Quantum Coherent Feedback Control With Photons

Haijin Ding and Guofeng Zhang, Member, IEEE

Abstract—The purpose of this article is to study two-photon dynamics induced by the coherent feedback control of a cavity quantum electrodynamics (cavity-QED) system coupled to a waveguide. In this setup, the two-level system in the cavity can work as a photon source, and the photon emitted into the waveguide can reinteract with the cavity-QED system many times after the transmission and reflection in the waveguide, during which the feedback can tune the number of the photons in and out of the cavity. We analyze the dynamics of two-photon processes in this coherent feedback network in two scenarios: the continuous mode coupling scheme and the discrete periodic mode coupling scheme between the waveguide and cavity. The difference of these coupling schemes is due to their relative scales and the number of semitransparent mirrors for coupling. Specifically, in the continuous mode coupling scheme, the generation of two-photon states is influenced by the length of the feedback loop by the waveguide and the coupling strength between the waveguide and the cavity-QED system. By tuning the length of the waveguide and the coupling strength, we are able to generate two-photon states efficiently. In the discrete periodic mode coupling scheme, the Rabi oscillation in the cavity can be stabilized and there are no notable two-photon states in the waveguide.

Index Terms—Cavity–waveguide interaction, photon feedback, quantum coherent feedback control.

I. INTRODUCTION

In recent years quantum feedback control has attracted much attention due to its wide applications in quantum information processing (QIP), such as quantum error correction [1], [2], [3], [4], quantum optical fields amplification [5], [6], stabilization of Rabi oscillation [7], robust stabilization of quantum states [8], [9], [10], entanglement generation [11], [12], and so on. Depending on whether or not the quantum state is measured to yield classical information for feedback control design, quantum feedback control can be divided into two categories: measurement feedback control and coherent feedback control [13], [14]. Measurement feedback control has been widely used in the generation of quantum states and error corrections in quantum computation. For example, the three-qubit “bit-flip” method is the simplest method for correcting error bits [1], [2]. In measurement feedback control, measuring the quantum state induces external disturbances and normally alters quantum dynamics [15], [16], [17], [18], [19], [20], [21], [22]. On the other hand, in a coherent feedback control network [5], [14], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], no measurement is involved, and thus the quantum coherence can be preserved during the time evolution of the quantum network.

In a coherent feedback network, the quantized field (e.g., a photon) is scattered from the quantum system (e.g., an atom or atomic ensemble), then it can be redirected to the quantum system to realize the desired evolution [33], [43], [47], [48]. For example, as illustrated in [49], an atom emits a photon into a semi-infinite waveguide, the emitted photon is later reflected by the end of the waveguide and interacts with the atom again. Coherent feedback has been realized on various experimental platforms, such as superconducting circuits, optical systems, nitrogen-vacancy (NV) center in diamond, and so on. For example, in a superconducting circuit, coherent feedback can be used to stabilize the dynamics of the quantum state [7] and implement a bistable state [50]. In an optical system, coherent feedback can enhance the squeezing of the optical beam [51], [52]. For quantum computation based on trapped ions, the real-time coherent feedback control on the target ion can be designed by measuring another correlated ion, and the control efficiency and coherence of the measurement-free target qubit can be enhanced [44]. Coherent feedback has also been realized in NV centers [53], atomic spins [43], and cavity-assisted cooling of mechanical oscillators [45], among others.

Compared with feedforward interactions where the emitted coherent field or the measurement record is not fed back to the system via a feedback channel, the dynamics of a coherent feedback network is more complex to analyze because of the delay induced by the feedback loop. Different from traditional classical feedback loops, the analysis of the delay in a quantum coherent feedback network is often inadequate if we only consider the wave velocity and the length of the round trip. As already analyzed in [54], [55], the existence of the delay not only induces phase shifts among different nodes, but also

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Haijin Ding is with the Shenzhen Research Institute, Hong Kong Polytechnic University, Shenzhen 518057, China, and also with the Laboratoire des Signaux et Systèmes (L2S), CNRS-CentraleSupélec-Université Paris-Sud, Université Paris-Saclay, 91190 Gif-sur-Yvette, France (e-mail: dhj17@tsinghua.org.cn).

Guofeng Zhang is with the Shenzhen Research Institute, Hong Kong Polytechnic University, Shenzhen 518057, China, and also with the Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong (e-mail: dhj17@tsinghua.org.cn).

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influences the coherence properties, such as correlations by modulating the evolution of quantum states. Hence, delay can work as a control mechanism. For example, when an atomic system is coupled with a waveguide, the evolution of the atomic population is affected by the round trip propagation time of the photon in the waveguide [56]. When there are two atoms coupled with the waveguide, which is ended with a totally reflecting mirror on one side (namely, a semi-infinite waveguide), the dynamics of the two-atom system is affected by the locations of the atoms in the waveguide [57]; a similar study can be found in [58]. The steady-state output two-photon state of a coherent feedback network composed of two atoms and driven by two input photons is derived in [59]. In addition to loop delay, coupling strength also influences the interaction and exchange of photons among components in a quantum network, and thus affects the performance of the coherent feedback network.

The analysis of coherent feedback with photons becomes even more complicated as the number of excitations increases. Practically, a multiphoton state can be generated with a multi-level atom [60], [61], [62], a single two-level atom that is repeatedly driven [63], or multiple two-level atoms [64]. Take the generation and absorption of a two-photon state as an example [63], [65], [66], [67], which is a generalization of the spontaneous emission of single photons [68], [69], [70], the two-photon interference varies according to the level structure of atoms [62], [71]. As theoretically analyzed and experimentally demonstrated in [72] on a quantum dot platform, which can be generalized to other platforms, such as ion traps [73] and quantum circuits [74], the number of emitted photons from a two-level atom is determined by the width of the driving pulse and the two-photon state can be generated by re-exciting the atom right after the first photonic wavepacket is emitted.

In the theoretical and experimental results abovementioned, waveguide has been commonly used to enable coherent feedback control due to its advantages in the transportation of quantum states [75] and the preservation of quantum coherence [76], [77]. For instance, the emitted photon from a cavity coupled with a two-level atom (cavity-QED) can be perfectly reflected by the end of the waveguide and fed back to the cavity. If the loop delay is properly designed, the photon is able to stabilize the Rabi oscillation in the cavity [49]. Generally speaking, photons transported in the waveguide and reflected by the terminal of the waveguide affect the evolution of the quantum system coupled to the waveguide. The length of the waveguide and the location of the quantum system determine the delay and phase shift induced by the feedback loop, thus regulating the evolution and the steady state of the quantum system. Besides, the effects of coherent feedback are different depending on whether the waveguide and the cavity are coupled through a continuous-mode scheme or a discrete-mode scheme [49],[56]. Specifically, different coherent feedback control mechanisms can be designed according to the size of the waveguide and the coupling methods and strengths to fulfil the required control performance, e.g., to maintain the Rabi oscillation [49], generate required atomic or photonic states [53], [78], or produce entangled quantum states [12].

In most studies of quantum coherent feedback control, it is always assumed that signals’ propagation time over the network is small and thus the effect of time delay on the control performance is ignored or treated as a constant phase shift, see for example [79, Section II-B], [23, Fig. 15], [80, Section VI], and the survey papers [13], [29], [46]. Thus, the system under study is essentially Markovian. If propagation time is treated in a more rigorous way, the system dynamics is non-Markovian and is conceivably more complex. For example, when two or more atoms are coupled with an infinite waveguide [59], [81], photons can bounce back and forth between two atoms, accordingly the system dynamics is complicated by the round-trip transmission delay of the photons between two atoms. Moreover, if a coherent feedback network is closed by a semi-infinite waveguide, photons reflected by the terminal mirror of the waveguide can reinteract with the atom, thus providing another feedback mechanism besides that induced by photons bouncing between two atoms. For this type of coherent feedback networks where a few atoms interacting with a few photons and the whole loop is closed by a semi-infinite waveguide, due to system complexity most existing studies focus on the single-excitation case, see for example [49],[56],[57]. In this article, we study the two-excitation case. Specifically, we study how to use coherent feedback to control the atomic evolution and the generation of two-photon states in the coherent feedback network shown in Fig. 1, where the Jaynes–Cummings system [82] is coupled to a waveguide. Based on the derivation of the relationship between quantum state evolution and the feedback loop length, we propose the optimal parameter design in the continuous coupling scheme to generate two photon states, and illustrate why the two photons cannot be simultaneously observed in the waveguide in the discrete coupling scheme.

The rest of this article is organized as follows. Section II concentrates on the feedback interaction when the waveguide and the cavity are coupled with continuous modes, especially on
the control performance, such as the two-photon generation and entanglement influenced by parameter design. The circumstance of the coupling with periodic discrete modes is explored in Section III, which is much different from the continuous coupling scheme studied in Section II. Finally, Section IV concludes this article.

II. COHERENT FEEDBACK WITH THE WAVEGUIDE VIA CONTINUOUS COUPLED MODES

We study the quantum coherent feedback network, as shown in Fig. 1. In (a), the cavity of length $l$ is coupled with a waveguide of the feedback loop length $2L$ through two semitransparent mirrors located at $z = 0$, as shown by the blue bar. An arbitrary mode in the waveguide can interact with the cavity that contains a two-level atom (represented with the two-way blue arrows). When $L \gg l$, the cavity and the waveguide are coupled with all the continuous modes of the waveguide [56]. While if the cavity can be only coupled with the reservoir confined at the area $z < 0$ through the mirror at $z = 0$, as shown in Fig. 1(b), then the waveguide functions similarly as another cavity. When $L$ and $l$ are comparable, the waveguide and cavity are coupled with discrete modes [83], [84], [85], [86].

In this section, we consider the circumstance that $L > l$ and arbitrary waveguide modes can be coupled with the atomic system independent of the feedback loop length. The Hamiltonian of the whole quantum system reads

$$H = H_A + H_I + \hbar \omega_c a^\dagger a + \int \omega_k d_k^\dagger d_k dk$$

(1)

where, $H_A = \frac{\gamma}{2} \omega_c \langle e | e | \rangle - \langle g | g \rangle$ is the Hamiltonian of the two-level atom with the atomic transition frequency $\omega_c$. $H_I$ represents the interaction Hamiltonian between the atom and the cavity as well as that between the cavity and the waveguide. The last two terms of $H$ are the quantized fields in the single-mode cavity and waveguide, respectively, where $\omega_c$ is the resonant frequency of the cavity, $\omega_k = ck$ is the frequency of the waveguide mode $k$ with $c$ being the velocity of the coherent fields in the waveguide. $a(a^\dagger)$ and $d_k(d_k^\dagger)$ are the annihilation(creation) operators of the cavity and waveguide modes, respectively. In this article, it is assumed that the atom is resonant with the cavity mode; i.e., $\omega_c = \omega_\gamma$.

Assumption 1: Initially, the atom is in the excited state and there is one photon in the cavity.

The quantum state of the whole system is of the form

$$\Psi(t) = c_e(t) |e, 1, \{0\}\rangle + \int_0^\infty c_{ek}(t)|e, 0, \{k\}\rangle dk$$

$$+ c_g(t) |g, 2, \{0\}\rangle + \int_0^\infty c_{gk}(t)|g, 1, \{k\}\rangle dk$$

$$+ \int_0^\infty \int_0^\infty c_{gkk}(t,k_1,k_2)|g, 0, \{k_1\} \{k_2\}\rangle dk_1 dk_2.$$  (2)

The meaning of each component on the right-hand side of (2) is as follows.

1) $|e, 1, \{0\}\rangle$: the atom is in the excited state $|e\rangle$ and there is one photon in the cavity.

2) $|e, 0, \{k\}\rangle$: the atom is in the excited state and there is one photon in the waveguide with the mode $k$.

3) $|g, 2, \{0\}\rangle$: the atom is in the ground state $|g\rangle$ and there are two photons in the cavity.

4) $|g, 1, \{k\}\rangle$: the atom is in the ground state, one photon is in the cavity and the other is in the waveguide with the mode $k$.

5) $|g, 0, \{k_1\} \{k_2\}\rangle$: the atom is in the ground state and there are two photons in the waveguide with modes $k_1$ and $k_2$, respectively.

Accordingly, the coefficients (probability amplitudes) $c_e(t)$, $c_g(t)$, $c_{ek}(t)$, $c_{gk}(t)$, $c_{gkk}(t,k_1,k_2)$ represent the time-varying amplitudes of these five states of the quantum system. According to Assumption 1, the initial state of the system is $|\Psi(0)\rangle = |e, 1, \{0\}\rangle$. Thus, the initial amplitudes are $c_e(0) = 1$, and $c_{ek}(0) = c_{gk}(0) = c_{gkk}(0) = 0$.

The evolution of the quantum system is governed by the Schrödinger equation

$$\frac{\partial}{\partial t} |\Psi(t)\rangle = -iH |\Psi(t)\rangle$$

where, the reduced Planck constant $\hbar$ is set to be 1 throughout this article. As given in [49], the interaction Hamiltonian $H_I$ in (1) reads

$$H_I = -\gamma (\sigma^a a^\dagger + \sigma^+ a)$$

$$- \int_0^\infty dk [G(k,t)a^\dagger d_k + G^*(k,t)ad_k^\dagger]$$

(3)

where, $\sigma^- = |g\rangle \langle e|$ and $\sigma^+ = |e\rangle \langle g|$ are the lowering and raising operators of the atom, respectively, $\gamma$ is the coupling strength between the atom and the cavity, $G(k,t) = G_0 \sin(kL)e^{-i(\omega - \Delta_0)t}$ describes the coupling between the cavity and the waveguide of the mode $k = \frac{2\pi}{\lambda}$, where $G_0$ is the magnitude and $\Delta_0 = \omega_\gamma$ is the central mode of the emitted photon [56]. The theoretical analysis to be conducted in this section is based on the following assumptions which can be fulfilled, as has been discussed in [72].

Assumption 2: The time evolution of the populations are continuous.

Assumption 3: The photon statistics of infinitely large modes is zero; in other words, $\lim_{k_1,k_2 \rightarrow \infty} \langle |c_{gkk}(t,k_1,k_2)|^2 \rangle = 0$.

Simply speaking, Assumption 2 is due to the fact that the Schrödinger equation generates the unitary evolution of a quantum system and hence its state evolution is continuous in time, while Assumption 3 is due to the fact that atomic emissions produce Lorentzian photonic pulses. Both Assumptions 2 and 3 are used in the proof of Lemma 4.

Substituting (2) into the Schrödinger equation yields a system of integro-differential equations

$$c_e(t) = i\sqrt{2} \gamma c_g(t) + i \int_0^\infty c_{ek}(t,k)G(k,t) dk$$

$$c_{ek}(t,k) = \iota c_e(t)G^*(k,t) + i \gamma c_{gk}(t,k)$$

$$c_g(t) = i\sqrt{2} \gamma c_e(t) + i \sqrt{2} \int_0^\infty c_{gk}(t,k)G(k,t) dk$$

$$c_{gk}(t,k) = \iota c_e(t,k) + i \sqrt{2} \gamma c_{gk}(t,k)G^*(k,t)$$

$$+ i \int_0^\infty G(k_2,t)c_{gkk}(t,k_2) dk_2$$

$$+ i \int_0^\infty G(k_1,t)c_{gkk}(t,k_1,k_2) dk_1$$

$$c_{gkk}(t,k_1,k_2) = \iota c_{gk}(t,k_2)G^*(k_1,t)$$

$$+ i \gamma c_{gk}(t,k_1)G^*(k_2,t)$$

(4a) - (4e)

where, the time evolutions of the amplitudes are continuous. Here, (4a) indicates that the state $|e, 1, \{0\}\rangle$ can be acquired
in two ways: 1) the atom absorbs one photon from the cavity, which contains two photons, or 2) the atom is initially excited in an empty cavity and the cavity can further absorb one photon from the waveguide. Equation (4b) indicates that: 1) the cavity can emit one photon into the waveguide when the atom is in the excited state and there is one photon in the cavity, or 2) the atom can emit one photon into the cavity. Equation (4c) represents two processes: the spontaneous emission of the excited atom and the absorption of one photon by the cavity from the waveguide. Equation (4d) shows the exchange of a photon between the waveguide and the cavity. Finally, (4e) means that the waveguide initially having a photon can absorb another one from the cavity to generate a two-photon state.

Generalizing the single-photon feedback scheme studied in [56], we are able to derive the control equation for the amplitude of the eigenstate \(|e, 1, \{k\}\) for \(t \geq 0\) as follows:

\[
\dot{c}_e(t) = i\sqrt{2}\gamma_c g(t) - \frac{G^2_0\pi}{2c}(c_e(t) - e^{i\Delta_0}\tau c_e(t - \tau))
\]

where \(\Theta\) represents the Heaviside step function and \(\tau = \frac{2\pi}{\omega}\) is the delay induced by the coherent feedback loop. Similarly, the amplitude of the eigenstate \(|g, 1, \{k\}\) is

\[
c_{gk}(t, k) = i\sqrt{2}\int_0^t c_{gk}(\nu)G^*(k, \nu)d\nu
\]

\[
- \gamma \int_0^t \int_0^\nu c_{e}(\nu)G^*(k, \nu)d\nu d\nu
\]

\[
- \gamma^2 \int_0^t \int_0^\nu c_{gk}(\nu, k)d\nu d\nu
\]

\[
- \frac{G^2_0\pi}{c} \int_0^t |c_{gk}(\nu, k) - c_{gk}(\nu - t, k)e^{i\Delta_0}\tau|d\nu
\]

and that for \(|g, 2, \{0\}\) is

\[
\dot{c}_g(t) = i\sqrt{2}\gamma_c g(t) - i\sqrt{2}\gamma c \frac{G^2_0\pi}{2c} c_e(t - \tau)e^{i\Delta_0}\tau\Theta(t - \tau)
\]

\[
- \frac{G^2_0\pi}{c} (c_g(t) - e^{i\Delta_0}\tau c_g(t - \tau)\Theta(t - \tau)).
\]

The derivation of (5)-(7) is given in Appendix A.

Let \(\kappa = \frac{G^2_0\pi}{2c}\) denote the coupling strength between the cavity and the waveguide. By means of (5)-(7), the control equations in (4) become

\[
\begin{align*}
\dot{c}_e(t) &= i\sqrt{2}\gamma_c g(t) - \kappa c_e(t) + \kappa e^{i\Delta_0}\tau c_e(t - \tau)\Theta(t - \tau) \\
\dot{c}_{ek}(t, k) &= ic_{ek}(t)G^*(k, t) + \gamma_c c_{gk}(t, k) \\
\dot{c}_g(t) &= i\sqrt{2}\gamma_c c_e(t) - \kappa c_g(t) + \kappa e^{i\Delta_0}\tau c_g(t - \tau)\Theta(t - \tau) \\
&- i\sqrt{2}\gamma_c \kappa c_e(t - \tau)\Theta(t - \tau)e^{i\Delta_0}\tau \\
\dot{c}_{gk}(t, k) &= -2\kappa c_{gk}(t, k) + 2\kappa c_{gk}(t - \tau, k)\Theta(t - \tau)e^{i\Delta_0}\tau \\
&+ i\sqrt{2}\gamma_c c_g(t)G^*(k, t) - \gamma \int_0^t c_{ek}(t)G^*(k, t)dt \\
&- \int_0^t \gamma_c^2 c_{gk}(t, k)dt \\
\dot{c}_{gkk}(t, k_1, k_2) &= i\gamma c_{ek}(t, k_2)G^*(k_1, t) \\
&+ i\gamma c_{gk}(t, k_1)G^*(k_2, t).
\end{align*}
\]

Laplace transforming the amplitudes \(c_e(t)\) in (8a) and \(c_g(t)\) in (8c) to get their frequency-domain counterparts \(C_e(s)\) and \(C_g(s)\) of the form

\[
C_e(s) = \frac{s + \kappa(1 - e^{i\Delta_0}\tau e^{-s\tau})}{(s + \kappa(1 - e^{i\Delta_0}\tau e^{-s\tau}))^2 + 2\gamma^2(1 - \kappa e^{i\Delta_0}\tau e^{-s\tau})}
\]

and

\[
C_g(s) = \frac{i\sqrt{2}\gamma(1 - \kappa e^{i\Delta_0}\tau e^{-s\tau})}{(s + \kappa(1 - e^{i\Delta_0}\tau e^{-s\tau}))^2 + 2\gamma^2(1 - \kappa e^{i\Delta_0}\tau e^{-s\tau})}
\]

respectively.

In the following sections, three different scenarios categorized by the length of the feedback loop are studied. In Section II-A, the length of the feedback loop is close to zero. In this case, the quantum state evolves within the transmission time of a single round trip in the feedback loop, after that the two photons are both emitted into the waveguide. Finally, in Section II-D, two-photon entanglement is analyzed.

### A. Feedback Control With a Waveguide of Small Length

When the length of the waveguide \(L\) is small, the induced delay \(\tau = \frac{2\pi}{\omega}\approx 1\). In this case, \(e^{-s\tau} \approx 1\) and

\[
\kappa(1 - e^{i\Delta_0}\tau e^{-s\tau}) \approx \kappa(1 - e^{i\Delta_0}).
\]

Consequently, from (9) and (10) we get

\[
C_e(s) \approx \frac{s + \kappa(1 - e^{i\Delta_0})}{(s + \kappa(1 - e^{i\Delta_0}))^2 + 2\gamma^2}
\]

\[
C_g(s) \approx \frac{i\sqrt{2}\gamma(1 - \kappa e^{i\Delta_0})}{(s + \kappa(1 - e^{i\Delta_0}))^2 + 2\gamma^2}
\]

Equations (12), (13), and (8d) yield

\[
\dot{c}_{gk}(t, k) \approx -2\kappa[\dot{c}_{gk}(t, k) - \dot{c}_g(t - \tau, k)e^{i\Delta_0}\tau] - \gamma^2 c_{gk}(t, k) + G_0\sin(kL)\{\{D - \frac{3\gamma^2}{2}\}e^{iE + i(\omega - \Delta_0 + F + \sqrt{2}\gamma)t} \\
&- (D + \frac{3\gamma^2}{2})e^{iE + i(\omega - \Delta_0 + F + \sqrt{2}\gamma)t}\}
\]

where

\[
D = \sqrt{2}\frac{|\Delta_0 - \omega - \sin(\Delta_0\tau) + i\kappa(\cos(\Delta_0\tau) - 1)|}{\gamma^2}
\]

\[E = \kappa(\cos(\Delta_0\tau) - 1)
\]

\[F = \sin(\Delta_0\tau).
\]

Denote

\[
R = \kappa(e^{i\Delta_0\tau} - 1).
\]

Then the Laplace transform of \(c_{gk}(t, k)\) w.r.t. \(t\) gives the frequency-domain function \(C_{gk}(s, k)\) of the from

\[
C_{gk}(s, k) = -3G_0\sin(kL)
\]
where, the parameters \( H(\omega) \), \( I(\omega) \), \( J(\omega) \), and \( K(\omega) \) are given in Appendix B.

Applying the inverse Laplace transform to \( C_c(s), C_g(s) \), and \( C_{gk}(s, k) \) obtained previously gives \( c_c(t), c_g(t) \), and \( c_{gk}(t, k) \) in the time domain, which are

\[
\begin{align*}
&c_c(t) = e^{-\kappa t} e^{i(\omega t - \Delta_0 t)} \cos(\sqrt{2} \gamma t) \\
&c_g(t) = i e^{-\kappa t} e^{i(\omega t - \Delta_0 t)} \sin(\sqrt{2} \gamma t) \\
&c_{gk}(t, k) \approx -3G_0^2 \sin(kL) \sinh(\Delta_0 t) \\
&\quad \times \left\{ H(\omega)e^{-\gamma t} + I(\omega)e^{\gamma t} \right\} \\
&\quad \times \sqrt{\frac{H(\omega)}{s - R + \sqrt{R^2 - \gamma^2}}} + \sqrt{\frac{I(\omega)}{s - R - \sqrt{R^2 - \gamma^2}}} \\
&\quad \times J(\omega) \left( s - E - i(\omega - \Delta_0 + F) + i\sqrt{2} \gamma \right) \\
&\quad \times K(\omega) \left( s - E - i(\omega - \Delta_0 + F) - i\sqrt{2} \gamma \right)
\end{align*}
\]

respectively.

Before presenting the main result in this section, we state the following lemma first.

**Lemma 1:** When \( \kappa \ll 1 \), for the mode \( k \) with frequency \( \omega = \kappa c \in [0, 2\Delta_0] \), we have \( H(\omega) = I(2\Delta_0 - \omega)^* \), and \( J(\omega) = K(2\Delta_0 - \omega)^* \), where \( * \) represents the complex conjugate.

**Proof:** See Appendix B. \( \square \)

The following is the main result of this section.

**Theorem 1:** When \( \kappa \ll 1 \) and \( k_1 + k_2 = \frac{2\Delta_0}{c} \), the amplitude \( c_{gk} \) of the two-photon state in the waveguide is purely imaginary.

**Proof:** Consider \( \hat{c}_{gk}(t, k_1, k_2) \) in (4e) and denote \( \omega_1 = k_1 c \) and \( \omega_2 = k_2 c \). When \( \kappa \ll 1 \), we have \( F \approx 0 \) in (15). Hence

\[
\begin{align*}
&i c_{gk}(t, k_2) G^*(k_1, t) \\
&= -3iG_0^2 \sin(k_2 L) \sin(k_1 L) \gamma \left\{ H(\omega_2)e^{i(\omega_1 - \Delta_0 - \gamma)t} \\
&\quad + I(\omega_2)e^{i(\omega_1 + \Delta_0 + \gamma)t} + J(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + F) + i\sqrt{2} \gamma t} \right\} \\
&\quad + K(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)} \\
&\approx -3iG_0^2 \sin(k_2 L) \sin(k_1 L) \gamma \left\{ H(\omega_2)e^{i(\omega_1 - \Delta_0 - \gamma)t} \\
&\quad + I(\omega_2)e^{i(\omega_1 + \Delta_0 + \gamma)t} + J(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)} \right\} \\
&\quad + K(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)}
\end{align*}
\]

When \( \omega_1 + \omega_2 = 2\Delta_0 \),

\[
\begin{align*}
&i c_{gk}(t, k_2) G^*(k_1, t) \\
&\approx -3iG_0^2 \sin(k_2 L) \sin(k_1 L) \gamma \left\{ H(\omega_2)e^{i(\omega_1 - \Delta_0 - \gamma)t} \\
&\quad + I(\omega_2)e^{i(\omega_1 + \Delta_0 + \gamma)t} + J(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)} \right\} \\
&\quad + K(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)}
\end{align*}
\]

According to Lemma 1,

\[
\begin{align*}
H(\omega_2) &= I(\omega_1)^* \\
\omega_1 - \Delta_0 - \gamma &= -(\omega_2 - \Delta_0 + \gamma) \\
J(\omega_2) &= K(\omega_1)^*
\end{align*}
\]

Therefore

\[
\begin{align*}
H(\omega_2)e^{i(\omega_1 - \Delta_0 - \gamma)t} + I(\omega_1)e^{i(\omega_2 - \Delta_0 + \gamma)t} \\
&= 2\Re(H(\omega_2)e^{i(\omega_1 - \Delta_0 - \gamma)t}) \\
H(\omega_1)e^{i(\omega_1 - \Delta_0 - \gamma)t} + I(\omega_2)e^{i(\omega_1 + \Delta_0 + \gamma)t} \\
&= 2\Re(H(\omega_1)e^{i(\omega_2 - \Delta_0 - \gamma)t}) \\
J(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)} + K(\omega_1)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)} \\
&= 2\Re(K(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)}) \\
K(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)} + J(\omega_1)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)} \\
&= 2\Re(K(\omega_2)e^{i(\omega_1 + \omega_2 - 2\Delta_0 + i\sqrt{2} \gamma t)})
\end{align*}
\]

which is a purely imaginary number. Because the initial condition is \( c_{gk}(0, k_1, k_2) = 0 \) as given in Assumption 1, \( c_{gk}(t, k_1, k_2) \) is also a purely imaginary number for all \( t \geq 0 \).

The numerical simulations are shown in Fig. 2, where \( k_1, k_2 \in [0, 100] \), \( \Delta_0 = 50 \), and \( \gamma = 2k_s \). As the length of the waveguide \( L \) as well as the coupling between the waveguide and the cavity \( \kappa \) [defined before (8)] is small, it can be seen from Fig. 2(a) that the atom oscillates between its excited state and ground state, where \( |c_0(t)|^2 \) and \( |c_D(t)|^2 \) are the numerical simulations with (8) and the dash-dot lines represent the fitting results of the populations with (12)–(13) based on the approximation in (11). Therefore, the populations of the single-photon state in Fig. 2(b) and the generated two-photon state in Fig. 2(e)–(h) are much smaller than the oscillation amplitude in Fig. 2(a). Fig. 2(c) and (d) shows the real and imaginary parts of \( c_{gk} \) when \( t = 80 \tau \), which agree with the conclusion in Theorem 1.

**Remark 1:** Recall that \( \gamma \) is the coupling strength between the atom and the cavity. If the atom is initialized in the excited state, a large \( \gamma \) will enhance the efficiency of spontaneously emitting a photon into the cavity. This, together with a large \( G_0 \), will further induce a big probability of detecting two photons in the...
waveguide. This agrees with (22), which says that large $\gamma$ and $G_0$ induce large amplitudes of $\dot{g}_{kk}$.

Remark 2: As $\Delta_0$ is the central mode of the continuous-mode photonic field, which should be theoretically much larger than the scale of $\tau$. In the numerical simulations, $\Delta_0 = 50$, which is large enough to ensure that $\Delta_0\tau \gg \tau$. As a result, the approximation in (11) is much easier to be satisfied than that in Lemma 1. In addition, a large range of $\Delta_0\tau$ provides an approach to controlling the feedback dynamics by tuning the length of the waveguide to get the suitable $\tau$.

### B. Feedback Control With a Waveguide of Finite Length

In the case of coherent feedback control via a waveguide of finite length, we have the following result.

Theorem 2: When $\Delta_0\tau = n\pi$ and $\tau \ll 1$, $n = 1, 2, \cdots$, the two-photon amplitude $c_{gkk}$ is purely imaginary provided that $\omega_1 + \omega_2 = 2\Delta_0$.

Proof: 1) When $n$ is an even number, $\cos(\Delta_0\pi) = 1$ and $\sin(\Delta_0\pi) = 0$. The calculation is the same as the proof of Theorem 1.

2) When $n$ is an odd number, $\cos(\Delta_0\pi) = -1$, and $\sin(\Delta_0\pi) = 0$. By (15) and (16), $R = k(\cos\Delta_0\tau - 1) = -2k\tau$, $E = k(\cos\Delta_0\tau - 1) = -2k\tau$, $F = \sin\Delta_0\tau = 0$, and $D = \frac{\sqrt{2}}{\tau}[\Delta_0 - \omega - \sin(\Delta_0\tau)] = \frac{\Delta_0}{\tau}$. Denote $M = \omega - \Delta_0 + F$. Then

$$C_{gkk}(s, k) = \frac{G_0\sin(kL)}{(s^2 - 2Rs + \gamma^2)} + \frac{H(\omega)}{s - R + \sqrt{R^2 - \gamma^2}} + \frac{I(\omega)}{s - R - \sqrt{R^2 - \gamma^2}}$$

Moreover, it is easy to show that (59) still holds. As a result, by (22) in the proof of Theorem 1, $c_{gkk}$ is purely imaginary.

Physically, Theorem 2 means that when $\Delta_0\tau = n\pi$ and $\omega_1 + \omega_2 = 2\Delta_0$, the amplitude of the two-photon state component of the whole state $\Psi(t)$ in (2) is a pure phase.

Theorem 3: When $\Delta_0\tau = 2n\pi$, $n = 1, 2, \cdots$, the single-photon state $|g, 1, \{k\}\rangle$ oscillates and does not decay to zero. When $\Delta_0\tau \neq 2n\pi$, eventually there are two photons in the waveguide.

Proof:

1) When $\Delta_0\tau = 2n\pi$, $E = \kappa(\cos\Delta_0\tau - 1) = 0$, and $R = \kappa(\sin\Delta_0\tau - 1) = 0$. In this case, by (17), the amplitude of the single-photon state $|g, 1, \{k\}\rangle$ is

$$c_{gk}(t, k) \approx -3G_0\sin(kL)\gamma[H(k)e^{-\gamma t} + I(\omega) + J(\omega)e^{iT}\{I(\omega) + F(t) - \gamma T\}] + K(k)e^{[I(\omega) - \gamma T]}.$$ (24)

Clearly, the four components of $c_{gk}(t, k)$ oscillate and

$$\lim_{t \to \infty} c_{gk}(t, k) \neq 0.$$ (25)

Moreover, by (4), $\lim_{t \to \infty} c_{g}(t, k) = \lim_{t \to \infty} \dot{c}_{gkk}(t, k) = 0$. Consequently, the dominant population of the steady state of the quantum system is $|g_{kk}|^2$.

Remark 3: When $\Delta_0\tau = (2n - 1)\pi$, $E = R = -2k\tau$. This means that the two-photon state is most easily generated when $\Delta_0\tau = (2n - 1)\pi$. Therefore, the population terms in the state as well as the number of photons in the waveguide can be controlled by tuning the length of the waveguide. For example, if the length of the waveguide $L$ is chosen to be close to the discrete periodic sequence $\frac{n\pi}{\Delta_0}$, then from the proof of Theorem 3, it is easy to see that two-photon states are difficult to generate. On the other hand, if we choose $L = \frac{2(n - 1)\pi}{\Delta_0}$, then due to $\Delta_0 \gg 1$, the two-photon state can be generated even with a short waveguide. These agree with the comparisons of the simulations in Fig. 3.

The numerical simulations are shown in Fig. 3, where $\omega_{1,2} \in [0, 100]$, $\Delta_0 = 50$, $G_0 = 0.5$, $\gamma = 2\pi$, $\Delta_0\tau = \pi, 2\pi$, respectively. When $\Delta_0\tau = \pi$ as shown in the upper two subfigures of Fig. 3, $E = \Re(R) < 0$ in (23), then $\lim_{t \to \infty} c_{g}(t) = \lim_{t \to \infty} c_{kk}(t) = \lim_{t \to \infty} c_{k}(t) = 0$, and $\dot{c}_{g}(t) \approx 0$ according to (8a). Therefore, $c_{g}(t, k) \approx 0$. This means there will be two photons in the waveguide, as indicated by the peak of the amplitude $|c_{gkk}|$ when $t = 80\tau$. However, the results are different at $\Delta_0\tau = 2\pi$, as shown in the lower two subfigures of Fig. 3, where both $|c_{g}(t)|^2$ and $|c_{k}(t)|^2$ oscillate persistently and the amplitude of the two-photon state $c_{gkk}$ is close to zero. The comparison of the simulation results in Fig. 3 confirms Theorem 3 and Remark 3.

### C. Feedback Control With a Long Waveguide

When the coherent feedback loop is long enough that the evolution time of the quantum system is shorter than the transmission delay from the cavity to the terminal end of the waveguide, $\delta(t - t' + \tau) = 0$ in (41). In this case, by (5) and (57), when $\tau \to \infty$ the equation of the amplitude with no photons in the waveguide is

$$\dot{c}_{g}(t) + 3\kappa \dot{c}_{g}(t) + [2\gamma^2 + 2\kappa^2]c_{g}(t) = 0$$ (25)

where $c_{g}(t) = c_{g}(t)$ and $c_{k}(t)$, respectively.

The following is the main result of this section.

Theorem 4: When the coherent feedback loop is long enough and the cavity-QED system’s evolution time is much shorter than the loop delay, there are two photons in the waveguide.
when $t \to \frac{L}{c}$, and the two-photon emission rate is maximized when $\kappa > 2\sqrt{2}\gamma$.

**Proof:** Because $\kappa > 0$, by (25) we have $\lim_{t \to \infty} c_{r}(t) = \lim_{t \to \infty} c_{g}(t) = 0$. Denote $\Omega_0 = \sqrt{[(\frac{\kappa}{2})^2 - 2\gamma^2]}$. When $\kappa > 2\sqrt{2}\gamma$, $0 < \Omega_0 < \frac{\kappa}{2}$. Solving (25) we get $c_{r}(t) = A_{1}e^{-\frac{\kappa}{2}t} + B_{1}e^{-\frac{\kappa}{2}t}$, which decays to zero without oscillations. On the other hand, when $\kappa = 2\sqrt{2}\gamma$, $c_{r}(t) = (A_{2} + B_{2}t)e^{-\frac{\kappa}{2}t}$; and when $\kappa < 2\sqrt{2}\gamma$, $c_{r}(t) = A_{2}e^{-\frac{\kappa}{2}t} + B_{2}e^{-\frac{\kappa}{2}t}$. In both of these two cases there are oscillations in the evolution of $c_{r}(t)$.

Denote $\tilde{p}(t, k) = c_{r}(t)G^{*}(t, k)$, $\tilde{q}(t, k) = c_{g}(t)G^{*}(t, k)$, and take $\Theta(t - \tau) = 0$ in (8d). Then we have

$$
\lim_{s \to 0} sC_{gkk}(s, k) = \lim_{s \to 0} \frac{i\sqrt{2s}\tilde{Q}(s, k) - \gamma \tilde{P}(s, k)}{s^2 + \kappa s + \gamma^2} = 0
$$

where, $\tilde{Q}(s, k)$ and $\tilde{P}(s, k)$ are the Laplace transform of $\tilde{p}(t, k)$ and $\tilde{q}(t, k)$, respectively. Then $\lim_{t \to \infty} c_{gkk}(t, k) = 0$, and $\lim_{t \to \infty} c_{kk}(t, k) = 0$ according to (8b). As the populations of one-photon states in the waveguide are close to zero, eventually there are two photons in the waveguide.

In Fig. 4, we take $L = 5$ m, which is around 80 times larger than the simulation with $L = 0.0628$ m in the bottom-right subfigure of Fig. 3. $\Delta_0 = 50$, $G_0 = 0.5$, and compare these three circumstances in Theorem 4 by setting $\kappa = 4\sqrt{2}\gamma$, $\kappa = 2\sqrt{2}\gamma$, and $\kappa = \sqrt{2^2}\gamma$, respectively. The simulations indicate that the populations $|c_{r}(t)|^2$ and $|c_{g}(t)|^2$ oscillate when $\kappa \leq 2\sqrt{2}\gamma$, finally while there are two photons in the waveguide in all the parameter settings of $\kappa$ and $\gamma$ if only the waveguide is long enough, and this agrees with the conclusion in Theorem 4.

**D. Two-Photon Entanglement**

In this section, we study entanglement property of the two photons in the waveguide.

According to [87], [88], [89], [62], [90], [91], a two-photon state is an entangled state if the amplitude $c_{gkk}(k_1, k_2)$ in (2) cannot be factorized as a product of two functions, one with argument $k_1$ and the other with argument $k_2$.

For the coherent feedback scheme in Fig. 1(a), according to Theorem 3 and Remark 3, the most efficient way for the generation of two-photon states occurs when $\kappa \tau = (2n + 1)\pi$. In this case, the steady-state amplitude is

$$
c_{gkk}(\infty, k_1, k_2) = \lim_{t \to \infty} c_{gkk}(t, k_1, k_2) = -iG_0^2 \sin(k_1 L) \sin(k_2 L)
$$

$$
\times \left[ \frac{H(k_1)}{R - \sqrt{R^2 - \gamma^2} + i\delta(k_1)} + \frac{J(k_1)}{R - \sqrt{R^2 - \gamma^2} + i\delta(k_2)} + \frac{K(k_2)}{R + i(\delta(k_1) + \delta(k_2) - \sqrt{2}\gamma)} + \frac{J(k_1)}{R + i(\delta(k_1) + \delta(k_2) + \sqrt{2}\gamma)} \right]
$$

where, $\Delta = -2\kappa, \delta(k_1) = ck_1 - \Delta_0, \delta(k_2) = ck_2 - \Delta_0$, and the other parameters $H, I, J, K$ can be calculated according to (58). Obviously, $c_{gkk}(\infty, k_1, k_2)$ is not factorizable in terms of $k_1$ and $k_2$, thus the generated two photons by the coherent feedback are entangled. The conclusion also applies to the circumstance that the waveguide is infinitely long because the factorizability will not be changed with the rising of the feedback loop length. Finally, the degree of entanglement is largely determined by the poles of $c_{gkk}(\infty, k_1, k_2)$ in (26). As the last four terms of the right-hand side of (26) indicate that the nonfactorizable $c_{gkk}(t, k_1, k_2)$ is dominant when $\kappa$ is close to zero, the generated two photons are more easily entangled when $\kappa$ is sufficiently small.

**Remark 4:** Practically, when the coupling between the waveguide and the cavity is small, the two correlated photons are slowly emitted into the waveguide. This also agrees with the simulation results in Fig. 4 where $|c_{g}(t)|^2$ oscillates more significantly for smaller $\kappa$.

**III. QUANTUM FEEDBACK CONTROL WITH DISCRETE MODES**

The discrete coupling between the cavity and the waveguide can be realized by tuning the length $l$ of the cavity and $L$ of the waveguide in Fig. 1(b). Thus, the setting $l \ll L$ used in Section II need not necessarily hold in the discrete coupling scheme. In this section, we study the feedback control of the system in Fig. 1(b) with discrete modes as discussed in [56], and the coupling strength is

$$
G_q(t) = G(k_q, t) = G_0 \sin(k_q L)e^{-i(\omega_q - \Delta_0)t}
$$

where, $k_q = \frac{(2q + 1)\pi}{L}$ and $q = 0, \pm 1, \pm 2, \ldots$. This parameter design of the coupling between the cavity and waveguide is in the most efficient fashion because the amplitude of the cavity field induced by the waveguide field via the coherent feedback is maximized, as theoretically analyzed in [83]; see also Appendix D. This scheme has been widely used in the analysis.
of the quasimodes of the cavity coupled with various systems, see, e.g., [83], [84], [85], [86].

The interaction Hamiltonian reads

\[ H_I = - \gamma (\sigma^- a + \sigma^+ a) \]

\[ - \sqrt{\frac{\pi}{2L}} \sum_{q=-\infty}^{\infty} (G_q(t) a^d_q + G^*_q(t) a^d_q a) \]  

(28)

where, “\(|\cdot|\)” means the adjoint of an operator while “*” means the complex conjugate of a complex number.

In contrast to (2), the overall system state under the discrete coupling is

\[ \Psi(t) = c_e(t)|e, 1, \{ 0 \} \rangle + \sum_{q=1}^{\infty} c_{eq}(t)|e, 0, \{ k_q \} \rangle \]

\[ + c_g(t)|g, 2, \{ 0 \} \rangle + \sum_{q=-\infty}^{\infty} c_{gq}(t)|g, 1, \{ k_q \} \rangle \]

\[ + \sum_{p,q=-\infty}^{\infty} c_{gqp}(t, k_p, k_q)|g, 0, \{ k_p \} \{ k_q \} \rangle dk_p dk_q \]  

(29)

where, \( c_e(t) \) is the amplitude that the atom is in the excited state and there is one photon in the cavity, \( c_g(t) \) is the amplitude that the atom is in the ground state and there are two photons in the cavity, \( c_{eq}(t) \), \( c_{gq}(t) \), and \( c_{gqp}(t, k_p, k_q) \) represent the amplitudes with the discrete modes of photons with modes \( k_q \) and \( k_p \), respectively.

In analogy to (4), the system of control equations with the delayed feedback loop is

\[ \dot{c}_e(t) = i\sqrt{2} \gamma c_g(t) + i \sqrt{\frac{\pi}{2L}} G_0 \]

\[ \times \sum_{q=-\infty}^{\infty} c_{eq}(t, k_q)(-1)^q e^{-i(\omega_q - \Delta_0)t} \]  

(30a)

\[ \dot{c}_{eq}(t, k_q) = i \sqrt{\frac{\pi}{2L}} G_0(1-1)^q e^{i(\omega_q - \Delta_0)t} \]

\[ + i\gamma c_{gq}(t, k_q) \]  

(30b)

\[ \dot{c}_g(t) = i\sqrt{2} c_{eq}(t) \]

\[ + \sqrt{\frac{\pi}{2L}} G_0 \times \sum_{q=-\infty}^{\infty} c_{gq}(t, k_q)(-1)^q e^{-i(\omega_q - \Delta_0)t} \]  

(30c)

\[ \dot{c}_{gq}(t, k_q) = i \gamma c_{eq}(t, k_q) \]

\[ + \sqrt{\frac{\pi}{2L}} G_0 \times \sum_{q=-\infty}^{\infty} c_{gq}(t, k_q)(-1)^q e^{-i(\omega_q - \Delta_0)t} \]

\[ + i \sqrt{\frac{\pi}{2L}} G_0 \sum_{k_2=-\infty}^{\infty} \Delta_0 k_2 e^{-i(\omega_{k_2} - \Delta_0)t} \]

\[ \times c_{gkk}(t, k_2, k_q) \]

\[ + i \sqrt{\frac{\pi}{2L}} G_0 \sum_{k_1=-\infty}^{\infty} \Delta_0 k_1 e^{-i(\omega_{k_1} - \Delta_0)t} \]

\[ \times c_{gqp}(t, k_1, k_q) \]  

(30d)

\[ \dot{c}_{gqp}(t, k_p, k_q) = i \sqrt{\frac{\pi}{2L}} c_{gq}(t, k_p) G^*(k_q, t) \]

\[ + i \sqrt{\frac{\pi}{2L}} c_{gq}(t, k_q) G^*(k_p, t) \]  

(30e)

Lemma 2: [84] The amplitude \( c_{gq} \) of the mode \( k_q \) transmitted from the cavity to the waveguide is proportional to

\[ \frac{1}{\sqrt{(\omega_q - \Delta_0)^2 + \Gamma^2}} \]

where, \( \Gamma = \frac{c(\pi - r)}{2L} \) with \( r \) being the reflection coefficient of the mirror at \( z = 0 \). Lemma 2 has been proved in [84]. A brief introduction of the interaction fields between the cavity and the waveguide of the discrete mode \( k_q \) is given in Appendix D. Lemma 2 reveals that the quantum field in the waveguide, i.e., \( c_{gq}(t, k_q) \), is Lorentzian with a narrowband in the frequency domain, and the major frequency component of \( k_q \) is around \( \Delta_0/c \) because

\[ \frac{1}{\sqrt{(\omega_q - \Delta_0)^2 + \Gamma^2}} \]

is peaked at \( k_q = \frac{\Delta_0}{c} \). See also Appendix D for more details.

Assumption 4: The cavity’s decay rate is small and the reflection rate \( r \) of the mirror at \( z = 0 \) is close to 1.

With the selected discrete modes in (27) which can maximize the feedback coupling efficiency between the cavity and the waveguide, we concentrate on the feedback design with the cavity of high quality as stated in Assumption 4, which is enough to induce efficient feedback when combined with (27), and this has been widely adopted in the feedback design, see, e.g., [84], [86].

Theorem 5: The integral \( \int_0^t c_{gq}(u, k_q) du = f(t) \delta(c_{k_q} - \Delta_0) \) when \( 1 - r \ll \frac{2\pi}{\Gamma} \) and the time domain envelope \( f(t) \) satisfies \( f(t) \approx 0 \).

Proof: For the semitransparent mirror with \(|r| \leq 1 \), and \( 1 - r \) represents the field transmitted from the cavity to the waveguide. When \( 1 - r \ll \frac{2\pi}{\Gamma} \)

\[ c_{k_q + 1} - c_{k_q} = \frac{c\pi}{L} \gg \frac{c(1 - r)}{2L} = \Gamma. \]  

(31)

Similarly, \( |c_{k_q - 1} - c_{k_q}| \gg \Gamma \). Hence, for the discrete modes \( k_q \neq \frac{\Delta_0}{c} \), \( |c_{k_q} - \Delta_0| \gg \Gamma^2 \) and \( \frac{1}{\sqrt{(\omega_{k_q} - \Delta_0)^2 + \Gamma^2}} \approx 0 \), thus the photon transmitted from the cavity to the waveguide satisfies that \( c_{gq}(t, k_q) \propto \delta(c_{k_q} - \Delta_0) \). Denote \( \int_0^t c_{gq}(u, k_q) du = f(t) \delta(c_{k_q} - \Delta_0) \) where the envelope \( f(t) \) is the function of time. Thus

\[ f(t) = \int_0^t c_{gq}(u, k_q) du \]  

\[ \delta(c_{k_q} - \Delta_0) \]  

(32)

and its derivative \( f'(t) = \frac{c_{gq}(t, k_q)}{\delta(c_{k_q} - \Delta_0)} \), when \( c_{k_q} = \Delta_0 \), \( f'(t) \approx 0 \) because \( \delta(c_{k_q} - \Delta_0) \gg 1 \) and \( c_{gq}(t, k_q) \) is finite. When \( k_q \neq \Delta_0 \), by Lemma 2 we also have \( f'(t) \approx 0 \).

Theorem 5 reflects the physical fact that the coupling between the cavity and waveguide through the semitransparent mirror located at \( z = 0 \) is maximized when the discrete mode in the waveguide is resonant with the cavity mode. As a result, the linewidth of the radiation field is usually narrow and rapidly decreases with the increasing of the detuning between the discrete waveguide mode and the cavity mode. More discussions can be found in [83].

Lemma 3: For the discrete mode control (30c)

\[ \sum_{q=-\infty}^{\infty} c_{gq}(t, k_q)(-1)^q e^{-i(\omega_q - \Delta_0)t} = 0 \]  

(33)

when \( 1 - r \ll \frac{2\pi}{\Gamma} \).

Proof: Notice that

\[ \frac{d}{dt} \sum_{q=-\infty}^{\infty} \int_0^t c_{gq}(u, k_q) du(-1)^q e^{-i(\omega_q - \Delta_0)t} \]
\[
\begin{align*}
&= \sum_{q=-\infty}^{\infty} c_{gq}(t, k_q)(-1)^q e^{-i(\omega_q - \Delta_0)t} \\
&\quad - i(\omega_q - \Delta_0) \sum_{q=-\infty}^{\infty} \int_0^t c_{gq}(u, k_q)du(-1)^q e^{-i(\omega_q - \Delta_0)t}.
\end{align*}
\]

We have
\[
\begin{align*}
\sum_{q=-\infty}^{\infty} c_{gq}(t, k_q)(-1)^q e^{-i(\omega_q - \Delta_0)t} \\
&= \left[ \sum_{q=-\infty}^{\infty} \int_0^t c_{gq}(u, k_q)du(-1)^q e^{-i(\omega_q - \Delta_0)t} \right]'
\end{align*}
\]

\[
+ i(\omega_q - \Delta_0) \sum_{q=-\infty}^{\infty} \int_0^t c_{gq}(u, k_q)du(-1)^q e^{-i(\omega_q - \Delta_0)t}
\]

\[
\begin{align*}
&= \left[ \sum_{q=-\infty}^{\infty} f(t)\delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t} \right]'
\end{align*}
\]

\[
+ i(\omega_q - \Delta_0) \sum_{q=-\infty}^{\infty} \int_0^t c_{gq}(u, k_q)du(-1)^q e^{-i(\omega_q - \Delta_0)t}
\]

\[
= -i \sum_{q=-\infty}^{\infty} (\omega_q - \Delta_0) f(t)\delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t}
\]

\[
+ f'(t) \left[ \sum_{q=-\infty}^{\infty} \delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t} \right]
\]

\[
+ i \sum_{q=-\infty}^{\infty} (\omega_q - \Delta_0) f(t)\delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t}
\]

\[
\begin{align*}
&\quad - i \sum_{q=-\infty}^{\infty} (\omega_q - \Delta_0) f(t)\delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t} \\
&\quad + \sqrt{\frac{\pi}{2L}} G_0^2 \sum_{q=0}^{\infty} c_e(t - q\tau) e^{i(\Delta_0 - \frac{\pi}{2})q\tau} \\
&\quad - \sqrt{\frac{\pi}{2L}} G_0 \delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t}.
\end{align*}
\]

where, \([\Delta_0 L]\) represents the integer nearest to \(\frac{\Delta_0 L}{c}\), the first and third terms cancel each other, and the second term is approximately zero when \(1 - r \ll \frac{2\pi}{\omega}\) as \(f'(t) \approx 0\) by Theorem 5. Thus, (33) holds.

By Theorem 5, (30a) and (30b), we get
\[
\begin{align*}
\hat{c}_c(t) &= i\sqrt{2}\gamma c_g(t) - \frac{\pi}{2L} G_0^2 \sum_{q=0}^{\infty} c_e(t - q\tau) e^{i(\Delta_0 - \frac{\pi}{2})q\tau} \\
&\quad - \sqrt{\frac{\pi}{2L}} G_0 \delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t}.
\end{align*}
\]

Please refer to Appendix C for a detailed derivation of (36).

Based on (36) and Lemma 3, we can obtain the following result.

**Theorem 6:** In the discrete modes case, the amplitudes \(c_c(t)\) and \(c_g(t)\) oscillate with damping close to zero, and there are almost no two-photon states in the waveguide.

**Proof:** Denote \(q' = \frac{\Delta_0 L}{c}\). Integrating both sides of (36) gives
\[
c_c(t) = i\sqrt{2}\gamma \int_0^t c_g(t)dt
\]

\[
\begin{align*}
&= \sum_{q=-\infty}^{\infty} c_{gq}(t, k_q)(-1)^q e^{-i(\omega_q - \Delta_0)t} \\
&\quad - i(\omega_q - \Delta_0) \sum_{q=-\infty}^{\infty} \int_0^t c_{gq}(u, k_q)du(-1)^q e^{-i(\omega_q - \Delta_0)t}.
\end{align*}
\]

Hence, by differentiating (37) twice we get
\[
\begin{align*}
\frac{\partial^2 \hat{c}_c(t)}{\partial t^2} &= -2\gamma^2 c_c(t) - \frac{\pi}{2L} G_0^2 \sum_{q=0}^{\infty} \hat{c}_c(t - q\tau) e^{i(\Delta_0 - \frac{\pi}{2})q\tau} \\
&\quad - \sqrt{\frac{\pi}{2L}} G_0 \delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t}.
\end{align*}
\]

Applying the Laplace transform to (38) we get
\[
\begin{align*}
s^2 C_c(s) - s c_c(0) - \hat{c}_c(0) &= -2\gamma^2 C_c(s) - \frac{\pi}{2L} G_0^2 \sum_{q=0}^{\infty} [s C_c(s) - c_c(0)] e^{q\tau} e^{i(\Delta_0 - \frac{\pi}{2})q\tau} \\
&\quad - \sqrt{\frac{\pi}{2L}} G_0 \delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t}.
\end{align*}
\]

\[
\begin{align*}
&= -2\gamma^2 C_c(s) - \frac{\pi}{2L} G_0^2 \sum_{q=0}^{\infty} [s C_c(s) - c_c(0)] e^{q\tau} e^{i(\Delta_0 + is\tau)q\tau} \\
&\quad - \sqrt{\frac{\pi}{2L}} G_0 \delta(q - \frac{\Delta_0 L}{c\pi})(-1)^q e^{-i(\omega_q - \Delta_0)t}.
\end{align*}
\]

According to Assumption 1 and (30a), \(c_c(0) = 1\). Moreover, \(c_c(0) = 0\) because \(c_g(0) = c_{eq}(0, k_q) = 0\). Thus, \(C_c(s) \approx \frac{\gamma^2}{\pi + 2\gamma^2}\) in (39). That is, \(c_c(t)\) oscillates with damping close to zero. Finally, according to (30c) and Lemma 3, \(\hat{c}_g(t) = i\sqrt{2}\gamma c_g(t)\), therefore \(c_g(t)\) also oscillates with damping close to zero.

In the numerical simulations for the discrete coupling scheme based on (30), as shown in Fig. 5, \(k\) is uniformly sampled as \(k_q = \frac{2\pi L}{2L}\), where \(q = 1, 2, \ldots, 39\), and \(L = 0.1m\). The simulations in Fig. 5(a) and (b) show that the discrete-coupling scheme can maintain the Rabi oscillation in the cavity no matter whether the coupling between the atom and cavity is weak or strong, which agrees with Theorem 6. This is different from the continuous coupling scheme, as shown in (5). Moreover, the damped amplitudes of \(c_g(t)\) and \(c_c(t)\) in the continuous
coupling scheme also show that there are generated two-photon states in the waveguide, while the oscillating $c_e(t)$ and $c_g(t)$ in the discrete coupling scheme reveal that the two photons cannot simultaneously exist in the waveguide. Fig. 5(c) further shows that when the coupling between the atom and cavity is much larger than that between the cavity and waveguide, the atom will oscillate between its ground and excited states no matter whether the coupling between the waveguide and cavity is continuous or discrete. All the three simulations under different parameter settings have illustrated the fact that the discrete coupling scheme can maintain the Rabi oscillations of the two-level atom in the cavity.

IV. CONCLUSION

In this article, we have studied a coherent feedback control scheme in the architecture that a waveguide is coupled with a cavity containing a two-level atom. The control performance depends critically on whether the coupling mode between the cavity and the waveguide is continuous or periodically discrete. For the continuous modes case, the generation of two-photon states can be controlled by tuning the length of the waveguide as well as the coupling between the waveguide and the cavity. The populations of the two-photon states in the waveguide can be maximized when the length of the waveguide is well designed. Moreover, for the waveguide that is long enough and the round-trip delay of the feedback loop is longer than the evolution time of the quantum state, the generation rate of the two photons can be optimized by engineering the coupling strength between the waveguide and the cavity as well as that between the cavity and the atom. The major difference between the discrete mode scheme and the continuous mode scheme is that, the discrete mode can stabilize the Rabi oscillation in the cavity and there are no stable two-photon states in the waveguide. The results in this article are useful for the coherent feedback design of quantum optical systems to generate multiphoton states or Rabi oscillations, and can be further applied in quantum networks for QIP.

APPENDIX A

QUANTUM FEEDBACK CONTROL WITH THE WAVEGUIDE CONTINUOUSLY COUPLED TO THE CAVITY

In this appendix, we give the derivation of (5)–(7) in the main text. The populations of the quantum state are governed by (4). As argued in the main text, using (5)–(7), we can rewrite (4) as (8), which demonstrates clearly the influence of the round trip delay $\tau$ on population distributions.

The rough idea of derivation is as follows. We first derive (5) by taking the integration of (4b) into (4a) in the main text. Then, (6) can be derived by taking the integration of (4e) into (4d). Based on this, finally (7) can be similarly derived. In the procedure, the properties of Dirac delta functions and Assumptions 2 and 3 are used.

i) Derivation of (5).

Integrating both sides of (4b) and substituting the resultant $c_{ek}$ into (4a) yields

$$
\dot{c}_e(t) = i \sqrt{2} \gamma c_g(t) - \int_0^\infty \int_0^t [c_e(t')G^*(k, t')] dt'dk + \gamma c_{gk}(t', k)G(k, t)dt'dk
$$

where the second term on the right-hand side can be further simplified, specifically

$$
\int_0^\infty \int_0^t c_e(t')G^*(k, t')G(k, t)dt'dk = G_0^2 \int_0^\infty \int_0^t \sin^2(kL)e^{-i(\omega - \Delta_0)t}e^{i(\omega - \Delta_0)t'}c_e(t')dt'dk
$$

$$
= G_0^2 \int_0^\infty \int_0^t (2 - e^{i\omega t} - e^{-i\omega t})e^{-i(\omega - \Delta_0)t}e^{i(\omega - \Delta_0)t'}c_e(t')dt'd\omega
$$

$$
= G_0^2 \int_0^\infty \int_0^t (2e^{-i(\omega - \Delta_0)(t-t')} - e^{-i\Delta_0 t}e^{-i(\omega - \Delta_0)(t-t')} - e^{-i\Delta_0 t}e^{-(t-t'+\tau)}c_e(t')dt'd\omega
$$

$$
= G_0^2 \int_0^\infty \int_0^t (2\delta(t-t') - e^{i\Delta_0 \tau} \delta(t-t'-\tau) - e^{-i\Delta_0 \tau} \delta(t-t'+\tau)c_e(t')dt'
$$

$$
= G_0^2 \frac{\pi}{2c} (c_e(t) - e^{i\Delta_0 \tau} c_e(t-\tau) \Theta(t-\tau))
$$

(41)

where, $\tau = \frac{2L}{c}$ is the round-trip delay.

On the other hand, the third term on the right-hand side of (40) reads

$$
\int_0^\infty \int_0^t \gamma c_{gk}(t', k)G(k, t)dt'dk
$$

$$
= \gamma \int_0^\infty \int_0^t c_{gk}(t', k)G_0 \sin(kL)e^{-i(\omega - \Delta_0)t}dt'dk
$$

$$
= \gamma G_0 e^{i\Delta_0 t} \int_0^\infty \int_0^t c_{gk}(t', k) \sin \left( \frac{\omega t'}{2} \right) e^{-i\omega t'} dt'dk
$$

$$
= \gamma G_0 e^{i\Delta_0 t} \int_0^\infty \int_0^t c_{gk} \left( t', \frac{\omega}{c} \right) \left( e^{i\omega (\frac{t'}{2} - \tau) - e^{-i\omega (\frac{t'}{2} + \tau)}} \right) d\omega dt'.
$$

(42)

By means of the following lemma, it can be shown that this term is 0.

Lemma 4: By Assumption 3, we have

$$
\int_0^t \int_0^\infty c_{gk} \left( t', \frac{\omega}{c} \right) \left( e^{i\omega (\frac{t'}{2} - \tau) - e^{-i\omega (\frac{t'}{2} + \tau)}} \right) d\omega dt' = 0.
$$

Proof: Because $\delta(t + \tau) = 0$ for $t \geq 0$ and $\lim_{k \to \infty} c_{gk}(t, k) = 0$ as given in Assumption 3, we have

$$
\int_0^t \int_0^\infty c_{gk} \left( t', \frac{\omega}{c} \right) \left( e^{i\omega (\frac{t'}{2} - \tau) - e^{-i\omega (\frac{t'}{2} + \tau)}} \right) d\omega dt' = 0.
$$

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\[
\begin{align*}
&= - \left( \delta \left( \frac{T}{2} - t \right) - \delta \left( \frac{T}{2} + t \right) \right) \int_0^t c_{gk}(t', 0) dt' \\
&\quad - \left( \delta \left( \frac{T}{2} - t \right) - \delta \left( \frac{T}{2} + t \right) \right) \int_0^t \int_0^\infty \frac{\partial c_{gk}(t', \omega)}{\partial \omega} d\omega dt' \\
&\quad - \delta \left( \frac{T}{2} - t \right) \int c_{gk}(t', 0) dt' + \delta \left( \frac{T}{2} - t \right) \int_0^t c_{gk}(t', 0) dt'.
\end{align*}
\]
\begin{equation}
(43)
\end{equation}

Notice that \( \delta \left( \frac{T}{2} - t \right) = 0 \) for all \( t \neq \frac{T}{2} \), and hence from (43) we have that, for \( t \neq \frac{T}{2} \)
\[
\int_0^t \int_0^\infty c_{gk} \left( t', \frac{\omega}{c} \right) \left( e^{i\omega (\frac{T}{2} - t)} - e^{-i\omega (\frac{T}{2} + t)} \right) d\omega dt' = 0.
\]
\begin{equation}
(44)
\end{equation}

Furthermore, the integrals in (43) are continuous when \( t \) varies around \( \frac{T}{2} \) because the evolution of amplitudes is continuous according to Assumption 2, thus (44) holds when \( t \geq 0 \).

Consequently, (40) can be written as (5) after combined with (41) and (44), thus we finish the derivation of (5).

ii) Derivation of (6).

According to (4d) in the main text, and together with (4b) and (4e), we have
\[
\dot{c}_{gk}(t, k) = i \sqrt{2} c_{g}(t) G^*(k, t) - \gamma \int_0^t c_{e}(t) G^*(k, t) dt \\
- \int_0^t \gamma^2 c_{gk}(t, k) dt \\
- 2 \int_0^t \int_0^\infty c_{gk}(t', k) G(k', t) G^*(k', t') dk' dt' \\
- 2 \int_0^t \int_0^\infty c_{gk}(t', k') G(k', t) G^*(k, t') dk' dt'.
\]
\begin{equation}
(45)
\end{equation}

It can be shown that
\[
\int_0^t \int_0^\infty c_{gk}(t', k) G(k', t) G^*(k', t') dk' dt' \\
= \int_0^t \int_0^\infty c_{gk}(t', k) G_0 \sin(k' L) e^{-i(\omega - \Delta_0) t} G_0 \\
x \sin(k' L) e^{i(\omega - \Delta_0) t} dk' dt' \\
= \frac{G_0^2}{c} \int_0^t \int_0^\infty c_{gk}(t', k) \sin^2(k' L) e^{-i(\omega - \Delta_0)(t-t')} dk' d\omega' dt' \\
= \frac{G_0^2 \pi}{2c} [c_{gk}(k, k) - c_{gk}(k - \tau, k)] \Theta(t - \tau)
\]
and
\[
\int_0^t \int_0^\infty c_{gk}(t', k') G(k', t) G^*(k, t') dk' dt' \\
= \int_0^t \int_0^\infty c_{gk}(t', k') G_0 \sin(k' L) e^{-i(\omega - \Delta_0) t} G_0 \\
x \sin(k' L) e^{i(\omega - \Delta_0) t} dk' dt' \\
= \frac{G_0^2 \sin(k L) e^{i\Delta_0 t}}{c} \int_0^t \int_0^\infty c_{gk}(t', k') \sin(k' L) \\
x e^{-i\omega t} e^{i(\omega - \Delta_0) t} d\omega' dt'.
\]
\begin{equation}
(47)
\end{equation}

where, \( \omega' = ck' \). Moreover, notice that
\[
\int_0^\infty \sin(k' L) e^{-i\omega' t} d\omega' \\
= \frac{1}{2i} \left[ e^{i\frac{k' L}{c} - e^{-i\omega' t}} e^{-i\omega' t} d\omega' \\
- \frac{1}{2i} \left[ e^{i\frac{k' L}{c} - e^{-i\omega' t}} e^{-i\omega' t} d\omega' \\
= \frac{1}{2i} \left[ e^{i\omega' (\frac{T}{2} - t)} - e^{-i\omega' (\frac{T}{2} + t)} \right] \\
= \frac{1}{2i} \left( \delta (\frac{T}{2} - t) + \delta (\frac{T}{2} + t) \right).
\]
\begin{equation}
(48)
\end{equation}

We have
\[
\int_0^\infty c_{gk}(t', k') \sin(k' L) e^{-i\omega' t} d\omega' \\
= \int_0^\infty c_{gk} \left( t', \frac{\omega'}{c} \right) \sin(k' L) e^{-i\omega' t} d\omega' \\
= \frac{1}{2i} \left[ \delta (\frac{T}{2} - t) + \delta (\frac{T}{2} + t) \right] c_{gk}(t', \frac{\omega'}{c}) \int_0^\infty \left( \frac{T}{2} - t \right) + \delta (\frac{T}{2} + t) \right] \frac{\partial c_{gk}(t', \frac{\omega'}{c})}{\partial \omega'} d\omega'.
\]
\begin{equation}
(49)
\end{equation}

Because
\[
\int_0^\infty c_{gk}(t, k) \sin^2(k, k) dk \leq 1
\]
\begin{equation}
(50)
\end{equation}

and
\[
\lim_{k \to \infty} c_{gk}(t, k) = 0
\]
\begin{equation}
(51)
\end{equation}

according to Assumption 3 in the main text, (49) becomes
\[
\int_0^\infty c_{gk}(t', k') \sin(k' L) e^{-i\omega' t} d\omega' \\
= - \frac{1}{2i} \int_0^\infty \left( \delta (\frac{T}{2} - t) + \delta (\frac{T}{2} + t) \right) \frac{\partial c_{gk}(t', \frac{\omega'}{c})}{\partial \omega'} d\omega'.
\]
\begin{equation}
(52)
\end{equation}

Substituting (52) into (47) yields
\[
\int_0^t \int_0^\infty c_{gk}(t', k') G(k', t) G^*(k, t') dk' dt' \\
= \frac{G_0^2 \sin(k L) e^{i\Delta_0 t}}{c} \int_0^t \int_0^\infty c_{gk}(t', k') \sin(k' L) \\
x e^{-i\omega t} e^{i(\omega - \Delta_0) t} d\omega' dt'.
\]
Thus, by continuity, the last term in (55) is 0.

In addition, denote \( \dot{c}_{g_k}(u, k) = \int_0^u c_{g_k}(\nu, k) d\nu \). Then, the integrant in the third line of (55)
\[
\int_0^\infty \int_0^t \int_0^u c_{g_k}(\nu, k) d\nu d\nu \sin(kL)e^{-i(\omega - \Delta_0)t} dk = 0
\]
which can be proved by replacing \( c_{g_k}(\nu, k) \) in (56) with \( \dot{c}_{g_k}(u, k) \). Hence, (55) can be simplified as follows:
\[
\dot{c}_g(t) = i\sqrt{2} \gamma c_g(t) - 2G_0^2 \int_0^t \int_0^u c_g(\nu)(e^{i(\omega - \Delta_0)(t - \nu)} \sin^2(kL) d\nu dk
\]
\[
- i\sqrt{2} \gamma G_0^2 \int_0^t \int_0^u c_g(\nu)e^{i(\omega - \Delta_0)(t - \nu)} d\nu d\nu \sin^2(kL) dk
\]
\[
\int_0^\infty \int_0^t \int_0^u c_g(\nu)e^{i(\omega - \Delta_0)(t - \nu)} d\nu d\nu \sin(kL)e^{-i(\omega - \Delta_0)t} dk
\]
\[
\int_0^\infty \int_0^t \int_0^u c_g(\nu - \tau, k) e^{i(\omega - \Delta_0)t} d\nu dk
\]
\[\int_0^\infty c_{g_k}(\nu, k) e^{i(\omega - \Delta_0)t} d\nu = 0
\]
and
\[
\int_0^\infty c_{g_k}(\nu, k) e^{i(\omega - \Delta_0)t} d\nu = 0
\]
which is (7).


**Appendix B**

**Proof of Lemma 1**

*Proof:* When $\kappa \tau \ll 1$, namely $L \ll 1$, the parameters defined in (15)–(16) satisfy that $E = R = \kappa (\cos(\Delta \tau) - 1) + i \kappa \sin(\Delta \tau) \approx 0$. Let $s = \Delta / 2 \left[ \Delta - \omega - \sin(\Delta \tau) + i \kappa (\cos(\Delta \tau) - 1) \right] \approx \frac{\Delta}{2} (\Delta - \omega)$. Then, $\sqrt{R^2 - \gamma^2} \approx i \gamma, \gamma \ll \Delta_0, F \approx 0$ in (15). Consequently,

$$
H(\omega) = \lim_{s \to -i \gamma} \frac{s - E - i(\omega - \Delta_0 + F + 2 \sqrt{2} D)}{(s - i \gamma)[(s - E - i(\omega - \Delta_0 + F)]^2 + 2 \gamma^2}
$$

$$
\approx \frac{\gamma + \frac{1}{2} \left(\omega - \Delta_0\right)}{2 \gamma (-\gamma + \omega - \Delta_0)^2 + 2 \gamma^2}
$$

$$
I(\omega) = \lim_{s \to -i \gamma} \frac{s - E - i(\omega - \Delta_0 + F + 2 \sqrt{2} D)}{(s - i \gamma)[(s - E - i(\omega - \Delta_0 + F)]^2 + 2 \gamma^2}
$$

$$
\approx \frac{\gamma - \frac{1}{2} \left(\omega - \Delta_0\right)}{2 \gamma (-\gamma - \omega + \Delta_0)^2 + 2 \gamma^2}
$$

$$
J(\omega) = \lim_{s \to E + i(\omega - \Delta_0 + F) + i \sqrt{2} \gamma} \frac{s - E - i(\omega - \Delta_0 + F + 2 \sqrt{2} D)}{(s - i \gamma)[(s - E - i(\omega - \Delta_0 + F) + i \sqrt{2} \gamma)]^2 + 2 \gamma^2}
$$

$$
\times \frac{s - E - i(\omega - \Delta_0 + F + 2 \sqrt{2} D)}{(s - i \gamma)[(s - E - i(\omega - \Delta_0 + F) + i \sqrt{2} \gamma)]^2 + 2 \gamma^2}
$$

$$
= \frac{\sqrt{2} \gamma + \frac{1}{2} (\Delta_0 - \omega)}{2 \sqrt{2} \gamma (- \omega - \Delta_0 - 2 \gamma)^2 + \gamma^2}
$$

$$
K(\omega) = \lim_{s \to E + i(\omega - \Delta_0 + F) + i \sqrt{2} \gamma} \frac{s - E - i(\omega - \Delta_0 + F + 2 \sqrt{2} D)}{(s - i \gamma)[(s - E - i(\omega - \Delta_0 + F) + i \sqrt{2} \gamma)]^2 + 2 \gamma^2}
$$

$$
\times \frac{s - E - i(\omega - \Delta_0 + F + 2 \sqrt{2} D)}{(s - i \gamma)[(s - E - i(\omega - \Delta_0 + F) + i \sqrt{2} \gamma)]^2 + 2 \gamma^2}
$$

$$
= \frac{- \sqrt{2} \gamma + \frac{1}{2} (\Delta_0 - \omega)}{- 2 \sqrt{2} \gamma (- \omega - \Delta_0 + 2 \gamma)^2 + \gamma^2}. \quad (58)
$$

In summary

$$
\begin{align*}
H(\omega) &= I(2 \Delta_0 - \omega)^* \\
J(\omega) &= K(2 \Delta_0 - \omega)^*.
\end{align*} \quad (59)
$$

**Appendix C**

**Quantum Feedback Control for the Discrete Method Scheme**

Equation (36) is proved as follows. Integrating both sides of (30b) in the main text yields:

$$
c_{eq}(t, k_q) = i \sqrt{\frac{\pi}{2L}} G_0 (-1)^q \int_0^t c_e(u) e^{i(\omega_q - \Delta_0)u} du \\
+ i \gamma \int_0^t c_{gq}(u, k_q) du. \quad (60)
$$

Substituting (60) into (30a) gives

$$
\dot{c}_e(t) = i \sqrt{2} \gamma c_q(t)
$$

$$
= - \frac{\pi}{2 L} G_0^2 \sum_{q=-\infty}^{\infty} \int_0^t c_e(u) e^{i(\omega_q - \Delta_0)(u-t)} du
$$

$$
- \sqrt{2} \frac{\pi}{2 L} G_0 \gamma \sum_{q=-\infty}^{\infty} \int_0^t c_{gg}(u, k_q) du (-1)^q e^{-i(\omega_q - \Delta_0)t}. \quad (61)
$$

Notice that

$$
\sum_{q=-\infty}^{\infty} e^{i(\omega_q - \Delta_0)(u-t)} = e^{i(\frac{\pi}{2} - \Delta_0)(u-t)} \sum_{q=-\infty}^{\infty} e^{i(\omega_q - \frac{\pi}{2})(u-t)}
$$

$$
= e^{i(\frac{\pi}{2} - \Delta_0)(u-t)} \sum_{q=-\infty}^{\infty} e^{i(2\pi + 1)(u-t)}
$$

$$
= e^{i(\frac{\pi}{2} - \Delta_0)(u-t)} \sum_{q=-\infty}^{\infty} e^{i2\pi u \frac{\omega_q}{\omega_q}}. \quad (62)
$$

According to the property of the Dirac comb

$$
\frac{1}{T} \sum_{q=-\infty}^{\infty} e^{i2\pi q \frac{u}{T}} = \sum_{q=-\infty}^{\infty} \delta(u - t - qT).
$$

Equation (62) becomes

$$
\sum_{q=-\infty}^{\infty} e^{i(\omega_q - \Delta_0)(u-t)}
$$

$$
= \tau e^{i(\frac{\pi}{2} - \Delta_0)(u-t)} \sum_{q=-\infty}^{\infty} \delta(u - t - qT). \quad (63)
$$

As a result

$$
\sum_{q=-\infty}^{\infty} \int_0^t c_e(u) e^{i(\omega_q - \Delta_0)(u-t)} du
$$

$$
= \int_0^t c_e(u) \sum_{q=-\infty}^{\infty} e^{i(\omega_q - \Delta_0)(u-t)} du
$$

$$
= \int_0^t c_e(u) \tau e^{i(\frac{\pi}{2} - \Delta_0)(u-t)} \sum_{q=-\infty}^{\infty} \delta(u - t - qT) du
$$

$$
= \sum_{q=-\infty}^{\infty} \int_0^t c_e(u) \tau e^{i(\frac{\pi}{2} - \Delta_0)(u-t)} \delta(u - t - qT) du
$$

$$
= \sum_{q=0}^{\infty} c_e(t + qT) \tau e^{i(\frac{\pi}{2} - \Delta_0)(t+qT-t)}
$$

$$
= \sum_{q=0}^{\infty} c_e(t + qT) \tau e^{i(\Delta_0 - \frac{\pi}{2})qT}. \quad (64)
$$

On the other hand, according to Theorem 5,

$$
\sum_{q=0}^{\infty} \int_0^t c_{gg}(u, k_q) du (-1)^q e^{-i(\omega_q - \Delta_0)t}
$$

$$
\int_0^t c_{gg}(u, k_q) du (-1)^q e^{-i(\omega_q - \Delta_0)t}
$$
\begin{equation}
\delta \left( q - \frac{\Delta_0 L}{c \tau} \right) \left( -1 \right)^q e^{-i(q\omega_0 - \Delta_0)t}.
\end{equation}

By means of (64)–(65), \( \dot{c}_c(t) \) in (61) can be rewritten as (36) in the main text.

**APPENDIX D**

**COUPLING SCHEME BETWEEN THE CAVITY AND WAVEGUIDE**

The electromagnetic field with discrete modes in the waveguide and cavity can be represented as [84] follows:

\begin{equation}
E(z, t) = \sum_q (\zeta_q(t) U_q(z) + \zeta_q^*(t) U^*_q(z))
\end{equation}

where, \( \zeta_q(t) \) is the amplitude of the field of the mode \( k_q \). Because the electromagnetic field in the waveguide and cavity are standing waves, the mode function \( U_q(z) \) with \( \xi^2_q = 1 \) can be represented as [83], [84], [85], [86]

\begin{equation}
U_q(z) = \begin{cases}
\xi_q \sin(k_q(z + L)) & z < 0 \\
M_q \sin(k(z - l)) & z > 0
\end{cases}
\end{equation}

where, \( \xi_q \) and \( M_q \) represent the amplitudes of the coupled mode \( k_q \) in the waveguide and cavity, which are divided by the semiconductor mirror at \( z = 0 \). \( U_q(z) \) is governed by the Maxwell equation

\begin{equation}
d^2U_q(z)/dz^2 + \left[1 + \eta(z)\right]k^2_qU_q(z) = 0
\end{equation}

with \( \eta \) being the transmissivity of the mirror at \( z = 0 \), and the boundary conditions are

\begin{equation}
\begin{align*}
U_q(0^+) &= U_q(0^-) \\
U_q(z)|_{z=-L,l} &= 0 \\
U_q(0^+) - U_q^*(0^-) &= -\eta\Delta^2_0 U_q(0)
\end{align*}
\end{equation}

where, the resonant frequency of the cavity \( \omega_c = \Delta_0 \gg 1 \). Solving the Maxwell (68) with \( U_q(z) \) in (67) gives the following equations at \( z = 0 \):

\begin{equation}
\begin{cases}
-M_q \sin(k_ql) = \xi_q \sin(k_qL) \\
M_q \cos(k_ql) - \xi_q \cos(k_qL) = -\eta\Delta^2_0 \xi_q \sin(k_qL)
\end{cases}
\end{equation}

As a result

\begin{equation}
\tan(k_qL) = \frac{\tan(k_ql)}{\eta\Delta^2_0 \tan(k_ql) - 1}.
\end{equation}

Using the boundary conditions in (69), the feedback coupling strength can be evaluated with the amplitude of the field in the cavity which is induced by the unit field in the waveguide as

\begin{equation}
\frac{M^2_q}{\xi^2_q} = \frac{\sin^2(k_qL)}{\sin^2(k_ql)} = \frac{\tan^2(k_ql)}{\tan^2(k_qL)} + 1
\end{equation}

and \( \xi^2_q = 1 \).

Obviously, the feedback coupling is maximized when \( \sin^2(k_qL) = 1 \). Thus, in the coherent feedback control with discrete coupled modes discussed in Section III, we choose the discrete modes as \( k_q = \frac{(2q+1)\pi}{2L} \). See also the early illustration on the set of discrete modes of the cavity with small leakage in [92].

Combining (71) and (72) with \( \sin^2(k_qL) = \frac{\tan^2(k_qL)}{\tan^2(k_qL)+1} \), yields

\begin{equation}
\frac{M^2_q}{\xi^2_q} = \frac{\tan^2(k_ql) + 1}{\tan^2(k_ql) + [\eta\Delta^2_0 \tan(k_qL) - 1]^2}
\end{equation}

\begin{equation}
\approx \frac{c^2 \Gamma/l}{(ck_q - \Delta_0)^2 + \Gamma^2}
\end{equation}

where, \( \Gamma = \frac{c(1-r)}{2L} \) with \( r \) being the reflection coefficient of the mirror at \( z = 0 \). Crudely speaking, (73) means that the transmitted field from the waveguide to the cavity is Lorentzian. Thus, when \( ck_q - \Delta_0 = 0 \), the transmitted field is maximized. More details can be found in [84].

The electromagnetic field \( E(z,t) \) can be quantized in the waveguide and cavity, respectively. The field amplitude of the mode \( k_q \) in the waveguide at \( z < 0 \) can be quantized as the operator \( d_q \), and the field in the cavity at \( z > 0 \) can be quantized as the operator \( a \), as in (28). The coupling strength of the mode \( k_q \) between the cavity and the waveguide \( G_q(t) \) is equivalent with \( \zeta_q(t)\sqrt{\frac{c^2 \Gamma/l}{(ck_q - \Delta_0)^2 + \Gamma^2}} \) according to the abovementioned analysis. Thus the single photon amplitude \( c_qq(t,k_q) \propto \sqrt{(ck_q - \Delta_0)^2 + \Gamma^2} \). Then generalized from (73), when the discrete mode \( \omega_q = ck_q = \Delta_0 \) in (27), the coupling between the cavity and the waveguide is maximized, as shown in Lemma 2 in the main text, the single photon amplitude is maximized at the Lorentzian peak around \( \Delta_0 \).

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Hailin Ding received the Ph.D. degree in quantum control from Tsinghua University, Beijing, China, in 2021.

From September to December 2021, he was a Research Assistant with the Hong Kong Polytechnic University, Hong Kong, and a Research Assistant with the Hong Kong Polytechnic University Shenzhen Research Institute, Shenzhen, China, in 2022. He is currently a Postdoctoral Researcher with the Laboratoire des Signaux et Systèmes (L2S), CNRS-Université Paris-Saclay, CentraleSupélec-Université Paris-Sud, Université Paris-Saclay. His research interests include quantum control and quantum optics.

Guofeng Zhang (Member, IEEE) received the B.Sc. and M.Sc. degrees from Northeastern University, Shenyang, China, in 1998 and 2000, respectively, and the Ph.D. degree from the University of Alberta, Edmonton, AB, Canada, in 2005, all in applied mathematics.

During 2005–2006, he was a Postdoctoral Fellow with the Department of Electrical and Computer Engineering, University of Windsor, Windsor, ON, Canada. In 2007, he was with the School of Electronic Engineering, University of Technology Chongqing. Since 2013, he has been with the School of Mechatronics Engineering, University of Alberta, Edmonton, AB, Canada. His research interests include quantum control, quantum algorithms, and tensor computation.

Dr. Zhang is an Associate Editor for IET Control Theory and Applications, 2015 IEEE Multi-Conference on Systems and Control, Sydney, Australia, September 21–23, 2015, and the Managing Guest Editor for the Special Issue on Quantum Control and Quantum Machine Learning for the Journal of The Franklin Institute.