Elementary Discrete and Continuous Interplay

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Abstract
We illustrate the interplay between certain discrete and continuous problems, by presenting a method for the study of the asymptotics of a divergent sequence, through consideration of the asymptotics of its continuous analogue.

1 Introduction

The alternating harmonic series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \) is convergent and its sum is \( \ln 2 \). The non-linear difference equation

\[ a_{n+1} = a_n + \frac{1}{a_n}, a_1 = 1, \]

has a divergent solution which grows like \( \sqrt{2n} \) as \( n \to \infty \). The first fact is common knowledge usually proved by going to the Maclaurin expansion of \( \ln(1 + x) \) and then laboring a bit to show that it remains valid for \( x = 1 \). The second fact would probably strike most readers as nothing more than a curiosity, perhaps obtained by some clever manipulation. Both problems belong in a discrete setting, though their answers derive somehow from the continuous setting. But this is no strange thing, since the continuous is constructed from the discrete through the completeness property. What is not clear is that these seemingly disparate mathematical facts, can be viewed on a common platform, and their solutions can be obtained from a method that exploits what we shall call here the discrete-continuous interplay. The method, which is the subject of this article, is elementary, requiring no prerequisite beyond calculus and a little knowledge of differential equations. It can be used in any course where the theory of convergence is given, and allows both instructor and student to go beyond the dichotomy of convergence-divergence of a given sequence or series. Specifically, it will make it possible to enrich the study of convergence theory, by opening the possibility of treating novel, and useful, problems by a method that, from the start, suggests a line of attack and a possible answer. The idea is that
a problem in the discrete setting is translated, by use of a simple dictionary, into
a problem in the continuous setting, where a solution is sought. If a solution
of the continuous problem is found, it will suggest an answer and a possible
approach for the solution of the discrete problem.

2 The integral test

One of the earliest exposures to the interplay between the discrete and the
continuous occurs in the integral test for convergence of series, familiar to all
beginners in calculus. The integral test is based on the two inequalities
\[
\int_1^{n+1} f(t)dt < f(1) + f(2) + \cdots + f(n) < f(1) + \int_1^n f(t)dt,
\]
which are valid for any function \( f \) defined on \([1, \infty)\), positive, and non-increasing
there. The middle sum is the \( n^\text{th} \) partial sum \( S_n \) of the series \( \sum_{k=1}^\infty f(k) \), while
the integral on the right is the ”continuous analogue” of \( S_n \). These inequalities
imply that the infinite series \( \sum_{k=1}^\infty f(k) \), converges if and only if the improper
integral \( \int_1^\infty f(t)dt \) converges. But as we shall see readily, these inequalities
supply one of many bridges between the discrete and continuous. As a warm up
for what is to come, we shall show how it can be employed to find the sum of
a convergent series from the divergence of another. To illustrate, consider the
case where \( f(t) = \frac{1}{t} \). Then we have
\[
\int_1^{n+1} \frac{1}{t}dt < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{t}dt,
\]
and we conclude that the harmonic series \( \sum_{k=1}^\infty \frac{1}{k} \) is divergent. But it is crucial
not to be content with this conclusion. Indeed, if we introduce the sequence \( H_n \)
defined by
\[
H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \int_1^n \frac{1}{t}dt,
\]
then \( H_n \) is a positive, monotone decreasing sequence, and hence convergent. Its
limit is usually denoted by \( \gamma \), and called the Euler-Mascheroni constant. Having
this new information at hand, and noticing that
\[
\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = H_{2n} - H_n + \int_1^2 \frac{1}{t}dt,
\]
we are led immediately, and without any appeal to a Maclaurin series, to the
sum of the alternating series
\[
\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} = \int_1^2 \frac{1}{t}dt =: \ln 2.
\]
So the divergence of the harmonic series, as seen through the two inequalities between the discrete and the continuous, led to a convergent sequence, which in turn led to the convergence and evaluation of the sum of another infinite series. The value of \( \ln 2 \) for the sum of the series, turned out to be independent of the limit of the sequence that helped find it.

The point made in the previous example deserves another illustration, so let us consider the series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k} \). The relevant function here is \( \frac{\ln t}{t} \) which is decreasing on \([e, \infty)\), so that the sequence \( T_n \) defined by

\[
T_n = \frac{\ln 1}{1} + \frac{\ln 2}{2} + \cdots + \frac{\ln n}{n} - \frac{1}{2} \ln^2 n
\]

is convergent. Once again we compute that

\[
\sum_{k=1}^{2n} \frac{(-1)^{k+1} \ln k}{k} = \sum_{k=1}^{2n} \frac{\ln k}{k} - \sum_{k=1}^{n} \frac{\ln 2k}{k}
\]

\[
= T_{2n} + \frac{1}{2} \ln^2 2n - \ln 2 \left( H_n + \ln n \right) - T_n - \frac{1}{2} \ln^2 n,
\]

and it follows, painlessly, that

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k} = \frac{1}{2} \ln^2 2 - \gamma \ln 2.
\]

Observe that the sum depends on the limit of the sequence \( H_n \), but not on that of \( T_n \). The reader can now, if he so wishes, evaluate the sums \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln^p k}{k} \), where \( p \) is a non-negative integer. He will be led to the introduction of constants \( \gamma_{\alpha} \) defined as limits

\[
\gamma_{\alpha} = \lim_{n \to \infty} \frac{\ln^\alpha 1}{1} + \frac{\ln^\alpha 2}{2} + \cdots + \frac{\ln^\alpha n}{n} - \frac{1}{\alpha + 1} \ln^{\alpha+1} n,
\]

and will be gratified to know they are called Stieltjes constants, appear as coefficients in the Laurent expansion of the zeta function of Riemann about the pole 1, and continue to be the subject of research [4]. But they will not be considered any further in this article.

It should be clear that the previous discussion can be summed up in the following simple proposition whose proof is left to the reader.

**Proposition 2.1** Let \( f \) be a positive function defined on \([1, \infty)\), and monotone non-increasing on \((a, \infty)\), where \( a \geq 1 \). Then the sequences

\[
A_n = \sum_{k=1}^{n} f(k) - \int_{1}^{n} f(t)dt, \quad B_n = \sum_{k=1}^{n} f(2k) - \int_{1}^{n} f(2t)dt
\]

are convergent. If, in addition, the series \( \sum_{k=1}^{\infty} f(k) \) is divergent, and \( \lim_{k \to \infty} f(k) = 0 \), then the series \( \sum_{k=1}^{\infty} (-1)^{k+1} f(k) \) is convergent, and its sum is

\[
\sum_{k=1}^{\infty} (-1)^{k+1} f(k) = L + \int_{1}^{2} f(t)dt.
\]
where \( L = \lim_{n \to \infty} (A_{2n} - 2B_n) = \lim_{n \to \infty} (A_n - 2B_n) \).

## 3 From discrete to continuous

The integral \( \int_1^n f(t)dt \) is here proposed as the "continuous analogue" of the "discrete" sum \( \sum_{k=1}^n f(k) \), but the two are not equal. However, see [3] for the case \( n = \infty \). In addition, we propose, the derivative \( f'(k) \) as the continuous analogue of the difference \( f(k+1) - f(k) \). The first step in the method that we propose here is to make a translation of the problem in the discrete setting into a continuous problem, using the dictionary as in the following table:

| Item   | Discrete | Continuous |
|--------|----------|------------|
| variable | \( n \) | \( t \) |
| function | \( a_n \) | \( f(t) \) |
| derivative | \( a_{n+1} - a_n \) | \( f'(t) \) |
| integral | \( \sum_{k=1}^n a_k \) | \( \int_1^n f(t)dt \) |

No claim is made as to how "faithful" this translation would be. For example, we know that \( a_n \to 0 \), if \( \sum_{k=1}^n a_k \) converges, but \( f \) may not have a limit at \( \infty \) if the integral \( \int_1^{\infty} f(t)dt \) converges. If \( a_n \to a \), as \( n \to \infty \), then the difference \( a_{n+1} - a_n \to 0 \), but this is not true of a function and its derivative. Nevertheless, use of this dictionary back and forth, will invariably lead to interesting novel problems. For example, it is known that for a function \( f \), differentiable on \((a, \infty)\), the condition

\[
\alpha > 0, \quad f'(t) + \alpha f(t) \to 0, \quad t \to \infty,
\]

implies that \( f(t) \to 0 \), as \( t \to \infty \). A translation of this gives the following problem on a real sequence \( a_n \): suppose that

\[
a_{n+1} - a_n + \alpha a_n = a_{n+1} + \beta a_n \to 0, \quad n \to \infty,
\]

does it follow that \( a_n \to 0 \), as \( n \to \infty \)?

Once a translation is made from the discrete to the continuous, the task shifts to a search for a solution of the continuous problem, which might involve a simple differential equation with a readily computable solution, or a very difficult integro-differential equation, or just a brief limit statement. But in solving any one of these, if that is possible, due attention must be made to the steps taken, in order to be able to translate them, usually in reverse order, back into the discrete case. Thus when \( f' \) is integrated to get \( f \), the corresponding step on the sequence would be to sum the differences \( a_{n+1} - a_n \). This means that some, sometimes not so simple, maneuvering must be used in order to obtain a proof in the discrete case. So no royal road is being paved here. The most important aspect of the method remains in its ability to suggest a possible line of attack and a possible answer. In addition, its range of applicability is reasonably wide. We shall illustrate various aspects of this method throughout.
the rest of this article, supplying enough details to elucidate its usefulness, as well as the subtleties involved in the process.

Consider the result stated in the introduction about the asymptotics of the sequence $a_n$ defined by

$$a_{n+1} = a_n + \frac{1}{a_n}, a_1 = 1.$$ 

This is an example of a first order difference equation. Were it linear, it would have been possible to solve for $a_n$ explicitly. But it is a non-linear difference equation, and so instead, we search for the asymptotic behaviour of $a_n$ as $n \to \infty$. This will give us the order of growth of $a_n$. It will tell us whether $a_n$ grows like some power $n^\alpha$, or like $\ln n$, etc...

Using the dictionary, with $a_n = f(t), a_{n+1} - a_n = f'(t)$, and ignoring for the moment the initial condition, the corresponding continuous problem involves a positive function satisfying the differential equation

$$f'(t) = \frac{1}{f(t)},$$

which is easily solved to give $f^2(t) = 2t$ as one particular solution. So we guess, from the "continuous" answer, that the sequence $a_n$ possibly satisfies $a_n \sim \sqrt{2n}$ as $n \to \infty$. We also see, from the continuous solution, that we should look at $a_n^2$. With these two insights at hand we can proceed to a proof. Squaring we first obtain

$$a_{n+1}^2 - a_n^2 = 2 + \frac{1}{a_n^2} \geq 2, a_2^2 - a_1^2 = 3.$$ 

Summing the first inequalities from 2 to $n$, gives us the first key inequality $a_n^2 \geq 2n$, for $n \geq 2$. But now this inequality gives us that

$$a_{n+1}^2 - a_n^2 = 2 + \frac{1}{a_n^2} \leq 2 + \frac{1}{2n}, n \geq 2,$$

and again summation gives us the second key inequality

$$a_{n+1}^2 \leq 2(n+1) + \frac{1}{2} \ln n, n \geq 2.$$ 

Putting the two together we obtain that

$$\lim_{n \to \infty} \frac{a_n^2}{2n} = 1, \lim_{n \to \infty} \frac{a_n}{\sqrt{2n}} = 1.$$ 

If instead of the term $\frac{1}{a_n}$ we had $\frac{1}{a_n^2}$, we would get $f^3(t) = 3t$, guess that $a_n \sim \sqrt[3]{3n}$, and start our proof by looking at $a_{n+1}^3 - a_n^3$. In fact we can replace the term $\frac{1}{a_n}$ by $\frac{1}{f(a_n)}$ thereby obtaining the following

**Theorem 3.1** Let $f$ be a positive non-decreasing continuous function defined on $[0, \infty)$. Let $a_n$ be a solution of

$$a_{n+1} - a_n = \frac{1}{f(a_n)}$$

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satisfying $a_1 > 0$. Put

$$F(x) = 1 + \int_0^x f(t)dt, \ x \geq 0.$$ 

Then

$$a_n \sim F^{-1}(n) \text{ as } n \to \infty.$$ 

**Proof.** The sequence $\{a_n\}$ is positive and increasing. Since $f$ is non-decreasing,

$$F(a_{n+1}) - F(a_n) = \int_{a_n}^{a_{n+1}} f(t)dt \geq f(a_n)(a_{n+1} - a_n) = 1,$$

for $n = 1, 2, 3, \ldots$. Summing from 1 to $n$ and noting that $F(a_1) > 1$, we obtain $F(a_{n+1}) \geq n + 1$ and so $a_n \geq F^{-1}(n)$, since the inverse function $F^{-1}$ is well-defined. But then $f(a_n) \geq f \circ F^{-1}(n)$ because $f$ is non-decreasing and this may be incorporated in the original difference equation to give

$$a_{n+1} - a_n \leq \frac{1}{g(n)} \text{ where } g = f \circ F^{-1}, \text{ and } n = 1, 2, 3, \ldots.$$ 

Once again, if we sum from 1 to $n$ we obtain

$$a_{n+1} - a_1 \leq \sum_{k=1}^{n} \frac{1}{g(k)}.$$ 

The composite function $g = f \circ F^{-1}$ is positive, non-decreasing, and continuous, and $F^{-1}$ is differentiable, so the change of variable formula [1] may be used to get the inequality

$$\sum_{k=1}^{n} \frac{1}{g(k)} \leq \frac{1}{g(1)} + \int_1^n \frac{dx}{g(x)} = \frac{1}{g(1)} + F^{-1}(n) - F^{-1}(1).$$

We have $g(1) = f(F^{-1}(1)) = f(0) > 0$, and so

$$a_{n+1} \leq a_1 + \frac{1}{g(1)} + F^{-1}(n) - F^{-1}(1) \leq c + F^{-1}(n + 1)$$

for a constant $c$ independent of $n$. We have thus shown that $F^{-1}(n) \leq a_n \leq F^{-1}(n) + c$, and the result follows since $F^{-1}(n) \to \infty$ as $n \to \infty$. 

**Example 3.1** If $a_{n+1} - a_n = a_n^{-\alpha}$, where $a_1, \alpha > 0$, then $a_n \sim (\alpha+1)^{-1} n^{\frac{1}{\alpha+1}}$ as $n \to \infty$.

**Example 3.2** If $a_{n+1} - a_n = \exp(-a_n)$, where $a_1 > 0$, then $a_n \sim \ln n$ as $n \to \infty$. 

6
4 Extensions

Returning to the solved example of the previous section, if instead of the term \( \frac{1}{a_n} \) we had \( \frac{1}{2a_n} \), we would get \( f^2(t) = \ln t \), guess that \( a_n \sim \sqrt{\ln n} \), and again start from the square of the given sequence. Thus we see that various generalizations of Theorem 1 are readily obtained. Here is a simple one whose proof is left to the reader.

Let \( f \) and \( g \) be positive, non-decreasing, and continuous functions defined on \([0, \infty)\). Let \( a_n \) be a solution of

\[
a_{n+1} - a_n = \frac{1}{f(a_n)g(n)}
\]

satisfying \( a_1 > 0 \). Put

\[
F(x) = 1 + \int_0^x f(t)dt, \quad x \geq 0,
\]

and assume that \( \int_1^n \frac{1}{g(t)} dt \to \infty \) as \( n \to \infty \). Then

\[
a_n \sim F^{-1}(\int_1^n \frac{1}{g(t)} dt) \quad \text{as} \quad n \to \infty.
\]

We emphasize that our aim here is not complete generality. In fact it is advisable to treat a given problem on its own merits, as consideration of the following example, where \( g \) is actually decreasing, will reveal.

**Example 4.1** If the sequence \( a_n \) is defined by the recurrence

\[
a_{n+1} - a_n = \frac{n^\alpha}{3a_n^2}, \quad a_1 > 0, \alpha > 0,
\]

then

\[
a_n \sim \frac{1}{\sqrt[3]{\alpha + 1}} n^{\frac{\alpha + 1}{3}}, \quad n \to \infty.
\]

5 Second order differences

The results in the previous section involved a sequence and its first differences. In other words we considered the problem of obtaining the asymptotic behaviour of a sequence satisfying a first order non-linear difference equation. Since the equation was not linear, it was not possible to solve it explicitly and that led to the question of obtaining its asymptotic behaviour. It is natural to ask whether the method could be of help in the case where second order differences arise. Of course, there will be added difficulties. The next example, with all its details, illustrates both the utility of the method and the difficulties that are bound to arise as we try to obtain an argument in the discrete setting from the usually easier argument in the continuous setting.
Problem 5.1 If $a_n$ is a sequence satisfying

$$a_{n+1} = a_n + \frac{1}{a_n} \sum_{k=1}^{n} a_k, a_1 = 1,$$

obtain the asymptotic behaviour of $a_n$.

If we put $A_n = \sum_{k=1}^{n} a_k$, it becomes clear that this equation involves first and second differences of $A_n$. Thus if we use the dictionary with the function $F$ corresponding to $A_n$ and $f$ corresponding to $a_n$ or $F'$, then the given difference equation and its continuous analogue are

$$a_{n+1} = a_n + \frac{A_n}{a_n}, F''(x)F'(x) = F(x).$$

In the continuous equation, multiplying by $F'$ and integrating, we are led, successively, to the equations

$$f^3(x) = \frac{F^2(x)}{2}, F^{1/3}(x) = \frac{1}{3} \left( \frac{3}{2} \right)^{1/3} x, F(x) = \frac{x^3}{18}, f(x) = \frac{x^2}{6}.$$ 

Thus we guess that $a_n \sim \frac{n^2}{6}$, and we have to start by obtaining a relationship between $2a_n^3$ and $3A_n^3$, as suggested by the continuous case.

Let us see what can be done from very simple considerations. We have, successively, $a_n \geq 1$, $a_{n+1} - a_n \geq 1$, $a_{n+1} \geq n + 1$, $A_n \geq \frac{n(n+1)}{2} > \frac{n^2}{2}$. These inequalities, are quite far from the expected result, but could be useful in the analysis, notably in achieving a certain necessary matching of the indices as we shall see presently. But at least they tell us that both $a_n$ and $A_n$ tend to $\infty$ as $n \to \infty$, and, in addition, that

$$\frac{nA_{n-1}}{A_{n+1}^2} \leq \frac{2}{n}, n \geq 2.$$ 

We shall use the two identities:

$$a_{n+1}^3 - a_n^3 = 3a_nA_n + \frac{2A_n^2}{a_n}, a_{n+1}^3 - a_n^3 = 3a_nA_n + \frac{A_n^3}{a_n^3}. $$

In the first identity, the positivity of all terms involved gives us

$$a_{n+1}^3 - a_n^3 \geq 3a_nA_n, 2(a_{n+1}^3 - a_n^3) \geq 3a_nA_n + 3a_nA_{n-1} = 3(A_n^2 - A_{n-1}^2)$$

which, upon summing, with $A_0 = 0$, leads to

$$2a_{n+1}^3 \geq 3A_n^2 + 2a_1^3 \geq 3A_n^2.$$ 

In a similar manner, the second identity, leads to

$$2(a_{n+1}^3 - a_n^3) \leq 3a_{n+1}A_n + 3a_{n+1}A_{n+1} + 2A_{n+1}^3 = 3(A_n^2 - A_{n+1}^2) + 2A_{n+1}^3.$$


and

\[ 2a_{n+1}^3 - 2a_1^3 \leq 3A_{n+1}^2 - 3A_1^2 + 2 \sum_{k=1}^{n} \frac{A_k^3}{a_k^3}. \]

Since \( A_n^{1/3} \geq 1 \), for \( n \geq 1 \),

\[ \left( \frac{A_n}{a_n} \right)^3 = \left( 1 + \frac{A_{n-1}}{a_n} \right)^3 \leq \left( 1 + \left( \frac{2}{3} \right)^{1/3} A_{n-1}^{1/3} \right)^3 \leq 8A_{n-1}, n \geq 2. \]

It follows then that

\[ 2a_{n+1}^3 - 2 \leq 3A_{n+1}^2 + 16 \sum_{k=2}^{n} A_{k-1} \leq 3A_{n+1}^2 + \frac{32}{n} A_{n+1}^2, \quad n \geq 2. \]

Thus we have, successively,

\[ \limsup_{n \to \infty} \frac{2a_n^3}{3A_n^2} \leq 1, \quad \lim_{n \to \infty} \frac{a_n}{A_n} = 0, \quad \lim_{n \to \infty} \frac{A_{n+1}}{A_n} = 1, \quad \liminf_{n \to \infty} \frac{2a_n^3}{3A_n^2} \geq 1, \quad \lim_{n \to \infty} \frac{2a_n^3}{3A_n^2} = 1, \]

and the first asymptotic relation is proved. It remains to obtain the behaviour of \( A_n \) and then \( a_n \). To this end we cast our first asymptotic result in the two equivalent forms

\[ \lim_{n \to \infty} \frac{A_{n+1} - A_n}{A_{n+1}^{2/3}} = \left( \frac{3}{2} \right)^{1/3}, \quad \lim_{n \to \infty} \frac{A_{n+1} - A_n}{A_n^{2/3}} = \left( \frac{3}{2} \right)^{1/3}. \]

Next, a use of the inequalities

\[ \frac{1}{3} \cdot \frac{A_{n+1} - A_n}{A_{n+1}^{2/3}} \leq \frac{1}{3} \int_{A_n}^{A_{n+1}} x^{-2/3} dx \leq \frac{1}{3} \cdot \frac{A_{n+1} - A_n}{A_n^{2/3}}, \]

along with the abelian result mentioned previously, readily leads to

\[ \lim_{n \to \infty} \frac{A_n^{1/3} - 1}{n} = \frac{1}{3} \left( \frac{3}{2} \right)^{1/3}. \]

Finally we have

\[ A_n \sim \frac{1}{18} n^3, \quad a_n \sim \frac{1}{6} n^2, \quad n \to \infty. \]

### 6 Tauberian results

Suppose \( T \) is a transform, whose exact form need not concern us here, that takes sequences into sequences, and we are given the existence of \( \lim_{n \to \infty} T a_n \). It is often necessary to find the asymptotic behaviour of the sequence \( a_n \). One of the difficulties encountered in such problems is that it appears that very little
is given, and one doesn’t have a clue as to how to start an attack on such a
problem. But if the problem admits of a translation into a continuous analogue,
then a line of attack might be suggested by the corresponding solution of the
continuous problem. Before we present an illustration of this, it will be useful
to have at hand the following well-known abelian result on sequences.

If \( a_n \) is a real or complex sequence, and \( a_n \to a \), as \( n \to \infty \), then

\[
\frac{a_1 + \cdots + a_n}{n} \to a, \ n \to \infty.
\]

Our next example illustrates the use of the method in a Tauberian problem.

**Problem 6.1** If \( a_n \) is a sequence of positive real numbers, and

\[
\lim_{n \to \infty} a_n \sum_{k=1}^{n} a_k^2 = 1,
\]

determine the order of growth of \( a_n \).

We employ the dictionary to move from the given problem to its continuous
analogue. So we have a positive function \( f \) satisfying

\[
\lim_{x \to \infty} f(x) \int_1^x f^2(t) dt = 1.
\]

If we recast this in the equivalent form

\[
\lim_{x \to \infty} f^2(x) \left( \int_1^x f^2(t) dt \right)^2 = 1,
\]

then with the introduction of \( F(x) = \int_1^x f^2(t) dt \), we have that \( \lim_{x \to \infty} F'(x) F^2(x) = 1 \), which, by L’Hospital’s rule, \( \downarrow \), implies that \( \lim_{x \to \infty} \frac{F^3(x)}{3x} = 1 \), so that \( \lim_{x \to \infty} \sqrt[3]{3x f(x)} = 1 \). So we guess that our sequence \( a_n \) satisfies \( \lim_{n \to \infty} \sqrt[3]{3n} a_n = 1 \), i.e. the sequence decays to zero like \( \frac{1}{\sqrt[3]{3n}} \). The continuous analogue also suggests a line of attack: introduce the sums \( A_n = \sum_{k=1}^{n} a_k^2 \), and look at \( A_n^3 \). Before proceeding any further, let us note that, since the method anticipates that the sequence \( a_n \) decays to zero like a power of \( n \), we should see if we can first establish the weaker result that the sequence \( a_n \) does actually tend to 0. To this end note that, if the series \( \sum_{k=1}^{\infty} a_k^2 \) converges then \( \lim_{n \to \infty} a_n A_n = 0 \) contrary to the given hypothesis. Thus the series diverges, and then the sequence \( A_n \) being monotone increasing and unbounded must tend to \( \infty \), and so \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n A_n \cdot \frac{1}{A_n} = 0 \). So \( a_n \to 0 \), and we can proceed to investigate its order of decay to 0. Motivated by the continuous analogue we compute

\[
A_{n+1}^3 - A_n^3 = a_{n+1}^2 \left( A_{n+1}^2 + A_{n+1} A_n + A_n^2 \right).
\]

We are given that \( \lim_{n \to \infty} a_n A_n = 1 \), so that we only need to find the limit

\[
\lim_{n \to \infty} a_{n+1} A_n.
\]
Since \( a_{n+1} A_n = a_{n+1} A_{n+1} - a_{n+1}^3 \), we also obtain that \( \lim_{n \to \infty} a_{n+1} A_n = 1 \), so that \( \lim_{n \to \infty} \left( \frac{A_{n+1}^3}{A_n} - A_n^3 \right) = 3 \). But then it follows, by the abelian result mentioned earlier, that

\[
\lim_{n \to \infty} \frac{A_{n+1}^3 - A_n^3}{n} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \left( \frac{A_{k+1}^3}{A_k} - A_k^3 \right)}{n} = 3.
\]

This gives \( \lim_{n \to \infty} \frac{A_n^3}{n} = 3 \), and we conclude that \( \lim_{n \to \infty} \sqrt[3]{3n} \ a_n = 1 \), as expected.

The reader will have no difficulty in formulating a generalization of this if we are given that \( a_n > 0 \), and

\[
\lim_{n \to \infty} \ a_n^p \sum_{k=1}^{n} a_k^q = 1,
\]

where \( p, q \) are positive integers.

Well, whether \( p, q \) are positive integers or just positive real numbers, the continuous analogue will be the problem

\[
\lim_{x \to \infty} f^p(x) \int_1^x f^q(t) \, dt = 1,
\]

which, when recast in the equivalent form

\[
\lim_{x \to \infty} F'(x) F^{q/p}(x) = 1, \quad F(x) = \int_1^x f^q(t) \, dt,
\]

tells us that \( \lim_{x \to \infty} \frac{F^{p+1}(x)}{(p+1)x} = 1 \), leads us to guess the appropriate decay of the sequence \( a_n \), and suggests that we start by putting \( A_n = \sum_{k=1}^{n} a_k^q \), and compute first differences of the \( A_n^{p+1} \). But here, when \( p + 1 \) is not a positive integer, there is no simple factorization of the difference. To proceed, we have to appeal to the continuous setting. Indeed, using that the derivative of \( t^{p+1} \) is \( (p+1) t^p \), we can write

\[
A_{n+1}^{p+1} - A_n^{p+1} = \int_{A_n}^{A_{n+1}} \left( \frac{q}{p} + 1 \right) t^p \, dt.
\]

Once we have this we obtain immediately a double inequality that replaces the identity in the case of cubes:

\[
\left( \frac{q}{p} + 1 \right) A_n^p a_{n+1}^q \leq A_{n+1}^{p+1} - A_n^{p+1} \leq \left( \frac{q}{p} + 1 \right) A_{n+1}^p (A_{n+1} - A_n) = \left( \frac{q}{p} + 1 \right) A_{n+1}^p a_{n+1}^q.
\]

On the right-hand side we have \( A_{n+1}^p a_{n+1}^q = (A_{n+1}a_{n+1}^p)^{\frac{q}{p}} \to 1 \) by hypothesis.

On the left-hand side we have \( A_n^p a_{n+1}^q = (A_n a_{n+1}^p)^{\frac{q}{p}} = (A_{n+1} a_{n+1}^p - a_{n+1}^{p+1})^{\frac{q}{p}} \to \)
1. We can thus proceed as before and obtain that
\[
\lim_{n \to \infty} \frac{A_{n+1}^q}{n} = \left( \frac{q}{p} + 1 \right),
\]
thereby obtaining the expected decay of \(A_n\), and so of \(a_n\).

7 Coupled systems

For problems involving a complex sequence \(z_n = a_n + ib_n\), the first thing that comes to mind is to separate into real and imaginary parts, to obtain a system of two real problems. As a simple example, the problem of the asymptotic behaviour of
\[
z_{n+1} = z_n + i n z_n, \quad z_1 = 1 + i,
\]
is equivalent to that of the coupled real system
\[
a_{n+1} = a_n + \frac{b_n}{n(a_n^2 + b_n^2)}, \quad b_{n+1} = b_n + \frac{a_n}{n(a_n^2 + b_n^2)}, \quad a_1 = b_1 = 1.
\]

Such systems are expected to be much more involved. But the method of the continuous analogue, can still be utilized as we hope to demonstrate in the result that follows. Exploration of the many possible generalizations is left to the reader.

Let \(a_n\) and \(b_n\) be the two sequences defined by the coupled system
\[
a_{n+1} - a_n = \frac{b_n}{b_n^2}, \quad b_{n+1} - b_n = \frac{a_n}{a_n^2}, \quad a_1 > 0, b_1 > 0.
\]

Our purpose here is to study the asymptotic behavior of each of these two sequences.

The continuous analogue, ignoring the initial conditions, consists of two positive functions satisfying the equations
\[
f'(t) = \frac{1}{g^2(t)}, \quad g'(t) = \frac{1}{f^2(t)},
\]
from which the one particular solution \(f(t) = g(t) = (3t)^{1/3}\) is easily derived. In particular \(f^3(t)g^3(t) = 9t^2\). So we guess that the sequences satisfy the asymptotic relations \(a_n \sim (3n)^{1/3}, b_n \sim (3n)^{1/3}, n \to \infty\). We also see that we might start with \(a_0^n\) and \(b_0^n\). Any inequality between these two will lead to a decoupling of the system and allow us to work separately on each sequence. We start by some straightforward observations:

The sequences \(a_n\) and \(b_n\) are positive increasing sequences. Denote their limits by \(a\) and \(b\) respectively. Then \(0 < a \leq \infty\), and \(0 < b \leq \infty\).

If \(0 < a < \infty\), then \(\frac{a_n}{n} \to \frac{a}{a^2}\), and \(a - a_n \sim \frac{a^n}{n}\), as \(n \to \infty\). If \(0 < b < \infty\), then \(\frac{b_n}{n} \to \frac{b}{b^2}\), and \(b - b_n \sim \frac{b^n}{n}\), as \(n \to \infty\). In particular, the limits \(a\) and \(b\) cannot both be finite.
If both $a$ and $b$ are infinite, then the asymptotic behavior of $a_n$ and $b_n$ is described in the following result:

**Theorem 7.1** Let $a_n$ and $b_n$ be the two sequences defined above, let $a$ and $b$ be their limits, and assume that $a = b = \infty$. Put

\[
\alpha = \limsup_{n \to \infty} n^{-\frac{2}{3}} a_n, \quad \alpha' = \liminf_{n \to \infty} n^{-\frac{2}{3}} a_n, \quad \beta = \limsup_{n \to \infty} n^{-\frac{2}{3}} b_n, \quad \beta' = \liminf_{n \to \infty} n^{-\frac{2}{3}} b_n.
\]

Then $\alpha' = \alpha = \beta = \beta'$, and

\[
a_n \sim (3n)^{1/3}, \quad b_n \sim (3n)^{1/3}, \quad n \to \infty.
\]

**Proof.** As suggested by the continuous case we consider the cubic powers of the two sequences with a view to obtain an inequality for their product. The monotonicity of $a_n$ and $b_n$, gives the elementary inequalities

\[
a_{n+1}^3 - a_n^3 \geq 3a_n^2 (a_{n+1} - a_n) = 3 a_n^2 b_{n+1}^3 - b_n^3 \geq 3b_n^2 (b_{n+1} - b_n) = 3 b_n^2 a_n^3.
\]

Upon summation and multiplication, these in turn imply

\[
(a_{n+1}^3 - a_1^3) (b_{n+1}^3 - b_1^3) \geq 9 \left( \sum_{k=1}^{n} \frac{a_k^2}{b_k^3} \right) \left( \sum_{k=1}^{n} \frac{b_k^2}{a_k^3} \right) \geq 9n^2, \quad n \geq 1,
\]

giving the "uncertainty" inequality

\[
a_{n+1}^3 \cdot b_{n+1}^3 \geq 9n^2, \quad n \geq 1.
\]

Of course, the matching is not perfect, but this is sort of in the nature of such problems. Rewrite the uncertainty inequality in the equivalent form

\[
a_{n+1}^{-2} \cdot b_{n+1}^{-2} \leq 9^{-\frac{2}{3}} n^{-\frac{4}{3}}, \quad n \geq 1,
\]

and use it to get

\[
a_{n+1}^{-1} - a_{n+2}^{-1} = \int_{a_{n+1}}^{a_{n+2}} t^{-2} dt \leq a_{n+1}^{-2} (a_{n+2} - a_{n+1}) \leq 9^{-\frac{2}{3}} n^{-\frac{4}{3}}.
\]

Now sum this up from $n \geq 2$, to $m \geq n + 2$ to obtain

\[
a_{n+1}^{-1} - a_{m+1}^{-1} \leq 9^{-\frac{2}{3}} \sum_{k=n}^{m-2} k^{-\frac{4}{3}}.
\]

We now let $m \to \infty$, and use the hypothesis that $a_m \to \infty$ as $m \to \infty$ to obtain
\[ a_{n+1}^{-\frac{1}{2}} \leq 9^{-\frac{1}{4}} \sum_{k=n}^{\infty} k^{-\frac{1}{2}} \leq 3 \cdot 9^{-\frac{1}{4}} (n-1)^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{3(n-1)}}, n \geq 2. \]

A similar result holds for \( b_n \) and we conclude that
\[
\lim \inf_{n \to \infty} n^{-\frac{1}{2}} a_n \geq 1, \lim \inf_{n \to \infty} n^{-\frac{1}{2}} b_n \geq 1.
\]

To get inequalities in the opposite direction, we return to the defining equations of the sequences and incorporate in them these new inequalities. Thus, for the first sequence, we get
\[
a_{n+1} - a_n \leq \frac{1}{(3(n-1))^{2/3}}, n \geq 2,
\]
which upon summation and estimation of the resulting sum with the corresponding integral, leads to
\[
a_{n+1} \leq a_2 + c + (3(n-1))^{1/3}, n \leq 2,
\]
where \( c \) is a constant. It follows that \( \lim \sup_{n \to \infty} n^{-\frac{1}{2}} a_n \leq 1 \), and hence the limit exists and equals 1. The same applies to the sequence \( b_n \). This completes the proof of the theorem.

8 Comparison of sequences

The constant sequence \( x_n = 1 \), and the sequence \( y_n = \sin^2 n \), are both positive and bounded, with \( y_n \leq x_n \), for all \( n \). Is there a way of quantifying, how much smaller is \( y_n \)? We propose to answer this question as follows: introduce two sequences \( a_n, b_n \) defined by
\[
a_{n+1} = a_n + \frac{x_n}{a_n}, b_{n+1} = b_n + \frac{y_n}{b_n}, a_1 = b_1 = 1.
\]

Then our method here may be used to obtain the asymptotic relations
\[
a_n \sim \sqrt{2n}, b_n \sim \sqrt{n}, n \to \infty.
\]
Thus the "excess" in magnitude of \( x_n \) over \( y_n \) is quantified by the constant \( \sqrt{2} \) present in these asymptotic relations.

9 Beyond the first term

So far all illustrations involved what is called the first term describing the asymptotic behavior of a given sequence. It is often desirable, and very useful, to find what is called the second term in the asymptotic expansion of the sequence. Of course we have not defined what an asymptotic expansion is, but we hope it will be clear from the next discussion what is intended. Let us go directly to an example, which we think, is quite suitable for our purpose.
Example 9.1 The sequence $x_n$ is defined by the recurrence

$$x_{n+1} = x_n - x_n^2, \, 0 < x_1 < 1.$$ 

Obtain the asymptotic behaviour of $x_n$.

First of all, it is easy to see that this sequence is positive and decreasing. That is $0 < x_{n+1} < x_n < 1$. So it is convergent, and it is immediate that its limit is 0. The corresponding continuous equation, $f'(t) = -f^2(t)$ has $f(t) = t^{-1}$ as one solution and we guess that $x_n \sim \frac{1}{n}$, or, put differently, that $nx_n - 1 \to 0$ as $n \to \infty$. If this can be proved, then we can go one step further and ask about the rate at which $nx_n - 1$ decays to 0. We trust that the reader will be able to show that $x_n \sim \frac{1}{n}$, and, by induction, that $nx_n < 1$ for all $n$. So let us attend to the question of the rate of decay of $nx_n - 1$. In order to apply our method, we need to put $a_n = nx_n$, and find a new equation satisfied by $a_n$ and then apply the continuous method to this equation. Perhaps the simplest way to do this is to multiply the given recurrence equation for $x_n$ by $n + 1$ and then write everything in terms of $a_n$. We are thus led to the problem of the asymptotic behaviour of $a_n$ where

$$a_{n+1} = a_n + \frac{a_n}{n} - \frac{(n+1)a_n^2}{n^2}, \, 0 < a_1 < 1.$$ 

The continuous analogue is a positive function $f$ satisfying

$$f'(t) = \frac{f(t)}{t} - (t + 1) \left( \frac{f(t)}{t} \right)^2.$$ 

The introduction of $g(t) = \frac{f(t)}{t}$, leads to the differential equation

$$g'(t) = -\frac{t + 1}{t} g^2(t),$$

which is solved by $g(t) = \frac{1}{t + \ln t}$, so that $f(t) = \frac{t}{t + \ln t}$. Thus $1 - f(t) = \frac{\ln t}{t + \ln t}$, and we guess that

$$1 - nx_n \sim \frac{\ln n}{n + \ln n} \sim \frac{\ln n}{n}, \, n \to \infty.$$ 

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