INTERNAL NEIGHBOURHOOD STRUCTURES III: THE HAUSDORFF REFLECTION

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Abstract. This paper develops further on internal preneighbourhood spaces [see 7, 8] in presenting the notion of internal Hausdorff spaces and the Hausdorff reflection. The development depends on the notions of proper morphisms, separated morphisms and separated objects, which are also developed in this paper. The Hausdorff reflection is described in three different ways: firstly as the largest subobject of the binary product whose components are indistinguishable by any internal Hausdorff space valued preneighbourhood morphism, secondly as the smallest effective equivalence relation whose quotient is an internal Hausdorff space and thirdly in admissibly well powered categories by transfinite induction on quotients by the diagonal.

1. Introduction

The notion of internal preneighbourhood spaces were initiated in [7]. The sole purpose of this paper is to introduce the notion of an internal Hausdorff space and the Hausdorff reflection of an internal preneighbourhood space. The paper is organised as follows:

(a) A context (see Definition 2.1(a)) is a triplet \( \mathcal{A} = (A, E, M) \), where \( A \) is a finitely complete category with finite coproducts, \((E, M)\) is a proper factorisation system on \( A \) such that for each object \( X \) the set \( \text{Sub}_M(X) \) of all \( M \)-subobjects (herein also called admissible subobjects) of \( X \) is a complete lattice. Contexts are ubiquitous, the most generic example being of a small complete small cocomplete well-powered category \( A \) with \( E \) being the set of all epimorphisms and \( M \) being the set of all extremal monomorphisms; other examples are listed on page 4. Contexts were introduced in [7] as a tool to introduce the notion of an internal neighbourhood space; in [8] the notion of closure and closed morphisms were developed and characterisations obtained for the sum admissible or closed morphisms to be again admissible or closed. In \( \S 2 \) all the definitions and facts necessary for this paper are recalled from [7, 8].

(b) The first set of results of this paper appear from \( \S 3 \) where the notion of a proper morphism (see Definition 3.1) is described. Every morphism is assumed here to reflect zero (see Definition 2.2). It is known in a category with pullbacks and initial object, every morphism reflects zero if and only if the initial object is strict (see Proposition 2.2(c), [8, Proposition 2.2.6]). Hence in the description of proper morphisms all pointed contexts (i.e., a context \( A \) in which the category \( A \) is pointed) get excluded. A preneighbourhood morphism \( f \) is shown to be proper if and only if every corestriction (see equation (1) and terminology that follow) of the product \( f \times 1_T \) for any internal preneighbourhood space \((T, \tau)\) is a closed morphism (see Theorem 3.1). The (possibly large) set \( \mathcal{A}_{pr} \) of proper morphisms contain all closed embeddings, is pullback stable, is closed under compositions and has nice cancellative properties (see Theorem 3.2).

(c) The description of proper morphisms make way for separated morphisms in \( \S 4 \), again in reflecting zero contexts. A preneighbourhood morphism \( f \) is separated if the diagonal morphism to its kernel pair is a proper morphism (see Definition 4.1). The (possibly large) set \( \mathcal{A}_{sep} \) of separated morphisms contain all monomorphisms, is pullback stable, is closed under compositions and also has nice cancellative properties similar to proper morphisms (see Theorem 4.2). Since pullbacks are none else than products in categories of bundles the notion of separated morphisms via this route leads to the description of separated objects.

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(d) The study of separated objects is done in §5, again in a reflecting zero context. An internal preneighbourhood space \((X, \mu)\) is separated (herein termed as an internal Hausdorff space) if the unique morphism from \(X\) to the terminal object is separated (see Definition 5). It is shown an internal preneighbourhood space \((X, \mu)\) is an internal Hausdorff space if and only if the diagonal is a closed embedding (see Theorem 5). Several other elegant properties are shown equivalent to internal Hausdorffness — for instance every preneighbourhood morphism with \((X, \mu)\) as domain is separated, or the product projection \(X \times T \not\to X\) to any other internal preneighbourhood space \((T, \tau)\) is separated, to mention two; for the complete list of eight see Theorem 5. The equivalent conditions imply the full subcategory \(\text{Haus}[A]\) of internal Hausdorff spaces a finitely complete subcategory of the category \(\text{pNbd}[A]\) of internal preneighbourhood spaces, closed under images of preneighbourhood morphisms stably in \(E\). Finally, given the characterisation of closure under sums in \([8, \text{Theorem 4.3.1 and Theorem 4.4.1}]\) in an extensive context the category \(\text{Haus}[A]\) is closed under finite sums if and only if \(1 + 1\) is an internal Hausdorff space (see Theorem 5.1).

(e) Finally in §6 the Hausdorff reflection of an internal preneighbourhood space is exhibited in three ways. Firstly, it is shown as the quotient of the largest subobject of the binary product whose components are indistinguishable by any internal Hausdorff space valued preneighbourhood morphism (see (19) in the proof of Theorem 6). Although this description requires quantification over (possibly large) sets, the subobject lattices are (possibly large) complete lattices, making the description plausible. Secondly, the effective equivalence relation generated by the subobject in (19) is shown to be the smallest effective equivalence relation whose quotient is an internal Hausdorff space (see Theorem 6.1). This gives a second description using the correspondence between quotients and effective equivalence relations. The second description takes recourse to the completeness of the (possibly large) lattice of admissible subobjects. Thirdly, in case when each object has a small set of admissible subobjects (herein called admissibly well-powered), the effective equivalence relation from the subobject in (19) is provided a construction using transfinite induction (see Theorem 6.2). The material of this last section is primarily a version of similar construction in \([11, 12]\).

The notation and terminology adopted in this paper are largely in line with the usage in \([10]\) or \([1]\). Apart from this, some specific notations and terms are explained here.

(f) In a category \(\mathcal{X}\), if \(X \overset{f}{\times} X \overset{g}{\times} Y\) be the coproduct of \(X\) and \(Y\) and \(X \overset{f}{\times} Z \overset{g}{\times} Y\) be morphisms then the unique morphism from \(X + Y\) to \(Z\) factoring through the coproduct injections is denoted by \(X + Y \overset{[f, g]}{\longrightarrow} Z\).

(g) The pullback of a morphism \(g\) along a morphism \(f\) is usually depicted by a diagram

\[
\begin{阵列}{ccc}
X \times_Z Y & \overset{f_g}{\longrightarrow} & Y \\
g \downarrow & & \downarrow g \\
X & \overset{f}{\longrightarrow} & Z
\end{阵列}
\]

Thus \(f_g\) is the pullback of \(f\) along \(g\) and \(f_g\) is the pullback of \(g\) along \(f\). In case of kernel pairs, the kernel pair of \(f\) is usually denoted by

\[\kerp f\]

\[\begin{阵列}{ccc}
X & \overset{f_1}{\longrightarrow} & X \\
f \downarrow & & \downarrow f \\
Y & \overset{f_2}{\longrightarrow} & Y
\end{阵列}\]

Morphisms \(T \overset{t_1}{\longrightarrow} X \times_Z Y\) are determined by pairs \((T \overset{t_1}{\longrightarrow} X, T \overset{t_2}{\longrightarrow} Y)\) where \(f_1 \cdot t_1 = g_1 \cdot t_2\), likewise, morphisms \(T \overset{t_1}{\longrightarrow} \ker f\) are determined by pairs \(T \overset{t_1}{\longrightarrow} X\) such that \(f_1 \cdot t_1 = f_1 \cdot t_2\).

(h) Given the proper \((E, M)\)-factorisation system, the morphisms of \(E\) are depicted as \(\cdot \rightarrow \cdot \rightarrow \cdot\) while the morphisms of \(M\) are depicted as \(\cdot \rightarrow \cdot \rightarrow \cdot\). The image and
The notion of an internal preneighbourhood space was initiated in [7]. The notion of a closure operator on an internal preneighbourhood space derived from its internal preneighbourhood system was introduced and detailed in [8]. The present section lists all definitions and results necessary for this paper; for details see [7, 8].

2. Preliminaries

The notion of an internal preneighbourhood space was initiated in [7]. The notion of a closure operator on an internal preneighbourhood space derived from its internal preneighbourhood system was introduced and detailed in [8]. The present section lists all definitions and results necessary for this paper; for details see [7, 8].

2.1. Internal Preneighbourhood Structures.

Definition. (a) A context is given by \((\mathcal{A}, E, M)\) where:

(i) \(\mathcal{A}\) is a finitely complete category with finite coproducts.

(ii) \((E, M)\) is a proper factorisation system (see [3]) on \(\mathcal{A}\) such that for each object \(X\) of \(\mathcal{A}\) the (possibly large) set \(\text{Sub}_M(X)\) is a complete lattice.

(b) If \(X\) is an object of \(\mathcal{A}\) then \(\text{Fil}(X)\) denotes the (possibly large) set of all filters in the lattice \(\text{Sub}_M(X)\).

(c) If \(X \xrightarrow{f} Y\) is a morphism of \(\mathcal{A}\), \(A \in \text{Fil}(X), B \in \text{Fil}(Y)\) then:

\[
\text{Fil}(f) A = \{y \in \text{Sub}_M(Y) : f^{-1} y \leq B \} \in \text{Fil}(Y),
\]

\[
\text{Fil}(f) B = \{ x \in \text{Sub}_M(X) : (\exists y \in B)(f^{-1} y \leq x) \} \in \text{Fil}(X).
\]

The filter \(\text{Fil}(f) A\) is the image filter of \(A\) under \(f\) and \(\text{Fil}(f) B\) is the inverse image filter of \(B\) under \(f\).

(d) An order preserving map \(\text{Sub}_M(X)^{op} \xrightarrow{\mu} \text{Fil}(X)\) is said to be:

(i) a preneighbourhood system on \(X\) if \(p \in \mu(m) \Rightarrow m \leq p\).

(ii) a weak neighbourhood system on \(X\) if it is a preneighbourhood system such that \(p \in \mu(m) \Rightarrow (\exists q \in \mu(m))(p \in \mu(q))\).

(iii) a neighbourhood system on \(X\) if it is a weak neighbourhood system such that for each \(S \subseteq \text{Sub}_M(X), \mu(\bigvee S) = \bigcap_{s \in S} \mu(s)\).

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1. The set \(\text{Fil}(X)\) is a complete lattice [see 7, Corollary 2.8], distributive if and only if \(\text{Sub}_M(X)\) is distributive [see 7, Proposition 2.7] or [see 9, Theorem 1.2].

2. Each morphism \(X \xrightarrow{f} Y\) induces the adjunctions \(\text{Sub}_M(X) \xleftarrow{\text{Fil}(f)} \text{Sub}_M(Y)\) as well as

\[
\text{Fil}(Y) \xleftarrow{\text{Fil}(f)} \text{Fil}(X)\] [see 7, Proposition 2.4 & Proposition 2.9].
(e) A pair \((X, \mu)\), where \(X\) is an object of \(\mathbb{A}\) and \(\mu\) is a preneighbourhood system on \(X\) is called an internal preneighbourhood space. Likewise for internal weak neighbourhood space and internal neighbourhood space.

(f) If \((X, \mu)\) and \((Y, \phi)\) are internal preneighbourhood spaces then a morphism \(X \xrightarrow{f} Y\) is a preneighbourhood neighbourhood morphism if \(p \in \phi(u) \Rightarrow f^{-1}p \in \mu(f^{-1}u)\).

If \(\mu\) and \(\phi\) are neighbourhood systems and \(f\) further satisfies the property:
\[
 f^{-1}(\bigvee_{t \in T} f^{-1}t) \text{ for each } T \subseteq \text{Sub}_M(Y)
\]
then \(f\) is a neighbourhood morphism.

(g) If \(X\) is an object of \(\mathbb{A}\) then \(\text{nbd}[X]\) is the set of all preneighbourhood systems on \(X\). Likewise for the symbols \(\text{wnbd}[X], \text{nbd}[X]\).

(h) The symbol \(\text{pwnbd}[\mathbb{A}]\) shall denote the category of internal preneighbourhood spaces and their morphisms; \(\text{pnbd}[\mathbb{A}]\) is the full subcategory of internal weak neighbourhood spaces. \(\text{nbd}[\mathbb{A}]\) is the subcategory of internal neighbourhood spaces and their morphisms.

Contexts abound:

(a) \((\text{FinSet}, \text{Surjections}, \text{Injections})\), see [7, Example 3.7].
(b) \((\text{Set}, \text{Surjections}, \text{Injections})\), see [7, Example 3.8].
(c) \((\text{Grp}, \text{Epi}, \text{Mono})\), see [7, Example 3.9 & Proposition 3.10].
(d) \((\text{Top}, \text{Epi}, \text{Mono})\), see [7, Example 3.11 & Proposition 3.12].
(e) \((\text{Loc}, \text{Epi}, \text{ExtMono})\), see [7, Example 3.13].
(f) \((\text{Loc}, \text{Epi}, \text{RegMono})\), see [7, Example 3.14].

Every topos with its usual factorisation structure, see [7, page 5, (iii)].

Every extension category (see [2]) with a proper factorisation structure, see [7, page 6, (v)]. This includes \(\text{Cat}\) the category of small categories, \(\text{CRing}^{op}\) of affine schemes and \(\text{Sch}\) the category of schemes.

(i) Given any context \((\mathbb{A}, E, M)\) and any object \(X\) of \(\mathbb{A}\), \(((\mathbb{A} \downarrow X), (E \downarrow X), (M \downarrow X))\) is the context where
\[
(\mathbb{E} \downarrow X) = \{ (X, x) \xrightarrow{e} (Y, y) : e \in E \} \\
(M \downarrow X) = \{ (X, x) \xrightarrow{m} (Y, y) : m \in M \}.
\]

See [7, page 5, (iv)] and [4, §2.10] for details.

(j) If \(\mathbb{A}\) is finitely complete, finitely cocomplete and has all intersections (respectively, cointersections) then it is well known that \(\mathbb{A}\) has a \((\text{Epi}(\mathbb{A}), \text{ExtMon}(\mathbb{A}))\)-factorisation system (respectively, \((\text{ExtEpi}(\mathbb{A}), \text{Mono}(\mathbb{A}))\)-factorisation system). Hence every small complete, small cocomplete category, well powered (respectively, co-well powered) category \(\mathbb{A}\) produces an example of a context, \(\mathcal{E} = (\mathbb{A}, \text{Epi}(\mathbb{A}), \text{ExtMon}(\mathbb{A}))\) (respectively, \(\mathcal{M} = (\mathbb{A}, \text{ExtEpi}(\mathbb{A}), \text{Mono}(\mathbb{A}))\)).

**Proposition.** Let \(\mathbb{A} = (\mathbb{A}, E, M)\) be a context.

(a) [7, Theorem 3.17] For any object \(X\) the set \(\text{pnbd}[X]\) of all preneighbourhood systems on \(X\) is a complete lattice. The finest and coarsest preneighbourhood systems are \(\uparrow m\) and \(\nabla(m)\) respectively, where
\[
\uparrow m = \{ p \in \text{Sub}_M(X) : m \leq p \} \text{ and } \nabla(m) = \begin{cases} 
\text{Sub}_M(X), & \text{if } m = \sigma_X \\
\{1_X\}, & \text{otherwise}
\end{cases}.
\]

(b) [7, Theorem 3.40] Given the internal preneighbourhood spaces \((X, \mu)\), \((Y, \phi)\) and a morphism \(X \xrightarrow{f} Y\) of \(\mathbb{A}\), the following are equivalent
(i) \(f\) is a preneighbourhood morphism.
(ii) For each \(n \in \text{Sub}_M(Y)\), \(\overset{f}{\sim} \phi(n) \subseteq \mu(f^{-1}n)\).
(iii) For each \(n \in \text{Sub}_M(Y)\), \(\phi(n) \subseteq \overset{f}{\sim} \mu(f^{-1}n)\).
(iv) For each \(m \in \text{Sub}_M(X)\), \(\overset{f}{\sim} \phi(\exists m) \subseteq \mu(m)\).

(c) [7, Theorem 4.8(a)] If \(\text{pwnbd}[\mathbb{A}] \xrightarrow{U} \mathbb{A}\) is the functor defined by \(U(X, \mu) = X\) on objects then \(U\) is a topological functor.

As a consequence of Proposition 2.1(c), the following special cases appear.

(a) If \((X, \mu)\) is an internal preneighbourhood space and \(M \xrightarrow{m} X\) with \(m \in \text{Sub}_M(X)\) then \(M\), unless otherwise specified is endowed with the smallest internal preneighbourhood system \((\mu|_M)\) which makes \(m\) a morphism of internal preneighbourhood spaces, where
\[
(\mu|_M)(u) = \{ v \in \text{Sub}_M(M) : (\exists x \in \mu(m \cdot u))(x \wedge m \leq m \cdot v) \}.
\]
(b) Given the preneighbourhood morphisms \((X, \mu) \xrightarrow{f} (Z, \psi) \xleftarrow{g} (Y, \phi)\) and the pullback of \(f\) along \(g\), unless otherwise specified, \(X \times_Z Y\) is endowed with the smallest internal preneighbourhood system \(\mu \times \phi\) which make the pullback projections \(g_f\) and \(f_g\) preneighbourhood morphisms, where

\[
(\mu \times \phi)(u) = \{ v \in \text{Sub}_M(X \times_Z Y) : (\exists x_f \in \mu(u^M_f))(\exists x_g \in \phi(u^M_g)) \\
\quad (v \geq g_f^{-1}x^M_f \land f_g^{-1}x^M_g) \}, \quad (7)
\]

\[
\exists_{i_f} u = u^M_f \text{ and } \exists_{i_g} u = u^M_g.
\]

(c) The terminal object \(1\) is always considered with its sole preneighbourhood system \(\uparrow\), and a special case of the pullback construction for \((X, \mu) \xrightarrow{f} (1, \uparrow) \xleftarrow{g} (Y, \phi)\) is the product \((X, \mu) \times (X \times Y, \mu \times \phi) \xrightarrow{p_1} (Y, \phi)\), where

\[
(\mu \times \phi)(u) = \{ v \in \text{Sub}_M(X \times Y) : (\exists x_1 \in \mu(u^M_1))(\exists x_2 \in \mu(u^M_2)) \\
\quad (v \geq p_1^{-1}x^M_1 \land p_2^{-1}x^M_2) \}, \quad (8)
\]

\[
\exists_{p_1} t_i = t_i^M \text{ for } i = 1, 2.
\]

In this regard, the following simple facts shall be of use:

**Lemma.** In any context \(\mathcal{A}\):

(a) If \(A \xrightarrow{a} X\) and \(B \xrightarrow{b} Y\) are admissible subobjects then \(p_1^{-1}a \land p_2^{-1}b = a \times b\).

(b) If every product projection is a morphism from \(E\) then:

\[
\exists_{p_1}(a \times b) = a \quad \text{ and } \quad \exists_{p_2}(a \times b) = b. \quad (9)
\]

(d) Given an internal preneighbourhood space \((X, \mu)\) and a morphism \(X \xrightarrow{f} Y\), the assignment \(p \mapsto \mu_f(p) = \overleftarrow{f}\mu(f^{-1}p)\) (for \(p \in \text{Sub}_M(Y)\)) defines a preneighbourhood system on \(Y\) and is the largest preneighbourhood system on \(Y\) which makes \(f\) a preneighbourhood morphism. The preneighbourhood system \(\mu_f\) is called the quotient preneighbourhood system of \(\mu\) induced by \(f\). If \(f\) is a regular epimorphism in \(\mathcal{A}\) then \((X, \mu) \xrightarrow{f} (Y, \mu_f)\) is a regular epimorphism of \(\text{pNbd}\)[\(\mathcal{A}\)] [see 7, Theorem 5.1].

The internal preneighbourhood space \((Y, \mu_f)\) shall be called a quotient preneighbourhood space of \((X, \mu)\) and it would be customary to denote \(Y\) by \(\left[\frac{X}{\ker f}\right]\).

In cases when \(f\) is the coequaliser of the pair \((R, \rho) \xrightarrow{r_1} (X, \mu)\) of preneighbourhood morphisms, then \(Y\) shall also alternatively denoted by \([\frac{X}{\ker f}]\).

(e) Given the coproduct \(X \xleftarrow{i_1} X + Y \xrightarrow{i_2} Y\) in \(\mathcal{A}\) of the internal preneighbourhood spaces \((X, \mu)\) and \((Y, \phi)\), unless otherwise specified, \(X + Y\) is endowed with the largest internal preneighbourhood system \(\mu + \phi\) which make the coproduct injections \(i_1, i_2\) preneighbourhood morphisms, where

\[
(\mu + \phi)(u) = \{ v \in \text{Sub}_M(X + Y) : (\exists x_1 \in \mu(u^M_1))(\exists x_2 \in \phi(u^M_2)) \\
\quad (v \geq \exists_{i_1} x_1 \lor \exists_{i_2} x_2) \}. \quad (10)
\]

2.2. Morphisms reflecting zero. Let \(\mathcal{A} = (\mathcal{A}, E, M)\) be a given context.

**Definition** ([see 8, Definition 2.2.1]). A morphism \(X \xrightarrow{f} Y\) of \(\mathcal{A}\) is said to reflect zero if \(f^{-1}\sigma_Y = \sigma_X\).

**Proposition.** ([see 8, Lemma 2.2.2]) A morphism \(X \xrightarrow{f} Y\) reflects zero if and only if

\[
\exists_x x = \sigma_Y \Rightarrow x = \sigma_X. \quad (11)
\]

(b) ([see 8, Theorem 2.2.3]) The set of all morphisms reflecting zero contain all admissible monomorphisms, is closed under composition and satisfies the following two properties:

(a) If \(g \circ f\) reflects zero then \(f\) reflects zero.

(b) For any morphism \(X \xrightarrow{f_n} Y\) reflecting zero and \(n \in \text{Sub}_M(Y)\), the corestriction \(f\) on \(N\) reflects zero.

(c) ([see 8, Proposition 2.2.6]) Every morphism of a category \(\mathcal{A}\) with an initial object and pullbacks reflect zero if and only if the initial object is strict.
Remark. A context $\mathcal{A} = (\mathbb{A}, \mathbb{E}, \mathbb{M})$ in which every morphism reflects zero shall be called a reflecting zero morphism.

2.3. Closure operation on admissible subobjects. Let $\mathcal{A} = (\mathbb{A}, \mathbb{E}, \mathbb{M})$ be a context.

Definition ([see 8, Definition 3.1.1]). Given any admissible subobject $p$ of an internal preneighbourhood space $(X, \mu)$, define:
\[
\cl_{\mu} p = \bigvee \{ x \in \text{Sub}_{\mu}(X) : u \in \mu(x) \Rightarrow u \land p \neq \sigma_X \}. \tag{12}
\]
An admissible subobject $p \in \text{Sub}_{\mu}(X)$ is $\mu$-closed if $p = \cl_{\mu} p$; $\mathcal{C}_\mu \subseteq \text{Sub}_{\mu}(X)$ is the (possibly large) set of $\mu$-closed subobjects of $(X, \mu)$.

Proposition. (a) ([see 8, Theorem 3.1.7]) For any internal preneighbourhood space $(X, \mu)$, $\text{Sub}_{\mu}(X) \xrightarrow{\cl_{\mu}} \text{Sub}_{\mu}(X)$ is an order preserving, idempotent, extensional function such that $\cl_{\mu} \sigma_X = \sigma_X$, i.e., $\cl_{\mu}$ is an idempotent closure operation on $\text{Sub}_{\mu}(X)$.

Furthermore, if $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a preneighbourhood morphism with $f$ reflecting zero then $f$ is continuous with respect to this closure operation, i.e., for every $p \in \text{Sub}_{\mu}(X)$, $\exists f \cl_{\mu} p \leq \cl_{\phi} \exists p$, or equivalently for every $q \in \text{Sub}_{\mu}(Y)$, $\exists f \cl_{\mu} q \leq f^{-1} \cl_{\phi} q$.

(b) ([see 8, Theorem 3.3.1]) Given the admissible subobjects $A \xrightarrow{a} M \xrightarrow{m} X$ of an internal preneighbourhood space $(X, \mu)$ with $m \in \mathcal{C}_\mu$:
\[
m \cdot \cl_{\mu} (m \cdot a) = \cl_{\mu} (m \cdot a). \tag{13}
\]
In particular: $a \in \mathcal{C}_{\{m\}} \Rightarrow m \cdot a \in \mathcal{C}_\mu$, i.e., the closure operation is transitive.

(c) ([see 8, Theorem 3.4.1]) If $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a preneighbourhood morphism with $\cl_{\phi} \exists f = \exists f$.

Example. In $(\text{Set}, \text{Surjections}, \text{Injections})$ the internal neighbourhood spaces are precisely the topological spaces; the closure defined here is precisely the usual closure of topological spaces. However, an internal preneighbourhood space in this context is a set $X$ equipped with an order reversing assignment $S \mapsto \mu(S)$ on the power set of $X$, where $\mu(S)$ is a filter of subsets of $X$ containing $S$. The definition of $\cl_{\mu} S$ in (12) provides an idempotent extensional closure operation in this general setup. This paper develops the notion of a Hausdorff space as well as Hausdorff reflection for such objects which are far from being a topological space.

In the case of the context $(\text{Loc}, \text{Epi}, \text{RegMon})$, the functorial neighbourhood system utilised in [5, 6], henceforth referred to as $T$-neighbourhood system:
\[
\tau_X(S) = \{ T \in \text{Sub}_{\text{RegMon}}(X) : (\exists a \in X)(S \subseteq \phi[a] \subseteq T) \}, \tag{14}
\]
where $\phi[a] = \{ x \in X : x = a \to x \}$ is the open sublocale of $X$ containing $a$, induces the same closure on locales as the classical one: since every locale $S = \bigvee_{a \in S} (a)$, where $(a)$ is the smallest sublocale of $X$ containing $a$, it is enough to consider $\tau_X(a) = \tau_X(\langle a \rangle)$, and $a \in \cl_{\tau_X} S \iff \langle a \rangle \subseteq \cl_{\tau_X} S$
\[
\iff (a \in \phi[x] \Rightarrow \phi[x] \land S \neq \sigma_X)
\]
\[
\iff (\phi[x] \land S = \sigma_X \Rightarrow a \notin \phi[x])
\]
\[
\iff (S \subseteq \uparrow x \Rightarrow a \geq x)
\]
\[
\iff (x \leq \bigwedge S \Rightarrow a \geq x)
\]
\[
\iff a \geq \bigwedge S,
\]
proving $\cl_{\tau_X} S = \uparrow(\bigwedge S)$ the usual closure of $S$. The proof is similar to the usual proof in topological spaces and rests on the intrinsic fact: in the Heyting lattice $\text{Sub}_{\text{RegMon}}(X)$ of sublocales of $X$, the open sublocale $\phi[x]$ and the closed sublocale $\uparrow x$ are complements.

2.4. Closed morphisms. Let $\mathcal{A} = (\mathbb{A}, \mathbb{E}, \mathbb{M})$ be a context.

Definition. A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a closed morphism if $p \in \mathcal{C}_\mu \Rightarrow \exists q, p \in \mathcal{C}_\phi$. 
Proposition. (a) ([see 8, Theorem 3.5.2, Corollary 3.5.3, 3.5.4]) If \( (X, \mu) \xrightarrow{f} (Y, \phi) \) is preneighbourhood morphism then \( f \) is a closed morphism if and only if for every admissible subobject \( p \) of \( X \), \( \text{cl}_\mu \exists \mu p \leq \exists \mu \text{cl}_\mu p \). If further \( f \) reflects zero then \( f \) is a closed morphism if and only if \( f \) preserves the closure operation, i.e., for every admissible subobject \( p \) of \( X \), \( \text{cl}_\mu \exists \mu p = \exists \mu \text{cl}_\mu p \). In particular, an admissible subobject \( M >^m > X \) is a closed morphism if and only if \( m \in \mathcal{C}_\mu \).

(b) ([see 8, Theorem 3.6.1]) Let \( \mathcal{A} = \bigcup_{\mu \in \text{pbd}(\mathcal{A})} \mathcal{C}_\mu \) be the (possibly large) set of all closed morphisms. The set \( \mathcal{A}_c \) contains all isomorphism, is closed under compositions and has the following two properties:

(i) If \( (X, \mu) \xrightarrow{f} (Y, \phi) \) be a closed morphism with \( f \) reflecting zero, then for each \( m \in \mathcal{C}_\mu \), the corestriction \( f_m \in \mathcal{A}_c \).

(ii) If \( g \circ f \in \mathcal{A}_c \) with \( f \) reflecting zero and formally surjective (i.e., \( \exists \mu f^{-1} = 1_{\text{cod}(f)} \) then \( g \in \mathcal{A}_c \).

Remark. In view of the last sentence in (a) of Proposition, \( m \in \mathcal{C}_\mu \) shall be called a closed embedding.

3. Proper morphisms

Let \( \mathcal{A} = (\mathcal{A}, E, M) \) be a reflecting zero context.

3.1. Definition of proper morphisms.

Definition. A preneighbourhood morphism \( (X, \mu) \xrightarrow{f} (Y, \phi) \) is said to be proper if it is stably in \( \mathcal{A}_c \), i.e., for every preneighbourhood morphism \( (T, \tau) \xrightarrow{h} (Y, \phi) \), the pullback \( f_h \) of \( f \) along \( h \) is a closed morphism. The symbol \( \mathcal{A}_pr \) denotes the (possibly large) set of proper morphisms in \( \mathcal{A} \).

Evidently \( \mathcal{A}_pr \) is the largest pullback stable set of closed preneighbourhood morphisms.

Theorem. Given any preneighbourhood morphism \( (X, \mu) \xrightarrow{f} (Y, \phi) \), \( f \) is a proper morphism if and only if for any internal preneighbourhood space \( (T, \tau) \) every corestriction of \( X \times T \xrightarrow{f \times 1_T} Y \times T \) is a closed morphism.

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{f_h} & T \\
\downarrow{h_f} & \searrow \downarrow{h} & \nearrow \downarrow{h_T} \\
X \times T & \xrightarrow{f \times 1_T} & Y \times T \\
\downarrow{p_1} & \searrow \downarrow{p_1} & \nearrow \downarrow{p_1} \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

in which \( p_1 \)’s are product projections and \( (T, \tau) \xrightarrow{h} (Y, \phi) \) is a preneighbourhood morphism. The horizontal square is the pullback of \( p_1 \) along \( f \), which proves the only if part. Using properties of pullbacks, the front vertical square is the pullback of \( f \) along \( h \) if and only if the top slanting square is the pullback of \( f \times 1_T \) along \( (h, 1_T) \). Hence \( f_h = (f \times 1_T)(h, 1_T) \), i.e., the corestriction of \( f \) along \( h \) is same as the corestriction of \( f \times 1_T \) along \( (h, 1_T) \). Hence \( f_h \) being a corestriction of \( f \times 1_T \) is closed by hypothesis, proving the if part.

3.2. Properties of Proper Morphisms.

Theorem. Let \( \mathcal{G} \) be the (possibly large) set of all proper morphisms \( f \) stably in \( E \), i.e., for any morphism \( g \), the pullback \( f_g \) of \( f \) along \( g \) is in \( E \).

The set \( \mathcal{A}_pr \) is a pullback stable set containing the closed embeddings, is closed under compositions and has the following two properties:

(a) If \( g \circ f \in \mathcal{A}_pr \) and \( f \in \mathcal{G} \) then \( g \in \mathcal{A}_pr \).

(b) If \( g \circ f \in \mathcal{A}_pr \) and \( g \in \text{Mono}(\mathcal{A}) \) then \( f \in \mathcal{A}_pr \).
Proof. Evidently, $\mathcal{A}_{pr}$ is the largest pullback stable subset of $\mathcal{A}_d$ and Proposition 2.3(c) implies every closed embedding is a proper morphism.

Assume $(X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)$ are proper preneighbourhood morphisms. Consider the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{f_w} & S \xrightarrow{g_w} W \\
\downarrow w & & \downarrow w \\
X & \xrightarrow{f} & Y \xrightarrow{g} Z
\end{array}
$$

where $(W, \omega) \underset{w}{\rightarrow} (Z, \psi)$ is a preneighbourhood morphism, the right hand square is the pullback of $w$ along $g$ and the left hand square is the pullback of $w_g$ along $f$. Hence the outer square is the pullback of $w_g \circ f$. If $g$ and $f$ are proper morphisms, $g_w$ and $f_w$ are both closed morphisms and hence their composite $g_w \circ f_w$ (Proposition 2.4(b)), proving $g \circ f$ is a proper morphism. On the other hand if the composite $g \circ f$ is a proper morphism then $g_w \circ f_w$ is a closed morphism. Further if $f$ is a proper morphism stably in $\mathcal{E}$, $f_w$ is a closed morphism stably in $\mathcal{E}$. Hence $g_w$ is a closed morphism (Proposition 2.4(b)(ii)) proving $g$ to be a proper morphism. This proves (a).

Let $(X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)$ be preneighbourhood morphisms such that $g \circ f$ is a proper morphism and $g$ is a monomorphism. Consider the commutative diagram

$$
\begin{array}{ccc}
T' & \xrightarrow{v} & T \\
\downarrow (u, v) & & \downarrow (g \circ h, 1_T) \\
X \times T & \xrightarrow{f \times 1_T} & Y \times T \xrightarrow{g \times 1_T} Z \times T
\end{array}
$$

in which $(T, \tau) \xrightarrow{h} (Y, \phi)$ is a preneighbourhood morphism and the left hand square is the pullback of $f \times 1_T$ along $(h, 1_T)$. Since $g$ is a monomorphism the right hand square is a pullback square. Hence the outer square is the pullback of $(g \circ h, 1_T)$ along $(g \circ f) \times 1_T$. Since $g \circ f$ is proper, using Theorem 3.1 on the outer pullback square, $v$ is a closed morphism. Hence using Theorem 3.1 again, $f$ is a proper morphism. This proves (b).

\[\square\]

Remark. In [4, §3] the notion of proper morphisms is also achieved as the largest stable set of closed morphisms. The set of proper morphisms in this work satisfy all the properties as described in [4] with the exception of right cancellability property in (a). The proper morphisms of [4, Proposition 3.2] are right cancellable with respect to all morphisms which are stably in $\mathcal{E}$.

Example. In the context $(\text{Set}, \text{Surjections}, \text{Injections})$ the proper maps of an internal neighbourhood space (i.e., a topological space) are precisely the familiar proper maps of topological spaces. The same is true in the context $(\text{Loc}, \text{Epi}, \text{RegMon})$ for locales with the $T$-neighbourhood system (see Example 2.3).

4. Separated morphisms

Let $\mathcal{A} = (\mathcal{A}, \mathcal{E}, \mathcal{M})$ be a reflecting zero context.

4.1. Definition of separated morphisms. Let $(X, \mu) \xrightarrow{f} (Y, \phi)$ be a preneighbourhood morphism and consider its kernel pair $\ker f \xrightarrow{f_1} X$. There exists the unique split monomorphism $X \xrightarrow{d_f} \ker f$ with $d_f = (1_X, 1_X)$ and hence $X$ is an admissible subobject of $\ker f$. The object $\ker f$ is equipped with the internal preneighbourhood system $\mu \times \phi \mu$ (see equation (7)). Any morphism $T \xrightarrow{t} \ker f$ is determined by the pair $T \xrightarrow{t_1} X \xrightarrow{t_2}$ of morphisms such that $f_1 \circ t = t_i$ $(i = 1, 2)$ and $t_i = t_i^E \circ t_i^M$ is the $(\mathcal{E}, \mathcal{M})$-factorisation of $t_i$ $(i = 1, 2)$.
Lemma. If \((X, \mu) \xrightarrow{f} (Y, \phi)\) is a preneighbourhood morphism, \(t = (t_1, t_2) \in \text{Sub}_M(\ker f)\) and \([t_1 = t_2] > m \xrightarrow{\mu} T \xrightarrow{t_1} X\) is the equaliser of the pair \((t_1, t_2)\), then:
\[
\begin{align*}
d_f^{-1} t &= t_1 \circ m_t = t_2 \circ m_t, \\
d_f \land t &= (t_1 \circ m_t, t_2 \circ m_t),
\end{align*}
\]
and
\[
\mu(d_f^{-1} t) \supseteq \mu(t_1^M) \lor \mu(t_2^M).
\]
In particular, \(\mu = (\mu \times _\phi \mu)_{|X}\).

Proof. The first two are trivial computations; for (17), \(t_1 \circ m_t = t_i^M \circ t_i \circ m_t\) implies \(t_i \circ m_t \subseteq t_i^M\), \((i = 1, 2)\) yielding the result from (15). An use of (17) shows \((\mu \times _\phi \mu)_{|X}\) \(\subseteq \mu\); for the reverse, the diagram below:
\[
\begin{array}{c}
\xymatrix{ \mu(a_1 = a_2) > m \ar[r] & \ker(f \circ \mu) = f_1^{-1} V \land f_2^{-1} V \ar[r] & f_2 V \ar[r] & V \\
& v_1 V \ar[r] & a = f_1^{-1} v \land f_2^{-1} v \ar[r] & v \ar[r] & \ker f \ar[r] & X \\
& f_1^{-1} V \ar[r] & f_1^{-1} v \ar[r] & \ker f \ar[r] & f_2 \ar[r] & X \ar[r] & Y \\
V > v \ar[r] & X \ar[r] & Y \\
\end{array}
\]

in which the squares are all pullbacks, shows for any \(v \in \mu(u)\), \(d_f \land f_1^{-1} v \land f_2^{-1} v \leq (v, v)\), so that \(v \in (\mu \times _\phi \mu)_{|X}(u)\). \(\square\)

Definition. A preneighbourhood morphism \((X, \mu) \xrightarrow{f} (Y, \phi)\) is said to be a separated morphism if \(d_f\) is a proper morphism. The symbol \(\mathcal{A}_{\text{sep}}\) denotes the (possibly large) set of all separated morphisms of \(\mathcal{A}\).

Remark. In view of Lemma, since every closed embedding is a proper map (Theorem 3.2), a preneighbourhood morphism \(f\) is separated if and only if \(d_f\) is a closed embedding.

4.2. Properties of Separated Morphisms.

Theorem. Let \(\mathcal{G}\) be the (possibly large) set of all proper morphisms which are stably in \(\mathcal{E}\). The set \(\mathcal{A}_{\text{sep}}\) of all separated morphisms of \(\mathcal{A}\) is a pullback stable set containing all monomorphisms, is closed under composition and satisfies the properties:
\[
\begin{align*}
(\text{a}) & \text{ If } g \circ f \in \mathcal{A}_{\text{sep}} \text{ and } f \in \mathcal{G} \text{ then } g \in \mathcal{A}_{\text{sep}}, \\
(\text{b}) & \text{ If } g \circ f \in \mathcal{A}_{\text{sep}} \text{ then } f \in \mathcal{A}_{\text{sep}}.
\end{align*}
\]

Proof. Since the kernel pair of a monomorphism \(f\) is trivial, \(d_f\) is an isomorphism. Hence every monomorphism is separated.

Towards the proof of composition closed, and the statements in (a) and (b), consider the preneighbourhood morphisms \((X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)\) and their kernel pairs. Evidently, \(\ker f \circ (f_1, f_2) \supseteq \ker g \circ (f_1 \circ h_1, f_2 \circ h_2)\).

Consider the commutative diagram
\[
\begin{array}{c}
\xymatrix{ X \ar[r]^{d_f} & \ker f \ar[r]^{f \circ f_2} & V \ar[r]^{v_2} & V \\
& (f_1, f_2) \ar[r]^{d_g} & (f_1 \circ h_1, f_2 \circ h_2) \ar[r]^{g_1 \circ g_2} & Y \times Y \\
& (h_1, h_2) \ar[r]^{v \times f} & X \times X \ar[r]^{f \circ f} & Y \times Y \\
\end{array}
\]

The top right hand square is a pullback square — if \(P \xrightarrow{p} Y\) and \(P \xrightarrow{q} X\) be morphisms such that \(h \circ q = h \circ r\) and \((p, p) = d_g \circ p = (f \circ h_1, f \circ h_2) \circ (g, r)\) then \(f \circ q = p = f \circ r\).

Hence, \(P \xrightarrow{(q, r)} \ker f\) is the unique morphism such that \((f_1, f_2) \circ (q, r) = (q, r)\) and
$f \circ f_2 \circ (q, r) = f \circ r = p$, proving the assertion. On the other hand the top outer square is trivially a pullback square. Hence using properties of pullback squares the pullback of $(f_1, f_2)$ along $d_h$ is $1_X$. Finally the bottom right hand square is trivially a pullback square.

(A) If $f$ be a proper morphism then each corestriction of both the morphisms $f \times 1_Y$ and $1_X \times f$ are closed morphisms (Theorem 3.1). Since any morphism $Z \xrightarrow{(g, h)} Y \times Y$ factors as $Z \xrightarrow{(1_Z, h)} Z \times Y \xrightarrow{g \times 1_Y} Y \times Y$, $f \times f = (f \times 1_Y) \circ (1_X \times f)$ is a proper morphism. Consequently, from the right hand bottom pullback square $(f_1, f_2) \circ (f_1, f_2) = (f, f_2) \circ (f_1, f_2)$ is a proper morphism. From the top outer square in $(\ast)$, since $d_g \circ f = (f \circ h_1, f \circ h_2) \circ d_h$, if $h$ is a separated morphism then $d_g \circ f$ is a proper morphism (Theorem 3.2). Hence, if $f$ is a proper morphism stably in $\mathcal{E}$, from Theorem 3.2 (a), $d_g$ is proper morphism, i.e., $g$ is separated, proving (a).

(B) If $h$ is separated then $d_h$ is a proper morphism. Hence from the left hand pullback square, $d_f$ being the pullback of $d_h$ along $(f_1, f_2)$ is a closed morphism. Hence $f$ is a separated, proving (b).

(C) If $g$ is separated, $d_g$ is a proper morphism and hence $(f_1, f_2)$ is closed. Further if $f$ is separated, $d_h = (f_1, f_2) \circ d_f$ is a closed morphism (Proposition 2.4(b)). Hence $h$ is a separated morphism proving $\alpha_{\text{sep}}$ is closed under compositions.

Finally, towards the pullback stability of separated morphisms, consider the preneighbourhood morphisms $(X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)$ with $f$ a separated morphism and the diagram above in which the front vertical and the right hand vertical squares depict the pullback of $f$ along $g$, the base horizontal square is the kernel pair of $f$ and the top horizontal square is the kernel pair of $f_g$ the pullback of $f$ along $g$.

Since $f \circ g \circ (f_2)_2 = g \circ f_2 \circ (f_2)_2 = g \circ (f_2)_1 = f \circ f_g \circ (f_2)_1$, there exists the unique morphism $\ker f_g \xrightarrow{v} \ker f$ such that

$$
\begin{align*}
&f_1 \circ v = g \circ (f_2)_1 \\
&f_2 \circ v = g \circ (f_2)_2
\end{align*}
$$

Furthermore, using properties of pullbacks squares, all the faces of the cube are pullback squares.

Since $d_f$ is the equaliser of $(f_1, f_2)$ and $f_1 \circ v \circ d_f = f_2 \circ v \circ d_f$, there exists a unique morphism $P \xrightarrow{w} X$ such that $d_f \circ w = v \circ d_f$. Hence $w = f_1 \circ d_f \circ w = f_1 \circ v \circ d_f = g \circ (f_2)_1 \circ d_f = g \circ v$. Furthermore, since the left most square is trivially a pullback square it follows from properties of pullbacks that $g \circ v$ is the pullback of $v$ along $d_f$.

Consequently, if $f$ is separated, then $d_f$, being the pullback of $d_f$ along $v$ is also a proper morphism. This proves $f_g$ a separated morphism whenever $f$ is a separated morphism. Hence $\alpha_{\text{sep}}$ is stable under pullbacks. □

**Example.** In the context (**Set**, Surjections, Injections) the separated maps between internal neighbourhood spaces are precisely those continuous maps in whose fibres distinct points are separated by disjoint neighbourhoods.

4.3. Recall: given a proper factorisation system ($\mathcal{E}, \mathcal{M}$) on $\mathcal{A}$, for any object $Y$ of $\mathcal{A}$ the category ($\mathcal{A} \downarrow Y$) of bundles on $X$ has ($\mathcal{E} \downarrow Y$), ($\mathcal{M} \downarrow Y$) as its proper factorisation system (see equation (4) & [see 4, page 5, (iv) for details]). Further, for any morphism $X \xrightarrow{f} Y$,
\[
\text{Sub}_M(X \downarrow Y)(X,f) = \text{Sub}_M(X), \text{Fil}(X,f) = \text{Fil}(X),
\]
as lattices, so that every internal preneighbourhood system \( \text{Sub}_M(X)^{op} \xrightarrow{\mathcal{L}} \text{Fil}(X) \) is also an internal preneighbourhood system on the object \((X,f)\) of \((A \downarrow Y)\). Since pullbacks in \(A\) are none else than products in the category of bundles, given the morphism \((X,\mu) \xrightarrow{\mathcal{L}} (Y,\phi)\) of internal preneighbourhood spaces, the kernel pair \(\ker p f = (\ker f, \mu \times_\phi \mu)\) is none else than the diagonal morphism \((X,\mu) \xrightarrow{\mathcal{L}} (X,\mu \times \mu)\). Hence \(f\) is a separated morphism if and only if the diagonal morphism \(d(X,\mu) \xrightarrow{\mathcal{L}} (X,\mu \times \mu)\) is a closed morphism in the context \((A \downarrow Y)\).

5. Internal Hausdorff Spaces

Let \(A = (A, E, M)\) be a reflecting zero context.

**Definition.** An internal preneighbourhood space \((X,\mu)\) of \(A\) is said to be an internal Hausdorff space if the unique morphism \(X \xrightarrow{\mu} 1\) is a separated morphism. The symbol \(\text{Haus}[A]\) shall denote the full subcategory of \(\text{pNbd}[A]\) consisting of internal Hausdorff spaces.

**Theorem.** The following are equivalent for any internal preneighbourhood space \((X,\mu)\) of \(A\):

(a) \((X,\mu)\) is an internal Hausdorff space.

(b) The diagonal morphism \((X,\mu) \xrightarrow{\mathcal{L}} (X \times X, \mu \times \mu)\) is a closed morphism.

(c) Every preneighbourhood morphism with \((X,\mu)\) as domain is separated.

(d) There exists a separated preneighbourhood morphism from \((X,\mu)\) to an internal Hausdorff space.

(e) The product projection \((X \times Y, \mu \times \phi) \xrightarrow{\mathcal{L}} (Y,\phi)\) is a separated morphism for every internal preneighbourhood space \((Y,\phi)\).

(f) For every internal Hausdorff space \((Y,\phi)\) the product \((X \times Y, \mu \times \phi)\) is an internal Hausdorff space.

(g) For every proper morphism \((X,\mu) \xrightarrow{\mathcal{L}} (Y,\phi)\) with \(f\) stably in \(E\), \((Y,\phi)\) is an internal Hausdorff space.

(h) If \((E, (\psi|_E))\) be the equaliser diagram for \(f\) and \(g\) then \(e\) is a closed morphism.

**Proof.**

(i) Evidently (a) and (b) are equivalent.

(ii) Given any preneighbourhood morphism \((X,\mu) \xrightarrow{\mathcal{L}} (Y,\phi)\) since \(t_X = t_Y \circ f\), an use of Theorem 4.2(b), shows (a) implies (c). On the other hand, (c) evidently implies (a).

(iii) Since \((1, \uparrow)\) is already an internal Hausdorff space, (a) automatically implies (d).

On the other hand if \((Y,\phi)\) be an internal Hausdorff space and \((X,\mu) \xrightarrow{\mathcal{L}} (Y,\phi)\) is a separated preneighbourhood morphism then \(t_X = t_Y \circ f\) implies from Theorem 4.2, \(X\) is an internal Hausdorff space. Hence (d) implies (a).

(iv) Given any proper morphism \((X,\mu) \xrightarrow{\mathcal{L}} (Y,\phi)\) with \(f\) stably in \(E\) an use of Theorem 4.2(a) prove from \(t_X = t_Y \circ f\) the implication of (g) from (a). On the contrary, assuming (g) and considering \(Y = X, f = 1_X, (a)\) follows.

(v) Since the product projection \((X \times Y, \mu \times \phi) \xrightarrow{\mathcal{L}} (Y,\phi)\) is the pullback of \(t_X\) along \(t_Y\), (a) implies (e) from pullback stability of separated morphisms (Theorem 4.2).

(vi) If \((Y,\phi)\) is an internal Hausdorff space and \((X \times Y, \mu \times \phi) \xrightarrow{\mathcal{L}} (Y,\phi)\) is the product projection, then \(t_{X \times Y} = t_Y \circ p_1\) implies (f) from (e) Theorem 4.2.

(vii) Since any internal preneighbourhood space isomorphic to an internal Hausdorff space is also an internal Hausdorff space, (f) evidently implies (a).
(viii) Since \( E \overset{e}{\to} Z \overset{f}{\to} X \) is an equaliser diagram if and only if the square
\[
\begin{array}{c}
\begin{array}{c}
E \\
\downarrow^e
\end{array} \quad \begin{array}{c}
\begin{array}{c}
Z \\
\downarrow^f
\end{array} \\

\end{array}
\end{array}
\]
is a pullback square, (b) implies (h) from the pullback stability of separated morphisms proved in Theorem 4.2. On the other hand, since \( d_X \) is the equaliser of the product projections, (h) implies (b).

\[\square\]

**Corollary.** The category \( \text{Haus}[\mathcal{A}] \) is a finitely complete subcategory of \( \text{plfd}[\mathcal{A}] \) closed under subobjects and images of preneighbourhood morphisms stably in \( \mathcal{E} \).

**Proof.** Since \((1, \uparrow)\) is an internal Hausdorff space, from (f) & (h) of Theorem the category \( \text{Haus}[\mathcal{A}] \) is closed under finite products and regular subobjects. Hence \( \text{Haus}[\mathcal{A}] \) is finitely complete. Let \((X, \mu)\) be an internal Hausdorff space and \( Y \to X \) be a monomorphism. Let \( \mu_f \) be the smallest preneighbourhood system on \( Y \) making \( f \) a preneighbourhood morphism. Then \((Y, \mu_f) \to (X, \mu)\) is separated morphism (Theorem 4.2) and hence \((Y, \mu_f)\) is an internal Hausdorff space from (d) of Theorem. Finally from (g) of Theorem if \((X, \mu)\) is an internal Hausdorff space and \((Y, \phi)\) be an internal preneighbourhood space such that \( Y = \exists_f X \) for some preneighbourhood morphism \( f \) stably in \( \mathcal{E} \) then \((Y, \phi)\) is also an internal Hausdorff space.

\[\square\]

### 5.1. Sum of internal Hausdorff spaces.

In this section assume an extensive context. Since in any extensive category the initial object is strict, it follows from Proposition 2.2(c) that it a reflecting zero context also. It is known:

(A) The admissible monomorphisms are closed under finite sums if and only if the monomorphisms in \( \mathcal{E} \) between finite sums are stable under pullbacks along coproduct injections ([see 8, Theorem 4.3.1]).

(B) The closed embeddings are closed under finite sums if and only if the coproduct injections are closed and a finite sum of closed embeddings is an admissible monomorphism ([see 8, Theorem 4.4.1]).

**Theorem.** In an extensive context, if the closed embeddings are closed under finite sums then the sum \( f + g \) of preneighbourhood morphisms is separated if and only if both \( f \) and \( g \) are separated.

In particular, the full subcategory \( \text{Haus}[\mathcal{A}] \) of internal Hausdorff spaces is closed under finite sums if and only if \( 1 + 1 \) is an internal Hausdorff space.

**Proof.** Consider the preneighbourhood morphisms \((X, \mu) \to (A, \alpha)\) and \((Y, \phi) \to (B, \beta)\). The diagram in (18) depicts the kernel pairs of \( f, g \) and \( f + g \) and the corresponding diagonal morphisms \( df, dg \) and \( d_{f+g} \). Since \( \mathcal{A} \) is extensive, all the squares are pullback squares, implying \( \kerp (f + g) = \kerp f + \kerp g \), \((f + g)_i = f_i + g_i\) for \( i = 1, 2 \) and \( d_{f+g} = df + dg \).

\[\begin{array}{c}
\begin{array}{c}
Y \\
\downarrow^e
\end{array} \quad \begin{array}{c}
\begin{array}{c}
Z \\
\downarrow^f
\end{array} \\

\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
E \\
\downarrow^e
\end{array} \quad \begin{array}{c}
\begin{array}{c}
X \\
\downarrow^f
\end{array} \\

\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
X \\
\downarrow^f
\end{array} \quad \begin{array}{c}
\begin{array}{c}
Y \\
\downarrow^g
\end{array} \\

\end{array}
\end{array}
\]
Hence if \( f, g \) are separated then \( d_{f+g} \) being the sum of two closed embeddings is also closed, and hence \( f + g \) is separated. Conversely, if \( f + g \) is separated, \( d_{f+g} \) is closed and \( d_f, d_g \) being the pullback of \( d_{f+g} \) are both closed. Hence both \( f \) and \( g \) are separated.

For the second part it is enough to prove the if part, in which case taking \( A = 1 = B \) and hence \( f = f_X, g = f_Y \), where \((X, \mu)\) and \((Y, \phi)\) are internal Hausdorff spaces, \( f_X \) and \( f_Y \) are separated morphisms implying \( f_X + f_Y \) is a separated morphism from the first part. Since \( 1 + 1 \) is an internal Hausdorff space, \( 1 + 1 \) is separated, so that \( f_{(X + Y)} = f_{1+1} \circ (f_X + f_Y) \) is a separated morphism from Theorem 4.2, completing the proof. \( \square \)

6. Hausdorff reflection

Let \( \mathcal{A} = (\mathfrak{A}, \mathfrak{E}, \mathfrak{M}) \) be a reflecting zero context with finite product projections in \( \mathfrak{E} \).

**Theorem.** In any reflecting zero context with product projections in \( \mathfrak{E} \), the full subcategory of internal Hausdorff spaces is a regular-epimorphic reflective subcategory (i.e., the reflections are regular epimorphisms).

**Proof.** Let \((X, \mu)\) be an internal preneighbourhood space. \( d = (d_1, d_2) = cl_{\mu \times \mu} d_X \) and \( R \supseteq (r_1, r_2) \times X \times X \) be the admissible subobject of \( X \times X \) defined by:

\[
(r_1, r_2) = \bigvee \{(u_1, u_2) \in \text{Sub}_M(X \times X) : (\forall \text{ internal Hausdorff space}(Y, \phi))
\]

\[
\forall (X, \mu) \xrightarrow{f} (Y, \phi)(f \circ u_1 = f \circ u_2) \} \in \text{Sub}_M(X \times X). \tag{19}
\]

Evidently, \( d_X \leq (r_1, r_2) \), implying \((r_1, r_2)\) is not trivial. Further, let \( X \xrightarrow{\mu} [\frac{X}{\mu}] \) be the coequaliser of the pair \((r_1, r_2)\) and \( \mu \) be the quotient preneighbourhood system on \([\frac{X}{\mu}]\) (see (d), page 5), i.e., for any \( u \in \text{Sub}_M([\frac{X}{\mu}]) \):

\[
\mu(u) = \{ v \in \text{Sub}_M([\frac{X}{\mu}]) : q^{-1} v \in \mu(q^{-1} u) \}. \tag{20}
\]

Firstly, if \((Z, \psi)\) is an internal Hausdorff space and \((X, \mu) \xrightarrow{f} (Z, \psi)\) be a preneighbourhood morphism then for each admissible subobject \((u_1, u_2) \leq (r_1, r_2), f \circ u_1 = f \circ u_2\), and hence \((u_1, u_2) \leq (f_1, f_2)\), where \((f_1, f_2)\) is the kernel pair of \( f \). Using (19), \((r_1, r_2) \leq (f_1, f_2)\), i.e., \( f \circ r_1 = f \circ r_2 \). Consequently, from the coequaliser there exists a unique morphism \([\frac{X}{\mu}] \xrightarrow{f} Z\) such that \( f = \bar{f} \circ q \). If for some \( z \in \text{Sub}_M(Z) \), \( w \in \psi(z) \) then \( f^{-1} w = f^{-1} \bar{f}^{-1} z \); since \( f^{-1} w = q^{-1} f^{-1} w \), from (20) \( \bar{f}^{-1} z \in \mu(q^{-1} u) \), proving existence of the unique preneighbourhood morphism \((\frac{X}{\mu}) \xrightarrow{f} \frac{Z}{\mu} \) such that \( f = \bar{f} \circ q \).

Now, for any \((u_1, u_2) \in \text{Sub}_M(X \times X)\) with both \( u_1 \) and \( u_2 \) not equal to \( 1_X \):

\[
(u_1, u_2) \leq cl_{\mu \times \mu} d_X \Leftrightarrow (v \in (\mu \times \mu)(u_1, u_2) \Rightarrow v \land d_X \neq \sigma_X \times X)
\]

\[
(\forall (v_1 \in \mu(u_1^M), v_2 \in \mu(u_2^M) \Rightarrow (v_1 \times v_2) \land d_X \neq \sigma_X \times X)
\]

\[
(\forall (v_1 \in \mu(u_1^M), v_2 \in \mu(u_2^M) \Rightarrow v_1 \land v_2 \neq \sigma_X)
\]

\[
\Leftrightarrow \mu(u_1^M) \lor \mu(u_2^M) \neq 1
\]

\[
\Leftrightarrow u_1^M \land u_2^M \leq cl_{\mu \times \mu} d_X,
\]

the last equivalence coming as a consequence of product projections being morphisms from \( \mathfrak{E} \). Since \( \mu_q(u) = \mu_q(\exists_q q^{-1} u) \) for all \( u \in \text{Sub}_M([\frac{X}{\mu}]) \), for any \((y_1, y_2) \in \text{Sub}_M([\frac{Y}{\mu}] \times [\frac{Y}{\mu}])\) with both \( y_1, y_2 \) not equal to \( 1_{\frac{Y}{\mu}} \):

\[
(y_1, y_2) \leq cl_{\mu_q \times \mu_q} d_{[\frac{X}{\mu}]} \Leftrightarrow \mu_q(y_1^M) \lor \mu_q(y_2^M) \neq 1
\]

\[
\Leftrightarrow \mu_q(\exists_q q^{-1} y_1^M) \lor \mu_q(\exists_q q^{-1} y_2^M) \neq 1
\]

\[
\Leftrightarrow (\exists_q q^{-1} y_1^M) \times (\exists_q q^{-1} y_2^M) \leq cl_{\mu_q \times \mu_q} d_{[\frac{X}{\mu}]}.
\]

Hence, in considering admissible subobjects smaller than \( cl_{\mu_q \times \mu_q} d_{[\frac{X}{\mu}]} \) it suffices to consider subobjects of the form \((\exists_u u_1, e_1, \exists_u u_2, e_2)\), where \( e_1, e_2 \in \mathfrak{E} \) have common domain, \( u_1, u_2 \in \text{Sub}_M(X) \) with \( u_i = q^{-1} \exists_u u_i \), \( i = 1, 2 \). Furthermore, \( cl_{\mu_q \times \mu_q} d_{[\frac{X}{\mu}]} \) is a suprema of subobjects of this special kind, since \( \left(1_{\frac{X}{\mu}}, 1_{\frac{X}{\mu}}\right) \leq cl_{\mu_q \times \mu_q} d_{[\frac{X}{\mu}]} \) and for every \( y \in \text{Sub}_M([\frac{X}{\mu}]) \) there exists a \( x \in \text{Sub}_M(X) \) such that \( y \leq \exists_q x \).
Choose and fix a $y = ((\exists u_1), e_1, (\exists u_2), e_2)$, where $e_1, e_2 \in E$ have common domain, $u_1, u_2 \in \text{Sub}_M(X)$ with $u_i = q^{-1}\exists u_i$, $i = 1, 2$. Since $(x_1, x_2) \leq (r_1, r_2)$ implies $(x_1, x_2) \leq (q_1, q_2)$ (where $(q_1, q_2)$ is the kernel pair of $q$), which in turn implies $\exists_y x_1^M = \exists_y x_2^M$, if $y$ be such that $\exists_y u_1 \neq \exists_y u_2$ then $u_1 \times u_2 \nless (r_1, r_2)$. For such a $y$, there exists an internal Hausdorff space $(Z, \psi)$ and a preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Z, \psi)$ such that $f \circ u_1 \circ p_1 \neq f \circ u_2 \circ p_2$. Hence if $u_1 \circ p_1, f \circ u_2 \circ p_2 \nless d_Z = \text{cl}_{\psi \times \psi}d_Z$, implies the existence of a $v_i \in \psi(\exists y u_i)$ ($i = 1, 2$) such that $v_1 \vee v_2 = \sigma_Z$. From above, there exists a unique preneighbourhood morphism $((\exists_y x_1), \mu_q) \xrightarrow{f} (Z, \psi)$ such that $f = f \circ q$. Since $f$ reflects zero, $f^{-1}v_1 \land f^{-1}v_2 = \sigma_{\exists y x}$; further, $f^{-1}\exists u_i = f^{-1}\exists_y x_i \geq \exists_y x_i$ implies $f^{-1}v_i \in \mu_q(\exists_y x_i)$ for $i = 1, 2$. Hence from (21), $y \nless \exists_y x_1 \times \exists_y x_2$. Thus, for any $y = ((\exists u_1), e_1, (\exists u_2), e_2)$, where $e_1, e_2 \in E$ have common domain, $u_1, u_2 \in \text{Sub}_M(X)$ with $u_i = q^{-1}\exists u_i$ for $i = 1, 2$, $y \leq \mu_q(u_1, u) \Rightarrow \exists_y u_1 \Rightarrow u_1 = u_2$. In such a case, taking $u = u_1 = u_2$, $y = \exists_y (u, u) \Rightarrow y \leq d_{\exists y x_i}$. This proves $d_{\exists y x}$ is closed and hence $((\exists_y x), \mu_q)$ is an internal Hausdorff space from Theorem 5. The quotient preneighbourhood morphism $(X, \mu) \xrightarrow{q} ((\exists_y x), \mu_q)$ is the reflection, completing the proof of the theorem.

Remark. The reflection is denoted by $\text{pNbd}[A] \xrightarrow{H} \text{Haus}[A]$ with $H(X) = ((\exists_y x), \mu_q)$ as above. The natural transformation $1_{\text{pNbd}[A]} \xrightarrow{\mu} I \circ H$ is the unit of the reflection (where $\text{Haus}[A] \xrightarrow{I} \text{pNbd}[A]$ is the inclusion functor) and has components $(X, \mu) \xrightarrow{q} ((\exists_y x), \mu_q)$ as above. Further if $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a preneighbourhood morphism then $H(X) \xrightarrow{Hf} H(Y)$ is the unique preneighbourhood morphism which makes the diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{q_X} & H(X) \\
\downarrow f & & \downarrow Hf \\
Y & \xrightarrow{\phi} & H(Y)
\end{array}
$$

(22)

to commute. Since $\text{Haus}[A]$ is a reflective subcategory of $\text{pNbd}[A]$, it has as many (co)limits as does $\text{pNbd}[A]$; since $\text{pNbd}[A]$ is a topological functor, $\text{pNbd}[A]$ has many (co)limits as in $A$. Hence $\text{Haus}[A]$ is as much (co)complete as the category $A$ itself.

6.1. This section shows the kernel pair of the coequaliser of $R \xrightarrow{r_1} X \xrightarrow{r_2} X$ in (19) is the smallest effective equivalence relation on $X$ with internal Hausdorff quotient. The context $A = (A, E, M)$ is a reflecting zero context with finite product projections in $E$.

**Theorem.** Let $(X, \mu) \xrightarrow{f} (Y, \phi)$ be a preneighbourhood morphism with $f$ a regular epimorphism, $\phi$ is the quotient preneighbourhood system on $Y$ and $(Y, \phi)$ is an internal Hausdorff space. Then $\ker f$ is a closed subobject of $X \times X$.

In particular, for the regular epimorphism $X \xrightarrow{q} [X]$ defined in (19), $\ker q$ is the smallest internal effective equivalence relation on $X$ such that its quotient preneighbourhood space is an internal Hausdorff space.

**Proof.** Since $(f_1, f_2)$ is the pullback of $d_Y$ along $f \times f$, if $(Y, \phi)$ is an internal Hausdorff space, using Theorem 5(b), $d_Y$ is a closed morphism and hence $(f_1, f_2)$ is a closed morphism (Proposition 2.3(c)). Let $S$ be the set of all effective equivalence relations on $X$ with quotient an internal Hausdorff space. From Theorem 6 and above, $\ker q \in S$. If $(X, \mu) \xrightarrow{f} (Y, \mu_f)$ be a regular epimorphism and $\ker f \in S$, then from Theorem 6, there exists a unique preneighbourhood morphism $((\exists_y x), \mu_q) \xrightarrow{f} (Y, \mu_f)$ such that $f = f \circ q$, implying in turn $\ker q \leq \ker f$. Hence $\ker q$ is the smallest element of $S$. □

6.2. Let $(X, \mu)$ be an internal preneighbourhood space. In the context of (19), using transfinite induction on ordinals define the transfinite sequence $(q_\alpha : \alpha \in \text{On})$ as follows:

**Step 1.** Take $q_0 = 1_X$; evidently, $\ker q_0 = d_X \leq \ker q$. 
**Step 2.** Assume \( \alpha \in \text{On} \) is a non-limit ordinal, \( \alpha = \beta + 1 \) and for each \( \gamma \leq \beta \), \( q_{\gamma} \) is defined with \( \ker p q_{\gamma} \leq \ker p q \) and \( \gamma \leq \gamma' \leq \beta \Rightarrow \ker p q_{\gamma} \leq \ker p q_{\gamma'} \). Consider the diagram below:

\[
\begin{array}{c}
v_{\beta} \\
\ker p q_{\beta} \\
\ker p q_{\alpha} \\
X \\
\end{array}
\]

\[
\begin{array}{c}
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
q_{\beta,2} \\
q_{\beta,1} \\
q_{\alpha,2} \\
q_{\alpha,1} \\
\end{array}
\]

\[
\begin{array}{c}
k_{\beta,1} \\
k_{\beta,2} \\
t_{\beta,\alpha} \\
\end{array}
\]

\[
\begin{array}{c}
k_{\beta,1} \\
k_{\beta,2} \\
\end{array}
\]

\[
\begin{array}{c}
\ker p \left( y_{\beta} \circ q_{\beta} \right) \\
\ker p \left( y_{\beta} \circ q_{\beta} \right) \\
\ker p \left( y_{\beta} \circ q_{\beta} \right) \\
\end{array}
\]

\[
\begin{array}{c}
D_{\beta} = \beta \\
D_{\beta} = \beta \\
D_{\beta} = \beta \\
\end{array}
\]

\[
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\]

where \( (d_{\beta,1}, d_{\beta,2}) \) is the subobject \( cl_{k_{\beta,1} \times k_{\beta,2}} d_{Y_{\beta}}, y_{\beta} \) is the coequaliser of this pair, the morphisms \( v_{\beta}, v_{\gamma} \) are the unique monomorphisms such that \( q_{\beta,1} = k_{\beta,1} \circ u_{\beta}, y_{\beta} \), \( q_{\beta,1} \circ v_{\beta} = q_{\beta} \circ k_{\beta,1} \) for \( i = 1, 2 \) and \( q_{\alpha} \) is the coequaliser of the pair \( (k_{\beta,1}, k_{\beta,2}) \).

Since \( q_{\alpha} = q_{\beta,1} = q_{\alpha} \circ q_{\beta,2} \), there exist the unique morphisms \( t_{\beta,\alpha} \) and \( s_{\beta,\alpha} \) such that \( q_{\alpha} = t_{\beta,\alpha} \circ q_{\beta} \) and \( q_{\beta,1} = q_{\alpha,1} \circ s_{\beta,\alpha} \) for \( i = 1, 2 \). Hence \( \ker p q_{\beta} \leq \ker p q_{\alpha} \). Furthermore, since \( \ker p q_{\beta} \leq \ker p q \), there exists a morphism \( w_{\beta} \) such that \( q = w_{\beta} \circ q_{\beta} \).

Using the Hausdorff reflection on \( (Y_{\beta}, \mu_{y_{\beta}}) \), it follows that the morphism \( w_{\beta} \) factors through \( y_{\beta} \), since \( (d_{\beta,1}, d_{\beta,2}) \) is smaller than the internal equivalence relation producing the Hausdorff reflection of \( (Y_{\beta}, \mu_{y_{\beta}}) \). Hence, \( q \circ k_{\beta,1} = w_{\beta} \circ q_{\beta} \circ k_{\beta,1} = w_{\beta} \circ q_{\beta} \circ k_{\beta,2} = q \circ k_{\beta,2} \) produces from the coequaliser \( q_{\alpha} \) the unique morphism \( w_{\alpha} \) such that \( q = w_{\alpha} \circ q_{\alpha} \), i.e., \( \ker p q_{\alpha} \leq \ker p q \).

**Step 3.** Assume \( \alpha \in \text{On} \) is a limit ordinal and for each \( \beta < \alpha \), \( q_{\beta} \) is defined with \( \ker p q_{\beta} \leq \ker p q \) and \( \gamma \leq \gamma' < \alpha \Rightarrow \ker p q_{\gamma} \leq \ker p q_{\gamma'} \). Consider the diagram below:

\[
\begin{array}{c}
u_{\beta} \\
\ker p q_{\beta} \\
\ker p q_{\alpha} \\
X \\
\end{array}
\]

\[
\begin{array}{c}
k_{\beta,1} \\
k_{\beta,2} \\
q_{\beta,1} \\
q_{\beta,2} \\
q_{\alpha,2} \\
q_{\alpha,1} \\
\end{array}
\]

\[
\begin{array}{c}
k_{\beta,1} \\
k_{\beta,2} \\
t_{\beta,\alpha} \\
\end{array}
\]

\[
\begin{array}{c}
k_{\beta,1} \\
k_{\beta,2} \\
\end{array}
\]

\[
\begin{array}{c}
\ker p q_{\beta} \\
\ker p q_{\beta} \\
\ker p q_{\alpha} \\
\end{array}
\]

\[
\begin{array}{c}
\ker p q_{\beta} \\
\ker p q_{\beta} \\
\ker p q_{\alpha} \\
\end{array}
\]

\[
\begin{array}{c}
Y_{\alpha} \\
Y_{\beta} \\
\end{array}
\]

where the admissible subobject \( K_{\alpha} = \bigvee_{\beta < \alpha} \ker p q_{\beta} \) is given by the pair \( (k_{\alpha,1}, k_{\alpha,2}) \), \( q_{\alpha} \) is the coequaliser of this pair and since \( \ker p q_{\beta} \leq K_{\alpha} \), there is the morphism \( u_{\beta} \) such that \( q_{\beta,1} = k_{\beta,1} \circ u_{\beta} \). Hence \( q_{\alpha} \circ q_{\beta,1} = q_{\alpha} \circ q_{\beta,2} \) implying the existence of the unique morphisms \( s_{\beta,\alpha} \) and \( t_{\beta,\alpha} \) such that \( q_{\alpha} = t_{\beta,\alpha} \circ q_{\beta} \) and \( q_{\beta,1} = q_{\alpha,1} \circ s_{\beta,\alpha} \) for \( i = 1, 2 \). Since for each \( \beta < \alpha \), \( \ker p q_{\beta} \leq \ker p q \), \( K_{\alpha} \leq \ker p q_{\beta} \), implying \( q_{\alpha} \circ k_{\alpha,1} = q_{\alpha} \circ k_{\alpha,2} \), i.e., \( \ker p q_{\alpha} \leq \ker p q \).

The procedure defines the sequence \( \left( q_{\alpha} : \alpha \in \text{On} \right) \) of regular epimorphisms such that \( \ker p q_{\alpha} \leq \ker p q_{\beta} \leq \ker p q \) for each \( 0 \leq \gamma \leq \beta \) in \( \text{On} \). In **Step 2.**, \( y_{\beta} \) is an isomorphism if and only if \( D_{\beta} = d_{Y_{\beta}}, \) i.e., using Theorem 5(b), \( (Y_{\beta}, \mu_{y_{\beta}}) \) is an internal Hausdorff space, in which case \( q_{\beta} = q_{\beta+1} \). On the other hand, if \( q_{\beta} = q_{\beta+1} \) then \( \ker p (y_{\beta} \circ q_{\beta}) = \ker p q_{\beta} \) and hence \( y_{\beta} \) is an isomorphism, implying \( (Y_{\beta}, \mu_{y_{\beta}}) \) is an internal Hausdorff space. Thus the transfinite sequence is eventually constant if and only if from some stage \( Y_{\beta} \) is an internal Hausdorff space.

**Theorem.** If each \( \text{Sub}_{\text{M}}(X) \) is a small set then there exists a \( \beta \in \text{On} \) such that \( q = q_{\beta} \).

**Proof.** Since \( \text{Sub}_{\text{M}}(X \times X) \) is a small set, there exists \( \beta, \gamma \in \text{On} \) such that \( \beta < \gamma \) and \( \ker p q_{\beta} = \ker p q_{\gamma} \). Hence \( \ker p q_{\beta+1} = \ker p q_{\beta} \) implying \( q_{\beta} = q_{\beta+1} \), i.e., \( (Y_{\beta}, \mu_{y_{\beta}}) \) is an internal Hausdorff space. Let \( \beta \) be the smallest such; consequently, \( \ker p q \leq \ker p q_{\beta} \leq \ker p q \) — the first inequality ensured by Theorem 6.1, completing the proof. \( \square \)

**Remark.** In any admissibly well-powered (i.e., when each lattice of admissible subobjects is a small set) reflecting zero context \( \mathcal{A} \) with finite product projections in \( E \), the Hausdorff reflection can be constructed from the diagonal using transfinite induction, on taking finer and finer quotients. This construction is well known for the context \( \text{Set} \), \( \text{Surjections, Injections} \) with its internal neighbourhood spaces (see [11] for details).
The present paper provides two fold generalisation of the construction — firstly to an arbitrary context as stipulated above, and secondly for arbitrary internal preneighbourhood spaces, which are not even internal neighbourhood spaces. In particular, there is now a construction of the Hausdorff reflection of locales using the $T$-neighbourhood systems.

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