Geometric boundary data for the gravitational field

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Abstract
An outstanding issue in the treatment of boundaries in general relativity is the lack of a local geometric interpretation of the necessary boundary data. For the Cauchy problem, the initial data is supplied by the 3-metric and extrinsic curvature of the initial Cauchy hypersurface, subject to constraints. This Cauchy data determine a solution to Einstein’s equations which is unique up to a diffeomorphism. Here, we show how three pieces of unconstrained boundary data, which are associated locally with the geometry of the boundary, likewise determine a solution of the initial-boundary value problem which is unique, up to a diffeomorphism. Two pieces of this data constitute a conformal class of rank-2 metrics, which represent the two gravitational degrees of freedom. The third piece, constructed from the extrinsic curvature of the boundary, determines the dynamical evolution of the boundary.

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1. Introduction
There exists a well posed Cauchy problem for Einstein’s equation which has the important property that local geometric data representing the 3-metric and extrinsic curvature of the initial Cauchy hypersurface determine a spacetime metric $g_{ab}$ which is unique up to diffeomorphism. Presently, there are two formulations of the initial-boundary value problem (IBVP) which are strongly well posed, the Friedrich–Nagy formulation [1] and the harmonic formulation [2–4], but neither provide a local geometric interpretation of the boundary data. For the harmonic formulation, there exists a nonlocal (in time) version in which the geometric interpretation
of the boundary data depends upon a background metric constructed from the initial Cauchy data [5]. As a result, the boundary data have a geometric interpretation which is nonlocal in time. In this work, we show how boundary data for the gravitational field can be posed which are locally determined by the geometry of the boundary, in the same sense as the Cauchy data.

In a Cauchy problem, initial data on a spacelike hypersurface \( S_0 \) determine a solution in the domain of dependence \( D(S_0) \) (which consists of those points whose past directed characteristics all intersect \( S_0 \)). In the IBVP, data on a timelike boundary \( T \) transverse to \( S_0 \) are used to extend the solution to the domain of dependence \( D(S_0 \cup T) \). Strong well-posedness [6] guarantees the existence of a unique solution which depends continuously on both the initial data and the boundary data.

The primary application of the gravitational IBVP is the simulation of an isolated astrophysical system containing neutron stars and black holes. The standard approach in numerical relativity, as in computational studies of other hyperbolic systems, is to introduce an artificial outer boundary \( T \), which is coincident with the boundary of the computational grid and whose cross-sections are spheres surrounding the system. The ability to compute the details of the gravitational radiation produced by compact astrophysical sources, such as coalescing black holes, is of major importance to the success of gravitational wave astronomy. If the simulation of such systems is not based upon a strongly well posed IBVP then the results cannot be trusted in the domain of dependence of the outer boundary. For comprehensive reviews of the gravitational IBVP see [7, 8].

For hyperbolic systems which are stable under lower order perturbations, the global solution in the spacetime manifold \( M \) can be obtained by patching together local solutions, i.e. the problem can be localized. Thus, for purposes of treating the underlying geometrical nature of the boundary data, it suffices to concentrate on the local problem in the neighborhood of a point on the boundary. That is the approach taken in this paper.

In the Friedrich–Nagy formulation, there are three essential pieces of boundary data which have geometrical or physical significance. One is the trace \( K \) of the extrinsic curvature \( K_{ab} \) of the boundary, which geometrically determines the location of the boundary. (Note that the coordinate specification of the location of the boundary is pure gauge information and does not determine its location in the same geometric sense that a curve is determined by its acceleration (curvature), given its initial position and velocity.) Two other pieces of data in the Friedrich–Nagy formulation, which are related to the gravitational radiation degrees of freedom, are supplied by a combination of the Weyl tensor components \( \Psi_0 \) and \( \Psi_4 \) in the Newman–Penrose notation [9]. The remaining boundary data specify the gauge freedom.

The Friedrich–Nagy formulation is based upon a symmetric hyperbolic Einstein-Bianchi system, with evolution variables consisting of an orthonormal tetrad, the associated connection coefficients and the Weyl curvature components. Although it differs from the metric based formulations used in numerical relativity, the requirement of three pieces of geometric boundary data should be universally applicable. (Statements found in the literature that only two pieces of boundary data suffice to specify the physical or geometrical properties of the gravitational field are misleading. They are only true when the boundary has been geometrically specified, e.g. for an \( r = \text{const} \) boundary in a background Schwarzschild geometry.)

The outgoing null vector \( K^a \) and ingoing null vector \( L^a \) used in defining \( \Psi_0 \) and \( \Psi_4 \), respectively, are determined by the unit normal to the boundary \( N^a \) and a choice of unit timelike vector \( T^a \) tangent to the boundary according to

\[
K^a = T^a + N^a, \quad L^a = T^a - N^a. \tag{1.1}
\]
The choice of direction of $T^a$ represents gauge freedom in this data. Friedrich and Nagy are careful to point out that this gauge freedom prevents interpreting $\Psi_0$ and $\Psi_4$ as purely geometric data.

This shortcoming could perhaps be avoided by choosing these null vectors to be principal null directions of the Weyl tensor. However, in a general spacetime this would lead to four possible choices which would then have to be incorporated (in some yet unknown way) into a well posed problem. An alternative, proposed in [10], is to base the data on the eigenvectors $V^a$ determined by the trace-free part of the extrinsic curvature according to

$$
(K_{ab} - \frac{1}{2}H_{ab}K)v^b = \lambda H_{ab}V^b, \tag{1.2}
$$

where $H_{ab}$ is the intrinsic 3-metric of the boundary. For a spherical worldtube in Minkowski space, this picks out a locally preferred timelike direction $\tilde{T}^a$. This suggests that the approach might extend to a suitably round outer boundary of an isolated system. However, it is again not clear whether such an approach can be properly incorporated into the evolution system.

Here we consider geometric boundary data for metric based formulations of the IBVP. Our main result is that, along with the initial Cauchy data, a spacetime metric satisfying Einstein’s equations is uniquely determined up to diffeomorphism by three pieces of boundary data related locally to the intrinsic metric $H_{ab}$ and extrinsic curvature $K_{ab}$ of the boundary. More specifically, two pieces of boundary data consist of a conformal class $\{Q_{ab}\}$ of rank-2 metrics of signature $(0++)$, which represent the two gravitational degrees of freedom. The null eigendirection of $\{Q_{ab}\}$ uniquely determines a flow of streamlines on the boundary. The third piece of data, which determines the dynamical evolution of the boundary, is a component of the extrinsic curvature of the boundary picked out by the unit vector to these streamlines. In section 2, we discuss the underlying geometry and present our main result as a local geometric data theorem.

The demonstration that this local geometric data leads to a strongly well posed IBVP is carried out using the harmonic reduction of Einstein’s equations to ten wave equations, as was the method used in establishing the analogous result for the Cauchy problem [11]. In doing so, the three pieces of local geometric boundary data must be supplemented by seven additional boundary conditions. Four of these conditions are supplied by the harmonic coordinate conditions. The other three fix the remaining freedom in the boundary values of the harmonic coordinates. In section 3, the resulting harmonic IBVP is reduced to a set of partial differential equations (PDEs) in the frozen coefficient formalism, which are subject to a combination of Dirichlet and Neumann boundary conditions.

The demonstration that the strong well-posedness of the frozen coefficient version of the harmonic IBVP extends to the full quasilinear problem was given in [2, 3] for the case of Sommerfeld boundary conditions. The application of these methods to more general PDEs and their application to boundary conditions for isolated systems was presented in [4]. In section 4, we demonstrate how this approach can be extended to Dirichlet and Neumann conditions.

The strong well-posedness of the frozen coefficient harmonic IBVP with local geometric boundary data is established in section 5. The key idea is that the full set of boundary conditions can be applied sequentially, similar to the approach followed in [2, 3] except now applied to a set of Dirichlet and Neumann conditions rather than Sommerfeld conditions. Our main result is then established in section 5.4.

Beyond the issue of a local geometrical characterization of the boundary data, there are other important aspects of the IBVP which remain unresolved. For the Cauchy problem, it has been shown that a given initial data set has a maximal development [12]. Two such maximal developments corresponding to the same initial data set must be related by a diffeomorphism.
This property, and the related issue of geometric uniqueness, have recently been discussed in the context of the IBVP in [8, 10]. For the Cauchy problem, an essential ingredient of geometric uniqueness is that two solutions of Einstein’s equations with the same initial data are related by a diffeomorphism. Since both solutions can be transformed to harmonic coordinates without changing the local geometric data, this result follows from the uniqueness of the solution in harmonic coordinates. The same argument applies, at least locally in time, to the solutions of the IBVP with the same initial-boundary data, as specified in the Local Geometric Data Theorem. Another geometric property of the Cauchy problem is that two diffeomorphic solutions must have diffeomorphic initial data. It is unlikely that such a strong property holds for the gravitational IBVP. Even for a scalar wave equation, the analogous result does not hold since the same solution can be specified, say, by either Dirichlet or Neumann boundary data. We do not address here the question whether it is possible to give a geometric classification of those initial-boundary data sets which give rise to diffeomorphically equivalent solutions of Einstein’s equations. However, our main result that three pieces of local geometric boundary data, along with the initial data, determine a geometrically unique solution is important input to the resolution of this question.

When the emphasis is on geometric issues we use abstract tensor indices, e.g. $v^a$ to denote a vector field, and when the specific spacetime coordinates $x^\mu = (t, x^i)$ are introduced we use the corresponding coordinate indices, e.g. $v^\mu = (v^t, v^i)$.

2. The initial-boundary data

We begin with a review of the initial data for the Cauchy problem. The standard treatment of the Cauchy problem introduces a time foliation $S_t$, with future directed unit normal $n_a$. The embedding of $S_t$ in the spacetime manifold $\mathcal{M}$ then gives rise to the decomposition of the spacetime metric

$$g_{ab} = -n_an_b + h_{ab},$$

(2.1)

where $h_{ab}$ is the 3-metric intrinsic to $S_t$. Geometric initial data are determined by the intrinsic metric $h_{ab}$ and extrinsic curvature $k_{ab} = h^c_a \nabla_c n_b$ of the initial Cauchy hypersurface $S_0$, where $\nabla_a$ is the covariant derivative associated with $g_{ab}$. These data are subject to the Hamiltonian and momentum constraints

$$2 G^{ab} n_a n_b = R - k_{ab}k^{ab} + k^2 = 0,$$

(2.2)

$$h^b_a G^{ba} n_a = D_b (k^{ab} - h^{ab} k) = 0,$$

(2.3)

where $R$ is the curvature scalar and $D_b$ is the covariant derivative associated with $h_{ab}$.

The remaining initial data necessary to determine a unique spacetime metric consist of gauge information, i.e. data that affect the solution only by a diffeomorphism. In the $3+1$ formulation of Einstein’s equations, the gauge freedom in the metric is governed by the choice of an evolution field

$$v^a = \alpha n^a + \beta^a, \quad \beta^a n_a = 0,$$

(2.4)

with lapse $\alpha$ and shift $\beta^a$. The lapse relates the unit future-directed normal to the time foliation, according to

$$n_a = -\alpha \nabla_a t.$$

(2.5)

The evolution field is transverse but not in general normal to the Cauchy hypersurfaces so that it determines the shift according to

$$\beta^a = h_b^a \beta^b.$$

(2.6)
The initial data required for the formulation of a well posed Cauchy problem depend upon the choice of hyperbolic reduction of Einstein’s equations. Here we consider the hyperbolic reduction associated with harmonic coordinates, as used in the classic work of Choquet-Bruhat [11]. Generalized harmonic coordinates $x^\mu = (t, x^i) = (t, x, y, z)$ are functionally independent solutions of the curved space scalar wave equation

$$g^{ab} \nabla_a \nabla_b x^\mu = - \hat{\Gamma}^\mu,$$  \hspace{1cm} (2.7)

where $\hat{\Gamma}^\mu (g, x)$ are harmonic gauge source functions [13]. Thus the harmonic coordinates can be determined by the initial data

$$x^\mu \big|_{S_0} = (0, x'), \quad \partial_t x^\mu \big|_{S_0} = \delta^\mu_t .$$  \hspace{1cm} (2.8)

In terms of the connection coefficients $\Gamma^\mu_{\rho\sigma}$, the harmonic coordinate conditions are

$$C^\mu := \Gamma^\mu - \hat{\Gamma}^\mu = 0,$$  \hspace{1cm} (2.9)

where

$$\Gamma^\mu = g^{\sigma\rho} \Gamma^\mu_{\rho\sigma} = - \frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} g^\rho^\mu), \quad g = \det g_{\mu\nu}.$$  \hspace{1cm} (2.10)

The hyperbolic reduction of the Einstein tensor results from setting

$$E^{\mu\nu} := G^{\mu\nu} - \nabla^\nu (\mu C^\nu) + \frac{1}{2} g^{\mu\nu} \nabla^\rho C^\rho = 0,$$  \hspace{1cm} (2.11)

where $C^\nu$ is treated as a vector field in constructing the covariant derivatives.

When the harmonic conditions (2.9) are satisfied, the principal part of (2.11) reduces to a curved space wave operator acting on the densitized metric, i.e.

$$E^{\mu\nu} = \frac{1}{2\sqrt{-g}} g^{\rho\beta} \partial_\rho \partial_\beta (\sqrt{-g} g^{\mu\nu}) + \text{lower order terms} = 0.$$  \hspace{1cm} (2.12)

Thus the harmonic evolution equations (2.11) are quasilinear wave equations for the components of the densitized metric $\sqrt{-g} g^{\mu\nu}$. The well-posedness of the Cauchy problem for the harmonic system (2.11) follows from known results for systems of quasilinear wave equations [11]. Such results are local in time since there is no general theory for the global existence of solutions to nonlinear equations.

Constraint preservation results from applying the contracted Bianchi identity $\nabla_\nu G^{\mu\nu} = 0$ to (2.11), which leads to the homogeneous wave equation

$$\nabla^\nu \nabla_\nu C^\mu + R^\mu_{\nu\rho} C^\nu = 0.$$  \hspace{1cm} (2.13)

If the initial data enforce

$$C^\mu \big|_{S_0} = 0$$  \hspace{1cm} (2.14)

and

$$\partial_t C^\mu \big|_{S_0} = 0$$  \hspace{1cm} (2.15)

then $C^\nu = 0$ is the unique solution of (2.13). It is straightforward to satisfy (2.14) by algebraically determining the initial values of $\partial_t g^{\mu\nu}$ in terms of the initial values of $g^{\mu\nu}$ and their spatial derivatives. In order to see how to satisfy (2.15) note that the reduced equations (2.11) imply

$$G^{\mu\nu} n_\nu = n_\nu \nabla^{\mu\nu} C^\nu - \frac{1}{2} g^{\mu\nu} \nabla_\nu C^\nu .$$  \hspace{1cm} (2.16)

As a result, if

$$G^{\mu\nu} n_\nu \big|_{S_0} = 0,$$  \hspace{1cm} (2.17)
i.e. if the Hamiltonian and momentum constraints are satisfied by the initial data, and if the reduced equations (2.11) are satisfied, then
\[ [n_i \nabla^i C^\nu - \frac{1}{2} n^\mu \nabla_\mu C^\nu] |_{S_0} = 0. \] (2.18)

It is straightforward to check that if \( C^\mu |_{S_0} = 0 \) then (2.18) implies (2.15).

By standard results, the Hamiltonian and momentum constraints on the initial data, along with the reduced evolution equations (2.11), imply that the initial conditions (2.14) and (2.15) required for preserving the harmonic conditions are satisfied. Conversely, if the Hamiltonian and momentum constraints are satisfied initially then (2.16) ensures that they will be preserved under harmonic evolution. In this way, the conditions \( C^\mu = 0 \) substitute for the constraints of the generalized harmonic formulation. This result extends to the harmonic formulation of the IBVP. If in addition to (2.14) and (2.15) the harmonic conditions are enforced on the boundary, i.e.
\[ C^\mu |_T = 0, \] (2.19)
then \( C^\mu = 0 \) is again the unique solution of (2.13) and the Hamiltonian and momentum constraints remain satisfied.

The free initial gauge data in the harmonic formulation consist of the initial values of the lapse and shift. For simplicity, we set the initial lapse to unity and the initial shift to zero so that the metric components satisfy
\[ g^\mu = -1, \quad g^\mu_\nu = 0, \quad t = 0. \] (2.20)

Along with the initial geometric data \( h^i_\nu \) and \( k^i_\nu \), these determine a unique solution to the Cauchy problem in the harmonic gauge. In geometric terms, \( h_{ab} \) and \( k_{ab} \) determine a solution which is unique up to a diffeomorphism.

We now formulate the additional geometric data necessary for the IBVP. In the IBVP, there is another natural decomposition of the metric at the boundary \( T \),
\[ g_{ab} = N_a N_b + H_{ab}, \] (2.21)
where \( N_a \) is the unit outward normal and \( H_{ab} \) is the 3-metric intrinsic to \( T \). The boundary \( T \) intersects the initial Cauchy hypersurface \( S_0 \) at a two-dimensional edge \( B_0 \). In general, the spacelike normal \( N_a \) to \( T \) is not orthogonal to the timelike normal \( n_a \) to \( S_0 \). As a result, the geometric initial data must also include the hyperbolic angle \( \theta_0 \) at the edge given by
\[ \sinh \theta_0 = N_a n_a |_{B_0}. \] (2.22)

The initial velocity of the boundary with respect to the initial Cauchy hypersurface is governed by \( \theta_0 \).

On the three-dimensional boundary \( T \), we represent the local geometric data which encode the two gravitational degrees of freedom by a conformal class \( \{ Q_{ab} \} \) of rank 2 metrics of signature \((0 + +)\) defined by the equivalence relation \( Q_{ab} \equiv \Omega^2 Q_{\tilde{a} \tilde{b}}, \Omega > 0 \). On the boundary, \( \{ Q_{ab} \} \) picks out an eigendirection, with null eigenvalue, i.e. it determines up to extension a non-vanishing vector field \( \tilde{T}^a \) tangent to \( T \) which satisfies \( Q_{ab} \tilde{T}^b = 0 \). In turn, \( \tilde{T}^a \) determines a flow of streamlines on \( T \), which is unique modulo parametrization. We pick the direction of the flow to point away from \( B_0 \).

In the region of \( T \) disjoint from the edge \( B_0 \), no additional properties of \( \tilde{T}^a \) are assumed. In particular, properties such as the hypersurface orthogonality of \( \tilde{T}^a \) cannot be determined without reference to a specific 3-metric on \( T \). However, as a compatibility condition at the edge \( B_0 \), we identify a member \( Q_{ab} \) of \( \{ Q_{ab} \} \) with the intrinsic 2-metric \( q_{ab} \) induced on \( B_0 \) by the initial data \( h_{ab} \),
\[ Q_{ab} |_{B_0} = q_{ab} |_{B_0}, \] (2.23)
As a consequence, \( \tilde{T}^a \) is normal to \( B_0 \).
Although $\tilde{T}^a$ is not in general hypersurface orthogonal, it is always possible to introduce local coordinates $(\tau, y^A)$ on the boundary satisfying

$$\mathcal{L}_{\tilde{T}} \tau = 1, \quad \mathcal{L}_{\tilde{T}} y^A = 0,$$

i.e. by Lie transport along the streamlines of $\tilde{T}^a$. Independent of the freedom in the choice of extension for $\tilde{T}^a$, in these coordinates $\tilde{T}_A = 0$ so that $Q_{\tau \tau} = Q_{\tau A} = 0$, i.e. the non-vanishing components are $Q_{AB}(\tau, y^A)$. Thus the conformal class $\{Q_{ab}\}$ is represented by $Q_{AB}/\sqrt{\det(Q_{CD})}$. The gauge freedom in this coordinate representation is the parametrization $\tau$ of the streamlines of $\tilde{T}^a$ (corresponding to the choice of extension) and the streamline coordinatization $y^A$ (corresponding to the diffeomorphisms on the factor space of streamlines obtained by identifying points on each streamline). This local representation allows comparison of different conformal data sets. Although, such coordinates would be useful in setting up a $3+1$ evolution problem, they are not useful in the approach adopted here for the construction of a solution via harmonic coordinates.

The plan here is to construct a metric $g_{ab}$ satisfying Einstein’s equations such that the intrinsic boundary metric $H_{ab}$ obtained from the $3+1$ boundary decomposition (2.21) has the further $2+1$ decomposition

$$H_{ab} = -T_a T_b + Q_{ab},$$

where $Q_{ab}$ belongs to $\{Q_{ab}\}$. A priori to the construction of a solution, the only condition on $T_a$ is that $T_a \tilde{T}^a < 0$, so that $H_{ab}$ has signature $(-++)$ and $T_a$ is future directed. The inverse to the 3-metric (2.25) can be expressed in the usual form

$$H^{ab} = -T^a T^b + Q^{ab}, \quad T^a T_a = -1, \quad T^a Q_{ab} = 0,$$

from which it follows that

$$T^a = -\frac{\tilde{T}^a}{\tilde{T}^b T_b}. \quad (2.27)$$

Thus, after the construction of a solution, $T^a$ is the future directed unit timelike vector tangent to the boundary which is geometrically picked out as an eigenvector of $\{Q_{ab}\}$ with null eigenvalue.

The data $\{Q_{ab}\}$, along with gauge conditions, are not sufficient to determine a unique solution. The remaining geometric data on the boundary $\mathcal{T}$ are obtained from its extrinsic curvature

$$K_{ab} = H^c_a \nabla_c N_b. \quad (2.28)$$

In the Friedrich–Nagy formulation of the IBVP, the trace $K = H^{ab} K_{ab}$ forms part of the boundary data. Using the fact that $H_{ab}$ has signature $(-++)$, Friedrich and Nagy show that when $K$ is expressed in terms of a boundary defining function it gives rise to a wave equation for that function which geometrically determines the location of the boundary. For the method we use here to establish the strong well-posedness of a metric formulation of the IBVP, there does not appear to be a way to incorporate $K$ into the boundary data. However, the alternative component

$$L = (H^{ab} - T^a T^b) K_{ab}$$

does supply the data in the required form. Because $(H^{ab} - T^a T^b)$ also has signature $(-++)$, $L$ geometrically determines the location of the boundary by the same construction used by Friedrich and Nagy.

We now state our main result.
Theorem (Local geometric data theorem). The Cauchy data \( h_{ab} \) and \( k_{ab} \) on \( \mathcal{S}_0 \), along with edge data \( \Theta_0 \) on \( \mathcal{B}_0 \) and boundary data \{\( Q_{ab} \)\} and \( L \) on \( T \), determine a metric which satisfies the vacuum Einstein equations (locally in time) such that \( T \) has induced metric of the form (2.25) and extrinsic curvature component (2.29). The solution is unique, up to a diffeomorphism. All data are assumed to be smooth and compatible.

Here the Cauchy data must satisfy the Hamiltonian and momentum constraints but the boundary data are constraint free subject to compatibility with the Cauchy data. The restriction of \{\( Q_{ab} \)\} and \( h_{ab} \) to \( \mathcal{B}_0 \) are required to lead to conformally equivalent 2-metrics via (2.23), which fulfills the lowest order compatibility condition. For a \( C^\infty \) solution, the compatibility conditions involve matching all derivatives of the initial data and boundary data at points on \( \mathcal{B}_0 \). This is a complicated requirement which we assume has been satisfied. Compatibility conditions pose no restriction on the boundary data in the region of \( T \) disjoint from \( \mathcal{B}_0 \).

Together \{\( Q_{ab} \)\} and \( L \) supply three pieces of boundary data which have the above local geometric interpretation after the construction of a solution. As for the case of the Cauchy problem, additional data, which control the gauge degrees of freedom, are necessary to determine a unique solution. This gauge data depend upon the particular hyperbolic reduction used to formulate the IBVP. In the formulation of a strongly well posed harmonic IBVP, the Einstein equations reduce to ten wave equations for the components of the metric, so that ten boundary conditions are necessary. In addition to the three pieces of boundary data \{\( Q_{ab} \)\} and \( L \), the harmonic conditions (2.9) supply four boundary conditions, as will be described in section 3. There are three more pieces of gauge data on the boundary which are necessary to specify completely the harmonic coordinate freedom. These data pin down the values of the harmonic coordinates on the boundary, as described below. Together with these harmonic coordinate conditions, the geometric data determine a strongly well posed problem with a unique solution.

A non-zero value of the hyperbolic angle \( \Theta_0 \) presents a technical complication in prescribing the three pieces of harmonic gauge data. However, the value of \( \Theta_0 \) can be adjusted to zero by carrying out a Cauchy evolution in the neighborhood of \( \mathcal{B}_0 \) to a new choice of \( \mathcal{S}_0 \), which keeps \( \mathcal{B}_0 \) unchanged. Since the Cauchy problem is well posed, the initial data for this modified problem depend continuously on the initial data for the original problem. Consequently, the original IBVP is strongly well posed if the IBVP for the modified problem with \( \Theta_0 = 0 \) is strongly well posed. In the following, we assume that this has been carried out. (Otherwise, the technical details in constructing a convenient gauge for establishing a well posed IBVP become more complicated; cf [3] where the case \( \Theta_0 \neq 0 \) is treated.) Referring to (2.22), the requirement that \( \Theta_0 = 0 \) implies

\[
N^a n^a|_{\mathcal{B}_0} = 0
\]

so that the compatibility condition (2.23) implies

\[
T^a|_{\mathcal{B}_0} = n^a|_{\mathcal{B}_0}.
\]

Since harmonic coordinates are solutions of the curved space scalar wave equation, they are determined by the initial data (2.8) along with boundary data for a scalar wave. The boundary data for these coordinates can be specified in any form which leads to a strongly well posed IBVP. For our present purpose, we consider homogeneous Dirichlet or Neumann boundary data. In order to investigate the possible choices, consider Gaussian normal coordinates \( \tilde{\xi}^\mu = (\tilde{t}, \tilde{x}, \tilde{\chi}^A) \) tailored to the boundary at \( \tilde{\xi} = 0 \) with \( T \) coordinatized by \( \tilde{t} \geq 0 \) and \( \tilde{\chi}^A = (\tilde{\gamma}, \tilde{\zeta}) \). In these coordinates, the metric has the form

\[
g_{\tilde{\mu} \tilde{\nu}} \, d\tilde{\xi}^\mu \, d\tilde{\xi}^\nu = d\tilde{\gamma}^2 + H_{\tilde{I}\tilde{J}} \, d\tilde{\xi}^\tilde{I} \, d\tilde{\xi}^\tilde{J}, \quad \chi^A = (\tilde{t}, \tilde{\chi}^A)
\]

in the neighborhood of the boundary. Thus \( g^{\tilde{\mu} \tilde{\nu}} = g^{\tilde{\chi} \tilde{\chi}} = 0 \) on the boundary.
Harmonic coordinates \( x^\mu = (t, x, x^A) \), \( x^A = (y, z) \), can now be introduced by solving an IBVP for the scalar wave equation (2.7). On the boundary we prescribe the homogeneous Dirichlet data \( x = \hat{x} = 0 \), so that the boundary is given by \( x = 0 \). For the remaining harmonic coordinates we prescribe the homogeneous Neumann data

\[
\frac{\partial x^A}{\partial \hat{x}} = 0, \quad \frac{\partial t}{\partial \hat{x}} = 0, \quad x = 0,
\]

so that on the boundary

\[
g^{\alpha A} = \frac{\partial x}{\partial \hat{x}} \frac{\partial x^A}{\partial \hat{x}} g^{\alpha \beta} = \frac{\partial x}{\partial \hat{x}} \frac{\partial x^A}{\partial \hat{x}} = 0.
\]

Similarly \( g^{\alpha A} = 0 \) on the boundary, which is consistent with the initial condition (2.30) at the edge \( \mathcal{B}_0 \). In summary, we use the boundary freedom in the choice of harmonic coordinates to set

\[
g^{\alpha A}|_T = g^{\alpha A}|_T = 0, \quad x|_T = 0.
\]

### 3. Reduction to PDEs

In order to reduce the IBVP to a set of PDEs for the metric with the initial-boundary data described in section 2, we express the harmonic Einstein equations (2.12) in the form

\[
g^{\alpha \rho} \partial_\alpha \partial_\rho (\sqrt{-g} g^{\mu \nu}) = F^{\mu \nu},
\]

where the forcing \( F^{\mu \nu} \) represents lower order terms which do not enter the principal part. Since the harmonic gauge source functions play no essential role in establishing well-posedness, we set \( \hat{\Gamma}^{\alpha \mu} = 0 \).

In the harmonic coordinates constructed in section 2, the initial data at \( t = 0 \), with the gauge conditions (2.20), consist of

\[
\begin{align*}
g^{ij} &= h^{ij}, & g^{ij} &= 0, & g^{ij} &= -1, \\
\partial_t g^{ij} &= -\frac{1}{2} k^{ij}, & \partial_t (\sqrt{-g} g^{ij}) &= -\partial_j (\sqrt{h} h^{ij}), & \partial_t (\sqrt{-g} g^{ij}) &= 0.
\end{align*}
\]

The boundary data at \( x = 0 \) consist of the gauge data (2.33),

\[
g^{ij} = 0, & g^{\alpha A} = 0,
\]

and the geometric boundary data consisting of the conformal class \( \{Q_{ab}\} \) and the field \( L \), which is the extrinsic curvature component

\[
L = (H^{ab} - T^a T^b) K_{ab} = -\frac{1}{2} \sqrt{-g} (H^{\mu \nu} - T^\mu T^\nu) \partial_\mu g_{\mu \nu}.
\]

Here (3.3) supplies three Dirichlet boundary conditions, \( \{Q_{ab}\} \) supplies two additional Dirichlet conditions and (3.4) supplies a Neumann condition on a combination of metric components. Four additional boundary conditions result from enforcing the harmonic constraints (2.19) on the boundary, which take the form

\[
\partial_\mu (\sqrt{-g} g^{ij})|_{x=0} = 0.
\]

We now formulate the PDEs for the frozen coefficient version of the problem. The material in sections 4 and 5 shows that the strong well-posedness of this frozen coefficient problem extends to the quasilinear problem. Following the approach used in [2, 3] for Sommerfeld boundary conditions, we localize the problem in the neighborhood of a point \( p \) on the boundary and the wave operator in (3.1) is frozen to its value at \( p \),

\[
g^{\rho \beta} (x_p) \partial_\rho \partial_\beta.
\]
By a constant linear transformation of the harmonic coordinates which keeps the $x$-direction fixed, we can then set $g^{ab}(x_p) = \eta^{ab}$ (the Minkowski metric). In doing so, the $x$-direction remains aligned with $N^i$ and we can further Lorentz transform the $t$-direction into the $T^a$ direction picked out by $\{Q_{ab}\}$, so that $T^a(x_p)\partial_a = \partial_t$. In these $(t,x,x^3)$ coordinates, with $x^3 = (y, z)$, we extend the Minkowski metric to a neighborhood of $p$ and linearize the equations in terms of the variable

$$\gamma^{\mu\nu} = \sqrt{-g_{\mu\nu}} = -\eta^{\mu\nu}. \quad (3.7)$$

The system (3.1) then takes the frozen coefficient form

$$(- \partial_t^2 + \partial^2 + \partial^2_3 + \partial^2_5) \begin{pmatrix} \gamma^{tt} & \gamma^{tx} & \gamma^{ty} & \gamma^{tz} \\ \\
\gamma^{tx} & \gamma^{xx} & \gamma^{xy} & \gamma^{xz} \\ \\
\gamma^{ty} & \gamma^{yx} & \gamma^{yy} & \gamma^{yz} \\ \\
\gamma^{tx} & \gamma^{zx} & \gamma^{zy} & \gamma^{zz} \end{pmatrix} = F, \quad x \geq 0, \quad t \geq 0, \quad (3.8)$$

with forcing matrix $F$.

In the neighborhood of $p$ we require that the data be sufficiently close to Minkowski data to allow the iterative construction of a solution to the quasilinear problem. This can be arranged by considering the rescaled metric $g'_{\mu\nu} = \lambda^{-2}g_{\mu\nu}$, where $\lambda \ll 1$ is a positive constant; cf p. 262 of [14]. Then $L' = L\lambda$ and, in the stretched coordinates $x^\mu = x^\mu_p + \lambda^{-1}(x^\mu - x^\mu_p)$, the transformed metric has components $g'_{\mu\nu}(x') = g_{\mu\nu}(x) = \eta_{\mu\nu} + O(\lambda)$.

In these coordinates, $g_{AB} = Q_{AB} + O(\lambda^2)$ in the neighborhood of $p$ and the conformal boundary data consist of

$$\tilde{Q}_{AB} = Q^{-1/2}Q_{AB}, \quad Q = \det Q_{AB}. \quad (3.9)$$

In the linearized approximation, this reduces to

$$\tilde{Q}^a_{\mu\nu} - \eta^{AB} = \gamma^{\mu\nu} - \frac{1}{\lambda} \eta^{AB} \eta_{CD} \gamma^{CD}, \quad (3.10)$$

(3.4) reduces to

$$L = -\frac{i}{2} \partial_\mu(2\gamma_{\mu\nu} + \gamma_{\nu\nu} + \gamma_{\nu\nu}) \quad (3.11)$$

and the harmonic constraints reduce to

$$\tilde{\partial}_\mu \gamma^{\mu\nu} = 0. \quad (3.12)$$

The boundary conditions for the linearized system now take the form

$$\frac{1}{2}(\gamma_{yy} - \gamma_{zz}) = q_1(t, y, z), \quad (3.13)$$

$$\gamma_{xz} = q_2(t, y, z), \quad (3.14)$$

$$\tilde{\partial}_\alpha(\gamma_{xx} + \frac{1}{2}(\gamma_{yy} + \gamma_{zz})) = q_4(t, y, z), \quad (3.15)$$

$$\gamma_{tt} = 0, \quad (3.16)$$

$$\gamma_{ty} = 0, \quad (3.17)$$

$$\gamma_{xz} = 0, \quad (3.18)$$

$$\tilde{\partial}_x \gamma_{xz} + \tilde{\partial}_y \gamma_{xy} + \tilde{\partial}_y \gamma_{yz} + \tilde{\partial}_z \gamma_{xz} = 0, \quad (3.19)$$

$$\tilde{\partial}_y \gamma_{yz} + \tilde{\partial}_y \gamma_{yx} + \tilde{\partial}_y \gamma_{yz} + \tilde{\partial}_z \gamma_{yz} = 0, \quad (3.20)$$

$$\tilde{\partial}_x \gamma_{xz} + \tilde{\partial}_y \gamma_{xz} + \tilde{\partial}_y \gamma_{yz} + \tilde{\partial}_y \gamma_{yz} = 0, \quad (3.21)$$

$$\tilde{\partial}_y \gamma_{zt} + \tilde{\partial}_x \gamma_{xt} + \tilde{\partial}_y \gamma_{yt} + \tilde{\partial}_z \gamma_{zt} = 0. \quad (3.22)$$

The Dirichlet data $q_1$ and $q_2$ are determined from the two conformally invariant degrees of freedom contained in (3.10). The Neumann data $q_3$ are determined from the data (3.11) prescribed by $L$. The Dirichlet conditions (3.16)–(3.18) arise from the boundary conditions (3.3) on the harmonic coordinates. The boundary conditions (3.19)–(3.22) arise from the harmonic constraints (3.12).
4. Energy estimates for quasilinear wave problems with Sommerfeld, Dirichlet or Neumann boundary conditions

We establish the strong well-posedness of the IBVP for quasilinear wave equations with Dirichlet and Neumann boundary conditions by an approach similar to that carried out in [3] for Sommerfeld boundary conditions. We begin by reviewing how to obtain energy estimates for the Sommerfeld case.

4.1. Sommerfeld boundary conditions

The energy estimates in sections 1–4 of [3] established that the solution of the frozen coefficient problem treated in [3]. We show that local existence theorems and energy estimates for second order quasilinear wave problems can be obtained in the same way as for first order symmetric hyperbolic systems. It all depends on a priori estimates for arbitrarily high derivatives of the solutions of linear equations with variable coefficients.

Consider the half-plane problem
\[ u_t = Pu + Ru + F, \quad x \geq 0, \quad t \geq 0, \quad -\infty < y < \infty, \]
with Sommerfeld-type boundary conditions at \( x = 0, \)
\[ \alpha (u_t + \gamma u) = u_x + q, \quad \alpha > 0, \quad \gamma > 0 \]
smooth boundary data \( q(t, y) \) and smooth initial data
\[ u(0, x, y) = f_1(x, y), \quad u_t(0, x, y) = f_2(x, y). \]

Here the subscripts \((t, x, y, z)\) denote partial derivatives, e.g. \( u_t = \partial_t u, \)
\[ Pu = (au_x)_x + (bu_y)_y - 2\gamma u_t - \gamma^2 u \]
and
\[ Ru = c_1 u_x + c_2 u_t + c_3 u_y + c_4 u. \]

\( Ru \) are terms of lower (first and zeroth) differential order. Also, we use the notation
\[ (u, v), \quad \|u\|^2 = (u, u); \quad (u, v)_B, \quad \|u\|^2_B = (u, u)_B \]
to denote the \( L^2 \) scalar product and norm over the half-plane and boundary, respectively.

All coefficients and data are smooth real functions and \( a \geq a_0 > 0, b \geq b_0 > 0, \) where \( a_0, b_0 \) are strictly positive constants. The initial data are compatible with the boundary conditions. In the above, \( \gamma > 0 \) is a constant obtained by the change of variables \( u \to e^{\gamma t} u' \) and then deleting the ‘prime’. This introduces the term \( \gamma^2 \|u\|^2 \) in the energy
\[ E := \|u_t\|^2 + (u_t, au_x) + (u_t, bu_y) + \gamma^2 \|u\|^2, \]
which provides an estimate of \( \|u\|^2. \)

**Lemma.** There is an energy estimate which is stable against lower order perturbations.

**Proof.** Integration by parts gives
\[
\partial_t E = \partial_t (\|u_t\|^2 + (u_x, au_x) + (u_y, bu_y) + \gamma^2 \|u\|^2)
\]
\[ = -4\gamma \|u_t\|^2 + 2(u_t, F) + 2(u_t, Ru) - 2(u_t, au_x)_B + a_t \|u_t\|^2 + b_t \|u_t\|^2 \]
\[ \leq \text{const} (E + \|F\|^2) - 2(u_t, au_x)_B. \]
Here, and below, the inequalities follow from the basic inequality
\((u, v) \leq \frac{1}{2}(A^2\|u\|^2 + A^{-2}\|v\|^2)\).

Using the boundary conditions, we obtain
\[-(ut, au_x)_B = -(ut, a\alpha u_t)_B - (u_t, a\alpha y u)_B + (u_t, aq)_B \leq -\frac{1}{2}a_0\alpha \gamma \delta \|u\|_B^2 + \text{const} (\|u\|_B^2 + \|q\|_B^2).\]

Therefore (4.5) implies
\[\partial_t (E + a_0\alpha \gamma \|u\|_B^2) \leq \text{const} (E + \|F\|^2 + \|u\|_B^2 + \|q\|_B^2).\] (4.6)

This proves the lemma. □

Now we can estimate the derivatives. Let \(v = u_x, w = u_t\). Differentiation of the differential equation gives
\[v_{tt} = P v + R v + R_t u + (a_x u_x)_x + (b_t v)_y + F_t,\]
\[w_{tt} = P w + R w + R_t u + (a_t u_t)_x + (b_t v)_y + F_t.\] (4.7)

Here \(R_t u\) and \(R_t u\) are linear combinations of first derivatives of \(u\) which we have already estimated and can be considered part of the forcing.

The differential equation (4.1) tells us that
\[au_{xx} = w_t - b v_y + \text{terms we have already estimated}.\]
Thus \(u_{xx}\) is lower order with respect to \(v\) and \(w\) and, except for lower order terms, \(v\) and \(w\) are solutions of the same differential equation as \(u\). They obey the same boundary conditions with data \(q_i(t, y)\) and \(q_i(t, y)\), respectively. Therefore we can estimate all second derivatives.

Repeating the process, we can estimate any number of derivatives.

We can now proceed in the same way as in [6], where we have considered first order systems to obtain existence theorems for equations with variable coefficients. We approximate the differential equation by a stable difference approximation and prove, using summation by parts, that the corresponding estimates for the divided differences hold independently of gridsize. In the limit of vanishing gridsize, we obtain the existence theorem. Since we can estimate any number of derivatives, it is well known, using Sobolev’s theorem, that we can obtain similar, although local in time, estimates for quasilinear systems. By the same iterative methods as for first order symmetric hyperbolic systems it follows that strong well-posedness extends locally in time to the quasilinear case, as well as other standard results such as the principle of finite speed of propagation.

**Remark.** There are no difficulties to extend the results to three spatial dimensions.

4.2. Dirichlet and Neumann conditions

If we replace the Sommerfeld boundary conditions by homogeneous Dirichlet or Neumann conditions, then the boundary term \((u, a u_x)_B\) in (4.5) vanishes. Thus the energy estimates in section 4.1 clearly hold for homogeneous Dirichlet or Neumann conditions with boundary data \(q = 0\).

Now we consider the half-plane problem for wave equations with inhomogeneous Dirichlet or Neumann boundary conditions. As we will show, we can transform these problems into problems with homogeneous boundary conditions by changing the forcing and the initial data. As a model problem, we consider the half-plane problem
\[u_t = (a(t, x, y)u_x)_x + (b(t, x, y)u_t)_y + F(t, x, y),\]
\[x \geq 0, \quad t \geq 0, \quad -\infty < y < \infty,\] (4.8)
with initial conditions
\[ u(0, x, y) = f_1(x, y), \quad u_t(0, x, y) = f_2(x, y), \]  
and Dirichlet boundary condition
\[ u(t, 0, y) = q(t, y). \]  
We assume that all coefficients and data are compatible and smooth. We make a change of variable
\[ \tilde{u}(t, x, y) = u(t, x, y) - \varphi(x)q(t, y). \]  
Here \( \varphi(x) \) is a smooth function, with \( \varphi(0) = 1 \), which decays exponentially. Then
\[ \tilde{u}(t, 0, y) = 0, \quad \text{i.e. } \tilde{u} \text{ satisfies homogeneous Dirichlet boundary conditions.} \]  
By (4.11),
\[ \tilde{u}_t = u_t - (\varphi(x)q(t, y))_t, \]
\[ (a\tilde{u}_x)_x = (au_x)_x - (a(\varphi(x)q(t, y))_x)_x, \]
\[ (b\tilde{u}_y)_y = (bu_y)_y - (b(\varphi(x)q(t, y))_y)_y. \]  
Finally, by (4.8), (4.12) and (4.13) we obtain the differential equation with modified forcing term
\[ \tilde{u}_t = (a(t, x, y)\tilde{u}_x)_x + (b(t, x, y)\tilde{u}_y)_y + F + \tilde{F}, \]  
which satisfies homogeneous Dirichlet boundary conditions. By assumption, \( F \) is a smooth function and \( \tilde{F} \) is composed of \( a, b, \varphi \) and \( q \) and their first two derivatives. Since derivatives are smooth functions, \( \tilde{F} \) is also a smooth function. Therefore \( \tilde{u}(t, x, y) \) satisfies the energy estimates arrived at in section 4.1.

Now we consider (4.8) with the Neumann boundary condition
\[ u_t(t, 0, y) = q(t, y). \]  
We make again the transformation (4.11) but now with \( \varphi_t(0) = 1 \), and obtain the corresponding energy estimate.

As an illustration of how the estimates extend to higher derivatives, consider the half-plane problem (4.8) with a homogeneous Dirichlet boundary condition for \( a = b = 1 \) (which poses no restriction),
\[ u_{tt} = u_{xx} + u_{yy} + F, \quad u_t(0, 0, y) = 0. \]  
Since \( F(t, x, y) \) and the data \( q(t, y) \), \( f_1(x, y) \) and \( f_2(x, y) \) are smooth functions, we can obtain energy estimates for the derivatives of \( u \) by differentiating (4.16). We obtain
\[ u_{tt} = u_{xxx} + u_{xxy} + F_x, \quad u_t(0, 0, y) = 0, \]  
\[ u_{tt} = u_{xxx} + u_{xyy} + F_y, \quad u_t(0, 0, y) = 0. \]  
As in (4.7), we introduce the variables
\[ v = u_x, \quad w = u_y. \]  
Then (4.17), (4.18) become
\[ v_{tt} = v_{xx} + v_{yy} + F_x, \quad v(t, 0, y) = 0, \]  
\[ w_{tt} = w_{xx} + w_{yy} + F_y, \quad w(t, 0, y) = 0. \]
Integration by parts then gives us an energy estimate for
\[ \|v_t\|^2 + \|v_x\|^2 + \|v_y\|^2 + \|w_t\|^2 + \|w_x\|^2 + \|w_y\|^2. \] (4.22)

By (4.16) and (4.19) we obtain
\[ u_{xx} + F = u_{tt} - u_{yy} = w_t - v_y. \]

Therefore, by (4.22), we obtain a bound for \(\|u_{xx}\|^2\).

We obtain a bound for \(\|u_{xxx}\|^2\) in the same way by replacing \(v\) and \(w\) by \(v(1) = uy_y, w(1) = u_{tt}\). (4.23)

Now we obtain the differential equations
\[ v_{tt}^{(1)} = v_{xx}^{(1)} + v_{yy}^{(1)} + F_{yy}, \quad (1, t, 0, y) = 0, \]
\[ u_{tt}^{(1)} = u_{xx}^{(1)} + u_{yy}^{(1)} + F_{tt}, \quad (1, t, 0, y) = 0, \] (4.24)

and we obtain energy estimates for
\[ \|v_{tt}^{(1)}\|^2 + \|v_{xx}^{(1)}\|^2 + \|v_{yy}^{(1)}\|^2 + \|u_{tt}^{(1)}\|^2 + \|u_{xx}^{(1)}\|^2 + \|u_{yy}^{(1)}\|^2. \] (4.25)

which we can express in terms of \(u\) according to
\[ \|u_{yy}\|^2 + \|u_{yyyy}\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \|u_{tttt}\|^2. \] (4.26)

By differentiation of (4.16) with respect to \(x\),
\[ u_{xxx} = u_{ttt} - u_{yyyy} - F_{y}. \] (4.27)

From (4.26), we already have estimates for \(\|u_{ttt}\|^2\) and \(\|u_{yyyy}\|^2\). Therefore we also obtain an estimate for \(\|u_{tttt}\|^2.\) This process can be continued.

Our result is not restricted to the model problem but is valid in general. For example, we can replace (4.8) by the corresponding half-plane problem in three spatial dimensions.

**Remark.** In many problems, surface waves, glancing waves and other waves specific to the boundary are important. In that case, there is no energy estimate and the above technique does not activate these phenomena. Instead, in such cases, we split the problem into two problems; one with homogeneous boundary conditions and another where only the boundary conditions do not vanish, i.e. the forcing and the initial values are zero. The first is covered by the results in this section. The second we treat by Fourier-Laplace techniques. For examples, see [15, 16].

5. **The strong well-posedness of the IBVP for the harmonic Einstein equations**

In section 5.4 we establish the strong well-posedness of the gravitational IBVP for the system \( (3.8) \) with boundary conditions \( (3.13) \)–\( (3.22) \) determined by local geometric boundary data and harmonic coordinate conditions. In order to illustrate how the estimates in section 4 apply we first progress through a sequence of model problems which illustrate a rich variety of acceptable boundary conditions.

5.1. **Model problem I: The harmonic Einstein equations in one spatial dimension**

First consider the half-plane problem in the frozen coefficient formalism of the harmonic Einstein equations for the system of wave equations in one space variable
\[ \left( -\partial_t^2 + \partial_x^2 \right) \left( \begin{array}{cc} \gamma_{tt} & \gamma_{tx} \\ \gamma_{tx} & \gamma_{xx} \end{array} \right) = F; \quad x \geq 0, \quad t \geq 0, \] (5.1)
with forcing matrix $F$. In standard notation, we treat the system in the sequential order

\begin{align}
(1) & \quad \partial_t^2 \gamma^{tx} = \partial_x^2 \gamma^{tx} + F_1, \\
(2) & \quad \partial_t^2 \gamma^{xx} = \partial_x^2 \gamma^{xx} + F_2, \\
(3) & \quad \partial_t^2 \gamma^{tt} = \partial_x^2 \gamma^{tt} + F_3.
\end{align}

(5.2)

Here $\gamma^{tx}(t, x)$, $\gamma^{xx}(t, x)$, $\gamma^{tt}(t, x)$ denote the dependent variables which we want to determine on the half-plane. The forcing terms $F_1(t, x)$, $F_2(t, x)$, $F_3(t, x)$ are smooth functions of $(t, x)$.

The solution of our problem is determined by the initial data corresponding to (3.2) along with

the Dirichlet boundary condition $\gamma^{tx}(t, 0) = q(t)$

or the Neumann boundary condition $\partial_t \gamma^{tx}(t, 0) = q(t)$, \quad (5.3)

and the harmonic boundary conditions applied in the sequential order

\begin{align}
& \partial_t \gamma^{tx}(t, 0) + \partial_x \gamma^{xx}(t, 0) = 0, \quad (5.4) \\
& \partial_t \gamma^{tt}(t, 0) + \partial_x \gamma^{tt}(t, 0) = 0. \quad (5.5)
\end{align}

We start with the wave equation for $\gamma^{tx}$ with smooth boundary data (5.3) and smooth compatible initial data. By means of the transformation (4.11) in section 4.2, we modify the forcing so that the variables satisfy homogeneous boundary conditions, which we denote by

$q(t) \equiv 0$. \quad (5.6)

Then we can estimate $\gamma^{tx}$ and its derivatives on the boundary, as well as in the interior $x > 0$, in terms of the data. The problem is strongly well posed and we can solve the wave equation for $\gamma^{tx}$.

Next, since $\gamma^{tx}(t, 0)$ is a known smooth function, we use the harmonic boundary condition (5.4) and obtain smooth Neumann boundary data $\partial_x \gamma^{xx}(t, 0)$ for $\gamma^{xx}$. We again use the transformation (4.11) so that $\partial_x \gamma^{xx}(t, 0) \equiv 0$, using the notation (5.6). The resulting wave problem for $\gamma^{xx}$ with homogeneous Neumann data is strongly well posed so that we can estimate $\gamma^{xx}(t, x)$ and its derivatives. Finally, we obtain the same result for $\gamma^{tt}$, using the harmonic boundary condition (5.5) and the transformation (4.11).

Remark. Alternatively, instead of (5.3), we could obtain a strongly well posed problem by prescribing Dirichlet or Neumann data for $\gamma^{tx}$ and using the harmonic boundary conditions to solve for the remaining components in the sequential order ($\gamma^{tx}$, $\gamma^{tt}$).

5.2. Model problem II: The harmonic Einstein equations in two spatial dimensions

Now consider the half-plane problem in frozen coefficient formalism in two spatial dimensions,

\[
\begin{pmatrix}
\partial_t^2 + \partial_x^2 + \partial_y^2 \\
& & & \\
0 & \partial_t & \partial_x & \partial_y \\
& & & \\
0 & 0 & 0 & \gamma^{vy} \\
& & & \\
0 & 0 & 0 & \gamma^{vy} \\
& & & \\
0 & 0 & 0 & \gamma^{vy}
\end{pmatrix}
\begin{pmatrix}
\gamma^{tt} \\
\gamma^{tx} \\
\gamma^{xx} \\
\gamma^{yy}
\end{pmatrix} = F, \quad x \geq 0, t \geq 0, -\infty < y < \infty,
\]

where $F$ again represents the forcing. The components $\gamma^{vy}$, $\gamma^{tx}$ and $\gamma^{vy}$ satisfy Dirichlet or Neumann boundary conditions. The initial data correspond to (3.2).

The harmonic boundary conditions are applied in the sequential order

\begin{align}
& \partial_t \gamma^{tx}(t, 0, y) + \partial_x \gamma^{xx}(t, 0, y) + \partial_y \gamma^{vy}(t, 0, y) = 0, \quad (5.7) \\
& \partial_t \gamma^{vy}(t, 0, y) + \partial_x \gamma^{vy}(t, 0, y) + \partial_y \gamma^{vy}(t, 0, y) = 0. \quad (5.8)
\end{align}
We now consider the half-plane problem for the linearized harmonic equations in three spatial dimensions. For model problem I, we use the transformation (4.11) so that the wave equations for $\gamma^{0\nu}$, $\gamma^{\nu x}$ and $\gamma^{\nu y}$ satisfy homogeneous Dirichlet or Neumann boundary conditions. Then the corresponding wave problems are well posed and there are energy estimates for these variables and their derivatives.

We use the harmonic boundary conditions to obtain estimates for the remaining variables. First, the boundary condition (5.7) determines smooth Neumann boundary data for $\gamma^{xx}$ in terms of previously estimated quantities. After using the transformation (4.11), it reduces to

$$\partial_t \gamma^{xx}(t, 0, y) = 0$$

and the resulting wave problem for $\gamma^{xx}$ is strongly well posed. Thus we can estimate $\gamma^{xx}(t, x, y)$ and its derivatives. Similarly, the boundary condition (5.8) determines smooth Dirichlet boundary data for $\gamma^{xy}(t, 0, y)$ in terms of previously estimated quantities. After the transformation (4.11), it reduces to

$$\partial_t \gamma^{xy}(t, 0, y) = 0$$

so that the resulting wave problem is strongly well posed and we can estimate $\gamma^{xy}(t, x, y)$ and its derivatives. Finally, the boundary condition (5.9) determines Dirichlet boundary data for $\gamma^{tt}(t, 0, y)$ in terms of previously estimated quantities and we can use the transformation (4.11) to obtain a strongly well posed problem for $\gamma^{tt}$.

5.3. Model problem III: The harmonic Einstein equations in three spatial dimensions

We now consider the half-plane problem for the linearized harmonic equations in three spatial dimensions

$$(-\partial^2 + \partial_x^2 + \partial_y^2 + \partial_z^2) \begin{pmatrix} \gamma^{tt} & \gamma^{tx} & \gamma^{ty} & \gamma^{tz} \\ \gamma^{tx} & \gamma^{xx} & \gamma^{xy} & \gamma^{xz} \\ \gamma^{ty} & \gamma^{yx} & \gamma^{yy} & \gamma^{yz} \\ \gamma^{tz} & \gamma^{zx} & \gamma^{zy} & \gamma^{zz} \end{pmatrix} = F,$$

$$x \geq 0, \quad t \geq 0, \quad -\infty < y < \infty, \quad -\infty < z < \infty,$$

(5.12)

where $F$ represents the forcing, $\gamma^{0\nu}$, $\gamma^{\nu x}$, $\gamma^{\nu y}$, $\gamma^{\nu z}$ and $\gamma^{\nu\nu}$ satisfy Dirichlet or Neumann boundary conditions and the initial data correspond to (3.2).

The harmonic constraints are applied on the boundary in the sequential order

$$\partial_t \gamma^{xx} + \partial_x \gamma^{xx} + \partial_y \gamma^{yx} + \partial_z \gamma^{xz} = 0,$$

(5.13)

$$\partial_t \gamma^{xy} + \partial_x \gamma^{xy} + \partial_y \gamma^{yy} + \partial_z \gamma^{yz} = 0,$$

(5.14)

$$\partial_t \gamma^{xz} + \partial_x \gamma^{xz} + \partial_y \gamma^{yz} + \partial_z \gamma^{zz} = 0,$$

(5.15)

$$\partial_t \gamma^{yy} + \partial_x \gamma^{yy} + \partial_y \gamma^{yy} + \partial_z \gamma^{yz} = 0.$$  

(5.16)

We proceed in the same way as in two space dimensions. We use the transformation (4.11) so that the six wave equations for $\gamma^{0\nu}$, $\gamma^{\nu x}$, $\gamma^{\nu y}$, $\gamma^{\nu z}$ and $\gamma^{\nu\nu}$ satisfy homogeneous Dirichlet or Neumann boundary conditions. Then there is an energy estimate for these variables and their derivatives. We then use the constraints to obtain estimates for the remaining variables.

The constraints (5.13)–(5.15) determine Neumann boundary data for $\partial_t \gamma^{xx}(t, 0, y, z)$ and Dirichlet boundary data for $\gamma^{xy}(t, 0, y, z)$ and $\gamma^{xz}(t, 0, y, z)$ in terms of the previously estimated variables. After using the transformation (4.11), the resulting wave problems are strongly well
posed so that we can estimate $\gamma_{xx}$, $\gamma_{ty}$ and $\gamma_{tz}$ and their derivatives. The constraint (5.16) then provides Dirichlet data $\gamma''(t, 0, y, z)$ for the remaining variable in terms of previously estimated variables. After the transformation (4.11), the resulting wave problem for $\gamma''$ is strongly well posed.

### 5.4. The harmonic Einstein equations with local geometric data

Now we turn to the three-dimensional harmonic Einstein system (5.12) with boundary conditions (3.13)–(3.22) determined by the local geometric boundary data and harmonic coordinate conditions, as prescribed in section 3. After applying the transformation (4.11), the conformal metric data (3.13)–(3.14) reduce to the homogeneous Dirichlet form

$$\gamma_{tt}(t, 0, y, z) \equiv 0,$$

and the extrinsic curvature data $L$ (3.15) reduce to the homogeneous Neumann form

$$\partial_x(\gamma_{yy} + \gamma_{zz}) (t, 0, y, z) \equiv 0.$$  \hspace{1cm} (5.18)

The boundary gauge data (3.16)–(3.18) are already in the homogeneous Dirichlet form

$$\gamma_{ty}(t, 0, y, z) = \gamma_{tz}(t, 0, y, z) = \gamma_{xx}(t, 0, y, z) = 0.$$  \hspace{1cm} (5.19)

The remaining boundary conditions are supplied by the harmonic constraints (3.19)–(3.22).

The situation is similar to model problem III but simpler since the gauge conditions (5.19) are already homogeneous and imply that the harmonic constraint (3.19) has the homogeneous form

$$\partial_x\gamma_{xx}(t, 0, y, z) = 0,$$

so that (5.18) reduces to

$$\partial_x(\gamma_{yy} + \gamma_{zz})(t, 0, y, z) \equiv 0.$$  \hspace{1cm} (5.20)

Together, the homogeneous boundary conditions (5.17), (5.19), (5.20) and (5.21) determine strongly well posed wave problems for the variables $(\gamma_{yy} - \gamma_{zz})$, $\gamma_{ty}$, $\gamma_{tx}$, $\gamma_{xy}$, $\gamma_{xz}$ and $(\gamma_{yy} + \gamma_{zz})$, respectively. Thus we can estimate those variables and their derivatives. Now we can proceed as in Model problem III to use the harmonic constraints (3.20)–(3.22) in sequential order to determine the required estimates for the remaining three independent variables $\gamma_{yx}$, $\gamma_{yz}$ and $\gamma''$. This determines a unique solution to the frozen coefficient problem. Along with the applicability to the quasilinear problem outlined in section 4, it establishes the Local Geometric Data Theorem proposed in section 2.

### 6. Discussion

We have shown how a conformal class of rank-2 metrics $\{Q_{ab}\}$ and an associated extrinsic curvature component $L$ supply local geometric boundary data for a solution of Einstein’s equations which is unique up to a diffeomorphism. The result was obtained by introducing harmonic coordinates to formulate boundary conditions for a strongly well posed IBVP. This method also broadens the possible formulation of strongly well posed harmonic IBVPs. The technique in [2, 3] based upon Sommerfeld conditions has been extended to include Dirichlet and Neumann conditions, subject to the sequential structure necessary to enforce the harmonic constraints. For computational applications, Sommerfeld conditions are most benevolent because they allow numerical error to leave the grid. It is therefore somewhat discordant with numerical application that a treatment of the boundary based upon local
geometric data must apparently include at least two Dirichlet conditions, associated with \( \{ Q_{ab} \} \), and one Neumann condition associated with the extrinsic curvature, such as the component \( L \).

There are many options in formulating a suitable combination of Dirichlet, Neumann and Sommerfeld conditions for a strongly well posed problem, provided the sequential structure is maintained. However, the only locally geometric boundary data allowed by the sequential method used here are \( \{ Q_{ab} \} \) and \( L \). For example, had we used the trace \( K \) of the extrinsic curvature of the boundary instead of the component \( L \) then (5.21) would have been replaced by

\[
\partial_t (y^{yy} + y^{zz} - y^{tt}) (t, 0, y, z) \equiv 0,
\]

which does not fit into the sequential structure for applying the constraints. It remains an open question whether a different analytic approach can be used to show that trace \( K \) boundary data can replace \( L \) in a strongly well posed harmonic IBVP.

An additional issue of high practical importance is the formulation of a strongly well posed IBVP for the \( 3+1 \) approach which has historically played a major role in numerical relativity \[17\]. In the \( 3+1 \) formalism, instead of the 10 wave equations of the harmonic system, Einstein’s equations are reduced to a pair of 6 first order in time equations for \( h_{ab} \) and \( k_{ab} \), supplemented by 4 conditions which determine the lapse and shift. Perhaps the geometric insight provided by our results can shed light on this outstanding problem.

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