Random matrix models of stochastic integral type for free infinitely divisible distributions

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Abstract

The Bercovici-Pata bijection maps the set of classical infinitely divisible distributions to the set of free infinitely divisible distributions. The purpose of this work is to study random matrix models for free infinitely divisible distributions under this bijection. First, we find a specific form of the polar decomposition for the Lévy measures of the random matrix models considered in Benaych-Georges [6] who introduced the models through their measures. Second, random matrix models for free infinitely divisible distributions are built consisting of infinitely divisible matrix stochastic integrals whenever their corresponding classical infinitely divisible distributions admit stochastic integral representations. These random matrix models are realizations of random matrices given by stochastic integrals with respect to matrix-valued Lévy processes. Examples of these random matrix models for several classes of free infinitely divisible distributions are given. In particular, it is shown that any free selfdecomposable infinitely divisible distribution has a random matrix model of Ornstein-Uhlenbeck type $\int_0^\infty e^{-t}d\Psi_t^d$, $d \geq 1$, where $\Psi_t^d$ is a $d \times d$ matrix-valued Lévy process satisfying an $I_{\log}$ condition.

1 Introduction

An ensemble of random matrices is a sequence $(M_d)_{d \geq 1}$ where $M_d$ is a $d \times d$ matrix whose entries are random variables. The empirical spectral distribution of $M_d$ is the uniform distribution of its spectrum $\lambda_1, \lambda_2, ..., \lambda_d$, that is the (random) probability measure $\mu_{M_d}$ defined as $\mu_{M_d} = \frac{1}{d} \sum_{i=1}^d \delta_{\lambda_i}$. A random matrix model for a probability measure $\mu$ is an ensemble $(M_d)_{d \geq 1}$ for which the empirical spectral distribution $\mu_{M_d}$ converges weakly to $\mu$.

Bercovici and Pata [7] introduced a bijection $\Lambda$ from the set of classical infinitely divisible distributions to the set of free infinitely divisible distributions to study relations between

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classical and free infinitely divisible aspects. Under this bijection Benaych-Georges [6] and Cavanal-Duvillard [8] construct for any classical one-dimensional infinitely divisible distribution $\mu$ a random matrix model for the corresponding free infinitely divisible distribution $\Lambda(\mu)$. This include the Wigner and Marchenko-Pastur results, which provide random matrix models of Gaussian Unitary Ensembles and Wishart random matrices for the semicircle and Marchenko-Pastur distributions, respectively.

Specifically, it is shown in [6] that for any infinitely divisible distribution $\mu$ on $\mathbb{R}$ there is a random matrix model of Hermitian matrices $(M_d)_{d \geq 1}$ for $\Lambda(\mu)$. Moreover, for each $d \geq 1$ the Fourier transform of $M_d$ is given by

$$E[\exp(itr(M_dA))] = \exp[dE_uC_\mu(\langle u, Au \rangle)]$$  \hspace{1cm} (1)

where $C_\mu$ is the cumulant transform of $\mu$, $u$ is a uniformly distributed column random vector in the unit sphere of $\mathbb{C}^d$ and $\langle \cdot, \cdot \rangle$ is the usual Hermitian product of $\mathbb{C}^d$.

Any $\mathbb{R}$-valued Lévy process $\{X^{(\mu)}_t : t \geq 0\}$ with law $\mu$ at time $t = 1$ has associated an $\mathbb{R}$-valued independently scattered random measure. The stochastic integral of a real-valued function $h$ on $[0, \infty)$ with respect to $X^{(\mu)}_t$ written as

$$\int_0^\infty h(t)dX^{(\mu)}_t,$$  \hspace{1cm} (2)

is an $\mathbb{R}$-valued infinitely divisible random variable defined in the sense of integrals of non random functions with respect to scattered random measures, see Urbanik and Woyczynski [21] and Rajput and Rosiński [15]; and Sato [19] for the $\mathbb{R}^d$ case. Several important classes of infinitely divisible distributions having this stochastic integral representation in law have been studied recently, see [4], [2], [1], [3] and [9].

Let $I_{\log}(\mathbb{R})$ be the class infinitely divisible distributions $\mu$ on $\mathbb{R}$ whose Lévy measures $\nu_\mu$ satisfy the condition $\int_{|x| > 2} \log |x| \nu_\mu(dx) < \infty$. It is shown in Jurek and Vervaat [11], Sato and Yamazato [20] and Sato [18], that the class of selfdecomposable distributions on $\mathbb{R}$ is characterized by the stochastic integrals of the form (2) where $h(t) = e^{-t}$ and $\mu \in I_{\log}(\mathbb{R})$ in the following sense. For any $\mu \in I_{\log}(\mathbb{R})$ there exists a selfdecomposable distribution $\tilde{\mu}$ such that

$$\tilde{\mu} = \mathcal{L}\left(\int_0^\infty e^{-t}dX^{(\mu)}_t\right)$$  \hspace{1cm} (3)

and vice versa, to any selfdecomposable distribution $\tilde{\mu}$ corresponds a distribution $\mu$ in $I_{\log}(\mathbb{R})$ such that [3] holds. This characterization of selfdecomposable distributions as stochastic integrals is related to Ornstein-Uhlenbeck type processes through the Langevin equation. The Langevin equation $dY_t = dX^{(\mu)}_t - Y_t dt$ has stationary solution $\{Y_t : t \geq 0\}$ if and only if $\mu \in I_{\log}(\mathbb{R})$. This stationary solution $\{Y_t\}$ is unique and $\mathcal{L}\left(Y_t\right) = \mathcal{L}\left(\int_0^\infty e^{-t}dX^{(\mu)}_t\right)$ for all $t \geq 0$. The process $\{Y_t\}$ is called the stationary Ornstein-Uhlenbeck type process, see Sato [18] and Rocha-Arteaga and Sato [16].

In this work we are concerned with random matrix models for free infinitely divisible distributions corresponding to the image $\Lambda$ of classical infinitely divisible distributions. We
show that for every classical one-dimensional infinitely divisible distribution representable as the stochastic integral (2) there exists a random matrix model for the corresponding free infinitely divisible distribution, consisting of a realization as matrix stochastic integral similar to (2) with respect to an appropriate matrix-valued Lévy process. In particular, the free self-decomposable distribution \( \Lambda(\tilde{\mu}) \) corresponding to the classical selfdecomposable distribution \( \tilde{\mu} \) with stochastic integral representation (3) where \( \mu \in I_{\log}^\mu \), has a realization as random matrix model of Ornstein-Uhlenbeck type

\[
\left( \int_0^\infty e^{-t} d\Psi_{t}^{\mu,d} \right)_{d \geq 1}
\]

where \( \Psi_{t}^{\mu,d} \) is a \( d \times d \) matrix-valued Lévy process satisfying an \( I_{\log} \)-condition. Recently, Pérez-Abreu and Sakuma [14] studied random matrix models for free infinitely divisible distributions as matrix stochastic integrals of Wiener-Gamma type. They considered free infinitely divisible distributions corresponding to the image \( \Lambda \) of classical generalized Gamma convolutions distributions with the so called Wiener-Gamma representation, a type of stochastic integral representation (2) with respect to the Gamma process.

This paper is organized as follows. In Section 2 we find the polar decomposition for the Lévy measures of the Hermitian matrices of the random matrix models (1). In Section 3 we construct random matrix models for free infinitely divisible distributions as realizations of classical matrix stochastic integrals with respect to matrix-valued Lévy processes. We prove that these matrix stochastic integrals have the Fourier transform (1). In Section 4 we provide examples of random matrix models of matrix stochastic integrals for several classes of free infinitely divisible distributions. In particular, the class of free selfdecomposable distributions corresponding to the image \( \Lambda \) of the class of selfdecomposable distributions in (3), has random matrix models of Ornstein-Uhlenbeck type matrix integrals.

### 2 Polar decomposition of Lévy measures of certain random matrix models

Let \( \mathcal{M}_d = \mathcal{M}_{d \times d}(\mathbb{C}) \) denote the linear space of \( d \times d \) Hermitian matrices with scalar product \( \langle A, B \rangle = \text{tr}(AB^*) \) and norm \( \| A \| = |\text{tr}(AA^*)|^{1/2} \) where \( \text{tr} \) denotes trace. Let \( \mathcal{S}_{\mathcal{M}_d} = \{ V \in \mathcal{M}_d : \text{rank}(V) = 1, \| V \| = 1 \} \) the set of Hermitian matrices of rank 1 on the unit sphere of \( \mathcal{M}_d \) and let \( \mathcal{S}_{\mathcal{M}_d}^+ = \{ V > 0 : \text{rank}(V) = 1, \| V \| = 1 \} \) the set of positive definite matrices of rank 1 on the unit sphere of \( \mathcal{M}_d \).

We recall the polar decomposition of Lévy measures on \( \mathbb{R} \), see [17] and [4]. The Lévy measure \( \nu \) of an infinitely divisible distribution \( \mu \) on \( \mathbb{R} \) with \( 0 < \nu(\mathbb{R}) \leq \infty \), can be expressed as

\[
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr),
\]

where \( \lambda \) is a measure on the unit sphere \( S = \{-1, 1\} \) of \( \mathbb{R} \) such that \( 0 < \lambda(S) \leq \infty \) and \( \nu_\xi \) is a measure on \( (0, \infty) \) for each \( \xi \in S \) such that \( 0 < \nu_\xi((0, \infty)) \leq \infty \). Here \( \lambda \) and \( \nu_\xi \) are called the spherical and radial components of \( \nu \), respectively.

Let \( \hat{\mu} \) and \( \mathcal{C}_\mu \) denote the Fourier transform and the cumulant transform of an infinitely
divisible distribution \( \mu \) on \( \mathbb{R} \), respectively. That is, \( C_\mu \) is the unique continuous function from \( \mathbb{R} \) into \( \mathbb{C} \) such that \( C_\mu(0) = 0 \) and \( \hat{\mu}(z) = \exp(C_\mu(z)) \) for every \( z \in \mathbb{R} \).

In [6, Theorem 6.1] it is established that for any infinitely divisible distribution \( \mu \) on \( \mathbb{R} \) there exists a random matrix model of Hermitian matrices \( (M_d)_{d \geq 1} \) for \( \Lambda(\mu) \). Moreover, in [6, Theorem 3.1] the Fourier transform of \( M_d \), for each \( d \geq 1 \), is given by

\[
\mathbb{E}[\exp(itr(M_dA))] = \exp\{d\mathbb{E}_u(C_\mu(\langle u, Au \rangle))\} \quad A \text{ Hermitian}
\]

where \( C_\mu \) is the cumulant transform of \( \mu \), \( u \) is a uniformly distributed column random vector in the unit sphere of \( \mathbb{C}^d \) and \( \langle \cdot, \cdot \rangle \) is the usual Hermitian product of \( \mathbb{C}^d \). In the sequel we denote by \( \omega_d \) the probability measure on the set of matrices of rank 1 induced by the transformation

\[ u \rightarrow V = uu^* \]

In the following result we find the specific form of the polar decomposition for the Lévy measures of the random matrix models considered in [6].

**Theorem 1** Let \( \mu \) be an infinitely divisible distribution on \( \mathbb{R} \) with Lévy measure \( \nu \) and let \( (M_d)_{d \geq 1} \) a random matrix model for \( \Lambda(\mu) \) where \( M_d \) has the Fourier transform (5). Then the Lévy measure \( \nu_{M_d} \) of \( M_d \) is expressed as

\[
\nu_{M_d}(B) = d \int_{\tilde{S}_d} \int_0^\infty 1_B(rV) \nu_V(dr) \Pi(dV) \quad B \in \mathcal{B}(\mathbb{M}_d \setminus \{0\}),
\]

where \( \nu_V = \nu^+ \text{ or } \nu^- \) according to \( V > 0 \) or \( V < 0 \) and \( \Pi(dV) \) is a measure on \( \tilde{S}_d \) such that

\[ \int_{\tilde{S}_d} 1_D(V) \Pi(dV) = \int_{\tilde{S}_d^+ \setminus \{-1,1\}} \int_{\tilde{S}_d^+ \setminus \{-1,1\}} \int_D(\xi \lambda(d\xi) \omega_d(dV)) \quad D \in \mathcal{B}(\tilde{S}_d), \]

where \( \lambda \) is the spherical measure of \( \nu \) and \( \omega_d \) is the probability measure on \( \tilde{S}_d^+ \) given by (6).

**Proof.** Let \( \lambda(d\xi) \) and \( \nu_V \) be the spherical and radial components of \( \nu \) given by (4), respectively. For every \( z \in \mathbb{R} \), \( C_\mu(z) = i\gamma z + \int_\mathbb{R} [e^{iz} - 1 - iz1_{|z| \leq 1}(x)] \nu(dx) \) where \( \gamma \in \mathbb{R} \) and where we have omitted the Gaussian term without loss of generality. From (5) we have for every Hermitian matrix \( A \),

\[
\log \mathbb{E}[\exp(itr(M_dA))] = d\mathbb{E}_u \left\{ i\gamma \langle u, Au \rangle + \int_\mathbb{R} \left[ e^{i(u, Au)x} - 1 - ix \langle u, Au \rangle 1_{|x| \leq 1}(x) \right] \nu(dx) \right\}
\]

\[ = i\gamma r_d + d \int_{\tilde{S}_d^+} \int_\mathbb{R} \left[ e^{itr(Auu^*)x} - 1 - ixtr(Auu^*) 1_{|x| \leq 1}(x) \right] \nu(dx) \omega_d(dV) \]
almost surely for every
sets in \([0, \infty)\) with respect to \(X\)
the infinitely divisible \(M\) which is defined as the limit in probability of
For any infinitely divisible matrix \(\Psi\) in \(M\)
a real-valued independently scattered random measure
Any real-valued Lévy process

\[ 3 \text{ Random matrix models of stochastic integral type} \]

Any real-valued Lévy process \(\{X_t^{(\mu)}: t \geq 0\}\) with law \(\mu\) at time \(t = 1\), uniquely induces
a real-valued independently scattered random measure \(\{M^{(\mu)}(B): B \in \mathcal{B}_0([0, \infty))\}\) such that
\(M^{(\mu)}([0, t]) = X_t^{(\mu)}\) almost surely, where \(\mathcal{B}_0([0, \infty))\) is the family of bounded Borel
sets in \([0, \infty)\). We will consider \(M^{(\mu)}\)-integrable (or \(X_t^{(\mu)}\)-integrable) real-valued functions \(h\)
on \([0, \infty)\) in the sense of [15] and [19]. Then \(\int_B h(t)M^{(\mu)}(dt)\) (or \(\int_B h(t)dX_t^{(\mu)}\)) is defined
almost surely for every \(B \in \mathcal{B}_0([0, \infty))\). The stochastic integral of a real-valued \(h\) on \([0, \infty)\)
with respect to \(X_t^{(\mu)}\) is a real-valued infinitely divisible random variable written as
\[
\eta = \int_0^\infty h(t)dX_t^{(\mu)},
\]
which is defined as the limit in probability of \(\int_{[0,s]} h(t)dX_t^{(\mu)}\) as \(s \to \infty\) whenever the limit
exists. Furthermore, its cumulant transform is given by
\[
\mathcal{C}_\eta(z) = \int_0^\infty \mathcal{C}_\mu(h(t)z)dt \quad z \in \mathbb{R}.
\]

For the complex matrix case we have a similar result; see [5] for the case of real matrices.
For any infinitely divisible matrix \(\Psi\) in \(M_d\) with associated matrix Lévy process \(\{\Psi_t^d: t \geq 0\}\),
the infinitely divisible \(d \times d\) matrix valued stochastic integral
\[
M = \int_0^\infty h(t)d\Psi_t^d,
\]

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whenever exists, has cumulant transform

\[ C_M (A) = \int_0^\infty C_{\Psi} (h(t)A) \, dt \quad A \in \mathbb{M}_d. \]  

(10)

In this work we consider matrix Lévy processes \( \{ \Psi^d_t \} \) corresponding to Lévy measures of the form

\[ \nu^d_{\Psi} (B) = d \int_{\mathbb{R}^+} \omega_d (dV) \int_{\mathbb{R}} 1_B (xV) \nu_\mu (dx), \]  

(11)

where\( \omega_d \) is the probability measure induced by the transformation \( u \to V = uu^* \) in (6) and \( \nu_\mu \) is a Lévy measure of an infinitely divisible distribution \( \mu \) on \( \mathbb{R} \). Observe that \( \nu^d_{\Psi} \) is Lévy measure supported in the subset of rank one matrices in \( \mathbb{M}_d \).

The following is the main result of this work, it provides random matrix models for free infinitely divisible distributions on \( \mathbb{R} \) given by matrix stochastic integrals of the form (9) when the corresponding classical infinitely divisible distributions under \( \Lambda \) are representable as the random integrals of the form (7).

**Theorem 2** Let \( \mu_h \) be an infinitely divisible distribution on \( \mathbb{R} \) given by the stochastic integral representation

\[ \mu_h = \mathcal{L} \left( \int_0^\infty h(t) dX_t^{(\mu)} \right), \]  

(12)

where \( X_t^{(\mu)} \) is a Lévy process on \( \mathbb{R} \) with law \( \mu \) at time \( t = 1 \) and Lévy measure \( \nu_\mu \). The free infinitely divisible distribution \( \Lambda (\mu_h) \) has a random matrix model given by the ensemble of infinitely divisible matrix stochastic integrals

\[ \left( M^d_h = \int_0^\infty h(t) d\Psi^d_t \right)_{d \geq 1}, \]  

(13)

where \( \Psi^d_t \) is the \( \mathbb{M}_d \)-valued Lévy process with Lévy measure \( \nu^d_{\Psi} \) given by (11) in terms of \( \omega_d \) and \( \nu_\mu \).

**Proof.** We will prove that the random matrices of the ensemble \( (M^d_h)_{d \geq 1} \) in (13) and the random matrices of the random matrix model \( (M_d)_{d \geq 1} \) for \( \Lambda (\mu_h) \) given in [6, Theorem 6.1] have the same laws. Recall from [6] that the Fourier transform of \( M_d \) is given by

\[ \mathbb{E} [\exp (itr (M_d A))] = \exp \left[ d \mathbb{E}_u C_{\mu_h} (\langle u, A u \rangle) \right] \quad A \text{ Hermitian}. \]

We will prove that this Fourier transform of \( M_d \) coincides with the Fourier transform of \( M^d_h \).
For that, we first calculate the cumulant transform of $\Psi_1^d$ at time 1 using (11),

$$C_{\Psi_1^d}(A) = \int_{\mathbb{M}_d^+} \left[ e^{i\text{tr}(AX)} - 1 - i\text{tr}(AX)1_{||X|| \leq 1}(X) \right] \nu_{\Psi_1^d}(dX)$$

$$= d \int_{\bar{\mathbb{S}}_{\mathbb{M}_d}^+} \int_{\mathbb{R}} \left[ e^{i\text{tr}(AV)x} - 1 - ix\text{tr}(AV)1_{|x| \leq 1}(x) \right] \omega_d(dV) \nu_{\mu}(dx)$$

$$= d \int_{\mathbb{R}} \mathbb{E}_V \left[ e^{i\text{tr}(AV)x} - 1 - ix\text{tr}(AV)1_{|x| \leq 1}(x) \right] \nu_{\mu}(dx)$$

$$= d\mathbb{E}_u \int_{\mathbb{R}} \left[ e^{i\text{tr}(Au^*)x} - 1 - ix\text{tr}(Au^*)1_{|x| \leq 1}(x) \right] \nu_{\mu}(dx)$$

$$= d\mathbb{E}_u C\mu \langle \langle u, Au \rangle \rangle. \quad (14)$$

Now we calculate the cumulant transform of $M_h^d$ from (10) and (14),

$$C_{M_h^d}(A) = \int_0^\infty C_{\Psi_1^d}(h(t)A) dt = \int_0^\infty d\mathbb{E}_u C\mu \langle \langle u, h(t)Au \rangle \rangle dt$$

where in the last equality we have used the relation (8) between the cumulant transforms of $\mu_h$ and $\mu$ corresponding to the stochastic integral representation (12).

**Remark 3** Let $\nu_{\mu}, \nu_{\mu_h}, \nu_{M_h^d}$ and $\nu_{\Psi^d}$ denote the Lévy measures of $\mu, \mu_h, M_h^d$ and $\Psi_1^d$ at time 1 in Theorem 2, respectively. From [19] the Lévy measures of $\mu$ and $\mu_h$ are related as follows

$$\nu_{\mu_h}(B) = \int_0^\infty dt \int_{\mathbb{R}} 1_B(h(t)x)\nu_{\mu}(dx) \quad B \in \mathcal{B} (\mathbb{R} \setminus \{0\})$$

and from [12] it is obtained a similar relation between the Lévy measures of $M_h^d$ and $\Psi^d$,

$$\nu_{M_h^d}(B) = \int_0^\infty dt \int_{\mathbb{M}_d^+} 1_B(h(t)X)\nu_{\Psi^d}(dX) \quad B \in \mathcal{B} (\mathbb{M}_d \setminus \{0\}) .$$

These two relations combined with (11) yield the Lévy measure of $M_h^d$ in terms of the Lévy measure of $\mu_h$

$$\nu_{M_h^d}(B) = d\int_{\bar{\mathbb{S}}_{\mathbb{M}_d}^+} \omega_d(dV) \int_{\mathbb{R}} \nu_{\mu_h}(dx) \int_0^\infty 1_B(h(t)xV) dt$$

$$= d\int_{\bar{\mathbb{S}}_{\mathbb{M}_d}^+} \omega_d(dV) \nu_{\mu_h}(dx) 1_B(xV), \quad B \in \mathcal{B} (\mathbb{M}_d \setminus \{0\}) . \quad (15)$$
4 Examples

In the following \( I(\mathbb{R}) \) denotes the class of infinitely divisible distributions on \( \mathbb{R} \) and \( L(\mathbb{R}) \) denotes the class of selfdecomposable distributions on \( \mathbb{R} \).

Let \((\lambda, \nu_\xi)\) denote the polar decomposition of the Lévy measure \( \nu \) of any \( \mu \in I(\mathbb{R}) \) given by (4), that is
\[
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr),
\]
where \( \lambda \) and \( \nu_\xi \) are the spherical and radial components of \( \nu \), respectively and \( S = \{-1, 1\} \).

Moreover, let \((\lambda, k_\xi)\) denote the following description of the above decomposition for self-decomposable distributions, see [18]. If \( \mu \in L(\mathbb{R}) \) the radial component of its Lévy measure \( \nu \) is expressed as
\[
\nu_\xi(dr) = 1_{(0,\infty)}(r)\frac{k_\xi(r)}{r}dr,
\]
where \( k_\xi(r) \) is a nonnegative measurable function in \( \xi \in S \) and decreasing, right continuous in \( r \in (0, \infty) \). Here \( k_\xi \) is called the \( k \)-function of \( \nu \).

Throughout this section we provide examples of random matrix models for free infinitely divisible distributions \( \Lambda(\mu_h) \) given by the ensembles of matrix stochastic integrals (13), for several classes infinitely divisible distributions \( \mu_h \) with stochastic integral representation (12). In most of examples we express the Lévy measure \( \nu_{\mu_h} \) in terms of \( \nu_{\mu_h} \). We emphasize, according to (15), that these Lévy measures \( \nu_{\mu_h} \) are supported in the subset of rank one matrices in \( \mathbb{M}_d \).

In all examples below \( \nu_\mu, \nu_{\mu_h}, \nu_{\mu_{M^d_h}} \) and \( \nu_{\Psi^d_t} \) denote the Lévy measures of \( \mu, \mu_h, M^d_h \) and \( \Psi^d_t \) at time 1 in Theorem 2 respectively.

The following two examples recover the random matrix models of Theorems 4.1 and 4.3 in [14] and find their corresponding Lévy measures.

Example 1. The class of Generalized Gamma Convolutions \( T(\mathbb{R}_+) \) is the smallest class of infinitely divisible distributions on \( \mathbb{R}_+ \) that contains all Gamma distributions and that is closed under convolution and weak convergence.

In [9], any distribution in \( T(\mathbb{R}_+) \) has stochastic integral representation (12) where \( \mu \) is a Gamma distribution and \( h(t) \) is a Borel measurable function \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \int_0^\infty \log (1 + h(t)) dt < \infty \). Such a representation is called the Wiener-Gamma integral representation.

Let \( \mu_h \in T(\mathbb{R}_+) \) has the Wiener-Gamma integral representation. By (8) in [14] the Lévy measure of \( \mu_h \) is expressed as
\[
\nu_{\mu_h}(dx) = 1_{(0,\infty)}(x) \int_0^\infty \frac{e^{-x/h(t)}}{x}dt dx.
\]
The corresponding random matrix models (13) for the free Generalized Gamma Convolutions \( \Lambda(\mu_h) \) consist of matrix stochastic integrals of Wiener-Gamma type
\[
\left( M^d_h = \int_0^\infty h(t)d\Psi^d_t \right)_{d \geq 1}.
\]
with Lévy measures

\[ \nu_{M_h}(B) = d \int_{\mathbb{R}^+_d} \omega_d \, (dV) \int_0^\infty dx 1_B(xV) \int_0^\infty \frac{e^{-x/h(t)}}{x} \, dt. \]

**Example 2.** The class of Thorin distributions \( T(\mathbb{R}) \) is the smallest class of infinitely divisible distributions on \( \mathbb{R} \) which contains all distributions in \( T(\mathbb{R}_+) \) and is closed under convolution, weak convergence and reflection.

It is shown in [4] that this class is characterized by the stochastic integral representation (12) where \( \mu \in ID_{\log}(\mathbb{R}) \) and \( h(t) \) is the inverse function of the incomplete Gamma function \( g(t) = \int_0^\infty e^{-s} s^{-1} \, ds \).

Let \( \mu_h \in T(\mathbb{R}) \) with such a representation. The random matrix models (13) for the corresponding free Thorin distributions \( \Lambda(\mu_h) \) are given by

\[ \left( M^d_h = \int_0^\infty g^*(t) d\Psi^d_t \right)_{d \geq 1} \quad g^* \text{ inverse function of } g. \]

Let \( (\lambda, \nu_\xi) \) and \( (\lambda_h, \nu_{h\xi}) \) be the polar decompositions of the Lévy measures of \( \mu \) and \( \mu_h \), respectively. It is proved in [1] that the corresponding \( k \)-function of \( \mu_h \) is \( k_{h\xi}(r) = \int_0^\infty e^{-r/s} \nu_\xi(ds) \). Therefore the Lévy measures of these matrix stochastic integrals are given by

\[ \nu_{M^d_h}(B) = d \int_{\mathbb{R}^+_d} \omega_d \, (dV) \int_s^\infty \frac{dr}{r} \left( \int_0^\infty e^{-r/s} \nu_\xi(ds) \right) 1_B(xV) \]

**Example 3.** The class of Bondesson distributions \( B(\mathbb{R}) \) is the smallest class of infinitely divisible distributions on \( \mathbb{R} \) that contains all mixtures of exponential distributions and that is closed under convolution, weak convergence and reflection.

This class is characterized by the stochastic integral representation (12) where \( \mu \in I(\mathbb{R}) \) and \( h(t) = 1_{(0,1)}(t) \log (1/t) \), see [4]. If \( \mu_h \in B(\mathbb{R}) \) has such a representation then by (2.17) in [4]

\[ \nu_{\mu_h}(dx) = \int_0^\infty e^{-s} \nu_{\mu}(s^{-1} dx) \, ds. \]

Therefore the random matrix models (13) for the free Bondesson distributions \( \Lambda(\mu_h) \) are given by

\[ \left( M^d_h = \int_0^1 \log (1/t) \, d\Psi^d_t \right)_{d \geq 1} \]

with Lévy measures

\[ \nu_{M^d_h}(B) = d \int_{\mathbb{R}^+_d} \omega_d \, (dV) \int_{-\infty}^\infty \int_0^\infty e^{-s} \nu_{\mu}(s^{-1} dx) \, ds 1_B(xV). \]

**Example 4.** The class of Thorin distributions \( T(\mathbb{R}) \) is also characterized (see Example 2) by the stochastic integral representation (12) where \( \mu \in L(\mathbb{R}) \) and \( h(t) = 1_{(0,1)}(t) \log (1/t), \)
If \( \mu_h \in T(\mathbb{R}) \) has this alternative representation, the corresponding random matrix models \( M^d_h \) for the free Thorin distributions \( \Lambda(\mu_h) \) are

\[
M^d_h = \int_0^1 \log \left( \frac{1}{t} \right) d\Psi_t^d \quad (d \geq 1).
\]

Let \( (\lambda, k_\xi) \) and \( (\lambda_h, k_{h\xi}) \) be the corresponding polar decompositions of the Lévy measures of \( \mu \) and \( \mu_h \), respectively. It is shown in [4] that \( \lambda = \lambda_h \) and

\[
k_{h\xi}(r) = \int_0^\infty k_\xi(rs^{-1})e^{-s}ds
\]

and hence the Lévy measures for these random matrix models are of the form

\[
\nu_{M^d_h}(B) = d \int_{\mathbb{S}^1_{M^d_d}} \omega(dV) \int_{-\infty}^\infty 1_B(xV) \nu_{\mu_h}(dx)
\]

\[
= d \int_{\mathbb{S}^1_{M^d_d}} \omega(dV) \int_{S} \lambda(d\xi) \int_0^\infty \frac{dr}{r} \left( \int_0^\infty k_\xi(rs^{-1})e^{-s}ds \right) 1_B(r\xi V).
\]

**Example 5.** The class of selfdecomposable distributions \( L(\mathbb{R}) \) is characterized by the stochastic integral representation (12) where \( \mu \in I_{\log}(\mathbb{R}) \) and \( h(t) = 1_{(0, \infty)}(t)e^{-t} \), see [11], [20] and [18].

Let \( \mu_h \in L(\mathbb{R}) \) with such a Ornstein-Uhlenbeck type integral representation. The random matrix models (13) for the corresponding free selfdecomposable distributions \( \Lambda(\mu_h) \) are given by

\[
M^d_h = \int_0^\infty e^{-t}d\Psi_t^d \quad (d \geq 1)
\]

and satisfy an \( I_{\log} \)-condition, that is

\[
\int_{||X||>2} \log ||X|| \nu^d_{\Phi}(dX) < \infty,
\]

which follows easily from (11) and the fact that \( \mu \in I_{\log}(\mathbb{R}) \). These Ornstein-Uhlenbeck type matrix integrals are supported on the subset of rank one matrices in \( M^d_d \).

Let \( (\lambda, \nu_\xi) \) and \( (\lambda_h, k_{h\xi}) \) be the corresponding polar decompositions of the Lévy measures of \( \mu \) and \( \mu_h \). It is proved in [1] that the \( k \)-function of \( \nu_{\mu_h} \) is given by

\[
k_{h\xi}(r) = \nu_\xi((r, \infty)).
\]

Therefore, the Lévy measures of these random matrix models of Ornstein-Uhlenbeck type for free selfdecomposable distributions \( \Lambda(\mu_h) \) are given by

\[
\nu_{M^d_h}(B) = d \int_{\mathbb{S}^1_{M^d_d}} \omega(dV) \int_{-\infty}^\infty 1_B(xV) \nu_{\mu_h}(dx)
\]

\[
= d \int_{\mathbb{S}^1_{M^d_d}} \omega(dV) \int_{S} \lambda_h(d\xi) \int_0^\infty \frac{dr}{r} \nu_\xi((r, \infty)) 1_B(r\xi V).
\]
Example 6. The class of type $G$ distributions $G_{sym}(\mathbb{R})$ is the class of symmetric distributions on $\mathbb{R}$ which are variance mixtures of the standard Gaussian distribution with positive infinitely divisible mixing distributions.

In [2] is proved that this class is characterized by the stochastic integral representation (12) where $\mu \in I(\mathbb{R})$ and $h(t)$ is the inverse function of $f(t) = \int_t^\infty \varphi(u) du$ where $\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$.

Let $\mu_h \in G_{sym}(\mathbb{R})$ with such a representation. In this case the random matrix models (13) for the corresponding free type $G$ distributions $\Lambda(\mu_h)$ are of the form

$$
(M^d_h = \int_0^\infty f^*(t) d\Psi^d_t)_{d \geq 1},
$$

where $f^*$ is the inverse function of $f$.

Let $(\lambda_h, \nu_{h_\xi})$ be the polar decomposition of the Lévy measure of the type $G$ distribution $\mu_h$. It is known that the radial component $\nu_{h_\xi}$ can be written as

$$
\nu_{h_\xi} (dr) = g_{h_\xi}(r^2)dr
$$

where $g_{h_\xi}(r)$ is measurable in $\xi \in S$ and completely monotone in $r \in (0, \infty)$. It is proved in [1] that

$$
g_{h_\xi}(r) = \int_0^\infty \varphi(r^{1/2}/s)s^{-1}\nu_\xi(ds),
$$

where $\nu_\xi$ is the radial component of $\nu_\mu$. Therefore the Lévy measures for these random matrix models for free type $G$ distributions $\Lambda(\mu_h)$ are given by

$$
\nu_{M^d_h}(B) = d \int_{S^+_d} \omega_d (dV) \int_S \lambda_h (d\xi) \int_0^\infty dr \left( \int_0^{\infty} \varphi(r/s)s^{-1}\nu_\xi(ds) \right) 1_B(r\xi V).
$$

Example 7. The class of $M$ distributions $M(\mathbb{R})$ is a subclass of selfdecomposable distributions of $G_{sym}(\mathbb{R})$. It is the class of symmetric infinitely divisible distributions on $\mathbb{R}$ whose Lévy measures have polar decomposition $(\lambda, \nu_\xi)$ such that

$$
\nu_\xi (dr) = \frac{g_\xi(r^2)}{r}dr,
$$

where $g_{h_\xi}(r)$ is measurable in $\xi \in S$ and completely monotone in $r \in (0, \infty)$.

In [3] is shown that the class $M(\mathbb{R})$ is characterized by the stochastic integral representation (12) where $\mu \in I_{log}(\mathbb{R})$ and $h(t)$ is the inverse function of $m(t) = \int_t^\infty \varphi(u) du$.

Let $\mu_h \in M(\mathbb{R})$ with such a representation. The random matrix models (13) for the corresponding free $M$ distributions $\Lambda(\mu_h)$ are

$$
(M^d_h = \int_0^\infty m^*(t) d\Psi^d_t)_{d \geq 1},
$$

where $m^*$ is the inverse function of $m$. 

11
Let \((\lambda_h, \nu_{h_\xi})\) be the polar decomposition of the Lévy measure of \(\mu_h\) and let \(g_{h_\xi}\) be the corresponding \(g\)-function in (10) of the radial component \(\nu_{h_\xi}\). It is proved in [1] that

\[
g_{h_\xi}(r) = \int_0^\infty \varphi(r^{1/2}/s) \nu_{\xi}(ds),
\]

where \(\nu_{\xi}\) is the radial component of \(\nu_{\mu}\). Therefore the Lévy measures for these random matrix models for the free \(M\) distributions \(\Lambda(\mu_h)\) can written as

\[
\nu_{M_d^h}(B) = d \int_{\mathbb{S}_{\mu_d}^+} \omega_d(dV) \int_{\mathbb{S}} \lambda_h(d\xi) \int_0^\infty dr \left( \int_0^\infty \varphi(r/s) \nu_{\xi}(ds) \right) 1_B(r\xi V).
\]

**Example 8.** The class of Jurek distributions \(U(\mathbb{R})\) is the class of infinitely divisible distributions on \(\mathbb{R}\) whose Lévy measures have polar decomposition \((\lambda, \nu_{\xi})\) such that

\[
\nu_{\xi}(dr) = l_{\xi}(r)dr,
\]

where \(l_{\xi}\) is measurable in \(\xi \in S\) and decreasing in \(r \in (0, \infty)\).

It is shown in [10] that this class is characterized by the stochastic integral representation (12) where \(\mu \in I(\mathbb{R})\) and \(h(t) = 1_{[0,1]}(t)t\).

Let \(\mu_h \in U(\mathbb{R})\) with such a representation. The corresponding random matrix models (13) for the free Jurek distributions \(\Lambda(\mu_h)\) are of the form

\[
\left( M_d^h = \int_0^1 td\Psi_t \right)_{d \geq 1}.
\]

Let \((\lambda, \nu_{\xi})\) and \((\lambda_h, \nu_{h_\xi})\) be the polar decompositions of the Lévy measures of \(\mu\) and \(\mu_h\), respectively. It is proved in [1] that the corresponding \(l\)-function in (17) for the radial component \(\nu_{h_\xi}\) is expressed as

\[
l_{h_\xi}(r) = \int_r^\infty x^{-1} \nu_{\xi}(dx).
\]

Therefore the Lévy measures of these random matrix models are expressed as

\[
\nu_{M_d^h}(B) = d \int_{\mathbb{S}_{\mu_d}^+} \omega_d(dV) \int_{\mathbb{S}} \lambda_h(d\xi) \int_0^\infty dr \left( \int_r^\infty x^{-1} \nu_{\xi}(dx) \right) 1_B(r\xi V).
\]

**Example 9.** The class of \(A\) distributions \(A(\mathbb{R})\) introduced in [12] is bigger than the Jurek class \(U(\mathbb{R})\). It is the class of infinitely divisible distributions on \(\mathbb{R}\) with Lévy measures \(\nu\) of the form

\[
\nu(B) = \int_{\mathbb{R}\setminus\{0\}} \rho(dx) \int_0^\infty a_1(r; |x|) 1_B \left( \frac{r}{|x|} \right) dr \quad B \in \mathcal{B}(\mathbb{R}),
\]

where \(\rho\) is a Lévy measure on \(\mathbb{R}\); and \(a_1(r; s)\) is the one-sided arcsine density with parameter \(s > 0\), that is,

\[
a_1(r; s) = \begin{cases} 2\pi^{-1} (s - r^2)^{-1/2} & 0 < r < s^{1/2} \\ 0 & \text{otherwise}. \end{cases}
\]
It is shown in [12] that the class $A(\mathbb{R})$ is characterized by the stochastic integral representation (12) where $\mu \in I(\mathbb{R})$ and $h(t) = 1_{[0,1]}(t) \cos \left( \frac{\pi}{2} t \right)$.

Let $\mu_h \in A(\mathbb{R})$ with such a representation. In this case the random matrix models (13) for the corresponding free $A$ distributions $\Lambda(\mu_h)$ are

$$
\left( M^d_t = \int_0^1 \cos \left( \frac{\pi}{2} t \right) d\Psi_t \right)_{d \geq 1}
$$

with Lévy measures, see (15),

$$
\nu_{M^d_t}(B) = d \int_{\mathbb{R}_+} \omega_d(dV) \int_{\mathbb{R}} \nu_{\mu}(dx) \int_0^1 1_B \left( xV \cos \left( \frac{\pi}{2} t \right) \right) dt.
$$

**Example 10.** The class of Type G distributions $G_{sym}(\mathbb{R})$ is also characterized (see Example 6) by the stochastic integral representation (12) where $\mu \in A(\mathbb{R})$ and $h(t) = 1_{(0,1/2)}(t) \left( \log \frac{1}{t} \right)^{1/2}$, see [12]. Let $\mu_h \in G_{sym}(\mathbb{R})$ has this alternative representation, that is

$$
\mu_h = \mathcal{L} \left( \int_0^{1/2} \left( \log \frac{1}{t} \right)^{1/2} dX_t^\mu \right) \quad \mu \in A(\mathbb{R}).
$$

The corresponding random matrix models (13) for the free Type G distributions $\Lambda(\mu_h)$ are given by

$$
\left( M^d_t = \int_0^{1/2} \left( \log \frac{1}{t} \right)^{1/2} d\Psi_t \right)_{d \geq 1}
$$

with Lévy measures

$$
\nu_{M^d_t}(B) = d \int_{\mathbb{R}_+} \omega_d(dV) \int_{\mathbb{R}} \nu_{\mu}(dx) \int_0^{1/2} 1_B \left( xV \left( \log \frac{1}{t} \right)^{1/2} \right) dt.
$$

**Example 11.** The class of generalized Type G distributions $G(\mathbb{R})$ introduced in [13] is the class of infinitely divisible distributions on $\mathbb{R}$ whose Lévy measures have polar decomposition $(\lambda, \nu_\xi)$ such that

$$
\nu_\xi(dr) = g_\xi(r^2)dr,
$$

where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in $r \in (0, \infty)$. Distributions of this class are not necessarily symmetric. If $\mu \in G(\mathbb{R})$ is symmetric then $\mu \in G_{sym}(\mathbb{R})$, that is $\mu$ is of Type G.

It is proved in [12] that the class of generalized type G distributions is characterized by the stochastic integral representation (12) where $\mu \in A(\mathbb{R})$ and $h(t) = 1_{(0,1)}(t) \left( \log \frac{1}{t} \right)^{1/2}$.

Let $\mu_h \in G(\mathbb{R})$ has such a representation, that is

$$
\mu_h = \mathcal{L} \left( \int_0^1 \left( \log \frac{1}{t} \right)^{1/2} dX_t^\mu \right) \quad \mu \in A(\mathbb{R}).
$$
This representation (19) is not necessarily a symmetric generalization of (18).

In this case the corresponding random matrix models (13) for the free generalized type \( G \) distributions \( \Lambda(\mu_h) \) are

\[
M_h^d = \int_0^1 \left( \log \frac{1}{t} \right)^{1/2} d\Psi^d_t \geq 1.
\]

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