An effective Hamiltonian for phase fluctuations on a lattice: an extended XY model

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We derive an effective Hamiltonian for phase fluctuations in an s-wave superconductor starting from the attractive Hubbard model on a square lattice. In contrast to the common assumption, we find that the effective Hamiltonian is not the usual XY model but is of an extended XY type. This extended feature is robust and leads to essential corrections in understanding phase fluctuations on a lattice. The effective coupling in the Hamiltonian varies significantly with temperature.

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The physics underlying the pseudogap phenomena in the cuprates remains unclear and is a topic of intense debate. One possibility is that the pseudogap is a precursor of the superconducting gap. This picture finds support in ARPES experiments, which show that the pseudogap seems to evolve into the superconducting gap as the temperature is lowered through the superconducting transition temperature $T_c$. In this picture, the Cooper pairs are formed above $T_c$ but without phase coherence and the superfluid stiffness is destroyed by the phase fluctuations. A different precursor scenario based on an extension of the BCS theory which induces finite momentum pairs has also had some success. Since the superconductivity is believed to resides in the CuO$_2$ planes and interlayer coupling is weak in the cuprates, it is conceivable that the high-$T_c$ superconductors reflect importantly their two-dimensional nature. Since no spontaneous breaking of continuous symmetry is allowed in 2D, a reasonable expectation is that the superconducting transition, at least, in the underdoped cuprates is not of the BCS kind but is rather of the Berezinskii-Kosterlitz-Thouless (BKT) type.

It is well-known that the effective Hamiltonian for phase fluctuations in an s-wave superconductor is equivalent to $H_{XY} = J_{XY} \sum d \nabla \psi^2$, where $\psi$ describes phase fluctuations. On a lattice, one has to either discretize $H_{XY}$ or derive the effective Hamiltonian from first principle. A common recipe to discretize $H_{XY}$ is the assumption that the continuum limit still holds and, therefore, the lattice version of the XY Hamiltonian ($H_{XY}$) has been uncritically accepted to describe phase fluctuations. In this paper, we derive an effective Hamiltonian for phase fluctuations in an s-wave superconductor from first principle. We find that the effective Hamiltonian is not of $H_{XY}$ type but rather it is an extended XY; however, its critical behavior is still BKT-like. In contrast to the usual phenomenological assumption, the effective coupling constant in the Hamiltonian now depends significantly on temperature.

We begin with a 2D attractive Hubbard model on a square lattice as the simplest case conceivable

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} C_{i\sigma}^+ C_{j\sigma} + t' \sum_{\langle\langle i,j \rangle\rangle} C_{i\sigma}^+ C_{j\sigma}$$

where $C_{i\sigma}$ is a fermion field with spin $\sigma$, $t$ is a chemical potential, $U(>0)$ is the pairing interaction, and $t'$ describe, respectively, the nearest neighbor (n.n) and the next nearest neighbor (n.n.n) hopping. The symbol $\langle i,j \rangle$ means a sum over the n.n pairs and $\langle\langle i,j \rangle\rangle$ indicates a sum over the n.n.n pairs. We will set the lattice constant $a = 1$ and also use units such that $\hbar = k_B = 1$. The partition function ($Z$), base on the Hamiltonian $\mathcal{H}$ and written in the language of path integral, is given by $Z = \int D\psi \exp[-S]$, where the action $S = \sum_{\tau} \int \left( \sum_{\langle i,j \rangle} C_{i\sigma}^+ \partial_\tau C_{i\sigma} + \mathcal{H} \right)$. The range for the integral over the imaginary time $\tau$ in the action is from 0 to $1/T$. We first consider only n.n hopping ($t$) with $t' = 0$ and derive an effective local Hamiltonian for the phase of the order parameter. At finite temperature we obtain, in this case, an extended XY model Hamiltonian which includes not only the n.n spin-spin interaction but also the second and third neighbor interaction. Inclusion of the n.n.n hopping makes this extended feature of the effective Hamiltonian even more robust in the sense that this property is now manifested even at zero temperature.

Introducing the Hubbard-Stratonovich transformation with an auxiliary field $\phi_i = \Delta_i e^{i\phi_i}$, and making a gauge transformation $\psi_i = \exp[i\tilde{r}_i \theta_j/2] \chi_i$, where $\chi_i$ is a spinor for neutral fermions and $\tilde{r}_i$ are Pauli matrices, after integrating out the fermion fields, we obtain $Z = \int D\phi^* D\phi \exp[-S_{eff}]$, where an effective action is

$$S_{eff} = \int \tau \sum_i \left[ U|\phi_i|^2 - \text{Tr} \ln[G^{-1}] \right].$$

Here, $\text{Tr}$ means the trace over the functional space, depending on the representation, and spin space. We will use a symbol $\text{tr}$ for a trace over spin space only. In the real (lattice) space, the Green function can be represented as $G^{-1}(i,j) = G_0^{-1}(i,j) - \Sigma(i,j)$, with

$$G_0^{-1}(i,j) = (-\tilde{r}_0 \partial_\tau + \tilde{r}_3 \Delta) \delta_{i,j} + \tilde{r}_3 t \sum_\delta \delta_{j,i+\delta}$$

and the self energy has the form

$$\Sigma(i,j) = \frac{\tilde{r}_3 \partial_\tau \theta_j}{2} \delta_{i,j} + \tilde{r}_0 \Sigma^{(0)}(i,j) + \tilde{r}_3 \Sigma^{(3)}(i,j),$$

where $C_{i\sigma}$ is a fermion field with spin $\sigma$, $\mu$ is a chemical potential, $U(>0)$ is the pairing interaction, $t$ and $t'$ describe, respectively, the nearest neighbor (n.n) and the next nearest neighbor (n.n.n) hopping. The symbol $\langle i,j \rangle$ means a sum over the n.n pairs and $\langle\langle i,j \rangle\rangle$ indicates a sum over the n.n.n pairs. We will set the lattice constant $a = 1$ and also use units such that $\hbar = k_B = 1$. The partition function ($Z$), base on the Hamiltonian $\mathcal{H}$ and written in the language of path integral, is given by $Z = \int D\psi \exp[-S]$, where the action $S = \sum_{\tau} \int \left( \sum_{\langle i,j \rangle} C_{i\sigma}^+ \partial_\tau C_{i\sigma} + \mathcal{H} \right)$. The range for the integral over the imaginary time $\tau$ in the action is from 0 to $1/T$. We first consider only n.n hopping ($t$) with $t' = 0$ and derive an effective local Hamiltonian for the phase of the order parameter. At finite temperature we obtain, in this case, an extended XY model Hamiltonian which includes not only the n.n spin-spin interaction but also the second and third neighbor interaction. Inclusion of the n.n.n hopping makes this extended feature of the effective Hamiltonian even more robust in the sense that this property is now manifested even at zero temperature.

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$$G_0^{-1}(i,j) = (-\tilde{r}_0 \partial_\tau + \tilde{r}_3 \mu + \tilde{r}_3 \Delta) \delta_{i,j} + \tilde{r}_3 t \sum_\delta \delta_{j,i+\delta}$$

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$$\Sigma(i,j) = \frac{\tilde{r}_3 \partial_\tau \theta_j}{2} \delta_{i,j} + \tilde{r}_0 \Sigma^{(0)}(i,j) + \tilde{r}_3 \Sigma^{(3)}(i,j),$$
where $\Sigma^{(0)}(i,j) = it \sum_{\delta} \delta \cos(\theta_{i,j}/2)$ and $\Sigma^{(3)}(i,j) = t \sum_{\delta} \delta \cos(\theta_{i,j}/2)$. Here $\theta_{i,j} - \theta_{i,j}$ is a phase difference between sites $i$ and $j$, and $\delta = \pm \hat{x}$ and $\pm \hat{y}$ on a square lattice. In the expression for $G^{-1}$ we consider only fluctuations of the phase and assume no fluctuation in the magnitude of the gap. Magnitude fluctuation will also change the self energy $\Sigma$. However, we ignore this and concentrate on the effects of phase fluctuations. For simplicity we consider the static case ($\partial_K \theta_i = 0$). A generalization to the time-dependent case is straightforward.

The effective action $S_{\text{eff}}$ can be separated into a mean-field part ($S^{(0)}$) and a phase fluctuation part ($S^{(3)}$), namely, $S_{\text{eff}} = S^{(0)} + S^{(3)}$, where

$$S^{(0)} = \frac{1}{2} \sum_{K,K'} \Lambda^{(0)}_{K,K'} \Sigma^{(0)}(K,K') \Sigma^{(0)}(K',K) + \frac{1}{2} \sum_{K,K'} \Lambda^{(3)}_{K,K'} \Sigma^{(3)}(K,K') \Sigma^{(3)}(K',K),$$

where $\Lambda^{(ij)}_{K,K'}$ is the static contribution to $\tilde{G}(K)\Sigma(K')\tilde{G}(K')$. The mixed terms do not contribute to $S^{(2)}$ in our consideration for a local effective action. Since in the momentum space $A_4(X_k) = \sum_0 \hat{\Lambda}_4(Q)e^{iQX_k}$, where $Q$ is a four vector $(q,\Omega_m)$ with a bosonic Matsubara frequency $\Omega_m$, the first term of $S^{(2)}$ is

$$-\frac{1}{2} \sum_{\delta,\delta'} \sum_{Q} \hat{\Delta}(Q) \hat{\Delta}(Q') e^{-iQx} e^{iQx} = -\frac{1}{2} \sum_{\delta,\delta'} \sum_{Q} \hat{\Delta}(Q) \hat{\Delta}(Q') e^{-iQx} e^{iQx} \Lambda^{(0)}_{K,K}.$$

A similar expression can be obtained for the second term. Since, as we mentioned earlier, we are concerned only with the local effective action for the phase and wish to see if the equivalence between the phase-only Hamiltonian and the usual $H_{XY}$ found for the continuum limit also holds on a lattice, we expand $S^{(2)}$ about $Q = 0$ and keep the leading order. This means that we neglect the so-called Landau terms associated with damping effects, which have been shown not to be important in a case of an $s$-wave superconductor in the continuum limit for $T \lesssim 0.6T_{MF}$, where $T_{MF}$ is a temperature at which $\Delta(T_{MF}) = 0$. In the $s$-wave case, the importance of the damping effects appears when there is a large suppression of $\Delta(T)$ so that $\Delta(T)$ is comparable to $T$, and the fraction of thermally excited quasiparticles is not negligible. However, it has been pointed out that for a $d$-wave superconductor the Landau terms have strong effects even at low temperature because of the nodal structure of the order parameter.

For simplicity, we will use four vector notation: $K = (k, \omega_n), \Sigma_K = T \sum_{\omega_n} \Sigma_k$, where $n$ is a fermionic Matsubara frequency, and $dX_k = \int d\tau \sum_i \tilde{\Sigma}_k$. It can be shown that $\langle K|G(K')\rangle = \delta(K - K')G(K')$ and $\langle K'|\Sigma_K \rangle = \Sigma_K \Sigma_K$. Note that $\Sigma^{(0)}(K')$ includes no contribution to $S^{(1)}$. Since $S^{(1)}$ includes the phase difference only between the n.n sites and the first non-trivial term is $\theta^2_{ij}$, we obtain the usual XY model action; however, $\theta_i^2$ terms will also appear in $S^{(2)}$. Moreover, the phase difference between the n.n sites is also obtained from the second trace. As we will see later, if we include the n.n hopping, even in $S^{(1)}$ one has the phase difference between the n.n sites.

Similarly, $S^{(2)} = \frac{1}{12} \sum_{K,K'} G(K)\Sigma(K,K')G(K')\Sigma(K',K)$, where $\Sigma = \tilde{\Sigma}_0^{(0)} + \tilde{\Sigma}_0^{(3)}$. After some manipulations, one arrives at

$$S^{(2)} = \frac{1}{2} \sum_{K,K'} \Lambda^{(0)}_{K,K'} \Sigma^{(0)}(K,K') \Sigma^{(0)}(K',K) + \frac{1}{2} \sum_{K,K'} \Lambda^{(3)}_{K,K'} \Sigma^{(3)}(K,K') \Sigma^{(3)}(K',K),$$

where $\Lambda^{(ij)}_{K,K'} = \text{tr}[G(K)\Sigma(K')\tilde{G}(K'-\tau_i)]$. The mixed terms do not contribute to $S^{(2)}$ in our consideration for a local effective action. Since in the momentum space $A_4(X_k) = \sum_0 \hat{\Lambda}_4(Q)e^{iQX_k}$, where $Q$ is a four vector $(q,\Omega_m)$ with a bosonic Matsubara frequency $\Omega_m$, the first term of $S^{(2)}$ is

$$\frac{1}{2} \sum_{\delta,\delta'} \sum_{Q} \hat{\Delta}(Q) \hat{\Delta}(Q') e^{-iQx} e^{iQx} \Lambda^{(0)}_{K,K}.$$
matrix $\hat{M}$ is not diagonal. This means that the effective Hamiltonian $\mathcal{H}_\theta$ is not equivalent to the usual $\mathcal{H}_{XY}$, which would be pure-diagonal; namely, we have not only terms like $\theta_{t,i+\pi}^2$ and $\theta_{t,i+\pi}^2$, but also terms of the form $\theta_{t,i+\pi}^2 \theta_{t,i+\pi}^2$ in $\mathcal{H}_\theta$. We will next investigate effects of these off-diagonal terms.

It is worthwhile making sure our new result for $\mathcal{H}_\theta$ reduces to the well known XY-type Hamiltonian in the continuum limit. For this purpose, we need to recover the explicit value of the lattice constant $a$ in the expression for $\mathcal{H}_\theta$ in order to track orders of $a$ in the limit $a \to 0$ and the phase difference becomes a derivative of the phase $(\nabla \theta)$ while $ta^2 \to \frac{1}{2m}$, where $m$ is an effective mass of an electron. It is straightforward to show that, in the continuum limit with $\xi = \frac{k^2}{\alpha} - \mu$, $\mathcal{H}_\theta \to H^{(1)}_{\theta} + H^{(2)}_{\theta}$, where $H^{(1)}_{\theta} = \frac{1}{8m} \sum_i \kappa \Gamma (3) \int \nabla \theta$ and $H^{(2)}_{\theta} = \frac{1}{16m^2} \sum_i \kappa \Gamma (0) k^2 \int \nabla \theta^2$. Consequently, the effective Hamiltonian $\mathcal{H}_\theta$ we derive does reduce to $\mathcal{H}_{XY}$.

Let us now consider effects of the off-diagonal terms in $\mathcal{H}_\theta$. At $T = 0$, $\mathcal{H}_\theta$ again becomes equivalent to the XY-type Hamiltonian because the off-diagonal terms of $M$, $\alpha$, and $\gamma$ vanish. However, at finite temperature they are nonzero so that $\mathcal{H}_\theta$ is no longer of the usual XY type. Instead

$$\mathcal{H}_\theta = \alpha \sum_{<ij>} \theta_{i,j}^2 + 2\beta \sum_i (\theta_{i,i+\pi} \theta_{i,i-\pi} + \theta_{i,i+\gamma} \theta_{i,i-\gamma}) + 2\gamma \sum_i (\theta_{i,i+\pi} + \theta_{i,i-\pi})(\theta_{i,i+\gamma} + \theta_{i,i-\gamma}).$$

Since the phase fluctuation between two sites is small by assumption, one can make the approximation: $\theta_{i,i+\pi} \approx \sin(\theta_{i,i+\pi})$. Introducing a 2D classical spin $S_i = (\cos(\theta_i), \sin(\theta_i))$ at site $i$, it can be shown within the approximation we have made, that, for example, $\theta_{i,i+\pi} \approx (S_i \times S_{i+\pi}) \cdot (S_i \times S_{i-\pi})$. Apply a vector identity to this expression, we then obtain $S_i \cdot S_{i+\pi} \cdot S_i \cdot S_{i-\pi} = S_i \cdot S_{i+\pi} - (S_i \cdot S_{i+\pi})(S_i \cdot S_{i-\pi})$, where we used the fact that $S_i \cdot S_i = 1$. As one can see, a spin at the site $i + \pi$ couples to a spin at $i - \pi$, which is a n.n spin-spin interaction. Note that the only assumption we have made is that terms higher than $\theta_{t,i}^2$ for a given $(i, j)$ are negligible. This does not mean, however, that only the n.n phase differences are important. Following the procedure we briefly described, one can show that the second term of $\mathcal{H}_\theta$, which is proportional to $\beta$, becomes $\beta \sum_{<ij>} \theta_{i,j}^2 + \beta \sum_{<ij>_{\gamma}} S_i \cdot S_j$ within a constant, where the symbol $<ij>_{\gamma}$ indicates a sum over the next-nearest neighbors. Similarly, the third term of $\mathcal{H}_\theta$ proportional to $\gamma$ turns out to be $2\gamma \sum_{<ij>_{\gamma}} \theta_{i,j}^2 + 2\gamma \sum_{<ij>_{\gamma}} S_i \cdot S_j$. Since $\theta_{t,i}^2$ can also be represented in term of $S_i \cdot S_j$, the effective Hamiltonian $\mathcal{H}_\theta$ can be written as

$$\mathcal{H}_\theta = -J_1 \sum_{<ij>} S_i \cdot S_j + J_2 \sum_{<ij>_{\gamma}} S_i \cdot S_j + J_3 \sum_{<ij>_{\gamma}} S_i \cdot S_j,$$

where $J_1 = 2(\alpha + \beta + 2\gamma)$, $J_2 = 2\gamma$, and $J_3 = \beta$. It is clear that this Hamiltonian is not of the usual XY type but instead of an extended XY type. A geometrical explanation for the appearance of the n.n.n and the next-nearest neighbor (n.n.n.n) term is illustrated in Fig. 1(a). Indeed, these terms come from the second trace proportional to $t^2$, which has a factor $\Sigma(0)\Sigma(0)$. Each self-energy $\Sigma(0)$ picks up $\delta = \pm \pi, \pm \gamma$, and the second trace gives terms with resulting vectors $\delta + \delta = \delta_2$, where $\delta_2 = \pm \pi \pm \gamma$ and $\delta_3 = \pm 2\pi \pm 2\gamma$. This geometrical picture also works when we include the n.n hopping ($t'$) in the electron dispersion curves.

The physics of $\mathcal{H}_\theta$ depends on the relative magnitudes of coefficients $J_1$, $J_2$, and $J_3$ as well as their relative signs. For example, if $\beta$ is negligible, and $\alpha$ and $\gamma$ are both positive so that $J_1 > 2J_2$, $\mathcal{H}_\theta$ describe a non-frustrated XY model, and its critical behavior can be understood [12] in terms of the usual XY Hamiltonian with an effective coupling constant $J_{eff} = (J_1 - 2J_2)$. However, this does not mean that in this case $\mathcal{H}_\theta$ is equivalent to $-J_{eff} \sum_{<ij>} S_i \cdot S_j$ because the local behaviors of these two Hamiltonians are different. In general, however, as long as $J_1$ is dominant, the large length scale behavior is of the usual XY type. To calculate $J_1$, $J_2$, and $J_3$, we need to know how $\Delta$ and $\mu$ change with increasing temperature. We choose the pairing interaction $U = 1.4t$ and the filling factor $n = 0.9$ and self-consistently solve the equations for gap and filling factor to determine $\Delta(T)$ and $\mu(T)$. What we obtain in the numerical calculation is that $J_1$ is negligible from $T = 0$ to $T = T_{MF}$, and ii) the inequality $J_1 > 2J_2$ holds almost all the way up to $T = T_{MF}$. Near $T_{MF}$, $J_1$ can be a little less than $2J_2$; however, as we mentioned before the local effective Hamiltonian for phase fluctuations may not be valid in this temperature regime, where effects of thermally excited quasiparticles cannot be neglected. Consequently, for the interesting temperature range $J_{eff} = J_1 - 2J_2$; namely, $J_{eff} = 2(\alpha + \beta)$. Interestingly, $J_{eff}$ does not depend on the coefficients of the n.n.n and the n.n.n.n interaction terms. However, this is not the case if we include the n.n hopping.

When we include the n.n hopping ($t'$), the Green function $G_G(i, j)$ and the self energy change as follows: $G_G^{-1}(i, j) \to G_G^{-1}(i, j) + \frac{\pi}{2} \sum_{\delta} \delta \delta_{i,i+\delta} + \Sigma^{(3)}(i, j) \to \Sigma^{(1)}(i, j) + it' \sum_{\delta} \delta \delta_{i,i+\delta} \sin(\theta_{i,i+\delta}/2)$, and a similar expression for $\Sigma^{(3)}(i, j)$ holds. In the same way that the $t^2$ term gives interactions of the form $S_i \cdot S_{i+\delta} + S_i \cdot S_{i+\delta}$, the $tt'$ term induces the form $S_i \cdot S_{i+\delta} + S_i \cdot S_{i-\delta} + S_i \cdot S_{i+\delta} \delta_3$, because $\delta + \delta_3 = \delta_4$, and $S_i \cdot S_{i+\delta} \delta_3$ because $\delta + \delta_\delta = \delta_3$, where $\delta_3 = \pm \pi \pm 22$, $\pm 2\pi \pm \gamma$, and $\delta_\delta = \pm 2\pi \pm 2\gamma$. Geometrical descriptions for these terms are also presented in Fig. 1(b) and 1(c). Following the same manipulation as before, we obtain the effective Hamiltonian including the n.n hopping contribution:

$$\mathcal{H}_\theta = -J_1 \sum_{<ij>} S_i \cdot S_j + J_2 \sum_{<ij>_{\gamma}} S_i \cdot S_j + J_3 \sum_{<ij>_{\gamma}} S_i \cdot S_j + J_4 \sum_{<ij>_{\gamma}} S_i \cdot S_j$$

where $J_4 = 2(\alpha + \beta + 2\gamma)$.
where $\sum_{<ij>\neq e(5)}$ means $\sum_{i} \delta_{\alpha(5)}$. It is obvious that the extended features of $H_{\theta}$ are robust in band structure parameters. Moreover, contrary to what we found before, now $J_2$ is finite even at $T = 0$ because the n.n.n hopping ($t'$) contributes to the first trace as well as to the second trace\[8\]. In order to calculate $J_i$ ($i = 1, 2, 3, 4$) we choose $t' = -0.2t$ with the same values of parameters as before ($U = 1.4t$ and $n = 0.9$). Since our numerical calculation indicates that $J_1$ and $J_2$ are dominant and $J_1 > 2J_2$ in most of the temperature range, we neglect $J_3$, $J_4$, and $J_5$. In this case as we mentioned earlier, the critical behavior is still of the usual XY type so that we introduce an effective coupling $J_{\text{eff}} = J_1 - 2J_2$. In Fig. 2, we plot $\frac{\pi}{2} J_{\text{eff}}$ vs $T$ scaled by $T_{MF}$. Since $J_{\text{eff}} < 0$ near $T_{MF}$, one might think this implies a frustrated XY case; however, we again point out that $\Delta(T)$ is comparable to $T$ in this region and, therefore, we estimate the validity of the local effective Hamiltonian to be $0 < T < T_0$, where $T_0$ is a temperature such that $J_{\text{eff}} > 0$. We also considered a case of a stronger interaction ($U = 2t$) and a lower filling factor ($n = 0.4$) with the same $t'$. The behavior of $J_{\text{eff}}$ is similar to the one in Fig. 2. We emphasize that $J_{\text{eff}}$ approaches zero earlier than does the gap $\Delta(T)$. Note that in the continuum limit\[9\] the coupling constant vanishes at $T_{MF}$.

In order to find out the BKT transition temperature ($T_{BKT}$), one has to solve self-consistently $T_{BKT} = \frac{\pi}{2} J_{\text{eff}}(T_{BKT})$ with equations for the gap and the filling factor. Since $\Delta(T)$ and $\mu(T)$ have been self-consistently obtained for given parameters, the solution of $T = \frac{\pi}{2} J_{\text{eff}}(T)$ (the crossing point between the dashed line and the solid curve) gives $T_{BKT}$, which is indicated by an arrow in Fig. 2. Suppose one use the usual XY Hamiltonian including only $J_1$ term in Eq. $[\text{3}]$ as seen (dotted curve) in Fig. 2. $T_{BKT}$ is larger than $T_{MF}$, which is unreasonable. Corrections to the usual XY model are essential in our approach.

In summary, An effective Hamiltonian for phase fluctuations has been derived starting from an attractive Hubbard model on a lattice. Unlike the common assumption, we found that the effective Hamiltonian is not of the usual XY type but is extended XY. This extended feature is robust and has important consequences. It is reinforced when the next nearest neighbor hopping is considered. The critical behavior is still BKT-like but with an effective coupling which depends significantly on the temperature.

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FIG. 1 (Kim and Carbotte)

FIG. 1: Geometrical illustrations of induced interactions for the next-nearest and the further-neighbor spin pairs.
FIG. 2: $\frac{\pi}{2} J_{\text{eff}}$(solid curve) as a function of temperature ($T$) scaled by $T_{MF}$. The BKT temperature $T_{BKT}$ is indicated by an arrow. If one use the usual XY Hamiltonian including only $J_1$ term, $T_{BKT}$ is larger than $T_{MF}$ (See the dotted curve).