., every formula of 3-variable first-order logic (of binary relations) can be converted to an equivalent relation-algebraic equation. For details, consult [28, 59].

Naturally, Tarski included the associative law as an axiom for relation algebras. With regard to the other axioms, Tarski found that he could not only express them with three variables, but also prove them with only three variables. On the other hand, Tarski’s proof of the associative law used four variables. Could it be proved with only three variables? J. C. C. McKinsey had invented an algebra that satisfies all of Tarski’s axioms for relation algebras except the associative law, thus proving that the associative law is independent of the other axioms. Tarski used McKinsey’s algebra to prove that the associative law for relative multiplication cannot be proved in first-order logic with only three variables.

This is how things stood in 1975, when Tarski asked, “whether this definition of relation algebra ... is justified in any intrinsic sense”. Tarski had proved that every equation true in all relation algebras, i.e., every equation that follows from his axioms by the rules of equational logic (equality is transitive and symmetric, and equals may be substituted for equals) can be proved in first-order logic with four variables. Since the associative law is the only axiom requiring four variables to prove, Tarski asked whether deleting it would result in an equational theory equivalent to 3-variable logic in means of proof as well as expression. If not, could the associative be replaced with a weaker version to yield an equational theory equivalent to 3-variable logic?

These problems were included in the draft of the Tarski-Givant book [59], which was being written at the time of Tarski’s talk. This book started life as an unpublished manuscript by Tarski from the early 1940s. Work on the revision was begun in 1971. It was planned to become Tarski’s contribution to the Proceedings of the Tarski Symposium [6, 9], held in honor of his 70th birthday, but grew into a project not published until four years after his death.

Around this time of Tarski’s talk it was proved that the answers are “no” and “yes”, i.e., deleting the associative law leaves an axiom set that is too weak, but a weakened associative law, dubbed the “semi-associative law” can replace the associative law to produce an equational theory that is a precise correlate of first-order logic of binary relation symbols and only three individual variables—every sentence of 3-variable logic is equivalent to an equation, and every provable sentence of 3-variable logic is equivalent to an equation provable from the weakened axiom set. (Algebras satisfying this weakened axiom set are now called semi-associative relation algebras, but their initial name was “Tarski algebras”.) Furthermore, the equations true in all relation algebras are exactly those that are equivalent to a statement in 3-variable logic of binary relations and can be proved with four variables. For details see [25, 26, 28, 59].

This last result provides a potential answer to Tarski’s question, “whether this definition of relation algebra ... is justified in any intrinsic sense”. The justification of Tarski’s axioms would be that their consequences are the equations that are

- equivalent to statements in first-order logic of binary relations, restricted to three variables, and
- are provable with four variables.

Certainly one can dispute whether this characterization is “intrinsic”, but any true equation not provable with four variables must require at least five, and finding
such formulas is difficult. The shortest ones known are quite complicated. It is a
safe bet that no such formula was ever encountered for any other purpose prior to
Lyndon’s proof that Tarski’s axioms are incomplete.

Tarski never published his proof that the associative law requires four variables
to prove; see [10, p. 65]. Henkin [8] published such a proof, but for cylindric algebras
rather than relation algebras. The connections between these two subjects had been
studied from the early 1960s by Monk [36, 37].

A search for Henkin [8] led to the same volume containing Routley-Meyer [49].
What Routley and Meyer define as a “relevant model structure” in that paper was
immediately recognized as nearly the same as the atom structure of an integral
relation relation, but with one property missing and two more added. The atom
structures of relation algebras with the two additional properties (density and com-
mutativity) form particularly nice relevant model structures. They have two other
additional properties, one called “normal” by Routley and Meyer, the other called
“tagging” by Dunn; for more details, see [30, §7].

Indeed, making use of the database of finite relation algebras compiled for [28],
one can see that out of 4527 integral relation algebras with five or fewer atoms, all
of them are “normal”, all of them have “tagging”, 3885 of them are commutative
(satisfy \(x \cdot y = y \cdot x\)), 822 of them are dense (satisfy \(x \leq x \cdot x\)), and 626 are both
commutative and dense. Many of these 626 relevant model structures are the atom
structures of proper relation algebras. The elements of proper relation algebras are
binary relations, and their operations are the usual set-theoretic operations on re-
lations: union, intersection, complementation (with respect to the largest relation),
relative multiplication (or composition), and conversion (forming the converse of a
binary relation). This allows Routley-Meyer semantics to be deciphered into
ordinary mathematical concepts in common use.

Routley and Meyer refer to the objects in a relevant model structure at first
as “worlds”, but settle on “set-ups” (which might be, as they suggest, “sets of
beliefs”). Other words have been employed on these objects, such as “situations”
or “points”. However, in relevant model structures arising from proper relation
algebras, the set-ups (or worlds, or situations, or points) are clearly identified; they
are simply binary relations.

The logical connectives considered by Routley and Meyer are conjunction \(\&\),
disjunction \(\lor\), negation \(\sim\), and implication \(\rightarrow\) [49, p. 204, §1]. Every valuation
\(v\) determines a map that sends each propositional variable and set-up to a truth
value, either \(T\) or \(F\) [49, p. 206, §3]. A valuation extends to an interpretation \(I\)
defined for all formulas and set-ups. An interpretation in turn determines a map,
we call it \(J\), from formulas to sets of set-ups. Conditions ii and iii [49, p. 206]
defining the extension show how the connectives are interpreted: conjunction as
intersection and disjunction as union. The treatment of negation involves (what
has become known as) the Routley star. The Routley star in a relevant model
structure matches up with the unary operation of forming the converse of the atom
in the atom structure of a relation algebra. In a proper relation algebra, this is
simply the ordinary converse of a binary relation—the result of turning all the
pairs around. Condition vi [49, p. 206] shows that negation is to be treated as the
converse of the complement (or, what is the same thing, the complement of the
converse). We call this simply converse-complementation. Condition v [49, p. 206]
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shows that the binary connective $\circ$, defined by

$$(D1) \quad A \circ B = \sim(A \rightarrow \sim B)$$

in [49, p. 204] (later called “fusion”), is interpreted as relative multiplication in the opposite order, that is,

$$(5) \quad A \circ B = B|A.$$

This is a good place to notice a notational coincidence. In the case where $A$ and $B$ are unary functions, [49] shows that $\circ$ denotes the usual operation of composing these two functions. The use of $\circ$ for functional composition is common in a wide range of mathematical literature, including calculus textbooks. Instead of writing $\langle x, y \rangle \in A$ in case $A$ is a function, it is customary to write $A(x) = y$, since there is no other ordered pair in $A$ whose first component is $x$, i.e., $y$ is uniquely determined by $A$ and $x$. Composing $A$ and $B$ produces a function denoted $A \circ B$, defined by

$$(A \circ B)(x) = A(B(x)).$$

This is an abbreviated way of describing the relative product of $A$ and $B$ in the opposite order. It says, in more detail, that $\langle x, B(x) \rangle \in B$ and $\langle B(x), A(B(x)) \rangle \in A$. Combining these two statements according to (1) yields $\langle x, A(B(x)) \rangle \in B|A$, establishing (5) in case $A$ and $B$ are functions. The notational coincidence is that the same symbol was (inadvertantly, as it turns out) chosen for the same thing.

A discussion of definition (D1), incorporating remarks of Anderson, Nelnap, Dunn, Woodruff, and Meyer, occurs in [2, §27.1.4], where the “memorable and delightful” properties of $\circ$ are mentioned, including associativity (see Lemma 38 below). However, they ask [2, p. 345],

“3. How then to interpret $\circ$? We confess puzzlement.

In some ways $\circ$ looks like conjunction . . .

But $\circ$ fails to have the property $A \circ B \rightarrow A$; so it isn’t conjunction.”

Perhaps the proper interpretation of $\circ$ is identified in [49]. In this context Meyer’s remarks seem remarkably insightful:

“The term ‘fusion’ is, I believe, due to Fine, and it is a good one; previous tries were ‘intensional conjunction’, ‘relevant conjunction’, ‘consistency’, and ‘cotenability’. But the first two invite confusion with the extensional conjunction ‘&’, while the latter two depend on properties of the negation-of $R$ that have not, so far, generalized to related logics. The notion, in one guise or another, has been invented or re-invented by Lewis, Nelson, Anderson-Belnap, Church, Dunn, Curry, Meredith, Powers, Routley, Urquhart, Fine and the author, no doubt among several score others. It is to be attributed accordingly to Tarski, on the ground that, when it comes to unifying principles, no one is likely to have anticipated him. Except, maybe, Peirce.” [32, Note 4, p. 85]

Condition iv [49, p. 206] shows that implication should be interpreted as residuation, defined as an operation on binary relations by [52], (1), and

$$(6) \quad A \rightarrow B = A^{-1}[B],$$
or, in expanded form

\[(A \rightarrow B)(x, y) \leftrightarrow \forall z(A(z, x) \rightarrow B(z, y)).\]

A good example of residuation is the subset relation between sets—it is the residual of the membership relation with itself. Formulas of relevance logic may be interpreted as subsets of a relevant model structure, i.e., as sets of atoms in the atom structure of a relation algebra, i.e., as elements of an atomic relation algebra (since the elements are joins of sets of atoms), or, and this is the most important case, as binary relations in a proper relation algebra. This includes an interpretation each connective in any relation algebra, and in proper relation algebras those interpretations are disjunction as union, conjunction as intersection, negation as converse-complementation, and implication as residuation.

What remains is to figure out, from the Routley-Meyer definition of verification in a relevant model structure, how a formula is verified in a proper relation algebra. Routley and Meyer explain,

“The real world plays a distinguished role in our semantical postulates. (Accordingly we call it 0 rather than G; not only does the former look better [this is supposed to be, remember, a mathematical semantics], but it correctly hints that 0 will play the formal role of an identity.) It’s necessary to distinguish 0 for the following reason: Logical truth does not turn out to be truth in all set-ups; for the strategy which dispatches the paradoxes lies in allowing even logical identities to turn out sometimes false. (What, after all, could be better grounds for denying that \(q\) entails \(p \rightarrow p\) than to admit that sometimes \(q\) is true when, essentially on grounds of relevance, \(p \rightarrow p\) isn’t?)

“What then is logical truth? Truth in all set-ups, of course, in which all the logical truths are true!”  [49, p. 202]

“Truth at 0 is as noted earlier what counts in verifying logical truths; accordingly we say simply that \(A\) is verified on \(v\), or on the associated \(I\), just in case \(I(A, 0) = T\), and otherwise that \(A\) is falsified on \(v\).”  [49, p. 207]

In other words, if the map determined by an interpretation sends a formula \(A\) to a set of set-ups that includes 0, then that formula is verified. The distinguished element in the atom structure of an integral relation algebra is the identity element. Integral relation algebras are exactly the ones in which the identity element is an atom. The identity element matches up with the distinguished 0 of a relevant model structure. In a proper relation algebra, the identity element is the identity relation on the underlying set whose pairs make up the binary relations belonging to the proper relation algebra.

Assign the binary relation symbols of Tarski’s extended first-order logic to binary relations in a proper relation algebra. According to the Routley-Meyer definition, a formula is verified under this assignment if and only if it evaluates (under the interpretation of its connectives as operations on binary relations) to a relation that contains the identity relation. Therefore a formula \(A\) is verified in a proper relation algebra if and only if

\[\forall x(A(x, x))\]
is true under this assignment. What does this mean for an implication? An implication $A \rightarrow B$ is verified if and only if

$$\forall x((A \rightarrow B)(x, x)),$$

or the equivalent sentence

$$\forall x \forall y(A(x, y) \rightarrow B(x, y)),$$

is true. These sentences assert that the relation denoted by $A$ is included in the relation denoted by $B$. The verified implications are the inclusions between binary relations obtained by interpreting the connectives as operations on binary relations.

All this is standard operating procedure in the theory of relation algebras. Ever since Tarski’s and Lyndon’s work in the 1950s, it has been a relevant question to ask for every relation algebra, is it isomorphic to a proper relation algebra (i.e., representable)? And if it is, what does that say about the binary relations in it?

Theorems asserting that relation algebras are representable are among the most important parts of the subject. Tarski’s early QRA Theorem is a prime example. If Tarski’s axioms for relation algebras had turned out to be complete, then his long and difficult theorem would have become pointless. Tarski’s QRA Theorem (see [58 VII or [59 8.4(iii)] or [28 Theorem 427]) asserts that if a relation algebra contains a pair of quasi-projections (elements that behave like projection functions) then it is representable. The QRA Theorem follows from the main result of the Tarski-Givant book, called the Main Mapping Theorem for $L^\times$ and $L^+_n$ [59 4.4(xxxiii)(xxxiv)], [28 Theorem 574]. The Main Mapping Theorem says that if a theory, formalized in first-order logic, proves the existence of a pair of functions acting sufficiently like projection functions (from ordered pairs to their components), then that theory can be formalized as a equational theory in the language of relation algebras. This enables Tarski’s formalization of set theory without variables ([57], [59 §4.6]).

To recall the characterization of Tarski’s axioms, let $\mathcal{E}_4$ be the equations provable in Tarski’s extended system of first-order logic of equality and other binary relations restricted to four variables. This class of equations is axiomatized by Tarski’s axioms for relation algebras together with the rules of deduction for equational logic. These equations contain the entire range of operations used by the nineteenth century algebraic logicians: union, intersection, complementation, converse, relative multiplication, and a distinguished identity relation. We might call $\mathcal{E}_4$ “Tarski’s equational logic” (for relation algebras).

Applying this characterization with the reduced set of operations available in relevance logic produces Tarski’s relevance logic $\mathcal{L}_4$. By definition, $\mathcal{L}_4$ consists of those formulas for which $\forall x(A(x, x))$ is provable in first-order logic of binary relations restricted to four variables. Unlike $\mathcal{E}_4$, the formulas in $\mathcal{L}_4$ contain only the operations corresponding to the connectives of relevance logic: union, intersection, converse-complementation, and residuation. Note that relative multiplication and residuation can be defined from each other using converse-complementation. On the other hand, complementation and converse (the Routley star) do not occur in the formulas in $\mathcal{L}_4$.

This definition of $\mathcal{L}_4$ is precise enough to demonstrate what formulas are in $\mathcal{L}_4$, what derived rules of inference it is closed under, and what formulas are not in $\mathcal{L}_4$. The exact choice of logical axioms for first-order logic doesn’t really matter, as experience has shown. One can quibble about what 4-variable logic should be. For example, respelling of bound variables is usually presented as a consequence of
the logical axioms, but its proof requires extra variables not occurring in a given sentence, and these may not exist if all four variables already occur in a sentence. Respelling of bound variables can be excluded or explicitly included, but the result is the same. For the sake of avoiding such questions and the notational complexities of quantifiers, a sequent calculus was employed in [24], as will be done here. (Another good option are proofs by natural deduction, restricted to examination of at most four objects at once.) Some of the rules from [24] can be used directly (the structural rules and ones for $\land$ and $\lor$), while new rules are formulated for the connectives $\sim$ and $\rightarrow$. The resulting proofs are close in appearance to informing reasoning using at most four objects.

By [24, Theorems 2] for $n = 4$ (or $n = 3$), a formula is provable with four (or three) variables, using the complete set of rules in [24], if and only if the corresponding equation is true in all relation algebras by [24, Theorems 5] (or semi-associative relation algebras by [24, Theorems 4]). The rules used here are a proper subset of the rules in [24], or are the result of the combined application of two rules from [24], as is the case for $\rightarrow|$, $|\rightarrow$, $\sim|$, and $|\sim$. Consequently every formula in $\mathcal{L}_4$ (or $\mathcal{L}_3$) corresponds to an equation true in all relation algebras (or semi-associative relation algebras). The correspondence is quite direct in [24]. An inclusion $A \subseteq B$ is true in all relation algebras if and only if the sequent $A_{01} \Rightarrow B_{01}$ is provable in 4-variable logic. The equivalent condition here is that $\Rightarrow(A \rightarrow B)_{00}$ is provable in 4-variable logic. These two sequents are interderivable, corresponding to the fact that one relation is a subset of another if and only if their residual contains the identity relation: $A \subseteq B$ iff $\text{Id} \subseteq A \rightarrow B$.

2. The sequent calculus

Definition 1.

- $\mathsf{Pv}$ is a countable set whose elements $p, q, r, \cdots \in \mathsf{Pv}$ are called propositional variables.
- $\mathsf{Fmla}$ is the closure of $\mathsf{Pv}$ under three binary operations $\lor$, $\land$, and $\rightarrow$, and one unary operation $\sim$.
- The elements $A, B, C, D, \cdots \in \mathsf{Fmla}$ are called formulas.
- The four elements of $\{0, 1, 2, 3\}$ are called objects or indices.
- An assertion $A_{ij}$ is a formula $A$ together with an ordered pair of individual objects $i, j \in \{0, 1, 2, 3\}$, added to the formula as subscripts.

Parentheses are omitted according to the convention that the operations are applied in this order: $\sim$, $\land$, $\lor$, and finally $\rightarrow$. An assertion $A_{ij}$ should be read as if it said $(i, j) \in A$, that is, $A$ is a relation that holds between objects $i$ and $j$. In first-order logic an assertion might more commonly be written $A(i, j)$, as was done earlier. The subscript style of writing an assertion was common in nineteenth century algebraic logic, and it reduces the burden of parentheses.

Definition 2.

- A sequent $\Gamma \Rightarrow \Delta$ is an ordered pair $(\Gamma, \Delta)$ of sets of assertions $\Gamma$ and $\Delta$.
- The sequent $\Gamma \Rightarrow \Delta$ is an Axiom if $\Gamma \cap \Delta \neq \emptyset$.
- A 4-proof is a finite sequence of sequents in which every sequent is either an Axiom or follows from one or two previous sequents by one of the rules of inference shown in Figure 2: Cut, Weakening, $\lor|$, $|\lor$, $\land$, $\sim|$, $|\sim$, $\rightarrow|$, and, if $k$ does not appear in $\Gamma \cup \Delta$, $|\rightarrow$.
The restriction to finite proofs in Definition 2 is motivated by the fact that a rule can “do nothing”. For example, every sequent follows from itself by Weakening (take $\Gamma' = \Delta' = \emptyset$). Without the restriction, the infinite $\mathbb{Z}$-indexed sequence in which every sequent is $A_{01} \Rightarrow B_{01}$ would be a “proof” of $A_{01} \Rightarrow B_{01}$. Abbreviations used in the notation for sequents is standard. For example, $\Delta, A_{ij} \Rightarrow \Delta', B_{ij} \Rightarrow \Delta, C$ is short for $\Delta \cup A \cup \{A_{ij}\} \Rightarrow \Delta' \cup \Delta' \cup \{B_{ij}, A, B, C\}$.

A sequent $\Gamma \Rightarrow \Delta$ should be read, “If all the assertions in $\Gamma$ are true, then one of the assertions in $\Delta$ is true.” For example, the sequent $A_{ij} \Rightarrow B_{ij}$ should be read, “If $(i,j) \in A$ then $(i,j) \in B$”. Under this reading, together with the intended interpretation of the connectives as set-theoretical operations, it is easy to see why all the rules in Figure 1 are correct. In particular, the rule $\rightarrow$ requires that $k \neq i,j$ and $k$ does not occur as a subscript in any assertion in $\Gamma$ or $\Delta$, as indicated by the notation “no $k$”. The reason for this is the universal quantifier in the definition of residuation, and is reflected in one of the common logical validities used in axiomatizations of first-order logic, namely $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi)$, where it is
Lemma | Objects | Formula
--- | --- | ---
$L(1)$ | $\{0\}$ | $A \lor \neg A$
$L(2)$ | $\{0, 1\}$ | $A \to A$
$L(3)$ | $\{0, 1\}$ | $A \land B \to A$
$L(4)$ | $\{0, 1\}$ | $A \land B \to B$
$L(5)$ | $\{0, 1\}$ | $A \to A \lor B$
$L(6)$ | $\{0, 1\}$ | $B \lor A \to A \lor B$
$L(7)$ | $\{0, 1\}$ | $B \land A \to A \land B$
$L(8)$ | $\{0, 1\}$ | $(A \land B) \land C \to A \land (B \land C)$
$L(9)$ | $\{0, 1\}$ | $(A \lor B) \lor C \to A \lor (B \lor C)$
$L(10)$ | $\{0, 1\}$ | $A \land (B \lor C) \to (A \land B) \lor (A \land C)$
$L(11)$ | $\{0, 1\}$ | $(A \to \neg C) \land (B \to C) \to \neg (A \land B)$
$L(12)$ | $\{0, 1\}$ | $(A \to \neg B) \land (\neg A \to \neg C) \to \neg B \lor \neg C$
$L(13)$ | $\{0, 1\}$ | $\neg \neg A \to A$
$L(14)$ | $\{0, 1\}$ | $A \to \neg \neg A$
$L(15)$ | $\{0, 1\}$ | $(A \lor B) \to \neg A \land \neg B$
$L(16)$ | $\{0, 1\}$ | $\neg (A \land B) \to \neg A \lor \neg B$
$L(17)$ | $\{0, 1\}$ | $\neg A \land \neg B \to \neg (A \lor B)$
$L(18)$ | $\{0, 1\}$ | $\neg A \lor \neg B \to \neg (A \land B)$
$L(19)$ | $\{0, 1\}$ | $((A \to A) \to B) \to B$

Table 1. Formulas in Tarski’s relevance logic, provable with 1 or 2 objects

required that $x$ does not occur free in $\varphi$. In proofs that a formula belongs to $\mathcal{L}_4$, the notation “no $k$” will accompany every application of rule $\to$, explicitly identifying the universally quantified object.

Definition 3.

- **A 4-proof of the sequent** $\Gamma \Rightarrow \Delta$ is a 4-proof in which $\Gamma \Rightarrow \Delta$ appears. We write
  $\vdash^4 \Gamma \Rightarrow \Delta$
  just in case $\Gamma \Rightarrow \Delta$ has a 4-proof.
- **A 4-proof of the formula** $A$ is a 4-proof of the sequent $\Rightarrow A_{00}$.
- **$\mathcal{L}_4$** is the set of formulas that have 4-proofs:
  $$\mathcal{L}_4 = \{ A : \vdash^4 \Rightarrow A_{00} \}.$$
are not the previous one or two. The second entry in Tables 1 and 2 is a list of the right. Line numbers in 4-proofs are included whenever the justifying sequents one or two sequents immediately preceding it according to the rule mentioned to the rule.

| Lemma | Objects | Formula |
|-------|---------|---------|
| L[21] | {0, 1, 2} | \((A \rightarrow B) \land (A \rightarrow C) \rightarrow (A \rightarrow B \land C)\) |
| L[22] | {0, 1, 2} | \((A \rightarrow C) \land (B \rightarrow C) \rightarrow (A \lor B \rightarrow C)\) |
| L[23] | {0, 1, 2} | \((A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \land C \rightarrow B \land D)\) |
| L[24] | {0, 1, 2} | \((A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \lor C \rightarrow B \lor D)\) |
| L[25] | {0, 1, 2} | \((A \rightarrow B) \lor (C \rightarrow D) \rightarrow (A \land C \rightarrow B \lor D)\) |
| L[26] | {0, 1, 2} | \(A \rightarrow (\sim B \rightarrow \sim(A \rightarrow B))\) |
| L[27] | {0, 1, 2} | \(A \rightarrow (B \rightarrow \sim(A \rightarrow B))\) |
| L[28] | {0, 1, 2} | \(A \rightarrow ((\sim B \rightarrow \sim A) \rightarrow B)\) |
| L[29] | {0, 1, 2} | \(A \rightarrow ((B \rightarrow \sim A) \rightarrow \sim B)\) |
| L[30] | {0, 1, 2} | \(\sim((A \rightarrow B) \rightarrow \sim A) \rightarrow B\) |
| L[31] | {0, 1, 2} | \((A \rightarrow B) \circ A \rightarrow B\) |
| L[32] | {0, 1, 2} | \(\sim A \rightarrow ((B \rightarrow A) \rightarrow \sim B)\) |
| L[33] | {0, 1, 2} | \(\sim(A \rightarrow B) \land C \rightarrow \sim((A \land \sim D) \rightarrow \sim B)\) |
| L[34] | {0, 1, 2} | \(\lor \sim(A \rightarrow \sim(B \land \sim(D \rightarrow \sim C)))\) |
| L[35] | {0, 1, 2} | \((A \circ B) \land C \rightarrow ((A \land \sim D) \circ B) \lor (A \circ (B \land (D \circ C)))\) |
| L[36] | {0, 1, 2} | \((A \rightarrow B) \land \sim(C \rightarrow \sim D) \rightarrow \sim(C \land B \rightarrow \sim D)\) |
| L[37] | {0, 1, 2} | \(\lor \sim(C \rightarrow \sim(D \land \sim A))\) |
| L[38] | {0, 1, 2} | \((A \rightarrow B) \land (C \land D) \rightarrow ((C \land B) \circ D) \lor (C \circ (D \land \sim A))\) |
| L[39] | {0, 1, 2} | \(((A \rightarrow B) \circ (C \rightarrow A) \rightarrow (C \rightarrow B))\) |
| L[40] | {0, 1, 2} | \((B \rightarrow (C \rightarrow A)) \rightarrow (\sim(B \rightarrow \sim C) \rightarrow A)\) |
| L[41] | {0, 1, 2} | \((B \rightarrow (C \rightarrow A)) \rightarrow ((B \circ C) \rightarrow A)\) |
| L[42] | {0, 1, 2} | \((\sim(A \rightarrow \sim B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))\) |
| L[43] | {0, 1, 2} | \(((A \circ B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))\) |
| L[44] | {0, 1, 2} | \((A \rightarrow B) \rightarrow (\sim(A \rightarrow C) \rightarrow \sim(B \rightarrow C))\) |
| L[45] | {0, 1, 2} | \((A \rightarrow B) \rightarrow ((A \circ D) \rightarrow (B \circ D))\) |
| L[46] | {0, 1, 2} | \((A \circ B) \circ C \rightarrow A \circ (B \circ C)\) |

Table 2. Formulas in Tarski’s relevance logic, provable with 3 or 4 objects

3. Tables of formulas and rules

Tables 1 and 2 show more than three dozen formulas in \(L_4\). Each entry begins with the number in parentheses, preceded by “L”, of the lemma in which that formula is shown to have a 4-proof. For example, the proof of Lemma 1 is a 4-proof of formula L[1]. In a 4-proof, every sequent is either an Axiom or follows from the one or two sequents immediately preceding it according to the rule mentioned to the right. Line numbers in 4-proofs are included whenever the justifying sequents are not the previous one or two. The second entry in Tables 1 and 2 is a list of the
objects that are actually used in the 4-proof of the formula. This provides a rough classification of the formulas into those belonging to what we might call $L_1$, $L_2$, $L_3$, and $L_4$, depending on the number of objects needed for their 4-proofs.

All of the formulas in $L_4$ make assertions about binary relations that are universally true. As was observed earlier, the verification of a formula of the form $A \rightarrow B$ in every proper relation algebra confirms that $A \subseteq B$, no matter how the propositional variables in $A$ and $B$ are interpreted as binary relations. For example, formula $L(2)$ asserts the universal truth that for every binary relation $A$, $A \subseteq A$, while $L(3)$ asserts that for all binary relations $A$ and $B$, $A \cap B \subseteq A$, as one would expect if the interpretation of $\wedge$ is intersection.

Table 3 shows more than a dozen derived rules of inference in Tarski’s relevance logic.
“Old friends of our project will be surprised to find that we were forced to split the book into two volumes – in order, of course, to avoid weighing the reader down either literally or financially – when we finally realized that the universe of relevance logics had expanded unnoticed overnight.” [2, p. xxiii]

“This book mentions or discusses so many different systems (Meyer claims the count exceeds that of the number of ships in Iliad II) that we have been driven . . . to try to devise a reasonably rational nomenclature.” [2, p. xxv]

“Additional axiom schemes drawn from the following lists may be added to basic system $\mathcal{B}$ . . . singly or in combination to yield a wealth of stronger systems:–” [50, p. 288]

“The following postulates are added . . . , singly or in combination, to provide modellings for the wealth of further systems of sentential logics introduced in the previous section.” [50, p. 300]

“In this chapter we first present algebraic analyses for an important and extensive class of affixing systems: the class comprises not only a great many relevant logics including all the more standard systems but also all the usual irrelevant logics and some unusual ones as well” [4, p. 72]

Tarski’s relevance logic $\mathcal{L}_4$ does not have this sort of variation. There is no list of formulas and rules from which to choose “singly or in combination”. The only available parameter is the number of variables used to prove any particular formula or deductive rule expressing a property of binary relations. The most interesting cases are when the number of variables is 1, 2, 3, or 4. The logic $\mathcal{L}_1$ already has the Law of the Excluded Middle, and among its rules are Adjunction, $\textit{modus ponens}$, and Disjunctive Syllogism. The logic $\mathcal{L}_2$ picks up all the formulas in Table 1 (many of which are part of various systems of Basic Logic), plus some more rules from Table 3 such as the Rules of Transitivity, Contraposition, and Cut. The logic $\mathcal{L}_3$ adds to this list the Rule of Suffixing, for example, along with some key formulas governing conjunction, disjunction, and fusion. However, the associative law for fusion is missing from $\mathcal{L}_3$, along with those axioms (such as Suffixing) and rules (such as Prefixing) from Tables 3 and 4 whose sequent proofs require four objects. (These omissions can be proved by examining semi-associative relation algebras that are not associative, hence not relation algebras, which fail to satisfy the the appropriate rules and equations). The logic $\mathcal{L}_5$, however, is (or, at least, has been) well beyond the consideration of even the most ardent inventors of systems.

Even $\mathcal{L}_4$ misses standard axioms used in various relevance logics. Such axioms can be added, perhaps yielding a “wealth of systems”. $\mathcal{L}_4$ is a “naturally occurring” system. The motivation for studying $\mathcal{L}_4$ certainly involves relevance logics. But $\mathcal{L}_4$ arises from entirely different considerations. Indeed, Tarski’s relevance logic $\mathcal{L}_4$ may satisfy van Benthem’s [60] suggestion that

“. . . the Routley semantics still has to prove its mettle. On the realistic side, its model structures ought to admit of, if not a natural linguistic anchoring, then at least one mathematical ‘standard example’, providing some food for independent reflection.”
Perhaps Tarski’s relevance logic should be considered as a “standard mathematical example.”

5. Basic logic

Tarski’s relevance logic contains the Basic Logic \( B \) of [4] and [50]. The axioms of Basic Logic in [50, pp. 287–8] are A1–A9, and its rules are R1–R5, with R3’ as an alternative to rules R3 and R4. In Tables [4] and [1], A1 is L(2), A2 is L(3), A3 is L(4), A4 is L(21), A5 is L(5), A6 is L(6), A7 is L(22), A8 is L(11), and A9 is L(14). The rules of Basic Logic in [50, pp. 287–8] are derived rules of inference in Tarski’s relevance logic. In Table [3], R1 is L(40), R2 is L(39), R3’ is L(50), R3 is L(47), R4 is L(49), and R5 is L(44). The axioms of Basic Logic in [4, pp. 192–3] are A1–A9, the same as axioms A1–A9 of [50, pp. 287–8]. The rules of Basic Logic in [4, p. 193] are R1–R4, where R1 is L(40), R2 is L(39), R3 is L(50), and R4 is L(44). Rule R5 in [4, p. 192–3] is part of systems \( E \) and \( EW \); R5 is L(16). Axiom A13 of system TW in [4, p. 193] is L(13). Axiom A17 of systems \( DK \) and \( TK \) in [4, p. 193] is L(1).

6. Properties of binary relations

All the formulas and rules of inference in Tarski’s relevance logic are true for arbitrary binary relations. They are verified in all proper relation algebras. More generally, they hold in every algebra of the form

\[ \mathcal{A} = (K, \cup, \cap, \rightarrow, \sim) \]

where \( K \) is a set of binary relations on some set \( U \), and \( K \) is closed under union \( \cup \), intersection \( \cap \), residuation \( \rightarrow \), and converse-complementation \( \sim \). This means that if the propositional variables in a formula \( A \) in Tarski’s relevance logic are assigned to binary relations in \( K \), then the binary relation assigned to \( A \) will contain the identity relation on \( U \). Conversely, any formula that holds in every such algebra \( \mathcal{A} \) will be part of Tarski’s relevance logic if it can be proved by looking at no more than four objects at a time. Thus, formulas not belonging to Tarski’s relevance logic are of two kinds. They are either valid for all binary relations but require more than five objects to prove, or else they postulate properties of binary relations that do not hold in general.

Here are some examples of formulas expressing special properties of binary relations; for details see [15]. The axiom of contraposition,

\[(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)\]

holds in \( \mathcal{A} \) if and only if the relations in \( K \) commute with each other under relative multiplication, i.e., fusion is commutative. The same applies to the axiom of permutation,

\[(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))\]

Commutativity of \( \mathcal{A} \) is enough to insure that the suffixing and modus ponens axioms

\[(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\]
\[A \rightarrow ((A \rightarrow B) \rightarrow B)\]

hold in \( \mathcal{A} \), but neither of them is equivalent to assuming \( \mathcal{A} \) is commutative. The contraction axiom and the reductio axiom

\[(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)\]
(13) \((A \rightarrow \sim A) \rightarrow \sim A\)

are each equivalent to assuming every relation in \(K\) is dense. The \(R\)-mingle axiom

(14) \(A \rightarrow (A \rightarrow A)\)

holds in \(R\) if and only if every relation in \(K\) is transitive.

7. System \(R\)

An axiom set for the Anderson-Belnap system \(R\) of relevant implication is presented by Routley-Meyer [49, p. 204]. It contains axioms A1–A13 along with axioms A14 and A15 [49, p. 224] when fusion \(\circ\) is included as primitive rather than defined as in (17). Eleven of these fifteen axioms occur in Tarski’s relevance logic. In Tables 1 and 2, A1 is L(2), A5 is L(6), A6 is L(4), A7 is L(21), A8 is L(5), A9 is L(9), A10 is L(22), A11 is L(11), A13 is L(14), A14 is L(27), and A15 is L(35). The remaining four axioms of \(R\) (A2, A3, A4, A12) do not occur in Tarski’s relevance logic: A2 is (11), A3 is (10), A4 is (12), and A12 is (8). The rules for \(R\) are L(39) and L(40), both part of Tarski’s relevance logic. If Tarski’s relevance logic is extended by adding axioms A1–A15, then all formulas of the logic \(R\) of Anderson-Belnap [2] become provable; see [50, Corollary 5.2(i)].

Adding axioms to \(L_4\) may be done by supplementing the rules in Figure 1. For example, to add (8), one may include the rule

\[ \Gamma, (A \rightarrow \sim B) ij \Rightarrow \Delta, (B \rightarrow \sim A) ij \quad \text{Contraposition} \]

8. System \(R\)-mingle

If Tarski’s relevance logic is extended by adding the axioms (8), (10), (11), (12), and (13), the result is the Dunn-McCall system \(R\)-mingle. \(R\)-mingle contains every formula valid for transitive dense commutative binary relations, no matter how many objects are needed for its proof. This is just a restatement of [50, Theorem 6.2]. (See [15] for another proof.) In more detail, \(A \rightarrow B\) is a theorem of \(R\)-mingle if and only if the inclusion \(A \subseteq B\) is true whenever all its propositional variables are interpreted as relations in a set \(K\) of dense transitive binary relations, where \(K\) is closed under union, intersection, residation, and converse-complementation (the interpretations of the connectives in \(A \rightarrow B\)) and \(K\) is commutative under relative multiplication. The underlying reason is that, as Meyer proved [2, Corollaries 3.1, 3.5, p. 413–4], the theorems of \(R\)-mingle are the formulas valid in all Sugihara matrices, and all Sugihara matrices are representable as sets of transitive dense binary relations, commutative under relative multiplication [15, 50]. An informal mnemonic for this result might be

\[ R\text{-mingle} = L_\infty + \text{all relations are dense, transitive, and commute under } \circ. \]

This completeness result involves binary relations and their natural operations. \(R\)-mingle is the set of laws (expressible with \(\cap, \cup, \rightarrow, \sim\)) that hold for all transitive, dense, commutative binary relations. Perhaps this provides another standard mathematical example of a relevance logic, as van Benthem suggested, although sometimes \(R\)-mingle is not regarded as a true relevance logic because of Meyer’s result [2, RM84, p. 417] that \(R\)-mingle has only the weak variable sharing property that if \(A \rightarrow B\) is a theorem of \(R\)-mingle then either \(A\) and \(B\) share a propositional variable or \(\sim A\) and \(B\) are both theorems of \(R\)-mingle.
9. System $\mathcal{L}_5$

Given Lyndon’s initial result and the non-finite axiomatizability results that followed, starting with Monk’s proof that the equational theory of representable relation algebras is not finitely based, it was easy to suspect that formulas must exist that are valid for all binary relations but are not in Tarski’s relevance logic because they require more than five objects to prove; see [29], [30] (Q1), p. 52. Indeed, Mikulás proved such a non-finite axiomatizability result for relevance logic. The formulas involved are complicated and not generally considered as potential axioms for relevance logics. Two formulas that are not theorems of $R$ are given in [30, Theorem 8.2]. Here is the shorter one. (Because of the associativity of $\circ$, one set of parentheses has been omitted from the final term.)

\[
\begin{align*}
\left( (A_{34} \circ A_{23}) & \land A_{24} \right) \circ \left( (A_{12} \circ A_{01}) \land A_{02} \right) \land A_{04} \\
\rightarrow \left( (A_{34} \circ A_{23}) & \land A_{24} \right) \circ \left( (A_{12} \circ [A_{01} \land \neg A_{01}]) \land A_{02} \right) \land A_{04} \\
\lor \left( ([A_{34} \land \neg A_{34}] \circ A_{23}) & \land A_{24} \right) \circ \left( (A_{12} \circ A_{01}) \land A_{02} \right) \land A_{04} \\
\lor A_{34} \circ \left( (A_{23} \circ A_{12}) & \land \left( ((A_{23} \circ A_{02}) \land (A_{43} \circ A_{04})) \circ A_{10} \right) \\
\land \left( A_{43} \circ ((A_{04} \circ A_{10}) \land (A_{24} \circ A_{12})) \right) \right) \circ A_{01}
\end{align*}
\]

The subscripts on the variables indicate which objects should appear as subscripts in assertions based on that formula. For example, the assertion $(A_{24})_{24}$ will appear in a properly constructed 5-proof.

10. Relevant model structures

Relevant model structures [49, §2] provide sound and complete semantics for system $R$. We will use them to show various formulas are not in $R$ or not in $\mathcal{L}_4$.

**Definition 4.** A relevant model structure $\mathcal{R} = (K, R, *, 0)$ consists of a non-empty set $K$, a ternary relation $R \subseteq K^3$, a unary operation $*: K \to K$, and a distinguished element $0 \in K$, such that postulates (p1)–(p6) hold for all $a, b, c \in K$, where

- \( R^{2}abcd \iff \exists_x(Rabx \land Rxcd) \)
- \( R^2a(bc)d \iff \exists_x(Rbcx \land Rad) \)
- \( R0aa \) (0-reflexivity)
- \( Raaa \) (density)
- \( R^2abcd \Rightarrow R^2acdb \)
- \( R^20abc \Rightarrow Rabc \) (0-cancellation)
- \( Rabc \Rightarrow Rac^*b^* \)
- \( a^{**} = a \) (involution)
By [30] Theorem 7.1, \( \mathfrak{R} = \langle K, R, *, 0 \rangle \) is a relevant model structure if and only if it satisfies (p1), (p2), (p3'), (p4), (p5'), (p6), and (comm), where

\[
\begin{align*}
\text{(comm)} & : Rabc \Rightarrow Rbac & \text{(commutativity)} \\
\text{(p3')} & : R^2 abed \Rightarrow R^2 a(bc)d & \text{(associativity)} \\
\text{(p5')} & : Rbc \Rightarrow Rc^* ab^* & \text{(rotation)}
\end{align*}
\]

**Definition 5.** Let \( \mathfrak{R} = \langle K, R, *, 0 \rangle \) be a relevant model structure. A **valuation** in \( \mathfrak{R} \) is a function \( \nu : \mathbf{P} \nu \times K \rightarrow \{T, F\} \) such that, for all \( a, b \in K \) and \( p \in \mathbf{P} \nu \), if \( R^p ab \) and \( \nu(p, a) = T \) then \( \nu(p, b) = T \). \( I \) is the interpretation associated with \( \nu \) if \( I : \mathbf{Fmla} \times K \rightarrow \{T, F\}, \) and for all \( A, B \in \mathbf{Fmla} \) and \( c \in K \),

1. \( I(p, c) = \nu(p, c) \),
2. \( I(A \wedge B, c) = T \) iff \( I(A, c) = T \) and \( I(B, c) = T \),
3. \( I(A \vee B, c) = T \) iff \( I(A, c) = T \) or \( I(B, c) = T \),
4. \( I(A \rightarrow B, c) = T \) iff for all \( a, b \), if \( R^p ab \) and \( I(A, a) = T \) then \( I(B, b) = T \),
5. \( I(A \circ B, c) = T \) iff for some \( a, b \), \( R^p ab \), \( I(A, a) = T \), and \( I(B, b) = T \),
6. \( I(\sim A, c) = T \) iff \( I(A, c^*) = F \).

A formula \( A \) is **true** on a valuation \( \nu \), or on the associated \( I \), at \( c \in K \) if \( I(A, c) = T \), and **false** on \( \nu \) at \( c \) if \( I(A, c) = F \). A formula \( A \) is **verified** on \( \nu \), or on the associated \( I \), if \( I(A, 0) = T \), otherwise **falsified**. A formula \( A \) is **valid** in \( \mathfrak{R} \) if \( A \) is verified on every valuation in \( \mathfrak{R} \), and **R-valid** if \( A \) is valid in every relevant model structure, otherwise **R-invalid**.

Condition [vi] follows from [iv] and [vi] when definition [11] is used instead of taking \( \circ \) as primitive; see [49] footnote 10, p. 206. By [49] Theorem 2, all theorems of \( \mathfrak{R} \) are R-valid, and by [49] Theorem 3, all R-valid formulas are theorems of \( \mathfrak{R} \). A relevant model structure \( \mathfrak{R} = \langle K, R, *, 0 \rangle \) is **normal** if \( 0^* = 0 \) [49] p. 218]. By [49] Theorem 4], a formula \( A \) is a theorem of \( \mathfrak{R} \) if and only if \( A \) is valid in every normal relevant model structure.

**Definition 6.** Given a relevant model structure \( \mathfrak{R} = \langle K, R, *, 0 \rangle \), define operations \( \circ, \rightarrow, *, \) and \( \sim \) on subsets \( X, Y \subseteq K \) by

\[
\begin{align*}
X \circ Y & = \{ z : Rxyz \text{ for some } x \in X \text{ and } y \in Y \}, \\
X \rightarrow Y & = \{ z : Rzxy \text{ and } x \in X \text{ then } y \in Y \}, \\
X^* & = \{ z^* : z \in X \}, \\
\sim X & = K \setminus X^*.
\end{align*}
\]

For any valuation \( \nu \) in \( \mathfrak{R} \) with associated interpretation \( I \), let \( J_\nu(A) = J(A) = \{ c : I(A, c) = T \} \) for every formula \( A \).

These operations (and their notation) are designed for the following consequences of Definition 5. For all formulas \( A \) and \( B \),

\[
\begin{align*}
J(A \wedge B) & = J(A) \cap J(B), \\
J(A \vee B) & = J(A) \cup J(B), \\
J(A \rightarrow B) & = J(A) \rightarrow J(B), \\
J(A \circ B) & = J(A) \circ J(B),
\end{align*}
\]
and \( K \) is the complex algebra. The elements of the complex algebra of \( K \) can be computed by using the distributive laws listed above. The triples can be read from the tables. For example, \( \langle L \rangle \) this formula is already part of \( A \) namely

\[ \text{Lemma 32 shows that } L(32) \text{ in Table 2 is in Tarski's relevance logic} \]

\( \circ \) [\( \begin{array}{cccc}
\emptyset & \{0\} & \{a\} & \{b\} & \{b^*\} \\
\{0\} & \{0\} & \{a\} & \{b\} & \{b^*\} \\
\{a\} & \{a\} & \{0, a, b\} & \{b, b^*\} & \{a, b, b^*\} \\
\{b\} & \{b\} & \{b, b^*\} & \{a, b, b^*\} & \{0, a, b^*\} \\
\{b^*\} & \{b^*\} & \{a, b, b^*\} & \{0, a, b^*\} & \{a, b^*\} \\
\end{array} \] \]

\( \mathcal{R}_1 = \)

\( \circ \) [\( \begin{array}{cccc}
\emptyset & \{0\} & \{a\} & \{b\} & \{b^*\} \\
\{0\} & \{0\} & \{a\} & \{b\} & \{b^*\} \\
\{a\} & \{a\} & \{0, a, b, b^*\} & \{a, b^*\} & \{a, b, b^*\} \\
\{b\} & \{b\} & \{a, b^*\} & \{0, a, b^*\} & \{a, b^*\} \\
\{b^*\} & \{b^*\} & \{a, b^*\} & \{0, a, b^*\} & \{a, b^*\} \\
\end{array} \] \]

\( \mathcal{R}_2 = \)

\( \circ \) [\( \begin{array}{cccc}
\emptyset & \{0\} & \{a\} & \{b\} & \{b^*\} \\
\{0\} & \{0\} & \{a\} & \{b\} & \{b^*\} \\
\{a\} & \{a\} & \{0, a, b, b^*\} & \{a, b^*\} & \{a, b, b^*\} \\
\{b\} & \{b\} & \{a, b^*\} & \{0, a, b^*\} & \{a, b^*\} \\
\{b^*\} & \{b^*\} & \{a, b^*\} & \{0, a, b^*\} & \{a, b^*\} \\
\end{array} \] \]

\( \mathcal{R}_3 = \)

\( J(\sim A) = \sim J(A) \),

and \( A \) is valid in \( \mathcal{R} \) if \( 0 \in J(A) \) for every valuation on \( \mathcal{R} \). Some useful observations to make at this point are, for all \( X, Y \subseteq K \),

- \( X \circ \emptyset = \emptyset \circ X = \emptyset \),
- \( X \circ (Y \cup Z) = X \circ Y \cup X \circ Z \),
- \( (Y \cup Z) \circ X = Y \circ X \cup Z \circ X \),
- \( X \rightarrow Y = \sim (X \circ \sim Y) \).

Every relevant model structure \( \mathcal{R} = \langle K, R, *, 0 \rangle \) has an associated algebra, called its “complex algebra”. The elements of the complex algebra of \( \mathcal{R} \) are all the subsets of \( K \), and the operations of the complex algebra are \( \cup, \cap \), and the operations \( \circ, \rightarrow \), and \( \sim \) from Definition 6.

11. A FORMULA IN \( \mathcal{L}_3 \) BUT NOT \( \mathcal{R} \)

Lemma 32 shows that \( L(32) \) in Table 2 is in Tarski’s relevance logic \( \mathcal{L}_3 \). In fact, this formula is already part of \( \mathcal{L}_3 \). However, \( L(32) \) is not a theorem of \( \mathcal{R} \). Here we present two normal relevant model structures that invalidate an instance of \( L(32) \), namely \( A \rightarrow B \), where \( p, q, r, s \in P_v \),

\[ A = (p \circ q) \land r, \]

\[ B = ((p \land \sim s) \circ q) \lor (p \circ (q \land (s \circ r))). \]

Let \( K = \{0, a, b, b^*\} \), where \( |K| = 4, 0^* = 0, a^* = a \), and \( * \) interchanges \( b \) and \( b^* \), as suggested by the notation. Two relevant model structures on \( K \), \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), are obtained by using two ternary relations \( R \) on \( K \). The ternary relation for \( \mathcal{R}_1 \) has 34 triples, while the ternary relation for \( \mathcal{R}_2 \) has 36 triples. Table 4 lists the \( \circ \)-products of all singleton subsets of \( K \) in both structures. The products for larger sets can be computed by using the distributive laws listed above. The triples can be read from the tables. For example, \( \langle a, a, b^* \rangle \) is a triple in the ternary relation
of \( R_2 \) but not \( R_1 \) because \( b^* \in \{0, a, b^*\} = \{a\} \cup \{a\} \) in the table for \( R_2 \), while \( b^* \notin \{0, a, b\} = \{a\} \cup \{a\} \) in the table for \( R_1 \).

Neither \( R_1 \) nor \( R_2 \) is the atom structure of a relation algebra. Their ternary relations fail to have the property, possessed by all atom structures of relation algebras, that \( R_{xyz} \leftrightarrow R_{zyx} \). In \( R_1 \), the triples \( \langle a, a, b \rangle \), \( \langle a, b^*, a \rangle \), and \( \langle b^*, a, a \rangle \) are present, but \( \langle b, a, a \rangle \), \( \langle a, b, a \rangle \), and \( \langle a, a, b^* \rangle \) are missing. In \( R_2 \), \( \langle b, b, b^* \rangle \) is present but \( \langle b^*, b^*, b \rangle \) is missing. Adding the missing triples to either structure produces \( R_3 \) in Table [4].

Now we proceed to use \( R_1 \) and \( R_2 \) to show that \( L_{32} \) in Table [2] is not a theorem of \( R \). For \( R_1 \), choose valuation \( \nu \) so that \( J(p) = J(s) = \{a\} \), that is,

\[
\nu(p, a) = \nu(s, a) = T,
\]

\[
\nu(p, 0) = \nu(s, 0) = \nu(p, b^*) = \nu(s, b) = \nu(s, b^*) = F.
\]

Then \( J(\sim s) = \{0, b, b^*\} \) so

\[
J(p \land \sim s) = \{a\} \cap \{0, b, b^*\} = \emptyset,
\]

hence

\[
J((p \land \sim s) \circ q) = J(p \land \sim s) \circ J(q) = \emptyset \land J(q) = \emptyset,
\]

regardless of the action of \( \nu \) on \( q \). Let \( J(q) = \{a\} \). Then

\[
J(p \circ q) = J(p) \circ J(q) = \{a\} \cap \{a\} = \{0, a, b\}.
\]

Next, let \( J(r) = \{b\} \). Then

\[
J(s \circ r) = J(s) \circ J(r) = \{a\} \cap \{b\} = \{b, b^*\},
\]

\[
J(q \land (s \circ r)) = J(q) \cap J(s \circ r) = \{a\} \cap \{b, b^*\} = \emptyset.
\]

This last equation gives us

\[
J(p \circ (q \land (s \circ r))) = J(p) \circ J(q \land (s \circ r)) = J(p) \circ \emptyset = \emptyset,
\]

regardless of our choice for \( J(p) \), and this, together with \( J((p \land \sim s) \circ q) = \emptyset \), gives us

\[
J(B) = J((p \land \sim s) \circ q) \cup J(p \circ (q \land (s \circ r))) = \emptyset \cup \emptyset = \emptyset.
\]

However, we also have

\[
J(A) = J((p \circ q) \land r) = J(p \circ q) \cap J(r) = \{0, a, b\} \cap \{b\} = \{b\}.
\]

Now, by definition, \( A \rightarrow B \) is verified if \( I(A \rightarrow B, 0) = T \). This means that for all \( x, y \in K \), if \( R_0xy \) and \( I(A, x) = T \) then \( I(B, y) = T \). However, from the table we have \( \{0\} \circ \{b\} = \{b\} \), which tells us that \( R_0bb \) by the definition of the operation \( \circ \), and \( I(A, b) = T \) since \( J(A) = \{b\} \), so we ought to have \( I(B, b) = T \) if \( A \rightarrow B \) were verified, but we don’t, because \( J(B) = \emptyset \). By the Routley-Meyer completeness results mentioned earlier, we conclude that \( A \rightarrow B \) is not a theorem of \( R \).

For \( R_3 \), we repeat all these steps, put with different values. Choose \( \nu \) so that \( J(p) = \{b\} \) and \( J(s) = \{b^*\} \). Then \( J(\sim s) = \{0, a, b^*\} \),

\[
J(p \land \sim s) = \{b\} \cap \{0, a, b^*\} = \emptyset,
\]
hence $$J((p \land \sim s) \circ q) = J(p \land \sim s) \circ J(q) = \emptyset \circ J(q) = \emptyset,$$
regardless of the action of $$\nu$$ on $$q$$. Let $$J(q) = \{b\}$$. Then $$J(p \circ q) = J(p) \circ J(q) = \{b\} \circ \{b\} = \{a,b,b^*\}.$$ Let $$J(r) = \{b^*\}$$. Then $$J((p \circ q) \land r) = J(p \circ q) \cap J(r) = \{a,b,b^*\} \cap \{b\} = \emptyset.$$ We also have $$J(A) = J((p \circ q) \land r) = J(p \circ q) \cap J(r) = \{a,b,b^*\} \cap \{b\} = \{b\}.$$ From the table we have $$\{0\} \circ \{b\} = \{b\},$$ hence $$R0bb$$, and $$I(A,b) = T$$ since $$J(A) = \{b\}$$. We ought to get $$I(B,b) = T$$ if $$A \to B$$ were verified, but we don’t since $$J(B) = \emptyset$$. By the Routley-Meyer completeness results, $$A \to B$$ is not a theorem of $$R$$.

By the way, all three relevant model structures $$\mathcal{R}_1$$, $$\mathcal{R}_2$$, and $$\mathcal{R}_3$$ validate formulas $$\text{(5)}, \text{(9)}, \text{(10)}, \text{(11)}, \text{(12)}, \text{and (13)},$$ but $$\text{(14)}$$ is invalidated in many ways. For example, choosing $$\nu$$ so that $$J(p) = \{a\}$$ yields the same calculations in all three structures:

$$J(p \to p) = J(p) \to J(p)$$
$$= \sim\{\{a\} \circ \sim\{a\}\}$$
$$= \sim\{\{a\} \circ \{0,b,b^*\}\}$$
$$= \sim\{a,b,b^*\}$$
$$= \{0\}$$
$$J(p \to (p \to p)) = \sim(J(p) \circ \sim(J(p \to p)))$$
$$= \sim\{\{a\} \circ \sim\{0\}\}$$
$$= \sim\{\{a\} \circ \{a,b,b^*\}\}$$
$$= \sim\{0,a,b,b^*\}$$
$$= \emptyset$$

12. A normal relevant model structure on a 21-element group

The two normal relevant model structures $$\mathcal{R}_1$$ and $$\mathcal{R}_2$$ are the simplest of several found by Prover9/Mace4 [31]. Although neither of them is the atom structure of a relation algebra, they turn out to be very nearly the same proper relation algebra. If three more triples are added to $$\mathcal{R}_1$$, namely $$\langle b,a,a \rangle$$, $$\langle a,b,a \rangle$$, and $$\langle a,a,b^* \rangle$$, or if
one more triple is added to \( \mathcal{R}_2 \), namely \( (b^*, b^*, b) \), they both become the normal relevant model structure \( \mathcal{R}_3 \), also shown in Table 5. \( \mathcal{R}_3 \) coincides with the relation algebra called \( 37 \) by Stephen D. Comer \[5\], and is actually isomorphic to a proper relation algebra whose base set is a 21-element group, first shown by Peter Jipsen.

Let \( G \) be the group generated by \( f \) and \( g \), subject to the relations \( f^3 = g^7 = 1 \) and \( gf = fg^2 \), where 1 is the identity element of \( G \). Alternatively, let \( G \) be the group generated by the permutations of \( \{1, \cdots, 21\} \) defined by

\[
\begin{align*}
 f &= (3, 6, 12)(5, 8, 14)(7, 10, 16)(9, 18, 15)(11, 20, 17)(13, 21, 19), \\
 g &= (2, 20, 17, 14, 11, 8, 5)(4, 16, 7, 19, 10, 21, 13).
\end{align*}
\]

Up to isomorphism, there are 8 ways (found by GAP \[7\]) to obtain \( \mathcal{R}_3 \) from \( G \). First let \( 0 = \{1\} \) be the singleton containing the identity element of \( G \). Next, choose one of the 8 partitions listed in Table 5 of the 20 non-identity elements into 3 sets \( a, b, \) and \( b^* \). Then 0 and \( a \) are closed under the formation of inverses in \( G \), \( b \) is the set of inverses of elements in \( b^* \), and vice versa,

\[
0 = \{1^{-1}\} \quad a = \{h^{-1}: h \in a\}, \quad b^* = \{h^{-1}: h \in b\}, \quad b = \{h^{-1}: h \in b^*\}.
\]

Here the Routley star * is the operation of forming all the inverses of the elements in a subset of \( G \). For any \( x, y, z \in K = \{0, a, b, b^*\} \), let the ternary relation \( R \) hold on the triple \( (x, y, z) \) just in case \( z \) is included in the set products of elements from \( x \) and \( y \), that is,

\[
R_{xyz} \leftrightarrow xy \supseteq z \leftrightarrow z \subseteq \{hk: h \in x, k \in y\},
\]

where \( hk \) is the product in \( G \) of the two group elements \( h, k \in G \) and \( xy \) is the set of products in \( G \) of pairs of elements, one from \( x \) and one from \( y \), in that order. This completes the construction of \( \mathcal{R}_3 \) from \( G \). (Ininitely many other groups can be used in a similar way; \( G \) is just the smallest one.) Every choice of partition from Table 5 produces that same table for the operation \( \circ \) in \( \mathcal{R}_3 \), as defined in \[15\] and shown in Table 4.

To show that this relevant model structure \( \mathcal{R}_3 \) is isomorphic to a proper relation algebra we use the right regular representation of the group \( G \), as is done in the

| \( a \) | \( b \) | \( b^* \) |
|---|---|---|
| \{\( f, f^2, g, g^2, g^3, g^5, g^6 \}\} | \{\( f^2g, f^2g^2, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} | \{\( g^3, f^2g^2, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} |
| \{\( fg, fg^2, fg^3, fg^4, fg^5, fg^6 \}\} | \{\( f^2g, f^2g^2, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} | \{\( g^3, f^2g^2, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} |
| \{\( fg, f^2g^2, g^3, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} | \{\( f^2g, f^2g^2, g^3, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} | \{\( g^3, f^2g^2, g^3, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} |
| \{\( fg, f^2g^2, g^3, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} | \{\( f^2g, f^2g^2, g^3, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} | \{\( g^3, f^2g^2, g^3, f^2g^3, f^2g^4, f^2g^5, f^2g^6 \}\} |

Table 5. Representations on \( \mathcal{R}_3 \) on a 21-element group.
proof of the Cayley representation theorem (every group is isomorphic to a group of permutations). First we recall that subsets of a group were once called “complexes”, and the set of subsets of the group $G$ forms an algebra called its “complex algebra”, whose operations are union, intersection, complementation with respect to $G$, multiplication of complexes as defined above, and the operation of forming all the inverses of elements in a subset of $G$, here denoted by the Routley star $\ast$. The complex algebra also has, as a distinguished element, the singleton consisting of just the identity element of the group. For every $x \subseteq G$, define the binary relation $\sigma(x)$ on $G$ by

$$\sigma(x) = \{\langle k, kh \rangle : k \in G, h \in x \} \subseteq G \times G.$$ 

Then $\sigma$ is an injective homomorphism from the complex algebra of $G$ into the proper relation algebra of all binary relations on $G$, in the sense that, for all $x, y \subseteq G$, recalling definitions (1) and (4), we have

$$\sigma(x \cup y) = \sigma(x) \cup \sigma(y),$$

$$\sigma(x \cap y) = \sigma(x) \cap \sigma(y),$$

$$\sigma(G \setminus x) = (G \times G) \setminus \sigma(x),$$

$$\sigma(xy) = \sigma(x)\sigma(y),$$

$$\sigma(x^\ast) = \sigma(x)^{-1},$$

$$\sigma(\{1\}) = \{\langle h, h \rangle : h \in G\}.$$ 

If $h \in G$ then $\sigma(\{h\})$ is the permutation used in the proof of Cayley’s theorem. The right regular representation has a property required by representations of relation algebras: the permutations associated with $\{h\}$ and $\{k\}$ must be disjoint (as sets) whenever $h \neq k$, simply because $\{h\} \cap \{k\} = \emptyset$ and this fact must be reflected in any representation. Applying $\sigma$ to the elements of $\mathcal{R}_3$ produces four binary relations on the 21-element set $G$,

$$\sigma(0) = \{\langle h, h \rangle : h \in G\}, \quad A = \sigma(a), \quad B = \sigma(b), \quad B^{-1} = \sigma(b^\ast).$$

These four relations form a partition of $G \times G$. One of them is the identity relation on $G$, and the converse of any one of them is either itself or another one of them. In particular, $(\sigma(0))^{-1} = \sigma(0)$ and $A^{-1} = A$, while $B$ and $B^{-1}$ are converses of each other. The table for $\mathcal{R}_3$ yields these conclusions about the relative products of these relations: $\sigma(0)\mid x = x\sigma(0) = x$ for all $x \in \{\sigma(0), A, B, B^{-1}\}$,

$$A|A = B|B^{-1} = B^{-1}|B = G \times G,$$

and all other products are equal to $(G \times G) \setminus \sigma(0)$. Any four relations with these properties gives us yet another representation of the relation algebra $37_{37}$, alias $\mathcal{R}_3$.

13. Counterexample to a theorem of Kowalski

Kowalski [14, Theorem 8.1] proved, “The relevant logic $R$ is sound and complete with respect to square-increasing, commutative, integral relation algebras.” $L(32)$ is a counterexample to this theorem. It is not a theorem of $R$ (because it is invalid the the relevant model structures $\mathcal{R}_1$ and $\mathcal{R}_2$) and yet holds in all relation algebras (including the square-increasing, commutative, integral ones). In fact, Lemma 32 shows that it is in $\mathcal{L}_3$ and is true in all semi-associative relation algebras (because it is provable with only three objects).
Theorem 8.1] is obtained as an immediate consequence of [14, Theorem 7.1], that “Every normal De Morgan monoid is embeddable as a bare [no constants] De Morgan monoid into a square-increasing, commutative, integral relation algebra.” The complex algebras of $K_1$ and $K_2$ are counterexamples to Theorem 7.1.

Part of the proof of Theorem 7.1 reads, “By definition of $\varepsilon$, it is an embedding of the lattice reduct of [the normal De Morgan monoid] $M$ into the lattice reduct of [the relation algebra] $U_M$; in particular, $\varepsilon$ is injective. Lemma 7.1 shows that the multiplication, implication and De Morgan negation are preserved as well, . . . ”. The proof of Theorem 7.1 uses Lemma 5.4 to show half of the preservation of multiplication, namely $\varepsilon(ab) \subseteq \varepsilon(a) \circ \varepsilon(b)$.

The difficulty arises in the proof of Lemma 5.4(1) at this point: “Since $M$ is a distributive lattice, $R$ is a prime filter and $R' = R$, proving (1).” At this stage in the proof, $\langle R, R' \rangle$ is known to be a maximal disjoint pair in $S$ (hence $R$ a proper filter and $R'$ a proper ideal disjoint from $R$, and they satisfy two additional technical conditions). If the desired conclusion that $R = R'$ were to fail, there would be some element $x \notin R \cup R'$. A desired contradiction could then be attained by showing that $x$ could be added to $R$ or to $R'$, i.e., either the filter generated by $x$ and $R$ is disjoint from the ideal $R'$, or else the ideal generated by $R'$ and $x$ is disjoint from $R$. The distributivity of $M$ insures that one of these two possibilities happens. For example, if the filter $R^+$ generated by $x$ and $R$ is disjoint from the ideal $R'$, but the ideal generated by $R'$ and $x$ is not disjoint from $R$, then $\langle R^+, R' \rangle$ would be a strictly larger disjoint filter-ideal pair. This would yield a contradiction if $\langle R, R' \rangle$ were maximal among all disjoint filter-ideal pairs, but it is only known to be maximal in $S$ (and subject to the technical conditions). The goal would be to show that $\langle R^+, R' \rangle$ is actually a strictly larger pair in $S$ (that this pair also satisfies the technical conditions). The difficulty in achieving this goal would be revealed by a more detailed examination of this situation for the complex algebras of $K_1$ and $K_2$.

14. Deriving $L(32)$ from Tarski’s axioms

$L(32)$ in Table 2 is shown to be in $L_3$ by Lemma 32. It is the translation into relevance logic notation of the following equation, which is true in all relation algebras.

$\ x; y \cdot z \leq (x \cdot \bar{w})y + x(y \cdot (w; z))$.

The equation (19) can therefore be derived from Tarski’s ten axioms for relation algebras [59, 8,2(i)] (treated as algebras of the form $\langle U, +, -, ;, \cdot, 1' \rangle$). Tarski’s axioms are

(R1) \quad x + y = y + x,
(R2) \quad x + (y + z) = (x + y) + z,
(R3) \quad \bar{x} + \bar{y} + \bar{x} + y = x,
(R4) \quad x; (y; z) = (x; y); z,
(R5) \quad (x + y); z = x; z + y; z,
(R6) \quad x; 1' = x,
(R7) \quad \bar{x} = x,
(R8) \quad (x + y)' = \bar{x} + \bar{y},
These are the axioms about which Tarski asked “whether this definition of relation algebra ... is justified in any intrinsic sense.”

The first three axioms are a set of postulates for Boolean algebras (treated as algebras of the form \( \langle U, +, \cdot \rangle \)) due to E. V. Huntington [11, 12, 13]. Proving all the usual equations true in Boolean algebras from the Huntington axioms is an interesting and challenging homework problem. One must first prove \( x + \overline{x} = y + \overline{y} \) in order to define the maximum element 1 by \( 1 = x + \overline{x} \). (See [27] for a solution.)

We will prove the following purely relation-algebraic facts directly from Tarski’s axioms. In the following proofs, any step that requires only Boolean algebra is marked “BA.”

\[
\begin{align*}
\text{(R}_9\text{)} & \quad (x:y)\bar{y} = \bar{y};x, \\
\text{(R}_10\text{)} & \quad \bar{x};\bar{y} + \bar{y} = \bar{y}.
\end{align*}
\]

Proof of (20):

\[
\begin{align*}
x & \leq y \iff x:z \leq y:z \\
\rightarrow & \quad (x + y):z = y:z \\
\leftrightarrow & \quad x:z + y:z = y:z \quad \text{(R}_5\text{)} \\
\leftrightarrow & \quad x:z \leq y:z \\
& \quad \text{BA}
\end{align*}
\]

Proof of (21):

\[
\begin{align*}
z:(x + y) & = z:x + z:y \\
\iff & \quad (x + y);z = z:z \\
\rightarrow & \quad x:z + y:z = y:z \quad \text{(R}_5\text{)} \\
\leftrightarrow & \quad x:z \leq y:z \\
& \quad \text{BA}
\end{align*}
\]

Proof of (22):

\[
\begin{align*}
z:(x + y) & = (\bar{z}:(x + y))\bar{y} \\
& = ((\bar{z} + \bar{y});\bar{z})\bar{y} \\
& = (\bar{x};\bar{y} + \bar{y};\bar{z})\bar{y} \\
& = ((\bar{x}:y)^\bar{y} + (z:y)^\bar{y})\bar{y} \\
& = (z:x + z:y)^\bar{y} \\
& = z:x + z:y \\
& \quad \text{(R}_7\text{)} \quad \text{BA}
\end{align*}
\]

The proof of (22) is like the proof of (21), but turned around in the obvious way.

Proof of (23):

\[
\begin{align*}
1 & = 1 + \bar{1} \\
& = \bar{1} + \bar{1} \quad \text{(R}_7\text{)} \\
& = (1 + 1)^\bar{y} \\
& = \bar{1} \quad \text{BA}
\end{align*}
\]
For (24), first note that, for any \( y \), the following statements are equivalent.

\[
\begin{align*}
\overline{\bar{x}} &\leq y \\
\overline{\bar{x}} + y &= y \\
\overline{\bar{x}} + \overline{\bar{y}} &= \overline{\bar{y}} \\
x + \overline{\bar{y}} &= 1 \\
x + \overline{\bar{y}} &= 1 \quad \text{(R}_7\text{), (R}_8\text{), (23)} \\
\overline{\bar{x}} &\leq y \\
\end{align*}
\]

We need only two instances of these equivalences. When \( y \) is either \( \overline{\bar{x}} \) or \( \overline{\bar{y}} \), we deduce that \( \overline{\bar{x}} \leq \overline{\bar{y}} \) and \( \overline{\bar{y}} \leq \overline{\bar{x}} \), respectively, hence (24) holds. Proof of (25):

\[
\begin{align*}
(x \cdot y)^\sim &= (\overline{x + y})^\sim \\
&= (\overline{x + y}) \quad \text{(24)} \\
&= \overline{x + y} \quad \text{(R}_8\text{)} \\
&= \overline{x + \overline{\bar{y}}} \quad \text{(24)} \\
&= \overline{x} \cdot \overline{\bar{y}} \quad \text{BA}
\end{align*}
\]

Proof of (26):

\[
\begin{align*}
x; y &= x; (y \cdot (\overline{\bar{x}}; z + \overline{\bar{x}}; z)) \\
&= x; (y \cdot \overline{\bar{x}}; z + y \cdot \overline{\bar{x}}; z) \\
&= x; (y \cdot \overline{\bar{x}}; z) + x; (y \cdot \overline{\bar{x}}; z) \quad \text{(21)} \\
&\leq x; (y \cdot \overline{\bar{x}}; z) + x; (\overline{\bar{x}}; z) \quad \text{(22)} \\
&\leq x; (y \cdot \overline{\bar{x}}; z) + \overline{\bar{x}} \quad \text{(R}_10\text{)}
\end{align*}
\]

From the previous equation we get

\[
\begin{align*}
x; y \cdot z &\leq (x; (y \cdot \overline{\bar{x}}; z) + \overline{\bar{x}}) \cdot z \\
&= x; (y \cdot \overline{\bar{x}}; z) \cdot z + \overline{\bar{x}} \cdot z \\
&= x; (y \cdot \overline{\bar{x}}; z) \cdot z + 0 \\
&= x; (y \cdot \overline{\bar{x}}; z) \cdot z
\end{align*}
\]

Associativity is not needed in any form for the proof of (19). Consequently (19) holds in all non-associative relation algebras (the class of algebras obtained by dropping (R_4) from the list of axioms). Here is a direct equational proof of (19).

\[
\begin{align*}
x; y \cdot z &= (x \cdot (\overline{\bar{w}} + \overline{\bar{w}})) \cdot y \cdot z \\
&= (x \cdot \overline{\bar{w}} + x \cdot \overline{\bar{w}}) \cdot y \cdot z \\
&= (((x \cdot \overline{\bar{w}}); y + (x \cdot \overline{\bar{w}}); y) \cdot y \cdot z \\
&= (x \cdot \overline{\bar{w}}) \cdot y \cdot z + (x \cdot \overline{\bar{w}}) \cdot y \cdot z \\
&\leq (x \cdot \overline{\bar{w}}) \cdot y + (x \cdot \overline{\bar{w}}) \cdot y \cdot z \\
&\leq (x \cdot \overline{\bar{w}}) \cdot y + (x \cdot \overline{\bar{w}}) \cdot (y \cdot \overline{\bar{w}}) \cdot z \quad \text{(26)} \\
&= (x \cdot \overline{\bar{w}}) \cdot y + (x \cdot \overline{\bar{w}}) \cdot (y \cdot (\overline{\bar{x}} \cdot \overline{\bar{w}})) \cdot z \quad \text{(26), (R}_7\text{)}
\end{align*}
\]
\[ \mathcal{R}_4 = \begin{array}{ccc} \circ & \{\emptyset\} & \{a\} & \{a^*\} \\ \{\emptyset\} & \{\emptyset\} & \{a\} & \{a^*\} \\ \{a\} & \{a\} & \{a\} & \{0, a, a^*\} \\ \{a^*\} & \{a^*\} & \{0, a, a^*\} & \{a^*\} \end{array} \]

Table 6. Belnap’s normal relevant model structure.

\[ \leq (x \cdot w); y + x; (y \cdot w; z) \quad (20), (22) \]

15. VARIABLE-SHARING

Tarski’s relevance logic has the variable-sharing property, even if extended beyond \( R \) by adding axioms insuring commutativity and density. Belnap’s \[3\] original proof of this fact for the logic \( E \) of Anderson-Belnap \[1\] applies with no changes. Belnap’s construction and proof are presented in this section. Belnap gave matrices for \( \wedge, \vee, \rightarrow, \sim \), and two defined unary connectives, \( N(A) = (A \rightarrow A) \rightarrow A \) and \( M(A) = \sim(N(\sim A)) \).

From the matrices for \( \wedge \) and \( \vee \) it is apparent that the eight values appearing in them, namely \(-3, -2, -1, 0, +1, +2, \) and \(+3\) (the last four are the designated values), form a lattice isomorphic to the lattice of subsets of the 3-element set \( \{-1, +0, -2\} \), with \(+3\) at the top and \(-3\) at the bottom, if \( \wedge \) and \( \vee \) are interpreted as intersection and union. This observation does not occur in \[3\], but in subsequent literature they are usually portrayed this way; see, for example, [2, pp. 198, 252], [50, p. 178], and \[4\, p. 102\].

What took nearly half a century after their introduction in 1960 was the realization in \[29\] that Belnap’s matrices define a proper relation algebra; see also \[15, 30\]. This proper relation algebra was known to Lyndon \[23\] in 1950, and became well known in the 1980s under the name “The Point Algebra”, because it describes the ways two points on the real line can be related to each other; the three atomic relations between two real numbers are \( x < y, x = y, \) and \( x > y \). The joins of pairs of these relations are \( \leq, \geq, \) and \( \neq \).

Two formulas \( A, B \in \text{Sent} \) are said to share a variable if some propositional variable \( p \in \text{Pr} \) occurs in both \( A \) and \( B \). To show \( A \) and \( B \) share a propositional variable whenever \( A \rightarrow B \in L_4 \), we use the normal relevant model structure \( \mathcal{R}_4 \) shown in Table 6. Choose a valuation \( \nu \) so that

\[ J(p) = \begin{cases} \{a\} & \text{if } p \text{ occurs in } A, \\ \{a^*\} & \text{if } p \text{ does not occur in } A. \end{cases} \]

One key feature of \( \mathcal{R}_4 \) is that \( \{a\}, \{0, a\} \) and \( \{a^*, \{0, a^*\}\} \) are both closed under \( \cup, \cap, \rightarrow, \circ, \) and \( \sim \). This is obvious for \( \cup \) and \( \cap \), clear for \( \circ \) from Table 6 easy to check for \( \sim \), and therefore is also true for \( \rightarrow \). The other key feature is that \( X \rightarrow Y = \emptyset \) whenever \( X \in \{a\}, \{0, a\} \) and \( Y \in \{a^*, \{0, a^*\}\} \). For this we provide two sample computations.

- \( \{a\} \rightarrow \{a^*\} = \sim(\{a\} \circ \sim\{a^*\}) = \sim(\{a\} \circ \{0, a^*\}) = \sim(0, a, a^*) = K \setminus \{0, a, a^*\} = \emptyset \)
- \( \{0, a\} \rightarrow \{0, a^*\} = \sim(\{0, a\} \circ \sim\{0, a^*\}) = \sim(\{0, a\} \circ \{a^*\}) = \sim(0, a, a^*) = K \setminus \{0, a, a^*\} = \emptyset \).
Table 7. A non-commutative normal relevant model structure

| $\mathcal{R}_5$ | $\{0\}$ | $\{a\}$ | $\{b\}$ | $\{b^*\}$ |
|-----------------|---------|---------|---------|---------|
| $\{0\}$        | $\{0\}$ | $\{a\}$ | $\{b\}$ | $\{b^*\}$ |
| $\{a\}$        | $\{a\}$ | $\{0, a, b, b^*\}$ | $\{a, b\}$ | $\{a\}$ |
| $\{b\}$        | $\{b\}$ | $\{a\}$ | $\{b\}$ | $\{0, a, b, b^*\}$ |
| $\{b^*\}$      | $\{b^*\}$ | $\{a, b^*\}$ | $\{0, b, b^*\}$ | $\{b^*\}$ |

By the choice of $\nu$, the closure of $\{\{a\}, \{0, a\}\}$ give us

$$J(A) \in \{\{a\}, \{0, a\}\}.$$  

Suppose that $B$ is a formula whose propositional variables do not occur in $A$. Then, by the choice of $\nu$ and the closure of $\{\{a^*\}, \{0, a^*\}\}$,

$$J(B) \in \{\{a^*\}, \{0, a^*\}\}.$$  

By the second key feature, we conclude that $J(A \rightarrow B) = J(A) \rightarrow J(B) = \emptyset$. Since $0$ is not in $\emptyset$, $A \rightarrow B$ is not valid in $K_4$. The contrapositive of what we have just proved is that if $A \rightarrow B$ is valid in $K_4$, then $A$ and $B$ must share a variable.

16. Representing Belnap’s normal relevant model structure

A representation of $K_4$ as the atom structure of a proper relation algebra can be obtained as follows. Let $\mathbb{Q}$ be the set of rational numbers. Let

$$\sigma(a) = \{(x, y) : x < y, x, y \in \mathbb{Q}\},$$  
$$\sigma(a^*) = \{(x, y) : x > y, x, y \in \mathbb{Q}\},$$  
$$\sigma(0) = \{(x, y) : x = y, x, y \in \mathbb{Q}\}.$$  

Extend $\sigma$ to all subsets of $K_4 = \{0, a, a^*\}$, by sending each subset of $K$ to the union of the images of its elements under $\sigma$. For example,

$$\sigma(\{a\}) = \sigma(a),$$  
$$\sigma(\{a, a^*\}) = \{(x, y) : x \neq y, x, y \in \mathbb{Q}\},$$  
$$\sigma(\{0, a\}) = \{(x, y) : x \leq y, x, y \in \mathbb{Q}\}.$$  

Thus $\sigma$ maps the complex algebra of $\mathbb{R}_4$ onto the proper relation algebra whose universe consists of the eight binary relations on the rationals usually denoted in a more colloquial notation as $=, \neq, <, >, \leq, \geq, 0, \text{and } \mathbb{Q} \times \mathbb{Q}$.

17. Four axioms of $\mathbb{R}$ not in $L_4$

Table 7 shows the atom structure of a non-commutative proper relation algebra called $13_{37}$ in [28]. It satisfies conditions (p1), (p2), (p4), and (p6) in Definition 4, plus (p3') and (p5'), and therefore has all the required properties to be a relevant model structure except (comm). It is normal since $0^* = 0$. It is therefore called a “non-commutative normal relevant model structure”.

Although condition (p1) is called “0-reflexivity”, it insures that the proper relation algebra $13_{37}$ is dense, i.e., satisfies $x \leq x^2$, where $x^2 = x \cdot x$. Condition (p1) should therefore be called “density”, but the term “square-increasing” is commonly used instead because it describes the shape of the equation that defines density. Since $\mathbb{R}_5$ satisfies (p1), it validates the formulas that assert density for all relations,
namely the contraction axiom (12) and the reductio axiom (13). On the other hand, since it is not commutative, the axioms depending on that assumption are invalid in \( K_5 \), namely contraposition (8), permutation (9), suffixing (10), and modus ponens (11). These formulas are invalidated in many ways, but in rather few ways if the valuations are restricted so the propositional variables are mapped to singletons and the formulas are mapped to the empty set. Here is a complete list of such valuations (calculated with GAP [7]).

- (8) is invalid in \( K_5 \) because
  \[
  J((p \rightarrow \neg q) \rightarrow (q \rightarrow \neg p)) = \emptyset
  \]
  whenever \( \nu \) is chosen so that one of these three sets of equations holds:
  \[
  \begin{align*}
  J(p) &= \{a\} & J(q) &= \{b\} \\
  J(p) &= \{b\} & J(q) &= \{b^*\} \\
  J(p) &= \{b^*\} & J(q) &= \{a\}
  \end{align*}
  \]

- (9) is invalid in \( K_5 \) because
  \[
  J((p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))) = \emptyset
  \]
  whenever \( \nu \) is chosen so that one of these two sets of equations holds:
  \[
  \begin{align*}
  J(p) &= \{a\} & J(q) &= \{b\} & J(r) &= \{a\} \\
  J(p) &= \{b^*\} & J(q) &= \{a\} & J(r) &= \{a\}
  \end{align*}
  \]

- (10) is invalid in \( K_5 \) because
  \[
  J((p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))) = \emptyset
  \]
  whenever \( \nu \) is chosen so that one of these four sets of equations holds:
  \[
  \begin{align*}
  J(p) &= \{0\} & J(q) &= \{a\} & J(r) &= \{a\} \\
  J(p) &= \{0\} & J(q) &= \{b\} & J(r) &= \{a\} \\
  J(p) &= \{b\} & J(q) &= \{a\} & J(r) &= \{a\} \\
  J(p) &= \{b\} & J(q) &= \{b\} & J(r) &= \{a\}
  \end{align*}
  \]

- (11) is invalid in \( K_5 \) because
  \[
  J(p \rightarrow ((p \rightarrow q) \rightarrow q)) = \emptyset
  \]
  whenever \( \nu \) is chosen so that one of these two sets of equations holds:
  \[
  \begin{align*}
  J(p) &= \{a\} & J(q) &= \{a\} \\
  J(p) &= \{b\} & J(q) &= \{a\}
  \end{align*}
  \]

18. Representing \( K_5 \) as a proper relation algebra

As with Belnap’s normal relevant model structure, there is a representation of \( K_5 \) as the atom structure of a proper relation algebra. Again, \( \mathbb{Q} \) is the set of rational numbers. Let \( U \) be the set of finite sequences of one or more rational numbers, in which the first is arbitrary and all others are positive. Define a binary relation \( B \subseteq U \times U \) as follows.
Think of each element of $U$ as representing a location, from which it is possible to either travel some positive distance in “the same direction”, or to “branch off” and travel some positive distance in “the new direction”. If $s$ is the new point at which one arrives by moving as described, then the pair $(r, s)$ is in the relation $B$. Finally, $B$ is the transitive closure of the set of all pairs obtained from this description.

More formally, an ordered pair $⟨r, s⟩$ of sequences $r, s ∈ U$ is in $B_0$ if and only if $r ≠ s$ and either $s$ can be obtained from $r$ by adding a nonnegative rational to the last entry of $r$ (travel in the same direction by that amount) or appending a positive rational number to the end of $r$ (travel in the new direction by that amount). Let $B$ be the transitive closure of $B_0$. Since $B$ is a partial ordering, we will symbolize it with “$<$” in these examples:

$⟨−8⟩ < ⟨0⟩ < ⟨1⟩ < ⟨1, 2, 3⟩ < ⟨1, 2, 4⟩ < ⟨1, 2, 4, 5⟩ < ⟨1, 2, 4, 5, 6⟩ < \ldots$

Let

$$σ(b^*) = B,$$
$$σ(b) = B^{-1},$$
$$σ(a) = U \times U \setminus (B ∪ B^{-1}),$$
$$σ(0) = \{⟨x, y⟩ : x = y, x, y ∈ U\},$$

and extend $σ$ to all subsets of $K = \{0, a, b, b^*\}$ by sending each subset to the union of the images of its elements under $σ$.

19. Axiomatizing classical relevant logic

In [34, p. 183], Meyer and Routley define a CR* model structure $\mathfrak{K} = (K, R, ^*, 0)$. Their definition is the same as that of a normal relevant model structure except that conditions (p1), (p4), and $0^* = 0$ are replaced by $R_0ab ↔ a = b$, from which the three conditions can be derived (using the remaining conditions (p2), (p3), (p5), and (p6)). Hence every CR* model structure is a normal relevant model structure (but not conversely).

Their language contains connectives $→, ∧, ¬$, and $^*$ [34, p. 184], while $∨$ is recovered by the definition $A ∨ B = ¬(¬∧¬B)$ [34, d5., p. 187] and ~ is defined by $¬A = (A^*)$ [34, d4., p. 186]. The notions of valuation and interpretation in Definition 5 are suitably altered by retaining the conditions pertaining to the connectives $→$ and $∧$, adding the conditions $I(¬A, c) = T$ iff $I(A, c) = F$ and $I(A^*, c) = T$ iff $I(A, c^*) = T$, and deriving the conditions for $∨, ◦$, and ~ through their definitions. Their system CR* of classical relevant logic is defined as the set of formulas valid in all CR* model structures [34, (9), p. 185]. In section III they

“... show that the system CR*, characterized so that its set of theorems is exactly the CR* valid formulas, exactly contains the system R of relevant implication on the definition of ~ by d4.” [34, p. 187]

This means that a formula $A$, written in the language of the connectives $∧, ∨, →, ◦$, and $∼$, is a theorem of $R$ if and only if the same formula, but with the connectives $∨, ◦$, and $∼$ defined in terms of $∧, →, ¬$, and $^*$, is valid in all CR* model structures. Their concluding remarks concern axiomatization.
"In conclusion, it will be noted that we have neglected to axiomatize \(\text{CR}^*\). The reason isn’t that it’s unaxiomatizable or anything like that; indeed, we presume that just putting together the axiomatization of \(\text{CR}\) in [33] and of \(\text{R}\) in [49] or [2] one would have an axiomatization of \(\text{CR}^*\), near enough, reversing d4 by then defining \(A^*\) as \(\neg\neg A\). Frankly, however, we can’t at this point stomach yet another completeness proof on ground that we have been over so often before; any readers that have stuck with us through the series of papers that began with [49] feel as we do, no doubt, letting the semantic characterization of \(\text{CR}^*\) above suffice. But the case is now pretty strong that \(\neg\) was just left out of Anderson-Belnap formulations of their logics, and evidence is building that the entire project of relevant logic is unified and simplified when the semantic \(\neg\), with a different function from the deduction-theoretic \(\neg\) that has been present from the start, is added. This paper is part of that evidence."

Meyer and Routley [33, p. 53] axiomatize the system \(\text{R}^+\) with axioms A1–A11, A14, and A15, and rules R1 and R2 from [49, p. 204]. These axioms and rules are the ones that do not mention negation. To combine these with the axioms and rules of [49], as they suggest, would seem to do nothing more than restore axioms A12 and A13 that involve negation. All these axioms and rules are recounted in [2, pp. 340-1]. Since they require defining \(A^*\) as \(\neg\neg A\), they may intend that the axioms involving negation appear twice, once with \(\neg\), and once with \(\neg\). This is also suggested by Meyer [32]. It would have been interesting if Meyer and Routley had attempted a more explicit axiomatization, for once the language includes the full range of connectives \(\wedge, \vee, \rightarrow, \circ, \neg, \neg\), and \(\ast\), either primitively or by definition, the opportunity exists to axiomatize classical relevant logic with Tarski’s axioms.

We might describe classical relevant logic as the system obtained from Tarski’s ten axioms \(\text{R}_1\)–\(\text{R}_{10}\), suitable renotated using these translations:

\[
\begin{align*}
A \vee B &= A + B, \\
A \wedge B &= A \cdot B, \\
B \circ A &= A; B, \\
A \rightarrow B &= \overline{A^{-1}; B}, \\
\neg A &= \overline{A}, \\
\sim A &= \overline{A}, \\
A^* &= A^{-1}.
\end{align*}
\]

Be that as it may, they (and their readers, perhaps) feel that the semantic characterization of \(\text{CR}^*\) suffices. Certainly that is all we need to observe that \(L(32)\) is not a theorem of \(\text{CR}^*\), because it is invalidated in the \(\text{CR}^*\) model structures \(\mathfrak{R}_1\) and \(\mathfrak{R}_2\). Indeed, all five normal relevant model structures \(\mathfrak{R}_1\)–\(\mathfrak{R}_5\) used in this paper satisfy the condition \(R0ab \leftrightarrow a = b\), and are therefore \(\text{CR}^*\) model structures.

20. Formulas in \(L_4\)

The 38 lemmas presented next establish membership in Tarski’s relevance logic of the formulas in Tables [1] and [2].
Lemma 1. $A \lor \sim A$

Proof.

1. $A_{o0} \Rightarrow A_{o0}$  \hspace{1cm} \text{Axiom}
2. $\Rightarrow A_{o0}, (\sim A)_{o0}$  \hspace{1cm} |\sim
3. $\Rightarrow (A \lor \sim A)_{o0}$  \hspace{1cm} |\lor  \hfill \Box

Lemma 2. $A \rightarrow A$

Proof.

1. $A_{10} \Rightarrow A_{10}$  \hspace{1cm} \text{Axiom}
2. $\Rightarrow (A \rightarrow A)_{o0}$  \hspace{1cm} |\rightarrow, no 1  \hfill \Box

Lemma 3. $A \land B \rightarrow A$

Proof.

1. $A_{10}, B_{10} \Rightarrow A_{10}$  \hspace{1cm} \text{Axiom})
2. $(A \land B)_{10} \Rightarrow A_{10}$  \hspace{1cm} \land|  
3. $\Rightarrow ((A \land B) \rightarrow A)_{o0}$  \hspace{1cm} |\rightarrow, no 1  \hfill \Box

Lemma 4. $A \land B \rightarrow B$

Proof.

1. $A_{10}, B_{10} \Rightarrow B_{10}$  \hspace{1cm} \text{Axiom}
2. $(A \land B)_{10} \Rightarrow B_{10}$  \hspace{1cm} 1, \land|
3. $\Rightarrow ((A \land B) \rightarrow B)_{o0}$  \hspace{1cm} 2, |\rightarrow, no 1  \hfill \Box

Lemma 5. $A \rightarrow A \lor B$

Proof.

1. $A_{10} \Rightarrow A_{10}, B_{10}$  \hspace{1cm} \text{Axiom}
2. $A_{10} \Rightarrow (A \lor B)_{10}$  \hspace{1cm} |\lor
3. $\Rightarrow (A \rightarrow A \lor B)_{o0}$  \hspace{1cm} |\rightarrow, no 1  \hfill \Box

Lemma 6. $B \rightarrow A \lor B$

Proof.

1. $B_{10} \Rightarrow A_{10}, B_{10}$  \hspace{1cm} \text{Axiom}
2. $B_{10} \Rightarrow (A \lor B)_{10}$  \hspace{1cm} |\lor
3. $\Rightarrow (B \rightarrow A \lor B)_{o0}$  \hspace{1cm} |\rightarrow, no 1  \hfill \Box
Lemma 7. $B \lor A \rightarrow A \lor B$

Proof.

1. $A_{10} \Rightarrow A_{10}, B_{10}$  
   Axiom
2. $A_{10} \Rightarrow (A \lor B)_{10}$  
   $\lor$
3. $B_{10} \Rightarrow A_{10}, B_{10}$  
   Axiom
4. $B_{10} \Rightarrow (A \lor B)_{10}$  
   $\lor$
5. $(B \lor A)_{10} \Rightarrow (A \lor B)_{10}$  
   2, 4, $\lor$
6. $\Rightarrow (B \lor A \rightarrow A \lor B)_{00}$  
   $\rightarrow$, no 1

\[\square\]

Lemma 8. $B \land A \rightarrow A \land B$

Proof.

1. $A_{10} \Rightarrow A_{10}$  
   Axiom
2. $B_{10} \Rightarrow B_{10}$  
   Axiom
3. $A_{10}, B_{10} \Rightarrow (A \land B)_{10}$  
   $\land$
4. $(B \land A)_{10} \Rightarrow (A \land B)_{10}$  
   $\land$
5. $\Rightarrow (B \land A \rightarrow A \land B)_{00}$  
   $\rightarrow$, no 1

\[\square\]

Lemma 9. $(A \land B) \land C \rightarrow A \land (B \land C)$

Proof.

1. $A_{10}, B_{10}, C_{10} \Rightarrow B_{10}$  
   Axiom
2. $A_{10}, B_{10}, C_{10} \Rightarrow C_{10}$  
   Axiom
3. $A_{10}, B_{10}, C_{10} \Rightarrow (A \land B)_{10}$  
   $\land$
4. $A_{10}, B_{10}, C_{10} \Rightarrow (B \land C)_{10}$  
   $\land$
5. $A_{10}, B_{10}, C_{10} \Rightarrow (A \land (B \land C))_{10}$  
   $\land$
6. $(A \land B)_{10}, C_{10} \Rightarrow (A \land (B \land C))_{10}$  
   $\land$
7. $(A \land B) \land C \Rightarrow (A \land (B \land C))_{10}$  
   $\land$
8. $\Rightarrow ((A \land B) \land C \rightarrow A \land (B \land C))_{00}$  
   $\rightarrow$, no 1

\[\square\]

Lemma 10. $(A \lor B) \lor C \rightarrow A \lor (B \lor C)$

Proof.

1. $A_{10} \Rightarrow A_{10}, B_{10}, C_{10}$  
   Axiom
2. $B_{10} \Rightarrow A_{10}, B_{10}, C_{10}$  
   Axiom
3. $(A \lor B)_{10} \Rightarrow A_{10}, B_{10}, C_{10}$  
   $\lor$
4. $C_{10} \Rightarrow A_{10}, B_{10}, C_{10}$  
   Axiom
5. $(A \lor B) \lor C \Rightarrow A_{10}, B_{10}, C_{10}$  
   $\lor$
6. $((A \lor B) \lor C)_{10} \Rightarrow A_{10}, (B \lor C)_{10}$  
   $\lor$

\[\square\]
Lemma 11. \( A \land (B \lor C) \rightarrow (A \land B) \lor (A \land C) \)

Proof.

1. \( A_{10} \Rightarrow A_{10} \) Axiom
2. \( B_{10} \Rightarrow B_{10} \) Axiom
3. \( C_{10} \Rightarrow C_{10} \) Axiom
4. \( A_{10}, B_{10} \Rightarrow (A \land B)_{10} \) 1, 2, |∧
5. \( A_{10}, C_{10} \Rightarrow (A \land C)_{10} \) 1, 3, |∧
6. \( A_{10}, (B \lor C)_{10} \Rightarrow (A \land B)_{10}, (A \land C)_{10} \lor |∧
7. \( (A \land (B \lor C))_{10} \Rightarrow (A \land B)_{10}, (A \land C)_{10} \land |∧
8. \( (A \lor (B \lor C))_{10} \Rightarrow ((A \land B) \lor (A \land C))_{10} \lor |∨
9. \( \Rightarrow ((A \land (B \lor C)) \lor ((A \land B) \lor (A \land C)))_{10} \rightarrow, \text{no 1} \)

Lemma 12. \( (A \rightarrow \sim C) \land (B \rightarrow C) \rightarrow \sim (A \land B) \)

Proof.

1. \( A_{01} \Rightarrow A_{01} \) Axiom
2. \( B_{01} \Rightarrow B_{01} \) Axiom
3. \( C_{00} \Rightarrow C_{00} \) Axiom
4. \( \sim C_{00}, C_{00} \Rightarrow \sim |\)
5. \( (A \rightarrow \sim C)_{10}, C_{00}, A_{01} \Rightarrow 1, 4, \rightarrow |\)
6. \( (A \rightarrow \sim C)_{10}, (B \rightarrow C)_{10}, A_{01}, B_{01} \Rightarrow 2, 5, \rightarrow |\)
7. \( (A \rightarrow \sim C)_{10}, (B \rightarrow C)_{10}, (A \land B)_{01} \Rightarrow \land |\)
8. \( (A \rightarrow \sim C)_{10}, (B \rightarrow C)_{10} \Rightarrow (\sim (A \land B))_{10} \lor |\sim
9. \( ((A \rightarrow \sim C) \land (B \rightarrow C))_{10} \Rightarrow (\sim (A \land B))_{10} \land |\)
10. \( \Rightarrow (((A \rightarrow \sim C) \land (B \rightarrow C)) \rightarrow \sim (A \land B))_{10} \rightarrow, \text{no 1} \)

Lemma 13. \( (A \rightarrow \sim B) \land (\sim A \rightarrow \sim C) \rightarrow (\sim B \lor \sim C) \)

Proof.

1. \( A_{11} \Rightarrow A_{11} \) Axiom
2. \( \Rightarrow A_{11}, (\sim A)_{11} \lor |\sim
3. \( C_{01} \Rightarrow C_{01} \) Axiom
4. \( \sim C_{10}, C_{01} \Rightarrow \sim |\)
5. \( (\sim A \rightarrow \sim C)_{10}, C_{01} \Rightarrow A_{11} 2, 4, \rightarrow |\)
6. \( B_{01} \Rightarrow B_{01} \) Axiom
7. \( \sim B_{10}, B_{01} \Rightarrow \sim |\)
Lemma 14. \( \sim \sim A \rightarrow A \)

\begin{proof}

1. \( A_{10} \rightarrow A_{10} \) \quad \text{Axiom}

2. \( \rightarrow A_{10}, (\sim A)_{01} \) \quad \sim

3. \( (\sim \sim A)_{10} \rightarrow A_{10} \) \quad \sim|

4. \( \rightarrow (\sim \sim A \rightarrow A)_{00} \) \quad \rightarrow, \text{ no 1}

\end{proof}

Lemma 15. \( A \rightarrow \sim \sim A \)

\begin{proof}

1. \( A_{01} \rightarrow A_{01} \) \quad \text{Axiom}

2. \( A_{10}, (\sim A)_{01} \rightarrow \) \quad \sim|

3. \( A_{10} \rightarrow (\sim \sim A)_{10} \) \quad \sim|

4. \( \rightarrow (A \rightarrow \sim \sim A)_{00} \) \quad \rightarrow, \text{ no 1}

\end{proof}

Lemma 16. \( \sim (A \lor B) \rightarrow (\sim A \land \sim B) \)

\begin{proof}

1. \( A_{01} \rightarrow A_{01} \) \quad \text{Axiom}

2. \( \rightarrow (\sim A)_{10}, A_{01} \) \quad \sim|

3. \( B_{01} \rightarrow B_{01} \) \quad \text{Axiom}

4. \( \rightarrow (\sim B)_{10}, B_{01} \) \quad \sim|

5. \( \rightarrow (\sim A \land \sim B)_{10}, A_{01}, B_{01} \) \quad 2, 4, \land

6. \( \rightarrow (\sim A \land \sim B)_{10}, (A \lor B)_{01} \) \quad \lor

7. \( (\sim (A \lor B))_{10} \rightarrow (\sim A \land \sim B)_{10} \) \quad \sim|

8. \( \rightarrow (\sim (A \lor B) \rightarrow (\sim A \land \sim B))_{00} \) \quad \rightarrow, \text{ no 1}

\end{proof}

Lemma 17. \( \sim (A \land B) \rightarrow (\sim A \lor \sim B) \)

\begin{proof}

1. \( A_{01} \rightarrow A_{01} \) \quad \text{Axiom}

2. \( B_{01} \rightarrow B_{01} \) \quad \text{Axiom}

\end{proof}
3. \( A_{01}, B_{01} \Rightarrow (A \land B)_{01} \)  
| \( \land \)  
4. \( A_{01}, B_{01}, (\neg (A \land B))_{10} \Rightarrow \)  
| \( \neg \)  
5. \( A_{01}, (\neg (A \land B))_{10} \Rightarrow (\neg B)_{10} \)  
| \( \neg \)  
6. \( (\neg (A \land B))_{10} \Rightarrow (\neg A)_{10}, (\neg B)_{10} \)  
| \( \neg \)  
7. \( (\neg (A \land B))_{10} \Rightarrow (\neg A \lor \neg B)_{10} \)  
| \( \lor \)  
8. \( \Rightarrow ((\neg (A \land B) \Rightarrow (\neg A \lor \neg B))_{00} \Rightarrow \rightarrow, \text{ no 1} \)

\[ \square \]

Lemma 18. \( (\neg A \land \neg B) \Rightarrow \neg (A \lor B) \)

Proof.

1. \( A_{01} \Rightarrow A_{01} \)  
   Axiom  
2. \( B_{01} \Rightarrow B_{01} \)  
   Axiom  
3. \( (A \lor B)_{01} \Rightarrow A_{01}, B_{01} \)  
   \( \lor \)  
4. \( (\neg B)_{10}, (A \lor B)_{01} \Rightarrow A_{01} \)  
   \( \neg \)  
5. \( (\neg A)_{10}, (\neg B)_{10}, (A \lor B)_{01} \Rightarrow \)  
   \( \neg \)  
6. \( (\neg A \land \neg B)_{10}, (A \lor B)_{01} \Rightarrow \land \)  
7. \( (\neg A \land \neg B)_{10} \Rightarrow (\neg (A \lor B))_{10} \)  
   \( \neg \)  
8. \( \Rightarrow ((\neg A \land \neg B) \Rightarrow (\neg (A \lor B))_{00} \Rightarrow \rightarrow, \text{ no 1} \)

\[ \square \]

Lemma 19. \( (\neg A \lor \neg B) \Rightarrow (\neg A \land \neg B) \)

Proof.

1. \( A_{01} \Rightarrow A_{01} \)  
   Axiom  
2. \( \Rightarrow (A \rightarrow A)_{11} \)  
   \( \rightarrow \)  
3. \( B_{10} \Rightarrow B_{10} \)  
   Axiom  
4. \( ((A \rightarrow A) \rightarrow B)_{10} \Rightarrow B_{10} \)  
   \( \rightarrow \)
Lemma 21. \((A \rightarrow B) \land (A \rightarrow C) \rightarrow (A \rightarrow (B \land C))\)

Proof.

1. \(A_{21} \Rightarrow A_{21}\)  \hspace{1cm} Axiom
2. \(B_{20} \Rightarrow B_{20}\)  \hspace{1cm} Axiom
3. \((A \rightarrow B)_{10}, A_{21} \Rightarrow B_{20}\) \hspace{1cm} 1, 2, \(\rightarrow\)
4. \(C_{20} \Rightarrow C_{20}\)  \hspace{1cm} Axiom
5. \((A \rightarrow C)_{10}, A_{21} \Rightarrow C_{20}\) \hspace{1cm} 1, 4, \(\rightarrow\)
6. \((A \rightarrow B)_{10}, (A \rightarrow C)_{10}, A_{21} \Rightarrow (B \land C)_{20}\) \hspace{1cm} 3, 5, \(\land\)
7. \(((A \rightarrow B) \land (A \rightarrow C))_{10}, A_{21} \Rightarrow (B \land C)_{20}\) \hspace{1cm} \(\land\)
8. \(((A \rightarrow B) \land (A \rightarrow C))_{10} \Rightarrow (A \rightarrow (B \land C))_{10}\) \hspace{1cm} \(\Rightarrow\), no 2
9. \(\Rightarrow (((A \rightarrow B) \land (A \rightarrow C)) \rightarrow (A \rightarrow (B \land C)))_{00}\) \hspace{1cm} \(\Rightarrow\), no 1

Lemma 22. \((A \rightarrow C) \land (B \rightarrow C) \rightarrow ((A \lor B) \rightarrow C)\)

Proof.

1. \(A_{21} \Rightarrow A_{21}\)  \hspace{1cm} Axiom
2. \(C_{20} \Rightarrow C_{20}\)  \hspace{1cm} Axiom
3. \((A \rightarrow C)_{10}, A_{21} \Rightarrow C_{20}\) \hspace{1cm} \(\Rightarrow\)
4. \(B_{21} \Rightarrow B_{21}\)  \hspace{1cm} Axiom
5. \(C_{20} \Rightarrow C_{20}\)  \hspace{1cm} Axiom
6. \((B \rightarrow C)_{10}, B_{21} \Rightarrow C_{20}\) \hspace{1cm} \(\Rightarrow\)
7. \((A \rightarrow C)_{10}, (B \rightarrow C)_{10}, (A \lor B)_{21} \Rightarrow C_{20}\) \hspace{1cm} 3, 6, \(\lor\)
8. \(((A \rightarrow C) \land (B \rightarrow C))_{10}, (A \lor B)_{21} \Rightarrow C_{20}\) \hspace{1cm} \(\land\)
9. \(((A \rightarrow C) \land (B \rightarrow C))_{10} \Rightarrow ((A \lor B) \rightarrow C)_{10}\) \hspace{1cm} \(\Rightarrow\), no 2
10. \(\Rightarrow (((A \rightarrow C) \land (B \rightarrow C)) \rightarrow ((A \lor B) \rightarrow C))_{00}\) \hspace{1cm} \(\Rightarrow\), no 1

Lemma 23. \((A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \land C \rightarrow B \land D)\)

Proof.

1. \(A_{21}, C_{21} \Rightarrow C_{21}\)  \hspace{1cm} Axiom
2. \((A \land C)_{21} \Rightarrow C_{21}\) \hspace{1cm} \(\land\)
3. \(B_{20}, D_{21} \Rightarrow B_{20}\)  \hspace{1cm} Axiom
4. \(B_{20}, D_{20} \Rightarrow D_{20}\)  \hspace{1cm} Axiom
5. \(B_{20}, D_{20} \Rightarrow (B \land D)_{20}\) \hspace{1cm} \(\land\)
6. \((B \rightarrow (C \rightarrow D))_{10}, (A \land C)_{21} \Rightarrow (B \land D)_{20}\) \hspace{1cm} 2, 5, \(\rightarrow\)
7. \(A_{21}, C_{21} \Rightarrow A_{21}\)  \hspace{1cm} Axiom
8. \((A \land C)_{21} \Rightarrow A_{21}\) \hspace{1cm} \(\land\)
Lemma 25. \((A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \lor C \rightarrow B \lor D)\)

Proof.

1. \(A_{21} \Rightarrow A_{21}\) Axiom
2. \(B_{20} \Rightarrow B_{20}, D_{20}\) Axiom
3. \(B_{20} \Rightarrow (B \lor D)_{20}\) \(|\lor|
4. \((A \rightarrow B)_{10}, A_{21} \Rightarrow (B \lor D)_{20}\) 4, 8, \(|\lor|
5. \(C_{21} \Rightarrow C_{21}\) Axiom
6. \(D_{20} \Rightarrow B_{20}, D_{20}\) Axiom
7. \(D_{20} \Rightarrow (B \lor D)_{20}\) \(|\lor|
8. \((C \rightarrow D)_{10}, C_{21} \Rightarrow (B \lor D)_{20}\) 5, 7, \(|\lor|
9. \((A \rightarrow B)_{10}, (C \rightarrow D)_{10}, (A \lor C)_{21} \Rightarrow (B \lor D)_{20}\) 4, 8, \(|\lor|
10. \((A \rightarrow B)_{10}, (C \rightarrow D)_{10} \Rightarrow (A \lor C \rightarrow B \lor D)_{10}\) \(|\Rightarrow|, \text{no } 2
11. \((A \rightarrow B)_{10}, (A \lor C)_{21} \Rightarrow (B \lor D)_{20}\) \(|\land|
12. \Rightarrow ((A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \lor C \rightarrow B \lor D))_{00}\) \(|\Rightarrow|, \text{no } 1

\[\square\]

Lemma 24. \((A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \lor C \rightarrow B \lor D)\)

Proof.

1. \(A_{21} \Rightarrow A_{21}\) Axiom
2. \(B_{20} \Rightarrow B_{20}, D_{20}\) Axiom
3. \((A \rightarrow B)_{10}, A_{21} \Rightarrow (B \lor D)_{20}\) \(|\lor|
4. \((A \rightarrow B)_{10}, A_{21} \Rightarrow (B \lor D)_{20}\) \(|\lor|
5. \((A \rightarrow B)_{10}, (A \lor C)_{21} \Rightarrow (B \lor D)_{20}\) \(|\land|
6. \((A \rightarrow B)_{10} \Rightarrow ((A \lor C) \rightarrow (B \lor D))_{10}\) no 2
7. \(C_{21} \Rightarrow C_{21}\) Axiom
8. \(D_{20}, A_{21} \Rightarrow B_{20}, D_{20}\) Axiom
9. \((C \rightarrow D)_{10}, A_{21}, C_{21} \Rightarrow B_{20}, D_{20}\) \(|\Rightarrow|
10. \((C \rightarrow D)_{10}, A_{21}, C_{21} \Rightarrow (B \lor D)_{20}\) \(|\lor|
11. \((C \rightarrow D)_{10}, (A \lor C)_{21} \Rightarrow (B \lor D)_{20}\) \(|\land|
12. \((C \rightarrow D)_{10} \Rightarrow ((A \lor C) \rightarrow (B \lor D))_{10}\) no 2
13. \(((A \rightarrow B) \lor (C \rightarrow D))_{10} \Rightarrow ((A \lor C) \rightarrow (B \lor D))_{10}\) 6, 12, \(|\lor|
14. \Rightarrow (((A \rightarrow B) \lor (C \rightarrow D)) \rightarrow ((A \lor C) \rightarrow (B \lor D)))_{00}\) \(|\Rightarrow|, \text{no } 1

\[\square\]
Lemma 26. $A \rightarrow (\sim B \rightarrow \sim(A \rightarrow B))$

Proof.

1. $A_{10} \Rightarrow A_{10}$  
   Axiom
2. $B_{12} \Rightarrow B_{12}$  
   Axiom
3. $A_{10}, (A \rightarrow B)_{02} \Rightarrow B_{12}$  
   $\rightarrow$|
4. $A_{10} \Rightarrow (\sim(A \rightarrow B))_{20}, B_{12}$  
   $|\sim$
5. $A_{10}, (\sim B)_{21} \Rightarrow (\sim(A \rightarrow B))_{20}$  
   $\sim$|
6. $A_{10} \Rightarrow (\sim B \rightarrow \sim(A \rightarrow B))_{10}$  
   $\rightarrow$, no 2
7. $\Rightarrow (A \rightarrow (\sim B \rightarrow \sim(A \rightarrow B)))_{00}$  
   $\rightarrow$, no 1

□

Lemma 27. $A \rightarrow (B \rightarrow \sim(A \rightarrow \sim B))$

Proof.

1. $A_{10} \Rightarrow A_{10}$  
   Axiom
2. $B_{21} \Rightarrow B_{21}$  
   Axiom
3. $B_{21}, (\sim B)_{12} \Rightarrow$  
   $\sim$|
4. $A_{10}, B_{21}, (A \rightarrow \sim B)_{02} \Rightarrow$  
   1, 3, $\rightarrow$|
5. $A_{10}, B_{21} \Rightarrow (\sim(A \rightarrow \sim B))_{20}$  
   $\sim$|
6. $A_{10} \Rightarrow (B \rightarrow \sim(A \rightarrow \sim B))_{10}$  
   $\rightarrow$, no 2
7. $\Rightarrow (A \rightarrow (B \rightarrow \sim(A \rightarrow \sim B)))_{00}$  
   $\rightarrow$, no 1

□

Lemma 28. $A \rightarrow ((\sim B \rightarrow \sim A) \rightarrow B)$

Proof.

1. $B_{20} \Rightarrow B_{20}$  
   Axiom
2. $\Rightarrow B_{20}, (\sim B)_{02}$  
   $\sim$|
3. $A_{10} \Rightarrow A_{10}$  
   Axiom
4. $A_{10}, (\sim A)_{01} \Rightarrow$  
   $\sim$|
5. $A_{10}, (\sim B \rightarrow \sim A)_{21} \Rightarrow B_{20}$  
   2, 4, $\rightarrow$|
6. $A_{10} \Rightarrow ((\sim B \rightarrow \sim A) \rightarrow B)_{10}$  
   $\rightarrow$, no 2
7. $\Rightarrow (A \rightarrow ((\sim B \rightarrow \sim A) \rightarrow B))_{00}$  
   $\rightarrow$, no 1

□

Lemma 29. $A \rightarrow ((B \rightarrow \sim A) \rightarrow \sim B)$

Proof.

1. $A_{10} \Rightarrow A_{10}$  
   Axiom
2. $A_{10}, (\sim A)_{01} \Rightarrow$  
   $\sim$|
3. $B_{02} \Rightarrow B_{02}$  
   Axiom
4. $A_{10}, (B \rightarrow \sim A)_{21}, B_{02} \Rightarrow$  
   $\rightarrow$|
Lemma 30. $\neg((A \rightarrow B) \rightarrow \neg A) \rightarrow B$

Proof.

1. $A_{12} \Rightarrow A_{12}$ Axiom
2. $A_{01} \Rightarrow A_{01}$ Axiom
3. $A_{10} \Rightarrow (\neg A)_{21}$ $\sim$
4. $B_{20} \Rightarrow (\neg A)_{21}, B_{10}$ Axiom
5. $\Rightarrow ((A \rightarrow B)_{20} \Rightarrow (\neg A)_{21}, B_{10}) \Rightarrow \sim$
6. $\Rightarrow ((A \rightarrow B)_{20} \equiv (\neg A)_{21})_{10} \Rightarrow B_{10} \Rightarrow \sim$
7. $\Rightarrow ((A \rightarrow B)_{20} \Rightarrow (\neg A)_{21})_{10} \Rightarrow B_{10} \Rightarrow \sim$

Lemma 31. $\neg A \rightarrow ((B \rightarrow A) \rightarrow \neg B))$

Proof.

1. $B_{02} \Rightarrow B_{02}$ Axiom
2. $A_{01} \Rightarrow A_{01}$ Axiom
3. $(B \rightarrow A)_{21}, B_{02} \Rightarrow A_{01} \Rightarrow \sim$
4. $(\neg A)_{10}, (B \rightarrow A)_{21}, B_{02} \Rightarrow \sim$
5. $(\neg A)_{10}, (B \rightarrow A)_{21} \Rightarrow (\neg B)_{20} \Rightarrow \sim$
6. $(\neg A)_{10} \Rightarrow ((B \rightarrow A)_{21} \Rightarrow \sim B)_{10} \Rightarrow \sim$
7. $\Rightarrow ((A \rightarrow ((B \rightarrow A)_{21} \rightarrow \sim B))_{01} \Rightarrow \sim$

Lemma 32.

$(A \circ B) \land C \rightarrow ((A \land \sim D) \circ B) \lor (A \circ B \land (D \circ C))$

$\neg(A \rightarrow \sim B) \land C \rightarrow \neg((A \land \sim D) \rightarrow \sim B) \lor \neg(A \rightarrow \sim B \land (D \rightarrow \sim C)))$

Proof.

1. $A_{20} \Rightarrow A_{20}$ Axiom
2. $D_{02} \Rightarrow D_{02}$ Axiom
3. $\Rightarrow (\sim D)_{20}, D_{02} \Rightarrow \sim$
4. $A_{20} \Rightarrow (A \land \sim D)_{20}, D_{02}$ 1, 3, $\land$
5. $B_{12} \Rightarrow B_{12}$ Axiom
6. $B_{12}, (\sim B)_{21} \Rightarrow \sim$
7. $A_{20}, B_{12}, ((A \land \sim D) \rightarrow \sim B)_{01} \Rightarrow D_{02}$ 4, 6, $\rightarrow$
8. $C_{10} \Rightarrow C_{10}$ Axiom
Lemma 33. \((A \rightarrow B) \wedge (C \circ D) \rightarrow ((C \circ B) \circ D) \vee (C \circ (D \wedge A))\)

\((A \rightarrow B) \wedge \neg (C \rightarrow \neg D) \rightarrow \neg ((C \wedge B) \rightarrow \neg D) \vee \neg (C \rightarrow \neg (D \wedge A))\)

Proof.

1. \(A_{21} \Rightarrow A_{21}\)  

   Axiom

2. \(B_{20} \Rightarrow B_{20}\)  

   Axiom

3. \((A \rightarrow B)_{10}, A_{21} \Rightarrow B_{20}\)  

   \(\rightarrow|\)

4. \((A \rightarrow B)_{10} \Rightarrow (\neg A)_{12}, B_{20}\)  

   \(\neg|\)

5. \(C_{20} \Rightarrow C_{20}\)  

   Axiom

6. \((A \rightarrow B)_{10}, C_{20} \Rightarrow (\neg A)_{12}, (C \wedge B)_{20}\)  

   \(|\wedge|\)

7. \(D_{12} \Rightarrow D_{12}\)  

   Axiom

8. \((A \rightarrow B)_{10}, C_{20}, D_{12} \Rightarrow (D \wedge \neg A)_{12}, (C \wedge B)_{20}\)  

   \(|\wedge|\)

9. \((A \rightarrow B)_{10}, C_{20} \Rightarrow (D \wedge \neg A)_{12}, (C \wedge B)_{20}, (\neg D)_{21}\)  

   \(\neg|\)

10. \((A \rightarrow B)_{10}, (\neg (D \wedge A))_{21}, C_{20} \Rightarrow (C \wedge B)_{20}, (\neg D)_{21}\)  

   \(\sim|\)

11. \((\neg D)_{21} \Rightarrow (\neg D)_{21}\)  

   Axiom

12. \((A \rightarrow B)_{10}, ((C \wedge B) \rightarrow \neg D)_{01}, (\neg (D \wedge A))_{21}, C_{20} \Rightarrow (\neg D)_{21}\)  

   \(\rightarrow|\)

13. \(C_{20} \Rightarrow C_{20}\)  

   Axiom

14. \((A \rightarrow B)_{10}, ((C \wedge B) \rightarrow \neg D)_{01}, (C \rightarrow \neg (D \wedge A))_{01}, C_{20} \Rightarrow (\neg D)_{21}\)  

   \(\rightarrow|\)

15. \((A \rightarrow B)_{10}, ((C \wedge B) \rightarrow \neg D)_{01}, (C \rightarrow \neg (D \wedge A))_{01} \Rightarrow (C \rightarrow \neg D)_{01}\)  

   \(\rightarrow|, \text{ no } 2\)

16. \((A \rightarrow B)_{10}, (C \rightarrow \neg (D \wedge A))_{01} \Rightarrow (\neg ((C \wedge B) \rightarrow \neg D))_{10}, (C \rightarrow \neg D)_{01}\)  

   \(\sim|\)

17. \((A \rightarrow B)_{10} \Rightarrow (\neg (C \wedge B) \rightarrow \neg D))_{10}, (C \rightarrow \neg (D \wedge A))_{10}\)  

   \(\neg|\)

18. \((A \rightarrow B)_{10}, (\neg (C \rightarrow \neg D))_{10} \Rightarrow (\neg (C \wedge B) \rightarrow \neg D))_{10}, (C \rightarrow \neg (D \wedge A))_{10}\)  

   \(\sim|\)

19. \(((A \rightarrow B) \wedge (C \rightarrow \neg D))_{10} \Rightarrow (\neg (C \wedge B) \rightarrow \neg D))_{10}, (C \rightarrow \neg (D \wedge A))_{10}\)  

   \(\wedge|\)
Lemma 34. \((A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))\)

Proof.

1. \(C_{32} \Rightarrow C_{32}\) Axiom
2. \(A_{31} \Rightarrow A_{31}\) Axiom
3. \((C \rightarrow A)_{21}, C_{32} \Rightarrow A_{31}\) \(\rightarrow\|
4. \(B_{30} \Rightarrow B_{30}\) Axiom
5. \((A \rightarrow B)_{10}, (C \rightarrow A)_{21}, C_{32} \Rightarrow B_{30}\) \(\rightarrow\|
6. \((A \rightarrow B)_{10}, (C \rightarrow A)_{21} \Rightarrow (C \rightarrow B)_{20}\) \(\rightarrow\), no 3
7. \((A \rightarrow B)_{10} \Rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))_{10}\) \(\rightarrow\), no 2
8. \(\Rightarrow ((A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B)))_{00}\) \(\rightarrow\), no 1

Lemma 35. \((B \rightarrow (C \rightarrow A)) \rightarrow (\neg (B \rightarrow \neg C) \rightarrow A)\)

Proof.

1. \(B_{31} \Rightarrow B_{31}\) Axiom
2. \(C_{23} \Rightarrow C_{23}\) Axiom
3. \(A_{20} \Rightarrow A_{20}\) Axiom
4. \((C \rightarrow A)_{30}, C_{23} \Rightarrow A_{20}\) \(\rightarrow\|
5. \((B \rightarrow (C \rightarrow A))_{10}, B_{31}, C_{23} \Rightarrow A_{20}\) 1, 4, \(\rightarrow\|
6. \((B \rightarrow (C \rightarrow A))_{10}, B_{31} \Rightarrow A_{20}, (\neg C)_{32}\) \(\sim\)
7. \((B \rightarrow (C \rightarrow A))_{10} \Rightarrow A_{20}, (B \rightarrow \neg C)_{12}\) \(\rightarrow\), no 3
8. \((B \rightarrow (C \rightarrow A))_{10}, (\neg (B \rightarrow \neg C))_{21} \Rightarrow A_{20}\) \(\sim\)
9. \((B \rightarrow (C \rightarrow A))_{10} \Rightarrow ((\neg (B \rightarrow \neg C) \rightarrow A))_{10}\) \(\rightarrow\), no 2
10. \(\Rightarrow ((B \rightarrow (C \rightarrow A)) \rightarrow (\neg (B \rightarrow \neg C) \rightarrow A))_{00}\) \(\rightarrow\), no 1

Lemma 36. \((\neg (A \rightarrow \neg B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))\)

Proof.

1. \(B_{32} \Rightarrow B_{32}\) Axiom
2. \(A_{21} \Rightarrow A_{21}\) Axiom
3. \(B_{32}, (\neg B)_{23} \Rightarrow \sim\|
4. \(A_{21}, B_{32}, (A \rightarrow \neg B)_{13} \Rightarrow \sim\|
5. \(A_{21}, B_{32} \Rightarrow (\neg (A \rightarrow \neg B))_{31}\) \(\sim\|
6. \(C_{30} \Rightarrow C_{30}\) Axiom
7. \((\neg (A \rightarrow \neg B) \rightarrow C)_{10}, A_{21}, B_{32} \Rightarrow C_{30}\) \(\rightarrow\|
8. \((\neg (A \rightarrow \neg B) \rightarrow C)_{10}, A_{21} \Rightarrow (B \rightarrow C)_{20}\) \(\rightarrow\), no 3
9. \((\sim (A \to \sim B) \to C)_{10} \Rightarrow (A \to (B \to C))_{10}\) \(\rightarrow\), no 2
10. \(\Rightarrow ((\sim (A \to \sim B) \to C) \to (A \to (B \to C)))_{00}\) \(\rightarrow\), no 1

\[\square\]

Lemma 37. \((A \to B) \to (\sim (A \to C) \to \sim (B \to C))\)
\[(A \to B) \to ((A \circ D) \to (B \circ D))\]

Proof.
1. \(C_{32} \Rightarrow C_{32}\) Axiom
2. \(B_{30} \Rightarrow B_{30}\) Axiom
3. \(B_{30}, (B \to C)_{02} \Rightarrow C_{32}\) \(\rightarrow\)
4. \(A_{31} \Rightarrow A_{31}\) Axiom
5. \((A \to B)_{10}, (B \to C)_{02}, A_{31} \Rightarrow C_{32}\) \(\rightarrow\)
6. \((A \to B)_{10}, (B \to C)_{02} \Rightarrow (A \to C)_{12}\) \(\rightarrow\), no 3
7. \((A \to B)_{10} \Rightarrow (\sim (B \to C))_{20}, (A \to C)_{12}\) \(\sim\)
8. \((A \to B)_{10}, (\sim (A \to C))_{21} \Rightarrow (\sim (B \to C))_{20}\) \(\sim\)
9. \((A \to B)_{10} \Rightarrow (\sim (A \to C) \to \sim (B \to C))_{10}\) \(\rightarrow\), no 2
10. \(\Rightarrow ((A \to B) \to (\sim (A \to C) \to \sim (B \to C)))_{00}\) \(\rightarrow\), no 1

\[\square\]

Lemma 38. \((A \circ B) \circ C \to A \circ (B \circ C)\)

Proof.
1. \((B \to \sim C)_{31} \Rightarrow (B \to \sim C)_{31}\) Axiom
2. \(\Rightarrow (\sim (B \to \sim C))_{13}, (B \to \sim C)_{31}\) \(\sim\)
3. \((\sim (B \to \sim C))_{31} \Rightarrow (B \to \sim C)_{31}\) \(\sim\)
4. \(A_{30} \Rightarrow A_{30}\) Axiom
5. \((A \to \sim (B \to \sim C))_{01}, A_{30} \Rightarrow (B \to \sim C)_{31}\) \(\rightarrow\)
6. \(B_{23} \Rightarrow B_{23}\) Axiom
7. \(C_{12} \Rightarrow C_{12}\) Axiom
8. \(C_{12}, (\sim C)_{21} \Rightarrow\)
9. \((B \to \sim C)_{31}, B_{23}, C_{12} \Rightarrow\) 6, 8, \(\rightarrow\)
10. \((A \to \sim (B \to \sim C))_{01}, A_{30}, B_{23}, C_{12} \Rightarrow\) 5, 9, Cut
11. \((A \to \sim (B \to \sim C))_{01}, C_{12}, A_{30} \Rightarrow (\sim B)_{32}\) \(\sim\)
12. \((A \to \sim (B \to \sim C))_{01}, C_{12} \Rightarrow (A \to \sim B)_{02}\) \(\rightarrow\), no 3
13. \((A \to \sim (B \to \sim C))_{01}, (\sim (A \to \sim B))_{20} \Rightarrow (\sim C)_{21}\) \(\sim\)
14. \((A \to \sim (B \to \sim C))_{01} \Rightarrow (\sim (A \to \sim B) \to \sim (C)_{01}\) \(\rightarrow\), no 2
15. \((A \to \sim (B \to \sim C))_{01}, (\sim (A \to \sim B) \to \sim (C))_{10} \Rightarrow\)
16. \((\sim (A \to \sim B) \to \sim C))_{10} \Rightarrow (\sim (A \to \sim (B \to \sim C)))_{10}\) \(\sim\)
17. \(\Rightarrow ((A \to \sim B) \to \sim C) \to (A \to \sim (B \to \sim C)))_{00}\) \(\rightarrow\), no 1
18. \(\Rightarrow (A \circ B \to \sim C) \to (A \to \sim (B \circ C)))_{00}\) \(\text{def} \circ\)
19. \( \Rightarrow ((A \circ B) \circ C \rightarrow A \circ (B \circ C)) \) \( \text{def } \circ \)

21. **Deductive rules of** \( \mathcal{L}_4 \)

The next 13 lemmas establish the derived rules of inference for Tarski’s relevance listed Table 3.

**Lemma 39** (adjunction). If \( A, B \in \mathcal{L}_4 \) then \( A \land B \in \mathcal{L}_4 \).

**Proof.** If \( A \) and \( B \) have 4-proofs we may concatenate them and add one more sequent to get a 4-proof of \( A \land B \), as follows.

1. \( \Rightarrow A_0 \) by some 4-proof
2. \( \Rightarrow B_0 \) by some 4-proof
3. \( \Rightarrow (A \land B)_0 \) \( | \land \)

**Lemma 40** (modus ponens). If \( A \rightarrow B \in \mathcal{L}_4 \) and \( A \in \mathcal{L}_4 \) then \( B \in \mathcal{L}_4 \).

**Proof.** Assume \( A \) and \( A \rightarrow B \) have 4-proofs. Concatenate a 4-proof of \( A \) with a 4-proof of \( A \rightarrow B \) and continue the sequence as follows, obtaining a 4-proof of \( B \), showing \( B \in \mathcal{L}_4 \).

1. \( \Rightarrow (A \rightarrow B)_0 \) by some 4-proof
2. \( \Rightarrow A_0 \) by some 4-proof
3. \( B_0 \Rightarrow B_0 \) Axiom
4. \( (A \rightarrow B)_0 \Rightarrow B_0 \) \( \rightarrow | \)
5. \( \Rightarrow B_0 \) 1, 4, Cut

**Lemma 41** (disjunctive syllogism). If \( A \lor B \in \mathcal{L}_4 \) and \( \sim A \in \mathcal{L}_4 \) then \( B \in \mathcal{L}_4 \).

**Proof.** If \( A \lor B \) and \( \sim A \) have 4-proofs, they may be continued to obtain a 4-proof of \( B \).

1. \( \Rightarrow (A \lor B)_0 \) by a 4-proof
2. \( \Rightarrow (\sim A)_0 \) by a 4-proof
3. \( A_0 \Rightarrow A_0 \) Axiom
4. \( B_0 \Rightarrow B_0 \) Axiom
5. \( (A \lor B)_0 \Rightarrow A_0, B_0 \) \( \lor | \)
6. \( \Rightarrow A_0, B_0 \) 1, 5, Cut
7. \( (\sim A)_0 \Rightarrow B_0 \) \( \sim | \)
8. \( \Rightarrow B_0 \) 2, 7, Cut

**Lemma 42** (transitivity). If \( A \rightarrow B \in \mathcal{L}_4 \) and \( B \rightarrow C \in \mathcal{L}_4 \) then \( A \rightarrow C \in \mathcal{L}_4 \).
Proof. Assume $A \rightarrow B \in \mathcal{L}_4$ and $B \rightarrow C \in \mathcal{L}_4$. Then the sequents $\Rightarrow (A \rightarrow B)_{00}$ and $\Rightarrow (B \rightarrow C)_{00}$ have 4-proofs that can be concatenated with steps 2–5 inserted between them, followed by sequents 7–12, yielding a 4-proof of $A \rightarrow C$, hence $A \rightarrow C \in \mathcal{L}_4$.

1. $\Rightarrow (A \rightarrow B)_{00}$ by a 4-proof
2. $A_{10} \Rightarrow A_{10}$ Axiom
3. $B_{10} \Rightarrow B_{10}$ Axiom
4. $(A \rightarrow B)_{00}, A_{10} \Rightarrow B_{10}$ $\rightarrow$
5. $A_{10} \Rightarrow B_{10}$ 1, 4, CUT
6. $\Rightarrow (B \rightarrow C)_{00}$ by a 4-proof
7. $B_{10} \Rightarrow B_{10}$ Axiom
8. $C_{10} \Rightarrow C_{10}$ Axiom
9. $(B \rightarrow C)_{00}, B_{10} \Rightarrow C_{10}$ $\rightarrow$
10. $B_{10} \Rightarrow C_{10}$ 6, 9, Cut
11. $A_{10} \Rightarrow C_{10}$ 5, 10, Cut
12. $\Rightarrow (A \rightarrow C)_{00}$ $\rightarrow$, no 1

\[ \square \]

Lemma 43 (contraposition). If $A \rightarrow B \in \mathcal{L}_4$ then $\sim B \rightarrow \sim A \in \mathcal{L}_4$.

Proof. Assume $A \rightarrow B \in \mathcal{L}_4$. By interchanging 0 and 1 throughout any 4-proof of $\Rightarrow (A \rightarrow B)_{00}$, we obtain a 4-proof of $\Rightarrow (A \rightarrow B)_{11}$, which may be continued as follows to obtain a 4-proof of $\sim B \rightarrow \sim A$.

1. $\Rightarrow (A \rightarrow B)_{11}$ by a (01)-permuted 4-proof
2. $A_{01} \Rightarrow A_{01}$ Axiom
3. $B_{01} \Rightarrow B_{01}$ Axiom
4. $A_{01}, (A \rightarrow B)_{11} \Rightarrow B_{01}$ $\rightarrow$
5. $A_{01} \Rightarrow B_{01}$ 1, 4, Cut
6. $\Rightarrow B_{01}, (\sim A)_{10}$ $\sim$
7. $(\sim B)_{10} \Rightarrow (\sim A)_{10}$ $\sim$
8. $\Rightarrow (\sim B \rightarrow \sim A)_{00}$ $\rightarrow$, no 1

\[ \square \]

Lemma 44 (contraposition.2). If $A \rightarrow \sim B \in \mathcal{L}_4$ then $B \rightarrow \sim A \in \mathcal{L}_4$.

Proof. Assume $A \rightarrow \sim B$ has a 4-proof. Obtain a 4-proof of $\Rightarrow (A \rightarrow \sim B)_{11}$ by interchanging 0 and 1 in a 4-proof of $\Rightarrow (A \rightarrow \sim B)_{00}$. Continue this 4-proof as follows to obtain a 4-proof of $B \rightarrow \sim A$.

1. $\Rightarrow (A \rightarrow \sim B)_{11}$ by a (01)-permuted 4-proof
2. $A_{01} \Rightarrow A_{01}$ Axiom
3. $B_{10} \Rightarrow B_{10}$ Axiom
4. $(\sim B)_{01}, B_{10} \Rightarrow \sim$
5. $(A \rightarrow \sim B)_{11}, B_{10}, A_{01} \Rightarrow 2, 4, \rightarrow$
Lemma 45 (cut). If $A \land B \rightarrow C \in \mathcal{L}_4$ and $B \rightarrow C \lor A \in \mathcal{L}_4$ then $B \rightarrow C \in \mathcal{L}_4$.

Proof. The Cut Rule in relevance logic is a derived rule in Basic Logic, called DR2 [50, p. 291]. To prove this simplified version of DR2, construct a 4-proof of $B \rightarrow C$ from 4-proofs of $A \land B \rightarrow C$ and $B \rightarrow C \lor A$ as follows. It is interesting that Cut for sequents is used five times.

1. $B_{10} \Rightarrow B_{10}$ Axiom
2. $(C \lor A)_{10} \Rightarrow (C \lor A)_{10}$ Axiom
3. $(B \rightarrow C \lor A)_{00}, B_{10} \Rightarrow (C \lor A)_{10}$ →|
4. $\Rightarrow (B \rightarrow C \lor A)_{00}$ by a 4-proof
5. $B_{10} \Rightarrow (C \lor A)_{10}$ Cut
6. $C_{10} \Rightarrow C_{10}$ Axiom
7. $A_{10} \Rightarrow A_{10}$ Axiom
8. $(C \lor A)_{10} \Rightarrow C_{10}, A_{10}$ ∨|
9. $B_{10} \Rightarrow C_{10}, A_{10}$ 5, 8, Cut
10. $(A \land B)_{10} \Rightarrow (A \land B)_{10}$ Axiom
11. $C_{10} \Rightarrow C_{10}$ Axiom
12. $(A \land B \rightarrow C)_{00}, (A \land B)_{10} \Rightarrow C_{10}$ →|
13. $\Rightarrow (A \land B \rightarrow C)_{00}$ by a 4-proof
14. $(A \land B)_{10} \Rightarrow C_{10}$ Cut
15. $A_{10} \Rightarrow A_{10}$ Axiom
16. $B_{10} \Rightarrow B_{10}$ Axiom
17. $A_{10}, B_{10} \Rightarrow (A \land B)_{10}$ |∧
18. $A_{10}, B_{10} \Rightarrow C_{10}$ 14, 17, Cut
19. $B_{10} \Rightarrow C_{10}$ 9, 18, Cut
20. $\Rightarrow (B \rightarrow C)_{00}$ |→, no 1

□

Lemma 46 (E-rule, BR1, R5). If $A \in \mathcal{L}_4$ then $(A \rightarrow B) \rightarrow B \in \mathcal{L}_4$.

Proof. The E-rule [4, p. 8] is also called BR1 [50, p. 289] and R5 [4, p. 193]. If $A$ has a 4-proof, then we obtain a 4-proof of $(A \rightarrow B) \rightarrow B$ by appending sequents to a 4-proof of $\Rightarrow A_{11}$, as follows.

1. $\Rightarrow A_{11}$ by a (01)-permuted 4-proof
2. $B_{10} \Rightarrow B_{10}$ Axiom
3. $(A \rightarrow B)_{10} \Rightarrow B_{10}$ →|
4. $\Rightarrow ((A \rightarrow B) \rightarrow B)_{00}$ |→, no 1

□
Lemma 47 (suffixing). If \( A \to B \in \mathcal{L}_4 \) then \((B \to C) \to (A \to C) \in \mathcal{L}_4 \).

Proof. Assume \( A \to B \) is 4-provable. Interchange 0 and 1 throughout a 4-proof of \( \Rightarrow (A \to B)_{00} \), obtaining a 4-proof of \( \Rightarrow (A \to B)_{11} \), and continue it as follows to obtain a 4-proof of \( (B \to C) \to (A \to C) \).

1. \( \Rightarrow (A \to B)_{11} \) by a \((01)\)-permuted 4-proof
2. \( A_{21} \Rightarrow A_{21} \) Axiom
3. \( B_{21} \Rightarrow B_{21} \) Axiom
4. \( (A \to B)_{11}, A_{21} \Rightarrow B_{21} \) \( \to\) |
5. \( C_{20} \Rightarrow C_{20} \) Axiom
6. \( (B \to C)_{10}, B_{21} \Rightarrow C_{20} \) 3, 5, \( \to\) |
7. \( A_{21} \Rightarrow B_{21} \) 1, 4, Cut
8. \( (B \to C)_{10}, A_{21} \Rightarrow C_{20} \) Cut
9. \( (B \to C)_{10} \Rightarrow (A \to C)_{10} \) \( \to\), no 2
10. \( \Rightarrow ((B \to C) \to (A \to C))_{00} \) \( \to\), no 1

Lemma 48 (cycling). If \( A \to (B \to C) \in \mathcal{L}_4 \) then \( B \to (\sim C \to \sim A) \in \mathcal{L}_4 \)

Proof. If \( A \to (B \to C) \) is 4-provable, then there is a 4-proof of \( \Rightarrow (A \to (B \to C))_{22} \), which may be incorporated into a 4-proof of \( B \to (\sim C \to \sim A) \) as follows.

1. \( A_{02} \Rightarrow A_{02} \) Axiom
2. \( (B \to C)_{02} \Rightarrow (B \to C)_{02} \) Axiom
3. \( (A \to (B \to C))_{22}, A_{02} \Rightarrow (B \to C)_{02} \) \( \to\) |
4. \( \Rightarrow (A \to (B \to C))_{22} \) by a \((02)\)-permuted 4-proof
5. \( A_{02} \Rightarrow (B \to C)_{02} \) Cut
6. \( B_{10} \Rightarrow B_{10} \) Axiom
7. \( C_{12} \Rightarrow C_{12} \) Axiom
8. \( (B \to C)_{02}, B_{10} \Rightarrow C_{12} \) \( \to\) |
9. \( A_{02}, B_{10} \Rightarrow C_{12} \) 5, 8, Cut
10. \( B_{10} \Rightarrow C_{12}, (\sim A)_{20} \) \( \sim\) |
11. \( B_{10}, (\sim C)_{21} \Rightarrow (\sim A)_{20} \) \( \sim\) |
12. \( B_{10} \Rightarrow (\sim C \to \sim A)_{10} \) \( \to\), no 2
13. \( \Rightarrow (B \to (\sim C \to \sim A))_{00} \) \( \to\), no 1

Lemma 49 (prefixing rule). If \( A \to B \in \mathcal{L}_4 \) then \( (C \to A) \to (C \to B) \in \mathcal{L}_4 \).

Proof. By Lemma 46 (\( A \to B \) \( \Rightarrow ((C \to A) \to (C \to B)) \in \mathcal{L}_4 \), so if \( A \to B \in \mathcal{L}_4 \) then \( ((C \to A) \to (C \to B)) \in \mathcal{L}_4 \) by Lemma 46. □

Lemma 50 (affixing). If \( A \to B \in \mathcal{L}_4 \) and \( C \to D \in \mathcal{L}_4 \) then \( (B \to C) \to (A \to D) \in \mathcal{L}_4 \).
Proof. By $C \rightarrow D \in L_4$ and Lemma 49,

$$(A \rightarrow C) \rightarrow (A \rightarrow D) \in L_4.$$  

by $A \rightarrow B \in L_4$ and Lemma 47,

$$(B \rightarrow C) \rightarrow (A \rightarrow C) \in L_4.$$  

Hence, by Lemma 42,

$$(B \rightarrow C) \rightarrow (A \rightarrow D) \in L_4.$$  

□

**Lemma 51** (monotonic fusion). If $A \rightarrow B \in L_4$ and $C \rightarrow D \in L_4$ then

$$(A \circ C) \rightarrow (B \circ D) = \sim(A \rightarrow \sim C) \rightarrow \sim(B \rightarrow \sim D) \in L_4,$$

Proof.

1. $A \rightarrow B \in L_4$ Assumption
2. $C \rightarrow D \in L_4$ Assumption
3. $\sim D \rightarrow \sim C \in L_4$ Lemma 44
4. $(B \rightarrow \sim D) \rightarrow (A \rightarrow \sim D) \in L_4$ 1, Lemma 47
5. $(A \rightarrow \sim D) \rightarrow (A \rightarrow \sim C) \in L_4$ 3, Lemma 49
6. $((B \rightarrow \sim D) \rightarrow (A \rightarrow \sim D)) \rightarrow ((B \rightarrow \sim D) \rightarrow (A \rightarrow \sim C)) \in L_4$ Lemma 49
7. $(B \rightarrow \sim D) \rightarrow (A \rightarrow \sim C) \in L_4$ 4, 6, Lemma 10
8. $\sim(A \rightarrow \sim C) \rightarrow \sim(B \rightarrow \sim D) \in L_4$ Lemma 44
9. $(A \circ C) \rightarrow (B \circ D) \in L_4$ definition  

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