Gauss Bonnet dark energy Chaplygin Gas Model

E. Karimkhani and A. Alaii and A. Khodam-Mohammadi

Department of Physics, Faculty of Science,
Bu-Ali Sina University, Hamedan 65178, Iran

Abstract

In this work we incorporate GB dark energy density and its modification, MGB, with Chaplygin gas component. We show that, presence of Chaplygin gas provides us a feature to obtain an exact solution for scalar field and potential of scalar field. Investigation on squared of sound speed provides a lower limit for constant parameters of MGB model. Also, we could find some bounds for free parameters of model.
I. INTRODUCTION

Astrophysical data out coming from distant Ia supernova [1], [2], [3], Large Scale Structure (LSS) [4], [5] and Cosmic Microwave Background (CMB) [6], [7] indicate that the expanding universe is in an accelerating phase. A missing energy component with negative pressure, so called, dark energy may cause this kind of expansion. The universe consists of about %73 of homogeneously distributed dark energy (DE), ~ %23 pressureless dark matter (DM), and ~ %4 is denoted for the usual baryonic matter. Lots of models have been proposed for explaining dark energy [8], [9]. Among many models, dynamical DE model with time varying equation of state (EOS) parameter have been given to solve some problems of standard cosmology such as coincidence problem. One of these series of models is Chaplygin gas (CG), first proposed by [10] and [11]. The initial idea of CG comes from aerodynamics [12] and regards it as a perfect fluid which plays a dual role in the history of the universe: it behaves as dark matter in the first epoch of evolution of the universe and as a dark energy at the late time. But the problem is that CG has some inconsistency with observational data like SNIa, BAO, CMB etc. [13], [14] and [15]. So Generalized Chaplygin gas (GCG) [16] and Modified Chaplygin gas (MCG) [17] was introduced in order to establish a viable cosmological model. In the other way, as a way it would be beneficial to involve a Gauss Bonnet term (GB) in the action and coupled it with scalar field which modified four dimensional gravity [18], [19]. Maybe this question arises why despite that the GB term does not contribute to the equations of motion, it has been involved in action? In fact, when it coupled to scalar field, which varies with time, it affects cosmological dynamics and as it has been shown in [20] at the background level a coupled Gauss-Bonnet term allows for a viable cosmology which could describe the early-time inflation and late-time acceleration.

In GB model, we propose density of dark energy as $\rho_\Lambda \propto L^{-4}$, because we want to consider density of dark energy in a natural way i.e. proportional to its volume without involving the black hole bound, as it has been studied thoroughly in [21]. But as we would show in this paper, the EoS parameter for GB dark energy model does not give rise to phantom phase of universe. Besides in [21], author shows that presence of matter drastically converts Friedmann equation into a nonlinear differential equation which changes the behavior of the EoS parameter and leads to $w_\Lambda \sim -1.17$ and allows for quintom behavior. However, in this paper, we incorporate GB dark energy density with a CG component without adding any
matter content. Our motivation for this incorporation is that CG eases in order to have an ultraviolet modifications of the general relativity. The presence of the CG component also provides a smooth crossing of the cosmological constant line in this scenario \[22\]. In addition, corporation GB or MGB with different CG models (i.e. CG, GCG and MCG) would be helpfull in order that obtain exact solution for scalar field and potential and would relieve us in order to determine some bounds for free parameters of model. So in the present paper, we investigate different cosmological solution for different composition of GB and CG models and we would show the traits for each one. Also, we would succeed in the frame work where $\kappa^2 = 8\pi G = M_p^{-2} = 1$ and in natural unit where $(\hbar = c) = 1$ and finally find that the best results which is consistent whit observation comes out when we consider MGB and GCG or MCG.

The outline of this paper is as follows: In next section, we introduce the GB dark energy and calculate the deceleration and EoS parameters. Then, in subsections 2.1 , 2.2 and 2.3 we investigate corporation GB with CG, GCG and MCG ,in turn, and then, scalar field and scalar potential is obtain by exact solution. Modification of GB leads to interesting cosmological consequences. Thus, In section III, the same procedure like section II has done by this difference that GB substitute by Modified Gauss Bonnet (MGB) energy density. In section IV, we would investigate Adiabatic Sound Speed, $v^2$, which is one of the critical physical quantity in the theory of linear perturbation. In section V, we discuss on behavior of scalar field, scalar potential and deceleration parameter versus $x$ for GB and MGB models and we gain some bounds for free parameters of models. Finally, we summarize our results in Sec. VI.

II. GAUSS BONNET DARK ENERGY IN A FLAT UNIVERSE

Gauss-Bonnet is invariant in 4-dimension and minimally coupled with curvature. Dark energy density in Gauss-Bonnet models is given by \[21\] as

$$\rho_D = \alpha G$$  \hspace{1cm} (1)

where $\alpha$ is a positive dimensionless parameter and $G$ is the 4-dimensional Gauss-Bonnet invariant

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\eta\gamma}R^{\mu\nu\eta\gamma}$$  \hspace{1cm} (2)
In a FRW universe with flat spatial part
\[ ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \] (3)
the Eq. (1) takes the form
\[ \rho_D = 24\alpha H^2 \left( H^2 + \dot{H} \right) \] (4)
By using the energy density \( \rho_D \), in absence of matter, Friedman equation in flat universe, yields
\[ H^2 = \frac{1}{3} \rho_D = 8\alpha H^2 \left( H^2 + \dot{H} \right) \] (5)
Defining the e-folding \( x \) with definition \( x = \ln a = \ln(1 + z) \), where \( z \) is the redshift parameter and using \( d/d(x) = \frac{1}{H} d/d(t) \), above equation becomes
\[ H^2 + \frac{1}{2} \frac{dH^2}{dx} - \frac{1}{8\alpha} = 0 \] (6)
Eq. (6) has the solution as
\[ H(x)^2 = \frac{1}{8\alpha} (1 + \xi e^{-2x}) \] (7)
Where \( \xi = 8\alpha H^2_0 - 1 \). Using the conservation equation
\[ \dot{\rho}_D + 3H (1 + w_D) \rho_D = 0 \] (8)
and Eqs. (4),(5), we can obtain the equation of state (EoS) parameter as
\[ w_D = -1 - \frac{\dot{\rho}_D}{3H \rho_D} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = -1 - \frac{2}{3} \left( \frac{1}{8\alpha H^2} - 1 \right) \] (9)
By using Eq.(7) the EoS parameter in term of \( x \) or \( z \) is given by
\[ w_D = -1 + \frac{2}{3} \left( \frac{\xi e^{-2x}}{1 + \xi e^{-2x}} \right) = -1 + \frac{\xi (1 + z)^2}{1 + \xi (1 + z)^2}. \] (10)
We see that the constant \( \xi \) plays a crucial role in the behavior of the EoS parameter. As for \( \xi = 0 \) (i.e. \( 8\alpha H^2_0 = 1 \)), the EoS parameter for \( \Lambda CDM \) model \( (w_\Lambda = -1) \) is retrieved and for other values of \( \xi \), the EOS parameter may get all values. As for \( \xi \neq 0 \) the universe may evolve in matter dominated phase, \( w = 0 \) (for large values of \( z \) and \( \xi \)), or Quintessence phase ,\( w < -\frac{1}{3} \), or even phantom phase, \( w < -1 \). Here it would be usefull to determine the signature of the deceleration parameter \( q \). By uses of Eqs. (3) and (7) we have
\[ q = -1 - \frac{\dot{H}}{H^2} = - \frac{1}{8\alpha H^2} = - \frac{1}{(1 + \xi e^{-2x})} \] (11)
Because $\alpha$ and $H_0^2$ are positive parameters so $\xi$ always must be greater than $-1$. Therefore, the deceleration parameter is always negative except for $-1 < \xi < 0$. In this way, the universe which is characterize by GB dark energy model could not exhibit a transition from deceleration to acceleration phase for $\xi \geq 0$ against what we expect from observations.

**A. Gauss Bonnet Chaplygin Gas Model**

Chaplygin gas is a perfect fluid with equation of state (EOS) as below

$$p_{CH} = -\frac{A}{\rho}$$  \hspace{1cm} (12)

where $p$ and $\rho$ are pressure and energy density and $A$ is a positive constant. By substituting Eq.(12) into the continuity equation, energy density is obtained as

$$\rho = \sqrt{A + Be^{-6\xi}}$$  \hspace{1cm} (13)

where $B$ is an integration constant[26]. By using standard scalar field DE model, the energy density and pressure of scalar field are written as

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) = \sqrt{A + Be^{-6\xi}}$$  \hspace{1cm} (14)

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) = -\frac{A}{\sqrt{A + Be^{-6\xi}}}$$  \hspace{1cm} (15)

By equating $p_{CH} = p_\phi$ and $\rho_{CH} = \rho_\phi$, the scalar potential and the kinetic energy terms for the CG model:

$$V(\phi) = \frac{2A + Be^{-6\xi}}{2\sqrt{A + Be^{-6\xi}}}$$  \hspace{1cm} (16)

$$\dot{\phi}^2 = \frac{Be^{-6\xi}}{\sqrt{A + Be^{-6\xi}}}$$  \hspace{1cm} (17)

The EOS parameter can be gained from Eqs. (14) and (15) as

$$w_{CH} = \frac{p}{\rho} = -\frac{A}{A + Be^{-6\xi}}$$  \hspace{1cm} (18)

By combining GB dark energy density with CG model we would calculate $A$ and $B$ constants. Equating Eqs. (14) and (13) and using Eq.(5) we could obtain constant $B$ as

$$B = e^{6\xi} (9H_0^4 - A) = e^{6\xi} \left[ \left( \frac{3}{8\alpha}(1 + \xi e^{-2\xi}) \right)^2 - A \right]$$  \hspace{1cm} (19)
Inserting Eq. (19) in Eq. (18), and using Eq.(10) the positive constant A is obtained as

\[ A = \frac{3}{(8\alpha)^2} [(2 + \xi e^{-2x})^2 - 1] \]  \tag{20}

Substituting A and B in Eqs. (16) and (17), we re-construct the scalar potential term

\[ V(x) = \frac{1}{8\alpha} (3 + 2\xi e^{-2x}) \]  \tag{21}

in this way kinetic energy term is obtained as

\[ \dot{\phi} = \frac{1}{2} \sqrt{\frac{\xi e^{-2x}}{\alpha}} \]  \tag{22}

According to \( \phi' = \frac{\dot{\phi}}{H} \), where prime means derivatives with respect to \( x = \ln a \), so we have

\[ \phi' = \sqrt{\frac{2\xi}{\xi + e^{2x}}} \]  \tag{23}

We want to understand behavior of scalar field so we integrate Eq. (23) to gain scalar field in terms of \( x \) as

\[ \phi = -\sqrt{2} \arctan h \left( \sqrt{\frac{e^{2x}}{\xi} + 1} \right) \]  \tag{24}

Eventually we rewrite the scalar potential with respect to the scalar field as

\[ V(\phi) = -\frac{1}{8\alpha} \left( 2 \cosh^2 \left( \frac{\phi}{\sqrt{2}} \right) - 3 \right) \]  \tag{25}

Here we established the connection between GB and CG models and reconstructed the potential and the dynamics of GB model to describe flat SCG universe. As it is seen, Eq. (21) just permits negative values for scalar field for all values of \( x \) variable. Now, by use of Eq.(10) and (24) we obtain EoS parameter regarding scalar field as

\[ w(\phi) = -1 + \frac{2}{3} \left[ \tanh^2 \left( \frac{\phi}{\sqrt{2}} \right) \right]^{-1} \]  \tag{26}

This equation reveals that the EoS parameter for a universe with dark energy density in the form of GB DE model reaches to \(-\frac{1}{3}\) for infinity value of \( \phi \) which represents a Quintessence phase of universe but never attains phantom phase.

B. Gauss Bonnet Generalized Chaplygin Gas

It’s possible to extend CG to the so-called generalized Chaplygin gas and it would be usefull in order to have more consistency with observation as mentioned in introduction.
Hence, in the present case, we consider a GB dark energy density with a GCG fluid in background and discuss on the connection between GB and GCG models and rewrite the potential and dynamics of scalar fields in flat universe. The Equation of state of GCG is given by

\[ p = -\frac{A}{\rho^{\delta-1}} \]  

where \( A \) is constant and \( 1 \leq \delta \leq 2 \). If \( \delta = 2 \) generalized chaplygin gas changes to chaplygin gas. As before, the energy density is obtained as

\[ \rho_{CH} = (A + Be^{-3\delta x})^{\frac{1}{\delta}} \]  

The scalar field model gives energy density and pressure of GCG as

\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) = (A + Be^{-3\delta x})^{\frac{1}{\delta}} \]  

\[ p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) = -A (A + Be^{-3\delta x})^{-\frac{\delta-1}{\delta}} \]  

So the scalar potential and kinetic energy terms take the form

\[ V(x) = \frac{2A + Be^{-3\delta x}}{2(A + Be^{-3\delta x})^{\frac{2}{\delta-1}}} \]  

\[ \dot{\phi}^2 = \frac{Be^{-3\delta x}}{(A + Be^{-3\delta x})^{\frac{2}{\delta-1}}} \]  

The EOS parameter can be obtained as

\[ w_{CH} = \frac{p}{\rho} = -\frac{A}{A + Be^{-3\delta x}} \]  

Like it is done in previous subsection, we can obtain constant B by equating Eq. (1) and (28) as

\[ B = e^{3\delta x} \left[ \left( \frac{3}{8\alpha} (1 + \xi e^{-2x}) \right)^{\delta} - A \right] \]  

Equating the EoS parameters: Eqs. (33) and (1) and Eq. (34), constant A obtain as follows

\[ A = \frac{3 + \xi e^{-2x}}{(8\alpha)^{\delta}} [3 (1 + \xi e^{(-2x)})]^{\delta-1} \]  

Inserting Eqs. (34) and (35) in Eqs. (31) and (32) we can express the potential and dynamics of GB Generalized Chaplygin Gas in flat universe as

\[ V(x) = \frac{3 + 2\xi e^{-2x}}{8\alpha} \]
\[ \dot{\phi} = \frac{1}{2} \sqrt{\frac{\xi e^{-2x}}{\alpha}} \]  
\[ (37) \]

As it is seen, potential and dynamics of GB Generalized CG are not function of \( \delta \) variable. Such as previous part, \( \phi' \) can be calculated as below and by integration with respect to \( x \) variable, scalar field would be gained as

\[ \phi' = \sqrt{\frac{2\xi}{e^{2x} + \xi}} \]  
\[ (38) \]

\[ \phi = -\sqrt{2} \arctan h \left( \sqrt{\frac{e^{2x}}{\xi} + 1} \right) \]  
\[ (39) \]

Simple calculation for scalar potential leads to

\[ V(\phi) = -\frac{1}{8\alpha} \left( 2\cosh^2 \left( \frac{\phi}{\sqrt{2}} \right) - 3 \right) \]  
\[ (40) \]

Two last equations are alike Eqs. (24) and (25), so the EoS parameter in term of \( \phi \) would be the same as Eq. (26).

C. Gauss Bonnet Modified Chaplygin Gas

For obtaining the gas equation in a more satisfactory way, it is possible to consider a modification for GCG by adding a positive term which have a linear relation with density to the EoS as it is known modified Chaplygin gas (MCG). Hence, the relation between pressure and energy density for the MCG is given by

\[ p = B\rho - \frac{A}{\rho^\epsilon} \]  
\[ (41) \]

where \( A, B \) are positive constants and \( 0 \leq \epsilon \leq 1 \). This EoS reduces to the GCG model for \( B = 0 \). By setting \( \epsilon = -1 \) and \( A = B + 1 \) cosmological constant \( \Lambda \) as a particular case is obtained. Continuity equation for energy density leads to

\[ \rho = \left[ \frac{A}{1+B} + Ce^{-3(B+1)(\epsilon+1)x} \right]^{\frac{1}{1+\epsilon}} \]  
\[ (42) \]

Initial condition for present time leads to \( C = \rho_0^{\epsilon+1} - \frac{A}{1+B} \). So the scalar potential and kinetic energy for MCG obtain as

\[ V(\phi) = \frac{1}{2} \left( (1 - B) \rho + A \rho^{-\epsilon} \right) \]  
\[ (43) \]
\[ \phi = (1 + B) \rho - A \rho^{\epsilon} \]  

These two parameters \( V(\phi) \) and \( \dot{\phi} \) are the same as Eqs. (36) and (37) because we equate dark energy density of each of CG, GCG or MCG models with GB dark energy density and also equate the EoS parameters of them, too. Hence scalar potential and kinetic energy for GB dark energy combined with each of CG, GCG or MCG are alike. And so on, Eqs. (39) and (40) are the same as previous sections. In addition that by use of this feature we could calculate constant parameter \( A \) in term of \( B \) for MCG model. It is worthwhile to mention that CG, GCG and MGC act on GB or MGB as a Catalyst which ease acquiring an exact solution for scalar potential and scalar field. Accordingly, by equating density of MCG and GB dark energy density Eqs. (42), (7) and (5) we could obtain \( A \) constant as

\[
A = \frac{1}{(1 - e^{-3(B+1)(\epsilon+1)x})} \left[ \left( \frac{3}{8\alpha} \left( 1 + \xi e^{-2x} \right) \right)^{1+\epsilon} - \left( \rho_\circ e^{-3x(1+B)} \right)^{1+\epsilon} \right] (1 + B) 
\]  

### III. MODIFIED GAUSS BONNET DARK ENERGY

The energy density of modified GB models (MGB) is given by

\[ \rho_D = H^2(\gamma H^2 + \lambda \dot{H}) \]  

where \( \gamma \) and \( \lambda \) are dimensionless constant. Using friedmann equation in flat universe and in absence of matter, we have

\[ \gamma H^2 + \frac{1}{2} \lambda \left( \frac{dH^2}{dx} \right) - 3 = 0 \]  

By solving above equation \( H^2 \) is obtained as

\[ H(x)^2 = \frac{1}{\gamma} (3 + \eta e^{-\frac{2\gamma x}{\lambda}}) \]  

Where \( \eta = \gamma H_0^2 - 3 \). For obtaining the equation of state (EoS) parameter for MGB model one can use Eqs. (46), (48) and (5), so we have

\[ \dot{H} = \frac{3 - \gamma H^2}{\lambda}, \quad \ddot{H} = \frac{-2\gamma \dot{H}}{\lambda} \]  

With simple calculation as before we have

\[ w_D = -1 - 2 \frac{\dot{H}}{3 H^2} = -1 + \frac{2\gamma}{3\lambda} \left( \frac{\eta e^{-\frac{2\gamma x}{\lambda}}}{3 + \eta e^{-\frac{2\gamma x}{\lambda}}} \right) \]
and we can write the EoS parameter regarding redshift, so it would be

\[ w_D = -1 + \frac{2\gamma}{3\lambda} \left( \frac{\eta(1+z)^{\frac{2}{\lambda}}}{3 + \eta(1+z)^{\frac{2}{\lambda}}} \right) \]  

(51)

Again we would try to examine the sign of deceleration parameter for MGB. As before, using Eqs. (49) and (48), \( q \) gains as

\[ q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{\gamma}{\lambda} \left( \frac{\eta e^{-\frac{2z}{\lambda}}}{3 + \eta e^{-\frac{2z}{\lambda}}} \right) \]  

(52)

Because \( \gamma \) and \( H_0^2 \) are positive parameters so \( \eta \) always must be greater than \(-3\). But using above equation would relieve us in order to determine some bound for free parameters of model. It is worth notice here that our universe has transition from deceleration to acceleration phase just for \( \eta > 0 \) and \( \frac{2z}{\lambda} > 1 \). we would discuss on it in more details and improved the bounds at discussion section.

**A. Modified Gauss Bonnet And Chaplygin Gas**

Such as before we want to establish correspondence between MGB and CG model and reconstruct the potential and dynamics of scalar field in flat cosmology. For obtaining integration constant \( B \), we equate Eqs. (13) and (46), so we have

\[ B = e^{6x} \left[ \left( \frac{3}{\gamma} (3 + \eta e^{-\frac{2z}{\lambda}}) \right)^2 - A \right] \]  

(53)

By using Eqs. (12), (13) from CG model and (50) from MGB model we can obtain constant \( A \) as

\[ A = \frac{9}{\gamma^2} \left( 3 + \eta e^{-\frac{2z}{\lambda}} \right) \left[ 3 + (1 - \frac{2\gamma}{3\lambda})\eta e^{-\frac{2z}{\lambda}} \right] \]  

(54)

Substituting \( A \) and \( B \) in Eqs. (16), (17) we have

\[ V(x) = \frac{3}{\gamma} \left[ 3 + (1 - \frac{2\gamma}{3\lambda})\eta e^{-\frac{2z}{\lambda}} \right] \]  

(55)

\[ \dot{\phi} = \sqrt{\frac{2\eta}{\lambda} e^{-\frac{2z}{\lambda}}} \]  

(56)

Integrating with respect to \( x \) variable potential is obtained as

\[ \phi = -\sqrt{\frac{2\lambda}{\gamma}} \arctan \left[ \sqrt{1 + \frac{3e^{\frac{2z}{\lambda}}}{\eta}} \right] \]  

(57)
Scalar potential term takes the form

\[ V(\phi) = \left( \frac{3}{\lambda} - \frac{9}{\gamma} \right) \cosh^2 \left[ \phi \sqrt{\frac{\gamma}{2\lambda}} \right] + \frac{9}{\gamma} \]  

(58)

Here, we could gain scalar field and potential of scalar field with an exact solution for MGB model beside that the cosine hyperbolic potential is able to describe both dark matter and dark energy within a tracker framework \[29\] and we discuss on it in more details at discussion section. Again, by use of Eqs. (57) and (50) we obtain EoS parameter in term of \( \phi \) as

\[ w_\Lambda(\phi) = -1 + \frac{2\gamma}{3\lambda} \left[ \tanh^2 \left( \phi \sqrt{\frac{\gamma}{2\lambda}} \right) \right]^{-1} \]  

(59)

Then, above equation shows that for MGB DE model, the EoS parameter for infinity value of \( \phi \) reaches to \(-1 + \frac{2\gamma}{3\lambda}\) and just for limiting case where \( \lambda \gg \gamma \) the EoS parameter for \( \Lambda CDM \) model \((w_\Lambda = -1)\) is regained.

**B. Modified Gauss Bonnet And Generalized Chaplygin Gas**

In this section we discuss correspondence between MGB model and GCG model then rewrite potential and dynamics of scalar field in flat universe. By equating Eqs. (28) and (46), constant \( B \) takes the form

\[ B = e^{3\delta x} \left[ \left( \frac{3}{\gamma} \right) (3 + \eta e^{-\frac{2\phi}{x}}) \right]^{\delta} - A \]  

(60)

Using Eqs. (27), (28) and (50) constant \( A \) is gained as

\[ A = \left( \frac{3}{\gamma} \right)^{\delta} \left( 3 + \eta e^{-\frac{2\phi}{x}} \right)^{\delta-1} \left[ 3 + \left( 1 - \frac{2\gamma}{3\lambda} \right) \eta e^{-\frac{2\phi}{x}} \right] \]  

(61)

Substituting \( A \) and \( B \) in Eqs. (31), (32), for potential and dynamics of MGB GCG we have

\[ V(\phi) = \frac{3}{\gamma} \left[ 3 + \left( 1 - \frac{\gamma}{3\lambda} \right) \eta e^{-\frac{2\phi}{x}} \right] \]  

(62)

\[ \dot{\phi} = \sqrt{\frac{2\eta e^{-\frac{2\phi}{x}}}{\lambda}} \]  

(63)

As before, \( \phi' \) is obtained as

\[ \phi' = \sqrt{\frac{2\gamma}{\lambda} \left[ \frac{\eta}{3 e^{-\frac{2\phi}{x}}} + \eta \right]} \]  

(64)
Integrating with respect to $x$ variable have

$$\phi = -\sqrt{\frac{2\lambda}{\gamma}} \tanh \left( \sqrt{\frac{3e^{\frac{2x}{\lambda}}}{\eta}} + 1 \right)$$

(65)

Scalar potential takes the form as

$$V(\phi) = \left( \frac{3}{\lambda} - \frac{9}{\gamma} \right) \cosh^2 \left[ \phi \sqrt{\frac{\gamma}{2\lambda}} + \frac{9}{\gamma} \right]$$

(66)

As we could see from these equations $V(\phi)$ and $\phi$ are independent from $\delta$ that comes from CG or GCG models. In this way, $w_D(\phi)$ is the same as Eq.(57), too.

C. Modified Gauss Bonnet And Modified Chaplygin Gas

In this section, we consider EoS parameter of MCG Eq. (41) which leads to Eq. (67) for energy density. Then, as before, equating it with MGB energy density Eq. (46), constant $A$ would be calculate as

$$A = \frac{1}{(1 - e^{-3(\rho + 1)(\epsilon + 1)x})} \left[ \left( \frac{3}{\gamma} \right) \left( 1 + \eta e^{-\frac{2x}{\lambda}} \right) \right]^{1+\epsilon} - \left( \rho_0 e^{-3x(1+B)} \right)^{1+\epsilon}(1 + B)$$

(67)

Because of the reason mentioned in section 2.3, $V(\phi)$ and $\phi$ are the same as Eqs. (55) and (56) and so on Eqs. (65) and (66) are confirmed in this section, likewise.

IV. ADIABATIC SOUND SPEED

Above models can be considered as models which unified dark matter and dark energy so we have to study perturbations and the structure formation. Investigation the squared of sound speed, $v^2$, would help us in order to determine the growth of perturbation in linear theory [30]. The sign of $v_s^2$ plays a crucial role in determining the stability of the background evolution. Positive sign of $v^2$ shows the periodic propagating mode for a density perturbation and probably represents an stable universe against perturbations. The negative sign of it shows an exponentially growing mode for a density perturbation, and can show sounds of instability for a given mode. Squared of sound speed define as

$$v^2 = \frac{dP}{d\rho} = \frac{\dot{P}}{\rho}$$

(68)
Differentiating the equation of state, $p = w_D \rho$ with respect to time, we have

$$\dot{p} = \dot{w_D} \rho + w_D \dot{\rho}$$

(69)

As it has been calculated for the EoS parameter we have $w_\Lambda = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}$ (even for GB or MGB Eqs. (9) and (50)). Then using Eqs. (8), (68) and (69) we could gain a relation for the squared sound speed in flat universe as

$$v^2 = -1 - \frac{1}{3} \left( \frac{\ddot{H}}{H} \right)$$

(70)

For GB dark energy model by using Eq. (5) $v^2$ is obtained as

$$v^2 = -\frac{1}{3}$$

It is seen that the squared of sound speed is independent of scale factor and constant for any time and its sign shows instability against the density perturbation. For MGB, Eq.(49) gives

$$v^2 = -1 + \frac{2\gamma}{3\lambda}$$

(71)

Which reveals that $v^2$ can be positive if $\frac{2\gamma}{3\lambda} > \frac{3}{2}$. Thus an stable DE dominated universe may be achieved in this model. In the next section we would improve this bound for $\frac{2\gamma}{3\lambda}$ in a proper way.

V. DISCUSSION

In this section we just focus on MGB dark energy model which it can give an stable DE dominated universe. At first, we start with Eq. (52) and plot deceleration parameter with respect to $x$ in Fig. (1). It shows that the deceleration parameter transits from deceleration ($q > 0$) to acceleration ($q < 0$) at some point in the past. The parameters $\eta$ and $\frac{x}{\chi}$ play a crucial rule for this point: As $\eta$ or $\frac{x}{\chi}$ adopt bigger values, the transition point approaches to us and propels in time. By choosing the best values for $q_0(\sim -0.6)$ and also for transition point as ($x \simeq -0.5$) which has been parameterized recently by [31], [32] and also is consistent with observations, [33], we obtain some bounds for $\eta$ and $\frac{x}{\chi}$ so that $0 < \eta < 2.5$ and $1.5 \leq \frac{x}{\chi} \leq 3$. By the way, MGB dark energy model has this feature to
predict transition from deceleration to acceleration as anticipated. In Fig. (2) by use of Eq. (55) for MGB model we plot $\gamma V(\phi) = V(\tilde{\phi})$ in versus $x$ for different values of $\frac{\lambda}{\chi}$ and $\eta = 1.5$. This figure shows that as time goes, $V(\tilde{\phi})$ inclines to small values and at late time, when $x \rightarrow \infty$, the scalar potential will reach to a constant. In addition, by increasing the ratio of $\frac{\lambda}{\chi}$ the scalar field potential adopts bigger values.

As it is seen, Eq. (58) represent a cosine hyperbolic form for potential and makes it able to describe both dark matter and dark energy within a tracker framework. According to
the quintessential tracker solution, our universe undergoes a phase from \( w = 0 \) to \( w = -1 \) and the effective equation of state value is \( w_{\text{eff}} = -0.75 \), \([34]\). The quintessence component tracks the background density for most of the history of the universe, then only recently grows to dominate the energy density and drives the universe into a period of accelerated expansion. The big advantage of the tracker solution is that it allows the quintessence model to be insensitive to initial conditions, \([35]\). So we use this feature in order to improve obtained bounds for parameters of our interest. In this way, Eqs. \((52)\) and \((62)\), for matter dominate universe \( (w = 0) \), lead to an equation for scalar potential as \( V(\phi) = \frac{9\gamma}{2\gamma-3\lambda} \) which prohibits \( \frac{\gamma}{\lambda} = \frac{3}{2} \) that causes infinite values for potential, and contrary to what is expected.

On the other hand these equations for quintessence phase \( (w = -\frac{1}{3}) \) give rise to \( V(\phi) = \frac{6\gamma}{\gamma-\lambda} \), which as before prevents \( \frac{\gamma}{\lambda} = 1 \) and this consists with what we obtained from investigation on deceleration parameter. Finally, given these limitations for \( \frac{\gamma}{\lambda} \), above bound for it changes to \( 1.5 < \frac{\gamma}{\lambda} \leq 3 \).

**VI. CONCLUSION**

In this paper we incorporate GB and MGB energy density combined with each of CG, GCG or MCG. This combination provides us a feature to obtain scalar field potential and scalar field with an exact solution. (Eqs. \((21)\), \((25)\) and \((24)\) for GB model and Eqs. \((36)\), \((39)\) and \((40)\) for MGB model). In addition, by use of this trait we could calculate constant parameter \( A \) and \( B \) for GC, CGC and MCG models. Premier, it would ease in order to determine some bound for free parameters of model. The EoS and deceleration parameters for both GB and MGB models was calculated. For GB model, deceleration parameter is always negative except for \( -1 < \xi < 0 \), thus our universe which is characterize by GB dark energy model could not exhibit a transition from deceleration to acceleration phase. It is obtained that the EoS parameter for a universe with dark energy density in the form of GB DE model reaches to \( -\frac{1}{3} \) for infinity value of scalar field which represents a Quintessence phase of universe but never achieves phantom phase. Also, for \( \xi = 0 \) (i.e. \( S_0 H_0^2 = 1 \)), the EoS parameter for \( \Lambda CDM \) model is retrieved. Investigation on the squared of sound showed instability against the density perturbation for GB model of dark energy density. For MGB model, we found that deceleration parameter has transition from deceleration to acceleration just for a limited range of values for \( \eta \) and \( \frac{\gamma}{\lambda} \) as when these two quantity adopt bigger values.
the transition point approaches to present time and propels in time. Choosing the best values for deceleration parameter at present time and also for transition point, according to observations, we could find bounds for $\eta$ and $\frac{\gamma\lambda}{\lambda}$ such as $0 < \eta < 2.5$ and $1.5 \leq \frac{\gamma\lambda}{\lambda} \leq 3$ and other calculations improve these bounds in more proper way. By redefining $\gamma V(\phi) = \tilde{V}(\phi)$ we showed that as time goes, the scalar potential inclines to small values and at late time will reach to a constant. Our investigation on $V(\phi)$ for matter dominate and quintessence phase of universe showed that $\frac{\gamma\lambda}{\lambda}$ could not be $1$ and $\frac{3}{2}$. Study of squared of sound speed for MGB model revealed that in order to have a stable DE dominated universe we need $\frac{\gamma\lambda}{\lambda} > \frac{3}{2}$. Hence our obtained bound for $\frac{\gamma\lambda}{\lambda}$ got more efficient and changed to $1.5 < \frac{\gamma\lambda}{\lambda} \leq 3$.

[1] A. G. Riess, et al., Astron. J. 116, 1009 (1998); [arxiv:astro-ph/9805201].
[2] S. Perlmutter, et al., Nature 391, 51 (1998).
[3] M. Hicken, et al., Astrophys. J. 700, 1097, (2009); [arxiv:astro-ph/0901.4804].
[4] M. Tegmark, et al, Astrophys. J. 606, 702 (2004).
[5] K. Abazajian, et al. [SDSS Collaboration] Astron. J. 129, 1755 (2005); [arxiv:astro-ph/0410239].
[6] D. N. Spergel, et al., Astrophys. J. Suppl. 148, 175 (2003).
[7] E. Komatsu, et al. [WMAP Collaboration], Astrophys. J. Suppl. 180, 330 (2009); [arXiv:astro-ph/0803.0547].
[8] E. J. Copeland, M. Sami and shinji Tsujikawa, Int. J. Mod. phys. D 15 (2006) 1753-1936 ; arXiv:hep-th/0603057.
[9] S. Nojiri, S. D. Odintsov, Phys. Rept. 505, 59 (2011); [arXiv:gr-qc/1011.0544].
[10] A. Y. Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B 511 (2001) 265.
[11] M. C. Bento, O. Bertolami, and A. A. Sen, Phys. Rev. D 66 (2002) 043507; [arXiv:grqc/0202064].
[12] S. Chaplygin, Sci. Mem. Moscow Univ. Math. Phys. 21 (1904) 1.
[13] V. Gorini, A. Kamenshchik, U. Moschella, V. Pasquier; arXiv:gr-qc/0403062.
[14] Z. H. Zhu Astron. Astrophys., 423 (2004) 421.
[15] M.C. Bento, O. Bertolami and A. A. Sen , Phys. Lett. B 575 (2003) 172.
[16] N. Bilic, G. B.Tupper and R. D. Viollier, Phys. Lett. B 535 (2001) 17.
[17] U. Debnath, A. Banerjee, and S. Chakraborty, Class. Quantum Grav. 21 (2004) 5609.

[18] G. Kofinas, R. Maartens and E. Papantonopoulos, JHEP 0310 (2003) 066; [arXiv:hep-th/0307138].

[19] R. A. Brown, R. Maartens, E. Papantonopoulos and V. Zamarias, JCAP 0511, 008 (2005); [arXiv:gr-qc/0508116];
R. A. Brown, Gen. Rel. Grav. 39 (2007) 477; [arXiv:gr-qc/0602050];
R. G. Cai, H. S. Zhang and A. Wang, Commun. Theor. Phys. 44 (2005) 948; [arXiv:hep-th/0505186].

[20] S. Nojiri, S. D. Odintsov and M. Sasaki, Gauss-Bonnet dark energy, Phys. Rev. D 71 (2005) 123509; [arXiv:hep-th/0504052].

[21] L. N. Granda; [arXiv:gr-qc/1308.6565v1].

[22] K. Nozari, et al. Mon. Not. R. Astron. Soc. 412 (2011) 2125–2136 .

[23] A. Khodam Mohammadi, M. Malekjani, M. Monshizade, Mod. Phys. Lett. A 27 (2012) 1250100; [arXiv:1201.3200].

[24] S. Nojiri, S.D. Odinstov, Phys. Rev. D 74 (2006) 086005.

[25] K. Karami, M.S. Khaledian, [arXiv:1004.1805]

[26] M. Malekjani, A. khodam-mohammadi, Int. J. Mod. Phys. D 20 (2011) 281-297 .

[27] M.C. Bento, O. Bertolami and A.A. Sen, Phys.Rev. D 70 (2004) 083519.

[28] R.R. Caldwell, Phys. Lett. B 545 (2002) 23.

[29] Urena-L´opez, L.A. and Liddle, A., Phys. Rev. D 66 (2002) 083005; [arXiv:astro-ph/0207493].

[30] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75 (2003) 559.

[31] D. Pavon et al., Phys. Rev. D 86 (2012) 083509.

[32] A. Khodam-mohammadi, E. Karimkhani., IJMPD Vol. 23, No. 10 (2014) 1450081.

[33] R. A. Daly et al., Astrophys. J. 677 (2008) 1.

[34] P. J. Steinhardt et al., Phys.Rev. D 59 (1999) 123504; [arXiv:astro-ph/9812313].

[35] J. Yoo and Y. Watanabe., Theoretical Models of Dark Energy; [arXiv:1212.4726v1].