Gabber’s presentation lemma over noetherian domains

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Abstract. Following Schmidt and Strunk, we give a proof of Gabber’s presentation lemma over a noetherian domain with infinite residue fields.

1. Introduction

Gabber’s presentation lemma, initially proved by O. Gabber in [Gab] (see also [CTHK], [HK]) plays a fundamental role in the study of \( \mathbb{A}^1 \)-homotopy theory, especially as developed by Morel in [Mor2]. This lemma may be thought of as an algebro-geometric analogue of the tubular neighbourhood theorem in differential geometry. In [SS], this lemma was generalized by J. Schmidt and F. Strunk to the case where the base is a spectrum of a Dedekind domain with infinite residue fields. The goal of this paper is to show that the arguments given in [SS] can, in fact, be modified to obtain a proof of Gabber’s presentation lemma over a general Noetherian domain with infinite residue fields. The following is the main result of this paper.

**Theorem 1.1.** Let \( R \) be a noetherian domain of dimension \( d \) such that all the residue fields are infinite and \( \text{Spec}(R) = S \). Then \( S \) has the shifted stable \( \mathbb{A}^1 \)-connectivity property that is if \( E \in \mathcal{S}H^i_{S}(S) \) then \( L^{d-i}E \in \mathcal{S}H^i_{S}(S) \).

The proof of Theorem 1.1 is exactly the same except for the input from Gabber’s presentation lemma, the required generality of which is available once Theorem 1.1 is proved.

An important ingredient of the proof of the main result of [SS] is [Kai, Theorem 4.1], which states that given an equi-dimensional scheme \( Y \) over a Dedekind scheme \( B \) with infinite residue fields, 

**Theorem 1.2.** Let \( R \) be a noetherian domain with infinite residue fields and \( S = \text{Spec}(R) \). Let \( X = \text{Spec}(A)/S \) be a smooth, affine, irreducible scheme of relative dimension \( d \). Let \( Z = \text{Spec}(B) \subset X \) be a closed subscheme of positive co-dimension, \( z \in Z \) be a point. Then Nisnevich locally on \( S \), there exists a map \( \Phi = (\Psi, \nu) : X \to \mathbb{A}^{d-1}_S \times \mathbb{A}^1_S \), an open subset \( V \subset \mathbb{A}^{d-1}_S \) and an open subset \( U \subset \Psi^{-1}(V) \) containing \( z \) such that

1. \( Z \cap U = Z \cap \Psi^{-1}(V) \)
2. \( \Psi|_Z : Z \to \mathbb{A}^{d-1}_S \) is finite
3. \( \Phi|_U : U \to \mathbb{A}^d_S \) is étale
4. \( \Phi|_{Z \cap U} : Z \cap U \to \mathbb{A}^1_V \) is a closed immersion
5. \( \Phi^{-1}(\Phi(Z \cap U)) \cap U = Z \cap U \).

In [SS] J. Schmidt and F. Strunk, use the presentation lemma to generalize the \( \mathbb{A}^1 \)-connectivity result of F. Morel ([Mor1, Theorem 6.1.8]) over Dedekind schemes with infinite residue fields. As an application of Theorem 1.1, we observe that the connectivity result holds over noetherian domains with infinite residue fields. To state this result we recall the following standard notation: For a base scheme \( S \), let \( \mathcal{S}H^i_{S}(S) \) be the model category of sheaves of \( S^1 \)-spectra over \( S \). For an integer \( i \), let \( \mathcal{S}H^i_{S}(S) \) be the full subcategory of \( i \)-connected spectra. Let \( \mathcal{S}H^i_{S}(S) \xrightarrow{L^{d-i}} \mathcal{S}H^i_{S}(S) \) be the \( \mathbb{A}^1 \)-fibrant replacement functor. Then

**Theorem 1.3.** Let \( R \) be a noetherian domain of dimension \( d \) such that all the residue fields are infinite and \( \text{Spec}(R) = S \). Then \( S \) has the shifted stable \( \mathbb{A}^1 \)-connectivity property that is if \( E \in \mathcal{S}H^i_{S}(S) \) then \( L^{d-i}E \in \mathcal{S}H^i_{S}(S) \).

The proof of Theorem 1.3 is exactly the same except for the input from Gabber’s presentation lemma, the required generality of which is available once Theorem 1.1 is proved.

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fields, Nisnevich locally on $B$ there exists a projective closure $\overline{Y}$ of $Y$ in which $Y$ is fiber-wise dense. Unfortunately, we are unable to prove such a result over a general base. However, we observe that a slightly weaker result (see Theorem 2.1) can be proved which suffices for our purpose. As in Gabber’s original proof of the presentation lemma, as well as in [SS], the condition of residue fields being infinite in Theorem 1.1 is required in order to make suitable generic choices. We are currently working on removing the condition of residue fields being infinite taking inputs from [HK].

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2. Fiber-wise denseness

In this section, we prove a technical result which is crucial to the proof of our main theorem. It is essentially [Kai, Theorem 4.1] with minor modifications (see also [Lev, Theorem 10.2.2]). Throughout this section, $\dim_B(Y)$ denotes the supremum of dimensions of all the fibers of $Y \to B$.

Theorem 2.1. Let $B$ be the spectrum of a noetherian domain. Let $Y/B$ be either a smooth scheme or a divisor in a smooth scheme $X$. Let $y \in Y$ be a point lying over a point $b \in B$ with $\dim_B(Y_b) = n$. Assume $k(b)$ is an infinite field. Then there exist Nisnevich neighborhoods $(Y', y) \to (Y, y)$ and $(B', b) \to (B, b)$, fitting into the following commutative diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B 
\end{array}
$$

and a closed immersion $Y' \to A^N_B$, for some $N \geq 0$ such that if $\overline{Y'}$ is its closure in $\mathbb{P}^N_B$, then $Y'_y$ is dense in the union of $n$-dimensional irreducible components of $(\overline{Y'})_y$.

Remark 2.2. The above theorem is a weaker statement than [Kai, Theorem 4.1] (see also [Lev, Theorem 10.2.2]) but over a general base. In the proof of [Kai, Theorem 4.1] the author mentions that the base is assumed to be Dedekind to ensure that the projective closure of an equi-dimensional scheme remains equi-dimensional.

We begin with an intermediary lemma which will be used repeatedly (see also [Lev, 10.1.4]).

Lemma 2.3. Let $X$ be an affine scheme. Choose a closed embedding $X \to A_B^N$ and a point $x \in X$. Let $\overline{X}$ be the projective closure of $X$ in $\mathbb{P}^N_B$ with fiber dimension $n$. Then, there exists

1. a projective scheme $\tilde{X}$,
2. an open neighbourhood $X_0$ of $x$ (in $X$),
3. an open immersion $X_0 \hookrightarrow \tilde{X}$ and
4. a projective morphism $\psi : \tilde{X} \to \mathbb{P}^n_B$

such that $\psi$ has fiber dimension one.

Proof. We follow the arguments given in [Kai, Theorem 4.1] verbatim (see also [Lev, Theorem 10.1.4]). After possibly shrinking $B$, we can find $n$ hyperplanes $\Psi = \{\psi_1, \ldots, \psi_n\}$ which are part of a basis of $\Gamma(\mathbb{P}^N_B, \mathcal{O}(1))$ as a $B$-module. The choice is such that $V(\Psi)$ does not contain $x$ and it meets $X$ fiber-wise properly over $B$, so that $\overline{X} \cap V(\Psi)$ is finite over $B$. Let $p : \mathbb{P}^N_B \to \mathbb{P}^n_B$ be the blowup of $\mathbb{P}^N_B$ along $V(\Psi)$, and $\tilde{X}$ the strict transform of $\overline{X}$ in the blowup. This gives us a map $\psi : \mathbb{P}^N_B \to \mathbb{P}^n_B$. Let $X_0 := \overline{X} \setminus V(\Psi)$. We have the following commutative diagram:
We claim that $\psi : \tilde{X} \to \mathbb{P}_B^{n-1}$ has fiber dimension one. To see this, choose any point $y \in \mathbb{P}_B^{n-1}$, and consider the composite $a : \text{Spec}(\Omega) \xrightarrow{y} \mathbb{P}_B^{n-1} \to B$. Then, the fiber of $\psi$ over $y$ may be identified with a linear subscheme $V(y)$ of $\mathbb{P}_a^n$, of dimension $N - n + 1$. Furthermore, $V(y)$ contains the base change $V(\Psi)_a$, which has dimension $N - n$, by construction. Again by construction, the intersection $V(y) \cap \tilde{X} \cap V(\Psi)_a$ is finite in $\mathbb{P}_a^n$. This means that $V(y) \cap \tilde{X}$ has dimension 1 in the projective space $V(y)$.

Further note that for $x \in V(\Psi)$, $p^{-1}(y) \simeq \mathbb{P}_B^{n-1}$. Also, the exceptional divisor of $\tilde{X}$ is an irreducible subscheme. Therefore, for any point $x \in V(\psi) \cap X$, the fiber $\tilde{X}_x$ is an irreducible subscheme of $\mathbb{P}_B^{n-1}$ of dimension $n - 1$. Therefore, $p^{-1}(\tilde{X}) = \tilde{X}$, so that $p : \psi^{-1}(y) \cap \tilde{X} \to V(y) \cap \tilde{X}$ is a bijection. Thus, $\psi : \tilde{X} \to \mathbb{P}_B^{n-1}$ has 1-dimensional fibers.

**Proof of 2.1.** We first prove the case when $Y = X$ is a smooth scheme. The proof is by induction on $n$. The case $n = 0$ follows from a version of Hensel’s lemma.

**Step 1:** As $X$ is smooth, Zariski locally on $B$, we write $X$ as a hypersurface in some $\mathbb{A}_B^n$. Let $\overline{X}$ denote its reduced closure in $\mathbb{P}_B^n$. Note that $\overline{X}$ also has fiber-dimension $n$ over $B$. By applying, Lemma 2.3, we get a projective morphism $\psi : \tilde{X} \to \mathbb{P}_B^{n-1}$ with 1-dimensional fibers.

**Step 2:** Set $T = \mathbb{P}_B^{n-1}$ and $t = \psi(x)$. Choose any projective embedding $\tilde{X} \hookrightarrow \mathbb{P}_T^{N_2}$. Let $(\tilde{X})_t$ and $(X_0)_t$ denote the fibers over $t$ of $\tilde{X}$ and $X_0$ respectively. Then choose a hyperplane $H_t \subset \mathbb{P}_T^{N_2}$ satisfying the next three conditions.

1. (if $x$ is closed point in $(X_0)_t$) $x \in H_t$.
2. $(\tilde{X})_t$ and $H_t$ meet properly in $\mathbb{P}_T^{N_2}$.
3. $H_t$ does not meet $(X_0)_t \setminus (X_0)_t$.

Now after restricting to a suitable Nisnevich neighbourhood of $T$, which we denote again by $T$ (and after base changing everything to $T$) using the hyperplane $H_t$, we can choose a Cartier divisor $D$ which fits into the following diagram

For sufficiently large $m$ we can find a section $s_0$ of $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(mD))$ which maps to nowhere vanishing section of $\Gamma(D, \mathcal{O}_D)$. Let $s_1 : \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}(mD)$ be the canonical inclusion. Since the zero-loci of $s_0$ and $s_1$ are disjoint we get a map

$$f = (s_0, s_1) : \tilde{X} \to \mathbb{P}_T^1.$$

Since the quasi-finite locus of a morphism is open, shrink $T$ around $t$ such that $D$ is contained in the quasi finite locus of $f$ after the base change. Let $X'_0$ be the quasi-finite locus of the base change.

$$f^{-1}(\infty_T) = D \xrightarrow{\text{quasi-finite}} X'_0 \xrightarrow{\text{quasi-finite}} \tilde{X}$$

$$\infty_T \xrightarrow{\text{quasi-finite}} \mathbb{P}_T^1$$
Then the subset $W = f(\bar{X} \setminus X_0') \subset \mathbb{P}^1_{\mathcal{T}}$ is proper over $T$ and is contained in $\mathbb{P}^1_{\mathcal{T}} \setminus H_1 = \mathbb{A}^1_T$, hence it is finite over $T$. The map $\bar{X} \setminus f^{-1}(W) \to \mathbb{P}^1_{\mathcal{T}} \setminus W$, being proper and quasi-finite, is finite. By condition (i) we see that $\bar{X} \setminus f^{-1}(W)$ contains $x$.

**Step 3:** Now by induction there exist Nisnevich neighborhoods $B_1 \to B$ and $T_1 \to T$ such that the projective compactification $T_1 \to \mathcal{T}$ is fiber-wise dense in union of $n$-dimensional irreducible components over $B_1$. Take a factorization of $f$ of the form $\bar{X} \hookrightarrow \mathbb{P}^N_{T_1} \times_{T_1} \mathbb{P}^1_{T_1} \to \mathbb{P}^1_{T_1}$. Let $\mathcal{X}_1$ denote the reduced closure of $\bar{X}$ in $\mathbb{P}^N_{T_1} \times_{T_1} \mathbb{P}^1_{T_1}$. We get the following diagram where every square is cartesian:

$$
\begin{array}{ccc}
X_2 := \bar{X} \setminus f^{-1}(W) & \hookrightarrow & \bar{X} \\
\downarrow & & \downarrow \\
\mathbb{P}^N_{T_1} \times_{T_1} (\mathbb{P}^1_{T_1} \setminus W) & \hookrightarrow & \mathbb{P}^N_{T_1} \times_{T_1} \mathbb{P}^1_{T_1} \\
\downarrow & & \downarrow \\
\mathbb{P}^1_{T_1} \setminus W & \hookrightarrow & \mathbb{P}^1_{T_1} \\
\end{array}
$$

By Stein factorization we decompose the map $\mathcal{f}_1 : \mathcal{X}_1 \to \mathbb{P}^1_{T_1}$ as

$$
\mathcal{f}_1 : \mathcal{X}_1 \to \mathcal{X}_2 \xrightarrow{\text{finite}} \mathbb{P}^1_{T_1},
$$

where the first map has geometrically connected fibers. Since $\mathcal{f}_1$ is finite over the open sen $\mathbb{P}^1_{T_1} \setminus W$, $\mathcal{X}_2 \times_{\mathcal{X}_1} (\mathbb{P}^1_{T_1} \setminus W)$ is isomorphic to $X_2 := \bar{X} \setminus f^{-1}(W)$. Since $X_2$ is open in $\mathcal{X}_2$, the fiber dimension of $\mathcal{X}_2$ is at least $n$. Combining this with the fact that $\mathcal{X}_2$ is finite over $\mathbb{P}^1_{T_1}$ we conclude that the fiber dimension of $\mathcal{X}_2$ over $B_1$ is exactly $n$.

We observe that since $T_1$ is fiberwise dense in the union of $n$-dimensional irreducible components of $\mathcal{T}$, so is $\mathbb{P}^1_{T_1}$ (in $\mathcal{X}_1$). Also as $W$ is finite over $T_1$, $\mathbb{P}^1_{T_1} \setminus W$ is fiberwise dense in $\mathbb{P}^1_{T_1}$. Hence it is dense in the union of $n$-dimensional irreducible components of $\mathbb{P}^1_{T_1}$. Now we claim $X_2$ intersects the fiber of $\mathcal{X}_2$ over any point $b_1$ of $B_1$. Let $X'_2$ be the irreducible component of dimension $n$ of the fiber $(\mathcal{X}_2)_{b_1}$. Then the induced map $X'_2 \to (\mathbb{P}^1_{T_1})_{b_1}$ is a finite morphism of schemes of the same dimension. Hence it is a surjection to an irreducible component say, $U$ of $(\mathbb{P}^1_{T_1})_{b_1}$. Further $\mathbb{P}^1_{T_1} \setminus W$ intersects $U$ by denseness. Taking inverse image of its intersection with irreducible component proves that $X_2$ intersects $Y$.

As $\mathcal{X}_2$ is projective over $B_1$, we choose any embedding of it in projective space $\mathbb{P}^N_{B_1}$. Then for the closed subscheme $\mathcal{X}_2 \setminus X_2$ (with reduced structure) there exists a hypersurface $H$ of $\mathbb{P}^N_{B_1}$ of degree, say $d$, containing $\mathcal{X}_2 \setminus X_2$, not containing the point $x$ and such that $H_{b_1}$ intersects $(X_2)_{b_1}$ properly in $\mathbb{P}^N_{B_1}$. Hence by discussion in previous paragraph, $H_{b_1}$ also intersects $(\mathcal{X}_2)_{b_1}$ properly. Replacing $X_2$ by $\mathcal{X}_2 \setminus H$ and taking $d$ fold Vernose embedding we may assume $H$ to be $\mathbb{P}^{N-1}_{\infty}$. Now we have the embedding $\mathcal{X}_2 \setminus H \hookrightarrow \mathbb{A}^N_{\infty} = \mathbb{P}^N_{B_1} \setminus \mathbb{P}^{N-1}_{\infty}$ thereby proving the smooth case.

We shall now consider the case when $Y$ is a divisor in a smooth scheme.

**Step 4:** Let $Y$ be a divisor in a smooth scheme $X$. We will produce a map, $\psi : \tilde{Y} \to \mathbb{P}^{d-1}$ whose fibers are $1$-dimensional.

Since $X$ is smooth, by Steps 1-3, Nisnevich locally, we have a closed embedding $Y \to X \to \mathbb{A}^N_{B_1}$ such that all fibers of $\tilde{Y} \to B$ are $n$-dimensional. Then by Lemma 2.3, we have a commutative diagram,
Then as in Step 2 of the theorem, we obtain a morphism Nisnevich on $Y$, $\phi : Y \to \mathbb{P}^1_B$, where $T$ is a Nisnevich neighbourhood of $\mathbb{P}^{n-1}$. Since $T$ is a smooth $B$-scheme, our theorem holds for $T$. The rest of the proof is the same as in Step 3.

3. Relative version of Gabber’s Presentation Lemma

We now prove Theorem 1.1. We follow [SS] to prove Theorem 1.1. The only difference being, that in their version of Theorem 2.1 (which is for Henselian DVR), they have the stronger condition of fiberwise denseness; which they use to construct a finite map $\Psi_Z : Z \to \mathbb{A}^{d-1}_S$. However, we observe that their proof still goes through with our weaker condition of denseness in $n$ dimensional components, which we illustrate in propositions 3.5 and 3.8. The rest of the proof doesn’t require any new inputs and we just state those results from [SS] which is essentially an application of the proof from [CTHK].

First we reduce to the case that $z$ is a closed point and $Z$ is a principal divisor.

**Lemma 3.1.** (See [CTHK, Lemma 3.2.1]) With the notation as in Theorem 1.1, there exists a closed point $z' \in X$ such that $z'$ is a specialization of $z$ and there exists a non-zero $f \in \Gamma(X, \mathcal{O}_X)$ such that $Z \subset V(f)$.

**Remark 3.2.** Since Theorem 1.1 is Nisnevich locally true so, henceforth we assume that the ring $R$ is Henselian local with the closed point $\sigma$ and an infinite residue field $k$.

Let $S = \text{Spec}(R)$ with $\mathbb{A}^n_S = R[x_1, \ldots, x_n]$. Let $E$ be $R$ span of $\{x_1, \ldots, x_n\}$ and consider $\mathcal{E} := \text{Spec}({\text{Sym}^*E'})$ (note that $\mathcal{E}(R) = E$). For any integer $d > 0$ and $R$ algebra $A$, $\mathcal{E}(A)$ parametrizes all linear morphisms $v = (v_1, \ldots, v_d) : \mathbb{A}^n_S \to \mathbb{A}^d_S$, where $T = \text{Spec}(A)$. Considering $\mathbb{A}_S^n \hookrightarrow \mathbb{P}^n_S = \text{Proj} S[x_1, \ldots, x_n]$, as a distinguished open subscheme $D(x_0)$, we extend such a linear morphism to a rational map $\overline{\tau} : \mathbb{P}^n_S \dashrightarrow \mathbb{P}^d_S$ whose locus of indeterminacy $L_\tau$ is given by $V_+(x_0, v_1, \ldots, v_d) \subset \mathbb{P}^d_S$ (We will use this notation throughout what follows). Given any closed subscheme $Y$ in $\mathbb{A}^n_S$ we denote by $\overline{Y}$ its projective closure in $\mathbb{P}^n_S$. For the following lemma we refer to Lemma 2.3 of [SS]

**Lemma 3.3.** In the setting of previous paragraph if $L_\tau \cap \overline{Y} = \emptyset$, then $\overline{\tau} : \overline{Y} \to \mathbb{P}^d_S$ and $v : Y \to \mathbb{A}^d_S$ are finite maps.

Following lemma is standard.

**Lemma 3.4.** Let $W$ be a closed subscheme of $\mathbb{P}^d_k$ then there exists a hyperplane $H \subset \mathbb{P}^d_k$ such that $\dim_k(H \cap W) = \dim_k(W) - 1$.

**Proof.** Let $\zeta_1, \ldots, \zeta_r$ be the generic points of $W$ corresponding to homogeneous prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Viewing $\mathfrak{p}_i$’s and $\Gamma(\mathcal{O}(1), \mathbb{P}^d_k)$ as vector spaces over the infinite field $k$, we find a hyperplane $H$ not containing $\zeta_i$’s; as no non trivial vector space over an infinite field can be written as a finite union of proper subspaces. Hence by Krull’s principal ideal theorem $\dim_k(H \cap W) = \dim_k(W) - 1$. □

**Proposition 3.5.** Let $Y$ be as in theorem 2.1 and $\overline{Y}$ be its projective closure, then there exist $v_1, \ldots, v_n$ in $k$-span of $\{X_1, \ldots, X_N\}$ such that $(\overline{Y})_\sigma \cap L_v = \emptyset$, where $L_v = V_+(x_0, v_1, \ldots, v_n)$.

**Proof.** Without loss of generality, we assume $\mathbb{A}^n_k = D(x_0)$. Let $H_{\infty} = V_+(x_0)$ denote the hyperplane at infinity of $\mathbb{P}^n_k$. Generic points of irreducible components of $\overline{Y}_\sigma$ lie in $\mathbb{A}^n_k = D(x_0)$. Therefore $\dim(\overline{Y}_\sigma \cap H_{\infty}) = n - 1$. By theorem 2.1, we have $\dim((\overline{Y})_\sigma \cap H_{\infty}) = n - 1$. Now applying lemma 3.4 repeatedly, proves the claim. □
Theorem 3.6. Let $X = \text{Spec}(A)/S$ be a smooth, affine, irreducible scheme of relative dimension $d$, let $Z = \text{Spec}(B)$ be a principal divisor of $X$ and $z$ be a closed point in $Z$. Then there exists an open subset $\Omega \subset E^d$ with $\Omega(k) \neq \emptyset$ such that for all $\Phi = (\Psi, \nu) \in \Omega(k)$ the following hold

1. $\Psi|_Z : Z \to \mathbb{A}^{d-1}_S$ is finite.
2. $\Psi$ is étale at all points of $F := \psi^{-1}(\psi(z)) \cap Z$.
3. $\Phi|_F : F \to \Phi(F)$ is +.

Recall that $\Phi : F \to \Phi(F)$ is said to be radicial [Sta, Tag 01S2] if $\Phi$ is injective and for all $x \in F$ the residue field extension $k(x)/k(\Phi(x))$ is trivial.

To prove this theorem, we first get an open set of finite maps in proposition 3.8. Then we get a non-empty open set of étale and radicial maps in Lemma 3.9.

Remark 3.7. By proposition 2.6 and lemma 2.7 of [SS] we have a closed embedding $X \hookrightarrow \mathbb{A}^N_S$ such that $Z$ (Nisnevich locally around $z$) satisfies Theorem 2.1.

Proposition 3.8. Let $X$ and $Z$ be as in Theorem 3.6 with $S$ a spectrum of a Henselian ring $R$. Then there exists an open subset $\Omega \subset E^d$ with $\Omega(R) \neq \emptyset$ such that for all $\Psi \in \Omega(R)$, $\Psi|_Z : Z \to \mathbb{A}_S^{d-1}$ is finite.

Proof. We proceed as in Lemma 2.11 of [SS]. By remark 3.7 we have closed embedding $X \hookrightarrow \mathbb{A}^N_S$. Viewing $E^{d-1}$ as a closed subscheme of $E^d$ by taking first $d-1$ factors we consider the closed subscheme

$$V = E^{d-1} \times_S H_\infty \hookrightarrow E^d \times_S H_\infty$$

where $H_\infty$ is the hyperplane at infinity in $\mathbb{P}^N_S$. Note that $V \to E^d$ has fiber $V_v = L(v_1, \ldots, v_d)$ for any $v = (v_1, \ldots, v_d) \in E^d(R)$. Consider the open subscheme $\Omega$ of $E^d$ defined as

$$E^d \setminus p_1(V \cap (E^d \times_S (Z \cap H_\infty))),$$

where $p_1$ is projection of $E^{d-1}$ onto the first factor. By construction every point in $\Omega(R)$ consists of a linear map $v = (v_1, \ldots, v_d) : \mathbb{A}^N_S \to \mathbb{A}^1_S$ such that $L_{v'} \cap Z = \emptyset$, where $v' = (v_1, \ldots, v_{d-1})$.

By lemma 3.3, this will be our required finite map, thus proving $\Omega(R) \neq \emptyset$ will finish the proposition.

As $R$ is Henselian, the induced map from $\Omega(R)$ to $\Omega(k)$ is surjective, hence it suffices to prove $\Omega(k) = \Omega_v(k) \neq \emptyset$. By construction we have, $\Omega_v(k) = E^d_R \setminus p_1(V_v \cap (E^d \times_S (Z_v \cap H_\infty))))$ and any point in $\Omega(k)$ gives a linear map $u = (u_1, \ldots, u_d) : \mathbb{A}^N_k \to \mathbb{A}^1_k$ such that $L_u \cap (Z_v) = \emptyset$, where $u' = (u_1, \ldots, u_{d-1})$. By lemma 3.5 such a map exists.

Proof. See lemma 2.12 of [SS].

Proposition 3.9. Let $\phi = (\psi, \nu) = (u_1, \ldots, u_d) : X \to \mathbb{A}^{d-1}_S \times \mathbb{A}^1_S$ and $F := \psi^{-1}(\psi(z)) \cap Z$. There exists an open set $\Omega_2 \subset E^d$ such that $\Omega_2(R) \neq \emptyset$ and for any $\phi \in \Omega_2(R)$

1. $\phi$ is étale at all points of $F$.
2. $\phi|_F : F \to \phi(F)$ is radicial.

Proof. See [SS].

Proof of Theorem 3.6. Let $\Omega_1$ and $\Omega_2$ be as in the Propositions 3.8 and 3.9. Then the set $\Omega = (\Omega_1 \times E) \cap \Omega_2$ satisfies all the required conditions.

Now we obtain the sets $U$ and $V$. The sets $U$ and $V$ are constructed to satisfy all the conditions of Theorem 1.1.

Lemma 3.10. Let $\Phi = (\Psi, \nu)$ satisfy the condition of Theorem 3.6. Then there exists an open neighborhood $V \subset \mathbb{A}^{d-1}_S$ of $\Psi(z)$ such that

1. $\Phi$ is étale at all points of $Z \cap \Psi^{-1}(V)$.
2. $\Phi|_{Z \cap \Psi^{-1}(V)} : \mathbb{A}^1_V \to \mathbb{A}^1_V$ is a closed immersion.

Proof. Lemma 2.13 of [SS].
Lemma 3.11. There exists a closed subset $U \subset \Psi^{-1}(V)$ such that

1. $U_1 = \Psi^{-1}(V) \setminus U$ contains $z$
2. $U_1$ satisfies $Z \cap \Psi^{-1}(V) = Z \cap U_1$ and $\Phi^{-1}(\Phi(Z \cap U_1)) \cap U_1 = Z \cap U_1$.

Proof. See Lemma 2.14 of [SS]

Proof of Theorem 1.1. Let $U_2$ be the open locus where $\Phi$ is étale. From Lemma 3.10 $z \in U_2$ and $Z \cap \Psi^{-1}(V) \subset U_2$. Now let $U = U_1 \cap U_2$, with $U_1$ as in Lemma 3.11. Then $U$ also satisfies conditions (2) and (3) of Lemma 3.11. Furthermore $\Psi_U$ is étale. Hence we get $\Phi, \Psi, U, V$ satisfying all the conditions of Theorem 1.1.

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