HOW STRONG A LOGISTIC DAMPING CAN PREVENT BLOW-UP FOR THE MINIMAL KELLER-SEGEL CHEMOTAXIS SYSTEM?

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Abstract. We study nonnegative solutions of parabolic-parabolic Keller-Segel minimal-chemotaxis-growth systems with prototype given by

\[
\begin{align*}
    u_t &= \nabla \cdot (d_1 \nabla u - \chi u \nabla v) + \kappa u - \mu u^2, \quad x \in \Omega, t > 0, \\
    v_t &= d_2 \Delta v - \beta v + \alpha u, \quad x \in \Omega, t > 0
\end{align*}
\]

in a smooth bounded smooth but not necessarily convex domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 3)\) with nonnegative initial data \( u_0, v_0 \) and homogeneous Neumann boundary data, where \( d_1, d_2, \alpha, \beta, \mu > 0, \chi, \kappa \in \mathbb{R} \).

We provide quantitative and qualitative descriptions of the competition between logistic damping and other ingredients, especially, chemotactic aggregation to guarantee boundedness and convergence. Specifically, we first obtain an explicit formula \( \mu_0 = \mu_0(n, d_1, d_2, \alpha, \chi) \) for the logistic damping rate \( \mu \) such that the system has no blow-ups whenever \( \mu > \mu_0 \).

In particular, for \( \Omega \subset \mathbb{R}^3 \), we get a clean formula for \( \mu_0 \):

\[
\mu_0(3, d_1, d_2, \alpha, \chi) = \begin{cases} 
    \frac{3}{4} \alpha \chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex}, \\
    \frac{3}{\sqrt{10}} \left( \frac{1}{d_1} + \frac{2}{d_2} \right) \alpha |\chi|, & \text{otherwise}
\end{cases}
\]

This offers a quantized effect of the logistic source on the prevention of blow-ups. Our result extends the fundamental boundedness principle by Winkler [53] with \( d_1 = 1, d_2 = \alpha = \beta = 1/\tau \), \( \Omega \) being convex and sufficiently large values of \( \mu \) beyond a certain number not explicitly known (except the simple case \( \tau = 1 \) and \( \chi > 0 \)) and quantizes the qualitative result of Yang et al. [63]. Besides, in non-convex domains, since \( \mu_0(3, 1, 1, 1, \chi) = (7.743416 \cdots) \chi \), the recent boundedness result, \( \mu > 20\chi \), of Mu and Lin [34] is greatly improved.

Then we derive another explicit formula: \( \mu_1 = 0 \) for \( \kappa \leq 0 \) and

\[
\mu_1 = \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) = \frac{\alpha |\chi|}{4} \sqrt{\frac{\kappa}{d_1 d_2 \beta}} \quad \text{for } \kappa > 0
\]

for the logistic damping rate so that convergence of bounded solutions is ensured and the respective convergence rates are explicitly calculated out whenever \( \mu > \mu_1 \). Recent convergence results of He-Zheng [12] are hence completed and refined. Furthermore, both \( \mu_0(n, d_1, d_2, \chi, \alpha) \) and \( \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) \) tend to infinity as \( d_1 \to 0 \) or \( d_2 \to 0 \) and they are decreasing in \( d_1 \) and \( d_2 \); therefore, small diffusion, especially, degenerate or nonlinear diffusion, enhances the possibility of the occurrence of "unbounded" solutions. This gives a clue on how to produce blow-up solutions for Keller-Segel chemotaxis models with logistic source.

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1. Introduction and outline of the main results

Chemotaxis, the tendency of cells, bacteria and similarly tiny organisms to orient the direction of movement (otherwise random) toward increasing or decreasing concentration of a signaling substance, has been attracting great attention in biological and mathematical community. A celebrated mathematical model, initially proposed by Keller-Segel [23, 24], makes up of two parabolic equations reflecting chemotactic movement through a nonlinear advective-diffusive term as its most defining characteristic. Their pioneering works have initiated vast investigations of the K-S model and its various forms of variants since 1970. We refer to the beautiful survey papers [3, 16, 19, 57], where a broad survey on the progress of various chemotaxis models and rich selection of references can be found.

If biological processes in which chemotaxis plays a role are modeled not only on small timescales, often the spontaneous growth of the population, whose density we will denote by $u$, should be incorporated. A prototypical choice to achieve this is the logistic type source $\kappa u - \mu u^2$ with birth and death rates $\kappa$ and $\mu$, respectively. Let us then begin with the most simplest perhaps the most interesting so-called minimal-chemotaxis-growth model

\[
\begin{cases}
  u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + \kappa u - \mu u^2, & x \in \Omega, t > 0, \\
  \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega,
\end{cases}
\]

(1.1)

where $\Omega \subset \mathbb{R}^n, n \geq 1$ is a smooth bounded domain, $v$ denotes the concentration of the chemical signal, $\kappa \geq 0, \mu \geq 0, \tau \geq 0$ and $\chi \in \mathbb{R}$. The nonlinear term $\chi u \nabla v$, the defining term in chemotaxis models, is called chemotactic term: in the case $\chi > 0$, it models the cells movement toward the higher concentrations of the chemical signal, which is called (positive) chemotaxis, in the case $\chi < 0$, it models the cells movement away from the higher concentrations of the chemical signal, which is called negative chemotaxis.

Model (1.1) with $\kappa = 0, \mu = 0$ corresponds to the classical Keller-Segel minimal model [23, 24], which and whose variants have been extensively explored since 1970. The striking feature of KS type models is the possibility of blow-up of solutions in a finite/infinite time (see, e.g., [19, 37, 57]), which strongly depends on the space dimension. A finite/infinite time blow-up never occurs in 1-dimension [15, 39, 60] (except in some extreme nonlinear degenerate diffusion model [8]), a critical mass blow-up occurs in 2-dimension: when the initial mass lies below the threshold solutions exist globally, while above the threshold solutions blow up in finite time [18, 35, 41], and generic blow-up in $\geq 3$-D [52, 57]. The knowledge about the classical KS type models appears to be rather complete, see the aforementioned surveys for more.

The blow up solution or a $\delta$ function is surely connected to the phenomenon of cell aggregation; on the other hand, various mechanisms proposed to the model have manifested that the blow-up solutions is fully precluded.
CHEMOTAXIS SYSTEM, LOGISTIC SOURCE, AND BOUNDEDNESS

while pattern formation arises \([7, 16, 30, 62]\). Among those mechanisms (see the introduction in [62]), inclusion of a growth source of cells is a common choice. In particular, the presence of logistic source has been shown to have an effect of preventing ultimate growth of populations. Indeed, in the case \(n = 1, 2\), even arbitrarily small \(\mu > 0\) will be sufficient to suppress blow-up by ensuring all solutions to (1.1) are global-in-time and uniformly bounded for all reasonably initial data \([15, 38, 39, 61]\). This is even true for a two-dimensional parabolic-elliptic chemotaxis system with singular sensitivity \([13]\). Whereas, in the case \(n \geq 3\), the first boundedness and global existence were obtained for a parabolic-elliptic simplification of (1.1), i.e., \(\tau = 0\), under the condition that \(\mu > \frac{(n-2)}{n} \chi\) \([47]\). Recently, this result was improved to the borderline case \(\mu > \frac{n-2}{n} \chi\) \([50]\). See also the existence of very weak solutions under more general conditions \([51]\). For the full parabolic-parabolic minimal chemotaxis-growth mode, fundamental findings were obtained by Winkler \([53]\). Under the additional assumptions that \(\Omega\) is convex and \(\mu\) is beyond a certain number \(\mu_0\) not explicitly known (except the case \(\tau = 1\) and \(\chi > 0\), where \(\mu > \frac{n}{4} \chi\) is sufficient to prevent blow-ups), he proved the existence and uniqueness of global, smooth, bounded solutions to (1.1). Recently, a progress on global boundedness to (1.1) with \(\chi > 0\) was derived as long as \(\mu > \theta_0 \chi\) for some implicit positive constant \(\theta_0\) depending on Sobolev embedding constants \([63]\). In 2015, an explicit lower bound for a 3-D chemotaxis-fluid system with logistic source was obtained by Tao and Winkler \([44]\), when applied to the chemotaxis system (1.1) with \(\chi = \tau = 1\), their result states that \(\mu \geq 23\) is enough to prevent blow-ups. This bound was further improved by Mu and Lin \([34]\) (2016) in three dimensions, wherein they replaced the logistic source in (1.1) by the damping term \(u - \mu u^r\) with \(r \geq 2\) to derive the boundedness under

\[
\mu > 20 \chi.
\]

Their arguments were done to the case that \(\Omega\) is convex by remarking that they could be adapted to non-convex domains by virtue of the papers \([22, 33]\). Of course, when \(r > 2\), this result was already implied by \([53, 65]\). Moreover, for the particular choices \(\kappa = \tau = 1\), under certain largeness condition on the ration \(\mu/\chi\), the stabilization of bounded solution \((u, v)\) of (1.1) to the constant equilibrium \((1/\mu, 1/\mu)\) as \(t \to \infty\) occurs \([14, 34, 58]\). While, for arbitrarily small \(\mu > 0\), only existence of global weak solutions to (1.1) is available \([26]\) in convex 3-D domains. Other dynamical properties of (1.1) can be found in \([6, 12, 17, 27, 45, 59]\). Finally, we observe that enormous variants of (1.1) have been considered to provide conditions on diffusion, degradation, chemo-sensitivity and mostly on the growth source ensuring the boundedness of the proposed models \([1, 2, 26, 36, 38, 43, 48, 49, 61, 64, 65]\) and the references therein, and that explosion of solutions is possible in chemotaxis systems despite logistic growth restriction \([50, 60]\). Therefore, it is meaningful to detect more circumstances where no blow-up is allowed for the minimal KS model with logistic growth (1.1).
It is widely known that the KS minimal model \((1.1)\) with \(\kappa = \mu = 0\) admits both bounded and unbounded solutions, identified via the critical chemotactic sensitivity \(u^2 \in [10, 20, 32]\). Therefore, the model \((1.1)\) is simply a supercritical case with the balance of logistic damping and aggregation effects, for which the property of solutions should be not only qualitatively but also quantitatively determined by the parameters involved. Motivated by the works \([34, 44, 53, 63]\), we attempt to provide a quantitative description of the competition between logistic damping and other ingredients, especially, chemotactic aggregation, and, in particular, we aim to find a full picture on how the lower bound \(\mu_0\) of the logistic damping rate \(\mu\) is affected by all the involving parameters so that no blow-up is allowed for \(\mu > \mu_0\). Therefore, in this paper, we will consider a full parameter K-S minimal system with a growth source covering the standard logistics source as follows:

\[
\begin{align*}
    u_t &= \nabla \cdot (d_1 \nabla u - \chi u \nabla v) + f(u), & x \in \Omega, t > 0, \\
    v_t &= d_2 \Delta v - \beta v + \alpha u, & x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^n, n \geq 1\) is a smooth bounded domain but not necessarily convex, \(d_1, d_2, \alpha, \beta > 0, \chi \in \mathbb{R}\) and \(f : \mathbb{R} \to \mathbb{R}\) is smooth and satisfies \(f(0) \geq 0\) as well as

\[
f(s) \leq a - \mu s^2, \quad \forall s \geq 0
\]

for some \(a \geq 0\) and \(\mu > 0\).

With the aid of the boundedness criteria on how the growth source affects the boundedness for a general class of chemotaxis-growth systems than \((1.3)\) \([3, 61]\), we provide a detailed algorithm to derive an explicit formula for the lower bound \(\mu_0\) of the logistic damping rate \(\mu\) such that the system \((1.3)\) admits only globally bounded solutions whenever \(\mu > \mu_0\). For nonconvex domains, our procedure is mainly carried out in physically relevant settings \((n = 3, 4, 5)\), where we have a clean and compact formula for \(\mu_0\). Precisely, our main quantitative findings in this regard read as follows:

**Theorem 1.1.** [How strong a logistic damping can prevent blow-up for \((1.3)\)]  
Let \(\Omega \subset \mathbb{R}^n (n \geq 3)\) be a bounded smooth domain, \(f\) satisfy the logistic condition \((1.4)\) and \(d_1, d_2, \alpha, \beta > 0, a \geq 0\) and \(\chi \in \mathbb{R}\).

(i) For \(n = 3\), let the lower logistic damping rate \(\mu_0 = \mu_0(3, d_1, d_2, \alpha, \chi)\) be explicitly given by

\[
\mu_0 = \left\{ \begin{array}{ll}
\frac{3}{4d_2} \alpha \chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex,} \\
\frac{3}{\sqrt{10 - 2}} \left( \frac{1}{d_1} + \frac{2}{d_2} \right) \alpha |\chi|, & \text{otherwise;}
\end{array} \right.
\]
(ii) for $n = 4, 5$, let the lower logistic damping rate $\mu_0 = \mu_0(n, d_1, d_2, \alpha, \chi)$ be explicitly given by

$$
\mu_0 = \begin{cases} 
\frac{n}{d_1} \alpha \chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex}, \\
\max \left\{ \frac{1}{2} h(n, d_1, d_2), \frac{n}{\sqrt{2n+4}} \left( \frac{1}{d_1} + \frac{3}{d_2} \right) \alpha |\chi| \right\}, & \text{otherwise} 
\end{cases}
$$

with

$$
h(n, d_1, d_2) = \min_{0 < \epsilon < d_1, 0 < \eta < d_2} \left\{ \sqrt{\frac{n}{18d_2}} + \sqrt{\frac{1}{2\epsilon} \left( \frac{1}{\eta} + \frac{n}{2d_2} \right)} + \sqrt{\frac{1}{(d_2-\eta)} \left( \frac{2}{\eta} + \frac{n}{2d_2} \right)} \right\}.
$$

Then, whenever $\mu > \mu_0$, the chemotaxis-growth system \[ \text{(1.3)} \] has a unique global solution $(u, v)$ for which both $u$ and $v$ are nonnegative and bounded in $\Omega \times (0, \infty)$.

**Remark 1.2.** [Notes on how strong a logistic damping can prevent blow-up]

(P1) The explicit logistic damping rate $\mu_0$ given in \[ \text{(1.5)} \] or \[ \text{(1.6)} \] exhibits the contributions of the degradation, creation and diffusion rates, etc in respective of boundedness of solutions of \[ \text{(1.3)} \]. That is, it shows how strong a logistic damping is needed to prevent blow-ups for \[ \text{(1.3)} \].

(P2) The formula for $\mu_0$ is not only explicitly expressible but also is independent of the degradation rate $\beta$ of signals, the birth rate $a$ of cells, the size of domain $\Omega$, initial data $u_0, v_0$ and Sobolev embedding constants. This gives a quantized effect of the logistic source on preventing blow-ups, and hence improves the boundedness principles \[ \text{[53, 61]} \] and the qualitative result \[ \text{[63]} \].

(P3) The form of $\mu_0(n, d_1, d_2, \alpha, \chi)$ captures our common understanding, since either $\chi = 0$ (no chemotaxis, cf. \[ \text{[67, Proposition 2.6]} \]) or $\alpha = 0$ (decoupled), the boundedness and global existence are easily seen for any $\mu > 0$.

(P4) The chemo-repulsion case, i.e., $\chi < 0$ is allowed as well in $\mu_0$.

(P5) $\mu_0(n, d_1, d_2, \alpha, \chi) \to \infty$ as $d_1 \to 0$ or $d_2 \to 0$ (and it is decreasing in $d_1$ and $d_2$); thus small diffusion, especially, degenerate or nonlinear diffusion, enhances the possibility of the occurrence of blow-ups.

(P6) In the case that $\Omega$ is nonconvex, we have

$$
\mu_0^{\text{(nonconvex)}}(3, 1, 1, 1, \chi) = \frac{9}{\sqrt{10} - 2} \chi = (7.743416 \cdots) \chi.
$$

Hence, the very recent boundedness result, $\mu > 20 \chi$ obtained from \[ \text{(1.2)} \] with $r = 2$, of Mu and Lin \[ \text{[34]} \], and the byproduct boundedness, $\mu \geq \mu_0(3, 1, 1, 1, 1) = 23$ of Tao and Winkler \[ \text{[44]} \], as their studied 3-D fluid system coupled with the minimal chemotaxis system \[ \text{(1.1)} \], are greatly improved.

For mathematical completeness, one may wonder such an explicit formula is also available in $\geq 6$ dimensions. Indeed, our algorithm suggests that an
explicit formula for $\mu_0$ in $n$-D ($n \geq 6$) of the form $\theta_0(n, d_1, d_2)\alpha|\chi|$, which is not clean but enjoys the first 5 properties (P1)-(P5), would be also available:

\[
\mu_0(n, d_1, d_2, \alpha, \chi) = \begin{cases} 
\frac{n}{4d_1}\alpha\chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex,} \\
\theta_0(n, d_1, d_2)\alpha|\chi|, & \text{otherwise}
\end{cases}
\]

(1.7)

with some explicit (perhaps messy) formula $\theta_0$ depending only on $n, d_1$ and $d_2$ with the property that $\theta_0 \to \infty$ as either $d_1 \to 0$ or $d_2 \to 0$.

Indeed, the first case of (1.7) has been shown in [53] for the KS model (1.1) with $\tau = 1$, see Lemma 4.8 for the full-parameter model (1.3). Based on [63] and carefully and painfully scrutinizing our procedure, it is quite possible to trace out the formula $\theta_0$ in (1.7). Here, we leave the rigorous justification for future investigations.

In nonconvex domains, the formula $\mu_0(3, d_1, d_2, \alpha, \chi)$ is the smallest damping rate that we could obtain using this procedure. While, in convex domains, we have $\mu_0^{(\text{convex})}(3, 1, 1, 1, 1) = 0.75$. A comparison to $\mu_0^{(\text{nonconvex})}(3, 1, 1, 1, 1) = 7.743416 \cdots$ as computed in (P3), indicates that convexity makes a big difference and that the formula $\mu_0(n, d_1, d_2, \alpha, \chi)$ may not be optimal even through it meets the expected properties described in (P2) and (P3), on the other hand.

This discussions lead us to ask other challenging questions left for the minimal-chemotaxis-growth model (1.3):

(Q2) Is there a critical damping rate $\mu_0^c$ that distinguishes between occurrence and impossibility of blow-up for (1.3)? That is, when the logistic source satisfies $f(u) \leq a - \mu u^2$ with $\mu < \mu_0^c$, blow-up occurs; when the source satisfies $f(u) \leq a - \mu u^2$ with $\mu > \mu_0^c$, blow-up is impossible.

(Q3) What happens for small logistic damping $\mu < \mu_0$, boundedness or blow-up?

Questions akin to (Q2) and (Q3) may have been indicated by existing literature [53, 61]. To explore them, a combination of the references [27, 42, 45, 47, 50, 56, 58, 59] may be of some help. A complete (quantitative or qualitative) description of the competition between chemotactic aggregation and logistic damping is definitely worthwhile for future explorations.

When logistic damping wins over chemotactic aggregation, i.e., $\mu > \mu_0$ or equivalently $|\chi| < \frac{a}{\theta_0(n, d_1, d_2)}$, we wish to see the explicit effects of each term in (1.3) on the long time dynamical properties of bounded solutions. To this end, we study the large time behavior for the minimal chemotaxis
model with a standard logistic source:

\[
\begin{aligned}
&u_t = \nabla \cdot (d_1 \nabla u - \chi u \nabla v) + \kappa u - \mu u^2, \quad x \in \Omega, t > 0, \\
v_t = d_2 \Delta v - \beta v + \alpha u, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega.
\end{aligned}
\]  

(1.8)

For \(d_1 = d_2 = \alpha = \beta = \kappa = 1\), the uniform convergence of bounded solution to \((1/\mu, 1/\mu)\) is first proved in \([58, \text{Theorem 1.1}]\) for \(\mu/\chi > 0\) sufficiently large and then in \([34, \text{Theorem 1.3}]\) for \(\mu > 20\chi\); for \(d_1 = d_2 = \alpha = \beta = 1\), He and Zheng \([14]\) modified the energy functional method from \([2, 21]\) to obtain the stabilities of the constant equilibria \((0,0)\) and \((\kappa/\mu, \kappa/\mu)\) with convergence estimates \([14, \text{Theorem 3}]\). Next, for completeness and to see the role of other parameters in the large time behavior of the solutions, we combine the energy functional method from \([2, 14, 21]\) to show the stability of the constant equilibria \((0,0)\) and \((\kappa/\mu, \kappa/\mu)\) for the full-parameter KS model \((1.8)\). Our precise results on the large time limit of bounded solutions of \((1.8)\) are collected in the following theorem.

**Theorem 1.3.** Let \(\Omega \subset \mathbb{R}^n (n \geq 3)\) be a bounded smooth domain, \(u_0 \geq, \neq 0\), \(d_1, d_2, \alpha, \beta, \mu > 0\), \(\kappa, \chi \in \mathbb{R}\) and, finally, let \(\mu > \mu_0\) as obtained in Theorem \([17]\) so that \((u, v)\) be a global and bounded solution of \((1.8)\).

(i) When \(\kappa > 0\), assume additionally that

\[
\mu > \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) = \frac{\alpha |\chi|}{4} \sqrt{\frac{\kappa}{d_1 d_2 \beta}}.
\]  

(1.9)

Then the solution \((u, v)\) of \((1.8)\) converges exponentially:

\[
\|u(\cdot, t) - \frac{\kappa}{\mu} L^\infty(\Omega) + \|v(\cdot, t) - \frac{\alpha \kappa}{\beta \mu} L^\infty(\Omega) \leq C e^{-\gamma t}
\]  

(1.10)

for all \(t \geq 0\) and some large constant \(C\) independent of \(t\) and

\[
\gamma = \min \left\{ \left( \mu - \frac{\alpha \kappa^2}{4 d_1 d_2 \mu} \right), \frac{\kappa^2}{4 d_1 d_2 \mu} \left( \beta - \frac{\alpha}{\chi} \right) \right\}, \quad \epsilon_0 = \frac{1}{2} \left( \frac{\alpha}{4 \beta} + \frac{4 d_1 d_2 \mu^2}{\alpha \kappa^2} \right).
\]

(ii) When \(\kappa = 0\), the solution \((u, v)\) of \((1.8)\) converges algebraically:

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C (t + 1)^{-\frac{\alpha}{4 \beta}}
\]  

(1.11)

for all \(t \geq 0\) and some large constant \(C\) independent of \(t\).

(iii) When \(\kappa < 0\), the solution \((u, v)\) of \((1.8)\) converges exponentially:

\[
u(\cdot, t) \leq C e^{\frac{\beta}{\kappa + t}}, \quad \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C e^{-\frac{\beta}{\kappa + t} \min \{ \beta - \kappa \} t}
\]  

(1.12)

for all \(t \geq 0\) and some large constant \(C\) independent of \(t\).

**Remark 1.4.** [Explicit effects on convergence and convergence rate]
The formula \( \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) \) exhibits the explicit effects of each gradient in (1.3) on convergence and it enjoys the property described in (P3). Besides, our convergence completes and refines [14, Theorem 3 for \( d_1 = d_2 = \alpha = \beta = 1 \)] by computing out the explicit rates of convergence.

From Remark 1.2, we find that \( \kappa \) and \( \beta \) do not play any role in boundedness, while, they play big roles in the long time behavior as seen in (1.9). These effects have not been detected yet in the literature. In particular, \( \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) \) is decreasing in \( d_1, d_2 \) and \( \beta \), and \( \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) \to \infty \) as \( d_1 \to 0 \) or \( d_2 \to 0 \) or \( \beta \to 0 \). Therefore, small diffusion, especially, degenerate or small degradation, makes the stabilization harder.

In chemotaxis-growth systems, the most challenging and interesting wide-open question is to detect the possibility of finite/infinite-time physical blow-ups \( (n = 3) \) (nonphysical blow-ups has been demonstrated in [56, 66] for \( n \geq 5 \)). Remarks 1.2 and 1.4, especially, (P5) and (P8) suggest certain clues on how to produce unbounded solutions [4, 9, 10, 28, 54, 55]. To attack such a challenging problem, one may try the following chemotaxis system and perhaps its simplified version:

\[
\begin{align*}
    u_t &= \nabla \cdot (\epsilon_1(u) \nabla u - u \nabla v) + \kappa u - \mu u^2, \quad x \in \Omega, t > 0, \\
    \tau v_t &= \epsilon_2 \Delta v - \epsilon_3 v + u, \quad x \in \Omega, t > 0 
\end{align*}
\]

with \( \epsilon_i > 0 \) \( (i = 2, 3) \) being sufficiently small and either \( \epsilon_1(u) \) being a sufficiently small positive constant or \( \epsilon_1(u) \to 0 \) as \( u \to \infty \) (very slow diffusion at point of high densities). Indeed, for \( \tau = 0, \epsilon_1(u) = 0, \epsilon_2 = \epsilon_3 = 1 \) in (1.13), verifications can be found in [27, 50]. We leave the challenging exploration of the possibility of blow-ups to (1.13) (and hence (Q2) and (Q3)) for our future studies.

2. Preliminaries and subtle inequalities for (1.1)

For convenience, we start with Young’s inequality, which states, for any positive numbers \( p \) and \( q \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), that

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b \geq 0.
\]

This immediately implies the so-called Young’s inequality with \( \epsilon \):

**Lemma 2.1.** (Young’s inequality with \( \epsilon \)) Let \( p \) and \( q \) be two given positive numbers with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, for any \( \epsilon > 0 \), it holds

\[
ab \leq \epsilon a^p + \frac{b^q}{(\epsilon p)^{\frac{q}{p}}}, \quad \forall a, b \geq 0.
\]

The local solvability and extendibility of the parabolic-parabolic chemotaxis system (1.3) is well-established by using a suitable fixed point argument and standard parabolic regularity theory; see, for example, [20, 29, 43, 53].
Lemma 2.2. Let $d_1, d_2, \alpha, \beta > 0$, $a \geq 0$, $\chi \in \mathbb{R}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Suppose that the initial data $(u_0, v_0)$ satisfies $u_0 \in C(\Omega)$ and $v_0 \in W^{1,p}(\Omega)$ with some $p > n$ and that $f \in W^{1,\infty}_{loc}(\mathbb{R})$ with $f(0) \geq 0$. Then there is a unique, nonnegative, classical maximal solution $(u, v)$ of the IBVP (1.3) on some maximal interval $[0, T_{\text{max}})$ with $0 < T_{\text{max}} \leq \infty$ such that

$$u \in C(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})),$$

$$v \in C(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})) \cap L^\infty_{loc}([0, T_{\text{max}}); W^{1,p}(\Omega)).$$

In particular, if $T_{\text{max}} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,p}(\Omega)} \to \infty \quad \text{as} \ t \to T_{\text{max}}. \quad (2.1)$$

Lemma 2.3. Let $f$ satisfy the logistic condition (1.4). Then the solution $(u, v)$ of (1.3) satisfies

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u^p + (p-1)(d_1 - \epsilon) \int_\Omega u^{p-2} |\nabla u|^2 + \mu \int_\Omega u^{p+1} \leq \frac{(p-1)\chi^2}{4\epsilon} \int_\Omega u^p |\nabla v|^2 + a \int_\Omega u^{p-1} \quad (2.2)$$

for all $t \in (0, T_{\text{max}})$ and for any $\epsilon \in (0, d_1)$.

Proof. For any $p \geq 1$, multiplying the $u$-equation in (1.3) by $u^{p-1}$ and integrating over $\Omega$ by parts, using Young’s inequality with $\epsilon$ and the logistic
condition (1.4), we conclude that
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p + (p-1)d_1 \int_\Omega u^{p-2} |\nabla u|^2
\]
\[
= (p-1) \chi \int_\Omega u^{p-1} \nabla u \nabla v + \int_\Omega f(u) u^{p-1}
\]
\[
\leq (p-1) \epsilon \int_\Omega u^{p-2} |\nabla u|^2 + \frac{(p-1)^2}{4\epsilon} \int_\Omega u^p |\nabla v|^2 + \int_\Omega u^{p-1} (a - \mu u^2),
\]
which gives the desired inequality (2.2).

\[\Box\]

**Lemma 2.5.** For any \( q \geq 1 \), the solution \((u,v)\) of (1.3) satisfies
\[
\frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} + (q-1)(d_2 - \eta) \int_\Omega |\nabla v|^{2(q-2)} |\nabla |\nabla v||^2 + 2\beta \int_\Omega |\nabla v|^{2q}
\]
\[
\leq \left[\frac{(q-1)\alpha^2}{\eta} + \frac{n\alpha^2}{2d_2}\right] \int_\Omega u^2 |\nabla v|^{2(q-1)} + d_2 \int_\Omega \nabla v^{2(q-1)} \frac{\partial}{\partial v} |\nabla v|^2 \qquad (2.3)
\]
for all \( t \in (0, T_{\text{max}}) \) and for any \( \eta \in (0, d_2) \).

**Proof.** For any \( q \geq 1 \), by using the \( v \)-equation in (1.3) and integrating by parts, we deduce that
\[
\frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} = 2 \int_\Omega |\nabla v|^{2(q-1)} \nabla v \cdot (d_2 \nabla \Delta v - \beta \nabla v + \alpha \nabla u)
\]
\[
= d_2 \int_\Omega |\nabla v|^{2(q-1)} \Delta |\nabla v|^2 - 2d_2 \int_\Omega |\nabla v|^{2(q-1)} |D^2 v|^2 - 2\beta \int_\Omega |\nabla v|^{2q}
\]
\[
+ 2\alpha \int_\Omega \nabla u \nabla v |\nabla v|^{2(q-1)}
\]
\[
= -(q-1)d_2 \int_\Omega |\nabla v|^{2(q-2)} |\nabla |\nabla v||^2 + d_2 \int_\Omega |\nabla v|^{2(q-1)} \frac{\partial}{\partial v} |\nabla v|^2
\]
\[
- 2d_2 \int_\Omega |\nabla v|^{2(q-1)} |D^2 v|^2 - 2\beta \int_\Omega |\nabla v|^{2q} + 2\alpha \int_\Omega \nabla u \nabla v |\nabla v|^{2(q-1)},
\]
where, from the first to the second line, we have used the point-wise identity
\[
2 \nabla v \cdot \nabla \Delta v = \Delta |\nabla v|^2 - 2|D^2 v|^2, \quad |D^2 v|^2 = \sum_{i,j=1}^n |v_{xixj}|^2. \quad (2.4)
\]
Now, we utilize integration by parts, Young’s inequality with \( \epsilon \) and the fact that
\[
|\Delta v|^2 = \left(\sum_{i=1}^n v_{xixi}\right)^2 \leq n \sum_{i=1}^n |v_{xixi}|^2 \leq n|D^2 v|^2 \quad (2.5)
\]
to estimate the last integral as follows:

\[
2\alpha \int_{\Omega} \nabla u \nabla v |\nabla v|^2(q-1) \\
= -2(q-1)\alpha \int_{\Omega} u|\nabla v|^{2(q-2)} \nabla v \cdot \nabla |\nabla v|^2 - 2\alpha \int_{\Omega} u|\nabla v|^{2(q-1)} \Delta v \\
\leq (q-1)\eta \int_{\Omega} |\nabla v|^{2(q-2)} |\nabla |\nabla v|^2|^2 + \frac{(q-1)\alpha^2}{\eta} \int_{\Omega} u^2|\nabla v|^{2(q-1)} \\
+ 2d_2 \int_{\Omega} |\nabla v|^{2(q-1)} |D^2 v|^2 + \frac{n\alpha^2}{2d_2} \int_{\Omega} u^2|\nabla v|^{2(q-1)}.
\]

Combining these two results, we obtain the desired inequality (2.3). □

**Lemma 2.6.** Let \( f \) satisfy the logistic condition (1.4). Then, for any \( p \geq 1, q \geq 1 \), the solution \((u, v)\) of (1.3) verifies

\[
\frac{d}{dt} \int_{\Omega} u^p|\nabla v|^{2q} + (p-1)(d_1 - \epsilon)p \int_{\Omega} u^{p-2}|\nabla u|^2|\nabla v|^{2q} + 2\beta q \int_{\Omega} u^p|\nabla v|^{2q} \\
+ (q-1)(d_2 - \eta)q \int_{\Omega} u^p|\nabla v|^{2(q-2)} |\nabla |\nabla v|^2|^2 + \mu p \int_{\Omega} u^{p+1}|\nabla v|^{2q} \\
\leq \frac{\chi^2(p-1)p}{4\epsilon} \int_{\Omega} u^p|\nabla v|^{2(q+1)} + \chi pq \int_{\Omega} u^p|\nabla v|^{2(q-1)} \nabla v \nabla |\nabla v|^2 \\
- (d_1 + d_2)pq \int_{\Omega} u^{p-1}|\nabla v|^{2(q-1)} \nabla u \nabla |\nabla v|^2 \\
+ \frac{q\alpha^2}{(p+1)^2} \left[ \frac{(q-1)}{\eta} + \frac{n}{2d_2} \right] \int_{\Omega} u^{p+2}|\nabla v|^{2(q-1)} \\
+ ap \int_{\Omega} u^{p-1}|\nabla v|^{2q} + d_2 q \int_{\partial \Omega} u^p|\nabla v|^{2(q-1)} \frac{\partial |\nabla v|^2}{\partial \nu} \\
\tag{2.6}
\]

for all \( t \in (0, T_{max}) \) and for any \( \epsilon \in (0, d_1) \) and \( \eta \in (0, d_2) \).
Proof. We use the $u$ and $v$-equations in [183], no flux boundary conditions for $u$ and $v$ and integration by parts to compute honestly that

$$
\frac{d}{dt} \int_\Omega u^p |\nabla v|^{2q} = p \int_\Omega u^{p-1} |\nabla v|^{2q} |\nabla \cdot (d_1 \nabla u - \chi u \nabla v) + f(u)|
$$

$$
+ 2q \int_\Omega u^p |\nabla v|^{2(q-1)} \nabla v \cdot (d_2 \nabla \Delta v - \beta \nabla v + \alpha \nabla u)
$$

$$
= -p \int_\Omega (d_1 \nabla u - \chi u \nabla v) \left[(p-1)u^{p-2} \nabla u |\nabla v|^{2q} + qu^{p-1} |\nabla v|^{2(q-1)} |\nabla v|^2 \right]
$$

$$
+ p \int_\Omega u^{p-1} |\nabla v|^{2q} f(u) + d_2 q \int_\Omega u^p |\nabla v|^{2(q-1)} \Delta |\nabla v|^2 - 2\beta q \int_\Omega u^p |\nabla v|^{2q}
$$

$$
- 2d_2 q \int_\Omega u^p |\nabla v|^{2(q-1)} |D^2 v|^2 + 2\alpha q \int_\Omega u^p |\nabla v|^{2(q-1)} \nabla u \nabla v
$$

$$
= -(d_1 + d_2) pq \int_\Omega u^{p-1} |\nabla v|^{2(q-1)} \nabla u |\nabla v|^2 - d_1 (p-1) p \int_\Omega u^{p-2} |\nabla u|^2 |\nabla v|^{2q}
$$

$$
+ p \int_\Omega u^{p-1} |\nabla v|^{2q} f(u) + \chi (p-1) p \int_\Omega u^{p-1} |\nabla v|^{2q} \nabla u \nabla v
$$

$$
+ \chi pq \int_\Omega u^p |\nabla v|^{2(q-1)} \nabla v \nabla |\nabla v|^2 - d_2 q (q-1) \int_\Omega u^p |\nabla v|^{2(q-2)} |\nabla |\nabla v|^2|^2
$$

$$
- 2d_2 q \int_\Omega u^p |\nabla v|^{2(q-1)} |D^2 v|^2 - 2\beta q \int_\Omega u^p |\nabla v|^{2q}
$$

$$
+ 2\alpha q \int_\Omega u^p |\nabla v|^{2(q-1)} \nabla u \nabla v + d_2 q \int_{\partial \Omega} u^p |\nabla v|^{2(q-1)} \frac{\partial |\nabla v|^2}{\partial \nu}. \quad (2.7)
$$

The logistic condition $f(u) \leq a - \mu u^2$ gives rise to

$$
p \int_\Omega u^{p-1} |\nabla v|^{2q} f(u) \leq ap \int_\Omega u^{p-1} |\nabla v|^{2q} - \mu p \int_\Omega u^{p+1} |\nabla v|^{2q}. \quad (2.8)
$$

A simple use of Young’s inequality with $\epsilon$ shows

$$
\chi (p-1) p \int_\Omega u^{p-1} |\nabla v|^{2q} \nabla u \nabla v
$$

$$
\leq \epsilon (p-1) p \int_\Omega u^{p-2} |\nabla u|^2 |\nabla v|^{2q} + \frac{\chi^2 (p-1) p}{4\epsilon} \int_\Omega u^p |\nabla v|^{2(q+1)}. \quad (2.9)
$$
Upon integration by parts, applications of Young’s inequality with \( \epsilon \) and (2.5), we find that
\[
2\alpha q \int_{\Omega} u^p |\nabla v|^{2(q-1)} \nabla u \nabla v = - \frac{2(q-1)q\alpha}{p+1} \int_{\Omega} u^{p+1} |\nabla v|^{2(q-2)} \nabla |\nabla v|^2 - \frac{2q\alpha}{p+1} \int_{\Omega} u^{p+1} |\nabla v|^{2(q-1)} \Delta v \\
\leq q(q-1)\eta \int_{\Omega} u^p |\nabla v|^{2(q-2)} |\nabla |\nabla v|^2|^2 + \frac{(q-1)q\alpha^2}{(p+1)^2\eta} \int_{\Omega} u^p+2 |\nabla v|^{2(q-1)} \\
+ 2d_2q \int_{\Omega} u^p |\nabla v|^{2(q-1)} |D^2 v|^2 + \frac{q\alpha^2}{2d_2(p+1)^2} \int_{\Omega} u^p+2 |\nabla v|^{2(q-1)}.
\] (2.10)
Substituting (2.8), (2.9) and (2.10) into (2.7), after suitable rearrangements, we obtain the key inequality (2.6).

3. Logistic damping prevents blow-up in 3-D setting
To get a clear and better understanding about how strong a logistic damping can prevent blowup phenomenon in chemotaxis systems with logistic sources, we first explore it in our physical world, i.e., in a three dimensional bounded smooth domain \( \Omega \). In this section, we will give details to the algorithm leading to the main result (i) of Theorem 1.1.
For the full-parameter chemotaxis-growth system (1.3) in 3-D, we observe that, to show the \( L^\infty \)-boundedness of \( u \), it is enough to show the \( L^{3/2} \)-boundedness of \( u \) for some \( \epsilon > 0 \), thanks to the boundedness criterion obtained in [61] via Moser iteration and in [3] via semigroup theory. This enables us to find out how large should \( \mu \) be so that blow-up is impossible.
To achieve our goal, we need to carefully collect the appearing constants in each derived inequalities. We shall indeed prove that \( \|u(t)\|_{L^2} \) is bounded.
For this purpose, inspired by [53], our analysis consists of deriving a delicate Gronwall inequality for the coupled functional
\[
z(t) := \delta_1 \int_{\Omega} u^2(\cdot, t) + \delta_2 \int_{\Omega} u(\cdot, t)|\nabla v(\cdot, t)|^2 + \delta_3 \int_{\Omega} |\nabla v(\cdot, t)|^4
\] (3.1)
of the form
\[
z'(t) + \epsilon z(t) \leq C_t, \quad t \in (0, T_{\max})
\]
for some carefully chosen positive constants \( \delta_1, \delta_2, \delta_3, \epsilon > 0 \) and perhaps large \( C_t \) independent of \( t \). Once this is done, the \( L^2 \)-and hence the \( L^\infty \)-boundedness of \( u \) will be obtained.
Upon easy applications of Lemma 2.4 with \( p = 2 \) and Lemma 2.5 with \( q = 2 \), we obtain the following two lemmas.

**Lemma 3.1.** Let \( f \) satisfy the logistic condition (1.4). Then the solution \((u,v)\) of (1.3) satisfies
\[
\frac{d}{dt} \int_{\Omega} u^2 + 2(d_1 - \epsilon_1) \int_{\Omega} |\nabla u|^2 + 2\mu \int_{\Omega} u^3 \leq \frac{\chi^2}{2\epsilon_1} \int_{\Omega} u^2 |\nabla v|^2 + 2a \int_{\Omega} u
\] (3.2)
for all \( t \in (0, T_{\max}) \) and for any \( \epsilon_1 \in (0, d_1) \).
Lemma 3.2. The solution $(u, v)$ of (1.3) satisfies
\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^4 + 2(d_2 - \epsilon_2) \int_{\Omega} |\nabla |\nabla v|^2|^2 + 4\beta \int_{\Omega} |\nabla v|^4 \\
\leq 2\left(\frac{n}{2d_2} + \frac{1}{\epsilon_2}\right) \alpha^2 \int_{\Omega} u^2 |\nabla v|^2 + 2d_2 \int_{\partial\Omega} |\nabla v|^2 \partial_{\nu} |\nabla v|^2
\]
for all $t \in (0, T_{\text{max}})$ and for any $\epsilon_2 \in (0, d_2)$.

Lemma 3.3. Under the logistic condition (1.4), the solution $(u, v)$ of (1.3) satisfies
\[
\frac{d}{dt} \int_{\Omega} u |\nabla v|^2 + 2(\mu - \frac{\alpha^2}{4\epsilon_3}) \int_{\Omega} u^2 |\nabla v|^2 + 2\beta \int_{\Omega} u |\nabla v|^2 \\
\leq \frac{(d_1 + d_2)^2}{4\epsilon_4} \int_{\Omega} |\nabla u|^2 + (\epsilon_3 + \epsilon_4) \int_{\Omega} |\nabla |\nabla v|^2|^2 + \frac{n\alpha^2}{8d_2} \int_{\Omega} u^3 \\
+ a \int_{\Omega} |\nabla v|^2 + d_2 \int_{\partial\Omega} u \partial_{\nu} |\nabla v|^2
\]
for all $t \in (0, T_{\text{max}})$ and for any $\epsilon_3, \epsilon_4 > 0$.

Proof. Lemma 2.6 with $p = 1 = q$ reads as
\[
\frac{d}{dt} \int_{\Omega} u |\nabla v|^2 + 2\beta \int_{\Omega} u |\nabla v|^2 + \mu \int_{\Omega} u^2 |\nabla v|^2 \\
\leq \chi \int_{\Omega} u |\nabla v|^2 \nabla v |\nabla v|^2 - (d_1 + d_2) \int_{\Omega} |\nabla u |\nabla v|^2 \\
+ \frac{n\alpha^2}{8d_2} \int_{\Omega} u^3 + a \int_{\Omega} |\nabla v|^2 + d_2 \int_{\partial\Omega} u \partial_{\nu} |\nabla v|^2
\]
for all $t \in (0, T_{\text{max}})$. Now, applying repeatedly Young’s inequality with $\epsilon$ and taking into account (3.2) and (3.3), we infer that
\[
\chi \int_{\Omega} u |\nabla v|^2 \nabla v |\nabla v|^2 \leq \frac{\chi^2}{4\epsilon_3} \int_{\Omega} u^2 |\nabla v|^2 + \epsilon_3 \int_{\Omega} |\nabla |\nabla v|^2|^2
\]
for any $\epsilon_3, \epsilon_4 > 0$. Then we substitute (3.6) and (3.7) into (3.5) to conclude (3.4). □

We are ready to study the coupled functional (3.1) by means of Lemmas 3.1 - 3.3.
Lemma 3.4. Under the logistic condition \( \mu \), the solution \((u, v)\) of (1.3) satisfies the inequality

\[
\frac{d}{dt}\left\{ \delta_1 \int_{\Omega} u^2 + \delta_2 \int_{\Omega} u|\nabla v|^2 + \delta_3 \int_{\Omega} |\nabla v|^4 \right\} + (2\mu\delta_1 - \frac{n\alpha^2}{8d_2}\delta_2) \int_{\Omega} u^3 \\
+ [2(d_1 - \epsilon_1)\delta_1 - \frac{(d_1 + d_2)^2}{4\epsilon_4}\delta_2] \int_{\Omega} |\nabla u|^2 + 4\beta\delta_3 \int_{\Omega} |\nabla v|^4 \\
+ [2(d_2 - \epsilon_2)\delta_3 - (\epsilon_3 + \epsilon_4)\delta_2] \int_{\Omega} |\nabla|\nabla v|^2|^2 + 2\delta_2 \int_{\Omega} u|\nabla v|^2 \\
+ [(\mu - \frac{\chi^2}{4\epsilon_3})\delta_2 - 2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2\delta_3 - \frac{\chi^2}{2\epsilon_1}\delta_1] \int_{\Omega} v^2|\nabla v|^2 \\
\leq 2a\delta_1 \int_{\Omega} u + a\delta_2 \int_{\Omega} |\nabla v|^2 + 2d_2\delta_3 \int_{\Omega} |\nabla|\nabla v|^2|^2 + d_2\delta_2 \int_{\Omega} u \frac{\partial}{\partial \nu} |\nabla v|^2
\]

for any \( t \in (0, T_{\text{max}}) \) and any positive constants \( \delta_1, \delta_2 \) and \( \delta_3 \) and \( \epsilon_1 \in (0, d_1), \epsilon_2 \in (0, d_2) \) and \( \epsilon_3, \epsilon_4 > 0 \).

Proof. By evident multiplications and additions from Lemmas 3.1 - 3.3, one can readily derive the inequality (3.8).

Motivated by (3.8), to find the possibly smallest lower bound \( \mu_0 \) for the damping rate \( \mu \) that could be obtained using such method, we wish to choose \( \epsilon_i \) and \( \delta_i \) to satisfy

\[
\begin{align*}
2(d_1 - \epsilon_1)\delta_1 - \frac{(d_1 + d_2)^2}{4\epsilon_4}\delta_2 > 0, \\
2\mu\delta_1 - \frac{n\alpha^2}{8d_2}\delta_2 > 0, \\
2(d_2 - \epsilon_2)\delta_3 - (\epsilon_3 + \epsilon_4)\delta_2 > 0, \\
(\mu - \frac{\chi^2}{4\epsilon_3})\delta_2 - 2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2\delta_3 - \frac{\chi^2}{2\epsilon_1}\delta_1 \geq 0.
\end{align*}
\]

This can be easily satisfied and \( \delta_2 \) may be taken to be one. The minimizer of the minimization problem (3.9) in \( \mu \) will give us the smallest damping rate \( \mu_0 \) we are seeking. Our goal is then to choose \( \epsilon_i \) and \( \delta_i \) so that \( \mu \) is minimized. By eliminations from (3.9), we end up with

\[
\begin{align*}
\mu > \frac{n\alpha^2}{16d_2}\delta_1, \\
\mu > \frac{\chi^2}{4\epsilon_3} + 2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2 - \frac{\epsilon_3 + \epsilon_4}{2(d_2 - \epsilon_2)} + \frac{\chi^2}{2\epsilon_1} \cdot \frac{(d_1 + d_2)^2}{2(d_1 - \epsilon_1)}
\end{align*}
\]

for any \( \epsilon_1 \in (0, d_1), \epsilon_2 \in (0, d_2) \) and \( \epsilon_3, \epsilon_4 > 0 \). Next, we shall first minimize the second expression on the right-hand side of (3.10). Notice that

\[
\frac{\chi^2}{2\epsilon_1} \cdot \frac{(d_1 + d_2)^2}{2(d_1 - \epsilon_1)} \geq \frac{\chi^2}{d_1^2} \cdot \frac{(d_1 + d_2)^2}{4\epsilon_4}
\]
with equality if and only if \( \epsilon_1 = d_1/2 \), and
\[
2\left(\frac{n}{2d_2} + \frac{1}{\epsilon_2}\right)\alpha^2 \cdot \epsilon_3 + \epsilon_4 \geq \left[\frac{n\alpha}{(\sqrt{2n} + 4 - 2)d_2}\right]^2 (\epsilon_3 + \epsilon_4)
\] (3.12)
with equality if and only if
\[
\epsilon_2 = \frac{(\sqrt{2n} + 4 - 2)}{n} d_2.
\]

Therefore, by algebraic calculations from (3.10)-(3.12) we deduce that the second expression on the right-hand side of (3.10) achieves its minimum
\[
\frac{n}{\sqrt{2n} + 4 - 2} \left(\frac{1}{d_1} + \frac{2}{d_2}\right)\alpha|\chi|
\] (3.13)
if and only if
\[
\left\{
\begin{array}{l}
\epsilon_1 = d_1, \\
\epsilon_2 = \frac{(\sqrt{2n} + 4 - 2)}{n} d_2, \\
\epsilon_3 = \frac{(\sqrt{2n} + 4 - 4\alpha)}{2d_1}\alpha|\chi|, \\
\epsilon_4 = \frac{(\sqrt{2n} + 4 - 2)(d_1 + d_2)}{2nd_1}\alpha|\chi|.
\end{array}
\right.
\] (3.14)

For such well-chosen \( \epsilon_i \) according to (3.14), we choose \( \delta_i \) to fulfill the first two inequalities in (3.9) as well as
\[
\frac{n\alpha^2 \delta_2}{16d_2 \delta_1} \leq \frac{n}{\sqrt{2n} + 4 - 2} \left(\frac{1}{d_1} + \frac{2}{d_2}\right)\alpha|\chi|,
\]
then we infer from (3.10)+(3.13) that
\[
\mu > \mu_0(n, d_1, d_2, \alpha, \chi) := \frac{n}{\sqrt{2n} + 4 - 2} \left(\frac{1}{d_1} + \frac{2}{d_2}\right)\alpha|\chi|.
\] (3.15)
Indeed, for any \( \mu \) satisfying (3.15), we fix \( \epsilon_i \) according to (3.14) and \( \delta_i \) according to
\[
\left\{
\begin{array}{l}
\delta_1 = \max \left\{ \frac{(d_1 + d_2)^2}{16d_2 (d_1 - 1)}, \frac{\alpha^2}{16d_2 (\frac{1}{d_1} + \frac{2}{d_2})|\chi|} \right\} + 1, \\
\delta_2 = 1, \\
\delta_3 = \frac{\epsilon_3 + \epsilon_4}{2(d_2 - \epsilon_2)} + 1,
\end{array}
\right.
\] (3.16)
then the inequality (3.9) will be satisfied.

For any \( \mu > \mu_0 \) as defined in (3.15), we shall illustrate that all integrals on the right-hand side of (3.8) can be controlled by the dissipative terms on the left and, as a result, yielding the assertion that \( \|u(t)\|_{L^2} \) is uniformly bounded and thus (i) of Theorem 1.1 in physically relevant setting \( n = 3 \) by the \( L^{\frac{n+2}{n}} \)-criterion in [3, 61].

**Lemma 3.5.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, \( f \) satisfy the logistic condition (1.4) and \( d_1, d_2, \alpha, \beta > 0, a \geq 0 \) and \( \chi \in \mathbb{R} \). Then, for any \( \mu > \mu_0(n, d_1, d_2, \alpha, \chi) \) as given in (3.14), there exists a constant \( C(u_0, v_0) \) such that the unique nonnegative solution \( (u, v) \) of the chemotaxis-growth system (1.3) satisfies
\[
\int_{\Omega} u^2(\cdot, t) + \int_{\Omega} u(\cdot, t)|\nabla v(\cdot, t)|^2 + \int_{\Omega} |\nabla v(\cdot, t)|^4 \leq C(u_0, v_0), \quad t \in (0, T_{\text{max}}).
\]
Thus, if \( n \leq 3 \), then the solution \((u, v)\) exists globally in time, i.e., \( T_{\text{max}} = \infty \) and \((u(\cdot, t), v(\cdot, t))\) is uniformly bounded in \( L^\infty(\Omega) \times W^{1,\infty}(\Omega) \) for all \( t \in (0, \infty) \).

Proof. It follows from Lemma 3.3 that the quantity
\[
z(t) := \delta_1 \int_\Omega u^2 + \delta_2 \int_\Omega u|\nabla v|^2 + \delta_3 \int_\Omega |\nabla v|^4, \quad t \in (0, T_{\text{max}}),
\]
fulfills
\[
z'(t) + \beta z(t) + [2(d_1 - \epsilon_1)\delta_1 - \frac{(d_1 + d_2)^2}{4\epsilon_1} \delta_2] \int_\Omega |\nabla u|^2
+ [2(d_2 - \epsilon_2)\delta_3 - (\epsilon_3 + \epsilon_4)\delta_2] \int_\Omega |\nabla |\nabla v|^2|^2 + 3\beta\delta_3 \int_\Omega |\nabla v|^4
+ \beta\delta_2 \int_\Omega u|\nabla v|^2 + (2\mu\delta_1 - \frac{\alpha^2}{8\delta_2} \delta_2) \int_\Omega u^3
\leq 2a\delta_1 \int_\Omega u + a\delta_2 \int_\Omega |\nabla v|^2 + \beta\delta_1 \int_\Omega u^2
+ 2d\delta_3 \int_\Omega |\nabla v|^2 \frac{\partial u}{\partial \nu} |\nabla v|^2 + 2d\delta_2 \int_\Omega u \frac{\partial u}{\partial \nu} |\nabla v|^2. \tag{3.17}
\]
In the sequel, we bound the integrals on the right-hand side of \( (3.17) \) in terms of the dissipative terms on its left-hand side.

Let us first focus on controlling the boundary integrals in \( (3.17) \). So far, there are a couple of existing ways to handle these boundary integrals, see \([22, 29, 44, 64]\), for instance. Here we would like to provide an alternative and transparent way to remove the technical assumption that the domain \( \Omega \) be convex. The starting point is based on the pointwise geometric inequality
\[
\frac{\partial |\nabla w|^2}{\partial \nu} \leq K_1(\Omega)|\nabla w|^2 \quad \text{on } \partial \Omega, \tag{3.18}
\]
which holds for any bounded smooth domain \( \Omega \subset \mathbb{R}^n \) and any \( w \) satisfying \( \frac{\partial w}{\partial \nu} = 0 \) on \( \partial \Omega \), cf. \([33, \text{Lemma 4.2}]\). Here and below, \( K_1 \) will denote some inessential constants. Notice also a user-friendly version of trace inequality with \( \epsilon \) (cf. \([10, \text{Remark 52.9}]\)): for any \( \epsilon > 0 \), one has
\[
\|w\|_{L^2(\partial \Omega)} \leq \epsilon \|\nabla w\|_{L^2(\Omega)} + C_\epsilon \|w\|_{L^2(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{3.19}
\]
Indeed, this is immediately implied, upon a use of Young’s inequality with \( \epsilon \), by the following version of trace inequality:
\[
\|w\|_{L^2(\partial \Omega)} \leq K_2 \|w\|_{L^2(\Omega)}^{\frac{1}{2}} \|w\|_{H^1(\Omega)}^{\frac{1}{2}}, \quad \forall w \in H^1(\Omega). \tag{3.20}
\]
As a matter of fact, by the property of trace inequality (the trace operator \( T \) maps \( H^\frac{3}{2}(\Omega) \) continuously onto \( L^2(\partial \Omega) \)), one has
\[
\|w\|_{L^2(\partial \Omega)} \leq K_3 \|w\|_{H^\frac{3}{2}(\Omega)}, \quad \forall w \in H^\frac{3}{2}(\Omega).
\]
On the other hand, it follows from the fact that \( H^\frac{3}{2} \) interpolates the spaces \( H^0 = L^2 \) and \( H^1 \) that
\[
\|w\|_{H^\frac{3}{2}(\Omega)} \leq K_4 \|w\|_{L^2(\Omega)}^{\frac{1}{4}} \|w\|_{H^1(\Omega)}^{\frac{3}{4}}, \quad \forall w \in H^1(\Omega).
\]
Collecting these two estimates directly leads to \( (3.20) \).
Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ by Kondrachov and that $L^2(\Omega)$ is continuously embedded in $L^1(\Omega)$, Lion’s lemma says, for any $\eta > 0$, that

$$\|w\|_{L^2(\Omega)} \leq \eta \|\nabla w\|_{L^2(\Omega)} + \eta \|w\|_{L^2(\Omega)} + C_\eta \|w\|_{L^1(\Omega)}, \quad \forall w \in H^1(\Omega).$$

Combing this with (3.19), we conclude, for any $\epsilon > 0$,

$$\|w\|_{L^2(\partial \Omega)} \leq \epsilon \|\nabla w\|_{L^2(\Omega)} + C_\epsilon \|w\|_{L^1(\Omega)}, \quad \forall w \in H^1(\Omega).$$

(3.21)

This gives another elementary proof for the equivalent trace inequality stated in [44, P. 13, line -4].

Now, with (3.18) and (3.21) at hand, we can cope with the boundary integrals in (3.17) as follows: for any $\epsilon > 0$,

$$2d_2 \delta_2 \int_{\partial \Omega} |\nabla v|^2 \frac{d_2 \delta_3}{\partial \nu} |\nabla v|^2 + d_2 \delta_3 \int_{\partial \Omega} u \frac{d_2 \delta_3}{\partial \nu} |\nabla v|^2$$

$$\leq K_5 \delta_2 \int_{\partial \Omega} |\nabla v|^4 + d_2 \delta_3 \int_{\partial \Omega} u |\nabla v|^2$$

$$\leq K_5 \delta_2 \int_{\partial \Omega} |\nabla v|^4 + K_6 \int_{\partial \Omega} u^2 = K_5 \|\nabla v\|^2 L^2(\partial \Omega) + K_6 \|u\|^2_{L^2(\partial \Omega)}$$

$$\leq \epsilon \int_{\Omega} |\nabla v|^2 + C_\epsilon \int_{\Omega} |\nabla u|^2 + \epsilon \int_{\Omega} |u|^2$$

(3.22)

For any $\mu > \mu_0(n, d_1, d_2, \alpha, \chi)$ as given in (3.15), we first fix $\epsilon_1, i = 1, 2, 3, 4$ according to (3.14), and then fix $\delta_1, \delta_2$ and $\delta_3$ complying with (3.10). Then the inequality (3.9) is satisfied. Upon such selections, we then fix $\epsilon$ according to

$$\epsilon = \frac{1}{2} \min \left\{ 2(d_1 - \epsilon_1) \delta_1 - \frac{(d_1 + d_2)^2}{4 \epsilon_4} \delta_3, \quad 2(d_2 - \epsilon_2) \delta_2 - (\epsilon_3 + \epsilon_4) \delta_2 \right\}.$$

Then the boundary integrals in (3.22) will be absorbed by the terms on the left-hand side of (3.17) and Lemma 2.3.

Next, notice that

$$\beta \delta_1 \int_{\Omega} u^2 \leq (2d_1 - \epsilon_1) \delta_1 - \frac{(d_1 + d_2)^2}{8d_2} \delta_3, \quad 2(d_2 - \epsilon_2) \delta_2 - (\epsilon_3 + \epsilon_4) \delta_2.$$

(3.23)

where

$$K_7 = \max \{ \beta \delta_1 u \|^2 - (2d_1 - \frac{n \alpha^2}{8d_2} \delta_2) u^3 |u \geq 0\} < \infty.$$

Finally, we substitute (3.22) and (3.23) into (3.17) and use Lemma 2.3 to conclude

$$z'(t) + \beta z(t) \leq C(u_0, v_0), \quad t \in [0, T_{\text{max}}),$$

which together the definition of $z$ simply leads to

$$z(t) = \delta_1 \int_{\Omega} u^2 + \delta_2 \int_{\Omega} u \nabla v|^2 + \delta_3 \int_{\Omega} |\nabla v|^4$$

$$\leq \delta_1 \int_{\Omega} u^2 + \delta_2 \int_{\Omega} u \nabla v|^2 + \delta_3 \int_{\Omega} |\nabla v|^4 + \frac{C(u_0, v_0)}{\delta_1}, \quad t \in [0, T_{\text{max}}).$$

This in particular shows the uniform spatial $L^2$-boundedness of $u(\cdot, t)$ for any $t \in [0, T_{\text{max}})$. Hence, by Moser iteration, cf. the $L^{\frac{n}{n-1}}$-criterion in [61, R1, Theorem 1.1] or [3, Lemma 3.2] with $n = 3$, we infer that $T_{\text{max}} = \infty$ and that $(u(\cdot, t), v(\cdot, t))$ is uniformly bounded in $L^{\infty}(\Omega) \times W^{1,\infty}(\Omega)$ for all $t \in (0, \infty)$. \[\square\]
4. How strong a logistic damping can prevent blowups in \( n \)-D

4.1. Blow-up prevention by logistic source in nonconvex domains.

In this subsection, based on the detailed algorithm in 3-D, we wish to provide a clue on how to compute the explicit logistic damping rate \( \mu_0 \) that suppresses blow-up whenever \( \mu > \mu_0 \) in \( \geq 4 \)-D. We will mainly do it for \( n = 4, 5 \). Our procedure suggests that an explicit logistic damping rate \( \mu_0 \) suppressing blow-up whenever \( \mu > \mu_0 \), which enjoys the properties as described in Remark 1.2, is also available in more higher dimensions.

When \( n \leq 5 \), it amounts to ensuring the uniform boundedness of \( \| u \|_{L^3(\Omega)} \). In such setup, the core analysis then lies in deriving a subtle estimate for

\[
z(t) := \delta_1 \int_\Omega u^3 + \delta_2 \int_\Omega u^2 |\nabla v|^2 + \delta_3 \int_\Omega u |\nabla v|^4 + \delta_4 \int_\Omega |\nabla v|^6, \quad t \in (0, T_{\text{max}})
\]

of the form

\[
z'(t) + \zeta z(t) \leq C\zeta, \quad t \in (0, T_{\text{max}})
\]

for some well chosen positive constants \( \delta_i \) and \( \zeta > 0 \) and perhaps large \( C_\zeta \) independent of \( t \). Once this is done, the \( L^3 \)-and hence the \( L^\infty \)-boundedness of \( u \) will be obtained, by the boundedness principles [3, 61].

Upon trivial applications of Lemma 2.4 with \( p = 3 \) and Lemma 2.5 with \( q = 3 \), we get the following two lemmas.

**Lemma 4.1.** Let \( f \) satisfy the logistic condition (1.4). Then the solution \( (u, v) \) of (1.3) satisfies

\[
\frac{d}{dt} \int_\Omega u^3 + 6(d_1 - \epsilon) \int_\Omega u |\nabla u|^2 + 3\mu \int_\Omega u^4 \leq \frac{3\chi^2}{2\epsilon} \int_\Omega u^3 |\nabla v|^2 + 3a \int_\Omega u^2 \quad (4.3)
\]

for all \( t \in (0, T_{\text{max}}) \) and for any \( \epsilon \in (0, d_1) \).

**Lemma 4.2.** The solution \( (u, v) \) of (1.3) satisfies

\[
\frac{d}{dt} \int_\Omega |\nabla v|^6 + 6(d_2 - \eta) \int_\Omega |\nabla v|^2 |\nabla |\nabla v||^2 + 6\beta \int_\Omega |\nabla v|^6 \leq 3 \left( \frac{2\alpha^2}{\eta} + \frac{n\alpha^2}{2d_2} \right) \int_\Omega u^2 |\nabla v|^4 + 3d_2 \int_{\partial\Omega} |\nabla v|^4 \frac{\partial}{\partial v} |\nabla v|^2 \quad (4.4)
\]

for all \( t \in (0, T_{\text{max}}) \) and for any \( \epsilon_2 \in (0, d_2) \).
Lemma 4.3. Let $f$ satisfy the logistic condition \((\text{1.4})\). Then the solution $(u, v)$ of \((\text{1.3})\) verifies

$$\frac{d}{dt} \int_{\Omega} u |\nabla v|^{4} + 4\beta \int_{\Omega} u |\nabla v|^{4}$$

$$+ (\mu - \frac{\chi^2}{2\epsilon_1}) \int_{\Omega} u^2 |\nabla v|^{4} + 2(d_2 - \eta) \int_{\Omega} u |\nabla|\nabla v|^{2}|^{2}$$

$$\leq 2(\epsilon_1 + \epsilon_2) \int_{\Omega} |\nabla v|^{2} |\nabla|\nabla v|^{2}|^{2} + \frac{(d_1 + d_2)^2}{2\epsilon_2} \int_{\Omega} |\nabla u|^{2} |\nabla v|^{2}$$

$$+ \frac{\alpha^2}{2} (\frac{1}{\eta} + \frac{n}{2d_2}) \int_{\Omega} u^3 |\nabla v|^{2}$$

$$+ a \int_{\Omega} |\nabla v|^{4} + 2d_2 \int_{\partial\Omega} u |\nabla v|^{2} \frac{\partial|\nabla v|^{2}}{\partial\nu}$$

\((4.5)\)

for all $t \in (0, T_{\text{max}})$ and for any $\epsilon \in (0, d_1)$, $\eta \in (0, d_2)$ and $\epsilon_1, \epsilon_2 > 0$.

Proof. Lemma 2.6 with $p = 1$ and $q = 2$ reads as

$$\frac{d}{dt} \int_{\Omega} u |\nabla v|^{4} + 4\beta \int_{\Omega} u |\nabla v|^{4}$$

$$+ 2(d_2 - \eta) \int_{\Omega} u |\nabla|\nabla v|^{2}|^{2} + \mu \int_{\Omega} u^2 |\nabla v|^{4}$$

$$\leq 2\chi \int_{\Omega} u |\nabla v|^{2} \nabla v \nabla|\nabla v|^{2}$$

$$- 2(d_1 + d_2) \int_{\Omega} |\nabla v|^{2} \nabla u \nabla|\nabla v|^{2}$$

$$+ \frac{\alpha^2}{2} (\frac{1}{\eta} + \frac{n}{2d_2}) \int_{\Omega} u^3 |\nabla v|^{2}$$

$$+ a \int_{\Omega} |\nabla v|^{4} + 2d_2 \int_{\partial\Omega} u |\nabla v|^{2} \frac{\partial|\nabla v|^{2}}{\partial\nu}.$$  \((4.6)\)

Taking into account \((\text{1.4})\) of Lemma 4.2 we estimate

$$2\chi \int_{\Omega} u |\nabla v|^{2} \nabla v \nabla|\nabla v|^{2} \leq 2\epsilon_1 \int_{\Omega} |\nabla v|^{2} |\nabla|\nabla v|^{2}|^{2} + \frac{\chi^2}{2\epsilon_1} \int_{\Omega} u^2 |\nabla v|^{4}$$ \((4.7)\)

and

$$- 2(d_1 + d_2) \int_{\Omega} |\nabla v|^{2} \nabla u \nabla|\nabla v|^{2}$$

$$\leq 2\epsilon_2 \int_{\Omega} |\nabla v|^{2} |\nabla v|^{2}|^{2} + \frac{(d_1 + d_2)^2}{2\epsilon_2} \int_{\Omega} |\nabla u|^{2} |\nabla v|^{2}.$$ \((4.8)\)

Then combining \((4.6)\), \((4.7)\) and \((4.8)\), we get the desired inequality \((4.5)\). $\square$
Lemma 4.4. Let \( f \) satisfy the logistic condition \( (1.4) \). Then the solution \((u, v)\) of \( (1.3) \) verifies

\[
\frac{d}{dt} \int_{\Omega} u^2 |\nabla v|^2 + 2(d_1 - \epsilon) \int_{\Omega} |\nabla u|^2 |\nabla v|^2 + 2\beta \int_{\Omega} u^2 |\nabla v|^2 + (2\mu - \frac{\chi^2}{2\epsilon_3}) \int_{\Omega} u^3 |\nabla v|^2 + 2\epsilon \int_{\Omega} u^2 |\nabla u|^2 |\nabla v|^2 
\]

\[
\leq \frac{\chi^2}{2\epsilon} \int_{\Omega} u^2 |\nabla v|^4 + 2\epsilon_4 \int_{\Omega} u|\nabla u|^2 + \frac{\alpha^2}{2d_2} \int_{\Omega} u^4 + 2a \int_{\Omega} u|\nabla v|^2 + d_2 \int_{\partial\Omega} u^2 \frac{\partial |\nabla v|^2}{\partial \nu},
\]

(4.9)

for all \( t \in (0, T_{\max}) \) and for any \( \epsilon \in (0, d_1), \eta \in (0, d_2) \) and \( \epsilon_3, \epsilon_4 > 0 \).

Proof. It follows from \( (2.6) \) with \( p = 2 \) and \( q = 1 \) in Lemma 2.6 that

\[
\frac{d}{dt} \int_{\Omega} u^2 |\nabla v|^2 + 2(d_1 - \epsilon) \int_{\Omega} |\nabla u|^2 |\nabla v|^2 + 2\beta \int_{\Omega} u^2 |\nabla v|^2 + 2\mu \int_{\Omega} u^3 |\nabla v|^2 
\]

\[
\leq \frac{\chi^2}{2\epsilon} \int_{\Omega} u^2 |\nabla v|^4 + 2\epsilon_4 \int_{\Omega} u|\nabla u|^2 + \frac{\alpha^2}{2d_2} \int_{\Omega} u^4
\]

\[
+ 2a \int_{\Omega} u|\nabla v|^2 + d_2 \int_{\partial\Omega} u^2 \frac{\partial |\nabla v|^2}{\partial \nu}.
\]

(4.10)

Taking into consideration \( (4.5) \) and \( (4.3) \), we bound

\[
2\chi \int_{\Omega} u^2 \nabla v \nabla |\nabla v|^2 \leq 2\epsilon_3 \int_{\Omega} u|\nabla |\nabla v|^2|^2 + \frac{\chi^2}{2\epsilon_3} \int_{\Omega} u^3 |\nabla v|^2
\]

(4.11)

and

\[
-2(d_1 + d_2) \int_{\Omega} u \nabla u \nabla |\nabla v|^2 \leq 2\epsilon_4 \int_{\Omega} u|\nabla u|^2 + \frac{d_1 + d_2}{2\epsilon_4} \int_{\Omega} u|\nabla |\nabla v|^2|^2.
\]

(4.12)

Then using \( (4.10), (4.11) \) and \( (4.12) \), we deduce \( (4.9) \) and hence the lemma. \( \square \)

At this moment, we are well-prepared to estimate the coupled functional \( (4.1) \) by means of Lemmas 4.1-4.4
Lemma 4.5. Under the logistic condition \(22\), the solution \((u,v)\) of \((1.3)\) satisfies the inequality

\[
\frac{d}{dt}\left\{ \delta_1 \int_{\Omega} u^3 + \delta_2 \int_{\Omega} u^2 |\nabla v|^2 + \delta_3 \int_{\Omega} u |\nabla v|^4 + \delta_4 \int_{\Omega} |\nabla v|^6 \right\} + 6\beta_4 \int_{\Omega} |\nabla v|^6 \\
+ 2[3(d_2 - \eta)\delta_4 - (\epsilon_1 + \epsilon_2)\delta_3] \int_{\Omega} |\nabla v|^2 |\nabla |\nabla v|^2|^2 + (3\mu_0 - \frac{n\alpha^2}{18\delta_2}) \int_{\Omega} u^2 \\
+ [2(d_1 - \epsilon)\delta_2 - \frac{(d_1 + d_2)^2}{2\epsilon_2}\delta_3] \int_{\Omega} |\nabla u|^2 |\nabla v|^2 + 2\beta_2 \int_{\Omega} u^2 |\nabla v|^2 \\
+ [2(d_2 - \eta)\delta_3 - (2\epsilon_3 + \frac{(d_1 + d_2)^2}{2\epsilon_4})\delta_2] \int_{\Omega} u |\nabla |\nabla v|^2|^2 \\
+ 4\beta_3 \int_{\Omega} |\nabla v|^4 + 2[3(d_1 - \epsilon)\delta_1 - \epsilon_4\delta_2] \int_{\Omega} u|\nabla v|^2 \\
+ [2\mu - \frac{\chi^2}{2\epsilon_3}\delta_2 - \frac{3\chi^2}{2\epsilon_1}\delta_1 - \frac{\alpha^2}{2}(\eta + \frac{n}{2\delta_2})\delta_3] \int_{\Omega} u^2 |\nabla v|^2 \\
+ [\mu - \frac{\chi^2}{2\epsilon_1}\delta_3 - \frac{\chi^2}{2\epsilon_2}\delta_2 - 3(\frac{2\alpha^2}{\eta} + \frac{n\alpha^2}{2\delta_2})\delta_4] \int_{\Omega} u^2 |\nabla v|^4 \\
\leq 3a_1 \int_{\Omega} u^2 + 2a_2 \int_{\Omega} u |\nabla v|^2 + a_3 \int_{\Omega} |\nabla v|^4 + 3d_2 \delta_4 \int_{\partial\Omega} |\nabla v|^4 \frac{\partial}{\partial v} |\nabla v|^2 \\
+ 2d_2 \delta_2 \int_{\partial\Omega} u |\nabla v|^2 \frac{\partial}{\partial v} |\nabla v|^2 + d_2 \delta_2 \int_{\partial\Omega} u^2 \frac{\partial}{\partial v} |\nabla v|^2 \\
}(4.13)

for all \(t \in (0,T_{max})\) and any positive constants \(\epsilon, \delta_1\) and \(\epsilon \in (0,d_1)\) and \(\eta \in (0,d_2)\).

Proof. By honest computations from Lemmas \(4.1\) - \(4.4\) and by evident multiplications and additions, one can readily derive the lemma. \(\square\)

As to the boundary integrals on the right-hand side of \((4.13)\), we deduce from \((3.18), (3.21)\) and Young’s inequality with epsilon that

\[
3d_2 \delta_4 \int_{\partial\Omega} |\nabla v|^4 \frac{\partial}{\partial v} |\nabla v|^2 + 2d_2 \delta_3 \int_{\partial\Omega} u |\nabla v|^2 \frac{\partial}{\partial v} |\nabla v|^2 + d_2 \delta_2 \int_{\partial\Omega} u^2 \frac{\partial}{\partial v} |\nabla v|^2 \\
\leq 3d_2 \delta_4 K_1 \int_{\partial\Omega} |\nabla v|^6 + 2d_2 \delta_3 K_1 \int_{\partial\Omega} u |\nabla v|^4 + d_2 \delta_2 K_1 \int_{\partial\Omega} u^2 |\nabla v|^2 \\
\leq K_8 \int_{\partial\Omega} u^3 + K_8 \int_{\partial\Omega} |\nabla v|^6 = K_8 ||u^\frac{3}{2}||^2_{L^2(\partial\Omega)} + + K_8 ||\nabla v|^3||^2_{L^2(\partial\Omega)} \\
\leq \xi \int_{\Omega} u \nabla u^2 + C\xi \left( \int_{\Omega} u^\frac{3}{2} \right)^2 + \sigma \int_{\Omega} |\nabla v|^2 |\nabla \nabla v|^2 + C\sigma \left( \int_{\Omega} |\nabla v|^3 \right)^2 \\
}(4.14)

for any \(\xi > 0\) and \(\sigma > 0\).
Based on this boundary integral estimate and (4.13), we wish to select \( \epsilon, \eta, \epsilon_i, \delta_i \) and \( \mu \) such that

\[
\begin{align*}
2[3(d_1 - \epsilon)\delta_1 - \epsilon_4\delta_2] &> 0, \\
2[3(d_2 - \eta)\delta_4 - (\epsilon_1 + \epsilon_2)\delta_3] &> 0, \\
2(d_1 - \epsilon)\delta_2 - \frac{(d_1 + d_2)^2}{2\epsilon_4}\delta_3 &\geq 0, \\
2(d_2 - \eta)\delta_3 - (2\epsilon_3 + \frac{(d_1 + d_2)^2}{2\epsilon_4})\delta_2 &\geq 0, \\
3\mu_0 - \frac{n\alpha^2}{18\epsilon_2}\delta_2 &\geq 0, \\
(2\mu - \frac{\chi^2}{2\epsilon_3})\delta_2 - \frac{3\chi^2}{2\epsilon_1}\delta_1 - \frac{n^2}{2\epsilon_2} + \frac{n\alpha^2}{2\epsilon_2} + \frac{3\alpha^2}{\epsilon_2} + \frac{n^2}{2\epsilon_2}\delta_3 &\geq 0, \\
(\mu - \frac{\chi^2}{2\epsilon_1})\delta_3 - \frac{\chi^2}{2\epsilon_1}\delta_2 - 3\left(\frac{2\alpha^2}{\epsilon_2} + \frac{n^2}{2\epsilon_2}\right)\delta_4 &\geq 0.
\end{align*}
\]

Then algebraic manipulations from (4.15) and (4.16) show that

\[
3\mu \geq \frac{n\alpha^2}{54\epsilon_2}\delta_1 + \frac{\chi^2}{4\epsilon_3} + \frac{3\chi^2}{2\epsilon_2}\delta_1 + \frac{n^2}{4\epsilon_2}\delta_2 + \frac{\chi^2}{2\epsilon_1}\delta_1 + \frac{\chi^2}{2\epsilon_1}\delta_2 + \frac{3\alpha^2}{\epsilon_2} + \frac{n^2}{2\epsilon_2}\delta_3 + 3\left(\frac{2\alpha^2}{\epsilon_2} + \frac{n^2}{2\epsilon_2}\right)\delta_4
\]

\[
> \sqrt{\frac{n}{18\epsilon_2} + \frac{1}{\epsilon_2}}|\chi| + \sqrt{\frac{n}{18\epsilon_2} + \frac{1}{\epsilon_2}}|\alpha|\chi + \sqrt{\frac{n}{18\epsilon_2} + \frac{1}{\epsilon_2}} + \sqrt{\frac{1}{(d_2-\eta)^2} + \frac{n}{2\epsilon_2}} + \frac{(d_1 + d_2)}{2\epsilon_2 - \eta} \right) |\chi|
\]

(4.17)

**Lemma 4.6.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded smooth domain, \( f \) satisfy the logistic condition (1.1) and \( d_1, d_2, \alpha, \beta > 0, a \geq 0 \) and \( \chi \in \mathbb{R} \). Then, for any

\[
\mu > \mu_0(n, d_1, d_2, \alpha, \chi) := \max\left\{ \frac{1}{3}h(n, d_1, d_2), \frac{1}{\sqrt{2n + 4}}\left( \frac{1}{d_1} + \frac{2}{d_2} \right) \right\} |\chi|
\]

(4.18)

with

\[
h(n, d_1, d_2) = \min_{0 < \epsilon < d_1, 0 < \eta < d_2} \left\{ \sqrt{\frac{n}{18\epsilon_2}} + \sqrt{\frac{1}{\epsilon_2}} + \frac{n}{2\epsilon_2} \right\} + \sqrt{\frac{1}{(d_2-\eta)^2} + \frac{n}{2\epsilon_2}} \right) \left[ \sqrt{2} + \frac{(d_1 + d_2)}{2\epsilon_2 - \eta} \right].
\]

there exists a constant \( C(u_0, \nu_0) \) such that the unique nonnegative solution \((u, v)\) of the chemotaxis-growth system (1.3) satisfies

\[
\int_{\Omega} u^3(t) + \int_{\Omega} u^2(\cdot, t)|\nabla v(\cdot, t)|^2 + \int_{\Omega} u|\nabla v|^4 + \int_{\Omega} |\nabla v(\cdot, t)|^6 \leq C(u_0, \nu_0), \forall t \in (0, T_{\text{max}}).
\]

Thus, if \( n \leq 5 \), then the solution \((u, v)\) exists globally in time, i.e., \( T_{\text{max}} = \infty \) and \((u(\cdot, t), v(\cdot, t)) \) is uniformly bounded in \( L^\infty(\Omega) \times W^{1,\infty}(\Omega) \) for all \( t \in (0, \infty) \).
Proof. Due to $\mu > \mu_0$ given in (4.13), we first know from Lemma 3.5 that

$$3a_1 \int_{\Omega} u^2 + 2a_2 \int_{\Omega} u |\nabla v|^2 + a_3 \int_{\Omega} |\nabla v|^4 \leq K_0(u_0, v_0)$$

(4.19)

for all $t \in (0, T_{\text{max}})$. Moreover, due to (4.17), the fact that $\mu > \mu_0$ allows us to fix $\epsilon, \eta, \epsilon_1, \delta_1$ satisfying (4.16) and (4.15). Indeed, we first choose $(\epsilon, \eta) = (\epsilon_0, \eta_0)$ so that $h$ is minimized, then we choose $\epsilon_1$ to be the minimizer of

$$\frac{\chi^2}{2\epsilon_1} + \left(\frac{2\alpha^2}{\eta} + \frac{n\alpha^2}{2d_2}\right) \delta_1 - \eta,$$

take $\epsilon_3 = 1$ and then fix $\epsilon_2$ and $\epsilon_4$ so that (4.16) is satisfied. Then, based on (4.15), (4.16) and (4.17), all $\delta_i$ can be chosen readily.

Upon such well chosen $\epsilon, \eta, \epsilon_1$ and $\delta_1$, using (4.13), (4.14) and arguing as Lemma 3.5, we can easily deduce a Gronwall inequality for the coupled quantity $z$ as defined by (4.11) of the form (4.2). As a matter of fact, by (4.13) and (4.14), we have

$$z'(t) + \beta z(t) + 2[3(d_1 - \epsilon)\delta_1 - \epsilon_4 \delta_2 - \xi] \int_{\Omega} u |\nabla u|^2 + (3\mu\delta_1 - \frac{n\alpha^2}{2d_2}\delta_2) \int_{\Omega} u^4$$

$$+ 2[3(d_2 - \eta)\delta_1 - (\epsilon_1 + \epsilon_2)\delta_3 - \sigma] \int_{\Omega} |\nabla u|^2 |\nabla v|^2| \nabla v|^2|^2$$

$$\leq 3a_1 \int_{\Omega} u^2 + 2a_2 \int_{\Omega} u |\nabla v|^2 + a_3 \int_{\Omega} |\nabla v|^4$$

$$+ \beta \delta_1 \int_{\Omega} u^3 + C\delta \int_{\Omega} u^\frac{3}{2} + C\sigma \int_{\Omega} |\nabla v|^3. \quad (4.20)$$

Now, by fixing $\xi$ and $\sigma$ sufficiently small, we deduce from (4.19), (4.20) and Holder inequality that

$$z'(t) + \beta z(t) \leq C(u_0, v_0),$$

giving the desired inequality stated in the lemma. In particular, it assures that $\|u(\cdot, t)\|_{L^2(\Omega)}$ is uniformly bounded. So the $L^\frac{3}{2} +$-criterion [3, 61] shows, if $n \leq 5$, that $(u(\cdot, t), v(\cdots, t))$ is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ for all $t \in (0, \infty)$.

The algorithm for proving $L^2$ and $L^3$-boundedness of $u$ may be in principle carried over to obtain $L^r$-boundedness of $u$ for $r = 4, 5, \cdots$ inductively by establishing a Gronwall inequality for the coupled quantity (motivated by [3] again)

$$z(t) := \sum_{k=0}^r \delta_i \int_{\Omega} u^k |\nabla v|^{2(r-k)}, \quad t \in (0, T_{\text{max}}) \quad (4.21)$$

of the form

$$z'(t) + \epsilon z(t) \leq C\epsilon, \quad t \in (0, T_{\text{max}})$$

for some carefully chosen positive constants $\delta_i$ and small $\epsilon > 0$ and perhaps large $C\epsilon$ independent of $t$. In this process, as $r$ becomes large, we will have more terms to cope with and more boundary integrals will appear. While simplification of the process is possible. In fact, for $r = 2, 3, \cdots$, suppose that $\mu > \mu_0^{(n)}(d_1, d_2, \alpha, \chi)$ is the condition under which $L^r$-boundedness of
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u (indeed, uniform boundedness of \( z(t) \) as in (4.21)) is guaranteed. Then, based on the procedures for \( r = 2 \) and \( r = 3 \), if \( \mu > \mu^{(r-1)}_0(n, d_1, d_2, \alpha, \chi) \), the process for \( r \) may be continued with the assumption that \( a = 0 \) and \( \Omega \) is convex.

Once this is done, the \( L^r \)-boundedness of \( u \) will be obtained. Further, by choosing \( r = \lfloor \frac{n}{2} \rfloor + 1 \), the \( L^\infty \)-criterion in \([3, 61]\) yields the desired \( L^\infty \)-boundedness of \( u \). Here, we state the following expected general result that offers a quantitative description on when logistic damping dominates over cheomtactic aggregation for (1.3) in \( \Omega \subset \mathbb{R}^n \). While, we have to leave a rigorous examination for future study.

**Proposition 4.7.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, \( f \) satisfy the logistic condition (1.4) and \( d_1, d_2, \alpha, \beta, \chi > 0 \), \( a \geq 0 \) and \( \chi \in \mathbb{R} \). Then, for any natural number \( r \geq 2 \), there exist a function \( \theta^{(r)}_0(n, d_1, d_2) \) which tends to infinity as \( d_1 \to 0 \) or \( d_2 \to 0 \) (and is decreasing in \( d_1, d_2 \)) and constant \( C(u_0, v_0) \) such that, for any

\[
\mu > \theta^{(r)}_0(n, d_1, d_2) \alpha |\chi|,
\]

the nonnegative solution \((u, v)\) of the chemotaxis-growth system (1.3) satisfies

\[
\sum_{k=0}^{r} \int_{\Omega} u^k |\nabla v|^{2(r-k)} \leq C(u_0, v_0), \quad t \in (0, T_{\text{max}}).
\]

As a result, if \( n \leq 2r - 1 \), then \((u, v)\) exists globally in time, i.e., \( T_{\text{max}} = \infty \) and \((u(\cdot, t), v(\cdot, t))\) is uniformly bounded in \( L^\infty(\Omega) \times W^{1,\infty}(\Omega) \) for all \( t \in (0, \infty) \).

4.2. **The special case that \( d_1 = d_2 \) and \( \Omega \subset \mathbb{R}^n \) is convex.** In the special case that \( \tau = 1 \) in (1.1), \( \chi > 0 \) (positive chemotaxis) and \( \Omega \subset \mathbb{R}^n \) is convex, the explicit lower bound

\[
\mu > \frac{n}{4} \chi
\]

ensuring global boundedness to the solution of (1.1) has been elucidated in \([53]\) by establishing a parabolic inequality for a combination of \( u \) and \( |\nabla v|^2 \). Here, in this subsection, for the full-parameter chemotaxis model (1.3), we will write down all the details for convenience and completeness.

**Lemma 4.8.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth convex domain, \( f \) satisfy the logistic condition (1.4) and \( d_1, d_2, \alpha, \beta, \chi > 0 \), \( a \geq 0 \). Assume that

\[
d_1 = d_2, \quad \mu > \frac{n}{4d_1} \alpha \chi,
\]

the unique nonnegative solution \((u, v)\) of the chemotaxis-growth system (1.3) exists globally in time, i.e., \( T_{\text{max}} = \infty \) and is bounded in the following way:

\[
u \leq \max_{\Omega \times [0,1]} \left\{ \max_{\Omega \times [0,1]} \frac{1}{2\alpha} \max_{\Omega \times [0,1]} \left( \max_{\Omega}(2\alpha u(\cdot, 1) + \chi |\nabla v(\cdot, 1)|^2) \right) \frac{\alpha \alpha}{\beta} + \frac{\alpha \beta}{\mu - \frac{\alpha \alpha}{4\alpha \chi}} \right\}
\]
and 
\[ v \leq \max \left\{ \max_{\Omega \times [0,1]} v, \frac{1}{2\beta} \max_{\Omega} \left\{ \max_{\Omega} (2a\alpha u(\cdot,1)+\chi|\nabla v(\cdot,1)|^2, \frac{a\alpha}{\beta} + \frac{\alpha\beta}{\mu-\frac{n\alpha}{2d_1\chi}}) \right\} \right\} \]

on \( \bar{\Omega} \times [0,\infty) \). In addition, for any \( \epsilon > 0 \), 
\[ |\nabla v| \leq \frac{1}{\chi} \max_{\Omega} \left\{ \max_{\Omega} (2a\alpha u(\cdot,\epsilon)+\chi|\nabla v(\cdot,\epsilon)|^2, \frac{a\alpha}{\beta} + \frac{\alpha\beta}{\mu-\frac{n\alpha}{2d_1\chi}}) \right\} \]
on \( \bar{\Omega} \times [\epsilon,\infty) \).

Moreover, if \( \|\nabla v_0\|_{L^\infty(\Omega)} < \infty \), then \( \epsilon \) can be chosen to be zero.

**Proof.** Taking the gradient first and then multiplying it by \( \nabla v \) in the \( v \)-equation of (1.3) and then using (2.4) and the fact that \( \alpha u > 0 \) for any solution \( u, v \) we then deduce from the logistic condition (4.22) by \( \chi \) yields (4.23)

\[ (2a\alpha + \chi|\nabla v|^2)_t = d_1 \Delta (2a\alpha + \chi|\nabla v|^2) - 2\beta |\nabla v|^2 + 2a \nabla u \nabla v. \] (4.22)

Multiplying the \( u \)-equation of (1.3) by \( 2\alpha \) and the equation (4.22) by \( \chi \) yields (4.23)

\[ (2a\alpha + \chi|\nabla v|^2)_t = d_1 \Delta (2a\alpha + \chi|\nabla v|^2) - 2\beta |\nabla v|^2 + 2a \nabla u \nabla v. \]

Notice from Cauchy-Schwarz inequality and (2.5) that 
\[ -2\alpha \chi u \Delta v \leq \frac{n\chi \alpha^2}{2d_1} u^2 + \frac{2d_1}{n} \chi |\Delta v|^2 \leq \frac{n\chi \alpha^2}{2d_1} u^2 + 2d_1 \chi |D^2 v|^2, \]
we then deduce from the logistic condition \( f(u) \leq a - \mu u^2 \) that

\[ (2a\alpha + \chi|\nabla v|^2)_t \leq \frac{(2a\alpha + \chi|\nabla v|^2) - 2\beta \chi |\nabla v|^2 + 2a \chi u \Delta v + 2a f(u)}{\mu - \frac{n\alpha}{4d_1 \chi}} u^2. \] (4.23)

Recall that, for a convex domain \( \Omega \), we have the well-known fact that \( \frac{\partial}{\partial \nu}(|\nabla v|^2) \leq 0 \) for any function satisfying \( \frac{\partial}{\partial \nu} = 0 \) on \( \partial \Omega \); see Matano \([31] \) Lemma 5.3 or \([53] \) or \([11] \). Consequently, by the homogeneous boundary conditions for \( u \) and \( v \), we get

\[ \frac{\partial}{\partial \nu} w := \frac{\partial}{\partial \nu} (2a\alpha + \chi|\nabla v|^2) = 2a \frac{\partial u}{\partial \nu} + 2\chi \frac{\partial}{\partial \nu} |\nabla v|^2 = 2\chi \frac{\partial}{\partial \nu} |\nabla v|^2 \leq 0 \] on \( \partial \Omega \).

Now, since \( \mu > \frac{n\alpha}{4d_1} \alpha \chi \), using elementary calculations we find that

\[ w_t - d_1 \Delta w + 2\beta w \leq M, \] (4.25)

where

\[ M = \max \left\{ 2a\alpha - 2a \left( \mu - \frac{n\alpha}{4d_1} \chi \right) u^2 + 4\alpha \beta u |u \geq 0 \right\} = 2a \alpha + \frac{2\alpha \beta^2}{\mu - \frac{n\alpha}{4d_1} \chi} < \infty. \]

Recall that from Lemma 2.2 that \( (u, v) \in C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \). So, we will perform a small time shift and treat any positive time as the “initial time”. Then, by (1.25) and (1.24), we conclude from the maximum principle and the Hopf boundary point lemma, for any \( \epsilon \in (0, T_{\text{max}}) \), that

\[ w = 2a\alpha + \chi|\nabla v|^2 \leq \max \left\{ \max_{\Omega} (2a\alpha(\cdot, \epsilon) + \chi|\nabla v(\cdot, \epsilon)|^2, \frac{M}{2\beta}) \right\} \text{ on } \bar{\Omega} \times [\epsilon, T_{\text{max}}]. \]
This gives the uniform boundedness for the coupled quantity $2\alpha u + \chi |\nabla v|^2$. The positiveness of $\chi$ then directly implies the $L^\infty$-boundedness of $u$ and $\nabla v$:

$$u \leq \max\left\{ \max_{\bar{\Omega} \times [0,\epsilon]} u, \frac{1}{2\alpha} \max_{\bar{\Omega}} \left\{ \max(2\alpha u(\cdot, \epsilon) + \chi |\nabla v(\cdot, \epsilon)|^2), \frac{M}{2\beta} \right\} \right\}$$

on $\bar{\Omega} \times [\epsilon, T_{\text{max}})$ and

$$|\nabla v| \leq \frac{1}{\chi} \max_{\bar{\Omega}} \left\{ \max(2\alpha u(\cdot, \epsilon) + \chi |\nabla v(\cdot, \epsilon)|^2), \frac{M}{2\beta} \right\}$$
on $\bar{\Omega} \times [\epsilon, T_{\text{max}})$.

These combined with the blow-up criterion (2.1) of Lemma 2.2 show that $(u, v)$ exists globally in time, i.e., $T_{\text{max}} = \infty$.

Finally, an application of maximum principle to the $v$-equation of (1.3) on the region $\bar{\Omega} \times [1, \infty)$ gives the bound for $v$. Indeed, $|\nabla v|$ is bounded if $|\nabla v_0|$ is. □

5. **Proof of the large time behavior for the KS model (1.5)**

In this section, we show the proof of Theorem 1.3 which relies on finding so-called Lyapounov functionals. Here, we will present all the necessary details for the clarity of obtaining the explicit convergence rates.

**Proof.** We modified the functional in [14] as

$$H(t) = \int_{\Omega} \left( u - \frac{\kappa}{\mu} - \frac{\kappa}{\mu} \ln\left( \frac{\mu}{\kappa u} \right) + \delta \int_{\Omega} (v - \frac{\alpha\kappa}{\beta\mu})^2 \right), \quad \delta = \frac{\kappa \chi^2}{8d_1 d_2 \mu}. \quad (5.1)$$

Differentiating $H$, using (1.8) and integrating by parts, we deduce from Cauchy-Schwarz inequality that
\[
\frac{d}{dt}H(t) = \int_{\Omega} \frac{u - \frac{\kappa}{\mu}}{u} u_t + 2\delta \int_{\Omega} (v - \frac{\alpha K}{\beta \mu}) v_t \\
+ 2\delta \int_{\Omega} (v - \frac{\alpha K}{\beta \mu}) (d_2 \Delta v - \beta v + \alpha u) \\
= \int_{\Omega} \frac{u - \frac{\kappa}{\mu}}{u} \left( \nabla \cdot (d_1 \nabla u - \chi u \nabla v) + u(\kappa - \mu u) \right) \\
- 2d_2\delta \int_{\Omega} |\nabla v|^2 + 2\delta \int_{\Omega} (v - \frac{\alpha K}{\beta \mu})(-\beta v + \alpha u) \\
= -\frac{\kappa d_1}{\mu} \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{\kappa}{\mu} \int_{\Omega} \frac{\nabla u}{u} \cdot \nabla v - \mu \int_{\Omega} (u - \frac{\kappa}{\mu})^2 \\
- 2d_2\delta \int_{\Omega} |\nabla v|^2 - 2\beta\delta \int_{\Omega} (v - \frac{\alpha K}{\beta \mu})^2 + 2\alpha\delta \int_{\Omega} (u - \frac{\alpha K}{\beta \mu})(v - \frac{\alpha K}{\beta \mu}) \\
\leq -2d_2(\delta - \frac{\kappa\chi^2}{8d_1d_2\mu}) \int_{\Omega} |\nabla v|^2 - (\mu - 2\alpha\delta) \int_{\Omega} (u - \frac{\kappa}{\mu})^2 \\
- 2\delta(\beta - \frac{\alpha}{4\epsilon}) \int_{\Omega} (v - \frac{\alpha K}{\beta \mu})^2 \\
= -(\mu - 2\alpha\delta) \int_{\Omega} (u - \frac{\kappa}{\mu})^2 - 2\delta(\beta - \frac{\alpha}{4\epsilon}) \int_{\Omega} (v - \frac{\alpha K}{\beta \mu})^2
\]

(5.2)

for any \( \epsilon > 0 \). Now, we wish to minimize \( \mu \) by choosing

\[
\left\{ \begin{array}{l}
\mu - 2\alpha\delta \epsilon > 0, \\
\beta - \frac{\alpha}{4\epsilon} > 0
\end{array} \right. \quad \iff \quad \left\{ \begin{array}{l}
\mu > \frac{\kappa\chi^2}{4d_1d_2\mu} \epsilon, \\
\frac{\alpha}{4\beta} < \epsilon < \frac{\kappa\chi^2}{4d_1d_2\mu} \alpha\chi^2
\end{array} \right. \quad \iff \quad \mu^2 > \frac{\kappa\alpha^2\chi^2}{16d_1d_2\beta}.
\]

(5.3)

The latter is guaranteed by our assumption (1.9). Now, for fixed \( \epsilon \) obeying (5.3), we set

\[
\eta = \min\{(\mu - 2\alpha\delta\epsilon), 2\delta(\beta - \frac{\alpha}{4\epsilon})\} = \min\{(\mu - \frac{\kappa\chi^2}{4d_1d_2\mu} \epsilon), \frac{\kappa\chi^2}{4d_1d_2\beta} (\beta - \frac{\alpha}{4\epsilon})\},
\]

and then we infer from (5.2), (5.3) and (5.4) that

\[
\frac{d}{dt}H(t) \leq -\eta \left( \int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha K}{\beta \mu})^2 \right).
\]

(5.5)

Since \( H(t) \geq 0 \), an integration of (5.5) from any \( t_0 \geq 0 \) to \( t \) yields

\[
\eta \int_{t_0}^t \left( \int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha K}{\beta \mu})^2 \right) \leq \frac{H(t_0)}{\eta}
\]

giving trivially

\[
\int_{t_0}^{\infty} \left( \int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha K}{\beta \mu})^2 \right) \leq \frac{H(t_0)}{\eta}.
\]

(5.6)
Thanks to Theorem 1.1, the condition $\mu > \mu_0$ ensures that $(u, v)$ is globally bounded and classical. Then from the parabolic regularity, we see that $\int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha \kappa}{\beta \mu})^2$ is uniformly bounded and uniformly continuous in $t$. This allows one to deduce from (5.6) that
\[
\int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha \kappa}{\beta \mu})^2 \to 0, \quad \text{as } t \to \infty. \tag{5.7}
\]
Since $u, v$ are smooth and bounded, the standard parabolic regularity for parabolic equations (cf. [25]) shows the existence of $\sigma \in (0, 1)$ and $C$ such that
\[
\|u\|_{C^{2+\sigma,1+\sigma}(\bar{\Omega} \times [t_1, t+1])}^2 + \|v\|_{C^{2+\sigma,1+\sigma}(\bar{\Omega} \times [t_1, t+1])}^2 \leq C, \quad \forall t \geq 1. \tag{5.8}
\]
Hence, in view of the Gagliardo-Nirenberg inequality, (5.7) and (5.8), we obtain
\[
\|u(\cdot, t) - \frac{\kappa}{\mu}\|_{L^\infty(\Omega)} \leq C_{GN} \|u(\cdot, t) - \frac{\kappa}{\mu}\|_{W^{1,\infty}(\Omega)}^2 \|u - \frac{\kappa}{\mu}\|_{L^2(\Omega)}^2 \leq C \|u(\cdot, t) - \frac{\kappa}{\mu}\|_{L^2(\Omega)}^2 \to 0, \quad \text{as } t \to \infty. \tag{5.9}
\]
In the same way, we get
\[
\|v(\cdot, t) - \frac{\alpha \kappa}{\beta \mu}\|_{L^\infty(\Omega)} \leq C \|v(\cdot, t) - \frac{\alpha \kappa}{\beta \mu}\|_{L^2(\Omega)}^2 \to 0, \quad \text{as } t \to \infty. \tag{5.10}
\]
Based on the definition of $H$ in (5.1), (5.9) and (5.5), we calculate via the L’Hospital rule that
\[
\lim_{u \to \frac{\kappa}{\mu}} \frac{u - \frac{\kappa}{\mu} - \frac{\kappa}{\mu} \ln \left( \frac{\mu}{\kappa} u \right)}{(u - \frac{\kappa}{\mu})^2} = \frac{\mu}{2\kappa}.
\]
This together with (5.5) allows one to find $t_1 \geq 0$ such that
\[
\frac{\mu}{4\kappa}(u - \frac{\kappa}{\mu})^2 \leq u - \frac{\kappa}{\mu} - \frac{\kappa}{\mu} \ln \left( \frac{\mu}{\kappa} u \right) \leq \frac{\mu}{\kappa}(u - \frac{\kappa}{\mu})^2, \quad t \geq t_1,
\]
and then (5.1) entails
\[
\min \left\{ \frac{\mu}{4\kappa}, \delta \right\} \left( \int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha \kappa}{\beta \mu})^2 \right) \leq H(t), \quad t \geq t_1 \tag{5.11}
\]
and
\[
H(t) \leq \max \left\{ \frac{\mu}{\kappa}, \delta \right\} \left( \int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha \kappa}{\beta \mu})^2 \right), \quad t \geq t_1. \tag{5.12}
\]
Combining (5.5) and (5.12), we obtain an ordinary differential inequality for $H$:
\[
\frac{d}{dt} H(t) \leq - \frac{\eta}{\max \left\{ \frac{\mu}{\kappa}, \delta \right\}} H(t), \quad t \geq t_1,
\]
directly yielding
\[
H(t) \leq H(t_1)e^{-\frac{\eta}{\max \left\{ \frac{\mu}{\kappa}, \delta \right\}}(t-t_1)}, \quad t \geq t_1.
\]
This in conjunction with (5.12) concludes that
\[
\int_{\Omega} (u - \frac{\kappa}{\mu})^2 + \int_{\Omega} (v - \frac{\alpha \kappa}{\beta \mu})^2 \leq \frac{H(t_1)}{\min\{\mu, \delta\}} e^{-\frac{\eta}{n+1} (t-t_1)}, \quad t \geq t_1.
\]
With this decay estimate at hand, we then derive from (5.9) and (5.10) that there exists a large constant $C > 0$ such that
\[
\|u(\cdot, t) - \frac{\kappa}{\mu} \|_{L^\infty(\Omega)} + \|v(\cdot, t) - \frac{\alpha \kappa}{\beta \mu} \|_{L^\infty(\Omega)} \leq C e^{-\frac{\eta}{n+1} (t-t_1)}, \quad t \geq t_1.
\]
Substituting into the definitions of $\delta$ in (5.1), $\eta$ in (5.4) and taking $\epsilon = \epsilon_0$ in (5.3), we obtain the exponential decay estimate (1.10) by choosing large constant $C$.

In the case of $\kappa = 0$, the uniform boundedness and global existence of solution does not affect as long as $\mu > \mu_0$. We integrate the first equation in (1.8) and use Holder inequality to obtain
\[
\frac{d}{dt} \int_{\Omega} u = -\mu \int_{\Omega} u^2 \leq -\mu |\Omega|^{-1} \left( \int_{\Omega} u \right)^2, \quad t > 0,
\]
which entails
\[
\int_{\Omega} u \leq \left[ \left( \int_{\Omega} u_0 \right)^{-1} + \mu |\Omega|^{-1} t \right]^{-1} \leq \frac{c_1}{t+1}, \quad t > 0.
\]
Hence, the Gagliardo-Nirenberg inequality coupled with the boundedness of $u$ shows that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{GN} \|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} \|u(\cdot, t)\|_{L^1(\Omega)} \leq C \left[ \left( \int_{\Omega} u_0 \right)^{-1} + \mu |\Omega|^{-1} t \right]^{-1/n+1}, \quad t > 0.
\]
An integration of the second equation in (1.8) shows
\[
\frac{d}{dt} \int_{\Omega} v = -\beta \int_{\Omega} v + \alpha \int_{\Omega} u \leq -\beta \int_{\Omega} v + \frac{c_2}{t+1}.
\]
Solving this Gronwall inequality shows
\[
\int_{\Omega} v \leq \|v_0\|_{L^1} e^{-\beta t} + c_2 \int_0^t \frac{e^{\beta s}}{s+1} \leq \|v_0\|_{L^1} e^{-\beta t} + \frac{c_3}{t+1} \leq \frac{c_4}{t+1},
\]
where we used the fact that
\[
\lim_{t \to \infty} \frac{(t+1)^{\frac{\beta}{s+1}}}{e^{\beta t}} = 1 < \infty, \quad \lim_{t \to \infty} (t+1)e^{-\beta t} = 0.
\]
Then we conclude from (5.14) with $u$ replaced by $v$ that, $t > 0$,
\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{c_4}{(t+1)^{n+1}}.
\]
In the case of $\kappa < 0$, we integrate the first equation in (1.8) to get
\[
\frac{d}{dt} \int_{\Omega} u = \kappa \int_{\Omega} u - \mu \int_{\Omega} u^2 \leq \kappa \int_{\Omega} u, \quad t > 0,
\]
and thus
\[ \int_{\Omega} u \leq e^{\kappa t} \int_{\Omega} u_0, \quad t > 0. \]  
(5.17)

Then the GN inequality (5.14) implies
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_5 e^{\kappa t}, \quad t > 0. \]  
(5.18)

Combining (5.17) and (5.15), we have
\[ \|v(\cdot, t)\|_{L^1} \leq \left\{ \begin{array}{ll}
\|v_0\|_{L^1} e^{-\beta t} + c_6 e^{-\beta t}, & \text{if } \beta = -\kappa, \\
\|v_0\|_{L^1} e^{-\beta t} + c_7 e^{\kappa t - e^{-\beta t}}, & \text{if } \beta \neq -\kappa.
\end{array} \right. \]

With this decay estimate at hand, the GN inequality (5.14) gives rise to
\[ \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq c_9 e^{-\frac{1}{2} \min\{\beta, -\kappa\} t}, \quad \forall t \geq 0. \]  
(5.19)

Extracting the essential ingredients of the estimates (5.14), (5.16), (5.18) and (5.19), we readily conclude the decay estimates (1.10), (1.11) and (1.12). □

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