Cobordism of immersions of surfaces in non-orientable 3-manifolds.

Rosa Gini

January 2000

Abstract

Some properties of non-orientable 3-manifolds are shown. The semi–group of cobordism of immersions of surfaces in such manifolds is computed and proven actually to be a group. Explicit invariants are provided.

Introduction

Following the definitions of [Wel66], [Pin85], [BS95] we will say that two immersions $f$ and $f'$ defined on compact closed surfaces not necessarily connected $F$ and $F'$ and taking values in the same 3-manifold $M$ are cobordant if there exists a cobordism $X$ between $F$ and $F'$ and an immersion $\Phi$ of $X$ in $M \times I$ that restricts to $f \times \{0\}$ and to $f' \times \{1\}$. Once fixed the manifold $M$ the set $N_2(M)$ of cobordism classes of immersions of surfaces in $M$ is a semi-group with the composition law given by disjoint union. That this be a group when $M = \mathbb{R}^3$ is given a priori by the fact that inverses are provided by composition with a reflection in a plane. In fact $N_2(\mathbb{R}^3)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$, as is proved in [Wel66]; explicit invariants are given in [Bro72] and [Pin85]; a generator is the so-called right immersion of Boy of the projective space, see [Pin85]. That $N_2(M)$ be a group when $M$ is a generic manifold is not straightforward; it is proven in [BS95] for orientable 3-manifolds that $N_2(M)$ is the finite set $H_2(M, \mathbb{Z}/2\mathbb{Z}) \times H_1(M, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/8\mathbb{Z}$ with a composition law that twists the compositions of the factors. We adapt here their
constructions to the non-orientable case, and obtain in theorem 2.1.2 that again $N_2(M)$ is a finite group, with support the set $H_2(M, \mathbb{Z}/2\mathbb{Z}) \times H_1(M, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ and a composition law similar to the one of the orientable case. The crucial point that causes a non-orientable 3-manifold to have a “smaller” group than an orientable 3-manifold with same homology is proposition 2.3.7, and comes from the fact that in a non-orientable environment there exist isotopies that reverse orientation.

The first section is essentially a review of results of [HH85], [BS95] and [KT89]; some remarks and properties are original and are used in the following; in particular in proposition 1.2.1 we compute explicitly the isotropy group for the action of adding kinks introduced in [HH85], and in theorem 1.4.4 we classify bands (that is, immersions of annuli or Moebius bands) in a non-orientable 3-manifolds up to regular homotopy. In the second section we develop the computation of the cobordism group.

This paper is an extended version of the talk given in Palermo in September 1999, during the Congress “Proprietà Geometriche delle Varietà Reali e Complesse: Nuovi Contributi Italiani”.

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1 Some properties of non-orientable 3-manifolds.

From now on a loop $c$ in a non-orientable manifold $X$ will be said to be orientable in $X$ if it preserves the orientation of $X$, that is, if $TX|_c$ is orientable, and will be said to be non-orientable in $X$ otherwise.

1.1 Projective and anti-equivariant framings.

It is well-known that orientable 3-manifolds are parallelizable. A non-orientable 3-manifold obviously is not, but its tangent bundle is still as simple as a non-orientable vector bundle can be, that is, its structure group can be reduced to $\{1, -1\}$. A calculus of characteristic classes in [HH85] proves in fact that if $M$ is a non-orientable 3-manifold then

$$TM \cong \det M \oplus \det M \oplus \det M.$$ 

This means that it is possible to define on $M$ a projective framing, i.e. a triple of linearly independent vector fields, well-defined up to sign. A triple $\{v_1, v_2, v_3\}$ of linearly independent vector fields on the orientation covering space $\tilde{M}$ of $M$ is said to be an anti-equivariant framing if

$$v_i(\sigma(P)) = -\sigma_*(v_i(P)) \quad \forall P \in \tilde{M}, i = 1, \ldots, 3,$$

where $\sigma$ is the covering translation of $\tilde{M}$; an anti-equivariant framing projects on a projective framing. Given a projective framing of $M$ one can always find an anti-equivariant framing of $\tilde{M}$ that projects on it: we call such an anti-equivariant framing a lifting of the projective framing.

We will assume, from now on, that all framings are orthonormal with respect to a fixed metric.
1.2 Isotropy in adding kinks.

The main result of [HH85] is to set up an action $\ast$ of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ on the set $\text{Imm}_\xi(F, M)$ of regular homotopy classes of immersions in a homotopy class $\xi$ of immersions of a surface $F$ in a 3-manifold $M$. This action is explicitly described in a beautiful geometric way: if $f$ is a representative of an element of $\text{Imm}_\xi(F, M)$ and $\alpha$ is in $H^1(F, \mathbb{Z}/2\mathbb{Z})$ then a representative of $f \ast \alpha$ is obtained by modifying $f$ in a tubular neighborhood $N$ of a dual curve to $\alpha$, in a way the authors describe as adding a kink: in case both of the curve in $F$ and its image in $M$ are orientable this local modification is the (local) composition of $f$ with the immersion of an annulus in a solid torus shown in figure 1; in the other cases the modification is similar. If $c$ is a dual curve to $\alpha$ then $f \ast \alpha$ will also be denoted by $f \ast c$.

The action of adding kinks is proven to be transitive in any homotopy class. Isotropy depends on a property of the class: we say a homotopy class $\xi$ is odd if both of $F$ and $M$ are non orientable and if there exists a self-homotopy $H$ of a map $f \in \xi$ such that $H(x, -)$ is a loop non-orientable in $M$ for $x \in F$; we call odd such a homotopy; the action of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ on $\text{Imm}_\xi(F, M)$ has then a group of isotropy of order 2 in any point if $\xi$ is odd. If $\xi$ is not odd we say it is even, and isotropy is trivial in this case. This means, in particular, that the set $\text{Imm}_\xi(F, M)$ has the cardinality of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ if $\xi$ is even, and half its cardinality if $\xi$ is odd.
We compute explicitly the isotropy group for the action on odd classes.

**Proposition 1.2.1** Let $\xi$ be a odd homotopy class of immersions of a non-orientable surface $F$ in a non-orientable 3-manifold $M$. Then the isotropy group of any regular homotopy class in $\xi$ is the vector subspace of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ generated by $w_1(F)$.

**Proof.** The action of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ on $Imm_\xi(F, M)$ is not explicitly defined in [HH85]. What is stated there is that, given a point $f$ of $Imm_\xi(F, M)$, it is possible to define a correspondence $C_f$ between $Imm_\xi(F, M)$ and $H^1(F, \mathbb{Z}/2\mathbb{Z})$ which is 1-1 if $\xi$ is even and 1-2 if $\xi$ is odd; remark that $C_f^{-1}$ is then always a function. This action is explicitly defined in [BS95] as

$$
*: \quad Imm_\xi(F, M) \times H^1(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow Imm_\xi(F, M)
$$

$$(f, \alpha) \quad \mapsto \quad \alpha \ast f := C_f^{-1}(\alpha).
$$

We prove that, if $\xi$ is odd, then $C_f(f) = \{1, w_1(F)\}$. The correspondence $C_f$ is constructed in several steps. The first is to associate to every $g \in Imm_\xi(F, M)$ the homotopy class of its differential: that this be a bijection is a consequence of [Hir59]; to $f$ is then associated $df$. The second step is where non-injectivity appears: to each differential one associates two elements of the set $Bun_f(TF, TM)$ of homotopy classes of bundle map that commute with $f$, via the choice of a projective framing of $M$. It follows from the construction given in [HHS85] that the classes associated to $df$ are $df$ itself and $-df$. The following steps give a bijection between $Bun_f(TF, TM)$ and $H^1(F, \mathbb{Z}/2\mathbb{Z})$, that associates the identity to $df$; we call $w$ the element of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ associated to $-df$: we are then left to prove that $w = w_1(F)$.

Let $c$ be a curve on $F$. We describe how to determine $w(c)$. Consider on $c$ the fiber bundle $\tau := f^*(TM) = TF|_c \oplus \nu_c$ and the automorphism of $\tau$ given by $-1_{TF|_c} \oplus 1_c$. If $f(c)$ is orientable in $M$ then the projective framing determines two opposite framings of the trivial bundle $\tau$; the map $-1_{TF|_c} \oplus 1_c$ read in one of these framings gives a closed path in $SO(3)$; $w(c)$ is then defined to be the class of this path in $H_1(SO(3), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, that does not depend on the choice between the two framings (nor on the choice of $c$ between representatives of its class modulo 2). If $f(c)$ is not orientable then $\tau$ is not trivial; the projective framing of $M$ restricted to $\tau$ defines two triples of orthonormal
vector fields discontinuous in a point; choose one of them \( \{v_1, v_2, v_3\} \); define a path of \( SO(3) \) by

\[
P \mapsto \text{linear transformation between } \{v_1(P), v_2(P), v_3(P)\} \text{ and } -1_{TP_c} \oplus 1_c(v_1(P), v_2(P), v_3(P)),
\]

for all \( P \in c \); thought the vector fields are discontinuous this path is well-defined, continuous and closed, and its homology class is independent of the choice between the two opposite triple of discontinuous vector fields. The value of \( w(c) \) is this homology class, again considered as an element of \( \mathbb{Z}/2\mathbb{Z} \).

We must now compute these values. Recall that a point of \( SO(3) \) is identified by a point of \( S^2 \), which indicates the oriented rotation axis, and a number between 0 and \( \pi \), which indicates the rotation angle taken in the positive sense given by orientation; the facts that the null rotation around any axis is the identity and that two rotations of \( \pi \) along opposite axes coincide give a bijection between \( SO(3) \) and the projective space \( \mathbb{P}^3 \), which is in fact a well-known homeomorphism. The point in \( SO(3) \) associated to a point \( P \) of \( c \) is the rotation of \( \pi \) with rotation axis rotating itself of \( \pi \) in the \( xz \) plane, from \((1,0,0)\) to \((-1,0,0)\). Consider first the case that \( f(c) \) is orientable in \( M \), that is \( \tau \) is framed by the projective framing. We then consider a homeomorphism of the standard torus in \( \mathbb{R}^3 \) to a tubular neighborhood of \( f(c) \) which sends the standard framing to the framing of \( \tau \). The image of \( f(N) \) results as a band in this standard torus, and the normal vector to this bands is parallel to the \( xz \) plane (in points where this intersection may be considered transverse) a number of times which is even if the band is orientable and odd if the band is a Moebius band: so we have shown that \( w(c) \) coincides with \( w_1(F)(c) \) when \( f(c) \) is orientable in \( M \).

We are then left to the case \( f(c) \) non-orientable in \( M \). We may suppose to have fixed a lifting to \( \tilde{M} \) of the projective framing of \( M \). Consider the non trivial double covering \( \tilde{N} \) of \( N \), let \( \tilde{f} \) be a map of this covering to \( \tilde{M} \) that commutes with \( f \) restricted to \( N \), let \( \tilde{\tau} \) be \( \tilde{f}^*(TM) \): this is a trivial bundle framed by the fixed framing of \( \tilde{M} \). Again we fix a homeomorphism of the standard torus with standard framing to a tubular neighborhood
of $\tilde{f}(\tilde{c})$ in $\tilde{M}$ with the fixed framing; the image of the band $\tilde{f}(\tilde{N})$ in this torus gives count two times of the band $f(N)$, so consider its (transverse) intersection with, say, the semi-space $y \geq 0$; the number we are looking for is the number of times, modulo 2, that the normal vector to this part of the band is parallel to the $xz$ plane (again this intersection can be taken to be transverse); again this number is even if $N$ is orientable and odd if $N$ is a Moebius band, so that again $w(c) = w_1(F)(c)$, and this ends the proof. △

1.3 Bands in orientable 3-manifolds.

The meaning of the main theorem in [HH83] is that the regular homotopy class of a map in its homotopy class is determined by the behaviour of the map in tubular neighborhoods of the curves of the surface. These tubular neighborhoods are either annuli or Moebius bands; we call band in a 3-manifold the immersion of an annulus or a Moebius band. In [BS95] the rôle of bands in this subject becomes more evident, and their properties are also a necessary brick in the computation of cobordism in the case of orientable 3-manifolds.

We review the properties of bands in orientable 3-manifolds, then we look at the case of bands in a non-orientable 3-manifold.

1.3.1 Bands in $\mathbb{R}^3$.

Consider an embedded band in $\mathbb{R}^3$, and consider the linking number of its core with its (possibly non-connected) boundary. We think of this linking number in the following way: we pick on the boundary of a tubular neighborhood $N$ of the core of the band a preferred basis for $H_1(\partial N, \mathbb{Z})$, with the orientation of the longitude coherent with an orientation of the core; we then give to the meridian the orientation that makes the global orientation off $\partial N$ compatible with the orientation of the environment; the intersection of the band with $\partial N$ represents a homology class, and the linking number is the meridian coordinate of this class in the preferred basis. This number is even if the band is orientable and odd if it is a Moebius band, so that its class modulo 2 is a total topological invariant of the band.
Given any band in $\mathbb{R}^3$ there exists a regular homotopy between its core and the standard circle in the $xy$ plane in $\mathbb{R}^3$; extend it to a regular homotopy between a tubular neighborhood of the core of the band and the standard torus of $\mathbb{R}^3$: we obtain a band regularly homotopic to the original and that differs from the annulus in the $xy$ plane by a number of half twists given by the linking number. Now if we isolate 4 half twists there exists a regular homotopy relative to the rest of the band which transforms 4 half twists in a couple of kinks and then in a piece of band with no twisting, see figures 2 and 3; this local construction reduces by regular homotopy the original band to one of four models, classified by the linking number modulo 4. This number is in fact an
invariant in $\mathbb{Z}/4\mathbb{Z}$ of the regular homotopy class of the band (see [GM86], page 114), so it coincides with the original linking number (modulo 4). This proves the following:

**Proposition 1.3.1** In $\mathbb{R}^3$ the linking number modulo 4 between the core of a band and the boundary oriented in a coherent way is a total invariant of the regular homotopy class of the band. $\triangle$

We call this invariant in $\mathbb{Z}/4\mathbb{Z}$ number of half twists of the band.

### 1.3.2 Spin structures and preferred longitudes in orientable 3-manifolds.

The problem in extending the definition of half twists of a band to generic 3-manifolds is that in a 3-manifold there is no preferred basis for the homology of the boundary of the tubular neighborhood of a knot: a meridian is still defined, and the orientation of the environment can give it an orientation as it does in $\mathbb{R}^3$, but there is no way, a priori, to distinguish between longitudes. Remark that the local modification of figure 3 can be performed in any manifold, so that up to regular homotopy we are still interested in the class of rest modulo 4 of the integer coordinate; this means that it is enough to choose a homology class of longitudes modulo 2, since longitudes that differ by two meridians give integer coordinates belonging to the same class modulo 4 (remark that the boundary of a band covers the core twice). In [KT89] it is shown how to use a Spin structure on an orientable 3-manifold to determine this choice, that reduces to the classical one when $M = \mathbb{R}^3$ with its unique Spin structure. We give here a slightly different way to define the same choice.

Orientable vector bundles on a circle are always trivial. Framings of an oriented vector bundle of rank 3 on a circle are 2, up to homotopy. This is because a homotopy class of framings can be considered as a homotopy class of sections of the associated principal bundle, which, being trivial, is homeomorphic to $S^1 \times SO(3)$: and $\pi_1(S^1 \times SO(3))$ contains four classes, only two of which can be realized as sections of the bundle.

Consider $S^1$ embedded in $\mathbb{R}^3$ in the standard way, and consider on it the fiber bundle $\tau$ given by restriction of the tangent bundle to $\mathbb{R}^3$. Define on $\tau$ two framings containing in each point the tangent vector to $S^1$: let $e_0$ be the oriented framing containing the
vector parallel to the $z$-axis as second element; let $e_1$ be the framing containing in the point $(\cos \theta, \sin \theta, 0)$ of $S^1$ the vector $\cos \theta(0, 0, 1) + \sin \theta(\cos \theta, \sin \theta, 0)$ as second element. These two framings are not equivalent, since the curves they describe in the principal bundle of $\tau$ differ by a generator of the homotopy group of $SO(3)$: this means that any other framing of $\tau$ is equivalent either to $e_0$ or to $e_1$. In particular the standard framing is homotopic to $e_1$, since $e_1$ extends to a framing of the tangent bundle to $\mathbf{R}^3$ restricted to the disc, by

$$(\rho \cos \theta, \rho \sin \theta, 0) \mapsto ((-\sin \rho \theta, \cos \rho \theta, 0), (\sin \rho \theta \cos \rho \theta, \sin^2 \rho \theta, \cos \rho \theta), \text{third orthonormal}),$$

and remark that the framing of the trivial bundle on the disc is unique up to homotopy, the disc being contractile. But now the second vector of either $e_0$ or $e_1$ frames the normal bundle to $S^1$ in $\mathbf{R}^3$, and in particular detects a longitude; so we get close to what we are looking for. Remark that the longitude which is preferred in the classical definition (that is, the one having linking number 0 with the core) is the one given by $e_0$, and that the longitude detected by $e_1$ belongs to the other class.

On the other side, $Spin$ structures on an oriented vector bundle on a manifold $B$ are acted on simply transitively by $H^1(B, \mathbf{Z}/2\mathbf{Z})$, so that $Spin$ structures on an oriented vector bundle of rank 3 on the circle are again 2. It is also true that a framing of a trivial bundle induces naturally a $Spin$ structure: in fact a $Spin$ structure on a vector bundle can be defined as a double covering of the associated principal $SO(n)$-bundle which be non trivial when restricted to the fiber (see [Mil62]), and a framing, giving a bundle equivalence between the principal bundle and $SO(n) \times B$, induces such a covering by pull-back of the standard covering by $Spin(n) \times B$:

$\text{induced spin structure} \quad \quad \quad Spin(n) \times B$

$\downarrow$

principal bundle

framing

$SO(n) \times B.$

The definition of $Spin$ structure via double covering implies that $Spin$ structures can be described by the elements of $H^1(\text{principal bundle}, \mathbf{Z}/2\mathbf{Z})$ that restrict to the fiber as generators of $H^1(SO(n), \mathbf{Z}/2\mathbf{Z})$. Two $Spin$ structures are equivalent if the two double coverings are equivalent, i.e. if the associated cohomology classes are the same.
Now, \( \pi_1(S^1 \times SO(3)) \) and \( H^1(S^1 \times SO(3), \mathbb{Z}/2\mathbb{Z}) \) are isomorphic, and the correspondence we gave between homotopy classes of sections of the principal bundle and cohomology classes that restrict to a fiber as a generator is bijective in this case, so that speaking of \( Spin \) structures or of framings of an oriented vector bundle of rank 3 on \( S^1 \) is the same thing.

We are now ready for the definition. Let \( M \) be an oriented 3-manifold with a fixed \( Spin \) structure, let \( K \) be a knot in \( M \); the \( Spin \) structure restricted on \( TM|_K \) gives a framing; choose a diffeomorphism of the standard \( S^1 \) in \( \mathbb{R}^3 \) with standard framing to \( K \) in \( M \) with this framing, deform by a homotopy the standard framing to \( e_1 \) and pull this deformation back to \( M \); the second vector of the framing, with usual identification of the normal bundle with the tubular neighborhood \( N \) of the knot, gives now a curve on \( \partial N \); the preferred longitude on \( \partial N \) will be the homology class in \( H_1(\partial N, \mathbb{Z}/2\mathbb{Z}) \) that doesn’t contain this curve. It is straightforward to verify that this choice doesn’t depend on the diffeomorphism nor on the deformation: in fact an oriented framing of \( T\mathbb{R}^3 \) restricted to the standard \( S^1 \) and containing the tangent vector to \( S^1 \) is homotopic either to \( e_0 \) or to \( e_1 \) according to the homology class modulo 2 of the curve they pick on the standard torus is equal to the class to the standard longitude or not. This proves the following characterization:

**Proposition 1.3.2** Any deformation of the framing of \( TM|_K \) that takes it to a framing containing the tangent vector to \( K \) chooses a curve in \( \partial N \) which is the non-preferred longitude. \( \triangle \)

This description of the preferred longitude is particularly useful when the \( Spin \) structure of the manifold is itself associated to a framing of the whole manifold. In this case to assign the preferred longitude it is even not necessary to mention the \( Spin \) structure.

We prove that our definition coincide with the definition of *even framing* in [KT89], page 209:

**Lemma 1.3.3** Let \( M \) be an oriented 3-manifold with a fixed \( Spin \) structure, and let \( K \) be a knot in \( M \); then any even framing of the normal bundle to \( K \) represents a preferred longitude.
The choice of even framing only depends on the Spin structure of $TM$ restricted to $K$, so it coincides either with the choice of preferred longitude or with its opposite. But the two choices coincide in the case of the standard circle in $\mathbb{R}^3$, where they both assign the class of the standard longitude of the torus, so they coincide in every case. △

If the knot represents the trivial class in $H_1(M, \mathbb{Z}/2\mathbb{Z})$ the preferred longitude does not depend on the choice of the Spin structure, since any two structures differ by the action of an element of $H^1(M, \mathbb{Z}/2\mathbb{Z})$, that doesn’t affect this knot. In this case the preferred longitude can be realized in a geometric way:

**Proposition 1.3.4** If a knot $K$ represents the trivial class in $H_1(M, \mathbb{Z}/2\mathbb{Z})$ then take any embedded surface $F$ such that $\partial F = K$; the (transverse) intersection of $F$ and $\partial N$ is a preferred longitude.

**Proof.** This property is the content of theorem 4.3 in [KT89]. △

**Corollary 1.3.5** If $M = \mathbb{R}^3$ the choice of preferred longitude reduces to the classical. △

Remark that the definition of preferred longitude is invariant under regular homotopy. For more details on this subject see [KT89].

In [BS95] the definition of even longitude allows to extend the notion of number of half twists to bands in generic oriented 3-manifolds. What is (implicitly) obtained is that given a knot $K$ there are 4 regular homotopy classes of bands having $K$ (or a knot homotopic to it) as core, and that the number of half twists is a total invariant.

### 1.4 Bands in non-orientable 3-manifolds.

#### 1.4.1 Equivariant Spin structure.

The first and less expensive attempt to extend the definition of half twists to bands immersed in a non-orientable 3-manifold $M$ is to look only at bands whose core is orientable in $M$, that is, bands that have two homeomorphic preimages in the orientation
double covering \( \tilde{M} \) of \( M \): the \textit{number of half twist} of the band can be defined as the number of half twists of one of its preimages, provided we fix on \( \tilde{M} \) a \textit{Spin} structure that makes the choice between the two preimages inessential. Call \( \sigma \) the covering translation of \( \tilde{M} \): we say a \textit{Spin} structure on \( \tilde{M} \) is \textit{equivariant} if, given an embedded circle \( K \) in \( \tilde{M} \) such that \( \sigma K \cap K = \emptyset \), whenever \( l \) is a curve on the boundary of a tubular neighborhood of \( K \) that represents a preferred longitude, then \( \sigma l \) represents a preferred longitude.

**Lemma 1.4.1** The \textit{Spin} structure on \( \tilde{M} \) induced by an anti-equivariant framing is equivariant.

**Proof.** Remark that \( \sigma \) restricted to a tubular neighborhood \( N \) of \( K \) does not preserve the homotopy class of the framing, since \( \sigma \) is orientation reversing. Call \( \{v_1, v_2, v_3\} \) the given anti-equivariant framing: the longitude \( l \) belongs to the class which doesn’t contain the curve detected on \( \partial N \) by a deformation of \( \{v_1, v_2, v_3\}\vert_K \); the same deformation composed with \( \sigma \) gives a deformation of \( \{\sigma v_1, \sigma v_2, \sigma v_3\}\vert_{\sigma K} = \{-v_1, -v_2, -v_3\}\vert_{\sigma K} \), which picks on the boundary of the tubular neighborhood of \( \sigma K \) the longitude \( \sigma l \).

We are then reduced to show that the curve on the standard torus in \( \mathbb{R}^3 \) detected by the non-oriented framing \( -e_1 \) belongs to the class which doesn’t contain the standard longitude (recall we are talking of classes modulo 2). But this is evident. \( \triangle \)

Now put on \( \tilde{M} \) an equivariant \textit{Spin} structure. Let \( \Sigma \) be a band in \( M \) whose core is orientable in \( M \), let \( \tilde{\Sigma} \) be one of its liftings to \( \tilde{M} \); the other is \( \sigma \tilde{\Sigma} \). We have

**Proposition 1.4.2** The \textit{number of half twists} of \( \tilde{\Sigma} \) is opposite to the \textit{number of half twists} of \( \sigma \tilde{\Sigma} \).

**Proof.** Call \( K \) the core of \( \Sigma \), with a fixed orientation, and call \( N \) a tubular neighborhood of \( K \), see figure 4. Let \( \{m, l\} \) be a preferred basis for \( H_1(\partial N, \mathbb{Z}) \): recall the orientation of \( l \) is chosen according to the orientation of \( K \), and the orientation of \( m \) is the one that, together with the orientation of \( l \), gives to \( \partial N \) the orientation coming from \( \tilde{M} \). This choice makes the meridian coordinate of a curve in \( \partial N \) independent of the choice of orientation for \( K \).
Recall that $\sigma$ is orientation reversing. This means that $\{\sigma m, \sigma l\}$ is not a preferred basis for the homology of a tubular neighborhood of $\sigma K$, whereas $\{-\sigma m, \sigma l\}$ is one. This gives the thesis. $\triangle$

But an even number and its opposite are congruent modulo 4; this proves:

**Corollary 1.4.3** The number of half twists is well defined for orientable bands whose core is orientable in $M$. $\triangle$

**1.4.2 Regular homotopies between bands in non-orientable 3-manifolds.**

In trying to extend further the definition of number of half twists to generic bands in non-orientable 3-manifold we look at regular homotopies in a non-orientable environment: we will see that in a non-orientable environment there are “more” regular homotopies. First recall the notion of odd self-homotopy in a non-orientable manifold $X$ (see §1.2): a self-homotopy $H$ is odd if the closed path $H(x, -)$ is non-orientable in $X$ for any $x$.

**Theorem 1.4.4** Let $M$ be a non-orientable 3-manifold, and let $K$ be an embedded circle in $M$; then

1. if $K$ is orientable in $M$ and doesn’t admit any odd self-homotopy then there are 4 regular homotopy classes of bands having a circle homotopic to $K$ as core;
2. if $K$ is orientable in $M$ and admits an odd self-homotopy then there are 3 regular homotopy classes of bands having a circle homotopic to $K$ as core, in particular any two non-orientable bands of this type are regularly homotopic;

3. if $K$ is non-orientable in $M$ then there are 3 regular homotopy classes of bands having a circle homotopic to $K$ as core, in particular any two non-orientable bands of this type are regularly homotopic; the two classes of orientable bands differ by a reparametrization.

Proof. Let $\Sigma$ be a band whose core is homotopic to $K$. Take the core of $\Sigma$ to $K$ with a regular homotopy and extend it (for example via an exponential map) to a tubular neighborhood: this takes $\Sigma$ to a band regularly homotopic and having $K$ as core. We now think of $\Sigma$ as a band having $K$ as core. By means of the local modification shown in figure 5 we can reduce any band having $K$ as core to one of four models; call $\Sigma_0$ one of the models of orientable band, the others differ from it only locally: call adding a local twist the operation of substituting to a piece looking as in the left side of figure 5 a piece looking as in the right side of the same figure; $\Sigma_i$ has $i$ local half twists, $i$ being 1 or 2, $\Sigma_{-1}$ has one local half twist in the opposite direction to the half twist of $\Sigma_1$. We have to decide under which conditions these models can or cannot admit regular homotopies. Recall that bands with number of half twists that differ modulo 2 are defined on non-homeomorphic domains, so that the notion of regular homotopy is only
possible between $\Sigma_0$ and $\Sigma_2$ and between $\Sigma_1$ and $\Sigma_{-1}$.

Let now $K$ be a curve orientable in $M$. Its tubular neighborhood $N$ is then a solid torus. The embedding $K$ lifts to $\tilde{M}$ in two embeddings $\tilde{K}$ and $\sigma\tilde{K}$; any homotopy between the two liftings projects to an odd self-homotopy of $K$, any self-homotopy of one of the liftings projects to an even self-homotopy of $K$. The liftings of the embedding of $N$ are two solid tori, and the covering projection restricted to either of these tori gives opposite orientations to $N$, see figure [4]. If we extend a homotopy between $\tilde{K}$ and $\sigma\tilde{K}$ to tubular neighborhoods it projects to an odd self-homotopy of $N$ that reverses the orientation, if we extend a self-homotopy of either $\tilde{K}$ or $\sigma\tilde{K}$ to a tubular neighborhood it projects to an even self-homotopy that preserves the orientation. On the other side any self-homotopy of $K$ lifts either to a self-homotopy of $\tilde{K}$, if it is even, or to a homotopy between $\tilde{K}$ and $\sigma\tilde{K}$, if it is odd.

Now if $K$ does not admit any self-homotopy (for example if $\tilde{K}$ and $\sigma\tilde{K}$ are not homotopic) then any regular homotopy between two different models would lift to a regular homotopy between bands having $\tilde{K}$ as core and different number of half twists, and this is not possible by the classification of bands in orientable manifolds: this proves the first part of the theorem.

If $K$ admits an odd self-homotopy then take a regular odd self-homotopy and extend it to a regular self-homotopy $H$ of $N$. Call $\tilde{\Sigma}_i$ the lifting of $\Sigma_i$ to a band having $\tilde{K}$ as core; we might suppose that $\tilde{\Sigma}_i$ has $i$ half twists. The lifted regular homotopy $\tilde{H}$ takes $\tilde{\Sigma}_i$ to a band having $\sigma\tilde{K}$ as core, and the same number of half twists. Recall that, by [1.4.2], $\sigma\tilde{\Sigma}_i$ has $-i$ half twists, so, if $i$ is odd it is not the final image of $\tilde{H}$. But $\sigma\tilde{\Sigma}_i$ projects to $\Sigma_i$, so that the final image of $\Sigma_1$ under $H$ is the other model, that is, $\Sigma_{-1}$. The same construction applied to $\Sigma_0$ shows that a odd homotopy preserves the model when the band is orientable, and that even homotopy preserve comes from the first part. This proves the second assertion.

We are left to the case when $K$ is a non-orientable curve in $M$. Its tubular neighborhood $N$ is then a solid Klein bottle. We can fix a diffeomorphism of $N$ to the following model of solid Klein bottle: consider $D^2 \times I$ modulo the relation $(P,0) \sim (\rho(P),1)$, $\rho$ being the reflection of $\mathbb{R}^2$ in the $x$ axis restricted to $D^2$. We can consider that $\Sigma_0$ is $[-1,1] \times \{0\} \times I / \sim$. Remark now that, when we rotate $[-1,1] \times \{0\} \times \{0\}$ in the
positive direction, the other copy of this segment, that is \([-1, 1] \times \{0\} \times \{1\}\), must rotate in the negative direction, see figure 6. The effect on the band is that, after a rotation of \(\pi\), \(\Sigma_0\) becomes a band having the same support as \(\Sigma_2\), and differing from it only by a reparametrization. Remark that the two boundary components of \(\Sigma_0\) belong to different homology classes.

On the other side if we apply this isotopy to \(\Sigma_1\) and rotate of \(\pi/2\) we obtain a band with support \(\{0\} \times [-1, 1] \times I/\sim\); if we apply the opposite rotation to \(\Sigma_{-1}\) we obtain a band with the same support: but this time sliding the band across the orientation disk we obtain exactly \(\Sigma_1\), and this ends the proof. \(\triangle\)

This theorem shows that it is not possible to extend the definition of number of half twists to generic bands in non-orientable \(M\), since they can be equivalent up to regular homotopy plus reparametrization or simply up to regular homotopy.

2 Computation of cobordism.

We adapt this proof to the non-orientable case.

Figure 6: The twisting of a band with core non-orientable in \(M\).
2.1 Statement and scheme of proof.

Consider an immersion $f$ of a compact closed surface $F$, non necessarily connected nor orientable, in a non-orientable connected 3-manifold non necessarily compact nor closed. Up to regular homotopy $f$ can be considered generic: in this case this means it has a finite number of curves of double points and a finite number of triple points. We define 3 invariants associated to $f$: let $H_f$ be the homology class represented by $f$ in $H_2(M, \mathbb{Z}/2\mathbb{Z})$; let $\delta_f$ be the homology class represented by the locus of double points of $f$, that is, the points of $M$ having 2 or 3 preimages, in $H_1(M, \mathbb{Z}/2\mathbb{Z})$; let $n_f$ be the Euler characteristic of $F$ modulo 2 (if $F$ is not connected we consider the sum of the Euler characteristics of its connected components).

**Lemma 2.1.1** These functions are invariant up to cobordism.

**Proof.** Homology between cobordant immersions is provided by the immersion of the cobordism, so that $H_-$ is well-defined up to cobordism. Homology between the loci of double points of cobordant immersions is provided by the locus of double points of the immersion of the cobordism, so that $\delta_-$ is well-defined up to cobordism. As long as $n_-$ is concerned, recall that the Euler class modulo 2 is the total invariant of classical cobordism of surfaces, so that in particular it is invariant up to cobordism of immersions. △

If $Y$ is a cobordism class of immersions we can then define $H_Y$, $\delta_Y$ and $n_Y$ as the invariants associated to any generic representative of $Y$. We will prove that the function

$$\Psi : N_2(M) \rightarrow H_2(M; \mathbb{Z}/2\mathbb{Z}) \times H_1(M; \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$$

$$Y \mapsto (H_Y, \delta_Y, n_Y)$$

is a total invariant. First remark that

$$\Psi(Y + Y') = (H_Y + H_{Y'}, \delta_Y + \delta_{Y'}, H_Y \cdot H_{Y'}, n_Y + n_{Y'})$$

where $\cdot$ is the bilinear form of intersection; it is straightforward to check that the composition law

$$(H, \delta, n) * (H', \delta', n') = (H + H', \delta + \delta' + H \cdot H', n + n')$$

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makes $H_2(M, \mathbb{Z}/2\mathbb{Z}) \times H_1(M, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ a commutative group, and $\Psi$ is clearly an homomorphism for this group structure.

**Theorem 2.1.2** $\Psi$ is an isomorphism of groups between $N_2(M)$ and $(H_2(M, \mathbb{Z}/2\mathbb{Z}) \times H_1(M, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}, \ast)$.

**Proof.** We are left to prove that $\Psi$ is bijective.

We first see surjectivity. Given a triple $(H, \delta, n)$ consider: an embedding $f$ that represents $H$; an embedded circle $K$ representing $\delta$, its tubular neighborhood $N$ and the immersion $h$ obtained by adding a kink along any longitude of the inclusion of $\partial N$ (remark this is the immersion of a torus or of a Klein bottle according to $w_1(M)(\delta_Y)$ being 0 or 1); if $n_f = n$ then $\Psi(f + h) = (H, \delta, n)$; if $n_f \neq n$ then consider $g = \phi \circ \gamma$, $\gamma$ being a generator of $N_2(\mathbb{R}^3)$ and $\phi$ a diffeomorphism of $\mathbb{R}^3$ to a ball $B$ in $M$: then $\Phi(f + h + g) = (H, \delta, n)$; and this proves surjectivity.

Consider an immersion $f$ of $F$ in $M$ and let $l$ be a closed circle in $F$; we denote by $q_f(l)$ the number of half twists of the band given by the restriction of $f$ to a tubular neighborhood of $l$ in $F$, possibly perturbed by a regular homotopy. Remark that, since $M$ is non-orientable, this is only defined when $l$ is orientable in $F$ and $f(c)$ is orientable in $M$. To prove injectivity we show in lemma 2.2.1, adapting the argument in [BS95], that any generic representative of a class $Y$ can be deformed via surgeries to a representative $f + h + g$ such that: $f$ is an embedding; $h$ is the immersion obtained from the inclusion of the boundary of a tubular neighborhood of a curve representing $\delta_Y$ by adding a kink along a longitude, possibly such that $q_h(l) = 0$ if $\delta_Y$ is orientable; $g$ is an immersion contained in a ball in $M$. This decomposition splits $\Psi$ in its 3 components, that is $\Psi(Y) = (H_f, \delta_h, n_f + n_g)$, so that if $\Psi(Y) = \Psi(Y')$ we are left to show that:

$$H_f = H_{f'} \Rightarrow f \sim_c f'$$  
$$\delta_h = \delta_{h'} \Rightarrow h \sim_c h'$$  
$$n_g = n_{g'} \Rightarrow g \sim_c g'$$

where $\sim_c$ means cobordism relation. That (1) is true is the content of lemma 2.3.1; that (2) is true is the content of lemmas 2.3.4 and 2.3.6; that (3) is true is the content of lemma 2.3.8. $\triangle$
2.2 Decomposition lemma.

**Lemma 2.2.1** Any generic representative of a class $Y \in N_2(M)$ can be deformed via surgeries to a representative $f + h + g$ such that: $f$ is an embedding; $h$ is the immersion obtained from the inclusion of the boundary of a tubular neighborhood of a curve representing $\delta_Y$ by adding a kink along a longitude, possibly such that $q_h(l) = 0$ if $\delta_Y$ is orientable; $g$ is an immersion contained in a ball in $M$.

**Proof.** Let $Y$ be a cobordism class, and let $f_1$ be a generic representative of $Y$: it is an immersion with a finite number of curves of double points and a finite number of triple points. We call $C(M)$ the subgroup of $N_2(M)$ given by classes which admit a representative immersed in a disk of $M$.

First we eliminate triple points. To do so call $\gamma$ the right immersion of Boy in $\mathbb{R}^3$; recall $\gamma$ is an immersion with a single triple point; consider the connected sum of $f_1$ and $\gamma$ in a chart of $M$; it is possible to deform by regular homotopy this immersion to an immersion with one triple point less (remark that $\gamma$ in a chart of $M$ is cobordant to its inverse, see proposition 2.3.7). By recursively repeating this construction we obtain an immersion $f_2$ without triple points and cobordant to $f_1$ up to an element of $C(M)$.

The immersion $f_2$ has a finite number of curves of double points. Take two such curves, connect them with a path, then substitute the path with a couple of tubes as in figure 7.
By recursively repeating this operation we are then left with an immersion $f_3$ representing $Y$ up to an element of $C(M)$ and with a single curve $K$ of double points; clearly $K$ represents $\delta_Y$.

The intersection of the image of $f_3$ with a tubular neighborhood of $K$ is a bundle on $S^1$ with fiber a figure $X$, orientable or non-orientable according to the orientability of $\delta_Y$. If we number orderly the edges of the figure $X$ we can identify the its group of isometries with a subgroup of $S_4$; in this framework we can reduce the structure group of our fiber bundle to one and only one of the following:

1. $G_0 = 1$
2. $G_1 = \langle (1234) \rangle$
3. $G_2 = \langle (13)(24) \rangle$
4. $G_3 = \langle (1432) \rangle$
5. $G_4 = \langle (12)(34) \rangle$
6. $G_5 = \langle (24) \rangle$
7. $G_6 = \langle (14)(23) \rangle$
8. $G_7 = \langle (13) \rangle$

call $L_0, \ldots, L_7$ the corresponding bundles; remark the first 4 are orientable, the last 4 are not.

Now remark that the groups with even index act also on the figure 8 obtained by connecting with arcs edges 1 and 4 and edges 2 and 3; this implies that in the fiber bundles with even index it is possible to substitute the fiber, and so we obtain respectively immersions of a torus in a solid torus, of a Klein bottle in a solid torus, of a torus in a solid Klein bottle and of a Klein bottle in a solid Klein bottle.

Go back to the fiber bundle on the curve of double points. If it is isomorphic to $L_2$ or to $L_4$ we immerse along a curve homotopically trivial in $M$ the fiber bundle in figure 8’s obtained from $L_2$ by substitution of fiber and we connect its curve of double points with $K$: the curve of double points of the new immersion has tubolar neighborhood
isomorphic to $L_0$ or to $L_6$, respectively, and still represents $Y$ up to elements of $C(M)$. We can then perform a Rohlin surgery as in figure 8, and obtain from one side an embedding $f$ and from the other a map $h$ obtained from the inclusion of the boundary of a tubular neighborhood of a curve representing $\delta_Y$ by adding a kink along a longitude $l$; and the sum of the class of $f$ and $h$ differs from $Y$ by an element $g \in C(M)$. So if $\delta_Y$ is non-orientable we are done. If $\delta_Y$ is orientable instead we have to consider the case that $q_h(l) = 2$. If it is so consider in a disk of $M$ an immersion $h'$ obtained from the standard by adding a kink along a longitude $l'$ such that $q_{h'}(l') = 2$; if we connect the curve of double points of our immersion to the curve of double points of $h'$ in the usual manner we obtain an immersion we call again $h$ that satisfies $q_h(l) = 0$.

Now go to the case when the tubular neighborhood of $K$ is isomorphic to a $L_i$ with $i$ odd. We have to consider another auxiliary immersion: take on $[-1, 1]$ the fibration with fibers as in figure 8, and complete it to the immersion of a closed surface in a disk of $M$ by identifying the two figure 8’s at the two edges (without torsions) and close the hole of the fiber on 1 with a 2-disk $D$; the resulting immersion has two curves of double points meeting in a triple point in $D$; the first has tubular neighborhood isomorphic to $L_1$; by connecting this curve to $K$ in a new curve $K'$ we get an immersion $f_4$, still representing $Y$ up to elements of $C(M)$, with the tubular neighborhood of $K'$ isomorphic to an $L_i$ with $i$ even; and repeating the previous construction we can assume $i$ equal to 0. The other curve of double points of $f_4$, say $K''$, is contained in $D$ and is shown
Figure 9: Fibers on $[-1, 1]$. 

Figure 10: The curve of double points $K''$ of the immersion $f_4$. 

in figure 10. Perform Rohlin surgery on $K'$: on $D$ the result is shown in figure 11: to the two curves so created one can again perform Rohlin surgery and obtain elements of $C(M)$. So, up to again applying the previous construction when $\delta_Y$ is orientable, we have a decomposition satisfying the required properties. △

2.3 Other lemmas.
Lemma 2.3.1 Let $f$ and $f'$ be two embeddings of compact closed surfaces $F$ and $F'$, respectively, in the same non-orientable $3$-manifold $M$. Then $f \sim_c f'$ if and only if $H_f = H_{f'}$.

Proof. The “if” part is invariance of $H_-$ up to cobordism, and has already be proved.

For the “only if” part let $H$ be $H_f = H_{f'}$. If $M$ is compact and without boundary $f$ and $f'$ are essentially the loci of zeros of two transverse sections $s$ and $s'$ of the same line bundle $L$, Poincaré dual to $H$, and this allows to construct the cobordism this way: consider a smooth homotopy $s_t$ between $s$ and $s'$, that be constant for $t$ close to 0 and to 1; consider on $M \times I$ the line bundle pull-back of $L$, and consider $s_t$ as a section of this line bundle: $s_t$ is then transverse and its locus of zeros is the desired cobordism. If $M$ is not compact or closed in order to apply Poincaré duality consider a compact $3$-manifold $N$ with boundary, contained in $M$ and containing in its interior a $3$-chain that bounds $f + f'$; consider the compact closed $3$-manifold $\bar{N}$ obtained by gluing to $N$ a second copy of $N$ itself; again $f$ and $f'$ represent the same element $H$ in the homology of $\bar{N}$, so they are loci of zeros of transverse sections $s$ and $s'$ of the same line bundle $L$, Poincaré dual (in $\bar{N}$) to $H$; we can assume that $s$ and $s'$ coincide outside a compact contained in $N$; we construct as before a smooth homotopy $s_t$, that we pretend relative to a compact containing the complementary of $N$; then the construction runs as before, and gives a cobordism which is in fact contained in $N \times I$, hence in $M \times I$. $\triangle$

Figure 11: Result of Rohlin surgery on the disk $D$. 
We now prove (2). We first introduce some notation. We denote $[f]$ the class of $f$ up to reparametrizations of $F$, that is, $g$ belongs to $[f]$ if there exists a diffeomorphism $\phi$ of $F$ such that $g = f \circ \phi$; remark that regular homotopy equivalence relation is well defined up to reparametrization, and that immersions that differ by a reparametrization are cobordant (see [BS95], pages 657–8). Finally denote by $[f] * K$ the reparametrization class of $f * f^{-1}(K)$, that is well defined. The key of the proof are propositions 10.5 and 10.7 of [BS95], that we adapt to a non-orientable situation:

**Lemma 2.3.2** Let $f$ be an immersion of $F$ in $M$. Let $K$ be a closed circle in $f(F)$ such that $q_f(f^{-1}(K)) = 0$. Then:

1. if $K$ is trivial in $\pi_1(M)$ then $[f]$ and $[f] * K$ are regularly homotopic;
2. if $K$ is trivial in $H_1(M, \mathbb{Z}/2\mathbb{Z})$ then $[f]$ and $[f] * K$ are cobordant.

**Proof.** 1. Let $c = f^{-1}(K)$; $c$ is orientable in $F$. Let $\phi$ be a twist of Dehn of $F$ along $c$, we prove that $f * c$ and $f \circ \phi$ are regularly homotopic. We make direct use of the result of [HH85].

First remark that, since $K$ is homotopically trivial, $f \circ \phi$ is homotopic to $f$, that on its side is homotopic to $f * c$. We can then consider on $f * c$ and $f \circ \phi$ the correspondence $C_f$, as defined in the proof of theorem 1.2.1; we see that $C_f(f * c) = C_f(f \circ \phi)$.

Consider an embedding of the standard torus in $\mathbb{R}^3$ with the Spin structure induced by restriction of the unique Spin structure on $\mathbb{R}^3$ as a tubular neighborhood of $K$, in such a way that the Spin structure of either of the two preimages of $K$ in $\tilde{M}$ is respected.

The hypothesis $q_f(c) = 0$ implies that this embedding can be chosen in such a way that $f$ restricted to a tubular neighborhood of $c$ can be read as the embedding in the standard torus of a band with no half twists. In the same model then $f \circ \phi$ and $f * c$ are read as shown in figure 12, and far from $K$ they coincide with $f$.

Now take a representative $d$ of an element $H_1(F, \mathbb{Z}/2\mathbb{Z})$, transverse to $c$: following again the proof of the main theorem of [HH85] one shows that if $d \cdot c = 0$ then $C_f(f \circ \phi)(d) = C_f(f * c)(d) = 0$, and if $d \cdot c = 1$ then $C_f(f \circ \phi)(d) = C_f(f * c)(d) = 1$; and this ends the proof.

2. The proof of [BS95] can be used, by means of 1.△
We remark that, since immersions that differ by a reparametrization are cobordant, the second part of the lemma implies:

**Corollary 2.3.3** If $K$ is in the image of an immersion $f$ and is trivial in $H_1(M, \mathbb{Z}/2\mathbb{Z})$ then $f$ is cobordant to $f * f^{-1}(K)$. △

We are now ready for the lemmas that prove (2).

**Lemma 2.3.4** Let $\delta \in H_1(M, \mathbb{Z}/2\mathbb{Z})$ such that $w_1(M)(\delta) = 0$; let $K$ and $K'$ be closed circles both representing $\delta$; let $i$ and $i'$ be the inclusions of the boundaries of tubular neighborhoods of $K$ and $K'$, respectively. Let $h$ and $h'$ be two immersions obtained from $i$ and $i'$ by adding kinks along longitudes $l$ and $l'$ such that $q_i(l) = q_{i'}(l') = 0$. Then $h \sim_c h'$.

**Proof.** This is proven in [BS95], page 672. We review their argument. Remark that $i$ and $i'$ are immersions of tori, representing the identity of the semi-group structure; their connected sum $\tilde{i} = i \# i'$ still represents the identity. The sum of the cobordism classes of $h$ and $h'$ is represented by their connected sum $\bar{h} = h \# h'$; remark that $\bar{h}$ is obtained from $\tilde{i}$ by adding a kink along a curve $\bar{l}$ homologous to $l + l'$, so that $q_{\tilde{i}}(\bar{l}) = q_i(l) + q_{i'}(l') = 0$. Moreover the homology class of $\tilde{i}(\bar{l})$ is $2\delta$, that is $0$, in $H_1(M, \mathbb{Z}/2\mathbb{Z})$. So we can apply corollary 2.3.3 and say that $\bar{h}$ is cobordant to $\tilde{i}$, that is the identity; so $h'$ belongs to the opposite of the class of $h$. 26
But now do the same construction with two copies of $h$, and conclude that $h$ itself belongs to its own opposite class, hence the claim. $\triangle$

To settle the non-orientable case we first need a remark:

**Lemma 2.3.5** Let $K$ be a curve non-orientable in $M$, let $i$ be the inclusion of the boundary of a tubular neighborhood $N$ of $K$, let $l$ and $m$ be respectively a longitude and the meridian of $\partial N$; then

$$i \ast l \sim_c i \ast (l + m).$$

**Proof.** The homotopy class of $i$ is odd: in fact $i$ can be homotopically deformed to the inclusion of $K$, this can slide across a surface representing $w_1(M)$ and then going back to $i$, and these three steps give a odd self-homotopy of $i$. But now $m$ represents the dual to $w_1(\partial N)$, so that by proposition [1.2.1] adding a kink along $m$ doesn't change regular homotopy class, hence cobordism class. $\triangle$

**Lemma 2.3.6** Let $\delta \in H_1(M, \mathbb{Z}/2\mathbb{Z})$ such that $w_1(M)(\delta) \neq 0$; let $K$ and $K'$ be closed circles both representing $\delta$; let $i$ and $i'$ be the inclusions of the boundaries of tubular neighborhoods of $K$ and $K'$, respectively. Let $h$ and $h'$ be two immersions obtained from $i$ and $i'$ by adding kinks along longitudes $l$ and $l'$. Then $h \sim_c h'$.

**Proof.** The inclusions $i$ and $i'$ are immersions of Klein bottles, both representing the identity of the cobordism semi-group. Their connected sum $\tilde{i} = i \# i'$ still represents the identity. The sum of the cobordism classes of $h$ and $h'$ is represented by their connected sum $h \# h'$, which is an immersion regularly homotopic to the one obtained from $\tilde{i}$ by adding a kink along a curve $\tilde{l}$ homologous to $l + l'$; this is cobordant to the immersion obtained from $\tilde{i}$ by adding a kink along the curve $\tilde{l}'$ homologous to $l + l' + m$, $m$ being the meridian of the tubular neighborhood of $K$, because of lemma 2.3.5. But both of $\tilde{l}$ and $\tilde{l}'$ represent $2\delta$, hence 0, in $H_1(M, \mathbb{Z}/2\mathbb{Z})$, and one of $q_i(\tilde{l})$ or $q_i(\tilde{l}')$ must be 0, see figure [13]. So we can apply corollary 2.3.3, and obtain that $\tilde{h}$ is cobordant to $\tilde{i}$, that is the identity class. This implies that $h'$ belongs to the opposite class to $h$. 

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Figure 13: The curves $\bar{l}$ and $\bar{l}'$ on the image of the embedding $\bar{i}$.

But now applying the same construction to two copies of $h$ we obtain that $h$ itself belongs to its own opposite class, hence the claim. △

We are left now to the part that is in some way more characteristic of the non-orientable case, that is, the point that gives account of the fact that the cobordism groups of non-orientable 3-manifolds are “smaller” than the groups of the orientable case. The key is again the fact that in a non-orientable environment there is a kind of isotopy that doesn’t exist in an orientable manifold, that is, an isotopy that reverses orientation.

**Proposition 2.3.7** Let $C(M)$ be the subgroup of $N_2(M)$ given by classes which admit a representative immersed in a disk of $M$. $C(M)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

**Proof.** Consider a coordinate chart $(U, \phi)$ of $M$. The diffeomorphism $\phi$ induces a homomorphism

$$\phi^* : N_2(\mathbb{R}^3) \longrightarrow C(M)$$

$$f \mapsto \phi \circ f$$

that is evidently surjective. Recall that $N_2(\mathbb{R}^3)$ is a cyclic group of order 8 generated by the right immersion of Boy of $\mathbb{P}^2$; we call $\gamma$ this immersion. This implies that $C(M)$ is a cyclic group generated by $\phi^*(\gamma)$, that can’t be trivial since $\mathbb{P}^2$ generates the cobordism group of surfaces.
Let now $f_t$ be a self-isotopy of $B$ that reverses the orientation, for example that crosses once a surface representing $w_1(M)$. Recall that the inverse of $\gamma$ in $N_2(\mathbb{R}^3)$ is given by composition with a reflection in a plane, so that $f_1 \circ \phi^*(\gamma)$ is the image via $\phi^*$ of the canonical representative of the opposite class to $\gamma$, hence represents the opposite class to $\phi^*(\gamma)$. But $f_1 \circ \phi^*(\gamma)$, being isotopic to $\phi^*(\gamma)$, is cobordant to it, so that $\gamma$ belongs to its own opposite class, hence has order 2. △

**Lemma 2.3.8** Let $g$ and $g'$ be immersions whose image is contained in a disk of $M$; then $g \sim_c g'$ if and only if $n_g = n_{g'}$.

**Proof.** The “if” part is invariance of $n_-$ up to cobordism, and has already be proved. For the “only if” it is enough to remark that $n_-$ realizes the homomorphism of proposition 2.3.7. △

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Rosa Gini, gini@dm.unipi.it,
Dipartimento di Matematica “Leonida Tonelli”,
via Filippo Buonarroti 2,
I–56127 Pisa, Italy.