Stationary Points of O’Hara’s Knot Energies

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Abstract

In this article we study the regularity of stationary points of the knot energies $E^{(\alpha)}$ introduced by O’Hara in [14, 15, 16] in the range $\alpha \in (2, 3)$. In a first step we prove that $E^{(\alpha)}$ is $C^1$ on the set of all regular embedded curves belonging to $H^{(\alpha+1)/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and calculate its derivative. After that we use the structure of the Euler-Lagrange equation to study the regularity of stationary points of $E^{(\alpha)}$ plus a positive multiple of the length. We show that stationary points of finite energy are of class $C^\infty$ — so especially all local minimizers of $E^{(\alpha)}$ among curves with fixed length are smooth.

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1 Introduction

The motion of a knotted charged fiber within a viscous liquid served as model for the definition of so-called knot energies introduced by Fukuhara [11]. One hopes that it will reach a stationary point minimizing its electrostatic energy and that the resulting shape will help to determine its knot type. The general idea is that this procedure leads to a “nicer shape” for a given knot in the same knot class, i. e. a representative that is as little entangled as possible with preferably large distances between different strands.

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For a general definition and an outline of different knot energies we refer the reader to O’Hara [17]. Recent developments include the investigation of geometric curvature energies such as the integral Menger curvature, see Strzelecki, Śzmielew, and von der Mosel [21, 22], which also extends to surfaces [23], or tangent-point energies [24] whose domains can be characterized via Sobolev-Slobodeckij spaces [4]. Attraction phenomena may also be modeled by a corresponding “inverse knot energy”, see Alt et al. [2] for an example from mathematical biology discussing interaction between pairs of filaments via cross-linkers.

In this paper, we investigate stationary points of the most prominent family of knot energies $E^{(\alpha)}: C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to [0, \infty)$,

$$
\gamma \mapsto \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left( \frac{1}{\gamma(u + w) - \gamma(u)^\alpha} - \frac{1}{d_{1}(u + w, u)^\alpha} \right) |\gamma'(u + w)| |\gamma'(u)| \, du \, dw,
$$

where $\alpha \in (2, 3)$, which goes back to O’Hara [14, 15, 16]. Here $d_{1}(u + w, u)$ denotes the intrinsic distance between $\gamma(u + w)$ and $\gamma(u)$ on the curve $\gamma$. More precisely, $d_{1}(u + w, u) := \min(\mathcal{L}(\gamma|_{[u,u+w]}), \mathcal{L}(\gamma) - \mathcal{L}(\gamma|_{[u,u+w]}))$ provided $|u| \leq \frac{1}{2}$ where $\mathcal{L}(\gamma) := \int_{0}^{1} |\gamma'(\theta)| \, d\theta$ is the length of $\gamma$.

The energy $E^{(2)}$ was thoroughly studied by Freedman, He, and Wang [10] who coined the name “Möbius energy” due to the Möbius invariance of this energy. While the existence of minimizers of the Möbius energy is ensured in prime knot classes only, O’Hara [15, 16] proved the existence of minimizers within any knot class if $\alpha \in (2, 3)$. Abrams et al. [1] proved that circles are the global minimizers of all these energies among all curves.

As to the regularity of stationary points, the first result was obtained by He [13] for $\alpha = 2$ who initially assuming $H^{2,3}$-regularity obtained $C^\infty$ by a bootstrapping argument. Together with a purely geometric result by Freedman, He, and Wang [10] heavily relying on the Möbius invariance, this gives $C^\infty$-regularity for all local $E^{(2)}$-minimizers (which exist at least in prime knot classes). An outline is given in [18]. Moreover, He was able to show that under suitable conditions any planar (i.e. $n = 2$) stationary point of $E^{(2)}$ is a circle [13, Thm. 6.3]. The argument highly relies on the Möbius invariance of $E^{(2)}$.

In [19], parts of these results were carried over to the energies $E^{(\alpha)}$ for $\alpha \in (2, 3)$. It was shown that stationary points in $H^{0,2} \cap H^{2,3}$ of the energy $E^{(\alpha)} + \lambda \mathcal{L}$, where $\mathcal{L}$ denotes the length functional and $\lambda > 0$ is a constant, are smooth. Here $H^{s,p}$ denote the Bessel potential spaces. Unfortunately, one does not know whether local minimizers of $E^{(\alpha)}$ belong to $H^{0,2} \cap H^{2,3}$ since the techniques used by Freedman, He, and Wang [10] to show the regularity of local minimizers completely break down in these cases.

In this article we will close this gap by proving a much stronger result. We will extend the results in [19] and [16] and prove smoothness of stationary points of the functionals $E^{(\alpha)} + \lambda \mathcal{L}$ under very natural conditions: We will only assume that the curve $\gamma$ we are looking at is parametrized by arc-length (which means no loss of generality as $E^{(\alpha)}$ is invariant of parametrization) and satisfies $E^{(\alpha)}(\gamma) < \infty$.

The first step to show this result is to extract as much information regarding the regularity of $\gamma$ out of the finiteness of the energy $E^{(\alpha)}$ as possible. After some partial result [7] in this direction, in [3] a classification of all curves with finite energy was given: An embedded curve parametrized by arc-length has finite energy $E^{(\alpha)}$ if and only if it belongs to the fractional Sobolev space $H^{(\alpha+1)/2,2}$.
Since formulas for the first variation of $E^{(\alpha)}$ are only known under the assumption that $\gamma \in H^2$, we then have to extend these to injective curves in $H^{(1+\alpha)/2}$ parameterized by arc-length. In fact our method even allows us to show that $E^{(\alpha)}$ is continuously differentiable on this space. To state the result, let

$$U_\varepsilon := \mathbb{R}/\mathbb{Z} \times \left\{ [1/2, \varepsilon] \cup [\varepsilon, 1/2] \right\}.$$  

**Theorem 1.1.** Let $\alpha \in (2, 3)$. The energies $E^{(\alpha)}$ are $C^1$-differentiable on the space of all injective regular curves $\gamma \in H^{(1+\alpha)/2}$. Furthermore, if $\gamma$ is parametrized by arc-length, the derivative at $\gamma$ in direction $h$ is given by

$$\delta E^{(\alpha)}(\gamma; h) = \lim_{\varepsilon \searrow 0} \int_{U_\varepsilon} \left( (\alpha - 2) \frac{\langle \gamma'(u), h'(u) \rangle}{|u|^\alpha} + 2 \frac{\langle \gamma'(u), h'(u) \rangle}{|\gamma(u + w) - \gamma(u)|^{\alpha/2}} - \alpha \frac{\langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle}{|\gamma(u + w) - \gamma(u)|^{\alpha + 2}} \right) \, dw \, du.$$  

Note that this is a principle value integral, i.e. we may not replace $U_\varepsilon$ by $U_0$.

Now we are in the position to state the main result of this article.

**Theorem 1.2.** Let $\alpha \in (2, 3)$ and $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a curve parametrized by arc-length with $E^{(\alpha)}(\gamma) < \infty$. If $\gamma$ is furthermore a stationary point of $E^{(\alpha)} + \lambda \mathcal{L}$, i.e. if

$$\delta E^{(\alpha)}(\gamma; h) + \lambda \int_{\mathbb{R}/\mathbb{Z}} \langle \gamma', h' \rangle = 0 \quad \forall h \in H^{(1+\alpha)/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n),$$

then $\gamma \in C^{0}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

The gradient flow of the Möbius energy $E^{(2)}$ was first discussed by He [13] where he states short time existence results for smooth initial data. In [6], the short time existence was proven for all initial data in $C^{2,\beta}$, $\beta > 0$, and first long time existence results for this gradient flow near local minimizers were derived. For a discussion of gradient flow for $E^{(\alpha)} + \lambda \mathcal{L}$ for positive $\lambda$ and $\alpha \in (2, 3)$ or the gradient flow of $E^{(\alpha)}$ with respect to fixed length we refer the reader to [5].

The energies $E^{(\alpha)}$ represent only the one-parameter range $p = 1$ of the larger family of knot energies

$$E^{\alpha,p}(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{1/2}^{1/2} \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^\alpha} - \frac{1}{d_j(u + w, u)^\alpha} \right)^p \left| \gamma'(u + w) \right| \left| \gamma'(u) \right| \, dw \, du,$$

where $\alpha p \geq 2$ and $(\alpha - 2)p < 1$, see O’Hara [15, 16] and [3, 7]. We do not expect that our results or the results for the gradient flow of the energies carry over to $p > 1$ as we expect the first variation to be a degenerate elliptic operator in this case.

Let us close this introduction by briefly introducing some notation and the Sobolev spaces of fractional order which are also referred to as Bessel potential spaces. For $s \in \mathbb{R}$ and $p \in [1, \infty]$ let $H^{s,p} := (\text{id} - \Delta)^{-s/2}L^p$ where $\Delta$ denotes the Laplacian. There are several equivalent definitions, e.g. by interpolation. In case $p = 2$, which mainly applies to our situation, the Bessel potential spaces coincide with the Sobolev spaces. This gives rise to the following fundamental characterization of $H^{s,2}$, $s \in (0, \infty) \setminus \mathbb{N}$.

$^{1}$If $\gamma$ belongs to $H^{s+1}$ one can use partial integration to obtain a formula for the $L^2$ gradient like in [19, Thm. 2.24].
Let $f \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. For $s \in (0, 1)$ we define the seminorm

$$[f]_{H^s} := \left( \int_{\mathbb{R}/\mathbb{Z}} \left( \int_{-1/2}^{1/2} \frac{|f(u + w) - f(u)|^2}{|w|^{1+2s}} \, dw \right)^{1/2} \, du \right).$$

Then the Sobolev space $H^{k+s,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $k \in \mathbb{N} \cup \{0\}$, is the set of all functions $H^{k+s,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for which the norm

$$\|f\|_{H^{k+s,2}} := \|f\|_{H^s} + \left[ f^{(k)} \right]_{H^s}$$

is finite. We will frequently use the embedding

$$H^{k+s,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \hookrightarrow C^{k+1/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \quad s \in \left( \frac{1}{2}, 1 \right), \quad (1.1)$$

see, e.g., Taylor [25, Chap. 4, Prop. 1.5]. For further information on Sobolev spaces we refer to the books by Grafakos [12, Chap. 6], Runst and Sickel [20, Chap. 2], and Taylor [25, Chap. 4 and 13].

For some space $X \subset C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ we will denote by $X_{inj}$ the (open) subspace consisting of all injective (embedded) and regular curves in $X$.

The standard scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, for complex vectors $a, b \in \mathbb{C}^n$ we define $(a, b)_{C^2} := \sum_{k=1}^d a_k b_k$. The $L^2$-scalar product is, as usual, given by $\langle f, g \rangle_{L^2} := \int_0^1 \langle f(u), g(u) \rangle \, du$.

Unless stated otherwise, we will assume

$$\alpha \in (2, 3)$$

throughout this paper.

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## 2 Continuous differentiability

In this section, we want to prove the following proposition from which Theorem 1.1 will follow quite easily. Recall that $U_\varepsilon = (\mathbb{R}/\mathbb{Z}) \times [-1/2, -\varepsilon] \cup [\varepsilon, 1/2]$.

**Proposition 2.1.** For $\alpha \in (2, 3)$ the energies $E^{(\alpha)}$ are continuously differentiable on $H^{a+1/2,2}_u$. The derivative of $E^{(\alpha)}$ at $\gamma \in H^{a+1/2,2}_u$ in direction $h \in H^{a+1/2,2}_u$ is given by

$$\delta E^{(\alpha)}(\gamma; h) =$$

$$\lim_{\varepsilon \searrow 0} \int_{U_\varepsilon} \left\{ 2 \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^2} - \frac{1}{d_\gamma(u + w, u)^2} \right) \left( \frac{\gamma'(u)}{\gamma'(u)} - h'(u) \right) \right.$$

$$- \alpha \left( \frac{\gamma(u + w) - \gamma(u), h(u + w) - h(u)}{|\gamma(u + w) - \gamma(u)|^{2+2}} - \frac{\int_{|w|<\varepsilon} d_\gamma(u + w, u)^{2+1}}{|\gamma(u + w)||\gamma'(u)|} \right) \left. \right\} \, dw \, du. \quad (2.1)$$
Note that since \( \gamma \in C^1 \) the derivative \( \frac{d}{dt}|_{t=0} d_{t,\epsilon} \gamma(u + w, u) \) is well defined for almost all \((u, w) \in \mathbb{R}/\mathbb{Z} \times [-1/2, 1/2]\). More precisely, we can deduce from
\[
d_{\gamma}(u + w, u) = \min \{ \mathcal{L}(\gamma|_{[u,u+w]}), \mathcal{L}(\gamma) - \mathcal{L}(\gamma|_{[u,u+w]}) \}
\]
that
\[
\frac{d}{dt}|_{t=0} d_{t,\epsilon} \gamma(u + w, u) = \begin{cases} 
|u| \int_0^1 \frac{\gamma(u + w) - \gamma(u)}{|\gamma(u) + \eta(w)|^2} \gamma'(u + \eta(w)) \, d\eta, & \text{if } \mathcal{L}(\gamma|_{[u,u+w]}) < \frac{1}{2}\mathcal{L}(\gamma), \\
-|u| \int_0^1 \frac{\gamma(u + w) - \gamma(u)}{|\gamma(u) + \eta(w)|^2} \gamma'(u + \eta(w)) \, d\eta, & \text{if } \mathcal{L}(\gamma|_{[u,u+w]}) > \frac{1}{2}\mathcal{L}(\gamma).
\end{cases}
\] (2.2)

To prove Proposition 2.1, we will first show that the following approximations of the energy \( E_{\epsilon}^{(\alpha)} \), in which we cut off the singular part, are continuously differentiable and give a formula for the derivative. For \( \epsilon > 0 \) we set
\[
E_{\epsilon}^{(\alpha)}(\gamma) := \iint_{U_{\epsilon}} \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^2} - \frac{1}{d_{\gamma}(u + w, u)^2} \right) |\gamma'(u + w)| \, dw \, du.
\]

To be more precise, we will show that \( E_{\epsilon}^{(\alpha)} \) is \( C^1 \) on the space of all embedded regular curves of class \( C^1 \), which due to the embedding (1.1) especially implies the continuous differentiability on \( H_{\alpha}^{(\alpha+1/2,2)} \).

The general strategy of the proof will be fairly standard. We first derive a formula for the pointwise variation of the integrand in the definition of \( E_{\epsilon}^{(\alpha)} \) and \( E^{(\alpha)} \) which holds almost everywhere. After that we will carefully analyse this formula in order to prove that the integrand defines a continuously differentiable map from \( C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) to \( L^1(U_{\epsilon}) \). This allows us to deduce that \( E_{\epsilon}^{(\alpha)} \) is continuously differentiable.

**Lemma 2.2.** The functional \( E_{\epsilon}^{(\alpha)} \) is continuously differentiable on the space of all injective regular curves in \( C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \). The directional derivative at \( \gamma \) in direction \( h \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) is given by
\[
\delta E_{\epsilon}^{(\alpha)}(\gamma; h) = \iint_{U_{\epsilon}} \left\{ \frac{1}{|\gamma(u + w) - \gamma(u)|^2} - \frac{1}{d_{\gamma}(u + w, u)^2} \right\} \frac{\gamma'(u)}{|\gamma'(u)|^2} \frac{h'(u)}{|h'(u)|^2} \, dw \, du
\]
\[
- \alpha \left\{ \frac{|\gamma(u + w) - \gamma(u), h(u + w) - h(u)|}{|\gamma(u + w) - \gamma(u)|^2 |h(u)|^2} - \frac{\frac{d}{dt}|_{t=0} d_{t,\epsilon} \gamma(u + w, u)}{d_{\gamma}(u + w, u)^{\alpha+1}} \right\} |\gamma'(u + w)||\gamma'(u)| \, dw \, du.
\] (2.3)

**Proof.** Let \( \gamma_0 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) be injective and regular and \( U \subset C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) be an open neighbourhood of \( \gamma_0 \) such that there is a constant \( c > 0 \) with
\[
\min\{||\gamma(u + w) - \gamma(u), d_{\gamma}(u + w, u)|| \geq c \, |w|, \quad |\gamma'(u)| \geq c\}
\] (2.4)

for all \( \gamma \in U \) and \((u, w) \in \mathbb{R}/\mathbb{Z} \times [-1/2, 1/2]\).

We will show that the integrand used to define the energies \( E^{(\alpha)} \) and \( E_{\epsilon}^{(\alpha)} \), i. e.

\[
(F\gamma)(u, w) := \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^2} - \frac{1}{d_{\gamma}(u + w, u)^2} \right) |\gamma'(u + w)||\gamma'(u)|,
\]

is continuously differentiable on the space of all injective regular curves in \( C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \). The directional derivative at \( \gamma \) in direction \( h \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) is given by
\[
\delta(F\gamma)(\gamma; h) = \iint_{U_{\epsilon}} \left\{ \frac{1}{|\gamma(u + w) - \gamma(u)|^2} - \frac{1}{d_{\gamma}(u + w, u)^2} \right\} \frac{\gamma'(u)}{|\gamma'(u)|^2} \frac{h'(u)}{|h'(u)|^2} \, dw \, du
\]
\[
- \alpha \left\{ \frac{|\gamma(u + w) - \gamma(u), h(u + w) - h(u)|}{|\gamma(u + w) - \gamma(u)|^2 |h(u)|^2} - \frac{\frac{d}{dt}|_{t=0} d_{t,\epsilon} \gamma(u + w, u)}{d_{\gamma}(u + w, u)^{\alpha+1}} \right\} |\gamma'(u + w)||\gamma'(u)| \, dw \, du.
\] (2.3)
defines a continuously differentiable operator from $U$ into $L^1(U_c)$ for any $\varepsilon > 0$ with directional derivative

$$
\frac{d}{d\tau} \left. (I(\gamma + \tau h))(u, w) \right|_{\tau=0} = \frac{1}{|\gamma(u + w) - \gamma(u)|^\varepsilon} - \frac{1}{d_s(u + w, u)^\varepsilon} \left| \frac{\gamma'(u + w)}{|\gamma'(u + w)|} \right| h'(u + w) \left| \gamma'(u + w) \right| - \alpha \left( \frac{(\gamma(u + w) - \gamma(u), h(u + w) - h(u))}{|\gamma(u + w) - \gamma(u)|^{\varepsilon + 2}} - \frac{\frac{d}{d\tau}}{d_s(u + w, u)^{\varepsilon + 1}} \right) \left| \gamma'(u + w) \right| \left| \gamma'(u) \right|.
$$

The statement then follows from the chain rule and the fact that the operator

$$
L^1(U_c) \rightarrow \mathbb{R},
\quad g \mapsto \int_{U_c} g(u, w) \, du \, dw,
$$

is continuously differentiable as it is a bounded linear operator.

The only non-trivial thing here is to deal with the intrinsic distance $d_s$ in the integrand that defines $E^{(\varepsilon)}_s$. Obviously $d_s(u, w)$ defines a continuous operator from $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ to $L^\infty(\mathbb{R}/\mathbb{Z} \times [-1/2, 1/2])$.

Using the fact that one has

$$
d_s(u + w, u) = \min \left\{ \mathcal{L}(\gamma|_{[a,b+u]}), \mathcal{L}(\gamma) - \mathcal{L}(\gamma|_{[a,b+u]}) \right\}
$$

and that $\gamma$ is regular, one can see that

$$
\left. \frac{d}{d\tau} \right|_{\tau=0} d_{s+\tau}(u + w, u) = D(\gamma, h)(u, w)
$$

for all $u, w$ with $\mathcal{L}(\gamma|_{[a,b+u]}) \neq \frac{1}{2} \mathcal{L}(\gamma)$ where

$$
D(\gamma; h)(u, w) := \begin{cases} 
[w]_0 \int_0^1 \left( \frac{\gamma(u + crw)}{|\gamma(u + crw)|}, h'(u + crw) \right) \, dr, & \text{if } \mathcal{L}(\gamma|_{[a,b+u]}) < \frac{1}{2} \mathcal{L}(\gamma), \\
-[w]_0 \int_0^1 \left( \frac{\gamma(u + crw)}{|\gamma(u + crw)|}, h'(u + crw) \right) \, dr, & \text{if } \mathcal{L}(\gamma|_{[a,b+u]}) \geq \frac{1}{2} \mathcal{L}(\gamma).
\end{cases}
$$

Since $\gamma$ is regular, the set $\mathcal{L}(\gamma|_{[a,b+u]}) = \frac{1}{2} \mathcal{L}(\gamma)$ is a compact $C^1$ submanifold of $\mathbb{R}/\mathbb{Z} \times [-1/2, 1/2]$ and hence a null set. Thus (2.6) and (2.5) hold almost everywhere.

Obviously, $D$ defines a continuous operator from the space $C^1_1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \times C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ to $L^1(U_c)$.

From Equation (2.5) we can read off that

$$
(DI(\gamma))(h) := \left. \frac{d}{d\tau} I(\gamma + \tau h)(u, w) \right|_{\tau=0}
$$

defines an operator $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \rightarrow L^1(U_c)$ that continuously depends on $\gamma$. Hence $I$ is a continuously differentiable operator from $U$ to $L^1(U_c)$.

Integrating and using a suitable reparametrization we then derive (2.3) from (2.5). \qed
Unfortunately, the energies $E^{(0)}_x$ do not form a Cauchy sequence in $C^1(H^{(a+1)/2, 2}_w)$ actually not even in $C^0(H^{(a+1)/2, 2}_w)$ basically due to the fact that bounded sequences in $L^1$ are not uniformly integrable. We will deduce Proposition 2.1 from Lemma 2.2 which roughly speaking shows that $E^{(0)}_x$ is nearly a Cauchy sequence in $C^1(X_0)$ for certain subsets $X_0 \subset H^{(a+1)/2, 2}$, $\delta \geq 0$. We will allow subsets $X_\delta \subset H^{(a+1)/2, 2}$ which satisfy the following substitute of the uniform integrability property

$$\limsup_{\epsilon \to 0} \sup_{\gamma \in X_\delta} \left( \int_{R/Z [-\epsilon, \epsilon]} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{|u|^{\alpha}}dwdu \right)^{1/2} \leq \delta. \quad (2.8)$$

Recall that $\text{lip}_\gamma E = \sup_{f, \neq \gamma} \frac{|E(f) - E(f)|}{\|f - \gamma\|}$ for some functional $E$ with $Y \subset \text{domain } E$.

**Lemma 2.3.** Let $\gamma_0 \in H^{(a+1)/2, 2}_w$. Then there is an open neighborhood $U \subset H^{(a+1)/2, 2}$ of $\gamma_0$ and a constant $C < \infty$, such that $E^{(0)}_x$ satisfies

$$\limsup_{\epsilon_i, \delta_i \to 0} \sup_{\gamma \in X_{\delta_i}} \text{lip}_\gamma (E^{(0)}_{\epsilon_i} - E^{(0)}_{\epsilon_i}) \leq C \delta \quad (2.9)$$

for all subsets $X_\delta \subset H^{(a+1)/2, 2}$ satisfying (2.8) with $\delta \in [0, 1]$.

For fixed $\gamma_0 \in H^{(a+1)/2, 2}_w$ and $h \in H^{(a+1)/2, 2}$ we will apply this lemma later to the sets

$$X_0 := \{ \gamma_0 + \tau h : \tau \in (-a, a) \}, \quad 0 < a < \infty,$$

and

$$X_\delta := \{ \gamma \in H^{(a+1)/2, 2}_w : \|\gamma - \gamma_0\|_{H^{(a+1)/2, 2}} \leq \delta \}, \quad \delta > 0.$$

Of course we have for $\gamma \gamma := \gamma + \tau h$, $|\tau| \leq a,$

$$\left( \int_{R/Z [-\epsilon, \epsilon]} \frac{|\gamma'_s(u+w) - \gamma'_s(u)|^2}{|u|^{\alpha}}dwdu \right)^{1/2} \leq \left( \int_{R/Z [-\epsilon, \epsilon]} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{|u|^{\alpha}}dwdu \right)^{1/2} + a \left( \int_{R/Z [-\epsilon, \epsilon]} \frac{|h'(u+w) - h'(u)|^2}{|u|^{\alpha}}dwdu \right)^{1/2} \to 0$$

as $\epsilon \to 0$ and for $\gamma \in X_\delta$ we have

$$\left( \int_{R/Z [-\epsilon, \epsilon]} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{|u|^{\alpha}}dwdu \right)^{1/2} \leq \left( \int_{R/Z [-\epsilon, \epsilon]} \frac{|\gamma'_0(u+w) - \gamma'_0(u)|^2}{|u|^{\alpha}}dwdu \right)^{1/2} + \delta \to \delta$$

so both satisfy (2.8).
Proof. Using that $H^{(\alpha+1)/2,2}$ continuously embeds into $C^1$ and $\gamma_0$ is a injective regular curve, we can find an open neighborhood $U \subset H^{(\alpha+1)/2,2}$ of $\gamma_0$ and a constant $c > 0$ such that (2.4) holds for all $\gamma \in U$ and $(u, w) \in \mathbb{R}/\mathbb{Z} \times [-1/2, 1/2]$. Making $U$ smaller if necessary, we can also achieve that there is an $\epsilon_0 > 0$ such that

$$d_f(u + w, u) = \mathcal{L}(\gamma|_{u+u})$$

for all $\gamma \in U$ and $w \in [-\epsilon_0, \epsilon_0]$. Let now $\epsilon_0 > \epsilon_2 > \epsilon_1$ and let us set

$$F^{(a)} := E^{(a)}_{\epsilon_2} - E^{(a)}_{\epsilon_1}.$$ 

We will now rewrite this difference in a more convenient form. For this let us introduce the function

$$g^{(a)}(\xi, \eta, \theta, t) := \frac{\xi^a - \eta^a}{\eta^2 - \xi^2}$$

which is Lipschitz continuous and positive on $[\tilde{c}, \infty)^4$ for any $\tilde{c} > 0$. We define for $u \in \mathbb{R}/\mathbb{Z}$, $w \in [-\epsilon, \epsilon]$

$$\mathcal{G}^{(a)}_f : (u, w) \mapsto g^{(a)}\left(\int_0^1 \gamma'(u + \theta u) \, d\theta_1 \cdot \int_0^1 |\gamma'(u + \theta w)| \, d\theta_2, |\gamma'(u + w)|, |\gamma'(u)|\right).$$

We have chosen $U$ in such a way that the arguments in $\mathcal{G}^{(a)}$ are uniformly bounded away from zero.

We decompose the integrand in the definition of $E^{(a)}$ for $|w| \leq \epsilon_0$ into

$$\left(\frac{1}{|\gamma(u + w) - \gamma(u)|^a} - \frac{1}{d_f(u + w, u)^a}\right) |\gamma'(u + w)| |\gamma'(u)|$$

$$= \frac{1}{|w|^a} \left( \int_0^1 |\gamma'(u + \theta_1 w)| \, d\theta_1 \right) - \frac{1}{|w|^a} \left( \int_0^1 |\gamma'(u + \theta_2 w)| \, d\theta_2 \right) |\gamma'(u + w)| |\gamma'(u)|$$

$$= \mathcal{G}^{(a)}_f(u, w) \left( \int_0^1 |\gamma'(u + \theta_1 w)| \, d\theta_1 \right)^2 \left( \int_0^1 |\gamma'(u + \theta_2 w)| \, d\theta_2 \right)^2 - \frac{1}{|w|^a} \int_0^1 |\gamma'(u + \theta_1 w)| \, d\theta_1 \int_0^1 |\gamma'(u + \theta_2 w)| \, d\theta_2.$$

Using $2|a - b| = 2(a, b) = |a - b|^2 - |a| - |b|^2$ for $a, b \in \mathbb{R}^n$ this can be written as

$$\frac{\mathcal{G}^{(a)}_f(u, w) \int_0^1 |\gamma'(u + \theta_1 w)| \, d\theta_1 \int_0^1 |\gamma'(u + \theta_2 w)| \, d\theta_2}{|w|^a}$$

$$- \frac{1}{4} \mathcal{G}^{(a)}_f(u, w) \int_0^1 |\gamma'(u + \theta_1 w)| \, d\theta_1 \int_0^1 |\gamma'(u + \theta_2 w)| \, d\theta_2.$$
Hence,

\[ F^{(\alpha)}(\gamma) = \frac{1}{2} \int_{\mathbb{R}/2} \int_{\mathbb{R}^2} \mathcal{G}^{(\alpha)}(u, w) \frac{\|\gamma'(u + \theta_1 w) - \gamma'(u + \theta_2 w)\|^2}{|u|^\alpha} \, d\theta_1 \, d\theta_2 \]

\[ - \frac{1}{2} \int_{\mathbb{R}/2} \int_{\mathbb{R}^2} \mathcal{G}^{(\gamma)}(u, w) \frac{\|\gamma'(u + \theta_1 w) - \gamma'(u + \theta_2 w)\|^2}{|w|^\alpha} \, d\theta_1 \, d\theta_2 \]

\[ =: \frac{1}{4} F^{(\alpha)}(\gamma) - \frac{1}{4} F^{(\gamma)}(\gamma). \]

To estimate the difference \( F^{(\alpha)}(\gamma) - F^{(\alpha)}(\gamma) \), we first consider

\[ \| \mathcal{G}^{(\alpha)}(u, w) - \mathcal{G}^{(\gamma)}(u, w) \| \]

\[ \leq C \left( \left\| \int_0^1 \gamma'(u + \theta_1 w) \, d\theta_1 - \int_0^1 \gamma'(u + \theta_2 w) \, d\theta_2 \right\| \right. \]

\[ + C \left( \left\| \int_0^1 (\|\gamma'(u + \theta w)\| - |\gamma'(u + \theta_1 w)|) \, d\theta \right\| \right. \]

\[ + C \left( |\gamma'(u + w) - |\gamma'(u + w)|) + C \left\| \gamma'(u) - |\gamma'(u)| \right\| \]

\[ \leq C \int_0^1 \|\gamma'(u + \theta w) - \gamma'(u + \theta w)\| \, d\theta + C \|\gamma'(u + w) - \gamma'(u + w)\| + C \left\| \gamma'(u) - \gamma'(u) \right\| \]

\[ \leq C \|\gamma' - \gamma\|_{L^1}. \]

We arrive at

\[ \| F^{(\alpha)}(\gamma) - F^{(\alpha)}(\gamma) \|

\[ \leq \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \mathcal{G}^{(\alpha)}(u, w) \frac{\|\gamma'(u + \theta_1 w) - \gamma'(u + \theta_2 w)\|^2}{|u|^\alpha} \, d\theta_1 \, d\theta_2 \]

\[ + \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \mathcal{G}^{(\gamma)}(u, w) \frac{\|\gamma'(u + \theta_1 w) - \gamma'(u + \theta_2 w)\|^2}{|w|^\alpha} \, d\theta_1 \, d\theta_2 \]

\[ \leq C \|\gamma' - \gamma\|_{L^1} \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \|\gamma'(u + \theta_1 w) - \gamma'(u + \theta_2 w)\|^2 \, d\theta_1 \, d\theta_2 \]

\[ + C \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \|\gamma' - \gamma\|_{L^1} \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \int_{\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]} \|\gamma' - \gamma\|_{L^1} \frac{\|\gamma'(u + \theta_1 w) - \gamma'(u + \theta_2 w)\|^2}{|w|^\alpha} \, d\theta_1 \, d\theta_2 \]

\[ \leq C \|\gamma'\|_{H^{(\alpha-1)/2}([\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]])} \|\gamma' - \gamma\|_{L^1} \]

\[ + C \|\gamma' + \gamma\|_{H^{(\alpha-1)/2}([\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]])} \|\gamma' - \gamma\|_{H^{(\alpha-1)/2}([\mathbb{R}/2 \setminus [-2\varepsilon, \varepsilon]])}. \]
where we set for a subset \( S \subset \mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}] \)

\[
\| f \|_{H^{(\alpha-1)/2} S} := \left( \int_S \frac{|f(u + w) - f(w)|^2}{|w|^\alpha} \, dw \, du \right)^{1/2}.
\]

For the second term we compute

\[
\left| F_2^{(\alpha)} (\gamma) - F_2^{(\alpha)} (\gamma') \right| \\
\leq \int_{\mathbb{R} / \mathbb{Z} \times [-\varepsilon, \varepsilon]} \left| \mathcal{G}_\gamma^{(\alpha)} (u, w) - \mathcal{G}_{\gamma'}^{(\alpha)} (u, w) \right| \frac{\int_{[0,1]} |(\gamma' (u + \theta_1 w) - \gamma' (u + \theta_2 w))|^2 \, d\theta_1 \, d\theta_2}{|w|^\alpha} \, dw \, du \\
+ \int_{\mathbb{R} / \mathbb{Z} \times [-\varepsilon, \varepsilon]} \left| \mathcal{G}_\gamma^{(\alpha)} (u, w) \right| \frac{\int_{[0,1]} |(\gamma' (u + \theta_1 w) - \gamma' (u + \theta_2 w))|^2 \, d\theta_1 \, d\theta_2}{|w|^\alpha} \, dw \, du
\]

\[
\leq C \| \gamma' - \gamma \|_{L^\infty} \int_{[0,1]} \left( \int_{\mathbb{R} / \mathbb{Z} \times [-\varepsilon, \varepsilon]} \frac{|(\gamma' (u + \theta_1 w) - \gamma' (u + \theta_2 w))|^2}{|w|^\alpha} \, dw \, du \, d\theta_1 \, d\theta_2 \right) \\
+ C \int_{[0,1]} \left( \int_{\mathbb{R} / \mathbb{Z} \times [-\varepsilon, \varepsilon]} \frac{|(\gamma' (u + \theta_1 w) - \gamma' (u + \theta_2 w))|^2}{|w|^\alpha} \, dw \, du \, d\theta_1 \, d\theta_2 \right)
\]

\[
\leq C \| \gamma' - \gamma \|_{L^\infty} \| \gamma' \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z} \times [-2\varepsilon, 2\varepsilon])}^2 \\
+ C \| \gamma' \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z} \times [-2\varepsilon, 2\varepsilon])} \| \gamma' - \gamma \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z} \times [-2\varepsilon, 2\varepsilon])}^2.
\]

Using the chain and product rule for Sobolev spaces and the formula

\[
|\gamma' - \gamma| = \frac{|\gamma' + \gamma' - \gamma|}{|\gamma'| + |\gamma'|},
\]

we get

\[
\| \gamma' - \gamma \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z} \times [-2\varepsilon, 2\varepsilon])} \leq \| \gamma' \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z})} \| \gamma' \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z})} \\
\leq C \| \gamma' - \gamma \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z}, \mathbb{R}^n)}
\]

and hence

\[
\| F_2^{(\alpha)} (\gamma) - F_2^{(\alpha)} (\gamma') \| \leq C \left( \| \gamma' \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z} \times [-2\varepsilon, 2\varepsilon])}^2 + \| \gamma' \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z} \times [-2\varepsilon, 2\varepsilon])} \right) \\
\| \gamma' - \gamma \|_{H^{(\alpha-1)/2} (\mathbb{R} / \mathbb{Z}, \mathbb{R}^n)}^2.
\]

From this the claim follows. \( \square \)

**Proof of Proposition 2.1.** From the classification of all embedded regular curves of finite energy in [3] we get \( E^{(\alpha)} (\gamma_0) < \infty \) for all \( \gamma_0 \in H^{(\alpha+1)/2} \). From this we deduce immediately that \( E^{(\alpha)}_\varepsilon \) converges to \( E^{(\alpha)} \) pointwise as \( \varepsilon \) tends to 0.
We begin by proving that directional derivatives exist for all directions \( h \in H^{(\alpha+1)/2,2} \). Let us fix \( \gamma_0 \in H^{(\alpha+1)/2,2}_u \) and let \( U \subset H^{(\alpha+1)/2,2}_u \) and \( C < \infty \) be as in Lemma 2.3. Applying first Lemma 2.3 with \( X_0 = \{ \gamma_0 + \tau h : \tau \in (-\tau_0, \tau_0) \} \) for \( \tau_0 \) small enough, we deduce for 

\[
f_{E}: \tau \mapsto E^{(\alpha)}_{\tau}(\gamma_0 + \tau h)
\]

that

\[
\left| f'_{\tau}(\tau) \right| = \left| \frac{f_{E}(\gamma_0 + \tau h; h) - f_{E}(\gamma_0; h)}{\tau} \right| \leq \limsup_{0 \to 0} \frac{E^{(\alpha)}_{E_2}(\gamma_0 + (\tau + \delta h) - E^{(\alpha)}_{E_1}(\gamma_0 + \tau h)} - E^{(\alpha)}_{E_2}(\gamma_0 + \tau h)}{\delta}
\]

\[
\leq \limsup_{\epsilon \to 0} \frac{E^{(\alpha)}_{E_1}(\gamma_0 + (\tau + \delta h) - E^{(\alpha)}_{E_1}(\gamma_0 + \tau h)} - E^{(\alpha)}_{E_2}(\gamma_0 + (\tau + \delta h) - E^{(\alpha)}_{E_2}(\gamma_0 + \tau h))}{\delta}
\]

\[
\leq \limsup_{\epsilon \to 0} \frac{\text{lip}_{U \cap X_0} (E^{(\alpha)}_{E_1} - E^{(\alpha)}_{E_2}) \| h \|_{H^{(\alpha+1)/2,2}}}{\delta h_{\| h \|_{H^{(\alpha+1)/2,2}}}}
\]

(2.12) 

As \( E^{(\alpha)}_{E_2} \to E^{(\alpha)} \) pointwise this proves that \( (f_{E})_{E \cap 0} \) is a Cauchy sequence in \( C^1((-\tau_0, \tau_0)) \) converging to \( E^{(\alpha)}(\gamma_0 + \tau h) = \lim_{E \cap 0} E^{(\alpha)}_{E_2}(\gamma_0 + \tau h) \) as \( \epsilon \to 0 \). Hence especially all directional derivatives of \( E^{(\alpha)} \) exist and

\[
\delta E^{(\alpha)}(\gamma_0; h) = \lim_{\epsilon \to 0} \delta E^{(\alpha)}_{E}(\gamma_0; h)
\]

for all \( \gamma_0 \in H^{(\alpha+1)/2,2}_u, h \in H^{(\alpha+1)/2,2}_u \).

The next step is to establish Gâteaux differentiability. To this end we merely have to show \( \delta E^{(\alpha)}(\gamma_0, \cdot) \in \left[H^{(\alpha+1)/2,2}_u\right]^* \) for \( \gamma_0 \in H^{(\alpha+1)/2,2}_u \). Linearity carries over from \( E^{(\alpha)}_{E} \).

For boundedness we choose \( \delta \in (0, 1] \) such that

\[
X_\delta := \{ \gamma \in H^{(\alpha+1)/2,2}_u : \| \gamma - \gamma_0 \| \leq \delta \} \subset U.
\]

Now

\[
\delta E^{(\alpha)}(\gamma_0; h) = \delta E^{(\alpha)}_{E}(\gamma_0; h) + \delta E^{(\alpha)}(\gamma_0; h) - \delta E^{(\alpha)}_{E}(\gamma_0; h) = \delta E^{(\alpha)}_{E}(\gamma_0; h) + \lim_{\epsilon \to 0} \delta E^{(\alpha)}_{E}(\gamma_0; h)
\]

and thus, arguing as in (2.12) and recalling \( \delta E^{(\alpha)}_{E}(\gamma_0; \cdot) \in \left[H^{(\alpha+1)/2,2}_u\right]^* \),

\[
|\delta E^{(\alpha)}(\gamma_0; h)| \leq \left| \delta E^{(\alpha)}_{E}(\gamma_0; h) \right| + \limsup_{\epsilon \to 0} \text{lip}_{U \cap X_\delta} (E^{(\alpha)}_{E_1} - E^{(\alpha)}_{E_2}) \| h \|_{H^{(\alpha+1)/2,2}}
\]

\[
\leq C \| h \|_{H^{(\alpha+1)/2,2}}
\]

for all \( \gamma_0 \in H^{(\alpha+1)/2,2}_u \) and \( h \in H^{(\alpha+1)/2,2}_u \). Hence, \( E^{(\alpha)} \) is Gâteaux differentiable and the differential \( \left(E^{(\alpha)} \right)'(\gamma_0) \in \left(H^{(\alpha+1)/2,2}_u\right)^* \) is given by

\[
\left(E^{(\alpha)} \right)'(\gamma_0) = \delta E^{(\alpha)}(\gamma_0, \cdot)
\]

for all \( \gamma_0 \in H^{(\alpha+1)/2,2}_u, h \in H^{(\alpha+1)/2,2}_u \).

Finally, to see that the differential is continuous, let \( \sigma > 0 \) be given and let us choose \( \delta > 0 \) and \( \epsilon > 0 \) so small that

\[
\text{lip}_{U \cap X_\delta} (E^{(\alpha)}_{E_1} - E^{(\alpha)}_{E_2}) \leq C \delta \leq \frac{1}{2} \epsilon
\]

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for all $\varepsilon_1, \varepsilon_2 < \varepsilon$. Then we have for $\gamma \in X_0 \cap U$ and any $h \in H^{(a+1)/2,2}$

$$|\delta E(\gamma; h) - \delta E(\gamma_0; h)| \leq |\delta E^0(\gamma; h) - \delta E^0_\varepsilon(\gamma; h) + |\delta E^0_\varepsilon(\gamma; h) - \delta E^0(\gamma_0; h)|$$

$$+ |\delta E^0_\varepsilon(\gamma_0; h) - \delta E^0(\gamma_0; h)|$$

(2.12) $\leq |\delta E^0_\varepsilon(\gamma; h) - \delta E^0(\gamma_0; h)| + \frac{2}{3} \varepsilon \|h\|_{H^{(a+1)/2,2}}.$

Since $E^0_\varepsilon$ is $C^1$ we deduce that there is an open neighborhood $V \subset X_0$ of $\gamma_0$ such that

$$|\delta E^0_\varepsilon(\gamma; h) - \delta E^0(\gamma_0; h)| \leq \frac{1}{3} \varepsilon \|h\|_{H^{(a+1)/2,2}}$$

and hence

$$|\delta E^0(\gamma; h) - \delta E^0(\gamma_0; h)| \leq \varepsilon \|h\|_{H^{(a+1)/2,2}}.$$

This proves that $(E^0(\gamma))'$ is continuous from $H^{(a+1)/2,2}$ into $(H^{(a+1)/2,2})^*$ and hence $E^0$ is $C^1(H^{(a+1)/2,2})$.

Proof of Theorem 1.1. The only thing left to do is to show that for curves $\gamma \in H^{(a+1)/2,2}$ parametrized by arc-length and $h \in H^{(a+1)/2,2}$ the derivative can be given in the form stated in the theorem. Using that $\gamma$ is parametrized by arc-length, we get from Proposition 2.1 and (2.2) that

$$\delta E^0(\gamma; h) \leftarrow_{\varepsilon \searrow 0} \int_{U_\varepsilon} \left[ 2 \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^p} - \frac{1}{|w|^p} \right) \langle \gamma'(u), h'(u) \rangle \right.$$}

$$- \alpha \left( \frac{\langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle}{|\gamma(u + w) - \gamma(u)|^{p+2}} - \frac{D(\gamma, h)(u, w)}{|w|^{p+1}} \right) \big] d|w|$$

where now

$$D(\gamma, h)(u, w) = |w| \int_0^1 \langle \gamma'(u + \theta w), h'(u + \theta w) \rangle d\theta$$

for all $(u, w) \in \mathbb{R}/\mathbb{Z} \times (-1/2, 1/2)$. Hence,

$$\delta E^0(\gamma; h) \leftarrow_{\varepsilon \searrow 0} \int_{U_\varepsilon} \left[ 2 \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^p} - \frac{1}{|w|^p} \right) \langle \gamma'(u), h'(u) \rangle \right.$$}

$$- \alpha \left( \frac{\langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle}{|\gamma(u + w) - \gamma(u)|^{p+2}} - \frac{D(\gamma, h)(u, w)}{|w|^{p+1}} \right) \big] d|w|$$

$$= \int_{U_\varepsilon} \left[ 2 \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^p} - \frac{1}{|w|^p} \right) \langle \gamma'(u), h'(u) \rangle \right.$$}

$$- \alpha \left( \frac{\langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle}{|\gamma(u + w) - \gamma(u)|^{p+2}} - \frac{D(\gamma, h)(u, w)}{|w|^{p+1}} \right) \big] d|w|$$

$$= \int_{U_\varepsilon} \left( \alpha - 2 \right) \frac{\langle \gamma'(u), h'(u) \rangle}{|w|^p} + 2 \frac{\langle \gamma'(u), h'(u) \rangle}{|\gamma(u + w) - \gamma(u)|^p}$$

$$- \alpha \left( \frac{\langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle}{|\gamma(u + w) - \gamma(u)|^{p+2}} - \frac{D(\gamma, h)(u, w)}{|w|^{p+1}} \right) \big] d|w|.$$
3 Regularity of stationary points

In this section we prove Theorem 1.2 so we are looking at embedded curves \( \gamma \in H^{1+\alpha/2} \) parametrized by arc-length that satisfy

\[
\delta E^{(\alpha)}(\gamma; h) + \lambda \int_{\mathbb{R}/\mathbb{Z}} \langle \gamma', h' \rangle = 0 \quad \forall h \in H^{(1+\alpha)/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)
\]

(3.1)

where \( \lambda > 0 \) and

\[
\delta E^{(\alpha)}(\gamma; h) = \lim_{\varepsilon \searrow 0} \int_{U_\varepsilon} \left( \frac{3}{\varepsilon^{1+\alpha}} \frac{\langle \gamma'(u), h'(u) \rangle}{|u|^\alpha} + 2 \frac{\langle \gamma'(u), h'(u) \rangle}{|\gamma(u + w) - \gamma(u)|^2} \right) dw du.
\]

To prove that \( \gamma \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \), we first decompose

\[
\delta E^{(\alpha)}(\gamma; h) = \alpha Q^{(\alpha)}(\gamma, h) + R^{(\alpha)}(\gamma, h)
\]

(3.2)

where

\[
Q^{(\alpha)}(\gamma, h) := \lim_{\varepsilon \searrow 0} \int_{U_\varepsilon} \left( \frac{\langle \gamma'(u), h'(u) \rangle}{|u|^\alpha} - \frac{\langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle}{|u|^\alpha + 2} \right) dw du
\]

and \( R^{(\alpha)}(\gamma, h) \) is given by

\[
R^{(\alpha)}(\gamma, h) := 2 \lim_{\varepsilon \searrow 0} \int_{U_\varepsilon} \langle \gamma'(u), h'(u) \rangle \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^2} - \frac{1}{|u|^\alpha} \right) dw du
\]

\[
- \alpha \lim_{\varepsilon \searrow 0} \int_{U_\varepsilon} \langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^2 + 2} - \frac{1}{|u|^\alpha + 2} \right) dv du.
\]

Later on, it will become evident that, in contrast to \( Q^{(\alpha)} \), the integral defining \( R^{(\alpha)} \) is not a principle value, i. e. we may write \( U_0 \) instead of \( U_\varepsilon \).

It was already observed by He in [13] and the second author in [19] that \( Q^{(\alpha)}(\gamma, h) \) is a lower order perturbation of the \( L^2 \) product of \((-\Delta)^{\alpha/2} \gamma \) and \((-\Delta)^{\alpha/2} h \). To see this, let us first extend \( Q^{(\alpha)} \) to complex valued functions by exchanging the scalar product on \( \mathbb{R}^n \) to the scalar product on \( \mathbb{C}^n \). We denote by \( \hat{f}(k) = \int_{\mathbb{R}/\mathbb{Z}} f(u) e^{-2\pi i ku} du \) the \( k \)-th Fourier coefficient of \( f \).

Proposition 3.1 (cf. [13, Lemma 2.3], [19, Proposition 1.4]). There is a sequence of real numbers \( q_k, k \in \mathbb{Z} \), converging to a positive constant for \( |k| \to \infty \) such that for all \( \gamma, h \in H^{1+\alpha/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) we have

\[
Q^{(\alpha)}(\gamma, h) = \sum_{k \in \mathbb{Z}} q_k |k|^{\alpha+1} \gamma(k) \overline{h(k)}.
\]

(3.3)

Apart from this observation, the proof of Theorem 1.2 relies on the following estimate regarding the term \( R^{(\alpha)}(\gamma, h) \). Basically it lets us treat this term like a lower order perturbation.

Proposition 3.2. Let \( \gamma \in H^{(\alpha+1)/2+\sigma}_w \) be parametrized by arc-length, \( \sigma \geq 0 \).
(i) In the case $\sigma = 0$ we have $R^{(\sigma)}(\gamma, \cdot) \in \left(H^{3/2+\varepsilon.2}\right)^*$ for any $\varepsilon > 0$.

(ii) If $\sigma > 0$ we have $R^{(\sigma)}(\gamma, \cdot) \in \left(H^{3/2-\varpi.2}\right)^*$ for all $\varpi < \sigma$.

We will prove Proposition 3.2 using Sobolev embeddings and the fractional Leibniz rule for Bessel potential spaces (cf. Lemma A.1).

First we will show, that the two summands building $R^{(\sigma)}$ can be brought into a common form and can thus be dealt with simultaneously. For that we use the fundamental theorem of calculus to get

$$
\langle \gamma(u + w) - \gamma(u), h(u + w) - h(u) \rangle \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^{\alpha + 2}} - \frac{1}{|u|^{\alpha + 2}} \right)
$$

$$
= w^2 \int_0^1 \int_0^1 \langle \gamma'(u + s_1 w), h'(u + s_2 w) \rangle \left( \frac{1}{|\gamma(u + w) - \gamma(u)|^{\alpha + 2}} - \frac{1}{|u|^{\alpha + 2}} \right) ds_1 ds_2.
$$

Furthermore, for $\beta > 0$,

$$
\frac{1}{|\gamma(u + w) - \gamma(u)|^\beta} - \frac{1}{|u|^\beta} = \frac{|u|^\beta}{|\gamma(u + w) - \gamma(u)|^\beta} - \frac{|u|^\beta}{|u|^\beta} = \frac{1 - |u|^\beta}{|\gamma(u + w) - \gamma(u)|^\beta} - \frac{1 - |u|^\beta}{|u|^\beta} = \frac{1 - |u|^\beta}{|\gamma(u + w) - \gamma(u)|^\beta} - \frac{1 - |u|^\beta}{|u|^\beta}
$$

$$
= \int_0^1 \int_0^1 G^{(\beta)} \left( \frac{\gamma(u + w) - \gamma(u)}{w} \right) \frac{1}{|\gamma(u + w) - \gamma(u)|^\beta} - \frac{1 - |u|^\beta}{|u|^\beta} \frac{1}{|\gamma(u + w) - \gamma(u)|^\beta} - \frac{1 - |u|^\beta}{|u|^\beta}
$$

$$
= \int_0^1 \int_0^1 G^{(\beta)} \left( \frac{\gamma(u + w) - \gamma(u)}{w} \right) \frac{1}{|\gamma(u + w) - \gamma(u)|^\beta} \left( |\gamma'(u + \tau_1 w) - \gamma'(u + \tau_2 w)|^2 / |w|^\beta \right) d\tau_1 d\tau_2
$$

where

$$
G^{(\beta)}(z) := \frac{1}{2|z|^\beta} \cdot \frac{1 - |z|^\beta}{1 - |z|^\beta}
$$

is an analytic function away from the origin. Defining

$$
g^{(\alpha,\beta)}_{\tau_1,\tau_2}(u, w) := G^{(\beta)} \left( \frac{\gamma(u + w) - \gamma(u)}{w} \right) \frac{1}{|\gamma(u + w) - \gamma(u)|^\beta} \frac{|\gamma'(u + \tau_1 w) - \gamma'(u + \tau_2 w)|^2}{|w|^\alpha} \gamma'(u + s_1 w)
$$

we thus get

$$
R^{(\alpha)}(\gamma, h) = \lim_{\varepsilon \searrow 0} \left\{ 2 \iint_{U_{\varepsilon}} \iint_{(0,1]^2} g^{(\alpha,\varepsilon)}_{\tau_1,\tau_2}(u, w, h'(u)) d\tau_1 d\tau_2 dw du 
$$

$$
- \alpha \iint_{U_{\varepsilon}} \iint_{(0,1]^2} \left\{ g^{(\alpha,\varepsilon+2)}_{\tau_1,\tau_2}(u, w, h'(u + s_2 w)) \right\} ds_1 ds_2 d\tau_1 d\tau_2 dw du \right\}
$$

(3.4)

Thus using Hölder’s inequality we get the estimate

$$
R^{(\sigma)}(\gamma, h) \leq C ||h||_{L^{\infty}} \sup_{\beta \in (0,\alpha+2)} \iint_{U_{\varepsilon}} \left\{ g^{(\alpha,\beta)}_{\tau_1,\tau_2}(\gamma, w) \right\} ||g^{(\alpha,\beta)}_{\tau_1,\tau_2}(\gamma, w) ||_{L^1} dw
$$

$$
\leq C ||h||_{H^{3/2+\varpi.2}} \sup_{\beta \in (0,\alpha+2)} \iint_{U_{\varepsilon}} \left\{ g^{(\alpha,\beta)}_{\tau_1,\tau_2}(\gamma, w) \right\} ||g^{(\alpha,\beta)}_{\tau_1,\tau_2}(\gamma, w) ||_{L^1} dw
$$

(3.5)

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for any $\varepsilon > 0$.

For $\sigma \in \mathbb{R}$ let

$$D^\sigma := (-\Delta)^{\sigma/2}. \quad (3.6)$$

By partial integration we infer for $\tilde{\sigma} \in \mathbb{R}$

$$\int_{\mathbb{R}/\mathbb{Z}} \left( g^{(a,\beta)}_{s_1,\tau_1,\tau_2}(u, w), H'(u + s_2 w) \right) \, du = \int_{\mathbb{R}/\mathbb{Z}} \left( D^\sigma g^{(a,\beta)}_{s_1,\tau_1,\tau_2}(u, w), D^{-\tilde{\sigma}} H'(u + s_2 w) \right) \, du$$

and we can estimate the absolute value by

$$\left\| D^{-\tilde{\sigma}} H' \right\|_{L^{\infty}} \int_{\mathbb{R}/\mathbb{Z}} \left| D^\sigma g^{(a,\beta)}_{s_1,\tau_1,\tau_2}(u, w) \right| \, du \leq C \left\| h \right\|_{H^{3/2-\sigma+\varepsilon,2}} \left\| g^{(a,\beta)}_{s_1,\tau_1,\tau_2}(\cdot, w) \right\|_{H^{\sigma,1}}$$

for any $\varepsilon > 0$. Combining this with Equation (3.4) we get

$$\left| R^{(a)}(\gamma, h) \right| \leq C \left\| h \right\|_{H^{3/2-\sigma+\varepsilon,2}} \sup_{\beta \in \{a,\alpha+2\}, s_1, \tau_1, \tau_2 \in [0,1]} \int_{-1/2}^{1/2} \left\| g^{(a,\beta)}_{s_1,\tau_1,\tau_2}(\cdot, w) \right\|_{H^{\sigma,1}} \, dw$$

for all $\tilde{\sigma}$ and $\varepsilon > 0$. To prove Proposition 3.2, given $\sigma > \tilde{\sigma}$ we set $\tilde{\sigma} = (\sigma + \tilde{\sigma})/2 > \tilde{\sigma}$ and $\varepsilon = \sigma - \tilde{\sigma}$ in the calculations above, to get

$$\left| R^{(a)}(\gamma, h) \right| \leq C \left\| h \right\|_{H^{3/2-\sigma+\varepsilon,2}} \sup_{\beta \in \{a,\alpha+2\}, s_1, \tau_1, \tau_2 \in [0,1]} \int_{-1/2}^{1/2} \left\| g^{(a,\beta)}_{s_1,\tau_1,\tau_2}(\cdot, w) \right\|_{H^{\sigma,1}} \, dw. \quad (3.7)$$

Proposition 3.2 now immediately follows from Estimate (3.5), Estimate (3.7) and the succeeding lemma.

**Lemma 3.3.** Let $\gamma \in H^{(a+1)/2+\sigma,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $\sigma \geq 0$ and $\beta > 0$.

(i) If $\sigma < 0$ then $g^{(a,\beta)}_{s_1,\tau_1,\tau_2} \in L^1(\mathbb{R}/\mathbb{Z} \times (-\frac{1}{2}, \frac{1}{2}), \mathbb{R}^n)$. Furthermore, there is a constant $C < \infty$ independent of $\tau_1, \tau_2$, and $s_1$ such that

$$\left\| g^{(a,\beta)}_{s_1,\tau_1,\tau_2} \right\|_{L^1} \leq C.$$

(ii) If $\sigma > 0$ then $g^{(a,\beta)}_{s_1,\tau_1,\tau_2} \in L^1((-\frac{1}{2}, \frac{1}{2}), H^{a,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ for all $\tilde{\sigma} < \sigma$ and there is a constant $C < \infty$ independent of $\tau_1, \tau_2$, and $s_1$ such that

$$\int_{-1/2}^{1/2} \left\| g^{(a,\beta)}_{s_1,\tau_1,\tau_2}(\cdot, w) \right\|_{H^{\sigma,1}} \, dw \leq C.$$

**Proof.** Let us first deal with the case $\sigma = 0$. We get

$$\left\| g^{(a,\beta)}_{s_1,\tau_1,\tau_2} \right\|_{L^1(\mathbb{R}/\mathbb{Z} \times (-\frac{1}{2}, \frac{1}{2}), \mathbb{R}^n)} \leq \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} G^{(a,\beta)}(\gamma(u + w) - \gamma(u)) \left| \frac{\gamma'(u + \tau_1 w) - \gamma'(u + \tau_2 w)}{w} \right|^2 \left| \gamma'(u + s_1 w) \right| \, dw \, du$$

$$\leq C \left\| \gamma' \right\|_{L^{\infty}} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left| \frac{\gamma'(u + \tau_1 w) - \gamma'(u + \tau_2 w)}{w} \right|^2 \, dw \, du$$

$$\leq C \left\| \gamma' \right\|_{L^{\infty}} \left\| \gamma' \right\|_{H^{a+1}/2,2}^2$$

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which proves the statement for \( \sigma = 0 \).

Since there is no suitable product rule for \( p = 1 \), we will estimate \( \|g^{(a,b)}_{\gamma_1,\tau_1}(\cdot, w)\|_{H^{\sigma,p}} \) for \( p > 1 \) sufficiently small. For this we will use a small \( \tilde{p} > p \) and let \( q \) be such that

\[
\frac{1}{p} = \frac{1}{2\tilde{p}} + \frac{1}{2q} + \frac{1}{q},
\]

i.e. we set

\[
q = 2 \frac{\tilde{p} p}{\tilde{p} - p}.
\]

Using that

\[
\gamma(u + w) - \gamma(u) = \int_0^1 \gamma'((u + \tau w)d\tau,
\]

that \( \gamma \) is bi-Lipschitz, and that \( G^{(b)} \) is analytic away from the origin, we get that

\[
\left\| G^{(b)} \left( \frac{\gamma(\cdot + w) - \gamma(\cdot)}{w} \right) \right\|_{H^{\sigma,q}} \leq C \|\gamma\|_{H^{\sigma + 1,q}} \leq C
\]

by the Sobolev embedding.

Using the fractional Leibniz rule (Lemma A.1) three times, we derive for \( \sigma' \in (0, \sigma') \)

\[
\left\| g^{(a,b)}_{\gamma_1,\tau_1}(\cdot, w)\right\|_{H^{\sigma',p}} \leq C \left\| G^{(b)} \left( \frac{\gamma(\cdot + w) - \gamma(\cdot)}{w} \right) \right\|_{H^{\sigma',q}} \|\gamma'\|_{H^{\sigma',q}} \left\| \frac{\gamma'(\cdot + \tau_1 w) - \gamma'(\cdot + \tau_2 w)}{w} \right\|_{H^{\sigma',2,p}} \|\gamma'\|_{H^{\sigma',2,p}}
\]

\[
\leq C \left\| \frac{\gamma'(\cdot + \tau_1 w) - \gamma'(\cdot + \tau_2 w)}{w} \right\|_{H^{\sigma',2,p}}.
\]

We now choose \( \tilde{p} > 1 \) so small that \( H^{\sigma,2} \) embeds into \( H^{\sigma + 2,\tilde{p}} \) and hence

\[
\left\| g^{(a,b)}_{\gamma_1,\tau_1}(\cdot, w)\right\|_{H^{\sigma,p}} \leq C \left\| \frac{\gamma'(\cdot + \tau_1 w) - \gamma'(\cdot + \tau_2 w)}{w} \right\|_{H^{\sigma,2}}.
\]

Thus, recalling (3.6),

\[
\int_{-1/2}^{1/2} \left\| g^{(a,b)}_{\gamma_1,\tau_1}(\cdot, w)\right\|_{H^{\sigma,p}} dw \leq C \int_{-1/2}^{1/2} \frac{\left\| \gamma'(\cdot + \tau_1 w) - \gamma'(\cdot + \tau_2 w) \right\|_{L^2}}{w^p} dw
\]

\[
+ C \int_{-1/2}^{1/2} \frac{\left\| D^{\sigma + 1} \gamma(\cdot + \tau_1 w) - D^{\sigma + 1} \gamma(\cdot + \tau_2 w) \right\|_{L^2}}{w^p} dw
\]

\[
\leq C \int_{-1/2}^{1/2} \frac{\left\| \gamma'(\cdot) - \gamma'(\cdot + \tau_2 - \tau_1) w) \right\|_{L^2}}{w^p} dw
\]

\[
+ C \int_{-1/2}^{1/2} \frac{\left\| D^{\sigma + 1} \gamma(\cdot) - D^{\sigma + 1} \gamma(\cdot + \tau_2 - \tau_1) w) \right\|_{L^2}}{w^p} dw
\]

\[
\leq C |\tau_2 - \tau_1| \int_{-1}^{1} \frac{\left\| \gamma'(\cdot) - \gamma'(\cdot + w) \right\|_{L^2}}{w^p} dw
\]

\[
+ C |\tau_2 - \tau_1| \int_{-1}^{1} \frac{\left\| D^{\sigma + 1} \gamma(\cdot) - D^{\sigma + 1} \gamma(\cdot + w) \right\|_{L^2}}{w^p} dw
\]

\[
\leq C \left\| \gamma'(\cdot) \right\|_{H^{(a-1)/2,2}} + C \left\| D^{\sigma + 1} \gamma(\cdot) \right\|_{H^{(a-1)/2,2}} \leq C.
\]
This proves Lemma 3.3. □

Proof of Theorem 1.2. Recall that any finite-energy curve belongs to $H^{(α+1)/2,2}$ by [3]. Let us assume that $γ ∈ H^{α+1/2,2}(R/Z, R^n)$ for $α ≥ 0$ is a stationary point of the energy $E^{(α)} + λL$. As the first variation of the length functional gives rise to a linear lower order term, Proposition 3.1 also applies to
\[
\tilde{Q}^{(α)}_A(γ, h) := αQ^{(α)}(γ, h) + λ \int_{R/Z} ⟨γ', h'⟩.
\]
In the case that $σ = 0$ we get from the Euler-Lagrange Equation (3.1) using the decomposition (3.2) and Proposition 3.2
\[
\tilde{Q}^{(α)}_A(γ, ·) ∈ \left(H^{3/2 + ε, 2}\right)^{*}
\]
for any $ε > 0$. Using Proposition 3.1 we hence get
\[
(q_k |k|^{α+1−3/2−ε}γ(k))_{k ∈ Z} ∈ ℓ^2.
\]
Together with the fact that $q_k$ converge to a positive constant as $k → ∞$ we get
\[
γ ∈ H^{3/2 + ε, 2}(R/Z, R^n).
\]
For $σ > 0$ we get using Proposition 3.2
\[
\tilde{Q}^{(α)}_A(γ, ·) ∈ \left(H^{3/2−σ, 2}\right)^{*} \quad \text{for all } σ < α
\]
and arguing as above
\[
γ ∈ H^{(α+1−σ)/2, 2−σ, 2}(R/Z, R^n)
\]
for all $σ < α$.

If we now initially assume that $γ ∈ H^{α+1, 2}$ we deduce by induction and since $α+1 > 0$ that
\[
γ ∈ H^{t, 2}(R/Z, R^n)
\]
for all $t ≤ α$ and thus $γ ∈ C^∞(R/Z, R^n)$. This proves Theorem 1.2 □

A Results on fractional Sobolev spaces

Let us gather two results we used in the article: The product and chain rule which go back to Coifman and Meyer [9] and Christ and Weinstein [8].

Lemma A.1 (Leibniz Rule, cf. [9]). Let $p_i, q_i, r ∈ (1, ∞)$, be such that $1/p_i + 1/q_i = 1/r$, for $i = 1, 2$ and $s > 0$. Then
\[
∥f ∙ g∥_{H^{s,r}} ≤ C (∥f∥_{L^{p_1}} ∥g∥_{H^{s,q_1}} + ∥f∥_{H^{s,p_2}} ∥g∥_{L^{q_2}}).
\]
We also refer to Runst and Sickel [20, Lem. 5.3.7/1 (i)]. — For the following statement, one mainly has to treat $\| (D^k\psi) \circ f \|_{H^{r,p}}$ for $k \in \mathbb{N} \cup \{0\}$ and $r \in (0, 1)$ which is e. g. covered by [20, Thm. 5.3.6/1 (i)].

**Lemma A.2 (Chain rule, cf. [9]).** Let $f \in H^{s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $s > 0$, $p \in (1, \infty)$. If $\psi \in C^\infty(\mathbb{R})$ such that $\psi$ and all its derivatives vanish at 0 then $\psi \circ f \in H^{s,p}$ and

$$\| \psi \circ f \|_{H^{s,p}} \leq C \| \psi \|_{C^k} \| f \|_{H^{s,p}}$$

where $k$ is the smallest integer greater or equal to $s$.

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