Asymptotic approach to the problem identification of a fringe delamination from the base

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Abstract. The paper presents an effective scheme for studying the direct and inverse problems of identifying the delamination of the elastic layer from the base. The proposed approach is based on the problem’s asymptotic analysis, based on the assumption that the relative size of the defect is small. In the framework of the asymptotic approach, the transcendental equations for reconstructing the bundle parameters are obtained. Identification is carried out using the additional information about the amplitude values of the displacement field measured at the upper boundary of the layer. The numerical results of the restoration are presented.

1. Introduction

The problems of the elastic bodies’ oscillations with internal defects have long attracted the attention of many scientists because of the wide range of problems in which these problems are encountered - construction, non-destructive testing, geophysics, seismology, bioengineering, etc.

It is worth noting that ultrasonic methods for diagnosing defects that are popular in industry make it possible to accurately and quickly determine the presence and size of defects in the body, but they are “powerless” in the case of small relative sizes’ defects, the presence of which under certain loading conditions can lead to a significant loss of structural stability and, as a consequence, to its further destruction. In this case, a simplified model of wave propagation in a body with a small crack can be very productive in solving both direct and inverse problems. The basis of such a simplified formulation is an asymptotic analysis of the problem.

At present, the direct and inverse problems for finite bodies with defects have been studied in sufficient detail [1-5]. As a rule, the main and most effective method for studying such problems is the finite element method (FEM) and various combinations of the FEM, the boundary integral equation method (BIEM), and the boundary element method (BEM). To solve the inverse problem of identifying the defect parameters, a so-called residual functional is compiled, which is the norm of the difference between the measured input data and the analytical data obtained in the process of solving the problem, i.e. the measured and analytical data. The residual functional depends on the crack parameters; it is a non-negative function that has a minimum at a point corresponding to the true values of the initial defect’s parameters. Therefore, as a result, the inverse identification problem reduces to the problem of minimizing the residual functional. Further successful solution of the inverse problem is directly related to the optimization algorithms’ efficiency, which are represented by...
a wide range of diverse algorithms, based on modern genetic algorithms and the mathematical apparatus of neural networks.

So, in [6], the effective work of a combination of FE, BE, and regularization algorithms for solving the systems of boundary integral equations is presented. The technique was applied to solve the problem of reconstructing two types of defects in layered composites under steady-state oscillations of the object of study — delamination and layers’ breaks. The inverse problem of identifying the surface cavities and cracks in a flat formulation was considered in [7] for a rectangular region. Reconstruction is carried out according to a given set of first four natural resonant frequencies. The solution of the inverse problems is reduced to minimizing the residual functional between the measured input information and the direct problems calculated with the specified defect parameters calculated numerically. Direct problems are solved using the finite element apparatus, and the functional minimization is carried out using the genetic algorithms. In [8], the inverse problem of identifying an inclined crack in a viscoelastic orthotropic layer by the data on displacement fields measured on a part of the upper boundary layer was investigated.

The asymptotic approach applied by the wave fields in an orthotropic strip with an internal defect was implemented in [9, 10]. In [9], an elastic band weakened by a “small” inclined internal crack was considered. The direct and inverse problems are solved on the basis of an asymptotic analysis of the problem. In [10], the problem of identifying a straight crack of a small relative length in a composite elastic layer was investigated, the defect is located at the two half-layers’ junction. A comparative reconstruction analysis results from the perspective of two approaches to solving the identification problems is carried out - the asymptotic approach and the BIEM link with genetic algorithms.

In this paper, we study the inverse the problem of identifying a strip bundle from the base in the framework of the asymptotic approach proposed [9], with a priori information on the smallness of the relative size of the defect. Such defects are quite common and arise as a result of “peeling”, “non-gluing” and “non-welding” of the object from the base, due to technical reasons.

2. The problem statement

Let us consider the steady-state oscillations of an elastic isotropic strip of thickness $h$ in anti-plane deformation mode. The lower edge of the strip is rigidly pinched. Fluctuations in the strip are caused by a concentrated load acting on the upper boundary. At the lower boundary of the strip there is a bundle of length $l = 2l_0$, located at the distance $d_0$ from the load application point. We direct the coordinate axes so that the axis $x_1$ coincided with the lower edge of the strip, and the axis $x_3$ direct perpendicular upward, passing through the load application point. The formulation in which the stratification in the process of oscillations remains open, i.e. the crack surface is stress free, is considered. Steady-state oscillations are described by the component of the shear stress vector $\sigma_{23} = p_0 e^{i\omega t}$, which makes it possible to separate the time factor and present the nonzero component of the displacement field in the form $u_2(x_1, x_3, t) = u(x_1, x_3)e^{-i\omega t}$.

After separation of the time factor, the equations of motion and boundary conditions take the form

$$\sigma_{2j,j} + k^2 u = 0, \ j = 1,3 \quad (2.1)$$

$$\sigma_{23} = \mu \frac{\partial u}{\partial x_1}, \ \sigma_{21} = \mu \frac{\partial u}{\partial x_1}, \ k^2 = \frac{\rho c^2}{\mu}$$

$$x_3 = 0, \ \begin{cases} u = 0, & x_1 \notin [d_0 - l_0, \ d_0 + l_0], \\ \sigma_{23} = \mu \frac{\partial u}{\partial x_3} = 0, & x_1 \in [d_0 - l_0, \ d_0 + l_0] \end{cases} \quad (2.2)$$
The inverse problem is to identify the bundle parameters \( d_0, l_0 \) according to the information about the displacement field measured at a point \( x_1 \) upper the band boundary in frequency sensing mode

\[
x_1 = h, \quad u(x_1, \omega) = u^*(x_1, \omega), \quad x_1 \in [c, d], \quad \omega \in [\omega_1, \omega_2]
\]

The formulation of the problem closes the conditions for the waves’ emission at infinity [11].

3. **The solution to the direct problem. Asymptotic Analysis**

To solve the direct problem, we consider an auxiliary problem, for this we introduce the “crack opening” function, which characterizes the displacement vector steps on the defect surface

\[
x_1 = 0, \quad u(x_1) = \chi(x_1) = \begin{cases} 0, & x_1 \not\in [d_0 - l_0, d_0 + l_0], \\ \neq 0, & x_1 \in [d_0 - l_0, d_0 + l_0], \end{cases}
\]

Thus, it becomes possible to use the Fourier transform in the coordinate \( x_1 \)

\[
\tilde{u}(x_1, \alpha) = \int_{-\infty}^{+\infty} u(x_1, x_3)e^{i\alpha x_3}dx_3.
\]

As a result of applying the direct and inverse Fourier transforms to the problem (2.1) under condition (3.1), after a series of mathematical actions, we obtain an integral representation of the displacement wave field in the strip

\[
u(x_1, x_3) = u^{cr}(x_1, x_3) + u^{\alpha}(x_1, x_3)
\]

\[
u^{cr}(x_1, x_3) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{p}sh\lambda x_3}{\lambda ch\lambda h} e^{-i\alpha x_3}d\alpha = \frac{1}{2\pi} \int_{\alpha}^{\alpha} \frac{\tilde{p}sh\lambda x_3}{\lambda ch\lambda h} e^{-i\alpha x_3}d\alpha
\]

\[
u^{\alpha}(x_1, x_3) = \frac{1}{2\pi} \int_{\alpha}^{\alpha} \frac{\tilde{\chi}ch(\lambda(x_3 - h))}{ch\lambda h} e^{-i\alpha x_3}d\alpha.
\]

Here is the integration loop \( \sigma \) envelopes in some way the integrand features are found from the equation \( ch\lambda h = 0 \).

\[
\alpha_m = \pm \left( k^2 - \frac{1}{h^2} \left( \frac{\pi}{2} + \pi m \right)^2 \right)^{1/2}, \quad m = 0, 1, 2,..\]

Among \( \alpha_m \) a finite number of the material (determine the number of traveling waves \( N \) in the strip) and a countable set of purely imaginary values.

The offset field is the sum of two fields \( u^{cr}, u^{\alpha} \) - is a reference field and a field determined by the delamination presence in the strip. The field \( u^{\alpha} \) is expressed by the crack opening function, to find which we formulate the integral equation.

We consider the initial formulation of the problem, \( x_1 \) in the expression (3.2) on the delamination surface and satisfy the condition for the stresses’ absence on the crack surface (2.2), as a result, we obtain the integral equation for the crack opening function
\[ \int_{d_0-l_0}^{d_0+l_0} \chi(\xi)K(x_i,\xi)d\xi = f(x_i), \quad x_i \in [d_0-l_0, d_0 + l_0] \quad (3.4) \]

\[ f(x_i) = \int_{\sigma} \frac{\overline{p}}{ch(\lambda h)} e^{-iGx_i}d\sigma = \frac{p_0\pi}{h^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha_n} \left( \frac{\pi}{2} + \pi n \right) e^{-i\alpha_n|x_i|}, \quad \alpha_n^* = |\alpha_n|. \]

\[ K(x_i,\xi) = \int_{\sigma} \frac{\lambda sh(\lambda h)}{ch(\lambda h)} e^{i\alpha(x-x_i)}d\alpha \]

We select the main and regular parts of the kernel \( K(x_i,\xi) \)

\[ K(x_i,\xi) = K_0(x_i,\xi) + K_1(x_i,\xi) \]

\[ K_0(x_i,\xi) = \int_{\sigma} |\alpha| e^{i\alpha(x-x_i)}d\alpha = -2 \frac{\alpha}{(\xi-x_i)^2} \]

\[ K_1(x_i,\xi) = \int_{\sigma} \left( \frac{\lambda sh(\lambda h)}{ch(\lambda h)} e^{i\alpha(x-x_i)} - |\alpha| e^{i\alpha(x-x_i)} \right)d\alpha \]

So, the integral equation (3.4) can be rewritten in the form

\[ -2 \int_{d_0-l_0}^{d_0+l_0} \chi(\xi-x_i)^2 d\xi + \int_{d_0-l_0}^{d_0+l_0} \chi(\xi)K_1(x_i,\xi)d\xi = f(x_i), \quad x_i \in [d_0-l_0, d_0 + l_0] \quad (3.5) \]

The integral equation (3.5) has a hypersingular singularity and is understood in the sense of a finite value according to Hadamard.

For further calculations, we carry out the parameterization of the bundle

\[ \xi = d_0 + l_0 t, \quad t, \tau \in [-1,1] \quad (3.6) \]

We carry out an asymptotic analysis of the problem under study provided that the relative size of the bundle is small \( (l_0 \to 0) \). Note that the second term in (3.5), taking into account the introduced parametrization, is proportional to the crack length, which allows to neglect further calculations under the condition \( l_0 \to 0 \). The right-hand side of (3.5) with \( l_0 \to 0 \) is a constant determined by the expression

\[ f_0 = \frac{p_0\pi}{h^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha_n} \left( \frac{\pi}{2} + \pi n \right) e^{-i\alpha_n|x_i|}, \quad \alpha_n^* = -\left( k^2 - \frac{1}{h^2} \left( \frac{\pi}{2} + \pi m \right)^2 \right)^{1/2} \]

As a result, we have a hypersingular integral equation with a constant right-hand side

\[ -2 \frac{1}{l_0} \int_{-1}^{1} \frac{\chi(t)}{(t-\tau)^2}dt = f_0, \quad \tau \in [-1,1] \quad (3.7) \]

Integral equation (3.7) has a solution in the smooth functions class

\[ \chi(t) = (1-t^2)^{1/2} W_0(l_0, d_0) \]

\[ W_0 = -\frac{l_0 f_0(d_0)}{2\pi}, \quad t \in [-1,1] \quad (3.8) \]
As a result, for bundles of small relative length, an analytical expression was obtained for
determining the crack opening function.

The asymptotic behavior of the wave field (3.2) at the upper boundary of the strip has the following
representation

\[
u^{cr}(x_i, h) = \left( \sum_{m=0}^{N} + \sum_{n=N+1}^{N} \right) A_n e^{-i\omega_n[k_0-x_i]}
\]

(3.9)

\[A_m = l_0^2 \frac{p_0 \pi}{8h^4} \sum_{n=0}^{\infty} \left( -1 \right)^{n+m+1} \frac{(\pi/\alpha_n^2) \pi n}{2 + \pi n} \]

Based on the results obtained, we calculated the displacement wave field measured at the upper
boundary of the layer using the boundary element method with respect to (3.2), (3.5) (exact solution of
the direct problem) and the asymptotic approach (2.8), (2.9) (approximate solution) for the thickness
layer \( h=1 \). The results are shown in Fig. 1.

**Figure 1.** Graphs of the displacement field on the upper bound of layer

\( k=6, \ l_0=0.05h, \ d_0=h \)

4. The inverse identification problem in the framework of the asymptotic approach

Let us consider the solution of the inverse problem of identifying the bundle parameters in the
presence of a priori information about the smallness of the defect’s relative size. For definiteness, we
consider the case when the point of data collection \( x_i^* \) to the right of the bundle \( d_0 - x_i^* < 0 \)

Let us consider the formulation of the inverse problem, in which the amplitude values are given as
additional information \( A^{(m)}(m=0,1) \) movement fields in the far zone of the upper boundary at the point \( x_i^* \) at the frequency corresponding to the wave number \( k_1 \), in which there are two traveling
waves \( A^{(0)}(k_1), A^{(1)}(k_1) \). Such an assumption does not contradict the given condition (2.3). Based on
the expression for the amplitudes (3.9), it is possible to reduce the problem of identifying the bundle to
the stepwise determination of the parameters by solving the transcendental equations. Note that the
parameters \( d_0, l_0 \) in the expression (3.9) are “separated” and act as factors
Stage 1. Determination of the bundle parameter $d_0$. For determining $d_0$ we consider the amplitudes ratio of the first and second waves, which we denote as

$$
\mu_1 + i\mu_2 = \frac{A^{(0)}(k_i)}{A^{(1)}(k_i)}
$$

(4.1)

Then to determine $d_0$ after highlighting the real (or imaginary) part in (3.9) we have the expression

$$
d_0 = -\frac{1}{\alpha_0 - \alpha_1} \arccos \frac{3\alpha_0^*\mu_1}{\alpha_1}
$$

(4.2)

It is worth noting that at some frequency values, when the values $\mu_1, \mu_2$ are close, the phantom parameter values appear $d_0$, in such cases, it is necessary to measure the amplitude values at a different frequency, or measure the amplitudes at three frequencies and select the arithmetic mean of the obtained parameters, which are close in value, eliminating the excess.

Stage 2. Determination of the bundle length $l_0$. After finding the depth of the defect, from the expression (3.9), which is proportional to the bundle half-length square, we can determine the parameter $l_0$, for example, as the arithmetic mean of imaginary (real) values $A^{(0)}(k_i)$ and $A^{(1)}(k_i)$.

Table 1 shows the numerical results of the bundle parameters’ reconstruction.

| Wave number | Exact value $l_0$ | Identified value $l_0$ | Exact value $d_0$ | Identified value $d_0$ |
|-------------|-----------------|------------------|-----------------|------------------|
| $k=6$       | 0.1             | 0.0989           | 1.5             | 1.405            |
|             | 0.01            | 0.0099           | 1.5             | 1.496            |
| $k=7$       | 0.01            | 0.0099           | 1.5             | 1.499            |
|             | 0.05            | 0.0498           | 1.5             | 1.495            |
|             | 0.05            | 0.0506           | 0.5             | 0.503            |
|             | 0.05            | 0.0499           | 0.2             | 0.2107           |
|             | 0.01            | 0.0099           | 0.2             | 0.2004           |

5. Summary

In the framework of the asymptotic approach, the simple transcendental expressions are obtained for the displacement wave field and field amplitude values provided that the relative defect size is small. The obtained numerical results indicate the effectiveness of the proposed approach. The asymptotic method can be used to identify the bundles as “close” and “remote” from the source of oscillations, the length of which is less than 0.2h, while the error in the parameters’ restoration is not more than 1-3%.

For a unique reconstruction of the parameters, it is sufficient to know the amplitude values of the displacement field at a frequency at which there are two waves in the layer.

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