Quantum Fields on the Light Front,
Formulation in Coordinates close to the Light Front,
Lattice Approximation

E.-M. Ilgenfritz, S. A. Paston, H.-J. Pirner, E. V. Prokhvatilov, V. A. Franke

Abstract

We review the fundamental ideas of quantizing a theory on a Light Front including the Hamiltonian approach to the problem of bound states on the Light Front and the limiting transition from formulating a theory in Lorentzian coordinates (where the quantization occurs on spacelike hyperplanes) to the theory on the Light Front, which demonstrates the equivalence of these variants of the theory. We describe attempts to find such a form of the limiting transition for gauge theories on the Wilson lattice.
1. Introduction

The idea of quantizing relativistic fields on the Light Front (LF) was proposed by Dirac [1], who introduced the LF coordinates

\[ x^\pm = (x^0 \pm x^3)/\sqrt{2}, \quad x^\perp \equiv x^k, \quad k = 1, 2. \]  

(1)

instead of the Lorentzian coordinates \( x^0, x^1, x^2, x^3 \). Here, \( x^+ \) plays the role of time, \( x^- \) (the "lightlike" coordinate) plays the role of one of the spatial coordinates, and \( x^k \) are "transverse" coordinates, \( k = 1, 2 \).

The field theory is quantized on the hyperplane \( x^+ = 0 \), which is tangent to the light cone and therefore corresponds to the LF. The role of the Hamiltonian is here played by the generator of translations along \( x^+ \), i.e., the operator \( P_+ = (P_0 + P_3)/\sqrt{2} \), and the role of one of the momentum components (the "lightlike" component) is played by the generator of translations along \( x^- \), i.e., the operator \( P_- = (P_0 - P_3)/\sqrt{2} \).

One advantage of quantizing on the LF is the formal simplification of the problem of describing the quantum vacuum state in field theory. Standardly, fields are quantized at a fixed time in the Lorentzian coordinates, for example, at \( x^0 = 0 \), and their Fourier transforms are merely related to "bare" creation and annihilation operators \( a^+(p) \) and \( a(p) \). A "bare" (or "mathematical") vacuum is determined by the condition

\[ a(p)|0\rangle = 0. \]  

(2)

Such a state corresponds to the free theory vacuum and does not coincide with the physical vacuum of the interacting theory. When solving the stationary Schrödinger equation in the Fock space over this mathematical vacuum, we must also describe the physical vacuum state in terms of the "bare" vacuum or "bare" creation operators. Such a description is possible in principle if we introduce ultraviolet and infrared regularizations. But this description is immensely complicated (authors often confine themselves to the "Gaussian" approximation in the simplest models [2]).

For quantization in the LF coordinates, it is essential that the lightlike component of the momentum \( P_- \) be nonnegative, and \( P_- > 0 \) for states with positive squared mass. If massless physical particles are not present, then the state with the momentum \( p_- = 0 \) formally describes the physical vacuum, i.e., it also corresponds to the minimum of the operator \( P_+ \). Introducing the "bare" creation and annihilation operators on the LF, we can use them to define the corresponding mathematical vacuum. By virtue of the structure of the momentum operator \( P_- \), the mathematical vacuum corresponds to the minimum of this operator, i.e., it coincides with the physical vacuum (still formally because there are divergences in the theory). We demonstrate this below in the example of the scalar field theory.

The Fock space constructed over this mathematical vacuum on the LF can be used to describe solutions of the corresponding analogue of the stationary Schrödinger equation determining the masses \( M \) of bound states at fixed values of the momenta \( P_- \) and \( P_\perp \equiv (P_1, P_2) \),

\[ P_+|\psi\rangle = H|\psi\rangle = p_+|\psi\rangle, \]
\[ P_-|\psi\rangle = p_-|\psi\rangle, \]
\[ P_\perp|\psi\rangle = 0, \]  

(3)

where \( M^2 = 2p_+p_- \).
As an example, we consider the theory of a scalar field (with the mass $m$) governed by the Lagrangian density $L$:

$$L = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - U(\varphi) = \partial_+ \varphi \partial_- \varphi - \frac{1}{2} \partial_k \varphi \partial_k \varphi - \frac{1}{2} m^2 \varphi^2 - U(\varphi), \quad \partial_\mu \varphi \equiv \frac{\partial}{\partial x^\mu} \varphi. \quad (4)$$

To construct the canonical formalism at $x^+ = 0$, we introduce the Fourier transform of the field $\varphi(x)$ w.r.t. the coordinate $x^-$, taking the nonnegativity of the momentum $p_-$ into account:

$$\varphi(x^-, x^\perp) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dp_- (2p_-)^{-1/2} \left( a^+(p_-, x^\perp) e^{ip_- x^-} + a(p_-, x^\perp) e^{-ip_- x^-} \right). \quad (5)$$

Then

$$\int dx^- \partial_+ \varphi \partial_- \varphi = \int_0^\infty dp_- \frac{1}{2i} \left( \partial_+ a^+(p_-, x^\perp) a(p_-, x^\perp) - a^+(p_-, x^\perp) \partial_- a(p_-, x^\perp) \right) \quad (6)$$

which is the canonical form in which $a(p_-, x^\perp)$ and $ia^+(p_-, x^\perp)$ are canonically conjugate variables. The quantum operators $a(p_-, x^\perp)$ and $a^+(p_-, x^\perp)$ satisfy the commutation relations

$$[a(p_-, x^\perp), a^+(q_-, y^\perp)] = \delta(p_- - q_-) \delta^2(x^\perp - y^\perp),$$

$$[a(p_-, x^\perp), a(q_-, y^\perp)] = [a^+(p_-, x^\perp), a^+(q_-, y^\perp)] = 0. \quad (7)$$

(at $x^+ = 0$). Using the expression for the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} L, \quad (8)$$

we can find the operator $P_-$,

$$P_- = \int d^2 x^\perp \int dx^- T_{-} = \int d^2 x^\perp \int dx^- (\partial_- \varphi)^2 = \int d^2 x^\perp \int_0^\infty dp_- p_- a^+(p_-, x^\perp) a(p_-, x^\perp), \quad (9)$$

where we drop an infinite constant. We define the mathematical vacuum state $|0\rangle$ the conditions

$$a(p_-, x^\perp)|0\rangle = 0, \quad p_- \geq 0. \quad (10)$$

Then $P_- |0\rangle = 0$, and the state $|0\rangle$ can be interpreted as the physical vacuum. For this to hold, we drop the infinite constant in Eq. (9).

The Hamiltonian can be obtained standardly from the Lagrangian written in terms of the variables $a(p_-, x^\perp)$ and $a^+(p_-, x^\perp)$. In the theory under consideration, it coincides with the expression

$$P_+ = \int d^2 x^\perp \int dx^- T_{+} = \int d^2 x^\perp \int dx^- \left( \frac{1}{2} \partial_\nu \varphi \partial_k \varphi + \frac{1}{2} m^2 \varphi^2 + U(\varphi) \right). \quad (11)$$

(up to a constant).

We now briefly formulate the difficulties in quantizing on the LF.

1. Ultraviolet singularities in the quantum field theory require introducing a regularization and subsequently renormalizing the theory. In the LF quantization, it is difficult to introduce
a regularization preserving the Lorentz and gauge symmetries. Noninvariant regularizations make the renormalization procedure more difficult and, in particular, result in introducing unusual counterterms and relating new arbitrary constants to them. When using numerical methods to solve the Schroedinger equation on the LF nonperturbatively, we must work with a regularized theory and fit constants such that the calculation results depend only weakly on the regularization parameters.

2. The quantization on the LF is related to special features and divergences when the momentum $p_-$ of "bare" quanta tends to zero. From the standpoint of Lorentzian coordinates, these divergences can be interpreted as ultraviolet ones as the limit $p_- \to 0$ is reached in the domain $p_3 \to \infty$ under the condition $p^2 = m^2$. But if we regularize the theory by cutting off the momentum $p_-$ ($p_- \geq \varepsilon > 0$), then we also exclude vacuum effects. The physical vacuum then coincides with the "bare" vacuum, which forbids condensates and spontaneous vacuum symmetry breaking.

All the vacuum effects must be taken into account by additional terms in the Hamiltonian on the LF. Obtaining these terms is difficult. For instance, their role can be played by the counterterms that renormalize singularities as $p_- \to 0$ and reconstruct the results of the Lorentz-covariant perturbation theory w.r.t. the interaction constant in all orders [3–5]. We can introduce these terms based on semiphenomenological considerations, for instance, by studying the limiting transition on the LF starting from the theory quantized on a spacelike surface close to the LF (see [6, 7]). We describe this method below.

There were also attempts to take vacuum effects into account when quantizing gauge theories on the LF by analyzing the exact operatorial solution of the Schwinger model [8].

3. When calculating the mass spectra of bound states, we must restrict the consideration to a finite number of degrees of freedom of quantum fields. For this, we can restrict the coordinates together with introducing periodic boundary conditions on the fields in these coordinates, $|x^-| \leq L$, $x^2 \leq L^2_\perp$, and also restrict the obtained discrete set of momenta. The canonical formalism for such a formulation contains involved constraints because there are zero Fourier modes of fields w.r.t. the coordinate $x^-$. For gauge fields on the LF, this problem was considered in [9, 10]. Solving the Schroedinger equation in the Fock space on the LF with a fixed total momentum of bound states ($p_- = p_\perp = (\pi n)/L$, $p_\perp = 0$), with the condition $p_n > 0$, and with the introduced cutoff over transverse momenta of separate excitations, we obtain finite-dimensional subspaces of the Fock space depending on $n$ for each integer $n$ (and for the restricted set of transverse momenta). Bound-state masses can be obtained as the limit of values of $m_n^2 = 2p_{n,+}p_{n,-}$ found in each of the subspaces as $n \to \infty$. For a number of (1+1)-dimensional models, the values of $m_n^2$ already become practically independent of $n$ at not too large $n$ [11–13]. But for (3+1)-dimensional theories, the dimensions of the Fock subspaces depend on the whole set of momenta $p_\perp$, which makes the calculations considerably more involved [14]. Moreover, this regularization breaks the Lorentz (and gauge) symmetry, complicating the renormalization procedure and the restoration of symmetry in the limit of removed regularization.

Another regularization method is to restrict the number of "bare" quanta participating in constructing the bound-state wave function in the Fock space on the LF, i.e., to introduce a "cutoff" in the Fock space w.r.t. the total number of "particles" (the Tamm-Dankoff method on the LF) [15, 16]. We can then leave coordinates unrestricted. But passing to the Fock subspace then encounters serious technical troubles when the number of "particles" (i.e., bare quanta) increases.
Quantizing a theory on the LF can be interpreted as the formal limit of quantization in a Lorentzian reference frame moving with a speed close to the speed of light w.r.t. desired bound states. We parameterize the corresponding Lorentz coordinate transformation $x \to x'$ with the parameter $\eta > 0, \eta \to 0$:

$$x'^{+} = \frac{\sqrt{2}}{\eta} x^{+}, \quad x'^{-} = \frac{\eta}{\sqrt{2}} x^{-}, \quad x'^{\perp} = x^{\perp},$$  \hspace{1cm} (12)

where $x'^{\pm} = (x^{0} \pm x^{3})/\sqrt{2}$. The plane of the field quantization $x^{0} = 0$ approximates the LF plane as $\eta \to 0$.

For the further discussion, we pass from the "fast moving" Lorentzian reference frame $x'$ to the convenient coordinates $\tilde{x}$:

$$\tilde{x}^{+} = \eta x^{0} = x^{+} + \frac{\eta^{2}}{2} x^{-}, \quad \tilde{x}^{-} = \eta^{-1} (x^{0} - x^{3}) = x^{-}, \quad \tilde{x}^{\perp} = x^{\perp}. \hspace{1cm} (13)$$

The plane $\tilde{x}^{+} = 0$ then coincides with the plane $x^{0} = 0$, and the coordinates $\tilde{x}$ become the LF coordinates as $\eta \to 0$. The metric tensor corresponding to these coordinates has the nonzero components

$$g_{+-} (\tilde{x}) = g_{-+} (\tilde{x}) = 1, \quad g_{--} (\tilde{x}) = -\eta^{2}, \quad g_{kk} (\tilde{x}) = -1,$$

$$g^{+-} (\tilde{x}) = g^{-+} (\tilde{x}) = 1, \quad g^{++} (\tilde{x}) = \eta^{2}, \quad g^{kk} (\tilde{x}) = -1. \hspace{1cm} (14)$$

We now demonstrate the limiting-transition method in the example of the scalar field theory written in these coordinates. We define the Lagrangian density as

$$L(\tilde{x}) = \sqrt{g(\tilde{x})} \left( \frac{1}{2} \partial_{\mu} \varphi (\tilde{x}) \partial^{\mu} \varphi (\tilde{x}) - \frac{1}{2} m^{2} \varphi^{2} (\tilde{x}) - \lambda \varphi^{4} (\tilde{x}) \right) =$$

$$= \tilde{\partial}_{+} \varphi (\tilde{x}) \tilde{\partial}_{-} \varphi (\tilde{x}) + \frac{\eta^{2}}{2} \left( \tilde{\partial}_{+} \varphi (\tilde{x}) \right)^{2} - \frac{1}{2} \left( \tilde{\partial}_{-} \varphi (\tilde{x}) \right)^{2} - \frac{1}{2} m^{2} \varphi^{2} (\tilde{x}) - \lambda \varphi^{4} (\tilde{x}), \hspace{1cm} (15)$$

where $\lambda$ is the coupling constant. We define the canonical variables at $\tilde{x}^{+} = 0$ as

$$\varphi (\tilde{x}), \quad \Pi (\tilde{x}) = \frac{\delta L}{\delta (\tilde{\partial}_{+} \varphi)} = \eta^{2} \tilde{\partial}_{+} \varphi + \tilde{\partial}_{-} \varphi. \hspace{1cm} (16)$$

Then the Hamiltonian is

$$H(\eta) = \int d^{2} x^{\perp} \int d \tilde{x}^{-} \left( \frac{(\Pi - \tilde{\partial}_{-} \varphi)^{2}}{2 \eta^{2}} + \frac{1}{2} \left( \tilde{\partial}_{-} \varphi \right)^{2} + \frac{1}{2} m^{2} \varphi^{2} + \lambda \varphi^{4} \right). \hspace{1cm} (17)$$

We define the "bare" creation and annihilation operators using the Fourier transformation,

$$\varphi (\tilde{x}) = (2\pi)^{-3/2} \int d^{2} \tilde{p} d^{2} \tilde{p}^{-} \frac{1}{\sqrt{2 \omega_{p}}} [a(\tilde{p}) + a^{+} (-\tilde{p})] e^{-i \tilde{p}^{\perp}},$$

$$\Pi (\tilde{x}) = (2\pi)^{-3/2} \int d^{2} \tilde{p} d^{2} \tilde{p}^{-} \sqrt{\omega_{p}} \frac{1}{2} [a(\tilde{p}) - a^{+} (-\tilde{p})] e^{-i \tilde{p}^{\perp}}. \hspace{1cm} (18)$$
where $\tilde{p}\tilde{x} \equiv p_\perp x^+ + \tilde{p}_-\tilde{x}^-$ and $\omega_p = \left(\tilde{p}_-^2 + \eta^2(p_\perp^2 + m^2)\right)^{1/2}$. At $\tilde{x}^+ = 0$, these operators satisfy the commutation relations

$$[a(\tilde{p}), a^+(\tilde{q})] = \delta^3(\tilde{p} - \tilde{q}), \quad [a(\tilde{p}), a(\tilde{q})] = [a^+(\tilde{p}), a^+(\tilde{q})] = 0. \quad (19)$$

The free part of the Hamiltonian can be written in the form

$$H_0 = \int d^2p_\perp d\tilde{p}_- \frac{\omega_p - \tilde{p}_-}{\eta^2} a^+(\tilde{p})a(\tilde{p}). \quad (20)$$

If we restrict the field modes in $\tilde{p}_-$ by the cutoff $|\tilde{p}_-| \geq \varepsilon$ and assume that the interaction Hamiltonian is normally ordered (which was done in (20)), then we can easily see that in the limit $\eta \to 0$ under the finite-energy condition, we obtain the canonical formulation of the theory on the LF on the subspace $|f_0\rangle$ of the Fock space determined by the conditions

$$a(\tilde{p})|f_0\rangle = 0, \quad \tilde{p}_- \leq -\varepsilon. \quad (21)$$

In the limit, this subspace becomes the Fock space on the LF. To take the vacuum effects into account approximately, we also consider the vicinity of $\tilde{p}_- \to 0$, including the field modes with $|\tilde{p}_-| \leq \Lambda\eta$, where $\Lambda$ is some quantity with the dimension of momentum. At $\eta \to 0$, we neglect field modes in the interval $\Lambda\eta < |\tilde{p}_-| < \varepsilon$. We assume that such a distortion of the theory is inessential in the limit $\eta \to 0$ if we set $\Lambda \to \infty$ upon passing to this limit (after the corresponding ultraviolet renormalization of the theory).

We now perform the limiting transition in the framework of the perturbation theory in the small parameter $\eta$ for the solutions of the Schroedinger equation with Hamiltonian (17),

$$\tilde{H}(\eta)|f(\eta)\rangle = \tilde{E}(\eta)|f(\eta)\rangle, \quad (22)$$

describing states with finite energy (and mass). For this, we expand the Hamiltonian in powers of $\eta$:

$$\tilde{H}(\eta) = \frac{1}{\eta^2}H^{(0)} + \frac{1}{\eta}H^{(1)} + H^{(2)} + \ldots. \quad (23)$$

The terms of this expansion can be obtained by substituting the field expansions corresponding to the above mode splitting

$$\varphi(\tilde{x}) \approx \varphi_\varepsilon(\tilde{x}) + \varphi_{\Lambda\eta}(\tilde{x}), \quad (24)$$

in the Hamiltonian, where the term $\varphi_\varepsilon(\tilde{x})$ contains only modes with $|\tilde{p}_-| \geq \varepsilon$ and $\varphi_{\Lambda\eta}(\tilde{x})$ contains those with $|\tilde{p}_-| \leq \Lambda\eta$. We assume the analogous expansion for $\Pi(\tilde{x})$.

The free part (quadratic in fields) of Hamiltonian (23) can be written in the form

$$H_0(\Pi, \varphi) \approx H_0(\Pi_\varepsilon, \varphi_\varepsilon) + H_0(\Pi_{\Lambda\eta}, \varphi_{\Lambda\eta}), \quad (25)$$

and the interacting part is

$$H_I = \lambda \int d^2x^+ d\tilde{x}^- \varphi^4(\tilde{x}) \approx H_I(\varphi_\varepsilon) + H_I(\varphi_{\Lambda\eta}) + \lambda \int d^2x^+ d\tilde{x}^- \left(6\varphi_{\Lambda\eta}^2 \varphi_\varepsilon^2 + 4\varphi_{\Lambda\eta} \varphi_\varepsilon^3\right). \quad (26)$$

We neglect the term $\varphi_{\Lambda\eta}^3 \varphi_\varepsilon$ in the integrand because in the desired limit $\eta \to 0$ the quantity $\varphi_{\Lambda\eta}^3$ becomes constant and the integral over $\tilde{x}^-$ of $\varphi_\varepsilon$ therefore vanishes.
Representing the fields in terms of the creation and annihilation operators, we obtain

\[ H_0(\tilde{\Pi}_\varepsilon, \tilde{\varphi}_\varepsilon) = \frac{1}{\eta^2} \int d^2p_\perp \int_{|\tilde{\rho}_-|\geq \varepsilon} d\tilde{\rho}_- \left[ \sqrt{\tilde{p}_\perp^2 + \eta^2 (p^2_\perp + m^2)} - \tilde{\rho}_- \right] a^+(\tilde{p})a(\tilde{p}) = \]

\[ = \frac{2}{\eta^2} \int d^2p_\perp \int_{p_- \leq -\varepsilon} dp_- |p_-| a^+(p)a(p) + \int d^2p_\perp \int_{p_- \geq \varepsilon} dp_- \frac{m^2 + p^2_\perp}{2p_-} a^+(p)a(p) + O(\eta^2). \quad (27) \]

On the other hand, the fields \( \varphi_{\Lambda\eta}, \Pi_{\Lambda\eta} \) can be represented in a “fast moving” Lorentzian reference frame in the form

\[ \varphi_{\Lambda\eta}(\tilde{x}) \bigg|_{\tilde{x}_+ = 0} = \varphi_{\Lambda}(-x^3, x^\perp) \bigg|_{x^0 = 0}, \]

\[ \Pi_{\Lambda\eta}(\tilde{x}) \bigg|_{\tilde{x}_+ = 0} = \eta \Pi_{\Lambda}(-x^3, x^\perp) \bigg|_{x^0 = 0}, \quad (28) \]

where \( \Lambda \) is the momentum cutoff, \( |p'_3| \leq \Lambda \). Hence, it is easy to obtain

\[ \tilde{H}(\Pi_{\Lambda\eta}, \varphi_{\Lambda\eta}) = H_0(\Pi_{\Lambda\eta}, \varphi_{\Lambda\eta}) + H_I(\varphi_{\Lambda\eta}) = \frac{1}{\eta} (P_0 + P_3)_{\Lambda, x^0 = 0}, \quad (29) \]

up to a possible additive constant that makes the quantum operator in the r.h.s. of the equality nonnegative and its minimum equal to zero. As a result, we obtain

\[ H^{(0)} = 2 \int d^2p_\perp \int_{p_- \leq -\varepsilon} dp_- |p_-| a^+(p)a(p), \quad (30) \]

\[ H^{(1)} = (P_0 + P_3)_{\Lambda, x^0 = 0}, \quad (31) \]

\[ H^{(2)} = \int d^2p_\perp \int_{p_- \geq \varepsilon} dp_- \frac{m^2 + p^2_\perp}{2p_-} a^+(p)a(p) + \]

\[ + \lambda \int d^2x^\perp d_3x^3 \left( 6\varphi^2_\Lambda(x) \bigg|_{x^0 = 0} \tilde{\varphi}_\varepsilon^2(x) + 4\varphi_\Lambda(x) \bigg|_{x^0 = 0} \tilde{\varphi}_\varepsilon^3(x) + \varphi_\Lambda(x) \tilde{\varphi}_\varepsilon^4(x) \right). \quad (32) \]

We now construct the perturbation theory for Schrödinger equation (22), introducing expansions in powers of \( \eta \),

\[ |f(\eta)\rangle = |f^{(0)}\rangle + \eta |f^{(1)}\rangle + \eta^2 |f^{(2)}\rangle + \ldots, \]

\[ \tilde{E}(\eta) = E + O(\eta), \quad (33) \]

where we take the finiteness condition for the energy \( E(\eta) \) in the limit \( \eta \to 0 \) into account. In the lowest order in \( \eta \), we have

\[ H^{(0)} |f^{(0)}\rangle = 0. \quad (34) \]

Hence,

\[ a(p) |f^{(0)}\rangle = 0, \quad p_- \leq -\varepsilon. \quad (35) \]
Further,
\[ H^{(1)}|f^{(0)}\rangle + H^{(0)}|f^{(1)}\rangle = 0. \] (36)

By virtue of equality (34), we hence have
\[ \langle f^{(0)}|H^{(1)}|f^{(0)}\rangle = 0 \] (37)
or, equivalently,
\[ \langle f^{(0)}|(P_0 + P_3)_{\Lambda,x^\alpha=0}|f^{(0)}\rangle = 0 \] (38)

We therefore conclude that the dependence of the state \(|f^{(0)}\rangle\) on the field modes \(\varphi_{\Lambda \eta}\) and \(\Pi_{\Lambda \eta}\) must correspond to the minimum of the operator \((P_0 + P_3)_{\Lambda,x^\alpha=0}\). Introducing the notation \(|\text{vac}_{\Lambda}\rangle\) for this dependence, we obtain the general form of the basis for the states \(|f^{(0)}\rangle\):

\[ \{|f^{(0)}\rangle\} = \left\{ \prod_{n,p_\mu \geq \epsilon} a^+(p_n)|0\rangle \right\} \otimes |\text{vac}_{\Lambda}\rangle_{x^\alpha=0}. \] (39)

In the next order in \(\eta\), we have
\[ H^{(2)}|f^{(0)}\rangle + H^{(1)}|f^{(1)}\rangle + H^{(2)}|f^{(0)}\rangle = E|f^{(0)}\rangle. \] (40)

Hence, the values of \(E\) are determined by the set of eigenvalues of the operator

\[ \mathcal{P}_0 H^{(2)} \mathcal{P}_0 = H_{\text{LF}} = \int d^2 p_\perp \int_{p^- \geq \epsilon} dp_- \frac{m^2 + p_\perp^2}{2p_-} a^+(p)a(p) + \]
\[ + \lambda \int d^2 x^+ d x^- \left( \bar{\varphi}^3_\epsilon(x) + 4\langle \text{vac}_{\Lambda} | \varphi_\Lambda | \text{vac}_{\Lambda} \rangle \varphi^3_\epsilon(x) + 6\langle \text{vac}_{\Lambda} | \varphi^2_\Lambda | \text{vac}_{\Lambda} \rangle \varphi^2_\epsilon(x) \right) \xrightarrow{\Lambda \to \infty} \]
\[ \longrightarrow H_{\text{LF}}^{\text{can}}(\varphi_\epsilon) + \lambda \int d^2 x^+ d x^- \left( 4\langle \varphi | \text{vac} \varphi^3_\epsilon(x) + 6\langle \varphi^2 | \text{vac} \varphi^2_\epsilon(x) \right). \] (41)

where we let \(\mathcal{P}_0\) denote the projection on the subspace of states \(|f^{(0)}\rangle\) and \(\langle \varphi^n | \text{vac}\) is the result of averaging contributions of the corresponding field modes over \(|\text{vac}_{\Lambda}\rangle\). In the framework of the described limiting transition, we cannot find the constants \(\langle \varphi | \text{vac}\) and \(\langle \varphi^2 | \text{vac}\), which hence remain free parameters.

The last relation determines the desired approximate expression for the Hamiltonian on the LF and its difference from the expression \(H_{\text{LF}}^{\text{can}}\) obtained by directly quantizing on the LF. We note that this result for the Hamiltonian on the LF is consistent with the covariant perturbation theory in the coupling constant in all orders [3].

But the above method for constructing the Hamiltonian on the LF meets difficulties in the gauge field case, where dynamical variables are gauge-noninvariant fields. It is difficult to introduce regularizations preserving Lorentz as well as gauge invariance. Moreover, the proposed splitting of one part of the Fourier modes of these fields from the other also breaks these symmetries if we neglect intermediate modes. The presence of constraints between canonical variables due to gauge symmetry complicates using the above procedure to study the limiting transition on the LF because these constraints are nonlinear in fields and the result may depend on which variables are taken as independent (and are separated into parts in passing to the Fourier modes).
3. Gauge theory on the lattice

We now restrict ourselves to describing a gauge-invariant approach for constructing Hamiltonians in the coordinates $\tilde{x}$, which are close to the LF coordinates, as a preliminary step to formulating and investigating the limiting transition on the LF for the gauge field theory. This approach uses the space-time lattice [17] in the above coordinates and is analogous to the method in [18].

We introduce the lattice in the coordinates $\tilde{x}^+$, $\tilde{x}^-$ and $\tilde{x}^\perp$ with the parameters $a_+$, $a_-$ and $a_\perp$ denoting the distances between lattice sites along the corresponding coordinates in the scale of those coordinates. Gauge fields are described by unitary $N \times N$ matrices $U_\mu(\tilde{x})$ from the group $SU(N)$. These matrices are set into correspondence to the lattice links as shown in the diagram:

\[ U_\mu(\tilde{x}) \]

\[ \tilde{x} - a_\mu \quad \tilde{x} \]

\[ U^+(\tilde{x}) = U^{-1}(\tilde{x}) \]

\[ \tilde{x} - a_\mu \quad \tilde{x} \quad \text{axis } \tilde{x}^\mu \]

Under the gauge transformations $\Omega(\tilde{x})$ corresponding to the fundamental representation of the group $SU(N)$, the matrices transform as

\[ U_\mu(\tilde{x}) \xrightarrow{\Omega} \Omega(\tilde{x})U_\mu(\tilde{x})\Omega^{-1}(\tilde{x} - a_\mu). \]

The formal transition to the theory in the continuous space with the gauge fields $\tilde{A}_\mu(\tilde{x}) = \tilde{A}_\mu(\tilde{x})\lambda^\alpha/2$, where $\lambda^\alpha/2$ are analogues of the Gell-Mann matrices for the group $SU(N)$, corresponds to the representation

\[ U_\mu(\tilde{x}) \approx e^{i a_\mu \tilde{A}_\mu(\tilde{x})} \approx 1 + i a_\mu \tilde{A}_\mu(\tilde{x}) + \ldots \]

under the condition $a_\mu \to 0$, where $\tilde{A}_\mu$ are the vectors related to the coordinates $\tilde{x}$.

We also define the "plaquette" variables $U_{\mu\nu}(\tilde{x})$:

\[ U_{\mu\nu}(\tilde{x}) = U_\mu(\tilde{x})U_\nu(\tilde{x} - a_\mu)U^{-1}_\mu(\tilde{x} - a_\nu)U^{-1}_\nu(\tilde{x}), \]

\[ U_{\mu\nu}(\tilde{x}) = U^+_{\nu\mu}(\tilde{x}) = U^{-1}_{\nu\mu}(\tilde{x}). \]

Under the gauge transformations, we have

\[ U_{\mu\nu}(\tilde{x}) \xrightarrow{\Omega} \Omega(\tilde{x})U_{\mu\nu}(\tilde{x})\Omega^{-1}(\tilde{x}), \]

and the quantity $\text{tr}U_{\mu\nu}(\tilde{x})$ is hence gauge invariant. The analogue of formula (43) for the plaquette variables is

\[ U_{\mu\nu}(\tilde{x}) \approx e^{i a_\mu a_\nu \tilde{F}_{\mu\nu}(\tilde{x})} \approx 1 + i a_\mu a_\nu \tilde{F}_{\mu\nu}(\tilde{x}) - \frac{1}{2} a_\mu^2 a_\nu^2 \tilde{F}_{\mu\nu}^2(\tilde{x}) + \ldots, \]

\[ \text{(46)} \]
where $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - i[\tilde{A}_\mu, \tilde{A}_\nu]$.

Formulas (43) and (46) yield an expression approximating the continuous-theory Lagrangian action in terms of lattice variables using the prescriptions

$$a_\mu a_\nu \tilde{F}_{\mu\nu}(\tilde{x}) \rightarrow \frac{1}{2i} \left( U_{\mu\nu}(\tilde{x}) - U_{\mu\nu}^+(\tilde{x}) \right) \equiv \text{Im} U_{\mu\nu}(\tilde{x}),$$

$$a_\mu^2 a_\nu^2 \text{tr} \tilde{F}_{\mu\nu}^2(\tilde{x}) \rightarrow \text{tr} \left( 2 - U_{\mu\nu}(\tilde{x}) - U_{\mu\nu}^+(\tilde{x}) \right) \equiv 2 \text{tr} \left( 1 - \text{Re} U_{\mu\nu}(\tilde{x}) \right).$$

In coordinates (13) close to the LF, we have

$$S = \int d^4 \tilde{x} L(\tilde{x}),$$

$$L(\tilde{x}) = -\frac{1}{4g^2} \tilde{F}_{\mu\nu}^a(\tilde{x}) \tilde{F}_\rho^\lambda(\tilde{x}) \tilde{g}^{\rho\mu}(\tilde{x}) \tilde{g}^{\lambda\nu}(\tilde{x}) =$$

$$= \frac{1}{2g^2} \tilde{F}_{-+}^a(\tilde{x}) \tilde{F}_{++}^a(\tilde{x}) + \sum_{k=1,2} \frac{1}{g^2} \left( \frac{\eta^2}{2} \tilde{F}_{+k}^a(\tilde{x}) \tilde{F}_{+k}^a(\tilde{x}) + \tilde{F}_{-k}^a(\tilde{x}) \tilde{F}_{-k}^a(\tilde{x}) \right) +$$

$$+ \frac{1}{2g^2} \tilde{F}_{12}^a(\tilde{x}) \tilde{F}_{12}^a(\tilde{x}).$$

in the continuous case. For the lattice action, we obtain

$$S_{\text{lat}} = \frac{2a_+ a_- a_1^2}{g^2} \sum_x \text{tr} \left( \frac{\text{Re} \left( 1 - U_{-+}(\tilde{x}) \right)}{a_+^2 a_-^2} + \sum_{k=1,2} \frac{\eta^2 \text{Re} \left( 1 - U_{+k}(\tilde{x}) \right)}{a_+^2 a_-^2} + \right.$$

$$\left. + \sum_{k=1,2} \frac{\text{Im} U_{+k}(\tilde{x}) \text{Im} U_{-k}(\tilde{x}) - \text{Re} \left( 1 - U_{12}(\tilde{x}) \right)}{a_+^2 a_-^2} \right).$$

To derive the analogue of the Hamiltonian, we consider the lattice representation of the functional integral [17] for the matrix elements of the evolution operator [18] (see [19] for an analogous consideration in the continuous case). We use the coordinate $\tilde{x}^+$ as the evolution parameter (subsequently considering the limit as $a_+ \to 0$) and also introduce the gauge condition

$$U_+(\tilde{x}) = 1.$$ 

Under such a condition, it is easy [18] to find the form of the quantum evolution operator $\tilde{T}_+$ relating the basis states (the eigenvalues corresponding to these states are the integration variables in the functional integral) at the time instants $\tilde{x}^+$ differing by the quantity $a_+$. The operator $\tilde{T}_+$ is in turn related to the Hamiltonian $\tilde{H}$ in the coordinates $\tilde{x}$ in the limit as $a_+ \to 0$:

$$\tilde{T}_+ = \exp \left( -i a_+ \tilde{H} + O(a_+^2) \right).$$

By analogy with [18], we introduce the basis of states used when defining the functional integral,

$$|U\rangle = \prod_{i,\tilde{x}} |U_i(\tilde{x})\rangle, \quad i = -, 1, 2,$$

$$\tilde{U}_i(\tilde{x})|U\rangle = U_i(\tilde{x})|U\rangle, \quad \langle U'|U\rangle = \delta(U', U), \quad 1 \equiv \int dU \langle U|U\rangle.$$
where $dU$ is the invariant measure on the group of matrices $U$.

We define the unitary operators $\hat{R}_{i,\bar{z}}(g_i)$ by analogy with the "shift" operation on the matrix group $U$:

$$\hat{R}_{i,\bar{z}}(g_i)|U_i(\bar{x})\rangle = |g_iU_i(\bar{x})\rangle,$$

where $g_i$ is an arbitrary $N \times N$ $SU(N)$ matrix.

It is easy to verify that the operator $\tilde{T}^+_i$ can be written as

$$\tilde{T}^+_i = \prod_k \prod_{\bar{x} - \bar{x}_-} \int dg_+ dg_- \hat{R}_{i,\bar{z}}(g_+)\hat{R}_{-i,\bar{x}}(g_-) \times$$

$$\times \exp \left[ -\frac{2i}{g^2} \text{tr} \left( \frac{g_- + g_+}{2a_+a_-} \right) - \frac{2i\eta^2}{g^2a_-} \text{tr} \left( \frac{g_k + g_k^+}{2} \right) - \frac{2i}{g^2} \text{tr} \left( \frac{g_k - g_k^+}{2i} \text{Im} U_{-k}(\bar{x}) + \frac{2ia_+a_-}{g^2a^2} \text{Re} U_{12}(\bar{x}) \right) \right].$$

We parameterize the matrices $g_i$ using the real parameters $\theta_i^a$:

$$g_i = \exp \left( i\theta_i^a \frac{\lambda^a}{2} \right).$$

For the operators $\hat{R}_{i,\bar{z}}(g_i)$, we have

$$\hat{R}_{i,\bar{z}}(g_i) = \exp \left( i\theta_i^a \tilde{\Pi}^a_i(\bar{x}) \right)$$

where the operators $\tilde{\Pi}^a_i(\bar{x})$ realize the representation of the matrices $\lambda^a/2$ and satisfy the commutation relations

$$[\tilde{\Pi}^a_i(\bar{x}), \tilde{\Pi}^b_i(\bar{x})] = if^{abc} \tilde{\Pi}^c_i(\bar{x})$$

and $f^{abc}$ are the structure constants of the group $SU(N)$,

$$\left[ \frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = if^{abc} \frac{\lambda^c}{2}.$$

In the limit as $a_+ \to 0$, we can evaluate the integrals over $g_i$ in the expression for $\tilde{T}^+_i$ in full analogy with [18]. As a result, we obtain the expression for the Hamiltonian:

$$\tilde{H}(\eta) = \sum_{\bar{x} - \bar{x}'} \left[ \frac{g^2}{2a^2} \tilde{\Pi}^a_i(\bar{x})\tilde{\Pi}^a_i(\bar{x}) + \frac{g^2}{2\eta^2a_-} \sum_{k=1,2} \left( \tilde{\Pi}^a_k(\bar{x}) - \frac{1}{g^2} \text{tr} \left( \lambda^a \text{Im} U_{-k}(\bar{x}) \right) \right)^2 - \frac{2a_-}{g^2a^2} \text{tr} \text{Re} U_{12}(\bar{x}) \right].$$

By virtue of formulas (53) and (56), we have the relation

$$e^{-i\theta_i^a \hat{\Pi}^a_i } \hat{U}_i e^{i\theta_i^a \hat{\Pi}^a_i } = e^{i\theta_i^a \lambda^a/2} \hat{U}_i,$$
whence we can obtain the commutation relations for the operators $\hat{\Pi}^a_i(\tilde{x})$ and $\hat{U}_i(\tilde{x})$:

$$\left[\hat{\Pi}^a_i(\tilde{x}), \hat{U}_i(\tilde{x})\right] = -\frac{\lambda^a}{2} \hat{U}_i(\tilde{x}).$$  \hspace{1cm} (61)

It is easy to verify that the form of these relations is invariant under the gauge transformations of variables if the quantity $\hat{\Pi}^a_i(x)$ obeys the transformation law

$$\hat{\Pi}_i(\tilde{x}) \xrightarrow{\Omega} \Omega(\tilde{x})\hat{\Pi}_i(\tilde{x})\Omega^{-1}(\tilde{x}),$$  \hspace{1cm} (62)

where $\hat{\Pi}_i(\tilde{x}) = (\lambda^a/2)\hat{\Pi}^a_i(\tilde{x})$. To obtain this result, it suffices to note that the matrices $\lambda^a$ are vectors in the adjoint representation of the group $SU(N)$.

Gauge condition (50) preserves the symmetry under the gauge transformations independent of $\tilde{x}^+$. The generators of these transformations correspond to the canonical constraints, which are regarded in the quantum theory as conditions determining the physical subspace of states [20]. Vectors of this subspace described by functionals of fields must be gauge invariant. On a space lattice of finite size, these states correspond to functions of traces of products of the matrices $U_i(\tilde{x})$ along all possible closed contours on the links of the lattice. The Hamiltonian $\check{H}(\eta)$ must be considered on only these states. The physical vacuum must correspond to the minimum of the Hamiltonian. Moreover, the vacuum state is assumed to be invariant under shifts along the coordinates $\tilde{x}^-$ and $\tilde{x}^\perp$. Analogously, when finding the spectrum of bound states, we can fix the subspace of states that are invariant under shifts along $\tilde{x}^\perp$ and acquire a phase factor under shifts along $\tilde{x}^-$ (this corresponds to using Eqs. (3) in the continuous theory).

The exact solution of this problem is obviously as difficult as the one in the standard Lorentzian coordinates with the Hamiltonian at the fixed time $x^0$.

Passing to the coordinates $\tilde{x}$ close to the LF coordinates with a small parameter $\eta$ allows using the smallness of $\eta$ if we make simple assumptions about a possible approximation to the exact solution, as demonstrated above for the scalar field theory in the continuous space. For example, we can develop the perturbation theory in the parameter $\eta$ for the equation for the eigenvalues of the Hamiltonian $\check{H}(\eta)$ fixing the lattice parameter values and spatial size (imposing periodic boundary conditions in spatial variables on fields). Investigating such a perturbation theory in the continuous space (with the fixed cutoff parameter $|x^-| \leq L$ and periodic boundary conditions in the coordinate $x^-$) for two-dimensional quantum electrodynamics demonstrated [21–23, 6] that using the phenomenologically fitted modification of the Hamiltonian terms containing zero modes of the field, we can attain an adequate inclusion of vacuum effects. This feeds the hope to find an analogous result in the lattice approach.

On the other hand, using the perturbation theory in $\eta$, we can construct the lattice vacuum state described by the functional of gauge fields, restricting ourself to just a few lowest orders in $\eta$, and then use this functional to calculate quantum correlation functions approximately, preserving the finite, but small, value of the parameter $\eta$. Such an approach can be useful when analyzing hadron scattering at high energies.

Acknowledgments. The authors thank the UNESCO Regional Bureau for Science and Culture in Europe for supporting the V. A. Fock International School of Physics. This work was supported in part (S. A. P. and E. V. P.) by the Russian Foundation for Basic Research (Grant No. 05-02-17477) and the Russian Federal Education Agency (Project No. RNP.2.1.1.1112).
References

[1] P. A. M. Dirac. Rev. Mod. Phys. 1949. V. 21. P. 392.
[2] P.M. Stevenson. Phys. Rev. D. 1985. V. 32. P. 1389.
[3] S. A. Paston, V. A. Franke. Theor. Math. Phys. 1997. V. 112. P. 1117. hep-th/9901110.
[4] S. A. Paston, E. V. Prokhlativol, V. A. Franke. Theor. Math. Phys. 1999. V. 120. P. 1164. hep-th/0002062.
[5] S. A. Paston, E. V. Prokhlativol, V. A. Franke. Theor. Math. Phys. 2002. V. 131. P. 516. hep-th/0302016.
[6] E. V. Prokhlativol, H. W. L. Naus, H.-J. Pirner. Phys. Rev. D. 1995. V. 51. P. 2933. hep-ph/9406275.
[7] E. V. Prokhlativol, V. A. Franke. Phys. Atomic Nucl. 1996. V. 59. P. 1030.
[8] S. Dalley, G. McCartor. hep-ph/0406287.
[9] V. A. Franke, Yu. V. Novozhilov, E. V. Prokhlativol. Lett. Math. Phys. 1981. V. 5. P. 239.
[10] V. A. Franke, Yu. V. Novozhilov, E. V. Prokhlativol. Lett. Math. Phys. 1981. V. 5. P. 437.
[11] A. M. Annenkova, E. V. Prokhlativol, V. A. Franke. Vestn. Leningr. Gos. Univ. 1985. No. 4. P. 80.
[12] S. A. Paston, E. V. Prokhlativol, V. A. Franke. Phys. Atomic Nucl. 2005. V. 68. P. 67. hep-th/0501186.
[13] S. J. Brodsky, H.-C. Pauli, S. S. Pinsky. Phys. Rep. 1997. V. 301. P. 299. hep-ph/9705477.
[14] A. M. Annenkova, E. V. Prokhlativol, V. A. Franke. Phys. Atomic Nucl. 1993. V. 56. P. 813.
[15] K.G. Wilson, T. Walhout, A. Harindranath, W.M. Zhang, R.J. Perry, S. Glazek. Phys. Rev. D. 1994. V. 49. P. 6720. hep-th/9401153.
[16] S. J. Brodsky, V. A. Franke, J. R. Hiller, G. McCartor, S. A. Paston, E. V. Prokhlativol. Nuclear Physics B. 2004. V. 703. P. 333. hep-ph/0406325.
[17] K. Wilson. Phys. Rev. D. 1974. V. 10. P. 2445.
[18] M. Creutz. Phys. Rev. D. 1977. V. 15. P. 1128.
[19] A. A. Slavnov, L. D. Faddeev. Introduction to Quantum Theory of Gauge Fields [in Russian], Nauka, Moscow (1988); English transl.: L. D. Faddeev and A. A. Slavnov, Gauge Fields: Introduction to Quantum Theory, Benjamin, London (1990).
[20] P. A. M. Dirac. Lectures on Quantum Mechanics, Belfer Grad. Sch. of Sci., Yeshiva Univ., New York (1964).
[21] E. V. Prokhvatilov, V. A. Franke. Sov. J. Nucl. Phys. 1988. V. 47. P. 559.

[22] E. V. Prokhvatilov, V. A. Franke. Sov. J. Nucl. Phys. 1989. V. 49. P. 688.

[23] A. B. Bylev, E. V. Prokhvatilov, V. A. Franke. Vestn. Leningr. Gos. Univ. 1989. Ser. 4. Iss. 2. No. 11. P. 66.