STABILITY OF TRANSLATING SOLUTIONS TO MEAN CURVATURE FLOW

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Abstract. We prove stability of rotationally symmetric translating solutions to mean curvature flow. For initial data that converge spatially at infinity to such a soliton, we obtain convergence for large times to that soliton without imposing any decay rates.

1. Introduction

We consider solutions \( u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R} \), \( n \geq 2 \), to the graphical mean curvature flow equation

\[
\dot{u} = \sqrt{1 + |\nabla u|^2} \, \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).
\]

We will consider rotationally symmetric, strictly convex, translating solutions \( U \). These solutions arise as parabolic rescalings of type II singularities [11]. They have constant time derivatives \( \dot{U} > 0 \). By scaling we may assume that \( \dot{U} \equiv 1 \), so we can write \( U(x, t) = U(x, 0) + t \). Our main theorem states that these translating solutions are dynamically stable.

**Theorem 1.1.** Let \( U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \), \( n \geq 2 \), be an entire rotationally symmetric, strictly convex solution to mean curvature flow, translating with speed \( \dot{U} \equiv 1 \). Let \( u_0 : \mathbb{R}^n \to \mathbb{R} \) be continuous such that the distance to \( U(\cdot, 0) \) tends to zero at infinity

\[
\lim_{|x| \to \infty} |u_0(x) - U(x, 0)| = 0.
\]

Then there exists a function \( u \in C^\infty (\mathbb{R}^n \times (0, \infty)) \cap C^0 (\mathbb{R}^n \times [0, \infty)) \) solving (MCF) for positive times with \( u(\cdot, 0) = u_0 \).

As time tends to infinity,

\[
u(\cdot, t) - U(\cdot, t) \to 0
\]

uniformly on \( \mathbb{R}^n \).

Note that no decay rate is imposed on \( u_0 - U(\cdot, 0) \). Our convergence result implies that \( u \) converges to a translating solution as \( t \to \infty \). Moreover, it converges to precisely the translating solution we perturbed initially.

The strategy of the proof is as follows. Known results [1] are easily extended to higher dimensions and establish the existence of a solution \( U \) as in Theorem 1.1, see
Lemma 2.2. Such rotationally symmetric translating solutions fulfill an ordinary differential equation. We will derive it in Section 2. Besides these solutions, this equation has also solutions which correspond to rotationally symmetric, graphical translating solutions which are defined in the complement of a ball, see Figure 1 for a cross-section of both types of solutions.

We denote such a solution by $W^+_R$, if it is defined in $(\mathbb{R}^n \setminus B_R(0)) \times \mathbb{R}$ and $\lim_{|x| \searrow R} \langle \nabla W^+_R, -x \rangle = \infty$. Similarly, we call it $W^-_R$ if $\lim_{|x| \searrow R} \langle \nabla W^-_R, -x \rangle = -\infty$. Adding appropriate constants to $W^+_R$ and $W^-_R$, we see that these solutions become asymptotic to $U$ at infinity.

We will use these solutions as barriers. They exist in the complement of a ball of any radius. Compare these solutions to $n$-catenoids, $n \geq 3$, which are asymptotic to hyperplanes, see also Appendix C. Hyperplanes are translating solutions to mean curvature flow. They move with velocity zero. Similarly, the translating solutions defined in the complement of a ball are at infinity asymptotic to entire translating solutions $U$ with $U$ as in Theorem 1.1.

In order to prove stability of a translating solution, we proceed as follows. Let $u_0$ be a graphical perturbation of $U(\cdot, 0)$ such that $U(\cdot, 0) - u_0$ tends to zero at infinity. Then there exists a solution to graphical mean curvature flow (MCF) which stays between two rotationally symmetric translating solutions. We obtain interior a priori estimates for this solution. Translating solutions $W^+_R$ and $W^-_R$ as indicated above in Figure 1 are then shifted vertically such that $W^+_R(x,t) - U(x,t)$ and $U(x,t) - W^-_R(x,t)$ tend to $\varepsilon$ for $|x| \to \infty$. Calling these shifted solutions again $W^+_R$ and $W^-_R$, we may choose $R$ and $\varepsilon$ such that $W^-_R(x,0) \leq u(x,0) \leq W^+_R(x,0)$ wherever these functions are defined. As the gradient of $W^\pm_R(x,t)$ becomes unbounded for $|x| \downarrow R$, we can apply the maximum principle and obtain that $W^-_R(x,t) \leq u(x,t) \leq W^+_R(x,t)$ holds for all times. Thus $|u(x,t) - U(x,t)|$ can attain a maximum of size larger than $2\varepsilon$ only in a bounded set. The strong maximum principle and the a priori estimates mentioned above can be used to show that $|u(x,t) - U(x,t)|$ will everywhere be smaller than $2\varepsilon$ for large $t$. Thus $u(x,t) - U(x,t)$ converges uniformly to zero as $t \to \infty$.

In [1], Steve Altschuler and Lani Wu have shown that entire rotationally symmetric translating mean curvature flow solutions exist. They can be obtained by
rescaling a type II singularity parabolically [11]. Xu-Jia Wang [14] found other con-
convex translating solutions without rotational symmetry. Stability of non-compact
gradient Kähler-Ricci solitons is investigated in [5]. Stability of the grim reaper,
a translating curve solving mean curvature flow, is considered in [13]. Richard
Hamilton [10] mentions non-convex, complete translating solutions to mean curva-
ture flow. We prove the existence of such solutions in Lemma 2.3.

The rest of the paper is organized as follows. In Section 2 we study the ordinary
differential equation for rotationally symmetric translating solutions. The existence
of a well behaved solution is shown in Section 3. Convergence to a translating
solution is proved in Section 4.

For the reader’s convenience, we have collected in appendices some results that
we use. We have a comparison principle in Appendix A and present interior esti-
mates and an existence result in Appendix B.

In Appendix C we show that our method implies directly that hyperplanes in
\( \mathbb{R}^{n+1} \), \( n \geq 3 \), are stable under mean curvature flow. Finally, we strengthen a
stability result for gradient Kähler-Ricci solitons in Appendix D.

2. Rotationally Symmetric Translating Solutions

Let \( V \) be a solution to graphical mean curvature flow. For solutions that translate
with speed 1, we can write \( V(x, t) = V(x, 0) + t \). In the rotationally symmetric case, (MCF)
reduces to an ordinary differential equation for \( \tilde{V}(r) = V(x, 0) \), where \( r = |x| \). Writing \( V(r) = \tilde{V}(r) \), this ordinary differential equation is

\[
1 = \frac{V''}{1 + V'^2} + (n - 1) \frac{V'}{r},
\]

where ‘ denotes derivatives with respect to \( r \). For our purposes, it will be convenient
to consider the ordinary differential equation for \( \varphi = V' \),

\[
\varphi' = (1 + \varphi^2) \left( 1 - (n - 1) \frac{\varphi}{r} \right).
\] (2.1)

Knowledge of asymptotic behavior allows us to find translating solutions which
become close to each other at infinity. Computer algebra calculations suggest that

\[
\varphi = \frac{r}{n - 1} - \frac{1}{r} + (n - 1)(n - 4) \left( \frac{1}{r^3} - (n - 1)^2 \left( n^2 - 12n + 31 \right) \frac{1}{r^5} + (n - 1)^3 \left( n^3 - 24n^2 + 164n - 330 \right) \frac{1}{r^7} - (n - 1)^4 \left( n^4 - 40n^3 + 510n^2 - 2554n + 4315 \right) \frac{1}{r^9} + O(r^{-11}) \right).
\]

For us, the first three terms of the expansion suffice.

**Lemma 2.1.** For any \( R > 0 \) and any \( \varphi_0 \in \mathbb{R} \), the boundary value problem

\[
\begin{aligned}
\varphi'(r) &= (1 + \varphi^2) \left( 1 - (n - 1) \frac{\varphi}{r} \right), \quad r \geq R, \\
\varphi(R) &= \varphi_0,
\end{aligned}
\]

has a unique \( C^\infty \)-solution \( \varphi \) on \([R, \infty)\). Moreover, as \( r \to \infty \), we have the asymp-
totic expansion

\[
\varphi(r) = \frac{r}{n - 1} - \frac{1}{r} + O(r^{-2}).
\] (2.2)
Proof. We show that a solution exists for all \( r > R \).

Assume \( 1 - \varphi(n-1)/r \) is negative. Then \( \varphi' < 0 \), that is, \( \varphi \) is decreasing as a function of \( r \). Therefore, for sufficiently large \( r \), \( 1 - \varphi(n-1)/r \) becomes positive and remains so. We may therefore assume that \( \varphi(r) < r/(n-1) \) for \( r > R \geq R \). As \( \varphi' > 0 \) in this region, \( \varphi \) is bounded from below. It follows that \( \varphi \) cannot become infinite for finite \( r \), and we obtain existence for all \( r \geq R \).

Next, we examine the asymptotic behaviour of this solution. We claim that for every given \( \varepsilon > 0 \) and \( r_0 > R \), there exists \( r_1 > r_0 \) such that

\[
\varphi(r_1) \geq \frac{r_1}{n-1}(1 - \varepsilon).
\]

If this is not the case, we estimate \( \varphi'(r) \geq (1 + \varphi^2)\varepsilon \) for \( r > r_0 \). This is impossible, however, as our solution \( \varphi \) exists for \( r \in [R, \infty) \).

Now observe that the function \( \zeta = \frac{r}{n-1}(1 - \varepsilon) \) satisfies

\[
\zeta' \leq (1 + \zeta^2)\left(1 - \frac{n-1}{r}\right)
\]

for \( r \) sufficiently large. Thus enlarging \( r_1 \), if necessary,

\[
\frac{r}{n-1}(1 - \varepsilon) \leq \varphi(r) \leq \frac{r}{n-1}
\]

for \( r > r_1 \). Since this works for every \( \varepsilon > 0 \) we obtain

\[
\varphi(r) = \frac{r}{n-1} + o(r).
\]

We can write \( \varphi = r/(n-1) + \psi \) where \( \psi \) is sublinear, so that \( |\psi(r)| < cr \) for all \( c > 0 \) and sufficiently large \( r \). This \( \psi \) satisfies

\[
\psi' = -(n-1)\psi \left(1 + \left(\frac{r}{n-1} + \psi\right)^2\right) - \frac{1}{n-1}.
\]

Observe that \( \psi \) is non-positive for \( r \) sufficiently large. We now show that \( \psi \to 0 \) as \( r \to \infty \). Fix \( \varepsilon > 0 \). Consider points \( r > r_2 \) where \( \psi(r) \leq -\varepsilon \). By the sublinearity of \( \psi \), we can also assume that \( -r/(2n-2) < \psi(r) \). Then

\[
\psi'(r) \geq \varepsilon(n-1)\frac{r^2}{r^2 + 4(n-1)^2} - \frac{1}{n-1} \geq c > 0
\]

for \( r_2 \) chosen large enough. Thus \( \psi(r) \geq -\varepsilon \) for \( r \) large enough.

Now set \( \lambda(r) = r\psi(r) \). We will show that \( \lambda \to -1 \). Since \( \psi = \lambda/r \to 0 \), for all \( \mu > 0 \) and sufficiently large \( r \) we have \( |\lambda(r)| \leq \mu r \).

Suppose \( \lambda(r) \geq -1 + \varepsilon \) for some \( \varepsilon > 0 \) and \( r \geq r_3 \). Then

\[
\lambda'(r) = \frac{r}{n-1} \left[1 + \lambda + 2(n-1)\frac{\lambda^2}{r^2}\right] - (n-2)\frac{\lambda}{r} - (n-1)\frac{\lambda^3}{r^3},
\]

so

\[
\lambda'(r) \leq -\frac{r}{n-1}(1 + \lambda) + \mu(n-2) + \mu^3(n-1) \leq -c
\]

for some \( c > 0 \) and sufficiently large \( r_3 \). We obtain that \( \lambda(r) \leq -1 + \varepsilon \) for \( r \) sufficiently large. Similarly if we assume \( \lambda(r) \leq -1 - \varepsilon \), we have

\[
\lambda'(r) \geq -\frac{r}{n-1} \left[-\varepsilon + 2\mu^2(n-1)\right] - \mu(n-2) - \mu^3(n-1) \geq c
\]

for some \( c > 0 \) and sufficiently large \( r_3 \). As above, we get \( \lambda(r) \geq -1 - \varepsilon \) for \( r \) sufficiently large. We conclude that \( \lambda \to -1 \).
Now set \( \lambda(r) = -1 + \eta(r)/r^2 \). We will show that \( \eta \to (n - 4)(n - 1) \). Since \( \eta/r^2 \to 0 \), for all \( \mu > 0 \) and sufficiently large \( r \) we have \( |\eta(r)| \leq \mu r^2 \).

Suppose \( \eta(r) > (n - 4)(n - 1) + \varepsilon \) for \( r > r_4 \) and \( \varepsilon > 0 \). As

\[
\eta'(r) = \frac{r}{n-1} \left[ (n-4)(n-1) - \eta \right] + \frac{1}{r} \left[ (8-n)\eta + (n-1) \right] - \frac{\eta}{r^3} [2\eta + 3(n-1)] + 3(n-1) \frac{\eta^2}{r^5} - (n-1) \frac{\eta^3}{r^7},
\]

we have

\[
\eta'(r) \leq \frac{r}{n-1} \left[ (n-4)(n-1) - \eta \right] + |8-n|\mu r + \frac{n-1}{r} - \frac{\eta}{r^3} \left[ 3(n-1) + 3(n-1) \frac{\eta^2}{r^5} - (n-1) \frac{\eta^3}{r^7} \right] \leq \frac{r}{n-1} \left[ -\varepsilon + \mu |8-n|(n-1) \right] + O(1)
\]

for \( r_4 \) large enough, if we choose \( \mu \) small compared to \( \varepsilon \). We obtain that \( \eta'(r) \leq (n-4)(n-1) + \varepsilon \) for \( r \) sufficiently large. Assume now that \( \eta(r) < (n-4)(n-1) - \varepsilon \) for \( r > r_4 \). Then

\[
\eta'(r) \geq \frac{r}{n-1} \left[ (n-4)(n-1) - \eta \right] - |8-n|\mu r + \frac{n-1}{r} - 2\mu^2 r - 3(n-1) \frac{\eta^2}{r^5} + 3(n-1) \frac{\eta^3}{r^7} \geq \frac{r}{n-1} \left[ \varepsilon - \mu |8-n|(n-1) - 2(n-1)\mu^2 \right] + O(1)
\]

\[
\geq c
\]

for small \( \mu \) and large \( r_4 \). We conclude that \( \eta \) converges to \((n - 4)(n - 1)\). \( \square \)

2.1. Existence of convex, rotationally symmetric translating solutions.

We mention, and slightly extend, a result of Altschuler and Wu:

**Lemma 2.2.** There exists an entire rotationally symmetric, strictly convex graphical solution to mean curvature flow, \( U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}, n \geq 2 \), translating with speed 1. We have the following asymptotic expansion as \( r \) approaches infinity:

\[
U(r,t) = t + \frac{r^2}{2(n-1)} - \ln r + O(r^{-1}).
\]

**Proof.** The existence of such solutions was shown by Altschuler and Wu in [1] for \( n = 2 \); their argument is in fact valid for \( n \geq 2 \). They also showed that such solutions (which we denote by \( U \)) are asymptotic to the paraboloid \( r^2/2(n-1) \). The finer asymptotic behaviour of \( U \) is a direct consequence of Lemma 2.1. \( \square \)

2.2. Existence of “winglike” translating solutions. We will prove that non-convex translating solutions, made up of the union of two graphs as shown in Figure 2, exist.

**Lemma 2.3.** For every \( R > 0 \), there exist rotationally symmetric graphical solutions to mean curvature flow, \( W^+_R, W^-_R : \mathbb{R}^n \setminus B_R \times [0, \infty) \to \mathbb{R}, n \geq 2 \), translating
with speed 1. We have the following asymptotic expansion as $r$ approaches infinity:

$$W^\pm_R(r, t) = t + \frac{r^2}{2(n-1)} - \ln r + O(r^{-1}) + C^\pm.$$ 

Moreover, the union of these graphs forms a complete non-convex translating solution to mean curvature flow.

**Proof.** Consider a hypersurface that is invariant under rotations around the $e_{n+1}$-axis. Assume that this hypersurface translates with velocity 1 in direction $e_{n+1}$. At points where the tangent plane is not orthogonal to $e_{n+1}$, we can locally represent it as a graph

$$h(x^{n+1}, t) \cdot S^{n-1} \times \{x^{n+1}\} \subset \mathbb{R}^n \times \mathbb{R}.$$ 

Observe that $h(x^{n+1}, t) = h(x^{n+1} - t, 0)$. We make the local ansatz

$$x^n = \left( h^2(x^{n+1} - t, 0) - \sum_{j=1}^{n-1} (x^j)^2 \right)^{\frac{1}{2}}$$

and, after a tedious calculation, obtain the following ordinary differential equation for $h(\cdot, t)$ at any fixed time $t$

$$\left( \frac{n-1}{h} - h' \right) \left( 1 + h'^2 \right) = h'',$$

where $'$ denotes differentiation with respect to $x^{n+1}$. For the rest of the proof we will suppress the $t$-argument.

Fix $y_0 \in \mathbb{R}$; a different choice of $y_0$ corresponds to translating the hypersurface in the $e_{n+1}$-direction. Starting with $h'(y_0) = 0$, $h(y_0) = R$, we obtain a strictly convex solution $h$ in a small interval around $y_0$. Returning to our original coordinate system and using Lemma 2.1, we can extend both branches all the way to infinity.

We denote the upper branch by $W^+_R$ and the lower branch by $W^-_R$. (This unintuitive naming will make sense when we use $W^+_R$ as an upper barrier.) We reintroduce...
the time dependence according to
\[ W^\pm_R(x, t) := W^\pm_R(x) + t. \]
The asymptotic behavior follows from Lemma 2.1.

3. Existence of a Solution

**Theorem 3.1.** Let \( u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuous function with \( |u_0 - U(x, 0)| \leq d \). Then there exists a solution \( u \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap C^0(\mathbb{R}^n \times [0, \infty)) \) to graphical mean curvature flow (MCF) with initial data \( u_0 \). Furthermore
\[
\|u(\cdot, t) - U(\cdot, t)\|_{C^0}
\]
is uniformly bounded for all \( t \in [0, \infty) \). All derivatives of \( u - U \) are bounded locally uniformly in \( x \), uniformly in \( t \in [\varepsilon, \infty) \).

If \( u_0 \) is more regular, we can apply Theorem A.1 to obtain a uniqueness result.

**Proof.** We will need a local oscillation bound in order to prove uniform \( C^k \) a priori estimates. Therefore, we use Theorem B.1 in such a way that we get the oscillation bound already for the approximating solutions.

Our strategy is similar to [8]. Our initial data \( u_0 \) is at finite vertical distance from the convex rotationally symmetric translating solution \( U(\cdot, 0) \). Let \( \underline{U}(x, t) := U(x, t) - d - 2 \) and \( \overline{U}(x, t) := U(x, t) + d + 1 \). Let \( R > 0 \). Define a continuous function \( u^R_0 : \overline{B}_R \rightarrow \mathbb{R} \) such that \( u^R_0 = u_0 \) in \( B_{R/2} \) and \( u^R_0(x) = U(x, 0) \) for all \( x \in \partial B_R \) and \( |u^R_0(x) - U(x, 0)| \leq d \) for all \( x \in B_R \). Theorem B.1 yields that there exists a solution \( u^R \) to graphical mean curvature flow with \( u^R = u^R_0 \) on \( B_R \times \{0\} \) and \( u^R = u^R_0 \) on \( \partial B_R \times [0, 1] \). According to the definition of \( \underline{U} \) and \( \overline{U} \) we obtain for \( (x, t) \in \partial B_R \times [0, 1] \)
\[
\underline{U}(x, t) < u^R(x, t) < \overline{U}(x, t).
\]
We apply the standard maximum principle and deduce that
\[
\underline{U}(x, t) < u^R(x, t) < \overline{U}(x, t).
\]
Note that upper and lower bounds do not depend on \( R \), and so this gives us a locally uniform in \( x \), uniform in \( t \in [0, 1] \), oscillation bound. Applying Theorem B.2 provides locally uniform in \( x \), uniform in \( t \in [\varepsilon, 1] \), \( C^\alpha \)-bounds on \( u^R \):
\[
|D^n u^R(x, t)| \leq C(d, \varepsilon, n, \alpha)
\]
for \( |x| < R/2 \). We let \( R \rightarrow \infty \) and apply Arzelà-Ascoli to find a solution \( u : \mathbb{R}^n \times (0, 1] \rightarrow \mathbb{R} \).

The initial data is attained for \( t \downarrow 0 \) via the local argument used in [2], which survives the limiting process.

A smooth solution for all positive times is obtained by iterating the above process. By the interior estimates of [8], this solution is smooth for positive times. We call it \( u \). It is not immediate that this solution has uniform bounds on derivatives as \( t \rightarrow \infty \).

We may now apply the comparison principle, Theorem A.1, and deduce that
\[
\underline{U}(x, t) \leq u(x, t) \leq \overline{U}(x, t) \quad \text{for all } (x, t) \in \mathbb{R}^n \times [0, \infty),
\]
so the oscillation of \( u(x, t) - U(x, t) \) is uniformly bounded in \( x \) and \( t \in [0, \infty) \). As above, this implies that we have bounds on \( u(x, t) - U(x, t) \) and all derivatives, locally uniformly in \( x \), uniformly in \( t \in [\varepsilon, \infty) \). \( \square \)
Remark 3.4. Our stability result does not require that we have a solution as constructed in Theorem 3.1. Suppose we have a $C^{2,1}(\mathbb{R}^n \times (0, \infty)) \cap C^0(\mathbb{R}^n \times [0, \infty))$-solution to graphical mean curvature flow (MCF) with $u(x, 0) = u_0(x)$ for $x \in \mathbb{R}^n$ and $u_0$ as in Theorem 1.1. Then the following lemma (and a similar estimate from below) ensure that the given solution grows at most quadratically on bounded time intervals. Thus we can apply the maximum principle, Theorem A.1 and the rest of the argument works also for these solutions.

Lemma 3.3. Let $u(x, t)$ define an entire graph moving under mean curvature flow that initially lies beneath a parabola $C|x|^2$, $C > 0$. Then $u(x, t) \leq C|x|^2 + 2Cn\tau$ for $0 \leq t \leq \tau$.

Proof. Fix a point $x$ with $|x| = r$, and a time $\tau$. The sphere with radius $R$ satisfying $R^2 = 2n\tau + r^2 + (2C)^{-2}$ centred over the origin at height $h = (2C)^{-1} + C(2n\tau + r^2)$ lies above our initial graph $u(\cdot, 0)$. Under mean curvature flow, the centre of the sphere remains fixed and the radius at time $t$ is $\sqrt{R^2 - 2n\tau}$. The sphere remains above graph $u$ for the duration of its existence, up until $T = R^2/(2n) = \tau + (2n)^{-1}(r^2 + (2C)^{-2})$. In particular it still exists at time $\tau$, at which time its height at radius $r$ is given by $Cr^2 + 2Cn\tau$. □

Remark 3.4. In the above proof, we cannot directly apply the comparison principle Theorem A.1 as we do not have the growth bounds required. On the other hand, we can compare with spheres as they are compact.

4. Convergence to a translating solution

We first see that our wings act as barriers for $u$.

Lemma 4.1. Let $\varepsilon > 0$, $R > 0$. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that
$$|u_0(x) - U(x, 0)| \leq \varepsilon \quad \text{for } |x| > R.$$ Let $u$ be a solution as in Theorem 3.1. Let $W_R^+$ and $W_R^-$ be “$\varepsilon$-shifted half-wing solutions” as in Section 2.2. More precisely, we assume that $|\nabla W_R^+|$ and $|\nabla W_R^-|$ become unbounded for $|x| \downarrow R$ and
$$\lim_{|x| \to \infty} W_R^+(x, 0) - U(x, 0) = \pm \varepsilon.$$ Then we obtain that
$$W_R^-(x, t) \leq u(x, t) \leq W_R^+(x, t) \quad \text{for } |x| > 2R$$ is preserved for all times.

Proof. We will only prove the upper bound $u(x, t) \leq W_R^+(x, t)$. The lower bound is obtained similarly. In order to use Corollary A.2, we need to show that
$$u(x, t) < W_R^+(x, t) \quad \text{for } |x| = R.$$ For small times, we may consider small spheres centred at $(x, W_R^+(x, 0))$ for $|x| = R$. These serve as barriers and show that (4.1) is preserved for a small time interval. We also obtain that
$$u(x, t) < W_R^+(x, t) \quad \text{for } |x| = R + \delta,$$ as long as $R > \delta > 0$ is chosen small enough. This inequality holds up to some small time $t_0 > 0$. 


According to our a priori estimates, we get uniform estimates on $Du(x,t)$ for $(x,t) \in B_{2R}(0) \times [t_0, \infty)$. Assume that $\delta$ is so small that we have $(DW^+_R(x,t),x) < 0$ and
\[
|DW^+_R(x,t)| > 2|Du(x,t)| \quad \text{for } (x,t) \in \partial B_{R+\delta}(0) \times [t_0, \infty).
\]
Our result follows if $u(x,t) < W^+_R(x,t)$ for $|x| = R + \delta$ and all times $t \geq 0$. This is not the case, there is a first time $t_1 > t_0 > 0$ such that $u(x_1,t_1) = W^+_R(x_1,t_1)$ for some $|x_1| = R + \delta$. As $|Du(x_1,t_1)| < |DW^+_R(x_1,t_1)|$ and $(DW^+_R(x_1,t_1),x) < 0$, we deduce that there is some $x_2$ with $|x_2| > R + \delta$ such that
\[
u(x_2,t_1) > W^+_R(x_2,t_1).
\]
As $t_1 > 0$ is minimal with that property, there exists $0 < t_2 < t_1$ such that $u(x,t_2) < W^+_R(x,t_2)$ on $\{|x| = R + \delta\}$ and $u(x_2,t_2) > W^+_R(x_2,t_2)$. This contradicts Corollary A.2. Therefore (4.1) is established, and with the application once again of Corollary A.2, our claim follows.

Thus, we obtain that $u$ and $U$ are close to each other at infinity. Next, we show that $u$ becomes everywhere close to $U$ for large times.

**Lemma 4.2.** Let $\varepsilon > 0$, $R > 0$. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that
\[
|u_0(x) - U(x,0)| \leq \varepsilon \quad \text{for } |x| > R.
\]
Let $u$ be a solution as in Theorem 3.1. Then
\[
|u(x,t) - U(x,t)| \leq 2\varepsilon
\]
for $t$ sufficiently large.

**Proof.** Let $W^+_R$ and $W^-_R$ be as in Lemma 4.1. Then
\[
W^-_R(x,t) \leq u(x,t) \leq W^+_R(x,t)
\]
for $|x| > 2R$. We only show that $u(x,t) \leq U(x,t) + 2\varepsilon$ for $t$ sufficiently large; the lower bound is obtained similarly. Define
\[
\Omega_t := \{x \in \mathbb{R}^n : u(x,t) - U(x,t) > 2\varepsilon\}.
\]
Note that $\Omega_t$ is uniformly bounded in $t$. We claim that $\Omega_t = \emptyset$ if $t$ is sufficiently large. As $\Omega_t$ is precompact, $u(\cdot, t) - U(\cdot, t)$ attains its maximum somewhere in $\Omega_t$. As in [9, Theorem 17.1], the difference $w(x,t) := u(x,t) - U(x,t)$ fulfills a parabolic equation of the form $\dot{w} = a^{ij}w_{ij} + b^iw_i$. If $\Omega_t \neq \emptyset$, $w(\cdot, t)$ attains its maximum, after which time we use the strong maximum principle to deduce that this maximum is strictly decreasing in time. If $\Omega_t \neq \emptyset$ for all $t > 0$, we define $w_k(x,t) := w(x,t + t_k)$, $w_k(x,t) : \mathbb{R}^n \times [-t_k, \infty) \to \mathbb{R}$ for a sequence $t_k \to \infty$. Then a subsequence of $\{w_k\}$ converges locally uniformly in any $C^l$-norm to a function $w^\infty : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $w^\infty + U$ evolves according to (MCF). As $\sup_{\mathbb{R}^n} u(\cdot, t) - U(\cdot, t)$ is strictly decreasing to a positive constant (again by the strict maximum principle) and $\Omega_t \neq \emptyset$, we deduce that $\sup_{\mathbb{R}^n} w^\infty(\cdot, t)$ is time-independent and not smaller than $2\varepsilon$. For each time $t$, the supremum is attained somewhere. Thus the strong maximum principle implies that $w$ equals a constant which is not smaller than $2\varepsilon$, contradicting (4.2). We obtain that $u(\cdot, t) - U(\cdot, t) < 2\varepsilon$ for sufficiently large values of $t$. \qed
Combining the results of this section with the existence results of Section 3 gives the proof of Theorem 1.1.

Appendix A. Maximum principle via viscosity solutions

A special case of a comparison principle by Guy Barles, Samuel Biton, Mariane Bourgoing and Olivier Ley in [3] is as follows:

**Theorem A.1.** Let \( u_1, u_2 : \mathbb{R}^n \times [0, T] \to \mathbb{R} \) be viscosity solutions to graphical mean curvature flow (MCF) with at most polynomial growth, that is,

\[
\frac{u_i(x,t)}{1 + |x|^k} \to 0 \quad \text{as} \quad |x| \to \infty, \quad \text{uniformly in } t.
\]

Let either \( u_1(\cdot,0) \) or \( u_2(\cdot,0) \) be locally Lipschitz continuous and fulfill

\[
|Du_i(x,0)| \leq C(1 + |x|^{\nu}) \quad \text{for almost all } x \in \mathbb{R}^n,
\]

where \( \nu < \left( 1 + \sqrt{5} \right) / 2 \).

If \( u_1(x,0) \leq u_2(x,0) \) then \( u_1 \leq u_2 \) in \( \mathbb{R}^n \times [0, T] \).

By direct inspection of the proof in [3] we see that this result also holds if we replace \( \mathbb{R}^n \) by \( \mathbb{R}^n \setminus K \), \( K \) a compact set, if \( u_1(x,t) < u_2(x,t) \) for \( x \in \partial K \).

**Corollary A.2.** Let \( u_1, u_2 \) fulfill the assumptions of Theorem A.1 with \( \mathbb{R}^n \) replaced by \( \mathbb{R}^n \setminus K \) for some compact set \( K \). If

\[
u < \left( 1 + \sqrt{5} \right) / 2
\]

on \( \partial K \times [0, T] \) and

\[
|u_i(x,t)| \leq C(1 + |x|^{\nu+1})
\]

for all \( (x,t) \in (\mathbb{R}^n \setminus K) \times [0, T] \), where \( \nu < \left( 1 + \sqrt{5} \right) / 2 \), we obtain that

\[
u < \left( 1 + \sqrt{5} \right) / 2
\]

on \( (\mathbb{R}^n \setminus K) \times [0, T] \).

For positive times the solution \( u \) obtained in Theorem 3.1 is a classical solution to mean curvature flow (MCF). It is continuous up to \( t = 0 \). As the locally uniform limit of a sequence of classical (and hence viscosity) solutions \( \{u(x,t+1/i)\} \), \( u \) itself is a viscosity solution for \( t \in [0,T] \).

Appendix B. Interior estimates

We use the following special case of a result in [2]:

**Theorem B.1.** Let \( u_0 \) be a continuous function on \( \overline{B_R} \) which equals a constant \( b \) on \( \partial B_R \). Then there exists a unique \( u \in C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\overline{B_R} \times (0,\infty)) \cap C^0(\overline{B_R} \times [0,\infty)) \) satisfying graphical mean curvature flow for positive times with \( u(x,0) = u_0(x) \) for all \( x \in \overline{B_R} \) and \( u(x,t) = b \) for \( x \in \partial B_R \).

For graphical solutions to mean curvature flow with locally in \( x \) and \( t \) uniformly bounded oscillation, there is the following interior estimate, see [6–8]:
Theorem B.2. Let \( u \) be a smooth solution to graphical mean curvature flow in \( B_R(0) \times [0, T] \) with oscillation bound \( M \). Then we have
\[
|Du(0,t)| \leq c(R, M, t, n),
\]
for \( 0 < t < T \). We also have
\[
|D^\alpha u(0,t)| \leq c(R, M, t, n, \alpha),
\]
where \( 0 < t < T \) and \( \alpha \) is a multi-index denoting a combination of spatial and temporal derivatives.

**Appendix C. Stability of the Plane**

Our method extends directly to the solution \( U(x, t) \equiv 0 \), where \((x, t) \in \mathbb{R}^n \times \mathbb{R} \) and \( n \geq 3 \). An \( n \)-catenoid, see e.g. [12], is a minimal hypersurface with two ends asymptotic to two parallel hyperplanes. Shifted and rescaled appropriately, these \( n \)-catenoids act as barriers and imply stability of \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \), \( n \geq 3 \), under mean curvature flow.

**Theorem C.1.** Let \( u_0: \mathbb{R}^n \to \mathbb{R} \) be a continuous function decaying at infinity,
\[
\lim_{|x| \to \infty} u_0(x) = 0.
\]
Then there exists a solution \( u \in C^\infty((\mathbb{R}^n \times (0, \infty)) \cap C^0((\mathbb{R}^n \times [0, \infty])) \to (\text{MCF}) \) for positive times with \( u(\cdot, 0) = u_0 \). For \( t \to \infty \), \( u(x, t) \) converges uniformly to zero.

Amusingly, we use \( n \)-catenoids, which are unstable minimal hypersurfaces [4], in order to prove stability.

**Appendix D. Stability of Gradient Kähler-Ricci Solitons**

In [5], Albert Chau and the second author proved stability for gradient Kähler-Ricci flow solitons. These solitons are analogous to the translating solutions considered here.

The main theorem of that paper, in which a decay rate is imposed, may be extended. We can drop the decay condition (2) in [5, Theorem 1.2] in favour of the requirement that the solution \( u \) to Kähler-Ricci flow initially tends to zero at infinity.

It suffices to add a small positive constant \( \varepsilon \) to an upper barrier used in the proof there and to argue as in the proof of Theorem 1.1 above. In this way, we show that \( u \) is eventually smaller than \( 2\varepsilon \).

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