The Relativistically Spinning Charged Sphere

D. Lynden-Bell
The Institute of Astronomy, The Observatories, Madingley Road, Cambridge, CB3 0HA, UK
& Clare College, Cambridge
(Dated: February 15, 2022)

When the equatorial spin velocity, \( v \), of a charged conducting sphere approaches \( c \), the Lorentz force causes a remarkable rearrangement of the total charge \( q \).

Charge of that sign is confined to a narrow equatorial belt at latitudes \( b \approx \sqrt{3} (1 - v^2/c^2)^{1/2} \) while charge of the opposite sign occupies most of the sphere’s surface. The change in field structure is shown to be a growing contribution of the ‘magic’ electromagnetic field of the charged Kerr-Newman black hole with Newton’s \( G \) set to zero. The total charge within the narrow equatorial belt grows as \((1 - v^2/c^2)^{-3/4}\) and tends to infinity as \( v \) approaches \( c \). The electromagnetic field, Poynting vector, field angular momentum and field energy are calculated for these configurations.

Gyromagnetic ratio, g-factor and electromagnetic mass are illustrated in terms of a 19th Century electron model. Classical models with no spin had the small classical electron radius \( e^2/mc^2 \sim \) a hundredth of the Compton wavelength, but models with spin take that larger size but are so relativistically concentrated to the equator that most of their mass is electromagnetic.

The method of images at inverse points of the sphere is shown to extend to charges at points with imaginary co-ordinates.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

When a charged conductor rotates steadily, any small resistivity in the conductor will cause dissipation, unless the charge rotates with the conductor. Thus, in the steady state the surface current is due to the rotation of the surface charge. For a sphere rotating with equatorial speed \( v \) much less than \( c \), the charge is almost uniformly distributed over the surface and the resulting magnetic field is almost uniform within the sphere and externally dipolar. If the charge were frozen to be uniform over the surface and the sphere were an insulator then those would be the forms of the fields, however fast the sphere rotated. (Appendix 1).

By contrast, when the sphere has a conducting surface the Lorentz force pushes the charge along the surface towards the equator, but the effect is only pronounced at relativistic speeds. Beyond \( v = 0.93c \) the effect is so strong that the charge density at the poles is of the opposite sign to the imposed total charge \( q \). As \( v \) gets still nearer to \( c \) more and more of the sphere is covered by such reversed charge, while the charge of the same sign as \( q \) is confined to an ever narrowing belt around the equator. However this belt now contains a charge considerable greater than \( q \) since the excess is needed to cancel the reversed charge elsewhere. Indeed we show that as \( v \) approaches \( c \) the total charge in the belt becomes \( \frac{2}{\sqrt{3}} (1 - v^2/c^2)^{-3/4} \) which tends to infinity. The boundaries of the equatorial belt are at latitudes \( |b| \approx \sqrt{3} \sqrt{1 - v^2/c^2} \) so the width of the belt tends to zero.

Defining \( u \) by \((1 - u)/(1 + u) = \sqrt{1 - v^2/c^2}\) we find that as \( v \) increases, the external field is changed by the addition of the electromagnetic field of a charge at the imaginary point \((0, 0, ia\sqrt{u})\). This is the Magic electromagnetic field discussed earlier (Newman, 1973, 2002; Lynden-Bell, 2003) which may also be found as the field left by a charged Kerr-Newman black hole (Newman et al., 1965) when \( G \) is set equal to zero. In our context it is being generated quite naturally as \( v \to c \). The corresponding internal field is that due to a charge at the complex image point under inversion in the sphere \((0, 0, -ia/\sqrt{u})\). Wave equations for particle motion in the Magic field have remarkable separability (Carter, 1968b; Chandrasekhar, 1976; Page, 1976; Lynden-Bell, 2000). All the Kerr-Newman metrics have the same gyromagnetic ratio as the Dirac electron (Carter, 1968a) and so do all the conformastationary metrics and others remarked upon by Pfister and King (2002).

There are several points of principle that arise in this problem which we raise but do not discuss in detail.

1. If a real physical sphere with a conducting surface is rotated relativistically, what shape would it become? One might suppose that it ought to be a sphere in the rotating axes, but the metric in such axes is anisotropic, so it is not clear what shape should be called spherical in such axes. We sidestep all such discussion by taking the surface to be spherical in fixed axes however fast the body is rotated. By this we eliminate all discussion of Fitzgerald contraction, the physical constitution of the sphere and its distortion under centrifugal and Lorentz forces. While this may reduce the physical reality of the problem, it gives a well defined uncomplicated problem.

2. For a slowly rotating conductor the charge resides on the surface so it does not matter whether the...
whole layer of the sphere is a conductor or only a
surface layer. However, once the magnetic part
of the Lorentz force is significant, the condition
that the Lorentz force density must vanish inside
the sphere leads to a requirement for internal charges
and the currents due to their rotation. Thus in
the relativistic régime the conducting solid sphere
gives a problem that differs from the conducting
spherical shell problem solved here.

3. We do not assume that our conductor has zero resis-
tivity, indeed we rely on some resistivity to ensure
that in the steady state the surface charge rotates
with the body. In particular, we do not assume
that either the surface or the body of the sphere is
a superconductor, so questions of the expulsion of
magnetic flux by the Meissner effect do not occur
in our problem.

4. A number of very interesting problems arise if it
is assumed that the charge carriers in the conduc-
tor have mass, because at relativistic speeds the
inertia of the charge carriers will itself cause them
to concentrate toward the equator. However the
Maxwell equations themsevles already account for
that part of the inertia contained in the electro-
magnetic field. When an electron is accelerated
the electromagnetic field is accelerated with it, so
at least part of its mass is due to the inertia of its
field energy. To put in the full mass of the electron
as well as the Maxwell stresses of the field distri-
bution would involve some double counting, not to
mention the larger effect of inertia on the heavy
nuclei that form the conductor’s lattice. We shall
again sidestep these problems by assuming that the
Lorentz force component along the conducting sur-
face is zero, so that there is no effect of inertial
forces on the charge carriers (other than those on
their electromagnetic fields as implied by Maxwell’s
equations).

II. THE MAXWELL EQUATIONS FOR THE
PROBLEM

We call the radius of the sphere a and rotate it with
angular velocity Ω so Ωa = v. We shall find it convenient
to use ω = Ωa/c = v/c. In the steady state, the Maxwell
equations reduce to Curl E = 0, div E = 4πρ, div B = 0
and Curl B = 4πJ. On the sphere’s surface there is a
surface charge σ(θ) and a sheet current Jφ = σΩac−1 sin θ
but away from that surface there are neither charges nor
currents so we may write E = −∇Φ and B = −∇χ where
∇²Φ = 0 = ∇²χ. The z → −z symmetry shows E even
and B odd.

The solutions to Laplace’s equation obeying the
boundary conditions at r = 0 and ∞ are with μ = cos θ

\[ \Phi = \frac{q}{a} \left\{ \sum_{n} \left( \frac{a}{r} \right)^{2n+1} \Phi_{2n} P_{2n}(\mu) \right\} r \geq a \]

and

\[ \chi = \frac{q}{a} \left\{ \sum_{n} \frac{1}{2n+1} \left( \frac{a}{r} \right)^{2n+2} \chi_{2n+1} P_{2n+1}(\mu) \right\} r > a \]

Here q is the total charge on the sphere and the co-
efficients have been arranged so that Φ and ∂χ/∂r are
both continuous across r = a. This corresponds to the
requirements that the surface component of E and the
normal component of B must be continuous.

From the discontinuity in \( E_r \) we have the expression
for the surface density of charge

\[ 4\pi \sigma = qa^{-2} \sum_{n=0}^{\infty} (4n+1) \Phi_{2n} P_{2n}(\mu) \ , \ (1) \]

so the condition that q is the total charge gives \( \Phi_0 = 1 \)
on integration over the sphere. The discontinuity in \( B_\theta \)
gives the surface current so

\[ 4\pi J_\phi = q \sin \theta a^{-2} \sum_{n=0}^{\infty} \frac{4n+3}{(2n+2)(2n+1)} \chi_{2n+1} P_{2n+1}'(\mu) \ , \ (2) \]

where \( P_{2n+1}' = d/d\mu (P_{2n+1}) \).

The condition that the current is solely due to the ro-
tation of the surface charge combines the above equations
to give

\[ \sum_{n=0}^{\infty} \frac{4n+3}{(2n+2)(2n+1)} \chi_{2n+1} P_{2n+1}' = \omega \sum_{n=0}^{\infty} (4n+1) \Phi_{2n} P_{2n} \ . \ (3) \]

Our final physical requirement is that at equilibrium
there is no surface component of the Lorentz force (be-
cause the surface is a conductor)

\[ E_\theta + \omega \sin \theta B_r = 0 \ , \]

which implies in terms of our potentials

\[ \sum_{n=0}^{\infty} \Phi_{2n} P_{2n}'(\mu) = -\omega \sum_{n=0}^{\infty} \chi_{2n+1} P_{2n+1}(\mu) \ . \ (4) \]

Our problem has now been reduced to the mathematical
case of solving equations [3] and [4] for the coefficients
\( \Phi_{2n} \) and \( \chi_{2n+1} \) when \( \omega \leq 1 \) and is given, and \( \Phi_0 = 1 \).
III. MATHEMATICAL SOLUTION

We need the following mathematical properties of the Legendre Polynomials

\[
\int_{-1}^{+1} P'_{2m+1} P_{2m} \, d\mu = \begin{cases} 2 & m \leq n \\ 0 & m > n \end{cases}
\]

\[
\int_{-1}^{+1} P'_{2m} P_{2m+1} \, d\mu = \begin{cases} 2 & m \leq n - 1 \\ 0 & m > n - 1 \end{cases},
\]

together with their orthogonality relation. Multiplying (4) by \( \frac{1}{2} P_{2m+1} \) and integrating from \(-1\) to \(+1\) we find using the above

\[
\sum_{n=m+1}^{\infty} \Phi_{2n} = -\omega \eta_{2m+1},
\]

where we have defined \( \eta_{2m+1} \) as \( \chi_{2m+1} / (4m + 3) \) because it proves to be a more convenient variable to use. Similarly multiplying (3) by \( \frac{1}{2} P_{2m} \) and integrating we find on writing the result in terms of \( \eta \)

\[
\sum_{n=m}^{\infty} \frac{(4n + 3)^2}{(2n + 2)(2n + 1)} \eta_{2n+1} = \omega \Phi_{2m},
\]

We shall give two different methods of solving (5) and (6), which are complementary in that one gives enlightenment on what is happening in the other.

A. First Method

Subtracting from (6) the same equation with \( m + 1 \) written for \( m \) we obtain

\[
\Phi_{2m+2} = \omega (\eta_{2m+3} - \eta_{2m+1}),
\]

performing the same operation on (6) and dividing by 4

\[
\frac{(m + \frac{3}{2})^2}{(m + 1)(m + \frac{5}{2})} \eta_{2m+1} = -\frac{\omega}{4} (\Phi_{2m+2} - \Phi_{2m}).
\]

Now the coefficient of \( \eta_{2m+1} \) is remarkably close to 1: indeed it is

\[
m^2 + \frac{3}{2} m + \frac{3}{4} = 1 + \frac{1}{16(m + 1)} \left( m + \frac{1}{2} \right) = 1 + \delta_m,
\]

where \( \delta_0 = \frac{1}{8}, \delta_1 = \frac{1}{16}, \delta_2 = \frac{1}{120}, \delta_3 = \frac{1}{24}, \delta_4 = \frac{1}{360} \)

e etc, thus the \( \delta_m \) quite rapidly become small compared to one which suggests that if only 1% accuracy is needed we might neglect the \( \delta_m \) for \( m \geq 2 \). This will form the basis for a rapidly convergent approximation scheme. Using (6) to eliminate the \( \Phi \) from (8) we find for \( m \geq 1 \)

\[
(1 + \delta_m) \eta_{2m+1} = -\frac{\omega^2}{4} (\eta_{2m+3} - 2\eta_{2m+1} + \eta_{2m-1}),
\]

hence we obtain a recurrence relation with almost constant coefficients

\[
\eta_{2m+3} + 2 [\omega^{-2} (1 + \delta_m) - 1] \eta_{2m+1} + \eta_{2m-1} = 0.
\]

Thus \( \eta_5 + 2 [\omega^{-2} (1 + \frac{1}{48}) - 1] \eta_3 + \eta_1 = 0 \). Now, since \( \omega \leq 1 \), \( 2\omega^{-2} - 1 \geq 1 \), so with a less than 2% error in the middle coefficient we can set \( \delta_m = 0 \) for \( m \geq 2 \). (9) is then reduced to a recurrence relationship with constant coefficients.

The solution is

\[
\eta_{2m+1} = C(-u)^m + D(-u)^{-m} \quad m \geq 2,
\]

where \( u \) and \( u^{-1} \) are the roots for \( t \) of

\[
t^2 - 2(2\omega^{-2} - 1)t + 1 = 0,
\]

notice that

\[
\omega^2 = 4u/(1 + u)^2 \quad \text{and} \quad \sqrt{1 - \omega^2} = \sqrt{1 - \frac{u^2}{(1 + u)^2}} = 1 - u.
\]

Evidently the product of the roots is one and the sum of the roots is positive, so both roots are positive with \( u < 1 \), \( u^{-1} > 1 \), for \( \omega < 1 \). However, the expression (11) will diverge oscillatory due to the \( u^{-1} \) term unless \( D = 0 \). Hence,

\[
\eta_{2m+1} = C(-u)^m \quad m \geq 2,
\]

where,

\[
u = 2\omega^{-2} \left[ 1 - \frac{1}{2} \omega^2 - \sqrt{1 - \omega^2} \right].
\]

Notice that

\[
\omega \rightarrow 0, \quad u \rightarrow \frac{1}{4} \omega^2,
\]

but as

\[
\omega \rightarrow 1, \quad u \rightarrow 1 - 2\sqrt{1 - \omega^2}.
\]

In particular

\[
\eta_5 = Cu^2,
\]
and

\[
\sum_{n=m}^{\infty} (1 + \delta_m) \eta_{2n+1} = C \frac{u^2}{1 + u} \Phi_{m+1} .
\]  

We still need to determine the constant \( C \) as well as \( \eta_3, \eta_1 \) and the \( \Phi_{2m} \). Our strategy is to return to the exact equations (6) and (7) for \( m = 0, 1, 2 \) and to combine them with the results (15) and (16) to determine the unknowns \( \Phi_2, \Phi_4, \eta_3, \eta_1 \) and \( C \).

We rewrite (6)

\[
\sum_{n=m}^{\infty} (1 + \delta_m) \eta_{2n+1} = \frac{\omega}{4} \Phi_{2m} ,
\]  

\( m = 0 \) gives

\[
C \frac{u^2}{1 + u} + (1 + \delta_1) \eta_3 + (1 + \delta_0) \eta_1 = \omega/4 ,
\]

(18)

\[ m = 1 \text{ gives } C \frac{u^2}{1 + u} + (1 + \delta_1) \eta_3 = \omega/4 \Phi_2 , \]

(19)

\[ m = 2 \text{ gives } C \frac{u^2}{1 + u} = \omega/4 \Phi_4 , \]

(20)

with \( m = 0 \) (17) gives

\[ \Phi_2 = \omega \left( \eta_3 - \eta_1 \right) , \]

(21)

with \( m = 1 \) (7) gives

\[ \Phi_4 = \omega \left( Cu^2 - \eta_3 \right) . \]

So subtracting (18) from (19) we have

\[ (1 + \delta_0) \eta_1 = \frac{\omega}{4} \left[ 1 - \omega \left( \eta_3 - \eta_1 \right) \right] , \]

(22)

while (19) and (20) become

\[ C \frac{u^2}{1 + u} + \left( 1 + \delta_1 - \frac{\omega^2}{4} \right) \eta_3 + \frac{\omega^2}{4} \eta_1 = 0 , \]

and

\[ C \left( \frac{u^2}{1 + u} - \frac{\omega^2}{4} + u^2 \right) = -\frac{\omega^2}{4} \eta_3 , \]

using (16) to re-express \( \frac{u^2}{1 + u} \) as \( \frac{u}{(1 + u)^2} \), we find

\[ \eta_3 = -Cu \]  

(23)

putting (23) into (22) we find

\[ \eta_1 = \left[ 1 + \delta_1 (1 + u)^2 \right] C , \]

(24)

finally putting expressions (23) and (24) for \( \eta_3 \) and \( \eta_1 \) into (21) we have our equation for the constant \( C \) again using (18)

\[ C = \frac{1}{2} \frac{u^2}{\Delta} , \]

(25)

where

\[ \Delta = 1 + (1 + u) \left[ \delta_0 + \delta_1 (1 + u + u^2) + \delta_2 \delta_1 (1 + u)^2 \right] , \]

(26)

with \( \delta_0 = \frac{1}{8} \) and \( \delta_1 = \frac{1}{18} \) and \( u \) given by (14).

With \( C \) so determined \( \eta_1 \) is given by (24) and \( \eta_3 \) by (23) may be incorporated into the more general formula

\[ \eta_{2m+1} = C (-u)^m \text{ for } m \geq 1 . \]

For \( u < 0.9 \) a somewhat more accurate solution may be shown to be

\[ \eta_{2m+1} = C (-u)^m \left[ 1 + \frac{(1 + u)}{16(1 - u)} \frac{1}{(m + \frac{1}{2})} \right] . \]

Its accuracy is 1.5% at \( u = .8 \) but diminishes to 7% at \( u = .9 \). Beyond that we give a separate treatment in section 3.2.

With all the \( \eta \) known the \( \Phi_{2m} \) are given by (7) with \( \omega \) given by (13). Thus

\[ \Phi_0 = 1 , \]

\[ \Phi_2 = -u \left[ 1 + \delta_1 (1 + u) \right] / \Delta , \]

\[ \Phi_{2m} = (-u)^m / \Delta \text{ for } m \geq 2 . \]

Hence

\[ \Sigma = \sum_{m=1}^{\infty} \Phi^2_{2m} \]

\[ = \frac{u^2}{\Delta^2} \left[ \frac{1}{(1 - u^2)} + 2 \delta_1 (1 + u) + \delta_2 (1 + u)^2 \right] .(27) \]

Those who wish for a much more accurate calculation including the first order perturbation theory for the terms neglected above, will find it in the appendix. However, there is always some tension between simple explanation and a detailed calculation in which the reader may get lost in the mass of formulae, so I have chosen the path of simplicity for the main paper which helps to keep the physics to the fore and to suppress the minutiae of mathematical manipulation. The above approximate solutions
for the potentials are
\[
\Phi = \begin{cases} 
\left( \frac{\alpha}{r} \left(1 - \frac{1}{\Delta} \right) + \frac{\eta}{\Delta} \right) \left( \frac{\alpha}{r^2} \right)^n P_{2n}(\mu) & r \geq a. \\
-\delta_1 u(1 - n + u) a^2 \frac{\partial}{\partial r} P_2(\mu) & r \geq a \\
\frac{\alpha}{a} \left(1 - \frac{1}{\Delta} \right) + \frac{\eta}{\Delta} \sum_{n=0}^{\infty} \left( \frac{\alpha}{r^2} \right)^n P_{2n}(\mu) & r \leq a. 
\end{cases}
\] (28)

Now
\[
(r^2 - 2\mu r + b^2)^{\frac{3}{4}} = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{b}{r} \right)^n P_n(\mu), \quad r \geq |b|.
\] (29)

So if we put \( b = i\alpha \sqrt{u} \), the real part is
\[
\frac{1}{r} \sum_{n=0}^{\infty} (-u)^n \left( \frac{\alpha}{r^2} \right)^n P_{2n}(\mu), \quad r \geq a,
\]
which should be compared to the first of the sums in \( \Phi \).

We see that our potential is that due to charges \( \frac{1}{2} q / \Delta \) at the imaginary points \( z = \pm i\alpha \sqrt{u} \) plus a quadrupolar correction term in \( \delta_1 P_2 \) and a charge \( q \left(1 - \frac{1}{\Delta} \right) \) at the origin. Whereas that construction gives us the field outside the sphere, the field inside the sphere can be thought of as arising from charges at the imaginary points conjugate to them under inversion in the sphere \( r = a \) plus a quadrupolar correction term in \( \delta_1 P_2 \).

The magnetostatic potential \( \chi \) takes the form for \( r \geq a \)
\[
\chi = \frac{q u \alpha}{\Delta} \left\{ \sum_n \frac{2n + \frac{3}{2}}{2n + 2} \left( \frac{\alpha}{r} \right)^{2n+2} (-u)^n + 3 \frac{1}{4} \delta_1 (1 + u)^2 a^2 \frac{\partial}{\partial r} \right\}.
\]

We notice that putting \( b = i\alpha \sqrt{u} \) and taking the imaginary part of (29) yields for \( r \geq a \)
\[
\text{Im} \left( r^2 - 2i\mu \sqrt{u} \ a - a^2 u \right)^{\frac{3}{4}} = \frac{u \alpha}{a} \sum (-u)^n \left( \frac{\alpha}{r^2} \right)^{2n+2},
\]
also
\[
\text{Im} \int_0^a \left( r^2 - 2i\mu \sqrt{u} \ ar - a^2 u \right)^{\frac{3}{4}} a^{-1} da = \frac{u \alpha}{a} \sum \frac{(-u)^n}{2n + 1} \left( \frac{\alpha}{r^2} \right)^{2n+2}.
\]
Hence at least for \( r \geq a \), \( \chi \) is of the form
\[
\chi = \frac{q \alpha}{\Delta} \text{Im} \left[ \left( r^2 - 2i\mu \sqrt{u} \ ar - a^2 u \right)^{\frac{3}{4}} - \frac{1}{4} \int_0^a \left( r^2 - 2i\mu \sqrt{u} \ ar - a^2 u \right)^{\frac{3}{4}} \frac{da}{a} \right] + 3 \delta_1 (1 + u)^2 \frac{q \alpha}{4 \Delta} \frac{u \alpha}{r^2} \mu.
\] (30)

within the square bracket the first term is clearly the magnetic field due to the charge \( \frac{q}{\Delta} \) at the imaginary point \( i\sqrt{u} \ a\hat{z} \) with the convention that \( \mathbf{E} + \mathbf{v} \mathbf{B} = -\nabla \left[ \Phi + i \chi \right] \) this fits nicely with our result for the potential. The second term in the square bracket may be interpreted at the magnetic potential of the charge distribution \( \frac{q}{\Delta} \frac{da}{a} \) at \( i\sqrt{u} \ a\hat{z} \) between \( a' = 0 \) and \( a' = a \). However, there is no corresponding part in the electrical potential.

**B. The Limit \( v = c \)**

When \( v/c \) is within half of a percent of 1 (ie \( \gamma > 20 \)) even the amazingly good perturbation theory of the Appendix which allows linearly for all the terms we have neglected, breaks down; it generates terms in \( \frac{1}{u} \ln(1 - u) \) which can no longer be treated as perturbations; we therefore need a new approach for that very small region.

Our exact difference equation for \( \eta_{2n+1} \)
\[
\eta_{2n+3} + \left[ u + \frac{1}{u} + \frac{(u + 1)^2}{16u} \right] \eta_{2n+1} + \eta_{2n-1} = 0,
\]
at \( u = 1 \) the solution when the \( \frac{1}{u} \) term is neglected is \( \eta_{2n+1} = C(-1)^n \). To solve the equation without any such neglect, we write \( \eta_{2n+1} = (-1)^n v_n \) and obtain
\[
\left[ (u - 1)^2 + \frac{(u + 1)^2}{16u} \right] v_n = 0.
\] (31)

Let us look for solutions in which \( v_n \) varies sufficiently slowly with \( n \) that we may approximate \( v_n \) as a continuous function \( v(n) \). Then \( (v_{n+1} - v_n) - (v_n - v_{n-1}) \)
\[
d^2 v/dn^2 = \left[ \frac{(u - 1)^2}{u} + \frac{(u + 1)^2}{16u} \right] \frac{1}{(n + \frac{3}{4})} v(n).
\] (32)

The solution that converges at large \( n \) is a Bessel function of imaginary argument. Setting \( N = n + \frac{3}{4} \)
\[
v(n) = C \sqrt{\frac{2}{\pi}} \left( \frac{1 - u}{u^2} N \right)^{\frac{1}{4}} K_{\nu} \left( \frac{1 - u}{u^2} N \right),
\]
where
\[
\nu^2 = \frac{1}{4} \left( 1 + \frac{1}{4} \left( \frac{u + 1}{u} \right)^2 \right)
\]
\[
v(n) \rightarrow C \pi^{\nu - \frac{1}{2}} \Gamma(\nu) \left( \frac{1 - u}{2u^2} N \right)^{\nu - \frac{1}{2}}.
\]
for small \( \frac{1}{\sqrt[3]{3}} N \) and to \( C \exp \left( -\frac{1}{\sqrt[3]{3}} N \right) \) for large.

In the relevant régime with \( u \sim 1 \) then \( \nu = \frac{1}{\sqrt[3]{3}} \).

It is more helpful to derive the correct form for \( n \) large, as we shall do presently. However, before that we should assure ourselves that our technique works by applying it to the equation in which the \((n + \frac{1}{3})^{-2}\) terms are absent. Then the exact convergent solution is \( v_n = C u^n \) and near \( u = 1 \) this may be re-expressed as

\[
v_n = C \exp \{ n \ln [1 - (1 - u)] \} \approx C \exp [-n(1 - u)] + 0(1 - u)^2.
\]

Equation (32) gives a solution

\[
v(n) = C \exp \left[ -n(1 - u) u^{-\frac{1}{2}} \right],
\]

which is the same to order \((1 - u)^2\). Thus our approximation only works in the neighbourhood of \( u = 1 \) but that is where we need it. When \( u = 1 \) equation (32) has the exact solution \( v(n) = C_1 (n + \frac{1}{3})^{-\alpha} \) where \( \alpha = 1 / (2\sqrt{2} + 2) \approx 0.207107 \) so this must be the asymptotic form of the solution of the difference equation for \( v_n \) when \( n \) is large.

Writing \( N = n + \frac{1}{3} \) the exact equation at \( u = 1 \) is

\[
v_{n+1} + v_{n-1} = \left[ 2 + \frac{1}{4(N^2 - \frac{1}{16})} \right] v_n,
\]

\[\text{to be exact away from } u = 1 \text{ an extra term}\]

\[
\left[ \frac{(u-1)^2}{n} \left( 1 + \frac{1}{16(N^2 + \frac{1}{16})} \right) v_n \text{ is needed on the right} \right],
\]

We look for solutions of the form

\[
v_n = C_1 N^{-\alpha} (1 + AN^{-2} + BN^{-4})
\]

Terms of order \( N^{-2-\alpha} \) in the recurrence relation give us the value of \( \alpha \) found above and the terms of order \( N^{-4-\alpha} \) can be used to give us an equation for \( A \). This is

\[
A = -\frac{1}{96} \left[ 2 - \frac{1}{2(2\alpha + 3)} \right] = -0.0193078.
\]

We chose the form with only even powers, as then the odd powers cancel. The next terms are of order \( N^{-6} \) times the leading terms. These give

\[
(4\alpha + 10) B = -\frac{1}{28} \left( 130\alpha + 134 \frac{\pi}{120} \right) \left( \frac{1}{120} + A \right) + \frac{1}{2048} + \frac{4}{120}.
\]

\[
\therefore \quad B = 0.0068443.
\]

We have checked by computer that the asymptotic formula with these values of \( A \) and \( B \) works down to \( n = 0 \) with an error with less than one part in 1000, so

\[
\eta_{2n+1} = C_1 (1-\alpha) N^{-\alpha} \left[ 1 + AN^{-2} + BN^{-4} \right]
\]

where \( N = n + \frac{1}{3} \) is the accurate convergent solution with the above values of \( \alpha, A, B \). The constant \( C_1 \) is determined via equation (6) with \( m = 0 \), \( \Phi_0 = 1 \), \( \omega = 1 \)

\[
C_1 = \frac{1}{4} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n N^{-\alpha}}{1 + \frac{1}{16(N^2 + \frac{1}{16})}} \left[ 1 + AN^{-2} + BN^{-4} \right] \right\} - \frac{1}{4} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n N^{-\alpha}}{1 + \frac{1}{16(N^2 + \frac{1}{16})}} \right\} = 0.699 \approx 0.358.
\]

The convergence of the sum is slow \( 2 \times 10^6 \) terms gave \( 0.6926 \) but comparison with \( 2 \times (10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8) \) terms show the increments decrease in geometrical progression by a factor 1.608, so extrapolation is accurate.

In this régime we can now calculate \( \eta_1 = 0.375 = \frac{3}{8} \).

IV. EXPLICIT EXPRESSIONS FOR POTENTIALS, CHARGES AND CURRENTS

Using (29) we write expression (28) for \( \Phi \) in the form

\[
\Phi = \left\{ \begin{array}{ll}
\frac{\pi}{4} \left[ 1 - \frac{\mu}{\Delta} \left( 1 + \frac{\mu^2 (1+u) u P_2(\mu)}{48} \right) \right] & \quad r \geq a \\
+q \frac{\mu}{\Delta} Re \left( r^2 - 2iu \frac{a r - \mu - u a^2}{\mu - u a} \right)^{-\frac{1}{2}} & \quad r \leq a
\end{array} \right.
\]

(35)

Notice that the points \((0, 0, iu a)\) and \((0, 0, -iu a)\) are inverse points for the sphere.

The surface density of charge is given in terms of the discontinuity in \( \partial \Phi / \partial r \) at \( r = a \).

\[
4\pi \sigma = qa^{-2} \left\{ \frac{1}{\Delta} \left( 1 + \frac{5}{48} u (1 + u) P_2 \right) + \frac{1}{\Delta} (1+u) Re \left( 1 + 2iu \frac{\mu - u}{\mu} \right)^{-\frac{1}{2}} \right\}.
\]

Putting \( \mu = 0 \) we obtain the values on the equator

\[
\sigma_{\text{equ}} = \frac{q}{4\pi a^2} \left[ \frac{1}{\Delta} \left( 1 - \frac{5}{96} u (1 + u) \right) + \frac{1 + u}{\Delta (1 - u) \frac{1}{2}} \right].
\]

(36)
this clearly diverges as \( u \) approaches 1 because \( \Delta \) remains finite. Similarly putting \( \mu = 1 \) and evaluating \( (1 + iu^{\frac{1}{3}})^{-3} \)

\[
\sigma_{\text{pole}} = \frac{q}{4\pi a^2} \left[ 1 - \frac{1}{\Delta} \left( 1 + \frac{5}{48} (1 + u) \right) + \frac{1 - 3u}{\Delta(1 + u)^2} \right].
\]  

(37)

This density changes sign as \( u \) increases. Taking account of the variation of \( \Delta \) given in (26), we find \( \sigma_{\text{pole}} = 0 \) at \( u = 0.454, \gamma = 2.66, \omega = 0.93 \). The surface current is solely due to the advection of the surface charge so \( J_\phi = \sigma \omega \sin \theta \).

We may get a good idea of the value of \( \theta \) or \( (\mu) \) at which the charge density changes sign as follows; \( \frac{1}{\Delta} \) is not far from 1. In fact \( 1 - \frac{1}{\Delta} = \frac{u}{1 + u} \), where \( \frac{1}{\Delta} \) is solely due to the advection of the surface charge so \( \frac{1}{\Delta} = \frac{u}{1 + u} \), i.e.

\[ f = \frac{1}{8}(1 + u) \left[ 1 + \frac{1}{6} (1 + u + u^2) + \frac{1}{48}(1 + u)^2 \right], \]

so, \( \frac{u}{1 + u} \) varies between \( \frac{1}{3} \) and \( \frac{5}{6} \). The major variation \( \sigma \) with \( \mu \) is caused by

\[ \left( 1 + u \right) Re \left( 1 + 2iu^{\frac{1}{3}} \mu - u \right)^{-\frac{3}{2}}, \]

which becomes \( > 6 \) for \( (1 - u) < \frac{1}{3} \). We write

\[ \left( 1 + 2iu^{\frac{1}{3}} \mu - u \right) = M e^{i\beta}, \]

where

\[ \beta = \tan^{-1} \left( 2u^{\frac{1}{3}} \frac{\mu}{1 - u} \right), \]

and

\[ M^2 = (1 - u)^2 + 4u\mu^2. \]

Evidently \( M^{-\frac{3}{2}} \cos \left( \frac{3}{2}\beta \right) \) will become zero when \( \beta = \frac{2\pi}{3} \) i.e. where \( 2u^{\frac{1}{3}} \mu/(1 - u) = \tan \frac{\pi}{3} = \sqrt{3}, \) so this corresponds to \( \mu = \frac{1}{2}\sqrt{3}/(1 - u) \) which becomes small when \( u \) approaches one. Thus the charge of the same sign as \( q \) becomes confined to a belt between latitudes

\[ \pm \sin^{-1} \left( \frac{1}{2} \sqrt{\frac{3}{u}} (1 - u) \right) \to \sqrt{3} \sqrt{1 - \frac{v^2}{c^2}}, \]

and the total width of the belt around the equator \( \to \sqrt{3}a(1 - u). \)

By then most of the sphere's surface is covered with charge of the opposite sign leaving only the equatorial belt with a much enhanced charge of the same sign as \( q \). The charge in that belt is approximately

\[ q \left[ 1 + \frac{1}{2\Delta} \left( \gamma^\frac{1}{2} - \frac{\gamma u}{\gamma_0} \right) U(\gamma - \gamma_0) \right], \]

where \( U \) is one for positive values and zero otherwise and \( \gamma_0 \) is the value of \( \gamma \) at which the charge changes sign at the pole i.e. \( \gamma_0 = 1.454 = 2.66. \)

Although all our calculations thus far have been in terms of the magnetostatic potential \( \chi \) we shall find that several global properties of the electromagnetic field are more nicely expressed in terms of the vector potential \( \mathbf{A} \).

Since \( \mathbf{B} \) lies in meridional planes and is axially symmetrical, we may take \( \mathbf{A} \) in the form \( -\mathbf{\zeta} \nabla \phi \) where \( \phi \) is the azimuthal angle of spherical polar co-ordinates. Then

\[ \mathbf{B} = \text{Curl} \, \mathbf{A} = -\nabla \zeta \times \nabla \phi \]

\[ B_r = -\partial \chi/\partial r = -\frac{1}{r \sin \theta} \partial \zeta/\partial \theta = +r^{-2} \partial \zeta/\partial \mu. \]

Inserting our expansion for \( \chi \) in terms of the \( P_{2n+1} \)

\[ q \sum \left( \frac{a}{r} \right)^{2n+3} \chi_{2n+1} P_{2n+1} = +r^{-2} \frac{\partial \zeta}{\partial \mu}, \]

but the \( P_n \) obey Legendre’s equation

\[ -\frac{1}{(2n + 1)(2n + 2)} \frac{d}{d\mu} \left( 1 - \mu^2 \right) \frac{dP_{2n+1}}{d\mu} = P_{2n+1}, \]

so inserting this into the expression above and using the boundary condition that \( \zeta = 0 \) when \( \mu = +1 \) we find

\[ q \sum \left( \frac{a}{r} \right)^{2n+1} \chi_{2n+1} P_{2n+1} = -\zeta \]

for \( r \gg a \) and a very similar condition for \( r \ll a \).

Now

\[ (1 - \mu^2)dP_n/d\mu = n(P_{n-1} - \mu P_n), \]

and

\[ \mu P_n = [(n + 1)P_{n+1} + nP_{n-1}]/(2n + 1), \]
FIG. 1: The electric lines of force for a relativistically rotating conducting spherical shell with \( u = 0.9 \) i.e. \( \gamma = 19 \), \( v/c = 0.9986 \). Notice the charge concentration at the equator and the diminution of the field strength as the lines cross the sphere again, due to the opposing charge there. In contrast the static shell has a radial field outside and none inside.

so

\[
(1 - \mu^2) dP_n/d\mu = \frac{n(n + 1)}{2n + 1} (P_{n-1} - P_{n+1})
\]

thus

\[
q \sum \left( \frac{a}{r} \right)^{2m+1} \frac{\chi_{2n+1} (P_{2n+2} - P_{2n})}{4n + 3} = \zeta
\]

Notice that \( \chi_{2n+1}/(4n + 3) \) is precisely our \( \eta_{2n+1} \) so

\[
\zeta = -q \left\{ \sum \left( \frac{a}{r} \right)^{2m+1} \eta_{2n+1} (P_{2n-2} - P_{2n+2}) \right\} \quad r \geq a
\]

\[
+ \sum \left( \frac{a}{r} \right)^{2n+1} \eta_{2n+1} (P_{2n-2} - P_{2n+2}) \quad r \leq a
\]

A similar change of potential shows that the electric field can be written as \( \nabla \phi \times \nabla \zeta \) where

\[
\zeta = q \left\{ \mu \left( 1 - \frac{1}{\Delta} \right) + \frac{a^2 u (1 + u)}{\Delta r^2 32} \mu (1 - \mu^2) \right\} + \frac{q \Delta}{2} \text{Re} \left\{ \frac{z - iu 1/2}{(r - i u 1/2)^{1/2}} \right\} \quad r \geq a
\]

The above expression is useful as the contours of constant \( \zeta \) give the electric lines of force. Whereas these are radial for a non-rotating sphere they are more interesting for high rotation speeds and are plotted for \( u = 0.9, \gamma = 19 \) as Figure 1.

V. GLOBAL PROPERTIES OF THE ELECTROMAGNETIC FIELD

The Poynting vector is \( \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} \) and as this flows around the axis it gives an angular momentum density

\[
\frac{1}{4\pi c} \mathbf{r} \times (\mathbf{E} \times \mathbf{B})
\]

Thus the total angular momentum in the electromagnetic field is

\[
\mathbf{L} = \frac{1}{4\pi c} \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d^3 x = \frac{c}{4\pi} \int \mathbf{r} \times [\nabla \Phi \times (\nabla \zeta \times \nabla \phi)] d^3 x,
\]

where we have used axial symmetry and the fact that \( \mathbf{B} \) lies in meridional planes to write the vector potential in the form \( -\zeta \nabla \phi \) where \( \phi \) is the spherical polar coordinate.

Since \( \nabla^2 \Phi = 0 \) we deduce that

\[
\mathbf{L} = \frac{-1}{4\pi c} \int \mathbf{r} \times \nabla \phi \nabla \Phi \nabla \zeta d^3 r = \frac{-\hat{z}}{4\pi c} \int \nabla \Phi \nabla \zeta d^3 r,
\]

where \( \mathbf{R} \times \nabla \phi = \hat{z} \) and \( \mathbf{R} = (x, y, 0) \) radial components of \( \mathbf{r} \times \nabla \phi \) average to zero around a circle about the axis, by symmetry.

Now \( \nabla \Phi \nabla \zeta = \text{div}(\zeta \nabla \Phi) - \zeta \nabla^2 \Phi \) and \( \nabla^2 \Phi \) is zero both within and outside the sphere. Hence the angular momentum in the field within the sphere is

\[
\mathbf{L}_i = \frac{-\hat{z}}{4\pi c} \int \text{div}(\zeta \nabla \Phi) d^3 r = \frac{-\hat{z}}{4\pi c} \int \zeta \nabla \Phi d^2 S,
\]

\[
= \frac{-\hat{z}}{2c} a^2 \int_{-1}^{+1} \zeta \frac{\partial \Phi}{\partial r} \bigg|_{a} a^2 d\mu,
\]

where the gradient \( \partial \Phi/\partial r \) is evaluated on the inside of \( r = a \). Similarly for the angular momentum external to the sphere

\[
\mathbf{L}_o = \frac{\hat{z} a^2}{2c} \int_{-1}^{+1} \zeta \frac{\partial \Phi}{\partial r} \bigg|_{a} d\mu,
\]

so the total angular momentum depends on \( \left( \frac{\partial \Phi}{\partial r} \right)_{a \rightarrow} - \left( \frac{\partial \Phi}{\partial r} \right)_{a \leftarrow} = -4\pi \sigma \) and we find

\[
\mathbf{L} = \mathbf{L}_o + \mathbf{L}_i = -\hat{z} 2\pi a^2 c^{-1} \int_{-1}^{+1} \zeta \sigma d\mu.
\]

Inserting the expressions for \( \zeta \) and \( \sigma \) from the last section

\[
\mathbf{L} = \frac{q^2 \hat{z}}{2c} \int_{-1}^{+1} \sum_{n=0}^{\infty} \eta_{2n+1} (P_{2n} - P_{2n+2}) \times \sum_{m=0}^{\infty} (4m + 1) \Phi_2 P_{2m} d\mu.
\]
Most of the terms vanish due to the orthogonality of the $P_n(\mu)$ and we are left with (defining $\eta_{-1} \equiv 0$)

$$L = \frac{d}{c} \sum_{n=0}^{\infty} \left( (\eta_{2n+1} - \eta_{2n-1}) \Phi_{2n} \right).$$

But by equation (7) this is just (remembering $\Phi_0 = 1$)

$$L = \frac{d}{c} \left( \Sigma + \omega \eta_1 \right) \quad \text{where} \quad \Sigma = \sum_{n=1}^{\infty} \Phi_{2n}^2. \quad (40)$$

These formulae are exact without any approximation. In a rather similar manner one may derive exact formulae for the electrical energy $\epsilon_e = \int (E^2/8\pi) d^3r$ and the total both inside and out gives

$$\epsilon_e = \frac{1}{8\pi} \int_{-1}^{1} \Phi_4 \sigma_2 \pi a^2 d\mu = \frac{d}{2a} \sum_{n=0}^{\infty} \Phi_{2n}^2 = \frac{d^2}{2a} (1 + \Sigma) \quad (41)$$

For the magnetic energy we have

$$\epsilon_m = \frac{1}{8\pi} \int (\text{curl} A)^2 d^3r = \frac{1}{8\pi} \int \text{div}(A \times B) d^3r$$

$$= \frac{1}{8\pi} \left[ \int (A \times B)_0 dS \right.$$

$$- \int (A \times B)_0 dS \left. \right]$$

$$= \frac{1}{8\pi} \int A_\phi [B_0 - B_1]_0 2\pi a^2 d\mu$$

$$= \frac{a^2}{4} \int A_\phi 4\pi J_\phi d\mu = -\frac{a^2}{4} \int a \sin \theta d\mu$$

$$= \frac{-\omega q}{4a} \int_{-1}^{+1} \sum_{m=1}^{\infty} (4m + 1) \Phi_{2m} P_{2m} d\mu$$

$$= \frac{1}{2} \left( \Sigma + \omega \eta_1 \right) = \frac{1}{2} \frac{Lc\omega}{a} \quad (42)$$

Thus

$$\epsilon = \epsilon_e + \epsilon_m = \frac{d^2}{2a} \left[ 1 + \Sigma + \frac{1}{2} \omega \eta_1 \right]. \quad (43)$$

$\Sigma$ is given by (27) and $\eta_1$ by (24) in our first approximation.

**VI. THE DIPOLE AND QUADRUPOLE MOMENTS**

Directly from our expressions for $\Phi$ and $\chi$ as series we may read off the dipole and quadrupole moments of the field. There is no electric dipole and the magnetic dipole is

$$\mu_m = \frac{1}{2} qa \chi_1 = \frac{3}{2} qa \eta_1 \quad (44)$$

similarly there is no magnetic quadrupole moment but the electric quadrupole is

$$Q_e = qa^2 \Phi_2$$

similarly the odd moments are missing from electric field and the even moments are missing from the magnetic field. We shall not detail the $(2n + 1)^{th}$ order moments but we note that as $v \to c$ they only fall as $(n + 3/4)^{-\alpha}$ with $\alpha$ close to 0.2. This slow fall leads to a divergent total field energy as $v \to c$. This may be seen by comparing the series in the sum $\Sigma$ with that for Riemann’s zeta function $\zeta(1)$ which is infinite.

The gyromagnetic ratio is $\mu_m/L$ and the g-factor

$$\frac{2mc}{q} \frac{\mu_m}{L}$$

so

$$g = \frac{3\eta_1}{(\epsilon/mc^2)} \left( \frac{1}{2} + \Sigma + \frac{1}{2} \omega \eta_1 \right) \left( \eta_1 + \frac{1}{2u_0 \eta_1} \Sigma \right)$$

This formula along with (40) assumes that all the angular momentum is in the moment of the Poynting vector of the electromagnetic field. When the system is highly relativistic $\Sigma$ dominates so

$$g \to \frac{2\eta_1}{(\epsilon/mc^2)} \approx \frac{9}{8} \left( \frac{mc^2}{\epsilon} \right)$$

where $\epsilon$ is the energy in the field. Notice that the presence of non-electromagnetic mass will increase $g$ above 9/8 if we have only electromagnetic angular momentum.

**VII. OLD FASHIONED ELECTRON MODELS**

A century ago it would have been natural to apply the foregoing results to Mie’s ‘model’ of the electron. Now everyone knows better (and some like me understand less). However in the spirit of those times let us set the charge $e$ in our first approximation to $e \hbar/2mc$ = const. Then

$$\mu_m = \frac{e \hbar}{2mc} = \frac{3}{2} e a \eta_1 \quad (45)$$

$$\frac{Lc}{e^2} = \frac{\hbar c}{2e^2} = \eta_1 + \frac{1 + u}{2u^{1/2}} \Sigma \quad (46)$$

and the total electromagnetic energy is then

$$\epsilon = \frac{e^2}{a} \left( \frac{1}{2} + \Sigma + \frac{u^{1/2}}{1 + u} \eta_1 \right)$$

$$= \frac{mc^2}{2} \left( \frac{1}{2} + \Sigma + \frac{1}{2u^{1/2}} \eta_1 \right) \eta_1 \quad (47)$$
From [16] we see that fixing the angular momentum and charge determines the relativistic factor. Since $\eta_1$ is never large and is only 0.375 in the extreme relativistic limit where $\Sigma$ becomes large the fact that the left hand side is $\simeq \frac{1}{137}$ shows us that we need the highly relativistic regime with $u$ close to 1 and $\Sigma \sim 68$. In that regime the terms in $\Sigma$ dominate both numerator and denominator of the right hand side of [47] so $\epsilon/mc^2 \simeq \frac{2}{3}h_1 = \frac{2}{137}$ where we use the limiting value of $\eta_1$ for this extremely relativistic case $\frac{2}{137} \sim 265, \gamma \sim 530$. Returning to equation [45] we find $a = \frac{2}{3}h/mc$ so the radius of the sphere is a little over $\frac{1}{2\pi}$ Compton wavelengths. The fact that at this large radius over half the rest mass is in the field energy shows that it was neglect of the effects of spin and charge polarization that led to the smaller radius $c^2/mc^2$ being considered classically.

Is there anything to be learned from these naive calculations? Perhaps yes. Firstly polarization changes in the highly relativistic regime are much higher than the net charge. This helps classical motions to give angular momenta much higher than $q^2/c$. Carter already pointed out that $g = 2$ for the Kerr metrics in relativity but did not draw attention to the polarization (Lynden-Bell 2003). $g = 2$ is a generic property of all the conformastationary metrics in General Relativity as well as the Kerr Metrics and Pfister & King (2002) suggest it is a rather general result. When the charges are uniform and fixed to the sphere the $g$ factor varies from $3/2$ to $11/6$ as $v$ increases (see Appendix I). Secondly the value of the electromagnetic angular momentum depends on $q^2/c$ and on how relativistically the system rotates but at given $v/c$; it does not depend on the radius of the configuration. The electromagnetic field energy and the dipole moment do depend on size, the latter gives a size related to the Compton wavelength but although this is two orders of magnitude larger than the classical radius of the electron the polarization due to rotation is so strong that more than half the rest mass can be electromagnetic even at this large radius.

Finally the fact that $\epsilon/mc^2$ is quite near $\frac{1}{2}$ suggests that a model with a somewhat different structure might yield exactly $\frac{1}{2}$. If the energy consists of an electromagnetic part $\epsilon \propto 1/a$ and a string with $\epsilon' = Ta$ then equilibrium will occur when $d/da(\epsilon + \epsilon') = 0$ i.e. $\epsilon = \epsilon'$ in which case $mc^2$ would be $\epsilon + \epsilon'$ and $\epsilon/mc^2 = \frac{3}{2}$. We note that a string loop will have no angular momentum about the axis of the loop so a purely magnetic angular momentum would then be correct. However my most recent disc models suggest that all the energy is electromagnetic.

**Acknowledgments**

I thank H. Ardavan for his enthusiastic interest in all singularities due to relativistic motion and for discussion of the convergence of the solutions near $v = c$ both in the main paper and the Appendix. The work leading to the approximate solutions with the higher $\delta_u$ neglected was done at the Chateaux de Mons conference in 2001 in honour of D.O. Gough cf (Lynden-Bell 2003) when it became too hot to attend the sessions. I thank N.W. Evans for reading and discussing the manuscript.

APPENDIX A: THE RAPIDLY ROTATING UNIFORMLY CHARGED SPHERE

The surface charge $\sigma = \frac{2\mu}{r^3}$ is fixed in the surface. The resulting surface current is $e^{-1}Qa \sin \theta = J$. We show that a field inside of $2\mu a^{-3}$ and an external field of a dipole $(3\mu \hat{r}r - \mu)/r^3$ satisfies the boundary conditions on the sphere.

Firstly $B_r = 2\mu a^{-3} \cos \theta$ and is continuous across the sphere from inside to outside. Secondly $[B_\theta]_{\text{out}} = \frac{9}{2}\mu a^{-3} \sin \theta (1 + 2)$ this has to equal $4\pi J = e^{-1}Qa^{-3} q \sin \theta$ so the boundary condition is satisfied provided $\mu = \frac{9}{4}q a^{-3}$. The electric field is $E = q \hat{r}/r^2$ outside and zero inside.

The angular momentum in the field is found from Poynting’s vector

$$L = \frac{1}{4\pi c} \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) dV = \frac{1}{4\pi c} \int \mathbf{r} \times \left(\mathbf{E} \times \frac{\mu}{r^3}\right) dV$$

$$= \frac{1}{4\pi c} \int \left(\frac{2\mu}{r^2} \mathbf{E} \times \frac{q \mathbf{B}}{r}\right) dV$$

$$= L = \frac{2}{2c}q \mu \int \int (2 \cos^2 \theta - 3 \cos^2 \theta + 1) \frac{1}{r^4} r^2 \sin \theta d\theta dr$$

$$= \frac{2}{3} \mu q/ac = \frac{2}{9} \frac{q^2}{c} \frac{v}{c}$$

The energy in the electric field is $\frac{1}{2}q^2/a = \epsilon_e$. The energy in the magnetic field is $\epsilon_m$ where
\[ \epsilon_m = \frac{1}{8\pi} \int B^2 dV = \frac{1}{8\pi} \left[ \frac{(2\mu}{a^3} \right]^2 \frac{4}{3} \pi a^3 \\
+ \mu^2 \int (3 \cos^2 \theta + 1) r^{-6} dV \right] = \mu^2 \left( \frac{2}{3} + \frac{1}{3} \right) = \frac{\mu^2}{a^3} \frac{1}{9} \frac{q^2 v^2}{c^2}. \]

Thus \( \epsilon = \epsilon_e + \epsilon_m = \frac{1}{2} \frac{q^2}{a} \left( 1 + \frac{3}{8} \frac{v^2}{c^2} \right). \) The gyromagnetic ratio is \( \mu/L = \frac{q}{2} ec/q. \) If all the angular momentum is in the field then the \( g \)-factor is

\[ g = \frac{2mc \mu}{q} = \frac{3}{2} \left( m \frac{e^2}{\epsilon} \right) \left( 1 + \frac{2}{9} \frac{v^2}{c^2} \right). \]

Notice two results of this calculation

1. \( \frac{2ec}{q} \leq \frac{\epsilon}{2} \) for \( v \leq c \) so putting \( L = \frac{h}{2} \) and \( q = e \) there could not be an angular momentum in the field as great as \( \frac{\epsilon}{2}. \) The maximum seems to be short by the large factor \( \sim \frac{4}{3} \times 137. \) But the conducting sphere behaves differently.

2. The \( g \) factor is always less than 2 if the energy is entirely electromagnetic but it is always greater than 3/2 and when \( v = c \) it becomes 11/6. Of course if the total energy is not all electromagnetic this result will change. If we fix the relativistic factor then \( \epsilon \propto 1/a \) if the other energy \( \epsilon_n \propto a^n \) then the condition of equilibrium gives \( \epsilon = n \epsilon_n \) so \( mc^2 = \epsilon = \epsilon_n = (1 + 1/n) \epsilon. \) Then the factor \( \frac{mc^2}{q^2} \) is \( (1 + 1/n). \) Cases could be made for \( n = 1 \) a string holding the electron together, \( n = 2 \) a tension over an area or \( n = 3 \) a tension throughout a volume so the factor might be 2, 3/2, 4/3 or of course 1 and it is of some interest to decide between these factors. However in the light of \( (1) \) above the insulating sphere with charge uniformly fixed to its surface is a bad model.

**APPENDIX B: SOLUTION BY GENERATING FUNCTIONS**

Here we give a second method of solving the mathematical problem which allows a greater accuracy to be obtained by providing a first order perturbation treatment to the terms totally neglected in the first method. Also this second method provides the basis for solving the problem of the relativistically rotating disk which we treat elsewhere. We have refrained from replacing the first treatment by the second because the second, though more powerful, is less direct and guidance from the first method proved vital in understanding some of the steps essential to the development of the second.

The equations to be solved are \( \delta \) and \( \gamma \) and we rewrite the latter in the form

\[ \sum_{n=m}^{\infty} (1 + \delta_n)\eta_{2n+1} = \frac{\omega}{4} \Phi_{2m}. \]

We multiply by \( U^m \) and sum from \( m = 0 \) to \( \infty. \) We define the generating functions \( \Phi(U) \) and \( \eta(U) \) by

\[ \Phi(U) = \sum_{m=0}^{\infty} \Phi_{2m} U^m \quad \text{and} \quad \eta(U) = \sum_{m=0}^{\infty} \eta_{2m+1} U^m, \]

and so discover

\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} U^m (1 + \delta_n) \eta_{2n+1} = \frac{\omega}{4} \Phi(U). \]

Now

\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} U^m = (1 - U^{n+1})/(1 - U), \]

thus

\[ \sum_{n=0}^{\infty} (1 - U^{n+1}) \eta_{2n+1} + \sum_{n=0}^{\infty} (1 - U^{n+1}) \delta_n \eta_{2n+1} \]

\[ = (1 - U) \frac{\omega}{4} \Phi(U) \]

so

\[ \eta(1) - U \eta(U) + \sum_{n=0}^{\infty} (1 - U^{n+1}) \delta_n \eta_{2n+1} \]

\[ = \frac{\omega}{4} (1 - U) \Phi(U). \quad (B1) \]

We now perform similar operations on equation \( \delta \) multiplying by \( U^m \) and summing from \( m = 0 \) to \( \infty. \) Reversing the order of summation

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} U^m \Phi_{2n} = \omega \eta(U). \]

But

\[ \sum_{m=0}^{n-1} U^m = (1 - U^n)/(1 - U), \]

so

\[ \Phi(1) - \Phi(U) = -\omega (1 - U) \eta(U). \quad (B2) \]
Notice that the lack of an \( n = 0 \) term in the sum does not matter as it cancels out between \( \Phi(1) \) and \( \Phi(U) \). Putting \( U = 0 \) in the result gives

\[
\Phi(1) - 1 = -\omega \eta_1 , \tag{B3}
\]

so we get an expression for \( \Phi(U) \) in terms of \( \eta \)

\[
\Phi(U) = \omega(1 - U)\eta(U) + 1 - \omega \eta_1 .
\]

Inserting this into (B3) and solving for \( \eta(U) \)

\[
\eta(U) = \frac{4}{\omega^2 Q} \left[ \eta(1) + (1 - U) \left( \frac{\omega^2}{4} \eta_1 - \frac{\omega}{4} \right) \right. \\
+ \left. \sum_{n=0}^{\infty} (1 - U^{n+1}) \delta_n \eta_{2n+1} \right] \tag{B4}
\]

where \( Q \) is the quadratic obtained from (12) by writing \( t = -U \)

\[ Q = U^2 + 2(2\omega^2 - 1)U + 1 = (U + u)(U + u^{-1}). \]

In our former approximation we neglected the \( \delta_n \) for \( n \geq 2 \) and kept only \( \delta_0 \) and \( \delta_1 \). Doing that in (B3) leads, via the method we shall shortly explain, back to our former solution which gave \( \eta_{2n+1} = C(-u)^n \) for \( n \geq 2 \). We shall adopt this approximate form of solution only for the small terms with \( \delta_n \), \( n \geq 2 \) in the sum in (B4) but we shall not assume that the value of \( C \) is exactly that given in equations (25) and (26) and we shall derive refined values for the \( \eta_{2n+1} \). Thus the sum in (B4) becomes

\[
\sum_{n=0}^{\infty} (1 - U^{n+1}) \delta_n \eta_{2n+1} = (1 - U)\delta_0 \eta_1 + \\
+ (1 - U^2)\delta_1 \eta_3 + C \sum_{n=2}^{\infty} (1 - U^{n+1}) \delta_n (-u)^n \tag{B5}
\]

We now define a function \( W(t) \)

\[
W(t) = \sum_{n=0}^{\infty} \delta_n (-t)^n = \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(2n + 1)(2n + 2)}(-t)^n .
\]

Writing \( w^2 \) for \( t \) we find

\[
\frac{d}{dw} [wW(w^2)] = \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2n + 2} (-w^2)^n
\]

and

\[
\frac{d}{dw} \left[ w^2 \frac{d}{dw} (wW) \right] = \frac{w}{4} \sum_{n=0}^{\infty} (-w^2)^n = \frac{w}{4(1 + w^2)} .
\]

Integrating and ensuring convergence at \( w = 0 \) gives us

\[
d/dw(wW) = \frac{1}{8w^2} ln(1 + w^2)
\]

and integration by parts then gives, again keeping \( W \) finite at \( w = 0 \)

\[
W = -\frac{1}{8w^2} ln(1 + w^2) + \frac{1}{4w} \int_0^w (1 + w^2)^{-1} dw .
\]

Replacing \( w^2 \) by \( t \) and taking account of the possibility that \( t \) may be negative we find the alternative forms of expression of a single analytic function

\[
W(t) = -\frac{1}{8t} ln(1 + t) + \left\{ \frac{1}{4\sqrt{t}} tan^{-1} \sqrt{t} \quad t \geq 0 \right\} .
\]

Notice that the series expansion of the final functions only involve integer powers of \( t \) so the square roots are only involved in the formal expression in terms of \( \tan^{-1} \) & ln. \( W(t) \) is analytic in the complex \( t \) plane in \( |t| < 1 \). Our former approximation retains only \( \delta_0 \) and \( \delta_1 \) so replaces \( W \) by the linear approximation \( W_L = \frac{1}{8} - \frac{1}{48} t \).

At \( t = 0 \), \( W = W_L = \frac{1}{8}, \Delta W = 0 \)

At \( t = 1 \), \( W = \frac{1}{16} - \frac{1}{8} \ln 2 = 0.1097, W_L = 0.1042, \Delta W = 0.0055 \)

At \( t = -1 \), \( W \to \frac{1}{4} \ln 2 = 0.1733, W_L = 0.1458, \Delta W = 0.0275 \)

So while that approximation for \( W \) remains good to 5% in the more physical range of positive \( t \), the accuracy falls to only 13% at \( t = -1 \) and our arguments will involve \( W \) there. It is therefore of importance to get the more accurate result to which this Appendix is devoted. Re-expressing (B5) in terms of \( W \)

\[
\sum_{n=0}^{\infty} (1 - U^{n+1}) \delta_n \eta_{2n+1} = (1 - U)\delta_0 (\eta_1 - C) + \\
+ (1 - U^2)\delta_1 (\eta_3 - Cu) + C[W(u) - UW(uU)]
\]

so our expression (B4) for \( \eta(U) \) becomes

\[
\eta(U) = \frac{4}{\omega^2 Q}(\eta(1) + (1 - U) \left[ \frac{\omega^2}{4} \eta_1 - \frac{\omega}{4} \delta_0 (\eta_1 - C) \right] + \\
(1 - U^2)\delta_1 (\eta_3 - Cu) + C[W(u) - UW(uU)]) . \tag{B7}
\]

(\( \eta_{2n+1} \) is the coefficient of \( U^n \) in this re-expressed as a power series in \( U \). However in our solution (11) of the recurrence relation (10) we had to choose the convergent solution and throw away the divergent one and only then did we have enough equations to yield a solution. We now have to find the analogous step in our new approach. The numerator of \( \eta(U) \) is analytic everywhere that \( W(uU) \) is analytic i.e. within \( |U| = u^{-1} \). Thus any poles that
can give us divergences in the high order coefficients of $U^l$ can only come from the zeros of $Q$. One such pole is at $U = -u^{-1}$ which is outside the $|U| = 1$ circle, inside which $\eta(U)$ should be analytic, but the other is within that circle at $U = u$. It is this pole that leads to the divergent solution of the recurrence relation whose coefficient $D$ had to be chosen as zero in (11). Thus the requirement of convergence in our coefficients $\eta_{2n+1}$ can be replaced by the requirement that $\eta(U)$ should be analytic with no poles within the circle $|U| = 1$. This in turn translates into the requirement that the numerator of $\eta(U)$ must vanish at $U = -u$ so

$$\eta(1) + (1 + u) \left[ \frac{\omega^2}{4} \eta_1 - \frac{\omega}{4} + \delta_0(\eta_1 - C) \right] +$$

$$+ (1 - u^2) \delta_1(\eta_3 + Cu) + C[W(u) + uW(-u^2)] = 0$$

without first having the solution by the recurrence relationship it would have been difficult to see this requirement. We now use this form of $\eta(1)$ in (177). At the same time we replace $\omega^2, Q$ by their values in terms of $u$ given in equation (18). Then

$$\eta(U) = (1 + Uu)^{-1} \left[ -\eta_1 + \frac{1}{2}u^2(1 + u) +$$

$$+ (1 + u)^2 \{-\delta_0(\eta_1 - C) + (u - U)\delta_1(\eta_3 + Cu) -$$

$$- C[uW(-u^2) + UW(u)]/(u + U) \right]$$

(B8)

Now $\eta(0) = \eta_1$ hence

$$\eta_1 = -u\eta_1 + \frac{1}{2}u^{1/2}(1 + u) + (1 + u)^2 \{-\delta_0(\eta_1 - C) +$$

$$+ u\delta_1(\eta_3 + Cu) - CW(-u^2)\}$$

(B9)

We use this expression to simplify (B8) into

$$\eta(U) = (1 + Uu)^{-1} \left[ \eta_1 + (1 + u)^2U \times$$

$$\times \left\{ \delta_1(\eta_3 + Cu) + C\frac{W(-u^2) - W(u)}{1 + Uu} \right\} \right]$$

(B10)

Notice that the term involving the $W$s has no singularity when $U = -u$ and that term in the overall numerator takes the value

$$-u[W(-u^2) - W(-1)]/(1 - u^2)$$

when $U = -1/u$. This is finite for $u < 1$. Thus isolating the term with a pole at $U = -1/u$ we have

$$\eta(U) = \frac{C}{1 + Uu} - \frac{(1 + u)^2}{u} \delta_1(\eta_3 + Cu) + C(1 + u)^2 \times$$

$$\times \left\{ \frac{W(-u^2) - W(u)}{1 + Uu} \right\}$$

(B11)

(B11)

where the numerator of the last term is zero at $U = -1/u$ so it has no pole there either. The first pole comes from the first term which gives the asymptotic expansion $C\Sigma(-Uu)^n$ as expected from the asymptotic form of the recurrence relation. This justifies the nomenclature of calling its coefficient $C$ which was shorthand for (B10)

$$C = \eta_1 + (1 + u)^2u^{-1} \times$$

$$\left\{ \delta_1(\eta_3 + Cu) + CU\frac{W(-u^2) - W(-1)}{1 - u^2} \right\}$$

(B12)

Now from the definition of $\eta(U)$ we have $\eta'(0) = \eta_3$. Applying this to (B11) we find the third equation which together with (B9) and (B12) we solve for $\eta_1, \eta_3$ and $C$

$$\eta_3 = -Cu + C(1 + u)^2 \times$$

$$\left\{ \frac{W(-u^2) - W(-1)}{1 - u^2} + \frac{W(-u^2) - W(0)}{u} \right\} =$$

$$-Cu + C(1 + u)^2g_0$$

(B13)

The resulting value of $\eta_3 + Cu$ we substitute into (B12) to obtain

$$\eta_1 = C\{1 + (1 + u)^2[g_1 + \delta_1g_0(1 + u^2)]\}$$

(B14)

where $g_1 = \frac{W(-u^2) - W(-1)}{1 - u^2}$ and $g_0 = g_1 + \frac{W(-u^2) - W(0)}{u^2}$ when only $\delta_0$ and $\delta_1$ are retained $g_1 = -\delta_1$ and $g_0$ is zero.

We now use (B13) and (B14) in (B9) to obtain an expression for $C$

$$C = \frac{1}{2}u^{1/2}/\Delta$$

where

$$\Delta = 1 - (1 + u)^2\{g_1[1 + (1 + u)\delta_0] +$$

$$+ (1 + u)\delta_1g_0[1 + u + u^2] + (1 + u)^2\delta_0\} +$$

$$+ (1 + u)W(-u^2)$$

(B15)

which reduces to (20) in the appropriate limit. With $\eta_1, \eta_3$ and $C$ now determined $\eta(U)$ is known from (B11).

We are now in a position to obtain a more accurate value for the surface density of charge at the poles and the equatorial velocity at which it changes sign. According to (11) we need

$$\sum_{n=0}^{\infty} (4n + 1)\Phi_{2n} = 4\Phi'(1) + \Phi(1)$$

But by (22) $\Phi'(1) = -\omega\eta(1)$ and by (18) $\Phi(1) = 1 - \omega\eta_0$ so $\sigma_p = \frac{1}{4\pi a^2} \{1 - \omega[\eta_1 + 4\eta(1)]\}$

Evaluating this expression as a function of $u$ we find that $\sigma_p = 0$ when $u = 0.455$ which is in better than expected agreement with the crude calculation of section
4 which gave \( u = .454 \) or 93% of \( c \) for the equatorial speed.

We now have our solution: \( C \) is given by \( \text{(B14)} \) and \( \text{(B15)} \). Let us recap the notation \( \omega = \Omega a/c = \frac{v}{c} \)

\[
u = 2\omega^2 \left[ 1 - \frac{1}{2} \omega^2 - \sqrt{1 - \omega^2} \right],
\]
so \( \gamma = \left( 1 - \frac{\nu^2}{c^2} \right)^{-1/2} = \frac{1 + u}{1 - u} \)

\( W(t) \) is the function defined by \( \text{(B5)} \),
\( g_1 \) is the expression \( [W(-u^2) - W(-1)]/(1 - u^2) \),
\( g_0 \) is \( g_1 + u^{-2}[W(-u^2) - W(0)] \)
\( \gamma_0 = W(0) = \frac{1}{8}, \quad \delta_1 = \frac{1}{48} \)

Thus all terms in \( C \) are known once \( \omega \) or \( u \) are given.
\( \text{(B13)} \) then gives \( \eta_1 \) and \( \text{(B12)} \) gives \( \eta_3 \).

The function \( \eta(U) \) is now given by \( \text{(B7)} \) with all terms therein known.

The coefficient of \( U^n \) in the expansion of \( \eta(U) \) as a power series in \( U \) is our refined solution for \( \eta_{2n+1} \) but with the first two \( \eta_1 \) and \( \eta_3 \) now explicitly known the others may also be found by direct repeated application of the exact recurrence relation \( \text{(10)} \) e.g.

\[
\eta_5 = C \left\{ u^2 + (1 + u)^2 [g_1 + \delta_1 - g_0(1 + u)^2] \right\}
\]

In practice the most interesting region is the relativistic region. \( u = \frac{1}{2} \) gives \( \gamma = 3 \), \( u = 0.8 \) gives \( \gamma = 9 \) while \( u = 0.9 \) gives \( \gamma = 19 \). Evaluating \( W(-u^2) \) for such values we find

\[
W(-u^2) = \frac{1}{8u^2} [(1 - u)\ln(1 - u) + (1 + u)\ln(1 + u)]
\]

\[
W(-1) = \frac{1}{4}\ln 2 = .173, \quad W(0) = \frac{1}{8}
\]

Near \( u = 1 \) \( g_1 = + \frac{1}{8u^2(1+u)} \times \)

\[
\times [(1 - u)\ln(1 - u) - (\ln 2 - 1) + (1 - u)(\ln 2 - \frac{1}{2})]
\]

\[
g_0 - g_1 = \frac{1}{8}u^{-4}[(1 - u)\ln(1 - u) + (1 + u)\ln(1 + u) - u^2]
\]

the dominant term is the \( \ln(1 - u) \) term in \( g_1 \).

Notice that near \( u = 1 \) these terms become large which invalidates our perturbation theory beyond \( \gamma \sim 20 \). The \( \gamma > 20 \) region is treated differently in the main body of the paper.