Abstract

In the lightly bound Skyrme model, several Skyrmions having particularly strong binding are clusters of unit baryon number Skyrmions arranged as truncated tetrahedra. Their baryon numbers form the sequence \( B = 4, 16, 40, 80, 140, 224 \). This is the standard sequence of tetrahedral numbers multiplied by four, and therefore agrees with the sequence of magic proton and neutron numbers \( 2, 8, 20, 40, 70, 112 \) that occurs in the nuclear shell model in the absence of strong spin-orbit coupling. This sequence includes several of the magic numbers that are predicted for tetrahedrally deformed nuclei, and also appears in the context of the FCC lattice geometry investigated long ago by Wigner and revived more recently.
1 Introduction

The Skyrme model proposes that baryons are topological solitons in a field theory of pions \[1, 2\]. Protons and neutrons are spin-half quantum states of the basic Skyrmion with unit baryon number, and they combine into an isospin-half doublet of nucleons \[3\]. The model incorporates approximate chiral symmetry and has had considerable success in modelling not only nucleons, but also the ground states and excited states of larger nuclei. Many Skyrmion solutions with higher baryon numbers are known. The simplest quantization technique is rigid body quantization of the classical Skyrmions of minimal energy, but recent work has considered some of the low-energy deformation modes of Skyrmions; this gives further states, and improved fits to the spectra of nuclei, in particular Carbon-12 \[4\] and Oxygen-16 \[5\].

However, two problems with the standard Skyrme model are that the classical Skyrmion binding energies are rather high, and that there is little evidence for a conventional sequence of magic numbers that should be a feature of a model of nuclei.

There have been various attempts to resolve the first problem by devising variants of the standard Skyrme model, and the variant we consider here is the one devised by the group in Leeds \[6, 7, 8\]. This adds a quartic potential term to the usual pion mass term, and also substantially changes the ratio between the quadratic and quartic terms in field derivatives (the sigma model term and Skyrme term). The model has much reduced classical binding energies, although this desirable feature is rather spoiled by quantum effects.

The Leeds group found the classical energy minima for baryon numbers up to \(B = 8\) in \[7\]. In addition to the low binding energies, their most striking result was that the Skyrmions were clusters of almost undeformed \(B = 1\) Skyrmions centred very close to vertices of a face centred cubic (FCC) lattice. In \[8\] they showed that an accurate approximation to their model is obtained by treating the \(B = 1\) Skyrmions as particles located precisely on the FCC vertices. They could study Skyrmions up to baryon number \(B = 23\) using this approximation.

It was not really a novel discovery that the optimal way to arrange large numbers of \(B = 1\) Skyrmions in three dimensions is to place them at the vertices of an FCC lattice. This was first noted by Kugler and Shtrikman \[9\].

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[^1]: We use the notation \(B\) for baryon number, as is standard in Skyrmion research. It is identical to atomic mass number, usually denoted by \(A\).
and Castillejo et al. [10]. A similar lattice of instantons probably occurs in
the holographic approach to multi-baryon systems [11] [12]. The FCC lattice
has four sublattices, and the Skyrmions have a uniform orientation on each
sublattice. Nearest neighbour Skyrmions, which are on distinct sublattices,
have a relative orientation that is maximally attractive (i.e. one Skyrmion
is rotated relative to the other by $\pi$ around a line perpendicular to the line
joining them), and this is why the overall energy is minimised.

In the standard Skyrme model with zero pion mass one can find an infi-
nite, periodic Skyrme crystal. If the lattice spacing is forced to be relatively
large, the crystal structure is an FCC lattice of $B = 1$ Skyrmions, but as
the lattice spacing decreases, it reaches a critical value where the Skyrmions
partially merge and the symmetry is enhanced. The crystal then becomes a
primitive cubic lattice of half-Skyrmions. The true energy minimum of the
Skyrme model occurs at a lattice spacing smaller than the critical value, so it
is a crystal of half-Skyrmions. At least, this is the case if one just considers
the static solutions for zero pion mass. The half-Skyrmion crystal struc-
ture is remarkable, but for massive pions the enhanced symmetry (a kind of
discrete chiral symmetry) is lost and the minimal energy crystal reverts to
having FCC structure. The FCC lattice is also almost certainly the minimal
energy crystal structure in the lightly bound Skyrme model.

Ma and Rho [13] have recently reconsidered the transition between the
FCC crystal of Skyrmions and the half-Skyrmion crystal with enhanced sym-
metry. By taking into account pion mass effects and quantum effects, they
argue that at normal nuclear densities, the Skyrmions are in the FCC phase,
but at higher densities (more than twice normal nuclear densities) the half-
Skyrmion phase will occur. This has consequences for neutron stars and
other dense nuclear systems that are not normally accessible in laboratory
experiments. It suggests that it is reasonable to study variants of the Skyrme
model with FCC arrangements of $B = 1$ Skyrmions, where the Skyrmions
are close to merging. The lightly bound model is just one such variant.

Related to the transition from the FCC crystal to the half-Skyrmion crys-
tal is what happens for baryon number $B = 4$. The optimal way to arrange
four $B = 1$ Skyrmions is to place them at the vertices of a regular tetrahe-
dron. The orientations are distinct and are those that occur on the four FCC
sublattices. All six pairs of Skyrmions maximally attract, and the field con-
figuration has tetrahedral symmetry. Remarkably, in the standard Skyrme
model, both for zero mass pions and for pions of realistic mass, the true
$B = 4$ Skyrmion of minimal energy has an enhanced cubic symmetry [14].
The four Skyrmions at the vertices of the tetrahedron merge into a cubic structure with half-Skyrmions at the eight vertices. However, for the lightly bound model, and probably many other variant models, the minimal energy solution remains tetrahedral. Note that a cubic structure easily deforms into a tetrahedral structure, and in two ways, because the vertices of a cube naturally split into two subsets forming tetrahedra. The relevant tetrahedral symmetry group is $T_d$, a subgroup of the cubic group $O_h$. It is therefore not surprising that the $B = 4$ Skyrmion has a vibrational mode that oscillates between two dual tetrahedra, and that this is one of the lowest frequency modes [15].

The existence of the cubic $B = 4$ Skyrmion in the standard Skyrme model has influenced much of the recent research into Skyrmions of higher baryon numbers. Skyrmion solutions, for baryon numbers a multiple of four, have been found by bringing several $B = 4$ cubes together [16]. The results are similar to those found in the alpha-cluster models of nuclei. The Skyrmion with $B = 8$ has two cubes touching along a face. Solutions with $B = 12$ have been found with three cubes arranged in an equilateral triangle, and also in a linear chain; these have similar energies. Four $B = 4$ cubes can be arranged tetrahedrally to give a $B = 16$ solution. Eight cubes can be arranged into a large cube with $B = 32$, and twenty seven $B = 4$ cubes produce the largest known standard Skyrmion, with $B = 108$ [17].

However, the study of higher baryon number Skyrmions as clusters of $B = 4$ cubes, in the standard Skyrme model, has reached an impasse. Quantizing the cubic $B = 32$ and $B = 108$ Skyrmions as rigid bodies doesn’t work well. It has also not been possible to construct a $B = 40$ solution from ten $B = 4$ cubes, to obtain a good model for the magic nucleus Calcium-40. More generally, the standard Skyrme model has not yet yielded obvious structures compatible with the known magic numbers for nuclei, beyond the $B = 16$ solution modelling Oxygen-16.

More promising, then, is the lightly bound Skyrme model, with its symmetries inherited from the FCC lattice. The maximal symmetry of clusters cut out from the FCC lattice is cubic, but these cubically symmetric clusters are not exceptionally tightly bound. The most tightly bound clusters appear to have tetrahedral symmetry, and have a single tetrahedral cluster of four $B = 1$ Skyrmions at their centre. We shall describe these next. Their baryon numbers match the magic numbers established in other nuclear models.
2 Clusters with Tetrahedral Symmetry

In the FCC lattice, each vertex has 12 nearest neighbours. Equivalently, the coordination number is 12. In the lightly bound Skyrme model there is one baryon (i.e. one $B = 1$ Skyrmion) at each vertex, and it is an accurate approximation to say that the binding is predominantly due to the pair interactions between nearest neighbours, which are all of the same strength. We shall use this approximation, and ignore longer-range contributions to the interactions. We refer to the binding between each nearest neighbour pair as a bond, so the total binding energy is a constant times the number of bonds. Let us normalise the energy so that this constant is unity, and identify the number of bonds as the binding energy $E$.

Using this approach, we see that as each baryon in the FCC lattice is bonded to twelve others, and each bond has two baryons at its ends, the binding energy per baryon of the complete lattice is $E/B = 6$. We are interested in finite clusters of baryons arranged as subsets of the FCC lattice. $E/B$ is then obviously less than 6, because some baryons, especially those on the surface of a cluster, have fewer than twelve nearest neighbours.

The smallest clusters beyond a single pair have baryon numbers $B = 3$ and $B = 4$. The $B = 3$ cluster is an equilateral triangle of baryons, with 3 bonds and $E/B = 1$; the $B = 4$ cluster is a tetrahedron of baryons, with 6 bonds and $E/B = 1.5$. The next highly symmetric cluster is the $B = 6$ octahedron with 12 bonds, for which $E/B = 2$. For baryon numbers less than 16 it is not possible for $E/B$ to be as large as 3. There are highly symmetric clusters with $B = 13$ (a single baryon surrounded by all twelve of its nearest neighbours) and $B = 14$ (a $B = 6$ octahedron with each face completed into a tetrahedron by adding one more baryon outside). These are both cubically symmetric, and have 36 and 40 bonds respectively, giving $E/B$ values of 2.77 and 2.86. Notice that the cubic symmetry is differently realised in these two cases, as the first cluster has a baryon at its centre, but the second does not.

For $B = 16$ there is a tetrahedrally symmetric cluster of baryons with 48 bonds, so $E/B = 3$. This has the basic $B = 4$ tetrahedron at the centre, attached to a triangular $B = 3$ cluster above each face. It therefore consists of four $B = 6$ octahedra each sharing one face with the central tetrahedron, and these octahedra have some shared edges and vertices. Four outer faces are hexagons with seven baryons, and four are triangles with three baryons (see Figure 1).
Figure 1: $B = 16$ cluster in the lightly bound Skyrme model.

Notice that this cluster inherits a basic property of the FCC lattice. The complete lattice can be decomposed into alternating $B = 4$ tetrahedra and $B = 6$ octahedra. Each tetrahedron shares faces with four neighbouring octahedra, and each octahedron shares faces with eight neighbouring tetrahedra. The octahedra all have the same spatial orientation, whereas the tetrahedra occur in two orientations related by inversion. The $B = 16$ cluster has a single central tetrahedron, together with its four neighbouring octahedra and six more tetrahedra between the octahedra. (The $B = 14$ cluster mentioned above has a single central octahedron together with its eight neighbouring tetrahedra.)

The larger clusters we shall consider are mainly those built on the $B = 16$ core, and the most symmetric ones have tetrahedral symmetry. One can also consider larger clusters with cubic symmetry, built on the $B = 13$ or $B = 14$ cores. For a discussion of these, and some further clusters, see the Appendix. The tetrahedral clusters have large $E/B$ values, but they have competitors. For example, at $B = 19$, there is a large octahedron (with a baryon at the centre) with 60 bonds; there is also a cluster where a $B = 3$ triangle is attached to one of the hexagonal faces of the $B = 16$ cluster, also with 60 bonds, but less symmetry. It is the second cluster that easily allows further
baryons to be attached, so as to achieve higher $E/B$ values.

At this point it is helpful to introduce Cartesian coordinates for the vertices of the FCC lattice. We choose the origin to be the centre of one of the $B = 4$ tetrahedra, and orient and scale this tetrahedron so that its vertices are at $(1, 1, 1), (1, -1, -1), (-1, 1, -1)$ and $(-1, -1, 1)$. The vertices of the entire FCC lattice are then at the positions $(a, b, c)$ where $a$, $b$ and $c$ are odd integers, and also $a + b + c = 3 \mod 4$. The vectors from any vertex to its nearest neighbours are $(0, \pm 2, \pm 2), (\pm 2, 0, \pm 2)$ and $(\pm 2,\pm 2, 0)$, and have squared length 8. Note that the sum of the coordinates of these vectors is always 0 mod 4. The points $(a, b, c) = (1, 1, 1) \mod 4$ form one of the four sublattices of the FCC lattice, and $B = 1$ Skyrmions located at these points all have the same orientation. Similarly for the points equal to $(1, -1, -1), (-1, 1, -1)$ or $(-1, -1, 1) \mod 4$.

It is straightforward to classify larger, tetrahedrally symmetric clusters using these coordinates. For example, the $B = 16$ cluster has baryons at all the allowed vertices with coordinates $(\pm 1, \pm 1, \pm 1)$, and the vertices whose coordinates are $(\pm 3, \pm 1, \pm 1)$ or its permutations. The constraint $a + b + c = 3 \mod 4$ allows half of the sign combinations here, so there are four vertices of the first type and twelve of the second type. These are at squared distances 3 and 11 from the origin, respectively. The next largest tetrahedrally symmetric cluster adds baryons at the twelve allowed vertices with coordinates $(\pm 3, \pm 3, \pm 1)$ and its permutations, all at squared distance 19, to produce $B = 28$. This efficiently adds four triangular $B = 3$ clusters above each hexagonal face of the $B = 16$ cluster, and adds 48 bonds, producing 96 bonds in total. (Note that the squared distances are always of the form $8k + 3$.)

At squared distance 27 there are two sets of vertices. There are twelve vertices $(\pm 5, \pm 1, \pm 1)$ and its permutations, and four vertices $(\pm 3, \pm 3, \pm 3)$. We discover here that tetrahedral symmetry is not a sufficient criterion for achieving a large value of $E/B$. It is optimal to add twelve baryons on the first set of vertices, to produce $B = 40$, but not optimal to add four baryons on the second set either before or after adding the twelve. The twelve baryons of the first set occur in six pairs; each pair attaches to two touching square faces of the $B = 28$ cluster, adding 9 bonds per pair, and 54 bonds altogether. The four baryons of the second set are isolated and add only 3 bonds each.

Therefore, there are tetrahedrally symmetric clusters with $B = 32$ and with $B = 44$, but the most strongly bound cluster in this region of baryon numbers is the $B = 40$ cluster, which has $96 + 54 = 150$ bonds, and $E/B =$.
3.75 (see Figure 2). It has the form of a truncated tetrahedron, as does the $B = 16$ cluster. Baryon numbers 4, 16 and 40 are magic numbers, corresponding to the nuclei Helium-4, Oxygen-16 and Calcium-40, which is encouraging, so we shall study these truncated tetrahedra further. For their baryon numbers, they have maximal or almost maximal values of $E/B$.

Figure 2: Truncated tetrahedral $B = 40$ cluster.

3 Magic, Truncated Tetrahedra

We now discuss in more generality the infinite family of truncated tetrahedral clusters that give exceptionally large values of the binding energy per baryon, $E/B$, and shall refer to their baryon numbers as magic.

The FCC lattice has complete, pure tetrahedra as subclusters. These are obtained by intersecting four planes parallel to the faces of the basic $B = 4$ cluster at the centre. The first of these clusters beyond $B = 4$ has $B = 10$, but this does not have a $B = 4$ tetrahedron at its centre, so more interesting is the next one, with $B = 20$. When four baryons are truncated from its vertices we recover the magic $B = 16$ cluster. More generally, complete tetrahedra have too much of a pointed shape to have an exceptional value
for $E/B$. What we need to do is to truncate the tetrahedra by slicing off four smaller tetrahedra to bring the cluster closer to a spherical shape.

To calculate the total baryon numbers of these truncated tetrahedra we require the pure tetrahedral numbers. A tetrahedron is built up from layers of equilateral triangles, so a tetrahedral number $T_N$ is the sum of a sequence of triangular numbers,

$$
T_N = \sum_{n=1}^{N} \frac{1}{2} (n+1)n = \frac{1}{6} (N+2)(N+1)N.
$$

(1)

The first few of these are $1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364$. (An amusing appearance of these numbers is in the song Twelve Days of Christmas. If one takes literally that on the first day the gift is a Partridge in a Pear Tree, and on the second it is two Turtle Doves and a Partridge in a Pear Tree, and so on, then by the 12th day the total number of gifts is $T_{12} = 364$.)

If we require our truncated tetrahedra to have a central $B = 4$ cluster, which appears desirable, then we must start with a complete tetrahedron whose edge has an even number of baryons. We then truncate this by removing four equal tetrahedra of baryons, leaving the shortest edge with just two baryons. This produces the truncated structure that is closest to spherical. We therefore start with a tetrahedron with $2N$ baryons on an edge, and remove four tetrahedra with $N-1$ baryons on an edge. The total baryon number remaining is

$$
M_N = T_{2N} - 4T_{N-1} = \frac{1}{6} (2N+2)(2N+1)2N - \frac{2}{3} (N+1)N(N-1)
= \frac{2}{3} (N+1)N(2N+1-N+1)
= \frac{2}{3} (N+2)(N+1)N.
$$

(2)

There are a pleasing number of common factors in the two contributions to $M_N$, and the result, surprisingly, is four times the tetrahedral number $T_N$. The cluster has $T_N$ baryons in each of the four orientations, matching the equal distribution of orientations that occurs for the central $B = 4$ tetrahedron, but the baryons with a given orientation are not arranged as a pure tetrahedron.

The first few of the numbers $M_N$ are $4, 16, 40, 80, 140, 224$, and we refer to these as tetrahedral magic (baryon) numbers. Assuming a nucleus with
one of these baryon numbers has equal numbers of protons and neutrons, then these form the sequence 2, 8, 20, 40, 70, 112. The first three of these are the standard magic numbers of nuclear physics.

We calculate next the total bond numbers of the truncated tetrahedra. We again start with a complete, pure tetrahedron, and consider slicing it up into parallel equilateral triangles. A triangle with \( n \) baryons along an edge has (the triangular number) \( \frac{1}{2}n(n-1) \) bonds in each of three directions, so there are \( \frac{3}{2}n(n-1) \) bonds within this triangle. The triangle is also bonded to the next smaller triangle by 3 bonds for each baryon in the smaller triangle, so there are \( \frac{3}{2}n(n-1) \) such bonds. This triangle therefore contributes \( 3n(n-1) \) bonds overall, which is six times a triangular number. Summing these up, we find that a complete tetrahedron with \( N \) baryons along each edge has \( (N+1)N(N-1) \) bonds in total, six times a tetrahedral number.

Our truncated tetrahedron starts as a complete tetrahedron with \( 2N \) baryons on an edge, and then four tetrahedra with \( N-1 \) baryons along an edge are removed. The bonds of the removed tetrahedra are lost, as are the bonds connecting these tetrahedra to what remains. The total number of bonds of the truncated tetrahedron is therefore

\[
E_N = (2N+1)2N(2N-1) - 4 \left( N(N-1)(N-2) + \frac{3}{2}N(N-1) \right)
\]

\[
= 2(2N-1)(N+2)N.
\]

The first few of these bond numbers are 6, 48, 150, 336, 630, 1056, and these are also the approximate binding energies. The binding energies per baryon are \( E/B = 1.5, 3, 3.75, 4.2, 4.5, 4.72 \), and the general algebraic formula is

\[
E/B = E_N/M_N = \frac{2(2N-1)(N+2)N}{3(N+2)(N+1)N} = \frac{2N-1}{N+1} = 6 - \frac{9}{N+1}.
\]

This slowly approaches 6 as expected, but only reaches 5 for \( N = 8 \), when \( B = 480 \), much larger than the baryon number of any observed nucleus.

An alternative way to find the bond number of a truncated tetrahedron is to note that baryons occur in four types of position. There are interior baryons, face baryons (not on an edge), edge baryons (not at a vertex), and
vertex baryons. These have coordination numbers 12, 9, 7 and 5, respectively. As each bond has two ends, the total bond number is half of the total coordination number. For example, the $B = 140$ truncated tetrahedron (which has edge lengths of 2 baryons and 5 baryons) has 40 interior baryons, 52 face baryons, 36 edge baryons and 12 vertex baryons, and the total bond number is 630.

Some of the truncated tetrahedra have another interesting property. When $N$ is odd, the truncated tetrahedron has an even number of parallel layers with triangular symmetry, and in this case it can be decomposed into a set of disjoint $B = 4$ tetrahedra that account for all the baryons. These $B = 4$ tetrahedra are connected by octahedra sharing the faces. The interpretation is that the truncated tetrahedra are clusters of alpha particles. In particular, the truncated tetrahedra with baryon numbers 40 and 140 decompose into 10 and 35 disjoint $B = 4$ tetrahedra, respectively, arranged with tetrahedral symmetry. The case $B = 40$ is particularly interesting, because until now it was not known how to arrange ten alpha particles into a Calcium-40 nucleus in any Skyrmion model.

The orientations of the $B = 4$ tetrahedra alternate. The arrangement for $B = 40$ is that six tetrahedra have one orientation and four the other. The six occur at vertices of an large octahedron, and the four at vertices of a large tetrahedron. The ten together do not form a pure tetrahedron. For $B = 140$ there are nineteen in one orientation, at the vertices of a larger octahedron, and sixteen in the other orientation, at the vertices of a larger truncated tetrahedron. It is curious that the symmetry group of one set is cubic, and larger than that of the other set.

4 Physics of Truncated Tetrahedra

We propose that truncated tetrahedra composed of lightly bound Skyrmions are models for some types of magic nuclei, and present some of the evidence here. Recall that the magic baryon numbers we have obtained are 4, 16, 40, 80, 140, 224. The first three of these clearly correspond to Helium-4, Oxygen-16 and Calcium-40. The fourth corresponds to Zirconium-80. This is conjectured to be magic, despite being close to the proton drip line, because 40 appears to be a magic number for both protons and neutrons in tetrahedrally deformed nuclei [18 [19]. Baryon number 140 is only conjecturally magic, based on the evidence that 70 is a magic number for protons/neutrons.
in tetrahedrally deformed nuclei. However, no nucleus exists with both 70 protons and 70 neutrons (the largest, short-lived nucleus with equal proton and neutron numbers is Tin-100, or possibly Tellurium-104). Baryon number 224 allows for Radium-224, which is octupole-deformed and possibly tetrahedral [20], with 88 protons and 136 neutrons.

The binding energy per baryon, eq. (4), matches the two leading terms in the Bethe–Weizsäcker, liquid drop mass formula

\[ E/B = a_V - a_S B^{-\frac{3}{2}}, \]

(5)

where empirically, \(a_S/a_V \simeq 1.1\). Using eq. (2), we see that it is a very good approximation to write \(B \simeq \frac{2}{3}(N + 1)^3\), and then eq. (4) becomes

\[ E/B = 6 - 9 \left( \frac{3}{2} B \right)^{-\frac{1}{3}}. \]

(6)

The prediction from the lightly bound Skyrme model is therefore that \(a_S/a_V = (\frac{3}{2})^{\frac{3}{2}} \simeq 1.3\).

One feature of magic nuclei, according to the shell model, is that there is a large energy gap between the highest filled level and the lowest unfilled level. In other words, it takes more than the usual energy to excite a single nucleon. The truncated tetrahedral Skyrmions have an analogous feature. For a truncated tetrahedron, the minimal coordination number of a baryon is 5. This occurs for the baryons at the twelve vertices. For other cluster shapes there is usually a baryon with a coordination number smaller than this. For example, the baryons at the vertices of pure tetrahedra have coordination number 3, and baryons at the vertices of untruncated octahedra have coordination number 4. (However, there are truncated octahedra where all coordination numbers are 6 or more – see Appendix). The energy required to remove one baryon from a truncated tetrahedron is therefore large, since 5 bonds need to be broken, and this is evidence for it being magic. Moreover, all the elementary faces of a truncated tetrahedron are triangles, so the optimal way to relocate the baryon is to attach it by 3 bonds to one of these triangles. Moving a single baryon from a vertex to one of these new locations is therefore at the cost of 2 bonds, so the one-baryon excitation energy (the energy of a nuclear particle-hole excitation) is 2 bond units. By identifying 6 bond units with \(a_V = 15.6\) MeV, we see that the bond unit is 2.6 MeV, so the one-baryon separation energy is predicted to be 13 MeV, and the one-baryon excitation energy to be 5.2 MeV.
It is also interesting to consider what can happen if a few baryons are added to a truncated tetrahedral core. A single baryon can be attached with 3 bonds (as just mentioned), but two neighbouring baryons can be added with 7 bonds, and it is rather efficient to attach a triangular cluster of three baryons to an underlying face, which adds 12 bonds. This significantly increases $E/B$ in the case that the core has $B = 16$ or $B = 40$. Attaching a triangular $B = 3$ cluster in this way could provide a model for Fluorine-19 (the only stable isotope of Fluorine), or Scandium-43 \[21\]. The larger faces of the $B = 40$ core accommodate attaching a hexagonal cluster of 7 baryons, which adds 33 bonds. This could model Scandium-47 or Titanium-47, which are moderately stable compared to neighbouring isotopes. Finally, the $B = 80$ core accommodates attaching a triangular cluster of 10 baryons to a large face, which adds 48 bonds, and could be related to 50 being a magic number.

The simplest, original version of the shell model of nuclei \[22\] supposed that the individual protons and neutrons move in a mean field potential that is a three-dimensional isotropic harmonic oscillator. The harmonic oscillator energy levels have high degeneracies, and a magic nucleus is one where all the states up to a given energy are filled. Allowing for two spin states, the magic proton and neutron numbers are precisely 2, 8, 20, 40, 70, 112, double the tetrahedral numbers. The appearance of tetrahedral numbers here is well known, but still rather surprising, because the mean field potential is spherically symmetric. Moreover, there seems no obvious connection with the spatial structure of a truncated tetrahedron that we have discussed.

The refined version of the shell model introduces a more sophisticated energy-dependence on the orbital angular momentum of each nucleon, and includes a strong spin-orbit force. The net effect is to raise the magic numbers 40, 70, 112 to 50, 82, 126. Magic nuclei do not have to have a magic baryon number. It is sufficient if either the proton or neutron numbers are magic. We do not yet understand this in the context of Skyrmions, but are still encouraged to find that Skyrmion magic numbers overlap those of nuclear physics in the cases of equal proton and neutron numbers. Particularly encouraging is to find that in the lightly bound Skyrme model, the Skyrmion with baryon number 40 is magic, and can be used to model Calcium-40. In the standard Skyrme model, a $B = 40$ solution of the field equations with similar symmetry probably exists, and the search for it is underway.

The shell model usually assumes a spherical mean field potential, but there have been substantial investigations over many decades of deformed
shapes. These deformations are usually assumed to be quadrupolar, producing an ellipsoidal shape with $D_{2h}$ symmetry [23], but octupolar deformations, including the special octahedral deformations that preserve tetrahedral symmetry have also been analysed [18, 19]. The magic numbers that would occur for such tetrahedrally deformed shapes appear to be much closer to the sequence one finds in the original shell model. The tetrahedral deformation seems to suppress the effect of the spin-orbit force. The magic proton and neutron numbers for tetrahedral nuclei include 40, 56 and 70. If one could ignore Coulomb effects, this would lead to magic baryon numbers 80 and 140, just as for the Skyrmions that are truncated tetrahedra. The relation between the (quantum) shell model calculations exploring tetrahedral deformations and the (classical) spatial structures we have found is not clear, despite the commonality of an underlying tetrahedral symmetry.

In the discussions of tetrahedrally deformed nuclei using the shell model, there is little consideration of the small proton and neutron magic numbers 2, 8 and 20, because the corresponding magic nuclei are usually supposed to be intrinsically spherical. However, a small tetrahedral deformation would not spoil these magic numbers, and cluster models of nuclei suggest that Oxygen-16 at least has a tetrahedral form.

Rigid body quantization of a tetrahedral structure leads to a non-standard rotational band, with spin/parities $J^P = 0^+, 3^-, 4^+, 6^-, 6^+, 7^-$, ... (see [24], for example). The existence of appropriately spaced $3^-$ and $4^+$ states, and absence of a lower-lying $2^+$ state is therefore a key indicator of a tetrahedral structure. Magic nuclei like Oxygen-16, Calcium-40 and Lead-208 are well known for having low-lying $3^-$ states and no such $2^+$ states, and encourage the interpretation of these magic nuclei as tetrahedrally deformed, although alternative interpretations in terms of octupole vibrations have also to be considered.

Within the standard Skyrme model, there have been a number of investigations of the quantum states of tetrahedrally and cubically symmetric Skyrmions [25]. There are constraints on the spin/parities determined by the symmetry of the Skyrmion and the topological Finkelstein–Rubinstein sign factors related to the symmetry group elements. Provided the baryon number is a multiple of four, and one seeks states with isospin zero, then the Finkelstein–Rubinstein signs are all $+$, and the rotational states of a tetrahedral Skyrmion have the same spin/parities $J^P$ as above, and energies proportional to $J(J+1)$. Oxygen-16 can be modelled this way, starting with a Skyrmion that consists of a tetrahedral cluster of $B = 4$ subunits. Recent
work has gone beyond rigid body quantization and this gives further states [5], and a better fit to the experimental spectrum of Oxygen-16. Even if Calcium-40 is intrinsically tetrahedral, its spectrum will combine vibrational and rotational states.

Within the lightly bound Skyrme model, the Finkelstein–Rubinstein signs can be calculated for rigidly rotating clusters of arbitrary shape and any baryon number [8], and the spins of some low-lying quantum states have been determined. There is no doubt that, using rigid body quantization, the same spin/parities would be obtained for Calcium-40 as for Oxygen-16 if one modelled the $B = 40$ Skyrmion as a truncated tetrahedron. It is less clear what would result if one quantized Skyrmions with higher baryon numbers. Coulomb effects need to be considered, and more importantly, the related asymmetry between neutron and proton numbers.

5 Wigner’s Model and Tetrahedral Symmetry

We will not review Wigner’s model of nuclei [26] in detail. Wigner made the simplyfying assumption of an SU(4) symmetry in nuclear physics, and treated the four states of the proton or neutron with spin up or spin down as a fundamental quartet of SU(4). Larger nuclei are then classified by irreducible representations (irreps) of SU(4). The weight diagrams of suitable irreps resemble the truncated tetrahedral clusters we have been discussing, and the weight labels include spin and isospin labels.

SU(4) is a Lie group of rank 3, so its root lattice and weight lattice are three-dimensional [27]. The root lattice is an FCC lattice. The weight lattice is reciprocal to this, so it is a BCC lattice, and it has four times as many points. There are four cosets of the root lattice in the weight lattice, and the weights of each irrep lie in just one of these. The cosets are shifted FCC lattices, but in thinking about them we do not shift the origin. The truncated tetrahedra of interest to us are all in the coset that contains the weights of the fundamental 4-dimensional irrep. Further clusters with their centres at the origin are in other cosets. For example, the $B = 13$ cluster mentioned earlier is in the root lattice, and the $B = 6$ octahedron and the cubic $B = 14$ cluster are in the coset of the 6-dimensional irrep (the vector of SO(6)). Their additional symmetry arises from the $\mathbb{Z}_2$ reflection symmetry
of the SU(4) Dynkin diagram.

An important feature of weight diagrams of SU(4) is that typically, the interior weights in a diagram have multiplicities greater than one. The 4-dimensional irrep, with its tetrahedral weight diagram, accommodates four nucleons – one of each type – and filling the four states gives an alpha particle. This is analogous to the $B = 4$ tetrahedron in the lightly bound Skyrme model modelling an alpha particle. The next irrep whose weight diagram has a truncated tetrahedral shape is 20-dimensional. The shape is the same as the truncated tetrahedron modelling Oxygen-16 in the lightly bound Skyrme model, but in Wigner’s SU(4) model it accommodates 20 nucleons, because the inner four weights have multiplicity two.

Despite its simplicity, Wigner’s model has lasting interest, and the more sophisticated variants that Wigner discussed in his original paper, with partial breaking of the SU(4) symmetry, capture phenomena of physical significance. However, Wigner does not seem to have argued that his weight diagrams have a spatial interpretation, despite being three-dimensional. The weight labels are internal quantum numbers.

Cook, Dallacasa and collaborators [28, 29], as well as others [30, 31], have rediscovered Wigner’s model, and have reinterpreted the FCC lattice as a model of the spatial structure of nuclei. Nuclei are clusters with at most one nucleon at each lattice site, but the nucleons acquire labels similar to those of Wigner. The labels combine the principal quantum number of the isotropic harmonic oscillator with the total angular momentum, together with spin and isospin labels $\pm \frac{1}{2}$. The most stable nuclei, in which complete shells of the isotropic harmonic oscillator are filled, have the shapes of truncated tetrahedra. Despite these models of nuclei appearing to be static, individual nucleons have angular momenta that increase as one moves away from a chosen axis – which is physically reasonable – and interestingly, spin up and spin down nucleons occur in complete, alternating planar layers. Similarly, protons and neutrons (isospin up and isospin down) occur in complete, alternating planar layers in an orthogonal direction. The layer structure is inherited from Wigner’s classification, where it occurs in the three-dimensional weight space, but Cook et al. argue that it occurs in physical space.

One might criticise the spatial interpretation as having little physical justification, but it is interesting to compare it with the lightly bound Skyrme model. The Skyrme model has a physical basis as an effective field theory of pions, with solitons representing the nucleons. The truncated tetrahedra arise as particularly strongly bound arrangements of $B = 1$ Skyrmions. A
key difference from both Wigner’s model and Cook’s reinterpretation is that
the $B = 1$ Skyrmions occur in four distinct orientations, rather than as four
distinct nucleon states. Static Skyrmions are not yet nuclei. Only after quanti-
zation of the complete Skyrmion structure, using rigid body quantization
or something more sophisticated, does one get a nucleus with an overall spin
and isospin.

From the perspective of Skyrmions, it might therefore appear that there
is no spin and isospin layering. However, that is not the case. To see this one
should consider, not a nucleus with spin and isospin zero, like Calcium-40,
but a nucleus with a small net spin, or a small net isospin. It is a useful
approximation to model such nuclei as classically spinning or isospinning
Skyrmions. Such an approximation can even be used for $B = 1$ Skyrmions.
By finding the semi-classical approximations to the Adkins, Nappi and Wit-
ten quantum states of $B = 1$ Skyrmions [3], Gisiger and Paranjape [32]
noted that protons always spin clockwise relative to a particular body axis of the
$B = 1$ Skyrmion (the body axis defined by the neutral pion field), and neu-
trons always spin anticlockwise. The axis is free to point in any direction in
space, which therefore allows protons or neutrons to be spin up or down rel-
ative to any spatial axis. This classical approximation has been found useful
for studying the collisions of two nucleons in the Skyrme model [32, 33].

Now suppose a truncated tetrahedral Skyrmion has a small net isospin.
In one set of planar layers, the $B = 1$ Skyrmion constituents occur in two
orientations, but for both of them the body axes point up. In the alternating
set of layers, the constituents again have two orientations, and for both of
them the body axes point down. Because of the (classical) isospin, the $B = 1$
Skyrmions are all spinning clockwise around these body axes, which produces
an excess of protons over neutrons (or anticlockwise, producing an excess
of neutrons). That means that in the first set of layers, the spins are all
down, and in the second set of layers, the spins are all up (or vice versa).
Similarly, if there is a net spin aligned with the preferred body axes, then
the isospins alternate between the layers. This spin and isospin layering
has much similarity to what Cook et al. describe, but for the Skyrmions
it requires some dynamics, and is present only in the sense of a quantum
superposition if there is no net spin or isospin, as for example in Calcium-40.
6 Remarks on Skyrmion Quantization

For the Skyrmions that are truncated tetrahedra, with magic baryon numbers, low-energy quantum states are probably best found using collective quantization of the rotational and isorotational degrees of freedom. Additional states arise from collective vibrational modes. The evidence for this comes from previous work on the $B = 4$ Skyrmion \[15\], and also the $B = 32$ Skyrmion \[34\], and on the $B = 12$ and $B = 16$ Skyrmions where detailed spectra match those found experimentally in Carbon-12 and Oxygen-16 \[4, 5\].

Theoretically, in the shell model, one may interpret a magic nucleus as having a rather rigid quantum state, because all the available one-particle states up to some level are occupied, and this rigidity is enhanced by the short-range nucleon-nucleon repulsion. The Pauli principle allows one-particle excitations only if a particle is excited to the next shell up, and this takes considerable energy. Similarly, the quantum ground state of the Skyrmion, with spin and isospin zero, involves little relative motion of the $B = 1$ Skyrmion constituents. The $B = 1$ Skyrmion locations and orientations are highly organised, as in a crystal. Also, similarly as in the shell model, the energy required to move a single Skyrmion from a complete truncated tetrahedral cluster up to the next layer is rather large, as previously mentioned.

But now consider the quantum state of a Skyrmion with baryon number just one or two greater than a magic number. The Skyrmion will have a truncated tetrahedral core, to which will be attached one or two additional $B = 1$ Skyrmions. There is considerable freedom as to where these additional Skyrmions are. Typically there are numerous locations in the FCC lattice where one additional Skyrmion can be attached with 3 bonds, and the energy is almost the same for all of these. This suggests that the additional Skyrmion should be treated like a valence nucleon, free to move in the outer shell it occupies. Rigid body quantization of a particular configuration, as considered in \[8\], is not justified here. Similarly, if there are two additional $B = 1$ Skyrmions, they are free to move fairly independently, although there is some preference for them to be close together, as this can create one additional bond.

The physics of such Skyrmions is therefore quite similar to the usual shell model physics of one or two additional nucleons interacting with a magic nucleus as core. The additional nucleons are fairly free, but there is an important, attractive residual interaction between them \[35\]. The residual interaction becomes more significant if there are three valence nucleons, as
it produces a significant spatial correlation between them, and the simplest shell model picture starts to break down. This matches what we have seen for lightly bound Skyrmions, where we saw that it was favourable to attach three $B = 1$ Skyrmions in the form of a triangle, because this adds 12 extra bonds. The three Skyrmions are strongly correlated spatially.

There remain some challenges for the Skyrme model here. It is important to see if the quantization of a single $B = 1$ Skyrmion outside a core leads to a strong spin-orbit coupling. The orientation of the Skyrmion varies with its location, so it is plausible that as it moves across the surface of the core it has to spin too. An analysis of this coupling has been carried out in the simpler Baby Skyrme model in two dimensions [36], but not yet in the context of the three-dimensional model. One should probably allow the $B = 1$ Skyrmion to move freely around the core, but a possible simplification is to constrain the $B = 1$ Skyrmion to occupy one of the FCC lattice sites (of which there are just a finite number in the layer outside a truncated tetrahedral core). The Hamiltonian for the $B = 1$ Skyrmion would then be a hopping Hamiltonian, as used frequently in condensed matter contexts. A further challenge is to allow for rotations of the core. It is presumably necessary to parametrise the orientation of the core using continuous coordinates (Euler angles) even if the $B = 1$ Skyrmion outside is treated as hopping.

**Appendix: Rectangular Bipyramids and Octahedra**

In this paper, we mainly considered the Skyrmions obtained from a tetrahedron with $2N$ baryons along an edge, truncated by removing four tetrahedra with $N - 1$ baryons along an edge. The eight faces of such a Skyrmion alternate between equilateral triangles of baryons and slightly larger hexagons with one bond along each short edge. To pass from one truncated tetrahedron to the next, it is sufficient to attach the next larger equilateral triangle of baryons to each of the four hexagonal faces. Each pair of these triangles is joined by a single bond.

If just two of these equilateral triangles are attached, then the baryon number is half-way between that of the truncated tetrahedron one starts with, and the next one. The number of bonds is just two less than half-way between, because the two triangles are joined by a single bond, whereas if all
four triangles are attached, they are joined by 6 bonds. The baryon number sequence obtained this way is therefore \( B = 10, 28, 60, 110, 182, 280 \) and the bond numbers (binding energies) are \( E = 25, 97, 241, 481, 841, 1345 \). The binding energies per baryon are \( E/B = 2.5, 3.46, 4.02, 4.37, 4.62, 4.80 \). We do not give the algebraic formulae, but these are easily deduced from those for the truncated tetrahedra. They are not especially simple.

The shapes of these clusters are rather elegant. They are rectangular bipyramids, with \( D_{2h} \) symmetry. This is best seen through their slicings into rectangles of baryons. For example, the \( B = 28 \) bipyramid is sliced into \( 2+6+12+6+2 \) baryons. The rectangles have sides that differ by one baryon, and they all have the same orientation (see Figure 3). For comparison, the \( B = 16 \) and \( B = 40 \) truncated tetrahedra slice into \( 2+6+6+2 \) baryons and \( 2+6+12+12+6+2 \) baryons, respectively (see Figures 1 and 2). Here, the lower rectangles (the second half of the sequence) are rotated by \( \pi/2 \) relative to the upper rectangles.

![Figure 3: B = 28 rectangular bipyramid.](image)

These bipyramids are interesting because of their high bond numbers. The 25 bonds of the \( B = 10 \) bipyramid is the maximum possible for this baryon number [8]. The \( B = 28 \) example has one more bond than the tetrahedrally symmetric cluster obtained by attaching four \( B = 3 \) triangles.
to the $B = 16$ truncated tetrahedron, as discussed in Section 2.

We now turn to more symmetric Skyrmions, with cubic symmetry. There is an infinite sequence of complete, pure octahedra. They alternate between having a baryon at the centre and not. The sequences of baryon numbers, bond numbers (binding energies), and the energies per baryon are $B = 1, 6, 19, 44, 85, 146, 231, E = 0, 12, 60, 168, 360, 660, 1092$, and $E/B = 0, 2, 3, 8, 2, 4, 24, 4, 52, 4, 73$. The baryon numbers are found by slicing the octahedra into squares. For example, the $B = 85$ octahedron has square slices $1 + 4 + 9 + 16 + 25 + 16 + 9 + 4 + 1$. The bond numbers are high. For example, the $B = 85$ octahedron has 24 bonds more than the $B = 80$ truncated tetrahedron. Generally, the numbers above slightly exceed those in the sequences for the truncated tetrahedra (starting with $B = 0, E = 0$). At each step, the difference in baryon number increases by 1, and the difference in bond number increases by 6. The algebraic formulae for the baryon number and bond number of an octahedron with $N$ baryons along an edge are

$$B = \frac{1}{3}(2N^2 + 1)N, \quad E = 2(2N - 1)N(N - 1).$$

Note that $B = T_{2N-1} - 4T_{N-1}$, because an octahedron can be obtained by suitably truncating a complete tetrahedron with $2N - 1$ baryons along an edge. For example, the $B = 19$ octahedron is a truncation of the $B = 35$ tetrahedron. (A minimal truncation of the $B = 35$ tetrahedron gives a $B = 31$ truncated tetrahedron with $E/B = 3.48$, a high value.)

For the larger octahedra, an even higher binding energy per baryon is achieved by truncating the six corners, removing one baryon from each. The truncation removes six baryons and 24 bonds, leaving six square faces and eight hexagonal faces. The sequences of baryon numbers and bond numbers (binding energies) for the truncated octahedra are $B = 13, 38, 79, 140, 225$ and $E = 36, 144, 336, 636, 1068$, and the binding energies per baryon are $E/B = 2.77, 3.79, 4.25, 4.54, 4.75$. The first of these corresponds to a truncated $B = 19$ octahedron, with a central baryon and its twelve nearest neighbours. Note that the truncated octahedron with $B = 140$ has 6 bonds more than the truncated tetrahedron with the same baryon number.

The truncated octahedra, starting with $B = 38$ (see Figure 4), have the following interesting property. The vertices have coordination number 6, so removing a vertex baryon to infinity requires breaking 6 bonds. This is the maximum possible for a convex vertex, which is mathematically related to the fact that the vectors from a vertex baryon to its six nearest neighbours
possess the same geometry as the six positive roots of SU(4). Note also that the surface of a truncated octahedron is topologically a sphere, but its curvature is concentrated at the vertices. The curvature is particularly small for these special vertices, and to compensate, there are 24 of them.

Figure 4: $B = 38$ truncated octahedron.

It is tempting to think of the complete octahedra and truncated octahedra as magic, but the calculations in [8] show that, at least for the examples with baryon numbers 6, 13 and 19, the binding energies are not exceptionally large when the interactions between all of the $B = 1$ Skyrmions are allowed for. This appears to be because the $B = 1$ Skyrmions are not distributed between the four orientations as equally as possible.

A more radical truncation of an octahedron, removing five baryons from each corner, is not optimal until the baryon numbers becomes larger than those relevant to nuclei.

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